Non-Landau Fermi Liquid induced by Bose Metal

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Understanding non-Landau Fermi liquids in dimensions higher than one, has been a subject of great interest. Such phases may serve as parent states for other unconventional phases of quantum matter, in a similar manner that conventional broken symmetry states can be understood as instabilities of the Landau Fermi liquid. In this work, we investigate the emergence of a novel non-Landau Fermi liquid in two dimensions, where the fermions with quadratic band-touching dispersion interact with a Bose metal. The bosonic excitations in the Bose metal possess an extended nodal-line spectrum in momentum space, which arises due to the subsystem symmetry or the restricted motion of bosons. Using renormalization group analysis and direct computations, we show that the extended infrared (IR) singularity of the Bose metal leads to a line of interacting fixed points of novel non-Landau Fermi liquids, where the anomalous dimension of the fermions varies continuously, akin to the Luttinger liquid in one dimension. Further, the multi-patch generalization of the model is used to explore other unusual features of the resulting ground state.

Introduction–Classification of gapless quantum ground states of interacting fermions is an outstanding question in modern theory of quantum matter. Deciphering the origin and instability patterns of such phases holds the key for understanding quantum critical phases and novel broken symmetry or topological phases that may arise thereof [1–3]. In one dimension, the Luttinger liquid [4] is a well-known example going beyond the paradigm of the Landau Fermi liquid, the standard model of conventional metals. It does not have well-defined quasiparticles and is characterized by a continuously-varying exponent that describes the algebraic correlations in space-time [4]. In higher dimensions, there have been numerous studies of non-Landau Fermi liquids that may arise from the long-range interaction between gapless fermions and critical bosonic excitations [5–22]. Nonetheless, these systems are often in the strong coupling limit, which makes it hard to find a controlled theoretical framework [17–22].

Considering the fermion-boson interactions in two and three dimensions, the bosons would condense at zero temperature if they are gapless at a specific momentum (unless they are Goldstone modes or gauge fields). This is the reason why such interactions are mostly studied near a quantum critical point for the bosons, where they remain gapless down to zero temperature. It would be interesting to consider an interacting boson system that remains gapless even without the quantum critical point and consider their interaction with fermions. The so-called Bose metal is precisely such a system. Here the elementary excitations are gapless in an extended region in the momentum space, which prohibits the condensation [23–26]. It is a consequence of the subsystem symmetry, where the boson number for each row and column of the underlying lattice is separately conserved [23, 27]. This phase was shown to arise in a model of bosons with a ring-exchange interaction in two-dimensional square lattice [23], where the correlated two-particle hoping preserves the subsystem symmetry mentioned above. Because of the restricted motion of the bosons and the subsystem symmetry, the Bose metal can be considered as an example of fracton phases [27–40].

In this work, we investigate non-Landau Fermi liquids that may arise from the interaction between the fermions with a quadratic dispersion and Bose metal in two dimensions. In the ring-exchange model of Bose metal on the square lattice [23], the dispersion of the low energy excitations are given by \( \omega_q \sim |\sin(q_x/2)\sin(q_y/2)| \) (Fig. 1(a)). The IR singularity along the nodal lines at \( q_x = 0 \) and \( q_y = 0 \) may provide a seed for non-Landau Fermi liquids when the Bose metal is coupled to fermions.

We first consider the interaction between the fermions with a quadratic dispersion at the Brillouin zone center and the Bose metal. Focusing on the low-energy continuum limit of the bosonic excitations, we show that the one-loop fermion self-energy acquires the logarithmic correction, \( \text{Re} \Sigma \sim \omega \ln \omega \), while the bosonic dispersion is not renormalized. Motivated by this discovery, we then consider the continuum limit of the two band model of fermions with \( p_x, p_y \) orbitals and the quadratic band-touching (Fig. 1(b)), which interact with the Bose metal. Using the renormalization group analysis, we find that a line of interacting fixed points arises (see Fig. 2(a)), where the fermions acquire an anomalous dimension that

![FIG. 1. Plots of the dispersions in the Bose metal and two-band model of fermions with \( p_x \) and \( p_y \) orbitals. (a) The Bose metal has nodal lines at \( k_x = 0 \) and \( k_y = 0 \) (red lines). (b) The dispersion of the fermions with \( A_1/A_0 = 0.1 \) and \( A_2/A_0 = 0.4 \). The blue and red bands are the top and bottom bands touching at \( (k_x, k_y) = (0, 0) \).](image-url)
varies continuously with the fixed point values of the dimensionless interaction parameter. In addition, the Bose metal is not renormalized or the subsystem symmetry is preserved at the fixed point. The continuously-varying anomalous dimension is akin to the Luttinger liquid [4].

In order to further explore the influence of the IR singularity from the nodal lines, we also consider the quadratic-band-touching fermions located at four different symmetric positions in the Brillouin zone, namely $k = (\pm k_0, 0), (0, \pm k_0)$ (see Fig. 2(b)). The lattice boson dispersion, $\omega_q \sim |\sin(q_x/2)|\sin(q_y/2)|$, allows two kinds of IR singularities. The first kind acts mainly within the same patch, defined as a small area in momentum space around each quadratic band-touching point. This involves the small momentum transfer $q = (\delta q_x, 2k_0)$, where $|\delta q_x|, |\delta q_y| \ll k_0$, with the relevant bosonic excitations described by $\omega_q \sim |\delta q_x|\sin(k_0)|$. The second kind involves the large momentum transfer (in one of the two directions) between the reflection-related patches, namely $q = (2k_0, \delta q_y)$, with the corresponding bosonic soft mode $\omega_q \sim |\delta q_x|\sin(k_0)$ or $\omega_q \sim |\sin(k_0)|\delta q_y|$. Considering the one-loop self-energy correction, it is found that the small momentum process in each patch dominates the low energy scaling. Hence the same low energy fixed point found earlier should apply in the scaling limit. On the other hand, the large momentum transfer in one of the momentum directions leads to a non-analytic correction to the fermion self-energy. For example, at the one-loop level, we find $\text{Re} \Sigma \sim \omega \ln \omega + \omega^{3/2}$ and $\text{Im} \Sigma \sim \omega + \omega^{-3/2}$. Here the subleading $\omega^{-3/2}$ is from the large momentum transfer between the patches. Hence the second derivative $\partial^2 \text{Im} \Sigma/\partial \omega^2 \sim \omega^{-1/2}$ diverges, which is in principle visible in the fermion scattering rate measurement in ARPES [41]. In this sense, the large momentum inter-patch process is dangerously irrelevant, and may play an important role in the characterization of the underlying fermionic ground state.

