Unitarity of the Higher Dimensional Standard Model

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We study the unitarity of the standard model (SM) in higher dimensions. We show that the essential features of SM unitarity remain after compactification, and place bounds on the highest Kaluza-Klein (KK) level $N_{KK}$ and the Higgs mass $m_H$ in the effective four-dimensional (4d) low-energy theory. We demonstrate these general observations by explicitly analyzing the effective 4d KK theory of a compactified 5d SM on $S^1/\mathbb{Z}_2$. The nontrivial energy cancellations in the scattering of longitudinal KK gluons or KK weak bosons, a consequence of the geometric Higgs mechanism, are verified. In the case of the electroweak gauge bosons, the longitudinal KK states also include a small mixture from the KK Higgs excitations. With the analyses before and after compactification, we derive the strongest bounds on $N_{KK}$ from gauge KK scattering. Applying these bounds to higher-dimensional SUSY GUTs implies that only a small number of KK states can be used to accelerate gauge coupling unification. As a consequence, we show that the GUT scale in the 5d minimal SUSY GUT cannot be lower than about $10^{14}$ GeV.

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1. Introduction

The conventional Higgs mechanism[1] provides the simplest way of perturbatively generating gauge boson masses while ensuring the unitarity of massive gauge boson scattering at high energies[2-6]. Kaluza-Klein (KK) compactification[7] of extra spatial dimensions, on the contrary, can geometrically realize vector boson mass generation without invoking a scalar Higgs particle. In this geometric mechanism the longitudinal components of the massive vector bosons in the effective four-dimensional (4d) low-energy theory arise from the extra components of the higher-dimensional gauge field. The scattering of massive KK gauge bosons has recently been demonstrated to respect low-energy unitarity in generic 5d Yang-Mills theories[8] up to energies inversely proportional to the square of the 5d coupling constant. In the present Letter, extending this study[8], we analyze the unitarity of the standard model (SM) in $D=(4+\delta)$ dimensions, present a number of general observations, and derive their physical consequences.

2. Unitarity in $D$-Dimensions and Compactification

We begin with the study of unitarity in the uncompacted higher-dimensional SM, where the computation of the high energy scattering amplitudes is simpler than in the compactified theory. Yang-Mills theories in $4+\delta$ dimensions are not renormalizable. The gauge coupling constant $g$ has mass dimension $-\delta/2$, and therefore we expect such a theory can only be an effective theory valid up to an ultraviolet (UV) cutoff $\Lambda$ of order $g^{-2/\delta}$. This effective description can only hold so long as the relevant high energy scattering amplitudes remain unitary, and we may estimate $\Lambda$ by determining the scale at which the tree-level scattering amplitudes violate unitarity.

We start by considering the QCD sector of the $D=(4+\delta)$ dimensional SM, in which the gauge bosons and the Higgs doublet propagate in the extra $\delta$ dimensions. The analysis of $D$-dimensional QCD (D-QCD) is a direct application of the study of 5d Yang-Mills theory[8]. The Yang-Mills symmetry is $SU(k)$ with $k=3$, and the gauge Lagrangian is,

$$\mathcal{L}_{QCD}^D = -\frac{1}{2} \text{Tr}(\hat{G}_{MN}\hat{G}^{MN}),$$

where, as noted above, the $D$-dimensional gauge coupling $g_D$ has a negative mass dimension $-\delta/2$. The gauge-fixing and Faddeev-Popov ghost terms $\mathcal{L}_{gf} + \mathcal{L}_{FP}$ can be constructed accordingly.

Consider gluon scattering, $\hat{G}_{j_1}^a \hat{G}_{j_2}^b \to \hat{G}_{j_3}^c \hat{G}_{j_4}^d$, where the index $j \in (1,2,\ldots,D-2)$ denotes the polarization states of gluon field $\hat{G}^a_M$. We expect that scattering amplitude will behave at high-energies as a constant of $O(\hat{g}_D^2)$. As in [8], we expect a large elastic scattering amplitude in the spin-0 and gauge-singlet two-particle state,

$$|\Psi_0\rangle = \frac{1}{\sqrt{(D-2)(k^2-1)}} \sum_{j=1}^{D-2} \sum_{a=1}^{k^2-1} |G_{j_3}^a G_{j_4}^a\rangle .$$

After a lengthy calculation[12], we derive the scattering amplitude for this spin-0 and gauge-singlet channel,

$$\tilde{F}_0(|\Psi_0\rangle \to |\Psi_0\rangle) = \frac{2k\hat{g}_D^2}{D-2} \left[ \frac{4(D-2)}{\sin^2 \theta} - (D-4)^2 \right].$$

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where $\theta$ is the scattering angle. The $D$-dimensional s-partial wave amplitude [9] is thus deduced as

$$\tilde{a}_{00} = \frac{(\sqrt{s})^{\delta}}{2(16\pi)^{1+\delta}} \text{Im} \left[ \frac{1}{\Gamma \left( 1 + \frac{\delta}{2} \right)} \int_{0}^{\pi} d\theta \left( \sin \theta \right)^{1+\delta} \tilde{f}_{0} \right]$$

$$= \frac{2k\sqrt{s}}{(16\pi)^{1+\delta/2}(2+\delta/1+\delta)} \left( \hat{g}_{s}^{2} s^{\frac{\delta}{2}} \right).$$

(4)

With the unitarity condition $|\text{Re} \tilde{a}_{00}| < 2!/2$ (where the factor $2!$ is due to the identical particles in the final state), we arrive at

$$\sqrt{s} < \left[ \left( \frac{16\pi}{2k\sqrt{s}} \right)^{1+\frac{\delta}{2}(2+\delta)} \Gamma \left( 1 + \frac{\delta}{2} \right) \frac{1}{\Lambda^\delta} \right]^{1/\delta},$$

(5)

which shows that tree-level unitarity violation in such a $D$-dimensional theory indeed occurs at an energy of order its intrinsic ultraviolet (UV) scale $O(\hat{g}_{s}^{-2/\delta})$.

