REGRESSION ADJUSTMENT IN RANDOMIZED EXPERIMENTS WITH A DIVERGING NUMBER OF COVARIATES

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Extending R. A. Fisher and D. A. Freedman’s results on the analysis of covariance, Lin [2013] proposed an ordinary least squares adjusted estimator of the average treatment effect in completely randomized experiments. His results are appealing because he uses Neyman’s randomization model without imposing any parametric assumptions, and the consistency and asymptotic normality of his estimator hold even if the linear model is misspecified. His results hold for a fixed dimension $p$ of the covariates with the sample size $n$ approaching infinity. However, it is common for practitioners to adjust for a large number of covariates to improve efficiency. Therefore, it is important to consider asymptotic regimes allowing for a diverging number of covariates. We show that Lin [2013]’s estimator is consistent when $\kappa \log p \to 0$ and asymptotically normal when $\kappa p \to 0$ under mild moment conditions, where $\kappa$ is the maximum leverage score of the covariate matrix. In the favorable case where leverage scores are all close, his estimator is consistent when $p = o(n/\log n)$ and is asymptotically normal when $p = o(n^{1/2})$. In addition, we propose a bias-corrected estimator that is consistent when $\kappa \log p \to 0$ and is asymptotically normal, with the same variance in the fixed-$p$ regime, when $\kappa^2 p \log p \to 0$. In the favorable case, the latter condition reduces to $p = o(n^{2/3}/(\log n)^{1/3})$. Our simulation confirms that $n^{2/3}$ is the phase-transition threshold for the confidence intervals based on Lin [2013]’s and our estimators. Similar to Lin [2013], we do not assume that the linear model is correct. Our analysis requires novel analytic tools for sampling without replacement, which complement and potentially enrich the theory in other areas such as survey sampling, matrix sketching, and transductive learning.

1. Introduction.

1.1. Potential outcomes and Neyman’s randomization model. We use potential outcomes to define causal effects [Neyman, 1923/1990]. Let $Y_i(1)$ and $Y_i(0)$ be the potential outcomes if unit $i \in \{1, \ldots, n\}$ receives the treatment...
and control, respectively. Neyman [1923/1990] treated all the potential outcomes as fixed quantities, and defined the average treatment effect (ATE) as \( \tau \triangleq n^{-1} \sum_{i=1}^{n} \tau_i \), where \( \tau_i = Y_i(1) - Y_i(0) \) is the individual treatment effect for unit \( i \). In a completely randomized experiment, the experimenter randomly assigns \( n_1 \) units to the treatment group and \( n_0 \) units to the control group, with \( n = n_1 + n_0 \). Let \( T_i \) denote the assignment of the \( i \)-th unit where \( T_i = 1 \) corresponds to the treatment and \( T_i = 0 \) corresponds to the control. For unit \( i \), only \( Y_{i}^{\text{obs}} = Y_i(T_i) \) is observed while the other potential outcome \( Y_i(1-T_i) \) is missing. Although \((Y_i(1), Y_i(0))_{i=1}^{n}\) are fixed, the \( Y_{i}^{\text{obs}} \)'s are random due to the randomization of the \( T_i \)'s.

Scheffé [1959, Chapter 9] called the above formulation the randomization model, under which all potential outcomes are fixed and the randomness comes solely from the treatment indicators. This is a finite-population perspective, which has been the gold standard for analyzing randomized experiments [Neyman, 1923/1990, 1935, Kempthorne, 1952, Imbens and Rubin, 2015, Mukerjee et al., 2018, Fogarty, 2018, Middleton, 2018]. In contrast, the super-population perspective assumes that the potential outcomes and other individual characteristics are independent and identically distributed (i.i.d.) draws from some distribution. Conditional on the potential outcomes, super-population inferences reduce to finite-population inferences. In general, finite-population inferences are theoretically more challenging than the super-population analogues. Finite-population inferences are still understudied because of the lack of analytic tools (e.g., central limit theorems and concentration inequalities for sampling without replacement).

We use the conventional notation \( O(\cdot), o(\cdot), O_p(\cdot) \) and \( o_p(\cdot) \). Let \( \mathbf{1} \) denote the vector with all entries 1, \( \mathbf{I} \) denote an identity matrix, and \( \mathbf{V} = \mathbf{I} - \left( \mathbf{1}^T \mathbf{1} \right)^{-1} \mathbf{1} \mathbf{1}^T \) denote the projection matrix orthogonal to \( \mathbf{1} \), with appropriate dimensions depending on the context. Let \( \| \cdot \|_q \) be the vector \( q \)-norm, i.e. \( \| \alpha \|_q = \left( \sum_{i=1}^{n} |\alpha_i|^q \right)^{1/q} \) and \( \| \alpha \|_{\infty} = \max_{1 \leq i \leq n} |\alpha_i| \). Let \( \| \cdot \|_\text{op} \) denote operator norm and \( \| \cdot \|_F \) denote the Frobenius norm of matrices. Let \( N(0,1) \) denote the standard normal distribution, and \( t(\nu) \) denote standard \( t \) distribution with degrees of freedom \( \nu \) with \( t(1) \) being the standard Cauchy distribution. Let \( \overset{d}{\rightarrow}, \overset{p}{\rightarrow}, \overset{a.s.}{\rightarrow} \) denote convergences in distribution, in probability, and almost surely, respectively.

1.2. Regression-adjusted average treatment effect estimates. Let \( \mathcal{T}_t = \{i : T_i = t\} \) be the indices and \( n_t = |\mathcal{T}_t| \) be the fixed sample size for treatment arm \( t \in \{0,1\} \). We consider a completely randomized experiment in which \( \mathcal{T}_t \) is a random size-\( n_t \) subset of \( \{1, \ldots, n\} \) uniformly over all \( \binom{n}{n_t} \) subsets.
The simple difference-in-means estimator

$$\hat{\tau}_{\text{unadj}} = \frac{1}{n_1} \sum_{i \in T_1} Y_{i}^{\text{obs}} - \frac{1}{n_0} \sum_{i \in T_0} Y_{i}^{\text{obs}} = \frac{1}{n_1} \sum_{i \in T_1} Y_i(1) - \frac{1}{n_0} \sum_{i \in T_0} Y_i(0)$$

is unbiased with variance $S^2_1/n_1 + S^2_0/n_0 - S^2_{\tau}/n$ [Neyman, 1923/1990], where $S^2_1, S^2_0$ and $S^2_{\tau}$ are the finite-population variances of the $Y_i(1)$’s, $Y_i(0)$’s and $\tau_i$’s, respectively.

The experimenter usually collects pretreatment covariates. If the covariates are predictive to the potential outcomes, it is intuitive to incorporate them in the analysis to improve the estimation efficiency. Suppose unit $i$ has a $p$-dimensional vector of pre-treatment covariates $x_i \in \mathbb{R}^p$. Early works on the analysis of covariance assumed constant treatment effects [Fisher, 1935, Kempthorne, 1952, Hinkelmann and Kempthorne, 2007], under which a commonly-used treatment effect estimate is the coefficient of the treatment indicator of the ordinary least squares (OLS) fit of the $Y_i^{\text{obs}}$’s on $T_i$’s and $x_i$’s. Freedman [2008] criticized this standard approach, showing that (a) it can be even less efficient than $\hat{\tau}_{\text{unadj}}$ in the presence of treatment effect heterogeneity, and (b) the estimated standard error based on the OLS can be inconsistent for the true standard error under the randomization model.

Lin [2013] proposes a simple solution. Without loss of generality, we center the covariates at $n^{-1} \sum_{i=1}^n x_i = 0$ because otherwise we can replace $x_i$ by $x_i - n^{-1} \sum_{i=1}^n x_i$. His estimator for the ATE is the coefficient of the treatment indicator in the OLS fit of the $Y_i^{\text{obs}}$’s on $T_i$’s, $x_i$’s and the interaction terms $T_i x_i$’s. He further shows that the Huber–White standard error [MacKinnon, 2013] is consistent for the true standard error. Lin [2013]’s results hold under the finite-population randomization model, without assuming that the linear model is correct.

We use an alternative formulation of regression adjustment and consider the following family of covariate-adjusted ATE estimator:

\begin{equation}
\hat{\tau}(\beta_1, \beta_0) = \frac{1}{n_1} \sum_{i \in T_1} (Y_i(1) - x_i^T \beta_1) - \frac{1}{n_0} \sum_{i \in T_0} (Y_i(0) - x_i^T \beta_0).
\end{equation}

Because $E(n_t^{-1} \sum_{i \in T_t} x_i^T \beta_t) = 0$, the estimator in (1) is unbiased for any fixed coefficient vectors $\beta_t \in \mathbb{R}^p$ ($t = 0, 1$). It is the difference-in-means estimator with potential outcomes replaced by $(Y_i(1) - x_i^T \beta_1, Y_i(0) - x_i^T \beta_0)_{i=1}^n$.

Let $Y(t) = (Y_1(t), \ldots, Y_n(t))^T \in \mathbb{R}^n$ denote the vector of potential outcomes under treatment $t$ ($t = 0, 1$), $X = (x_1, \ldots, x_n)^T$ denote the matrix of covariates. Without loss of generality, we assume

\begin{equation}
1^T X = 0 \text{ and } \text{rank}(X) = p,
\end{equation}
i.e., the covariate matrix has centered columns and full column rank. Otherwise, we transform $X$ to $VX$ and remove the redundant columns to ensure the full column rank condition. Let $\beta_t$ be the population OLS coefficient of regressing $Y(t)$ on $X$ with an intercept:

$$
(\mu_t, \beta_t) = \arg\min_{\mu \in \mathbb{R}, \beta \in \mathbb{R}^p} \|Y(t) - \mu 1 - X\beta\|^2_2
$$

(3)

$$
= \left( \frac{1}{n} \sum_{i=1}^{n} Y_i(t), (X^T X)^{-1} X^T Y(t) \right),
$$

(4)

where (4) holds because $X$ is orthogonal to $1$. Li and Ding [2017, Example 9] show that the OLS coefficients $(\beta_1, \beta_0)$ in (3) minimize the variance of the estimator defined in (1).

The classical analysis of covariance chooses $\beta_1 = \beta_0 = \hat{\beta}$, the coefficient of the covariates in the OLS fit of the $Y_i^{obs}$’s on $T_i$’s and $x_i$’s with an intercept. This strategy implicitly assumes away treatment effect heterogeneity, and can lead to inferior properties when $\beta_1 \neq \beta_0$ [Freedman, 2008]. Lin [2013] chooses $\beta_1 = \hat{\beta}_1$ and $\beta_0 = \hat{\beta}_0$, the coefficients of the covariates in the OLS fit of $Y_i^{obs}$’s on $x_i$’s with an intercept, in the treatment and control groups, respectively. Numerically, this is identical to the estimator obtained from the regression with interactions discussed before. Ignoring the sampling variability of $\hat{\beta}_1$ and $\hat{\beta}_0$, Li and Ding [2017, Example 9] show that Lin [2013]’s estimator is optimal within the family of estimators defined in (1).

1.3. Our contributions. In practice, it is common to have many covariates. A better asymptotic approximation of the sampling distribution is to allow the dimension of covariates $p$ to grow with the sample size $n$ at certain rate. Under the finite-population randomization model, Bloniarz et al. [2016] discussed a high dimensional regime with possibly larger $p$ than $n$ but assumed that the potential outcomes could be well approximated by a sparse linear combination of the covariates, under the ultra sparse regime (termed, for example, by Cai and Guo [2017]) where the number of non-zero coefficients is many fewer than $n^{1/2}/\log p$. Under a super-population framework, Wager et al. [2016] discussed covariate adjustment using the OLS and some other machine learning techniques assuming linear models for both potential outcomes.

We study Lin [2013]’s estimator under the finite-population perspective in the regime where $p < n$ but $p$ grows with $n$ at certain rate. We focus on this estimator because (a) it is widely used in practice because of its simplicity, and (b) it does not require any tuning parameter unlike other high dimensional or machine learning methods. As in the classic linear regression,
asymptotic properties depend crucially on the maximum leverage score $\kappa = \max_{1 \leq i \leq n} H_{ii}$, where the $i$-the leverage score $H_{ii}$ is $i$-th diagonal entry of the hat matrix $H = X(X^TX)^{-1}X^T$ [Huber, 1973]. To keep the statements clean, we consider the regime $\kappa \log p \to 0$. Under this regime, we prove the consistency of Lin [2013]'s estimator under mild moment conditions on the population OLS residuals. In the favorable case where all leverage scores are close to their average, the consistency holds if $p = o(n/\log n)$.

In addition, we prove that Lin [2013]'s estimator is asymptotically normal under $\kappa p \to 0$ and extra mild conditions, with the same variance formula as the fixed-$p$ regime. Furthermore, we proposed a debiased estimator, which is asymptotically normal under an even weaker assumption $\kappa^2 p \log p \to 0$, with the same variance as before. In the favorable case where all leverage scores are close to their average, Lin [2013]'s estimator is asymptotically normal when $p = o(n^{1/2})$, but the debiased estimator is asymptotically normal when $p = o(n^{2/3}/(\log n)^{1/3})$. Lin [2013]'s estimator can also be asymptotically normal in the latter regime, but it requires an extra condition.

More interestingly, our simulation shows that the condition $p = o(n^{2/3})$ is the intrinsic theoretical limit of the confidence intervals based on Lin [2013]'s and our debiased estimators. It is not an artifact of our proof. We present a simple experiment to illustrate the phenomenon. We generate $X \in \mathbb{R}^{n \times n}$ with i.i.d. $N(0,1)$ entries and generate $Y^{(1)}, Y^{(0)} \in \mathbb{R}^n$ with i.i.d. entries from $N(0,1)$ or $t(2)$ or the standard Cauchy distribution. In the following steps, we keep them fixed. For each exponent $\gamma \in \{0, 1/60, \ldots, 50/60\}$, let $p = \lceil n^{\gamma} \rceil$ and take the first $p$ columns of $X$ as the covariate matrix. We randomly select $n/2$ units into the treatment group and calculate Lin [2013]'s and the debiased estimator. We repeat it using 1000 random seeds and normalize the estimates into z-scores using the asymptotic standard deviation derived by Lin [2013] based on the true $Y^{(1)}$ and $Y^{(0)}$. Finally, we estimate the empirical 95% coverage by the proportion of the 1000 z-scores in $[-1.96, 1.96]$. Figure 1 reports the results for the debiased estimator with $n \in \{2^9, 2^{10}, 2^{11}, 2^{12}, 2^{13}\}$, omitting the almost identical results for Lin [2013]'s estimator. The coverage drops significantly when $\gamma \geq 2/3$ regardless of the distribution of potential outcomes. Appendix F.1 in Supplementary Material III shows more simulation.

For statistical inference, we propose several asymptotically conservative variance estimators, which yield valid asymptotic confidence intervals for the ATE. We prove the results under the same regime $\kappa \log p \to 0$ with the same conditions as required for asymptotic normality.

Importantly, our result allows the second moment of the potential outcomes to diverge at a fast rate. In particular, our asymptotic normality and
Fig 1: Empirical 95% coverage of the debiased estimator over 1000 replicates. Columns correspond to sample sizes from \(2^9\) to \(2^{13}\). Rows correspond to different distributions of potential outcomes.

confidence interval require only a Lindeberg–Feller-type condition, without any other strong moment conditions [e.g., Lin, 2013, Bloniarz et al., 2016]. Technically, we prove novel vector and matrix concentration inequalities for sampling without replacement. These tools are particularly useful for finite population causal inference, and can also complement and potentially enrich the theory in other areas such as survey sampling [e.g., Cochran, 2007], matrix sketching [e.g., Woodruff et al., 2014] and transductive learning [e.g., El-Yaniv and Pechyony, 2009].

2. Regression Adjustment.

2.1. Point Estimators. We reformulate Lin [2013]'s estimator. The ATE is the difference between the two intercepts of the population OLS coefficients in (4):

\[
\tau = \frac{1}{n} \sum_{i=1}^{n} Y_i(1) - \frac{1}{n} \sum_{i=1}^{n} Y_i(0) = \mu_1 - \mu_0.
\]

Therefore, we focus on estimating \(\mu_1\) and \(\mu_0\). Let \(X_t \in \mathbb{R}^{n_t \times p}\) denote the sub-matrix formed by the rows of \(X\), and \(Y_{t}^{\text{obs}} \in \mathbb{R}^{n_t}\) the subvector of \(Y^{\text{obs}} = (Y_1^{\text{obs}}, \ldots, Y_n^{\text{obs}})^T\), with indices in \(T_t\) (\(t = 0, 1\)). The regression-adjusted estimator follows two steps. First, for \(t \in \{0, 1\}\), we regress \(Y_t^{\text{obs}}\) on
\( X_t \) with an intercept, and obtain the fitted intercept \( \hat{\mu}_t \in \mathbb{R} \) and coefficient of the covariate \( \hat{\beta}_t \in \mathbb{R}^p \). Second, we estimate \( \tau \) by

\[
\hat{\tau}_{\text{adj}} = \hat{\mu}_1 - \hat{\mu}_0.
\]

In general, \( \hat{\tau}_{\text{adj}} \) is biased in finite samples. Correcting the bias gives stronger theoretical guarantees as our later asymptotic analysis suggests. Here we give an explicit formula of the bias-corrected estimator. Define the potential residuals based on the population OLS as

\[
e(t) = Y(t) - \mu_t - X_t \beta_t, \quad (t = 0, 1).
\]

The property of the OLS guarantees that \( e(t) \) is orthogonal to \( 1 \) and \( X \):

\[
1^T e(t) = 0, \quad X^T e(t) = 0, \quad (t = 0, 1).
\]

Let \( \hat{e} \in \mathbb{R}^n \) be the vector residuals from the sample OLS:

\[
\hat{e}_i = \begin{cases} 
Y_{i}^{\text{obs}} - \hat{\mu}_1 - x_{i}^T \hat{\beta}_1, & (i \in \mathcal{T}_1), \\
Y_{i}^{\text{obs}} - \hat{\mu}_0 - x_{i}^T \hat{\beta}_0, & (i \in \mathcal{T}_0).
\end{cases}
\]

For any vector \( \alpha \in \mathbb{R}^n \), let \( \alpha_t \) denote the subvector of \( \alpha \) with indices in \( \mathcal{T}_t \) (e.g. \( Y_t(1), e_t(1), \hat{e}_t, \) etc.).

Let \( H = X(X^T X)^{-1}X^T \) be the hat matrix of \( X \), and \( H_t = X_t(X_t^T X_t)^{-1}X_t^T \) be the hat matrix of \( X_t \). Let \( H_{ii} \) be the \( i \)-th diagonal element of \( H \), also termed as the leverage score, which measures the distance between the \( i \)-th observation and other observations. Define

\[
\Delta_t = \frac{1}{n_t} \sum_{i=1}^{n_t} e_i(t) H_{ii}, \quad \tilde{\Delta}_t = \frac{1}{n_t} \sum_{i \in \mathcal{T}_t} \hat{e}_i H_{ii}.
\]

We introduce the following debiased estimator:

\[
\hat{\tau}_{\text{adj}}^{\text{de}} = \hat{\tau}_{\text{adj}} - \left( \frac{n_1}{n_0} \hat{\Delta}_0 - \frac{n_0}{n_1} \hat{\Delta}_1 \right).
\]

The form of the debiased estimator (10) seems mysterious at this moment, but it will be clearer in our theoretical analysis in the next section. In finite-population inference, Cochran [2007] discussed bias-correction for regression estimator in survey sampling, and Lin [2013] discussed the analogue in causal inference. Both focused on univariate covariate and finite sample properties of estimators. In contrast, we discuss more general bias correction and prove better asymptotic guarantees with a diverging \( p \).
2.2. Variance estimators. For fixed $p$, Lin [2013] proved that $n^{1/2}(\hat{\tau}_{\text{adj}} - \tau)$ is asymptotically normal with variance

\begin{equation}
\sigma^2_n = \frac{1}{n_1} \sum_{i=1}^{n_1} e_i^2(1) + \frac{1}{n_0} \sum_{i=1}^{n_0} e_i^2(0) - \frac{1}{n} \sum_{i=1}^{n} (e_i(1) - e_i(0))^2
\end{equation}

\begin{equation}
= \sum_{i=1}^{n} \left( \sqrt{\frac{n_0}{n_1n}} e_i(1) + \sqrt{\frac{n_1}{n_0n}} e_i(0) \right)^2.
\end{equation}

The second form (12) follows from some simple algebra and shows that $\sigma^2_n$ is always non-negative. The first form (11) motivates conservative variance estimators. The third term in (11) has no consistent estimator without further assumptions on $e(1)$ and $e(0)$. Ignoring it and estimating the first two terms in (11) by their sample analogues, we have the following variance estimator:

\begin{equation}
\hat{\sigma}^2 = \frac{n}{n_1(n_1 - 1)} \sum_{i \in T_1} \hat{e}^2_i + \frac{n}{n_0(n_0 - 1)} \sum_{i \in T_0} \hat{e}^2_i.
\end{equation}

Although (13) appears to be conservative due to the neglect of the third term in (12), we find in numerical experiments that it typically underestimates $\sigma^2_n$ in the cases beyond our theoretic limit with many covariates or many influential observations. The classic linear regression literature suggests rescaling the residual as

\begin{equation}
\tilde{e}_i = \begin{cases} 
\hat{e}_i & \text{(HC0)} \\
\sqrt{\frac{n-1}{n-p}} \hat{e}_i & \text{(HC1 correction)} \\
\sqrt{1-H_{t,ii}} \hat{e}_i & \text{(HC2 correction)} \\
\frac{\hat{e}_i}{1-H_{t,ii}} & \text{(HC3 correction)}
\end{cases}, \quad (i \in T_t)
\end{equation}

where $H_{t,ii}$ is the diagonal element of $H_t$ corresponding to unit $i$. HC0 corresponds to the estimator (13) without corrections. Previous literature has shown that the above corrections, especially HC3, are effective in improving the finite sample performance of the Huber–White variance estimator in linear regression under independent super-population sampling [e.g. MacKinnon, 2013]. More interestingly, it is also beneficial to borrow these HC $j$’s to the context of a completely randomized experiment. This motivates the following variance estimators

\begin{equation}
\hat{\sigma}^2_{HCj} = \frac{n}{n_1(n_1 - 1)} \sum_{i \in T_1} \tilde{e}^2_{i,j} + \frac{n}{n_0(n_0 - 1)} \sum_{i \in T_0} \tilde{e}^2_{i,j}
\end{equation}

where $\tilde{e}_{i,j}$ is the residual in (14) with $j$ corresponding to HC $j$ for $j = 0, 1, 2, 3$. 


