Transonicity in black hole accretion – A mathematical study using the generalized Sturmf chains

Shilpi Agarwal,1⋆ Tapas K. Das2† and Rukmini Dey2‡
1Faculty of Science, Banaras Hindu University, Varanasi 221005, India.
2Harish–Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211019, India

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ABSTRACT
We apply the theory of algebraic polynomials to analytically investigate the transonic properties of hydrodynamic accretion onto non-rotating astrophysical black holes. We first construct the equation describing the space gradient of the dynamical flow velocity of accreting matter. Such equation is isomorphic to a first order autonomous dynamical system. Application of the fixed point condition enables us to construct an $n$th degree algebraic equation for the space variable along which the flow streamlines are defined to possess certain first integrals of motion. The constant coefficients for each term in that equation are functions of certain specified initial boundary conditions. Such initial boundary conditions span over a certain domain on the real line $\mathbb{R}$ – effectively, as individual sub-domain of $\mathbb{R} \times \mathbb{R}$ (spherical flow) and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ (accretion disc) for the polytropic accretion, and of $\mathbb{R}$ (spherical flow) and $\mathbb{R} \times \mathbb{R}$ (accretion disc) for isothermal accretion. The solution of aforesaid equation would then provide the critical (and consequently, the sonic) point $r_c$. The critical points itself are permissible only within a certain open interval $[r_g, L_{\rightarrow \infty}]$, where $r_g$ is the radius of the event horizon and $L_{\rightarrow \infty}$ is the physically acceptable maximally allowed limit on the value of a critical point.

For polynomials of degree $n > 4$, analytical solutions are not available. We use the Sturm’s theorem (a corollary of Sylvester’s theorem), to construct the Sturm’s chain algorithm, which can be used to calculate the number of real roots (lying within a certain sub-domain of $\mathbb{R}$) for a polynomial of any countably finite arbitrarily large integral $n$, subjected to certain sub-domains of constant co-efficients. The problem now reduces to identify the polynomials in $r_c$ with the Sturm’s sequence, and to find out the maximum number of physically acceptable solution an accretion flow with certain geometric configuration, space-time metric, and equation of state can have, and thus to investigate its multi-critical properties completely analytically, where the polynomials in $r_c$ are of $n > 4$ (for complete general relativistic axisymmetric flow, for example, where $n = 14$), and thus, for which the critical points can not be computed analytically.

Our work, as we believe, has significant importance, because for the first time in the literature, we provide a purely analytical method, by applying certain powerful theorem of algebraic polynomials in pure mathematics, to check whether certain astrophysical hydrodynamic accretion may undergo more than one sonic transitions. Our work can be generalized to analytically calculate the maximal number of equilibrium points certain autonomous dynamical system can have in general.

Key words: accretion, accretion discs – black hole physics – hydrodynamics – gravitation

⋆ shilpiagarwal.2006@gmail.com
† tapas@mri.ernet.in
‡ rkmn@mri.ernet.in
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1 INTRODUCTION

Black holes are the vacuum solutions of Einstein’s field equations in general relativity. Classically, a black hole is conceived as a singularity in space time, censored from the rest of the Universe by a mathematically defined one way surface, the event horizon. In astrophysics, black holes are the end points of the gravitational collapse of massive celestial objects. ‘Observed’ astrophysical black holes may be broadly classified into three different categories, the stellar mass (\( M_{\text{BH}} \sim \text{a few} \ M_\odot \)), the intermediate mass (significantly more massive than the stellar mass black holes but far less massive than the super massive black holes) and super massive (\( M_{\text{BH}} \gtrsim 10^6 M_\odot \)) black holes. All of the above mentioned candidates accrete matter from the surroundings, provided that the sources for such infalling material do exist. Depending on the intrinsic angular momentum content of accreting material, either spherically symmetric (zero angular momentum flow of matter), or axisymmetric (matter flow with non-zero finite angular momentum) flow geometry may be invoked to study an accreting black hole system. Since the black holes manifest their presence only gravitationally, and no spectral information can directly be obtained for these candidates, one must rely on the accretion processes to understand their observational signature (Pringle 1981; Kato, Fukue & Mineshige 1998; Frank et al. 2002).

The local Mach number \( M \) of the accreting fluid can be defined as the ratio of the local dynamical flow velocity to the local velocity of propagation of the acoustic perturbation embedded inside the accreting matter. The flow will be locally subsonic or supersonic according to \( M(r) < 1 \) or \( > 1 \). The flow is transonic if at any moment it crosses \( M = 1 \). At a distance far away from the black hole, accreting material almost always remains subsonic (except possibly for the supersonic stellar wind fed accretion) since it possesses negligible dynamical flow velocity. On the other hand, the flow velocity will approach the velocity of light \( c \) while crossing the event horizon, whereas the maximum possible value of sound speed (even for the steepest possible equation of state) would be \( c/\sqrt{3} \), resulting \( M > 1 \) close to the event horizon. In order to satisfy such inner boundary condition imposed by the event horizon, accretion onto black holes exhibit transonic properties in general.

A sonic/transonic transition in black hole accretion occurs when a subsonic to supersonic or supersonic to subsonic transition takes place either continuously (usually from a subsonic to a supersonic transition) or discontinuously (usually from a supersonic to a subsonic transition). The particular value of the spatial location where such transition takes place continuously is called a transonic point or a sonic point, and where such crossing takes place discontinuously are called shocks or discontinuities. In supersonic black hole accretion, perturbation of various kinds may produce shocks, where some dynamical and thermodynamic accretion variables changes discontinuously as such shock surfaces are crossed. Certain boundary conditions are to be satisfied across the shock, and according to those conditions, shocks in black hole accretion discs are classified into various categories. Such shock waves are quite often generated in supersonic accretion flows having small amount of intrinsic angular momentum, resulting the final subsonic state of the flow. This is because the repulsive centrifugal potential barrier experienced by such flows is sufficiently strong to brake the infalling motion and a stationary solution could be introduced only through a shock. Rotating, transonic astrophysical fluid flows are thus believed to be ‘prone’ to the shock formation phenomena. The study of steady, standing, stationary shock waves produced in black hole accretion and related phenomena thus acquired an important status in recent years (Fukue 1983, 1987, 2004; Chakrabarti 1989; Kafatos & Yang 1994; Yang & Kafatos 1995; Cadiz & Tsuruta 1998; Fukumura & Tsuruta 2004; Takahashi et al. 1992; Das 2002; Das et al. 2003; Abraham et al. 2006; Das et al. 2007; Lu et al. 1997; Lu & Gu 2004; Nakavama & Fukue 1989; Nakakura & Yamada 2008; Nakavama 1996; Nakakura & Yamada 2009; Tóth, Keppens, & Botchev 1998; Das & Czerny 2009).

On the other hand, a physical transonic accretion solutions can mathematically be realized as critical solution on the phase portrait (spanned by dynamical flow velocity/Mach number and the radial distance) of the black hole accretion. This is because, from analytical perspective, problems in black hole accretion fall under the general class of nonlinear dynamics (Ray & Bhattacharjee 2002; Afshordi & Paczynski 2003; Ray 2003a,b; Ray & Bhattacharjee 2005a; Chaudhury et al. 2006; Ray & Bhattacharjee 2006, 2007; Bhattacharjee & Ray 2007; Goswami et al. 2007; Bhattacharjee et al. 2009), since accretion describes the dynamics of a compressible astrophysical fluid, governed by a set of nonlinear differential equations. Such non-linear equations describing the steady, inviscid axisymmetric flow can further be tailored to construct a first order autonomous dynamical system. Physical transonic solution in such flows can be represented mathematically as critical solutions in the velocity (or Mach number) phase plane of the flow – they are associated with the critical points (alternatively known as the fixed points or the equilibrium points, see Jordan & Smith (1999) and Chicone (2006) for further details about the fixed point analysis techniques). To maintain the transonicity such critical points will perforce have to be saddle points, which will enable a solution to pass through themselves.

Hereafter, ‘multi-critical’ flow refers to the category of the accretion flow configuration which can have multiple critical points accessible to the accretion flow. For low angular momentum axisymmetric black hole accretion, it may so happen that the critical features are exhibited more than once in the phase portrait of a stationary solution describing such flow (Liang & Thompson 1980; Abramowicz & Zurek 1981; Muchotrzeb & Paczynski 1982; Muchotrzeb 1983; Fukue 1983, 1987, 2004a; Li 1985, 1986; Muchotrzeb-Czerny 1986; Abramowicz & Kato 1989; Abramowicz & Chakrabarti 1991; Kafatos & Yang 1994; Yang & Kafatos 1995; Cadiz & Tsuruta 1998; Das 2002; Barai et al. 2004; Abraham et al. 2006; Das et al. 2007, 2009; and Czerny 2009), and accretion becomes multi-critical. In reality, such weakly rotating sub-Keplerian flows are exhibited in various physical situations, such as detached binary systems fed by accretion from OB stellar winds (Illarionov & Sunyaev 1975; Liang & Nolan 1980), semi-detached low-mass non-magnetic binaries (Biskalo et al. 1998), and super-massive black holes fed by accretion from slowly rotating central stellar clusters (Illarionov 1988; Ho 1999 and references therein). Even
Two issues are, thus, of utmost importance in studying a shocked accretion flow around astrophysical black holes. Firstly, the inner boundary condition near the horizon dictates the presence of a saddle type sonic point through which the flow must pass to become supersonic before it finally plunges through the event horizon. Secondly, shocks in supersonic flow forces the flow to become subsonic. Hence, for accretion onto black hole, presence of at least two saddle type sonic points are thus a necessary, but not sufficient, condition for the shock formation. Since a sonic point is associated with the formation of a critical point in the accretion flow, multi-criticality, thus, plays a crucial role in studying the physics of shock formation and related phenomena in connection to the black hole accretion processes. However, depending on the symmetry of the flow and the equation of state of the accreting material, such critical points may or may not be isomorphic to the corresponding sonic points (see, e.g., Das & Czerny (2009) for further detail about the clear distinction of the sonic points and the critical points). Hence the investigation of the multi-transonic shocked accretion flow around astrophysical black holes finally boils down to the following set of operations:

One constructs the corresponding autonomous dynamical systems, then identifies the saddle type critical points of the phase trajectory of the flow. Lastly, the global understanding of the flow topologies are performed – which necessitates a complete numerical integration of the nonlinear stationary equations describing the velocity phase space behaviour of the flow.