Bose metal—We first consider the ring-exchange model of bosons on the square lattice [23],

$$H = \sum_r \left[ \frac{U}{2} (n_r - \bar{n})^2 - K \cos(\Delta_{xy} \phi_r) \right], \tag{1}$$

where $\Delta_{xy} \phi_r = \phi_{r+\hat{x}} - \phi_{r+\hat{y}} + \phi_{r+\hat{x}+\hat{y}}$ is defined on each plaquette. Here $n_r$ and $\phi_r$ correspond to the boson number and the phase of the boson wave function at each lattice site $r = (x, y)$, which satisfy the canonical commutation relation, $[\phi_r, n_r] = i \delta_{rr'}$. $\bar{n}$ is the mean boson density. The phase $\phi_r$ is $2\pi$ periodic, $\phi_r = \phi_r + 2\pi n_r$, so that $n_r$ takes the integer values. The ring-exchange interaction correspond to the two-particle correlated hopping in each plaquette such that the boson number in each row and column of the lattice is preserved.

It was shown that the Bose metal phase is a stable ground state of this model for a range of parameters [23, 27], where the effective low energy action can be obtained via the expansion, $\cos(\Delta_{xy} \phi) \sim 1 + \frac{1}{2}(\Delta_{xy} \phi)^2$,

$$S = \frac{1}{2} \int d^2 q \frac{d^2 q}{(2\pi)^2} \int_k \frac{d\omega}{2\pi} \left( \omega_n^2 + \omega_q^2 \right) |\phi(\omega_n, q)|^2, \tag{2}$$

where $\omega_q \equiv 4u|\sin(q_x/2)|\sin(q_y/2)|$ for $q_x, q_y \in (-\pi, \pi)$. Here we set $U = 1$ and $u = \sqrt{\frac{E}{2}} K$. The nodal lines at $q_x = 0, q_y = 0$ reflect the presence of infinite number of conserved quantities (Fig. 1(a)). The corresponding symmetries involve the invariance of the action under $\phi(x, y) \to \phi(x, y) + \Phi_x(x) + \Phi_y(y)$, where $\Phi_x(x), \Phi_y(y)$ are arbitrary functions of $x$ and $y$ [23, 27]. The lattice action for the Bose metal is symmetries under $C_4$ rotation, inversion, time reversal, and reflection about $x$ and $y$ axes. The continuum limit is rather subtle and $\Delta_{xy} \phi \to \partial_x \phi \partial_y \phi$ is well defined while $\partial_x \phi$ and $\partial_y \phi$ are not [27]. The resulting continuum action, $S \sim \int d\tau \int d^2 x \left[ |\partial_x \phi|^2 + u^2 (\partial_x \phi \partial_y \phi)^2 \right]$ is also consistent with $\omega_q = u|q_x q_y|$ for small momentum $q_x, q_y$. The symmetries mentioned above imply that the coupling to fermion bilinears, $\psi^\dagger \psi \Phi$, where $\Phi$ is a symmetry-allowed matrix representation, involves the form factor, $\mathcal{F}(q) = 4\sin(q_x/2)\sin(q_y/2)$, in the lattice model. Thus, the interaction vertex has the form $\sum_{k,q} \psi^\dagger_k M \Phi_k \mathcal{F}(q) \phi_{k+q}$, which leads to $\psi^\dagger M \psi (\partial_x \phi \partial_y \phi)$ in the continuum limit.

$p$-wave orbital model and preliminary analysis—Let us consider the fermions in $p_x, p_y$ orbitals on the two-dimensional square lattice. In the continuum limit, the model can be written as

$$H = \sum_k \psi_k^\dagger \mathcal{H}(k) \psi_k,$$

$$\mathcal{H}(k) = A_0 k^2 \sigma_0 + A_2 (2k_x k_y) \sigma_x + A_4 (k_x^2 - k_y^2) \sigma_z, \tag{3}$$

where $\psi^\dagger = (c_x, c_y)$ is a two-component spinor, $c_{x,y}$ is the fermion annihilation operator for $p_x, p_y$ orbital, $\sigma_i$ is the Pauli matrix, and $\sigma_0$ is the $2 \times 2$ identity matrix. The corresponding fermion dispersion is $E(k) = A_0 k^2 \pm \sqrt{4A_2^2 k_x^2 k_y^2 + A_4^2 (k_x^2 - k_y^2)^2}$. We assume $A_0, A_2$ are positive. We focus on the case of $A_0 > \max(A_2, A_4)$, in which both bands are quadratically touching at $k = (0, 0)$ (Fig. 1(b)). The Hamiltonian is invariant under $C_4$ rotation, $U_{C_4} \psi = i\sigma_y \psi$, and the reflection about $x$ and $y$ axes, $U_{R_{xy}} = \pm \sigma_x \psi$.

