Exceptional Photon Blockade

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Exceptional points (EPs) are special degeneracies in non-Hermitian systems at which both eigenvalues and their corresponding eigenvectors simultaneously coalesce. Despite numerous EP-enabled novel phenomena and applications have been revealed in classical optics, purely quantum EP effects, especially peculiar features of quantum correlations at EPs, has been elusive. Here, for a nonlinear resonator, we show that EPs can lead to up to 4 orders of magnitude difference in optical second-order quantum correlations. As a result, single-photon blockade emerges when the system is at an EP while photon-induced tunneling occurs when the system is far from EPs. Moreover, two-photon blockade can also appear by tuning the system around EPs. Our findings pave the way towards devising exotic quantum EP devices and, on a more fundamental level, shed new light on the effects of non-Hermitian degeneracies on quantum fluctuations and quantum states of photons.

Unconventional effects in non-Hermitian physics, accompanied by spectral degeneracies known as exceptional points (EPs) [1–8], have led to a variety of novel applications, such as wireless power transfer [9], unidirectional light flow [10–13], single-mode lasers [14, 15], and ultra-sensitive sensors [16–18]. EPs are experimentally demonstrated not only in optics [10–19], but also in e.g., electronics [20–24], mechanics [25–30], and thermology [31]. These studies, however, have been focused on the classical regime, and the influence of EPs on purely quantum effects has not been extensively explored.

Very recently, pioneer experiments exploring non-Hermiticity and EPs have been performed in quantum systems or with quantum states of light [32–34]. A surprising outcome of Ref. [34] was that in an atomic gas, quantum correlations can disappear when mode coalescence leads to vanishing nonlinearity. Here, we propose to manipulate quantum effects using periodic EPs in an optical Kerr resonator. In contrast to Ref. [34], we show that in this system, nonlinear mode coupling always exists and quantum correlations can be significantly altered at EPs. Specifically, we show that photon blockade (PB) can emerge at EPs while photon-induced tunneling (PIT) occurs when the system is far away from EPs, thus paving the way towards utilizing the power of EPs to steer quantumness of light.

As a quantum phenomenon in which the absorption of one photon blocks the absorption of subsequent photons [35], PB plays a key role in achieving single-photon devices [36–39]. On the contrary, in PIT, absorption of the first photon enhances the absorption of a second photon [40]. These effects were experimentally demonstrated in Kerr-type cavity-QED systems [40, 41]. Recently, two-photon blockade (the presence of two photons prevents a third one from being absorbed) [42–44] and unconventional photon blockade (UPB, i.e., optical antibouncing induced by quantum interference) [45–48] have also been observed. This opens a route for creating two-photon devices, and fabricating quantum devices with weak nonlinearity. Our work here, as far as we know, is the first to reveal the possibility of tuning PB and PIT by harnessing the EPs.

In this work, we consider periodic EPs in whispering-gallery-mode (WGM) resonators, as demonstrated in recent experiments [12, 16]. In those experiments [12, 16, 49], two Rayleigh scatterers (e.g., nanotips) were placed in the evanescent field of the resonator and their relative size and distance were tuned to realize periodic EPs (Fig. 1a). The first nanotip induces coupling between the two frequency-degenerate counter-propagating modes of the resonator. This lifts the degeneracy and hence leads to optical mode splitting. The second nanotip then induces asymmetric coupling between the modes. By tuning the relative size and the positions of the nanotips, one can steer the system to an EP, or away from it. Periodic EPs emerge by tuning the relative distance between the nanotips along the boundary of the resonator. Here we show that inducing such periodic EPs in a Kerr resonator can be used to engineer purely quantum effects at single-photon levels.

The scatterer-induced coupling of the clockwise- (CW) and counterclockwise- (CCW) traveling lights can be described by $\hat{H}_j = \hbar J_{2j} \hat{a}_2^\dagger \hat{a}_1 + \hbar J_{2j} \hat{a}_2^\dagger \hat{a}_1$, where $\hat{a}_1$ ($\hat{a}_1^\dagger$) and $\hat{a}_2$ ($\hat{a}_2^\dagger$) are the annihilation (creation) operators of the CW and CCW modes, respectively, and the scattering rate from the CW (CCW) mode to the CCW (CW) mode is [16]: $J_{12(21)} = \epsilon_1 + \epsilon_2 e^{\pm i2\sigma}$. Here $\sigma$ is the azimuthal
mode number, $\beta$ is the relative angular position of the two scatterers, $2\xi_j$ ($j = 1, 2$) is the complex frequency splitting that is introduced by $j$th scatterer alone. The effective non-Hermitian Hamiltonian describing our system is then: $H_1 = \omega \hat{a}^\dagger \hat{a}_1 + \omega \hat{a}_2^\dagger \hat{a}_2 + H_j + H_k$, where $\omega = \omega_0 + \epsilon_1 + \epsilon_2$, $\omega_0$ being the frequency of the bare system, when $J_{12(21)} = 0$. Also $H_k = \sum_{j=1,2} h \chi \hat{a}_j^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_j$, where $\chi = \hbar \omega^2 c n_2/(n_0^2 V_{\text{eff}})$ [50] denotes the single-photon Kerr frequency shift, $n_2$ ($n_0$) is the nonlinear (linear) refraction index, $V_{\text{eff}}$ is effective cavity-mode volume, $c$ is the speed of light in vacuum, and $\omega$ is the pump frequency.

Denoting $(m,n)$ ras $m$ photons in the bare CW mode and $n$ photons in the bare CCW mode, in the single- and two-excitation subspaces, we can write the eigenenergies of this system as: $E_1^\pm = \omega \pm \delta_1$, and $E_2^\pm = 2\omega \pm 2\chi + \delta_2$, with the eigenvectors: $\psi_1^\pm = \sqrt{J_{12(21)}} |1,0\rangle \pm \sqrt{J_{21}} |0,1\rangle$. 

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**Figure 1. Hamiltonian exceptional points (HEPs) and Liouvillian exceptional points (LEPs) in a non-Hermitian system.**

- **a.** Steering periodic EPs in an optical microresonator. Centre, Two nanotips ($S_1$ and $S_2$) simulating Rayleigh scatterers are coupled to a Kerr resonator, which is driven by a laser with frequency $\omega_L$ and amplitude $\xi$. By tuning relative phase angle $\beta$ between the scatterers, one can control the coupling between the counter-propagating modes, resulting in periodical revival and suppression of mode splitting and coalesce. Left, Eigenenergy spectrum versus $\beta/\pi$ in the subspace with $N$-photon ($N = 0, 1, 2$). $\psi_N$, eigenstates of the system; $\omega$ or $\chi$, single-photon frequency or its Kerr shift; $\omega_L = \omega$, single-photon frequency index, $\delta_1(\beta)$ is effective cavity-mode volume, $c$ is the speed of light in vacuum, and $\omega$ is the pump frequency.

- **b.** Dependence of the normalized cavity excitation spectrum on the optical detuning $\Delta_0/\gamma$ for different $\beta/\pi$. The left column is analytical solution based on quantum-trajectory method and the right is numerical results of a full quantum simulation based on a “hybrid” Lindblad master equation. For the weak optical drive, $\xi_1/\gamma = 0.25$, the excitation spectrum is dominated by single-photon events and shows that single photon is emitted for $\omega_L = \omega$ ($\omega_L = \omega \pm \delta_1$) when the system is at (away from) EPs. For the other parameters, see the main text.
and \( \psi_2 = \sqrt{2} J_{12} |2, 0\rangle + \delta_2^\pm |1, 1\rangle + \sqrt{2} J_{21} |0, 2\rangle \). The one-particle sector has two eigenstates \( \psi_1^\pm \), and \( \delta_1 = \sqrt{J_{12}/J_{21}} \); instead, the two-particle sector has three eigenstates \( \psi_2^s \), indicating the states \( \pm 0 \); \( \delta_2^s = -\chi \pm \sqrt{\chi^2 + 4 J_{12} J_{21}} \) while \( \delta_2^0 = 0 \). The eigenmode structure, depending on the asymmetry of the scattering rates, can be tuned by the relative angular position \( \beta \) between the nanotips (Fig. 1a). \( \tilde{H}_t \) has two EPs for which \( E_{12}^\pm = \omega \). The first one emerges for \( J_{21} = 0 \), where \( \psi_{12}^\pm = |1, 0\rangle \), corresponding to solely CW propagation, EP\(_{CW}^\text{CCW}\). The second one is associated to the CCW propagation (EP\(_{CCW}^\text{CCW}\)) and emerges when \( J_{12} = 0 \). From \( J_{12(21)} = 0 \) we determine the corresponding \( \beta \) as

\[
\beta_{\text{EP}}^\text{CCW,CW} = \frac{z \pi}{2 \gamma} \mp \frac{\arg(\epsilon_1) - \arg(\epsilon_2)}{2 \gamma} \quad (z = \pm 1, \pm 3, \ldots).
\]

In practice, one should also consider photon losses by introducing the dissipation rate \( \gamma = \omega_0/Q \), where \( Q \) is the cavity quality factor, and continuously probe the system by applying a coherent CW drive. In the frame rotating at the drive frequency \( \omega_L = \omega_0 \) the total Hamiltonian is

\[
\hat{H} = \Delta \left( \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \right) + \hat{H}_t + \hat{H}_c + \xi \left( \hat{a}_1 + \hat{a}_1^\dagger \right),
\]

where \( \Delta = \Delta_0 + \epsilon_1 + \epsilon_2 = \omega - \omega_L \) is the cavity-to-laser detuning and \( \xi = \sqrt{\gamma^d \rho_{in}/(\hbar \omega L)} \) is the driving amplitude with driving power \( P_{in} \) and cavity-waveguide coupling rate \( \gamma^d \).

The experimentally accessible parameters we used here are [51–56]: \( \lambda = 1550 \text{ nm}, \quad Q = 5 \times 10^9, \quad V_{\text{eff}} = 150 \mu \text{m}^3, \quad n_0 = 1.4, \quad P_{in} = 4 \text{ fW} \) (i.e., \( \xi/\gamma \sim 0.25 \)). \( V_{\text{eff}} \) is typically \( 10^{-2} - 10^{-1} \mu \text{m}^3 \) [51, 52], \( Q \) has been increased up to \( 10^{12} \) in WGM microresonators [53, 54], and \( P_{in} \) has reached 0.5 fW [55]. The Kerr coefficient \( n_2 \sim 10^{-14} \text{m}^2/\text{W} \) for typical materials [56], which can be further enhanced [57–62], e.g., via feedback control [61, 62] or squeezing [59, 60]. Also in the experiment [12], \( \sigma = 1, \epsilon_1/\gamma = 1.5 - 0.1i, \quad \epsilon_2/\gamma = 1.485 - 0.14i, \) thus we have \( \delta_1 = 0 \) at \( \beta_{\text{EP}}^\text{CW} = \pi/2 \) and \( \beta_{\text{EP}}^\text{CCW} = 3\pi/2 \).

We first study the excitation spectrum for CW mode, i.e., \( S_1(D) = \lim_{t \to \infty} \langle \hat{a}_1^\dagger \hat{a}_1 \rangle/n_0 \), where \( n_0 = \xi^2/\gamma^2 \).

Figure 2. EP-engineered photon blockade (PB) and photon-induced tunneling (PIT). a. Second-order correlation function with zero-time delay of the CW mode \( g_{11}^{(2)}(0) \) versus \( \beta \) for \( \omega_L = \omega \) and \( \xi/\gamma = 0.25\gamma \). b. Dependence of the photon probability distribution. Unconventional photon blockade (UPB, green markers) occurs near EPs, where two-photon probability \( P_{20} \) reaches the minima at \( \beta = 0.4\pi, 0.6\pi, 1.4\pi, 1.6\pi \). Also, PIT (blue markers) emerges when the system is far away from EPs, since \( P_{10} \) is suppressed at \( \beta = \pi \). In contrast, due to the enhancement of \( P_{10} \), conventional photon blockade (CPB, yellow markers) is observed when the system is at EPs (i.e., \( \beta_{\text{EP}} = \pi/2, 3\pi/2 \)). In a and b, the black solid curves represent the analytical solution, and the dots and circles correspond to results of the full quantum simulations. c. The energy-level diagram shows the origins of UPB, CPB, and PIT. When the system is near (far away from) EPs, UPB (PIT) emerges since \( \xi \) is suppressed at \( 10 \). From \( \xi \) is typically \( 0.03 \), the unevenly spaced levels of the eigenenergy spectrum prevent reaching the two photon states for the single-photon frequency \( \omega_L = \omega \) (yellow arrows); i.e., resulting in CPB. d. This EP-induced switching of quantum effects can also be recognized from the relative photon distributions \( \mathcal{R}(m) = [P_m - \rho_{m}] / P_m \), i.e., the deviation of the photon distribution \( P_m \) from the standard Poisson distribution \( \rho_m \) with the same mean number \( m \) in the CW mode. For all panels, the other parameters are the same as those in Fig. 1.
According to the quantum trajectory method, the semiclassical optical decay (i.e., no quantum jump involved) can be included in the effective Hamiltonian \( \hat{H}_{\text{eff}} = \hat{H} - \sum_j (\gamma_j/2) \hat{a}_j^\dagger \hat{a}_j [63] \). In the subspace with two-excitations, a general state of the system can be expressed as \( \psi(t) = \sum_{N=0}^2 \sum_{m=-N}^{N} C_{m,N-m} |m,N-m\rangle \), where \( C_{m,n} \) are probability amplitudes corresponding to state \( |m,n\rangle \). By solving the Schrödinger equation with \( \hat{H}_{\text{eff}} \) and \( \psi(t) \), we can obtain the solutions \( C_{00}(\infty) = 1 \), \( C_{10}(\infty) = 2\xi\Delta_1/\eta_1 \), \( C_{01}(\infty) = -4\xi J_{21}/\eta_1 \), and

\[
\begin{align*}
C_{20}(\infty) &= 2\sqrt{2}\xi^2 \left( \Delta_1^2 + 4J_{12}J_{21}\chi/\Delta_2 \right)/\eta_1\eta_2, \\
C_{11}(\infty) &= -4J_{21}\xi^2 \left( \Delta_1 + \Delta_2 \right)/\eta_1\eta_2, \\
C_{02}(\infty) &= 4\sqrt{2}\xi J_{21}\xi^2 \left( \Delta_1/\Delta_2 + 1 \right)/\eta_1\eta_2,
\end{align*}
\]

where \( \Delta_1 = 2\Delta - i\gamma \), \( \Delta_2 = \Delta_1 + 2\chi \), and \( \eta_1 = 4J_{12}J_{21} - \Delta_1^2 \), \( \eta_2 = 4J_{12}J_{21} - \Delta_1\Delta_2 \). Denoting the probabilities of finding \( m \) and \( n \) particles respectively in the CW and CCW modes by \( P_{mn} = |C_{mn}|^2 \), we can obtain the excitation spectrum of CW mode as \( S_{11}(\Delta_0) = \sum_{N=0}^2 \sum_{m=0}^{N} m P_{mn}/n_0 \). For a weak driving field (\( \xi < \gamma \)), the photon number involved is very small, and the dominant contribution for \( S_{11} \) arises from terms of probability amplitudes which are linear in \( \xi \), i.e., single-photon probability amplitudes \( C_{10} \). Thus, the cavity excitation spectrum can provide the evidence of EP which originates from the coalescence of \( E_1 \) and \( \psi_1 \).

In the previous analysis, we did not take into account the effect of quantum noise on the system. Indeed, the cavity losses and asymmetric backscattering from the nanotips were treated only as non-Hermitian terms in the effective Hamiltonian. To confirm the validity of the previous results, here we consider a more refined approximation in which the cavity losses are treated fully quantum and the backscattering continues to be semiclassical. We thus obtain a “hybrid” formalism for Lindblad master equation [64, 65],

\[
\dot{\rho} = -i[\hat{H}, \rho] - i(\hat{H}_-, \rho) + \sum_j \delta_j(\rho, \hat{A}_j) + i2\text{tr} (\hat{\rho} \hat{H}_- \rho),
\]

where \( \rho(t) \) is the normalized density matrix of the system at time \( t \), with \( \text{tr}(\rho) = 1 \), and \( D(\rho, \hat{A}_j) = \hat{A}_j \rho \hat{A}_j^\dagger - \hat{A}_j^\dagger \hat{A}_j \rho/2 - \rho \hat{A}_j^\dagger \hat{A}_j/2 \) are the dissipators associated with the jump operators \( \hat{A}_j = \sqrt{\gamma_j} \hat{a}_j \). Here \( \hat{H}_\pm \) are the Hermitian and anti-Hermitian parts of the total Hamiltonian \( \hat{H} \), with \( \hat{H}_\pm = \pm \hat{H}_\mp \). The photon-number probability

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**Figure 3. Frequency tunable photon blockade in non-Hermitian system.** a. Dependence of the second-order correlation function \( g_2(0) \) on the optical detuning \( \Delta_0/\gamma \) for different relative phase angles \( \beta = \{\pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2\} \). Tunable CPB with different frequencies \( \omega \pm \delta_1(\beta) \equiv \omega \pm \delta_1 \) is observed by driving with \( \omega_L = \omega \pm \delta_1 \). The solid curves and dots respectively represent the analytical solution and the numerical results of the full quantum simulation. b. Single-photon probability \( P_{10} \) versus \( \Delta_0/\gamma \) and \( \beta \). For different \( \beta \), \( P_{10} \) reaches the maxima by pumping with single-photon resonance frequencies matching those in a. c. The energy-level diagram shows the origin of this tunable CPB. The shifts on \( N \)-photon energy \( \delta_N(\beta) \equiv \delta_N \) are dependent on \( \omega \), \( \beta \), and \( \omega \) by controlling nanotips. d. For example, by setting \( \beta = \pi \), CPB with \( \omega - \delta_1 \) (red) and \( \omega + \delta_1 \) (orange) are observed at \( \Delta_0/\gamma = -6 \) and \( \Delta_0 = 0 \), respectively. Results in b and d are obtained from the full quantum simulation. For all plots, the other parameters are the same as in Fig. 1.
$$P_{mn} = \langle m, n| \hat{\rho}_{ns}|m, n \rangle$$ can be obtained for the normalized steady-state solutions $\hat{\rho}_{ns}$ of the master equation. We find an excellent agreement between our previous analytical results and the numerical solutions, as shown in Fig. 1b.

