CHOQUET OPERATORS ASSOCIATED TO VECTOR CAPACITIES

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Abstract. The integral representation of Choquet operators defined on a space $C(X)$ is established by using the Choquet-Bochner integral of a real-valued function with respect to a vector capacity.

1. Introduction

Choquet’s theory of integrability (as described by Denneberg [10] and Wang and Klir [30]) leads to a new class of nonlinear operators called Choquet operators because they are defined by a mix of conditions representative for Choquet’s integral. Its technical definition is detailed as follows.

Given a Hausdorff topological space $X$, we will denote by $F(X)$ the vector lattice of all real-valued functions defined on $X$ endowed with the pointwise ordering. Two important vector sublattices of it are

$$C(X) = \{ f \in F(X) : f \text{ continuous} \}$$

and

$$C_b(X) = \{ f \in F(X) : f \text{ continuous and bounded} \}.$$  

With respect to the sup norm, $C_b(X)$ becomes a Banach lattice. See the next section for details concerning the ordered Banach spaces.

As is well known, all norms on the $N$-dimensional real vector space $\mathbb{R}^N$ are equivalent. See Bhatia [3], Theorem 13, p. 16. When endowed with the sup norm and the coordinate wise ordering, $\mathbb{R}^N$ can be identified (algebraically, isometrically and in order) with the space $C(\{1,\ldots,N\})$, where $\{1,\ldots,N\}$ carries the discrete topology.

Suppose that $X$ and $Y$ are two Hausdorff topological spaces and $E$ and $F$ are respectively ordered vector subspaces of $F(X)$ and $F(Y)$. An operator $T : E \to F$ is said to be a Choquet operator (respectively a Choquet functional when $F = \mathbb{R}$) if it satisfies the following three conditions:

Ch1) (Sublinearity) $T$ is subadditive and positively homogeneous, that is, $T(f + g) \leq T(f) + T(g)$ and $T(af) = aT(f)$ for all $f, g$ in $E$ and $a \geq 0$;

Ch2) (Comonotonic additivity) $T(f + g) = T(f) + T(g)$ whenever the functions $f, g \in E$ are comonotone in the sense that $(f(s) - f(t)) \cdot (g(s) - g(t)) \geq 0$ for all $s, t \in X$.

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(Ch3) **(Monotonicity)** $f \leq g$ in $E$ implies $T(f) \leq T(g)$.

The linear Choquet operators acting on ordered Banach spaces are nothing but the linear and positive operators acting on these spaces; see Corollary 1. While they are omnipresent in the various fields of mathematics, the nonlinear Choquet operators are less visible, their study beginning with the seminal papers of Schmeidler [28, 29] in the 80’s. An important step ahead was done by the contributions of Zhou [31], Marinacci and Montrucchio [17] and Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio [4, 5], which led to the study of vector-valued Choquet operators in their own. See [14] and [15].

Interestingly, the condition of comonotonic additivity (the substitute for additivity) lies at the core of many results concerning the real analysis. Indeed, its meaning in the context of real numbers, can be easily understood by identifying each real number $x$ with the affine function $\alpha_x(t) = tx, t \in \mathbb{R}$. As a consequence, two real numbers $x$ and $y$ are comonotone if and only if the functions $\alpha_x$ and $\alpha_y$ are comonotone, equivalently, if either both $x$ and $y$ are nonnegative or both are nonpositive. This yields the simplest example of Choquet functional from $\mathbb{R}$ into itself which is not linear, the function $x \rightarrow x^+$. At the same time one can indicate a large family of nonlinear Choquet operators from $C([-1,1])$ into an arbitrary ordered Banach space $E$,

$$T_{\varphi,U}(f) = U \left( \int_{-1}^{1} f^+(tx) \varphi(x) dx \right),$$

where $\varphi \in C([-1,1])$ is any nonnegative function such that $\varphi(0) = 0$ and $U : C([-1,1]) \rightarrow E$ is any monotonic linear operator.

Based on previous work done by Zhou [31], Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio [4], proved that a larger class of functionals defined on a space $C(X)$ (where $X$ is a Hausdorff compact space) admit a Choquet analogue of the Riesz representation theorem. The aim of our paper is to further extend their results to the case of operators by developing a Choquet-Bochner theory of integration relative to monotone set functions taking values in ordered Banach spaces.

Section 2 is devoted to a quick review of some basic facts from the theory of ordered Banach spaces. While the particular case of Banach lattices is nicely covered by a series of textbooks such as those by Meyer-Nieberg [18] and Schaefer [26], the general theory of ordered Banach spaces is still waiting to become the subject of an authoritative book.

In Section 3 we develop the theory of Choquet-Bochner integral associated to a vector capacity (that is, to a monotone set function $\mu$ taking values in an ordered Banach space such that $\mu(\emptyset) = 0$). As is shown in Theorem 1, this integral has all nice features of the Choquet integral: monotonicity, positive homogeneity and comonotonic additivity. The transfer of properties from vector capacities to their integrals also works in a number of important cases such as the upper/lower continuity and submodularity. See Theorem 1. In the case of submodular vector capacities with values in a Banach lattice, the integral analogue of the modulus inequality also holds. See Theorem 2.

Section 4 deals with the integral representation of the Choquet operators defined on spaces $C(X)$ ($X$ being compact and Hausdorff) and taking values in a Banach lattice with order continuous norm. The main result, Theorem 3, shows that each such operator is the Choquet-Bochner integral associated to a suitable
upper continuous vector capacity. In Section 5, this representation is generalized to the framework of comonotonic additive operators with bounded variation. See Theorem 4. The basic ingredient is Lemma 12 which shows that every comonotonic additive operator with bounded variation can be written as the difference of two positively homogeneous, translation invariant and monotone operators.

The paper ends with a short list of open problems.

2. Preliminaries on ordered Banach spaces

An ordered vector space is a real vector space $E$ endowed with an order relation $\leq$ such that the following two conditions are verified:

- $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in E$; and
- $x \leq y$ implies $\lambda x \leq \lambda y$ for $x, y \in E$ and $\lambda \in \mathbb{R}_+ = [0, \infty)$.

In this case the set $E_+ = \{x \in E : x \geq 0\}$ is a convex cone, called the positive cone. A real Banach space endowed with an order relation that makes it an ordered vector space is called an ordered Banach space if the norm is monotone on the positive cone, that is,

$$0 \leq x \leq y \implies \|x\| \leq \|y\|.$$  

Note that in this paper we will consider only ordered Banach spaces whose positive cones are closed (in the norm topology), proper ($-E_+ \cap E_+ = \{0\}$) and generating ($E = E_+ - E_+$).

A convenient way to emphasize the properties of ordered Banach spaces is that described by Davies in [8]. According to Davies, a real Banach space $E$ endowed with a closed and generating cone $E_+$ such that

$$\|x\| = \inf \{\|y\| : y \in E, -y \leq x \leq y\} \quad \text{for all } x \in E,$$

is called a regularly ordered Banach space. Examples are the Banach lattices and some other spaces such as $\text{Sym}(n, \mathbb{R})$, the ordered Banach space of all $n \times n$-dimensional symmetric matrices with real coefficients. The norm of a symmetric matrix $A$ is defined by the formula

$$\|A\| = \sup_{\|x\| \leq 1} |\langle Ax, x \rangle|,$$

and the positive cone $\text{Sym}^+(n, \mathbb{R})$ of $\text{Sym}(n, \mathbb{R})$ consists of all symmetric matrices $A$ such that $\langle Ax, x \rangle \geq 0$ for all $x$.

Lemma 1. Every ordered Banach space can be renormed by an equivalent norm to become a regularly ordered Banach space.

For details, see Namioka [19]. Some other useful properties of ordered Banach spaces are listed below.

Lemma 2. Suppose that $E$ is a regularly ordered Banach space. Then:

(a) There exists a constant $C > 0$ such that every element $x \in E$ admits a decomposition of the form $x = u - v$ where $u, v \in E_+$ and $\|u\|, \|v\| \leq C \|x\|$.

(b) The dual space of $E$, $E^*$, when endowed with the dual cone

$$E_+^* = \{x^* \in E^* : x^*(x) \geq 0 \text{ for all } x \in E_+\}$$

is a regularly ordered Banach space.

(c) $x \leq y$ in $E$ is equivalent to $x^*(x) \leq x^*(y)$ for all $x^* \in E_+^*$.

(d) $\|x\| = \sup \{x^*(x) : x^* \in E_+^*, \|x^*\| \leq 1\}$ for all $x \in E_+$. 

If \((x_n)\) is a decreasing sequence of positive elements of \(E\) which converges weakly to 0, then \(\|x_n\| \to 0\).

The assertion \((e)\) is a generalization of Dini’s lemma in real analysis; see [7], p. 173.

**Proof.** The assertion \((a)\) follows immediately from Lemma 1. For \((b)\), see Davies [8], Lemma 2.4. The assertion \((c)\) is an easy consequence of the Hahn-Banach separation theorem; see [24], Theorem 2.5.3, p. 100.

The assertion \((d)\) is also a consequence of the Hahn-Banach separation theorem; see [27], Theorem 4.3, p. 223.