Based on normal approximations, we can construct Wald-type confidence intervals for the ATE based on point estimators \( \hat{\tau}_{\text{adj}} \) and \( \hat{\tau}_{\text{de}} \) with estimated standard errors \( \hat{\sigma}_{HC,j} \).

3. Main Results.

3.1. Regularity conditions. We embed the finite population of quantities \( \{(x_i, Y_i(1), Y_i(0))\}_{i=1}^n \) into a sequence, and invoke regularity conditions on this sequence. The first condition is on the sample sizes.

**Assumption 1.** \( n/n_1 = O(1) \) and \( n/n_0 = O(1) \).

Assumption 1 holds automatically if treatment and control groups have fixed proportions (e.g., \( n_1/n = n_0/n = 1/2 \) for balanced experiments). It is not essential and can be removed at the cost of complicating the statements.

The second condition is on \( \kappa = \max_{1 \leq i \leq n} H_{ii} \), the maximum leverage score, which also plays a crucial role in classic linear models [e.g. Huber, 1973, Mammen, 1989, Donoho and Huo, 2001].

**Assumption 2.** \( \kappa \log p = o(1) \).

The maximum leverage score satisfies

\[
\frac{p}{n} = \frac{\text{tr}(H)}{n} \leq \kappa \leq \|H\|_{op} = 1 \implies \kappa \in [p/n, 1].
\]

Assumption 2 permits influential observations as long as \( \kappa = o(1/\log p) \). In the favorable case where \( \kappa = O(p/n) \), it reduces to \( p \log p/n \to 0 \), which permits \( p \) to grow almost linearly with \( n \). Moreover, it implies

\[
\frac{p}{n} \leq \kappa = o \left( \frac{1}{\log p} \right) = o(1) \implies p = o(n).
\]

Assumptions 1 and 2 are useful for establishing consistency. The following two extra conditions are useful for variance estimation and asymptotic normality. The third condition is on the correlation between the potential residuals from the population OLS in (6).

**Assumption 3.** There exist a constant \( \eta > 0 \) independent of \( n \) such that

\[
\rho_e \triangleq \frac{e(1)^T e(0)}{\|e(1)\|_2\|e(0)\|_2} > -1 + \eta.
\]
Assumption 3 is rather mild because it is unlikely to have the perfect negative correlation between the treatment and control residual potential outcomes in practice. We will show in Lemma 6.6 that Assumption 3 implies that \( \sigma_n^2 \) is not too small to avoid super-efficiency.

The fourth condition is on the following two measures of the potential residuals based on the population OLS in (6).

\[
\mathcal{E}_2 = n^{-1} \max \left\{ \|e(0)\|_2^2, \|e(1)\|_2^2 \right\}, \quad \mathcal{E}_\infty = \max \left\{ \|e(0)\|_\infty, \|e(1)\|_\infty \right\}.
\]

**Assumption 4.** \( \mathcal{E}_\infty^2 / (n \mathcal{E}_2) = o(1) \).

Assumption 4 is a Lindeberg–Feller-type condition requiring no single residual dominates the vector of residuals. A similar form appeared in Hájek [1960]'s finite population central limit theorem. Unlike Assumption 4, previous works usually require more stringent assumptions on the fourth moment [Lin, 2013, Bloniarz et al., 2016].

Although the above assumptions are about fixed quantities in the finite population, it is insightful to consider the case where the quantities are realizations of random variables. This approach is powerful to justify assumptions because it connects them to more comprehensible conditions on the data generating process. See Portnoy [1984, 1985], Lei et al. [2018] for examples in other contexts.

For Assumption 2, we consider the case where \((x_i)_{i=1}^n\) are realizations of i.i.d. random vectors. Anatolyev and Yaskov [2017] show that under mild conditions each leverage score concentrates around \(p/n\). Here we further consider the magnitude of the maximum leverage score \(\kappa\).

**Proposition 3.1.** Let \(Z_i\) be i.i.d. random vectors in \(\mathbb{R}^p\) with arbitrary mean. Assume that \(Z_i\) has independent entries with \(\max_{1 \leq j \leq p} \mathbb{E}|Z_{ij} - \mathbb{E}Z_{ij}|^\delta \leq M = O(1)\) for some \(\delta > 2\). Define \(Z = (Z_1^T, \ldots, Z_n^T)^T \in \mathbb{R}^{n \times p}\) and \(X = VZ\) so that \(X\) has centered columns. If \(p = O(n^\gamma)\) for some \(\gamma < 1\), then over the randomness of \(Z\),

\[
\max_{1 \leq i \leq n} \left| H_{ii} - \frac{p}{n} \right| = O_p \left( \frac{p^2/\min\{\delta, 4\}}{n(\delta-2)/\delta} + \frac{p^{3/2}}{n^{3/2}} \right), \quad \kappa = O_p \left( \frac{p}{n} + \frac{p^{2/\min\{\delta, 4\}}}{n^{(\delta-2)/\delta}} \right).
\]

When \(\delta > 4\), Proposition 3.1 implies that \(\kappa = O_p(p/n + n^{-(\delta-4)/2\delta}(p/n)^{1/2})\). In this case, Assumption 2 holds with high probability if \(p = O(n^\gamma)\) for any \(\gamma < 1\). In particular, the fixed-p regime corresponds to \(\gamma = 0\).

The hat matrix of \(X\) is invariant to any nonsingular linear transformation of the columns. In other words, \(X\) and \(XA\) have the same leverage scores
for any invertible $A \in \mathbb{R}^{n \times n}$. Thus we can extend Proposition 3.1 to random matrices with correlated columns in the form of $VZA$. In particular, when $Z_i \overset{i.i.d.}{\sim} \mathcal{N}(\mu, \mathbf{I})$ and $A = \Sigma^{1/2}$, $Z_iA \overset{i.i.d.}{\sim} \mathcal{N}(\Sigma^{1/2}\mu, \Sigma)$. The previous argument implies that Proposition 3.1 holds for $X = VZA$. We will revisit Proposition 3.1 when imposing further conditions on the $H_{ii}$’s and $\kappa$.

For Assumption 4, we consider the case where the $Y_i(t)$’s are realizations of i.i.d. random variables, and make a connection with the usual moment conditions. This helps to understand the growth rates of $\mathcal{E}_2$ and $\mathcal{E}_\infty$.

**Proposition 3.2.** Let $Y(t) \in \mathbb{R}^n$ be a non-constant random vector with i.i.d. entries, and $X$ be any fixed matrix with centered columns. If for some $\delta > 0$, $\mathbb{E}|Y_i(t) - \mathbb{E}Y_i(t)|^\delta < \infty$ for $t = 0, 1$, then

$$
\mathcal{E}_2 = \begin{cases} 
O_p(1) & (\delta \geq 2) \\
O_p(n^{2/\delta - 1}) & (\delta < 2)
\end{cases}, \quad \mathcal{E}_\infty = O_p(n^{1/\delta}).
$$

Furthermore, $\mathcal{E}_2^{-1} = O_p(1)$ if $Y_i(1)$ or $Y_i(0)$ is not a constant.

When $\delta > 2$, Proposition 3.2 implies $\mathcal{E}_\infty^2/(n\mathcal{E}_2) = O_p(n^{2/\delta - 1}) = o_p(1)$, and thus Assumption 4 holds with high probability. We will revisit Proposition 3.2 for the consistency of $\hat{\tau}_{adj}$ and $\hat{\tau}_{de adj}$.

Importantly, we do not need the i.i.d. assumptions in Propositions 3.1 and 3.2 for our theory, but use them to aid interpretation.

### 3.2. Non-Asymptotic Representations

We start from a non-asymptotic representation of $\hat{\tau}_{adj}$.

**Theorem 3.1.** Under Assumptions 1 and 2,

$$
\hat{\tau}_{adj} - \tau = \left( \frac{1^T e_1(1)}{n_1} - \frac{1^T e_0(0)}{n_0} \right) + \left( \frac{n_1}{n_0} \Delta_0 - \frac{n_0}{n_1} \Delta_1 \right) \\
+ O_p \left( \frac{\mathcal{E}_2^2 \kappa p \log p}{n} + \frac{\mathcal{E}_2 \kappa}{n} \right).
$$

(18)

The first term in (18) is the difference-in-means estimator of the residual potential outcomes based on the population OLS. The second term is non-standard and behaves as a “bias,” which motivates the debiased estimator $\hat{\tau}_{de adj}$ by subtracting its empirical analogue from $\hat{\tau}_{adj}$.

We need to carefully analyze $\Delta_t$ and $\hat{\Delta}_t - \Delta_t$ to simplify Theorem 3.1 and to derive a non-asymptotic representation of $\hat{\tau}_{de adj}$. Define $\Delta = \max\{|\Delta_1|, |\Delta_0|\}$. 


The Cauchy–Schwarz inequality implies

\[
|\Delta_t| = \max_{t=0,1} |\Delta_t| \leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} H_{ii} \times \max_{t=0,1} \frac{1}{n} \sum_{i=1}^{n} e_i^2(t) H_{ii}} \leq \sqrt{\frac{\varepsilon_2 \kappa p}{n}}.
\]

This helps us to obtain the following expansions.

**Corollary 3.1.** Under Assumptions 1 and 2,

\[
\hat{\tau}_{\text{adj}} - \tau = \frac{1}{n_1} \mathbf{1}^T e_1(1) - \frac{1}{n_0} \mathbf{1}^T e_0(0) + O_P \left( \Delta + \sqrt{\frac{\varepsilon_2 \kappa^2 p \log p}{n}} + \sqrt{\frac{\varepsilon_2 \kappa}{n}} \right),
\]

\[
\hat{\tau}_{\text{de adj}} - \tau = \frac{1}{n_1} \mathbf{1}^T e_1(1) - \frac{1}{n_0} \mathbf{1}^T e_0(0) + O_P \left( \sqrt{\frac{\varepsilon_2 \kappa^2 p \log p}{n} + \frac{\varepsilon_2 \kappa}{n}} \right).
\]

Expansion (20) follows from (18) and Assumption 1, and (21) holds because the upper bound in (19) dominates the third term of (18). Expansion (22) shows that our de-biasing strategy works because \(|\hat{\Delta}_t - \Delta_t|\) is of higher order compared to the third term of (22). These non-asymptotic expansions in Corollary 3.1 are crucial for our later analysis.

**3.3. Consistency.** Because the first term in (18) is the difference-in-means of the potential residuals, Neyman [1923/1990] implies that it has mean 0 and variance \(\sigma_n^2/n\). We then use Chebyshev’s inequality to obtain

\[
\frac{\mathbf{1}^T e_1(1)}{n_1} - \frac{\mathbf{1}^T e_0(0)}{n_0} = O_P \left( \sqrt{\frac{\sigma_n^2}{n}} \right) = O_P \left( \frac{\varepsilon_2}{n} \right).
\]

Coupled with (23) and \(\kappa \leq 1\), Corollary 3.1 implies that

\[
\hat{\tau}_{\text{adj}} - \tau = O_P \left( \sqrt{\frac{\varepsilon_2 (\kappa p + 1)}{n}} \right),
\]

\[
\hat{\tau}_{\text{de adj}} - \tau = O_P \left( \sqrt{\frac{\varepsilon_2 (\kappa^2 p \log p + 1)}{n}} \right).
\]

These expansions immediately imply the following consistency result. We essentially require the right-hand sides of the above two identities go to 0.
Theorem 3.2. Under Assumptions 1 and 2,

(i) \( \hat{\tau}_{\text{adj}} \) is consistent if \( \mathcal{E}_2 = o \left( \frac{n}{\kappa p + 1} \right) \).

(ii) \( \hat{\tau}_{\text{de adj}} \) is consistent if \( \mathcal{E}_2 = o \left( \frac{n^{2/3}}{p \log p} \right) \).

In the classical fixed-\( p \) regime, Theorem 3.2 implies that both \( \hat{\tau}_{\text{adj}} \) and \( \hat{\tau}_{\text{de adj}} \) are consistent when \( \mathcal{E}_2 = o(n) \) because \( \kappa \leq 1 \). From Proposition 3.2, the condition \( \mathcal{E}_2 = o(n) \) corresponds to the existence of finite first moment under a super-population i.i.d sampling. In the favorable case where \( \kappa = O(p/n) \), the same condition \( \mathcal{E}_2 = o(n) \) is sufficient for the consistency of \( \hat{\tau}_{\text{adj}} \) if \( p = O(n^{1/2}) \) and for the consistency of \( \hat{\tau}_{\text{de adj}} \) if \( p = O(n^{2/3}/(\log n)^{1/3}) \). Thus, both estimators are robust to the heavy-tailedness of the potential residuals.

Moreover, when the residuals are not extremely heavy-tailed such that \( \mathcal{E}_2 = o(n/p) \), Theorem 3.2 implies that both estimators are always consistent, without any further assumption on \( \kappa \) (except Assumption 2). The consistency can hold without a uniformly bounded second moment of the potential residuals.

3.4. Asymptotic normality. The first term of (18) is the difference-in-means estimator with potential residuals. We can use the classical finite population central limit theorem to show that it is asymptotically normal with mean 0 and variance \( \sigma^2_n/n \). Therefore, the asymptotic normalities of \( \hat{\tau}_{\text{adj}} \) and \( \hat{\tau}_{\text{de adj}} \) hold if the the remainders of (20) and (22) are asymptotically vanishing after multiplied by \( n^{1/2}/\sigma_n \).

We first consider \( \hat{\tau}_{\text{adj}} \).

Theorem 3.3. Under Assumptions 1–4, \( n^{1/2}(\hat{\tau}_{\text{adj}} - \tau)/\sigma_n \overset{d}{\to} N(0, 1) \) if \( \kappa^2 p \log p = o(1) \) and \( n\Delta^2 = o(\mathcal{E}_2) \).

Replacing \( \Delta \) in Theorem (3.3) by the upper bound \( |\Delta| \leq \sqrt{\mathcal{E}_2 p \kappa / n} \) in (19), we obtain the following looser but cleaner result.

Corollary 3.2. Under Assumptions 1–4, \( n^{1/2}(\hat{\tau}_{\text{adj}} - \tau)/\sigma_n \overset{d}{\to} N(0, 1) \) if \( \kappa p = o(1) \).

In the favorable case where \( \kappa = O(p/n) \), the condition \( \kappa p = o(1) \) reduces to \( p^2/n \to 0 \), i.e., \( p = o(n^{1/2}) \). In this case, Corollary 3.2 extends Lin [2013]'s result to \( p = o(n^{1/2}) \). However, this does not explain the numerical phenomenon in Figure 1 that Lin [2013]'s result seems to hold with \( p \) growing up to \( n^{2/3} \) even without debiasing. This is because the bound...
(19) is loose. In fact, because \( e(t) \) has mean zero, we can rewrite \( \Delta_t \) as
\[
\Delta_t = n^{-1} \sum_{i=1}^{n} e_i(t) (H_{ii} - p/n).
\]
The Cauchy–Schwarz inequality implies
\[
\Delta = \max_{t=0,1} |\Delta_t| \leq \max_{1 \leq i \leq n} \left| H_{ii} - \frac{p}{n} \right| \times \max_{t=0,1} \frac{1}{n} \sum_{i=1}^{n} |e_i(t)| \leq \max_{1 \leq i \leq n} \left| H_{ii} - \frac{p}{n} \right| \sqrt{\mathbb{E}_2}.
\]
Therefore, the condition \( \Delta = o\left(\sqrt{\mathbb{E}_2/n}\right) \) in Theorem 3.3 holds whenever
\[
\max_{1 \leq i \leq n} \left| H_{ii} - \frac{p}{n} \right| = o\left(n^{-1/2}\right).
\]
That is, under (24), the asymptotic normality of \( \hat{\tau}_{adj} \) holds when the other condition in Theorem 3.3 holds, i.e., \( \kappa^2 p \log p \to 0 \). In the favorable case where \( \kappa = O(p/n) \), the condition reduces to \( p^3 \log p / n^2 \to 0 \), which further implies \( p = o(n^{2/3} / (\log n)^{1/3}) \). This fills the gap to \( n^{2/3} \) up to a log-factor. Under \( p = o(n^{2/3} / (\log n)^{1/3}) \), we can use Proposition 3.1 to verify that (24) holds with high probability if entries of \( X \) are independent and have finite 12-th moments; the simulation setting for Fig. 1 satisfies the moment condition because the \( x_{ij} \)'s are i.i.d. \( N(0,1) \).

Although we can fill the theoretical gap, it is not ideal to impose an extra condition on the leverage scores. In contrast, the debiased estimator is asymptotically normal without any further condition.

**Theorem 3.4.** Under Assumptions 1–4, \( n^{1/2}(\hat{\tau}_{adj} - \tau) / \sigma_n \xrightarrow{d} N(0,1) \) if \( \kappa^2 p \log p = o(1) \).

Therefore, the debiased estimator has better theoretical guarantees. Under \( p = o(n^{2/3} / (\log n)^{1/3}) \), we can use Proposition 3.1 to verify that the condition \( \kappa^2 p \log p = o(1) \) holds if entries of \( X \) are independent and have finite \( (6 + \epsilon) \)-th moments.

3.5. **Variance estimation.** The variance estimators \( \hat{\sigma}_{HCj}^2 \)'s are all asymptotically equivalent because the correction terms in (14) are negligible under our asymptotic regime. We can prove that the \( \hat{\sigma}_{HCj}^2 \)'s for all \( j \) are asymptotically conservative estimators of \( \sigma_n^2 \).

**Theorem 3.5.** Under Assumptions 1–4, \( \hat{\sigma}_{HCj}^2 / \sigma_n^2 \geq 1 + o_p(1) \) for all \( j \in \{0,1,2,3\} \).

Therefore, the Wald-type confidence intervals for the ATE are all asymptotically conservative.
3.6. Related works. Theoretical analyses under the finite-population randomization model are non-trivial due to the lack of probability tools. The closest work to ours is Bloniarz et al. [2016], which allows $p$ to grow with $n$ and potentially exceed $n$. However, they assume that the potential outcomes have sparse linear representations based on the covariates, and require $s = o(n^{1/2}/\log p)$ where $s$ is a measure of sparsity. Under additional regularities conditions, they show that $\hat{\tau}(\hat{\beta}_{\text{lasso}}^1, \hat{\beta}_0)$ is consistent and asymptotically normal with $(\hat{\beta}_{\text{lasso}}^1, \hat{\beta}_0)$ being the LASSO coefficients of the covariates. Although the LASSO-adjusted estimator can handle ultra-high dimensional case where $p > n$, it has three limitations. First, the requirement $s \ll n^{1/2}/\log p$ is stringent. For instance, the PAC-man dataset considered by Bloniarz et al. [2016] has $n = 1013$ and $p = 1172$, so the condition reads $s \ll 4.5$, which implicitly imposes a strong sparse modelling assumption. Second, the penalty level of the LASSO depends on unobserved quantities. Although they use the cross-validation to select the penalty level, the theoretical properties of this procedure is still unclear. Third, their “restrictive eigenvalue condition” imposes certain non-singularity on the submatrices of the covariate matrix. However, (submatrices of) the covariate matrix can be ill-conditioned especially when interaction terms are included in practice. In addition, this condition is computationally challenging to check. Admittedly, our results cannot deal with the case of $p > n$. Nevertheless, we argue that $p < n$ is an important regime in many applications, ranging from randomized clinical trials to program evaluation.

The growth rate $n^{2/3}$ also appears in the context of linear models under the i.i.d. sampling framework. For instance, Portnoy [1985] shows a necessary condition $p = o(n^{2/3}/\log n)$ for particular designs with complicated conditions. Extending Portnoy [1985]’s result to general matrices, Mammen [1989] improves the condition to $\kappa n^{1/3}(\log n)^{2/3} = o(1)$, which, in the favorable case with $\kappa = O(p/n)$, reduces to $p = o(n^{2/3}/(\log n)^{2/3})$. Similar to Fig. 1, when $p/n^{2/3}$ is bounded away from 0, the usual expansion-based techniques might fail for deriving the asymptotics; see Fan et al. [2017] for evidence in logistic regressions and Lei et al. [2018] for a qualitatively different technique to derive the asymptotic normality for M-estimators.

