Deformation of D-branes in three-dimensional anti-de Sitter black holes

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Abstract. This talk discusses the possible extension of the relation between string theory and noncommutative geometry to the case of BTZ black holes. After emphasizing some geometrical properties of these black holes, we show that the non-rotating massive BTZ black holes supports winding D1-brane configurations, which cross the black hole horizons and singularities. We then discuss strict deformation quantization of these D-branes worldvolumes.

1. Motivations and summary

The emergence of noncommutative geometry in string theory is yet a beautiful manifestation of the close interplay between mathematical and physical concepts. The idea that the spacetime coordinates do not commute is of course not new, and can be traced back to [28]. In the context of string theory, one can roughly express this appearance as follows. When considering open strings propagating in flat space-time, ending on a D-brane carrying a magnetic field, one observes that the open strings’ end behave like they were living in a noncommutative space, with noncommutativity parameter $\theta^{\mu\nu}$ related to the presence of the B-field. Alternatively, one may observe that the pointwise multiplication of functions on the brane worldvolume is deformed into a Moyal product [26, 27]. At the next level, it has been shown by Seiberg and Witten[27] that the entire string dynamics, in a certain regime, corresponding to the decoupling limit of the massive string modes, can be described in terms of a noncommutative Yang-Mills theory. Their analysis also led to state an equivalence, called the Seiberg-Witten map, between ordinary and noncommutative gauge fields.

A natural prolongation of these works would consists in extending this analysis to strings evolving no longer in flat space-time, but now in curved backgrounds. The Wess-Zumino-Witten (WZW) models, describing strings on group manifolds, give a variety of exact string backgrounds (and also their gauged versions and marginal deformations). Among these backgrounds, it is

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quite remarkable to notice that some of them describe *black holes*. We will focus on one of them, the so-called *BTZ black holes* [3, 2], which can be seen geometrically as quotients of *AdS*$_3$, and are therefore described by a SL(2, $\mathbb{R}$) WZW model [18]. It is interesting to remark that some BTZ black holes (in this talk we considered the non-rotating massive one) support D-brane configuration on which open strings could end, as we will show in Sect. 4. One may then wonder how their worldvolume geometry will be modified due to the presence of the WZW B-field, which has to supplement the BTZ gravitational background to make it exact.

This partly motivates the present contribution$^3$ in which we outline some aspects of our work on strict (in the sense of Rieffel) deformations of BTZ spaces, based on [7, 8, 10] (see Sect. 3). We show how to construct, on each D-brane worldvolume, an invariant deformation of the algebra of functions in the direction of the Poisson bracket associated with the B-field, and adapted to the symmetries of the (curved) D-branes in BTZ spaces. The non-vanishing curvature (actually, the fact that these D-branes admit an action of a non-abelian Lie group) is essentially the reason why the Moyal product would have to be abandoned. The construction highly relies on the geometric properties of the non-rotating BTZ black hole, which we present in Sect. 2.

To close the circle, one would now have to be able to see how these deformations emerge in studying strings on BTZ black holes. From a string theoretic point of view, the ability to solve the underlying conformal field theory is closely related to the geometry of the background. In order of increasing difficulty, we find for the $SU(2)$ WZW model (rational CFT, compact background, strings on $S^3$), the $H^+_3$ model (non-rational CFT, euclidian, non-compact background, strings on euclidian $AdS_3$), the $SL(2, R)$ WZW model (non-rational CFT, lorentzian, non-compact background, strings on euclidian AdS$_3$), and finally orbifolds of $SL(2, R)$ WZW model (strings on BTZ black holes). These models are understood at very different levels. The $SU(2)$ model has been completely solved. The $H^+_3$ model has been studied in details by Teschner [31, 30, 32]. A spectrum for the $SL(2, R)$ WZW model has been proposed in [21, 23, 22], and revealed a more intricate structure than its euclidian counterpart. This could of course have been expected since, even if the two backgrounds are related by a Wick rotation, the harmonic analysis and representation theory associated to these spaces are sensibly different. Finally, not much is unfortunately known about the spectrum of strings on BTZ black holes (see however [24, 25, 33, 16, 17, 15], and this of course constitutes the main obstacle to the derivation of the underlying noncommutative geometry.

