MULTIFRACTAL TUBES

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Abstract. Tube formulas refer to the study of volumes of r neighbourhoods of sets. For sets satisfying some (possible very weak) convexity conditions, this has a long history. However, within the past 20 years Lapidus has initiated and pioneered a systematic study of tube formulas for fractal sets. Following this, it is natural to ask to what extent it is possible to develop a theory of multifractal tube formulas for multifractal measures. In this paper we propose and develop a framework for such a theory. Firstly, we define multifractal tube formulas and, more generally, multifractal tube measures for general multifractal measures. Secondly, we introduce and develop two approaches for analysing these concepts for self-similar multifractal measures, namely:

(1) Multifractal tubes of self-similar measures and renewal theory. Using techniques from renewal theory we give a complete description of the asymptotic behaviour of the multifractal tube formulas and tube measures of self-similar measures satisfying the Open Set Condition.

(2) Multifractal tubes of self-similar measures and zeta-functions. Unfortunately, renewal theory techniques do not yield “explicit” expressions for the functions describing the asymptotic behaviour of the multifractal tube formulas and tube measures of self-similar measures. This is clearly undesirable. For this reason, we introduce and develop a second framework for studying multifractal tube formulas of self-similar measures. This approach is based on multifractal zeta-functions and allow us obtain “explicit” expressions for the multifractal tube formulas of self-similar measures, namely, using the Mellin transform and the residue theorem, we are able to express the multifractal tube formulas as sums involving the residues of the zeta-function.

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Part 1:
Statements of Results

Tube formulas refer to the study of volumes of \( r \) neighbourhoods of sets. For sets satisfying some (possible very weak) convexity conditions, this has a long history going back to Steiner in the early 19th century. However, within the past 20 years Lapidus has initiated and pioneered a systematic study of tube formulas for fractal sets. Following this line of investigation, it is natural to ask to what extent it is possible to develop a theory of multifractal tube formulas for multifractal measures. The purpose of this paper is to propose a framework for developing such a theory. Firstly, we define multifractal tube formulas and, more generally, multifractal tube measures for general multifractal measures. Secondly, we introduce and develop two approaches for analysing these concepts for self-similar multifractal measures, namely:

1. Multifractal tubes of self-similar measures and renewal theory. Using techniques from renewal theory we give a complete description of the asymptotic behaviour of the multifractal tube formulas and tube measures of self-similar measures satisfying the Open Set Condition. This is presented in Section 3 (for tube formulas) and Section 4 (for tube measures).

2. Multifractal tubes of self-similar measures and zeta-functions. While renewal theory techniques are powerful tools, they do not yield “explicit” expressions for the functions describing the asymptotic behaviour of the multifractal tube formulas and tube measures of self-similar measures. This is clearly undesirable. For this reason, we introduce and develop a second framework for studying multifractal tube formulas of self-similar measures. This approach is based on multifractal zeta-functions and allow us obtain “explicit” expressions for the multifractal tube formulas of self-similar measures, namely, using the Mellin transform and the residue theorem, we are able to express the multifractal tube formulas as sums involving the residues of the zeta-function. This is done in Section 5.

1. Fractal tubes

Let \( E \) be a subset of \( \mathbb{R}^d \) and \( r > 0 \). We will write \( B(E, r) \) for the open \( r \) neighbourhood of \( E \), i.e.

\[
B(E, r) = \left\{ x \in \mathbb{R}^d \mid \text{dist}(x, E) < r \right\}.
\]

(1.1)

Intuitively we think of the set \( B(E, r) \) as consisting of the \( E \) surrounded by a “tube” of width \( r \). Our main interest is to compute the volume of the “tube” of width \( r \) surrounding
for equivalently computing the volume of the set \( B(E, r) \) and then subtracting the volume of \( E \). To make this formal, we define the Minkowski volume \( V_r(E) \) of \( E \) by

\[
V_r(E) = \frac{1}{r^d} \mathcal{L}^d(B(E, r));
\]

here and below \( \mathcal{L}^d \) denotes the \( d \)-dimensional Lebesgue measure in \( \mathbb{R}^d \) and the normalising factor \( \frac{1}{r^d} \) is included to make the subsequent results simpler - we note that different authors use different normalising factors. Tube formulas refer to formulas for computing the Minkowski volume \( V_r(E) \) as a function of the width \( r \) of the “tube” surrounding \( E \). In particular, one is typically interested in the following two types of results:

- Asymptotic behaviour: finding a formula for the asymptotic behaviour of \( V_r(E) \) as \( r \searrow 0 \);
- Explicit formulas: finding an explicit formula for \( V_r(E) \) valid for all sufficiently small \( r \).

For convex sets \( E \), this problem has a rich and fascinating history starting with the work of Steiner in the early 19th century. Indeed, Steiner showed that if \( C \) is a bounded convex subset of \( \mathbb{R}^d \), then there are constants \( \kappa^0(C), \kappa^1(C), \ldots, \kappa^d(C) \) such that

\[
\mathcal{L}^d(B(C, r)) = \sum_{l} \kappa^l(C) r^{d-l}
\]

for \( r > 0 \). The coefficients \( \kappa^l(C) \) are called the Quermassintegrale or mixed volumes, and the polynomial \( \sum_l \kappa^l(C) r^{d-l} \) is called the Steiner polynomial. We also note that the coefficients have clear geometric interpretations. For example, \( \kappa^d(C) \) equals the volume of \( C \) and \( \kappa^{d-1}(C) \) is equal to the surface area of \( C \). Steiner’s formula has subsequently been extended to more general classes of sets. For example, in the late 1930’s Weyl proved that a similar result holds for compact oriented \( d \)-dimensional Riemannian manifolds \( C \) (with or without boundary) isometrically embedded into Euclidean space. This theory reached its mature form in the 1960’s where Federer [Fed1,Fed2] unified the tube formulas of Steiner for convex bodies and of Weyl for smooth submanifolds, as described in [BeGo,Gray,We], and extended these results to sets of positive reach. Federer’s tube formula has since been extended in various directions by a number of researchers in integral geometry and geometric measure theory, including [Fu1,Fu2,Schn1,Schn2,St,Zä1,Zä2] and most recently (and most generally) in [HuLaWe]. The books [Gray,Mo,Schn2] contain extensive endnotes with further information and many other references. While the above references investigate tube formulas for sets that satisfy some (possibly very weak) convexity and/or smoothness conditions, very recently there has been significant interest in developing a theory of tube formulas for fractal sets and a number of exciting works have appeared. Indeed, in the early 1990’s Lapidus introduced the notion of “complex dimensions” and has during the past 20 years very successfully pioneered the use of “complex dimensions” to obtain explicit tube formulas for certain classes of fractal subsets of (mainly) the real line; this exciting theory is described in detail in Lapidus & van Frankenhuyzen’s intriguing books [Lap-vF1,Lap-vF2]. In a parallel development, and building on earlier work by Lalley [Lal1,Lal2,Lal3] and Gatzouras [Ga] (see also [Fa3]), Winter [Wi] has initiated the systematic study of curvatures of fractal sets and applied this theory to study the asymptotic behaviour of the Minkowski volume \( V_r(E) \) of fractal sets \( E \) using methods from renewal theory. The work in this paper may be viewed as a natural higher dimensional multifractal development of this line of research.

The Minkowski volume \( V_r(E) \) is closely related to various notions from Fractal Geometry. Indeed, using the Minkowski volume \( V_r(E) \), we define the lower and upper Minkowski
The link with Fractal Geometry is now explained as follows. Namely, box dimensions play an important role in Fractal Geometry and it is not difficult to see that the lower Minkowski dimension equals the lower box dimension and that the upper Minkowski dimension equals the upper box dimension; for the definition of the box dimensions the reader is referred to Falconer’s text book [Fa1].

It is clearly also of interest to analyse the behaviour of the Minkowski volume \( V_{r}(E) \) itself as \( r \searrow 0 \). Indeed, if, for example, \( a_{1}, \ldots, a_{d}, b_{1}, \ldots, b_{d} \) are real numbers with \( a_{i} \leq b_{i} \) for all \( i \), and \( U \) denotes the rectangle \([a_{1}, b_{1}] \times \cdots \times [a_{d}, b_{d}]\) in \( \mathbb{R}^{d} \), then it is clear that \( V_{r}(U) = ((b_{1} + r) - (a_{1} - r)) \cdots ((b_{d} + r) - (a_{d} - r)) \rightarrow (b_{1} - a_{1}) \cdots (b_{d} - a_{d}) = \mathcal{L}^{d}(U) \). This suggests that if \( t \) is a real number, then the limit \( \lim_{r \searrow 0} \frac{1}{r^{t}} V_{r}(E) \) (if it exists) may be interpreted as the \( t \)-dimensional volume of \( E \). Motivated by this, for a real number \( t \), we define the lower and upper \( t \)-dimensional Minkowski content of \( E \) by

\[
\begin{align*}
M_{\downarrow}(E) &= \liminf_{r \searrow 0} \frac{1}{r^{t}} V_{r}(E), \\
M_{\uparrow}(E) &= \limsup_{r \searrow 0} \frac{1}{r^{t}} V_{r}(E).
\end{align*}
\]  \hspace{1cm} (1.5)

If \( M_{\downarrow}(E) = M_{\uparrow}(E) \), i.e. if the limit \( \lim_{r \searrow 0} \frac{1}{r^{t}} V_{r}(E) \) exists, then we say the \( E \) is \( t \)-Minkowski measurable, and we denote the common value of \( M_{\downarrow}(E) \) and \( M_{\uparrow}(E) \) by \( M^{t}(E) \), i.e. we write

\[
M^{t}(E) = M_{\downarrow}(E) = M_{\uparrow}(E).
\]  \hspace{1cm} (1.6)

Of course, a set \( E \) may not be Minkowski measurable, i.e. the limit \( \lim_{r \searrow 0} \frac{1}{r^{t}} V_{r}(E) \) may not exist. In this case it is natural to study the limiting behaviour of suitably defined “averages” of \( \frac{1}{r^{t}} V_{r}(E) \). We therefore define the lower and upper average \( t \)-dimensional Minkowski content of \( E \) by

\[
\begin{align*}
M_{\text{ave}}^{\downarrow}(E) &= \liminf_{r \searrow 0} \frac{1}{r^{t}} \int_{r}^{1} \frac{1}{s^{t}} V_{s}(E) \, ds \\
M_{\text{ave}}^{\uparrow}(E) &= \limsup_{r \searrow 0} \frac{1}{r^{t}} \int_{r}^{1} \frac{1}{s^{t}} V_{s}(E) \, ds.
\end{align*}
\]  \hspace{1cm} (1.7)

If \( M_{\text{ave}}^{\downarrow}(E) = M_{\text{ave}}^{\uparrow}(E) \), i.e. if the limit \( \lim_{r \searrow 0} \frac{1}{r^{t}} \int_{r}^{1} \frac{1}{s^{t}} V_{s}(E) \, ds \) exists, then we say the \( E \) is \( t \)-averagely Minkowski measurable, and we denote the common value of \( M_{\text{ave}}^{\downarrow}(E) \) and \( M_{\text{ave}}^{\uparrow}(E) \) by \( M_{\text{ave}}^{t}(E) \), i.e. we write

\[
M_{\text{ave}}^{t}(E) = M_{\text{ave}}^{\downarrow}(E) = M_{\text{ave}}^{\uparrow}(E).
\]  \hspace{1cm} (1.8)

While the Minkowski dimensions in many cases can be computed rigorously relatively easy, it is a notoriously difficult problem to compute the Minkowski content. In fact, it is only within the past 15 years that the Minkowski content of non-trivial examples have been computed. Indeed, using techniques from complex analysis, Lapidus and collaborators [Lap1-vF1,Lap1-vF2] have computed the Minkowski content of certain self-similar subsets of the real line,
and using ideas from the theory of Mercerian theorems, Falconer [Fa3] have obtained similar results.

It is our intention to extend the notion of Minkowski volume $V_r(E)$ to multifractals and investigate the asymptotic behaviour of the corresponding multifractal Minkowski volume as $r \searrow 0$ for self-similar multifractals. In order to motivate our definitions we will now explain what the term “multifractal analysis” covers.

2. Multifractals.

2.1. Multifractal spectra. Distributions with widely varying intensity occur often in the physical sciences, e.g. the spatial-temporal distribution of rainfall, the spatial distribution of oil and gas in the underground, the distribution of galaxies in the universe, the dissipation of energy in a highly turbulent fluid flow, or the occupation measure on strange attractors. Such distributions are called multifractals and have recently been the focus of much attention in the physics literature.

For a Borel measure $\mu$ on a $\mathbb{R}^d$ and a real number $\alpha$, let us consider the set $\Delta_\mu(\alpha)$ of those points $x$ in $\mathbb{R}^d$ for which the measure $\mu(B(x,r))$ of the ball $B(x,r)$ with center $x$ and radius $r$ behaves like $r^\alpha$ for small $r$, i.e. the set

$$\Delta_\mu(\alpha) = \left\{ x \in \mathbb{R}^d \left| \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\}.$$

If the intensity of the measure $\mu$ varies very widely, it may happen that the sets $\Delta_\mu(\alpha)$ display a fractal-like character for a range of values of $\alpha$. If this is the case, then the measure is called a multifractal measure or simply a multifractal, and it is natural to study the sizes of the sets $\Delta_\mu(\alpha)$ as $\alpha$ varies. We do this by studying the Hausdorff dimension of $\Delta_\mu(\alpha)$. More precisely, we define the multifractal spectrum $f_\mu: \mathbb{R} \to \mathbb{R}$ of $\mu$ by

$$f_\mu(\alpha) = \dim \Delta_\mu(\alpha),$$

(2.1)

of the sets $\Delta_\mu(\alpha)$ as a function of $\alpha$ where $\dim$ denotes the Hausdorff dimension. The function in (2.1) and similar functions are generically known as “the multifractal spectrum of $\mu$”, “the singularity spectrum of $\mu$” or “the spectrum of scaling indices”, and one of the main problems in multifractal analysis is to study these and related functions. The function $f_\mu(\alpha)$ was first explicitly defined by the physicists Halsey et al. in 1986 in their seminal paper [HaJeKaPrSh]. The concepts underlying the above mentioned multifractal decompositions go back to two early papers by Mandelbrot [Man1,Man2] from 1972 and 1974, respectively, where Mandelbrot suggested that the bulk of intermittent dissipation of energy in a highly turbulent fluid flow occurs over a set of fractal dimension. The ideas introduced in [Man1,Man2] were taken up by Frisch & Parisi [FrPa] in 1985 and finally by Halsey et al. [HaJeKaPrSh] in 1986. Of course, for many measures the limit $\lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r}$ may fail to exist for all or many $x$, in which case we need to work with lower or upper limits as $r$ tends to 0 and (perhaps) replace “$= \alpha$” in the definition of $\Delta_\mu(\alpha)$ with “$\leq \alpha$” or “$\geq \alpha$”.

2.2. Renyi dimensions. Based on a remarkable insight together with a clever heuristic argument Halsey et al. [HaJeKaPrSh] suggested that the multifractal spectrum $f_\mu(\alpha)$ can be computed using a principle known as the Multifractal Formalism. The Multifractal Formalism involves the so-called Renyi dimensions which we will now define. Let $\mu$ be a Borel measure on $\mathbb{R}^d$. For $q \in \mathbb{R}$ and $r > 0$, we define the $q$-th moment $I^q_{\mu,r}(E)$ of a subset $E$ of $\mathbb{R}^d$ with respect to $\mu$ at scale $r$ by

$$I^q_{\mu,r}(E) = \int_E \mu(B(x,r))^q d\mu(x).$$

(2.2)
Next, the lower and upper Renyi dimensions of $E$ with respect to $\mu$ are defined by

$$\dim_{q,\mu}^L(E) = \liminf_{r \searrow 0} \frac{\log I_{q,\mu}^L(E)}{-\log r},$$

$$\dim_{q,\mu}^U(E) = \limsup_{r \searrow 0} \frac{\log I_{q,\mu}^U(E)}{-\log r}. \quad (2.3)$$

In particular, the Renyi dimensions of the support $\text{supp} \mu$ of $\mu$ play an important role in the statement of the Multifractal Formalism. For this reason it is useful to denote these dimensions by a separate notation, and we therefore define the lower and upper Renyi spectra $\tau_{q,\mu}, \tau_{q,\mu}^*: R \rightarrow [-\infty, \infty]$ of $\mu$ by

$$\tau_{q,\mu}(\alpha) = \dim_{q,\mu}^L(\text{supp} \mu) = \liminf_{r \searrow 0} \frac{\log I_{q,\mu}^L(\text{supp} \mu)}{-\log r},$$

$$\tau_{q,\mu}^*(\alpha) = \dim_{q,\mu}^U(\text{supp} \mu) = \limsup_{r \searrow 0} \frac{\log I_{q,\mu}^U(\text{supp} \mu)}{-\log r}. \quad (2.4)$$

### 2.3. The Multifractal Formalism

We can now state the Multifractal Formalism. Loosely speaking the Multifractal Formalism says that the multifractal spectrum $f_\mu$ and the Renyi dimensions $\tau_{q,\mu}(\alpha)$ and $\tau_{q,\mu}^*(\alpha)$ carry the same information. More precisely, the Multifractal Formalism says that the multifractal spectrum equals the Legendre transform of the Renyi dimensions. Before stating this formally, we remind the reader that if $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a real valued function, then the Legendre transform $\varphi^*: \mathbb{R} \rightarrow [-\infty, \infty]$ of $\varphi$ is defined by

$$\varphi^*(x) = \inf_y (xy + \varphi(y)). \quad (2.5)$$

We can state the Multifractal Formalism.

**The Multifractal Formalism – A Physics Folklore Theorem.** The multifractal spectrum $f_\mu$ of $\mu$ equals the Legendre transforms, $\tau_{q,\mu}^*$ and $\tau_{q,\mu}^*$, of the Renyi dimensions, i.e.

$$f_\mu(\alpha) = \tau_{q,\mu}^*(\alpha) = \tau_{q,\mu}^*(\alpha)$$

for all $\alpha \geq 0$.

The Multifractal Formalism is a truly remarkable result: it says that the locally defined multifractal spectrum $f_\mu$ can be computed in terms of the Legendre transforms of the globally defined moment scaling functions $\tau_{q,\mu}^*$ and $\tau_{q,\mu}^*$. There is apriori no reason to expect that the Legendre transforms of the moment scaling function $\tau_{q,\mu}^*$ and $\tau_{q,\mu}^*$ should provide any information about the fractal dimension of the set of points $x$ such that $\mu(B(x, r)) \approx r^\alpha$ for $r \approx 0$. In some sense the Multifractal Formalism is a genuine mystery.

During the past 20 years there has been an enormous interest in verifying the Multifractal Formalism and computing the multifractal spectra of measures in the mathematical literature. In the mid 1990’s Cawley & Mauldin [CaMa] and Arbeiter & Patzschke [ArPa] verified the Multifractal Formalism for self-similar measures satisfying the OSC, and within the last 20 years the multifractal spectra of various classes of measures in Euclidean space $\mathbb{R}^d$ exhibiting some degree of self-similarity have been computed rigorously, cf. the textbooks [Fa2,Pes2] and the references therein.
3. Multifractal tubes

3.1. Multifractal tubes of general measures. Motivated by Lapidus & van Frankenhuyzen investigations [Lap-vF1,Lap-vF2] of tube formulas for fractal sets, it is natural to seek to develop a theory of multifractal tube formulas for multifractal measures. We will now present a framework for developing such a theory and as an application illustrating these ideas we give a complete description of the asymptotic behaviour of the multifractal tube formulas for self-similar measures satisfying the Open Set Condition.

Multifractal tube formulas are defined as follows. First note that if \( r > 0 \) and \( E \) is a subset of \( \mathbb{R}^d \), then the Minkowski volume \( V_r(E) \) is given by

\[
V_r(E) = \frac{1}{r^d} \mathcal{L}^d(B(E, r))
\]

where we have rewritten the Lebesgue measure \( \mathcal{L}^d(B(E, r)) \) of \( B(E, r) \) as the integral \( \int_{B(E, r)} d\mathcal{L}^d(x) \). Motivated by the Renyi dimensions (i.e. (2.2) and (2.3)) and the above expression for \( V_r(E) \), we define the multifractal Minkowski volume as follows. Namely, let \( r > 0 \) and \( E \) be a subset of \( \mathbb{R}^d \). For real number \( q \) and a Borel measure \( \mu \) on \( \mathbb{R}^d \), we define the multifractal \( q \) Minkowski volume \( V_{\mu, r}^q(E) \) of \( E \) with respect to the measure \( \mu \) by

\[
V_{\mu, r}^q(E) = \frac{1}{r^d} \int_{B(E, r)} \mu(B(x, r))^q d\mathcal{L}^d(x).
\] (3.1)

Note, that if \( q = 0 \), then the \( q \) multifractal Minkowski volume \( V_{\mu, r}^q(E) \) reduces to the usual Minkowski volume, i.e

\[
V_{\mu, r}^0(E) = V_r(E).
\]

The importance of the Renyi dimensions in multifractal analysis together with the formal resemblance between the multifractal Minkowski volume \( V_{\mu, r}^q(E) \) and the moments \( I_{\mu, r}^q(E) \) used in the definition the Renyi dimensions may be seen as a justification for calling the quantity \( V_{\mu, r}^q(E) \) for the multifractal Minkowski volume; a further justification for this terminology will be provided below.

Using the multifractal Minkowski volume we can define multifractal Minkowski dimensions. For real number \( q \) and a Borel measure \( \mu \) on \( \mathbb{R}^d \), we define the lower and upper multifractal \( q \) Minkowski dimension of \( E \), by

\[
\dim^q_{\mu, \text{inf}}(E) = \liminf_{r \searrow 0} \frac{\log V_{\mu, r}^q(E)}{-\log r},
\]

\[
\dim^q_{\mu, \text{sup}}(E) = \limsup_{r \searrow 0} \frac{\log V_{\mu, r}^q(E)}{-\log r}.
\] (3.2)

Again we note the close similarity between the multifractal Minkowski dimensions and the Renyi dimensions. Indeed, the next proposition shows that this similarity is not merely a formal resemblance. In fact, for \( q \geq 0 \), the multifractal Minkowski dimensions and the Renyi dimensions coincide. This clearly provides a further justification for calling the quantity \( V_{\mu, r}^q(E) \) for the multifractal Minkowski volume.
Proposition 3.1. Let \( \mu \) be a Borel measure on \( \mathbb{R}^d \) and \( E \subseteq \mathbb{R}^d \). If \( q \geq 0 \), then
\[
\dim_{M,\mu}^q(E) = \dim_{R,\mu}^q(E),
\]
\[
\overline{\dim}_{M,\mu}^q(E) = \overline{\dim}_{R,\mu}^q(E).
\]
In particular, if \( q \geq 0 \), then
\[
\dim_{M,\mu}(\text{supp } \mu) = \tau_{\mu}(q),
\]
\[
\overline{\dim}_{M,\mu}(\text{supp } \mu) = \overline{\tau}_{\mu}(q).
\]

Proof
This follows easily from the definitions and the proof is therefore omitted. \( \square \)

Having defined multifractal Minkowski dimensions, we also define multifractal Minkowski content and average multifractal Minkowski content. For real numbers \( q \) and \( t \), we define the lower and upper \((q, t)\)-dimensional multifractal Minkowski content of \( E \) with respect to \( \mu \) by
\[
M_{\mu}^{q, t}(E) = \liminf_{r \to 0} \frac{1}{r^{-t}} V_{q, \mu}^r(E),
\]
\[
\overline{M}_{\mu}^{q, t}(E) = \limsup_{r \to 0} \frac{1}{r^{-t}} V_{q, \mu}^r(E).
\]
(3.3)

If \( M_{\mu}^{q, t}(E) = \overline{M}_{\mu}^{q, t}(E) \), i.e. if the limit \( \lim_{r \to 0} \frac{1}{r^{-t}} V_{q, \mu}^r(E) \) exists, then we say the \( E \) is \((q, t)\) multifractal Minkowski measurable with respect to \( \mu \), and we denote the common value of \( M_{\mu}^{q, t}(E) \) and \( \overline{M}_{\mu}^{q, t}(E) \) by \( M_{\mu}^{q, t}(E) \), i.e. we write
\[
M_{\mu}^{q, t}(E) = \overline{M}_{\mu}^{q, t}(E) = \overline{M}_{\mu}^{q, t}(E).
\]
(3.4)

Of course, sets may not be multifractal Minkowski measurable, and it is therefore useful to introduce a suitable averaging procedure when computing the multifractal Minkowski content. Motivated by this we define the lower and upper \((q, t)\)-dimensional average multifractal Minkowski content of \( E \) with respect to \( \mu \) by
\[
M_{\mu, \text{ave}}^{q, t}(E) = \liminf_{r \to 0} \frac{1}{-\log r} \int_r^1 s^{-t} V_{q, \mu, s}^r(E) \frac{ds}{s},
\]
\[
\overline{M}_{\mu, \text{ave}}^{q, t}(E) = \limsup_{r \to 0} \frac{1}{-\log r} \int_r^1 s^{-t} V_{q, \mu, s}^r(E) \frac{ds}{s}.
\]
(3.5)

If \( M_{\mu, \text{ave}}^{q, t}(E) = \overline{M}_{\mu, \text{ave}}^{q, t}(E) \), i.e. if the limit \( \lim_{r \to 0} \frac{1}{-\log r} \int_r^1 s^{-t} V_{q, \mu, s}^r(E) \frac{ds}{s} \) exists, then we say the \( E \) is \((q, t)\) averagely multifractal Minkowski measurable with respect to \( \mu \), and we denote the common value of \( M_{\mu, \text{ave}}^{q, t}(E) \) and \( \overline{M}_{\mu, \text{ave}}^{q, t}(E) \) by \( M_{\mu, \text{ave}}^{q, t}(E) \), i.e. we write
\[
M_{\mu, \text{ave}}^{q, t}(E) = \overline{M}_{\mu, \text{ave}}^{q, t}(E) = \overline{M}_{\mu, \text{ave}}^{q, t}(E).
\]
(3.6)

Note that definitions (3.3), (3.4), (3.5) and (3.6) reduce to (1.5), (1.6), (1.7) and (1.8), respectively, for \( q = 0 \). We will now give a complete description of the multifractal Minkowski contents for self-similar measures \( \mu \).

3.2. Multifractal tubes of self-similar measures. We will now compute the multifractal Minkowski content of self-similar measures. We begin by recalling the definition
of a self-similar measure. Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, N$ be contracting similarities and let $(p_1, \ldots, p_N)$ be a probability vector. We denote the Lipschitz constant of $S_i$ by $r_i \in (0, 1)$. The self-similar set $K$ and the self-similar measure $\mu$ associated with the list $(S_1, \ldots, S_N, p_1, \ldots, p_N)$ are defined as follows. Namely, $K$ is the unique non-empty compact subset of $\mathbb{R}^d$ such that

$$K = \bigcup_i S_i(K),$$  

and $\mu$ the unique Borel probability measure on $\mathbb{R}^d$ such that

$$\mu = \sum_i p_i \mu \circ S_i^{-1},$$

cf. [Hu]. We note that it is well-known that $\text{supp} \mu = K$.

We will frequently assume that the list $(S_1, \ldots, S_N)$ satisfies certain “disjointness” conditions, viz. the Open Set Condition (OSC) or the Strong Separation Condition (SSC) defined below.

**The Open Set Condition**: There exists an open non-empty and bounded subset $U$ of $\mathbb{R}^d$ with $\bigcup_i S_iU \subseteq U$ and $S_iU \cap S_jU = \emptyset$ for all $i, j$ with $i \neq j$.

**The Strong Separation Condition**: There exists an open non-empty and bounded subset $U$ of $\mathbb{R}^d$ with $\bigcup_i S_iU \subseteq U$ and $S_iU \cap S_jU = \emptyset$ for all $i, j$ with $i \neq j$.

Multifractal analysis of self-similar measures has attracted an enormous interest during the past 20 years. For example, using methods from ergodic theory, Peres & Solomyak [PeSo] have recently shown that for any self-similar measure $\mu$, the Renyi dimensions always exist, i.e. the limit $\lim_{r \to 0} \frac{\log I^q_r(K)}{-\log r}$ always exists, regardless of whether or not the OSC is satisfied provided $q \geq 0$. If in addition the OSC is satisfied, an explicit expression for the limits

$$\tau_{\mu}(q) = \liminf_{r \to 0} \frac{\log I^q_r(K)}{-\log r}$$

and

$$\tau_{\mu}(q) = \limsup_{r \to 0} \frac{\log I^q_r(K)}{-\log r},$$

for $q \in \mathbb{R}$, where $\beta(q)$ is defined by

$$\sum_i p_i^q \beta(q) = 1.$$  

Arbeiter & Patzschke [ArPa] and Cawley & Mauldin [CaMa] also verified the Multifractal Formalism for self-similar measures satisfying the OSC. Namely, in [ArPa,CaMa] it is proved that if the OSC is satisfied, then

$$\tau_{\mu}(q) = \liminf_{r \to 0} \frac{\log I^q_r(K)}{-\log r} = \beta(q),$$

$$\tau_{\mu}(q) = \limsup_{r \to 0} \frac{\log I^q_r(K)}{-\log r} = \beta(q),$$

for $q \in \mathbb{R}$, where $\beta(q)$ is defined by

$$\sum_i p_i^q \beta(q) = 1.$$  

Arbeiter & Patzschke [ArPa] and Cawley & Mauldin [CaMa] also verified the Multifractal Formalism for self-similar measures satisfying the OSC. Namely, in [ArPa,CaMa] it is proved that if $\mu$ is a self-similar measure satisfying the OSC, then

$$f_\mu(\alpha) = \beta^*(\alpha)$$

for all $\alpha \geq 0$; recall, that the definition of the Legendre transform is given in (2.5). We will now compute the multifractal Minkowski dimensions and multifractal Minkowski content of self-similar measures satisfying various separation conditions. First, we note that the multifractal Minkowski dimensions coincide with $\beta(q)$. This is not a deep fact and is included mainly for completeness.
Theorem 3.2. Let $K$ and $\mu$ be given by (3.7) and (3.8). Fix $q \in \mathbb{R}$ and assume that one of the following conditions is satisfied:

(i) The OSC is satisfied and $0 \leq q$;
(ii) The SSC is satisfied.

Then we have

$$\dim_{M,\mu}^q(K) = \dim_{M,\mu}^q(K) = \beta(q)$$

for all $q \in \mathbb{R}$.

Proof

As noted above, this is not a deep fact and can be proven directly from the definitions using standard arguments (similar to those in [ArPa] or Falconer’s textbook [Fa2]). The result also follows immediately from the main Theorem 3.3 below.

Next, we give a complete description of the asymptotic behaviour of the multifractal tube formulas for self-similar measures satisfying the OSC. In particular, we prove that if the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then $K$ is $(q, \beta(q))$ multifractal Minkowski measurable with respect to $\mu$, and if the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is contained in a discrete additive subgroup of $\mathbb{R}$, then $K$ is $(q, \beta(q))$ average multifractal Minkowski measurable with respect to $\mu$. This is the content of Theorem 3.2. The proof of Theorem 3.2 is based on renewal theory and will be discussed after the statement of the theorem.

Theorem 3.3. Let $K$ and $\mu$ be given by (3.7) and (3.8). Fix $q \in \mathbb{R}$ and assume that one of the following conditions is satisfied:

(i) The OSC is satisfied and $0 \leq q$;
(ii) The SSC is satisfied.

Define $\lambda_q : (0, \infty) \to \mathbb{R}$ by

$$\lambda_q(r) = V_{\mu,r}^q(K) - \sum_i p_i^q 1_{(0,r_i]}(r) V_{\mu,r_i^{-1}}^q(K)$$

Then we have:

1. The non-arithmetic case: If the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then

$$\frac{1}{r^{-\beta(q)}} V_{\mu,r}^q(K) = c_q + \varepsilon_q(r)$$

where $c_q \in \mathbb{R}$ is the constant given by

$$c_q = \frac{1}{-\sum_i p_i^q r_i^{-\beta(q)} \log r_i} \int_0^1 r^{-\beta(q)} \lambda_q(r) \frac{dr}{r}$$

and $\varepsilon_q(r) \to 0$ as $r \searrow 0$. In addition, $K$ is $(q, \beta(q))$ multifractal Minkowski measurable with respect to $\mu$ with

$$M_{\mu}^{q,\beta(q)}(K) = \frac{1}{-\sum_i p_i^q r_i^{-\beta(q)} \log r_i} \int_0^1 r^{-\beta(q)} \lambda_q(r) \frac{dr}{r}.$$
(2) The arithmetic case: If the set \( \{ \log r_1^{-1}, \ldots, \log r_N^{-1} \} \) is contained in a discrete additive subgroup of \( \mathbb{R} \) and \( \langle \log r_1^{-1}, \ldots, \log r_N^{-1} \rangle = u \mathbb{Z} \) with \( u > 0 \), then

\[
\frac{1}{r^{-\beta(q)}} V_{\mu,r}^q(K) = \pi_q(r) + \varepsilon_q(r)
\]

where \( \pi_q : (0, \infty) \to \mathbb{R} \) is the multiplicatively periodic function with period equal to \( e^u \) (i.e. \( \pi_q(e^u r) = \pi_q(r) \) for all \( r \in (0, \infty) \)) given by

\[
\pi_q(r) = \frac{1}{-\sum_i p_i^q r_i^{\beta(q)} \log r_i} \sum_{n \in \mathbb{Z}} \sum_{re^{nu} \leq 1} (re^{nu})^{\beta(q)} \lambda_q(re^{nu}) u \]

and \( \varepsilon_q(r) \to 0 \) as \( r \searrow 0 \). In addition, \( K \) is \((q,\beta(q))\)-averagely multifractal Minkowski measurable with respect to \( \mu \) with

\[
M_{\mu,\text{ave}}^{q,\beta(q)}(K) = \frac{1}{-\sum_i p_i^q r_i^{\beta(q)} \log r_i} \int_0^1 r^{\beta(q)} \lambda_q(r) \frac{dr}{r}.
\]

It is instructive to consider the special case \( q = 0 \). Indeed, since the multifractal \( q \) Minkowski volume for \( q = 0 \) equals the usual Minkowski volume and since the \((q,t)\)-dimensional multifractal Minkowski content for \( q = 0 \) equals the usual \( t \)-dimensional Minkowski content, the following corollary, providing formulas for the asymptotic behaviour of the Minkowski volume of self-similar sets, follows immediately from Theorem 3.3 by putting \( q = 0 \). This result was first obtained by Gatzouras [Ga] and later by Winter [Wi].

**Corollary 3.4 [Ga].** Let \( K \) be given by (3.7). Assume that the OSC is satisfied. Let \( t \) denote the common value of the box dimensions and the Hausdorff dimension of \( K \), i.e. \( t \) is the unique number such that \( \sum_i r_i^t = 1 \) (see [Fa2] or [Hu]). Define \( \lambda : (0, \infty) \to \mathbb{R} \) by

\[
\lambda(r) = V_r(K) - \sum_i 1_{(0,r_i)}(r) V_{r_i}^{-1}(K)
\]

Then we have:

(1) The non-arithmetic case: If the set \( \{ \log r_1^{-1}, \ldots, \log r_N^{-1} \} \) is not contained in a discrete additive subgroup of \( \mathbb{R} \), then

\[
\frac{1}{r^{-t}} V_r(K) = c + \varepsilon(r)
\]

where \( c \in \mathbb{R} \) is the constant given by

\[
c = \frac{1}{-\sum_i r_i^t \log r_i} \int_0^1 r^t \lambda(r) \frac{dr}{r}
\]

and \( \varepsilon(r) \to 0 \) as \( r \searrow 0 \). In addition, \( K \) is \( t \)-Minkowski measurable with

\[
M_t(K) = \frac{1}{-\sum_i r_i^t \log r_i} \int_0^1 r^t \lambda(r) \frac{dr}{r}.
\]
(2) The arithmetic case: If the set \( \{ \log r_1^{-1}, \ldots, \log r_N^{-1} \} \) is contained in a discrete additive subgroup of \( \mathbb{R} \) and \( \langle \log r_1^{-1}, \ldots, \log r_N^{-1} \rangle = u\mathbb{Z} \) with \( u > 0 \), then

\[
\frac{1}{r-t} V_r(K) = \pi(r) + \varepsilon(r)
\]

where \( \pi : (0, \infty) \to \mathbb{R} \) is the multiplicatively periodic function with period equal to \( e^u \) (i.e. \( \pi(e^u r) = \pi(r) \) for all \( r \in (0, \infty) \)) given by

\[
\pi(r) = \frac{1}{-\sum_i r_i^t \log r_i} \sum_{n \in \mathbb{Z}} \sum_{re^u \leq 1} (re^u)^t \lambda(re^u) u
\]

and \( \varepsilon(r) \to 0 \) as \( r \searrow 0 \). In addition, \( K \) is \( t \) averagely Minkowski measurable with

\[
M_{\text{ave}}^t(K) = \frac{1}{-\sum_i r_i^t \log r_i} \int_0^1 r^t \lambda(r) \frac{dr}{r}.
\]

Proof
Since \( \beta(0) = \dim_{\mathcal{H}}(K) = \dim_{\mathcal{H}}(K) = \dim(K) = t \) (see [Fa2] or [Hu]) and \( V^0_{\mu, \varepsilon}(K) = V_r(K) \), this follows from Theorem 3.3 by putting \( q = 0 \). \( \square \)

3.3. How does one prove Theorem 3.3? How does one prove Theorem 3.3 on the asymptotic behaviour of multifractal tubes of self-similar measures? The proof is based on renewal theory and, in particular, on a very recent renewal theorem by Levitin & Vassiliev [LeVa]. Below we state Levitin & Vassiliev’s renewal theorem.

**Theorem 3.5.** Levitin & Vassiliev’s renewal theorem [LeVa]. Let \( t_1, \ldots, t_N > 0 \) and \( p_1, \ldots, p_N > 0 \) with \( \sum_i p_i = 1 \). Define the probability measure \( P \) by

\[
P = \sum_i p_i \delta_{t_i}.
\]

Let \( \lambda, \Lambda : \mathbb{R} \to \mathbb{R} \) be real valued functions satisfying the following conditions:

(i) The function \( \lambda \) is piecewise continuous;

(ii) There are constants \( c, k > 0 \) such that

\[
|\lambda(t)| \leq ce^{-k|t|}
\]

for all \( t \in \mathbb{R} \);

(iii) We have

\[
\Lambda(t) \to 0 \quad \text{as} \quad t \to -\infty;
\]

(iv) We have

\[
\Lambda(t) = \int_0^t \Lambda(t-s) dP(s) + \lambda(t)
\]

for all \( t \in \mathbb{R} \).
Then the following holds:

1. The non-arithmetic case: If \( \{t_1, \ldots, t_N\} \) is not contained in a discrete additive subgroup of \( \mathbb{R} \), then

\[
\Lambda(t) = c + \varepsilon(t)
\]

for all \( t \in \mathbb{R} \) where

\[
c = \frac{1}{s} \int ds \; dP(s) \int \lambda(s) \; ds
\]

and \( \varepsilon(t) \to 0 \) as \( t \to \infty \). In addition,

\[
\frac{1}{T} \int_0^T \Lambda(t) \; dt \to c = \frac{1}{s} \int ds \; dP(s) \int \lambda(s) \; ds \quad \text{as} \quad T \to \infty.
\] (3.11)

2. The arithmetic case: If \( \{t_1, \ldots, t_N\} \) is contained in a discrete additive subgroup of \( \mathbb{R} \) and \( \langle t_1, \ldots, t_N \rangle = u \mathbb{Z} \) with \( u > 0 \), then

\[
\Lambda(t) = \pi(t) + \varepsilon(t)
\]

for all \( t \in \mathbb{R} \) where \( \pi: \mathbb{R} \to \mathbb{R} \) is the periodic function with period equal to \( u \), (i.e. \( \pi(t + u) = \pi(t) \) for all \( t \in \mathbb{R} \)) given by

\[
\pi(t) = \frac{1}{s} \int ds \; dP(s) u \sum_{n \in \mathbb{Z}} \lambda(t + nu)
\]

and \( \varepsilon(t) \to 0 \) as \( t \to \infty \). In addition

\[
\frac{1}{T} \int_0^T \Lambda(t) \; dt \to c = \frac{1}{s} \int ds \; dP(s) \int \lambda(s) \; ds \quad \text{as} \quad T \to \infty.
\] (3.12)

Proof

All statements, except (3.11) and (3.12), follow [LeVa], and statements (3.11) and (3.12) are easily proved from the remaining parts of the theorem. □

The key difference between Levitin & Vassiliev’s renewal theorem and the classical renewal theorem from Feller’s books [Fel1,Fel2] is the conclusion in the arithmetic case. While the assumptions in the classical renewal theorem are weaker, the conclusion in the arithmetic case is also weaker. More precisely, in the arithmetic case Levitin & Vassiliev’s renewal theorem says that the error-term \( \varepsilon(t) \) tends to 0 as \( t \) tends to infinity, i.e.

\[
\lim_{t \to \infty} \varepsilon(t) = 0,
\]

whereas the classical renewal theorem only allows us to conclude that the error-term \( \varepsilon(t) \) tends to 0 as \( t \) tends to infinity through “steps” of length \( u \), i.e.

\[
\lim_{n \in \mathbb{N}} \varepsilon(nu + s) = 0
\]

for all \( s \in \mathbb{R} \).

Using Levitin & Vassiliev’s renewal theorem (Theorem 3.5) we can now prove Theorem 3.3. Below is a sketch of the proof; the detailed arguments are presented in Sections 6–9. In order to prove Theorem 3.3, we will apply Levitin & Vassiliev’s Renewal Theorem to the
probability measure $P = P_q$ and the functions $\lambda = \lambda^0_q$ and $\Lambda = \Lambda^0_q$ defined below. First, recall that $\lambda_q : (0, \infty) \to \mathbb{R}$ is defined by

$$\lambda_q(r) = V^q_{\mu,r}(K) - \sum_i p^q_i \mathbf{1}_{(0, r^i)}(r) V^q_{\mu, r^{-1}}(K).$$

Next, define $\Lambda_q : (0, \infty) \to \mathbb{R}$ by

$$\Lambda_q(r) = V^q_{\mu,r}(K).$$

We now define the functions $\lambda^0_q, \Lambda^0_q : \mathbb{R} \to \mathbb{R}$ as follows. Namely, define $\lambda^0_q : \mathbb{R} \to \mathbb{R}$ by

$$\lambda^0_q(t) = 1_{[0, \infty)}(t) e^{-t \beta(q)} \lambda_q(e^{-t}),$$

and define $\Lambda^0_q : \mathbb{R} \to \mathbb{R}$ by

$$\Lambda^0_q(t) = 1_{[0, \infty)}(t) e^{-t \beta(q)} \Lambda_q(e^{-t}).$$

Finally, define the probability measure $P_q$ by

$$P_q = \sum_i p^q_i r^i \beta(q) \delta_{\log r^i}. $$

The crux of the matter now is to show that the probability measure $P = P_q$ and the functions $\lambda = \lambda^0_q$ and $\Lambda = \Lambda^0_q$ satisfy Conditions (i)–(iv) in Levitin & Vassiliev’s renewal theorem. Conditions (i), (iii) and (iv) are not difficult to verify. The main difficulty is to prove that Condition (ii) is satisfied. This is highly technical and requires a number very delicate estimates. Finally, applying Levitin & Vassiliev’s renewal theorem to the probability measure $P = P_q$ and the functions $\lambda = \lambda^0_q$ and $\Lambda = \Lambda^0_q$ yields Theorem 3.3.

4. Multifractal tubes measures

4.1. Multifractal tube measures of general measures. The statement in Theorem 3.3 is a global one: it provides information about the limiting behaviour of the suitably normalized multifractal Minkowski volume

$$\frac{1}{r^{-\beta(q)}} V^q_{\mu,r}(K)$$

of the entire support $K$ of $\mu$ as $r \searrow 0$. However, it is equally natural to ask for local versions of Theorem 3.3 describing the limiting behaviour of the normalized multifractal Minkowski volume

$$\frac{1}{r^{-\beta(q)}} V^q_{\mu,r}(E)$$

of (well behaved) subsets $E$ of the support of $\mu$ as $r \searrow 0$. In order to address this question, we now introduce multifractal tube measures. A further motivation for introducing multifractal tube measures comes from convex geometry and will be discussed below.

The multifractal tube measures are defined as follows. Fix a Borel measure $\mu$ on $\mathbb{R}^d$ and $r > 0$. For a real number $q$, we define the multifractal Minkowski tube measure $\mathcal{I}^q_{\mu,r}$ by

$$\mathcal{I}^q_{\mu,r}(E) = \frac{1}{p^d} \int_{E \cap B(\text{supp } \mu, r)} \mu(B(x, r))^q d\mathcal{L}^d(x)$$

(4.1)
for Borel subsets $E$ of $\mathbb{R}^d$; recall, that $\text{supp}\, \mu$ denotes the support of $\mu$. Of course, the measures $T^q_{\mu,r}$ will, in general, not converge weakly as $r \searrow 0$ (indeed, it follows immediately from Theorem 3.3 that, in general, $T^q_{\mu,r}(\mathbb{R}^d) = \mathbb{V}^q_{\mu,r}(K)$ does not converge as $r \searrow 0$). Hence in order to ensure weak convergence of $T^q_{\mu,r}$ as $r \searrow 0$ it is necessary to normalize the measures $T^q_{\mu,r}$. There are two natural ways to normalized. Firstly, we can normalize by volume. More precisely, we define the volume normalized multifractal tube measure $\mathbb{V}^q_{\mu,r}$ by

$$\mathbb{V}^q_{\mu,r} = \frac{1}{T^q_{\mu,r}(\mathbb{R}^d)} T^q_{\mu,r} ,$$

(4.2)

Secondly, we can normalize by scaling. More precisely, we defined the lower and upper scaling normalized multifractal tube measures $\mathcal{S}^q_{\mu,r}$ and $\underline{\mathcal{S}}^q_{\mu,r}$ by

$$\mathcal{S}^q_{\mu,r} = \frac{1}{r^{-\dim_{M,\mu}(\text{supp}\, \mu)}} T^q_{\mu,r} ,$$

$$\underline{\mathcal{S}}^q_{\mu,r} = \frac{1}{r^{-\dim_{\underline{M},\mu}(\text{supp}\, \mu)}} T^q_{\mu,r} ;$$

(4.3)

recall, that $\dim_{M,\mu}$ and $\dim_{\underline{M},\mu}$ denote the lower and upper multifractal $q$ Minkowski dimension, respectively, see (3.2).

It is instructive to consider the particular case $q = 0$. To discuss this case we first make the following definition. Namely, if $U$ is a closed subset of $\mathbb{R}^d$ and $r > 0$, the parallel volume measure $\mathbb{V}^0_{U,r}$ of $U$ is defined by

$$\mathbb{V}^0_{U,r}(E) = \frac{\mathbb{L}^d(E \cap B(U,r))}{\mathbb{L}^d(B(U,r))} ,$$

(4.4)

see, for example, the texts [Gray, Mo, Schn2]. We now note that if $q = 0$ and $\mu$ is any Borel measure with $\text{supp}\, \mu = U$, then the volume normalized multifractal tube measure $\mathbb{V}^q_{\mu,r}$ simplifies to

$$\mathbb{V}^0_{\mu,r}(E) = \frac{\mathbb{L}^d(E \cap B(\text{supp}\, \mu,r))}{\mathbb{L}^d(B(\text{supp}\, \mu,r))}$$

$$= \frac{\mathbb{L}^d(E \cap B(U,r))}{\mathbb{L}^d(B(U,r))}$$

$$= \mathbb{V}^0_{U,r}(E) .$$

(4.5)

This observation provides a further motivation for introducing multifractal tube measures. Namely, the measure $\mathbb{V}^0_{\mu,r}(E) = \mathbb{V}^0_{U,r}(E)$ is closely related to the notion of curvature measures in convex geometry. Curvature measures were introduced in the 1950’s and are now recognized as a very powerful tool for analyzing geometric properties of convex sets, see [Gray, Mo, Schn2]. Indeed, if $U$ is a closed convex subset of $\mathbb{R}^d$ with non-empty interior and $l = 0, 1, 2, \ldots, d$, then the $l$-th order curvature measure $\mathcal{V}^l_{U,r}$ associated with $U$ is defined as the weak limit $\mathcal{V}^l_{U} = \lim_{r \searrow 0} \mathcal{V}^l_{U,r}$ of a certain family $(\mathcal{V}^l_{U,r})_{r>0}$ of measures. While we will not provide the reader with the definition of the measures $\mathcal{V}^l_{U,r}$ for a general integer $l = 0, 1, 2, \ldots, d$ (instead the interested reader can find the definition in previously mentioned texts [Gray, Mo, Schn2]), we do note that if $l = d$, then $\mathcal{V}^d_{U,r} = \mathbb{V}^0_{U,r}$. In particular,
the $d$-th order curvature measure $V_U^d$ is defined by
\[
V_U^d = \lim_{r \downarrow 0} V_{U,r}^d = \lim_{r \downarrow 0} V_{U,r} = \lim_{r \downarrow 0} V_{\mu,r}^0,
\]
where we have used the fact that $V_{\mu,r}^0 = V_{U,r}$ (see (4.5)) and $\lim$ denotes the limit with respect to the weak topology. This shows that the weak limit
\[
\lim_{r \downarrow 0} V_{\mu,r}^q
\]
(if it exists) may be viewed as a $d$-th order multifractal curvature measure and the study of multifractal tube measures can therefore be seen as a first attempt to create a theory of multifractal curvatures.