4. Numerical Experiments. We perform extensive numerical experiments to confirm and complement our theory. We examine the performance of the estimators $\hat{\tau}_{\text{adj}}$ and $\hat{\tau}_{\text{adj}}^{\text{de}}$ as well as the variance estimators $\hat{\sigma}_{\text{HCC}}^2$ for $j = 0, 1, 2, 3$.

4.1. Data Generating Process. We examine the moderate sample performance of the estimators. We set $n = 500, n_1 = 250$ and generate a matrix
$X \in \mathbb{R}^{n \times n}$ with i.i.d. entries from $N(0, 1)$. We keep the matrix fixed. For each exponent $\gamma \in \{0, 1/30, \ldots, 25/30\}$, we let $p = \lceil n^{\gamma} \rceil$ and take the first $p$ columns of $X$ as the covariate matrix. In Supplementary Material III, we also simulate $X$ with entries from $t(2)$ or $t(1)$, and take $X$ from two real datasets. With $X$, we construct the potential outcomes as

$$Y(1) = X\beta^*_1 + \sigma^*_1 \epsilon(1), \quad Y(0) = X\beta^*_0 + \sigma^*_0 \epsilon(0),$$

with $\beta^*_1 = \beta^*_0 = 0 \in \mathbb{R}^p$, $\sigma^*_1 = \sigma^*_0 = 1$, and $\epsilon(1), \epsilon(0) \in \mathbb{R}^n$. We consider two types of $(\epsilon(1), \epsilon(0))$ are realizations of random vectors with i.i.d. entries from $N(0, 1)$, or $t(2)$, or $t(1)$. The second type of $(\epsilon(1), \epsilon(0))$ corresponds to the “most biased” situation which maximizes the “bias term” (second term) in the expansion (18) in some sense. Specifically, we consider the case where $\epsilon(0) = \epsilon$ and $\epsilon(1) = 2\epsilon$ for some vector $\epsilon$ that satisfies (7) with sample variance 1. To maximize the bias term, we take $\epsilon$ as the solution of

$$\max_{\epsilon \in \mathbb{R}^n} \left| \frac{n_1}{n_0} \Delta_0 - \frac{n_0}{n_1} \Delta_1 \right| = \left| \frac{1}{n} \sum_{i=1}^{n} H_{ii} \epsilon_i \right|, \quad \text{s.t.} \ |\epsilon|_2^2 / n = 1, X^T \epsilon = 1^T \epsilon = 0.$$

We give more details of constructing $\epsilon$ in Section F of Supplementary Material III. We consider this peculiar type of residuals to illustrate the effectiveness of bias reduction.

Given $X \in \mathbb{R}^{n \times p}$ and potential outcomes $Y(1), Y(0) \in \mathbb{R}^p$, we generate 5000 binary vectors $T \in \mathbb{R}^n$ with $n_1$ units assigned to treatment. For each assignment vector, we observe half of the potential outcomes.

4.2. Repeated Sampling Evaluations. Based on the observe data, we obtain two estimates $\hat{\tau}_{\text{adj}}$ and $\hat{\tau}_{\text{de}}$, as well as five variance estimates $\hat{\sigma}^2_{\text{HC}j} (j = 0, 1, 2, 3)$ and $\hat{\sigma}_n^2$. Technically, $\hat{\sigma}_n^2$ is not an estimate because it is the theoretical asymptotic variance. Below $\hat{\tau}$ can be either $\hat{\tau}_{\text{adj}}$ or $\hat{\tau}_{\text{de}}$, and $\hat{\sigma}^2$ can be any of the five estimates.

Let $\hat{\tau}_1, \ldots, \hat{\tau}_{5000}$ denote the estimates in 5000 replicates, and $\tau$ denote the true ATE. The empirical absolute bias is $\text{bias}(\hat{\tau}) = \left| \frac{1}{5000} \sum_{k=1}^{5000} \hat{\tau}_k - \tau \right|$. In each experiment, we compute the ratio $\text{bias}(\hat{\tau}_{\text{de}})/\text{bias}(\hat{\tau}_{\text{adj}})$ to measure the effectiveness of bias reduction.

Similarly, let $\hat{\sigma}^2_1, \ldots, \hat{\sigma}^2_{5000}$ denote the variance estimates obtained in 5000 replicates, and $\hat{\sigma}_n^2$ denote the empirical variance of $(\hat{\tau}_1, \ldots, \hat{\tau}_{5000})$. We compute the standard deviation inflation ratio $\text{SDR}(\hat{\sigma}) = 5000^{-1} \sum_{k=1}^{5000} \hat{\sigma}_k / \hat{\sigma}_n$. Note that $\hat{\sigma}_n^2$ is an unbiased estimate of true sampling variance of $\hat{\tau}$, which can be different from the theoretical asymptotic variance $\sigma_n^2$. 


For each estimate and variance estimate, we compute the $t$-statistic $n^{1/2}(\hat{\tau} - \tau) / \hat{\sigma}$ and the z-score $n^{1/2}(\hat{\tau} - \tau) / \sigma_*$. For each $t$-statistic and the z-score, we estimate the empirical 95% coverage by the proportion within $[-1.96, 1.96]$, the 95% quantile range of $N(0, 1)$.

In sum, we compute three measures defined above: bias ratios, standard deviation inflation ratios, and 95% coverage. We repeat 50 times using different random seeds and record the medians of them. Fig. 2 summarizes the results.

4.3. Results. From Fig. 2a, $\hat{\tau}_{\text{adj}}^{de}$ reduces a non-negligible proportion of bias regardless of the distribution of potential outcomes, especially for large $p$. In the “most biased” scenario, the bias reduction can be dramatic. This remains to be true when $X$ is generated with heavy-tailed entries; see Appendix F.2 in Supplementary Material III for more details.

For standard deviation inflation ratios, we find that the true sampling variances of $\hat{\tau}_{\text{adj}}$ and $\hat{\tau}_{\text{adj}}^{de}$ are almost identical and thus we set the sampling variance of $\hat{\tau}_{\text{adj}}$ as the baseline variance $\hat{\sigma}_2$. Fig. 2b shows an interesting phenomenon that the theoretical asymptotic variance $\sigma_n^2$ tends to underestimate the true sampling variance for large $p$ in finite samples. Corollary 3.1 partially suggests this. The theoretical asymptotic variance is simply the variance of the first term while the finite sample variance also involves the second term and, more importantly, the error term, which can be large in the presence of high dimensional or influential observations. All variance estimators overestimate $\sigma_n^2$ because they all ignore the third term of $\sigma_n^2$. However, all estimators, except the HC3 estimator, tend to underestimate the true sampling variance for large $p$. In contrast, the HC3 estimator does not suffer from anti-conservatism.

Fig. 2b shows that HC0 and HC1 variance estimates lie between the theoretical asymptotic variance and the HC2 variance estimate. For better visualization, we only plot the 95% coverage of $t$-statistics computed from $\sigma_n^2$, $\hat{\sigma}_{HC2}^2$ and $\hat{\sigma}_{HC3}^2$ in Fig. 2c. In addition, we plot the coverage for the z-score which uses the true sampling variance $\sigma_n^2$. We draw the following conclusions from Fig. 2c. First, the coverage of two ATE estimates are almost identical. This indicates that the absolute bias does not affect the coverage, though $\hat{\tau}_{\text{adj}}^{de}$ reduces a non-negligible proportion of bias. Second, as Fig. 2b suggests, the $t$-statistic with HC3 variance estimate has the best coverage. Surprisingly, it protects the coverage against the increasing dimension. In contrast, the theoretical asymptotic variance and HC$j$ ($j = 0, 1, 2$) variance estimates yield significantly lower coverage for large $p$. Therefore, we advocate using $\hat{\sigma}_{HC3}^2$ for variance estimation.
(a) Ratio of bias between $\hat{\tau}_{\text{adj}}$ and $\hat{\tau}_{\text{adj}}$

(b) Ratio of standard deviation between five standard deviation estimates, $\sigma_n, \hat{\sigma}_{\text{HC0}}, \hat{\sigma}_{\text{HC1}}, \hat{\sigma}_{\text{HC2}}, \hat{\sigma}_{\text{HC3}}$, and the true standard deviation of $\hat{\tau}_{\text{adj}}$

(c) Empirical 95% coverage of $t$-statistics derived from two estimators and four variance estimators (“truth” for the true sampling variance (of $\hat{\tau}_{\text{adj}}$), “theoretical” for $\sigma_n^2$, “HC2” for $\hat{\sigma}_{\text{HC2}}^2$ and “HC3” for $\hat{\sigma}_{\text{HC3}}^2$)

Fig 2: Simulation. $X$ contains i.i.d. $N(0, 1)$ entries. Each column corresponds to a distribution of potential outcomes.
Finally, it is interesting that the z-score, with the true sampling variance, has almost perfect 95% coverage for all types of potential outcomes and for all dimensions, even for $p$ larger than $n^{2/3}$. This provides evidence that, at least in this case, both estimators are still asymptotically normal for $p \gg n^{2/3}$ although they have larger asymptotic variances than $\sigma^2_n$. We explore this aspect further in Appendix F.2 in Supplementary Material III by checking the normality of the estimates. Note that this does not contradict our statement that $n^{2/3}$ is the fundamental limit of Lin [2013]'s confidence interval because in this regime the asymptotic variance is different and thus involves a different central limit theorem. We leave this to future research.

5. Conclusions and Practical Suggestions. Fisher [1935] advocated using the analysis of covariance under treatment-unit additivity. Freedman [2008] highlighted its dangers under treatment effect heterogeneity. Lin [2013] proposed a simple OLS estimator with treatment-covariate interactions accounting for potential heterogeneity. Lin [2013]'s estimator is consistent and asymptotically normal over complete randomizations even if the linear model is incorrect. His estimator is likely to attract attentions from practitioners because of its simplicity and transparency. Therefore, it is important to explore its theoretical limit by allowing for a growing dimension of the covariates, which reflects practical situations with many covariates. Our asymptotic analyses and empirical results show that $p = O(n^{2/3})$ appears to be the fundamental threshold.

The maximum leverage score $\kappa$ plays a critical role in the theory. Fortunately, this quantity depends only on the covariates. Therefore, in practice, we can transform the covariate matrix to have a smaller $\kappa$ without looking at the outcomes. Proposition 3.1 suggests that $\kappa$ depends on the moments of the covariates. For example, we can truncate covariates within a bounded interval or transform each column of covariates to the rank vector divided by sample size $n$, to ensure infinite order moments. Because the consistency and asymptotic normality do not rely on a correctly-specified linear model, these transformations will not affect the validity of the procedure although they are likely to affect the efficiency. We will explore this direction in future research.

As a minimal requirement, practitioners should always report $p/n^{2/3}$, $\kappa \log p$, and $\kappa^2 p \log p$ before applying Lin [2013]'s and our debiased estimators because the results are more credible when these quantities are close to zero. Nevertheless, we emphasize that we only show the sufficiency, not the necessity, of these requirements. From the extra experimental results in Appendix F.2 in Supplementary Material III, we find scenarios with the
central limit theorem holds even when $\kappa \approx 1$. This suggests the possibility to further relax the regularity conditions on leverage scores, or more generally, on $X$. We leave this to future research.

6. Technical Lemmas.

6.1. Some general results for sampling without replacement. Completely randomized experiments have deep connections with sampling without replacement because the treatment and control groups are simple random samples from a finite population of $n$ units. Below we use $T$ to denote a random size-$m$ subset of $\{1, \ldots, n\}$ over all $\binom{n}{m}$ subsets, and $S^{p-1} = \{ (\omega_1, \ldots, \omega_p)^T : \omega_1^2 + \cdots + \omega_p^2 = 1 \}$ to denote the $(p-1)$-dimensional unit sphere.

The first lemma gives the mean and variance of the sample total from sampling without replacement. See Cochran [2007, Theorem 2.2] for a proof.

**Lemma 6.1.** Let $(w_1, \ldots, w_n)$ be fixed scalars with mean $\bar{w} = n^{-1} \sum_{i=1}^n w_i$. Then $\sum_{i \in T} w_i$ has mean $m \bar{w}$ and variance

$$\text{Var} \left( \sum_{i \in T} w_i \right) = \frac{m(n-m)}{n(n-1)} \sum_{i=1}^n (w_i - \bar{w})^2.$$

The second lemma gives the Berry–Esseen-type bound for the finite population central limit theorem. See Bikelis [1969] and Höglund [1978] for proofs.

**Lemma 6.2.** Let $(w_1, \ldots, w_n)$ be fixed scalars with $\bar{w} = n^{-1} \sum_{i=1}^n w_i$ and $S_w^2 = \sum_{i=1}^n (w_i - \bar{w})^2$. Let $m = nf$ for some $f \in (0, 1)$. Then

$$d_K \left( \frac{\sum_{i \in T} (w_i - \bar{w})}{S_w \sqrt{f(1-f)}}, N(0, 1) \right) \leq \frac{C}{\sqrt{f(1-f)}} \frac{\sum_{i=1}^n (w_i - \bar{w})^2}{S_w^3} \leq \frac{C}{\sqrt{f(1-f)}} \frac{\max_{1 \leq i \leq n} |w_i - \bar{w}|}{S_w},$$

where $d_K$ denotes the Kolmogorov distance between two distributions, and $C$ is a universal constant.

The following two lemmas give novel vector and matrix concentration inequalities for sampling without replacement.

**Lemma 6.3.** Let $(u_1, \ldots, u_n)$ be a finite population of $p$-dimensional vectors with $\sum_{i=1}^n u_i = 0$. Then for any $\delta \in (0, 1)$, with probability $1 - \delta$

$$\left\| \sum_{i \in T} u_i \right\|_2 \leq \|U\|_F \sqrt{\frac{m(n-m)}{n(n-1)}} + \|U\|_{op} \sqrt{8 \log \frac{1}{\delta}}.$$
where $u_i^T$ is the $i$-th row of the matrix $U \in \mathbb{R}^{n \times p}$.

**Lemma 6.4.** Let $(V_1, \ldots, V_n)$ be a finite population of $(p \times p)$-dimensional Hermitian matrices with $\sum_{i=1}^n V_i = 0$. Let $C(p) = 4(1 + \lceil 2 \log p \rceil)$, and

$$\nu^2 = \left\| \frac{1}{n} \sum_{i=1}^n V_i^2 \right\|_{\text{op}}, \quad \nu^2 = \sup_{\omega \in \mathbb{S}^{p-1}} \frac{1}{n} \sum_{i=1}^n (\omega^T V_i \omega)^2, \quad \nu_+ = \max_{1 \leq i \leq n} \| V_i \|_{\text{op}}.$$

Then for any $\delta \in (0, 1)$, with probability $1 - \delta$,

$$\left\| \sum_{i \in T} V_i \right\|_{\text{op}} \leq \sqrt{nC(p)\nu + C(p)\nu_+} + \sqrt{8n \log \frac{1}{\delta} \nu_{-}}.$$

The following lemma gives the mean and variance of the summation over randomly selected rows and columns from a deterministic matrix $Q \in \mathbb{R}^{n \times n}$.

**Lemma 6.5.** Let $Q \in \mathbb{R}^{n \times n}$ be a deterministic matrix, and $Q_T \equiv \sum_{i,j \in T} Q_{ij}$. Assume $n \geq 4$. Then

$$\mathbb{E}Q_T = \frac{m(n-m)}{n(n-1)} \text{tr}(Q) + \frac{m(m-1)}{n(n-1)} 1^T Q 1.$$

If $Q$ further satisfies $1^T Q = Q 1 = 0$, then

$$\text{Var}(Q_T) \leq \frac{m(n-m)}{n(n-1)} \|Q\|_{F}^2.$$

Lemmas 6.3–6.5 are critical for our proofs. The proofs are relegated to Supplementary Material I. They are novel tools to the best of our knowledge and potentially useful in other contexts such as survey sampling, matrix sketching, and transductive learning.

### 6.2. Some results particularly useful for our setting

We first give an implication of Assumption 3, a lower bound on $\sigma_n^2$ under Assumption 1.

**Lemma 6.6.** Under Assumptions 1 and 3, $\sigma_n^2 \geq \eta \min \{n_1/n_0, n_0/n_1\} \mathcal{E}_2$.

The following explicit formula is the starting point of our proof.

**Lemma 6.7.** Recall $H_t = X_t(X_t^T X_t)^{-1} X_t^T$ ($t = 0, 1$). Then

$$(26) \hat{\tau}_{\text{adj}} - \tau = \frac{1^T e_1(1)/n_1 - 1^T H_1 e_1(1)/n_1}{1 - 1^T H_1 1/n_1} - \frac{1^T e_0(0)/n_0 - 1^T H_0 e_0(0)/n_0}{1 - 1^T H_0 1/n_0}.$$
The quantities $\mu_t, e(t)$, and our estimators $(\hat{\tau}_{\text{adj}}, \hat{\tau}_{\text{de}})$ are all invariant if $X$ is transformed to $XZ$ for any full rank matrix $Z \in \mathbb{R}^{p \times p}$, provided that (2) holds. Thus, without loss of generality, we assume

$$n^{-1}X^TX = I.$$  

Otherwise, suppose $X$ has the singular value decomposition $U \Sigma V^T$ with $U \in \mathbb{R}^{n \times p}, \Sigma, V \in \mathbb{R}^{p \times p}$, then we can replace $X$ by $\frac{n^{1/2}}{2}U = X(\frac{n^{1/2}}{2}V \Sigma^{-1})$ to ensure (27). We can verify that the key properties in (7) still hold. Assuming (27), we can rewrite the hat matrix and the leverage scores as

$$H = n^{-1}XX^T, \quad H_{ii} = n^{-1}\|x_i\|^2_2, \quad H_{ij} = n^{-1}x_i^T x_j.$$ 

Note that the invariance property under the standardization (27) is a feature of the OLS-based regression adjustment. It does not hold for many other estimators [e.g., Bloniarz et al., 2016, Wager et al., 2016].

We will repeatedly use the following results to obtain the stochastic orders of the terms in (26). They are consequences of Lemmas 6.3 and 6.4.

**Lemma 6.8.** Under Assumption 1, for $t = 0, 1,$

\[
\frac{1^T e_t(t)}{n_t} = O_P \left( \frac{\sqrt{\mathcal{E}_2}}{n} \right), \quad \left\| \frac{X^T 1}{n_t} \right\|_2 = O_P \left( \frac{\sqrt{p}}{n} \right),
\]

\[
\left\| \frac{X^T e_t(t)}{n_t} \right\|_2 = O_P \left( \sqrt{\mathcal{E}_2 \kappa} \right).
\]

**Lemma 6.9.** Under Assumptions 1, 2 and (27), for $t = 0, 1,$

\[
\left\| \frac{X^T X_t}{n_t} - I \right\|_{op} = O_P \left( \sqrt{\kappa \log p} \right), \quad \left\| \left( \frac{X^T X_t}{n_t} \right)^{-1} \right\|_{op} = O_P(1),
\]

\[
\left\| \left( \frac{X^T X_t}{n_t} \right)^{-1} - I \right\|_{op} = O_P \left( \sqrt{\kappa \log p} \right).
\]

The following lemma states some key properties of an intermediate quantity, which will facilitate our proofs.

**Lemma 6.10.** Define $Q(t) = H \text{diag} (e(t)) = (H_{ij} e_j(t))_{i,j=1}^n$. It satisfies

$$1^T Q(t) = 0, \quad Q(t) 1 = 0, \quad 1^T Q(t) 1 = 0,$$

$$\text{tr}(Q(t)) = n\Delta_t, \quad \|Q(t)\|_F^2 = \sum_{i=1}^n e_i^2(t) H_{ii} \leq n\mathcal{E}_2 \kappa.$$
7. Proofs of the main results.

