Backward stochastic partial differential equations with quadratic growth✩

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Abstract
This paper is concerned with the existence and uniqueness of weak solutions to the Cauchy-Dirichlet problem of backward stochastic partial differential equations (BSPDEs) with nonhomogeneous terms of quadratic growth in both the gradient of the first unknown and the second unknown. As an example, we consider a non-Markovian stochastic optimal control problem with cost functional formulated by a quadratic BSDE, where the corresponding value function satisfies the above quadratic BSPDE.

Keywords: Backward stochastic partial differential equations, quadratic growth, change of variables, weak solutions, stochastic HJB equations.

1. Introduction
Denote by \( \mathcal{T} \) the fixed time duration \([0,T]\). Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{T}}, \mathbb{P})\) be a complete filtered probability space on which a \(d_0\)-dimensional standard Wiener process \(W_t = (W^1_t, \ldots, W^{d_0}_t)'\) is defined such that \(\{\mathcal{F}_t\}_{t\in\mathcal{T}}\) is the natural filtration generated by \(W\) and augmented by all the \(\mathbb{P}\)-null sets in \(\mathcal{F}\). We denote by \(\mathcal{P}\) the predictable \(\sigma\)-algebra associated with \(\{\mathcal{F}_t\}_{t\in\mathcal{T}}\).

In this paper we consider the Cauchy-Dirichlet problem for the following parabolic quadratic backward stochastic partial differential equation (BSPDE in short)
\[
du = -\left[ (a^{ij} u_{x^j} + \sigma^{ik} q^k)_{x^i} + f(t,x,u,u_{x^i},q) \right] dt + q^kdW^k_t, \tag{1.1}
\]
with the terminal-boundary condition
\[
\begin{aligned}
  u(t,x) &= 0, \quad t \in \mathcal{T}, \ x \in \partial \mathcal{D}, \\
  u(T,x) &= \varphi(x), \quad x \in \mathcal{D}.
\end{aligned} \tag{1.2}
\]

Here \(\mathcal{D}\) is a simply connected and bounded region in the Euclidean space \(\mathbb{R}^d\) and we use the convention that repeated indices imply summation. By the terminology “super-parabolic” (resp., “degenerate”) we mean the condition that there exist positive constants

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κ and K such that
\[
\kappa I_d + (\sigma^{ik})(\sigma^{jk})^* \leq 2(a^{ij}) \leq K I_d.
\]
(resp., \(2a^{ij} - (\sigma^{ik})(\sigma^{jk})^* \geq 0\))

And “quadratic” means that
\[
|f(t, x, v, p, r)| \leq \lambda_0(t, x) + \lambda_1|v| + \gamma(|v|)(|p|^2 + |r|^2),
\]
for some positive constant \(\lambda_1\), bounded predictable field \(\lambda_0\), and increasing function \(\gamma(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+\). We refer to \((f, \varphi)\) as the parameters of BSPDE (1.1)-(1.2).

BSPDEs are generalized backward stochastic differential equations (BSDEs in short) with values in function spaces. Linear BSDEs were initiated by Bismut [2] as the adjoint equations in 1973 when he studied the stochastic maximum principle of stochastic optimal control problems. In 1990, Pardoux and Peng [28] introduced the general nonlinear BSDEs with Lipschitz continuous generators. In the last two decades, extensive research on such kind of equations has indicated that BSDEs can serve as a powerful tool in many fields such as mathematical finance, stochastic control, and partial differential equations (PDEs in short). See among others [17], [27], [29], [30]. Since this paper is inspired by the study of quadratic BSDEs, a kind of BSDEs with generators of quadratic growth in the martingale term, we mainly introduce the development in this direction. The motivation of studying quadratic BSDEs derives from the feedback representation of the optimal control in the setting of linear quadratic stochastic control problem where the related backward stochastic Riccati equation (BSRE in short) turns out to be a quadratic BSDE. Bismut [3] first considered BSREs in a special case where the generator depends on the second unknown in a linear way. Later Peng [31] applied Bellman’s principle of quasi-linearization to deal with relatively general BSREs. In 2000, Kobylanski [18] developed a quite useful technique, the idea of which is from the Cole-Hopf transformation in PDE theory, to overcome the difficulty from the quadratic growth of the generator in the martingale term and obtained the existence result in one-dimensional case. Numerous literatures later were devoted to solving the challenging problem concerning the existence and uniqueness of solutions to BSREs in multidimensional case, among which we refer [3], [19], [20], [21], [22], [23] and the references therein to show the theoretical developments. Until 2003, Tang [33] gave a complete solution to this long standing problem by a new constructive method. In addition, general BSDEs with quadratic features are also applied to describe the value functions and corresponding optimal trading strategies in utility maximization problems (see e.g. [15], [33]) and appear naturally in the study of the BSDEs on manifolds (see e.g. [4], [5]). For the recent theoretical progress on the existence and uniqueness of solutions to quadratic BSDEs, one can refer to [6], [7].

As the infinite dimensional counterparts of BSDEs, BSPDEs also arise from stochastic optimal control theory. For instance, they serve as the adjoint equations in the formulation of the stochastic maximum principle for controlled stochastic differential equations (SDEs) with partially observed information (see e.g. [1], [34]) or controlled stochastic parabolic partial differential equations (see e.g. [26], [40]). Value functions of the stochastic optimization problem of controlled non-Markovian SDEs, according to Bellman’s optimal principle and Itô-Wentzell’s formula, have been shown to satisfy the so-called stochastic Hamilton-Jacobi-Bellman (HJB) equations, a class of fully nonlinear BSPDEs (see e.g.
As for their important applications to issues from financial models with random parameters, we refer to [14] and [24].

The theory of BSPDEs is more rich since such equations have features of both BSDEs and PDEs. The theory on existence, uniqueness and regularity of solutions to Cauchy problem of BSPDEs is fairly complete. See [11], [39], [10] for non-degenerate BSPDEs, and [13], [16], [25], [36] for the more difficult degenerate case. However, discussions on Cauchy-Dirichlet problem are relatively less, and one can refer to [12] and [37]. Methods mainly applied to handle BSPDEs include: techniques of semigroup of operators in the case of BSPDEs with deterministic coefficients, adjoint arguments closely related to the theory of forward SPDEs, probabilistic representation methods depending on the theory of forward-backward stochastic differential equations (FBSDEs), and PDE’s techniques, such as frozen coefficient method and continuation method. The last two methods are proved to be powerful to handle degenerate BSPDEs. As indicated in Remark 4.1, our approaches and results can be easily extended to the case of the whole space \( \mathbb{R}^d \), that is, the Cauchy problem.

The rest of this paper is organized as follows. In section 2 we introduce some notations and preliminary results. Section 3 and section 4 are devoted to the existence and uniqueness of solutions to the Cauchy-Dirichlet problems of quadratic BSPDEs, respectively. In section 5, an example of quadratic BSPDEs is demonstrated in the context of a stochastic optimal control problem with cost functional formulated by a quadratic BSDE.

2. Notations and preliminaries

For a given Banach space \( \mathcal{B} \) and a constant \( p \in [1, \infty] \), we denote by \( L^p_{\mathcal{B}}(\Omega \times \mathcal{T}; \mathcal{B}) \) the space of all \( \mathcal{B} \)-valued predictable processes \( X : \Omega \times \mathcal{T} \to \mathcal{B} \) such that \( \mathbb{E} \int_0^\infty \| X_t \|^2_{\mathcal{B}} dt < \infty \). We also denote by \( C(\mathcal{T}; \mathcal{B}) \) the space of all \( \mathcal{B} \)-valued continuous adapted processes \( X : \Omega \times \mathcal{T} \to \mathcal{B} \) such that \( \mathbb{E} \sup_{t \in \mathcal{T}} \| X_t \|^2_{\mathcal{B}} < \infty \) and by \( L^p(E) \) the space of all real valued measurable functions \( f \) defined on a measure space \( (E, \mathcal{E}, \mu) \) such that \( \int_E |f|^p d\mu < \infty \). For simplicity we denote

\[ L^p := L^p_{\mathcal{B}}(\Omega \times \mathcal{T}; L^p(\mathcal{D})). \]

For a vector \( q \in \mathbb{R}^d \), \( q^i \) means its \( i \)-th component, \( i = 1, 2, \cdots, d \). For a function \( u \) defined on \( \mathbb{R}^d \), \( u_{x^i} \) or \( D_i u \) means the derivative of \( u \) with respect to \( x^i \). \( u_x \) or \( Du \) stands for the the gradient of \( u \), and \( D^2 u \) stands for the Hessian of \( u \).

For a integer \( m \), we simply denote by \( H^m(\mathcal{D}) \) and \( H^m_0(\mathcal{D}) \) the Sobolev spaces \( W^{m,2}(\mathcal{D}) \) and \( W^{m,2}_0(\mathcal{D}) \), respectively, with inner product \( \langle \cdot , \cdot \rangle_m \). With the above notations, we
simply denote
\[
H^m(D) := L^2_\mathcal{P}(\Omega \times \mathcal{T}; H^m(D)), \quad m = -1, 0, 1, 2, \ldots,
\]
\[
H^n_0(D) := L^2_\mathcal{P}(\Omega \times \mathcal{T}; H^n_0(D)), \quad n = 1, 2, 3, \ldots,
\]
\[
H^m(D; \mathbb{R}^{d_0}) := (H^m(D))^{d_0}.
\]
And we denote by \(C_0^\infty(D)\) the space of infinitely differential real functions with compact support defined on \(D\).

We first introduce the notation of weak solution to the BSPDE (1.1).

Definition 2.1. A pair of random fields \((u, q) \in H^1_0(D) \times H^0(D; \mathbb{R}^{d_0})\) is said to be a weak solution to BSPDE (1.1) if for every \(\eta \in C_0^\infty(D)\),

\[
\int_D u(t, x)\eta(x)dx = \int_D \varphi(x)\eta(x)dx + \int_t^T \int_D \left[ -\left( a^{ij}(s, x)u_{x^j} + \sigma^{jk}(s, x)q^k(s, x) \right)\eta_{x^i}(x) + f(s, x, u(s, x), u_{x^i}(s, x), q(s, x))\eta(x) \right] dxds - \int_t^T \int_D q^k(s, x)\eta(x)dxdW^k_s, \quad d\mathbb{P} \times dt \text{-a.e.}
\]

We present a generalized Itô’s formula and a comparison principle for weak solutions to BSPDEs, the proof of which one can refer to [9] or [32].

