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Magnus Expansion Approach to Parametric Oscillator Systems in a Thermal Bath

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Abstract: We develop a Magnus formalism for periodically driven systems which provides an expansion both in the driving term and in the inverse driving frequency, applicable to isolated and dissipative systems. We derive explicit formulas for a driving term with a cosine dependence on time, up to fourth order. We apply these to the steady state of a classical parametric oscillator coupled to a thermal bath, which we solve numerically for comparison. Beyond dynamical stabilisation at second order, we find that the higher orders further renormalise the oscillator frequency, and additionally create a weakly renormalised effective temperature. The renormalised oscillator frequency is quantitatively accurate almost up to the parametric instability, as we confirm numerically. Additionally, a cut-off dependent term is generated, which indicates the breakdown of the hierarchy of time scales of the system, as a precursor to the instability. Finally, we apply this formalism to a parametrically driven chain, as an example for the control of the dispersion of a many-body system.

Keywords: Dissipative; Magnus Expansion; Periodically Driven System.

1 Introduction

The study of periodically driven systems has experienced renewed interest in recent times. Both in solid state and ultra cold atom systems, strong periodic driving has been used to control nonequilibrium states. In ultra-cold atom systems, periodic lattice driving has been used to realise an effective, synthetic gauge field, see [1]. In solid-state systems, pump–probe experiments [2], on high-\(T_c\) superconductors and on graphene have been performed, see [3–5]. Theoretical studies on light-induced superconductivity were reported in [6–11].

Remarkably, in both cases, external high-frequency driving is used to control the low-frequency behaviour of each system. The quintessential example for this phenomenon is the Kapitza effect [12]. In the case of the effective synthetic field in an ultra-cold atom system, this process is explicitly described by an approximate, effective low-energy Hamiltonian, which, in contrast to the original, nondriven Hamiltonian, has a synthetic field. In the case of a driven high-\(T_c\) superconductor, the near-resonant driving of an optical phonon mode results in a modified response in the low-frequency optical conductivity. Both of these observations exemplify the development of a new field of emergence in driven many-body systems.

In this article, we give a systematic expansion of the emergent low-energy description of a driven system. This discussion applies and extends the Magnus formalism, as discussed in [13–16]. Our formalism provides a systematic expansion both in the driving amplitude and in the inverse driving frequency and is applicable to closed and open classical systems, to closed quantum systems. We derive explicit, general expressions for the leading terms beyond second order. As a key example, we apply this formalism to a parametrically driven oscillator, coupled to a thermal bath [17, 18], and determine the properties of its steady state. An insightful discussion of parametric oscillators was given in [19, 20], as well as in [21]. We then apply our results to a chain of parametrically driven oscillators. This provides insight in how the dispersion of a system can be controlled via parametric driving.

This article is organised as follows: In Section 2, we describe the dissipatively coupled, parametrically driven oscillator, and give a discussion of its properties using elementary ansatz functions. In Section 3, we develop the Magnus expansion in full generality first and then apply it to the parametric oscillator in Section 4. In Section 5, we discuss the control of the dispersion of a parametrically driven chain of oscillators, and in Section 6, we conclude.


2 Parametric Oscillator

As the key example to which we apply the Magnus expansion, we consider a parametrically driven oscillator, described by the Hamiltonian

\[ H = H_0 + H_{dr} \]

with

\[ H_0 = \frac{p^2}{2m} + \frac{m\omega_0^2}{2} x^2 \]

(2)

\[ H_{dr} = \frac{m\omega_0^2}{2} A \cos(\omega_m t) x^2. \]

(3)

\( p \) and \( x \) are the momentum and spatial coordinate of the oscillator, \( m \) the mass, and \( \omega_0 \) the bare oscillator frequency. \( A \) is the amplitude of the parametric driving term and \( \omega_m \) the driving frequency.

We assume that this oscillator is coupled to a thermal bath of temperature \( T \), via a dissipative term. The resulting equations of motion are of the Langevin form:

\[ \frac{dx}{dt} = \frac{-p}{m} \]

(4)

\[ \frac{dp}{dt} = -m\gamma x - \gamma p + \xi. \]

(5)

\( \gamma \) is the damping rate, and \( \xi \) describes white noise, with the correlation function \( \langle \xi(t_1)\xi(t_2) \rangle = 2\gamma/k_B T \delta(t_1-t_2) \), where \( k_B \) is the Boltzmann constant. In thermal equilibrium, in the absence of driving, the system is described by the canonical distribution \( \rho_0(x, p) = \exp(-\beta H_0(x, p))/Z \), with \( \beta = 1/(k_B T) \). \( Z \) is the partition function, which normalises this probability distribution. For this distribution, the variances of \( x \) and \( p \) are \( \langle x^2 \rangle = \gamma^2/\omega_0^2 \) and \( \langle p^2 \rangle = \gamma^2/\omega_0^2 \), with \( \gamma^2 = m k_B T \) and \( x = \sqrt{m k_B T} \). Furthermore, we have \( \langle \xi \rangle = \langle p \rangle = \langle xp \rangle = 0 \). We note that for a classical oscillator, \( x \) and \( p \), can be used to rescale \( x \) and \( p \). With this choice, the temperature does not appear in any of the remaining quantities and simply provides an energy scale for the system. For a quantum mechanical oscillator, this rescaling cannot be performed. Here, an additional regime appears in which quantum fluctuations dominate, for \( k_B T \ll \hbar \omega_0 \). A full discussion of the driven, dissipative quantum mechanical oscillator will be given elsewhere. The analysis presented here addresses isolated quantum systems, in addition to dissipative classical systems.

