A GENERALIZATION OF ESTEVES–HOMMA’S EXAMPLE OF TANGENTIALLY DEGENERATE CURVES

SATORU FUKASAWA

Abstract. This paper presents a method of a construction of tangentially degenerate curves with a birational Gauss map, focusing on the non-classicality of automorphisms. This method describes a generalization of Esteves–Homma’s example of this kind. In addition, this paper presents a smooth projective curve with a birational Gauss map such that a general tangent line contains three or more points of the curve, which answers a question raised by Kaji in the affirmative.

1. Introduction

An irreducible projective curve $C \subset \mathbb{P}^N$ with $N \geq 3$ not contained in any plane over an algebraically closed field $k$ of characteristic $p \geq 0$ is said to be tangentially degenerate if $(T_P C \setminus \{P\}) \cap C \neq \emptyset$ for a general point $P \in C$, where $T_P C \subset \mathbb{P}^N$ is the projective tangent line at $P$. This terminology is due to Kaji [6]. In characteristic zero, the existence of tangentially degenerate curves was asked by Terracini in 1932 ([13, p.143], see also [2]). For the case where $p = 0$ and the morphism $X \to \mathbb{P}^N$ induced by the normalization $\varphi : X \to C$ is unramified, the nonexistence of such curves was proved by Kaji [6] in 1986. Generalizations of Kaji’s theorem were obtained in this century ([1, 8]).

In positive characteristic, there exist many examples of tangentially degenerate curves. In most of such known cases, the Gauss map $\gamma : C \dashrightarrow \mathbb{G}(1, \mathbb{P}^N)$, which sends a smooth point $P \in C$ to the tangent line $T_P C \subset \mathbb{G}(1, \mathbb{P}^N)$, is not separable. Very surprisingly, in 1994, Esteves and Homma [3] presented an embedding

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^3; \; (1 : t) \mapsto (1 : t : t^2 - t^p : t^3 + 2t^p - 3t^{p+1})$$

of $\mathbb{P}^1$ such that $\varphi(\mathbb{P}^1)$ is tangentially degenerate and the Gauss map of $\varphi(\mathbb{P}^1)$ is birational onto its image. This is only known example of a tangentially degenerate
curve with a separable Gauss map. The existence of another example has been unknown for a long time. One reason is that the separability of the Gauss map is equivalent to the reflexivity with respect to projective dual (\cite{4, 5, 7, 14}), and that several pathological phenomena in positive characteristic do not occur under the assumption that the reflexivity holds (\cite{4}).

This paper proves the following:

**Theorem 1.** Let $p > 2$, let $q$ be a power of $p$, and let $\mathbb{F}_q \subset k$ be a finite field of $q^n$ elements containing the set $\{\alpha \in k \mid \alpha^{q-2} = 1\}$. We consider the morphism

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^4; \; (1 : t) \mapsto (1 : t^2 - t^q : t^{q^n} - t^{q^{2n}} : t(t^{q^n} - t^{q^{2n}})).$$

Then the following hold.

(a) $\varphi$ is an embedding.

(b) For any point $P \in \mathbb{P}^1 \setminus \{(0 : 1)\}$, the set $(T_{\varphi(P)}\varphi(\mathbb{P}^1) \setminus \{\varphi(P)\}) \cap \varphi(\mathbb{P}^1)$ consists of exactly $q - 2$ points.

(c) The Gauss map of $\varphi(\mathbb{P}^1)$ is birational onto its image.

Theorem \cite{1} answers the following question raised by Kaji \cite{9} in the affirmative.

**Question 1.** Does there exist a tangentially degenerate space curve $C \subset \mathbb{P}^N$ with a birational Gauss map such that a general tangent line $T_P C$ of $C$ contains two points of $C$ other than $P$?

In the proof of Theorem \cite{1} it is important to show that $q - 2$ automorphisms of $\mathbb{P}^1$ are non-classical, in the sense of Levcovitz \cite{10} p.136. In this paper, this term is used only in an extremal case.

