A HIGHER INDEX THEOREM FOR FOLIATED MANIFOLDS WITH BOUNDARY

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Abstract. Following Gorokhovsky and Lott and using an extension of the b-pseudodifferential calculus of Melrose, we give a formula for the Chern character of the Dirac index class of a longitudinal Dirac type operators on a foliated manifold with boundary. For this purpose we use the Bismut local index formula in the context of noncommutative geometry. This paper uses heavily the methods and technical results developed by E.Leichtnam and P.Piazza.

1. Introduction

Since its formulation, the Atiyah-Singer index formula [3, 4] has been, and still is, the subject of a large amount of significant and interesting researches. The subject of this paper is closely related to the Family Index Theorem which is stated and proved by Atiyah and Singer [5]. Consider a fibration of even dimensional spin manifolds $F \to M \to B$. Let $D$ be a family of Dirac type operators acting on smooth sections of fiberwise spin bundle twisted by a hermitian vector bundle $E \to B$. This family gives rise to a continuous map from $B$ into the space of Fredholm operators. This last space is the classifying space for topological $K$-theory. So $D$ determines a class $\text{ind}(D)$ in $K_0(B)$ and its Chern character $\text{Ch} (\text{ind}(D))$ determines an element in $H^{\text{dr}}_*(B)$. Using a connection $\nabla^{T(M|F)}$ for the vertical tangent bundle $M$ and a Clifford hermitian connection $\nabla^E$, one can define characteristic forms $\hat{A}(TF, \nabla^{T(M|F)})$ and $\text{Ch}(E, \nabla^E)$ which represent elements in $H^{\text{dr}}_*(M)$. The family index formula states the following equality in $H^{\text{dr}}_*(B)$

\begin{equation}
\text{Ch} (\text{ind}(D)) = \frac{1}{(2\pi i)^{p/2}} \int_{M|F} \hat{A}(TF, \nabla^{T(M|F)}) \wedge \text{Ch}(E, \nabla^E).
\end{equation}

Here $p = \dim F$ and the integration is performed along the fibers and produces a closed differential form which represents an element of $H^{\text{dr}}_*(B)$. If $B$ is a single point this reduces to the Atiyah-Singer formula which has a local proof based on the McKean-Singer formula [25] and the short time asymptotic of the trace density of the heat operator. So it is natural to ask whether there is a local proof for the family index theorem. J.M. Bismut proved this family local index theorem by generalizing the Daniel Quillen superconnection formalism to infinite dimension [7, 30]. The index formula for a family of spin manifolds with boundary was established by Bismut and Cheeger in [8] and generalized significantly by R. Melrose and P. Piazza in [27] using the $b$-pseudodifferential calculus of Melrose. The index formula in this case, with the Atiyah-Patodi-Singer boundary condition, is the following equality in $H^{\text{dr}}_*(B)$

\begin{equation}
\text{Ch} (\text{ind}(D)) = \frac{1}{(2\pi i)^{p/2}} \int \hat{A}(TF, \nabla^{T(M|F)}) \wedge \text{Ch}(E, \nabla^E) - \frac{1}{2} \eta(D_0).
\end{equation}

Here $D_0$ is the boundary part of the Dirac family $D$, which is assumed to be invertible, and the differential form $\eta(D_0)$ is a spectral invariant of the boundary family $D_0$. The $b$-calculus approach is

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particularly suitable to handle the Atiyah-Patodi-Singer boundary conditions.
The family index theorem was partially generalized by Connes \cite{Connes1985}, to state a numerical index theorem for a longitudinal foliation family of Dirac type operators. The foliation is assumed to carry a holonomy invariant measure. The underlying space for this theorem is the foliation groupoid $G$ associated to the foliated manifold. The generalization of the Connes’ theorem to foliated manifolds with boundary is due to Ramachandran \cite{Ramachandran1987}. In this generalization the foliation is assumed to be transverse to the boundary. In the foliation case the groupoid structure provides a convolution algebra structure on the space of compactly supported function on $G$. A suitable completion of this algebra form a $C^*$-algebra $C^*(G)$. Later, the higher family index theorem in $K_0(C^*_r(G))$ was established by Connes and Skandalis \cite{Connes1985b}. The $C^*$-algebra of a fibration family is Morita equivalence to the algebra of continuous functions on base manifold. So this last index theorem reduces to Atiyah-Singer index theorem for algebra $G$ operator where A. Gorokhovsky and Lott have proved a more general index theorem for a family of \cite{Gorokhovsky2018}. In fact A. Gorokhovsky and Lott have proved a more general index theorem for a family of $G$-invariant Dirac operator where $G$ is a smooth foliation groupoid. Our work can be easily extended to include this more general case in the presence of boundary. They assume that the foliation carries a holonomy invariant closed current. This current is used to average the higher degree transversal differential forms to get a number-valued index. The novelty in their work is to introduce a generalized Chern-Weil construction for the Chern character of the Dirac index class. Then they use the Bismut local index theorem prove their index formula. In this paper we follow Gorokhovsky and Lott \cite{Gorokhovsky2018} to state and to prove a higher index formula for foliated manifolds with boundary. Of course we assume some hypothesis on the foliation and its holonomy groupoid (see hypothesis \[1\]) and an invertibility condition for the boundary family (see hypothesis \[2\]). In the appendix we recall briefly the generalized Chern-Weil construction proposed by Gorokhovsky and Lott in \cite{Gorokhovsky2018}. In section \[2\] we fix the notation and set up the geometric structures. We investigate also some differential geometric properties of $b$-foliations which are required in sequel. In section \[3\] we deal with the algebraic and geometric structures of the holonomy groupoid and prove in proposition \[3\] that the generalized curvature of the Bismut superconnection is $G$-invariant. This is crucial to prove the Bismut local formula for $G$-invariant family of Dirac operators. In section \[4\] we state, in a $G$-invariant manner, those aspects of the Melrose’s $b$-calculus which are necessary in the forthcoming sections. This calculus is the analytic framework for our work. In particular the $G$-invariant $b$-calculus with bounds is the carrier for the good $G$-invariant parametrices. In this section we state the $G$-invariant Bismut $b$-density theorem and prove the defect formula for the $b$-trace in proposition \[8\]. In section \[5\] we define the Chern character and study the long time behavior of the Chern character in proposition \[9\]. The proof of this proposition as well as that of transgression formula for Chern character in proposition \[12\] are based on the defect formula for $b$-trace. The convergence of eta form results from some estimates on the heat kernel for $Cl(1)$-superconnection which are taken from \cite{Bismut1987}.

In this paper we assume that the leaves and the holonomy group of the foliation $(M, F)$ are of polynomial growth. Let $E$ be a longitudinal Clifford bundle on $M$ and let $D = r^*\tilde{D}$ be the $G$-invariant Dirac operator acting on the smooth sections of the $G$-invariant vector bundle $r^*E$, where $r : G \to M$ denotes the target map. Let $s : G \to M$ denote the source map and for each $x \in M$ put $G_x := s^{-1}(x)$. It turns out that $G_x$ is a manifold with boundary and the restriction of $r^*E$ to the boundary of this manifold has a natural decomposition $r^*E|_{G^*_x} = E_{0x} \oplus E_{0x}$. The restriction of the family of Dirac operators to the boundary of $G_x$’s defines a family $D_0 = (D_{0x})_x$ of Dirac operators acting on sections of $r^*E_{0x} \to G^*_x$. We assume the family $D_0$ is invertible. With this assumption, the analytical index $ind(D)$ can be defined as an element in the $K$-theory of a certain subalgebra $R_{b,G}$ of the algebra of $b$-pseudodifferential smoothing operators with vanishing indicial family. Given a horizontal connection $\nu := T h M$, we define a generalized Chern character of this index class taking its value in $C^\infty(M, \Omega^1(F) \otimes \Lambda^*\nu^*)$. Let $\rho : C^\infty(M, \Omega^1(F) \otimes \Lambda^*\nu^*) \to \mathbb{C}$ be the linear functional defined by the pairing with a holonomy invariant current. For example the integration of function with respect to a holonomy invariant transversal measure provides such a functional on horizontal differential 0-forms on $M$. Let $\mathbb{B}_s$ be the rescaled $Cl(1)$-superconnection acting on $C^\infty(G^0, r^*E_{0\beta})$. With the above hypothesis, the following integral is proved to be convergent and defines the eta
invariant of the family $D_0$ \[
\rho(\eta_0) = \frac{1}{\sqrt{s}} \int_0^\infty \rho \left( ST r_\alpha \left( \frac{dE_\alpha}{ds} e^{-(s^2-l_0)} \right) \right) \, ds.
\]
Here $B^2 - l_0$ is the $G$-invariant part of $B^2$, i.e. it is the pull back of an operator on $(\partial M, \partial F)$ by $r$. Define the operation $\{ \cdot \}_p$ by the following relation where the longitudinal bundle $\Omega^1(F)$ of $b$-density of order 1 is identified with the longitudinal bundle $\Lambda^p(\partial F)$ \[
\{ \cdot \}_p : \Lambda^*(\partial M) \to \Lambda^*(\partial M) \otimes \Lambda^p(\partial F) \cong \Lambda^*(\partial M) \otimes \Omega^1(F).
\]
Under above conditions, results of sections 4 and 5 imply together the following index theorem (c.f. theorem 16).

**Theorem 1** (b-index theorem for $G$-invariant Dirac operators). Let $G$ denote the holonomy groupoid associated to the foliated manifold $(M, F)$. Let $p = \text{dim } F$ is an even number and let $E$ be a longitudinal Clifford bundle over $(M, F)$ with associated Dirac operator $D$. If $p$ is a linear functional as explained in above then the following index formula holds \[
\rho(\text{Ch}(\text{ind}(D))) = \frac{1}{(2\pi i)^{p/2}} \int_M \rho(x) \{ \hat{A}(M, F)(x) \text{Ch}(E/S)(x) \}_{p} - \frac{1}{2} \rho(\eta_0).
\]

The integrand is the Atiyah-Singer characteristic differential form coming out from asymptotic behavior of the heat kernel of Bismut superconnection for foliation $(M, F)$. The precise definition is given by relations (1.5) and (1.7).

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**Dedication:** The author would like to dedicate this paper to Jamal Amoo Rezaii, a great mathematics teacher in Bijar-Iran.

2. Geometric setting and notation

2.1. Dirac longitudinal structures on foliated manifolds with boundary. Let $(M, F)$ be a smooth foliated compact manifold with boundary with even dimensional leaves which intersect transversally the boundary $\partial M$. So there exists a neighborhood $U$ of $\partial M$ and a diffeomorphism $\phi : U \to \partial M \times [0, 1)$ such that

\[
\phi_*(F_U) \simeq (\partial M \times [0, 1), \partial F \times [0, 1))
\]

where $\phi_*(F_U)$ denotes the push-forward of the foliation $F_U$ by $\phi$. We denote the coordinate of the interval $[0, 1)$ by $t$. We denote by $p$ the dimension of the leaves and by $q = n - p$ their co-dimension. In what follows we refer to [26] (See also [27] and [18]) for notation and basic concepts from $b$-calculus and to [20] for basic facts in spin geometry. We denote by $\gamma F$ the longitudinal b-tangent bundle of $M$, i.e. the vector bundle on $M$ whose restriction to each leaf is the b-tangent bundle of that leaf. A longitudinal exact b-metric on $M$ is a smooth bilinear form on $\gamma F$ which falls down to an exact b-metric when it is restricted to each leaf. We suppose that it takes the following form in the neighborhood $U$

\[
g = \frac{dt^2}{t^2} + g_0
\]

where $g_0$ is a longitudinal Riemannian metric on $T\partial M$. The set of the smooth sections of the longitudinal b-density bundle equipped with the $C^\infty$-topology is denoted by $C^\infty(M, \Omega^1 \frac{1}{b})$, whereas $\hat{C}^\infty(M, \Omega^1 \frac{1}{b})$ consists of those elements in $C^\infty(M, \Omega^1 \frac{1}{b})$ which vanish to all order at $\partial M$. The dual spaces of these spaces are denoted, respectively, by $\hat{C}^\infty(M, \Omega^1 \frac{1}{b})$ and $C^\infty(M, \Omega^1 \frac{1}{b})$.

Associated to the riemannian structure $g$ on $\gamma F$ there is the longitudinal b-Clifford bundle on $M$ denoted by $\text{Cliff}(\gamma F)$ which is, due to the parity of the dimension of the leaves, a $\mathbb{Z}_2$-graded longitudinal vector bundle (in accordance with [27] page 4], our convention for the Clifford relation is $v.v=g(v,v)$). Now let $E \to M$ be a smooth bundle on $M$ which is a $\text{Cliff}(\gamma F)$ bundle, so it is $\mathbb{Z}_2$-graded and $E = E^+ \oplus E^-$ with respect to this grading. Let $E$ be equipped with a hermitian b-connection $\nabla^E$ which is compatible with the Clifford action of $\text{Cliff}(\gamma F)$. We denote by
\( \hat{D} : C^\infty(M, E) \to C^\infty(M, E) \) the longitudinal b-Dirac operator corresponding to these data. This operator is grading reversing with respect to the grading \( E = E^+ \oplus E^- \) and takes the following form

\[
\hat{D} = \begin{pmatrix} 0 & \hat{D}^- \\ \hat{D}^+ & 0 \end{pmatrix}; \quad (\hat{D}^\dagger)^* = \hat{D}^-.
\]

The longitudinal Riemannian metric \( g_0 \) defines the longitudinal Clifford bundle on the boundary. We denote this bundle by \( \text{Cliff}(T^*\partial F) \). The following application extends to a natural algebra morphism on this Clifford bundle

\[
(2.3) \quad T^*(\partial F) \xrightarrow{j} \text{Cliff}(T^*\partial F)|\partial M
\]

Identify \( E^+|\partial M \) by \( E^0 \). Then \( \sigma^- = c l(i \frac{dt}{t}) \) is an isomorphism between \( E^-|\partial M \) and \( E^0 \).

The bundle \( E = E^+ \oplus E^- \), restricted to the boundary \( \partial M \), is also a longitudinal Clifford bundle over \( (\partial M, \partial F) \). So we can consider the boundary grading reversing Dirac operator \( \hat{D}_\partial = \hat{D}_\partial^+ \oplus \hat{D}_\partial^- \). It is easy to verify the relation \( \hat{D}_\partial^- \circ (\sigma^-)^{-1} = \sigma^- \circ \hat{D}_\partial^+ \). Put \( \hat{D}_\partial = \hat{D}_\partial^- \circ (\sigma^-)^{-1} = \sigma^- \circ \hat{D}_\partial^+ \). With respect to the isomorphism \( \sigma := id \oplus \sigma^- : E|\partial M \to E_0 \oplus E_0 \), the Dirac operator \( \hat{D} \) takes the following form as an operator on \( C^\infty(\partial M \times [0, 1], E_0 \oplus E_0) \)

\[
(2.4) \quad \hat{D}_{|\partial M \times [0, 1]} = \begin{pmatrix} 0 & -t\partial_t + \hat{D}_0 \\ t\partial_t + \hat{D}_0 & 0 \end{pmatrix}.
\]

2.2. Differential geometry of foliated manifolds. As an additional structure, we suppose that there is given a horizontal distribution \( T^h M \), that is a sub bundle of \( TM \) such that \( TM \approx T^h M \oplus T^b F \).

The first and second component of a vector \( x \in TM \) with respect to this decomposition is denoted, respectively, by \( x^h \) and \( x^b \). In what follows, for simplifying the notation, we denote sometimes the horizontal bundle by \( \nu \). We suppose that \( T^h_x M \subset T_x \partial M \) for \( x \in \partial M \), so the sections of \( \nu \) are b-vector fields and the differential operators defined by them are b-differential operators. The previous decomposition of the tangent bundle \( TM \) gives rise to the following isomorphism between graded algebras

\[
(2.5) \quad \Omega^*(T^*M) \approx \Omega^*(T^h M) \otimes \Omega^*(T^b F).
\]

Let \( \omega \in \Omega^*(T^*M) \) be of degree \((k, l)\) with respect to this decomposition. The covariant derivation in the horizontal direction gives another differential form \( d^h(\omega) \) of degree \((k + 1, l)\). We can define \( d^h(\mu) \), where \( \mu \) is a density, by identifying the b-densities with powers of longitudinal differential forms of degree \( p \). In this way we get the graded derivation \( d^h : (M, \Lambda^*\nu^* \otimes \Omega^b \Sigma) \to (M, \Lambda^*\nu^* \otimes \Omega^b \Sigma) \) by putting

\[
d^h(\omega \otimes \mu) := d^h(\omega \otimes \mu + (-1)^{\deg(\omega)} \omega \wedge d^b(\mu).
\]

To the horizontal distribution \( T^h M \) is associated its curvature \( T^h \). It is a longitudinal vector-valued horizontal 2-form defined by the following formula

\[
(2.6) \quad T^h(y, z)(x) = [y^h, z^h]^\nu(x) \in T^b_x F.
\]

The Lie derivation along \( T^h \) defines an operator \( L_h : \Omega^*(M) \to \Omega^{*+2}(M) \). Using this operator one can define obviously another operator, denoted by the same symbol,

\[
L_h : C^\infty(M; \Lambda^*\nu^* \otimes \Omega^b \Sigma) \to C^\infty(M; \Lambda^{*+2}\nu^* \otimes \Omega^b \Sigma).
\]

The definition of \( L_h \) can be directly extended to sections of \( E \) with differential form coefficients.