### 2. Geometry of BTZ black holes

#### 2.1. General construction

Vacuum Einstein’s equations in 2+1 dimensions with negative cosmological constant admit black hole solutions, as first quoted in [12]. A remarkable property of these solutions lies in the fact that they arise from identifications of points of *AdS*$_3$ by a discrete subgroup of its isometry group, as shown by Bañados, Henneaux, Teitelboim and Zanelli [3, 2]. They are therefore usually referred to as *BTZ black holes* in the physics literature. According to the type of subgroup, the vacuum, extremal or generic (rotating or non-rotating) massive black holes are obtained. Here, we will essentially focus on the latter type of black hole.

The space $AdS_3$ is identified with the (universal covering of) the simple Lie group $SL(2, \mathbb{R})$ :

$$AdS_3 \cong SL(2, \mathbb{R}) = \{ g = \begin{pmatrix} u + x & y + t \\ y - t & u - x \end{pmatrix} | x, y, u, t \in \mathbb{R}, \ det \ g = 1 \} . \tag{1}$$

endowed with its bi-invariant Killing metric $\beta$. This metric is defined at the identity $e$ using the

$^3$ Partly, because the construction could also be used, e.g. in the definition of spectral triples for noncommutative, non-compact lorentzian manifolds
Killing form $B$ of the Lie algebra $sl(2, \mathbb{R})$, identified with the tangent space at the identity:

$$\beta_e(X, Y) = B(X, Y) = \frac{1}{2} \text{Tr}(X.Y), \; X, Y \in sl(2, \mathbb{R}),$$

(2)

The Lie algebra

$$sl(2, \mathbb{R}) = \left\{ \begin{pmatrix} z^H & z^E \\ z^F & -z^H \end{pmatrix} : z^H, z^E, z^F \right\},$$

(3)

is expressed in terms of the generators $\{H, E, F\}$ satisfying the commutation relations:

$$[H, E] = 2E, \; [H, F] = -2F, \; [E, F] = H.$$

(4)

For later purpose, we also define the one-parameter subgroups of $SL(2, \mathbb{R})$:

$$A = \exp(\mathbb{R}H), \; N = \exp(\mathbb{R}E), \; K = \exp(\mathbb{R}T),$$

(5)

which are the building blocks of its Iwasawa decomposition (see [4, 35]):

$$K \times A \times N \to SL(2, \mathbb{R}) : (k, a, n) \to kan \text{ or ank}, \; k \in K, \; a \in A, \; n \in N,$$

(6)

the mapping being an analytic diffeomorphism of the product manifold $K \times A \times N$ onto $SL(2, \mathbb{R})$ (see [35], p.234).

From the bi-invariance of the Killing metric, the isometry group of $AdS_3$ is locally isomorphic to $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. Any Killing vector of $AdS_3$ can thus be written in the form

$$\Xi = (X, Y), \; X, Y \in sl(2, \mathbb{R}),$$

(7)

and generates a one-parameter subgroup of isometries $\Psi_t$:

$$\Psi_t(g) = \exp(tX) g \exp(-tY), \; g \in SL(2, \mathbb{R}), t \in \mathbb{R}.$$

(8)

BTZ black holes are obtained as quotients of $AdS_3$ under actions of the form $\Psi_n, n \in \mathbb{Z}$, for well-chosen generators $X$ and $Y$ [3, 2, 7].

