Instantons and Solitons in Heterotic String Theory

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Abstract

This is a transcript of lectures given at the Sixth Jorge Andre Swieca Summer School in Theoretical Physics. The subject of these lectures is soliton solutions of string theory. We construct a class of exact conformal field theories possessing a spacetime soliton or instanton interpretation and present a preliminary discussion of their physical properties.

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1. Introduction and Motivation

For the purposes of these lectures we are going to assume that string theory is identical to two-dimensional conformal field theory. As is well-known, conformal field theories of the right central charge and field content describe solutions of the classical equations of motion of string field theory. To obtain true quantum physics one has to sum over higher-genus worldsheets, an important, but difficult, task which we will not attempt here.

Static solutions of the classical equations of motion of conventional field theory can represent either candidate vacuum states, solitons or instantons. The soliton and instanton solutions are distinguished from a vacuum by being non-invariant to translations and boosts and by having a finite mass (which goes to infinity as the coupling goes to zero). Vacuum solutions may have a non-zero energy density, but this must be interpreted as a cosmological constant rather than a mass. In short, soliton solutions identify new physical particles which do not appear in the spectrum of small fluctuations about the vacuum and may have exotic quantum numbers. The magnetic monopole solution of spontaneously broken gauge theory is a classic example. An important development in the study of ordinary quantum field theory was the demonstration that solitons and their associated conserved charges survive quantization, and are not just artifacts of the classical analysis. It is obviously important to repeat this development for string theory: essentially stringy physics could lead to even more exotic types of soliton or it could lead to weak instability of the old ones. Either outcome could have important phenomenological consequences.

Most of what we have learned about string theory so far comes from conformal field theories with a vacuum interpretation. In order to make contact with reality, it is necessary to compactify six of the ten spacetime dimensions of superstring theory and the most carefully studied nontrivial conformal field theories have precisely this interpretation. The geometry of the six compactified dimensions may be very complicated, but the remaining four dimensions are perfectly flat, and the compactification theory manifests no meaningful energy or mass: it is a vacuum.

To address the soliton issue then, it will be necessary to find conformal field theories describing localized energy density embedded in asymptotically flat spacetime (that is, we want to find non-compactifications!) Once we find such a solution, we need to develop appropriate conformal field theory methods for studying its non-vacuum properties, most notably its mass. As we shall see, these are difficult problems, but some progress has been made on the first issue.

There are two ways to proceed in the study of this problem. The first is the ”strings in background field” method [1,2,3], in which the usual spacetime fields (graviton, dilaton and antisymmetric tensor) appear as coupling constant functions in a worldsheet nonlinear sigma model and the spacetime equations of motion for these fields arise from the condition that the conformal invariance beta functions should vanish. The beta functions are computed as an expansion in powers of the string tension $\alpha'$ and, in the leading approximation, yield standard Einstein-Yang-Mills-like spacetime equations of motion. Not-so-standard stringy effects arise from higher-order corrections. It is conceptually straightforward to look for asymptotically flat solutions of these field equations and most previous attempts to study string solitons have taken this approach. The problem is that the $\alpha'$ expansion is only valid if curvatures are everywhere small, a condition which is not met in many inter-
testing solutions. The second way to proceed is to use purely algebraic methods to generate exact conformal field theories in the hope that it will be possible to generate some that have a solitonic spacetime interpretation. This approach is nonperturbative and perfectly capable of handling cases with strong curvature, but no useful exact theories have been found in this way.

In these lectures we will tackle the string soliton problem by a hybrid of the above two approaches. We will first construct a soliton solution of the lowest-order spacetime equations and we will then use supersymmetry arguments to show that this solution gets no corrections to any order in $\alpha'$. Finally, we will use purely algebraic methods to discuss a limiting case of our general solution and to argue that no nonperturbative effects have been missed. At the end we will draw conclusions and make some proposals for further development. The work on which these lectures are based has been done in collaboration with J. Harvey and A. Strominger and is reported in [4,5]. The presentation given here is a pedagogical elaboration of those references and will, in places, follow them quite closely.