Lemma 2.1. Suppose \(f^0 \in L^1(D)\), \(f^i, q^k \in H^0(D)\), \(i = 1, \ldots, d\), \(k = 1, \ldots, d_0\), and \(u \in H^1_0(D) \cap C(\mathcal{T}; L^2(D))\). If for any \(\eta \in C_0^\infty(D)\),

\[
(u(t), \eta)_0 = (u(T), \eta)_0 + \int_t^T \left[ (f^0(s), \eta_0 - (f^i(s), \eta_{x^i})) ds - \int_t^T \psi(s, x)\eta_k dW^k_s \right], \quad \forall t \in \mathcal{T}, \ a.s.,
\]

it holds that for every \(\psi\) such that \(\psi'\) and \(\psi''\) are bounded and \(\psi'(0) = 0\),

\[
\int_D \psi(u(t, x))dx - \int_D \psi(u(T, x))dx = \int_t^T \int_D \left[ \psi f^0 - \psi''(u)u_{x^i}f^i - \frac{1}{2}\psi''\right](s, x)dxds
\]

\[
- \int_t^T \int_D \psi(s, x)dxdW^k_s, \quad \forall t \in \mathcal{T}, \ a.s. \tag{2.2}
\]

Lemma 2.2. Let \((u_1, q_1), (u_2, q_2) \in H^1_0(D) \times H^0(D; \mathbb{R}^{d_0})\) be weak solutions to BSPDEs with parameters \((f_1, \varphi_1)\) and \((f_2, \varphi_2)\), respectively. Assume

(i) For any \((v, p, r) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d_0}\), \(f_i(\cdot, \cdot, \cdot, v, p, r)\) is \(\mathcal{P} \times \mathcal{B}(D)\) measurable, \(i = 1, 2\). Moreover, there exists a constant \(L > 0\) such that for any \((v_1, p_1, r_1), (v_2, p_2, r_2) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d_0}\),

\[
|f_i(t, x, v_1, p_1, r_1) - f_i(t, x, v_2, p_2, r_2)| \leq L(|v_1 - v_2| + |p_1 - p_2| + |r_1 - r_2|), \quad \forall (\omega, t, x) \in \Omega \times \mathcal{T} \times D, \quad i = 1, 2.
\]

(ii) \(f_i(\cdot, 0, 0, 0) \in H^0(D)\), \(i = 1, 2\).

(iii) \(\varphi_i : \Omega \times D \rightarrow \mathbb{R}\) is \(\mathcal{F}_T \times \mathcal{B}(D)\) measurable, and \(\varphi_i \in L^2(\Omega \times D)\), \(i = 1, 2\).

If \(\varphi_1 \leq \varphi_2\) and \(f_1 \leq f_2\), we have \(u_1 \leq u_2\).
Using a similar procedure in Proposition 3.2 we have

**Corollary 2.3.** Let the parameters \((f, \varphi)\) of BSPDE (1.1)-(1.2) satisfy the assumptions in Lemma 2.2 and let \((u, q)\) be a weak solution to BSPDE (1.1)-(1.2) with parameters \((f, \varphi)\). Suppose \(\zeta : \mathcal{T} \to [0, \infty)\) satisfies the ODE \(\zeta(t) = -g(t, \zeta(t))\). Then, if \(f(\omega, t, x, \zeta(t), 0, 0) \leq g(t, \zeta(t))\), we have
\[
 u(t, x) \leq \zeta(t), \quad d\mathbb{P} \times dx \text{ a.e., } \forall t \in \mathcal{T}.
\]

Finally, we give a simple but useful result, which will be used frequently in the subsequent argument.

**Lemma 2.4.** Let \(\mu_0 = \frac{\kappa}{1+2R}\). Then for any vectors \(p \in \mathbb{R}^d\) and \(r \in \mathbb{R}^{d_0}\), it holds that
\[
 2a^{ij}p_ip_j + 2\sigma^{ik}p_ip^kr^k + |r|^2 \geq \mu_0(|p|^2 + |r|^2).
\]

### 3. The existence of solutions

Throughout this paper we always assume that coefficients \(a^{ij} = a^{ji}\) and \(\sigma^{ik}\) are \(\mathcal{B}(\mathcal{D})\) measurable and bounded functions, \(i, j = 1, \ldots, d, \; k = 1, \ldots, d_0\). As for coefficients \(f\) and \(\varphi\), we assume in this section

\((H1)\): (i) For every \((v, p, r) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+^d\), \(f(\cdot, \cdot, \cdot, v, p, r)\) is \(\mathcal{B}(\mathcal{D})\) measurable. And for every \((\omega, t, x)\), \(f\) is continuous with respect to \((v, p, r)\).

(ii) There exist a positive function \(\lambda_0 \in \mathbb{L}^\infty\) and a increasing function \(\gamma(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+\) such that for every \((\omega, t, x, v, p, r)\),
\[
 |f(t, x, v, p, r)| \leq \lambda_0(t, x) + \lambda_1|v| + \gamma(|v|)(|p|^2 + |r|^2).
\]

\((H2)\): \(\varphi : \Omega \times \mathcal{D} \to \mathbb{R}\) is \(\mathcal{F}_T \times \mathcal{B}(\mathcal{D})\) measurable and \(\varphi \in L^\infty(\Omega \times \mathcal{D})\) \(\cap \mathbb{L}^2(\Omega \times \mathcal{D})\).

The main theorem of this section is

**Theorem 3.1.** Suppose (1.3), (H1) and (H2) hold. Then there exists a weak solution \((u, q) \in \mathbb{H}_0^1(\mathcal{D}) \times \mathbb{H}_0^0(\mathcal{D}; \mathbb{R}_+^d)\) to BSPDE (1.1)-(1.2), and \(u \in L^2(\Omega; C(\mathcal{T}; L^2(\mathcal{D})))\) \(\cap \mathbb{L}^\infty\).

#### 3.1. Boundedness and convergence

To prove Theorem 3.1, we need to establish a prior estimates. To the end of this subsection, we first strengthen the condition (ii) in (H1) to the case
\[
 |f(t, x, v, p, r)| \leq \lambda_0(t, x) + \lambda_1|v| + \lambda\mu_0(|p|^2 + |r|^2),
\]
where \(\lambda\) is a positive constant.

**Proposition 3.2.** Let (1.3), (3.1) and (H2) be satisfied. Suppose \((u, q) \in \mathbb{H}_0^1(\mathcal{D}) \times \mathbb{H}_0^0(\mathcal{D}; \mathbb{R}_+^d)\) is a weak solution to BSPDE (1.1)-(1.2) and in addition \(u \in C(\mathcal{T}; L^2(\mathcal{D}))\) \(\cap \mathbb{L}^\infty\). Then,
\[
 \|u(t, \cdot)\|_{L^\infty(\Omega \times \mathcal{D})} \leq \frac{\|\lambda_0\|_{\mathbb{L}_\infty}}{\lambda_1}(e^{\lambda_1(T-t)} - 1) + e^{\lambda_1(T-t)}\|\varphi\|_{L^\infty(\Omega \times \mathcal{D})}, \; \forall t \in \mathcal{T}.
\]  
Moreover, there exists a constant \(C_1\) depending only on \(\|u\|_{L^\infty}, \|\varphi(x)\|_{L^2(\Omega \times \mathcal{D})}, \|\lambda_0\|_{L^2}, \mu_0, \lambda, \lambda_1\) and \(T\), such that
\[
 \|u_x\|_{\mathbb{H}_0^0(\mathcal{D})}^2 + \|q\|_{\mathbb{H}_0^0(\mathcal{D})}^2 \leq C_1.
\]
The next result shows that the existence of solution to BSPDE (3.1)-(3.2) can be obtained by an approximation scheme.

**Proposition 3.3.** Suppose that a sequence of functions \((f^n)_{n \geq 1}\) and \(f\) satisfy (H1) and that a sequence of functions \((\varphi^n)_{n \geq 1}\) and \(\varphi\) satisfy (H2). Furthermore, we assume

(a) For every \((\omega, t, x)\), the sequence \((f^n)_n\) converges to \(f\) on \(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d\) locally uniformly and the sequence \((\varphi^n)_n\) converges to \(\varphi\) in \(L^\infty(\Omega \times D)\) as \(n \to \infty\).

(b) There exist a positive constant \(\lambda\) and a function \(\lambda_2 \in L^\infty \cap L^2\) such that
\[
|f^n(t, x, v, p, r)| \leq \lambda_2(t, x) + \lambda \nu_0(|p|^2 + |r|^2), \quad \forall n \in \mathbb{N}, \ \forall (\omega, t, x, v, p, r).
\]

(c) For every \(n \in \mathbb{N}\), BSPDE with parameters \((f^n, \varphi^n)\) has a weak solution \((u^n, q^n)\) in \(H_0^1(D) \times \mathbb{H}^0(D; \mathbb{R}^d)\), \(u^n \in L^2(\Omega; C(T; L^2(D))) \cap L^\infty\), and \((u^n)_n\) is a monotone sequence. Moreover, there exists a positive constant \(M\) such that \(\|u^n\|_{L^\infty} \leq M\) for all \(n \in \mathbb{N}\).

Then, BSPDE (3.1)-(3.2) has a weak solution \((u, q)\) in \(H_0^1(D) \times \mathbb{H}^0(D; \mathbb{R}^d)\) which satisfies
\[
\lim_{n \to \infty} u^n = u \text{ uniformly on } \Omega \times T \times D,
\]
\[
\lim_{n \to \infty} q^n = q \text{ in } \mathbb{H}^0(D; \mathbb{R}^d),
\]
and moreover \(u \in L^2(\Omega; C(T; L^2(D))) \cap L^\infty\).

The proofs of the above two Propositions are both technical and lengthy and thus are arranged in the appendix section.

### 3.2. Change of variables

This section is devoted to the change of variables between two weak solution. To be more precise, we justify that the exponential change of variables of a weak solution to some BSPDE is also a weak solution to another corresponding BSPDE. This technique is crucial in the proof of Theorem 3.1 in the next section.

