An upper bound for the minimum modulus in a covering system with squarefree moduli

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Abstract

Based on work of P. Balister, B. Bollobás, R. Morris, J. Sahasrabudhe and M. Tiba, we show that if a covering system has distinct squarefree moduli, then the minimum modulus is at most 118. We also show that in general the \(k\)-th smallest modulus in a covering system with distinct moduli (provided it is required for the covering) is bounded by an absolute constant.

Keywords: covering system, squarefree, minimum modulus problem

1 Introduction

A covering system (or a covering) is a finite system of congruences \(x \equiv a_j \pmod{m_j}, j \in \{1, 2, \ldots, r\}\), such that every integer satisfies at least one of the congruences. Of particular interest is the case in which all the moduli are distinct. In 1950, P. Erdős [1] wrote, “It seems likely that for every \(c\) there exists such a system all the moduli of which are > \(c\).” In other words, Erdős felt that the minimum modulus in a covering system having distinct moduli can be arbitrarily large. Establishing whether that is indeed the case has become known as the minimum modulus problem for covering systems, and Erdős offered $1000 for a resolution to the problem [2, Section F13]. The minimum modulus problem has only fairly recently been resolved by R. Hough [3] who showed, contrary to what Erdős believed, the minimum modulus is
bounded and in particular $\leq 10^{16}$. More recent work by P. Balister, B. Bollobás, R. Morris, J. Sahasrabudhe and M. Tiba [4] has brought to light some new ideas which have led to a simpler argument producing an upper bound of 615999 on the minimum modulus.

This paper concerns the related question of covering systems with distinct moduli that are all squarefree, that is each modulus is not divisible by the square of a prime. In [3] and [5], Hough and Balister, Bollobás, Morris, Sahasrabudhe and Tiba, respectively, use the case of squarefree moduli to illustrate their approaches to the minimum modulus problem. In particular, the latter give a simple exposition of their distortion method by showing that the minimum modulus in the case of distinct squarefree moduli is bounded. They end their paper by indicating that a direct application of what is written there gives a “(fairly terrible) bound of roughly $\exp(10200)$ for the minimum modulus in a covering system” having distinct squarefree moduli. As they note, finding a small bound in this case is not really the point, given the more interesting minimum modulus problem in which the moduli are not necessarily squarefree is resolved now with a bound of 615999 on the minimum modulus.

Nevertheless, Erdős pointed out that the congruences

\[
\begin{align*}
n &\equiv 0 \pmod{2} & n &\equiv 1 \pmod{10} & n &\equiv 4 \pmod{35} \\
n &\equiv 0 \pmod{3} & n &\equiv 1 \pmod{14} & n &\equiv 5 \pmod{42} \\
n &\equiv 0 \pmod{5} & n &\equiv 2 \pmod{15} & n &\equiv 59 \pmod{70} \\
n &\equiv 1 \pmod{6} & n &\equiv 2 \pmod{21} & n &\equiv 104 \pmod{105}
\end{align*}
\]

provide an example of a covering in which the moduli are distinct squarefree numbers, and it is unknown as to whether there is such an example where the minimum modulus is $> 2$. This problem, due to J. Selfridge, goes back to at least 1981, being cited with the example above in [6, Section F13]. Thus, the problem of obtaining a good upper bound on the minimum modulus problem in the case of distinct squarefree moduli is an interesting one.

In this paper, we make the following progress on this question.

**Theorem 1.1** Every covering system with distinct squarefree moduli has a minimum modulus which is $\leq 118$.

Our arguments are not novel in that we basically take the eloquent exposition given by Balister, Bollobás, Morris, Sahasrabudhe and Tiba in [5] and simply refine the arguments to produce our upper bound.

There is a natural question of whether a bound can be given on the second, third, etc., smallest modulus of a covering system with distinct (possibly squarefree) moduli. We give a simple elementary argument that the following is a consequence of R. Hough’s initial work on this topic [3].
Theorem 1.2 Fix a non-negative integer $k$. Then there exists a $B(k + 1)$ satisfying the following. Let $C$ be a covering with moduli $m_1 < m_2 < \cdots < m_r$ and congruences

\[ x \equiv a_j \pmod{m_j}, \quad 1 \leq j \leq r, \tag{1} \]

satisfying $r \geq k + 1$ and the first $k$ congruencies in (1) do not form a covering of the integers. Then $m_{k+1} \leq B(k + 1)$.

This leads to some further questions. Our proof can be adjusted easily to be constructive, though we will make no attempt to find explicit bounds on $B(k)$ as our approach is undoubtedly not optimal. How small can one make such bounds? In particular, what is the smallest possible value of $B(k)$? Theorem 1.2 implies the same result holds in the case that the moduli $m_j$, $1 \leq j \leq k$, are all squarefree. What are the answers to these questions in the case of squarefree moduli? Knowing the various moduli in a covering system are bounded as in Theorem 1.2 but taking into account that the number of moduli in a covering system can be arbitrarily large, is it possible to give a complete classification of all possible covering systems involving distinct moduli? A similar question can be asked in the case that the moduli are not distinct, though this will necessarily be a larger classification.

Before leaving this introduction, we mention some related literature. Some other recent work on this subject include [7–12]. More information related to covering systems with distinct squarefree moduli can be found in Kruckenberg’s dissertation [13]. For example, he shows that the only covering systems with distinct squarefree moduli where the least common multiple $L$ of the moduli is the product of 4 or fewer primes is when $L = 210$ and that there is a covering system with distinct squarefree moduli which uses the modulus 2 but not the modulus 3. A number of applications of covering systems beyond the paper by Erdős [1] can be found in the references given in [14].

2 Preliminary background

In this section, as well as the next, our approach is based on the work of P. Balister, B. Bollobás, R. Morris, J. Sahasrabudhe, and M. Tiba in [5] where they give the basic idea behind their method in [4] but restrict to the case where the moduli are distinct squarefree numbers. Our goal is simply to refine the ideas to allow for the bound 118 in Theorem 1.1. However, we will make use of weighted sums instead of probabilities to give a slightly different perspective without any real change in content.

We begin with the following.

Lemma 2.1 Let $C$ be a system of congruences consisting of moduli $m_1, \ldots, m_r$, and set $L = \text{lcm}(m_1, \ldots, m_r)$. Then, $C$ is a covering system if and only if every integer in $[1, L]$ is satisfied by a congruence in $C$. 

For example, consider the covering:

\[
\begin{align*}
    n &\equiv 0 \pmod{2} & n &\equiv 3 \pmod{8} \\
    n &\equiv 0 \pmod{3} & n &\equiv 7 \pmod{12} \\
    n &\equiv 1 \pmod{4} & n &\equiv 23 \pmod{24}.
\end{align*}
\]

Here \( L = 24 \). Lemma 2.1 implies that we only need to check that each integer from 1 to 24 satisfies at least one congruence to verify that the above is a covering system, and this can easily be done.

**Proof of Lemma 2.1.** Let \( C \) be a system of congruences with the least common multiple of the moduli equal to \( L \). If \( C \) is a covering system, then every integer in \([1, L]\) satisfies a congruence in \( C \). Now, suppose every integer in \([1, L]\) satisfies a congruence in \( C \). Let \( n \) be an integer. Let \( a \equiv n \pmod{L} \) where \( 1 \leq a \leq L \). Then, there exists a congruence \( x \equiv b \pmod{m} \) in \( C \) such that \( a \equiv b \pmod{m} \). Since \( a \equiv n \pmod{L} \) and \( m \) divides \( L \), we have \( n \equiv a \equiv b \pmod{m} \). Therefore, \( n \) satisfies the congruence \( x \equiv b \pmod{m} \) in \( C \). Thus, \( C \) is a covering system, and the lemma follows. \( \square \)

We return to the example given in the introduction involving squarefree moduli, where the least common multiple of the moduli is \( L = 210 = 2 \cdot 3 \cdot 5 \cdot 7 \). One can use Lemma 2.1 to quickly verify that the congruences given there form a covering system of the integers. We can also think of covering the associated elements of \( Q = S_1 \times S_2 \times S_3 \times S_4 \) where \( S_1 = \{1, 2\} \), \( S_2 = \{1, 2, 3\} \), \( S_3 = \{1, 2, 3, 4, 5\} \), and \( S_4 = \{1, 2, 3, 4, 5, 6, 7\} \). With this approach each congruence covers a portion of \( Q \). For example, the congruence \( n \equiv 0 \pmod{2} \) covers \( \{2\} \times S_2 \times S_3 \times S_4 \subset Q \). By the Chinese Remainder Theorem, each integer in \([1, L]\) uniquely corresponds to an element of \( Q \). For example, the integer 77 corresponds to \((1, 2, 2, 7)\) in \( Q \) since \( 77 \equiv 1 \pmod{2} \), \( 77 \equiv 2 \pmod{3} \), \( 77 \equiv 2 \pmod{5} \), and \( 77 \equiv 7 \pmod{7} \). Given Lemma 2.1, for distinct squarefree moduli, we can view a covering system as a system of congruences with the product of the moduli \( L \) for which each element of \( Q \) corresponds to an integer that satisfies at least one of the congruences.