**Definition 1** (Levcovitz \cite{10}). Let $X$ be a smooth projective curve, and let $\varphi : X \to \mathbb{P}^N$ with $N \geq 2$ be a morphism, which is birational onto its image. An automorphism $\sigma \in \text{Aut}(X) \setminus \{1\}$ is said to be non-classical with respect to $\varphi$ if

$$\varphi \circ \sigma(P) \in \left\langle \varphi(P), \frac{d\varphi}{dx}(P) \right\rangle$$

for a general point $P$ of $X$, where $x$ is a local parameter at some point of $X$.

**Remark 1.** If $\sigma \in \text{Aut}(X) \setminus \{1\}$ is non-classical with respect to a birational morphism $\varphi : X \to \mathbb{P}^N$ with $N \geq 3$ onto its image, then $\varphi(X)$ is tangentially degenerate.
This paper presents a method of a construction of tangentially degenerate curves with a birational Gauss map, focusing on the non-classicality of automorphisms. In particular, this paper proves the following:

**Theorem 2.** Let $p > 2$ and $N \geq 3$. If a smooth projective curve $X$ admits a local parameter $x \in k(X)$ at some point $P \in X$, a function $y \in k(X)$, and an automorphism $\sigma \in \text{Aut}(X)$ such that

(a) $y \notin \langle 1, x \rangle$,
(b) $k(X) = k(x, y)$,
(c) $\sigma^*x = x + \alpha$ for some $\alpha \in k \setminus \{0\}$, and
(d) $\sigma^*y - y = \alpha \frac{dy}{dx},$

then there exists a morphism $\varphi : X \to \mathbb{P}^N$, which is birational onto its image, such that

(1) $\varphi(X)$ is not contained in any hyperplane of $\mathbb{P}^N$,
(2) the automorphism $\sigma$ is non-classical with respect to $\varphi$, and
(3) the Gauss map of $\varphi(X)$ is birational onto its image.

In the case where the order of $\sigma$ is $p$, assumptions can be simplified.

**Theorem 3.** Let $p > 2$ and $N \geq 3$. If a smooth projective curve $X$ admits a local parameter $x \in k(X)$ at some point $P \in X$ and an automorphism $\sigma \in \text{Aut}(X)$ such that

(a) $\sigma^*x - x \in k \setminus \{0\}$, and
(b) the order of $\sigma$ is $p$,

then the same assertion as in Theorem 2 holds.

**Remark 2.** If we take $k(X) = k(t)$, $\alpha = 1$, $x = t$ and $y = t^2 - t^p$ (and $g_3 = t^3 + 2t^p - 3t^{p+1}$ in the proof of Theorem 2), then we can recover Esteves–Homma’s embedding.

**Remark 3.** Conditions (a) and (b) in Theorem 3 are satisfied for many examples of Artin–Schreier curves. A curve $X$ with a function field $k(x, y)$ given by an irreducible polynomial $x^q - x - g(y) \in k[x, y]$ such that $g(y) \in k[y]$ and $g_y \neq 0$ is such an example.

### 2. Proofs of main theorems

This paper introduces a vector subspace $V_{\sigma, x} \subset k(X)$ and investigates it, inspired by Esteves–Homma’s functions $t^2 - t^p$ and $t^3 + 2t^p - 3t^{p+1}$. Let $x$ be a local parameter
of a smooth projective curve $X$ at a point $P \in X$, let $\sigma \in \text{Aut}(X) \setminus \{1\}$, and let $f := \sigma^* x - x \in k(X)$. We define the set

$$V_{\sigma,x} := \left\{ g \in k(X) \mid \sigma^* g - g = f \frac{dg}{dx} \right\}.$$ 

**Proposition 1.** Assume that $p > 0$ and $f \neq 0$. Then the following hold.

(a) The set $V_{\sigma,x}$ is a vector subspace of $k(X)$ over $k$ with $x \in V_{\sigma,x}$.