We consider the following equation
\[
\begin{align*}
\frac{dv}{t \times \partial D} = 0, & \quad v(T) = e^{\lambda \varphi} - 1, \\
\end{align*}
\]
where \(F\) is a function defined on \(\Omega \times T \times D\). Set
\[
u = \frac{1}{\lambda} \ln(v + 1), \quad q = \frac{r}{(\nu + 1)}. \tag{3.5}
\]
\[
\text{Applying Itô's formula formally to } u, \text{ we obtain } (u, q) \text{ satisfies the BSPDE}
\]
\[
\begin{align*}
\frac{du}{t \times \partial D} = 0, & \quad u(T) = \varphi, \\
\end{align*}
\]
where
\[
f(t, x, u, u_x, q) = \lambda e^{-\lambda u} F(t, x) + \lambda (a^{ij} u_{x^i} + \sigma^{ij} q^k) u_{x^j} + \frac{1}{2} |q|^2. \tag{3.7}
\]

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Lemma 3.4. Suppose $F$ and $\varphi$ are both bounded functions. Let $(v, r)$ be a weak solution to BSPDE (3.4) and satisfy $0 < \gamma \leq v + 1 \leq \Gamma < \infty$, where $\gamma$, $\Gamma$ are constants. Then the pair of random fields $(u, q)$ defined by (3.5) is a weak solution to BSPDE (3.6).

Proof. For any given test function $\eta \in C_0^\infty(D)$, set $K := \text{supp}(\eta)$, $\varepsilon_0 = \text{dist}(K, \partial D)$.

We further choose a nonnegative function $\zeta \in C_0^\infty(D)$ such that:

$$\supp(\zeta) \subset \{|x| \leq 1\}, \quad \int_{\mathbb{R}^d} \zeta(x)dx = 1,$$

and define

$$\zeta^\varepsilon(x) = \varepsilon^{-d} \zeta(x/\varepsilon), \quad \forall \varepsilon \in (0, \varepsilon_0),$$

$$v^\varepsilon(x) = \zeta^\varepsilon * v(x), \quad \forall x \in K,$$

$$r^\varepsilon(x) = \zeta^\varepsilon * r(x), \quad \forall x \in K.$$

Since $(v, r)$ is a weak solution to equation (3.4), we know from the definition of $(v^\varepsilon, r^\varepsilon)$ that the pair $(v^\varepsilon, r^\varepsilon)$ satisfies

$$v^\varepsilon(t, x) = v^\varepsilon(T, x) + \int_t^T \left\{ D_i [\zeta^\varepsilon * (a^{ij} D_j v + \sigma^{ik} r^k)] + \zeta^\varepsilon * F \right\}(s, x)ds$$

$$- \int_t^T r^\varepsilon, k(s, x)dW^k_s, \quad \forall(t, x) \in T \times K.$$

Setting

$$u^\varepsilon = \frac{1}{\lambda} \ln(v^\varepsilon + 1), \quad q^\varepsilon = \frac{r^\varepsilon}{\lambda(v^\varepsilon + 1)},$$

then we have $u^\varepsilon_x = \frac{v^\varepsilon}{\lambda(v^\varepsilon + 1)}$. Applying Itô’s formula to $u^\varepsilon$, we get

$$u^\varepsilon(t, x) - u^\varepsilon(T, x)$$

$$= \int_t^T \frac{1}{\lambda(v^\varepsilon + 1)} \cdot \left\{ D_i [\zeta^\varepsilon * (a^{ij} D_j v + \sigma^{ik} r^k)] + \zeta^\varepsilon * F \right\}(s, x)ds$$

$$+ \int_t^T \frac{\lambda}{2} |q^\varepsilon(t, x)|^2 dt - \int_t^T q^\varepsilon(s, x)dW_s, \quad \forall(t, x) \in T \times K.$$

Multiplying $\eta$ on both sides of the above equality and integrating over $K$, applying Fubini’s theorem and the fact that $K = \text{supp}(\eta)$, we have

$$\int_K u^\varepsilon(t, x)\eta(x)dx - \int_K u^\varepsilon(T, x)\eta(x)dx$$

$$= \int_t^T \int_K \frac{1}{\lambda(v^\varepsilon + 1)} \cdot \left\{ D_i [\zeta^\varepsilon * (a^{ij} D_j v + \sigma^{ik} r^k)] + \zeta^\varepsilon * F \right\}(s, x)\eta(x)dxdxds$$

$$(3.8)$$

$$+ \frac{\lambda}{2} \int_t^T \int_K |q^\varepsilon(s, x)|^2 \eta(x)dxdxds - \int_t^T \int_K q^\varepsilon(s, x)\eta(x)dxdW_s.$$
Green’s formula yields
\[
\int_K u^\varepsilon(t, x)\eta(x) \, dx - \int_K u^\varepsilon(T, x)\eta(x) \, dx = -\int_t^T \int_K \left[ \xi^\varepsilon * (a^{ij} v_{x^j} + \sigma^{ik} r^k) \right] \frac{\partial}{\partial x^i} \left[ \frac{\eta}{\lambda(v^\varepsilon + 1)} \right] (s, x) \, dx \, ds
\]
\[
+ \int_t^T \int_K \left\{ \frac{1}{\lambda(v^\varepsilon + 1)} \left( \xi^\varepsilon * F \right) + \frac{\lambda}{2} |q|^2 \right\} (s, x)\eta(x) \, ds - \int_t^T \int_K q^\varepsilon(s, x)\eta(x) \, dx \, dW_s. \tag{3.9}
\]

In what follows we will take limits as \( \varepsilon \to 0 \) on both sides of (3.9). First, for every \( h \in L^p(\mathbb{R}^d) \) with \( p \in [1, \infty) \), \( \xi^\varepsilon \ast h \) converges strongly to \( h \) in \( L^p(\mathbb{R}^d) \) as \( \varepsilon \to 0 \). Moreover, we know that

(i) It is easy to verify that \( \lambda^{-1} \ln \gamma \leq u \leq \lambda^{-1} \ln \Gamma \);  
(ii) The fact that \( (v, r) \) is a weak solution to equation (3.4) indicates that \( v \in \mathbb{H}^1(\mathcal{D}), r \in \mathbb{H}^0(\mathcal{D}) \), which implies \( a^{ij} v_{x^j} + \sigma^{ik} r^k \in \mathbb{H}^0(\mathcal{D}) \);  
(iii) Since \( \varepsilon \to 0 \) in the equality (3.9), we have

\[
\int_K u(t, x)\eta(x) \, dx - \int_K u(T, x)\eta(x) \, dx = -\int_t^T \int_K (a^{ij} v_{x^j} + \sigma^{ik} r^k) \frac{\partial}{\partial x^i} \left[ \frac{\eta}{\lambda(v^\varepsilon + 1)} \right] (s, x) \, dx \, ds
\]
\[
+ \int_t^T \int_K \left\{ \frac{F}{\lambda(v^\varepsilon + 1)} + \frac{\lambda}{2} |q|^2 \right\} (s, x)\eta(x) \, ds - \int_t^T \int_K q(s, x)\eta(x) \, dx \, dW_s. \tag{10.9}
\]

Substituting \( e^{\lambda u} - 1 \) and \( \lambda e^{\lambda u} q \) for \( v \) and \( r \) respectively, we have

\[
\int_D u(t, x)\eta(x) \, dx - \int_D u(T, x)\eta(x) \, dx = -\int_t^T \int_D (a^{ij} v_{x^j} + \sigma^{ik} r^k)(s, x)\eta_{x^j}(x) \, dx \, ds
\]
\[
+ \int_t^T \int_D f(s, x, u, u_x, q)\eta(x) \, dxds - \int_t^T \int_D q(s, x)\eta(x) \, dx \, dW_s, \tag{10.11}
\]

where \( f(s, x, u, u_x, q) \) is given in (3.7).

We can deduce from the arbitrariness of \( \eta \) in (3.11) that \((u, q)\) is a weak solution to equation (3.6). \( \square \)

3.3. Proof of Theorem 3.1

We are now in a position to prove Theorem 3.1. We first assume (3.1) holds, i.e.,

\[ |f(t, x, v, p, r)| \leq \lambda_0(t, x) + \lambda_1 |v| + \lambda \mu_0(|p|^2 + |r|^2). \]

Denote \( M = \frac{\|\lambda\|_{L^\infty}}{\lambda_1}(e^{\lambda T} - 1) + e^{\lambda T} \|\varphi\|_{L^\infty(\Omega \times \mathcal{D})} \). By Proposition 3.2 if \((u, q)\) is a weak solution to BSPDE (1.1)-(1.2) and \( u \) is bounded, we have \( u(\cdot) \leq M \).

Set
\[ v = e^{2\lambda u} - 1, \quad r = 2\lambda e^{2\lambda u} q. \]
Then \((v, r)\) formally satisfies
\[
\begin{aligned}
\left\{
\begin{array}{l}
dv = -\left[(a^{ij}v_{x^j} + \sigma^{ik}r_{x^i})_{x^j} + F(t, x, v, v_r, r)\right] dt + r^k dW^k_t, \\
v|_{T \times \partial \mathcal{D}} = 0, \quad v(T) = e^{2\lambda \varphi} - 1,
\end{array}
\right.
\end{aligned}
\tag{3.12}
\]
where
\[
F(t, x, v, p, r) = 2\lambda(v + 1)f\left(t, x, \frac{1}{2\lambda} \ln(v + 1), \frac{p}{2\lambda(v + 1)}, \frac{r}{2\lambda(v + 1)}\right) - \frac{1}{2(v + 1)}(2a^{ij}p^ip^j + 2\sigma^{ik}p^i r^k + |r|^2).
\]

Take a function \(\psi \in C^\infty\) such that
\[
\psi(z) = \begin{cases} 
1, & z \in [e^{-2\lambda M}, e^{2\lambda M}], \\
0, & z \notin [e^{-2\lambda(M+1)}, e^{2\lambda(M+1)}],
\end{cases}
\]
and denote \(\tilde{F}(t, x, v, p, r) = \psi(v + 1)F(t, x, v, p, r)\). From (3.11) and Lemma 2.4, we know
\[
-\psi(v + 1)\left[2\lambda(\|\lambda_0\|_{L^\infty} + \lambda_1 + \lambda_1 M)(v + 1) + \frac{C_{K,\mu_0}}{v + 1}(|p|^2 + |r|^2)\right] \\
\leq \tilde{F}(t, x, v, p, r) \leq 2\lambda(\|\lambda_0\|_{L^\infty} + \lambda_1 + \lambda_1 M)\psi(v + 1)(v + 1),
\tag{3.13}
\]
where \(C_{K,\mu_0}\) is a constant depending on \(K\) and \(\mu_0\).

