Width and Magnetic Field Dependence of Transition Temperature in Ultranarrow Superconducting Wires

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We calculate the transition temperature in ultranarrow superconducting wires as a function of resistance, wire width and applied magnetic field. We compare the results of first-order perturbation theory and the non-perturbative resummation technique developed by Oreg and Finkel’stein. The latter technique is found to be superior as it is valid even in the strong disorder limit. In both cases, the predicted additional suppression of the transition temperature due to the reduced dimensionality is strongly dependent upon the boundary conditions used. When we use the correct (zero-gradient) boundary conditions, we find that theory and experiment are consistent, although more experimental data is required to verify this systematically. We calculate the magnetic field dependence of the transition temperature for different wire widths and resistances in the hope that this will be measured in future experiments. The predicted results have a rich structure – in particular we find a dimensional crossover which can be tuned by varying either the width of the wire or its resistance per square.

I. INTRODUCTION

Recent experiments on disordered superconducting wires have led to renewed interest in the interplay of interaction and disorder in low-dimensional systems. These experiments have been made possible by the development of fabrication techniques using either stencils or carbon nanotubes, and involve wires of width as low as 50 Å and thickness as low as 10 Å. An obvious question to ask in such systems is how the transition temperature is affected by wire width and thickness. Decreasing the thickness of the wire increases the normal state resistance per square which is a measure of the amount of disorder in the system. This increase in disorder reduces the transition temperature by causing the electrons to move diffusively rather than ballistically, as discussed below. Decreasing the width of the wire leads to a crossover from two- to one-dimensional behavior, where increased fluctuation effects further reduce the transition temperature. Ultimately the combined disorder and fluctuation effects completely suppress superconductivity, and a superconductor-insulator transition results. The effect of dimensionality on this quantum phase transition is obviously very interesting from a theoretical point of view. As a first step to understanding this transition, we will compare predictions of transition temperature to experimental results. Unfortunately there is very little data available for comparison because each data point requires the fabrication of a new wire which is very costly in terms of experimental effort. We therefore suggest the application of a perpendicular magnetic field as a way to generate more data points for each wire and make predictions for this effect.

The most basic model used consists of electrons moving in a disordered background interacting via the Coulomb repulsion and a featureless phonon-mediated attraction. Simple as this model is, significant difficulties arise even when calculating the suppression of transition temperature in two-dimensional films. These difficulties will be compounded when looking at quasi one-dimensional wires, which are in a crossover regime, so it is most important that they are fully understood in the simpler two-dimensional case. The basic physics involved is not in dispute, and can be understood from the BCS-MacMillan formula for the transition temperature, $T_c$,

$$T_c = 1.13 \omega_D \exp \left( - \frac{1}{N(0) [\gamma - \mu^*]} \right),$$  \hspace{1cm} (1)

where $\omega_D$ is the Debye frequency (a typical phonon frequency), $N(0)$ is the electronic density of states per spin at the Fermi surface, $\gamma$ is the featureless attraction, and $\mu^*$ is the Coulomb pseudopotential, a measure of the Coulomb repulsion. The disorder causes the electrons to move diffusively rather than ballistically making them less efficient at screening the Coulomb repulsion. This increases $\mu^*$ and decreases $N(0)$, both effects leading to a decrease in $T_c$. We shall assume that the disorder does not affect phonon properties such as $\omega_D$ and $\gamma$, an assumption that is backed up by detailed calculation on at least one model of electron-phonon interaction. Other authors11 have questioned this assumption, and we intend to address this point in future work.

The first difficulty which occurs in two dimensions relates to the low-momentum singularity of the disorder-screened Coulomb potential, which is given by
\[ V_C(q, \omega_m) = \frac{4\pi e^2}{q^2} \frac{Dq^2 + |\omega_m|}{Dq^2 + Dq^2 + |\omega_m|} \approx \frac{1}{2N(0)} \frac{Dq^2 + |\omega_m|}{Dq^2}, \]  

where \( q_{TF} \) is the Thomas-Fermi screening wavenumber defined by \( q_{TF}^2 = 8\pi e^2 N(0) \), \( D = \frac{v_F^2 \tau}{2} \) is the diffusion constant, \( v_F \) is the Fermi velocity, and \( \tau \) is the elastic scattering time. This singularity leads to a strong suppression of the one-particle density of states which is seen experimentally in tunneling measurements. Naively inserting this density of states into Eqn. [1] predicts a large suppression of \( T_c \) which is not seen experimentally. This prediction is, in fact, incorrect due to a cancellation between the various diagrammatic contributions in perturbation theory. The apparently mysterious cancellation turns out to be due to gauge invariance and is thus expected to be a very robust phenomenon. The end result is that the Coulomb interaction is effectively featureless with strength \( N(0)V_C = 1/2 \). The fact that we can legitimately use a featureless interaction makes calculations much simpler. Our first task is therefore to check that this replacement is still valid in quasi-1D systems.

The other difficulty is that perturbation theory cannot take us to the superconductor-insulator transition as it is only valid for weak disorder, and \textit{ad hoc} attempts to extend its range of validity lead to unphysical reentrance problems. Oreg and Finkel’stein overcame this problem by developing a resummation technique, which has recently been shown to be related to strong-coupling theory. We reexamine their predictions for wires, and find, rather surprisingly, that the problem is extremely sensitive to the boundary conditions used. This unexpected sensitivity leads us to check all details and approximations used in the resummation technique. Upon satisfactorily completing the necessary checks, we compare theory and experiment. Although good agreement is obtained when we focus on the suppression of \( T_c \) due to increasing disorder, it is less good when we focus on the additional suppression due to decreasing wire width. This is largely due to the limited amount of experimental data (only about four data points for each of four wire widths). We propose the application of a perpendicular magnetic field as a means of generating more data with a minimum of effort. We extend the resummation technique to include a magnetic field, and make predictions for transition temperature as a function of magnetic field for different wire widths and thicknesses. The most interesting feature which emerges is a crossover from two- to one-dimensional behavior which can be tuned by either decreasing wire width or increasing the resistance per square. We hope that the appropriate experiments will be carried out in the near future.

The remainder of the paper is organised in the following manner. In section (II) we consider first-order perturbation theory in the absence of a magnetic field. This is the simplest approach possible, and we will use it as a testing ground for the validity of the approximations used. Specifically we will show that it is legitimate to use a featureless Coulomb interaction and ignore self-energy effects, but that the details of boundary conditions are important. In section (III) we make predictions for transition temperature using the resummation technique, all the necessary approximations having been justified in section (II), and then compare theory and experiment. In section (IV) we consider the effect of an applied magnetic field on the transition temperature. The dimensions of the wires are such that we must consider both orbital and Zeeman (spin) effects. As yet there is no experimental data available for comparison to the theory.

II. FIRST ORDER PERTURBATION THEORY

We will now calculate the transition temperature, \( T_c \), which we identify as the temperature at which the pair propagator, \( L(q, \Omega = 0) \), diverges. The latter is defined in Fig. 1(a) in terms of the BCS interaction, \( \gamma \), and the pair polarization function, \( P(q, 0) \), by

\[ L(q, 0)^{-1} = \gamma^{-1} - P(q, 0), \]

Let us first consider mean field theory, which corresponds to zeroth order in disorder perturbation theory, for which the expansion parameter is \( 1/k_F \ell \), where \( k_F \) is the Fermi wavenumber and \( \ell = v_F \tau \) is the elastic mean free path. The mean field pair polarization function is shown in Fig. 1(b), and leads to the pair propagator

\[ L_0(q, 0)^{-1} = N(0) \left[ \ln \left( \frac{T}{T_{c0}} \right) + \psi \left( \frac{1}{2} + \frac{Dq^2}{4\pi T} \right) - \psi \left( \frac{1}{2} \right) \right]. \]

The first-order perturbative corrections to the pair polarization function, \( \delta P(q, 0) \), are shown in Fig. 1(c), and lead to a suppression in transition temperature

\[ \ln \left( \frac{T_c}{T_{c0}} \right) = \frac{\delta P(q, 0)}{N(0)}. \]
Note that we evaluate the pair propagator at non-zero external momentum, \( q \), since we will eventually consider the effect of a magnetic field.

