RIEMANN SURFACES OF SOME STATIC DISPERSION MODELS AND PROJECTIVE SPACES

The S-matrix in the static limit of a dispersion relation has a finite order N and is a matrix of meromorphic functions of energy $\omega$ in the plane with cuts $(-\infty, -1], [+1, +\infty)$. In the elastic case it reduces to N functions $S_i(\omega)$ connected by the crossing symmetry matrix A. The problem of analytical continuation of $S_i(\omega)$ from the physical sheet to unphysical ones can be treated as a nonlinear system of difference equations. It is shown that a global analysis of this system can be carried out effectively in projective spaces $P_N$ and $P_{N+1}$. The connection between spaces $P_N$ and $P_{N+1}$ is discussed.

1 Introduction

The low-energy hadron scattering problem remains in the focus of attention [1]. The successful development of QCD poses the question of the validity of the analytic properties of hadron-hadron process amplitudes previously proved for strong interactions. In a series of works by Oehme [2], it was recently shown that they remain valid in QCD as well. We consider the nonrelativistic limit of the dispersion relations, which is known as static equations [3], and confine ourselves to studying the equations of this type by reducing them to a nonlinear boundary-value problem [4]. It has the form of a series of conditions on the S-matrix elements $S_i$.

Conditions 1.
A) \( S_i(z) \) – are meromorphic functions in the complex \( z \) plan with the cuts with the cuts \((-\infty, -1], [+1, +\infty)\), i.e., the only singularities of these functions in this domain are their zeros and poles.

\[
\text{B) } S_{i}^{*}(z) = S_{i}(z^{*}),
\]

\[
\text{C) } |S_{i}(\omega + i0)|^{2} = 1 \text{ for } \omega \geq 1 \quad S_{i}(\omega + i0) = \lim_{\epsilon \to +0} S_{i}(\omega + i\epsilon),
\]

\[
\text{D) } S_{i}(-z) = \sum_{j=1}^{N} A_{ij} S_{j}(z).
\]

The real values of the variable \( z \) are the total energy \( \omega \) of a relativistic particle scattered by a fixed center. The meromorphy requirement for the functions \( S_i(z) \) arises as a consequence of the static limit of the scattering problem [5]. Elastic unitarity condition 1C holds only on the right cut in the \( z \) plane. On the left cut, the functions \( S_i(z) \) are determined by crossing-symmetry conditions 1D. The crossing-symmetry matrix \( A \) is determined by the group that leaves the \( S \)-matrix invariant; the matrix \( A \) is known for some groups [4]. The aim in this paper is to formulate a method for studying the Riemann surfaces of some static dispersion models.

### 2 Analytic continuation of the \( S \)-matrix to nonphysical sheets

We write Conditions 1 in a matrix form. For this, we introduce the column

\[
S^{(0)}(z) = [S_1(z), S_2(z), \ldots, S_N(z)]^T,
\]

where the upper index denotes the physical sheet of the \( S \)-matrix Riemann surface. Conditions 1A, 1B, and 1D hold on the physical sheet, and unitarity condition 1C can be extended to the complex values of \( \omega \), and just like condition 1C, the
extension has the component form

\[ S_i^{(0)}(z) S_i^{(1)}(z) = 1 \]

and analytically continues the \( S \)-matrix to the first nonphysical sheet of the Riemann surface. To rewrite unitarity conditions 1C in the matrix form, we introduce the nonlinear inversion transformation \( I \) by the formula

\[ IS(z) = [1/S_1(z), 1/S_2(z), \cdots, 1/S_N(z)] . \]

As a result, Conditions 1 take the following form.

