Generalized field-transforming metamaterials

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**Abstract.** In this paper, we introduce a generalized concept of field-transforming metamaterials, which perform field transformations defined as linear relations between the original and transformed fields. These artificial media change the fields in a prescribed fashion in the volume occupied by the medium. We show what electromagnetic properties of transforming medium are required. The coefficients of these linear functions can be arbitrary scalar functions of position and frequency, which makes the approach quite general and opens a possibility to realize various unusual devices.
1. Introduction

It has been recently realized that metamaterials—artificial electromagnetic materials with engineered properties—can be designed to control electromagnetic fields in rather general ways. The concept of ‘transformation optics’, which is based on finding artificial materials that create the desired configuration of electromagnetic fields, has been developed by several research teams, see e.g. [1]–[3].

In the known approaches one starts from a certain transformation of spatial coordinates and possibly also time, which corresponds to a desired transformation of electromagnetic fields. It has been shown that transformation of spatial coordinates (accompanied by the corresponding transformation of electromagnetic fields) can be mimicked by introducing electromagnetic materials with specific electromagnetic properties into the domain where the coordinates have been transformed [1, 2, 4]. As an example of such a ‘coordinate-transforming’ device an ‘invisibility cloak’ has been suggested [1, 4]. In addition to perfect invisibility devices, similar approaches were applied to perfect lenses, description of the Aharonov–Bohm effect and artificial black holes. It has been known for a long time that the material relations of an isotropic magnetodielectric transform into certain bi-anisotropic relations if the medium is moving with a constant velocity (e.g. [5]). It was also suggested that this effect of time transformation can possibly be mimicked by a metamaterial [2, 6, 7].

We have recently introduced an alternative paradigm of transformation optics, where the required material properties are defined directly from the desired transformation of electromagnetic fields. This concept of field-transforming metamaterials was proposed in [8],
where it was shown what metamaterial properties are required in order to perform the field transformation of the form $E_0 \rightarrow E = F(r, \omega)E_0$, $H_0 \rightarrow H = G(r, \omega)H_0$, where $F$ and $G$ are arbitrary differentiable functions of position and frequency (we work in the frequency domain and the field vectors as well as the transformation coefficients are in general complex numbers).

In this concept, we start directly from a desired transformation of electromagnetic fields and do not involve any space nor time transforms. A special case of similar transformations was proposed earlier as a numerical technique for termination of computational domain by modulating fields with a function decaying to zero at the termination [9]. The physical interpretation of this numerical technique in terms of a slab of a material with the moving-medium material relations was presented in [6], and the pulse propagation in a slab of this medium backed by a boundary was studied in [10].

In this paper, we generalize the concept of field-transforming metamaterials introducing methods to perform general bi-anisotropic field transformations, where the desired fields depend, in the general linear fashion, on both electric and magnetic fields of the original field distribution. Due to the generality of the approach, the concept can be applied to a great variety of field transformations, not limited to cloaking devices.

2. Field transformations with metamaterials

Let us assume that in a certain volume $V$ of free space there exist electromagnetic fields $E_0(r)$ and $H_0(r)$ created by sources located outside volume $V$ (we work in the frequency domain, and these vectors are complex amplitudes of the fields). The main idea is to fill volume $V$ with a material in such a way that after the volume is filled, the original fields $E_0$ and $H_0$ would be transformed to other fields, according to the design goals. Here, we consider metamaterials performing the general linear field transformation defined as

$$E(r) = F(r, \omega)E_0(r) + \sqrt{\frac{\mu_0}{\varepsilon_0}} A(r, \omega)H_0(r), \quad (1)$$

$$H(r) = G(r, \omega)H_0(r) + \sqrt{\frac{\varepsilon_0}{\mu_0}} C(r, \omega)E_0(r). \quad (2)$$

Here, scalar functions $F(r, \omega)$, $G(r, \omega)$ $A(r, \omega)$ and $C(r, \omega)$ are arbitrary differentiable complex functions. In this paper, we consider only the case of scalar coefficients in the above relations, although the method can be extended to the most general linear relations between original and transformed fields by replacing scalar coefficients by arbitrary dyadics.