When these fermions interact with the bosonic excitations in the Bose metal, the action for the interacting model can be written as

$$S = S_0 + S_{\text{int}},$$

$$S_0 = \int_{x, \tau} \psi^\dagger (\partial_\tau + H) \psi \left[ \frac{1}{2} |(\partial_x \phi)|^2 + u^2 (\partial_x \phi \partial_y \phi)^2 \right],$$

$$S_{\text{int}} = g \int_{x, \tau} (\partial_x \phi \partial_y \psi) (\psi^\dagger \sigma_y \psi), \tag{4}$$

where $\phi$ is the phase field of the bosons, $\psi$ and $\mathcal{H}$ are the fermion fields and their Hamiltonian introduced earlier. The tree-level scaling dimensions of fields and parameters are $[\psi] = d/2, [\phi] = (d - z)/2, [A_4] = [u] = z - 2$. In
particular, the scaling dimension of $g$ is $|g| = (3z - d - 4)/2$. Hence, for $d = 2$ and $z = 2$, this interaction is marginal, $|g| = 0$. Before going further, we introduce the dimensionless parameters, $\alpha_g \equiv g^2/\pi^2u^3$, and $a_0 = A_i/u$ where $i = 0, x, z$. Below, we use the fermion and boson propagators given by $G(i\omega_n, k) = (-i\omega_n\sigma_0 + \mathcal{H})^{-1}$ and $D(i\omega_n, q) = (\omega_n^2 + u^2q^2)^{-1}$, respectively.

We first consider the model with $A_0 \ne 0$, but all other $A_x, A_z = 0$. The one-loop boson self-energy is

$$\Pi(i\omega_n, q) = -g^2a_0^2\frac{2}{\Lambda(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\Omega_m}{2\pi} \times Tr[\sigma_x G(i\Omega_m + i\omega_n, p + q)\sigma_x G(i\Omega_m, p)]$$

where $\Lambda$ is the UV cutoff. The integration over $\Omega_m$ gives $\Pi = 0$. Therefore, the bosonic action is not renormalized. The one-loop fermion self-energy is given by

$$\Sigma(i\omega_n) = \frac{g^2}{2} \int_{-\infty}^{\infty} \sigma_x G(i\omega_n + i\Omega_m, p)\sigma_x D(i\Omega_m, p)[\mathcal{F}(p)]^2$$

$$= \frac{g}{8} \int_{-\infty}^{\infty} \frac{d\nu}{\pi} \left(1 - \frac{|\nu|}{\sqrt{\Omega^2 + x^2/\Lambda^2}}\right) + a_0^2\sigma_0$$

where $\int_{-\infty}^{\infty} \frac{d\nu}{\pi} \left(1 - \frac{|\nu|}{\sqrt{\Omega^2 + x^2/\Lambda^2}}\right) = 0.33$ slices. Here, the initial value $a_{x\text{,init}} = 0$ is used. The red solid line represents the line of fixed points, $(a_0^*, a_x^*, a_z^*) = (1.4518, 0, 0, a_z^*)$, where $a_z^*$ depends on the initial values of other parameters. (b) The location of the fermion band-touching points, in momentum space, in the multi-patch model. The locations of band-touching points are $k = (\pm k_0, 0)$ and $(0, \pm k_0)$.

$$\delta\Pi = -g^2a_0^2\frac{2}{\Lambda} \int_{-\infty}^{\infty} \sigma_x G(i\Omega_m + i\omega_n, p + q)\sigma_x G(i\Omega_m, p)]$$

where $\int_{-\infty}^{\infty} \sigma_x G(i\Omega_m + i\omega_n, p + q)\sigma_x G(i\Omega_m, p)] = C_{r,1} i\omega + C_{r,0} \log \left|\frac{|\omega|}{a_0^2}\right| \sigma_0$.

$$\delta\Sigma(i\omega_n, k) = g^2 \int_{-\infty}^{\infty} \sigma_x G(i\Omega_m + i\omega_n, p + q)\sigma_x G(i\Omega_m, p)]$$

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$$\Sigma(i\omega_n) = \frac{g^2}{2} \int_{-\infty}^{\infty} \sigma_x G(i\omega_n + i\Omega_m, p)\sigma_x D(i\Omega_m, p)[\mathcal{F}(p)]^2$$

$$= \frac{g}{8} \int_{-\infty}^{\infty} \frac{d\nu}{\pi} \left(1 - \frac{|\nu|}{\sqrt{\Omega^2 + x^2/\Lambda^2}}\right) + a_0^2\sigma_0$$

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Fig. 2(a). This means there exists a line of stable fixed points, \((a_0^\dagger, a^\dagger, a^\dagger, \alpha^\dagger) = (1.4518, 0, 0, \alpha^\dagger)\), characterized by continuously varying fixed point values of \(\alpha^\dagger\). Some numerical solutions of the RG equations are shown in Fig. S1 in Supplementary Material.

Along the line of stable fixed points, the anomalous dimension of the fermions is finite as \(\alpha_g \neq 0\) and is given by \(\eta_f = \alpha_g C_f\), where \(C_f = F(\omega_0) = 0.07438\). This means the fermion propagator behaves as \(G(\omega_0, \mathbf{k}) \sim 1/(\omega_0 + \mathbf{E}_0)\) in the low-energy limit, and there is no quasiparticle pole. The absence of the quasiparticle pole is the signature of the non-Fermi liquid behavior [22]. Moreover, the anomalous dimension of the fermions changes continuously as \(\alpha_g\) is varied along the line of stable fixed points. This behavior is reminiscent of the Luttinger liquids in one dimension [4]. In this sense, we may consider the line of fixed points as a two-dimensional version of the Luttinger liquids.

