Abstract. Classical solutions of initial boundary value problems for infinite systems of quasilinear parabolic differential functional equations are considered. Two type of difference schemes are constructed. We prove that solutions of infinite difference schemes approximate solutions of our differential functional problem. In the second part of the paper we show that solutions of infinite differential functional systems can be approximated by solutions of difference systems with initial boundary conditions and the systems are finite.

A complete convergence analysis for the methods is presented. The proof of the stability is based on a comparison technique with nonlinear estimates of the Perron type for given functions.

1. Introduction

For any metric spaces $X$ and $Y$ we denote by $C(X,Y)$ the class of all continuous functions from $X$ into $Y$. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers respectively. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Denote by $l^{\infty}$ the class of all real sequences $p = \{p_\mu\}_{\mu \in \mathbb{N}}$ such that

$$||p||_{l^{\infty}} = \sup \{|p_\mu| : \mu \in \mathbb{N}\} < \infty.$$ 

Let the symbol $M_{n \times n}$ denotes the set of all real $n \times n$ matrices. For $A = [a_{ij}]_{i,j=1,...,n}$ we write $||A||_{n \times n} = \sum_{i,j=1}^{n} |a_{ij}|$. Denote by $M^{\infty}$ the set of all $P = \{P_\mu\}_{\mu \in \mathbb{N}}$ such that $P_\mu \in M_{n \times n}$, $\mu \in \mathbb{N}$, and

$$||P||_{l^{\infty}} = \sup \{||P_\mu||_{n \times n} : \mu \in \mathbb{N}\} < \infty.$$ 

Let $a > 0$, $b = (b_1, \ldots, b_n) \in R^n$, $b_j > 0$ for $j = 1, \ldots, n$ and $\tau_0 \in R_+$.

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\(\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n_+\), \(R_+ = [0, +\infty)\), be given. Define the sets
\(E = [0, a] \times (-b, b),\ \ D = [-\tau_0, 0] \times [-\tau, \tau],\ \ E_0 = [-\tau_0, 0] \times [-b - \tau, b + \tau],\ \ \partial_0 E = ([0, a] \times [-b - \tau, b + \tau]) \setminus E,\ \ \Omega = E_0 \cup E \cup \partial_0 E.\)

For a function \(z : \Omega \to l^\infty\) and for \((t, x) \in \bar{E}\) where \(\bar{E}\) is the closure of \(E\), we consider the function \(z(t,x) : D \to l^\infty\) given by
\[z(t,x)(s,y) = z(t + s, x + y),\ \ (s,y) \in D.\]

If \(w \in C(D, l^\infty)\) then we write
\[\|w\|_D = \max \{||w(s,y)||_{l^\infty} : (s,y) \in D\}.\]

Let \(\Xi = E \times C(D, l^\infty)\) and \(\Sigma = E \times C(D, l^\infty) \times \mathbb{R}^n\). Suppose that
\(\varphi : \Xi \to \mathcal{M}^\infty,\ \ \varphi = \{\varphi^{(\mu)}\}_{\mu \in \mathbb{N}},\ \ \varphi^{(\mu)} = \{\varphi^{(\mu)}_{ij}\}_{i,j=1,\ldots,n},\ \ \mu \in \mathbb{N},\ \ f : \Sigma \to l^\infty,\ \ f = \{f^{(\mu)}\}_{\mu \in \mathbb{N}},\ \ \varphi : E_0 \cup \partial_0 E \to l^\infty,\ \ \varphi = \{\varphi_\mu\}_{\mu \in \mathbb{N}},\)
are given functions. For \(z : \Omega \to l^\infty,\ \ z = \{z_\mu\}_{\mu \in \mathbb{N}},\) and for \((t,x) \in E,\ \ x = (x_1, \ldots, x_n)\), we write
\[\partial_t z(t,x) = \{\partial_{t}\mu z(t,x)\}_{\mu \in \mathbb{N}},\ \ F[z](t,x) = \{F^{(\mu)}[z](t,x)\}_{\mu \in \mathbb{N}}\]
and
\[F^{(\mu)}[z](t,x) = \sum_{i,j=1}^{n} \varphi^{(\mu)}_{ij}(t,x,z(t,x))\partial_{x_i x_j} z_\mu(t,x) + f^{(\mu)}(t,x,z(t,x), \partial_x z_\mu(t,x))\]
where \(\partial_{x_i z_\mu} = (\partial_{x_1} z_\mu, \ldots, \partial_{x_n} z_\mu)\) and \(\mu \in \mathbb{N}\). We consider the infinite system of quasilinear differential functional equations
\[(1)\ \ \ \ \ \ \ \ \ \ \ \partial_t z(t,x) = F[z](t,x)\]
with the initial boundary condition
\[(2)\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ z(t,x) = \varphi(t,x) \ \ on \ E_0 \cup \partial_0 E.\]

We consider classical solutions of the above problem. A function \(v : \Omega \to l^\infty,\ \ v = \{v_\mu\}_{\mu \in \mathbb{N}},\) is a classical solution of problem (1), (2) if
(i) \(v \in C(\Omega, l^\infty),\) derivatives \(\partial_t v_\mu, \partial_{x_i} v_\mu, \partial_{x_i x_j} v_\mu, 1 \leq i, j \leq n,\) exist and they are continuous on \(E\) for all \(\mu \in \mathbb{N},\)
(ii) \(v\) satisfies (1) on \(E\) and the condition (2) holds.

We are interested in establishing a method of approximating of classical solutions to problem (1), (2) by solutions of associated difference equations. The classical difference methods for partial differential equations or systems consist of replacing partial derivatives by suitable difference operators. Solutions of difference functional equations are defined on the mesh. Thus we
need also some interpolating operators. Then solutions of difference equations or systems approximate, under suitable assumptions on given functions and on the mesh, solutions of the original problem.

The main task in these investigations is to find a finite difference approximation which is stable. The method of difference inequalities and theorems on linear recurrent inequalities are used in the investigation of the stability of difference schemes. Convergence results are also based on a general theorem on an error estimate of approximate solutions to functional difference equations of the Volterra type with initial or initial boundary conditions and with an unknown function of several variables.

The problems mentioned above have an extensive bibliography. It is not our aim to show a full review of papers concerning difference methods for parabolic differential functional problems. We will mention only those which contain such reviews. They are [2], [4], [6], [7].

Theorems on the existence and uniqueness of solutions to infinite systems of parabolic differential functional equations can be found in [1], [8], [9].

Two types of difference schemes for problem (1), (2) are constructed. We show that solutions of infinite difference schemes approximate solutions of our differential functional problem. In the second part of the paper we show that solutions of infinite differential functional systems can be approximated by solutions of difference systems with initial boundary conditions and the systems are finite. Results presented in the paper are new also in the case of infinite systems without a functional dependence.

The present paper is organized in the following way. In Section 2 we construct the infinite system of functional difference equations generated by (1), (2). The difference operators approximating mixed derivatives depend on local properties of coefficients of the differential equations. We prove a theorem on the convergence of the difference method. In Section 3 we consider finite systems of functional differential equations. Components of the unknown function \( z = \{z_\mu\}_{\mu \in \mathbb{N}} \) with the numbers greater than fixed \( k \in \mathbb{N} \), are replaced by the respective components of an extension of the initial boundary function. The difference method used to such differential systems has the following property. The solution of the difference scheme approximates the solution of differential problem (1), (2) if the step of the mesh is tending to zero and if the number of equations used in this scheme is increasing to infinity.

