ON THE LARGE TIME ASYMPTOTICS OF SCHRÖDINGER TYPE EQUATIONS WITH GENERAL DATA

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ABSTRACT. For the Schrödinger equation with a general interaction term, which may be linear or nonlinear, time dependent and including charge transfer potentials, we prove the global solutions are asymptotically given by a free wave and a weakly localized part. The proof is based on constructing in a new way the Free Channel Wave Operator, and further tools from the recent works [22, 23, 35]. This work generalizes the results of the first part of [22, 23] to arbitrary dimension, and non-radial data.

1. Introduction

The analysis of dispersive wave equations and systems is of critical importance in the study of evolution equations in Physics and Geometry. It is well known that the asymptotic solutions of such equations, if they exist, show a dizzying zoo of possible solutions. Besides the ”free wave”, which corresponds to a solution of the equation without interaction terms, a multitude of other solutions may appear. Such solutions are localized around possibly moving center of mass. They include nonlinear bound states, solitons, breathers, hedgehogs, vortices etc... The analysis of such equations is usually done on a case by case basis, due to this complexity. [34] A natural question then follows: is it true that in general, solutions of dispersive equations converge in appropriate norm ($L^2$ or $H^1$) to a free wave and independently moving localized parts (localized in space)? In fact this is precisely the statement of Asymptotic Completeness in the case of $N$-body Scattering. In this case the possible outgoing clusters are clearly identified, as bound states of subsystems. But when the interaction term includes time dependent potentials (even localized in space) and more general nonlinear terms, we do not have an a-priori knowledge of the possible asymptotic states.

In the case of time independent interaction terms, one can use spectral theory. The scattering states evolve from the continuous spectrum, and the localized part is formed by the point spectrum. Once the interaction is time dependent/nonlinear that is not possible. In fact, there are no general scattering results for localized time dependent potentials. The exceptions are charge transfer Hamiltonians [41, 11, 40, 24, 26], decaying in time potentials and small potentials [13, 27], time periodic potentials [42, 13] and random (in time) potentials [2]. See also[4, 5]. For potentials with asymptotic energy distribution more could be done [33]. A recent progress for more general localized potentials without smallness assumptions is obtained in [35].

Turning to the nonlinear case, Tao [37, 38, 39] has shown that the asymptotic decomposition holds for NLS with inter-critical nonlinearities, in 3 or higher space dimensions, in the case of radial initial data. In particular, in a sufficiently high dimension, and with an

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interaction that is a sum of a smooth compactly supported potential and a repulsive nonlinearity, Tao was able to show that the localized part is smooth and localized. In other cases, Tao showed the localized part is only weakly localized and smooth. Tao’s work uses direct estimates of the incoming and outgoing parts of the solution to control the nonlinear part, via Duhamel representation. In a certain sense, it is in the spirit of Enss’ work. See also [25].

In contrast, the new approach of Liu-Soff er [22, 23] is based on proving a-priori estimates on the full dynamics, which hold in a suitably localized regions of the extended phase-space. In this way it was possible to show the asymptotic decomposition for general localized interactions, including time and space dependent ones, which are localized. Radial initial data is assumed. More detailed information is obtained on the localized part of the solution. Besides being smooth, its expanding part (if it exists) can grow at most like $|x| \leq C \sqrt{t}$ (for $t \geq 1$ and some constant $C > 0$), and furthermore, is concentrated in a thin set of the extended phase-space. The free part of the solution concentrates on the propagation set where $x = vt, v = 2P$, and $P$ being the dual to the space variable, the momentum, is given by the operator $-i\nabla_x$. The weakly localized part is found to be localized in the regions where

$$|x|/t^\alpha \sim 1 \quad \text{and} \quad |P| \sim t^{-\alpha}, \quad \forall \ 0 < \alpha \leq 1/2.$$ 

It therefore shows that the spreading part follows a self similar pattern. The method of proof is based on three main parts: first, construct the Free Channel Wave Operator. Then prove localization of the remainder localized part, and use it to prove the smoothness of the localized part. Finally, by using further propagation estimates which are adapted to localized solutions, Liu and Soffer proved the concentration on thin sets of the phase-space corresponding to self similar solutions. It should be emphasized that the spreading localized solutions, if they exist, were shown to have a non-small nuclei part around the origin. This is true for both the results of Tao [37, 38, 39] and Liu-Soffer [22, 23]. Therefore, these are not purely self-similar solutions, as appear in the special cases of critical nonlinearities. See e.g. [36, 8].

We will follow here this point of view. The key tool from scattering theory that is used to study multichannel scattering is the notion of channel wave operator, which we denote by

$$\Omega^a_\psi(0) \equiv s-\lim_{t \to \infty} e^{iH_at} U(t, 0)\psi(0), \quad \psi(0) \in \mathcal{H}, \tag{1.1}$$

where $U(t, 0)$ denotes the evolution operator of the full dynamics on a Hilbert space $\mathcal{H}$ and $H_a$ stands for the generator of one possible asymptotic dynamics, for example, $H_a$ can be $-\Delta_x$. Here, (1.1) is well-defined whenever the strong limit exists.

Here the limit is taken in the strong sense in $L^2$. Note that since $U(t, 0)$ is nonlinear in general, then so is the wave operator $\Omega^a_\psi$. $U(t, 0)\psi(0)$ is the solution of the dispersive equation with initial data $\psi(0)$ and dynamics (linear or nonlinear) $U(t) \equiv U(t, 0)$ generated by a Hamiltonian $H(t)$. The asymptotic dynamics is generated by a Hamiltonian $H_a$ for a given channel denoted by $a$. In this work we will only construct the free channel wave operator, where $H_a = -\Delta$.

A crucial observation is that one can modify the definition of the Channel wave operators to

$$\Omega^a_\psi(0) \equiv s-\lim_{t \to \infty} e^{iH_at} J_a U(t, 0)\psi(0), \quad \psi(0) \in \mathcal{H}, \tag{1.2}$$

where $J_a$ is a suitable projection operator on the channel $a$.
provided
\[ w\lim_{t \to \infty} e^{iH_0t}(1 - J_a)U(t, 0)\psi(0) = 0. \]

Here \( J_a \) denotes some operator with norm 1 and will be chosen later. See [29]. This construction can be easily generalized to the case where the asymptotic dynamics is nonlinear. In practice, we should choose \( J_a \) to be a member of a partition of unity which is supported on the extended phase space where the solution is expected to converge; to be useful, it should also be decaying (in some vague sense) on the support of the interaction that couples the channel \( a \) to the rest of the solution.

Now, to prove that the limit exists we use Cook’s method. For this, we need to show the integrability of the derivative w.r.t. time of the vector \( e^{iH_0t}J_aU(t, 0)\psi(0) \) in \( \mathcal{H} \). Taking the derivative (w.r.t. time) gives two types of terms: with \( \partial_t[U(t, 0)] = (-i)H(t)U(t, 0) \),
\[ \partial_t[e^{iH_0t}J_aU(t, 0)]\psi(0) = e^{iH_0t}D_{H_a}(J_a)U(t, 0)\psi(0) - e^{iH_0t}iJ_a(H(t) - H_a)U(t, 0)\psi(0). \]

Here the operator \( D_H(B) \) denotes
\[ D_H(B) \equiv i[H, B] + \frac{\partial B}{\partial t}. \]

For example, with \( \psi(t) \equiv U(t, 0)\psi(0) \), when \( H_a = H_0 \equiv -\Delta \), and the interaction \( N(x, t, |\psi(t)|) = H(t) - (-\Delta) \), \( \partial_t[e^{iH_0t}J_aU(t, 0)]\psi(0) \) reads
\[ \partial_t[e^{iH_0t}J_aU(t, 0)]\psi(0) = e^{iH_0t}D_{H_0}(J_a)\psi(t) - ie^{iH_0t}J_aN(x, t, |\psi(t)|)\psi(t). \]

By choosing
\[ J_a = F\left(\frac{|x|}{t^\alpha} \geq 1\right), \]
where \( F \) denotes a smooth-cut off function or a smooth characteristic function, it is easy to see that such \( J_a \) satisfies our requirement, as on its support the interaction term vanishes like \( t^{-m} \) for a localized interaction vanishing like \( |x|^{-m} \) at infinity. Furthermore, it is not hard to prove that the identity (1.3) holds true by using duality and the dispersive estimate of free flows. However, the Heisenberg Derivative part coming from \( D_H \) is not necessarily integrable in time, under the full dynamics. The solution can have a part that stays on the boundary of the support of \( F \), or revisit it for infinitely many times. To resolve this problem, as was done in the \( N \)-body case [29] and in the general nonlinear case [22, 23], we further microlocalize the partition of unity, such that on the boundary, the solution can be shown to decay sufficiently fast in time \( t \) (by propagation estimates). In [29] these boundaries are cones in the configuration space, and then one needs to microlocalize the momentum to point either out or into the cone. In [22, 23] one microlocalizes the partition \( F \) by localizing on the incoming/outgoing parts of the solution. This microlocalization needs to be done in a way that allows proving propagation estimates there [29, 7]. It should be clear by now, that this method is tied to a distinguished point in space, and requires the interaction term to be localized around it. The function \( F \) can only annihilate a localized term, and the notion of incoming and outgoing is tied to the choice of origin. Therefore, in order to go to the general initial data case, we need a more general type of constructions. This is the content of this work.

The key new construction is a free channel wave operator, with a different type of localization in the phase space. This localization is constructed by projecting in the phase-space on a neighborhood of the thin propagation set in the extended phase space. As
the free wave concentrates where $P := -i\nabla_x$ and $x = 2Pt$, we use the projection, with $H_0 = H_0 \equiv -\Delta_x$,

$$J_{\text{free}} \equiv J_a = \mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right), \quad \text{for some } \alpha \in (0, 1) \text{ and } t > 1,$$

where $\mathcal{F}_c$ denotes a smooth cut-off function. Here, the subscript of $\mathcal{F}_c$ stands for the ”conformal multiplier” and we define $\mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right) as an operator on $L^2$ or $\mathcal{H}$ by using the spectral theorem and the self-adjointness of $|x - 2Pt|$. It is a property of the free dynamics that the solution vanishes outside the support of $\mathcal{F}_c$ as time goes to infinity. The fundamental property of this operator that we use is the following equation:

$$(1.7) \quad e^{-itH_0} \mathcal{F}_c\left(\frac{|x|}{t^a} \leq 1\right)e^{itH_0} = \mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right).$$

See Section A for detailed computations.

Throughout this paper, $C$ will denote a constant and may vary from one line to another. We write $\leq$ or $\geq$ whenever $A \leq CB$ or $CA \geq B$ for some constant $C > 0$. We write $A \lesssim B$ or $A \gtrsim B$ if $A \leq CB$ or $CA \gtrsim B$ for some constant $C_a > 0$ which depends on parameter $a$.

A useful property of $\mathcal{F}_c$ is the following inequality,

$$||\mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right)\phi||_{L^p(\mathbb{R}^n)} = ||e^{-\frac{|x|^2}{4t}}\mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right)\phi||_{L^p(\mathbb{R}^n)}$$

$$\leq ||Pe^{-\frac{|x|^2}{4t}}\mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right)\phi||_{L^p(\mathbb{R}^n)}^{1-a}||e^{-\frac{|x|^2}{4t}}\mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right)\phi||_{L^p(\mathbb{R}^n)}^a$$

$$\leq |||1/t||2Pt - x||\mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right)\phi||_{L^p(\mathbb{R}^n)}^{1-a}||\mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right)\phi||_{L^p(\mathbb{R}^n)}^a$$

$$\leq t^{(-1+a)a}||\mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right)\phi||_{L^p(\mathbb{R}^n)}$$

$$\lesssim t^{(-1+a)a}||\mathcal{F}_c\left(\frac{|x|}{t^a} \leq 1\right)e^{itH_0}\phi||_{L^p(\mathbb{R}^n)},$$

where $n$ is the space dimension, $|x|$ denotes the length of $x$ in $\mathbb{R}^n$ and we have used Nirenberg-Sobolev type inequality, the unitarity of $e^{-itH_0}$ on $L^2(\mathbb{R}^n)$ and Eqs.

$$(1.9) \quad Pe^{-\frac{|x|^2}{4t}} = e^{-\frac{|x|^2}{4t}}P - e^{-\frac{|x|^2}{4t}}\frac{x}{2} \cdot P = \frac{e^{-\frac{|x|^2}{4t}}}{2}(P - \frac{x}{2t})$$

and (1.7). Here, the constants $p > 2$ and $a$ depend on the dimension of the space. For example, in three space dimensions, $p = 6, a = 1$. Furthermore, the Heisenberg Derivative of this operator is positive:

$$(1.10) \quad D_{H_0} \mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right) = -a\frac{|x - 2Pt|}{t^{1+a}}\mathcal{F}_c' \geq 0.$$

This is due to the fact that $D_{H_0}(x - 2Pt)^2 = 0$.

The operator $\mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right)$ and its functions have a long history. In fact, the operator $|x - 2Pt|^2$ is the multiplier that gives the conformal identity for Schrödinger equations. Then $\mathcal{F}_c\left(\frac{|x - 2Pt|}{t^a} \leq 1\right)$ was used to prove sharp propagation estimates in [28, 30, 31, 32, 10, 6]. In a completely different way it was used in [19, 20]. Using propagation estimates similar
to [29], the problem of showing the existence of the free channel wave operator, defined
in terms of the above $F_c$, is reduced to proving the propagation estimate that follows from
using $F_c$ as a propagation observable. Since the Heisenberg derivative is positive, it remains
to verify for what interaction terms the following is true:
\[
\int_1^\infty \|F_c N(x, t, |\psi|)|U(t, 0)\psi(0)\|_{L^2_c} dt < \infty.
\]
See Section 3 for detailed discussions.

2. The Scattering Problem and Results

Let $H_0 := -\Delta_x$. We consider the general class of Nonlinear Schrödinger type equa-
tions of the form:
\[
\begin{aligned}

i\partial_t \psi(x, t) &= H_0\psi(x, t) + N(x, t, \psi(x, t))\psi(x, t) \\
\psi(x, 0) &= \psi_0 \in \mathcal{H}_a^\delta(\mathbb{R}^n),
\end{aligned}
\]
with space dimension $n \geq 1$, where $\mathcal{H}_a^\delta \equiv \mathcal{H}_a^\delta(\mathbb{R}^n)$, $a \in [0, 1]$, denotes the $L^2$ Sobolev space
of order $a$. $N$ is NOT assumed to be real.

2.1. Assumptions and examples. We consider solutions $\psi(t) \equiv \psi(x, t)$ of system (2.1)
which exist globally in $t \in \mathbb{R}$ and are uniformly bounded in $\mathcal{H}_a^\delta$:

Assumption 2.1. There exists a positive constant $C > 0$ such that
\[
E := \sup_{t \in \mathbb{R}} \|\psi(t)\|_{\mathcal{H}_a^\delta} \leq C < \infty
\]
is valid for some $a \in [0, 1]$.

Let $\langle \cdot \rangle : \mathbb{R}^n \to \mathbb{R}, x \mapsto \sqrt{|x|^2 + 1}$, denote the Japanese brackets. We consider the
interaction $N(x, t, \psi(t))$, which falls into one of the following categories:

1. (Space localized potentials): For $n \geq 1$ and some $\delta > 1$,
\[
\langle x \rangle^\delta N(x, t, \psi(t)) \in L_c^\infty(\mathbb{R}^{n+1}).
\]
2. (L^p potentials): For $n \geq 3$,
\[
N(x, t, \psi(t))\psi(t) \in L_t^\infty L_{x}^1(\mathbb{R}^{n+1}).
\]

Our $L^p$ potentials cover the following models, which are proved in Examples 6.1
and 6.2 of Section 6.1:

(a) (Charge transfer Hamiltonians): Let Assumption 2.1 hold. When the space
dimension $n \geq 3$, the charge transfer interaction $N(x, t, \psi) = \sum_{j=1}^{N} V_j(x - tv_j, t)$,
where $V_j(x, t) \in L_t^\infty L_x^2(\mathbb{R}^{n+1})$, $j = 1, \ldots, N$, with $v_j \neq v_l$ if $j \neq l$, satisfies
condition (2.4).

(b) (Purely nonlinear systems): Let Assumption 2.1 hold with $a = 1$. When space
dimension $n \geq 3$, $N(x, t, \psi) = I(|\psi|)$, with $I(|\psi|)$ satisfying the estimate
\[
\|I(|\psi|)|\psi|\|_{L_c^\infty(\mathbb{R}^n)} \leq \|\psi\|_{H_a^1} 1,
\]
satisfies condition (2.4).
Here, typical examples for purely nonlinear interactions (interactions which depend only on \( \psi(t) \)) are polynomial nonlinearities (see Example 6.3 of Section 6.1):

\[
I(|\psi|) = P(|\psi|), \quad n \geq 3,
\]

where \( P(z) \) denotes a polynomial of degree \( N \) with \( P(0) = 0 \). The main assumption to be verified in this case, is that the energy identity implies the \( L^1 \) condition (see (2.5)). See Example 6.3 for a detailed discussion.

**Assumption 2.2 (Space localized potentials).** Assume Eq. (2.3) is valid for some \( \delta > 1 \).

**Assumption 2.3 (\( L^p \) potentials).** Assume Eq. (2.4) is valid.

Let \( W_x^{k,p}(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \), denote the \( L^p \) Sobolev space of order \( k \). We refer to \( \mathcal{N}(x,t,\psi(t)) \) as a charge-transfer interaction if \( \mathcal{N} \) is linear and if there exists a positive integer \( N \geq 2 \) and \( N \) vectors \( v_j \in \mathbb{R}^n \), \( j = 1, \ldots, N \), \( (v_j \neq v_l \text{ if } j \neq l) \) such that

\[
\mathcal{N}(x,t,\psi(t)) = \sum_{j=1}^{N} V_j(x-tv_j, t),
\]

where \( V_j(x,t), j = 1, \ldots, N \), are functions localized in \( x \) variable (see Assumption 2.4).

**Assumption 2.4 (Charge-transfer potentials).** There exists \( \delta \geq n + 1 \) such that \( \mathcal{N}(x,t,\psi(t)) \) satisfies (2.7) with \( V_j(x,t), j = 1, \ldots, N \), satisfying

\[
\sup_{t \in \mathbb{R}} \|(x)^\delta V_j(x,t)\|_{W_x^{1,\infty}(\mathbb{R}^n)} < \infty.
\]

### 2.2. Main results.

Let \( \mathcal{F}_j(\lambda), j = c, 1, 2 \), denote smooth characteristic functions of the interval \([1, +\infty)\) satisfying

\[
\mathcal{F}_j(\lambda) = \begin{cases} 
1 & \text{when } \lambda \geq 1, \\
0 & \text{when } \lambda < \frac{1}{2},
\end{cases} \quad j = c, 1, 2.
\]

For each \( j \in \{c, 1, 2\} \), we define

\[
\mathcal{F}_j(\lambda > a) := \mathcal{F}_j(\lambda/a)
\]

and

\[
\mathcal{F}_j(\lambda \leq a) := 1 - \mathcal{F}_j(\lambda/a).
\]

In this paper, we restrict our discussion to the case where \( t \geq 0 \). The case where \( t < 0 \) is treated similarly. Here are our main results. When \( \mathcal{N}(x,t,\psi(t)) \) is a \( L^p \) potential, the adapted free channel wave operator, defined in (2.12), exists on \( L^2_x(\mathbb{R}^n), n \geq 3 \).

**Theorem 2.1 (\( L^p \) potentials).** Consider \( \psi(t) \) as the solution to system (2.1). Let Assumptions 2.1 and 2.3 hold. When the space dimension \( n \geq 3 \), for all \( \alpha \in (0, 1 - 2/n) \), the adapted free channel wave operator acting on the initial data \( \psi(0) \), defined as

\[
\Omega^\alpha_x \psi(0) := s - \lim_{\varepsilon \to 0} e^{iH_0 \varepsilon} \mathcal{F}_c(\frac{|x - 2tP|}{\varepsilon^\alpha} \leq 1) \psi(t)
\]

exists in \( L^2_x(\mathbb{R}^n) \). Furthermore, \( \Omega^\alpha_x \psi(0) \) is independent on the choice of \( \alpha \) in the following sense: for all \( \alpha, \alpha' \in (0, 1 - 2/n) \),

\[
\Omega^\alpha_x \psi(0) = \Omega^\alpha_{\alpha'} \psi(0).
\]
Remark 2.2. The condition on the nonlinearity here is not sharp. One expects that the same proof can be applied to the abstract version, which involves a more general interaction. See Appendix.

Remark 2.3. To control the non-free part, which has localization properties to be defined later, even in three or higher spatial dimensions, we require that the interaction term be localized or of the charge transfer type, as detailed in Theorems 2.6 and 2.9. It is important to note that the interaction can also be nonlinear. This is based on the application of radial Sobolev embedding theorems of an $H^1$ function in three or higher dimensions.

Remark 2.4. In three or higher space dimensions, since we don’t need the space localization to control the interaction term, the theorem applies to the general nonlinear systems without using the spherical symmetry assumption.

When $N(x, t, \psi(t))$ is localized in $x$ variable, we can establish the existence of the adapted free channel wave operators in all space dimensions and provide some useful properties for the non-free part. We find that the non-free part of the solution is weakly localized in the following sense.

Definition 2.5 (The weakly localized part of the solution.). We say that part of the solution defined as $\psi_{wl}(x, t)$ is weakly localized if it has non-zero mass and spreads slowly in the following sense: there exists $\beta \in (0, 1)$ such that for all $t \geq 1$,

$$\left(\psi_{wl}(x, t), |x|\psi_{wl}(x, t)\right)_{L^2_x(\mathbb{R}^n)} \lesssim t^\beta$$

holds true.

Theorem 2.6 (Space localized potentials). Consider $\psi(t)$ as the solution to system (2.1). Let Assumptions 2.1 and 2.2 hold. Let $\delta$ be as in Assumption 2.2. Then for all $n \geq 1$, there exist two positive constants $c_1, c_2$, which depend on $n$ and $\delta$ (see Eq. (3.21)), such that for all $\alpha \in (0, c_1)$ and for all $\beta \in (0, \min\{c_2, \alpha\})$, the adapted free channel wave operator acting on $\psi(0)$, defined as

$$\Omega_{\alpha, \beta}^*(\psi(0)) := \lim_{t \to \infty} e^{itH_0^s} \mathcal{F}_c\left(\frac{|x-2itP|}{|t^\beta_P|} \leq 1\right) \mathcal{F}_1(1) \psi(t),$$

exists in $L^2_s(\mathbb{R}^n)$ and $\Omega_{\alpha, \beta}^* \psi(0)$ is independent on the choices of $\alpha$ and $\beta$: for all $(\alpha, \beta), (\alpha', \beta') \in (0, c_1) \times (0, c_2)$ with $\beta < \alpha$ and $\beta' < \alpha'$,

$$\Omega_{\alpha, \beta}^* \psi(0) = \Omega_{\alpha', \beta'}^* \psi(0).$$

Furthermore, if $\delta > 2$, for every $\epsilon \in (0, 1/2)$, there exists a weakly localized part of $\psi(t)$, denoted by $\psi_{wl}(t) \equiv \psi_{wl, \epsilon}(t)$, such that

1. the equation

$$\lim_{t \to \infty} \|\psi(t) - e^{-itH_0^s} \Omega_{\alpha, \beta}^* \psi(0) - \psi_{wl}(t)\|_{L^2_x(\mathbb{R}^n)} = 0$$

holds true;

2. $\psi_{wl}(t)$ is weakly localized satisfying

$$\left(\psi_{wl}(t), |x|\psi_{wl}(t)\right)_{L^2_x(\mathbb{R}^n)} \lesssim t^{1/2+\epsilon}, \quad t \geq 1.$$
Remark 2.7. One expects that when \( N(x,t,\psi(t)) = \partial_j g_{ij}(x,t) \partial_i + I(x,t,\psi(t)) \), where \( g \) denotes the metric tensor and \( I(x,t,\psi(t)) \) satisfies Assumption 2.2, the adapted free channel wave operator acting on \( \psi(0) \), defined by

\[
(2.19) \quad \tilde{\Omega}^*_\alpha,\beta \psi(0) := s\lim_{t \to 0} e^{itH_0} \mathcal{F}_\epsilon \left( \frac{|x - 2tP|}{\rho} \leq 1 \right) \mathcal{F}_1 (t^\beta |P| > 1) \mathcal{F}_1 (t^\beta |P| \leq 1) \psi(t)
\]

exists in \( L^2_1(\mathbb{R}^n) \), \( n \geq 1 \) by taking \( \alpha, \beta > 0 \) sufficiently small.

When \( n \geq 5 \) and \( N(x,t,\psi(t)) \) is a charge-transfer interaction, we find that the non-free component consists of several moving, weakly localized parts, as defined below.