In Figure 1, we depict the time averaged magnitude of \( \langle x^2(t) \rangle / \langle x^2 \rangle \) in the steady state as a function of driving frequency and driving amplitude. Panels (A) and (B) depict the same data on different scales. The system displays a power broadened instability emerging from \( -2\omega_m \) and a dynamical stabilisation for large \( \omega_m \) and \( A \). In panel (A), we show the comparison to (8), in panel (B) to (7).

\[ \omega_\text{m}/\omega_0 \] Here, and in the examples throughout this article, we choose \( \gamma/\omega_0 = 0.1 \). The most striking feature of this plot is the parametric resonance that appears near \( \omega_m = 2\omega_\text{m} \) for small \( A \). This feature is then power broadened for increasing \( A \). In this regime, the magnitude of \( \langle x(t)^2 \rangle \) is increased by orders of magnitude, compared to the equilibrium value. In addition to this strong heating effect, there is a regime for large \( \omega_m/\omega_\text{m} \) and large amplitude, for which a reduction of \( \langle x(t)^2 \rangle \) is observed. Here, the parametric driving leads to a dynamic stabilisation of the fluctuations of \( x \). It is this counterintuitive and quintessential example of reducing fluctuations via high-frequency driving that we study systematically in this article.

In Figure 2, we show the same quantity in the steady state as a function of \( A \), for a fixed value of the driving frequency, \( \omega_m/\omega_\text{m} = 20 \), to give a clearer insight into the quantitative behaviour. The magnitude of these fluctuations is visibly reduced with increasing driving amplitude. However, eventually this trend of decreasing fluctuations is rapidly reverted, resulting in a steep increase of the
fluctuations. As visible from Figure 1, this steep increase is due to the power-broadened parametric instability. The onset of this instability determines the location of the minimal amount of fluctuations that can be achieved with this type of driving. It is therefore imperative to understand the origin of this steep increase of the fluctuations and provide a systematic approach to determine its behaviour.

In Figure 3, we show a histogram of the distribution $\rho_{\text{dr}}$ in the steady state, in comparison to the equilibrium distribution $\rho_0$; we show $\rho_{\text{dr}} - \rho_0$. The distribution $\rho_{\text{dr}}$ is generated from trajectories of the Langevin equation, which have been low-frequency filtered via

$$x(s) = \int ds G(s-t)x(t),$$

and similarly for $p(t)$, derived from $p(t)$. $G(s)$ is a normalised Gaussian, with a time scale $\sigma$, for which we choose $\sigma = 1/\omega_0$. In Figure 3, furthermore, we choose $A = 10$ and $\omega_m/\omega_0 = 20$. We observe that the width of the distribution along $x$-direction is reduced, due to the dynamical stabilisation that is described in the following sections. Along the $p$-direction, the distribution is only weakly affected. We emphasise that for a quantitative comparison of the driven state to the effective, low-frequency predictions, the exclusion of the high-frequency contributions in the numerics is essential. We elaborate on this point in Appendix A.

### 2.1 Elementary Approach

Before we develop the renormalisation of the oscillator due to the periodic driving systematically in the following section, we give estimates of its behaviour using various ansatz functions.

We start out by giving an estimate for the instability regime, and note that a more detailed discussion is given in Appendix B. We consider the equation of motion of the isolated system,

$$2\omega_0 (1 + \cos(\omega t)) x = 0.$$

We consider the ansatz

$$x(t) = a_0 \cos(\omega_{\text{eff}} t) + a_1 \cos((\omega_m - \omega_{\text{eff}}) t),$$

where $a_0$ and $a_1$ are constant coefficients. We solve for the effective frequency $\omega_{\text{eff}}$, which gives

$$\omega_{\text{eff}} = \frac{\omega_m - \sqrt{\omega_m^2 + 4\omega_0^2 - 2\omega_m \sqrt{A^2 \omega_0^2 + 4\omega_m^2}}}{2} \quad (6)$$

The instability regime is reached when the expression under the outer square root becomes negative. This occurs at

$$\frac{\omega_{\text{m,eff}}}{\omega_0} = \sqrt{2A + 4}, \quad (7)$$

which simplifies to

$$\frac{\omega_{\text{m,eff}}}{\omega_0} = \sqrt{2A}, \quad (8)$$

for large $A$. This provides an estimate for the instability regime for large driving amplitudes and frequencies, which we show in Figure 1, and which gives good agreement.

To give an estimate for the renormalisation of $\omega_{\text{eff}}$, we extend this ansatz to include not only the frequencies $\omega_{\text{eff}}$,
and $\omega_m - \omega_{\text{eff}}$, but also the next three contributing terms, corresponding to the frequencies $\omega_m + \omega_{\text{eff}}, 2\omega_m - \omega_{\text{eff}}$, and $2\omega_m + \omega_{\text{eff}}$. This ansatz is explicitly written in (B5). This ansatz results in (B6) for the effective frequency. We solve this equation iteratively in the amplitude $A$, which gives the expansion

$$\omega_{\text{eff}}^2 = \omega_0^2 + \frac{A^2 \omega_0^6}{2(\omega_m^2 - 4\omega_0^2)} + \frac{25A^4 \omega_0^8}{32\omega_m^6}$$  \hspace{1cm} (9)

At second order in $A$ and at second order in the inverse driving frequency, this is

$$\frac{\omega_{\text{eff}}^2}{\omega_0^2} = 1 + \frac{A^2 \omega_0^2}{2\omega_m^2}$$  \hspace{1cm} (10)

This approximation for the effective frequency is shown in Figure 2. We note that the fourth-order term in (9) is positive. This is indeed confirmed further down by the systematic Magnus expansion. However, the Magnus expansion determines the correct prefactor, which differs from the one found here.