(b) The field $(k(X)^\sigma)^p$ is a subspace of $V_{\sigma,x}$. Furthermore, if the order of $\sigma$ is finite, then the dimensions of $(k(X)^\sigma)^p$ and $V_{\sigma,x}$ are infinite over $k$.

(c) If the order of $\sigma$ is prime, then $k(X) = k(x, y)$ for some $y \in ((k(X)^\sigma)^p) \setminus k$. If $f = \alpha$, that is, $\sigma^* x = x + \alpha$, then $x^2 + \beta x^p \in V_{\sigma,x}$.

(d) Let $n \geq 1$ be an integer, let $\alpha \in k \setminus \{0\}$, and let $\beta \in k$ with $\beta \alpha^p + \alpha^2 = 0$. Then the order of $\sigma$ is finite, and it is divisible by $p$.

(e) If $f \in k \setminus \{0\}$, then the order of $\sigma$ is finite, and it is divisible by $p$.

**Proof.** Let $g, h \in V_{\sigma,x}$ and $a, b \in k$. Then

$$\sigma^*(ag + bh) - (ag + bh) = a\sigma^* g + b\sigma^* h - (ag + bh) = a(\sigma^* g - g) + b(\sigma^* h - h) = af \frac{dg}{dx} + bh \frac{dh}{dx} = f \frac{d}{dx}(ag + bh).$$

On the other hand,

$$\sigma^* x - x = f = \int \frac{dx}{dx}.$$

Assertion (a) follows.

For each $g \in (k(X)^\sigma)^p$,

$$\sigma^* g - g = 0, \quad \frac{dg}{dx} = 0.$$

This implies that $(k(X)^\sigma)^p \subset V_{\sigma,x}$. Assume that the order of $\sigma$ is finite. Then $[k(X) : k(X)^\sigma]$ is finite. Since $[k(X)^\sigma : (k(X)^\sigma)^p]$ is finite, it follows that $[k(X) : (k(X)^\sigma)^p]$ is finite, and hence, the dimension of $(k(X)^\sigma)^p$ is infinite. Assertion (b) follows.

Assume that the order of $\sigma$ is prime. Since $[k(X) : k(X)^\sigma]$ is prime and $x \notin k(X)^\sigma$, it follows that $k(X) = (k(X)^\sigma)(x)$. It is inferred that $k(X)^p \subset (k(X)^\sigma)^p(x)$. Since $k(X)/k(x)$ is finite and separable, there exists $z \in k(X)^p$ such that $k(X) = k(x, z)$ (see, for example, [11, Proposition 3.10.2]). Therefore, there exist $y_1, \ldots, y_n \in (k(X)^\sigma)^p$ such that $k(X) = k(x)(y_1, \ldots, y_n)$. Since $k(X)/k(x)$ is separable, there exists $y \in (k(X)^\sigma)^p$ such that $k(X) = k(x, y)$. If $y \in k$, then $k(X) = k(x)$, and we
can take another \( y \in (k(X)^p)^p \setminus k \) with \( k(x) = k(x, y) \). Since \((k(X)^p)^p \cap \langle 1, x \rangle = k\), it follows that \( y \notin \langle 1, x \rangle \). Assertion (c) follows.

Let \( g = x^2 + \beta x^{p^n} \). Then
\[
\sigma^* g - g = (x + \alpha)^2 + \beta(x + \alpha)^{p^n} - (x^2 + \beta x^{p^n})
= 2\alpha x + (\alpha^2 + \beta \alpha^{p^n}) = 2\alpha x = f \frac{dg}{dx},
\]
and hence, \( g \in V_{\sigma, x} \). Assertion (d) follows.

