An Elementary Canonical Classical and Quantum Dynamics for General Relativity

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Abstract. A consistent canonical classical and quantum dynamics in the framework of special relativity was formulated by Stueckelberg in 1941, and generalized to many body theory by Horwitz and Piron in 1973 (SHP). In this paper, using local coordinate transformations, following the original procedure of Einstein, this theory is embedded into the framework of general relativity (GR) both for potential models (where the potential appears as a spacetime mass distribution with dimension of mass) and for electromagnetism (emerging as a gauge field on the quantum mechanical Hilbert space). The canonical Poisson brackets of the SHP theory remain valid (invariant under local coordinate transformations) on the manifold of GR, and provide the basis, following Dirac’s quantization procedure, for formulating a quantum theory. The theory is developed both for one and many particles.

Keywords: Relativistic dynamics, General relativity, Quantum theory on curved space, U(1) gauge, many body theory in general relativity.

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1. Introduction
Einstein’s procedure to obtain general relativity was to assume that a free particle (event) with coordinates \( \{ \xi^\mu \} \) in Minkowski spacetime travels on a worldline described by (timelike, metric \((-+:+,+))\)

\[
ds^2 = -\eta_{\mu\nu} d\xi^\mu d\xi^\nu, \tag{1}
\]

where \( ds \) is the interval of proper time, so that the “proper velocity” satisfies

\[
\dot{\xi}^\mu \dot{\xi}^\nu \eta_{\mu\nu} = -1 \tag{2}
\]

and that for no forces acting

\[
\ddot{\xi}^\mu = 0. \tag{3}
\]

He then assumed that this motion is in a locally flat tangent space of a manifold with nontrivial metric and connection form with coordinates \( \{ x^\mu \} \); it then follows from (2) and (3), by transformation of coordinates, that the physical motion follows geodesic curves, providing the theory of general relativity as we know it. The structure of the coordinates, to be determined by solutions of the Einstein equations, was obtained by the construction of a second rank geometrical tensor set equal to the energy momentum tensor.
We study here this structure in case the motion in the locally flat coordinate system follows the relativistic canonical Hamiltonian dynamics of Stueckelberg, Horwitz and Piron (SHP) [1] with world scalar potential. The theory was originally formulated for a single particle by Stueckelberg in 1941 [7]. Stueckelberg envisaged the motion of a particle along a world line in spacetime that can curve and turn to flow backward in time, resulting in the phenomenon of pair annihilation in classical dynamics. The world line was then described by an invariant monotonic parameter $\tau$. The theory was generalized by Horwitz and Piron in 1973 [3] (see [1]) to be applicable to many body systems by assuming that the parameter $\tau$ is universal (as for Newtonian time [4]), enabling them to solve the two body problem classically, and later, a quantum solution was found by Arshansky and Horwitz [5], both for bound state and scattering theory [6].

Performing a coordinate transformation to general coordinates, along with the corresponding transformation of the momenta (the cotangent space of the original Minkowski manifold), we obtain the SHP theory in a curved space of general coordinates and momenta with a canonical Hamilton-Lagrange (symplectic) structure; the method is also generalized here to a $U(1)$ Abelian gauge theory. This procedure provides a fundamental derivation of the framework assumed by Horwitz, Gershon and Schiffer [7] in their discussion of the Bekenstein-Sanders fields [8] introduced into the TeVeS theory of Bekenstein and Milgrom [9], a geometrical way of obtaining the MOND theory introduced by Milgrom [10] to explain the rotation curves of galaxies.

Birrell and Davies [11] have discussed fields on curved spacetime, and considerable progress has been made, as discussed by Poisson [12] in the formulation of Hamiltonian dynamics of such dynamical fields using Lagrangian functionals associated with the curvature of spacetime. The approach used in this paper is fundamentally different in that it studies a canonical dynamics (both Hamiltonian and Lagrangian) of particles on a curved spacetime.

This method is applied also to the many body case, for which the SHP Hamiltonian is a sum of terms quadratic in four momentum with a many body potential term. Each particle is assumed to move locally in a flat Minkowski space, the tangent space of the general manifold of motions at that point; these local motions can then be mapped at each point $x^\mu$ by coordinate transformation into the curvilinear coordinates reflecting the curvature induced by the Einstein equations.

Throughout most of our development, we assume a $\tau$ independent background gravitational field; the local coordinate transformations from the flat Minkowski space to the curved space are taken to be independent of $\tau$, consistently with an energy momentum tensor that is $\tau$ independent. In a more dynamical setting, when the energy momentum tensor depends on $\tau$, the spacetime evolves nontrivially in $\tau$; the transformations from the local Minkowski coordinates to the curved space coordinates then depend on $\tau$. We discuss this situation in an Appendix; many of the results for the $\tau$ independent case remain (such as the Poisson bracket relations), but the geodesic type equations, for example, are modified.

We also discuss the quantum theory that emerges from a quantization of this structure, and the electromagnetic theory for both single particle and many body systems.

2. Single particle in external potential

We write the SHP Hamiltonian as

$$K = \frac{1}{2M} \eta^{\mu\nu} \pi_\mu \pi_\nu + V(\xi)$$

(4)

where $\eta^{\mu\nu}$ is the flat Minkowski metric $(-+++)$ and $\pi_\mu, \xi^\mu$ are the spacetime canonical momenta and coordinates in the local tangent space, following Einstein’s use of the equivalence

1 A non-Abelian gauge was discussed there, and then an Abelian limit was taken, leaving a term that could cancel caustic singularities.
principle. The existence of a potential term (which may be Lorentz invariant), representing
non-gravitational forces, implies that the “free fall” condition is replaced by a local dynamics
carried along by the free falling system (an additional force acting on the particle within the
“elevator” according to the coordinates in the tangent space)).

The canonical equations are

\[ \dot{\xi}^\mu = \frac{\partial K}{\partial \pi_\mu}, \quad \dot{\pi}_\mu = -\frac{\partial K}{\partial \xi^\mu} = -\frac{\partial V}{\partial \xi^\mu}, \tag{5} \]

where the dot here indicates \( \frac{d}{d\tau} \), with \( \tau \) the invariant universal “world time”. Since then

\[ \dot{\xi}^\mu = \frac{1}{M} \eta^{\mu\nu} \pi_\nu, \tag{6} \]

or \( \pi_\nu = \eta_{\nu\mu} M \dot{\xi}^\mu \),

the Hamiltonian can then be written as

\[ K = \frac{M}{2} \eta_{\mu\nu} \dot{\xi}^\mu \dot{\xi}^\nu + V(\xi). \tag{8} \]

We now transform the local coordinates (contravariantly) according to the diffeomorphism

\[ d\xi^\mu = \frac{\partial \xi^\mu}{\partial x^\lambda} dx^\lambda \tag{9} \]

to attach small changes in \( \xi \) to corresponding small changes in the coordinates \( x \) on the curved
space, so that

\[ \dot{\xi}^\mu = \frac{\partial \xi^\mu}{\partial x^\lambda} \dot{x}^\lambda. \tag{10} \]

The Hamiltonian then becomes

\[ K = \frac{M}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + V(x), \tag{11} \]

where \( V(x) \) is the potential at the point \( \xi \) corresponding to the point \( x \) (actually the function
\( V(\xi) \) could be labeled \( V_x(\xi) \), a function of \( \xi \) in a small neighborhood of the point \( x \)), and

\[ g_{\mu\nu} = \eta_{\lambda\sigma} \frac{\partial \xi^\lambda}{\partial x^\mu} \frac{\partial \xi^\sigma}{\partial x^\nu}. \tag{12} \]

Since \( V \) has significance as the source of a force in the local frame only through its derivatives,
we can make this pointwise correspondence with a globally defined function \( V(x) \)\(^2\). We shall assume in most of the work of this paper that the geometric structure does not depend on \( \tau \), and
is concerned with the study of the covariant dynamical evolution of a system in a background
gravitational field. We study the case of a \( \tau \) dependent metric in the Appendix.