3 Magnus Expansion

We now turn to the Magnus expansion of the system. This expansion provides a time-independent approximation of the low-frequency sector of the system, derived from the original, time-dependent Hamiltonian that describes all frequencies. After deriving general expressions for the Magnus terms beyond second order, we ask the question if and how the key features of the parametric oscillator, the dynamical stabilisation and the instability regime, can be captured within this approach. We note that these features, as they were described in the previous section, might suggest that such an approach might not be possible in a consistent fashion for the fourth-order correction. This is due to the following two observations. We observed, as shown in (9), that the fourth-order correction has a positive prefactor, which results in an addition stabilisation of the oscillator. This term would be derived from a term in an effective Hamiltonian that is of the form $-A^4/\omega_m^6$, with a positive prefactor. On the other hand, if the instability of (8) is derived from an effective Hamiltonian, it also needs to be derived from a term of the form $-A^4/\omega_m^6$, but now with a negative prefactor.

Interestingly, as we discuss below, the Magnus expansion provides two types of terms at fourth order. One of them is cut-off independent and features a positive prefactor. The resulting renormalisation due to this term is in agreement with the numerically obtained result. The other term is cut-off dependent. It indicates that the hierarchy of time scales that is required for the Magnus expansion breaks down. We interpret this as a precursor of the instability regime, and indeed find that the scaling for this regime, as shown in (8), is predicted correctly.

3.1 Kramers Equation

To apply the Magnus expansion, we formulate the time evolution of the system (4 and 5), as a time evolution of the phase space distribution $\rho(x, p, t)$. This is given by the Kramers equation

$$\partial_t \rho = L(t)\rho,$$  \hspace{1cm} (11)

Here, $L(t) = L_0 + L_{\omega}(t)$ with

$$L_\omega \rho = -v \partial_x \rho + \omega_m^2 x \partial_p \rho + \gamma \left( \partial_t (v \rho) + \frac{kT}{m} \partial_p \rho \right)$$  \hspace{1cm} (12)

and

$$L_{\omega} = L_{\omega,0} \cos(\omega_m t)$$  \hspace{1cm} (13)

with

$$L_{\omega,0} = A \omega_m^2 x \partial_x \omega_m.$$  \hspace{1cm} (14)

We refer to (11) as the Kramers equation to distinguish it from the Fokker–Planck equation, which we reserve for the over-damped limit, in accordance with the terminology of [22].

3.2 General Expansion

We now derive the expansion of the low-energy description in full generality. We consider a general, dynamical system that is described by the same equation of motion

$$\partial_t \rho = L(t) \rho,$$  \hspace{1cm} (15)

as before, without the assumption of the specific form of the equation of motion, as in the previous section. The parametrically driven oscillator will serve as the example to which we apply our results further down. The system under consideration can be either a closed or an open classical system or a closed quantum system. For a closed quantum system, we interpret the operator $L(t)$ as a Hamiltonian, divided by $\hbar$, i.e. $L(t) = H(t)/\hbar$. For an open
system, we also include dissipative terms, as in (11). We again assume that $L(t)$ has the form

$$L(t) = L_0 + L_{\text{dr}}(t),$$

where $L_0$ describes the time-independent part of the system and $L_{\text{dr}}(t)$ is the driving term, again of the form

$$L_{\text{dr}}(t) = L_{\text{dr},0} \cos(\omega_0 t).$$

We perform the Magnus expansion in the interaction picture. In this picture, the order of the Magnus expansion coincides with the order of the driving term. In the case of the parametric oscillator, this is the order of the driving amplitude $A$. For the interaction picture, we define

$$L_{\text{dr},i}(t, s) = \exp(-L_0 s) L_{\text{dr}}(t) \exp(L_0 s)$$

where the standard interaction picture term is $L_{\text{dr},i}(t) = L_{\text{dr},i}(t, t)$. Then the equation of motion is

$$\partial_t \rho_i = L_{\text{dr},i}(t) \rho_i,$$

Its solution is

$$\rho_i(t) = e^{-\int_{t_0}^t \rho_i(s) ds}$$

where $T_i$ is the time ordering operator and $\rho_i(t_0)$ the initial state at $t_0$. The Magnus expansion consists of re-expressing this solution in the form $\exp(\sum_i M_i)$, where $M_i$ is the Magnus term of $i$-th order, see [15].

We time average each of these terms over a time interval $[t_0, t]$. The time interval is long compared to the driving period but short compared to the dynamics that is created by $H_0$. For the parametric oscillator, we demand $1/\omega_0 \gg t - t_0 \gg 1/\omega_0$. The time interval $\Delta t = t - t_0$ is also the inverse of a frequency cut-off $\omega = 2\pi/\Delta t$, for which we equivalently demand $\omega \ll \omega_c \ll \omega_0$. For a general system, the frequency $\omega_0$ has to be replaced by a typical frequency that is characteristic for the dynamics of $H_0$.

The second-order Magnus term in the interaction picture is given by

$$M_{2,i} = -\frac{1}{2} \int_{t_0}^t \int_{t_0}^t \partial_t [L_{\text{dr},i} (t_1), L_{\text{dr},i} (t_2)] ds$$

We transfer this expression back to the Schrödinger picture and project this term on the frequency range below $\omega_c$. The resulting effective $L^{(2)}_{\text{eff}}$ is time independent because all the oscillatory contributions oscillate with a frequency above the cut-off frequency. The time evolution that results from this term is of the form $\exp(L_{\text{eff}}^{(2)} \Delta t)$. Therefore, we can simplify (21) by taking the time derivative with respect to $t$, which reduces the number of integrations. The resulting second-order term is therefore

$$L^{(2)}_{\text{eff}} = \left[-\frac{1}{2} \int_{t_0}^t dt_i [L_{\text{dr},i} (t_i), L_{\text{dr},i} (t)] \right]_{t \ll \omega_c}$$

with $\tilde{t}_i = t_i - t$. We expand the expression in (18) to first order:

$$L_{\text{dr},i}(t, s) = L_{\text{dr},i}(t) - s [L_{\text{dr},0}, L_{\text{dr},i}(t)]$$

We use this first-order expansion, with $s \to \tilde{t}_i$ and $t \to t_i$, and the time dependence of the driving term (17),

$$L^{(2)}_{\text{eff}} = \frac{1}{2} [L_{\text{dr},0}, L_{\text{dr},i}], L_{\text{dr},i}$$

The low-frequency part of the time integral, which refers to frequencies below $\omega_c$, is

$$\left[ \int_{t_0}^t dt_i \tilde{t}_i \cos(\omega_0 t) \cos(\omega_0 t_i) \right]_{t \ll \omega_c} = \frac{1}{2\omega_c^2}$$

Therefore, we obtain

$$L^{(2,2)}_{\text{eff}} = \frac{1}{4\omega_c^2} [L_{\text{dr},0}, L_{\text{dr},0}]$$

Here, and throughout the article, we use the notation $L^{(n,m)}_{\text{eff}}$ refer to the $n$-th order of the Magnus expansion and to the $m$-th order in the inverse driving frequency.