Using the same method as [18, pp. 572], we can construct a sequence of functions \(\{F^n(t, x, v, p, r) : n \geq 1\}\) such that
(a) For every \(n\) and any \((\omega, t, x)\), \(F^n(t, x, v, p, r)\) is uniformly Lipschitz continuous with respect to \((v, p, r)\).
(b) The sequence \(\{F^n\}_n\) is decreasing, and for almost every \((\omega, t, x)\), \(F^n(t, x, v, p, r)\) locally uniformly converges to \(\tilde{F}(t, x, v, p, r)\) on \(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d\). Additionally,
\[
\tilde{F}(t, x, v, p, r) \leq F^n(t, x, v, p, r) \\
\leq 2\lambda(\|\lambda_0\|_{L^\infty} + \lambda_1 + \lambda_1 M)\psi(v + 1)(v + 1) + 2^{-n}.
\]

By Lemma 2.3 in [12], BSPDE
\[
\begin{aligned}
\left\{
\begin{array}{l}
dv^n = -\left[(a^{ij}v^n_{x^j} + \sigma^{ik}r^n_{x^i})_{x^j} + F^n(t, x, v^n, v^n_r, r^n)\right] dt + r^n dW^k_t, \\
v^n|_{T \times \partial \mathcal{D}} = 0, \quad v^n(T) = e^{2\lambda \varphi} - 1,
\end{array}
\right.
\end{aligned}
\]
has a unique weak solution \((v^n, r^n) \in \mathbb{H}^1_0(\mathcal{D}) \times \mathbb{H}^0(\mathcal{D})\). Moreover, on account of Lemma 2.2 we know that for every \(n \in \mathbb{N}\), \(v^{n+1} \leq v^n\). On the other hand, applying Corollary 2.3 and meanwhile noticing inequality (3.13), we have
\[
e^{-2\lambda(M+1)} - 1 \leq v^n \leq e^{2\lambda(M+1)}.
\]
Setting
\[
u^n = \frac{\ln(v^n + 1)}{2\lambda}, \quad q^n = \frac{r^n}{2\lambda(v^n + 1)},
\]

9
we can deduce from Lemma 3.4 that \((u^n, q^n)\) is a weak solution to the equation
\[
\begin{aligned}
da^n &= -[(a^{ij}u^n_{x^j} + \sigma^{ik}q^n_k)_{x^i} + f^n(t, x, u^n, q^n)]
dt + q^n_k dW^k_t,
\end{aligned}
\]
\[
u^n|_{\partial T \times \partial D} = 0, \quad u^n(T) = \varphi,
\]
where
\[
f^n(t, x, v, p, r) = \frac{1}{2\lambda e^{2\lambda v}} F^n(t, x, e^{2\lambda v} - 1, 2\lambda e^{2\lambda v}p, 2\lambda e^{2\lambda v}r) + \lambda(2a^{ij}p^i p^j + 2\sigma^{ik}p^i r^k + |r|^2).
\]

In view of the properties (a) and (b) of \(F^n\), it is easy to verify that \(f^n\) satisfies the conditions in Proposition 3.3 and the corresponding limit is
\[
\widetilde{f}(t, x, v, p, r) = \frac{1}{2\lambda e^{2\lambda v}} \widetilde{F}(t, x, e^{2\lambda v} - 1, 2\lambda e^{2\lambda v}p, 2\lambda e^{2\lambda v}r) + \lambda(2a^{ij}p^i p^j + 2\sigma^{ik}p^i r^k + |r|^2) = \psi(e^{2\lambda v})f(t, x, v, p, r) + [1 - \psi(e^{2\lambda v})]\lambda(2a^{ij}p^i p^j + 2\sigma^{ik}p^i r^k + |r|^2).
\]

Therefore, from Proposition 3.3 we know the following equation
\[
\begin{aligned}
da\tilde{u} &= -[(a^{ij}\tilde{u}_{x^j} + \sigma^{ik}\tilde{q}_k)_{x^i} + \tilde{f}(t, x, \tilde{u}, \tilde{u}_x, \tilde{q})] 
dt + \tilde{q}_k dW^k_t,
\end{aligned}
\]
\[
\tilde{u}|_{\partial T \times \partial D} = 0, \quad \tilde{u}(T) = \varphi,
\]
has at least a weak solution \((\tilde{u}, \tilde{q}) \in H^1_0(D) \cap L^\infty \times H^0(D; \mathbb{R}^d)\). Furthermore, according to Proposition 3.2 we have \(|\tilde{u}(\cdot)| \leq M\). We notice that when \(|v| \leq M\),
\[
\tilde{f}(t, x, v, p, r) = f(t, x, v, p, r).
\]

Therefore \((\tilde{u}, \tilde{q})\) is also a weak solution to BSPDE (1.1) - (1.2).

Finally we prove the existence of solution to BSPDE (1.1) - (1.2) in the general case, i.e., condition (3.1) is replaced by (H1),
\[
|f(t, x, v, p, r)| \leq \lambda_0(t, x) + \lambda_1|v| + \gamma(|v|)(|p|^2 + |r|^2).
\]

Since the estimate (3.2) in Proposition 3.2 is independent of \(\lambda\), we can use the truncation technique to complete the proof.

Denote the sets
\[
E^+ := \{ (\omega, t, x, v, p, r) \mid f > \lambda_0 + \lambda_1|v| + \gamma(M)(|p|^2 + |r|^2) \},
\]
\[
E^- := \{ (\omega, t, x, v, p, r) \mid -f > \lambda_0 + \lambda_1|v| + \gamma(M)(|p|^2 + |r|^2) \},
\]
and recall \(M = \|\lambda_0\|_{\mathcal{L}_1}(e^{\lambda_1 T} - 1) + e^{\lambda_1 T}\|\varphi\|_{L^\infty(\Omega \times D)}\).

Let
\[
\tilde{f}(\omega, t, x, v, p, r) = \begin{cases}
\lambda_0 + \lambda_1|v| + \gamma(M)(|p|^2 + |r|^2), & (\omega, t, x, v, p, r) \in E^+, \\
f(\omega, t, x, v, p, r), & (\omega, t, x, v, p, r) \notin E^+ \cup E^-, \\
-\lambda_0 - \lambda_1|v| - \gamma(M)(|p|^2 + |r|^2), & (\omega, t, x, v, p, r) \in E^-.
\end{cases}
\]

Obviously \(\tilde{f}\) satisfies condition (3.1). It follows from the previous arguments that BSPDE
\[
\begin{aligned}
da\tilde{u} &= -[(a^{ij}\tilde{u}_{x^j} + \sigma^{ik}\tilde{q}_k)_{x^i} + \tilde{f}(t, x, \tilde{u}, \tilde{u}_x, \tilde{q})] 
dt + \tilde{q}_k dW^k_t,
\end{aligned}
\]
\[
\tilde{u}|_{\partial T \times \partial D} = 0, \quad \tilde{u}(T) = \varphi,
\]
has at least a weak solution \((\tilde{u}, \tilde{q}) \in H^1_0(D) \times H^0(D; \mathbb{R}^d)\) and \(|\tilde{u}| \leq M\). However, when \(|v| \leq M\),

10
\[ \hat{f}(t, x, v, p, r) = f(t, x, v, p, r). \]

Therefore, \((\hat{u}, \hat{q})\) is also a weak solution to BSPDE (1.1)-(1.2). The proof is complete.

4. The uniqueness of solutions

Let \( M > 0 \) be a fixed constant. For simplicity, we denote by \( z = (p, r) \in \mathbb{R}^{d+2d_0} \) for vectors \( p \in \mathbb{R}^d \) and \( r \in \mathbb{R}^{d_0} \). We assume \( f(\omega, t, x, u, z) = f(\omega, t, x, u, p, r) \) satisfies

\( (H3) \) there exist functions \( l(\cdot) \in L^1(\mathcal{T} \times \mathcal{D}) \cap L^\infty(\mathcal{T} \times \mathcal{D}) \), \( k(\cdot) \in L^2(\mathcal{T}) \) and a positive constant \( \Lambda \) such that for any \((t, x) \in \mathcal{T} \times \mathcal{D}, u \in [-M, M], \) and \( z \in \mathbb{R}^{d+2d_0}, \)

\[
|f(t, x, u, z)| \leq l(t) + \Lambda|z|^2, \quad \text{a.s.,}
\]
\[
|f_z(t, x, u, z)| \leq k(t) + \Lambda|z|, \quad \text{a.s.}
\]

Moreover, for any \( \varepsilon > 0 \), there exists \( l_\varepsilon(\cdot) \in L^1(\mathcal{T}) \) such that for every \((t, x) \in \mathcal{T} \times \mathcal{D}, u \in \mathbb{R}, z \in \mathbb{R}^{d+2d_0}, \)

\[
|f_u(t, x, u, z)| \leq l_\varepsilon(t) + \varepsilon|z|^2, \quad \text{a.s.}
\]

The main theorem of this section concerns the uniqueness of solutions to BSPDE (1.1)-(1.2).

**Theorem 4.1.** Let condition \((H3)\) be satisfied. Suppose \((u^1, q^1)\) and \((u^2, q^2)\) are both weak solutions to BSPDE (1.1)-(1.2) and \( |u^1|, |u^2| \leq M \). Then \( u^1 = u^2 \).

**Proof.** The first step. We first prove this theorem under a more stringent condition. Assume

\( (H4) \) there exist a positive constant \( a \) and a function \( b(\cdot) \in L^1(\mathcal{T}) \) such that for every \((t, x) \in \mathcal{T} \times \mathcal{D}, u \in [-M, M], z \in \mathbb{R}^{d+2d_0}, \)

\[
f_u(t, x, u, z) + a|f_z(t, x, u, z)|^2 \leq b(t), \quad \text{a.s.}
\]

Denote \( \hat{u} = u^1 - u^2 \) and \( \hat{q} = q^1 - q^2 \). Set \( \hat{u}^+ = \max(0, \hat{u}) \). Applying Itô’s formula (Lemma 2.1), we have for any \( m \geq 2 \) and \( m \in \mathbb{N}, \)

\[
\int_\mathcal{D} [\hat{u}^+(t, x)]^{2m} \, dx
+ m(2m - 1) \int_t^T \int_\mathcal{D} (\hat{u}^+)^{2(m-1)} (2a^{ij} \hat{u}_{x^i} \hat{u}_{x^j} + 2\sigma^{ik} \hat{u}_{x^i} \hat{q} + |\hat{q}|^2)(s, x) \, dxds
= 2m \int_t^T \int_\mathcal{D} (\hat{u}^+)^{2m-1} \hat{f}(s, x) \, dxds - 2m \int_t^T \int_\mathcal{D} (\hat{u}^+)^{2m-2} \hat{q}(s, x) \, dxdW_s.
\]

