Babai’s analysis of his algorithm [1] uses the Classification of Finite Simple Groups (CFSG) in Section 8. In particular, he uses Schreier’s Hypothesis (a well-known consequence of CFSG), which states that the outer automorphism groups of finite simple groups are solvable. We will make Babai’s bound worse, but still quasipolynomial, and remove CFSG entirely from the analysis. Our short proof relies on a little-known result of Wielandt [5].

More precisely, we give a CFSG-free proof of a weaker version of the key group-theoretic result of Babai, Lemma 8.5 from [1]. See Lemma 12.

All groups in this note are finite. Throughout log means logarithm to the base 2. We use the following notation:

- |G| - the order of a group G;
- Sym(Ω) - the symmetric group on the set Ω;
- Alt(n) - the alternating group of degree n;
- For G < Sym(Ω), G_x denotes the pointstabiliser of a point x ∈ Ω (G will always be transitive and all pointstabilisers equivalent as permutation groups);
- If G < Sym(Ω), and Δ is a subset of Ω fixed by G, then the image of the action of G on Δ is called the permutation group induced on Δ by G, and written G^Δ;
- Comp_A(G) - the set of isomorphism types of abelian composition factors of a group G. Note that we have |Comp_A(G)| ≤ log |G|.

For notation and definitions unexplained here and below see the monograph of Dixon and Mortimer [4].

We will first prove a useful result for groups of odd order (without using even the Feit-Thompson theorem). We call the orbits of a pointstabiliser in a transitive group suborbits. We need the following basic lemma of Jordan (see [12, 18.2]):

**Lemma 1.** Let G be a primitive group of degree n and G_x a pointstabiliser. If a prime p divides |G_x| then p divides |G_x^Δ| for all suborbits Δ of length at least 2.
Recall that the exponent of a group is the smallest common multiple of the orders of its elements.

**Lemma 2.** Let $G$ be a transitive group of degree $n$. If $G$ has odd order then the exponent of $G$ is at most $n^{(\log n)^2}$.

**Proof.** We will actually prove the following claim:

**Claim.** The product $\pi(G)$ of all different primes dividing $|G|$ is at most $n^{\log n}$.

Since any element of $G$ of prime power order has order at most $n$, the claim implies our statement.

Assume first that $G$ is primitive. Since $G$ has odd order, $G$ cannot be doubly transitive. The smallest non-trivial suborbit $\Delta$ has size $< \frac{n}{2}$. Lemma 1 implies that $\pi(G_x) = \pi(G_x^\Delta)$. It is clear that $\pi(G)$ divides $n \cdot \pi(G_x)$, hence by induction $\pi(G) \leq n \cdot \left(\frac{n}{2}\right)^{\log \frac{n}{2}} \leq n^{\log n}$.

Assume now that $G$ is imprimitive. Let $B_1$ be a minimal non-trivial block of imprimitivity (of size $b$) and $B_1, \ldots, B_k$ the corresponding system of blocks of imprimitivity. Then $G$ permutes these blocks transitively, let $K$ denote the kernel of this action. By induction $\pi(G/K) \leq k^{\log k}$.

We may assume that $K \neq 1$. Then the minimality of $B_1$ implies that $K$ acts transitively on each $B_i$, and the corresponding transitive groups are permutation equivalent. Hence $\pi(K) = \pi(K^{B_1}) \leq b^{\log b}$.

Using $kb = n$ we obtain that $\pi(G) \leq n^{\log n}$. This completes the proof. \[ \square \]

We now prove an easy result which implies that to obtain a polylogarithmic bound for $|\text{Comp}_A(G)|$ in terms of the degree of the transitive group $G$ it is sufficient to obtain such a bound for $|\text{Comp}_A(G_x)|$ (where $G_x$ is a pointstabiliser).

**Proposition 3.** Let $G < \text{Sym}(\Omega)$ be a transitive group of degree $n$ and $G_x$ a pointstabiliser. Then

$$|\text{Comp}_A(G) \setminus \text{Comp}_A(G_x)| < \log n.$$  

**Proof.** Let $G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \ldots$ be a composition chain of $G$. Then the subgroups $G_i \cap G_x$ form a subnormal chain $G_x \supseteq G_1 \cap G_x \supseteq G_2 \cap G_x \ldots$ (possibly with repetitions). Now $G/G_1 \geq G_1 \cdot G_x / G_1 \cong G_x / (G_1 \cap G_x)$ and similarly $G_i / G_{i+1} \geq G_{i+1} \cdot (G_i \cap G_x) / G_{i+1} \cong (G_i \cap G_x) / (G_{i+1} \cap G_x)$ for $i \geq 1$.