**Multi-patch model**— In order to examine further the influence of the nodal lines in the Bose metal, we consider multiple fermions at symmetry-related locations in the momentum space. Let us assume that the fermions with the quadratic dispersion, \(A_0 k^2\), are located at \(\mathbf{k} = (\pm k_0, 0,0, \pm k_0)\) (Fig. 2(b)). We use the term “patch” to call a small area in momentum space near each band-touching point. For small momentum transfer in both \(k_x\) and \(k_y\) directions, we only need to consider the scattering processes within the same patch. However, when the momentum transfer is large in one of the directions and small in the other, two patch areas related by the reflection about \(k_x\) or \(k_y\) axis, can be connected by such a momentum transfer in the low energy limit.

To illustrate this point, let us focus on the patch near \(\mathbf{k} = (k_0, 0,0,0)\) and compute the fermion self-energy. In the case of the small momentum transfer, \(\mathbf{q} = (\delta q_x, \delta q_y)\), the dispersion and form factor for the bosonic excitations are given by \(\omega_q = u|\delta q_x, \delta q_y|\) and \(F(\mathbf{q}) = \delta q_x, \delta q_y\). The one-loop fermion self-energy due to small momentum transfer within the same patch has the same form as Eq. 6, and after analytic continuation,

\[
\Sigma_{\text{small}}(\omega) \approx \alpha_g \left[ C_{R,1}^{\text{small}} \omega + C_{R,\log}^{\text{small}} \omega \ln \left( \frac{|\omega|}{u \Lambda^2} \right) + C_{I,1}^{\text{small}} i \omega \right].
\]

Now let us consider the large momentum transfer between the patches near \(\mathbf{k} = (k_0, 0,0,0)\) and \(\mathbf{k} = (-k_0, 0,0,0)\). The corresponding momentum transfer between these patches for low energy bosonic excitations can be written as \(\mathbf{q} = (-2k_0, \delta q_y)\). The dispersion and form factor are given by \(\omega_q = 2u|\delta q_y| \sin k_0|\) and \(F(\mathbf{q}) = 2|\delta q_y| \sin k_0\). Note that we do not consider the large momentum transfer between the patches located near \(\mathbf{k} = (0, \pm k_0,0,0)\) as it is not a low energy process. Now the one-loop fermion self-energy at \(\mathbf{k} = (k_0, 0,0,0)\) due to the momentum transfer \(\mathbf{q} = (-2k_0, \delta q_y)\) is given by

\[
\Sigma_{\text{large}}(i \omega_n)
\]

\[
= g^2 \int \frac{d\mathbf{p}}{\Lambda} G(i \Omega_n + i \omega_n, \mathbf{p}) D(i \Omega_n, \mathbf{p} - 2k_0 \hat{x}) F(\mathbf{p} - 2k_0 \hat{x})^2
\]
\[
= \alpha_g \frac{u^2 \Lambda^2}{8} \int_0^1 dx \int_1^1 dy \frac{1 - |\Omega|/\sqrt{\Omega^2 + \varepsilon x}}{-(i \Omega + (\omega_n/u \Lambda^2)) + a_0 x}
\]
\[
\approx \alpha_g \left[ C_{R,1}^{\text{large}} \omega_{\lambda}^{3/2} + C_{I,1}^{\text{large}} \omega_{\lambda} \right] + C_{R,3/2}^{\text{large}} \frac{\ln(|\omega_n/u \Lambda^2|)}{u \Lambda^2} + C_{R,1}^{\text{large}} \frac{\omega_{\lambda}^{3/2}}{u \Lambda^2},
\]

where \(\varepsilon \equiv 4 \Lambda^{-2} \sin^2(k_0)\), \(D(i \Omega_n, \mathbf{p} - 2k_0 \hat{x}) = (\omega_n^2 + 4p_y^2 \sin^2 k_0)^{-1}\).

After analytic continuation, the total fermion self-energy in real frequency, \(\Sigma_{\text{tot}}(\omega) \equiv \Sigma_{\text{small}}(\omega) + \Sigma_{\text{large}}(\omega)\), is obtained. The real part of this fermion self-energy is

\[
\text{Re}\Sigma_{\text{tot}}(\omega) = \alpha_g \left[ C_{R,1}^{\text{tot}} \omega + C_{R,\log}^{\text{tot}} \omega \ln \left( \frac{|\omega|}{u \Lambda^2} \right) \right. \]
\[
\left. + C_{R,3/2}^{\text{tot}} \omega_{\lambda}^{3/2} \left( \frac{1}{u \Lambda^2} \right)^{1/2} \right].
\]

The leading contribution, \(\omega \ln \omega\), comes from the small momentum transfer process within the same patch. It implies that the same renormalization group fixed point obtained earlier for the single patch model would describe the low energy properties of this non-Fermi liquid state.

On the other hand, the imaginary part of the total fermion self-energy is given by

\[
\text{Im}\Sigma_{\text{tot}}(\omega) = \alpha_g \left[ C_{I,1}^{\text{tot}} \omega_{\lambda}^{3/2} + C_{I,3/2}^{\text{tot}} \omega_{\lambda} \left( \frac{1}{u \Lambda^2} \right)^{1/2} \right].
\]

While the leading behavior is linear in \(\omega\), the non-analytic correction \(\omega^{3/2}\) comes from the large momentum process. Note that the second derivative, \(\partial^2 \text{Im}\Sigma(\omega)/\partial^2 \omega \propto \omega^{-1/2}\), is singular in the low energy limit. If we only kept the leading order contribution from the small momentum transfer within the same patch, we would not be able to find such a singular behavior. In this sense, the large momentum transfer process is dangerously irrelevant. It should also be noted that both the small and large momentum processes do not renormalize the bosonic action of the Bose metal (see Supplementary Material).