It is, of course, also possible to define versions of the parallel volume measure analogous to $S_{U,r}^q$ and $S_{\mu,r}^q$. Indeed, if $U$ is a closed subset of $\mathbb{R}^d$ and $r > 0$, we define the lower and upper scaling parallel volume measures $S_{U,r}^q$ and $S_{\mu,r}^q$ of $U$ by
\[
S_{U,r}^q(E) = \frac{1}{r^{-\dim(M(U) + d)}} L^d(E \cap B(U, r)),
\]
\[
S_{\mu,r}^q(E) = \frac{1}{r^{-\dim(M(U) + d)}} L^d(E \cap B(U, r));
\]
recall, that $\dim$ and $\overline{\dim}$ denote the lower and upper Minkowski dimension, respectively, see (1.4). As above, we note that if $q = 0$ and $\mu$ is any probability measure with $\text{supp} \mu = U$, then the scaling normalized multifractal tube measure $S_{\mu,r}^q$ and $S_{\mu,r}^q$ simplify to
\[
S_{\mu,r}^0(E) = S_{U,r}^q(E),
\]
\[
S_{\mu,r}^0(E) = S_{U,r}^q(E).
\]

4.2. Multifractal tube measures of self-similar measures. For self-similar measures $\mu$ satisfying the OSC, we will now investigate the existence of the weak limits of the multifractal tube measures $V_{\mu,r}^q$, $S_{U,r}^q$, and $S_{\mu,r}^q$ as $r \downarrow 0$. In fact, in many cases these limits exist and equal normalized the multifractal Hausdorff measure (defined below) restricted to the support of $\mu$.

The multifractal Hausdorff measure is defined as follows. Namely, in an attempt to develop a theoretical framework for studying the multifractal structure of general Borel measures, Olsen [Ol1], Pesin [Pes1] and Peyrière [Pey] introduced a two parameter family $\{H_{\mu}^{q,t} | q, t \in \mathbb{R} \}$ of measures based on certain generalizations of the Hausdorff measure. The measures $H_{\mu}^{q,t}$ have subsequently been investigated further by a large number of authors, including [Col,Da1,Da2,HoRaSt,Ol2,O’N1,O’N2,Sche], and are defined as follows. Let $E \subseteq \mathbb{R}^d$ and $\delta > 0$. A countable family $(B(x_i, r_i))_i$ of closed balls in $\mathbb{R}^d$ is called a centered $\delta$-covering of $E$ if $E \subseteq \cup_i B(x_i, r_i)$, $x_i \in E$ and $0 < r_i < \delta$ for all $i$. For $E \subseteq \mathbb{R}^d$, $q,t \in \mathbb{R}$ and $\delta > 0$ write
\[
\overline{H}_{\mu,\delta}^{q,t}(E) = \inf \left\{ \sum_i \mu(B(x_i, r_i))^q (2r_i)^t \left| (B(x_i, r_i))_i \text{ is a centered } \delta \text{-covering of } E \right. \right\},
\]
\[
\overline{H}_{\mu}^{q,t}(E) = \sup_{\delta > 0} \overline{H}_{\mu,\delta}^{q,t}(E),
\]
\[
H_{\mu}^{q,t}(E) = \sup_{F \subseteq E} \overline{H}_{\mu}^{q,t}(F).
\]
It follows from [Ol1] that $\mathcal{H}^{q,t}_\mu$ is a measure on the family of Borel subsets of $\mathbb{R}^d$. The measure $\mathcal{H}^{q,t}_\mu$ is, of course, a multifractal generalization of the centered Hausdorff measure. In fact, it is easily seen that if $t \geq 0$, then $2^{-t}\mathcal{H}^{0,t}_\mu \leq \mathcal{H}^{q,t}_\mu \leq \mathcal{H}^{0,t}_\mu$ where $\mathcal{H}^t$ denotes the $t$-dimensional Hausdorff measure. It is also easily seen that the measure $\mathcal{H}^{q,t}_\mu$ in the usual way assign a dimension to each subset $E$ of $\mathbb{R}^d$ (see [Ol1]): there exist a unique number $\dim^0_\mu(E) \in [-\infty, \infty]$ such that

$$\mathcal{H}^{q,t}_\mu(E) = \begin{cases} \infty & \text{for } t < \dim^0_\mu(E) \\ 0 & \text{for } \dim^0_\mu(E) < t \end{cases}. $$

The number $\dim^0_\mu(E)$ is an obvious multifractal analogue of the Hausdorff dimension $\dim(E)$ of $E$. In fact, it follows immediately from the definitions that $\dim(E) = \dim^0_\mu(E)$. One of the main importances of the multifractal Hausdorff measure $\mathcal{H}^{q,t}_\mu$ is its relationship with the multifractal spectrum of $\mu$. Indeed, if we define the dimension function $b_\mu : \mathbb{R} \to [-\infty, \infty]$ by

$$b_\mu(q) = \dim^0_\mu(\text{supp } \mu),$$

then it follows from [Ol1] that the multifractal spectra $f_\mu$ of $\mu$ (recall, that the multifractal spectrum $f_\mu$ is defined in (2.1)) is bounded above by the Legendre transform $\phi^*_{\mu} (\alpha)$ for all $\alpha \geq 0$, i.e.

$$f_\mu(\alpha) \leq \phi^*_{\mu} (\alpha)$$

for all $\alpha \geq 0$, see [Ol1]; recall, that the definition of the Legendre transform $\phi^*$ of a real valued function $\varphi : \mathbb{R} \to \mathbb{R}$ is given in section 2.3. This inequality may be viewed as a rigorous version of the Multifractal Formalism. Furthermore, for many natural families of measures we have $f_\mu(\alpha) = b^*_{\mu}(\alpha)$ for all $\alpha \geq 0$, cf. [Col, Da1, Da2, Ol1, Ol2].

Using the multifractal Hausdorff measures $\mathcal{H}^{q,t}_\mu$, we will now explicitly identify the weak limits of the multifractal tube measures $\mathcal{V}^{q,t}_\mu, \mathcal{S}^{q,t}_{\mu,r}$ and $\mathcal{S}^{q,t}_{\mu,r} \rightarrow \mu - \text{additive subgroup of } \mathbb{R}$ for self-similar measures $\mu$. The first result shows that the weak limit of $\mathcal{V}^{q,t}_\mu$ as $r \searrow 0$ always exists and equals the normalized multifractal Hausdorff measure. In Theorem 4.1 and the subsequent parts of the paper we use the following notation. Namely, if $\mathcal{M}$ is a Borel measure on $\mathbb{R}^d$ and $E$ is a Borel subset of $\mathbb{R}^d$, then we denote the restriction of $\mathcal{M}$ to $E$ by $\mathcal{M} \llcorner E$, i.e.

$$ (\mathcal{M} \llcorner E)(B) = \mathcal{M}(E \cap B) $$

(4.8)

for all Borel subsets $B$ of $\mathbb{R}^d$. We can now state Theorem 4.1.

**Theorem 4.1.** Let $K$ and $\mu$ be given by (3.7) and (3.8). Fix $q \in \mathbb{R}$ and assume that one of the following conditions is satisfied:

(i) The OSC is satisfied and $0 \leq q$;

(ii) The SSC is satisfied.

Then we have

$$ \mathcal{V}^{q,t}_\mu \rightarrow 1 \mathcal{H}^{q,\beta(q)}_{\mu} \mathcal{L} K \quad \text{weakly}. $$

Next, we study the limiting behaviour of $\mathcal{S}^{q,t}_{\mu,r}$ and $\mathcal{S}^{q,t}_{\mu,r}$ as $r \searrow 0$ for self-similar measures $\mu$. Contrary to Theorem 4.1, the weak limits of $\mathcal{S}^{q,t}_{\mu,r}$ and $\mathcal{S}^{q,t}_{\mu,r}$ as $r \searrow 0$ may not exist. Indeed, if the set $\{ \log r_1^{-1}, \ldots, \log r_N^{-1} \}$ is contained in a discrete additive subgroup of $\mathbb{R}$, then the weak limits of $\mathcal{S}^{q,t}_{\mu,r}$ and $\mathcal{S}^{q,t}_{\mu,r}$ as $r \searrow 0$ do not necessarily exist; however the weak limits of certain averages of $\mathcal{S}^{q,t}_{\mu,r}$ and $\mathcal{S}^{q,t}_{\mu,r}$ exist and equal a multiple of the normalized multifractal Hausdorff measure. On the other hand, if the set $\{ \log r_1^{-1}, \ldots, \log r_N^{-1} \}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then the weak limits of $\mathcal{S}^{q,t}_{\mu,r}$ and $\mathcal{S}^{q,t}_{\mu,r}$ as $r \searrow 0$ always exist and equal a multiple of the normalized multifractal Hausdorff measure.
Theorem 4.2. Let \( K \) and \( \mu \) be given by (3.7) and (3.8). Fix \( q \in \mathbb{R} \) and assume that one of the following conditions is satisfied:

(i) The OSC is satisfied and \( 0 \leq q \); 
(ii) The SSC is satisfied.

Then the following holds.

1. We have
   \[
   S^q_{\mu,r} = S^q_{\mu,r} = \frac{1}{r^{\beta(q)}} T^q_{\mu,r}.
   \]
   Write \( S^q_{\mu,r} \) for the common value of \( S^q_{\mu,r} \) and \( S^q_{\mu,r} \), i.e. write
   \[
   S^q_{\mu,r} = \frac{1}{r^{\beta(q)}} T^q_{\mu,r}.
   \]
   Also, define the average measure \( S^q_{\mu,r,ave} \) by
   \[
   S^q_{\mu,r,ave} = \frac{1}{-\log r} \int_r^1 \frac{1}{s^{\beta(q)}} T^q_{\mu,s} \, ds.
   \]
   Then the following holds.

2. The non-arithmetic case: If the set \( \{\log r_1^{-1}, \ldots, \log r_N^{-1}\} \) is not contained in a discrete additive subgroup of \( \mathbb{R} \), then
   \[
   S^q_{\mu,r,ave} \rightarrow M^{q,\beta(q)}(K) = \frac{1}{\mathcal{H}^{q,\beta(q)}_{\mu}(K)} \mathcal{H}^{q,\beta(q)}(K) \mu K \quad \text{weakly,}
   \]
   \[
   S^q_{\mu,r,ave} \rightarrow M^{q,\beta(q,ave)}(K) = \frac{1}{\mathcal{H}^{q,\beta(q,ave)}_{\mu}(K)} \mathcal{H}^{q,\beta(q,ave)}(K) \mu K \quad \text{weakly;}
   \]
   recall, that \( K \) is \( (q,\beta(q)) \) multifractal Minkowski measurable with respect to \( \mu \) and \( (q,\beta(q)) \) average multifractal Minkowski measurable with respect to \( \mu \) by Theorem 3.3 and the multifractal Minkowski content \( M^{q,\beta(q)}(K) \) and the average multifractal Minkowski content \( M^{q,\beta(q,ave)}(K) \) are therefore well-defined.

3. The arithmetic case: If the set \( \{\log r_1^{-1}, \ldots, \log r_N^{-1}\} \) is contained in a discrete additive subgroup of \( \mathbb{R} \), then
   \[
   S^q_{\mu,r,ave} \rightarrow M^{q,\beta(q,ave)}(K) = \frac{1}{\mathcal{H}^{q,\beta(q,ave)}_{\mu}(K)} \mathcal{H}^{q,\beta(q,ave)}(K) \mu K \quad \text{weakly;}
   \]
   recall, that \( K \) is \( (q,\beta(q)) \) average multifractal Minkowski measurable with respect to \( \mu \) by Theorem 3.3 and the average multifractal Minkowski content \( M^{q,\beta(q,ave)}(K) \) is therefore well-defined.

As with Theorem 3.3, it is instructive to consider the special case \( q = 0 \). Indeed, we first note that if \( K \) and \( \mu \) are given by (3.7) and (3.8), then (see (4.5))
\[
\gamma^0_{\mu,r}(E) = \frac{\mathcal{L}^d(E \cap B(K,r))}{\mathcal{L}^d(B(K,r))} = \gamma_{K,r}(E)
\]
Let $K$ be given by (3.7). Assume that the OSC is satisfied. Let $t$ denote the common value of the box dimensions and the Hausdorff dimension of $K$, i.e. $t$ is the unique number such that $\sum r_i = 1$. For $r > 0$, the normalised parallel body measure $V_{K,r}$ is given by

$$V_{K,r}(E) = \frac{1}{r^{-t+d}} \mathcal{L}^d(E \cap B(K,r)).$$

Then we have

$$V_{K,r} \to \frac{1}{H^t(K)} H^t \mu^t K \quad \text{weakly.}$$

Proof

Recall, that $K$ is given by (3.7), and let $\mu$ be given by (3.8). Since $V^{0}_{\mu,r} = V_{K,r}$, the statement now follows from Theorem 4.1 by putting $q = 0$.

Corollary 4.4. [Wi]. Let $K$ be given by (3.7). Assume that the OSC is satisfied. Let $t$ denote the common value of the box dimensions and the Hausdorff dimension of $K$, i.e. $t$ is the unique number such that $\sum r_i = 1$.

1. We have

$$S_{K,r}(E) = S_{K,r}(E) = \frac{1}{r^{-t+d}} \mathcal{L}^d(E \cap B(K,r)).$$

Write $S_{K,r}$ for the common value of $S_{K,r}$ and $S_{K,r}$, i.e. write

$$S_{K,r}(E) = \frac{1}{r^{-t+d}} \mathcal{L}^d(E \cap B(K,r)).$$

Also, define the average measure $S_{K,r,ave}$ by

$$S_{K,r,ave}(E) = \frac{1}{-\log r} \int_r^1 \frac{1}{s^{-t+d}} \mathcal{L}^d(E \cap B(K,s)) \frac{ds}{s}.$$

Then the following holds.

1. The non-arithmetic case: If the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then

$$S_{K,r} \to M^t(K) \frac{1}{H^t(K)} H^t \mu^t K \quad \text{weakly},$$

$$S_{K,r,ave} \to M^t_{ave}(K) \frac{1}{H^t(K)} H^t \mu^t K \quad \text{weakly}.$$
Recall, that $K$ is $t$ Minkowski measurable and $t$ average Minkowski measurable by Corollary 3.4 and the Minkowski content $M^t(K)$ and the average Minkowski content $M^t_{\text{ave}}(K)$ are therefore well-defined.

(3) The arithmetic case: If the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is contained in a discrete additive subgroup of $\mathbb{R}$ then

$$S_{K,r,\text{ave}} \to M^t_{\text{ave}}(K) \frac{1}{H^t(K)} H^t \bigcup K \quad \text{weakly;}$$

recall, that $K$ is $t$ average Minkowski measurable by Corollary 3.4 and the average multifractal Minkowski content $M^t_{\text{ave}}(K)$ is therefore well-defined.

\textbf{Proof}

Recall, that $K$ is given by (3.7), and let $\mu$ be given by (3.8). Since $S_{\mu,t}^0 = S_{\mu,t}^0 = S_{K,r}$, the statement now follows from Theorem 4.2 by putting $q = 0$. \hfill \Box

In Section 4.1 it was suggested that the limiting behaviour of the multifractal tube measures $V^t_{\mu,t}(K)$ may be viewed as providing a local version of Theorem 3.3. Indeed, Theorem 3.3 clearly describes the limiting behaviour of $\frac{1}{\log r} V^t_{\mu,t}(K)$ as $r \to 0$ whereas the main results in Section 4.1 equally clearly provide information about the the limiting behaviour of $\frac{1}{\log r} V^t_{\mu,t}(E)$ as $r \to 0$ for “well-behaved” subsets $E$ of $K$. The viewpoint is made even more explicit (and precise) in the next corollary. Below we use the following notation, namely, if $E$ is a subset of $\mathbb{R}^d$, then we denote the boundary of $E$ by $\partial E$.

\textbf{Corollary 4.5.} Let $K$ and $\mu$ be given by (3.7) and (3.8). Fix $q \in \mathbb{R}$ and assume that one of the following conditions is satisfied:

(i) The OSC is satisfied and $0 \leq q$;

(ii) The SSC is satisfied.

Let $E \subseteq \mathbb{R}^d$ be a Borel set with:

\begin{itemize}
  \item[(a)] $H^t_{\mu} \beta(q) (E \cap K) > 0$.
  \item[(b)] $H^t_{\mu} \beta(q) (\partial E \cap K) = 0$.
  \item[(c)] $E \cap B(K, r) = B(E \cap K, r)$ for $r$ small enough.
\end{itemize}

(Observe that, for example, the set $E = \mathbb{R}^d$ satisfies the above conditions, and if $K = L \cup M$ with $\text{dist}(L, M) > 0$ and $H^t_{\mu} \beta(q)(L) > 0$ and $0 < \delta < \text{dist}(L, M)$, then the set $E = B(L, \delta)$ satisfies the above conditions.)

Then the following holds.

(1) The non-arithmetic case: If the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then $E \cap K$ is $(q, \beta(q))$ multifractal Minkowski measurable with respect to $\mu$ with

$$M^q_{\mu} \beta(q) (E \cap K) = M^q_{\mu} \beta(q) (K) \frac{H^q_{\mu} \beta(q) (E \cap K)}{H^q_{\mu} \beta(q) (K)} ;$$

recall, that $K$ is $(q, \beta(q))$ average multifractal Minkowski measurable with respect to $\mu$ by Theorem 3.3 and the multifractal Minkowski content $M^q_{\mu} \beta(q)(K)$ is therefore well-defined.

(2) The arithmetic case: If the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is contained in a discrete additive subgroup of $\mathbb{R}$ then $E \cap K$ is $(q, \beta(q))$ average multifractal Minkowski measurable with respect to $\mu$ with

$$M^q_{\mu, \text{ave}} \beta(q) (E \cap K) = M^q_{\mu, \text{ave}} \beta(q) (K) \frac{H^q_{\mu} \beta(q) (E \cap K)}{H^q_{\mu} \beta(q) (K)} ;$$
recall, that $K$ is $(q, \beta(q))$ average multifractal Minkowski measurable with respect to $\mu$ by Theorem 3.3 and the average multifractal Minkowski content $M_{\mu, \text{ave}}^{q, \beta(q)}(K)$ is therefore well-defined.

**Proof**

This follows immediately from Theorem 4.2 since the condition $E \cap B(K, r) = B(E \cap K, r)$, implies that $T_{\mu,r}^q(E) = \frac{1}{r^d} \int_{E \cap B(K, r)} \mu(B(x,r))^q \, d\mathcal{L}^d(x) = \frac{1}{r^d} \int_{B(E \cap K, r)} \mu(B(x,r))^q \, d\mathcal{L}^d(x) = V_{\mu,r}^q(E \cap K)$. □

Note that Corollary 4.5 is a genuine extension of Theorem 3.3: namely, if we let $E = K$ in Corollary 4.5, then Corollary 4.5 simplifies to Theorem 3.3.

### 5. Symbolic multifractal tubes of self-similar measures: Multifractal zeta-functions and explicit formulas

Throughout this section we will let $K$ and $\mu$ denote the self-similar set and the self-similar measure given by (3.7) and (3.8), respectively. While Theorem 3.3 provides complete information about the asymptotic behaviour of the multifractal Minkowski volume $V_{\mu,r}^q(K)$ of $K$, it does not provide “explicit” formulas for the multifractal Minkowski content $M_{\mu, \text{ave}}^{q, \beta(q)}(K)$. Indeed, the formulas in Theorem 3.3 for multifractal Minkowski content of $K$ involve the integral of an auxiliary function $\lambda_q$. Even in very simple cases it is highly unlikely that this integral can be computed explicitly. This is clearly unsatisfactory and it would be desirable if more explicit expressions could be found. In fact, even in the fractal case, the problem of finding explicit formulas for the Minkowski content is highly non-trivial. However, despite, or perhaps in spite, of the difficulties, this problem has recently attracted considerable interest. In particular, Lapidus and collaborators [LapPea1, LapPea2, LapPeaWi, Lap-vF1, Lap-vF2] have during the past 20 years and with spectacular success pioneered the use of zeta-functions to obtain explicit formulas for the Minkowski content of self-similar subsets of the line and certain self-similar sets in higher dimensions. It would clearly be desirable if analogous formulas for the multifractal Minkowski content could be found. However, the significant difficulties encountered by Lapidus and collaborators when computing the Minkowski content of self-similar subsets of the line suggests that this problem is exceptionally difficult. For this reason we introduce “symbolic” multifractal Minkowski volumes. The “symbolic” multifractal Minkowski volumes are defined in such a way that they are “compatible” with the usual Minkowski volumes (see Theorem 5.1 below for a precise formulation of this) and such that the zeta-function technique can be applied to give explicit formulas for the corresponding “symbolic” multifractal Minkowski content. A multifractal zeta-function is a meromorphic function whose residues are closely related to the asymptotic behaviour of the “symbolic” multifractal Minkowski volume. Namely, using the residue theorem it is possible to relate the “symbolic” multifractal Minkowski volume to the residues of the zeta-function, and a careful analysis of the residues will then provide explicit formulas for the “symbolic” multifractal Minkowski volume. The idea of using zeta-functions in order to obtain explicit formulas for the “symbolic” multifractal Minkowski content has classical origins. For example, the “standard” proofs of the Prime Number Theorem is based on applying this technique to the Riemann zeta-function, see [Ed,Pat]. The zeta-function technique for is not only restricted to problems in number theory, but has also been used successfully to obtain explicit formulas for “counting functions” in many other areas in mathematics. For example, in dynamical systems, Parry &
Pollicott [ParPo1, ParPo2] obtained asymptotic formulas for the number of closed geodesics whose length is less than \(x\) as \(x \to \infty\) by applying this technique to Ruelle’s zeta-function for Axiom A flows. For other applications of this technique in dynamical systems the reader is referred to Ruelle’s text [Rue2].

5.1. Symbolic multifractal tubes of self-similar measures. We will now define the symbolic multifractal Minkowski volume. We first introduce the following notation. Let \(\Sigma = \{1, \ldots, N\}\) and write

\[
\Sigma^m = \{1, \ldots, N\}^m, \\
\Sigma^* = \bigcup_m \Sigma^m.
\]  

(5.1)
i.e. \(\Sigma^m\) is the family of all strings \(i = i_1 \ldots i_m\) of length \(m\) with \(i_j \in \{1, \ldots, N\}\) and \(\Sigma^*\) is the family of all finite strings \(i = i_1 \ldots i_m\) with \(i_j \in \{1, \ldots, N\}\). Also, for \(i = i_1 \ldots i_m \in \Sigma^*\), we will write

\[
\check{i} = i_1 \ldots i_{m-1}.
\]

Next, for brevity, put

\[
r_{\min} = \min_{i=1, \ldots, N} r_i, \quad r_{\max} = \max_{i=1, \ldots, N} r_i.
\]

(5.2)

Finally, if \(i = i_1 \ldots i_m \in \Sigma^*\), then we will write \(\check{i}\) for the “parent” of \(i\), i.e. we will write

\[
\check{i} = i_1 \ldots i_{m-1}.
\]

We can now define the symbolic multifractal Minkowski volume. We provide several comments discussing the motivation behind the definition of the symbolic multifractal Minkowski volume immediately after the statement of the definition.

**Definition.** Symbolic multifractal Minkowski volume. Fix \(q \in \mathbb{R}\) and \(l = 0, 1, \ldots, d\). For brevity write

\[
\sigma_{q,l} = \sum_{i=1}^N p_i^q r_i^{l-dq},
\]

(5.3)

and let

\[
C_{\mu,r}^{q,l,\text{sym}}(K) = \sum_{r_i < r < r_{i-1}} p_i^q r_i^{l-dq} + \frac{1+\sigma_{q,l}}{2} \sum_{r=r_{i-1}} p_i^q r_i^{l-dq}.
\]

(5.4)

Let \(\kappa_{\mu}^{q,0}(K), \kappa_{\mu}^{q,1}(K), \ldots, \kappa_{\mu}^{q,d}(K)\) be real numbers satisfying the following consistency condition

\[
\sum_l \kappa_{\mu}^{q,l}(K) (\sigma_{q,l} - 1) = 0.
\]

(5.5)

For \(r > 0\), we define the symbolic \(q\) multifractal Minkowski volume \(V_{\mu,r}^{q,\text{sym}}(K)\) of \(K\) with respect to the measure \(\mu\) by

\[
V_{\mu,r}^{q,\text{sym}}(K) = \frac{1}{r^d} \sum_l \kappa_{\mu}^{q,l}(K) C_{\mu,r}^{q,l,\text{sym}}(K) r^{(d-l)+d} q
\]

\[
= \sum_l \kappa_{\mu}^{q,l}(K) C_{\mu,r}^{q,l,\text{sym}}(K) r^{-l+d} q.
\]

(5.6)
Comment. Motivating the definition of $V_{q,sym}^{q,sym}(K)$. We will now make a number of comments explaining the motivation behind the definition of the symbolic $q$ multifractal Minkowski volume $V_{q,sym}(K)$.

1) Motivating definition (5.6): It is clear that definition (5.6) is motivated by Steiner’s formula (1.3): the quantity

$$C_{(\mu,r)}^{q,l,sym}(K) = \sum_{i=1}^{\infty} \sum_{r_{i-1} < r < r_i} p_i^{q,r-dq} + \frac{1}{2} \sum_{i} p_i^{q,r-dq}$$

clearly corresponds to the term $r^{d-l}$ in Steiner’s formula, and the quantities

$$\kappa_{q,0}(K), \kappa_{q,1}(K), \ldots, \kappa_{q,d}(K)$$
correspond to the Quermassintegrale $\kappa_{0}(C), \kappa_{1}(C), \ldots, \kappa_{d}(C)$ in Steiner’s formula.

2) Motivating consistency condition (5.5): Consistency condition (5.5) is motivated by the following argument. If $C$ is a bounded convex subset of $\mathbb{R}^d$ with smooth boundary $\partial C$, then it follows from Weyl’s extension of Steiner’s formula (to compact oriented $d$-dimensional Riemannian manifolds isometrically embedded into Euclidean space) applied to $\partial C$ that there are constants $k_{q,0}(C), k_{q,1}(C), \ldots, k_{q,d}(C)$ such that the volume of $B(\mathbb{R}^d \setminus C, r) \cap C = B(\partial C, r) \cap C$ is given by

$$\mathcal{L}^d(B(\mathbb{R}^d \setminus C, r) \cap C) = \sum_{l} k_{q,l}(C) r^{d-l}$$

(5.7)

for all $r$ less than the inner radius $r_{inner}$ of $C$. Trivially, we also have

$$\mathcal{L}^d(B(\mathbb{R}^d \setminus C, r) \cap C) = \mathcal{L}^d(C)$$

(5.8)

for all $r$ greater than the inner radius $r_{inner}$ of $C$. Since the volume $\mathcal{L}^d(B(\mathbb{R}^d \setminus C, r) \cap C)$ is a continuous function of $r$, it now follows from (5.7) and (5.8) that the constants $k_{q,0}(C), k_{q,1}(C), \ldots, k_{q,d}(C)$ must satisfy the following consistency condition, namely, $\sum_{l<d} k_{q,l}(C) r_{inner}^{d-l} = \mathcal{L}^d(C)$. Writing $k_{q,d}(C) = -\mathcal{L}^d(C)$, this condition can be rewritten as

$$\sum_{l<d} k_{q,l}(C) r_{inner}^{d-l} = -k_{q,d}(C) r_{inner}^{d-d}$$

i.e.

$$\sum_{l} k_{q,l}(C) r_{inner}^{d-l} = 0.$$  

(5.9)

However, typically the set $K$ has zero $d$-dimensional volume, and the symbolic multifractal Minkowski volume $V_{q,sym}^{q,sym}(K)$ of $B(K, r)$ can therefore also be thought of as being equal to the symbolic multifractal Minkowski volume of $B(K, r) \cap (\mathbb{R}^d \setminus K)$. Comparison with (5.9) therefore suggests that the coefficients $\kappa_{q,0}(K), \kappa_{q,1}(K), \ldots, \kappa_{q,d}(K)$ must satisfy the following consistency condition, namely,

$$\sum_{l} \kappa_{q,l}(K) (\sigma_{q,l} - 1) = 0.$$  

(5.10)

While consistency condition (5.5) is motivated by the informal discussion above, we note that it, nevertheless, plays a crucial role in the proofs in Section 16.
Motivating the weight factor $\frac{1+\frac{1}{q}}{2}$ in (5.4): Recall definition (5.4) of $C^q_{\mu,r}(K)$, namely,

$$C^q_{\mu,r}(K) = \sum_{r_i < r < r_i} p_i^q r_i^{1-dq} + \frac{1+\frac{1}{q}}{2} \sum_{r = r_i}^{r} p_i^q r_i^{1-dq}.$$ 

In this definition, the “boundary” term $\sum_{r_i < r < r_i} p_i^q r_i^{1-dq}$ has been weighted by the factor $\frac{1+\frac{1}{q}}{2}$. The motivation behind this is the following. Namely, below we intend to apply the Mellin transform to the function $r \to C^q_{\mu,r}(K)$. However, the Mellin transform only applies to piecewise continuous functions $f$ for which $\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x) = f(x_0)$ at all discontinuity points $x_0$. Weighting the “boundary” term $\sum_{r_i < r < r_i} p_i^q r_i^{1-dq}$ by the factor $\frac{1+\frac{1}{q}}{2}$ ensures that the function $r \to C^q_{\mu,r}(K)$ satisfies this condition. A similar practice is also commonly used in number theory where analogous “boundary” terms are weighted by $\frac{1}{2}$. Borrowing terminology from physics where the parameter $q$ is interpreted as the inverse temperature of the physical system associated with $\mu$ (see, for example, [BaPo, pp. 128–132; BeSc, pp. 114–126; Ot, pp. 309–910]), we may therefore, somewhat poetically, say that the factor $\frac{1+\frac{1}{q}}{2}$ represents the usual weight factor $\frac{1}{2}$ “raised to the temperature $\frac{1}{q}$.”

Comment. Comparing $V^q_{\mu,r}(K)$ and $V^q_{\mu,r}^{\text{sym}}(K)$. The definition of the symbolic multifractal Minkowski volume may be viewed as a natural multifractal analogue of the usual multifractal Minkowski volume $V^q_{\mu,r}(K)$ given by

$$V^q_{\mu,r}(K) = \frac{1}{r^d} \int_{B(K,r)} \mu(B(x,r))^q \, d\mathcal{L}^d(x).$$

Indeed, even though the symbolic $q$ multifractal Minkowski volume $V^q_{\mu,r}^{\text{sym}}(K)$ does not necessarily equal the usual $q$ multifractal Minkowski volume $V^q_{\mu,r}(K)$, it is nevertheless “compatible” with $V^q_{\mu,r}(K)$. More precisely, the usual $q$ multifractal Minkowski volume and the symbolic $q$ multifractal Minkowski volume give rise to the same dimensions. This is the content of Theorem 5.1. While this is not a deep fact it may be seen as providing further justification for the study the symbolic $q$ multifractal Minkowski volume.

Theorem 5.1. Let $q \in \mathbb{R}$. Recall that we define the lower and upper multifractal $q$ Minkowski dimension of $K$ by

$$\dim^q_{M,\mu}(K) = \liminf_{r \searrow 0} \frac{\log V^q_{\mu,r}(K)}{-\log r},$$

$$\overline{\dim}^q_{M,\mu}(K) = \limsup_{r \searrow 0} \frac{\log V^q_{\mu,r}(K)}{-\log r}.$$ 

Similarly, we define the symbolic lower and upper multifractal $q$ Minkowski dimension of $K$ by

$$\dim_{M,\mu}^q(K) = \liminf_{r \searrow 0} \frac{\log V^q_{\mu,r}(K)}{-\log r},$$

$$\overline{\dim}_{M,\mu}^q(K) = \limsup_{r \searrow 0} \frac{\log V^q_{\mu,r}(K)}{-\log r}.$$
Assume that one of the following conditions is satisfied:

(i) The OSC is satisfied and \(0 \leq q\);

(ii) The SSC is satisfied.

Then we have

\[
\dim_{M,\mu}^q(K) = \dim_{M,\mu}^{q,\text{sym}}(K),
\]

\[
\dim_{M,\mu}^d(K) = \dim_{M,\mu}^{q,\text{sym}}(K).
\]

**Proof**

As noted above, this is not a deep fact and follows from the definitions using standard arguments. \(\square\)

In analogy with the usual multifractal Minkowski content (see (3.3)–(3.6)) we also define symbolic multifractal Minkowski content. For real numbers \(q\) and \(t\), we define the lower and upper \((q, t)\)-dimensional symbolic multifractal Minkowski content of \(K\) with respect to \(\mu\) by

\[
M_{\mu}^{q,t,\text{sym}}(K) = \liminf_{r \searrow 0} \frac{1}{r^{-t}} V_{\mu,r}^{q,\text{sym}}(K),
\]

\[
M_{\mu}^{q,t,\text{sym}}(K) = \limsup_{r \searrow 0} \frac{1}{r^{-t}} V_{\mu,r}^{q,\text{sym}}(K).
\]

If \(M_{\mu}^{q,t,\text{sym}}(K) = M_{\mu}^{q,t,\text{sym}}(K)\), i.e. if the limit \(\lim_{r \searrow 0} \frac{1}{r^{-t}} V_{\mu,r}^{q,\text{sym}}(K)\) exists, then we say that \(K\) is \((q, t)\) symbolic multifractal Minkowski measurable with respect to \(\mu\), and we denote the common value of \(M_{\mu}^{q,t,\text{sym}}(K)\) and \(M_{\mu}^{q,t,\text{sym}}(K)\) by \(M_{\mu}^{q,t,\text{sym}}(K)\), i.e. we write

\[
M_{\mu}^{q,t,\text{sym}}(K) = M_{\mu}^{q,t,\text{sym}}(K) = M_{\mu}^{q,t,\text{sym}}(K).
\]

Of course, \(K\) may not be symbolic multifractal Minkowski measurable, and it is therefore useful to introduce a suitable averaging procedure when computing the symbolic multifractal Minkowski content. Motivated by this we define the lower and upper \((q, t)\)-dimensional symbolic average multifractal Minkowski content of \(K\) with respect to \(\mu\) by

\[
M_{\mu,\text{ave}}^{q,t,\text{sym}}(K) = \liminf_{r \searrow 0} \frac{1}{-\log r} \int_r^1 \frac{1}{s^{-t}} V_{\mu,s}^{q,\text{sym}}(K) \frac{ds}{s},
\]

\[
M_{\mu,\text{ave}}^{q,t,\text{sym}}(K) = \liminf_{r \searrow 0} \frac{1}{-\log r} \int_r^1 \frac{1}{s^{-t}} V_{\mu,s}^{q,\text{sym}}(K) \frac{ds}{s}.
\]

If \(M_{\mu,\text{ave}}^{q,t,\text{sym}}(K) = M_{\mu,\text{ave}}^{q,t,\text{sym}}(K)\), i.e. if the limit \(\lim_{r \searrow 0} \frac{1}{-\log r} \int_r^1 \frac{1}{s^{-t}} V_{\mu,s}^{q,\text{sym}}(K) \frac{ds}{s}\) exists, then we say the \(K\) is \((q, t)\) averagely symbolic multifractal Minkowski measurable with respect to \(\mu\), and we denote the common value of \(M_{\mu,\text{ave}}^{q,t,\text{sym}}(K)\) and \(M_{\mu,\text{ave}}^{q,t,\text{sym}}(K)\) by \(M_{\mu,\text{ave}}^{q,t,\text{sym}}(K)\), i.e. we write

\[
M_{\mu,\text{ave}}^{q,t,\text{sym}}(K) = M_{\mu,\text{ave}}^{q,t,\text{sym}}(K) = M_{\mu,\text{ave}}^{q,t,\text{sym}}(K).
\]

How does one obtain explicit expressions for \(M_{\mu,\text{ave}}^{q,t,\text{sym}}(K)\) and/or \(M_{\mu,\text{ave}}^{q,t,\text{sym}}(K)\)? The main tool for this is the notion of a multifractal zeta-function. A multifractal zeta-function is a certain meromorphic function whose residues are closely related to the asymptotic behaviour of \(V_{\mu,r}^{q,\text{sym}}(K)\) as \(r \searrow 0\). In the next section we define multifractal zeta-functions, and the subsequent sections explain how multifractal zeta-functions can be used to analyse the symbolic multifractal Minkowski volume.
5.2. Multifractal zeta-functions of self-similar measures – a tool for finding explicit formulas for the symbolic multifractal Minkowski volume of self-similar measures. Informally, the multifractal zeta-function $\zeta^q_\mu$ is defined by

$$\zeta^q_\mu(s) = \sum_i p_i^q r_i^s$$

for those complex numbers $s$ for which the series $\sum_i p_i^q r_i^s$ converges. Formally, we proceed as described in the definition below. In the definition below and in the later sections of the paper we use the following notation, namely, if $f : \mathbb{C} \to \mathbb{C}$ is a complex valued function on $\mathbb{C}$, then we write $Z(f)$ for the zeros of $f$, i.e. we write

$$Z(f) = \left\{ s \in \mathbb{C} \mid f(s) = 0 \right\}.$$

**Definition.** The multifractal zeta-function $\zeta^q_\mu$ and the modified multifractal zeta-function $Z^q_\mu$. Fix $q \in \mathbb{R}$. For $s \in \mathbb{C}$ with $\text{Re } s > \beta(q)$, the series $\sum_i p_i^q r_i^s$ is convergent with

$$\sum_i p_i^q r_i^s = \frac{\sum_i p_i^q r_i^s}{1 - \sum_i p_i^q r_i^s}. \quad (5.15)$$

We can therefore define the multifractal zeta-function $\zeta^q_\mu$ by

$$\zeta^q_\mu(s) = \sum_i p_i^q r_i^s \quad \text{for } s \in \left\{ w \in \mathbb{C} \mid \text{Re } s > \beta(q) \right\}. \quad (5.16)$$

It follows from (5.15) that $\zeta^q_\mu$ can be extended to $\mathbb{C} \setminus Z(w \to 1 - \sum_i p_i^q r_i^w)$ by

$$\zeta^q_\mu(s) = \frac{\sum_i p_i^q r_i^s}{1 - \sum_i p_i^q r_i^s} \quad \text{for } s \in \mathbb{C} \setminus Z \left( w \to 1 - \sum_i p_i^q r_i^w \right). \quad (5.17)$$

We define the modified zeta-function $Z^q_\mu$ by

$$Z^q_\mu(s) = \left( \sum_t \frac{\kappa^{q,t}(K) (\sigma_{ql} - 1)}{s - (1 - dq)} \right) \zeta^q_\mu(s) \quad \text{for } s \in \mathbb{C} \setminus Z \left( w \to 1 - \sum_i p_i^q r_i^w \right). \quad (5.18)$$

**Proof**

For $s \in \mathbb{C}$ with $\text{Re } s > \beta(q)$, we have $|\sum_i p_i^q r_i^s| \leq \sum_i p_i^q |r_i^s| = \sum_i p_i^q r_i^{\text{Re } s} < 1$, and the series $\sum_i p_i^q r_i^s = \sum_n \sum_{|i|=n} p_i^{|q|} r_i^{|s|} = \sum_n (\sum_i p_i^{|q|} r_i^{|s|})^n$ is therefore convergent with $\sum_i p_i^q r_i^s = \frac{\sum_i p_i^q r_i^s}{1 - \sum_i p_i^q r_i^s}$. \(\square\)

Zeta-functions similar to $\zeta^q_\mu$ have appear frequently in the study of dynamical systems, see, for example, [Lap-vF2,ParPo1,ParPo2,Rue1,Rue2] and the references therein. In addition, we note that Lapidus and collaborators have introduced various intriguing multifractal zeta-functions [LapRo,LapLe-Ver]. However, the multifractal zeta-functions in [LapRo,LapLe-Ver] serve very different purposes and are significantly different from the multifractal zeta-function $\zeta^q_\mu$ introduced above. Indeed, our motivation for introducing the function $\zeta^q_\mu$ is that explicit formulas for $V^q_{\mu,r}^{\text{sym}}(K)$ involving the residues of $\zeta^q_\mu$ can be obtained by first use the Mellin transform to write $V^q_{\mu,r}^{\text{sym}}(K)$ as a complex contour integral of $\zeta^q_\mu$ and then
use the residue theorem to express this integral as a sum involving the residues of $\zeta^q_P$. As with all applications of the residue theorem, this requires information about the poles and residues of $\zeta^q_P$. For this reason, the next section lists some of the main properties of the poles and residues of $\zeta^q_P$.

5.3. An intermezzo: the poles and residues of $\zeta^q_P$ and the sequence $(t_{q,n})_n$. For $q \in \mathbb{R}$, define $\alpha(q)$ by

$$\alpha(q) = \inf \left\{ t \in \mathbb{R} \mid \sum_{r_i = r_{\text{min}}} p_i^q r_i^t \leq 1 + \sum_{r_i > r_{\text{min}}} p_i^q r_i^t \right\}. \tag{5.19}$$

Also, recall that $\beta(q)$ is defined by

$$\sum_i p_i^q r_i^{\beta(q)} = 1.$$

Using the numbers $\alpha(q)$ and $\beta(q)$ we can now describe the location of the poles of $\zeta^q_P$. Recall, that if $f$ is a meromorphic function, then $Z(f)$ denotes the set of zeros of $f$, i.e.

$$Z(f) = \left\{ s \in \mathbb{C} \mid f(s) = 0 \right\};$$

in addition, we let $P(f)$ denotes the set of poles of $f$, i.e.

$$P(f) = \left\{ s \in \mathbb{C} \mid s \text{ is a pole of } f \right\}.$$

Proposition 5.2. The poles of $\zeta^q_P$. Fix $q \in \mathbb{R}$.

1. We have $-\infty < \alpha(q) \leq \beta(q) < \infty$.
2. We have

$$P(\zeta^q_P) = Z \left( s \to 1 - \sum_i p_i^q r_i^s \right).$$

3. We have

$$P(\zeta^q_P) \subseteq \left\{ s \in \mathbb{C} \mid \alpha(q) \leq \text{Re}(s) \leq \beta(q) \right\}.$$

4.1 Poles $\omega$ with Re$(\omega) = \beta(q)$ in the non-arithmetic case: If the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then

$$P(\zeta^q_P) \cap \left\{ s \in \mathbb{C} \mid \text{Re}(s) = \beta(q) \right\} = \{ \beta(q) \}.$$

4.2 Poles $\omega$ with Re$(\omega) = \beta(q)$ in the arithmetic case: If the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is contained in a discrete additive subgroup of $\mathbb{R}$ and $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\} = u\mathbb{Z}$ with $u > 0$, then

$$P(\zeta^q_P) \cap \left\{ s \in \mathbb{C} \mid \text{Re}(s) = \beta(q) \right\} = \beta(q) + \frac{2\pi}{u} i \mathbb{Z},$$

and for each $i$, there is a unique integer $k_i$ such that $\log r_i^{-1} = k_i u$ and, in addition,

$$P(\zeta^q_P) = \left( \beta(q) + \frac{2\pi}{u} i \mathbb{Z} \right) \cup \bigcup_{w \in Z(\sum_{i \neq \beta(q)} p_i^q r_i^{k_i})} \left( -\log \frac{|w|}{u} - \frac{\text{Arg} w}{u} i + \frac{2\pi}{u} i \mathbb{Z} \right).$$
(where \( \text{Arg} \, z \) denotes the principal argument of \( z \in \mathbb{C} \).

5. Density of poles of \( \zeta_q^\mu \): Writing \( \gamma = -\frac{1}{2} \log r_{\text{min}} \), then we have

\[
\{ \omega \in P(\zeta_q^\mu) \mid | \text{Im}(\omega) | \leq t \} = \gamma t + \mathcal{O}(\log t).
\]

Proposition 5.2 play an important role in the proofs of the main results in this section and is proved in Section 13: statements (1)–(4.2) are proved in Proposition 13.1 and the density statement (5) is proved in Theorem 13.8. Statements (1)–(4.2) are not deep and the proofs of those statements are straightforward. However, the density statement (5) requires a more involved and careful proof based on the Argument Principle (see [Con, p. 123]) together with various variants of Jensen’s formula from complex analysis (see [Con, p. 280]), namely Proposition 13.6 and Proposition 13.7. We note that the intriguing books by Lapidus & van Frankenhuysen [Lap-vF1,Lap-vF2] prove related density results. Indeed, in the first editor of their book [Lap-vF1], Lapidus & van Frankenhuysen prove a density result similar to (5) for the poles of a zeta-function related to \( \zeta \) in the first editor of their book [Lap-vF1], Lapidus & van Frankenhuysen prove a density result similar to (5) for the poles of a zeta-function related to \( \zeta_q^\mu \) involving an error term of the form \( \mathcal{O}(\sqrt{t}) \). However, in the second editor of their book [Lap-vF2], Lapidus & van Frankenhuysen present a density result for a larger class of zeta-functions with an improved error term of the form \( \mathcal{O}(1) \). While we have not been able to prove the density statement (5) with an error term of the form \( \mathcal{O}(1) \), we note that the error term in (5), namely \( \mathcal{O}(\log t) \), in consistent with the results in [JoLaGo, Theorem 6.2 and pp. 58–59] where related results (also involving error terms of the form \( \mathcal{O}(\log t) \)) are considered in very general and abstract settings.

It follows from Proposition 5.2 that there is a critical strip

\[
\mathcal{S}_q^{\text{crit}} = \left\{ s \in \mathbb{C} \mid \alpha(q) \leq \text{Re}(s) \leq \beta(q) \right\}
\]

and a critical line

\[
\mathcal{L}_q^{\text{crit}} = \left\{ s \in \mathbb{C} \mid \text{Re}(s) = \beta(q) \right\}
\]

such that all poles of \( \zeta_q^\mu \) belong to \( \mathcal{S}_q^{\text{crit}} \) and lie to the left of the line \( \mathcal{L}_q^{\text{crit}} \). We also observe that the nature of the poles on the critical line is determined by the algebraic properties of \( \zeta_q^\mu \) the logarithms of \( r_1, \ldots, r_N \).

Below we first use the Mellin transform to obtain an explicit formula for \( V_q,\text{sym}^{\mu,r}(K) \), namely Theorem 5.4, expressing \( V_q,\text{sym}^{\mu,r}(K) \) as a complex contour integral of \( \zeta_q^\mu \). Next, we use the residue theorem to “compute” the contour integral from Theorem 5.4 thus obtaining the second explicit formula for \( V_q,\text{sym}^{\mu,r}(K) \), namely Theorem 5.5, expressing \( V_q,\text{sym}^{\mu,r}(K) \) as a sum of residues of \( \zeta_q^\mu \). However, since all the all the poles of \( \zeta_q^\mu \) lie in the critical strip \( \mathcal{S}_q^{\text{crit}} \), any application of the residue theorem to \( \zeta_q^\mu \) is likely to involve integrating \( \zeta_q^\mu \) over line segments crossing the critical strip \( \mathcal{S}_q^{\text{crit}} \). For this reason precise information about the poles and the growth of \( \zeta_q^\mu \) on line segments that cross the critical strip \( \mathcal{S}_q^{\text{crit}} \) is needed. Such estimates are provided by the next result, i.e. Theorem 5.3 below. This result says that there is a sequence of horizontal line segments crossing the critical strip \( \mathcal{S}_q^{\text{crit}} \) without hitting the poles of \( \zeta_q^\mu \) and such that \( \zeta_q^\mu \) is uniformly bounded on these line segments.