In general, we let \( S_1, \ldots, S_n \) be finite sets. We define a hyperplane to be \( A = Y_1 \times \cdots \times Y_n \) where \( Y_j \subseteq S_j \) and \( |Y_j| \in \{1, |S_j|\} \) for \( j \in \{1, \ldots, n\} \). We also define two hyperplanes \( A \) and \( A' \) to be parallel if \( F(A) = F(A') \) where \( F(A) = \{ j : |Y_j| = 1 \} \). We call \( F(A) \) the set of fixed coordinates of \( A \). We consider the set of natural numbers \( \mathbb{N} \) to be the set of positive integers.

**Theorem 2.2** (P. Balister, B. Bollobás, R. Morris, J. Sahasrabudhe, and M. Tiba [5]) For every sequence of finite sets \( S_1, S_2, \ldots, \) each of size at least 2, satisfying \( \liminf_{k \to \infty} |S_k|/k > 3 \), there is a positive integer \( C \) such that the following holds. Let \( \mathcal{A} \) be a collection of hyperplanes that cover \( Q = S_1 \times \ldots \times S_n \) for some \( n \in \mathbb{N} \), (that is, every element of \( Q \) is on some hyperplane in \( \mathcal{A} \)).
Suppose no two hyperplanes in \( A \) are parallel. Then, there exists a hyperplane \( A \in A \) with \( F(A) \subseteq \{1, \ldots, C\} \).

The above result is the main result in [5]. To connect it to our earlier discussion, let \( \mathcal{C} \) be a covering system with distinct squarefree moduli. Let \( p_j \) denote the \( j^{\text{th}} \) prime, and set \( S_j = \{1, \ldots, p_j\} \). Observe that each \( S_j \) is of size at least 2, and we have \( \liminf_{k \to \infty} |S_k|/k = \lim_{k \to \infty} p_k/k = \infty > 3 \). Let \( C \) be a positive integer as in Theorem 2.2. Let \( p_n \) be the largest prime dividing a modulus in \( \mathcal{C} \). Set \( Q = S_1 \times \cdots \times S_n \). Each congruence \( x \equiv a \pmod{m} \) in \( \mathcal{C} \) corresponds to a hyperplane \( A_m = Y_1 \times \cdots \times Y_n \subseteq Q \) where (i) if \( p_j \) divides \( m \), then \( Y_j = \{b\} \) with \( b \equiv a \pmod{p_j} \) and \( b \in S_j \), and (ii) if \( p_j \) does not divide \( m \), then \( Y_j = S_j \). Then, the covering \( \mathcal{C} \) corresponds to a finite collection of hyperplanes \( A \) which covers \( Q \). Note that the moduli of \( \mathcal{C} \) are distinct, so the hyperplanes in \( A \) are pairwise non-parallel. Thus, by Theorem 2.2, there exists an \( A_m \in A \) with \( F(A_m) \subseteq \{1, \ldots, C\} \). Observe that this \( m \) divides \( p_1 \cdots p_C \). Therefore, we can obtain that the minimum modulus must be \( \leq p_1 \cdots p_C \).

Our goal is to slightly alter the proof of Theorem 2.2 in [5] to give our proof of Theorem 1.1.

3 Further background

We will prove Theorem 2.2 by expanding upon the method outlined in [5]. Let \( S_1, S_2, \ldots \) be an infinite sequence of finite sets. As discussed at the end of the previous section, we will be connecting these sets to our covering system in Theorem 1.1 by taking \( S_j = \{1, \ldots, p_j\} \), for each \( j \), where \( p_j \) denotes the \( j^{\text{th}} \) prime. For a positive integer \( k \), define \( Q_k = S_1 \times \cdots \times S_k \). Fix a positive integer \( n \). Let \( A \) be a collection of hyperplanes, pairwise non-parallel, that cover \( Q_n \).

We define a \textit{weight} on a set \( X = \{x_1, \ldots, x_k\} \) to be a function mapping each \( x_i \) to \( q_i \geq 0 \) for \( 1 \leq i \leq k \) such that \( q_1 + \cdots + q_k = 1 \). As we are setting the sum of the weights \( q_i \) equal to 1, these weights can be viewed as a probability assigned to the elements of \( X \), as done in [5].

Let \( Q = Q_n \). We define weights \( w_n(x) \) on the elements \( x \) of \( Q \). From the previous section, for covering systems, these weights correspond to weights on the integers in the interval \([1, L]\) where \( L = p_1 \cdots p_n \). The weight of a subset \( T \subseteq Q \) is defined as the sum of the weights of the elements in \( T \), so \( w_n(T) = \sum_{x \in T} w_n(x) \). We interpret this to mean \( w_n(\emptyset) = 0 \). If \( x = (a_1, \ldots, a_{k-1}) \in S_1 \times \cdots \times S_{k-1} \) and \( y \in S_k \), then we write \( w_k(x, y) = w_k((a_1, \ldots, a_{k-1}, y)) = w_n(A) \) where \( A \) is the hyperplane

\[ A = \{a_1\} \times \{a_2\} \times \cdots \times \{a_{k-1}\} \times \{y\} \times S_{k+1} \times \cdots \times S_n. \]  

In general, if \( X \subseteq S_1 \times \cdots \times S_k \), then we identify \( w_k(X) \) with \( w_n(X \times S_{k+1} \times \cdots \times S_n) \). The fiber \( F_x \) associated to \( x = (a_1, \ldots, a_{k-1}) \in S_1 \times \cdots \times S_{k-1} \) is the set of tuples \( (a_1, \ldots, a_{k-1}, y) \) with \( y \in S_k \). At the \( k^{\text{th}} \) stage, we will determine
the weights of the hyperplanes in the form of (2). We define

$$A_k = \{ A \in A : \max(F(A)) = k \}.$$ 

In the languages of congruences, $A_1$ corresponds to the set of congruences modulo $p_1 = 2$, $A_2$ corresponds to the set of congruences modulo $p_2 = 3$ and $p_1p_2 = 6$, and so on. We also define

$$B_k = \bigcup_{A \in A_k} A.$$ 

With regard to coverings, $B_k$ corresponds to the elements of $Q_n$ which are covered by a congruence with a modulus whose largest prime divisor is $p_k$. Note that if $A \in A_k$, then $F(A) \subseteq \{ 1, \ldots, k \}$, so $B_k$ can also be thought of as a subset of $Q_k = S_1 \times \ldots \times S_k$.

In our proof of Theorem 2.2, by assigning weights to elements of $Q$ in the manner below and supposing $F(A) \not\subseteq \{ 1, \ldots, C \}$ for every hyperplane $A \in A$, we prove that the collection of hyperplanes $A$ does not cover $Q$ to obtain our result by contradiction.