**Discussion**— In the two-dimensional Bose metal phase, one can define the currents, \(J_0 = \partial_x \phi\), \(J_{xy} = u^2 \partial_y \partial_y \phi\), and the continuity equation, \(\partial_t J_0 = \partial_x J_{xy}\), can be derived from the equation of motion, \(\partial_t^2 \phi = \partial_x \partial_y (u^2 \partial_y \partial_y \phi)\) [27, 39, 40]. Noting that the conjugate momentum \(\pi_0 = \partial_x \phi \sim n(x, y, \tau)\) is the boson number, one can obtain the conserved quantities, \(Q_\tau(x, y, \tau) = \int dx J_0(x, y, \tau),\)

\[
Q_\tau(x, y, \tau) = \int dx J_0(x, y, \tau),\]
showed that the non-Landau Fermi (or non-Fermi for short) liquid fixed point exists, where the bosonic action is not renormalized as long as $A_0 > \max(A_x, A_z)$ in the microscopic model. Therefore, at the fixed point $g = g^*$, the continuity equation is simply modified to $\partial_x J_0 = \partial_x \partial_y \tilde{J}_{xy}$, where $\tilde{J}_{xy} = J_{xy} + g^* \partial_x \partial_y (\Psi^\dagger \sigma_x \Psi)$. This leads to $\partial Q_x / \partial \tau = \int dy \partial_x \partial_y \tilde{J}_{xy}$, $\partial Q_y / \partial \tau = \int dx \partial_x \partial_y \tilde{J}_{xy}$, where the additional contribution from the fermion also vanishes because it is a total derivative of the fermion bilinear. Thus, $Q_x$ and $Q_y$ remain conserved so that the subsystem symmetry of the Bose metal is intact for the non-Fermi liquid fixed point.

In the multi-patch model, where the quadratic-band-touching fermions are located at multiple places in the momentum space, we showed that the nodal-line excitations can influence the large momentum (in one of the two momentum directions) scattering processes between different patches. While the contribution from the inter-patch scattering process is irrelevant at the $T = 0$ fixed point, the non-analytic $T^{3/2}$ correction from such processes may appear in the fermion self-energy, $\text{Im} \Sigma \sim T + T^{3/2}$, at finite temperature, before one reaches the low temperature scaling regime. For example, the behavior, $\partial^2 \text{Im} \Sigma / \partial T^2 \propto T^{-1/2}$, could be seen in the ARPES measurement [41].

In the current work, we investigated the interaction between the quadratic-band-touching fermions and the Bose metal. In particular, we focus on the case where there is no renormalization of the nodal-line spectrum of the Bose metal. It would be interesting to investigate the interaction between fermions with a Fermi surface and the Bose metal. Here one may expect that the low energy excitations of the Bose metal would be strongly influenced by the particle-hole excitations of the Fermi sea. This would be an interesting subject of future study.

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Supplemental Material for “Non-Landau Fermi Liquid induced by Bose Metal”

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1. TIGHT-BINDING HAMILTONIAN FOR p-ORBITALS

Here, we introduce the tight-binding Hamiltonian of the fermions with p-orbitals on the square lattice. First, the nearest neighbor hopping Hamiltonian is given by

\[
H_{NN} = \sum_i t_\sigma (c_{x,i}^\dagger c_{x,i+\hat{x}} + c_{y,i}^\dagger c_{y,i+\hat{y}} + \text{h.c.}) + \sum_i t_\pi (c_{x,i}^\dagger c_{y,i} + c_{y,i}^\dagger c_{x,i}^\dagger + \text{h.c.})
\]

\[
= \sum_k \left( \begin{array}{cc} z_x & \sigma^\dagger \\ \sigma & z_y \end{array} \right) \left( \begin{array}{cc} 2t_\sigma \cos(k_x) + 2t_\pi \cos(k_y) & 0 \\ 0 & 2t_\sigma \cos(k_y) + 2t_\pi \cos(k_x) \end{array} \right) \left( \begin{array}{c} \sigma_{x,k} \\ \sigma_{y,k} \end{array} \right),
\]

where h.c. stands for the hermitian conjugate and \(c_{(x,y),i}\) and \(c_{(x,y),k}\) are the fermion annihilation operators in the position and momentum space for \(x\) (\(y\)) orbital, respectively. \(t_\sigma\) and \(t_\pi\) are the hopping parameters for the \(\sigma\) and \(\pi\) bonds.

The next nearest neighbor hopping Hamiltonian is given by

\[
H_{NNN} = \sum_i \sum_{\alpha=x,y} \hat{t} (c_{\alpha,i}^\dagger c_{\alpha,i+\hat{x}} + c_{\alpha,i}^\dagger c_{\alpha,i+\hat{y}} + c_{\alpha,i}^\dagger c_{\alpha,i-\hat{x}} + c_{\alpha,i-\hat{y}} c_{\alpha,i} + \text{h.c.})
\]

\[
+ \sum_i \sum_{\alpha \neq \beta} \hat{t} (c_{\alpha,i}^\dagger c_{\beta,i} + c_{\alpha,i}^\dagger c_{\beta,i} - c_{\alpha,i}^\dagger c_{\beta,i} - c_{\alpha,i}^\dagger c_{\beta,i} + c_{\alpha,i}^\dagger c_{\beta,i} + \text{h.c.})
\]

\[
= \sum_k \left( \begin{array}{cc} z_x & \sigma^\dagger \\ \sigma & z_y \end{array} \right) \left( \begin{array}{cc} 4\hat{t} \cos(k_x) \cos(k_y) & -4\hat{t} \sin(k_x) \sin(k_y) \\ -4\hat{t} \sin(k_x) \sin(k_y) & 4\hat{t} \cos(k_x) \cos(k_y) \end{array} \right) \left( \begin{array}{c} \sigma_{x,k} \\ \sigma_{y,k} \end{array} \right).
\]