### 3.3 Fourth Order in $\omega_m^{-1}$

We now derive the next order term in the inverse frequency. We consider the expansion in (18) to third order

$$L_{\text{dr},i}(t, s) = L_{\text{dr},i}(t) - s [L_{\text{dr},0}, L_{\text{dr},i}(t)] + \frac{s^2}{2} \text{ad}^2_{\omega_i} L_{\text{dr},i}(t) - \frac{s^3}{3!} \text{ad}^3_{\omega_i} L_{\text{dr},i}(t)$$

where we introduced the notation of the adjoint derivative $\text{ad}^n_{\omega_i} L_{\text{dr},i}(t)$. It is defined via $\text{ad}^n_{\omega_i} L_{\text{dr},i}(t) = [L_{\text{dr},0}, \text{ad}^{n-1}_{\omega_i} L_{\text{dr},i}(t)]$, and $\text{ad}^0_{\omega_i} L_{\text{dr},i}(t) = L_{\text{dr},i}(t)$. The term that is quadratic in $s$ gives no low-frequency contribution, therefore $L^{(2,3)}_{\text{eff}} = 0$. The fourth-order term is

$$L^{(2,4)}_{\text{eff}} = \left[ \frac{1}{2} \text{ad}^1_{\omega_i} L_{\text{dr},i}, L_{\text{dr},0} \right]$$

$$\times \left[ \int_{t_0}^t dt_i \tilde{t}_i \cos(\omega_0 t) \cos(\omega_0 t_i) \right]_{t \ll \omega_c}$$
We use the integral property
\[
\int_{t_0}^{t_i} dt_i \int_{t_i}^{t_j} dt_i \int_{t_i}^{t_j} dt_i \int_{t_i}^{t_j} dt_i \cos(\omega_m t) \cos(\omega_m t_i) = -\frac{3}{\omega_m^4}
\]
which results in
\[
I^{(2,0)}_{\text{eff}} = -\frac{1}{4\omega_m^2} \left[ \text{ad}^3 \left( L_{t_i}, L_{t_i}, L_{t_i}, L_{t_i} \right) \right]_{t_i < t_0}
\]
(29)

Higher order terms of the form \( I^{(2,n)}_{\text{eff}} \) can be derived in a similar manner.

### 3.4 Fourth-Order Magnus Expansion

For the fourth-order term in the driving term, we proceed along the same lines as for the quadratic term in the previous sections. The fourth-order term in the interaction picture has the form
\[
M_{k_i} = -\frac{1}{12} \int_{t_0}^{t_i} dt_i \int_{t_i}^{t_j} dt_i \int_{t_i}^{t_j} dt_i \int_{t_i}^{t_j} dt_i \cos(\omega_m t) \cos(\omega_m t_i) \cos(\omega_m t_j) \cos(\omega_m t_k)
\]
\[
\left[ \left[ L_{t_j}(t), \left[ L_{t_j}(t), L_{t_j}(t), L_{t_j}(t) \right], L_{t_j}(t) \right] \right] + \left[ \left[ L_{t_j}(t), L_{t_j}(t), L_{t_j}(t) \right], L_{t_j}(t) \right]
\]
\[
+ \left[ \left[ L_{t_j}(t), L_{t_j}(t), L_{t_j}(t) \right], L_{t_j}(t) \right] + \left[ \left[ L_{t_j}(t), L_{t_j}(t), L_{t_j}(t) \right], L_{t_j}(t) \right] \]
\]
\[
+ \left[ \left[ L_{t_j}(t), L_{t_j}(t), L_{t_j}(t) \right], L_{t_j}(t) \right] \right]_{t_i < t_0}
\]
(30)

Again, we transform this expression to the Schrödinger picture. We project this term on the low-frequency regime. Interestingly, we find two contributions, as we show below. The first is proportional to \( \Delta t \). Therefore, it lends itself to an interpretation as an effective low-energy description. The second term is cubic in \( \Delta t \), which means that we can write
\[
[M_{k_i}]_{t_i < t_0} = I^{(4,0)}_{\text{eff}} \Delta t_i + I^{(4,1)}_{\text{eff}} \Delta t_i^3
\]
(31)

and we also introduce the definition \( I^{(4,0)}_{\text{eff}} = L^{(4,0)}_{\text{eff}} \Delta t_i \). We again obtain the operators \( I^{(4,0)}_{\text{eff}} \) and \( I^{(4,1)}_{\text{eff}} \) by considering the low-frequency sector of the time derivative of \( M_{k_i} \), i.e.
\[
L^{(4,0)}_{\text{eff}} + 3L^{(4,1)}_{\text{eff}} = \left[ -\frac{1}{12} \int_{t_0}^{t_i} dt_i \int_{t_i}^{t_j} dt_i \int_{t_i}^{t_j} dt_i \int_{t_i}^{t_j} dt_i \left[ \left[ L_{t_j}(t), \left[ L_{t_j}(t), L_{t_j}(t), L_{t_j}(t) \right], L_{t_j}(t) \right] \right] \right]_{t_i < t_0}
\]
\[
+ \left[ \left[ L_{t_j}(t), L_{t_j}(t), L_{t_j}(t) \right], L_{t_j}(t) \right] + \left[ \left[ L_{t_j}(t), L_{t_j}(t), L_{t_j}(t) \right], L_{t_j}(t) \right] + \left[ \left[ L_{t_j}(t), L_{t_j}(t), L_{t_j}(t) \right], L_{t_j}(t) \right] + \left[ \left[ L_{t_j}(t), L_{t_j}(t), L_{t_j}(t) \right], L_{t_j}(t) \right] \right]_{t_i < t_0}
\]
\[
+ \left[ \left[ L_{t_j}(t), L_{t_j}(t), L_{t_j}(t) \right], L_{t_j}(t) \right] \right]_{t_i < t_0}
\]
(32)