The covariant derivation in the horizontal direction defines the operator \( \nabla^E : C^\infty(M, E) \to C^\infty(M, \nu^* \otimes E) \). The definition of this operator can be extended to \( C^\infty(M, \Lambda^*\nu^* \otimes \Omega^b \Sigma \otimes E) \) by putting

\[
\nabla^E(\omega \otimes \mu \otimes \xi) := d^h(\omega \otimes \mu \otimes \xi) + (-1)^{\deg(\omega)} \omega \otimes d^h(\mu \otimes \xi) + (-1)^{\deg(\omega)} \omega \otimes \mu \otimes \nabla^E(\xi).
\]

This means that \( \nabla^E \) is a connection.
The following relations hold between various operators defined above (for proofs see, e.g., [14] section III.7.a] or [29] (3.13))
\[
(d^h)^2 = -\mathcal{L}_h ; \quad (\nabla^E)^2 = -\mathcal{L}_h.
\]
Using the tensor product of the longitudinal Clifford connection on $E$ and the Levi-Civita connection on differential forms, one can construct the longitudinal Dirac operator acting on distributional sections of $E$. From now on we suppose that the degree of a longitudinal density is zero, so it commutes with the Clifford action of the longitudinal (co)tangent bundle. We denote this operator again by $\hat{D}$.

Now we are ready to define the Bismut superconnection $\tilde{\mathcal{A}}$ by
\[
\tilde{\mathcal{A}} : C^\infty(M, \Lambda^*\nu^* \otimes \Omega^{\frac{1}{2}} \otimes E) \to C^\infty(M, \Lambda^*\nu^* \otimes \Omega^{\frac{1}{2}} \otimes E)
\]
\[
\tilde{\mathcal{A}} = \hat{D} + \nabla^E - \frac{1}{4i} \text{cl}(T^h).
\]

**Remark 1.** Let $\mu'$ denote a longitudinal $p$-form. We recall from [6] lemma 10.4] that $d^h(\mu') = \kappa \wedge \mu' + i_{T^h}(\mu')$. Here $\kappa$ denotes the mean curvature associated to the degenerated metric $g = 0 \oplus g$ on $b^TM$. In particular $d^h(\mu') = \kappa \wedge \mu'$. So we get the following relation where $\mu$ is a longitudinal half $b$-density
\[
d^h(\mu) = \frac{1}{2} \kappa \wedge \mu.
\]
This relation shows that the mean curvature term in the definition of the Bismut superconnection in [6] lemma 10.4] exists implicitly in (2.8) in the action of $\nabla^E$ on half densities.

### 3. Holonomy Groupoid

**3.1. Geometric structures of the holonomy groupoid.** In this section we establish some differential geometric constructions which will be used in the forthcoming sections. The basic reference for the following matters are [13] and [29]. Let $G$ be the holonomy groupoid associated to the foliation $(M, F)$ and denote the source and the range maps respectively by $s$ and $r$. The groupoid $G$ is a manifold with corner and a typical trivializing chart for it is $(U_x, [\gamma], U_y)$. Here $x$ and $y$ are two points in $M$ which belong to the same leaf of the foliation $(M, F)$ while $U_x$ and $U_y$ are trivializing charts of $M$ around $x$ and $y$ respectively. $[\gamma]$ is the holonomy class of a path $\gamma$ between $x$ and $y$ where $[\gamma] = [\gamma']$ if the holonomy of $\gamma$ and $\gamma'$ are equal. It is clear from this discussion that $G_x := s^{-1}(x)$ and $G^y := r^{-1}(y)$ are smooth manifolds with boundary and the boundary of $G_x$ is $G^\partial_x$ where
\[
G^\partial_x := \{ u \in G | s(u) = x \text{ and } r(u) \in \partial M \}.
\]
The definition of $G^\partial_y$ is similar and we have $\partial G^\partial = G^\partial_G$. In this paper we use the standard notation of the theory of groupoid used, e.g. in [29]. For example $G^\partial$ consists of those elements $u \in G$ with $r(u) \in \partial M$. Clearly $G^\partial$ is the union of $G^\partial_x$ for $x \in M$. The similar comments apply to $G_\partial$. The boundary $\partial M$ can be used to define the sub-groupoid $G^\partial_\partial$ of $G$ by
\[
G^\partial_\partial := \{ u \in G | s(u) \in \partial M \text{ and } r(u) \in \partial M \}.
\]
The source and range maps of this groupoid are the restriction of the source and range maps of the groupoid $G$ and take their values in $\partial M$. The units of the groupoid $G^\partial_\partial$ are naturally identified with the points of $\partial M$.

Take $u \in G$ with $s(u) = x$ and $r(u) = y$. Let $V$ and $V'$ be local charts for the same leaf around, respectively, $x$ and $y$. Let $T, T'$ be sufficiently small local transversal passing through $x$ and $y$. We denote by $\text{Hol}_u : T \to T'$ and $u_\ast : T_x \to T_y$ respectively, the local holonomy diffeomorphism induced by $u$ and its derivative. The sets of the form $U = V \times V' \times T \times_{\text{Hol}_u} T'$ define a basis for the topology of $G$. With this topology, $G$ has the structure of a smooth manifold with corners which is foliated by manifolds with boundary $\{ G_x \}_{x \in M}$. Given a connection $T^HM$ for the foliation $(M, F)$, there is a unique horizontal distribution $T^H M$ on $G$ defined by $T^H_u G = (d_x)^{-1} T^h_x M \cap (d_y)^{-1} T^h_y M$. The following relation holds
\[
T_u G = T_u G_x \oplus T_u G^\partial \oplus T^H_u G.
\]
Here the first summand is, the so-called, $G$-invariant part of $TG$. It is given by $r^*T_yF$ when $r$ is restricted to $G_x$. The direct sum $T_uG_x \oplus T_uG^h$ is the vertical while $T_u^hG$ is the horizontal part of $T_uG$. The application $dr \oplus ds$ gives an isomorphism between the vertical part of $T_uG$ and $T_uF \oplus T_xF$.

Concerning the horizontal part, we have the following isomorphisms given again by $ds$ and $dr$

$$T^h_xM \xrightarrow{ds} T^h_uG \xrightarrow{dr} T^h_yM.$$  

Here the map from the left end to the right end is given by the holonomy $u_*$. Therefore a horizontal vector $y$ in $T_uG$ is determined uniquely by two vector $y_x \in T^h_xM$ and $y_y \in T^h_yM$ such that $u_*(y_x) = y_y$. On the other hand a vertical vector $x$ in $T_uG$ is determined uniquely by two vector $x_x \in T_xF$ and $x_y \in T_yF$. A smooth function $f$ on $G$ get the form $f_V \times f_V \times f_T \times f_T$ in the local chart $U$. The holonomy invariance condition reads $f_T \circ Hol_u = f_T$. The following formulas hold and are usually more suitable to perform local computations

$$x.f = x_x.f_V + x_y.f_T; \quad y.f = y_y.f_T = y_y.f_T.$$  

Now let $y$ and $z$ be two sections of the distribution $T^H G$ and let $T^H(y, z)$ denote the vertical component of the vector field $[y, z]$. $T^H$ is in fact a tensor and is called the curvature of the connection $T^H G$. From the above discussion and using the relation (2.6) it is clear that

$$T^H(y, z) = r^*T^h(y, z) \oplus s^*T^h(y, z).$$  

So the $G$-invariant part of $T^H$ is given by $r^*T^h$.

The manifold with corners $G$ is foliated by manifolds $\{G_x\}_{x \in M}$, this is the foliation that we are interested in. We denote this foliation by $(G, G_x)$. In the sequel we speak about the longitudinal structures on $G$ with respect to this foliation. Since the target map $r|_{G_x}$ is a covering map on its range, we can lift all longitudinal structures on the foliation $(M, F)$ to the holonomy foliation $(G, G_x)$. Such structures are $G$-invariant with respect to the left action of the groupoid $G$ on itself. We denote by $D$ the lifting of the Dirac operator $D$. For $0 \leq k \leq q$ put

$$\Omega^k \mathcal{B} := C_c^{\infty}(G, \Lambda^k r^* r^* \otimes s^*(\Omega^1)) \otimes \Omega^k \mathcal{B}$$

$$\Omega^k \mathcal{E} := C_c^{\infty}(G, \Lambda^k s^* r^* \otimes r^*(\Omega^2 \otimes E)) \otimes \Omega^k \mathcal{E}.$$  

Now we introduce some differential operators on these spaces. As in the definition of $d^h$, the covariant derivative along the horizontal direction $T^H G$ defines the operator $d^H$ on $\Omega^* \mathcal{B}$ and the operator $\nabla^E$ on $\Omega^* \mathcal{E}$. Let $r^*(\omega) \otimes s^*(\mu)$ be an element in $\Omega^* \mathcal{B}$, and let $y$ be a horizontal vector field. With the notation of the relations (3.3) the following formula holds

$$d^H(y)(r^*(\omega) \otimes s^*(\mu)) = r^*(d^h y_y)(\omega) \otimes s^*(\mu) + \langle -1 \rangle^{\deg} \omega r^*(\omega) \otimes s^*(d^h y_x)(\mu).$$  

Take an element in $\Omega^* \mathcal{E}$ of the form $s^*(\omega) \otimes r^*(\mu \otimes \xi)$, a similar local formula holds for the action of $\nabla^E$

$$\nabla^E(y)(s^*(\omega) \otimes r^*(\mu \otimes \xi)) = s^*(d^h y_x)(\omega) \otimes r^*(\mu \otimes \xi)$$

$$+ \langle -1 \rangle^{\deg} \omega r^*(\omega) \otimes r^*(d^h y_y)(\mu) \otimes r^*(\xi)$$

$$+ \langle -1 \rangle^{\deg} \omega s^*(\omega) \otimes r^*(\mu) \otimes r^* \nabla^E(y_x)(\xi).$$  

As in the previous section, the Lie derivative along the curvature vector $T^H (\text{with differential form coefficients})$ defines two operators on $\Omega^* \mathcal{B}$ and $\Omega^* \mathcal{E}$ that we denote by the same symbol $\mathcal{L}_H$. The first formula of (3.3) with (3.4) give the following local formula

$$\mathcal{L}_H(y, z)(r^*(\omega) \otimes s^*(\mu)) = r^* \mathcal{L}_h (y_y, z_y) (\omega) \otimes s^*(\mu) + r^*(\omega) \otimes s^* \mathcal{L}_h (y_x, z_x) (\mu)$$  

There is a similar local formula for the action of $\mathcal{L}_H$ on $\Omega^* \mathcal{E}$ . Finally we define the operator $l : \Omega^k \mathcal{E} \rightarrow \Omega^{k+2} \mathcal{E}$ by

$$l(y, z)(s^*(\omega) \otimes r^*(\mu \otimes \xi)) = s^*(\omega) \otimes r^* \mathcal{L}_h (y_x, z_x)(\mu \otimes \xi)$$  

It is clear that $\mathcal{L}_H - l = r^* (\mathcal{L}_h)$, so $\mathcal{L}_H - l$ is a longitudinal $G$-invariant differential operator.

**Lemma 2.** The following formulas hold

1. $\mathcal{L}_H - l = r^* (\mathcal{L}_h)$. 

\[ (2) \quad (d^H)^2 = -\frac{1}{2} \mathcal{L}_H \quad ; \quad (\nabla^E)^2 = -\mathcal{L}_H. \]
\[ (3) \quad \nabla^E \circ l = l \circ \nabla^E \quad ; \quad D \circ l = l \circ D. \]
In particular \((d^H)^2 \neq 0\) if the connection \(T^hM\) is not integrable.

\textbf{Proof.} The first part is already proved in the above discussion. The second part follows from the first part and relations (2.7). Comparing with the relations (2.7), the extra factor \(\frac{1}{2}\) is coming from the action on half densities. To prove the third part, notice that \(D = r^* \tilde{D}\) so \(D(s^*(\omega) \otimes r^*(\mu \otimes \xi)) = (-1)^{\deg(\omega)} s^*(\omega) \otimes r^* \tilde{D}(\mu \otimes \xi)\). The commutativity relation \(D \circ l = l \circ D\) is clear from this relation and local expression (3.7). Using the relations (3.6) and (3.7), the third part reduces to relations \(d^h \circ \mathcal{L}_h = \mathcal{L}_h \circ d^h\) and \(\nabla^E \circ \mathcal{L}_h = \mathcal{L}_h \circ \nabla^E\) which follows from (2.7). \(\square\)

For the future use we put a Cliff\((TG)\)-module structure on \(\Omega^* \mathcal{E}\). At first we equip the tangent bundle \(TG\) with a degenerate Riemannian structure as follows. From (3.11) at each point \(u \in G\) we have \(T_uG = T_uG_x \oplus T_uG_y \oplus T_uH\). The first summand has the \(G\)-invariant Riemannian structure \((\gamma|_{G_x})^*g\). Let the two other summands have the null metric and equip the total space by the direct sum Riemannian structure. Now take a section of \(TG\) which is of the form \(r^*(\mathbf{x}) + s^*(\mathbf{x}') + \mathbf{v}\) with respect to the previous direct sum decomposition. Let \(\omega \otimes \mu \otimes \mathbf{x}\) be an element in \(\Omega^* \mathcal{E}\) and put
\[
(3.8) \quad s^*(\mathbf{x}')(\omega \otimes \mu \otimes \xi) = 0 \quad ; \quad \mathbf{v}^*. (\omega \otimes \mu \otimes \xi) = \mathbf{v}^* \wedge \omega \otimes \mu \otimes \xi
\]
\[
(3.9) \quad \mathbf{x}. (\omega \otimes \mu \otimes \xi) = (-1)^{\deg(\omega)} \omega \otimes \mu \otimes \text{cl}(\text{Cliff}(TG))\text{x}.\xi
\]
It is clear that \(\mathbf{x}' \cdot \mathbf{x}' = \mathbf{v} \cdot \mathbf{v} = 0\), and \(\mathbf{x} \cdot \mathbf{x} = g(\mathbf{x}, \mathbf{x})\) so these relations define a Clifford-module structure on \(\Omega^* \mathcal{E}\). This definition for the Clifford structure and the relation (3.11) imply the \(G\)-invariance relation \(\text{cl}(T^H) = r^* \text{cl}(T^h)\).

Now we are ready to introduce the Bismut superconnection \(\mathbf{A} : \Omega^* \mathcal{E} \to \Omega^* \mathcal{E}\) by the following formula
\[
\mathbf{A} = D + \nabla^E - \frac{1}{4i} \text{cl}(T^H).
\]
The following property of the Bismut superconnection is crucial for our purpose.

\textbf{Proposition 3.} The following relation holds,
\[
(3.10) \quad \mathbf{A}^2 - l = r^*(\mathbf{A}^2)
\]
where \(\mathbf{A}\) denotes the Bismut \(b\)-superconnection on the foliated manifold \((M, F)\).

\textbf{Proof.} From the definition of the superconnection we have
\[
(3.11) \quad \mathbf{A}^2 - l = D^2 + \frac{1}{16} \text{cl}(T^H)^2 + \frac{1}{4i} \{D \circ \text{cl}(T^H) + \text{cl}(T^H) \circ D\}
\]
\[
+ \{\nabla^E D + D \nabla^E\} + \frac{1}{4i} \{\nabla^E \text{cl}(T^H) + \text{cl}(T^H) \nabla^E\}
\]
\[
+ \{(\nabla^E)^2 - l\}.
\]
The following similar relation holds for the Bismut superconnection \(\mathbf{A}\)
\[
(3.12) \quad \mathbf{A}^2 = D^2 + \frac{1}{16} \text{cl}(T^h)^2 + \frac{1}{4i} \{D \circ \text{cl}(T^h) + \text{cl}(T^h) \circ D\}
\]
\[
+ \{\nabla^E D + D \nabla^E\} + \frac{1}{4i} \{\nabla^E \text{cl}(T^h) + \text{cl}(T^h) \nabla^E\}
\]
\[
+ \{(\nabla^E)^2\}.
\]
We have already shown the relation \(\text{cl}(T^H) = r^* \text{cl}(T^h)\). On the other hand, from the very definition one has \(D = r^* \tilde{D}\). Thus each term in the right hand side of (3.11) is the pull-back, by \(r^*\), of the corresponding term in (3.12). On the other hand, the relations (2.7), with the first and second part of the lemma (B) imply the relation \((\nabla^E)^2 - l = r^*(\nabla^E)^2\). Now consider the general element \(s^\omega \otimes r^*(\mu \otimes \xi)\) in \(\Omega^* \mathcal{E}\) with \(\deg(\omega) = k\). Using the local expression for \(\nabla^E\) one has
\[
\nabla^E D(s^*(\omega) \otimes r^*(\mu \otimes \xi)) = (-1)^k s^* \nabla^E(\omega) \otimes r^* \tilde{D}(\mu \otimes \xi) + s^* \omega \otimes r^* \nabla^E \tilde{D}(\mu \otimes \xi)
\]
\[
D \nabla^E(s^*(\omega) \otimes r^*(\mu \otimes \xi)) = (-1)^{k+1} s^* \nabla^E \omega \otimes r^* \tilde{D}(\mu \otimes \xi) + s^* \omega \otimes r^* \tilde{D} \nabla^E(\mu \otimes \xi).
\]
Adding these relations gives the desired $G$-invariance relation
\[ \nabla^E D + D \nabla^E = r^*(\nabla^E \tilde{D} + \tilde{D} \nabla^E). \]
A quit similar computation shows the following $G$-invariance relation and finishes the proof of the proposition
\[ \nabla^E \text{cl}(T^H) + \text{cl}(T^H) \nabla^E = r^*(\nabla^E \text{cl}(T^h) + \text{cl}(T^h) \nabla^E) \]

