ON THE REGULARIZATION OF FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND

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Abstract. In this paper the problem of recovering a regularized solution of the Fredholm integral equations of the first kind with Hermitian and square-integrable kernels, and with data corrupted by additive noise, is considered. Instead of using a variational regularization of Tikhonov type, based on a priori global bounds, we propose a method of truncation of eigenfunction expansions that can be proved to converge asymptotically, in the sense of the $L^2$-norm, in the limit of noise vanishing. Here we extend the probabilistic counterpart of this procedure by constructing a probabilistically regularized solution without assuming any structure of order on the sequence of the Fourier coefficients of the data. This probabilistic approach allows us to use the statistical tools proper of time-series analysis, and in this way we attain a new regularizing algorithm, which is illustrated by some numerical examples. Finally, a comparison with solutions obtained by the means of the variational regularization exhibits how some intrinsic limits of the variational-based techniques can be overcome.

Key words. integral equations, inverse problems, regularization, information theory.

AMS subject classifications. 45B05, 45Q05

1. Introduction. We consider the Fredholm integral equations of the first kind

\begin{equation}
(Af)(x) = \int_a^b K(x,y)f(y)\,dy = g(x) \quad (a \leq x \leq b)
\end{equation}

whose kernel $K(x,y)$ is supposed to be Hermitian and square integrable; i.e.,

\begin{equation}
K(x,y) = \overline{K(y,x)}
\end{equation}

and

\begin{equation}
\int_a^b \left\{ \int_a^b |K(x,y)|^2\,dx \right\} \,dy < \infty.
\end{equation}

Then $A : L^2(a,b) \to L^2(a,b)$ is a self-adjoint compact operator.

For simplicity we shall suppose hereafter that the kernel $K$, the function $g$, and the unknown function $f$ are real-valued functions; in addition, we assume that the interval $[a,b]$ is a bounded and closed subset of the real line.

The Hilbert-Schmidt Theorem guarantees that the integral operator $A$ admits a set of eigenfunctions $\{\psi_k\}_{1}^{\infty}$ and, accordingly, a countably infinite set of eigenvalues $\{\lambda_k\}_{1}^{\infty}$. The eigenfunctions form an orthonormal basis of the orthogonal complement of the null space of the operator $A$ and therefore an orthonormal basis of $L^2(a,b)$ when $A$ is injective. For the sake of simplicity only this case will be considered, although this assumption can be easily relaxed with slight technical modifications.

The Hilbert-Schmidt theorem also guarantees that $\lim_{k \to \infty} \lambda_k = 0$. Furthermore, we shall suppose hereafter that the eigenvalues are ordered as follows: $\lambda_1 > \lambda_2 > \lambda_3 > ...$. 

\footnote{http://www.siam.org/journals/sima/29-4/30174.html}
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In view of the Hilbert-Schmidt theorem we associate with the integral equation (1.1) the following eigenfunction expansion:

\[ f(x) = \sum_{k=1}^{\infty} \left( \frac{g_k}{\lambda_k} \right) \psi_k(x), \tag{1.4} \]

where \( g_k = (g, \psi_k), \) \((\cdot, \cdot)\) denoting the scalar product in \( L^2(a,b) \). The series (1.4) converges in the sense of \( L^2 \).

**Remark.** If the support of the data does not coincide with that of the solutions, i.e., \( A : L^2(a,b) \rightarrow L^2(c,d) \) with \([a,b]\) different from \([c,d]\), the problem can be worked out in terms of singular values and singular functions of the operator \( A \), and all of the following results can be easily reformulated.

In view of the fact that there always exists some inherent noise in the data, instead of (1.1) we have to deal with the following equation:

\[ Af + n = \overline{g} \quad (\overline{g} = g + n), \tag{1.5} \]

where \( n \) represents the noise. Therefore, instead of expansion (1.4) we have to consider the following expansion:

\[ \sum_{k=1}^{\infty} \left( \frac{\overline{g}_k}{\lambda_k} \right) \psi_k, \tag{1.6} \]

where \( \overline{g}_k = (\overline{g}, \psi_k) \). Expansion (1.6) is generally diverging because \( \overline{g} \) does not belong, in general, to the range of the operator \( A \). This is precisely a manifestation of the ill-posed character of the Fredholm integral equation of the first kind.

Several methods of regularization have been proposed (see [10, 14, 16] and references therein); all of them modify one of the elements of the triplet \( \{A, X, Y\} \), where \( A \) is the integral operator defined by (1.1), whereas \( X \) and \( Y \) are, respectively, the solution and the data space (in our case \( X \equiv Y \equiv L^2(a,b) \)). Among these methods the procedure, which is probably the most popular, consists in admitting only those solutions that belong to a compact subset of the solution space \( X \). In particular the famous method of Tikhonov leads to the construction of “regularizing operators” by the minimization of “smoothing functionals”. In this latter functional the smoothing term is obtained precisely by restricting the admitted solutions to a compact subset of the space \( X \); then the continuity of \( A^{-1} \) follows from compactness. This restriction is realized by the use of a priori bounds which can be written assuming some prior knowledge of the solution. Therefore, in addition to the inequality

\[ \| Af - \overline{g} \| \leq \epsilon \tag{1.7} \]

which corresponds to a bound on the noise (\( \| \cdot \| \) denoting the norm in \( L^2(a,b) \)), one also considers an a priori bound on the solution of the following form:

\[ \| Cf \|_Z \leq E, \tag{1.8} \]

where \( Z \) denotes the “constraint space” and, accordingly, \( C \) is the “constraint operator”. From the bounds (1.7) and (1.8) we are led to define the regularized solution as the minimum of the following functional:

\[ \Phi(f) = \| Af - \overline{g} \|^2 + \alpha^2 \| Cf \|_Z^2, \quad \alpha = \left( \frac{\epsilon}{E} \right). \tag{1.9} \]
In spite of several significant merits, this procedure is not free from defects. Concerning the possibility of writing suitable a priori bounds on the solution, we want to remark strongly that two different types of problems must be distinguished:

a) synthesis problems;

b) inverse problems,

and to note that both are frequently solved by the use of Fredholm integral equations of the first kind. In the first class of problems, that basically consists in finding the source that produces a prescribed effect (e.g., prescribed boundary values), the a priori bounds are intrinsic of the problem itself, whereas this is not always the case for the second class. As typical examples we can consider:

a') the antenna synthesis;

b') the signal recovery.

The problem of the antenna synthesis consists in determining, within a certain degree of approximation, the current intensity that generates a desired radiation pattern. It can be formulated in terms of Fredholm equation of the first kind [18, 24] and, consequently, it presents the typical pathology of the ill-posed problems. In this problem the a priori bound on the ohmic losses associated with the current intensity is necessary and can be regarded as a natural constraint intrinsic of the problem. Conversely, in the case of the signal recovery problem, the a priori bounds can be written only if prior knowledge on the signal is assumed. Generally, it is possible to have some a priori information regarding, for instance, the support of the signal or requiring the function representing the signal to be nonnegative. But even in these cases the prior knowledge could be insufficiently specific to be peculiar of the function to be reconstructed, and arbitrary, though reasonable, constraints must be added to solve the problem. Strictly connected with this question there is the crux of the matter: the practical choice of the regularization parameter $\alpha$ (see formula (1.9)) for a fixed $g$, when the a priori bound (1.8) is unknown or it is not sufficiently precise.

Moreover, let us note that the functional (1.9) works as a filter whose action is smoothing the Fourier components $g_k$ for high values of $k$. But it is easy to exhibit examples of signals whose Fourier components are small, or even zero, for low values of $k$, while the significant contributions of the signal are brought by those components at intermediate values of $k$, which are smoothed out by the action of the filter. In these situations the standard regularization method fails, showing that the only existence of the minimum of functional (1.9) does not guarantee the bulk of the signal had been really recovered. This delicate point will be illustrated with numerical examples in section 4.

We suggest a different approach which is based on the following observation: for the moment, suppose that the moduli of the noiseless Fourier coefficients $|g_k|$ are monotonically decreasing as $k$ increases; then, although the formal series (1.6) diverges, nevertheless the effect of the error remains limited in the beginning of the expansion, and there exists a point (a certain value of $k$) where divergence sets in. Thus, the idea is to stop the expansion at the point where it turns to diverge. This rough and qualitative description can be put in rigorous form by proving that even if the series (1.6) diverges, nevertheless it converges (in the sense of $L^2$-norm) as $\epsilon$ (i.e., the bound on the noise) tends to zero. This result, which has been proved by two of us (see [17]), does not give (except in very particular cases) a practical numerical method for finding out the truncation point (i.e., the value of $k$) where to stop expansion (1.6). However, here we prove a probabilistic generalization of the results presented in [17] by removing the quite restrictive assumption that the Fourier
coefficients $|g_k|$ of the signal to be recovered are monotonically decreasing. Compared to the significance of the new results is relevant. First, the hypothesis made in [17] on the order of the coefficients $|g_k|$ leads to a regularization procedure that essentially works as an ideal low-pass filter, and, as previously discussed, this does not guarantee to recover correctly the signals whose bulk is localized at intermediate frequencies. Conversely, in this paper it will be shown how to construct a regularized solution without assuming any kind of order on the coefficients $|g_k|$ by exploiting the tools supplied by the information theory. This result will lead to a more effective regularizing algorithm which is based on a suitable statistical analysis of the data and whose main feature is indeed the frequency selectivity. Second, from the application point of view, the hypothesis on the order of the coefficients $|g_k|$ is too restrictive; thus, by removing it, a much larger class of real signals can be practically analyzed. These questions are precisely the contents of sections 3 and 4. We will prove, indeed, in Section 3 that it is possible to split the noisy Fourier coefficients $g_k$ into two classes:

1) the Fourier coefficients $g_k$ from which a significant amount of information on $f_k = (f, \psi_k)$ can be extracted;
2) the Fourier coefficients $g_k$ that can be regarded as random numbers because the noise prevails on the coefficients $g_k$.