Conditions 2.

\[ A) \quad S^{(0)}(z) \quad \text{is a column of } N \text{ meromorphic functions in the complex} \]
\[ \text{plane } z \text{ with the cuts } (-\infty, -1], [+1, +\infty), \text{i.e., the only singularities} \]
\[ \text{of these functions in this domain are their zeros and poles.} \quad (2) \]

\[ B) \quad S^{(0)*}(z) = S^{(0)}(z^*), \]

\[ C) \quad S^{(1)}(z) = IS^{(0)}(z), \]

\[ D) \quad S^{(0)}(-z) = AS^{(0)}(z). \]

We define the analytic continuation to nonphysical sheets as

\[ S^{(p)}(z) = (IA)^p S^{(0)}(z(-1)^p). \quad (3) \]

By definition (3), unitarity condition 2C and crossing-symmetry condition 2D are easily extended to non-physical sheets:

\[ IS^{(p)}(z) = S^{(1-p)}(z), \quad AS^{(p)}(z) = S^{(-p)}(-z), \quad (4) \]

and we have the formula

\[ (IA)^q S^{(p)}(z) = S^{(q+p)}(z(-1)^q). \quad (5) \]

Definition (3) is motivated by the well-known solution [5] of the problem defined by Conditions 1 for the two-row matrix
This solution for the \( S \)-matrix \( S(z) \) is given by

\[
S(z) = \left( \frac{W(W - 2)/(W^2 - 1)}{W(W + 1)/(W^2 - 1)} \right) D(z),
\]

where \( W = w + i\sqrt{z^2 - 1}\beta(z) \), \( w = 1/\pi \arcsin z \), \( \beta(z) = -\beta(-z) \) is a meromorphic function, and \( D(z) = D(-z) \) is the Blaschke function of the variable \( \zeta = \frac{1+i\sqrt{z^2 - 1}}{z} \).

The Blaschke function is given by

\[
D(\zeta[z]) = \zeta^\lambda \prod_n \frac{|\zeta_n|}{\zeta_n-\zeta} \frac{\zeta_n-\zeta}{1-\zeta_n^*\zeta}
\]

where \( \lambda \) is the order of zero, and the set of zeros \( \{\zeta_n\} \), \(|\zeta_n| < 1\), is symmetric with respect to the origin and the axes \( \text{Im}\zeta = 0 \), \( \text{Re}\zeta = 0 \). In addition to solution (6), Conditions 1 allow a trivial solution: the column of identical Blaschke functions

\[
S(z) = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) D(z).
\]

Conditions 2 therefore do not determine the form of the Riemann surface of \( S(z) \) uniquely. For solution (6), the Riemann surface of \( S(z) \) is infinite-sheeted because of the function \( w \), and the equalities

\[
S^{(0)}(z) = S(W)|_{|w| \leq 1/2}, \quad S^{(\pm n)}(z(-1)^{(\pm n)}) = S(W)|_{|w \pm n| \leq 1/2},
\]

hold, which allow rewriting Eqs.(5) as

\[
(IA)^n S(W) = S(W + n),
\]

\[
(AI)^n S(W) = S(W - n).
\]

Equations (7) are a system of nonlinear autonomous difference equations and can naturally be called the dynamic form of the static dispersion relations. The
same term can therefore be used for Eqs.(5) as well. Unlike Eqs.(7), they form a system of nonlinear difference equations in which the number of a sheet of the Riemann surface serves as an argument and the energy variable $z$ is a parameter.

3 Formulation of the problem
in projective spaces

The example of two-row solution (6) shows that, in general, the solution of the problem defined by Conditions 1 is determined by $N + 1$ entire functions, among which $N$ functions satisfy crossing-symmetry condition 1D and the last one is symmetric with respect to $z$ and ensures the validity of unitarity condition 1C. Conditions 1A, 1B, and 1D are homogeneous and can be considered in the projective spaces $P_{N-1}$ and $P_N$. We define the nonlinear inversion transformation $I_p$ such that it is correct in these spaces [6]: $I_p$,

$$I_p = \Pi_{j=1, i \neq j}^m S_j,$$

$$m = N - 1, N.$$

We reformulate the problem defined by Conditions 1 for these spaces. For the space $P_{N-1}$, the crossing-symmetry matrix has the form specified by Conditions 1; for the space $P_N$, its dimensionality increases by one, i.e.,

$$A_{N-1} = A, A_N = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

where $A_N$ is a block matrix. As a result, instead of Conditions 1, we obtain the following set of requirements on a column of $m$ functions.