where \(\hat{t}\) and \(\hat{t}'\) are the hopping parameters for the next nearest neighbor hopping. The total tight-binding Hamiltonian is

\[
H_{tb} = \sum_k \left( \begin{array}{cc} z_x & \sigma^\dagger \\ \sigma & z_y \end{array} \right) \mathcal{H}(k) \left( \begin{array}{c} \sigma_{x,k} \\ \sigma_{y,k} \end{array} \right),
\]

where

\[
\mathcal{H}(k) = \left( \begin{array}{cc} 2t_\sigma \cos(k_x) + 2t_\pi \cos(k_y) + 4\hat{t} \cos(k_x) \cos(k_y) & -4\hat{t} \sin(k_x) \sin(k_y) \\ -4\hat{t} \sin(k_x) \sin(k_y) & 2t_\sigma \cos(k_y) + 2t_\pi \cos(k_x) + 4\hat{t} \cos(k_x) \cos(k_y) \end{array} \right).
\]

Then, near \(k = (0,0),\)

\[
\mathcal{H} = 2(t_\sigma + t_\pi + 2\hat{t}) \sigma_0 + \left( \begin{array}{cc} -(t_\sigma + 2\hat{t}) & 0 \\ 0 & -(t_\sigma + 2\hat{t}) \end{array} \right) + E_1 \sigma_0 + \left( \begin{array}{cc} A_1(k_x^2 + k_y^2) & A_x(2k_xk_y) \\ A_x(2k_xk_y) & A_z(k_x^2 + k_y^2) \end{array} \right) \sigma_0
\]

\[
= E_1 \sigma_0 + \left( \begin{array}{cc} A_1(k_x^2 + k_y^2) & A_x(2k_xk_y) \\ A_x(2k_xk_y) & A_z(k_x^2 + k_y^2) \end{array} \right) \sigma_0
\]

where \(E_1 \equiv 2(t_\sigma + t_\pi + 2\hat{t}), A_0 = -(t_\sigma + t_\pi + 4\hat{t})/2, A_x = -2\hat{t}, \) and \(A_z = (t_\pi - t_\sigma)a^2/2\)

2. DEFINITION OF \(F_\omega, F_0\) FUNCTIONS

The functions \(F_{\omega,a_\omega,x_\omega,z_\omega}\) in Eq. 9 and 10 in the main text are defined as

\[
F_{\omega}(a_\omega, a_\omega, a_\omega) = \int_{-\infty}^{\infty} d\omega \int_0^{\pi/4} d\theta \frac{\omega^2 S(\theta)(\omega^2 + a_\omega^2 + a_\omega^2 S(\theta) + a_\omega^2 C(\theta))}{(\omega^2 + S(\theta)/4)(\omega^2 + 2(a_\omega^2 + a_\omega^2 S(\theta) + a_\omega^2 C(\theta))(a_\omega^2 - a_\omega^2 S(\theta) - a_\omega^2 C(\theta))^2)}
\]

\[
F_0(a_\omega, a_\omega, a_\omega) = \int_{-\infty}^{\infty} d\omega \int_0^{\pi/4} d\theta \frac{\omega^2(-\omega^2 + 3S(\theta)/4)(\omega^2 + a_\omega^2 S(\theta) - a_\omega^2 C(\theta))^2}{(\omega^2 + S(\theta)/4)^2(\omega^2 + 2(a_\omega^2 + a_\omega^2 S(\theta) + a_\omega^2 C(\theta))(a_\omega^2 - a_\omega^2 S(\theta) - a_\omega^2 C(\theta))^2)}
\]
\[ F_x(a_0, a_x, a_z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{\pi/4} \frac{d\theta}{d\theta} \left( 4\omega^2 S(\theta)(-\omega^2 + S(\theta)/4)(\omega^2 - a_0^2 + a_x^2 S(\theta) + a_z^2 C(\theta)) \right) \]

\[ F_z(a_0, a_x, a_z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{\pi/4} \frac{d\theta}{d\theta} \left( \omega^2 C(\theta)(-\omega^2 + 3S(\theta)/4)(\omega^2 - a_0^2 + a_x^2 S(\theta) + a_z^2 C(\theta)) \right) \]

\[ F_g(a_0, a_x, a_z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{\pi/4} \frac{d\theta}{d\theta} \left( \frac{\omega^2 C(\theta)}{\omega^2 + S(\theta)/4}(\omega^2 + 2a_0^2 + a_x^2 S(\theta) + a_z^2 C(\theta)) + (a_0^2 - a_x^2 S(\theta) - a_z^2 C(\theta))^2 \right) \]

where \( S(\theta) = \sin^2(2\theta) \) and \( C(\theta) = \cos^2(2\theta) \). Here, \( F_{\omega,0,g} \) are positive, but \( F_z \) is negative for \( a_0 > a_x, a_z \).

When \( a_z \to 0 \), \( F_{\omega,0,\omega,0} \) can be written as

\[ F_0(a_0, a_x, 0) = \frac{-1 + 4a_0^2 - 4a_x^2}{(1 - 2a_0)^2 - 4a_x^2} \]

\[ + \left[ \frac{1 - 2a_0 - 2a_x}{\sqrt{4a_0^2 - (1 - 2a_x)^2}} \right] \left[ 1 + 2a_x \right] - \left[ \frac{1 - 2a_0 + 2a_x}{\sqrt{4a_0^2 - (1 - 2a_x)^2}} \right] \left[ 1 - 2a_0 \right], \]

\[ F_x(a_0, a_x, 0) = \frac{6(-1 + 4a_0^2 - 4a_x^2 + 4a_x^4 - 8a_0^2(1 + 2a_x^2))}{(16a_0^4 + (1 - 4a_x)^2 - 8a_0^2(1 + 4a_x^2))^2} \]

\[ - \frac{2(8a_0^4 + a_x(1 + 2a_x)(1 - 2a_x)^2 + 4a_0^2(1 + a_x - 4a_x^2))}{(16a_0^4 + (1 - 4a_x)^2 - 8a_0^2(1 + 4a_x^2))^2} \left[ \frac{1 - 2a_0 - 2a_x}{\sqrt{4a_0^2 - (1 - 2a_x)^2}} \right] \left[ 1 + 2a_x \right] - \left[ \frac{1 - 2a_0 + 2a_x}{\sqrt{4a_0^2 - (1 - 2a_x)^2}} \right] \left[ 1 - 2a_0 \right], \]

\[ F_g(a_0, a_x, 0) = F_\omega(a_0, a_x, 0). \]