2.1. Required constitutive parameters

Substituting (1) and (2) into the Maxwell equations and demanding that the original fields $E_0$ and $H_0$ satisfy the free-space Maxwell equations, one finds that the transformed fields $E$ and $H$ satisfy Maxwell equations $\nabla \times E = -j\omega B$ and $\nabla \times H = j\omega D$ in a medium with the following material relations:

$$B = \sqrt{\frac{\mu_0}{\varepsilon_0} \frac{j}{\omega} \frac{1}{FG - AC}} (F\nabla A - A\nabla F) \times H + \mu_0 \frac{1}{FG - AC} \left( F^2 + A^2 \right) H$$

$$+ \frac{j}{\omega} \frac{1}{FG - AC} (G\nabla F - C\nabla A) \times E - \sqrt{\varepsilon_0\mu_0} \frac{1}{FG - AC} \left( AG + CF \right) E, \quad (3)$$

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\[
D = \sqrt{\frac{\varepsilon_0}{\mu_0 \omega}} \left( C \nabla G - G \nabla C \right) \times E + \varepsilon_0 \frac{1}{FG - AC} \left( G^2 + C^2 \right) E \\
+ \frac{j}{\omega} \frac{1}{FG - AC} \left( A \nabla C - F \nabla G \right) \times H - \sqrt{\varepsilon_0 \mu_0} \frac{1}{FG - AC} \left( AG + CF \right) H. 
\]

Relations (3) and (4) describe a bi-anisotropic medium, whose constitutive relations can be conveniently written as [7]

\[
B = \bar{\mu} \cdot H + \sqrt{\varepsilon_0 \mu_0}\left( \bar{\chi} + j\bar{\kappa} \right)^T \cdot E, 
\]

\[
D = \bar{\varepsilon} \cdot E + \sqrt{\varepsilon_0 \mu_0}\left( \bar{\chi} - j\bar{\kappa} \right) \cdot H. 
\]

Comparing with (3) and (4), we can identify the required material parameters of the field-transforming medium.

The permittivity dyadic

\[
\bar{\varepsilon} = \varepsilon_0 \frac{1}{FG - AC} \left( G^2 + C^2 \right) \bar{I} + \sqrt{\varepsilon_0 \mu_0} \frac{1}{FG - AC} \left( C \nabla G - G \nabla C \right) \times \bar{I}, 
\]

the permeability dyadic

\[
\bar{\mu} = \mu_0 \frac{1}{FG - AC} \left( F^2 + A^2 \right) \bar{I} + \sqrt{\varepsilon_0 \mu_0} \frac{1}{FG - AC} \left( F \nabla A - A \nabla F \right) \times \bar{I}, 
\]

the nonreciprocity dyadic \(\bar{\chi}\)

\[
\bar{\chi} = -\frac{1}{FG - AC} \left( AG + CF \right) \bar{I} + \frac{1}{\sqrt{\varepsilon_0 \mu_0} \omega} \frac{j}{FG - AC} \left[ \left( A \nabla C - F \nabla G \right) - \left( G \nabla F - C \nabla A \right) \right] \times \bar{I}, 
\]

and the reciprocal magnetoelectric coupling dyadic (also called chirality dyadic) \(\bar{\kappa}\)

\[
\bar{\kappa} = \frac{1}{\sqrt{\varepsilon_0 \mu_0} \omega} \frac{1}{FG - AC} \left[ -\left( A \nabla C - F \nabla G \right) - \left( G \nabla F - C \nabla A \right) \right] \times \bar{I}. 
\]

In the above equations, \(\bar{I}\) is the unit dyadic.

3. Classification of generalized field-transforming metamaterials

In this section, we classify the various cases of different materials that may need to be obtained depending on the required transformation (1) and (2).