### 2.2. The non-rotating massive BTZ black hole

We will here concentrate on a particular black hole, the non-rotating massive one. In this case, we define the $BHTZ$ subgroup as the one-parameter subgroup of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ generated by the Killing vector $\Xi == (aH, aH), a \in \mathbb{R}_0$:

$$\psi_t(z) = \exp(taH) z \exp(-taH), \; z \in SL(2, \mathbb{R}).$$

(9)

A non-rotating massive BTZ black hole is then obtained as the quotient $SL(2, \mathbb{R})/\psi_Z$ by an isometric action of $Z$, that is, by performing the following identifications:

$$z \sim \exp(naH) z \exp(-naH), \; n \in \mathbb{Z}.$$

(10)

The resulting space-time displays a black hole structure [2], as we will shortly discover. In particular, it exhibits horizons. Note that, to avoid closed time-like curves in the quotient space, the regions of $AdS_3$ where the orbits of the BHTZ action (9) are time-like, or equivalently where $\beta_z(\Xi, \Xi) < 0$, must be excluded. We will henceforth consider an open and connected domain
\( U \subseteq AdS_3 \), called a safe region, where \( \beta_z(aH, aH) > 0, \forall z \in U \). The black holes singularities \( S \) are defined as the surfaces where the identifications becomes light-like:

\[
S = \{ z \in AdS_3 \mid \beta_z(\Xi, \Xi) = 0 \} \quad \text{(11)}
\]

These may be easily visualized in the conformal representation of \( AdS_3 \), where \( AdS_3 \) is identified with the three-dimensional Einstein static cylinder, see Fig.1.

It is possible to find a global description of the non-rotating massive black hole, in which the identifications (10) are easily implemented. The construction (see [7, 8, 6]) is based on a foliation of \( AdS_3 \) by two-dimensional leaves \( O_\rho \), parameterized by a real parameter \( \rho \), which are stable under the identifications (10) and all diffeomorphic, see Fig.2. This essentially reduces the study of the black hole from 3 to 2 dimensions.

**Figure 1.** A finite-time section of the conformal representation of \( AdS_3 \). In this figure, time flows upwards. The light-like surfaces correspond to the singularities, demarcating a diamond-shaped safe region.

The leaves of the foliation are actually twisted conjugacy classes, meaning that the leaf through a point \( g \in \text{SL}(2, \mathbb{R}) \) is

\[
O_g = \{ h g \sigma(h^{-1}) \mid h \in \text{SL}(2, \mathbb{R}) \} \quad \text{(12)}
\]

where \( \sigma \) is the external automorphism of \( \text{SL}(2, \mathbb{R}) \) corresponding to the Lie algebra automorphism (also denoted by \( \sigma \)) acting on \( H, E, F \) as

\[
\sigma(H) = H, \quad \sigma(E) = -E, \quad \sigma(F) = -F \quad \text{(13)}
\]

We may introduce the twisted action \( \tau \) of \( \text{SL}(2, \mathbb{R}) \) on itself:

\[
\tau : \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \to \text{SL}(2, \mathbb{R}) : (g, z) \to \tau_g(z) = g z \sigma(g^{-1}) \quad \text{(14)}
\]

The above foliation then relies on the observation [7, 10] that the application

\[
\phi : K \times A \times N \to \text{SL}(2, \mathbb{R}) : (k, a, n) \to \phi(k, a, n) = \tau_{kn}(a) \quad \text{(15)}
\]
is a global diffeomorphism, which can be seen as a \textit{twisted Iwasawa decomposition}. The decomposition (15) shows that the two-dimensional leaves of the foliation are identified with the submanifold $KN$, or equivalently to the homogeneous space $G/A \sim \text{AdS}_2$. Using the latter representation, it is possible to show ([7], see also [6]) that a global parametrization of a safe region (see Fig.1) is given by

$$g(\rho, \theta, \tau) = \tau \exp(\frac{\rho}{2} H) \exp(-\frac{\tau}{T})(\exp(\rho H)), \quad (\tau, \rho, \theta) \in \mathbb{R}^3. \quad (16)$$

The action of the BHTZ subgroup (9) reads in these coordinates

$$ (\tau, \rho, \theta) \rightarrow (\tau, \rho, \theta + 2na) . \quad (17)$$

By restricting the safe region to a fundamental domain of the BHTZ action, one gets a three-dimensional picture of the evolution of the black hole, from its initial to its final singularities $S_i$ and $S_f$, as shown in Fig.3. By drawing the black hole horizons, one gets a three-dimensional Penrose diagram (Fig. 5), while the usual Penrose diagram is recovered by taking a $\theta = \text{cst.}$ section (Fig.6).

