Parseval frames with $n+1$ vectors in $\mathbb{R}^n$

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Abstract

We prove a uniqueness theorem for triangular Parseval frame with $n+1$ vectors in $\mathbb{R}^n$. We also provide a characterization of unit-norm frames that can be scaled to a Parseval frame.

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1 Introduction

Let $B = \{v_1, ..., v_N\}$ be a set of vectors in $\mathbb{R}^n$. We say that $B$ is a frame if it contains a basis of $\mathbb{R}^n$, or equivalently, if there exist constants $A$, $B > 0$ for which $A||v||^2 \leq \sum_{j=1}^{N} <v,v_j>^2 \leq B||v||^2$ for every $v \in \mathbb{R}^n$. Here and throughout the paper, $<,>$ and $||,||$ are the usual scalar product and norm in $\mathbb{R}^n$. In general $A < B$, but we say that a frame is tight if $A = B$, and is Parseval if $A = B = 1$.

Parseval frames are nontrivial generalizations of orthonormal bases. Vectors in a Parseval frame are not necessarily orthogonal or linearly independent, and do not necessarily have the same length, but the Parseval identities $v = \sum_{j=1}^{N} <v,v_j>v_j$ and $||v||^2 = \sum_{j=1}^{N} <v,v_j>^2$ still hold. In the applications, frames are more useful than bases because they are resilient against the corruptions of additive noise and quantization, while providing numerically stable reconstructions ([1], [7], [9]). Appropriate frame decomposition may reveal hidden signal characteristics, and have been employed as detection devices. Specific types of finite tight frames have been studied to solve problems in information theory. The references are too many to cite, but see [3], the recent book [1] and the references cited there.

In recent years, several inquiries about tight frames have been raised. In particular: how to characterize Parseval frames with $N$ elements in $\mathbb{R}^n$ (or Parseval $N$ frames), and whether it is possible to scale a given frame so that the resulting frame is Parseval.

Following [2] and [11], we say that a frame $B = \{v_1, ..., v_N\}$ is scalable if there exists positive constants $\ell_1, ..., \ell_N$ such that $\{\ell_1v_1, ..., \ell_Nv_N\}$ is a Parseval frame. Two Parseval $N$-frames are equivalent if one can be transformed into the other with a rotation of coordinates and the reflection of one or more vectors. A frame is nontrivial if no two vectors are parallel. In
the rest of the paper, when we say "unique" we will always mean "unique up to an equivalence", and we will often assume without saying that frames are nontrivial.

It is well known that Parseval n-frames are orthonormal (see also Corollary 3.3). Consequently, for given unit vector $w$, there is a unique Parseval n-frame that contains $w$. If $||w|| \neq 1$, no Parseval n-frame contains $w$.

When $N > n$ and $||w|| \leq 1$, there are infinitely many non-equivalent Parseval $N$-frames that contain $w$. By the main theorem in [5], it is possible to construct a Parseval frame $\{v_1, ..., v_n\}$ with vectors of prescribed lengths $0 < \ell_1, ..., \ell_N \leq 1$ that satisfy $\ell_1^2 + ... + \ell_N^2 = n$. We can let $\ell_N = ||w||$ and, after a rotation of coordinates, assume that $v_N = w$, thus proving that the Parseval frames that contain $w$ are as many as the sets of constants $\ell_1, ..., \ell_{N-1}$.

But when $N = n + 1$, there is a class of Parseval frames that can be uniquely constructed from a given vector: precisely, all triangular frames, that is, frames $\{v_1, ..., v_N\}$ such that the matrix $(v_1, ..., v_n)$ whose columns are $v_1, ..., v_n$ is right triangular. We recall that a matrix $\{a_{i,j}\}_{1 \leq i,j \leq n}$ is right-triangular if $a_{i,j} = 0$ if $i > j$.

The following theorem will be proved in Section 3.