3.1. Reciprocity and nonreciprocity

Considering the required permittivity and permeability, we see that in the general case they both contain symmetric and anti-symmetric parts, so the material is nonreciprocal. The nonreciprocal parts of permittivity and permeability vanish, if coefficients \(A\) and \(C\) in (1) and (2) equal zero, meaning that the desired field transformation does not involve magnetoelectric coupling, as in [8]. Note that in this case, the medium can be still nonreciprocal due to its magnetoelectric properties, see below. The other important case when the permittivity and permeability are
Table 1. Classification of reciprocal bi-anisotropic media.

| Coupling parameters | Class                          |
|---------------------|--------------------------------|
| \(\kappa \neq 0, \vec{N} = 0, \vec{J} = 0\) | Isotropic chiral medium        |
| \(\kappa \neq 0, \vec{N} \neq 0, \vec{J} = 0\) | Anisotropic chiral medium      |
| \(\kappa = 0, \vec{N} \neq 0, \vec{J} = 0\) | Pseudochiral medium            |
| \(\kappa = 0, \vec{N} = 0, \vec{J} \neq 0\) | Omega medium                   |
| \(\kappa \neq 0, \vec{N} = 0, \vec{J} \neq 0\) | Chiral omega medium            |
| \(\kappa = 0, \vec{N} \neq 0, \vec{J} \neq 0\) | Pseudochiral omega medium      |
| \(\kappa \neq 0, \vec{N} \neq 0, \vec{J} \neq 0\) | General reciprocal bi-anisotropic medium |

Symmetric (actually scalar in the present case) is when the coefficients \(F\) and \(G\) vanish. These transformations define a new electric field as a function of the original magnetic field and vice versa. Furthermore, we notice that if \(A = C = 0\) and, in addition, \(G = F\), the wave impedance of the medium does not change. This is expected, because this is the case when the electric and magnetic fields are transformed in the same way.

Next, let us consider what types of magnetoelectric coupling in field-transforming metamaterials are required for various transformations of fields. Here, we will use the general classification of bi-anisotropic media according to [7, 11]. This classification is based on splitting the coupling dyadics \(\vec{\kappa}\) and \(\vec{\chi}\) into three components, e.g. for \(\vec{\kappa}\) we write

\[
\vec{\kappa} = \kappa \vec{I} + \vec{N} + \vec{J},
\]

where \(\vec{N}\) and \(\vec{J}\) denote the symmetric and antisymmetric parts, respectively. The scalar coefficient \(\kappa\) is equal to the trace of \(\vec{\kappa}\), and it is the chirality parameter. In our case it is identically zero, meaning that field-transforming metamaterials (with scalar transformation coefficients) are nonchiral (their microstructure is mirror-symmetric). Also, the symmetric part \(\vec{N}\) is identically zero, and only the antisymmetric part

\[
\vec{\kappa} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \frac{1}{2\omega \sqrt{FG - AC}} \left[ -(A \nabla C - F \nabla G) - (G \nabla F - C \nabla A) \right] \times \vec{I}
\]

remains.

Classification of reciprocal bi-anisotropic media is given by table 1 [7]. Field-transforming metamaterials possess reciprocal magnetoelectric coupling as in omega media. The required effects can be realized by introducing inclusions in form of pairs of the letter \(\Omega\), arranging them in pairs forming ‘hats’ [11, 12].

Similarly, we split the nonreciprocity dyadic \(\vec{\chi}\) as

\[
\vec{\chi} = \chi \vec{I} + \vec{Q} + \vec{S},
\]

where \(\vec{Q}\) and \(\vec{S}\) denote the symmetric and antisymmetric parts, respectively, and \(\chi\) is the trace of dyadic \(\vec{\chi}\). From (9), we can identify the parts of \(\vec{\chi}\) as

\[
\chi = -\frac{1}{FG - AC} (AG + CF),
\]

\[
\vec{Q} = 0,
\]

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Table 2. Classification of nonreciprocal bi-anisotropic media.

| Coupling parameters | Class                      |
|---------------------|----------------------------|
| \(\chi \neq 0, \vec{Q} = 0, \vec{S} = 0\) | Tellegen medium            |
| \(\chi \neq 0, \vec{Q} \neq 0, \vec{S} = 0\) | Anisotropic Tellegen medium |
| \(\chi = 0, \vec{Q} \neq 0, \vec{S} = 0\) | PseudoTellegen medium      |
| \(\chi = 0, \vec{Q} = 0, \vec{S} \neq 0\) | Moving medium              |
| \(\chi \neq 0, \vec{Q} = 0, \vec{S} \neq 0\) | Moving Tellegen medium     |
| \(\chi = 0, \vec{Q} \neq 0, \vec{S} \neq 0\) | Moving pseudoTellegen medium |
| \(\chi \neq 0, \vec{Q} \neq 0, \vec{S} \neq 0\) | Nonreciprocal (nonchiral) medium |

\[
\vec{S} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \frac{j}{2\omega} \frac{1}{FG - AC} [ (A\nabla C - F\nabla G) - (G\nabla F - C\nabla A) ] \times \vec{I}. \tag{16}
\]

