Common Permutation Problem

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Abstract

In this paper we show that the following problem is \textit{NP}-complete: Given an alphabet \(\Sigma\) and two strings over \(\Sigma\), the question is whether there exists a permutation of \(\Sigma\) which is a subsequence of both of the given strings.

1 Introduction

In computer science, efficient algorithms for various string problems are studied. One of such problems is a well-known \textit{Longest Common Subsequence} problem. For two given strings, the problem is to find the longest string which is a subsequence of both the strings. A survey of efficient algorithms for this problem can be found in [1].

Let us consider a modification of the \textit{Longest Common Subsequence} problem. Instead of finding any longest common subsequence, we restrict ourselves to subsequences in which symbols do not repeat, i.e., every symbol occurs at most once. We call this problem \textit{Longest Restricted Common Subsequence}.

Example. For strings “bcaba” and “babcca”, the longest common subsequence is “baba” while the longest restricted common subsequence is “bca”.

\textit{Longest Restricted Common Subsequence} is an optimization problem. In this paper we consider its special case which is the following decision problem: Suppose that the two strings are formed over an alphabet \(\Sigma\). The question is, do the two strings contain a restricted common subsequence of the maximal possible length, i.e., a string that contains every symbol of \(\Sigma\) exactly once? Such a string is a permutation of \(\Sigma\). Therefore, we call this problem the \textit{Common Permutation} problem.

\textbf{Common Permutation}

\textit{Instance}: An alphabet \(\Sigma\) and two strings \(a, b\) over \(\Sigma\).

\textit{Question}: Is there a permutation of \(\Sigma\) which is a common subsequence of \(a\) and \(b\)?

We will show that \textit{Common Permutation} is \(NP\)-complete. Moreover, we will show that \textit{Common Permutation} is \(NP\)-complete even if the input strings contain every symbol of \(\Sigma\) at most twice.

\textit{Common Permutation} can be reduced to \textit{Longest Restricted Common Subsequence} by asking whether the longest restricted common subsequence of the two strings is equal to the size of the alphabet. Since \textit{Common Permutation} will be shown to be \(NP\)-complete, it follows that \textit{Longest Restricted Common Subsequence} is \(NP\)-hard.

In the next section we define the terms used in this paper. Section 3 introduces \textit{alignments} as a way to visualize the \textit{Common Permutation} problem. Finally, Section 4 presents the proof of \(NP\)-completeness by reducing \textit{3SAT} to \textit{Common Permutation}.

\footnote{A more general version of this problem (with the same name) appeared in [3] together with its efficient solution. Unfortunately, that solution is incorrect. Our result in this paper indeed shows that an efficient (polynomial) solution for this problem does not exist unless \(P = NP\).}
2 Preliminaries

An alphabet is a finite set of symbols. A string over an alphabet $\Sigma$ is a finite sequence $a = a_1a_2 \ldots a_N$ where $N$ is a length of the string and $a_i \in \Sigma$ for all $i \in \{1, \ldots, N\}$. We say that $a_i$ is a symbol on a position $i$ in the string $a$. For a given symbol $x \in \Sigma$, occurrences of $x$ in $a$ are all positions $i$ such that $a_i = x$.

A subsequence of a string $a = a_1a_2 \ldots a_N$ over $\Sigma$ is a string $b = a_{i_1}a_{i_2} \ldots a_{i_n}$ where $n \in \{0, 1, \ldots, N\}$ and $1 \leq i_1 < i_2 < \cdots < i_n \leq N$. A common subsequence of two strings $a$ and $b$ is a string which is a subsequence of both $a$ and $b$.

A permutation of a finite set $A = \{x_1, \ldots, x_n\}$ is a string $x_{i_1}x_{i_2} \ldots x_{i_n}$ (note that the length of the string is the same as the number of elements in $A$) where $i_j \in \{1, \ldots, n\}$ for $j \in \{1, \ldots, n\}$ and for all $k, l \in \{1, \ldots, n\}$ if $k \neq l$ then $i_k \neq i_l$.

The above definitions give a formal basis for the statement of the problem from Section 1.

For the proof of $NP$-completeness in Section 4 we use the reduction from 3-Satisfiability ($3SAT$ for short). The following definitions are from [2].