Next, we analyze the second-order correlation function with zero-time delay, i.e., $g^{(2)}(0) = \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle / \langle \hat{a} \hat{a} \rangle^2$, with $\hat{m} = \hat{a}^\dagger \hat{a}$. The conditions $g^{(2)}(0)$ is a local minimum and $g^{(2)}(0) \ll 1$ characterize 1PB [41], while $g^{(2)}(0) > 1$ and $g^{(2)}(0)$ is a local maximum characterize PIT [40]. Considering the correlation function of CW mode $g^{(2)}_{11}(0)$, based on Eq. (2), we find

$$g^{(2)}_{11}(0) = \frac{2P_{20}}{W_2} \approx \frac{\gamma_2^2 (\Delta_1^2 + 4J_{12} J_{21} \chi / \Delta_2)^2}{2\Delta_1 \eta_2^2},$$

where $W_2 = (P_{10} + P_{11} + 2P_{20})^2$. As seen in Figs. 2a and 2b, there is again an excellent agreement between the analytical results in Eq. (10) and the numerical ones. This confirms that the quantum noise stemming from photon losses barely affects the EP. We notice that in the weak-driving regime $\zeta \ll \gamma$ the overall effect of drive and dissipation will be that of coupling the eigenstates of $\hat{H}_{\text{eff}}$. For $\omega_L = \omega$, i.e., $\Delta_0 / \gamma \sim -3$, when the system is near an EP, UPB emerges at $\beta = 0.4\pi, 0.6\pi, 1.4\pi, 1.6\pi$, since $\beta$ fulfills the condition $C_{20} = 0$, i.e.,

$$\cos(2\sigma \beta) = \frac{-\text{Re}(\epsilon_1) \text{Im}(\epsilon_1) + \text{Re}(\epsilon_2) \text{Im}(\epsilon_2) + \kappa^3/(8\chi)}{\text{Re}(\epsilon_1) \text{Im}(\epsilon_2) + \text{Im}(\epsilon_1) \text{Re}(\epsilon_2)}.$$}

This can be interpreted as destructive quantum interference between the following two paths from the one to the two photon state: (i) the direct excitation $|1, 0 \rangle \xrightarrow{\sqrt{\chi}} |2, 0 \rangle$ and (ii) tunnel-coupling-mediated transition $|1, 0 \rangle \xrightarrow{J_{12}} |0, 1 \rangle \xrightarrow{\xi} |1, 1 \rangle \xrightarrow{\sqrt{\chi} J_{21}} |2, 0 \rangle$, as shown in Fig. 2c. Similarly, the destructive interference between $|0, 0 \rangle \xrightarrow{\xi} |1, 0 \rangle$ and $|0, 1 \rangle \xrightarrow{J_{12}} |1, 0 \rangle$, which can suppress the single-photon probability $C_{10} \to 0$, i.e., $\cos(2\sigma \beta) \to 1$, and enhance the relative two- and three-photon probabilities (see Fig. 2d). Thus, by virtually absorbing one photon, PIT is observed when the system is far away from EPs, i.e., $\beta = \pi$.

More interestingly, PB can occur when the system is at EPs i.e., $\beta_{\text{EP}} = \pi/2, 3\pi/2$. This phenomenon is fully due to the asymmetric backscattering between the CW- and CCW-travelling waves, since only one of the traveling directions is dominant. For example, in the case of $J_{21} = 0$, the transition matrix element between $\psi_1$ and $\psi_2^-$ is completely dependent on the $J_{12}$ term since the action of the drive is to couple only the states of the form $|n, m\rangle$ with those $|n+1, m\rangle$. Thus, even if the drive is resonantly coupled to the transition from $\psi_0$ and $\psi_2^-$, the population on $\psi_2^-$ is suppressed by the very small coupling induced by the drive (see Supplementary Information for numerical analysis), and the only states which are effectively coupled by the driving are those out-of-resonance, namely $\psi_{1,2}^{0,1+}$. In this regard, the presence of a fully asymmetric backscattering together with the EPs is the key ingredient to produce an effective CPB effect which cannot be observed in Hermitian settings (see more details in Supplementary Information Sec. S2). This exceptional photon blockade proves the effect of an EP (even if in the semiclassical picture) on the quantum properties of a system.

We observe a completely different quantum behavior when the system operates near or far away from EPs. By tuning $\beta$, we can bring the system to an EP where two-photon resonance of the CW mode is prohibited, leading to CPB and hence to the generation of strongly antibunched light with $g^{(2)}_{11,\text{EP}}(0) \sim 0.001$. Similarly, we can steer the system far away from EPs where two-photon resonance of the CW mode is allowed, leading to PIT and hence to the generation of bunched light with $g^{(2)}_{11}(0) \sim 0.93$. This EP-induced switching between PB and PIT and the prediction of up to four orders of magnitude difference in second-order quantum correlations are fundamentally different than already observed and studied non-Hermitian effects, and open a new venue for studying quantum mechanical processes in non-Hermitian settings.

Figure 3 shows the frequency tunable photon blockade in non-Hermitian system. The drive with frequency $\omega_L = \omega \pm \delta_1$ can resonantly couple the transitions from $\psi_0$ to $\psi_1^\pm$, the transitions from $\psi_1^+$ to $\psi_2^-$, however, are approximately detuned by $\delta_1$ for $\chi \gg J_{12,21}$; being out-of-resonance, the state containing two photons cannot be populated. For example, when the system is far away from EPs ($\beta = \pi$), the input lights with frequencies $\omega \pm \delta_1$ can resonantly couple to the transitions from the zero-photon state to single-photon states (i.e., the CW mode can be excited around $\Delta_0 = 0$ and $\Delta_0 = -6\gamma$) which corresponds to CPB and PIT and the prediction of up to four orders of magnitude difference in second-order quantum correlations are fundamentally different than already observed and studied non-Hermitian effects, and open a new venue for studying quantum mechanical processes in non-Hermitian settings.

Finally, we consider the full Liouvillian picture (no approximation) and still we find that the position of the EPs is unaffected by considering quantum noise. Liouvillian EPs are defined as degeneracies of Liouvillian superoperator $\mathcal{L}$ including the effects of quantum jumps [66]. Liouvillian EPs can be obtained by numerically calculating the spectra of $\mathcal{L}$, whose eigenmatrix $\hat{\rho}$ and eigenvalue $\lambda$ are defined via the relation $\mathcal{L}\hat{\rho} = \lambda\hat{\rho}$ with

$$\mathcal{L}\hat{\rho} = -i(\hat{H}_1^\dagger \hat{\rho} - \hat{\rho} \hat{H}_1^\dagger) + \sum_j D(\hat{\rho}, \hat{A}_j) + D(\hat{\rho}, \hat{\Gamma}),$$

where $\hat{\Gamma} = \sqrt{-2i\hat{H}_1^\dagger}$ is the additional jump operator, $\hat{H}_1^\dagger$ are the Hermitian and anti-Hermitian parts of the effective Hamiltonian $\hat{H}_1$. As shown in Fig. 1a, we find that the critical values of $\beta$ are the same for both Hamiltonian EPs and Liouvillian EPs [66].

In this work, we have shown that non-Hermitian spectral degeneracies known as EPs substantially affect photonic quantum correlations including PB and PIT. By
controlling the asymmetry and the strength of coupling between its CW and CCW modes, a ring resonator can be operated at an EP (e.g., ideal nonreciprocal coupling such as CW mode couples to the CCW mode but CCW mode does not couple to the CW mode) or far from an EP (e.g., ideal symmetric or reciprocal coupling). This in turn helps steering the system from PB regime (i.e., at an EP) to PIT regime (i.e., far from an EP) and vice versa. Similarly, one can steer the system from single-photon blockade regime to two-photon blockade regime by operating it at an EP or away from an EP, respectively (see Supplement for more results). Our results open a new direction in the studies of non-Hermitian physics, in particular they show how one can use EPs to control quantum correlations and achieve non-classical light switching. As such, EPs can be used to achieve novel quantum devices and play a key role in quantum engineering [67–69], quantum metrology [70–72], and quantum information processing [73, 74]. Our work reveals the basic mechanism of non-Hermitian systems with EPS operated in the deep quantum regime. Our approach and results can be applied further to study such a wide range of quantum effects as quantum entanglement, photon bundles [75], and dynamical blockade [76], aiming to improve the performance of quantum sensors [77, 78] and quantum unidirectional devices [38, 39].

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Supplementary Information for “Exceptional photon blockade”

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Here, we present technical details on exceptional-point- (EP) engineered photon blockade and photon-induced tunneling in a nonlinear optical micro-toroid resonator with two nanotips as Rayleigh scatterers. Our discussion includes: (1) periodic EPs in Kerr micro-toroid resonator, including Hamiltonian and Liouvillian EPs; (2) derivation of probability distribution, excitation spectrum, and quantum correlations functions; (3) quantum correlation properties with EPs, including conventional and unconventional photon blockade, photon-induced tunneling, two-photon blockade, quantum correlations in clockwise- and counterclockwise-travelling waves, and quantum correlations in weak nonlinear regime; (4) thermal response in quantum correlations.

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### S1. Periodic Exceptional Points in Kerr Micro-toroid Resonator

We consider a Kerr micro-toroid resonator coupled with two silica nano-tips as Rayleigh scatterers, the non-Hermitian optical coupling of the clockwise- (CW) and counterclockwise- (CCW) travelling waves induced by the nanotips, can be described by the scattering rate \( S_{12(21)} \)

\[
J_{12(21)} = \epsilon_1 + \epsilon_2 e^{\pm i2\sigma\beta},
\]

where \( J_{12} \) (\( J_{21} \)) corresponds to the scattering from the CCW (CW) mode to the CW (CCW) mode, \( \sigma \) is the azimuthal mode number, \( \beta \) is the relative angular position of the two scatterers, \( 2\epsilon_1 \) (\( 2\epsilon_2 \)) is the complex frequency splitting that is introduced by first (second) scatterer alone. This non-Hermitian system can be described by the following non-Hermitian Hamiltonian

\[
\hat{H} = \hat{H}_0 + \hat{H}_1 + \hat{H}_k + \hat{H}_d,
\]

with

\[
\begin{align*}
\hat{H}_0 &= \sum_{j=1,2} \hbar \omega \hat{a}_j \hat{a}_j, \\
\hat{H}_1 &= \hbar J_{12} \hat{a}_1 \hat{a}_2 + \hbar J_{21} \hat{a}_2 \hat{a}_1, \\
\hat{H}_k &= \sum_{j=1,2} \hbar \chi \hat{a}_j \hat{a}_j \hat{a}_j \hat{a}_j \\
\hat{H}_d &= \hbar \xi (\hat{a}_1^\dagger e^{-i\omega_L t} + \hat{a}_1 e^{i\omega_L t}),
\end{align*}
\]

where \( \omega = \omega_0 + \epsilon_1 + \epsilon_2 \), \( \omega_0 \) is the frequency of the unperturbed resonance mode, \( \hat{a}_1 \) and \( \hat{a}_2 \) are the annihilation operators of the CW and CCW modes, respectively. \( \chi = \hbar \omega^2 c n_2 / (\eta_0^2 V_{\text{eff}}) \) is the strength of the Kerr nonlinear interaction with the nonlinear (linear) refraction index \( n_2 \) (\( n_0 \)), effective cavity-mode volume \( V_{\text{eff}} \), and the speed of light in vacuum \( c \). \( \xi = \sqrt{\gamma_{\text{ex}} P_m / (\hbar \omega_L)} \) is the driving amplitude with driving power \( P_m \) and cavity-waveguide coupling rate \( \gamma_{\text{ex}} \).

In a frame rotating with the driving frequency \( \omega_L \), the Hamiltonian, given in Eq. (S1.2), is transformed to

\[
\hat{H}_r = i\hbar \frac{d\hat{D}}{dt} \hat{D} + \hat{D}^\dagger \hat{H} \hat{D},
\]

with \( \hat{D} = \exp[-i\omega_L (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2)t] \). Then we have the effective non-Hermitian Hamiltonian as follows

\[
\hat{H}_r = \hbar \Delta (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) + \hbar J_{12} \hat{a}_1 \hat{a}_2 + \hbar J_{21} \hat{a}_2 \hat{a}_1 + \hbar \chi (\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1) + \hbar \xi (\hat{a}_1^\dagger + \hat{a}_1),
\]

where \( \Delta = \Delta_0 + \epsilon_1 + \epsilon_2 \), and the optical detuning is given by \( \Delta_0 = \omega_0 - \omega_L \).

To study exceptional-point- (EP) engineered quantum effects in this system, we consider both Hamiltonian EPs and Liouvillian EPs [S3]. Hamiltonian EPs are usually defined as degeneracies of non-Hermitian Hamiltonians, such that at least two eigenfrequencies are identical and the corresponding eigenstates coalesce. However, for a fully quantum approach, the effects of quantum jumps should be included; thus, we analyze the EPs defined via degeneracies of a Liouvillian superoperator, i.e., Liouvillian EPs [S3].

#### A. Hamiltonian exceptional points

Hamiltonian EPs are special spectral degeneracies of non-Hermitian Hamiltonians governing the dynamics of open systems. At Hamiltonian EPs eigenvalues and the corresponding eigenstates of the non-Hermitian Hamiltonian coalesce [S4–S8]. Therefore, we investigate the eigensystem of non-Hermitian Hamiltonian to study Hamiltonian EPs. For the isolated system, i.e., the Hamiltonian, given in Eq. (S1.2), without the driving term \( \hat{H}_d \),

\[
\hat{H}_1 = \hat{H}_0 + \hat{H}_1 + \hat{H}_k = \hbar \omega \hat{a}_1^\dagger \hat{a}_1 + \hbar \omega \hat{a}_2^\dagger \hat{a}_2 + \hbar J_{12} \hat{a}_1^\dagger \hat{a}_2 + \hbar J_{21} \hat{a}_2^\dagger \hat{a}_1 + \hbar \chi (\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1),
\]

we have \( [\hat{a}_1^\dagger \hat{a}_1^\dagger + \hat{a}_2^\dagger \hat{a}_2^\dagger, \hat{H}_1] = 0 \) and \( [\hat{a}_i^\dagger, \hat{H}_k] = 0 \) for \( i = 1, 2 \), i.e., the total excitation number \( N = \langle \hat{n} \rangle + \langle \hat{n} \rangle \), where \( \hat{n} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \) is a conserved quantity. Thus, we can obtain the eigensystem with the Hilbert space spanned by the basis states \( |m, n\rangle \), where \( |m, n\rangle \) represents the Fock state with \( m \) particles in the CW mode and \( n \) particles in the CCW mode, and the total excitation number is \( N = m + n \).