Definition 2.8 (A moving weakly localized part). Let \( t \geq 0 \) and \( v \in \mathbb{R}^n \). We say that part of the solution defined as \( \psi_{wl,v}(x,t) \) is moving and weakly localized if it has mass and spreads slowly around \( tv \), as \( t \to \infty \), in the following sense: there exists \( \beta \in (0,1) \) such that for all \( t \geq 1 \),

\[
(2.20) \quad (\psi_{wl,v}(x,t), |x - tv| \psi_{wl,v}(x,t))_{L^2_2(\mathbb{R}^n)} \leq t^\beta
\]

holds true.

Theorem 2.9. Let \( \psi(t) \) be the solution to system (2.1) satisfying Assumption 2.1 with \( a = 1 \). Let Assumption 2.4 be satisfied. Then when \( n \geq 5 \), for any \( \epsilon \in (0,1/2) \), there exist \( N \) moving weakly localized parts, \( \psi_{wl,j}(t) \equiv \psi_{wl,e,j}(t) \), \( j = 1, \cdots, N \), such that

1. the equation

\[
(2.21) \quad \lim_{t \to \infty} \| \psi(t) - e^{-itH_0} \Omega^*_\alpha \psi(0) - \sum_{j=1}^{N} \psi_{wl,j}(t) \|_{L^2_2(\mathbb{R}^n)} = 0,
\]

holds true;

2. \( \psi_{wl,j}(t) \), \( j = 1, \cdots, N \), are moving weakly localized parts around \( tv_j \) satisfying

\[
(2.22) \quad (\psi_{wl,j}(t), |x - tv_j| \psi_{wl,j}(t))_{L^2_2(\mathbb{R}^n)} \leq t^{1/2+\epsilon}, \quad t \geq 1.
\]

We the non-free part is small, we obtain scattering for system (2.1):

Proposition 2.10. Let \( \psi(t) \) be the solution to system (2.1) in \( n = 3 \) space dimensions. Let us suppose that the following conditions are satisfied:

1. Assumptions 2.1 and 2.3 are valid with \( a = 1 \);
2. \( N(x,t,\psi(t)) \) is given by \( N(x,t,\psi(t)) = I(\psi(t)) \) where \( I : [0,\infty) \to \mathbb{R} \), is a function satisfying the following condition: for any pair \( f(x), g(x) \in \mathcal{H}_1(\mathbb{R}^3) \),

\[
\| I(f(x))f(x) - I(g(x))g(x) \|_{L^6_1(\mathbb{R}^3)} \leq C_{11} \| f(x) - g(x) \|_{\mathcal{H}_1} \| f(x) - g(x) \|_{L^6_1(\mathbb{R}^3)} + C_{12} \| g(x) \|_{L^6_1(\mathbb{R}^3)},
\]

where \( C_{11} = C_{11}(\| f \|_{\mathcal{H}_1}, \| f \|_{\mathcal{H}_1}) > 0 \) and \( C_{12} = C_{12}(\| f \|_{\mathcal{H}_1}, \| g \|_{\mathcal{H}_1}) > 0 \) are two positive constants dependent on \( \| f \|_{\mathcal{H}_1} \) and \( \| g \|_{\mathcal{H}_1} \).

Then there exists \( m > 0 \) such that

\[
(2.24) \quad \limsup_{t \to \infty} \| \psi(t) - e^{-itH_0} \Omega^*_\alpha \psi(0) \|_{\mathcal{H}_1} < m
\]

implies

\[
(2.25) \quad \lim_{t \to \infty} \| \psi(t) - e^{-itH_0} \Omega^*_\alpha \psi(0) \|_{L^2_2(\mathbb{R}^3)} = 0.
\]
Remark 2.11. The above proposition implies that if the non-free part is small, then it vanishes asymptotically. When \( I(|\psi|) = \pm \lambda |\psi|^p \) with \( \lambda > 0 \) for all \( 4/3 \leq p \leq 4 \), condition (2.23) holds true by the following argument:

\[
(2.26) \quad |f|^p g - |g|^p g = (|f|^p - |g|^p)(f - g) + (|f|^p - |g|^p)g + (f - g)|g|^p,
\]

by Hölder’s inequality, with either \( 2(p - 1) \in [2, 6] \) or \( 6(p - 1) \in [2, 6] \).

\[
\|(|f|^p - |g|^p)(f - g)\|_{L^6_{t,x}(\mathbb{R}^3)} \leq \| |f|^p - |g|^p \|_{L^6_{t,x}(\mathbb{R}^3)} \| f - g \|_{H^s_t} \| f - g \|_{L^6_{t,x}(\mathbb{R}^3)}
\]

\[
\leq \| f^{p-1} \|_{L^6_{t,x}(\mathbb{R}^3)} + \| g^{p-1} \|_{L^6_{t,x}(\mathbb{R}^3)} \| f - g \|_{H^s_t} \| f - g \|_{L^6_{t,x}(\mathbb{R}^3)}
\]

(2.27)

and with \( 3p/2 \in [2, 6] \),

\[
\|(|f|^p - |g|^p)g + (f - g)|g|^p \|_{L^6_{t,x}(\mathbb{R}^3)} \leq \| f^{p-1} \|_{L^6_{t,x}(\mathbb{R}^3)} \| g \|_{H^s_t} \| f - g \|_{L^6_{t,x}(\mathbb{R}^3)}
\]

(2.28)

where \( C_1 \) and \( C_2 \) are two positive constants dependent on \( \| f \|_{H^s_t} \) and \( \| g \|_{H^s_t} \).

3. Propagation Estimate, Relative Propagation Estimate, \( tT \) potentials, estimates for interaction terms and commutator estimates

We let \( b \in \mathbb{R} \) denote the lower bound on the time interval of interest.

3.1. Propagation Estimate. Given a family of self-adjoint operators \( \{B(t)\}_{t \geq b} \), we define

\[
(3.1) \quad \langle B \rangle_t := \langle \psi(t), B(t)\psi(t) \rangle_{L^2_{t,x}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \psi(t)^* B(t) \psi(t) d^n x, \quad t \geq b,
\]

where \( \psi(t) \) is the solution to (2.1) and \( f^* \) denotes the conjugate of \( f \) for any function \( f \). Suppose \( \langle B \rangle_t \), for \( t \geq b \), satisfies a boundedness condition which is uniform over \( t \in [b, \infty) \):

\[
(3.2) \quad \sup_{t \geq b} \| \langle B \rangle_t \| < \infty,
\]

and \( \partial_t \| \langle B \rangle_t \| \) satisfies the decomposition, for all \( t \geq b \),

\[
(3.3) \quad \partial_t \| \langle B \rangle_t \| = \pm \langle \psi(t), C^* C \psi(t) \rangle_{L^2_{t,x}(\mathbb{R}^n)} + g(t)
\]

(3.4)

\[
g(t) \in L^1_{t}[b, \infty), \quad C^* C \geq 0.
\]

We then refer to the family \( \{B(t)\}_{t \geq b} \) as a Propagation Observable (PROB). See, for example, [16], [29] and [33].

From a PROB, by using Eqs. (3.3) and (3.4), we derive a Propagation Estimate (PRES) for all \( t_2 \geq t_1 \geq b \):

\[
(3.5) \quad \int_{t_1}^{t_2} \| C(t) \psi(t) \|_{L^2_{t,x}(\mathbb{R}^n)} dt = \pm \langle \psi(t_2), B(t_2) \psi(t_2) \rangle_{L^2_{t,x}(\mathbb{R}^n)} + \langle \psi(t_1), B(t_1) \psi(t_1) \rangle_{L^2_{t,x}(\mathbb{R}^n)} - \int_{t_1}^{t_2} g(s) ds \leq \sup_{t \in [t_1, t_2]} \| \langle \psi(t), B(t) \psi(t) \rangle_{L^2_{t,x}(\mathbb{R}^n)} \| + \| g(t) \|_{L^1_{t}[b, \infty)}.
\]
3.2. **Relative Propagation Estimate.** We use a modified PRES as well. Given a family of self-adjoint operators \( \{ \hat{B}(t) \}_{t \in \mathbb{R}} \) and a flow \( \phi(t) \), we define

\[
(3.6) \quad \langle \hat{B} : \phi(t) \rangle_t := (\phi(t), \hat{B}(t)\phi(t))_{L^2_x(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \phi(t)^* \hat{B}(t)\phi(t) d^n x.
\]

Suppose \( \langle \hat{B} : \phi(t) \rangle_t \), for \( t \geq b \), satisfies a boundedness condition uniform in \( t \in [b, \infty) \):

\[
(3.7) \quad \sup_{t \geq b} |\langle \hat{B} : \phi(t) \rangle_t| < \infty,
\]

and assume that there exists a positive integer \( N \in \mathbb{N}^+ \) such that \( \partial_t [\langle \hat{B} : \phi(t) \rangle_t] \) satisfies the decomposition, for all \( t \geq b \),

\[
(3.8) \quad \partial_t [\langle \hat{B} : \phi(t) \rangle_t] = \sum_{j=1}^N (\psi(t), C_j^* C_j \psi(t))_{L^2_x(\mathbb{R}^n)} + g(t)
\]

\[
(3.9) \quad g(t) \in L^1_x([b, \infty)), \quad C_j^* C_j \geq 0.
\]

We then refer to the family \( \{ \hat{B}(t) \}_{t \geq b} \) as a **Relative Propagation Observable** (RPROB).

From a RPROB, by using Eqs. (3.8) and (3.9), we derive a **Relative Propagation Estimate** (RPRES) for all \( t_2 \geq t_1 \geq b \):

\[
(3.10) \quad \int_{t_1}^{t_2} \|C_j(t)\phi(t)\|_{L^2_x(\mathbb{R}^n)}^2 dt \leq \sup_{t \geq b} |\langle \phi(t), \hat{B}(t)\phi(t) \rangle_{L^2_x(\mathbb{R}^n)}| + \|g(t)\|_{L^1_x([b, \infty])}.
\]

We will use \( \phi(t) = e^{iHt}\psi(t) \) in the proof of Theorems 2.1 and 2.6.

We conclude this subsection by presenting an abstract version of the main proposition concerning the existence of the Free Channel Wave Operator. One expects that the proof of Proposition 3.1 can be derived using arguments similar to those used in Theorems 2.1 and 2.6. A sketch of the proof is provided in Appendix C.

**Proposition 3.1.** Let \( H_0 = \omega(P) \) be the generator of the free evolution operator \( U_0(t) \equiv e^{-iH_0t} \) acting on a Hilbert space \( \mathcal{H} = L^2_x(\mathbb{R}^n), n \geq 1 \). Let \( \psi(t) \) be the solution of a Schrödinger type equation

\[
\frac{i}{\hbar} \frac{\partial \psi(t)}{\partial t} = (H_0 + N(x, t, \psi(t)))\psi(t), \quad t > 0.
\]

Assume that for initial data \( \psi(0) = \psi_0 \) the solution of the above (possibly nonlinear) equation is global, uniformly bounded in \( \mathcal{H}_1^{1/2} \) (that is, estimate (2.2)).

1. If the group \( U_0(t) \) is bounded from \( L^p_x(\mathbb{R}^n) \) into \( L^{p'}(\mathbb{R}^n) \) with a bound that decays faster than \( 1/t^{1+\epsilon} \) for some \( \epsilon > 0 \), where \( 1 \leq p < 2 \) and \( p' \) is the conjugate of \( p \), then the following strong limit, defining the Free Channel Wave Operator acting on \( \psi(0) \) exists in \( L^2_x(\mathbb{R}^n) \):

\[
(3.11) \quad \Omega^*_{\text{free}, \alpha} \omega_0(0) = \lim_{t \to \infty} U_0(-t)\mathcal{F}_x \left( \frac{|x - tv(P)|}{p^a} \right) \leq 1 \psi(t)
\]

provided the interaction term satisfies the following estimate:

\[
(3.12) \quad \sup_{t \in \mathbb{R}^n} \|N(x, t, \psi(t))\psi(t)\|_{L^2_x(\mathbb{R}^n)} \leq 1.
\]

Furthermore, \( \Omega^*_{\alpha} \psi(0) \) is independent on the choice of \( \alpha \) in the following sense: for all \( \alpha, \alpha' \in (0, \frac{2p}{(2-p)n}) \),

\[
(3.13) \quad \Omega^*_{\text{free}, \alpha'} \omega_0(0) = \Omega^*_{\text{free}, \alpha} \omega_0(0).
\]
(2) If there exist \(k > 0, \rho \in [1, 2)\) and \(\rho' \in (2, \infty]\) such that the group \(U_\rho(t)\) is bounded from \(W^{k, \rho}_x(\mathbb{R}^n)\) into \(L^\rho_{\rho'}(\mathbb{R}^n)\) with a bound that decays faster than \(1/t^{1+\varepsilon}\) for some \(\varepsilon > 0\), then for all \(\alpha \in (0, \min(\{2\rho/(p-2), \rho/n\})\) and all \(\beta \in (0, \min(\alpha, \frac{\rho-n\alpha(1/2-1/\beta)}{pk})\)), the free channel wave operator acting on \(\psi(0)\), defined by

\[
\Omega^*_{\text{free}, \alpha, \beta}(0) = s\lim_{t \to \infty} e^{itH_0} \mathcal{F}_c \left( \frac{|x - tv(P)|}{t^\alpha} \right) \leq 1 \mathcal{F}_1(|P| \leq t^\rho)\psi(t),
\]

exists in \(L^\rho_{\rho'}(\mathbb{R}^n)\). Furthermore, \(\Omega^*_{\text{free}, \alpha, \beta}(0)\) is independent on the choice of \(\alpha\) and \(\beta\) in the following sense: for all \(\alpha, \alpha' \in (0, \min(\{2\rho/(p-2), \rho/n\})\) and all \(\beta \in (0, \frac{\rho-n\alpha(1/2-1/\beta)}{pk})\), \(\beta' \in (0, \frac{\rho-n\alpha'(-1/2-1/\beta')}{pk})\),

\[
\Omega^*_{\text{free}, \alpha, \beta}(0) = \Omega^*_{\text{free}, \alpha', \beta'}(0).
\]

3.3. Estimates for interaction terms. We need the following dispersive estimates for the free flow:

(1) \(L^p\) decay estimates, see for example Eq. (1.1) of [26]:

\[
\|e^{-itH_0} f(x)\|_{L^p_t(\mathbb{R}^n)} \lesssim_{\alpha, \beta} \frac{1}{|t|^{\frac{n}{2} + \frac{1}{p'}}} \|f(x)\|_{L^p_t(\mathbb{R}^n)}, \quad f \in L^p_t(\mathbb{R}^n), \quad t \in \mathbb{R} - \{0\}
\]

where

\[
\frac{1}{p} + \frac{1}{p'} = 1, \quad 2 \leq p \leq \infty.
\]

(2) Local decay estimates, see the first proof in Appendix B for its proof: For \(0 < \alpha < 1 - \beta\) and \(\beta \in (0, 1/2)\),

\[
\|\mathcal{F}_c \left( \frac{|x|}{t^\alpha} \right) \|_{L^\rho_t(\mathbb{R}^n)} \leq 1 \mathcal{F}_1(|P| > 1) e^{itH_0} \langle x \rangle^{-\sigma} \|L^{\rho}_{\rho'}(\mathbb{R}^n) \|_{L^{\rho}_{\rho'}(\mathbb{R}^n)} \lesssim \frac{1}{t^{\sigma(1-\beta)}}, \quad t \geq 1, \quad \sigma \geq 0.
\]

In this section, we use the following notations

\[
\mathcal{F}_1 = \mathcal{F}_1(|P| > 1), \quad \mathcal{F}_c = \mathcal{F}_c \left( \frac{|x|}{t^\alpha} \right) \leq 1
\]

and

\[
\mathcal{F}^{(1)}_1 = \mathcal{F}^{(1)}_1(k) := \frac{d}{dk} [\mathcal{F}_1(k)]
\]

when it does not lead to confusion.

3.3.1. Space localized \(N(x, t, \psi(t))\).

**Proposition 3.2.** Let \(\delta\) be as in Assumption 2.2. Take

\[
c_1 = \frac{\delta + 1}{2\delta} < 1 \quad \text{and} \quad c_2 = \frac{\delta - 1}{4\delta} < \frac{1}{4}.
\]

If Assumption 2.2 is satisfied, then for all \(\alpha \in (0, c_1)\) and \(\beta \in (0, c_2)\), the estimate

\[
\|\mathcal{F}_c \mathcal{F}_1 e^{itH_0} N(x, t, \psi(t))\psi(t)\|_{L^\infty_t(\mathbb{R}^n)} \leq \frac{1}{t^{\alpha\delta}} \|\langle x \rangle^{\delta} N(x, t, \psi(t))\psi(t)\|_{L^\infty_t(\mathbb{R}^{n+1})}, \quad t \geq 1
\]

is valid.
Proof. Since $\alpha < c_1 < 1 - c_2 < 1 - \beta$, by local decay estimates (3.18), we obtain

$$\|\mathcal{F}_\epsilon e^{it\mathcal{H}_0}[N(x, t, \psi(t))\psi(t)]\|_{L^2_{\epsilon}(\mathbb{R}^n)} \lesssim \frac{1}{t^{\delta(1-\beta)}} \|\mathcal{F}_\epsilon \langle x \rangle^\delta\|_{L^2_{\epsilon}(\mathbb{R}^n) \rightarrow L^2_{\epsilon}(\mathbb{R}^n)}$$

(3.23)

$$\lesssim \frac{1}{t^{\delta(1-\beta)}(1-\alpha)} \lesssim \frac{1}{t^{\delta(1-\alpha)}} \lesssim \frac{1}{t^{\delta+1}}.$$

\[\square\]

Remark 3.3. The space localization for $N(x, t, \psi(t))$ is not needed in three or more space dimensions. This is because the dispersive estimate implies a decay rate faster than $\frac{1}{t^{\frac{\alpha}{2}}}$.

See Proposition 3.4. Here, the discussion is mainly for the one and two space dimensional cases.

3.3.2. $L^p$ potentials.

Proposition 3.4. If Assumptions 2.1 and 2.3 are valid, then for all $\alpha \in (0, 1 - 2/n)$, $n \geq 3$ and $t \geq 1$,

$$\|\mathcal{F}_\epsilon e^{it\mathcal{H}_0}[N(x, t, \psi(t))\psi(t)]\|_{L^2_{\epsilon}(\mathbb{R}^n)} \lesssim \frac{1}{t^{1+\beta}} \|N(x, t, \psi(t))\psi(t)\|_{L^p_{\epsilon}L^1_{\epsilon}(\mathbb{R}^{n+1})}$$

(3.24)

where $\beta$ is given by

$$\beta := \frac{n(1-\alpha)}{2} - 1 > 0.$$

Proof. By using H"older’s inequality and $L^\infty$ decay (estimate (3.16) with $p = \infty$), we obtain

$$\|\mathcal{F}_\epsilon e^{it\mathcal{H}_0}[N(x, t, \psi(t))\psi(t)]\|_{L^2_{\epsilon}(\mathbb{R}^n)}$$

$$\lesssim \|\mathcal{F}_\epsilon\|_{L^2_{\epsilon}(\mathbb{R}^n)} \|e^{it\mathcal{H}_0}\|_{L^{1}_{\epsilon}(\mathbb{R}^n) \rightarrow L^{\infty}_{\epsilon}(\mathbb{R}^n)} \|N(x, t, \psi(t))\psi(t)\|_{L^1_{\epsilon}(\mathbb{R}^n)}$$

$$\lesssim n t^{\frac{n}{2}} \|N(x, t, \psi(t))\psi(t)\|_{L^\infty_{\epsilon}L^1_{\epsilon}(\mathbb{R}^{n+1})}$$

$$\lesssim n \frac{1}{t^{1+\beta}} \|N(x, t, \psi(t))\psi(t)\|_{L^\infty_{\epsilon}L^1_{\epsilon}(\mathbb{R}^{n+1})}.$$

(3.26)

\[\square\]

Remark 3.5. Based on the proof of Proposition 3.4, $L^\infty$ decay estimates of the free flow are not necessary in 3 or higher dimensions. For example, $L^{6+\epsilon}$ decay in $t$ will be sufficient in 3 space dimensions provided $\alpha \ll 1$.

Commutator estimates are required for identifying a positive term. Roughly speaking, consider an expression of the form $F(x)G(P) + G(P)F(x)$. In our applications both variables $x, P$ are scaled with a fractional power of $t$. Suppose $F$ and $G$ are both positive, bounded and smooth. Then, the positive term, which is corresponding to $(\phi(t), C_jC_j^\star \phi(t))_{L^2_{\epsilon}(\mathbb{R}^n)}$ in Eq. (3.8), can be constructed as follows:

$$F(x)G(P) + G(P)F(x) = 2 \sqrt{F}G \sqrt{F} + \sqrt{F}, \sqrt{G}, G]$$

(3.27)

or

$$F(x)G(P) + G(P)F(x) = 2 \sqrt{G}F \sqrt{G} + \sqrt{G}, \sqrt{G}, F].$$

(3.28)

The double commutator can be estimated using the commutator estimates provided below.
Lemma 3.6 (Commutator estimates). For all \( \beta < \alpha, l = 0, 1, \) and \( t \geq 1, \) the commutator estimate
\[
\| [F_c, F_1^{(l)}] \|_{L_a^2(\mathbb{R}^n) \rightarrow L_a^2(\mathbb{R}^n)} \lesssim_n \frac{1}{p^a - \beta}
\]
holds true, where \( F_1^{(0)} \equiv F_1. \)

Proof. Let \( \hat{F}_1(\xi) \) denote the Fourier transform of \( F_1(x) \) in \( x \) variable:
\[
\hat{F}_1(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} F_1(x) d^n x.
\]
To compute \([F_c, F_1^{(l)}] \), we find, with \( F_c = F_c(\frac{|x|}{\rho^a} \leq 1) \),
\[
[F_c, F_1^{(l)}] = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{F}_1^{(l)}(\xi) e^{i\beta \cdot P \cdot \xi} \times [e^{-i\beta \cdot P \cdot \xi} F_c e^{i\beta \cdot P \cdot \xi} - F_c] \, d^n \xi
\]
\[
= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{F}_1^{(l)}(\xi) e^{i\beta \cdot P \cdot \xi} \left( F_c \left( \frac{|x - \rho^a \xi|}{\rho^a} \leq 1 \right) - F_c \left( \frac{|x|}{\rho^a} \leq 1 \right) \right) \, d^n \xi.
\]
Using that by the mean-value Theorem,
\[
\left| \frac{F_c \left( \frac{|x - \rho^a \xi|}{\rho^a} \leq 1 \right) - F_c \left( \frac{|x|}{\rho^a} \leq 1 \right) }{\rho^a \xi} \right| \leq \sup_{x \in \mathbb{R}^n} |F_c'(|x| \leq 1)| \leq 1,
\]
we estimate
\[
\| [F_c, F_1^{(l)}] \|_{L_a^2(\mathbb{R}^n) \rightarrow L_a^2(\mathbb{R}^n)} \lesssim_n \frac{1}{p^a - \beta} \int |\hat{F}_1^{(l)}(\xi)| |\xi| d^n \xi \lesssim_n \frac{1}{p^a - \beta}.
\]

4. Proofs of Theorem 2.1 and Theorem 2.6

In this section we prove Theorems 2.1 and 2.6. The proof of Theorem 2.6 requires the concept of forward/backward propagation waves. We adopt the following notations
\[
F_1^{(l)} = F_1^{(l)}(|\beta| P > 1) \quad \text{or} \quad F_1^{(0)} = F_1^{(0)}(|\beta| |P| > 1), \quad l = 0, 1,
\]
and
\[
F_c = F_c \left( \frac{|x|}{\rho^a} \leq 1 \right) \quad \text{or} \quad F_c = F_c \left( \frac{|x|}{S^a} \leq 1 \right)
\]
provided that it does not lead to any confusion.

4.1. Forward/backward propagation waves. We start by discussing the concept of forward and backward propagation waves. These waves are analogous to the incoming and outgoing waves first introduced by Enss [9].