### 3. RG FLOW EQUATIONS OF PARAMETERS

From the one-loop corrections introduced in the main text, we obtain the following RG flow equations for the parameters of the model,

\[ \frac{1}{A_i} \frac{dA_i}{d\ell} = (z - 2) - \alpha_g(F_\omega - F_i), \]

\[ \frac{1}{u_i} \frac{du_i}{d\ell} = (z - 2), \]

\[ \frac{1}{g_i} \frac{dg_i}{d\ell} = \frac{1}{2}(3z - d - 4) - \alpha_g(F_\omega - F_g). \]

Combining the RG flow equations above, we obtain the RG flow equations of the dimensionless parameters,

\[ \frac{1}{a_i} \frac{da_i}{d\ell} = -\alpha_g(F_\omega - F_i), \]
self-energy. The Callan-Symanzik equation for the fermion propagator is given by

$$
\frac{1}{\alpha_g} \frac{d\alpha_g}{d\ell} = (2 - d) - 2\alpha_g (F_\omega - F_g).
$$

(S18)

As mentioned in the main text, the RG flow equations Eq. S17 and S18 lead to a line of stable fixed points, 
$$(a_0, a_x, a_x^*, \alpha_g^*) = (1.4518, 0, 0, \alpha_g^*),$$
where $\alpha_g^*$ depends on the initial values of the dimensionless parameters. For example, the RG flows for several initial values of the parameters are shown in Fig. S1(a) and S1(b). The fixed point values of the parameters in Fig. S1(a) and S1(b) belong to the line of stable fixed points. As $a_z \to 0$, $a_0$ and $a_x$ approach the fixed point values, $(a_0^*, a_x^*) = (1.4518, 0)$, represented by the red point in Fig. S1(c).

4. ANOMALOUS DIMENSION BY FIELD-THEORETICAL RG

Here, using the field-theoretical RG, we find the anomalous dimension of the fermions from the one-loop fermion self-energy. The Callan-Symanzik equation for the fermion propagator is given by

$$
\left( \frac{\Lambda}{\partial \Lambda} \beta_g + \beta_g \frac{\partial}{\partial g} + \sum_i \beta_{A_i} \frac{\partial}{\partial A_i} + \beta_u \frac{\partial}{\partial u} + \eta_f \right) G = 0,
$$

(S19)

where $\eta_f$ is the anomalous dimension of the fermions, and $\Lambda$ is the UV cutoff. $\beta_g, \beta_{A_i},$ and $\beta_u$ are the beta equations of $g, A_i,$ and $u$, and they are defined as

$$
\beta_g = -\Lambda \frac{\partial \ln Z_{\psi}^{-1}}{\partial \Lambda}, \quad \beta_{A_i} = \Lambda \frac{\partial A_i}{\partial \Lambda}, \quad \beta_u = \Lambda \frac{\partial u}{\partial \Lambda}, \quad \eta_f = -\Lambda \frac{\partial \ln Z_{\psi}^{-1}}{\partial \Lambda},
$$

(S20)

respectively, and $Z_{\psi}$ is wavefunction the renormalization constant of the fermions. Assuming that the parameters are near the fixed point values, we have $\beta_{A_i} = \beta_u = \beta_g = 0$. Then, Eq. S19 can be written as $\eta_f = -\frac{\Lambda \frac{\partial G}{\partial \Lambda}}{G}$. The one-loop fermion self-energy has the form, $\Sigma(i\omega_n) \approx -i\omega_n\alpha_g C_{\log} \ln \left( \frac{\omega_n}{\Lambda} \right) + \text{regular + \cdots}$, as shown in the main text (where $C_{\log} > 0$). From this, we obtain $\frac{\Lambda \frac{\partial G}{\partial \Lambda}}{G} = -2\alpha_g C_{\log}$. Hence the anomalous dimension of the fermions is $\eta_f = 2\alpha_g C_{\log}$. For example, when we have the fixed point value $(a_0, a_x, a_z) = (1.4518, 0, 0)$, using $C_{\log} = 0.03719$, we obtain $\eta_f = 0.07438\alpha_g$. Note that along the line of stable fixed points, $C_{\log}$ does not change.

From the perturbative Wilsonian RG, we can also obtain the anomalous dimension of the fermions, $\eta_f = -\frac{\Lambda \frac{\partial \ln Z_{\psi}^{-1}}{\partial \Lambda}}{C_{\log}} = \alpha_g F_\omega$, where $Z_{\psi} = 1 + \alpha_g F_\omega \ln(\Lambda/\mu)$ and $F_\omega(1.4518, 0, 0) = C_f = 0.07439$ at the fixed point. Then, we can check that the anomalous dimension of the fermions from the field-theoretical RG and Wilsonian RG is consistent, and $2C_{\log} \approx C_f$. 
where \( \Sigma \) comes from the last term in Eq. S23. Note that in the real parts of \( \Sigma \), the logarithmic correction comes from the last term in Eq. S21.