In contrast to the reciprocal magnetoelectric coupling, in this case the trace of the dyadic \(\vec{\chi}\) is generally nonzero. Looking at table 2 [7], we see that the transformation medium can be either a Tellegen medium, a moving medium, or a moving Tellegen medium.

If all the transformation coefficients do not depend on the spatial coordinates, the field-transforming metamaterial is the Tellegen medium [7, 11], except the special case when \(AG = -CF\). Tellegen media can be synthesized as composites containing magnetized ferrite inclusions coupled with small metal strips or wires [13] or as mechanically bound particles having permanent electric and magnetic moments [14, 15].

3.2. Losses and gain

A bi-anisotropic medium is lossless if the permittivity and permeability dyadics are Hermitian:

\[
\vec{\varepsilon} = \vec{\varepsilon}^\dagger, \quad \vec{\mu} = \vec{\mu}^\dagger. \tag{17}
\]

In addition, the magnetoelectric dyadics for lossless media satisfy

\[
\vec{\chi} + j \vec{\kappa} = (\vec{\chi} - j\vec{\kappa})^*, \tag{18}
\]

where * denotes complex conjugate [7].

Analyzing formulae (7)–(10) for the required material parameters we see that if the transformation coefficients are complex numbers, in general the required material can be a lossy or gain medium. For real-valued coefficients \(F, G, A\) and \(C\) the permittivity permeability, and reciprocal magnetoelectric coupling coefficient correspond to lossless media. However, the antisymmetric part of the nonreciprocity dyadic (describing effects of 'moving' media) may correspond to lossy or gain media, depending on how the transformation coefficients depend on the position.

3.3. Presence or absence of sources

In the derivation of the material relations in field-transforming metamaterials (3) and (4), we have assumed that in the medium there are no source currents, that is, the field equations are
∇ × \textbf{E} = -jω\textbf{B} and ∇ × \textbf{H} = jω\textbf{D}. In addition, conditions
\[ \nabla \cdot \textbf{D} = 0, \quad \nabla \cdot \textbf{B} = 0, \tag{19} \]
should also be satisfied in source-free media. For the general case, we have no proof that conditions (19) always hold. However, we have checked that they are satisfied in many important special cases, which include, for example, position independent transformation coefficients and exponential dependence of the coefficients on the position vector.

4. Particular cases

Let us next consider some particular cases of field-transforming metamaterials to reveal the physical meaning of such transformations of fields.

4.1. Backward-wave medium

Consider the case of the field transformation \( \textbf{E}_0 \rightarrow \sqrt{\frac{\mu_0}{\varepsilon_0}} \textbf{H} \) and \( \textbf{H}_0 \rightarrow \sqrt{\frac{\varepsilon_0}{\mu_0}} \textbf{E} \), which corresponds to \( F = G = 0 \) and \( A = C = 1 \) in (1) and (2). Transforming electric field into magnetic field and vice versa, we reverse the propagation direction of plane waves in the transformation volume, so we expect that this would correspond to a backward-wave material filling the volume. And indeed we see that the material relations (3) and (4) reduce to
\[ \textbf{B} = -\mu_0 \textbf{H}, \tag{20} \]
\[ \textbf{D} = -\varepsilon_0 \textbf{E}, \tag{21} \]
which are the material relations of the Veselago medium [16].