For the gluon scattering being described by this effective higher-dimensional gauge theory, the scattering energy $\sqrt{s}$ cannot exceed the cutoff $\Lambda$ of the effective theory, $\sqrt{s} < \Lambda$. Therefore, for the highest possible scattering energy, the scale $\Lambda$ is also bounded by the right hand side of Eq. (5), which results in a constraint on $\hat{g}_{s}^{2}$,

$$\hat{g}_{s}^{2} < \left( \frac{(16\pi)^{1+\frac{\delta}{2}(2+\delta)} \Gamma \left( 1 + \frac{\delta}{2} \right) \frac{1}{\Lambda^\delta}}{2k\sqrt{s}} \right)^{\delta}$$

(6)

The uncompactified higher-dimensional SM by itself is, of course, of no direct phenomenological relevance. Rather, we would like to consider the situation when the $n$ extra spatial dimensions are compact. In this case, each extra dimension is manifested as Kaluza-Klein towers of the given fields, with mass spectra characterized by $|n|/R$, where $1/R$ is the compactification scale and $n$ the KK-level. The details of the spectra and interactions of these modes depend on the extra-dimensional boundary conditions imposed at the scale $1/R$. In the next section, we will explicitly investigate the unitarity of the 5d SM compactified on $S^{1}/\mathbb{Z}_{2}$. However, we expect the essential properties of unitarity – which arise from the high-energy/short-distance behavior ($E \gg 1/R$) of the higher-dimensional gauge theory – should be insensitive to the details of the compactification. In particular, we expect that Eq. (6) should hold at least approximately after the extra dimensions are compactified.

In the following, we investigate the consequences of the bound Eq. (6) as applied to a toroidal compactification of the $\delta$ extra dimensions on $T^{d} = (S^{1}/\mathbb{Z}_{2})^{d}$, with a common radius $R$. (For $\delta \geq 2$, the current analysis may be readily extended to the case of asymmetric compactification [10] or with nontrivial shape moduli [11], with which some of the constraints could be relaxed [12].)

The $D$-dimensional gauge coupling $g_{s}^{2}$ is connected to the dimensionless coupling $\hat{g}_{s}^{2}$ in the compactified effective 4d KK theory via, $g_{s}^{2} = V_{\delta} \hat{g}_{s}^{2}$ with the $D$-dimensional volume $V_{\delta} = (\pi R)^{\delta}$. As described previously, we expect the coupling $\hat{g}_{s}^{2}$ to be bounded for a given value of the cutoff $\Lambda$. The effective 4d KK theory can only contain modes of a mass below the cutoff $\Lambda$. Therefore, the highest KK level in this compactification is fixed by $N_{KK} = \Lambda R$. We can thus convert Eq. (6) into a constraint on $N_{KK}$,

$$N_{KK} < \left[ \frac{\left( \frac{4^{2+\delta}}{2k} \right) \delta \left( 2 + \delta \right) \Gamma \left( \frac{1+\delta}{2} \right) \hat{g}_{s}^{2}}{1+\delta} \right]^{1/\delta},$$

(7)

which, as written, applies to any compactified $SU(k)$ Yang-Mills theory.

Eq. (7) shows that for a given value of the four-dimensional gauge coupling, the highest KK level in the effective 4d theory is bounded from above. The emergence of this bound reflects the fact that, as shown in Ref. [8], the bad high-energy behavior of the underlying higher-dimensional gauge theory manifests itself through the appearance of the myriad KK excitations which couple together to enhance the scattering amplitude and thus speed up unitarity saturation.

To illustrate these features, we first apply the bound (7) to the QCD sector. Taking a sample value of $\alpha_{s} \simeq 0.1$ at $O$(TeV) scale, we derive the following bounds from Eq. (7),

$$N_{KK} < (2, 3, 3, 4, 6, 4),$$

(8)

for $\delta = D - 4 = (1, 2, 3, 4, 5, 6, 7)$, respectively. It is also straightforward to apply the condition (7) to the scattering of weak gauge bosons in the $SU(2)_{W} \otimes U(1)_{Y}$ electroweak (EW) gauge sector of the $D$-dimensional SM. Ignoring the small $W_{\pm}B$ mixing and setting $SU(k) = SU(2)_{W}$, we deduce the following bounds on $N_{KK}$, for $\delta = (1, 2, 3, 4, 5, 6, 7)$,

$$N_{KK} < (9, 6, 5, 6, 8, 6),$$

(9)

where we have replaced $g_{s}^{-2}$ by $g^{-2} = (v/2m_{W})^{2}$ in (7), with $v$ the Higgs vacuum expectation value (VEV) and $m_{W}$ the mass of weak gauge bosons. For $\delta \leq 3$, this is significantly weaker than (8) from the QCD sector. The non-monotonic behavior of $N_{KK}$ bounds (8) and (9) as a function of $\delta$ is because in the right-hand side of Eq. (7) the denominator has a maximum at $\delta \simeq 2.5$ and flips sign at $\delta \simeq 6.2$.