For each $k$, we will choose $\delta_k \in [0, 1/2]$. We define weights $w_k$ inductively as follows. As noted above, we can view $B_k$ as a subset of $Q_k = S_1 \times \ldots \times S_k$, and do so. When $k = 1$, if $y \in S_1$ and $|B_1|/|S_1| \leq \delta_1$, we set

$$w_1(y) = \begin{cases} 0 & \text{if } y \in B_1 \\ \frac{1}{|S_1| - |B_1|} & \text{if } y \notin B_1. \end{cases}$$

If $y \in S_1$ and $|B_1|/|S_1| > \delta_1$, we set

$$w_1(y) = \begin{cases} \frac{(|B_1|/|S_1|) - \delta_1}{(|B_1|/|S_1|)(1 - \delta_1)} \cdot \frac{1}{|S_1|} & \text{if } y \in B_1 \\ \frac{1}{1 - \delta_1} \cdot \frac{1}{|S_1|} & \text{if } y \notin B_1. \end{cases}$$

Observe that in both cases, we have $\sum_{y \in S_1} w_1(y) = 1$; for example, if $|B_1|/|S_1| > \delta_1$, then

$$\sum_{y \in S_1} w_1(y) = \sum_{y \in B_1} w_1(y) + \sum_{y \notin B_1} w_1(y)$$

$$= \sum_{y \in B_1} \frac{(|B_1|/|S_1|) - \delta_1}{(|B_1|/|S_1|)(1 - \delta_1)} \cdot \frac{1}{|S_1|} + \sum_{y \notin B_1} \frac{1}{1 - \delta_1} \cdot \frac{1}{|S_1|}$$

$$= \frac{(|B_1|/|S_1|) - \delta_1}{(|B_1|/|S_1|)(1 - \delta_1)} \cdot \frac{|B_1|}{|S_1|} + \frac{1}{1 - \delta_1} \cdot \frac{|S_1| - |B_1|}{|S_1|} = 1.$$
The above weights correspond to setting $k = 1$ and replacing $\alpha_1(x)$ with $|B_1|/|S_1|$ and $w_0(x)$ with 1 in the discussion below.

Suppose $k \geq 2$ and $w_{k-1}$ is defined on $Q_{k-1}$. For each $x \in Q_{k-1}$, we define

$$
\alpha_k(x) = \frac{|\{y \in S_k : (x, y) \in B_k\}|}{|S_k|} = \frac{|F_x \cap B_k|}{|S_k|},
$$

which is the proportion of the fiber $F_x = \{(x, y) : y \in S_k\}$ that is covered by one or more hyperplanes in $A_k$. If $\alpha_k(x) \leq \delta_k$, we set

$$
w_k(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in B_k \\
\frac{1}{1 - \alpha_k(x)} \cdot \frac{w_{k-1}(x)}{|S_k|} & \text{if } (x, y) \notin B_k.
\end{cases}
$$

If $\alpha_k(x) > \delta_k$, we set

$$
w_k(x, y) = \begin{cases} 
\frac{\alpha_k(x) - \delta_k}{\alpha_k(x)(1 - \delta_k)} \cdot \frac{w_{k-1}(x)}{|S_k|} & \text{if } (x, y) \in B_k \\
\frac{1}{1 - \delta_k} \cdot \frac{w_{k-1}(x)}{|S_k|} & \text{if } (x, y) \notin B_k.
\end{cases}
$$

In both the cases $\alpha_k(x) \leq \delta_k$ and $\alpha_k(x) > \delta_k$, we justify that

$$
\sum_{y \in S_k} w_k(x, y) = w_{k-1}(x),
$$

so weight is preserved along the fibers with each increase of $k$. If $x$ is an element of $Q_{k-1}$ and $\alpha_k(x) \leq \delta_k$, we have

$$
\sum_{y \in S_k} w_k(x, y) = \sum_{\substack{y \in S_k \\ (x, y) \in B_k}} w_k(x, y) + \sum_{\substack{y \in S_k \\ (x, y) \notin B_k}} w_k(x, y)
= \sum_{\substack{y \in S_k \\ (x, y) \in B_k}} 0 + \sum_{\substack{y \in S_k \\ (x, y) \notin B_k}} \frac{1}{1 - \alpha_k(x)} \cdot \frac{w_{k-1}(x)}{|S_k|}
= \frac{1}{1 - \alpha_k(x)} \cdot \frac{w_{k-1}(x)}{|S_k|} \sum_{y \in S_k \atop (x, y) \notin B_k} 1
= \frac{1}{1 - \alpha_k(x)} \cdot \frac{w_{k-1}(x)}{|S_k|} \cdot (|S_k| - |S_k|\alpha_k(x))
= w_{k-1}(x),
$$

where we have used the definition of $\alpha_k(x)$ in the second from the last equality. Also, if $x$ is an element of $Q_{k-1}$ and $\alpha_k(x) > \delta_k$, we then have

$$
\sum_{y \in S_k} w_k(x, y) = \sum_{\substack{y \in S_k \\ (x, y) \in B_k}} w_k(x, y) + \sum_{\substack{y \in S_k \\ (x, y) \notin B_k}} w_k(x, y)
= \sum_{\substack{y \in S_k \\ (x, y) \in B_k}} 0 + \sum_{\substack{y \in S_k \\ (x, y) \notin B_k}} \frac{\alpha_k(x) - \delta_k}{\alpha_k(x)(1 - \delta_k)} \cdot \frac{w_{k-1}(x)}{|S_k|}
= \frac{\alpha_k(x) - \delta_k}{\alpha_k(x)(1 - \delta_k)} \cdot \frac{w_{k-1}(x)}{|S_k|} \sum_{y \in S_k \atop (x, y) \notin B_k} 1
= \frac{\alpha_k(x) - \delta_k}{\alpha_k(x)(1 - \delta_k)} \cdot \frac{w_{k-1}(x)}{|S_k|} \cdot (|S_k| - |S_k|\alpha_k(x))
= w_{k-1}(x),
$$

where we have used the definition of $\alpha_k(x)$ in the second from the last equality.
Thus, weight is preserved along the fibers with each increase of $k$, so that in particular we have $\sum_{x \in Q} w_k(x) = 1$ since as already noted the equation holds for $k = 1$. In other words, as we extend our definition of the weights from $w_{k-1}(x)$ for $x \in Q_{k-1}$ to $w_k(x)$ for $x \in Q_k$, we maintain the property that the sum of all the weights is 1. With $A$ and $Q = Q_n$ as in Theorem 2.2, note that each hyperplane $A \in A$ belongs to exactly one set $B_k$ for $1 \leq k \leq n$. Thus, the following holds.

**Lemma 3.1** Let $A$ be a collection of hyperplanes in $Q = S_1 \times \ldots \times S_n$. If

$$\sum_{k=1}^{n} w_k(B_k) < 1,$$

then $A$ does not cover $Q$.

The basic idea therefore is to show the inequality in Lemma 3.1 when $A$ comes from a set of congruences with distinct squarefree moduli $> 118$. This idea describes the basic approach of the authors in [4, 5] as well.

### 4 Upper bounds on $w_k(B_k)$

For any element $x \in Q_{k-1}$ and any element $y \in S_k$, we justify that

$$w_k(x, y) \leq \frac{1}{1 - \delta_k} \cdot \frac{w_{k-1}(x)}{|S_k|} \quad \text{for } k \geq 2. \quad (3)$$

In the case that $k = 1$, by considering $|B_1|/|S_1| \leq \delta_1$ and $|B_1|/|S_1| > \delta_1$ separately, similar to the argument which follows for $k \geq 2$, one can easily
verify (3) with \( w_0(x) \) replaced by 1. For \( k \geq 2 \) and \( \alpha_k(x) \leq \delta_k \), we have
\[
w_k(x, y) \leq \frac{1}{1 - \alpha_k(x)} \cdot \frac{w_{k-1}(x)}{|S_k|} \leq \frac{1}{1 - \delta_k} \cdot \frac{w_{k-1}(x)}{|S_k|}.
\]
If \( k \geq 2 \) and \( \alpha_k(x) > \delta_k \) and \( (x, y) \not\in B_k \), our result holds by the definition of \( w_k(x, y) \). If \( k \geq 2 \) and \( \alpha_k(x) > \delta_k \) and \( (x, y) \in B_k \), then we obtain
\[
w_k(x, y) \leq \frac{\alpha_k(x) - \delta_k}{\alpha_k(x)(1 - \delta_k)} \cdot \frac{w_{k-1}(x)}{|S_k|}
= \left( \frac{1}{1 - \delta_k} - \frac{\delta_k}{\alpha_k(x)(1 - \delta_k)} \right) \cdot \frac{w_{k-1}(x)}{|S_k|}
\leq \frac{1}{1 - \delta_k} \cdot \frac{w_{k-1}(x)}{|S_k|}.
\]
Thus, (3) holds.