2.1 3-Satisfiability

Let $U = \{u_1, u_2, \ldots, u_n\}$ be a set of Boolean variables. A truth assignment for $U$ is a function $t : U \rightarrow \{T, F\}$. If $t(u) = T$ we say that $u$ is true under $t$; if $t(u) = F$ we say that $u$ is false. If $u$ is a variable in $U$, then $u$ and $\overline{u}$ are literals over $U$. The literal $u$ is true under $t$ if and only if the variable $u$ is true under $t$; the literal $\overline{u}$ is true if and only if the variable $u$ is false.

A clause over $U$ is a set of literals over $U$, for example $\{u_1, \overline{u}_3, u_8\}$. It represents the disjunction of those literals and is satisfied by a truth assignment if and only if at least one of its members is true under that assignment. In other words, the clause is not satisfied if and only all its literals are false. The clause above will be satisfied by $t$ unless $t(u_1) = F$, $t(u_3) = T$, $t(u_8) = F$. A collection $C$ of clauses over $U$ is satisfiable if and only if there exists some truth assignment for $U$ that simultaneously satisfies all the clauses in $C$. Such a truth assignment is called a satisfying truth assignment for $C$.

3SAT

Instance: A set $U$ of variables and a collection $C$ of clauses over $U$ with exactly three literals per clause.

Question: Is there a satisfying truth assignment for $C$?

Theorem 2.1 3SAT is $NP$-complete.

See [2] for the definition of $NP$-completeness and for the proof of this theorem.

3 Alignments

In Section 4 we will use a notion of alignments. Imagine the two input strings of Common Permutation written in two rows, one string per row. For every symbol of the alphabet $\Sigma$ we want to find exactly one occurrence of that symbol in both strings, such that we can “align” those occurrences.

Example. For two strings “bcaba” and “babcca”, one of the possible alignments is depicted below (the aligned occurrences are bold)

```
|   | b | c | a | b | ab | c |
|---|---|---|---|---|----|---|
|   | a |   |   |   |     |   |
```
Formally, let \(a\) and \(b\) be strings over an alphabet \(\Sigma\). Let \(n\) be the number of symbols in \(\Sigma\). An alignment (denoted \(A\)) of \(a\) and \(b\) is a sequence of ordered pairs \(A = \langle i_1, j_1 \rangle, \langle i_2, j_2 \rangle, \ldots, \langle i_n, j_n \rangle\) such that for all \(k\), the value of \(i_k\) is a position in the string \(a\), \(j_k\) is a position in the string \(b\), and \(a_{i_k} = b_{j_k}\). Moreover, \(i_1 < \cdots < i_n\), \(j_1 < \cdots < j_n\), and \(a_{i_1}a_{i_2} \cdots a_{i_n} (= b_{j_1}b_{j_2} \cdots b_{j_n})\) is a permutation of \(\Sigma\).

For all \(k\) we say that, in the alignment \(A\), the position \(i_k\) in the string \(a\) is aligned with the position \(j_k\) in the string \(b\). We also say that the symbol \(a_{i_k} (= b_{j_k})\) is aligned at the position \(i_k\) in \(a\), and at the position \(j_k\) in \(b\). Positions \(i_k\) and \(j_k\) are aligned occurrences of \(a_{i_k}\).

Notice that once a position \(i\) (in \(a\)) is aligned with a position \(j\) (in \(b\)), positions less than \(i\) (in \(a\)) cannot be aligned with positions greater than \(j\) (in \(b\)) and vice versa. In other words, the aligned occurrences of different symbols cannot “cross”.

**Lemma 3.1** Let \(a\) and \(b\) be two strings over \(\Sigma\). A permutation of \(\Sigma\) which is a common subsequence of \(a\) and \(b\) exists if and only if there exists one or more alignments of \(a\) and \(b\).

**Proof.** An alignment corresponds to subsequences in \(a\) and \(b\) which comprise a (common) permutation of \(\Sigma\). \(\Box\)

According to this lemma, an alignment of two strings is an existence proof (of a polynomial size with respect to lengths of the strings) for an instance of **Common Permutation**. Therefore, **Common Permutation** is in \(NP\). The proof of \(NP\)-completeness follows in the next section.

### 4 Reduction

In this section we will reduce 3SAT to **Common Permutation**.

**Theorem 4.1** Common Permutation is \(NP\)-complete.

**Proof.** Let \(U\) be a finite set of variables and \(C = \{c_1, c_2, \ldots, c_n\}\) be a set of clauses over \(U\). We have to construct an alphabet \(\Sigma\) and two strings \(a, b\) over \(\Sigma\) such that there exists a permutation of \(\Sigma\) which is a common subsequence of both \(a\) and \(b\) if and only if \(C\) is satisfiable.