However, for all the importance of transonic flows, there exists as yet no general mathematical prescription allowing one a direct analytical understanding of the nature of the multi-criticality without having to take recourse to the existing semi-analytic approach of numerically finding out the total number of physically acceptable critical points the accretion flow can have. Das, Chattopadhyay & Chakrabarti (2001) analytically computed the location of the critical points in an axisymmetric accretion, by being restricted to a specific single case where only the polytropic accretion was studied under the influence of a specific potential, the Paczyński & Wiita (1980) pseudo-Schwarzschild potential, and did not provide any generalized scheme for predicting the number of critical points an accretion flow can have. In general, one needs to introduce a first integral of motion in the form of a polynomial of the critical point, and involving the initial boundary conditions. Then one makes attempt to analytically solve that polynomial to find the roots (the location of the critical points). For adiabatic flow in Paczyński & Wiita (1980) potential, such a polynomial is of degree four, and hence can be solved analytically. However, the specific choice of the pseudo potential or departure from the pseudo-Newtonian framework may provide a polynomial in the critical point of degree higher than four, and a polynomial of degree $n > 4$ is non-analytically solvable for its roots. Such higher degree polynomials are indeed encountered for some specific flow configuration, for pseudo-Newtonian flow under the Nowak & Wagoner (1991) potential, or for complete general relativistic axisymmetric flow in the Schwarzschild metric where the polynomial will be of degree fourteen. Hence analytical solution of the polynomial (as has been done in Das, Chattopadhyay & Chakrabarti (2001)) is not a useful method to explore the multi-transonicity for realistic flow structures. Instead of explicitly solving the polynomial, one should rather investigate how many physically acceptable roots the polynomial can have.

This is precisely the main achievement of our work presented in this paper. Using the theory of algebraic polynomials, we developed a mathematical algorithm capable of finding the number of physically acceptable solution the polynomial can have, for any arbitrary large value of $n$ ($n$ is the degree of the polynomial as mentioned earlier). Hence for a specified set of values of the initial boundary conditions, it can mathematically be predicted whether the flow will be multi-critical (more than one real physical roots for the polynomial) or not. This paper, thus, purports to address that particular issue of investigating the transonicity of a generalized flow structure without encountering the usual semi-analytic numerical techniques, and to derive some predictive insights about the qualitative character of the flow, and in relation to that, certain physical features of the multi-criticality of the flow will also be addressed. In our work, we would like to develop a complete analytical formalism to investigate the critical behaviour of the flow structure for all possible flow configuration around a non rotating black hole. Both spherically symmetric as well as axisymmetric flow governed by the polytropic as well as the isothermal equation of state will be studied in the Newtonian, the post-Newtonian pseudo-Schwarzschild and in complete general relativistic framework.

### 2 FIRST INTEGRAL OF MOTION AS A POLYNOMIAL IN CRITICAL RADIUS

In this section, spherical and axisymmetric flows will be revisited using the Newtonian, post-Newtonian pseudo-Schwarzschild and complete general relativistic framework, to construct certain first integrals of motion in the form of a polynomial, a subset of real positive roots of which will provide the critical points of the flow. Both polytropic (for Newtonian, pseudo-Schwarzschild and general relativistic accretion in spherical symmetry and axisymmetry) as well as isothermal (for Newtonian and pseudo-Schwarzschild accretion in spherical symmetry and axisymmetry) will be considered. Some part of the content of this section is essentially a conglomerated overview of Sarkar & Das (2001); Sarkar & Das (2001); Das (2002); Das et al. (2003); Das (2004); Das et al. (2007). We believe that such repetition will be useful to have a proper understanding of the subsequent sections.
2.1 Newtonian Accretion

2.1.1 Spherically Symmetric Accretion

To researchers in astrophysics, physical models manifesting spherical symmetry carry an abiding appeal – studies in black hole accretion are no exception to this trend, since it is indeed a worthwhile exercise to consider such simplified geometric configuration in purely Newtonian formalism to begin with, and then to delve into more intricate problems. The pioneering work in this field was due to Bondi (1952), where the formal fluid dynamical equations in the Newtonian construct of space and time was introduced to study the stationary accretion problem.

Accretion flow described in this section is $\theta$ and $\phi$ symmetric and possesses only radial inflow velocity. In this section, we use the gravitational radius $r_g$ as $r_g = \frac{2GM_B}{c^2}$. The radial distances and velocities are scaled in units of $r_g$ and $c$ respectively and all other derived quantities are scaled accordingly; $G = c = M_B = 1$ is used hereafter. We assume the dynamical in-fall time scale to be short compared with any dissipation time scale during the accretion process.

The non-relativistic equation of motion for spherically accreting matter in a gravitational potential denoted by $\Phi$ may be written as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\partial \Phi}{\partial r} = 0,$$  \hspace{1cm} (1)

The first term in (1) is the Eulerian time derivative of the dynamical velocity, the second term is the ‘advective’ term, the third term is the momentum deposition due to the pressure gradient and the last term is the gravitational force. Another equation necessary to describe the motion of the fluid is the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( \rho u r^2 \right) = 0.$$  \hspace{1cm} (2)

To integrate the above set of equations, one also needs the equation of state that specifies the intrinsic properties of the fluid. We will study accretion described by either a polytropic or an isothermal equation of state.

We employ a polytropic equation of state of the form $p = K \rho^\gamma$. The sound speed $c_s$ is defined by

$$c_s^2 \equiv \left. \frac{\partial p}{\partial \rho} \right|_{\text{constant entropy}} = \frac{\gamma}{\gamma - 1} \frac{2}{\rho}.$$  \hspace{1cm} (3)

Assuming stationarity of the flow, we find the following conservation equations:

1) Conservation of energy implies constancy of the specific energy $E$

$$E = \frac{u^2}{2} + \frac{c_s^2}{\gamma - 1} + \Phi.$$  \hspace{1cm} (4)

2) Conservation of the baryon number implies constancy of the accretion rate $\dot{M}$

$$\dot{M} = 4\pi \rho u r^2.$$  \hspace{1cm} (5)

The corresponding entropy accretion rate comes out to be:

$$\Xi = 4\pi c_s^2 \frac{2}{\gamma - 1} \rho u r^2.$$  \hspace{1cm} (6)

Equation (4) is obtained from (1), and (5) follows directly from (2).

Substituting $\rho$ in terms of $c_s$ and differentiating (5) with respect to $r$, we obtain

$$c_s^2 = \frac{c_s (1 - \gamma)}{2} \left( \frac{u'}{u} + \frac{2}{r} \right),$$  \hspace{1cm} (7)

where $'$ denotes the derivative with respect to $r$. Next we differentiate (4) and eliminating $c_s'$ with the help of (7) we obtain

$$u' = \frac{2c_s^2/r - \Phi'}{u - c_s^2/u}.$$  \hspace{1cm} (8)

A real physical transonic flow must be smooth everywhere, except possibly at a shock. Hence, if the denominator of (8) vanishes at a point, the numerator must also vanish at that point to ensure the physical continuity of the flow. Borrowing the terminology from the dynamical systems theory (see, e.g., Jordan & Smith (1999)), one therefore arrives at the critical point conditions by making the numerator and the denominator of (8) simultaneously equal to zero. One thus finds the critical point conditions as

$$u_{c=r_c} = c_{s=r_c} = \sqrt{\frac{r_c \Phi'}{2 \Phi_{r=r_c}}},$$  \hspace{1cm} (9)

The critical point and the sonic points are thus equivalent. Hereafter, the subscript $c$ indicates that a particular quantity is evaluated at $r_c$. The location of the critical point/acoustic horizon is obtained by solving the algebraic equation

$$E - \frac{1}{4} \left( \frac{\gamma + 1}{\gamma - 1} \right) r_c \Phi_{r=r_c} - \Phi_c = 0.$$  \hspace{1cm} (10)
Using the system of units as described above, we substitute the value of $\Phi$ in purely Newtonian construct as $-1/r$ and obtain:

$$4E (\gamma - 1) r - (5 - 3\gamma) = 0 \tag{11}$$

The above equation is a polynomial of degree one in $r$, with constant real co-efficient parametrized by $[E, \gamma]$.

We now set the appropriate limits on $E$ and $\gamma$ to model the realistic situations encountered in astrophysics. For relativistic flow, $E$ is scaled in terms of the rest mass energy and includes the rest mass energy, $E < 1$ corresponds to the negative energy accretion state where radiative extraction of rest mass energy from the fluid is required. For such extraction to be made possible, the accreting fluid has to possess viscosity or other dissipative mechanisms, on which we would not like to focus in this work. On the other hand, almost any $E > 1$ is mathematically allowed, large values of $E$ represents flows starting from infinity with extremely high thermal energy (see section 13.4 for further detail), and $E > 2$ accretion represents enormously hot flow configurations at very large distance from the black hole, which are not properly conceivable in realistic astrophysical situations. Hence one sets $1 \lesssim E \lesssim 2$ for relativistic flows, and $0 \lesssim E \lesssim 1$ for non relativistic flows. Now, $\gamma = 1$ corresponds to isothermal accretion where accreting fluid remains optically thin. This is the physical lower limit for $\gamma$, and $\gamma < 1$ is not realistic in accretion astrophysics. On the other hand, $\gamma > 2$ is possible only for superdense matter with substantially large magnetic field (which requires the accreting material to be governed by general relativistic magneto-hydrodynamic equations, dealing with which is beyond the scope of this article) and direction dependent anisotropic pressure. One thus sets $1 \lesssim \gamma \lesssim 2$ as well. However, one should note that the most preferred values of $\gamma$ for realistic black hole accretion ranges from $4/3$ to $5/3$ [Frank et al. 2002].

Eq. (11) being a polynomial of degree one, multi-criticality is ruled out in this situation. Also to note that only the real positive solutions of (11) in the domain $r > 1$ are physically acceptable, since $r = 1$ is the radius of the event horizon. We will observe in subsequent sections that due to the choice of the modified gravitational potential, or due to the departure from the spherical symmetry, the defining polynomials may be of degree greater than one, and the analytical solution will not be straight forward.

For our purpose, (11) is the fundamental equation for the polytropic accretion, because its roots will provide the information about the multi-criticality. Irrespective of the geometric configuration, and the kind of space time in which the flow is being studied, our main aim is to construct the above kind of energy first integral polynomial for the critical point(s). For this purpose, the fundamental requirements are the expressions for the $E$, $\dot{M}$ and $\dot{M}_s$. Once we have all such expressions, we can arrive at the energy polynomial, and can either solve it analytically (if the degree $n$ of the polynomial is less than or equal to four) to find out the roots explicitly, or we can apply a mathematical scheme in general (irrespective of the degree of the energy polynomial), which will enable us to find out the number of physically acceptable roots (real, positive and greater than the Schwarzschild radius) corresponding to the polynomial. In this way we will be able to explore the transonicity of any generic flow completely analytically, which has never been done before in the literature.

For isothermal flow, equation of state of the form

$$p = \frac{RT}{\mu} \rho = c_s^2 \rho \tag{12},$$

is employed to study the accretion, where $T$ is the temperature, $R$ and $\mu$ are the universal gas constant and the mean molecular weight, respectively. The quantity $c_s$ is the isothermal sound speed defined by

$$c_s^2 = \left. \frac{\partial p}{\partial \rho} \right|_T = \Theta T \tag{13},$$

where the derivative is taken at fixed temperature and the constant $\Theta = \kappa_B/(\mu m_H)$ with $m_H \approx m_p$ being the mass of the hydrogen atom. In our model we assume that the accreting matter is predominantly hydrogen, hence $\mu \simeq 1$.

