The Alon-Tarsi number of planar graphs

Xuding Zhu

October 1, 2018

Abstract

This paper proves that the Alon-Tarsi number of any planar graph is at most 5, which gives an alternate proof of the 5-choosability as well as the 5-paintability of planar graphs.

Keywords: planar graph; list colouring; on-line list colouring; Alon-Tarsi number.

1 Introduction

Assume $G$ is a graph. We associate to each vertex $v$ of $G$ a variable $x_v$. The graph polynomial $P_G(\vec{x})$ of $G$ is defined as

$$P_G(\vec{x}) = \prod_{u,v,u<v} (x_v - x_u),$$

where $\vec{x} = \{x_v : v \in V(G)\}$ and “$<$” is an arbitrary fixed ordering of the vertices of $G$. It is easy to see that a mapping $\phi : V \to R$ is a proper colouring of $G$ if and only if $P_G(\phi) \neq 0$, where $P_G(\phi)$ means to evaluate the polynomial at $x_v = \phi(v)$ for $v \in V(G)$. Thus to find a proper colouring of $G$ is equivalent to find an assignment of $\vec{x}$ so that the polynomial evaluated at this assignment is non-zero. The Combinatorial Nullstellensatz gives a sufficient condition for the existence of such an assignment.

Assume $P(\vec{x})$ a polynomial with variable set $X$. An index function $\eta$ for $P(\vec{x})$ is a mapping which assigns to each variable $x$ a non-negative integer $\eta(x)$. Given an index function $\eta$, we denote by $\vec{x}^\eta$ the monomial $\prod_{x \in X} x^{\eta(x)}$, and denote by $c_{P,\eta}$ the coefficient of $\vec{x}^\eta$ in the expansion of $P(\vec{x})$. The Combinatorial Nullstellensatz asserts that if $\eta$ is an index function with $\sum_{x \in X} \eta(x)$ equals the degree of $P(\vec{x})$ and $c_{P,\eta} \neq 0$, and $A_x$ is a set of $\eta(x) + 1$ real numbers (or elements of a field) for each $x \in X$, then there is an assignment $\phi$ such that $\phi(x) \in A_x$ for each $x \in X$ and $P(\phi) \neq 0$. In particular, if $c_{P_G,\eta} \neq 0$ and $\eta(x_v) < k$ for all $v \in V$, then $G$ is $k$-choosable. This method developed by Alon and

*Department of Mathematics, Zhejiang Normal University, China. E-mail: xudingzhu@gmail.com. Grant Number: NSFC 11571319.
Tarsi is now a powerful tool in the study of list colouring of graphs. Jensen and Toft [3] defined the Alon-Tarsi number of $G$ as

$$AT(G) = \min\{k : c_{P, \eta} \neq 0 \text{ for some index function } \eta \text{ with } \eta(x_v) < k \text{ for all } v \in V(G)\}.$$  

As observed in [2], $AT(G)$ has some distinct features and it is of interest to study $AT(G)$ as a separate graph invariant. Let $ch(G)$ be the choice number of $G$ and $\chi_P(G)$ be the paint number (or the online choice number) of $G$ (cf. [4] and [7]). It follows from a result of Alon- and Tarsi [1] and a generalization of this result by Schauz [5] that $ch(G) \leq \chi_P(G) \leq AT(G)$ for any graph $G$. There are graphs for which both inequalities are strict. However, upper bounds for the choice number of many natural classes of graphs are also upper bounds for their Alon-Tarsi number. Thomassen [6] proved that every planar graph $G$ has $ch(G) \leq 5$, and Schauz [4] showed that every planar graph $G$ has $\chi_P(G) \leq 5$. A natural question (cf. [2]) is whether $AT(G) \leq 5$ for every planar graph $G$. In this note we answer this question in affirmative.

**Theorem 1** If $G$ is a planar graph, then $AT(G) \leq 5$.

### 2 A proof of Theorem 1

The proof of Theorem 1 is parallel to Thomassen’s proof of the 5-choosability of planar graphs in [6].

For simplicity, we write $c_{G, \eta}$ for $c_{P, \eta}$, and say $\eta$ is an index function of $G$ instead of $P_G(\vec{x})$.