2. Infinite systems of difference equations

We formulate a difference problem corresponding to (1), (2). We denote by \( \mathcal{F}(A, B) \) the class of all functions defined on \( A \) and taking values in \( B \), where \( A \) and \( B \) are arbitrary sets. We define a mesh on the set \( \Omega \) in
the following way. Suppose that \((h_0, h')\) where \(h' = (h_1, \ldots, h_n)\), \(h_i > 0\), 
\(0 \leq i \leq n\), stand for steps of the mesh. We write \(||h'|| = h_1 + \ldots + h_n\). For 
\(h = (h_0, h')\) and \((r, m) \in \mathbb{Z}^{1+n}\) where \(m = (m_1, \ldots, m_n)\) we define nodal 
points as follows:

\[
    t^{(r)} = r h_0, \quad x^{(m)} = (x_1^{(m_1)}, \ldots, x_n^{(m_n)}) = (m_1 h_1, \ldots, m_n h_n).
\]

Denote by \(\Delta\) the set of all \(h = (h_0, h')\) such that there are \(\tilde{N}_0 \in \mathbb{N}, N = (N_1, \ldots, N_n) \in \mathbb{N}^n\) with the properties: \(\tilde{N}_0 h_0 = \tau_0\) and \((N_1 h_1, \ldots, N_n h_n) = \tau\). There is \(N_0 \in \mathbb{N}\) such that \(N_0 h_0 \leq a < (N_0 + 1) h_0\). Let

\[
    R_h^{1+n} = \{(t^{(r)}, x^{(m)}) : (r, m) \in \mathbb{Z}^{1+n}\},
\]

\[
    D_h = D \cap R_h^{1+n}, \quad E_h = E \cap R_h^{1+n}, \quad E_{0, h} = E_0 \cap R_h^{1+n}, \quad \partial_0 E_h = \partial_0 E \cap R_h^{1+n}
\]

and

\[
    \Omega_h = E_h \cup E_{0, h} \cup \partial_0 E_h.
\]

If \(z : \Omega_h \to l^\infty\) and \(\omega : \Omega_h \to R\) then we write \(z^{(r,m)} = z(t^{(r)}, x^{(m)})\) and

\[
    \omega^{(r,m)} = \omega(t^{(r)}, x^{(m)})
\]

on \(\Omega_h\). For a function \(z : \Omega_h \to l^\infty\) and for a point \((t^{(r)}, x^{(m)}) \in E_h\) we define the function \(z_{[r,m]} : D_h \to l^\infty\) by

\[
    z_{[r,m]}(s, y) = z(t^{(r)} + s, x^{(m)} + y), \quad (s, y) \in D_h.
\]

It is clear that the operator \((t^{(r)}, x^{(m)}) \mapsto z_{[r,m]}\) is a discrete version of the 
Hale operator \((t, x) \mapsto z(t, x)\). Put

\[
    E'_h = \{(t^{(r)}, x^{(m)}) \in E_h : (t^{(r)} + h_0, x^{(m)}) \in E_h\}.
\]

The motivation for the definition of the set \(E'_h\) is the following. Approximate 
solutions of problem (1), (2) are functions \(u_h\) defined on \(E_h\). We will write 
a difference system generated by (1) at each point of the set \(E'_h\).

In the sequel we will need the following interpolating operator \(T_h : F(D_h, l^\infty) \to C(D, l^\infty)\). Let us denote by \((\chi_1, \ldots, \chi_n)\) the family of sets 
defined by

\[
    \chi_i = \{0, 1\} \quad \text{if} \quad \tau_i > 0 \quad \text{and} \quad \chi_i = \{0\} \quad \text{if} \quad \tau_i = 0 \quad \text{where} \quad 1 \leq i \leq n.
\]

Set \(\vartheta = (\vartheta_1, \ldots, \vartheta_n) \in \mathbb{Z}^n\) and \(\vartheta_i = 1\) if \(\tau_i > 0\) and \(\vartheta_i = 0\) if \(\tau_i = 0\) where 
\(1 \leq i \leq n\). Write

\[
    S_+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i \in \chi_i \text{ for } 1 \leq i \leq n\}.
\]

Let \(w \in F(D_h, l^\infty)\) and \((t, x) \in D\). Suppose that \(\tau_0 > 0\). There exists 
\((t^{(r)}, x^{(m)}) \in D_h\) such that \((t^{(r+1)}, x^{(m+\vartheta)}) \in D_h\) and \(t^{(r)} \leq t \leq t^{(r+1)}\),
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$$x^{(m)} \leq x \leq x^{(m+\vartheta)}$$. Write

$$F(t) = \frac{t - t^{(r)}}{h_0} \sum_{\lambda \in S_+} w^{(r+1,m+\lambda)} (\frac{x - x^{(m)}}{h'})^\lambda \left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\lambda}$$

$$+ \left(1 - \frac{t - t^{(r)}}{h_0}\right) \sum_{\lambda \in S_+} w^{(r,m+\lambda)} (\frac{x - x^{(m)}}{h'})^\lambda \left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\lambda}$$

where

$$\left(\frac{x - x^{(m)}}{h'}\right)^\lambda = \prod_{i=1}^{n} \left(\frac{x_i - x_i^{(m)}}{h_i}\right)^{\lambda_i}$$

$$\left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\lambda} = \prod_{i=1}^{n} \left(1 - \frac{x_i - x_i^{(m)}}{h_i}\right)^{1-\lambda_i}$$

and we take $0^0 = 1$ in the above formulas. If $\tau_0 = 0$ and $x^{(m)} \leq x \leq x^{(m+\vartheta)}$ then we put

$$(T_h w)(t, x) = \sum_{\lambda \in S_+} w^{(0,m+\lambda)} (\frac{x - x^{(m)}}{h'})^\lambda \left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\lambda}$$

Then we have defined $T_h w$ on $D$. The above interpolating operator has been proposed in [5] for the construction of difference schemes corresponding to first order partial differential functional equations. We will use the following property of $T_h$.

**Lemma 1.** Suppose that $w : D \to l^\infty$ and

1) $w(: x) : [-\tau_0, 0] \to l^\infty$ is of class $C^1$ for $x \in [-\tau, \tau]$ and

$$||\partial_t w(t, x)||_D \leq \bar{c}, \quad (t, x) \in D,$$

2) $w(t, :) : [-\tau, \tau] \to l^\infty$ is of class $C^2$ for $t \in [-\tau_0, 0]$ and

$$||\partial_{x_i,x_j} w(t, x)||_D \leq \bar{C}, \quad (t, x) \in D, \quad 1 \leq i, j \leq n.$$

Then

$$||T_h w - \omega||_D \leq \bar{c} h_0 + \bar{C}||h'||^2$$

where $w_h$ is the restriction of $w$ to the set $D_h$.

The proof of the above lemma is similar to the proof of Theorem 3.18 in [5]. Details are omitted.

Put

$$J = \{(i, j) : 1 \leq i, j \leq n, i \neq j\}.$$

Write $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n$ with 1 standing on i-th place. We define the difference operators $\delta_0, \delta = (\delta_1, \ldots, \delta_n)$ and $\delta^+_i, \delta^-_i, 1 \leq i \leq n,$
in the following way. For \( \omega : \Omega_h \rightarrow R \) and \( (t^{(r)}, x^{(m)}) \in E'_h \) set

\[
\delta_0 \omega(r,m) = \frac{1}{h_0} (\omega^{(r+1,m)} - \omega^{(r,m)}),
\]

\[
\delta_i \omega(r,m) = \frac{1}{2h_i} (\omega^{(r,m+e_i)} - \omega^{(r,m-e_i)}),
\]

\[
\delta^+ \omega(r,m) = \frac{1}{h_i} (\omega^{(r,m+e_i)} - \omega^{(r,m)}),
\]

\[
\delta^- \omega(r,m) = \frac{1}{h_i} (\omega^{(r,m)} - \omega^{(r,m-e_i)}),
\]

where \( 1 \leq i \leq n \). For \( z : \Omega_h \rightarrow l^\infty, z = \{z_\mu\}_{\mu \in N} \), and for \( (t^{(r)}, x^{(m)}) \in E_h \) we put

\[
\delta_0 z^{(r,m)} = \{\delta_0 z_\mu^{(r,m)}\}_{\mu \in N}, \quad F_h[z]^{(r,m)} = \{F_h^{(\mu)}[z]^{(r,m)}\}_{\mu \in N}
\]

and

\[
F_h^{(\mu)}[z]^{(r,m)} = \sum_{i,j=1}^n \varrho_{ij}^{(\mu)} (t^{(r)}, x^{(m)}, T_h z^{(r,m)}) \delta_{ij} z_\mu^{(r,m)}
\]

\[
+ f^{(\mu)} (t^{(r)}, x^{(m)}, T_h z^{(r,m)}, \delta z_\mu^{(r,m)}), \quad \mu \in N,
\]

where the difference operators \( \delta_{ij}, 1 \leq i, j \leq n, \) are define as follows. Write

\[
\delta_{ij} z_\mu^{(r,m)} = \delta^+ \delta^- z_\mu^{(r,m)}, \quad 1 \leq i \leq n.
\]