If $p$ is a prime not dividing $n$, then the product of the $p$-parts of the orders $|G_i / G_{i+1}|$ must be the same as the product of the $p$-parts...
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of the orders $|(G_i \cap G_x)/(G_{i+1} \cap G_x)|$. This implies that if $G$ has a composition factor of order $p$ then $G_x$ also has such a composition factor. Our statement follows.

□

The following result of Wielandt, that appeared first in [5] (which the author found in the insightful survey of Cameron [3]) explains why we have to concentrate on pointstabilisers of transitive groups, rather than on transitive groups themselves (see [3] p 110–111) for a more detailed explanation).

Since Wielandt’s beautiful result is at the heart of our argument, we will reproduce here his short proof from [5].

For this we need some more notation.

The orbits of $G \leq Sym(\Omega)$ on $\Omega \times \Omega$ are called the orbitals of $G$ on $\Omega$. For each orbital $\Delta$ there is a paired orbital denoted $\Delta'$, where $(y, x) \in \Delta'$ if and only if $(x, y) \in \Delta$. For each orbital $\Delta$ of $G$ and each $x \in \Omega$ we define $\Delta(x) = \{y \in \Omega | (x, y) \in \Delta\}$. Such sets are exactly the suborbits of $G_x$. If $\Delta$ and $\Delta'$ are paired orbitals, then $\Delta(x)$ and $\Delta'(x)$ are called paired suborbits. A suborbit $\Delta(x)$ is called self-paired if $\Delta(x) = \Delta'(x)$. (To simplify notation we often denote $\Delta(x)$ by $\Delta$.)

Lemma 4. (Wielandt) Let $G < Sym(\Omega)$ be a nonregular primitive group and let $\Delta(x)$ be a $G_x$-orbit $\neq \{x\}$, let $y$ be an element of $\Delta(x)$ and let $y'$ be an element of $\Delta'(x)$, where $\Delta'(x)$ is the $G_x$-orbit paired with $\Delta(x)$. Let $T(x)$ be the kernel of the action of $G_x$ on $\Delta(x)$. Then all composition factors of $T(x)$ appear as composition factors of $(G_{x,y})^{\Delta(x)}$ or of $(G_{x,y}')^{\Delta'(x)}$.

Proof. For a subgroup $H$ of $G$, denote by $H^*$ the smallest subnormal subgroup of $H$ such that every composition factor between $H$ and $H^*$ is a composition factor of $G_{x,y}$ or of $G_{x,y}'$; $H^*$ is a characteristic subgroup of $H$ (Wielandt [11] Th. 13, p. 220). Now $G_{x,y}^{\Delta(x)} \cong G_{x,y}/T(x)$ and therefore $G_{x,y}^{*} = T(x)^*$. Similarly $G_{x,y}'^{*} \cong U(x)^*$, where $U(x)$ denotes the pointwise stabiliser of $\{x\} \cup \Delta'(x)$. We can choose the notation so that $\Delta(x)^g = \Delta(x^g)$ for all $x \in \Omega, g \in G$. Then $\Delta'(x)^g = \Delta'(x^g)$ and $y \in \Delta(x)$ implies $x \in \Delta'(y)$ so $G_{x,y}^{*} = U(y)^*$, since a non-trivial normal subgroup of a primitive group must be transitive, and the theorem is proved. □

It was a great surprise to the present author that one can prove a useful result for pointstabilisers of arbitrary transitive groups by an
elementary induction argument (for transitive groups themselves using induction is quite standard). However a paper of Isaacs [6], where abelian pointstabilisers in transitive groups are shown to have a distinctive numerical property gave a strong indication that such an argument may exist and may not rely on CFSG. (The theorem of Isaacs in [6] in turn may be traced back to some results and questions in [2]). Note also that Theorem 7 below is not very hard to prove using well-known consequences of CFSG.

Definition 1. Let $D$ be a direct product of the groups $D_1, D_2, \ldots, D_t$. A subdirect product subgroup of $D$ is a subgroup $G$ which projects onto all the constituents $D_1, D_2, \ldots, D_t$.

The following well-known result (see [10, Ch.2 (4.19)]) describes the structure of subdirect product subgroups of two groups.

Lemma 5. Let $G$ be a subdirect product subgroup of $D = D_1 \times D_2$. Then setting $N_1 = D_1 \cap G$ and $N_2 = D_2 \cap G$ we have $N_1 \times N_2 \triangleleft G$, and $G/(N_1 \times N_2) \cong D_1/N_1 \cong D_2/N_2$.

