ON THE FINITENESS OF BASS NUMBERS OF LOCAL COHOMOLOGY MODULES AND COMINIMAXNESS

KAMAL BAHMANPOUR, REZA NAGHIPOUR* AND MONIREH SEDGHI

DEDICATED TO PROFESSOR HYMAN BASS

Abstract. In this paper, we continue the study of cominimaxness modules with respect to an ideal of a commutative Noetherian ring (cf. [2]), and Bass numbers of local cohomology modules. Let $R$ denote a commutative Noetherian local ring and $I$ an ideal of $R$. We first show that the Bass numbers $\mu^0(p, H^2_I(R))$ and $\mu^1(p, H^2_I(R))$ are finite for all $p \in \text{Spec } R$, whenever $R$ is regular. As a consequence, it follows that the Goldie dimension of $H^2_I(R)$ is finite. Also, for a finitely generated $R$-module $M$ of dimension $d$, it is shown that the Bass numbers of $H^{d-1}_I(M)$ are finite if and only if $\text{Ext}^i_R(R/I, H^{d-1}_I(M))$ be minimax for all $i \geq 0$. Finally, we prove that if $\dim R/I = 2$, then the Bass numbers of $H^n_I(M)$ are finite if and only if $\text{Ext}^i_R(R/I, H^n_I(M))$ be minimax, for all $i \geq 0$, where $n$ is a non-negative integer.

1. Introduction

Throughout this paper, let $R$ denote a commutative Noetherian ring (with identity) and $I$ an ideal of $R$. For an $R$-module $M$, the $i^{th}$ local cohomology module of $M$ with respect to $I$ is defined as

$$H_i^I(M) = \lim_{n \to \infty} \text{Ext}^i_R(R/I^n, M).$$

We refer the reader to [10] and [6] for more details about local cohomology. An important problem in commutative algebra is to determine when the Bass numbers of the $i^{th}$ local cohomology module $H_i^I(M)$ is finite. In [12] Huneke conjectured that for any ideal $I$ in a regular local ring $(R, m, k)$, the Bass numbers

$$\mu^i(p, H_i^I(R)) = \dim_{k(p)} \text{Ext}^i_{R_p}(k(p), H_i^I(R_p))$$

are finite for all $i$ and $j$, where $k(p) := R_p/pR_p$. In particular the injective resolution of the local cohomology has only finitely many copies of the injective hull of $R/p$ for any $p$.

There is evidence that this conjecture is true. It is shown that by Huneke and Sharp [13] and Lyubeznik [16] [17] that the conjecture holds for a regular local ring containing

*Corresponding author: e-mail: naghipour@ipm.ir (Reza Naghipour).

Key words and phrases. Bass numbers, cominimax modules, Krull dimension, local cohomology, minimax modules.

2000 Mathematics Subject Classification: 13D45, 14B15, 13E05.

This research was in part supported by a grant from IPM (No. 89130053, 89130048).

*Corresponding author: e-mail: naghipour@ipm.ir (Reza Naghipour).
a field. We remark that the Bass numbers might be infinite if $R$ is not regular. For
example, if $I := (x, y)R \subseteq R := k[x, y, z, w]/(xz - yw)$, then $\mu^0(m, H^2_I(R)) = \infty$ for $m := (x, y, z, w)R$ (see [11]).

We say that $M$ is a minimax module if there is a finitely generated submodule $N$ of $M$, such that $M/N$ is Artinian. The interesting class of minimax modules was introduced by H. Zöshinger in [23] and he has in [23] and [24] given many equivalent conditions for a module to be minimax. Finally, the $R$-module $M$ is said to be an $I$-cominimax if support of $M$ is contained in $V(I)$ and $\text{Ext}_R^i(R/I, M)$ is minimax for all $i \geq 0$. The concept of the $I$-cominimax modules were introduced in [2] as a generalization of important notion of $I$-cofinite modules.

The main subject of the paper is to continue the study of $I$-cominimaxness properties and the finiteness of Bass numbers of local cohomology modules. First, we provide a partial answer to Huneke’s conjecture. Namely, it will be shown that the Bass numbers of $H^i_I(M)$ are finite if and only if the $I$-cofinite modules.

Let $R$ be a local (Noetherian) ring, $I$ an ideal of $R$ and $M$ a finitely generated $R$-module of dimension $d$. Then the Bass numbers of $H^i_I(M)/\Gamma_m(H^i_I(M))$ are finite.

One of our tools for proving Theorem 1.1 is the following:

**Proposition 1.2.** Let $(R, m)$, $I$ and $M$ be as in Theorem 1.1. Then $H^i_m(H^i_I(M))$ is an Artinian $R$-module, where $d = \dim M$.

Pursuing this point of view further we derive the following consequence of Theorem 1.1.

**Theorem 1.3.** Let $(R, m)$ be a local (Noetherian) ring, $I$ an ideal of $R$ and $M$ a finitely generated $R$-module of dimension $d$. Then the following conditions are equivalent:

(i) The Bass numbers of $H^i_I(M)$ are finite.

(ii) $\text{Soc}(H^i_I(M))$ is finitely generated.

(iii) $H^d_I(M)$ is $I$-cominimax.

As the second main result of this paper we obtain a characterization of the finiteness of Bass numbers of $i^{th}$ local cohomology modules of $M$ with respect to $I$ of dimension 2, i.e. $\dim R/I = 2$. More precisely we shall show that:

**Theorem 1.4.** Let $(R, m)$ be a local (Noetherian) ring and let $I$ be an ideal of $R$ with $\dim R/I = 2$. Let $M$ be a finitely generated $R$-module and $i$ a non-negative integer. Then the Bass numbers of $H^i_I(M)$ are finite if and only if the $R$-module $H^i_I(M)$ is $I$-cominimax.
The proof of Theorem 1.4 is given in 2.17. As an application, we derive the following consequences of Theorem 1.4, which is a characterization of $I$-cominimaxness of $H^i_I(M)$ in terms of finiteness of Bass numbers of local cohomology modules $H^i_I(M)$ for certain ideal $I$ of $R$.

**Theorem 1.5.** Let $(R, m)$ be a local (Noetherian) ring, and let $I$ be an ideal of $R$ such that $\dim R/I = 2$. Let $M$ be a finitely generated $R$-module and $t$ a non-negative integer. Then the following conditions are equivalent:

(i) $\text{Soc} H^i_I(M)$ is finitely generated for all $i \leq t$.

(ii) $\text{Ext}^j_R(R/m, H^i_I(M))$ is finitely generated for all $j \geq 0$ and all $i \leq t - 1$.

(iii) $H^i_I(M)$ is $I$-cominimax for all $i \leq t - 1$.

Using Theorem 1.5 we obtain some results as following:

**Corollary 1.6.** Let $(R, m)$ be a regular local ring of dimension $d \geq 2$ containing a field and let $I$ be an ideal of $R$ with $\dim R/I = 2$. Then the $R$-module $H^i_I(R)$ is $I$-cominimax for all $i \geq 0$.

**Corollary 1.7.** Let $R$ be a Noetherian ring, $M$ a finitely generated $R$-module and $I$ an ideal of $R$ such that $\dim M/IM = 2$. Let

$$\Sigma := \{p \in \text{Spec } R : \mu^i(p, H^i_I(M)) = \infty \text{ for some integers } i \geq 0 \text{ and } j \geq 0\}.$$

Then $\Sigma$ is countable and $\Sigma \subseteq \text{Max}(R)$.

**Corollary 1.8.** Let $(R, m)$ be a local (Noetherian) ring, $M$ a finitely generated $R$-module and $I$ an ideal of $R$ such that $\dim M/IM = 3$. Then the set

$$\Sigma := \{p \in \text{Spec } R : \mu^i(p, H^i_I(M)) = \infty \text{ for some integers } i \geq 0 \text{ and } j \geq 0\},$$

is countable.