In the weak driving regime, i.e., the driving strength is smaller than the resonator damping rate \( \xi \ll \gamma \), the Hilbert space of this system can be restricted within a subspace with few photons, i.e., subspace spanned by the basis states
Figure S1. Periodic EPs in a Kerr micro-toroid resonator coupled with two silica nano-tips. The left column is the result, given in Eqs. (S1.5) and (S1.6), in the subspace with 1 excitation (green) and the right is in the subspace with 2 excitations (yellow). a, Real (solid curves) and b, imaginary (dashed curves) parts of the frequency splitting as a function of relative angular position $\beta/\pi$. c, Scalar product (circles) between the eigenstates associated to the Hamiltonian EPs as a function of relative angular position $\beta/\pi$. At Hamiltonian EPs, both eigenvalues and the corresponding eigenstates of $\hat{H}$ coalesce, i.e., frequency splitting (Re[$\Delta E$]) and difference in linewidth (Im[$\Delta E$]) vanish; also, scalar product between the eigenstates equal to 1. By considering experimentally accessible parameters [S1, S9–S13], $\sigma = 1$, $\epsilon_1/\gamma = 1.5 - 0.1i$, $\epsilon_2/\gamma = 1.485 - 0.14i$, $\lambda = 1550$ nm, $Q = 5 \times 10^9$, $V_{\text{eff}} = 150 \mu$m$^3$, $n_0 = 1.4$, and $n_2 = 3 \times 10^{-14}$ m$^2$/W, EPs can be obtained at $\beta/\pi = 0.5$ and $\beta/\pi = 1.5$.

\[ |\pm \rangle \left[ \begin{array}{c} \pm \psi_1 \\ \pm \psi_2 \end{array} \right] \]

Then, the eigensystem of the non-Hermitian Hamiltonian $\hat{H}$, given in Eq. (S1.4), in this subspace can be obtained as follows:

\[
E = 0, \quad E_1^\pm = h (\omega \pm \delta_1), \quad E_2^s = h (2\omega + 2\chi + \delta_2^s), \quad \omega = \sqrt{\frac{\chi}{\epsilon}}, \quad \chi = \frac{2J_{21}}{\epsilon}, \quad \delta_2 = \frac{3}{2} \left( \frac{2J_{21}}{\epsilon} \right)
\]

\[
\psi_0 = |0,0 \rangle, \quad \psi_1^\pm = \left( \begin{array}{c} J_{21} \\ \pm \delta_1 \end{array} \right), \quad \psi_2^s = \left( \begin{array}{c} \delta_2 \\ \sqrt{\frac{2J_{21}}{\epsilon}} \end{array} \right)
\]

where $\delta_1 = \sqrt{J_{12}J_{21}}$; the superscript $s$ indicate ± or 0, and $\delta_2^s = -\chi \pm \sqrt{\chi^2 + 4J_{12}J_{21}}$, $\delta_0^s = 0$. Thus, the complex frequency splittings are

\[
\Delta E_1 = E_1^+ - E_1^-, \quad \Delta E_2 = E_2^+ - E_2^0
\]

for $N = 1$ and $N = 2$, respectively, and Hamiltonian EPs correspond to the points with $\Delta E_1 = 0$ and $\Delta E_2 = 0$. In Figs. S1a and S1b, we plot the real and imaginary part of the complex frequency splittings proving that the system admits periodic EPs at $\beta = \pi/2$ and $3\pi/2$ where eigenenergies coalesce. Figure S1c shows that the corresponding eigenstates have scalar products equal to 1 for $\beta = \pi/2$ and $3\pi/2$, proving that indeed the bifurcations are produced by EPs. Moreover, the critical values of $\beta$ at Hamiltonian EPs are given by:

\[
\beta_{\text{EP}} = \frac{2\pi}{2\sigma} \mp \frac{\arg (\epsilon_1) - \arg (\epsilon_2)}{2\sigma} (\mp = \pm 1, \pm 3, \ldots),
\]

where $\mp$ correspond to the backscattering coefficients $J_{21} = 0$ and $J_{12} = 0$, respectively. For the case with $J_{12} = 0$, by considering the experimentally accessible parameters [S1, S9, S10]: $Q = \omega_0/\gamma = 5 \times 10^9$, $\sigma = 1$, $\epsilon_1/\gamma = 1.5 - 0.1i$, $\epsilon_2/\gamma = 1.485 - 0.14i$, $\lambda = 1550$ nm, $Q = 5 \times 10^9$, $V_{\text{eff}} = 150 \mu$m$^3$, $n_0 = 1.4$, and $n_2 = 3 \times 10^{-14}$ m$^2$/W, Hamiltonian EPs correspond to the points with $\Delta E_1 = 0$ and $\Delta E_2 = 0$. In Figs. S1a and S1b, we plot the real and imaginary part of the complex frequency splittings proving that the system admits periodic EPs at $\beta = \pi/2$ and $3\pi/2$ where eigenenergies coalesce. Figure S1c shows that the corresponding eigenstates have scalar products equal to 1 for $\beta = \pi/2$ and $3\pi/2$, proving that indeed the bifurcations are produced by EPs. Moreover, the critical values of $\beta$ at Hamiltonian EPs are given by:

\[
\beta_{\text{EP}} = \frac{2\pi}{2\sigma} \mp \frac{\arg (\epsilon_1) - \arg (\epsilon_2)}{2\sigma} (\mp = \pm 1, \pm 3, \ldots),
\]
\(\epsilon_2/\gamma = 1.485 - 0.14i\), we have \(\beta_{EP} = 0.496\pi\) and \(\beta_{EP} = 1.496\pi\) with \(z = 1\) and \(z = 3\), respectively, which is agree well with the results shown in Fig. S1.

More details of the calculation are as follows. The subspace with \(N\) excitations is spanned by the bases

\[
|0, N\rangle, |1, N\rangle, |2, N-2\rangle, \ldots, |m, N-m\rangle, \ldots, |N-1, 1\rangle, |N, 0\rangle,
\]

and the dimension of this subspace is \(N + 1\). To obtain the matrix of the isolated Hamiltonian \(\hat{H}_1\), given in Eq. (S1.4), to above bases

\[
\hat{H}_1 |m, N-m\rangle = \left[\hbar \omega \hat{a}_1^\dagger \hat{a}_1 + \hbar \omega \hat{a}_2^\dagger \hat{a}_2 + \hbar J_{12} \hat{a}_1^\dagger \hat{a}_2 + \hbar J_{21} \hat{a}_2^\dagger \hat{a}_1 + \hbar \chi (\hat{a}_1^\dagger \hat{a}_1^\dagger + \hat{a}_2^\dagger \hat{a}_2^\dagger)\right] |m, N-m\rangle
\]

then, we have

\[
\langle m', N-m'| \hat{H}_1 |m, N-m\rangle = \left\{\hbar \omega N + \hbar \chi \left[(N-m)^2 - N + m^2\right]\right\} \delta_{m',m} + \hbar J_{12} \sqrt{(m+1)(N-m)} \langle m', N-m'| (m+1, N-m-1) \rangle + \hbar J_{21} \sqrt{m(N-m+1)} \langle m', N-m'| (m-1, N-m+1) \rangle
\]

i.e., the elements of the matrix are given by

\[
\langle m', N-m'| \hat{H}_1 |m, N-m\rangle = A^m \delta_{m',m} + B^m_{12} \delta_{m',m+1} + B^m_{21} \delta_{m',m-1}, \tag{S1.8}
\]

where

\[
A^m = \hbar \omega N + \hbar \chi \left[(N-m)^2 - N + m^2\right],
B^m_{12} = \hbar J_{12} \sqrt{(m+1)(N-m)},
B^m_{21} = \hbar J_{21} \sqrt{m(N-m+1)}, \quad (m = 0, 1, 2, \ldots, N).
\]

1. Subspace with no photons

According to the time-independent Schrödinger equation, when \(N = 0\), we have

\[
\hat{H}_1 \psi_0 = E_0 \psi_0, \tag{S1.9}
\]

and

\[
\hat{H}_1 |0, 0\rangle = \left[\hbar \omega \hat{a}_1^\dagger \hat{a}_1 + \hbar \omega \hat{a}_2^\dagger \hat{a}_2 + \hbar J_{12} \hat{a}_1^\dagger \hat{a}_2 + \hbar J_{21} \hat{a}_2^\dagger \hat{a}_1 + \hbar \chi (\hat{a}_1^\dagger \hat{a}_1^\dagger + \hat{a}_2^\dagger \hat{a}_2^\dagger)\right] |0, 0\rangle
\]

\[
= 0 |0, 0\rangle, \tag{S1.10}
\]

i.e., in this subspace with 0 photons, the eigenstate of \(\hat{H}_1\) is given by \(\psi_0 = |0, 0\rangle\) with the eigenenergy \(E_0 = 0\).
2. Subspace with one photon

When $N = 1$, the bases are $|0, 1\rangle$ and $|1, 0\rangle$. If we define

$$|0, 1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1, 0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then in this subspace, according to Eq. (S1.8), the Hamiltonian can be expressed as

$$\hat{H}_i = \hbar \begin{pmatrix} \omega & J_{21} \\ J_{12} & \omega \end{pmatrix}.$$ (S1.12)

If $(a_i \ b_i)^T$ are the eigenvectors of $\hat{H}_i/\hbar$ with eigenvalues $\lambda_i$, we have

$$\begin{pmatrix} \omega & J_{21} \\ J_{12} & \omega \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \lambda_i \begin{pmatrix} a_i \\ b_i \end{pmatrix}.$$ (S1.13)

The eigenvalues can be obtained from the characteristic equation

$$\begin{vmatrix} \omega - \lambda_i & J_{21} \\ J_{12} & \omega - \lambda_i \end{vmatrix} = 0,$$ (S1.14)

i.e.,

$$0 = (\omega - \lambda_i)^2 - J_{12}J_{21},$$
$$0 = \Lambda_1^2 - J_{12}J_{21},$$
$$\Lambda_1 = \sqrt{J_{12}J_{21}},$$

where $\Lambda_1 = \omega - \lambda_i$. Thus the eigenvalues are

$$\lambda_{\pm} = \omega \pm \sqrt{J_{12}J_{21}} = \omega \pm \delta_1,$$ (S1.15)

where $\delta_1 = \sqrt{J_{12}J_{21}}$. According to Eq. (S1.13), we have

$$0 = \Lambda_1 a_i + J_{21} b_i,$$
$$b_i = -\frac{\Lambda_1}{J_{21}} a_i.$$ For $\lambda_i = \lambda_{\pm} = \omega \pm \delta_1$, i.e., $\Lambda_1 = \mp \delta_1$, we can obtain $a_i = \pm \sqrt{J_{21}}$, $b_i = \sqrt{J_{12}}$, and the unnormalized eigenvectors

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} \pm \sqrt{J_{21}} \\ \sqrt{J_{12}} \end{pmatrix}.$$ (S1.16)

In this subspace with 1 photon, with the normalization factor $N_1 = (|\sqrt{J_{21}}|^2 + |\sqrt{J_{12}}|^2)^{-1/2}$, the eigenstates of $\hat{H}_i$ are given by

$$\psi_i^\pm = C_{01}^\pm |0, 1\rangle + C_{10}^\pm |1, 0\rangle,$$ (S1.17)

where

$$C_{01}^\pm = \pm \sqrt{J_{21}N_1},$$
$$C_{10}^\pm = \sqrt{J_{12}N_1},$$

and the corresponding eigenenergies are

$$E_i^\pm = \hbar (\omega \pm \delta_1).$$ (S1.18)
3. Subspace with two photons

When \( N = 2 \), the bases are \( |0, 2\rangle, |1, 1\rangle, \) and \( |2, 0\rangle \). If we define

\[
|0, 2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1, 1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |2, 0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\]

then in this subspace, according to Eq. (S1.8), the Hamiltonian can be expressed as

\[
\hat{H}_i = \hbar \begin{pmatrix} 2\omega + 2\chi & \sqrt{2}J_{21} & 0 \\ \sqrt{2}J_{12} & 2\omega & \sqrt{2}J_{21} \\ 0 & \sqrt{2}J_{12} & 2\omega + 2\chi \end{pmatrix}.
\]

If \( (a_i \ b_i \ c_i)^T \) are the eigenvectors of \( \hat{H}_i/\hbar \) with eigenvalues \( \lambda_i \), we have

\[
\begin{pmatrix} 2\omega + 2\chi & \sqrt{2}J_{21} & 0 \\ \sqrt{2}J_{12} & 2\omega & \sqrt{2}J_{21} \\ 0 & \sqrt{2}J_{12} & 2\omega + 2\chi \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \lambda_i \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}, \tag{S1.21}
\]

The eigenvalues can be obtained from the characteristic equation

\[
\begin{vmatrix} 2\omega + 2\chi - \lambda_i & \sqrt{2}J_{21} & 0 \\ \sqrt{2}J_{12} & 2\omega - \lambda_i & \sqrt{2}J_{21} \\ 0 & \sqrt{2}J_{12} & 2\omega + 2\chi - \lambda_i \end{vmatrix} = 0, \tag{S1.22}
\]

i.e.,

\[
0 = (2\omega + 2\chi - \lambda_i)^2(2\omega - \lambda_i) - 4J_{12}J_{21}(2\omega + 2\chi - \lambda_i),
\]

\[
0 = (2\omega + 2\chi - \lambda_i)(2\omega + 2\chi - \lambda_i)(2\omega - \lambda_i) - 4J_{12}J_{21},
\]

\[
0 = (2\omega + 2\chi - \lambda_i)[(2\omega - \lambda_i)^2 + 2\chi(2\omega - \lambda_i) - 4J_{12}J_{21}],
\]

\[
0 = (2\omega + 2\chi - \lambda_i)(\lambda_i^2 - 4\omega^2 - 4\omega\lambda_i + 4\chi\omega - 2\chi\lambda_i - 4J_{12}J_{21}),
\]

\[
0 = (2\omega + 2\chi - \lambda_i)(\lambda_i^2 - (4\omega + 2\chi)\lambda_i + 4(\omega^2 + \chi\omega - J_{12}J_{21})],
\]

and

\[
2\omega + 2\chi - \lambda_i = 0, \text{ or } \lambda_i^2 - c_1\lambda_i + c_0 = 0,
\]

where \( c_0 = 4(\omega^2 + \chi\omega - J_{12}J_{21}), \) \( c_1 = 2(2\omega + \chi) \). For the latter equation, we have

\[
\lambda_\pm = \frac{1}{2} \left[ 2(2\omega + \chi) \pm \sqrt{4(2\omega + \chi)^2 - 16(\omega^2 + \chi\omega - J_{12}J_{21})} \right]
\]

\[
= 2\omega + \chi \pm \sqrt{(2\omega + \chi)^2 - 4(\omega^2 + \chi\omega - J_{12}J_{21})}
\]

\[
= 2\omega + \chi \pm \sqrt{4\omega^2 + 2\chi^2 + 4\chi\omega - 4(\omega^2 + \chi\omega - J_{12}J_{21})}
\]

\[
= 2\omega + \chi \pm \sqrt{\chi^2 + 4J_{12}J_{21}}.
\]

Thus the eigenvalues are \( \lambda_0 = 2\omega + 2\chi \), and \( \lambda_\pm = 2\omega + 2\chi + \delta_2^\pm, \text{ i.e.,} \)

\[
\lambda_\pm = 2\omega + 2\chi + \delta_2^s, \tag{S1.23}
\]

where the subscript and superscript \( s \) indicate \pm or 0, the corresponding terms are \( \delta_2^\pm = -\chi \pm \sqrt{\chi^2 + 4J_{12}J_{21}}, \) and \( \delta_2^0 = 0 \). According to Eq. (S1.21), we can obtain

\[
0 = (2\omega + 2\chi - \lambda_i) a_i + \sqrt{2}J_{21}b_i,
\]

\[
0 = \Lambda_2 a_i + \sqrt{2}J_{21}b_i,
\]

\[
b_i = -\frac{\Lambda_2}{\sqrt{2}J_{21}} a_i,
\]
and

\[
0 = \sqrt{2}J_{12}b_i + (2\omega + 2\chi - \lambda_i)c_i, \\
0 = \sqrt{2}J_{12}b_i + \Lambda_2c_i, \\
c_i = -\frac{\sqrt{2}J_{12}}{\Lambda_2}b_i = \frac{J_{12}}{J_{21}}a_i,
\]

where \(\Lambda_2 = 2\omega + 2\chi - \lambda_i\). For \(\lambda_i = \lambda_s\), i.e., \(\Lambda_2 = -\delta_s^2\), we have \(a_i = \sqrt{2}J_{21}\), \(b_i = \delta_s^2\), \(c_i = \sqrt{2}J_{12}\), and the unnormalized eigenvectors

\[
\begin{pmatrix}
a_i \\
b_i \\
c_i
\end{pmatrix} = \begin{pmatrix}
\sqrt{2}J_{21} \\
\delta_s^2 \\
\sqrt{2}J_{12}
\end{pmatrix}.
\]

(S1.24)

In this subspace with 2 photons, with the normalization factor \(N_s^2 = (2|J_{12}|^2 + |\delta_s^2|^2 + 2|J_{21}|^2)^{-1/2}\), the eigenstates of \(\hat{H}\) are given by

\[
\psi_s^s = C_{02}^s|0, 2\rangle + C_{11}^s|1, 1\rangle + C_{20}^s|2, 0\rangle,
\]

where

\[
C_{02}^s = \sqrt{2}J_{21}N_s^2, \\
C_{11}^s = \delta_s^2N_s^2, \\
C_{20}^s = \sqrt{2}J_{12}N_s^2,
\]

and the corresponding eigenenergies are

\[
E_s^s = \hbar(2\omega + 2\chi + \delta_s^2).
\]

(S1.25)

We note that EPs require fully asymmetric backscattering [S1, S2, S15, S16], i.e.,

\[
J_{21} = 0 \text{ and } J_{12} \neq 0, \text{ or } J_{12} = 0 \text{ and } J_{21} \neq 0,
\]