Let \( S^{n-1} \) denote the unit sphere in \( \mathbb{R}^n \). We define a class of functions on \( S^{n-1} \), \( \{ F^\beta(\xi) \}_{\beta \in I} \), as a smooth partition of unity with an index set
\[
I = \{ \hat{h}_1, \cdots, \hat{h}_N \} \subseteq S^{n-1}
\]
for some \( N \in \mathbb{N}^+ \), satisfying that there exists \( c > 0 \) such that for every \( \hat{h}_i \in I \),
\[
F^\beta(\xi) = \begin{cases} 
1 & \text{when } |\xi - \hat{h}| < c \\
0 & \text{when } |\xi - \hat{h}| > 2c
\end{cases}, \quad \xi \in S^{n-1}.
\]
Given \( \hat{h} \in I \), we define \( \tilde{F}^\hat{h} : S^{n-1} \to \mathbb{R} \), as another smooth cut-off function satisfying

\[
\tilde{F}^\hat{h}(\xi) = \begin{cases} 
1 & \text{when } |\xi - \hat{h}| < 4c, \quad \xi \in S^{n-1}. \\
0 & \text{when } |\xi - \hat{h}| > 8c.
\end{cases}
\]

For \( h \in \mathbb{R}^n \setminus \{0\} \), we define \( \hat{h} := h/|h| \) and \( \hat{h} = 0 \) when \( h = 0 \). We also assume that \( c > 0 \), defined in (4.4) and (4.5), is properly chosen such that for all \( x, q \in \mathbb{R}^n \) with \( x \neq 0 \) and \( q \neq 0 \),

\[
F^\hat{h}(\hat{x})\tilde{F}^\hat{h}(\hat{q})|x + q| \geq \frac{1}{10}(|x| + |q|),
\]

and

\[
F^\hat{h}(\hat{x})(1 - \tilde{F}^\hat{h}(\hat{q}))|x - q| \geq \frac{1}{10^\varepsilon}(|x| + |q|).
\]

Now let us define the projection on forward/backward propagation set in terms of the phase-space \((r, v) \in \mathbb{R}^{n+m}:\)

**Definition 4.1** (Projection on the forward/backward propagation set). The projections onto the forward and backward propagation sets, in terms of \((r, v) \in \mathbb{R}^{n+m} \), are defined as follows:

\[
P^+(r, v) := \sum_{b=1}^{N} F^{\hat{h}_0}(\hat{r})\tilde{F}^{\hat{h}_0}(\hat{v}),
\]

and

\[
P^-(r, v) := 1 - P^+(r, v),
\]

respectively.

4.1.1. *Estimates for Schrödinger operators.* With \( P = -i\nabla_s \), we define the projections \( P^\pm \) as

\[
P^\pm \equiv P^\pm(x, 2P).
\]

We need following estimates and their proofs can be found in Appendix B.

**Lemma 4.2.** For all \( \varepsilon \in (0, 1/2) \) and \( s, t, \sigma \geq 0 \), the estimates

\[
\|\mathcal{F}_s\left(\frac{|x|}{(t + 1)^{1/2 + \varepsilon}}\right) \geq 1\|P^\pm e^{\pm it\hat{h}_0}\mathcal{F}_1\left(\sqrt{t + 1}|P| \geq 1\right)\langle x \rangle^{-\sigma} \|_{L^2_{\sigma}(\mathbb{R}^n)} \to L^2_{\sigma}(\mathbb{R}^n)} \lesssim_{\varepsilon} \frac{1}{(t + 1)^{1/2 + \varepsilon}}
\]

and

\[
\|P^\pm e^{\pm it\hat{h}_0}\mathcal{F}_1\left((s + 1)^{1/2 - \varepsilon}|P| \geq 1\right)\langle x \rangle^{-\sigma} \|_{L^2_{\sigma}(\mathbb{R}^n)} \to L^2_{\sigma}(\mathbb{R}^n)} \lesssim_{\varepsilon} \frac{1}{(s)^{\sigma/2}}
\]

hold true.

**Lemma 4.3.** For all \( \varepsilon \in (0, 1/2), t, \sigma \geq 0 \) and \( s \in [0, t] \), the estimate

\[
\|\mathcal{F}_s\left(\frac{|x|}{(t + 1)^{1/2 + \varepsilon}}\right) \geq 1\|P^+ e^{-it\hat{h}_0}\mathcal{F}_1\left(\sqrt{t + 1}|P| < 1\right)\langle x \rangle^{-\sigma} \|_{L^2_{\sigma}(\mathbb{R}^n)} \to L^2_{\sigma}(\mathbb{R}^n)} \lesssim_{\varepsilon} \frac{1}{(t + 1)^{1/2 + \varepsilon + \varepsilon}}
\]

\[
\|\mathcal{F}_s\left(\frac{|x|}{(t + 1)^{1/2 + \varepsilon}}\right) \geq 1\|P^- e^{it\hat{h}_0}\mathcal{F}_1\left(\sqrt{t + 1}|P| \geq 1\right)\langle x \rangle^{-\sigma} \|_{L^2_{\sigma}(\mathbb{R}^n)} \to L^2_{\sigma}(\mathbb{R}^n)} \lesssim_{\varepsilon} \frac{1}{(t + 1)^{1/2 + \varepsilon}}
\]
holds true.

**Lemma 4.4.** For all \( f \in L^2_\Omega(\mathbb{R}^n) \), we have

\[
\lim_{s \to \infty} \|P^+_s e^{i s H_0} f\|_{L^2_\Omega(\mathbb{R}^n)} = 0.
\]

**Lemma 4.5.** For all \( f \in L^2_\Omega(\mathbb{R}^n) \), \( \alpha \in (0, 1) \) and \( s \geq 0 \), we have

\[
\lim_{s \to \infty} \|\chi(|x| \leq s^\alpha P^+_s e^{i s H_0} f\|_{L^2_\Omega(\mathbb{R}^n)} = 0.
\]

### 4.1.2. Estimates for translated Schrödinger operators

For \( \eta \in \mathbb{R}^n \) and \( t \geq 0 \), we define the projections \( P^\pm_{t\eta} \equiv P^\pm( x - t\eta, 2P - \eta) \).

We need following estimates for charge-transfer problems and their proofs can be found in Appendix B.

**Lemma 4.6.** For all \( \eta \in \mathbb{R}^n \), and \( t, \sigma \geq 0 \), the estimates,

\[
\|\mathcal{F}_e \left( \frac{|x - t\eta|}{(t + 1)^{1/2 + \varepsilon}} \right) \geq 1 \| P^\pm_{t\eta} e^{i(s - t)H_0} \mathcal{F}_\Omega \left( \sqrt{t + 1} |2P - \eta| \geq 1 \right) (x - s\eta)^{\sigma} \|_{L^2_\Omega(\mathbb{R}^n) \to L^2_\Omega(\mathbb{R}^n)} \lesssim \frac{1}{(t + 1)^{\sigma/2}}
\]

holds true.

**Lemma 4.7.** For all \( \varepsilon \in (0, 1/2) \), \( \eta \in \mathbb{R}^n \), \( t, \sigma \geq 0 \) and \( s \in [0, t] \), the estimate

\[
\|\mathcal{F}_e \left( \frac{|x - t\eta|}{(t + 1)^{1/2 + \varepsilon}} \right) \geq 1 \| P^\pm_{t\eta} e^{-i(s-t)H_0} \mathcal{F}_\Omega \left( \sqrt{t + 1} |2P - \eta| < 1 \right) (x - s\eta)^{\sigma} \|_{L^2_\Omega(\mathbb{R}^n) \to L^2_\Omega(\mathbb{R}^n)} \lesssim \frac{1}{(t + 1)^{\sigma/2}}
\]

holds true.

**Lemma 4.8.** For all \( f \in L^2_\Omega(\mathbb{R}^n) \) and \( \eta \in \mathbb{R}^n \), we have

\[
\lim_{t \to \infty} \|P^\pm_{t\eta} e^{-i H_0} f\|_{L^2_\Omega(\mathbb{R}^n)} = 0.
\]

**Lemma 4.9.** For all \( f \in L^2_\Omega(\mathbb{R}^n) \), \( \eta \in \mathbb{R}^n \), \( \alpha \in (0, 1) \) and \( t \geq 0 \), we have

\[
\lim_{t \to \infty} \|\chi(|x - t\eta| \leq t^\alpha) P^\pm_{t\eta} e^{-i H_0} f\|_{L^2_\Omega(\mathbb{R}^n)} = 0.
\]

### 4.2. Proof of Theorem 2.1

**Proof of Theorem 2.1.** We define

\[
\Omega^*_\alpha(t)\psi(0) := e^{i H_0} \mathcal{F}_e \left( \frac{|x - 2tP|}{t^\alpha} \right) \leq 1) \psi(t).
\]

By Eqs. (1.7) and (4.22), \( \Omega^*_\alpha(t)\psi(0) \) reads, with \( \mathcal{F}_e \equiv \mathcal{F}_e \left( \frac{|x|}{t^\alpha} \leq 1 \right),
\]

\[
\Omega^*_\alpha(t)\psi(0) = \mathcal{F}_e e^{iH_0} \psi(t).
\]
In what follows, we use
\begin{equation}
\mathcal{F}_c = \mathcal{F}_c\left(\frac{|x|}{t^\alpha} \leq 1\right) \quad \text{or} \quad \mathcal{F}_c = \mathcal{F}_c\left(\frac{|x|}{s^\alpha} \leq 1\right),
\end{equation}
when it does not lead to confusion. Using Cook’s method to expand \(\Omega_{\alpha}^*(t)\psi(0)\), we obtain
\begin{equation}
\Omega_{\alpha}^*(t)\psi(0) = \Omega_{\alpha}^*(1)\psi(0) + (-i) \int_1^t \mathcal{F}_c e^{itH_0} \mathcal{N}(x, s, \psi(s))\psi(s)ds + \int_1^t \partial_t [\mathcal{F}_c] e^{itH_0} \psi(s)ds
\end{equation}
(4.25) 
\[=: \Omega_{\alpha}^*(1)\psi(0) + \psi_{\text{ini}}(t) + \psi_{\rho}(t).\]
By the unitarity of \(e^{itH_0}\), Assumption 2.1 and Eq. (4.23),
\begin{equation}
\Omega_{\alpha}^*(1)\psi(0) \text{ exists in } L^2_s(\mathbb{R}^n).
\end{equation}
By \(\alpha \in (0, 1 - 2/n), n \geq 3 \) and Proposition 3.4, \(\psi_{\text{ini}}(t)\) satisfies the estimate, with \(\beta = \frac{m(1-\alpha)}{2} - 1\),
\begin{equation}
\|\psi_{\text{ini}}(t)\|_{L^2_s(\mathbb{R}^n)} \leq \int_1^t \|\mathcal{F}_c e^{itH_0} \mathcal{N}(x, s, \psi(s))\psi(s)\|_{L^2_s(\mathbb{R}^n)}ds
\end{equation}
\[\leq \|N(x, s, \psi(s))\|_{L^\infty_n L^1_{s}(\mathbb{R}^{n+1})}ds.
\end{equation}
(4.27)
which implies that
\begin{equation}
\psi_{\text{ini}}(\infty) \text{ exists in } L^2_s(\mathbb{R}^n).
\end{equation}
For \(\psi_{\rho}(t)\), we use \textbf{RPRES} by taking \(b = 1\) and
\begin{equation}
\begin{cases}
B(t) := \mathcal{F}_c \\
\phi(t) = e^{itH_0}\psi(t)
\end{cases}
\end{equation}
(4.29)
\begin{equation}
\partial_t \langle B : \phi(t) \rangle = (\phi(t), \partial_t \mathcal{F}_c \phi(t))_{L^2_s(\mathbb{R}^n)} + (-i)(\phi(t), \mathcal{F}_c e^{itH_0} \mathcal{N}(x, t, \psi(t))\psi(t))_{L^2_s(\mathbb{R}^n)}
\end{equation}
\[+ \iota \mathcal{F}_c e^{itH_0} \mathcal{N}(x, t, \psi(t))\psi(t), \phi(t)\}_{L^2_s(\mathbb{R}^n)}
\end{equation}
(4.30)
\[=: (\phi(t), C^*C \phi(t))_{L^2_s(\mathbb{R}^n)} + g(t)
\end{equation}
where \(C^*C\) and \(g(t)\) are given by, with \(\mathcal{F}_c' (\lambda \leq 1) \equiv \frac{d}{ds} [\mathcal{F}_c (\lambda \leq 1)].\)
\begin{equation}
C^*C := \partial_t [\mathcal{F}_c] = \mathcal{F}_c' \left(\frac{|x|}{t^\alpha} \leq 1\right) \times \frac{-\alpha}{t} \times \frac{|x|}{t^\alpha} \geq 0
\end{equation}
(4.31)
and
\begin{equation}
g(t) := (-i)(\phi(t), \mathcal{F}_c e^{itH_0} \mathcal{N}(x, t, \psi(t))\psi(t))_{L^2_s(\mathbb{R}^n)} + \iota \mathcal{F}_c e^{itH_0} \mathcal{N}(x, t, \psi(t))\psi(t), \phi(t)\}_{L^2_s(\mathbb{R}^n)}
\end{equation}
(4.32)
We observe that \(\langle B : \phi(t) \rangle\) is uniformly bounded over \(t\). Utilizing the Cauchy-Schwarz inequality, the unitarity of \(e^{itH_0}\), and Assumption 2.1, we have:
\begin{equation}
|\langle B : \phi(t) \rangle| = (e^{itH_0} \psi(t), \mathcal{F}_c e^{itH_0} \psi(t))_{L^2_s(\mathbb{R}^n)} \leq \|e^{itH_0} \psi(t)\|_{L^2_s(\mathbb{R}^n)}^2 = \|\psi(t)\|_{L^2_s(\mathbb{R}^n)}^2 \leq E^2.
\end{equation}
(4.33)
Furthermore, \( g(t) \in L^1_t[1, \infty) \). To be precise, by applying the Cauchy-Schwarz inequality, and using Assumptions 2.1 and 2.3, along with Proposition 3.4, \( g(t) \) satisfies the following estimate:

\[
|g(t)| \leq 2\|\phi(t)\|_{L^2_t(\mathbb{R}^n)}\|F_c e^{iH_0} N(x, t, \psi(t))\|_{L^2_t(\mathbb{R}^n)}
\]

(4.34)

\[
\leq n^\frac{E}{1+\beta} \|N(x, t, \psi(t))\|_{L^\infty_t L^{\infty+1}_r(\mathbb{R}^n)} \in L^1_t[1, \infty).
\]

Hence, the family \( \{B(t)\}_{t \in [1, \infty)} \) is a RPORB with respect to \( \phi(t) = e^{iH_0} \psi(t) \) and by Eqs. (3.10), (4.33) and (4.34), we obtain

\[
\int_1^\infty |(\phi(t), \partial_t[F_c] \phi(t))_{L^2_t(\mathbb{R}^n)}| dt = \int_1^\infty |(\phi(t), \partial_t[F_c] \phi(t))_{L^2_t(\mathbb{R}^n)}| dt \leq \sup_{t \in [1, \infty)} |\langle B : \phi \rangle_t| + \|g(t)\|_{L^2_t[1, \infty)}
\]

(4.35)

\[
\leq n^2 E^2 + E \|N(x, t, \psi(t))\|_{L^\infty_t L^{\infty+1}_r(\mathbb{R}^n)} < \infty.
\]

By Cauchy-Schwarz inequality and the non-negativity of \( \partial_t[F_c] \) (see (4.31)), \( \psi(t) \) satisfies the estimate, for \( T_2 \geq T_1 \geq 1, \)

\[
\|\psi_p(T_2) - \psi_p(T_1)\|_{L^2_t(\mathbb{R}^n)} \leq \left( \int_{T_1}^{T_2} |\partial_t[F_c] \phi(t)| dt \right)^{1/2} \left( \int_{T_1}^{T_2} |\partial_t[F_c] \phi(t)|^2 dt \right)^{1/2} \|L^2_t(\mathbb{R}^n).
\]

(4.36)

By estimates

\[
\int_{T_1}^{T_2} \partial_t[F_c] dt = F_c (\frac{|x|}{T_2 t} \leq 1) \int_{T_1}^{T_2} \partial_t[F_c] (\frac{|x|}{T_2 t} \leq 1) \leq 1 \leq 1
\]

and (4.35), estimate (4.36) leads to

\[
\|\psi_p(T_2) - \psi_p(T_1)\|_{L^2_t(\mathbb{R}^n)} \leq \left( \int_{T_1}^{T_2} |\partial_t[F_c] \phi(t)|^2 dt \right)^{1/2} \|L^2_t(\mathbb{R}^n) = \left( \int_{\mathbb{R}^n} \int_{T_1}^{T_2} |\partial_t[F_c] \phi(t)|^2 dt dx \right)^{1/2}
\]

(4.38)

\[
= \left( \int_{T_1}^{T_2} |(\phi(t), \partial_t[F_c] \phi(t))_{L^2_t(\mathbb{R}^n)}| dt \right)^{1/2} \to 0
\]

as \( T_1 \to \infty \), where we have used

\[
\int_{\mathbb{R}^n} \int_{T_1}^{T_2} |\partial_t[F_c] \phi(t)|^2 dt dx \leq (T_2 - T_1) \sup_{t \in \mathbb{R}^n} \|\psi(t)\|_{L^2_t(\mathbb{R}^n)}^2 < \infty
\]

and Fubini’s Theorem. Hence, \( \{\psi_p(t)\}_{t \geq 1} \) is Cauchy in \( L^2_t(\mathbb{R}^n) \) and therefore

(4.40)

\[ \psi_p(\infty) \text{ exists in } L^2_t(\mathbb{R}^n). \]

Eq. (4.40), together with Eqs. (4.26) and (4.28), implies that

(4.41)

\[ \Omega^\alpha_n \psi(0) \equiv \Omega^\alpha_n (\infty) \psi(0) \text{ exists in } L^2_t(\mathbb{R}^n). \]

We also have that for all \( \alpha, \alpha' \in (0, 1 - 2/n) \) and \( \varphi \in L^2_t(\mathbb{R}^n) \), by Cauchy-Schwarz inequality, Assumption 2.1 and the unitarity of \( e^{iH_0} \),

\[
|\langle \phi, \Omega^\alpha_n(t) \psi(0) - \Omega^\alpha_{n'}(t) \psi(0) \rangle_{L^2_t(\mathbb{R}^n)}| = \left| \langle (F_c (\frac{|x|}{n'} \leq 1) - F_c (\frac{|x|}{n'} \leq 1)) \phi, e^{iH_0} \psi(t) \rangle_{L^2_t(\mathbb{R}^n)} \right|
\]

\[
\leq \| (F_c (\frac{|x|}{n'} \leq 1) - F_c (\frac{|x|}{n'} \leq 1)) \phi \|_{L^2_t(\mathbb{R}^n)} \| e^{iH_0} \psi(t) \|_{L^2_t(\mathbb{R}^n)}
\]

\[
\leq \| (F_c (\frac{|x|}{n'} \leq 1) - F_c (\frac{|x|}{n'} \leq 1)) \phi \|_{L^2_t(\mathbb{R}^n)} \| e^{iH_0} \psi(t) \|_{L^2_t(\mathbb{R}^n)}.
\]
\[ E\|F_c(\frac{x}{t^\alpha} \leq 1) - F_c(\frac{|x|}{t^\alpha} \leq 1)\|_{L^2(\mathbb{R}^n)} \leq 0 \]

as \( t \to \infty \). This implies

\[ \text{w- lim}_{t \to \infty} \Omega_\alpha(t) = 0, \quad \Omega_\alpha(t) \in L^2(\mathbb{R}^n) \]

and therefore, due to the existence of \( \Omega_\alpha(t) \) and \( \Omega_\alpha'(t) \) in \( L^2(\mathbb{R}^n) \) in strong sense, Eq. (2.13).

4.3. **Proof of Theorem 2.6.** The proof of Theorem 2.6 requires following proposition and lemma.

**Proposition 4.10.** Let \( \alpha, \beta \) and \( n \) be as in Theorem 2.6. Let Assumptions 2.1 and 2.2 be satisfied. Then \( \Omega_\alpha(t) \), defined in (2.15), exists in \( L^2(\mathbb{R}^n) \) and Eq. (2.16) is valid for all \( (\alpha, \beta) \), \( (\alpha', \beta') \in (0, c_1) \times (0, c_2) \) with \( \beta < \alpha \) and \( \beta' < \alpha' \).

**Proof.** Let

\[ \Omega_\alpha(t) = e^{itH_0}F_c(\frac{|x|}{t^\alpha} \leq 1)F_1(t^n|P| > 1)\psi(t). \]

By Eq. (1.7), \( \Omega_\alpha(t) \psi(t) \) reads, with \( F_c \equiv F_c(\frac{|x|}{t^\alpha} \leq 1) \) and \( F_1 \equiv F_1(t^n|P| > 1) \),

\[ \Omega_\alpha(t) \psi(t) = F_cF_1 e^{itH_0} \psi(t). \]

In what follows, we use

\[ F_c = F_c(\frac{|x|}{t^\alpha} \leq 1) \quad \text{or} \quad F_c = F_c(\frac{|x|}{t^\alpha} \leq 1) \]

and

\[ F_1 = F_1(t^n|P| > 1) \quad \text{or} \quad F_1 = F_1(t^n|P| > 1), \]

when it does not lead to confusion. By Cook’s method to expand \( \Omega_\alpha(t) \psi(t) \), \( \Omega_\alpha'(t) \psi(t) \) can be rewritten as

\[ \Omega_\alpha(t) \psi(t) = \Omega_\alpha(1) \psi(t) + (-i) \int_1^t F_cF_1 e^{itH_0} N(x, s, \psi(s)) \psi(s) ds + \int_1^t \partial_s [F_c] F_1 e^{itH_0} \psi(s) ds + \int_1^t \partial_s [F_1] F_c e^{itH_0} \psi(s) ds + \int_1^t [F_c, \partial_s [F_1]] e^{itH_0} \psi(s) ds \]

\[ = \Omega_\alpha(1) \psi(t) + \psi_{int}(t) + \psi_{p,1}(t) + \psi_{p,2}(t) + \psi_c(t). \]

By the unitarity of \( e^{itH_0} \), Assumption 2.1 and Eq. (4.44),

\[ \Omega_\alpha(1) \psi(t) \text{ exists in } L^2(\mathbb{R}^n). \]

By Proposition 3.2, we obtain that

\[ \|F_cF_1 e^{itH_0} N(x, t, \psi(t)) \psi(t) \|_{L^2(\mathbb{R}^n)} \in L^1_{\infty} \]

which implies that

\[ \psi_{int}(\infty) \text{ exists in } L^2(\mathbb{R}^n). \]
For $\psi_{p,1}(t)$ and $\psi_{p,2}(t)$, we use R PRES by taking, with $b = 1$,
\begin{align}
B_1(t) := \mathcal{F}_c \mathcal{F}_1 t \\
\phi(t) = e^{itH_0} \psi(t)
\end{align}
and
\begin{align}
B_2(t) := \sqrt{\mathcal{F}_c} \mathcal{F}_1^2 \sqrt{\mathcal{F}_c} \\
\phi(t) = e^{itH_0} \psi(t)
\end{align}
respectively.