7.1. Proof of the non-asymptotic representation.

Proof of Theorem 3.1. We need to analyze the terms in (26). First, by Lemmas 6.8 and 6.9,
\[
\frac{1^TH_t\mathbf{1}}{nt} = \frac{1^TX_t}{nt} \left( \frac{X^TX}{nt} \right)^{-1} \frac{X^T\mathbf{1}}{nt} \leq \left\| \left( \frac{X^TX}{nt} \right)^{-1} \right\|_{op} \left\| \frac{X^T\mathbf{1}}{nt} \right\|_2^2 = O_P \left( \frac{p}{n} \right).
\]

Using (17) that \( p = o(n) \), we obtain that
\[
(29) \frac{1}{1 - \frac{1^TH_t\mathbf{1}}{nt}} = 1 + O_P \left( \frac{p}{n} \right).
\]

Second,
\[
\frac{1^TH_te_t(t)}{nt} = \frac{1^TX_t}{nt} \left( \frac{X^TX}{nt} \right)^{-1} \frac{X^Te_t(t)}{nt} = \frac{1^TX_t}{nt} \frac{X^Te_t(t)}{nt} + \frac{1^TX_t}{nt} \left( \left( \frac{X^TX}{nt} \right)^{-1} - I \right) \frac{X^Te_t(t)}{nt} \triangleq R_{t1} + R_{t2}.
\]

Note that here we do not use the naive bound for \( \frac{1^TH_te_t(t)}{nt} \) as for \( \frac{1^TH_t\mathbf{1}}{nt} \) in (29) because this gives weaker results. Instead, we bound \( R_{t1} \) and \( R_{t2} \) separately. Lemmas 6.8 and 6.9 imply
\[
(31) R_{t2} \leq \left\| \left( \frac{X^TX}{nt} \right)^{-1} - I \right\|_{op} \left\| \frac{X^T\mathbf{1}}{nt} \right\|_2 \left\| \frac{X^Te_t(t)}{nt} \right\|_2 = O_P \left( \sqrt{\frac{\varepsilon_2\kappa^2p\log p}{n}} \right).
\]

We apply Chebyshev’s inequality to obtain that
\[
(32) R_{t1} = \mathbb{E}R_{t1} + O_P \left( \sqrt{\text{Var}(R_{t1})} \right).
\]

Therefore, to bound \( R_{t1} \), we need to calculate its first two moments. Recalling (28) and the definition of \( Q(t) \) in Lemma 6.10, we have
\[
R_{t1} = \frac{1}{n_t^2} \left( \sum_{i \in T_t} x_i^T \right) \left( \sum_{i \in T_t} x_ie_t(t) \right) = \frac{1}{n_t^2} \sum_{i \in T_t} \sum_{j \in T_t} x_i^T x je(t)
\]
\[
= \frac{1}{n_t^2} \sum_{i \in T_t} \sum_{j \in T_t} nH_{ij}e(t) = \frac{n}{n_t^2} \sum_{i \in T_t} \sum_{j \in T_t} Q_{ij}(t).
\]
Lemmas 6.5 and 6.10 imply the expectation of $R_{t1}$:

$$
\mathbb{E}R_{t1} = \frac{n}{n_t^2} \left( \frac{n_1 n_0}{n(n-1)} \text{tr}(Q(t)) + \frac{n_t(n_t-1)}{n(n-1)} 1^T Q(t) 1 \right)
$$

(34)

$$
= \frac{n n_1 n_0}{n_t^2 (n-1)} \Delta_t = \frac{n_1 n_0}{n_t^2} \Delta_t + O \left( \frac{|\Delta_t|}{n} \right).
$$

We then bound the variance of $R_{t1}$:

$$
\text{Var}(R_{t1}) = \frac{n^2}{n_t^4} \text{Var} \left( \sum_{i,j \in T_t} Q_{ij}(t) \right) \leq \frac{n^2}{n_t^4} \frac{n_1 n_0}{n(n-1)} \|Q(t)\|_F^2
$$

(35)

$$
\leq \frac{n^2}{n_t^4} \frac{n_1 n_0}{n(n-1)} n \mathcal{E}_2 \kappa = O \left( \frac{\mathcal{E}_2 \kappa}{n} \right),
$$

(36)

where (35) follows from Lemma 6.5, (36) follows from Lemma 6.10 and Assumption 1. Putting (30)–(34) and (36) together, we obtain that

$$
\frac{1^T H_t e(t)}{n_t} = \frac{n_1 n_0}{n_t^2} \Delta_t + O_p \left( \frac{\mathcal{E}_2 \kappa^2 p \log p}{n} + \frac{|\Delta_t|}{n} + \sqrt{\frac{\mathcal{E}_2 \kappa}{n}} \right)
$$

(37)

By (19) and (17) that $p = o(n)$, (37) further simplifies to

$$
\frac{1^T H_t e(t)}{n_t} = \frac{n_1 n_0}{n_t^2} \Delta_t + O_p \left( \frac{\mathcal{E}_2 \kappa^2 p \log p}{n} + \sqrt{\frac{\mathcal{E}_2 \kappa}{n}} \right).
$$

(38)

Using Lemma 6.8, (38), and the fact that $\kappa \leq 1$, we have

$$
\frac{1^T e(t)}{n_t} - \frac{1^T H_t e(t)}{n_t} = O_p \left( \sqrt{\frac{\mathcal{E}_2}{n}} + \Delta + \sqrt{\frac{\mathcal{E}_2 \kappa^2 p \log p}{n}} \right).
$$

(39)

Finally, putting (29), (38) and (39) together into (26), we obtain that

$$
\hat{\tau}_{adj} - \tau = \left( \frac{1^T e_1(1)}{n_1} - \frac{1^T H_1 e_1(1)}{n_1} \right) \left( 1 + O_p \left( \frac{p}{n} \right) \right)
$$

$$
- \left( \frac{1^T e_0(0)}{n_0} - \frac{1^T H_0 e_0(0)}{n_0} \right) \left( 1 + O_p \left( \frac{p}{n} \right) \right)
$$

$$
= \frac{1^T e_1(1)}{n_1} - \frac{1^T e_0(0)}{n_0} + \frac{1^T H_0 e_0(0)}{n_0} - \frac{1^T H_1 e_1(1)}{n_1}
$$

$$
+ O_p \left( \sqrt{\frac{p^2 \mathcal{E}_2}{n^3}} + \frac{p \Delta}{n} + \sqrt{\frac{\mathcal{E}_2 \kappa^2 p^3 \log p}{n^3}} \right).
$$
\[ T_1^\top e_1(1) - T_0^\top e_0(0) + \frac{n_1}{n_0} \Delta_0 - \frac{n_0}{n_1} \Delta_1 + O_P \left( \frac{\sqrt{p^2 \mathbb{E}_2}}{n} + \frac{p \Delta}{n} + \frac{\mathbb{E}_2 \kappa^2 p \log p}{n} + \frac{\mathbb{E}_2 \kappa}{n} \right). \]

(40)

where (40) uses (17) that \( p = o(n) \). The fourth terms dominates the first term in (40) because \( p = o(n) \) and \( \kappa \geq p/n \). The third term dominates the second term in (40) because, by (19),

\[ \frac{p \Delta}{n} \leq \kappa \Delta \leq \sqrt{\kappa} \Delta = O \left( \frac{\sqrt{\mathbb{E}_2 \kappa^2 p}}{n} \right). \]

Deleting the first two terms in (40), we complete the proof. \( \Box \)

Proof of Corollary 3.1. Assumption 1 implies \( \frac{n_1}{n_0} \Delta_0 - \frac{n_0}{n_1} \Delta_1 = O(\Delta) \), which, coupled with Theorem 3.1, implies (20).

The key is to prove the result for the debiased estimator. By definition,

\[ \hat{\tau}_{\text{adj}} - \tau = \frac{T_1^\top e_1(1)}{n_1} - \frac{T_0^\top e_0(0)}{n_0} + \frac{n_1}{n_0} (\Delta_0 - \hat{\Delta}_0) - \frac{n_0}{n_1} (\Delta_1 - \hat{\Delta}_1) + O_p \left( \frac{\sqrt{\mathbb{E}_2 \kappa^2 p \log p}}{n} + \frac{\mathbb{E}_2 \kappa}{n} \right), \]

and therefore, the key is to bound \( |\Delta_t - \hat{\Delta}_t| \).

We introduce an intermediate quantity \( \hat{\Delta}_t = n_t^{-1} \sum_{i \in T_t} H_{ii} e_i(t) \). It has mean \( \mathbb{E} \hat{\Delta}_t = \Delta_t \) and variance

\[ \text{Var}(\hat{\Delta}_t) \leq \frac{1}{n_t^2} \frac{n_1 n_0}{n(n-1)} \sum_{i=1}^{n_t} H_{ii}^2 \mathbb{E}_2 e_i^2(t) \leq \frac{\mathbb{E}_2 \kappa^2}{n_t} = O \left( \frac{\mathbb{E}_2 \kappa^2}{n} \right), \]

from Lemma 6.1 and Assumption 1. Equipped with the first two moments, we use Chebyshev’s inequality to obtain

\[ |\hat{\Delta}_t - \Delta_t| = O_p \left( \frac{\mathbb{E}_2 \kappa^2}{n} \right). \]

(42)

Next we bound \( |\hat{\Delta}_t - \tilde{\Delta}_t| \). The Cauchy–Schwarz inequality implies

\[ |\hat{\Delta}_t - \tilde{\Delta}_t| \leq \frac{1}{n_t} \sum_{i \in T_t} H_{ii} |\hat{e}_i - e_i(t)| \leq \sqrt{\frac{1}{n_t} \sum_{i \in T_t} H_{ii}^2} \sqrt{\frac{1}{n_t} \sum_{i \in T_t} (\hat{e}_i - e_i(t))^2}. \]

(43)
First,

\[ \frac{1}{n_t} \sum_{i \in T_t} H_{ii}^2 \leq \frac{n \kappa}{n_t} \left( \frac{1}{n} \sum_{i=1}^{n} H_{ii} \right) = O \left( \frac{\kappa p}{n} \right). \tag{44} \]

Second, using the fact \( \hat{e}_t = (I - H_t)e_t(t) \), we have

\[ \frac{1}{n_t} \sum_{i \in T_t} (\hat{e}_i - e_i(t))^2 = \frac{1}{n_t} \| \hat{e}_t - e_t(t) \|_2^2 = \frac{1}{n_t} e_t^T(t) H_t e_t(t) \]

\[ = \left( \frac{X_t^T e_t(t)}{n_t} \right)^T \left( \frac{X_t^T X_t}{n_t} \right)^{-1} \frac{X_t^T e_t(t)}{n_t} \]

\[ \leq \left\| \frac{X_t^T X_t}{n_t} \right\|_{op}^{-1} \left\| \frac{X_t^T e_t(t)}{n_t} \right\|_2^2 = O_P(\mathcal{E}_2 \kappa), \tag{45} \]

where the last line follows from Lemma 6.8. Putting (44) and (45) into (43), we obtain

\[ |\hat{\Delta}_t - \tilde{\Delta}_t| = O_P \left( \sqrt{\frac{\mathcal{E}_2 \kappa^2 p}{n}} \right). \tag{46} \]

Combining (42) and (46) together, we have

\[ |\hat{\Delta}_t - \Delta_t| = O_P \left( \sqrt{\frac{\mathcal{E}_2 \kappa^2 p}{n}} \right). \]

We complete the proof by invoking Theorem 3.1. \( \square \)

7.2. Proof of asymptotic normality.

Proofs of Theorems 3.3 and 3.4. We first prove the asymptotic normality of the first term in the expansions:

\[ \frac{n^{1/2}}{\sigma_n} \left( \frac{1^T e_1(1)}{n_1} - \frac{1^T e_0(0)}{n_0} \right) \xrightarrow{d} N(0,1). \tag{47} \]

Recalling \( 0 = 1^T e(0) = 1^T e_1(1) + 1^T e_0(0) \), we obtain that

\[ \frac{n^{1/2}}{n_1} 1^T e_1(1) - \frac{n^{1/2}}{n_0} 1^T e_0(0) = \frac{n^{1/2}}{n_1} 1^T e_1(1) + \frac{n^{1/2}}{n_0} 1^T e_1(0) \]

\[ = \sum_{i \in T_t} \left( \frac{n^{1/2}}{n_1} e_i(1) + \frac{n^{1/2}}{n_0} e_i(0) \right) \equiv \sum_{i \in T_t} w_i. \tag{48} \]
where \( w_i = \frac{n_{1}}{n_1} e_i(1) + \frac{n_{0}}{n_0} e_i(0) \). Based on (12), we can verify that

\[
S_w^2 \triangleq \sum_{i=1}^{n} (w_i - \bar{w})^2 = \sum_{i=1}^{n} w_i^2 = n \sum_{i=1}^{n} \left( \frac{e_i(1)}{n_1} + \frac{e_i(0)}{n_0} \right)^2 = \frac{n^2}{n_1 n_0} \sigma_n^2.
\]

Applying Lemma 6.2 to the representation (48), we have

\[
d_K \left( \frac{n_{1}/2}{\sigma_n} \left( \frac{1^T e_1(1)}{n_1} - \frac{1^T e_0(0)}{n_0} \right), N(0, 1) \right) = O \left( \max_{1 \leq i \leq n} |w_i| \right).
\]

Lemma 6.6 and Assumption 4 imply

\[
S_w^{-1} = O \left( \mathcal{E}_2^{-1/2} \right), \quad \max_{1 \leq i \leq n} |w_i| = O \left( \frac{\mathcal{E}_\infty}{n^{1/2}} \right) = o \left( \frac{\mathcal{E}_2^{1/2}}{n^{1/2}} \right).
\]

Therefore, (47) holds because convergence in Kolmogorov distance implies weak convergence.

We then prove the asymptotic normalities of the two estimators. Corollary 3.1 and Lemma 6.6 imply

\[
n_{1/2} \left( \hat{\tau}_{\text{adj}} - \tau \right) \\
= n_{1/2} \left( \frac{1^T e_1(1)}{n_1} - \frac{1^T e_0(0)}{n_0} \right) + O_P \left( \frac{\sqrt{\mathcal{E}_2} \kappa^2 p \log p}{\sigma_n} + \frac{n^{1/2} \Delta}{\sigma_n} + \frac{\sqrt{\mathcal{E}_2}}{\sigma_n} \right)
\]

\[
= n_{1/2} \left( \frac{1^T e_1(1)}{n_1} - \frac{1^T e_0(0)}{n_0} \right) + O_P \left( \sqrt{\kappa^2 p \log p} + \sqrt{n \mathcal{E}_2} \Delta + \sqrt{\mathcal{E}_2} \right).
\]

We complete the proof by noting that \( \kappa = o(1) \) in (17) under Assumption 2. The same proof carries over to \( \hat{\tau}_{\text{adj}} \).

**7.3. Proof of asymptotic conservatism of variance estimators.**

**Proof of Theorem 3.5.** First, we prove the result for \( j = 0 \). Recalling \( \hat{\varepsilon}_t = (I - H_t) e_t(t) \), we have

\[
\frac{1}{n_t} \sum_{i \in T_t} \varepsilon_i^2 = \frac{1}{n_t} e_t(t)^T (I - H_t) e_t(t)
\]

\[
= \frac{1}{n_t} \sum_{i \in T_t} \hat{\varepsilon}_i^2(t) - \left( \frac{X_t^T e_t(t)}{n_t} \right)^T \left( \frac{X_t^T X_t}{n_t} \right)^{-1} \frac{X_t^T e_t(t)}{n_t} \triangleq S_{t1} - S_{t2}.
\]
Lemma 6.8 and the fact $\kappa = o(1)$ in (17) together imply a bound for $S_{t2}$:

$$S_{t2} \leq \left\| \left( \frac{X_t^T X_t}{n_t} \right)^{-1} \right\|_{\text{op}} \left\| \frac{X_t \epsilon_t(t)}{n_t} \right\|_2^2 = O_p(\mathcal{E}_2 \kappa) = o_P(\mathcal{E}_2).$$

The first term, $S_{t1}$, has mean $E S_{t1} = n^{-1} \sum_{i=1}^{n} e_i^2(t)$ and variance

$$\text{Var}(S_{t1}) \leq \frac{1}{n_t} \frac{n_1 n_0}{n(n-1)} \sum_{i=1}^{n} e_i^4(t)$$

$$\leq \frac{n}{n_t^2} \mathcal{E}_\infty^2 \mathcal{E}_2 = O \left( \frac{\mathcal{E}_\infty^2 \mathcal{E}_2}{n} \right)$$

$$= o_P(\mathcal{E}_2^2),$$

where (51) follows from Lemma 6.1, (52) follows from the definitions of $\mathcal{E}_2$ and $\mathcal{E}_\infty$ and Assumption 1, and (53) follows from Assumption 4 that $\mathcal{E}_\infty^2 = o(n \mathcal{E}_2)$. Therefore, applying Chebyshev’s inequality, we obtain

$$S_{t1} = E S_{t1} + O_P \left( \sqrt{\text{Var}(S_{t1})} \right) = \frac{1}{n} \sum_{i=1}^{n} e_i^2(t) + o_P(\mathcal{E}_2).$$

Combining the bounds for $S_{t1}$ in (54) and $S_{t2}$ in (50), we have

$$\frac{1}{n_t} \sum_{i \in T_t} e_i^2 = \frac{1}{n} \sum_{i=1}^{n} e_i^2(t) + o_P(\mathcal{E}_2).$$

Using the formula of $\hat{\sigma}_n^2$ in (13) and Assumption 1, we have

$$\hat{\sigma}_{HC0}^2 = \frac{n}{n_1 - 1} \left( \frac{1}{n} \sum_{i=1}^{n} e_i^2(1) + o_P(\mathcal{E}_2) \right) + \frac{n}{n_0 - 1} \left( \frac{1}{n} \sum_{i=1}^{n} e_i^2(0) + o_P(\mathcal{E}_2) \right)$$

$$= \frac{1}{n_1} \sum_{i=1}^{n} e_i^2(1) + \frac{1}{n_0} \sum_{i=1}^{n} e_i^2(0) + o_P(\mathcal{E}_2).$$

Using the formula of $\sigma_n^2$ in (11), we have

$$\hat{\sigma}_{HC0}^2 \geq \sigma_n^2 + \frac{1}{n} \sum_{i=1}^{n} (e_i(1) - e_i(0))^2 + o_P(\mathcal{E}_2) \geq \sigma_n^2 + o_P(\mathcal{E}_2),$$

which, coupled with Lemma 6.6, implies that $\hat{\sigma}_{HC0}^2 / \sigma_n^2 \geq 1 + o_P(1)$. 

Next we prove that the $\hat{\sigma}_{HC}^2$’s are asymptotically equivalent. It suffices to show that

\begin{equation}
\min_{j=1,2,3} \min_{1 \leq i \leq n} \frac{\hat{e}_{i,j}}{|\hat{e}_i|} = 1 + o_P(1). \tag{56}
\end{equation}

The proof for $j = 1$ follows from $p/n = o(1)$ in (17). To prove (56) for $j = 2, 3$, we need to prove that $\max_{t=0,1} \max_{i \in T_t} H_{t,ii} = o_P(1)$. This follows from Lemma 6.9 and the fact that $\kappa = o(1)$ in (17):

$$\max_{i \in T_t} H_{t,ii} = \max_{i \in T_t} \frac{1}{n_t} \left( X_{t}^T X_t \right)^{-1} x_i = O_P \left( \frac{\max_{1 \leq i \leq n} \|x_i\|^2_2}{n_t} \right) = O_P(\kappa).$$

\begin{center}
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\end{center}

**Supplementary Materials.** Supplementary Material I gives the proofs of the lemmas in Section 6. Supplementary Material II gives the proofs of the propositions in Section 3.1. Supplementary Material III gives extensive numerical experiments. We post the programs to replicate all the experimental results at [https://github.com/lihualei71/RegAdjNeymanRubin/](https://github.com/lihualei71/RegAdjNeymanRubin/).

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Supplementary Materials of “Regression adjustment in randomized experiments with a diverging number of covariates”

Supplementary Material I gives the proofs of the lemmas in Section 6. Supplementary Material II gives the proofs of the propositions in Section 3.1. Supplementary Material III gives extensive numerical experiments. We post the programs to replicate all the experimental results at https://github.com/lihualei71/RegAdjNeymanRubin/.
Supplementary Material I: Technical Tools

APPENDIX A: CONCENTRATION INEQUALITIES FOR SAMPLING WITHOUT REPLACEMENT

A.1. Some existing tools. The proofs rely on concentration inequalities for sampling without replacement. Hoeffding [1963, Theorem 4] proved the following result that sampling without replacement is more concentrated in convex ordering than i.i.d. sampling.

**Proposition A.1.** Let $C = (c_1, \ldots, c_n)$ be a finite population of fixed elements. Let $Z_1, \ldots, Z_m$ be a random sample with replacement from $C$ and $W_1, \ldots, W_m$ be a random sample without replacement from $C$. If the function $f(x)$ is continuous and convex, then

$$
\mathbb{E} f \left( \sum_{i=1}^{m} Z_i \right) \geq \mathbb{E} f \left( \sum_{i=1}^{m} W_i \right).
$$

From this result, most concentration inequalities for independent sampling carry over to sampling without replacement. Later a line of works, in different contexts, showed an even more surprising phenomenon that sampling without replacement can have strictly better concentration than independent sampling [e.g., Serfling, 1974, Diaconis and Shahshahani, 1987, Lee and Yau, 1998, Bobkov, 2004, Cortes et al., 2009, El-Yaniv and Pechyony, 2009, Bardenet et al., 2015, Tolstikhin, 2017]. In particular, Tolstikhin [2017, Theorem 9] proved a useful concentration inequality for the empirical processes for sampling without replacement.

**Proposition A.2.** Let $C = (c_1, \ldots, c_n)$ be a finite population of fixed elements, and $W_1, \ldots, W_m$ be a random sample without replacement from $C$. Let $\mathcal{F}$ be a class of functions on $C$, and

$$
S(\mathcal{F}) = \sup_{f \in \mathcal{F}} \sum_{i=1}^{m} f(W_i), \quad \nu(\mathcal{F})^2 = \sup_{f \in \mathcal{F}} \text{Var}(f(W_1)).
$$

Then

$$
\mathbb{P}(S(\mathcal{F}) - \mathbb{E}[S(\mathcal{F})] \geq t) \leq \exp \left\{ -\frac{(n + 2)t^2}{8n^2 \nu(\mathcal{F})^2} \right\}.
$$

Proposition A.2 gives a sub-gaussian tail of $S(\mathcal{F})$ with the sub-gaussian parameter depending solely on the variance. In contrast, the concentration
inequalities in the standard empirical process theory for independent sampling usually requires the functions in \( F \) to be uniformly bounded and the tail is either sub-gaussian with the sub-gaussian parameter being the uniform bound on \( F \) or sub-exponential with Bernstein-style behaviors; see Boucheron et al. [2013] for instance. Therefore, Proposition A.2 provides a more precise statement that sampling without replacement is more concentrated than independent sampling for a large class of statistics.