For a hyperplane \( A = Y_1 \times \ldots \times Y_n \) and a set \( U \subseteq \{1, \ldots, n\} \), we define
\[A^U = Y_1^U \times \ldots \times Y_n^U\] to be the hyperplane with \( Y_i^U = Y_i \) if \( i \in U \) and \( Y_i^U = S_i \) if \( i \not\in U \). We set \( A' = A^{\{1, \ldots, k-1\}} \). For each \( J \subseteq \{1, \ldots, n\} \), we define
\[
\nu(J) = \prod_{j \in J} \frac{1}{(1 - \delta_j)|S_j|} \quad \text{and} \quad \|J\| = \prod_{j \in J} |S_j|.
\]
To clarify, we set, as usual, empty products to be 1 so that \( \|\emptyset\| = 1 \).

If \( A \) is a hyperplane corresponding to some congruence with squarefree modulus \( m \) in a covering system, then \( \|F(A)\| = m \). We are interested in showing that the modulus of some congruence is bounded above by \( C_0 = 118 \). We use \( C_0 \) instead of 118 for the moment to clarify that most of what is done below is independent of the value of \( C_0 \), so the reader can view this for the time being as a variable to be determined. We assume
\[
\|F(A)\| > C_0 \quad \text{for all} \ A \in \mathcal{A}, \quad \text{(4)}
\]
with a goal of obtaining a contradiction.

Our next estimate will help us formulate a bound on \( w_k(B_k) \) which we will use when \( k \) is small.

**Lemma 4.1** Let \( \mathcal{A} \) be a collection of hyperplanes, pairwise non-parallel. Then, for \( k \geq 1 \), we have
\[
w_k(B_k) \leq \sum_{A \in \mathcal{A}_k} w_k(A) \leq \sum_{A \in \mathcal{A}_k} \nu(F(A)) = \sum_{A \in \mathcal{A}_k} \prod_{j \in F(A)} \frac{1}{(1 - \delta_j)|S_j|}.
\]
Proof. Since \( B_k = \bigcup_{A \in \mathcal{A}_k} A \), we have \( w_k(B_k) \leq \sum_{A \in \mathcal{A}_k} w_k(A) \). We will induct on \( k \) to prove \( w_k(A) \leq \nu(F(A)) \) for \( A \in \mathcal{A}_k \). For the induction, we will want more generally to look at hyperplanes not necessarily in \( \mathcal{A} \) as well. For this reason, we denote by \( \mathcal{A}^\text{all}_k \) the set of all hyperplanes \( A \) in \( S_1 \times S_2 \times \cdots \times S_k \) (or equivalently in \( S_1 \times S_2 \times \cdots \times S_n \)) for which \( \max(F(A)) = k \). We justify by induction that

\[
w_k(A) \leq \nu(F(A)) = \prod_{j \in F(A)} \frac{1}{(1 - \delta_j)|S_j|} \quad \text{for all } A \in \mathcal{A}^\text{all}_k. \tag{5}\]

For our base case, consider \( k = 1 \). With \( k = 1 \) and \( A \in \mathcal{A}^\text{all}_1 \), we see that \( F(A) = \{1\} \). Since \( F(A) = \{1\} \), we obtain \( A = \{y\} \) (or equivalently \( A = \{y\} \times S_2 \times \cdots \times S_n \)) for some \( y \in S_1 \). The comment about the case \( k = 1 \) after (3) now implies

\[
w_1(A) = w_1(y) \leq \frac{1}{1 - \delta_1} \cdot \frac{1}{|S_1|} = \nu(\{1\}).
\]

Thus, (5) holds when we restrict to \( A \in \mathcal{A}^\text{all}_1 \).

For our inductive step, suppose that for some \( k \in \{2, \ldots, n\} \), we have \( w_j(A) \leq \nu(F(A)) \) whenever \( A \in \mathcal{A}^\text{all}_j \), where \( 1 \leq j < k \). Let \( A \in \mathcal{A}^\text{all}_k \). As before, we have \( k \in F(A) \). With \( A' = A^{\{1, \ldots, k-1\}} \), we obtain from (3) that

\[
w_k(A) \leq \frac{1}{(1 - \delta_k)|S_k|} \cdot w_{k-1}(A').
\]

Since \( A' \subseteq \{1, \ldots, k-1\} \) and \( F(A') = F(A) \setminus \{k\} \), then by our inductive hypothesis, we have \( w_{k-1}(A') \leq \nu(F(A) \setminus \{k\}) \). Thus, for \( k \geq 1 \), we have

\[
w_k(A) \leq \frac{1}{(1 - \delta_k)|S_k|} \cdot w_{k-1}(A') \leq \frac{1}{(1 - \delta_k)|S_k|} \cdot \nu(F(A) \setminus \{k\}) = \nu(F(A)).
\]

This completes the induction argument.

We are now able to conclude

\[
w_k(B_k) \leq \sum_{A \in \mathcal{A}_k} w_k(A) \leq \sum_{A \in \mathcal{A}_k} \nu(F(A)) = \sum_{A \in \mathcal{A}_k} \prod_{j \in F(A)} \frac{1}{(1 - \delta_j)|S_j|},
\]

which completes our proof. \( \square \)
Corollary 4.2 Let $A$ be a collection of hyperplanes, pairwise non-parallel, satisfying (4). Then

$$w_k(B_k) \leq \frac{1}{(1 - \delta_k)|S_k|} \sum_{J \subseteq \{1, \ldots, k-1\}} \nu(J)$$

$$= \frac{1}{(1 - \delta_k)|S_k|} \sum_{J \subseteq \{1, \ldots, k-1\}} \prod_{j \in J} \frac{1}{(1 - \delta_j)|S_j|}.$$

Proof. For $A \in \mathcal{A}_k$, we write

$$F(A) = J \cup \{k\},$$

where $J \subseteq \{1, \ldots, k-1\}$. For such $A$ and $J$, we have

$$\|F(A)\| = \|J\| \cdot |S_k|.$$ In particular, $\|F(A)\| > C_0$ is equivalent to $\|J\| > C_0/|S_k|$. Also, since the hyperplanes in $\mathcal{A}$ are pairwise non-parallel, different $A \in \mathcal{A}_k$ correspond to different $J \subseteq \{1, \ldots, k-1\}$. Since every $A \in \mathcal{A}_k$ satisfies (4), the result follows from Lemma 4.1. \qed

For our next bound, we define the weighted sum

$$E_{k-1} = \begin{cases} \sum_{x \in Q_{k-1}} \alpha_k(x)^2 w_{k-1}(x) & \text{if } k \geq 2 \\ \left(|B_1|/|S_1\right)^2 & \text{if } k = 1. \end{cases}$$

This weighted sum can be viewed as the expected value of $\alpha_k(x)^2$ and, for this reason, was denoted $E_{k-1}[\alpha_k(x)^2]$ in [5]. We will only treat the weighted sum through the definition above. We state our next result for $k \geq 1$, but note that the result and separate argument for $k = 1$ is not needed in the rest of the paper.