Throughout this paper, $R$ will always be a commutative Noetherian ring with non-zero identity and $I$ will be an ideal of $R$. For each $R$-module $L$, we denote by $\text{Ass}_R L$ (resp. $m\text{Ass}_R L$) the set $\{p \in \text{Ass}_R L : \dim R/p = \dim L\}$ (resp. the set of minimal primes of $\text{Ass}_R L$). We shall use $\text{Max } R$ to denote the set of all maximal ideals of $R$. Also, for any ideal $a$ of $R$, we denote $\{p \in \text{Spec } R : p \supseteq a\}$ by $V(a)$. Finally, for any ideal $b$ of $R$, the radical of $b$, denoted by $\text{Rad}(b)$, is defined to be the set $\{x \in R : x^n \in b \text{ for some } n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [6] and [18].

2. **The Results**

Let $(R, m)$ be a local (Noetherian) ring and $I$ an ideal of $R$. As noted in the introduction, it is well known that the Bass numbers $\mu^i(p, H^2_I(R))$ are finite whenever $R$ is a regular ring containing a field. Our first result shows that if $R$ is regular, then $\mu^i(p, H^2_I(R))$ is finite for $i \in \{0, 1\}$.

**Theorem 2.1.** Let $(R, m)$ be a regular local ring and $I$ an ideal of $R$. Then the Bass numbers $\mu^0(p, H^2_I(R))$ and $\mu^1(p, H^2_I(R))$ are finite for all $p \in \text{Spec } R$. 
Proof. In view of [5] Proposition 2.20 the $R_p$-modules $\text{Ext}^i_{R_p}(\kappa(p), H^1_{R_p}(R_p))$ are finitely generated for all $j \geq 0$ and $i = 0, 1$. Hence it follows from [15] Corollary 3.5] that $\mu^0(p, H^1_p(R))$ and $\mu^1(p, H^1_p(R))$ are finite. \hfill $\square$

As a corollary of the previous theorem, we deduce that the Goldie dimension of $H^2_p(R)$ is finite.

**Corollary 2.2.** Let $(R, \mathfrak{m})$ be a regular local ring and $I$ an ideal of $R$. Then the $R$-module $H^2(I)_p$ has finite Goldie dimension.

**Proof.** As $\text{Ass}_R(H^2_p(R))$ is finite, by [4] Theorem 2.4, it follows from Theorem 2.1 and the definition that $H^2_p(R)$ has finite Goldie dimension. \hfill $\square$

For an arbitrary $R$-module $M$, the next result gives us a necessary and sufficient condition for the finiteness of $\mu^i(p, M)$, the Bass numbers of $M$, in terms of the minimaxness of $\text{Ext}^i_R(R/I, M)$. Recall that for any module $L$ over a local ring $(R, \mathfrak{m})$, the socle of $L$, denoted by $\text{Soc}(L)$, is defined to be the $R$-module $\text{Hom}_R(R/\mathfrak{m}, L)$.

**Theorem 2.3.** Let $(R, \mathfrak{m})$ be a local (Noetherian) ring and $I$ an ideal of $R$ such that $\dim R/I = 1$. Suppose that $M$ is an $R$-module and $n \geq 0$ an integer. Then the following conditions are equivalent:

(i) $\mu^i(p, M)$ is finite for all $p \in V(I)$ and for all $i \leq n$.

(ii) $\text{Ext}^i_R(R/I, M)$ is minimax for all $i \leq n$.

**Proof.** First we show (i)$\implies$(ii). Let $q \in \text{mAss}_R(R/I)$ and $i \leq n$. Then $q \neq \mathfrak{m}$, and so $\text{Rad}(q + Rx) = \mathfrak{m}$ for all $x \in \mathfrak{m} \setminus q$. Hence it follows from [7] Proposition 1] that the $R$-module $\text{Ext}^i_R(R/q + Rx, M)$ is finitely generated for all $i \leq n$. (Note that $\text{Ext}^i_R(R/\mathfrak{m}, M)$ is finitely generated for all $i \leq n$.) Now, for $x \in \mathfrak{m} \setminus q$ the exact sequence

$$0 \to R/q \to R/q \to R/q + Rx \to 0,$$

provides the following exact sequence:

$$\text{Ext}^i_R(R/q + Rx, M) \to \text{Ext}^i_R(R/q, M) \to \text{Ext}^i_R(R/q, M),$$

which implies that the $R$-module $\text{Soc}(\text{Ext}^i_R(R/q, M))$ is finitely generated, for all $i \leq n$. On the other hand, if $L := \text{Ext}^i_R(R/q, M)$, then as by assumption the $R_q$-module $L_q$ is finitely generated, it follows that there exists a finitely generated submodule $K$ of $L$, such that $(L/K)_q = 0$, and so $\text{Supp}(L/K) \subseteq V(m)$. Finally, the exact sequence

$$0 \to K \to L \to L/K \to 0,$$

induces the exact sequence

$$\text{Soc}(L) \to \text{Soc}(L/K) \to \text{Ext}^i_R(R/\mathfrak{m}, K).$$

Hence the $R$-module $\text{Soc}(L/K)$ is finitely generated. Consequently in view of [19] Proposition 4.1] the $R$-module $L/K$ is Artinian, that is $L$ is a minimax $R$-module. Now, the assertion follows from [2] Corollary 2.8].

In order to prove (ii)$\implies$(i), use [2] Corollary 2.8] and the definition of the minimax
Corollary 2.4. Let $R$, $I$ and $M$ be as in Theorem 2.3. Then the following conditions are equivalent:

(i) $\mu^i(p, M)$ is finite for all $i \geq 0$ and all primes $p \in V(I)$.

(ii) $\text{Ext}^i_R(R/I, M)$ is minimax for all $i \geq 0$.

Proof. The result follows from Theorem 2.3.

Corollary 2.5. Let $(R, m)$ be a local (Noetherian) ring. Let $I$ and $J$ be two ideals of $R$ such that $\text{dim} R/I \leq 1$ and $\text{dim} R/J = 1$. Then the $R$-modules $\text{Ext}^i_R(R/J, H^j_I(R))$ are minimax for all integers $i, j \geq 0$.

Proof. The assertion follows from [5, Corollary 2.10] and Theorem 2.3.

Before we state the next corollary, recall that for an ideal $I$ of a commutative ring $R$, the ideal transform $D_I(R)$ is defined by

$$D_I(R) := \lim_{n \to \infty} \text{Hom}_R(I^n, R).$$

See [6, Section 2.2] for the basic properties of ideal transforms.

Corollary 2.6. Let $(R, m)$ be a regular local ring and let $I, J$ be two ideals of $R$ such that $\text{dim} R/I = 1$. Then

(i) For any integer $i \geq 0$, the $R$-modules $\text{Ext}^i_R(R/I, D_J(R))$ and $\text{Ext}^i_R(R/I, H^j_J(R))$ are minimax.

(ii) The $R$-modules $\text{Hom}_R(R/I, H^j_I(R))$ and $\text{Ext}^1_R(R/I, H^j_J(R))$ are minimax.

(iii) For all integers $i, j \geq 0$, the $R$-module $\text{Ext}^i_R(R/I, H^j_J(R))$ is minimax, whenever $R$ contains a field.

Proof. (i) follows from [5, Proposition 2.20] and Theorem 2.3. In order to prove (ii) use Theorems 2.3 and 2.1. Finally, (iii) follows from Corollary 2.4 and [13, 17].

In order to state next results, we need here some preliminary results about the Artinianness of local cohomology modules.