In section 4 it will be shown how it is possible to separate practically the coefficients $g_k$ into these two classes by the use of statistical tools supplied by the so-called “time-series” analysis. Therefore, we can practically construct an approximation which converges to the real solution, and furthermore we can have some confidence that the bulk of the function $f$ has been effectively recovered.

The paper is organized as follows. In the first part of section 2 a short sketch of the variational method based on the minimization of functional (1.9) is given. This will be done in order to have explicitly the formulae which will be used in section 4, where our procedure and the variational one will be compared. The second part of section 2 is devoted to the probabilistic formulation of the regularization problem in a quite general setting. In section 3 we start illustrating the asymptotic convergence of the eigenfunction expansion (in the sense of $L^2$-norm) as $\epsilon$ tends to zero; then this result is reconsidered from the viewpoint of probability and information theory. Here a key role will be played by the Bayes formula: it will provide the various terms of our approximation, which will be proved to be a probabilistically regularized solution of (1.1). The first part of section 4 is devoted to the discussion of the statistical tools that are necessary for practically recovering the regularized solution from finite samples of noisy data. Finally, some numerical examples are given in the second part of section 4.

2. Variational and probabilistic regularization.

2.1. Variational regularization. After the classical book of Tikhonov and Arsenine [23] the literature on the theory and applications of the variational regularization has been rapidly growing (see, for instance, [14]). In order to compare our algorithm with this classical one, some formulae and results of the variational regularization will be here recalled here (see [19, 20, 22] for proofs and details).

Let us characterize, first of all, the constraint operator $C$ and, accordingly, the constraint space $Z$. Let us take a constraint operator $C$ such that $C^*C$ and $A^*A$ commute (this assumption does not restrict the theory and the applications significantly [5, 15]). Then, the space $Z$ is composed by those functions $f \in L^2(\alpha, \beta)$ such that
\[ \| Cf \|_Z \text{ is finite}; \text{i.e.,} \]

\[ \| Cf \|_Z^2 = (C^* Cf, f) = \sum_{k=1}^{\infty} c_k^2 |f_k|^2 < \infty. \tag{2.1} \]

Now we consider the ball \( \mathcal{U}_Z = \{ f \in Z \| \sum_{k=1}^{\infty} c_k^2 |f_k|^2 \leq E^2 \} \), and the restriction \( A_0 \) of the operator \( A \) (see (1.1)) to the ball \( \mathcal{U}_Z \). Then, the following propositions can be proved.

**Proposition 2.1.** If \( \lim_{k \to \infty} c_k^2 = +\infty \) the operator \( A_0^{-1} \) is continuous.

**Proposition 2.2.** The functional \( \Phi(f) \), with \( \alpha = (\epsilon/E) \), has a unique minimum which is given by:

\[ f_* = [A^* A + \left( \frac{\epsilon}{E} \right)^2 C^* C]^{-1} A^* g. \tag{2.2} \]

By expanding \( g \) in terms of \( \psi_k \) (eigenfunctions of the operator \( A \)), we have:

\[ f_* = \sum_{k=1}^{\infty} \frac{\lambda_k g_k}{\lambda_k^2 + c_k^2 \left( \frac{\epsilon}{E} \right)^2} \psi_k. \tag{2.3} \]

Next, we have the following proposition.

**Proposition 2.3.** The following limit holds true for any function \( f \) satisfying the bounds (1.7) and (1.8):

\[ \lim_{\epsilon \to 0} \| f - f_* \| = 0 \text{ (} E \text{ fixed).} \tag{2.4} \]

In numerical computations it is often convenient to use truncated approximations. For instance, one can derive from the smoothed solution \( f_{1(1)} \) the following truncated approximation:

\[ f_{1(1)}^{(1)} = \sum_{k=1}^{k_{\alpha}} \frac{g_k}{\lambda_k} \psi_k, \tag{2.5} \]

where \( k_{\alpha} \) is the largest integer such that

\[ \lambda_k \geq \left( \frac{\epsilon}{E} \right) |c_k|. \tag{2.6} \]

**Proposition 2.4.** The following limit holds true for any function \( f \) satisfying bounds (1.7) and (1.8):

\[ \lim_{\epsilon \to 0} \| f - f_{1(1)}^{(1)} \| = 0 \text{ (} E \text{ fixed).} \tag{2.7} \]

In several problems a weaker a priori bound should be used by setting \( C = I \) (the identity operator). Therefore, instead of bound (1.8), we have

\[ \| f \| = \left( \sum_{k=1}^{\infty} |f_k|^2 \right)^{1/2} \leq E. \tag{2.8} \]
In this case the unique minimum of functional (1.9) is given by

\[ f^{(2)}_* = \sum_{k=1}^{\infty} \frac{\lambda_k \overline{g}_k}{\lambda_k^2 + (\overline{E})^2} \psi_k, \]

and, accordingly, the following truncated approximation can be introduced:

\[ f^{(3)}_* = \sum_{k=1}^{k_\beta} \frac{\overline{g}_k}{\lambda_k} \psi_k, \]

where \( k_\beta \) is the largest integer such that

\[ \lambda_k \geq \epsilon E. \]

Both \( f^{(2)}_* \) and \( f^{(3)}_* \) converge to \( f \) as \( \epsilon \to 0 \) in a weak sense. In fact, as shown in [20, 21], the following proposition can be proved.

**Proposition 2.5.** For any function \( f \) which satisfies the bounds (1.7) and (2.8), the following limits hold true:

\[ \lim_{\epsilon \to 0} \left| \left| \left( f - f^{(2)}_* \right), v \right| \right| = 0 \quad (\|v\| \leq 1, \ E \text{ fixed}), \]

\[ \lim_{\epsilon \to 0} \left| \left| \left( f - f^{(3)}_* \right), v \right| \right| = 0 \quad (\|v\| \leq 1, \ E \text{ fixed}). \]

**2.2. Probabilistic regularization.** Here we want to reconsider (1.5) from a probabilistic point of view. With this in mind we rewrite (1.5) in the following form:

\[ A\xi + \zeta = \eta, \]

where \( \xi, \zeta \) and \( \eta \), which correspond to \( f, n \) and \( \overline{g} \) respectively, are Gaussian weak random variables (w.r.v.) in the Hilbert space \( L^2(a,b) \) [2]. A Gaussian w.r.v. is uniquely defined by its mean element and its covariance operator; in the present case we denote by \( R_{\xi\xi}, R_{\zeta\zeta} \) and \( R_{\eta\eta} \) the covariance operators of \( \xi, \zeta \) and \( \eta \) respectively. Next, we make the following assumptions:

I) \( \xi \) and \( \zeta \) have zero mean; i.e. \( m_\xi = m_\zeta = 0; \)

II) \( \xi \) and \( \zeta \) are uncorrelated, i.e. \( R_{\xi\zeta} = 0; \)

III) \( R^{-1}_{\zeta\zeta} \) exists.

The third assumption is the mathematical formulation of the fact that all the components of the data function are affected by noise. As it is shown by Franklin (see formula (3.11) of [11]), if the signal and the noise satisfy assumptions I) and II), then

\[ R_{\eta\eta} = AR_{\xi\xi}A^* + R_{\zeta\zeta} \]

and the cross-covariance operator is given by

\[ R_{\xi\eta} = R_{\xi\xi}A^*. \]

We also assume that \( R_{\zeta\zeta} \) will depend on a parameter \( \epsilon \) that tends to zero when the noise vanishes; i.e.,

\[ R_{\zeta\zeta} = \epsilon^2 N, \]
where $N$ is a given operator (e.g., $N = I$ for the white noise).

Now we are faced with the following problem.

Problem. Given a value $\mathbf{g}$ of the w.r.v. $\eta$ find an estimate of the w.r.v. $\xi$.

A linear estimate of $\xi$ will be any w.r.v. $\xi_{L} = L\eta$, where $L : Y \to X$, is an arbitrary linear continuous operator. Then, from a value $\mathbf{g}$ of $\eta$ one obtains the linear estimate $L\mathbf{g}$ of the w.r.v. $\xi$. Now a measure of the reliability of the estimator $L$ is given by

$$
\delta^{2}(\epsilon, v; L) = \mathbb{E}\{||\xi - L\eta, v||^{2}\}, \quad (v \in X = L^{2}(a, b)),
$$

where $\mathbb{E}\{\cdot\}$ denotes the expectation value. Then, we have the following proposition.

Proposition 2.6. If the covariance operator $R_{\xi \xi}$ has a bounded inverse, then there exists a unique operator $L_{0}$ that minimizes $\delta^{2}(\epsilon, v; L)$ for any $v \in X$, and it is given by

$$
L_{0} = R_{\xi \eta}R_{\eta \eta}^{-1} = R_{\xi \xi}A^{*}[AR_{\xi \xi}A^{*} + R_{\xi \xi}]^{-1}.
$$

Proof. See [4, 5].