**Theorem 5.3.** Growth estimates of \( \zeta_q^\mu \) inside the critical strip \( \alpha(q) \leq \text{Re}(s) \leq \beta(q) \) and the sequence \( (t_{q,n})_n \). Fix \( q \in \mathbb{R} \). Then there is an increasing sequence \( (t_{q,n})_n \) of positive real numbers with \( t_{q,n} \to \infty \) satisfying the following: for all real numbers \( c \), there is a constant \( k_c \) such that for all \( \sigma \leq c \) and all \( n \), we have

\[
|\zeta_q^\mu(\sigma \pm it_{q,n})| \leq k_c.
\]
Theorem 5.3 is proven in Section 13. The statement in Theorem 5.3 is deep and relies on a number of very delicate estimates. The reader is also referred to [JoLaGo] for related results in very general and abstract settings and to [Lap-vF2] for somewhat related results for $q = 0$.

5.4. Symbolic multifractal tubes of self-similar measures: the first explicit formula. Using the Mellin transform, we first obtain an explicit formula for $V_{\mu,r}^q,\text{sym}(K)$ expressing $V_{\mu,r}^q,\text{sym}(K)$ as a complex contour integral of $\zeta_{\mu}^q$. Schematically this part can be represented as follows:

The Mellin transform

$V_{\mu,r}^q,\text{sym}(K)$ equals a complex contour integral of $\zeta_{\mu}^q$

More precisely, using the Mellin transform technique, we obtain the first explicit formula for $V_{\mu,r}^q,\text{sym}(K)$, namely Theorem 5.4 below, expressing $V_{\mu,r}^q,\text{sym}(K)$ as a complex contour integral of the zeta-function $\zeta_{\mu}^q$.

Theorem 5.2. The first explicit formula for $V_{\mu,r}^q,\text{sym}(K)$. Fix $q \in \mathbb{R}$. For $c > \max\left(-dq, 1-dq, \ldots, d-dq, \beta(q)\right)$ and $0 < r < r_{\min}$, we have

$$V_{\mu,r}^q,\text{sym}(K) = \sum_l \kappa_{\mu,l}^q(K) \sigma_{q,l} r^{-l+dq} + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z_{\mu}^q(s) r^{-s} ds.$$ 

The proof of Theorem 5.4 is given in Section 14.

5.5. Symbolic multifractal tubes of self-similar measures: the second explicit formula. Next, using the first explicit formula for $V_{\mu,r}^q,\text{sym}(K)$, namely Theorem 5.4 below, expressing $V_{\mu,r}^q,\text{sym}(K)$ as a complex contour integral of the zeta-function $\zeta_{\mu}^q$, and the residue theorem, we obtain an explicit formula for $V_{\mu,r}^q,\text{sym}(K)$, namely Theorem 5.5 below, expressing $V_{\mu,r}^q,\text{sym}(K)$ as a sum of residues of $\zeta_{\mu}^q$. Schematically this part can be represented as follows:

The residue theorem

$V_{\mu,r}^q,\text{sym}(K)$ equals a sum of residues of $\zeta_{\mu}^q$

More precisely, we now use the first explicit formula for $V_{\mu,r}^q,\text{sym}(K)$, i.e. Theorem 5.4, and the residue theorem together with the growth estimate provided by Theorem 5.3 to obtain the second explicit formula for $V_{\mu,r}^q,\text{sym}(K)$, namely Theorem 5.5 below, expressing $V_{\mu,r}^q,\text{sym}(K)$ as the sum of a series involving the residues of $\zeta_{\mu}^q$.

Theorem 5.5. The second explicit formula for $V_{\mu,r}^q,\text{sym}(K)$: a multifractal Steiner formula. Fix $q \in \mathbb{R}$. Assume that $\beta(q) \notin \{-dq, 1-dq, \ldots, d-dq\}$. Let $(t_{q,n})_n$ be the sequence from Theorem 5.3. For all $0 < r < r_{\min}$, we have

$$V_{\mu,r}^q,\text{sym}(K) = \lim_n \sum_{\omega \in P(\zeta_{\mu}^q)} \text{res}\left( s \rightarrow Z_{\mu}^q(s) r^{-s}; \omega \right).$$
The proof of Theorem 5.5 is given in Section 15. It is clear that Theorem 5.5 may be viewed as a multifractal Steiner formula for the symbolic multifractal Minkowski volume of $V_{\mu,r}^{q,\text{sym}}(K)$. This viewpoint is perhaps even more clear in the following special case.

**Corollary 5.6.** The second explicit formula for $V_{\mu,r}^{q,\text{sym}}(K)$: a multifractal Steiner formula. Fix $q \in \mathbb{R}$. Assume that $\beta(q) \notin \{-dq, 1-dq, \ldots, d-dq\}$ and that all the poles of $\zeta_{\mu}^q$ are simple. Let $(t_{q,n})_n$ be the sequence from Theorem 5.3. For all $0 < r < r_{\min}$, we have

$$V_{\mu,r}^{q,\text{sym}}(K) = \lim_{n} \sum_{\omega \in P(\zeta_{\mu}^q)} \left( \sum_{l} \frac{\kappa_{\mu,l}(K) (\sigma_{q,l} - 1)}{\omega - (l - dq)} \right) \text{res}(\zeta_{\mu}^q; \omega) r^{-\omega}.$$ 

**Proof**

If $\omega$ is a simple pole of $\zeta_{\mu}^q$, then clearly $\text{res}(s \rightarrow Z_{\mu}^q(s) r^{-s}; \omega) = \text{res}(s \rightarrow (\sum_{l} \frac{\kappa_{\mu,l}(K) (\sigma_{q,l} - 1)}{s - (l - dq)}) \zeta_{\mu}^q(s) r^{-s}; \omega) = (\sum_{l} \frac{\kappa_{\mu,l}(K) (\sigma_{q,l} - 1)}{\omega - (l - dq)} \text{res}(\zeta_{\mu}^q; \omega) r^{-\omega}$. The desired result follows immediately from this observation and Theorem 5.5.

The formula in Corollary 5.6 has an even closer resemblance to Steiner’s formula. Namely, the symbolic multifractal Minkowski volume of $V_{\mu,r}^{q,\text{sym}}(K)$ is written as a “sum” of powers of $r$.

**5.6. Symbolic multifractal tubes of self-similar measures: the third explicit formula.** Finally, we use the second explicit formula for $V_{\mu,r}^{q,\text{sym}}(K)$, i.e. Theorem 5.3, expressing $V_{\mu,r}^{q,\text{sym}}(K)$ as a sum of residues of $\zeta_{\mu}^q$ together with a very careful analysis of the residues of $\zeta_{\mu}^q$ to prove Theorem 5.7 providing explicit formulas for the limiting behaviour of $V_{\mu,r}^{q,\text{sym}}(K)$ as $r \searrow 0$. Schematically this part can be represented as follows:

$$V_{\mu,r}^{q,\text{sym}}(K) \text{ equals a sum of residues of } \zeta_{\mu}^q \downarrow$$

a careful analysis of the residues of $\zeta_{\mu}^q$ \hspace{1cm} \hspace{1cm} an explicit formula for $\lim_{r \searrow 0} V_{\mu,r}^{q,\text{sym}}(K)$

Recall, that the second explicit formula for $V_{\mu,r}^{q,\text{sym}}(K)$, i.e. Theorem 5.5, expressing $V_{\mu,r}^{q,\text{sym}}(K)$ as a sum of residues of $\zeta_{\mu}^q$ says that if $q \in \mathbb{R}$ and $\beta(q) \notin \{-dq, 1-dq, \ldots, d-dq\}$, then

$$V_{\mu,r}^{q,\text{sym}}(K) = \lim_{n} \sum_{\omega \in P(\zeta_{\mu}^q)} \text{res}(s \rightarrow Z_{\mu}^q(s) r^{-s}; \omega)$$

for all $0 < r < r_{\min}$, and so

$$\frac{1}{r^{-\beta(q)}} V_{\mu,r}^{q,\text{sym}}(K) = \frac{1}{r^{-\beta(q)}} \lim_{n} \sum_{\omega \in P(\zeta_{\mu}^q)} \text{res}(s \rightarrow Z_{\mu}^q(s) r^{-s}; \omega). \quad (5.20)$$

for all $0 < r < r_{\min}$. The following heuristics suggests a plausible approach for analyzing the asymptotic behaviour of $\frac{1}{r^{-\beta(q)}} V_{\mu,r}^{q,\text{sym}}(K)$ as $r \searrow 0$. Since all poles $\omega$ of $\zeta_{\mu}^q$ lie on or to the
left of the critical line $\mathcal{L}_{\text{crit}}^q$ (see Section 5.3), i.e. since $\Re(\omega) \leq \beta(q)$, it is tempting to split
the sum in (5.20) into following two parts, namely, the sum of those $\omega \in P(\zeta_{p}^q)$ for which $\Re(\omega) < \beta(q)$ and the sum of those $\omega \in P(\zeta_{p}^q)$ for which $\Re(\omega) = \beta(q)$, i.e. we attempt to write

$$\frac{1}{r^{-\beta(q)}} V_{\mu, r}^{q, \text{sym}}(K) = \frac{1}{r^{-\beta(q)}} \lim_{n} \sum_{\omega \in P(\zeta_{p}^q)} \text{res} \left( s \rightarrow Z_{\mu}^{q}(s) r^{-s}; \omega \right) \bigg| \text{Im(\omega)} \leq t_{q, n} \bigg| \text{Re(\omega)} = \beta(q)$$

$$= \pi_{\text{sym}}^q(r) + E_{\text{sym}}^q(r)$$

(5.21)

where

$$\pi_{\text{sym}}^q(r) = \frac{1}{r^{-\beta(q)}} \lim_{n} \sum_{\omega \in P(\zeta_{p}^q)} \text{res} \left( s \rightarrow Z_{\mu}^{q}(s) r^{-s}; \omega \right) \bigg| \text{Im(\omega)} \leq t_{q, n} \bigg| \text{Re(\omega)} < \beta(q)$$

(5.22)

$$E_{\text{sym}}^q(r) = \frac{1}{r^{-\beta(q)}} \lim_{n} \sum_{\omega \in P(\zeta_{p}^q)} \text{res} \left( s \rightarrow Z_{\mu}^{q}(s) r^{-s}; \omega \right) \bigg| \text{Im(\omega)} \leq t_{q, n} \bigg| \text{Re(\omega)} < \beta(q)$$

(5.23)

of course, at the moment we do not even know if the limits (5.22) and (5.23) exist. For poles $\omega$ of $\zeta_{p}^q$ with $\Re(\omega) < \beta(q)$ we have $\Re(\beta(q) - \omega) > 0$, and it therefore seems plausible that if $\omega$ is a pole of $\zeta_{p}^q$ with $\Re(\omega) < \beta(q)$, then

$$\text{res} \left( s \rightarrow Z_{\mu}^{q}(s) r^{\beta(q) - s}; \omega \right) \rightarrow 0 \text{ as } r \searrow 0,$$

suggesting that

$$E_{\text{sym}}^q(r) = \frac{1}{r^{-\beta(q)}} \lim_{n} \sum_{\omega \in P(\zeta_{p}^q)} \text{res} \left( s \rightarrow Z_{\mu}^{q}(s) r^{-s}; \omega \right) \bigg| \text{Im(\omega)} \leq t_{q, n} \bigg| \text{Re(\omega)} < \beta(q)$$

$$= \lim_{n} \sum_{\omega \in P(\zeta_{p}^q)} \text{res} \left( s \rightarrow Z_{\mu}^{q}(s) r^{\beta(q) - s}; \omega \right) \bigg| \text{Im(\omega)} \leq t_{q, n} \bigg| \text{Re(\omega)} < \beta(q)$$

$$\rightarrow \lim_{n} \sum_{\omega \in P(\zeta_{p}^q)} 0 \bigg| \text{Im(\omega)} \leq t_{q, n} \bigg| \text{Re(\omega)} < \beta(q)$$

$$= 0 \text{ as } r \searrow 0.$$}

(5.24)

Finally, combining (5.21) and (5.24) indicates that the asymptotic behaviour of

$$\frac{1}{r^{-\beta(q)}} V_{\mu, r}^{q, \text{sym}}(K) = \pi_{\text{sym}}^q(r) + E_{\text{sym}}^q(r)$$
is determined by \( \pi_q^{\text{sym}}(r) \). In Section 16 we will prove that the heuristic argument outlined above is correct. As suggested by the expressions for \( \pi_q^{\text{sym}}(r) \) and \( E_q^{\text{sym}}(r) \), a rigorous analysis depends on a very careful analysis of the structure of the poles of \( \zeta_q^{\text{sym}} \) on and “critical line” \( L_{\text{crit}}^q = \{ s \in \mathbb{C} | \text{Re}(s) = \beta(q) \} \). Indeed, in order to prove that the limit \( \pi_q^{\text{sym}}(r) \) exists, a good understanding of the poles \( \omega \) of \( \zeta_q^{\text{sym}} \) with \( \text{Re}(\omega) = \beta(q) \) is needed and in order to prove that \( E_q^{\text{sym}}(r) \) tends to zero as \( r \searrow 0 \), a good understanding of the poles \( \omega \) of \( \zeta_q^{\text{sym}} \) with \( \text{Re}(\omega) = \beta(q) \) is needed. While a good understanding of the poles \( \omega \) of \( \zeta_q^{\text{sym}} \) with \( \text{Re}(\omega) = \beta(q) \) is easily obtained (and is, in fact, provided by Proposition 5.2), it is highly non-trivial to obtain a good understanding of the poles \( \omega \) of \( \zeta_q^{\text{sym}} \) with \( \text{Re}(\omega) \) close to but not equal to \( \beta(q) \). Indeed, a very substantial part of Section 13 is devoted to this problem. In particular, Proposition 13.2, Proposition 13.3 and Theorem 13.5 provide detailed information about the structure of the poles and residues of \( \zeta_q^{\text{sym}} \) near the critical line. Finally, in Section 16 these results are used to prove Theorem 5.7 showing that \( E_q^{\text{sym}}(r) \) tends to zero as \( r \searrow 0 \). More precisely, Theorem 5.7 below provides a complete description of the limiting behaviour of \( V_q^{\mu,\text{sym}}(K) \) as \( r \searrow 0 \). Namely, if the set \( \{ \log r_1^{-1}, \ldots, \log r_N^{-1} \} \) is not contained in a discrete additive subgroup of \( \mathbb{R} \), then \( \frac{1}{r^{\beta(q)}} V_q^{\mu,\text{sym}}(K) \) behaves asymptotically as a multiplicatively periodic function \( \pi_q^{\text{sym}} \) and we provide an explicit formula for \( \pi_q^{\text{sym}} \), and if the set \( \{ \log r_1^{-1}, \ldots, \log r_N^{-1} \} \) is contained in a discrete additive subgroup of \( \mathbb{R} \), then \( \frac{1}{r^{\beta(q)}} V_q^{\mu,\text{sym}}(K) \) converges to a constant \( c_q^{\text{sym}} \) and we provide an explicit formula for \( c_q^{\text{sym}} \). We will now give the precise statement of Theorem 5.7. In Theorem 5.7 we write \( \text{frac}(x) \) for the fractional part of a real number \( x \).

**Theorem 5.7.** Fix \( q \in \mathbb{R} \). Assume that \( \beta(q) \notin \{-dq, 1-dq, \ldots, d-dq\} \).

1. **The non-arithmetic case:** If the set \( \{ \log r_1^{-1}, \ldots, \log r_N^{-1} \} \) is not contained in a discrete additive subgroup of \( \mathbb{R} \), then
   \[
   \frac{1}{r^{\beta(q)}} V_q^{\mu,\text{sym}}(K) = \frac{1}{r^{\beta(q)}} V_q^{\mu,\text{sym}}(K) = \pi_q^{\text{sym}}(r) + \varepsilon_q^{\text{sym}}(r)
   \]
   for all \( 0 < r < r_{\min} \) where \( \varepsilon_q^{\text{sym}}(r) \) is the constant given by
   \[
   \varepsilon_q^{\text{sym}} = -\frac{1}{\sum_l p_{r_{\min}}^{\mu}(q) \log r_l} \sum_l \frac{\kappa_{r_{\min}}^{\mu}(K) (q, r_{\min})}{\beta(q) - (l - dq)}
   \]
   and \( \varepsilon_q^{\text{sym}}(r) \to 0 \) as \( r \searrow 0 \). In addition, \( K \) is \( (q, \beta(q)) \) symbolic multifractal Minkowski measurable with respect to \( \mu \) with
   \[
   M_{\mu, q, \beta(q), \text{sym}}(K) = -\frac{1}{\sum_l p_{r_{\min}}^{\mu}(q) \log r_l} \sum_l \frac{\kappa_{r_{\min}}^{\mu}(K) (q, r_{\min})}{\beta(q) - (l - dq)}.
   \]

2. **The arithmetic case:** If the set \( \{ \log r_1^{-1}, \ldots, \log r_N^{-1} \} \) is contained in a discrete additive subgroup of \( \mathbb{R} \) and \( \{ \log r_1^{-1}, \ldots, \log r_N^{-1} \} = u\mathbb{Z} \) with \( u > 0 \), then
   \[
   \frac{1}{r^{\beta(q)}} V_q^{\mu,\text{sym}}(K) = \pi_q^{\text{sym}}(r) + \varepsilon_q^{\text{sym}}(r)
   \]
   for all \( 0 < r < r_{\min} \) where \( \pi_q^{\text{sym}} : (0, \infty) \to \mathbb{R} \) is the multiplicatively periodic function
with period equal to $e^u$ (i.e. $\pi^\text{sym}_q(e^u r) = \pi^\text{sym}_q(r)$ for all $r \in (0, \infty)$) given by

$$
\pi^\text{sym}_q(r) = -\frac{1}{\sum_i p_i^q r_i^{\beta(q)}} \log r_i 
\times \sum_{l=0,1,\ldots,d} \left( \frac{\kappa_{q,l}^\text{sym}(K) (\sigma_{q,l} - 1)}{1 - e^{-u(\beta(q) - (l - dq))}} \right) 
\times \left\{ \begin{array}{ll} 
\frac{e^{-u(\beta(q) - (l - dq))} + 1}{2} & \text{for } r \in e^Z_u; \\
-1 \left( e^{-u(\beta(q) - (l - dq)) \frac{\log r_u}{u}} \right) & \text{for } r \notin e^Z_u 
\end{array} \right. 
$$

and $\varepsilon_q(r) \to 0$ as $r \searrow 0$. In addition, $K$ is $(q, \beta(q))$ average symbolic multifractal Minkowski measurable with respect to $\mu$ with

$$
M_{q,\beta(q),\text{sym}}(K) = -\frac{1}{\sum_i p_i^q r_i^{\beta(q)}} \log r_i 
\times \sum_{l=0,1,\ldots,d} \kappa_{q,l}^\text{sym}(K) (\sigma_{q,l} - 1) 
\times \left\{ \begin{array}{ll}
\frac{e^{-u(\beta(q) - (l - dq))} + 1}{2} & \text{for } r \in e^Z_u; \\
-1 \left( e^{-u(\beta(q) - (l - dq)) \frac{\log r_u}{u}} \right) & \text{for } r \notin e^Z_u 
\end{array} \right.
$$

Theorem 5.7 is proved in Section 16. Theorem 5.7 is clearly a “symbolic” version of Theorem 3.3. However, the key difference between Theorem 5.7 and Theorem 3.3 is that Theorem 5.7 (in contrast to Theorem 3.3) has explicit formulas for the constant $c^\text{sym}_q$ and the function $\pi^\text{sym}_q$. 
Part 2:
Proofs of the Results from Section 3

6. Proving that
\[ \lambda_{q,m}(r) \leq \sum_{|i|=|j|=m} Q_{ij}^q(r) \]

The main purpose of this section is to prove Proposition 6.3. However, we begin by introducing some notation. Let \( \Sigma = \{1, \ldots, N\} \) and write
\[
\Sigma^m = \{1, \ldots, n\}^m, \\
\Sigma^* = \bigcup_m \Sigma_m, \\
\Sigma^N = \{1, \ldots, n\}^N,
\]
i.e. \( \Sigma^m \) is the family of all strings \( i = i_1 \ldots i_m \) of length \( m \) with \( i_j \in \{1, \ldots, N\} \), \( \Sigma^* \) is the family of all finite strings \( i = i_1 \ldots i_m \) with \( i_j \in \{1, \ldots, N\} \) and \( \Sigma^* \) is the family of all infinite strings \( i = i_1 i_2 \ldots \) with \( i_j \in \{1, \ldots, N\} \). For \( i \in \Sigma^m \), we write \( |i| = m \) for the length of \( i \). and for a positive integer \( n \) with \( n \leq m \), we write \( i|_n = i_1 \ldots i_n \) for the truncation of \( i \) to the \( n \)-th place. Also, for \( i = i_1 \ldots i_m, j = j_1 \ldots j_n \in \Sigma^* \), let \( ij = i_1 \ldots i_m j_1 \ldots j_n \) denote the concatenation of \( i \) and \( j \). Next, if \( i = i_1 \ldots i_m \in \Sigma^* \), we will write \( S_i = S_{i_1} \circ \cdots \circ S_{i_m} \), \( r_i = r_{i_1} \cdots r_{i_m} \) and \( p_i = p_{i_1} \cdots p_{i_m} \). Finally, write
\[
r_{\text{min}} = \min_{i=1,\ldots,N} r_i, \\
r_{\text{max}} = \max_{i=1,\ldots,N} r_i.
\]

We now introduce the two key quantities in this (and the subsequent) sections, namely, \( Q_{ij}^q(r) \) and \( \lambda_{q,m}(r) \). For \( E \subseteq \mathbb{R}^d \) and \( r > 0 \), recall that \( B(E, r) \) denotes the (open) \( r \)-neighbourhood of \( E \), i.e. \( B(E, r) = \{ x \in \mathbb{R}^d | \text{dist}(x, E) < r \} \). For \( q \in \mathbb{R} \) and \( i, j \in \Sigma^* \) and \( r > 0 \), write
\[
Q_{ij}^q(r) = \frac{1}{r^d} \int_{B(S_i K, r) \cap B(S_j K, r)} \mu(B(x, r))^q d\mathcal{L}^d(x).
\]
Next, for \( q \in \mathbb{R} \) and a positive integer \( m \) and \( r > 0 \), write
\[
\lambda_{q,m}(r) = T^q_{\mu,r}(B(K,r)) - \sum_{|i|=m} p^q_{i} 1_{(0, r_i]}(r) T^q_{\mu,r_i^{-1}}(B(K,r_i^{-1}r))
\]
\[
= V^q_{\mu,r}(K) - \sum_{|i|=m} p^q_{i} 1_{(0, r_i]}(r) V^q_{\mu,r_i^{-1}}(K),
\]
(6.3)

If \( m = 1 \), then we will write \( \lambda(r) \) for \( \lambda_{q,1}(r) \), i.e. we will write
\[
\lambda_{q}(r) = T^q_{\mu,r}(B(K,r)) - \sum_{|i|=1} p^q_{i} 1_{(0, r_i]}(r) T^q_{\mu,r_i^{-1}}(B(K,r_i^{-1}r))
\]
\[
= V^q_{\mu,r}(K) - \sum_{|i|=1} p^q_{i} 1_{(0, r_i]}(r) V^q_{\mu,r_i^{-1}}(K),
\]
(6.4)

The main result in the section is Proposition 6.3 providing an upper bound for the difference \( \lambda_{q,m}(r) = V^q_{\mu,r}(K) - \sum_{|i|=m} p^q_{i} 1_{(0, r_i]}(r) V^q_{\mu,r_i^{-1}}(K) \) in terms of \( Q^q_{1,3}(r) \); namely, in Proposition 6.3 we will prove that if \( r > 0 \) is sufficiently small, then
\[
|\lambda_{q,m}(r)| \leq \sum_{|i|=|j|=m, i \neq j} Q^q_{1,3}(r).
\]
(6.5)

We now turn towards the proof of Proposition 6.3. We begin with two lemmas.

**Lemma 6.1.** Fix \( q \in \mathbb{R} \). Let \( m \in \mathbb{N} \).

1. For \( r > 0 \), we have
\[
T^q_{\mu,r}(B(K,r)) \leq \sum_{|i|=m} T^q_{\mu,r}(B(S_i K, r)).
\]

2. For \( r > 0 \), we have
\[
- \sum_{|i|=|j|=m, i \neq j} Q^q_{1,3}(r) + \sum_{|i|=m} T^q_{\mu,r}(B(S_i K, r)) \leq T^q_{\mu,r}(B(K,r)).
\]

**Proof**

1. Fix \( r > 0 \). Since \( K = \cup_{|i|=m} S_i K \) we obtain
\[
T^q_{\mu,r}(B(K,r)) = \frac{1}{r^d} \int_{B(K,r)} \mu(B(x, r))^q d\mathcal{L}^d(x)
\]
\[
= \frac{1}{r^d} \int_{B(\cup_{|i|=m} S_i K, r)} \mu(B(x, r))^q d\mathcal{L}^d(x)
\]
\[
\leq \frac{1}{r^d} \int_{\cup_{|i|=m} B(S_i K, r)} \mu(B(x, r))^q d\mathcal{L}^d(x)
\]
\[
\leq \sum_{|i|=m} \frac{1}{r^d} \int_{B(S_i K, r)} \mu(B(x, r))^q d\mathcal{L}^d(x)
\]
\[
\leq \sum_{|i|=m} T^q_{\mu,r}(B(S_i K, r)).
\]
(2) Fix \( r > 0 \). For \( i \in \Sigma^* \) with \( |i| = m \), write

\[
G_i = B(S_i K, r) \setminus \bigcup_{\substack{|j| = |i| \\
j \neq i}} B(S_j K, r) .
\]

Also, for \( i, j \in \Sigma^* \) with \( |i| = |j| = m \), write

\[
H_{i,j} = B(S_i K, r) \cap B(S_j K, r) .
\]

Since clearly \((G_i)_{|i|=m}\) is a family of pairwise disjoint sets with \( \bigcup_{|i|=m} G_i \subseteq B(K, r) \), we conclude that

\[
T_{\mu,r}^g \left( B(K, r) \right) = \frac{1}{r^d} \int_{B(K, r)} \mu(B(x, r))^g \, d\mathcal{L}^d(x) 
\geq \frac{1}{r^d} \int_{\bigcup_{|i|=m} G_i} \mu(B(x, r))^g \, d\mathcal{L}^d(x) 
= \sum_{|i|=m} \frac{1}{r^d} \int_{G_i} \mu(B(x, r))^g \, d\mathcal{L}^d(x) .
\tag{6.6}
\]

Next, note that

\[
B(S_i K, r) \subseteq G_i \cup \bigcup_{\substack{|j| = |i| \\
j \neq i}} H_{i,j} .
\]

It follows from this that

\[
\frac{1}{r^d} \int_{G_i} \mu(B(x, r))^g \, d\mathcal{L}^d(x) + \sum_{\substack{|j| = |i| \\
j \neq i}} \frac{1}{r^d} \int_{H_{i,j}} \mu(B(x, r))^g \, d\mathcal{L}^d(x) 
\geq \frac{1}{r^d} \int_{G_i \cup \bigcup_{|j| = |i|, j \neq i} H_{i,j}} \mu(B(x, r))^g \, d\mathcal{L}^d(x) 
\geq \frac{1}{r^d} \int_{B(S_i K, r)} \mu(B(x, r))^g \, d\mathcal{L}^d(x) ,
\]

whence

\[
\frac{1}{r^d} \int_{G_i} \mu(B(x, r))^g \, d\mathcal{L}^d(x) 
\geq \frac{1}{r^d} \int_{B(S_i K, r)} \mu(B(x, r))^g \, d\mathcal{L}^d(x) - \sum_{\substack{|j| = |i| \\
j \neq i}} \frac{1}{r^d} \int_{H_{i,j}} \mu(B(x, r))^g \, d\mathcal{L}^d(x) .
\tag{6.7}
\]
Finally, combining (6.6) and (6.7) gives

$$T^q_{\mu, r}(B(K, r)) \geq \sum_{|i|=m} \frac{1}{r^d} \int_{G_i} \mu(B(x, r))^q \, d\mathcal{L}^d(x)$$

$$= \sum_{|i|=m} \frac{1}{r^d} \int_{B(S_i K, r)} \mu(B(x, r))^q \, d\mathcal{L}^d(x) - \sum_{|j|=|i|} \frac{1}{r^d} \int_{H_{i,j}} \mu(B(x, r))^q \, d\mathcal{L}^d(x)$$

$$= \sum_{|i|=m} T^q_{\mu, r}(B(S_i K, r)) - \sum_{|i|=|j|=m} \sum_{i \neq j} Q^q_{i,j}(r).$$

This completes the proof. \(\square\)

**Lemma 6.2.** Fix \(q \in \mathbb{R}\). Let \(i \in \Sigma^*\).

1. For \(r > 0\), we have

$$T^q_{\mu, r}(B(S_i K, r)) \leq p^q_{i} T^q_{\mu, r^{-1}, r}(B(K, r^{-1} r)) + \sum_{|j|=|i|} \sum_{j \neq i} Q^q_{i,j}(r).$$

2. For \(0 \leq q \) and \(r > 0\), we have

$$p^q_{i} T^q_{\mu, r^{-1}, r}(B(K, r^{-1} r)) \leq T^q_{\mu, r}(B(S_i K, r)).$$

**Proof**

(1) Fix \(r > 0\). Write

$$G = B(S_i K, r) \setminus \bigcup_{|j|=|i|} B(S_j K, r).$$

Also, for \(j \in \Sigma^*\) with \(|j| = |i|\), write

$$H_{j} = B(S_i K, r) \cap B(S_j K, r).$$

Note that

$$B(S_i K, r) \subseteq G \cup \bigcup_{|j|=|i|} H_{j}.$$
It follows from this and the equation \( \mu(B(x,r)) = \sum_{|j|=|i|} p_j \mu(S_j^{-1}B(x,r)) \), that

\[
T^q_{\mu,r}(B(S_i K, r)) \\
= \frac{1}{r^d} \int_{B(S_i K, r)} \mu(B(x,r))^q \, dL^d(x) \\
\leq \frac{1}{r^d} \int_{G \cup \cup_{|j|=|i|, j \neq i} H_j} \mu(B(x,r))^q \, dL^d(x) \\
\leq \frac{1}{r^d} \int_G \mu(B(x,r))^q \, dL^d(x) + \sum_{|j|=|i|} \frac{1}{r^d} \int_{H_j} \mu(B(x,r))^q \, dL^d(x) \\
= \frac{1}{r^d} \int_G \left( \sum_{|j|=|i|} p_j \mu(S_j^{-1}B(x,r)) \right)^q \, dL^d(x) + \sum_{|j|=|i| \atop j \neq i} \frac{1}{r^d} \int_{H_j} \mu(B(x,r))^q \, dL^d(x).
\]

(6.8)

Now observe that \( S_j^{-1}B(x,r) = \emptyset \) for all \( x \in G \) and all \( j \in \Sigma^* \) with \( |j| = |i| \) and \( j \neq i \). It follows from this that \( \sum_{|j|=|i|} p_j \mu(S_j^{-1}B(x,r)) = p_i \mu(S_i^{-1}B(x,r)), \) and (6.8) therefore simplifies to

\[
T^q_{\mu,r}(B(S_i K, r)) \leq p_i \frac{1}{r^d} \int_G \mu(S_i^{-1}B(x,r))^q \, dL^d(x) + \sum_{|j|=|i| \atop j \neq i} \frac{1}{r^d} \int_{H_j} \mu(B(x,r))^q \, dL^d(x) \\
= p_i \frac{1}{r^d} \int_G \mu(S_i^{-1}B(x,r))^q \, dL^d(x) + \sum_{|j|=|i| \atop j \neq i} Q_{i,j}(r) \\
= p_i \frac{1}{r^d} \int_G \mu(B(S_i^{-1}x,r_i^{-1}r))^q \, dL^d(x) + \sum_{|j|=|i| \atop j \neq i} Q_{i,j}(r) \\
= p_i \frac{1}{r^d} \int_{S_i^{-1}G} \mu(B(x,r_i^{-1}r))^q \, dL^d(x) + \sum_{|j|=|i| \atop j \neq i} Q_{i,j}(r).
\]

(6.9)

Finally, using the fact that \( S_i^{-1} G \subseteq S_i^{-1}B(S_i K, r) \subseteq B(K, r_i^{-1}r) \), we conclude from (6.9) that

\[
T^q_{\mu,r}(B(S_i K, r)) \leq p_i \frac{1}{(r_i^{-1}r)^d} \int_{B(K, r_i^{-1}r)} \mu(B(x,r_i^{-1}r))^q \, dL^d(x) + \sum_{|j|=|i| \atop j \neq i} Q_{i,j}(r) \\
= p_i T^q_{\mu,r_i^{-1}r}(B(K, r_i^{-1}r)) + \sum_{|j|=|i| \atop j \neq i} Q_{i,j}^q(r).
\]

(2) Fix \( r > 0 \). Write

\[
G = B(S_i K, r) \setminus \bigcup_{|j|=|i| \atop j \neq i} B(S_j K, r).
\]

Also, write

\[
H = B(S_i K, r) \cap \bigcup_{|j|=|i| \atop j \neq i} B(S_j K, r).
\]
Note that \( B(S_i K, r) = G \cup H \) and \( G \cap H = \emptyset \). It follows from this and the identity \( \mu(B(x, r)) = \sum |j|=|i| p_j \mu(S_j^{-1} B(x, r)) \), that

\[
T_{\mu,r}^q \left( B(S_i K, r) \right) = \frac{1}{p^d} \int_{B(S_i K, r)} \mu(B(x, r))^q dL^d(x)
\]

\[
= \frac{1}{p^d} \int_G \mu(B(x, r))^q dL^d(x) + \frac{1}{p^d} \int_H \mu(B(x, r))^q dL^d(x)
\]

\[
= \frac{1}{p^d} \int_G \left( \sum |j|=|i| p_j \mu(S_j^{-1} B(x, r)) \right) dL^d(x) + \frac{1}{p^d} \int_H \mu(B(x, r))^q dL^d(x).
\]

(6.10)

Now observe that \( S_j^{-1} B(x, r) = \emptyset \) for all \( x \in G \) and all \( j \in \Sigma^* \) with \( |j| = |i| \) and \( j \neq i \). It follows from this that \( \sum |j|=|i| p_j \mu(S_j^{-1} B(x, r)) = p_i \mu(S_i^{-1} B(x, r)) \), and (6.10) therefore simplifies to

\[
T_{\mu,r}^q \left( B(S_i K, r) \right) = p_i^q \frac{1}{p^d} \int_G \mu(S_i^{-1} B(x, r))^q dL^d(x) + \frac{1}{p^d} \int_H \mu(B(x, r))^q dL^d(x).
\]

(6.11)

Once more using the fact that \( B(S_i K, r) = G \cup H \) and \( G \cap H = \emptyset \), we conclude from (6.11) that

\[
T_{\mu,r}^q \left( B(S_i K, r) \right) = p_i^q \frac{1}{p^d} \int_{G \cup H} \mu(S_i^{-1} B(x, r))^q dL^d(x)
\]

\[
= p_i^q \frac{1}{p^d} \int_H \mu(S_i^{-1} B(x, r))^q dL^d(x) + \frac{1}{p^d} \int_H \mu(B(x, r))^q dL^d(x)
\]

\[
= p_i^q \frac{1}{p^d} \int_{B(S_i K, r)} \mu(S_i^{-1} B(x, r))^q dL^d(x)
\]

\[
= \frac{1}{p^d} \int_H \left( \mu(B(x, r))^q - p_i^q \mu(S_i^{-1} B(x, r))^q \right) dL^d(x).
\]

(6.12)

Finally, note that since \( 0 \leq q \), we conclude that \( \mu(B(x, r))^q = (\sum |j|=|i| p_j \mu(S_j^{-1} B(x, r)))^q \geq p_i^q \mu(S_i^{-1} B(x, r))^q \) for all \( x \) and all \( r > 0 \). It therefore follows from (6.12) that

\[
T_{\mu,r}^q \left( B(S_i K, r) \right) \geq p_i^q \frac{1}{p^d} \int_{B(S_i K, r)} \mu(S_i^{-1} B(x, r))^q dL^d(x)
\]

\[
= p_i^q \frac{1}{p^d} \int_{B(S_i K, r)} \mu(B(S_i^{-1} x, r_i^{-1} r))^q dL^d(x)
\]

\[
= p_i^q \frac{1}{p^d} \int_{S_i^{-1} B(S_i K, r)} \mu(B(x, r_i^{-1} r))^q dL^d(x)
\]

\[
= p_i^q \frac{1}{(r_i^{-1} r)^d} \int_{B(K, r_i^{-1} r)} \mu(B(x, r_i^{-1} r))^q dL^d(x)
\]

\[
= T_{\mu,r_i^{-1},r}^q \left( B(K, r_i^{-1} r) \right).
\]
This completes the proof. 

\[ \square \]

Proposition 6.3. Fix \( q \in \mathbb{R} \). Let \( m \in \mathbb{N} \).

1. If \( 0 \leq q \) and the OSC is satisfied, then for \( r > 0 \), we have

\[
\left\| T^q_{\mu,r} \left( B(K,r) \right) \right\| \quad \sum_{i=0}^{m} p_i^q \left\| T^q_{\mu_i,r^{-1}_i} \left( B(K,r^{-1}_i) \right) \right\| \leq \sum_{i=|j|=m, i \neq j} Q^q_{i,j}(r).
\]

In particular, for \( r > 0 \) with \( r < r_{min}^m \), we have

\[
|\lambda_{q,m}(r)| \leq \sum_{i=|j|=m, i \neq j} Q^q_{i,j}(r).
\]

2. If the SSC is satisfied, then for \( r > 0 \) with \( r < \frac{1}{2} \min_{|i|=m, i \neq j} \text{dist}(S_i K, S_j K) \), we have

\[
T^q_{\mu,r} \left( B(K,r) \right) - \sum_{i=0}^{m} p_i^q T^q_{\mu_i,r^{-1}_i} \left( B(K,r^{-1}_i) \right) = 0.
\]

In particular, for \( r > 0 \) with \( r < \min \left( r_{min}^m, \frac{1}{2} \min_{|i|=m, i \neq j} \text{dist}(S_i K, S_j K) \right) \), we have

\[
\lambda_{q,m}(r) = 0.
\]

Proof

1. This follows immediately from Lemma 6.1 and Lemma 6.2.

2. Since \( r < \frac{1}{2} \min_{|i|=m, i \neq j} \text{dist}(S_i K, S_j K) \) and \( K = \cup_{|i|=m} S_i K \), we clearly have

\[
T^q_{\mu,r} \left( B(K,r) \right) = \frac{1}{p^d} \int_{B(K,r)} \mu(B(x,r))^q \, d\mathcal{L}^d(x)
\]

\[
= \frac{1}{p^d} \int_{B(\cup_{|i|=m} S_i K,r)} \mu(B(x,r))^q \, d\mathcal{L}^d(x)
\]

\[
= \frac{1}{p^d} \int_{\cup_{|i|=m} B(S_i K,r)} \mu(B(x,r))^q \, d\mathcal{L}^d(x)
\]

\[
= \sum_{|i|=m} \frac{1}{p^d} \int_{B(S_i K,r)} \mu(B(x,r))^q \, d\mathcal{L}^d(x)
\]

\[
= \sum_{|i|=m} \frac{1}{p^d} \int_{B(S_i K,r)} \left( \sum_{|j|=|i|} p_j \mu(S_j^{-1} B(x,r)) \right)^q \, d\mathcal{L}^d(x).
\]

(6.13)

Next, observe that since \( r < \frac{1}{2} \min_{|i|=m, i \neq j} \text{dist}(S_i K, S_j K) \), we conclude that \( S_j^{-1} B(x,r) = \emptyset \) for all \( x \in B(S_i K,r) \) for all \( i,j \in \Sigma^* \) with \( |j| = |i| \) and \( j \neq i \). It follows from this that
\[ \sum_{|j|=|i|} p_j \mu(S_i^{-1} B(x, r)) = p_i \mu(S_{i}^{-1} B(x, r)), \text{ and } (6.13) \text{ therefore simplifies to} \\
\mathcal{I}^q_{\mu,r}(B(K, r)) = \sum_{|i|=m} p_i^q \frac{1}{r_i^d} \int_{B(S_i K, r)} \mu(S_i^{-1} B(x, r)) d \mathcal{L}^d(x) \\
= \sum_{|i|=m} p_i^q \frac{1}{r_i^d} \int_{B(S_i K, r)} \mu(B(x, r_i^{-1} r)) d \mathcal{L}^d(x) \\
= \sum_{|i|=m} p_i^q \frac{1}{r_i^d} r_i^d \int_{S_i^{-1} B(S_i K, r)} \mu(B(x, r_i^{-1} r)) d \mathcal{L}^d(x) \\
= \sum_{|i|=m} p_i^q T_i^{q, r_i^{-1} r}(B(K, r_i^{-1} r)) . \\
\]

This completes the proof. \( \square \)

7. Proving that
\[ Q_{i,j}^q(r) \leq \text{constant } Z_{i}^q(r) \]

The main purpose of this section is to prove Proposition 7.8. However, we begin by introduction some notation. For \( i, h \in \Sigma^* \), we write
\[ i \preceq h \]
if and only if \( i \) is a substring of \( h \), i.e. if and only if there are strings \( s, t \in \Sigma^* \) such that
\[ h = sit. \]

Next, if \( (S_1, \ldots, S_n) \) satisfies the OSC, then it follows from a result by Schief [Schi] that there exists an open, bounded and non-empty subset \( U \) of \( \mathbb{R}^d \) with
\[ \bigcup_i S_i U \subseteq U , \]
\[ S_i U \cap S_j U = \emptyset \quad \text{for all } i, j \text{ with } i \neq j , \quad (7.1) \]
\[ U \cap K \neq \emptyset . \]

In addition, it is easily seen that \( S_i K \subseteq S_i \overline{U} \) for all \( i \in \Sigma^* \), and that \( S_i K \cap S_j U = \emptyset \) for all \( i, j \in \Sigma^* \) with \( |i| = |j| \) and \( i \neq j \), cf. [Hu]. Also, since \( U \) is open and bounded there exist \( \rho_1, \rho_2 > 0 \) such that \( U \) contains a ball of radius \( \rho_1 \), and \( U \) is contained in a ball of radius \( \rho_2 \). Since \( U \cap K \neq \emptyset \), we can choose \( l \in \Sigma^* \) such that
\[ S_l K \subseteq U , \quad (7.2) \]
and the compactness of \( S_l K \) now implies that
\[ d_0 = \text{dist}(S_l K, \mathbb{R}^d \setminus U) > 0 . \quad (7.3) \]
For brevity write $D_0 = \text{diam} K$. Also, for a positive integer $m \in \mathbb{N}$, let $M_m \in \mathbb{N}$ be chosen such that
\[ \frac{1}{r_{M_m} - 1} \geq 2 \frac{D_0}{d_0} \frac{1}{r_{M_m - 1}} \]  
(7.4)
(recall that $r_{\text{min}} = \min_{i=1, \ldots, N} r_i$ and $r_{\text{max}} = \max_{i=1, \ldots, N} r_i$). Now put
\[ r_m = r_{M_m + m}, \]  
(7.5)
\[ a_m = \frac{1}{D_0} \frac{r_{\text{min}}}{r_{M_m + m}}, \]  
(7.6)
\[ b_m = \frac{1}{D_0} \frac{1}{r_{\text{min}} + m}, \]  
(7.7)
and define $Z_m^q : (0, \infty) \to \mathbb{R}$ by
\[ Z_m^q(r) = \sum_{\substack{h \in \Sigma^* \mid |h| \geq \ell \leq r_m r_m \mid \hat{h} \prec h \}} \phi_h. \]  
(7.8)
If $m = 1$, then we will write $Z^q(r)$ for $Z_1^q(r)$, i.e. we will write
\[ Z^q(r) = \sum_{\substack{h \in \Sigma^* \mid |h| \geq \ell \leq r_1 r_1 \mid \hat{h} \prec h \}} \phi_h. \]  
(7.9)

The main purpose of this section to prove Proposition 7.8 saying if $m$ is a positive integer and $i, j \in \Sigma^*$ with $|i| = |j| = m$ and $i \neq j$, then there is a constant $c_{i,j,m} > 0$ such that for $r > 0$ with $r < \frac{1}{2} r_m$, we have
\[ Q_{i,j}^q(r) \leq \begin{cases} c_{i,j,m} Z_m^q \left( \frac{1}{2} r \right) & \text{for } q < 0; \\ c_{i,j,m} Z_m^q (2r) & \text{for } 0 \leq q. \end{cases} \]

We now turn towards the proof of Proposition 7.8. However, we first make the following definition. Namely, for a string $i = i_1 \ldots i_m \in \Sigma^*$, let
\[ \hat{i} = i_1 \ldots i_{m-1} \]
denote the “parent” of $i$, and for $r > 0$ write
\[ \Sigma^*_r = \left\{ i \in \Sigma^* \mid r_1 \text{ diam } K < r \leq r_1 \text{ diam } K \right\}. \]

**Lemma 7.1.** Assume that the OSC is satisfied.
Let $l$ and $d_0$ be as in (7.2) and (7.3). Assume that $u, w \in \Sigma$ and $u, w, h \in \Sigma^*$ satisfy the following conditions:
(i) $u \neq w$;  
(ii) $\text{dist}(S_{uw}K, S_{uw}K) \leq r_{uw}d_0$.  


Then \( l \not \prec h \).

Proof

Let \( U \) be the open set in (7.1). Assume, in order to obtain a contradiction, that \( l \) is a substring of \( h \), i.e. there exist strings \( s, t \in \Sigma^* \) such that \( h = stl \). Hence

\[
\text{dist} \left( S_{uh}K, \mathbb{R}^d \setminus S_{us}U \right) = \text{dist} \left( S_{uslt}K, \mathbb{R}^d \setminus S_{us}U \right) \\
\geq \text{dist} \left( S_{usl}K, \mathbb{R}^d \setminus S_{us}U \right) \\
= r_{usl} \text{dist} \left( S_{l}K, \mathbb{R}^d \setminus U \right) \\
\geq \text{dist} \left( S_{ui}K, \mathbb{R}^d \setminus S_{u}U \right) \\
= r_{us}d_0 \\
> r_{uh}d_0. 
\]

(7.10)

Also, \( u \not \equiv w \) (by (i)), we conclude that \( S_{us}U \cap S_{uw}K = \emptyset \), i.e. \( S_{uw}K \subseteq \mathbb{R}^d \setminus S_{us}U \), whence (using (ii))

\[
\text{dist} \left( S_{uh}K, \mathbb{R}^d \setminus S_{us}U \right) \leq \text{dist} \left( S_{uh}K, S_{uw}K \right) \\
\leq r_{uh}d_0. 
\]

(7.11)

Inequalities (7.10) and (7.11) give the desired contradiction. \( \square \)

Lemma 7.2. Assume that the OSC is satisfied.

Let \( l \) be as in (7.2). Let \( m \in \mathbb{N} \). Let \( i, j \in \Sigma^* \) with \( |i| = |j| = m \) and \( i \neq j \). Assume that \( r > 0 \), \( x \in \mathbb{R}^d \) and \( k \in \Sigma^* \) satisfy the following conditions:

(i) \( 0 < r < r_m \);
(ii) \( x \in B(S_iK, r) \cap B(S_jK, r) \);
(iii) \( k \in \Sigma_r^* \) and \( x \in B(S_kK, r) \).

Then there are strings \( u, h, v \in \Sigma^* \) with \( k = uhv \) such that

\[
|u| = m, \\
|v| = M_m, \\
|h| \geq |l|, \\
l \not \prec h.
\]

Proof

Recall, that \( D_0 = \text{diam} K \). Since \( k \in \Sigma_r^* \), we conclude that \( r_{\min}^{(|k|} \leq r_kD_0 \leq r \leq r_m = r_{\min}^{M_m+m+|l|} \), whence \( |k| \geq M_m + m + |l| \). It follows from this that there are \( u, h, v \in \Sigma^* \) with \( k = uhv \) such that \( |u| = m, |v| = M_m \) and \( |h| \geq |l| \). We must now show that

\[
l \not \prec h.
\]

Note that \( |i| = m = |u| \leq |k| \) and \( |j| = m = |u| \leq |k| \). Hence, since \( i \neq j \), we can find \( w \in \{i, j\} \) such that

\[
w \neq k|m = u.
\]

Write \( u = u_1 \ldots u_m \) and \( w = w_1 \ldots w_m \). Since \( w \neq u \), there is \( s \in \{1, \ldots, m\} \) such that

\[
w_i = u_i \text{ for } i = 1, \ldots, s - 1, \\
w_s \neq u_s.
\]
Next, we prove the following two claims.