Next, we analyze the scalar Higgs sector of the $D$-dimensional SM. The Higgs Lagrangian is

$$\mathcal{L}_{h}^{D} = D_{M} \hat{\Phi}^{1} D^{M} \hat{\Phi} - \left[ -\mu^{2} \hat{\Phi}^{1} \hat{\Phi} + \lambda (\hat{\Phi}^{1} \hat{\Phi})^{2} \right],$$

(10)

where $D_{M} \hat{\Phi} = (\partial_{M} - i\hat{g}^{-1} \hat{W}_{M} + i\hat{g}^{-1} \hat{B}_{M}) \hat{\Phi}$, and the $D$-dimensional Higgs (gauge) coupling $\lambda (\hat{g}, \hat{g})$ has a mass
dimension of $-\delta$ ($-\delta/2$). Due to $-\mu^2 < 0$, the Higgs doublet $\Phi$ develops a VEV so that $\Phi = (-i\pi^+, (\bar{v} + \bar{h}_0 + i\pi^0)/\sqrt{2})^T$, and the neutral boson $h_0$ acquires its mass at tree-level
\[ m_H = \sqrt{2\lambda v^2}. \] (11)

Consider the $D$-dimensional scalar-scalar scattering, $|\phi^2 \phi^2| \rightarrow |\phi^2 \phi^2|$, via the neutral channels $|\phi^2 \phi^2|, |\phi^2 \phi^2| = i\frac{1}{(2\pi)^2} \frac{1}{2} (\phi^2 \phi^2), \frac{1}{2} (\phi^2 \phi^2), \frac{1}{2} (\phi^2 \phi^2)$, in analogy with the customary analysis in the usual 4d SM Higgs sector [5]. Their scattering amplitudes form a $4 \times 4$ matrix and, for $s > m_H^2$, approach a constant matrix which arises from the four-scalar contact interaction. (The channel $h_0 \pi^0$ actually decouples from the other three.) The eigenvalues of this constant matrix are readily worked out as $-\hat{\lambda} \cdot (6, 2, 2, 2)$ so that the maximal eigenvalue amplitude is $T_{\text{max}} = -6\hat{\lambda}$. Thus, using the first formula in Eq. (4), we derive the $D$-dimensional s-partial wave amplitude,
\[ \hat{a}_0 = -C_0 \left( \frac{3\sqrt{3}}{(16\pi)^{1/2}} \Gamma \left( \frac{3}{2} \right) \right). \] (12)

The relevant unitarity condition is $|\text{Re} \hat{a}_0| < \frac{1}{2}$, where the possible identical particle factors are included [5] in the normalization of in/out state $|\phi^2 \phi^2|$ or $|\phi^2 \phi^2|$ mentioned above. So, we deduce,
\[ \sqrt{\hat{\delta}} < \left( \frac{2C_0 \hat{\lambda}}{\Gamma(\frac{1}{2} + \frac{\delta}{2})} \right)^{\frac{1}{2}}, \] (13)
which again shows that the unitarity violation in such a $D$-dimensional theory indeed occurs at the intrinsic UV scale, of $O(\lambda^{-1/3})$ in this case. Similar to Eq. (6), we may interpret Eq. (13) as an upper bound on the cutoff $\Lambda$, and translate this into a condition on $\hat{\lambda}$,
\[ \hat{\lambda} < \frac{1}{2C_0 \sqrt{\hat{\delta}}} = \frac{1}{2C_0 \Lambda^{\frac{1}{2}}}. \] (14)

In parallel with our analysis of the gauge sector, we interpret (14) in the context of compactification on $T^\delta = (S^1/\mathbb{Z}_2)^D$ and examine the consequences for the effective 4d KK-theory. The $D$-dimensional coupling $\hat{\lambda}$ is related to the dimensionless coupling $\hat{\lambda}$ in the 4d KK-theory via,
\[ \hat{\lambda} = V_\delta \hat{\lambda} = (\pi R)^\delta \lambda, \] and Eq. (14) can be rewritten as
\[ \lambda < \frac{1}{2C_0 V_\delta \Lambda^{\delta}} = \frac{1}{2C_0 \pi^\delta (N_{\text{KK}})^\delta}, \] (15)
where $N_{\text{KK}} \approx \Lambda R$ represents the highest KK levels associated with each compactified extra dimension in the low-energy effective theory. Here, for simplicity, we have ignored the small $O(m_\pi^2 R^2)$ correction to the KK mass spectrum in the EW sector.

The 4d KK theory contains a zero-mode Higgs doublet $\Phi_0 = (-i\pi^+, (v + h_0 + i\pi^0)/\sqrt{2})^T$, and its KK excitations $\Phi_n = (-i\pi^+, (v + H_n + i\pi^0)/\sqrt{2})^T$, where the VEV of the 4d Higgs doublet $\Phi_0$, $v = (\sqrt{2}G_F)^{-1/2} \approx 246$ GeV, is related to that of $\Phi$ via $\bar{v} = v/\sqrt{\delta}$. The corresponding mass of the zero-mode neutral Higgs boson $h_0^0$ is then given by
\[ m_H = \sqrt{2\lambda v^2} = \sqrt{2\lambda \bar{v}^2} = m_H, \] (16)
which is unchanged under the compactification. Hence, we can deduce, from (15), the unitarity bound on the physical mass of the 4d SM Higgs boson $h_0^0$,
\[ m_H < \frac{v}{\sqrt{C_0 V_\delta \Lambda^\delta}} \approx \frac{v}{\sqrt{C_0 \pi^\delta (N_{\text{KK}})^\delta}}, \] (17)
where for $\delta = (1, 2, 3, 4, 5, 6, 7)_ijkl$,
\[ \frac{1}{\sqrt{C_0 \pi^\delta}} \approx (4.6, 8.0, 14.7, 28.5, 57.6, 121, 260). \]
Alternatively, for a given $m_H$,
\[ N_{\text{KK}} < \left( \frac{v^2}{C_0 \pi^\delta m_H^2} \right)^{\frac{1}{2}}. \] (18)