The five diagrams of Fig. 1(c) can be understood in physical terms: diagrams 1 and 2 are self-energy diagrams and give corrections to the electronic density of states, \( N(0) \); diagrams 3 and 4 are pseudopotential diagrams and give corrections to the effective Coulomb repulsion, \( \mu^* \); diagram 5 involves the effect of Coulomb interaction on superconducting fluctuations. We will first perform the calculations exactly, and demonstrate the cancellation of the low-momentum singularity in the Coulomb interaction which occurs between diagrams 1–4 and 5. We show that this is equivalent to considering only diagrams 1–4 with a featureless Coulomb interaction of strength \( N(0)V_C = 1/2 \).

In other words, the only purpose of diagram 5 is to remove the low-momentum singularity, as required by gauge invariance. Finally we show that the self-energy diagrams 1 and 2 can be ignored as they give a much smaller contribution than diagrams 3 and 4. The contributions from the five diagrams are as follows:

\[ \delta P_1 = -4\pi N(0)T \sum_{\epsilon_l} T \sum_{\omega_m} \frac{1}{aL} \sum_{q'} \frac{Dq^2 + Dq'^2 + 3|\epsilon_l| + |\epsilon_l + \omega_m|}{(Dq^2 + 2|\epsilon_l|)(Dq^2 + 2|\epsilon_l + |\epsilon_l + \omega_m|)\frac{1}{(Dq^2 + 2|\epsilon_l|)(Dq^2 + 2|\epsilon_l + \omega_m|)(D(q + q')^2 + |\epsilon_l + \omega_m|\epsilon_l)} V_C(q', \omega_m) \theta(-\epsilon_l + \omega_m)} \]

\[ \delta P_2 = 4\pi N(0)T \sum_{\epsilon_l} T \sum_{\omega_m} \frac{1}{aL} \sum_{q'} \frac{1}{(Dq^2 + 2|\epsilon_l|)(Dq^2 + 2|\epsilon_l + \omega_m|)(D(q + q')^2 + |\epsilon_l + \omega_m|\epsilon_l)} V_C(q', \omega_m) \theta(-\epsilon_l + \omega_m)} \]

\[ \delta P_3 = -4\pi N(0)T \sum_{\epsilon_l} T \sum_{\omega_m} \frac{1}{aL} \sum_{q'} \frac{Dq^2 + Dq'^2 + |\epsilon_l| + |\epsilon_l + \omega_m|}{(Dq^2 + 2|\epsilon_l|)(Dq^2 + 2|\epsilon_l + \omega_m|)(D(q + q')^2 + |\epsilon_l + \omega_m|\epsilon_l)} V_C(q', \omega_m) \theta(-\epsilon_l + \omega_m)} \]

\[ \delta P_4 = 4\pi^2 N(0)^2 T \sum_{\omega_m} \frac{1}{aL} \sum_{q'} \left[ T \sum_{\epsilon_l} \left\{ \frac{|\omega_m|}{(Dq^2 + 2|\epsilon_l|)(Dq^2 + 2|\epsilon_l + \omega_m|)(D(q + q')^2 + |\epsilon_l + \omega_m|\epsilon_l)} V_C(q', \omega_m) \right\} \right] \]

\[ (7) \]

where \( \epsilon_l = (2l + 1)\pi T \) is a Fermi Matsubara frequency, \( \omega_m = 2m\pi T \) is a Bose Matsubara frequency, and \( a, L \) are the width and length of the wire respectively. The diffusive form of the denominators in Eqn. (6) is valid only for \( |\omega_m| < 1/\tau \), so we introduce the upper cutoff, \( 1/\tau \), in all these sums.

For the rest of this section we will work at zero magnetic field, so we set \( q = 0 \). The terms \( \delta P_1-5 \) have previously been summed for 2D films to give the expression

\[ \ln \left( \frac{T_c}{T_{c0}} \right) = -\frac{1}{2\pi T} \sum_{\omega_m} \frac{Dq^2}{2\omega_m} \psi \left( \frac{1}{2} + \frac{\omega_m}{2\pi T} \right) \]

\[ + \frac{4(Dq^2)^2}{|\omega_m - (Dq^2)^2|^2} \left[ \psi \left( \frac{1}{2} + \frac{\omega_m}{2\pi T} \right) - \psi \left( \frac{1}{2} + \frac{1}{2} \right) \right] \]

The 1/q^2 singularity in the screened Coulomb potential \( V_C(q', \omega_m) \) is cancelled since there is an overall factor of \( Dq^2 \) in the above equation. Note that there is, in fact, no divergence at \( Dq^2 = \omega_m \), but this can only be seen by Taylor expanding to second order. The need to perform this expansion makes the calculation involved, especially at the level of numerics. To allow simplification, we define \( y = Dq^2 / 2\pi T \) and \( m = \omega_m / 2\pi T \), which leads to the result

\[ \ln \left( \frac{T_c}{T_{c0}} \right) = -\frac{1}{4\pi^2 N(0)T} \sum_{m=1}^{M} \frac{1}{aL} \sum_{q'} \left\{ \frac{2}{m} \left[ \psi \left( \frac{1}{2} + m \right) - \psi \left( \frac{1}{2} \right) \right] + \psi' \left( \frac{1}{2} + m \right) \right\} \frac{1}{y + m} \]

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where we have split off the most divergent term at large momentum and frequency, and $M = 1/2\pi T\tau$. To proceed further we must consider the $q'$-sum in various dimensions. In two dimensions the sum becomes an integral, and it is most sensible to perform the cut-off in a circularly symmetric manner

$$\frac{1}{aL} \sum_{q'} \rightarrow \int_{0}^{1/\tau} \frac{2\pi q' dq'}{(2\pi)^2} = \frac{T}{2D} \int_{0}^{M} dy.$$  

(9)

In quasi-1D wires, we must replace the integral in the transverse direction by a sum,

$$\frac{1}{aL} \sum_{q'} \rightarrow \frac{1}{a} \sum_{q'_a} \int_{0}^{1/\tau} \frac{dq'_a}{2\pi},$$  

(10)

where $q'^2 = q'^2 + q_a^2$. At this point we have to decide which boundary conditions to use in the transverse direction. To illustrate the strong effect this has we will compare periodic and zero-gradient boundary conditions, the details of which are listed in Table 1. The zero-gradient boundary conditions are the physically correct ones for a superconductor-insulator boundary as discussed in Ref. [2]. To perform the sum over $q'$ note that we can write $y = y_L + y_a$ where $y_L = Dq'^2/2\pi T$ and $y_a = Dq_a^2/2\pi T$, and if we define $z = \sqrt{y_L}$, the $q'_L$-integral becomes

$$\int_{0}^{1/\tau} dq'_L = \int_{0}^{\sqrt{M}} \frac{dz}{z}.$$  

(11)

where $L_T = \sqrt{2\pi D/T}$ is the thermal length. Note that $y = z^2 + 4p(a/L_T)^2$ for zero-gradient boundary conditions and $y = z^2 + p(a/L_T)^2$ for periodic boundary conditions. The final result of perturbation theory is then given by Eqn. (8) with the prefactor and sums becoming

$$\frac{-4y}{(m+y)(m-y)^2} \left\{ \begin{array}{l} \left[ \psi \left( \frac{1}{2} + m \right) - \psi \left( \frac{1}{2} \right) \right]^2 - \left[ \psi \left( \frac{1}{2} + m \right) - \psi \left( \frac{1}{2} \right) \right] + \frac{1}{2} (y - m) \psi' \left( \frac{1}{2} + m \right) \end{array} \right\}$$  