**Claim 1:** We have \( \text{dist}(S_{u_1 \ldots u_m}K, S_{w_1 \ldots w_m}K) \leq 2 \frac{1}{r_{\min}} r. \)

**Proof of Claim 1.** Since \( w_1 = u_1, \ldots, w_{s-1} = u_{s-1} \), we deduce that

\[
\text{dist}(S_{u_{s} \ldots u_{m}}K, S_{w_{s} \ldots w_{m}}K) = \frac{1}{r_{u_1 \ldots u_{s-1}}} \text{dist}(S_{u_1 \ldots u_m}K, S_{w_1 \ldots w_m}K) \\
\leq \frac{1}{r_{\min}^{s-1}} \text{dist}(S_{u_1 \ldots u_m}K, S_{w_1 \ldots w_m}K) \\
\leq \frac{1}{r_{\min}^{m-s}} \text{dist}(S_{u_{s} \ldots u_{m}}K, S_{w_{s} \ldots w_{m}}K) \\
= \frac{1}{r_{\min}^{m-1}} \text{dist}(S_{u_{s}}K, S_{w_{s}}K).
\]

(7.12)

Next, since \( S_{u_{s}h}K \subseteq S_{u_{s}}K \), it follows from (7.12) that

\[
\text{dist}(S_{u_1 \ldots u_m}hK, S_{w_1 \ldots w_m}K) \leq \frac{1}{r_{\min}^{m-1}} \text{dist}(S_{u_{s}h}K, S_{w_{s}}K) \\
= \frac{1}{r_{\min}^{m-1}} \text{dist}(S_{h}K, S_{w_s}K) \\
\leq \frac{1}{r_{\min}^{m-1}} \left( \text{dist}(S_{h}K, x) + \text{dist}(x, S_{w_s}K) \right).
\]

(7.13)

However, since \( x \in B(S_iK, r) \cap B(S_jK, r) \subseteq B(S_wK, r) \), we have \( \text{dist}(x, S_wK) \leq r \). Similarly, since \( x \in B(S_kK, r) \), we have \( \text{dist}(S_kK, x) \leq r \). It therefore follows from (7.13) that

\[
\text{dist}(S_{u_1 \ldots u_m}hK, S_{w_1 \ldots w_m}K) \leq 2 \frac{1}{r_{\min}} r
\]

This completes the proof of Claim 1.

**Claim 2:** We have \( d_0r_{u_1 \ldots u_m} \geq d_0 \frac{1}{r_{\max}} r_{u_{s}h} \).

**Proof of Claim 2.** We have (using the fact that \( k \in \Sigma_r^* \))

\[
d_0r_{u_1 \ldots u_m} \geq d_0r_{u_{s}h} \\
\geq d_0 \frac{1}{r_{\max}} r_{u_{s}h} \\
\geq d_0 \frac{1}{D_0^r \alpha_{\max}^{M_{\max}} M_{\max} r_{\max}}.
\]

This completes the proof of Claim 2.

It now follows from Claim 1, Claim 2 and (7.4) that

\[
\text{dist}(S_{u_1 \ldots u_m}hK, S_{w_1 \ldots w_m}K) \leq r_{u_1 \ldots u_m}d_0,
\]

and we therefore deduce from Lemma 7.1 that \( l \not\prec h \). This completes the proof. \( \square \)
Lemma 7.3. Assume that the OSC is satisfied. For $i \in \Sigma^*$, we have

$$\mu(S_i K) = p_i.$$ 

Proof
This lemma is proved in [Graf]. □

Lemma 7.4 below is a slight modification of a result due to Hutchinson [Hu] and the proof is therefore omitted. Moreover, Lemma 7.5 is a standard result and the proof of Lemma 7.5 is therefore also omitted.

Lemma 7.4. Let $r, k, k_1, k_2 > 0$, and let $(V_i)_i$ be a family of pairwise disjoint open subsets of $\mathbb{R}^d$. Assume that each set $V_i$ contains a ball of radius $k_1 r$ and is contained in a ball of radius $k_2 r$. Then

$$| \{ i | V_i \cap B(x, kr) \neq \emptyset \} | \leq \left( \frac{k + 2k_2}{k_1} \right)^d$$

for all $x \in \mathbb{R}^d$.

Lemma 7.5. Fix $q \in \mathbb{R}$. Let $c > 0$ and $(a_i)_i \subseteq I$ be a family of real numbers with $|I| \leq c$. Then $(\sum_{i \in I} a_i)^q \leq \max(1, c^{q-1}) \sum_{i \in I} a_i^q$.

Before stating and proving the next proposition we need the following definition. Namely, for $r > 0$, we will say that a subset $F$ of $\mathbb{R}^d$ is $r$-separated if

$$B(x, r) \cap B(y, r) = \emptyset$$

for all $x, y \in F$.

Proposition 7.6. Fix $q \in \mathbb{R}$ and assume that the OSC is satisfied.

Let $m \in \mathbb{N}$. Let $i, j \in \Sigma^*$ with $|i| = |j| = m$ and $i \neq j$.

There exists a constant $k_{1,j,m} > 0$ such that if $r > 0$ with $r < r_m$ and $F \subseteq B(S_i K, r) \cap B(S_j K, r)$ is a $(1 + \frac{\rho_2}{r^2})r$-separated set, then we have

$$\sum_{x \in F} \mu(B(x, r))^q \leq k_{1,j,m} Z_m^q(r)$$

(recall, that $\rho_2$ is a positive real number such that the non-empty and open set $U$ from (7.1) contains a ball of radius equal to $\rho_2$, and that $Z_m^q(r)$ is defined in (7.8)).

Proof
Let $U, l$ and $d_0$ be as in (7.1), (7.2) and (7.3), respectively. Fix $0 < r < r_m$. For each $x \in F$ we may choose $k(x) \in \Sigma^*_r$ such that $x \in S_{k(x)} K$. We clearly have

$$S_{k(x)} K \subseteq B(x, r) \cap K \subseteq \bigcup_{k \in \Sigma^*_r \text{ dist}(x, S_k K) \leq r} S_k K,$$
for all \( x \in F \), whence
\[
\mu(B(x, r))^q \leq \begin{cases} 
\mu(S_k(x)K)^q & \text{for } q \leq 0; \\
\mu \left( \bigcup_{k \in \Sigma^*_r \text{ dist}(x, S_kK) \leq r} S_kK \right)^q & \text{for } 0 \leq q; 
\end{cases}
\]
for all \( x \in F \). This implies that
\[
\sum_{x \in F} \mu(B(x, r))^q \leq \begin{cases} 
\sum_{x \in F} \mu(S_k(x)K)^q & \text{for } q \leq 0; \\
\left( \sum_{k \in \Sigma^*_r \text{ dist}(x, S_kK) \leq r} \mu(S_kK) \right)^q & \text{for } 0 \leq q,
\end{cases}
\]
for all \( r > 0 \) and all \( x \in \mathbb{R}^d \).

**Proof of Claim 1.** Recall that \( U \) contains a ball of radius \( \rho_1 \) and is contained in a ball of radius \( \rho_2 \).

For \( k \in \Sigma^*_r \), we therefore conclude that \( S_kU \) contains a ball of radius \( r_k \rho_1 \) and that \( r_k \rho_1 \geq r_k r_{\min} \rho_1 \geq \frac{\mu(S_kK)}{\rho_1} r \) (because \( k \in \Sigma^*_r \)). We deduce from this that \( S_kU \) contains a ball of radius \( \frac{\mu(S_kK)}{\rho_1} r \).

For \( k \in \Sigma^*_r \), we also conclude that \( S_kU \) is contained in a ball of radius \( r_k \rho_2 \) and that \( r_k \rho_2 \leq \frac{\rho_2}{2\alpha} r \) (because \( k \in \Sigma^*_r \)). We deduce from this that \( S_kU \) is contained in a ball of radius \( \frac{\rho_2}{2\alpha} r \).

Next, since \((S_kU)_{k \in \Sigma^*_r}\) is a pairwise disjoint family of sets with \( S_kK \subseteq \overline{S_kU} \), Lemma 7.4 therefore implies that
\[
\left| \left\{ k \in \Sigma^*_r \mid \text{dist}(x, S_kK) \leq r \right\} \right| \leq \left| \left\{ k \in \Sigma^*_r \mid S_kK \cap B(x, r) \neq \emptyset \right\} \right| \leq \left| \left\{ k \in \Sigma^*_r \mid \overline{S_kU} \cap B(x, r) \neq \emptyset \right\} \right| \leq C_0
\]
(7.15)
for all \( x \), where \( C_0 = ((1 + 2^{\frac{\rho_2}{D_0}})/\rho_{\min})^d \).

Finally, we deduce from (7.15) and Lemma 7.5 that

\[
\left( \sum_{k \in \Sigma^*_r} \frac{\mu(S_kK)}{\text{dist}(x,S_kK) \leq r} \right)^q \leq C_1 \sum_{k \in \Sigma^*_r} \frac{\mu(S_kK)^q}{\text{dist}(x,S_kK) \leq r},
\]

where \( C_1 = \max(1, C_0^{q-1}) \). This completes the proof of Claim 1.

**Claim 2.** There is a constant \( C_2 > 0 \) such that

\[
\sum_{x \in F \setminus S_kU} \sum_{k \in \Sigma^*_r} \frac{\mu(S_kK)^q}{\text{dist}(x,S_kK) \leq r} \leq C_2 \sum_{|u|=m} \sum_{|v|=M_m} \sum_{h \in \Sigma^*_r} \mu(S_{uhv}K)^q
\]

for all \( r > 0 \).

**Proof of Claim 2.** Again, recall that \( U \) contains a ball of radius \( \rho_1 \) and is contained in a ball of radius \( \rho_2 \).

Fix \( y \in \mathbb{R}^d \).

We first prove that if \( y \in \mathbb{R}^d \), then

\[
\bigcup_{k \in \Sigma^*_r} S_kU \subseteq B(y, (1 + \frac{\rho_2}{\rho_0})r).
\]  

(7.16)

Indeed, this follows from the fact that \( S_kU \) is contained in a ball of radius \( r_k \rho_2 \) and \( r_k \rho_2 \leq \frac{\rho_2}{\rho_0} r_k \) for all \( k \in \Sigma^*_r \). This proves (7.16).

Next, for \( k \in \Sigma^*_r \), we conclude that \( S_kU \) contains a ball of radius \( r_k \rho_1 \) and that \( r_k \rho_1 \geq r_k \rho_{\min} \rho_1 \geq \frac{\rho_{\min}}{D_0} r_k \) (because \( k \in \Sigma^*_r \)). We deduce from this that:

\[
\bigcup_{k \in \Sigma^*_r} S_kU \text{ contains a ball of radius } \frac{\rho_{\min}}{D_0} r.
\]  

(7.17)

Also, we deduce from (7.16) that:

\[
\bigcup_{k \in \Sigma^*_r} S_kU \text{ is contained in a ball of radius } \left(1 + \frac{\rho_2}{\rho_0}\right) r.
\]  

(7.18)

Next, we prove that:

\[
\left( \bigcup_{k \in \Sigma^*_r} S_kU \right)_{y \in F} \text{ is a pairwise disjoint family of open sets.}
\]  

(7.19)

Indeed, this follows from (7.16) and the fact that \( F \) is \((1 + \frac{\rho_2}{\rho_0})r\)-separated. This proves (7.19).
Finally, we deduce from (7.17), (7.18), (7.19) and Lemma 7.4 that if \( x \in F \), then

\[
\left\{ y \in F \left| \left( \bigcup_{k \in \Sigma^*_K} S_k K \right) \cap \left( \bigcup_{\text{dist}(y,S_k K) \leq r} S_k K \right) \neq \emptyset \right. \right\} 
\leq \left\{ y \in F \left| \left( \bigcup_{k \in \Sigma^*_K} S_k K \right) \cap B(x, (1 + \frac{D_0}{L_0})r) \neq \emptyset \right. \right\}
\leq \left\{ y \in F \left| \left( \bigcup_{k \in \Sigma^*_K} S_k U \right) \cap B(x, (1 + \frac{D_0}{L_0})r) \neq \emptyset \right. \right\}
\leq \left\{ y \in F \left| \left( \bigcup_{k \in \Sigma^*_K} S_k U \right) \cap B(x, (1 + \frac{D_0}{L_0})r) \neq \emptyset \right. \right\}
\leq C_2
\]

where \( C_2 = (\left((1 + \frac{D_0}{L_0}) + 2(1 + \frac{D_0}{D_0})\right)/(\frac{D_0}{L_0}))^d \).

We deduce from (7.20) that each term in the sum

\[
\sum_{x \in F} \sum_{\text{dist}(x,S_k K) \leq r} \mu(S_k K)^q
\]

is repeated at most \( C_2 \) times. This observation and Lemma 7.2 (which is applicable since \( 0 < r < r_m \)) now gives

\[
\sum_{x \in F} \sum_{\text{dist}(x,S_k K) \leq r} \mu(S_k K)^q \leq C_2 \sum_{|u|=m} \sum_{|v|=M_m} \sum_{h \in \Sigma^*_K} \sum_{\text{uhv} \in \Sigma^*_K} \mu(S_{uhv} K)^q.
\]

This completes the proof of Claim 2.
Combining (7.14), Claim 1 and Claim 2 now yields

\[
\sum_{x \in F} \mu(B(x, r))^q \leq \sum_{x \in F} \left( \sum_{k \in \Sigma^*_{\delta(x, S_k K) \leq r}} \mu(S_k K) \right)^q \quad \text{for } q \leq 0;
\]

\[
\leq \sum_{x \in F} \sum_{k \in \Sigma^*_{\delta(x, S_k K) \leq r}} \mu(S_k K)^q \quad \text{for } q \leq 0;
\]

\[
\leq C_1 \sum_{x \in F} \sum_{k \in \Sigma^*_{\delta(x, S_k K) \leq r}} \mu(S_k K)^q \quad \text{for } q \leq 0;
\]

\[
\leq C_1 C_2 \sum_{|u|=m} \sum_{|v|=M_m} \sum_{h \in \Sigma^*_{uhv}} \sum_{|b| \geq |l|} \sum_{1 \neq h} \mu(S_{uhv} K)^q,
\]

\[\text{(7.21)}\]

where the last equality is due to Lemma 7.3. However, if \(u, h, v \in \Sigma^*\) with \(|u|=m, |v|=M_m\) and \(uhv \in \Sigma^*\), then \(r_h = \frac{\max a}{r_{a,r}} \leq \frac{x}{D_{\min}} = b_m r\) and \(r_h = \frac{\max a}{r_{a,r}} \geq \frac{r_{a,r}}{D_{\max}} = a_m r\). We deduce from this and (7.21) that

\[
\sum_{x \in F} \mu(B(x, r))^q \leq C_1 C_2 \sum_{|u|=m} \sum_{|v|=M_m} \sum_{h \in \Sigma^*_{a_m \leq r_h \leq b_m r}} \sum_{|b| \geq |l|} \sum_{1 \neq h} p_{uhv}^q p_h^q p_v^q,
\]

\[
\leq C_1 C_2 N^{m+M_m} \sum_{h \in \Sigma^*_{a_m \leq r_h \leq b_m r}} \sum_{|b| \geq |l|} \sum_{1 \neq h} p_h^q
\]

\[
= k_{1,j,m} z_m^q(r)
\]

where \(k_{1,j,m} = C_1 C_2 N^{m+M_m} (\max_i p_i^q)^{m+M_m}\). □

We now turn towards the proof of the main result in this section, namely, Proposition 7.8. However, in order to deduce Proposition 7.8 from Proposition 7.6 we need the following simple covering lemma.
Lemma 7.7. Let $d \in \mathbb{N}$ and $u \geq 1$. Then there exists a positive integer $\chi \in \mathbb{N}$ satisfying the following: if $E \subseteq \mathbb{R}^d$ is a bounded set and $s > 0$, then there are sets $F_1, \ldots, F_\chi \subseteq E$ such that

1. The set $F_i$ is $us$-separated for each $i$.
2. We have $E \subseteq \bigcup_{i=1}^\chi \bigcup_{y \in F_i} B(y, s)$.

Proof
This is easily proved and the proof is therefore omitted. □

We can now prove Proposition 7.8.

Proposition 7.8. Fix $q \in \mathbb{R}$ and assume that the OSC is satisfied.

Let $m \in \mathbb{N}$. Let $i, j \in \Sigma^*$ with $|i| = |j| = m$ and $i \neq j$.

There exists a constant $c_{i,j,m} > 0$ such that if $r > 0$ with $r < \frac{1}{2} r_m$, then we have

$$Q^q_{i,j}(r)\leq \begin{cases} c_{i,j,m} Z^q_m \left(\frac{1}{2} r\right) & \text{for } q < 0; \\ c_{i,j,m} Z^q_m (2r) & \text{for } 0 \leq q. \end{cases}$$

(recall, that $Z^q_m (r)$ is defined in (7.8)).

Proof
It follows from Lemma 7.7 that there is a positive integer $\chi$ such that for all $r > 0$ we can find sets $F_{r,1}, \ldots, F_{r,\chi} \subseteq B(S_i K, r) \cap B(S_j K, r)$ satisfying:

the set $F_{r,i}$ is $(1 + \frac{\rho D}{2})2r$-separated for each $i$, \hfill (7.22)

and

$$B(S_i K, r) \cap B(S_j K, r) \subseteq \bigcup_{i=1}^\chi \bigcup_{y \in F_{r,i}} B(y, \frac{1}{2} r).$$ \hfill (7.23)

Fix $0 < r < \frac{1}{2} r_m$. It follows from (7.23) that

$$Q^q_{i,j}(r) = \frac{1}{r^d} \int_{B(S_i K, r) \cap B(S_j K, r)} \mu(B(x, r))^q \, d\mathcal{L}^d(x)$$

$$\leq \frac{1}{r^d} \sum_{i=1}^\chi \sum_{y \in F_{r,i}} \int_{B(y, \frac{1}{2} r)} \mu(B(x, r))^q \, d\mathcal{L}^d(x).$$ \hfill (7.24)

Now, note that for $x \in B(y, \frac{1}{2} r)$, we clearly have $B(y, \frac{1}{2} r) \subseteq B(x, r) \subseteq B(y, 2r)$, whence

$$\mu(B(x, r))^q \leq \begin{cases} \mu(B(y, \frac{1}{2} r))^q & \text{for } q < 0; \\ \mu(B(y, 2r))^q & \text{for } 0 \leq q. \end{cases}$$ \hfill (7.25)

Next, writing $\Omega_d = \mathcal{L}^d(B(0, 1))$ for the volume of the unit ball in $\mathbb{R}^d$ and combining (7.24)
and (7.25) gives

\[
Q_{1,i,j}(r) \leq \begin{cases}
\frac{1}{r_d} \sum_{i=1}^{\chi} \sum_{y \in F_{r,i}} \mu(B(y, \frac{1}{2}r))^q d\mathcal{L}^d(x) & \text{for } q < 0; \\
\frac{1}{r_d} \sum_{i=1}^{\chi} \sum_{y \in F_{r,i}} \mu(B(y, 2r))^q d\mathcal{L}^d(x) & \text{for } 0 \leq q;
\end{cases}
\]

(7.26)

However, since both \( \frac{1}{2}r < r_m \) and \( 2r < r_m \) and, in addition, the set \( F_{r,i} \) is \( (1 + \frac{\kappa_0}{D_0}) \frac{1}{2}r \)-separated (cf. (7.22)), and therefore, in particular, \( (1 + \frac{\kappa_0}{D_0}) \frac{1}{2}r \)-separated, we conclude from Proposition 7.6 that

\[
\sum_{y \in F_{r,i}} \mu(B(y, \frac{1}{2}r))^q \leq k_{1,i,m} Z_m^q (\frac{1}{2}r)
\]

(7.27)

and

\[
\sum_{y \in F_{r,i}} \mu(B(y, 2r))^q \leq k_{1,i,m} Z_m^q (2r)
\]

(7.28)

for all \( i \) where \( k_{1,i,m} \) is the constant in Proposition 7.6. Finally, we deduce from (7.26), (7.27) and (7.28) that

\[
Q_{1,i,j}(r) \leq \begin{cases}
\frac{\Omega_d}{2^d} \sum_{i=1}^{\chi} k_{1,i,m} Z_m^q (\frac{1}{2}r) & \text{for } q < 0; \\
\frac{\Omega_d}{2^d} \sum_{i=1}^{\chi} k_{1,i,m} Z_m^q (2r) & \text{for } 0 \leq q.
\end{cases}
\]

\[
= \begin{cases}
c_{i,j,m} Z_m^q (\frac{1}{2}r) & \text{for } q < 0; \\
c_{i,j,m} Z_m^q (2r) & \text{for } 0 \leq q,
\end{cases}
\]

where \( c_{i,j,m} = \frac{\Omega_d}{2^d} \chi k_{1,i,m} \). This completes the proof. \( \square \)
8. Proving that

\[ Z_m^q(r) \leq \text{constant } r^{-\gamma(q)} \]

The main purpose of this section is to prove Proposition 8.2. Let \( l \) be as in (7.2) and fix \( q \in \mathbb{R} \). Observe that that function \( \Xi_q : s \to \sum_{|i|=|l|, i \neq l} p_i^q r_i^{\gamma(q)} \) is continuous and strictly decreasing with \( \lim_{s \to -\infty} \Xi_q(s) = \infty \) and \( \lim_{s \to \infty} \Xi_q(s) = 0 \). Hence, there exists a unique \( \gamma(q) \in \mathbb{R} \) such that

\[ \sum_{|i|=|l|, i \neq l} p_i^q r_i^{\gamma(q)} = 1. \quad (8.1) \]

Also, note that since \( \sum_{|i|=|l|, i \neq l} p_i^q r_i^{\gamma(q)} = 1 = \sum_{|i|=|l|} p_i^q r_i^{\beta(q)} \) it follows that

\[ \gamma(q) < \beta(q). \quad (8.2) \]

The main purpose of this section is to prove Proposition 8.2 saying that for each positive integer \( m \), there is a constant \( c_m \) such that for all \( r > 0 \), we have

\[ Z_m^q(r) \leq c_m r^{-\gamma(q)} \]

(recall, that \( Z_m^q(r) \) is defined in (7.8)). We begin with a small lemma.

**Lemma 8.1.** Fix \( q \in \mathbb{R} \). Let \( m \in \mathbb{N} \). For \( c > 0 \), we have

\[ \sup_{c \leq r} r^{\gamma(q)} Z_m^q(r) < \infty \]

(recall, that \( Z_m^q(r) \) is defined in (7.8)).

**Proof**

Let

\[ \Gamma_m = \left\{ h \in \Sigma^* \mid a_m c \leq r_h \right\}. \]

Observe that if \( h \in \Gamma_m \), then \( a_m c \leq r_h \leq |h| \leq \frac{\log a_m c}{\log r_{\max}} \), and since there are only finitely many strings \( h \in \Sigma^* \) with \( |h| \leq \frac{\log a_m c}{\log r_{\max}} \), we deduce from this that

\[ |\Gamma_m| < \infty. \quad (8.3) \]

Next, note that if \( h \in \Sigma^* \) satisfies \( a_m r \leq r_h \leq b_m r \), then \( r^{\gamma(q)} \leq \frac{1}{a_m^{\gamma(q)} b_m^{\gamma(q)}} r^{\gamma(q)} \) if \( \gamma(q) \geq 0 \) and \( r^{\gamma(q)} \leq \frac{1}{b_m^{\gamma(q)} b_m^{\gamma(q)}} r^{\gamma(q)} \) if \( \gamma(q) \leq 0 \). We conclude from this that if \( h \in \Sigma^* \) satisfies \( a_m r \leq r_h \leq b_m r \), then

\[ r^{\gamma(q)} \leq \max \left( \frac{1}{a_m^{\gamma(q)}}, \frac{1}{b_m^{\gamma(q)}} \right) r^{\gamma(q)} \quad (8.4) \]

for all \( q \).
Now combining (8.3) and (8.4) gives

$$\sup_{\gamma \leq r} r^{\gamma(q)} Z_m^q(r) \leq \sup_{\gamma \leq r} \sum_{h \in \Sigma^*} r^{\gamma(q)} p_h^q$$

$$\leq \max \left( \frac{1}{a_m^{\gamma(q)}}, \frac{1}{b_m^{\gamma(q)}} \right) \sup_{\gamma \leq r} \sum_{h \in \Sigma^*} r_h^{\gamma(q)} p_h^q$$

$$\leq \max \left( \frac{1}{a_m^{\gamma(q)}}, \frac{1}{b_m^{\gamma(q)}} \right) \sum_{h \in \Sigma^*} r_h^{\gamma(q)} p_h^q$$

$$\leq \max \left( \frac{1}{a_m^{\gamma(q)}}, \frac{1}{b_m^{\gamma(q)}} \right) \sum_{h \in \Gamma_m} r_h^{\gamma(q)} p_h^q$$

$$\leq \infty,$$

where the sum $\sum_{h \in \Gamma_m} r_h^{\gamma(q)} p_h^q$ is finite since the set $\Gamma_m$ is finite by (8.3). \qed

**Proposition 8.2.** Fix $q \in \mathbb{R}$. Let $l$ be as in (7.2). Let $m \in \mathbb{N}$. Then there exists a constant $c_m > 0$ such that for $r > 0$, we have

$$Z_m^q(r) \leq c_m r^{-\gamma(q)}$$

(recall, that $Z_m^q(r)$ is defined in (7.8)).

*Proof*

Choose $\delta_m > 0$ such that $b_m^\delta \leq l_{\min}$. Next, define $W_m^q : (0, \infty) \to [0, \infty)$ by $W_m^q(r) = r^{\gamma(q)} Z_m^q(r)$.

Observe that for all $r > 0$, we have

$$Z_m^q(r) = \sum_{h \in \Sigma^*} p_h^q$$

$$= \sum_{|i|=|l|} \sum_{j \in \Sigma^*} p_{ij}^q$$

$$\leq \sum_{|i|=|l|} \sum_{j \in \Sigma^*} p_{ij}^q.$$  \hspace{1cm} (8.5)

However, if $0 < r < \delta_m$ and $i, j \in \Sigma^*$ satisfy $|i| = |l|$ and $r_j \leq b_m \delta_i$, then $r_i^{\beta} \leq r_j \leq b_m \delta_i \leq b_m \delta_i \leq b_m \delta_i = b_m \delta_i \leq r_j \leq b_m \delta_i \leq b_m \delta_i \leq r_i^{\beta}$, whence $|j| \geq |l|$. We conclude from this and (8.5) that if
0 < r < \delta_m, then

\[ Z^q_m(r) \leq \sum_{|i| = |l|} p_i \sum_{j \in \Sigma^* \atop |j| \geq |i|} p^q_j \]

\[ = \sum_{|i| = |l|} p_i Z^q_m(\frac{r_i}{r_l}) \cdot \]

Hence, for 0 < r < \delta_m we obtain

\[ W^q_m(r) = r^{\gamma(q)} Z^q_m(r) \]

\[ \leq \sum_{|i| = |l|} p_i r^{\gamma(q)}(\frac{r_i}{r_l})^{\gamma(q)} Z^q_m(\frac{r_i}{r_l}) \]

\[ = \sum_{|i| = |l|} p_i r^{\gamma(q)} W^q_m(\frac{r_i}{r_l}). \quad (8.6) \]

Let \( \Delta = r^{||l||}_{\max} \). It follows from (8.6) and definition (8.1) of \( \gamma(q) \) that, if 0 < a < \delta_m, then

\[ \sup_{a \Delta < r < \delta_m} W^q_m(r) \leq \sup_{a \Delta < r < \delta_m} \sum_{|i| = |l|} p_i r^{\gamma(q)} W^q_m(\frac{r_i}{r_l}) \]

\[ \leq \sum_{|i| = |l|} p_i r^{\gamma(q)} \sup_{a \Delta < r < \delta_m} W^q_m(\frac{r_i}{r_l}) \]

\[ \leq \sum_{|i| = |l|} p_i r^{\gamma(q)} \sup_{a \leq s} W^q_m(s) \]

\[ = \sup_{a \leq s} W^q_m(s), \quad (8.7) \]

We deduce from (8.7) that if 0 < a < \delta_m, then

\[ \sup_{a \Delta \leq r} W^q_m(r) \leq \max \left( \sup_{a \Delta < r < \delta_m} W^q_m(r), \sup_{\delta_m < r} W^q_m(r) \right) \]

\[ \leq \max \left( \sup_{a \leq r} W^q_m(r), \sup_{\delta_m < r} W^q_m(r) \right) \]

\[ = \sup_{a \leq r} W^q_m(r). \quad [\text{since } a < \delta_m] \quad (8.8) \]

Next, choose a positive integer \( k_m \) such that \( \Delta^k_m < \delta_m \). Repeated applications of (8.8)
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now gives

\[
\sup_{0 < r} W^q_m(r) = \sup_{k \in \mathbb{N}} \sup_{\Delta^{k+m} \leq r} W^q_m(r) \\
\leq \sup_{k \in \mathbb{N}} \sup_{\Delta^{k+m-1} \leq r} W^q_m(r) \\
\vdots \\
\leq \sup_{k \in \mathbb{N}} \sup_{\Delta^m \leq r} W^q_m(r) \\
\leq \sup_{\Delta^m \leq r} r^\gamma(q) Z^q_m(r)
\]

Finally, combining (8.9) and Lemma 8.1 shows that \( r^\gamma(q) Z^q_m(r) = W^q_m(r) \leq \sup_{s > 0} W^q_m(s) \leq \sup_{\Delta^m \leq s} s^\gamma(q) Z^q_m(s) < \infty \). This completes the proof of Proposition 8.2. \( \square \)

9. Proof of Theorem 3.3

The purpose of this section is to prove Theorem 3.3. The proof are based on renewal theory and, in particular, a recent renewal theorem by Levitin & Vassiliev [LeVa]. Below we state Levitin & Vassiliev’s renewal theorem.

**Theorem 9.1.** Levitin & Vassiliev’s renewal theorem [LeVa]. Let \( t_1, \ldots, t_N > 0 \) and \( p_1, \ldots, p_N > 0 \) with \( \sum_i p_i = 1 \). Define the probability measure \( P \) by

\[
P = \sum_i p_i \delta_{t_i}.
\]

Let \( \lambda, \Lambda : \mathbb{R} \to \mathbb{R} \) be real valued functions satisfying the following conditions:

(i) The function \( \lambda \) is piecewise continuous;

(ii) There are constants \( c, k > 0 \) such that

\[
|\lambda(t)| \leq ce^{-k|t|}
\]

for all \( t \in \mathbb{R} \);

(iii) We have

\[
\Lambda(t) \to 0 \text{ as } t \to -\infty;
\]

(iv) We have

\[
\Lambda(t) = \int \Lambda(t - s) dP(s) + \lambda(t)
\]

for all \( t \in \mathbb{R} \).

The following holds:

(1) The non-arithmetic case: If \( \{t_1, \ldots, t_N\} \) is not contained in a discrete additive subgroup of \( \mathbb{R} \), then

\[
\Lambda(t) = c + \varepsilon(t)
\]
for all \( t \in \mathbb{R} \) where
\[
c = \frac{1}{s} \int \lambda(s) ds ,
\]
\[
\varepsilon(t) \to 0 \text{ as } t \to \infty .
\]

In addition,
\[
\frac{1}{T} \int_0^T \Lambda(t) dt \to c = \frac{1}{s} \int \lambda(s) ds \text{ as } T \to \infty . \tag{9.1}
\]

(2) The arithmetic case: If \( \{t_1, \ldots, t_N\} \) is contained in a discrete additive subgroup of \( \mathbb{R} \) and \( \langle t_1, \ldots, t_N \rangle = u\mathbb{Z} \) with \( u > 0 \), then
\[
\Lambda(t) = \pi(t) + \varepsilon(t)
\]
for all \( t \in \mathbb{R} \) where
\[
\pi(t) = \frac{1}{s} \int \lambda(s) ds,
\]
\[
\varepsilon(t) \to 0 \text{ as } t \to \infty .
\]

In addition, the function \( \pi \) is \( u \)-periodic and
\[
\frac{1}{T} \int_0^T \Lambda(t) dt \to c = \frac{1}{s} \int \lambda(s) ds \text{ as } T \to \infty . \tag{9.2}
\]

Proof
All statements, except (9.1) and (9.2), follow [LeVa]. Below we prove (9.1) and (9.2). Indeed, (9.1) follows immediately and (9.2) is proved as follows. Namely, since \( \pi \) is \( u \)-periodic we conclude that
\[
\frac{1}{T} \int_0^T \Lambda(t) dt = \frac{1}{T} \int_0^T \pi(t) dt + \frac{1}{T} \int_0^T \varepsilon(t) dt
\]
\[
\to \frac{1}{u} \int_0^u \pi(t) dt
\]
\[
= \frac{1}{s} \int dP(t) \int_0^u \sum_{n \in \mathbb{Z}} \lambda(t + nu) dt . \tag{9.3}
\]

Now observe that since \( |\lambda(t)| \leq ce^{-k|t|} \) for all \( t \in \mathbb{R} \) and \( \int ce^{-k|t|} dt < \infty \), it follows from two applications of Lebesgue’s Dominated Convergence Theorem and the fact that \( \pi \) is \( u \)-periodic that
\[
\int_0^u \sum_{n \in \mathbb{Z}} \lambda(t + nu) dt = \sum_{n \in \mathbb{Z}} \int_0^u \lambda(t + nu) dt
\]
\[
= \sum_{n \in \mathbb{Z}} \int_{nu}^{(n+1)u} \lambda(t) dt
\]
\[
= \sum_{n \in \mathbb{Z}} \int_{nu}^{(n+1)u} 1_{[nu,(n+1)u)}(t) \lambda(t) dt
\]
\[
= \int \sum_{n \in \mathbb{Z}} 1_{[nu,(n+1)u)}(t) \lambda(t) dt
\]
\[
= \int \lambda(t) dt . \tag{9.4}
\]
Finally, combining (9.3) and (9.4) shows that
\[
\frac{1}{T} \int_0^T \Lambda(t) \, dt \to \frac{1}{\int t \, dP(t)} \int \lambda(t) \, dt .
\]

This completes the proof. \(\blacksquare\)

In order to prove Theorem 3.3, we will apply Levitin \& Vassiliev’s renewal theorem to the probability measure \(P = P_q\) and the functions \(\lambda = \lambda_q\) and \(\Lambda = \Lambda_q\) defined as follows. Namely, first recall that \(\lambda_q : (0, \infty) \to \mathbb{R}\) is defined by
\[
\lambda_q(r) = V_{\mu,r}^q(K) - \sum_i p_i^q \mathbf{1}_{(0,r_i]}(r) V_{\mu,r_i^{-1}r}^q(K)
\]
and define \(\Lambda_q : (0, \infty) \to \mathbb{R}\) by
\[
\Lambda_q(r) = V_{\mu,r}^q(K).
\]

Also, define \(\lambda_0^q : \mathbb{R} \to \mathbb{R}\) by
\[
\lambda_0^q(t) = \mathbf{1}_{[0,\infty)}(t) e^{-t\beta(q)} \lambda_q(e^{-t})
\]
and define \(\Lambda_0^q : \mathbb{R} \to \mathbb{R}\) by
\[
\Lambda_0^q(t) = \mathbf{1}_{[0,\infty)}(t) e^{-t\beta(q)} \Lambda_q(e^{-t}) .
\]

Finally, define the probability measure \(P_q\) by
\[
P_q = \sum_i p_i^q \delta_k^{-1} \log r_i .
\]

Next, we show (in Propositions 9.2–9.5) that probability measure \(P = P_q\) and the functions \(\lambda = \lambda_0^q\) and \(\Lambda = \Lambda_0^q\) satisfy conditions (i)–(iv) in Levitin \& Vassiliev’s renewal theorem.

**Proposition 9.2.** Fix \(q \in \mathbb{R}\) and assume that one of the following conditions is satisfied:

(i) The OSC is satisfied and \(0 \leq q\);

(ii) The SSC is satisfied.

Then the function \(\lambda_0^q\) is piecewise continuous.

**Proof**

Define \(f : (0, \infty) \to \mathbb{R}\) by
\[
f(r) = \int_{B(K,r)} \mu(B(x,r))q \, dL^d(x) .
\]

It clearly suffices to show that \(f\) is continuous. We therefore fix \(r_0 > 0\) and prove that \(f\) is continuous at \(r_0\). For \(0 < h < 1\), define \(\varphi, \varphi_h^-, \varphi_h^+ : \mathbb{R}^d \to \mathbb{R}\) by
\[
\varphi(x) = \mathbf{1}_{B(K,r_0)}(x) \mu(B(x,r_0))q,
\]
\[
\varphi_h^-(x) = \mathbf{1}_{B(K,r_0-h)}(x) \mu(B(x,r_0-h))q,
\]
\[
\varphi_h^+(x) = \mathbf{1}_{B(K,r_0+h)}(x) \mu(B(x,r_0+h))q .
\]
Since $\cup_{0<h<1}B(x, r_0 - h) = B(x, r_0)$, we conclude that $\mu(B(x, r_0 - h)) \to \mu(B(x, r_0))$ as $h \to 0$, whence $\varphi_h^-(x) \to \varphi(x)$ for all $x$ as $h \to 0$. We also have $|\varphi_h^+(x)| \leq 1_{B(K,r_0)}(x)$ for all $x$ where $\int 1_{B(K,r_0)}(x) \, d\mathcal{L}^d(x) = \mathcal{L}^d(B(K,r_0)) < \infty$, and we therefore conclude from Lebesgue’s Dominated Convergence Theorem that

$$f(r_0 - h) = \int \varphi_h^-(x) \, d\mathcal{L}^d(x) \to \int \varphi(x) \, d\mathcal{L}^d(x) = f(r_0) \quad (9.5)$$

as $h \to 0$.

Similarly, since $\cap_{0<h<1}B(x, r_0 + h) = B(x, r_0)$, we conclude that $\mu(B(x, r_0 + h)) \to \mu(B(x, r_0))$ as $h \to 0$, whence $\varphi_h^+(x) \to 1_{B(K,r_0)}(x) \, \mu(B(x, r_0))$ for all $x$ as $h \to 0$. We also have $|\varphi_h^+(x)| \leq 1_{B(K,r_0+1)}(x)$ for all $x$ where $\int 1_{B(K,r_0+1)}(x) \, d\mathcal{L}^d(x) = \mathcal{L}^d(B(K,r_0+1)) < \infty$, and we therefore conclude from the Dominated Convergence Theorem that

$$f(r_0 + h) = \int \varphi_h^+(x) \, d\mathcal{L}^d(x) \to \int 1_{B(K,r_0)}(x) \, \mu(B(x, r_0)) \, d\mathcal{L}^d(x) \quad (9.6)$$

as $h \to 0$. Since clearly $\mathcal{L}^d(B(K,r_0)) = \mathcal{L}^d(B(K,r_0))$, we deduce from (9.6) that

$$f(r_0 + h) \to \int 1_{B(K,r_0)}(x) \, \mu(B(x, r_0)) \, d\mathcal{L}^d(x) \quad (9.7)$$

as $h \to 0$. Next, it is proved in [Mat] that either $K$ lies in a $l$-dimensional affine subspace of $\mathbb{R}^d$ with $l < d$ or $\mu(K \cap \Gamma) = 0$ for every $l$ dimensional $C^1$-submanifold $\Gamma \subseteq \mathbb{R}^d$ with $0 < l < d$. This implies that $\mu(B(x, r_0) \setminus B(x, r_0)) = 0$ for all $x$, and we therefore deduce from (9.7) that

$$f(r_0 + h) \to \int 1_{B(K,r_0)}(x) \, \mu(B(x, r_0)) \, d\mathcal{L}^d(x) = f(r_0) \quad (9.8)$$

as $h \to 0$.

Finally, it follows from (9.5) and (9.8) that $f$ is continuous at $r_0$. \hfill \Box

**Proposition 9.3.** Fix $q \in \mathbb{R}$ and assume that one of the following conditions is satisfied:

(i) The OSC is satisfied and $0 \leq q$;

(ii) The SSC is satisfied.

Then there is a constant $c > 0$ such that $|\lambda^q(t)| \leq ce^{-(\beta(q) - \gamma(q))|t|}$ for all $t \in \mathbb{R}$.

**Proof**

For a positive integer $m$ and $i, j \in \Sigma^*$, let $r_m$ be the number defined in (7.5). Also, let $c_{i,j,m}$ be the constant in Proposition 7.8 and let $c_{m}$ be the constant in Proposition 8.2.

Choose $t_0 > 0$ such that $e^{-t} < \min(r_{\min}, \frac{1}{2}r_1)$ for $t \geq t_0$, and observe that for $t \geq t_0$, we
have

\[ |\lambda^0_q(t)| = 1_{(0, \infty)}(t) \cdot e^{-t\beta(q)} |\lambda_q(e^{-t})| \]

\[ = e^{-t\beta(q)} \left| V^q_{\mu, e^{-t}}(K) - \sum_i p_i^q 1_{(0, r_i]}(e^{-t}) V^q_{\mu, e^{-t}}(K) \right| \]

\[ = e^{-t\beta(q)} \left| V^q_{\mu, e^{-t}}(K) - \sum_i p_i^q V^q_{\mu, e^{-t}}(K) \right| \quad \text{[since } e^{-t} < r_{\min} \leq r_i \text{]} \]

\[ \leq e^{-t\beta(q)} \sum_{|i| = |j| = 1 \atop i \neq j} \lambda^0_{q, i,j} (e^{-t}) \quad \text{[by Proposition 6.3]} \]

\[ \leq \begin{cases} e^{-t\beta(q)} \sum_{|i| = |j| = 1 \atop i \neq j} c_{i,j} \cdot Z^q_i \left( \frac{1}{2} e^{-t} \right) & \text{for } q < 0; \\ e^{-t\beta(q)} \sum_{|i| = |j| = 1 \atop i \neq j} c_{i,j} \cdot Z^q_i \left( 2e^{-t} \right) & \text{for } 0 \leq q \quad \text{[by Proposition 7.8]} \end{cases} \]

\[ \leq \begin{cases} e^{-t\beta(q)} \sum_{|i| = |j| = 1 \atop i \neq j} c_{i,j} \cdot \left( \frac{1}{2} e^{-t} \right)^{-\gamma(q)} & \text{for } q < 0; \\ e^{-t\beta(q)} \sum_{|i| = |j| = 1 \atop i \neq j} c_{i,j} \cdot \left( 2e^{-t} \right)^{-\gamma(q)} & \text{for } 0 \leq q \quad \text{[by Proposition 8.2]} \end{cases} \]

\[ = ce^{-\beta(q)\gamma(q)t} \quad (9.9) \]

where \( c = c_1 \max(\left( \frac{1}{2} \right)^{-\gamma(q)}, 2^{-\gamma(q)}) \sum_{|i| = |j| = 1 \atop i \neq j} c_{i,j}.1. \)

Next, since \( \lambda^0_q \) is piecewise continuous (by Proposition 9.2), we conclude that \( \lambda^0_q \) is bounded on the compact interval \([0, t_0]\), i.e. there is a constant \( M_0 \) such that \( |\lambda^0_q(t)| \leq M_0 \) for all \( t \in [0, t_0] \). It follows from this and (9.9) that

\[ |\lambda^0_q(t)| \leq \max \left( \frac{M_0}{e^{-\beta(q)\gamma(q)t}}, c \right) e^{-\beta(q)\gamma(q)t} \quad (9.10) \]

for all \( t \geq 0. \)

The statement now follows from (9.10) and the fact that \( \lambda^0_q(t) = 0 \) for all \( t < 0. \) \( \square \)

**Proposition 9.4.** Fix \( q \in \mathbb{R} \). Then \( \Lambda^0_q(t) \to 0 \) as \( t \to -\infty. \)

**Proof**

Indeed, this follows trivially from the fact that \( \Lambda^0_q(t) = 0 \) for all \( t < 0. \) \( \square \)

**Proposition 9.5.** Fix \( q \in \mathbb{R} \). Then \( \Lambda^0_q(t) = \int \Lambda^0_q(t - s) dP_q(s) + \lambda^0_q(t) \) for all \( t \in \mathbb{R}. \)

**Proof**
It follows immediately from the definitions of $\Lambda^0_q$, $\Lambda^0_q$ and $P_q$ that
\[
\Lambda^0_q(t) = 1_{[0, \infty)}(t) e^{-t \beta(q)} \Lambda_q(e^{-t}) \\
= 1_{[0, \infty)}(t) e^{-t \beta(q)} \left( \sum_i p^q_i 1_{(0, r_i)}(e^{-t}) V^q_{\mu_i r_i^{-1} e^{-t}}(K) + \lambda_q(e^{-t}) \right) \\
= \sum_i p^q_i e^{-t \beta(q)} 1_{(0, r_i)}(e^{-t}) 1_{[0, \infty)}(t) V^q_{\mu_i r_i^{-1} e^{-t}}(K) + \lambda_q(t) \\
= \sum_i p^q_i r_i^{\beta(q)} 1_{[0, \infty)}(t - \log r_i^{-1}) 1_{[0, \infty)}(t) e^{-\beta(q)(t-\log r_i^{-1})} V^q_{\mu_i e^{-(t-\log r_i^{-1})}}(K) + \lambda_q(t) \\
= \sum_i p^q_i r_i^{\beta(q)} 1_{[0, \infty)}(t - \log r_i^{-1}) V^q_{\mu_i e^{-(t-\log r_i^{-1})}}(K) + \lambda_q(t) \\
= \sum_i p^q_i r_i^{\beta(q)} \Lambda^0_q(t - \log r_i^{-1}) + \lambda_q(t) \\
= \int \Lambda^0_q(t - s) dP_q(s) + \lambda^0_q(t)
\]
for all $t \in \mathbb{R}$. $\square$

We can now prove Theorem 3.3.

Proof of Theorem 3.3

It follows from Propositions 9.2–9.5 that Theorem 9.1 can be applied to the probability measure $P = P_q$ and the functions $\lambda^0_q$ and $\Lambda^0_q$. We divide the proof into two cases.

Case 1: If $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$, is not contained in a discrete additive subgroup of $\mathbb{R}$. If $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$, is not contained in a discrete additive subgroup of $\mathbb{R}$, then Theorem 9.1 implies that
\[
\Lambda^0_q(t) = c_q + \varepsilon^0_q(t)
\]
where $c_q \in \mathbb{R}$ is the constant given by
\[
c_q = \frac{1}{\int_{0}^{\infty} dP_q(s)} \int \Lambda^0_q(s) ds \\
= \frac{1}{- \sum_i p^q_i r_i^{\beta(q)} \log r_i} \int_{0}^{\infty} e^{-s \beta(q)} \lambda_q(e^{-s}) ds \\
= \frac{1}{- \sum_i p^q_i r_i^{\beta(q)} \log r_i} \int_{0}^{1} r^{\beta(q)} \lambda_q(r) \frac{dr}{r}
\]
and
\[
\varepsilon^0_q(t) \to 0 \text{ as } t \to \infty.
\]
In particular, we have
\[
r^{\beta(q)} V^q_{\mu, r}(K) = \Lambda^0_q(\log \frac{1}{r}) = c_q + \varepsilon_q(r)
\]
(9.11)
where $\varepsilon_q(r) = \varepsilon^0_q(\log \frac{1}{r}) \to 0$ as $r \searrow 0$.

Finally, it follows from (9.11) that
\[
r^{\beta(q)} V^q_{\mu, r}(K) \to c_q \text{ as } r \searrow 0.
\]
Case 2: If \( \{\log r_1^{-1}, \ldots, \log r_N^{-1}\} \) is contained in a discrete additive subgroup of \( \mathbb{R} \). If \( \{\log r_1^{-1}, \ldots, \log r_N^{-1}\} \) is contained in a discrete additive subgroup of \( \mathbb{R} \) and \( (t_1, \ldots, t_N) = u\mathbb{Z} \) with \( u > 0 \), then Theorem 9.1 implies that
\[
\Lambda_q^0(t) = \pi_q^0(r) + \varepsilon_q^0(t)
\]
where \( \pi_q^0 : \mathbb{R} \to \mathbb{R} \) is the function given by
\[
\pi_q^0(t) = \frac{1}{t} \sum_{n \in \mathbb{Z}} \Lambda_q^0(t + nu)
\]
and
\[
\varepsilon_q^0(t) \to 0 \text{ as } t \to \infty.
\]
Moreover, we have
\[
\pi_q^0(t + u) = \pi_q^0(t)
\]
for all \( t \in \mathbb{R} \), i.e. \( \pi_q^0 \) is additively periodic with period equal to \( u \). In particular, we have
\[
r^\beta(q) V_{\mu,r}^q(K) = \Lambda_q^0(\log \frac{1}{r}) = \pi_q(r) + \varepsilon_q(r)
\]
where \( \pi_q : \mathbb{R} \to \mathbb{R} \) is the function given by
\[
\pi_q(r) = \pi_q^0(\log \frac{1}{r})
\]
and \( \varepsilon_q(r) = \varepsilon_q^0(\log \frac{1}{r}) \to 0 \) as \( r \searrow 0 \). Moreover, since \( \pi_q^0 \) is additively periodic with period equal to \( u \), we have
\[
\pi_q(e^u r) = \pi_q^0(\log \frac{1}{e^u r}) = \pi_q^0(\log \frac{1}{r} - u) = \pi_q^0(\log \frac{1}{r}) = \pi_q(r)
\]
for all \( r > 0 \), i.e. \( \pi_q \) is multiplicatively periodic with period equal to \( e^u \).