Two first integrals of motion can be obtained as:

$$\frac{u^2}{2} + \Theta T n \rho + \Phi = C \tag{14}$$

$$\dot{M} = 4\pi \rho u r^2 \tag{15}$$

Where $C$ is a constant, $u$ and $\dot{M}$ being the dynamical flow velocity and the mass accretion rate respectively. The space gradient of the dynamical velocity can be expressed as:

$$\frac{du}{dr} = \frac{2\Theta T}{u} - \Phi' u + \frac{2\Phi'}{u} \tag{16}$$

The critical point condition thus comes out to be:

$$u_s = \sqrt{\frac{r_c \Phi'}{\Theta}} = C_s = \Theta T \tag{17}$$

Putting the value of $\Phi = -1/r$, one obtains:

$$2\Theta^2 T^2 r - 1 = 0 \tag{18}$$
which is to be solved to obtain the critical point. Once again, we obtain a polynomial of degree one, and hence the multi-criticality is ruled out. The generalization of the above determining equation is quite straightforward—we need to establish a polynomial relation in \(r_c\) parametrized by the flow temperature \(T\) (for spherical accretion) as well as flow angular momentum (for axisymmetric flow).

### 2.1.2 Axisymmetric Accretion

For all categories (Newtonian, pseudo-Schwarzschild and general relativistic) of axisymmetric accretion, we assume that the disc has a radius-dependent local thickness \(H(r)\), and its central plane coincides with the equatorial plane of the black hole. It is a standard practice in accretion disc theory (see, e.g., Matsumoto et al. [1984]; Paczyński [1987]; Abramowicz et al. [1988]; Chen & Taam [1993]; Kafatos & Yang [1994]; Artemova et al. [1996]; Narayan, Kato & Honma [1997]; Witti [1998], Hawley & Krolik [2001]; Armitage, Reynolds & Chiang [2001]) to use the vertically averaged model in describing the black-hole accretion discs where the equations of motion apply to the equatorial plane of the black hole. We follow the same procedure here. The thermodynamic flow variables are averaged over the disc height, i.e., a thermodynamic quantity \(\Phi\) where the effective potential \(\Phi_{\text{eff}}\) is the summation of the gravitational potential and the ‘centrifugal potential’ due to the conserved specific angular momentum \(\lambda\) (see Das [2002] for further detail), leads to the following energy conservation equation (on the equatorial plane of the disc) in the steady state:

\[
\mathcal{E} = \frac{1}{2} u^2 + \frac{c_s^2}{\gamma - 1} + \frac{x}{2} + \Phi
\]

and the continuity equation:

\[
\frac{d}{dr} [u r H(r)] = 0
\]

can be integrated to obtain the baryon number conservation equation:

\[
\dot{M} = \sqrt{\frac{\gamma - 1}{\gamma}} c_s r \Phi^\frac{3}{2} (\Phi')^{-\frac{1}{2}}.
\]

The entropy accretion rate \(\dot{\Xi}\) can be expressed as:

\[
\dot{\Xi} = \sqrt{\frac{\gamma - 1}{\gamma}} c_s (\frac{\gamma + 1}{\gamma - 1}) \Phi^\frac{3}{2} (\Phi')^{-\frac{1}{2}}
\]

For a particular value of \([\mathcal{E}, \lambda, \gamma]\), it is now quite straightforward to derive the space gradient of the acoustic velocity \(\frac{dc_s}{dr}\) and the dynamical flow velocity \(\frac{du}{dr}\) for flow as:

\[
\left(\frac{dc_s}{dr}\right) = c_s \left(\frac{\gamma - 1}{\gamma} + \frac{1}{2} \frac{\Phi''}{\Phi'} - \frac{3}{2r} \frac{1}{u}\frac{du}{dr}\right)
\]

and,

\[
\left(\frac{du}{dr}\right) = \frac{\frac{\lambda^2}{\gamma} + \Phi'(r) - \frac{c_s^2}{\gamma + 1} \left(\frac{3}{r} + \frac{\Phi''(r)}{\Phi'(r)}\right)}{u - \frac{2c_s^2}{u(r + 1)}}
\]

where \(\Phi''\) represents the derivative of \(\Phi'\). Hence the critical point condition comes out to be:

\[
[c_s]_{r=r_c} = \sqrt{\frac{1 + \gamma}{2}} [u]_{r=r_c} = \left[\Phi'(r) + \gamma \Phi'(r) \left(\frac{\lambda^2 + r^2 \Phi'(r)}{3 \Phi'(r) + r \Phi''(r)}\right)\right]_{r=r_c}
\]

Note that the Mach number \(M_c\) at the critical point is not equal to unity, rather:

\[
M_c = \sqrt{\frac{2}{\gamma + 1}}
\]

Hence, unlike the spherical flow, the critical points and the sonic points are not equivalent. The sonic points (where the flow makes a transition from the subsonic to a supersonic state) are formed at a smaller (compared to the critical points) radial distance for the accretion. We will see
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in subsequent discussion that this is true in general for polytropic flow in the axisymmetric configuration if the disc height is taken not to be a constant quantity.

For any fixed set of \([E, \lambda, \gamma]\), the critical points can be obtained by solving the following polynomial of \(r\):

\[
E - \left[ \frac{\lambda^2}{2r^2} + \Phi \right]_{r=r_c} - \frac{2\gamma}{\gamma - 1} \left[ \frac{\Phi'(r) + \gamma \Phi(r)}{r^2} \right]_{r=r_c} = 0.
\] (28)

Similarly, for the isothermal accretion, the critical point conditions come out to be (see Das et al. (2003) for the derivation and further detail):

\[
|u|_{r=r_c} = \Theta T^{\frac{2}{\gamma}} = \sqrt{\frac{\Phi'}{\Theta + \frac{2\lambda^2}{T^2}}}_{r=r_c} - \frac{\lambda^2}{\gamma - 1}.
\] (29)

Note that the Mach number at the critical point is exactly equal to unity, hence, unlike the polytropic axisymmetric flow, the critical points and the sonic points are identical for isothermal accretion disc.

The solution of the following equation parametrized by \([T, \lambda]\) provides the critical/sonic points:

\[
\Phi''_{r=r_a} + \frac{2}{\Theta T^2} \left( \Phi' \right)^2_{r=r_a} - \left[ \frac{3}{r_c \Theta} + \frac{2\lambda^2}{T^2 r_c^2 \Theta} \right] \Phi'_{r=r_a} = 0.
\] (30)

Putting the Newtonian potential \((\phi = -1/r)\) in (28) and (30), we find that for polytropic as well as for the isothermal flow, the determining polynomials are quadratic equations:

\[
2\gamma r_c^2 + \left( 2 - \frac{20\gamma}{\gamma - 1} \right) r_c + \lambda^2 \left( \frac{20\gamma}{\gamma - 1} - 1 \right) = 0 \quad \text{Polytropic}
\] (31)

\[-5\Theta^2 T r_c^2 + 2\gamma r_c - 2\lambda^2 = 0 \quad \text{Isothermal}
\] (32)

Hence, the maximum number of the critical points such a flow can have is two. However, real physical flow does not allow such configuration. The theory of dynamical systems asserts that no two successive critical points can be of similar type (both saddle, or both centre type, for example). Hence, out of the two solutions, flow will have one saddle type and one centre type critical point. However, it has been demonstrated that Muchotrzeb & Paczyński (1982; Muchotrzeb 1983; Muchotrzeb-Czerny 1986; Chaudhury et al. 2006) a saddle type critical point which is accompanied by a centre-type critical point but without another saddle type critical point, the flow solutions will all curl about the centre-type critical point by forming a homoclinic orbit passing through the saddle type critical point, and by no means connecting the event horizon with infinity through such solutions will be possible. To avoid this globally invalid situation, and to make accretion a feasible proposition, purely Newtonian axisymmetric flow must be mono-transonic. This finding also enable us to introduces a generic statement of the form:

**Proposition 2.1.** No physically acceptable accretion flow can have two critical points.

Hereafter in this section we provide (after citing proper references) only the expressions for the critical point conditions and the form of the corresponding first integral polynomials for the critical point(s), parametrized by \([E, \gamma]\) (spherical flow) or \([E, \lambda, \gamma]\) (axisymmetric flow) for polytropic accretion, and by \([T]\) (spherical flow) or \([T, \lambda]\) (axisymmetric flow) for the isothermal accretion.

### 2.2 Post-Newtonian pseudo-Schwarzschild Accretion

Rigorous investigation of the complete general relativistic transonic black hole accretion disc structure is extremely complicated. At the same time it is understood that, as relativistic effects play an important role in the regions close to the accreting black hole (where most of the gravitational potential energy is released), purely Newtonian gravitational potential cannot be a realistic choice to describe transonic black hole accretion in general. To compromise between the ease of handling of a Newtonian description of gravity and the realistic situations described by complicated general relativistic calculations, a series of ‘modified’ Newtonian potentials have been introduced to describe the general relativistic effects that are most important for accretion disk structure around Schwarzschild and Kerr black holes (see Artemova et al. (1998); Das (2003), and references therein for further discussion).

Introduction of such potentials allows one to investigate the complicated physical processes taking place in disc accretion in a semi-Newtonian framework by avoiding pure general relativistic calculations so that most of the features of spacetime around a compact object are retained and some crucial properties of the analogous relativistic solutions of disc structure could be reproduced with high accuracy. Hence, those potentials describing the space time around a non rotating black hole might be designated as ‘post-Newtonian pseudo- Schwarzschild’ potentials (Artemova et al. (1998); Das (2003), and references therein). As long as one is not interested in astrophysical processes extremely close (within \(1 - 2\) gravitational radius) to a black hole horizon, it is safe to use the pseudo-Schwarzschild potentials to study accretion on to a Schwarzschild black hole. Among all available pseudo-Schwarzschild potentials, the Paczyński & Witt (1980) potential of the form...
The critical point condition for such flow comes out to be (Das 2004):

2.4.1 Spherical Flow

Following the procedure described in §2.1.1, the corresponding polynomials in \( r_c \) parametrized by \([E, \gamma]\) and \([T]\) for the polytropic and isothermal flow respectively can be expressed as:

\[
8E(\gamma - 1)r_c^2 + (16E - 16E\gamma + 3\gamma - 5)r_c + (8E\gamma - 8E - 4\gamma + 4) = 0 \quad \text{(Polytropic)}
\]

\[
4\lambda^2\Theta^2T^2 - (1 + 8\lambda^2 T^2)r_c + 4\Theta^2T^2 = 0 \quad \text{(Isothermal)}
\]

According to the proposition (2.1), the only physically acceptable flow will be mono-transonic for spherical accretion under the influence of the Paczyński & Wiita (1980) potential.

2.4.2 Axisymmetric Accretion

Following the procedure described in §2.1.2, we find that for the polytropic accretion the energy first integral polynomial is of the form:

\[
a_4r_c^4 + a_3r_c^3 + a_2r_c^2 + a_1r_c + a_0 = 0
\]

where:

\[
a_4 = -10E, \ a_3 = 16E - 5 + \frac{2\gamma}{(\gamma - 1)}, \ a_2 = -6E + 3 + 5\lambda^2 - \frac{4\lambda^2\gamma}{(\gamma - 1)}, \ a_1 = \frac{8\lambda^2}{(\gamma - 1)}, \ a_0 = -\lambda^2(\frac{\gamma + 3}{\gamma - 1})
\]

The first integral polynomial for isothermal flow is:

\[
2r_c^4 - (2 + \Theta^2T)r_c^3 + (3\Theta^2T)r_c^2 - 2\lambda^2r_c + 2\lambda^2 = 0
\]

Both of the above polynomials are of degree \( n = 4 \). However, we will prove in the next section and in §3 that at most three physically acceptable (real, positive and greater than unity) roots may be found. Hence the accretion flow governed by (35) or (37) is multi-critical.