**Definition 2** Assume $G$ is a plane graph and $e = v_1v_2$ is a boundary edge of $G$. An index function $\eta$ of $G - e$ is a nice for $(G, e)$ if the following hold:

- $c_{G-e, \eta} \neq 0$.
- $\eta(v_1) = \eta(v_2) = 0$, $\eta(v) \leq 2$ for every other boundary vertex $v$, and $\eta(v) \leq 4$ for every interior vertex $v$.

If $\eta$ is a nice index function for $(G, e)$, then let $\eta'(x) = \eta(x)$ except that $\eta'(x_{v_1}) = 1$. As $P_G(\vec{x}) = (x_{v_1} - x_{v_2})P_{G-e}(\vec{x})$ and $\eta'(v_2) = 0$, we know that $c_{G, \eta'} = c_{G-e, \eta} \neq 0$. Note that $\eta'(x_v) < 5$ for each vertex $v$. Thus Theorem 1 follows from Theorem 3 below.

**Theorem 3** Assume $G$ is a plane graph and $e = v_1v_2$ is a boundary edge of $G$. Then there exists a nice index function $\eta$ for $(G, e)$.

A variable $x$ is a dummy variable in $P(\vec{x})$ if $x$ does not really occur in $P(\vec{x})$, or equivalently, $\eta(x) = 0$ for each monomial $\vec{x}^n$ in the expansion of $P$ with a nonzero $c_{P, \eta}$. We shall frequently need to consider the summation and the product of polynomials. By introducing dummy variables, we assume the involved polynomials in the sum or the
product have the same set of variables. For example, we may view $x_2^2$ be the same as $x_1^0x_2^1x_3^0 \ldots x_n^0$, i.e., $x_2^2 = x^n$, where the variable set is $X = \{x_1, x_2, \ldots, x_n\}$ and $\eta(x_2) = 2$, $\eta(x_i) = 0$ for $i \neq 2$. We denote by $X$ the set of variables for polynomials in concern. For two index functions $\eta_1, \eta_2$, we write $\eta_1 \leq \eta_2$ if $\eta_1(x) \leq \eta_2(x)$ for all $x \in X$, and $\eta = \eta_2 - \eta_1$ means that $\eta(x) = \eta_2(x) - \eta_1(x)$ for all $x \in X$.

**Observation 4**

1. If $P(\bar{x}) = \alpha P_1(\bar{x}) + \beta P_2(\bar{x})$, then $c_{P,\eta} = \alpha c_{P_1,\eta} + \beta c_{P_2,\eta}$.

2. If $P(\bar{x}) = \bar{x}^{\eta'} P_1(\bar{x})$, then $c_{P,\eta} = c_{P_1,\eta - \eta'}$.

3. If $P(\bar{x}) = \bar{x}^{\eta'} P_1(\bar{x})$ and $\eta' \notin \eta$, then $c_{P,\eta} = 0$.

4. If $P(\bar{x}) = P_1(\bar{x})P_2(\bar{x})$ and for any $\eta'$ with $c_{P_2,\eta'} \neq 0$, there is a dummy variable $x$ of $P_1(\bar{x})$ such that $\eta'(x) \neq \eta(x)$, then $c_{P,\eta} = 0$.

5. If $G$ is a graph and $c_{G,\eta} \neq 0$, then $\sum_{x \in X} \eta(x) = |E(G)|$.

**Proof of Theorem** Assume the theorem is not true and $G$ is a minimum counterexample.

First we consider the case that $G$ has a chord $e' = xy$. Let $G_1, G_2$ be the two $e'$-components (i.e., $G_1, G_2$ are induced subgraphs of $G$ with $V(G) = V(G_1) \cup V(G_2)$ and $V(G_1) \cap V(G_2) = \{x, y\}$) with $e \in G_1$.

By the minimality of $G$, there exist a nice index function $\eta_1$ for $(G_1, e)$, and a nice index function $\eta_2$ for $(G_2, e')$. Let $\eta = \eta_1 + \eta_2$. Note that

$$P_{G-e}(\bar{x}) = P_{G_1-e}(\bar{x})P_{G_2-e'}(\bar{x}).$$

Let $R(\bar{x}) = c_{G_2-e',\eta_2} \bar{x}^{\eta_2}, Q(\bar{x}) = P_{G_2-e'}(\bar{x}) - R(\bar{x}), P_1(\bar{x}) = R(\bar{x})P_{G_1-e}(\bar{x})$ and $P_2(\bar{x}) = Q(\bar{x})P_{G_1-e}(\bar{x})$. Then $P_{G-e}(\bar{x}) = P_1(\bar{x}) + P_2(\bar{x})$.