Let \( f = \alpha \in k \). Then \((\sigma^p)^* x = x + p\alpha = x\), and hence, \( x \in k(X)^{\sigma^p} \). Since \( x \in k(X) \) is transcendental over \( k \), it follows that \( [k(X) : k(X)^{\sigma^p}] \) is finite. This implies that \((\sigma^p)^m = 1 \) for some positive integer \( m \). On the contrary, if \( \sigma^m = 1 \), then \( x = (\sigma^m)^* x = x + ma \). This implies that \( m \) is divisible by \( p \). Assertion (e) follows. \( \square \)

**Remark 4.** For any automorphism \( \sigma \in \text{Aut}(X) \setminus \{1\} \) and any point \( P \in X \), there exists a local parameter \( x \in k(X) \) at \( P \) such that \( \sigma^* x - x \neq 0 \).

We prove main theorems.

**Proof of Theorem 1.** Let \( P_\infty = (0 : 1) \in \mathbb{P}^1 \). By the expression, \( \mathbb{P}^1 \setminus \{P_\infty\} \cong \varphi(\mathbb{P}^1 \setminus \{P_\infty\}) \) as affine varieties. We consider a neighborhood of \( P_\infty \). The morphism \( \varphi \) is represented by
\[
\varphi(s : 1) = \left( s^{2^n+1} : s^{q^{2^n}} : s^{q^{2^n}-1} - s^{q^{2^n+1-q^n}} : s^{q^{2^n+1-q^n}} - s : s^{q^{2^n-q^n}} - 1 \right)
= \left( \frac{s^{q^{2^n}+1}}{s^{q^{2^n}-q^n} - 1} : \frac{s^{q^{2^n}}}{s^{q^{2^n}-q^n} - 1} : \frac{s^{q^{2^n}-1} - s^{q^{2^n+1-q^n}}}{s^{q^{2^n}-q^n} - 1} : s : 1 \right).
\]
Therefore, \( \varphi(P_\infty) = (0 : 0 : 0 : 1) \), and the order of the hyperplane defined by \( X_3 = 0 \) at \( P_\infty \) is equal to 1. This implies that \( \varphi(P_\infty) \) is a smooth point. Assertion (a) follows.

Let \( \sigma_\alpha : \mathbb{P}^1 \to \mathbb{P}^1 \) be an automorphism given by \( t \mapsto t + \alpha \) with \( \alpha^{q^2-2} = 1 \). Since \( \alpha \in \mathbb{F}_{q^n} \), it follows that \( \alpha^{q^n} - \alpha^{q^{2n}} = (\alpha - \alpha^{q^n})q^n = 0 \). Then
\[
\frac{d\varphi}{dt} = (0, 1, 2t, 0, t^{q^n} - t^{q^{2n}}),
\]
\[
\varphi \circ \sigma_\alpha = (1, t + \alpha, t^2 - t^q + 2\alpha t, t^{q^n} - t^{q^{2n}}, (t + \alpha)(t^{q^n} - t^{q^{2n}})).
\]
Let \( P = (1 : t) \in \mathbb{P}^1 \setminus \{P_\infty\} \). It follows that
\[
\varphi(\sigma_\alpha(P)) = \varphi(P) + \alpha \frac{d\varphi}{dt}(P) \in \left\langle \varphi(P), \frac{d\varphi}{dt}(P) \right\rangle.
\]
for any \( \alpha \in k \) with \( \alpha^{q-2} = 1 \). Therefore, there exist \( q-2 \) points of \((T_{\varphi(P)}\varphi(\mathbb{P}^1) \setminus \{\varphi(P)\}) \cap \varphi(\mathbb{P}^1)\). It is not difficult to check that \( \varphi(P_\infty) \notin T_{\varphi(P)}\varphi(\mathbb{P}^1) \). Assume that \( \varphi(P') \in T_{\varphi(P)}\varphi(\mathbb{P}^1) \) for a point \( P' = (1 : u) \in \mathbb{P}^1 \setminus \{P_\infty\} \). Then, for some \( \beta \in k \),

\[
    u = t + \beta \quad \text{and} \quad u^2 - u^q = t^2 - t^q + 2\beta t.
\]

These imply that \( u = t + \beta \) and \( \beta^2 - \beta^q = 0 \). Assertion (b) follows.