The corresponding Lagrangian is then

\[ L = \frac{M}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - V(x), \tag{13} \]

\(^2\) Since \( V(x) \) has dimension of mass, one can think of this function as a scalar mass field, reflecting forces acting
in the local tangent space at each point. It may play the role of “dark energy” \([7]\). If \( V = 0 \), our discussion
reduces to that of the usual general relativity, but with a well-defined canonical momentum variable.
In the locally flat coordinates in the neighborhood of $x^\mu$, the symplectic structure of Hamiltonian mechanics (e.g. da Silva [13]) implies that the momentum $\pi_\mu$, lying in the cotangent space of the manifold $\{\xi^\mu\}$, transforms covariantly under the local transformation (9), i.e., as does $\frac{\partial}{\partial \xi^\mu}$, so that we may define

$$p_\mu = \frac{\partial \xi^\lambda}{\partial x^\mu} \pi_\lambda. \quad (14)$$

This definition is consistent with the transformation properties of the momentum defined by the Lagrangian (13):

$$p_\mu = \frac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu}, \quad (15)$$
yielding

$$p_\mu = M g_{\mu\nu} \dot{x}^\nu. \quad (16)$$
The second factor in the definition (12) of $g_{\mu\nu}$ acts on $\dot{x}^\nu$; with (10) we then have (as in (14))

$$p_\mu = M \eta_{\lambda\sigma} \frac{\partial \xi^\lambda}{\partial x^\mu} \xi^\sigma = \frac{\partial \xi^\lambda}{\partial x^\mu} \pi_\lambda. \quad (17)$$

We now discuss the geodesic equation obtained by studying the condition

$$\ddot{\xi}^\mu = -\frac{1}{M} \ddot{\pi}_\mu = -\frac{1}{M} \eta^{\mu\nu} \frac{\partial V(\xi)}{\partial \xi^\nu}. \quad (18)$$

On the other hand,\(^3\)

$$\ddot{\xi}^\mu = \frac{d}{d\tau} \left( \frac{\partial \xi^\mu}{\partial x^\lambda} \dot{x}^\lambda \right)$$
$$= \frac{\partial^2 \xi^\mu}{\partial x^\lambda \partial x^\gamma} \ddot{x}^\gamma \dot{x}^\lambda$$
$$+ \frac{\partial \xi^\mu}{\partial x^\lambda} \ddot{x}^\lambda$$
$$= -\frac{1}{M} \eta^{\mu\nu} \frac{\partial x^\lambda}{\partial \xi^\nu} \frac{\partial V(x)}{\partial x^\lambda}, \quad (19)$$
so that, after multiplying by $\frac{\partial x^\sigma}{\partial \xi^\nu}$ and summing over $\mu$, we obtain

$$\ddot{x}^\sigma = -\frac{\partial x^\sigma}{\partial \xi^\mu} \frac{\partial^2 \xi^\mu}{\partial x^\lambda \partial x^\gamma} \ddot{x}^\gamma \dot{x}^\lambda$$
$$- \frac{1}{M} \eta^{\mu\nu} \frac{\partial x^\lambda}{\partial \xi^\nu} \frac{\partial x^\sigma}{\partial \xi^\mu} \frac{\partial V(x)}{\partial x^\lambda}. \quad (20)$$

Finally, with (12) and the usual definition of the connection

$$\Gamma^\sigma_{\lambda\gamma} = \frac{\partial x^\sigma}{\partial \xi^\mu} \frac{\partial^2 \xi^\mu}{\partial x^\lambda \partial x^\gamma}, \quad (21)$$

\(^3\) Here we assume $\xi^\mu$ locally a function of $x(\tau)$ only. If the spacetime evolves ($\tau$ dependent energy momentum tensor), then it is an explicit function of $\tau$ as well, i.e., $\xi(x(\tau), \tau)$. We show in the Appendix how $\frac{\partial \xi(x(\tau), \tau)}{\partial \tau}$ is related to $\frac{\partial g_{\mu\nu}(x(\tau), \tau)}{\partial \tau}$.
we obtain the modified geodesic type equation
\[ \ddot{x}^\sigma = -\Gamma^\sigma_{\lambda\gamma} \dot{x}^\gamma x^\lambda - \frac{1}{M} g^{\sigma\lambda} \frac{\partial V(x)}{\partial x^\lambda}, \tag{22} \]
from which we see that the derivative of the potential \( V(\xi) \) is mapped, under this coordinate transformation into a “force” resulting in a modification of the acceleration along the geodesic-like curves.

The procedure that we have carried out here provides a canonical dynamical structure for the motions in the curvilinear coordinates. We first remark that the Poisson bracket remains valid for the coordinates \( \{x, p\} \). In the local coordinates \( \{\xi, \pi\} \), the \( \tau \) derivative of a function \( F(\xi, \pi) \) is
\[ \frac{dF(\xi, \pi)}{d\tau} = \frac{\partial F(\xi, \pi)}{\partial x^\mu} \dot{x}^\mu + \frac{\partial F(\xi, \pi)}{\partial \pi_\mu} \dot{\pi}_\mu \]
\[ = \frac{\partial F(\xi, \pi)}{\partial x^\mu} \frac{\partial K}{\partial \pi_\mu} - \frac{\partial F(\xi, \pi)}{\partial \pi_\mu} \frac{\partial K}{\partial \xi^\nu} \]
\[ = [F, K]_{PB}(\xi, \pi). \tag{23} \]
If we replace in this formula
\[ \frac{\partial}{\partial \xi^\mu} = \frac{\partial x^\lambda}{\partial \xi^\mu} \frac{\partial}{\partial x^\lambda}, \]
\[ \frac{\partial}{\partial \pi_\mu} = \frac{\partial x^\lambda}{\partial \pi_\mu} \frac{\partial}{\partial p_\lambda}, \tag{24} \]
we immediately (as assured by the invariance of the Poisson bracket under local coordinate transformations) obtain
\[ \frac{dF(\xi, \pi)}{d\tau} = \frac{\partial F}{\partial x^\mu} \frac{\partial K}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial K}{\partial x^\nu} \equiv [F, K]_{PB}(x, p) \tag{25} \]
In this definition of Poisson bracket we have, as for the \( \xi^\mu, \pi_\nu \) relation,
\[ [x^\mu, p_\nu]_{PB}(x, p) = \delta^\mu_\nu. \tag{26} \]
Furthermore (we drop the \( (x, p) \) label henceforth),
\[ [p_\mu, F(x)]_{PB} = -\frac{\partial F}{\partial x^\mu}, \tag{27} \]
so that \( p_\mu \) acts infinitesimally as the generator of translation along the coordinate curves and
\[ [x^\mu, F(p)]_{PB} = \frac{\partial F(p)}{\partial p_\mu}, \tag{28} \]
so that \( x^\mu \) is the generator of translations in \( p_\mu \).

This structure clearly provides a phase space which could serve as the basis for statistical mechanics, and lends itself to the construction of a canonical quantum theory on the curved spacetime, as we discuss below.

We now turn to a discussion of the dynamics introduced into the curved space by the procedure outlined above. We start by developing the relation between \( \dot{p}_\mu \) and the geodesic equations for \( x^\mu \), and show that the result agrees with the direct Hamiltonian calculation.