**Corollary 1.** Every ordered Banach space \(E\) can be embedded into a space \(C(X)\), where \(X\) is a suitable compact space.

**Proof.** According to the Alaoglu theorem, the set \(X = \{x^* \in E^*_e : \|x^*\| \leq 1\}\) is compact relative to the \(w^*\) topology. Taking into account the assertions \((c)\) and \((d)\) of Lemma 2 one can easily conclude that \(E\) embeds into \(C(X)\) (algebraically, isometrically and in order) via the map

\[
\Phi : E \to C(X), \quad (\Phi(x))(x^*) = x^*(x).
\]

The following important result is due to V. Klee [16]. A simple proof of it is available in [23].

**Lemma 3.** Every positive linear operator \(T : E \to F\) acting on ordered Banach spaces is continuous.

Sometimes, spaces with a richer structure are necessary.

A vector lattice is any ordered vector space \(E\) such that \(\sup\{x, y\}\) and \(\inf\{x, y\}\) exist for all \(x, y \in E\). In this case for each \(x \in E\) we can define \(x^+ = \sup\{x, 0\}\) (the positive part of \(x\)), \(x^- = \sup\{-x, 0\}\) (the negative part of \(x\)) and \(|x| = \sup\{-x, x\}\) (the modulus of \(x\)). We have \(x = x^+ - x^-\) and \(|x| = x^+ + x^-.\) A vector lattice endowed with a norm \(\| \cdot \|\) such that

\[
|x| \leq |y| \quad \text{implies} \quad x \leq \|y\|
\]

is called a normed vector lattice; it is called a Banach lattice when in addition it is metric complete.

Examples of Banach lattice are numerous: the discrete spaces \(\mathbb{R}^n, c_0, c\) and \(\ell^p\) for \(1 \leq p \leq \infty\) (endowed with the coordinate-wise order), and the function spaces \(C(K)\) (for \(K\) a compact Hausdorff space) and \(L^p(\mu)\) with \(1 \leq p \leq \infty\) (endowed with pointwise order). Of a special interest are the Banach lattices with order continuous norm, that is, the Banach lattices for which every monotone and order bounded sequence is convergent in the norm topology. So are \(\mathbb{R}^n, c_0\) and \(L^p(\mu)\) for \(1 \leq p < \infty\).

**Lemma 4.** Every monotone and order bounded sequence of elements in a Banach lattice \(E\) with order continuous norm admits a supremum and an infimum and all closed order intervals in \(E\) are weakly compact.

For details, see Meyer-Nieberg [18], Theorem 2.4.2, p. 86.
3. The Choquet-Bochner Integral

This section is devoted to the extension of Choquet’s theory of integrability to the framework with respect to a monotone set function with values in the positive cone of a regularly ordered Banach space $E$. This draws a parallel to the real-valued case already treated in full details by Denneberg [10] and Wang and Klar [30].

Given a nonempty set $X$, a lattice of subsets of $X$ means any collection $\Sigma$ of subsets that contains $\emptyset$ and $X$ and is closed under finite intersections and unions. A lattice $\Sigma$ is an algebra if in addition it is closed under complementation. An algebra which is closed under countable unions and intersections is called a $\sigma$-algebra.

Of a special interest is the case where $X$ is a compact Hausdorff space and $\Sigma$ is either the lattice $\Sigma^+_c(X)$, of all upper contour closed sets $S = \{x \in X : f(x) \geq t\}$, or the lattice $\Sigma^+_o(X)$ of all upper contour open sets $S = \{x \in X : f(x) > t\}$, associated to pairs $f \in C(X)$ and $t \in \mathbb{R}$.

When $X$ is a compact metrizable space, $\Sigma^+_c(X)$ coincides with the lattice of all closed subsets of $X$ (and $\Sigma^+_o(X)$ coincides with the lattice of all open subsets of $X$).

In what follows $\Sigma$ denotes a lattice of subsets of an abstract set $X$ and $E$ is a regularly ordered Banach space.

**Definition 1.** A set function $\mu : \Sigma \to E_+$ is called a vector capacity if it verifies the following two conditions:

1. $\mu(\emptyset) = 0$; and
2. $\mu(A) \leq \mu(B)$ for all $A, B \in \Sigma$ with $A \subset B$.

Notice that any vector capacity $\mu$ is positive and takes values in the order interval $[0, \mu(X)]$.

An important class of vector capacities is that of additive (respectively $\sigma$-additive) vector measures with positive values, that is, of capacities $\mu : \Sigma \to E_+$ with the property

$$
\mu \left( \bigcup_n A_n \right) = \sum_{n=1}^{\infty} \mu(A_n),
$$

for every finite (respectively infinite) sequence $A_1, A_2, A_3, \ldots$ of disjoint sets belonging to $\Sigma$ such that $\bigcup_n A_n \in \Sigma$.

Some other classes of capacities exhibiting various extensions of the property of additivity are listed below.

A vector capacity $\mu : \Sigma \to E_+$ is called submodular if

$$
\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B) \quad \text{for all } A, B \in \Sigma
$$

and it is called supermodular when the inequality (3.1) works in the reversed way. Every additive measure taking values in $E_+$ is both submodular and supermodular.

A vector capacity $\mu : \Sigma \to E_+$ is called lower continuous (or continuous by ascending sequences) if

$$
\lim_{n \to \infty} \mu(A_n) = \mu \left( \bigcup_{n=1}^{\infty} A_n \right)
$$

for every nondecreasing sequence $(A_n)_n$ of sets in $\Sigma$ such that $\bigcup_{n=1}^{\infty} A_n \in \Sigma$; $\mu$ is called upper continuous (or continuous by descending sequences) if

$$
\lim_{n \to \infty} \mu(A_n) = \mu \left( \bigcap_{n=1}^{\infty} A_n \right)
$$
for every nonincreasing sequence \((A_n)_n\) of sets in \(\Sigma\) such that \(\cap_{n=1}^{\infty} A_n \in \Sigma\). If \(\mu\) is an additive capacity defined on a \(\sigma\)-algebra, then its upper/lower continuity is equivalent to the property of \(\sigma\)-additivity.

When \(\Sigma\) is an algebra of subsets of \(X\), then to each vector capacity \(\mu\) defined on \(\Sigma\), one can attach a new vector capacity \(\overline{\mu}\), the dual of \(\mu\), which is defined by the formula

\[
\overline{\mu}(A) = \mu(X) - \mu(X \setminus A).
\]

Notice that \(\overline{\overline{\mu}} = \mu\).

The dual of a submodular (supermodular) capacity is a supermodular (submodular) capacity. Also, dual of a lower continuous (upper continuous) capacity is an upper continuous (lower continuous) capacity.

**Example 1.** There are several standard procedures to attach to a \(\sigma\)-additive vector measure \(\mu : \Sigma \to E_+\) certain not necessarily additive capacities. So is the case of distorted measures, \(\nu(A) = T(\mu(A))\), obtained from \(\mu\) by applying to it a continuous nondecreasing distortion \(T : [0, \mu(X)] \to [0, \mu(X)]\). The vector capacities \(\nu\) so obtained are both upper and lower continuous.

For example, this is the case when \(E = \text{Sym}(n, \mathbb{R})\), \(\mu(X) = I\) (the identity of \(\mathbb{R}^n\)), and \(T : [0, I] \to [0, I]\) is the distortion defined by the formula \(T(A) = A^2\).

Taking into account the assertions (c) and (d) of Lemma 2, some aspects (but not all) of the theory of vector capacities are straightforward consequences of the theory of \(\mathbb{R}_+\)-valued capacities.

**Lemma 5.** A set function \(\mu : \Sigma \to \mathbb{R}_+\) is a submodular (supermodular, lower continuous, upper continuous) vector capacity if and only if \(x^* \circ \mu\) is a submodular (supermodular, lower continuous, upper continuous) \(\mathbb{R}_+\)-valued capacity whenever \(x^* \in E^*_+\).

When \(E = \mathbb{R}^n\), this assertion can be formulated via the components \(\mu_k = \text{pr}_k \circ \mu\) of \(\mu\).

In what follows the term of (upper) measurable function refers to any function \(f : X \to \mathbb{R}\) whose all upper contour sets \(\{x \in X : f(x) \geq t\}\) belong to \(\Sigma\). When \(\Sigma\) is a \(\sigma\)-algebra, this notion of measurability is equivalent to the Borel measurability.

We will denote by \(B(\Sigma)\) the set of all bounded measurable functions \(f : X \to \mathbb{R}\).

In general \(B(\Sigma)\) is not a vector space (unless the case when \(\Sigma\) is a \(\sigma\)-algebra). However, even when \(\Sigma\) is only an algebra, the set \(B(\Sigma)\) plays some nice properties of stability: if \(f, g \in B(\Sigma)\) and \(\alpha, \beta \in \mathbb{R}\), then

\[
\inf \{f, g\}, \, \sup \{f, g\} \quad \text{and} \quad \alpha + \beta f
\]

also belong to \(B(\Sigma)\). See [17], Proposition 15.