The w.r.v. $L_{0}\eta$ is called the best linear estimate of $\xi$, and, given a value $\mathbf{g}$ of $\eta$, the best linear estimate $f_{(4)}^{*}$ for the value of $\xi$ is

$$
f_{(4)}^{*} = \frac{R_{\xi \xi}A^{*}}{AR_{\xi \xi}A^{*} + R_{\xi \xi}}\mathbf{g}, \quad (A^{*} = A).
$$

If $\xi$ and $L\eta$ have finite variance, then the global mean-square error may be defined as follows:

$$
\delta^{2}(\epsilon, L) = \mathbb{E}\{||\xi - L\eta||^{2}\}.
$$

When the operator $L_{0}$ which minimizes (2.18) does exist, it also minimizes the global error (2.21) if $L_{0}\eta$ has finite variance; i.e., if $\text{Tr}(L_{0}R_{\eta \eta}L_{0}^{*}) < \infty$, then the following proposition can be proved.

Proposition 2.7. If the following assumptions

i) $R_{\xi \xi}$ is an operator of trace class;

ii) $R_{\xi \xi} = \epsilon^{2}N$ has bounded inverse;

iii) the equation $Af = 0$, where $f \in \text{Range}\left(R_{\xi \xi}^{1/2}\right)$, has only the trivial solution $f = 0$

are satisfied, then the following limit holds true:

$$
\lim_{\epsilon \to 0}\delta^{2}(\epsilon, L) = 0,
$$

where $\delta^{2}(\epsilon) = \inf_{L}\delta^{2}(\epsilon; L)$.

Proof. See [4, 5].

Let us note that $\delta^{2}(\epsilon) = \delta^{2}(\epsilon; L_{0})$ when $L_{0}$ does exist and is unique.

If we want to compare the probabilistic results obtained above with the variational ones, which have been obtained by the use of eigenfunction expansions, we must expand $\xi$ and $\zeta$ in terms of the eigenfunctions of the operator $A$ (i.e., $\{\psi_{k}\}^{\infty}_{0}$). Their Fourier components are the random variables $\xi_{k} = (\xi, \psi_{k})$ and $\zeta_{k} = (\zeta, \psi_{k})$, whose variances are given respectively by $\rho_{k}^{2}$ and $\epsilon^{2}\nu_{k}^{2}$. Next, in addition to the assumptions I)-III) made before, we make the following hypothesis in spite of the fact that it turns out to be completely unrealistic (see section 3):
the Fourier components of $\xi$ are mutually uncorrelated as well as the Fourier components of $\zeta$.

Therefore, if $R^{-1}_{\xi\xi}$ is bounded (i.e., $\sup_k (1/E^2 \nu_k^2) < \infty$), then the operator $L_0$ exists and the best linear estimate (2.20) can be written as

$$f^{(4)} = \sum_{k=1}^{\infty} \frac{\lambda_k \rho_k^2}{\bar{\lambda}_k \rho_k^2 + \epsilon^2 \nu_k^2} \gamma_k \psi_k.$$  

Finally, the quantities $\delta^2(\epsilon, v; L_0)$ and $\delta^2(\epsilon)$ become

$$\delta^2(\epsilon, v; L_0) = \text{E} \{ ||(\xi - L_0 \eta, v) ||^2 \} = (R_{\xi\xi} - L_0 R_{\eta\eta} L_0^*) v, v = \epsilon^2 \sum_{k=1}^{\infty} \frac{\rho_k^2 \nu_k^2}{\lambda_k \rho_k^2 + \epsilon^2 \nu_k^2} |v_k|^2$$

and

$$\delta^2(\epsilon) = \delta^2(\epsilon; L_0) = \text{Tr} [R_{\xi\xi} - L_0 R_{\eta\eta} L_0^*] = \epsilon^2 \sum_{k=1}^{\infty} \frac{\rho_k^2 \nu_k^2}{\lambda_k \rho_k^2 + \epsilon^2 \nu_k^2},$$

and we have the following proposition.

**Proposition 2.8.** The following statements hold true:

i) for any $v \in X$ ($X = L^2(a, b)$)

$$\lim_{\epsilon \to 0} \delta^2(\epsilon, v; L_0) = 0,$$

ii) if $\text{Tr} R_{\xi\xi} < \infty$, then

$$\lim_{\epsilon \to 0} \delta^2(\epsilon) = 0.$$

3. Information theory and regularization.

3.1. Asymptotic convergence, in the $L^2$-norm, of the eigenfunction expansion. In the variational regularization, use is made of global a priori bounds (e.g., formulae (1.8) or (2.8)), which are the natural constraints in the case of synthesis problems where the variational approach is certainly appropriate. But these global bounds are not necessarily given in the case of inverse problems where the prior knowledge on the solution can be, in several cases, rather poor. Moreover, in the truncated solutions derived by the methods of variational regularization, the point at which to stop the expansion is obtained by comparing the eigenvalues $\lambda_k$ with the ratio $(\epsilon/E)$ (i.e., formula (2.11), or with $(\epsilon/E)|c_k|$ (see formula (2.6)). In both cases this approach appears quite unnatural from the viewpoint of the experimental or physical sciences, whose methodology would rather suggest to stop the expansions at the value $k_0$ of $k$ such that for $k > k_0$ the Fourier coefficients $g_k$ of the noiseless data are smaller or at most of the same order of magnitude of $\epsilon$, and, consequently, it is impossible to extract information from the corresponding coefficients $\gamma_k$. With this in mind, and assuming that the noise is represented by a bounded and integrable function $n(x)$ which satisfies the following condition:

$$\sup |n(x)| \leq \epsilon, \quad x \in [a, b],$$

...
the following results have been proved by two of us:

**Lemma 3.1.** The following statements hold true:

\[
\sum_{k=1}^{\infty} \left( \frac{g_k}{\lambda_k} \right)^2 = \|f\|^2 = C_1 \quad (C_1 = \text{constant}),
\]

(3.2)

\[
\sum_{k=1}^{\infty} \left( \frac{\overline{g}_k}{\lambda_k} \right)^2 = +\infty \quad \text{if} \ \overline{g} \not\in \text{Range } (A),
\]

(3.3)

\[
\lim_{\epsilon \to 0} \overline{g}_{k_0} = g_k, \ \forall k.
\]

(3.4)

If \( k_0(\epsilon) \) is defined by

\[
k_0(\epsilon) = \max \left\{ m \in \mathbb{N} : \sum_{k=1}^{m} \left( \frac{\overline{g}_k}{\lambda_k} \right)^2 \leq C_1 \right\},
\]

(3.5)

then

\[
\lim_{\epsilon \to 0} k_0(\epsilon) = +\infty.
\]

(3.6)

**Proof.** See [17]. \( \Box \)

Now we can introduce the following approximation

\[
f^{(\epsilon)}_0 = \sum_{k=1}^{k_0(\epsilon)} \frac{\overline{g}_k}{\lambda_k} \psi_k
\]

(3.7)

and prove the following theorem.

**Theorem 3.2.** The following equality holds true:

\[
\lim_{\epsilon \to 0} \|f - f^{(\epsilon)}_0\| = 0.
\]

(3.8)

**Proof.** See [17]. \( \Box \)

If we consider a sequence of noisy data \( \overline{g} \) which tends to \( g \) for \( \epsilon \to 0 \) in the sense of the \( L^2 \)-norm (i.e., \( \lim_{\epsilon \to 0} \|\overline{g} - g\| = 0 \)), then \( f^{(\epsilon)}_0 \) will tend to \( f \) as \( \epsilon \to 0 \) in the sense of the \( L^2 \)-norm (i.e., \( \lim_{\epsilon \to 0} \|f^{(\epsilon)}_0 - f\| = 0 \)). In fact, since \( \|\overline{g} - g\|^2 = \sum_{k=1}^{\infty} |\overline{g}_k - g_k|^2 \), the \( \lim_{\epsilon \to 0} \|\overline{g} - g\| = 0 \) implies that for any \( k \), \( \lim_{\epsilon \to 0} \overline{g}_k = g_k \), and in view of Lemma 3.1 and Theorem 3.2, it can be concluded that \( \lim_{\epsilon \to 0} \|f^{(\epsilon)}_0 - f\| = 0 \). Therefore, from approximation (3.7) we can derive an operator \( \overline{B} \) defined by:

\[
\overline{B} \overline{g} = \sum_{k=1}^{k_0(\epsilon)} \frac{\overline{g}_k}{\lambda_k} \psi_k,
\]

(3.9)

which continuously maps (i.e., preserving the convergence) the data \( \overline{g} \) into the solution space \( X \). Thus, continuity has been restored without requiring compactness.

Two types of difficulties still remain:
a) how to determine numerically the truncation point $k_0(\epsilon)$, if the norm of the function $f$ (i.e., the constant $C_1 = \|f\|^2$) is unknown;

b) in any case the convergence of approximation (3.7) is not sufficient to guarantee that the bulk of the unknown function $f$ has been really recovered.

We can give a satisfactory answer to these questions only in very specific and peculiar cases, as we will explain below. Suppose that the moduli of the Fourier coefficients $|g_k|$ are monotonically decreasing for increasing values of $k$. Since $\sum_{k}\geq k_0$ and $\sum_{k}\geq n_k$, it turns out that at a certain value $k_0$ of $k$ we have $|g_k| \approx |n_k| \leq \epsilon$. The Fourier coefficients of the noiseless data are of the same order of magnitude as the Fourier components of the noise, and at this point we cannot extract any information from the noisy Fourier coefficients $\sum_{k}\geq n_k$. Let us now introduce the function $M(m) = \sum_{k=1}^{m} (\sum_{k}\geq n_k / \lambda_k)^2$, whose relevant properties are:

1) It is an increasing function of $m$;

2) If $\epsilon$ is sufficiently small and the values of $|g_k|$ are monotonically decreasing for increasing $k$, $M(m)$ presents a “plateau” when it reaches the value $C_1$.