with \( \tilde{t}_i = t_i - t \). The factor of 3 in front of \( I^{(4,1)}_{\text{eff}} \) is due to the derivative of (31). We use the expansion of \( L_{t_i} \) given in (27). We order the resulting terms according to the combined order of the times \( \tilde{t} \), i.e. \( \tilde{t}^3 \tilde{t}^3 \tilde{t}^3 \), and \( k = k_1 + k_2 + k_3 \). The first- and second-order terms with \( k = 1 \) and \( k = 2 \) give no contribution. For the \( k = 3 \) term, a number of contributions are generated in this expansion. These contain time integrals of the form
\[
c_{k_1,k_2,k_3} = \int_{t_0}^{t_i} dt_i \int_{t_i}^{t_j} dt_i \int_{t_i}^{t_j} dt_i \int_{t_i}^{t_j} dt_i \cos(\omega_m t_i) \cos(\omega_m t_j)
\]
(33)

The integrals that are necessary to derive \( I^{(4,0)}_{\text{eff}} \) are given in Table 1. All the terms that scale as \( 1/\omega_m^k \) contribute to \( I^{(4,0)}_{\text{eff}} \). These are written out and simplified in Appendix C. We obtain \( I^{(4,0)}_{\text{eff}} \) to be
\[
I^{(4,0)}_{\text{eff}} = \frac{1}{12\omega_m} \left( \frac{39}{64} \left[ L_{t_i}, L_{t_i}, \left[ L_{t_i}, L_{t_i}, L_{t_i}, L_{t_i} \right], L_{t_i} \right] \right)
\]
\[
+ \frac{61}{64} \left[ L_{t_i}, \left[ L_{t_i}, L_{t_i}, L_{t_i}, L_{t_i} \right], L_{t_i} \right]
\]
\[
+ \frac{87}{32} \left[ \left[ L_{t_i}, L_{t_i}, L_{t_i}, L_{t_i} \right], \left[ L_{t_i}, L_{t_i}, L_{t_i}, L_{t_i} \right] \right]
\]
\[
- \frac{3}{32} \left[ \left[ L_{t_i}, L_{t_i}, L_{t_i}, L_{t_i} \right], \left[ L_{t_i}, L_{t_i}, L_{t_i}, L_{t_i} \right] \right] \right) \right)
\]
(34)

The term that scales as \( \Delta t^2 / \omega_m^3 \), which is due to the \( c_{2,3,1} \) integral, gives \( 3L^{(4,1)}_{\text{eff}} \). Therefore, the cut-off-dependent contribution is
\[
I^{(4,1)}_{\text{eff}} = \frac{\Delta t^2}{144\omega_m^3} \left[ L_{t_i}, \left[ L_{t_i}, L_{t_i}, L_{t_i}, L_{t_i} \right] \right]
\]
(35)

| \( c_{k_1,k_2,k_3} \) | \( c_{2,3,1} \) | \( c_{3,2,1} \) | \( c_{2,1,1} \) | \( c_{2,3,3} \) | \( c_{1,3,2} \) | \( c_{1,2,0} \) | \( c_{2,2,1} \) | \( c_{2,1,2} \) |
|---|---|---|---|---|---|---|---|---|
| \( -\frac{9}{8\omega_m^3} \) | \( \frac{7}{16\omega_m^3} \) | \( \frac{1}{32\omega_m^3} \) | \( \frac{\Delta t^2}{4\omega_m^3} \) | \( \frac{45}{64\omega_m^6} \) | \( -\frac{15}{64\omega_m^6} \) | \( \frac{27}{64\omega_m^6} \) | \( -\frac{33}{32\omega_m^6} \) | \( \frac{21}{64\omega_m^6} \) |
We emphasise again that for any system that can be written in the form of (15–17), the results given in (26, 29, 34, and 35) apply. They constitute the main conceptual result of this paper.

4 Magnus Expansion of the Parametric Oscillator

We now apply our results to the case of the parametric oscillator, introduced above. For the \( L_{2,2}^{(2,2)} \) correction, we use (26) and find

\[
L_{2,2}^{(2,2)} = \frac{A^2 \omega_0^5}{2 \omega_m^2} x \partial_y
\]  

(36)

This implies a renormalisation of the oscillator frequency of the form

\[
\frac{\omega_{2,2}^2}{\omega_0^2} = 1 + \frac{A^2 \omega_0^2}{2 \omega_m^2}
\]  

(37)

This coincides with the second-order term that was obtained in (10). At the fourth in the inverse driving frequency, we have

\[
L_{4,4}^{(2,2)} = \frac{A^2 \omega_0^5}{4 \omega_m^2} (2(4 \omega_0^2 - \gamma^2)x \partial_y + 8\gamma(T/m) \partial_y)
\]  

(38)

where we applied (29). Interestingly, in addition to a further renormalisation of the oscillator frequency, a renormalisation of the temperature is created:

\[
\frac{\omega_{4,4}^2}{\omega_0^2} = 1 + \frac{A^2 \omega_0^2}{2 \omega_m^2} + \frac{A^2 \omega_0^2}{2 \omega_m^2} (4 \omega_0^2 - \gamma^2)
\]  

\[
\frac{T_{4,4}}{T_0} = 1 + \frac{2A^2 \omega_0^6}{\omega_m^6}
\]  