![Figure 3](image1.png)

\textbf{Figure 3.} Dynamical evolution of the black hole, from its initial singularity $S_i$ to the final singularity $S_f$. The shaded region represents a timelike section of a fundamental domain of the BHTZ action.

2.3. \textit{Extensions of the non-rotating massive black hole}

If one forgets for a moment the requirement that no closed time-like curves may be present in the quotient space, there are two inequivalent possible extensions of the regular BTZ space-time preserving the separable Hausdorf manifold structure of the quotient. The situation, in each leaf, is analog to that in Misner space, obtained by performing discrete identifications in two-dimensional Minkowski space along boosts ([36], Sect.5.7, and [8] for the analogy). One of the extensions, represented in Fig.7, has an interesting geometric property. Indeed, points in this extension can be represented by

$$g(\rho, \phi, w) = \tau r(\phi, w) \exp(-\pi/4 T)(\exp(\rho H)), \quad r(\phi, w) = \exp(\phi H) \exp(w E) \in AN, \quad (w, \rho, \phi) \in \mathbb{R}^3. \quad (18)$$

![Figure 4](image2.png)

\textbf{Figure 4.} A $\rho = \text{cst}$ section of Fig.3, where the coordinate lines are represented. The orbits of the BHTZ subgroup correspond to $\tau = \text{cst.}$.
This shows that each leaf in this extension admits an action of the minimal parabolic (solvable) subgroup $AN \subset SL(2, \mathbb{R})$, expressed in terms of the twisted action introduced in eq.(14):

\[\tau : AN \times \mathcal{O}_\rho \rightarrow \mathcal{O}_\rho : (r, g) \rightarrow \tau_r(g).\] (19)

This is a left action, since $\tau_{r_2} \circ \tau_{r_1} = \tau_{r_2 r_1}$.

Remark that the $AN$-subgroup and its Cartan conjugated $A\bar{N}$ ($\bar{N} = \exp(\mathbb{R} F)$) are intrinsically related to the structure of the non-rotating massive BTZ black hole, through the two following propositions[10, 11]:

**Proposition 1.** In $G = AdS_3$, the non-rotating BTZ black hole singularities are given by a union of minimal parabolic subgroups of $G$:

\[S = Z(G)AN \cup Z(G)A\bar{N},\] (20)

where $Z(G) = \{ e, -e \}$ denotes the center of $G = SL(2, \mathbb{R})$.

**Proposition 2.** In $G = AdS_3$, the non-rotating BTZ black hole horizons correspond to a union of lateral classes of minimal parabolic subgroups of $G$:

\[\mathcal{H} = Z(G)ANJ \cup Z(G)A\bar{N}J,\] (21)

where $J = \exp(\frac{3\pi}{2} \mathbf{T}) \in K$ satisfies $J^2 = e$.

Moreover, higher-dimensional generalizations of the BTZ construction (i.e. black holes obtained as quotients of $AdS_l$, $l \geq 3$, see [14] and references therein), appear to be completely determined by minimal parabolic subgroups of the corresponding $AdS_l$ isometry group $SO(2, l-1)$ [11].

3. **Strict deformation quantization of extended BTZ spaces**

Roughly said, the concept of quantization consists in assigning a quantum system to a classical one. On the classical side, observables are identified with smooth real-valued functions on a Poisson manifold $M$ (the phase space), whose points are the states of the system. These observables form a commutative Poisson algebra. On the quantum side, the observables become
linear self-adjoint operators acting on a complex Hilbert space $\mathcal{H}$ whose rays represent the states. The algebra of operators is now non-commutative, and is endowed with an additional structure provided by the commutator. Furthermore, if $G$ is a symmetry group of the classical system, represented by an action of $G$ by symplectomorphisms on the manifold, then it is required that the observables form a unitary representation of $G \mathcal{H}$.