Theorem 2.7. Let $(R, m)$ be a local (Noetherian) ring and $I$ an ideal of $R$ such that $\text{dim} R/I = 2$. Then for any finitely generated $R$-module $M$, the $R$-module $H^1_m(H^i_I(M))$ is Artinian for all $i \geq 0$.

Proof. Since $\text{dim} R/I = 2$, there exists $x \in m$ such that $\text{dim} R/I + Rx = 1$. Let $J := I + Rx$. As $H^1_m(H^0_I(M))$ is Artinian, we may assume that $i \geq 1$. Then by [21, Corollary 1.4], there exists the following exact sequence,

$$0 \to H^1_J(H^i_{I-1}(M)) \to H^i_J(M) \to H^0_J(H^i_I(M)) \to 0.$$
Now, as \( \dim H_I^1(H_I^{-1}(M)) \leq 1 \), it follows from this exact sequence that the sequence
\[
H_m^0(H_I^1(M)) \to H_m^0(H_I^0(H_I^1(M))) \to 0,
\]
is exact, and so by [5, Corollary 2.16] the \( R \)-module \( H_m^0(H_I^0(H_I^1(M))) \) is Artinian. On the other hand, since \( \dim R/J = 1 \) there exists \( y \in m \) such that \( J + Ry \) is \( m \)-primary and so
\[
H_{J+Ry}^1(H_I^0(H_I^1(M))) = H_m^1(H_I^0(H_I^1(M))),
\]
is Artinian \( R \)-module. Moreover, using again [21, Corollary 1.4] it follows that the sequences:
\[
0 \to H_I^1(H_I^j(M)) \to H_I^{j+1}(M) \to H_I^0(H_I^{j+1}(M)) \to 0,
\]
\[
0 \to H_{J+Ry}^1(H_I^0(H_I^1(M))) \to H_m^1(H_I^1(M)) \to H_J^0(H_I^1(M))) \to 0,
\]
are exact. Now, since by [5, Corollary 2.16], \( H_J^0(H_I^1(M)) \) is Artinian, it follows that \( H_{J+Ry}^0(H_I^1(M)) \) is Artinian. Consequently the \( R \)-module \( H_m^1(H_I^1(M)) \) is Artinian, as required. \qed

**Theorem 2.8.** Let \((R, m)\) be a local (Noetherian) ring, \( I \) an ideal of \( R \) and \( M \) a finitely generated \( R \)-module of dimension \( d \geq 1 \). Then \( H_m^1(H_I^{d-1}(M)) \) is an Artinian \( R \)-module.

**Proof.** We use induction on \( d \). When \( d = 1 \), there is nothing to prove. Now suppose that \( d > 1 \) and the result has been proved for non-zero finitely generated \( R \)-modules of dimension \( d - 1 \). By replacing \( M \) by \( M/\Gamma_I(M) \) we may assume that \( M \) is a non-zero finitely generated \( I \)-torsion-free \( R \)-module. Then, by [6, Lemma 2.1.1], the ideal \( I \) contains an element \( x \) which is a non-zerodevisor on \( M \). Hence the exact sequence
\[
0 \to M \to M \to M/xM \to 0
\]
induces an exact sequence
\[
\cdots \to H_I^j(M) \to xH_I^j(M) \to H_I^j(M/xM) \to H_I^{j+1}(M) \to xH_I^{j+1}(M) \to \cdots.
\]
Therefore we have the following exact sequence,
\[
0 \to H_I^j(M)/xH_I^j(M) \to H_I^j(M/xM) \to (0 : H_I^{j+1}(M)) \to 0,
\]
and so it follows from \( \dim M/xM = d - 1 \) that \( H_I^{d-1}(M)/xH_I^{d-1}(M) \) is Artinian. Moreover, by the inductive hypothesis, the \( R \)-module \( H_m^1(H_I^{d-2}(M/xM)) \) is Artinian. Next, in view of [21, Corollary 3.3], \( \text{Supp} H_I^{d-2}(M/xM) \) is finite. Hence we have \( \dim H_I^{d-2}(M/xM) \leq 1 \). Therefore the exact sequence
\[
0 \to H_I^{d-2}(M)/xH_I^{d-2}(M) \to H_I^{d-2}(M/xM) \to (0 : H_I^{d-1}(M)) \to 0,
\]
provides the exact sequence
\[
H_m^1(H_I^{d-2}(M)/xH_I^{d-2}(M)) \to H_m^1(H_I^{d-2}(M/xM)) \to H_m^1((0 : H_I^{d-1}(M))) \to 0.
\]
Hence the \( R \)-module \( H_m^1((0 : H_I^{d-1}(M))) \) is Artinian. Now, from the exact sequences
\[
0 \to (0 : H_I^{d-1}(M)) \to H_I^{d-1}(M) \to xH_I^{d-1}(M) \to 0,
\]
we obtain the following exact sequences

\[ \text{Ker}(H^1(R/I, T)) \rightarrow H^1(R/I, T) \rightarrow H^1(R/I, T)/\text{im}(\text{Ker}(H^1(R/I, T)) \rightarrow 0, \]

we form the following exact sequences

\[ H^1_m((0 : H^1(R/I, T)) \rightarrow H^1_m(H^1(R/I, T)) \rightarrow H^1_m(H^1(R/I, T))/H^1_m((0 : H^1(R/I, T)) \rightarrow 0, \]

Then \( \text{Ker}(H^1_m(f)) \) and \( \text{Ker}(H^1_m(g)) \) are Artinian (note that \( H^1_m((0 : H^1(R/I, T)) \) and \( H^1(R/I, T)/H^1_m((0 : H^1(R/I, T)) \) are Artinian), and the sequence

\[ 0 \rightarrow \text{Ker}(H^1_m(f)) \rightarrow \text{Ker}(H^1_m(g)) \rightarrow H^1_m(g), \]

is exact. Since \( \text{Ker}(H^1_m(g \circ f)) = (0 : H^1_m(H^1(R/I, T)) \) and \( \text{Ker}(H^1_m(g \circ H^1_m(f)) \) is Artinian, it follows that \( (0 : H^1_m(H^1(R/I, T)) \) is also Artinian. Whence according to Melkersson’s Theorem \[19\] Proposition 1.4 the \( R \)-module \( H^1_m(H^1(R/I, T)) \) is Artinian, and this completes the inductive step.

\[ \square \]

**Theorem 2.9.** Let \((R, m)\) be a local (Noetherian) ring, \(I\) an ideal of \(R\) and \(M\) a finitely generated \(R\)-module of dimension \(d\). Then the Bass numbers of \(H^1(R/I, T)\) of \(H^1(R/I, T)\) are finite.

**Proof.** In view of Theorem 2.8, the \(R\)-module \(H^1(R/I, T)\) is Artinian. On the other hand, since \(\text{Supp} \ H^1(R/I, T)\) is finite (see \[20\] Corollary 3.3), it follows that \(\dim H^1(R/I, T) \leq 1\), and so for \(i \geq 2\), \(H^i_m(H^1(R/I, T)/\Gamma_m(H^1(R/I, T))) = 0\). Therefore for all \(i \neq 1\) we have \(H^1_m(H^1(R/I, T)/\Gamma_m(H^1(R/I, T))) = 0\). Consequently, in view of \[19\] Corollary 3.10 the \(R\)-modules \(\text{Ext}^i_R(R/m, H^1(R/I, T)/\Gamma_m(H^1(R/I, T)))\) are finitely generated. (Note that a module is \(m\)-cofinite if and only if it is Artinian.) Furthermore, because of \(\dim H^1(R/I, T) \leq 1\), it follows that \(\dim R/p = 1\) for all \(p \in \text{Supp} \ H^1(R/I, T)/\Gamma_m(H^1(R/I, T))\) with \(p \neq m\). Since

\[ (H^1(R/I, T)/\Gamma_m(H^1(R/I, T)))_p \cong (H^1(R/I, T))_p, \]

for all \(p \in \text{Supp} \ H^1(R/I, T)/\Gamma_m(H^1(R/I, T))\) with \(p \neq m\), it follows that \(H^1(R/I, T)_p \neq 0\), and so \(\dim M_p = d - 1\). Consequently \(H^1(R/I, T)_p\) is an Artinian \(R_p\)-module. Hence for each \(i \geq 0\) and each \(p \in \text{Spec} R\), the \(i^\text{th}\) Bass number

\[ \mu_i(p, H^1(R/I, T)/\Gamma_m(H^1(R/I, T))) = \dim_{K(p)} \text{Ext}^i_R(p, (H^1(R/I, T)/\Gamma_m(H^1(R/I, T))))_p, \]

is finite, where \(K(p) = R_p/pR_p\).

\[ \square \]

The following lemma and proposition will be useful in the proof of the next main result of this paper. An \(R\)-module \(M\) is said to be weakly Laskerian if the set of associated primes of any quotient module of \(M\) is finite (cf. \[8\] \[10\]).