**Theorem 1.1.** Let $B = \{v_1, ..., v_n, w\}$ be a triangular Parseval frame, with $||w|| < 1$. Then $B$ is unique, in the sense that if $B' = \{v'_1, ..., v'_n, w\}$ is another triangular Parseval frame, then $v'_j = \pm v_j$.

Every frame is equivalent, through a rotation of coordinates $\rho$, to a triangular frame, and so Theorem 1.1 implies that every Parseval $(n+1)$-frame that contains a given vector $w$ is equivalent to one which is uniquely determined by $\rho(w)$. However, that does not imply that the frame itself is uniquely determined by $w$ because the rotation $\rho$ depends also on the other vectors of the frame.

We also study the problem of determining whether a given frame $B = \{v_1, ..., v_n, v_{n+1}\} \subset \mathbb{R}^n$ is scalable or not. Assume $||v_j|| = 1$, and let $\theta_{i,j} \in [0, \pi)$ be the angle between $v_i$ and $v_j$.

If $B$ contains an orthonormal basis, then the problem has no solution, so we assume that this not the case. We prove the following

**Theorem 1.2.** $B$ is scalable if and only there exist constants $\ell_1, ..., \ell_{n+1}$ such that for every $i \neq j$

$$
(1 - \ell_i^2)(1 - \ell_j^2) = \ell_i^2 \ell_j^2 \cos \theta_{i,j}^2. 
$$

(1.1)

1We are indebted to P. Casazza for this remark.
The identity (1.1) has several interesting consequences (see corollary 3.2). First of all, it shows that \( \ell_2^2 \leq 1 \); if \( \ell_i = 1 \) for some \( i \), we also have \( \cos \theta_{i,j} = 0 \) for every \( j \), and so \( v_i \) is orthogonal to all other vectors. This interesting fact is also true for other Parseval frames, and is a consequence of the following

**Theorem 1.3.** Let \( \mathcal{B} = \{v_1, \ldots, v_N\} \) be a Parseval frame. Let \( ||v_j|| = \ell_j \).

Then
\[
\sum_{j=i}^N \ell_j^2 \cos^2 \theta_{ij} = 1 \quad \sum_{j=i}^N \ell_j^2 \sin^2 \theta_{ij} = n - 1 \quad (1.2)
\]

The identities (1.2) are probably known, but we did not find a reference in the literature. It is worthwhile to remark that from (1.2) follows that
\[
\sum_{j=i}^N \ell_j^2 \cos^2 \theta_{ij} - \sum_{j=i}^N \ell_j^2 \sin^2 \theta_{ij} = \sum_{j=i}^N \ell_j^2 \cos(2 \theta_{ij}) = 2 - n.
\]

When \( n = 2 \), this identity is proved in Proposition 2.1.

Another consequence of Theorem 1.1 is the following

**Corollary 1.4.** If \( \mathcal{B} = \{v_1, \ldots, v_n, v_{n+1}\} \) is a scalable frame, then, for every \( 1 \leq i \leq n+1 \), and every \( j \neq k \neq i \) and \( k' \neq j' \neq i \),
\[
\frac{\left| \cos \theta_{k,j} \right|}{\cos \theta_{k,j} + \cos \theta_{k,i} \cos \theta_{j,i}} = \frac{\left| \cos \theta_{k',j'} \right|}{\cos \theta_{k',j'} + \cos \theta_{k',i} \cos \theta_{j',i}} \quad (1.3)
\]

We prove Theorems 1.1, 1.2 and 1.3 and their corollaries in Section 3. In Section 2 we prove some preliminary results and lemmas.

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### 2 Preliminaries

We refer to [4] or to [10] for the definitions and basic properties of finite frames.

We recall that \( \mathcal{B} = \{v_1, \ldots, v_N\} \) is a Parseval frames in \( \mathbb{R}^n \) if and only if rows of the matrix \( (v_1, \ldots, v_N) \) are orthonormal. Consequently, \( \sum_{i=1}^N ||v_i||^2 = n \). If the vectors in \( \mathcal{B} \) have all the same length, then \( ||v_i|| = \sqrt{n/N} \). See e.g. [4].
We will often let $\vec{e}_1 = (1, 0, \ldots, 0), \ldots, \vec{e}_n = (0, \ldots, 0, 1)$, and we will denote by $(v_1, \ldots, \hat{v}_k, \ldots, v_N)$ the matrix with the column $v_k$ removed.