Given a capacity \(\mu : \Sigma \to \mathbb{R}_+\), the Choquet integral of a measurable function \(f : X \to \mathbb{R}\) on a set \(A \in \Sigma\) is defined as the sum of two Riemann improper integrals,

\[
(C) \int_A f \, d\mu = \int_0^{+\infty} \mu(\{x \in A : f(x) \geq t\}) \, dt
\]

\[
+ \int_{-\infty}^{0} \left[ \mu(\{x \in A : f(x) \geq t\}) - \mu(A) \right] \, dt.
\]

Accordingly, \(f\) is said to be Choquet integrable on \(A\) if both integrals above are finite. See the seminal paper of Choquet [6].
If \( f \geq 0 \), then the last integral in the formula (3.2) is 0. When \( \Sigma \) is a \( \sigma \)-algebra, the inequality sign \( \geq \) in the above two integrands can be replaced by \( > \); see [30], Theorem 11.1, p. 226.

Every bounded measurable function is Choquet integrable. The Choquet integral coincides with the Lebesgue integral when the underlying set function \( \mu \) is a \( \sigma \)-additive measure defined on a \( \sigma \)-algebra.

The theory of Choquet integral is available from numerous sources including the books of Denneberg [10], and Wang and Klir [30].

The concept of integrability of a measurable function with respect to a vector capacity \( \mu : \Sigma \to E_+ \) can be introduced by a fusion between the Choquet integral and the Bochner theory of integration of vector-valued functions.

Recall that a function \( \psi : \mathbb{R} \to E \) is Bochner integrable with respect to the Lebesgue measure on \( \mathbb{R} \) if there exists a sequence of step functions \( \psi_n : \mathbb{R} \to E \) such that

\[
\lim_{n \to \infty} \psi_m(t) = \psi(t) \quad \text{almost everywhere and} \quad \int_{\mathbb{R}} \| \psi - \psi_n \| \, dt \to 0.
\]

In this case, the (Bochner) integral of \( \psi \) is defined by

\[
\int_{\mathbb{R}} \psi \, dt = \lim_{n \to \infty} \int_{\mathbb{R}} \psi_n \, dt.
\]

Notice that if \( T : E \to F \) is a bounded linear operator, then

\[(3.3) \quad T \left( \int_{\mathbb{R}} \psi \, dt \right) = \int_{\mathbb{R}} T \circ \psi \, dt.
\]

Details about Bochner integral are available in the books of Diestel and Uhl [11] and Dinculeanu [12].

**Definition 2.** A measurable function \( f : X \to \mathbb{R} \) is called Choquet-Bochner integrable with respect to the vector capacity \( \mu : \Sigma \to E_+ \) on the set \( A \in \Sigma \) if for every \( A \in \Sigma \) the functions \( t \to \mu(\{x \in A : f(x) \geq t\}) \) and \( t \to \mu(\{x \in A : f(x) \geq t\}) - \mu(A) \) are Bochner integrable respectively on \( [0, \infty) \) and \( (-\infty, 0] \). Under these circumstances, the Choquet-Bochner integral over \( A \) is defined by the formula

\[
(CB) \int_A f \, d\mu = \int_0^{+\infty} \mu(\{x \in A : f(x) \geq t\}) \, dt \\
+ \int_{-\infty}^0 [\mu(\{x \in A : f(x) \geq t\}) - \mu(A)] \, dt.
\]

According to the formula (3.3), if \( f \) is Choquet-Bochner integrable, then

\[(3.4) \quad x^* \left( (CB) \int_A f \, d\mu \right) = (C) \int_A f \, d(x^* \circ \mu),
\]

for every positive linear functional \( x^* \in E^* \).

A large class of Choquet-Bochner integrable is indicated below.

**Lemma 6.** If \( f : X \to \mathbb{R} \) is a bounded measurable function, then it is Choquet-Bochner integrable on every set \( A \in \Sigma \).

**Proof.** Suppose that \( f \) takes values in the interval \([0, M]\). Then the function \( \varphi(t) = \mu(\{x \in A : f(x) \geq t\}) \) is positive and nonincreasing on the interval \([0, M]\) and null
outside this interval. As a consequence, the function \( \varphi \) is the uniform limit of the sequence of step functions defined as follows:

\[
\varphi_n(t) = 0 \text{ if } t \notin [0, M] \\
\varphi_n(t) = \varphi(t) \text{ if } t = M
\]

and

\[
\varphi_n(t) = \varphi\left(\frac{k}{n}M\right)
\]

if \( t \in \left[\frac{k}{n}M, \frac{k+1}{n}M\right) \) and \( k = 0, \ldots, n - 1 \). A simple computation shows that

\[
\int_0^\infty \| \varphi(t) - \varphi_n(t) \| \, dt = \int_0^M \| \varphi(t) - \varphi_n(t) \| \, dt \\
\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \| \varphi(t) - \varphi_n(t) \| \, dt \\
\leq \frac{2M}{n} \mu(X) \to 0
\]

as \( n \to \infty \), which means the Bochner integrability of the function \( \varphi \). See [11], p. 44.

The other cases, when \( f \) takes values in the interval \([M, 0]\) or in an interval \([m, M]\) with \( m < 0 < M \), can be treated in a similar way. □

The next lemma collects a number of simple (but important) properties of the Choquet-Bochner integral.

**Lemma 7.** Suppose that \( E \) is an ordered Banach space, \( \mu : \Sigma \to E_+ \) is a vector capacity and \( A \in \Sigma \).

(a) If \( f \) and \( g \) are Choquet-Bochner integrable functions, then

\[
f \geq 0 \text{ implies } (\text{CB}) \int_A f \, d\mu \geq 0 \quad (\text{positivity})
\]

\[
f \leq g \text{ implies } (\text{CB}) \int_A f \, d\mu \leq (\text{CB}) \int_A g \, d\mu \quad (\text{monotonicity})
\]

\[
(\text{CB}) \int_A af \, d\mu = a \cdot (\text{CB}) \int_A f \, d\mu \quad \text{for all } a \geq 0 \quad (\text{positive homogeneity})
\]

\[
(\text{CB}) \int_A 1 \, d\mu = \mu(A) \quad (\text{calibration}).
\]

(b) In general, the Choquet-Bochner integral is not additive but if \( f \) and \( g \) are Choquet-Bochner integrable functions and also comonotonic in the sense of Dellacherie [9] (that is, \( (f(\omega) - f(\omega')) \cdot (g(\omega) - g(\omega')) \geq 0 \), for all \( \omega, \omega' \in X \)), then

\[
(\text{CB}) \int_A (f + g) \, d\mu = (\text{CB}) \int_A f \, d\mu + (\text{CB}) \int_A g \, d\mu.
\]

In particular, the Choquet-Bochner integral is translation invariant,

\[
(\text{CB}) \int_A (f + c) \, d\mu = (\text{CB}) \int_A f \, d\mu + c \mu(A),
\]

for all Choquet-Bochner integrable functions \( f \), all sets \( A \in \Sigma \) and all numbers \( c \in \mathbb{R} \).
Proof. According to Lemma 2 (c), and formula (3.3), applied in the case of an arbitrary functional $x^* \in E_+^*$, the proof of both assertions (a) and (b) reduce to the case of capacities with values in $\mathbb{R}_+$, already covered by Proposition 5.1 in [10], pp. 64-65. □

Corollary 2. The equality

$$\text{(CB)} \int_A (\alpha f + c) d\mu = \alpha \left( \text{(CB)} \int_A f d\mu \right) + c \cdot \mu(A),$$

holds for all Choquet-Bochner integrable functions $f$, all sets $A \in \Sigma$ and all numbers $\alpha \in \mathbb{R}_+$ and $c \in \mathbb{R}$.

The next result describes how the special properties of a vector capacity transfer to the Choquet-Bochner integral.

Theorem 1. (a) If $\mu$ is an upper continuous capacity, then the Choquet-Bochner integral is an upper continuous operator, that is,

$$\lim_{n \to \infty} \left\| \left( \text{(CB)} \int_A f_n d\mu - \text{(CB)} \int_A f d\mu \right) \right\| = 0,$$

whenever $(f_n)_n$ is a nonincreasing sequence of Choquet-Bochner integrable functions that converges pointwise to the Choquet-Bochner integrable function $f$ and $A \in \Sigma$.

(b) If $\mu$ is a lower continuous capacity, then the Choquet-Bochner integral is lower continuous in the sense that

$$\lim_{n \to \infty} \left\| \left( \text{(CB)} \int_A f_n d\mu - \text{(CB)} \int_A f d\mu \right) \right\| = 0$$

whenever $(f_n)_n$ is a nondecreasing sequence of Choquet-Bochner integrable functions that converges pointwise to the Choquet-Bochner integrable function $f$ and $A \in \Sigma$.

(c) If $\Sigma$ is an algebra and $\mu : \Sigma \to E_+$ is a submodular capacity, then the Choquet-Bochner integral is a submodular operator in the sense that

$$\text{(CB)} \int_A \sup \{ f, g \} d\mu + \text{(CB)} \int_A \inf \{ f, g \} d\mu \leq \text{(CB)} \int_A f d\mu + \text{(CB)} \int_A g d\mu$$

whenever $f$ and $g$ are Choquet-Bochner integrable and $A \in \Sigma$.