Recall from (5) that
\[ p_\mu = M g_{\mu\lambda} \dot{x}^\lambda, \tag{29} \]
so that
\[ \dot{p}_\mu = M \left( \frac{\partial g_{\mu\lambda}}{\partial x^\gamma} \dot{x}^\gamma \dot{x}^\lambda + g_{\mu\sigma} \dot{x}^\sigma \right). \]  
(30)

Since, by Eq. (22),
\[ \ddot{x}^\sigma = -\Gamma^\sigma_{\lambda\gamma} \dot{x}^\gamma \dot{x}^\lambda - \frac{1}{M} g_{\sigma\lambda} \frac{\partial V(x)}{\partial x^\lambda}, \]
(31)

Eq. (1.25) becomes
\[ \dot{p}_\mu = -\frac{\partial V(x)}{\partial x^\mu} + M \left( \frac{\partial g_{\mu\lambda}}{\partial x^\gamma} \dot{x}^\gamma \dot{x}^\lambda - g_{\mu\sigma} \Gamma^\sigma_{\lambda\gamma} \dot{x}^\gamma \dot{x}^\lambda \right). \]
(32)

We now use the relation
\[ \Gamma^\sigma_{\lambda\gamma} = \frac{1}{2} g^{\sigma\eta} \left( \frac{\partial g_{\eta\lambda}}{\partial x^\gamma} + \frac{\partial g_{\eta\gamma}}{\partial x^\lambda} - \frac{\partial g_{\lambda\gamma}}{\partial x^\eta} \right) \]
(33)

to obtain
\[ \dot{p}_\mu = -\frac{\partial V(x)}{\partial x^\mu} + M \left( \frac{\partial g_{\mu\lambda}}{\partial x^\gamma} \dot{x}^\gamma \dot{x}^\lambda - \frac{1}{2} \left( \frac{\partial g_{\mu\lambda}}{\partial x^\gamma} + \frac{\partial g_{\mu\gamma}}{\partial x^\lambda} - \frac{\partial g_{\lambda\gamma}}{\partial x^\mu} \right) \dot{x}^\gamma \dot{x}^\lambda \right). \]
(34)

The first term in the brackets with coefficient $M$, symmetrized under multiplication by $\dot{x}^\gamma \dot{x}^\lambda$, cancels the first two terms of the contribution from the connection form leaving the rather remarkable result
\[ \dot{p}_\mu = -\frac{\partial V(x)}{\partial x^\mu} + M \frac{\partial g_{\mu\lambda}}{\partial x^\gamma} \dot{x}^\gamma \dot{x}^\lambda. \]
(35)

The 3D form of the second term plays an important role in the study of (3D) Hamiltonian stability analysis in the work of Horowitz, Ben Zion, Lewkowicz and Levitan [14], where it is called the “reduced connection” form.

Eq. (35) displays the “force” induced from the local potential function along with the acceleration induced by the geometry. Since the derivative of the metric diverges in the neighborhood of the black hole solution, (35) provides a direct interpretation of the geometrical configuration as resulting in an “actual”, from the point of view of the SHP theory, very large force on the particle at the black hole horizon as would be seen in coordinates of this type. We do not study here Kruskal type coordinates [15] which do not display a singularity at the horizon.

We may also write (35) in terms of the full connection form by noting that with (12),
\[ \frac{\partial g_{\lambda\gamma}}{\partial x^\mu} = \eta_{\alpha\beta} \left( \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu \partial x^\gamma} + \frac{\partial \xi^\alpha}{\partial x^\lambda} \frac{\partial \xi^\beta}{\partial x^\gamma} \frac{\partial^2 \xi^\beta}{\partial x^\mu} \right). \]
(36)

Multiplying by $\dot{x}^\gamma \dot{x}^\lambda$, the two terms combine to give a factor of two. We then return to the original definition of $\Gamma$ in (21) in the form
\[ \frac{\partial^2 \xi^\alpha}{\partial x^\gamma \partial x^\mu} = \frac{\partial \xi^\alpha}{\partial x^\sigma} \Gamma^\sigma_{\lambda\mu}, \]
(37)

so we can write
\[ \frac{\partial g_{\lambda\gamma}}{\partial x^\mu} \dot{x}^\gamma \dot{x}^\lambda = 2 \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\gamma} \frac{\partial \xi^\beta}{\partial x^\lambda} \Gamma^\sigma_{\lambda\mu} \dot{x}^\gamma \dot{x}^\lambda = 2 g_{\sigma\gamma} \Gamma^\sigma_{\lambda\mu} \dot{x}^\gamma \dot{x}^\lambda. \]
(38)

We therefore have
\[ \dot{p}_\mu = -\frac{\partial V(x)}{\partial x^\mu} + M g_{\mu\sigma} \Gamma^\sigma_{\lambda\mu} \dot{x}^\gamma \dot{x}^\lambda. \]
(39)
We now return to the Hamiltonian (11) and carry out the calculation directly. Since $\dot{x}^\mu$ is, in general, a function of $x^\mu$, we write the Hamiltonian (using (16)) in terms of the momenta, assured by the canonical structure to be independent variables,

$$K = \frac{1}{2M} g^{\alpha\beta} p_\alpha p_\beta + V(x).$$

Then,

$$\dot{p}_\mu = -\frac{1}{2M} \frac{\partial g^{\alpha\beta}}{\partial x^\mu} p_\alpha p_\beta - \frac{\partial V(x)}{\partial x^\mu}.\tag{41}$$

Returning to the form with $\dot{x}$ again, we have, with $p_\alpha = M g_{\alpha\lambda} \dot{x}^\lambda$,

$$\dot{p}_\mu = \frac{M}{2} \frac{\partial g^{\alpha\beta}}{\partial x^\mu} g_{\alpha\lambda} g_{\beta\gamma} \dot{x}^\gamma \dot{x}^\lambda - \frac{\partial V(x)}{\partial x^\mu}.\tag{42}$$

Since, however,

$$\frac{\partial g^{\alpha\beta}}{\partial x^\mu} g_{\alpha\lambda} g_{\beta\gamma} = -\frac{\partial g_{\lambda\gamma}}{\partial x^\mu},\tag{43}$$

we recover the form obtained in (35).

We now turn to discussion of the many body problem.

3. The many body system with interaction potential

The many body Hamiltonian of the SHP theory is

$$K = \sum_{i=1}^{N} \frac{1}{2M_i} \eta^{\mu\nu} \pi_\mu \pi_\nu + V(\xi_1, \xi_2, \ldots \xi_N),$$

where the potential $V(\xi_1, \xi_2, \ldots \xi_N)$ is a function of the locally flat coordinates in the neighborhood of each of the particles at $\{x_i\}$. Although this function is Lorentz scalar, Poincaré invariance is, in general, inapplicable (even in the two-body case), unless all of the particles are in a sufficiently small neighborhood to be able to neglect the effects of curvature.

The Hamilton equations are (in the tangent space in the neighborhood of each particle at the point $x_i$)

$$\dot{\xi}_i^\mu = \frac{\partial K}{\partial \pi_\mu i}, \quad \dot{\pi}_i = -\frac{\partial K}{\partial \xi_\mu i}, \quad \frac{\partial V}{\partial \xi_\mu i}.\tag{45}$$

We then have

$$\dot{\xi}_i^\mu = \frac{1}{M_i} \eta^{\mu\nu} \pi_\nu, \tag{46}$$

or

$$\pi_\nu = \eta_{\nu\mu} M_i \dot{\xi}_i^\mu.$$  

Following the procedure we used for the one-body case above, we may substitute this expression into the Hamiltonian to obtain

$$K = \sum_{i=1}^{N} \frac{M_i}{2} \eta_{\mu\nu} \dot{\xi}_i^\mu \dot{\xi}_i^\nu + V(\xi_1, \xi_2, \ldots \xi_N),\tag{47}$$

At the location $x^\mu_i$ of the $i^{th}$ particle, since in this neighborhood, $\xi_\mu^i$ is a function locally of $x^\mu_i$, we can then make a local coordinate transformation