To the best of our knowledge, the following proposition is due to P. Casazza (unpublished-2000) but can also be found in [8] and in the recent preprint [6].

**Proposition 2.1.** $B = \{v_1, \ldots, v_N\} \subset \mathbb{R}^2$ is a tight frame if and only if for some index $i \leq N$,

$$\sum_{j=1}^{N} ||v_j||^2 e^{2i\theta_{i,j}} = 0. \quad (2.1)$$

It is easy to verify that if (2.1) is valid for some index $i$, then it is valid for all other $i$'s.

**Proof.** Let $\ell_j = ||v_j||$. After a rotation, we can let $v_i = v_1 = (\ell_1, 0)$ and $\theta_{1,j} = \theta_j$, so that $v_j = (\ell_j \cos \theta_j, \ell_j \sin \theta_j)$.

$B$ is a tight frame with frame constant $A$ if and only if the rows of the matrix $(v_1, \ldots, v_N)$ are orthogonal and have length $A$. That implies

$$\sum_{j=1}^{N} \ell_j^2 \cos^2 \theta_j = \sum_{j=1}^{N} \ell_j^2 \sin^2 \theta_j = A \quad (2.2)$$

and

$$\sum_{j=1}^{N} \ell_j^2 \cos \theta_j \sin \theta_j = 0. \quad (2.3)$$

From (2.2) follows that $\sum_{j=1}^{N} \ell_j^2 (\cos^2 \theta_j - \sin^2 \theta_j) = \sum_{j=1}^{N} \ell_j^2 \cos(2\theta_j) = 0$, and from (2.3) that $\sum_{j=1}^{N} \ell_j^2 \sin(2\theta_j) = 0$, and so we have proved (2.1).

If (2.1) holds, then (2.2) and (2.3) hold as well, and from these identities follows that $B$ is a tight frame.

$\square$

**Corollary 2.2.** Let $B = \{v_1, v_2, v_3\} \subset \mathbb{R}^2$ be a tight frame. Assume that the $v_i$'s have all the same length. Then, $\theta_{1,2} = \pi/3$, and $\theta_{1,3} = 2\pi/3$.

So, every such frame $B$ is equivalent to a dilation of the "Mercedes-Benz frame" $\{(1, 0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2})\}$.
Proof. Let \( v_1 = (1, 0) \), and \( \theta_{1,i} = \theta_i \) for simplicity. By Proposition 2.1, 
\( 1 + \cos(2\theta_2) + \cos(2\theta_3) = 0 \), and \( \sin(2\theta_2) + \sin(2\theta_3) = 0 \). It is easy to verify that these equations are satisfied only when \( \theta_2 = \frac{\pi}{3} \) and \( \theta_3 = \frac{2\pi}{3} \) or vice versa.

The following simple proposition is a special case of Theorem 1.1, and will be a necessary step in the proof.

**Lemma 2.3.** Let \( w = (\alpha_1, \alpha_2) \) be given. Assume \( ||w|| < 1 \). There exists a unique nontrivial Parseval frame \( \{v_1, v_2, w\} \subset \mathbb{R}^2 \), with \( v_1 = (a_{11}, 0) \), \( v_2 = (a_{12}, a_{22}) \), and \( a_{1,1}, a_{2,2} > 0 \).