Conditions 3.
A) $S^{(0)}(z)$ is a column of $m$ meromorphic functions in the complex $z$ plane with the cuts $(-\infty, -1], [+1, +\infty)$, i.e., the only singularities of these functions in this domain are their zeros and poles

B) $S^{(0)*}(z) = S^{(0)}(z^*)$,

C) $S^{(1)}(z) = I_p S^{(0)}(z),$

D) $S^{(0)}(-z) = A_m S^{(0)}(z)$.

We illustrate the scheme of the solution for two-row case in terms of the projective spaces $P_1, P_2$. We let $(x_o, x_1) = (S_1, S_2)$ denote the coordinates of the point $x$ in the space $P_1$. We introduce the affine coordinate $X = x_0 / x_1$ on the projective line $P_1$. Setting $z = 0$ in (3), we obtain the law for continuing the coordinate $X^{(0)}$ from the physical sheet to the first nonphysical sheet:

$$X^{(1)} = \frac{2X^{(0)} + 1}{-X^{(0)} + 4}.$$  \hspace{1cm} (9)

Taking the $n$th power of linear fractional transformation (8) and using crossing-symmetry condition 3D, we find that

$$X^{(0)} = -2 \quad X^{(n)} = \frac{n - 2}{n + 1}.$$ \hspace{1cm} (10)

On of crossing-symmetry conditions 3D thus proves unnecessary. This conclusion remains valid for 3 × 3 crossing-symmetry matrices. The solution of the two-row problem for the line $P_1$ allows finding only the ratio of the functions $S_1$ and $S_2$. The functions themselves can be found from the solution for the projective plane $P_2$. We write the projective coordinates of the point $(x) = (x_0, x_1, x_2)$ in $P_2$ in a basis explicitly taking the crossing symmetry into account:

$$x_0 = s - 2a$$

$$x_1 = s + a$$

$$x_2 = c,$$

$$x_3 = t.$$  \hspace{1cm} (11)
where $s$ and $c$ are symmetric functions of $z$ and $a$ is an antisymmetric function of $z$.

Considering the transformation $(I_pA_2)^n$ in the basis $s, a, c$, we can easily see that $s, a,$ and $c$ are related by

$$s^2 - a^2 - sc = 0,$$

(12)

which is invariant under the transformations $I_p$ and $A_2$. In other words, Eq.(12) in $P_2$ defines an invariant curve $C$ whose points do not leave $C$ under the action of the transformations $I_p$ and $A_2$. In the basis $(x_0, x_1, x_2),$ the equation of the curve $C$ is given by

$$x_1^2 + 2x_0x_1 - 2x_1x_2 - x_0x_2 = 0.$$  

(13)

Using Eqs.(10) and (13), we can easily find that

$$\frac{x_1}{x_2} = \frac{n}{n - 1}$$

(14)

and thus completely define the functions $S_1$ and $S_2$. Taking unitarity condition 1C (which has not been used yet) into account, we can recover formula (4) completely.

We discuss the relation between the descriptions of the two-row problem defined by Conditions 1 for the spaces $P_1$ and $P_2$. In the projective plane $P_2$, the solution is given by invariant curve (13). It is irreducible and rational as is any algebraic curve of the second order. In the affine coordinates, it becomes

$$x = \frac{x_0}{x_2}, \ y = \frac{x_1}{x_2}, \ x^2 + 2xy - 2x - y = 0.$$  

If we construct a bundle of lines of the form $\lambda_0g_0 + \lambda_1g_1$ with the base point $(x_0, y_0)$ in curve (13), then the coordinates of the second intersection of the lines in the bundle with curve (10) are rational functions of $k = \lambda_1/\lambda_0$:

$$x = \frac{-(x_0 + 2y_0) + 2 + k}{1 + 2k}, \ y = y_0 + k(x - x_0).$$  

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The functions $x$ and $y$ are reduced to formulas (10) and (14) by the specially chosen parametrization

$$k = \frac{(-x_0 - 2y_0 + 1)n + x_0 + 2y_0 - 2}{n + 1},$$

which depends on the base point of the bundle. A bundle of lines behaves as the projective space $P_1$ under collineations (linear transformations with nonzero determinants) in the space $P_2$. The projective space $P_1$ is thus represented by any bundle of lines whose base point lies on invariant curve (13) of the space $P_2$. In [4], the invariant manifolds for the problem defined by Conditions 1 with dimensionalities $N \geq 3$ were studied and constructed using series in a neighborhood of the rest points of dynamic systems (5). Using projective spaces, we can reconsider this problem from a new standpoint. We consider the problem defined by Conditions 1 with the three-row matrix