The difference expressions \( \delta_{ij} z_\mu^{(r,m)} \) for \( (i, j) \in J \) are defined in the following way:

\[
\delta_{ij} z_\mu^{(r,m)} = \frac{1}{2} (\delta^+ \delta^- z_\mu^{(r,m)} + \delta^- \delta^+ z_\mu^{(r,m)}) \quad \text{if} \quad \varrho_{ij}^{(\mu)} (t^{(r)}, x^{(m)}, T_h z^{(r,m)}) \geq 0,
\]

\[
\delta_{ij} z_\mu^{(r,m)} = \frac{1}{2} (\delta^+ \delta^- z_\mu^{(r,m)} + \delta^- \delta^+ z_\mu^{(r,m)}) \quad \text{if} \quad \varrho_{ij}^{(\mu)} (t^{(r)}, x^{(m)}, T_h z^{(r,m)}) < 0.
\]

We will approximate solutions of (1), (2) by means of solutions of the problem

\[
\delta_0 z^{(r,m)} = F_h[z]^{(r,m)},
\]

\[
z^{(r,m)} = \varphi^{(r,m)}_h \quad \text{on} \quad E_{0,h} \cup \partial_0 E_h,
\]

where \( \varphi_h : E_{0,h} \cup \partial_0 E_h \rightarrow l^\infty \) is a given function. It is easy to see that there exists exactly one solution \( u_h : \Omega_h \rightarrow l^\infty \) of problem (11), (12).

**Assumption H[\( \varrho, f, \Delta \)]**. The functions \( \rho : \Xi \rightarrow M^\infty \) and \( f : \Sigma \rightarrow l^\infty \) satisfy the conditions:

1) for \( \mu \in N \) there exists \( \partial_q f^{(\mu)} = (\partial_{q_1} f^{(\mu)}, \ldots, \partial_{q_n} f^{(\mu)}) \) on \( \Sigma \) and \( \partial_q f^{(\mu)}(t, x, w, \cdot) \in C(R^n, R^n) \) where \( (t, x, w) \in \Xi, \)
2) for $h \in \Delta$ the inequalities

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} q_{ii}^{(\mu)}(Q) + h_0 \sum_{(i,j) \in J} \frac{1}{h_i h_j} |q_{ij}^{(\mu)}(Q)| \geq 0,$$

and

$$\frac{1}{h_i} q_{ii}^{(\mu)}(Q) - h_0 \sum_{j=1, j \neq i}^{n} \frac{1}{h_j} |q_{ij}^{(\mu)}(Q)| - \frac{1}{2} |\partial_{q_i} f^{(\mu)}(P)| \geq 0$$

where $1 \leq i \leq n$, hold with $Q = (t, x, w) \in \Xi$, $P = (t, x, w, q) \in \Sigma$ and $\mu \in \mathbb{N}$.

We first prove that solutions of (11), (12) are uniformly bounded on $\Omega_h$.

**Theorem 1.** Suppose that Assumption $H[\varrho, f, \Delta]$ is satisfied and

1) there is $B_0 \in \mathbb{R}_+$ such that $\|f(t, x, w, 0)\|_{\infty} \leq B_0$, $(t, x, w) \in \Xi$,

2) there is $A_0 \in \mathbb{R}_+$ such that $\|\varphi^{(r,m)}_h\|_{\infty} \leq A_0$ on $E_{0,h} \cup \partial_0 E_h$, $h \in \Delta$.

Then the solution $u_h : \Omega_h \to l^\infty$, $u_h = \{u_{h,\mu}\}_{\mu \in \mathbb{N}}$, of (11), (12) satisfies the condition

$$\|u_{h}^{(r,m)}\|_{\infty} \leq A_0 + aB_0 \text{ on } \Omega_h, \; h \in \Delta.$$

**Proof.** Let $\zeta_h^{(r)} = \{\zeta_{h,\mu}^{(r)}\}_{\mu \in \mathbb{N}}, \; 0 \leq r \leq N_0$, be given by

$$\zeta_{h,\mu}^{(r)} = \max \{|u_{h,\mu}^{(\nu,m)}| : (t^{(\nu)}, x^{(m)}) \in \Omega_h \cap \{[-\tau_0, t^{(r)}] \times \mathbb{R}^n\}\}.$$

We prove that for each $r$, $0 \leq r < N_0$,

$$\zeta_{h,\mu}^{(r+1)} \leq \max \{A_0, \; \zeta_{h,\mu}^{(r)} + h_0 B_0\}, \; \mu \in \mathbb{N}.$$

We have on $E_h^r$

$$\delta_0 u_{h,\mu}^{(r,m)} = \sum_{i,j=1}^{n} q_{ij}^{(\mu)}(Q^{(r,m)}[u_h]) \delta_{ij} u_{h,\mu}^{(r,m)}$$

$$+ \sum_{i=1}^{n} \partial_{q_i} f^{(\mu)}(P^{(r,m)}[u_h]) \delta_{i} u_{h,\mu}^{(r,m)} + f^{(\mu)}(t^{(r)}, x^{(m)}, T_h(u_h)_{[r,m]}, 0), \; \mu \in \mathbb{N},$$

where

$$Q^{(r,m)}[u_h] = (t^{(r)}, x^{(m)}, T_h(u_h)_{[r,m]}),$$

and $P^{(r,m)}[u_h] = (t^{(r)}, x^{(m)}, T_h(u_h)_{[r,m]} ; \xi \delta u_{h,\mu}^{(r,m)}), \; \xi \in (0, 1)$. For a point
(t(r), x(m)) ∈ E_h we write
\[ J_{\mu,+}^{(r,m)}[u_h] = \{(i,j) ∈ J : \varphi_{ij}^{(\mu)}(t(r), x(m), T_h(u_h)[r,m]) ≥ 0\}, \]
\[ J_{\mu,-}^{(r,m)}[u_h] = J \setminus J_{\mu,+}^{(r,m)}[u_h]. \]

We conclude from (16) that
\[
\begin{align*}
(u^{(r+1,m)}_{h,\mu}) = & u^{(r,m)}_{h,\mu} + h_0 \sum_{i=1}^{n} \varphi_{ii}^{(\mu)}(Q^{(r,m)}[u_h]) \frac{1}{h_i^2} (u^{(r,m+e_i)}_{h,\mu} - 2u^{(r,m)}_{h,\mu} + u^{(r,m-e_i)}_{h,\mu}) \\
& + h_0 \sum_{(i,j) ∈ J_{\mu,+}^{(r,m)}[u_h]} \frac{1}{2h_i h_j} \varphi_{ij}^{(\mu)}(Q^{(r,m)}[u_h]) \cdot (2u^{(r,m)}_{h,\mu} + u^{(r,m+e_i+e_j)}_{h,\mu} + u^{(r,m-e_i-e_j)}_{h,\mu}) \\
& - u^{(r,m-e_i)}_{h,\mu} - u^{(r,m+e_i)}_{h,\mu} - u^{(r,m-e_j)}_{h,\mu} - u^{(r,m+e_j)}_{h,\mu}) \\
& + h_0 \sum_{(i,j) ∈ J_{\mu,-}^{(r,m)}[u_h]} \frac{1}{2h_i h_j} \varphi_{ij}^{(\mu)}(Q^{(r,m)}[u_h]) \cdot (-2u^{(r,m)}_{h,\mu} - u^{(r,m+e_i-e_j)}_{h,\mu} - u^{(r,m-e_i-e_j)}_{h,\mu}) \\
& + u^{(r,m-e_i)}_{h,\mu} + u^{(r,m- e_j)}_{h,\mu} + u^{(r,m+ e_i)}_{h,\mu} + u^{(r,m + e_j)}_{h,\mu}) \\
& + h_0 \sum_{i=1}^{n} \partial_{q_i}f^{(\mu)}(P^{(r,m)}[u_h]) \frac{1}{2h_i} (u^{(r,m+e_i)}_{h,\mu} - u^{(r,m-e_i)}_{h,\mu}) \\
& + h_0 f^{(\mu)}(t(r), x(m), T_h(u_h)[r,m], 0).
\end{align*}
\]