A few comments are in order. We note that Eq. (18) shares similar features to the gauge KK bound (7), except that in the Higgs sector the coupling $\lambda$ (or mass $m_H$) is not fixed by observation. If we impose the existing direct Higgs search limit at LEP-2, $m_H > 114.3$ GeV (95% C.L.), we can deduce, from (18),
\[ N_{\text{KK}} \lesssim (98, 17, 10, 7, 6, 6, 6), \] (19)
for $\delta = (1, 2, 3, 4, 5, 6, 7)$. The limits (19) are significantly weaker than the bounds (8)-(9) from the gauge KK sector, especially for $\delta \lesssim 3$. The bound (17) on the Higgs boson mass is listed in Table I for various values of $\delta$ and $N_{\text{KK}}$. The entries marked by $\times$ are excluded by direct Higgs searches at LEP-2, cf. Eq. (19). We see that $N_{\text{KK}}$ is more severely bounded by the unitarity of the gauge boson scattering, especially for gluons as long as they propagate in the bulk [cf. (8)]. Imposing the stronger bound (8), we can examine how Eq. (17) would constrain $m_H$. The corresponding upper limits are displayed in the entries of Table I marked by $\ast$.

| $N_{\text{KK}}$ | 3 | 4 | 5 | 6 | 8 | 10 | 12 | 15 | 20 |
|---------------|---|---|---|---|---|----|----|----|----|
| $\delta = 1$  | 656 | 568 | 508 | 464 | 402 | 359 | 328 | 293 | 254 |
| $\delta = 2$  | 656* | 492 | 394 | 328 | 246 | 197 | 164 | 131 | 98  |
| $\delta = 3$  | 698* | 453 | 324 | 247 | 160 | 115 | 87  | 62  | 41  |
| $\delta = 4$  | 780* | 438* | 281 | 195 | 110 | 70  | 49  | 31  | 18  |
| $\delta = 5$  | 909* | 443* | 254 | 161 | 78  | 45  | 28  | 16  | 8   |
| $\delta = 6$  | 1098* | 463* | 237* | 137* | 58  | 30  | 17  | 9   | 4   |
| $\delta = 7$  | 1368* | 500* | 229 | 121 | 44  | 20  | 11  | 5   | 2   |
Finally, we stress that the strong bounds we have derived for either the coupling constants \(g^2, \lambda\) or the Higgs mass \(m_H\) are due to the large extra dimensional volume \(V_5 \sim R^5\) which appears in relating the \(D\)-dimensional and four-dimensional physics. Equivalently, following \cite{8}, the strong bounds arise because of the appearance of many KK excitations in the compactified theory which couple together to enhance the scattering amplitude and speed up unitarity saturation. Therefore, it is clear that if any field is restricted on a brane, the corresponding unitarity bound on its couplings would reduce back to that of the customary 4d SM.

In the scenario with universal extra dimensions \cite{14}, all SM fields propagate in the extra dimensions and are thus subject to the unitarity limits discussed above. In particular, these results suggest that the “self-broken” standard model \cite{15} would not be realized consistently. For a given \(N_{KK}\) and under the compactification \((S^1/Z_2)^d\), the total number of states consistent with the unitarity constraint is approximately given by

\[
n(N_{KK}) \approx \frac{\pi^{d/2}}{\Gamma \left(1 + \frac{d}{2}\right)} \left(\frac{N_{KK}}{2}\right)^{\frac{d}{2}} . \tag{20}\]

For \(D = 6\), our constraints in \((8)\) yield a total number of KK modes \(n \leq 7\), and for \(D = 8\) we find \(n \leq 79\). Neither appears likely to yield a sufficiently light top quark mass at the observed value, for a wide range of compactification scales \cite{15}.

For theories with extra dimensional perturbative gauge unification \cite{13}, all gauge fields live in the bulk and again these bounds apply. We will return to implications for GUTs in Sec. 4. Finally, we note that our analysis may be extended to bounds on gauge KK scattering and graviton KK scattering \cite{12} in a warped 5d SM à la Randall-Sundrum (RS1) \cite{16}.

3. KK Theory: E-Cancellations and Unitarity Limits

We now turn to an explicit analysis of the effective 4d KK theory arising from compactifying a 5d SM on \(S^1/Z_2\). The Lagrangian of this 4d KK theory will contain interactions involving purely physical fields \cite{17, 18} and interactions involving additional geometric and ordinary would-be Goldstone fields \cite{8, 19}. In Ref. \cite{8}, the scattering of the longitudinal components of the KK excitations of Yang-Mills fields was systematically computed. Here we will further compute all amplitudes involving transversely polarized gauge KK fields, and in the case of the EW gauge bosons, include the effect of EW symmetry breaking of the SM.

We begin by considering gluon KK scattering \(G_{a,n}^j G_{b,n}^{j'} \to G_{a,n}^{j'} G_{b,n}^{j}\), where \(j \in (+, 0, -)\) denotes the three helicity states and \((n, \ell)\) the KK levels. We may define a spin-0, gauge-singlet state, \(\lvert \Psi_0^a \rangle = \frac{1}{\sqrt{3(k^2 - 1)}} \sum_{j=-1}^{+1} \sum_{a=1}^{k^2-1} |G_{a,n}^j G_{a,n}^{j'}\rangle\),

where \(k = 3\) for QCD SU(3)_c. The corresponding \(\lvert \Psi_0^a \rangle \to \lvert \Psi_0^b \rangle\) scattering channel consists of 9 helicity amplitudes, but only 4 of them are independent under the discrete \((P, C, T)\) symmetries. We arrive at

\[
\mathcal{T}_0 \{ |\Psi_0^a \rangle \to |\Psi_0^b \rangle \} = \frac{1}{3(k^2 - 1)} \sum_{j,j'=-1}^{+1} \sum_{a,c=1}^{k^2-1} \{ \mathcal{T}_{00,00}^{aa,cc} + 2 \mathcal{T}_{00,00}^{aa,cc} \}
\]