Indeed, from formula (3.6) in Lemma 3.1 it follows that $M(m)$ remains nearly constant when it attains the value $C_1$. An explicit numerical example of this “plateau” is given in Figure 4.1D in section 4.

This “plateau” corresponds to the order-disorder transition in the coefficients $\sum_{k}\geq n_k$: for $k < k_0(\epsilon)$ the data $g_k$ prevail on $n_k$ whereas for $k > k_0(\epsilon)$ the noise components $n_k$ are larger or, at least, of the same order of magnitude of the noiseless data. However, it must be remarked that in practical cases to single out the plateau which does really correspond to the order-disorder transition in the coefficients $\sum_{k}\geq n_k$ can be made difficult by the presence of other spurious plateaux due to the erratic behavior of the noise. Furthermore, if the coefficients $g_k$ are negligible for low values of $k$, and the actual bulk of information is located only at intermediate values of $k$, there could be no numerical evidence of such a plateau in spite of the fact that the convergence guaranteed by Theorem 4.2 remains true. Then we are forced to look for other methods that overcome these difficulties. This issue will be investigated by means of probabilistic methods, as will be illustrated in the next subsection.

3.2. Bayes formula, information theory, and regularization. Here our goal is to find a probabilistic extension of the result of Theorem 4.2 in which the assumption requiring the Fourier coefficients $|g_k|$ to be monotonically decreasing will be removed. In fact, we will show how to construct a regularizing solution from the noisy data, disregarding the order of the coefficients $|g_k|$. For this purpose, we turn (2.14) into an infinite sequence of one-dimensional equations by means of orthogonal projections:

$$\lambda_k \xi_k + \zeta_k = \eta_k, \quad (k = 1, 2, \ldots),$$

where $\xi_k = (\xi_k, \psi_k)$, $\zeta_k = (\zeta_k, \psi_k)$, $\eta_k = (\eta_k, \psi_k)$ are Gaussian random variables. Here we retain assumptions I-III) made in section 2.2 but we remove assumption IV).

In fact, there is no reason to assume that the basis $\{\psi_k\}_{k=1}^{\infty}$ which diagonalizes the operator $A$ also diagonalizes the covariance operators $R_{\xi_k, \xi_k}$, $R_{\xi_k, \zeta_k}$, $R_{\eta_k, \eta_k}$ in $H$. Therefore, we can introduce the variances $\rho_k^2 = (R_{\xi_k, \psi_k}, \psi_k)$, $\epsilon^2 \nu_k^2 = (R_{\xi_k, \psi_k}, \psi_k)$, $\lambda_k^2 \rho_k^2 + \epsilon^2 \nu_k^2 = (R_{\eta_k, \psi_k}, \psi_k)$, without assuming that the Fourier components $\xi_k$ of $\xi$ (and analogously also for $\zeta_k$ and $\eta_k$) are mutually uncorrelated. In view of the assumptions I) and III) the following probability densities for $\xi_k$ and $\zeta_k$ can be assumed:

$$p_{\xi_k}(x) = \frac{1}{\sqrt{2\pi \rho_k}} \exp \left\{ - \left( \frac{x^2}{2\rho_k^2} \right) \right\}, \quad (k = 1, 2, \ldots)$$
and
\begin{equation}
\tag{3.12}
p_{\xi_k}(x) = \frac{1}{\sqrt{2\pi \nu_k}} \exp \left\{ -\left( \frac{x^2}{2\epsilon^2 \nu_k^2} \right) \right\}, \quad (k = 1, 2, ...).
\end{equation}

By the use of the (3.10), we can also introduce the conditional probability density \( p_{\eta_k}(y|x) \) of the random variable \( \eta_k \) for fixed \( \xi_k = x \), which reads
\begin{equation}
\tag{3.13}
p_{\eta_k}(y|x) = \frac{1}{\sqrt{2\pi \nu_k}} \exp \left\{ -\left( \frac{(y - \lambda_k x)^2}{2\epsilon^2 \nu_k^2} \right) \right\} = \frac{1}{\sqrt{2\pi \nu_k}} \exp \left\{ -\frac{\lambda_k^2}{2\epsilon^2 \nu_k^2} \left( x - \frac{y}{\lambda_k} \right)^2 \right\}.
\end{equation}

Now let us apply the Bayes formula that provides the conditional probability density of \( \xi_k \) given \( \eta_k \) through the following expression:
\begin{equation}
\tag{3.14}
p_{\xi_k}(x|y) = \frac{p_{\xi_k}(x)p_{\eta_k}(y|x)}{p_{\eta_k}(y)}.
\end{equation}

Thus, if a realization of the random variable \( \eta_k \) is given by \( g_k \) (see the formulation of the problem in section 2.2), formula (3.14) becomes
\begin{equation}
\tag{3.15}
p_{\xi_k}(x|g_k) = A_k \exp \left\{ -\frac{x^2}{2\rho_k^2} \right\} \exp \left\{ -\frac{\lambda_k^2}{2\epsilon^2 \nu_k^2} \left( x - \frac{g_k}{\lambda_k} \right)^2 \right\} \quad (A_k = \text{const.}).
\end{equation}

Now the amount of information on the variable \( \xi_k \) which is contained in the variable \( \eta_k \) can be evaluated. We have
\begin{equation}
\tag{3.16}
J(\xi_k, \eta_k) = -\frac{1}{2} \log(1 - r_k^2),
\end{equation}
where
\begin{equation}
\tag{3.17}
r_k^2 = \frac{|E\{\xi_k \eta_k\}|^2}{E\{|\xi_k|^2\} E\{|\eta_k|^2\}} = \frac{(\lambda_k \rho_k)^2}{(\lambda_k \rho_k)^2 + (\epsilon \nu_k)^2}.
\end{equation}

Thus,
\begin{equation}
\tag{3.18}
J(\xi_k, \eta_k) = \frac{1}{2} \log \left( 1 + \frac{\lambda_k^2 \rho_k^2}{\epsilon^2 \nu_k^2} \right).
\end{equation}

From equality (3.18) it follows that \( J(\xi_k, \eta_k) < \frac{1}{2} \log 2 \), if \( \lambda_k \rho_k < \epsilon \nu_k \). Thus, we are naturally led to introduce the following sets:
\begin{equation}
\tag{3.19}
J_k = \{ k : \lambda_k \rho_k \geq \epsilon \nu_k \},
\end{equation}
\begin{equation}
\tag{3.20}
N_k = \{ k : \lambda_k \rho_k < \epsilon \nu_k \}.
\end{equation}

Reverting to the conditional probability density (3.15), it can be regarded as the product of two Gaussian probability densities: \( p_1(x) = A_k^{(1)} \exp \left\{ -\frac{x^2}{2\rho_k^2} \right\} \) and \( p_2(x) = A_k^{(2)} \exp \left\{ -\frac{(\lambda_k^2 / 2\epsilon^2 \nu_k^2) (x - (g_k / \lambda_k))^2}{2} \right\} \), \( A_k = A_k^{(1)} \cdot A_k^{(2)} \), whose variances...
are respectively given by $\rho_k$ and $(\epsilon \nu_k / \lambda_k)$. Let us note that if $k \in I_k$, the variance associated with the density $p_2(x)$ is smaller than the corresponding variance of $p_1(x)$, and vice versa if $k \in N_k$. Therefore, it is reasonable to consider as an acceptable approximation of $\langle \xi_k \rangle$ the mean value given by the density $p_2(x)$ if $k \in I_k$, or the mean value given by the density $p_1(x)$ if $k \in N_k$. We can write the following approximation:

$$\langle \xi_k \rangle = \begin{cases} \frac{g_k}{\lambda_k} & (k \in I_k), \\ 0 & (k \in N_k). \end{cases} \quad (3.21)$$

Consequently, given the value $g$ of the w.r.v. $\eta$, we are led to consider the following estimate of $\xi$:

$$\hat{B}g = \sum_{k \in I_k} \frac{g_k}{\lambda_k} \psi_k. \quad (3.22)$$

However, these are only heuristic considerations based on plausible arguments. They will become rigorous statements only if it will proved that they lead to a solution $\hat{B}g$ which is probabilistically regularized. For this purpose, the global mean-square error associated with the operator $\hat{B}$, i.e., $E \{ \| \xi - \hat{B}\eta \|^2 \}$, must be evaluated, and we have the following proposition.

**Proposition 3.3.**

i) If $\lim_{k \to \infty} (\lambda_k \rho_k / \nu_k) = 0$, then the set $I_k$ is finite for any fixed positive value of $\epsilon$;

ii) assuming that the limit stated in i) holds true, and, in addition, that $R_{\xi\xi}$ is an operator of trace class, then the following relationship holds:

$$E \{ \| \xi - \hat{B}\eta \|^2 \} = \sum_{k \in N_k} \rho_k^2 + \sum_{k \in I_k} \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} < \infty. \quad (3.23)$$

**Proof.** The proof of statement i) is obvious if we recall the definition of the set $I_k$ (formula (3.19)). Statement ii) follows easily from the equality

$$E \{ \| \xi - \hat{B}\eta \|^2 \} = \text{Tr} (R_{\xi\xi} - R_{\xi\xi} A^* \hat{B}^* - \hat{B} A R_{\xi\xi} + \hat{B} R_{\eta\eta} \hat{B}^*) \quad (3.24)$$

and by the use of formulae (2.15), (2.17), and (3.22).

In order to prove that approximation (3.22) is regularized, we need the following auxiliary lemma.