For Q = (t, x, w) ∈ Ξ, P = (t, x, w, q) ∈ Σ, (t(r), x(m)) ∈ E_h and μ ∈ N we write
\begin{align*}
(18) & \quad A^{(r,m)}_{\mu} = 1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \varphi_{ii}^{(\mu)}(Q) + h_0 \sum_{(i,j) ∈ J} \frac{1}{h_i h_j} |\varphi_{ij}^{(\mu)}(Q)|, \\
(19) & \quad B^{(r,m)}_{\mu,i} = \frac{h_0}{h_i^2} \varphi_{ii}^{(\mu)}(Q) - \sum_{j=1, j≠i}^{n} \frac{h_0}{h_i h_j} |\varphi_{ij}^{(\mu)}(Q)| \\
& \quad + \frac{h_0}{2h_i} \partial_{q_i}f^{(\mu)}(P), \ 1 ≤ i ≤ n, \\
(20) & \quad C^{(r,m)}_{\mu,i} = \frac{h_0}{h_i^2} \varphi_{ii}^{(\mu)}(Q) - \sum_{j=1, j≠i}^{n} \frac{h_0}{h_i h_j} |\varphi_{ij}^{(\mu)}(Q)| \\
& \quad - \frac{h_0}{2h_i} \partial_{q_i}f^{(\mu)}(P), \ 1 ≤ i ≤ n,
\end{align*}
\[ D^{(r,m)}_{\mu,ij} = \frac{h_0}{2h_i h_j} |g_{ij}^\mu(Q)|, \quad (i, j) \in J. \]

Then

\[
\begin{align*}
\zeta^{(r+1,m)}_{h,\mu} &= u^{(r,m)}_{h,\mu} A_{\mu}^{(r,m)} + \sum_{i=1}^{n} u^{(r,m+e_i)}_{h,\mu} B_{\mu,i}^{(r,m)} + \sum_{i=1}^{n} u^{(r,m-e_i)}_{h,\mu} C_{\mu,i}^{(r,m)} \\
&+ \sum_{(i,j) \in j^{(r,m)}_{\mu,+}[u_h]} (u^{(r,m+e_i+e_j)}_{h,\mu} + u^{(r,m-e_i-e_j)}_{h,\mu}) D_{\mu,ij}^{(r,m)} \\
&+ \sum_{(i,j) \in j^{(r,m)}_{\mu,-}[u_h]} (u^{(r,m+e_i-e_j)}_{h,\mu} + u^{(r,m-e_i+e_j)}_{h,\mu}) D_{\mu,ij}^{(r,m)} \\
&+ h_0 f(\mu)^{(r)}(t^{(r)}, x^{(m)}, T_{h}(u_h)_{[r,m], 0})
\end{align*}
\]

where \( A_{\mu}^{(r,m)} \), \( B_{\mu,i}^{(r,m)} \), \( C_{\mu,i}^{(r,m)} \), \( D_{\mu,ij}^{(r,m)} \) are defined by (18) - (21) with \( Q = Q^{(r,m)}[u_h] \) and \( P = P^{(r,m)}_{\mu}[u_h] \). Thus we obtain (15). Since \( ||\zeta^{(0)}_{h}||_{\infty} \leq A_0 \) we have for \( 0 \leq r \leq N_0 \)

\[ ||\zeta^{(r)}_{h}||_{\infty} \leq A_0 + r h_0 B_0 \leq A_0 + a B_0. \]

The proof of Theorem 1 is complete. 

Now we formulate general conditions for the convergence of the method (11), (12). If \( w \in C(D, l^\infty) \), \( w = \{w_{\mu}\}_{\mu \in N} \), then we write

\[ |w|_D = \{||w_{\mu}||_{0}\}_{\mu \in N} \]

where \( ||w_{\mu}||_{0} = \max\{|w_{\mu}(s, x)| : (s, x) \in D\} \). Put

\[
\begin{align*}
l_{\infty}^+ &= \{p \in l_{\infty} : p = \{p_{\mu}\}_{\mu \in N}, p_{\mu} \in R_{+}, \mu \in N\}, \\
l_{0}^+ &= \{p \in l_{+}^\infty : p = \{p_{\mu}\}_{\mu \in N}, \lim_{\mu \to \infty} p_{\mu} = 0\}.
\end{align*}
\]

**Assumption H[\( g, f, \sigma \).** The functions \( g : E \to M^\infty \), \( f : \Sigma \to l^\infty \) are continuous and there exists a comparison function \( \sigma \in C([0, a] \times l_{\infty}^+, l_{\infty}^+) \) of variables \((t, p)\), nondecreasing with respect to both variables, such that

1) there is \( L \in l_{\infty}^+ \) such that

\[ \sigma(t, p) \leq L \text{ on } [0, a] \times l_{\infty}^+, \]

and \( \sigma(t, 0) = 0 \) for \( t \in [0, a] \), where \( 0 \) is the sequence of zeros,

2) for each \( C \geq 1 \) a function \( \tilde{\omega}(t) = 0, \ t \in [0, a] \), is the maximum solution of the problem

\[ \omega'(t) = C \sigma(t, \omega(t)), \ \omega(0) = 0, \]

and for each \( c \in l_0^\infty \) there is a maximum solution \( \hat{\omega} = \{\hat{\omega}_{\mu}\}_{\mu \in N} \) of the problem
\[ \omega'(t) = C\sigma(t, \omega(t)) + c, \quad \omega(0) = 0, \]

and \( \lim_{\mu \to \infty} \tilde{\omega}_\mu(t) = 0 \) uniformly on \([0,a]\),

3) the estimates

\[
\| \varphi^{(\mu)}(t, x, w) - \varphi^{(\mu)}(t, x, \bar{w}) \|_{n \times n} \leq \sigma_{\mu}(t, |w - \bar{w}|_D), \quad \mu \in \mathbb{N},
\]

\[
| f^{(\mu)}(t, x, w, q) - f^{(\mu)}(t, x, \bar{w}, q) | \leq \sigma_{\mu}(t, |w - \bar{w}|_D), \quad \mu \in \mathbb{N},
\]

are satisfied on \( \Sigma \) and \( \Sigma \) respectively.

**Remark 1.** Let \( a_{\mu j} \in R_+, \mu, j \in \mathbb{N} \), be such that the series \( S_{\mu} = \sum_{j=1}^{\infty} a_{\mu j}, \mu \in \mathbb{N} \), are convergent and the sequence \( S = \{ S_{\mu} \} \) tends to zero. Fix the sequence \( \tilde{p} \in l_+^\infty, \tilde{p} = \{ \tilde{\rho}_\mu \}, \) such that \( \tilde{\rho}_\mu > 0 \) for \( \mu \in \mathbb{N} \). Put

\[
I[\tilde{\rho}] = \{ p \in l_+^\infty : p \leq \tilde{p} \}.
\]

Then the function \( \sigma : [0, a] \times l_+^\infty \to l_+^\infty, \sigma = \{ \sigma_{\mu} \} \), given by

\[
\sigma_{\mu}(t, p) = \sum_{j=1}^{\infty} a_{\mu j} p_j, \quad \mu \in \mathbb{N}, \quad t \in [0, a], \quad p \in I[\tilde{\rho}],
\]

\[
\sigma_{\mu}(t, p) = \sum_{j=1}^{\infty} a_{\mu j} \tilde{\rho}_j, \quad \mu \in \mathbb{N}, \quad t \in [0, a], \quad p \in l_+^\infty \setminus I[\tilde{\rho}]
\]

satisfies the conditions 1) and 2) of Assumption \( H[\varrho, f, \sigma] \).