2. Soliton Solutions: Spacetime Approach

Let us first discuss the problem of finding string solitons via the "strings in background fields" spacetime approach. The beta functions for strings propagating in a background of massless fields are the equations of motion of a certain master spacetime action which can be computed as an expansion in the string tension $\alpha'$. For the heterotic string, the leading terms in this action are identical to the $D=10$, $N=1$ supergravity and super Yang-Mills action. The bosonic part of this action reads

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{g} e^{-2\phi} \left( R + 4(\nabla\phi)^2 - \frac{1}{3} H^2 - \frac{\alpha'}{30} \text{Tr} F^2 \right),$$

(2.1)

where the three-form antisymmetric tensor field strength is related to the two-form potential by the familiar anomaly equation [3]

$$H = d \wedge B + \alpha' \left( \omega_3^L(\Omega_-) - \frac{1}{30} \Omega_3^M(\Omega) \right) + \ldots$$

(2.2)

(where $\omega_3$ is the Chern-Simons three-form) so that

$$d \wedge H = \alpha' (\text{tr} R \wedge R - \frac{1}{30} \text{Tr} F \wedge F).$$

(2.3)

The trace is conventionally normalized so that $\text{Tr} F \wedge F = \sum_i F^i \wedge F^i$ with $i$ an adjoint gauge group index. An important, and potentially confusing, point is that the connection $\Omega_\pm$ appearing in (2.2) is a non-Riemannian connection related to the usual spin connection $\omega$ by

$$\Omega_{\pm M} = \omega_{M}^{AB} \pm H_{M}^{AB}. \quad (2.4)$$

Since the antisymmetric tensor field plays a crucial role in all of our solutions, this subtlety will be crucial.
Rather than directly solve the equations of motion for this action, it is much more convenient to look for bosonic backgrounds which are annihilated by some of the N=1 supersymmetry transformations (only the vacuum is annihilated by all the the supersymmetries). This is a fairly standard trick which has been applied to the magnetic monopole problem and to string problems in [7,8]. It is highly nontrivial that any such solutions can be found at all, but if they can, they are automatically solutions of the usual equations of motion. The Fermi field supersymmetry transformation laws which follow from (2.1) are

\[
\delta \chi = F_{MN} \gamma^{MN} \epsilon \\
\delta \lambda = (\gamma^M \partial_M \phi - \frac{1}{6} H_{MNP} \gamma^{MNP}) \epsilon \\
\delta \psi_M = (\partial_M + \frac{1}{4} \Omega^{AB}_M \gamma_{AB}) \epsilon,
\] (2.5)

and it is apparent that to find backgrounds for which all of (2.5) vanish, it is only necessary to solve first-order equations, rather than the more complicated second-order equations which follow from varying the action. We will shortly construct a simple ansatz for the bosonic fields which does just this.

First, however, we have to specify what type of soliton we are hoping to construct. In the standard four-dimensional context, we are acquainted with solitonic solutions of many dimensionalities: Instantons are localized at a point in Euclidean spacetime and trace out a zero-dimensional worldsheet; magnetic monopoles are localized at a point in three-dimensional space and trace out a one-dimensional worldsheet; cosmic strings are localized on a line in three-dimensional space and trace out a two-dimensional worldsheet and so on. Generically, we call a soliton whose instantaneous time slice has p-dimensional extension a p-brane. In string theory, since the space-time dimension is ten, we could in principle find solutions with p anywhere between zero and nine. In these lectures, we will study solutions with \( p = 5 \): five-branes. They are of particular interest because, as was shown some time ago by Teitelboim and Nepomechie [9], five-branes are dual to fundamental strings in ten dimensions in much the same way that magnetic monopoles are dual to electric charge in four dimensions. Dirac’s monopole argument has only to be modified by replacing the Maxwell field \( A_\mu \) by the antisymmetric tensor field \( B_{\mu \nu} \) and changing the dimensionality of spacetime. The duality argument doesn’t guarantee that the dual objects actually exist, of course, but Strominger has shown that, at least perturbatively, the heterotic string does have soliton solutions with five-brane structure [7]. In these lectures we will concentrate on constructing five-brane solitons: they are in some sense maximally “stringy” and, more to the point, many of them are amenable to exact conformal field theory analysis. How such objects would manifest themselves phenomenologically is an interesting question which depends on the details of compactification down to four dimensions. This question has yet to be studied in any detail, and we will have little to say about it here.