**Lemma 2.10.** Let \(R\) be a Noetherian ring and \(I\) an ideal of \(R\). Then, for any \(R\)-module \(T\), the following conditions are equivalent:

(i) \(\text{Ext}^n_R(R/I, T)\) is weakly Laskerian for all \(n \geq 0\).
(ii) For any finitely generated R-module N with support in \( V(I) \), \( \text{Ext}_R^n(N, T) \) is weakly Laskerian for all \( n \geq 0 \).

**Proof.** For proving the assertion we use the proof of [14, Lemma 1] and [9, Lemma 2.2]. \( \square \)

**Proposition 2.11.** Let \((R, \mathfrak{m})\) be a local (Noetherian) ring and \( M \) a finitely generated \( R \)-module. Let \( I \subseteq J \) be two ideals of \( R \) such that \( \dim R/I = 2 \). Then, for all \( i, j \geq 0 \), there exists a finitely generated submodule \( L \) of \( \text{Ext}_R^j(R/J, H_I^j(M)) \) such that \( \text{Supp}(\text{Ext}_R^j(R/J, H_I^j(M))/L) \subseteq V(\mathfrak{m}) \).

**Proof.** In view of [5, Corollary 3.3] and Lemma 2.10 the set \( \text{Ass}_R \text{Ext}_R^j(R/J, H_I^j(M)) \) is finite. Let

\[ \text{Ass}_R \text{Ext}_R^j(R/J, H_I^j(M)) \backslash \{m\} = \{p_1, \ldots, p_n\}. \]

Since \( p_k \neq \mathfrak{m} \) for all \( k = 1, \ldots, n \) and height \( \mathfrak{m}/I = 2 \), it follows that height \( p_k/I \leq 1 \), and so \( \dim R_{p_k}/IR_{p_k} \leq 1 \) for all \( k = 1, \ldots, n \). Hence by [5, Corollary 2.7] the \( R_{p_k} \)-modules of \( (\text{Ext}_R^j(R/I, H_I^j(M)))_{p_k} \) are finitely generated for all \( k = 1, \ldots, n \). Therefore, by [14, Lemma 1], the \( R_{q_t} \)-modules of \( (\text{Ext}_R^j(R/J, H_I^j(M)))_{q_t} \) are finitely generated for all \( k = 1, \ldots, n \). Consequently for every \( k = 1, \ldots, n \), there exists a finitely generated submodule \( T_k \) of \( \text{Ext}_R^j(R/J, H_I^j(M)) \) such that \( (T_k)_{q_t} = (\text{Ext}_R^j(R/J, H_I^j(M)))_{q_t} \). Let \( K := T_1 + \cdots + T_n \). Then \( K \) is a finitely generated submodule of \( \text{Ext}_R^j(R/J, H_I^j(M)) \) and

\[ \text{Supp}(\text{Ext}_R^j(R/J, H_I^j(M))/K) \cap \{p_1, \ldots, p_n\} = \emptyset. \]

So that \( \dim \text{Ext}_R^j(R/J, H_I^j(M))/K \leq 1 \). Now if \( \text{Supp}(\text{Ext}_R^j(R/J, H_I^j(M))/K) \subseteq \{m\} \), then the result follows for \( L = K \). We may therefore assume that

\[ \dim \text{Ext}_R^j(R/J, H_I^j(M))/K = 1. \]

Let \( \text{Ass}_R \text{Ext}_R^j(R/J, H_I^j(M))/K \backslash \{m\} = \{q_1, \ldots, q_s\} \). Then, as the above, the \( R_{q_t} \)-module of \( (\text{Ext}_R^j(R/J, H_I^j(M))/K)_{q_t} \) is finitely generated for \( t = 1, \ldots, s \). Hence there exists a finitely generated submodule \( L_t/K \) of \( \text{Ext}_R^j(R/J, H_I^j(M))/K \) such that

\[ (\text{Ext}_R^j(R/J, H_I^j(M))/K)_{q_t} = (L_t)_{q_t}. \]

Let \( L := L_1 + \cdots + L_s + K \). Then \( L \) is a finitely generated submodule of \( \text{Ext}_R^j(R/J, H_I^j(M)) \) and

\[ \text{Supp}(\text{Ext}_R^j(R/J, H_I^j(M))/L) \cap \{q_1, \ldots, q_s\} = \emptyset. \]

Since \( \text{Supp} \text{Ext}_R^j(R/J, H_I^j(M))/L \subseteq \text{Supp} \text{Ext}_R^j(R/J, H_I^j(M))/K \), it follows that

\[ \text{Supp} \text{Ext}_R^j(R/J, H_I^j(M))/L \subseteq \{m\}, \]

as required. \( \square \)

We are now ready to state and prove the second main theorem of the paper, which is a characterization of the finiteness of the Bass numbers of \((d - 1)^{th}\) local cohomology of \( M \) with respect to an arbitrary ideal \( I \) in terms of the \( I \)-cominimaxness of \( H_I^{d-1}(M) \), where
Theorem 2.12. Let \((R, \mathfrak{m})\) be a local (Noetherian) ring, \(I\) an ideal of \(R\) and \(M\) a finitely generated \(R\)-module of dimension \(d\). Then the following conditions are equivalent:

(i) The Bass numbers of the \(R\)-module \(H_I^{d-1}(M)\) are finite.

(ii) \(\text{Soc}(H_I^{d-1}(M))\) is finitely generated.

(iii) \(H_I^{d-1}(M)\) is \(I\)-cominimax.

Proof. The conclusion (i) \(\implies\) (ii) is obviously true. In order to prove (ii) \(\implies\) (i), let \(\text{Soc}(H_I^{d-1}(M))\) be a finitely generated \(R\)-module. Then, according to Melkersson’s Theorem [19, Proposition 1.4], \(\Gamma_m(H_I^{d-1}(M))\) is Artinian. Therefore the Bass numbers of \(\Gamma_m(H_I^{d-1}(M))\) are finite. Now, it follows from Theorem 2.9 that the Bass numbers of \(H_I^{d-1}(M)\) are finite, as required.

In order to prove the implication (iii) \(\implies\) (ii), it follows from [2, Corollary 2.8] that the \(R\)-module \(\text{Ext}^j_R(R/\mathfrak{m}, H_I^{d-1}(M))\) is minimax for all \(j \geq 0\). Now it easily seen from definition that \(\text{Ext}^j_R(R/\mathfrak{m}, H_I^{d-1}(M))\) is of finite length for all \(j \geq 0\). In particular the \(R\)-module \(\text{Hom}_R(R/\mathfrak{m}, H_I^{d-1}(M))\) is finitely generated, as required.