$$A = \begin{pmatrix}
1/3 & -1 & 5/3 \\
-1/3 & 1/2 & 5/6 \\
1/3 & 1/2 & 1/6
\end{pmatrix}, \quad (15)$$

which describes the scattering of two particles whose angular momenta are equal to unity. In the space $P_3$, the matrix $A_3$ has three eigenvalues equal to +1 and one eigenvalue equal to -1. The coordinates of the point $(x)$ in $P_3$ can be expressed in terms of three symmetric functions $s_1, s_2$ and $s_3$ of $z$ and one antisymmetric function $a$ of $z$ by an ordinary collineation (an automorphism of the projective space):

$$x_i = b_{ij}s_j + b_{i4}a.$$ 

We construct a plane in $P_3$ that is invariant under the linear transformation of the coordinates of $x$ determined by the matrix $A_3$. It is given by

$$c_0x_0 + c_1x_1 + (2c_0 + c_1)x_2 + c_2x_3 = 0. \quad (16)$$
It is easy to see that the plane \( x_1 + x_2 = 0 \) is a particular case of plane (16) and is invariant under the transformation \( I_p \). This plane is the space \( P_2 \) in which the problem defined by Conditions 1 with matrix (15) is reduced to the solvable two-row problem [7]. The plane \( x_1 + x_2 = 0 \) does not contain the rest point \( \bar{x} = (1, 1, 1, 1) \) of the dynamic system defined by Conditions 3, i.e., the fixed point of transformation (5). If we require the point \( \bar{x} \) to lie in plane (16), then we obtain the equation

\[
c_0 x_0 + c_1 x_1 + (2c_0 + c_1)x_2 - (3c_0 - 2c_1)x_3 = 0.
\]  

(17)

The transformation \( I_p \) maps plane (17) onto the cubic surface

\[
0 x_1 x_2 x_3 + c_1 x_0 x_2 x_3 + (2c_0 - c_1)x_0 x_1 x_3 - (3c_0 + 2x_1)x_0 x_1 x_2 = 0
\]  

(18)

in \( P_3 \), which is not invariant under the transformation \( A_3 \).

The intersection of plane (17) and surface (18) determines a planar spatial curve \( C \), which is not invariant under the transformation \( A_3 \) in general. Indeed, excluding \( x_3 \) from Eqs.(17) and (18), we obtain a third-degree homogeneous equation \( G(x_0, x_1, x_2) = 0 \). In the basis \( s_1, s_2, a \), the function \( G \) on the space \( P_2 \) contains odd powers of the antisymmetric function \( a \) for any \( c_0 \) and \( c_1 \). The coefficient of \( a \) is a quadratic form with respect to \( s_1, s_2, \) and \( a \). The invariance of the planar spatial curve \( C \) under the transformation \( A_3 \) implies that this quadratic form should vanish. As any second-degree equation, it defines rational functions \( s_1, s_2 \) and \( a \) of some parameter \( t \). Substituting them in the even part (with respect to \( a \)) of the function \( G(x_0, x_1, x_2) \), we obtain a third-degree equation with respect to \( t \), which has three solutions in general. An invariant curve exists only if this equation is identically zero, i.e., if \( G \) is reducible. The equation determining the coefficients \( c_0 \) and \( c_1 \) is given by

\[
R_{x_0}(G, G'_{x_1}) = 0
\]  

(19)
where is the resultant of $G$ and $G'_x$ with respect to $x_0$. From Eq. (19), we obtain $c_0 = -1, c_1 = 3$ and find the function

$$G(x_0, x_1, x_2) = (-3x_1^2 + x_0x_1 + 3x_0x_2 - x_1x_2)(-x_0 + x_2) = 0 \ , \quad (20)$$

which defines the reducible curve $C$. The first factor in Eq. (20) is invariant under the transformations $I_p$ and $A_2$ and together with Eq. (17) defines the well-known solution [8] with a finite number of poles with respect to $w$. It is represented in $P_3$ as the intersection of the plane