The tangent line is spanned by the row vectors of the matrix

\[
\left( \begin{array}{cccc}
\varphi \\
\frac{d\varphi}{dt}
\end{array} \right) \sim \left( \begin{array}{cccc}
1 & 0 & -t^2 - t^q & t^{q^n} - t^{q^{2n}} & 0 \\
0 & 1 & 2t & 0 & t^{q^n} - t^{q^{2n}}
\end{array} \right).
\]

The function field \( k(\gamma \circ \varphi(\mathbb{P}^1)) \) of the image of the Gauss map \( \gamma \) contains the function \( \frac{d(t^2-t^q)}{dt} = 2t \). Since \( p > 2 \), it follows that \( t \in k(\gamma \circ \varphi(\mathbb{P}^1)) \). This implies that the Gauss map \( \gamma \) is birational onto its image. \( \square \)

**Proof of Theorem 2.** Assume that condition (c) is satisfied for an automorphism \( \sigma \in \text{Aut}(X) \). Let \( V_{\sigma,x} \) be the vector space as in Proposition 1 with \( f = \alpha \), and let \( g_3, \ldots, g_N \in V_{\sigma,x} \). We consider the rational map

\[
\varphi : X \dashrightarrow \mathbb{P}^N; \quad (1 : x : y : g_3 : \cdots : g_N).
\]

By condition (b), \( \varphi \) is birational onto its image. By conditions (a) and (d), \( y \notin \langle 1, x \rangle \) and \( y \in V_{\sigma,x} \). According to Proposition 1(b) and (e), we can take \( g_3, \ldots, g_N \in V_{\sigma,x} \) such that \( \dim \langle 1, x, y, g_3, \ldots, g_N \rangle = N + 1 \). This implies assertion (1). Then

\[
\frac{d\varphi}{dx} = \left( 0, 1, \frac{dy}{dx}, \ldots, \frac{dg_i}{dx}, \ldots \right),
\]

\[
\varphi \circ \sigma = (1, x + \alpha, \sigma^*y, \ldots, \sigma^*g_i, \ldots).
\]

Since \( 1, x, y, g_3, \ldots, g_N \in V_{\sigma,x} \), it follows that

\[
\varphi(\sigma(P)) = \varphi(P) + \alpha \frac{d\varphi}{dx}(P) \in \left( \varphi(P), \frac{d\varphi}{dx}(P) \right)
\]

for a general point \( P \) of \( X \). Assertion (2) follows. The tangent line is spanned by the row vectors of the matrix

\[
\left( \begin{array}{cccc}
1 & 0 & y - x \frac{dy}{dx} & \cdots & g_i - x \frac{dg_i}{dx} & \cdots \\
0 & 1 & \frac{dy}{dx} & \cdots & \frac{dg_i}{dx} & \cdots
\end{array} \right).
\]

If we take \( g_i = x^2 + \beta x^{p^n} \) with \( \beta \alpha^{p^n} + \alpha^2 = 0 \) for some \( i \), as in Proposition 1(d), then the function field \( k(\gamma \circ \varphi(X)) \) of the image of the Gauss map \( \gamma \) contains the function \( \frac{dg_i}{dx} = 2x \). Since \( p > 2 \), it follows that \( x \in k(\gamma \circ \varphi(X)) \). In this case,
since \( dy/dx, y - x(dy/dx) \in k(\gamma \circ \varphi(X)) \), it is inferred that \( y \in k(\gamma \circ \varphi(X)) \) and the Gauss map \( \gamma \) is birational onto its image. Assertion (3) follows.