Essentially the same computations in the proof of the last relation prove the vanishing relation
\[ \nabla^E \text{cl}(T^h) + \text{cl}(T^h) \nabla^E = 0. \]
Now for $s > 0$ define the rescaled Bismut superconnection by $\tilde{A}_s = s (D + \nabla^E - \frac{1}{s} \text{cl}(T))$. This vanishing formula shows that $\tilde{A}_s^2 - l = s^2 D^2 + sF + J$, where $F$ and $J$ are $G$-invariant differential operators with differential form coefficients of positive degrees. In particular these operators are nilpotent. For each $x \in M$, put $G^0 := \partial G_x$ and let $G^0$ be the union of all $G^0_x$, equipped with the induced topology from $G$. We recall the isomorphisms $E_0 = E^\perp_{\partial M}$ and $E_0 \simeq E_{\partial M}$. Using these isomorphisms and the fact that the connection $T^h_M$ is tangent to the boundary at boundary points, it turns out that $\nabla^E$, $T^H$ and $l$ (c.f. \[5.7\]) define differential operators on $C^\infty_c(G^0, \Lambda^k s^* \nu^* \otimes r^*(\Omega^\perp \otimes E_0))$ and satisfy the relations of the lemma \[2\]. We denote these operators by $\nabla^E$, $T^H$ and $l_0$. The following linear spaces will be used later
\[ \Omega^k \mathcal{E}_0 = C^\infty_c(G^0, \Lambda^k s^* \nu^* \otimes r^*(\Omega^\perp \otimes (E_0 \otimes E_0))) ; \quad \Omega^* \mathcal{E}_0 = \oplus_k \Omega^k \mathcal{E}_0. \]
Here $\Omega^\perp$ refers to the longitudinal bundle of half densities of the foliation $(\partial M, \partial F)$, which is, in fact, the restriction of the longitudinal bundle $\lambda^\perp \Omega^\perp$ to $(\partial M, \partial F)$, c.f. \[26\] relation 4.47. Now we define the $\text{Cl}(1)$-superconnection $\mathbb{B}$, acting on $\Omega^* \mathcal{E}_0$, by the following formula (c.f. \[27\] page 28).
\[ \mathbb{B} = a. D_0 + \text{Id} \cdot \nabla^{E_0} - \frac{1}{4i}a. \text{cl}(T^H) \]
where $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\text{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with respect to the obvious direct sum decomposition of $\Omega^* \mathcal{E}_0$. In fact $\mathbb{B}$ is the restriction of $\Lambda$ to $\Omega^* \mathcal{E}_0$. Like $\Lambda$ in \[2.2\] the $\text{Cl}(1)$-superconnection $\mathbb{B}$ can be defined for the foliation $(\partial M, \partial F)$. In fact $\mathbb{B}$ is the restriction of the superconnection $\Lambda$ to the boundary foliation $(\partial M, \partial F)$. The invariance relation $\mathbb{B}^2 - l = r^*\mathbb{B}^2$ holds and its proof is completely similar to the proof of the proposition \[3\]. We have $\mathbb{B}_0^2 = s^2 D_0^2 + sF_0 + J_0$ where $F_0$ and $J_0$ are differential operators with differential forms coefficient of positive degree.

3.2. Algebraic structures of the holonomy groupoid. Groupoid action of $G$ on itself provides an algebra structure on $\Omega^* \mathcal{B}$ and a left $\Omega^* \mathcal{B}$-module structure on $\Omega^* \mathcal{E}$. Let $\phi$ and $\psi$ be two elements of $\Omega^* \mathcal{B}$ and put
\[ \phi \cdot \psi (u) := \int_{G^ru(u)} \phi(v) \wedge (v^*)^{-1} \psi(v^{-1}u) \]
where $v^*$ denotes the dual of $v_*$. Notice that the integration is taken with respect to the density component of the first factor. Now let $\xi \in \Omega^* \mathcal{E}$ and define
\[ \phi \cdot \xi (u) := \int_{G^ru(u)} \phi(v) (v^*)^{-1} \xi(uv). \]
Here again the integration is taken with respect to the density component of $\phi$. With respect to these operations $\Omega^* \mathcal{E}$ is a left-module on the algebra $\Omega^* \mathcal{B}$. In what follows we characterize the structure of $\text{End}_{\partial M}(\Omega^* \mathcal{E})$ and describe its algebra structure. Let $P : \mathcal{E} \to \Omega^* \mathcal{E}$ be a linear operator which is defined through its smooth kernel by
\[ P(\xi)(u) = \int_{G^ru(u)} \xi(v) v^* K(v, u); \]
\[ K(v, u) \in r^* (\wedge^* \nu^* \otimes \Omega^\perp) \otimes r^* (\Omega^\perp) \otimes \text{Hom}(E_{r(v)}, E_{r(u)}). \]
In above relation the integration is taken with respect to the product of half densities coming from \( \xi \) and from the end point of \( v \). It is easy to verify that \( B \)-linearity is equivalent to the following invariance property

\[
K(v\gamma, u\gamma) = K(v, u) \quad \forall \gamma \in G^s(u).
\]

Therefore the operator \( P \) may be defined as follows:

\[
P(\xi)(u) = \int_{G^s(u)} \xi(v) v^* K(vu^{-1});
\]

\[
K(\gamma) \in r_\gamma^*(\Lambda^*\nu^* \otimes \Omega^2) \otimes H om(E_{r(\gamma)}, E_{s(\gamma)}).
\]

The operators defined in this way are called \( G \)-invariant. We denote the set of all these operators by \( Hom_G(\mathcal{E}, \Omega^* \mathcal{E}) \). By applying \( (vu^{-1})^* \) to \( \xi(v) \) in relation (3.19), the action of \( P \) can be extended to \( \Omega^* \mathcal{E} \) and in this way \( P \) defines an element in \( End_{\Omega^* \mathcal{B}}(\Omega^* \mathcal{E}) \). In particular we get the induced algebra structure on \( Hom_G(\mathcal{E}, \Omega^* \mathcal{E}) \).

Notice that in relation (3.17) if \( \xi \) belongs to \( \Omega^* \mathcal{E}_0 \) then \( \phi \xi \) belongs to \( \Omega^* \mathcal{E}_0 \) too. So \( \Omega^* \mathcal{E}_0 \) is a right module on \( \Omega^* \mathcal{B} \). Let \( Q \) be a linear operator on \( \Omega^* \mathcal{E}_0 \) defined through its smooth kernel

\[
Q(\xi)(u) = \int_{G^q(u)} \xi(v) v^* K(v, u);
\]

\[
K(v, u) \in r_\gamma^*(\Lambda^*\nu^* \otimes \Omega^2) \otimes H om(E_{r(\gamma)}, E_{s(\gamma)}).
\]

As in above, if the operator \( Q \) is \( \Omega^* \mathcal{B} \)-linear, i.e. \( Q \in End_{\Omega^* \mathcal{B}}(\Omega^* \mathcal{E}_0) \), then its kernel takes the form \( K(v, u) := K_Q(vu^{-1}) \). Since in this case \( v \) and \( u \) are in \( G^s(u) \), we have \( vu^{-1} \in G^0_q \). So \( Q \) may be defined through a kernel which is defined on \( G_q^0 \)

\[
Q(\xi)(u) = \int_{G^q(u)} \xi(v) v^* K_Q(vu^{-1});
\]

\[
K_Q(\gamma) \in r_\gamma^*(\Lambda^*\nu^* \otimes \Omega^2) \otimes H om(E_{r(\gamma)}, E_{s(\gamma)}); \gamma \in G_q^0.
\]

Following [10 relation 58], we shall define a supertrace for operators \( Q \) in \( End_{\Omega^* \mathcal{B}}(\Omega^* \mathcal{E}_0) \) with smooth kernels, i.e. the \( G \)-invariant smoothing operator on \( \Omega^* \mathcal{E}_0 \). Let \( \tau \) be an element in \( C^\infty_G(\mathcal{G}, s^*(\Omega^1)) \) satisfying the following relation (c.f. [32 proposition 6.11] for the proof of the existence of \( \tau \)):

\[
\int_{M^y} \tau(v) = 1; \forall y \in M.
\]

Taking into the account the symmetries coming from the action of the groupoid \( G \) on \( G^q \), following [10 relation 58], we define \( STr(Q) \in C^\infty(M, \Lambda^{1} \otimes \Lambda^*\nu^*) \) by

\[
STr(Q)(x) = \int_{G_q^q} \tau(v) \text{str}(v^* K_Q(vu^{-1})) .
\]

Here \( \text{str} \) is calculated with respect to the grading \( r^*E_{\partial M} = r^*E_0 \oplus r^*E_0 \) and the integration is taken with respect to the product of two half densities coming from the trace of the Schwartz kernel, c.f. relation (3.22). Notice that in above relation \( x \) is a point in \( M \), so even though \( Q \) is a boundary operator, \( STr(Q) \) belongs to \( C^\infty(M, \Lambda^{1} \otimes \Lambda^*\nu^*) \). The following proposition is the counterpart of the proposition 2 of [10] in our situation.

**Proposition 4.**

a) Let \( \rho \) be a distributional linear functional on \( C^\infty(M, \Lambda^k \nu) \) such that the linear functional \( \eta \) defined on \( \Omega^* \mathcal{B} \) by

\[
\eta(\phi) = \rho(\phi_M)
\]

be a graded supertrace on \( \Omega^* \mathcal{B} \). Then \( \rho \circ STr \) is independent of \( \tau \) and defines a supertrace on \( End_{\Omega^* \mathcal{B}}(\Omega^* \mathcal{E}_0) \), i.e. vanishes on supercommutators.

b) If in addition \( \eta \) is a closed graded trace on \( \Omega^k \mathcal{B} \), i.e. \( \eta \circ dH = 0 \), then for all \( Q \in End_{\Omega^* \mathcal{B}}(\Omega^* \mathcal{E}_0) \)

\[
\rho \circ STr(\nabla Q, Q) = 0 .
\]
Proof. It suffices to prove this proposition for $Q \in \text{End}_G(\mathcal{E}_0)$, the general case follows easily from this special case. We show at first the independence from $\tau$. Let $\chi(G_0)$ denote the characteristic function of $G_0$ which can be approximated by continuous functions. From (3.16) we have

$$\chi(G_0).\phi(x) = \int_{G^*} \chi(G_0)(v) \wedge \phi(v^{-1}) = \int_{G^*_0} \phi(v^{-1}).$$

On the other hand it is easy to verify that $\phi.\chi(G_0)(x) = 0$ if $x \notin \partial M$ while

$$\phi.\chi(G_0)(x) = \int_{G^*} \phi(v) \chi(G_0)(v^{-1}) = \int_{G^*} \phi(v); \text{ for } x \in \partial M.$$ 

The trace property of $\eta$ reads $\eta(\chi(G_0).\phi) = \eta(\phi.\chi(G_0))$ for each $\phi \in \mathcal{B}$, so the above relations imply the following one

$$(3.25) \quad \int_M \rho(x) \int_{G^*_2} \phi(v) = \int_{\partial M} \rho(y) \int_{G^*_y} \phi(v).$$

Here we have used the fact that $\rho$ is distributional and can be approximated by distributions defined by smooth functions $\rho(x)$ (e.g. see [23] theorem 6.3]). If we replace $\chi(G_0)$ by $1 - \chi(t \geq \epsilon)$ we obtain

$$(3.26) \quad \int_M \rho(x) \int_{G^*_2} \phi(v) = \int_{M^t \geq \epsilon} \rho(y) \int_{G^*_y} \phi(v).$$

Here $t \geq 0$ is a $G$-invariant defining function for the boundary of leaves $G_j$'s. Now let $Q$ be a smoothing $G$-invariant operator with kernel $K_Q$. Using (3.25) we obtain the following equalities where $O_y^y$ denotes the unite elements of the group $G^y$ for each $y \in M$

$$\rho \circ \text{Str} Q = \int_M \int_{G^*_2} \rho(x) \tau(v) \text{str} K_Q(vv^{-1})$$

$$= \int_{\partial M} \rho(y) \int_{G^*_y} \tau(v) \text{str} K_Q(vv^{-1})$$

$$= \int_{\partial M} \rho(y) \text{str} K_Q(O_y^y) \int_{G^*_y} \tau(v)$$

$$= \int_{\partial M} \rho(y) \text{str} K_Q(O_y^y).$$

The last expression does not depend on $\tau$ therefore $\rho \circ \text{Str} Q$ is independent of $\tau$. Moreover this formula can be used to show that $\rho \circ \text{Str} r$ is a supertrace. For this purpose it suffices to show that $\rho \circ \text{Str} r(Q, Q') = 0$ when $Q$, $Q'$ belong to $\text{End}_G(\mathcal{E}_0)$ with $\text{dim}_C \mathcal{E}_0 = 1$. In this situation $K_Q$ and $K_{Q'}$ are compactly supported smooth functions on $G^y$ and we consider them as $G$-invariant compactly supported functions on $G$, i.e. elements in $\mathcal{B}$. By (3.16) and (3.21) we have

$$K_{QQ'}(O_y^y) = \int_{G^*_y} K_{QQ'}(\gamma) K_Q(\gamma^{-1}) = (K_{QQ'}.K_Q)(O_y^y)$$

where $K_{QQ'}.K_Q$ denotes the multiplication of two elements of the algebra $\mathcal{B}$. Using (3.27) and above relation we obtain

$$\rho \circ \text{Str} r(Q, Q') = \int_{\partial M} \rho(y)(K_{QQ'}.K_Q - K_Q.K_{QQ'})(O_y^y)$$

$$= \eta((K_{QQ'}.K_Q - K_Q.K_{QQ'});M)$$

But the last expression vanishes by the assumption on $\eta$ and this completes the proof of the first part of the proposition.

Part (b) of this proposition follows from the following relation that we are going to prove.

$$(3.28) \quad \rho \circ \text{Str} r(\nabla \mathcal{E}_0, Q) = \eta(d^H \circ r^*(\text{Str} r Q)).$$

Using the Leibnitz rule for $\nabla \mathcal{E}_0$, it suffices to prove this relation for $Q \in \text{End}_G(\mathcal{E}_0)$. Let $T^H_1G$ be a connection which is integrable in an small neighborhood $V$ of $v$ (that is $T^G_1$ is the tangent bundle of local transversal sub manifolds of $M$). A horizontal vector $y_1 \in T^H_1G$ determines uniquely a horizontal vector $y \in T^H_1G$ such that $y_1 - y$ is a vertical vector. Denote by $d^H_1$ and $\nabla \mathcal{E}_0$ the associated operators to
the integrable connection $T^H G$. It follows from the above discussion that $d^H - d^H$ is the differentiation with respect to vertical vectors which implies the vanishing of $(d^H - d^H)^\ast (Tr Q)$. On the other hand, $V_{10}^2 - V_{10}^1$ is an element of $Hom_G(E_0, \Omega^2 \otimes E_0)$ which implies $STr[V_{10}^2, Q] = STr[V_{10}^1, Q]$. Thus to prove relation (3.28), we may assume that $T^H G$ is integrable. So one can find local coordinate system $T \times V$ around $v \in G$ such that $T^H M$ is the tangent bundle of transversal $T$. Denote the coordinate of $T$ by $z$. The horizontal differentiation is nothing else than the partial derivatives $\partial_z$. Considering $K_Q$ as a local matrix-valued function it is easy to see that $str(\partial_z, K_Q)(v) = \partial_z(str K_Q(v,v))$, that means $str[V_{10}, K_Q](v) = d^H(str K_Q)(v) \in \Lambda^\ast \nu^\ast$ which with (3.30) and (3.29) imply the following equalities and complete the proof

$$
\rho \circ STr[V_{10}, K_Q] = \int_M \rho(x) \int_{G^2_0} \tau(v) str[V_{10}, Q](vv^{-1})
= \int_M \rho(x) \int_{G^2_0} \tau(v) d^H(str K_Q)(vv^{-1})
= \int_{\partial M} \rho(y) d^h(str K_Q(\Omega^0_y)) \int_{G^2_0} \tau(v)
= \int_{\partial M} \rho(y) d^h(\int_{G^2_0} \tau(v) str K_Q(vv^{-1}))
= \int_M \rho(x) d^h(\int_{G^2_0} \tau(v) str K_Q(vv^{-1}))
= \eta(d^H \circ r^\ast(St R Q))
$$

Example 1. Let $C^k$ be a holonomy invariant $k$-current acting on $C^\infty(M, \Lambda^k \nu^\ast)$ and let $\{U_i\}$, denote a finite covering for $(M, F)$ by the flowboxes $U_i = T_i \times V_i$ where $dim(V_i) = p$. Denote by $C^k_i$ the restriction of $C^k$ to $T_i$. Let $\{\phi_i\}$ be a partition of unity subordinate to this covering. The linear functional $\rho$ given by

$$
\rho : C^\infty(M, \Lambda^1 \nu^\ast) \to \mathbb{C}
\rho(\omega) = \sum_i < \int_{V_i} \phi_i \omega, C^k_i >
$$

(3.29)

satisfies the condition of the above proposition, c.f. examples 6 and 7 of [10]. In above the integration on $V_i$ is taken with respect to the $\Lambda^1$-factor of $\omega$ while the integration on $V_i \cap \partial M$ is taken with respect to the $\Omega^1$-factor of $\omega$.