**Proof.** We find \( a_{1,1}, a_{1,2} \) and \( a_{2,2} \) so that the rows of the matrix
\[
\begin{pmatrix}
  a_{11} & a_{12} & \alpha_1 \\
  0 & a_{22} & \alpha_2
\end{pmatrix}
\]
are orthonormal. That is,
\[
\alpha_1^2 + a_{1,1}^2 + a_{1,2}^2 = 1, \quad \alpha_2^2 + a_{2,2}^2 = 1, \quad \alpha_1 a_2 + a_{1,2} a_{2,2} = 0.
\]  
(2.4)

From the second equation, \( a_{2,2} = \pm \sqrt{1 - \alpha_2^2} \); if we can chose \( a_{2,2} > 0 \), from the third equation we obtain \( a_{1,2} = -\frac{\alpha_1 a_2}{\sqrt{1-a_2^2}} \) and from the first equation
\[
a_{1,1}^2 = 1 - a_1^2 - a_{12}^2 = 1 - a_1^2 - \frac{\alpha_1^2 a_2^2}{1-a_2^2} = \frac{1-a_1^2 - a_2^2}{1-a_2^2}.
\]

Note that \( a_{1,1}^2 > 0 \) because \( ||w||^2 = \alpha_1^2 + \alpha_2^2 < 1 \). We can chose then
\[
a_{1,1} = \sqrt{1 - a_2^2 - \alpha_1^2}.
\]

Note also that \( v \) and \( v_2 \) cannot be parallel; otherwise, \( \frac{a_{1,2}}{a_{2,2}} = -\frac{\alpha_1 a_2}{1-a_2^2} = \frac{\alpha_1}{\alpha_2} \iff -a_2^2 = 1 - a_2^2 \), which is not possible. \( \Box \)

**Remark.** The proof shows that \( v_1 \) and \( v_2 \) are uniquely determined by \( w \). It shows also that if \( ||w|| = 1 \), then \( a_{1,1} = 0 \), and consequently \( v_1 = 0 \).

### 3 Proofs

In this section we prove Theorem 1.1 and some of its corollaries.
Proof of Theorem 1.1. Let \( w = (\alpha_1, \ldots, \alpha_n) \). We construct a nontrivial Parseval frame \( \mathcal{M} = \{v_1, \ldots, v_n, w\} \subset \mathbb{R}^n \) with the following properties: the matrix \((v_1, \ldots, v_n) = \{a_{i,j}\}_{1 \leq i, j \leq n}\) is right triangular, and

\[
a_{j,j} = \begin{cases} \sqrt{1 - \alpha_j^2} & \text{if } j = n \\ \sqrt{1 - \sum_{k=j}^{n} a_k^2} / \sqrt{1 - \sum_{k=j+1}^{n} \alpha_k^2} & \text{if } 1 \leq j < n. \end{cases}
\]  

(3.1)

The proof will show that \( \mathcal{M} \) is unique, and also that the assumption that \( \|w\| < 1 \) is necessary in the proof.

To construct the vectors \( v_j \) we argue by induction on \( n \). When \( n = 2 \) we have already proved the result in Lemma 2.3. We now assume that the lemma is valid in dimension \( n - 1 \), and we show that it is valid also in dimension \( n \).

Let \( \tilde{w} = (\alpha_2, \ldots, \alpha_n) \). By assumptions, there exist vectors \( \tilde{v}_2, \ldots, \tilde{v}_n \) such that the set \( \mathcal{M} = \{\tilde{v}_2, \ldots, \tilde{v}_n, \tilde{w}\} \subset \mathbb{R}^{n-1} \) is a Parseval frame, and the matrix \((\tilde{v}_2, \ldots, \tilde{v}_n, \tilde{w})\) is right triangular and invertible. If we assume that the elements of the diagonal are positive, the \( \tilde{v}_j \)'s are uniquely determined by \( w \). We let \( \tilde{v}_j = (a_{2,j}, \ldots, a_{n,j}) \), with \( a_{k,j} = 0 \) if \( k < j \) and \( a_{j,j} > 0 \).