Lemma 4.3 Let $A$ be a collection of hyperplanes in $Q = S_1 \times \ldots \times S_n$. Let $k \geq 1$, and suppose $\delta_k \in (0, 1/2]$ (i.e., $\delta_k \neq 0$). Then

$$w_k(B_k) \leq \frac{1}{4\delta_k(1 - \delta_k)} E_{k-1}.$$


Proof. First, consider $w_k(B_k)$ with $k \geq 2$. We have

$$w_k(B_k) = \sum_{x \in Q_{k-1}} \sum_{y \in S_k} w_k(x, y)$$

$$\leq \sum_{x \in Q_{k-1}} |F_x \cap B_k| \cdot \max \left\{ 0, \frac{\alpha_k(x) - \delta_k}{\alpha_k(x)(1 - \delta_k)} \right\} \cdot \frac{w_{k-1}(x)}{|S_k|}.$$ 

Since $\alpha_k(x) = |F_x \cap B_k|/|S_k|$, we obtain

$$w_k(B_k) \leq \frac{1}{1 - \delta_k} \sum_{x \in Q_{k-1}} \max\{0, \alpha_k(x) - \delta_k\} \cdot w_{k-1}(x).$$

An important observation from [5] is that

$$4\delta_k^2 - 4\delta_k \alpha_k(x) + \alpha_k(x)^2 = (2\delta_k - \alpha_k(x))^2 \geq 0,$$

so $\alpha_k(x)^2/4\delta_k \geq \alpha_k(x) - \delta_k$. Thus,

$$w_k(B_k) \leq \frac{1}{1 - \delta_k} \sum_{x \in Q_{k-1}} \frac{\alpha_k(x)^2}{4\delta_k} \cdot w_{k-1}(x)$$

$$= \frac{1}{4\delta_k(1 - \delta_k)} \sum_{x \in Q_{k-1}} \alpha_k(x)^2 \cdot w_{k-1}(x)$$

$$= \frac{1}{4\delta_k(1 - \delta_k)} E_{k-1}.$$ 

In the case that $k = 1$, we have

$$w_1(B_1) = \sum_{y \in B_1} w_1(y) \leq |B_1| \cdot \max \left\{ 0, \frac{|B_1|/|S_1| - \delta_1}{(|B_1|/|S_1|)(1 - \delta_1)} \right\} \cdot \frac{1}{|S_1|}.$$ 

Following the arguments above with $\alpha_k(x)$ replaced by $|B_1|/|S_1|$, we obtain

$$w_1(B_1) \leq \frac{1}{4\delta_1(1 - \delta_1)} \left( \frac{|B_1|}{|S_1|} \right)^2 = \frac{1}{4\delta_1(1 - \delta_1)} E_0.$$ 

The lemma follows. \qed

Lemma 4.4 Let $\mathcal{A}$ be a collection of hyperplanes, pairwise non-parallel, in $Q$ satisfying (4) for some constant $C_0 \geq 0$. Then, for each integer $k \in [1, n]$, we
have
\[ E_{k-1} \leq \frac{1}{|S_k|^2} \sum_{F_1,F_2 \subseteq \{1,\ldots,k-1\}} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|}. \]

**Proof.** Similar to the proof of Corollary 4.2, for \( A \) in \( \mathcal{A}_k \), we write
\[ F(A) = F_0(A) \cup \{k\} \]
for some \( F_0(A) \) in \( \{1, \ldots, k-1\} \). Then
\[ \|F(A)\| = \|F_0(A)\| \cdot |S_k|. \]
Observe that the condition \( \|F(A)\| > C_0 \) in (4) is equivalent to \( \|F_0(A)\| > C_0/|S_k| \).

From the definition of \( \alpha_k(x) \), we obtain
\[ \alpha_k(x) = \frac{1}{|S_k|} \sum_{y \in S_k} 1 \leq \frac{1}{|S_k|} \sum_{y \in S_k} \sum_{A \in \mathcal{A}_k} \sum_{(x,y) \in A} 1 = \frac{1}{|S_k|} \sum_{A \in \mathcal{A}_k} \sum_{(x,y) \in A} 1. \]
Recall the notation \( A' = A^{1,\ldots,k-1} \). Since for each \( x \in Q_{k-1} \) and \( A \in \mathcal{A}_k \), there exists a unique \( y \in S_k \) with \( (x,y) \in A \) if and only if \( x \in A' \), then we have
\[ \alpha_k(x) \leq \frac{1}{|S_k|} \sum_{A \in \mathcal{A}_k} \sum_{x \in A'} 1. \]
We then deduce
\[ \alpha_k(x)^2 \leq \frac{1}{|S_k|^2} \sum_{A_1,A_2 \in \mathcal{A}_k} \sum_{x \in A_1 \cap A_2} 1, \]
so that
\[ \sum_{x \in Q_{k-1}} w_{k-1}(x) \alpha_k(x)^2 \leq \frac{1}{|S_k|^2} \sum_{A_1,A_2 \in \mathcal{A}_k} \sum_{x \in Q_{k-1}} \sum_{x \in A_1 \cap A_2} w_{k-1}(x). \]
Thus, we deduce
\[ E_{k-1} \leq \frac{1}{|S_k|^2} \sum_{A_1,A_2 \in \mathcal{A}_k} w_{k-1}(A_1' \cap A_2'). \]
If the intersection of $A'_1$ and $A'_2$ is empty, then $w_{k-1}(A'_1 \cap A'_2) = 0$. If the intersection of $A'_1$ and $A'_2$ is non-empty, then the intersection is a hyperplane with 
$$(F(A_1) \setminus \{k\}) \cup (F(A_2) \setminus \{k\}) = F_0(A_1) \cup F_0(A_2)$$
as its set of fixed coordinates. Let $F_1 = F_0(A_1)$ and $F_2 = F_0(A_2)$. Recall that $\|F_i\| > C_0/|S_k|$ for $i \in \{1, 2\}$. Note that $F_1$ and $F_2$ uniquely determine $A_1$ and $A_2$ in $A_k$, respectively, since no two hyperplanes in $A$ are parallel. From (5), we obtain

$$E_{k-1} \leq \frac{1}{|S_k|^2} \sum_{A_1, A_2 \in A_k} \nu(F(A_1' \cap A_2'))$$
$$= \frac{1}{|S_k|^2} \sum_{A_1, A_2 \in A_k} \nu(F_0(A_1) \cup F_0(A_2))$$
$$\leq \frac{1}{|S_k|^2} \sum_{F_1, F_2 \subseteq \{1, \ldots, k-1\}} \nu(F_1 \cup F_2)$$
$$= \frac{1}{|S_k|^2} \sum_{F_1, F_2 \subseteq \{1, \ldots, k-1\}} \prod_{j \in F_1 \cup F_2} \frac{1}{1 - \delta_j |S_j|},$$

finishing the proof. \hfill \Box

As a consequence of Lemma 4.3 and Lemma 4.4, we immediately obtain the following.

**Corollary 4.5** Fix a constant $C_0 \geq 0$. Let $A$ be a collection of hyperplanes, pairwise non-parallel, in $Q = S_1 \times \ldots \times S_n$ satisfying (4). Then, for each integer $k \in \{1, 2, \ldots, n\}$, we have

$$w_k(B_k) \leq \frac{1}{4\delta_k (1 - \delta_k)|S_k|^2} \sum_{F_1, F_2 \subseteq \{1, \ldots, k-1\}} \prod_{j \in F_1 \cup F_2} \frac{1}{1 - \delta_j |S_j|}.$$

We also indicate a different way to express the same bound on $w_k(B_k)$ which leads however to easier computations.

**Corollary 4.6** Fix a constant $C_0 \geq 0$. Let $A$ be a collection of hyperplanes, pairwise non-parallel, in $Q = S_1 \times \ldots \times S_n$ such that for every hyperplane $A \in A$ we have $\|F(A)\| > C_0$. Fix $k \in \{1, 2, \ldots, n\}$. Let $r$ be the minimal positive integer such that $|S_t| > C_0/|S_k|$ for all $t \geq r$, and suppose $r \leq k - 1$. 

An upper bound for the minimum modulus in a squarefree covering
Define
\[ U = \sum_{F_1 \subseteq \{1, \ldots, r-1\}} \sum_{F_2 \subseteq \{1, \ldots, r-1\}} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|} \]
and
\[ V = \sum_{F_1 \subseteq \{1, \ldots, r-1\}} \sum_{F_2 \subseteq \{1, \ldots, r-1\}} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|} \]
Then
\[ w_k(B_k) \leq \frac{1}{4\delta_k(1 - \delta_k)|S_k|^2} \left( \prod_{j=1}^{k-1} \left( 1 + \frac{3}{(1 - \delta_j)|S_j|} \right) - 2(U + V) \prod_{j=r}^{k-1} \left( 1 + \frac{1}{(1 - \delta_j)|S_j|} \right) + U \right). \]

Before going to the proof, we clarify the interpretation of $U$ and $V$ when $r = 1$. With $r = 1$, the double sum in the definition of $U$ has either zero terms or exactly one term corresponding to $F_1 = F_2 = \emptyset$. Since $\|\emptyset\| = 1$, the term exists precisely when $C_0/|S_k| \geq 1$. The empty product is 1 so that in this case $U = 1$. On the other hand, if $C_0/|S_k| < 1$, then $\emptyset$ does not satisfy the conditions on $F_1$ and $F_2$ in the double sum; hence, in this case, the double sum has no terms and is 0. Since we cannot have both $\|\emptyset\| \leq C_0/|S_k|$ and $\|\emptyset\| > C_0/|S_k|$, we deduce $V = 0$ whenever $r = 1$.