We begin with R PRES for $\psi_{p,1}(t)$. We find that
\begin{align}
\partial_t \langle B_1 : \phi(t) \rangle_t &= (\phi(t), \partial_t[\mathcal{F}_c \mathcal{F}_1] \phi(t))_{L^2(\mathbb{R}^n)} + (-i)(\phi(t), \mathcal{F}_c \mathcal{F}_1 e^{itH_0} N(x, t, \psi(t)) \psi(t))_{L^2(\mathbb{R}^n)} \\
&+ i(\mathcal{F}_c \mathcal{F}_1 e^{itH_0} N(x, t, \psi(t)) \psi(t), \phi(t))_{L^2(\mathbb{R}^n)}
\end{align}
By
\begin{align}
\partial_t[\mathcal{F}_c \mathcal{F}_1] &= \mathcal{F}_c \partial_t[\mathcal{F}_1] + \mathcal{F}_1 \partial_t[\mathcal{F}_c] + \mathcal{F}_1 \partial_t[\mathcal{F}_1] \\
&= \mathcal{F}_c \partial_t[\mathcal{F}_1] + \mathcal{F}_1 \partial_t[\mathcal{F}_c] + \mathcal{F}_1 \partial_t[\mathcal{F}_1] \\
&+ [\mathcal{F}_c, \mathcal{F}_1] \partial_t[\mathcal{F}_1]
\end{align}
where $F_1(t)$ is given by
\begin{align}
F_1(t) &= [\partial_t[\mathcal{F}_1], \mathcal{F}_c] \mathcal{F}_1 \\
&+ [\mathcal{F}_c, \partial_t[\mathcal{F}_1]] \mathcal{F}_1
\end{align}
Eq. (4.54) implies
\begin{align}
\partial_t \langle B_1 : \phi(t) \rangle_t &= \sum_{j=1}^{2} (\phi(t), C_j C_1 \phi(t))_{L^2(\mathbb{R}^n)} + g(t)
\end{align}
where $C_j C_1$, $j = 1, 2$, and $g(t)$ are given by, with $\mathcal{F}_c'(\lambda \leq 1) \equiv \frac{d}{d\lambda}[\mathcal{F}_c(\lambda \leq 1)]$,
\begin{align}
C_1 C_1 := \mathcal{F}_c \partial_t[\mathcal{F}_1] \mathcal{F}_1
&= \mathcal{F}_c \mathcal{F}_1' \frac{|x|}{t^a} \leq 1 \times \frac{-\alpha}{t} \times \frac{|x|}{t^a} \mathcal{F}_1 \\
&\geq 0
\end{align}
and
\begin{align}
C_2 C_2 := 2 \sqrt{\mathcal{F}_c} \partial_t[\mathcal{F}_1] \mathcal{F}_1 \sqrt{\mathcal{F}_c}
&= \sqrt{\mathcal{F}_c} \mathcal{F}_1' (p^j |P| \geq 1) \times \frac{B}{t} \times \frac{|P|}{\sqrt{\mathcal{F}_c}} \\
&\geq 0
\end{align}
and
\[
g(t) := (-i)(\phi(t), \mathcal{F}_c \mathcal{F}_1 \mathcal{F}_2 e^{i \mathcal{H}_0} \mathcal{N}(x, t, \psi(t)) \psi(t))_{L_2^2(\mathbb{R}^n)}
+ i(\mathcal{F}_c \mathcal{F}_1 \mathcal{F}_2 e^{i \mathcal{H}_0} \mathcal{N}(x, t, \psi(t)) \psi(t), \phi(t))_{L_2^2(\mathbb{R}^n)} + (\phi(t), F_1(t) \phi(t))_{L_2^2(\mathbb{R}^n)},
\]
respectively. Here, we note that \(|B_1 : \phi(t)\rangle_i|\) is uniformly bounded in \(t\) by Cauchy-Schwarz inequality, unitarity of \(e^{i \mathcal{H}_0}\) and Assumption 2.1,
\[
|\langle B_1 : \phi(t) \rangle| = (e^{i \mathcal{H}_0} \psi(t), \mathcal{F}_c \mathcal{F}_1 \mathcal{F}_2 e^{i \mathcal{H}_0} \psi(t))_{L_2^2(\mathbb{R}^n)}
\leq \|e^{i \mathcal{H}_0} \psi(t)\|_{L_2^2(\mathbb{R}^n)}^2
= \|\psi(t)\|_{L_2^2(\mathbb{R}^n)}^2
\leq E^2,
\]
and \(g(t) \in L_1^1[1, \infty)\): by Lemma 3.6 and Eq. (4.56), with \(\alpha > \beta\),
\[
|\langle \phi(t), F_1(t) \phi(t) \rangle|_{L_2^2(\mathbb{R}^n)}|
\leq E \|\partial_t \mathcal{F}_1\| \sqrt{\mathcal{F}_c} \|L_2^2(\mathbb{R}^n) \rightarrow L_2^2(\mathbb{R}^n)} + \|\partial_t \mathcal{F}_1\|_{L_2^2(\mathbb{R}^n)} \|\sqrt{\mathcal{F}_c} \|_{L_2^2(\mathbb{R}^n) \rightarrow L_2^2(\mathbb{R}^n)}
+ \|\sqrt{\mathcal{F}_c} \|_{L_2^2(\mathbb{R}^n) \rightarrow L_2^2(\mathbb{R}^n)} \|\partial_t \mathcal{F}_1\|_{L_2^2(\mathbb{R}^n) \rightarrow L_2^2(\mathbb{R}^n)}
\leq E, n \frac{1}{t^{1+\alpha - \beta}}
\]
(4.62) \(\leq E, n \in L_1^1[1, \infty)\),
where the extra factor \(\frac{1}{t}\) comes from \(\partial_t \mathcal{F}_1\). Estimate (4.62), together with Cauchy-Schwarz inequality, Assumptions 2.1 and 2.2 and Proposition 3.2, implies the estimate, with \(\delta > 1\) and \(\alpha > \beta\),
\[
|g(t)| \leq 2\|\phi(t)\|_{L_2^2(\mathbb{R}^n)} \|\mathcal{F}_c \mathcal{F}_1 e^{i \mathcal{H}_0} \mathcal{N}(x, t, \psi(t)) \psi(t)\|_{L_2^2(\mathbb{R}^n)} + |\langle \phi(t), F_1(t) \phi(t) \rangle|_{L_2^2(\mathbb{R}^n)}|
\leq E, n \frac{1}{t^{\frac{\alpha}{2}}} \|\mathcal{N}(x, t, \psi(t))\|_{L_2^2(\mathbb{R}^{n+1})} + \frac{1}{t^{1+\alpha - \beta}}
\]
(4.63) \(\in L_1^1[1, \infty)\).
Hence, the family \(\{B_1(t)\}_{t \in [1, \infty)}\) is a RPROB with respect to \(\phi(t) = e^{i \mathcal{H}_0} \psi(t)\) and by Eqs. (3.10), (4.61) and (4.63), we obtain
\[
\int_1^{\infty} |\langle \phi(t), \mathcal{F}_1 \partial_t \mathcal{F}_1 \mathcal{F}_2 \phi(t) \rangle|_{L_2^2(\mathbb{R}^n)} dt
= \int_1^{\infty} (\phi(t), \mathcal{F}_1 \partial_t \mathcal{F}_1 \mathcal{F}_2 \phi(t))_{L_2^2(\mathbb{R}^n)} dt
\leq \sup_{t \in [1, \infty)} |\langle B_1 : \phi(t) \rangle| + \|g(t)\|_{L_1^1[1, \infty)}
\leq E, n + \|\mathcal{N}(x, t, \psi(t))\|_{L_2^2(\mathbb{R}^{n+1})}
< \infty.
\]
(4.64)
By Cauchy-Schwarz inequality and the non-negativity of \(\partial_t \mathcal{F}_1\) (see (4.31)), \(\psi_{p,1}(t)\) satisfies the estimate, for \(T_2 \geq T_1 \geq 1\),
\[
\|\psi_{p,1}(T_2) - \psi_{p,1}(T_1)\|_{L_2^2(\mathbb{R}^n)}
\]
as \( T \) and Fubini’s Theorem. Hence,

\[
\text{(4.67)}
\]

\[
\text{(4.68)}
\]

Next, we obtain the RPRES for which implies, for all \( \text{(4.70)} \)

\[
\text{(4.71)}
\]

as \( T_1 \to \infty \), where we have used

\[
\text{(4.69)}
\]

\[
\text{(4.70)}
\]

which implies, for all \( T_2 \geq T_1 \geq 1 \),

\[
\text{(4.71)}
\]
as $T_1 \to \infty$. Hence, $\{\psi_{p,2}(t)\}_{t \geq 1}$ is Cauchy in $L^2_1(\mathbb{R}^n)$ and
\begin{equation}
\psi_{p,2}(\infty) \text{ exists in } L^2_1(\mathbb{R}^n).
\end{equation}

By Lemma 3.6, with $\alpha > \beta$,
\begin{equation}
\| \mathcal{F}_e, \partial_t |\mathcal{F}_1| \| e^{itH_0} \psi(t) \|_{L^2_1(\mathbb{R}^n)} \lesssim_{\alpha, \beta} \frac{1}{t^{1+\alpha-\beta}} \in L^1_t[1, \infty),
\end{equation}
which implies that
\begin{equation}
\psi_\epsilon(\infty) \text{ exists in } L^2_1(\mathbb{R}^n).
\end{equation}

\begin{align}
\text{(4.49), (4.51), (4.68), (4.72) and (4.74), together with Eq. (4.48), imply that}
\end{align}

\begin{equation}
\Omega^{*}_{\alpha, \beta} \psi(0) \text{ exists in } L^2_1(\mathbb{R}^n).
\end{equation}

Take $\phi \in L^2_1(\mathbb{R}^n)$. We also have that for all $(\alpha, \beta), (\alpha', \beta') \in (0, c_1) \times (0, c_2)$ with $\beta < \alpha$ and $\beta' < \alpha'$, by Cauchy-Schwarz inequality, Assumption 2.1 and the unitarity of $e^{itH_0}$,
\begin{align}
&\left| \left( \phi, \Omega^{*}_{\alpha, \beta}(t) \psi(0) - \Omega^{*}_{\alpha', \beta'}(t) \psi(0) \right)_{L^2_1(\mathbb{R}^n)} \right|
\leq \left| \left( \mathcal{F}_e \left( \frac{|x|}{t^{\alpha}} \leq 1 \right) \mathcal{F}_1 \left( \frac{|x|}{t^{\beta}} \leq 1 \right) - \mathcal{F}_e \left( \frac{|x|}{t^{\alpha'}} \leq 1 \right) \mathcal{F}_1 \left( \frac{|x|}{t^{\beta'}} \leq 1 \right) \right) \phi, e^{itH_0} \psi(t) \right|_{L^2_1(\mathbb{R}^n)}
\leq E \left| \left( \mathcal{F}_e \left( \frac{|x|}{t^{\alpha}} \leq 1 \right) - \mathcal{F}_e \left( \frac{|x|}{t^{\alpha'}} \leq 1 \right) \right) \phi \right|_{L^2_1(\mathbb{R}^n)}
\leq E \left| \left( \mathcal{F}_e \left( \frac{|x|}{t^{\alpha}} \leq 1 \right) - \mathcal{F}_e \left( \frac{|x|}{t^{\alpha'}} \leq 1 \right) \right) \phi \right|_{L^2_1(\mathbb{R}^n)}
\to 0
\end{align}
as $t \to \infty$. This implies
\begin{equation}
\text{w- lim}_{t \to \infty} \Omega^{*}_{\alpha, \beta}(t) \psi(0) - \Omega^{*}_{\alpha', \beta'}(t) \psi(0) = 0, \quad \text{in } L^2_1(\mathbb{R}^n)
\end{equation}
and therefore, due to the existence of $\Omega^{*}_{\alpha, \beta} \psi(0)$ and $\Omega^{*}_{\alpha', \beta'} \psi(0)$ in $L^2_1(\mathbb{R}^n)$ in strong sense, Eq. (2.16). \hfill \Box

We define
\begin{equation}
\psi_{wl}(t) := \mathcal{F}_e \left( \frac{|x|}{(t+1)^{1/2+\epsilon}} < 1 \right) \psi(t).
\end{equation}

**Lemma 4.11.** Let $\psi(t)$ be the solution to system (2.1). Let Assumptions 2.1 and 2.2 be satisfied. Then
\begin{equation}
\lim_{t \to \infty} \| \psi(t) - e^{-itH_0} \Omega^{*}_{\alpha, \beta} \psi(0) - \psi_{wl}(t) \|_{L^2_1(\mathbb{R}^n)} = 0.
\end{equation}

We find that $\mathcal{F}_e \left( \frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1 \right) \psi(t) \equiv \psi(t) - \psi_{wl}(t)$ satisfies the decomposition
\begin{equation}
\mathcal{F}_e \left( \frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1 \right) \psi(t) = \psi_1(t) + \psi_2(t) + \psi_3(t),
\end{equation}
where $\psi_j(t), j = 1, 2, 3$, are given by
\begin{equation}
\psi_1(t) := \mathcal{F}_e \left( \frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1 \right) P^+ \mathcal{F}_1( \sqrt{t+1}|P| \geq 1) \psi(t),
\end{equation}
We approximate \(e^{-itH_0}\Omega_{\alpha,\beta}\psi(0)\) and arrive at (4.79) by showing that

\[
\lim_{t \to \infty} ||\psi_j(t)||_{L_2^2(\mathbb{R}^n)} = 0, \quad j = 2, 3,
\]

by Lemma 4.4. \(\psi_2(t)\) and \(\psi_3(t)\) read

\[
\psi_2(t) = \mathcal{F}^{-1} \left( \frac{|x|}{(t+1)^{1/2+\varepsilon}} \right) \geq 1)P^- e^{-itH_0}\psi(0)
\]

\[
+ (-i) \int_0^t \mathcal{F}^{-1} \left( \frac{|x|}{(t+1)^{1/2+\varepsilon}} \right) \geq 1)P^- e^{-it(\varepsilon+\delta)H_0} \mathcal{N}(x, s, \psi(s))\psi(s)ds
\]

\[
\psi_3(t) = \mathcal{F}^{-1} \left( \frac{|x|}{(t+1)^{1/2+\varepsilon}} \right) \geq 1)P^+ \mathcal{F}_1 (\sqrt{t+1}|P| \geq 1)e^{-itH_0}\Omega_{\alpha,\beta}\psi(0)
\]

\[
+ (-i) \int_0^t \mathcal{F}^{-1} \left( \frac{|x|}{(t+1)^{1/2+\varepsilon}} \right) \geq 1)P^+ \mathcal{F}_1 (\sqrt{t+1}|P| \geq 1)e^{-it(\varepsilon+\delta)H_0} \mathcal{N}(x, s, \psi(s))\psi(s)ds
\]

(4.87) \(\Rightarrow \psi_{21}(t) + \psi_{22}(t)\),

respectively. By Lemma 4.4, \(\psi_{21}(t)\) satisfies

\[
||\psi_{21}(t)||_{L_2^2(\mathbb{R}^n)} \leq ||P^- e^{-itH_0}\psi(0)||_{L_2^2(\mathbb{R}^n)} \to 0
\]

as \(t \to \infty\). By Lemma 4.2 and Assumptions 2.1 and 2.2, \(\psi_{22}(t)\) satisfies, with \(\delta > 2\),

\[
||\psi_{22}(t)||_{L_2^2(\mathbb{R}^n)} \leq \int_0^t ||\mathcal{F}^{-1} \left( \frac{|x|}{(t+1)^{1/2+\varepsilon}} \right) \geq 1)P^- e^{-it(\varepsilon+\delta)H_0} \mathcal{F}_1 (\sqrt{t+1}|P| \geq 1)|x|^{-\delta}||_{L_2^2(\mathbb{R}^n)} \to L_2^2(\mathbb{R}^n)ds
\]

\[
\times \sup_{u \in \mathbb{R}} ||\langle x \rangle^\delta \mathcal{N}(x, u, \psi(u))\psi(u)||_{L_2^2(\mathbb{R}^n)}ds
\]

\[
\lesssim \varepsilon \int_0^t \frac{1}{(t+1)^{1/2+\varepsilon} + s/\sqrt{t+1})^\delta} \sup_{u \in \mathbb{R}} ||\langle x \rangle^\delta \mathcal{N}(x, u, \psi(u))\psi(u)||_{L_2^2(\mathbb{R}^n)}ds
\]

\[
\lesssim \varepsilon \sup_{u \in \mathbb{R}} ||\langle x \rangle^\delta \mathcal{N}(x, u, \psi(u))\psi(u)||_{L_2^2(\mathbb{R}^n)}
\]

(4.90) \(\to 0\).
as $t \to \infty$. (4.89) and (4.90), together with Eq. (4.87), imply
\[
\lim_{t \to \infty} \|\psi_2(t)\|_{L^2_\alpha(\mathbb{R}^n)} = 0. 
\]
By
\[
s-\lim_{t \to \infty} \mathcal{F}_1(\sqrt{t^s + 1} |P| < 1) = 0, \quad \text{on } L^2_\alpha(\mathbb{R}^n),
\]
\(\psi_{31}(t)\) satisfies
\[
\lim_{t \to \infty} \sup_{s} \|\psi_{31}(t)\|_{L^2_\alpha(\mathbb{R}^n)} \leq \lim_{t \to \infty} \|\mathcal{F}_1(\sqrt{t^s + 1} |P| < 1)\psi(0)\|_{L^2_\alpha(\mathbb{R}^n)} = 0.
\]
By Lemma 4.3 and Assumptions 2.1 and 2.2, $\psi_{32}(t)$ satisfies, with $\delta > 2$,
\[
\|\psi_{32}(t)\|_{L^2_\alpha(\mathbb{R}^n)} \leq \int_0^t \|\mathcal{F}_c\left(\frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1\right)P^+ e^{-it(\epsilon-\delta)H_0}\mathcal{F}_1(\sqrt{t^s + 1} |P| < 1)\langle x \rangle^{-\delta}\|_{L^2_\alpha(\mathbb{R}^n)} - L^2_\alpha(\mathbb{R}^n)} ds
\]
\[
\leq \epsilon \int_0^t \frac{1}{(t+1)^{1/2+\epsilon}} \sup_{\nu \in \mathbb{R}} \|\langle x \rangle^{\delta} N(x, u, \psi(u))\|_{L^2_\alpha(\mathbb{R}^n)} ds
\]
\[
\leq \epsilon \frac{t}{(t+1)^{1/2+\epsilon}} \sup_{\nu \in \mathbb{R}} \|\langle x \rangle^{\delta} N(x, u, \psi(u))\|_{L^2_\alpha(\mathbb{R}^n)} \to 0
\]
as $t \to \infty$. (4.93) and (4.94), together with Eq. (4.88), imply
\[
\lim_{t \to \infty} \|\psi_3(t)\|_{L^2_\alpha(\mathbb{R}^n)} = 0.
\]
Eqs. (4.91) and (4.95) imply Eq. (4.84).

Next, we prove (4.85). Let
\[
\Omega^*\psi(0) := w-\lim_{t \to \infty} \psi(0) + (-i) \int_0^\infty e^{itH_0} N(x, s, \psi(s))\psi(s) ds \quad \text{in } L^2_\alpha(\mathbb{R}^n). 
\]
By Cook’s method, $\Omega^*\psi(0)$ exists in $L^2_\alpha(\mathbb{R}^n)$. By Eqs. (2.15) and (2.16), this implies $\Omega^*\psi(0) = \Omega^*_{\alpha, \beta}\psi(0)$ for any $\alpha \in (0, c_1)$ and $\beta \in (0, c_2)$. This is because the channel wave operator exists in the strong sens, and on the complement support, where $|x| \geq t^\epsilon$, the weak limit exists and is equal to zero. By Duhamel’s expansion and $\Omega^*\psi(0) = \Omega^*_{\alpha, \beta}\psi(0)$, we have
\[
\mathcal{F}_c\left(\frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1\right)P^+ \mathcal{F}_1(\sqrt{t^s + 1} |P| > 1) e^{-itH_0} \Omega^*_{\alpha, \beta}\psi(0) - \psi_1(t)
\]
\[
= (-i) \mathcal{F}_c\left(\frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1\right)P^+ \int_t^\infty \mathcal{F}_1(\sqrt{t^s + 1} |P| > 1) e^{i(s-t)H_0} N(x, s, \psi(s))\psi(s) ds.
\]
By Lemma 4.2 and Assumptions 2.1 and 2.2, (4.97) implies, with $\delta > 2$,
\[
\|\mathcal{F}_c\left(\frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1\right)P^+ \mathcal{F}_1(\sqrt{t^s + 1} |P| > 1) e^{-itH_0} \Omega^*_{\alpha, \beta}\psi(0) - \psi_1(t)\|_{L^2_\alpha(\mathbb{R}^n)}
\]
\[
\leq \int_t^\infty \|\mathcal{F}_c\left(\frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1\right)P^+ \mathcal{F}_1(\sqrt{t^s + 1} |P| > 1) e^{i(s-t)H_0} \langle x \rangle^{-\delta}\|_{L^2_\alpha(\mathbb{R}^n)} - L^2_\alpha(\mathbb{R}^n)} ds
\]
\[
\times \|\langle x \rangle^{\delta} N(x, s, \psi(s))\|_{L^2_\alpha(\mathbb{R}^n)} ds
\]
\[
\leq \epsilon \int_t^\infty \frac{1}{((t+1)^{1/2+\epsilon} + (s-t)/\sqrt{t^s + 1})^{\delta}} \|\langle x \rangle^{\delta} N(x, s, \psi(s))\|_{L^2_\alpha(\mathbb{R}^n)} ds
\]
as $t \to \infty$, (4.98) implies (4.85).

Now we prove (4.86). Eq. (4.92), together with Lemmas 4.4 and 4.5, implies

\[
\|e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0) - \mathcal{F}_c\left(\frac{|x|}{(t + 1)^{1/2 + \epsilon}}\right) \geq 1\| P^+ \mathcal{F}_1(\sqrt{t + 1}|P| \geq 1)e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0)\|_{L^2_\psi}\]
\[
\leq\|\mathcal{F}_c\left(\frac{|x|}{(t + 1)^{1/2 + \epsilon}}\right) < 1\| P^+ \mathcal{F}_1(\sqrt{t + 1}|P| < 1)e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0)\|_{L^2_\psi} + \| P^+ e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0)\|_{L^2_\psi}
\]
\[
\leq\|\mathcal{F}_1(\sqrt{t + 1}|P| < 1)e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0)\|_{L^2_{\psi}}\]
\[
\leq\|\mathcal{F}_1(\sqrt{t + 1}|P| < 1)e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0)\|_{L^2_{\psi}}\leq 1\| P^+ e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0)\|_{L^2_\psi}
\]
\[
(4.99)
\]
\[
+ \| P^+ e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0)\|_{L^2_\psi} \to 0
\]
as $t \to \infty$, (4.99) implies (4.86). By (4.84), (4.85), (4.86) and Eq. (4.80), together with Eq. (4.78), we arrive at

\[
\|\psi(t) - \psi_{\mathcal{F}_c}(t) - e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0)\|_{L^2_{\psi}}
\leq\|\mathcal{F}_c\left(\frac{|x|}{(t + 1)^{1/2 + \epsilon}} \geq 1\| \psi(t) - e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0)\|_{L^2_{\psi}}
\]
\[
\leq\|\mathcal{F}_c\left(\frac{|x|}{(t + 1)^{1/2 + \epsilon}} \geq 1\| \psi(t) - e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0)\|_{L^2_{\psi}}
\]
\[
\leq\|\psi_1(t) - \mathcal{F}_c\left(\frac{|x|}{(t + 1)^{1/2 + \epsilon}} \geq 1\| \psi(t) - e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0)\|_{L^2_{\psi}}
\]
\[
+ \| e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0) - \mathcal{F}_c\left(\frac{|x|}{(t + 1)^{1/2 + \epsilon}} \geq 1\| \psi(t) - e^{-itH_0}\Omega_{\alpha,\beta}^*\psi(0)\|_{L^2_{\psi}}
\]
\[
(4.100) + \|\psi_2(t)\|_{L^2_{\psi}} + \|\psi_3(t)\|_{L^2_{\psi}} \to 0,
\]
as $t \to \infty$.

**Proof of Theorem 2.6.** It follows from Proposition 4.10, Lemma 4.11 and Eq. (4.78).

\[
\square
\]

5. Charge Transfer Potentials

In this section, we prove Theorem 2.9. The proof of Theorem 2.9 requires the use of the following proposition and lemmas. Let

\[
\psi_j(t) := (-i) \int_0^t e^{i(-t+s)H_0} V_j(x - sv_j, s)\psi(s)ds, \quad j = 1, \ldots, N.
\]

Then $\psi_j(t), j = 1, \ldots, N$, satisfy the differential equations

\[
i\partial_t \psi_j(t) = H_0 \psi_j(t) + V_j(x - tv_j, t)\psi(t), \quad t > 0.
\]
**Proposition 5.1.** If all the assumptions of Theorem 2.9 are satisfied, then for every \( j = 1, \cdots, N \), \( \psi_j(t) \) is uniformly bounded in \( L^2_{\alpha}(\mathbb{R}^n) \) for all \( t \in [0, \infty) \):

\[
\sup_{t \geq 0} \| \psi_j(t) \|_{L^2_{\alpha}(\mathbb{R}^n)} \leq 1.
\]

The proof of Proposition 5.1 requires Lemmas 5.2 and 5.3, listed below.

**Lemma 5.2.** Let \( v_j \in \mathbb{R}^n, j = 1, \cdots, N \), be \( N \) non-congruent vectors: \( |v_j - v_{j'}| \geq \epsilon, \forall j \neq j' \). For all \( j, j' \in \{1, \cdots, N\} \) with \( j \neq j' \), and all \( \sigma \geq n + 1 \), the estimate

\[
\| (x)^{\sigma} (P)^{-2} e^{i((s_1 - s_2)H_0 - s_1v_j \cdot x + s_2v_{j'} \cdot x)} \|_{L^2_{\alpha}(\mathbb{R}^n)} \lesssim |v_j - v_{j'}|rac{1}{(s_1 - s_2)^{n/2-2}} \frac{1}{\langle s_1 \rangle \langle s_2 \rangle},
\]

holds.