$$d\xi_\sigma^i = \frac{\partial \xi_\sigma^i}{\partial x^\mu_i} dx^\mu_i.$$ \tag{48}
Defining
\[ g_{\mu\nu}(x_i) = \eta_{\sigma\lambda} \frac{\partial \xi^\sigma_i}{\partial x^\mu_i} \frac{\partial \xi^\lambda_i}{\partial x^\nu_i}, \]
(49)
one obtains the Hamiltonian in terms of the four-velocities; changing notation for the arguments of the potential,
\[ K = \sum_{i=1}^{N} M_i \frac{1}{2} g_{\mu\nu}(x_i) \dot{x}_i^\mu \dot{x}_i^\nu + V(x_1, x_2, \ldots x_N), \]
(50)
with corresponding Lagrangian
\[ L = \sum_{i=1}^{N} M_i \frac{1}{2} g_{\mu\nu}(x_i) \dot{x}_i^\mu \dot{x}_i^\nu - V(x_1, x_2, \ldots x_N). \]
(51)
As for the one body case, we can find the equations for the geodesic motion of the particles. Since (10) is valid for each of the particle coordinates,
\[ \dot{\xi}^\mu_i = \frac{\partial \xi^\mu_i}{\partial x^\lambda_i} \dot{x}_i^\lambda, \]
(52)
from which we similarly obtain
\[ \ddot{\xi}^\mu_i = \frac{d}{d\tau}\left( \frac{\partial \xi^\mu_i}{\partial x^\lambda_i} \dot{x}_i^\lambda \right) = \frac{\partial^2 \xi^\mu_i}{\partial x^\lambda_i \partial x^\gamma_i} \dot{x}_i^\lambda \dot{x}_i^\gamma + \frac{\partial \xi^\mu_i}{\partial x^\lambda_i} \ddot{x}_i^\lambda = \frac{1}{M_i} \eta^\mu\nu \frac{\partial x^\lambda_i}{\partial \xi^\sigma_i} \frac{\partial V(x_1, x_2, \ldots x_N)}{\partial x^\lambda_i}. \]
(53)
The many body geodesic curves are therefore described by
\[ \ddot{x}_i^\sigma = -\frac{\partial x^\sigma_i}{\partial \xi^\mu_i} \frac{\partial^2 \xi^\mu_i}{\partial x^\lambda_i \partial x^\gamma_i} \dot{x}_i^\lambda \dot{x}_i^\gamma - \frac{1}{M_i} \eta^\mu\nu \frac{\partial x^\lambda_i}{\partial \xi^\sigma_i} \frac{\partial V(x_1, x_2, \ldots x_N)}{\partial x^\lambda_i}. \]
(54)
We can consider the Jacobian for the local mapping (48) as a field, a mapping defined over all \( \{x^\mu\} \), in (48) evaluated at the point \( x^\mu_i \) where the \( i^{th} \) particle is found.
We then define a local connection form at the point \( x_i \) as
\[ \Gamma_{\lambda\gamma}^\sigma(x_i) = \frac{\partial x^\sigma_i}{\partial \xi^\mu_i} \frac{\partial^2 \xi^\mu_i}{\partial x^\lambda_i \partial x^\gamma_i} \]
(55)
also, since it is a property of the manifold, as a field evaluated at the point \( x^\mu_i \), so that the geodesic equations can be written as
\[ \ddot{x}_i^\sigma = -\Gamma_{\lambda\gamma}^\sigma(x_i) \dot{x}_i^\gamma \dot{x}_i^\lambda - \frac{1}{M_i} \eta^\mu\nu \frac{\partial V(x_1, x_2, \ldots x_N)}{\partial x^\lambda_i}. \]
(56)
\footnote{We assume that (general covariance) \( V(x_1, x_2, \ldots x_N) \) is a scalar function under local diffeomorphisms of any of the variables.}
The connection form in this case then also satisfies (33) at each point \( x_i \). Since this connection form coincides with Einstein’s, the same method can be used to construct a Ricci tensor; the resulting Einstein equations will therefore have the same form, although there will necessarily be differences in the structure of the energy momentum tensor.\(^5\) The empty space solution [16] will be applicable in this framework as well, providing an interesting example for application [17], and the homogeneous case of Robertson Friedman and Walker [18] would have a similar form to the well-known solution. Applications of this type will be investigated in succeeding papers.

Following the same procedure as for (25), with general functions \( F(x_1, x_2, \ldots, x_N, p_1, p_2, \ldots p_N) \) with variables \( \xi_1, \xi_2, \ldots, \xi_N, \pi_1, \pi_2, \ldots, \pi_N \) in the tangent bundles assigned to the points \( x_1, x_2, \ldots, x_N, p_1, p_2, \ldots, p_N \) in the general phase space for the \( N \)-body system, the Poisson bracket is defined by

\[
\frac{dF(\xi_1, \xi_2, \ldots, \xi_N, \pi_1, \pi_2, \ldots, \pi_N)}{d\tau} = \frac{\partial F(\{\xi, \pi\})}{\partial \xi_i} \dot{\xi}_i + \frac{\partial F(\{\xi, \pi\})}{\partial \pi_{\mu i}} \dot{\pi}_{\mu i} = \frac{\partial F(\{\xi, \pi\})}{\partial \xi_i} \frac{\partial K}{\partial \pi_{\mu i}} - \frac{\partial F(\{\xi, \pi\})}{\partial \pi_{\mu i}} \frac{\partial K}{\partial \xi_i} \quad (57)
\]

The local transformations on the differentials cancel as for the one particle case (at each point \( x_i \)), so the Poisson bracket remains in the same form on the \( \{x_i, p_i\} \) phase space. We therefore have

\[
\frac{dF(x_1, x_2, \ldots, x_N, p_1, p_2, \ldots, p_N)}{d\tau} = \sum_i \frac{\partial F(\{x, p\})}{\partial x_i} \frac{\partial K}{\partial p_{\mu i}} - \frac{\partial F(\{x, p\})}{\partial p_{\mu i}} \frac{\partial K}{\partial x_i} \quad (58)
\]

In general, for two functions \( A(\{x, p\}) \) and \( B(\{x, p\}) \), the many body Poisson bracket is then

\[
[A, B]_{PB} = \sum_i \left( \frac{\partial A(\{x, p\})}{\partial x_i} \frac{\partial B(\{x, p\})}{\partial p_{\mu i}} - \frac{\partial A(\{x, p\})}{\partial p_{\mu i}} \frac{\partial B(\{x, p\})}{\partial x_i} \right) \quad (59)
\]

Since the variables \( x_1, x_2, \ldots, x_N, p_1, p_2, \ldots, p_N \) are to be considered as kinematically independent, we obtain the canonical bracket

\[
[x_{\mu i}^\lambda, p_{\mu \nu}]_{PB} = \delta_{ij} \delta^{\mu \nu} \quad (60)
\]

We now turn to the equations of motion for \( p_{\mu i} \). At the point \( x_i ^\mu \), as we have argued above,

\[
p_{\mu i} = \frac{\partial \xi_i^\lambda}{\partial x_i^\mu}(x_i) \pi_{\lambda i} \quad (61)
\]

It therefore follows, in the same way that we found (39), that

\[
\dot{p}_{\mu i} = -\frac{\partial V(x_1, x_2, \ldots, x_N, p_1)}{\partial x_i^\mu} + M_i g_{\sigma \gamma}(x_i) \Gamma^\sigma_{\lambda \mu}(x_i) \dot{x}_i^\gamma \dot{x}_i^\lambda \\
= -\frac{\partial V(x_1, x_2, \ldots, x_N, p_1)}{\partial x_i^\mu} + M_i \frac{1}{2} (g_{\sigma \gamma}(x_i) \Gamma^\sigma_{\lambda \mu}(x_i) + g_{\sigma \lambda}(x_i) \Gamma^\sigma_{\gamma \mu}(x_i)) \dot{x}_i^\gamma \dot{x}_i^\lambda. \quad (62)
\]

As for the one body case, this result also follows from the Lagrangian (51).

The time rate of change of the momentum is coupled, as for the geodesic motions of the \( x_i \), to the other \( N - 1 \) particles through the potential function.

\(^5\) Note that even in the absence of a potential function, the solutions for \( g_{\mu \nu}(x) \) would reflect the many body structure of the energy momentum tensor through the Einstein equations.
4. Quantum Theory on the Curved Space

The Poisson bracket formulas (27) and (28) can be considered as a basis for defining a quantum theory with canonical commutation relations

\[ [x^\mu, p_\nu] = i\hbar \delta^\mu_\nu, \quad (63) \]

so that

\[ [p_\mu, F(x)] = -i\hbar \frac{\partial F}{\partial x^\mu}, \quad (64) \]

and

\[ [x^\mu, F(p)] = i\hbar \frac{\partial F(p)}{\partial p_\mu}. \quad (65) \]

The transcription of the SHP Schrödinger equation for a wave function \( \psi_\tau(x) \) can be taken to be (see also Schwinger and deWitt [19])

\[ i \frac{\partial}{\partial \tau} \psi_\tau(x) = K \psi_\tau(x), \quad (66) \]

where the operator valued Hamiltonian can be taken to be the Hermitian form, on a Hilbert space defined with scalar product (with invariant measure; we write \( g = -\det \{ g^{\mu\nu} \} \)),

\[ (\psi, \chi) = \int d^4 x \sqrt{g} \psi^*_\tau(x) \chi_\tau(x), \quad (67) \]

and

\[ K = \frac{1}{2M \sqrt{g}} p_\mu g^{\mu\nu} p_\nu + V(x). \quad (68) \]

The normalization condition over the manifold \( \{ x \} \) is not a trivial transcription of the Euclidean condition on the SHP quantum theory [1]. If we think of the integral (67) as constructed from summing over coordinate components, a large excursion along a coordinate of the curved space may bring one, perhaps many times, to a nearby neighborhood of some point. The integration (67) must be considered as a total volume sum on the invariant measure on the whole space, consistent with the idea that the norm is the sum of probability measures on every subset contained. The procedure for carrying out such integrals would, of course, depend on the geometrical structure of the manifold.