We show that \( \mathcal{M} \) is the projection on \( \mathbb{R}^{n-1} \) of a Parseval frame in \( \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} \) that satisfies the assumption of the theorem. To this aim, we prove that there exist scalars \( x_1, \ldots, x_n \) so that the vectors \( \{v_1, \ldots, v_{n+1}\} \) which are defined by

\[
v_1 = (x_1, 0, \ldots, 0), \quad v_j = (x_j, \tilde{v}_j) \text{ if } 2 \leq j \leq n, \quad v_{n+1} = w \quad (3.2)
\]

form a Parseval frame of \( \mathbb{R}^n \). The proof is in various steps: first, we construct a unit vector \((y_2, \ldots, y_{n+1})\) which is orthogonal to the rows of the matrix \((\tilde{v}_2, \ldots, \tilde{v}_n, \tilde{w})\). Then, we show that there exists \(-1 < \lambda < 1\) so that \( \lambda y_{n+1} = \alpha_1 \). Finally, we chose \( x_1 = \sqrt{1 - \lambda^2} \), \( x_j = \lambda y_j \), and we prove that the vectors \( v_1, \ldots, v_{n+1} \) defined in (3.2) form a Parseval frame that satisfies the assumption of the lemma.

First of all, we observe that \( \{v_1, \ldots, v_{n+1}\} \) is a Parseval frame if and only if \( \bar{x} = (x_1, x_2, \ldots, x_n, \alpha_1) \) is a unit vector that satisfies the orthogonality conditions:

\[
(\tilde{v}_2, \ldots, \tilde{v}_n, \tilde{w})\bar{x} = \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2,n} & \alpha_2 \\ 0 & a_{33} & \cdots & a_{3,n} & \alpha_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n,n} & \alpha_n \end{pmatrix} \begin{pmatrix} x_2 \\ \vdots \\ x_n \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (3.3)
\]
By a well known formula of linear algebra, the vector
\[ \vec{y} = y_2 \vec{e}_2 + \ldots + y_{n+1} \vec{e}_{n+1} = \det \begin{pmatrix} \vec{e}_2 & \vec{e}_3 & \ldots & \vec{e}_n & \vec{e}_{n+1} \\ a_{2,2} & a_{2,3} & \ldots & a_{2,n} & \alpha_2 \\ 0 & a_{3,3} & \ldots & a_{3,n} & \alpha_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & a_{n-1,n} & \alpha_{n-1} \\ 0 & 0 & \ldots & a_{n,n} & \alpha_n \end{pmatrix} \] (3.4)
is orthogonal to the rows of the matrix in (3.3), and so it is a constant multiple of \( \vec{x} \). That is, \( \vec{x} = \lambda \vec{y} \) for some \( \lambda \in \mathbb{R} \).

Let us prove that \( ||\vec{y}|| = 1 \). The rows of the matrix \( (\vec{v}_2, \ldots, \vec{v}_n, \vec{w}) \) are orthonormal, and so after a rotation \( (\vec{v}_2, \ldots, \vec{v}_n, \vec{w}) = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \end{pmatrix} \). (3.5)
The formula in (3.4) applied with the matrix in (3.5) produces the vector \( \vec{e}_1 = (1, 0, \ldots, 0) \). Thus, \( \vec{y} \) in (3.4) is a rotation of \( \vec{e}_1 \), and so it is a unit vector as well.

We now prove that \( |\lambda| < 1 \). From \( \vec{x} = (x_2, \ldots, x_n, \alpha_1) = \lambda(y_2, \ldots, y_n, y_{n+1}) \), we obtain \( \lambda = \alpha_1/y_{n+1} \). By (3.4),
\[ y_{n+1} = (-1)^{n+1} \det \begin{pmatrix} a_{2,2} & a_{2,3} & \ldots & a_{2,n-1} & a_{2,n} \\ 0 & a_{3,3} & \ldots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \ldots & 0 & a_{n,n} \end{pmatrix} = (-1)^{n+1} \prod_{j=2}^{n} a_{j,j}, \]
Recalling that by (3.1), \( a_{j,j} = \sqrt{1 - \sum_{k=j+1}^{n} \alpha_k^2} \), we can see at once that
\[ y_{n+1} = (-1)^{n+1} \prod_{j=2}^{n} a_{j,j} = (-1)^{n+1} \sqrt{1 - \alpha_2^2 - \ldots - \alpha_{n-1}^2 - \alpha_n^2} \]
\[ = (-1)^{n+1} \sqrt{1 - ||w||^2 + \alpha_1^2}. \] (3.6)
In view of $\lambda y_{n+1} = \alpha_1$, we obtain