Let us attempt to construct a five-brane solution to (2.3). The supersymmetry variations are determined by a positive chirality Majorana-Weyl \( SO(9,1) \) spinor \( \epsilon \). Because of the five-brane structure, it is useful to note that \( \epsilon \) decomposes under \( SO(9,1) \supset SO(5,1) \otimes SO(4) \) as

\[
16 \rightarrow (4_+, 2_+) \oplus (4_-, 2_-)
\] (2.6)
where the $\pm$ subscripts denote the chirality of the representations. Denote world indices of the four-dimensional space transverse to the fivebrane by $\mu, \nu = 6 \ldots 9$ and the corresponding tangent space indices by $m, n = 6 \ldots 9$. We assume that no fields depend on the longitudinal coordinates (those with indices $\mu = 0 \ldots 5$) and that the nontrivial tensor fields in the solution have only transverse indices. Then the gamma matrix terms in (2.5) are sensitive only to the $SO(4)$ part of $\epsilon$ and, in particular, to its $SO(4)$ chirality.

One immediately sees how to make the gaugino variation vanish (in what follows we treat $\epsilon$ as an $SO(4)$ spinor and let all indices be four-dimensional): As a consequence of the four-dimensional gamma-matrix identity $\gamma^{mn}\epsilon_\pm = \mp \frac{1}{2} \epsilon^{mnrs}\gamma^{rs}\epsilon_\pm$ one has $F_{mn}\gamma^{mn}\epsilon_\pm = \mp \tilde{F}_{mn}\gamma^{mn}\epsilon_\pm$, where the dual field strength is defined by $\tilde{F}_{mn} = \frac{1}{2} \epsilon^{mnrs}F^{rs}$. Therefore, $\delta \chi$ vanishes if $F_{mn} = \pm \tilde{F}_{mn}$ and $\epsilon = (4_+, 2_-)$ which is to say that if the gauge field is taken to be an instanton, then $\delta \chi$ vanishes for all supersymmetries with positive $SO(4)$ chirality!

To deal with the other supersymmetry variations, we must adopt an ansatz for the non-trivial behavior of the metric and antisymmetric tensor fields in the four dimensions transverse to the five-brane (the specific form is inspired by the work of Dabholkar and Harvey [10] on string-like solitons). For the metric tensor we write

$$g_{\mu\nu} = e^{-2\phi}\delta_{\mu\nu} \quad \mu, \nu = 6 \ldots 9$$

and for the antisymmetric tensor field strength

$$H_{\mu\nu\lambda} = \sqrt{g} \epsilon_{\mu\nu\lambda\sigma}\partial^\sigma \phi = e^{-2\phi}\epsilon_{\mu\nu\lambda\sigma}\partial^\sigma \phi$$

where $\phi$ is to be identified with the dilaton field. With this ansatz and the rather obvious vierbein choice $e^m_\mu = \delta_m^\mu e^{-\phi}$, we can also calculate the generalized spin connections (2.4) which appear in (2.5) and (2.2):

$$\Omega_{\pm}^{mn} = \delta_m^\mu \partial_{\eta} \phi - \delta_n^\mu \partial_{m} \phi \pm \epsilon_{mn\eta\rho}\partial^\rho \phi$$

Now consider the $\delta \lambda$ term in (2.5). Because of the ansatz, both terms are linear in $\partial \phi$. By standard four-dimensional gamma-matrix algebra, the relative sign of the two terms is proportional to the $SO(4)$ chirality of the spinor $\epsilon$. We have chosen the sign and normalization of the ansatz for $H$ so that $\delta \lambda$ vanishes for $\epsilon \in (4_+, 2_+)$. Finally, consider the gravitino variation in (2.5). A crucial fact, following from (2.9), is that while $\Omega_{\pm}$ would in general be an $SO(4)$ connection, with the chosen ansatz it is actually pure $SU(2)$. To be precise,

$$\Omega_{\pm}^{mn}\gamma^{mn}\epsilon_\eta = (\gamma^\mu\nu\partial_\mu \phi)(1 \mp \eta)\epsilon_\eta$$

so that $\Omega_{\pm}$ annihilates the $(4_+, 2_-)$ spinor. Since (2.5) involves only $\Omega_{\pm}$, it suffices to take $\epsilon$ to be a constant $(4_+, 2_+)$ spinor to make the gravitino variation vanish.