$$-x_0 + 3x_1 + x_2 - 3x_3 = 0 \ ; \quad (21)$$

and the surface

$$-3x_1^2 + x_0x_1 + 3x_0x_2 - x_1x_2 = 0 . \quad (22)$$

Using Eq. (22) and writing Eq. (21) in the form

$$x_1x_3 = x_0x_2,$$

we can easily verify the invariance of (21) under the transformation $I_p$. Under the action of the transformation $A_3$, the second factor in (20) becomes $(-x_1 + x_2)$; as a result, we have the degenerate quadratic form

$$(-x_0 + x_2)(-x_1 + x_2) = 0 ,$$

which is invariant under the transformations $I_p$ and $A_3$. It determines two bundles of lines that are invariant under the transformation $I_p$ and pass into each other under the transformation $A_3$:

$$x_0 = x_2, \quad \frac{x_0}{x_1} = \frac{n + 1/6}{n - 7/6}, \quad x_1 = x_2, \quad \frac{x_0}{x_1} = \frac{n - 3/2}{n + 1/2}.$$
4 Conclusion

The nonlinear boundary-value problem of constructing an $N$-dimensional (condition 1A), elastically unitary (condition 1C), and crossing-symmetric (condition 1D) $S$-matrix is formulated in the projective spaces $P_{N-1}$ and $P_N$. In the space $P_{N-1}$, it can be considered the result of projecting (ignoring unitarity condition 1C) the initial problem defined by Conditions 1 from the affine space $A_N$ to the projective space $P_{N-1}$. The condition for the analytic continuation of the $S$-matrix to nonphysical sheets is represented as a nonlinear autonomous system of difference equations, i.e., in the dynamic form. It can also be considered a nonlinear transformation in the spaces $A_N$, $P_{N-1}$, and $P_N$. In particular, among its fixed points, there is a point corresponding to the $S$-matrix without interaction. In the neighborhood of this point, the $S$-matrix was studied using power series in $1/w$, which can sometimes be summed [4]. The use of the projective space technique allows analyzing the solutions globally, i.e., constructing the invariant subspaces containing the solutions to be found. The invariant subspaces are determined by functions that are homogeneous in the projective spaces $P_{N-1}$ and $P_N$ but not in the affine space $A_N$. This statement disagrees with the conclusion in [9], according to which the invariant subspaces in the affine space $A_N$ are also determined by homogeneous functions. The above geometric interpretation of the boundary-value problem defined by Conditions 1 in the projective spaces $P_{N-1}$ and $P_N$ and the examples considered in [5] and [8] indicate that the homogeneity requirement on the functions defining the invariant subspaces of $A_N$ should be rejected. Concrete applications of the described procedure for solving the nonlinear boundary-value problem are demonstrated in Appendices 1 and 2.
Appendix 1

The two-row crossing-symmetry matrix for the group SU(2) is given by

\[ A_2 = \frac{1}{2l+1} \begin{pmatrix} -1 & 2l+2 \\ 2l & 1 \end{pmatrix}, \quad l \in \mathbb{N}. \]

The matrix considered in the paper is a particular case of it for \( l = 1 \). We give the calculation scheme for the general case. In the projective line \( P_1 \), the first affine coordinate \( X = x_0/x_1 \) is continued to the first nonphysical sheet according to the rule

\[ X^{(i)} = \frac{2lX^{(0)} + 1}{-X^{(0)} + (2l + 2)} \]

and together with the crossing-symmetry condition yields the value of \( X^{(n)} \):

\[ X^{(n)} = \frac{n - (l + 1)}{n + l}, \quad X^{(0)} = -(1 + 1/l). \quad (23) \]