(39)

(40)

It is generated because the white-noise dissipative term contains fluctuations at all frequencies, in particular at the driving frequency \( \omega_0 \). This results in an additional renormalisation of the low-frequency regime, via time averaging, of the system at this higher order. For the example presented here, the magnitude of the renormalisation is small. However, nonlinear systems will in general create nonlinear effective dissipative terms at this order. Finally, we determine the two terms at order \( A^4 \). The cut-off independent term is

\[
L_{4,4}^{(4,4)} = \frac{107 A^4 \omega_0^8}{96 \omega_m^9} x \partial_y
\]  

(41)

This term generates an additional renormalisation of the oscillator frequency, resulting in

\[
\frac{\omega_{4,4}^2}{\omega_0^2} = 1 + \frac{A^2 \omega_0^2}{2 \omega_m^2} + \frac{A^2 \omega_0^2}{2 \omega_m^2} (4 \omega_0^2 - \gamma^2)
\]  

+ \frac{107 A^4 \omega_0^8}{96 \omega_m^9}
\]  

(42)

We note that this renormalisation at fourth order in \( A \) has a positive prefactor, as in the estimate in (9). However, the systematic Magnus expansion gives the correct magnitude of the prefactor.

In Figure 4, we depict the power spectrum \( S_p(\omega) \) as a function of the driving amplitude \( A \), depicted on a logarithmic scale. For the driving frequency, we use \( \omega_m/\omega_0 = 20 \). We show the second-order estimate of the effective frequency \( \omega_{2,2}^{(eff)} \), which refers to (37). In addition, we show the fourth-order estimate \( \omega_{4,4}^{(eff)} \) based on (42).
driving amplitude $A$, and for the fixed driving frequency $\omega_d / \omega_0 = 20$. The power spectrum is defined via
\begin{equation}
S_\omega(\omega) = \langle \rho(-\omega) \rho(\omega) \rangle
\end{equation}
with $\rho(\omega) = (1/\sqrt{T_s}) \int dt' \exp(-i\omega t') \rho(t')$, where $T_s$ is the sampling interval. At $A = 0$, the power spectrum reduces to that of a harmonic oscillator, with a single peak at $\omega_c$. As the driving is turned on, additional peaks appear at $n\omega_m \pm \omega_c$, where $n$ is an integer describing the Floquet band. We note that these frequencies are approximately the ones that were used in the ansatz functions in Section 2.1 and Appendix B. With increasing driving amplitude, the effective oscillator frequency increases. We compare this increase to the second-order prediction, given in (37), and the fourth-order prediction (42). The fourth-order estimate describes the oscillator frequency well almost up to the instability, which is reached around $A = 180$, in this example. We emphasise that the orange bar at $A = 180$ is numerical data. Here, the magnitude of power spectrum increases rapidly by many orders of magnitude.

The cut-off dependent term is
\begin{equation}
\mathcal{L}_{\text{eff},c}^{(a,b)} = -\frac{A^2 \omega_m^2 \Delta t^2}{18 \omega_m^2} x^2 \partial_x^2
\end{equation}

This term competes with the previously discussed terms that stabilise the oscillator. For simplicity, we only consider the dominant term of the effective frequency (36). We relate the time scale $\Delta t$ to a frequency cut-off via $\Delta t = 2\pi / \omega_c$. We assume to be in the strongly renormalised regime, $\omega_d / \omega_m = A^2 \omega_m / 2$ $\omega_m$. Therefore, $\mathcal{L}_{\text{eff},c}^{(a,b)}$ competes with this renormalisation if
\begin{equation}
\frac{A^2 \omega_m^2}{2 \omega_m^2} = \frac{2 \pi^2 A^2 \omega_m^2}{9 \omega_m^2 \omega_c^2}
\end{equation}

This results in the criterium
\begin{equation}
\sqrt{\frac{\omega_c \omega_m}{\omega_0}} = \sqrt{A}
\end{equation}

If we consider a cut-off frequency chosen as fraction of the driving frequency, and therefore $\omega_c - \omega_m$, we recover
\begin{equation}
\frac{\omega_m}{\omega_0} = \sqrt{A}
\end{equation}

which displays the same scaling as in (8). The scaling displayed in (46) can also be motivated by comparing the cut-off frequency $\omega_c$ to $\omega_m / \omega_0 \sim A \omega_d / \omega_c$. Again, this condition indicates that the originally assumed hierarchy of energy scales is no longer valid. This property of the system derives from the cut-off dependent term $\mathcal{L}_{\text{eff},c}^{(a,b)}$. While this term in itself cannot be interpreted as a contribution to the effective equation of motion, it can give an insight into the breakdown of the necessary hierarchy of time scales of the system.

5 Parametrically Driven Chain

We apply this formalism to the stabilisation of a chain of oscillators via parametric driving. This, and related mechanisms have been considered in the context of light enhanced superconductivity, with the following motivation. If we imagine a complex order parameter field describing fluctuating superconducting order, a key feature of this system is its phase stiffness. In equilibrium, it controls the superconducting stability and the critical current. The phase stiffness in turn is related to how steeply the dispersion of the system increases with increasing momentum. Therefore, one possible explanation of light enhanced superconductivity might entail stabilising and steepening the dispersion of the system.