A systematic calculation shows that the amplitudes \(\mathcal{T}_{00,00}^{aa,cc}\) and \(\mathcal{T}_{00,00}^{aa,cc}\) vanish to \(O(g^2 E^0)\), while the amplitudes \(\mathcal{T}_{00,00}^{aa,cc}\) and \(\mathcal{T}_{00,00}^{aa,cc}\) both have nonzero \(O(g^2 E^0)\) contributions. We have verified the nontrivial energy cancellations at \(O(\varepsilon)\) \([\mathcal{O}(E^2)]\) for the amplitudes \(\mathcal{T}_{00,00}^{ab,cd}\) \((\mathcal{T}_{00,00}^{ab,cd})\) involving four \([two]\) external longitudinal KK gluon states, and the consistency with the Kaluza-Klein Equivalence Theorem (KK-ET) \cite{8}. We then compute the s-wave partial amplitude for \((22)\) with \(n \neq \ell\),

\[
a_{00} = \frac{k g_s^2}{24\pi} \left[ -1 + 6 \ln \frac{N_s^2}{|n^2 - \ell^2|} \right] , \tag{23}\]

where \(N_s \equiv \sqrt{s} R \leq N_{KK}\). This explicitly shows that due to the exact E-cancellations, the partial wave amplitude indeed behaves as constant at the leading order.

To maximize the scattering amplitude, we define a normalized state consisting of KK-levels up to \(N_0\),

\[
|\Omega\rangle = \frac{1}{\sqrt{N_0}} \sum_{n=1}^{N_0} |\Psi_0^n\rangle , \tag{24}\]

where the kinematics of \(2 \to 2\) scattering requires \(N_0 < N_s/2\). For \((N_0)_{\text{max}} = (N_s)_{\text{max}}/2 = N_{KK}/2\), we deduce the maximal s-wave amplitude for \(|\Omega\rangle \to |\Omega\rangle\), to leading order in \(N_{KK}\),

\[
a_{00}|\Omega\rangle = \frac{k g_s^2}{8\pi} \left[ -\frac{N_{KK}}{6} + \frac{4}{N_{KK}} \sum_{n \neq 1}^{N_{KK}} \ln \frac{N_{KK}^2}{|n^2 - \ell^2|} \right] . \tag{25}\]

From the unitarity condition \(|a_{00}| < \sqrt{2} / 2\) (where the 2! arises from identical particles in the final state \([12]\)), we derive the following numerical bound for the QCD sector,

\[
N_{KK} \leq 4 . \tag{26}\]

We have performed a similar analysis for KK scattering in the EW gauge sector, where a small mixing \cite{19} arises between the geometric KK Goldstone bosons \(W_{n}^{+}\) and the KK excitation modes \(\pi_n^a\) of ordinary Goldstone bosons in the Higgs doublet. This mixing is described by a mixing angle \(\sin \theta_n = m_{\ell}/M_{\ell} \ll 1\), and \(m_{\ell} = m_{w,z}\) is the mass of the zero-mode gauge bosons \((W_{0}^{\pm}, Z_{0}^{0})\). As a result, the “eaten” KK Goldstone field is \(\tilde{\pi}_n^a = \cos \theta_n V_n^{a,5} + \sin \theta_n \pi_n^a\) \((V_n^{a,5} = W_n^{a,5}, Z_n^{0,5})\), and the gauge KK modes \(V_{a,\mu}^n = (W_{n}^{a,\mu}, Z_{n}^{0,\mu})\) have mass \(M_n = \sqrt{(n/R)^2 + m_{\ell}^2} \simeq n/R\) \((m_{\ell} \ll n/R)\). There are
three types of physical KK Higgs states \((H_n^\pm, P_0^0)\) which are just orthogonal to the “eaten” KK Goldstone fields \((\pi_n^+, \pi_n^-)\).

From direct calculation \[12\], we find that this small mixing causes extra \(E^2\) contributions to individual contributions to the scattering amplitude with four external longitudinal gauge KK states, but they exactly cancel to \(O(E^0)\) for each process. This is consistent with the \(E^2\)-counting for the corresponding KK Goldstone amplitude based on the KK-ET \[8\] which, unlike the conventional ET for the 4d SM \([4, 5, 20–22]\), involves the geometric Higgs mechanism from compactification. Analogous to our analysis in the QCD sector and ignoring the tiny constant terms suppressed by \(m_w^2 R^2\) or \(m_w^4 R^4\), we derive the unitarity bound on \(N_{KK}\) from a coupled channel analysis for the \(2 \to 2\) EW gauge KK scattering,

\[ N_{KK} \leq 11, \quad (27) \]

where we have replaced \(g^2\) in Eq. (25) by the weak gauge coupling \(g^2 = (2m_w/v)^2\).

It is interesting to compare the bounds (26) and (27) with those estimated from the uncompactified \(D\)-dimensional scattering analysis for \(D = 5\). We see that (27) agrees with (9) quite well where the \(N_{KK}\) upper limits are about \(9 \sim 11\), while (26) agrees with (8) up to a factor of \(2\) where the \(N_{KK}\) is constrained to be no higher than the range of \(2 \sim 4\). This is as expected, however, since for very low values of \(N_{KK}\) the kinematic effects due to the finite KK masses (which are absent in the uncompactified analysis) would become more important. Also, the subleading terms ignored in the 4d amplitude (25) are suppressed by a factor \(1/N_{KK}\) relative to the leading terms and imply a larger uncertainty for very low values of \(N_{KK}\). Hence, we see that the two independent analyses are consistent with each other, and they provide consistent estimates for the unitarity bounds.