Proof. (a) Since $\mu$ is an upper continuous capacity and $(f_n)_n$ is a nonincreasing sequence of measurable functions that converges pointwise to the measurable function $f$ it follows that

$$\mu (\{ x \in A : f_n(x) \geq t \}) \searrow \mu (\{ x \in A : f(x) \geq t \})$$

in the norm topology. Taking into account the property of monotonicity of the Choquet-Bochner integral (already noticed in Lemma 7(a)) we have

$$\text{(CB)} \int_A f_n d\mu \geq \text{(CB)} \int_A f_n d\mu \geq \cdots \geq \text{(CB)} \int_A f d\mu,$$

so by Bepo Levi’s monotone convergence theorem from the theory of Lebesgue integral (see [12], Theorem 2, p. 133) it follows that

$$x^* \left( \text{(CB)} \int_A f_n d\mu \right) = \left( x^* \left( \text{(CB)} \int_A f_n d\mu \right) \right) \to \left( x^* \left( \text{(CB)} \int_A f d\mu \right) \right)$$

for all $x^* \in E_+^*$. The conclusion of the assertion (c) is now a direct consequence of the generalized Dini’s lemma (see Lemma 2(e)).
(b) The argument is similar to that used to prove the assertion (a).
(c) Since $\Sigma$ is an algebra, both functions $\inf\{f, g\}$ and $\sup\{f, g\}$ are measurable. The fact that $\mu$ is submodular implies

$$\mu(\{x : \sup\{f, g\}(x) \geq t\}) + \mu(\{x : \inf\{f, g\}(x) \geq t\})$$

$$= \mu(\{x : f(x) \geq t\} \cup \{x : g(x) \geq t\}) + \mu(\{x : f(x) \geq t\} \cap \{x : g(x) \geq t\})$$

$$\leq \mu(\{x : f(x) \geq t\}) + \mu(\{x : g(x) \geq t\}),$$

and the same works when $\mu$ is replaced by $\mu^* = x^* \circ \mu$, where $x^* \in E^*_+$. Integrating side by side the last inequality it follows that

$$(C) \int_A \sup\{f, g\} d\mu^* + (C) \int_A \inf\{f, g\} d\mu^* \leq (C) \int_A f d\mu^* + (C) \int_A g d\mu^*,$$

which yields (via Lemma 2 (c)) the submodularity of the Choquet-Bochner integral.

□

The property of subadditivity of the Choquet-Bochner integral makes the objective of the following result:

**Theorem 2.** (The Subadditivity Theorem) If $\mu$ is a submodular capacity, then the associated Choquet-Bochner integral is subadditive, that is,

$$(CB) \int_A (f + g) d\mu \leq (CB) \int_A f d\mu + (CB) \int_A g d\mu$$

whenever $f$, $g$ and $f + g$ are Choquet-Bochner integrable functions and $A \in \Sigma$.

In addition, when $E$ is a Banach lattice and $f$, $g$, $f - g$ and $g - f$ are Choquet-Bochner integrable functions, then the following integral analog of the modulus inequality holds true:

$$\left| (CB) \int_A f d\mu - (CB) \int_A g d\mu \right| \leq (CB) \int_A |f - g| d\mu$$

for all $A \in \Sigma$. In particular,

$$\left| (CB) \int_A f d\mu \right| \leq (CB) \int_A |f| d\mu$$

whenever $f$ and $-f$ are Choquet-Bochner integrable functions and $A \in \Sigma$.

**Proof.** According to Lemma 5 if $\mu$ is submodular, then every real-valued capacity $\mu^* = x^* \circ \mu$ also is submodular, whenever $x^* \in E^*_+$. Then

$$(C) \int_A f d\mu^* + (C) \int_A g d\mu^* \leq (C) \int_A (f + g) d\mu^*,$$

as a consequence of Theorem 6.3, p. 75, in [10]. Therefore the first inequality in the statement of Theorem 2 is now a direct consequence of Lemma 2 (c) and the formula (3.4).

For the second inequality, notice that the subadditivity property implies

$$(CB) \int_A f d\mu = (CB) \int_A (f - g + g) d\mu$$

$$\leq (CB) \int_A (f - g) d\mu + (CB) \int_A g d\mu,$$
and taking into account that \( f - g \leq |f - g| \) we infer that
\[
(CB) \int_A f \, d\mu - (CB) \int_A g \, d\mu \leq (CB) \int_A |f - g| \, d\mu.
\]
Interchanging \( f \) and \( g \) we also obtain
\[
\pm ((CB) \int_A f \, d\mu - (CB) \int_A g \, d\mu) \leq (CB) \int_A |f - g| \, d\mu
\]
and the proof is done. \( \square \)

4. The integral representation of Choquet operators defined on a space \( C(X) \)

The special case when \( X \) is a compact Hausdorff space and \( \Sigma = \mathcal{B}(X) \), the \( \sigma \)-algebra of all Borel subsets of \( X \), allows us to shift the entire discussion concerning the Choquet-Bochner integral from the vector space \( B(\Sigma) \) (of all real-valued Borel measurable functions) to the Banach lattice \( C(X) \) (of all real-valued continuous functions defined on \( X \)). Indeed, in this case \( C(X) \) is a subspace of \( B(\Sigma) \) and all nice properties stated in Lemma 2 and in Theorem 2 remain true when restricting the integral to the space \( C(X) \). As a consequence the integral operator

\[
(CB) \mu : C(X) \rightarrow E, \quad (CB) \mu(f) = (CB) \int_X f \, d\mu,
\]

associated to a vector-valued submodular capacity \( \mu : B(X) \rightarrow E \), is a Choquet operator.

The linear and positive functionals defined on \( C(X) \) can be represented either as integrals with respect to a unique regular Borel measure (the Riesz-Kakutani representation theorem) or as Choquet integrals (see Epstein and Wang [13]).

**Example 2.** The space \( c \) of all convergent sequences of real numbers can be identified with the space of continuous functions on \( \hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\} \) (the one-point compactification of the discrete space \( \mathbb{N} \)). The functional

\[
I : c \rightarrow \mathbb{R}, \quad I(x) = \lim_{n \to \infty} x(n)
\]

is linear and monotone (therefore continuous by Lemma 3) and its Riesz representation is

\[
I(x) = \int_{\mathbb{N}} x(n) d\delta_\infty(n),
\]

where \( \delta_\infty \) is the Dirac measure concentrated at \( \infty \), that is, \( \delta_\infty(A) = 1 \) if \( A \in \mathcal{P}(\hat{\mathbb{N}}) \) and \( \{\infty\} \in A \) and \( \delta_\infty(A) = 0 \) if \( \{\infty\} \notin A \). Meantime, \( I \) admits the Choquet representation

\[
I(x) = (C) \int_{\mathbb{N}} x(n) d\mu(n)
\]

where \( \mu \) is the capacity defined on the power set \( \mathcal{P}(\hat{\mathbb{N}}) \) by \( \mu(A) = 0 \) if \( A \) is finite and \( \mu(A) = 1 \) otherwise.

As the Riesz-Kakutani representation theorem also holds in the case of linear and positive operators \( T : C(X) \rightarrow E \), it seems natural to search for an analogue in the case of Choquet operators. The answer is provided by the following representation theorem that extend a result due to Epstein and Wang [13] from functionals to operators:
Theorem 3. Let $X$ be a compact Hausdorff space and $E$ be a Banach lattice with order continuous norm. Then for every comonotonic additive and monotone operator $I : C(X) \to E$ with $I(1) > 0$ there exists a unique upper continuous vector capacity $\mu : \Sigma_{up}^+(X) \to E_+$ such that

\begin{equation}
I(f) = (\text{CB}) \int_X f d\mu \quad \text{for all} \ f \in C(X).
\end{equation}

Moreover, $\mu$ admits a unique extension to $\mathcal{B}(X)$ (also denoted $\mu$) that fulfills the following two properties of regularity:

(R1) $\mu(A) = \sup \{ \mu(K) : K \text{ closed}, \ K \subset A \}$ for all $A \in \mathcal{B}(X)$;

(R2) $\mu(K) = \inf \{ \mu(O) : O \text{ open}, \ O \supseteq K \}$ for all closed sets $K$.

Recall that $\Sigma_{up}^+(X)$ represents the lattice of upper contour sets associated to the continuous functions defined on $X$. This lattice coincides with the lattice of all closed subsets of $X$ when $X$ is compact and metrizable.

The proof of Theorem 3 needs several auxiliary results and will be detailed at the end of this section.

Lemma 8. Let $E$ be an ordered Banach space. Then every monotone and translation invariant operator $I : C(X) \to E$ is Lipschitz continuous.

Proof. Given $f, g \in C(X)$, one can choose a decreasing sequence $(\alpha_n)_n$ of positive numbers such that $\alpha_n \downarrow \|f - g\|$. Then $f \leq g + \|f - g\| \leq g + \alpha_n \cdot 1$, which implies

$I(f) \leq I(g + \alpha_n \cdot 1) = I(g) + \alpha_n I(1)$

due to the properties of monotonicity and translation invariance and positive homogeneity of $I$. Since the role of $f$ and $g$ is symmetric, this leads to the fact that $|I(f) - I(g)| \leq I(1) \cdot \alpha_n$ for all $n$, whence by passing to the limit as $n \to \infty$, we conclude that

$|I(f) - I(g)| \leq I(1) \cdot \|f - g\|.
$

Lemma 9. Let $E$ be an ordered Banach space. Then every comonotonic additive and monotone operator $I : C(X) \to E$ is positively homogeneous.