(8)

\[ \sum_{m=1}^{M} \frac{1}{aL} \sum_{q'} = -\frac{1}{4\pi^2 N(0) T} \sum_{m=1}^{M} \frac{1}{aL} \sum_{q'} \rightarrow -t \sum_{m=1}^{M} \int_{0}^{M} dy \\
\rightarrow -t \sum_{m=1}^{M} \int_{0}^{M} dy,$$

\[ \rightarrow -t \sum_{m=1}^{M} \int_{0}^{M} dy, \]  

(12)

In the above, $t$ is a dimensionless measure of disorder given by

$$t = \frac{1}{8\pi^2 N(0) D} = \frac{R_a}{R_0}$$  

(13)

where $R_a$ is the normal state resistance per square, and $R_0 = 2\pi h/e^2 \approx 162k\Omega$. Note that $t$ only differs from the previously discussed parameter $1/k_F\ell$ by a factor $2\pi$.

Having shown that the low-momentum singularity in the screened Coulomb potential is cancelled by diagram 5, we recalculate the suppression of transition temperature using a featureless Coulomb potential of magnitude $N(0)V_C = 1/2$, and including only diagrams 1–4. We have previously shown[2] that the above equations simplify to give

$$\delta P_{12} = -T \sum_{\omega_m > 0} \left[ \psi \left( \frac{1}{2} + \frac{\omega_m}{2\pi T} \right) - \psi \left( \frac{1}{2} \right) \right] \frac{1}{(Dq'^2 + \omega_m)^2},$$

$$\delta P_{14} = -2T \sum_{\omega_m > 0} \left[ \psi \left( \frac{1}{2} + \frac{\omega_m}{2\pi T} \right) - \psi \left( \frac{1}{2} \right) \right] \frac{1}{\omega_m} \sum_{\omega_m} \frac{1}{(Dq'^2 + \omega_m)^2}.$$  

(14)

For convenience we define dimensionless sums $I_1(m)$ and $I_2(m)$ as

$$I_1(m) = \frac{4\pi D}{aL} \sum_{q'} \frac{1}{(Dq'^2 + |\omega_m|)^2},$$

$$I_2(m) = \frac{4\pi D}{aL} \sum_{q'} \frac{2\pi T}{(Dq'^2 + |\omega_m|)^2}.$$  

(15)
which are evaluated in two dimensions and quasi-one dimension in Appendix A. When we combine Eqns. (14) and (17) we obtain the formula for transition temperature suppression assuming a featureless Coulomb potential,

$$\ln \left( \frac{T_c}{T_{c0}} \right) = -t \sum_{m=1}^{M} \left[ \psi \left( \frac{1}{2} + m \right) - \psi \left( \frac{1}{2} \right) \right] \left[ I_2(m) + \frac{2I_1(m)}{m} \right].$$

If we ignore the self-energy diagrams ($\delta P_{12}$), the $I_2(m)$ term in the above equation vanishes to leave

$$\ln \left( \frac{T_c}{T_{c0}} \right) = -2t \sum_{m=1}^{M} \left[ \psi \left( \frac{1}{2} + m \right) - \psi \left( \frac{1}{2} \right) \right] \frac{2I_1(m)}{m}.$$  

Having derived expressions for the suppression of transition temperature using various approximations and boundary conditions, we perform a detailed numerical analysis, the results of which can be summarised as follows. The transition temperature is very sensitive to the boundary conditions used – periodic boundary conditions do not yield a dependence of transition temperature on wire width, whilst zero-gradient boundary conditions show a systematic suppression as wire width is decreased. The predictions are also sensitive to the value of the thermal length, $L_T$, since the width, $a$, enters only in the dimensionless combination $a/L_T$. Experimentally $L_T$ may be known only roughly and it may be necessary to fit its value. On the other hand, the predictions are not very sensitive to either the form of the Coulomb interaction used (Eqns. (8) and (16)), or the inclusion of self-energy corrections (Eqns. (16) and (17)). The slight difference between these approaches vanishes when we fit the initial slope of the $T_c(R_a)$ curve to experiment, which sets the value of the BCS attraction, $\gamma$. This demonstrates that the simplest approach – featureless Coulomb potential and no self-energy diagrams – is valid. This is important since these simplifications are necessary for the resummation technique discussed in the next section. It is necessary to use this technique to cure the unphysical re-entrance problems which beset perturbation theory at low temperature.

### III. THE RESUMMATION TECHNIQUE

In this section we will review the resummation technique developed by Oreg and Finkel’stein (OF) as applied to the problem of $T_c$ suppression in quasi-1D wires. This method involves calculating the pair-amplitude, $\Gamma(\epsilon_n, \epsilon_l)$, which is a matrix with Fermi Matsubara frequencies labelling its rows and columns. The transition temperature is identified as the temperature at which this matrix becomes singular. OF obtained an equation for $\Gamma$ by including only pseudopotential diagrams, and solved it using matrix perturbation theory with zero-gradient boundary conditions. We have performed calculations to show that it is legitimate to include only these diagrams, and that it is essential to use zero-gradient boundary conditions, as described in section II for perturbation theory. In a previous paper we have shown that the equation for $\Gamma$ can be solved exactly at no extra numerical cost, so the matrix perturbation theory is unnecessary, and indeed gives incorrect results at large disorder. Therefore we shall include only pseudopotential diagrams and solve the equation for $\Gamma$ using an exact diagonalization method with zero-gradient boundary conditions.

Before we proceed further, let us briefly describe the details of the technique. The ladder summation involved is shown in Fig. 2. The block $t\Lambda$, an effective Coulomb pseudopotential, contains diagrams 3 and 4 of Fig. 1(c); the block $\Sigma$, the self-energy of the Cooperon impurity ladder, $C$, contains diagrams 1 and 2 of Fig. 1(c). We have included the self-energy diagrams for completeness, although their effects can be ignored in calculations, as discussed above. The equation for $\Gamma$ is seen to be

$$\Gamma(\epsilon_n, \epsilon_l) = -|\gamma| + t\Lambda(\epsilon_n, \epsilon_l) - 2\pi T \sum_{m=0}^{M} \left[ -|\gamma| + t\Lambda(\epsilon_n, \epsilon_m) \right] \frac{1}{|\epsilon_m|} \Gamma(\epsilon_m, \epsilon_l)$$

where the contributions to $t\Lambda$ are

$$t\Lambda_3(\epsilon_n, \epsilon_m) = \frac{1}{2\pi N(0)} \frac{1}{aL} \sum_{q'} \frac{Dq'^2 + Dq^2 + |\epsilon_n| + |\epsilon_m|}{|Dq'^2 + |\epsilon_n| + |\epsilon_m||} \theta(-\epsilon_n\epsilon_m)$$

$$t\Lambda_4(\epsilon_n, \epsilon_m) = \frac{1}{2\pi N(0)} \frac{1}{aL} \sum_{q'} \frac{1}{|Dq'^2 + |\epsilon_n| + |\epsilon_m||} \theta(\epsilon_n\epsilon_m).$$