Here

\[
\hat{f}(s, x) = f(s, x, u^1, u^1_x, q^1) - f(s, x, u^2, u^2_x, q^2)
= \left( \int_0^1 f_u(\Xi) \, d\lambda \right) \hat{u} + \left( \int_0^1 f_z(\Xi) \, d\lambda \right) (\hat{u}_x, \hat{q})',
\]

where

\[
\Xi := (s, x, \lambda u^1 + (1 - \lambda) u^2, \lambda u^1_\lambda + (1 - \lambda) u^2_\lambda, \lambda q^1 + (1 - \lambda) q^2).
\]
Cauchy-Schwarz’s inequality yields
\[(\hat{u}^+)^{2m-1} \leq \left( \int_0^1 (f_u + a|f_x|^2)(\Xi)d\lambda \right)(\hat{u}^+)^{2m} + \frac{1}{4a} (\hat{u}^+)^{2(m-1)}(|\hat{u}_x|^2 + |\check{q}|^2).\]

Noticing the inequality (2.3) and assumption (H4), we can deduce from (4.1) that
\[
\int_D [\hat{u}^+(t, x)]^{2m} dx + m \left[ \mu_0(2m - 1) - \frac{1}{2a} \right] \int_D (\hat{u}^+)^{2(m-1)}(|\hat{u}_x|^2 + |\check{q}|^2)(s, x) dx ds
\leq 2m \int_T \int_D b(s)(\hat{u}^+)^{2m}(s, x) dx ds - 2m \int_T \int_D (\hat{u}^+)^{2m-1} \check{q}(s, x) dx dW_s.
\]

Taking expectation on both sides of the above inequality, we have
\[
\mathbb{E} \int_D [\hat{u}^+(t, x)]^{2m} dx + m \left[ \mu_0(2m - 1) - \frac{1}{2a} \right] \mathbb{E} \int_T \int_D (\hat{u}^+)^{2(m-1)}(|\hat{u}_x|^2 + |\check{q}|^2)(s, x) dx ds
\leq 2m \int_T \int_D b(s)\mathbb{E}(\hat{u}^+)^{2m}(s, x) dx ds.
\]

Choosing \(m\) large enough such that \(\mu_0(2m - 1) - \frac{1}{2a} \geq 0\), together with Gronwall’s inequality, we know that \(\mathbb{E} \int_D [\hat{u}^+(t, x)]^{2m} dx = 0\), for all \(t \in T\). So \(u^1 \leq u^2\).

In the same way we can prove \(u^2 \leq u^1\). Hence \(u^1 = u^2\).

**The second step.** We will search for an appropriate change of variables to convert BSPDE (1.1)-(1.2) satisfying (H3) to another BSPDE satisfying condition (H4). Let
\[
\tilde{u} = \phi^{-1}(u), \quad \check{q} = q/w(u),
\]
where \(\phi\) is a smooth and increasing function to be determined with the condition \(\phi(0) = 0\) and \(w(u) = \phi'(\tilde{u}) = \phi'(\phi^{-1}(u))\).

Suppose \((u, q)\) is a weak solution to BSPDE (1.1)-(1.2). Analogous to the proof of Lemma 3.4, it is easy to verify that \((\tilde{u}, \check{q})\) is a weak solution to the equation
\[
\begin{align*}
\left\{ \begin{array}{l}
d\tilde{u} = -[(a^{ij}\tilde{u}_{x^j} + \sigma^{ik}\check{q}^k)_{x^i} + F(t, x, \tilde{u}, \tilde{u}_x, \check{q})] dt + \check{q}^k dW^k_t, \\
\tilde{u}|_{T \times \partial D} = 0, \quad \tilde{u}(T) = \phi^{-1}(\varphi),
\end{array} \right.
\end{align*}
\]
(4.2)

where
\[
F(t, x, \tilde{u}, \tilde{u}_x, \check{q}) = \frac{1}{\phi'(\tilde{u})} \left[ f(t, x, \phi(\tilde{u}), \phi'(\tilde{u})\tilde{u}_x, \phi'(\tilde{u})\check{q})
+ \frac{1}{2} \phi''(\tilde{u})(2a^{ij}\tilde{u}_{x^j}\tilde{u}_{x^i} + 2\sigma^{ik}\check{q}_x^k\check{q}_x^k + |\check{q}|^2) \right].
\]
(4.3)

Therefore it is sufficient to prove the equation (4.2) has a unique bounded weak solution.

We still denote \(z = (u_x, q)\) and \(\tilde{z} = (\tilde{u}_x, \check{q})\). Obviously \(z = \phi'(\tilde{u})\tilde{z}\). We denote by \(\langle A(t, x)\tilde{z}, \tilde{z} \rangle\) the positive definite quadratic form
\[
2a^{ij}\tilde{u}_{x^j}\tilde{u}_{x^i} + 2\sigma^{ik}\check{q}_x^k\check{q}_x^k + |\check{q}|^2,
\]
where $A$ is a function with value in the space of symmetric positive definite matrices. We can deduce from Lemma 2.4 that
\[ \langle A(t, x) \tilde{z}, \tilde{z} \rangle = \tilde{z} A(t, x) \tilde{z}' \geq \mu_0 |\tilde{z}|^2. \]
So (4.3) can be rewritten as
\[ F(t, x, \tilde{u}, \tilde{z}) = \frac{1}{\phi'(\tilde{u})} \left[ f(t, x, \phi(\tilde{u}), \phi'(\tilde{u}) \tilde{z}) + \frac{1}{2} \phi''(\tilde{u}) \langle A(t, x) \tilde{z}, \tilde{z} \rangle \right]. \]
By simple computation,
\[ F_{\tilde{u}}(t, x, \tilde{u}, \tilde{z}) = -\frac{w'}{w} f(t, x, u, z) + f_u(t, x, u, z) + \frac{w''}{2w} \langle A(t, x) z, z \rangle \]
\[ + \frac{w'}{w} \langle Az, z \rangle + \frac{w'}{w} (zf_z - f) + f_u, \]
\[ F_{\tilde{z}}(t, x, \tilde{u}, \tilde{z}) = f_z(t, x, u, z) + \frac{w'}{w} z. \]
If we can choose an appropriate $\phi$ such that $w > 0$, $w' > 0$ and $w'' < 0$, from (H3) we have
\[ \left( F_{\tilde{u}} + a |F_{\tilde{z}}|^2 \right)(t, x, \tilde{u}, \tilde{z}) \]
\[ = \frac{w''}{2w} \langle Az, z \rangle + \frac{w'}{w} (zf_z - f) + f_u + a \left| f_z + \frac{w'}{w} z \right|^2 \]
\[ \leq \frac{\mu_0 w''}{2w} |z|^2 + \frac{w'}{w} \left| l(t) + k(t) |z| + 2\Lambda |z|^2 \right| + l_\varepsilon(t) + \varepsilon |z|^2 \]
\[ + a \left[ k(t) + \left( \Lambda + \frac{w'}{w} \right) |z| \right]^2 \]
\[ \leq |z|^2 \left[ \frac{\mu_0 w''}{2w} + 2\Lambda \frac{w'}{w} + \varepsilon + a \left( \Lambda + \frac{w'}{w} \right)^2 \right] \]
\[ + |z|^2 \left[ k(t) \frac{w'}{w} + 2ak(t) \left( \Lambda + \frac{w'}{w} \right) \right] + l(t) \frac{w'}{w} + l_\varepsilon(t) + a[k(t)]^2 \]
\[ \leq |z|^2 \left[ \frac{\mu_0 w''}{2w} + 2\Lambda \frac{w'}{w} + \left( \frac{w'}{w} \right)^2 + \varepsilon + 2a \left( \Lambda + \frac{w'}{w} \right)^2 \right] \]
\[ + l(t) \frac{w'}{w} + l_\varepsilon(t) + (1 + 2a)[k(t)]^2. \]
Once we find a function $\phi$ such that besides $w(u) > 0$, $w'(u) > 0$ and $w''(u) < 0$ on $[-M, M]$,
\[ \frac{\mu_0 w''}{2w} + 2\Lambda \frac{w'}{w} + \left( \frac{w'}{w} \right)^2 \leq -\delta < 0, \]
we can choose $a$ and $\varepsilon$ small enough to assure that $F(t, x, \tilde{u}, \tilde{z})$ satisfies condition (H4). Then we will obtain the desired result.
Set
\[ u = \phi(\tilde{u}) = \frac{1}{\beta} \ln \left( \frac{(Be^\beta - 1)e^{\beta B\tilde{u}} + 1}{Be^\beta} \right), \]
where \( B > 1 \) and \( \beta > 0 \) are constants to be determined. Obviously \( \phi \) is a strictly increasing function and \( \phi(0) = 0 \). By computation we know that for any \( u \in [-M, M] \),
\[
\begin{align*}
    w(u) &= B - e^{-\beta(u+M)} > 0, \\
    w'(u) &= \beta e^{-\beta(u+M)} > 0, \\
    w''(u) &= -\beta^2 e^{-\beta(u+M)} < 0.
\end{align*}
\]
Furthermore,
\[
\frac{\mu_0}{2} \frac{w''}{w} + 2\Lambda \frac{w'}{w} + \left( \frac{w'}{w} \right)^2 = -\frac{\beta e^{-\beta(u+M)}}{(B - e^{-\beta(u+M)})^2} \left[ \left( \frac{\mu_0}{2} \beta - 2\Lambda \right) B + \left( 2\Lambda B - \frac{\mu_0}{2} + \frac{2}{2} \right) e^{-\beta(u+M)} \right].
\]
We can choose appropriate \( \beta \) and \( B \) to assure the above equality negative. The proof is complete.

**Remark 4.1.** In the case \( \mathcal{D} = \mathbb{R}^d \), we claim that the conclusions concerning the existence and uniqueness of solutions to BSPDEs in bounded domains are still valid. Indeed, setting \( g^i = f^i - u_{x^i} \), we know that \( (u, q) \) is a weak solution to the BSPDE
\[
\begin{align*}
    du &= - (\Delta u + f^0 + \sum_{i=1}^d g^i_{x^i}) dt + q^k dW_t^k, \\
    u(T, x) &= \varphi(x).
\end{align*}
\]
Approximating the coefficients \( f^0, g^i, i = 1, \cdots, d \), and \( \varphi \) by sequences of functions in the space \( C_0^\infty(\mathbb{R}^d) \), and applying Corollary 3.4 in [12], we can prove that the Itô’s formula in Lemma 2.1 is still valid. Once the Itô’s formula is established, we can obtain the claim since in addition to the assumptions on the boundedness of coefficients, we require their corresponding integrability in appropriate spaces to avoid the item \( \text{meas}(\mathcal{D}) \) appearing in the estimates.