Next, we perform a coupled channel analysis for the \(2 \to 2\) Higgs KK scattering and derive the \(m_H\) bounds in the effective 4d KK theory. There are four types of processes (and their crossing channels) which appear relevant to this coupled channel analysis: (i) \(\Pi_n \Pi_n \to \Pi_k \Pi_k\), (ii) \(h_0 \Pi_{2k} \to \Pi_k \Pi_k\), (iii) \(h_0 h_0 \to \Pi_k \Pi_k\), (iv) \(V_0^0 V_0^0 \to \Pi_k \Pi_k\), where \(\Pi_n \in (H_0, P_0^0, H_0^\pm)\) represents three types of physical Higgs KK states and \(V_0^0 \in (W_{zL}^0, Z_{zL}^0)\) denotes the zero-modes of the longitudinal weak gauge bosons. As will be clear shortly, we find that the only important processes for our coupled channel analysis are type-(i) which involves only KK scalars (without zero-mode) for the in/out states of the \(2 \to 2\) scattering.

For the type-(i) channels, we will consider the scattering \((\Pi_n \Pi_n \to \Pi_k \Pi_k)\), via electrically neutral KK channels \((\Pi_n \Pi_n), (\Pi_k \Pi_k) = (H_0^\pm, P_0^0), (V_0^0, V_0^0, V_0^0, V_0^0)\) in analogy with the customary analysis of the 4d SM \([5]\). Again, the channel \((H_\ell^\pm P_\ell^0)\) decouples from the other three channels and their scattering amplitudes form a \(4 \times 4\) matrix which approaches constant for \(s \gg M_H^2\). We then derive the eigenvalues of this matrix as, \(-\lambda \cdot (6, 2, 2, 2)\) for \(n \neq k\), and \(-(3\lambda)/2 \cdot (6, 2, 2, 2)\) for \(n = k\). Thus, the maximal eigenvalue amplitudes are

\[ T_{\text{max}}[nn, kk] = -6\lambda, \quad T_{\text{max}}[nn, nn] = -9\lambda, \quad (28) \]

where \(n \neq k\). Defining a normalized state consisting of \(N_0\) pairs of KK states,

\[ |S\rangle = \frac{1}{\sqrt{N_0}} \sum_{n=1}^{N_0} |nn\rangle, \quad (29) \]

we deduce the \(s\)-wave amplitude for \(|S\rangle \to |S\rangle\), at the leading order of \(N_0\),

\[ a_0[|S\rangle] = \frac{3N_0}{16\pi} \left( \frac{m_H}{v} \right)^2, \quad (30) \]

where the inelastic channels \(nn \to kk\) \((n \neq k)\) dominate while channels \(nn \to nn\) are only of \(O((N_0)^0)\). From the unitarity condition \(|\Re a_0| < 1/2\) and noting the kinematic requirement \(N_0 \leq N_{KK}/2 \sim AR/2\), we deduce the \(N_{KK}\) limit,

\[ N_{KK} < \frac{16\pi}{3} \left( \frac{v}{m_H} \right)^2, \quad (31) \]

or, the bound on the Higgs mass,

\[ m_H < \left( \frac{16\pi}{3} \right)^{1/2} \frac{v}{\sqrt{N_{KK}}}. \quad (32) \]

With these we can constrain \(N_{KK}\) by imposing the LEP-2 Higgs search limit \(m_H < 114.3\, \text{GeV}\),

\[ N_{KK} \leq 77. \quad (33) \]

Comparing our estimated bound (20) for \(D = 5\) with the above limit, we see that the difference is only about 21%. Using the condition (32), we further derive the Higgs mass limits,

\[ m_H < (581, 503, 450, 411, 356, 318, 291, 260, 225)\, \text{GeV}, \quad (34) \]

for the inputs \(N_{KK} = (3, 4, 5, 6, 8, 10, 12, 15, 20)\), respectively. Again, we notice that the estimated bounds in Table I \((\delta = 1)\) are in reasonable agreement.

Finally, we comment that for type-(ii), (iii) and (iv) processes we could define similar normalized in/out state to Eq. (29), but it is readily seen that the corresponding scattering amplitudes only have leading contributions at \(O(1), O(\sqrt{N_0})\) and \(O(N_0)\), respectively. The type-(i) channels with \(n = k\) also have amplitudes of \(O(1)\). So, the bounds from these other channels are too weak to be useful in comparison with that of type-(i) with \(n \neq k\).

4. Gauge Unification and Unitarity Constraints

In higher dimensional Grand Unified Theories (GUTs), the KK states will contribute to the running gauge coupling constants. For sufficiently many KK states, these
contributions mimic the power-law running expected in a higher dimensional theory and it has been proposed that this could substantially lower the GUT scale in these theories [13]. The unitarity bounds we derived from the gauge KK scattering in the previous sections can apply to any higher dimensional GUT (with/without supersymmetry) whose low energy theory contains the SM gauge bosons and their KK excitations as part of the spectrum. We will show that such unitarity constraints severely restrict the number of KK states which can consistently accelerate perturbative gauge-coupling unification. This bound prevents the unification scale in 5d minimal SUSY GUT from being lower than about 10^{14} \text{GeV}.