The proof consists of two parts. The first part presents the construction of \(\Sigma\) and the strings. The second part proves that the construction is correct in a sense that it satisfies the property described above.

**Construction** The alphabet \(\Sigma\) consists of a pair of symbols \(u^i\) and \(\overline{u}\) for every variable \(u \in U\) and every clause \(c_i\) for which either \(u \in c_i\) or \(\overline{u} \in c_i\). Additionally, \(\Sigma\) contains a special “boundary” symbol \(\bullet\).

The strings \(a\) and \(b\) have two parts: “truth-setting” part and “satisfaction testing” part. The parts are separated by the boundary symbol which ensures that occurrences from one part cannot be aligned with occurrences from the other part of the strings.

The “truth-setting” part consists of a concatenation of blocks, one for each variable. Let \(u\) be a variable from \(U\) and let \(\{i_1, \ldots, i_m\}\) be the indexes of clauses in which it appears. The strings contain the following block for variable \(u\):

\[
a = \ldots u^{i_1}u^{i_2} \ldots u^{i_m} \quad \overline{u}^{i_1}\overline{u}^{i_2} \ldots \overline{u}^{i_m} \ldots
\]

\[
b = \ldots \overline{u}^{i_1}\overline{u}^{i_2} \ldots \overline{u}^{i_m} \quad u^{i_1}u^{i_2} \ldots u^{i_m} \ldots
\]

This block is constructed in such a way that it is possible to simultaneously align all the symbols \(\{u^{i_1}, u^{i_2}, \ldots, u^{i_m}\}\) inside this block, or all the symbols \(\{\overline{u}^{i_1}, \overline{u}^{i_2}, \ldots, \overline{u}^{i_m}\}\). It is, however, not possible to simultaneously align both \(u^{i}\) and \(\overline{u}^{j}\) for some \(i\) and \(j\) inside this block.

The “satisfaction-testing” part consists of a concatenation of blocks, one for each clause. For a clause \(c_i \in C\), let \(x, y, \) and \(z\) be the literals in the clause \(c_i\), i.e., \(c_i = \{x, y, z\}\). We use the following notation:
• if $x = u$ for $u \in U$, then $x^i = u^i$ and $\overline{x}^i = \overline{u}^i$

• if $x = \overline{u}$ for $u \in U$, then $x^i = \overline{u}^i$ and $\overline{x}^i = u^i$

The strings contain the following block for the clause $c_i$:

\[
\begin{align*}
a & = \ldots x^i y^i z^i \overline{x}^i \overline{y}^i \overline{z}^i \ldots \\
b & = \ldots x^i y^i z^i \overline{y}^i \overline{x}^i \overline{y}^i \ldots 
\end{align*}
\]

The block has two parts. The left part is the same for both strings. The right part is constructed in such a way that the symbols \{+$, y, z$\} cannot be simultaneously aligned in this block. Notice that these are the symbols corresponding to the truth assignment for which the clause is false.

The alphabet $\Sigma$ contains $6n + 1$ symbols. The length of the string $a$ is $6n + 1 + 6n = 12n + 1$; the length of the string $b$ is $6n + 1 + 7n = 13n + 1$. Therefore, the size of the constructed Common Permutation instance is polynomial with respect to the original 3SAT instance. The construction can be carried out in polynomial time.

**Example.** For a set of variables $\{w, x, y, z\}$ and clauses $\{\{w, \overline{x}, y\}, \{\overline{x}, x, \overline{y}\}\}$ which represent the logical function

\[(w \lor \overline{x} \lor y) \land (\overline{x} \lor x \lor \overline{y})\]

we get the alphabet

\[\Sigma = \{w^1, \overline{w}^1, x^1, \overline{x}^1, x^2, \overline{x}^2, y^1, \overline{y}^1, y^2, \overline{y}^2, z^2, \overline{z}^2\}\]

and the following strings:

\[
\begin{align*}
a & = w^1 \overline{w}^1 \ldots x^1 y^1 \ldots y^2 \overline{y}^2 \ldots z^2 \overline{z}^2 \\
b & = \overline{y}^1 w^1 \ldots x^2 \overline{x}^2 \ldots \overline{y}^2 y^2 \ldots \overline{z}^2 z^2 \\
\text{for } w & \text{ for } x \text{ for } y \text{ for } z
\end{align*}
\]

for clause 1 for clause 2

**Correctness.** Now we verify that the constructed strings $a$ and $b$ contain a common permutation of $\Sigma$ if and only if $C$ is satisfiable.