Putting all this together, we see that if we choose the gauge field to be any instanton and fix the metric and antisymmetric tensor in terms of the dilaton according to the above ansatz, then the state is annihilated by all supersymmetry variations generated by a spacetime constant $(4_+, 2_+)$ spinor. Thus, half of the supersymmetries are unbroken, and the other half, by standard reasoning [11], will be associated with fermionic zero-modes bound to the soliton.
The one unresolved question concerns the functional form of the dilaton field. Notice that the ansatz for the antisymmetric tensor was given in terms of its three-form field strength $H_{mnp}$, rather than its two-form potential $B_{mn}$. This is potentially inconsistent, since the curl of the field strength must satisfy the anomalous Bianchi identity (2.3). Within the ansatz (2.8), the curl of $H$ is given by

$$d \land H = \frac{1}{4!} \epsilon_{rstuv} \partial_r \{ e^{-2\phi} \epsilon_{stu} \partial_v \phi \} \sim \Box e^{-2\phi} \quad (2.11)$$

(where $\Box$ is just the flat Laplacian) and one can thus, in principle, solve (2.3) for $\phi$. The slight problem with this approach is that (2.3) is only the leading order in $\alpha'$ approximation to the true anomaly and the best we can hope to do is to construct solutions as a power series in $\alpha'$. Since our goal is to find exact solutions, we adopt the different strategy of looking for special backgrounds where the $R \land R - F \land F$ anomaly on the r.h.s of (2.3) cancels. If that is possible, the equation for $d \land H$ becomes $\Box e^{-2\phi} = 0$, an equation which can be solved once and for all with no expansion in powers of $\alpha'$. The cancellation of the anomaly of course means that the underlying sigma model has been made effectively left-right symmetric, a property which will play a key role in the proof that our solutions are exact in $\alpha'$. It remains to show that desired cancellation can, in fact, be achieved.

What is required, according to (2.3), is that the curvature $R(\Omega_-)$ should cancel against the instanton Yang-Mills field $F$. We will take the instanton to be embedded in an $SU(2)$ subgroup of the gauge group (this is always the lowest-action instanton), so what is needed is that $\Omega_-$ be a self-dual $SU(2)$ connection. The $SU(2)$ condition we already know to be met, so the only issue is self-duality. Given the special ansatz and coordinate system of (2.7), it is easy to calculate the curvature of $\Omega_\pm$:

$$R(\Omega_\pm)^{mn}_{\mu\nu} = \delta_{m\nu} \nabla_m \nabla_\nu \phi - \delta_{m\mu} \nabla_m \nabla_\mu \phi - \delta_{m\nu} \nabla_n \nabla_\mu \phi + \delta_{m\mu} \nabla_n \nabla_\nu \phi$$

$$\pm \epsilon_{\mu mn\lambda} \nabla_\lambda \nabla_\mu \phi \mp \epsilon_{\nu mn\lambda} \nabla_\lambda \nabla_\nu \phi \quad (2.12)$$

where

$$\nabla_\mu \nabla_\nu \phi = \partial_\mu \partial_\nu \phi - \delta_{\mu\nu} \partial_m \partial_m \phi + 2 \partial_\mu \partial_\nu \phi$$

$$\nabla^2 \phi = 2 e^{2\phi} \Box e^{-2\phi} \quad (2.13)$$

It is then a trivial arithmetical exercise to show that, in four dimensions and under the condition that $\delta^{mn} \nabla_m \nabla_n \phi = 0$, $\Omega$ is self-dual:

$$R(\Omega_\pm)^{mn}_{\mu\nu} = \mp \frac{1}{2} \epsilon_{\mu\nu \lambda\sigma} R(\Omega_\pm)^{mn}_{\lambda\sigma} \quad (2.14)$$

Since a self-dual $SU(2)$ connection is an instanton connection, it will be possible to choose a gauge instanton which exactly matches the "metric" instanton $\Omega_-$ and makes the r.h.s of (2.3) vanish, thus making the whole solution self-consistent. In the next section, we will explore the qualitative properties of the solutions which we have constructed in the above rather roundabout manner.