5. **Application to non-Markovian stochastic control problems**

Analogous to [31], in this section we give an example, a stochastic control problem with a recursive cost functional formulated by a quadratic BSDE, to illustrate that the corresponding value function will formally satisfy a kind of stochastic Hamilton-Jacobi-Bellman equations with quadratic growth.

The stochastic HJB equation that we concern has the form
\[
\begin{align*}
    -du(t, x) &= \frac{1}{2} tr \left\{ [\sigma(t, x)\sigma^*(t, x) + \pi(t, x)\pi^*(t, x)]D^2_x u(t, x) \right\} dt \\
    &\quad + \inf_{v \in V} \left\{ f(t, x, u(t, x), \langle D_x u(t, x), \sigma(t, x) \rangle + q(t, x), \right. \\
    &\quad \left. \langle D_x u(t, x), \pi(t, x) \rangle, v \right\} + \langle b(t, x, v), D_x u(t, x) \rangle \right\} dt \\
    &\quad + \langle D_x q(t, x), \sigma(t, x) \rangle dt + q(t, x) dW_t, \\
    u(T, x) &= \phi(x).
\end{align*}
\]
where the coefficients satisfy the-algebra associated with the dimensional. Denote by \( P \) and augmented by all the \( \mathbb{P} \)-null sets in \( \mathcal{F} \). We also denote by \( \mathcal{F}^* \) the predictable \( \sigma \)-algebra associated with \( \{ \mathcal{F}_t^* \}_{t \in T} \).

We introduce the admissible control set \( \mathcal{V}_{t,T} := \{ v(\cdot) \mid v(\cdot) \text{ is a } V\text{-valued and } \mathcal{F}^* \text{ measurable process defined on } [t,T] \text{ and} \]
\[
\mathbb{E} \int_t^T v^2(s) ds < \infty \}
\]
where \( V \) is a compact set of \( \mathbb{R}^n \).

We consider the controlled system parameterized by the initial data \((t,x) \in T \times \mathbb{R}^n:\]
\[
\begin{cases}
  dX^{t,x,v}_s = b(s, X^{t,x,v}_s, v_s) ds + \sigma(s, X^{t,x,v}_s) dW_s + \pi(s, X^{t,x,v}_s) dB_s, \\
  X^{t,x,v}_t = x,
\end{cases}
\]
(5.2)

where the coefficients
\[
b: \Omega \times T \times \mathbb{R}^n \times V \rightarrow \mathbb{R}^n, \quad \sigma: \Omega \times T \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \pi: \Omega \times T \times \mathbb{R}^n \rightarrow \mathbb{R}^n,
\]
satisfy
\[
(A1) \text{ } b, \sigma \text{ and } \pi \text{ are bounded functions and for every } (x,v) \in \mathbb{R}^n \times V, \text{ } b(\cdot, x, v), \sigma(\cdot, x) \text{ and } \pi(\cdot, x) \text{ are } \mathcal{F} \text{ measurable processes.}
\]
\[
(A2) \text{ There exists } L > 0 \text{ such that } \\
|b(t,x,v) - b(t,x',v')| + |\sigma(t,x) - \sigma(t,x')| + |\pi(t,x) - \pi(t,x')| \leq L(|x - x'| + |v - v'|).
\]

For a given admissible control \( v(\cdot) \in \mathcal{V}_{t,T} \), we consider the following BSDE
\[
\begin{cases}
  dY^{t,x,v}_s = -f(s, X^{t,x,v}_s, Y^{t,x,v}_s, Z^{t,x,v}_s, v_s) ds + \tilde{Z}^{t,x,v}_s dW_s + \tilde{Z}^{t,x,v}_s dB_s, \\
  Y^{t,x,v}_T = \phi(X^{t,x,v}_T),
\end{cases}
\]
(5.3)

where we denote \( Z = (\tilde{Z}, \tilde{Z}) \). We assume that
\[
(A3) f: \Omega \times T \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^2 \times V \rightarrow \mathbb{R} \text{ satisfies condition } (H3).
\]
\[
(A4) \text{ The terminal value } \phi: \Omega \times \mathbb{R}^n \rightarrow \mathbb{R} \text{ is } \mathcal{F}_T \times \mathcal{B}(\mathbb{R}^n) \text{ measurable and } \phi \in L^2(\Omega \times \mathbb{R}^n) \cap L^\infty(\Omega \times \mathbb{R}^n).
\]

It is well known that under conditions (A1), (A2), (A3) and (A4), SDE (5.2) and BSDE (5.3) have unique solutions, respectively.

For a given admissible control \( v(\cdot) \in \mathcal{V}_{t,T} \), we introduce the associated cost functional
\[
J(t,x; v(\cdot)) = \mathbb{E}^{\mathcal{F}_t} Y^{t,x,v}_t.
\]

Thus the value function of the stochastic optimal control problem is
\[
u(t,x) := \text{ess inf}_{v(\cdot) \in \mathcal{V}_{t,T}} J(t,x; v(\cdot)).
Since the related coefficients \( b, \sigma, f \) and \( \phi \) are random functions, the value function \( u \) is a random field. We recall the generalized dynamic programming principle for the above control problem with recursive cost functional in [38]. For given initial data \((t, x) \in \mathcal{T} \times \mathbb{R}^n\), a positive number \( \delta \leq T - t \), and a random variable \( \eta \in L^2(\Omega, \mathcal{F}_t \mathbb{P}; \mathbb{R}) \), we denote a backward semigroup by
\[
G^{t,x,v}_t[\eta] := Y_t.
\]
Here \((Y_s, Z_s)_{s \in [t, t+\delta]}\) is the solution to the following BSDE
\[
\begin{cases}
  dY_s = -f(s, X^{t,x,v}_s, Y_s, Z_s, v_s) ds + \tilde{Z}_s dW_s + \bar{Z}_s dB_s, & s \in [t, t+\delta], \\
  Y_{t+\delta} = \eta,
\end{cases}
\]
where \(X^{t,x,v}\) is the solution to SDE (5.2).

Then we have the generalized dynamic programming principle (Theorem 6.6 in Section 2 in [38])
\[
\begin{aligned}
  u(t, x) &= \text{ess inf}_{v(\cdot) \in \mathcal{V}_{t,T}} \mathbb{E} \mathcal{F}_t G^{t,x,v}_t[u(t + \delta, X^{t,x,v}_{t+\delta})].
\end{aligned}
\]

Suppose \( u \) is smooth with respect to \((t, x)\), we can use the Itô-Wentzell’s formula (see, e.g. [31]) and a similar procedure in [31] to obtain that the value function \( u \) formally satisfies BSPDE (5.1). According to our theoretical results, \( u \) is a bounded random field.

6. Appendix

6.1. Proof of Proposition 3.2

Suppose \( \xi \) satisfies the following ODE
\[
\xi(t) = \|\varphi^+\|_{L^\infty(\Omega \times \mathcal{D})} + \int_t^T (\lambda_1 \xi(s) + \|\lambda_0\|_{L^\infty}) ds.
\]
Then for any \( t \in \mathcal{T} \), we have
\[
\xi(t) = \frac{\|\lambda_0\|_{L^\infty}}{\lambda_1} (e^{\lambda_1(T-t)} - 1) + e^{\lambda_1(T-t)} \|\varphi^+\|_{L^\infty(\Omega \times \mathcal{D})}.
\]
We will prove \( u(t, x) \leq \xi(t) \) a.e. \((\omega, x)\).

Denote \( M_1 := \|u\|_{L^\infty} + \|\varphi\|_{L^\infty(\Omega \times \mathcal{D})} \). Define a function \( \Psi_1 \) on \([-M_1, M_1]\) as follows
\[
\Psi_1(v) = \begin{cases} 
  e^{2\lambda v} - (1 + 2\lambda v + 2\lambda^2 v^2), & \text{when } v \in [0, M_1], \\
  0, & \text{when } v \in [-M_1, 0].
\end{cases}
\]
By simple computation, we know \( \Psi_1 \) has the properties: \( \forall v \in [-M_1, M_1] \),
\[
\Psi_1(v) \geq 0, \quad \Psi_1'(v) \geq 0, \\
\Psi_1(v) = 0 \iff v \leq 0, \\
0 \leq v \Psi_1'(v) \leq 2(M_1 + 3)\lambda \Psi_1(v), \\
\lambda \Psi_1' - \frac{1}{2} \Psi_1'' \leq 0.
\]
By Lemma 2.1, we have
\[
\int_{D} \Psi_1(u(t, x) - \xi(t))dx - \int_{D} \Psi_1(\varphi(x) - \xi(T))dx
\]
\[
= \int_{t}^{T} \int_{D} \Psi_1^\prime(u(s, x) - \xi(s)) \left( (a^{ij} u_{x^j} + \sigma^{ik} q^k)_{x^i}(s, x) + f(s, x, u(s, x), u_x(s, x), q(s, x)) - (\lambda_1 \xi(s) + \|\lambda_0\|_{L^\infty}) \right) dxds - \frac{1}{2} \int_{t}^{T} \int_{D} \Psi_1'' dxds
\]
\[
- \int_{t}^{T} \int_{D} \Psi_1^k(s, x) dx dW_s^k.
\]

According to the integration by parts and Lemma 2.3, we have
\[
\int_{t}^{T} \int_{D} \left( \Psi_1^{ij} u_{x^j} + \sigma^{ik} q^k \right)_{x^i}(s, x) dxds
\]
\[
= - \int_{t}^{T} \int_{D} \Psi_1''(u(s, x) - \xi(s)) \left( a^{ij} u_{x^j} + \sigma^{ik} u_x q^k + \frac{1}{2} |q|^2 \right) (s, x) dxds
\]
\[
\leq - \frac{\mu_0}{2} \int_{t}^{T} \int_{D} \Psi_1''(u(s, x) - \xi(s)) \left( |u_x|^2 + |q|^2 \right)(s, x) dxds.
\]

On the other hand, set \( \tilde{\lambda}_1 = \lambda_1 \text{ sgn}(u) \), then
\[
f(s, x, u, u_x, q) \leq \lambda_0(s, x) + \tilde{\lambda}_1 u + \lambda \mu_0(\|u_x\|^2 + |q|^2).
\]