We have kept non-zero external momentum $q$ so that we can include magnetic field effects later. For now we consider the case of zero magnetic field, so we set $q = 0$, and the two contributions to $t\Lambda$ are identical. It follows that $\Lambda(\epsilon_n, \epsilon_m)$ does not depend upon the relative sign of $\epsilon_n$ and $\epsilon_m$ and has the form
\[ \Lambda(\epsilon_n, \epsilon_m) = I_1(n + m + 1), \]  

where \( I_1 \) is defined in Eqn. (\?). Eqn. (\?) can then be written in matrix form as

\[ \hat{\Gamma} = -|\gamma|\hat{1} + t\hat{\Lambda} - [-|\gamma|\hat{1} + t\hat{\Lambda}]\hat{\gamma}^{-1}\hat{\Gamma}, \]  

where \( \hat{\Gamma}_{nm} = \Gamma(\epsilon_n, \epsilon_m), \hat{1}_{nm} = 1, \hat{\Lambda}_{nm} = \Lambda(\epsilon_n, \epsilon_m) \) and \( \hat{\epsilon}_{nm} = (n + 1/2)\delta_{nm} \). This has the solution

\[ \hat{\Gamma} = \hat{\epsilon}^{1/2}(\hat{1} - |\gamma|\hat{\Pi})^{-1}\hat{\epsilon}^{-1/2}(-|\gamma|\hat{1} + t\hat{\Lambda}), \]  

where

\[ \hat{\Pi} = \hat{\epsilon}^{-1/2}[\hat{1} - |\gamma|t\hat{\Lambda}]\hat{\epsilon}^{-1/2}, \]  

and \( \hat{I}_{nm} = \delta_{nm} \) is the identity matrix. It follows that when the matrix \( \hat{\Pi} \) has an eigenvalue equal to \( 1/|\gamma| \), the pair amplitude diverges, and we have found \( T_c \). In other words, we need to solve the eigenvalue equation

\[ [|\gamma|^{-1}\hat{I} - \hat{\Pi}(T_c)] |\psi\rangle = 0. \]  

We solve Eqn. (\?) by diagonalizing the matrix \([|\gamma|^{-1}\hat{I} - \hat{\Pi}(T)]\) numerically, and decreasing the temperature until its lowest eigenvalue equals zero. Since we only need the lowest eigenvalue, and the matrix has such a simple structure, the Lanczos method is very efficient here. The only minor complication is that the matrix \( \hat{\Pi}(T) \) depends upon \( T \) both through the dependence of \( \hat{\Lambda} \) upon \( T \) and through its rank \( M = 1/2\pi T \). We therefore start at the mean-field value of \( M \), which we will call \( M_0 \), and decrease the temperature by increasing \( M \) successively by one. We diagonalize the matrix \([|\gamma|^{-1}\hat{I} - \hat{\Pi}]\) for each value of \( M \) until the lowest eigenvalue changes sign. At this point we have found \( T_c \) for the given problem, and \( T_c/T_{c0} = M_0/M \). This is obviously the method of choice, since no approximation is involved.

We can now compare theory and experiment: there are three sets of experimental data that we know about – those of Graybeal et al. (1987)\( ^9 \) Sharifi et al. (1993)\( ^10 \) and Xiong et al. (1997)\( ^2 \). The data of Graybeal et al consists of \( T_c \) vs \( a \) for wires of constant thickness (and hence constant resistance per square). Their data shows that the relative suppression of \( T_c \) varies as \( 1/a^2 \) with a proportionality constant independent of \( R_a \). We have performed calculations which predict a dependence roughly of the form \( 1/a \), but with a proportionality constant dependent on \( R_a \). We will therefore not consider this data further. The data of Sharifi et al and Xiong et al are both of the form of \( T_c(R_a, a) \) vs \( R_a \) for wires of different width, but only the latter shows a consistent trend with decreasing width, so it is only this data that we shall attempt to fit. The fact that this is the most recent data is reassuring, since it suggests that more recent fabrication techniques have produced better quality wires.

In Fig. 3 we compare the data of Xiong et al with our theoretical predictions. In doing this we encounter two difficulties. The first is that experiment and theory do not agree for some of the 2D films used. We assume that this is due to some additional physical processes that will affect samples in the same manner independent of width. We therefore multiply our theoretical predictions by \( \delta_{nm} \). We will therefore assume a different \( T_c(2D, \text{expt})/T_c(2D, \text{theory}) \) to factor out this effect. The second difficulty is that wires of different widths do not appear to have the same value of \( T_c \) in the clean limit, \( T_c(0, a) \). We therefore assume a different \( T_c(0, a) \) for each wire. Upon making these two assumptions, we see that the agreement between theory and experiment is good. However plotting the data in this form emphasises the suppression of transition temperature due to disorder, whereas we are interested in the additional suppression due to the decrease in width. For this reason we need to replot the data to emphasise this latter point.

In Fig. 4 we replot Fig. 3 in terms of the additional relative suppression, \( \Theta(R_a, a) \), which is defined by

\[ \Theta(R_a, a) = \frac{T_c(R_a, a)/T_c(0, a)}{T_c(R_a, 2D)/T_c(0, 2D)}. \]  

This quantity focuses on the basic physical phenomenon we are interested in – the additional suppression of transition temperature caused by the reduced dimensionality in the quasi-1D wires. It has the additional benefit of factoring out the two difficulties discussed in the previous paragraph. When plotted in this form, the agreement between theory and experiment no longer appears to be as good. This is because our theory, and indeed any theory, predicts a smooth dimensional crossover whereas the experimental data shows an abrupt crossover between the 2D film and the 1000 Å wire, which lie on one curve, and the 580 Å., 350 Å., 250 Å., which lie on a different curve. The fact that the data for the three narrowest wires fall on top of each other might be due to them having the same effective dimensionless width \( a/L_T \). If we allowed ourselves the additional freedom of fitting this parameter separately for each wire, we
would obtain good agreement between theory and experiment, but we do not believe this is justified. We have used the value of the thermal length quoted in Sharifi et al (viz \( L_T = 400 \text{Å} \) at \( T = 4K \), which corresponds to \( L_T = 300 \text{Å} \) at \( T_0 = 6.9K \)).

The basic problem is clearly that there is not enough experimental data available – there are only 19 points in total, and each of these corresponds to a different wire. To carry out a more systematic comparison between theory and experiment, one would need wires of many different resistances (thicknesses) for each value of width. This would, however, be very costly in terms of experimental effort. A less costly method for generating more data would be to apply a magnetic field to a given wire. We therefore extend our theory to include the effect of such a field in the next section.

### IV. EFFECT OF AN APPLIED MAGNETIC FIELD

We will now consider the effect of an applied perpendicular magnetic field. There are two effects to consider here: the orbital effect, caused by the magnetic field scrambling the relative phase of the electrons in a Cooper pair, and the Zeeman effect, caused by the magnetic field lifting the degeneracy of up and down spins. Let us first consider the relative importance of the two effects. The orbital effect will destroy superconductivity when the magnetic dephasing rate, \( 1/\tau_H \), is of the order of \( k_B T_c \), which gives \( H \approx 4T \) for a Pb wire of width 250Å. The Zeeman splitting will destroy superconductivity when its magnitude \( \mu_B H \) is of the order of \( k_B T_c \), which corresponds to \( H \approx 10T \) for Pb. Since these fields have the same order of magnitude, we will need to consider both effects.