There is in general not a unique way to quantize a system. The formal deformation quantization approach, introduced in [5], consists in deforming the structure of the algebra of classical observables into a non-commutative algebra, to be able to read the composition of operators at the level of functions without reference to a particular Hilbert space representation. If $Q : \mathcal{A} \subseteq C^\infty(M) \to \text{End}(\mathcal{H}) : f \to Q_f$ denotes the "quantization map", the system will be quantized by deforming the pointwise product in $C^\infty(M)$ (or in a suitable subalgebra) to a family $\star_\lambda$ of products defined in such a way that $Q_{f \star_\lambda g} = Q_f \circ Q_g$. These products are required to be associative and to satisfy the following conditions:

1. $f \star_\lambda g \xrightarrow[\lambda \to 0]{} f g$
2. $\frac{f \star_\lambda g - g \star_\lambda f}{\lambda} \xrightarrow[\lambda \to 0]{} \{f, g\}$

where $\{,\}$ is the Poisson bracket on $M$. The ordinary product is said to be deformed in the direction of the Poisson bracket. This procedure produces in general a formal deformation, i.e. a map into the formal power series in the parameter $\lambda$:

$$\star_\lambda : \mathcal{A} \times \mathcal{A} \to \mathcal{A}[[\lambda]] : (f, g) \to f \star_\lambda g = \sum_{k=0}^{\infty} B_k(f, g) \lambda^k ,$$

where the $B_k$’s are bi-differential operators. Kontsevitch proved that formal deformations always exist, i.e. that any Poisson structure on a smooth manifold can be quantized, and gave an explicit formula allowing to compute the bi-differential operators [20].

A setting for deformation quantization which is compatible with $C^*$-algebras was developed by Rieffel [37] in the so-called strict deformation quantization.

A strict deformation quantization of $\mathcal{A} \subseteq C^\infty(M)$ (if $M$ is not compact, one usually takes $\mathcal{A}$ to be the algebra of complex-valued Schwartz functions on $M$) is obtained by assigning an associative product $\star_\lambda$, an involution $^*\lambda$ and a $C^*$-norm $||.||_\lambda$ on $\mathcal{A}$, for $\lambda \in I$ (some interval containing zero), which for $\lambda = 0$ are the original pointwise product, complex conjugation involution, and supremum norm, such that

1. $\forall f \in \mathcal{A}$, the function $\lambda \to ||f||_\lambda$ is continuous
2. $\forall f, g \in \mathcal{A}, ||\frac{f \star_\lambda g - g \star_\lambda f}{\lambda} - \{f, g\}||_\lambda$ converges to 0 for $\lambda \to 0$.

For our purposes, it will be sufficient to notice that the product of two functions is again a function, rather than a formal power series. We will not discuss the precise nature of the function algebra $\mathcal{A}_\lambda$ (for details, see [39, 8]). We will further be considering WKB quantizations in which the multiplication is defined by an oscillatory integral product formula of the type

$$f \star_\lambda g(x) = \int_{M \times M} a_\lambda(x, y, z) e^{iS(x, y, z)} f(y)g(z)dydz ,$$

where $a_\lambda$ and $S$ are called the amplitude and the phase of the WKB kernel.

We now sketch how to construct strict WKB quantizations of the extended BTZ space-time presented in the preceding section. We will proceed in two steps: first, we will show how to construct a deformation of a manifold admitting an action of a Lie group $G$, from the data of an invariant product on $G$. Then we will give an explicit formula for an $AN$-invariant WKB
oscillatory kernel on the group $AN$. Further details about the derivation, the function algebra $A_\lambda$, the geometrical interpretation of the phase and amplitude may be found in [39, 8, 10, 41].