By (5) of Observation 4, for any index function $\eta'$ with $c_{Q,\eta'} \neq 0$, we have $\eta' \not\in \eta_2$ and hence there is a vertex $v \in V(G_2) - \{x, y\}$ such that $\eta'(x_v) \neq \eta_2(x_v) = \eta(x_v)$. As $x_v$ is a dummy variable in $P_{G_1-e}(\bar{x})$, by (4) of Observation 4 we have $c_{P_2,\eta} = 0$. By (1) and (2) of Observation 4

$$c_{G-e,\eta} = c_{P_1,\eta} = c_{G_2-e',\eta_2} c_{G_1-e,\eta_1} \neq 0.$$  

So $\eta$ is a nice index function for $(G, e)$.

Assume $G$ has no chord and assume $B(G) = (v_1, v_2, \ldots, v_n)$.

Let $G' = G - v_n$. Let $v_1, u_1, u_2, \ldots, u_k, v_{n-1}$ be the neighbours of $v_n$. Let

$$S(\bar{x}) = (x_{v_n} - x_{v_1})(x_{v_{n-1}} - x_{v_n})(x_{u_1} - x_{v_n}) \ldots (x_{u_k} - x_{v_n}).$$

Then $P_{G-e}(\bar{x}) = S(\bar{x})P_{G'-e}(\bar{x})$. 

3
If $n = 3$, then let $\eta'$ be nice for $(G', e)$. Let $\eta(x_v) = \eta'(x_v)$ for $v \notin \{u_1, u_2, \ldots, u_k\}$ and $\eta(x_v) = \eta'(x_v) + 1$ for $v \notin \{u_1, u_2, \ldots, u_k\}$ and $\eta(x_v) = 2$. Let $\eta''(x_v) = 2$, $\eta''(x_u) = 1$ for $i = 1, 2, \ldots, k$ and $\eta''(x) = 0$ for other $x$. Then

$$S(\bar{x}) = -\bar{x}^{\eta''} + x_{v_1} A(\bar{x}) + x_{v_2} B(\bar{x}) + x_{v_3}^3 C(\bar{x})$$

for some polynomials $A(\bar{x}), B(\bar{x})$ and $C(\bar{x})$. Let $P_1(\bar{x}) = x_{v_1} A(\bar{x}) P_{G'-e}(\bar{x}), P_2(\bar{x}) = x_{v_2} B(\bar{x}) P_{G'-e}(\bar{x})$ and $P_3(\bar{x}) = x_{v_3}^3 C(\bar{x}) P_{G'-e}(\bar{x})$. As $\eta(x_{v_1}) = \eta(x_{v_2}) = 0$ and $\eta(x_{v_3}) = 2$, it follows from (3) of Observation 4 that $c_{P_1, \eta} = c_{P_2, \eta} = c_{P_3, \eta} = 0$. By (1) and (2) of Observation 4, $c_{G'-e, \eta} = -c_{G'-e, \eta'} \neq 0$. Hence $\eta$ is nice for $(G, e)$.

Assume $n \geq 4$.

We say an index function $\eta'$ for $G' - e$ special if $\eta'(v_{n-1}) \leq 1$, $\eta'(v_1) = \eta'(v_2) = 0$, $\eta'(u_j) \leq 3$ for $j = 1, 2, \ldots, k$, $\eta'(v) \leq 2$ for each other boundary vertex $v$ and $\eta'(v) \leq 4$ for each interior vertex $v$.

**Case 1.** $c_{G'-e, \eta'} \neq 0$ for some special index function $\eta'$ for $G' - e$.

Let $\eta(v) = \eta'(v)$ for $v \notin \{u_1, u_2, \ldots, u_k, v_{n-1}\}$ and $\eta(v) = \eta'(v)+1$ for $v \in \{u_1, u_2, \ldots, u_k, v_{n-1}\}$ and $\eta(x_{v}) = 1$. Let $\eta''$ be the index function defined as $\eta''(x_{v}) = \eta''(x_{v_{n-1}}) = \eta''(x_{u_1}) = \cdots = \eta''(x_{u_k}) = 1$ and $\eta''(x) = 0$ for other variables $x$. Then