Proof. Let $f \in C(X)_+$. Since $I$ is comonotonic additive, we get $I(0) = I(0 + 0) = 2 \cdot I(0)$, which implies $I(0) = 0$. As a consequence, $I(0 \cdot f) = 0 \cdot I(f)$. Then the same argument shows that $I(2f) = I(f + f) = 2I(f)$ and by mathematical induction we infer that $I(pf) = pI(f)$ for all $p \in \mathbb{N}$.

Now, consider the case of positive rational numbers $r = p/q$, where $p, q \in \mathbb{N}$. Then $I(f) = I\left(q \cdot \frac{1}{q} f\right) = qI\left(\frac{1}{q} f\right)$, which implies $I\left(\frac{1}{q} f\right) = \frac{1}{q} \cdot I(f)$. Therefore

$I\left(\frac{p}{q} f\right) = pI\left(\frac{1}{q} f\right) = \frac{p}{q} \cdot I(f)$.

Passing to the case of an arbitrary positive number $\alpha$, let us choose a decreasing sequence $(r_n)_n$ of rationals converging to $\alpha$. Then $r_n f \geq \alpha f$ for all $n$, which yields $r_n I(f) = I(r_n f) \geq I(\alpha f) \ n \in \mathbb{N}$. Passing here to limit (in the norm of $E$) it follows $\alpha I(f) \geq I(\alpha f)$. On the other hand, considering a sequence of positive rational numbers $s_n \nearrow \alpha$ and reasoning as above, we easily obtain $\alpha I(f) \leq I(\alpha f)$, which combined with the previous inequality proves the assertion of Lemma 3 in the case of nonnegative functions.
When \( f \in C(X) \) is arbitrary, one can choose a positive number \( \lambda \) such that \( f + \lambda \geq 0 \). By the above reasoning and the property of comonotonic additivity, for all \( \alpha \geq 0 \),

\[
\alpha I(f) + \alpha \lambda I(1) = \alpha I(f + \lambda) = I(\alpha (f + \lambda)) \\
= I(\alpha f + \alpha \lambda) = I(\alpha f) + \alpha \lambda I(1),
\]

which ends the proof of Lemma 9. \( \square \)

**Lemma 10.** Let \( E \) be a Banach lattice with order continuous norm. Then every monotone, positively homogeneous and translation invariant operator \( I : C(X) \to E \) is weakly compact and upper continuous.

**Proof.** The weak compactness of \( I \) follows from the fact that \( I \) maps the closed order interval in \( C(X) \) (that is, all closed balls) into closed order intervals in \( E \) and all such intervals are weakly compact in \( E \). See Lemma 3.

For the property of upper continuity, let \((f_n)_n\) be any nonincreasing sequence of functions in \( C(X) \), which converges pointwise to a continuous function \( f \). By Dini’s lemma, the sequence \((f_n)_n\) is convergent to \( f \) in the norm topology of \( C(X) \). Therefore, for each \( \varepsilon > 0 \) there is an index \( N \) such that \( f \leq f_n < f + \varepsilon \) for all \( n \geq N \). Since \( I \) is monotone it follows that

\[
\begin{align*}
(4.3) \quad I(f) & \leq I(f_n) \leq I(f + \varepsilon) = I(f) + \varepsilon \cdot I(1) \quad \text{for all } n \geq N,
\end{align*}
\]

where 1 is the unit of \( C(X) \). Taking into account that \( E \) has order continuous norm and \( I(f_n) \geq I(f_{n+1}) \geq I(f) \) for all \( n \) it follows that the limit \( \lim_{n \to \infty} I(f_n) \) exists in \( E \) and \( \lim_{n \to \infty} I(f_n) \geq I(f) \). Combining this fact with (4.3) we infer that \( I(f) \leq \lim_{n \to \infty} I(f_n) \leq I(f) + \varepsilon I(1) \). Since \( \varepsilon > 0 \) was chosen arbitrarily we conclude that \( \lim_{n \to \infty} I(f_n) = I(f) \). \( \square \)

The next result, was stated for real-valued functionals in [31], Lemma 1 (and attributed by him to Masimo Marinacci). For the convenience of the reader we include here the details.

**Lemma 11.** Let \( E \) be an ordered Banach space.

(a) Suppose that \( I : C(X) \to E \) is a monotone, positively homogeneous and translation invariant operator. The following two properties are equivalent:

\( (a_1) \lim_{n \to \infty} I(f_n) = I(f) \) for any nonincreasing sequence \((f_n)_n\) in \( C(X) \) that converges pointwise to a function \( f \) also in \( C(X) \);

\( (a_2) \lim_{n \to \infty} I(f_n) \leq I(f) \) for any nonincreasing sequence \((f_n)_n\) in \( C(X) \) and any \( f \) in \( C(X) \) such that for each \( x \in X \) there is an index \( n_x \in \mathbb{N} \) such that \( f_n(x) \leq f(x) \) whenever \( n \geq n_x \).

(b) For any vector capacity \( \mu : \mathcal{B}(X) \to E_+ \), the following two properties are equivalent:

\( (b_1) \lim_{n \to \infty} \mu(A_n) = \mu(A) \), for any nonincreasing sequence \((A_n)_n\) of sets in \( \mathcal{B}(X) \) such that \( A = \bigcap_{n=1}^\infty A_n \);

\( (b_2) \lim_{n \to \infty} \mu(A_n) \leq \mu(A) \), for any nonincreasing sequence \((A_n)_n\) of sets in \( \mathcal{B}(X) \) and any \( A \in \mathcal{B}(X) \) such that \( \bigcap_{n=1}^\infty A_n \subset A \).

All the limits above are considered in the norm topology of \( E \).

**Proof.** \((a_1) \Rightarrow (a_2)\). Let an arbitrary sequence \((f_n)_n\) in \( C(X) \) and \( f \in C(X) \), be such that \((f_n)_n\) is nonincreasing and, for all \( x \in X \), there is an \( n_x \) with \( f_n(x) \leq f(x) \) for all \( n \geq n_x \). Since the sequence \((\max\{f_n, f\})\) is also a nonincreasing
sequence in $C(X)$ and $\lim_{n \to \infty} \max \{f_n(x), f(x)\} = f(x)$ for all $x \in X$, $(a_i)$ implies
$\lim_{n \to \infty} I(\max \{f_n, f\}) = I(f)$. By the monotonicity of $I$ it follows $\lim_{n \to \infty} I(f_n) \leq \lim_{n \to \infty} I(\max \{f_n, f\}) = I(f)$.

$(a_2) \Rightarrow (a_1)$. Let $f_n, f \in C(X)$, $n \in \mathbb{N}$, be such that $(f_n)_n$ is nonincreasing and
$\lim_{n \to \infty} f_n(x) = f(x)$, for all $x \in X$. Fix $\varepsilon > 0$ arbitrary. Since for all $x \in X$, there exists $n_x \in \mathbb{N}$ such that $f_n(x) \leq f(x) + \varepsilon$, for all $n \geq n_x$, $(a_{ii})$ implies (from
monotonicity, positive homogeneity and translation invariance) $\lim_{n \to \infty} I(f_n) \leq I(f + \varepsilon \cdot 1) = I(f) + \varepsilon I(1)$. Passing with $\varepsilon \to 0$ it follows $\lim_{n \to \infty} I(f_n) \leq I(f)$. But since $(f_n)_n$ is nonincreasing and $I$ is monotone, we also have $\lim_{n \to \infty} I(f_n) \geq I(f)$, which combined with the previous inequality implies $\lim_{n \to \infty} I(f_n) = I(f)$.

The equivalence $(b_1) \Leftrightarrow (b_2)$ can be proved in a similar way. □

Recall that in the case of compact Hausdorff space $X$ the lattice $\Sigma^+_u(X)$ represents the lattice of upper contour sets associated to the continuous functions defined on $X$. This lattice coincides with the lattice of all closed subsets of $X$ when $X$ is compact and metrizable; indeed, if $d$ is the metric of $X$, then every closed subset $A \subset X$ admits the representation $A = \{x : -d(x, A) \geq 0\}$.

**Proof of Theorem 3.** Notice first that according to Lemma 9 and Lemma 10 the
operator $I$ is also positively homogeneous and upper continuous.

Every set $K \in \Sigma^+_u(X)$ admits a representation of the form

$$K = \{x : f(x) \geq \alpha\},$$

for suitable $f \in C(X)$ and $\alpha \in \mathbb{R}$. As a consequence its characteristic function
$\chi_K$ is the pointwise limit of a nonincreasing sequence $(f^K_n)_n$ of continuous and
nonnegative functions. For example, one may choose

$$(4.4) \quad f^K_n(x) = 1 - \inf \{1, n(\alpha - f)^+\} = \begin{cases} 
0 & \text{if } f(x) \leq \alpha - 1/n \\
\in (0, 1) & \text{if } \alpha - 1/n < \varphi(x) < \alpha \\
1 & \text{if } f(x) \geq \alpha \text{ (i.e., } x \in K\}. 
\end{cases}$$

See [11], p. 1814. Since $I$ is monotone, the sequence $(I(f^K_n))_n$ is also nonincreasing and bounded from below by 0, which implies (due to the order continuity of the norm of $E$), that it is also convergent in the norm topology of $E$.