Finally for the proof of (i) \(\implies\) (iii), let \(\dim R/I = n\). For \(n \leq 1\) this follows by [5, Corollary 2.7]. Hence we may assume that \(n \geq 2\). Let \(x_1, \ldots, x_{n-1} \in \mathfrak{m}\) be a filter regular sequence on \(R/I\). Let \(K_s := I + (x_1, \ldots, x_s)\) for \(s = 0, 1, \ldots, n - 1\). We first show by induction on \(t\), that the \(R\)-module \(\text{Ext}^j_R(R/K_{n-t-1}, H_I^{d-1}(M))\) is minimax for all \(j \geq 0\). Since \(x_1 + I, \ldots, x_{n-1} + I\) is a part of system of parameters for \(R/I\), it follows that \(\dim R/K_{n-t-1} = 1\), and so the result is evidently true for \(t = 0\), by Theorem 2.3. Suppose that \(n - 1 \geq t > 0\) and the case \(t - 1\) is settled. Since \(\Gamma_m(R/K_{n-t-1})\) is a finitely generated \(R\)-module with \(\text{Supp}(\Gamma_m(R/K_{n-t-1})) \subseteq \{\mathfrak{m}\}\), it follows that from [14, Lemma 1] and hypothesis (i) that the \(R\)-module \(\text{Ext}^j_R(\Gamma_m(R/K_{n-t-1}), H_I^{d-1}(M))\) is finitely generated for all \(j \geq 0\). Now, as \(\Gamma_m(R/K_{n-t-1}) = J/K_{n-t-1}\) for some ideal \(J\) of \(R\), from the exact sequence

\[
0 \longrightarrow J/K_{n-t-1} \longrightarrow R/K_{n-t-1} \longrightarrow R/J \longrightarrow 0,
\]

we get the following long exact sequence,

\[
\cdots \longrightarrow \text{Ext}^{j-1}_R(J/K_{n-t-1}, H_I^{d-1}(M)) \longrightarrow \text{Ext}^j_R(R/J, H_I^{d-1}(M)) \longrightarrow \text{Ext}^j_R(R/K_{n-t-1}, H_I^{d-1}(M)) \longrightarrow \text{Ext}^j_R(J/K_{n-t-1}, H_I^{d-1}(M)) \longrightarrow \cdots.
\]

Hence we get that the \(R\)-module \(\text{Ext}^j_R(R/K_{n-t-1}, H_I^{d-1}(M))\) is minimax if and only if the \(R\)-module \(\text{Ext}^j_R(R/J, H_I^{d-1}(M))\) is minimax for all \(j \geq 0\). Now, as \(x_1, \ldots, x_{n-1}\) is a filter regular sequence on \(R/I\) in \(\mathfrak{m}\), it follows from definition that

\[
x_{n-t} \notin \bigcup_{p \in \text{Ass}_R(R/K_{n-t-1}) \setminus \{\mathfrak{m}\}} p.
\]
Therefore, it follows from

\[ \text{Ass}_R(R/J) = \text{Ass}_R(R/K_{n-t-1}) \setminus \{m\} \]

that \( x_{n-t} \notin \bigcup_{p \in \text{Ass}_R(R/J)} p \), and so \( x_{n-t} \) is an \( R/J \)-regular element in \( m \). Consequently, the exactness

\[ 0 \rightarrow R/J \xrightarrow{x_{n-t}} R/J \rightarrow R/J + Rx_{n-t} \rightarrow 0, \]

implies that the sequence

\[ 0 \rightarrow \text{Ext}_R^{j-1}(R/J, H_t^{d-1}(M))/x_{n-t} \rightarrow \text{Ext}_R^{j-1}(R/J, H_t^{d-1}(M)) \rightarrow \]

\[ \rightarrow \text{Ext}_R^{j}(R/J + Rx_{n-t}, H_t^{d-1}(M)) \rightarrow \{0 : \text{Ext}_R^{j}(R/J, H_t^{d-1}(M)) \ x_{n-t}\} \rightarrow 0, \]

is exact. Since \( \text{Supp}(R/J + Rx_{n-t}) \subseteq V(K_{n-t}) \), (note that \( K_{n-t} \subseteq J + Rx_{n-t} \)) the induction hypothesis and [14 Lemma 1] yield that the \( R \)-module

\[ \text{Ext}_R^{j}(R/J + Rx_{n-t}, H_t^{d-1}(M)), \]

is minimax, for all \( j \geq 0 \). Therefore the \( R \)-module \( \{0 : \text{Ext}_R^{j}(R/J, H_t^{d-1}(M)) \ x_{n-t}\} \) is minimax. Now, let \( p \in \text{Supp} H_t^{d-1}(M) \setminus \{m\} \). Then, by [19, Proposition 5.1] and Vanishing theorem the \( R_p \)-module \( (H_t^{d-1}(M))_p \), is \( IR_p \)-cofinite. As \( I \subseteq J \subseteq J + Rx_{n-t} \), it follows from [14 Lemma 1] that the \( R_p \)-module \( (\text{Ext}_R^{j}(R/J, H_t^{d-1}(M)))_p \) is finitely generated. Moreover, as in view of [20, Corollary 3.3] the set \( \text{Supp} H_t^{d-1}(M) \) is finite, it follows that the set \( \text{Ass}_R \text{Ext}_R^{j}(R/J, H_t^{d-1}(M)) \) is finite. Therefore, by the proof of Lemma 2.11, there exists a finitely generated submodule \( L \) of \( \text{Ext}_R^{j}(R/J, H_t^{d-1}(M)) \) such that

\[ \text{Supp} \text{Ext}_R^{j}(R/J, H_t^{d-1}(M))/L \subseteq \{m\}. \]

Next, from the exact sequence

\[ 0 \rightarrow L \rightarrow \text{Ext}_R^{j}(R/J, H_t^{d-1}(M)) \rightarrow \text{Ext}_R^{j}(R/J, H_t^{d-1}(M))/L \rightarrow 0, \]

we get the exact sequence

\[ (0 : \text{Ext}_R^{j}(R/J, H_t^{d-1}(M)) \ x_{n-t}) \rightarrow (0 : \text{Ext}_R^{j}(R/J, H_t^{d-1}(M))/L \ x_{n-t}) \rightarrow \text{Ext}_R^{j}(R/Rx_{n-t}, L). \]

Hence, it yields that the \( R \)-module \( (0 : \text{Ext}_R^{j}(R/J, H_t^{d-1}(M))/L \ x_{n-t}) \) is minimax. Since \( \text{Supp}(0 : \text{Ext}_R^{j}(R/J, H_t^{d-1}(M))/L \ x_{n-t}) \subseteq \{m\} \), it follows from the definition that the \( R \)-module \( (0 : \text{Ext}_R^{j}(R/J, H_t^{d-1}(M))/L \ x_{n-t}) \) is Artinian. As \( (0 : \text{Ext}_R^{j}(R/J, H_t^{d-1}(M))/L \ x_{n-t}) \) is \( Rx_{n-t}-\)torsion, according to Melkersson [19 Proposition 1.4] \( \text{Ext}_R^{j}(R/J, H_t^{d-1}(M))/L \) is an Artinian \( R \)-module. That is \( \text{Ext}_R^{j}(R/J, H_t^{d-1}(M)) \) is a minimax \( R \)-module, as required.

**Corollary 2.13.** Let \((R, m)\) be a regular local ring of dimension \( d \geq 1 \) containing a field and let \( I \) be an ideal of \( R \). Then the \( R \)-module \( H_t^{d-1}(R) \) is \( I \)-cominimax.