Proposition 6. The composition factors of $G$ are among the composition factors of $D_1, D_2, \ldots, D_t$.

Proof. Easy induction using Lemma 5. \qed

The key result of this note is the following.

Theorem 7. Let $G$ be a transitive permutation group of degree $n$ and $G_x$ a pointstabiliser. Then $|\text{Comp}_A(G_x)| < 2(\log n)^2$.

The proof of Theorem 7 is based on two reduction lemmas: Lemma 8 and Lemma 10.

Definition 2. A group $G < \text{Sym}(\Omega)$ is called quasiprimitive if it is transitive and each of its non-trivial normal subgroups are also transitive. Primitive permutation groups are quasiprimitive (see [7] for more on quasiprimitive groups).

Lemma 8 (Reduction for transitive groups). Let $G$ be a transitive imprimitive group of degree $n$. We have

$$|\text{Comp}_A(G_x)| \leq |\text{Comp}_A(X_\alpha)| + |\text{Comp}_A(Y_\beta)| + \log n,$$

where $X$ and $Y$ are certain transitive groups acting on sets of size $t \geq 2$ and $m \geq 2$ respectively, where $tm \leq n$, and $X_\alpha$ and $Y_\beta$ are pointstabilisers.
Proof. a) Let us first assume that $G$ is quasiprimitive but not primitive. Since $G$ is not primitive $G_x$ is not a maximal subgroup of $G$. Consider a proper maximal subgroup $M$ of $G$ containing $G_x$. Let $N$ be the (unique) maximal normal subgroup of $G$ contained in $M$. If $N \neq 1$, then the quasiprimitivity of $G$ implies that $G = NG_x \leq M$, a contradiction. Hence the (primitive) representation of $G$ on the cosets of $M$ is faithful of degree, say, $t$.

Consider now the representation of $M$ on the cosets of $G_x$. The kernel of this action is a normal subgroup $K$ of $G_x$, hence all composition factors of $K$ appear among the composition factors of $M$, which by the above is the pointstabiliser of a transitive group of degree $|G : M| = t$. The “remaining” composition factors of $G_x$ are the composition factors of $G_x/K$, which is clearly a pointstabiliser of a transitive group of degree $|M : G_x| = n/t$.

b) Let us next assume that $G$ is a transitive group which has a nontrivial intransitive normal subgroup $N$. The orbits of $N$ form a system of imprimitivity $B = B_1, B_2, \ldots, B_t$ of $G$. Denote by $K$ the kernel of the action of $G$ on this system of blocks. Denote by $\bar{G}$ the image of this action and by $\bar{G}_B$ the stabiliser of the point $\bar{B}$. Clearly this stabiliser has index $t$ in $G$. Let $H$ be the inverse image of this stabiliser $\bar{G}_B$ in $G$, a subgroup of index $t$ in $G$. It is clear that $H$ contains $KG_b$ for an element $b$ of $B$. Since $K$ is transitive on $B$ we have $|B| = |K : K \cap G_b|$. But this is equal to $|KG_b : G_b|$ and therefore actually $H = KG_b$.

Now $H/K = KG_b/K$ is isomorphic to $G_b/(K \cap G_b)$ hence the composition factors in this quotient of $G_b$ are the composition factors of $\bar{G}_B$.

The remaining composition factors of $G_b$ are the composition factors of $K_b$. Let $J$ be the kernel of the action of $K_b$ on $B$. Now $J$ is also the kernel of the action of $K$ on $B$ hence a normal subgroup of $K$. Note that $K$ is a subdirect product subgroup of the $t$-th direct power of some transitive group $T$ of degree $n/t = m$.

Now $K_b/J$ is permutation equivalent to a pointstabiliser in $T$. By Proposition 6 the composition factors of $J$ are among the composition factors of $T$. By Proposition 3 at most $\log m$ of the composition factors of $T$ (hence $J$) do not occur among the composition factors of a pointstabiliser in $T$, hence the same is true for $K_b$.

$\square$
Recall that the socle $\text{Soc}(G)$ of a group $G$ is the product of its minimal normal subgroups. The following is well-known (see also [1]).

**Proposition 9.**

(1) The socle is a direct product of simple groups.

(2) The socle of a primitive permutation group is a direct product of isomorphic simple groups.

(3) If the socle of a primitive permutation group $G$ is abelian then it is elementary abelian of order $p^s = n$ and $G/\text{Soc}(G)$ embeds into $\text{GL}(s,p)$. In particular we have $|G/\text{Soc}(G)| < p^{(s^2)}$.

