SKYRMION LIQUID PHASE OF THE QUANTUM FERROMAGNET IN TWO DIMENSIONS

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Abstract

The two-dimensional quantum ferromagnet filled with a liquid of skyrmions is studied theoretically in the context of the quantum Hall effect near electronic filling factor $\nu = 1$. A cross-over between the classical ferromagnetic phase at $\nu = 1$ and a classical paramagnetic phase at $\nu \neq 1$ is obtained, which is consistent with recent Knight-shift measurements. A new collective mode associated with the skyrmion liquid that is of the cyclotron type is also identified.

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It has recently been proposed that skyrmion spin textures are tied to excess electronic charge with respect to the quantum Hall state at unit filling,\textsuperscript{1,2} the spin arrangement of which is known to be a two-dimensional (2D) ferromagnet. Crudely speaking, a skyrmion is simply a center of reversed spins within the 2D ferromagnet. The 2D environment is crucial, however, to insure the topological stability of these objects.\textsuperscript{3–5} Experimental evidence in support of the former proposal has come from nuclear magnetic resonance (NMR) studies of the quantum Hall state, where a rapid degradation in the total spin polarization as measured from the Knight shift has been observed as a function of the deviation from unit filling.\textsuperscript{6} Optical and magnetotransport measurements have reached similar conclusions.\textsuperscript{7} Furthermore, zero-temperature Hartree-Fock calculations based on a crystalline background of skyrmions have obtained quantitative agreement with the experimentally observed dependence of the spin polarization with the electronic filling factor at low temperature.\textsuperscript{8} It is natural to wonder, however, whether a skyrmion liquid\textsuperscript{9} (or gas\textsuperscript{10}) state is more likely to occur in such quantum Hall states at non-zero temperature. In fact, recent specific heat measurements,\textsuperscript{11} as well as theoretical calculations,\textsuperscript{12} indicate that the skyrmion crystal melts at a low temperature with respect to the Zeeman energy splitting in the quantum Hall system.

Motivated by the magnetic phenomena displayed by the quantum Hall state near unit filling, we shall study here the 2D quantum ferromagnet in the presence of a homogeneous net density of skyrmions. The residual Coulomb interaction inherited by the skyrmions in the context of the quantum Hall effect will be neglected, however, which means that the liquid nature of the system is exaggerated.\textsuperscript{13} The theoretical discussion begins with a meanfield analysis of the CP\textsuperscript{1} model for the quantum ferromagnet\textsuperscript{14,15} in two dimensions at constant skyrmion density.\textsuperscript{10} The density of skyrmions in the ferromagnet acts as a continuous dial between the following regimes (see Fig. 1): a skyrmion-poor 2D ferromagnet at extremely low density; a skyrmion-rich classical paramagnet at higher densities that is characterized by a ferromagnetic correlation length limited by the separation between skyrmions; and finally an ideal quantum paramagnet at extremely high density. Note that no phase transitions separate these regions. The central result of the mean-field analysis is the magnetization (see Fig. 2), which is found to give a fair comparison with available Knight-shift data on the quantum Hall state. A new collective mode connected with the
skyrmion liquid is also found, however, once fluctuations with respect to the meanfield saddle point are accounted for. It can be understood in physical terms as a collective precession of the boundary spins, which is equivalent to the orbit of a skyrmion around the edge of the ferromagnet. As expected then, no Goldstone mode appears in the paramagnetic skyrmion liquid.

Our task then is to determine the thermodynamic and dynamic properties of the 2D ferromagnet in the presence of both a homogeneous density of skyrmions and of an external magnetic field. As a means to this end, we introduce the partition function

\[ Z = \int \mathcal{D}a' \mathcal{D}a'z \mathcal{D}z \mathcal{D}\bar{z} \exp(-S/h) \]  

for the corresponding CP\(^1\) model\(^{14,15}\) whose action \(S\) in imaginary time \(\tau = it\) reads

\[ S = \int_0^{\hbar/k_B T} \tau d\tau \int d^2r \{ \hbar M_0[z(\partial_\tau - a'_{0}) + a'_{0} + c.c.] + 2\rho_s |(\nabla - i\vec{a}')z|^2 + M_0 E_Z(|z_-|^2 - |z_+|^2) \}. \]

