Sampling Theorem Associated with $q$-Dirac System

Fatma Hıra
Hitit University, Arts and Science Faculty, Department of Mathematics, 19030, Çorum, Turkey
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Abstract

This paper deals with $q$-analogue of sampling theory associated with $q$-Dirac system. We derive sampling representation for transform whose kernel is a solution of this $q$-Dirac system. As a special case, three examples are given.

1 Introduction

Consider the following $q$-Dirac system

\[
\begin{aligned}
-D_q y_2 + p(x) y_1 &= \lambda y_1, \\
D_q y_1 + r(x) y_2 &= \lambda y_2, \\
k_{11} y_1 (0) + k_{12} y_2 (0) &= 0, \\
k_{21} y_1 (a) + k_{22} y_2 (a q^{-1}) &= 0,
\end{aligned}
\]

where $k_{ij}$ $(i, j = 1, 2)$ are real numbers, $\lambda$ is a complex eigenvalue parameter, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $p(x)$ and $r(x)$ are real-valued functions defined on $[0, a]$ and continuous at zero and $p(x), r(x) \in L_q^1(0, a)$ (see [1, 2]).

The papers in $q$-Dirac system are few, see [1 – 3]. However sampling theories associated with $q$-Dirac system do not exist as far as we know. So that we will construct a $q$-analogue of sampling theorem for $q$-Dirac system (1.1)-(1.3), building on recent results in [1, 2]. To achieve our aim we will briefly give the spectral analysis of the problem (1.1)-(1.3). Then we derive sampling theorem using solution. In the last section we give three examples illustrating the obtained results.
2 Notations and Preliminaries

We state the $q$–notations and results which will be needed for the derivation of the sampling theorem. Throughout this paper $q$ is a positive number with $0 < q < 1$.

A set $A \subseteq \mathbb{R}$ is called $q$-geometric if, for every $x \in A$, $qx \in A$. Let $f$ be a real or complex-valued function defined on a $q$-geometric set $A$. The $q$-difference operator is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{x(1-q)}, \quad x \neq 0. \quad (2.1)$$

If $0 \in A$, the $q$-derivative at zero is defined to be

$$D_q f(0) := \lim_{n \to \infty} \frac{f(xq^n) - f(0)}{xq^n}, \quad x \in A, \quad (2.2)$$

if the limit exists and does not depend on $x$. Also, for $x \in A$, $D_{q^{-1}}$ is defined to be

$$D_{q^{-1}} f(x) := \begin{cases} 
\frac{f(x) - f(q^{-1}x)}{x(1-q^{-1})}, & x \in A \setminus \{0\}, \\
D_q f(0), & x = 0,
\end{cases} \quad (2.3)$$

provided that $D_q f(0)$ exists. The following relation can be verified directly from the definition

$$D_{q^{-1}} f(x) = (D_q f)(xq^{-1}). \quad (2.4)$$

A right inverse, $q$-integration, of the $q$-difference operator $D_q$ is defined by Jackson [4] as

$$\int_0^x f(t) d_q t := x(1-q) \sum_{n=0}^{\infty} q^n f(xq^n), \quad x \in A, \quad (2.5)$$

provided that the series converges. A $q$-analog of the fundamental theorem of calculus is given by

$$D_q \int_0^x f(t) d_q t = f(x), \quad \int_0^x D_q f(t) d_q t = f(x) - \lim_{n \to \infty} f(xq^n), \quad (2.6)$$

where $\lim_{n \to \infty} f(xq^n)$ can be replaced by $f(0)$ if $f$ is $q$-regular at zero, that is, if $\lim_{n \to \infty} f(xq^n) = f(0)$, for all $x \in A$. Throughout this paper, we deal only with functions $q$-regular at zero.

The $q$-type product formula is given by

$$D_q (fg)(x) = g(x) D_q f(x) + f(qx) D_q g(x), \quad (2.7)$$

and hence the $q$-integration by parts is given by

$$\int_0^a g(x) D_q f(x) d_q x = (fg)(a) - (fg)(0) - \int_0^a D_q g(x) f(qx) d_q x, \quad (2.8)$$
where \( f \) and \( g \) are \( q \)-regular at zero.