2.3 Analytical solutions

Subjected to a set of specific initial boundary conditions – a set of values of \([E, \lambda, \gamma]\) for the adiabatic flow and a set of values of \([T, \lambda]\) for the isothermal flow, equation (35) and (37) can be solved completely analytically to find the number of real physical roots. We use the Ferrari’s method for solving a quartic equation to find out the solutions for (35) and (37). The Ferrari’s method has been described in detail in Appendix - I.

As for an illustrative example, a choice of \([E = 0.005, \lambda = 1.65, \gamma = 4/3]\) provides four roots for the equation (36) as 51.54, 6.3713, 2.8315 and 0.7566 (all measured in the unit of \(2GM/c^2\)). The fourth root is located inside the event horizon, hence the accretion flow can have three real physical critical points. If one uses the eigenvalue analysis (Chaudhury et al. 2006; Goswami et al. 2007) to find out the nature of the critical points, then it can easily be shown that the outer and the inner saddle type critical points are located at 51.54 and 2.8315 and the centre type critical point is located at 6.3713. For isothermal flow, \([T = 2\times10^9, \lambda = 1.9]\) provides four roots as 205.198875, 4.328, 2.2689 and 0.7566 (all measured in the unit of \(2GM/c^2\)). Here again, the fourth root is located inside the event horizon, hence the accretion flow can have three real physical critical points. The above mentioned eigenvalue analysis can reveal the nature of the critical points, and once again it can be shown that there are two saddle type inner and the outer critical points and one centre type middle critical point. As mentioned earlier, for the isothermal flow the sonic points are identical with the critical points, but for the polytropic flow they are not, and one needs to numerically integrate the flow from the saddle type critical points to find the corresponding sonic points (the radial distance where the Mach number becomes exactly equal to unity).

2.4 General Relativistic flow in the Schwarzschild Metric

2.4.1 Spherical Flow

The critical point condition for such flow comes out to be (Das 2004):

\[
u_c = c_s|_{r=r_c} = \sqrt{\frac{1}{4r_c - 3}}
\]

Thus the critical points and the sonic points are isomorphic. The corresponding energy first integral polynomial is:

\[
A_3r^3 + A_2r^2 + A_1r + A_0 = 0
\]
where
\[
\begin{align*}
A_0 &= 64(\gamma - 1)^2 - 64\epsilon^2(\gamma - 1)^2 - 64\epsilon^2 + 128\gamma\epsilon^2 \\
A_1 &= 144(\gamma - 1)^2 + 196\epsilon^2\gamma - 64\epsilon^2 - 160\epsilon^2\gamma \\
A_2 &= 188(\gamma - 1)^2 - 36\epsilon^2\gamma - 16\epsilon^2 + 48\epsilon^2\gamma \\
A_3 &= -27(\gamma - 1)^2
\end{align*}
\] (40)

However, it can be shown that inspite of the fact that the above equation is a cubic one, general relativistic spherical accretion is only mono-transonic [Das 2004].

2.4.2 Axisymmetric Accretion

Using \( r_g = GM_{BH}/c^2 \), the critical point conditions are [Das et al. 2007]
\[
\begin{align*}
\dot{u}_c &= \pm \frac{f_2(r_c, \lambda)}{f_1(r_c, \lambda) + f_2(r_c, \lambda)}; \\
c_c &= \frac{\gamma + 1}{2} \left[ \frac{f_2(r_c, \lambda)}{f_1(r_c, \lambda)} \right],
\end{align*}
\] (41)
where \( u_c \equiv u(r_c) \) and \( c_c \equiv c_c(r_c) \), \( r_c \) being the location of the critical point. \( f_1(r_c, \lambda) \) and \( f_2(r_c, \lambda) \) are defined as:
\[
\begin{align*}
f_1(r_c, \lambda) &= \frac{3r_c^2 - 2\lambda^2r_c + 3\lambda^2}{r_c^2 - \lambda^2r_c(r_c - 2)}, \\
f_2(r_c, \lambda) &= \frac{2r_c - 3}{r_c^2 - \lambda^2r_c(r_c - 2)} - \frac{2r_c^3 - \lambda^2r_c + \lambda^2}{r_c^2 - \lambda^2r_c(r_c - 2)}
\end{align*}
\] (42)

Clearly, the critical points are not coincident with the sonic points.

We derive the energy first integral polynomial as:
\[
\begin{align*}
&\pm r_c^{14}\{(\gamma - 1)^2(-108)\} + r_c^{13}\{396(\gamma - 1)^2\} \\
&+ r_c^{12}\{(\gamma - 1)^2(252\lambda^2 - 360)\} \\
&+ r_c^{11}\{(\gamma - 1)^2(-1356\lambda^2)\} \\
&+ r_c^{10}\{(2424)(\gamma - 1)^2\lambda^2 + 32\lambda^2(\gamma + 1)^2 + 432\lambda^2(\gamma - 1)^2\lambda^4\} \\
&+ r_c^9\{(\gamma - 1)^2(1352\lambda^4 - 1440\lambda^4) - \lambda^2(\gamma + 1)^2(63\lambda^2 + 6)36\lambda^2(\gamma - 1)^2 - 864\lambda^2(\gamma - 1)^2\} \\
&+ r_c^8\{(\gamma - 1)^2(48\lambda^6 - 3598\lambda^4) + \lambda^2(\gamma + 1)^2(3\lambda^4 + 34\lambda^2) + 36\lambda^2(\gamma - 1)^2\lambda^2 - 864\lambda^2\lambda^2(\gamma - 1)^2\} \\
&+ r_c^7\{(\gamma - 1)^2(4160\lambda^4 - 416\lambda^6) - \lambda^2(\gamma + 1)^2(65\lambda^2 + 26\lambda^4) - 96\lambda^2(\gamma - 1)^2\lambda^2 + 302\lambda^2(\gamma - 1)^2\lambda^4\} \\
&+ r_c^6\{(\gamma - 1)^2(1448\lambda^6 - 1800\lambda^4) + \lambda^2(\gamma + 1)^2(90\lambda^4 + 42\lambda^2) - \lambda^2(\gamma - 1)^2(48\lambda^4 - 72\lambda^2) + \lambda^2(\gamma - 1)^2(576\lambda^4 - 2592\lambda^2)\} \\
&+ r_c^5\{(\gamma - 1)^2(-2512\lambda^6 + \lambda^2(\gamma + 1)^2(23\lambda^6 - 156\lambda^4) + 248\lambda^2(\gamma - 1)^2\lambda^2 - 288\lambda^2\lambda^4(\gamma - 1)^2\} \\
&+ r_c^4\{2152\lambda^6(\gamma - 1)^2 + \lambda^2(\gamma + 1)^2(136\lambda^4 - 30\lambda^6) - \lambda^2(\gamma - 1)^2(432\lambda^4 - 16\lambda^6) + \lambda^2(\gamma - 1)^2(480\lambda^4 - 128\lambda^6)\} \\
&+ r_c^3\{(\gamma - 1)^2(-720\lambda^6) + \lambda^2(\gamma + 1)^2(280\lambda^6 - 48\lambda^4) - \lambda^2(\gamma - 1)^2(112\lambda^6 - 252\lambda^4) + \lambda^2(\gamma - 1)^2(832\lambda^4 - 2592\lambda^6)\} \\
&+ r_c^2\{2\lambda^2(\gamma - 1)^2(-400\lambda^6) + 292\lambda^2\lambda^6(\gamma - 1)^2 - 204\lambda^2(\gamma - 1)^2\lambda^6\} \\
&+ r_c\{304\lambda^6(\gamma + 1)^2 + 336\lambda^2\lambda^4(\gamma - 1)^2\} \\
&+ 2208\lambda^2\lambda^6(\gamma - 1)^2 + \{ -\lambda^2(\gamma - 1)^2 + 144\lambda^6(\gamma - 1)^2 - 864\lambda^2\lambda^6(\gamma - 1)^2\} = 0
\end{align*}
\] (43)

The above equation, being an \( n = 14 \) polynomial, is non analytically solvable. In [4] we will demonstrate how we can analytically find out the number of physically admissible real roots for this polynomial, and can investigate the transonicity of the flow.

3 STURM THEOREM AND GENERALIZED STURM SEQUENCE (CHAIN)

In this section we will elaborate the idea of the generalized Sturm sequence/chain, and will discuss its application in finding the number of roots of a algebraic polynomial equations with real co-efficients. Since the central concept of this theorem is heavily based on the idea of the greatest common divisor of a polynomial and related Euclidean algorithm, we start our discussion by clarifying such concept in somewhat great detail for the convenience of the reader.

3.1 Greatest common divisor for two numbers

Let us start with the concept of the divisibility first.

Given two non-zero integers \( z_1 \) and \( z_2 \), one defines that \( z_1 \) divides \( z_2 \), if and only if there exists some integer \( z_3 \in \mathbb{Z} \) such that:
For any other $z$.

In other words, the greatest common divisor (or the ‘greatest common factor’ or the ‘highest common factor’) of the integers without leaving any remainder. Two numbers $z_1$ and $z_2$, denoted by $gcd(z_1, z_2)$, is the positive integer $z_d \in \mathbb{Z}$, which satisfies:

1) $z_d | z_1$ and $z_d | z_2$.

2) For any other $z_c \in \mathbb{Z}$, if $z_c | z_1$ and $z_c | z_2$ then $z_c | z_d$.

In other words, the greatest common divisor $gcd(z_1, z_2)$ of two non-zero integers $z_1$ and $z_2$ is the largest possible integer that divides both the integers without leaving any remainder. Two numbers $z_1$ and $z_2$ are called ‘co-prime’ (alternatively, ‘relatively prime’), if:

$$gcd(z_1, z_2) = 1$$

The idea of a greatest common divisor can be generalized by defining the greater common divisor of a non-empty set of integers. If $S_\mathbb{Z}$ is a non-empty set of integers, then the greatest common divisor of $S_\mathbb{Z}$ is a positive integer $z_d$ such that:

1) If $z_d | z_1$ for all $z_1 \in S_\mathbb{Z}$

2) If $z_d | z_1$, for all $z_1 \in S_\mathbb{Z}$, then $z_2 | z_d$.

then we denote $z_d = gcd(S_\mathbb{Z})$.

### 3.2 Euclidean algorithm

Euclidean algorithm (first described in detail in Euclid’s ‘Elements’ in 300 BC, and is still in use, making it the oldest available numerical algorithm still in common use) provides an efficient procedure for computing the greatest common divisor of two integers. Following Stark [1978], below we provide a simplified illustration of the Euclidean algorithm for two integers:

Let us first set a ‘counter’ $i$ for counting the steps of the algorithm, with initial step corresponding to $i = 0$. Let any $i$th step of the algorithm begins with two non-negative remainders $r_{i-1}$ and $r_{i-2}$ with the requirement that $r_{i-2} < r_{i-1}$, owing to the fact that the fundamental aim of the algorithm is to reduce the remainder in successive steps, to finally bring it down to the zero in the ultimate step which terminates the algorithm. Hence, for the dummy index $i$, at the first step we have:

$$r_{-2} = z_2$$

$$r_{-1} = z_1$$

the integers for which the greatest common divisor is sought for. After we divide $z_2$ by $z_1$ (operation corresponds to $i = 1$), since $z_2$ is not divisible by $z_1$, one obtains:

$$r_{-2} = q_0 r_{-1} + r_0$$

where $r_0$ is the remainder and $q_0$ be the quotient.