We need the following result from [Tropp, 2016, Theorem 5.1.(2)] to prove the matrix concentration inequality.

**Proposition A.3.** Let \( \tilde{V}_1, \ldots, \tilde{V}_m \) be independent \( p \times p \) random matrices with \( \mathbb{E} \tilde{V}_i = 0 \) for all \( i \). Let \( C(p) = 4(1 + [2 \log p]) \). Then

\[
\left( \mathbb{E} \left\| \sum_{i=1}^{n} \tilde{V}_i \right\|_{\text{op}}^2 \right)^{\frac{1}{2}} \leq \sqrt{C(p)} \left( \sum_{i=1}^{n} \mathbb{E} \tilde{V}_i^2 \right)^{\frac{1}{2}} + C(p) \left( \mathbb{E} \max_{1 \leq i \leq n} \| \tilde{V}_i \|_{\text{op}}^2 \right)^{\frac{1}{2}}.
\]

We will also use the fact that for any \( u \in \mathbb{R}^p \) and Hermitian \( V \in \mathbb{R}^{p \times p} \),

\[
\| u \|_2 = \sup_{\omega \in \mathcal{S}^{p-1}} u^T \omega, \quad \| V \|_{\text{op}} = \sup_{\omega \in \mathcal{S}^{p-1}} \omega^T V \omega.
\]

**A.2. Proofs of Lemmas 6.3 and 6.4.**

**Proof of Lemma 6.3.** Let

\[
C = (u_1, \ldots, u_n), \quad \text{and} \quad \mathcal{F} = \{ f_\omega(u) = u^T \omega : \omega \in \mathcal{S}^{p-1} \}.
\]

Further, let \( u \) be a vector that is randomly sampled from \( C \). Then

\[
\nu^2(\mathcal{F}) = \sup_{\omega \in \mathcal{S}^{p-1}} \text{Var}(u^T \omega) \leq \sup_{\omega \in \mathcal{S}^{p-1}} \mathbb{E}(u^T \omega)^2
\]

\[
= \sup_{\omega \in \mathcal{S}^{p-1}} \frac{1}{n} \sum_{i=1}^{n} (u_i^T \omega)^2 = \sup_{\omega \in \mathcal{S}^{p-1}} \left( \frac{1}{n} \sum_{i=1}^{n} u_i u_i^T \right) \omega
\]

\[
= \sup_{\omega \in \mathcal{S}^{p-1}} \omega^T \left( \frac{U^T U}{n} \right) \omega \leq \frac{\| U \|_{\text{op}}^2}{n}.
\]

By Proposition A.2,

\[
P \left( \left\| \sum_{i \in \mathcal{T}} u_i \right\|_2 \geq \mathbb{E} \left\| \sum_{i \in \mathcal{T}} u_i \right\|_2 + t \right) \leq \exp \left\{ -\frac{(n + 2)t^2}{8n \| U \|_{\text{op}}^2} \right\} \leq \exp \left\{ -\frac{t^2}{8 \| U \|_{\text{op}}^2} \right\},
\]
or, equivalently, with probability $1 - \delta$,

\[(S57) \quad \left\| \sum_{i \in T} u_i \right\|_2 \leq E \left\| \sum_{i \in T} u_i \right\|_2 + \left\| U \right\|_{op} \sqrt{8 \log \frac{1}{\delta}}.\]

By the Cauchy–Schwarz inequality,

\[\left( E \left\| \sum_{i \in T} u_i \right\|_2 \right)^2 \leq E \left( \sum_{i \in T} u_i \right)^2 = \sum_{j=1}^{p} E \left( \sum_{i \in T} u_{ij} \right)^2.\]

Lemma 6.1 implies

\[E \left( \sum_{i \in T} u_{ij} \right)^2 = \frac{m(n - m)}{n(n - 1)} \sum_{i=1}^{n} u_{ij}^2.\]

As a result,

\[(S58) \quad \left( E \left\| \sum_{i \in T} u_i \right\|_2 \right)^2 \leq \frac{m(n - m)}{n(n - 1)} \sum_{i=1}^{n} \| u_i \|^2 = \| U \|^2_F \frac{m(n - m)}{n(n - 1)}.\]

We complete the proof by using (S57) and (S58).

Proof of Lemma 6.4. Let

\[C = (V_1, \ldots, V_n), \quad \text{and} \quad \mathcal{F} = \{ f_\omega(V) = \omega^T V \omega : \omega \in S^{p-1} \}.\]

Further, let $V$ be a vector that is randomly sampled from $C$. Then

\[\nu^2(\mathcal{F}) = \sup_{\omega \in S^{p-1}} \text{Var}(\omega^T V \omega) \leq \sup_{\omega \in S^{p-1}} E(\omega^T V \omega)^2 = \sup_{\omega \in S^{p-1}} \frac{1}{n} \sum_{i=1}^{n} (\omega^T V_i \omega)^2 = \nu^2.\]

By Proposition A.2,

\[P \left( \frac{1}{n} \sum_{i \in T} V_i \right) \geq E \left\| \sum_{i \in T} V_i \right\|_{op} + t \leq \exp \left\{ -\frac{(n + 2)t^2}{8n^2\nu^2} \right\} \leq \exp \left\{ -\frac{t^2}{8n\nu^2} \right\},\]

or, equivalently, with probability $1 - \delta$,

\[(S59) \quad \left\| \sum_{i \in T} V_i \right\|_{op} \leq E \left\| \sum_{i \in T} V_i \right\|_{op} + \sqrt{8n \log \frac{1}{\delta}} \nu.\]
We then bound $\mathbb{E}\left\| \sum_{i \in T} V_i \right\|_{\text{op}}$. Let $\tilde{V}_1, \ldots, \tilde{V}_m$ be an i.i.d. random sample with replacement from $C$. We have

\begin{equation}
\mathbb{E}\left\| \sum_{i \in T} V_i \right\|_{\text{op}} \leq \mathbb{E}\left\| \sum_{i=1}^m \tilde{V}_i \right\|_{\text{op}} \leq \left( \mathbb{E}\left\| \sum_{i=1}^m \tilde{V}_i \right\|_{\text{op}}^2 \right)^{1/2} \leq \sqrt{nC(p)\nu + C(p)\nu_+},
\end{equation}

where the first inequality follows from Proposition A.1 due to the convexity of $\| \cdot \|_{\text{op}}$, the second inequality follows from the Cauchy–Schwarz inequality, and the third inequality follows from Proposition A.3.

Combining (S59) and (S60), we complete the proof. \hfill \Box

**APPENDIX B: MEAN AND VARIANCE OF THE SUM OF RANDOM ROWS AND COLUMNS OF A MATRIX**

We give a full proof of Lemma 6.5. When $m = 0$ or $m = n$, $Q_T$ is deterministic with zero variance and the inequality holds automatically. Thus we assume $1 \leq m \leq n - 1$.

Let $\sum_{[i_1, \ldots, i_k]}$ denote the sum over all $(i_1, \ldots, i_k)$ with mutually distinct elements in $\{1, \ldots, n\}$. For instance, $\sum_{[i,j]}$ denotes the sum over all pairs $(i, j)$ with $i \neq j$. We first state a basic result for sampling without replacement.

**Lemma B.1.** Let $i_1, \ldots, i_k$ be distinct indices in $\{1, \ldots, n\}$ and $T$ be a uniformly random subset of $\{1, \ldots, n\}$ with size $m$. Then

$$
\mathbb{P}(i_1, \ldots, i_k \in T) = \frac{m \cdots (m - k + 1)}{n \cdots (n - k + 1)}.
$$

By definition,

\begin{equation}
Q_T = \sum_{i=1}^n Q_{ii} I(i \in T) + \sum_{[i,j]} Q_{ij} I(i, j \in T).
\end{equation}

The mean of $Q_T$ follows directly from Lemma B.1:

\begin{align*}
\mathbb{E}Q_T &= \sum_{i=1}^n Q_{ii} \cdot \frac{m}{n} + \sum_{[i,j]} Q_{ij} \cdot \frac{m(m-1)}{n(n-1)} \\
&= \frac{m(n-m)}{n(n-1)} \operatorname{tr}(Q) + \frac{m(m-1)}{n(n-1)} (1^T Q 1).
\end{align*}
The rest of this section proves the result of the variance. Let
\[ c_1 = \frac{m(n-m)}{n(n-1)}, \quad c_2 = \text{Var}(I(1,2 \in T)) = \frac{c_1 (m-1)(n+m-1)}{n(n-1)}, \]
\[ c_3 = \text{Cov}(I(1,2 \in T), I(1,3 \in T)) = \frac{c_1 (m-1)(mn-2m-2n+2)}{n(n-1)(n-2)}, \]
\[ c_4 = \text{Cov}(I(1,2 \in T), I(3,4 \in T)) = \frac{c_1 (m-1)(-4mn+6n+6m-6)}{n(n-1)(n-2)(n-3)}, \]
\[ c_5 = \text{Cov}(I(1 \in T), I(1,2 \in T)) = \frac{c_1 m-1}{n}, \]
\[ c_6 = \text{Cov}(I(1 \in T), I(2,3 \in T)) = -c_1 \frac{2(m-1)}{n(n-2)}. \]

Using (S61), we have
\[
\text{Var}(Q_T) = \text{Var} \left( \sum_{i=1}^{n} Q_{ii} I(i \in T) \right) + \text{Var} \left( \sum_{[i,j]} Q_{ij} I(i,j \in T) \right) + 2 \text{Cov} \left( \sum_{i=1}^{n} Q_{ii} I(i \in T), \sum_{[i,j]} Q_{ij} I(i,j \in T) \right).
\]

The next subsection deals with the three terms in (S62), separately.

**B.1. Simplifying (S62).**

**Term V_1.** Lemma 6.1 implies
\[
V_1 = \text{Var} \left( \sum_{i=1}^{n} Q_{ii} I(i \in T) \right) = \frac{m(n-m)}{n(n-1)} \sum_{i=1}^{n} \left( Q_{ii} - \frac{1}{n} \sum_{i=1}^{n} Q_{ii} \right)^2
\]
\[
= c_1 \sum_{i=1}^{n} Q_{ii}^2 - \frac{c_1}{n} (\text{tr}(Q))^2.
\]

**Term V_II.** We expand V_II as
\[
V_{II} = \text{Var} \left( \sum_{[i,j]} Q_{ij} I(i,j \in T) \right) = \text{Cov} \left( \sum_{[i,j]} Q_{ij} I(i,j \in T), \sum_{[i',j']} Q_{i'j'} I(i',j' \in T) \right)
\]
\[
= \sum_{[i,j]} (Q_{ij}^2 + Q_{ij} Q_{ji}) \text{Var}(I(i,j \in T)) + \sum_{[i,j,k,\ell]} Q_{ij} Q_{k\ell} \text{Cov}(I(i,j \in T), I(k,\ell \in T)).
\]
\[ + \sum_{[i,j,k]} (Q_{ij}Q_{ik} + Q_{ij}Q_{ki}) \text{Cov}(I(i, j \in T), I(i, k \in T)) \]
\[ + \sum_{[i,j,k]} (Q_{ij}Q_{jk} + Q_{ij}Q_{kj}) \text{Cov}(I(i, j \in T), I(j, k \in T)) \]
\[ = c_2 \sum_{[i,j]} (Q_{ij}^2 + Q_{ij}Q_{ji}) + c_4 \sum_{[i,j,k,\ell]} Q_{ij}Q_{k\ell} \]
\[ + c_3 \sum_{[i,j,k]} (Q_{ij}Q_{ik} + Q_{ij}Q_{ki} + Q_{ij}Q_{jk} + Q_{ij}Q_{kj}) \].

We then reduce the summation over \([i,j,k]\) to summations over fewer indices. First,
\[
\left( \sum_{[i,j]} Q_{ij} \right)^2 = \sum_{[i,j]} (Q_{ij}^2 + Q_{ij}Q_{ji}) + \sum_{[i,j,k,\ell]} Q_{ij}Q_{k\ell} \]
\[ + \sum_{[i,j,k]} (Q_{ij}Q_{ik} + Q_{ij}Q_{ki} + Q_{ij}Q_{jk} + Q_{ij}Q_{kj}) .\]

Second, \(1^TQ1 = 0\) implies \(\sum_{[i,j]} Q_{ij} = -\sum_{i=1}^n Q_{ii} = -\text{tr}(Q)\), which further implies
\[
\sum_{[i,j,k,\ell]} Q_{ij}Q_{k\ell} = (\text{tr}(Q))^2 - \sum_{[i,j]} (Q_{ij}^2 + Q_{ij}Q_{ji}) \]
\[ - \sum_{[i,j,k]} (Q_{ij}Q_{ik} + Q_{ij}Q_{ki} + Q_{ij}Q_{jk} + Q_{ij}Q_{kj}) .\]

The above two facts simplify \(V_{II}\) to
\[
V_{II} = c_4 (\text{tr}(Q))^2 + (c_2 - c_4) \sum_{[i,j]} (Q_{ij}^2 + Q_{ij}Q_{ji}) \]
\[ + (c_3 - c_4) \sum_{[i,j,k]} (Q_{ij}Q_{ik} + Q_{ij}Q_{ki} + Q_{ij}Q_{jk} + Q_{ij}Q_{kj}) .\]

We then reduce the summation over \([i, j, k]\) to summations over fewer indices. Note that \(1^TQ = Q1 = 0\) implies \(\sum_{j=1}^n Q_{ij} = \sum_{i=1}^n Q_{ij} = 0\), which further implies
\[
\sum_{[i,j,k]} Q_{ij}Q_{ik} = \sum_{[i,j]} Q_{ij} \sum_{k \neq i,j} Q_{ik} = -\sum_{[i,j]} Q_{ij}(Q_{ii} + Q_{ij}) \]
\[ = -\sum_{i=1}^n Q_{ii} \sum_{j \neq i} Q_{ij} - \sum_{[i,j]} Q_{ij}^2 = \sum_{i=1}^n Q_{ii}^2 - \sum_{[i,j]} Q_{ij}^2 .\]
Similarly,

\[
\sum_{[i,j,k]} Q_{ij} Q_{kj} = \sum_{[i,j]} Q_{ij}^2 - \sum_{[i,j]} Q_{ij}^2,
\]

\[
\sum_{[i,j,k]} Q_{ij} Q_{ki} = \sum_{[i,j,k]} Q_{ij} Q_{jk} = \sum_{[i,j]} Q_{ii}^2 - \sum_{[i,j]} Q_{ij} Q_{ji}.
\]

Using the above three identities to simplify the third term in (S64), we obtain

(S65)

\[
V_{III} = c_4 (\text{tr}(Q))^2 + 4(c_3 - c_4) \sum_{i=1}^{n} Q_{ii}^2 + (c_2 - 2c_3 + c_4) \sum_{[i,j]} (Q_{ij}^2 + Q_{ij} Q_{ji}).
\]

**Term V_{III}**. The covariance term is

\[
V_{III} = \text{Cov} \left( \sum_{i=1}^{n} Q_{ii} I(i \in T), \sum_{[i,j]} Q_{ij} I(i, j \in T) \right)
\]

\[
= \sum_{[i,j]} Q_{ii} (Q_{ij} + Q_{ji}) \text{Cov} (I(i \in T), I(i, j \in T))
\]

\[
+ \sum_{[i,j,k]} Q_{ii} Q_{jk} \text{Cov} (I(i \in T), I(j, k \in T))
\]

\[
= c_5 \sum_{[i,j]} Q_{ii} (Q_{ij} + Q_{ji}) + c_6 \sum_{[i,j,k]} Q_{ii} Q_{jk}.
\]

Similar to previous arguments,

\[
\sum_{[i,j]} Q_{ii} (Q_{ij} + Q_{ji}) = \sum_{i=1}^{n} Q_{ii} \sum_{j \neq i} (Q_{ij} + Q_{ji}) = -2 \sum_{i=1}^{n} Q_{ii}^2,
\]

\[
\sum_{[i,j,k]} Q_{ii} Q_{jk} = \sum_{[i,j]} Q_{ii} \sum_{k \neq i,j} Q_{jk} = - \sum_{[i,j]} Q_{ii} (Q_{jj} + Q_{ji})
\]

\[
= - \sum_{i=1}^{n} Q_{ii} (Q_{jj} + Q_{ji}) = - \sum_{i=1}^{n} Q_{ii} (\text{tr}(Q) - Q_{ii} - Q_{ji})
\]

\[
= -(\text{tr}(Q))^2 + 2 \sum_{i=1}^{n} Q_{ii}^2.
\]

Using the above two identities, we can simplify \(V_{III}\) to

(S66)

\[
V_{III} = -c_6 (\text{tr}(Q))^2 - 2(c_5 - c_6) \sum_{i=1}^{n} Q_{ii}^2.
\]
Putting (S63), (S65) and (S66) together, we obtain that

\[
\text{Var}(Q) = \left( c_1 + 4c_3 - 4c_4 - 4c_5 + 4c_6 \right) c_1 \sum_{i=1}^{n} Q_{ii}^2 + \left( c_4 - \frac{c_1}{n} - 2c_6 \right) (\text{tr}(Q))^2 \\
+ (c_2 - 2c_3 + c_4) \sum_{[i,j]} (Q_{ij}^2 + Q_{ij}Q_{ji}).
\]

(S67)

We simplify (S67) in the next subsection by deriving bounds for the coefficients.

**B.2. Bounding the coefficients \(C_I, C_{II}\) and \(C_{III}\) in (S67).**

**Bounding \(C_1\).** We have

\[
C_1 = c_1 + 4c_3 - 4c_4 - 4c_5 + 4c_6 \\
= c_1 + 4c_1 \frac{m-1}{n} \left( \frac{mn - 2m - 2n + 2}{(n-1)(n-2)} + \frac{4mn - 6m - 6n + 6}{(n-1)(n-2)(n-3)} - 1 - \frac{2}{n-2} \right).
\]

Through tedious calculation, we obtain that

\[
\frac{mn - 2m - 2n + 2}{(n-1)(n-2)} + \frac{4mn - 6m - 6n + 6}{(n-1)(n-2)(n-3)} - 1 - \frac{2}{n-2} = -\frac{(n-m-1)n}{(n-2)(n-3)}.
\]

Thus, \(C_1 = c_1 \left( 1 - \frac{4(m-1)(n-m-1)}{(n-2)(n-3)} \right) \).

**Bounding \(C_{II}\).** We have

\[
C_{II} = c_4 - \frac{c_1}{n} - 2c_6 = -\frac{c_1}{n} + c_1 \frac{m-1}{n(n-2)} \left( \frac{4mn + 6m + 6n - 6}{(n-1)(n-3)} + 4 \right)
\]

\[
= -\frac{c_1}{n} \left( 1 + \frac{(m-1)(4n^2 - 4mn + 6m - 10n + 6)}{(n-1)(n-2)(n-3)} \right)
\]

\[
= -\frac{c_1}{n} \left( 1 - \frac{(m-1)(n-m-1)(4n-6)}{(n-1)(n-2)(n-3)} \right)
\]

\[
\leq c_1 \frac{(m-1)(n-m-1)(4n-6)}{n(n-1)(n-2)(n-3)} \leq \frac{c_1}{n} \frac{4(m-1)(n-m-1)}{n(n-2)(n-3)}.
\]

**Bounding \(C_{III}\).** We consider four cases.

- If \(m = 1\), then \(c_2 = c_3 = c_4 = 0\) and \(C_{III} \leq \frac{c_1}{2}\).
- If \(m = 2\), then

\[
C_{III} = c_1 \left( 1 + \frac{n}{n(n-1)} - \frac{4}{n(n-1)(n-2)} - \frac{2}{n(n-1)(n-2)} \right)
\]
\[= c_1 \left( \frac{n + 1}{n(n - 1)} + \frac{2}{n(n - 1)(n - 2)} \right) \leq \frac{c_1}{2}.\]

- If \( m = 3 \), then
  \[C_{III} = c_1 \left( \frac{2(n + 2)}{n(n - 1)} - \frac{4(n - 4)}{n(n - 1)(n - 2)} - \frac{12n - 24}{n(n - 1)(n - 2)(n - 3)} \right)\]
  \[= c_1 \left( \frac{2(n + 2)}{n(n - 1)} - \frac{4(n - 4)}{n(n - 1)(n - 2)} - \frac{12}{n(n - 1)(n - 3)} \right).\]
  If \( n \geq 7 \),
  \[C_{III} \leq c_1 \frac{2(n + 2)}{n(n - 1)} \leq \frac{c_1}{2}.\]
  For \( n = 4, 5, 6 \), we can also verify that \( C_{III} \leq \frac{c_1}{2} \).