$$\lambda = (-1)^{n+1} \frac{\alpha_1}{\sqrt{1-||w||^2 + \alpha_1^2}}.$$  

Clearly, $|\lambda| < 1$ because $||w|| < 1$. We now let

$$x_1 = \sqrt{1 - \lambda^2} = \frac{\sqrt{1-||w||^2}}{\sqrt{1-||w||^2 + \alpha_1^2}}, \quad (3.7)$$

and we define the $v_j$’s as in (3.2). The first rows of the matrix $(v_1, ..., v_{n+1})$ is $(\sqrt{1-\lambda^2}, \bar{x}) = (\sqrt{1-\lambda^2}, \lambda \bar{y})$, and so it is unitary and perpendicular to the other rows. Therefore, the $\{v_j\}$ form a tight frame that satisfies the assumption of the theorem.

The proof of Theorem 1.1 shows the following interesting fact: By (3.6) and (3.7)

$$\det(v_1, ..., v_n) = \prod_{j=1}^{n} a_{jj} = x_1 \prod_{j=2}^{n} a_{jj} = \sqrt{1-||w||^2}.$$  

This formula does not depend on the fact that $(v_1, ..., v_n)$ is right triangular, because every $n \times n$ matrix can be reduced in this form with a rotation that does not alter its determinant and does not alter the norm of $w$. This observation proves the following

**Corollary 3.1.** Let $\{w_1, ..., w_{n+1}\}$ be a Parseval frame. Then,

$$\det(w_1, ..., \hat{w}_j, ..., w_{n+1}) = \pm \sqrt{1-||w_j||^2}.$$

**Proof of Theorem 1.2.** Let $\ell_j = ||v_j||$. If $\{v_1, ..., v_{n+1}\}$ is a Parseval frame, then the rows of the matrix $B = (v_1, ..., v_{n+1})$ are orthonormal. While proving Theorem 1.1, we have constructed a vector $\vec{x} = (x_1, ..., x_{n+1})$, with $x_j = (-1)^{j+1} \lambda \det(v_1, ..., \hat{v}_j, ..., v_{n+1})$, which is perpendicular to the rows of $B$. By Corollary 3.1, $x_j = \pm \sqrt{1-\ell_j^2}$. Since $B$ is a Parseval frame, $\ell_1^2 + ... + \ell_{n+1}^2 = n$, and so

$$||\vec{x}||^2 = x_1^2 + ... + x_{n+1}^2 = (1-\ell_1^2) + ... + (1-\ell_{n+1}^2) = 1.$$
So, the \((n+1) \times (n+1)\) matrix \(\tilde{B}\) which is obtained from \(B\) with the addition of the row \(\tilde{y}\), is unitary, and therefore also the columns of \(\tilde{B}\) are orthonormal. For every \(i, j \leq n + 1\),

\[
\langle v_i, v_j \rangle \pm \sqrt{1 - \ell_i^2} \sqrt{1 - \ell_j^2} = \ell_i \ell_j \cos \theta_{ij} \pm \sqrt{1 - \ell_i^2} \sqrt{1 - \ell_j^2} = 0 \quad (3.8)
\]

which implies (1.1).

Conversely, suppose that (1.1) holds. By (3.8), the vectors \(\tilde{v}_j = (\pm \sqrt{1 - \ell_j^2}, v_j)\) are orthonormal for some choice of the sign \(\pm\); therefore, the columns of the matrix \(\tilde{B}\) are orthonormal, and so also the rows are orthonormal, and \(B\) is a Parseval frame.