Proof of Corollary 4.6. From Corollary 4.5, we obtain
\[ 4\delta_k(1 - \delta_k)|S_k|^2 w_k(B_k) \leq \sum_{F_1, F_2 \subseteq \{1, \ldots, k-1\}} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|} \]
\[ = \sum_{J \subseteq \{1, \ldots, k-1\}} \sum_{F_1, F_2 \subseteq \{1, \ldots, k-1\}} \prod_{j \in J} \frac{1}{(1 - \delta_j)|S_j|} \]
\[ - \sum_{F_1 \subseteq \{1, \ldots, k-1\}} \sum_{F_2 \subseteq \{1, \ldots, k-1\}} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|} \]
\[ - \sum_{F_2 \subseteq \{1, \ldots, k-1\}} \sum_{F_1 \subseteq \{1, \ldots, k-1\}} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|} \]
\[ + \sum_{F_1 \subseteq \{1, \ldots, k-1\}} \sum_{F_2 \subseteq \{1, \ldots, k-1\}} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|} \].
For $i \in \{1, 2\}$, if $\|F_i\| \leq C_0/|S_k|$, then $F_i \subseteq \{1, 2, \ldots, r-1\}$, which we obtain from the definition of $\|F_i\|$ and $r$. Hence, we deduce

$$4\delta_k(1 - \delta_k)|S_k|^2 w_k(B_k) \leq \sum_{J \subseteq \{1, \ldots, k-1\}} \sum_{F_1, F_2 \subseteq \{1, \ldots, k-1\} \atop F_1 \cup F_2 = J} \prod_{j \in J} \frac{1}{(1 - \delta_j)|S_j|}$$

$$- \sum_{F_1 \subseteq \{1, \ldots, r-1\}} \sum_{F_2 \subseteq \{1, \ldots, k-1\} \atop \|F_1\| \leq C_0/|S_k|} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|}$$

$$- \sum_{F_2 \subseteq \{1, \ldots, r-1\}} \sum_{F_1 \subseteq \{1, \ldots, k-1\} \atop \|F_1\| \leq C_0/|S_k|} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|}$$

$$+ \sum_{F_1 \subseteq \{1, \ldots, r-1\}} \sum_{F_2 \subseteq \{1, \ldots, r-1\} \atop \|F_1\| \leq C_0/|S_k| \|F_2\| \leq C_0/|S_k|} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|}.$$ 

The last double sum of a product above is equal to $U$. Considering the second double sum of a product on the right-hand side of the above inequality, we can express $F_2$ as $A \cup B$ where $A \subseteq \{1, \ldots, r-1\}$ and $B \subseteq \{r, \ldots, k-1\}$, so we obtain

$$\sum_{F_1 \subseteq \{1, \ldots, r-1\}} \sum_{F_2 \subseteq \{1, \ldots, k-1\} \atop \|F_1\| \leq C_0/|S_k|} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|}$$

$$= \sum_{F_1 \subseteq \{1, \ldots, r-1\}} \sum_{A \subseteq \{1, \ldots, r-1\} \atop \|F_1\| \leq C_0/|S_k|} \sum_{B \subseteq \{r, \ldots, k-1\} \atop \|F_1\| \leq C_0/|S_k|} \prod_{j \in F_1 \cup (A \cup B)} \frac{1}{(1 - \delta_j)|S_j|} \prod_{j \in B} \frac{1}{(1 - \delta_j)|S_j|}$$

$$= \sum_{F_1 \subseteq \{1, \ldots, r-1\}} \sum_{A \subseteq \{1, \ldots, r-1\} \atop \|F_1\| \leq C_0/|S_k|} \prod_{j \in F_1 \cup A} \frac{1}{(1 - \delta_j)|S_j|} \sum_{B \subseteq \{r, \ldots, k-1\} \atop \|F_1\| \leq C_0/|S_k|} \prod_{j \in B} \frac{1}{(1 - \delta_j)|S_j|},$$

where the second to last equality holds since $(F_1 \cup A) \cap B = \emptyset$. Observe that

$$\sum_{B \subseteq \{r, \ldots, k-1\}} \prod_{j \in B} \frac{1}{(1 - \delta_j)|S_j|} = \prod_{j=r}^{k-1} \left(1 + \frac{1}{(1 - \delta_j)|S_j|}\right).$$
We also have

\[
\sum_{F_1 \subseteq \{1, \ldots, r-1\}} \sum_{A \subseteq \{1, \ldots, r-1\}} \prod_{j \in F_1 \cup A} \frac{1}{(1 - \delta_j)|S_j|} = \sum_{F_1 \subseteq \{1, \ldots, r-1\}} \sum_{A \subseteq \{1, \ldots, r-1\}} \prod_{j \in F_1 \cup A} \frac{1}{(1 - \delta_j)|S_j|} + \sum_{F_1 \subseteq \{1, \ldots, r-1\}} \sum_{A \subseteq \{1, \ldots, r-1\}} \prod_{j \in F_1 \cup A} \frac{1}{(1 - \delta_j)|S_j|} = U + V.
\]

Thus, we deduce

\[
\sum_{F_1 \subseteq \{1, \ldots, r-1\}} \sum_{F_2 \subseteq \{1, \ldots, k-1\}} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|} = (U + V) \prod_{j=r}^{k-1} \left(1 + \frac{1}{(1 - \delta_j)|S_j|}\right).
\]

Observe that this last equation is equivalent to

\[
\sum_{F_2 \subseteq \{1, \ldots, r-1\}} \sum_{F_1 \subseteq \{1, \ldots, k-1\}} \prod_{j \in F_1 \cup F_2} \frac{1}{(1 - \delta_j)|S_j|} = (U + V) \prod_{j=r}^{k-1} \left(1 + \frac{1}{(1 - \delta_j)|S_j|}\right).
\]

Rearranging the order of our second sum and product below, we have

\[
\sum_{J \subseteq \{1, \ldots, k-1\}} \sum_{F_1, F_2 \subseteq \{1, \ldots, k-1\}} \prod_{j \in J} \frac{1}{(1 - \delta_j)|S_j|} = \sum_{J \subseteq \{1, \ldots, k-1\}} \prod_{j \in J} \frac{1}{(1 - \delta_j)|S_j|} \sum_{F_1, F_2 \subseteq \{1, \ldots, k-1\}} 1.
\]
Since each element of $J$ with $F_1 \cup F_2 = J$ is either in $F_1$ and not $F_2$, in $F_2$ and not $F_1$, or in both $F_1$ and $F_2$, we deduce

$$\sum_{F_1, F_2 \subseteq \{1, \ldots, k-1\}} 1 = 3^{|J|}.\)$$

Substituting, we obtain

$$\sum_{J \subseteq \{1, \ldots, k-1\}} \sum_{F_1, F_2 \subseteq \{1, \ldots, k-1\}} \prod_{j \in J} \frac{1}{(1 - \delta_j)|S_j|}$$

$$= \sum_{J \subseteq \{1, \ldots, k-1\}} 3^{|J|} \prod_{j \in J} \frac{1}{(1 - \delta_j)|S_j|}$$

$$= \sum_{J \subseteq \{1, \ldots, k-1\}} \prod_{j \in J} \frac{3}{(1 - \delta_j)|S_j|}$$