For any arbitrary $i$th step of the algorithm, the aim is to find a quotient $q_j$ and remainder $r_i$, such that:

$$r_{i-2} = q_j r_{i-1} + r_i$$

where $r_i < r_{i-1}$

at some step $i = j$ (common sense dictates that $j$ can not be infinitely large), the algorithm terminates because the remainder becomes zero. Hence the final non-zero remainder $r_{j-1}$ will be the greatest common divisor of the corresponding integers.

We will now illustrate the Euclidean algorithm for finding the greatest common divisor for two polynomials.

### 3.3 Greatest common divisor and related Euclidean algorithm for polynomials

Let us first define a polynomial to be ‘monic’ if the co-efficient of the term for the highest degree variable in the polynomial is unity (one). Let us now consider $p_1(x)$ and $p_2(x)$ to be two non-zero polynomials with co-efficient from a field $\mathbb{F}$ (field of real, complex, or rational numbers, for example). A greatest common divisor of $p_1(x)$ and $p_2(x)$ is defined to the the monic polynomial $p_d(x)$ of highest degree such that $p_d(x)$ divides both $p_1(x)$ and $p_2(x)$. It is obvious that $\mathbb{F}$ be field and $p_d(x)$ be a monic, are necessary hypothesis.

In more compact form, a greatest common divisor of two polynomials $p_1, p_2 \in \mathbb{R}[x]$ is a polynomial $p_d \in \mathbb{R}[x]$ of greatest possible degree which divides both $p_1$ and $p_2$. Clearly, $p_d$ is not unique, and is only defined up to multiplication by a non-zero scalar, since for a non-zero scalar $c \in \mathbb{R}$, if $p_d$ is a $gcd(p_1, p_2 \in \mathbb{R}[x])$, so as $cp_d$. Given polynomials $p_1, p_2 \in \mathbb{R}[x]$, the division algorithm provides polynomials $p_3, p_4 \in \mathbb{R}[x]$, with $deg(p_4) < deg(p_3)$ such that
Then, if \( p_d \) is \( \gcd(p_1, p_2) \), if and only if \( p_d \) is \( \gcd(p_2, p_4) \) as is obvious.

One can compute the \( \gcd \) of two polynomials by collecting the common factors by factorizing the polynomials. However, this technique, although intuitively simple, almost always create a serious practical threat while making attempt to factorize the large high degree polynomials in reality. Euclidean algorithm appears to be relatively less complicated and a faster method for all practical purposes. Just like the integers as shown in the previous subsection, Euclid’s algorithm can directly be applied for the polynomials as well, with decreasing degree for the polynomials at each step. The last non-zero remainder, after made monic if necessary, comes out to be the greatest common divisor of the two polynomials under consideration.

Being equipped with the concept of the divisibilty, \( \gcd \) and the Euclidean algorithm, we are now in a position to define the Sturm theorem and to discuss its applications.

### 3.4 Sturm Theorem: The purpose and the definition

Sturm theorem is due to Jacques Charles Francois Sturm (29\textsuperscript{th} September 1803 - 15\textsuperscript{th} December 1855), a Geneva born French mathematician and a close collaborator of Joseph Liouville (Sturm was the co-eponym of the great Sturm-Liouville problem, an eigenvalue problem in second order differential equation). In collaboration with his long term friend Jean-Daniel Colladon, Sturm performed the first ever experimental determination of the velocity of sound in water. Sturm theorem, published in 1829 in the eleventh volume of the ‘Buletin des Sciences de Ferussac’ under the title ‘Memoire sur la resolution des equations numeriques’ Sturm theorem, which is actually a root counting theorem, is used to find the number of real roots over a certain interval of a algebraic polynomial with real co-efficient. It can be stated as:

**Theorem 3.1.** The number of real roots of an algebraic polynomial with real coefficient whose roots are simple over an interval, the endpoints of which are not roots, is equal to the difference between the number of sign changes of the Sturm chains formed for the interval ends.

Hence, given a polynomial \( p \in \mathbb{R}[X] \), if we need to find the number of roots it can have in a certain open interval \([a, b]\), \( a \) and \( b \) not being the roots of \( f \), we then construct a sequence, called ‘Sturm chain’, of polynomials, called the generalized sturm chains. Such a sequence is derived from \( p \) using the Euclidean algorithm. For the polynomial \( p \) as described above, the Sturm chain \( p_0, p_1, ..., p_k \) can be defined as:

\[
\begin{align*}
p_0 &= p \\
p_1 &= p' \\
p_n &= -\text{rem} (p_{n-2}, p_{n-1}), n \geq 2
\end{align*}
\]

where \( \text{rem} (p_{n-2}, p_{n-1}) \) is the remainder of the polynomial \( p_{n-2} \) upon division by the polynomial \( p_{n-1} \). The sequence terminates once one of the \( p_i \) becomes zero. We then evaluate this chain of polynomials at the end points \( a \) and \( b \) of the open interval. The number of roots of \( p \) in \([a, b]\) is the difference between the number of sign changes on the chain of polynomials at the end point \( a \) and the number of sign changes at the end point \( b \). Thus, for any number \( t \), if \( N_p(t) \) denotes the number of sign changes in the Sturm chain \( p_0(t), p_1(t), ..., \) then for real numbers \( a \) and \( b \) that (both) are not roots of \( p \), the number of distinct real roots of \( p \) in the open interval \([a, b]\) is \( |N_p(a) - N_p(b)| \). By making \( a \to -\infty \) and \( b \to +\infty \), one can find the total number of roots \( p \) can have on the entire domain of \( \mathbb{R} \).

A more formal definition of the Strum theorem, as a corollary of the Sylvester’s theorem, is what follows:

**Definition** Let \( R \) be the real closed field, and let \( p \) and \( P \) be in \( R[X] \). The Sturm sequence of \( p \) and \( P \) is the sequence of polynomials \((p_0, p_1, ..., p_k)\) defined as follows:

\[
\begin{align*}
p_0 &= p, p_1 = p' P \\
p_i &= p_{i-1}q_i - p_{i-2}, \text{ with } q_i \in R[X] \text{ and } \deg(p_i) < \deg(p_{i-1}) \text{ for } i = 2, 3, ..., k, p_k \text{ is a greatest common divisor of } p \text{ and } P.
\end{align*}
\]

Given a sequence \((a_0, ..., a_k)\) of elements of \( R \) with \( a_0 \neq 0 \), we define the number of sign changes in the sequence \((a_0, ..., a_k)\) as follows: count one sign change if \( a_i a_l < 0 \) with \( l > i + 1 \) and \( a_i = 0 \) for every \( j, i < j < l \).

If \( a \in R \) is not a root of \( p \) and \((p_0, ..., p_k)\) is the Sturm sequence of \( p \) and \( P \), we define \( v(p, P; a) \) to be the number of sign changes in \((p_0(a), ..., p_k(a))\).

**Theorem 3.2. (Sylvester’s Theorem)** Let \( R \) be a real closed field and let \( p \) and \( P \) be two polynomials in \( R[X] \). Let \( a, b \in R \) be such that \( a < b \) and neither \( a \) nor \( b \) are roots of \( p \). Then the difference between the number of roots of \( p \) in the interval \([a, b]\) for which \( P \) is positive and the number of roots of \( p \) in the interval \([a, b]\) for which \( P \) is negative, is equal to \( v(p, P; a) - v(p, P; b) \).

**Corollary 3.3. (Sturm’s Theorem):** Let \( R \) be a real closed field and \( p \in R[X] \). Let \( a, b \in R \) be such that \( a < b \) and neither \( a \) nor \( b \) are roots of \( p \). Then the number of roots of \( p \) in the interval \([a, b]\) is equal to \( v(p, 1; a) - v(p, 1; b) \).

The proof of these two theorems are given in the Appendix.

In next section we will provide an illustrative application of Sturm theorem by applying it to compute the number of critical points for axisymmetric polytropic accretion in Paczyński and Wiita (1980) pseudo-Schwarzschild potential.

---

1 According to some historian, the theorem was originally discovered by Jean Baptist Fourier, well before Sturm, on the eve of the French revolution.
4 NUMBER OF CRITICAL POINTS FOR PSEUDO-SCHWARZSCHILD POLYTROPIC AXISYMMETRIC ACCRETION

In §2.2.1 we have shown that the energy first integral of motion polynomial in critical points is a fourth degree equation for this case, and hence analytical solution is possible to explicitly find the roots (which has been performed in §2.3). We compute the number of roots (for same set of initial boundary condition) as a sanity check against the known (already calculated, both analytically as well as numerically) to show that Sturm theorem can actually provide the exact number of sonic point, before we apply it to find out the number of roots for relativistic accretion where no such analytical solution exists (since it is a fourteenth degree equation as already been demonstrated in §2.4.2) for cross verification.

The energy first integral polynomial for such situation can be expressed as:

\[ p_0(r) = a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0 \]  

(54)

where

\[ a_4 = -10\mathcal{E}, \]
\[ a_3 = 16\mathcal{E} - 5 + \frac{2\gamma}{(\gamma - 1)}, \]
\[ a_2 = -6\mathcal{E} + 3 + 5\lambda^2 - \frac{4\lambda^2\gamma}{(\gamma - 1)}, \]
\[ a_1 = \frac{8\lambda^2}{(\gamma - 1)}, \]
\[ a_0 = -\lambda^2 \frac{\gamma + 3}{\gamma - 1}. \]  

(55)

(56)

Subsequently we obtain the corresponding sequences as:

\[ p_1(r) = 4a_4 r^3 + 3a_3 r^2 + 2a_2 r + a_1 \]
\[ p_2(r) = b_1 r^2 + b_2 r + b_3 \]
\[ p_3(r) = c_1 r + c_2 \]
\[ p_4(r) = -[b_3 - (b_2 - \frac{b_1 c_2}{c_1}) \frac{c_2}{c_1}] \]  

(57)

where

\[ b_1 = -\frac{a_2}{2} - \frac{3(a_3)^2}{16a_4}, \]
\[ b_2 = -\frac{3a_1}{4} - \frac{2a_2 a_3}{16a_4}, \]
\[ b_3 = -[a_0 - \frac{a_1 a_3}{16a_4}], \]
\[ c_1 = -\frac{a_2}{b_1} - \frac{4a_4 b_3}{b_1} - \frac{(3a_3 - 4a_4 b_2)}{b_1} \frac{b_3}{b_1}, \]
\[ c_2 = -[a_1 - (3a_3 - \frac{4a_4 b_2}{b_1} \frac{b_3}{b_1})] \]

For the parameter set \[ \mathcal{E} = 0.0001, \lambda = 1.75, \gamma = \frac{4}{3} \], one obtains

\[ p_0(r) = -0.001 r^4 + 3.00016 r^3 - 30.6881 r^2 + 73.5 r - 39.8125 \]
\[ p_1(r) = -0.004 r^3 + 9.00048 r^2 - 61.3762 r + 73.5 \]
\[ p_2(r) = -1672.256 r^2 + 11453.475 r - 13742.088 \]
\[ p_3(r) = -0.117 r + 0.2388 \]
\[ p_4(r) = -2668.4868 \]

We now consider the left boundary of the open interval to be the Schwarzschild radius, which is unity in our scaled unit of radial distance used to describe the flow equation for pseudo-Schwarzschild axisymmetric polytropic accretion, and the right boundary to be \(10^6\) (in our scaled unit of radial distance), which is such a large distance that beyond which practically no critical point is expected to form. We then calculate the sign changes of the Sturm sequence and show the results in the following table:
We find that \( N_f^{(1)} = 3 \) and \( N_f^{(10^6)} = 0 \), hence the number of critical point obtained is three, which is fully consistent with previous analytical and numerical computation.