Finally it follows from Theorem 9.1 that
\[
\frac{1}{T} \int_0^T \Lambda_q^0(t) \, dt \to c_q \text{ as } T \to \infty.
\]
However, since
\[
\frac{1}{T} \int_0^T \Lambda_q^0(t) \, dt = \frac{1}{T} \int_0^T e^{-t\beta(q)} V_{\mu,e^{-t}}^q(K) \, dt
\]
\[
= \frac{1}{-\log e^{-T}} \int_{e^{-T}}^{1} s^\beta(q) V_{\mu,s}^q(K) \, ds,
\]
we now conclude that
\[
\frac{1}{-\log r} \int_r^1 s^\beta(q) V_{\mu,s}^q(K) \, ds \to c_q \text{ as } r \searrow 0.
\]
This completes the proof of Theorem 3.3 in Case 2. \( \square \)
Part 3:
Proofs of the Results from Section 4

10. Analysis of $\mathcal{H}^{q,\beta(q)}_{\mu}$

The purpose of this section is twofold. Firstly, we prove Theorem 10.3. Secondly, we apply Theorem 10.3 to obtain an explicit formula for the multifractal Hausdorff measure $\mathcal{H}^{q,\beta(q)}_{\mu}(S_1K)$ of $S_1K$, cf. Proposition 10.5.(6); this formula plays an important part in Sections 11–12 when identifying the weak limit of the (suitably normalized) tube measure $I^{q}_{\mu, r}$. We now turn towards the proof of Theorem 10.3. We begin with two auxiliary lemmas.

Lemma 10.1. Let $\mu$ be a probability measure on $\mathbb{R}^d$. Fix $q, t \in \mathbb{R}$ and $E \subseteq \mathbb{R}^d$. Let $\varepsilon > 0$. Then there is a constant $c > 0$ and a sequence $(r_n)_n$ of positive real numbers with $r_n \to 0$ satisfying the following: for all $F \subseteq E$ and $n$, the inequality below is satisfied, namely,

for $q < 0$, we have

$$\frac{\mu^{q,t}}{r_n^{q,t}} (F) \leq c r_n^{t-(\dim_{\mathcal{M}_{\mu}(E)+\varepsilon})}.$$  \hspace{1cm} (10.1)

Proof

Since $\lim_{r \to 0} \frac{\log V_{\mu, r}^q(E)}{-\log r} < \dim_{\mathcal{M}_{\mu}(E)} + \varepsilon$, we can find a sequence $(r_n)_n$ of real numbers with $r_n \to 0$ such that $\frac{\log V_{\mu, r_n}^q(E)}{-\log r_n} < \dim_{\mathcal{M}_{\mu}(E)} + \varepsilon$ for all $n$, whence

$$V_{\mu, r_n}^q(E) \leq r_n^{-(\dim_{\mathcal{M}_{\mu}(E)}+\varepsilon)}$$

for all $n$.

Next, fix $F \subseteq E$ and let $\delta > 0$. It is clear that we can choose a countable centered covering $(B(x_n,i,r_n))_{i \in I_n}$ of $F$ such that there are $2^d$ subsets $I_{n,1}, \ldots, I_{n,2^d}$ of $I$ satisfying:

$$\bigcup_k I_{n,k} = I_n,$$

$$I_{n,k} \cap I_{n,l} = \emptyset \text{ for } k \neq l,$$

the sets $(B(x_n,i,r_n))_{i \in I_{n,k}}$ are pairwise disjoint.
Writing $\Omega_d$ for the Lebesgue measure of the unit ball in $\mathbb{R}^d$, we have
\[
\overline{\mathcal{H}}_{\mu, 2r_n}(F) \leq \sum_{i \in I_n} \mu(B(x_{n,i}, 2r_n))q(4r_n)^t
\]
\[
= \frac{4^d}{\Omega_d} \sum_{i \in I_n} \frac{1}{r_n^d} \int_{B(x_{n,i}, r_n)} \mu(B(x_{n,i}, 2r_n))q d\mathcal{L}^d(x) r_n^t
\text{ for } q < 0,
\]
(10.2)
\[
\overline{\mathcal{H}}_{\mu, \frac{1}{2}r_n}(F) \leq \sum_{i \in I_n} \mu(B(x_{n,i}, \frac{1}{2}r_n))q r_n^t
\]
\[
= \frac{2^d}{\Omega_d} \sum_{i \in I_n} \frac{1}{r_n^d} \int_{B(x_{n,i}, r_n)} \mu(B(x_{n,i}, \frac{1}{2}r_n))q d\mathcal{L}^d(x) r_n^t
\text{ for } 0 \leq q.
\]

However, if $x \in B(x_{n,i}, r_n)$, then clearly $B(x_{n,i}, \frac{1}{2}r_n) \subseteq B(x, r_n) \subseteq B(x_{n,i}, 2r_n)$, whence
\[
\mu(B(x_{n,i}, 2r_n))^q \leq \mu(B(x, r_n))^q \text{ for } q < 0,
\]
\[
\mu(B(x_{n,i}, \frac{1}{2}r_n))^q \leq \mu(B(x, r_n))^q \text{ for } 0 \leq q.
\]
(10.3)

It follows from (10.2) and (10.3) that
\[
\overline{\mathcal{H}}_{\mu, 2r_n}(F) \leq \frac{4^d}{\Omega_d} \sum_{i \in I_n} \frac{1}{r_n^d} \int_{B(x_{n,i}, r_n)} \mu(B(x, r_n))^q d\mathcal{L}^d(x) r_n^t
\]
\[
\leq c_0 \sum_{k=1}^{2^d} \sum_{i \in I_{n,k}} \frac{1}{r_n^d} \int_{B(x_{n,i}, r_n)} \mu(B(x, r_n))^q d\mathcal{L}^d(x) r_n^t
\text{ for } q < 0,
\]
(10.4)
\[
\overline{\mathcal{H}}_{\mu, \frac{1}{2}r_n}(F) \leq \frac{2^d}{\Omega_d} \sum_{i \in I_n} \frac{1}{r_n^d} \int_{B(x_{n,i}, r_n)} \mu(B(x, r_n))^q d\mathcal{L}^d(x) r_n^t
\]
\[
\leq c_0 \sum_{k=1}^{2^d} \sum_{i \in I_{n,k}} \frac{1}{r_n^d} \int_{B(x_{n,i}, r_n)} \mu(B(x, r_n))^q d\mathcal{L}^d(x) r_n^t
\text{ for } 0 \leq q,
\]

where $c_0 = \max(\frac{4^d}{\Omega_d}, \frac{2^d}{\Omega_d})$. Next, using the fact that the sets $(B(x_{n,i}, r_n))_{i \in I_{n,k}}$ are pairwise disjoint and $x_{n,i} \in F \subseteq E$, we conclude from (10.4) that:

for $q < 0$, we have
\[
\overline{\mathcal{H}}_{\mu, 2r_n}(F) \leq c_0 \sum_{k=1}^{2^d} \sum_{i \in I_{n,k}} \frac{1}{r_n^d} \int_{B(x_{n,i}, r_n)} \mu(B(x, r_n))^q d\mathcal{L}^d(x) r_n^t
\]
\[
= c_0 \sum_{k=1}^{2^d} \frac{1}{r_n^d} \int_{\bigcup_{i \in I_{n,k}} B(x_{n,i}, r_n)} \mu(B(x, r_n))^q d\mathcal{L}^d(x) r_n^t
\]
\[
\leq c_0 \sum_{k=1}^{2^d} \frac{1}{r_n^d} \int_{B(E, r_n)} \mu(B(x, r_n))^q d\mathcal{L}^d(x) r_n^t
\]
\[
= c_0 \sum_{k=1}^{2^d} V_{\mu, r_n}(E) r_n^t.
\]
(10.6)
Finally, combining (10.1) and (10.5) shows that

for $q < 0$, we have $H_{\mu,t}^{q,t}(F) \leq c r_n^{-\left(\dim_{M,\mu}^q(E) + \varepsilon\right)}$

for $0 \leq q$, we have $H_{\mu,t}^{q,t}(F) \leq c r_n^{-\left(\dim_{M,\mu}^q(E) + \varepsilon\right)}$

where $c = 2^d c_0$. \hfill $\Box$

**Lemma 10.2.** Let $\mu$ be a probability measure on $\mathbb{R}^d$. Fix $q, t \in \mathbb{R}$ and $E \subseteq \mathbb{R}^d$. Then

$$\dim_{H,\mu}^q(E) \leq \dim_{M,\mu}^q(E) \leq \dim_{M,\mu}^q(E).$$

**Proof**

It clearly suffices to prove that $\dim_{H,\mu}^q(E) \leq \dim_{M,\mu}^q(E)$. Let $\varepsilon > 0$ and write $t = \dim_{M,\mu}^q(E) + \varepsilon$. It follows from Lemma 10.1 that there is a constant $c > 0$ and a sequence $(r_n)_n$ of positive real numbers with $r_n \to 0$ satisfying the following: for all $F \subseteq E$ and $n$, the inequality below is satisfied, namely,

for $q < 0$, we have $H_{\mu,t}^{q,t}(F) \leq c r_n^{-\left(\dim_{M,\mu}^q(E) + \varepsilon\right)}$

for $0 \leq q$, we have $H_{\mu,t}^{q,t}(F) \leq c r_n^{-\left(\dim_{M,\mu}^q(E) + \varepsilon\right)}$

Hence for all $F \subseteq E$, we have

$$H_{\mu,t}^{q,t}(F) = \begin{cases} 
\limsup_n H_{\mu,t}^{q,t}(F) & \text{for } q < 0; \\
\limsup_n H_{\mu,t}^{q,t}(F) & \text{for } 0 \leq q \\
\leq c. 
\end{cases}$$

Since, $F \subseteq E$ was arbitrary, this implies that

$$H_{\mu,t}^{q,t}(E) = \sup_{F \subseteq E} H_{\mu,t}^{q,t}(F) \leq c,$$

whence $\dim_{H,\mu}^q(E) \leq t = \dim_{M,\mu}^q(E) + \varepsilon$. Finally, letting $\varepsilon \searrow 0$ gives the desired result. \hfill $\Box$

We can now prove Theorem 10.3.

**Theorem 10.3.** Fix $q \in \mathbb{R}$ and assume that the OSC is satisfied.

For $i, j \in \Sigma^*$ with $|i| = |j|$ and $i \neq j$, we have

$$\dim_{H,\mu}^q(S_i K \cap S_j K) \leq \dim_{M,\mu}^q(S_i K \cap S_j K) \leq \dim_{M,\mu}^q(S_i K \cap S_j K),$$

whence $\dim_{H,\mu}^q(E) \leq t = \dim_{M,\mu}^q(E) + \varepsilon$. Finally, letting $\varepsilon \searrow 0$ gives the desired result. \hfill $\Box$

**Proof**

For a positive integer $m$ and $i, j \in \Sigma^*$ with $|i| = |j| = m$, let $r_m$ be the number defined
in (7.5). Also, let \( c_{1,j,m} \) be the constant in Proposition 7.8 and let \( c_m \) be the constant in Proposition 8.2.

Now, fix \( i,j \in \Sigma^* \) with \( |i| = |j| \) and write \( m \) for the common value of \(|i|\) and \(|j|\), i.e. write \( |i| = |j| = m \). It clearly suffices to prove that \( \dim_{M,q}(S_i K \cap S_j K) \leq \gamma(q) \). In order to prove \( \dim_{M,q}(S_i K \cap S_j K) \leq \gamma(q) \), we note that for \( r < r_m \), we have

\[
V_{\mu,r}^q (S_i K \cap S_j K) = \frac{1}{r^d} \int_{B(S_i K \cap S_j K, r)} \mu(B(x,r))^q \, d\mathcal{L}^d(x) \\
\leq \frac{1}{r^d} \int_{B(S_i K \cap B(S_j K, r))} \mu(B(x,r))^q \, d\mathcal{L}^d(x) \\
= Q_{i,j}^q(r) \\
\leq \begin{cases} 
  c_{1,j,m} Z_m^q \left( \frac{1}{2} r \right) & \text{for } q < 0; \\
  c_{1,j,m} Z_m^q (2r) & \text{for } 0 \leq q 
\end{cases} \quad \text{[by Proposition 7.8]}
\leq \begin{cases} 
  c_{1,j,m} c_m \left( \frac{1}{2} r \right)^{-\gamma(q)} & \text{for } q < 0; \\
  c_{1,j,m} c_m (2r)^{-\gamma(q)} & \text{for } 0 \leq q 
\end{cases} \quad \text{[by Proposition 8.2]}
\leq c r^{-\gamma(q)} \tag{10.7}
\]

where \( c = \max((\frac{1}{2})^{-\gamma(q)}, 2^{-\gamma(q)}) c_m c_{1,j,m} \). It follows immediately from inequality (10.7) that \( \dim_{M,q}(S_i K \cap S_j K) \leq \gamma(q) \).

Next, we prove Proposition 10.5 providing an explicit formula for the multifractal Hausdorff measure \( \mathcal{H}^q_{\mu,\beta(q)}(S_i K) \) of \( S_i K \). We begin with some definitions and an auxiliary lemma. Let \( q \in \mathbb{R} \) and \( E \subseteq \mathbb{R}^d \). For a locally finite measure \( \nu \) on \( \mathbb{R}^d \) and a bi-measurable map \( T : \mathbb{R}^d \to \mathbb{R}^d \), the lower and upper \( q \)-th order Jacobian of \( T \) on \( E \) with respect to \( \nu \) are defined by

\[
J^q_\nu(T, E) = \liminf_{r \to 0} \inf_{x \in E} \left( \frac{\nu(T B(x,r))}{\nu(B(x,r))} \right)^q,
\]
and

\[
J^q_\nu(T, E) = \limsup_{r \to 0} \sup_{x \in E} \left( \frac{\nu(T B(x,r))}{\nu(B(x,r))} \right)^q,
\]
respectively. If \( J^q_\nu(T, E) \) and \( J^q_\nu(T, E) \) coincide, we write \( J^q_\nu(T, E) \) for the common value and call it the \( q \)-th order Jacobian of \( T \) on \( E \) with respect to \( \nu \). The main importance of the Jacobians for our purpose is that they determine the scaling behaviour of \( \mathcal{H}^q_{\nu,\ell} \) and \( \mathcal{P}^q_{\nu,\ell} \). This is stated formally in the next lemma.

**Lemma 10.4.** Let \( \nu \) be a probability measure on \( \mathbb{R}^d \) and let \( T : \mathbb{R}^d \to \mathbb{R}^d \) be a similarity map, i.e. there exists a constant \( c \in (0, \infty) \) such that \( |T x - T y| = c|x - y| \) for all \( x, y \in \mathbb{R}^d \). Assume that \( T(\supp \nu) \subseteq \supp \nu \). Let \( q, t \in \mathbb{R} \) and \( E \subseteq \supp \nu \). Then

\[
J^q_\nu(T, E)c^t \mathcal{H}^q_{\nu,\ell}(E) \leq \mathcal{H}^q_{\nu,\ell}(TE) \leq J^q_\nu(T, E)c^t \mathcal{H}^q_{\nu,\ell}(E).
\]

**Proof**
Following easily from the definitions. See also [Ol1, Lemma 4.3]. \( \square \)
Proposition 10.5. Fix $q \in \mathbb{R}$ and assume that one of the following conditions is satisfied:

(i) The OSC is satisfied and $0 \leq q$;
(ii) The SSC is satisfied.

For $i \in \Sigma^*$ and $\delta > 0$ write

$$\Delta_{i,\delta} = \bigcup_{|j|=|i|, j \neq i} S_j^{-1} B(S_j K, \delta).$$

(1) For $i, j \in \Sigma^*$ with $|i| = |j|$ and $i \neq j$, we have

$$\mathcal{H}_{\mu}^{q,\beta(q)}(S_i K \cap S_j K) = 0.$$

(2) For $i, j \in \Sigma^*$ with $|i| = |j|$ and $i \neq j$, we have

$$\mathcal{H}_{\mu}^{q,\beta(q)}(S_i^{-1} S_j K) = 0.$$

(3) For $i \in \Sigma^*$, we have

$$\mathcal{H}_{\mu}^{q,\beta(q)}(\Delta_{i,\delta}) \to 0 \quad \text{as } \delta \searrow 0.$$

(4) For $i \in \Sigma^*$, we have

$$\mathcal{H}_{\mu}^{q,\beta(q)}(S_i \Delta_{i,\delta}) \to 0 \quad \text{as } \delta \searrow 0.$$

(5) For $i \in \Sigma^*$ and $\delta > 0$, we have

$$\mathcal{J}^q_i(S_i, K \setminus \Delta_{i,\delta}) = \mathcal{J}^q_i(S_i, K \setminus \Delta_{i,\delta}) = p_i^q.$$

(6) For $q \geq 0$ and $i \in \Sigma^*$, we have

$$\mathcal{H}_{\mu}^{q,\beta(q)}(S_i K) = r_i^{\beta(q)} p_i^q \mathcal{H}_{\mu}^{q,\beta(q)}(K).$$

Proof

(1) It follows from (1) that $\dim_{\mathcal{H}_{\mu}}^{q}(S_i K \cap S_j K) < \beta(q)$, whence $\mathcal{H}_{\mu}^{q,\beta(q)}(S_i K \cap S_j K) = 0$.

(2) We divide the proof into two cases.

Case 1: The OSC is satisfied and $0 \leq q$. It follows from Lemma 10.4 that

$$\mathcal{H}_{\mu}^{q,\beta(q)}(S_i^{-1} S_j K) = \mathcal{H}_{\mu}^{q,\beta(q)}(S_i^{-1}(S_i K \cap S_j K)) \leq r_i^{-\beta(q)} \mathcal{J}_i^{q}(S_i^{-1}, K) \mathcal{H}_{\mu}^{q,\beta(q)}(S_i K \cap S_j K).$$

Since $\mathcal{H}_{\mu}^{q,\beta(q)}(S_i K \cap S_j K) = 0$ (by Theorem 10.3), it therefore suffices to show that $\mathcal{J}_i^q(S_i^{-1}, K) < \infty$. Below we prove that $\mathcal{J}_i^q(S_i^{-1}, K) < \infty$. Indeed, (using the fact that $q \geq 0$) we have

$$\mathcal{J}_i^q(S_i^{-1}, K) = \limsup_{r \searrow 0} \sup_{x \in K} \left( \frac{\mu(S_i^{-1} B(x, r))}{\mu B(x, r)} \right)^q \leq \limsup_{r \searrow 0} \sup_{x \in K} \left( \frac{p_i \mu(S_i^{-1} B(x, r))}{\sum_{|j|=|i|} p_j \mu(S_i^{-1} B(x, r))} \right)^q p_i^{-q} \leq p_i^{-q} < \infty.$$
Case 2: The SSC is satisfied. Indeed, if the SSC is satisfied, then \( S_i^{-1}S_jK \) for all \( i, j \in \Sigma^* \) with \( |i| = |j| \) and \( i \neq j \), and the result therefore follows immediately.

(3) Since \( (\Delta_{i,\delta})_{\delta>0} \) is a decreasing family of sets (i.e. if \( 0 < \delta \leq \rho \), then \( \Delta_{i,\delta} \subseteq \Delta_{i,\rho} \)) with \( \cap_{\rho>0}\Delta_{i,\rho} = \cup_{|j|=|i|, j \neq i} S_i^{-1}S_jK \) (because \( \cup_{|j|=|i|, j \neq i} S_i^{-1}S_jK \) is closed), it follows from (2) that

\[
\mathcal{H}_\mu^{q,\beta}(\Delta_{i,\delta}) \to \mathcal{H}_\mu^{q,\beta}(\cap_{\rho>0}\Delta_{i,\rho})
= \mathcal{H}_\mu^{q,\beta}(\cup_{|j|=|i|, j \neq i} S_i^{-1}S_jK)
= 0.
\]

(4) Since \( (S_i\Delta_{i,\delta})_{\delta>0} \) is a decreasing family of sets (i.e. if \( 0 < \delta \leq \rho \), then \( S_i\Delta_{i,\delta} \subseteq S_i\Delta_{i,\rho} \)) with \( \cap_{\rho>0}S_i\Delta_{i,\rho} = \cap_{\rho>0} \left( \cup_{|j|=|i|, j \neq i} (S_iK \cap B(S_jK, \rho)) \right) = \cup_{|j|=|i|, j \neq i} (S_iK \cap S_jK) \) (because \( \cup_{|j|=|i|, j \neq i} (S_iK \cap S_jK) \) is closed), it follows from (1) that

\[
\mathcal{H}_\mu^{q,\beta}(S_i\Delta_{i,\delta}) \to \mathcal{H}_\mu^{q,\beta}(\cap_{\rho>0}S_i\Delta_{i,\rho})
= \mathcal{H}_\mu^{q,\beta}(\cup_{|j|=|i|, j \neq i} (S_iK \cap S_jK))
= 0.
\]

(5) Let \( x \in K \setminus \Delta_{i,\delta} \). Then \( S_ix \notin \cup_{|j|=|i|, j \neq i} B(S_jK, \delta) \), whence \( S_iB(x, r) \cap K \subseteq S_iK \setminus \cup_{|j|=|i|, j \neq i} S_jK \) for \( 0 < r < \delta \), and so

\[
\frac{\mu S_iB(x, r)}{\mu B(x, r)} = \sum_{|j|=|i|, j \neq i} \frac{\mu(S_i^{-1}S_jB(x, r))}{\mu B(x, r)} = \frac{p_i \mu B(x, r)}{\mu B(x, r)} = p_i
\]
for all \( 0 < r < \delta \).

(6) We have

\[
\mathcal{H}_\mu^{q,\beta}(S_iK) = \lim_{\delta \to 0} \mathcal{H}_\mu^{q,\beta}(S_i(K \setminus \Delta_{i,\delta}))
= \lim_{\delta \to 0} r_i^{\beta(q)} v_i^{q}(S_iK \setminus \Delta_{i,\delta}) \mathcal{H}_\mu^{q,\beta}(K \setminus \Delta_{i,\delta})
= \lim_{\delta \to 0} r_i^{\beta(q)} p_i^q \mathcal{H}_\mu^{q,\beta}(K \setminus \Delta_{i,\delta})
= r_i^{\beta(q)} p_i^q \mathcal{H}_\mu^{q,\beta}(K).
\]

This completes the proof. \( \square \)

11. Proof of Theorem 4.1

The purpose of this section is to prove Theorem 4.1. We begin with some technical lemmas. For \( i \in \Sigma^* \), we define \( h_i^q : (0, \infty) \to \mathbb{R} \) by

\[
h_i^q(r) = \frac{V^q_{\mu, r^{-1}}(K)}{(r_i^{-1})^{-\beta(q)}} = \frac{V^q_{\mu, r}(K)}{r^{-\beta(q)}}.
\]
Lemma 11.1. Fix $q \in \mathbb{R}$ and assume that one of the following conditions is satisfied:

(i) The OSC is satisfied and $0 \leq q$;
(ii) The SSC is satisfied.

Let $i \in \Sigma^*$. Then

$$h_i^q(r) \rightarrow 1 \quad \text{as } r \searrow 0.$$ . Proof

This statement follows immediately from Theorem 3.3 in the non-arithmetic case. In the arithmetic case there exists for each $i = 1, \ldots, N$ a positive integer $n_i$ such that $\log \frac{1}{r} = u n_i$, and hence, if we write $i = i_1 \ldots i_m$ and $m_i = |\{k \mid i_k = i\}|$ for $i = 1, \ldots, m$, then Theorem 3.3 implies that

$$h_i^q(r) = \frac{\pi_q(r^{-1}) + \epsilon_q(r^{-1})}{\pi_q(r) + \epsilon_q(r)} = \frac{\pi_q(\prod r^{-m_i} r) + \epsilon_q(r^{-1})}{\pi_q(r) + \epsilon_q(r)} = \frac{\pi_q(\sum n_i m_i r) + \epsilon_q(r^{-1})}{\pi_q(r) + \epsilon_q(r)} \rightarrow 1 \quad \text{as } r \searrow 0. \quad \square$$

Lemma 11.2. Fix $q \in \mathbb{R}$ and assume that one of the following conditions is satisfied:

(i) The OSC is satisfied and $0 \leq q$;
(ii) The SSC is satisfied.

Then we have:

(1) For $i \in \Sigma^*$, we have

$$\frac{1}{I_{q, r}(\mathbb{R}^d)} T_{q, r}^i (B(S_1 K, r)) \rightarrow p_i^{q, \beta(q)}.$$ (2) For $i \in \Sigma^*$, we have

$$\frac{1}{H_{q, r}(K)} H_{q, r}^i (B(S_1 K, r)) \rightarrow p_i^{q, \beta(q)}.$$

Proof

(1) Since $V_{q, r}^{\mu, r^{-1}}(K) = T_{q, r}^i (B(K, r^{-1} r))$, it follows from Lemma 6.2 that

$$|I_{q, r}^i (B(S_1 K, r)) - p_i^{q, \mu, r^{-1}}(K)| = |I_{q, r}^i (B(S_1 K, r)) - p_i^{q, T_{q, r}^i} (B(K, r^{-1} r))| \leq \sum_{|j| = |i| \neq j} Q_{i,j}^q(r).$$

Next, since $V_{q, r}^{\mu, r^{-1}}(K) = I_{q, r}^i (\mathbb{R}^d)$, this implies that

$$\left| \frac{1}{I_{q, r}(\mathbb{R}^d)} T_{q, r}^i (B(S_1 K, r)) - p_i^{q, V_{q, r}^{\mu, r^{-1}}(K)} \right| \leq \sum_{|j| = |i| \neq j} Q_{i,j}^q(r). \quad (11.1)$$

However, $p_i^{q, V_{q, r}^{\mu, r^{-1}}(K)} = p_i^{q, \beta(q)} h_i^q(r)$, and it therefore follows from (11.1) that

$$\left| \frac{1}{I_{q, r}(\mathbb{R}^d)} T_{q, r}^i (B(S_1 K, r)) - p_i^{q, \beta(q)} h_i^q(r) \right| \leq \sum_{|j| = |i| \neq j} Q_{i,j}^q(r). \quad (11.2)$$
Write \( m = |i| \). Let \( r_m \) be the number defined in (7.5). Also, for \( j \in \Sigma^* \) with \( |i| = |j| = m \), let \( c_{i,j,m} \) be the constant in Proposition 7.8 and let \( c_m \) be the constant in Proposition 8.2. Observe that for \( 0 < r < \frac{1}{2}r_m \), we have

\[
Q_{1,j}^q(r) \leq \begin{cases} 
  c_{i,j,m} Z_{m}^q \left( \frac{1}{2}r \right) & \text{for } q < 0; \\
  c_{i,j,m} Z_{m}^q (2r) & \text{for } 0 \leq q
\end{cases}
\]

[by Proposition 7.8]

\[
\leq \begin{cases} 
  c_{i,j,m} c_m \left( \frac{1}{2}r \right)^{-\gamma(q)} & \text{for } q < 0; \\
  c_{i,j,m} c_m (2r)^{-\gamma(q)} & \text{for } 0 \leq q
\end{cases}
\]

[by Proposition 8.2]

\[
= c c_{i,j,m} c_m r^{-\gamma(q)}.
\]

where \( c = \max((1/2)^{-\gamma(q)}, 2^{-\gamma(q)}) \). Hence, (11.2) implies that

\[
\left| \frac{1}{T_{\mu,r}^q(\mathbb{R}^d)} T_{\mu,r}^q \left( B(S_i K, r) \right) - p_i^q r_i^\beta(q) \right|
\]

\[
\leq \left| \frac{1}{T_{\mu,r}^q(\mathbb{R}^d)} T_{\mu,r}^q \left( B(S_i K, r) \right) - p_i^q r_i^\beta(q) h_i^q(r) \right| + \left| p_i^q r_i^\beta(q) h_i^q(r) - p_i^q r_i^\beta(q) \right|
\]

\[
\leq \sum_{|j|=|i|} Q_{1,j}^q(r) V_{\mu,r}^q(K) + p_i^q r_i^\beta(q) |h_i^q(r) - 1|
\]

\[
\leq \sum_{|j|=|i|} c c_{i,j,m} c_m (2r)^{-\gamma(q)} V_{\mu,r}^q(K) + p_i^q r_i^\beta(q) |h_i^q(r) - 1|
\]

\[
= \sum_{|j|=|i|} c c_{i,j,m} c_m r^{\beta(q)-\gamma(q)} \left( \frac{1}{r^{-\beta(q)}} V_{\mu,r}^q(K) \right)^{-1} + p_i^q r_i^\beta(q) |h_i^q(r) - 1|
\]

(11.3)

for all \( 0 < r < \frac{1}{2}r_m \). Since \( r^{\beta(q)-\gamma(q)} \to 0 \) as \( r \searrow 0 \) (because \( \beta(q) - \gamma(q) > 0 \) by (8.2)) and \( \frac{1}{r^{-\beta(q)}} V_{\mu,r}^q(K) \) is bounded away from 0 for \( r \) small enough by Theorem 3.3, it follows from Lemma 11.1 and (11.3) that

\[
\frac{1}{T_{\mu,r}^q(\mathbb{R}^d)} T_{\mu,r}^q \left( B(S_i K, r) \right) \to p_i^q r_i^\beta(q).
\]

(2) For brevity write \( H = \frac{1}{H_{\mu,q}(K)} H_{\mu,tr}^q(K) \). Since \( (B(S_i K, r))_{r>0} \) is a decreasing family of sets (i.e. if \( 0 < s \leq t \), then \( B(S_i K, s) \subseteq B(S_i K, t) \)) with \( \cap_{r>0} B(S_i K, r) = S_i K \) (because \( S_i K \) is closed) and \( H(B(S_i K, r)) \leq 1 < \infty \) for all \( r > 0 \), we conclude that

\[
H \left( B(S_i K, r) \right) \to H \left( \bigcap_{r>0} B(S_i K, r) \right) = H(S_i K).
\]

(11.4)

Next, it follows from Proposition 10.5 that \( H(S_i K) = r_i^\beta(q) p_i^q \). Combining this and (11.4) shows that

\[
H \left( B(S_i K, r) \right) \to p_i^q r_i^\beta(q).
\]

(11.5)

This completes the proof. \( \square \)
Lemma 11.3. Fix $q \in \mathbb{R}$ and assume that one of the following conditions is satisfied:

(i) The OSC is satisfied and $0 \leq q$;

(ii) The SSC is satisfied.

For $m \in \mathbb{R}$ and $r > 0$, let $(E_i(r))_{|i|=m}$ be a family of Borel sets such that $E_i(r) \subseteq B(S_iK, r)$ for all $i$ and $\cup_{|i|=m} E_i(r) = \cup_{|i|=m} B(S_iK, r)$. Then we have:

(1) For $i \in \Sigma^*$, we have

$$
\frac{1}{\mathcal{T}^q_{\mu,r}(R)}\mathcal{T}^q_{\mu,r} \left( E_i(r) \right) \to p_1^q r_1^\beta(q).
$$

(2) For $i \in \Sigma^*$, we have

$$
\frac{1}{\mathcal{H}^q_{\mu,r}(K)}\mathcal{H}^q_{\mu,r} \left( E_i(r) \cap K \right) \to p_1^q r_1^\beta(q).
$$

Proof

(1) Recall, that we write $\mathcal{V}^q_{\mu,r} = \frac{1}{\mathcal{T}^q_{\mu,r}(R)}\mathcal{T}^q_{\mu,r}$ for $r > 0$, and note that Lemma 11.2 implies that

$$
\mathcal{V}^q_{\mu,r} \left( B(S_iK, r) \right) \to p_1^q r_1^\beta(q) \quad (11.6)
$$

for all $i$.

Since clearly $1 \leq \sum_{|i|=m} \mathcal{V}^q_{\mu,r} \left( E_i(r) \right)$ (because $1 = \mathcal{V}^q_{\mu,r} \left( B(K, r) \right) \leq \mathcal{V}^q_{\mu,r} \left( \cup_{|i|=m} B(S_iK, r) \right) = \sum_{|i|=m} \mathcal{V}^q_{\mu,r} \left( E_i(r) \right)$), we conclude from (11.6) that

$$
\sum_{|i|=m} \left( \mathcal{V}^q_{\mu,r} \left( B(S_iK, r) \right) - \mathcal{V}^q_{\mu,r} \left( E_i(r) \right) \right) = \sum_{|i|=m} \mathcal{V}^q_{\mu,r} \left( B(S_iK, r) \right) - \sum_{|i|=m} \mathcal{V}^q_{\mu,r} \left( E_i(r) \right)
\leq \sum_{|i|=m} \mathcal{V}^q_{\mu,r} \left( B(S_iK, r) \right) - 1
\to \sum_{|i|=m} p_1^q r_1^\beta(q) - 1 = 0.
$$

Next, since $\mathcal{V}^q_{\mu,r} \left( B(S_iK, r) \right) - \mathcal{V}^q_{\mu,r} \left( E_i(r) \right) \geq 0$ for all $i$ (because $E_i(r) \subseteq B(S_iK, r)$), it follows from (11.7) that

$$
\mathcal{V}^q_{\mu,r} \left( B(S_iK, r) \right) - \mathcal{V}^q_{\mu,r} \left( E_i(r) \right) \to 0 \quad (11.8)
$$

for all $i$.

Finally, we deduce from (11.6) and (11.8) that

$$
\mathcal{V}^q_{\mu,r} \left( E_i(r) \right) = \mathcal{V}^q_{\mu,r} \left( B(S_iK, r) \right) - \left( \mathcal{V}^q_{\mu,r} \left( B(S_iK, r) \right) - \mathcal{V}^q_{\mu,r} \left( E_i(r) \right) \right)
\to p_1^q r_1^\beta(q) - 0
= p_1^q r_1^\beta(q)
$$

for all $i$.

(2) The proof of (2) is similar to the proof of (1) and is therefore omitted. □

We can now prove Theorem 4.1.
Proof of Theorem 4.1
Fix \( q \geq 0 \). Recall, that we write \( \mathcal{V}^{q}_{\mu, r} = \frac{1}{m^{q, \beta(q)}_{\mu, r}} \mathcal{I}^{q, \beta(q)}_{\mu, r} \). Also, for brevity write \( H = \frac{1}{n^{q, \beta(q)}_{\mu, r}(K)} \mathcal{H}^{q, \beta(q)}_{\mu, r} \).

We therefore let \( m \to \infty \), and we must now prove that
\[
\mathcal{V}^{q}_{\mu, r} \to H \quad \text{weakly.}
\]

Below we prove (11.10).

Let \( \varepsilon > 0 \). Since \( f \) is uniformly continuous (because \( f \) is continuous with compact support) there is a real number \( \delta_0 > 0 \) such that if \( x, y \in \mathbb{R}^d \) satisfy \( |x - y| \leq \delta_0 \), then
\[
|f(x) - f(y)| \leq \frac{\varepsilon}{4}.
\]

Next, since \( \max_{|i|=m_0} \text{diam} \ S_i K \leq \rho_{\max}^m \text{diam} K \to 0 \) as \( m \to \infty \), there is a positive integer \( m_0 \) such that \( \text{diam} \ S_i K \leq \frac{\delta_0}{4} \) for all \( i \) with \( |i| = m_0 \).

Also, for each positive number \( r > 0 \), we may clearly choose a family \( \{ E_i(r) \}_{|i|=m_0} \) of pairwise disjoint Borel sets \( E_i(r) \) such that \( E_i(r) \subseteq B(S_i K, r) \) for all \( i \) and \( \cup_{|i|=m_0} E_i(r) = \cup_{|i|=m_0} B(S_i K, r) \).

Finally, fix \( i \) with \( |i| = m_0 \). It follows from Lemma 11.3 that \( H(E_i(r)) \to \rho_1^q r_1^\beta(q) \), and we can therefore choose a positive real number \( s_1 > 0 \) such that for \( 0 < r < s_1 \) we have
\[
\left| H(E_i(r)) - \rho_1^q r_1^\beta(q) \right| \leq \frac{\varepsilon}{4 \|f\|_\infty N^{m_0}},
\]

(11.12)

Similarly, it also follows from Lemma 11.3 that \( \mathcal{V}^{q}_{\mu, r}(E_i(r)) \to \rho_1^q r_1^\beta(q) \), and we can therefore choose a positive real number \( t_1 > 0 \) such that for \( 0 < r < t_1 \) we have
\[
\left| \mathcal{V}^{q}_{\mu, r}(E_i(r)) - \rho_1^q r_1^\beta(q) \right| \leq \frac{\varepsilon}{4 \|f\|_\infty N^{m_0}},
\]

(11.13)

\[
V^{q}_{\mu, r}(E_i(r)) \leq 2\rho_1^q r_1^\beta(q).
\]

Let
\[
r_0 = \min \left( \min_{|i|=m_0} s_1, \min_{|i|=m_0} t_1 \cdot \frac{\delta_0}{4} \right).
\]

We will now prove that if \( 0 < r < r_0 \), then
\[
\left| \int f \, d\mathcal{V}^{q}_{\mu, r} - \int f \, dH \right| \leq \varepsilon.
\]

Fix \( 0 < r < r_0 \). Since the family \( \{ E_i(r) \}_{|i|=m_0} \) consists of pairwise disjoint Borel sets \( E_i(r) \) such that \( E_i(r) \subseteq B(S_i K, r) \) for all \( i \) and \( \cup_{|i|=m_0} E_i(r) = \cup_{|i|=m_0} B(S_i K, r) \), we conclude that
\[
\left| \int f \, d\mathcal{V}^{q}_{\mu, r} - \int f \, dH \right| \leq \sum_{|i|=m_0} \left| \int E_i(r) f \, d\mathcal{V}^{q}_{\mu, r} - \int E_i(r) f \, dH \right|
\]

\[
\leq \sum_{|i|=m_0} \left| \int E_i(r) f \, d\mathcal{V}^{q}_{\mu, r} - \int E_i(r) f \, dH \right|.
\]

(11.14)
Next, for each \( i \in \Sigma^* \) with \( |i| = m_0 \), fix \( x_i \in E_i(r) \). We now have, using (11.12), (11.13) and (11.14) that

\[
\left| \int f \, d\mathcal{V}^q_{\mu,r} - \int f \, d\mathcal{H} \right| \leq \sum_{|i|=m_0} \left( \int_{E_i(r)} f \, d\mathcal{V}^q_{\mu,r} - \int_{E_i(r)} f \, d\mathcal{H} \right) \\
\leq \sum_{|i|=m_0} \left( \int_{E_i(r)} f \, d\mathcal{V}^q_{\mu,r} - \int_{E_i(r)} f(x_i) \, d\mathcal{V}^q_{\mu,r} \right) \\
+ \left| \int_{E_i(r)} f(x_i) \, d\mathcal{V}^q_{\mu,r} - \int_{E_i(r)} f(x_i) \, d\mathcal{H} \right| \\
+ \left| \int_{E_i(r)} f(x_i) \, d\mathcal{H} - \int_{E_i(r)} f \, d\mathcal{H} \right| \\
\leq \sum_{|i|=m_0} \left( \sup_{x,y \in E_i(r)} |f(x) - f(y)| \mathcal{V}^q_{\mu,r}(E_i(r)) \right) \\
+ \|f\|_\infty \left| \mathcal{V}^q_{\mu,r}(E_i(r)) - H(E_i(r)) \right| \\
+ \sup_{x,y \in E_i(r)} |f(x) - f(y)| H(E_i(r)) \\
\leq \sum_{|i|=m_0} \left( 2 \sup_{x,y \in E_i(r)} |f(x) - f(y)| 2p_i^q r_i^p \right) \\
+ \|f\|_\infty \left| \mathcal{V}^q_{\mu,r}(E_i(r)) - H(E_i(r)) \right| \\
\leq 4 \sum_{|i|=m_0} \sup_{x,y \in B(S_iK, r)} |f(x) - f(y)| p_i^q r_i^p \\
+ \|f\|_\infty \sum_{|i|=m_0} \left( \left| \mathcal{V}^q_{\mu,r}(E_i(r)) - p_i^q r_i^p \right| + \left| p_i^q r_i^p - H(E_i(r)) \right| \right) \\
\leq 4 \sum_{|i|=m_0} \sup_{x,y \in B(S_iK, r)} |f(x) - f(y)| p_i^q r_i^p \\
+ \|f\|_\infty \sum_{|i|=m_0} \left( \frac{\varepsilon}{4\|f\|_\infty N^{m_0}} + \frac{\varepsilon}{4\|f\|_\infty N^{m_0}} \right)
\[ \leq 4 \sum_{|i|=m_0} \sup_{x,y \in B(S_i K, r)} |f(x) - f(y)| p^q r^\beta(q) \]
\[ + \frac{\varepsilon}{2N m_0} \sum_{|i|=m_0} 1 \]
\[ \leq 4 \sum_{|i|=m_0} \sup_{x,y \in B(S_i K, r)} |f(x) - f(y)| p^q r^\beta(q) + \frac{\varepsilon}{2} \quad (11.15) \]

However, for each \( i \in \Sigma^* \) with \( |i| = m_0 \) and and for all \( x, y \in B(S_i K, r) \), we have \( |x - y| \leq 2(\text{diam} S_i K + r) \leq 2(\frac{1}{4} + r_0) \leq 2(\frac{1}{4} + \frac{1}{2}) = \delta_0 \), and it therefore follows from (11.11) that \( |f(x) - f(y)| \leq \frac{\varepsilon}{8} \). This and (11.15) imply that
\[
\int f d\nu^q_{\mu,r} - \int f dH \leq 4 \frac{\varepsilon}{8} \sum_{|i|=m_0} p^q r^\beta(q) + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

This completes the proof. \( \square \)

12. Proof of Theorem 4.2

The purpose of this section is to prove Theorem 4.2. We begin we a small lemma.

**Lemma 12.1.** Let \( \varphi, \Phi : (0, 1) \to (0, \infty) \) be measurable functions such that \( \int_r^1 \Phi(s) \frac{ds}{s} < \infty \)
for all \( r \) and \( \int_r^1 \varphi(s) \Phi(s) ds < \infty \) for all \( r \). Let \( c, C \geq 0 \). Assume that
\[
\frac{1}{-\log r} \int_r^1 \Phi(s) \frac{ds}{s} \to C \quad \text{as} \quad r \searrow 0,
\]
\[
\varphi(r) \to c \quad \text{as} \quad r \searrow 0.
\]

Then the following holds.

1. We have
\[
\frac{1}{-\log r} \int_r^1 \varphi(s) \Phi(s) \frac{ds}{s} \to cC \quad \text{as} \quad r \searrow 0.
\]

2. We have
\[
\frac{1}{-\log r} \int_r^1 \varphi(s) \frac{ds}{s} \to c \quad \text{as} \quad r \searrow 0.
\]

**Proof**

1. This follows by a standard argument and the proof is therefore omitted.

2. This follows from (1) by putting \( \Phi = 1 \) and noticing that \( \frac{1}{-\log r} \int_r^1 \frac{ds}{s} = 1 \) for all \( r > 0 \).

This completes the proof. \( \square \)

We can now prove Theorem 4.2.
Proof of Theorem 4.2
Recall, that the \((q, \beta(q))\) multifractal Minkowski content \(M_q^{q,\beta(q)}(K)\) and the \((q, \beta(q))\) average multifractal Minkowski content \(M_{\mu,\text{ave}}^{q,\beta(q)}(K)\) are defined by

\[
M_q^{q,\beta(q)}(K) = \lim_{r \to 0} \frac{1}{r^{-\beta(q)}} V_q^{\mu,r}(K),
M_{\mu,\text{ave}}^{q,\beta(q)}(K) = \lim_{r \to 0} \frac{1}{-\log r} \int_r^1 \frac{1}{s^{-\beta(q)}} V_q^{\mu,s}(K) \frac{ds}{s},
\]

provided the limits exist. Also, recall (from Theorem 3.2 or Theorem 3.3) that the multifractal Minkowski dimensions are given by \(\dim_q^{M,\mu}(K) = \dim_q^{M,\mu}(K) = \beta(q)\). In particular, this implies that the the tube measures \(S_q^{q,\beta(q)}(K)\) and \(S_{\mu,r}^{q,\beta(q)}\) coincide, and that the average tube measures \(S^{q,\beta(q)}_{\mu,r,\text{ave}}\) and \(S^{q,\beta(q)}_{\mu,r,\text{ave}}\) coincide. Below we write \(S_{\mu,r}^{q,\beta(q)}\) for the common value of \(S^{q,\beta(q)}_{\mu,r}\) and \(S^{q,\beta(q)}_{\mu,r}\), and we write \(S_{\mu,r,\text{ave}}^{q,\beta(q)}\) for the common value of \(S^{q,\beta(q)}_{\mu,r,\text{ave}}\) and \(S^{q,\beta(q)}_{\mu,r,\text{ave}}\), i.e. we write

\[
S_{\mu,r}^{q,\beta(q)} = \frac{1}{r^{-\beta(q)}} T_{\mu,r}^q,
S_{\mu,r,\text{ave}}^{q,\beta(q)} = \frac{1}{-\log r} \int_r^1 \frac{1}{s^{-\beta(q)}} T_{\mu,s}^q \frac{ds}{s}.
\]

In particular, we note that

\[
S_{\mu,r,\text{ave}}^{q,\beta(q)} = \frac{1}{-\log r} \int_r^1 S_{\mu,s}^{q,\beta(q)} \frac{ds}{s}.
\]

Finally, notice that if \(f : \mathbb{R}^d \to \mathbb{R}\) is a continuous function with compact support, then a standard argument shows that

\[
\int f \, d S_{\mu,r,\text{ave}}^{q,\beta(q)} = \frac{1}{-\log r} \int_r^1 \left( \int f \, d S_{\mu,s}^{q,\beta(q)} \right) \frac{ds}{s}. \tag{12.1}
\]

Recall, that we write \(V_{\mu,r} = \frac{1}{r^{-\beta(q)}} T_{\mu,r}^q\). Also, for brevity write \(H = \frac{1}{H_{\mu,\text{ave}}^{q,\beta(q)}} H_{\mu,\text{ave}}^{q,\beta(q)} \mathbf{L} K\). We can now prove the statements in Theorem 4.2.

(1) It follows from Theorem 3.2 that \(\dim_q^{M,\mu}(K) = \dim_q^{M,\mu}(K) = \beta(q)\) and this clearly implies the desired statement.

(2) Let \(f : \mathbb{R}^d \to \mathbb{R}\) is a continuous function with compact support. Since clearly \(S_{\mu,r}^{q,\beta(q)} = \frac{1}{r^{-\beta(q)}} V_{\mu,r}^q(K) V_{\mu,r}^q\), it now follows from Theorem 3.3 and Theorem 4.1 that

\[
\int f \, d S_{\mu,r}^{q,\beta(q)} = \frac{1}{r^{-\beta(q)}} V_{\mu,r}^q(K) \int f \, d V_{\mu,r}^q
\]

\[
\to M_{\mu,\text{ave}}^{q,\beta(q)}(K) \int f \, d H. \tag{12.2}
\]

We also deduce from Lemma 12.1.(2) (applied to the function \(\varphi : (0, \infty) \to (0, \infty)\) defined by \(\varphi(r) = \int f \, d S_{\mu,r}^q\), (12.1) and (12.2) that

\[
\int f \, d S_{\mu,r,\text{ave}}^{q,\beta(q)} = \frac{1}{-\log r} \int_r^1 \left( \int f \, d S_{\mu,s}^{q,\beta(q)} \right) \frac{ds}{s}
\]

\[
\to M_{\mu,\text{ave}}^{q,\beta(q)}(K) \int f \, d H
\]

\[
= M_{\mu,\text{ave}}^{q,\beta(q)}(K) \int f \, d H.
\]
This completes the proof.