First we follow [39, 38] and section 4 of [8].

Let $G$ be a Lie group. We define the left (resp. right) action, $L^*_g$ (resp. $R^*_g$), on Fun[$G, \mathbb{C}$] as

$$L^*_g[f](g) = f(hg) , \quad R^*_h[f](g) = f(gh) \quad \forall g, h \in G .$$

Suppose there exist a left action $\tau$ of $G$ on a manifold $M$, i.e. :

$$\tau : G \times M \to M : (g, x) \to \tau_g(x) , \quad \tau_g \circ \tau_h = \tau_{gh} .$$

This action on $M$ induces an action $\alpha$ on Fun[$M, \mathbb{C}$] :

$$\alpha : G \times \text{Fun}[M, \mathbb{C}] \to \text{Fun}[M, \mathbb{C}] : (g, u) \to \alpha_g[u] ,$$

with $\alpha_g[u](x) = u(\tau_{g^{-1}}(x))$, $\alpha_g \circ \alpha_h = \alpha_{gh}$, $\forall x \in M$.

For fixed $x \in M$, we may define a map $\tilde{\alpha}^x$ from Fun[$M, \mathbb{C}$] into Fun[$G, \mathbb{C}$] :

$$\text{Fun}[M, \mathbb{C}] \xrightarrow{\tilde{\alpha}^x} \text{Fun}[G, \mathbb{C}] : u \mapsto \tilde{\alpha}^x[u] \quad \forall g \in G : \tilde{\alpha}^x[u](g) = u(\tau_{g^{-1}}(x)) .$$

Let us also assume that on $G$ we have a left invariant star product, denoted by $\star_G$, i.e. a star product satisfying the relation

$$L^*_g[f_1 \star_G f_2] = L^*_g[f_1] \star_G L^*_g[f_2] .$$

From the left invariant star product on $G$, we induce a star product on $M$, denoted $\star_M$, defined as :

$$(u \star_M v)(x) := \left( \tilde{\alpha}^x[u] \right) \star_G \left( \tilde{\alpha}^x[v] \right) (e) ,$$

$e$ denoting the identity element of $G$. One can check that all the required properties are indeed satisfied by the newly defined product, thanks to the properties of $\star_G$.

As a second step, we observed in the preceding section that each leaf $\rho = \text{cst.}$, in the extended BTZ space-time, admits an action of the $AN$ group, see eq. (19). It turns out that an $AN$-invariant product of the form (23) can be constructed on $AN$. The phase and amplitude are given by

$$S(x, y, z) = \{ \sinh [(a_y - a_x) n_z + \sinh [(a_z - a_y)] n_x + \sinh [(a_x - a_z)] n_y] \} , \quad (31)$$

and

$$a_\lambda(x, y, z) = \frac{1}{(2 \pi \lambda)^2} \frac{\mathcal{P}(a_y - a_x)\mathcal{P}(a_x - a_z)}{\mathcal{P}(a_z - a_y)} \cosh(a_z - a_y) , \quad (32)$$

where $\mathcal{P}$ is any real even function satisfying $\mathcal{P}(0) = 1$, and with $x_i = (a_{x_i}, n_{x_i}) \in AN \cong \mathbb{R}^2$.

From these results, we are thus able to construct a strict WKB quantization of each leaf in the foliation (see [8] for explicit formulas and further discussions).
4. D-branes in BTZ black holes

4.1. D-branes in flat space in presence of a constant B-field

Let us first explore how noncommutativity shows up in string theory. Consider an open string propagating in a flat D-dimensional space-time $M^d$ with metric in presence of a constant $B_{\mu\nu}$. The embedding of the string worldsheet, is parameterized by a set of fields $X^\mu : \Sigma \to M^d$, where $\Sigma$ is the upper half plane (to which the strip can be mapped by a conformal transformation). Using complex coordinates $z$ and $\bar{z}$ on the (euclidian) string worldsheet, we find that the string equations of motion are

$$\partial \bar{\partial} X^\mu(z, \bar{z}) = 0 \quad .$$

(33)