**Lemma 10** (Reduction for primitive groups). Let $G$ be a primitive group of degree $n$ with a self-paired suborbit $\Delta$. Then one of the following holds:

1. $|\text{Comp}_A(G_x)| \leq (\log n)^2$
2. $|\text{Comp}_A(G_x)| \leq |\text{Comp}_A(P_y)|$ for some primitive permutation group $P$ acting on $\Delta$ and some $y \in \Delta$.
3. $|\text{Comp}_A(G_x)| \leq |\text{Comp}_A(X_\alpha)| + |\text{Comp}_A(Y_\beta)| + 2\log n$, where $X$ and $Y$ are certain transitive groups acting on sets of size $t \geq 2$ and $m \geq 2$ respectively, where $tm \leq |\Delta| \leq n$, and $X_\alpha$ and $Y_\beta$ are pointstabilisers.

**Proof.** We may assume that $G$ is nonregular. Let $\Delta$ be the smallest non-trivial self-paired orbit of $G_x$. Let $P$ denote the image of the action of $G_x$ on $\Delta$. By Lemma 1

(*) $\text{Comp}_A(G_x) \subseteq \text{Comp}_A(P) \cup \text{Comp}_A(P_y)$.

Assume first that $P$ is primitive. If $\text{Soc}(P)$ is abelian, then using Proposition 9(3) it is easy to see that (1) holds. If $\text{Soc}(P)$ is non-abelian, then $\text{Comp}_A(P) = \text{Comp}_A(P/\text{Soc}(P))$. Clearly $\text{Soc}(P) \cdot P_y = P$, hence

$P/\text{Soc}(P) \cong P_y/(P_y \cap \text{Soc}(P))$.

In particular, $\text{Comp}_A(P) \subseteq \text{Comp}_A(P_y)$, and (*) implies (2).

Assume next that $P$ is imprimitive. Proposition 9 and (*) imply that

$|\text{Comp}_A(G_x)| \leq |\text{Comp}_A(P_y)| + \log n$.

Applying Lemma 8 to $P$ acting on $\Delta$ we obtain (3). \qed

The proof of Theorem 7. The proof of the theorem will be simple induction based on Lemma 8 and Lemma 10.

Let us first assume that $G$ is primitive. If $G$ has odd order then our statement follows from the Claim in the proof of Lemma 2. Otherwise $G$ contains an element of order two, hence it has a self-paired suborbit [4, 3.2.5], and we can apply Lemma 10.
In case (1) and (2) of Lemma 10 our statement follows. In case (3) we may assume that $m \geq t$, and set $k = n^t$. Note that $k \geq \sqrt{n}$ and $t \geq 2$. We have $2(\log(tk))^2 = 2(\log t)^2 + 2(\log k)^2 + 4 \log t \log k \geq 2(\log t)^2 + 2(\log k)^2 + 2 \log n \geq |\text{Comp}_A(X_\alpha)| + |\text{Comp}_A(Y_\beta)| + 2 \log n \geq |\text{Comp}_A(G_\delta)|$. The inductive step follows.

The inductive step for transitive imprimitive groups (using Lemma 8) is similar, but easier. □

Next we will use Theorem 7 to give a CFSG-free proof of a weaker version of the key group-theoretic result of Babai [1]. We will also rely on the following well-known fact.

**Proposition 11.** The sum of the first $m$ primes is $O(m^2 \log m)$.

See [9] for a recent reference concerning sharp bounds.

Here is the promised weaker version of Lemma 8.5 of [1].

**Lemma 12.** Let $G$ be a primitive group of degree $n$. Assume that $\varphi : G \to \text{Alt}(k)$ is an epimorphism, where $k > (\log n)^5$. If $k$ is large enough then $\varphi$ is an isomorphism; hence $G \cong \text{Alt}(k)$.

**Proof.** Assume that $N = \ker(\varphi)$ is non-trivial. Then $N$ is transitive and $G/N \cong \text{Alt}(k)$ contains a cyclic subgroup $C$ whose order is a product of $c\sqrt{k/\log k}$ different primes (for some absolute constant $c > 0$). This follows easily using Proposition 11.

The extension $H$ of $N$ by $C$ is transitive (since $N$ itself is transitive) and we have $|\text{Comp}_A(H)| \geq c\sqrt{k/\log k}$, which contradicts Theorem 7 for $k$ large enough. □

**Remark 13.** Babai [1] proves the same statement for $k \geq \max\{8, 2 + \log n\}$ using Schreier’s Hypothesis.

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