(S1.26)

since it leads to coalescences for both eigenenergies \(E^\pm_s\) (\(E_2^{+,0}\)) and eigenstates \(\psi_1^\pm (\psi_2^{+,0})\) with \(\delta_1 = 0\) (\(\delta_2^+ = \delta_2^0\)). Since \(\epsilon_j = \text{Re}[\epsilon_j] + i\text{Im}[\epsilon_j] \ (j = 1, 2)\), for \(J_{21,12} \equiv \epsilon_1 + \epsilon_2e^{ \mp 2i\sigma \beta} = 0\), we have

\[
\begin{align*}
\text{Re}[\epsilon_1] + \text{Re}[\epsilon_2] &\cos(2\sigma \beta) \pm \text{Im}[\epsilon_2] \sin(2\sigma \beta) = 0, \\
\text{Im}[\epsilon_1] + \text{Im}[\epsilon_2] &\cos(2\sigma \beta) \mp \text{Re}[\epsilon_2] \sin(2\sigma \beta) = 0,
\end{align*}
\]

i.e.,

\[
\begin{align*}
\cos(2\sigma \beta) &= -(\text{Re}[\epsilon_1]\text{Re}[\epsilon_2] + \text{Im}[\epsilon_1]\text{Im}[\epsilon_2])/(\text{Re}[\epsilon_2]^2 + \text{Im}[\epsilon_2]^2) \\
\sin(2\sigma \beta) &= \mp(\text{Re}[\epsilon_1]\text{Im}[\epsilon_2] - \text{Re}[\epsilon_2]\text{Im}[\epsilon_1])/(\text{Re}[\epsilon_2]^2 + \text{Im}[\epsilon_2]^2).
\end{align*}
\]

(S1.27)

At EPs, the \(\beta_{EP}\) is satisfied to

\[
\begin{align*}
\cos(2\sigma \beta_{EP} + 2p\pi) &= -(\text{Re}[\epsilon_1]\text{Re}[\epsilon_2] + \text{Im}[\epsilon_1]\text{Im}[\epsilon_2])/(\text{Re}[\epsilon_2]^2 + \text{Im}[\epsilon_2]^2) \\
\sin(2\sigma \beta_{EP} + 2p\pi) &= \mp(\text{Re}[\epsilon_1]\text{Im}[\epsilon_2] - \text{Re}[\epsilon_2]\text{Im}[\epsilon_1])/(\text{Re}[\epsilon_2]^2 + \text{Im}[\epsilon_2]^2) \quad (p = 0, \pm 1, \pm 2, ...),
\end{align*}
\]

\[
\tan(2\sigma \beta_{EP} + 2p\pi + \pi) = \mp \frac{\text{Re}[\epsilon_2]\text{Im}[\epsilon_1] - \text{Re}[\epsilon_1]\text{Im}[\epsilon_2]}{\text{Re}[\epsilon_1]\text{Re}[\epsilon_2] + \text{Im}[\epsilon_1]\text{Im}[\epsilon_2]} \quad (p = 0, \pm 1, \pm 2, ...),
\]

\[
\beta_{EP} = \frac{z\pi}{2\sigma} \mp \frac{\arg(\epsilon_1) - \arg(\epsilon_2)}{2\sigma} \quad (z = \pm 1, \pm 3, ...).
\]

(S1.28)
B. Liouvillian exceptional points

According to the quantum trajectory method [S17], the optical decay can be included in the effective Hamiltonian

$$\hat{H}_e = \hat{H}_i - i \gamma (\hat{a}_1^{\dagger} \hat{a}_1 + \hat{a}_2^{\dagger} \hat{a}_2).$$

(S1.29)

We can also analyze Hamiltonian EPs by numerically diagonalizing the effective Hamiltonian $\hat{H}_{\text{eff}}$ via the relation

$$\hat{H}_e |\phi_i\rangle = h_i |\phi_i\rangle,$$

(S1.30)

where $h_i$ are the eigenvalues of $\hat{H}_e$ and $|\phi_i\rangle$ are corresponding eigenvectors. In Fig. S2 we plot the real and imaginary parts of the eigenvalues $h_i$ of $\hat{H}_e$, proving that the system admits periodic EPs by tuning the relative angular position at $\beta = \pi/2$ and $3\pi/2$, which agree well with the analytical results, as shown in Fig. S1. However, Hamiltonian EPs, given in the classical and semiclassical approaches, result from continuous, mostly slow, non-unitary evolution without quantum jumps.

For a fully quantum simulation, quantum jumps should be included in a fully quantum approach to make it equivalent to, e.g., the Lindblad master-equation approach with the so-called Liouvillian superoperator. By partitioning the Hamiltonian $\hat{H}_i$ into Hermitian and anti-Hermitian parts: $\hat{H}_i = \hat{H}_1 + \hat{H}_2$, and $\hat{H}_{\pm} = \pm \hat{H}_{\mp}$,

$$\hat{H}_1^\dagger = \hbar \text{Re}(\omega)(\hat{a}_1^{\dagger} \hat{a}_1 + \hat{a}_2^{\dagger} \hat{a}_2) + \hbar J_{12}^\dagger \hat{a}_1^{\dagger} \hat{a}_2 + \hbar J_{21}^\dagger \hat{a}_2^{\dagger} \hat{a}_1 + \hbar \chi (\hat{a}_1^{\dagger} \hat{a}_1 \hat{a}_1 \hat{a}_1 + \hat{a}_2^{\dagger} \hat{a}_2 \hat{a}_2 \hat{a}_2),$$

$$\hat{H}_2^\dagger = i\hbar \text{Im}(\omega)(\hat{a}_1^{\dagger} \hat{a}_1 + \hat{a}_2^{\dagger} \hat{a}_2) + i\hbar J_{12}^\dagger \hat{a}_1^{\dagger} \hat{a}_2 + i\hbar J_{21}^\dagger \hat{a}_2^{\dagger} \hat{a}_1,$$

(S1.31)

where $\text{Re}(\omega) = \omega_0 + \text{Re}(\epsilon_1 + \epsilon_2)$, $\text{Im}(\omega) = \text{Im}(\epsilon_1 + \epsilon_2)$, and

$$J_{12}^\dagger = \text{Re}(\epsilon_1) + \text{Re}(\epsilon_2)e^{i2\sigma\beta},$$

$$J_{21}^\dagger = i\text{Im}(\epsilon_1 + \epsilon_2)e^{2\sigma\beta},$$

Figure S2. Hamiltonian EPs (HEPs) and Liouvillian EPs (LEPs). a, Spectral properties of the non-Hermitian Hamiltonian $\hat{H}_i$ based on the numerical solution of Eq. (S1.30). HEPs can be obtained from the degeneracies of the eigenvalues $h_i$. b, Spectral properties of the full Liouvillian superoperator $\mathcal{L}$ based on the numerical solution of Eq. (S1.33). LEPs can be obtained from the degeneracies of the eigenvalues $\lambda_i$. Clearly, in this non-Hermitian system, the positions of HEPs and LEPs, i.e., the critical values of $\beta$, have great agreements, which is also confirmed in Ref. [S3] for the example of two coupled bosonic modes. Here, $\alpha = 1.215259 \times 10^9$, the other parameters are the same as those in Fig. S1.
Liouvillian superoperator $\mathcal{L}$ is given by

\[ \mathcal{L}\hat{\rho} = -i(\hat{H}_0^+ \hat{\rho} - \hat{\rho} \hat{H}_0^+) + \sum_j \mathcal{D}(\hat{\rho}, \hat{A}_j) + \mathcal{D}(\hat{\rho}, \hat{\Gamma}), \]  

(S1.32)

where $\mathcal{D}(\hat{\rho}, \hat{A}_j) = \hat{A}_j \hat{\rho} \hat{A}_j^\dagger - \hat{A}_j^\dagger \hat{A}_j \hat{\rho} / 2 - \hat{\rho} \hat{A}_j^\dagger \hat{A}_j / 2$ are the dissipators associated with the jump operators $\hat{A}_j = \sqrt{\tau} \hat{a}_j$, and $\hat{\Gamma} = \sqrt{-2i\hat{H}_0^+}$ is the additional jump operator. Then, Liouvillian EPs can be obtained via the degeneracies of Liouvillian superoperator, whose eigenvalues $\lambda_i$ and eigenmatrices $\hat{\rho}_i$ are defined via the relation \[\mathcal{L}\hat{\rho}_i = \lambda_i \hat{\rho}_i. \] (S1.33)

Figures S2b show the spectral analysis on the Liouvillian superoperator $\mathcal{L}$. More interestingly, the positions of Liouvillian EPs, i.e., the critical values of $\beta$, are the same as those of Hamiltonian EPs, which is also confirmed in Ref. [S3] for the example of two coupled bosonic modes. Therefore, in this non-Hermitian system, under the experimentally accessible parameters, EPs always occur at $\beta = \pi/2$ and $3\pi/2$, whether in semiclassical or fully quantum analysis.

More details of the calculation are as follows. Since $\epsilon_j = \text{Re}(\epsilon_j) + i\text{Im}(\epsilon_j)$, $J_{12,21} = \epsilon_1 + \epsilon_2 \exp(\pm 2i\sigma \beta)$, i.e.,

\[
\begin{align*}
J_{12} &= \text{Re}(\epsilon_1) + i \text{Im}(\epsilon_1) + [\text{Re}(\epsilon_2) + i \text{Im}(\epsilon_2)]\{\cos(2\sigma \beta) + i \sin(2\sigma \beta)\} \\
&= \text{Re}(\epsilon_1) + \text{Re}(\epsilon_2) \cos(2\sigma \beta) - \text{Im}(\epsilon_2) \sin(2\sigma \beta) + i \text{Im}(\epsilon_1) + i \text{Im}(\epsilon_2) \cos(2\sigma \beta) + i \text{Re}(\epsilon_2) \sin(2\sigma \beta), \\
J_{21} &= \text{Re}(\epsilon_1) + i \text{Im}(\epsilon_1) + [\text{Re}(\epsilon_2) + i \text{Im}(\epsilon_2)]\{\cos(2\sigma \beta) - i \sin(2\sigma \beta)\} \\
&= \text{Re}(\epsilon_1) + \text{Re}(\epsilon_2) \cos(2\sigma \beta) + \text{Im}(\epsilon_2) \sin(2\sigma \beta) + i \text{Im}(\epsilon_1) + i \text{Im}(\epsilon_2) \cos(2\sigma \beta) - i \text{Re}(\epsilon_2) \sin(2\sigma \beta),
\end{align*}
\]

the asymmetry coupling of CW and CCW travelling waves can be rewritten as

\[
\begin{align*}
J_{12} &= \text{Re}(J_{12}^+ + J_{12}^-) + i \text{Im}(J_{12}^+ + J_{12}^-), \\
J_{21} &= \text{Re}(J_{21}^+ + J_{21}^-) + i \text{Im}(J_{21}^+ + J_{21}^-),
\end{align*}
\]

(S1.34)

where

\[
\begin{align*}
\text{Re}(J_{12}^+) &= \text{Re}(J_{12}^-) = \text{Re}(\epsilon_1) + \text{Re}(\epsilon_2) \cos(2\sigma \beta), \quad \text{Re}(J_{12}^-) = -\text{Re}(J_{21}^+) = -\text{Im}(\epsilon_2) \sin(2\sigma \beta), \\
\text{Im}(J_{12}^+) &= \text{Im}(J_{12}^-) = \text{Im}(\epsilon_1) + \text{Im}(\epsilon_2) \cos(2\sigma \beta), \quad \text{Im}(J_{12}^-) = -\text{Im}(J_{21}^+) = -\text{Re}(\epsilon_2) \sin(2\sigma \beta).
\end{align*}
\]

Then we have

\[
\begin{align*}
\hat{H}_i &= \hbar[\omega_0 + \text{Re}(\epsilon_1 + \epsilon_2) + i \text{Im}(\epsilon_1 + \epsilon_2)](\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) + \hbar[\text{Re}(J_{12}^+) + \text{Re}(J_{12}^-) + i \text{Im}(J_{12}^+) + i \text{Im}(J_{12}^-)]\hat{a}_1^\dagger \hat{a}_2 \\
&\quad + \hbar[\text{Re}(J_{21}^+) + \text{Re}(J_{21}^-) + i \text{Im}(J_{21}^+) + i \text{Im}(J_{21}^-)]\hat{a}_2^\dagger \hat{a}_1 + \hbar(\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2),
\end{align*}
\]

\[
\begin{align*}
\hat{H}_i^\dagger &= \hbar[\omega_0 + \text{Re}(\epsilon_1 + \epsilon_2) - i \text{Im}(\epsilon_1 + \epsilon_2)](\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) + \hbar[\text{Re}(J_{21}^+) + \text{Re}(J_{21}^-) + i \text{Im}(J_{21}^+) + i \text{Im}(J_{21}^-)]\hat{a}_1^\dagger \hat{a}_2 \\
&\quad + \hbar[\text{Re}(J_{12}^+) + \text{Re}(J_{12}^-) - i \text{Im}(J_{12}^+) - i \text{Im}(J_{12}^-)]\hat{a}_2^\dagger \hat{a}_1 + \hbar(\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2).
\end{align*}
\]

The non-Hermitian Hamiltonian $\hat{H}_i$ can be partitioned into Hermitian and anti-Hermitian parts by considering

\[
\hat{H}_i^\pm = \frac{1}{2}(\hat{H}_i \pm \hat{H}_i^\dagger),
\]

(S1.35)

then we can obtain

\[
\begin{align*}
\hat{H}_i^+ &= \hbar[\omega_0 + \text{Re}(\epsilon_1 + \epsilon_2)](\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) + \hbar[\text{Re}(\epsilon_1) + \text{Re}(\epsilon_2)e^{2i\sigma \beta}]\hat{a}_1^\dagger \hat{a}_2 + \hbar[\text{Re}(\epsilon_1) + \text{Re}(\epsilon_2)e^{-2i\sigma \beta}]\hat{a}_2^\dagger \hat{a}_1 \\
&\quad + \hbar(\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2), \\
\hat{H}_i^- &= \hbar[\text{Im}(\epsilon_1 + \epsilon_2)](\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) + \hbar(\text{Im}(\epsilon_1) + \text{Im}(\epsilon_2)e^{-2i\sigma \beta}]\hat{a}_1^\dagger \hat{a}_2 + \hbar[\text{Im}(\epsilon_1) + \text{Im}(\epsilon_2)e^{2i\sigma \beta}]\hat{a}_2^\dagger \hat{a}_1.
\end{align*}
\]

(S1.36)
S2. DERIVATION OF PROBABILITY DISTRIBUTION, EXCITATION SPECTRUM, AND QUANTUM CORRELATION FUNCTIONS

For \( \text{Im}(\epsilon_{1,2}) < 0 \), we set \( \gamma' = -\text{Im}(\epsilon_1 + \epsilon_2) \); then, the Hamiltonian, given in Eq. (S1.3), can be rewritten as

\[
\hat{H}_t = \hbar [\Delta_0 + \text{Re}(\epsilon_1 + \epsilon_2) + i\text{Im}(\epsilon_1 + \epsilon_2)] (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) + \hat{H}_j + \hat{H}_k + \hat{H}_D
\]

\[
= \hbar [\Delta_0 + \text{Re}(\epsilon_1 + \epsilon_2)] (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) + \hat{H}_j + \hat{H}_k + \hat{H}_D - i\gamma' (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2)
\]

where

\[
\hat{H} = \hbar [\Delta_0 + \epsilon_1 + \epsilon_2] (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) + \hbar J_{12} \hat{a}_1^\dagger \hat{a}_2 + \hbar J_{21} \hat{a}_2^\dagger \hat{a}_1 + \hbar \chi (\hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2) + \hbar \xi (\hat{a}_1^\dagger + \hat{a}_1).
\]

Here, \( \hat{H}_D = \hbar \xi (\hat{a}_1^\dagger + \hat{a}_1) \), \( \gamma' \) can be considered as the scatterers-induced effective loss rate of the cavity field. In addition, we introduce \( \gamma \) as the rate of the cavity dissipation (the quality factor of the cavity is denoted by \( Q = \omega_0/\gamma \)); then, the total decay rate of the cavity field is \( \kappa = \gamma_\text{ex} + \gamma + 2\gamma' \). According to the quantum trajectory method [S17], the optical decay can be included in the following effective Hamiltonian

\[
\hat{H}_{\text{eff}} = \hat{H} - i\hbar \frac{\kappa}{2} (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) = \hat{H}_t - i\hbar \frac{\gamma}{2} (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2).
\]  

(S2.1)

Under the weak-driving condition, \( \xi \leq \gamma \), Hilbert space can be restricted within a subspace with few photons. In the subspace with \( N = m + n = 3 \) excitations, a general state of the system can be expressed as

\[
|\psi(t)\rangle = C_{00}(t) |00\rangle + C_{10}(t) |10\rangle + C_{01}(t) |01\rangle + C_{20}(t) |20\rangle + C_{11}(t) |11\rangle + C_{02}(t) |02\rangle + C_{30}(t) |30\rangle + C_{21}(t) |21\rangle + C_{12}(t) |12\rangle + C_{03}(t) |03\rangle.
\]  

(S2.2)