Variation of the open string action also imposes boundary conditions to be satisfied at the boundary of the string worldsheet. Dirichlet boundary conditions state that

$$\partial X^i = - \bar{\partial} X^i \quad \text{at} \quad \partial \Sigma \Leftrightarrow X^i = X_0^i \quad , \quad i = p + 1, \cdots, d - 1 \quad ,$$

(34)

that is, the endpoints of the string are restricted to move on a $p$-dimensional hyper-plane in $M^d$. This hyper-plane is called a $Dp$-brane. Let us focus on the directions along the brane. The boundary conditions then correspond to generalized Neumann BC, which are

$$\partial - \bar{\partial} X^\mu + 2\pi i a' B_{\mu\nu} X^\nu = 0 \quad , \quad \mu = 0, \cdots, p \quad \text{at} \quad \partial \Sigma.$$  

(35)

One would now like to determine what kind of field theory reproduces the low energy limit of the interactions of open string modes, by considering only the massless modes of the spectrum (i.e. by sending the string tension to infinity). This turns out to be a noncommutative Yang-Mills theory. Furthermore, it is shown in [27] that the open strings feel that they are living on a non-commutative space whose parameters depend upon the magnetic field present on the brane:

$$[X^\mu(\tau), X^\nu(\tau)] = i \Theta^{\mu\nu} \quad ,$$

(36)

with $\Theta^{\mu\nu} = \left( \frac{1}{1 + B} \right)^{\mu\nu}$, and $\tau = \text{Re}(z) = \text{Re}(\bar{z})$ parameterizes the boundary of the worldsheet. The corresponding product deforming the pointwise multiplication in the flat space whose coordinates satisfy (36), in the sense that $f \star g = f(g + \frac{i}{2} \Theta^{\mu\nu} \partial_\mu f \partial_\nu g + O(\Theta^2))$, appears to be the usual Moyal product. Note that the Moyal product admits the usual asymptotic expansion

$$(f \star g)(x) = e^{i \frac{\Theta^{\mu\nu} \partial_\mu \partial_\nu}{2}} f(x + \xi) g(x + \eta)|_{\xi = \eta = 0} \quad ,$$

(37)

the deformation parameter being included in $\Theta^{\mu\nu}$. But the Moyal product also enjoys the property to be strict, in the sense above, meaning that in a suitable functional framework, the product of two functions is again a function. Moreover, the Moyal product admits an integral form, sometimes referred to as the Weyl product, where it appears in the WKB form (23) with unit amplitude and phase given by $S^0(x, y, z) := \omega^0(x, y) + \omega^0(y, z) + \omega^0(z, x)$, where $\omega^0$ is the two-form associated with the bivector $\Theta^{\mu\nu} \partial_\mu \wedge \partial_\nu$ (see [39]).

4.2. D-branes in non-rotating massive BTZ black holes

An additional remarkable feature of BTZ black holes lies in the fact that they constitute exact string theory backgrounds, through the SL(2, $\mathbb{R}$) Wess-Zumino-Witten (WZW) model[18]. The latter models describe strings propagating on group manifolds in presence of a particular background B-field, and give an important class of exact string backgrounds, which are not yet understood in their generality. The action is expressed in terms of a map $g : \Sigma \to G$ expressing the embedding of the string worldsheet into the group manifold. The equations of motion state that

$$\partial \bar{J} = 0 \quad , \quad \bar{\partial} J = 0 \quad ,$$

(38)
where
\[ J = g^{-1} \partial g \quad \text{and} \quad \bar{J} = -\partial gg^{-1}. \]  
(39)