The relation \( x_0/x_1 \) is therefore defined on every nonphysical sheet for \( z = 0 \), and to construct the functions \( S_1 \) and \( S_2 \), it suffices to find \( x_1/x_2 \). We let \( \varphi = x_1/x_2 \) denote this ratio. It is determined by the system of functional equations

\[ \varphi^{(n)} \varphi^{(1-n)} = 1, \quad (24) \]

\[ \frac{\varphi^{(n)}}{\varphi^{(-n)}} = \frac{n + l}{n - l}, \quad (25) \]

which follow from the unitarity and crossing-symmetry conditions (4) on nonphysical sheets. We use those Eqs. (4) here that were not involved in deriving formulas (23). Equation (24) has an obvious solution in the ring of meromorphic functions,

\[ \varphi^{(n)} = \frac{G(n)}{G(1-n)}, \quad (26) \]

where \( G(n) \) is an arbitrary entire function. Solution (26) can be represented in another form, \( \log \varphi^{(n)} = g(n - 1/2) \), where \( g(n - 1/2) \) is an odd function of
its argument. This form of the function \( \log \varphi^{(n)} \) is convenient for solving Eq.\((25)\), which can be easily rewritten as

\[
g(n + 1) + g(n) = \log \frac{n + 1/2 + l}{n + 1/2 - l}. \tag{27}
\]

We can find a particular solution of inhomogeneous difference equation \((27)\) by consecutively changing the unknown function according to the formula

\[
g_m(n) = g_{m+1}(n) + \log \frac{n + (-1)^m \alpha_{m+1}}{n - (-1)^m \alpha_{m+1}},
\]

where \( \alpha_k = 1/2 + l - k \) and \( g_0(n) = g(n) \). The function \( g_k(n) \) satisfies the equation

\[
g_k(n + 1) + g_k(n) = \log \frac{n + 1/2 + (-1)^k(l - k)}{n + 1/2 - (-1)^k(l - k)}
\]

and we obviously have \( g_l(n + 1) + g_l(n) = 0 \). The general solution of this homogeneous equation determines the function \( D(z) \), which enters formula \((6)\) and places no restrictions on the form of the invariant constraints on \( x_0, x_1, x_2 \). We therefore set \( g_l = 0 \) and obtain the expression for \( \varphi^{(n)} \):

\[
\varphi^{(n)} = \prod_{m=1}^l \frac{n - 1/2 - (-1)^m(1/2 + l - m)}{n - 1/2 + (-1)^m(1/2 + l - m)}. \tag{28}
\]

Excluding the parameter \( n \) from Eqs.\((23)\) and \((28)\), we obtain an equation determined by a homogeneous polynomial in \( x_0, x_1, x_2 \) of degree \( l + 1 \); it gives Eq.\((13)\) for \( l = 1 \).

**Appendix 2**

We apply the developed method to the problem of scattering a pseudoscalar meson with unit angular momentum by a fixed nucleon with the same angular momentum. In this case, the crossing-symmetry matrix is given by expression \((15)\). We decompose the column \( S(z) \) into a sum of eigenvectors of the matrix \( A \):

\[
S(z) = s_1(z) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{4} s_2(z) \begin{pmatrix} 15 \\ -5 \\ 3 \end{pmatrix} + 2 \psi(z) \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}. \tag{29}
\]
For $q = 1$, $p = 0$, functional equation (5) in the limit $z \to \infty$ determines the fixed (rest) points of the problem. Returning from the basis $s_1(z), s_2(z), \psi(z)$ to the column $S(z)$, we have

$$S = \pm i \begin{pmatrix} -(2 \pm \sqrt{5}) \\ -\frac{1 \pm \sqrt{5}}{2} \\ \frac{1 \pm \sqrt{5}}{2} \end{pmatrix}.$$  

We can see from (30) that all rest points lie in the plane $S_2 + S_3 = 0$. This plane is invariant under the inversion transformation $I$ and the crossing-symmetry transformation $A$. In the plane $S_2 + S_3 = 0$, three-row crossing-symmetry matrix (15) passes into the two-row matrix $A_2$

$$A_2 = \frac{1}{3} \begin{pmatrix} 1 & -8 \\ -1 & -1 \end{pmatrix},$$

and the problem is thus reduced to finding two functions $S_1(z)$ and $S_2(z)$. Setting $z = 0$ and defining $X^{(k)} = S_1^{(k)}/S_2^{(k)}$, where $k$ is the number of the sheet of the Riemann surface, we see that the transition from the physical sheet to the sheet with the number $n$ is realized by the linear fractional transformation

$$X^{(n)} = \sqrt{5} \frac{\sqrt{5}(X^{(0)} - 2) (y_+ - y_-) + (X^{(0)} + 4) y_+ y_-}{(X^{(0)} + 4) (y_+ - y_-) + \sqrt{5}(X^{(0)} - 2) (y_+ y_-)},$$

where $y_\pm = (3 \pm \sqrt{5})/2$. The unitarity and crossing-symmetry requirements on $X^{(n)}$ give the condition

$$(X^{(0)} - 2)(X^{(0)} + 4) = 0$$

which determines $X^{(0)}$. Consequently, we obtain two different solutions, $X^{(0)} = 2$ and $X^{(0)} = -4$, which are compatible with the unitarity and crossing-symmetry requirements.