The one-loop fermion self-energy for the small momentum transfer within the same patch is

\[
\Sigma_{\text{small}}(i\omega_n) = g^2 \int \frac{d^4p}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} G(i\Omega_m + i\omega_n, p) D(i\Omega_m, p) \mathcal{F}(p - k_0)
\]

\[
= g^2 \int \frac{dp_x dp_y}{(2\pi)^2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{2\pi - i(\Omega_m + \omega_n)} + A_0 p^2 \Omega_m^2 + u^2 p^2 p_y^2
\]

\[
= g^2 \int \frac{dp_x dp_y}{(2\pi)^2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{2\pi - i(\Omega_m + \omega_n)} \frac{(p/\Lambda)^4 \sin^2(2\theta)/4}{4\Lambda^{-2} \sin^2(k_0) \sin^2(\theta(p/\Lambda)^2)}
\]

The coefficients \( C_i^{\text{large}} \) depend on \( a_0 \) and \( \xi \). The non-analytic \( |\omega_n|^{3/2} \) comes form the last term in Eq. S23. Note that in the real parts of \( \Sigma_{\text{small}} \) and \( \Sigma_{\text{large}} \) have constant terms which depend on the cutoff, but we drop those.

The comparison between the numerical values and fitting functions of \( \Sigma_{\text{small}}(i\omega_n) \) and \( \Sigma_{\text{large}}(i\omega_n) \) for \( a_0 = 1.4518 \) and \( \xi = 1 \) is shown in Fig. S2. The coefficients are \( C_{R,1}^{\text{small}} = -0.6842, C_{R,2}^{\text{small}} = -0.01951, C_{R,1}^{\text{large}} = 0.02305, \) and \( C_{R,1}^{\text{small}} = -0.03719 \) for the small momentum transfer, \( C_{R,3}^{\text{large}} = -0.6132, C_{R,2}^{\text{large}} = 1.466, C_{R,3}^{\text{large}} = 0.3905, C_{R,3}^{\text{large}} = -0.6110 \) for the large momentum transfer, respectively.

After analytic continuation, \( i\omega_n \to \omega + i\eta \),

\[
\Sigma_{\text{small}}(\omega) \approx \alpha g \left( C_{R,1}^{\text{small}} \omega + C_{R,1,\log}^{\text{small}} \ln \left( \frac{|\omega|}{u\Lambda^2} \right) + C_{R,2}^{\text{small}} \omega^2 u\Lambda^2 + C_{R,1}^{\text{small}} \omega \right)
\]

\[
\Sigma_{\text{large}}(\omega) \approx \alpha g \left( C_{R,1}^{\text{large}} \omega + C_{R,3}^{\text{large}} \frac{|\omega|^{3/2}}{(u\Lambda^2)^{1/2}} + C_{R,2}^{\text{large}} \omega^2 u\Lambda^2 + C_{R,3}^{\text{large}} \frac{i|\omega|^{3/2}}{(u\Lambda^2)^{1/2}} \right)
\]
The total one-loop fermion self-energy is

\[
\Sigma_{\text{tot}}(\omega) \approx \alpha_g \left( C_{R,1}^\text{tot} \omega + C_{R,\log}^\text{tot} \omega \ln \left( \frac{\omega}{u\Lambda^2} \right) + C_{R,3/2}^\text{tot} \frac{\omega^{3/2}}{(u\Lambda^2)^{1/2}} + C_{R,2}^\text{tot} \frac{\omega^2}{u\Lambda^2} 
+ C_{\Sigma,1}^\text{tot} i\omega + C_{\Sigma,3/2}^\text{tot} \frac{i\omega^{3/2}}{(u\Lambda^2)^{1/2}} \right)
\]  

(S27)

6. BOSON SELF-ENERGY IN MULTI-PATCH MODEL

Here, we show that both the small and large momentum transfer processes do not renormalize the bosonic excitations in the multi-patch model. The one-loop boson self-energy for the small momentum transfer is given by

\[
\Pi(i\omega_n, \mathbf{q}) = -g^2 q_x^2 q_y^2 \int_{\Lambda} \frac{d^2 p}{(2\pi)^2} \int_{-\infty}^{\infty} d\Omega_m G(i\Omega_m + i\omega_n, \mathbf{p} + \mathbf{q}) G(i\Omega_m, \mathbf{p}) 
= -g^2 q_x^2 q_y^2 \int_{\Lambda} \frac{d^2 p}{(2\pi)^2} \int_{-\infty}^{\infty} d\Omega_m \frac{1}{2\pi} \frac{1}{-i(\Omega_m + \omega_n) + A_0(\mathbf{p} + \mathbf{q})^2 - i\Omega_m + A_0 p^2} 
= 0.
\]
In terms of $\Omega_m$, the integrand has poles at $-\omega_n - iA_0(p + q)^2$ and $-iA_0 \Omega_m^2$. If we choose to close the $d\Omega_m$ contour in the upper half-plane, the integral becomes zero because two poles are in the lower half-plane. Therefore, $\Pi = 0$.

The one-loop boson self-energy for the large momentum transfer is given by

$$
\Pi(i\omega_n, q) = -g^2 q_x q_y \int \frac{d^2p}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\Omega_m}{2\pi} G(i\Omega_m + i\omega_n, p + k_0 + q) G(i\Omega_m, p)
$$

$$
= -g^2 q_x q_y \int \frac{d^2p}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\Omega_m}{2\pi} \frac{1}{\Omega_m - i(\Omega_m + \omega_n) + A_0(p + k_0 + q)^2 - i\Omega_m + A_0 \Omega_m^2}
$$

$$
= 0.
$$

It also vanishes in the same way that the one-loop boson self-energy for the small momentum transfer does. Therefore, the small and large momentum processes in the multi-patch model do not renormalize the bosonic action of the Bose metal.