These are the counterparts of (33). Similarly, a particular class of D-branes in WZW models is obtained by imposing *gluing conditions* on the chiral currents:
\[ J = R\bar{J}, \quad \text{at} \quad \partial \Sigma. \]  
(40)

where \( R \) is a metric preserving Lie algebra automorphism. These D-branes are called *symmetric*, because they describe configurations preserving the maximal amount of symmetry of the bulk theory, that is, conformal invariance and half of the current algebra \([29]\). Their geometry in the group manifold is completely encoded in the Lie algebra automorphism \( R \). D-branes corresponding to (40) can be identified to regular \((R = Id)\), translated \((R = \text{inner automorphism})\) or twisted \((R = \text{outer automorphism})\) conjugacy classes in \( G \). Symmetric D-branes of the \( Sl(2, \mathbb{R}) \) WZW model are of three types: two-dimensional hyperbolic planes \((H_2)\), de Sitter branes \((dS_2)\) and anti-de Sitter branes \((AdS_2)\). It was shown in \([1]\), that the \( AdS_2 \) worldvolumes, corresponding to twisted conjugacy classes, are the only physically relevant classical configurations.

We are now ready to make the link with the geometry of the non-rotating BTZ black-hole. We saw in the first section that the spinless BTZ black hole admits a foliation by leaves, the \( \rho = \text{cst.} \) surfaces, which are stable under the action of the BHTZ subgroup and constitutes twisted conjugacy classes in \( Sl(2, \mathbb{R}) \) \((AdS_2 \text{ spaces})\). From our previous discussion, these solutions correspond to projections of twisted \( AdS_3 \) conjugacy classes that wrap around BTZ space, and they may be interpreted as closed D1-branes in BTZ space. These branes are represented in Fig. 8. It is interesting to note that these D-branes seem not to worry about the presence of the singularity, nor about the fact that they can evolve in a region with closed time-like curves (behind the singularity). It can indeed be checked that the Dirac-Born-Infeld (DBI) action, governing the semi-classical motion of these branes is proportional to \( \cosh \rho_0 \), and so...
is constant everywhere for a given D1-brane $\rho = \rho_0$. This phenomenon has actually already been observed in another black hole solution in string theory, namely the two-dimensional black hole, described by a $\text{SL}(2, \mathbb{R})/U(1)$ gauged WZW model [34]. This space-time supports D0-brane configurations with the same peculiar properties. These two observations are in fact not disconnected: the gauged and ungauged $\text{SL}(2, \mathbb{R})$ WZW models are linked by a continuous line of marginal deformations driven by integrable marginal operators [19, 13]. The D-branes discussed here simply survive along the whole deformation line [40].

Notice that the construction presented here can also be performed in the case of the rotating black hole[9]. The foliation leading to a global description of the black hole is, however, sensibly different from that of the non-rotating case (Sect. 2.2). The two-dimensional leaves still admit an action of an $\text{AN}$ subgroup of $\text{SL}(2, \mathbb{R})$, but are no longer (twisted) conjugacy classes and display a Misner-space causal structure. Analysis of the DBI action for these surfaces reveal that they actually solve the equations of motion, but with an imaginary worldvolume electric field (similar to what happens for $dS_2$ branes in $AdS_3$ [1]).

Finally, let us comment on the relations between the two last sections. By pushing the analogy with the flat case, one would naturally expect that the D-branes worldvolume in BTZ space become noncommutative. The deformation would, in some sense, have to respect the symmetries acting on the branes, so that the Moyal product would be ruled out. Also, the magnetic field constituting part of the WZW backgrounds is $B \sim F(\rho)dw \wedge d\phi$, in the coordinates (18). It corresponds to the left-invariant Poisson bracket in the direction of which we have been able to deform the algebra of function on the leaves $\rho = \text{cst}$.

It would of course be interesting to analyze how noncommutative geometry emerges from string theory in this curved and non-compact situation, and if this would be related to the tools we developed in the preceding section. This clearly requires a deeper understanding of string theory on BTZ black hole backgrounds, whose difficulty is rooted in the non-compact and lorentzian nature of the underlying conformal field theory.

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