For more results and properties in \( q \)-calculus, readers are referred to the recent works [5–8].

The basic trigonometric functions \( \cos (z; q) \) and \( \sin (z; q) \) are defined on \( \mathbb{C} \) by

\[
\cos (z; q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (z (1-q))^{2n}}{(q; q)_{2n}},
\]

\[
\sin (z; q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} (z (1-q))^{2n+1}}{(q; q)_{2n+1}},
\]

and they are \( q \)-analogs of the cosine and sine functions. \( \cos (., q) \) and \( \sin (., q) \) have only real and simple zeros \( \{ \pm x_m \}_{m=1}^{\infty} \) and \( \{ 0, \pm y_m \}_{m=1}^{\infty} \), respectively, where

\[
x_m = (1-q)^{-1} q^{-m+1/2} \epsilon_m^{(1/2)} \quad \text{if} \quad q^3 < (1-q^2)^2, \quad (2.11)
\]

\[
y_m = (1-q)^{-1} q^{-m+\epsilon_m} (-1/2) \quad \text{if} \quad q < (1-q^2)^2. \quad (2.12)
\]

Moreover, for any \( q \in (0,1) \), (2.11) and (2.12) hold for sufficiently large \( m \), cf. [5, 9–11].

Let \( L^2_q (0, a) \) be the space of all complex valued functions defined on \( [0, a] \) such that

\[
\| f \| := \left( \int_0^a |f(x)|^2 \, dq \, x \right)^{1/2} < \infty. \quad (2.13)
\]

The space \( L^2_q (0, a) \) is a separable Hilbert space with the inner product (see [12])

\[
\langle f, g \rangle := \int_0^a f(x) \overline{g(x)} \, dq \, x, \quad f, g \in L^2_q (0, a). \quad (2.14)
\]

Let \( H_q \) be the Hilbert space

\[
H_q := \left\{ y (x) = \begin{pmatrix} y_1 (x) \\ y_2 (x) \end{pmatrix}, \quad y_1 (x), y_2 (x) \in L^2_q (0, a) \right\}.
\]

The inner product of \( H_q \) is defined by

\[
\langle y (.), z (.) \rangle_{H_q} := \int_0^a y^\top (x) z (x) \, dq \, x, \quad (2.15)
\]

where \( \top \) denotes the matrix transpose, \( y (x) = \begin{pmatrix} y_1 (x) \\ y_2 (x) \end{pmatrix} \), \( z (x) = \begin{pmatrix} z_1 (x) \\ z_2 (x) \end{pmatrix} \) \( \in \) \( H_q \), \( y_i (.) \), \( z_i (.) \) \( \in \) \( L^2_q (0, a) \) \( (i = 1, 2) \).
It is known [2] that the problem (1.1)-(1.3) has a countable number of eigenvalues \( \{ \lambda_n \}_{n=-\infty}^{\infty} \) which are real and simple, and to every eigenvalue \( \lambda_n \), there corresponds a vector-valued eigenfunction \( y_n^\top(x, \lambda_n) = (y_{n,1}(x, \lambda_n), y_{n,2}(x, \lambda_n)) \). Moreover, vector-valued eigenfunctions belonging to different eigenvalues are orthogonal, i.e.,

\[
\int_0^a y_n^\top(x, \lambda_n) y_m(x, \lambda_m) d_q x = \int_0^a \{ y_{n,1}(x, \lambda_n) y_{m,1}(x, \lambda_m) + y_{n,2}(x, \lambda_n) y_{m,2}(x, \lambda_m) \} d_q x = 0, \quad \text{for } \lambda_n \neq \lambda_m.
\]

Let \( y_1(x, \lambda_1) = \begin{pmatrix} y_{11}(x, \lambda_1) \\ y_{12}(x, \lambda_1) \end{pmatrix} \) and \( y_2(x, \lambda_2) = \begin{pmatrix} y_{21}(x, \lambda_2) \\ y_{22}(x, \lambda_2) \end{pmatrix} \) be solutions of (1.1): hence

\[
\begin{align*}
-\frac{1}{q^2} D_{q^{-1}} y_{12} + \{ p(x) - \lambda_1 \} y_{11} &= 0, \\
D_q y_{11} + \{ r(x) - \lambda_1 \} y_{12} &= 0,
\end{align*}
\]

and

\[
\begin{align*}
-\frac{1}{q^2} D_{q^{-1}} y_{22} + \{ p(x) - \lambda_2 \} y_{21} &= 0, \\
D_q y_{21} + \{ r(x) - \lambda_2 \} y_{22} &= 0.
\end{align*}
\]