We first examine the orbital effect. The magnetic field causes the eigenvalues of the Cooperon operator to change from the zero-field values

\[
Dq^2 = D \left[ \left( \frac{2\pi n}{L} \right)^2 + \left( \frac{\pi p}{a} \right)^2 \right], \quad n = 0, \pm 1, \pm 2 \ldots, \quad p = 0, 1, 2 \ldots
\]

to the eigenvalues, \( \lambda \), of the minimally coupled Cooperon operator

\[
D(-i\nabla + 2eA)^2\psi(x, y) = D \left[ \left( -i \frac{\partial}{\partial x} + 2eHy \right)^2 + \left( -i \frac{\partial}{\partial y} \right)^2 \right] \psi(x, y) = \lambda \psi(x, y),
\]

where we have chosen the gauge \( A = (-Hy, 0, 0) \). The eigenfunctions \( \psi(x, y) \) must satisfy zero-gradient boundary conditions on the sides of the wire, \( y = \pm a/2 \), and periodic boundary conditions at the ends of the wire, \( x = 0, L \). The solution has the separable form

\[
\psi(x, y) = \phi_{q_n, p}(y)e^{iqx}, \quad q_n = \frac{2\pi n}{L}, \quad n = 0, \pm 1, \pm 2 \ldots
\]

where \( \phi_{q_n, p} \) satisfies the equation

\[
D \left[ -\frac{d^2}{dy^2} + (q + 2eHy)^2 \right] \phi_{q_n, p}(y) = \lambda_{q_n, p} \phi_{q_n, p}(y), \quad \frac{d\phi_{q_n, p}}{dy}(\pm a/2) = 0,
\]

and \( p = 0, 1, 2 \ldots \) labels the eigenvalues in ascending order.

To solve this eigenvalue problem numerically, we convert it into a matrix diagonalization problem by working in a truncated function space spanned by eigenfunctions of the zero-field problem. The lowest eigenvalue, \( \lambda_{0,0} \), equals \( 2/\tau_H \), where \( 1/\tau_H \) is the magnetic dephasing rate which appears in the Abrikosov-Gor’kov equation for \( T_c(H) \). There are two asymptotic limits where \( 1/\tau_H \) is easily determined: the 2D limit, where the problem becomes that of a simple harmonic oscillator, with \( 1/\tau_H = DeH \); and the 1D limit, where the eigenfunction is simply a constant, and hence (using perturbation theory) \( 1/\tau_H = De^2H^2a^2/6 \). The 2D to 1D crossover occurs when the width, \( a \), is of the order of the magnetic length, \( L_H = (\hbar/eH)^{1/2} \), which occurs at the field \( H_0 = \hbar/ea^2 \). At larger magnetic fields the cyclotron orbit of the electron lies entirely within the wire, and hence we are in the 2D limit; at smaller magnetic fields the electron hits the sides of the wire before completing a cyclotron orbit, and hence we are in the 1D limit. To summarise,

\[
\frac{1}{\tau_H} = \begin{cases} DeH & H \gg H_0 \text{ (} a \gg L_H \text{)} \quad 2D \\ De^2H^2a^2/6 & H \ll H_0 \text{ (} a \ll L_H \text{)} \quad 1D \end{cases}
\]
These eigenvalues then enter into our equations for the pair propagator in two ways: (i) the term $Dq^2$ in the $q$-dependent pair propagator is replaced by the lowest eigenvalue $\lambda_{0,0}$; (ii) sums over functions of $Dq^2$ in diagrams which involve Cooperonic ladders (e.g. diagrams 2 and 4 in Fig. 1(c)) are replaced by sums over $\lambda_{p_n,p}$. Obviously sums over functions of $Dq^2$ in diagrams which involve diffusion ladders (e.g. diagrams 1 and 3 in Fig. 1(c)) remain unchanged. It is numerically very expensive to perform the sums over all the eigenvalues, so we would like to see how important it is to use these eigenvalues in any sums. To answer this question we again use perturbation theory as our testing ground. We perform the Cooperonic sum in two different ways - both correctly, and by replacing it by the corresponding diffusion sum. In both cases we still make the replacement of $Dq^2$ by $\lambda_{0,0} = 2/\tau T$. The perturbative equations are then

\[
\begin{align*}
\frac{\delta P_1}{N(0)} &= t \frac{M}{2} \sum_{m=1}^{M} \left\{ 2 \left[ \psi \left( \frac{1}{2} + m + \alpha \right) - \psi \left( \frac{1}{2} + \alpha \right) \right] I_2(m) + \left[ \psi' \left( \frac{1}{2} + \alpha \right) - \psi' \left( \frac{1}{2} + m + \alpha \right) \right] I_1(m) \right\} \\
\frac{\delta P_2}{N(0)} &= t \frac{\pi}{2} \sum_{m=1}^{M} \left[ \psi' \left( \frac{1}{2} + \alpha \right) - \psi' \left( \frac{1}{2} + m + \alpha \right) \right] C_1(m) \\
\frac{\delta P_3}{N(0)} &= -t \sum_{m=1}^{M} \frac{1}{m + 2\alpha} \left[ \psi \left( \frac{1}{2} + m + \alpha \right) - \psi \left( \frac{1}{2} + \alpha \right) \right] (I_1(m) + 2\alpha I_2(m)) \\
\frac{\delta P_4}{N(0)} &= -t \sum_{m=1}^{M} \frac{1}{m + 2\alpha} \left[ \psi \left( \frac{1}{2} + m + \alpha \right) - \psi \left( \frac{1}{2} + \alpha \right) \right] C_1(m)
\end{align*}
\]

where $\alpha = 1/2\pi T \tau H$. The diffusive sums $I_1(m)$ and $I_2(m)$ are defined as in Eqn. (14) except that here we perform the momentum sums in the longitudinal direction exactly to give

\[
\begin{align*}
I_1(m) &= \frac{4\pi D}{aL} \sum_q \frac{1}{Dq^2 + 2\pi T m} = \frac{L_T^2}{\pi L a} \sum_{n=-n_0}^{n_0} \sum_{k=0}^{k_0} \frac{1}{m + \left( \frac{k a}{r_0} \right)^2 + \left( \frac{n L}{r_0} \right)^2} \\
I_2(m) &= \frac{4\pi D}{aL} \sum_q \frac{2\pi T}{(Dq^2 + 2\pi T m)^2} = \frac{L_T^2}{\pi L a} \sum_{n=-n_0}^{n_0} \sum_{k=0}^{k_0} \frac{1}{m + \left( \frac{k a}{r_0} \right)^2 + \left( \frac{n L}{r_0} \right)^2}^2
\end{align*}
\]

and the Cooperonic sum, $C_1$, which is now distinct from $I_1$, is given by

\[
C_1(m) = \frac{4\pi D}{aL} \sum_q \frac{1}{Dq^2 + 2\pi T m} = \frac{L_T^2}{\pi L a} \sum_{n=-n_0}^{n_0} \sum_{k=0}^{k_0} \frac{1}{m + \lambda_{nk}/2\pi T}.
\]

We found essentially no difference between using the correct $C_1$, or replacing it by $I_1$, over a wide range of magnetic fields. It follows that the only replacement we need to make is of $Dq^2$ by $\lambda_{0,0}$, which greatly reduces the numerical cost.