Noticing that \((\tilde{\lambda}_1 - \lambda_1) \xi(s) \leq 0\), we have
\[
f(s, x, u(s, x), u_x(s, x), q(s, x)) - (\lambda_1 \xi(s) + \|\lambda_0\|_{L^\infty})
\]
\[
\leq \lambda_0(s, x) + \tilde{\lambda}_1 u(s, x) + \lambda \mu_0(\|u_x\|^2 + |q|^2)(s, x) - (\lambda_1 \xi(s) + \|\lambda_0\|_{L^\infty})
\]
\[
\leq \tilde{\lambda}_1 (u(s, x) - \xi(s)) + (\tilde{\lambda}_1 - \lambda_1) \xi(s) + \lambda \mu_0(\|u_x\|^2 + |q|^2)(s, x)
\]
\[
\leq \tilde{\lambda}_1 (u(s, x) - \xi(s)) + \lambda \mu_0(\|u_x\|^2 + |q|^2)(s, x).
\]

Thus,
\[
\int_{D} \Psi_1(u(t, x) - \xi(t))dx - \int_{D} \Psi_1(\varphi(x) - \xi(T))dx
\]
\[
\leq \int_{t}^{T} \int_{D} \tilde{\lambda}_1 \Psi_1'(u(s, x) - \xi(s))(u(s, x) - \xi(s)) dxds
\]
\[
+ \int_{t}^{T} \int_{D} \mu_0 (\lambda \Psi_1' - \frac{1}{2} \Psi_1'') (u(s, x) - \xi(s))(\|u_x\|^2 + |q|^2)(s, x) dxds
\]
\[
- \int_{t}^{T} \int_{D} \Psi_1^k(s, x) dx dW_s^k.
\]
In view of the properties that Ψ₁ possesses, we have

\[
0 \leq \int_{\mathcal{D}} \Psi_1(u(t, x) - \xi(t))dx \\
\leq \int_t^T \int_{\mathcal{D}} 2(M_1 + 3)\lambda \lambda_1 \Psi_1(u(s, x) - \xi(s))dxds \\
- \int_t^T \int_{\mathcal{D}} \Psi_1^{(k)}(s, x)dx dW^k_s, \quad a.s..
\]

Taking expectation on both sides of the above inequality, we get

\[
0 \leq \mathbb{E} \int_{\mathcal{D}} \Psi_1(u(t, x) - \xi(t))dx \\
\leq 2(M_1 + 3)\lambda \lambda_1 \int_t^T \mathbb{E} \left[ \int_{\mathcal{D}} \Psi_1(u(s, x) - \xi(s))dx \right] ds.
\]

Gronwall’s inequality yields

\[
\mathbb{E} \int_{\mathcal{D}} \Psi_1(u(t, x) - \xi(t))dx = 0, \quad \forall \ t \in \mathcal{T}.
\]

Due to Ψ₁(ν) ≥ 0, it holds that for every \( t \in \mathcal{T} \),

\[
Ψ_1(u(t, x) - \xi(t)) = 0, \quad a.e. (ω, x).
\]

The fact that Ψ₁(ν) = 0 ⇔ ν ≤ 0 implies that for every \( t \in \mathcal{T} \),

\[
u(t, x) \leq \xi(t), \quad a.e. (ω, x).
\]

In the same way we can also prove that for every \( t \in \mathcal{T} \),

\[
u(t, x) \geq -\|\lambda_0\|_{L^\infty}(e^{\lambda_1(T-t)} - 1) - e^{\lambda_1(T-t)}\|\varphi^-\|_{L^\infty(Ω \times \mathcal{D})}, \quad a.e. (ω, x).
\]

So we obtain (3.2).

Next we prove (3.3). Denote \( M_2 = \|u\|_{L^\infty} \). Define a function Ψ₂ on \([-M_2, M_2]\) as

\[
Ψ_2(ν) = \begin{cases} \\
\frac{1}{2} \lambda^{-2}[e^{2\lambda v} - (1 + 2\lambda v)], & \text{when } ν \in [0, M_2], \\
Ψ_2(-ν), & \text{when } ν \in [-M_2, 0].
\end{cases}
\]

It is easy to verify that Ψ₂ has the following properties: for every ν ∈ \([-M_2, M_2]\),

\[
Ψ_2(ν) \geq 0, \quad Ψ_2'(0) = 0, \quad |Ψ_2'(ν)| \leq \frac{e^{2\lambda M_2} - 1}{\lambda}, \\
\frac{1}{2} Ψ_2''(ν) - λ|Ψ_2'(ν)| = 1.
\]
Applying Itô’s formula to compute \( \int_D \Psi_2(u(t, x)) \, dx \), we have

\[
\int_D \Psi_2(u(t, x)) \, dx - \int_D \Psi_2(\varphi(x)) \, dx \\
\leq \int_t^T \int_D |\Psi_2'(u(s, x))|(\lambda_0(s, x) + \lambda_1|u(s, x)|) \, dx ds \\
+ \int_t^T \int_D \mu_0(\lambda|\Psi_2'| - \frac{1}{2}\Psi_2'')(u(s, x))(|u_x|^2 + |q|^2)(s, x) \, dx ds \\
- \int_t^T \int_D \Psi_2^k(s, x) \, dx dW^k.
\]

(6.2)

Since \( \Psi_2 \) and \( \Psi_2' \), defined on the finite duration \([−M_2, M_2]\), are of the same order as \( v^2 \) and \( v \) near the zero respectively, there exist positive constants \( k_1, k_2, k_3 \) and \( k_4 \) depending only on \( \lambda \) and \( M_2 \), such that

\[
k_1v^2 \leq \Psi_2(v) \leq k_2v^2, \quad k_3|v| \leq |\Psi_2'(v)| \leq k_4|v|.
\]

Thus,

\[
\int_t^T \int_D |\Psi_2'(u(s, x))|\lambda_0(s, x) \, dx ds \\
\leq \frac{k_4^2}{2} \int_t^T \int_D |u(s, x)|^2 \, dx ds + \frac{1}{2} \int_t^T \int_D \lambda_0^2(s, x) \, dx ds.
\]

Taking expectation on both sides of (6.2), we obtain

\[
\mu_0 \mathbb{E} \int_t^T \int_D (|u_x|^2 + |q|^2)(s, x) \, dx ds + k_1 \mathbb{E} \int_D |u(t, x)|^2 \, dx \\
\leq k_2 \mathbb{E} \int_D |\varphi(x)|^2 \, dx + \frac{1}{2} \mathbb{E} \int_0^T \int_D \lambda_0^2(s, x) \, dx ds \\
+ (\frac{k_4^2}{2} + k_4\lambda_1) \int_t^T \mathbb{E} \int_D |u(s, x)|^2 \, dx ds.
\]

(6.3)

Gronwall’s inequality yields

\[
\sup_{t \in T} \mathbb{E} \int_D |u(t, x)|^2 \, dx \leq \left( \frac{k_2}{k_1} \|\varphi(x)\|_{L^2(\Omega \times D)} + \frac{1}{2k_1}\|\lambda_0\|_{L^2} \right) e^{\frac{k_4^2 + 2k_4\lambda_1}{2k_1} T}.
\]

Again from (6.3) we deduce that

\[
\|u_x\|_{L^2(\Omega \times D)}^2 + \|q\|_{L^2(\Omega \times D)}^2 \leq C_1,
\]

where \( C_1 \) depends on \( \|\varphi(x)\|_{L^2(\Omega \times D)}, \|\lambda_0\|_{L^2}, \mu_0, \lambda, \lambda_1 \) and \( T \). The proof of Proposition 3.2 is complete.
6.2. Proof of Proposition 3.3

Since the sequence \((u^n)_n\) is monotone and bounded, there exists its limit function which we denote by \(u\). Obviously \(u \in L^\infty\). By the monotone convergence theorem, 
\[
\lim_{n \to \infty} \|u - u^n\|_{H^1(D)} = 0.
\]

We know from (3.3) in Proposition 3.2 that for any \(n \in \mathbb{N}\),
\[
\|u^n\|_{H^1(D)}^2 + \|q^n\|_{H^0(D)}^2 \leq C_1.
\]

So we can extract a subsequence \(\{n'\}\) of the sequence \(\{n\}\) and find functions \(v \in H^1(D; \mathbb{R}^d)\) and \(q \in H^0(D; \mathbb{R}^{d_0})\) such that
\[
u^n \to v \text{ weakly in } H^1(D),
\]
\[
q^n \to q \text{ weakly in } H^0(D).
\]

The uniqueness of limit implies \(v = u\).

Next we finish the proof by three steps.

**Step 1.** Due to the existence of the nonhomogeneous term \(f\), the weak convergence of \((u^n', q^n')_{n'}\) can not assure that the limit \((u, q)\) is a weak solution to BSPDE (1.1)-(1.2). Now we prove that the sequences \((u^n)_n\) and \((q^n)_n\) converge strongly in \(H^0(D)\).

We first deduce from condition (b) that for any \(n, m \in \mathbb{N}\),
\[
|f^n(t, x, u^n, u^n_x, q^n) - f^m(t, x, u^m, u^m_x, q^m)|
\leq 2\lambda_2(t, x) + 5\lambda_0(|u^n - u^m|^2 + |u^n_x - u^m_x|^2 + |u^m|^2) + |q^n - q^m|^2 + |q^n - q|^2 + |q|^2.
\]

Define a function \(\Psi_3\) on \([0, 2M]\) as follows
\[
\Psi_3(v) = \frac{1}{200\lambda^2}(e^{20\lambda v} - 20\lambda v - 1).
\]

It is easy to verify that \(\Psi_3\) is an increasing function and that
\[
\Psi_3(0) = \Psi_3'(0) = 0, \quad \frac{1}{2}\Psi_3'' - 10\lambda\Psi_3' \equiv 1.
\]

For notational simplicity, we denote \(u^\infty = u\), \(q^\infty = q\),
\[
\delta^{n,m}_u = u^n - u^m, \quad \delta^{n,m}_q = q^n - q^m.
\]

By Lemma 2.1 and the integration by parts, we have
\[
\int_D \Psi_3(\delta^{n,m}_u(0, x))dx - \int_D \Psi_3(\delta^{n,m}_u(T, x))dx
\]
\[=
\int_0^T \int_D \Psi_3'(\delta^{n,m}_u(t, x))[f^n(t, x, u^n, u^n_x, q^n) - f^m(t, x, u^m, u^m_x, q^m)]dxdt
\]
\[- \int_0^T \int_D \Psi_3'(\delta^{n,m}_u(t, x))[\sigma^{ij}(\delta^{n,m}_u)_x(\delta^{n,m}_u)_x + \sigma^{ik}(\delta^{n,m}_u)_x(\delta^{n,m}_u)_x]dxdt
\]
\[- \frac{1}{2} \int_0^T \int_D \Psi_3'(\delta^{n,m}_u(t, x))|\delta^{n,m}_u(t, x)|^2dxdt
\]
\[- \int_0^T \Psi_3'(\delta^{n,m}_u(t, x))\delta^{n,m}_u(t, x)dxdt.
\]