We substitute the above general state and the Hamiltonian given in Eq. (S2.1) into the Schrödinger equation

\[
i\hbar |\dot{\psi}(t)\rangle = \hat{H}_{\text{eff}} |\psi(t)\rangle,
\]

then we have

\[
i\hbar |\dot{\psi}(t)\rangle = \hbar \xi C_{10}(t) |00\rangle + \hbar \left( \Delta - i\frac{\gamma}{2} \right) C_{10}(t) |10\rangle + \hbar J_{12} C_{01}(t) |10\rangle + \hbar \xi C_{00}(t) |10\rangle + \sqrt{2} \hbar \xi C_{20}(t) |10\rangle
\]

\[
+ \hbar \left( \Delta - i\frac{\gamma}{2} \right) C_{20}(t) |20\rangle + \sqrt{2} \hbar J_{12} C_{11}(t) |20\rangle + 2 \hbar \chi C_{20}(t) |20\rangle + \sqrt{2} \hbar \xi C_{10}(t) |20\rangle + \sqrt{3} \hbar \xi C_{30}(t) |20\rangle
\]

\[
+ 2 \hbar \left( \Delta - i\frac{\gamma}{2} \right) C_{11}(t) |11\rangle + \sqrt{2} \hbar J_{12} C_{20}(t) |11\rangle + \sqrt{2} \hbar J_{12} C_{02}(t) |11\rangle + \hbar \xi C_{01}(t) |11\rangle + \sqrt{2} \hbar \xi C_{21}(t) |11\rangle
\]

\[
+ 2 \hbar \left( \Delta - i\frac{\gamma}{2} \right) C_{02}(t) |02\rangle + \sqrt{2} \hbar J_{12} C_{11}(t) |02\rangle + 2 \hbar \chi C_{02}(t) |02\rangle + \hbar \xi C_{12}(t) |02\rangle
\]

\[
+ 3 \hbar \left( \Delta - i\frac{\gamma}{2} \right) C_{30}(t) |30\rangle + \sqrt{3} \hbar J_{12} C_{21}(t) |30\rangle + 6 \hbar \chi C_{30}(t) |30\rangle + \sqrt{3} \hbar \xi C_{20}(t) |30\rangle
\]

\[
+ 3 \hbar \left( \Delta - i\frac{\gamma}{2} \right) C_{21}(t) |21\rangle + \sqrt{3} \hbar J_{12} C_{30}(t) |21\rangle + 2 \hbar J_{12} C_{12}(t) |21\rangle + 2 \hbar \chi C_{21}(t) |21\rangle + \sqrt{2} \hbar \xi C_{11}(t) |21\rangle
\]

\[
+ 3 \hbar \left( \Delta - i\frac{\gamma}{2} \right) C_{12}(t) |12\rangle + 2 \hbar J_{12} C_{21}(t) |12\rangle + \sqrt{3} \hbar J_{12} C_{03}(t) |12\rangle + 2 \hbar \chi C_{12}(t) |12\rangle + \hbar \xi C_{02}(t) |12\rangle
\]

\[
+ 3 \hbar \left( \Delta - i\frac{\gamma}{2} \right) C_{03}(t) |03\rangle + \sqrt{3} \hbar J_{12} C_{12}(t) |03\rangle + 6 \hbar \chi C_{03}(t) |03\rangle
\]

\[
+ 2 \hbar \xi C_{30}(t) |40\rangle + \sqrt{3} \hbar \xi C_{21}(t) |31\rangle + \sqrt{2} \hbar \xi C_{12}(t) |22\rangle + \hbar \xi C_{03}(t) |13\rangle.
\]
Due to the limits of the basis states, the terms including four particles, i.e., the terms of $2\hbar \xi C_{30}(t)\ket{40}$, $\sqrt{3}\hbar \xi C_{21}(t)\ket{31}$, $\sqrt{2}\hbar \xi C_{12}(t)\ket{22}$, and $\hbar \xi C_{03}(t)\ket{13}$, can be neglected. By comparing the coefficients of the same basis states in the above two equations, we have the following equations of motion for the probability amplitudes

$$
i\dot{C}_{00}(t) = \xi C_{10}(t),$$
$$
i\dot{C}_{10}(t) = \left(\Delta - i\frac{\gamma}{2}\right)C_{10}(t) + J_{12}C_{01}(t) + \xi C_{00}(t) + \sqrt{2}\xi C_{20}(t),$$
$$
i\dot{C}_{01}(t) = \left(\Delta - i\frac{\gamma}{2}\right)C_{01}(t) + J_{21}C_{10}(t) + \xi C_{11}(t),$$
$$
i\dot{C}_{20}(t) = 2\left(\Delta - i\frac{\gamma}{2}\right)C_{20}(t) + \sqrt{2}J_{12}C_{11}(t) + 2\chi C_{20}(t) + \sqrt{2}\xi C_{10}(t) + \sqrt{3}\xi C_{30}(t),$$
$$
i\dot{C}_{11}(t) = 2\left(\Delta - i\frac{\gamma}{2}\right)C_{11}(t) + \sqrt{2}J_{21}C_{20}(t) + \sqrt{2}J_{12}C_{02}(t) + \xi C_{01}(t) + \sqrt{2}\xi C_{21}(t),$$
$$
i\dot{C}_{02}(t) = 2\left(\Delta - i\frac{\gamma}{2}\right)C_{02}(t) + \sqrt{2}J_{21}C_{11}(t) + 2\chi C_{02}(t) + \xi C_{12}(t),$$
$$
i\dot{C}_{30}(t) = 3\left(\Delta - i\frac{\gamma}{2}\right)C_{30}(t) + \sqrt{3}J_{12}C_{21}(t) + 6\chi C_{30}(t) + \sqrt{3}\xi C_{20}(t),$$
$$
i\dot{C}_{21}(t) = 3\left(\Delta - i\frac{\gamma}{2}\right)C_{21}(t) + \sqrt{3}J_{21}C_{30}(t) + 2J_{12}C_{12}(t) + 2\chi C_{21}(t) + \sqrt{2}\xi C_{11}(t),$$
$$
i\dot{C}_{12}(t) = 3\left(\Delta - i\frac{\gamma}{2}\right)C_{12}(t) + 2J_{21}C_{21}(t) + \sqrt{3}J_{12}C_{03}(t) + 2\chi C_{12}(t) + \xi C_{02}(t),$$
$$
i\dot{C}_{03}(t) = 3\left(\Delta - i\frac{\gamma}{2}\right)C_{03}(t) + \sqrt{3}J_{21}C_{12}(t) + 6\chi C_{03}(t). \tag{S2.4}$$

If there is no driving field, the cavity field remains in the vacuum. When a weak-driving field is applied to the cavity, it may excite few photons in the cavity. Thus, we have the following approximate expressions: $C_{00} \sim 1$, $C_{10,01} \sim \xi/\gamma$, $C_{20,11,02} \sim \xi^2/\gamma^2$, and $C_{30,21,12,03} \sim \xi^3/\gamma^3$. Then, we can use a perturbation method to solve the above equations by discarding higher-order terms in each equation for lower-order variables, i.e., the above equations of motion for the probability amplitudes become

$$
i\dot{C}_{00}(t) = 0,$$
$$
i\dot{C}_{10}(t) = \left(\Delta - i\frac{\gamma}{2}\right)C_{10}(t) + J_{12}C_{01}(t) + \xi C_{00}(t),$$
$$
i\dot{C}_{01}(t) = \left(\Delta - i\frac{\gamma}{2}\right)C_{01}(t) + J_{21}C_{10}(t),$$
$$
i\dot{C}_{20}(t) = 2\left(\Delta - i\frac{\gamma}{2}\right)C_{20}(t) + \sqrt{2}J_{12}C_{11}(t) + 2\chi C_{20}(t) + \sqrt{2}\xi C_{10}(t),$$
$$
i\dot{C}_{11}(t) = 2\left(\Delta - i\frac{\gamma}{2}\right)C_{11}(t) + \sqrt{2}J_{21}C_{20}(t) + \sqrt{2}J_{12}C_{02}(t) + \xi C_{01}(t),$$
$$
i\dot{C}_{02}(t) = 2\left(\Delta - i\frac{\gamma}{2}\right)C_{02}(t) + \sqrt{2}J_{21}C_{11}(t) + 2\chi C_{02}(t),$$
$$
i\dot{C}_{30}(t) = 3\left(\Delta - i\frac{\gamma}{2}\right)C_{30}(t) + \sqrt{3}J_{12}C_{21}(t) + 6\chi C_{30}(t) + \sqrt{3}\xi C_{20}(t),$$
$$
i\dot{C}_{21}(t) = 3\left(\Delta - i\frac{\gamma}{2}\right)C_{21}(t) + \sqrt{3}J_{21}C_{30}(t) + 2J_{12}C_{12}(t) + 2\chi C_{21}(t) + \sqrt{2}\xi C_{11}(t),$$
$$
i\dot{C}_{12}(t) = 3\left(\Delta - i\frac{\gamma}{2}\right)C_{12}(t) + 2J_{21}C_{21}(t) + \sqrt{3}J_{12}C_{03}(t) + 2\chi C_{12}(t) + \xi C_{02}(t),$$
$$
i\dot{C}_{03}(t) = 3\left(\Delta - i\frac{\gamma}{2}\right)C_{03}(t) + \sqrt{3}J_{21}C_{12}(t) + 6\chi C_{03}(t). \tag{S2.5}$$

For the initially empty resonator, i.e., the initial state of the system is the vacuum state $\ket{00}$, the initial condition reads as: $C_{00}(0) = 1$. Due to the first equations of $i\dot{C}_{00}(t) = 0$, the solution of the probability amplitude for steady state $\ket{00}$ can be obtained as: $C_{00} = 1$. By considering infinite-time limit condition, $t \to \infty$, the steady-state solutions of other probability amplitudes can be obtained. Setting $\dot{C}_{mn}(t) = 0$, we can obtain the following equations of the
probability amplitudes for steady states

\[ 0 = \Delta_1 C_{10}(\infty) + 2J_{12}C_{01}(\infty) + 2\xi C_{00}(\infty), \]
\[ 0 = \Delta_1 C_{01}(\infty) + 2J_{21}C_{10}(\infty), \]
\[ 0 = \Delta_2 C_{20}(\infty) + \sqrt{2}J_{12}C_{11}(\infty) + \sqrt{2}\xi C_{10}(\infty), \]
\[ 0 = \Delta_1 C_{11}(\infty) + \sqrt{2}J_{21}C_{20}(\infty) + \sqrt{2}J_{12}C_{02}(\infty) + \xi C_{01}(\infty), \]
\[ 0 = \Delta_2 C_{02}(\infty) + \sqrt{2}J_{21}C_{11}(\infty), \]
\[ 0 = 3\Delta_3 C_{30}(\infty) + 2\sqrt{3}J_{12}C_{21}(\infty) + 2\sqrt{3}\xi C_{20}(\infty), \]
\[ 0 = \Delta_4 C_{21}(\infty) + 2\sqrt{3}J_{21}C_{30}(\infty) + 4J_{12}C_{12}(\infty) + 2\sqrt{2}\xi C_{11}(\infty), \]
\[ 0 = \Delta_4 C_{12}(\infty) + 4J_{21}C_{21}(\infty) + 2\sqrt{3}J_{12}C_{03}(\infty) + 2\xi C_{02}(\infty), \]
\[ 0 = 3\Delta_3 C_{03}(\infty) + 2\sqrt{3}J_{21}C_{12}(\infty), \]

(S2.6)

where \( \Delta_1 = 2\Delta - i\gamma, \Delta_2 = \Delta_1 + 2\chi, \Delta_3 = 2\Delta + 4\chi - i\gamma, \) and \( \Delta_4 = 3\Delta_3 - 8\chi. \) This linear equations can be represented as an augmented matrix in row reduction (also known as Gaussian elimination). The matrix can be modified by using elementary row operations until it reaches reduced row echelon form. Then, the solutions of the linear equations can be obtained. The first two equations of Eq. (S2.6) are self-consistent

\[
\begin{align*}
\Delta_1 C_{10} + 2J_{12}C_{01} &= -2\xi, \\
2J_{21}C_{10} + \Delta_1 C_{01} &= 0,
\end{align*}
\]

(S2.7)

and the augmented matrix of this linear equations is written as

\[
A_1 = \begin{pmatrix}
\Delta_1 & 2J_{12} & -2\xi \\
2J_{21} & \Delta_1 & 0
\end{pmatrix}.
\]

(S2.8)

We apply elementary row operations to above matrix

\[
A_1 \xrightarrow{r_2 \rightarrow r_2 - \frac{2J_{21}}{\eta_1} r_1} \left( \begin{array}{ccc}
\Delta_1 & 2J_{12} & -2\xi \\
0 & \frac{4J_{21}\xi}{\Delta_1} & \frac{2\xi\Delta^2_1}{\eta_1}
\end{array} \right) \xrightarrow{r_2 \rightarrow r_2 + \frac{\Delta_1}{\eta_1} r_1} \left( \begin{array}{ccc}
\Delta_1 & 0 & \frac{2\xi\Delta^2_1}{\eta_1}
\end{array} \right) \xrightarrow{r_2 \rightarrow (r_2 / \Delta_1)} \left( \begin{array}{ccc}
1 & 0 & \frac{2\xi\Delta_1}{\eta_1}
\end{array} \right),
\]

where \( \eta_1 = 4J_{12}J_{21} - \Delta_1^2. \) Then, we obtain the following solutions

\[ C_{10} = \frac{2\xi\Delta_1}{\eta_1}, \quad C_{01} = -\frac{4\xi J_{21}}{\eta_1}. \]

(S2.9)

By substituting the above solutions into the 3rd-5th equations of Eq. (S2.6), we have

\[
\begin{align*}
\Delta_2 C_{20} + \sqrt{2}J_{12}C_{11} &= -2\sqrt{2}\xi^2 \Delta_1 \\
\sqrt{2}J_{21}C_{20} + \Delta_1 C_{11} + \sqrt{2}J_{12}C_{02} &= \frac{4\xi J_{21}}{\eta_1} \\
\sqrt{2}J_{21}C_{11} + \Delta_2 C_{02} &= 0,
\end{align*}
\]

(S2.10)

and corresponding augmented matrix

\[
A_2 = \begin{pmatrix}
\Delta_2 & \sqrt{2}J_{12} & 0 & -2\sqrt{2}\xi^2 \Delta_1 \\
\sqrt{2}J_{21} & \Delta_1 & \sqrt{2}J_{12} & \frac{4\xi J_{21}}{\eta_1} \\
0 & \sqrt{2}J_{21} & \Delta_2 & 0
\end{pmatrix}.
\]

(S2.11)
Applying elementary row operations, then we obtain

\[
\begin{pmatrix}
\Delta_2 & \sqrt{2}J_{12} & 0 & -2\sqrt{2}\zeta^2 \Delta_1 \\
0 & -\frac{\lambda_1}{\Delta_2} & \sqrt{2}J_{12} & 4J_{21}\xi^2(\Delta_1 + \Delta_2) \\
0 & 0 & \sqrt{2}J_{21} & \Delta_2
\end{pmatrix}
\xrightarrow{r_3 \times \frac{\lambda_1}{\eta_2 \Delta_2}}
\begin{pmatrix}
\Delta_2 & \sqrt{2}J_{12} & 0 & -2\sqrt{2}\zeta^2 \Delta_1 \\
0 & -\frac{\lambda_1}{\Delta_2} & \sqrt{2}J_{12} & 4J_{21}\xi^2(\Delta_1 + \Delta_2) \\
0 & 0 & \sqrt{2}J_{21} & \Delta_2
\end{pmatrix}
\xrightarrow{r_2 - r_1 \lambda_2 \Delta_2}
\begin{pmatrix}
\Delta_2 & \sqrt{2}J_{12} & 0 & -2\sqrt{2}\zeta^2 \Delta_1 \\
0 & -\frac{\lambda_1}{\Delta_2} & \sqrt{2}J_{12} & 4J_{21}\xi^2(\Delta_1 + \Delta_2) \\
0 & 0 & \sqrt{2}J_{21} & \Delta_2
\end{pmatrix}
\xrightarrow{r_3 \times \frac{\lambda_1}{\eta_2 \Delta_2}}
\begin{pmatrix}
\Delta_2 & \sqrt{2}J_{12} & 0 & -2\sqrt{2}\zeta^2 \Delta_1 \\
0 & -\frac{\lambda_1}{\Delta_2} & \sqrt{2}J_{12} & 4J_{21}\xi^2(\Delta_1 + \Delta_2) \\
0 & 0 & \sqrt{2}J_{21} & \Delta_2
\end{pmatrix}
\xrightarrow{r_2 \times (-\frac{\lambda_2}{\Delta_2})}
\begin{pmatrix}
\Delta_2 & \sqrt{2}J_{12} & 0 & -2\sqrt{2}\zeta^2 \Delta_1 \\
0 & -\frac{\lambda_1}{\Delta_2} & \sqrt{2}J_{12} & 4J_{21}\xi^2(\Delta_1 + \Delta_2) \\
0 & 0 & \sqrt{2}J_{21} & \Delta_2
\end{pmatrix}
\xrightarrow{r_1 - \sqrt{2}J_{12} \times r_2}
\begin{pmatrix}
\Delta_2 & 0 & 0 & 2\sqrt{2}\zeta^2 \Delta_2 + 4J_{12}J_{21}\lambda \\
0 & 1 & 0 & -4J_{21}\xi^2(\Delta_1 + \Delta_2) \\
0 & 0 & 1 & 4\sqrt{2}\zeta^2 \xi^2(\Delta_1/\Delta_2 + 1)
\end{pmatrix}
\xrightarrow{r_1 / \Delta_2}
\begin{pmatrix}
1 & 0 & 0 & 2\sqrt{2}\zeta^2 \Delta_2 + 4J_{12}J_{21}\lambda \\
0 & 1 & 0 & -4J_{21}\xi^2(\Delta_1 + \Delta_2) \\
0 & 0 & 1 & 4\sqrt{2}\zeta^2 \xi^2(\Delta_1/\Delta_2 + 1)
\end{pmatrix},
\]

where \(\eta_2 = 4J_{12}J_{21} - \Delta_1 \Delta_2, \lambda_1 = 2J_{12}J_{21} - \Delta_1 \Delta_2\). Then we obtain the following solutions

\[
C_{20} = \frac{2\sqrt{2}\zeta^2 (\Delta_1^2 + 4J_{12}J_{21} \lambda/\Delta_2)}{\eta \eta_2}, \quad C_{11} = -\frac{4J_{21} \xi^2 (\Delta_1 + \Delta_2)}{\eta \eta_2}, \quad C_{02} = \frac{4\sqrt{2}J_{21}^2 \xi^2 (\Delta_1/\Delta_2 + 1)}{\eta \eta_2},
\]