We here give the simplest, yet generic, case of a one-dimensional chain of oscillators. The system is described by $H = H_0 + H_d(t)$, with
\begin{equation}
H_0 = \sum_i \left( \frac{p_i^2}{2m} + \omega_0^2 (x_i - x_{i+1})^2 \right)
\end{equation}

with $i = 1, \ldots, N$. The driving term is
\begin{equation}
H_d = \sum_i m \omega_k \cos(\omega_d t) (x_i - x_{i+1})^2
\end{equation}

We therefore have a parametrically driven lattice of oscillators. We Fourier transform the system via $x_i = \frac{1}{\sqrt{N}} \sum_k \exp(ikr) x_k$ and $p_i = \frac{1}{\sqrt{N}} \sum_k \exp(ikr) p_k$, and note that $[x_k, p_k] = -i \hbar \delta_{k,-k}$. The Langevin equations for the system are
\begin{equation}
\frac{dx_k}{dt} = \frac{p_k}{m}
\end{equation}

\begin{equation}
\frac{d\xi_k}{dt} = -m \omega_k^2 (1 + A \cos(\omega_d t)) \xi_k - \gamma \xi_k + \xi_k
\end{equation}

with $\langle \xi_k(t) \xi_k(t) \rangle = 2 \gamma \hbar \delta(t - t')$. In real space, this corresponds to $\langle \xi(t) \xi(t) \rangle = 2 \gamma \hbar \delta(t - t')$. The dispersion $\omega_{k,0}$ is
We observe that (50 and 51) are equivalent to (4 and 5), with the replacement $\omega_0 \rightarrow \omega_{k,0}$. We can therefore apply the results for the single oscillator (42), to each momentum mode and obtain the effective dispersion

$$\omega_{k,0} = \omega_0 \sqrt{2 - 2 \cos k} = 2\omega_0 |\sin(k/2)|$$

(52)

The second-order correction, derived from $\omega_{k,0}^{(2)}$, contains contributions of the form $\sim \cos(2k)$. This can be seen by substituting $\omega_{k,0} = 2\omega_0 |\sin(k/2)|$, see (52). This describes coupling to the next-nearest neighbor, induced by the periodic driving, because a next-nearest coupling term of the form $\sum k_{\pm 1,\pm 1}$ gives rise to $\cos(2k)$ terms in momentum space when Fourier transformed. When substituting (52), in the term that is quadratic in $\omega$, we obtain terms up to $-\cos(3k)$, which corresponds to coupling to the third neighbor. Finally, the term quartic in $\omega$ contains coupling to the fourth nearest neighbor.

In Figure 5, we show the two point correlation $G(k, \omega)$ in momentum and coordinate space in the steady state for a one-dimensional chain of parametrically driven oscillators. The two point correlation function is defined by

$$G(k, \omega) = \langle X(-k, -\omega) X(k, \omega) \rangle$$

(54)

with

$$X(k, \omega) = \frac{1}{\sqrt{K T_s}} \int dt' \int dr' x(r', t') \exp(-ikr') \exp(-i\omega t')$$

where $T_s$ and $K$ are sampling time interval and space interval, respectively. Compared with the non-driven situation, the driven dispersion line has a steeper slope, which means that the driving term stiffens the system significantly. We compare the numerics with effective dispersion $\omega_{k,\text{eff}}$ (53). It is clearly seen that at higher k modes, the second-order correction deviates from the numerics while the fourth-order correction describes the numerics precisely.

We also observe that the strongest renormalisation of the dispersion occurs at its upper edge. This includes the onset of the parametric instability. We now have the condition

$$\frac{\omega_m}{\omega_{k,0}} = \sqrt{A}$$

(55)

which is first reached for the maximum of the band. This sets the upper limit for the driving amplitude $A$ that can be used to stabilise the dispersion. However, we note that, depending on the physical system, the range of $A$ might be much more limited. For example, the value of the spring constant between neighboring oscillators might not allow for negative values, meaning that $A < 1$. With this constraint, the magnitude of the renormalisation is small, of the order of $\omega_{k,0}^2 / \omega_m^2$.

### 6 Conclusions

We have developed a systematic Magnus expansion in the driving term and the inverse driving frequency. In this formalism, we have derived explicit expressions for a system with a driving term with cosine time dependence. This system can be either a quantum mechanical system or a classical system including dissipative terms. The main, conceptual formulas are given in (29, 34, and 35), which are the terms beyond the widely discussed lowest order term in (26). At fourth order in the driving term, we find two contributions, one cut-off independent and one cut-off dependent. The cut-off independent term contributes to the effective Kramers or Hamilton operator, whereas the increasing magnitude of the cut-off dependent term indicates the breakdown of the hierarchy of time scales that was originally assumed. We apply this formalism to a parametrically driven oscillator, coupled to a thermal bath, and to a parametric oscillator chain. We obtain the magnitude of stabilisation that can be achieved for these systems and the onset of the instability. We
emphasise that our formalism can be applied to a wide range of driven systems, including nonlinear systems and many-body systems. It will be of particular interest to the emerging field of controlling many-body systems via external driving.

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**Appendix A: Frequency Cut-Off**

In this section, we discuss the comparison of the predictions of the effective description to the observables extracted from the full system. Because the effective description is a low-frequency description, it is, in general, imperative to apply a frequency cut-off on the observables, for a quantitative comparison. While for some observables depend only weakly on the introduction of this cut-off, in general, the low-pass-filtered observable will differ from the observable that includes all frequencies.

As discussed in Section 2, we have depicted the phase space distribution that is derived from the low-frequency-filtered trajectories \((x(t), p(t))\) in Figure 3. For comparison, we depict the phase space distribution that is derived from original trajectories \((x(t), p(t))\) that include all frequencies, in Figure 6. As is clearly visible, for this distribution a broadening of the distribution in the \(p\)-direction occurs, in contrast to Figure 3.

To elaborate on this further, we depict the time average of \(\langle p^2(t) \rangle\) and \(\langle p^4(t) \rangle\) in the steady state, as a function of \(A\), in Figure 7. \(\langle p^4(t) \rangle\) has a strong dependence on \(A\), which is approximately quadratic. \(\langle p^2(t) \rangle\), however, has only a very weak \(A\) dependence, only given the weak temperature renormalisation that was given in (40).