4.2. Field rotation

Consider the case of the field transformation \( \textbf{E}_0 \rightarrow \sqrt{\frac{\mu_0}{\varepsilon_0}} \textbf{H} \) and \( \textbf{H}_0 \rightarrow -\sqrt{\frac{\varepsilon_0}{\mu_0}} \textbf{E} \), which corresponds to \( F = G = 0 \) and \( A = -1, C = 1 \) in (1) and (2). This kind of transformation relates to changing the polarization of the field: fields of a plane wave are rotated by \(-90°\) around the propagation axis. The material relations (3) and (4) reduce to
\[ \textbf{B} = \mu_0 \textbf{H}, \tag{22} \]
\[ \textbf{D} = \varepsilon_0 \textbf{E}, \tag{23} \]
which are the material relations of free space. The reason why we do not see the change of polarization here is simply due to the fact that we wanted to consider \( F, G, A \) and \( C \) as scalars. The media (original or transformed) do not ‘feel’ the polarization of the fields, so the rotation does not have any influence on the material relations. For instance, unpolarized light is obviously invariant under this transformation. We want to remind the reader that by considering the transformation coefficients \( F, G, A \) and \( C \) as dyadics, one can obtain more general transformations, which tailor also the field polarization in the general way.

4.3. Isotropic Tellegen medium

Consider next a more general case when all the transformation coefficients do not depend on the position vector inside the transformation domain. In this case, gradients of these functions
vanish, and we see that the material relations are those of an isotropic Tellegen material:

\[ B = \mu_0 \frac{1}{FG - AC} \left( F^2 + A^2 \right) H - \sqrt{\varepsilon_0 \mu_0} \frac{AG + CF}{FG - AC} E, \]  

(24)

\[ D = \varepsilon_0 \frac{1}{FG - AC} \left( G^2 + C^2 \right) E - \sqrt{\varepsilon_0 \mu_0} \frac{AG + CF}{FG - AC} H. \]  

(25)

The medium is reciprocal only if \( \chi = -\frac{AG + CF}{FG - AC} = 0 \). We can note that if the transformation coefficients vary within the transformation volume, this generally requires simulation of a moving medium with the transformation metamaterial.

4.4. Tellegen nihility

Let us consider again the case where all the coefficients are position-independent, that is,

\[ \nabla F = \nabla G = \nabla A = \nabla C = 0, \]  

(26)

and also demand that the permittivity and permeability of the transforming medium become zero. This takes place when either

\[ A = jF, \quad C = jG \]  

(27)

or

\[ A = -jF, \quad C = -jG. \]  

(28)

The material relations of the medium become

\[ B = -j \sqrt{\varepsilon_0 \mu_0} E, \]  

(29)

\[ D = -j \sqrt{\varepsilon_0 \mu_0} H, \]  

(30)

for the case (27) or

\[ B = j \sqrt{\varepsilon_0 \mu_0} E, \]  

(31)

\[ D = j \sqrt{\varepsilon_0 \mu_0} H, \]  

for the case (28).

These constitutive relations look similar to that of chiral nihility [17], but in this case the only nonzero material parameter is the Tellegen parameter. The Maxwell equations corresponding to relations (30) take the form

\[ \nabla \times H = -k_0 H, \quad \nabla \times E = k_0 E, \]  

(31)

where \( k_0 = \omega \sqrt{\varepsilon_0 \mu_0} \) is the free-space wavenumber. Thus, solutions of equation (31) are eigenfunctions of operator \( \text{rot} = \nabla \times \vec{I} \). If \( \partial / \partial x = \partial / \partial y = 0 \) then for the waves propagating in the \( z \)-direction

\[ E_x = -e_0 \exp(-jk_0 z), \quad H_x = e_0 \eta^{-1} \exp(-jk_0 z), \]  

(32)

\[ E_y = je_0 \exp(-jk_0 z), \quad H_y = je_0 \eta^{-1} \exp(-jk_0 z), \]  

\[ E_z = 0, \quad H_z = 0, \]
where $\eta$ is the wave impedance. Substituting expressions (27) or (28) for $A$ and $C$ into the original definitions (1) and (2), we obtain the following formula for the wave impedance:

$$
\eta = \frac{F}{G} \eta_0,
$$

(33)

where $\eta_0$ is the wave impedance of vacuum.

As in the case of chiral nihility [17] and the Beltrami fields in chiral media [14, 18], the fields (32) correspond to circularly polarized waves. However, in contrast to those two cases, where the phase shift between electric and magnetic fields equals exactly to $\pi/2$, in the considered case it depends on relation between $F$ and $G$ according to (33). If $F = G$, the electric and magnetic fields oscillate in phase.