Let us next consider the effect of the Zeeman splitting. This leads to changes in both the mean field and perturbative terms, and we will show that only the former effect is important. To do this we will evaluate the mean-field pair polarization function and the first-order perturbation correction, $\delta P$, both with and without Zeeman splitting. We find that the latter is essentially unaffected. To treat the Zeeman effect, we need formulas for the Cooperon impurity ladder and Hikami boxes $\mathbb{F}$, where we note that the boxes in diagrams 1 and 3 of Fig. 1(c) will give different contributions. The non-zero elements of the spin-dependent Cooperon ladder, $C_{\alpha\beta\gamma\delta}$, are

\[
C_{\uparrow\uparrow\downarrow\downarrow} = \frac{1}{2\pi N(0)\tau^2} \frac{1}{Dq^2 + |\omega| + |\omega'| - 2i\Delta}, \quad C_{\downarrow\downarrow\uparrow\uparrow} = \frac{1}{2\pi N(0)\tau^2} \frac{1}{Dq^2 + |\omega| + |\omega'| + 2i\Delta} = C_{\uparrow\uparrow\downarrow\downarrow}^*
\]

where $\Delta = \mu_B H$ is the Zeeman splitting. The appropriate spin combination which occurs in an s-wave pair propagator is $C_{\uparrow\downarrow\uparrow\downarrow} + C_{\downarrow\uparrow\downarrow\uparrow} = 2\Re e C_{\uparrow\uparrow\downarrow\downarrow}$. This leads to the mean field formula for $T_c$,

\[
\ln \left( \frac{T_c}{T_c^0} \right) = \psi \left( \frac{1}{2} \right) - \Re e \left[ \psi \left( \frac{1}{2} + \kappa \right) \right]
\]

where $\kappa = \alpha - i\beta$ with $\beta = \Delta/2\pi T_c = \mu_B H/2\pi T_c$. To calculate the first-order pair polarization terms $\delta P_{\uparrow\downarrow\downarrow\downarrow}$, we note that: (a) the diffusion ladders on Coulomb vertices are unchanged; (b) Cooperon ladders pick up an extra $\mp 2i\Delta$ as
described above; (c) the Hikami box in \( \delta P_1 \) picks up an extra \( \mp 2i\Delta \), whilst the box in \( \delta P_3 \) is unaffected. This leads to the formulas

\[
\frac{\delta P_1}{\mathcal{N}(0)} = -\frac{t}{2} \Re \sum_{m=1}^{M} \left\{ 2 \left[ \psi \left( \frac{1}{2} + m + \kappa \right) - \psi \left( \frac{1}{2} + \kappa \right) \right] I_2(m) + \left[ \psi' \left( \frac{1}{2} + \kappa \right) - \psi' \left( \frac{1}{2} + m + \kappa \right) \right] I_1(m) \right\}
\]

\[
\frac{\delta P_2}{\mathcal{N}(0)} = +\frac{t}{2} \Re \sum_{m=1}^{M} \left\{ \psi' \left( \frac{1}{2} + \kappa \right) - \psi' \left( \frac{1}{2} + m + \kappa \right) \right\} \mathcal{C}_1(m)
\]

\[
\frac{\delta P_3}{\mathcal{N}(0)} = -t \Re \sum_{m=1}^{M} \frac{1}{m + 2\kappa} \left[ \psi \left( \frac{1}{2} + m + \kappa \right) - \psi \left( \frac{1}{2} + \kappa \right) \right] (I_1(m) + 2\alpha I_2(m))
\]

\[
\frac{\delta P_4}{\mathcal{N}(0)} = -t \Re \sum_{m=1}^{M} \frac{1}{m + 2\kappa} \left[ \psi \left( \frac{1}{2} + m + \kappa \right) - \psi \left( \frac{1}{2} + \kappa \right) \right] \mathcal{C}_1(m)
\]

where \( I_1(m) \) and \( I_2(m) \) are the same as before, whilst

\[
\mathcal{C}_1(m) = \frac{4\pi D}{aL} \sum_{q} \frac{1}{Dq^2 + 2\pi Tm - 2i\Delta} = \frac{L^2}{\pi aL} \sum_{n=-n_0}^{n_0} \sum_{k=0}^{k_0} \frac{1}{m + 2\kappa + \left( \frac{kL}{2a} \right)^2 + \left( \frac{qL}{2} \right)^2}
\]

Calculating the \( \delta P_{1-4} \) as above, with \( \beta \) set to zero, we find essentially no difference over a wide range of magnetic fields until we get to widths of order 10\( \Omega \). It follows that we only need consider the Zeeman splitting at the mean field level. Even this is a small effect, only noticeable for wires narrower than about 150\( \Omega \).

Let us finally perform calculations of \( T_c \) using the resummation technique. From the discussions above, we now know that to include magnetic field effects we need only replace \( Dq^2 \) in Cooperons by \( 2/\tau_H \), which must be calculated numerically, and include the Zeeman splitting at the mean field level. The Cooperon denominator \( Dq^2 + 2|\epsilon_m| \) then becomes \( 2|\epsilon_m| + 2/\tau_H - 2i\Delta \), and hence \( \epsilon_m = (n + 1/2 + \kappa)\delta_{nm} \), which is now complex. Because the two spin configurations which occur in an s-wave pair propagator are complex conjugates of each other, we end up taking the real part of the complex matrix equation. The equation for \( \Lambda_{nm} \) becomes

\[
\Lambda_{nm} = \begin{cases} 
I_1(n + m + 1 + 2\alpha) & \epsilon_n<\epsilon_m > 0 \\
I_1(n + m + 1 + 2\alpha) + 2I_2(n + m + 1 + 2\alpha) & \epsilon_n<\epsilon_m < 0 
\end{cases}
\]

where we have not included the Zeeman effect as we have shown this to be much smaller in perturbation theory.

We are now in a position to calculate the transition temperature, \( T_c \), as a function of resistance per square, \( R_\Omega \), wire width, \( a \), and applied magnetic field, \( H \). This obviously leads to a wide choice of ways to present results. Experimentally one would probably vary magnetic field on a sample of fixed width and thickness, and measure \( T_c(H) \). In Fig. 5 we therefore plot \( T_c(H) \) for wires of different resistances per square, \( R_\Omega = 0\Omega, 100\Omega, 200\Omega \), and widths \( a = 500\Omega, 600\Omega, 750\Omega, 2D \) film. There is far too much data on this plot, but some general trends are apparent. Three physical processes are occurring here: increasing \( R_\Omega \) increases the effective Coulomb repulsion which strongly suppresses \( T_c \); decreasing \( a \) means that a larger magnetic field is required to fit a flux quantum inside the wire, and hence \( H_{c2} \) increases at fixed \( R_\Omega \); decreasing \( a \) also causes a crossover from two to one-dimensional behavior, and so increases fluctuation effects, leading to a small reduction of \( T_c \) at fixed \( R_\Omega \). The last two effects mean that the \( T_c(H) \) curves for wires with the same \( R_\Omega \) and different \( a \) cross each other at non-zero field.

Another effect of the dimensional crossover is to change the shape of the \( T_c(H) \) curve. This occurs because the magnetic pair-breaking rate, \( 1/\tau_H \), is linear in magnetic field in the two-dimensional limit, but quadratic in magnetic field in the one-dimensional limit. This leads to two limiting forms for the dimensionless \( T_c(H) \) curve,

\[
\ln \left( \frac{T_c(H)}{T_c(0)} \right) = \psi \left( \frac{1}{2} \right) - \psi \left( \frac{1}{2} + e^{\phi(1/2)} \left[ \frac{H}{H_{c2}} \right]^p \right) \left[ \frac{T_c(0)}{T_c(H)} \right],
\]

where \( p = 1 \) in the 2D limit and \( p = 2 \) in the 1D limit. We can move from the 2D limit to the 1D limit either by decreasing \( a \) at fixed \( R_\Omega \), as shown in Fig. 6, or by increasing \( R_\Omega \) at fixed \( a \), as shown in Fig. 7. Considering first the width dependence, we see in Fig. 6 that the dimensional crossover occurs when \( a \sim L_T \). For the wider wires the \( T_c(H) \) curve joins the two-dimensional curve at the value of the magnetic field such that the magnetic length, \( L_T \), is of order the width, \( a \); for narrower wires this value of \( H \) exceeds the two-dimensional value of \( H_{c2} \), and the curves never meet. For ultranarrow wires, \( a < 250\Omega \), the Zeeman effect becomes important. This causes the \( T_c(H) \)
curves to change shape again, moving slowly back in the direction of the 2D limit. Now let us consider the resistance
dependence of the $T_c(H)$ curve. For narrow wires, $a \approx 250 \text{Å}$, all $T_c(H)$ curves fall on the 1D curve independent of $R$. For wide wires, $a \approx 10000 \text{Å}$, all $T_c(H)$ curves fall on the 2D curve independent of $R$. For wires of intermediate
width, the $T_c(H)$ curves can be changed from the 1D curve to the 2D curve by increas-
ing $R$, as shown in Fig. 7. Finally for ultranarrow wires, $a \approx 50 \text{Å}$, the $T_c(H)$ curves can again be tuned by varying $R$, in this case the tuning
being between one-dimensional orbit and Zeeman curves.