In next section, we illustrate how one can apply Sturm theorem to find out the number of critical point for general relativistic axisymmetric accretion in Schwarzschild metric.

5 NUMBER OF CRITICAL POINT FOR RELATIVISTIC ACCRETION

As has been demonstrated for the pseudo-Schwarzschild case, we first write down the complete expression for the Sturm chains. Then for a suitable parameter set \([E, \lambda, \gamma]\), we can find the difference of the sign change of the Sturm chains at the open interval left boundary, i.e., at the Schwarzschild radius, and at the right boundary, i.e., at \(10^6\) gravitational radius, to find the number of critical points the accretion flow can have.

The form of the original polynomial has already been explicitly expressed using (43). We now construct the Sturm chains as:

\[
\begin{align*}
 p_0(r) &= A_{14} r^{14} + A_{13} r^{13} + A_{12} r^{12} + \ldots + A_1 r + A_0 \\
p_1(r) &= 14A_{14} r^{13} + 13A_{13} r^{12} + \ldots + 2A_2 r + A_1 \\
p_2(r) &= B_{14} r^{12} + B_{13} r^{11} + \ldots + B_2 r + B_1 \\
p_3(r) &= C_{12} r^{11} + C_{11} r^{10} + \ldots + C_2 r + C_1 \\
p_4(r) &= D_{11} r^{10} + D_{10} r^{9} + \ldots + D_2 r + D_1 \\
p_5(r) &= E_{10} r^9 + E_9 r^8 + \ldots + E_2 r + E_1 \\
p_6(r) &= F_9 r^8 + F_8 r^7 + \ldots + F_2 r + F_1 \\
p(r) &= G r^7 + G r^6 + \ldots + G_2 r + G_1 \\
p_8(r) &= H_2 r^6 + H_1 r^5 + \ldots + H_2 r + H_1 \\
p_9(r) &= I_5 r^5 + I_4 r^4 + \ldots + I_2 r + I_1 \\
p_{10}(r) &= J_4 r^4 + J_3 r^3 + \ldots + J_2 r + J_1 \\
p_{11}(r) &= K_4 r^3 + K_3 r^2 + K_2 r + K_1 \\
p_{12}(r) &= L_3 r^2 + L_2 r + L_1 \\
p_{13}(r) &= M_2 r + M_1 \\
p_{14}(r) &= -(L_1 - (L_2 - \frac{L_3 M_1}{M_2} + M_1) M_2)
\end{align*}
\]

Where the explicit expression of the corresponding co-efficients \(A_i, B_i, \ldots\) has been provided in the equation (43) and in the Appendix - III.

It is important to note that direct application of the Sturm’s theorem may not always be sufficient since some of the roots may yield a negative energy for \(E\) (since the \(E\) equation was squared to get the polynomial). To get positive values of the energy, we must impose the condition that

\[
\gamma - (1 + c^2) \geq 0,
\]

which is the term present in \(E\) which could go negative. This introduces the condition that

\[
\frac{2 \gamma r^4 - 2 r^4 - 5 r^3 \gamma + 3 r^3 - 3 \lambda^2 r^2 \gamma + 5 \lambda^2 r^2 + 10 \lambda^2 r \gamma - 18 \lambda^2 r - 8 \lambda^2 \gamma + 16 \lambda^2}{2(r^4 - 2 \lambda^2 r^2 + 7 \lambda^2 r - 2 r^3 - 6 \lambda^2)} < 0
\]

This is equivalent to the condition that \(\frac{p(r)}{q(r)} \geq 0\) where \(p(r)\) and \(q(r)\) are 4th order polynomials. To find the region where this happens, one has to find the 4 roots of each of \(p(r)\) and \(q(r)\) – which is analytically possible since roots of quartics are analytically solvable. Once the roots are obtained it is a trivial matter to check for what regions the rational function is positive.

A simplified version for the above mentioned procedure to find the positivity condition is as follows:
We would like to find out the intervals in which \( p(r)/q(r) > 0 \) where \( p(r) \) and \( q(r) \) are quartic polynomials. We factorize \( p(r) = (r - r_1)(r - r_2)(r - r_3)(r - r_4) \) and \( q(r) = (r - s_1)(r - s_2)(r - s_3)(r - s_4) \) using the algorithm for finding roots of a quartic. If the roots are all real, we can just note down the sign changes of each factor to the right and left of each root and find out the intervals where the rational function is positive. If there are complex roots, they come in complex conjugates, since the coefficients of the polynomials are real. Say, if \( r_3 \) is complex and \( r_4 \) is its complex conjugate, then the part \( (r - r_3)(r - r_4) = r^2 - (r_3 + r_4)r + r_3 r_4 \) does not change sign since it is non-zero on the real line. It is easy to determine its sign.

As an illustrative example we take \([\varepsilon = 1.0001, \lambda = 3.4, \gamma = 4/3]\). We then get the number of sign changes to be three in the interval between the location of the event horizon and a representative distance \( 10^6 \) (in the units of \( GM/c^2 \)). Hence the accretion is multi-critical for this set of initial boundary condition. This is in complete agreement with the results found elsewhere (see, e.g., the figure 2. of [Das et al., 2007]) using numerical techniques.

6 DISCUSSION

Our methodology is based on the algebraic form of the first integral obtained by solving the radial momentum equation (the Euler equation to be more specific, since we are confined to the inviscid flow only). The structure for such a first integral has to be a formal polynomial with appropriate constant co-efficients. There are certain pseudo-potentials, introduced by Artemova et al. (1996), for example, for which the first integrals (neither for the polytropic nor for the isothermal flow) can not be written in a purely algebraic polynomial form. Also for general relativistic accretion in the Kerr metric, the expression for the energy first integral can not be reduced to such a polynomial form. Hence, the Sturm’s generalized chain can not be constructed for such accretion flow. Alternative methodology are required to investigate the multi-critical behaviour for such kind of accretion.

Using the method illustrated in this work, it is possible to find out how many critical points a transonic black hole accretion flow can have. It is thus possible to predict whether such accretion flow can have multi-critical properties for a certain specific value/domain of the initial boundary conditions. It is, however, not possible to investigate, using the eigenvalue analysis as illustrated in [Chaudhury et al., 2006; Goswami et al., 2007], the nature of such critical points - i.e., whether they are of saddle type or are of centre type, since such prediction requires the exact location of the critical points (the value of the roots of the polynomial). However, the theory of dynamical systems ensures that no two consecutive critical points be of same nature (both saddle or both centre). On the other hand, our experience predicts (it is rather a documented fact) that for all kind of black hole accretion, irrespective of the equation of state, the space time geometry or the flow configuration used, one has two saddle type critical points and one centre type critical point flanked by them. Hence if the application of Sturm’s generalized chain ensures the presence of three critical points, we can say that out of those three critical points, accretion flow will have two saddle type critical points, hence a specific subset of the solution having three roots corresponding to the first integral polynomial, can make transonic transition for more than one times, if appropriate conditions for connecting the flow through the outer critical point and for flow through the inner critical points are available, see, e.g., [Das & Czerny, 2009] for further discussion.

In this work we have considered only inviscid accretion. Our methodology of investigating the multi-critical properties, however, is expected to be equally valid for the viscous accretion disc as well. For the viscous flow, the radial momentum conservation equation involving the first order space derivative of the dynamical flow velocity will certainly provide a first integral of motion upon integration. Because of the fact that a viscous accretion disc is not a non-dissipative system, such constant of motion, however, can never be identified with the specific generalized Sturm chain can be made possible to find out how many critical points such an accretion flow can have subjected to the specific algebraic expression. Our work, as we believe, can have a broader perspective as well, in the field of the study of dynamical systems in general. For a first order autonomous dynamical system, provided one can evaluate the critical point conditions, the corresponding generalized \( n \)th degree algebraic equation involving the position co-ordinate and one (or more) first integral of motion can be constructed. If such algebraic equation can finally be reduced to a \( n \)th degree polynomial with well defined domain for the constant co efficient, one can easily find out the maximal number of fixed points of such dynamical systems.

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7 APPENDIX - I : ON FERRARI’S METHOD

Ferrari’s method is used to analytically solve a quartic equation for its roots. Given the quartic equation $Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$, $(A, B, C, D$ real or complex) its solution (i.e. the roots of the quartic) can be found by means of the following calculations:

$$\alpha = -\frac{B}{8A} + \frac{C}{4A}$$
$$\beta = \frac{B^2}{2A} - \frac{BC}{4A} + \frac{D}{2A}$$
$$\gamma = -\frac{B^3}{24A^2} + \frac{BC^2}{16A^3} - \frac{BD}{4A^2} + \frac{5E}{8A}$$

If $\beta = 0$ then $x = -\frac{\alpha}{A} \pm \sqrt{\frac{-\alpha^2 + \sqrt{\alpha^2 - 4\gamma}}{2}}$, where $\pm_s$ and $\pm_t$ are two distinct sets of plus and minuses , i.e., there are four possibilities, $(\pm_s, \pm_t) = (+, +), (+, -), (-, +), (-, -)$.

If $\beta \neq 0$ then continue with

$$P = -\frac{\alpha^2}{B} - \beta,$$
$$Q = -\frac{\alpha^3}{3B} + \frac{\alpha\beta}{3} - \frac{\gamma}{3},$$
$$R = -\frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}$$

(either sign of the square root will do)
$$U = R^{1/3}$$

(there are three complex roots, but any one of them will do).

To compute the cube-root of a complex number, we proceed as follows: Let $R = x_1 + ix_2 = re^{i\theta}$ where $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \tan^{-1}(\frac{x_2}{x_1})$. (Here $i = \sqrt{-1}$).

$$R^{1/3} = r^{1/3} e^{i\theta/3} = r^{1/3} (\cos(\theta/3) + i \sin(\theta/3)).$$

The other possible choices are $R^{1/3} \omega$ and $R^{1/3} \omega^2$ where $\omega = (-1 + \sqrt{3}i)/2$ is a cube root of unity.