Multiplying (2.16) by \( y_{21} \) and \( y_{22} \) and (2.17) by \(-y_{11}\) and \(-y_{22}\) respectively, and adding them together also using the formula (2.4) we obtain

\[
D_q \{ y_{11}(x, \lambda_1) y_{22}(x q^{-1}, \lambda_2) - y_{12}(x q^{-1}, \lambda_1) y_{21}(x, \lambda_2) \} = (\lambda_1 - \lambda_2) \{ y_{11}(x, \lambda_1) y_{21}(x, \lambda_2) + y_{12}(x, \lambda_1) y_{22}(x, \lambda_2) \}.
\]

Let \( y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \), \( z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \) \( \in H_q \). Then the Wronskian of \( y(x) \) and \( z(x) \) is defined by

\[
W(y, z)(x) := y_1(x) z_2(x q^{-1}) - y_2(x) z_1(x q^{-1}).
\]

Let us consider the next initial value problem

\[
\begin{align*}
-\frac{1}{q^2} D_{q^{-1}} y_2 + p(x) y_1 &= \lambda y_1, \\
D_q y_1 + r(x) y_2 &= \lambda y_2,
\end{align*}
\]

\( y_1(0) = k_{12}, \quad y_2(0) = -k_{11}. \)

By virtue of Theorem 1 in [1], this problem has a unique solution \( \phi(x, \lambda) = \begin{pmatrix} \phi_1(x, \lambda) \\ \phi_2(x, \lambda) \end{pmatrix} \). It is obvious that \( \phi(x, \lambda) \) satisfies the boundary condition (1.2) and this function is uniformly bounded on the subsets of the form \([0, a] \times \Omega\) where \( \Omega \subset \mathbb{C} \) is compact. The proof is similar to the one in the proof of Lemma 3.1 in [13]. To find the eigenvalues of the \( q \)-Dirac system (1.1)-(1.3) we have to insert this function into the boundary condition (1.3) and find the roots.
of the obtained equation. So, putting the function \( \phi(x, \lambda) \) into the boundary condition (1.3) we get the following equation whose zeros are the eigenvalues of the \( q \)-Dirac system (1.1)-(1.3)

\[
\omega(\lambda) = k_{21} \phi_1(a, \lambda) + k_{22} \phi_2(aq^{-1}, \lambda).
\]

(2.22)

It is also known that if \( \{ \phi_n(\cdot) \}_{n=-\infty}^{\infty} \) denotes a set of vector-valued eigenfunctions corresponding \( \{ \lambda_n \}_{n=-\infty}^{\infty} \), then \( \{ \phi_n(\cdot) \}_{n=-\infty}^{\infty} \) is a complete orthogonal set of \( H_q \). For more details about how to obtain the solutions and the eigenvalues for \( q \)-Dirac system see [1, 2], similar to the classical case of Dirac system [14] and \( q \)-Sturm-Liouville problems [15, 16].

3 The Sampling Theory

The WKS (Whittaker-Kotel’nikov-Shannon) [17–19] sampling theorem has been generalized in many different ways. The connection between the WKS sampling theorem and boundary value problems was first observed by Weiss [20] and followed by Kramer [21]. In [22], sampling theorem is introduced where sampling representations are derived for integral transforms whose kernels are solutions of one-dimensional regular Dirac systems. In recent years, the connection between sampling theorems and \( q \)-boundary value problems has been the focus of many research papers. In [12, 23], \( q \)-versions of the classical sampling theorem of WKS as well as Kramer’s analytic theorem were introduced. These results were extended to \( q \)-Sturm-Liouville problems in [13, 24], singular \( q \)-Sturm-Liouville problem in [25] and the \( q, \omega \)-Hahn-Sturm-Liouville problem in [26].