Here \(M_0 = sa^{-2}\) is the fully polarized magnetization for a density \(a^{-2}\) of spin \(s\) moments, \(\rho_s = s^2J\) is the spin stiffness of the ferromagnet, and \(E_Z = \mu gH\) is the Zeeman energy splitting. The complex doublet field \(z = (z_-, z_+)\) is connected to the normalized magnetization by \(\vec{m} = \bar{z}\vec{\sigma} z\), where \(\vec{\sigma}\) denotes the Pauli matrices with the \(z\)-axis directed along the external magnetic field. Integration (1) over the scalar potential \(a'_{0}\) and over the vector potential \(\vec{a}'\) in the Coulomb gauge \((\nabla \cdot \vec{a}' = 0)\) enforces the respective constraints \(|z|^2 = 1\) and \(\vec{a}' = i(\nabla \bar{z})z\) characteristic of the CP\(^1\) model for the 2D ferromagnet.\(^4,5,10\)

The 3-vector \(J^\mu_S = (n_S, \vec{J}_S)\) for the local skyrmion density, \(n_S\), and local current density, \(\vec{J}_S\), is also related to these potential fields by the manifestly gauge-invariant formula \(J^\mu_S = (2\pi)^{-1} \epsilon^{\mu\nu\gamma} \partial_\nu a'_{\gamma}\), where \(a'_{\mu} = (a'_{0}, \vec{a}')\) and \(\partial_\mu = (\partial_t, \nabla)\). (Repeated Greek indices are hereafter summed over.) This identity can be understood by noticing that \(n_S = (\partial_x a'_{0} - \partial_y a'_{x})/2\pi\) gives the local skyrmion density of the classical ferromagnet,\(^10\) while the 3-vector \(J^\mu_S\) formally satisfies the continuity equation, \(\dot{n}_S + \nabla \cdot \vec{J}_S = 0.\(^3,5\)

In the meanfield approximation we presume constant potentials \(a'_{0}\) and ignore fluctuations. Here, the fictitious magnetic field \(b_S\) is tied to the average skyrmion density by \(n_S = b_S/2\pi\), which is a conserved quantity. Integrating (1) over the \(z\)-fields and then minimizing the resulting free energy with respect to \(a'_{0}\), we obtain the averaged
constraint 1 = $\langle \bar{z}_-z_- \rangle + \langle \bar{z}_+z_+ \rangle$, where $\langle \bar{z}_\pm z_\pm \rangle = \frac{1}{2}(n_S/M_0) \sum_{n=0}^{\infty} \left[ \exp\left(\frac{E_n^\pm}{k_BT}\right) - 1 \right]^{-1}$ is the trace of the propagator for the $z_\pm$ quantum, with Landau-type energy levels $E_n^\pm = (a^2/s)(\xi^{-2} + 2b_{S0})\rho_s \mp \frac{1}{2}E_Z$. This sets the unknown $\xi^{-2} \propto a_0'$, which then allows us to compute the normalized magnetization along the $z$-direction, $m_z = \langle \bar{z}_+z_+ \rangle - \langle \bar{z}_-z_- \rangle$, as a function of temperature. The Landau levels in the present mean-field theory have an energy splitting of $\hbar\omega_{c}' = (n_S/M_0)E_S$, where $E_S = 4\pi\rho_s$ is the classical energy cost of a single skyrmion.\(^3\)\(^-\)\(^5\) Let us now analyze these mean-field equations along the skyrmion-density axis. In the skyrmion-poor limit $\hbar\omega_{c}' \ll k_BT$, for example, the sum over Landau levels may be converted into an energy integral. We thereby recover known results for the pure quantum 2D ferromagnet.\(^1\)\(^5\) In particular, at low fields, $E_Z < E_S$, this pure ferromagnet has three distinct phases (see Fig. 1): (i) a low-temperature ($k_BT \ll E_Z$) quantum activated regime characterized by the saturated magnetization $m_z \approx 1 - (k_BT/E_S)e^{-E_Z/k_BT}$; (ii) a high-temperature ($E_S \ll k_BT$) quantum-critical regime that follows the Curie law $m_z \approx E_Z/2k_BT$; (iii) and a classical regime at intermediate temperatures ($E_Z \ll k_BT \ll E_S$) that interpolates smoothly between the previous two behaviors.\(^1\)\(^5\) The ferromagnetic correlation length in the latter classical ferromagnetic phase notably diverges exponentially as $\xi_{FM} \propto e^{E_S/2k_BT}$. No critical phenomena separate the phases mentioned above.