$$S(\bar{x}) = -\bar{x}^{\eta''} + x_{v_1} A(\bar{x}) + x_{v_2}^2 B(\bar{x})$$

for some polynomials $A(\bar{x})$ and $B(\bar{x})$. Let $P(\bar{x}) = \bar{x}^{\eta''} P_{G'-e}(\bar{x}), P_1(\bar{x}) = x_{v_1} A(\bar{x}) P_{G'-e}(\bar{x})$ and $P_2(\bar{x}) = x_{v_2}^2 B(\bar{x}) P_{G'-e}(\bar{x})$. Then

$$P_{G', \eta'}(\bar{x}) = P(\bar{x}) + P_1(\bar{x}) + P_2(\bar{x}).$$

As $\eta(v_1) = 0$ and $\eta(v_2) = 1$, it follows from (3) of Observation 4 that $c_{P, \eta} = c_{P, \eta}$ = 0. By (1) and (2) of Observation 4 we have $c_{G-e, \eta} = c_{P, \eta} = c_{G'-e, \eta''} \neq 0$. Hence $\eta$ is nice for $(G, e)$.

**Case 2.** $c_{G'-e, \eta'} = 0$ for every special index function $\eta'$ for $G' - e$.

By the minimality of $G$, there is an index function $\eta''$ nice for $(G', e)$. Let $\eta(x_v) = \eta''(x_v)$ for $v \notin \{u_1, u_2, \ldots, u_k\}$ and $\eta(x_v) = \eta''(x_v) + 1$ for $v \notin \{u_1, u_2, \ldots, u_k\}$ and $\eta(x_v) = 2$. Let $\eta''(x_{v}) = 2$, $\eta''(x_{u}) = 1$ for $i = 1, 2, \ldots, k$ and $\eta''(x) = 0$ for other $x$. Then

$$S(\bar{x}) = -\bar{x}^{\eta''} + x_{v_1} A(\bar{x}) + x_{v_{n-1}} B(\bar{x}) + x_{v_3}^3 C(\bar{x})$$

for some polynomials $A(\bar{x}), B(\bar{x})$ and $C(\bar{x})$.

Let $P(\bar{x}) = \bar{x}^{\eta''} P_{G'-e}(\bar{x}), P_1(\bar{x}) = x_{v_1} A(\bar{x}) P_{G'-e}(\bar{x}), P_2(\bar{x}) = x_{v_{n-1}} B(\bar{x}) P_{G'-e}(\bar{x})$ and $P_3(\bar{x}) = x_{v_3}^3 C(\bar{x}) P_{G'-e}(\bar{x})$. As $\eta(x_{v}) = 0$ and $\eta(x_{v_{n-1}}) = 2$, it follows from (3) of Observation 4 that $c_{P_1, \eta} = c_{P_3, \eta} = 0$. As $\eta(v_{n-1}) \leq 2$, $\eta(u) = \eta''(u) + 1 \leq 3$, it follows from (2) of Observation 4 that $c_{P_2, \eta} = c_{G'-e, \eta}$ for a special index function $\eta'$ for $G' - e$ (note that $\eta'(v_{n-1}) = \eta(x_{v_{n-1}}) - 1$ and $\eta'(x) \leq \eta(x)$ for other $x$). By our assumption, $c_{G'-e, \eta'} = 0$. Therefore $c_{P, \eta'} = 0$. As

$$P_{G'-e}(\bar{x}) = P(\bar{x}) + P_1(\bar{x}) + P_2(\bar{x}) + P_3(\bar{x}),$$

by (1) and (2) of Observation 4 $c_{G-e, \eta} = c_{P, \eta} = c_{G'-e, \eta''} \neq 0$. □
3 An alternate proof

A digraph $D$ is Eulerian if $d^+_D(v) = d^-_D(v)$ for every vertex $v$. Assume $G$ is a graph and $D$ is an orientation of $G$. Let $EE(D)$ (respectively, $OE(D)$) be the set of spanning Eulerian sub-digraphs of $D$ with an even (respectively, an odd) number of edges. Alon and Tarsi [1] showed that for an index function $\eta$ of $G$, $c_{G,\eta} = \pm(|EE(D)| - |OE(D)|)$ for an orientation $D$ with $d^+_D(v) = \eta(v)$ for every $v \in V(G)$. Thus to prove that $c_{G,\eta} \neq 0$ is equivalent to show that there is an orientation $D$ of $G$ with $d^+_D(v) = \eta(v)$ for every $v \in V(G)$ for which $|EE(D)| \neq |OE(D)|$.