V. DISCUSSION AND CONCLUSIONS

In this paper we have considered the effect of localization and interaction on transition temperature in ultranarrow
superconducting wires. Our main conclusion is that, taking into account the relatively small amount of experimental
data, theory and experiment are consistent with each other. The agreement is not perfect: the experimental data
seems to show a very sudden crossover from 2D to 1D behavior when the wire width becomes of the order of the
thermal length, whilst theory predicts a smooth crossover. Indeed, when plotted as the additional relative suppression
of transition temperature due to finite width, the experimental data falls on two curves – one for the 2D film and
1000Å wire; the other for the 580Å, 350Å and 250Å wires. These conclusions are based on a single experiment, as
this is the only experiment which shows any consistent trend with wire width. More experimental data is obviously
needed to investigate this further. Given the difficulty in fabricating these wires, we suggest that the magnetic field
dependence of transition temperature be measured in future experiments. We have therefore made predictions for
magnetic field dependence of transition temperature in wires of varying width and resistance in the hope that such
measurements will be made in the future. Our main prediction is of a change in the shape of the reduced $T_c(H)$
curves due to the dimensional crossover. This may be tuned either by varying wire width or resistance.

Our work is based on the resummation technique developed by Oreg and Finkel’stein, which is a very powerful
method of going beyond perturbation theory, and hence giving believable results at strong disorder. The suppression
of the transition temperature is very sensitive to boundary conditions, due to the different relative weights of the
zero momentum mode. We have used the physically correct zero-gradient boundary conditions in our work. The
unexpected strong dependence of results on boundary conditions led us to worry about all details of the calculation.
We therefore also investigated the effects of using a featureless Coulomb interaction instead of the exact one, ignoring
self-energy contributions, and solving the OF matrix equation exactly instead of using an approximate method. We
found that it is legitimate to use a featureless Coulomb interaction and ignore self-energy contributions, but that our
matrix equations should be solved by exact diagonalization (which in any case is no more difficult).

In our calculations of the magnetic field dependence of transition temperature, we find that the orbital effect is
the most important, and can be treated by replacing the $Dq^2$ terms in the momentum dependent pair propagator,
$L(q,0)$, by twice the magnetic dephasing rate, $2/\tau_H$. The latter is the lowest eigenvalue of the minimally coupled
Cooperon operator, and must be found numerically. The Zeeman splitting is found to have a non-negligible effect
only at the mean field level, and even this effect is small. The perturbative term only becomes important in very
narrow wires (of order 10Å), which cannot currently be fabricated.

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APPENDIX A: DETAILS OF SOME MOMENTUM SUMS

In this appendix we derive expressions for the quantities $I_1(m)$ and $I_2(m)$ defined in Eqn. (13) as

$$I_1 \equiv \frac{4\pi D}{aL} \sum_{q'} \frac{1}{Dq'^2 + |\omega_m|}$$

$$I_2 \equiv \frac{4\pi D}{aL} \sum_{q'} \frac{2\pi T}{(Dq'^2 + |\omega_m|)^2}$$

(A1)
In two dimensions the sum becomes an integral, and cutting off in a circularly symmetric manner,

\[
\frac{1}{aL} \sum_{q'} = \int_0^{Dq^2=1/\tau} \frac{2\pi q' dq'}{(2\pi)^2} = \int_0^{1/\tau} \frac{d(Dq'^2)}{4\pi D},
\]

we obtain the results

\[I_1(m) = \ln \left(\frac{M}{m}\right), \quad I_2(m) = \frac{1}{m},\]

In quasi-1D wires we replace the integral in the transverse direction by a sum, so that \(I_1(m)\) becomes

\[I_1(m) = 4\pi D \frac{1}{a} \sum_{q''} \frac{1}{2\pi} \int_0^{Dq'^2=1/\tau} \frac{dq'}{Dq'^2 + Dq''^2 + 2\pi Tm} = \frac{4}{a} \sum_{q''} \sqrt{\frac{1}{2\pi Tm}} \frac{1}{q''^2 + 2\pi Tm} \tan^{-1} \left(\frac{1}{\sqrt{q''^2 + 2\pi Tm}}\right),\]

and \(I_2(m)\) can be found by differentiating the above with respect to \(m\) and dividing by \(-2\pi T\). Substituting in the two types of boundary condition listed in Table 1 yields

\[I_1(m) = \begin{cases} \frac{2}{\pi} \sum_{p=-p_0}^{p_0} (p^2 + mA^2)^{-\frac{1}{2}} \tan^{-1} \left(\frac{p_0}{\sqrt{p^2 + mA^2}}\right) & \text{PBC} \\ \frac{4}{\pi} \sum_{p=0}^{2p_0} (p^2 + 4mA^2)^{-\frac{1}{2}} \tan^{-1} \left(\frac{2p_0}{\sqrt{p^2 + 4mA^2}}\right) & \text{ZGBC} \end{cases}\]

and

\[I_2(m) = \begin{cases} \frac{A^2}{\pi} \sum_{p=-p_0}^{p_0} (p^2 + mA^2)^{-\frac{1}{2}} \left(\tan^{-1} \left(\frac{p_0}{\sqrt{p^2 + mA^2}}\right) + \frac{p_0 \sqrt{p^2 + mA^2}}{p^2 + (m + M)A^2}\right) & \text{PBC} \\ \frac{8A^2}{\pi} \sum_{p=0}^{2p_0} (p^2 + 4mA^2)^{-\frac{1}{2}} \left(\tan^{-1} \left(\frac{2p_0}{\sqrt{p^2 + 4mA^2}}\right) + \frac{2p_0 \sqrt{p^2 + 4mA^2}}{p^2 + 4(m + M)A^2}\right) & \text{ZGBC} \end{cases}\]

where \(A = a/L_T\) is a dimensionless width, the ratio of the wire width to the thermal length.

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1. J.M. Graybeal, P.M. Mankiewich, R.C. Dynes and M.R. Beasley, Phys. Rev. Lett. 59, 2697 (1987).
2. F. Sharifi, A.V. Herzog and R.C. Dynes, Phys. Rev. Lett. 71, 428 (1993).
3. P. Xiong, A.V. Herzog and R.C. Dynes, Phys. Rev. Lett. 78, 927 (1997).
4. A. Bezryadin, C.N. Lau and M. Tinkham, Nature 404, 971 (2000).
5. B. Reulet, Bull. Am. Phys. Soc. 45, 486 (2000).
6. D.B. Haviland, Y. Liu and A.M. Goldman, Phys. Rev. Lett. 62, 2180 (1989).
7. S. Maekawa and H. Fukuyama, J. Phys. Soc. Jpn. 51, 1380 (1982).
8. A.M. Finkel’stein, JETP Lett. 45, 46 (1987).
9. R.A. Smith, M.Y. Reizer and J.W. Wilkins, Phys. Rev. B 51, 6470 (1995).
10. A. Schmid, Z. Phys. 259, 421 (1973).
11. D. Belitz, Phys. Rev. B 35, 1636 (1987); ibid 35, 1651 (1987).
12. B.L. Altshuler, A.G. Aronov and P.A. Lee, Phys. Rev. Lett. 44, 1288 (1980).
13. J.M. Valles Jnr, R.C. Dynes and J.P. Garno, Phys. Rev. B 40, 6680 (1989).
14. R.A. Smith and V. Ambegaokar, Phys. Rev. B 62, 5913 (2000).
We note that Oreg and Finkel'stein also use the physically correct zero-gradient boundary conditions, except that their ratio $a/L_T$ is effectively half our value. This is unimportant as the experimental value of $L_T$ is known only roughly. It follows that our predictions and theirs are in agreement.