This construction can be carried over to the many body case directly, i.e., with the operator properties of the coordinates and momenta

\[ [x^\mu_i, p_{\nu i}] = i\hbar \delta^\mu_i \delta^\nu_j, \quad (69) \]

and therefore

\[ [p_{\mu i}, F(\{ x \})] = -i\hbar \frac{\partial F(\{ x \})}{\partial x^\mu_i}, \quad (70) \]

and

\[ [x^\mu_i, F(\{ p \})] = i\hbar \frac{\partial F(\{ p \})}{\partial p_{\mu i}}. \quad (71) \]

The scalar product is then (the flat space Lorentz invariant \( d^4 x \) goes over to the local diffeomorphism invariant \( d^4 x \sqrt{g} \))

\[ (\psi, \chi) = \int \Pi_i \{ d^4 (x_i) \sqrt{g(x_i)} \} \psi^*_\tau(x_1, x_2, \ldots x_N) \chi_\tau(x_1, x_2, \ldots x_N). \quad (72) \]
In this scalar product, the Hamiltonian
\[ K = \sum_i \frac{1}{2M\sqrt{g(x_i)}} p_\mu g^{\mu\nu}(x_i) p_{\nu i} + V(x_1, x_2, \ldots x_N) \] (73)
is formally Hermitian. The procedure of integration by parts in the scalar product involves a term by term cancellation of the factors \( \sqrt{g(x_i)} \), thus permitting, in each term, integration by parts for the \( p_i \).

We now turn to discuss the introduction of electromagnetism, for a single particle and for many particle systems.

5. Electromagnetism
As C.N. Yang [20] wrote, electromagnetism can be thought of as a \( U(1) \) fiber bundle. The electromagnetic potential vector field emerges as a section on the fiber bundle in the gauge transformations of the quantum theory. To illustrate this idea, consider what happens to Eq. (66) if we consider, instead of the function \( \psi(\tau(x)) \), the function \( \psi'(\tau(x)) \) resulting from a unitary transformation \( e^{i\Lambda(x, \tau)} \) defined locally on the Hilbert space at each value of \( \tau \). Since \( p_\mu \) acts like a derivative on \( x^\mu \), it differentiates \( \Lambda(x, \tau) \), just as for the corresponding computation in the flat Minkowski space. As for the flat space case, we must add a gauge compensation term so that
\[ (p_\mu - a'_\mu(x, \tau)) e^{i\Lambda(x, \tau)} \psi(\tau(x)) = e^{i\Lambda(x, \tau)} (p_\mu - a_\mu(x, \tau)) \psi(\tau(x)), \] (74)
i.e., assuring that \( (p_\mu - a_\mu(x, \tau)) \psi(\tau(x)) \) is an element of the Hilbert space and therefore undergoes the same unitary transformation as \( \psi(\tau(x)) \). Carrying out the derivative implied by the action of \( p_\mu \) (with \( \hbar = 1 \) here), we find the condition that
\[ a'_\mu(x, \tau) = a_\mu(x, \tau) + \frac{\partial\Lambda(x, \tau)}{\partial x^\mu}, \] (75)
the usual form of a gauge transformation. From the scalar nature of the wave function, we have implicitly assumed Einstein’s property of general covariance for the fields \( a_\mu(x, \tau) \).

Unless we restrict ourselves to the so-called “Hamilton gauge” (with \( \Lambda \) independent of \( \tau \)), the form of (66) implies the existence of a fifth field \( a_5(x, \tau) \), for which we must have
\[ \{ i \frac{\partial}{\partial \tau} + a_5'(x, \tau) \} \psi'(\tau(x)) = e^{i\Lambda(x, \tau)} \{ i \frac{\partial}{\partial \tau} + a_5(x, \tau) \} \psi(\tau(x)), \] (76)
By the same argument, we then have
\[ a_5'(x, \tau) = a_5(x, \tau) + \frac{\partial}{\partial \tau} \Lambda(x, \tau), \] (77)
The Stueckelberg-Schrödinger equation then becomes
\[ i \frac{\partial}{\partial \tau} \psi(\tau(x)) = \left\{ \frac{1}{2M\sqrt{g}} (p_\mu - a_\mu(x, \tau)) g^{\mu\nu}(p_\nu - a_\nu(x, \tau)) - a_5(x, \tau)(x) \right\} \psi(\tau(x)), \] (78)
where the scalar field of the potential model is now replaced by the generally \( \tau \) dependent \( a_5(x, \tau) \).

In the usual way, we can define in the flat tangent space, a gauge invariant field strength[1][21]
\[ f_{\alpha\beta}(\xi, \tau) = \partial_\alpha a_\beta(\xi, \tau) - \partial_\beta a_\alpha(\xi, \tau), \] (79)
where \( \alpha, \beta = (0, 1, 2, 3, 5) \), which satisfies the equation

\[
\partial^\alpha \tilde{f}_{\alpha\beta}(\xi, \tau) = j_\beta(\xi, \tau).
\] (80)

The first four components of the current have the form of a Jackson type construction [22] before integration over \( \tau \), and the fifth component is the density \( \rho(\xi, \tau) \propto \psi^* (\xi) \psi (\xi) \) in the SHP theory.

It is easy to see that a coordinate transformation leads to the rule of replacement of derivatives by covariant derivatives so that in the curved space

\[
f_{\mu \nu}(x, \tau) = a_{\mu ; \nu} - a_{\nu ; \mu} = \partial_{\xi} \frac{\partial \xi^\lambda}{\partial x^\mu} \partial_{\xi} \frac{\partial \xi^\sigma}{\partial x^\nu} \tilde{f}_{\lambda \sigma},
\] (81)

since \( a_5 \) is a Lorentz scalar.

For the four- and fifth -components, we have

\[
f^{\mu \nu}(\xi, \tau) + \partial^5 f_{5 \nu} = j_\nu(\xi, \tau);
\]

\[
f_{5 \mu}(x, \tau) = j_5(x, \tau) = \rho(x, \tau),
\] (82)

where the last is analogous to the non-relativistic \( \nabla \cdot E = \rho \). Clearly, the covariant divergence of \( j_\nu(\xi, \tau) \) vanishes.

We now study the structure of the corresponding current. To do this, we write an action for which the variation with respect to \( \psi^* (x) \) yields the Stueckelberg-Schrödinger equation (78),

\[
S = \int d\tau d^4 x \sqrt{g} \left\{ i \psi^* (x) \frac{\partial}{\partial \tau} \psi (x) - i \psi (x) \frac{\partial}{\partial \tau} \psi^* (x) + a_5(x, \tau) \psi^* (x) \psi (x) + \frac{1}{2M \sqrt{g}} (p_\mu - a_\mu(x, \tau)) g^{\mu \nu} (p_\nu - a_\nu(x, \tau)) - a_5(x, \tau) \psi (x) \right\} \psi (x)
\] (83)

where \( f^{\mu \nu}(x, \tau) = g^{\mu \lambda} g^{\nu \sigma} f_{\lambda \sigma}(x, \tau) \). We add to the action a purely electromagnetic part

\[
S_{em} = + \frac{1}{4 \sqrt{g}} (f^{\mu \nu} f_{\mu \nu} + f^{\mu 5} f_{\mu 5}),
\] (84)

where (since \( a_5 \) is scalar its covariant derivative is an ordinary derivative)

\[
f_{\mu 5} = \partial_{\mu} a_5 - \partial_{5} a_\mu,
\] (85)

and \( a^5 = \pm a_5 \) depending on the metric for \( O(4, 1) \) or \( O(3, 2) \) chosen for the 5D manifold. As for the nonrelativistic theory on 3D, where the gauge fields make accessible the (3, 1) manifold of Minkowski space, the gauge fields of the \( (3 + 1)D \) theory make accessible a (4, 1) or (3, 2) manifold. As we shall see in our discussion of the many body problem, the assumption of universality in \( \tau \) does not admit such a higher symmetry.