Definition 5 Assume $G$ is a plane graph and $e = v_1v_2$ is a boundary edge of $G$. An orientation $D$ of $G - e$ is a nice for $(G, e)$ if the following hold:

- $|EE(D)| \neq |OE(D)|$.
- $d^+_D(v_1) = d^+_D(v_2) = 0$, $d^+_D(v) \leq 2$ for every other boundary vertex $v$, and $d^-_D(v) \leq 4$ for every interior vertex $v$.

The following theorem is just a restatement of Theorem 3, and its proof is essentially the same as the proof of Theorem 3. However, the translation from calculating the coefficients of a polynomial to counting Eulerian subgraphs is not completely trivial. We include a proof of this statement for pedagogical reason.

Theorem 6 Assume $G$ is a plane graph and $e = v_1v_2$ is a boundary edge of $G$, then $(G, e)$ has a nice orientation.

Proof. Assume the theorem is not true and $G$ is a minimum counterexample.

First we consider the case that $G$ has a chord $e' = xy$. Let $G_1, G_2$ be the two $e'$-components with $e \in G_1$.

By the minimality of $G$, $(G_1, e)$ has a nice orientation $D_1$, and $(G_2, e')$ has a nice orientation $D_2$. Let $D = D_1 \cup D_2$. Edges in $D_2$ incident to $x, y$ are not contained in any directed cycles, and hence are not contained in any Eulerian sub-digraph of $D$. Therefore

\[
EE(D) = EE(D_1) \times EE(D_2) + OE(D_1) \times OE(D_2), \\
OE(D) = EE(D_1) \times OE(D_2) + OE(D_1) \times EE(D_2).
\]

So $|EE(D)| - |OE(D)| = (|EE(D_1)| - |OE(D_1)|)(|EE(D_2)| - |OE(D_2)|) \neq 0$. Hence $D$ is a nice orientation of $(G, e)$.

Assume $G$ has no chord. Assume $B(G) = (v_1, v_2, \ldots, v_n)$.

Let $G' = G - v_n$. If $n = 3$, i.e. $B(G)$ is a triangle, then let $D'$ be a nice orientation for $(G', e)$, and let $D$ be obtained from $D'$ by adding arcs $(v_3, v_1)$ and $(v_3, v_2)$. As $v_1, v_2$ are sinks, no edge incident to $v_n$ is contained in a directed cycle and hence $EE(D) = EE(D')$ and $OE(D) = OE(D')$. Therefore $D$ is a nice orientation for $(G, e)$.

Assume $n \geq 4$. Let $v_1, u_1, u_2, \ldots, u_k, v_{n-1}$ be the neighbours of $v_n$.

We call an orientation $D$ of $(G', e)$ special if the following hold:
• $v_1, v_2$ has out-degree 0, $v_{n-1}$ has out-degree at most 1, and each of $u_1, u_2, \ldots, u_k$ has out-degree at most 3, each other boundary vertex has out-degree at most 2.

• every interior vertex has out-degree at most 4.

**Case 1.** $(G', e)$ has a special orientation $D'$ with $|EE(D')| \neq |OE(D')|$. Let $D$ be the orientation of $G - e$ which is obtained from $D'$ by adding arcs

$$(v_n, v_1), (v_{n-1}, v_n), (u_1, v_n), \ldots, (u_k, v_n).$$

Then $D$ is a nice orientation of $(G, e)$, as $EE(D') = EE(D)$ and $OE(D') = OE(D)$.