(3) Let \( f : \mathbb{R}^d \to \mathbb{R} \) is a continuous function with compact support. As above, since
\[
S_{\mu,r}^q = \frac{1}{r^\beta(q)} \int f d\mathcal{V}_{\mu,r}^q (K) V_{\mu,r}^q,
\]
it now follows from Lemma 12.1.(1) (applied to the functions \( \phi, \Phi : (0, 1) \to (0, \infty) \) defined by
\[
\phi(r) = \frac{1}{r^\beta(q)} \int f d\mathcal{V}_{\mu,r}^q \\
\Phi(r) = \frac{1}{r^\beta(q)} \int f d\mathcal{V}_{\mu,r}^q (K)
\]
, Theorem 3.3, Theorem 4.1 and (12.1) that
\[
\int f dS_{\mu,r,ave}^q = \frac{1}{-\log r} \int_r^1 \left( \int f dS_{\mu,s}^q \right) \frac{ds}{s} = \frac{1}{-\log r} \int_r^1 \left( \frac{1}{r^\beta(q)} V_{\mu,r}^q (K) \int f d\mathcal{V}_{\mu,s}^q \right) \frac{ds}{s} \rightarrow M_{\mu,ave}^{q,\beta(q)} (K) \int f dH.
\]

This completes the proof. \( \square \)
Part 4: Proofs of the Results from Section 5

13. Analysis of the poles of $\zeta_q^\mu$

In this section we establish various technical growth estimates related to the poles and residues of the zeta function $\zeta_q^\mu$. These estimates will play important parts in the proofs of Theorem 5.4, Theorem 5.5 and Theorem 5.7. We begin by defining the number $\alpha(q)$; this number plays an important part in describing the location of the poles of $\zeta_q^\mu$. Fix $q \in \mathbb{R}$ and define $\alpha(q)$ by

$$\alpha(q) = \inf \left\{ t \in \mathbb{R} \left| \sum_{r_i = r_{\min}} p_i^q r_i^t \leq 1 + \sum_{r_i > r_{\min}} p_i^q r_i^t \right. \right\}. \quad (13.1)$$

Also, recall that $\beta(q)$ is defined by

$$\sum_i p_i^q r_i^{\beta(q)} = 1. \quad (13.2)$$

Using the numbers $\alpha(q)$ and $\beta(q)$ we can now describe the location of the poles of $\zeta_q^\mu$. We first prove the statements in Proposition 5.2; however, for the benefit of the reader, before proving Proposition 5.2 we repeat the statements of the proposition. Recall, that if $f$ is a meromorphic function, then $Z(f)$ denotes the set of zeros of $f$ and that $P(f)$ denotes the set of poles of $f$.

**Proposition 13.1 (i.e. statements (1)–(4.2) in Proposition 5.2).** The poles of $\zeta_q^\mu$. Fix $q \in \mathbb{R}$.

1. We have $-\infty < \alpha(q) \leq \beta(q) < \infty$.
2. We have
   $$P(\zeta_q^\mu) = Z\left( s \rightarrow 1 - \sum_i p_i^q r_i^{s} \right).$$
3. We have
   $$P(\zeta_q^\mu) \subseteq \left\{ s \in \mathbb{C} \left| \alpha(q) \leq \text{Re}(s) \leq \beta(q) \right. \right\}. \quad (4.1)$$
4. The poles $\omega$ with $\text{Re}(\omega) = \beta(q)$ in the non-arithmetic case: If the set $\{ \log r_{1}^{-1}, \ldots, \log r_{N}^{-1} \}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then
   $$P(\zeta_q^\mu) \cap \left\{ s \in \mathbb{C} \left| \text{Re}(s) = \beta(q) \right. \right\} = \{ \beta(q) \}.$$
The poles $\omega$ with $\text{Re}(\omega) = \beta(q)$ in the arithmetic case: If the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is contained in a discrete additive subgroup of $\mathbb{R}$ and $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\} = u\mathbb{Z}$ with $u > 0$, then

$$P(\zeta_0^q) \cap \left\{ s \in \mathbb{C} \mid \text{Re}(s) = \beta(q) \right\} = \beta(q) + \frac{2\pi}{u}i\mathbb{Z},$$

and for each $i$, there is a unique integer $k_i$ such that $\log r_i^{-1} = k_iu$ and, in addition,

$$P(\zeta_0^q) = \left( \beta(q) + \frac{2\pi}{u}i\mathbb{Z} \right) \cup \bigcup_{w \in \mathbb{Z}(z \rightarrow 1 - \sum_i p_i^{-1}z^{k_i})} \left( -\frac{\log |w|}{u} - \frac{\text{Arg } w}{u}i + \frac{2\pi}{u}i\mathbb{Z} \right)$$

(where $\text{Arg } z$ denotes the principal argument of $z \in \mathbb{C}$).

**Proof**

(1) It clearly suffices to prove that $\alpha(q) > -\infty$ and $\alpha(q) \leq \beta(q)$. We first prove that $\alpha(q) > -\infty$. Indeed, if we define $\Phi_q : \mathbb{R} \rightarrow \mathbb{R}$ by $\Phi_q(t) = 1 + \sum_{i, r_i > \min} p_i^q r_i^{q} - \sum_{i, r_i = \min} p_i^q r_i^{q}$, then $\lim_{t \rightarrow -\infty} \Phi_q(t) = -\infty$ and $\lim_{t \rightarrow \infty} \Phi_q(t) = 1$, whence $\alpha(q) = \inf\{ t \in \mathbb{R} \mid \Phi_q(t) \geq 0 \} \in \mathbb{R}$. Next, we show that $\alpha(q) \leq \beta(q)$. Indeed, it follows from the definition of $\beta(q)$ that

$$\sum_{i, r_i = \min} p_i^q r_i^{q} \leq \frac{\beta(q)}{1 - 1} + \sum_{i, r_i > \min} p_i^q r_i^{q},$$

and so $\beta(q) \geq \alpha(q)$.

(2) Since $\zeta_0^q(s) = \sum_{i, r_i = \min} p_i^q r_i^{q}$ for $s \in \mathbb{C} \setminus \mathbb{Z}(s \rightarrow 1 - \sum_i p_i^q r_i^{q})$, it follows immediately that $P(\zeta_0^q) = \mathbb{Z}(s \rightarrow 1 - \sum_i p_i^q r_i^{q})$.

(3) Let $s = \sigma + it \in \mathbb{C}$ with $\sigma, t \in \mathbb{R}$ be such that $\sum_i p_i^q r_i^{q} = 1$. We must now prove that $\alpha(q) \leq \sigma \leq \beta(q)$.

We first prove that $\sigma \leq \beta(q)$. Indeed, since $1 = \sum_i p_i^q r_i^{q} \leq \sum_i p_i^q |r_i^{q}| = \sum_i p_i^q r_i^{q}$, we conclude immediately from the definition of $\beta(q)$ that $\sigma \leq \beta(q)$.

Next, we prove that $\alpha(q) \leq \sigma$. To prove this inequality we note that $\sum_{i, r_i = \min} p_i^q r_i^{q} + \sum_{i, r_i > \min} p_i^q r_i^{q} = \sum_{i, r_i = \min} p_i^q r_i^{q} + \sum_{i, r_i > \min} p_i^q r_i^{q}$, from which we see that $\sum_{i, r_i = \min} p_i^q r_i^{q} = 1 - \sum_{i, r_i > \min} p_i^q r_i^{q}$. It follows immediately from this and the definition of $\alpha(q)$ that $\alpha(q) \leq \sigma$.

(4.1) We must prove that $P(\zeta_0^q) \cap \{ s \in \mathbb{C} \mid \text{Re}(s) = \beta(q) \} = \{ \beta(q) \}$. We first note that $\beta(q) \in \mathbb{Z}(s \rightarrow 1 - \sum_i p_i^q r_i^{q}) \cap \{ s \in \mathbb{C} \mid \text{Re}(s) = \beta(q) \} = P(\zeta_0^q) \cap \{ s \in \mathbb{C} \mid \text{Re}(s) = \beta(q) \}$. Next, we prove that $\omega \in P(\zeta_0^q) \cap \{ s \in \mathbb{C} \mid \text{Re}(s) = \beta(q) \}$, then $\omega = \beta(q)$. We therefore fix $\omega \in P(\zeta_0^q) \cap \{ s \in \mathbb{C} \mid \text{Re}(s) = \beta(q) \}$. It follows that there is $t \in \mathbb{R}$ such that $\omega = \beta(q) + it$. We must now show that $t = 0$. Observe that since $\beta(q) + it = \omega \in P(\zeta_0^q) = \mathbb{Z}(s \rightarrow 1 - \sum_i p_i^q r_i^{q})$, we have $1 = \sum_i p_i^q r_i^{q} e^{it} \log r_i = \sum_i p_i^q r_i^{q} e^{it} \log r_i$. We therefore deduce that $1 = \sum_i p_i^q r_i^{q} e^{it} \log r_i$. We conclude from (13.3) that the (2-dimensional planar) vectors given by $p_i^q r_i^{q} e^{it} \log r_i$, $\ldots, p_N^q r_N^{q} e^{it} \log r_N$ must be positive multiples of a common (2-dimensional planar) unit vector, i.e. there is a (2-dimensional planar) unit vector $e^{i\theta}$ with $\theta \in [-\pi, \pi]$ and positive numbers $\lambda_1, \ldots, \lambda_N \geq 0$ such that $p_i^q r_i^{q} e^{it} \log r_i = \lambda_i e^{i\theta}$ for all $i$. Since $p_i^q r_i^{q} > 0$,
this implies that $p_i^q r_i^{\beta(q)} = \lambda_i$, and consequently $e^{ik \log r_i} = e^{i\theta}$. It follows from this that $t \log r_i - \theta \in 2\pi \mathbb{Z}$ and we can therefore find integers $m_i$ such that

$$t \log r_i = \theta + 2\pi m_i.$$  

(13.4)

Next, since $1 = \sum_i p_i^q r_i^{\beta(q)} e^{ik \log r_i}$, we conclude from (13.4) that $1 = \sum_i p_i^q r_i^{\beta(q)} e^{ik (\theta + 2\pi m_i)} = (\sum_i p_i^q r_i^{\beta(q)} e^{i\theta})$, whence $\theta = 0$ (because $\theta \in [-\pi, \pi]$). This and another application of (13.4) shows that

$$t \log r_1 = 2\pi m_1, \ldots, t \log r_N = 2\pi m_N.$$  

(13.5)

Finally, if $t \neq 0$, then it follows from (13.5) that $\log r_i^{-1} = -\frac{2\pi}{i} m_i$ for all $i$, and the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is therefore contained in the discrete additive subgroup $\frac{-2\pi}{i} \mathbb{Z}$. However, this contracts the fact that $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is not contained in any discrete additive subgroup of $\mathbb{R}$. Consequently, we conclude that $t = 0$.

(4.2) Since $\log r_i^{-1} = k_i u$, we deduce that if $s \in \mathbb{C}$, then $1 - \sum_i p_i^q r_i^s = 1 - \sum_i p_i^q (e^{-us})^{k_i}$, whence

$$P(\zeta_q^s) = Z \left( s \rightarrow 1 - \sum_i p_i^q r_i^s \right)$$

$$= Z \left( s \rightarrow 1 - \sum_i p_i^q (e^{-us})^{k_i} \right)$$

$$= \bigcup_{w \in \mathbb{Z} \left( z \rightarrow 1 - \sum_i p_i^q z^{k_i} \right)} \left\{ s \in \mathbb{C} \mid w = e^{-us} \right\}$$

$$= (\beta(q) + \frac{2\pi}{u} \mathbb{1} \mathbb{Z}) \cup \bigcup_{w \in \mathbb{Z} \left( z \rightarrow 1 - \sum_i p_i^q z^{k_i} \right), w \neq e^{-us}} \left( -\frac{\log |w|}{u} - \frac{\arg w}{u} + \frac{2\pi}{u} \mathbb{1} \mathbb{Z} \right).$$

This completes the proof.

The next two propositions, i.e. Proposition 13.2 and Proposition 13.3, contain detailed information about the poles of $\zeta_q^s$ near the "critical line" $\text{Re}(s) = \beta(q)$.

**Proposition 13.2. The poles of $\zeta_q^s$ near the "critical line" $\text{Re}(s) = \beta(q)$.** Fix $q \in \mathbb{R}$.

Then there is a number $b(q) \in \mathbb{R}$ with the following properties:

1. We have $b(q) < \beta(q)$.
2. If:

   \[ \omega \text{ is a pole of } \zeta_q^s \text{ with } \omega \in \left\{ s \in \mathbb{C} \mid b(q) \leq \text{Re}(s) \leq \beta(q) \right\}, \]

then:

   (i) $\omega$ is a simple pole of $\zeta_q^s$;

   (ii) $\text{res}(\zeta_q^s; \omega) = \frac{1}{\sum_i p_i^q r_i^s \log r_i}$;

   (iii) $|\text{res}(\zeta_q^s; \omega)| \leq \frac{1}{\log r_{\text{max}}}$.  

**Proof**

Choose $i_*$ such that

$$r_{i_*} = \begin{cases} r_{\text{max}} & \text{if } \beta(q) \leq 0; \\ r_{\text{min}} & \text{if } 0 < \beta(q), \end{cases}$$

\[ \text{then:} \]

(i) $\omega$ is a simple pole of $\zeta_q^s$;

(ii) $\text{res}(\zeta_q^s; \omega) = \frac{1}{\sum_i p_i^q r_i^s \log r_i}$;

(iii) $|\text{res}(\zeta_q^s; \omega)| \leq \frac{1}{\log r_{\text{max}}}$. 


and define $b_*(q) \in \mathbb{R}$ by
\[
\sum_{i \neq i_*} p_i^q b_*(q) = 1.
\]
Observe that $\sum_{i \neq i_*} p_i^q b_*(q) = 1 = \sum_i p_i^q \beta(q) > \sum_{i \neq i_*} p_i^q \beta(q)$, whence $b_*(q) < \beta(q)$. Next, since $b_*(q) < \beta(q)$ we can choose a real number $b(q)$ with
\[
b_*(q) < b(q) < \beta(q)
\]such that $b(q) < \beta(q) \leq 0$ if $\beta(q) \leq 0$ and $0 \leq b(q) < \beta(q)$ if $0 < \beta(q)$.

We must now prove that if $\omega$ is a pole of $\zeta_\mu^n$ with $\omega \in \{ s \in \mathbb{C} \mid b(q) \leq \text{Re}(s) \leq \beta(q) \}$, then $\omega$ is a simple pole of $\zeta_\mu^n$ with $\text{res}(\zeta_\mu^n; \omega) = -\frac{1}{\sum p_i^q r_i^q \log r_i}$ and $|\text{res}(\zeta_\mu^n; \omega)| \leq -\frac{1}{\log r_{\max}}$. We therefore fix a pole $\omega$ of $\zeta_\mu^n$ with $\omega \in \{ s \in \mathbb{C} \mid b(q) \leq \text{Re}(s) \leq \beta(q) \}$. Define $f, g : \mathbb{C} \to \mathbb{C}$ by $g(s) = \sum_i p_i^q r_i^q$ and $f(s) = 1 - \sum_i p_i^q r_i^q$, and note that $\zeta_\mu^n = \frac{g}{f}$. Also, note that since $\omega$ is pole of $\zeta_\mu^n$, we conclude that $f(\omega) = 0$ and, consequently, $g(\omega) = 1 - f(\omega) = 1$. We now prove the following three claims.

Claim 1. $\text{Re}(p_i^q r_i^\omega) \geq 0$ for all $i$.

Proof of Claim 1. Indeed, otherwise there an index $j$ such that $\text{Re}(p_j^q r_j^\omega) < 0$, whence (using the fact that $g(\omega) = 1$)
\[
1 = g(\omega) = \text{Re}(g(\omega)) = \text{Re}\left(\sum_i p_i^q r_i^\omega\right) = \sum_i \text{Re}(p_i^q r_i^\omega) < \sum_{i \neq j} \text{Re}(p_i^q r_i^\omega) \leq \sum_{i \neq j} p_i^q r_i^\sigma.
\]
Next, note that if $\beta(q) \leq 0$, then $r_* = r_{\max} \geq r_j$ and $\sigma = \text{Re}(\omega) \leq \beta(q) \leq 0$, whence $r_*^\sigma \leq r_j^\sigma$, and so $\sum_{i \neq j} p_i^q r_i^\sigma \leq \sum_{i \neq i_*} p_i^q r_i^\sigma$. On the other hand, if $0 < \beta(q)$, then $r_* = r_{\min} \leq r_j$ and $0 \leq b(q) \leq \text{Re}(\omega) = \sigma$, whence $r_*^\sigma \leq r_j^\sigma$, and so $\sum_{i \neq j} p_i^q r_i^\sigma \leq \sum_{i \neq i_*} p_i^q r_i^\sigma$. Consequently, we always have $\sum_{i \neq j} p_i^q r_i^\sigma \leq \sum_{i \neq i_*} p_i^q r_i^\sigma$. We conclude from this and (13.6) that
\[
1 < \sum_{i \neq j} p_i^q r_i^\sigma \leq \sum_{i \neq i_*} p_i^q r_i^\sigma.
\]
However, since $\sigma = \text{Re}(\omega) \geq b(q) > b_*(q)$, we deduce that $r_*^\sigma < r_j^b_*$. The definition of $b_*(q)$ and (13.7) therefore imply that
\[
1 < \sum_{i \neq i_*} p_i^q r_i^\sigma \leq \sum_{i \neq i_*} p_i^q b_*(q) = 1.
\]
The desired contradiction follows immediately from (13.8). This completes the proof of Claim 1.

Claim 2. We have \( g(\omega) = 1 \). In particular, \( g(\omega) \neq 0 \).

**Proof of Claim 2.** Indeed, this has already been observed above. This completes the proof of Claim 2.

Claim 3. We have \( \Re(f'(\omega)) \geq -\log r_{\max} \). In particular, \( f(\omega) = 0 \) and \( f'(\omega) \neq 0 \).

**Proof of Claim 3.** We have

\[
\Re(f'(\omega)) = - \sum_i \Re(p_i^q r_i^\omega) \log r_i .
\]

Now observe that it follows from Claim 1 that \( \Re(p_i^q r_i^\omega) \geq 0 \) for all \( i \), and (13.9) therefore implies that

\[
\Re(f'(\omega)) \geq - \sum_i \Re(p_i^q r_i^\omega) \log r_i
\]

\[
\geq - \sum_i \Re(p_i^q r_i^\omega) \log r_{\max}
\]

\[
= - \Re \left( \sum_i p_i^q r_i^\omega \right) \log r_{\max}
\]

\[
= - \Re(g(\omega)) \log r_{\max}
\]

\[
= - \Re(1) \log r_{\max}
\]

\[
= - \log r_{\max} .
\]

This completes the proof of Claim 3.

Using the fact that \( \zeta_\mu^q = g/\ell \), we conclude from Claim 2 and Claim 3 that \( \omega \) is a simple pole of \( \zeta_\mu^q \) with \( \text{res}(\zeta_\mu^q; \omega) = g(\omega)/f(\omega) \), whence \( |\text{res}(\zeta_\mu^q; \omega)| = \frac{|g(\omega)|}{|f(\omega)|} \leq \frac{1}{|\Re(f'(\omega))|} \leq -\frac{1}{\log r_{\max}} \). This completes the proof of Proposition 13.2.

Next, we state and prove Proposition 13.3. The construction of the contour \( \Gamma \) in Proposition 13.3 is motivated by an argument in [DeKoÖzRaUr].

**Proposition 13.3.** The poles of \( \zeta_\mu^q \) near the “critical line” \( \Re(s) = \beta(q) \): construction of \( \Gamma \). Fix \( q \in \mathbb{R} \). Assume that \( \beta(q) \notin \{-dq, 1 - dq, \ldots, d - dq\} \). Let \( b(q) \) be as in Proposition 13.2. Then there are two real numbers \( b_0(q) \) and \( \beta_0(q) \), and two real valued sequences \( (u_n(q))_{n \in \mathbb{Z}} \) and \( (v_n(q))_{n \in \mathbb{Z}} \) satisfying the following conditions:

1. We have \( b(q) < b_0(q) < \beta_0(q) < \beta(q) \).
2. We have \( \{ -dq, 1 - dq, \ldots, d - dq\} \cap [b_0(q), \beta(q)] = \emptyset \).
3. We have \( 0 \notin [b_0(q), \beta_0(q)] \).
4. We have \( u_{-n}(q) = -v_n(q) \) and \( v_{-n}(q) = -u_n(q) \) for all \( n \). In addition, \( \ldots < u_{-1}(q) < v_{-1}(q) < u_0(q) < 0 < v_0(q) < u_1(q) < v_1(q) < \ldots \)

and

\[
\lim_{n \to -\infty} u_n(q) = \lim_{n \to \infty} v_n(q) = \infty ,
\]

\[
\lim_{n \to -\infty} u_n(q) = \lim_{n \to \infty} v_n(q) = -\infty .
\]

5. We have \( u_{n+1}(q) - u_n(q) \geq -\frac{\pi}{\log r_{\min}} \) for all \( n \).
Let

\[ \Pi_n^+ \text{ be the directed horizontal line segment from } b_0(q) + i v_n(q) \text{ to } \beta_0(q) + i v_n(q), \]

\[ \Gamma_n^+ \text{ be the directed “left” vertical line segment from } b_0(q) + i u_n(q) \text{ to } b_0(q) + i v_n(q), \]

\[ \Pi_n^- \text{ be the directed horizontal line segment from } \beta_0(q) + i u_n(q) \text{ to } b_0(q) + i u_n(q), \]

\[ \Gamma_n^- \text{ be the directed “right” vertical line segment from } \beta_0(q) + i v_{n-1}(q) \text{ to } \beta_0(q) + i u_n(q), \]

and

\[ \Gamma_n \text{ be the concatenation of } \Gamma_n^-, \Pi_n^-, \Gamma_n^+ \text{ and } \Pi_n^+, \]

\[ \Gamma \text{ be the concatenation of } \ldots, \Gamma_{-1}, \Gamma_0, \Gamma_1, \ldots. \]

Then the following holds:

(6) The path \( \Gamma \) does not intersect the sets of poles of \( \zeta_p^q \).

(7) We have \( \sup_{s \in \Gamma} |\zeta_p^q(s)| < \infty \).

(8) If \( M \) is a bounded subset of \( \mathbb{R} \), then \( \sup_n \sup_{s \in (M + u_n(q) \delta) \cup (M + v_n(q) \delta)} |\zeta_p^q(s)| < \infty \).

Below we sketch the contour \( \Gamma \).
Proof
Since $\beta(q) \notin \{-dq, 1-dq, \ldots, d-dq\}$, we can choose $b_0(q)$ such that $b(q) < b_0(q) < \beta(q)$. \{-dq, 1-dq, \ldots, d-dq\} \cap \{b_0(q), \beta(q)\} = \emptyset$ and $0 \notin \{b_0(q), \beta(q)\}$. As $b_0(q) < \beta(q)$, we conclude that $\sum_i p_i r_i^{b_0(q)} > 1$. This implies that there is a positive number $d_0 > 0$ such that

$$\sum_i p_i r_i^{b_0(q)} = 1 + d_0. \tag{13.10}$$

Next, note that we can choose $\theta_0 > 0$ such that the following four conditions are satisfied:

$$3\theta_0 \frac{\log r_{\min}}{\log r_{\max}} \leq \left(\frac{d_0}{1+d_0}\right)^2, \quad \theta_0 \frac{\log r_{\min}}{\log r_{\max}} < d_0, \quad \frac{d_0}{\log r_{\max}} < \frac{2\pi}{\log r_{\min}} \quad \text{and} \quad \theta_0 \leq \frac{\pi}{2}. \quad \text{Now, let}$$

$$\delta_0 = \frac{6\theta_0 \log r_{\min}(p_i r_i^{\beta(q)})}{8} \quad \text{and define} \quad \beta_0(q) \quad \text{by}$$

$$\sum_i p_i r_i^{\beta_0(q)} = 1 + \delta_0. \tag{13.11}$$

The statements in the proposition follow from the seven claims below.
Claim 1. We have \( b(q) < b_0(q) < \beta_0(q) < \beta(q) \).

Proof of Claim 1. It follows from the definition of \( b_0(q) \) that \( b(q) < b_0(q) \), and since
\[
0 < \delta_0 = \frac{\min_{i}(p_i r_i q)}{\delta} < d_0,
\]
we conclude that \( b_0(q) < \beta_0(q) < \beta(q) \). This concludes the proof of Claim 1.

Next, for each \( i \), we define \( \Theta_i : \mathbb{R} \to \mathbb{R} \) by
\[
\Theta_i(t) = \text{Arg}(r_i^t) = \text{Arg}(e^{t \log r_i})
\]
(here we write Arg \((z)\) for the principal argument of a complex number \( z \), i.e. Arg \((z)\) is the unique argument of \( z \) belonging to the interval \([-\pi, \pi)\)). Write
\[
G_i = \left\{ t \in \mathbb{R} \mid |\Theta_i(t)| < \theta_0 \right\}
\]
and note that
\[
G_i = \bigcup_{n \in \mathbb{Z}} B \left( -n \pi \frac{1}{\log r_i}, \rho_i \right)
\]
where \( \rho_i = -\frac{\theta_0}{\log r_i} \). Also write
\[
G = \left\{ t \in \mathbb{R} \mid \max_i |\Theta_i(t)| < \theta_0 \right\}
\]
and note that
\[
G = \bigcap_i G_i .
\]

We now prove the following claim describing the structure of the set \( G \). Below we use the following notation, namely, if \( M \) is a subset of \( \mathbb{R} \), then we will write \( -M = \{ -x \mid x \in M \} \).

Claim 2.

(1) The set \( G \) is open and \( 0 \in G \).
(2) The set \( G \) is unbounded.
(3) The set \( \mathbb{R} \setminus G \) is unbounded.
(4) \( G = -G \).

Proof of Claim 2. (1) This is clear since \( G_i \) is open for all \( i \) with \( 0 \in G_i \).
(2) In order to prove this statement we use Dirichlet’s theorem on simultaneous Diophantine approximation. For the benefit of the reader we will now state this result.

Dirichlet’s theorem on simultaneous Diophantine approximation [Te, Lemma 14.1]. Let \( n \) be a positive integer and let \( \xi_1, \ldots, \xi_n \in \mathbb{R} \). Fix \( Q \in \mathbb{N} \) and \( \tau > 0 \). Then there is \( t \in [\tau, \tau Q^n] \) and there are \( k_1, \ldots, k_n \in \mathbb{Z} \) such that \( |\xi_i - k_i| \leq \frac{1}{Q} \) for all \( i \).

We can now prove that \( G \) is unbounded. Let \( \tau > 0 \). We must now show that there is \( t \in G \) with \( t \geq \tau \). We first choose \( Q \in \mathbb{N} \) with \( \frac{1}{Q} < \frac{\delta_0}{2\pi} \). Next, it follows from Dirichlet’s theorem on simultaneous Diophantine approximation (applied to \( \xi_i = \frac{1}{2\pi} \log r_i \) for \( i = 1, \ldots, N \)) that there is \( t \in [\tau, \tau Q^N] \) and that there are \( k_1, \ldots, k_N \in \mathbb{Z} \) such that \( |\frac{1}{2\pi} \log r_i - k_i| \leq \frac{1}{Q} < \frac{\delta_0}{2\pi} \) for all \( i \), i.e. \( |t \log r_i - 2\pi k_i| \leq \theta_0 \) for all \( i \). This clearly implies that \( |\Theta_i(t)| = |\text{Arg}(e^{t \log r_i})| < \theta_0 \) for all \( i \), and so \( t \in G_i \) for all \( i \), whence \( t \in \bigcap_i G_i = G \). Also, since \( t \in [\tau, \tau Q^N] \), we see that \( t \geq \tau \).
$\log$ following holds. Namely, since $a_n$ implies that $−2n_1 < b_n < a_n + b_n < a_n$ and $a_n$ is unbounded. Thus, $G$ is an open unbounded set with an infinite number of connected components and that all connected components of $G$ are bounded. We therefore conclude that there are numbers $... < a_{-2} < b_{-2} < a_{-1} < b_{-1} < a_0 < b_0 < a_1 < b_1 < a_2 < b_2 < ...$ such that

$$G = \bigcup_n (a_n, b_n).$$

and

$$a_n \to \infty \text{ as } n \to \infty \text{ and } a_n \to -\infty \text{ as } n \to -\infty.$$

In addition, since $G = -G$, we see that $a_n = -b_n$ and $b_n = -a_n$ for all $n$. Now put

$$u_n(q) = a_n - \left( -2\theta_0 \frac{1}{\log r_{\max}} \right),$$

$$v_n(q) = b_n + \left( -2\theta_0 \frac{1}{\log r_{\max}} \right).$$

**Claim 3.** Let $n$ be an integer. Then the following statements hold.

1. $... < u_{n-1}(q) < v_{n-1}(q) < u_0(q) < 0 < v_0(q) < u_1(q) < v_1(q) < ...$
2. $u_{-n}(q) = -v_n(q)$ and $v_{-n}(q) = u_n(q)$.
3. $u_{n+1} - u_n \geq \frac{\pi}{\log r_{\min}}.$

**Proof of Claim 3.** First note that

$$a_{n+1} - b_n \geq \min_i \left( -2\pi \frac{1}{\log r_i} - 2\rho_i \right) = \min_i \left( -2\pi \frac{1}{\log r_i} - \left( -2\theta_0 \frac{1}{\log r_i} \right) \right) = -2(\pi - \theta_0) \frac{1}{\log r_{\min}}.$$ (13.12)

1. It is clear that $u_n(q) < a_n < b_n < v_n(q)$ and since $-\frac{4\theta_0}{\log r_{\max}} < -\frac{\pi - 2\theta_0}{\log r_{\min}}$, (13.12) implies that $-\frac{4\theta_0}{\log r_{\max}} < -\frac{\pi - 2\theta_0}{\log r_{\min}} \leq a_{n+1} - b_n$, whence $v_n(q) = b_n + \left( -2\theta_0 \frac{1}{\log r_{\max}} \right) - (-2\theta_0 \frac{1}{\log r_{\max}}) = u_{n+1}(q)$.
2. Since $a_{-n} = -b_n$ and $b_{-n} = -a_n$, this follows immediately from the definitions of $u_n(q)$ and $v_n(q)$.
3. Since $\theta_0 > \frac{\pi}{2}$, we conclude from (13.12) that $u_{n+1}(q) - u_n(q) = a_{n+1} - a_n \geq a_{n+1} - b_n \geq -\frac{2\pi - 2\theta_0}{\log r_{\min}} \geq -\frac{\theta_0}{\log r_{\min}}$. This completes the proof of Claim 3.

**Claim 4.** Let $i = 1, \ldots, N$ and $n$ be an integer. Then the following statements hold.

1. $-(\frac{\theta_0}{1 + \theta_0})^{\frac{1}{2}} \leq \Theta_i(t) \leq (\frac{\theta_0}{1 + \theta_0})^{\frac{1}{2}}$ for $t \in [u_n(q), v_n(q)]$.
2. $\theta_0 \leq \Theta_i(u_n(q)) \leq (\frac{\theta_0}{1 + \theta_0})^{\frac{1}{2}}$.
3. $-(\frac{\theta_0}{1 + \theta_0})^{\frac{1}{2}} \leq \Theta_i(v_n(q)) \leq -\theta_0$.

**Proof of Claim 4.** We first observe that if $t = s + \varepsilon$ with $s \in [a_n, b_n]$ and $\varepsilon \in \mathbb{R}$, then the following holds. Namely, since $s \in [a_n, b_n]$, we conclude that $|\arg(e^{i^s \log r_i})| = |\Theta_i(s)| < \theta_0$ for all $i$. This implies that for each $i$, we can find an integer $m_i \in \mathbb{Z}$ such that $-\theta_0 \leq s \log r_i - 2\pi m_i \leq \theta_0$, and since $t = s + \varepsilon$, we deduce from this that

$$-\theta_0 + \varepsilon \log r_{\min} \leq t \log r_i - 2\pi m_i \leq \theta_0 + \varepsilon \log r_{\min}$$

(13.13)

for all $i$. 

Let $t \in [u_n(q), v_n(q)]$. It is clear that $t = s + \varepsilon$ with $s \in [a_n, b_n]$ and $|\varepsilon| \leq -2\theta_0 \frac{1}{\log r_{\max}}$. It follows from this and (13.13) that for each $i$, we can find an integer $m_i \in \mathbb{Z}$ such that

$$-\theta_0 + \varepsilon \log r_{\min} \leq t \log r_i - 2\pi m_i \leq \theta_0 + \varepsilon \log r_{\min},$$

whence

$$|t \log r_i - 2\pi m_i| \leq |t - \theta_0| + |\varepsilon| \log r_{\min} \leq \theta_0 + 2\theta_0 \frac{\log r_{\min}}{\log r_{\max}} \leq 3\theta_0 \frac{\log r_{\min}}{\log r_{\max}} \leq \left( \frac{d_0}{1+d_0} \right)^2 \Theta. $$

This clearly implies that $|\Theta_i(t)| \leq \left( \frac{d_0}{1+d_0} \right)^2 \Theta$.

(2) We have $u_n(q) = a_n + \varepsilon$ where $\varepsilon = 2\theta_0 \frac{1}{\log r_{\max}}$. It follows from this and (13.13) that for each $i$, we can find an integer $m_i \in \mathbb{Z}$ such that

$$-\theta_0 + \varepsilon \log r_{\min} \leq u_n(q) \log r_i - 2\pi m_i \leq \theta_0 + \varepsilon \log r_{\min}. $$

Since $-\theta_0 + \varepsilon \log r_{\min} = -\theta_0 + 2\theta_0 \frac{\log r_{\min}}{\log r_{\max}} \geq -\theta_0 + 2\theta_0 = \theta_0$ and

$$\theta_0 - \varepsilon \log r_{\min} = \theta_0 + 2\theta_0 \frac{\log r_{\min}}{\log r_{\max}} \leq 3\theta_0 \frac{\log r_{\min}}{\log r_{\max}} \leq \left( \frac{d_0}{1+d_0} \right)^2 \Theta,$$

we therefore deduce that

$$\theta_0 \leq u_n(q) \log r_i - 2\pi m_i \leq \left( \frac{d_0}{1+d_0} \right)^2 \Theta. $$

This clearly implies that $\theta_0 \leq \Theta_i(u_n(q)) \leq \left( \frac{d_0}{1+d_0} \right)^2 \Theta$.

(3) We have $v_n(q) = b_n + \varepsilon$ where $\varepsilon = -2\theta_0 \frac{1}{\log r_{\max}}$. It follows from this and (13.13) that for each $i$, we can find an integer $m_i \in \mathbb{Z}$ such that

$$-\theta_0 + \varepsilon \log r_{\min} \leq v_n(q) \log r_i - 2\pi m_i \leq \theta_0 + \varepsilon \log r_{\min}. $$

Since $-\theta_0 + \varepsilon \log r_{\min} = -\theta_0 - 2\theta_0 \frac{\log r_{\min}}{\log r_{\max}} \leq -3\theta_0 \frac{\log r_{\min}}{\log r_{\max}} \leq -\left( \frac{d_0}{1+d_0} \right)^2 \Theta,$

and

$$\theta_0 - \varepsilon \log r_{\min} = \theta_0 - 2\theta_0 \frac{\log r_{\min}}{\log r_{\max}} \geq \theta_0 - 2\theta_0 = -\theta_0,$$

we therefore deduce that

$$-v_n(q) \log r_i - 2\pi m_i \leq -\theta_0. $$

This clearly implies that $-v_n(q) \Theta_i(v_n(q)) \leq -\theta_0$. This completes the proof of Claim 4.

In the claims below we use the following notation, namely, we define the function $f : \mathbb{C} \to \mathbb{C}$ by $f(s) = 1 - \sum_i p_i^q r_i^s$.

**Claim 5.** Let $n$ be an integer. Then the following statements hold.

1. $\inf_{s \in \Gamma_n^0} \text{Re}(f(s)) \geq \delta_0$.
2. $\sup_{s \in \Gamma_n^0} |f(s)| \leq \frac{\sum_i p_i^q r_i^\beta_0}{\delta_0}$.

**Proof of Claim 5.** (1) Let $s \in \Gamma_n^0$. Consequently, there is $t \in [v_{n-1}(q), u_n(q)]$ such that $s = \beta_0(q) + i t$. As $t \in [v_{n-1}(q), u_n(q)]$, we deduce that $t \not\in G$, whence $\max_i |\Theta_i(t)| \geq \theta_0$ and we can therefore find $i_0$ such that $|\Theta_{i_0}(t)| \geq \theta_0$. Using the fact that $\cos \theta \leq 1 - \frac{1}{4} \theta^2$ for all $\theta$ with $|\theta| \leq \frac{\pi}{2}$, this implies that

$$\cos \Theta_{i_0}(t) \leq \cos \theta_0 \leq 1 - \frac{1}{4} \theta_0^2 \leq 1 - 2\theta_0 \frac{1}{\min_i (p_i^q r_i^\beta(q))} \leq 1 - 2\theta_0 \frac{1}{p_i^q r_i^\beta(q)}.$$

and so

$$p_i^q r_i^\beta_0 \cos \Theta_{i_0}(t) \leq p_i^q r_i^\beta_0 - 2\theta_0 \frac{r_i^\beta_0}{r_i^\beta(q)}. \tag{13.14}$$

Next, since $\beta_0(q) < \beta(q)$, we conclude that $r_i^\beta(q) < r_i^\beta_0(q)$, whence $-\frac{r_i^\beta_0(q)}{r_i^\beta(q)} \leq -1$. It therefore follows from (13.14) that

$$p_i^q r_i^\beta_0 \cos \Theta_{i_0}(t) \leq p_i^q r_i^\beta_0 - 2\theta_0. \tag{13.15}$$
Inequality (13.15) now implies that

\[
\Re \left( \sum_i p_i^q r_i^s \right) = \Re \left( p_{i_0}^q r_i^{\beta_0(q)} e^{it \log r_i} \right) + \Re \left( \sum_{i \neq i_0} p_i^q r_i^{\beta_0(q)} e^{it \log r_i} \right) \\
\leq p_{i_0}^q r_i^{\beta_0(q)} \cos \Theta_i(t) + \sum_{i \neq i_0} p_i^q r_i^{\beta_0(q)} \\
\leq p_{i_0}^q r_i^{\beta_0(q)} - 2\delta_0 + \sum_{i \neq i_0} p_i^q r_i^{\beta_0(q)} \\
= \sum_i p_i^q r_i^{\beta_0(q)} - 2\delta_0 \\
= 1 + \delta_0 - 2\delta_0 \\
= 1 - \delta_0 .
\]  

(13.16)

We see from (13.16) that \(\Re(f(s)) = \Re(1 - \sum_i p_i^q r_i^s) = 1 - \Re(\sum_i p_i^q r_i^s) \geq 1 - (1 - \delta_0) = \delta_0\).

(2) Since \(\zeta_i^q = \frac{1}{f},\) we conclude from (1) that \(|\zeta_i^q(s)| \leq \frac{|1-f(s)|}{|f(s)|} \leq \frac{\sum_i p_i^q r_i^{\beta_0(q)}}{\delta_0}\) for all \(s \in \Gamma_n^-\). This completes the proof of Claim 5.

Claim 6. Let \(n\) be an integer. Then the following statements hold.

1. \(\sup_{s \in \Gamma_n^+} \Re(f(s)) \leq -\frac{d_0}{2}\).
2. \(\sup_{s \in \Gamma_n^+} |\zeta_i^q(s)| \leq \frac{\sum_i p_i^q r_i^{\beta_0(q)}}{\delta_0}\).

Proof of Claim 6. (1) Let \(s \in \Gamma_n^+\). Consequently, there is \(t \in [u_n(q), v_n(q)]\) such that \(s = b_0(q) + it\). As \(t \in [u_n(q), v_n(q)]\), we conclude from statement (1) in Claim 4 that \(-\frac{d_0}{1+d_0} \leq \Theta_i(t) \leq \frac{d_0}{1-d_0}\) whence \(\cos \Theta_i(t) \geq \cos\left(\frac{d_0}{1+d_0}\right)\) for all \(i\). Using the fact that \(\cos \theta \geq 1 - \frac{1}{2} \theta^2\) for all \(\theta\), this implies that

\[
\cos \Theta_i(t) \geq \cos \left(\frac{d_0}{1+d_0}\right) \\
\geq 1 - \frac{d_0}{2} 1 + d_0
\]

for all \(i\), and so

\[
\Re \left( \sum_i p_i^q r_i^s \right) = \Re \left( \sum_i p_i^q r_i^{b_0(q)} e^{it \log r_i} \right) \\
= \sum_i p_i^q r_i^{b_0(q)} \cos \Theta_i(t) \\
\geq \sum_i p_i^q r_i^{b_0(q)} \left(1 - \frac{1}{2} \frac{d_0}{1+d_0}\right) \\
= (1+d_0) \left(1 - \frac{1}{2} \frac{d_0}{1+d_0}\right) \\
= 1 + \frac{d_0}{2} .
\]  

(13.17)

We see from (13.17) that \(\Re(f(s)) = \Re(1 - \sum_i p_i^q r_i^s) = 1 - \Re(\sum_i p_i^q r_i^s) \leq 1 - (1 + \frac{d_0}{2}) = -\frac{d_0}{2}\).
(2) Since \( \zeta_n^q = \frac{1}{f} \), we conclude from (1) that
\[ |\zeta_n^q(s)| \leq \frac{|1-f(s)|}{|f(s)|} \leq \frac{\sum p_i^q r_i^{0(q)}}{\text{Im}(f(s))} \leq 2 \sum p_i^q r_i^{0(q)} \]
for all \( s \in \Gamma_n^+ \). This completes the proof of Claim 6.

Claim 7. Let \( n \) be an integer. Let \( M \) be a subset of \( \mathbb{R} \) and write \( c = \inf_{x \in M} \sum p_i^q r_i^{+} \) and \( C = \sup_{x \in M} \sum p_i^q r_i^{+} \). Then the following statements hold.

1. \( \inf_{s \in (M+u_n(q)) \cup (M+v_n(q))} \left| \text{Im}(f(s)) \right| \geq c \sin \theta_0 \).
2. \( \sup_{s \in (M+u_n(q)) \cup (M+v_n(q))} \left| \zeta_n^q(s) \right| \leq \frac{C}{c \sin \theta_0} \).

In particular, if \( M = [\beta_0(q), \beta_0(q)] \), then \( M + u_n(q) \mathbf{i} = \Pi^- \) and \( M + v_n(q) \mathbf{i} = \Pi^+_n \), and it therefore follows from (1) and (2) that

3. \( \inf_{s \in \Pi^- \cup \Pi^+_n} \left| \text{Im}(f(s)) \right| \geq \sin \theta_0 \).
4. \( \sup_{s \in \Pi^- \cup \Pi^+_n} \left| \zeta_n^q(s) \right| \leq \frac{\sum p_i^q r_i^{0(q)}}{\sin \theta_0} \).

Proof of Claim 7. (1) Let \( s \in (M + u_n(q) \mathbf{i}) \cup (M + v_n(q) \mathbf{i}) \). Consequently, there is \( \sigma \in M \) and \( t \in \{u_n(q), v_n(q)\} \) such that \( s = \sigma + \mathbf{t} \mathbf{i} \), and so

\[
\text{Im} \left( \sum_i p_i^q r_i^a \right) = \text{Im} \left( \sum_i p_i^q r_i^a e^{\mathbf{t} \log r_i} \right) = \sum_i p_i^q r_i^a \sin \Theta_i(t). \tag{13.18}
\]

We now make the following two observations. Namely, it follows from statement (2) in Claim 4 that \( \theta_0 \leq \Theta_i(u_n(q)) \leq \frac{\theta_0}{1+\theta_0} \) whence \( \sin \Theta_i(u_n(q)) \geq \sin \theta_0 \) for all \( i \), and it follows from statement (3) in Claim 4 that \( -\frac{\theta_0}{1+\theta_0} \leq \Theta_i(v_n(q)) \leq -\theta_0 \) whence \( \sin \Theta_i(v_n(q)) \leq \sin(-\theta_0) = -\sin \theta_0 \) for all \( i \). These observations and (13.18) imply that

\[
\text{Im} \left( \sum_i p_i^q r_i^a \right) = \begin{cases} 
\geq \sum_i p_i^q r_i^a \sin \theta_0 & \text{for } t = u_n(q); \\
\leq -\sum_i p_i^q r_i^a \sin \theta_0 & \text{for } t = v_n(q). 
\end{cases} \tag{13.19}
\]

Next, since \( \sigma \in M \), it follows that \( \sum_i p_i^q r_i^a \geq c \), and we therefore conclude from (13.19) that

\[
\text{Im} \left( \sum_i p_i^q r_i^a \right) = \begin{cases} 
\geq c \sin \theta_0 & \text{for } t = u_n(q); \\
\leq -c \sin \theta_0 & \text{for } t = v_n(q). 
\end{cases}
\]

It follows from this that \( |\text{Im}(f(s))| = |\text{Im}(1 - \sum_i p_i^q r_i^a)| = |\text{Im}(\sum_i p_i^q r_i^a)| \geq c \sin \theta_0 \).

(2) Since \( \zeta_n^q = \frac{1-f}{f} \), we deduce from (1) that \( |\zeta_n^q(s)| \leq \frac{|1-f(s)|}{|f(s)|} \leq \frac{C}{c \sin \theta_0} \) for all \( s \in (M + u_n(q) \mathbf{i}) \cup (M + v_n(q) \mathbf{i}) \). This completes the proof of Claim 7.

The proof now follows from Claims 1,3,5–7. \( \square \)

The next two results, i.e Lemma 13.4 and Theorem 13.5, give growth estimates of the zeta function \( \zeta_n^q \) outside and inside the critical strip, respectively. For \( i \) with \( r_i > r_{\min} \), write \( s_i = \frac{\log r_i}{r_i} \) and put \( s_{\max} = \max_{r_i > r_{\min}} s_i \).
Lemma 13.4. Growth estimates of \( \zeta'_\mu \) outside the critical strip \( \alpha(q) \leq \text{Re}(s) \leq \beta(q) \). Fix \( q \in \mathbb{R} \). Define \( f : \mathbb{C} \to \mathbb{C} \) by

\[
f(s) = 1 - \sum_i p_i^q r_i^s.
\]

There are two real numbers \( A(q) \) and \( B(q) \) and two constants \( c > 0 \) and \( k > 0 \) satisfying the following:

1. We have \( A(q) \leq \alpha(q) \leq \beta(q) \leq B(q) \).
2. For all \( \sigma \geq B(q) \) and \( t \in \mathbb{R} \), we have \( |f'(\sigma + it)| \leq cr_{\max}^\sigma \).
3. For all \( \sigma \leq A(q) \) and \( t \in \mathbb{R} \), we have \( |f'(\sigma + it)| - \log r_{\min} \leq cs_{\min}^{-\sigma} \).
4. For all \( \sigma \geq B(q) \) and \( t \in \mathbb{R} \), we have \( \frac{1}{2} \leq |f(\sigma + it)| \leq \frac{3}{2} \).
5. For all \( \sigma \leq A(q) \) and \( t \in \mathbb{R} \), we have \( |\zeta'_\mu(\sigma + it)| \leq k \).