In this section, we state and prove \( q \)-analogue of sampling theorem associated with \( q \)-Dirac system (1.1)-(1.3), inspired by the classical case [22].

**Theorem 1** Let \( f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \in H_q \) and \( F(\lambda) \) be the \( q \)-type transform

\[
F(\lambda) = \int_0^a f^T(x) \phi(x, \lambda) d_q x, \hspace{1cm} \lambda \in \mathbb{C}, \tag{3.1}
\]

where \( \phi(x, \lambda) \) is the solution defined above. Then \( F(\lambda) \) is an entire function that can be reconstructed using its values at the points \( \{ \lambda_n \}_{n=-\infty}^{\infty} \) by means of the sampling form

\[
F(\lambda) = \sum_{n=-\infty}^{\infty} F(\lambda_n) \frac{\omega(\lambda)}{(\lambda - \lambda_n) \omega'(\lambda_n)}, \tag{3.2}
\]

where \( \omega(\lambda) \) is defined in (2.22). The series (3.2) converges absolutely on \( \mathbb{C} \) and uniformly on compact subsets of \( \mathbb{C} \).
Proof. Since $\phi(x, \lambda)$ is in $H_q$ for any $\lambda$, we have

$$\phi(x, \lambda) = \sum_{n=-\infty}^{\infty} \hat{\phi}_n \frac{\phi_n(x)}{\|\phi_n\|_{H_q}},$$  

(3.3)

where

$$\hat{\phi}_n = \int_0^a \phi^T(x, \lambda) \phi_n(x) d_q x$$  

$$= \int_0 \left\{ \phi_1(x, \lambda) \phi_{n,1}(x) + \phi_2(x, \lambda) \phi_{n,2}(x) \right\} d_q x,$$  

(3.4)

$\phi^T(x, \lambda) = (\phi_1(x, \lambda), \phi_2(x, \lambda))$ and $\phi_n^T(x) = (\phi_{n,1}(x), \phi_{n,2}(x))$ is the vector-valued eigenfunction corresponding to the eigenvalue $\lambda_n$.

Since $f$ is in $H_q$, it has the Fourier expansion

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n \frac{\phi_n(x)}{\|\phi_n\|_{H_q}}.$$  

(3.5)

where

$$\hat{f}_n = \int_0^a f^T(x) \phi_n(x) d_q x$$  

$$= \int_0 \left\{ f_1(x) \phi_{n,1}(x) + f_2(x) \phi_{n,2}(x) \right\} d_q x.$$  

(3.6)

In view of Parseval’s relation and definition (3.1), we obtain

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(\lambda_n) \frac{\hat{\phi}_n^2}{\|\phi_n\|_{H_q}^2}.$$  

(3.7)

Let $\lambda \in \mathbb{C}, \lambda \neq \lambda_n$ and $n \in \mathbb{N}$ be fixed. From relation (2.18), with $y_{11}(x) = \phi_1(x, \lambda), y_{12}(x) = \phi_2(x, \lambda)$ and $y_{21}(x) = \phi_{n,1}(x), y_{22}(x) = \phi_{n,2}(x)$, we obtain

$$(\lambda - \lambda_n) \int_0^a \left\{ \phi_1(x, \lambda) \phi_{n,1}(x) + \phi_2(x, \lambda) \phi_{n,2}(x) \right\} d_q x$$  

$$= W(\phi(., \lambda), \phi_n(\cdot))|_{x=a} - W(\phi(., \lambda), \phi_n(\cdot))|_{x=0}.$$  

(3.8)

From (2.19) and the definition of $\phi(., \lambda)$, we have

$$(\lambda - \lambda_n) \int_0^a \left\{ \phi_1(x, \lambda) \phi_{n,1}(x) + \phi_2(x, \lambda) \phi_{n,2}(x) \right\} d_q x$$  

$$= \phi_1(a, \lambda) \phi_{n,2}(aq^{-1}) - \phi_{n,1}(a) \phi_2(aq^{-1}, \lambda).$$  

(3.9)