**Proof.** The result follows from Theorem 2.12 and the fact that the Bass numbers of the \( R \)-module \( H_t^{d-1}(R) \) are finite (see [13, 17]).
Corollary 2.14. Let \((R, \mathfrak{m})\) be a local (Noetherian) ring and \(I\) an ideal of \(R\). Let \(M\) be a finitely generated \(R\)-module of dimension \(d\). Then the Bass numbers of the \(R\)-module \(H^d_{\mathfrak{m}}(M)\) are finite, whenever \(\mathfrak{m} \notin \text{Ass}_R H^d_{\mathfrak{m}}(M)\).

**Proof.** Since \(\text{Ass}_R(\text{Soc} H^d_{\mathfrak{m}}(M)) = V(\mathfrak{m}) \cap \text{Ass}_R H^d_{\mathfrak{m}}(M)\), it follows from \(\mathfrak{m} \notin \text{Ass}_R H^d_{\mathfrak{m}}(M)\) that \(\text{Ass}_R(\text{Soc} H^d_{\mathfrak{m}}(M)) = \emptyset\). Hence \(\text{Soc} H^d_{\mathfrak{m}}(M) = 0\), and so the result now follows from Theorem 2.12. \(\square\)

Corollary 2.15. Let \((R, \mathfrak{m})\) be a local (Noetherian) ring. Let \(I\) be an ideal of \(R\) and \(x \in \mathfrak{m}\) such that \(I \subseteq Rx\). Then for each non-zero finitely generated \(R\)-module \(M\) of dimension \(d \geq 1\), the \(R\)-module \(H^d_{I}(M)\) is \(I\)-cominimax. In particular, the Bass numbers of \(H^d_{I}(M)\) are finite.

**Proof.** When \(d = 1\) there is nothing to prove. In the case that \(d = 2\), it follows from [3, Theorem 2.3] that the \(R\)-module \(\text{Hom}_R(R/I, H^1_I(M))\) is finitely generated and so the \(R\)-module \(\text{Soc}(H^1_I(M))\) is finitely generated. Now the assertion follows from Theorem 2.12. Finally, for \(d \geq 3\), it follows from [11, Lemma 2.5] that the map \(H^{d-1}_I(M) \rightarrowtail H^d_I(M)\) is an isomorphism, and so \(\text{Soc}(H^d_I(M)) = 0\). Now the assertion again follows from Theorem 2.12. \(\square\)

Corollary 2.16. Let \((R, \mathfrak{m})\) be a regular local ring and \(I\) an ideal of \(R\) such that \(\text{height}(I) = 1\). Then for each non-zero finitely generated \(R\)-module \(M\) of dimension \(d \geq 1\), the \(R\)-module \(H^d_{I}(M)\) is \(I\)-cominimax. In particular, the Bass numbers of \(H^d_{I}(M)\) are finite.

**Proof.** Since \(R\) is a UFD and \(\text{height}(I) = 1\), it follows that \(I\) is contained in a principal prime ideal. Now the assertion follows from Corollary 2.15. \(\square\)

We are now ready to show that the subjects of the previous results can be used to prove a characterization of the finiteness of the Bass numbers of \(i\)th local cohomology of \(M\) with respect to an ideal \(I\) of dimension 2, i.e., where \(\dim R/I = 2\), in terms of the \(I\)-cominimaxness of \(H^1_I(M)\). The main result is Theorem 2.19. The following theorem will serve to shorten the proof of the main theorem.

**Theorem 2.17.** Let \((R, \mathfrak{m})\) be a local (Noetherian) ring and let \(I\) be an ideal of \(R\) with \(\dim R/I = 2\). Let \(M\) be a finitely generated \(R\)-module and \(i\) a non-negative integer. Then the following conditions are equivalent:

(i) The Bass numbers of \(H^i_I(M)\) are finite.

(ii) The \(R\)-module \(H^i_I(M)\) is \(I\)-cominimax.

**Proof.** First we show the implication (i)\(\implies\) (ii). Suppose that \(\Gamma_m(R/I) := J/I\). Then \(\text{Supp} J/I \subseteq \{\mathfrak{m}\}\), and so it follows from [14, Lemma 1] and the assumption (i) that the
$R$-module $\text{Ext}_R^j(J/I, H^1_I(M))$ is finitely generated for all $j \geq 0$. Now, the exact sequence
\[ 0 \rightarrow J/I \rightarrow R/I \rightarrow R/J \rightarrow 0 \]
induces a long exact sequence
\[ \cdots \rightarrow \text{Ext}^1_R(J/I, H^1_I(M)) \rightarrow \text{Ext}^1_R(R/J, H^1_I(M)) \rightarrow \]
\[ \rightarrow \text{Ext}^1_R(R/I, H^1_I(M)) \rightarrow \text{Ext}^1_R(J/I, H^1_I(M)) \rightarrow \cdots . \]
Therefore $H^1_I(M)$ is $I$-cominimax if and only if the $R$-module $\text{Ext}^1_R(R/J, H^1_I(M))$ is minimax for all $j \geq 0$.

It is easy to see that $\Gamma_m(R/J) = 0$. Hence, by [6, Lemma 2.1.1] $m$ contains an $R/J$-regular element $x$. Then $\dim R/J/x(R/J) = \dim R/J - 1$, and so $\dim R/J + Rx = 1$. Furthermore, the exact sequence
\[ 0 \rightarrow R/J \xrightarrow{x} R/J \rightarrow R/J + Rx \rightarrow 0 \]
gives us an exact sequence
\[ 0 \rightarrow \text{Ext}^1_R(J/I, H^1_I(M))/x \text{Ext}^1_R(J/I, H^1_I(M)) \rightarrow \]
\[ \rightarrow \text{Ext}^1_R(R/J + Rx, H^1_I(M)) \rightarrow (0 : \text{Ext}^1_R(R/J, H^1_I(M)) x) \rightarrow 0 \]
Now, since $\dim R/J + Rx = 1$, it follows from the assumption (i) and Theorem 2.3 that the $R$-module $\text{Ext}^1_R(R/J + Rx, H^1_I(M))$ is minimax, for all $j \geq 0$. On the other hand, in view of Proposition 2.11, there exists a finitely generated submodule $L$ of $\text{Ext}^1_R(R/J, H^1_I(M))$ such that $\text{Supp}(\text{Ext}^1_R(R/J, H^1_I(M))/L) \subseteq \{m\}$. Now, from the exact sequence
\[ 0 \rightarrow L \rightarrow \text{Ext}^1_R(R/J, H^1_I(M)) \rightarrow \text{Ext}^1_R(R/J, H^1_I(M))/L \rightarrow 0 \]
we obtain the exact sequence
\[ (0 : \text{Ext}^1_R(R/J, H^1_I(M)) x) \rightarrow (0 : \text{Ext}^1_R(R/J, H^1_I(M))/L x) \rightarrow \text{Ext}^1_R(R/Rx, L), \]
which implies that the $R$-module $(0 : \text{Ext}^1_R(R/J, H^1_I(M))/L x)$ is minimax. As
\[ \text{Supp}(0 : \text{Ext}^1_R(R/J, H^1_I(M))/L x) \subseteq \{m\}, \]
it follows from the definition that the $R$-module $(0 : \text{Ext}^1_R(R/J, H^1_I(M))/L x)$ is Artinian. Since $(0 : \text{Ext}^1_R(R/J, H^1_I(M))/L x)$ is $Rx$-torsion, it follows from Melkersson’s result (see [19, Proposition 1.4]) that $\text{Ext}^1_R(R/J, H^1_I(M))/L$ is an Artinian $R$-module, and so $\text{Ext}^1_R(R/J, H^1_I(M))$ is a minimax $R$-module, as required.