**Proof**

1. These results follows from straightforward estimates and we are therefore omitted the proofs.

5. It suffices to show that there are numbers \( \sigma_q \) and \( c_q \) such that for all \( \sigma \leq \sigma_q \) and all \( t \in \mathbb{R} \), we have \( |\zeta'_\mu(\sigma + it)| \leq c_q \). First, observe that if we for a real number \( \sigma \), define \( f_\sigma : \mathbb{R} \to \mathbb{C} \) by \( f_\sigma(t) = \sum_i p_i^q r_i^{\sigma + it} \), then

\[
\left| 1 - \sum_i p_i^q r_i^{\sigma + it} \right| = r_{\min}^\sigma \left| f_\sigma(t) - \frac{1}{r_{\min}^\sigma} \right|.
\]

(13.20)

Next, note that

\[
\sup\limits_{t \in \mathbb{R}} \left| f_\sigma(t) - \frac{1}{r_{\min}^\sigma} \right| - \sum_{r_i=r_{\min}} p_i^q = \sup\limits_{t \in \mathbb{R}} \left| f_\sigma(t) - \frac{1}{r_{\min}^\sigma} \right| - \sum_{r_i=r_{\min}} p_i^q r_i^{\sigma + it} \\
\leq \sup\limits_{t \in \mathbb{R}} \left| f_\sigma(t) - \frac{1}{r_{\min}^\sigma} \right| - \sum_{r_i=r_{\min}} p_i^q r_i^{\sigma + it} \\
\leq \sup\limits_{t \in \mathbb{R}} \sum_{r_i=r_{\min}} p_i^q r_i^{\sigma + it} \frac{r_i}{r_{\min}} \left| f_\sigma(t) - \frac{1}{r_{\min}^\sigma} \right| + \frac{1}{r_{\min}^\sigma}.
\]

(13.21)

Since, clearly \( \sum_{r_i=r_{\min}} p_i^q r_i^{\sigma + it} \frac{r_i}{r_{\min}} \) - \( \sum_{r_i=r_{\min}} p_i^q r_i^{\sigma + it} \) = \( \sum_{r_i>r_{\min}} p_i^q r_i^{\sigma + it} \frac{r_i}{r_{\min}} \), we now deduce form (13.21) that

\[
\sup\limits_{t \in \mathbb{R}} \left| f_\sigma(t) - \frac{1}{r_{\min}^\sigma} \right| - \sum_{r_i=r_{\min}} p_i^q = \sup\limits_{t \in \mathbb{R}} \sum_{r_i>r_{\min}} p_i^q r_i^{\sigma + it} \frac{r_i}{r_{\min}} + \frac{1}{r_{\min}^\sigma} \\
\leq \sup\limits_{t \in \mathbb{R}} \sum_{r_i>r_{\min}} p_i^q r_i^{\sigma + it} \frac{r_i}{r_{\min}} + \frac{1}{r_{\min}^\sigma} \\
\leq \sum_{r_i>r_{\min}} p_i^q \frac{r_i}{r_{\min}} \sigma + \frac{1}{r_{\min}^\sigma} \\
\rightarrow 0 \text{ as } \sigma \to -\infty.
\]

(13.22)
It follows from (13.22) that we can choose a number \( \sigma_q \) with \( \sigma_q < 0 \) such that for all \( \sigma \leq \sigma_q \), we have \[ \sup_{t \in \mathbb{R}} \left| f_\sigma(t) - \frac{1}{\gamma^\sigma} \right| \leq \frac{1}{2} \sum_{r_i = r_{\text{min}}} p_i^\sigma. \] This clearly implies that for all \( \sigma \geq \sigma_q \) and all \( t \in \mathbb{R} \), we have

\[
\left| f_\sigma(t) - \frac{1}{\gamma^\sigma} \right| \geq \frac{1}{2} \sum_{r_i = r_{\text{min}}} p_i^\sigma. \tag{13.23}
\]

Finally, combining (13.20) and (13.23) shows that for all \( \sigma \leq \sigma_q \) and all \( t \in \mathbb{R} \), we have

\[
| \zeta^q_\mu(\sigma \pm i t) | = \frac{\left| \sum_i p_i^{\sigma} r_i^{\sigma+1} \right|}{1 - \sum_i p_i^{\sigma} r_i^{\sigma+1}} \leq \frac{1}{r_{\text{min}}^\sigma f_\sigma(t) - \frac{1}{\gamma^\sigma}} \leq \frac{1}{r_{\text{min}}^\sigma} \frac{2}{\sum_{r_i = r_{\text{min}}} p_i^\sigma} \leq \frac{2}{\sum_{r_i = r_{\text{min}}} p_i^\sigma}.
\]

This completes the proof. \( \square \)

**Theorem 13.5 (i.e. Theorem 5.3).** Growth estimates of \( \zeta^q_\mu \) inside the critical strip \( \alpha(q) \leq \Re(s) \leq \beta(q) \). Fix \( q \in \mathbb{R} \). Then there is an increasing sequence \( (t_{q,n})_n \) of positive real numbers with \( t_{q,n} \to \infty \) satisfying the following: for all real numbers \( c \), there is a constant \( k_c \) such that for all \( \sigma \leq c \) and all \( n \), we have

\[
| \zeta^q_\mu(\sigma \pm i t_{q,n}) | \leq k_c.
\]

**Proof**

For \( n \in \mathbb{Z} \), let \( u_n(q) \) and \( v_n(q) \) be defined as in Proposition 13.3. Now define \( t_{q,n} \) by

\[
t_{q,n} = u_n(q) \tag{13.24}
\]

for \( n \in \mathbb{N} \) and observe that it follows from Proposition 13.3 that

\[
-t_{q,n} = -u_n(q) = v_{-n}(q) \tag{13.25}
\]

for \( n \in \mathbb{N} \). It also follows from Proposition 13.3 that \( t_{q,n} \to \infty \). Next, fix a real number \( c \). We must now prove that \( \sup_{\sigma \leq c, n \in \mathbb{N}} | \zeta^q_\mu(\sigma \pm i t_{q,n}) | < \infty \). Letting \( A(q) \) be as is Lemma 13.4, it follows from Lemma 13.4.(4) and Proposition 13.3.(8) (applied to the set \( M = [A(q),c] \)) that \( \sup_{\sigma \leq A(q), t \in \mathbb{R}} | \zeta^q_\mu(\sigma \pm i t) | < \infty \) and \( \sup_n \sup_{\sigma \in ([A(q),c]+u_n(q)+v_n(q)+i]} | \zeta^q_\mu(s) | < \infty, \)
respectively, whence (using (13.24) and (13.25))

\[
\sup_{\sigma \leq c \atop n \in \mathbb{N}} |\zeta_q^\mu(\sigma \pm i t_{q,n})| = \max \left( \sup_{\sigma \leq A(q) \atop n \in \mathbb{N}} |\zeta_q^\mu(\sigma \pm i t_{q,n})| , \sup_{\sigma \in [A(q),c] \atop n \in \mathbb{N}} |\zeta_q^\mu(\sigma + i t_{q,n})| , \sup_{\sigma \in [A(q),c] \atop n \in \mathbb{N}} |\zeta_q^\mu(\sigma - i t_{q,n})| \right)
\leq \max \left( \sup_{\sigma \leq A(q) \atop i \in \mathbb{R}} |\zeta_q^\mu(\sigma \pm i t)| , \sup_{n \in \mathbb{Z}} \sup_{s \in [A(q),c]+u_n(q) i} |\zeta_q^\mu(s)| , \sup_{n \in \mathbb{Z}} \sup_{s \in [A(q),c]+v_n(q) i} |\zeta_q^\mu(s)| \right)
\leq \infty.
\]

This completes the proof. \(\square\)

The final result in this section, i.e. Theorem 13.8, provides an estimate for the density of the poles of the zeta function \(\zeta_q^\mu\). We first list two variants of Jensen’s formula from complex analysis (see [Con, p. 280]) that will be needed in order to prove Theorem 13.8. We begin with a definition. For a holomorphic function \(F\) and \(R > 0\) write

\[
M_F(R) = \sup_{s \in \mathbb{C} \atop |s|=R} |F(s)|.
\]

We can now state the two results from complex analysis needed to prove Theorem 13.8.

**Proposition 13.6 [Te, Corollary 11.2]**. Let \(F\) be a holomorphic function with \(F(0) = 1\) and let \(R \) and \(\rho\) be positive real numbers with \(0 < \rho < \frac{1}{2} R\). Then

\[
\sup_{s \in \mathbb{C} \setminus Z(F) \atop |s| \leq \rho} \left| F'(s) F(s) - \sum_{\omega \in Z(F) \atop |\omega| \leq \frac{1}{2} R} \frac{1}{s - \omega} \right| \leq \frac{4R}{(R - 2\rho)^2} \log M_F(R).
\]

**Proposition 13.7 [Te, Section 11; Rud, p. 309]**. Let \(F\) be a holomorphic function with \(F(0) = 1\) and let \(R\) be a positive real number. Then

\[
\left| \{ \omega \in Z(F) \mid |\omega| \leq \frac{1}{2} R \} \right| \leq \frac{1}{\log 2} \log M_F(R).
\]
Theorem 13.8 (i.e. statement (5) in Proposition 15.2). Density of poles of $\zeta^q$.

Fix $q \in \mathbb{R}$. Write $\gamma = -\frac{1}{2} \log r_{\text{min}}$. We have

$$\left| \{ \omega \in P(\zeta^q) \mid \text{Im}(\omega) \leq t \} \right| = \gamma t + O(\log t).$$

Proof

For brevity write $N_t = | \{ \omega \in P(\zeta^q) \mid \text{Im}(\omega) \leq t \} |$. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(s) = 1 - \sum_i p_i^q r_i^q$ and note that $\zeta^q = \frac{1-f}{1-\bar{f}}$. In particular, it follows from this that $P(\zeta^q) = Z(f)$, whence $N_t = | \{ \omega \in Z(f) \mid \text{Im}(\omega) \leq t \} |$.

Let the constants $r_{\text{max}}$ and $s_{\text{max}}$ be as in Lemma 13.4, and recall that it follows Lemma 13.4 that there is a constant $c > 0$ such that if $\sigma \geq B(q)$ and $t \in \mathbb{R}$, then

$$\left| \frac{f'(\sigma + it)}{f(\sigma + it)} \right| \leq cr_{\text{max}}^\sigma,$$

and if $\sigma \leq A(q)$ and $t \in \mathbb{R}$, then

$$\left| \frac{f'(\sigma + it)}{f(\sigma + it)} - \log r_{\text{min}} \right| \leq cs_{\text{max}}^{-\sigma}.$$

Also, for $t > 0$, we choose $\sigma_t^+$ such that $cr_{\text{max}}^\sigma_t = \frac{1}{t}$ and choose we $\sigma_t^-$ such that $cs_{\text{max}}^{-\sigma_t} = \frac{1}{t}$. It is clear that $\sigma_t^+ \to \infty$ as $t \to \infty$ and that $\sigma_t^- \to -\infty$ as $t \to \infty$. We can therefore find $t_0 > 0$ such that $\sigma_t^+ \geq B(q)$ for $t \geq t_0$ and $\sigma_t^- \leq A(q)$ for $t \geq t_0$.

Next, we fix a positive real number $t$ with $t \geq t_0$ such that $f(\sigma + it) \neq 0$ for all $\sigma \in \mathbb{R}$, and define paths $\Sigma_t^+, \Sigma_t^-, \Lambda_t, \Delta_t$ in $\mathbb{C}$ by

- $\Sigma_t^+$ is the directed line-segment from $\sigma_t^+ + it$ to $\sigma_t^- + it$,
- $\Sigma_t^-$ is the directed line-segment from $\sigma_t^- - it$ to $\sigma_t^+ - it$,
- $\Lambda_t$ is the directed line-segment from $\sigma_t^+ - it$ to $\sigma_t^+ + it$,
- $\Delta_t$ is the directed line-segment from $\sigma_t^- + it$ to $\sigma_t^- - it$,

and let $C_t$ denote the simple closed path obtained by concatenating the line segments $\Lambda_t$, $\Sigma_t^+$, $\Delta_t$ and $\Sigma_t^-$. Since $C_t$ does not pass through any of the zeros of $f$ (because $t$ is chosen such that $f(\sigma + it) \neq 0$ for all $\sigma \in \mathbb{R}$) and since all zeros $\omega$ of $f$ satisfy $\text{Re}(\omega) \in [\alpha(q), \beta(q)] \subset [A(q), B(q)]$ (see Lemma 13.4), it follows from the Argument Principle (see [Con, p. 123]) that $2\pi i N_t = 2\pi i \left| \{ \omega \in Z(f) \mid \text{Im}(\omega) \leq t \} \right| = \int_{C_t} \frac{f'(s)}{f(s)} \, ds$, whence

$$2\pi N_t = \text{Im} \int_{C_t} \frac{f'(s)}{f(s)} \, ds = \text{Im} \int_{\Sigma_t^+} \frac{f'(s)}{f(s)} \, ds + \text{Im} \int_{\Sigma_t^-} \frac{f'(s)}{f(s)} \, ds + \text{Im} \int_{\Lambda_t} \frac{f'(s)}{f(s)} \, ds + \text{Im} \int_{\Delta_t} \frac{f'(s)}{f(s)} \, ds.$$  

(13.26)
As $\int_{\Delta_t} ds = -2t \, \i$, and so $\int_{\Delta_t} (-\log r_{\text{min}}) \, ds = -2\pi \, \gamma t$, we conclude from (13.26) that

$$2\pi |N_t - \gamma t|$$

$$= \left| \text{Im} \int_{\Sigma_1^+} \frac{f'(s)}{f(s)} \, ds + \text{Im} \int_{\Sigma_1^-} \frac{f'(s)}{f(s)} \, ds + \text{Im} \int_{\Lambda_t} \frac{f'(s)}{f(s)} \, ds + \text{Im} \int_{\Delta_t} \frac{f'(s)}{f(s)} \, ds - 2\pi \gamma t \right|$$

$$= \left| \text{Im} \int_{\Sigma_1^+} \frac{f'(s)}{f(s)} \, ds + \text{Im} \int_{\Sigma_1^-} \frac{f'(s)}{f(s)} \, ds + \text{Im} \int_{\Lambda_t} \left( \frac{f'(s)}{f(s)} - \log r_{\text{min}} \right) \, ds \right|$$

$$\leq \left| \text{Im} \int_{\Sigma_1^+} \frac{f'(s)}{f(s)} \, ds \right| + \left| \text{Im} \int_{\Sigma_1^-} \frac{f'(s)}{f(s)} \, ds \right|$$

$$+ \left| \int_{\Lambda_t} \frac{f'(s)}{f(s)} \, ds \right| + \left| \int_{\Delta_t} \left( \frac{f'(s)}{f(s)} - \log r_{\text{min}} \right) \, ds \right| .$$

(13.27)

We now estimate the four integrals in (13.27).

**Claim 1.** We have $|\int_{\Lambda_t} \frac{f'(s)}{f(s)} \, ds| \leq 2$.

**Proof of Claim 1.** Write $\ell(\Lambda_t) = 2t$ for the length of $\Lambda_t$. For $s = \sigma + i \tau \in \Lambda_t$ with $\sigma, \tau \in \mathbb{R}$, we have $\sigma \geq \sigma_1^+$ whence $|\frac{f'(s)}{f(s)}| \leq c\sigma_{\text{max}} \leq \frac{1}{t}$, and consequently $|\int_{\Lambda_t} \frac{f'(s)}{f(s)} \, ds| \leq \ell(\Lambda_t) \sup_{s \in \Lambda_t} |\frac{f'(s)}{f(s)}| \leq 2t \frac{1}{t} = 2$. This proves Claim 1.

**Claim 2.** We have $|\int_{\Delta_t} (\frac{f'(s)}{f(s)} - \log r_{\text{min}}) \, ds| \leq 2$.

**Proof of Claim 2.** Write $\ell(\Delta_t) = 2t$ for the length of $\Delta_t$. For $s = \sigma + i \tau \in \Delta_t$ with $\sigma, \tau \in \mathbb{R}$, we have $\sigma \leq \sigma_1^-$ whence $|\frac{f'(s)}{f(s)} - \log r_{\text{min}}| \leq c\sigma_{\text{max}} \leq \frac{1}{t}$, and consequently $|\int_{\Delta_t} (\frac{f'(s)}{f(s)} - \log r_{\text{min}}) \, ds| \leq \ell(\Delta_t) \sup_{s \in \Delta_t} |\frac{f'(s)}{f(s)} - \log r_{\text{min}}| \leq 2t \frac{1}{t} = 2$. This proves Claim 2.

**Claim 3.** There are positive constants $c_0$ and $c_1$ such that $|\text{Im} \int_{\Sigma_1^\pm} \frac{f'(s)}{f(s)} \, ds| \leq c_0 + c_1 \log t$.

**Proof of Claim 3.** Write $\rho_t = \max(|B(q) - \sigma_1^-|, |B(q) - \sigma_1^+|)$ and $R_t = 3\rho_t$. Next, define
We now note that it follows from Proposition 13.6 that $\frac{f'(\sigma)}{f_1(\sigma)} = \frac{f'_{\sigma+B(q)}(\sigma+\iota t)}{f(\sigma+B(q)+\iota t)}$ for $\sigma \in \mathbb{R}$. Hence

$$\left| \Im \int_{\Sigma_+^+} \frac{f'(s)}{f(s)} \, ds \right| = \left| \Im \int_{\sigma^+_1-B(q)}^{\sigma^+_1-B(q)} \frac{f'(\sigma + B(q) + \iota t)}{f(\sigma + B(q) + \iota t)} \, d\sigma \right| = \left| \Im \int_{\sigma^+_1-B(q)}^{\sigma^+_1-B(q)} \frac{F_1'(\sigma)}{F_1(\sigma)} \, d\sigma \right|$$

$$\leq \left| \Im \int_{\sigma^+_1-B(q)}^{\sigma^+_1-B(q)} \left( \frac{F_1'(\sigma)}{F_1(\sigma)} - \sum_{\omega \in Z(F_t)} \frac{1}{\sigma - \omega} \right) \, d\sigma \right| + \left| \Im \int_{\sigma^+_1-B(q)}^{\sigma^+_1-B(q)} \sum_{\omega \in Z(F_t)} \frac{1}{\sigma - \omega} \, d\sigma \right|$$

$$\leq (\sigma^+_1 - \sigma^-_1) \sup_{\sigma \in [\sigma^-_1-B(q), \sigma^+_1-B(q)]} \left| \frac{F_1'(\sigma)}{F_1(\sigma)} - \sum_{\omega \in Z(F_t)} \frac{1}{\sigma - \omega} \right|$$

Using the fact that $[\sigma^-_1-B(q), \sigma^+_1-B(q)] \subseteq [-\rho_1, \rho_1]$ and $\sigma^+_1 - \sigma^-_1 \leq 2\rho_1 \leq R_t$, we therefore conclude that

$$\left| \Im \int_{\Sigma_+^+} \frac{f'(s)}{f(s)} \, ds \right| \leq R_t \sup_{\sigma \in [-\rho_1, \rho_1]} \left| \frac{F_1'(\sigma)}{F_1(\sigma)} - \sum_{\omega \in Z(F_t)} \frac{1}{\sigma - \omega} \right| + \sum_{\omega \in Z(F_t)} \left| \Im \int_{\sigma^-_1-B(q)}^{\sigma^-_1-B(q)} \frac{1}{\sigma - \omega} \, d\sigma \right|.$$  \hfill (13.28)

We now note that it follows from Proposition 13.6 that

$$\sup_{\sigma \in [-\rho_1, \rho_1]} \left| \frac{F_1'(\sigma)}{F_1(\sigma)} - \sum_{\omega \in Z(F_t)} \frac{1}{\sigma - \omega} \right| \leq \frac{4R_t}{(R_t - 2\rho_1)^2} \log M_{F_t}(R_t)$$

$$= \frac{9}{R_t} \log M_{F_t}(R_t).$$ \hfill (13.29)
Combining (13.28) and (13.29) gives
\[
\left| \operatorname{Im} \int_{\Sigma_1^\pm} \frac{f'(s)}{f(s)} \, ds \right| \leq 9 \log M_{F_1}(R_t) + \sum_{\omega \in Z(F_1)} \frac{\operatorname{Im} \int_{\sigma_i^B - B(q)} \frac{1}{\sigma - \omega} \, d\sigma}{\sigma_i^B - B(q)}.
\] (13.30)

However, a simple computation shows that if \( \omega \in \mathbb{C} \) with \( \operatorname{Im} \omega \neq 0 \), then \( \int_{\sigma_i^{B(q)} - B(q)} \frac{1}{\sigma - \omega} \, d\sigma = \int_{\sigma_i^{B(q)} - B(q)} \frac{1}{1 + \omega^2} \, dx \), and so \( \left| \int_{\sigma_i^{B(q)} - B(q)} \frac{1}{\sigma - \omega} \, d\sigma \right| \leq \pi \). We therefore conclude from (13.30) that
\[
\left| \operatorname{Im} \int_{\Sigma_1^\pm} \frac{f'(s)}{f(s)} \, ds \right| \leq 9 \log M_{F_1}(R_t) + \pi \left\{ \omega \in Z(F_1) \mid |\omega| \leq \frac{1}{2} R_t \right\}.
\] (13.31)

Now an application of Proposition 13.7 shows that \( \left| \{ \omega \in Z(F_1) \mid |\omega| \leq \frac{1}{2} R_t \} \right| \leq \frac{1}{\log 2} \log M_{F_1}(R_t) \), and it therefore follows from (13.31) that
\[
\left| \operatorname{Im} \int_{\Sigma_1^\pm} \frac{f'(s)}{f(s)} \, ds \right| \leq (9 + \frac{\pi}{\log 2}) \log M_{F_1}(R_t).
\] (13.32)

Next, we estimate \( M_{F_1}(R_t) \). Indeed, since \( |f(B(q) \pm it)| \geq \frac{1}{2} \) (cf. Lemma 13.4), we conclude that \( M_{F_1}(R_t) = \sup_{|s| = R_t} \left| \frac{f(s + B(q) \pm it)}{|f(B(q) \pm it)|} \right| \leq 2(1 + \sup_{s \in [-R_t, R_t]} \left| \int_{B(q) - \sigma_i^-}^{B(q) + \sigma_i^+} \frac{1}{\omega} \, d\sigma \right|) = 2 + 2 \sum_i B_i^{\pm} \int_{\sigma_i^-}^{\sigma_i^+} t_i^{-R_t} \leq 2 + C \log t \) where \( C = 2 \sum_i B_i^{\pm} \). Also, \( c_{\max}^{-\sigma_i^-} = \frac{1}{t} \) and \( c_{\max}^{\sigma_i^+} = \frac{1}{t} \). This clearly implies that there are positive constants \( c_0^-, c_0^+, c_1^-, c_1^+ \) such that \( -\sigma_i^- = c_0^- + c_1^- \log t \) and \( \sigma_i^+ = c_0^+ + c_1^+ \log t \). Consequently \( R_t = 3R_t = 3 \max(|B(q) - \sigma_i^-|, |B(q) - \sigma_i^+|) \leq 3(\sigma_i^- - \sigma_i^+) = 3(c_0^- + c_0^+) + 3(c_1^- + c_1^+) \log t \). We therefore conclude that \( M_{F_1}(R_t) \leq 2 + C \log t \) where \( C = C_{\min}^- + C_{\min}^+ \) and \( C_1 = \frac{1}{\log t} \). It follows from this and (13.32) that
\[
\left| \operatorname{Im} \int_{\Sigma_1^\pm} \frac{f'(s)}{f(s)} \, ds \right| \leq \left( 9 + \frac{\pi}{\log 2} \right) \log(2 + C_1 \log t).
\]

This completes the proof of Claim 3.

The proof of the theorem now follows from (13.27), Claim 1, Claim 2 and Claim 3. \( \square \)

14. PROOF OF THEOREM 5.4

The purpose of this section is to prove Theorem 5.4. Recall that for a real number \( q \) and \( l = 0, 1, \ldots, d \), we write
\[
\sigma_{q,l} = \sum_{l=1}^{N} p_i^{\bar{q}_l} t_i^{-dq},
\]
and
\[
C_{\mu,r}^{\bar{q},l}\text{sym}(K) = \sum_{r_1 < r < r_1} p_i^{\bar{q}_l} t_i^{-dq} + \sum_{r=r_1}^{N} p_i^{\bar{q}_l} t_i^{-dq}.
\]
Also, recall that we define the symbolic \( q \)-multifractal Minkowski volume of \( \mu \) by
\[
V_{q,\text{sym}}^{\mu}(K) = \frac{1}{r^d} \sum_{l} C_{q,l}^{\mu,\text{sym}}(K) r^{(d-l)+dq} = \sum_{l} C_{q,l}^{\mu,\text{sym}}(K) r^{-(d-l)+dq}.
\]

The key technique used for proving Theorem 5.4 is to apply the Mellin transform to (a suitably rescaled version of) the function \( r \to C_{q,l}^{\mu,\text{sym}}(K) \). The Mellin transform is a general method for expressing functions (satisfying various growth conditions) as complex contour integrals. The precise statement is given by the Mellin transform theorem below.

**Theorem 14.1. The Mellin transform theorem [Pat].** Let \( a, b \in [-\infty, \infty] \) with \( a < b \) and let \( f : (0, \infty) \to \mathbb{R} \) be a real valued function. Assume that the following conditions are satisfied:

(i) The function \( f \) is piecewise continuous on all compact subintervals of \((0, \infty)\), and at all discontinuity points \( x_0 > 0 \) of \( f \), we have \( f(x_0) = \lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x) \);

(ii) If \( s \in \mathbb{C} \) satisfies \( a < \text{Re}(s) < b \), then \( \int_0^\infty |x^{s-1} f(x)| \, dx < \infty \).

Then we have:

1. For \( s \in \mathbb{C} \) with \( a < \text{Re}(s) < b \) the integral
   \[
   \int_0^\infty x^{s-1} f(x) \, dx
   \]
   is well-defined.

   It follows from (1) that the function \( Mf : \{ s \in \mathbb{C} | a < \text{Re}(s) < b \} \to \mathbb{C} \) given by
   \[
   (Mf)(s) = \int_0^\infty x^{s-1} f(x) \, dx
   \]
   is well-defined. The function \( Mf \) is called the Mellin transform of \( f \).

2. For \( c \in \mathbb{R} \) with \( a < c < b \) and \( x > 0 \) the integral
   \[
   \int_{c-i\infty}^{c+i\infty} x^{-s} (Mf)(s) \, ds
   \]
   is well-defined.

3. For \( c \in \mathbb{R} \) with \( a < c < b \) and \( x > 0 \), we have
   \[
   f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} (Mf)(s) \, ds.
   \]

In order to prove Theorem 5.4, we apply the Mellin transform theorem to the function \( r \to C_{q,l}^{\mu,\text{sym}}(K) \). However, before applying the Mellin transform it is useful to “rescale” the function \( r \to C_{q,l}^{\mu,\text{sym}}(K) \). This is to ensure that all points of discontinuity satisfy Condition (i) in the Mellin transform theorem. In order to do this, we first define \( E : \mathbb{R} \to \mathbb{R} \) by
\[
E(t) = \begin{cases} 
0 & \text{for } t < 0; \\
\frac{1}{2} & \text{for } t = 0; \\
1 & \text{for } 0 < t.
\end{cases}
\]
For $q \in \mathbb{R}$ and $l = 0, 1, \ldots, d$, we now define the “rescaled” version $r \rightarrow B^q_{\mu} (r)$ of the function $r \rightarrow C^q_{\mu}(K)$ by

$$B^q_{\mu}(r) = \sum_i p_i^q \left( \frac{r_i}{r} \right)^{l-dq} E(r_i - r).$$

Proposition 14.2 below shows that $B^q_{\mu}(r)$ is, indeed, a “rescaled” version of $C^q_{\mu}(K)$.

**Proposition 14.2.** Fix $q \in \mathbb{R}$ and $l = 0, 1, \ldots, d$. For $0 < r < r_{\text{min}}$, we have

$$C^q_{\mu}(K)r^{-l+dq} = \sigma_{q,l}r^{-l+dq} + (\sigma_{q,l} - 1)B^q_{\mu}(r).$$

**Proof**

This follows from a straightforward but lengthy calculation which we are omitting. □

Next, we apply the Mellin Transform Theorem to the “rescaled” function $B^q_{\mu}$.

**Proposition 14.3.** Fix $q \in \mathbb{R}$ and $l = 0, 1, \ldots, d$. Write $H^q_{\mu} = \{ s \in \mathbb{C} \mid \text{Re}(s) > \max (l - dq, \beta(q)) \}$.

1. The function $B^q_{\mu}$ is piecewise continuous on all compact subintervals of $(0, \infty)$, and at all discontinuity points $r_0 > 0$ of $B^q_{\mu}$, we have $B^q_{\mu}(r_0) = \frac{\lim_{r \to r_0} B^q_{\mu}(r) + \lim_{r \to r_0} B^q_{\mu}(r)}{2}$.

2. For $s \in H^q_{\mu}$, we have $\int_0^\infty |B^q_{\mu}(r)r^{s-1}| dr < \infty$.

It follows from (1) and (2) that the Mellin transform $MB^q_{\mu} : H^q_{\mu} \to \mathbb{C}$ of $B^q_{\mu}$ given by

$$(MB^q_{\mu})(s) = \int_0^\infty B^q_{\mu}(r)r^{s-1} dr$$

is well-defined.

3. For $s \in H^q_{\mu}$, we have

$$(MB^q_{\mu})(s) = \frac{1}{s - (l - dq)} C^q_{\mu}(s).$$

4. For $c > \max (l - dq, \beta(q))$ and $r > 0$ the integral

$$\lim_{t \to \infty} \int_{c-i t}^{c+i t} (MB^q_{\mu})(s)r^{-s} ds$$

is well-defined.

5. For $c > \max (l - dq, \beta(q))$ and $r > 0$, we have

$$B^q_{\mu}(r) = \lim_{t \to \infty} \frac{1}{2\pi i} \int_{c-i t}^{c+i t} (MB^q_{\mu})(s)r^{-s} ds.$$

In particular, for $c > \max (l - dq, \beta(q))$ and $r > 0$, we have

$$B^q_{\mu}(r) = \lim_{t \to \infty} \frac{1}{2\pi i} \int_{c-i t}^{c+i t} \frac{1}{s - (l - dq)} C^q_{\mu}(s)r^{-s} ds.$$
Proof

(1) This follows immediately from the fact that \( E \) is piecewise continuous on all compact subintervals of \( (0, \infty) \) and that all discontinuity points \( r_0 > 0 \) of \( E \) satisfy \( E(r_0) = \lim_{\tau \to r_0^+} E(r) + \lim_{\tau \to r_0^-} E(r) \).

(2) Write \( s = \sigma + it \) with \( \sigma, t \in \mathbb{R} \). It follows from the definition of \( B^{q,l}_\mu(r) \) that

\[
\int_0^\infty |B^{q,l}_\mu(r)^{r^{s-1}}| \, dr = \int_0^\infty \left| \sum_i p^q_i \left( \frac{r_1}{r} \right)^{l-dq} E(r_1 - r) r^{s-1} \right| \, dr \\
\leq \int_0^\infty \sum_i p^q_i \left( \frac{r_1}{r} \right)^{l-dq} E(r_1 - r) \, r^{s-1} \, dr \\
= \int_0^\infty \sum_i \int_0^\infty p^q_i \left( \frac{r_1}{r} \right)^{l-dq} E(r_1 - r) \, r^{s-1} \, dr \\
= \sum_i \int_0^\infty p^q_i \int_0^{r_1} \left( \frac{r_1}{r} \right)^{l-dq} \, r^{s-1} \, dr .
\]

(14.1)

Since \( \sigma = \text{Re}(s) > l - dq \), we deduce that \( \int_0^\infty (\frac{r_1}{r})^{l-dq} \, r^{s-1} \, dr = \frac{1}{\sigma - (l - dq)} r^{\sigma} \), and we therefore conclude from (14.1) that

\[
\int_0^\infty |B^{q,l}_\mu(r)^{r^{s-1}}| \, dr \leq \frac{1}{\sigma - (l - dq)} \sum_i p^q_i r^{\sigma}_i \\
= \frac{1}{\sigma - (l - dq)} \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} p^q_i r^{\sigma}_i \\
= \frac{1}{\sigma - (l - dq)} \sum_{k=1}^{\infty} \left( \sum_{i} p^q_i r^{\sigma}_i \right)^k .
\]

(14.2)

Finally, since \( \sigma = \text{Re}(s) > \beta(q) \), we conclude that \( \sum_i p^q_i r^{\sigma}_i < 1 \), whence \( \sum_{k=1}^{\infty} \left( \sum_{i} p^q_i r^{\sigma}_i \right)^k < \infty \), and we therefore deduce from the previous inequality (14.2) that \( \int_0^\infty |B^{q,l}_\mu(r)^{r^{s-1}}| \, dr \leq \frac{1}{\sigma - (l - dq)} \sum_{k=1}^{\infty} \left( \sum_{i} p^q_i r^{\sigma}_i \right)^k < \infty \).

(3) Since the series \( B^{q,l}_\mu(r)^{r^{s-1}} = \sum_i p^q_i (\frac{r_1}{r})^{l-dq} E(r_1 - r) r^{s-1} \) only has finitely many non-zero terms we immediately conclude that

\[
(MB^{q,l}_\mu)(s) = \int_0^\infty B^{q,l}_\mu(r)^{r^{s-1}} \, dr \\
= \int_0^\infty \sum_i p^q_i \left( \frac{r_1}{r} \right)^{l-dq} E(r_1 - r) r^{s-1} \, dr \\
= \sum_i p^q_i \int_0^\infty \left( \frac{r_1}{r} \right)^{l-dq} E(r_1 - r) r^{s-1} \, dr \\
= \sum_i p^q_i \int_0^{r_1} \left( \frac{r_1}{r} \right)^{l-dq} r^{s-1} \, dr .
\]

(14.3)
As \( \text{Re}(s) > l - dq \), we deduce that \( \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s - (l - dq)} r^{-s} ds = \frac{1}{s - (l - dq)} r^s \), and it now follows from (14.3) that

\[
(\text{MB}^q_{\mu})_l(s) = \sum r_l^q \eta^q_{\mu}(s) = \frac{1}{s - (l - dq)} \frac{1}{s - (l - dq)} c^q_{\mu}(s).
\]

(4) This follows immediately from the Mellin Transform Theorem.
(5) This follows immediately from the Mellin Transform Theorem and (3). \(\square\)

We can now prove Theorem 5.4.

**Proof of Theorem 5.4**

Fix \( c > \max\{ - dq, 1 - dq, \ldots, d - dq, \beta(q) \} \) and \( 0 < r < r_{\min} \). Using Proposition 14.2 and Proposition 14.3, we have

\[
\begin{align*}
V_{\mu,r}^{q_{\text{sym}}}(K) &= \sum \mu^q_{\mu} \eta^q_{\mu, l}(K) C^{q_{\text{sym}}}_{\mu,l}(K) r^{-l - dq} \\
&= \sum \mu^q_{\mu} \sigma_{q,l} r^{-l + dq} + (\sigma_{q,l} - 1) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s - (l - dq)} c^q_{\mu}(s) r^{-s} ds \\
&= \sum \mu^q_{\mu} \sigma_{q,l} r^{-l + dq} + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \sum \mu^q_{\mu} \sigma_{q,l} - 1 \right) c^q_{\mu}(s) r^{-s} ds \\
&= \sum \mu^q_{\mu} \sigma_{q,l} r^{-l + dq} + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z^q_{\mu}(s) r^{-s} ds.
\end{align*}
\]

This completes the proof. \(\square\)

**15. PROOF OF THEOREM 5.5**

The purpose of this section is to prove Theorem 5.5. We first use the estimates from Section 11 together with the residue theorem to compute the complex contour integral \( \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z^q_{\mu}(s) r^{-s} ds \) appearing in Theorem 5.4.

**Proposition 15.1.** Fix \( q \in \mathbb{R} \) and \( c > \max\{ - dq, 1 - dq, \ldots, d - dq, \beta(q) \} \). Let \( (t_{q,n})_n \) be the sequence from Theorem 13.5. For all \( 0 < r < 1 \), we have

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z^q_{\mu}(s) r^{-s} ds = \lim_{\omega \to P(s \to Z^q_{\mu}(s) r^{-s})} \sum_{|\text{Im}(\omega)| \leq t_{q,n}} \text{res} \left( s \to Z^q_{\mu}(s) r^{-s}; \omega \right).
\]

**Proof**

Fix \( l = 0, 1, \ldots, d \). It clearly suffices to prove that for all \( 0 < r < r_{\min} \), we have

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s - (l - dq)} c^q_{\mu}(s) r^{-s} ds = \lim_{\omega \to P(s \to c^q_{\mu}(s) r^{-s})} \sum_{|\text{Im}(\omega)| \leq t_{q,n}} \text{res} \left( s \to \frac{1}{s - (l - dq)} c^q_{\mu}(s) r^{-s}; \omega \right).
\]
Let $A(q)$ be the constant in Lemma 13.4. Note that it follows from Lemma 13.4 that there is a constant $k$ such that if $\sigma \leq A(q)$ and $t \in \mathbb{R}$, then

$$|\zeta_q^\sigma(\sigma + it)| \leq k.$$ 

Also, note that it follows from Theorem 13.5 that there is a constant $k_c$ such that if $\sigma \leq c$ and $n \in \mathbb{N}$, then

$$|\zeta_q^\sigma(\sigma + it_{q,n})| \leq k_c.$$ 

Next, for all positive integers $n$ and $m$ with $-m \leq A(q)$, we define paths $\Sigma^+_n$, $\Sigma^-_n$, $\Lambda_n$, $\Delta_{n,m}$ in $\mathbb{C}$ by

- $\Sigma^+_n$ is the directed line-segment from $c + it_{q,n}$ to $-m + it_{q,n}$,
- $\Sigma^-_n$ is the directed line-segment from $-m - it_{q,n}$ to $c - it_{q,n}$,
- $\Lambda_n$ is the directed line-segment from $c - it_{q,n}$ to $c - it_{q,n}$,
- $\Delta_{n,m}$ is the directed line-segment from $-m + it_{q,n}$ to $-m - it_{q,n}$.

Below we sketch the paths $U_{n,m}$, $L_{n,m}$, $\Gamma_n$ and $\Lambda_{n,m}$.

*Fig. 15.1.* The paths $U_{n,m}$, $L_{n,m}$, $\Gamma_n$ and $\Lambda_{n,m}$. 
Since the paths \( \Sigma_{n,m}^+, \Sigma_{n,m}^-, \Lambda_n \) and \( \Delta_{n,m} \) enclose the region \{ \( s \in \mathbb{C} \mid -m < \text{Re}(s) < c, -t_{q,n} < \text{Im}(s) < t_{q,n} \} \) and since all poles \( \omega \) of the function \( s \to \frac{1}{s-(l-dq)} \mathcal{G}^q \) satisfy \( \text{Re}(\omega) \in [\alpha(q), \max(l-dq, \beta(q))] \subseteq (-m, c) \), it now follows from the residue theorem that

\[
\int_{c-iT_{q,n}}^{c+iT_{q,n}} \frac{1}{s-(l-dq)} \mathcal{G}^q \, ds
\]

Using the estimates from Section 11, we will now provide estimates for the three integrals: \( \int_{\Sigma_{n,m}^+} \frac{1}{s-(l-dq)} \mathcal{G}^q \, ds, \int_{\Sigma_{n,m}^-} \frac{1}{s-(l-dq)} \mathcal{G}^q \, ds \) and \( \int_{\Sigma_{n,m}} \frac{1}{s-(l-dq)} \mathcal{G}^q \, ds \) in (15.1) and show that they tend to 0 by first fixing \( n \) and letting \( m \to \infty \), and then letting \( n \to \infty \). Define \( f_{n,m}, f_n, g_{n,m} : (0,1) \to \mathbb{R} \) by

\[
f_{n,m}(r) = \frac{k_c \left( r^{-c} - r^{-m} \right) 1}{-\log r} t_{q,n},
\]

\[
f_n(r) = \frac{k_c r^{-c}}{-\log r} t_{q,n},
\]

\[
g_{n,m}(r) = 2k r^m \left( \log \left( \sqrt{t_{q,n}^2 + (m+(l-dq))^2} + t_{q,n} \right) - \log (m+(l-dq)) \right).
\]

Below we estimate the integrals \( \int_{\Sigma_{n,m}^+} \frac{1}{s-(l-dq)} \mathcal{G}^q \, ds, \int_{\Sigma_{n,m}^-} \frac{1}{s-(l-dq)} \mathcal{G}^q \, ds \) and \( \int_{\Sigma_{n,m}} \frac{1}{s-(l-dq)} \mathcal{G}^q \, ds \) using the functions \( f_{n,m}, f_n \) and \( g_{n,m} \).

**Claim 1.** For \( 0 < r < 1 \), we have

\[
\left| \int_{\Sigma_{n,m}^+} \frac{1}{s-(l-dq)} \mathcal{G}^q \, ds \right| \leq f_{n,m}(r).
\]
**Proof of Claim 1.** We have, using Theorem 13.5,

\[
\left| \int_{\Delta_{m,n}} \frac{1}{s-(l-dq)} \zeta^q_{\mu}(s) r^{-s} \, ds \right| \\
\leq \int_{-m}^{c} \frac{1}{|s-m|} \left| \zeta^q_{\mu}(-m - \sigma + \epsilon \pm i t_{q,n}) \right| \left| r^{\sigma + m - \epsilon \pm i t_{q,n}} \right| \, d\sigma \\
\leq \int_{-m}^{c} k \sigma r^{m+\sigma-\epsilon} \, d\sigma \\
= f_{n,m}(r). \]

This completes the proof of Claim 1.

**Claim 2.** For \(0 < r < 1\), we have

\[
\left| \int_{\Delta_{m,n}} \frac{1}{s} \zeta^q_{\mu}(s) r^{-s} \, ds \right| \leq g_{n,m}(r). \]

**Proof of Claim 2.** We have, using Lemma 13.4,

\[
\left| \int_{\Delta_{m,n}} \frac{1}{s-(l-dq)} \zeta^q_{\mu}(s) r^{-s} \, ds \right| \leq \int_{-m}^{c} \frac{1}{|s-m|} \left| \zeta^q_{\mu}(-m - \epsilon i t) \right| \left| r^{\epsilon + m - \epsilon i} \right| \, dt \\
\leq \int_{-m}^{c} \frac{1}{\sqrt{(m + (l-dq))^2 + t^2}} k \, r^{m} \, dt \\
= g_{n,m}(r). \]

This completes the proof of Claim 2.

Finally, we prove the following claim.

**Claim 3.** For \(0 < r < 1\), we have

\[
2\pi i \sum_{\omega \in P(s \to \frac{1}{s-(l-dq)} \zeta^q_{\mu}(s) r^{-s}) \, \text{res} \left( s \to \frac{1}{s-(l-dq)} \zeta^q_{\mu}(s) r^{-s} ; \omega \right)} - \int_{-c-i t_{q,n}}^{c+i t_{q,n}} \frac{1}{s-(l-dq)} \zeta^q_{\mu}(s) r^{-s} \, ds \to 0 \]

**Proof of Claim 3.** Let \(\varepsilon > 0\). Next, note that for each fixed \(n \in \mathbb{N}\), we have \(f_{n,m}(r) \to f_{n}(r)\) as \(m \to \infty\) and \(g_{n,m}(r) \to 0\) as \(m \to \infty\) (since \(r < 1\)). For each fixed \(n \in \mathbb{N}\), we can therefore choose a positive integer \(M_n\) such that if \(m \geq M_n\), then

\[
f_{n,m}(r) \leq 2f_{n}(r), \\
g_{n,m}(r) \leq \frac{\varepsilon}{2}. \]

Also, since \(f_{n}(r) \to 0\) as \(n \to \infty\), we can choose a positive integer \(N_0\) such that if \(n \geq N_0\), then

\[
f_{n}(r) \leq \frac{\varepsilon}{5}. \]
Fix \( n \geq N_0 \). Using (15.1), Claim 1 and Claim 2, we now conclude that

\[
\begin{align*}
2\pi i \sum_{\omega \in P(s \rightarrow \infty)} \frac{\mathrm{res} \left( s \rightarrow \frac{1}{s - (l - dq)} \zeta_\mu^q(s)^{-r} \right)}{s - (l - dq)} c_\mu^q(s)^{-r} ds & \leq 2f_{n,M,n}(r) + g_{n,M,n}(r) \\
& \leq 4f_n(r) + g_{n,M,n}(r) \\
& \leq 4\varepsilon + \frac{\varepsilon}{3} \\
& = \varepsilon.
\end{align*}
\]

This proves Claim 3.

Finally, we deduce from Claim 3 that for \( 0 < r < 1 \), we have

\[
2\pi i \sum_{\omega \in P(s \rightarrow \infty)} \frac{\mathrm{res} \left( s \rightarrow \frac{1}{s - (l - dq)} \zeta_\mu^q(s)^{-r} \right)}{s - (l - dq)} c_\mu^q(s)^{-r} ds \\
= \left( 2\pi i \sum_{\omega \in P(s \rightarrow \infty)} \frac{\mathrm{res} \left( s \rightarrow \frac{1}{s - (l - dq)} \zeta_\mu^q(s)^{-r} \right)}{s - (l - dq)} c_\mu^q(s)^{-r} ds \right) \\
+ \int_{c-i\infty}^{c+i\infty} \frac{1}{s - (l - dq)} c_\mu^q(s)^{-r} ds \\
\rightarrow 0 + \lim_{t \to \infty} \int_{c-i\infty}^{c+i\infty} \frac{1}{s - (l - dq)} c_\mu^q(s)^{-r} ds \\
= \int_{c-i\infty}^{c+i\infty} \frac{1}{s - (l - dq)} c_\mu^q(s)^{-r} ds.
\]

This completes the proof. \( \square \)

We can now prove the second explicit formula for \( V_{\mu,r}^{q,\text{sym}}(K) \), i.e. Theorem 5.5.

**Proof of Theorem 5.5**

It follows from Theorem 5.4 and Proposition 15.1 that if \( 0 < r < r_{\text{min}} \), then

\[
V_{\mu,r}^{q,\text{sym}}(K) = \sum_l \kappa_{\mu,l}^{q,l}(K) \sigma_{q,l} r^{-l+dq} + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z_\mu^q(s)^{-r} ds \\
= \sum_l \kappa_{\mu,l}^{q,l}(K) \sigma_{q,l} r^{-l+dq} + \lim_{n=0} \sum_{\omega \in P(s \rightarrow \infty)} \mathrm{res} \left( s \rightarrow \frac{1}{s - (l - dq)} \zeta_\mu^q(s)^{-r} \right) \omega.
\]

Also, if \( \beta(q) \neq l - dq \), then clearly \( P(s \rightarrow \frac{1}{s - (l - dq)} \zeta_\mu^q(s)^{-r}) = P(\zeta_\mu^q) \cup (l - dq) \) and
\[ P(\zeta^q_n) \cap \{ l - dq \} = \emptyset. \] It follows from this observation that (15.2) can be written as

\[
V_q^{q,\text{sym}}(K) = \sum_l \kappa^q_{\mu,l}(K) \sigma_{q,l} r^{-l+dq} + \sum_l \text{res } \left( s \to Z^q_\mu(s) r^{-s}; l - dq \right) + \lim_n \sum_{\omega \in P(\zeta^q_n)} \text{res } \left( s \to Z^q_\mu(s) r^{-s}; \omega \right). \]

(15.3)

Next, if \( \beta(q) \notin \{ -dq, 1 - dq, \ldots, d - dq \} \), then a simple calculation (using the fact that if \( f \) and \( g \) are meromorphic functions with \( f(\omega) \neq 0, g(\omega) = 0 \) and \( g'(\omega) \neq 0 \), then \( \omega \) is a pole of \( \frac{f}{g} \) and \( \text{res}(\frac{f}{g}; \omega) = \frac{f(\omega)}{g'(\omega)} \)) shows that

\[
\text{res } \left( s \to Z^q_\mu(s) r^{-s}; l - dq \right) = \text{res } \left( s \to \left( \sum_k \frac{\kappa^q_{\mu,k}(K) (\sigma_{q,k} - 1)}{s - (k - dq)} \right) \zeta^q_\mu(s) r^{-s}; l - dq \right)
= \text{res } \left( s \to \frac{\kappa^q_{\mu,l}(K) (\sigma_{q,l} - 1)}{s - (l - dq)} \zeta^q_\mu(s) r^{-s}; l - dq \right)
= \frac{\kappa^q_{\mu,l}(K) (\sigma_{q,l} - 1)}{s - (l - dq)} \zeta^q_\mu(l - dq) r^{-(l - dq)}
= -\frac{\kappa^q_{\mu,l}(K) (\sigma_{q,l} - 1)}{s - (l - dq)} \zeta^q_\mu(l - dq) r^{-(l - dq)}. \]

(15.4)

Combining (15.3) and (15.4) now yields

\[
V_q^{q,\text{sym}}(K) = \sum_l \kappa^q_{\mu,l}(K) \sigma_{q,l} r^{-l+dq} - \sum_l \kappa^q_{\mu,l}(K) \sigma_{q,l} r^{-l+dq} + \lim_n \sum_{\omega \in P(\zeta^q_n)} \text{res } \left( s \to Z^q_\mu(s) r^{-s}; \omega \right) + \lim_n \sum_{\omega \in P(\zeta^q_n)} \text{res } \left( s \to Z^q_\mu(s) r^{-s}; \omega \right).
\]

This completes the proof. \( \square \)

16. PROOF OF THEOREM 5.7

The purpose of this section is to prove Theorem 5.7. Since the proof is somewhat long and involved we will now give a brief description of the main outline of the argument. However, we first introduce some notation. Fix \( q \in \mathbb{R} \) and let \( \Gamma \) denote the path defined in Proposition 13.3. We now write \( G^q \) for the set of those \( s \in \mathbb{C} \) such that \( s \) lies strictly to the right of \( \Gamma \), i.e.

\[
G^q = \left\{ s \in \mathbb{C} \mid \sup_{z \in \Gamma, \text{Im}(z) = \text{Im}(s)} \text{Re}(z) < \text{Re}(s) \right\}. \quad (16.1)
\]

Assuming that \( \beta(q) \notin \{ -dq, 1 - dq, \ldots, d - dq \} \), the proof of Theorem 5.7 is divided into the following five parts:
Part 1. The behavior of $Z^q_N(s) r^{-s}$ on $\Gamma$:

Part 1.1: For all $0 < r < 1$, the following integral exists, namely

$$\int_{\Gamma} Z^q_N(s) r^{-s} \, ds .$$

Part 1.2: In addition

$$\frac{1}{r^{-\beta(q)}} \int_{\Gamma} Z^q_N(s) r^{-s} \, ds \to 0 \text{ as } r \searrow 0 .$$

Part 1 is proved in Lemma 16.1 and Theorem 16.2.