This allows us to define $\mu$ on the sets $K \in \Sigma^+_u(X)$ by the formula

$$(4.5) \quad \mu(K) = \lim_{n \to \infty} I(f^K_n).$$

The definition of $\mu(K)$ is independent of the particular sequence $(f^K_n)_n$ with the
aforementioned properties. Indeed, if $(g^K_n)_n$ is another such sequence, fix a positive
integer $m$, and infer from Lemma [11]$(a)$ that $\lim_{n \to \infty} I(g^K_n) \leq I(f^K_m)$. Taking the
limit as $m \to \infty$ on the right-hand side we get

$$\lim_{n \to \infty} I(g^K_n) \leq \lim_{m \to \infty} I(f^K_m).$$

Then, interchanging $(f^K_n)_n$ and $(g^K_n)_n$ we conclude that actually equality holds.

Clearly, the set function $\mu : \Sigma^+_u(X) \to E_+$ is a vector capacity and it takes
values in the order interval $[0, I(1)]$.

We next show that $\mu$ is upper continuous on $\Sigma^+_u(X)$, that is,

$$\mu(K) = \lim_{n \to \infty} \mu(K_n)$$
whenever \((K_n)_n\) is a nonincreasing sequence of sets in \(\Sigma^+_\up{up}(X)\) such that \(K = \cap_{n=1}^{\infty} K_n\). Indeed, using formula (4.1) one can choose a nonincreasing sequence of continuous function \(g_n\) such that \(g_n \geq \chi_{K_n}, g_n = 0\) outside the neighborhood of radius \(1/n\) of \(K_n\) and \(I(g_n) - \mu(K_n) \to 0\). See formula (4.1) and using analogous reasonings with those concerning relations (7)-(9) in [31], pp. 1814-1815. Then \(\mu(K) = \lim_{n \to \infty} I(g_n)\), which implies the equality \(\mu(K) = \lim_{n \to \infty} \mu(K_n)\).

The next goal is the representation formula (4.2). For this, let \(x^* \in E^*_+\) be arbitrarily fixed and consider the comonotonic additive and monotone functional

\[x^* \circ I : C(X) \to \mathbb{R}.\]

It verifies \(I^*(1) = x^*(I(1)) > 0\), so by Theorem 1 in [31] there is a unique upper continuous capacity \(\nu^* : \Sigma^+_\up{up}(X) \to [0, I^*(1)]\) such that

\[(x^* \circ I)(f) = (C) \int_X f d\nu^* \text{ for all } f \in C(X).\]

The capacity \(\nu^*\) is obtained via an approximation process similar to [41.5]. Therefore

\[\nu^*(K) = \lim_{n \to \infty} x^*(I(f_n^K)) = \lim_{n \to \infty} (I(f_n^K)) = x^*(\mu(K))\]

for all \(K \in \Sigma^+_\up{up}(X)\), which implies

\[x^*(I(f)) = (C) \int_X f d(x^* \circ \mu).\]

Since \(x^* \in E^*_+\) was arbitrarily fixed, an appeal to Lemma [2] (c) easily yields the equality

\[I(f) = (CB) \int_X f d\mu.\]

For the second part of Theorem 3 since \(\mu\) takes values in an order bounded interval, one can extend it to all Borel subsets of \(X\) via the formula

\[\mu(A) = \sup \{ \mu(K) : K \text{ closed, } K \subset A \}, \quad A \in \mathcal{B}(X).\]

The fact that the resulting set function \(\mu\) is a vector capacity is immediate.

This set function \(\mu\) also verifies the regularity condition \((R2)\). Indeed, given a closed set \(K\), we can consider the sequence of open sets

\[O_n = \{ x \in X : d(x,K) < 1/n \}.\]

Clearly, \(\mu(K) \leq \mu(O_n) \leq \mu(K_n)\) where

\[K_n = \{ x \in X : d(x,K) \leq 1/n \}.\]

Since \(\mu\) is upper continuous on closed sets, it follows that \(\lim_{n \to \infty} \mu(K_n) = \mu(K)\), whence \(\lim_{n \to \infty} \mu(O_n) = \mu(K)\). Therefore \(\mu\) verifies the regularity condition \((R2)\). The uniqueness of the extension of \(\mu\) to \(\mathcal{B}(X)\) is motivated by the condition \((R1)\).

\[\square\]

**Remark 1.** If the operator \(I\) is submodular, that is,

\[I(\sup \{ f, g \}) + I(\inf \{ f, g \}) \leq I(f) + I(g) \quad \text{for all } f, g \in C(X),\]

then the vector capacity \(\mu : \Sigma^+_\up{up}(X) \to E^*_+\) stated by Theorem 3 is submodular. This is a consequence of Theorem 13 (c) in [4]. For the convenience of the reader we will recall here the argument. Let \(A, B \in \Sigma^+_\up{up}(X)\) and consider the sequences
(f^A_n(x))_n and (f^B_n(x))_n of continuous functions associated respectively to A and B by the formula (4.4). Then
\[ \mu(A) = \lim_{n \to \infty} I(f^A_n), \quad \mu(B) = \lim_{n \to \infty} I(f^B_n), \quad \mu(A \cup B) = \lim_{n \to \infty} I(\sup \{f^A_n, f^B_n\}) \]
and \( \mu(A \cap B) = \lim_{n \to \infty} I(\inf \{f^A_n, f^B_n\}) \). Since I is submodular, it follows that
\[ \mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B), \]
and the proof is done.

5. The case of operators with bounded variation

The representation Theorem 3 can be extended outside the framework of monotone operators by considering the class of operators with bounded variation.

As above, X is a compact Hausdorff space and E is a Banach lattice with order continuous norm.

Definition 3. An operator I : C(X) → E has bounded variation over an order interval [f, g] if
\[ \vee I = \sup \sum_{k=0}^{n} |I(f_k) - I(f_{k-1})| \]
exists in E,
the supremum being taken over all finite chains \( f = f_0 \leq f_1 \leq \cdots \leq f_n = g \) of functions in the Banach lattice C(X). The operator I is said to have bounded variation if it has bounded variations on all order intervals [f, g] in C(X).

Clearly, if I is monotone, then \( \vee I = I(g) - I(f) \) for all \( f \leq g \) in C(X) and thus I has bounded variation.

More generally, every operator I : C(X) → E which can be represented as the difference I = I_1 - I_2 of two monotone operators I_1, I_2 : C(X) → E has bounded variation. This follows from the order completeness of E and the modulus inequality, which provides an upper bound for the sums appearing in formula (5.1):
\[ \sum_{k=0}^{n} |I(f_k) - I(f_{k-1})| \leq I_1(g) - I_1(f) + I_2(g) - I_2(f). \]

Remarkably, the converse also holds. The basic ingredient is the following result.

Lemma 12. Suppose that I : C(X) → E is a comonotonic additive operator with bounded variation. Then there exist two positively homogeneous, translation invariant and monotone operators I_1, I_2 : C(X) → E such that I = I_1 - I_2.

Moreover, if I is upper continuous, then both operators I_1 and I_2 can be chosen to be upper continuous.

Proof. The proof is done in the footsteps of Lemma 14 in [3] by noticing first the following four facts:
(a) According to our hypotheses,
\[ I(\alpha f + \beta) = \alpha I(f) + I(\beta) \]
for all \( f \in C(X) \), \( \alpha \in \mathbb{R}_+ \) and \( \beta \in \mathbb{R} \).
(b) \( \vee^{f+\alpha} I = \vee^{f-\alpha} I \) for all \( f \in C(X) \) and \( \alpha \in \mathbb{R} \) with \( f + \alpha \geq 0 \).
This follows from the definition of the variation.
(c) \( \vee^0 I = \alpha \vee^0 I \) for all \( f \in C(X)_+ \) and \( \alpha \in \mathbb{R}_+ \).
Indeed for every \( \varepsilon \in E \), \( \varepsilon > 0 \), there exists a chain \( 0 = f_0 \leq f_1 \leq \cdots \leq f_n = f \)
such that
\[ \sum_{k=0}^{n} |I(f_k) - I(f_{k-1})| \geq \vee^0 I - \varepsilon. \]
According to fact (a) the chain \(0 = \alpha f_0 \leq \alpha f_1 \leq \cdots \leq \alpha f_n = \alpha f\) verifies
\[
\vee_0^\alpha I \geq \sum_{k=0}^n |I(\alpha f_k) - I(\alpha f_{k-1})| \geq \alpha \vee_0^\alpha I - \alpha \varepsilon.
\]
As \(\varepsilon > 0\) was arbitrarily fixed it follows that \(\vee_0^\alpha I \geq \alpha \vee_0^\alpha I\). By replacing \(\alpha\) to \(1/\alpha\) and then \(f\) by \(\alpha f\), one obtains the reverse inequality, \(\vee_0^\alpha I \leq \alpha \vee_0^\alpha I\).