Now let $Q$ be as in above acting on $C^\infty_c(G^0, \Lambda^k \nu^\ast \otimes r^\ast(\Omega^0 \otimes (E_0 \oplus E_0)))$ with smooth kernel $K$ such that

$$
K(v, v) = Id K_0(v, v) + \alpha K_1(v, v) = \begin{pmatrix} K_0 & \alpha \\ K_1 & K_0 \end{pmatrix}
$$

(3.30)

where $K_0(v, v)$ and $K_1(v, v)$ are elements in $\Lambda^\ast \nu^\ast \otimes \Omega^1(\mathbb{R}^n F) \otimes End(E_0)$ with $x = s(v)$ and $y = r(v)$, while $Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Put $str_a K(v, v) := tr(Id K_1)$ and define the CI(1)-supertrace (or $\alpha$-supertrace) of $Q$, denoted by $STr_a(Q) \in C^\infty(M, \Lambda^1 \otimes \Lambda^\ast \nu^\ast)$, by the following formula

$$
STr_a(Q)(x) := \int_{G^2_0} \tau(v) str_a(v^\ast K_{G^2_0}(v, v))
$$

(3.31)

It is clear from above discussion that if $Q$ is an even operator on $C^\infty_c(G^0, \Lambda^k \nu^\ast \otimes r^\ast(\Omega^0 \otimes (E_0 \oplus E_0)))$ then $STr_a(Q)$ belong to $C^\infty(M, \Lambda^1 \otimes \Lambda^{odd} \nu^\ast)$

Remark 2. Let $r^\ast(\nu)$ be a $G$-invariant element of $\Omega^\ast B$. The proof of the relation (3.32) can be applied to prove the following relation which will be used later

$$
\int_M \int_{G^2_0} \rho(x) \tau(v) r^\ast(\nu)(v) = \int_M \rho(x) \omega(x).
$$

(3.32)
Similar to the formula \(3.24\) we could define \(STr(P)\) for \(P \in End_{\mathcal{G}}(\mathcal{E})\) with compact support disjoint from boundary. This condition on the support cannot be satisfied always, so we have to define a more general trace for a larger class of \(G\)-invariant operators. These are smoothing \(G\)-invariant \(b\)-pseudodifferential operators which are introduced by R. Melrose \(20\). In the following section we follow \(24\) and \(21\) to describe those aspects of Melrose’s \(b\)-calculus which are necessary for future uses in this work.

4. Some aspects of \(G\)-invariant \(b\)-calculus

The underlying space for small \(b\)-calculus is an appropriate \(b\)-stretched product. For each \(x \in M\), the leaf \(F_x\), passing through \(x\), with its exact \(b\)-metric is a \(b\)-manifold. The covering space with boundary \(G_x \rightarrow F_x\), equipped with the pull-back metric is a non compact \(b\)-manifold. Let \(B_x\) consists of those elements \((u, v)\) of \(G^2_x \times G^2_x\) such that \(u\) and \(v\) belong to the same connected component of \(G^2_x\). The \(b\)-stretched product \(G^2_{x, b}\) is obtained from \((G_x)^2\) by replacing \(B_x\) with inward-pointing unit normal bundle to \(B_x\), i.e. \(S_+(B_x)\), and putting an appropriate topology on it. \(S_+(B_x)\) is denoted by \(bf_x\) and is called the front face of \(b\)-stretched product space \(G^2_{x, b}\). It turns out that this space is equipped with a surjective smooth blow-down map \((\beta_x)_b : G^2_{x, b} \rightarrow (G_x)^2\) which restricts to a diffeomorphism \(G^2_{x, b}\backslash S_+(B_x) \rightarrow (G_x)^2\backslash B_x\). Using this diffeomorphism one can lift the subset \(G^2_x \times \hat{G}_x, G^2_x \times \hat{G}_x\) and \(\Delta_x = diag(G_x \times G_x)\) of \((G_x)^2\backslash B_x\) to \(G^2_{x, b}\). We denote the closure of the lifted sets, respectively, by \(lb_x, rb_x\) and \(\Delta_{x, b}\). These are left boundary, right boundary and the \(b\)-diagonal of \(b\)-stretched product space \(G^2_{x, b}\). The space \(G^2_{x, b}\) is the carrier for the kernel of \(b\)-pseudodifferential operators on \(G_x\).

As in the family case (cf. the appendix of \(24\)), the \(b\)-stretched product of the holonomy groupoid \(G\) is the smooth family, parameterized by \(x \in M\), of stretched product spaces \(G^2_{x, b}\). We denote this space by \(G^2_x\). Clearly each element of \(G^2_x\) provides a diffeomorphism from \(G^2_{y, b}\) to \(G^2_{x, b}\). An object defined on \(G^2_x\) is called \(G\)-invariant if it is invariant with respect to this action of \(G\) on \(G^2_x\). We denote the family \(\beta_{x, b} : G^2_{x, b} \rightarrow G^2_x\) of blow-down maps by \(\beta_y : G^2_y \rightarrow G^2_x\). Similarly we denote by \(lb\), \(rb\) and \(bf\) respectively the \(G\)-invariant left, right and the front face family of \(G^2_{x, b}\)’s, while \(\Delta_{b}\) denotes the \(G\)-invariant family of \(b\)-diagonals \(\Delta_{x, g}\). The function \(r := x + x'\) is a \(G\)-invariant defining function for \(bf\). For each \(x \in M\), the small \(b\)-calculus \(\Psi^*_{b}(G^2_{x, b}, r^*(\Omega^2_0 \otimes E))\) is defined exactly as in \(20\) chapter 4 through a precise description of the kernels on \(G^2_{x, b}\). By a \(G\)-invariant \(b\)-pseudodifferential operator \(P\), we mean a family \(P_x\) of \(b\)-pseudodifferential operators with kernels \(K_x\) such that \(K_x\) is \(G\)-invariant. \(P\) is of order \(m\) if for each \(x\) the operator \(P_x\) is of order \(m\). Similarly \(P\) is classical or one-step polyhomogeneous if each \(P_x\) is one step polyhomogeneous. We denote the set of all \(G\)-invariant \(b\)-pseudodifferential operators by \(\Psi^*_{b, G}(G^2_{b}, r^*(\Omega^2_0 \otimes E))\) and the set of all classical \(b\)-pseudodifferential operator by \(\Psi^*_{b, G, cl}(G^2_{b, b}, r^*(\Omega^2_0 \otimes E))\). The operator \(P\) is elliptic if \(P_x\) is elliptic for each \(x \in M\). For each \(x \in M\) the manifold \(G^2_x\) is smooth and without boundary and \(E_0 = r^*(E_{\partial M})\) is a vector bundle over it. The diagonal action of each element of \(G^2_x\) provides a diffeomorphism from \(G^2_y \times G^2_y\) to \(G^2_x \times G^2_x\). Now we can proceed as in above and define the set of all \(G\)-invariant pseudodifferential operators \(\Psi^*_G(G^\partial, r^*(\Omega^2_0 \otimes E_0))\). Unlike the compact case, the small \(b\)-calculus \(\Psi^*_{b, G}(G^2_b, \Omega^2_0)\) and the pseudodifferential calculus \(\Psi^*_{G}(G^\partial, r^*(\Omega^2_0) \otimes E_0)\) are not algebras. Consequently the principal symbol exact sequence can not be used to construct a parametrix for, e.g. an elliptic \(b\)-differential operator. To overcome this difficulty, following \(21\) definition 4.6], we make some assumption on the behavior of kernels far from \(b\)-diagonal, and also on the nature of the foliation groupoid.

Let \(0 < \epsilon < 1\) and set

\[\mathcal{O}_\epsilon(bf) = \bigcup_{x \in M} \{ u \in G^2_{b, x} | d(\beta_{x, b}(u), B_x) < \epsilon \}\]

Here and in what follows \(d(\cdot, \cdot)\) denotes the \(G\)-invariant distance function coming from an ordinary \(G\)-invariant metric \(\tilde{g}\) on \(G\). Let \(x\) and \(x'\) be two \(G\)-invariant defining functions, respectively, for \(lb\) and \(rb\) and put \(r := x + x'\) and \(\tau := (x - x')/(x + x')\) If \((v, v')\) is a local coordinate for \(B := \bigcup B_x\) then \((r, \tau, v, v')\) define a local coordinate system for \(\mathcal{O}_\epsilon(bf)\) around \(bf\).
Definition 1. Let \( K \in C^{-\infty}(G_b^2, \Omega^\Delta_b) \) be a distributional section of \( r^*\Omega^\Delta_b \otimes r^*\Omega^\Delta_b \) which is smooth outside a small neighborhood of \( b \)-diagonal \( \Delta_b \). One says \( K \) has the rapidly decreasing property outside \( \epsilon \)-neighborhood of lifted diagonal \( \Delta_b \) if

i) For each multi-index of derivations \( \alpha \) and any \( N \in \mathbb{N} \) there is a constant \( C_{\alpha,N} \) such that for all \( (u, u') \in G_b^2 \) satisfying \( d(u, u') > \epsilon \) one has

\[
|\nabla^\alpha K(u, u')(1 + d(u, u'))|^N < C_{\alpha,N}
\]

(4.1)

ii) There exists a constant \( D_{\alpha,N} \) such that for each \( (r, \tau, v, v') \) with \( d(v, v') > \epsilon \) one has

\[
|\nabla^\alpha K(r, \tau, v, v')(1 + d(v, v'))|^N < D_{\alpha,N}
\]

Notice that, due to the compactness of \( M \), this definition is independent of the \( G \)-invariant metric.

For each \( m \in \mathbb{R} \cup \{-\infty\} \) denote by \( \mathcal{R}^{m}_{b,G}(G; r^*E) \) the subspace of \( \Psi^{m}_{b,G}(G; r^* (\Omega^\Delta_b \otimes E)) \) consisting of the operators with rapidly decreasing kernels outside of a \( \epsilon \)-neighborhood of \( \Delta_b \), and put \( \mathcal{R}^{m}_{b,G}(G, r^*E) = \cup_m \mathcal{R}^{m}_{b,G}(G; r^*E) \). Similarly define \( \mathcal{R}^{m}_{b,G}(G^\partial; \mathcal{E}_0) \) as the subset of \( \Psi^{m}_{b,G}(G^\partial, \mathcal{E}_0) \) consisting of operators with rapidly decreasing kernels outside of a \( \epsilon \)-neighborhood of the diagonal of \( G^\partial \times G^\partial \) and put

\[
\mathcal{R}^{m}_{b,G}(G^\partial; \mathcal{E}_0) = \cup_m \mathcal{R}^{m}_{b,G}(G^\partial; \mathcal{E}_0); \quad \mathcal{R}^{m}_{G}(G^\partial; \mathcal{E}_0) = \cap_m \mathcal{R}^{m}_{b,G}(G^\partial; \mathcal{E}_0).
\]

The spaces \( \mathcal{R}^{*}_{b,G} \) and \( \mathcal{R}^{*}_{G} \) are algebras if for each \( x \in M \) and \( N \in \mathbb{N} \) sufficiently large, the following integrals are convergent

\[
\theta(x, y)(1 + d(x, y))^{-N} \int_{G_x \times G_x} \mu_d \theta'(x, y)(1 + d(x, y))^{-N} \int_{G_x \times G_x} \mu_d
\]

Here \( \theta \) and \( \theta' \) are respectively the characteristic function of the subsets of \( G_x \times G_x \) and \( G^\partial_x \times G^\partial_x \) consisting of the points \( (x, y) \) with \( d(y, y) \geq \epsilon \) for a positive real number \( \epsilon \). This is the case if the Riemannian manifolds \( (G_x, \hat{g}) \) and \( (G^\partial_x, \hat{g}) \) are of polynomial growth. Let \( \hat{X} \) be a Riemannian manifold (non compact and with boundary) and let \( \Gamma \) be a discrete group of isometries of this manifold which acts properly on \( \hat{X} \) such that \( X := \hat{X}/\Gamma \) is a Riemannian manifold. The proof of \([23]\) shows that if \( X \) and \( \Gamma \) are of polynomial growth then \( \hat{X} \) is of polynomial growth too. This discussion shows that the following hypothesis is made of polynomial growth, so the spaces \( \mathcal{R}^{m}_{b,G}(G, r^*E) \) and \( \mathcal{R}^{m}_{G}(G^\partial, r^*E|_{\partial M}) \) have the algebra structures. In particular the spaces of rapidly decreasing smoothing operators \( \mathcal{R}^{m}_{b,G}(G, r^*E) \) and \( \mathcal{R}^{m}_{G}(G^\partial, r^*E|_{\partial M}) \) have the algebra structure .

Hypothesis 1.

i) The leaves of the foliations \( (M, F) \) and \( (\partial M, \partial F) \) are of polynomial growth with respect to a (hence to all) smooth longitudinal Riemannian metric on \( M \) and \( \partial M \).

ii) For each \( x \in M \) the holonomy group \( G_x^\partial \) is of polynomial growth. Moreover, the actions of \( G_x^\partial \) on \( G_x \) and \( G^\partial_x \) are proper.

Lemma 5. The small space of \( G \)-invariant rapidly decreasing classical \( b \)-pseudodifferential operators \( \mathcal{R}^{m}_{b,G}(G, r^*E) \) is an algebra. Moreover, one has the following exact sequence where \( b_m \) denotes the principal symbol map.

\[
0 \to \mathcal{R}^{m-1}_{b,G,os} \to \mathcal{R}^{m}_{b,G,os} \xrightarrow{\partial_m} C^{\infty}(\mathcal{S}(\pi_{b,G}(r^*E)); End(\pi_{b,G}(r^*E))) \to 0.
\]

Proof. This short sequence is exact when is restricted to \( G_x \) for any \( x \in M \). In despite of the non compactness of the underlying manifold \( G_x \), this follows from \([24]\) proposition 4.23. \( \square \)

As in the ordinary pseudodifferential calculus, the above lemma implies the existence of parametrices for elliptic \( b \)-operators. More precisely, given an elliptic operator \( P \in \mathcal{R}^{m}_{b,G,os}(G, r^*E) \), there exists \( Q \in \mathcal{R}^{m}_{b,G,os}(G, r^*E) \) and the smoothing operator \( R \) in \( \mathcal{R}^{m}_{G}(G, r^*E) \) such that

\[
P \circ Q - Id = R.
\]

Moreover, since the principal symbol depends only on the behavior kernel near \( \Delta_b \), the kernel of the parametrix \( Q \) may be assumed supported in a small neighborhood of \( \Delta_b \). If \( P \) is a differential operator this implies that the kernel of \( R \) is also supported in a small neighborhood of \( \Delta_b \). Let \( P \) be
a $G$-invariant operator with rapidly decreasing kernel $K_p$ in the small b-calculus, $P \in \mathcal{R}_{b,G}^\ast (G, r^* E)$. The boundary operator of $P$ is the operator $P_{|G^0} \in \mathcal{R}_{G^0}^\ast (G^0, r^* E_{|\partial M})$ defined by the following relation
\begin{equation}
P_{|G^0} (\xi_0) = P (\xi)_{|G^0}.
\end{equation}

Here $\xi_0$ is an element in $C^\infty_c (G^0, r^* E_{|\partial M})$ and $\xi$ is an arbitrary extension of $\xi_0$ to an element in $C^\infty_c (G, r^* (\Omega^2 \otimes \otimes E))$. Since the kernel of operators in small b-calculus vanish smoothly at left and right boundaries of $G^2_{x,b}$, the above definition is independent of the extension $\xi$. This implies also the product formula $(P \circ Q)_{G^0} = P_{G^0} \circ Q_{G^0}$. We recall that $r = x + x'$ is a defining function for the front face $bf$ in $G^2_{x,b}$. If the boundary operator of $P$ vanishes then the kernel $K_p$ of $P$ has to vanish on $bf$. So $K_p$ takes the form $r K_{P'}$ where $P' \in \mathcal{R}_{b,G}^\ast (G, r^* E)$. This follows from the explicit description of the behavior of kernels near $bf$, see [20, relation 4.61].

> From now on we fix a $G$-invariant co-normal structure $n$ for $G^0 \subset G$. This consists of a $G$-invariant family $\{n_x\}_{x \in M}$ of sections of the normal bundle of $G^2_{x,b}$ in $G_x$. Let $\sigma$ be a defining function for $G^0 \subset G$ satisfying $d\sigma(n) = 1$. Following [26, Proposition 5.8], we define the indicial family $I_n (P, \lambda); \lambda \in \mathbb{C}$ of a $G$-invariant b-pseudodifferential operator $P$ by the following formula which is independent of $\sigma$ as long as it satisfies the condition $d\sigma(n) = 1$
\begin{equation}
I_n (P, \lambda) = (\sigma^{-i \lambda} P \sigma^{i \lambda})_{G^0}.
\end{equation}

We have clearly the product formula $I_n (P \circ P', \lambda) = I_n (P, \lambda) \circ I_n (P', \lambda)$. As an example one has
\begin{equation}
I_n (\mathfrak{B}, \lambda) = \lambda \gamma + \mathbb{B} ; \quad \gamma := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\end{equation}

It is clear from above discussion that $I_n (P, \lambda) = 0$ if and only if the kernel $K_p$ of $P$ vanishes on $bf$.

We recall from [13] that each $G$-invariant b-differential operator $P$ has a $G$-invariant parametrix $Q$ in small b-calculus such that the remainder $R = P - Q - Id$ is a smoothing b-pseudodifferential operator. In what follows we need to have a good parametrix with remainder having a vanishing indicial operator. Such a parametrix does not exist in small b-calculus but in a slightly extended calculus which is, the so-called, full calculus with bounds. We recall from [21, section 7] that an operator $P$ in the full calculus with bound $\mathcal{R}_{b,G}^{\alpha,\delta} (G_x, r^* (\Omega^2 \otimes E))$ is defined by a precise description of the singularities of its kernel on $G^2_{x,b}$. In this extended calculus, the singularity on $\Delta_b$ is the same as in the small b-calculus. This part of singularities determines the symbol and the degree of the operator. In contrast to the small b-calculus, $K_p$ is singular near the left, the right and the front face hypersurfaces of $G^2_{x,b}$. The number $\delta$ which determines the order of these singularities is determined by $I_n (P, \lambda)$. Let $\delta$ be a non negative number and the operators satisfy the rapidly decreasing property of definition refapdec. Then, with the hypothesis [1] the full calculus $\mathcal{R}_{b,G}^{\alpha,\delta} (G_x, r^* (\Omega^2 \otimes E))$ is an algebra. The full calculus with bounds $\mathcal{R}_{b,G}^{\alpha,\delta}$ consists of the $G$-invariant family of full calculus for $G_x$'s. To show the existence of a parametrix with remainder having a vanishing indicial operator we will suppose the following hypothesis

**Hypothesis 2.** The family of Dirac operators $D_0$ is $L^2$-invertible with bounded inverse. In other word, there exists $\varepsilon_0 > 0$ such that for each $x \in M$, the $L^2$-spectrum of the unbounded operator
\begin{equation}
D_0 (G_x) : L^2 (G^0_x, r^* E_{|\partial M}) \rightarrow L^2 (G^0_x, r^* E_{|\partial M})
\end{equation}
is disjoint from $[-\varepsilon_0, \varepsilon_0]$. Notice that this is a global hypothesis depending on the geometry of $G^0$. The indicial family of $D$ is given by $I_n (D, \lambda) = \lambda \gamma + \alpha D_0$. So, as a consequence of the above hypothesis, $I_n (D, \lambda)$ is invertible if $-\varepsilon_0 \leq \text{Im} (\lambda) \leq \varepsilon_0$.