On the other hand, in the opposing skyrmion-rich limit, $\hbar\omega_{c}' \gg k_BT$, it becomes sufficient to retain only the lowest Landau-level term ($n = 0$) in mean-field sums. The spin-spin correlation function is in general proportional to that of the $z$-quanta squared, which in the present case is given by\(^1\)\(^0\) $|\langle z(\vec{r})\bar{z}(\vec{r}') \rangle|^2 = \exp(-\frac{1}{2}\rho_s^{-2}b_S - |\vec{r} - \vec{r}'|^2b_S)$. This means that the ferromagnetic correlation length in the skyrmion-rich regime is limited by the separation between neighboring skyrmions; i.e., $\xi_{FM} \sim n_S^{-1/2}$. Upon further analysis of the low-field case $E_Z < E_S$ in this limit, the following regimes can be identified (see Fig. 1): (i) a quantum-activated regime at low temperature $k_BT \ll E_Z$ and at low skyrmion density $n_S/M_0 \ll 1$, with a saturated magnetization $m_z \approx 1 - (n_S/M_0)e^{-E_Z/k_BT}$; (ii) an ideal quantum paramagnet at extremely high skyrmion densities $n_S/M_0 \gg 1$, with a corresponding magnetization

$$m_z^{qp} = \tanh(E_Z/2k_BT);$$

(iii) and a classical paramagnet in the remainder of the phase diagram, $n_S/M_0 \ll 1$ and
\[ E_Z \ll k_B T \ll E_S, \] with a normalized magnetization

\[ m_z^{cp} = [1 + (T/T_*)^2]^{1/2} - T/T_*, \] (4)

where \( k_B T_* = (M_0/n_S)E_Z \) is the crossover scale (see ref. 10). Notice that the magnetization of the classical paramagnet follows a Curie law, \( m_z \cong T_*/2T \), for temperatures \( T \gg T_* \). Again, no critical phenomena separate the various regimes.

We shall now obtain the structure of the collective modes in the skyrmion liquid by allowing for fluctuations with respect the above meanfield saddlepoint. In particular, dynamical fluctuations in the fictitious gauge field, \( \delta a'_\mu \), which are equivalent to fluctuations in the skyrmion density and current, contribute the standard second-order term

\[ S_2 = \frac{i}{2\hbar} \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} [\Pi^+_{\mu\nu}(\vec{k}, \omega) + \Pi^-_{\mu\nu}(\vec{k}, \omega)] \delta a'_\mu(\vec{k}, \omega) \delta a'_\nu(-\vec{k}, -\omega) \] (5)

to the action (2), where

\[ \Pi^\pm_{\mu\nu}(\vec{k}, \omega) = V^{-1} \sum_{n,q} 2n_B(E_n^\pm) (\hat{x}_\mu \hat{x}_\nu + \hat{y}_\mu \hat{y}_\nu) \]

\[ - V^{-1} \sum_{n \neq n', q} \frac{n_B(E_n^\pm) - n_B(E_{n'}^\pm)}{\hbar \omega - E_n^\pm + E_{n'}^\pm} \langle n, q | j_\mu(\vec{k}) | n', q \rangle \langle n', q | j_\nu(-\vec{k}) | n, q \rangle \] (6)

is the Kubo-Lindhard formula corresponding to the \( z_\pm \) quantum for wavenumber \( \vec{k} \) chosen to be parallel to the \( x \)-axis. Here \( n_B(E_n^\pm) \) denotes the Bose distribution function, \( j_0(\vec{k}) = e^{ikx} \) and \( j'(\vec{k}) = 2(a^2/s)\rho_s e^{ikx}/(i \vec{\nabla} + \vec{a}')e^{ikx}/2 \) are the respective density and current operators, and \( \langle \vec{r}|n, q \rangle = \langle x'|n \rangle \) are the eigenstates for Landau levels in the gauge \( \vec{a}' = (0, b_S x) \), where \( |n \rangle \) denote the standard harmonic oscillator states, with \( x' = x - b_S^{-1} q \).