$$= \prod_{j=1}^{k-1} \left(1 + \frac{3}{(1 - \delta_j)|S_j|}\right).$$

Combining the above, we conclude that

$$w_k(B_k) \leq \left(\frac{1}{4\delta_k} \right) \left( \prod_{j=1}^{k-1} \left(1 + \frac{3}{(1 - \delta_j)|S_j|}\right) \right)^2$$

$$- 2(U + V) \prod_{j=r}^{k-1} \left(1 + \frac{1}{(1 - \delta_j)|S_j|}\right) + U,\)$$

which completes the proof. □

The idea is to apply the prior upper bounds for $w_k(B_k)$ to estimate the value of $w_k(B_k)$ for $k \leq N$, where in the end we will take $N = 10^6$. Next, we show how to find an upper bound for $w_k(B_k)$ for $k > N$ and then find an upper bound for

$$\sum_{k > N} w_k(B_k).$$

For this part we require $N \geq 61$ to be an integer and $k > N$. Note that we view $N$ as fixed, so we will allow constants below to depend on $N$. We set $\delta_j = 1/2$ for all $j > N$. As we will be using Corollary 4.6 to compute $w_k(B_k)$ for $k = N$, we will have already completed most of the calculation for

$$M_0 = \prod_{j=1}^{N} \left(1 + \frac{3}{(1 - \delta_j)|S_j|}\right),$$

(6)
so we make use of it. Finally, we denote the \( j \)th prime by \( p_j \) and the number of primes \( \leq x \) by \( \pi(x) \).

**Lemma 4.7** With the above notation, we set

\[
    c_1 = -\log \log p_N + \frac{1}{\log^2 p_N} \quad \text{and} \quad c_2 = 1 + \frac{3}{2\log p_N}.
\]

If \( |S_j| = p_j \) for every \( j > N \), then

\[
    \sum_{k>N} w_k(B_k) \leq \frac{2c_2M_0e^{6c_1}}{p_N} \left( \log^5 p_N + 5\log^4 p_N + 20\log^3 p_N 
    
    + 60\log^2 p_N + 120\log p_N + 120 \right).
\]

**Proof.** From \( \delta_k = 1/2 \) and Corollary 4.6 (with \( C_0 = 0 \) so \( r = 1 \) and \( U = V = 0 \)), we see that, for \( k > N \), we have

\[
    w_k(B_k) \leq \frac{1}{4\delta_k(1-\delta_k)|S_k|^2} \prod_{j=1}^{k-1} \left( 1 + \frac{3}{(1-\delta_j)|S_j|} \right)
    
    = \frac{1}{|S_k|^2} \prod_{j=1}^{k-1} \left( 1 + \frac{3}{(1-\delta_j)|S_j|} \right).
\]  

(7)

Since \( k > N \) and \( \delta_j = 1/2 \) for all \( j > N \), we obtain

\[
    \prod_{j=1}^{k-1} \left( 1 + \frac{3}{(1-\delta_j)|S_j|} \right) = M_0 \prod_{j=N+1}^{k-1} \left( 1 + \frac{6}{|S_j|} \right) \leq M_0 \exp \left( 6 \sum_{j=N+1}^{k-1} \frac{1}{|S_j|} \right),
\]

(8)

where we have used that \( 1 + x \leq e^x \) for all real numbers \( x \) (the function \( e^x \) is convex up and \( y = 1 + x \) is a tangent line to its graph at \( x = 0 \)).

We are now ready to make use of the specification that \( |S_j| = p_j \) for every \( j > N \). From the work of J. B. Rosser and L. Schoenfeld [15, Theorem 5], we have the estimates

\[
    \log \log x + B - \frac{1}{2\log^2 x} \leq \sum_{p \leq x} \frac{1}{p} < \log \log x + B + \frac{1}{2\log^2 x}, \quad \text{for } x \geq 286,
\]
for some constant $B \approx 0.2614972128$. As $N + 1 \geq 62$ and $p_{62} = 293 > 286$, we deduce that

$$\sum_{j=N+1}^{k-1} \frac{1}{|S_j|} = \sum_{j=N+1}^{k-1} \frac{1}{p_j} = \sum_{p \leq p_k-1} \frac{1}{p} - \sum_{p \leq p_N} \frac{1}{p} < \left( \log \log p_{k-1} + B + \frac{1}{2 \log^2 p_{k-1}} \right) - \left( \log \log p_N + B - \frac{1}{2 \log^2 p_N} \right)$$

$$= \log \log p_{k-1} - \log \log p_N + \frac{1}{2 \log^2 p_{k-1}} + \frac{1}{2 \log^2 p_N} \leq \log \log p_{k-1} - \log \log p_N + \frac{1}{2 \log^2 p_N} + \frac{1}{2 \log^2 p_N} = \log \log p_{k-1} + c_1.$$

From (8), we now see that

$$\prod_{j=1}^{k-1} \left( 1 + \frac{3}{(1-\delta_j)|S_j|} \right) \leq M_0 \exp \left( 6 \log \log p_{k-1} + 6c_1 \right) = M_0 e^{6c_1} \log^6 p_{k-1}.$$

From (7), we obtain the estimate for $w_k(B_k)$ for $k > N$ that we will want, namely

$$w_k(B_k) \leq M_0 e^{6c_1} \frac{\log^6 p_k}{p_k}.$$

Next, we want an estimate of the sum over $k > N$ of this bound for $w_k(B_k)$. We make use of a Riemann-Stieltjes integral to obtain

$$\sum_{k=N+1}^{\infty} \frac{\log^6 p_k}{p_k^2} \leq \int_{p_N}^{\infty} \frac{\log^6 t}{t^2} \, d \pi(t)$$

$$= \frac{\pi(t) \log^6 t}{t^2} \bigg|_{p_N}^{\infty} - \int_{p_N}^{\infty} \pi(t) \, d \left( \frac{\log^6 t}{t^2} \right)$$

$$\leq 2 \int_{p_N}^{\infty} \frac{\pi(t) \log^6 t}{t^3} \, dt,$$

where we have used that

$$d \left( \frac{\log^6 t}{t^2} \right) = \left( \frac{6 \log^5 t}{t^3} - \frac{2 \log^6 t}{t^3} \right) \, dt.$$
and ignored negative quantities. From J. B. Rosser and L. Schoenfeld [15, Theorem 1], we have
\[
\pi(x) < \frac{x}{\log x} \left( 1 + \frac{3}{2 \log x} \right)
\]
for all \( x > 1 \).

Thus, for \( t \geq p_N \), we obtain \( \pi(t) \leq c_2 t/\log t \). Thus,
\[
\sum_{k=N+1}^{\infty} \frac{\log^6 p_k}{p_k^2} \leq 2c_2 \int_{p_N}^{\infty} \frac{\log^5 t}{t^2} dt.
\]
The latter integral can be computed exactly to obtain
\[
\sum_{k=N+1}^{\infty} \frac{\log^6 p_k}{p_k^2} \leq \frac{2c_2}{p_N} \left( \log^5 p_N + 5 \log^4 p_N + 20 \log^3 p_N 
+ 60 \log^2 p_N + 120 \log p_N + 120 \right).
\]
Combining the above, the lemma follows. \( \square \)

### 4.1 Proof of Theorem 1.1

We will now prove Theorem 1.1, which states that every covering system with distinct squarefree moduli has a minimum modulus which is \( \leq 118 \).

**Proof.** Let \( A \) be a collection of hyperplanes covering \( Q = S_1 \times \ldots \times S_n \) corresponding to a covering system with distinct squarefree moduli. Recall that \( |S_k| = p_k \) and if \( A \) is a hyperplane corresponding to some congruence with squarefree modulus \( m \) in a covering system, then \( \|F(A)\| = m \). For the sake of contradiction, assume that \( \|F(A)\| > C_0 = 118 \) for all \( A \in A \). For our computations, we used Maple 2019.