**Theorem 2.** Suppose that Assumptions \( H[\varrho, f, \Delta] \) and \( H[\varrho, f, \sigma] \) are satisfied and

1) the function \( v : \Omega \to l^\infty, v = \{ v_{\mu} \}_{\mu \in \mathbb{N}}, \) is a classical solution of (1), (2) and there is \( c_0 \in R_+ \) such that

\[
|\partial_{x_i x_j} v_{\mu}(t, x)| \leq c_0, \quad (t, x) \in E, \quad 1 \leq i, j \leq n, \quad \mu \in \mathbb{N},
\]

2) the function \( u_h : \Omega_h \to l^\infty, u_h = \{ u_{h, \mu} \}_{\mu \in \mathbb{N}}, \) is a solution of problem (11), (12),

3) there exists a function \( \alpha_0 : \Delta \to R_+ \) such that

\[
||\varphi^{(r,m)}_h - \varphi(t^{(r)}, x^{(m)})||_\infty \leq \alpha_0(h) \text{ on } E_{0,h} \cup \partial_0 E_h
\]

and \( \lim_{h \to 0} \alpha_0(h) = 0. \)

Then there is \( \gamma : \Delta \to R_+ \) such that

\[
|| u_h^{(r,m)} - v(t^{(r)}, x^{(m)}) ||_\infty \leq \gamma(h) \text{ on } E_h
\]

and \( \lim_{h \to 0} \gamma(h) = 0. \)

**Proof.** Put \( \tilde{v}_h = \{ v_{h, \mu} \}_{\mu \in \mathbb{N}}, \) where \( v_{h, \mu} \) is the restriction of \( v_\mu \) to the set \( \Omega_h, \mu \in \mathbb{N}. \) We have on \( E'_h \)

\[
\delta_0 \tilde{v}_h^{(r,m)} = F_h[\tilde{v}_h]^{(r,m)} + \Gamma_h^{(r,m)}
\]
where $\Gamma_h^{(r,m)} = \{ \Gamma_{h,\mu}^{(r,m)} \}_{\mu \in \mathbb{N}}$ and there is $\alpha : \Delta \to R_+$ such that

$$||\Gamma_h^{(r,m)}||_\infty \leq \alpha(h) \text{ on } E'_h$$

and $\lim_{h \to 0} \alpha(h) = 0$. The function $\varepsilon_h : \Omega_h \to l^\infty$, $\varepsilon_h = \{ \varepsilon_{h,\mu} \}_{\mu \in \mathbb{N}}$, given by

$$\varepsilon_h = u_h - \tilde{v}_h$$

satisfies the difference functional system

$$(23) \quad \delta_0 \varepsilon_h^{(r,m)} = \sum_{i,j=1}^{n} \delta_{ij}^{(r,m)} (x^{(m)}, T_h(u_h)[r,m]) \delta_{ij} \varepsilon_h^{(r,m)}$$

$$+ f^{(r)}(x^{(m)}, T_h(u_h)[r,m]) \delta_{ij} \varepsilon_h^{(r,m)}$$

$$- f^{(r)}(x^{(m)}, T_h(u_h)[r,m]) \delta_{ij} \varepsilon_h^{(r,m)} + A_h^{(r,m)}$$

where $\mu \in \mathbb{N}$ and

$$A_h^{(r,m)} = \sum_{i,j=1}^{n} \left( \delta_{ij}^{(r,m)} (x^{(m)}, T_h(u_h)[r,m]) - f^{(r)}(x^{(m)}, T_h(u_h)[r,m]) \delta_{ij} \varepsilon_h^{(r,m)} + A_h^{(r,m)} \right)$$

We can write the system (23) in the form

$$\delta_0 \varepsilon_h^{(r,m)}$$

$$= \sum_{i,j=1}^{n} \delta_{ij}^{(r,m)} (Q^{(r,m)}[u_h]) \delta_{ij} \varepsilon_h^{(r,m)} + \sum_{i=1}^{n} \partial_i f^{(r,m)} (P^{(r,m)}[u_h], \tilde{v}_h) \delta_{i} \varepsilon_h^{(r,m)} + A_h^{(r,m)}$$

where $\mu \in \mathbb{N}$, $Q^{(r,m)}[u_h]$ is given by (17) and

$$P^{(r,m)}[u_h, \tilde{v}_h] = (t^{(r)}, x^{(m)}, T_h(u_h)[r,m], \delta v_{h,\mu}^{(r,m)} + \xi (\delta u_{h,\mu}^{(r,m)} - \delta v_{h,\mu}^{(r,m)}))$$

with some $\xi \in (0, 1)$. After regrouping we obtain

$$\varepsilon_{h,\mu}^{(r+1,m)} = \varepsilon_{h,\mu}^{(r,m)} A_h^{(r,m)} + \sum_{i=1}^{n} \varepsilon_{h,\mu}^{(r,m+e_i)} B_{\mu, i}^{(r,m)} + \sum_{i=1}^{n} \varepsilon_{h,\mu}^{(r,m-e_i)} C_{\mu, i}^{(r,m)}$$

$$+ \sum_{(i,j) \in J_h^{(r,m)}}^{(r,m)} [u_h] (\varepsilon_{h,\mu}^{(r,m+e_i+e_j)} + \varepsilon_{h,\mu}^{(r,m-e_i-e_j)}) D_{\mu, ij}^{(r,m)}$$

$$+ \sum_{(i,j) \in J_h^{(r,m)-}[u_h]}^{(r,m)} (\varepsilon_{h,\mu}^{(r,m+e_i)} + \varepsilon_{h,\mu}^{(r,m-e_i+e_j)}) D_{\mu, ij}^{(r,m)} + h_0 A_h^{(r,m)}$$
where $A^{(r,m)}_{\mu}, B^{(r,m)}_{\mu,i}, C^{(r,m)}_{\mu,i}, D^{(r,m)}_{\mu,i,j}$ are defined by (18) - (21) with $Q = Q^{(r,m)}[u_h], P = P^{(r,m)}_{\mu}[u_h, \tilde{v}_h]$. Define $\eta^{(r)}_h, 0 \leq r \leq N_0$, by $\eta^{(r)}_h = \{\eta^{(r)}_{h,\mu}\}_{\mu \in \mathbb{N}},
\eta^{(r)}_{h,\mu} = \max\{|e^{(r,\mu)}_{e^{(r,\mu)}_h}| : (t^{(r,\mu)}_e, x^{(r,\mu)}_e) \in \Omega_h \cap [(-\tau_0, t^{(r)}) \times \mathbb{R}^n]\}.$

Observe that for $A^{(r,m)}_{h,\mu} = \{A^{(r,m)}_{h,\mu}\}_{\mu \in \mathbb{N}}$ we obtain the estimate
\[|A^{(r,m)}_{h,\mu}| \leq (1 + c_0)\sigma_\mu(t^{(r)}, \eta^{(r)}_h) + \alpha(h).\]

Thus the following inequalities
\[(24)\quad \eta^{(r+1)}_{h,\mu} \leq \max\{\alpha_0(h), \eta^{(r)}_{h,\mu} + h_0(1 + c_0)\sigma_\mu(t^{(r)}, \eta^{(r)}_h) + h_0\alpha(h)\},\]
\[(25)\quad \eta^{(0)}_{h,\mu} \leq \alpha_0(h)\]
hold. Let $\omega_h = \{\omega_{h,\mu}\}_{\mu \in \mathbb{N}}$ be the maximum solution of the problem
\[\omega^{(r)}_h(t) = (1 + c_0)\sigma_\mu(t, \omega(t)) + \alpha(h), \quad \omega_\mu(0) = \alpha_0(h), \quad \mu \in \mathbb{N}.\]
The solution $\omega_h$ is defined on $[0, a]$ and $\lim_{t \to 0} \omega_h(t) = 0$ uniformly on $[0, a]$. For each $\mu \in \mathbb{N}$ the function $\omega_{h,\mu}$ is convex on $[0, a]$, therefore we have
\[\omega_{h,\mu}(t^{(r+1)}) \geq \omega_{h,\mu}(t^{(r)}) + h_0(1 + c_0)\sigma_\mu(t^{(r)}, \omega_h(t^{(r)})) + h_0\alpha(h)\]
where $0 \leq r \leq N_0 - 1$. Since the conditions (24), (25) are satisfied, we have
\[\eta^{(r)}_h \leq \omega_h(t^{(r)}), \quad 0 \leq r \leq N_0.\]
Thus we obtain the assertion (22) with $\gamma(h) = ||\omega_h(a)||_{\infty}$. The proof of Theorem 2 is complete. ■