In 1995, Land, Shnerb and Horwitz [23] studied the consequences of assuming covariant commutation relations between \( x^\mu \) and \( \dot{x}^\nu \) on a manifold using a theorem of Hojman and Sheply [24] extending and generalizing the work of Tanimura [25]. Their results, including the development of the 5D theory, agree in the one particle sector with what we have presented here.

To fully treat such a development with the methods we have used here, one would have to start with a one degree higher dimensional Stueckelberg equation; its gauge fields would open...
the possibility of a 6D manifold as a result of gauge invariance. We shall truncate this sequence here at the level of 4D, retaining $\tau$ as the universal invariant parameter of evolution.

We now obtain the current by variation of $a_\mu$ in the action. Integrating by parts in the kinetic term (for $p_\mu = -i \frac{\partial}{\partial x^\mu}$), we have

$$S_{\text{kin}} = -\frac{1}{2M} \int d\tau d^4x((p_\mu - a_\mu)\psi^* g^{\mu\nu}(p_\nu - a_\nu)\psi)$$

where

$$S_{\text{kin}} = +\frac{1}{2M} \int d\tau d^4x((p_\mu + a_\mu)\psi^* g^{\mu\nu}(p_\nu - a_\nu)\psi), \quad (86)$$

so that

$$\frac{\delta S_{\text{kin}}}{\delta a_\mu} = \frac{1}{2Mi} (\psi^* g^{\mu\nu}(\nabla_\nu - ia_\nu)\psi - ((\nabla_\nu + ia_\nu)\psi^*)g^{\mu\nu}\psi). \quad (87)$$

However, $\sqrt{g}\psi^*\psi(x,\tau)$ is the probability to find the particle (event) in the invariant volume element $\sqrt{g}d^4x$, so that $\psi^*\psi(x,\tau)$ must go over to $(\sqrt{g})^{-1}\delta^4(x - x')$ in the classical limit (see Weinberg [26]). Therefore, we must define the current as

$$j_\mu(x,\tau) = \frac{1}{2Mi\sqrt{g}} (\psi^* g^{\mu\nu}(\nabla_\nu - ia_\nu)\psi - ((\nabla_\nu + ia_\nu)\psi^*)g^{\mu\nu}\psi), \quad (89)$$

in agreement in form with the corresponding known nonrelativistic formula ($g \to 1$ in the nonrelativistic limit). Note further that the integral of the current over a hypersurface with the invariant measure $d^4x\sqrt{g}$ has well-defined physical meaning.

We now study the variation of the action with respect to $a_5$. The variation of the full action (both $S_m$ and $S_{em}$) with respect to $a_5$ then yields the field equation

$$f^{\mu\nu}(x,\tau) \equiv (\sqrt{g})^{-1}\rho(x,\tau) = \psi^*\psi(x,\tau), \quad (90)$$

in analogy to the standard Maxwell equation $\nabla \cdot E = \rho$.

Furthermore, the variation with respect to $a_\mu$, with the definition of the current (89), yields the covariant field equations (as one would conclude from the application of general covariance [26]) The variation of $a_\mu$ in (85) leads to the second field equation of (81).

$$f^{\mu\nu}(x,\tau) = j_\nu(x,\tau), \quad (4.17)(91)$$

or, equivalently,

$$\partial_\mu(\sqrt{g}f^{\mu\nu}) = \sqrt{g}j^\nu \quad (92)$$

The Lorentz force (see also [23]) follows by directly transcribing the flat space formula (for charge unity),

$$F^\mu = f^{\mu}_{\nu} \frac{dx^\nu}{d\tau} \quad (93)$$

6 Note that if we follow the method of Jackson [22], defining the macroscopic current

$$J^\mu(x) = (\sqrt{g})^{-1} \int d\tau \delta^4(x - x(\tau)), \quad (88)$$

then $\partial_\mu(\sqrt{g}J^\nu) = 0$. 


6. The many body problem for electromagnetism

In the following, we generalize this structure to the many body problem.

The many-body wave function can be written as the span of the direct product of wave functions associated with isomorphic one particle Hilbert spaces (which also may be used in the construction of the Fock space on the manifold). The norm and orthogonality follow from the properties of the one particle spaces as above (with the rule that corresponding elements of the sequences are contracted by scalar product). We may therefore write

\[ \psi_\tau(x_1, x_2, \ldots x_N) = \sum \alpha_{\alpha_1, \alpha_2, \ldots} \phi_{\alpha_1, \tau}(x_1) \phi_{\alpha_2, \tau}(x_2) \cdots \phi_{\alpha_N, \tau}(x_N). \]  

(94)

We now argue that a local unitary transformation of the form \( e^{i \Lambda(x, \tau)} \) should act, with the same function \( \Lambda(x, \tau) \) in each of the factor spaces.\(^7\) This construction provides a convenient mechanism for the gauge transformations of the Bose-Einstein or Fermi-Dirac Fock spaces (and, in general, for linear combinations). Furthermore, as we shall see below, it enables us to define a field \( a_\mu(x, \tau) \). We therefore define the gauge transformation \( \psi \to \psi' \) as

\[ \psi'_\tau(x_1, x_2, \ldots x_N) = \sum \alpha_{\alpha_1, \alpha_2, \ldots} \phi_{\alpha_1, \tau}(x_1) \phi_{\alpha_2, \tau}(x_2) \cdots \phi_{\alpha_N, \tau}(x_N) \]

\[ \times e^{i(\Lambda(x_1, \tau) + \Lambda(x_2, \tau) + \ldots + \Lambda(x_N, \tau))} \psi_\tau(x_1, x_2, \ldots x_N) \]  

(95)

The remaining argument is the same as for the one body case. At each point \( x_i \) of the wave function associated with the \( i^{th} \) factor, we have

\[ \left(-i \frac{\partial}{\partial x_i^\mu} - a'_\mu(x_i, \tau) \right) \psi'_\tau(x_1, x_2, \ldots x_N) = e^{i(\Lambda(x_1, \tau) + \Lambda(x_2, \tau) + \ldots + \Lambda(x_N, \tau))} \]

\[ \left(-i \frac{\partial}{\partial x_i^\mu} - a_\mu(x_i, \tau) \right) \psi_\tau(x_1, x_2, \ldots x_N), \]  

(96)

so that

\[ a'_\mu(x_i, \tau) = a_\mu(x_i, \tau) + \frac{\partial \Lambda(x_i, \tau)}{\partial x_i^\mu}, \]

(97)

It therefore follows that this procedure leads to a local covariant field for the electromagnetic potential vector.

If we call the compensation function for the \( \tau \) evolution \( a_5(x_1, x_2, \ldots x_N, \tau) \) and

\[ \Lambda(x_1, x_2, \ldots x_N, \tau) = \Lambda(x_1, \tau) + \Lambda(x_2, \tau) + \cdots + \Lambda(x_N, \tau) \]

(98)

then it follows that

\[ a'_5(x_1, x_2, \ldots x_N, \tau) = a_5(x_1, x_2, \ldots x_N, \tau) + \frac{\partial}{\partial \tau} \Lambda(x_1, x_2, \ldots x_N, \tau). \]

(99)

The fifth gauge function is clearly not a property of the individual particles, and in this sense it appears not to correspond to a local field on the individual particles (it is a local field, however, on the configuration space \((x_1, x_2, \ldots x_N)\)).