**Corollary 3.2.** Let \(B = \{v_1, ... v_{n+1}\}\) be a nontrivial Parseval frame. Then, \(\frac{1}{n+1} < \ell_j^2\) for every \(j\). Moreover, for all \(j\) with the possible exception of one, \(\frac{1}{2} < \ell_j^2\).

**Proof.** The identity (1.1) implies that, for \(i \neq j\),

\[
1 - \ell_j^2 - \ell_i^2 + \ell_i^2 \ell_j^2 \sin^2 \theta_{ij} = 0. \quad (3.9)
\]

That implies \(\ell_j^2 + \ell_i^2 \geq 1\) for every \(i \neq j\), and so all \(\ell_j^2\)’s, with the possible exception of one, are \(\geq \frac{1}{2}\). Recalling that \(\sum_{i=1}^{n+1} \ell_i^2 = n\),

\[
1 - (n+1)\ell_j^2 + \sum_{i=1}^{n+1} \ell_i^2 \ell_j^2 \sin^2 \theta_{ij} = 0,
\]

and so \(\ell_j^2 > \frac{1}{n+1}\). \(\Box\)

**Proof of Theorem 1.3.** After a rotation, we can assume \(v_i = v_1 = (\ell_1, 0, ..., 0)\). We let \(\theta_{1,j} = \theta_j\) for simplicity. With this rotation \(v_j = (\ell_j \cos \theta_j, \ell_j \sin \theta_j, w_j)\) where \(w_j\) is a unitary vector in \(\mathbb{R}^{n-1}\). The rows of the matrix \((v_1, ... v_N)\) are orthonormal, and so the norm of the first row is

\[
\sum_{j \geq 1} \ell_j^2 \cos^2 \theta_j + \ell_1^2 = 1 \quad (3.10)
\]
The projections of \(v_2, \ldots, v_N\) over a hyperplane that is orthogonal to \(v_1\) form a tight frame on this hyperplane. That is to say that \(\{\ell_2 \sin \theta_2 w_2, \ldots, \ell_N \sin \theta_N w_N\}\) is a tight frame in \(\mathbb{R}^{n-1}\), and so it satisfies

\[
\ell_2^2 \sin^2 \theta_2 ||w_2||^2 + \ldots + \ell_N^2 \sin^2 \theta_N ||w_N||^2 = \ell_2^2 \sin^2 \theta_2 + \ldots + \ell_N^2 \sin^2 \theta_N = n - 1. \tag{3.11}
\]

**Corollary 3.3.** \(\{v_1, v_2, \ldots, v_n\}\) is a Parseval frame in \(\mathbb{R}^n\) if and only if the \(v_i\)'s are orthonormal

**Proof.** By (3.10), all vectors in a Parseval frame have length \(\leq 1\). By (3.11)

\[
\sum_{j=1}^n \ell_j^2 \sin^2 \theta_{i,j} = n - 1
\]

which implies that \(\ell_j = 1\) and \(\sin \theta_{ij} = 1\) for every \(j \neq i\), and so all vectors are orthonormal.

**Proof of Corollary 1.4.** Assume that \(\mathcal{B}\) is scalable; fix \(i < n + 1\), and chose \(j \neq k \neq i\). By 1.1

\[
(1 - \ell_i^2)(1 - \ell_j^2) = \ell_i^2 \ell_j^2 \cos \theta_{i,j}^2,
\]

\[
(1 - \ell_i^2)(1 - \ell_k^2) = \ell_i^2 \ell_k^2 \cos \theta_{i,k}^2,
\]

\[
(1 - \ell_k^2)(1 - \ell_j^2) = \ell_k^2 \ell_j^2 \cos \theta_{k,j}^2.
\]

These equations are easily solvable for \(\ell_i^2, \ell_j^2\) and \(\ell_k^2\); we obtain

\[
\ell_i^2 = \frac{|\cos \theta_{k,j}|}{|\cos \theta_{k,j}| + |\cos \theta_{k,i} \cos \theta_{j,i}|}.
\]

This expression for \(\ell_i\) must be independent of the choice of \(j\) and \(k\), and so (1.3) is proved.

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