(S2.12)

Substituting the above solutions into the last four equations of Eq. (S2.6),

\[
\begin{align*}
3\Delta_3 C_{30}(t) + 2\sqrt{3} J_{12} C_{21}(t) & = -2\sqrt{3} \zeta C_{20}(t) \\
2\sqrt{3} J_{21} C_{30}(t) + \Delta_4 C_{21}(t) + 4J_{12} C_{12}(t) & = -2\sqrt{2} \zeta C_{11}(t) \\
4J_{21} C_{21}(t) + \Delta_4 C_{12}(t) + 2\sqrt{3} J_{12} C_{03}(t) & = -2\xi C_{02}(t) \\
2\sqrt{3} J_{21} C_{12}(t) + 3\Delta_3 C_{03}(t) & = 0
\end{align*}
\]

(S2.13)

to obtain the augmented matrix as follows

\[
B = \begin{pmatrix}
3\Delta_3 & 2\sqrt{3} J_{12} & 0 & 0 & -2\sqrt{3} \zeta C_{20}(t) \\
2\sqrt{3} J_{21} & \Delta_4 & 4J_{12} & 0 & -2\sqrt{2} \zeta C_{11}(t) \\
0 & 4J_{21} & \Delta_4 & 2\sqrt{3} J_{12} & -2\xi C_{02}(t) \\
0 & 0 & 2\sqrt{3} J_{21} & 3\Delta_3 & 0
\end{pmatrix}.
\]

We apply elementary row operations to this matrix

\[
B \xrightarrow{r_2 - \frac{\lambda_2}{\eta_3} \times r_1}
\begin{pmatrix}
3\Delta_3 & 2\sqrt{3} J_{12} & 0 & 0 & -2\sqrt{3} \zeta C_{20}(t) \\
0 & -\frac{\lambda_1}{\Delta_2} & 4J_{12} & 0 & -4J_{21} \xi^2 C_{20}(t) - 2\sqrt{2} \zeta C_{11}(t) \\
0 & 4J_{21} & \Delta_4 & 2\sqrt{3} J_{12} & -2\xi C_{02}(t) \\
0 & 0 & 2\sqrt{3} J_{21} & 3\Delta_3 & 0
\end{pmatrix}
\xrightarrow{r_2 \times (-\Delta_3)/\eta_3}
\begin{pmatrix}
3\Delta_3 & 2\sqrt{3} J_{12} & 0 & 0 & -2\sqrt{3} \zeta C_{20}(t) \\
0 & -\frac{\lambda_1}{\Delta_2} & 4J_{12} & 0 & -4J_{21} \xi^2 C_{20}(t) - 2\sqrt{2} \zeta C_{11}(t) \\
0 & 4J_{21} & \Delta_4 & 2\sqrt{3} J_{12} & -2\xi C_{02}(t) \\
0 & 0 & 2\sqrt{3} J_{21} & 3\Delta_3 & 0
\end{pmatrix}
\xrightarrow{r_3 - 4J_{21} \times r_2}
\begin{pmatrix}
3\Delta_3 & 2\sqrt{3} J_{12} & 0 & 0 & -2\sqrt{3} \zeta C_{20}(t) \\
0 & -\frac{\lambda_1}{\Delta_2} & 4J_{12} & 0 & -4J_{21} \xi^2 C_{20}(t) - 2\sqrt{2} \zeta C_{11}(t) \\
0 & 4J_{21} & \Delta_4 & 2\sqrt{3} J_{12} & -2\xi C_{02}(t) \\
0 & 0 & 2\sqrt{3} J_{21} & 3\Delta_3 & 0
\end{pmatrix}.
\]
with the definitions of $\Delta_i$ (i = 1, 2, 3, 4) being

$$
\Delta_1 = 2\Delta - i\gamma, \quad \Delta_2 = \Delta_1 + 2\chi, \\
\Delta_3 = \Delta_1 + 4\chi, \quad \Delta_4 = 3\Delta_3 - 8\chi.
$$

(S2.17)
1. Probability distribution

The probabilities of finding $m$ particles in the CW mode and $n$ particles in the CCW mode are given by

$$P_{mn} = |C_{mn}|^2 / \mathcal{M},$$  \hspace{1cm} (S2.18)

with the normalization coefficient in the subspace with $N = m + n = 3$ excitations

$$\mathcal{M} = \sum_{N=0}^{3} \sum_{m=0}^{N} |C_{mn}|^2.$$  \hspace{1cm} (S2.19)

2. Excitation spectrum

The excitation spectrum of CW and CCW modes are denoted by $S_{11}(\Delta_0)$ and $S_{22}(\Delta_0)$, respectively, and can be obtained based on above probability distribution and $n_0 = \xi^2 / \kappa^2$:

$$S_{11}(\Delta_0) = \frac{1}{n_0} \sum_{N=0}^{3} \sum_{m=0}^{N} mP_{mn},$$  \hspace{1cm} (S2.20)

$$S_{22}(\Delta_0) = \frac{1}{n_0} \sum_{N=0}^{3} \sum_{n=0}^{N} nP_{mn}.$$  \hspace{1cm} (S2.21)

3. Quantum correlation functions

The equal-time (namely zero-time-delay) second-order correlation function of CW and CCW modes are written as

$$g_{11}^{(2)}(0) = \frac{\langle \hat{a}_1^2 \hat{a}_1^2 \rangle - \langle \hat{a}_1^2 \rangle \langle \hat{a}_1^2 \rangle}{\langle \hat{a}_1^2 \rangle^2} = \frac{\langle \hat{n}^2 - \hat{n} \rangle}{\langle \hat{n} \rangle^2},$$  \hspace{1cm} (S2.22)

$$g_{22}^{(2)}(0) = \frac{\langle \hat{a}_2^2 \hat{a}_2^2 \rangle - \langle \hat{a}_2^2 \rangle \langle \hat{a}_2^2 \rangle}{\langle \hat{a}_2^2 \rangle^2} = \frac{\langle \hat{n}^2 - \hat{n} \rangle}{\langle \hat{n} \rangle^2},$$  \hspace{1cm} (S2.23)

respectively. The cross-correlation between CW mode and CCW mode is defined by

$$g_{12}^{(2)}(0) = \frac{\langle \hat{a}_1^2 \hat{a}_2^2 \rangle}{\langle \hat{a}_1^2 \rangle \langle \hat{a}_2^2 \rangle} = \frac{\langle \hat{nm} \rangle}{\langle \hat{n} \rangle \langle \hat{m} \rangle}.$$  \hspace{1cm} (S2.24)

When the cavity field is in the state given in Eq. (S2.2), we have

$$g_{11}^{(2)}(0) = \frac{\langle \psi(t) | \hat{n}^2 - \hat{n} | \psi(t) \rangle}{\langle \psi(t) | \hat{n} | \psi(t) \rangle^2} = \frac{\sum_{N=0}^{3} \sum_{m=0}^{N} (m^2 - m)P_{mn}}{\langle \psi(t) | \hat{m} | \psi(t) \rangle^2} = \frac{2P_{10} + 2P_{21} + 6P_{30}}{W_{11}},$$  \hspace{1cm} (S2.25)

$$g_{22}^{(2)}(0) = \frac{\langle \psi(t) | \hat{n}^2 - \hat{n} | \psi(t) \rangle}{\langle \psi(t) | \hat{n} | \psi(t) \rangle^2} = \frac{\sum_{N=0}^{3} \sum_{n=0}^{N} (n^2 - n)P_{mn}}{\langle \psi(t) | \hat{n} | \psi(t) \rangle^2} = \frac{2P_{12} + 2P_{12} + 6P_{30}}{W_{22}},$$  \hspace{1cm} (S2.26)

$$g_{12}^{(2)}(0) = \frac{\langle \psi(t) | \hat{n} \hat{m} | \psi(t) \rangle}{\langle \psi(t) | \hat{n} | \psi(t) \rangle \langle \psi(t) | \hat{m} | \psi(t) \rangle} = \frac{P_{11} + 2P_{21} + 2P_{12}}{W_{11} W_{22}},$$  \hspace{1cm} (S2.27)

where $W_{11} = P_{10} + P_{11} + P_{12} + 2P_{20} + 2P_{21} + 3P_{30}$, $W_{22} = P_{01} + P_{01} + P_{21} + 2P_{02} + 2P_{12} + 3P_{33}$. In the weak driving regime, we have the following approximate formulas: $C_{00} \sim 1$, $C_{10,01} \sim \xi / \gamma$, $C_{20,11,02} \sim \xi^2 / \gamma^2$, and $C_{30,21,12,03} \sim \xi^3 / \gamma^3$, i.e., $P_{10,01} \gg P_{20,11,02} \gg P_{30,21,12,03}$ and $\mathcal{M} \sim 1$. Therefore, the approximate equal-time second-order correlation function can be written as

$$g_{11}^{(2)}(0) \simeq \frac{2P_{20}}{P_{10}} \simeq \frac{16 \xi^4 (\Delta_1^2 + 4J_{12}J_{21} \chi / \Delta_2)^2}{\eta_1^2 \eta_2^2}, \quad \frac{\eta_1^4}{16 \xi^4 \Delta_1^4} = \frac{\eta_2^2 (\Delta_1^2 + 4J_{12}J_{21} \chi / \Delta_2)^2}{\Delta_1^2 \eta_2^2},$$  \hspace{1cm} (S2.28)
When the cavity field is in the state given in Eq. (S2.2), we have
\[ H^{(2)}_2(0) \approx \frac{2P_{02}}{P_{01}^2} \approx \frac{64J_{21}^4\xi^4(\Delta_1/\Delta_2 + 1)^2}{\eta_1^2\eta_2^2}, \]
\[ \eta_1^2 = \frac{4^4\xi^4J_{21}^4}{\eta_1^2\eta_2^2}, \]
\[ (\Delta_1/\Delta_2 + 1)^2. \]

(S2.29)

\[ g^{(2)}_{12}(0) \approx \frac{P_{11}}{P_{10}P_{01}} \approx \frac{16J_{21}^4\xi^4(\Delta_1 + \Delta_2)^2}{\eta_1^2\eta_2^2}, \]
\[ \eta_2^2 = \frac{4^2\xi^2\Delta_1}{\eta_1^2\eta_2^2}. \]

(S2.30)

The equal-time (namely zero-time-delay) third-order correlation functions are written as
\[ g^{(3)}_{11}(0) = \frac{\langle \hat{a}_{1}^{\dagger\dagger}\hat{a}_{1}^{\dagger} \rangle}{\langle \hat{a}_{1}^{\dagger} \hat{a}_{1} \rangle^2} = \frac{\langle \hat{n}_3 - 3\hat{n}\hat{n}_2 + 2\hat{m} \rangle}{\langle \hat{n} \rangle^2}, \]
\[ g^{(3)}_{22}(0) = \frac{\langle \hat{a}_{2}^{\dagger\dagger}\hat{a}_{2}^{\dagger} \rangle}{\langle \hat{a}_{2}^{\dagger} \hat{a}_{2} \rangle^2} = \frac{\langle \hat{n}_3 - 3\hat{n}_2^2 + 2\hat{n} \rangle}{\langle \hat{n}_2 \rangle^2}. \]

When the cavity field is in the state given in Eq. (S2.2), we have
\[ g^{(3)}_{11}(0) = \frac{\langle \psi(t)|\hat{n}_3 - 3\hat{n}\hat{n}_2 + 2\hat{m} \rangle \psi(t) \rangle}{\langle \psi(t)|\hat{n} \psi(t) \rangle^2} = \sum_{N=0}^{3} \sum_{m=0}^{N} (m^3 - 3m^2 + 2m)P_{mn} = \frac{6P_{30}}{W_{11}}, \]
\[ g^{(3)}_{22}(0) = \frac{\langle \psi(t)|\hat{n}_3 - 3\hat{n}_2^2 + 2\hat{m} \rangle \psi(t) \rangle}{\langle \psi(t)|\hat{n}_2 \psi(t) \rangle^2} = \sum_{N=0}^{3} \sum_{m=0}^{N} (m^3 - 3m^2 + 2m)P_{mn} = \frac{6P_{30}}{W_{22}}. \]

Considering the weak-driving case, the approximate equal-time third-order correlation functions are written as
\[ g^{(3)}_{11}(0) \approx \frac{6P_{30}}{P_{10}^2} \approx \frac{64^6(\mu\Delta_1^2 + 4J_{12}J_{21}\Gamma)^2}{\mu^2\eta_1^2\eta_2^2\Delta_3^2}, \]
\[ \eta_1^6 = \frac{4^6\xi^6\Delta_3^2}{\mu^2\eta_1^2\eta_2^2\Delta_3^2}, \]
\[ \frac{\Delta_1^2 + 4J_{12}J_{21}\Gamma)^2}{\mu^2\eta_1^2\eta_2^2\Delta_3^2}. \]

(S2.31)

\[ g^{(3)}_{22}(0) \approx \frac{6P_{30}}{P_{01}^2} \approx \frac{4^6J_{12}^6\xi^6\Gamma^2}{\mu^2\eta_1^2\eta_2^2\Delta_3^2}, \]
\[ \frac{\Delta_1^2 + 4J_{12}J_{21}\Gamma)^2}{\mu^2\eta_1^2\eta_2^2\Delta_3^2}. \]

(S2.32)

In order to confirm our analytical results based on quantum trajectory method, we numerically study the fully quantum dynamics of the system by considering the effects of quantum jumps via the Lindblad master-equation approach. However, for non-Hermitian system, the Lindblad master equation should be transformed into a “hybrid” formalism [S18, S19]:
\[ \dot{\rho} = -i[\hat{H}, \rho] - i\{\hat{H}, \rho\} + \sum_j \hat{D}(\rho, \hat{a}_j) + i2\text{tr}(\rho\hat{H}_-\rho). \]

(S2.33)

where \( \hat{D}(\rho, \hat{a}_j) = (\gamma/2)(2\hat{a}_j\rho\hat{a}_j^\dagger - \hat{a}_j^\dagger\hat{a}_j\rho - \rho\hat{a}_j^\dagger\hat{a}_j) \). Here \( \hat{H}_\pm \) are the Hermitian and anti-Hermitian parts of the effective Hamiltonian \( \hat{H}_r \), given in Eq. (S1.3),
\[ \hat{H}_+ = \hbar\text{Re}(\Delta)(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2) + \hbar J_{12}\hat{a}_1^\dagger\hat{a}_1\hat{a}_2 + \hbar\xi(\hat{a}_1^\dagger\hat{a}_1\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2\hat{a}_2) + \hbar\xi(\hat{a}_1^\dagger + \hat{a}_1), \]
\[ \hat{H}_- = i\hbar\text{Im}(\Delta)(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2) + i\hbar\xi(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2) + i\hbar\xi(\hat{a}_1^\dagger + \hat{a}_1), \]

(S2.34)

where \( \text{Re}(\Delta) = \Delta_0 + \text{Re}(\epsilon_1 + \epsilon_2), \text{Im}(\Delta) = \text{Im}(\epsilon_1 + \epsilon_2) \), and
\[ J_{12}^\dagger = \text{Re}(\epsilon_1) + \text{Re}(\epsilon_2)e^{i2\sigma\beta}, \quad J_{21}^\dagger = \text{Re}(\epsilon_1) + \text{Re}(\epsilon_2)e^{-i2\sigma\beta}, \]
\[ J_{12}^\dagger = \text{Im}(\epsilon_1) + \text{Im}(\epsilon_2)e^{i2\sigma\beta}, \quad J_{21}^\dagger = \text{Im}(\epsilon_1) + \text{Im}(\epsilon_2)e^{-i2\sigma\beta}. \]

Due to \( \hat{H}_r = \hat{H}_+ + \hat{H}_- \) and \( \hat{H}_r^\dagger = \pm\hat{H}_r \), the master equation given in Eq. (S2.33) can be written as
\[ \dot{\rho} = -i(\hat{H}_r\rho - \rho\hat{H}_r^\dagger) + \sum_j \hat{D}(\rho, \hat{a}_j) + i2\text{tr}(\rho\hat{H}_-\rho). \]

(S2.35)
S3. QUANTUM CORRELATION PROPERTIES WITH EXCEPTIONAL POINTS

A. Photon blockade and photon-induced tunneling

Photon blockade (PB) was first introduced by Imamoglu et al. [S20] as an analogue of Coulomb blockade for quantum-well electrons, which shows the emission of photons with sub-poissonian statistics. The laser drives a strong nonlinear cavity in resonance with the zero- to one-photon transition; however, due to the presence of a large nonlinearity-induced frequency shift, another photon cannot be added as the one- to two-photon transition is off resonant. In view of its important role in achieving single-photon devices, the realization of PB has been anticipated in Jaynes-Cummings model [S21] and Kerr cavity [S22, S23]. Moreover, PB has been demonstrated experimentally in diverse systems from cavity or circuit QED [S24–S29] to cavity-free devices [S30]. Besides, optomechanical PB [S31–S34] and and nonreciprocal PB [S35, S36] have also been explored.