(d) \(\vee^\alpha I = \vee_{-\alpha}^0 I + \vee_0^\alpha I = \vee_{-\alpha}^0 I + \vee_0^\alpha I\) for all \(f \in C(X)_+\) and \(\alpha \in \mathbb{R}_+\).

The fact that \(\vee^\alpha I \geq \vee_{-\alpha}^0 I + \vee_0^\alpha I \equiv \vee_0^\alpha I + \vee_0^\alpha I\) is a direct consequence of the definition of variation. For the other inequality, fix arbitrarily \(\varepsilon > 0\) in \(E\) and choose a chain \(-\alpha = f_0 \leq f_1 \leq \cdots \leq f_n = f\) such that
\[
\sum_{k=0}^n |I(f_k) - I(f_{k-1})| \geq \vee_{-\alpha}^\alpha I - \varepsilon.
\]
Then \(-\alpha = -f_0^- \leq -f_1^- \leq \cdots \leq -f_n^- = 0\) and \(0 = f_0^+ \leq f_1^+ \leq \cdots \leq f_n^+ = f\).
Since all pairs \(-f_k^-, f_k^+\) are comonotonic, we have
\[
\begin{align*}
\vee_{-\alpha}^0 I + \vee_0^\alpha I &\geq \sum_{k=0}^n |I(-f_k^-) - I(-f_{k-1}^-)| + \sum_{k=0}^n |I(f_k^+) - I(f_{k-1}^+)| \\
&\geq \sum_{k=0}^n |I(-f_k^-) - I(-f_{k-1}^-) + I(f_k^+) - I(f_{k-1}^+)| \\
&= \sum_{k=0}^n |I(f_k) - I(f_{k-1})| \geq \vee_{-\alpha}^\alpha I - \varepsilon
\end{align*}
\]
and it remains to take the supremum over \(\varepsilon > 0\).

Now we can proceed to the choice of the operators \(I_1\) and \(I_2\). By definition,
\[
I_1(f) = \vee_0^\alpha I \quad \text{if} \quad f \in C(X)_+,
\]
while if \(f \in C(X)\) is an arbitrary function, one choose \(\alpha \in \mathbb{R}_+\) such that \(f + \alpha \geq 0\) and put
\[
I_1(f) = \vee_0^{f + \alpha} I - \alpha \vee_0^1 I.
\]
The fact that \(I_1\) is well-defined (that is, independent of \(\alpha\)) can be proved as follows. Suppose that \(\alpha, \beta > 0\) are two numbers such that \(f + \alpha \geq 0\) and \(f + \beta \geq 0\). Without loss of generality we may assume that \(\alpha < \beta\). Indeed, according to the facts (b) – (d), we have
\[
\begin{align*}
\vee_0^{f + \beta} I - \beta \vee_0^1 I &= \vee_0^{f + \alpha + (\beta - \alpha)} I - \beta \vee_0^1 I \quad \text{fact (b)} \\
&= \vee_0^{f + \alpha} I - \beta \vee_0^1 I \quad \text{fact (d)} \\
&= \vee_0^{\beta - \alpha} I + \vee_0^{f + \alpha} I - \beta \vee_0^1 I \quad \text{fact (b)} \\
&= \vee_0^{f + \alpha} I - \beta \vee_0^1 I \quad \text{fact (c)} \\
&= \vee_0^{f + \alpha} I - \alpha \vee_0^1 I.
\end{align*}
\]
By definition,
\[
I_2 = I - I_1.
\]
Let \(f, g\) be two functions in \(C(X)\) such that \(f \leq g\) and let \(\alpha > 0\) such that \(f + \alpha \geq 0\). Since \(I\) is monotonic and has bounded variation,
\[
0 \leq I(g) - I(f) = I(g + \alpha) - I(f + \alpha) \leq \vee_0^{f + \alpha} I \leq \vee_0^{f + \alpha} I - \vee_0^{f + \alpha} I = I_1(g) - I_1(f),
\]
whence we infer that both operators \(I_1\) and \(I_2\) are monotonic.
Due to the fact (c), the operator \( I_1 \) is positively homogeneous. Therefore the same is true for \( I_2 \).

For the property of translation invariance, let \( f \in C(X) \), \( \beta \in \mathbb{R} \) and choose \( \alpha > 0 \) such that \( f + \beta + \alpha \geq 0 \) and \( \beta + \alpha \geq 0 \). Then
\[
I_1(f + \beta) = \nu^0_{f+\beta+\alpha} I - \alpha \nu^1_0 I
\]
\[
= I_1(f + \beta + \alpha) - \alpha \nu^1_0 I
\]
while from facts (b)&(d) we infer that \( I_1(f) = I_1(f + \beta + \alpha) - (\beta + \alpha) \nu^1_0 I \). Therefore
\[
I_1(f + \beta) = I_1(f) + I_1(\beta)
\]
which proves that indeed \( I_1 \) is translation invariant. The same holds for \( I_2 = I - I_1 \).

As concerns the second part of Lemma 12, it suffices to prove that \( I_1 \) is upper continuous when \( I \) has this property.

Let \( (f_n)_n \) be a decreasing sequence in \( C(X) \) which converges pointwise to a function \( f \) also in \( C(X) \). By Dini’s lemma, \( \|f_n - f\| \to 0 \). Since \( I_1 \) is a Lipschitz operator (see Lemma 3) we conclude that \( \|I_1(f_n) - I(f)\| \to 0 \).

This ends the proof of Lemma 12.

Now it is clear that the representation Theorem 3 can be extended to the framework of comonotonic additive operators \( I : C(X) \to E \) with bounded variation by considering Choquet-Bochner integrals associated to differences of vector capacities (that is, to set functions with bounded variation in the sense of Aumann and Shapley 1).

Let \( \Sigma \) be a lattice of sets of \( X \) and \( \mu : \pm \to E \) a set function taking values in a Banach lattice \( E \) with order continuous norm.

The \textit{variation} of the set function \( \mu \) is the set function \( |\mu| \) defined by the formula
\[
|\mu|(A) = \sup \sum_{k=0}^n |\mu(A_k) - \mu(A_{k-1})| \quad \text{for} \quad A \in \Sigma,
\]
where the supremum is taken over all finite chains \( A_0 = \emptyset \subset A_1 \subset \cdots \subset A_n \subset A \) of sets in the lattice \( \Sigma \).

The space of set functions with bounded variation,
\[
\text{bv}(\Sigma, E) = \{ \mu : \Sigma \to E : \mu(\emptyset) = 0 \text{ and } |\mu|(X) \text{ exists in } E \},
\]
is a normed vector space when endowed with the norm
\[
\|\mu\|_{\text{bv}} = \| |\mu|(X) \|.
\]

Associated to every set function \( \mu \in \text{bv}(\Sigma, E) \) are two positive vector-valued set functions, the \textit{inner upper variation} of \( \mu \), defined by
\[
\mu^+(A) = \sup \sum_{k=0}^n (\mu(A_k) - \mu(A_{k-1}))^+ \quad \text{for} \quad A \in \Sigma
\]
and the \textit{inner lower variation} of \( \mu \), defined by
\[
\mu^-(A) = \sup \sum_{k=0}^n (\mu(A_k) - \mu(A_{k-1}))^- \quad \text{for} \quad A \in \Sigma
\]
in both cases the supremum is taken over all finite chains \( A_0 = \emptyset \subset A_1 \subset \cdots \subset A_n \subset A \) of sets in \( \Sigma \). Notice that
\[
\mu = \mu^+ - \mu^- \text{ and } |\mu| = \mu^+ + \mu^-.
\]
Lemma 13. We assume that $E$ is a Banach lattice with order continuous norm. The following conditions are equivalent for a set function $\mu : \Sigma \to E$:

(a) $\mu \in \bv(\Sigma, E)$;
(b) $\mu^+$ and $\mu^-$ are vector capacities;
(c) there exist two vector capacities $\mu_1$ and $\mu_2$ on $\Sigma$ such that $\mu = \mu_1 - \mu_2$.

Moreover, for any such decomposition we have $\mu^+ \leq \mu_1$ and $\mu^- \leq \mu_2$.

The details are similar to those presented in [1], Ch. 1, §4.

We are now in a position to state the main result of this section.

Theorem 4. Let $X$ be a compact Hausdorff space and $E$ be a Banach lattice with order continuous norm. Then for every comonotonic additive operator $I : C(X) \to E$ with bounded variation there exists a unique upper continuous set function $\mu : \Sigma_{up}^+(X) \to E$ with bounded variation such that

$$I(f) = \int_0^{\pm \infty} \mu \left( \{ x : f(x) \geq t \} \right) dt$$

$$+ \int_{-\infty}^0 \mu \left( \{ x : f(x) \geq t \} \right) - \mu(A) dt.$$

for all $f \in C(X)$.