20
Noticing $\Psi'_3 \geq 0$, Lemma 2.3 and (6.4), we have
\[
\int_D \Psi_3(\delta^{n,m}_u(0, x))dx - \int_D \Psi_3(\delta^{n,m}_u(T, x))dx \\
\leq \int_0^T \int_D \Psi'_3(\delta^{n,m}_u) \times [2\lambda_2 + 5\lambda\mu_0(|(\delta^{n,m}_u)_x|^2 + |\delta^{n,m}_q|^2 \\
+ |(\delta^{n,\infty}_u)_x|^2 + |\delta^{n,\infty}_q|^2 + |u|^2 + |q|^2)](t, x)dxdt \\
- \frac{\mu_0}{2} \int_0^T \int_D \Psi''_3(\delta^{n,m}_u)[|(\delta^{n,m}_u)_x|^2 + |\delta^{n,m}_q|^2](t, x)dxdt \\
- \int_0^T \int_D \Psi'_3(\delta^{n,m}_u(t, x))\delta^{n,m}_q(t, x)dxdW_t.
\]

Taking expectation on both side of the above inequality, we get
\[
\mathbb{E} \int_D \Psi_3(\delta^{n,m}_u(0, x))dx - \mu_0\mathbb{E} \int_0^T \int_D 5\lambda \Psi'_3(\delta^{n,m}_u)[|(\delta^{n,\infty}_u)_x|^2 + |\delta^{n,\infty}_q|^2](t, x)dxdt \\
+ \mu_0\mathbb{E} \int_0^T \int_D \left[ \frac{1}{2} \Psi''_3 - 5\lambda \Psi'_3 \right](\delta^{n,m}_u)[|(\delta^{n,m}_u)_x|^2 + |\delta^{n,m}_q|^2](t, x)dxdt \\
\leq \mathbb{E} \int_D \Psi_3(\delta^{n,m}_u(T, x))dx + \mathbb{E} \int_0^T \int_D \Psi'_3(\delta^{n,m}_u)[2\lambda_2 + 5\lambda\mu_0(|u|^2 + |q|^2)](t, x)dxdt.
\]

Letting $m$ tend to infinity along the subsequence \{n\}, together with the fact that $u^n$ converges pointwise to $u$, we can deduce from the dominated convergence theorem that
\[
\mathbb{E} \int_D \Psi_3(\delta^{n,\infty}_u(0, x))dx - \mu_0\mathbb{E} \int_0^T \int_D 5\lambda \Psi'_3(\delta^{n,\infty}_u)[|(\delta^{n,\infty}_u)_x|^2 + |\delta^{n,\infty}_q|^2](t, x)dxdt \\
+ \lim_{\{n\}' \ni m \to \infty} \mu_0\mathbb{E} \int_0^T \int_D \left[ \frac{1}{2} \Psi''_3 - 5\lambda \Psi'_3 \right](\delta^{n,\infty}_u)[|(\delta^{n,\infty}_u)_x|^2 + |\delta^{n,\infty}_q|^2](t, x)dxdt \\
\leq \mathbb{E} \int_D \Psi_3(\delta^{n,\infty}_u(T, x))dx + \mathbb{E} \int_0^T \int_D \Psi'_3(\delta^{n,\infty}_u)[2\lambda_2 + 5\lambda\mu_0(|u|^2 + |q|^2)](t, x)dxdt.
\]

In virtue of the weak convergence of the two sequences \{(u^n)'\}_n' and \{(q^n)'\}_n', we have
\[
\mathbb{E} \int_0^T \int_D \Psi'_3(\delta^{n,\infty}_u)[|(\delta^{n,\infty}_u)_x|^2 + |\delta^{n,\infty}_q|^2](t, x)dxdt \\
\leq \lim_{\{n\}' \ni m \to \infty} \mathbb{E} \int_0^T \int_D \Psi'_3(\delta^{n,\infty}_u)[|(\delta^{n,\infty}_u)_x|^2 + |\delta^{n,\infty}_q|^2](t, x)dxdt.
\]

Since $\frac{1}{2} \Psi''_3 - 10\lambda \Psi'_3 \equiv 1$, we have
\[
\mathbb{E} \int_D \Psi_3(\delta^{n,\infty}_u(0, x))dx + \lim_{\{n\}' \ni m \to \infty} \mu_0\mathbb{E} \int_0^T \int_D [|(\delta^{n,m}_u)_x|^2 + |\delta^{n,m}_q|^2](t, x)dxdt \\
\leq \mathbb{E} \int_D \Psi_3(\delta^{n,\infty}_u(T, x))dx + \mathbb{E} \int_0^T \int_D \Psi'_3(\delta^{n,\infty}_u)[2\lambda_2 + 5\lambda\mu_0(|u|^2 + |q|^2)](t, x)dxdt.
\]
The resonance theorem yields that
\[ \mathbb{E} \int_{\mathcal{D}} \Psi_3(\delta^{n,\infty}_u(0, x)) dx + \mu_0 \mathbb{E} \int_0^T \int_{\mathcal{D}} \left[ |(\delta^{n,\infty}_u)_x|^2 + |\delta^{n,\infty}_q|^2 \right] (t, x) dx dt \]
\[ \leq \mathbb{E} \int_{\mathcal{D}} \Psi_3(\delta^{n,\infty}_u(T, x)) dx + \mathbb{E} \int_0^T \int_{\mathcal{D}} \Psi_3(\delta^{n,\infty}_u)[2\lambda_2 + 5\lambda \mu_0 (|u|^2 + |q|^2)] (t, x) dx dt. \]
Noticing again \( u^n \) converges pointwise to \( u \), \( \delta^{n,\infty}_u \) converges to 0. Therefore, the dominated convergence theorem yields
\[ \lim_{n \to \infty} \mathbb{E} \int_0^T \int_{\mathcal{D}} \left[ |(\delta^{n,\infty}_u)_x|^2 + |\delta^{n,\infty}_q|^2 \right] (t, x) dx dt = 0, \]
which implies that \( (u^n)_n \) and \( (q^n)_n \) converge strongly to \( u_x \) and \( q \) in \( \mathbb{H}^0(\mathcal{D}) \), respectively.

**Step 2.** We prove that \((u, q)\) is a weak solution to BSPDE (1.1)-(1.2). To this end, we need the following lemma, the proof of which can be obtained by the same argument as that used in Lemma 2.5 in [18].

**Lemma 6.1.** Suppose a sequence \((v^n)_n\) converges to \( v \) strongly in \( \mathbb{H}^0(\mathcal{D}) \). Then there exists a subsequence \((v^{n_k})_k\) such that \((v^{n_k})_k\) converges to \( v \) a.e. and \( \tilde{v} := \sup_k |v^{n_k}| \in \mathbb{H}^0(\mathcal{D}) \).

According to the above lemma, we can extract a subsequence \( \{n_k\} \) such that
\[ u^{n_k} \rightarrow u_x, \quad d\mathbb{P} \times dt \times dx \text{-a.e. and } \mathbb{E} \sup_k |u^{n_k}| \in \mathbb{H}^0(\mathcal{D}), \]
\[ q^{n_k} \rightarrow q, \quad d\mathbb{P} \times dt \times dx \text{-a.e. and } \mathbb{E} \sup_k |q^{n_k}| \in \mathbb{H}^0(\mathcal{D}). \]

Then it follows from condition (a) that for a.e. \((\omega, t, x) \in \Omega \times \mathcal{T} \times \mathcal{D}, \)
\[ \lim_{k \to \infty} f^{n_k}(t, x, u^{n_k}(t, x), u^{n_k}_x(t, x), q^{n_k}(t, x)) = f(t, x, u(t, x), u_x(t, x), q(t, x)). \]

On the other hand, we have
\[ |f^{n_k}(t, x, u^{n_k}, u^{n_k}_x, q^{n_k})| \leq \lambda_2(t, x) + \lambda \mu_0 \sup_k (|u^{n_k}|^2 + |q^{n_k}|^2) \leq \lambda_2(t, x) + \lambda \mu_0 (|\tilde{v}|^2 + |\tilde{q}|^2). \]

The dominated convergence theorem yields
\[ \lim_{k \to \infty} \mathbb{E} \int_0^T \int_{\mathcal{D}} \left| f^{n_k}(t, x, u^{n_k}(t, x), u^{n_k}_x(t, x), q^{n_k}(t, x)) \right| dt dx dt = 0. \]

In view of the strong convergence of \((u^{n_k})_k\) and \((q^{n_k})_k\) in \( \mathbb{H}^0(\mathcal{D}) \), we obtain that \((u, q)\) is a weak solution to BSPDE (1.1)-(1.2).

**Step 3.** Finally we prove that \( u \in L^2(\Omega; C(\mathcal{T}; L^2(\mathcal{D}))) \). Applying Itô’s formula to \( \|u^{n_k}(t, \cdot) - u^n(t, \cdot)\|_{L^2(\mathcal{D})} \) and proceeding several standard computation, we get that
\[ \mathbb{E} \sup_{t \in \mathcal{T}} \|u^{n_k}(t, \cdot) - u^n(t, \cdot)\|_{L^2}^2 \]
\[ \leq \mathbb{E} \int_0^T \int_{\mathcal{D}} |u^{n_k} - u^n| |f^{n_k} - f^n| dx ds + \mathbb{E} \sup_{t \in \mathcal{T}} \int_0^T \int_{\mathcal{D}} (u^{n_k} - u^n)(q^{n_k} - q^n) dx dW_s \]
\[ \leq 2M \mathbb{E} \int_0^T \int_{\mathcal{D}} |f^{n_k} - f^n| dx ds + \frac{1}{2} \mathbb{E} \sup_{t \in \mathcal{T}} \|u^{n_k}(t, \cdot) - u^n(t, \cdot)\|_{L^2}^2 + C \|q^{n_k} - q^n\|_{\mathbb{H}^0}^2. \]

22
Hence it is easy to see that
\[ \mathbb{E} \sup_{t \in \mathcal{T}} \| u^{n_k}(t, \cdot) - u^{n_l}(t, \cdot) \|_{L^2}^2 \to 0, \quad \text{as } k, l \to \infty, \]
which implies that \( \{u^{n_k}\} \) is a Cauchy sequence in \( L^2(\Omega; C(T; L^2(D))) \), and thus its limit \( u \in L^2(\Omega; C(T; L^2(D))) \). The proof of Proposition 3.3 is complete.

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