**Proposition 6.** Let the above hypothesis be satisfied by the boundary operator $D_0$ and let $0 < \delta < \varepsilon_0$. Then there exists a parametrix $Q \in \mathcal{R}_{b,os,G}^{\delta-1,\delta}$ such that
\begin{equation}
DQ - Id = R ; \quad QD - Id = R',
\end{equation}
where $R$ and $R'$ are in $\rho_{ad} \mathcal{R}_{b,G}^{\infty,\delta}$. In particular $I_n (R, \lambda) = I_n (R', \lambda) = 0$. 

Proof. Let $T$ be a small transversal submanifolds of $M$. We refer to \cite{21} theorem 7.1 for the construction of the smooth family of parametrices $Q_x$ for $D_{D_x}, x \in T$ such that

$$D_x Q_x - Id = R_x; \quad R \in \rho_{bf} R_{b}^{-\infty, \delta}(G_x, r^* (\Omega^Z \otimes E)).$$

In addition to the properties described in the proposition, the kernel of $Q_x$ and of $R$ are invariant with respect to the action of $G_x$ on $G_x$. Let $y \notin T$ and take $u \in G_y^\nu$ which provides a diffeomorphism $u : G_x \to G_y$. This diffeomorphism can be used to define $Q_y := u^* Q_x$ and $R_y := u^* R_x$ which belong, respectively, to $R_{b}^{-\infty, \delta}(G_y, r^* (\Omega^Z \otimes E))$ and $\rho_{bf} R_{b}^{-\infty, \delta}(G_y, r^* (\Omega^Z \otimes E))$ such that $D_y Q_y - Id = R_y$.

In this way we have extended the family of parametrices $Q_x, x \in T$ to a family on the set $G_T \cup G^g_T$. The groupoid $G$ can be covered by disjoint union of such sets. We can apply the above method to define, $G$-invariantly, the parametrix family on whole $M$. \hfill $\square$

For each $x \in M$, let $D_x$ denote the restriction $D_{D_x}$. For $s > 0$, one can define the heat operator $e^{-s \Delta_x}$ which belongs to $R_{b}^{-\infty, \delta}(G_x, r^* (\Omega^Z \otimes E))$. To see the construction of this heat operator we refer to \cite{21} Theorem 10.3. Putting together all these operators for all $x \in M$ we obtain the $G$-invariant heat operator $e^{-s \Delta}$. This operator is an element of $R_{b}^{-\infty, \delta}(G; r^* (\Omega^Z \otimes E))$ provided that $s > 0$. Now, using the Volterra formula (cf. \eqref{5.17} and \eqref{5.18}) we can define for $s > 0$ the heat kernel $e^{-(s^2 - l)}$ of the Bismut superconnection $A_s$. This operator belongs to $R_{b}^{-\infty, \delta}(G; r^* (\Omega^Z \otimes \Lambda^* r^* \otimes E))$.

Let $\pi_F : \mathcal{H}^\nu M \to \mathcal{H}^\nu F$ be the projection on the longitudinal $b$-tangent bundle with kernel $T^b M$. Using a scalar product $g_h$ on $T^h M$ one get the direct sum scalar product $g_h \oplus g$ on the bundle $\mathcal{H}^\nu M$. Let $\nabla$ be the Levi-Civita connection associated to this scalar product and put

$$\nabla^{\mathcal{H}/\mathcal{F}} = \pi_F \nabla \pi_F.$$

The connection $\nabla^{\mathcal{H}/\mathcal{F}}$ for $\mathcal{H}^\nu F$ is independent of the horizontal metric $g_h$, c.f. \cite{6} chapter 10. We denote by $\tilde{\nabla}^{\mathcal{H}/\mathcal{F}}$ the curvature of this connection which is a smooth section of $\mathcal{H} \Omega^2(T^F)$ with coefficients in $\Omega^2(M)$. The longitudinal $A$-genus of the foliation $(M, F)$ is an element of $\Omega^*(M)$ defined by

$$\tilde{A}(M, F) = \det \left( \frac{\tilde{\nabla}^{\mathcal{H}/\mathcal{F}}/2}{\sinh(\tilde{\nabla}^{\mathcal{H}/\mathcal{F}}/2)} \right)^{1/2} \quad (4.6)$$

Before stating the next proposition, we introduce some notation and concepts. We recall that $E$ is a longitudinal Clifford bundle. It turns out that $E$ has a Clifford connection $\nabla^E$ which is compatible with the Levi-Civita connection $\nabla^{\mathcal{H}/\mathcal{F}}$. Using Proposition 3.43 of \cite{6} the curvature of this connection $\tilde{\nabla}^{\mathcal{H}/\mathcal{F}}$ has the following decomposition

$$\tilde{\nabla}^{\mathcal{H}/\mathcal{F}} = \text{cl}(\tilde{\nabla}^{\mathcal{H}/\mathcal{F}}) + \tilde{\nabla}^{\mathcal{H}/\mathcal{F}}$$

where $\text{cl}(\tilde{\nabla}^{\mathcal{H}/\mathcal{F}})$ is the action of the Riemannian curvature of the connection $\nabla^{\mathcal{H}/\mathcal{F}}$ on the bundle $E$ through the Clifford action. More precisely we have

$$\text{cl}(\tilde{\nabla}^{\mathcal{H}/\mathcal{F}})(e_{\alpha}, e_{\beta}) = \frac{1}{4} \sum_{1 \leq i, j \leq S} (\tilde{\nabla}^{\mathcal{H}/\mathcal{F}}(e_{\alpha}, e_{\beta})e_i, e_j) \text{cl}(e_i) \text{cl}(e_j)$$

where $\{e_{\alpha}\}_\alpha$ is a basis for the tangent bundle $TM$ while $\{e_i\}_i$ is a basis for the longitudinal tangent bundle $T^F$. The term $\tilde{\nabla}^{\mathcal{H}/\mathcal{F}}$ is an invariant of the Clifford connection $\nabla^E$ which is called the twisting curvature of $\nabla^E$. The twisted Chern character of the vector bundle $E$ is a differential form on $M$ defined by

$$\text{Ch}(E/S) = \text{str} e^{-\tilde{\nabla}^{\mathcal{H}/\mathcal{F}}} \quad (4.7)$$

If $E = S \otimes V$ then the twisting curvature is the curvature of the twisting hermitian bundle $V$. Notice that because at the boundary point $\nabla_{\partial} = 0$, the differential forms $\tilde{A}(M, F)$ and $\text{Ch}(E/S)$ are tangential at boundary points. Using the decomposition \eqref{2.3} one defines the following projection

$$\{ \cdot \}_p : \Lambda^*(\mathcal{H}^\nu M) \to \Lambda^*(T^h M) \otimes \Lambda^p(\mathcal{H}^\nu F) \xrightarrow{\mathcal{H}^\nu M} \Lambda^*(T^h M) \otimes \Omega^1(F).$$

In above the isomorphism between the longitudinal bundle $\Lambda^p(\mathcal{H}^\nu F)$ and the bundle of longitudinal $b$-densities of order 1 is given by the longitudinal $b$-metric $g$. 
Proposition 7 (G-invariant Bismut b-density theorem). We have the following asymptotic relation for the restriction of the heat kernel of the Bismut superconnection to the diagonal $\Delta_0$. This relation occurs in $\Lambda^* (T_u^H G) \otimes \Omega^1 (F)$

\[
\lim_{s \to 0} \text{str} e^{-(\lambda_u^2 - l)}(u, u) = \frac{1}{(2\pi i)^{p/2}} r^* \{ \hat{A}(M, F) \text{Ch}(E/S)(y) \} p
\]

where $r(u) = y$.

Proof. Proposition asserts that $\lambda_u^2 - l = r^* \hat{A}_s$, so $e^{-(\lambda_u^2 - l)} = r^* (e^{\hat{A}_s})$. This proves the G-invariance of the restriction of the kernel $e^{-(\lambda_u^2 - l)}$ to the diagonal $\Delta_0(G)$. So

\[
\text{str} e^{-(\lambda_u^2 - l)}(u, u) = r^* (\text{str} e^{-\hat{A}_s}(y, y))
\]

where $y = r(u)$. Since $(M, F)$ is locally a fibration of b-manifolds and the assertion of the proposition is local, we may assume that $(M, F)$ is a fibration of b-manifolds and $E = S \otimes V$. For a fixed $y \in F$, the heat kernel $k_s(y, y) := e^{-\hat{A}_s}(y, y)$ is a smooth $s$-depending element in $\Omega^1_u \otimes \Lambda^* \nu_e \otimes \text{End}(S_y) \otimes \text{End}(V_y)$. The linear space $\text{End}(S_y)$ can be identified with the complex Clifford algebra $C(\tau_y F)$ and then with the complexified exterior algebra associated to $\tau_y F$, denoted by $\Lambda^* \tau_y F$. Using this identification $k_s(y, y)$ may be considered as a smooth function of $s \in \mathbb{R}^+$ with values in $\Omega^1_u \otimes \Lambda^* \nu_e \otimes \Lambda^* \tau_y F \otimes \text{End}(V_y)$. Using the riemannian 1-density $\mu_y$, at point $y$, this space can be identified to $\Lambda^* \nu_e \otimes \Lambda^* \tau_y F \otimes \text{End}(V_y)$ and the corresponding element to $k_s(y, y)$ in this last space is the heat kernel, at $(y, y)$ of the rescaled Bismut superconnection as is introduced in [6] page 333 without any use of the densities (see also remark [1]). Therefore, as long as the $k_s(y, y)$ is concerned, up to the multiplication by $\mu_y$ we may assume that our superconnection is the same of [6, page 333]. In this proof the vertical degree of a typical element $\omega \otimes \alpha \otimes T$ in $\Lambda^* \nu_e \otimes \Lambda^* \tau_y F \otimes \text{End}(V_y)$ refers to the degree of $\alpha$, its horizontal degree refers to the degree of $\omega$ and the total degree of such element is the sum of its vertical and horizontal degrees. Let $\delta_s$ be the rescale operator on $\Lambda^* \nu_e \otimes \Lambda^* \tau_y F \otimes \text{End}(V_y)$ that multiplies an element of horizontal degree $l$ by $s^{-l}$. Using the Bismut density theorem for family [6, theorem 10.21], one has the following asymptotic formula when $s$ goes toward 0

\[
k_s(y, y) \sim (4\pi s^2)^{-n/2} \sum_{j=0}^{\infty} s^{2j} \delta_s k_j(y)
\]

where $k_j(y)$ is a section of $\Lambda^* \nu_e \otimes \Lambda^* \tau_y F \otimes \text{End}(V_y)$ such that its total degree is less than or equal to $2j$. Moreover the following formula holds, where $F_y$ denotes the curvature of the bundle $V$ at point $y$ and $\sigma(k_j(y))$ denotes term of $k_j(y)$ with highest total degree

\[
\sum_{j=0}^{p/2} \sigma(k_j(y)) = (2\pi i)^{p/2} \hat{A}(M, F) e^{-F_y}.
\]

The left hand side of the above formula is called the full symbol of $k_s(y, y)$. The action of the supertrace on a typical element $\omega \otimes \alpha \otimes T$ equals to $\text{str}(\omega) \text{tr}(T)$. It is well known that the supertrace of $\alpha \in \Lambda^q T_y^* F$ as an element in $\text{End}(S)$ vanishes if $q \neq p$ and equals to its devision by the riemannian volume element if $q = p$. Consequently, in above asymptotic relation the terms giving rise to negative powers of $s$ have no contribution in $\text{str} k_s(y, y)$ for $j \leq p/2$. On the other hand it is clear that the contribution of those terms generating a positive power of $s$ vanishes when $t$ goes to 0. Therefore the supertrace $\text{str} k_s(y, y)$ is coming from constant terms with vertical degree $p/2$. These are exactly the terms in the full symbol of $k_s(y, y)$. So, using the above formula for the full symbol and the above discussion one get

\[
\lim_{s \to 0} \text{str} k_s(y, y) = (2\pi i)^{p/2} [\hat{A}(M, F) \otimes \text{tre}^{-F_y}]_p(y) = (2\pi i)^{p/2} [\hat{A}(M, F) \otimes \text{Ch}(E/S)]_p(y)
\]

where $[ \cdot ]_p$ denotes the part of with vertical degree $p$ divided by the volume elements of $T_y F$. Multiplying this asymptotic formula by $\mu_y$ is the desired relation (4.8). □
4.1. b-supertrace. With \( \tau \) defined by (3.23), the expression (3.24) can be used to define \( \text{STr}(Q) \) for \( Q \) in the algebra \( \mathcal{R}_G^{\infty}(G^0, s^*(\Lambda^* \mu^*) \otimes r^* (\Omega^+ \oplus E)) \), i.e. the algebra of the smoothing \( G \)-invariant operators on the boundary groupoid \( G^0 \) with rapidly decreasing kernel. The proposition \( \text{II} \) remains true in this case and \( \rho \circ \text{STr} \) defines a b-supertrace on this algebra. Due to the presence of the b-boundary, for \( P \in \mathcal{R}_G^{\infty}(G, s^*(\Lambda^* \mu^*) \otimes r^* (\Omega^+ \oplus E)) \) a similar expression in not usually convergent and cannot be used to define the trace of \( P \). The following b-supertrace is a combination of the Melrose b-trace defect formula \( \text{II} \) and the b-supertrace defect formula \( \text{II} \).

\[
\begin{align*}
\text{bSTr} : \mathcal{R}_G^{\infty}(G, s^*(\Lambda^* \mu^*) \otimes r^* (\Omega^+ \oplus E)) & \to C^\infty(M, \Lambda^* \mu^* \otimes \Omega^1) \\
\text{bSTr}(P)(x) & = \lim_{\epsilon \to 0} \left\{ \int_{G_\epsilon^*(v) \geq x} \tau(v) \text{str} K_{\hat{P}}(v) + \ln \int_{G_\epsilon^*(v)} \tau(v) \text{str} K_{\hat{P}}(v) \right\}.
\end{align*}
\]

Here the integrations are taken with respect to longitudinal densities coming from the supertrace of the Schwartz kernels. So \( \text{bSTr}(P)(x) \) belongs to \( \Omega^1(F) \), where \( F \) is the leaf passing by \( x \). For keeping the notation simple, we have skipped the holonomy action of \( v^* \) on the kernel (see relation (3.24) for the similar situation with explicit holonomy action action ). For each \( x \in M \), Kernel \( \tau K_{\hat{P}}(v) \) is compactly supported, so the proof of the lemma 4.62 of \( \text{II} \) shows that the above limit exists and is independent of the boundary defining function \( t \) satisfying \( dt(n) = 1 \).

**Remark 3.** If \( P \) is an operator as in above with kernel \( K_P \) then \( \text{bSTr}(P) \) is defined by relation (4.11) provided that \( \text{str} \) is replaced by \( \text{tr} \). The proof of the following proposition remain true if we replace \( \text{bSTr} \) by \( \text{bTr} \). If \( K_{\hat{P}} \neq 0 \) for each \( x \in M \), then the first integral in the right hand side of (4.11) defines \( \text{Tr}(P) \) provided that \( \text{str} \) is replaced by \( \text{tr} \). We will use this definition in subsection 4.1.

**Proposition 8.**

a) Let \( \rho \) be the distributional linear functional on \( C^\infty(M, \Omega^1 \otimes \Lambda^0 \nu) \) defined by (3.21). Assume that the linear functional \( \eta \) on \( \Omega^* \mathcal{B} \) defined by

\[
\eta(\phi) = \rho(\phi_M)
\]

is a supertrace on the foliation algebra \( \Omega^* \mathcal{B} \). The linear operator \( \rho \circ \text{bSTr} \) is independent of \( \tau \) and defines a b-supertrace, i.e. for each \( P \in \text{Diff}(G; r^*(\Omega^+ \oplus E)) \) and for each \( Q \in \mathcal{R}_G^{\infty}(G, r^*(\Omega^+ \oplus E)) \) the following defect formula holds

\[
\rho \circ \text{bSTr}[P, Q] = \frac{i}{2\pi} \rho \circ \int_R \text{STr}\{\partial_\lambda I_n(P, \lambda) \circ I_n(Q, \lambda)\} \, d\lambda.
\]

b) In addition, if \( \eta \) is a closed graded trace on \( \Omega^k \mathcal{B} \), then

\[
\rho \circ \text{bSTr}[\nabla^P, P] = 0,
\]

where \( P \in \Omega^{k-1}(r^*) \otimes \mathcal{R}_G^{\infty}(G, r^*(\Omega^+ \oplus E)) \).