3. Finite systems of difference equations

We consider the problem (1), (2). Let $\tilde{\varphi} : \Omega \to l^\infty, \tilde{\varphi} = \{\tilde{\varphi}_\mu\}_{\mu \in \mathbb{N}},$ be such that $\tilde{\varphi}(t, x) = \varphi(t, x), (t, x) \in E_0 \cup \partial_0 E.$ Fix $k \in \mathbb{N}$. For $w : D \to R^k, w = (w_1, \ldots, w_k),$ or $w : D \to l^\infty, w = \{w_\mu\}_{\mu \in \mathbb{N}},$ and for $\tilde{w} : D \to l^\infty, \tilde{w} = \{\tilde{w}_\mu\}_{\mu \in \mathbb{N}},$ put
\[\{w, \tilde{w}\}_k = \{\tilde{w}_\mu\}_{\mu \in \mathbb{N}} \text{ where } \tilde{w}_\mu = w_\mu \text{ for } 1 \leq \mu \leq k \text{ and } \tilde{w}_\mu = \tilde{w}_\mu \text{ for } \mu > k.\]

Consider the differential functional system
\[\partial_t z_\mu(t, x) = \sum_{i,j=1}^n e^{(\mu)}_{i,j}(t, x, [z_{(t,x)}], \tilde{\varphi}_{(t,x)}k)\partial_{x_ix_j} z_\mu(t, x) + f^{(\mu)}(t, x, [z_{(t,x)}], \tilde{\varphi}_{(t,x)}k, \partial x z_\mu(t, x)), \quad 1 \leq \mu \leq k;\]
where $z = (z_1, \ldots, z_k), \text{ with the initial boundary condition}\]
\[z_\mu(t, x) = \varphi_\mu(t, x), (t, x) \in E_0 \cup \partial_0 E, \quad 1 \leq \mu \leq k.\]
For \( r, \bar{r} \in M_{n \times n} \), \( r = [r_{ij}]_{i,j=1,...,n}, \bar{r} = [\bar{r}_{ij}]_{i,j=1,...,n} \), we write \( r \leq \bar{r} \) if \( \sum_{i,j=1}^{n} (r_{ij} - \bar{r}_{ij})\lambda_j\lambda_i \leq 0 \) for any \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \).

**Assumption** \( H[\varrho, f, \sigma, \varphi] \). The functions \( \varrho : \Xi \to \mathcal{M}^{\infty} \) and \( f : \Sigma \to l^{\infty} \) satisfy Assumption \( H[\varrho, f, \sigma] \) and

1) if \( r, \bar{r} \in M_{n \times n} \), \( r = [r_{ij}]_{i,j=1,...,n}, \bar{r} = [\bar{r}_{ij}]_{i,j=1,...,n} \), and \( r \leq \bar{r} \) then
\[
\sum_{i,j=1}^{n} \varrho_{ij}^{(\mu)}(t, x, w)(r_{ij} - \bar{r}_{ij}) \leq 0, \quad (t, x, w) \in \Xi, \quad \mu \in \mathbb{N},
\]

2) the function \( \varphi \in C(E_0 \cup \partial_0 E, l^{\infty}) \) is such that there exists \( \tilde{\varphi} \in C(\Omega, l^{\infty}) \), \( \tilde{\varphi} = \{\tilde{\varphi}_{\mu}\}_{\mu \in \mathbb{N}} \), with the properties
   
   (i) \( \tilde{\varphi}(t, x) = \varphi(t, x) \) for \( (t, x) \in E_0 \cup \partial_0 E \),
   
   (ii) for each \( \mu \in \mathbb{N} \) the function \( \tilde{\varphi}_{\mu}(\cdot, x) : [0, a] \to R \) is of class \( C^1 \) and \( \tilde{\varphi}_{\mu}(t, \cdot) : [-b, b] \to R \) is of class \( C^2 \), \( x \in [-b, b], t \in [0, a] \),
   
   (iii) there is \( d \in R^+ \) such that for each \( \mu \in \mathbb{N} \)
\[
|\partial_{x_i x_j} \tilde{\varphi}_{\mu}(t, x)| \leq d, \quad (t, x) \in E, \quad 1 \leq i, j \leq n,
\]

3) for each \( \mu \in \mathbb{N} \) there is \( C_{\mu} \in R^+ \) such that for \( (t, x) \in E \)
\[
|\sum_{i,j=1}^{n} \varrho_{ij}^{(\mu)}(t, x, \tilde{\varphi}(t, x)) \partial_{x_i x_j} \tilde{\varphi}_{\mu}(t, x) + f^{(\mu)}(t, x, \tilde{\varphi}(t, x), \partial_x \tilde{\varphi}_{\mu}(t, x)) - \partial_t \tilde{\varphi}_{\mu}(t, x)| \leq C_{\mu}
\]
and \( \lim_{\mu \to \infty} C_{\mu} = 0 \).

**Remark 2.** If we assume that for each \( \mu \in \mathbb{N} \) there are \( \tilde{\varphi}_{\mu}, \tilde{B}_{\mu}, \tilde{C}_{\mu} \in R^+ \) such that for \( (t, x) \in E \)
\[
|\varrho^{(\mu)}(t, x, \varphi(t, x))|_{n \times n} \leq \tilde{A}_{\mu}, \quad |f^{(\mu)}(t, x, \varphi(t, x), \partial_x \varphi(t, x))| \leq \tilde{B}_{\mu},
\]
and \( \lim_{\mu \to \infty} \tilde{A}_{\mu} = \lim_{\mu \to \infty} \tilde{B}_{\mu} = \lim_{\mu \to \infty} \tilde{C}_{\mu} = 0 \), then the condition 3) of Assumption \( H[\varrho, f, \sigma, \varphi] \) is satisfied.

**Lemma 2.** If Assumption \( H[\varrho, f, \sigma, \varphi] \) is satisfied and the function \( v : \Omega \to l^{\infty}, v = \{v_{\mu}\}_{\mu \in \mathbb{N}} \), is a classical solution of (1), (2) then for each \( \mu \in \mathbb{N} \) there exists \( \tilde{\omega}_{\mu} \in C([0, a], R^+) \) such that
\[
|v_{\mu}(t, x) - \varphi_{\mu}(t, x)| \leq \tilde{\omega}_{\mu}(t), \quad (t, x) \in E,
\]
and \( \lim_{\mu \to \infty} \tilde{\omega}_{\mu}(t) = 0 \) uniformly on \([0, a]\).

**Proof.** Define \( \tilde{v} : \Omega \to l^{\infty}, \tilde{v} = \{\tilde{v}_{\mu}\}_{\mu \in \mathbb{N}} \), by \( \tilde{v}(t, x) = v(t, x) - \varphi(t, x), \quad (t, x) \in \Omega \). Then \( \tilde{v}(t, x) = 0 \) on \( E_0 \cup \partial_0 E \) and
\[
\partial_t \tilde{v}_{\mu}(t, x) = F^{(\mu)}[\tilde{v} + \varphi](t, x) - \partial_t \varphi_{\mu}(t, x), \quad (t, x) \in E, \quad \mu \in \mathbb{N}.
\]
The function $G = \{G_\mu\}_{\mu \in \mathbb{N}}$ defined on $E \times C(D, l^\infty) \times \mathbb{R}^n \times M_{n \times n}$ by

$$G_\mu(t, x, w, q, r) = \sum_{i,j=1}^n \left( g^{(\mu)}_{ij}(t, x, w + \tilde{\varphi}(t, x)) (r_{ij} + \partial_{xi,xj} \tilde{\varphi}_\mu(t, x)) + f^{(\mu)}(t, x, w + \tilde{\varphi}(t, x), q + \partial_x \tilde{\varphi}_\mu(t, x)) - \partial_t \tilde{\varphi}_\mu(t, x), \mu \in \mathbb{N},
$$

where $r = [r_{ij}]_{i,j=1,\ldots,n}$, satisfies the following conditions. If $r, \bar{r} \in M_{n \times n}$ are such that $r \leq \bar{r}$ then