**Lemma 3.4.** Let $k_{\gamma}(\epsilon)$ be defined as follows:

$$k_{\gamma}(\epsilon) = \max \left\{ m \in \mathbb{N} : \sum_{k=1}^{m} \left( \rho_k^2 + \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} \right) \leq \Gamma \right\}, \quad (3.25)$$

where $\Gamma = \text{Tr} R_{\xi\xi}$ is finite. Then the following statements hold true:

i) $\lim_{\epsilon \to 0} k_{\gamma}(\epsilon) = +\infty,$

ii) $\lim_{\epsilon \to 0} \left( \sum_{k=k_{\gamma}+1}^{k_{\gamma}} \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} + \sum_{k=k_{\gamma}+1}^{\infty} \rho_k^2 \right) = 0. \quad (3.27)$
Proof. i) Let \( k_{\gamma_1} \) denote the sum \((k_{\gamma} + 1)\). Then suppose that the limit \(3.26\) does not hold. This latter assumption would imply that there exists a finite number \( M \), which does not depend on \( \epsilon \), such that \( k_{\gamma_1} < M \). Furthermore, this bound should remain true for any sequence \( \epsilon_i \) tending to zero. Then, we have the following inequalities:

\[
\Gamma < \sum_{k=1}^{k_{\gamma_1}} \left( \rho_k^2 + \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} \right) \leq \sum_{k=1}^{M} \left( \rho_k^2 + \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} \right).
\]

(3.28)

Now for any sequence \( \epsilon_i \) tending to zero, we have

\[
\Gamma < \sum_{k=1}^{M} \rho_k^2 \leq \sum_{k=1}^{\infty} \rho_k^2 = \Gamma,
\]

which is contradictory. Then, limit \(3.26\) holds.

ii) From \( \sum_{k=1}^{\infty} \rho_k^2 = \text{Tr} R_{\xi\xi} = \Gamma < \infty \), and in view of statement i), it follows that \( \lim_{\epsilon \to 0} \sum_{k=k_{\gamma_1}}^{\infty} \rho_k^2 = 0 \). Regarding the sum \( \sum_{k=1}^{k_{\gamma_1}} (\epsilon^2 \nu_k^2 / \lambda_k^2) \), we can proceed as follows. From formula \(3.25\) we have

\[
\sum_{k=1}^{k_{\gamma_1}} \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} + \sum_{k=1}^{k_{\gamma_1}} \rho_k^2 \leq \Gamma.
\]

(3.30)

Then

\[
\sum_{k=1}^{k_{\gamma_1}} \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} \leq \Gamma - \sum_{k=1}^{k_{\gamma_1}} \rho_k^2 = \sum_{k=k_{\gamma_1}}^{\infty} \rho_k^2,
\]

(3.31)

but in view of the fact that \( \lim_{\epsilon \to 0} \sum_{k=k_{\gamma_1}}^{\infty} \rho_k^2 = 0 \), we have \( \lim_{\epsilon \to 0} \sum_{k=1}^{k_{\gamma_1}} (\epsilon^2 \nu_k^2 / \lambda_k^2) = 0 \), and statement ii) is proved.

We can now prove the following theorem.

**Theorem 3.5.** If the covariance operator \( R_{\xi\xi} \) is of trace class, and if the set \( \mathcal{I}_k \) is finite (see Proposition 3.3), then the following limit holds true:

\[
\lim_{\epsilon \to 0} \delta^2(\epsilon, \hat{B}) = \lim_{\epsilon \to 0} E \left\{ \| \xi - \hat{B}\eta \|^2 \right\} = 0;
\]

(3.32)

i.e., approximation \(3.22\) is probabilistically regularized.

**Proof.** In view of formula \(3.23\) in Proposition 3.4, the proof of equality \(3.32\) reduces to the proof of the following limit:

\[
\lim_{\epsilon \to 0} \left\{ \sum_{k \in \mathcal{I}_k} \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} + \sum_{k \in \mathcal{N}_k} \rho_k^2 \right\} = 0.
\]

(3.33)

Regarding the first sum of \(3.33\), we divide the set \( \mathcal{I}_k \) into two subsets defined by

\[
\mathcal{I}_k^{(1)} = \{ k \in \mathcal{I}_k : k \leq k_{\gamma} \},
\]

(3.34)

\[
\mathcal{I}_k^{(2)} = \{ k \in \mathcal{I}_k : k > k_{\gamma} \}; \quad (\mathcal{I}_k = \mathcal{I}_k^{(1)} \cup \mathcal{I}_k^{(2)}).
\]

(3.35)
Accordingly, we can write

\[
\sum_{k \in J_k} \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} = \sum_{k \in J_k^{(1)}} \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} + \sum_{k \in J_k^{(2)}} \frac{\epsilon^2 \nu_k^2}{\lambda_k^2}. \tag{3.36}
\]

Then \( \sum_{k \in J_k^{(1)}} (\epsilon^2 \nu_k^2 / \lambda_k^2) \leq \sum_{k=1}^{k_\gamma} (\epsilon^2 \nu_k^2 / \lambda_k^2) \), and in view of Lemma 3.4 (where we proved that \( \lim_{\epsilon \to 0} \sum_{k=1}^{k_\gamma} (\epsilon^2 \nu_k^2 / \lambda_k^2) = 0 \)) it follows that

\[
\lim_{\epsilon \to 0} \sum_{k \in J_k^{(1)}} \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} = 0. \tag{3.37}
\]

Regarding the term \( \sum_{k \in J_k^{(2)}} (\epsilon^2 \nu_k^2 / \lambda_k^2) \), since \( k \in J_k \) then \( \rho_k^2 \geq (\epsilon^2 \nu_k^2 / \lambda_k^2) \) and therefore

\[
\sum_{k \in J_k^{(2)}} \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} \leq \sum_{k=k_{\gamma_1}}^{\infty} \rho_k^2. \tag{3.38}
\]

But, as we have seen in Lemma 3.4 \( \lim_{\epsilon \to 0} \sum_{k=k_{\gamma_1}}^{\infty} \rho_k^2 = 0 \), and consequently

\[
\lim_{\epsilon \to 0} \sum_{k \in J_k^{(2)}} \frac{\epsilon^2 \nu_k^2}{\lambda_k^2} = 0. \tag{3.39}
\]

We can conclude that \( \lim_{\epsilon \to 0} \sum_{k \in J_k} (\epsilon^2 \nu_k^2 / \lambda_k^2) = 0 \). Regarding the sum \( \sum_{k \in N_k} \rho_k^2 \), we proceed in an analogous way by splitting the set \( N_k \) into two subsets defined by

\[
N_k^{(1)} = \{ k \in N_k : k \leq k_{\gamma} \},
\]
\[
N_k^{(2)} = \{ k \in N_k : k > k_{\gamma} \}; \quad (N_k = N_k^{(1)} \cup N_k^{(2)}). \tag{3.40}
\]

Accordingly, we write

\[
\sum_{k \in N_k} \rho_k^2 = \sum_{k \in N_k^{(1)}} \rho_k^2 + \sum_{k \in N_k^{(2)}} \rho_k^2. \tag{3.42}
\]

If \( k \in N_k^{(1)} \) and by the use of inequality \( \rho_k^2 < (\epsilon^2 \nu_k^2 / \lambda_k^2) \) (because \( k \in N_k \)) we can write

\[
\sum_{k \in N_k^{(1)}} \rho_k^2 \leq \sum_{k=1}^{k_{\gamma}} \frac{\epsilon^2 \nu_k^2}{\lambda_k^2}. \tag{3.43}
\]

But in Lemma 3.4 we proved that \( \lim_{\epsilon \to 0} \sum_{k=1}^{k_{\gamma}} (\epsilon^2 \nu_k^2 / \lambda_k^2) = 0 \), and therefore we have \( \lim_{\epsilon \to 0} \sum_{k \in N_k^{(1)}} \rho_k^2 = 0 \). Regarding the second term on the right-hand side of formula (3.42), we have

\[
\sum_{k \in N_k^{(2)}} \rho_k^2 \leq \sum_{k=k_{\gamma_1}}^{\infty} \rho_k^2. \tag{3.44}
\]

But, again, \( \lim_{\epsilon \to 0} \sum_{k=k_{\gamma_1}}^{\infty} \rho_k^2 = 0 \), and then \( \lim_{\epsilon \to 0} \sum_{k \in N_k^{(2)}} \rho_k^2 = 0 \).  \( \Box \)
Remarks. i) It is worth it to notice that the proof of Theorem 3.5 does not require any type of order in the sum (3.22). In fact, the only assumption that \( \{ \lambda_k \} \) is a strictly decreasing sequence does not evidently imply that the terms \( (\lambda_k \rho_k / \epsilon \nu_k) \) have any type of monotonicity in \( k \), and, consequently, the sum (3.22) cannot, in general, be regarded as an ordered sum of terms up to a certain maximum value of \( k \). Thus, unlike the regularized solutions (2.3), (2.5), (2.9), (2.10) and also (3.9), \( \hat{B} \eta \) features frequency selectivity, which is obtained by evaluating the information content of the noisy Fourier coefficients.

ii) Notice that the estimate (3.22) associated with the operator \( \hat{B} \) represents a probabilistically regularized solution, in the sense of the formula (3.32), even if, in general, it does not minimize the global mean-square error (2.21).

At this point in order to apply the results of this section, statistical methods that allow for splitting the coefficients \( \eta_k \) into the sets \( I_k \) and \( N_k \) must be investigated. These methods will be illustrated in the next section.