**Proof.** The restriction of the kernel \( K_P \) of \( P \in \mathcal{R}_G^{\infty} \) to the b-diagonal of \( G^0_\epsilon \) defines a smooth section \( \tilde{K}_P \) of \( \text{End}(E) \) over the b-diagonal of \( M^\mu_\epsilon \) such that \( K_P(v) = \tilde{K}_P(r(v), r(v)) \). Using relations (3.25) and (3.26), the proof of Proposition \( \text{II} \) can be applied to get the following analogue of (3.27)

\[
\rho \circ \text{bSTr}(P) = \lim_{\epsilon \to 0} \left\{ \int_{M^\mu_\epsilon \geq x} \rho(x) \text{str} \tilde{K}_P(x, x) + \ln \int_{\partial M^\mu_\epsilon} \rho(y) \text{str} \tilde{K}_P(y, y) \right\}
\]

The right hand side of this expression is independent of \( \tau \). This proves the first part of (a). To prove the defect formula, notice that \( (PQ)_{G^0} = P_{G^0}Q_{G^0} \) for \( P, Q \in \mathcal{R}_G^{\infty} \). Using the proposition \( \text{II} \) the boundary part in relation (4.11) is a trace, i.e. it vanishes on supercommutators. In particular this boundary part has no contribution in the evaluation of \( \rho \circ \text{bSTr} \) on the supercommutators. Let \( \theta \in \text{End}(E) \) be the operator defined by point-wise multiplication by \( \tau \). Using (4.11) one has

\[
\rho \circ \text{bSTr}(P) = \rho \circ \text{bSTr}(\theta P)
\]

where \( \text{bSTr} \) is defined by the following relation provided that the kernel \( K_P \) is compactly supported

\[
\text{bSTr}(P)(x) = \lim_{\epsilon \to 0} \left\{ \int_{G_\epsilon^*(v) \geq x} \text{str} K_P(v) + \ln \int_{G_\epsilon^*(v)} \text{str} K_P(v) \right\}
\]
The second integral in above expression defines the operator $TR$ on $\text{End}_B(E_0)$ satisfying $\rho \circ TR \circ P_{G^\partial} = \rho \circ TR(\theta P|_{G^\partial})$. As a consequence of the relation (3.123) we prove the following equality

\begin{equation}
{\mathbf{STR}}(\theta QP) = {\mathbf{STR}}(Q\theta P)
\end{equation}

which implies

\begin{equation}
\rho \circ {\mathbf{STR}}[P, Q] = \rho \circ {\mathbf{STR}}[\theta P, Q].
\end{equation}

Since the boundary term in the definition of $\rho \circ {\mathbf{STR}}$ has no contribution in its evaluation on the commutators, the equality (4.10) is equivalent to the following one

\begin{equation}
\int_M \rho(x) \int_{G^\partial_z} \int_{G_x} du dv \tau(u) \text{str} K_Q(u, v) \circ K_P(v, u) = \int_M \rho(y) \int_{G^\partial_{z'}} \int_{G_y} du' dv' \tau(u') \text{str} K_Q(u', v') \circ K_P(v', u')
\end{equation}

that can be proved as follows

\begin{align*}
\int_M \rho(x) \int_{G^\partial_z} \int_{G_x} du dv \tau(u) &\text{str} K_Q(u, v) \circ K_P(v, u) \\
&= \int_M \rho(x) \int_{G^\partial_z} \int_{G_x} du dv \tau(u) \tau(vw^{-1}) \text{str} K_Q(u, v) \circ K_P(v, u) \\
&= \int_M \rho(x) \int_{G_x} du \int_{G^\partial_z} \int_{G_x} dv' \tau(u'w) \tau(v') \text{str} K_Q(u'w, v'w) \circ K_P(v'w, u'w) \\
&= \int_M \rho(y) \int_{G^\partial_{z'}} \int_{G_y} du' \int_{G_{z'(w)}} dv \tau(u'w) \tau(v') \text{str} K_Q(u', v') \circ K_P(v', u')
\end{align*}

It is clear from formula (4.15) that

\begin{equation}
{\mathbf{STR}}[\theta P, Q] = {\mathbf{STR}}[\theta P|_{G^\partial_{z'}}, Q|_{G^\partial_{z'}}].
\end{equation}

Because $\tau$ is compactly supported, for each $z \in M$, $(\theta P)_{G^\partial_z}$ is a $b$-differential operator with compactly supported Schwartz kernel. It turns out that the proof of Melrose’s trace defect formula [26] page 154 remain true in this compact support situation, so the supertrace defect formula [27] proposition 9 can be applied to $b$-manifold $G_x$, compactly supported smoothing $b$-pseudodifferential operator $\theta P$ and $b$-differential operator $Q$ to obtain the following leaf-wise defect formula

\begin{equation}
{\mathbf{STR}}_z[\theta P|_{G^\partial_z}, Q|_{G^\partial_z}) = \frac{i}{2\pi} \int_{\mathbb{R}} \text{str} \partial_\lambda I_n((\theta P|_{G^\partial_z}, \lambda) \circ I_n(Q|_{G^\partial_z})) \circ \lambda.
\end{equation}

Therefore

\begin{align*}
\rho \circ {\mathbf{STR}}[P, Q] &= \rho \circ {\mathbf{STR}}[\theta P, Q] \\
&= \frac{i}{2\pi} \int_{\mathbb{R}} \rho(x) \int_{\mathbb{R}} \text{str} \partial_\lambda I_n((\theta P|_{G^\partial_z}, \lambda) \circ I_n(Q|_{G^\partial_z})) \circ \lambda \\
&= \frac{i}{2\pi} \int_{\mathbb{R}} \rho(x) \text{str} (\partial_\lambda I_n(P, \lambda) \circ I_n(Q, \lambda)) \circ \lambda \\
&= \frac{i}{2\pi} \int_{\mathbb{R}} \rho(x) \text{str} (\partial_\lambda I_n(P, \lambda) \circ I_n(Q, \lambda)) \circ \lambda,
\end{align*}

and this is the desired defect formula (4.12). To prove part (b), it is enough to prove the following analogous of the relation (4.28)

\begin{equation}
\rho \circ {\mathbf{STR}}[\nabla^\partial, P] = \eta(d^H \circ \tau^*(\text{STR}(P))).
\end{equation}

The space $T^H_v G$ is included in $T_v G^\partial$ if $v \in G^\partial$, this implies the equality $\nabla^\partial G^\partial = \nabla G^\partial$ which proves the relation $\text{str} \nabla^\partial P|_{G^\partial} = \text{str} [\nabla G^\partial, P|_{G^\partial}]$. Now, by applying relation (4.28) we get

\begin{equation}
\rho \circ {\mathbf{STR}}[\nabla^\partial, P] = \eta(d^H \circ \tau^*(\text{STR}(P|_{G^\partial}))).
\end{equation}
This proves the boundary part of (4.13) which is implicit in the definition of \( b\text{Str} \). However, using (3.20), the proof of (3.23) can be slightly modified to get
\[
\rho \circ \text{Str}[\nabla^E, P_{G|G^\geq}] = \eta(dH \circ r^*(\text{Str} P_{G|G^\geq}).
\]
These last two equations and the defining relation (4.11) together prove part (b) of the proposition. \( \square \)

5. Chern character and eta invariant

5.1. Chern character. Assume that the Dirac operator \( D_0 \) satisfies the hypothesis (32) and fix \( \delta \in \mathbb{R} \) such that \( 0 < \delta < \epsilon_0 \). With respect to the decomposition \( E = E^+ \oplus E^- \), the parametrix \( Q \) given by proposition 4 is an odd operator of the form
\[
Q = \begin{pmatrix} 0 & Q^- \\ Q^+ & 0 \end{pmatrix}.
\]
In this section, for keeping the notation simple, we use the symbol \( Q \) to denote \( Q^- \). Therefore
\[
S_+ := \text{Id} - Q D^+ \in \rho(\mathcal{R}_{b,G}^{\infty,0}(G, r^*(\Omega_{1/2}^+ \otimes E^+))
\]
\[
S_- := \text{Id} - D^- Q \in \rho(\mathcal{R}_{b,G}^{\infty,0}(G, r^*(\Omega_{1/2}^+ \otimes E^-)).
\]
Since \( \delta > 0 \) we have \( I_n(S_{\pm}, \lambda) = 0 \) and
\[
I(Q, \lambda) = (i\lambda + D_0)^{-1}.
\]
Let the projections \( p \) and \( p_0 \) be given by
\[
p = \begin{pmatrix} S_+^2 & S_-(\text{Id} + S_+)Q \\ S_->D^+ & \text{Id} - S_-^2 \end{pmatrix} ; \quad p_0 = \begin{pmatrix} 0 & 0 \\ 0 & \text{Id} \end{pmatrix}
\]
and similar to [17] relation 5.16] define the analytical index class of the longitudinal Dirac operator by the following formula
\[
\text{ind}(\hat{D}) := [p - p_0] \in K_0(\mathcal{R}_{b,G}).
\]
Here \( \mathcal{R}_{b,G} \) is the sub algebra of \( \mathcal{R}_{b,G}^{\infty,0}(G, r^*(\Omega_{1/2}^+)) \) consisting of the operators with vanishing indicial family. So it is natural to define the Chern character of the index class by the following formula where \( \kappa^E := (\nabla^E)^2 - l \) denotes the curvature of the connection \( \nabla^E \) and \( Tr \) is defined in remark 3.

Definition 2. The Chern character of the index class is defined by
\[
\rho \circ \text{Ch}(\text{ind}(D)) = \rho \circ Tr(pe^{-p\kappa^E}.p - p_0 e^{-p_0\kappa^E}.p_0).
\]
In this definition the operator which is acted on by \( Tr \) is smoothing with vanishing indicial family, so it has finite \( Tr \). The index theorem that we want to establish gives a formula to calculate the Chern character of the index class \( \text{ind}(D) \). The first step toward this purpose is the following proposition which establishes a relationship between the Chern character of the index class and the Chern character of the Bismut superconnection given by
\[
(5.3) \quad \rho \circ b\text{Ch}(A_0) = \rho \circ b\text{Str} e^{-(A_0^2 - l)}.
\]
The rest of this subsection is devoted to the proof of this proposition. In fact this proposition is a consequence of the next two lemmas.

Proposition 9. Let \( \rho \) be a functional satisfying the conditions described in the proposition (3). The following asymptotic formula holds
\[
\lim_{s \to +\infty} \rho \circ b\text{Ch}(A_0) = \rho \circ \text{Ch}(\text{ind}(D)).
\]
To prove this proposition we follow the approach used in [17] section 5] and [22] sections 8-9]. Put
\[
J = \begin{pmatrix} S_+ & -(\text{Id} + S_+)Q \\ D^+ & S_- \end{pmatrix} ; \quad \rho_1 = \begin{pmatrix} \text{Id} & 0 \\ 0 & 0 \end{pmatrix}
\]
we have \( p = J\rho_1 J^{-1} \) where
\[
J^{-1} = \begin{pmatrix} S_+ & Q(\text{Id} + S_-) \\ -D^+ & S_- \end{pmatrix}.
\]
Now using the isomorphism $J$ we define the following connection on $\mathcal{E}$
\[
\tilde{\nabla}^\mathcal{E} = (p_1 J^{-1} \circ \nabla^\mathcal{E} \circ Jp_1) + (p_0 \nabla^\mathcal{E} p_0)
\]
or equivalently
\[
(5.4) \quad J \circ \tilde{\nabla}^\mathcal{E} \circ J^{-1} = p \circ \nabla^\mathcal{E} \circ p + (1 - p)J \circ \nabla^\mathcal{E} \circ J^{-1}(1 - p)
\]
Put $\nabla^\mathcal{E} = \tilde{\nabla}^\mathcal{E}_+ + \tilde{\nabla}^\mathcal{E}_-$ and $\tilde{\nabla}^\mathcal{E} = \tilde{\nabla}^\mathcal{E}_+ \oplus \tilde{\nabla}^\mathcal{E}_-$, then
\[
(5.5) \quad \tilde{\nabla}^\mathcal{E}_+ = p_0 \tilde{\nabla}^\mathcal{E} p_0 = \nabla^\mathcal{E}_-
\]
\[
(5.6) \quad \tilde{\nabla}^\mathcal{E}_- = p_1 \circ \tilde{\nabla}^\mathcal{E} \circ p_1 = S_+ \nabla^\mathcal{E}_+ S_+ + Q(Id + S_-)\nabla^\mathcal{E}_+ D^+.
\]
Finally, using this connection, we introduce the following family of superconnections from $\mathcal{E}$ into $\Omega^1 \mathcal{E}$
\[
A(s) = \left( \begin{array}{cc} \tilde{\nabla}^\mathcal{E}_+ & s D^- \\ s D^+ & \tilde{\nabla}^\mathcal{E}_- \end{array} \right).
\]
Denote its curvature by $K(s) = A^2(s) - I$ and put
\[
e^{-K(s)} = \begin{pmatrix} e^{-K(s)}_{11} & e^{-K(s)}_{12} \\ e^{-K(s)}_{21} & e^{-K(s)}_{22} \end{pmatrix}.
\]
With above notation the following formal equalities are easy to verify
\[
STr(e^{-K(s)}) = Tr(e^{-K(s)}_{11}) - Tr(e^{-K(s)}_{22}) = Tr(Jp_1 e^{-K(s)}_{11} J^{-1} - e^{-K(s)}_{22}) = Tr(D^+ e_{11} K(s) Q(Id + S_-) - e_{22} K(s))) + Tr(S_+ e_{11} K(s) S_+).
\]
Although neither $e_{11} K(s)$ nor $e_{22} K(s)$ have finite $Tr$ (they have non-vanishing indicial families), the operators in the last line, which are acted on by $Tr$, are rapidly decreasing and have vanishing indicial operator. Therefore we can define the Chern character of the superconnection $A(s)$ by the following formula with value in $C^\infty(M, \Lambda^* \nu^* \otimes \Omega^1)$
\[
(5.7) \quad \text{Ch}(A(s)) := Tr(S_+ e_{11} K(s) S_+) + Tr(D^+ e_{11} K(s) Q(Id + S_-) - e_{22} K(s))
\]
Our interest in $\text{Ch}(A(s))$ is justified by the following lemma showing $\rho \circ \text{Ch}(A(s)) = \rho \circ \text{Ch}(\text{ind}(D))$.

**Lemma 10.** Under the conditions of the above proposition

1. The following relation holds
   \[
   \rho \circ \text{Ch}(A(0)) = \rho \circ \text{Ch}(\text{ind}(D)).
   \]

2. For $s > 0$ one has
   \[
   \rho \circ \text{Ch}(A(s)) = \rho \circ \text{Ch}(A(0)).
   \]

**Proof.** Using the relations (5.4) and (5.6) we have
\[
pJe^{-K(0)}J^{-1}p = pe^{-pK(0)}p
\]
On the other hand $pJ = Jp_1$ and $p_0 e^{-K(0)} p_0 = p_0 e^{-p_0 K(0)} p_0$ so
\[
\text{Ch}(\text{ind}(D)) = Tr(pe^{-pK(0)}p - p_0 e^{-p_0 K(0)} p_0) = Tr(Jp_1 e^{-K(0)}_{11} J^{-1} - p_0 e^{-K(0)}_{11} p_0) = Tr(S_+ e_{11} K(0) S_+ + D^+ e_{11} K(0) Q(Id + S_-) - e_{22} K(0)) = \text{Ch}(K(0))
\]
This proves the first part of the lemma. Now we are going to prove the second part. Using the Duhamel formula we get

\[
\frac{d}{ds} \text{Str}(e^{-K(s)}) = -b \text{Str} \int_0^1 e^{-uK(s)} \circ \frac{d}{ds} K(s) \circ e^{-(1-u)K(s)} \, dz
\]

(5.8)

\[
= -b \text{Str} \int_0^1 [e^{-uK(s)} \frac{d}{ds} K(s)e^{-(1-u)K(s)}] \, du
\]

(5.9)

We simplify the expression (5.8) by means of the defect formula. From the very definitions and up to the smoothing operators with vanishing indicial families, we have the following formula in the ungraded notation

\[
A^2(s) = \left( \begin{array}{ccc}
(\nabla^2_{\xi})^2 + s^2 D^- D^+ & s\nabla^2_{\xi} D^- - sD^- \nabla^2_{\xi} \\
0 & s^2 D^+ D^-
\end{array} \right)
\]

so

\[
I(\frac{d}{ds} K(s), \lambda) = \frac{d}{ds} I(K(s), \lambda)
\]

(5.10)

\[
= \left( 2s(\lambda^2 + D_0^2) (\nabla^2_{\xi} - l_\pm)(-i\lambda + D_0) - (-i\lambda + D_0)(\nabla^2_{\xi} - l_-) \right) \frac{Z}{2s(\lambda^2 + D_0^2)}.\]

Using (5.6) one has also

\[
I(e^{-uK(s)}, \lambda) = \left( (i\lambda + D_0)^{-1} e^{-u(\{K_0(s) + s^2\lambda \lambda + s^2 D_0^2\})} (i\lambda + D_0) \right) \frac{Z}{e^{-u(\{K_0(s) + s^2(\lambda^2 + D_0^2)\})}).\]

(5.11)

Using the defect formula and ignoring the odd functions of \( \lambda \), one get the following equalities

\[
\frac{-i}{2\pi} \text{Str} \int_0^1 \int_\mathbb{R} \partial_\lambda I(e^{-uK_0(s)}, \lambda) I(\frac{d}{ds} K_0(s) e^{-(1-u)K_0(s)}, \lambda) \, d\lambda \, du
\]

(5.12)

\[
= -\frac{1}{2\pi} \text{Tr} \int_0^1 (i\lambda + D_0)^{-1} e^{-u(\{K_0 + s^2 + \hat{s}^2 \lambda^2\})} 2s(D_0^2 + \lambda^2) e^{-(1-u)(\{K_0 + s^2 \lambda^2 D_0 + s^2 \lambda^2\})} \, du \, d\lambda
\]

\[
+ \frac{1}{\pi} \text{Tr} \int_\mathbb{R} sD_0 e^{-(K_0 + s^2 D_0 + s^2 \lambda^2)} \, d\lambda
\]