$$G_\mu(t, x, w, q, r) - G_\mu(t, x, w, q, \bar{r}) = \sum_{i,j=1}^n \left( g^{(\mu)}_{ij}(t, x, w + \tilde{\varphi}(t, x)) (r_{ij} - \bar{r}_{ij}) \leq 0 \right)$$

where $(t, x, w, q) \in E \times C(D, l^\infty) \times \mathbb{R}^n, \mu \in \mathbb{N}$. Furthermore

$$|G_\mu(t, x, w, 0, 0)| \leq \sum_{i,j=1}^n \left| g^{(\mu)}_{ij}(t, x, w + \tilde{\varphi}(t, x)) - g^{(\mu)}_{ij}(t, x, \tilde{\varphi}(t, x)) \right| \cdot \left| \partial_{xi,xj} \tilde{\varphi}_\mu(t, x) \right| + \left| f^{(\mu)}(t, x, w + \tilde{\varphi}(t, x), \partial_x \tilde{\varphi}_\mu(t, x)) - f^{(\mu)}(t, x, \tilde{\varphi}(t, x), \partial_x \tilde{\varphi}_\mu(t, x)) \right| + \left| \sum_{i,j=1}^n g^{(\mu)}_{ij}(t, x, \tilde{\varphi}(t, x)) \partial_{xi,xj} \tilde{\varphi}_\mu(t, x) \right| + \left| f^{(\mu)}(t, x, \tilde{\varphi}(t, x), \partial_x \tilde{\varphi}_\mu(t, x)) - \partial_t \tilde{\varphi}_\mu(t, x) \right| \leq (1 + d) \sigma_\mu(t, |w|_D) + C_\mu$$

where $(t, x, w) \in E \times C(D, l^\infty)$ and $\mu \in \mathbb{N}$. We have

$$\partial_t \tilde{v}_\mu(t, x) = G_\mu(t, x, \tilde{v}(t, x), \partial_x \tilde{v}_\mu(t, x), \partial_{xx} \tilde{v}_\mu(t, x)) \text{ on } E, \mu \in \mathbb{N},$$

where $\partial_{xx} \tilde{v}_\mu(t, x) = [\partial_{xi,xj} \tilde{v}_\mu(t, x)]_{i,j=1,\ldots,n}, \mu \in \mathbb{N}$. It follows from the comparison theorem for infinite systems of parabolic functional differential equations (see [3]) that

$$|\tilde{v}_\mu(t, x)| \leq \tilde{\omega}_\mu(t) \text{ on } E$$

where $\tilde{\omega} = \{\tilde{\omega}_\mu\}_{\mu \in \mathbb{N}}$ is a solution of the problem

$$\omega'_\mu(t) = (1 + d) \sigma_\mu(t, \omega(t)) + C_\mu, \quad \omega_\mu(0) = 0, \quad \mu \in \mathbb{N}.$$

Since $\lim_{t \to \infty} C_\mu = 0$, the assertion of Lemma 2 follows from the assumption 2) of $H[\varrho, f, \sigma]$. 

If $p \in R^k, p = (p_1, \ldots, p_k), \text{ or } p \in l^\infty, p = \{p_\mu\}_{\mu \in \mathbb{N}}, \text{ and } q \in l^\infty, q = \{q_\mu\}_{\mu \in \mathbb{N}}, \text{ we put}$

$$\langle p, q \rangle_k = \{s_\mu\} \text{ where } s_\mu = p_\mu \text{ for } 1 \leq \mu \leq k \text{ and } s_\mu = q_\mu \text{ for } \mu > k.$$
For \( w \in C(D, \mathbb{R}^k), \) \( w = (w_1, \ldots, w_k), \) we define
\[
|w|_{k,D} = (||w_1||_0, \ldots, ||w_k||_0).
\]

**Lemma 3.** Suppose that Assumption \( H[\theta, f, \sigma, \varphi] \) is satisfied and

1) \( v : \Omega \to \ell^\infty, v = \{v_\mu\}_{\mu \in \mathbb{N}}, \) is a classical solution of (1), (2) and there is \( c_0 \in \mathbb{R}^+ \) such that
\[
|\partial_{x_i x_j} v_\mu(t,x)| \leq c_0 \text{ on } \overline{E}, \ 1 \leq i,j \leq n, \ \mu \in \mathbb{N},
\]

2) for each \( k \in \mathbb{N} \) the function \( v^{[k]} : \Omega \to \mathbb{R}^k, \) \( v^{[k]} = (v^{[k]}_1, \ldots, v^{[k]}_k), \) is a solution of (26), (27).

Then there exists \( \omega^{[k]} \in C([0, a], \mathbb{R}^+ \cap \ell^\infty) \) such that
\[
|v_\mu(t,x) - v_\mu^{[k]}(t,x)| \leq \omega^{[k]}(t), \ (t,x) \in E, \ 1 \leq \mu \leq k,
\]
and \( \lim_{k \to \infty} \omega^{[k]}(t) = 0 \) uniformly on \([0, a].\)

**Proof.** Let the function \( v^{[k]} : \Omega \to \mathbb{R}^k, \) \( v^{[k]} = (v^{[k]}_1, \ldots, v^{[k]}_k), \) be defined by
\[
v^{[k]}_\mu(t,x) = u^{[k]}_\mu(t,x) - v_\mu(t,x), \ (t,x) \in \Omega, \ 1 \leq \mu \leq k.
\]
Then \( v^{[k]}_\mu(t,x) = 0 \) on \( E_0 \cup \partial_0 E, \ 1 \leq \mu \leq k, \) and
\[
\frac{\partial_t v^{[k]}_\mu(t,x)}{} = \sum_{i,j=1}^n (\varphi^{(\mu)}(t,x, [u^{[k]}_\mu(t,x)]_k) \partial_{x_i x_j} u^{[k]}_\mu(t,x) - \varphi^{(\mu)}(t,x, v^{[k]}_\mu(t,x)) \partial_{x_i x_j} v_\mu(t,x))
\]
\[
+ f^{(\mu)}(t,x, [u^{[k]}_\mu(t,x)]_k, \partial_x u^{[k]}_\mu(t,x)) - f^{(\mu)}(t,x, v^{[k]}_\mu(t,x), \partial_x v_\mu(t,x))
\]
where \( 1 \leq \mu \leq k, \ (t,x) \in E. \) We define \( H : E \times C(D, \mathbb{R}^k) \times \mathbb{R}^n \times M_{n \times n} \to \mathbb{R}^k, \)
\( H = (H_1, \ldots, H_k), \) as follows:
\[
H_\mu(t,x,w,q,r) = \sum_{i,j=1}^n \varphi^{(\mu)}(t,x, [w + v(t,x), \varphi(t,x)]_k) r_{ij}
\]
\[
+ \sum_{i,j=1}^n (\varphi^{(\mu)}(t,x, [w + v(t,x), \varphi(t,x)]_k) - \varphi^{(\mu)}(t,x, v(t,x))) \partial_{x_i x_j} v_\mu(t,x)
\]
\[
+ f^{(\mu)}(t,x, [w + v(t,x), \varphi(t,x)]_k, q + \partial_x v_\mu(t,x)) - f^{(\mu)}(t,x, v(t,x), \partial_x v_\mu(t,x)).
\]
We have
\[
\partial_t v^{[k]}_\mu(t,x) = H_\mu(t,x, v^{[k]}_\mu(t,x), \partial_x v^{[k]}_\mu(t,x), \partial_{xx} v^{[k]}_\mu(t,x)) \text{ on } E, \ 1 \leq \mu \leq k.
\]
If \( r, \bar{r} \in M_{n \times n} \) are such that \( r \leq \bar{r} \) then
\[
H_\mu(t,x,w,q,r) - H_\mu(t,x,w,q,\bar{r}) \leq 0
\]
where \((t, x, w, q) \in E \times C(D, \mathbb{R}^k) \times \mathbb{R}^n, 1 \leq \mu \leq k\). If \((t, x, w) \in E \times C(D, \mathbb{R}^k), w = (w_1, \ldots, w_k), 1 \leq \mu \leq k,\) then

\[|H_\mu(t, x, w, 0, 0)|\]