If $U = 0$ then $y = -\frac{2\alpha}{B} - Q^{1/3}$

If $U \neq 0$ let $y = -\frac{2\alpha}{B} + U - \frac{\beta}{3y}$

Let $W = \sqrt{\alpha + 2y}$

Then, the four roots of the quartic are

$$x = -\frac{B}{4A} + \frac{\pm W \pm \sqrt{-3(\alpha + 2y) \pm \Delta}}{2}$$

The two $\pm_s$ must have the same sign and the $\pm_t$ are two independent $\pm$. To get all the roots, compute $x$ for the four possibilities of $\pm_s$ and $\pm_t$.

8 APPENDIX - II : PROOF OF SYLVESTER’S THEOREM:

First note that the Sturm sequence $(f_0, ..., f_k)$ is (up to signs) equal to the sequence obtained from the euclidean algorithm. Define a new sequence $(g_0, ..., g_k)$ by $g_i = f_i/f_k$ for $i \in \{0, ..., k\}$. Note that the number of sign changes in $(f_0(x), f_1(x))$ (resp. $(f_{i-1}(x), f_i(x), f_{i+1}(x))$) and the number of sign changes in $(g_0(x), g_1(x))$ (resp. $(g_{i-1}(x), g_i(x), g_{i+1}(x))$) coincide for any $x$ which is not a root of $f$. Note also that the roots of $g_0$ are exactly the roots of $f$ which are not roots of $g$. Observe that for $i \in 0, ..., k, g_{i-1}$ and $g_i$ are relatively prime. We consider, now, how $v(f, g; x)$ behaves when $x$ passes through a root $c$ of a polynomial $g_i$. If $c$ is a root of $g_0$, then it is not a root of $g_1$. We write $f'/(c_i) > 0$ (resp. < 0) if $f'$ is positive (resp. negative) immediately to the left of $c$. The sign of $f'(c_\pm)$ is defined similarly. Now we recall the following result: if $R$ is a real closed field, $f \in R[X], a, b \in R$ with $a < b$ and if the derivative $f'$ is positive (resp. negative) on $[a, b]$, then $f$ is strictly increasing (resp. strictly decreasing) on $[a, b]$. Then, according to the signs of $g(c), f'(c_-)$ and $f'(c_+)$ we have the following 8 cases:

| $g(c)$ | $f'(c_-)$ | $f'(c_+)$ |
|--------|------------|------------|
| $g(c) > 0$ | $f'(c_-) > 0$ | $f'(c_+) > 0$ |

$$f'g \begin{cases} + & \text{if } f'(c_-) > 0 \text{ and } f'(c_+) > 0 \\ - & \text{if } f'(c_-) < 0 \text{ and } f'(c_+) < 0 \\ 0 & \text{otherwise} \end{cases}$$
\[
\begin{array}{ccc}
c_- & c & c_+ \\
\hline
f & - & 0 & + \\
f'g & - & - & \\
\end{array}
\]

\( g(c) > 0, f'(c_-) < 0, f'(c_+) > 0 \)

\[
\begin{array}{ccc}
c_- & c & c_+ \\
\hline
f & + & 0 & + \\
f'g & - & - & \\
\end{array}
\]

\( g(c) < 0, f'(c_-) < 0, f'(c_+) > 0 \)

\[
\begin{array}{ccc}
c_- & c & c_+ \\
\hline
f & + & 0 & + \\
f'g & + & - & \\
\end{array}
\]

\( g(c) > 0, f'(c_-) > 0, f'(c_+) < 0 \)

\[
\begin{array}{ccc}
c_- & c & c_+ \\
\hline
f & - & 0 & - \\
f'g & + & - & \\
\end{array}
\]

\( g(c) < 0, f'(c_-) > 0, f'(c_+) < 0 \)

\[
\begin{array}{ccc}
c_- & c & c_+ \\
\hline
f & - & 0 & - \\
f'g & - & + & \\
\end{array}
\]

\( g(c) > 0, f'(c_-) < 0, f'(c_+) < 0 \)

\[
\begin{array}{ccc}
c_- & c & c_+ \\
\hline
f & + & 0 & - \\
f'g & - & - & \\
\end{array}
\]

\( g(c) < 0, f'(c_-) < 0, f'(c_+) < 0 \)
In every as $x$ passes through $c$, the number of sign changes in $(f_0(x), f_1(x))$ decreases by 1 if $g(c) > 0$, and increases by 1 if $g(c) < 0$. If $c$ is a root of $g_i$ with $i = 1, \ldots, k$, then it is neither a root of $g_{i-1}$ nor a root of $g_{i+1}$, and $g_{i-1}(c)g_{i+1}(c) < 0$, by the definition of the sequence. Passing through $c$ does not lead to any modification of the number of sign changes in $(f_{i-1}(x), f_i(x), f_{i+1}(x))$ in this case.

Proof of Sturm’s theorem: Using $g = 1$ in previous theorem.

9 APPENDIX - III: EXPLICIT EXPRESSIONS FOR THE CO-EFFICIENTS FOR THE STURM CHAIN CONSTRUCTED FOR THE RELATIVISTIC AXISYMMETRIC ACCRETION

\[
\begin{array}{ccc}
  c_- & c & c_+ \\
  f & + & 0 & - \\
  f'g & + & + & \\
\end{array}
\]

\[B_{13} = -\left\{ \frac{2A_{12}}{14} - \frac{13A_{13}^2}{196A_{14}} \right\} \]
\[B_{12} = -\left\{ \frac{3A_{11}}{14} - \frac{12A_{12}A_{13}}{196A_{14}} \right\} \]
\[B_{11} = -\left\{ \frac{4A_{10}}{14} - \frac{11A_{13}A_{11}}{196A_{14}} \right\} \]
\[B_{10} = -\left\{ \frac{5A_9}{14} - \frac{10A_{11}A_{10}}{196A_{14}} \right\} \]
\[B_9 = -\left\{ \frac{6A_8}{14} - \frac{9A_{13}A_9}{196A_{14}} \right\} \]
\[B_8 = -\left\{ \frac{7A_7}{14} - \frac{8A_{13}A_8}{196A_{14}} \right\} \]
\[B_7 = -\left\{ \frac{8A_6}{14} - \frac{7A_{13}A_7}{196A_{14}} \right\} \]
\[B_6 = -\left\{ \frac{9A_5}{14} - \frac{6A_{13}A_6}{196A_{14}} \right\} \]
\[B_5 = -\left\{ \frac{10A_4}{14} - \frac{5A_{13}A_5}{196A_{14}} \right\} \]
\[B_4 = -\left\{ \frac{11A_3}{14} - \frac{4A_{13}A_4}{196A_{14}} \right\} \]
\[B_3 = -\left\{ \frac{12A_2}{14} - \frac{3A_{13}A_3}{196A_{14}} \right\} \]
\[B_2 = -\left\{ \frac{13A_1}{14} - \frac{2A_{13}A_2}{196A_{14}} \right\} \]
\[B_1 = -\left\{ A_0 - \frac{A_{13}}{196A_{14}} \right\} \]

\[C_{12} = -\{12A_{12} - \frac{14A_{14}B_{11}}{B_{13}} - (13A_{13} - \frac{14A_{14}B_{12}}{B_{13}})B_{12}\} \]
\[C_{11} = -\{11A_{11} - \frac{14A_{14}B_{10}}{B_{13}} - (13A_{13} - \frac{14A_{14}B_{12}}{B_{13}})B_{11}\} \]
\[C_{10} = -\{10A_{10} - \frac{14A_{14}B_9}{B_{13}} - (13A_{13} - \frac{14A_{14}B_{12}}{B_{13}})B_{10}\} \]
\[C_9 = -\{9A_9 - \frac{14A_{14}B_8}{B_{13}} - (13A_{13} - \frac{14A_{14}B_{12}}{B_{13}})B_9\} \]
\[C_8 = -\{8A_8 - \frac{14A_{14}B_7}{B_{13}} - (13A_{13} - \frac{14A_{14}B_{12}}{B_{13}})B_8\} \]
\[C_7 = -\{7A_7 - \frac{14A_{14}B_6}{B_{13}} - (13A_{13} - \frac{14A_{14}B_{12}}{B_{13}})B_7\} \]
\[C_6 = -\{6A_6 - \frac{14A_{14}B_5}{B_{13}} - (13A_{13} - \frac{14A_{14}B_{12}}{B_{13}})B_6\} \]
\[C_5 = -\{5A_5 - \frac{14A_{14}B_4}{B_{13}} - (13A_{13} - \frac{14A_{14}B_{12}}{B_{13}})B_5\} \]