A completely different mechanism to realize the emission of sub-poissonian photons has been named unconventional photon blockade (UPB) [S37] as opposed to the conventional photon blockade (CPB) described above. UPB, the nonlinear cavity in resonance with the zero- to one-photon transition; however, due to the presence of a large nonlinearity-induced frequency shift, another photon cannot be added as the one- to two-photon transition is off resonant. In view of its important role in achieving single-photon devices, the realization of PB has been anticipated in Jaynes-Cummings model [S21] and Kerr cavity [S22, S23]. Moreover, PB has been demonstrated experimentally in diverse systems from cavity or circuit QED [S24–S29] to cavity-free devices [S30]. Besides, optomechanical PB [S31–S34] and and nonreciprocal PB [S35, S36] have also been explored.

In the past two decades, the concept of PB have extended from single photon to two-photon [S42] and multi-photon [S43]. Two-photon blockade (2PB) [S44–S55] has been studied in Kerr and cavity or circuit QED systems, and was first experimentally demonstrated by Hansen et al. in 2017 [S56]. The occurrence of PB is usually experimentally characterized by the $g^{(2)}(0)$ function, which is usually measured with extended Hanbury Brown-Twiss interferometers. Specially, the condition $g^{(2)}(0) < 1$ is satisfied for single photon blockade (1PB), and 2PB is characterized by the conditions of $g^{(2)}(0) > 1$ and $g^{(3)}(0) < 1$ (Table I).

Photon-induced tunneling (PIT), capturing the physics of a nonlinear optical system in which the absorption of one photon enhances the probabilities of subsequent photons, has been observed experimentally in Ref. [S14, S25, S46, S57, S58]. Table II shows that more refined criteria for PIT are sometimes applied based on higher-order correlation functions $g^{(l)}(0)$ with $l > 2$.

In this work, due to the weak-driving-induced small mean photon number $\langle \hat{n} \rangle < 1$, by considering quantum correlation functions $g^{(2)}(0)$ and $g^{(3)}(0)$, we refer to 1PB, 2PB, and PIT if the following conditions are satisfied respectively:

$$1\text{PB} : g^{(2)}(0) < 1 \text{ and } g^{(2)}(0) \text{ is a local minimum,} \quad (S3.1a)$$

$$2\text{PB} : g^{(2)}(0) > 1 \text{ and } g^{(3)}(0) < 1, \quad (S3.1b)$$

$$\text{PIT} : g^{(2,3)}(0) > 1 \text{ and } g^{(2,3)}(0) \text{ is a local maximum.} \quad (S3.1c)$$

We consider the quantum correlations in CW mode. As shown in Fig. S3, by using above criteria and resonantly driving the system, $\omega_0 = \omega$ (i.e., $\Delta_0/\gamma \sim -3$), UPB and PIT are observed when the system is not at EPs; while exceptional photon blockade (EPB) emerges when the system at EPs.

UPB can occur at $\beta = 0.4\pi, 0.6\pi, 1.4\pi, 1.6\pi$ in our system, since the relative phase angle $\beta$ fulfills the condition $C_{20} \rightarrow 0$. According to Eq. (S2.14), we have

$$C_{20} = \frac{2\sqrt{2}\xi^2[(2\Delta - i\gamma)^2(2\Delta + 2\chi - i\gamma) + 4J_{12}J_{21}\chi]}{[4J_{12}J_{21} - (2\Delta - i\gamma)^2[4J_{12}J_{21} - (2\Delta - i\gamma)(2\Delta + 2\chi - i\gamma)](2\Delta + 2\chi - i\gamma)]}\quad (S3.2)$$

where $\kappa = \gamma + 2\gamma', \gamma' = -\text{Im}(\epsilon_1 + \epsilon_2), \Delta = \Delta_0 + \epsilon_1 + \epsilon_2 = \Delta_0 + \text{Re}(\epsilon_1 + \epsilon_2) + i\text{Im}(\epsilon_1 + \epsilon_2)$, and the conditions to satisfy $C_{20} \rightarrow 0$

$$0 = [2\text{Re}(\Delta) - i\kappa]_3 + 2\chi[2\text{Re}(\Delta) - i\kappa]^2 + 4J_{12}J_{21}\chi$$

$$= 8\text{Re}(\Delta)^3 + 8\chi\text{Re}(\Delta)^2 - 6\kappa^2\text{Re}(\Delta) + 4\chi\text{Re}(J_{12}J_{21}) - 2\chi\kappa^2$$

$$- i12\text{Re}(\Delta)^2\kappa + i\kappa^3 + i8\chi\text{Re}(\Delta)\kappa + i4\chi\text{Im}(J_{12}J_{21}). \quad (S3.3)$$
TABLE I. Criteria of two-photon blockade (2PB) used in literature.

| Reference | Criteria of 2PB |
|-----------|-----------------|
| Kubanek et al. (2008) [S42] | $C^{(2)}(0)$ is a local maximum and $C^{(2)}(0) > 0$, where $C^{(2)}(0) \equiv |g^{(2)}(0) - 1| \times \langle \hat{a}^\dagger \hat{a} \rangle^2$ |
| Miranowicz et al. (2013) [S44] | $F_2 \approx 1$ and $F_m \ll 1$ for $m < 2$, where $F_N \equiv \sum_{m=0}^{N} F_m$ |
| Rundquist et al. (2014) [S46] | $g^{(2)}(0) > 1$ and $g^{(3)}(0) < 1$ |
| Hansen et al. (2017) [S56] | $g^{(2)}(0) > 1$ and $g^{(3)}(0) < 1$ |
| Zhu et al. (2017) [S50] | $g^{(2)}(0) > 1$ and $g^{(3)}(0) < 1$ |
| Felicetti et al. (2018) [S51] | $g^{(2)}(0) \geq 1$ and $g^{(3)}(0) < 1$ |
| Huang et al. (2018) [S35] | $g^{(2)}(0) \geq f^{(2)}$ and $g^{(3)}(0) < f$, where $f \equiv e^{-\langle \hat{n} \rangle}$ and $f^{(2)} \equiv e^{-\langle \hat{n} \rangle} + \langle \hat{n} \rangle \cdot g^{(3)}(0)$, i.e., $g^{(2)}(0) > 1$ and $g^{(3)}(0) < 1$ when the mean photon number $\langle \hat{n} \rangle$ is very small |
| Bin et al. (2018) [S52] | $g^{(2)}(0) > 1$ and $g^{(3)}(0) < 1$ |
| K.-Kudlaszyk et al. (2019) [S55] | $g^{(2)}(0) > 1$ and $g^{(3)}(0) < 1$ |

TABLE II. Criteria of photon-induced tunneling (PIT) used in literature.

| Reference | Criteria of PIT |
|-----------|----------------|
| Faraon et al. (2008) [S25] | $g^{(2)}(0)$ is a local maximum |
| Majumdar et al. (2012) [S14, S57] | $g^{(2)}(0) > 1$ |
| Xu et al. (2013) [S58] | $g^{(2)}(0) > 1$ (two-photon tunneling); $g^{(3)}(0) > g^{(2)}(0) > 1$ (three-photon tunneling) |
| Rundquist et al. (2014) [S46] | $g^{(3)}(0) > g^{(2)}(0)$ |
| Huang et al. (2018) [S35] | $g^{(l)}(0) > 1$ for $l = 2, 3, 4$ |
| K.-Kudlaszyk et al. (2019) [S55] | $g^{(2)}(0) > 1$ (two-photon tunneling); $g^{(3)}(0) > g^{(2)}(0) > 1$ (three-photon tunneling) |

Figure S3. Exceptional-points-engineered single photon blockade and photon-induced tunneling. a. Second-order correlation function of CW mode $g^{(2)}_{\omega_L}(0)$ versus optical detuning $\Delta \omega_L / \gamma$ at $\omega_L = \omega$. To characterize PIT, third-order correlation function $g^{(3)}_{\omega_L}(0)$ has been shown in a3. b. Dependence of the photon probabilities $P_{10}$ (circles) and $P_{20}$ (asterisks) on $\Delta \omega / \gamma$. For all plots, the black solid and dashed curves represent the analytical solution based on quantum-trajectory method. The dots, circles and asterisks correspond to results of a full quantum simulation based on “hybrid” formalism for Lindblad master equation. Table III gives the experimentally accessible parameters used in calculations.
from Eq. (S3.5), we have

\[
\Re(\Delta) = \Re(\Delta_0) + \Re(\Delta_L)
\]

By considering \(\Re(J_{12,21}) = D_1 + D_2 \cos(2\sigma\beta)\) and \(\Im(J_{12,21}) = D_3 + D_4 \cos(2\sigma\beta)\), where

\[
\begin{align*}
D_1 &= \Re(\epsilon_1)^2 - \Im(\epsilon_1)^2 + \Re(\epsilon_2)^2 - \Im(\epsilon_2)^2, \\
D_2 &= 2\Re(\epsilon_1) \Re(\epsilon_2) - 2\Im(\epsilon_1) \Im(\epsilon_2), \\
D_3 &= 2\Re(\epsilon_1) \Im(\epsilon_1) + 2\Re(\epsilon_2) \Im(\epsilon_2), \\
D_4 &= 2\Re(\epsilon_1) \Im(\epsilon_2) + 2\Im(\epsilon_1) \Re(\epsilon_2),
\end{align*}
\]

from Eq. (S3.5), we have

\[
\begin{align*}
\Im(J_{12,21}) &= \frac{3\Re(\Delta)^2 - 2\chi \Re(\Delta) - \chi^3/4}{\chi}, \\
\cos(2\sigma\beta) &= \frac{3\Re(\Delta)^2 - 2\chi \Re(\Delta) - \chi D_3 - \chi^3/4}{\chi D_4},
\end{align*}
\]

then

\[
\Re(J_{12,21}) = \frac{3\chi D_2 \Re(\Delta)^2 - 2\chi \Re(\Delta) + \chi (D_1 D_4 - D_2 D_3) - 3\chi^3 D_2/4}{\chi D_4}.
\]

Therefore, we have

\[
\begin{align*}
4\Re(\Delta)^3 + 4\chi \Re(\Delta)^2 - 3\chi^2 \Re(\Delta) + 2\chi \Re(J_{12,21}) &= \chi \kappa^2, \\
12\chi \Re(\Delta)^2 - 8\chi \chi \Re(\Delta) - 4\chi \Im(J_{12,21}) &= \kappa^3.
\end{align*}
\]

### Table III. The most important symbols and experimentally accessible values with Refs. used in this paper.

| Symbol | Meaning |
|--------|---------|
| \(\beta\) | Relative angular position of two nano scatterers |
| \(c\) | Speed of light in vacuum |
| \(\lambda\) | Light wavelength \(\lambda = 1550\text{nm}\) |
| \(\omega_0\) | Resonance frequency of the bare cavity without scatterers \(\omega_0 = 2\pi c/\lambda\) |
| \(\omega_L\) | Input laser frequency |
| \(\Delta_0\) | Detuning from bare cavity to input laser \(\Delta_0 = \omega_0 - \omega_L\) |
| \(\sigma\) | Azimuthal mode number \(\sigma = 1 [S1]\) |
| \(\gamma_{\text{ex}}\) | Optical waveguide coupling rate |
| \(\gamma\) | Effective dissipation rate induced by scatterers \(\gamma' = -\Re(\epsilon_1 + \epsilon_2)\) |
| \(\kappa\) | Total dissipation rate \(\kappa = \gamma_{\text{ex}} + \gamma + 2\gamma'\) |
| \(\omega\) | Resonance frequency of the non-Hermitian system \(\omega = \omega_0 + \epsilon_1 + \epsilon_2\) |
| \(\Delta\) | Cavity-to-laser detuning \(\Delta = \Delta_0 + \epsilon_1 + \epsilon_2 = \omega - \omega_L\) |

\(J_{12,21} = \epsilon_1 + \epsilon_2 e^{+i2\sigma\beta}\) asymmetry coupling rate between CW and CCW modes \(\hat{H}_i = hJ_{12,21} \hat{A}_i^\dagger \hat{A}_2 + hJ_{21,21} \hat{A}_1^\dagger \hat{A}_1\)

\(n_0\) | Linear refraction index \(n_0 = 1.4\) |
| \(n_2\) | Nonlinear refraction index \(n_2 = 10^{-14} \text{m}^2/W\) and \(10^{-15} \text{m}^2/W [S13]\) |
| \(V_{\text{eff}}\) | Effective cavity-mode volume \(V_{\text{eff}} = 150 \mu\text{m}^3 [S11, S12]\) |
| \(\chi\) | \(\chi = \hbar \omega^2 n_2 / (n_0 V_{\text{eff}})\) strength of the Kerr nonlinearity \(\hat{H}_k = \sum_{j=1,2} \hbar \chi \hat{a}_j^\dagger \hat{a}_j \hat{a}_j\) |
| \(P_{\text{in}}\) | Driving power \(P_{\text{in}} = 4 \text{W} [S59]\) |
| \(\xi\) | Driving amplitude \(\xi = \sqrt{\gamma_{\text{ex}} P_{\text{in}} / (\hbar \omega L)}\) |
| \(m, n\) | Photon number in CW, CCW modes \(m = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle, n = \langle \hat{a}_2^\dagger \hat{a}_2 \rangle\) |
| \(N\) | Total photon number \(N = m + n\) |
| \(P_{m,n}\) | Probability for finding \(m\) photon in CW mode and \(n\) photons in CCW mode |
| \(\mathcal{P}(m)\) | Standard Poisson distribution with \(m\)-photon in CW mode \(\mathcal{P}(m) \equiv \langle \hat{a}_1^\dagger \hat{a}_1 \rangle^m \exp(-\langle \hat{a}_1^\dagger \hat{a}_1 \rangle)/m!\) |
| \(\mathcal{R}(m)\) | Relative photon distributions \(\mathcal{R}(m) \equiv \frac{\mathcal{P}(m)}{\mathcal{P}(m)}\) |
| \(n_0\) | Normalized factor \(n_0 = \xi^2/\kappa^2\) |
| \(S_{ij}(\Delta_0)\) | Excitation spectrum of CW, CCW modes \(S_{ij}(\Delta_0) \equiv \frac{1}{n_0} \sum_{N=0}^\infty \sum_{m=0}^N \langle \hat{a}_j^\dagger \hat{a}_j \rangle P_{m,n}\) |
| \(g_{ij}^{(1)}(0)\) | \(g_{ij}^{(1)}(0) \equiv \langle \hat{a}_i^\dagger \hat{a}_j \rangle /\langle \hat{a}_j^\dagger \hat{a}_j \rangle\) \(l\)-th order correlation functions at zero-time delay |
| \(g_{12}^{(2)}(0)\) | Second-order cross-correlation functions at zero-time delay \(g_{12}^{(2)}(0) \equiv \langle \hat{a}_1^\dagger \hat{a}_2 \hat{a}_1^\dagger \hat{a}_1 \rangle /\langle \hat{a}_1^\dagger \hat{a}_1 \rangle \langle \hat{a}_2^\dagger \hat{a}_2 \rangle\) |
| \(w^*, u^*\) | \(w^* = |(\psi_2^* | \hat{H}_D | \psi_1^*)|^2\), \(u^* = |(\psi_2^* | \hat{H}_D | \psi_1^*)|^2\) transition matrix element from \(\psi_1^*\) to \(\psi_2^*\)
Figure S4. Mechanisms of exceptional-points-engineered single photon blockade and photon-induced tunneling. (a) Mechanism of UPB occurring at $\beta = 0.4\pi$. (b) Mechanism of PIT emerging at $\beta = \pi$. (c) Mechanism of PIT-induced UB based on (c) dependence of transition matrix elements $u^\pm = |\langle \psi^\pm | \hat{H}| \psi^\mp \rangle|^2$ on $\beta/\pi$, and (d) normalized cavity excitation spectrums of CW mode $S_{11}$ and $S_{22}$. The solid curves represent the analytical solution based on quantum-trajectory method. The circles and squares correspond to results of a fully quantum simulation based on “hybrid” formalism Lindblad master equation. The other parameters are the same as those in Fig. S3.

Substituting above equation into Eq. (S3.4),

$$\text{Re}(\Delta)^3 + (\chi + \frac{3\kappa D_2}{2D_4})\text{Re}(\Delta)^2 - \left(\frac{3}{4}\kappa^2 + \chi \frac{D_2}{D_4}\right)\text{Re}(\Delta) + \frac{1}{2}\chi(D_1 - \frac{D_2D_3}{D_4}) - \frac{\kappa^3D_2}{8D_4} - \frac{1}{4}\chi^2 = 0, \quad (S3.7)$$

we can obtain the optimal conditions of $\text{Re}(\Delta)$ and $\beta$ to satisfy $C_{20} \to 0$.