**Proof.** When \( |s_1 - s_2| \geq 1 \), the phase function \( f(q) = (s_1 - s_2)|q|^2 - s_1v_j \cdot q + s_2v_{j'} \cdot q \) is a non-degenerate quadratic form since

\[
|\det(H(f))| = 2^n|s_1 - s_2|^n \geq 1,
\]

where \( H(f) \) denotes the Hessian Matrix of \( f \). By the method of stationary phase in Fourier space, using \( \sigma \geq n + 1 \), we have

\[
\| (x)^{\sigma} (P)^{-2} e^{i((s_1 - s_2)H_0 - s_1v_j \cdot x + s_2v_{j'} \cdot x)} \|_{L^2_{\alpha}(\mathbb{R}^n)} \lesssim |v_j - v_{j'}| \frac{1}{(s_1 - s_2)^{n/2}} \frac{1}{\langle s_1 \rangle \langle s_2 \rangle}.
\]

When \( |s_1 - s_2| < 1 \) and \( s_2 \geq 1 \), the phase function \( f(q) \) can be rewritten as

\[
f(q) = (s_1 - s_2)|q|^2 - (s_1 - s_2)v_j \cdot q + s_2(v_{j'} - v_j) \cdot q.
\]

Let \( e_1 := (v_{j'} - v_j)/|v_{j'} - v_j| \) and \( q_1 := e_1 \cdot q \). By using

\[
e^{is_1|v_{j'} - v_j|q_1} = \frac{1}{is_2|v_{j'} - v_j|} \partial_{q_1} [e^{is_2|v_{j'} - v_j|q_1}],
\]

we integrate by parts twice in the Fourier space to obtain

\[
\| (x)^{\sigma} (P)^{-2} e^{i((s_1 - s_2)H_0 - s_1v_j \cdot x + s_2v_{j'} \cdot x)} \|_{L^2_{\alpha}(\mathbb{R}^n)} \lesssim |v_j - v_{j'}| \frac{1}{(s_1 - s_2)^{n/2}} \frac{1}{\langle s_1 \rangle \langle s_2 \rangle}.
\]

When \( |s_1 - s_2| < 1 \) and \( s_2 < 1 \), we have \( s_1 < 2 \) and therefore by the unitarity of \( e^{i((s_1 - s_2)H_0 - s_1v_j \cdot x + s_2v_{j'} \cdot x)} \), we arrive at

\[
\| (x)^{\sigma} (P)^{-2} e^{i((s_1 - s_2)H_0 - s_1v_j \cdot x + s_2v_{j'} \cdot x)} \|_{L^2_{\alpha}(\mathbb{R}^n)} \lesssim \frac{1}{|s_1 - s_2|^{n/2}} \frac{1}{\langle s_1 \rangle \langle s_2 \rangle},
\]

Estimates (5.6), (5.9) and (5.10) imply (5.4).
Lemma 5.3. Under the assumptions of Theorem 2.9, for all \( j, l \in \{1, \cdots, N\} \) with \( j \neq l \),
\[
\left| \langle \psi_j(t), \psi_l(t) \rangle_{L_2^2(\mathbb{R}^n)} \right| \leq E ||v_j||_1 ||v_l||_1 \quad t \geq 0.
\]

Proof. Take \( j, l \in \{1, \cdots, N\} \) with \( j < l \). By Cauchy-Schwarz inequality, the unitarity of \( e^{itH_0}, k = 1, 2 \), and Assumptions 2.1 and 2.4, we have
\[
\int_0^t \int_0^t \int_0^t |e^{-i(t-s_1)H_0}V_j(x-s_1v_j, s_1)\psi(s_1)||e^{-i(t-s_2)H_0}V_l(x-s_2v_l, s_2)\psi(s_2)|dxds_1ds_2
\]
\[
\leq \int_0^t \int_0^t \int_0^t \int_0^t ||e^{-i(t-s_1)H_0}V_j(x-s_1v_j, s_1)\psi(s_1)||^2 ||e^{-i(t-s_2)H_0}V_l(x-s_2v_l, s_2)\psi(s_2)||^2 ds_1ds_2
\]
\[
= \int_0^t \int_0^t \int_0^t \int_0^t ||V_j(x-s_1v_j, s_1)\psi(s_1)||^2 ||V_l(x-s_2v_l, s_2)\psi(s_2)||^2 ds_1ds_2
\]
\[
\leq t^2 ||V_j(x,s)||_{L_{n,w}^o(\mathbb{R}^n)}||V_l(x,s)||_{L_{n,w}^o(\mathbb{R}^n)} E^2 < \infty.
\]

By Eq. (5.12) and Fubini’s Theorem, \( \langle \psi_j(t), \psi_l(t) \rangle_{L_2^2(\mathbb{R}^n)} \) reads
\[
\langle \psi_j(t), \psi_l(t) \rangle_{L_2^2(\mathbb{R}^n)}
= \left( \int_0^t e^{-i(t-s_1)H_0}V_j(x-s_1v_j, s_1)\psi(s_1)ds_1 \right) \left( \int_0^t e^{-i(t-s_2)H_0}V_l(x-s_2v_l, s_2)\psi(s_2)ds_2 \right)_{L_2^2(\mathbb{R}^n)}
= \int_0^t \int_0^t \int_0^t \int_0^t (V_j(x-s_1v_j, s_1)\psi(s_1)) e^{i(s_2-s_1)H_0}V_l(x-s_2v_l, s_2)\psi(s_2) ds_1ds_2
\]
\[
= \int_0^t \int_0^t \int_0^t \int_0^t (V_j(x-s_1v_j, s_1)e^{is_1Pv_j}\psi(s_1)) e^{i(s_2-s_1)H_0}V_l(x-s_2v_l, s_2)e^{is_2Pv_l}\psi(s_2) ds_1ds_2,
\]
where we have used the unitarity of \( e^{-i(t-s_1)H_0} \) and the equations
\[
V_j(x-s_1v_j, s_1) = e^{-is_1Pv_j}V_j(x,s_1)e^{is_1Pv_j}
\]
and
\[
V_l(x-s_2v_l, s_2) = e^{-is_2Pv_l}V_l(x,s_2)e^{is_2Pv_l}.
\]

By Lemma B.3, we have
\[
\|\langle x \rangle^\delta \langle P \rangle V_j(x,s_1)e^{is_1Pv_j}\psi(s_1)\|_{L_2^2(\mathbb{R}^n)} \leq \delta \|\langle x \rangle^\delta V_j(x,s_1)e^{is_1Pv_j}\psi(s_1)\|_{H_1^1}
\]
\[
\leq E \sup_{t \in \mathbb{R}} \|\langle x \rangle^\delta V_j(x,t)\|_{W_{n,w}^{1,\infty}(\mathbb{R}^n)}
\]
and
\[
\|\langle x \rangle^\delta \langle P \rangle V_l(x,s_2)e^{is_2Pv_l}\psi(s_2)\|_{L_2^2(\mathbb{R}^n)} \leq \delta \|\langle x \rangle^\delta V_l(x,s_2)e^{is_2Pv_l}\psi(s_2)\|_{H_1^1}
\]
\[
\leq E \sup_{t \in \mathbb{R}} \|\langle x \rangle^\delta V_l(x,t)\|_{W_{n,w}^{1,\infty}(\mathbb{R}^n)}.
\]

By Lemma 5.2, Cauchy-Schwarz inequality and the estimates (5.16) and (5.17), together with assumptions of Theorem 2.9, we have, for \( n \geq 5 \),
\[
\|\langle \psi_j(t), \psi_l(t) \rangle_{L_2^2(\mathbb{R}^n)} \| \leq \int_0^t \int_0^t \|\langle x \rangle^\delta \langle P \rangle V_j(x,s_1)e^{is_1Pv_j}\psi(s_1)\|_{L_2^2(\mathbb{R}^n)}
\]
\[
\times \|\langle x \rangle^\delta \langle P \rangle^{-2} e^{i(s_2-s_1)H_0+is_1Pv_l-\frac{s_2}{s_1}P} \langle x \rangle^{-\delta} \|_{L_2^2(\mathbb{R}^n)} \rightarrow L_2^2(\mathbb{R}^n)}
\]
\[ \times \|(x)\delta(P)V(x, s_2)e^{i\pi P\cdot \gamma} \psi(s_2)\|_{L^2_t(\mathbb{R}^n)} ds_1 ds_2 \]
\[ \lesssim E_{\psi(s), \psi(s)} \sup_{s \in \mathbb{R}} \|\langle x \rangle^{\delta} V_j(x, t)\|_{W^{1,\infty}_x(\mathbb{R}^n)} \sup_{s \in \mathbb{R}} \|\langle x \rangle^{\delta} V_j(x, t)\|_{W^{1,\infty}_x(\mathbb{R}^n)} \]
\[ \times \int_0^t \int_0^t \frac{1}{|s_1 - s_2|^{n/2-2}} \left| \frac{1}{\langle s_1 \rangle \langle s_2 \rangle} \right| ds_1 ds_2 \]
(5.18) \[ \lesssim E_{\psi(s), \psi(s)} \sup_{s \in \mathbb{R}} \|\langle x \rangle^{\delta} V_j(x, t)\|_{W^{1,\infty}_x(\mathbb{R}^n)} \sup_{s \in \mathbb{R}} \|\langle x \rangle^{\delta} V_j(x, t)\|_{W^{1,\infty}_x(\mathbb{R}^n)}. \]

\[ \Box \]

**Proof of Proposition 5.1.** By Duhamel’s formula, \( \psi(t) - e^{-itH_0} \psi(0) \) reads

(5.19) \[ \psi(t) - e^{-itH_0} \psi(0) = \sum_{j=1}^{N} \psi_j(t). \]

Expanding \( (\psi(t) - e^{-itH_0} \psi(0), \psi(t) - e^{-itH_0} \psi(0))_{L^2_t(\mathbb{R}^n)} \) by Eq. (5.19), we obtain

\[ (\psi(t) - e^{-itH_0} \psi(0), \psi(t) - e^{-itH_0} \psi(0))_{L^2_t(\mathbb{R}^n)} = \sum_{j=1}^{N} (\psi_j(t), \psi_j(t))_{L^2_t(\mathbb{R}^n)} + \sum_{1 \leq j < l \leq N} 2\Re(\psi_j(t), \psi_l(t))_{L^2_t(\mathbb{R}^n)}, \]
(5.20) which is equivalent to

\[ \sum_{j=1}^{N} (\psi_j(t), \psi_j(t))_{L^2_t(\mathbb{R}^n)} \]
(5.21) \[ = (\psi(t) - e^{-itH_0} \psi(0), \psi(t) - e^{-itH_0} \psi(0))_{L^2_t(\mathbb{R}^n)} - \sum_{1 \leq j < l \leq N} 2\Re(\psi_j(t), \psi_l(t))_{L^2_t(\mathbb{R}^n)}. \]

By Cauchy-Schwarz inequality, the unitarity of \( e^{-itH_0} \) and Assumption 2.1, Eq. (5.21), together with Lemma 5.3, implies,

\[ \sum_{j=1}^{N} (\psi_j(t), \psi_j(t))_{L^2_t(\mathbb{R}^n)} \leq 2\|\psi(t)\|_{L^2_t(\mathbb{R}^n)}^2 + 2\|\psi(0)\|_{L^2_t(\mathbb{R}^n)}^2 \]
(5.22) \[ + 2 \sum_{1 \leq j < l \leq N} \left| (\psi_j(t), \psi_l(t))_{L^2_t(\mathbb{R}^n)} \right| \leq E 1 + N^2 \leq E 1, \quad t \geq 1. \]

This together with the positivity of \( (\psi_j(t), \psi_j(t))_{L^2_t(\mathbb{R}^n)} \), \( j = 1, \cdots, N \), yields estimate (5.3).

\[ \Box \]

**Proposition 5.4.** Let assumptions in Theorem 2.9 be satisfied. Then for every \( j = 1, \cdots, N \), \( \varepsilon \in (0, 1/2) \) and \( \alpha \in (0, 1 - 2/n) \), \( n \geq 5 \),

(1)

(5.23) \[ \psi_{j+}(x) := s \lim_{t \to \infty} e^{itH_0} F_c \left( \frac{|x - 2tP|}{r^\alpha} \right) \leq 1) \psi_j(t) \]

exists in \( L^2_t(\mathbb{R}^n) \) and for all \( \alpha, \alpha' \in (0, 1 - 2/n) \),

(5.24) \[ s \lim_{t \to \infty} e^{itH_0} F_c \left( \frac{|x - 2tP|}{r^\alpha} \right) \leq 1) \psi_j(t) = s \lim_{t \to \infty} e^{itH_0} F_c \left( \frac{|x - 2tP|}{r^{\alpha'}} \right) \leq 1) \psi_j(t); \]
(2) there exist $N$ moving weakly localized parts, $\psi_{wl,j}(t) \equiv \psi_{wl,j}(t)$ such that the equation

\begin{equation}
\lim_{t \to \infty} \| \psi_j(t) - e^{-itH_0}\psi_{j+}(x) - \psi_{wl,j}(t) \|_{L^2(\mathbb{R}^n)} = 0
\end{equation}

holds true, and $\psi_{wl,j}(t)$, $j = 1, \ldots, N$, are moving weakly localized parts around $t_0$.

Proof. Proof of the existence of $\psi_{j+}(x)$: We use Cook’s method and the process similar to the proof of Theorem 2.1 to show the existence of $\psi_{j+}(x)$ in $L^2_\alpha(\mathbb{R}^n)$.

Let

\begin{equation}
\psi_{j+}(t) := e^{itH_0} \mathcal{F}_{c}\left( \frac{|x - 2tP|}{r^\alpha} \leq 1 \right) \psi_j(t).
\end{equation}

By Eq. (1.7), $\psi_{j+}(t)$ reads

\begin{equation}
\psi_{j+}(t) = F_c\left( \frac{|x|}{r^\alpha} \leq 1 \right) e^{itH_0} \psi_j(t).
\end{equation}

Using Cook’s method to expand $\psi_{j+}(t)$ and, by Eqs. (5.1) and

\begin{equation}
\partial_t [e^{itH_0} \psi_j(t)] = e^{itH_0} (iH_0 - iH_0) \psi_j(t) + (-i) e^{itH_0} V_j(x - tv_j, t) \psi_j(t),
\end{equation}

we obtain

\begin{align*}
\psi_{j+}(t) &= \psi_{j+}(1) + (-i) \int_1^t F_c\left( \frac{|x|}{s^\alpha} \leq 1 \right) e^{isH_0} V_j(x - sv_j, s) \psi_j(s) ds \\
&\quad + \int_1^t \partial_s [F_c\left( \frac{|x|}{s^\alpha} \leq 1 \right) e^{isH_0} \psi_j(s)] ds \\
&= \psi_{j+}(1) + \psi_{j+}(1) + \psi_{j+}(t).
\end{align*}

By the unitarity of $e^{itH_0}$ and Lemma 5.1, $\psi_{j+}(1) \in L^2_\alpha(\mathbb{R}^n)$. By letting $\alpha \in (0, 1 - 2/n)$, $n \geq 5$, and using Cauchy-Schwarz inequality and Assumption 2.4, we find that $\psi_{j+}(1)$ satisfies the estimate ($\beta = \frac{n(1 - \alpha)}{2} - 1$),

\begin{align*}
\| \psi_{j+}(1) \|_{L^2(\mathbb{R}^n)} &\leq \int_1^t \| F_c\left( \frac{|x|}{s^\alpha} \leq 1 \right) e^{isH_0} V_j(x - sv_j, s) \psi_j(s) \|_{L^2(\mathbb{R}^n)} ds \\
&\leq \int_1^t \frac{1}{t^{1+\beta}} \| V_j(x - sv_j, s) \psi_j(s) \|_{L^2(\mathbb{R}^n)} ds \\
&\leq \epsilon \| \psi_j(\infty) \|_{L^2(\mathbb{R}^n)}
\end{align*}

which leads to the existence of $\psi_{j+}(\infty)$ in $L^2_\alpha(\mathbb{R}^n)$. For $\psi_{j+}(t)$, we use RPRES by taking $b = 1$ and

\begin{align*}
B(t) &:= \mathcal{F}_{c}\left( \frac{|x|}{r^\alpha} \leq 1 \right) \\
\phi(t) &:= e^{itH_0} \psi_j(t)
\end{align*}

We find that

\begin{align*}
\partial_t [B : \phi(t)]_1 &= (\phi(t), \partial_t [\mathcal{F}_{c}\left( \frac{|x|}{r^\alpha} \leq 1 \right) \phi(t)])_{L^2(\mathbb{R}^n)} + (-i) (\phi(t), \mathcal{F}_{c}\left( \frac{|x|}{r^\alpha} \leq 1 \right) e^{itH_0} V_j(x - tv_j, t) \psi_j(t))_{L^2(\mathbb{R}^n)}
\end{align*}
\[ + i\langle \mathcal{F}_e(\frac{|x|}{t^\alpha} \leq 1)e^{itH_0}V_j(x - tv_j, t)\psi(t), \phi(t) \rangle_{L^2(\mathbb{R})} \]

(5.33) \(=\langle \phi(t), C^*C\phi(t) \rangle_{L^2(\mathbb{R})} + g(t) \)

where \(C^*C\) and \(g(t)\) are given by, with \(\mathcal{F}_e'(\lambda \leq 1) \equiv \frac{d}{d\lambda}\{F_e(\lambda \leq 1)\},\)

\[ C^*C := \partial_t[\mathcal{F}_e(\frac{|x|}{t^\alpha} \leq 1)] \]

(5.34)

\[ = \mathcal{F}_e'(\frac{|x|}{t^\alpha} \leq 1) \times \frac{-\alpha}{t} \times \frac{|x|}{t^\alpha} \geq 0 \]

and

\[ g(t) := (-i)\langle \phi(t), \mathcal{F}_e(\frac{|x|}{t^\alpha} \leq 1)e^{itH_0}V_j(x - tv_j, t)\psi(t), \phi(t) \rangle_{L^2(\mathbb{R})} \]

(5.35)

\[ + i\langle \mathcal{F}_e(\frac{|x|}{t^\alpha} \leq 1)e^{itH_0}V_j(x - tv_j, t)\psi(t), \phi(t) \rangle_{L^2(\mathbb{R})}, \]

respectively. We note that \(\langle B : \phi(t) \rangle\) is uniformly bounded in \(t\). This follows from the Cauchy-Schwarz inequality, the unitarity of \(e^{itH_0}\), and Lemma 5.1:

\[ |\langle B : \phi(t) \rangle| = \langle e^{itH_0}\psi_j(t), F_e(\frac{|x|}{t^\alpha} \leq 1)e^{itH_0}\psi_j(t) \rangle_{L^2(\mathbb{R})} \]

(5.36)

\[ \leq ||e^{itH_0}\psi_j(t)||_{L^2(\mathbb{R})}^2 \|\psi_j(t)\|_{L^2(\mathbb{R})}^2 \leq E 1. \]

Additionally, \(g(t)\) satisfies the following estimate due to Assumptions 2.1 and 2.4, and the Cauchy-Schwarz inequality:

\[ |g(t)| \leq 2||\phi(t)||_{L^2(\mathbb{R})}||\mathcal{F}_e(\frac{|x|}{t^\alpha} \leq 1) e^{itH_0}V_j(x - tv_j, t)\psi(t)||_{L^2(\mathbb{R})} \]

(5.37)

\[ \leq n,E \frac{1}{t^{1+\beta}} ||V_j(x - tv_j, t)\psi(t)||_{L^pL^1(\mathbb{R}^{n+1})} \leq n,E \frac{1}{t^{1+\beta}} ||(x)^\beta V_j(x, t)||_{L^pL^1(\mathbb{R}^{n+1})} \|\psi(t)||_{L^pL^1(\mathbb{R}^{n+1})}. \]

This implies \(g \in L^1[1, \infty)\). Hence, the family \(\{B(t)\}_{t \in [1, \infty)}\) is a RPROB with respect to \(\phi(t) = e^{itH_0}\psi_j(t)\) and by Eq. (3.10), Eqs. (5.36) and (5.37), we obtain

\[ \int_1^\infty |\langle \phi(t), \partial_t[\mathcal{F}_e(\frac{|x|}{t^\alpha} \leq 1)]\phi(t) \rangle_{L^2(\mathbb{R})}| dt \]

\[ = \int_1^\infty \langle \phi(t), \partial_t[\mathcal{F}_e(\frac{|x|}{t^\alpha} \leq 1)]\phi(t) \rangle_{L^2(\mathbb{R})} dt \leq \sup_{t \in [1, \infty)} |\langle B : \phi(t) \rangle| + ||g(t)||_{L^1[1, \infty)} \]

(5.38)

\[ \leq n,E 1 + ||(x)^\beta V_j(x, t)||_{L^pL^1(\mathbb{R}^{n+1})} < \infty. \]

By Cauchy-Schwarz inequality and the non-negativity of \(\partial_t[\mathcal{F}_e(\frac{|x|}{t^\alpha} \leq 1)]\) (see (5.34)), \(\psi_{j,p}(t)\) satisfies the estimate, for \(T_2 \geq T_1\),

\[ ||\psi_{j,p}(T_2) - \psi_{j,p}(T_1)||_{L^2(\mathbb{R})} \]

\[ \leq \left( \int_{T_1}^{T_2} ||\partial_t[\mathcal{F}_e(\frac{|x|}{t^\alpha} \leq 1)]\phi(t) dt ||_{L^2(\mathbb{R})} \right)^{1/2} \left( \int_{T_1}^{T_2} ||\partial_t[\mathcal{F}_e(\frac{|x|}{t^\alpha} \leq 1)]||^2 dt \right)^{1/2} \]

(5.39)
By estimates (4.37) and (5.38), estimate (5.39) leads to
\[
\|\psi_{j,p}(T_2) - \psi_{j,p}(T_1)\|_{L^2_t(L^2_x)} \leq \left( \int_{T_1}^{T_2} \left( \int_{R^3} |\partial_t[\mathcal{F}_c(\frac{|x|}{\rho(t)})] |^2 \phi(t) \right) dt \right)^{1/2} \|L^2_t(L^2_x) \|
\]
(5.40)
\[
= \left( \int_{T_1}^{\infty} \|\phi(t)\|_{L^2_t(L^2_x)} dt \right)^{1/2} \|L^2_t(L^2_x) \| \to 0,
\]
as \(T_1 \to \infty\). Hence, \(\{\psi_{j,p}(t)\}_{t \geq 1}\) is Cauchy in \(L^2_t(L^2_x)\) and therefore \(\psi_{j,p}(\infty)\) exists in \(L^2_t(L^2_x)\).
This together with the existence of \(\psi_{j,lim}(\infty)\) in \(L^2_t(L^2_x)\) and that \(\psi_{j,lim}(1)\) is also in \(L^2_t(L^2_x)\), implies the existence of \(\psi_{j,lim}(x)\) in \(L^2_t(L^2_x)\). Here, we use the notation \(\psi_{j,lim}(x)\) similar to \(\Omega^2_\alpha \psi_0(0)\) (defined in Eq. (2.12)) since \(\psi_{j,lim}(x)\) is independent on \(\alpha \in (0, 1-2/n)\). See the proof of Theorem 2.1 in Section 4.2.

Existence of \(\psi_{wl,j}(t)\): Given \(\epsilon \in (0, 1/2)\), take
\[
\psi_{wl,j}(t) = \psi_{wl,j}(t) = \mathcal{F}_c \left( \frac{|x - lv_j|}{(t + 1)^{1/2+\epsilon}} < 1 \right) \psi_j(t), \quad j = 1, \ldots, N.
\]
Break \(\psi(t) - \psi_{wl,j}(t)\) into three pieces:
\[
\psi(t) - \psi_{wl,j}(t) = \psi_{j1}(t) + \psi_{j2}(t) + \psi_{j3}(t),
\]
where \(\psi_{j,k}(t), k = 1, 2, 3\), are given by
\[
\psi_{j1}(t) = \mathcal{F}_c \left( \frac{|x - lv_j|}{(t + 1)^{1/2+\epsilon}} \geq 1 \right) P_{n_j}^+ \mathcal{F}_1(\sqrt{t + 1}|2P - v_j| \geq 1) \psi_j(t),
\]
\[
\psi_{j2}(t) = \mathcal{F}_c \left( \frac{|x - lv_j|}{(t + 1)^{1/2+\epsilon}} \geq 1 \right) P_{n_j}^- \psi_j(t)
\]
and
\[
\psi_{j3}(t) = \mathcal{F}_c \left( \frac{|x - lv_j|}{(t + 1)^{1/2+\epsilon}} \geq 1 \right) P_{n_j}^+ \mathcal{F}_1(\sqrt{t + 1}|2P - v_j| < 1) \psi_j(t),
\]
respectively. We approximate \(\psi_{j1}(t)\) by \(e^{-itH_0} \psi_{j,lim}(x)\) and arrive at (5.25) by showing that
\[
\lim_{t \to \infty} \|\psi_{jl}(t)\|_{L^2_t(L^2_x)} = 0, \quad l = 2, 3,
\]
(5.47)
\[
\lim_{t \to \infty} \|\psi_{j1}(t) - \mathcal{F}_c(\frac{|x - lv_j|}{(t + 1)^{1/2+\epsilon}} \geq 1) P^+ \mathcal{F}_1(\sqrt{t + 1}|2P - v_j| \geq 1) e^{-itH_0} \psi_{j,lim}(x)\|_{L^2_t(L^2_x)} = 0
\]
and
\[
\lim_{t \to \infty} \|\mathcal{F}_c(\frac{|x - lv_j|}{(t + 1)^{1/2+\epsilon}} \geq 1) P^+ \mathcal{F}_1(\sqrt{t + 1}|2P - v_j| \geq 1) e^{-itH_0} \psi_{j,lim}(x)\|_{L^2_t(L^2_x)} = 0.
\]
(5.48)
By Lemma 4.6 and Assumptions 2.1 and 2.4, \(\psi_{j2}(t)\) satisfies, with \(\delta > 2\),
\[
\|\psi_{j2}(t)\|_{L^2_t(L^2_x)} \leq \int_0^t \|\mathcal{F}_c(\frac{|x - lv_j|}{(t + 1)^{1/2+\epsilon}} \geq 1) P_{n_j}^- e^{-i(t-s)H_0} \mathcal{F}_1(\sqrt{t + 1}|2P - v_j| \geq 1)
\]
\[
\times (x - sv_j)^{-\delta}\|L^2_t(L^2_x) \|_{L^2_t(L^2_x)})(x - sv_j)^{\delta} V_j(x - sv_j, s)\|L^2_t(L^2_x)\|\psi(x)\|L^2_t(L^2_x) ds
\]
\[
\leq E, \epsilon \int_0^t \frac{1}{(t + 1)^{1/2+\epsilon} + s/ \sqrt{t + 1}} \sup_{u \in R} \|\psi(x)\|L^2_t(L^2_x) ds
\]
(5.49) \[ \frac{t}{(t+1)^{1+2\epsilon}} \sup_{u \in \mathbb{R}} \| \langle x \rangle^\delta V_j(x,u) \|_{L^\infty_t(L^p_x)} \to 0, \]
as \( t \to \infty \). Eq. (5.49) implies

(5.50) \[ \lim_{t \to \infty} \| \psi_3(t) \|_{L^2_t(L^2_x)} = 0. \]

By Lemma 4.7 and Assumptions 2.1 and 2.4, \( \psi_{j_3}(t) \) satisfies, with \( \delta > 2 \),

\[
\| \psi_{j_3}(t) \|_{L^2_t(L^p_x)} \leq \int_0^t \left\| \mathcal{F}_c \left( \frac{|x-tv_j|}{(t+1)^{1/2+\epsilon}} \geq 1 \right) \right\|_{L^2_t(L^2_x)} \| \langle x \rangle^\delta V_j(x-sv_j,s) \|_{L^\infty_t(L^p_x)} \| \psi(s) \|_{L^2_t(L^2_x)} \, ds
\]

(5.51)

\[ \leq E, \epsilon \int_0^t \frac{1}{(t+1)^{1/2+2\epsilon}} \sup_{u \in \mathbb{R}} \| \langle x \rangle^\delta V_j(x,u) \|_{L^p_t(L^\infty_x)} \, ds \leq E, \epsilon \frac{t}{(t+1)^{1+2\epsilon}} \sup_{u \in \mathbb{R}} \| \langle x \rangle^\delta V_j(x,u) \|_{L^\infty_t(L^p_x)} \to 0 \]
as \( t \to \infty \). Eq. (5.51) implies

(5.52) \[ \lim_{t \to \infty} \| \psi_3(t) \|_{L^2_t(L^2_x)} = 0. \]

Eqs. (5.50) and (5.52) imply Eq. (5.46).