For prove (ii)$\implies$(i) first observe that the $R$-module $\text{Ext}^i_R(R/m, H^1_I(M))$ is minimax, for all $j \geq 0$, and so by the definition, $\text{Ext}^i_R(R/m, H^1_I(M))$ has finite length. Therefore $\mu^j(m, H^1_I(M))$ is finite, for all $j \geq 0$. Now, let $p \in \text{Supp} H^1_I(M) \setminus V(m)$. Then $\dim R_p/IR_p \leq 1$ as easily seen. Therefore the $R_p$-module of $H^1_I(M)_p$ is $IR_p$-cofinite, and so the Bass numbers of $H^1_I(M)_p$ are finite. Since the Bass numbers are stable under the localization, it follows that the Bass numbers of $H^1_I(M)$ are finite, as required. \qed
The next result is an immediate consequence of Theorem 2.17 and Huneke-Sharp-Lyubeznik’ results.

**Corollary 2.18.** Let \((R, \mathfrak{m})\) be a regular local ring of dimension \(d \geq 2\) containing a field and let \(I\) be an ideal of \(R\) with \(\dim R/I = 2\). Then the \(R\)-module \(H^i_I(R)\) is \(I\)-cominimax for all \(i \geq 0\).

**Proof.** The result follows from Theorem 2.17 and the fact that the Bass numbers of the \(R\)-module \(H^i_I(R)\) are finite (see \([13, 17]\)), for all \(i \geq 0\). \(\square\)

We are now ready to state and prove the third main theorem of the paper, which is a characterization of the finiteness of the Bass numbers of \(i\)-th local cohomology of \(M\) with respect to an ideal \(I\) of dimension 2, in terms of the \(I\)-cominimaxness of \(H^i_I(M)\).

**Theorem 2.19.** Let \((R, \mathfrak{m})\) be a local (Noetherian) ring, and let \(I\) be an ideal of \(R\) such that \(\dim R/I = 2\). Let \(M\) be a finitely generated \(R\)-module and \(t\) a non-negative integer. Then the following conditions are equivalent:

(i) \(\text{Soc} H^i_I(M)\) is finitely generated for all \(i \leq t\).

(ii) \(\text{Ext}^j_{R}(R/\mathfrak{m}, H^i_I(M))\) is finitely generated for all \(j \geq 0\) and for all \(i \leq t - 1\).

(iii) \(H^1_I(M)\) is \(I\)-cominimax, for all \(i \leq t - 1\).

**Proof.** The implication (ii)\(\implies\) (i) follows from \([13, \text{Corollary 3.5}]\). In order to prove (i)\(\implies\) (ii), as \(\text{Ass}_R H^1_I(M)\) is finite (see \([5, \text{Corollary 4.3}]\)), we can find an element \(x \in \mathfrak{m}\), such that \(x \notin \bigcup_{p \in \text{Ass}_R(R/I)} p\), and \(V(Rx) \cap (\bigcup_{i=0}^t \text{Ass}_R H^1_I(M)) \subseteq V(\mathfrak{m})\).

Now, let \(L := I + Rx\). Then it is easy to see that \(\dim R/L = 1\), and

\[
H^0_L(H^1_I(M)) = H^0_{R\mathfrak{m}}(H^1_I(M)) = H^0_{\mathfrak{m}}(H^1_I(M)),
\]

is an Artinian \(R\)-module for all \(i \leq t\), by \([19, \text{Proposition 4.1}]\) and assumption (i). Now, for all \(i \leq t - 1\), there exists the short exact sequence

\[
0 \rightarrow H^1_{R\mathfrak{m}}(H^1_I(M)) \rightarrow H^1_{L}(H^1_I(M)) \rightarrow H^0_{R\mathfrak{m}}(H^1_I(M)) \rightarrow 0,
\]

(cf. for instance \([21, \text{Corollary 3.5}]\)). Since \(H^1_I(M)\) is \(L\)-cofinite and

\[
H^0_{R\mathfrak{m}}(H^1_I(M)) = H^0_{\mathfrak{m}}(H^1_I(M))
\]

is Artinian, it follows from \([5, \text{Lemma 2.1}]\) that \(\text{Ext}^j_R(R/L, H^1_{R\mathfrak{m}}(H^1_I(M)))\) is minimax for all \(j \geq 0\) and \(i \leq t - 1\). Furthermore, since \(H^1_{R\mathfrak{m}}(H^1_I(M)) = H^1_K(H^1_I(M))\), for all \(k \geq 0\), it follows that \(H^k_K(H^1_I(M)) = 0\) for all \(k \geq 2\). Hence \(\text{Ext}^j_R(R/L, H^1_K(H^1_I(M)))\) is minimax for all \(j \geq 0\), \(k \geq 0\) and \(i \leq t - 1\), and so in view of \([19, \text{Proposition 3.9}]\), the \(R\)-module \(\text{Ext}^j_R(R/L, H^1_I(M))\) is minimax for all \(j \geq 0\) and \(i \leq t - 1\). Consequently, by \([2, \text{Theorem 2.7}]\) the \(R\)-module \(K_{i,j} := \text{Ext}^j_R(R/\mathfrak{m}, H^1_I(M))\) is minimax for all \(j \geq 0\) and \(i \leq t - 1\). Hence there is a finitely generated submodule \(N_{i,j}\) of \(K_{i,j}\), such that \(K_{i,j}/N_{i,j}\) is Artinian. As \(\mathfrak{m}K_{i,j} = 0\) and \(\mathfrak{m}(K_{i,j}/N_{i,j}) = 0\), it follows that \(K_{i,j}\) has finite length, as required.

For proving (ii)\(\implies\)(iii), first we use the proof of implication (ii)\(\implies\)(i) in Theorem 2.17, to see that the Bass numbers of \(H^1_I(M)\) are finite, for all \(i \leq t - 1\). Now, the
assertion follows from Theorem 2.17. Finally the implication \( (iii) \Rightarrow (ii) \) follows again from Theorem 2.17. \(\square\)

We end the paper by drawing some consequences of Theorem 2.19. The first consequence will help us to construct a certain subset of \(\text{Spec} \ R\) which is countable.

**Corollary 2.20.** Let \((R, \mathfrak{m})\) be a local (Noetherian) ring and let \(I\) be an ideal of \(R\) with \(\dim R/I = 2\). Let \(M\) be a finitely generated \(R\)-module such that \(\mathfrak{m} \not\in \bigcup_{i \geq 0} \text{Ass}_R(H_i^j(M))\). Then \(\text{Ext}^3_R(R/\mathfrak{m}, H_i^j(M))\) is finitely generated for all \(j \geq 0\) and all \(i \geq 0\).

**Proof.** Since \(\mathfrak{m} \not\in \bigcup_{i \geq 0} \text{Ass}_R(H_i^j(M))\), it follows that \(\text{Soc} H_i^j(M) = 0\) for all \(i \geq 0\). Now the assertion follows from Theorem 2.19. \(\square\)

**Proposition 2.21.** Let \(R\) be a Noetherian ring and \(I\) an ideal of \(R\) such that \(\dim R/I = 2\). For a finitely generated \(R\)-module \(M\), let

\[
\Sigma := \{ p \in \text{Spec} (R) : \mu^j(p, H_i^j(M)) = \infty \text{ for some integers } i \geq 0 \text{ and } j \geq 0 \}.
\]

Then \(\Sigma\) is countable and \(\Sigma \subseteq \text{Max}(R)\).

**Proof.** In order to show that \(\Sigma \subseteq \text{Max}(R)\), let \(p \in \Sigma \setminus \text{Max}(R)\). Then it follows from the definition that \(p \in V(I)\), and it is easy to see that \(\text{height}(p/I) \leq 1\). Hence it follows from [5, Corollary 2.10] that \(p \notin \Sigma\), which is a contradiction. Thus \(\Sigma \subseteq \text{Max}(R)\).