\[
\leq \sum_{i,j=1}^{n} \left[ \partial_{ij}^{(\mu)}(t, x, [w + v(t, x), \tilde{\varphi}(t, x)]_k) - \partial_{ij}^{(\mu)}(t, x, v(t, x)) \right] \cdot |\partial_{x_i x_j} v_\mu(t, x)| \\
+ \sum_{i,j=1}^{n} \left| \partial_{ij}^{(\mu)}(t, x, [v(t, x), \tilde{\varphi}(t, x)]_k) - \partial_{ij}^{(\mu)}(t, x, v(t, x)) \right| \cdot |\partial_{x_i x_j} v_\mu(t, x)| \\
+ \left| f^{(\mu)}(t, x, [w + v(t, x), \tilde{\varphi}(t, x)]_k, \partial_x v_\mu(t, x)) - f^{(\mu)}(t, x, v(t, x), \partial_x v_\mu(t, x)) \right| \\
\leq (1 + c_0)\sigma_{\mu}(t, (|w|_{k,D}, 0)_k) + \beta_k(t)
\]

where

\[\beta_k(t) = \max \{(1 + c_0)\sigma_{\mu}(t, (0, \tilde{\omega}(t))_k) : 1 \leq \mu \leq k\}\]

and \(\tilde{\omega} = \{\tilde{\omega}_\mu\}_{\mu \in \mathbb{N}}\) is given in the assertion of Lemma 2. It follows from the comparison theorem for systems of parabolic functional differential equations that

\[|v^{[k]}_\mu(t, x)| \leq \omega^{[k]}_\mu(t) \text{ on } E\]

where \(\omega^{[k]} = (\omega_1^{[k]}, \ldots, \omega_k^{[k]})\) is a solution of the problem

\[\omega'_\mu(t) = (1 + c_0)\sigma_{\mu}(t, (\omega(t), 0)_k) + \beta_k(a), \quad \omega_\mu(0) = 0, \quad 1 \leq \mu \leq k.\]

Since \(\lim_{k \to \infty} \beta_k(a) = 0\), we have for

\[\omega^{[k]}(t) = \max \{\omega^{[k]}_\mu(t) : 1 \leq \mu \leq k\}\]

that \(\lim_{k \to \infty} \omega^{[k]}(t) = 0\) uniformly on \([0, a]\) and this completes the proof of Lemma 3. \(\bullet\)

Consider the difference functional problem

\[\delta_0 z^{(r,m)}_\mu(t, x) = \sum_{i,j=1}^{n} \partial_{ij}^{(\mu)}(t^{(r)}, x^{(m)}), [T_h z^{[r,m]}, \tilde{\varphi}(t^{(r)}, x^{(m)})]_k)\delta_{ij} z^{(r,m)}_\mu \\
+ f^{(\mu)}(t^{(r)}, x^{(m)}, [T_h z^{[r,m]}, \tilde{\varphi}(t^{(r)}, x^{(m)})]_k, \delta z^{(r,m)}_\mu), \quad 1 \leq \mu \leq k,
\]

\[z^{(r,m)}_\mu(t, x) = \varphi^{(r,m)}_{h,\mu} \text{ on } E_{0, h} \cup \partial_0 E_h, \quad 1 \leq \mu \leq k,
\]

where \(z = (z_1, \ldots, z_k)\) and \(\varphi_h : E_{0, h} \cup \partial_0 E_h \to \mathbb{R}^k, \varphi_h = (\varphi_{h,1}, \ldots, \varphi_{h,k})\) is a given function. Difference operators \(\delta_0, \delta = (\delta_1, \ldots, \delta_n)\) and \(\delta_{ii}, 1 \leq i \leq n,\)
are given by (4), (5) and (8). The difference expressions \( \delta_{ij}z_{\mu} \) for \((i,j) \in J\) are defined by (9) and (10) with \((t^{(r)}, x^{(m)}), [T_hz_{[r,m]}, \tilde{\phi}_{(t^{(r)}, x^{(m)})}]_k\) instead of \((t^{(r)}, x^{(m)}), \mathcal{T}_{h}z_{[r,m]}\).

We are ready to prove the main theorem in this part of the paper.

**Theorem 3.** Suppose that Assumptions \(H[\varrho, f, \sigma, \varphi]\) and \(H[\varrho, f, \Delta]\) are satisfied and

1) the function \(v : \Omega \to \ell^\infty, v = \{v_\mu\}_{\mu \in \mathbb{N}}\), is a classical solution of (1), (2) and there is \(c_0 \in \mathbb{R}_+\) such that \(|\partial_{xi}x_j v_\mu(t, x)| \leq c_0\) on \(E, 1 \leq i, j \leq n, \mu \in \mathbb{N},\)

2) for each \(k \in \mathbb{N}\) the function

\[ u^{[k]} : \Omega \to \mathbb{R}^k, \quad u^{[k]} = (u^{[k]}_1, \ldots, u^{[k]}_k), \]

is a solution of (26), (27) and the constant \(c_k \in \mathbb{R}_+\) is such that

\[ |\partial_{xi}x_j u^{[k]}_\mu(t, x)| \leq c_k \quad \text{on} \quad E, 1 \leq i, j \leq n, 1 \leq \mu \leq k, \]

3) for each \(k \in \mathbb{N}\) and \(h \in \Delta\) the function \(u^{[k]}_h : \Omega_h \to \mathbb{R}^k, u^{[k]}_h = (u^{[k]}_{h,1}, \ldots, u^{[k]}_{h,k}),\) is a solution of (28), (29),

4) for each \(k \in \mathbb{N}\) there is \(a_0^{[k]} : \Delta \to \mathbb{R}_+\) such that

\[ |\varphi^{(r,m)}_{h,\mu} - \varphi_{\mu}(t^{(r)}, x^{(m)})| \leq a_0^{[k]}(h) \quad \text{on} \quad E_{0,h} \cup \partial_0E_h, 1 \leq \mu \leq k, \]

and \(\lim_{h \to 0} a_0^{[k]}(h) = 0.\)

Then for each \(k \in \mathbb{N}\) there exist \(\gamma^{[k]} : \Delta \to \mathbb{R}_+\) and \(\varepsilon_k \in \mathbb{R}_+\) such that

\[ |(u^{[k]}_{h,\mu})^{(r,m)} - v_\mu(t^{(r)}, x^{(m)})| \leq \gamma^{[k]}(h) + \varepsilon_k \quad \text{on} \quad E_h, 1 \leq \mu \leq k, \]

and \(\lim_{h \to 0} \gamma^{[k]}(h) = 0\) and \(\lim_{k \to \infty} \varepsilon_k = 0.\)

**Proof.** Let us fix \(k \in \mathbb{N}\). Using the methods from the proof of Theorem 2 we can prove that

\[ |(u^{[k]}_{h,\mu})^{(r,m)} - u^{[k]}_\mu(t^{(r)}, x^{(m)})| \leq \omega^{[k]}_h(t^{(r)}) \quad \text{on} \quad E_h, 1 \leq \mu \leq k, \]

where \(\omega^{[k]}_h(t) = \max \{\omega^{[k]}_{h,\mu}(t) : 1 \leq \mu \leq k\}\) and \(\omega^{[k]}_h = (\omega^{[k]}_{h,1}, \ldots, \omega^{[k]}_{h,k})\) is a solution of the problem

\[ \omega'_{\mu}(t) = (1 + c_k)\sigma_{\mu}(t, (\omega(t), 0)_k) + \alpha^{[k]}(h), \quad \omega(0) = a_0^{[k]}(h), 1 \leq \mu \leq k, \]

with \(\alpha^{[k]} : \Delta \to \mathbb{R}_+\) satisfying condition \(\lim_{h \to 0} \alpha^{[k]}(h) = 0.\) It follows from Lemma 3 that

\[ |u^{[k]}_\mu(t^{(r)}, x^{(m)}) - v_\mu(t^{(r)}, x^{(m)})| \leq \omega^{[k]}(t^{(r)}) \quad \text{on} \quad E_h, 1 \leq \mu \leq k. \]

Thus we obtain the assertion (30) with \(\gamma^{[k]}(h) = \omega^{[k]}_h(a)\) and \(\varepsilon_k = \omega^{[k]}(a).\)
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