4.5. Position-independent transformation coefficients

An interesting general conclusion can be drawn with regard to field transformations with all the coefficients being position independent. In this case, the wavenumber in the field-transforming metamaterial is the same as in the original medium (free space in our case). This can be shown by using the equation for the wavenumber $k$ in a Tellegen medium [14]

$$
k = k_0 \sqrt{\varepsilon_r \mu_r - \chi^2},
$$

(34)

where $\varepsilon_r$ and $\mu_r$ are the relative permittivity and permeability. Since

$$
\varepsilon_r \mu_r - \chi^2 = \frac{(G^2 + C^2)(F^2 + A^2)}{(FG - AC)^2} - \frac{(AG + CF)^2}{(FG - AC)^2} = 1,
$$

(35)

(34) simplifies to

$$
k = k_0.
$$

(36)

This means that for transformations of this class only the wave impedance of the transforming medium needs to be changed, but not the wavenumber. One can notice that this property indeed holds in all of the above special cases where the coefficients are position-independent.

4.6. Moving omega medium

Next, we consider another class of field-transforming media assuming $A(r, \omega) = C(r, \omega) = 0$. Transformations of this type correspond to moving omega materials with the constitutive relations of the form

$$
\mathbf{D} = \varepsilon_0 \frac{G}{F} \mathbf{E} - \frac{j}{\omega G} \nabla G \times \mathbf{H}, \quad \mathbf{B} = \mu_0 \frac{F}{G} \mathbf{H} + \frac{j}{\omega F} \nabla F \times \mathbf{E}.
$$

(37)

Both the nonreciprocity dyadic $\tilde{\chi}$ and the reciprocal magnetoelectric coupling dyadic $\tilde{\kappa}$ are antisymmetric

$$
\tilde{\chi} = -\frac{j}{2k_0} \left( \frac{\nabla F}{F} + \frac{\nabla G}{G} \right) \times \mathbf{I}, \quad \tilde{\kappa} = -\frac{1}{2k_0} \left( \frac{\nabla F}{F} - \frac{\nabla G}{G} \right) \times \mathbf{I},
$$

(38)

where the first quantity describes moving-media effects and the second one corresponds to omega-medium properties. If the two transformation coefficients are equal ($F = G$), the omega coupling vanishes.
The Maxwell equations for the transformed fields read
\[
\nabla \times E(\mathbf{r}) = -j\omega \mu_0 \frac{F(\mathbf{r})}{G(\mathbf{r})} H(\mathbf{r}) + \frac{1}{F(\mathbf{r})} \nabla F(\mathbf{r}) \times E(\mathbf{r}),
\]
\[
\nabla \times H(\mathbf{r}) = j\omega \epsilon_0 \frac{G(\mathbf{r})}{F(\mathbf{r})} E(\mathbf{r}) + \frac{1}{G(\mathbf{r})} \nabla G(\mathbf{r}) \times H(\mathbf{r}).
\]

(39)

4.7. Artificial moving medium

Let us consider a more special case when the transformation preserves the wave impedance of the medium \([F(\mathbf{r}) = G(\mathbf{r})]\) [8]. Material relations (37) take the form of material relations of slowly moving media (velocity \(v \ll c\), where \(c\) is the speed of light in vacuum), see, e.g. [19]
\[
D = \frac{1}{\sqrt{1 - v^2/c^2}} \left( \epsilon_0 E + \frac{1}{c^2} \mathbf{v} \times \mathbf{H} \right) \approx \epsilon_0 E + \frac{1}{c^2} \mathbf{v} \times \mathbf{H},
\]
\[
B = \frac{1}{\sqrt{1 - v^2/c^2}} \left( \mu_0 H - \frac{1}{c^2} \mathbf{v} \times E \right) \approx \mu_0 H - \frac{1}{c^2} \mathbf{v} \times E.
\]

(40)

In the last relation, we have neglected the second-order terms \(v^2/c^2\). The ‘velocity’ of the transforming medium can be identified by comparing with (37) as
\[
v = -j\frac{c^2}{\omega} \frac{\nabla F}{F}.
\]

(41)

To get more insight, let us analyze the one-dimensional case considering transformation functions depending only on one coordinate, \(z\) and assuming that \(F(z) = G(z)\) is an exponential function of \(z\). Two cases should be distinguished:

- \(\nabla F/F\) is imaginary, e.g. \(F(z) = e^{-\alpha z}\) with a real parameter \(\alpha\). In this case, the effective ‘velocity’ is real. This metamaterial simulates moving media. Note that the required material relations have the form of the relations for slowly moving media even if the equivalent velocity is not small.