- If \( m \geq 4 \), then
  \[4mn - 6m - 6n + 6 = (2m - 6)(n - 3) + 2(mn - 6) \geq (2m + 2)(n - 3).\]
  and thus
  \[c_4 \leq c_1 \frac{(2m + 2)(m - 1)}{n(n - 1)(n - 2)}.\]

Then we have
\[C_{III} \leq c_1 \frac{m - 1}{n(n - 1)} \left( n + m - 1 - \frac{2(mn - 2m - 2n + 2)}{n - 2} \right) - \frac{2m + 2}{n - 2}\]
\[= c_1 \frac{m - 1}{n(n - 1)} \left( n + m - 1 - \frac{2mn - 4n - 2m + 6}{n - 2} \right)\]
\[= c_1 \frac{m - 1}{n(n - 1)} \left( n - m + 3 - \frac{2m - 2}{n - 2} \right)\]
\[\leq c_1 \left( \frac{(m - 1)(n - m + 3)}{n(n - 1)} - \frac{2(m - 1)^2}{n(n - 1)(n - 2)} \right)\]
\[\leq c_1 \left( \frac{(n + 2)^2}{4n(n - 1)} - \frac{2(m - 1)^2}{n(n - 1)(n - 2)} \right)\]
\[\leq c_1 \left( \frac{(n + 2)^2}{4n(n - 1)} - \frac{18}{n(n - 1)(n - 2)} \right).\]

(S68)

If \( n \geq 7 \),
\[C_{III} \leq c_1 \frac{(n + 2)^2}{4n(n - 1)} \leq \frac{81c_1}{168} \leq \frac{c_1}{2}.\]

For \( n = 4, 5, 6 \), we can also verify that \( C_{III} \leq \frac{c_1}{2} \).
Therefore, we always have \( C_{III} \leq \frac{c_1}{2} \).
Using the above bounds for \((C_I, C_{II}, C_{III})\) in (S67), we obtain that
\[
\text{Var}(Q_T) \leq c_1 \left(1 - \frac{4(m - 1)(n - m - 1)}{(n - 2)(n - 3)}\right) \sum_{i=1}^{n} Q_{ii}^2 + c_1 \frac{4(m - 1)(n - m - 1)}{(n - 2)(n - 3)} \frac{(\text{tr}(Q))^2}{n} + \frac{c_1}{2} \sum_{[i,j]} (Q_{ij}^2 + Q_{ji}Q_{ji}).
\]
Because \((\text{tr}(Q))^2 \leq n \sum_{i=1}^{n} Q_{ii}^2\) and \(Q_{ij}Q_{ji} \leq (Q_{ij}^2 + Q_{ji}^2)/2\), we conclude that \(\text{Var}(Q_T) \leq c_1 \|Q\|^2_F\).

**APPENDIX C: PROOFS OF THE LEMMAS IN SECTION 6.2**

**Proof of Lemma 6.6.** Using the definitions of \(\sigma_n^2\) and \(\rho_e\), we have
\[
\sigma_n^2 = \left(\frac{1}{n_1} - \frac{1}{n}\right) \sum_{i=1}^{n} e_i^2(1) + \left(\frac{1}{n_0} - \frac{1}{n}\right) \sum_{i=1}^{n} e_i^2(0) + \frac{2}{n} \sum_{i=1}^{n} e_i(1)e_i(0)
= \frac{n_0}{n_1n} \sum_{i=1}^{n} e_i^2(1) + \frac{n_1}{n_0n} \sum_{i=1}^{n} e_i^2(0) + \frac{2\rho_e}{n} \sqrt{\sum_{i=1}^{n} e_i^2(1)} \sqrt{\sum_{i=1}^{n} e_i^2(0)}.
\]
If \(\rho_e \geq 0\), then
\[
\sigma_n^2 \geq \frac{n_0}{n_1n} \sum_{i=1}^{n} e_i^2(1) + \frac{n_1}{n_0n} \sum_{i=1}^{n} e_i^2(0) \geq \min \left\{ \frac{n_1}{n_0}, \frac{n_0}{n_1} \right\} \mathcal{E}_2.
\]
If \(\rho_e < 0\), then using the fact
\[
\left(\sqrt{\frac{n_0}{n_1}}a - \sqrt{\frac{n_1}{n_0}}b\right)^2 \geq 0 \iff 2ab \leq \frac{n_0}{n_1}a^2 + \frac{n_1}{n_0}b^2,
\]
we obtain that
\[
\sigma_n^2 \geq (1 + \rho_e) \left(\frac{n_0}{n_1n} \sum_{i=1}^{n} e_i^2(1) + \frac{n_1}{n_0n} \sum_{i=1}^{n} e_i^2(0)\right) \geq \eta \min \left\{ \frac{n_1}{n_0}, \frac{n_0}{n_1} \right\} \mathcal{E}_2.
\]
Putting the pieces together, we complete the proof.

**Proof of Lemma 6.7.** Recall that \(\hat{\mu}_t\) is the intercept from the OLS fit of \(Y_{t}^{\text{obs}}\) on \(1\) and \(X_t\). From the Frisch–Waugh Theorem, it is identical to the coefficient of the OLS fit of the residual \((I - H_t)Y_{t}^{\text{obs}}\) on the residual \((I - H_t)1\), after projecting onto \(X_t\):
\[
\hat{\mu}_t = \frac{1^T(I - H_t)(I - H_t)Y_{t}^{\text{obs}}}{\| (I - H_t)1 \|^2_2} = \frac{1^T(I - H_t)Y_{t}^{\text{obs}}}{1^T(I - H_t)1}.
\]
Using the definition (6) and the fact that \((I - H_t)X_t = 0\), we have
\[(I - H_t)Y_t^{\text{obs}} = (I - H_t)(\mu_t 1 + X_t \beta_t + e_t(t)) = \mu_t (I - H_t)1 + (I - H_t)e_t(t),\]
which further implies
\[
\Rightarrow \hat{\mu}_t = \mu_t + \frac{1^T(I - H_t)e_t(t)}{1^T(I - H_t)1} = \mu_t + \frac{1^T e_t(t)/n_t - 1^T H_t e_t(t)/n_t}{1 - 1^T H_t 1/n_t}.
\]
Recalling that \(\tau = \mu_1 - \mu_0\), we complete the proof.

**Proof of Lemma 6.8.** Because \(\|U\|_{\text{op}} \leq \|U\|_F\), Lemma 6.3 implies that with probability \(1 - \delta\),
\[
\left\| \sum_{i \in T} u_i \right\|_2 / \|U\|_F \leq \sqrt{\frac{m(n-m)}{n(n-1)}} + \sqrt{8 \log \frac{1}{\delta}},
\]
which further implies \(\left\| \sum_{i \in \tilde{T}} u_i \right\|_2 = O_P(\|U\|_F)\). This immediately implies the three results in Lemma 6.8 by choosing appropriate \(U\).

Let \(u_i = e_i(t)\) with \(\sum_{i=1}^n u_i = 0\), \(U = (u_1, \ldots, u_n)^T \in \mathbb{R}^{n \times 1}\), and \(\|U\|_F^2 = \sum_{i=1}^n u_i^2 = \sum_{i=1}^n e_i^2(t)\). Therefore,
\[
1^T e_t(t) = \left\| \sum_{i \in \tilde{T}} u_i \right\|_2 = O_P(\|U\|_F) = O_P\left(\sqrt{\sum_{i=1}^n e_i^2(t)}\right) = O_P\left(\sqrt{n \mathcal{E}_2}\right).
\]
Let \(u_i = x_i\) with \(\sum_{i=1}^n u_i = 0\), \(U = X\), and \(\|U\|_F = \|X\|_F = \sqrt{\text{tr}(X^TX)} = \text{tr}(nI) = np\). Therefore,
\[
\|X_t^T 1\|_2 = \left\| \sum_{i \in \tilde{T}} u_i \right\|_2 = O_P(\|U\|_F) = O_P\left(\sqrt{np}\right).
\]
Let \(u_i = x_i e_i(t)\) with \(\sum_{i=1}^n u_i = 0\) due to (7). Therefore,
\[
\|X_t^T e_t(t)\|_2 = \left\| \sum_{i \in \tilde{T}} u_i \right\|_2 = O_P\left(\sqrt{\sum_{i=1}^n \|x_i\|^2 e_i^2(t)}\right).
\]
Recalling (28) that \(\|x_i\|^2 = nH_{ii} \leq n\kappa\), we have \(\|X_t^T e_t(t)\|_2 = O_P\left(n\sqrt{\mathcal{E}_2}\kappa\right)\).

We need the following proposition below.

**Proposition C.1.** \(A\) and \(B\) are two symmetric matrices. \(A\) is positive definite and \(A + B\) is invertible. Then
\[
\|(A + B)^{-1} - A^{-1}\|_{\text{op}} \leq \frac{\|A^{-1}\|_{\text{op}}^2 \cdot \|B\|_{\text{op}}}{1 - \min\{1, \|A^{-1}\|_{\text{op}} \cdot \|B\|_{\text{op}}\}}.
\]
PROOF OF PROPOSITION C.1. Let $M = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and $\Lambda(M)$ be its spectrum. By definition, $\|M\|_{op} \leq \|A^{-1}\|_{op} \cdot \|B\|_{op}$. If $\|A^{-1}\|_{op} \cdot \|B\|_{op} \geq 1$, the inequality is trivial because the right-hand side of it is $\infty$. Without loss of generality, we assume $\|A^{-1}\|_{op} \cdot \|B\|_{op} < 1$, which implies $\|M\|_{op} < 1$.

Combining

$$\|(A + B)^{-1} - A^{-1}\|_{op} = \|A^{-\frac{1}{2}}((I + M)^{-1} - I)A^{-\frac{1}{2}}\|_{op} \leq \|A^{-1}\|_{op} \cdot \|I - (I + M)^{-1}\|_{op}$$

and

$$\|I - (I + M)^{-1}\|_{op} \leq \sup_{\lambda \in \Lambda(M)} \left| \frac{\lambda}{1 + \lambda} \right| = \frac{\|M\|_{op}}{1 - \|M\|_{op}} \leq \frac{\|A^{-1}\|_{op} \cdot \|B\|_{op}}{1 - \|A^{-1}\|_{op} \cdot \|B\|_{op}},$$

we have

$$\|(A + B)^{-1} - A^{-1}\|_{op} \leq \frac{\|A^{-1}\|_{op}^2 \cdot \|B\|_{op}}{1 - \|A^{-1}\|_{op} \cdot \|B\|_{op}}.$$

\[ \square \]

PROOF OF LEMMA 6.9. Let $V_i = x_i x_i^T - I$, then $\sum_{i=1}^{n} V_i = 0$. In the following, we repeatedly using the basic facts: $n^{-1}X^T X = I$, $\|x_i\|^2 = nH_{ii}$, and $\sum_{i=1}^{n} x_i x_i^T = XX^T = nH$. Recalling the definitions of $\nu, \nu_+$ and $\nu_-$ in Lemma 6.4, we have

$$\nu^2 = \left\| \frac{1}{n} \sum_{i=1}^{n} V_i \right\|_{op} = \left\| \frac{1}{n} \sum_{i=1}^{n} \left( \|x_i\|^2 \frac{2}{x_i^T x_i} - 2x_i x_i^T + I \right) \right\|_{op}$$

$$= \left\| \left( \frac{1}{n} \sum_{i=1}^{n} \|x_i\|^2 \frac{2}{x_i^T x_i} \right) I - 1 \right\|_{op} = \left\| \left( \sum_{i=1}^{n} H_{ii} x_i x_i^T \right) I - 1 \right\|_{op}$$

$$\leq \sum_{i=1}^{n} H_{ii} x_i x_i^T + 1 \leq \kappa \sum_{i=1}^{n} x_i x_i^T + 1 = n\kappa \|H\|_{op} + 1 = n\kappa + 1,$$

$$\nu_+ = \max_{1 \leq i \leq n} \|x_i x_i^T - I\|_{op} \leq \max_{1 \leq i \leq n} \|x_i\|^2 + 1 = n \max_{1 \leq i \leq n} H_{ii} + 1 = n\kappa + 1,$$

$$\nu_-^2 = \sup_{\omega \in S^{n-1}} \frac{1}{n} \sum_{i=1}^{n} (\omega^T V_i \omega)^2 = \sup_{\omega \in S^{n-1}} \frac{1}{n} \sum_{i=1}^{n} ((x_i^T \omega)^2 - 1)^2$$

$$= \sup_{\omega \in S^{n-1}} \frac{1}{n} \sum_{i=1}^{n} \left[ (x_i^T \omega)^4 - 2(x_i^T \omega)^2 + 1 \right]$$
\[
\begin{align*}
= \sup_{\omega \in S_{p-1}} \frac{1}{n} \sum_{i=1}^{n} (x_i^T \omega)^4 & - 2 \omega^T \left( \frac{X^T X}{n} \right) \omega + 1 \\
= \sup_{\omega \in S_{p-1}} \frac{1}{n} \sum_{i=1}^{n} (x_i^T \omega)^4 & - 1 \leq \sup_{\omega \in S_{p-1}} \frac{1}{n} \sum_{i=1}^{n} (x_i^T \omega)^4 \\
\leq \sup_{\omega \in S_{p-1}} \frac{1}{n} \sum_{i=1}^{n} \|x_i\|^2 \|x_i^T \omega\|^2 & = \left\| \sum_{i=1}^{n} H_i x_i x_i^T \right\|_{op} \leq n \kappa.
\end{align*}
\]

By Lemma 6.4,
\[
\left\| \frac{X_t^T X_t}{n_t} - I \right\|_{op} = \frac{1}{n_t} \left\| \sum_{i \in T_t} V_i \right\|_{op} = O_P \left( \frac{1}{n_t} \left[ n \sqrt{C(p) \kappa} + n C(p) \kappa + n \sqrt{\kappa} \right] \right) = O_P \left( \sqrt{\kappa \log p} + \kappa \log p \right).
\]

By Assumption 2, \( \kappa \log p = o(1) \), and therefore the first result holds:

\[(S69) \quad \left\| \frac{X_t^T X_t}{n_t} - I \right\|_{op} = O_P \left( \sqrt{\kappa \log p} \right) = o_P(1). \]

Thus with probability \( 1 - o(1) \),

\[(S70) \quad \left\| \frac{X_t^T X_t}{n_t} - I \right\|_{op} \leq \frac{1}{2} \implies \left\| \frac{X_t^T X_t}{n_t} \right\|_{op} \geq \frac{1}{2}, \]

where we use the convexity of \( \cdot \|_{op}. \) Note that for any Hermitian matrix \( A \), \( \|A^{-1}\|_{op} = \lambda_{\min}(A)^{-1} \) where \( \lambda_{\min} \) denotes the minimum eigenvalue. Thus with probability \( 1 - o(1) \),

\[(S71) \quad \left\| \left( \frac{X_t^T X_t}{n_t} \right)^{-1} \right\|_{op} \leq 2. \]

As a result,

\[\left\| \left( \frac{X_t^T X_t}{n_t} \right)^{-1} \right\|_{op} = O_P(1). \]

To prove the third claim, we apply Proposition C.1 with \( A = I \) and \( B = n_t^{-1} X_t^T X_t - I \). By (S70) and (S71), with probability \( 1 - o(1) \), \( A + B \) is invertible and \( \|B\|_{op} \leq 1/2 \). Together with (S69), we have

\[\left\| \left( \frac{X_t^T X_t}{n_t} \right)^{-1} - I \right\|_{op} = O_P \left( \frac{\|B\|_{op}}{1 - \|B\|_{op}} \right) = O_P(\|B\|_{op}) = O_P(\sqrt{\kappa \log p}). \]

\[\square\]
Proof of Lemma 6.10. First, (7) implies
\[ 1^T Q(t) = 1^T H \text{diag}(e(t)) = 1^T X (X^T X)^{-1} X^T \text{diag}(e(t)) = 0, \]
which further imply \( 1^T Q(t) 1 = 0 \). Second, (9) implies \( \text{tr}(Q(t)) = n \Delta_t \). Third,
\[ \|Q(t)\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} H_{ij}^2 e_{ij}^2(t) = \sum_{j=1}^{n} e_{jj}^2(t) \left( \sum_{i=1}^{n} H_{ij}^2 \right). \]
Because \( H \) is idempotent, \( H^T H = H \implies \sum_{i=1}^{n} H_{ij}^2 = H_{jj} \) for all \( j \). Thus,
\[ \|Q(t)\|_F^2 = \sum_{j=1}^{n} e_{jj}^2(t) H_{jj} \leq n \mathcal{E}_2 \kappa. \]
\( \square \)
Supplementary Material II: Propositions 3.1 and 3.2

The proofs rely on the following results.

**Proposition C.2.** [modified version of Corollary 3.1 of Yaskov [2014]]
Let $Z_i$ be i.i.d. random vectors in $\mathbb{R}^p$ with mean 0 and covariance $I$. Suppose

$$L(\delta) \triangleq \sup_{\nu \in S^{p-1}} \mathbb{E}|\nu^T Z_i|^\delta < \infty$$

for some $\delta > 2$. For any constant $C > 0$, with probability $1 - e^{-Cp}$,

$$\lambda_{\min} \left( \frac{Z^T Z}{n} \right) \geq 1 - 5 \left( \frac{pC}{n} \right)^{\frac{\delta}{\delta + 2}} L(\delta)^{\frac{2}{\delta + 2}} \left( 1 + \frac{1}{C} \right).$$

**Proof of Proposition C.2.** Write $y = p/n$ and $L = L(\delta)$. The proof of Corollary 3.1 of Yaskov [2014, page 6] showed that for any $a > 0$,

$$\mathbb{P} \left( \lambda_{\min} \left( \frac{Z^T Z}{n} \right) < 1 - 4La^{-\delta/2} - 5ay \right) \leq \exp \left\{ -La^{-1-\delta/2n} \right\}.$$

Let $a = (Cy/L)^{-2/(\delta+2)}$. Then the right-hand side is $1 - e^{-Cp}$. Thus with probability $1 - e^{-Cp}$,

$$\lambda_{\min} \left( \frac{Z^T Z}{n} \right) \geq 1 - y^{\delta+2} L^{\frac{2}{\delta + 2}} \left( 5C^{-\frac{2}{\delta + 2}} + 4C^{\frac{\delta}{\delta + 2}} \right)$$

$$\geq 1 - 5(1+C^{\frac{\delta}{\delta + 2}}) \left( 1 + \frac{1}{C} \right).$$

$$\square$$

**Proposition C.3 (Theorem 1 of Tikhomirov [2017]).** Let $Z_i$ be i.i.d. random vectors in $\mathbb{R}^p$ with mean 0 and covariance $I$. Suppose

$$L(\delta) \triangleq \sup_{\nu \in S^{p-1}} \mathbb{E}|\nu^T Z_i|^\delta < \infty$$

for some $\delta > 2$. Then with probability at least $1 - 1/n$,

$$\nu(\delta)^{-1} \left\| \frac{Z^T Z}{n} - I \right\|_{op} \leq \max_{1 \leq i \leq n} \frac{||Z_i||^2}{n} + L(\delta)^{\frac{2}{\delta + 2}} \left\{ \left( \frac{L}{n} \right)^{\delta-2} \frac{\log^4 \left( \frac{n}{p} \right)}{\min(\delta-2, 1)} + \left( \frac{L}{n} \right)^{\frac{\min(\delta, 4) - 2}{\min(\delta, 4)}} \right\},$$

for some constant $\nu(\delta)$ depending only on $\delta$.

**Proposition C.4 (Theorem 2 of von Bahr et al. [1965]).** Let $Z_i$ be independent mean-zero random variables. Then for any $r \in [1, 2]$,

$$\mathbb{E} \left( \sum_{i=1}^n Z_i \right)^r \leq 2 \sum_{i=1}^n \mathbb{E}|Z_i|^r.$$
APPENDIX D: PROOF OF PROPOSITION 3.1

D.1. A lemma. First we prove a more general result.

**Lemma D.1.** Let \( Z_i \) be i.i.d. random vectors in \( \mathbb{R}^p \) with mean \( \mu \in \mathbb{R}^p \) and covariance matrix \( \Sigma \in \mathbb{R}^{p \times p} \). Let \( \tilde{Z}_i = \Sigma^{-1/2}(Z_i - \mu) \), and assume

\[
\sup_{\nu \in S^{p-1}} \mathbb{E}[\nu^T \tilde{Z}_i]^\delta = O(1), \quad \text{and} \quad \max_{1 \leq i \leq n} ||\tilde{Z}_i||_2^2 - \mathbb{E}||\tilde{Z}_i||_2^2 = O_p(\omega(n,p)),
\]

for some \( \delta > 2 \) and some function \( \omega(n,p) \) increasing in \( n \) and \( p \). Further let \( Z = (Z_1^T, \ldots, Z_n^T)^T \) and \( X = VZ \) so that \( X \) has centered columns. If \( p = O(n^\gamma) \) for some \( \gamma < 1 \), then over the randomness of \( Z \),

\[
\kappa = \frac{p}{n} + O_p \left( \frac{\omega(n,p)}{n} + \left( \frac{p}{n} \right)^{2\delta - 2} \log^4 \left( \frac{n}{p} \right) + \left( \frac{p}{n} \right)^{\min\{2\delta - 2, 6\}} \right).
\]

**Proof of Lemma D.1.** Let \( \tilde{Z} = (\tilde{Z}_1^T, \ldots, \tilde{Z}_n^T)^T \) and \( \tilde{X} = V\tilde{Z} \). Then \( \tilde{X} = V(Z - 1\mu^T)\Sigma^{-1/2} = VZ\Sigma^{-1/2} \), and thus

\[
\tilde{X}(\tilde{X}^T\tilde{X})^{-1}\tilde{X}^T = VZ \left( Z^T V Z \right)^{-1} V^T = X(X^TX)^{-1}X^T.
\]

Therefore, we can assume \( \mu = 0 \) and \( \Sigma = I \) without loss of generality, in which case \( Z_i = \tilde{Z}_i \) has mean 0 and covariance matrix \( I \).