Expression (6) can be evaluated explicitly in the long-wavelength limit, \( \vec{k} \rightarrow 0 \). Taking the Landau gauge (\( \delta a'_x = 0 \)), one obtains a polarizibility tensor of the form \( \Pi^+_{\mu\nu} + \Pi^-_{\mu\nu} = \epsilon(\omega)\pi_{\mu\nu} \) at frequencies \( \omega \) near \( \omega_c' \), with a dielectric function \( \epsilon(\omega) = 4\rho_s/(\omega_c'^2 - \omega^2) \), and with components \( \pi_{yy} = c_1^2 k^2 - \omega^2, \pi_{0y} = i k \omega_c', \pi_{y0} = -\pi_{0y}, \) and \( \pi_{00} = -k^2 \). Here, \( c_1 = (3/8\pi)^{1/2}(n_S^{1/2}/M_0)(E_S/\hbar) \) is the dispersion velocity in the skyrmion-rich limit. The collective mode, by definition, is the null eigenstate of the previous polarizibility tensor. It appears at frequencies \( \omega'_c(k) = (\omega_c'^2 + c_1^2 k^2)^{1/2} \), with skyrmion density-current components \( (\delta n_S, \vec{J}_S) \propto |k, (\hat{x} + i\hat{y})\omega'_c| \) of the cyclotron type. Notice that the residue \( 1/\epsilon[\omega'_c(k)] \) of the
pole in the response function, $\langle \delta a'_\mu \delta a'_\nu \rangle$, corresponding to this collective mode varies as $c^2 k^2$, which vanishes in the long-wavelength limit.

The above result for the cyclotron-type collective mode connected with the skyrmion liquid is notably independent of temperature. This suggests that it has a mechanical basis, which we now outline. Consider the edge of the ferromagnet filled with skyrmions. A density $n_S$ of skyrmions in the bulk will then produce a normalized edge magnetization, $\delta \vec{m} \sim (n_S a^2)(\hat{e} \times \hat{z})$, where $\hat{e}$ and $\hat{z}$ are respectively the unit vectors along the edge and along the bulk magnetization. The presence of such a magnetization will then induce a precession about it by the spins at the edge, at an angular frequency $\omega'_c = \rho_s |\delta \vec{m}|/s \hbar \sim (n_S/M_0)(\rho_s/\hbar)$. Yet given that the spin in a symmetric skyrmion winds around precisely once along any straight line passing through the center of the skyrmion, then such a precession is equivalent to the orbit of skyrmions around the bulk at an angular frequency $\omega'_c$. Notice that this argument depends only on the presence of a non-zero density of skyrmions in bulk, which suggests that the collective mode is robust with respect to any tendency towards crystallization in the skyrmion liquid. Last, it should be remarked that the same argument may be employed to demonstrate that two skyrmions must orbit about one another.

The proposal that electron (hole) excitations with respect to the $\nu = 1$ quantum Hall state induce a background skyrmion spin-texture suggests that the present theory for the skyrmion-filled ferromagnet describes the magnetic properties of quantum Hall liquids in the vicinity of $\nu = 1$. However, the Chern-Simons term $S_{CS} = (4\pi)^{-1} \int dt \int d^2 r \epsilon^{\mu \nu \gamma} a_\mu \partial_\nu a_\gamma$ must first be added to the ferromagnetic action (2), where the statistical gauge field $a_\mu$ that transmutes the original electronic field $\Psi_\sigma$ into the bosonic CP$^1$ model field $z_\sigma$ is related by the identity $a'_\mu = a_\mu + (e/\hbar c)A_\mu$ to the skyrmionic gauge field $a'_\mu$ and to the external electro-magnetic potentials $A_\mu$. When this Chern-Simons term is combined with the one (5) that is dynamically generated by the skyrmion liquid, $S_2 = ... + (4\pi \nu_S)^{-1} \int dt \int d^2 r \epsilon^{\mu \nu \gamma} a'_\mu \partial_\nu a'_\gamma + ...$, where $\nu_S = n_S a^2$ is the skyrmionic filling factor (for $s = 1/2$), we obtain that (i) the Hall resistance is given by the Ioffe-Larkin composition formula $\rho_{xy} = (1 + \nu_S)(\hbar/e^2)$, while (ii) the prefactor corresponding to the dynamically generated Chern-Simons terms must be replaced by $\nu_S^{-1} \rightarrow \nu_S^{-1} + 1$ in the previous discussion concerning the skyrmionic collective mode. However, the skyrmionic
filling factor is equal to \( \nu_S = \nu^{-1} - 1 \) for electronic filling factors \( \nu < 1 \).\(^{10} \) This implies that the Hall resistance is that naively expected for free electrons in a partially-filled lowest Landau level, \( \rho_{xy} = \nu^{-1}(h/e^2) \), and that the nature of the previously discussed collective mode remains unchanged for electronic filling factors in the vicinity of \( \nu = 1 \), since \( \nu_S \ll 1 \) in such case. The latter implies that the skyrmion/charge hybrids orbit at a new cyclotron frequency, \( \omega'_c < \omega_c \). This conflicts with the fact that each electron (hole) must orbit at the true cyclotron frequency, \( \omega_c \). The paradox is resolved as follows: The skyrmion collective mode is indeed possible in the long wavelength limit, \( k \to 0 \), in which case it is purely transverse and shows no charge fluctuations. As mentioned before, it corresponds to a precession of the boundary spins at angular frequency \( \omega'_c \) in such case. The remaining skyrmion cyclotron modes at \( k \neq 0 \) are not possible, however, since they must oscillate at the true cyclotron frequency by charge conservation. This conflict must then be interpreted as a breakdown of the present mean-field approximation. We conclude, nevertheless, that the magnetic properties of the quantum Hall state are indeed well described by the ferromagnetic action (2) in the absence of the Chern-Simons term for low enough skyrmion densities.