**Case 2.** For any special orientation $D'$ of $(G', e)$, $|EE(D')| = |OE(D')|$. By the minimality of $G$, $(G', e)$ has a nice orientation $D''$. Let $D$ be the orientation of $G - e$ obtained from $D''$ by adding arcs $(v_n, v_1), (v_n, v_{n-1}), (u_1, v_n), \ldots, (u_k, v_n)$. For $i = 1, 2, \ldots, k$, let

$$EE_i(D) = \{ H \in EE(D) : (u_i, v_n) \in H \}, \quad OE_i(D) = \{ H \in OE(D) : (u_i, v_n) \in H \}.$$

For $i = 1, 2, \ldots, k$, if $EE_i(D) \cup OE_i(D) \neq \emptyset$, then let $C_i$ be a directed cycle in $D$ containing $(u_i, v_n)$. Note that every directed edge of an Eulerian digraph is contained in a directed cycle, and $C_i$ must contain $(v_n, v_{n-1})$. Let $D'_i$ be the orientation of $G'$ which is obtained from $D''$ by reversing the direction of edges in $C_i \cap D''$ (note that $C_i \cap D'' = C_i - \{(u_i, v_n), (v_n, v_{n-1})\}$ is a directed path from $v_{n-1}$ to $u_i$).

Observe that $D'_i$ is a special orientation of $(G', e)$. Hence $|EE(D'_i)| = |OE(D'_i)|$.

Now we show that $|EE_i(D)| = |OE_i(D)|$ for $i = 1, 2, \ldots, k$. If $EE_i(D) \cup OE_i(D) = \emptyset$, then this is trivially true.

Assume $EE_i(D) \cup OE_i(D) \neq \emptyset$.

For each $H \in EE_i(D) \cup OE_i(D)$, $H \Delta C_i^{-1}$ is the symmetric difference of $H$ and $C_i^{-1}$, i.e., the digraph obtained from the edge disjoint union of $H$ and $C_i^{-1}$ by deleting digons. Note that the symmetric difference of any two Eulerian digraphs is an Eulerian digraph. Moreover, any edge of $C_i$ contained in $H$ will form a digon with the corresponding edge in $C_i^{-1}$ and hence is deleted. In particular, $(u_i, v_n), (v_n, v_{n-1})$ are edges of $C_i$ contained in $H$ and are deleted. So $H \Delta C_i^{-1}$ is a sub-digraph of $D''$.

Similarly, for each $H \in EE(D'_i) \cup OE(D'_i)$, $H \Delta C_i^{-1} \in EE(D'_i) \cup OE(D'_i)$. As $(H \Delta C_i) \Delta C_i^{-1} = H, \phi(H) := H \Delta C_i^{-1} \Delta C_i$ is a one-to-one correspondence between $EE(D'_i) \cup OE(D'_i)$ and $EE_i(D) \cup OE_i(D)$. If $C_i$ is of even length, then $|E(\phi(H))|$ and $|E(H)|$ have the same parity; if $C_i$ is of odd length, then $|E(\phi(H))|$ and $|E(H)|$ have different same parities. So if $C_i$ is of even length, then $|EE_i(D)| = |OE(D_i')|, |OE_i(D)| = |OE(D_i')|$; if $C_i$ is of odd length, then $|EE_i(D)| = |OE(D_i')|, |OE_i(D)| = |EE(D_i')|$. In any case, $|EE_i(D)| = |OE_i(D)|$. 

6
Now

\[ EE(D) = EE(D'') \cup \bigcup_{i=1}^{k} EE_i(D), \quad OE(D) = OE(D'') \cup \bigcup_{i=1}^{k} OE_i(D). \]

The unions above are disjoint unions. So \(|EE(D)| - |OE(D)| = |EE(D'')| - |OE(D'')| \neq 0.\]

I would like to thank Grzegorz Gutowski for bringing this problem to my attention, and thank Jaroslaw Grytczuk for pointing out that the problem was contained in [2].

References

[1] N. Alon, M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (2) (1992) 125–134.

[2] D. Hefetz, On two generalizations of the Alon-Tarsi polynomial method, Journal of Combinatorial Theory Ser. B, 101 (2011) 403–414.

[3] T. Jensen, B. Toft, Graph Coloring Problems, Wiley, New York, 1995.

[4] U. Schauz, Mr. Paint and Mrs. Correct, Electron. J. Combin. 16 (1) (2009) R77.

[5] U. Schauz, A paintability version of the Combinatorial Nullstellensatz and list colorings of k-partite k-uniform hypergraphs, Electron. J. Combin. 17 (1) (2010) R176.

[6] C. Thomassen, Every planar graph is 5-choosable, Journal of Combinatorial Theory Ser. B, 62(1) (1994):180–181.

[7] X. Zhu, On-line list colouring of graphs, Electron. J. Combin. 16 (1) (2009) R127.