Apply again the Duhamel and the defect formula to get

\[
\frac{d}{ds} \text{Str}[J^{-1}, J_p e^{-K(s)p}] = \frac{d}{ds} \text{Str} e^{-K(s)} - \frac{d}{ds} \text{Ch}(A(s)),
\]

where the last equality is coming from relation (5.12). Thus we obtain the equality

\[
\frac{d}{ds} \rho \circ \text{Ch}(A(s)) = - \rho \frac{d}{ds} \text{Str} K(s) e^{-K(s)}
\]

(5.13)

\[
+ \frac{1}{\pi} \rho \circ \text{Tr} \int_\mathbb{R} sD_0 e^{-(K_0 + s^2 D_0 + s^2 \lambda^2)} \, d\lambda.
\]
The following relations are the direct consequences of the definition of the involved operators and of the fact that $A(s)$ is a grading reversing operator

$$
{^b \text{STr}} \frac{d}{ds} K(s) e^{-K(s)} = {^b \text{STr}} \{ \left( \frac{d}{ds} A(s) A(s) + A(s) \frac{d}{ds} A(s) \right) e^{-K(s)} \}
$$

$$
= {^b \text{STr}} [A(s), \frac{d}{ds} e^{-K(s)}]
$$

$$
= {^b \text{STr}} [\tilde{\nabla}^c - \nabla^c, \left( \begin{array}{c} 0 \\ D^- \\ 0 \end{array} \right)]
$$

$$
+ {^b \text{STr}} [\nabla^c, \frac{d}{ds} A(s) e^{-K(s)}]
$$

$$
+ {^b \text{STr}} \left[ \left( \begin{array}{c} 0 \\ sD^+ \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ D^- \\ 0 \end{array} \right) \right] e^{-K(s)}].
$$

The defect formula (4.12) and the relation (5.10) imply the vanishing of the first term of the right hand side of the last equality while the third term is equal to

$$
\frac{1}{\pi} Tr \int_{\mathbb{R}} sD_0 e^{-(K_0 + s^2 \hat{D}_0 + s^2 \lambda^2)} d\lambda.
$$

Summarizing we get the following relation

$$
\frac{d}{ds} \text{Ch}(A(s)) = {^b \text{STr}} [\tilde{\nabla}^c, \frac{d}{ds} A(s) e^{-K(s)}].
$$

$\rho$ satisfies the condition of the proposition (5) so the relation (4.13) implies the vanishing of the above expression when it is acted on by the functional $\rho$, i.e.

$$
\frac{d}{ds} \rho \cdot \text{Ch}(A(s)) = 0.
$$

This completes the proof of the second part of the lemma

Since $e^{-K(s)}$ is a smoothing $b$-pseudodifferential operator, similar to (5.3) it is natural to define the $b$-Chern character of the superconnection $A(s)$ by the following formula with value in $C^\infty(M, A^{*} \nu^* \otimes \Omega^{1})$

$$
{^b \text{Ch}}(A(s)) := {^b \text{STr}} (e^{-K(s)}).
$$

The following lemma shows that the asymptotic behavior of all Chern characters defined so far are the same when $s$ goes to $\infty$. Moreover this lemma with the previous one provide a proof for the proposition [3]. The following relation is easy to verify by using the defect formula for $^b \text{Tr}$ (see remark [3] and will be used in the following discussion

$$
(5.12)
{^b \text{Ch}}(A(s)) - \text{Ch}(A(s)) = {^b \text{Tr}} [J^{-1}, J p_1 e^{-K(s)} p_1].
$$

Lemma 11. Under the conditions of the above proposition

1. One has

$$
\rho \circ {^b \text{Ch}}(A(s)) - \rho \circ \text{Ch}(A(s)) = B_1(s),
$$

where $B_1(s)$ is a boundary depending term going to zero when $s \to \infty$.

2. The following relation holds

$$
\rho \circ {^b \text{Ch}}(A(s)) = \rho \circ {^b \text{Ch}}(A_\nu) + B(s)
$$

where $B_2(s)$ is a boundary depending term going to zero when $s \to \infty$.

Proof. Using the relations (3.8) and (3.9), the operator $K(s)$ has the following expansion in the ungraded notation

$$
K(s) = \left( (\tilde{\nabla}^c_s)^2 - I_+ + s^2 D^- D^+ \right) - s D^+ \tilde{\nabla}^- + s \tilde{\nabla}^- D^+ - s^2 D^- D^+ + (\tilde{\nabla}^c_s)^2 - I_-
$$
Using the relations (5.6) we get
\[ K_{21}(s) = -s D^+(S_+ \nabla^F_+ S_+ + Q(Id + S_-)\nabla^F_- D^+) + s \nabla^F_- D^+ \]
\[ = -s D^+ S_+ \nabla^F_+ S_+ - s(Id - S_2^2)\nabla^F_- D^+ + s \nabla^F_- D^+ \]
\[ = -s D^+ S_+ \nabla^F_+ S_+ + sS_2^2 \nabla^F_- D^+ \]
In particular \( I(K(s)_{21}, \lambda) = 0 \), so
\[ I(e^{-K(s)}, \lambda)_{11} = e^{-I(K(s), \lambda)_{11}} = e^{-(\nabla^F_0)^2 l_+ - s^2(-i\lambda + D_0)(i\lambda + D_0)} \]
\[ = e^{-K^-_0(s) - s^2(\lambda^2 + D_0^2)} , \]
where \( K^-_0(s) := (\nabla^F_0)^2 - l_+ \). Now using (5.12) and the defect formula (4.12) we obtain
\[ ^1\text{Ch}(A(s)) - \text{Ch}(A(s)) = \frac{i}{2\pi} Tr[J^{-1}, Jp_1 e^{-K(s)} p_1] \]
\[ = \frac{i}{2\pi} \int e^{-K^-_0(0) - s^2\lambda^2 - s^2 \nabla^F_0(\lambda + D_0)^{-1} d\lambda \}
\[ = \frac{1}{2\pi} \int e^{-K^-_0(0) - s^2\lambda^2} (i\lambda + D_0)^{-1} d\lambda \}
\]
Denote the last expression by \( B(s) \) which is a boundary depending term. The self adjoint Dirac operator \( D_0 \) verifies the invertibility condition of hypothesis (4.12) So the \( \lambda \)-depending family of operators \( (i\lambda + D_0)^{-1} \) is uniformly bounded. This implies the finiteness of the integral factor of \( B(s) \). In the other hand, using the Volterra development, we have
\[ e^{-K^-_0(0) - s^2 \nabla^F_0} = e^{-s^2 \nabla^F_0} - \int_0^1 e^{-u_0 s^2 \nabla^F_0} e^{-s^2 \nabla^F_0} + \ldots \]
\[ + (-1)^k \int_{\triangle_k} e^{-u_0 s^2 \nabla^F_0} e^{-u_0 s^2 \nabla^F_0} e^{-u_1 s^2 \nabla^F_0} e^{-u_2 s^2 \nabla^F_0} \ldots e^{-u_k s^2 \nabla^F_0} d\sigma(u_0, u_1, \ldots, u_k) \]
where \( \triangle_k \) denotes the k-simplex \( \triangle_k = \{(u_0, u_1, \ldots, u_k) \in [0, 1]^k : \sum_{j=0}^k u_j = 1 \} \).
We recall from [24] relation 29 that the following estimate holds for each finite order \( G \)-invariant differential operator \( P \)
\[ |P e^{-s^2 \nabla^F_0^2} D_0^2)| (u, v)| \leq const.F_N(\text{R}(u, v)) \sum Q(s, s^{-1}, \delta)e^{-s^2 \delta^2} \]
Here \( Q \) is a polynomial of \( s \) and \( s^{-1} \) depending on the differential operator \( P \), while \( F_N(r) \) is a function of \( r \) which is \( O(r^{N+1}) \) as \( r \to +\infty \). Using this estimate, the limit of each term in above expression is zero when \( s \to \infty \). Since \( K^-_0(0) \) is a differential two-form, one has only a finite number of non zero terms in this expression, so the vanishing of all terms implies the vanishing of \( B(s) \) when \( s \) goes toward \( +\infty \). This completes the proof of the first part of the lemma. The second part can be proved by applying essentially the same computations used in the first part and in the previous lemma. \( \square \)

5.2. The eta invariant. Fix \( \epsilon > 0 \) sufficiently small, put \( N = \lfloor \frac{\epsilon}{4} \rfloor + 1 \) and \( \text{R}(u, v) = d(u, v) - \epsilon \) where \( (u, v) \in G^3 \times_s G^3 \). One has the following uniform estimate for this family which is a consequence of the finite propagation speed estimate explained in [24] relation 30. This estimate holds only when the hypotheses (1) and (2) are satisfied.
\[ |\nabla^F_0 e^{-s^2 \nabla^F_0^2} D_0^2)| (u, v)| \leq \]
\[ \text{const.}(R/s^2)^{-1/2}[R^{-2(k+l)} + R^{-2(k+l) - 4N} + \]
\[ R^{2(k+l)} s^{-4(k+l)} + R^{2(k+l) - 4N} s^{-4(k+l) - 8N}]e^{-R^2/4s^2}. \]
(5.14)
If in addition the boundary Dirac operator $D_0$ satisfies the hypothesis \( (2) \), and if \( 0 < \delta < \epsilon_0 \), then the following estimate holds for each finite order $G$-invariant differential operator $P$ \cite{24} relation 29

\[
(5.15) \quad |P_n e^{-s^2 D_0^2} D_0^2(u, v)| \leq \text{const} \cdot F_N(R(u, v)) \sum Q(s, s^{-1}, \delta) e^{-s^2 k^2}.
\]

Here $Q$ is a polynomial of $s$ and $s^{-1}$ depending on the differential operator $P$, while $F_N(r)$ is a function of $r$ which is $O(r^{n+1})$ as $r \to +\infty$.

We need also some estimates for the heat kernel when the parameter $s$ is small. Using \cite{24} relation 31 there is $s_0 > 0$ and $\epsilon > 0$ such that for $0 < s < s_0$ and $d(u, v) < \epsilon$

\[
(5.16) \quad |\nabla_o^s e^{-s^2 D_0} u, v)| \leq \text{const} \cdot s^{-(p+|\alpha|)} e^{-d^2(u, v) s^2}.
\]

Given the heat kernel of the longitudinal Dirac operator $D_0$, the heat kernel for the superconnection is given by the Volterra development

\[
(5.17) \quad e^{-s^2 D_0^2 - s F_0 - J_0} = e^{-s^2 D_0^2} + \sum_{k>0} (-s^2)^k I_k
\]

where

\[
(5.18) \quad s^{2k} I_k(u, u') = \int_{\Delta_k} \int_{G^k} e^{-s^2 t_0 D_0^2(u, u_0)} s^2 P_0 e^{-s^2 t_1 D_0^2(u_0, u_1)} s^2 P_0 \cdots e^{-s^2 t_{k-1} D_0^2(u_{k-1}, u')} dt.
\]

Here $s^2 P_0 = s F_0 + J_0$ and $\Delta_k$ denote the $k$-simplex

\[
\Delta_k = \{(t_0, t_1, \ldots, t_k) \in [0, 1]^k : \sum_{j=0}^k t_j = 1 \}.
\]

If $k > n - p$ then $I_k = 0$, so we have only to prove that for each $s > 0$ and for each $0 \leq k \leq n - p$ the integral (5.18) is convergent and defines a smooth section on $(0, +\infty) \times G^0 \times_s G^0$. For a fixed $k + 1$-tuple $(t_0, t_1, \ldots, t_k)$, the rapid decay property expressed in relation (5.14) implies the absolute convergence of the integration on $G^k$. In the other hand, for $s \neq 0$ at least one of the heat operator appearing in the integrand is smoothing (since $t_i \neq 0$ for some $0 \leq i \leq k$), so the integrand is smooth with respect to all its variables. Thus for $s \neq 0$, $I_k$ defines a smoothing operator which depends smoothly on $s$.

It is clear from above Volterra development that the heat kernel of the rescaled Cl(1)-superconnection $B_s$ is of the form given by relation (3.30). So we can apply the Cl(1)-supertrace (3.31) to define the (rescaled) Cl(1)-Chern character by the following relation (see \cite{27} pages 28,33 ]

\[
\text{Ch}_\alpha(B_s) := \text{STr}_\alpha(e^{-(B_s^2 - l)}).
\]

The operator $B_s^2 - l$ is even (i.e. grading preserving) on $C^\infty_c(G^0, \Lambda^k \ast \nu^* \otimes r^*(\Omega^1_+ \otimes (E_0 \oplus E_0)))$, therefore $\text{Ch}_\alpha(B_s)$ is a sum of differential forms of odd orders as we have explained just after the relation \cite{27} 31. The following proposition shows the relation between the Chern character of the Bismut superconnection and of the Cl(1)-superconnection (see \cite{27} Proposition 11)

**Proposition 12.** Let $\rho$ satisfy the condition described in the first part of the proposition \cite{8}. We have

\[
d\rho \circ \hat{\text{Ch}}(A_s) = -\rho \circ \text{STr} \{ \nabla^E, \frac{d}{ds} \hat{\text{Ch}}_r(e^{-(\hat{\text{Ch}}_r^2 - l)}) \} - \frac{1}{2} \rho \circ \hat{\eta}(s)
\]

where the rescaled eta form is defined by

\[
\hat{\eta}(s) = \frac{1}{\sqrt{\pi}} \text{STr}_\alpha(\frac{d}{ds} e^{-(B_s^2 - l)})
\]

**Proof.** Using Duhamel’s formula we get

\[
(5.19) \quad \frac{d}{ds} \rho \circ \text{Ch}_\alpha(A_s) = -\int_0^1 \rho \circ \text{STr} \{ e^{-\tau(\hat{\text{Ch}}_r^2 - l)} \frac{d}{ds} \hat{\text{Ch}}_r^2(l) \} e^{-(1-\tau)(\hat{\text{Ch}}_r^2 - l)} dt.
\]
\[ \partial_\lambda I_n(e^{-t(\mathcal{A}_s^2 - l)}, \lambda) \circ I_n \left( \frac{d(\mathcal{A}_s^2 - l)}{ds} e^{-(1-t)(\mathcal{A}_s^2 - l)}, \lambda \right), \]

and the defect formula of the proposition gives the following expression for (5.15):

\[ \frac{d}{ds} \rho \circ \text{Ch}(\mathcal{A}_s) = -\rho \circ \text{Str} \frac{d(\mathcal{A}_s^2 - l)}{ds} e^{-(\mathcal{A}_s^2 - l)}. \]

\( \mathcal{A}_s \) is an odd operator which commutes with \( e^{-(\mathcal{A}_s^2 - l)} \), so we get the following relations where \( \mathcal{A}_{[1]} \) is nothing else than \( \nabla^\epsilon \):

\[
\frac{d(\mathcal{A}_s^2 - l)}{ds} e^{-(\mathcal{A}_s^2 - l)} = \frac{d\mathcal{A}_s}{ds} e^{-(\mathcal{A}_s^2 - l)} \mathcal{A}_s + \frac{\partial}{\partial s} \mathcal{A}_s e^{-(\mathcal{A}_s^2 - l)} = [\mathcal{A}_s, \frac{d\mathcal{A}_s}{ds} e^{-(\mathcal{A}_s^2 - l)}]_{\text{gt}}
\]

(5.20)

Notice that the operators

\[ \mathcal{A}_s - \mathcal{A}_{[1]} \quad \frac{d\mathcal{A}_s}{ds} \quad (\mathcal{A}_s^2 - l) \]

are all \( G \)-invariant b-differential operators. Using again the relation (4.55) and the defect formula, the first term in the expression (5.20), followed by operator \( \rho \circ \text{Str} \), is equal to

\[
-\frac{i}{2\pi} \rho \circ \text{Str} \int_{\mathbb{R}} \partial_\lambda I(\mathcal{A}_s - \mathcal{A}_{[1]}, \lambda) I\left( \frac{d\mathcal{A}_s}{ds} e^{-(\mathcal{A}_s^2 - l)}, \lambda \right) d\lambda
\]

\[
= -\frac{i}{2\pi} \rho \circ \text{Str} \int_{\mathbb{R}} s \gamma. \frac{d\mathcal{B}_s}{ds} e^{-(\mathcal{B}_s^2 - l_0 + s^2 \lambda^2)} d\lambda
\]

(5.21)

The second equality is a consequence of the fact that the even part of \( \frac{d}{ds} I(\mathcal{A}_s, \lambda) \), with respect to \( \lambda \), is \( \frac{d}{ds} \), while the other terms are all odd functions of \( \lambda \). On the other hand

\[ \text{Str} \circ \gamma = tr \circ R \circ \gamma = -i \text{Str}_\alpha, \]

where \( R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). So \( \text{Str} \circ \gamma = -i \text{Str}_\alpha \) and the expression (5.21) is equal to

\[
-\frac{1}{2\sqrt{\pi}} \rho \circ \text{Str}_\alpha \frac{d\mathcal{B}_s}{ds} e^{-(\mathcal{B}_s^2 - l_0)}
\]

which is, by definition, equal to \( \frac{-1}{2} \rho(\hat{\gamma}(s)) \). \( \square \)

**Corollary 13.** Let \( \rho \) satisfies the conditions described in part (b) of the proposition (3) then

\[ \frac{d}{ds} \rho \circ \text{Ch}(\mathcal{A}_s) = -\frac{1}{2} \rho(\hat{\gamma}(s)). \]

To establish the index theorem we need to integrate the above \( s \)-depending eta form over \([0, +\infty)\), so the integrability has to be proved. To prove the next proposition we follow the proof of [24, Proposition 25].