The field strengths associated with the fifth field form a set

\[ f_{5, \mu}(x_1, x_2, \ldots x_N) = \partial_\mu a_5(x_1, x_2, \ldots x_N) - \partial_\tau a_\mu(x_i). \]

(100)

\(^7\) One can think of this procedure as the action of an operator \( (e^{i \Lambda})^N = (e^{i \Lambda_N}) \) acting on the \( N \) particle state.
Under gauge transformation,
\[
\partial_\mu (a_5(x_1, x_2, \ldots, x_N, \tau) + \frac{\partial}{\partial \tau} \Lambda(x_1, x_2, \ldots, x_N, \tau)) - \partial_\tau (a_\mu(x_1, \tau) + \frac{\partial}{\partial x_1^\mu} \Lambda(x_1\tau)) \\
= \partial_\mu a_5(x_1, x_2, \ldots, x_N, \tau) - \partial_\tau a_\mu(x_1, \tau),
\]
(101)
since the derivative \( \frac{\partial}{\partial x_1^\mu} \) selects the term in the sum that cancels \( \frac{\partial}{\partial x_1^\mu} \Lambda(x_1\tau) \).

To be able to write the elements of this set in a uniform way in the arguments, we define a field on \( x_1, x_2, \ldots, x_N \) for each \( \tau \) such that the projection
\[
a^\mu_i(x_1, x_2, \ldots, x_N, \tau) = a_\mu(x_1, \tau).
\]
(102)
We can then write
\[
f^i_{\mu_5}(x_1, x_2, \ldots, x_N) = \partial_\mu a_5(x_1, x_2, \ldots, x_N) - \partial_\tau a^\mu_i(x_1, x_2, \ldots, x_N).
\]
(103)

### 7. Electromagnetic Currents and Field Equations for the Many Body System.

In the following we discuss the electromagnetic current and field equations for the \( N \) particle case.

We write the action for an \( N \) particle system in the presence of electromagnetism as
\[
S = \int d\tau \Pi_i d^4x_i \sqrt{g(x)} \left\{ i \psi^* \tau(x_1, x_2, \ldots, x_N) \right\}
\]
\[
\frac{\partial}{\partial \tau} \psi(x_1, x_2, \ldots, x_N) - i\psi(x_1, x_2, \ldots, x_N) \frac{\partial}{\partial \tau} \psi^* \tau(x_1, x_2, \ldots, x_N)
\]
\[
+ a_5(x_1, x_2, \ldots, x_N, \tau) \psi^* \tau(x_1, x_2, \ldots, x_N) \psi \tau(x_1, x_2, \ldots, x_N)
\]
\[
- \left\{ \psi^* \tau(x_1, x_2, \ldots, x_N) \Sigma_i \left\{ \frac{1}{2M_i \sqrt{g(x)}} (p_\mu - a_\mu(x_i, \tau)) g^{\mu\nu}(x_i)(p_\nu - a_\nu(x_i, \tau)) \right\}
\]
\[
- a_5(x_1, x_2, \ldots, x_N, \tau) \right\} \psi \tau(x_1, x_2, \ldots, x_N)
\]
(104)
As for the one-particle case, we add an electromagnetic part
\[
S_{em} = + \int d\tau \Sigma_i \int d^4x_i \frac{1}{4\sqrt{g}} \left\{ f^{\mu\nu}(x_i, \tau) f_{\mu\nu}(x_i, \tau) \right\}
\]
\[
+ f^i_{\mu_5}(x_1, x_2, \ldots, x_N) f^{\mu_5 i}(x_1, x_2, \ldots, x_N) \right\}.
\]
(105)
The variation with respect to \( a_5 \) (taking into account the factor \( g^{\mu\nu} \) in raising the index) yields the equation of motion
\[
f^i_{\mu_5}(x_1, x_2, \ldots, x_N, \tau) = \psi^* \tau(x_1, x_2, \ldots, x_N) \psi \tau(x_1, x_2, \ldots, x_N)
\]
\[
= \Pi_i g(x_i)^{-\frac{1}{2}} \rho(x_1, x_2, \ldots, x_N, \tau)
\]
(106)
the density on the full space \( (x_1, x_2, \ldots, x_N) \) at each \( \tau \).

Finally, the variation with respect to \( a_\mu(x_i, \tau) \) of the interaction term, since \( f^{\mu\nu}(x, \tau) \) is a one particle quantity, the field equations
\[
f_{\mu\nu}(x_i, \tau) = j_\mu(x_i, \tau),
\]
(107)
where (the variation in \( a_\mu(x_i, \tau) \) fixes \( \tau \) but not the coordinates except for \( x_j \))
\[
j_\mu(x_i, \tau) = \Pi_{j \neq i} d^4x_j \sqrt{g(x_j)} \frac{1}{2M_i \sqrt{g(x_i)}} (\psi^* \tau(x_1, x_2, \ldots, x_N) g^{\mu\nu}(x_i)(\partial_\mu - ia_\mu(x_i)) \psi \tau(x_1, x_2, \ldots, x_N)
\]
\[
- ((\partial_\mu + a_\mu(x_i)) \psi^* \tau(x_1, x_2, \ldots, x_N)) g^{\mu\nu}(x_i) \psi \tau(x_1, x_2, \ldots, x_N)).
\]
(108)
The local one-particle field equations are then
\[ f_{\mu\nu}(x_i, \tau) = j_{\nu}(x_i, \tau) \]  
(109)
or, as in the one-particle case,
\[ \partial_{\mu}(\sqrt{g(x_i)}f_{\mu\nu}(x_i, \tau)) = \sqrt{g(x_i)}j_{\nu}(x_i, \tau). \]  
(110)

The Lorentz force acting on a particle in this many body framework is then
\[ F^\mu(x_i, \tau) = f^\mu_{\nu}(x_i, \tau)\frac{dx_i^\nu}{d\tau}, \]  
(111)
providing a basis, for example, for writing Vlasov equations in general relativistic statistical mechanics.

8. Summary and Outlook
We have shown that the SHP theory can be embedded by local coordinate transformations into the framework of general relativity. The Minkowski spacetime coordinates of the SHP theory are considered to lie in the tangent space of a manifold with metric and connection form derived from the coordinate transformations on the equations of motion for particles moving on the locally flat Minkowski spacetime, parametrized by a universal monotonic world time \( \tau \). The four momentum is well-defined on the manifold, and a formula for its \( \tau \) derivative, which may be understood as a “force”, is obtained, displaying the effect of the potential as well as the curvature (through the connection form).

For the many body system, each particle, at the points \( \{x_i^\mu\} \) is assumed to move locally on a flat Minkowski space, which is then transformed by local coordinate transformation to the manifold of GR with coordinates \( x^\mu \). Since particles with (flat Minkowski) coordinates \( \xi_1, \xi_2 \ldots \xi_N \) lie in different local tangent spaces at the points \( x_1, x_2 \ldots x_N \) of the curvilinear coordinatization of GR, Poincaré invariance of the potential function is not applicable.

Since the Poisson bracket of the SHP theory is unchanged in form under local diffeomorphisms, it forms the basis of a quantum theory in which the momentum operator generates infinitesimal translations along the local coordinates, and \( p_\mu \) is therefore represented by \( -i\frac{\partial}{\partial x^\mu} \) on a Hilbert space of functions \( \psi_\tau(x) \), square integrable over the invariant measure \( d^4x\sqrt{g(x)} \).

This Hilbert space provides a basis for a local \( U(1) \) gauge, for which the compensation fields (sections on the bundle) correspond to the classical 5D electromagnetic fields [21]. We obtain field equations for the electromagnetic fields and associated currents from an action (\( \tau \) integrated Lagrangian).

The many body quantum theory is treated by constructing a tensor product space and the associated electromagnetic theory is developed assuming that each factor in the tensor product carries the same gauge transformations. This enables us to define a gauge compensation field \( a_\mu(x) \) for the four components which can be evaluated on each particle, but due to the universality of \( \tau \), the fifth component must depend on the coordinates of all of the particles as a locally defined function on the full configuration space, similar to the function \( V(x_1, x_2 \ldots x_N) \) of the potential model.