By definition, \( H_{ii} = x_i^T(X^TX)^{-1}x_i \), and therefore

\[
H_{ii} = \frac{1}{n} x_i^T \left( \left( \frac{X^TX}{n} \right)^{-1} - I \right) x_i + \frac{||x_i||_2^2}{n}
\]

\[
\leq \frac{||x_i||_2^2}{n} \left( 1 + \left( \left( \frac{X^TX}{n} \right)^{-1} - I \right)_{op} \right) \tag{S72}.
\]

To bound \( \kappa \), we need to bound two key terms below.

**Bounding** \( \left\| (n^{-1}X^TX)^{-1} - I \right\|_{op} \). Let \( \tilde{Z} = n^{-1}\sum_{i=1}^n Z_i \). Note that

\[
\mathbb{E}||\tilde{Z}||_2^2 = \frac{1}{n^2} \sum_{i=1}^n \mathbb{E}||Z_i||_2^2 = \frac{1}{n} \mathbb{E}||Z_1||_2^2 = \frac{p}{n}.
\]

By Markov’s inequality,

\[
||\tilde{Z}||_2^2 = O_p \left( \frac{p}{n} \right), \tag{S73}
\]

and
and more precisely,

$$P\left( \|\bar{Z}\|_2^2 \leq \sqrt{\frac{p}{n}} \right) = 1 - P\left( \|\bar{Z}\|_2^2 > \sqrt{\frac{p}{n}} \right) \geq 1 - \sqrt{\frac{p}{n}}. $$

Let $\mathcal{A}_1$ denote the above event. Then

$$P(\mathcal{A}_1) \geq 1 - \sqrt{\frac{p}{n}}. $$

By Proposition C.3,

$$\left\| \frac{Z^T Z}{n} - I \right\|_{op} = O_P\left( \frac{\max_{1 \leq i \leq n} \|Z_i\|_2^2}{n} + \left( \frac{p}{n} \right)^{\frac{\delta - 2}{\delta}} \log^4 \left( \frac{n}{p} \right) + \left( \frac{p}{n} \right)^{\min\left(\frac{\delta - 2}{\delta}, \frac{1}{2}\right)} \right). $$

By the condition of Lemma D.1,

$$\max_{1 \leq i \leq n} \|Z_i\|_2^2 = \frac{p}{n} + \max_{1 \leq i \leq n} \|Z_i\|_2^2 - \mathbb{E}\|Z_i\|_2^2 = \frac{p}{n} + O_P\left( \frac{\omega(n, p)}{n} \right).$$

Combining the above three equations, we have

$$\left\| \frac{X^T X}{n} - I \right\|_{op} = \left\| \frac{Z^T Z}{n} - I - \bar{Z} \bar{Z}^T \right\|_{op} \leq \left\| \frac{Z^T Z}{n} - I \right\|_{op} + \|\bar{Z}\|_2^2$$

$$= O_P\left( \frac{p}{n} + \frac{\omega(n, p)}{n} + \left( \frac{p}{n} \right)^{\frac{\delta - 2}{\delta}} \log^4 \left( \frac{n}{p} \right) + \left( \frac{p}{n} \right)^{\min\left(\frac{\delta - 2}{\delta}, \frac{1}{2}\right)} \right)$$

$$= O_P\left( \frac{\omega(n, p)}{n} + \left( \frac{p}{n} \right)^{\frac{\delta - 2}{\delta}} \log^4 \left( \frac{n}{p} \right) + \left( \frac{p}{n} \right)^{\min\left(\frac{\delta - 2}{\delta}, \frac{1}{2}\right)} \right), $$

where the last line uses the fact that the third term dominates the first term due to $p/n \to 0$. On the other hand, by Proposition C.2 with $C = \sqrt{n}/p$, with probability $1 - e^{-\sqrt{np}}$,

$$\lambda_{\min}\left( \frac{Z^T Z}{n} \right) \geq 1 - 5 \left( \sqrt{\frac{p}{n}} \right)^{\frac{\delta}{\delta + 2}} L(\delta)^{\frac{2}{\delta + 2}} \left( 1 + \sqrt{\frac{p}{n}} \right) \geq 1 - 10 \left( \sqrt{\frac{p}{n}} \right)^{\frac{\delta}{2(\delta + 2)}} L(\delta)^{\frac{2}{\delta + 2}}.$$

Let $\mathcal{A}_2$ denote the above event, then

$$P(\mathcal{A}_2) \geq 1 - e^{-\sqrt{np}}.$$
Note that for any Hermitian matrices $A$ and $B$, the convexity of $\| \cdot \|_{\text{op}}$ implies that
\[ |\lambda_{\min}(A) - \lambda_{\min}(B)| = |\lambda_{\max}(-A) - \lambda_{\max}(-B)| \leq \| -A - (-B) \|_{\text{op}} = \|A - B\|_{\text{op}}. \]

We have
\[ \lambda_{\min}\left(\frac{X^T X}{n}\right) \geq \lambda_{\min}\left(\frac{Z^T Z}{n}\right) - \|\bar{Z} \bar{Z}^T\|_{\text{op}} = \lambda_{\min}\left(\frac{Z^T Z}{n}\right) - \|\bar{Z}\|_2^2. \]

Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. Then on $\mathcal{A}$,
\[ \lambda_{\min}\left(\frac{X^T X}{n}\right) \geq 1 - 10\left(\frac{p}{n}\right)^{\frac{\delta^2}{2(\delta+2)}} L(\delta)^{\frac{\delta^2}{2(\delta+2)}} - \sqrt{\frac{p}{n}}. \]

Since $p/n \to 0$, for sufficiently large $n$,
\[ \lambda_{\min}\left(\frac{X^T X}{n}\right) \geq \frac{1}{2} \]
with probability
\[ \mathbb{P}(\mathcal{A}) \geq \mathbb{P}(\mathcal{A}_1) + \mathbb{P}(\mathcal{A}_2) - 1 \geq 1 - e^{-\sqrt{np}L(\delta)} - \sqrt{\frac{p}{n}} = 1 - o(1). \]

Finally, using Proposition C.1 with $A = I$ and $B = n^{-1}X^T X - I$, by Slusky's lemma, we have that
\[ (S77) \quad \left\|\left(\frac{X^T X}{n}\right)^{-1} - I\right\|_{\text{op}} = O_p\left(\frac{\omega(n,p)}{n} + \left(\frac{p}{n}\right)^{\frac{\delta^2}{2(\delta+2)}} \log^4\left(\frac{n}{p}\right) + \left(\frac{p}{n}\right)^{\frac{\min(\delta-2,2)}{\min(3,4)}}\right). \]

Because $p = O(n^\gamma)$ for some $\gamma < 1$,
\[ (S78) \quad \left\|\left(\frac{X^T X}{n}\right)^{-1} - I\right\|_{\text{op}} = O_p\left(\frac{\omega(n,p)}{n}\right) + o_p(1). \]

**Bounding** $\max_{1 \leq i \leq n} \|x_i\|_2^2$. Because $x_i = Z_i - \bar{Z}$, the Cauchy–Schwarz inequality implies
\[ \|x_i\|_2^2 = \|Z_i\|_2^2 - 2Z_i^T \bar{Z} + \|\bar{Z}\|_2^2 \leq \|Z_i\|_2^2 + 2\|Z_i\|_2 \|\bar{Z}\|_2 + \|\bar{Z}\|_2^2. \]

By (S75) and (S73),
\[ \frac{\max_{1 \leq i \leq n} \|x_i\|_2^2}{n} = \frac{\mathbb{E}\|Z_i\|_2^2}{n} + \frac{\max_i \|Z_i\|_2^2}{n} - \mathbb{E}\|Z_i\|_2^2 + 2\|Z_i\|_2 \|\bar{Z}\|_2 + \|\bar{Z}\|_2^2 \]
\[
\frac{\omega(n,p)}{n^3} + \frac{\sqrt{\omega(n,p)p}}{n} \quad \text{and} \quad \frac{\omega(n,p)}{n^3} + \frac{\sqrt{\omega(n,p)p}}{n^{3/2}}.
\]

Because \(\omega(n,p)\) is increasing and \(p/n \to 0\), we have

\[
\sqrt{\frac{\omega(n,p)p}{n^3}} = O\left(\frac{\omega(n,p)}{n} \left(\frac{p}{n}\right)^{1/2}\right) = o\left(\frac{\omega(n,p)}{n}\right).
\]

Thus, we obtain that

\[
\text{(S79) } \max_{1 \leq i \leq n} \|x_i\|_2^2 = \frac{p}{n} + O_P\left(\frac{\omega(n,p)}{n} + \frac{\sqrt{\omega(n,p)p}}{n^{3/2}}\right).
\]

Putting (S72), (S78) and (S79) together and using some tedious cancellations, we have

\[
\kappa = \frac{p}{n} + O_P\left(\frac{\omega(n,p)}{n} + \frac{\sqrt{\omega(n,p)p}}{n^{3/2}}\right).
\]

Because

\[
\left(\frac{p}{n}\right)^{1+\frac{\min\{\delta-2,2\}}{\min\{\delta,4\}}} \geq \left(\frac{p}{n}\right)^{3/2} \geq \frac{p}{n^{3/2}},
\]

(S80) further simplifies to

\[
\kappa = \frac{p}{n} + O_P\left(\frac{\omega(n,p)}{n} + \frac{\omega^2(n,p)}{n^2} + \frac{\left(\frac{p}{n}\right)^{\frac{2\delta-2}{\delta}} \log^4 \left(\frac{n}{p}\right)}{\left(\frac{p}{n}\right)^{\frac{\min\{2\delta-2,6\}}{\min\{\delta,4\}}}}\right).
\]

We complete the proof using \(\kappa \leq 1\).

**D.2. Use Lemma D.1 to prove Proposition 3.1.** As argued in the proof of Proposition D.1, we can assume \(\mu = 0\) without loss of generality. Because the hat matrix is invariant to rescaling, we further assume \(E Z_i^2 = 1\) without loss of generality. Based on Proposition D.1, it suffices to verify

\[
\sup_{\nu \in \mathbb{S}^{p-1}} E|\nu^T Z_i|^\delta = O(1),
\]

\[
\text{(S81)}
\]
\[ \max_{1 \leq i \leq n} \left| \| Z_i \|_2^2 - \mathbb{E} \| Z_i \|_2^2 \right| = O_p \left( n^{2} p \min(\delta, 4) \right). \]

If (S81) and (S82) hold, by Proposition D.1, we have that

\[
\kappa = \frac{p}{n} + O_p \left( \frac{p^{2/\min(\delta, 4)}}{n^{(\delta-2)/\delta}} + \left( \frac{p}{n} \right)^{\frac{2\delta-2}{\delta}} \log^{4} \left( \frac{n}{p} \right) + \left( \frac{p}{n} \right)^{\frac{\min(2\delta-2, 6)}{\min(\delta, 4)}} \right).
\]

Then we can prove Proposition 3.1 for two cases.

**Case 1.** If \( \delta > 4 \), then \( \frac{2\delta-2}{\delta} < \frac{3}{2} = \min \left( \frac{2\delta-2}{\delta}, \frac{2}{\delta} \right) \). Thus the third term dominates the second term in the above \( O_p(\cdot) \), implying

\[
\kappa = \frac{p}{n} + O_p \left( \frac{p^{1/2}}{n^{(\delta-2)/\delta}} + \left( \frac{p}{n} \right)^{\frac{3}{2}} \right).
\]

**Case 2.** If \( \delta \leq 4 \), then

\[
\kappa = \frac{p}{n} + O_p \left( \frac{p^{2/\delta}}{n^{(\delta-2)/\delta}} + \left( \frac{p}{n} \right)^{\frac{2\delta-2}{\delta}} \log^{4} \left( \frac{n}{p} \right) \right).
\]

Because

\[
\left( \frac{p}{n} \right)^{\frac{2\delta-2}{\delta}} n^{-\frac{\delta-2}{\delta}} = p^{\frac{2\delta}{\delta}} p^{\frac{\delta-4}{\delta}} \leq \frac{p^{2/\delta}}{n^{(\delta-2)/\delta} n},
\]

the first term dominates in the above \( O_p(\cdot) \), implying

\[
\kappa = \frac{p}{n} + O_p \left( \frac{p^{2/\delta}}{n^{(\delta-2)/\delta}} \right) = \frac{p}{n} + O_p \left( \frac{p^{2/\delta}}{n^{(\delta-2)/\delta}} + \left( \frac{p}{n} \right)^{3/2} \right).
\]

The last identity holds because \( p^{3/2}/n^{3/2} \) is of smaller order and thus we can add it back.

We will prove (S81) and (S82) below.

**D.2.1. Proving (S81).** By Rosenthal [1970]’s inequality,

\[
\mathbb{E} |\nu^T Z_i|^\delta = \mathbb{E} \left| \sum_{j=1}^{p} \nu_j Z_{ij} \right|^\delta \leq C \left( \sum_{j=1}^{p} |\nu_j|^\delta \mathbb{E} |Z_{ij}|^\delta + \left( \sum_{j=1}^{p} \nu_j^2 \mathbb{E} Z_{ij}^2 \right)^{\delta/2} \right),
\]

where \( C \) is a constant depending only on \( \delta \). Because \( \| \nu \|_2 = 1 \), we have \( \max_{1 \leq j \leq p} |\nu_j| \leq 1 \) and thus

\[
\sum_{j=1}^{p} |\nu_j|^\delta \mathbb{E} |Z_{ij}|^\delta \leq M \sum_{j=1}^{p} |\nu_j|^\delta \leq M \sum_{j=1}^{p} |\nu_j|^\delta = M.
\]
Hölder’s inequality implies $\mathbb{E}Z_{ij}^2 \leq \left(\mathbb{E} |Z_{ij}|^2 \right)^{2/\delta} \leq M^{2/\delta}$, which further implies
\[
\left( \sum_{j=1}^{p} \nu_j^2 \mathbb{E} Z_{ij}^2 \right)^{\delta/2} \leq \left( M^{2/\delta} \right)^{\delta/2} = M.
\]
Because the above two bounds hold regardless of $\nu$, we conclude that
\[
\sup_{\nu \in S^{p-1}} \mathbb{E} \nu^T Z_i \delta \leq 2CM = O(1).
\]

D.2.2. Proving (S82). Let $W_{ij} = Z_{ij}^2 - \mathbb{E} Z_{ij}^2$. Using Jensen’s inequality that $\mathbb{E}|(X + Y)/2|^r \leq (\mathbb{E}|X|^r + \mathbb{E}|Y|^r)/2$ for any random variables $X, Y$ and $r > 1$, we obtain that
\[
\mathbb{E}|W_{ij}|^{\delta/2} \leq 2^{\delta/2-1} \left( \mathbb{E}|Z_{ij}|^{\delta} + (\mathbb{E}Z_{ij}^2)^{\delta/2} \right) \leq 2^{\delta/2} \mathbb{E}|Z_{ij}|^{\delta} \leq 2^{\delta/2} M \triangleq \tilde{M}.
\]
We consider two cases.

Case 1: $\delta \geq 4$. By Hölder’s inequality, $\mathbb{E}W_{ij}^2 \leq \tilde{M}^{4/\delta}$. By Rosenthal [1970]’s inequality,
\[
\mathbb{E}\left\|Z_i\right\|^2_2 - \mathbb{E}\left\|Z_i\right\|^2_2 \delta^{2/\delta} = \mathbb{E}\left| \sum_{j=1}^{p} W_{ij} \right|^{\delta/2} \leq C \left( \sum_{j=1}^{p} \mathbb{E}|W_{ij}|^{\delta/2} + \left( \sum_{j=1}^{p} \mathbb{E}W_{ij}^2 \right)^{\delta/4} \right) \leq C \left( p\tilde{M} + p^{\delta/4}\tilde{M} \right) \leq CM^{\delta/4},
\]
which implies $\mathbb{E}\left\|Z_i\right\|^2_2 - \mathbb{E}\left\|Z_i\right\|^2_2 \delta^{2/\delta} = O \left( p^{\delta/4} \right)$. As a result,
\[
\mathbb{E}\left\{ \max_{1 \leq i \leq n} \left| \left\|Z_i\right\|^2_2 - \mathbb{E}\left\|Z_i\right\|^2_2 \right|^{\delta/2} \right\} \leq \sum_{i=1}^{n} \mathbb{E}\left\|Z_i\right\|^2_2 - \mathbb{E}\left\|Z_i\right\|^2_2 \delta^{2/\delta} = O \left( np^{\delta/4} \right).
\]

By Markov’s inequality, $\max_{1 \leq i \leq n} \left| \left\|Z_i\right\|^2_2 - \mathbb{E}\left\|Z_i\right\|^2_2 \right| = O_p \left( n^{2/\delta} p^{1/2} \right)$.

Case 2: $\delta < 4$. By Proposition C.4, with $\delta/2 \in (1, 2),$
\[
\mathbb{E}\left\|Z_i\right\|^2_2 - \mathbb{E}\left\|Z_i\right\|^2_2 \delta^{2/\delta} = \mathbb{E}\left| \sum_{j=1}^{p} W_{ij} \right|^{\delta/2} \leq 2^{\delta/2} \mathbb{E}|W_{ij}|^{\delta/2} \leq 2p\tilde{M}.
\]
Similar to Case 1, $\max_{1 \leq i \leq n} \left| \left\|Z_i\right\|^2_2 - \mathbb{E}\left\|Z_i\right\|^2_2 \right| = O_p \left( n^{2/\delta} p^{2/\delta} \right)$.
APPENDIX E: PROOF OF PROPOSITION 3.2

Let $\bar{Y}(t) = n^{-1} \sum_{i=1}^{n} Y_i(t)$. Note that $H1 = X(X^T X)^{-1}X^T 1 = 0$. By definition, $e(t) = (1-H)\{Y(t) - \bar{Y}(t)\} = (1-H)\{Y(t) - EY_i(t) 1\}$. Throughout the rest of the proof, we assume that $EY_i(t) = 0$ without loss of generality, and define $M(\delta) \triangleq \max_{t=0,1} E|Y_i(t)|^\delta$.

E.1. Bounding $E_2^2$. Let $Z_i = Y_i(t)^2$. Then the moment condition reads $E|Z_i|^\delta < \infty$. The Kolmogorov–Marcinkiewicz–Zygmund strong law of large number [Kallenberg, 2006, Theorem 4.23] implies

$$\frac{1}{n} \sum_{i=1}^{n} Z_i \overset{a.s.}{\to} EZ_1 = O_p(1), \quad \text{if } \delta \geq 2,$$

$$\frac{1}{n^{2/\delta}} \sum_{i=1}^{n} Z_i = o(1) \implies \frac{1}{n} \sum_{i=1}^{n} Z_i = o_p(n^{2/\delta - 1}), \quad \text{if } \delta < 2.$$  

On the other hand,

$$\frac{1}{n} \|e(t)\|_2^2 = \frac{1}{n} Y(t)^T (I-H)Y(t) \leq \frac{1}{n} \|Y(t)\|_2^2 = \frac{1}{n} \sum_{i=1}^{n} Z_i,$$

which further implies the bound for $E_2$.