4. Numerical analysis: the regularizing algorithm.

4.1. The correlation function of the noisy data. The application of the results of the previous section to a Fredholm equation of the first kind would involve using statistical tools for the determination of the two sets \( I_k \) and \( N_k \). In this section this issue is discussed and the basic steps of a numerical algorithm for constructing the regularized solution \( \hat{B} \eta \) from the noisy data \( \eta \) are outlined. For simplicity we shall work throughout only with data corrupted by white noise. However, provided the independence assumption between \( \xi \) and \( \zeta \), more general cases involving “colored” noise could be treated by using suitable methods, for instance, “prewhitening” transformations [4], whose discussion is beyond the scope of this section. Here our goal is to show that statistical estimates of the amount of information carried by the Fourier coefficients \( \eta_k \) can be sufficient to construct a satisfactory regularized solution. Furthermore, the direct comparison of the numerical results clearly evidences how some inherent limitations of the variational regularization scheme are overcome.

Following the analysis of the previous section, we are now faced with the problem of separating the Fourier coefficients \( \eta_k \) into two classes; one containing all the Fourier coefficients of the noisy data which are correlated, the other containing the \( \eta_k \) that can be regarded as random numbers. This task can be achieved by computing the correlation function of the random variables \( \eta_k \): i.e., the probabilistic counterpart of the coefficients \( \eta_k \):

\[
\Delta_\eta(k_1,k_2) = \frac{E[\eta_{k_1} - E(\eta_k_1)][\eta_{k_2} - E(\eta_{k_2})]}{E[\eta_{k_1} - E(\eta_k_1)]^2}^{1/2}E[\eta_{k_2} - E(\eta_{k_2})]^2)^{1/2}. \tag{4.1}
\]

In practice, just a finite realization \( \{ \eta_k \}_1^N \) of the random variables \( \eta_k \) is available, from which estimates of the autocorrelations can be obtained by regarding the data \( \{ \eta_k \}_1^N \) as a finite length record of a stationary random normal series. In principle the assumption of stationarity of the series \( \{ \eta_k \} \) is not correct because in general the moments of the random variables \( \eta_k \) will depend on \( k \), but from the practical point of view this is usually the only possible chance. In fact, in many areas of application, it is difficult or even impossible to have multiple independent realizations \( \{ \eta_k \}_1^N \) of the process \( \{ \eta_k \} \), so estimates of ensemble averages cannot be computed. Thus, we are forced to introduce the working hypothesis that the process \( \{ \eta_k \} \) is stationary in wide sense [9], that is \( \Delta_\eta(k_1,k_2) = \Delta_\eta(k_1 - k_2) \), and compute the estimates of
the autocorrelation coefficients by means of the ergodic relation between ensemble and time (i.e., the index \( k \) in our case) averages. Of course, such a restriction can be removed whenever many independent sets of data \( \{ \mathbf{g}_k \}_{k=1}^N \) would be available for evaluating ensemble averages. Anyway, we will see later in the discussion of the algorithm how an ambiguity in the reconstruction of the regularized solution \( \hat{\mathbf{g}} \) due to the assumed invariance for \( k \)-translation of \( \{ \eta_k \} \) will be removed.

A number of estimators of the autocorrelation function have been suggested by statisticians and their properties are discussed in detail in [15]. An estimate which is widely used by statisticians, and in the following examples as well, is given by

\[
\delta_g(n) = \frac{\sum_{k=1}^{N-n} (\overline{g}_k - \langle \overline{g}_k \rangle)(\overline{g}_{k+n} - \langle \overline{g}_{k+n} \rangle)}{\left\{ \sum_{k=1}^{N-n} (\overline{g}_k - \langle \overline{g}_k \rangle)^2 \sum_{k=1}^{N-n} (\overline{g}_{k+n} - \langle \overline{g}_{k+n} \rangle)^2 \right\}^{1/2}}, \quad n = 0, \ldots, N-1,
\]

where

\[
\langle \overline{g}_k \rangle = \frac{1}{N-n} \sum_{k=1}^{N-n} \overline{g}_k; \quad \langle \overline{g}_{k+n} \rangle = \frac{1}{N-n} \sum_{k=1}^{N-n} \overline{g}_{k+n}.
\]

Equation (4.2), which is based on the scatter diagram of \( \overline{g}_{k+n} \) against \( \overline{g}_k \) for \( k = 1, \ldots, N - n \), represents the maximum likelihood estimate of the autocorrelation coefficients of two random variables \( \eta_k \) and \( \eta_{k+n} \) whose joint probability distribution function is bivariate normal.

In order to identify the structure of the series \( \{ \overline{g}_k \}_{k=1}^N \) so that we can separate correlated components from the random ones, it is necessary to have a crude test on whether \( \delta_g(n) \) is effectively zero. It has been shown by Anderson [1] that the distribution of an estimated autocorrelation coefficient, whose theoretical value is zero, is approximately normal. Thus, on the hypothesis that the theoretical autocorrelation \( \Delta \eta(n) = 0 \), the estimate \( \delta_g(n) \) divided by its standard error \( \sigma_g(n) \) will be approximately distributed as a unit normal deviate. This fact may be used to provide a rough guide as to whether theoretical autocorrelations are essentially zero. To this purpose it is usually sufficient to remember that, for normal distribution, deviations exceeding two standard errors in either direction have a probability of about 0.05, so that the 95% confidence interval of the estimate is approximately \( \delta_g(n) \pm 1.96 \sigma_g(n) \).

Estimated autocorrelations can have rather large variances and can be highly correlated with each other [3, 12], so that care is required in the interpretation of individual autocorrelations. In particular, moderately large estimated autocorrelations can occur after the theoretical autocorrelation function has damped out and, in any case, it must be considered that an estimated autocorrelation function always exhibits less damping than the theoretical one, as the estimated autocorrelations are inflated by sampling fluctuations (see also the following Example 1). Thus, in order to avoid a purely empirical analysis of the autocorrelations, it is necessary to assume a rough model of the series that allows to evaluate the order of magnitude of the sampling errors \( \sigma_g(n) \) associated to the autocorrelation estimator.

According to the discussion of Section 3.2, since we are expected to find the set \( I_k \) to be finite, we are also expected that the autocorrelation function \( \Delta \eta(n) \) will vanish beyond a certain lag \( n_0 \). Thus, in what follows, it will be assumed that there exists an index \( n_0 \) such that \( \Delta \eta(n) = 0 \) for \( n > n_0 \). In this case, if the record length \( N \) is
large enough (i.e., such that \(O(1/N^2)\) terms can be neglected), use can be made of the Bartlett’s approximate expression for the variance of the estimated autocorrelations of a stationary normal process \([3]\):

\[
\text{var} [\hat{\delta}_g(n)] \sim \frac{1}{N - n} \left\{ 1 + 2 \sum_{v=1}^{n_0} \Delta^2_v (\hat{\theta}) \right\}, \quad \text{for } n > n_0.
\]

To use (4.4) in practice, the estimated autocorrelations \(\hat{\delta}_g\) are substituted for the theoretical ones \(\Delta_n\), and in this case we shall refer to the square root of (4.4) as the large-lag standard error \(\sigma_g(n; n_0)\) \([4]\).

The index \(n_0\) is actually recovered in a recursive way through an hypothesis generation-verification procedure. Starting from the assumption that the series is completely random, i.e., \(n_0 = 0\), the standard error \(\sigma_g(n; 0)\) is computed and the first index \(\pi > 0\) such that \(|\delta_g(\pi)| > 1.96 \sigma_g(n; 0)\) is searched for. If there exists such an index \(\pi\), it becomes the new candidate to be \(n_0\), i.e., we set \(n_0 = \pi\), \(\sigma_g(n; n_0)\) is computed, and again it is tested whether the series is compatible with the hypothesis that \(\Delta_n(n) = 0\) for \(n > n_0\). The whole procedure is repeated until no new index \(\pi\) is found. Formally, \(n_0\) is then defined as

\[
n_0 = \max \{ \pi \geq 0 : \forall n \in (\pi, N - 1], \ |\delta_g(n)| < 1.96 \sigma_g(n, \pi) \}.
\]

The set \(Q\) of the lags corresponding to autocorrelation values that are effectively different from zero and, consequently, indicating lack of randomness of the coefficients \(\theta_k\), is defined as:

\[
Q = \{ 0 < n \leq n_0 : |\delta_g(n)| > 1.96 \sigma_g(n, 0) \}.
\]

Let \(N_c\) be the number of elements of \(Q\).

As previously discussed, as a consequence of the inevitable assumption of stationarity of the process \(\{\eta_k\}\), the Fourier coefficients \(\theta_k\) that are correlated cannot be determined in a unique way from the set \(Q\). In fact, an integer \(n_i \in Q\) just indicates a strong correlation between at least two Fourier coefficients \(n_i\) apart. This means that, in principle, any couple \((\theta_{k_i}, \theta_{k_i + n_i})\) for any integer \(1 \leq k_i \leq (N - n_i)\) could have generated such a strong correlation at the lag \(n_i\). Thus, from the set \(Q\) we can construct \(N_c\) families \(F_i\) defined as

\[
F_i = \{ (\theta_{k_i}, \theta_{k_i + n_i}) \}^{N-n_i}_{k_i=1}, \quad i = 1, ..., N_c
\]

from which the couples of coefficients \(\theta_k\) that are likely to be correlated can be selected. In theory, that is for \(N \to \infty\), the \(N_c\) indices \(k_i\) and the \(N_c\) elements \(n_i \in Q\) are mutually dependent. In fact, any two coefficients \(\theta_{k_1}, \theta_{k_2}\) which are selected from the families \(F_i\) must satisfy the pairwise compatibility conditions requiring \(|k_1 - k_2| \in Q\). Or, in other words, it can be seen that, given the set \(Q\), the number \(N_j\) of admissible Fourier coefficients \(\theta_k\) is combinatorially constrained to be

\[
\frac{1}{2} \left( 1 + \sqrt{1 + 8N_c} \right) \leq N_j \leq N_c + 1.
\]

The left inequality in (4.8) follows directly from the observation that the maximum number of correlations among \(N_j\) coefficients is \(\binom{N_j}{2}\), then \(N_c \leq \binom{N_j}{2}\), whereas the
right inequality expresses that at least \((N_I - 1)\) distinct correlations can be computed among \(N_I\) coefficients (i.e., \(N_c \geq N_I - 1\)). For instance, if \(N_c = 2\), we have from inequalities (4.8) that there need to be \(N_I = 3\) coefficients \(\bar{g}_k\) to construct the set \(Q\), or, referring to (4.7), that the two indices \(k_1\) and \(k_2\) must coincide, i.e., \(k_1 \equiv k_2 \geq 1\).