Next, we prove (5.47). Let

(5.53) \[ \tilde{\psi}_{j,\pm}(x) := \lim_{t \to \infty} \langle -i \rangle \int_0^t e^{isH_0} V_j(x-sv_j,s) \psi(s) \, ds. \]

By the existence of \( \tilde{\psi}_{j,\pm}(x) \) and Eq. (5.24), \( \tilde{\psi}_{j,\pm}(x) \) exists in \( L^2_t(L^2_x) \) and \( \tilde{\psi}_{j,\pm}(x) = \psi_{j,\pm}(x) \). By Duhamel’s expansion and \( \tilde{\psi}_{j,\pm} = \psi_{j,\pm} \), we have

(5.54)

\[ = (-i) \mathcal{F}_c \left( \frac{|x-tv_j|}{(t+1)^{1/2+\epsilon}} \geq 1 \right) P^+ \left( \sqrt{t+1} |2P-v_j| < 1 \right) e^{-itH_0} \psi_{j,\pm} - \psi_j(t) \]

By Lemma 4.6 and Assumptions 2.1 and 2.4, (5.54) implies, with \( \delta > 2 \),

\[
\| \mathcal{F}_c \left( \frac{|x-tv_j|}{(t+1)^{1/2+\epsilon}} \geq 1 \right) P^+ \left( \sqrt{t+1} |2P-v_j| < 1 \right) e^{-itH_0} \psi_{j,\pm} - \psi_j(t) \|_{L^2_t(L^2_x)} \leq \int_t^\infty \| \mathcal{F}_c \left( \frac{|x-tv_j|}{(t+1)^{1/2+\epsilon}} \geq 1 \right) \right\|_{L^2_t(L^2_x)} \| \langle x \rangle^\delta V_j(x-sv_j,s) \|_{L^\infty_t(L^p_x)} \| \psi(s) \|_{L^2_t(L^2_x)} \, ds
\]

(5.55)

\[ \leq E, \epsilon \int_t^\infty \frac{1}{(t+1)^{1/2+2\epsilon} + (s-t) \sqrt{t+1} \delta} \sup_{u \in \mathbb{R}} \| \langle x \rangle^\delta V_j(x,u) \|_{L^p_t(L^\infty_x)} \, ds \]
as \( t \to \infty \). (5.55) implies (5.47).

Now we prove (5.48). Equation

(5.56) \[ s^- \lim_{t \to \infty} \mathcal{F}_1 ( \sqrt{t+1} |2P-\eta| < 1 ) = 0 \quad \text{on } L^2_t(L^2_x), \]
together with Lemmas 4.8 and 4.9, implies

\[
\|e^{-itH_0}\psi_{j,+}(x) - \mathcal{F}_c\left(\frac{|x - tv_j|}{(t + 1)^{1/2 + \epsilon}} \geq 1\right)P_{tv_j}^+ \mathcal{F}_1(\sqrt{t + 1}|2P - v_j|) \geq 1)e^{-itH_0}\psi_{j,+}(x)\|_{L^2(\mathbb{R}^n)} \\
\leq \|\mathcal{F}_c\left(\frac{|x - tv_j|}{(t + 1)^{1/2 + \epsilon}} \geq 1\right)P_{tv_j}^+ \mathcal{F}_1(\sqrt{t + 1}|2P - \eta| < 1)e^{-itH_0}\psi_{j,+}(x)\|_{L^2(\mathbb{R}^n)} \\
+ \|\mathcal{F}_c\left(\frac{|x - tv_j|}{(t + 1)^{1/2 + \epsilon}} < 1\right)P_{tv_j}^+ e^{-itH_0}\psi_{j,+}(x)\|_{L^2(\mathbb{R}^n)} + \|P_{tv_j}^- e^{-itH_0}\psi_{j,+}(x)\|_{L^2(\mathbb{R}^n)} \\
\leq \|\mathcal{F}_1(\sqrt{t + 1}|2P - \eta| < 1)\psi_{j,+}(x)\|_{L^2(\mathbb{R}^n)} + \|\mathcal{F}_c\left(\frac{|x - tv_j|}{(t + 1)^{1/2 + \epsilon}} < 1\right)P_{tv_j}^+ e^{-itH_0}\psi_{j,+}(x)\|_{L^2(\mathbb{R}^n)} \\
\leq \|P_{tv_j}^- e^{-itH_0}\psi_{j,+}\|_{L^2(\mathbb{R}^n)} \to 0
\]

(5.57) as \(t \to \infty\), (5.57) implies (5.48). By (5.46), (5.47), (5.48) and Eq. (5.42), together with Eq. (5.41), we arrive at

\[
\|\psi_j(t) - \psi_{\text{nl},j}(t) - e^{-itH_0}\psi_{j,+}\|_{L^2(\mathbb{R}^n)} \\
= \|\psi_j(t) - \mathcal{F}_c\left(\frac{|x - tv_j|}{(t + 1)^{1/2 + \epsilon}} < 1\right)\psi_j(t) - e^{-itH_0}\psi_{j,+}(x)\|_{L^2(\mathbb{R}^n)} \\
\leq \|\mathcal{F}_c\left(\frac{|x - tv_j|}{(t + 1)^{1/2 + \epsilon}} \geq 1\right)\psi_j(t) - e^{-itH_0}\psi_{j,+}(x)\|_{L^2(\mathbb{R}^n)} \\
\leq \|\psi_{j1}(t) - \mathcal{F}_c\left(\frac{|x - tv_j|}{(t + 1)^{1/2 + \epsilon}} \geq 1\right)P^+ \mathcal{F}_1(\sqrt{t + 1}|2P - v_j|) \geq 1)e^{-itH_0}\psi_{j,+}\|_{L^2(\mathbb{R}^n)} \\
+ \|e^{-itH_0}\psi_{j,+} - \mathcal{F}_c\left(\frac{|x - tv_j|}{(t + 1)^{1/2 + \epsilon}} \geq 1\right)P^+ \mathcal{F}_1(\sqrt{t + 1}|2P - v_j|) \geq 1)e^{-itH_0}\psi_{j,+}\|_{L^2(\mathbb{R}^n)} \\
\leq \|\psi_{j2}(t)\|_{L^2(\mathbb{R}^n)} + \|\psi_{j3}(t)\|_{L^2(\mathbb{R}^n)} \to 0
\]

(5.58) as \(t \to \infty\).

\[\square\]

**Proof of Theorem 2.9.** By Duhamel’s formula, \(\psi(t)\) reads

\[
(5.59) \quad \psi(t) = e^{-itH_0}\psi(0) + \sum_{j=1}^{N} \psi_j(t),
\]

which, together with Theorem 2.1, Eq. (5.1) and Eq. (5.23) of Proposition 5.4, implies

\[
\Omega_{\psi_0}^*\psi(0) = \lim_{t \to 0} e^{itH_0} \mathcal{F}_c\left(\frac{|x - 2tP|}{P^2} \leq 1\right)\psi(t) \\
= \lim_{t \to 0} e^{itH_0} \mathcal{F}_c\left(\frac{|x - 2tP|}{P^2} \leq 1\right)e^{-itH_0}\psi(0) + \sum_{j=1}^{N} \lim_{t \to 0} e^{itH_0} \mathcal{F}_c\left(\frac{|x - 2tP|}{P^2} \leq 1\right)\psi_j(t) \\
= \lim_{t \to 0} \mathcal{F}_c\left(\frac{|x|}{P^2} \leq 1\right)\psi(0) + \sum_{j=1}^{N} \sum_{j=1}^{N} \lim_{t \to 0} e^{itH_0} \mathcal{F}_c\left(\frac{|x - 2tP|}{P^2} \leq 1\right)\psi_j(t) \\
= \psi(0) + \sum_{j=1}^{N} \psi_j(x).
\]
By Proposition 5.4 and Eq. (5.60), we have
\[
\limsup_{t \to \infty} \| \psi(t) - e^{-itH_0} \Omega^*_\alpha \psi(0) - \sum_{j=1}^{N} \psi_{wl,j}(t) \|_{L^2_t(L^2_v)} \\
= \limsup_{t \to \infty} \| e^{-itH_0} \psi(0) + \sum_{j=1}^{N} \psi_j(t) - e^{-itH_0} \Omega^*_\alpha \psi(0) - \sum_{j=1}^{N} \psi_{wl,j}(t) \|_{L^2_t(L^2_v)} \\
\leq \sum_{j=1}^{N} \limsup_{t \to \infty} \| \psi_j(t) - e^{-itH_0} \psi_{j+}(x) - \psi_{wl,j}(t) \|_{L^2_t(L^2_v)} = 0. \\
\]
(5.61)

6. Examples and proof of Proposition 2.10

6.1. Examples.

Example 6.1. Let \( N \in \mathbb{N}^+ \) be a positive integer. When \( N(x,t,\psi) = \sum_{j=1}^{N} V_j(x - tv_j, t) \) with \( V_j(x,t) \in L_1^\infty L_2^2(\mathbb{R}^{n+1}), j = 1, \ldots, N \), Assumption 2.3 is satisfied if Assumption 2.1 is satisfied.

Proof. To compute \( \| N(x,t,\psi(t))\psi(t) \|_{L^\infty_t L^2_v(\mathbb{R}^{n+1})} \), we find, by Hölder’s inequality
\[
\| N(x,t,\psi(t))\psi(t) \|_{L^\infty_t L^2_v(\mathbb{R}^{n+1})} \leq \| N(x,t,\psi(t)) \|_{L^\infty_t L^2_v(\mathbb{R}^{n+1})} \| \psi(t) \|_{L^2_t L^2_v(\mathbb{R}^{n+1})} \\
\leq E \sum_{j=1}^{N} \| V_j(x,t) \|_{L^\infty_t L^2_v(\mathbb{R}^{n+1})} < \infty. \\
\]

Example 6.2. Suppose that Assumption 2.1 is valid with \( a = 1 \). When space dimension \( n \geq 3 \), \( N(x,t,\psi(t)) = I(|\psi(t)|) \), with \( I(|\psi|) \) satisfying the estimate
\[
\| I(|\psi|)\psi \|_{L^1_t(\mathbb{R}^n)} \lesssim \| \psi \|_{L^1_t(\mathbb{R}^n)}, \quad 1, \\
\]
satisfies condition (2.4).

Proof. It follows from that, by Hölder’s inequality, Assumption (2.1) and Eq. (6.2),
\[
\| N(x,t,\psi(t))\psi(t) \|_{L^\infty_t L^2_v(\mathbb{R}^{n+1})} = \| I(|\psi(t)|)\psi(t) \|_{L^\infty_t L^2_v(\mathbb{R}^{n+1})} \leq E 1. \\
\]

Example 6.3. Assume that Assumption 2.1 is valid with \( a = 1 \) and let \( N \in \mathbb{N}^+ \) be a positive integer. When space dimension \( n \geq 3 \), \( N(x,t,\psi(t)) = P(|\psi(t)|) \), where \( P(z) \) denotes a polynomial of degree \( N \) with \( P(0) = 0 \), satisfies condition (2.4) provided that
\[
\| \psi(0) \|_{L^{N+2}(\mathbb{R}^n)} < \infty. \\
\]

Proof. The energy of this purely nonlinear system is given by
\[
E(\psi(t)) := \langle \psi(t), (-\Delta_x + \tilde{P}(|\psi(t)|)\psi(t)) \rangle_{L^2_v(\mathbb{R}^n)}, \\
\]
(6.5)
where $\tilde{P}(z)$ is another polynomial of the same degree as $P(z)$, defined by, with $P'(k) \equiv \frac{d}{dk}[P(k)]$.

\begin{equation}
\tilde{P}(k) = P(k) - \int_{0}^{k} \frac{u^2 P'(u) du}{k^2}, \quad k \in \mathbb{R}.
\end{equation}

By condition (6.4) and interpolation inequality,

\begin{equation}
E(\psi(0)) < \infty.
\end{equation}

By Assumption 2.1, this implies that there is a energy conservation law. That is, $E(\psi(t)) = E(\psi(0))$ for all $t \in \mathbb{R}$. Let

\begin{equation}
p(z) = \sum_{j=1}^{N} a_{j} z^j.
\end{equation}

By Eqs. (6.6) and (6.8), Hölder’s inequality and interpolation inequality, we obtain

\begin{equation}
|a_{N}|^{1/(N+2)} \|\psi(t)\|_{L^{N+2}(\mathbb{R}^n)}^{1/(N+2)} \leq \left( \|\psi(t)\|_{H^1(\mathbb{R}^n)}^2 + \sum_{j=1}^{N-1} \|\psi(t)\|_{L^{N+2}(\mathbb{R}^n)}^{1+j} \right)^{1/(N+2)} \leq \left( \|\psi(t)\|_{H^1(\mathbb{R}^n)}^2 + \sum_{j=1}^{N-1} \|\psi(t)\|_{L^{N+2}(\mathbb{R}^n)}^{j(N+2)} \|\psi(t)\|_{L^{N+2}(\mathbb{R}^n)}^{(1-b_j)(N+2)} \right)^{1/(N+2)} \leq \|\psi(t)\|_{L^{N+2}(\mathbb{R}^n)}^{2(N+2)} + \|\psi(t)\|_{L^{N+2}(\mathbb{R}^n)}^{b_j} \|\psi(t)\|_{L^{N+2}(\mathbb{R}^n)}^{(1-b_j)} \right)^{1/(N+2)} \leq 1,
\end{equation}

where $b_j \in (0, 1), j = 1, \ldots, N - 1$, are numbers (determined by Hölder’s inequality) satisfying

\begin{equation}
\frac{1}{j+2} = \frac{b_j}{2} + \frac{1-b_j}{N+2}.
\end{equation}

With $a_{N} \neq 0$, since $\|\psi(t)\|_{L^{N+2}(\mathbb{R}^n)}$ in the right-hand side of Eq. (6.9) is sub-linear, Eq. (6.9) implies (The constant $E$ is defined in Assumption 2.1)

\begin{equation}
\|\psi(t)\|_{L^{N+2}(\mathbb{R}^n)} \leq E 1,
\end{equation}

which leads to estimate, by interpolation,

\begin{equation}
\|N(x, t, \psi(t))|_{L^0_{t}H^1_{x}(\mathbb{R}^{n+1})} = \|P(|\psi(t)|)\psi(t)|_{L^0_{t}H^1_{x}(\mathbb{R}^{n+1})} \leq \|\psi(t)|_{L^0_{t}L^{N+1}_{x}(\mathbb{R}^{n+1})}^{N+1} + \|\psi(t)|_{L^0_{t}L^{N+2}_{x}(\mathbb{R}^{n+1})}^{2} \leq E 1.
\end{equation}

This completes the proof.

\[\square\]

6.2. Proof of Proposition 2.10.

Proof for Proposition 2.10. We note that, due to Assumption 2.1 with $a = 1$,

\begin{equation}
\|\mathcal{F}_{\tau} \left| \frac{x}{\tau} \right| \leq 1 \right| e^{iH_0 \tau} \psi(t)\|_{H^1_{x}} \leq E 1, \quad \forall \tau \geq 1.
\end{equation}

This, together with the existence of $\Omega^0_x \psi(0) \in L^2_x(\mathbb{R}^n)$, implies

\begin{equation}
\Omega^0_x \psi(0) \in \mathcal{H}^1_x.
\end{equation}
Next, we prove that \( \psi(t) \) satisfies the endpoint Strichartz estimate
\[
\|\psi(t)\|_{L_2^2 L_t^5(\mathbb{R}^{3+1})} \lesssim E 1
\]
provided that
\[
\limsup_{t \to \infty} \|\psi(t) - e^{-it\mathcal{H}_0} \Omega^*_a \psi(0)\|_{\mathcal{H}_t^1} \text{ is sufficiently small.}
\]
We note that by Assumption 2.1,
\[
\|\psi(t)\|_{L_2^\infty L_t^5(\mathbb{R}^{3+1})} \lesssim E 1,
\]
which implies the endpoint Strichartz estimate locally in \( t \):
\[
\|\psi(t)\|_{L_2^2 L_t^5(\mathbb{R}^{3+1} \times [0,T])} \lesssim E, \quad T > 0.
\]
Next, we use estimate (6.18) to prove estimate (6.15). The endpoint Strichartz estimate of \( \psi(t) \) follows from a standard contraction argument. By Duhamel’s formula, \( \psi(t) - e^{-it\mathcal{H}_0} \Omega^*_a \psi(0) \) reads
\[
\psi(t) - e^{-it\mathcal{H}_0} \Omega^*_a \psi(0) = e^{-it\mathcal{H}_0} \psi(0) + (-i) \int_0^t e^{-i(t-s)\mathcal{H}_0} I(\psi(s)) \psi(s) ds - e^{-it\mathcal{H}_0} \Omega^*_a \psi(0)
\]
\[
= e^{-it\mathcal{H}_0} (\psi(0) - \Omega^*_a \psi(0)) + (-i) \int_0^t e^{-i(t-s)\mathcal{H}_0} I(\psi(s)) \psi(s) ds.
\]
By writing
\[
(-i) \int_0^t e^{-i(t-s)\mathcal{H}_0} I(\psi(s)) \psi(s) ds = (-i) \int_0^t e^{-i(t-s)\mathcal{H}_0} (I(\psi(s)) \psi(s) - I(\psi(0)) e^{-i\mathcal{H}_0} \Omega^*_a \psi(0)) ds
\]
\[
+ (-i) \int_0^t e^{-i(t-s)\mathcal{H}_0} I(\psi(0)) e^{-i\mathcal{H}_0} \Omega^*_a \psi(0) ds
\]
\[
=: \psi_f(t) + \psi_f(t),
\]
according to Eq. (6.19), we arrive at
\[
\psi(t) - e^{-it\mathcal{H}_0} \Omega^*_a \psi(0) = e^{-it\mathcal{H}_0} (\psi(0) - \Omega^*_a \psi(0)) + \psi_f(t) + \psi_f(t).
\]
By Strichartz estimate of free flows, we obtain
\[
\|e^{-it\mathcal{H}_0} (\psi(0) - \Omega^*_a \psi(0))\|_{L_2^2 L_t^5(\mathbb{R}^{3+1})} \lesssim \|\psi(0) - \Omega^*_a \psi(0)\|_{L_2^2(\mathbb{R}^3)}
\]
\[
\lesssim \|\psi(0)\|_{L_2^2(\mathbb{R}^3)}.
\]
By the dual homogeneous Strichartz estimates
\[
\| \int_{t > s} e^{is\mathcal{H}_0} F(s) ds \|_{L_2^2 L_t^5(\mathbb{R}^{3+1})} \lesssim \|F(s)\|_{L_2^2 L_t^5(\mathbb{R}^{3+1})},
\]
Hölder’s inequality and Eq. (2.23) (with \( g = 0 \)), \( \psi_f(t) \) satisfies
\[
\|\psi_f(t)\|_{L_2^2 L_t^5(\mathbb{R}^{3+1})} = \| \int_0^t e^{-i(t-s)\mathcal{H}_0} I(e^{-isH_0} \Omega^*_a \psi(0)) e^{-isH_0} \Omega^*_a \psi(0) ds \|_{L_2^2 L_t^5(\mathbb{R}^{3+1})}
\]
\[
\lesssim \|I(e^{-isH_0} \Omega^*_a \psi(0)) e^{-isH_0} \Omega^*_a \psi(0)\|_{L_2^2 L_t^6(\mathbb{R}^{3+1})}
\]
\[
\lesssim E \|e^{-isH_0} \Omega^*_a \psi(0)\|_{L_2^2 L_t^5(\mathbb{R}^{3+1})}
\]
\[
\lesssim E \|\psi(0)\|_{L_2^2(\mathbb{R}^3)}.
\]
By inequality (2.23), estimate (6.18) and the inhomogeneous Strichartz estimate

\begin{equation}
\| \int_{s \leq t} e^{-i(t-s)H_0} F(s) ds \|_{L^2_t L^6_x(\mathbb{R}^{3+1})} \leq \| F(s) \|_{L^4_t L^6_x(\mathbb{R}^{3+1})},
\end{equation}

\( \psi_r(t) \) satisfies, for \( T \geq T_1 \geq 0 \),

\[ \| (I(t) \psi(s)) \|_{L^2_t L^6_x(\mathbb{R}^{3+1})} \leq C_1 \| \psi(s) \|_{L^2_t L^6_x(\mathbb{R}^{3+1})} \]

where \( C_1 = C_1(E) > 0 \) and \( C_2 = C_2(E, T_1) > 0 \) denote two positive constants. Estimates (6.22), (6.24) and (6.26) imply, for all \( T \geq T_1 \),

\begin{equation}
\| \psi(t) - e^{-iH_0} \Omega^*_\alpha \psi(0) \|_{L^2_t L^6_x(\mathbb{R}^{3+1})} \leq C_1 \| \psi(s) - e^{-iH_0} \Omega^*_\alpha \psi(0) \|_{L^2_t L^6_x(\mathbb{R}^{3+1})} + C_2 \| \psi(0) \|_{L^2_x(\mathbb{R}^3)}
\end{equation}

where \( C_1 = C(E) > 0 \) and \( C_2 = C(E, T_1) > 0 \) are two positive constants. By taking

\begin{equation}
m := \frac{1}{C_1}
\end{equation}

and by taking \( T_1 \) large enough such that \( t \geq T_1 \) implies

\begin{equation}
\| \psi(t) - e^{-iH_0} \Omega^*_\alpha \psi(0) \|_{H^1(\mathbb{R}^3)} < m,
\end{equation}

we have, with

\begin{equation}
\tilde{C} := C_1 \| \psi(t) - e^{-iH_0} \Omega^*_\alpha \psi(0) \|_{L^\infty_t H^1_x([T_1, \infty))} < 1,
\end{equation}

for all \( T \geq T_1 \),

\begin{equation}
\| \psi(t) - e^{-iH_0} \Omega^*_\alpha \psi(0) \|_{L^2_t L^6_x(\mathbb{R}^{3+1})} \leq C_2 \| \psi(0) \|_{L^2_x(\mathbb{R}^3)} + \tilde{C} \| \psi(t) - e^{-iH_0} \Omega^*_\alpha \psi(0) \|_{L^2_t L^6_x(\mathbb{R}^{3+1})},
\end{equation}

which leads to

\begin{equation}
\| \psi(t) - e^{-iH_0} \Omega^*_\alpha \psi(0) \|_{L^2_t L^6_x(\mathbb{R}^{3+1})} \leq E \| \psi(0) \|_{L^2_x(\mathbb{R}^3)}, \quad \forall t \geq T_1.
\end{equation}

By taking \( T \to \infty \), we arrive at

\begin{equation}
\| \psi(t) - e^{-iH_0} \Omega^*_\alpha \psi(0) \|_{L^2_t L^6_x(\mathbb{R}^{3+1})} \leq E \| \psi(0) \|_{L^2_x(\mathbb{R}^3)}
\end{equation}

and therefore,

\begin{equation}
\| \psi(0) \|_{L^2_t L^6_x(\mathbb{R}^{3+1})} \leq E \| \psi(0) \|_{L^2_x(\mathbb{R}^3)}.
\end{equation}
By using Duhamel’s formula, the dual homogeneous Strichartz estimate
\begin{equation}
\| \int_{\mathbb{R}} e^{i s H_0} F(s) ds \|_{L^2_\infty(\mathbb{R}^3)} \leq \| F(s) \|_{L^2_t L^{6/5}_x(\mathbb{R}^{3+1})},
\end{equation}
Hölder’s inequality, Eq. (2.23) (with $f = \psi$ and $g = 0$) and Strichartz estimate of $\psi(t)$, we obtain
$$
\| \psi(t) - e^{-i t H_0} \Omega_\nu^* \psi(0) \|_{H^1} = \| \int_{-\infty}^{t} e^{-i (t-s) H_0} I(\psi(s)) \psi(s) ds \|_{L^2(\mathbb{R}^3)} \\
\leq \| \chi(|s| \geq t) I(\psi(s)) \psi(s) \|_{L^2_t L^{6/5}_x(\mathbb{R}^{3+1})} \\
\leq \| \chi(s \geq t) \psi(s) \|_{L^2_t L^{6/5}_x(\mathbb{R}^{3+1})} \\
\to 0
$$
(6.36)
as $t \to \infty$. We finish the proof.