E.2. Bounding $E_2^{-1}$. Without loss of generality, we assume that $Y_i(1)$ is not a constant with probability 1. First we show that

$$\frac{Y(1)^T H Y(1)}{Y(1)^T Y(1)} = o_p(1).$$

For any permutation $\pi$ on $\{1, \ldots, n\}$, let $H(\pi)$ denote the matrix with

$$H(\pi)_{ij} = H_{\pi(i),\pi(j)}.$$

Since $Y_i(1)$ are i.i.d., for any $\pi$,

$$(Y_1(1), \ldots, Y_n(1)) \overset{d}{=} (Y_{\pi^{-1}(1)}(1), \ldots, Y_{\pi^{-1}(n)}(1)),$$

and thus

$$\frac{Y(1)^T H(\pi) Y(1)}{Y(1)^T Y(1)} = \frac{\sum_{i,j=1}^{n} H_{\pi(i),\pi(j)} Y_i(1) Y_j(1)}{\sum_{i=1}^{n} Y_i(1)^2} = \frac{\sum_{i,j=1}^{n} H_{\pi^{-1}(i),\pi^{-1}(j)} Y_{\pi^{-1}(i)}(1) Y_{\pi^{-1}(j)}(1)}{\sum_{i=1}^{n} Y_{\pi^{-1}(i)}(1)^2} \overset{d}{=} \frac{Y(1)^T H Y(1)}{Y(1)^T Y(1)}.$$
On the other hand,
\[
\frac{Y(1)^THY(1)}{Y(1)^TY(1)} \leq 1
\]
and thus it has finite expectation. This implies that
\[
E \frac{Y(1)^THY(1)}{Y(1)^TY(1)} = \frac{1}{n!} \sum_\pi Y(1)^T H(\pi) Y(1) = \frac{1}{n!} \frac{Y(1)^T (\sum_\pi H(\pi)) Y(1)}{Y(1)^TY(1)}.
\]
Let \(H^* = \sum_\pi H(\pi)/n!\). It is straightforward to show that
\[
H^*_ii = \frac{1}{n} \sum_{i=1}^n H_{ii} = \frac{p}{n}, \quad H^*_ij = \frac{1}{n(n-1)} \sum_{i\neq j} H_{ij} = -\frac{1}{n(n-1)} \sum_{i=1}^n H_{ii} = -\frac{p}{n(n-1)},
\]
where the last equality uses the fact that \(\sum_{i,j=1}^n H_{ij} = 0\). Therefore,
\[
E \frac{Y(1)^THY(1)}{Y(1)^TY(1)} = E \frac{Y(1)^TH^*Y(1)}{Y(1)^TY(1)}
\]
\[
= E \frac{n^2 Y(1)^TY(1) - \frac{p}{n(n-1)} \sum_{i\neq j} Y_i(1)Y_j(1)}{Y(1)^TY(1)}
\]
\[
= \frac{p}{n-1} - \frac{p}{n(n-1)} E (\sum_{i=1}^n Y_i(1))^2 \leq \frac{p}{n-1}.
\]
By Markov inequality, with probability \(1 - \frac{2p}{n-1} = 1 - o(1)\),
\[
\frac{Y(1)^THY(1)}{Y(1)^TY(1)} \leq \frac{1}{2}.
\]
Let \(\mathcal{A}\) denote this event. Then
\[
P(\mathcal{A}^c) = o(1),
\]
and on \(\mathcal{A}\),
\[
\frac{1}{n} ||e(1)||_2^2 = \frac{1}{n} Y(1)^T(I-H)Y(1) \geq \frac{1}{2n} ||Y(1)||_2^2.
\]
On the other hand, fix \(k > 0\), and let \(\bar{Z}_i = Y_i(1)I(|Y_i(1)| \leq k)\). For sufficiently large \(k\), \(E\bar{Z}_i > 0\). By the law of large numbers, \(n^{-1} \sum_{i=1}^n \bar{Z}_i = E\bar{Z}_i \times (1 + o_P(1))\). Thus on \(\mathcal{A}\),
\[
\mathcal{E}_2 \geq \frac{1}{2n} \sum_{i=1}^n Y_i(1)^2 \geq \frac{1}{2n} \sum_{i=1}^n \bar{Z}_i = E\bar{Z}_i \times (1 + o_P(1))
\]
Since \(P(\mathcal{A}^c) = o(1)\), we conclude that
\[
\mathcal{E}_2^{-1} = O_P(1).
\]
E.3. Bounding $E_\infty$. We apply the triangle inequality to obtain

$$\|e(t)\|_\infty \leq \|Y(t)\|_\infty + \|HY(t)\|_\infty.$$  

We bound the first term using a standard technique and Markov’s inequality:

(S83) \[ \mathbb{E}\|Y(t)\|_\infty^\delta \leq \sum_{i=1}^{n} \mathbb{E}|Y_i(t)|^\delta = nM(\delta) \implies \|Y(t)\|_\infty = O_P(n^{1/\delta}). \]

Next we bound the second term $\|HY(t)\|_\infty$. Define $\tilde{Y}(t) = HY(t)$ with

$$\tilde{Y}_i(t) = \sum_{j=1}^{n} H_{ij}Y_j(t), \quad (i = 1, \ldots, n).$$

Fix $\epsilon > 0$ and define

$$D = \left( \frac{M(\delta)}{\epsilon} \right)^{1/\delta}.$$

We decompose $\tilde{Y}_i(t)$ into two parts:

$$\tilde{Y}_i(t) = \sum_{j=1}^{n} H_{ij}Y_j(t)I(|Y_j(t)| \leq Dn^{1/\delta}) + \sum_{j=1}^{n} H_{ij}Y_j(t)I(|Y_j(t)| > Dn^{1/\delta})$$

$$\triangleq R_{1,i}(t) + R_{2,i}(t).$$

The second term $R_{2,i}(t)$ satisfies

$$\mathbb{P}(\exists i, R_{2,i}(t) \neq 0) \leq \mathbb{P}(\exists j, |Y_j(t)| > Dn^{1/\delta}) \leq \sum_{j=1}^{n} \mathbb{P}(|Y_j(t)| > Dn^{1/\delta})$$

(S84) \[ \leq \sum_{j=1}^{n} \frac{1}{D^\delta n} \mathbb{E}|Y_j(t)|^\delta \leq \frac{M(\delta)}{D^\delta} = \epsilon. \]

To deal with the first term $R_{1,i}(t)$, we define

$$w_j(t) = Y_j(t)I(|Y_j(t)| \leq Dn^{1/\delta}) - \mathbb{E} \left\{ Y_j(t)I(|Y_j(t)| \leq Dn^{1/\delta}) \right\},$$

with $\mathbb{E}w_j(t) = 0$. Because

$$1^T H = 0 \implies \sum_{j=1}^{n} H_{ij} = 0 \implies \sum_{j=1}^{n} H_{ij} \mathbb{E} \left\{ Y_j(t)I(|Y_j(t)| \leq Dn^{1/\delta}) \right\} = 0,$$

we can rewrite $R_{1,i}(t)$ as

$$R_{1,i}(t) = \sum_{j=1}^{n} H_{ij}w_j(t).$$

The rest of the proof proceeds based on two cases.
Case 1: $\delta < 2$. First, the $w_j(t)$'s are i.i.d. with second moment bounded by
\[
\mathbb{E}w_j(t)^2 \leq \mathbb{E}\left\{Y_j(t)I(|Y_j(t)| \leq Dn^{1/\delta}\right\} \\
\leq (Dn^{1/\delta})^2 \mathbb{E}|Y_j(t)|^\delta \\
\leq n(2-\delta)^{\delta/2} \epsilon^{(2-\delta)/\delta} M(\delta)^{2/\delta}.
\]
Second, using the fact that $\sum_{i=1}^n H_{ij}^2 = H_{ii}$, we obtain
\[
\mathbb{E}R_{1,i}(t)^2 = \sum_{j=1}^n H_{ij}^2 \mathbb{E}w_j(t)^2 = \mathbb{E}w_1(t)^2 \left(\sum_{j=1}^n H_{ij}^2\right) = H_{ii}\mathbb{E}w_1(t)^2.
\]
Let $R_1(t)$ denote the vector $(R_{1,i}(t))_{i=1}^n$. Then
\[
\mathbb{E}\|R_1(t)\|_\infty^2 \leq \sum_{i=1}^n \mathbb{E}R_{1,i}(t)^2 = \left(\sum_{i=1}^n H_{ii}\right) \mathbb{E}w_1(t)^2 \leq pn(2-\delta)^{\delta/2} \epsilon^{(2-\delta)/\delta} M(\delta)^{2/\delta}.
\]
By Markov's inequality, with probability $1 - \epsilon$,
\[
(S85) \quad \|R_1(t)\|_\infty \leq \left(\frac{\mathbb{E}\|R_1(t)\|_\infty^2}{\epsilon}\right)^{1/2} = p^{1/2}n^{(2-\delta)/2} \epsilon^{-(4-\delta)/2\delta} M(\delta)^{1/\delta}.
\]
Combining (S84) and (S85), we obtain that with probability $1 - 2\epsilon$,
\[
\|HY(t)\|_\infty \leq p^{1/2}n^{(2-\delta)/2} \epsilon^{-(4-\delta)/2\delta} M(\delta)^{1/\delta}.
\]
Because this holds for arbitrary $\epsilon$, we conclude that if $\delta < 2$,
\[
\|HY(t)\|_\infty = O_\mathbb{P}\left(p^{1/2}n^{1/\delta-1/2}\right) = o_\mathbb{P}(n^{1/\delta}).
\]
Case 2: $\delta \geq 2$. Using the convexity of the mapping $|\cdot|^\delta$, we have
\[
\mathbb{E}\left|\frac{w_j(t)}{2}\right|^\delta \leq \mathbb{E}\left\{|Y_j(t)|^\delta I(|Y_j(t)| \leq Dn^{1/\delta}\right\} + \mathbb{E}\left\{Y_j(t)I(|Y_j(t)| \leq Dn^{1/\delta}\right\}\frac{1}{2} \mathbb{E}|Y_j(t)|^\delta.
\]
Applying Jensen’s inequality on the second term, we have
\[
\mathbb{E}|w_j(t)|^\delta \leq 2^\delta \mathbb{E}\left\{|Y_j(t)|^\delta I(|Y_j(t)| \leq Dn^{1/\delta}\right\} \leq 2^\delta \mathbb{E}|Y_j(t)|^\delta \leq 2^\delta M(\delta).
\]
By Rosenthal [1970]'s inequality, there exists a constant $C$ depending only on $\delta$, such that
\[
\mathbb{E}|R_{1,i}(t)|^\delta \leq C \left(\sum_{j=1}^n \mathbb{E}|H_{ij}w_j(t)|^\delta + \left(\sum_{j=1}^n \mathbb{E}|H_{ij}w_j(t)|^2\right)^{\delta/2}\right).
\]
\[
\begin{align*}
&\leq C \left( 2^\delta M(\delta) \sum_{j=1}^{n} |H_{ij}|^\delta + \left( 2^2 M(2) \sum_{j=1}^{n} H_{ij}^2 \right)^{\delta/2} \right) \\
&\leq C' 2^\delta \left( M(\delta) H_{ii}^{\delta/2-1} \sum_{j=1}^{n} H_{ij}^2 + M(2)^{\delta/2} H_{ii}^{\delta/2} \right) \\
&= C' 2^\delta (M(\delta) + M(2)^{\delta/2}) H_{ii}^{\delta/2} \leq C' 2^\delta (M(\delta) + M(2)^{\delta/2}) H_{ii}.
\end{align*}
\]

where the last two lines use \( \sum_{j=1}^{n} H_{ij}^2 = H_{ii} \), \( H_{ij}^2 \leq H_{ii} \), and \( H_{ii}^{\delta/2} \leq H_{ii} \) due to \( H_{ii} \leq 1 \) and \( \delta/2 > 1 \). As a result,

\[
\mathbb{E} \| R_1(t) \|_\infty^\delta \leq \sum_{i=1}^{n} \mathbb{E} |R_{1,i}(t)|^\delta \leq C' 2^\delta (M(\delta) + M(2)^{\delta/2}) \sum_{i=1}^{n} H_{ii} \\
= C' 2^\delta (M(\delta) + M(2)^{\delta/2}) p.
\]

Markov’s inequality implies that with probability \( 1 - \epsilon \),

\[
(S86) \quad \| R_1(t) \|_\infty \leq \left( \frac{\mathbb{E} \| R_1(t) \|_\infty^\delta}{\epsilon} \right)^{1/\delta} = p^{1/\delta} \left( C' 2^\delta (M(\delta) + M(2)^{\delta/2}) \right)^{1/\delta}.
\]

Combining (S84) and (S86), we obtain that with probability \( 1 - 2\epsilon \),

\[
\| HY(t) \|_\infty \leq p^{1/\delta} \left( C' 2^\delta (M(\delta) + M(2)^{\delta/2}) \right)^{1/\delta}.
\]

Because this holds for arbitrary \( \epsilon \), we conclude that if \( \delta \geq 2 \),

\[
\| HY(t) \|_\infty = O_p \left( p^{1/\delta} \right) = o_p(n^{1/\delta}).
\]
Supplementary Material III: Experiments

APPENDIX F: ADDITIONAL EXPERIMENTS

Using the following proposition, we know that the solution of $\epsilon$ in Section 4.1 is the rescaled OLS residual vector obtained by regressing the leverage scores $(H_{ii})_{i=1}^n$ on $X$ with an intercept.

**Proposition F.1.** Let $a \in \mathbb{R}^n$ be any vector, and $A \in \mathbb{R}^{n \times m}$ be any matrix with $H_A = A(A^T A)^{-1} A^T$ being its projection matrix. Define $e = (I - H_A)a$. Then $x^* = n^{1/2} e / \|e\|_2$ is the optimal solution of

$$\max_{x \in \mathbb{R}^n} |a^T x| \quad \text{s.t.} \quad \|x\|_2^2 = 1, A^T x = 0.$$

**Proof of Proposition F.1.** The constraint $A^T x = 0$ implies $H_A x = 0$. Thus, $|a^T x| = |a^T x - a^T H_A x| = |a^T (I - H_A)x| = |e^T x|$. The Cauchy–Schwarz inequality implies $|e^T x| \leq \|e\|_2 \|x\|_2 = n^{1/2} \|e\|_2$, with the maximum objective value achieved by $x = n^{1/2} e / \|e\|_2$. \qed

We give more simulation in the rest of this section.

**F.1. Phase-transition point of valid 95% coverage.** Section 1.3 shows that the 95% coverage turns invalid at $p = O(n^{2/3})$ when $X$ contains i.i.d. $N(0,1)$ entries, which have infinite moments. We can use Proposition 3.1 to verify that if $\delta = 6$ and $p = O(n^{2/3})$, then $\kappa = O_P(\frac{p}{n})$. This is a favorable case that all leverage scores are close to the average.

In this section we explore the less favorable scenarios where $X$ has more influential observations. In particular, we generate $X$ with i.i.d. entries from either t(2) or standard Cauchy distribution, with results in Fig. S3.

When $X$ has t(2) entries, the phase-transition point remains at $p = O(n^{2/3})$ and the coverage curves exhibit similar patterns as in the $N(0,1)$ case. However, when $X$ has Cauchy entries, the phase-transition point moves downward to $p = O(n^{1/2})$. In all cases, the coverage curves exhibit similar patterns for different distributions of potential outcomes. Our theory suggests that the difference in the upper and lower panels of Fig. S3 might be due to the behaviors of leverage scores.

**F.2. Other experimental results on synthetic datasets.** Section 4 shows the results for $X$ contains i.i.d. $N(0,1)$ entries. Here we plot the results for $X$ containing i.i.d. entries from heavy-tailed distributions, including t(2) and standard Cauchy distribution. The former exhibits almost the same qualitative pattern (see Fig. S4). However, for the latter, the bias reduction
Fig S3: Empirical 95% coverage of the debiased estimator over 1000 replicates. Rows correspond to different distributions of the potential outcomes.
is less effective and none of the variance estimates, including HC3 estimate, is able to protect against undercoverage when $p > n^{1/2}$ (see Fig. S5).

F.3. Experimental results on real datasets.

F.3.1. The LaLonde data. We use the dataset from a randomized experiment on evaluating the impact of National Supported Work Demonstration, a labor training program, on postintervention income levels [LaLonde, 1986, Dehejia and Wahba, 1999]. It is available at http://users.nber.org/~rdehejia/data/nswdata2.html, and has $n = 445$ units with $n_1 = 185$ units assigned in the program. It has 10 basic covariates: age, education, Black (1 if black, 0 otherwise), Hispanic (1 if Hispanic, 0 otherwise), married (1 if married, 0 otherwise), nodegree (1 if no degree, 0 otherwise), RE74/RE75 (earnings in 1974/1975), u74/u75 (1 if RE74/RE75 = 0, 0 otherwise). We form a $445 \times 49$ $X$ by including all covariates and two-way interaction terms, and removing the ones perfectly collinear with others. We generate potential outcomes which mimics the truth. Specifically, we first regress the observed outcomes on the covariates in each group separately to obtain the coefficient vectors $\hat{\beta}_1, \hat{\beta}_0 \in \mathbb{R}^{49}$ and the estimates $\hat{\sigma}_1, \hat{\sigma}_0$ of error standard deviation.

For each $p \in \{1, 2, \ldots, 49\}$, we randomly extract $p$ columns to form a $445 \times p$ submatrix. Then we generate potential outcomes from (25) by setting $\hat{\beta}_1, \hat{\beta}_0$ to be the subvector of $\beta_1, \beta_0$ corresponding to the positions of selected columns and setting $\sigma_1 = \hat{\sigma}_1/2$ and $\sigma_0 = \hat{\sigma}_0/2$. Then we perform all steps as for the synthetic datasets before. For each $p$ we repeat the above procedure using 50 random seeds and report the median of all measures. Fig. S6 shows the results.

Compared to the synthetic dataset in Section 4, this dataset is more adversarial to our theory in that even the HC3 variance estimate suffers from undercoverage for large $p$. It turns out that $\kappa = 0.887$ in this dataset while $\kappa = 0.184$ for random matrices with i.i.d. $N(0, 1)$ entries.

F.3.2. The STAR data. The second dataset is from the Student Achievement and Retention (STAR) Project, a randomized evaluation of academic services and incentives on college freshmen. It has 974 units with 118 units assigned to the treatment group. Angrist et al. [2009] give more details. We include gender, age, high school GPA, mother language, indicator on whether living at home, frequency on putting off studying for tests, education, mother education, father education, intention to graduate in four years and indicator whether being at the preferred school. We also include
the interaction terms between age, gender, high school GPA and all other variables. This ends up with 53 variables. Fig. S7 shows the results.

**F.4. Further explorations of asymptotic normality.** We investigate the reasons for undercoverage. There are two main possibilities. First, it is due to the underestimation of variance although the estimator is asymptotically normal. Second, it is due to the non-normal asymptotic distribution. We could modify the variance estimator to solve the first problem, but it is more challenging to solve the second problem. The second problem requires a deeper theory.

For each ATE estimates, we consider two measures: skewness and kurtosis. Fan et al. [2017] used \( p \)-value as a measure, but we use two direct measures of normality. For calibration, \( N(0, 1) \) has skewness 0 and kurtosis 3. For each measure, we plot the median as well as the upper and the lower 25% quartiles.

For synthetic datasets, when potential outcomes are light-tailed and the entries of the covariate matrix are not too heavy-tailed, the asymptotic normality holds even for \( p \gg n^{2/3} \). This suggests that the undercoverage is due to the underestimation of variance. Otherwise, the asymptotic normality does not hold. In these cases the asymptotic distribution is heavy-tailed, because the asymptotic kurtosis is much larger than 3, though it mostly remains symmetric. Figures S8–S10 show the results.

For real datasets, the undercoverage is mainly driven by the non-normal asymptotic distributions. Figures S11 and S12 show the results.
Fig S4: Simulation. $X$ contains i.i.d. $t(2)$ entries. Each column corresponds to a distribution of potential outcomes.
Fig S5: Simulation. X contains i.i.d. standard Cauchy entries. Each column corresponds to a distribution of potential outcomes.
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(a) Ratio of bias between \( \hat{\tau}^{de}_{\text{adj}} \) and \( \hat{\tau}_{\text{adj}} \)

(b) Ratio of standard deviation between five standard deviation estimates, \( \sigma_n, \hat{\sigma}_{HC0}, \hat{\sigma}_{HC1}, \hat{\sigma}_{HC2}, \hat{\sigma}_{HC3} \), and the true standard deviation of \( \hat{\tau}_{\text{adj}} \)

(c) Empirical 95% coverage of t-statistics derived from two estimators and four variance estimators (“truth” for the true sampling variance (of \( \hat{\tau}_{\text{adj}} \)), “theoretical” for \( \sigma_n^2 \), “HC2” for \( \hat{\sigma}_{HC2}^2 \) and “HC3” for \( \hat{\sigma}_{HC3}^2 \))

Fig S6: \( X \) is generated from the Lalonde dataset. Each column corresponds to a distribution of potential outcomes.
(a) Ratio of bias between $\hat{\tau}_{\text{adj}}^{\text{de}}$ and $\hat{\tau}_{\text{adj}}$

(b) Ratio of standard deviation between five standard deviation estimates, $\sigma_n$, $\hat{\sigma}_{\text{HC0}}$, $\hat{\sigma}_{\text{HC1}}$, $\hat{\sigma}_{\text{HC2}}$, $\hat{\sigma}_{\text{HC3}}$, and the true standard deviation of $\hat{\tau}_{\text{adj}}$

(c) Empirical 95% coverage of $t$-statistics derived from two estimators and four variance estimators (“truth” for the true sampling variance (of $\hat{\tau}_{\text{adj}}$), “theoretical” for $\sigma_n^2$, “HC2” for $\hat{\sigma}_{\text{HC2}}^2$ and “HC3” for $\hat{\sigma}_{\text{HC3}}^2$)

Fig S7: $X$ is generated from the STAR dataset. Each column corresponds to a distribution of potential outcomes.
Fig S8: Normality check. $X$ contains i.i.d. $N(0, 1)$ entries. Each column corresponds to a distribution of potential outcomes and each row corresponds to an estimator.
Fig S9: Normality check. $X$ contains i.i.d. $t(2)$ entries. Each column corresponds to a distribution of potential outcomes and each row corresponds to an estimator.
Fig S10: Normality check. $X$ contains i.i.d. standard Cauchy entries. Each column corresponds to a distribution of potential outcomes and each row corresponds to an estimator.
(a) Skewness of $\hat{\tau}_{adj}$ and $\hat{\tau}^{de}_{adj}$

(b) Kurtosis of $\hat{\tau}_{adj}$ and $\hat{\tau}^{de}_{adj}$

Fig S11: Normality check. $X$ is generated from the Lalonde dataset. Each column corresponds to a distribution of potential outcomes and each row corresponds to an estimator.
Fig S12: Normality check. $X$ is generated from the STAR dataset. Each column corresponds to a distribution of potential outcomes and each row corresponds to an estimator.