On the other hand, as for all integers \(n \geq 1\) and \(i \geq 0\), the sets \(\text{Ass}_R \text{Ext}^j_R(R/I^n, M)\) are finite and \(H_i^j(M) = \lim_{n \geq 1} \text{Ext}^j_R(R/I^n, M)\), it follows that the set \(\bigcup_{i \geq 0} \text{Ass}_R H_i^j(M)\) is countable. Now, in order to complete the proof, it is enough to show that \(\Sigma \subseteq \bigcup_{i \geq 0} \text{Ass}_R H_i^j(M)\). To do this suppose that the contrary is true. Then there is \(\mathfrak{m} \in \Sigma \setminus \bigcup_{i \geq 0} \text{Ass}_R H_i^j(M)\). Hence, by [5, Corollary 2.10] we have \(\text{height}(\mathfrak{m}/I) = 2\) and so it follows from Corollary 2.20 that \(\mathfrak{m} \not\in \Sigma\), which is a contradiction. \(\square\)

**Corollary 2.22.** Let \(R\) be a Noetherian ring, \(M\) a finitely generated \(R\)-module and \(I\) an ideal of \(R\) such that \(\dim M/IM = 2\). Let

\[
\Sigma := \{ p \in \text{Spec} (R) : \mu^j(p, H_i^j(M)) = \infty \text{ for some integers } i \geq 0 \text{ and } j \geq 0 \}.
\]

Then \(\Sigma\) is countable and \(\Sigma \subseteq \text{Max}(R)\).

**Proof.** As \(\dim M/IM = 2\) it follows from that \(\dim R/I + \text{Ann}_R M = 2\). Now the result follows from \(H_i^j(M) \cong H_i^j + \text{Ann}_R (M)\) and Proposition 2.21. \(\square\)

**Corollary 2.23.** Let \((R, \mathfrak{m})\) be a local (Noetherian) ring, and let \(I\) be an ideal of \(R\) such that \(\dim R/I = 3\). Let \(M\) be a finitely generated \(R\)-module. Then the set

\[
\Sigma := \{ p \in \text{Spec} (R) : \mu^j(p, H_i^j(M)) = \infty \text{ for some integers } i \geq 0 \text{ and } j \geq 0 \},
\]

is countable.
Proof. Let \( p \in \Sigma \setminus \{m\} \). Then by [5, Corollary 2.10] we have \( \text{height}(p/I) = 2 \), and so it follows from Corollary 2.20 that \( p \in \bigcup_{i \geq 0} \text{Ass}_R H^i_I(M) \). Thus \( \Sigma \setminus \{m\} \subseteq \bigcup_{i \geq 0} \text{Ass}_R H^i_I(M) \), and so the set \( \Sigma \) is countable. \( \square \)

Corollary 2.24. Let \( (R, m) \) be a local (Noetherian) ring, \( M \) a finitely generated \( R \)-module and \( I \) an ideal of \( R \) such that \( \text{dim} M/IM = 3 \). Then the set
\[
\Sigma := \{ p \in \text{Spec}(R) : \mu^i(p, H^j_I(M)) = \infty \text{ for some integers } i \geq 0 \text{ and } j \geq 0 \}
\]
is countable.

Proof. As \( \text{dim} M/IM = 3 \) it follows that \( \text{dim} R/I + \text{Ann}_R(M) = 3 \). Now the result follows from \( H^i_I(M) \cong H^i_{I+\text{Ann}_R(M)}(M) \) and Corollary 2.23 . \( \square \)

Acknowledgments

The authors are deeply grateful to the referee for a very careful reading of the manuscript and many valuable suggestions. We also would like to thank Professors Hossein Zakeri and Kamran Divaani-Aazar for their careful reading of the first draft and many helpful suggestions.

References

[1] N. Abazari and K. Bahmanpour, On the finiteness of Bass numbers of local cohomology modules, J. Alg. Appl. 10(2011), 783-791.
[2] J. Azami, R. Naghipour and B. Vakili, Finiteness properties of local cohomology modules for \( \alpha \)-minimax modules, Proc. Amer. Math. Soc. 137(2009), 439-448.
[3] K. Bahmanpour and R. Naghipour, On the cofiniteness of local cohomology modules, Proc. Amer. Math. Soc. 136(2008), 2359-2363.
[4] K. Bahmanpour and R. Naghipour, Associated primes of local cohomology modules and Matlis duality, J. Algebra, 320(2008), 2632-2641.
[5] K. Bahmanpour and R. Naghipour, Cofiniteness of local cohomology modules for ideals of small dimension, J. Algebra, 321(2009), 1997-2011.
[6] M.P. Brodmann and R.Y. Sharp, Local cohomology; an algebraic introduction with geometric applications, Cambridge University Press, Cambridge,1998.
[7] D. Delfino and T. Marley, Cofinite modules and local cohomology, J. Pure and Appl. Algebra 121(1997), 45-52.
[8] K. Divaani-Aazar and A. Mafi, Associated primes of local cohomology modules, Proc. Amer. Math. Soc. 133(2005), 655-660.
[9] K. Divaani-Aazar and A. Mafi, Associated primes of local cohomology modules of weakly Laskerian modules, Comm. Algebra 34(2006), 681-690.
[10] A. Grothendieck, Local cohomology, Notes by R. Hartshorne, Lecture Notes in Math., 862, Springer, New York, 1966.
[11] R. Hartshorne, Affine duality and cofiniteness, Invent. Math. 9(1970), 145-164.
[12] C. Huneke, Problems on local cohomology, Free resolutions in commutative algebra and algebraic geometry, Res. Notes Math. 2(1992), 93-108.
[13] C. Huneke and R.Y. Sharp, Bass numbers of local cohomology modules, Trans. Amer. Soc. 339(1993), 765-779.
[14] K.I. Kawasaki, On the finiteness of Bass numbers of local cohomology modules, Proc. Amer. Math. Soc. 124(1996), 3275-3279.
[15] K. Khashyarmanesh, On the finiteness properties of extension and torsion functors of local cohomology modules, Proc. Amer. Math. Soc. 135 (2007), 1319-1327.
[16] G. Lyubeznik, Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra), Invent. Math. 113(1993), 41-55.
[17] G. Lyubeznik, F-modules: applications to local cohomology modules and D-modules in characteristic p > 0, J. Reine Angew. Math. 491(1997), 65-130.
[18] H. Matsumura, Commutative ring theory, Cambridge Univ. Press, Cambridge, UK, 1986.
[19] L. Melkersson, Modules cofinite with respect to an ideal, J. Algebra, 285(2005), 649-668.
[20] R. Naghipour and M. Sedghi, A characterization of Cohen-Macaulay modules and local cohomology, Arch. Math., 87(2006), 303-308.
[21] P. Schenzel, Preregular sequences, local cohomology, and completion, Math. Scand. 92 (2003), 161-180.
[22] P. Schenzel, N.V. Trung and N.T. Cuong, Verallgemeinerte Cohen-Macaulay-Module, Math. Nachr. 85 (1978), 57-73.
[23] H. Zöschinger, Minimax moduln, J. Algebra. 102 (1986), 1-32.
[24] H. Zöschinger, Über die maximalbedingung für radikalvolle untermoduln, Hokkaido Math. J. 17 (1988), 101-116.

Department of Mathematics, Ardabil Branch, Islamic Azad University, Ardabil, Iran.
E-mail address: bahmanpour.k@gmail.com

Department of Mathematics, University of Tabriz, Tabriz, Iran; and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran.
E-mail address: naghipour@ipm.ir
E-mail address: naghipour@tabrizu.ac.ir

Department of Mathematics, Azarbaijan University of Shahid Madani, Tabriz, Iran.
E-mail address: sedghi@azaruniv.edu