P.G. de Gennes, “Superconductivity of Metals and Alloys”, (Addison-Wesley, Redwood City, 1989), p227.

R.A. Smith, B.S. Handy and V. Ambegaokar, Phys. Rev. B 61, 6352 (2000).

A.M. Clogston, Phys. Rev. Lett. 9, 266 (1962); B.S. Chandrasekhar, Appl. Phys. Lett. 1, 7 (1962).

A.A. Abrikosov and L.P. Gor’kov, Sov. Phys. JETP 12, 1243 (1961).

S. Hikami, Phys. Rev. B 24, 2671 (1981).
| Type          | Definition                        | Momenta, $q_a$ | Details of sum over $p$ |
|--------------|-----------------------------------|----------------|------------------------|
| Periodic     | $\psi(x + a) = \psi(x)$           | $\frac{2\pi p}{a}$ | $\sum_{p=-p_0}^{p_0}$ |
| Zero-Gradient| $\psi'(x = \pm a/2) = 0$          | $\frac{\pi p}{a}$  | $\sum_{p=0}^{2p_0}$ |

**TABLE I.** Details of the boundary conditions discussed in the text. The upper cut-off in momentum is given by $Dq_a^2 = 1/\tau$, so that $p_0 = a/2\pi\sqrt{D\tau} = \sqrt{M}(a/L_T)$ where $L_T = \sqrt{2\pi D/T}$ is the thermal length and $M = 1/2\pi T \tau$. 


FIG. 1. The diagrams needed to calculate the first-order perturbative correction to the pair-propagator $L(q, 0)$. (a) Definition of pair propagator, $L(q, 0)$, in terms of the pair polarization function, $P(q, 0)$, and the 4-point BCS interaction $-|\gamma|$. (b) Zeroth-order (mean-field) polarization function, $P_0(q, 0)$, in a dirty superconductor. The solid lines are electron Green functions; the dashed lines are impurity lines. (c) First-order perturbative correction to pair polarization function, $\delta P(q, 0)$. The wiggly line is the screened Coulomb interaction.
FIG. 2. Diagrammatic equation for the pair amplitude matrix $\Gamma(\epsilon_n, \epsilon_l)$. Block $\gamma$ is the BCS interaction. Block $t\Lambda$ is the correction to the effective interaction caused by the interplay of Coulomb interaction and disorder. Block $\Sigma$ is the correction to the bare Cooperon ladder caused by Coulomb interaction and disorder, which yields a negligible contribution and is thus ignored in our numerical calculations.
FIG. 3. Comparison of theory (solid lines) and experiment (symbols) for transition temperature, $T_c$, as a function of resistance per square, $R_e$, for various wire widths, $a = 250\text{Å}, 350\text{Å}, 580\text{Å}, 1000\text{Å}$. The experimental data is that of Xiong et al. The theory used is the one we have argued to be correct – resummation technique solved by exact diagonalization using zero-gradient boundary conditions. The theory plots are, from top to bottom, for 1000Å, 580Å, 350Å, and 250Å wires. Rescaling has been applied to take into account the fact that theory and experiment do not agree with each other in two dimensions, and that the transition temperature at zero resistance appears to depend upon width. The thermal length, $L_T$, has been taken to be $300\text{Å}$ at $T_{c0} = 6.9\text{K}$ throughout – this is the value quoted by Sharifi et al. Plotting the data in this form focuses on the suppression of transition temperature due to disorder, and we see that the agreement between theory and experiment is good, particularly for the 1000Å and 250Å wires.
FIG. 4. Comparison of theory (solid lines) and experiment (symbols) for the additional relative suppression of transition temperature, $\Theta(R, a)$, as a function of resistance per square, $R$, for various wire widths, $a = 250 \text{ Å}, 350 \text{ Å}, 580 \text{ Å}, 1000 \text{ Å}$. Plotting the data in this form focuses on the suppression of transition temperature due to width, and we see that the agreement between theory and experiment is not particularly good. This may have been anticipated from the experimental data which shows an abrupt crossover between the 2D film and the 1000 Å wire, which lie on one curve, and the 580 Å, 350 Å, 250 Å wires, which lie on a different curve. Our theory, and indeed any theory, predicts a smooth crossover.
FIG. 5. Predicted transition temperature, $T_c$, as a function of applied perpendicular magnetic field, $H$, for wires of various widths, $a = 500 \, \text{Å}, 600 \, \text{Å}, 750 \, \text{Å}, 2D$ film, and resistances per square, $R_a = 0 \Omega, 1000 \Omega, 2000 \Omega$. Three physical effects can be seen in this figure: increasing $R_a$ reduces $T_c$ dramatically due to the increased Coulomb repulsion; decreasing $a$ increases the upper critical field, $H_{c2}$, as a larger field is needed to fit a flux quantum into the wire; decreasing $a$ at fixed $R_a$ causes a small additional suppression of $T_c$ at zero field due to increased fluctuation effects caused by the crossover from two to one-dimensional behavior. The last two effects mean that the $T_c(H)$ curves for wires with the same $R_a$ and different $a$ cross each other at non-zero field. Finally the shape of the $T_c(H)$ curve changes as width $a$ is altered due to the dimensional crossover. There is obviously a large amount of data in this figure, and to analyse the individual trends we will need to re-plot some of this data in a different form.
FIG. 6. Predicted dependence of the shape of the $T_c(H)$ curve on wire width, $a$, at a fixed resistance per square, $R_\varphi = 1000\Omega$. We plot using dimensionless variables $T_c(H)/T_c(H=0)$ vs $H/H_{c2}$ in order to compare the curve shape. As width is reduced we see a crossover from the two-dimensional limit, where the magnetic pair-breaking rate $1/\tau_H \propto H$, to the one-dimensional limit where $1/\tau_H \propto H^2$. The crossover occurs when the wire width is of the order of the thermal length, $L_T$. For the wider wires the $T_c(H)$ curve joins the two-dimensional curve at the value of the magnetic field such that the magnetic length, $L_H$, is of order the width, $a$; for narrower wires this value of $H$ exceeds the two-dimensional value of $H_{c2}$, and the curves never meet. For ultranarrow wires, $a < 250\AA$, the Zeeman effect becomes important. This causes the $T_c(H)$ curves to change shape again, moving slowly back in the direction of the 2D limit.
FIG. 7. Predicted dependence of the shape of the $T_c(H)$ curve on resistance per square, $R$, at a fixed wire width, $a = 1000 \text{Å}$. We see a crossover from two-dimensional to one-dimensional behavior as $R$ is increased. This disorder-tuned crossover occurs only for wires of intermediate width; a similar plot for $a = 10000 \text{Å}$ shows all data falling on the two-dimensional curve independent of $R$, whilst that for $a = 250 \text{Å}$ shows all data falling on the one-dimensional curve.