In any case, the compatibility conditions are not sufficient to constrain in a unique way the selection of the coefficients \(\bar{g}_k\) and, consequently, the construction of the regularized solution.

In practice, that is when the record length \(N\) is finite and particularly when the signal-to-noise ratio (SNR) of the data \(g\) is small, the compatibility constraints cannot be assumed to be satisfied. In fact, because of the sampling fluctuations in the estimates \(\delta g(n)\), some correlations which are actually different from zero could be incorrectly detected by the procedure discussed above. However, we shall see later in the discussion of the numerical examples how the compatibility constraints can provide us with a confidence check on the reliability of the regularized solution \(\hat{\mathcal{B}}g\).

In order to recover in a unique way from the set \(Q\) the Fourier coefficients that are likely to be correlated, we adopt the following criterion suggested by the definition itself of the autocorrelation function: for any \(n_i \in Q\), \(i = 1, \ldots, N_c\), we select the pair \((\bar{g}_{k^*_i}, \bar{g}_{k^*_i + n_i})\) giving the maximum contribution to the autocorrelation estimate \(\delta g(n_i)\); i.e., we define \(k^*_i\) as

\[
(4.9) \quad k^*_i = \arg \max_{k \in [1, N-n_i]} \{|\bar{g}_k \bar{g}_{k+n_i}|\}, \quad i = 1, \ldots, N_c,
\]

and, accordingly, we can define the set of frequencies \(\mathcal{J}_k\) exhibiting correlated Fourier coefficients as

\[
(4.10) \quad \mathcal{J}_k = \{k^*_i\}_{i=1}^{N_c} \cup \{k^*_i + n_i\}_{i=1}^{N_c},
\]

where each element of \(\mathcal{J}_k\) is counted only once.

**4.2. Numerical examples.** Throughout this section we shall consider as a sample problem the integral equation (1.1) with kernel

\[
(4.11) \quad K(x, y) = \begin{cases} (1-x) y & \text{if } 0 \leq y \leq x \leq 1, \\ x (1-y) & \text{if } 0 \leq x \leq y \leq 1 \end{cases}
\]

whose eigenfunctions and eigenvalues are, respectively,

\[
(4.12) \quad \psi_k(x) = \sqrt{2} \sin(k \pi x), \\
(4.13) \quad \lambda_k = \frac{1}{k^2 \pi^2}.
\]

The data \(g(x)\) have been noised by adding white noise \(n(x)\), simulated by computer generated random numbers uniformly distributed in the interval \([-\epsilon, \epsilon]\) (see also [22] for a very preliminary numerical analysis of this problem). The examples shown hereafter differ for the choice of the input signal \(f(x)\) and for the values of the noise boundary \(\epsilon\), whereas the performances of the algorithm are evaluated by direct comparison of the reconstructed signal with the true signal \(f(x)\). In every example reported here, the approximations obtained through the variational scheme (see section 2.1) are computed by setting the constraint operator \(C\) such that \(c_k = k, \ (k = 1, 2, \ldots)\), the parameter \(\epsilon\) corresponding to the boundary on the noise equal to the dispersion of the noise \(D_\epsilon\) (see (1.7)), and the boundary \(E\) on the solution equal to the norm of the unknown function, i.e., \(E = \|f(x)\|\) (see (1.8)).
Fig. 4.1. Example 1: \( f_1(x) = (1 - x) \sin(3 \sin(3x)), \) \( \epsilon = 10^{-4}, \) SNR \( \simeq 25.7 \text{dB}, \) \( N = 512. \) (A) Noiseless Fourier coefficients \( g_k. \) (B) Modulus of the autoregression function. The horizontal dotted straight line indicates the 95\% confidence limit 1.96\( \sigma_g(n;0) \) for a purely random sequence. The solid curved line indicates the confidence limit 1.96\( \sigma_g(n;3), n > 3. \) From the analysis of \( \delta_g(n) \) we have \( Q = \{1, 2, 3\} \) and \( J_k = \{1, 2, 3, 4\}. \) (C) Regularized solutions. The solid line represents the actual solution \( f_1(x). \) The dots represent the reconstruction \( \hat{f}_k. \) The crosses represent the variational solution \( f_1^{(1)} \) obtained by using \( c_k = k; k_0 = 8 \) (see equations (2.5) and (2.6)). (D) Plot of the function \( M(m) = \sum_{k=1}^{m} (g_k/\lambda_k)^2. \) Notice that the value of \( M(m) \) corresponding to its first plateau, i.e., approximately for \( 4 \leq m \leq 10, \) is about the squared norm of the true solution.

In Figure 4.1 the analysis of the sample function \( f_1(x) = (1 - x) \sin(3 \sin(3x)) \) with noise boundary \( \epsilon = 10^{-4} \) is summarized. The global SNR, defined as the ratio of the mean power of the noiseless data to the noise variance, was SNR \( \simeq 25.7 \) dB. The function \( f_1(x) \) is characterized by having the bulk of information localized in the first few values of \( k \) (see the related noiseless coefficients \( g_k \) in Figure 4.1A) so that we expect that also a variational solution could provide a satisfactory reconstruction of the input signal. Figure 4.1B shows the behavior of the autoregression function \( \delta_g(n) \) along with the two lines indicating the statistical confidence limits we used to discriminate whether the autoregressions are essentially null. The dashed horizontal straight line represents the threshold that we would have under the hypothesis of purely random sequence \( \{g_k\}, \) whereas the solid line represents the threshold corresponding to the model of autoregression function of ideal damped type. In this example we found \( n_0 = 3, Q = \{1, 2, 3\}, \) and the autocorrelation at \( n = 4 \) was rejected in spite of its quite large value (see formula (4.6)). The direct inspection of the values of \( c_k \) and \( g_k \) in repeated realizations showed that for \( k = 5 \) the noise was usually larger than the Fourier coefficient, confirming hence the result that the autocorrelation \( \delta_g(4) \) was abnormally inflated by the large autocorrelations at \( n = 1, 2, 3. \)
According to the criteria (4.9) and (4.10), the set of frequencies whose corresponding
Fourier coefficients exhibit strong correlations is \( J_k = \{1, 2, 3, 4\} \). It is worth noticing
that in this case the elements of \( J_k \) satisfy all the compatibility constraints, i.e., any
difference between elements of \( J_k \) belongs to \( Q \), and \( N_j \) satisfies constraints (4.8).
This complete cross-consistency between \( Q \) and \( J_k \) gives a high level of confidence in
the result of the whole analysis. In Figure 4.1C, the true function \( f_1 \) (solid line), the
regularized solution \( \hat{B}_g \) (crosses) and the regularized function \( f^{(1)}_\hat{g} \) (dots) are com-
pared. The truncation point of \( f^{(1)}_\hat{g} \), obtained through the criterion (2.10), was \( \alpha = 8 
\). Figure 4.1C shows how in this case both regularization methods lead to comparable
results, which are quite satisfactory approximations of the “unknown” function
\( g \).

The plot of the function \( M(m) \), displayed in Figure 4.1D confirms the correctness of
the two approximations. In fact, it clearly exhibits a “plateau”, ranging from about
\( m = 3 \) to \( m = 10 \), that corresponds to the order-disorder transition of the coefficients
\( \hat{g}_k \). Then it could be argued that for any truncation point belonging to this “plateau”
the truncated approximation will hold coefficients \( \hat{g}_k \) whose information content is not
completely obscured by the noise. In every example discussed here, the regularized
solutions \( f^{(2)}_\hat{g} \) and \( f^{(3)}_\hat{g} \) (see (2.9) and (2.10)) have also been considered, providing in
cases worse results (not plotted).

The second and third examples, shown in Figure 4.2, are quite simple but a little
tricky, and show the deep differences between our approach and the variational one.
They consist of a finite linear combination of, respectively, 3 and 10 basis functions
\( \psi_k \) (see the legend for numerical details), and, indeed, they have been chosen as
typical signals in which the bulk of the information is not grouped in a single block of
consecutive low frequencies. In these cases, setting global constraints on the solution,
such as in the variational methods, leads inevitably to a failure, which is clearly evident
from Figure 4.2A, F, since the lack of selectivity necessarily causes the regularized
solution \( f^{(1)}_\hat{g} \) to contain pure noisy components. On the contrary, the selectivity
achieved through the analysis of the autocorrelation function overcomes this limit.
In both examples the analysis of the autocorrelation function (see Figure 4.2A,D)
led to the correct selection of the components that carry information in spite of the
quite small SNR (in the Example 2, SNR \( \approx 0.55 \text{dB} \)). Referring to the Example 2 depicted in Figure 4.2A,B,C, it can be observed that all the compatibility constraints
are indeed satisfied; however, it is worth to remark that, because of the sampling
fluctuation of the estimates \( \hat{g}(n) \), the autocorrelation \( \hat{g}(6) \) was not always detected
in different realizations of the noisy data \( \{\hat{g}_k\}_N \). In these cases the set \( J_k \), computed
from the set \( Q = \{4, 10\} \) missing \( n = 6 \), is still correct, i.e., \( J_k = \{3, 7, 13\} \), even
though one compatibility constraint is not fulfilled.