In the \( \overline{\text{MS}} \) scheme, the running gauge coupling may be expressed as [13, 23]

\[
\alpha_j^{-1}(\mu) = \alpha_j^{-1} - \frac{2\pi}{b_{ij}} \ln \frac{\mu}{m_z} - \frac{1}{2\pi} F(\delta, n(\mu)) + \kappa_j,
\]

where \( \alpha_j \equiv \alpha_j(m_z) \), \( \kappa_j \) represents corrections from the higher loop-levels and possible higher dimensional operators suppressed by GUT scale \( M_G \) [23]. The \( F(\delta, n(\mu)) \) term arises from the one-loop KK contributions,

\[
F(\delta, n(\mu)) = \sum_{n=1}^{n(\mu)} D_n \ln \frac{\mu}{M_n},
\]

where \( M_n \) is the mass of relevant KK excitations at level \( n \), \( n(\mu) \) is defined by \( M_{n(\mu)} < \mu < M_{n(\mu)+1} \), and \( 1 \leq n(\mu) \leq N_{\text{KK}} \). The \( D_n \) denotes the degeneracy at the KK level \( n \).

For \( \delta = 1 \) with compactification on \( S^1/\mathbb{Z}_2 \), we have \( D_n = 1 \) and

\[
F(1, N_{\text{KK}}) = N_{\text{KK}} \ln N_{\text{KK}} - \ln(N_{\text{KK}}!),
\]

where we set \( N_{\text{KK}} = R M_G \). Unification at the GUT scale imposes the conditions \( \alpha_1(M_G) = \alpha_2(M_G) = \alpha_3(M_G) \), where we have used the usual GUT normalization, \( \alpha_1 = (5/3)g_e^2/(4\pi) \). From this, we arrive at

\[
M_G = m_z \exp \left\{ \frac{2\pi}{\Delta b_{ij}} \left[ \Delta \alpha_{zij}^{-1} - \frac{\Delta b_{ij}}{2\pi} F(\delta, N_{\text{KK}}) + \Delta_{ij} \right] \right\},
\]

where \( \Delta \alpha_{zij}^{-1} = \alpha_{z_i}^{-1} - \alpha_{z_j}^{-1} \), \( \Delta b_{ij} = b_i - b_j \), \( \Delta b_{ij} = \tilde{b}_i - \tilde{b}_j \), \( \Delta_{ij} = \kappa_i - \kappa_j = O(10^{-2}) \), and \( i < j = 1, 2, 3 \). Eq. (38) contains three relations with \( (ij) = (12, 23, 13) \), two of which are independent.

As in the case of four-dimensions, the \( D \)-dimensional extension of the SM without supersymmetry (SUSY) does not realize perturbative gauge unification, unless additional fields are added to the SM particle spectrum [13]. The simplest example of the perturbative gauge unification is the minimal supersymmetric extension of the SM (MSSM). Following [13, 23, 24], we will consider a \( D \)-dimensional MSSM compactified on \( T^d = (S^1/\mathbb{Z}_2)^d \), with vector-supermultiplets and two Higgs supermultiplets propagating in the bulk, and chiral-supermultiplets for fermions sitting on the brane. For the simplest GUT group \( SU(5) \) [25], the SM fermions fill an entire \( SU(5) \) representation, their presence in the bulk would not effect unification, but does change the value of the unified coupling. Since matter contributes positively to the beta functions and drives the theory to stronger coupling, allowing matter in the bulk will only enhance the effect of KK excitations and strengthen our unitarity bounds. For this \( D \)-dimensional MSSM, the coefficients of one-loop beta functions are, \( (b_1, b_2, b_3) = (33/5, 1, -3) \), and \( (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) = (3/5, -3, -6) \). Substituting these into (38), we derive the numerical relation

\[
M_G \approx 10^4 \exp \left[ 28.3 \left( 1 - \frac{F(\delta, N_{\text{KK}})}{44.1} + 0.040\Delta_{12} \right) \right] \text{GeV},
\]

where we have imposed the condition \( \alpha_1(M_G) = \alpha_2(M_G) \) since the \( Z \)-pole values of \( \alpha_{1,2} \) are much more precisely known than the value of \( \alpha_3 \).

![Graph](image)

**FIG. 1:** (a) The GUT scale \( M_G \) as a function of required \( N_{\text{KK}} \) value for the 5d MSSM compactified on \( S^1/\mathbb{Z}_2 \). Note that, the bound \( N_{\text{KK}} < 11 \) in Eq. (27) restricts \( M_G \geq 10^{14} \text{GeV} \). (b) The evolution of gauge couplings with the scale \( \mu \) (in GeV) in the 5d MSSM, where \( R^{-1} = 10^{13} \text{GeV} \), \( N_{\text{KK}} = 10 \) and the unification occurs at \( \mu = 10^{14} \text{GeV} \).

Using (39), we plot the GUT scale \( M_G \) as a function of \( N_{\text{KK}} \) for the 5d MSSM in Fig.1(a). In [23], it was
argued that the term $\Delta_{ij}$ may receive contributions from the higher dimensional operators suppressed by the string scale which is of $O(10^{-2})$ in the MSSM and could be as large as 10% in its next-to-minimal extension (NMSSM). In Fig. 1(a), we have varied $\Delta_{12}$ from $+10\%$ to $-10\%$. Using the Eq. (35), we also plot in Fig. 1(b) the evolution of three gauge coupling constants with a typical high compactification scale $R^{-1} = 10^{14}$ GeV, where the higher order parameter $\kappa_j$ is varied within $\pm10\%$. In this case, the unification is accelerated to a lower scale $M_G = 10^{13}$ GeV due to the presence of $N_{KK} = 10$ KK states.