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\[
\begin{align*}
C_4 &= -\{4A_4 - \frac{14A_{14}B_3}{B_{13}} - (13A_{13} - \frac{14A_{14}B_{12}}{B_{13}} \frac{B_3}{B_{13}})\} \\
C_3 &= -\{3A_3 - \frac{14A_{14}B_2}{B_{13}} - (13A_{13} - \frac{14A_{14}B_{12}}{B_{13}} \frac{B_2}{B_{13}})\} \\
C_2 &= -\{2A_2 - \frac{14A_{14}B_1}{B_{13}} - (13A_{13} - \frac{14A_{14}B_{12}}{B_{13}} \frac{B_1}{B_{13}})\} \\
C_1 &= -\{A_1 - (13A_{13} - \frac{14A_{14}B_{12}}{B_{13}} \frac{B_1}{B_{13}})\} \\
D_{11} &= -\{B_{11} - \frac{B_{13}C_{10}}{C_{12}} - (B_{12} - \frac{B_{13}C_{11}}{C_{12}} \frac{C_{11}}{C_{12}})\} \\
D_{10} &= -\{B_{10} - \frac{B_{13}C_{9}}{C_{12}} - (B_{12} - \frac{B_{13}C_{11}}{C_{12}} \frac{C_{10}}{C_{12}})\} \\
D_9 &= -\{B_9 - \frac{B_{13}C_{8}}{C_{12}} - (B_{12} - \frac{B_{13}C_{11}}{C_{12}} \frac{C_9}{C_{12}})\} \\
D_8 &= -\{B_8 - \frac{B_{13}C_{7}}{C_{12}} - (B_{12} - \frac{B_{13}C_{11}}{C_{12}} \frac{C_8}{C_{12}})\} \\
D_7 &= -\{B_7 - \frac{B_{13}C_{6}}{C_{12}} - (B_{12} - \frac{B_{13}C_{11}}{C_{12}} \frac{C_7}{C_{12}})\} \\
D_6 &= -\{B_6 - \frac{B_{13}C_{5}}{C_{12}} - (B_{12} - \frac{B_{13}C_{11}}{C_{12}} \frac{C_6}{C_{12}})\} \\
D_5 &= -\{B_5 - \frac{B_{13}C_{4}}{C_{12}} - (B_{12} - \frac{B_{13}C_{11}}{C_{12}} \frac{C_5}{C_{12}})\} \\
D_4 &= -\{B_4 - \frac{B_{13}C_{3}}{C_{12}} - (B_{12} - \frac{B_{13}C_{11}}{C_{12}} \frac{C_4}{C_{12}})\} \\
D_3 &= -\{B_3 - \frac{B_{13}C_{2}}{C_{12}} - (B_{12} - \frac{B_{13}C_{11}}{C_{12}} \frac{C_3}{C_{12}})\} \\
D_2 &= -\{B_2 - \frac{B_{13}C_{1}}{C_{12}} - (B_{12} - \frac{B_{13}C_{11}}{C_{12}} \frac{C_2}{C_{12}})\} \\
D_1 &= -\{B_1 - (B_{12} - \frac{B_{13}C_{11}}{C_{12}} \frac{C_1}{C_{12}})\} \\
E_{10} &= -\{C_{10} - \frac{C_{12}D_9}{D_{11}} - (C_{11} - \frac{C_{12}D_{10}}{D_{11}} \frac{D_{10}}{D_{11}})\} \\
E_9 &= -\{C_9 - \frac{C_{12}D_8}{D_{11}} - (C_{11} - \frac{C_{12}D_{10}}{D_{11}} \frac{D_9}{D_{11}})\} \\
E_8 &= -\{C_8 - \frac{C_{12}D_7}{D_{11}} - (C_{11} - \frac{C_{12}D_{10}}{D_{11}} \frac{D_8}{D_{11}})\} \\
E_7 &= -\{C_7 - \frac{C_{12}D_6}{D_{11}} - (C_{11} - \frac{C_{12}D_{10}}{D_{11}} \frac{D_7}{D_{11}})\} \\
E_6 &= -\{C_6 - \frac{C_{12}D_5}{D_{11}} - (C_{11} - \frac{C_{12}D_{10}}{D_{11}} \frac{D_6}{D_{11}})\} \\
E_5 &= -\{C_5 - \frac{C_{12}D_4}{D_{11}} - (C_{11} - \frac{C_{12}D_{10}}{D_{11}} \frac{D_5}{D_{11}})\} \\
E_4 &= -\{C_4 - \frac{C_{12}D_3}{D_{11}} - (C_{11} - \frac{C_{12}D_{10}}{D_{11}} \frac{D_4}{D_{11}})\} \\
E_3 &= -\{C_3 - \frac{C_{12}D_2}{D_{11}} - (C_{11} - \frac{C_{12}D_{10}}{D_{11}} \frac{D_3}{D_{11}})\} \\
E_2 &= -\{C_2 - \frac{C_{12}D_1}{D_{11}} - (C_{11} - \frac{C_{12}D_{10}}{D_{11}} \frac{D_2}{D_{11}})\} \\
E_1 &= -\{C_1 - (C_{11} - \frac{C_{12}D_{10}}{D_{11}} \frac{D_1}{D_{11}})\} \\
F_9 &= -\{D_9 - \frac{D_{11}E_8}{E_{10}} - (D_{10} - \frac{D_{11}E_9}{E_{10}} \frac{E_9}{E_{10}})\} \\
F_8 &= -\{D_8 - \frac{D_{11}E_7}{E_{10}} - (D_{10} - \frac{D_{11}E_9}{E_{10}} \frac{E_8}{E_{10}})\} \\
F_7 &= -\{D_7 - \frac{D_{11}E_6}{E_{10}} - (D_{10} - \frac{D_{11}E_9}{E_{10}} \frac{E_7}{E_{10}})\}
\end{align*}
\]
\[
F_6 = - \left\{ \frac{D_6 - D_{11}E_5}{E_{10}} - \left( \frac{D_{11}E_9}{E_{10}} \right) E_6 \right\}
\]
\[
F_5 = - \left\{ \frac{D_5 - D_{11}E_4}{E_{10}} - \left( \frac{D_{11}E_9}{E_{10}} \right) E_5 \right\}
\]
\[
F_4 = - \left\{ \frac{D_4 - D_{11}E_3}{E_{10}} - \left( \frac{D_{11}E_9}{E_{10}} \right) E_4 \right\}
\]
\[
F_3 = - \left\{ \frac{D_3 - D_{11}E_2}{E_{10}} - \left( \frac{D_{11}E_9}{E_{10}} \right) E_3 \right\}
\]
\[
F_2 = - \left\{ \frac{D_2 - D_{11}E_1}{E_{10}} - \left( \frac{D_{11}E_9}{E_{10}} \right) E_2 \right\}
\]
\[
F_1 = - \left\{ \frac{D_1 - (D_{11}E_9)}{E_{10}} \right\} E_1
\]

\[
G_8 = - \left\{ \frac{E_8 - \frac{E_{10}F_7}{F_9}}{F_9} - \left( \frac{E_9 - \frac{E_{10}F_8}{F_9}}{F_9} \right) F_8 \right\}
\]
\[
G_7 = - \left\{ \frac{E_7 - \frac{E_{10}F_6}{F_9}}{F_9} - \left( \frac{E_9 - \frac{E_{10}F_8}{F_9}}{F_9} \right) F_7 \right\}
\]
\[
G_6 = - \left\{ \frac{E_6 - \frac{E_{10}F_5}{F_9}}{F_9} - \left( \frac{E_9 - \frac{E_{10}F_8}{F_9}}{F_9} \right) F_6 \right\}
\]
\[
G_5 = - \left\{ \frac{E_5 - \frac{E_{10}F_4}{F_9}}{F_9} - \left( \frac{E_9 - \frac{E_{10}F_8}{F_9}}{F_9} \right) F_5 \right\}
\]
\[
G_4 = - \left\{ \frac{E_4 - \frac{E_{10}F_3}{F_9}}{F_9} - \left( \frac{E_9 - \frac{E_{10}F_8}{F_9}}{F_9} \right) F_4 \right\}
\]
\[
G_3 = - \left\{ \frac{E_3 - \frac{E_{10}F_2}{F_9}}{F_9} - \left( \frac{E_9 - \frac{E_{10}F_8}{F_9}}{F_9} \right) F_3 \right\}
\]
\[
G_2 = - \left\{ \frac{E_2 - \frac{E_{10}F_1}{F_9}}{F_9} - \left( \frac{E_9 - \frac{E_{10}F_8}{F_9}}{F_9} \right) F_2 \right\}
\]
\[
G_1 = - \left\{ \frac{E_1 - \frac{E_{10}F_8}{F_9}}{F_9} \right\} F_1
\]

\[
H_7 = - \left\{ \frac{F_7 - \frac{F_8G_6}{G_8}}{G_8} - \left( \frac{F_8 - \frac{F_6G_7}{G_8}}{G_8} \right) G_7 \right\}
\]
\[
H_6 = - \left\{ \frac{F_6 - \frac{F_8G_5}{G_8}}{G_8} - \left( \frac{F_8 - \frac{F_6G_7}{G_8}}{G_8} \right) G_6 \right\}
\]
\[
H_5 = - \left\{ \frac{F_5 - \frac{F_8G_4}{G_8}}{G_8} - \left( \frac{F_8 - \frac{F_6G_7}{G_8}}{G_8} \right) G_5 \right\}
\]
\[
H_4 = - \left\{ \frac{F_4 - \frac{F_8G_3}{G_8}}{G_8} - \left( \frac{F_8 - \frac{F_6G_7}{G_8}}{G_8} \right) G_4 \right\}
\]
\[
H_3 = - \left\{ \frac{F_3 - \frac{F_8G_2}{G_8}}{G_8} - \left( \frac{F_8 - \frac{F_6G_7}{G_8}}{G_8} \right) G_3 \right\}
\]
\[
H_2 = - \left\{ \frac{F_2 - \frac{F_8G_1}{G_8}}{G_8} - \left( \frac{F_8 - \frac{F_6G_7}{G_8}}{G_8} \right) G_2 \right\}
\]
\[
H_1 = - \left\{ \frac{F_1 - \frac{F_8G_0}{G_8}}{G_8} \right\} G_1
\]

\[
I_6 = - \left\{ \frac{G_6 - \frac{G_8H_5}{H_7}}{H_7} - \left( \frac{G_7 - \frac{G_8H_6}{H_7}}{H_7} \right) H_6 \right\}
\]
\[
I_5 = - \left\{ \frac{G_5 - \frac{G_8H_4}{H_7}}{H_7} - \left( \frac{G_7 - \frac{G_8H_6}{H_7}}{H_7} \right) H_5 \right\}
\]
\[
I_4 = - \left\{ \frac{G_4 - \frac{G_8H_3}{H_7}}{H_7} - \left( \frac{G_7 - \frac{G_8H_6}{H_7}}{H_7} \right) H_4 \right\}
\]
\[
I_3 = - \left\{ \frac{G_3 - \frac{G_8H_2}{H_7}}{H_7} - \left( \frac{G_7 - \frac{G_8H_6}{H_7}}{H_7} \right) H_3 \right\}
\]
\[
I_2 = - \left\{ \frac{G_2 - \frac{G_8H_1}{H_7}}{H_7} - \left( \frac{G_7 - \frac{G_8H_6}{H_7}}{H_7} \right) H_2 \right\}
\]
\[
I_1 = - \left\{ \frac{G_1 - \frac{G_8H_0}{H_7}}{H_7} \right\} H_1
\]
\[ J_5 = -\{ H_5 - \frac{H_7 J_4}{I_6} - (H_6 - \frac{H_7 L_2}{I_6}) \frac{I_5}{I_6} \} \]

\[ J_4 = -\{ H_4 - \frac{H_7 J_3}{I_6} - (H_6 - \frac{H_7 L_2}{I_6}) \frac{I_4}{I_6} \} \]

\[ J_3 = -\{ H_3 - \frac{H_7 J_2}{I_6} - (H_6 - \frac{H_7 L_2}{I_6}) \frac{I_3}{I_6} \} \]

\[ J_2 = -\{ H_2 - \frac{H_7 J_1}{I_6} - (H_6 - \frac{H_7 L_2}{I_6}) \frac{I_2}{I_6} \} \]

\[ J_1 = -\{ H_1 - (H_6 - \frac{H_7 L_2}{I_6}) \frac{I_1}{I_6} \} \]

\[ K_4 = -\{ I_4 - \frac{I_6 J_3}{J_5} - (I_5 - \frac{I_6 J_4}{J_5}) \frac{J_4}{J_5} \} \]

\[ K_3 = -\{ I_3 - \frac{I_6 J_2}{J_5} - (I_5 - \frac{I_6 J_4}{J_5}) \frac{J_3}{J_5} \} \]

\[ K_2 = -\{ I_2 - \frac{I_6 J_1}{J_5} - (I_5 - \frac{I_6 J_4}{J_5}) \frac{J_2}{J_5} \} \]

\[ K_1 = -\{ I_1 - (I_5 - \frac{I_6 J_4}{J_5}) \frac{J_1}{J_5} \} \]

\[ L_3 = -\{ J_3 - \frac{J_5 K_2}{K_4} - (J_4 - \frac{J_5 K_3}{K_4}) \frac{K_3}{K_4} \} \]

\[ L_2 = -\{ J_2 - \frac{J_5 K_1}{K_4} - (J_4 - \frac{J_5 K_3}{K_4}) \frac{K_2}{K_4} \} \]

\[ L_1 = -\{ J_1 - (J_4 - \frac{J_5 K_3}{K_4}) \frac{K_1}{K_4} \} \]

\[ M_2 = -\{ K - \frac{K_4 L_1}{L_3} - (K_3 - \frac{K_4 L_2}{L_3}) \frac{L_2}{L_3} \} \]

\[ M_1 = -\{ K_1 - (K_3 - \frac{K_4 L_2}{L_3}) \frac{L_1}{L_3} \} \]

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