**Appendix B: Elementary Ansatz**

We consider the equation of motion for the isolated system

\[
\ddot{x} + \omega_0^2(1 + A \cos(\omega_m t)) x = 0. \tag{B1}
\]

To estimate the regime in which the instability of the system occurs, we consider the ansatz

\[
x(t) = a_0 \cos(\omega_{st} t) + a_1 \cos((\omega_m - \omega_{st}) t)
\]

where \(a_0\) and \(a_1\) are constant coefficients. \(\omega_{st}\) is the effective oscillation frequency, which we solve for. Substituting this ansatz in the equation of motion, and ignoring further frequencies, this results in the equations \((\omega_{st}^2 - \omega_0^2)a_0 = Aa_0^2a_1/2\) and \(((\omega_m - \omega_{st})^2 - \omega_0^2)a_1 = Aa_0^2a_1/2\). We eliminate \(a_0\) and \(a_1\) and obtain the equation

\[\omega_{cut} = 10 \quad \text{all freq.}\]
\[ (\omega_m^2 - \omega_0^2)((\omega_m - \omega_0)^2 - \omega_0^2) = \frac{A^2 \omega_0^4}{4} \]  

(B3)

The resulting \( \omega_{eff} \) is

\[ \omega_{eff} = \omega_m - \sqrt{\omega_m^2 + 4\omega_0^2 - 2A\sqrt{\omega_0^2 + 4\omega_0^2}} \frac{\omega_m - \omega_0}{2} \]  

(B4)

The parametric resonance is reached when the expression under the square root becomes negative. We note that the effective frequency \( \omega_{eff} \) increases monotonously, with increasing \( A \). The instability occurs when the two frequencies \( \omega_{eff} \) and \( \omega_m - \omega_{eff} \) equal each other. We confirm this behaviour by calculating the power spectrum of the driven state, which is shown in Figure 4.

To give a more accurate estimate of the renormalisation of \( \omega_{eff} \), we consider the following ansatz

\[ x(t) = a_0 \cos(\omega_0 t) + a_1 \cos((\omega_m - \omega_0) t) + a_2 \cos((\omega_m + \omega_{eff}) t) + a_3 \cos((2\omega_m - \omega_{eff}) t) \]  

(B5)

When we substitute this in the equation of motion, we obtain the following equation for \( \omega_{eff}^2 \):

\[ \omega_{eff}^2 = \frac{A^2 \omega_0^4}{4} \left( \frac{1}{\omega_0^2 - (\omega_m - \omega_{eff})^2} + \frac{A^2 \omega_0^4}{4(2\omega_m - \omega_{eff})^2 - \omega_0^2} \right) \]  

(B6)

We solve this equation iteratively in the driving amplitude \( A \), which gives

\[ \omega_{eff}^2 = \omega_m^2 + \frac{A^2 \omega_0^4}{2(\omega_m^2 - 4\omega_0^2)} + \frac{25A^2 \omega_0^8}{32 \omega_m^8} \]  

(B7)

Here, we kept the leading order in the inverse frequency \( 1/\omega_m \) for the fourth-order term, which scales as \( 1/\omega_m^4 \). We kept all orders in \( 1/\omega_m \) for the term that is second ordering \( A \).

Appendix C: Fourth-Order Term of the Magnus Expansion

After expanding (32) to the order \( k = 3 \), evaluating the integrals of the form of (33), and collecting the terms that scale as \( 1/\omega_m^4 \), we obtain for \( L^{(4)}_{\text{MAG}} \):

\[ J^{(4)}_{\text{MAG}} = \frac{1}{12\omega_0^4} \left\{ \frac{1}{64} \left[ L_{\text{MAG}}^{(0)}, \left[ \left[ \text{ad}_{\omega} L_{\text{MAG}}^{(0)}, \left[ L_{\text{MAG}}^{(0)}, \left[ L_{\text{MAG}}^{(0)}, L_{\text{MAG}}^{(0)} \right] \right] \right] \right] \right] + \frac{1}{16} \left[ L_{\text{MAG}}^{(0)}, \left[ \left[ \text{ad}_{\omega} L_{\text{MAG}}^{(0)}, \left[ L_{\text{MAG}}^{(0)}, \left[ L_{\text{MAG}}^{(0)}, L_{\text{MAG}}^{(0)} \right] \right] \right] \right] \right\} + \frac{1}{16} \left[ L_{\text{MAG}}^{(0)}, \left[ \left[ \text{ad}_{\omega} L_{\text{MAG}}^{(0)}, \left[ L_{\text{MAG}}^{(0)}, \left[ L_{\text{MAG}}^{(0)}, L_{\text{MAG}}^{(0)} \right] \right] \right] \right] \right\} \]  

(C1)
To simplify this expression, we first combine the terms that are related by commutation. In addition, we use that
\[
\begin{align*}
\text{(C2)} & \quad [\text{ad}_{t_a}, [[\text{ad}_{t_a}, L_{t_a,0}, L_{t_a,0}]], L_{t_a,0}] \\
\text{(C3)} & \quad +[\text{ad}_{t_a}^2 L_{t_a,0}, [[\text{ad}_{t_a}, L_{t_a,0}, L_{t_a,0}]], L_{t_a,0}] \\
\text{(C4)} & \quad = [[\text{ad}_{t_a}, L_{t_a,0}, [\text{ad}_{t_a}^2 L_{t_a,0}, L_{t_a,0}]], L_{t_a,0}] \\
\text{(C5)} & \quad + [[\text{ad}_{t_a}^2 L_{t_a,0}, [\text{ad}_{t_a}, L_{t_a,0}, L_{t_a,0}]], L_{t_a,0}] \\
\text{(C6)} & \quad \text{and} \\
\text{(C7)} & \quad \text{(C8)} \\
\end{align*}
\]

With these identities, we simplify the expression to the form given in (34).

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