- \(F(z)\) is a purely real function, e.g. \(F(z) = e^{-\alpha z}\), where \(\alpha\) is real. In this case, the effective ‘velocity’ of the medium is imaginary. Note that the material relations of a medium traveling faster than light formally have an imaginary vector coefficient in the second term of (40). However, one cannot say that this metamaterial simulates media moving faster than light, because in that case also the effective permittivity and permeability would be imaginary quantities.

Let us examine the source-free conditions \(\nabla \cdot D = 0\) and \(\nabla \cdot B = 0\) for this special case of \(F = G = e^{-\alpha z}\). Actually, \(\nabla F(\mathbf{r})/F(\mathbf{r}) = \nabla G(\mathbf{r})/G(\mathbf{r}) = -\alpha z_0\). Then it follows from (37):
\[
\nabla \cdot D = \nabla \cdot [F(\mathbf{r})\epsilon_0 E_0(\mathbf{r})] + \frac{j\omega}{\omega} \nabla \cdot [\alpha z_0 \times F(\mathbf{r}) H_0].
\]

(42)

Simple calculation using the free-space Maxwell equations relating \(E_0\) and \(H_0\) shows that \(\nabla \cdot D = 0\). Condition \(\nabla \cdot B = 0\) can be checked similarly.

Let us check the passivity condition for this medium. The period-average of the time derivative of the field energy is expressed as [7]:
\[
\left\langle \frac{dW}{dt} \right\rangle = \frac{j\omega}{4} f^* \cdot (M - M^\dagger) \cdot f,
\]

(43)
where
\[ f = \left( \begin{array}{c} E \\ H \end{array} \right), \quad M = \left( \begin{array}{cc} \varepsilon_0 \tilde{\varepsilon} & \varepsilon_0 \mu_0 \tilde{\chi} \\ \varepsilon_0 \mu_0 \tilde{\varepsilon} & \mu_0 \tilde{\mu} \end{array} \right). \] (44)

For real transformation coefficients, we take as an example \( F(z) = e^{-\alpha z} \), where \( \alpha > 0 \). For a plane wave propagating along \( z \), calculation leads to
\[ \left( \frac{dW}{dt} \right)_t = \frac{\alpha}{\eta} e^{-2\alpha z} > 0. \] (45)

This implies that the medium possesses losses. However, for a plane wave traveling in the opposite direction (along \(-z\)), the sign in the above expression is reversed, corresponding to a gain medium. This is expected because in the last case the field amplitude grows along the propagation direction. In the case of imaginary \( \alpha \) this value is equal to zero and the medium is lossless.

**5. Transmission through a slab of a field-transforming medium**

Let us consider an infinite slab of a field-transforming metamaterial in vacuum, which is excited by a normally incident plane wave. Here, we continue studying the case of moving-medium transformations, that is, \( F = G \) and \( A = C = 0 \). Let the thickness of the slab be \( d \), and the field-transforming properties be defined by a function of one variable \( z \)—the coordinate in the direction normal to the interfaces: \( F(z) = G(z) \). The general solution for the eigenwaves in the slab is, obviously,
\[ E(z) = F(z)(ae^{-jk_0z} + be^{jk_0z}), \] (46)
where \( E \) is the electric field amplitude, and \( a \) and \( b \) are constants, because this is how the free-space solution is transformed in this metamaterial slab. If \( F = G \), the wave impedance in the transforming medium does not change.