To test this idea, we shall apply our results to recent Knight-shift measurements\(^6 \) that probe the magnetization of the quantum Hall state near electronic filling factor \( \nu = 1 \) and at magnetic fields corresponding to a Zeeman splitting of \( E_Z \cong 2 \) K. First, the present theory predicts a cross-over from classical ferromagnetic behavior at \( \nu = 1 \) to classical paramagnetic behavior away from \( \nu = 1 \) at a skyrmionic filling factor of \( \nu_S^* = \frac{sk_B T}{E_S} \) (see Fig. 1). The latter regime is characterized by a magnetization that depends strongly on the skyrmion density. Fixing the temperature at the Zeeman energy, \( k_B T = E_Z \), then yields a cross-over electronic filling factor of \( \nu_s = 0.97 \), which is consistent with experiment.\(^6 \) Here the theoretical value for the spin-stiffness expected of an ideal 2D quantum Hall state at \( \nu = 1 \) is assumed,\(^{15} \) i.e., \( E_S = 40 \) K. In Fig. 2 is also shown the theoretically predicted magnetization for a density of skyrmions,\(^{10} \) \( n_S/M_0 = 2(1-\nu)/\nu \cong 0.3 \), that corresponds to an electronic filling-factor of \( \nu = 0.88 \). The experimental Knight-shift results, \( K_s(T, \nu) \), at this filling are also shown, where the formula \( M/M_0 = K_s(T, \nu)/\nu K_s(0, 1) \) has been used to compute the normalized magnetization. Although the experimental results fall within the bounds of the theory set by the classical paramagnetic result at \( E_S \to \infty \) and the quantum
paramagnetic result (3) at $E_S = 0$, they lie substantially below the theoretically expected curve. This suggests that the naive theoretical value for the ferromagnetic spin stiffness may be too high. Similar conclusions were reached by Read and Sachdev$^{15}$ from their theoretical fits to the magnetization at $\nu = 1$. Indeed, it is known that $\rho_s$ can be reduced by as much as 70% once corrections due to the non-zero thickness of typical quantum Hall effect devices are taken into account.$^1$

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13. The area of each skyrmion in the classical paramagnetic phase is known to be $A_S(T) = (k_B T / E_Z) a^2$ (see ref. 10). At temperatures $k_B T > E_Z$, it is then much larger than its true size at zero temperature, $A_S(0) \approx a^2$, which results from the balance of the Zeeman energy with the residual Coulomb interaction (see refs. 1 and 8). Hence, the neglect of the latter is valid at such elevated temperatures, where entropic forces dominate.
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Figure Caption

Fig. 1 Shown is the schematic phase diagram for the skyrmion liquid in the low-field limit. All perforated lines designate crossovers. In particular, the diagonal dashed line ($k_B T = \hbar \omega_c'$) separates skyrmion-poor and skyrmion-rich regimes. The classical paramagnet is the only phase in which the magnetization depends strongly on skyrmion density [see Eq. (4)].

Fig. 2 The normalized magnetization for the skyrmion liquid at relatively weak magnetic field is shown. The squares represent Knight-shift measurements for the quantum Hall state at electronic filling-factor $\nu = 0.88$ (see ref. 6, as well as the discussion in the text).
\( \left( \frac{n_s}{M_0} = 0.3, \ E_Z = 2 \ \text{K} \right) \)