Assume that $k_{22} \neq 0$. Since $\phi_n(.)$ is an eigenfunction, then it satisfies (1.3). Hence

$$\phi_{n,2}(aq^{-1}) = \frac{k_{21}}{k_{22}} \phi_{n,1}(a).$$  

(3.10)
Substituting from (3.10) in (3.9), we obtain
\[(\lambda - \lambda_n) \int_0^a \left\{ \phi_1(x, \lambda) \phi_{n,1}(x) + \phi_2(x, \lambda) \phi_{n,2}(x) \right\} dq \, dx \]
\[= -\phi_{n,1}(a) \left\{ \frac{k_{21}}{k_{22}} \phi_1(a, \lambda) + \phi_2(aq^{-1}, \lambda) \right\} \]
\[= -\frac{\omega(\lambda) \phi_{n,1}(a)}{k_{22}} \]  
(provided that \(k_{22} \neq 0\)). Similarly, we can show that
\[(\lambda - \lambda_n) \int_0^a \left\{ \phi_1(x, \lambda) \phi_{n,1}(x) + \phi_2(x, \lambda) \phi_{n,2}(x) \right\} dq \, dx \]
\[= -\frac{\omega(\lambda) \phi_{n,2}(aq^{-1})}{k_{21}} \]  
(provided that \(k_{21} \neq 0\)). Differentiating with respect to \(\lambda\) and taking the limit as \(\lambda \to \lambda_n\), we obtain
\[\|\phi_n\|^2_{H_q} = \int_0^a \phi_n^\top(x) \phi_n(x) \, dq \, dx \]
\[= -\frac{\omega'(\lambda_n) \phi_{n,1}(a)}{k_{22}}, \quad (3.13)\]
\[= -\frac{\omega'(\lambda_n) \phi_{n,2}(aq^{-1})}{k_{21}}. \quad (3.14)\]
From (3.4), (3.11) and (3.13), we have for \(k_{22} \neq 0\),
\[\frac{\hat{\phi}_n}{\|\phi_n\|^2_{H_q}} = \frac{\omega(\lambda)}{(\lambda - \lambda_n) \omega'(\lambda_n)}, \quad (3.15)\]
and if \(k_{21} \neq 0\), we use (3.4), (3.12) and (3.14) to obtain the same result. Therefore from (3.7) and (3.15) we get (3.2) when \(\lambda\) is not an eigenvalue. Now we investigate the convergence of (3.2). Using Cauchy-Schwarz inequality for \(\lambda \in \mathbb{C}\).
By Cauchy-Schwarz inequality

$$\Gamma_N (\lambda) \leq \|\phi (., \lambda)\|_{H_q} \left( \sum_{k=-N}^{N} \frac{|\hat{f}_k|^2}{\|\phi_k\|^2_{H_q}} \right)^{1/2}.$$ 

Since the function $\phi (., \lambda)$ is uniformly bounded on the subsets of $\mathbb{C}$, we can find a positive constant $C_\Omega$ which is independent of $\lambda$ such that $\|\phi (., \lambda)\|_{H_q} \leq C_\Omega$, $\lambda \in \Omega_M$. Thus

$$\Gamma_N (\lambda) \leq C_\Omega \left( \sum_{k=-N}^{N} \frac{|\hat{f}_k|^2}{\|\phi_k\|^2_{H_q}} \right)^{1/2} \rightarrow 0 \text{ as } N \rightarrow \infty.$$ 

Hence (3.2) converges uniformly on compact subsets of $\mathbb{C}$. Thus $F(\lambda)$ is an entire function and the proof is complete.

4 Examples

In this section we give three examples illustrating the sampling theorem of the previous section.