\section*{Appendix A. Phase-space operators}

\textit{Proof of Eq. (1.7).} Let $P = (P_1, \ldots, P_n)$ and $x = (x_1, \ldots, x_n)$. To compute $i[H_0, x_j]$, $j = 1, \ldots, n$, we find
\begin{equation}
i[H_0, x_j] = i[H_0 x_j] + 2i (P_j x_j) P_j = 2P_j,
\end{equation}
(A.1)
which implies
\begin{equation}
i[H_0, x] = 2P
\end{equation}
(A.2)
and therefore,
\begin{equation}
\partial_s [e^{-i s H_0} x e^{i s H_0}] = e^{-i s H_0} (-i) [H_0, x] e^{i s H_0} \\
= -2P.
\end{equation}
(A.3)
Eq. (A.3) implies
\begin{equation}
e^{-i s H_0} x e^{i s H_0} = e^{-i s H_0} x e^{i s H_0} |_{s=0} + \int_{0}^{s} \partial_s [e^{-i s H_0} x e^{i s H_0}] ds \\
= x - 2tP,
\end{equation}
(A.4)
which leads to
\begin{equation}
e^{-i t H_0} e^{i x \xi} e^{i t H_0} = e^{-i (x - 2t P) \xi}, \quad \xi \in \mathbb{R}^n.
\end{equation}
(A.5)
Therefore, by Fourier transform and Eq. (A.5), $e^{-i t H_0} F_\zeta(\frac{|\xi|}{t^\alpha} > 1) e^{i t H_0}$ reads
\begin{equation}
e^{-i t H_0} F_\zeta(\frac{|x|}{t^\alpha} > 1) e^{i t H_0} = \frac{1}{(2\pi)^n/2} \int_{\mathbb{R}^n} \hat{F}_\zeta(\xi) e^{-i t H_0} e^{i \xi \cdot \xi} e^{i t H_0} d^n \xi \\
= \frac{1}{(2\pi)^n/2} \int_{\mathbb{R}^n} \hat{F}_\zeta(\xi) e^{i \xi \cdot \frac{x-2tP}{t^\alpha}} d^n \xi
\end{equation}
(A.6)
which implies Eq. (1.7). This completes the proof.

\end{document}
Proof of estimate (3.18). Let
\[ O(t) := \mathcal{F}_c(\frac{|t|}{t^\alpha} \leq 1)\mathcal{F}_1(t^\beta|P| > 1)e^{\pm i t H_0} \langle x \rangle^{-\sigma}. \]

Break \( O(t) \) into two pieces:
\[ O(t) = O_1(t) + O_2(t), \]
where
\[ O_1(t) := \mathcal{F}_c(\frac{|t|}{t^\alpha} \leq 1)\mathcal{F}_1(t^\beta|P| > 1)e^{\pm i t H_0} \langle x \rangle^{-\sigma} \langle |x| \geq \frac{1}{10} t^{1-\beta} \rangle, \]
and
\[ O_2(t) := \mathcal{F}_c(\frac{|t|}{t^\alpha} \leq 1)\mathcal{F}_1(t^\beta|P| > 1)e^{\pm i t H_0} \langle x \rangle^{-\sigma} \langle |x| < \frac{1}{10} t^{1-\beta} \rangle. \]

By using the weight \( \langle x \rangle^{-\sigma} \) and the unitarity of \( e^{\pm i t H_0} \), we obtain, for \( t \geq 1 \),
\[ \|O_1(t)\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq \|\mathcal{F}_c(\frac{|t|}{t^\alpha} \leq 1)\mathcal{F}_1(t^\beta|P| > 1)e^{\pm i t H_0}\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \|\langle x \rangle^{-\sigma} \langle |x| \geq \frac{1}{10} t^{1-\beta} \rangle\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq \frac{1}{t^{\sigma(1-\beta)}}, \]

For \( O_2(t) \), take \( f \in L^2_\mathcal{F}(\mathbb{R}^n) \). By Fourier transform, \( O_2(t)f \) reads, with \( \mathcal{F}_c \equiv \mathcal{F}_c(\frac{|t|}{t^\alpha} \leq 1) \) and \( \mathcal{F}_1 \equiv \mathcal{F}_1(t^\beta|q| > 1) \),
\[ O_2f = \frac{1}{(2\pi)^n} \int \mathcal{F}_c e^{\pm i q \cdot x} \mathcal{F}_1(t^\beta|q| > 1)e^{-i y \cdot q} \langle y \rangle^{-\sigma} f(y)dydq, \]
where we have used
\[ \langle \langle x \rangle^{-\sigma} f \rangle \rangle(q) = \frac{1}{(2\pi)^{n/2}} \int e^{-i q \cdot y} \langle y \rangle^{-\sigma} f(y)dy. \]

When \( q \) is in the support of \( \mathcal{F}_1(t^\beta|q| > 1) \), \( x \) is in the support of \( \mathcal{F}_c(\frac{|t|}{t^\alpha} \leq 1) \) and \( |y| \leq \frac{1}{10} t^{1-\beta} \), \( x \pm 2tq - y \) satisfies, for \( t \gg 1 \) and \( \alpha \in (0, 1-\beta) \)
\[ |x \pm 2tq - y| \geq |2tq| - |x| - |y| \geq \frac{9}{10} t^{1-\beta} - 2t^\alpha \geq \frac{1}{2} t^{1-\beta} \]
and
\[ |x \pm 2tq - y| \geq |2tq| - |x| - |y| \geq |tq|. \]

We define an orthogonal basis \( \{e_1, \cdots, e_n\} \) in \( \mathbb{R}^n \) with \( e_1 \) satisfying
\[ \langle x \rangle^{-\sigma} = \frac{1}{(2\pi)^{n/2}} \int e^{-i q \cdot y} \langle y \rangle^{-\sigma} dy \]
for some constant \( C = C(n) > 0 \), where \( x_1 := x \cdot e_1, q_1 := q \cdot e_1 \) and \( y_1 := y \cdot e_1 \). Let \( \tilde{F}_1(k-1) \) denote a smooth cut-off function satisfying, with \( \mathcal{F}_1(k > 1) \equiv \frac{d}{dk} [\mathcal{F}_1(k > 1)] \) and \( [a, b] := \text{supp} \mathcal{F}_1(k > 1) \),
\[ \tilde{F}_1(k-1) = \begin{cases} 1 & \text{when } \text{kin}[a, b] \\ 0 & \text{when } \text{kin}(-\infty, a/2) \cup (2b, \infty) \end{cases}, \]
By using
\[ e^{\{x_1 q_1 \pm i \theta q_1^2 - y \}} = \frac{1}{i(x_1 \pm 2 t q_1 - y)} \partial_{q_1} \{ e^{\{x_1 q_1 \pm i \theta q_1^2 - y \}} \} \]
and estimates, by estimates (B.8) and (B.9),
\[ |\partial_{q_1} \{ e^{\{x_1 q_1 \pm i \theta q_1^2 - y \}} \}| \leq \frac{2t}{|x_1 \pm 2 t q_1 - y|^2} \leq \frac{1}{t^{1-\beta}|q| (\leq \frac{1}{t^{1-\beta}})}, \]
we integrate by parts the right-hand side of Eq. (B.12) up to a constant, and therefore, we obtain with \( \beta \in (0, 1/2) \),
\[ \|O_2(t)f\|_{L^2(R^n)} \lesssim \frac{1}{t^N\|f\|_{L^2(R^n)}}. \]
Estimates (B.5) and (B.16) imply
\[ \|O(t)\|_{L^2(R^n) \rightarrow L^2(R^n)} \leq \frac{1}{t^{\sigma(1-\beta)}}. \]
\( \square \)

**Proof of Lemma 4.2.** In this proof, we use notations
\[ \mathcal{F}_c = \mathcal{F}_c \left( \frac{|x|}{(t+1)^{1/2+\epsilon}} \geq 1 \right) \]
and
\[ \mathcal{F}_1 = \mathcal{F}_1 (\sqrt{t+1}|P| \geq 1) \text{ or } \mathcal{F}_1 = \mathcal{F}_1 (\sqrt{t+1}|q| \geq 1) \]
We start with proving estimate (4.11). Let
\[ A^\pm(t,s) := P^\pm e^{\pm i s H_0} \mathcal{F}_1 (x)^{-\sigma}. \]
Break \( A^\pm(t,s) \) into two pieces:
\[ A^\pm(t,s) = A^\pm_1(t,s) + A^\pm_2(t,s), \]
where \( A^\pm_j(t,s), j = 1, 2, \) are given by
\[ A^\pm_1(t,s) := P^\pm e^{\pm i s H_0} \mathcal{F}_1 (x)^{-\sigma} \chi(|x| > \frac{1}{10^{10}} ((t + 1)^{1/2 + \epsilon} + s/ \sqrt{t+1})) \]
and
\[ A^\pm_2(t,s) := P^\pm e^{\pm i s H_0} \mathcal{F}_1 (x)^{-\sigma} \chi(|x| \leq \frac{1}{10^{10}} ((t + 1)^{1/2 + \epsilon} + s/ \sqrt{t+1})). \]
\[ \|A^\pm_1(t,s)\|_{L^2(R^n) \rightarrow L^2(R^n)} \text{ satisfies} \]
\[ \|A^\pm_1(t,s)\|_{L^2(R^n) \rightarrow L^2(R^n)} \leq \frac{1}{10^{10}} ((t + 1)^{1/2 + \epsilon} + s/ \sqrt{t+1})) \]
For $A_2^+(t, s)$, choose $f \in L^2_s(\mathbb{R}^n)$ and by Fourier transform, $A_2^+(t, s)f$ reads

$$A_2^+(t, s)f = \frac{1}{(2\pi)^n} \int e^{ixq} P^\pm(x, 2q) e^{\pm i\omega} \mathcal{F}_1 e^{-iyq}$$

(B.25)

where we have used that $P^\pm = P^\pm(x, 2P)$ (see Eq. (4.10)). We note that the phase function in the right-hand side of Eq. (B.25) is given by:

$$f(q) = (x - y) \cdot q \pm s|q|^2.$$  

When

We choose an orthogonal basis $\{e_1, \cdots, e_n\}$ with $e_1 := \frac{x - y}{|x - y|}$. Let $x_1 := x \cdot e_1, y_1 := x \cdot e_1$ and $q_1 := q \cdot e_1$. We also have the estimate

|\nabla_q[f(q)]| = |x - y \pm 2sq| \geq \frac{1}{2}|x \pm 2sq| - |y| \geq \frac{1}{10^6}(|x| + 2|s||q|) - |y|

\begin{align}
&\geq \frac{1}{10^6} \left( \frac{(t + 1)^{1/2}}{2} + s/(\sqrt{1 + r}) \right) - |y|

\begin{align}
&\geq \frac{1}{10^7}(|x| + s|q|).
\end{align}

(B.30)

We choose an orthogonal basis $\{e_1, \cdots, e_n\}$ with $e_1 := \frac{x - y}{|x - y|}$. Let $x_1 := x \cdot e_1, y_1 := x \cdot e_1$ and $q_1 := q \cdot e_1$. We also have the estimate

$$\|\mathcal{F}_c A_2^+(t, s)f\|_{L^2(\mathbb{R}^n)} \leq \|\langle x \rangle^{-\frac{r}{2}}\|_{L^2(\mathbb{R}^n)} \|\langle x \rangle^n \mathcal{F}_c A_2^+(t, s)f\|_{L^\infty(\mathbb{R}^n)} \leq \|\langle x \rangle^n A_2^+(t, s)f\|_{L^\infty(\mathbb{R}^n)}.$$  

By estimate (B.30), we have

$$|\partial_{q_1}[\frac{1}{(x_1 - y_1 \pm 2sq_1)^2}]| \leq \frac{1}{s(|x| + s|q|)^2} \leq \frac{1}{|q|||x| + s|q||}.$$  

(B.32)

This, together with estimates

$$|\partial_{q_1}[\mathcal{F}_1(\sqrt{t + 1}|q| \geq 1)]| = \frac{\sqrt{t + 1}|q_1|}{|q|} |\mathcal{F}_1(\sqrt{t + 1}|q| \geq 1)|$$

(B.33)
and (recall that $P^\pm(r, v)$ is defined in terms of $\hat{r}$ and $\hat{v}$. See Eqs. (4.8) and (4.9).)

\begin{equation}
|\partial_q^j [\mathcal{F}_1(\sqrt{t + 1}|q| \geq 1)|] \leq \frac{1}{|q|^j}
\end{equation}

implies, by taking integration by parts in the right-hand side of estimate (B.36) and using estimates (B.24), (B.31) and (B.40) imply estimate (4.11). Similarly, we have estimate (4.12).
\textit{Proof of Lemma 4.3.} Let \( x \) and \( y \) denote the position variables on the left-hand side and the right-hand side, respectively. The velocity is given by \( \nabla \varphi[H_0] = 2P \). Let \( q \) denote the variable in the Fourier space. When \( |x| \geq (t + 1)^{1/2 + \epsilon}/2 \), \( s|q| \leq \frac{2}{\sqrt{t+1}} \) and \( |y| \leq (t + 1)^{1/2 + \epsilon}/4 \),
\begin{equation}
|x - y - 2tq| \geq |x| - |y| - 2t|q| \geq (t + 1)^{1/2 + \epsilon}/20.
\end{equation}
Therefore, by using a similar argument of Lemma 4.2, we get estimate (4.13). This completes the proof. \( \square \)

\textit{Proof of Lemma 4.4.} We fix \( s \geq 0 \). \( P^\pm e^{isH_0}f \) satisfies, for all \( M \geq 1 \) and \( \epsilon \in (0,1/2) \),
\begin{align}
\|P^\pm e^{isH_0}f\|_{L^2(\mathbb{R}^n)} &\leq \|P^\pm e^{isH_0}F_1(\sqrt{s + 1}|P| \geq 1)\chi(|x| \leq M)f\|_{L^2(\mathbb{R}^n)} \\
&+ \|P^\pm e^{isH_0}F_1(\sqrt{s + 1}|P| \geq 1)\chi(|x| > M)f\|_{L^2(\mathbb{R}^n)} \\
&+ \|P^\pm e^{isH_0}(1 - F_1(\sqrt{s + 1}|P| \geq 1))f\|_{L^2(\mathbb{R}^n)} \\
\leq &\|P^\pm e^{isH_0}F_1(\sqrt{s + 1}|P| \geq 1)\rangle^{-1}\|_{L^2(\mathbb{R}^n)}\langle 1\|f\|_{L^2(\mathbb{R}^n)} \\
&\times \|\chi \chi(|x| \leq M)f\|_{L^2(\mathbb{R}^n)} + \|\chi(|x| > M)f\|_{L^2(\mathbb{R}^n)} \\
&+ \|(1 - F_1(\sqrt{s + 1}|P| \geq 1))f\|_{L^2(\mathbb{R}^n)}.
\end{align}
By taking \( M = (1 + s)^{1/100} \) and by using Lemma 4.2, we obtain that
\begin{align}
\|P^\pm e^{isH_0}f\|_{L^2(\mathbb{R}^n)} &\leq \frac{1}{\langle s \rangle^{1/2}}(1 + s)^{1/100}\|f\|_{L^2(\mathbb{R}^n)} + \|\chi(|x| > (s + 1)^{1/100})f\|_{L^2(\mathbb{R}^n)} \\
&+ \|(1 - F_1(\sqrt{s + 1}|P| \geq 1))f\|_{L^2(\mathbb{R}^n)} \\
&\rightarrow 0
\end{align}
as \( s \rightarrow \infty \). \( \square \)

\textit{Proof of Lemma 4.5.} Let \( x \) and \( y \) denote the position variables on the left-hand side and the right-hand side, respectively. The velocity is given by \( \nabla \varphi[H_0] = 2P \). We break \( \chi(|x| \leq s^\alpha)P^\pm e^{isH_0}f \) into three pieces:
\begin{align}
\chi(|x| \leq s^\alpha)P^\pm e^{isH_0}f &= \chi(|x| \leq s^\alpha)P^\pm e^{isH_0}\mathcal{F}_1(s^{(1-\alpha)/100}|P| > 1)\chi(|x| \leq s^\alpha)f \\
&+ \chi(|x| \leq s^\alpha)P^\pm e^{isH_0}\mathcal{F}_1(s^{(1-\alpha)/100}|P| \leq 1)\chi(|x| \leq s^\alpha)f \\
&+ \chi(|x| \leq s^\alpha)P^\pm e^{isH_0}\chi(|x| > s^\alpha)f
\end{align}
We have
\begin{align}
\lim_{s \rightarrow \infty} \sup_{s} \|f_2(s)\|_{L^2(\mathbb{R}^n)} &\leq \lim_{s \rightarrow \infty} \sup_{s} \|\mathcal{F}_1(s^{(1-\alpha)/100}|P| \leq 1)f\|_{L^2(\mathbb{R}^n)} \\
&\leq \lim_{s \rightarrow \infty} \sup_{s} \|\mathcal{F}_1(s^{(1-\alpha)/100}|P| \leq 1)f\|_{L^2(\mathbb{R}^n)} + \lim_{s \rightarrow \infty} \sup_{s} \|\chi(|x| > s^\alpha)f\|_{L^2(\mathbb{R}^n)} \\
&= 0
\end{align}
and
\begin{align}
\lim_{s \rightarrow \infty} \sup_{s} \|f_3(s)\|_{L^2(\mathbb{R}^n)} &\leq \lim_{s \rightarrow \infty} \sup_{s} \|\chi(|x| > s^\alpha)f\|_{L^2(\mathbb{R}^n)} \\
&= 0.
\end{align}
To estimate $f_1(s)$, we follow a similar argument of Lemma 4.2. by Fourier transform, $A_2^\alpha(t, s)f$ reads

$$
 f_1(s) = \frac{1}{(2\pi)^d} \int \chi(|x| \leq s^\alpha) e^{ix\cdot q} P^\pm(x, 2q)e^{\pm isq^2} \mathcal{F}_1(s^{(1-\alpha)/100}|q| > 1)e^{-iy\cdot q}
$$

(B.47)

$$
 \times \chi(|y| \leq s^\alpha)f(y)d^n y d^n q,
$$

where we have used that $P^\pm = P^\pm(x, 2P)$ (see Eq. (4.10)). We note that the phase function in the right-hand side of Eq. (B.47) is given by:

(B.48) 
$$
 f(q) = (x - y) \cdot q + s|q|^2.
$$

When $|y| \leq s^\alpha$, $|x| \geq s^\alpha$ and $|q| \geq \frac{1}{2}s^{(\alpha-1)/100}$, by Eqs. (4.6)-(4.10) and estimate (B.49)

$$
 2s|q| \geq s^{(99+\alpha)/100} \geq s^\alpha, \quad s \geq 1,
$$

we have

$$
 |\nabla_q[f(q)]| = |x - y \pm 2sq| \geq \frac{1}{2}|x \pm 2sq| - |y| \geq \frac{1}{10^6}(|x| + 2s|q|) - |y| \geq \frac{1}{10^7}(|x| + s|q|).
$$

(B.50)

We choose an orthogonal basis $\{e_1, \cdots, e_n\}$ with $e_1 := \frac{x - y \pm 2sq}{|x - y \pm 2sq|}$. Let $x_1 := x \cdot e_1, y_1 := x \cdot e_1$ and $q_1 := q \cdot e_1$. We also have the estimate (B.51)

$$
 \|f_1(s)\|_{L^2(\mathbb{R}^n)} \leq \|(x)\|_{L^2(\mathbb{R}^n)}\|(x)^{\alpha}f_1(s)\|_{L^p(\mathbb{R}^n)} \lesssim \|(x)^{\alpha}f_1(s)\|_{L^p(\mathbb{R}^n)}.
$$

By estimate (B.30), we have

(B.52) 
$$
 |\partial_{q_1}[F_1(s^{1-\alpha/100}|q| \geq 1)]| = \frac{1}{(x_1 - y_1 \pm 2sq_1)} \leq \frac{s}{(|x| + s|q|)^2} \lesssim \frac{1}{|q|(|x| + s|q|)}.
$$

This, together with estimates (B.53) and (B.54)

$$
 |\partial_{q_1}[F_1(s^{1-\alpha/100}|q| \geq 1)]| = \frac{s^{(1-\alpha/100)}|q_1|}{|q|}|\mathcal{F}_1(s^{1-\alpha/100}|q| \geq 1)| \lesssim \frac{1}{|q|},
$$

(B.53)

and (recall that $P^\pm(r, \nu)$ is defined in terms of $\hat{r}$ and $\hat{\nu}$. See Eqs. (4.8) and (4.9).)

(B.54) 
$$
 |\partial_{q_1}[P^\pm(x, 2q)]| \lesssim \frac{1}{|q|^j}, \quad j = 1, 2, \cdots, \quad \text{implying, by taking integration by parts in } q_1 \text{ variable for } N \text{ times},
$$

implies, by taking integration by parts in $q_1$ variable for $N$ times,

(B.55) 

$$
 |(x)^{\alpha}f_1(s)| \lesssim \int \frac{\langle x \rangle^n}{|q|^N} \frac{\chi(|q| \geq \frac{1}{2}s^{(\alpha-1)/100})}{\langle x \rangle^N \langle |x| + s|q| \rangle^N} \chi(|y| \leq s^\alpha)|f(y)|d^n y.
$$

Taking the integral over $q$ in the right-hand side of estimate (B.56) and using estimates (with $|q| \geq \frac{1}{2}s^{(\alpha-1)/100}$)

(B.56) 

$$
 \frac{1}{\langle |x| + s|q| \rangle} \leq \frac{1}{\langle |x| + s^{(99+\alpha)/100} \rangle},
$$

(B.57)
and
\[ (B.58) \quad \frac{(x)^a}{(|x| + s^{(99+\alpha)/100})^N} \lesssim \frac{1}{(s^{(99+\alpha)/100})^{N-n}}. \]
we obtain
\[ \| \langle x \rangle^a f_1(s) \|_{L^p_{a}(\mathbb{R}^n)} \lesssim \int \frac{(x)^a \chi(|y| \geq \frac{1}{7} s^{(\alpha-1)/100})}{|y|^{N-1} (|x| + s|y|)^N} \chi(|y| \leq s^\sigma) |f(y)| d^n y \]
\[ \lesssim \int \frac{(s^{(99+\alpha)/100})^{N-n} \chi(|y| \leq s^\sigma) |f(y)| d^n y}{s^{(\alpha-1)(N-n)/100}} \]
\[ \lesssim \frac{1}{(s)^{N-n}} \int \chi(|y| \leq s^\sigma) |f(y)| d^n y. \]
(B.59)

By using Cauchy-Schwarz inequality in estimate (B.39), we arrive at, as \( s \to \infty \),
\[ \| \langle x \rangle^a f_1(s) \|_{L^p_{a}(\mathbb{R}^n)} \lesssim \frac{1}{(s)^{N-n}} \| f(y) \|_{L^p_{\alpha}(\mathbb{R}^n)} \| \chi(|y| \leq s^\sigma) \|_{L^2_{\alpha}(\mathbb{R}^n)} \]
\[ \leq \frac{1}{(s)^{N-n-an/2}} \| f(y) \|_{L^p_{\alpha}(\mathbb{R}^n)} \to 0 \]
(B.60)
with \( N = n + an/2 + 1 \). This, together with estimates (B.45) and (B.46), implies (4.15).