The work of this paper is primarily restricted to describing a relativistic dynamics in a \( \tau \) independent gravitation field, \( i.e., \) the metric is assumed independent of \( \tau \). Since the connection form has the same structure as the usual GR, one can write Einstein’s equations in the same way [26]. Therefore, the energy momentum tensor, determining \( g^{\mu\nu}(x) \), should be, in this case, independent of \( \tau \). To achieve this, one may use partially integrated currents, taking into account
correlations \[27\], or the zero modes extracted from full integration yielding 4D conserved currents \[1\][22]. In the more general case, where the structure of spacetime is dynamical (for example, collision between black holes or for unstable stars such as supernova) the energy momentum tensor would depend on \(\tau\). We show in the Appendix how, in such cases, the corresponding explicit dependence of the local transformations from the Minkowski space coordinates to the curved coordinates on \(\tau\) can be expressed in terms of the now \(\tau\) dependent metric tensor.

The classical results of this paper provide an eight dimensional phase space for general relativity, just as the SHP theory provides for special relativity, and therefore a general relativistic statistical mechanics can be formulated. The assumptions necessary to construct Gibbs ensembles \[28\] in this context must be carefully examined.

The existence of a two body quantum theory makes accessible the possibility of constructing a quantum scattering theory, for which the formulation of asymptotic conditions, as for the construction of the Gibbs ensemble, poses an interesting challenge. The many body problem, generalizing the work of Horwitz and Arshansky \[29\], can be formulated in this framework. The many body Hilbert space can be used to construct a Fock space as the basis for a quantum field theory.

We finally remark that the results of the work of this paper provide a basis for the approach of Bekenstein and Sanders \[8\] and the associated discussion of nonabelian gauge fields given by Horwitz, Gershon and Schiffer \[7\]; it will therefore be of interest to follow the development of the \(U(1)\) gauge theory given here with a study of non-Abelian gauge theories (see \[23\]).

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**Appendix**

We study here the effect of a \(\tau\) evolving spacetime, a situation which would occur if the energy momentum tensor depends on \(\tau\). For example, using \(\tau\) dependent momenta.

Generally, at a point \(x^\lambda\), the velocity of a particle is \(\dot{x}^\lambda\), just a motion on the coordinates \(\{x\}\). If the spacetime is changing, we think of the tangent space as reflecting this change; the particle moves on \(x^\lambda(\tau)\), even as the meaning of \(x^\lambda\) changes, but at every point there is a change in the local flat tangent space coordinates. Therefore, \(\xi\) changes as the particle moves and as the world coordinates evolve. At each \(\tau\), it is still true that

\[
d\xi^\mu = \frac{\partial \xi^\mu}{\partial x^\lambda} dx^\lambda, \quad (A.1)
\]

but \(\frac{\partial \xi^\mu}{\partial x^\lambda}\) changes as the particle moves and as the spacetime evolves. We can write

\[
\frac{\partial \xi^\mu}{\partial x^\lambda} = \frac{\partial \xi^\mu}{\partial x^\lambda}(x(\tau), \tau) \quad (A.2)
\]

so that

\[
\frac{d}{d\tau} \frac{\partial \xi^\mu}{\partial x^\lambda} = \frac{\partial^2 \xi^\mu}{\partial x^\lambda \partial x^\sigma} ((x(\tau), \tau) \dot{x}^\sigma + \frac{\partial}{\partial \tau} \frac{\partial \xi^\mu}{\partial x^\lambda}(x(\tau), \tau), \quad (A.3)
\]

where the second term is due to the change in orientation of the \(\xi^\mu\) coordinates in \(\tau\).

The canonical structure postulated in (4) and (5), and the definition of \(g_{\mu\nu}\) remains the same but, as a function of \(\frac{\partial \xi^\mu}{\partial x^\lambda}(x(\tau), \tau)\), it now becomes an function of \(\tau\). We therefore have

\[
K = \frac{M}{2} g_{\mu\nu}(x(\tau), \tau) \dot{x}^\mu \dot{x}^\nu + V(x), \quad (A.4)
\]
We now calculate from (A.1) the total \( \tau \) derivative

\[
\ddot{\xi}^\mu = \frac{d}{d\tau}\left( \frac{\partial \xi^\mu}{\partial x^\lambda} \dot{x}^\lambda \right)
\]

\[
= \frac{\partial^2 \xi^\mu}{\partial x^\lambda \partial x^\gamma} \dot{x}^\gamma \dot{x}^\lambda + \frac{\partial}{\partial \tau} \frac{\partial \xi^\mu}{\partial x^\lambda} \dot{x}^\lambda + \frac{\partial \xi^\mu}{\partial x^\lambda} \ddot{x}^\lambda
\]

\[
= -\frac{1}{M^{\mu\nu\lambda}} \frac{\partial x^\lambda}{\partial \xi^\nu} \frac{\partial V(x)}{\partial x^\mu},
\]

where the last equality, as before, follows from the canonical structure of the tangent space.

Multiplying by \( \frac{\partial x^\sigma}{\partial \xi^\mu} \), and solving for \( \ddot{x}^\sigma \), we find a geodesic type equation as before but with an additional (apparently dissipative type) term

\[
\ddot{x}^\sigma = -\Gamma^\sigma_{\lambda\gamma} \dot{x}^\gamma \dot{x}^\lambda - \frac{1}{M^{\mu\nu\lambda}} \frac{\partial x^\lambda}{\partial \xi^\nu} \frac{\partial V(x)}{\partial x^\mu} \frac{\partial \xi^\mu}{\partial x^\lambda}.
\]

We now show that \( \frac{\partial}{\partial \tau} \frac{\partial \xi^\mu}{\partial x^\lambda} \) can be expressed in terms of \( \frac{\partial}{\partial \tau} g_{\mu\nu}(x(\tau), \tau) \), which then carries the information, from the Einstein equations, about the evolution of the spacetime.

From the definition (12) we compute

\[
\frac{\partial g_{\mu\nu}(x(\tau), \tau)}{\partial \tau} = \eta_{\sigma\gamma} \left[ \frac{\partial}{\partial \tau} \left( \frac{\partial \xi^\sigma}{\partial x^\mu} \right) \frac{\partial \xi^\gamma}{\partial x^\nu} + \frac{\partial \xi^\gamma}{\partial x^\mu} \left( \frac{\partial \xi^\sigma}{\partial x^\nu} \frac{\partial \xi^\mu}{\partial \tau} \right) \right],
\]

where we have used the \( \sigma, \gamma \) symmetry of \( \eta_{\sigma\gamma} \) in the second term.

Now, define

\[
\frac{\partial}{\partial \tau} \left( \frac{\partial \xi^\mu}{\partial x^\lambda} \right) \equiv t^\mu_{\lambda}.
\]

Then we can write (A.7) as

\[
\frac{\partial g_{\mu\nu}(x(\tau), \tau)}{\partial \tau} = \eta_{\sigma\gamma} \left[ t^\sigma_{\mu} \frac{\partial \xi^\gamma}{\partial x^\nu} + t^\nu_{\sigma} \frac{\partial \xi^\gamma}{\partial x^\mu} \right]
\]

\[
= \eta_{\sigma\gamma} \left( \delta^\gamma_{\lambda} \frac{\partial \xi^\sigma}{\partial x^\mu} + \delta^\sigma_{\lambda} \frac{\partial \xi^\gamma}{\partial x^\mu} \right) t^\mu_{\lambda}
\]

\[
= M_{\mu\nu\lambda} t^\sigma_{\lambda}.
\]

This equation can be inverted with the matrix \( N_{\lambda}^{\sigma\mu\nu} \) satisfying

\[
N_{\lambda}^{\sigma\mu\nu} M_{\mu\nu\lambda} = \delta^\lambda_{\sigma} \delta^\sigma_{\lambda}.
\]

It is a possible exceptional case that \( N_{\lambda}^{\sigma\mu\nu} \) could be singular; this would correspond to a singular development of the transformation function \( \frac{\partial \xi^\mu}{\partial x^\sigma} \) in \( \tau \), which we do not treat here.

Multiplying (A.9) by \( N_{\lambda}^{\sigma\mu\nu} \), we obtain

\[
N_{\lambda}^{\sigma\mu\nu} \frac{\partial g_{\mu\nu}(x(\tau), \tau)}{\partial \tau} = t^\sigma_{\lambda} = \frac{\partial}{\partial \tau} \left( \frac{\partial \xi^\sigma}{\partial x^\lambda} \right).
\]
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