**Proof of Theorem 3.** Assume that conditions (a) and (b) are satisfied for an automorphism \( \sigma \). By Proposition 1 (b) and (c), there exists \( y \in V_{\sigma,x} \setminus \langle 1, x \rangle \) such that \( k(X) = k(x, y) \). Conditions (a), (b), (c) and (d) in Theorem 2 are satisfied.

\[ \square \]

**Remark 5.**

(a) The automorphism \( \sigma \) is non-classical with respect to the plane model given by conditions (a), (b), (c) and (d) in Theorem 2.

(b) The same assertion as Theorem 2 without condition (3) holds, if \( p = 2 \).

(c) Assume that \( p > 0 \) and \( N \geq 3 \). By Proposition 1 and the same method as in the proof of Theorem 2, it can be proved that for any automorphism \( \sigma \in \text{Aut}(X) \setminus \{1\} \) of prime order, there exists a birational embedding \( \varphi : X \to \mathbb{P}^N \) with conditions (1) and (2) as in Theorem 2.

Finally, this paper raises the following:

**Question 2.** Assume that a smooth projective curve \( C \subset \mathbb{P}^N \) not in any plane is tangentially degenerate and the Gauss map of \( C \) is birational onto its image. Then is it true that \( C \) is rational?

**Acknowledgements**

The author is grateful to Professor Seiji Nishioka for helpful conversations, by which the author was able to improve Proposition 1. The author thanks Professor Hajime Kaji for telling Question 1 and useful comments.

**References**

[1] M. Bolognesi and G. Pirola, Osculating spaces and Diophantine equations (with an Appendix by P. Corvaja and U. Zannier), Math. Nachr. 284 (2011), 960–972.
[2] C. Ciliberto, Review of [6], Mathematical Reviews, MR0850959 (87i:14027).
[3] E. Esteves and M. Homma, Order sequences and rational curves, In: Projective geometry with applications, Lecture Notes in Pure and Appl. Math. 166, Dekker, New York, 1994, pp.27–42.
[4] A. Hefez and S. Kleiman, Notes on the duality of projective varieties, in: Geometry Today, Progress in Mathematics Vol. 60, Birkhäuser, Boston, 1985, pp.143–183.
[5] A. Hefez and J. Voloch, Frobenius non classical curves, Arch Math. 54 (1990), 263–273.
[6] H. Kaji, On the tangentially degenerate curves, J. London Math. Soc. (2) 33 (1986), 430–440.
[7] H. Kaji, On the inseparable degrees of the Gauss map and the projection of the conormal variety to the dual of higher order for space curves, Math. Ann. 292 (1992), 529–532.
[8] H. Kaji, On the tangentially degenerate curves, II, Bull. Braz. Math. Soc. (N.S.) 45 (2014), 745–752.
[9] H. Kaji, Private communications, November 2021.
[10] D. Levcovitz, Bounds for the number of fixed points of automorphisms of curves, Proc. London Math. Soc. (3) 62 (1991), 133–150.
[11] H. Stichtenoth, Algebraic Function Fields and Codes, Graduate Texts in Mathematics 254, Springer-Verlag, Berlin Heidelberg, 2009.
[12] K.-O. Stöhr and J. F. Voloch, Weierstrass points and curves over finite fields, Proc. London Math. Soc. (3) 52 (1986), 1–19.
[13] A. Terracini, Sulla riducibilità di alcune particolari corrispondenze algebriche, Rend. Circ. Mat. Palermo 56 (1932), 112–143.
[14] J. Voloch, On the geometry of space curves, in: Proceedings of the 10th School of Algebra, Vitoria, Brazil, 1989 (IMPA, Rio de Janeiro, 1990), pp.121–122.

DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE, YAMAGATA UNIVERSITY, KOJIRAKAWA-MACHI 1-4-12, YAMAGATA 990-8560, JAPAN

Email address: s.fukasawa@sci.kj.yamagata-u.ac.jp