**Proof of Lemma 4.6.** By equations
\[ \mathcal{F}_c \left( \frac{|x - t\eta|}{(t + 1)^{1/2+\epsilon}} \right) \geq 1 \right) P^\pm_{tn} e^{i(s-t)H_0} \mathcal{F}_1(\sqrt{\lambda} + 1 |2P - \eta| \geq 1) \langle x - s\eta \rangle^{-\sigma} \]
\[ = e^{-i\eta P} \mathcal{F}_c \left( \frac{|x|}{(t + 1)^{1/2+\epsilon}} \right) \geq 1 \right) P^\pm(\eta, 2P - \eta) e^{i\eta P} e^{i(s-t)H_0} \mathcal{F}_1(\sqrt{\lambda} + 1 |2P - \eta| \geq 1) \]
\[ \times e^{i\eta P} \mathcal{F}_c(\frac{|x|}{(t + 1)^{1/2+\epsilon}} \geq 1 \right) P^\pm(\eta, 2P - \eta) e^{i(s-t)H_0} \mathcal{F}_1(\sqrt{\lambda} + 1 |2P - \eta| \geq 1) \langle x \rangle^{-\sigma} e^{i\eta P} \]
and
\[ P^\pm_{tn} e^{-itH_0} \mathcal{F}_1(\sqrt{\lambda} + 1 |2P - \eta| \geq 1) \langle x \rangle^{-\sigma} \]
\[ = e^{-i\eta P} P^\pm(\eta, 2P - \eta) e^{i\eta P} e^{-itH_0} \mathcal{F}_1(\sqrt{\lambda} + 1 |2P - \eta| \geq 1) \langle x \rangle^{-\sigma} \]
\[ = e^{-i\eta P} P^\pm(\eta, 2P - \eta) e^{-i(tH_0 - \eta P)} \mathcal{F}_1(\sqrt{\lambda} + 1 |2P - \eta| \geq 1) \langle x \rangle^{-\sigma}, \]
we obtain
\[ \| \mathcal{F}_c \left( \frac{|x - t\eta|}{(t + 1)^{1/2+\epsilon}} \right) \geq 1 \right) P^\pm_{tn} e^{i(s-t)H_0} \mathcal{F}_1(\sqrt{\lambda} + 1 |2P - \eta| \geq 1) \langle x - s\eta \rangle^{-\sigma} \|_{L^2_{a}(\mathbb{R}^n)} \]
(B.61)
\[ = \| \mathcal{F}_c \left( \frac{|x|}{(t + 1)^{1/2+\epsilon}} \right) \geq 1 \right) P^\pm(\eta, 2P - \eta) e^{i(s-t)(H_0 - \eta P)} \mathcal{F}_1(\sqrt{\lambda} + 1 |2P - \eta| \geq 1) \langle x \rangle^{-\sigma} \|_{L^2_{a}(\mathbb{R}^n)} \]
and
\[ \| P^\pm_{tn} e^{-itH_0} \mathcal{F}_1(\sqrt{\lambda} + 1 |2P - \eta| \geq 1) \langle x \rangle^{-\sigma} \|_{L^2_{a}(\mathbb{R}^n)} \]
\[ = \| P^\pm(\eta, 2P - \eta) e^{-i(tH_0 - \eta P)} \mathcal{F}_1(\sqrt{\lambda} + 1 |2P - \eta| \geq 1) \langle x \rangle^{-\sigma} \|_{L^2_{a}(\mathbb{R}^n)}, \]
(B.62)
By a similar argument of Lemma 4.2, we obtain, with \( u \geq 0 \),
\[
\| \mathcal{F}_c \left( \frac{|x|}{(t + 1)^{1/2 + \varepsilon}} \right) \geq 1 \| P^+ x (2P - \eta) e^{i(t - |x|)H_0} \mathcal{F}_1 (\sqrt{t + 1}|2P - \eta| \geq 1) (x)^{-\sigma} \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)}
\]
(B.63)
\[
\leq \frac{1}{(t + 1)^{1/2 + \varepsilon} + u/ \sqrt{t + 1})} 
\]
and
\[
\| P^- (x, 2P - \eta) e^{-i(t - |x|)H_0} \mathcal{F}_1 (\sqrt{t + 1}|2P - \eta| \geq 1) (x)^{-\sigma} \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq \frac{1}{(t + 1)^{1/2 + \varepsilon}}.
\]
(B.64)

Estimates (B.63) and (B.64), together with Eqs. (B.61) and (B.62), imply estimates (4.17) and (4.18). This completes the proof. \( \square \)

**Proof of Lemma 4.7.** By equation
\[
\mathcal{F}_c \left( \frac{|x - \eta|}{(t + 1)^{1/2 + \varepsilon}} \right) \geq 1 \| P^+_0 e^{-i(t - |x|)H_0} \mathcal{F}_1 (\sqrt{t + 1}|2P - \eta| < 1) (x - s\eta)^{-\sigma} \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)}
\]
(B.65)
\[
eq e^{-i\eta \cdot P} \mathcal{F}_c \left( \frac{|x|}{(t + 1)^{1/2 + \varepsilon}} \right) \geq 1 \| P^+_0 e^{-i(t - |x|)H_0} \mathcal{F}_1 (\sqrt{t + 1}|2P - \eta| < 1) (x - s\eta)^{-\sigma} e^{isp \cdot P} \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)}
\]
we obtain
\[
\| \mathcal{F}_c \left( \frac{|x - \eta|}{(t + 1)^{1/2 + \varepsilon}} \right) \geq 1 \| P^+_0 e^{-i(t - |x|)H_0} \mathcal{F}_1 (\sqrt{t + 1}|2P - \eta| < 1) (x - s\eta)^{-\sigma} \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)}
\]
(B.66)
\[
= \| \mathcal{F}_c \left( \frac{|x|}{(t + 1)^{1/2 + \varepsilon}} \right) \geq 1 \| P^+_0 e^{-i(t - |x|)H_0} \mathcal{F}_1 (\sqrt{t + 1}|2P - \eta| < 1) (x - s\eta)^{-\sigma} \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)}.
\]

By a similar argument of Lemma 4.3, we obtain, with \( s \in [0, t] \),
\[
\| \mathcal{F}_c \left( \frac{|x|}{(t + 1)^{1/2 + \varepsilon}} \right) \geq 1 \| P^+_0 e^{-i(t - |x|)H_0} \mathcal{F}_1 (\sqrt{t + 1}|2P - \eta| < 1) (x - s\eta)^{-\sigma} \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)}
\]
(B.67)
\[
\leq \frac{1}{(t + 1)^{1/2 + \sigma + \sigma \varepsilon}}.
\]
Estimate (B.67) and Eq. (B.66) imply estimate (4.19). This completes the proof. \( \square \)

**Proof of Lemma 4.8.** \( P^{-}_{\eta} e^{-iH_0} f \) satisfies, for all \( M \geq 1 \) and \( \epsilon \in (0, 1/2), \) with \( t \geq 0, \)
\[
\| P^{-}_{\eta} e^{-iH_0} f \|_{L^2(\mathbb{R}^n)} \leq \| P^{-}_{\eta} e^{-iH_0} F_1 (\sqrt{t + 1}|2P - \eta| \geq 1) c(|x| \leq M) \psi(0) \|_{L^2(\mathbb{R}^n)}
\]
\[
+ \| P^{-}_{\eta} e^{-iH_0} F_1 (\sqrt{t + 1}|2P - \eta| \geq 1) c(|x| > M) \psi(0) \|_{L^2(\mathbb{R}^n)}
\]
\[
+ \| P^{-}_{\eta} e^{-iH_0} (1 - F_1 (\sqrt{t + 1}|2P - \eta| \geq 1) \psi(0) \|_{L^2(\mathbb{R}^n)}
\]
\[
\leq \| P^{-}_{\eta} e^{-iH_0} F_1 (\sqrt{t + 1}|2P - \eta| \geq 1) (x)^{-1} \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)}
\]
\[
\times \| (x) \chi(|x| \leq M) \psi(0) \|_{L^2(\mathbb{R}^n)} + \| (x) \chi(|x| > M) \psi(0) \|_{L^2(\mathbb{R}^n)}
\]
(B.68)
\[
+ \| (1 - F_1 (\sqrt{t + 1}|2P - \eta| \geq 1) \psi(0) \|_{L^2(\mathbb{R}^n)}.
\]
By taking $M = (1 + t)^{1/100}$ and by using Lemma 4.6, we obtain that
\[ \|P_{\eta}e^{-itH_0}f\|_{L^2_x(\mathbb{R}^n)} \leq \frac{1}{(tT)^{1/2}} (1 + t)^{1/100}\|\psi(0)\|_{L^2_x(\mathbb{R}^n)} + \|\chi(|x| > (t + 1)^{1/100})\psi(0)\|_{L^2_x(\mathbb{R}^n)} \]
\[ + \| (1 - F_1(\sqrt{t + 1}2P - \eta) \geq 1)) \psi(0)\|_{L^2_x(\mathbb{R}^n)} \to 0 \]
(B.69) 

as $t \to \infty$. This completes the proof. \qed

**Proof of Lemma 4.9.** By equation
\[ \chi(|x - \eta| \leq t^{\alpha})P_{\eta}^\pm e^{-itH_0}f = e^{-it\alpha P} \chi(|x| \leq t^{\alpha})P^-(x, 2P - \eta)e^{it\alpha P}e^{-itH_0}f \]
(B.70) 
we obtain
\[ \|\chi(|x - \eta| \leq t^{\alpha})P_{\eta}^\pm e^{-itH_0}f\|_{L^2_x(\mathbb{R}^n)} = \|\chi(|x| \leq t^{\alpha})P^-(x, 2P - \eta)e^{-it(H_0 - \eta P)}f\|_{L^2_x(\mathbb{R}^n)} \]
(B.71)
This, together with Lemma 4.6, implies
\[ \|\chi(|x| \leq t^{\alpha})P^-(x, 2P - \eta)e^{-it(H_0 - \eta P)}f\|_{L^2_x(\mathbb{R}^n)} \]
\[ \leq \|P^-(x, 2P - \eta)e^{-it(H_0 - \eta P)}f\|_{L^2_x(\mathbb{R}^n)} + \|\chi(|x| < t^{1/10})f\|_{L^2_x(\mathbb{R}^n)} \]
\[ + \|\chi(|x| \geq t^{1/10})f\|_{L^2_x(\mathbb{R}^n)} + \|\chi(|x| \geq t^{1/10})f\|_{L^2_x(\mathbb{R}^n)} \]
\[ + \|F_1(\sqrt{t + 1}2P - \eta) \leq 1) f\|_{L^2_x(\mathbb{R}^n)} \]
(B.72) 
\[ \to 0 \]
as $t \to \infty$. This leads to
\[ \limsup_{t \to \infty} \|\chi(|x - \eta| \leq t^{\alpha})P_{\eta}^\pm e^{-itH_0}f\|_{L^2_x(\mathbb{R}^n)} = 0, \]
(B.73) 
which, together with
\[ \lim_{t \to \infty} \|\chi(|x - \eta| \leq t^{\alpha})e^{-itH_0}f\|_{L^2_x(\mathbb{R}^n)} = 0, \]
(B.74) 
implies
\[ \limsup_{t \to \infty} \|\chi(|x - \eta| \leq t^{\alpha})P_{\eta}^\pm e^{-itH_0}f\|_{L^2_x(\mathbb{R}^n)} = 0. \]
(B.75)
Both Eqs. (B.73) and (B.75) imply Eq. (4.21). We finish the proof. \qed

**Lemma B.1.** Let $\{e_1, \cdots, e_n\}$ denote an orthogonal basis in $\mathbb{R}^n$. For all $\sigma > 0$, we have
\[ \|\langle x \rangle^{\sigma \cdot \beta} - P_{\beta}\langle P \rangle^{-\sigma} \|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} \leq 1, \quad j = 1, \cdots, n. \]
(B.76)
Proof. We note that \([\langle x \rangle^\sigma, \frac{P_j}{\langle P \rangle}]\) reads

\[
\langle x \rangle^\sigma, \frac{P_j}{\langle P \rangle} = \frac{1}{(2\pi)^{n/2}} \int h_j(\xi)\langle x \rangle^\sigma, e^{iP_x}d\xi,
\]

where \(0 := \lim_{\epsilon \to 0}\)

\[
h_j(\xi) := \frac{1}{(2\pi)^{n/2}} \int e^{-iy\xi - 0y} \frac{y_j}{\langle y \rangle} dy.
\]

Note that since \(\frac{P_j}{\langle P \rangle}\) is a smooth multiplier, then (by repeated integrations by part) the Fourier transform is a fast-decaying distribution, for large \(\xi\). At the origin it has the singularity \(1/\xi\) (like that of the Fourier transform of a step function). Hence,

\[
\int |\xi|(1 + |\xi|)^{\sigma-1})|h_j(\xi)|d\xi < \infty, \quad j = 1, \ldots, n.
\]

To compute \([\langle x \rangle^\sigma, e^{i\xi P}]\), we find

\[
\langle x \rangle^\sigma, e^{i\xi P} = e^{i\xi P} \left(e^{-i\xi P} \langle x \rangle^\sigma e^{i\xi P} - \langle x \rangle^\sigma\right) = \int_0^{\xi} e^{-i\xi \hat{\xi} - (i\xi \hat{\xi}) \cdot P \cdot \hat{\xi}, \langle x \rangle^\sigma} e^{i\xi P} du = (-i)^2 e^{i\xi P} \langle x \rangle^\sigma \cdot \hat{\xi} \cdot \hat{\xi} e^{i\xi P}.
\]

We find that

\[
\langle x \rangle^\sigma \cdot \hat{\xi} \cdot \hat{\xi} \cdot e^{i\xi P} = \langle x \rangle^\sigma \cdot \hat{\xi} \cdot e^{i\xi P},
\]

which implies

\[
\langle x \rangle^\sigma \cdot \hat{\xi} \cdot e^{i\xi P} \leq \langle x \rangle^\sigma \cdot \hat{\xi} \cdot e^{i\xi P},
\]

(\(B.81\))

\[
\langle x \rangle^\sigma \cdot \hat{\xi} \cdot e^{i\xi P} \leq \langle x \rangle^\sigma \cdot \hat{\xi} \cdot e^{i\xi P},
\]

which implies

\[
\langle x \rangle^\sigma \cdot \hat{\xi} \cdot e^{i\xi P} \leq \langle x \rangle^\sigma \cdot \hat{\xi} \cdot e^{i\xi P},
\]

(\(B.82\))

\[
\langle x \rangle^\sigma \cdot \hat{\xi} \cdot e^{i\xi P} \leq \langle x \rangle^\sigma \cdot \hat{\xi} \cdot e^{i\xi P},
\]

(\(B.83\))

By Eq. (B.77) and estimate (B.79), this leads to estimate (B.76).

\(\square\)

Remark B.2. Here the weight \(\langle x \rangle^\sigma \cdot \hat{\xi} \cdot e^{i\xi P}\) in Eq. (B.82) is not optimal.

Lemma B.3. For all \(\delta > 0\), \(\langle x \rangle^\delta f \in \mathcal{H}_1^1\), we have

\[
\|\langle x \rangle^\delta (P)f\|_{L_2^2(\mathbb{R}^n)} \leq \|\langle x \rangle^\delta f\|_{\mathcal{H}_1^1}.
\]

(\(B.84\))
Proof. Using
\begin{equation}
\langle x \rangle^{\delta} \langle P \rangle f = \langle x \rangle^{\delta} \frac{P^2 + 1}{\langle P \rangle} f,
\end{equation}
we write \( \langle x \rangle^{\delta} \langle P \rangle f \) as
\begin{equation}
\langle x \rangle^{\delta} \langle P \rangle f = \sum_{j=0}^{n} v_j,
\end{equation}
where
\begin{equation}
v_0 := \langle x \rangle^{\delta} \frac{1}{\langle P \rangle} f
\end{equation}
and
\begin{equation}
v_j := \langle x \rangle^{\delta} \frac{p^2_j}{\langle P \rangle} f, \quad j = 1, \ldots, n.
\end{equation}
We note that \( \frac{1}{\langle P \rangle} \) is a Fourier multiplier and
\begin{equation}
\langle x \rangle^{\delta} \mathcal{F}^{-1} \left[ \frac{1}{\langle q \rangle} \right](x) \in L^1_s(\mathbb{R}^n),
\end{equation}
where \( \mathcal{F}^{-1}[h] \) denotes the inverse Fourier transform of \( h \). By Eq.
\begin{equation}
\langle x \rangle^{\delta} \frac{1}{\langle P \rangle} f = \langle x \rangle^{\delta} \int \mathcal{F}^{-1} \left[ \frac{1}{\langle q \rangle} \right](y)f(x-y)dy
\end{equation}
and estimate
\begin{equation}
\langle x \rangle^{\delta} \lesssim \langle x-y \rangle^{\delta}(y)^{\delta}, \quad \forall y \in \mathbb{R}^n,
\end{equation}
this implies
\begin{align*}
\|v_0\|_{L^2_s(\mathbb{R}^n)} &= \|\langle x \rangle^{\delta} \int \mathcal{F}^{-1} \left[ \frac{1}{\langle q \rangle} \right](y)f(x-y)dy\|_{L^2_s(\mathbb{R}^n)} \\
&\lesssim \|\langle x \rangle^{\delta} \mathcal{F}^{-1} \left[ \frac{1}{\langle q \rangle} \right](x)\|_{L^1_s(\mathbb{R}^n)} \|\langle x \rangle^{\delta} f(x)\|_{L^2_s(\mathbb{R}^n)} \\
&\lesssim \|\langle x \rangle^{\delta} f\|_{L^2_s(\mathbb{R}^n)}.
\end{align*}
By Lemma B.1 and using
\begin{equation}
v_j = \langle x \rangle^\sigma \frac{P_j}{\langle P \rangle} (P_j f) = \frac{P_j}{\langle P \rangle} \langle x \rangle^\sigma (P_j f) + [\langle x \rangle^\sigma, \frac{P_j}{\langle P \rangle}] P_j f,
\end{equation}
we find
\begin{align*}
\|v_j\|_{L^2_s(\mathbb{R}^n)} &\leq \left\| \frac{P_j}{\langle P \rangle} \langle x \rangle^\sigma P_j f \right\|_{L^2_s(\mathbb{R}^n)} + \left\| [\langle x \rangle^\sigma, \frac{P_j}{\langle P \rangle}] P_j f \right\|_{L^2_s(\mathbb{R}^n)} \\
&\leq \|\langle x \rangle^\sigma P_j f\| + \left\| [\langle x \rangle^\sigma, \frac{P_j}{\langle P \rangle}] \langle x \rangle^{-\sigma} \right\|_{L^2_s(\mathbb{R}^n)} \|\langle x \rangle^{\sigma} P_j f\|_{L^2_s(\mathbb{R}^n)} \\
&\leq \sigma \|\langle x \rangle^\sigma P_j f\|_{L^2_s(\mathbb{R}^n)}.
\end{align*}
Since
\begin{align*}
\|\langle x \rangle^\sigma P_j f\|_{L^2_s(\mathbb{R}^n)} &\leq \|P_j(\langle x \rangle^\sigma f)\|_{L^2_s(\mathbb{R}^n)} + \|P_j f\|_{L^2_s(\mathbb{R}^n)} \\
&\leq \|\langle x \rangle^\sigma f\|_{H^1} + \sigma \|\langle x \rangle^{\sigma-1} f\|_{L^2_s(\mathbb{R}^n)}
\end{align*}
Inequality (C.4) provided that
\[ \parallel y \parallel_{L^2_\mathbb{R}^n} \leq c \parallel \langle x \rangle^{\gamma} f \parallel_{H^1_\mathbb{R}^n}. \]

Estimates (B.92) and (B.96), together with Eq. (B.86), yield estimate (B.84). \(\square\)

**Appendix C. The sketch of Proof of Proposition 3.1**

The proof of Proposition 3.1 is the same as the proof of the original existence proof of the Free Channel Wave Operator. The use of Cook’s argument reduces as before the problem to proving the integrability in time of the contribution of the Interaction term. We use the dispersive estimate for \(U_0(t)\) similar to the one one we used for the free flow: for some \(p \in (2, \infty]\),
\[ \parallel U_0(-t)N(x, t, \psi(t))\psi(t)\parallel_{L^p_\mathbb{R}^n} \leq \frac{1}{t^{1+\gamma}}, \quad t \geq 1. \]
Then the integrability of the interaction term follows from, with \(1/q + 1/p = 1/2\),
\[ \parallel F_c(|x|/t^\alpha \leq 1)U_0(-t)N(x, t, \psi(t))\psi(t)\parallel_{L^p_\mathbb{R}^n} \leq \parallel F_c(|x|/t^\alpha \leq 1)\parallel_{L^p_\mathbb{R}^n} \parallel U_0(-t)N(x, t, \psi(t))\psi(t)\parallel_{L^p_\mathbb{R}^n} \]
\[ \leq \frac{1}{t^{1+\epsilon - \alpha/q - \beta k}} \in L^1_\mathbb{R}[1, \infty) \]
provided that
\[ \alpha < \frac{eq}{n} = \frac{\epsilon}{n(1/2 - 1/p)} = \frac{2 ep}{n(p - 2)}. \]
This, together with estimate (4.31) and propagation estimate, implies the existence of free channel wave operator in part (1) of Proposition 3.4.

When the dimension is low, and the thresholds of \(U_0(t)\) results in too low dispersive decay rate, or when the decay requires a smooth initial data (as is the case for the wave equation) we add to the definition of the Free Channel Wave operator cut-off functions that vanish in a shrinking in time neighborhood of the thresholds, and also cut off a neighborhood of infinite frequency. Then, the above estimates hold as well. This frequency cutoff does NOT change the wave operator. This is because on the complement, the wave operator is zero (by taking the weak limit in \(t\) that defines the operator.) This holds provided the solution is uniformly bounded in \(H^1_\mathbb{R}\). In part (2) of Proposition 3.4, we cut off a neighborhood of infinite frequency. The interaction term satisfies the estimate, with \(\frac{1}{q} + \frac{1}{p} = \frac{1}{2}\),
\[ \parallel F_c(|x|/t^\alpha \leq 1)P(|P| \leq t^\beta)U_0(-t)N(x, t, \psi(t))\psi(t)\parallel_{L^p_\mathbb{R}^n} \leq \parallel F_c(|x|/t^\alpha \leq 1)\parallel_{L^p_\mathbb{R}^n} \parallel P(|P| \leq t^\beta)U_0(-t)N(x, t, \psi(t))\psi(t)\parallel_{L^p_\mathbb{R}^n} \]
\[ \leq \frac{1}{t^{1+\epsilon - \alpha/q - \beta k}} \in L^1_\mathbb{R}[1, \infty) \]
provided that
\[ \epsilon - n\alpha/q - \beta k > 0. \]
Inequality (C.4) is satisfied since when \(\alpha \in (0, \min\{\frac{2\epsilon k}{p-2\beta}, \frac{\epsilon k}{\beta n}\})\) and \(\epsilon \in (0, \min\{\alpha, \frac{\epsilon k}{\beta n(1/2 - 1/p)}\})\),
\[ \epsilon \frac{\beta}{n} > 0, \]
\[ \epsilon \frac{\beta}{n} - n\alpha > 0, \]
\[ \epsilon \tilde{p} - n \tilde{p} \alpha (1/2 - 1/\tilde{p}) = \epsilon \tilde{p} - n \alpha (\tilde{p} - 2) / 2 = \epsilon \tilde{p} - n \tilde{p} \alpha / \tilde{q} > 0 \]
and
\[ \epsilon - n \alpha / \tilde{q} - \beta k > \epsilon - n \alpha / \tilde{q} - (\epsilon \tilde{p} - n \alpha) / \tilde{p} = 0. \]
\[ \text{(C.6)} \quad \text{and} \quad \epsilon - n \alpha / \tilde{q} - \beta k > \epsilon - n \alpha / \tilde{q} - (\epsilon \tilde{p} - n \alpha) / \tilde{p} = 0. \]
\[ \text{(C.7)} \]

Additionally, \( \beta < \alpha \) and both space and frequency cut-off functions satisfy the non-negativity property. Estimates (4.31) and

\[ \partial_t \left[ F_1(|q| \leq t^\beta) \right] = -\beta t^{-1-\beta} |q| F_1'(|q| \leq t^\beta) \geq 0 \]

\[ \text{(C.8)} \]

are valid. Then by using propagation estimates twice, we obtain the existence of the free channel wave operator.

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