A feature of the above development which could cause confusion is the intricate interplay of the two non-Riemannian connections $\Omega(\pm)$. To refresh the reader’s memory,
we will summarize the essentials of this phenomenon (we denote the \((4_+ , 2_+ )\) spinor by \(\epsilon_+ \)):

The gravitino supersymmetry variation equation boils down to

\[
\Omega^{[ab]}_{\mu \gamma} \epsilon_+^{ab} \epsilon_+ = 0
\]

which in turn implies that

\[
R_{\mu \nu}(\Omega_+)^{ab} \gamma^{ab} \epsilon_+ = 0.
\]

The index-pair interchange symmetry for a non-Riemannian connection, which reads

\[
R_{\mu \nu}(\Omega_+)^{ab, cd} \gamma^{ab} \epsilon_+ = 0.
\]

If we then make the identification

\[
F_{\mu \nu}^{[ab]} \sim R_{\mu \nu}(\Omega_-)^{[ab]},
\]

we see that we have reproduced the gaugino supersymmetry variation equation. This is simply to emphasize that, because of the crucial role of the antisymmetric tensor in these solutions, the precise way in which the \(\Omega_+\)'s and \(\Omega_-\)'s appear in the various equations we deal with is tightly constrained and quite critical.

### 3. Development and Interpretation of the Solutions

Now we will work out the geometry and physical interpretation of the solution described in the previous section. To recapitulate, we have found the following solution of the low-energy spacetime effective action of the heterotic string:

\[
ds^2 = e^{-2\phi(x)} \delta_{\mu \nu} dx^\mu dx^\nu + \eta_{\alpha \beta} dy^\alpha dy^\beta + \eta_{\alpha \beta} dy^\alpha dy^\beta
\]

\[
H_{\mu \nu} = \epsilon_{\mu mnp} \partial_p \phi
\]

\[
H_{\mu \nu \lambda} = -\frac{1}{2} \epsilon_{\mu \nu \lambda \sigma} \partial_\sigma e^{-2\phi}
\]

\[
F_{\mu \nu}^{[mn]} = \tilde{F}_{\mu \nu}^{[mn]} = R_{\mu \nu}(\Omega_-)^{[mn]},
\]

where \(\mu \nu = 6 \ldots 9\), \(\alpha \beta = 0 \ldots 5\) and \(\eta_{\alpha \beta}\) is the Minkowski metric. The last equation expresses the fact that the gauge field is a self-dual instanton with moduli chosen so that it coincides (up to gauge transformations of course) with the curvature of the generalized connection of the theory. The consistency condition for all this is just \(\Box e^{-2\phi} = 0\).

The solution of the consistency condition on \(\phi\) is just a constant plus a sum of poles:

\[
e^{-2\phi} = e^{-2\phi_0} + \sum_{i=1}^N \frac{Q_i}{(x - x_i)^2}
\]

The constant term is fixed by the (arbitrary) asymptotic value of the dilaton field, \(\phi_0\). In string theory, \(e^{-\phi}\) is identified with the local value of the string loop coupling constant, \(g_{str}\). For the solution described by (3.2), \(g_{str}\) goes to a constant at spatial infinity and goes to infinity at the locations of the poles! We shall worry about the physical interpretation of this fact in due course. Now, the metric of our solution is conformally flat with conformal factor given by (3.2). Since \(\phi\) goes to a constant at infinity, the geometry is asymptotically flat, which is precisely what we want for a soliton interpretation. In the neighborhood of a singularity, we can replace \(e^{-2\phi}\) by a simple pole \(Q/r^2\) and obtain the approximate line element

\[
ds^2 \sim \frac{Q}{r^2} (dr^2 + r^2 d\Omega^2)
\]

\[= dt^2 + Q d\Omega^2
\]
where $d\Omega^2_3$ is the line element on the unit three-sphere and we have introduced a new radial coordinate $t = \sqrt{Q} \log(r/\sqrt{Q})$. This expression becomes more and more accurate as $t \to -\infty$. In this same limit, the other fields are given by

$$\phi = t/\sqrt{Q} \quad H = Q \epsilon_3,$$  \hfill (3.4)

where $\epsilon_3$ is the volume form on the three sphere. In Sect. 5 we will see that the linear behavior of the dilaton field plays a crucial role in the underlying exact conformal field theory. The geometry described by (3.3) is a cylinder whose cross-section is a three-sphere of constant area $2\pi^2Q$. The global geometry is that of a collection of semi-infinite cylinders, or wormholes (one for each pole in $e^{-2\phi}$), glued into asymptotically flat four-dimensional space. The wormholes are semi-infinite since the approximation of (3.3) becomes better and better as $t \to -\infty$ and breaks down as $t \to +\infty$. It is these wormholes which we propose to interpret as solitons.