Specially, when $\omega_L = \omega$, i.e., $\text{Re}(\Delta) = 0$, we have

$$\cos(2\sigma\beta) = -\frac{2\text{Re} (\epsilon_1) \text{Im} (\epsilon_1) + 2\text{Re} (\epsilon_2) \text{Im} (\epsilon_2) + \kappa^3/4\chi}{2[\text{Re} (\epsilon_1) \text{Im} (\epsilon_2) + \text{Im} (\epsilon_1) \text{Re} (\epsilon_2)]}, \quad (S3.8)$$

i.e.,

$$\beta_{\text{UPB}} = \frac{p\pi}{\sigma} \pm \frac{1}{2\sigma} \arccos \left[ \frac{\text{Re}(\epsilon_1)^2 + \text{Im}(\epsilon_2)^2 - \text{Re}(\epsilon_2)^2 - \text{Re}(\epsilon_2)^2 + \kappa^2/2}{2\text{Re}(\epsilon_1) \text{Re}(\epsilon_2) - 2\text{Im}(\epsilon_1) \text{Im}(\epsilon_2)} \right], \quad (p = 0, \pm 1, \pm 2, ...) \quad (S3.9)$$

from Eq. (S3.5). Then, we can obtain the values of $\beta_{\text{opt}}$, where UPB happens, are $0.4045\pi, 0.5955\pi, 1.4045\pi$, and $1.5955\pi$, which are agree with our numerical results. This can be interpreted as destructive quantum interference between the following two paths from the one to the two photon state: (i) the direct excitation $|1, 0\rangle \sqrt{2J_{21}} |2, 0\rangle$ and (ii) tunnel-coupling-mediated transition $|1, 0\rangle \xrightarrow{J_{21}} |0, 1\rangle \xrightarrow{\xi} |1, 1\rangle \sqrt{2J_{21}} |2, 0\rangle$, as shown in Fig. S4a.

Similarly, Fig. S4b shows the destructive quantum interference between $|0, 0\rangle \xrightarrow{\xi} |0, 1\rangle$ and $|0, 1\rangle \xrightarrow{J_{21}} |1, 0\rangle$, which can suppress the single-photon probability $C_{10} \to 0$ (see Fig. S3b3). Thus, by virtually absorbing one photon, PIT is observed when the system is far away from EPs, i.e., $\beta = 0, \pi$, and $2\pi$, as shown in Fig. S3a3. Since

$$C_{10} = \frac{2\xi(2\Delta - i\gamma)}{4J_{12}J_{21} - (2\Delta - i\gamma)^2} \to 0,$$

we have $4J_{12}J_{21} - (2\Delta - i\gamma)^2 \gg 2\xi(2\Delta - i\gamma)$, i.e., $A \gg B$, where $A = 4\epsilon_1^2 + 4\epsilon_2^2 + 8\epsilon_1\epsilon_2 \cos(2\sigma\beta)$, $B = \text{Im}[\text{Re}(\Delta) - i\kappa][\text{Re}(\Delta) + 2\xi - i\kappa]$. For resonance-case $\text{Re}(\Delta) = 0$, we have $\max|A|$ when $\max[\text{Re}(2\sigma\beta)] = 1$, i.e.,

$$\beta_{\text{PIT}} = \frac{p\pi}{\sigma}, \quad (p = 0, \pm 1, \pm 2, ...), \quad (S3.10)$$
which is agree well with the numerical result.

The quantum phenomenon becomes particularly interesting at the EPs. First, let us consider the case $J_{21} = 0$. In this case, there exist a unique eigenvector $\psi_1 = |1, 0\rangle$ whose energy is exactly $\omega$. Moreover, the state $\psi_2$ has exactly energy $2\omega$. Naively, one would imagine that, since the state with one photon is resonant to the one with two photons, the presence of the two-photon state would be enhanced with respect to the PIT case. However, this is false. Indeed, the action of the drive is to couple only the states of the form $|n, m\rangle$ with those $|n + 1, m\rangle$. As it stems from Eq. (S1.5), the transition matrix element between $\psi_1$ and $\psi_2$ would be completely due to the $J_{12}$ term, and it is much smaller than the matrix element between $\psi_1$ and $\psi_2^{0, +}$, as shown in Fig. S4c. Thus, only the CW-travelling wave is dominant (Fig S4d), and even if the coupling is resonant, the presence of two photons is suppressed by the very small coupling induced by the drive. In this regard, the presence of an antisymmetric backscattering together with the EP produces an effective photon-blockade effect, since the only states which are effectively coupled by the driving are those out-of-resonance, namely $\psi_2^{0, +}$. Therefore, we observe an unlightening of the CW mode.

A very different situation appear for $J_{12} = 0$. Indeed, $\psi_1 = |0, 1\rangle$ means that the CW excitation introduced by the drive will be scattered to the CCW mode. This mode, however, will be strongly resonant with the mode $\psi_2^-$. As shown in Fig. S4c, the transition matrix element between $\psi_1$ and $\psi_2^-$ is larger than the matrix element between $\psi_1$ and $\psi_2^{0, +}$. However, since $\psi_2^-$ will contain no $|2, 0\rangle$ component, the presence of the EP will result in an effective PB for the CW mode. Even if in both cases the presence of an EP will result in an effective PB, these two phenomena are strikingly different. Indeed, for $J_{21} = 0$ the physics will be dominated by the single photon of CW mode. For $J_{12} = 0$, both the CW and CCW photons will be present in the resonator, and the state $|1, 1\rangle$ will be intensively populated. Nevertheless, in both of them we expect to observe an effective PB of the CW mode, associated to the suppression of the state $|2, 0\rangle$. This exceptional photon blockade (EPB) proves the effect of an EP (even if in the semiclassical picture) on the quantum properties of a system.

Moreover, 2PB is observed when the relative phase angle is tuned to $\beta \sim 0.3\pi, 0.7\pi, 1.3\pi$ and $1.7\pi$ (For example, Fig. S5). This $\beta$-tuned quantum effect can be understood from the from the relative photon distributions of CW mode $R(m) = |P_m - P_m|/P_m$, i.e., the deviation of the photon distribution $P_m$ from the standard Poisson distribution $P_m$ with the same mean photon number $m$, as shown in Fig. S5b. $R(m) > 0$ or $R(m) < 0$ denotes the probability for finding $m$-photon is enhanced or suppressed. Figure S5c shows the PB and PIT occur periodically by tuning relative phase angle $\beta$, which introduce periodic EPs and altered quantum correlations periodically.

More details of the calculation on transition matrix element are as follows. As we consider the weak-driving case ($\xi \ll \gamma$), the transition matrix element from $\psi_1^\pm$ to $\psi_2^\pm$ are given by

$$u^s = \left| \langle \psi_2^s | \hat{H}_D | \psi_1^s \rangle \right|^2, \quad w^s = \left| \langle \psi_2^{s\dagger} | \hat{H}_D | \psi_1^s \rangle \right|^2. \quad (S3.11)$$

**Figure S5.** Exceptional-points-induced two photon blockade. a, Correlation functions $g_{11}^{(2)}(0)$ (red curve) and $g_{11}^{(3)}(0)$ (purple curve) versus optical detuning $\Delta_0/\gamma$. b, The origin of this EPs-induced 2PB effect is shown from the relative photon distributions $R(m) = |P(m) - P(m)|/P(m)$, i.e., the deviation of the photon distribution $P(m)$ from the standard Poisson distribution with the same mean photon number $m$, where $P(m)$ is the probability of finding $m$ photons in the CW mode. c, $g_{11}^{(2)}(0)$ versus $\beta/\pi$ and $\Delta_0/\gamma$. The results in a and b are obtained from a fully quantum simulation based on “hybrid” formalism Lindblad master equation. The plot in c represent the analytical solution based on quantum-trajectory method. The other parameters are the same as those in Fig. S3.
Substituting Eq. (S1.5) into above equations, we can obtain

\[
\begin{align*}
    u^\pm &= \xi^2 \left| C_{11}^{\pm\pm} (1, 1| \hat{a}_1^\dagger |0, 1) + C_{10}^{\pm\pm} (2, 0| \hat{a}_1^\dagger \hat{a}_1 |1, 0) \right|^2 = \xi^2 \left| C_{11}^{\pm\pm} + \sqrt{2} C_{10}^{\pm\pm} \right|^2, \\
    w^\pm &= \xi^2 \left| C_{11}^{\pm\pm} (1, 1| \hat{a}_1^\dagger |0, 1) + C_{10}^{\pm\pm} (2, 0| \hat{a}_1^\dagger \hat{a}_1 |1, 0) \right|^2 = \xi^2 \left| C_{11}^{\pm\pm} + \sqrt{2} C_{10}^{\pm\pm} \right|^2, \\
    w^0 &= \xi^2 \left| C_{10}^{\pm\pm} (2, 0| \hat{a}_1^\dagger \hat{a}_1 |1, 0) \right|^2 = \xi^2 \left| \sqrt{2} C_{10}^{\pm\pm} \right|^2, \\
    w^0 &= \xi^2 \left| C_{10}^{\pm\pm} (2, 0| \hat{a}_1^\dagger \hat{a}_1 |1, 0) \right|^2 = \xi^2 \left| \sqrt{2} C_{10}^{\pm\pm} \right|^2. 
\end{align*}
\]

(S3.12)

**B. Quantum correlations in CW and CCW modes**

Here, we study quantum correlations in CW mode and CCW mode, and cross-correlation between CCW and CW modes. Figures S6a1 and S6a2 show that, for the CW mode, PIT occurs with \( g_{11}^{(2)}(0) = 33.9 \) and \( g_{11}^{(3)}(0) = 199.8 \) when the system is far away from EPs. This purely quantum effect happens because of the suppression of single-photon probability (see Fig. S6b1) and relative enhancement of \( m \)-photon probabilities (\( m > 1 \), see Fig. 2d in main article) when the relative phase angle \( \beta \) is tuned to 0, \( \pi \), or 2\( \pi \). UPB emerges with \( g_{11}^{(2)}(0) = 0.0115 \) [or \( g_{11}^{(2)}(0) = 0.0125 \) and

![Figure S6. Quantum correlations with exceptional points in CW and CCW modes. a1, and a2, Correlation functions \( g_{11}^{(2)}(0) \) (red curves) and \( g_{11}^{(3)}(0) \) (blue curves) versus \( \beta/\pi \) in CW mode. a3, Self-correlation in CCW modes \( g_{22}^{(2)}(0) \) (red curves) and cross-correlation between CCW and CW modes \( g_{12}^{(2)}(0) \) (blue curves) and a4, Dependence of \( g_{22}^{(3)}(0) \) on \( \beta/\pi \). b, Photon probabilities versus \( \beta/\pi \) in b1, CW mode, and b2, CCW mode. For all plots, the solid curves represent the analytical solution based on quantum-trajectory method. The circles, squares, and stars correspond to results of a fully quantum simulation based on “hybrid” formalism Lindblad master equation. Here, \( \omega_L = \omega \), the other parameters are the same as those in Fig. S3.](image)
$g^{(3)}_{11}(0) = 0.000588$ [or $g^{(3)}_{11}(0) = 0.000586$] when the system is near EPs which can be understood from the destructive quantum interference induced $C_{20} \rightarrow 0$ when $\beta = 0.4\pi$ (or $\beta = 0.6\pi$) and $1.4\pi$ (or $1.6\pi$). More interesting, when the system at EPs, i.e., $\beta = 0.5\pi$ and $1.5\pi$, EPB with $g^{(2)}_{11}(0) = 0.014$ and $g^{(3)}_{11}(0) = 5.0 \times 10^{-5}$ is observed associated to the suppression of the state $|2, 0\rangle$ induced by EPs.

These effects in CCW modes are completely different from above effects in CW modes. No PIT can occur in CCW mode since $g^{(2)}_{11}(0)$ are always smaller than 1 by tuning $\beta$, as shown in Figs S6a3 and S6a4. Only when the system at EPs, PB in CCW mode with $g^{(2)}_{22}(0) = 0.004$ and $g^{(3)}_{22}(0) = 4.5 \times 10^{-7}$ can be found because both of single-photon and two-photon probabilities are suppressed due to the physics is dominated by the CW-traveling wave, and $P_{02}$ is suppressed more strongly than $P_{01}$, as shown in Fig S6b2. As to the cross-correlation between CCW and CW modes, Fig. S6a3 shows $g^{(2)}_{12}(0) > 1$ when the system is far away from EPs, while $g^{(2)}_{12}(0) < 1$ at EPs.

C. Quantum correlations in weak nonlinear regime

Now, we consider the situations with weak nonlinearity, $\chi/\zeta = 0.65$. As shown in Fig. S7, PIT can happen when the system is far away from EPs ($\beta = \pi$). Single-photon blockade at EPs ($\beta = \pi/2$) is weaker than that in strong nonlinear case. At EPs, CW mode dominates the system, and the only states which are effectively coupled by driving are $\psi_{1}$ and $\psi_{2}$, $+$; however, these states are near-resonance since weak nonlinearity induced frequency shift is much smaller, which leads to weaker suppression of two photons. UPB occurs at $\Delta_{0}/\gamma = -2.76$ and $\beta/\pi = 0.35, 0.65, 1.35$, or 1.65. This effect can be interpreted as destructive quantum interference between different paths from the one to the two photon state, which leads to $C_{20}$ (or $P_{20} \rightarrow 0$).

![Graph of Quantum correlations in weak nonlinear regime](image-url)

**Figure S7. Quantum correlations in weak nonlinear regime.** Second-order correlation functions $g^{(2)}_{11}(0)$ versus optical detuning $\Delta_{0}/\gamma$ for different $\beta$ by considering weak nonlinearity $n_{2} = 0.1 \times 10^{-14}$ m$^{2}$/W, i.e., $\chi/\gamma = 0.65$. UPB occurs around $\beta = 0.35\pi, 0.65\pi, 1.35\pi, 1.65\pi$, which can be see more clearly in the inset (i). This quantum effect steering by tuning $\beta$ can be understood from the two-photon probability distribution $P_{20}$, as shown in the inset (ii). For all plots, the solid curves represent the analytical solution based on quantum-trajectory method. The markers correspond to results of a fully quantum simulation based on “hybrid” formalism Lindblad master equation. The other parameters are the same as those in Fig. S3.
S4. THERMAL RESPONSE IN QUANTUM CORRELATIONS

Figure S8. Thermal response in quantum correlations The second-order correlation function of CW mode \( g^{(2)}(0) \) versus mean thermal photon number \( n_{th} \) for different relative angular position \( \beta = 0.5\pi, 0.6\pi, \pi \). The inset plot shows dependence of \( g^{(2)}(0) \) on the environment temperature \( T \). Here, we consider the resonance case, \( \omega_L = \omega \). The other parameters are the same as those in Fig. S1. All results are obtained by using fully quantum simulation (b) based on the “hybrid” formalism Lindblad master equation including thermal effect terms, Eq. (S4.1).

In order to study thermal response in this system, we numerically calculate the following master equation

\[
\dot{\rho} = -i[H', \rho] - i\{H', \dot{\rho}\} + \sum_j \hat{D}(\rho', \hat{a}_j) + \sum_j \hat{D}'(\rho', \hat{a}_j) + i2\text{tr}(\rho' \hat{H}' - )\rho',
\] (S4.1)

where

\[
\hat{D}(\rho', \hat{a}_j) = \frac{\gamma}{2}(n_{th} + 1)(2\hat{a}_j\rho'\hat{a}^\dagger_j - \hat{a}^\dagger_j\hat{a}_j\rho' - \rho'\hat{a}^\dagger_j\hat{a}_j),
\]

\[
\hat{D}'(\rho', \hat{a}_j) = \frac{\gamma}{2}n_{th}(2\hat{a}^\dagger_j\rho'\hat{a}_j - \hat{a}_j\hat{a}^\dagger_j\rho' - \rho'\hat{a}_j\hat{a}^\dagger_j),
\]

and \( n_{th} = [\exp(h\omega/k_BT)]^{-1} \) is the average thermal photon number with environment temperature \( T \) and the Boltzmann constant \( k_B \).

Figure S8 shows that the thermal photons greatly affect the correlation \( g^{(2)}_{11}(0) \) and tend to destroy the PB and PIT effects. The cavity field of CW mode tends to be thermal state with \( g^{(2)}_{11}(0) \rightarrow 2 \) for \( n_{th} > 0.3 \), i.e., \( T > 9000 \) K. We introduce the critical thermal photon number where PB and PIT disappear. When the system is at EP, \( \beta_{EP} = \pi/2 \), \( g^{(2)}_{11}(0) \) becomes unit for \( n_{th} \sim 0.11 \), this critical thermal photon number of CPB is larger than that of UPB and PIT which are \( n_{th} \sim 0.03 \) and \( n_{th} \sim 0.003 \), respectively; thus, CPB is more stable under higher environment temperature. However, due to the large frequency of the light, \( n_{th} \) is altered slightly when \( T < 300 \) K, which makes the PB and PIT effects steady.
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