Part 2. The behavior of $Z^q_N(s) r^{-s}$ between $\Gamma$ and the critical line $\text{Re}(s) = \beta(q)$:

Part 2.1: For all $0 < r < 1$, the following limit exists, namely

$$\lim_n \sum_{\omega \in P^1(\mathbb{C}_N^q) \cap G^q \mid |\text{Im}(\omega)| \leq t_{q,n}} \text{res} \left( s \to Z^q_N(s) r^{-s}; \omega \right) .$$

Part 2.2: In addition

$$\frac{1}{r^{-\beta(q)}} \lim_n \sum_{\omega \in P^1(\mathbb{C}_N^q) \cap G^q \mid |\text{Im}(\omega)| \leq t_{q,n}} \text{res} \left( s \to Z^q_N(s) r^{-s}; \omega \right) \to 0 \text{ as } r \searrow 0 .$$

Part 2 is proved in Theorem 16.3.

Part 3. The behavior of $Z^q_N(s) r^{-s}$ on the critical line $\text{Re}(s) = \beta(q)$:

Part 3.1: For all $r > 0$, the following limit exists, namely

$$\lim_n \sum_{\omega \in P^1(\mathbb{C}_N^q) \cap G^q \mid |\text{Im}(\omega)| \leq t_{q,n}} \text{res} \left( s \to Z^q_N(s) r^{-s}; \omega \right) .$$

Part 3.2: In addition

$$\frac{1}{r^{-\beta(q)}} \lim_n \sum_{\omega \in P^1(\mathbb{C}_N^q) \cap G^q \mid |\text{Im}(\omega)| \leq t_{q,n}} \text{res} \left( s \to Z^q_N(s) r^{-s}; \omega \right)$$

is a multiplicatively periodic function of $r$. Part 3 is proved in Theorem 16.4.
Part 4. Computing \( \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z_{\mu}^q(s) r^{-s} \, ds \) using Part 1.1, Part 2.1 and Part 3.1:

For all \( 0 < r < 1 \), we have

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z_{\mu}^q(s) r^{-s} \, ds = - \sum_{l=0,1,\ldots,d} \kappa_{q,l}^\ell(K) \sigma_{q,l} r^{-(l-dq)} + \lim_n \sum_{\omega \in P(\zeta_q^q) \cap G^q} \text{Res} \left( s \to Z_{\mu}^q(s) r^{-s}; \omega \right) \]

\[
+ \lim_n \sum_{\omega \in P(\zeta_q^q) \cap G^q} \text{Res} \left( s \to Z_{\mu}^q(s) r^{-s}; \omega \right) + \frac{1}{2\pi i} \int_{\Gamma} Z_{\mu}^q(s) r^{-s} \, ds;\]

observe that both of the two limits and the integral on the right hand side of the above equality are well-defined by Part 1.1, Part 2.1 and Part 3.1. Part 4 is proved in Theorem 16.5.

Part 5. Proving Theorem 5.7 using Part 4, Part 1.2, Part 2.2 and Part 3.2:

Theorem 5.4 shows that for all \( 0 < r < r_{\text{min}} \), we have

\[
\frac{1}{r^{\beta(q)}} V_{q,\text{sym}}^{\mu,r}(K) = \frac{1}{r^{\beta(q)}} \sum_l \kappa_{q,l}^\ell(K) \sigma_{q,l} r^{l-dq} + \frac{1}{2\pi i} \frac{1}{r^{\beta(q)}} \int_{c-i\infty}^{c+i\infty} Z_{\mu}^q(s) r^{-s} \, ds,
\]

and Part 4, Part 1.2, Part 2.2 and Part 3.2 shows that for all \( 0 < r < 1 \), we have

\[
\frac{1}{r^{\beta(q)}} \sum_l \kappa_{q,l}^\ell(K) \sigma_{q,l} r^{l-dq} + \frac{1}{2\pi i} \frac{1}{r^{\beta(q)}} \int_{c-i\infty}^{c+i\infty} Z_{\mu}^q(s) r^{-s} \, ds = \pi_q^{\text{sym}}(r) + \varepsilon_q^{\text{sym}}(r)
\]

where \( \pi_q^{\text{sym}} \) is a multiplicatively periodic function and \( \varepsilon_q^{\text{sym}}(r) \to 0 \) as \( r \searrow 0 \). Consequently, for all \( 0 < r < r_{\text{min}} \), we have

\[
\frac{1}{r^{\beta(q)}} V_{q,\text{sym}}^{\mu,r}(K) = \pi_q^{\text{sym}}(r) + \varepsilon_q^{\text{sym}}(r).
\]

Part 5 is proved after the statement and proof of Theorem 16.5.

After this brief outline, we now state and prove the results in this section.

Lemma 16.1.

1. Let \( a_1, \ldots, a_n \) be complex numbers with \( \sum_i a_i = 0 \). Let \( x_1, \ldots, x_n \) be real numbers and let \( I \) be a compact interval with \( \{x_1, \ldots, x_n\} \cap I = \emptyset \). Then

\[
\sup_{z \in \mathbb{C}} \left| \frac{1}{z} \sum_i \frac{a_i}{z-x_i} \right| < \infty.
\]
(2) Fix \( q \in \mathbb{R} \). Assume that \( \beta(q) \not\in \{-dq, 1 - dq, \ldots, d - dq\} \). Let \( b_0(q) \) be as in Proposition 13.3. Then

\[
\sup_{s \in \mathbb{C}, \Re(z) \in [b_0(q), \beta(q)]} \left| s^2 \sum_{l=0}^{d} \frac{\kappa^q_l(K)(\sigma_{q,l} - 1)}{s - (l - dq)} \right| < \infty.
\]

**Proof**

(1) Below we will use the following notation, namely, if \( R \) is a polynomial, then we will write \( \deg R \) for the degree of \( R \). Let \( Q \) denote the polynomial defined by \( Q(z) = \prod_{i} (z - x_i) \). It is clear that there is a polynomial \( P \) with \( \deg P \leq n - 2 \) such that

\[
\sum_i \frac{a_i}{z - x_i} = \left( \sum_i a_i \right) z^{n-1} + P(z)
\]

for all \( z \in \mathbb{C} \setminus \{x_1, \ldots, x_n\} \), and since \( \sum_i a_i = 0 \), this shows that \( z^2 \sum_i \frac{a_i}{z - x_i} = \frac{z^2 P(z)}{Q(z)} \) for all \( z \in \mathbb{C} \setminus \{x_1, \ldots, x_n\} \). However, since \( \deg(z \to z^2 P(z)) \leq n = \deg Q \) (because \( \deg P \leq n - 2 \)), we conclude that \( \limsup_{|z| \to \infty} \left| \frac{z^2 P(z)}{Q(z)} \right| < \infty \). This clearly implies that there is a constant \( A > 0 \) such that

\[
\sup_{|z| \geq A} \left| z^2 \sum_i \frac{a_i}{z - x_i} \right| < \infty, \quad (16.2)
\]

Next, let \( C = \{z \in \mathbb{C} \mid |z| \leq A, \Re(z) \in I\} \). Since \( \{x_1, \ldots, x_n\} \cap I = \emptyset \), we conclude that the function \( \Phi : C \to \mathbb{C} \) defined by \( \Phi(z) = z^2 \sum_i \frac{a_i}{z - x_i} \) is well-defined and continuous on \( C \). Also, since \( C \) is compact (because \( I \) is compact), the continuity of \( \Phi \) implies that \( \Phi \) is bounded on \( C \), i.e., \( \sup_{|z| \leq A, \Re(z) \in I} \left| z^2 \sum_i \frac{a_i}{z - x_i} \right| = \sup_{|z| \leq A, \Re(z) \in I} \left| \Phi(z) \right| < \infty \). The desired result now follows from this and (16.2).

(2) This follows immediately from (1) since \( \sum_i \kappa^q_l(K)(\sigma_{q,l} - 1) = 0 \). \( \square \)

**Theorem 16.2.** The behaviour of \( Z^q_\mu(s)r^{-s} \) on \( \Gamma \). Fix \( q \in \mathbb{R} \). Assume that \( \beta(q) \not\in \{-dq, 1 - dq, \ldots, d - dq\} \). Let \( b_0(q), \beta_0(q) \) and \( \Gamma \) be as in Proposition 13.3.

(1) There is a constant \( c > 0 \) such that for all \( 0 < r < 1 \), we have

\[
\int_\Gamma |Z^q_\mu(s)r^{-s}| |ds| \leq cr^{-\beta_0(q)}.
\]

(2) For all \( 0 < r < 1 \), the function \( \Gamma \to \mathbb{C} : s \to Z^q_\mu(s)r^{-s} \) is integrable. In particular, for all \( 0 < r < 1 \), the following integral is well-defined, namely

\[
\int_\Gamma Z^q_\mu(s)r^{-s} |ds|.
\]

(3) We have

\[
\frac{1}{r^{-\beta(q)}} \int_\Gamma Z^q_\mu(s)r^{-s} |ds| \to 0 \quad \text{as} \quad r \searrow 0.
\]

**Proof**

(1) We first note that it follows from Lemma 16.1 and Proposition 13.3 that there is a constant \( M \) such that if \( s \in \mathbb{C} \) with \( b_0(q) \leq \Re(s) \leq \beta_0(q) \), then

\[
\left| \sum_i \kappa^q_l(K)(\sigma_{q,l} - 1) \right| \leq \frac{M}{|s|^2}, \quad (16.3)
\]
and if \( s \in \Gamma \), then

\[
|\xi^q_n(s)| \leq M. \quad (16.4)
\]

For an integer \( n \), let the numbers \( u_n(q) \) and \( v_n(q) \) and the paths \( \Gamma_n^-, \Pi_n^- \), \( \Gamma_n^+ \) and \( \Pi_n^+ \) be defined as in Proposition 13.3. Inequalities now (16.3) and (16.4) imply that

\[
\int_{\Gamma} |Z^q_n(s) r^{-s}| |ds| = \int_{\Gamma} \left| \sum_{l} \frac{k_{n,l}(K) (\sigma_{q,l} - 1)}{s - (l - dq)} \right| \xi^q_n(s) r^{-s} |ds|
\leq M^2 \int_{\Gamma} \frac{1}{|s|^2} r^{-Re(s)} |ds|
= M^2 \left( \sum_{n = -\infty}^{\infty} \int_{\Pi_n^-} \frac{1}{|s|^2} r^{-Re(s)} |ds| + \sum_{n = -\infty}^{\infty} \int_{\Pi_n^+} \frac{1}{|s|^2} r^{-Re(s)} |ds| \right), \quad (16.5)
\]

Below we analyse the sums \( \sum_{n = -\infty}^{\infty} \int_{\Pi_n^-} \frac{1}{|s|^2} r^{-Re(s)} |ds| + \sum_{n = -\infty}^{\infty} \int_{\Pi_n^+} \frac{1}{|s|^2} r^{-Re(s)} |ds| \) and \( \sum_{n = -\infty}^{\infty} \int_{\Gamma_n^-} \frac{1}{|s|^2} r^{-Re(s)} |ds| + \sum_{n = -\infty}^{\infty} \int_{\Gamma_n^+} \frac{1}{|s|^2} r^{-Re(s)} |ds| \) appearing on the left hand side of (16.5).

First we find an upper bound for the first of the two sums, namely, the following sum \( \sum_{n = -\infty}^{\infty} \int_{\Pi_n^-} \frac{1}{|s|^2} r^{-Re(s)} |ds| + \sum_{n = -\infty}^{\infty} \int_{\Pi_n^+} \frac{1}{|s|^2} r^{-Re(s)} |ds| \). For brevity write \( w = -\frac{1}{log r_{\min}} \). Since (see Proposition 13.3) \( \ldots < u_{-1}(q) < v_{-1}(q) < u_0(q) < 0 < v_0(q) < u_1(q) < v_1(q) < \ldots \) and \( u_{n+1}(q) - u_n(q) \geq w \) for all \( n \), we conclude \( v_n(q) \geq u_n(q) \geq (n-1)w + u_1(q) \) for all \( n \geq 1 \) and that \( |v_{-(n+1)}(q)| \geq |u_{-(n+1)}(q)| \geq |u_n(q)| \geq |u_0(q)| + |u_0(q)| \) for all \( n \geq 0 \). This clearly implies that \( \liminf_{n \to \pm \infty} \frac{|u_n(q)|}{|n+1|} \geq w \) and \( \liminf_{n \to \pm \infty} \frac{|v_n(q)|}{|n+1|} \geq w \). Hence, if we write \( w_0 = \min(\inf_{n} \frac{|u_n(q)|}{|n+1|}, \inf_{n} \frac{|v_n(q)|}{|n+1|}) \), then \( w_0 > 0 \) and

\[
|u_n(q)| \geq w_0(|n| + 1), \quad |v_n(q)| \geq w_0(|n| + 1) \quad (16.6)
\]
for all positive integers $n$. Using (16.6), we now deduce that if $0 < r < 1$, then

$$
\sum_{n=-\infty}^{\infty} \int_{H_n} \frac{1}{|s|^2} r^{-\text{Re}(s)} |ds| + \sum_{n=\infty}^{\infty} \int_{H_n^+} \frac{1}{|s|^2} r^{-\text{Re}(s)} |ds| \\
= \sum_{n=-\infty}^{\infty} \int_{b_0(q)}^{\beta_1(q)} \frac{1}{\sigma^2 + u_n(q)^2} r^{-\sigma} d\sigma \\
+ \sum_{n=-\infty}^{\infty} \int_{b_0(q)}^{\beta_1(q)} \frac{1}{\sigma^2 + v_n(q)^2} r^{-\sigma} d\sigma \\
\leq \sum_{n=-\infty}^{\infty} \int_{b_0(q)}^{\beta_1(q)} \frac{1}{u_n(q)^2} r^{-\beta_0(q)} d\sigma \\
+ \sum_{n=-\infty}^{\infty} \int_{b_0(q)}^{\beta_1(q)} \frac{1}{v_n(q)^2} r^{-\beta_0(q)} d\sigma \\
\leq 2 \frac{\beta_0(q) - b_0(q)}{u_0} \left( \sum_{n=-\infty}^{\infty} \frac{1}{(|n|+1)^2} \right) r^{-\beta_0(q)}. 
$$

(16.7)

Next, we find an upper bound for the second of the sums, namely, $\sum_{n=-\infty}^{\infty} \int_{G_n^+} \frac{1}{|s|^2} r^{-\text{Re}(s)} |ds|$. If $0 < r < 1$, then

$$
\sum_{n=-\infty}^{\infty} \int_{G_n^+} \frac{1}{|s|^2} r^{-\text{Re}(s)} |ds| + \sum_{n=\infty}^{\infty} \int_{G_n^+} \frac{1}{|s|^2} r^{-\text{Re}(s)} |ds| \\
= \sum_{n=-\infty}^{\infty} \int_{b_0(q)}^{\beta_1(q)} \frac{1}{b_0(q)^2 + t^2} r^{-b_0(q)} dt \\
+ \sum_{n=-\infty}^{\infty} \int_{b_0(q)}^{\beta_1(q)} \frac{1}{\beta_0(q)^2 + t^2} r^{-\beta_0(q)} dt \\
\leq \sum_{n=-\infty}^{\infty} \int_{b_0(q)}^{\beta_1(q)} \frac{1}{\min(b_0(q)^2, \beta_0(q)^2) + t^2} dt \max \left\{ r^{-b_0(q)}, r^{-\beta_0(q)} \right\} \\
\leq \int_{-\infty}^{\infty} \frac{1}{\min(b_0(q)^2, \beta_0(q)^2) + t^2} dt r^{-\beta_0(q)} \\
= \frac{\pi}{\min(|b_0(q)|, |\beta_0(q)|)} r^{-\beta_0(q)}. 
$$

(16.8)

Finally, combining (16.5), (16.7) and (16.8) now gives

$$
\int_{G} |Z^q_{\mu}(s) r^{-s}| |ds| \leq c r^{-\beta_0(q)}
$$

where $c = M^2 \left( \frac{2 \beta_0(q) - b_0(q)}{u_0} \sum_{n=-\infty}^{\infty} \frac{1}{(|n|+1)^2} + \frac{\pi}{\min(|b_0(q)|, |\beta_0(q)|)} \right)$.

(2) This statement follows immediately from (1).

(3) Since $\beta_0(q) < \beta(q)$, it follows immediately from (1) that $\left| \int_{G} Z^q_{\mu}(s) r^{-s} ds \right| \leq \frac{1}{r^{-\beta_0(q)}} \int_{G} |Z^q_{\mu}(s) r^{-s}| |ds| \leq c r^{\beta_0(q) - \beta_0(q)} \to 0$ as $r \searrow 0$. 

$\square$
Theorem 16.3. The behavior of $Z^q(s) r^{-s}$ between $\Gamma$ and the critical line $\text{Re}(s) = \beta(q)$. Fix $q \in \mathbb{R}$. Assume that $\beta(q) \notin \{-dq, 1 - dq, \ldots, d - dq\}$. Let $(t_{q,n})_n$ be the sequence from Theorem 13.5.

(1) For $\omega \in P(\zeta^q_\mu) \cap G^q$, define $f_\omega : (0, 1) \to \mathbb{C}$ by
$$f_\omega(r) = \frac{1}{r^{-\beta(q)}} \text{res} \left( s \to Z^q_\mu(s) r^{-s}; \omega \right).$$

Then
$$\sum_{\omega \in P(\zeta^q_\mu) \cap G^q} \|f_\omega\|_\infty < \infty.$$

(2) For all $0 < r < 1$, the following limit exists, namely
$$\lim_{n} \sum_{\omega \in P(\zeta^q_\mu) \cap G^q \atop \text{Im}(\omega) \leq t_{q,n} \atop \text{Re}(\omega) < \beta(q)} \text{res} \left( s \to Z^q_\mu(s) r^{-s}; \omega \right).$$

(3) We have
$$\lim_{n} \sum_{\omega \in P(\zeta^q_\mu) \cap G^q \atop \text{Im}(\omega) \leq t_{q,n} \atop \text{Re}(\omega) < \beta(q)} \text{res} \left( s \to Z^q_\mu(s) r^{-s}; \omega \right) \to 0 \text{ as } r \searrow 0.$$

Proof
We first note that if $\omega \in P(\zeta^q_\mu) \cap G^q$, then $\text{Re}(\omega) \in [b_0(q), \beta(q)]$, and since $\{-dq, 1 - dq, \ldots, d - dq\} \cap [b_0(q), \beta(q)] = \emptyset$, we therefore deduce that for all $l = 0, 1, \ldots, d$, we have $\omega \neq l - dq$. This implies that
$$\text{res}(s \to Z^q_\mu(s) r^{-s}; \omega) = \text{res}(s \to \left( \sum_l \frac{k^{q,l}_\mu(K) (\sigma_{q,l} - 1)}{s - (l-dq)} \right) \zeta^q_\mu(s) r^{-s}; \omega)$$
$$= \left( \sum_l \frac{k^{q,l}_\mu(K) (\sigma_{q,l} - 1)}{\omega - (l-dq)} \right) r^{-\omega} \text{res}(\zeta^q_\mu; \omega).$$

(1) Lemma 16.1 and Proposition 13.2 show that there is a constant $M > 0$ such that if $s \in \mathbb{C}$ with $\text{Re}(s) \in [b_0(q), \beta(q)]$, then
$$\left| \sum_l \frac{k^{q,l}_\mu(K) (\sigma_{q,l} - 1)}{s - (l-dq)} \right| \leq M \frac{1}{|s|^2},$$

and if $\omega \in P(\zeta^q_\mu) \cap G^q$, then
$$|\text{res}(\zeta^q_\mu; \omega)| \leq M.$$
Combining (16.9) and the inequalities (16.10) and (16.11) shows that
\[
\sum_{\omega \in P(\zeta_0^q) \cap G^q} \| f_\omega \|_\infty = \lim_{n} \sum_{\omega \in P(\zeta_0^q) \cap G^q} \| f_\omega \|_\infty \\
= \lim_{n} \sum_{\omega \in P(\zeta_0^q) \cap G^q} \sup_{0 < r < 1} \left| \frac{1}{r^\beta(q)} \text{res} \left( s \rightarrow Z^q_\mu(s) r^{-s}; \omega \right) \right| \\
\leq \lim_{n} \sum_{\omega \in P(\zeta_0^q) \cap G^q} \sup_{0 < r < 1} \left| \frac{1}{r^\beta(q)} \sum_{l} \frac{\kappa^q_{\mu,l}(K)(\sigma_{q,l} - 1)}{\omega - (l - d_q)} \right| |r^{-\omega}| |\text{res}(\zeta_0^q; \omega)| \\
\leq M^2 \lim_{n} \sum_{\omega \in P(\zeta_0^q) \cap G^q} \sup_{0 < r < 1} \left| \frac{1}{|\omega|^2} \right| r^{\beta(q) - \text{Re}(\omega)} \\
= M^2 \lim_{n} \sum_{\omega \in P(\zeta_0^q) \cap G^q} \frac{1}{|\omega|^2} \\
\leq M^2 \lim_{n} \left( \sum_{k \in \mathbb{N}} \sum_{\omega \in P(\zeta_0^q) \cap G^q} \frac{1}{|\omega|^2} \right) \\
+ \sum_{\omega \in P(\zeta_0^q) \cap G^q} \frac{1}{|\omega|^2} \left( \int(t_{q,n}) < |\text{Im}(\omega)| \leq t_{q,n} \right) \left( \text{Re}(\omega) < \beta(q) \right) \\
\leq M^2 \left( \sum_{k \in \mathbb{N}} \frac{1}{k^2} \right) \\
+ \frac{1}{\int(t_{q,n})^2} \left( \int(t_{q,n}) < |\text{Im}(\omega)| \leq t_{q,n} \right) \left( \text{Re}(\omega) < \beta(q) \right) \\
\leq M^2 \left( \sum_{k \in \mathbb{N}} \frac{1}{k^2} \right) \\
+ \frac{1}{\int(t_{q,n})^2} \left( \int(t_{q,n}) < |\text{Im}(\omega)| \leq t_{q,n} \right) \left( \text{Re}(\omega) < \beta(q) \right)
\[ M^2 \lim_n \sum_{k \in \mathbb{N}} \frac{1}{k^2} = M^2 \lim_n \sum_{k \in \mathbb{N}} \frac{1}{k^2} \]

\[ \sum_{k \leq \text{int}(t_n)} \left\{ \omega \in P(\zeta_q^0) \mid k < |\text{Im}(\omega)| \leq k + 1 \right\} \frac{1}{k^2}. \]

\[ (16.12) \]

For brevity write \( \Xi_t = \{ \omega \in P(\zeta_q^0) \mid |\text{Im}(\omega)| \leq t \} \) for \( t > 0 \), and note that it follows from Theorem 13.8 that \( |\Xi_t| = \gamma t + \mathcal{O}(\log t) \) where \( \gamma = -\frac{1}{\pi} \log r_{\text{min}} \). This clearly implies that there is a constant \( c > 0 \) such that

\[ \gamma t - c \log t \leq |\Xi_t| \leq \gamma t + c \log t \quad (16.13) \]

for all \( t > 0 \). Since \( \Xi_k \subseteq \Xi_{k+1} \), it follows from (16.13) that

\[ \left| \left\{ \omega \in P(\zeta_q^0) \mid k < |\text{Im}(\omega)| \leq k + 1 \right\} \right| = |\Xi_{k+1} \setminus \Xi_k| = |\Xi_{k+1}| - |\Xi_k| \]

\[ \leq (\gamma(k + 1) + c \log(k + 1)) - (\gamma k - c \log k) \]

\[ \leq \gamma + 2c \log(k + 1). \quad (16.14) \]

Inequality (16.14) and (16.15) now imply that

\[ \sum_{\omega \in P(\zeta_q^0) \cap G^q, \text{Re}(\omega) < \beta(q)} \| f_\omega \|_\infty \leq M^2 \lim_n \sum_{k \in \mathbb{N}} \frac{1}{k^2} \left\{ \omega \in P(\zeta_q^0) \mid k < |\text{Im}(\omega)| \leq k + 1 \right\} \]

\[ \leq M^2(\gamma + 2c) \frac{\sum_{k=1}^{\infty} \log(k + 1)}{k^2} < \infty. \]

(2) This follows immediately from (1).

(3) Observe that for each \( \omega \in P(\zeta_q^0) \cap G^q \) with \( \text{Re}(\omega) < \beta(q) \) we have (using (16.9))

\[ |f_\omega(r)| = \left| \frac{1}{r^{-\beta(q)}} \text{res} \left( s \to Z^0_p(s) r^{-\lambda}; \omega \right) \right| \]

\[ = \frac{1}{r^{-\beta(q)}} \left| \sum_l \frac{\kappa_{l,3}^0(K)(\sigma_{q,l} - 1)}{\omega - (l - dq)} \right| |r^{-\omega}|| \text{res}(\zeta_q^0; \omega)| \]

\[ = \frac{1}{r^{-\beta(q)}} \left| \sum_l \frac{\kappa_{l,3}^0(K)(\sigma_{q,l} - 1)}{\omega - (l - dq)} \right| r^{\beta(q) - \text{Re}(\omega)} | \text{res}(\zeta_q^0; \omega)| \]

\[ \to 0 \quad \text{as} \quad r \searrow 0. \quad (16.15) \]

Next, since it follows from part (1) of the theorem that \( \sum_{\omega \in P(\zeta_q^0) \cap G^q, \text{Re}(\omega) < \beta(q)} \| f_\omega \|_\infty < \infty, \)
we now conclude from (16.15) that
\[
\frac{1}{r-\beta(q)} \lim_n \sum_{\omega \in P(\zeta_q) \cap G^q \atop \Im(\omega) \leq t_{q,n}, \Re(\omega) < \beta(q)} \res \left( s \to Z^q_{\mu}(s) r^{-s}; \omega \right) = \lim_n \sum_{\omega \in P(\zeta_q) \cap G^q \atop \Im(\omega) \leq t_{q,n}, \Re(\omega) < \beta(q)} f_\omega(r) \\
\rightarrow \lim_n \sum_{\omega \in P(\zeta_q) \cap G^q \atop \Im(\omega) \leq t_{q,n}, \Re(\omega) < \beta(q)} 0
\]
\[
= 0 \text{ as } r \searrow 0.
\]

This completes the proof. \(\square\)

**Theorem 16.4.** The behavior of \(Z^q_{\mu}(s) r^{-s}\) on the critical line \(\Re(s) = \beta(q)\). Fix \(q \in \mathbb{R}\). Assume that \(\beta(q) \notin \{-dq, 1-dq, \ldots, d-dq\}\). Let \((t_{q,n})_n\) be the sequence from Theorem 13.5.

1. For \(r > 0\), the following limit exists, namely
\[
\frac{1}{r-\beta(q)} \lim_n \sum_{\omega \in P(\zeta_q) \cap G^q \atop \Im(\omega) \leq t_{q,n}, \Re(\omega) = \beta(q)} \res \left( s \to Z^q_{\mu}(s) r^{-s}; \omega \right).
\]
Write
\[
\pi_{q}^\text{sym}(r) = \frac{1}{r-\beta(q)} \lim_n \sum_{\omega \in P(\zeta_q) \cap G^q \atop \Im(\omega) \leq t_{q,n}, \Re(\omega) = \beta(q)} \res \left( s \to Z^q_{\mu}(s) r^{-s}; \omega \right).
\]

2. If the set \(\{\log r_1, \ldots, \log r_N\}\) is not contained in a discrete additive subgroup of \(\mathbb{R}\), then
\[
\pi_{q}^\text{sym}(r) = -\frac{1}{\sum_i P_i^q r_i^\beta(q) \log r_i} \sum_l \kappa_{q,l}^j(K) (\sigma_{q,l} - 1) \frac{1}{\beta(q) - (l-dq)}.
\]

3. If the set \(\{\log r_1, \ldots, \log r_N\}\) is contained in a discrete additive subgroup of \(\mathbb{R}\) and \(\langle \log r_1, \ldots, \log r_N \rangle = u\mathbb{Z}\) with \(u > 0\), then
\[
\pi_{q}^\text{sym}(r) = -\frac{1}{\sum_i P_i^q r_i^\beta(q) \log r_i} u
\]
\[
\times \sum_l \left( \frac{\kappa_{q,l}^j(K) (\sigma_{q,l} - 1)}{e^{u(\beta(q) - (l-dq))} - 1} \right)
\]
\[
\times \begin{cases} 
\frac{e^{u(\beta(q) - (l-dq)) + 1}}{2} & \text{for } r \in e^{Zu}, \\
\frac{e^{u(\beta(q) - (l-dq)) \frac{\log r}{u}}}{2} & \text{for } r \notin e^{Zu}
\end{cases}.
\]

Recall, that for a real number \(x\), we write \(\frac{\text{frac}(x)}{u}\) for the fractional part of \(x\).
Assume that $\beta(q) + it$ with $t \in \mathbb{R}$ is a pole of $\zeta_{\mu}^q$. It follows from Proposition 13.2 that $\beta(q) + it$ is a simple pole of $\zeta_{\mu}^q$, and since $\beta(q) \notin \{-dq, 1-dq, \ldots, d-dq\}$, we therefore conclude from the definition of $Z_{\mu}^q$ using the fact that if $f$ and $g$ are meromorphic functions and $\omega$ is a simple pole of $f$ and $g(\omega) \neq 0$, then $\omega$ is a pole of $fg$ and $\text{res}(fg; \omega) = g(\omega)\text{res}(f;\omega))$ that

$$\text{res} \left( s \rightarrow Z_{\mu}^q(s) r^{-s}; \beta(q) + it \right) = \text{res} \left( s \rightarrow \left( \sum_l \frac{\kappa_{\mu}^{q,l}(K)(\sigma_{q,l} - 1)}{s - (l - dq)} \right) \zeta_{\mu}^q(s) r^{-s}; \beta(q) + it \right)$$

$$= \left( \sum_l \frac{\kappa_{\mu}^{q,l}(K)(\sigma_{q,l} - 1)}{(\beta(q) + it) - (l - dq)} \right) r^{-(\beta(q) + it)} \text{res} \left( \zeta_{\mu}^q; \beta(q) + it \right).$$

(16.15)

It also follows from Proposition 13.2 that $\text{res} \left( \zeta_{\mu}^q; \beta(q) + it \right) = -\sum_i 1_{p_i q r_i^{(\beta(q) + it) \log r_i}}$, and we therefore conclude from (16.15) that

$$\text{res} \left( s \rightarrow Z_{\mu}^q(s) r^{-s}; \beta(q) + it \right)$$

$$= -\sum_i p_i q r_i^{(\beta(q) + it) \log r_i} \left( \sum_l \frac{\kappa_{\mu}^{q,l}(K)(\sigma_{q,l} - 1)}{(\beta(q) + it) - (l - dq)} \right) r^{-(\beta(q) + it)}.$$

(16.16)

If the set $\{\log r_1, \ldots, \log r_N\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then it follows from Proposition 13.1 that $\beta(q)$ is the only pole $\omega$ of $\zeta_{\mu}^q$ with $\text{Re}(\omega) = \beta(q)$, and it therefore follows from (16.16) that

$$\sum_{\omega \in P(\zeta_{\mu}^q) \cap G^s \mid \text{Im}(\omega) \leq t_{p_n}} \text{res} \left( s \rightarrow Z_{\mu}^q(s) r^{-s}; \omega \right) = \text{res} \left( s \rightarrow Z_{\mu}^q(s) r^{-s}; \beta(q) \right)$$

$$= -\sum_i 1_{p_i q r_i^{\beta(q) \log r_i}} \left( \sum_l \frac{\kappa_{\mu}^{q,l}(K)(\sigma_{q,l} - 1)}{\beta(q) - (l - dq)} \right) r^{-\beta(q)}$$

for all integers $n$. This proves the desired result.

If, on the other hand, the set $\{\log r_1, \ldots, \log r_N\}$ is contained in a discrete additive subgroup of $\mathbb{R}$ and $\log r_1, \ldots, \log r_N = u\mathbb{Z}$ with $u > 0$, then it follows from another application of Proposition 13.1 that the set of poles $\omega$ of $\zeta_{\mu}^q$ with $\text{Re}(\omega) = \beta(q)$ is given by
\( \beta(q) + \frac{2\pi}{a} i \mathbb{Z} \), and it therefore follows by a further application of (16.16) that
\[
\sum_{\substack{\omega \in \mathcal{P}(q)^\circ \cap \mathcal{G}^q \cap \mathbb{R}^q \\
\text{Re}(\omega) = \beta(q) \\
\text{Im}(\omega) \leq t_{q,n}}}
\text{res} \left( s \to Z^q_\mu(s) r^{-s}; \omega \right)
\]
\[
= \sum_{\substack{k \leq t_{q,n} \atop |k|}} \text{res} \left( s \to Z^q_\mu(s) r^{-s}; \beta(q) + \frac{2\pi}{a} i k \right)
\]
\[
= \sum_{\substack{k \leq t_{q,n} \atop |k|}} -\frac{1}{2\pi i} \sum_{l} \frac{\kappa_\mu^{q,l}(K) (\sigma_{q,l} - \frac{1}{2})}{(\beta(q) + \frac{2\pi}{a} i k) - (l - dq)} r^{-\beta(q) + \frac{2\pi}{a} i k}
\]
\[
= \sum_{\substack{k \leq t_{q,n} \atop |k|}} -\frac{1}{2\pi i} \sum_{l} \frac{\kappa_\mu^{q,l}(K) (\sigma_{q,l} - \frac{1}{2})}{(\beta(q) + \frac{2\pi}{a} i k) - (l - dq)} r^{-\beta(q) + \frac{2\pi}{a} i k}
\]
(16.17)
for all integers \( n \). Finally, a simple Fourier analysis argument shows that if \( a \) is a real number with \( a \neq 0 \), then \( e^{a \text{frac}(x)} = \sum_{k \in \mathbb{Z}} \frac{e^{\frac{2\pi}{a} i k x}}{a - 2\pi i k x} \) for \( x \in \mathbb{R} \setminus \mathbb{Z} \) and \( \frac{2\pi}{a x} = \sum_{k \in \mathbb{Z}} \frac{e^{\frac{2\pi}{a} i k x}}{a - 2\pi i k x} \) for \( x \in \mathbb{Z} \); recall, that for a real number \( x \), we write \( \text{frac}(x) \) for the fractional part of \( x \). The desired result now follows from this and (16.17).

**Theorem 16.5.** Fix \( q \in \mathbb{R} \) and \( c > \max \{ -dq, 1 - dq, \ldots, d - dq, \beta(q) \} \). Assume that \( \beta(q) \notin \{ -dq, 1 - dq, \ldots, d - dq \} \). Let \( (t_{q,n})_n \) be the sequence from Theorem 13.5 and let \( \Gamma \) be as in Proposition 13.3. For all \( 0 < r < 1 \), we have
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z^q_\mu(s) r^{-s} ds = -\sum_{l=0,\ldots,d} \kappa_\mu^{q,l}(K) \sigma_{q,l} r^{-(l-dq)}
\]
\[
+ \lim_{n} \sum_{\substack{\omega \in \mathcal{P}(q)^\circ \cap \mathcal{G}^q \\
\text{Re}(\omega) < \beta(q) \\
\text{Im}(\omega) \leq t_{q,n}}} \text{res} \left( s \to Z^q_\mu(s) r^{-s}; \omega \right)
\]
\[
+ \lim_{n} \sum_{\substack{\omega \in \mathcal{P}(q)^\circ \cap \mathcal{G}^q \\
\text{Re}(\omega) = \beta(q) \\
\text{Im}(\omega) \leq t_{q,n}}} \text{res} \left( s \to Z^q_\mu(s) r^{-s}; \omega \right)
\]
\[
+ \frac{1}{2\pi i} \int_{\Gamma} Z^q_\mu(s) r^{-s} ds ; \quad (16.18)
\]
observe that both of the two limits and the integral on the right hand side of (16.18) are well-defined by Theorems 13.2–13.4.

**Proof**
It is clear that the intersections \( \Gamma \cap (\mathbb{R} - i t_{q,n}) \) and \( \Gamma \cap (\mathbb{R} + i t_{q,n}) \) are compact line segments, i.e. there are compact intervals \( I_{\Gamma,n}^- \) and \( I_{\Gamma,n}^+ \) such that \( \Gamma \cap (\mathbb{R} - i t_{q,n}) = I_{\Gamma,n}^- - i t_{q,n} \) and \( \Gamma \cap (\mathbb{R} + i t_{q,n}) = I_{\Gamma,n}^+ + i t_{q,n} \). Now, define paths \( \Delta_n \), \( \Lambda_n \), \( \Sigma_n^- \) and \( \Sigma_n^+ \) by:
\[
\Delta_n \quad \text{is the part of } \Gamma \text{ that lies in the set } \left\{ s \in \mathbb{C} \middle| |\text{Im}(s)| \leq t_{q,n} \right\},
\]
\[
\Lambda_n \quad \text{is the directed line segment from } c - i t_{q,n} \text{ to } c + i t_{q,n},
\]
\[
\Sigma_n^- \quad \text{is the directed line segment from } (\max I_{\Gamma,n}^-) - i t_{q,n} \text{ to } c - i t_{q,n},
\]
\[
\Sigma_n^+ \quad \text{is the directed line segment from } c + i t_{q,n} \text{ to } (\max I_{\Gamma,n}^+) + i t_{q,n}.
\]
Using standard notation, we let \( -\Delta_n \) denote the path \( \Delta_n \) equipped with the opposite direction. Below we sketch the paths \( -\Delta_n \), \( \Lambda_n \), \( \Sigma_n^- \) and \( \Sigma_n^+ \).
Fig. 16.1. The paths $\Delta_n$, $\Lambda_n$, $\Sigma^-_n$ and $\Sigma^+_n$.

Next, we observe that since $\{-dq, 1-dq, \ldots, d-dq\} \cap [b_0(q), \beta(q)] = \emptyset$ and all the poles $\omega$ of $\zeta^q_\mu$ satisfy $\Re(\omega) \leq \beta(q) < c$, it follows that

$$
P(Z^q_\mu(s) r^{-s}) \cap \left( \{ s \in \mathbb{C} \mid \Re(s) < c, -t_{q,n} < \Im(s) < t_{q,n} \} \cap G^q \right)
$$

$$
= P \left( s \mapsto \left( \sum_{l} \frac{\kappa^q_{\mu,l}(K) (\sigma_{q,l} - 1)}{s - (l - dq)} \zeta^q_{\mu}(s) r^{-s} \right) \right)
$$

$$
\cap \left( \{ s \in \mathbb{C} \mid \Re(s) < c, -t_{q,n} < \Im(s) < t_{q,n} \} \cap G^q \right)
$$

$$
= \left( \bigcup_{l=0,1,\ldots,d} \{ l - dq \} \right) \cup \left\{ \omega \in P(\zeta^q_\mu) \cap G^q \mid \Re(\omega) < \beta(q), -t_{q,n} < \Im(\omega) < t_{q,n} \right\}
$$

$$
\cup \left\{ \omega \in P(\zeta^q_\mu) \cap G^q \mid \Re(\omega) = \beta(q), -t_{q,n} < \Im(\omega) < t_{q,n} \right\}.
$$

(16.19)

As the paths $\Delta_n$, $\Lambda_n$, $\Sigma^-_n$ and $\Sigma^+_n$ enclose the region $\{ s \in \mathbb{C} \mid \Re(s) < c, -t_{q,n} < \Im(s) < c \}$. 


Below we compute the sum \( \sum_{\kappa} \int_{\Sigma^+} \zeta^\mu_{q} (s) r^{-s} \, ds \) and show that the integrals \( \int_{\Sigma^+} \zeta^\mu_{q} (s) r^{-s} \, ds \) and \( \int_{\Sigma^+} \zeta^\mu_{q} (s) r^{-s} \, ds \) tend to 0 as \( r \to 0 \).

We first compute the sum \( \sum_{\kappa} \int_{\Sigma^+} \zeta^\mu_{q} (s) r^{-s} \, ds \) and show that \( \int_{\Sigma^+} \zeta^\mu_{q} (s) r^{-s} \, ds \) tend to 0 as \( r \to 0 \).

Next, we show that the integrals \( \int_{\Sigma^+} \zeta^\mu_{q} (s) r^{-s} \, ds \) and \( \int_{\Sigma^+} \zeta^\mu_{q} (s) r^{-s} \, ds \) tend to 0 as \( r \to 0 \). Indeed, it follows from Lemma 16.1 that there is a constant \( M > 0 \) such that

\[
| \sum \frac{\kappa^\mu_{q} (s) (\sigma, l - dq)}{s} \zeta^\mu_{q} (s) r^{-s} | \leq M | s |^{\sigma} \text{ for all } s \text{ with } \text{Re}(s) \in [b_0(q), \beta(q)].
\]

It also follows from Theorem 13.5 that there is a constant \( \kappa \) such that if \( \sigma \leq c \) and \( n \in \mathbb{N} \), then

\[
| \zeta^\mu_{q} (\sigma + i t_{q,n}) | \leq k_c.
\]

Writing \( \ell(\Sigma^+_{n}) \) for the length of the line segment \( \Sigma^+_{n} \), we now conclude that

\[
\left| \int_{\Sigma^+_{n}} \zeta^\mu_{q} (s) r^{-s} \, ds \right| \leq \ell(\Sigma^+_{n}) \sup_{s \in \Sigma^+_{n}} | \zeta^\mu_{q} (s) r^{-s} |
\]

\[
\leq (\beta_0(q) - b_0(q)) \sup_{s \in \Sigma^+_{n}} \left| \sum_{l} \frac{\kappa^\mu_{q} (s) (\sigma, l - dq)}{s - (l - dq)} \right| | \zeta^\mu_{q} (s) r^{-\beta_0(q)} |
\]

\[
\leq (\beta_0(q) - b_0(q)) M \left( \sup_{s \in \Sigma^+_{n}} \frac{1}{|s|^2} \right) k_c r^{-\beta_0(q)}
\]

\[
\leq (\beta_0(q) - b_0(q)) M \frac{1}{t_{q,n}^{\beta_0(q)}} k_c r^{-\beta_0(q)}
\]

\[
\to 0.
\]
Similarly, one can prove that

\[ \left| \int_{-\Delta_n} Z^q_{\mu}(s) r^{-s} \, ds \right| \to 0. \] (16.23)

Also, note that since \( \int_{\Gamma} |Z^q_{\mu}(s)| r^{-s} |ds| < \infty \) (by Theorem 16.2), it follows from the Dominated Convergence Theorem that

\[ \int_{-\Delta_n} Z^q_{\mu}(s) r^{-s} \, ds = - \int_{\Delta_n} Z^q_{\mu}(s) r^{-s} \, ds \]
\[ \to - \int_{\Gamma} Z^q_{\mu}(s) r^{-s} \, ds. \] (16.24)

Finally, the desired result now follows from (16.20–16.24) by letting \( n \to \infty. \) \[ \square \]

We can now prove Theorem 5.7.

**Proof of Theorem 5.7**

Fix \( q \in \mathbb{R} \) and assume that \( \beta(q) \notin \{-dq, 1-dq, \ldots, d-dq\}. \) Let \( c > \max(-dq, 1-dq, \ldots, d-dq, \beta(q)) \). It follows from Theorem 5.4 and Theorem 16.5 that for all \( 0 < r < r_{\min} \) we have

\[
\frac{1}{r^{-\beta(q)}} V^q_{\mu,r}(K) = \frac{1}{r^{-\beta(q)}} \sum_l \kappa^q_{\mu,l}(K) \sigma_{q,l} r^{-l+dq} + \frac{1}{2\pi i} \lim_{n \to \infty} \int_{c-\frac{1}{2}i}^{c+\frac{1}{2}i} Z^q_{\mu}(s) r^{-s} \, ds
\]
\[
= \frac{1}{r^{-\beta(q)}} \sum_l \kappa^q_{\mu,l}(K) \sigma_{q,l} r^{-l+dq} - \frac{1}{r^{-\beta(q)}} \sum_{l=0,1, \ldots, d} \kappa^q_{\mu,l}(K) \sigma_{q,l} r^{-(l-dq)}
\]
\[ + \frac{1}{r^{-\beta(q)}} \lim_{n \to \infty} \sum_{\omega \in \mathcal{P}(\zeta^q_{\mu}) \cap G^q_{\sigma}} \text{res} \left( s \to Z^q_{\mu}(s) r^{-s}; \omega \right)
\]
\[ + \frac{1}{r^{-\beta(q)}} \lim_{n \to \infty} \sum_{\omega \in \mathcal{P}(\zeta^q_{\mu}) \cap G^q_{\sigma}} \text{res} \left( s \to Z^q_{\mu}(s) r^{-s}; \omega \right)
\]
\[ + \frac{1}{r^{-\beta(q)}} \lim_{n \to \infty} \sum_{\omega \in \mathcal{P}(\zeta^q_{\mu}) \cap G^q_{\sigma}} \text{res} \left( s \to Z^q_{\mu}(s) r^{-s}; \omega \right)
\]
\[ + \frac{1}{2\pi i} \lim_{n \to \infty} \int_{\Gamma} Z^q_{\mu}(s) r^{-s} \, ds \]
\[
\begin{align*}
&= \frac{1}{r - \beta(q)} \sum_{l=0,1,\ldots,d} \kappa^{q,l}_{\mu}(K) \sigma_{q,l} \ r^{-(l-dq)} \\
&+ \frac{1}{r - \beta(q)} \lim_{n} \sum_{\omega \in P(\zeta^{q}_{\mu}) \cap G^{q}} \text{res} \left( s \rightarrow Z^{q}_{\mu}(s) \ r^{-s}; \omega \right) \\
&+ \frac{1}{r - \beta(q)} \lim_{n} \sum_{\omega \in P(\zeta^{q}_{\mu}) \cap G^{q}} \text{res} \left( s \rightarrow Z^{q}_{\mu}(s) \ r^{-s}; \omega \right) \\
&+ \frac{1}{2\pi i} \frac{1}{r - \beta(q)} \int_{\Gamma} Z^{q}_{\mu}(s) \ r^{-s} \ ds \\
&= \pi^{\text{sym}}_{q}(r) + \varepsilon^{\text{sym}}_{q}(r)
\end{align*}
\]

where (see Theorem 16.4)

\[
\pi^{\text{sym}}_{q}(r) = \frac{1}{r - \beta(q)} \lim_{n} \sum_{\omega \in P(\zeta^{q}_{\mu}) \cap G^{q}} \text{res} \left( s \rightarrow Z^{q}_{\mu}(s) \ r^{-s}; \omega \right)
\]

and

\[
\varepsilon^{\text{sym}}_{q}(r) = \varepsilon^{\text{sym}}_{q,\bullet}(r) + \varepsilon^{\text{sym}}_{q,\diamond}(r) + \varepsilon^{\text{sym}}_{q,\triangle}(r)
\]

with

\[
\varepsilon^{\text{sym}}_{q,\bullet}(r) = \frac{1}{r - \beta(q)} \lim_{n} \sum_{\omega \in P(\zeta^{q}_{\mu}) \cap G^{q}} \text{res} \left( s \rightarrow Z^{q}_{\mu}(s) \ r^{-s}; \omega \right),
\]

\[
\varepsilon^{\text{sym}}_{q,\diamond}(r) = \frac{1}{r - \beta(q)} \sum_{l=0,1,\ldots,d} \kappa^{q,l}_{\mu}(K) \sigma_{q,l} \ r^{-(l-dq)},
\]

\[
\varepsilon^{\text{sym}}_{q,\triangle}(r) = \frac{1}{2\pi i} \frac{1}{r - \beta(q)} \int_{\Gamma} Z^{q}_{\mu}(s) \ r^{-s} \ ds.
\]

Finally, we note that it is clear that \( \varepsilon^{\text{sym}}_{q,\bullet}(r) \rightarrow 0 \) as \( r \searrow 0 \), and that it follows from Theorem 16.2 and Theorem 16.3 that \( \varepsilon^{\text{sym}}_{q,\diamond}(r) \rightarrow 0 \) as \( r \searrow 0 \) and that \( \varepsilon^{\text{sym}}_{q,\triangle}(r) \rightarrow 0 \) as \( r \searrow 0 \). We therefore conclude that

\[
\varepsilon^{\text{sym}}_{q}(r) = \varepsilon^{\text{sym}}_{q,\bullet}(r) + \varepsilon^{\text{sym}}_{q,\diamond}(r) + \varepsilon^{\text{sym}}_{q,\triangle}(r) \rightarrow 0 \text{ as } r \searrow 0.
\]

Theorem 5.7 follows from this.
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