A more complex example is shown in Figure 4.3. Following the trace of the pre-
vious example, here we have the input function \( f_2 \) which is characterized by having
the significant Fourier components grouped in different ranges of the \( k \) axis. Conse-
quently, the Fourier coefficients \( g_k \) that clearly emerge from the noise (in this example
\( \epsilon = 10^{-4} \)) are quite sparse in the range \( 1 \leq k \leq 12 \) (see Figure 4.3A). The plot of the
regularized solution \( \hat{B}_g \), obtained from the analysis of the autocorrelation function
shown in Figure 4.3B, shows an acceptable agreement with the real solution \( f_2 \), even
though the procedure failed in detecting the coefficient at \( k = 5 \). On the contrary, the
“nontruncated” (in the sense that the sum runs up to \( N \)) solution \( f_\ast \) (see (2.8)),
which is displayed in Figure 4.3D, yields a rather poor reconstruction either because
the constraint operator \( C \) smooths out too many frequencies or because distortions are
introduced by those coefficients which are essentially noise (e.g., \( k = 2, 6, 7, 8, 9, 10 \)).
Fig. 4.2. Example 2: $f_2(x) = 5 \sin(3\pi x) + 10 \sin(7\pi x) + 15 \sin(13\pi x)$, $\epsilon = 3 \times 10^{-3}$, $\text{SNR} \approx 0.54 \text{dB}$, $N = 512$. (A) Modulus of the autocorrelation function. $Q = \{4, 6, 10\}$, $I_k = \{3, 7, 13\}$. (B) Comparison between the actual solution $f_2(x)$ (solid line) and the regularized solution $\hat{f}_2(x)$ (dots). (C) Comparison between the actual solution $f_2(x)$ (solid line) and the approximated solution $f_2^{(1)}(x)$ with $k_0 = 9$ (see criterion (2.6)). (D) Example 3: Modulus of the autocorrelation function. $f_3(x) = \sum_{j=1}^{10} a_j \sin(k_j \pi x)$, with $a_j = \{17, 23, 27, 33, 55, 68, 70, 77, 81\}$ and $k_j = \{5, 9, 13, 17, 18, 23, 24, 25, 31, 33\}$. $\epsilon = 10^{-3}$, $\text{SNR} \approx 9.79 \text{dB}$, $N = 1024$: $Q = \{4, 5, 8, 9, 12, 13, 14, 15, 16, 18, 19, 20, 22, 24, 26, 28\}$, $I_k = \{5, 9, 13, 17, 18, 23, 24, 25, 31, 33\}$. (E) Comparison between the actual solution $f_3(x)$ (solid line) and the regularized solution $\hat{f}_3(x)$ (dots). (F) Comparison between the actual solution $f_2(x)$ (solid line) and the approximated solution $f_2^{(1)}(x)$ with $k_0 = 27$.

Of course, the variational reconstruction could be considerably improved by choosing a more appropriate operator $C$ and different values for the parameters $\epsilon$ and $E$, but this would require more precise a priori knowledge on the actual solution.

In conclusion, some final remarks. The method of regularization based on the analysis of the correlation function of the data allows to pick out the Fourier components of the noisy data which are likely to carry exploitable information on the unknown solution, and at the same time, for rejecting the ones dominated by the
Example 4: \( f_4(x) = (1 - x) \sin(5 \sin(12x)), \epsilon = 10^{-4}, \text{SNR} \approx 4.6dB, N = 512. \) (A) Noiseless Fourier coefficients \( g_k. \) (B) Modulus of the autocorrelation function. \( Q = \{1, 2, 3, 5, 7, 8, 9\}, J_k = \{1, 3, 4, 9, 11, 12\}. \) (C) Comparison between the actual solution \( f_4(x) \) (solid line) and the regularized solution \( \hat{f}_4(x) \) (dots). (D) Comparison between the actual solution \( f_4(x) \) (solid line) and the variational solution \( f_\star(x) \) (see (2.3)).

noise. Frequency selectivity is not featured by methods of regularization that basically work as low-pass filters, and we have seen this inherent limit through examples in which frequency selectivity is essential for a satisfactory reconstruction.

The regularized solution \( \hat{f}_4 \) is founded only on a suitable analysis of the real data, that aims at holding only the data whose information content is significant. This approach naturally agrees with the methodology of the experimental physical science.

A moderate number of reasonable assumptions have been made in the construction of the regularized solution \( \hat{f}_4 \) (see Theorem 3.5), and, more important, the solution itself does not depend on unknown parameters. Even in the variational approach, methods to reduce the dependence of the solution on free parameters have been widely investigated, and several practical strategies for choosing the regularization parameter \( \alpha \) (see functional (1.9)) have been proposed (see, for instance, [8, 25] and references therein). Since the optimal parameter is impossible to determine because the exact solution is not known, many of these strategies can provide estimates of the asymptotically optimal rate of convergence of the regularized solution to the real solution when the noise vanishes.

The main difficulty of the method we have proposed regards the analysis of the correlation function. First, the correctness of the regularized solution depends on the capability of the correlation function to catch the information content of the data.
and to exhibit it in an effective way. Second, usually quite large data samples, i.e., \( N \) large, are necessary in order to limit sample fluctuations that could give rise to incorrect interpretation of the correlation function itself.

REFERENCES

[1] R. L. Anderson, *Distribution of the serial correlation coefficient*, Ann. Math. Stat., 13 (1942), pp. 1–13.

[2] A. V. Balakrishnan, *Applied Functional Analysis*, Springer-Verlag, New York, 1976.

[3] M. S. Bartlett, *Stochastic Processes: Methods and Applications*, 3rd ed., Cambridge University Press, Cambridge, UK, 1978.

[4] M. Bertero and G. A. Viano, *On probabilistic methods for the solution of improperly posed problems*, Boll. Un. Mat. Ital. B (5), 15 (1978), pp. 483-508.

[5] M. Bertero, C. de Mol and G. A. Viano, *On the problems of object restoration and image extrapolation in optics*, J. Math. Phys., 20 (1979), pp. 509-521.

[6] M. Bertero, C. De Mol and G. A. Viano, *The stability of inverse problem*, in *Inverse Scattering Problems in Optics*, Springer-Verlag, Berlin, 1980, pp. 161–212.

[7] G. E. P. Box and G. M. Jenkins, *Time Series Analysis*, Holden-Day, San Francisco, 1976.

[8] A. M. Davies, *Optimality in regularization*, in *Inverse Problems in Scattering and Imaging*, M. Bertero and E.R. Pike, eds., Adam Hilger, Bristol, UK, 1992, pp. 393–410.

[9] J. L. Doob, *Stochastic Processes*, John Wiley, New York, 1953.

[10] H. W. Engl, *Regularization methods for the stable solution of inverse problems*, Surveys Math. Indust., 3 (1993), pp. 71-143.

[11] J. N. Franklin, *Well-posed stochastic extensions of ill-posed linear problems*, J. Math. Anal. Appl., 31 (1970), pp. 682-716.

[12] W. A. Fuller, *Introduction to Statistical Time Series*, John Wiley, New York, 1976.

[13] I. M. Gel’fand and A. M. Yaïlom, *Calculation of the amount of information about a random function contained in another such function*, Amer. Math. Soc. Transl. Ser. 2, 12 (1959), pp. 199-246.

[14] C. W. Groetsch, *The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind*, Pitman, Boston, 1984.

[15] G. M. Jenkins and D. G. Watts, *Spectral Analysis and Its Applications*, Holden-Day, San Francisco, 1968.

[16] M. Hankel, *Conjugate Gradient Type Methods for Ill-Posed Problems*, Pitman Res. Notes Math. Ser. 327, Longman Sci. Tech., Harlow, 1995.

[17] N. Magnoli and G. A. Viano, *On the eigenfunction expansions associated with Fredholm integral equations of first kind in presence of noise*, J. Math. Anal. Appl., 197 (1996), pp. 188-206.

[18] N. Magnoli and G. A. Viano, *The source identification problem in electromagnetic theory*, J. Math. Phys., 38 (1997), pp. 2366–2388.

[19] D. Middleton, *An Introduction to Statistical Communication Theory*, McGraw-Hill, New York, 1960.

[20] K. Miller, *Least square methods for ill-posed problems with a prescribed bound*, SIAM J. Math. Anal., 1 (1970), pp. 52-74.

[21] K. Miller and G. A. Viano, *On the necessity of nearly-best-possible methods for analytic continuation of scattering data*, J. Math. Phys., 14 (1973), pp. 1037-1047.

[22] E. Scalas and G. A. Viano, *Resolving power and information theory in signal recovery*, J. Opt. Soc. Amer. A, 10 (1993), pp. 991-996.

[23] A. Tikhonov and V. Arsenine, *Méthodes de Résolution de Problèmes Mal Posés*, Mir, Moscow, 1976.

[24] G. A. Viano, *On the regularization of the antenna synthesis problem*, in *Partial Differential Equations and Applications*, P. Marcellini, G. T. Talenti, and E. Vesentini, eds., Marcel Dekker, 1996, pp. 313-318.

[25] G. Wahba, *Practical approximate solutions to linear operator equations when the data are noisy*, SIAM J. Numer. Anal., 14 (1977), pp. 651-667.