Let $t : U \rightarrow \{T, F\}$ be any satisfying truth assignment for $C$. We will show that there exists a permutation of $\Sigma$ which a common subsequence of both $a$ and $b$, i.e., that it is possible to align all symbols from $\Sigma$ in the strings.

There is only one choice how to align the boundary symbol. For a variable $u \in U$, if $t(u) = T$, we align $\overline{u}^i$ symbols for all $i$ in the “truth-assigning” part and $u^i$ in the “satisfaction-testing” part. If $t(u) = F$ we conversely align $u^i$ symbols in the “truth-assigning” part and $\overline{u}^i$ in the “satisfaction-testing” part.

As we noted during construction, the desired alignment in the “truth-assigning” part is always possible to find. To show that we can align the remaining symbols in the “satisfaction-testing” part, we use that $t$ is satisfying $C$.

Notice that the symbols which we have to align in the “satisfaction-testing” part correspond to the truth values of the variables. For example, if we have to align symbol $\overline{u}^i$ in this part, we know that $t(u) = F$. For every clause $c_i \in C$, $c_i = \{x, y, z\}$, we align the remaining symbols corresponding to clause $c_i$ in the block for $c_i$. The symbols $x^i$, $y^i$, and $z^i$ can be aligned in the first part of the block. It is easy to see that any pair of symbols from \{+$, y, z$\} can be aligned in the second part. Therefore, for all seven possibilities how $c_i$ can be satisfied, we can align the corresponding symbols in the block for the clause $c_i$.

For the proof in the opposite direction, suppose now that $a$ and $b$ have a common permutation of the symbols in $\Sigma$. We will construct a satisfying truth assignment for $C$. For that we look at “truth-setting” part of the strings. For a variable $u \in U$,

• if the symbol $\overline{u}^i$ for some $i$ is aligned in the “truth-setting” part of the strings, we set $t(u) = T$,
• if the symbol $u^i$ for some $i$ is aligned in the “truth-setting” part of the strings, we set $t(u) = F$,

• if none of the symbols $\{u^i, \overline{u}^i\}$ are aligned in the “truth-setting” part, we set $t(u)$ arbitrarily, say $t(u) = T$.

Notice that, according to the construction of the “truth-setting” part, this is a valid definition of the assignment, i.e., it cannot happen that we would want to assign $t(u)$ to both $T$ and $F$.

We now have to prove that $t$ is a satisfying truth assignment for $C$. For any clause $c_i \in C$, let $x, y, z$ be its literals, so $c_i = \{x, y, z\}$. We know that not all the symbols $\{\overline{x}^i, \overline{y}^i, \overline{z}^i\}$ can be aligned in “satisfaction-testing” part of the strings, so at least one them must be aligned in the “truth-setting” part. Without loss of generality say it is $\overline{x}^i$. Therefore, by the definition of $t$, we know that the literal $x$ is true, and therefore $c_i$ is true. □

Our construction in the proof used every symbol at most twice in the string $a$, but used some of the symbols three times in the string $b$. The following corollary shows that we can use a slightly different construction which uses every symbol at most twice in both the strings.

**Corollary 4.1** Common Permutation is NP-complete even if every symbol occurs at most twice in the given strings.

**Proof.** We will use the same construction as in the proof of Theorem 4.1 except for the definition of $\Sigma$, and blocks for clauses. For every clause $c_i \in C$ we will add three additional symbols $\triangle_i, \lozenge_i$, and $\square_i$ to the alphabet $\Sigma$. The strings contain the following block for the clause $c_i$:

$$a = \ldots \overline{x}^i \overline{y}^i \triangle_i \lozenge_i \square_i z^i \ldots$$

$$b = \ldots x^i \triangle_i \overline{y}^i \lozenge_i \square_i \triangle_i \lozenge_i \square_i z^i \ldots$$

One can verify that inside this block we can align symbols corresponding to the satisfying assignment for $c_i$, but we cannot align simultaneously $\overline{x}^i, \overline{y}^i, \overline{z}^i$.

The constructed strings contain every symbol exactly twice with the exception of • which they contain once. □

5 Acknowledgements

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