**Example 1.** Consider $q$–Dirac system (1.1)-(1.3) in which $p(x) = 0 = r(x)$:

$$\begin{cases} 
-\frac{1}{q} D_{q^{-1}} y_2 = \lambda y_1, \\
D_q y_1 = \lambda y_2, \\
y_1(0) = 0, \\
y_2(\pi q^{-1}) = 0.
\end{cases}$$

(4.1)

(4.2)

(4.3)

It is easy to see that a solution (4.1) and (4.2) is given by

$$\phi^T (x, \lambda) = (\sin (\lambda x; q), \cos (\lambda \sqrt{q} x; q)).$$

By substituting this solution in (4.3), we obtain $\omega (\lambda) = \cos (\lambda q^{-1/2} \pi; q)$, hence, the eigenvalues are $\lambda_n = \frac{q^{1-n+\varepsilon_n(1/2)}}{(1-q) \pi}$. Applying Theorem 1, the $q$–transforms

$$F(\lambda) = \int_{0}^{\pi} f^T (x) \phi (x, \lambda) d_q x$$

$$= \int_{0}^{\pi} \left\{ f_1 (x) \sin (\lambda x; q) + f_2 (x) \cos (\lambda \sqrt{q} x; q) \right\} d_q x,$$  

(4.4)
for some \( f_1 \) and \( f_2 \in L^2_q (0, \pi) \), then it has the sampling formula

\[
F (\lambda) = \sum_{n=-\infty}^{\infty} F (\lambda_n) \frac{\cos (\lambda q^{-1/2} \pi; q)}{(\lambda - \lambda_n) \omega' (\lambda_n)}. \tag{4.5}
\]

**Example 2.** Consider \( q \)-Dirac equation (4.1) together with the following boundary conditions

\[
y_2 (0) = 0, \tag{4.6}
\]
\[
y_1 (\pi) = 0. \tag{4.7}
\]

In this case \( \phi^T (x, \lambda) = (\cos (\lambda x; q), - \sqrt{q} \sin (\lambda \sqrt{q} x; q)) \). Since \( \omega (\lambda) = \cos (\lambda \pi; q) \), then the eigenvalues are given by \( \lambda_n = \frac{q^{-n+1/2} + \pi n (1/2)}{(1-q) \pi} \). Applying Theorem 1 above to the \( q \)-transform

\[
F (\lambda) = \frac{\pi}{0} \int \{ f_1 (x) \cos (\lambda x; q) - f_2 (x) \sqrt{q} \sin (\lambda \sqrt{q} x; q) \} d_q x, \tag{4.8}
\]

for some \( f_1 \) and \( f_2 \in L^2_q (0, \pi) \), then we obtain

\[
F (\lambda) = \sum_{n=-\infty}^{\infty} F (\lambda_n) \frac{\cos (\lambda \pi; q)}{(\lambda - \lambda_n) \omega' (\lambda_n)}. \tag{4.9}
\]

**Example 3.** Consider \( q \)-Dirac equation (4.1) together with the following boundary conditions

\[
y_1 (0) + y_2 (0) = 0, \tag{4.10}
\]
\[
y_2 (\pi q^{-1}) = 0. \tag{4.11}
\]

In this case

\[
\phi^T (x, \lambda) = (\cos (\lambda x; q) - \sin (\lambda x; q), - \sqrt{q} \sin (\lambda \sqrt{q} x; q) - \cos (\lambda \sqrt{q} x; q)).
\]

Since \( \omega (\lambda) = - \sqrt{q} \sin (\lambda q^{-1/2} \pi; q) - \cos (\lambda q^{-1/2} \pi; q) \), then the eigenvalues of this problem are the solutions of equation

\[
\sqrt{q} \sin \left( \lambda q^{-1/2} \pi; q \right) = - \cos \left( \lambda q^{-1/2} \pi; q \right). \tag{4.12}
\]

Applying Theorem 1 above to the \( q \)-transform

\[
F (\lambda) = \frac{\pi}{0} \int \{ f_1 (x) (\cos (\lambda x; q) - \sin (\lambda x; q))
\]
\[
- f_2 (x) (\sqrt{q} \sin (\lambda \sqrt{q} x; q) + \cos (\lambda \sqrt{q} x; q)) \} d_q x, \tag{4.13}
\]

for some \( f_1 \) and \( f_2 \in L^2_q (0, \pi) \), then we obtain

\[
F (\lambda) = \sum_{n=-\infty}^{\infty} F (\lambda_n) \frac{- \sqrt{q} \sin \left( \lambda q^{-1/2} \pi; q \right) - \cos \left( \lambda q^{-1/2} \pi; q \right)}{(\lambda - \lambda_n) \omega' (\lambda_n)}. \tag{4.14}
\]
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