For a compactification size $R$ and a GUT scale $M_G$, perturbative unification can only occur if the KK modes of level $N_{KK} = R M_G$ satisfy the unitarity bounds in Sec. 3. We note that when $M_G$ is close to the conventional GUT scale, the running coupling $\alpha_2 = g^2/4\pi$ in the range between $R^{-1}$ and $M_G$ has about the same size as [or slightly larger than] the weak scale value $\alpha_2(m_Z)$ [cf. Fig. 1(b)]. Hence, we can apply the unitarity limit $N_{KK} \leq 11$ in Eq. (27), and find that, because of $N_{KK} = M_G/R$, the scales $R^{-1}$ and $M_G$ cannot be separated by more than one order of magnitude. This is a generic feature for any higher-dimensional GUT theory, when the bound (27) can be applied directly. Therefore, no substantial acceleration of four-dimensional perturbative gauge unification is possible from embedding the theory in higher dimensions. In particular, the unification scale $M_G$ in the 5d minimal SUSY GUT and similar theories cannot be lower than about $10^{14}$ GeV.

As this work was being completed a related study [27] appeared, which considered bounds on $m_{H}$ from the scattering of $W_{OL}^+, W_{OL}^-$ into the scalar KK states by using the equivalence theorem.

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5. Conclusions

In this Letter we have extended the study of the unitarity of compactified 5d Yang-Mills theories, and investigated the unitarity of the standard model (SM) in higher dimensions. Analyzing gauge-boson scattering in the uncompactified $D$-dimensional theory, we derive an upper bound on the UV cutoff of the theory. Using this estimate in the compactified case yields bounds on the highest Kaluza-Klein (KK) level $N_{KK}$ and the Higgs mass $m_H$ allowed in the effective 4d low-energy theory. We demonstrated the validity of these general observations by explicitly analyzing the effective 4d Kaluza-Klein theory from a compactified five-dimensional SM on $S^1/Z_2$. With the analyses before and after compactification, we derive the strongest bounds on $N_{KK}$ from the gauge KK scattering. Applying these bounds to higher-dimensional supersymmetric GUTs, we show that only a small number of KK states can be used to accelerate the perturbative gauge coupling unification. In particular, the unification scale $M_G$ in the 5d minimal SUSY GUT and similar theories cannot be lower than about $10^{14}$ GeV.

[1] P. W. Higgs, Phys. Lett. 12, 132 (1964).
[2] C. H. Llewellyn Smith, Phys. Lett. 446, 233 (1973).
[3] D. A. Dicus, V. S. Mathur, Phys. Rev. D7, 3111 (1973).
[4] J. M. Cornwall, D. N. Levin, and G. Tiktopoulos, Phys. Rev. D10, 1145 (1974).
[5] B. W. Lee, C. Quigg, and H. B. Thacker, Phys. Rev. D16, 1519 (1977).
[6] M. J. G. Veltman, Acta Phys. Polon. B8, 475 (1977).
[7] T. Kaluza, Sitz. Preuss. Akad. Wiss K1, 966 (1921); O. Klein, Z. Phys. 37, 895 (1926).
[8] R. S. Chivukula, D. A. Dicus, H.-J. He, Phys. Lett. B525, 175 (2002) [hep-ph/0111016]; R. S. Chivukula and H.-J. He, Phys. Lett. B532, 121 (2002) [hep-ph/0201164].
[9] M. Soldate, Phys. Lett. B186, 321 (1987); M. Chaichian and J. Fischer, Nucl. Phys. B303, 557 (1988).
[10] J. Lykken and S. Nandi, Phys. Lett. B485, 224 (2000) [hep-ph/9908505].
[11] K. R. Dienes, Phys. Rev. Lett. 88, 011601 (2002) [hep-ph/0108115].
[12] R. S. Chivukula, D. A. Dicus, H.-J. He, and S. Nandi, (2003) in preparation.
[13] K. R. Dienes, E. Dudas, and T. Gherghetta, Phys. Lett. B436, 55 (1998) [hep-ph/9803466]; Nucl. Phys. B537, 47 (1999) [hep-ph/9806292].
[14] T. Appelquist, H. C. Cheng and B. A. Dobrescu, Phys. Rev. D64, 035002 (2001) [hep-ph/0012100].
[15] N. Arkani-Hamed, H. C. Cheng, B. A. Dobrescu, and L. J. Hall, Phys. Rev. D62, 096006 (2000) [hep-ph/0006238].
[16] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999) [hep-ph/9905221].
[17] D. A. Dicus, C. D. McMullen, and S. Nandi, Phys. Rev. D65, 076007 (2002) [hep-ph/0112259].
[18] C. T. Hill, S. Pokorski, and J. Wang, Phys. Rev. D64, 105005 (2001) [hep-th/0104035].
[19] A. Muck, A. Pilaftsis, and R. Ruckl, Phys. Rev. D65, 085037 (2002) [hep-ph/0110391].
[20] M. S. Chanowitz and M. K. Gaillard, Nucl. Phys. B261, 379 (1985).
[21] H.-J. He, Y.-P. Kuang, X. Li, Phys. Rev. Lett. 69, 2619 (1992); Phys. Rev. D49, 4842 (1994); Phys. Lett. B329, 278 (1994); H.-J. He, Y.-P. Kuang, C.-P. Yuan, Phys. Rev. D51, 6463 (1995), and comprehensive review in DESY-97-056 [hep-ph/9704276] with references therein.
[22] H.-J. He and W. B. Kilgore, Phys. Rev. D55, 1515 (1997).
[23] H. C. Cheng, B. A. Dobrescu, C. T. Hill, Nucl. Phys. B573, 597 (2000) [hep-ph/9906327].
[24] I. Antoniadis, Phys. Lett. B 246, 377 (1990).
[25] H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438
(1974).

[26] L. J. Hall and Y. Nomura, Phys. Rev. D66, 075004 (2002) [hep-ph/0205067], and references therein.

[27] For a related extension of Ref. [8], S. De Curtis, D. Dominici, J.R. Pelaez, hep-ph/0211353, hep-ph/0301059.