Proof. By Lemma 12, there exist two functionals $I_1, I_2 : C(X) \to E$ that are monotone, translation invariant, positively homogeneous, upper continuous and such that $I = I_1 - I_2$. Define $\mu : \Sigma_{up}^+(X) \to E$ by $\mu = \mu_1 - \mu_2$, where $\mu_1, \mu_2$ are associated to the functionals $I_1$ and $I_2$ via Theorem 3. Then $\mu$ is upper continuous and by Lemma 13 it also has bounded variation. We now prove that the representation formula (5.2) holds.

Suppose that $f \in C(X)_+$. The integral

$$\int_0^{\pm \infty} \mu \left( \{ x : f(x) \geq t \} \right) dt$$

is well defined as a Bochner integral (see Lemma 6). Given $\epsilon > 0$, one can choose an equidistant division $0 = t_0 < \cdots < t_m = \| f \|$ of $[0, \| f \|]$ such that

$$\left\| \int_0^{\pm \infty} \mu \left( \{ x : f(x) \geq t \} \right) dt - \sum_{k=0}^{m-1} \mu \left( \{ x : f(x) \geq t_k \} \right) (t_{k+1} - t_k) \right\| < \epsilon$$

and $\| f \| / m < \epsilon$.

Denote $C_k = \{ x : f(x) \geq t_k \}$ for $k = 0, \ldots, m - 1$. By the definition of $\mu_1$ and $\mu_2$ (see the proof of Theorem 3) one can choose functions $f_n^C \in C(X)_+$ such that

$$\left\| \mu \left( C_k \right) - I(f_n^C) \right\| < \epsilon / \| f \| \quad \text{and} \quad n^{-1} < \| f \| / m.$$

Because the functions $f_n^C$ are defined by the formula (4.3) and $n^{-1} < \| f \| / m$, it follows that the functions $f_n^C(t_{i+1} - t_i)$ and $\sum_{k=i+1}^{m-1} f_n^C(t_{k+1} - t_k)$ are comonotonic.
for all indices $i$, so that
\[
I\left(\sum_{k=0}^{m-1} f_n^{C_k}(t_{k+1} - t_k)\right) = I(f_n^{C_k})(t_1 - t_0) + I\left(\sum_{k=1}^{m-1} f_n^{C_k}(t_{k+1} - t_k)\right)
\]
\[
= \cdots = m \cdot I(f_n^{C_k})(t_{k+1} - t_k);
\]
the property of positive homogeneity of $I$ is assured by Lemma 12.

Notice that
\[
f(x) \leq \sum_{k=0}^{m-1} f_n^{C_k}(x)(t_{k+1} - t_k) \leq f(x) + 2\varepsilon \quad \text{for all } x \in X.
\]
Since the operators $I_1$, $I_2$ are monotone and translation invariant, it follows that
\[
I_j(f) \leq I_j\left(\sum_{k=0}^{m-1} f_n^{C_k}(t_{k+1} - t_k)\right) \leq I_j(f) + 2\varepsilon I_j(1) \quad \text{for } j \in \{1, 2\}.
\]
whence
\[
(5.5) \quad \left\| I_j\left(\sum_{k=0}^{m-1} f_n^{C_k}(t_{k+1} - t_k)\right) - I_j(f) \right\| \leq 2\varepsilon \|I_j(1)\| \quad \text{for } j \in \{1, 2\}.
\]
Therefore
\[
\left\| \int_0^{+\infty} \mu(\{x : f(x) \geq t\}) dt - I(f) \right\|
\leq \left\| \int_0^{+\infty} \mu(\{x : f(x) \geq t\}) dt - \sum_{k=0}^{m-1} I(f_n^{C_k})(t_{k+1} - t_k) \right\|
\]
\[
+ \left\| \sum_{k=0}^{m-1} I(f_n^{C_k})(t_{k+1} - t_k) - I(f) \right\|
\leq \left\| \int_0^{+\infty} \mu(\{x : f(x) \geq t\}) dt - \sum_{k=0}^{m-1} I(f_n^{C_k})(t_{k+1} - t_k) \right\|
\]
\[
+ 2\varepsilon \|I_1(1)\| + 2\varepsilon \|I_2(1)\|
\]
\[
\leq \left\| \int_0^{+\infty} \mu(\{x : f(x) \geq t\}) dt - \sum_{k=0}^{n-1} \mu(C_k)(t_{k+1} - t_k) \right\|
\]
\[
+ \left\| \sum_{k=0}^{n-1} \mu(C_k)(t_{k+1} - t_k) - \sum_{k=0}^{n-1} I(f_n^{C_k})(t_{k+1} - t_k) \right\|
\]
\[
+ 2\varepsilon \|I_1(1)\| + 2\varepsilon \|I_2(1)\|
\]
\[
\leq 2\varepsilon (1 + \|I_1(1)\| + \|I_2(1)\|).
\]
Since $\varepsilon > 0$ was arbitrarily fixed, the above reasoning yields formula (5.2) in the case of positive functions.
If \( f \not\in C(X)_+ \), then \( f + \|f\| \in C(X)_+ \) and by the preceding considerations we have
\[
I(f) + \|f\| I(1) = I(f + \|f\|) = \int_{0}^{+\infty} \mu(\{ x : f(x) + \|f\| \geq t \}) \, dt
\]
\[
= \int_{0}^{+\infty} \mu(\{ x : f(x) \geq t \}) \, dt + \int_{-\|f\|}^{0} \mu(\{ x : f(x) \geq t \}) \, dt
\]
\[
= \int_{0}^{+\infty} \mu(\{ x : f(x) \geq t \}) \, dt + \int_{-\|f\|}^{0} \mu(\{ x : f(x) \geq t \}) - \mu(X) \, dt + \|f\| \mu(X)
\]
\[
= \int_{0}^{+\infty} \mu(\{ x : f(x) \geq t \}) \, dt + \int_{-\|f\|}^{0} \mu(\{ x : f(x) \geq t \}) - \mu(X) \, dt + \|f\| I(1).
\]
The proof of the representation formula (5.2) is now complete.

As concerns the uniqueness of \( \mu \), suppose that \( \nu \) is another upper continuous monotone set function with bounded variations for which the formula (5.2) holds. Given a set \( K = \{ x : f(x) \geq t \} \in \Sigma_{up}(X) \), it is known that the functions \( f^K_n : X \to [0, 1] \) defined by the formula (4.4) decrease to \( \chi_K \). This implies that \( \{ x : f^K_n(x) \geq t \} \) is decreasing to \( K \) for each \( t \in [0, 1] \). Consider the sequence of functions
\[
\varphi_n : [0, 1] \to E, \quad \varphi_n(t) = \nu(\{ x : f^K_n(x) \geq t \}).
\]
Notice that all these functions have bounded variation and their variation is bounded by the variation of \( \nu(X) \).

By Lebesgue dominated convergence,
\[
\lim_{n \to \infty} \int_{0}^{1} \varphi_n(t) \, dt = \nu(K).
\]
On the other hand, by (5.2) and the definition of \( \mu \),
\[
\mu(K) = \lim_{n \to \infty} I(f^K_n) = \lim_{n \to \infty} \int_{0}^{1} \varphi_n(t) \, dt = \nu(K),
\]
which ends the proof of the uniqueness. \( \Box \)

6. Open problems

We end our paper by mentioning few open problems that might be of interest to our readers.

**Problem 1.** Is the order continuity of the norm of \( E \) a necessary condition for the validity of Theorems 3 and 4?

As was noticed by Bartle, Dunford and Schwartz [2], much of the theory of weakly compact linear operators defined on a space \( C(X) \) is dominated by the concept of absolute continuity. For more recent contributions see [20], [21] and [22].

Suppose that \( \mathcal{A} \) is a \( \sigma \)-algebra and \( E \) is a Banach lattice with order continuous norm.

A vector capacity \( \mu : \mathcal{A} \to E_+ \) is called absolutely continuous with respect to a capacity \( \lambda : \mathcal{A} \to [0, \infty) \) (denoted \( \mu \ll \lambda \)) if for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that
\[
A \in \mathcal{A}, \quad \lambda(A) < \delta \implies \|\mu(A)\| < \varepsilon.
\]
The following lemma extends a result proved by Pettis in the context of \( \sigma \)-additive measures.
Lemma 14. If $\mu$ is upper continuous and $\lambda$ is upper continuous and supermodular then the condition $\mu \ll \lambda$ is equivalent to the following one:

$$A \in \mathcal{A} \text{ and } \lambda(A) = 0 \implies \mu(A) = 0.$$ 

The proof is immediate, by reductio ad absurdum.

Problem 2. Suppose $\mu : \mathcal{A} \to E_+$ is an upper continuous vector measure. Does there exist a capacity $\lambda : \mathcal{A} \to [0, \infty)$ such that $\mu \ll \lambda$?

If Yes, is it possible to choose $\lambda$ of the form $\lambda = x^* \circ \mu$ for a suitable $x^* \in E_+^*$?

It would be also interesting the following operator analogue of Problem 2:

Problem 3. Does there exist for each Choquet operator $I : C(X) \to E$ a functional $x^* \in E_+^*$ such that for every $\varepsilon > 0$ there is $\delta > 0$ with the property

$$\|I(f)\| \leq \varepsilon \|f\| + \delta x^* (|f|) \text{ for all } f \in C(X).$$

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