**Proposition 14.** Assume that the \( G \)-invariant boundary Dirac operator \( D_0 \) satisfies the hypothesis (3) and that the holonomy groupoid \( G \) satisfies the polynomial growth conditions of hypothesis (4). Then the following integral, taking its value in \( C^\infty(\mathcal{M}, \Lambda^* \nu^*) \), is convergent.

\[
\eta_0 = \int_0^{+\infty} \hat{\gamma}(s) ds
\]

(5.22)
In the relation (5.16) take
\[ s \chi_s \] and prove that it is integrable over \([0, +\infty)\).

(5.24) \[ \chi_s \]

In particular, using the polar coordinates of \( R \), the above expression is integrable at +\( \infty \). So the hypothesis \([5.16] \) implies the finiteness of the first factor of the above expression. The second factor is trivially finite. Moreover it is dominated by \( e^{-s^2 \delta^2} \), so for each \( u \in G^0 \), the integration over this piece provides a function of \( s \) which is absolutely integrable at +\( \infty \).

Now let one of the \( t_j \)'s, say \( t_0 \), be less than or equal to \( s_0 \) for all \( j \). Using the estimate \([5.16] \) the above integral will be bounded from above by a constant multiple of

\[ \int R^{k+1} \int G_1^0 \int G_2^0 \int \chi(s^2 - \sum j t_j) (s_0 D_0 + \frac{1}{4\delta^2} \text{cl}(T_0^H)) e^{-t_0 D_0^k} (u, u_1) P^+ e^{-t_0 D_0^k} (u_1, u_2) P^+ e^{-t_0 D_0^k} (u_2, u_3) \cdots \cdots dt_0 \cdots dt_k. \]

For a fixed \( s > 0 \) take \( s^2 t_j \) as a new variable and denote it again by the symbol \( t_j \). We rewrite the above expression in the following form where \( \chi \) is the characteristic function of the set \( \{ 0 \} \subset \mathbb{R} \)

\[ \int R^{k+1} \int G_1^0 \int G_2^0 \int \chi(s^2 - \sum j t_j) (s_0 D_0 + \frac{1}{4\delta^2} \text{cl}(T_0^H)) e^{-t_0 D_0^k} (u, u_1) P^+ e^{-t_0 D_0^k} (u_1, u_2) P^+ e^{-t_0 D_0^k} (u_2, u_3) \cdots \cdots dt_0 \cdots dt_k. \]

In the relation \([5.16] \) take \( s_0 \leq \frac{1}{2\delta^2} \). Divide the domain of the integration into \( 2^{k+1} \) pieces such that on each one the value of \( t_j \) be less than or equal to, or greater than \( s_0 \). We begin by the piece on which \( t_j > s_0 \) for all \( j \). Using the estimate \([5.16] \) the above integral will be bounded from above by a constant multiple of

\[ \int R^{k+1} \int G_1^0 \int G_2^0 \int F_N(d(u_1, u_2)) F_N(d(u_1, u_3)) \cdots \cdots F_N(d(u_k, u)) du_1 \cdots dt_0 \cdots dt_k. \]

To obtain this expression we have used the fact that \( \text{cl}(T^H) \) commutes with \( D_0^k \) for putting together the operator \( 1/s^2 \text{cl}(T^H) \) and \( e^{-t_0 \delta^2} \) where \( t_j \geq s_0^2/k \). As before, the second factor is a uniformly bounded function of \( (u_1, u) \), while the third factor is bounded by a constant multiple of \( e^{-s^2 \delta^2}/2 \). For \( d(u, u_1) \leq 2\epsilon \), using \([5.16] \), the first factor can be bounded from above by \( (\epsilon 24 \text{ relation } 42) \)

\[ \text{const.} \int t_0^{p+1/2} e^{-\frac{d(u, u_1)^2}{t_0}} dt_0 \leq \text{const.} \frac{1}{(d(u, u_1))^{p-1}}. \]

In particular, using the polar coordinates of \( \mathbb{R}^p \), the left hand side of the above expression is integrable with respect to \( u_1 \). If \( d(u, u_1) \geq 2\epsilon \), using the estimate \([5.16] \), the expression \([5.25] \) is bounded by

\[ Q'(R, R^{-1}, t_0^{-1}) e^{-R^2/4t_0} \]

where \( Q' \) is a polynomial function of its variables. In particular this expression defines an integrable function of \( u_1 \) and goes to zero faster than any power in \( d(u, u_1) \) as \( d(u, u_1) \to 0 \). Therefore in
this case, for each \( u \in G^0 \), the expression (5.24) defines a differential form depending on \( s \) which is integrable at \( +\infty \). Clearly this argument can be applied when \( t_j \geq s_0 \) for some \( 0 \leq j \leq k \). But such a \( j \) always exists because \( \sum_j t_j = s^2 \) and \( s^2 \geq (k+1)s_0 \). This proves the integrability at \( +\infty \). The following lemma proves the convergence of the differential form (5.24) when \( s \to 0 \). This prove the integrability at 0 and completes the proof of the proposition. \( \square \)

**Lemma 15.** The eta form \( \eta(s) \) has a limit when \( s \to 0 \).

**Proof.** In what follows we use the notation and the content of the proof of the proposition \([1]\). At first we prove an asymptotic relation for the \( Cl(1) \)-supertrace of the \( Cl(1) \)-superconnection \( \mathcal{B} \). For this purpose consider the foliation \( (\partial M \times \mathbb{R}, \partial F \times \mathbb{R}) \) which is equipped with the longitudinal product metric \( g_0 + d^2t \), where \( t \) denotes the coordinate of the \( \mathbb{R} \)-factor. The connection \( T^h \), lifted along the \( R \)-factor, defines a connection for this foliation. The vector bundle \( E_{|\partial M} \simeq E_0 \oplus E_0 \) also can be lifted along \( \mathbb{R} \)-factor, and we denote the lifted bundle by the same symbol \( E_0 \oplus E_0 \). For \( j = 1, 2 \) let \( (\gamma_j, \delta_j) \) be two paths in a leaf of this foliation with common initial and end points. These paths are assumed to be equivalent if the holonomy of \( \gamma_1 \) and \( \gamma_2 \), with respect to the foliation \( (M, F) \), are equal. The set of all equivalence classes is a groupoid \( \mathcal{G} \) over the base manifold \( M \times \mathbb{R} \). We have for example \( \mathcal{G}_{(x, t)} = G^0_x \times \mathbb{R} \) and so on. Let \( \mathcal{A} \) denote the Bismut superconnection associated to the groupoid \( \mathcal{G} \) and let \( \mathcal{A}_s \) denote the associated rescaled operator. From the above construction it is clear that the groupoid \( \mathcal{G} \) and the operator \( \mathcal{A}_s \) are models for a collar neighborhood of \( G^0 \subset G \) and the restriction of \( \mathcal{A}_s \) to this neighborhood. As differential operators on \( C^\infty(\mathcal{G}, r^* (\Omega^{1/2} \otimes \Lambda^* \nu^* \otimes E)) \) we have

\[
\mathcal{A}_s^2 = \mathbb{R}_s^2 - s^2 \frac{\partial^2}{\partial t^2}.
\]

Because all involved operators in above commute with \( \partial / \partial t \) we have

\[
e^{-((\mathbb{B}_s^2 - 1)((u, t)), (u, t))} = e^{-((\mathbb{B}_s^2 - 1)(u, u))} e^{s^2(\partial_t)^2}(t, t) = r^* (e^{-\mathbb{B}_s^2}(y, y)) e^{s^2(\partial_t)^2}(t, t)
\]

Since the above expression is local when \( t \) goes toward 0, we can and will assume that \( (M \times \mathbb{R}, F \times \mathbb{R}) \) is a fibration and that \( E \) is the lifting of \( (S_0 \oplus S_0) \otimes V \) along \( \mathbb{R} \). Therefore using asymptotic relation (4.10) and the fact that \( e^{s^2(\partial_t)^2}(t, t) = (4 \pi s^2)^{-1/2} \sum_{j=0}^\infty s^{2j} \delta_j k_j(y) \)

Here \( \mathbb{B}_s^2(y, y) \), the heat kernel of \( \mathbb{B}_s^2 \), and \( k_j(y) \)'s are elements of \( \Omega^1 \otimes \Lambda^* \nu^* \otimes \text{End}(S_0 \oplus S_0) \otimes \text{End}(V_\nu) \). Moreover, using the definition (5.15) of \( \mathbb{B} \) and Duhamel’s formula, it turns out that the \( \text{End}(S_0 \oplus S_0) \otimes \text{End}(V_\nu) \)-factor has the following form

\[
(\text{Id} \ R + \alpha \ S) \otimes T = \left( \begin{array}{cc} R & S \\ S & R \end{array} \right) \otimes T
\]

where \( R, S \in \text{End}(S_0) \) and \( T \in \text{End}(V_\nu) \). By definition the \( Cl(1) \)-supertrace (denoted by \( \text{str} \alpha \)) of such an element is

\[
\text{str} \alpha ((\text{Id} \ R + \alpha \ S) \otimes T) := 2 \text{tr}(S) \cdot \text{tr}(T).
\]

With respect to the identification \( \text{End}(S_0 \oplus E_0) \simeq \Lambda^* T^*_y \partial F \otimes T^*_y \mathbb{R} \) the elements of the form \( \text{Id} \ R \) correspond to \( (\Lambda^* T^*_y \partial F) \wedge \partial_t \) while the elements of the form \( \alpha \ S \) correspond to \( \Lambda^* T^*_y \partial F \). Therefore the \( Cl(1) \)-supertrace of an element in \( \Lambda^* T^*_y \partial F \otimes T^*_y \mathbb{R} \) is non zero only if this element is the volume element of \( \Lambda^* T^*_y \partial F \) and in this case its \( Cl(1) \)-supertrace is equal to 1. Consequently, applying the \( \text{str} \alpha \) on both side of the above asymptotic relation and using the fact that the total degree of \( k_j(y) \) is a most 2j, we get the following asymptotic formula when \( s \) goes to 0

\[
(5.26) \quad \text{str} \alpha e^{-((\mathbb{B}_s^2 - 1))(u, u)} \sim r^* (B_0 + sB_1 + s^2 B_j + \ldots)
\]

Here \( B_j \)'s are elements in the finite dimensional linear space \( \Omega^1(T_y \partial F) \otimes \Lambda^* \nu^*_y \) and above asymptotic relation occurs in this space. Let \( \mathbb{R}^+ \) denote the positive real line and consider the foliation \( (\partial M \times \mathbb{R}^+, \partial F) \). Let \( t \) denote the parameter of \( \mathbb{R}^+ \), then \( \bar{g}(x, t) := t^{-1} g_0(x) \) define a longitudinal Riemannian metric for this foliated manifold where \( g_0 \) is the longitudinal Riemannian structure of \( (\partial M, \partial F) \). Moreover this foliation is...
equipped with the horizontal distribution $T^b\partial M$ and the lifted (along $\mathbb{R}^+$) Clifford bundle $E$. The holonomy groupoid associated to this foliation is $G \times \mathbb{R}^+$. The $\mathbb{C}l(1)$-superconnection $\mathcal{B}$ associated to this foliation satisfies the asymptotic formula \((5.26)\). Using this asymptotic formula instead of the Bismut local index formula in the proof of \cite[Theorem 10.32]{16}, we get the convergence of the eta form when $s$ goes toward $0$. \hfill $\Box$

The above proposition justifies the following definition

**Definition 3.** The eta invariant, depending on the boundary Dirac operator $D_0$, the boundary horizontal distribution $T^b_{\partial M}$ and the functional $\rho$ is defined by the following relation

$$\rho(\eta_0) = \rho \circ \int_0^{+\infty} \tilde{\eta}(s) \, ds$$

6. The index theorem

Now we can formulate the main theorem of the paper which is a direct conclusion of the relation \((3.32)\), the corollary \cite{13} and the propositions \cite{7} and \cite{9}.

**Theorem 16** \((b\text{-}index\text{ }theorem\text{ }for\text{ }G\text{-}invariant\text{ }Dirac\text{ }operators)\). Let $G$ denote the holonomy groupoid associated to the boundary foliated manifold $(M, F)$ satisfying the hypothesis \cite{11}. Assume that $\rho = \dim F$ is an even number and let $E$ be a longitudinal Clifford bundle over $(M, F)$ with the associated Dirac operator $\tilde{D}$. Let $D = r^*D$ be the $G$-invariant Dirac operator, acting on the smooth sections of the $G$-invariant vector bundle $r^*E \to G$. We assume that the boundary family satisfies the invertibility hypothesis \cite{2}. Let $\rho$ be a functional satisfying the conditions described in the proposition \cite{8}. Then the following index formula holds

$$\rho(\text{Ch}(\text{ind}(D))) = \frac{1}{(2\pi i)^{p/2}} \int_M \rho(x) \{ \tilde{A}(M, F) \text{Ch}(E/S)(x) \}^p - \frac{1}{2} \rho(\eta_0).$$

In this formula the Chern character of the analytical index $\text{Ch}(\text{ind}(D))$ is defined by \((5.2)\) while the eta form $\eta_0$ is given by \((5.22)\). The integrand is defined by relations \((4.6)\) and \((4.7)\), and the operation $\{\ldots\}^p$ is defined just before the proposition \cite{7}.

**Remarks:**

i) When $\partial M = \emptyset$ and under the polynomial growth condition of hypothesis \cite{11}, this index theorem implies the index theorem established by A.Gorokhovsky and J.Lott in \cite{16} Theorem 1. The extra coefficient $\frac{1}{(2\pi i)^{p/2}}$ appeared in our formula is a consequence of our definition for the Chern character (Compare the relation \((A.1)\) in the appendix with the relation \((34)\) of \cite{16}).

ii) Let $C^0$ be an invariant zero current, i.e. an holonomy invariant measure. Using this invariant measure, the relation \((5.29)\) defines a functional $\rho : C_0(M, \Omega^1) \to \mathbb{R}$ satisfying the conditions of the proposition \cite{8}. In this case, again with hypothesis \cite{11} the above theorem reduces to the A.Connes’ foliation index theorem \cite{13} when $\partial M = \emptyset$.

7. Appendix

In this appendix we describe briefly a general construction for the Chern character which is proposed by Gorokhovsky and Lott \cite{16}. This is a generalization of the classical Chern-Weil construction.

Let $(\mathcal{B}^*, d)$ be a $\mathbb{Z}$-graded algebra where $d$ is a linear operator of grading one and $d^2 \neq 0$. We denote the operator $d^2$ by $\alpha$. The algebra $\mathcal{B}$ can be viewed as a $\mathbb{Z}_2$-graded algebra with the even sub algebra $\mathcal{B}^e = \oplus \mathcal{B}^{2i}$ and the odd sub space $\mathcal{B}^o = \oplus \mathcal{B}^{2i+1}$. It is clear that $d$ is a grading reversing operator on $\mathcal{B}$. Let $E$ be a $\mathbb{Z}_2$-graded left $\mathcal{B}$-module equipped with a superconnection $\nabla$. This means that $\nabla : E \to E$ is an odd operator satisfying the following relation for all $\phi \in \mathcal{B}$ and $\xi \in E$

$$\nabla(\phi \xi) = d(\phi) \xi + \phi \nabla(\xi).$$

The algebra $\text{End}_{\mathcal{B}}(E)$ is naturally a $\mathbb{Z}_2$-graded algebra and is equipped with a canonical supertrace

$$\text{str} : \text{End}_{\mathcal{B}}(E) \to \mathcal{B}_{ab}$$

$$\text{str}(\xi^* \otimes \xi) = \xi^*(\xi) ; \xi^* \in E^* , \xi \in E$$

where $\mathcal{B}_{ab} := \mathcal{B}/[\mathcal{B}, \mathcal{B}].$
In the classical case, when $\alpha = 0$, one defines the Chern character of this superconnection by the following relation

\begin{equation}
\text{Ch}_p(\mathcal{E}, \nabla) = p \circ \text{str}(e^{-K}) \in A
\end{equation}

where $K := \nabla^2 \in \text{End}_B(\mathcal{E})$ denotes the curvature of the superconnection $\nabla$ and $p$ is an appropriate graded trace on $B$ which takes its values in an Abelian group $A$. A graded trace means a linear function which vanishes on the supercommutators as well as on the image of $\alpha$. When $\alpha \neq 0$ we assume the existence of an even linear map $l : \mathcal{E} \to \mathcal{E}$ satisfying the following relations

\begin{align}
l(\phi \xi) &= \alpha(\phi)\xi + \phi(l(\xi)) \\
l \circ \nabla &= \nabla \circ l
\end{align}

and define the curvature of the connection $\nabla$ to be $K = \nabla^2 - l$. The following lemma shows that $K$ shares the common properties with the classical curvature (for the proof see lemmas 2 and 7 of [16]).

**Lemma 17.**

a) $K : \mathcal{E} \to \mathcal{E}$ is $B$-linear.

b) $K$ is a flat $B$-linear map, i.e $[\nabla, K] = 0$.

We can now define the Chern character by the formula (A.1). Let $d^t$ denote the adjoint of $d$. Given a graded trace $p$, it is clear that $d^t(p) := p \circ d$ is a graded trace too, moreover $d^t \circ d^t = 0$. Thus the set of all graded traces form a complex with differential $d^t$. The trace $p$ is closed if $d^t(p) = 0$. For the proof of the following lemma we refer to [10] lemma 8.

**Lemma 18.**

a) For a fixed closed graded trace $p$, the Chern character defined in (A.1) is independent of the connection $\nabla$, so one can denote the Chern character by $\text{Ch}_p(\mathcal{E})$.

b) If $p_1$ is homologous with $p_2$ then $\text{Ch}_{p_1}(\mathcal{E}) = \text{Ch}_{p_2}(\mathcal{E})$.

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