The ratio $S_1/S_2$ is thus determined for $z = 0$ on every nonphysical sheet of the Riemann surface defined by Conditions 2 with matrix (31), and to construct $S_1$ and
\( S_2 \), it suffices to find any of these functions. We set \( S_2(n) = \Phi(n) = -s_2(n) + \psi(n) \), where \( s_2 \) and \( \psi \) are the functions introduced in (29). This function satisfies the system of functional equations

\[
\Phi(1-n)\Phi(n) = 1 ,
\]

\( \Phi(n) = (1)n^{n+1/2} \frac{chlogy_{+}}{chlogy_{+}}, \quad X^{(0)} = 2 \) \hspace{1cm} \( (35) \)

\[
\Phi(n) = (1)n^{n+1/2} \frac{shlogy_{+}}{shlogy_{+}}, \quad X^{(0)} = -4 \)

Relation (32) is used in deriving Eq.(35). Equation (34) has the solution

\[
\Phi(n) = e^{g(n-1/2)},
\]

where \( g(n) \) is an arbitrary odd function, \( g(n) = -g(-n) \). Substituting (37) in (35) and changing \( n \to n + 1/2 \), we obtain the difference equation

\[
g(n + 1) + g(n) = log(-1)n^{n+1/2} \frac{ch(n+1)logy_+}{ch(nlogy_+)}, \quad X^{(0)} = 2 \]

\[
g(n + 1) + g(n) = log(-1)n^{n+1/2} \frac{sh(n+1)logy_+}{sh(nlogy_+)}, \quad X^{(0)} = -4 \]

for the unknown function \( g(n) \).

Solving Eq.(37) by the method of consecutive functional changes, we obtain

\[
g(n) = g_1(n) + g_\infty(n) + \sum_{m=0}^{\infty} G_m(n) \]

where \( g_\infty(n) = nlogy_+ \) and

\[
G_m(n) = log\frac{ch(n+1+2m)logy_+ch(n-2(m+1)logy_+)}{ch(n-1-2m)logy_+ch(n+2(m+1)logy_+)}, \quad X^{(0)} = 2 \]

\[
G_m(n) = log\frac{sh(n+1+2m)logy_+sh(n-2(m+1)logy_+)}{sh(n-1-2m)logy_+sh(n+2(m+1)logy_+)}, \quad X^{(0)} = -4 \]
The term \( g_{-1}(n) \) is introduced to take the factor -1 in Eq.(37) into account. We set \( e^{g_{-1}(n)} = \xi(n) \). The function \( \xi(n) \) solves the system of functional equations

\[
\xi(n+1)\xi(n) = -1, \quad \xi(n)\xi(-n) = 1.
\] (43)

The general solution of this system is expressed in terms of \( \theta \)-functions. We confine ourselves to the degenerate case here,

\[
\xi(n) = \tan \frac{\pi}{2}(n + \frac{1}{2})
\] (44)

We now use unitarity condition 1C. As a result, the function \( n \) considered as a function of the complex variable \( z \) solves the boundary-value problem and is given by

\[
n(z) = \frac{1}{\pi} \arcsin z + i\sqrt{z^2 - 1}\beta(z),
\] (45)

where \( \beta(z) = -\beta(-z) \) is an arbitrary meromorphic function. It follows from Eq.(42) that the Riemann surface of the model under consideration has algebraic ramification points at \( z = \pm 1 \) and a logarithmic ramification point at infinity. Formulas (32),(33),(37)-(39),(41), and (42) now give the general solution of the problem defined by Conditions 1 for crossing-symmetry matrix (31).
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