We choose \( \delta_j \) as below:

\[
\begin{align*}
\delta_1 &= \cdots = \delta_7 = 0, \quad \delta_8 = 0.171, \quad \delta_9 = 0.190, \quad \delta_{10} = 0.199, \\
\delta_{11} &= 0.210, \quad \delta_{12} = 0.210, \quad \delta_{13} = 0.224, \quad \delta_{14} = 0.233, \\
\delta_{15} &= 0.237, \quad \delta_{16} = 0.237, \quad \delta_{17} = 0.237, \quad \delta_{18} = 0.252, \\
\delta_{19} &= 0.252, \quad \delta_{20} = 0.255, \quad \delta_{21} = 0.260, \quad \delta_{22} = 0.261, \\
\delta_{23} &= 0.263, \quad \delta_{24} = 0.264, \quad \delta_{25} = 0.262, \quad \delta_{26} = 0.265, \quad \delta_{27} = 0.269, \\
\delta_j &= 0.279 \text{ (for } 28 \leq j \leq 35), \quad \delta_j = 0.289 \text{ (for } 36 \leq j \leq 45), \\
\delta_j &= 0.297 \text{ (for } 46 \leq j \leq 60), \quad \delta_j = 0.307 \text{ (for } 61 \leq j \leq 99), \\
\delta_j &= 0.331 \text{ (for } 100 \leq j \leq 1000), \quad \delta_j = 0.372 \text{ (for } 1001 \leq j \leq 10000),
\end{align*}
\]
\[ \delta_j = 0.418 \quad \text{(for } 10001 \leq j \leq 1000000), \quad \delta_j = 0.5 \quad \text{(for } j \geq 1000001). \]

Using Corollary 4.2, we compute that \( w_1(B_1) = w_2(B_2) = w_3(B_3) = 0, \)
\[ w_4(B_4) \leq 1/210, \quad w_5(B_5) \leq 3/110, \quad w_6(B_6) \leq 50/1001, \quad \text{and} \quad w_7(B_7) \leq 43/715, \]
so we have
\[
\sum_{k=1}^{7} w_k(B_k) \leq \frac{194}{1365} = 0.142124542124\ldots .
\]

Using Corollary 4.6 and taking into account the comments before its proof, we calculate
\[
\sum_{k=8}^{10^6} w_k(B_k) \leq 0.856857558639\ldots .
\]

The computations for (10), in particular for the upper bound on \( w_N(B_N) \)
where \( N = 10^6 \), provide us with all but the last factor (where \( j = N \)) of
the product for \( M_0 \) in (6). Including that factor and applying Lemma 4.7, we
obtain
\[
\sum_{k>10^6} w_k(B_k) \leq 0.000402960685\ldots .
\]

Combining (9), (10), and (11), we obtain
\[
\sum_{k=1}^{\infty} w_k(B_k) \leq 0.999385061449\ldots < 1.
\]

Thus, by Lemma 3.1, \( A \) does not cover \( Q \), which is a contradiction. Therefore,
every covering system with distinct squarefree moduli has a minimum modulus
which is \( \leq 118. \)

5 Proof of Theorem 1.2

We proceed by induction on \( k \), with the case \( k = 0 \) being initially covered in
[3]. Suppose that we know the result holds for \( k - 1 \), and assume that it does
not hold for \( k \). In other words, we assume we know that \( m_1, m_2, \ldots, m_k \) are
necessarily bounded in a covering system as in the statement of the theorem
but that \( m_{k+1} \) can be arbitrarily large. Note that by the conditions in the
theorem, the congruences modulo \( m_1, m_2, \ldots, m_k \) do not by themselves form a
covering system (that is, at least one more congruence is needed). Since there
are a finite number of choices for the minimal \( k \) moduli in a covering system
and an infinite number of choices for the \((k+1)\)st smallest modulus, there is
some fixed choice of \( m_1, m_2, \ldots, m_k \) for which there exist an infinite number
of covering systems \( C_1, C_2, \ldots \) satisfying:
(i) For each \( i \geq 1 \), the smallest \( k \) moduli appearing in congruences in \( C_i \) are \( m_1, m_2, \ldots, m_k \).

(ii) For each \( i \geq 1 \), the congruences modulo \( m_1, m_2, \ldots, m_k \) in \( C_i \) do not by themselves form a covering system.

(iii) Let \( m_t(C_i) \) be the \( t \)th smallest modulus of \( C_i \) and \( M(C_i) \) the maximal modulus of \( C_i \). Then \( m_{k+1}(C_1) > B(1) \) and

\[
m_{k+1}(C_i) > \max\{B(1), M(C_{i-1})\} \quad \text{for each} \quad i \geq 2.
\]

In (iii), the condition \( m_{k+1}(C_1) > B(1) \) implies \( M(C_1) > B(1) \) and, consequently, \( \max\{B(1), M(C_{i-1})\} = M(C_{i-1}) \) for each \( i \geq 2 \). Nevertheless, the inequality as written in (iii) serves the purpose of emphasizing the information we want to use in our argument.

The idea is to obtain a contradiction by constructing a covering of the integers using congruences with distinct moduli all \( > B(1) \), which will contradict the definition of \( B(1) \) (and [3]). We will do this by covering one residue class modulo \( m_1m_2 \cdots m_k \) for each covering system \( C_1, C_2, \ldots, C_{m_1m_2 \cdots m_k} \). More precisely, for each \( i \), we will show that the congruences in \( C_i \) with moduli \( > m_k \) can be used to cover one residue class modulo \( m_1m_2 \cdots m_k \). Given (iii) above, the set of moduli \( > m_k \) in \( C_i \) are disjoint for different \( i \), so that the congruences used as \( i \) varies involve distinct moduli.

Fix \( i \), and write the congruences in \( C_i \) as in (1). Note that the \( a_j \) and \( m_j \) appearing there depend on \( i \), but we will suppress that dependence here noting again that \( i \) is fixed. Let \( b \) be a fixed arbitrary integer, and suppose we wish to find \( r - k \) congruences with moduli \( m_{k+1}, \ldots, m_r \) such that every integer which is \( b \) modulo \( m_1m_2 \cdots m_k \) satisfies at least one of the \( r - k \) congruences.

Each congruence \( x \equiv a_j \pmod{m_j} \) in (1) restricted to \( 1 \leq j \leq k \) is equivalent to \( m_1m_2 \cdots m_k/m_j \) congruences modulo \( m_1m_2 \cdots m_k \). In other words, each congruence \( x \equiv a_j \pmod{m_j} \), with \( 1 \leq j \leq k \), covers precisely \( m_1m_2 \cdots m_k/m_j \) residue classes of integers modulo \( m_1m_2 \cdots m_k \). By (ii) above, the first \( k \) congruences in (1) do not form a covering system, so there is an \( a \in \mathbb{Z} \) for which no integer satisfying

\[
x \equiv a \pmod{m_1m_2 \cdots m_k}
\]

satisfies one of the first \( k \) congruences in (1). Each integer satisfying (13) therefore satisfies at least one of the congruences

\[
x \equiv a_j \pmod{m_j}, \quad \text{where} \quad k + 1 \leq j \leq r.
\]

We claim that each integer which is \( b \) modulo \( m_1m_2 \cdots m_k \) necessarily satisfies at least one of the congruences

\[
x \equiv a_j - a + b \pmod{m_j}, \quad \text{where} \quad k + 1 \leq j \leq r.
\]
An upper bound for the minimum modulus in a squarefree covering

Indeed, for each integer \( t \), we know that \( a + tm_1m_2 \ldots m_k \) satisfies (13), so that

\[
a + tm_1m_2 \ldots m_k \equiv a_j \pmod{m_j}, \quad \text{for some } k + 1 \leq j \leq r.
\]

By rewriting, for each \( t \in \mathbb{Z} \), we get

\[
b + tm_1m_2 \ldots m_k \equiv a_j - a + b \pmod{m_j}, \quad \text{for some } k + 1 \leq j \leq r.
\]

This implies what was claimed.

Thus, for each \( i \), we can choose a residue class modulo \( m_1m_2 \ldots m_k \) and cover the integers in that residue class using congruences with the moduli from \( C_i \) which are \( > m_k \). We can therefore cover all the residue classes modulo \( m_1m_2 \ldots m_k \) by using the moduli \( > m_k \) from the congruences in

\[C_1, C_2, \ldots, C_{m_1m_2 \ldots m_k}.
\]

As noted earlier, we deduce from (iii) above that these moduli are all distinct and \( > B(1) \), contradicting the case \( k = 0 \) established in [3] and finishing the proof.

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