Writing the boundary conditions on two interfaces, reflection and transmission coefficients can be found in the usual way. As a result, we have found that the reflection coefficient does not depend on function \( F(z) \) at all, and it is given by the standard formula for a reflection coefficient from an isotropic slab at normal incidence. In particular, if the permittivity and permeability of the slab equal to that of free space, the reflection coefficient \( R = 0 \). The transmission coefficient in this matched case reads
\[ T = \frac{F(d)}{F(0)} e^{-jk_0d}, \] (47)
where \( F(d) \) and \( F(0) \) are the values of the transformation function at the two sides of the slab.

The nonreciprocal nature of field-transforming metamaterials is clearly visible here. If function \( F(z) \) is real and it decays with \( z \), then fields of waves traveling in the positive direction of \( z \) decay, whereas fields of waves traveling in the opposite directions grow. Obviously, such performance cannot be realized by passive media (see the calculation of the absorbed power in section 4.7). If \( F(d) = F(0) = 1 \), the slab is ‘invisible’, although inside the slab the field distribution can be pretty arbitrary, as dictated by the given function \( F(z) \).

**6. Conclusions**

Two different approaches to design of metamaterials which transform fields in a desired way are known. The first one has been proposed and developed in [1, 2], and it is based on

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coordinate transformations. The second one has been introduced in [8] and developed here, and it transforms the fields directly, according to a desired prescription. This technique can be potentially applied to a very wide variety of engineering problems, where a certain field transformation is required. One of the most important properties of the field transformation approach is that a medium with prescribed parameters turns out to be nonreciprocal with the exception of some special cases. Most often, a metamaterial simulating moving media is required. Potential physical realizations of metamaterials with the properties required for general linear field transformations (such as artificial moving media) are discussed in [6, 7, 13, 15, 20].

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References

[1] Pendry J B, Shurig D and Smith D R 2006 Science 312 1780
[2] Leonhardt U and Philbin T 2006 New J. Phys. 8 247
[3] Cai W, Chettiar U K, Kildishev A V and Shalaev V M 2007 Nat. Photonics 1 224
[4] Leonhardt U 2006 Science 312 1777
[5] Landau L D and Lifshits E M 1984 Electrodynamics of Continuous Media 2nd edn (Oxford: Pergamon)
[6] Tretyakov S A 1998 IEEE Microw. Guid. Wave Lett. 8 321
[7] Serdyukov A N, Semchenko I V, Tretyakov S A and Sihvola A 2001 Electromagnetics of Bi-Anisotropic Materials: Theory and Applications (Amsterdam: Gordon and Breach Science)
[8] Tretyakov S A and Nefedov I S 2007 Proc. Metamaterials 2007 p 474
[9] Peng J and Balanis C A 1997 IEEE Microw. Guid. Wave Lett. 7 347
[10] Maslovski S I and Tretyakov S A 1999 Microw. Opt. Technol. Lett. 23 59
[11] Tretyakov S A, Sihvola A H, Sochava A A and Simovski C R 1998 J. Electromagn. Waves Appl. 12 481
[12] Tretyakov S A and Sochava A A 1993 Electron. Lett. 29 1048
[13] Tretyakov S A, Maslovski S I, Nefedov I S, Viitanen A J, Belov P A and Sanmartin A Electromagnetics 23 665
[14] Lindell I V, Sihvola A H, Tretyakov S A and Viitanen A J 1994 Electromagnetic Waves in Chiral and Bi-Isotropic Media (London: Artech House)
[15] Ghosh A, Sheridon N K and Fischer P 2007 Janus particles with coupled electric and magnetic moments make a disordered magneto-electric medium arXiv:0708.1126v1
[16] Veselago V G 1968 Sov. Phys.—Usp. 10 509 (Engl. Transl.)
Veselago V G 1967 Usp. Fiz. Nauk 92 517–26 (In Russian)
[17] Tretyakov S, Nefedov I, Sihvola A, Maslovski S and Simovski C 2003 J. Electromagn. Waves Appl. 17 595
[18] Lakhkia A 1994 Beltrami Fields in Chiral Media (Singapore: World Scientific)
[19] Kong J A 2005 Electromagnetic Wave Theory (Cambridge, MA: EMW Publishing)
[20] Zagriadski S V and Tretyakov S A 2002 Electromagnetics 22 85

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