A further crucial fact is that the residues, $Q$, are quantized. Consider an $S_3$ which surrounds a single pole, of residue $Q$, in $e^{-2\phi}$. The net flux of $H$ through this $S_3$ is entirely due to the enclosed pole and can easily be calculated:

$$H_{ijk} = -\frac{1}{2} \epsilon_{ijkl} \partial_l e^{-2\phi} = Q \epsilon_{ijkl} \partial_l x^l / x^4$$

$$\int_{S_3} H = 2\pi^2 Q$$  \hfill (3.5)

The flux of $H$ through the $S_3$ at infinity is thus proportional to the sum of all the residues.

By a familiar cohomology argument, however, the flux of $H$ through any $S_3$ must be an integer multiple of some basic unit $\lbrack 12 \rbrack$. The point is that, if the flux of $H$ is non-zero, then there cannot be a unique two-form potential $B$ covering the whole sphere. The best one can do is to have two sections, $B^\pm$, covering the upper and lower halves of the $S_3$ and related to each other by a gauge transformation in an overlap region which is topologically an $S_2$. Since the sigma model action involves $B$, not $H$, the non-uniqueness of $B$ could lead to an ill-defined sigma model path integral. It is possible to show that, with our definitions of the sigma model action, this danger is avoided if and only if the flux of $H$ is an integral multiple of $\alpha'$. The details of this argument can be found in $\lbrack 12 \rbrack$. The consequence for us is that the residues of the poles in $e^{-2\phi}$ are discretely quantized: $Q_i = n_i \alpha'$. As a result, the cross-sectional areas of the individual wormholes are quantized in units of $2\pi^2 \alpha'$ and there is thus a minimal transverse scale size of the fivebrane. (This fact may be useful in future attempts to quantize the transverse fluctuations of the fivebrane.)

Finally, we want to characterize the instanton component of this solution. The key point is that, when the dilaton field satisfies $\Box e^{-2\phi} = 0$, we can construct a self-dual $SU(2)$ connection out of the scalar field $\phi$ and we want to identify this connection with the gauge instanton. But there is a well-known ansatz $\lbrack 13 \rbrack$ for constructing multi-instanton solutions which has precisely this character: if we define an $SU(2)$ connection by

$$A_\mu(x) = \Sigma_{\mu\nu} \partial_\nu \log f \quad \Sigma_{\mu\nu} = \frac{1}{2} \bar{\eta}_{\mu\nu i} \sigma^i$$  \hfill (3.6)

where $\bar{\eta}_{\mu\nu i}$ is 'tHooft’s anti-self-dual symbol, then the condition of self-duality of the connection reduces to $\Box f = 0$, an equation which has the general solution

$$f = 1 + \sum_{i=1}^{N} N \frac{\rho_i^2}{(x - x_i)^2}$$  \hfill (3.7)
(we normalize the behavior at infinity to unity since $A_\mu$ is invariant to rescaling $f$ by an overall constant). An important point is that the total instanton number of the solution built on $f$ is $N$, the number of poles. The gauge potential which follows from taking $f$ to have a single pole can be shown to be

$$A_\mu = -2\rho^2 \Sigma_{\mu\nu} \frac{x^\nu}{x^2(x^2 + \rho^2)} , $$

(3.8)

an expression which one immediately recognizes as the singular gauge instanton of scale size $\rho$ centered at $x = 0$. The only way this can match our construction of a self-dual generalized connection is if we make the identification

$$f(x) = e^{2\phi_0} e^{-2\phi} = 1 + \sum_{i=1}^{N} \frac{e^{2\phi_0} Q_i}{(x - x_i)^2} .$$

(3.9)

Thus, given the solution (3.2) for the dilaton field, we can assert that the associated instanton has instanton number $N$, with instantons of scale size $e^{2\phi_0} Q_i$ localized at positions $x_i$. Since the $Q_i$ are quantized, so are the instanton scale sizes. The only free parameters (moduli) are the $4N$ center locations of the instantons. In ten dimensions, the multiple instanton solution corresponds to multiple fivebranes with the locations in the transverse four-dimensional space of the individual fivebranes given by the center coordinates of the individual instantons.

An important fact about the solution we have just constructed is that it is not perturbative in $\alpha'$. As we saw in the discussion following (3.1), the wormhole associated with a pole of residue $Q = n\alpha'$ has a cross-section which is a sphere of area $2\pi^2 Q$ and therefore has curvature $R \sim 1/Q$. In the perturbative sigma model approach to strings in background fields, one finds that the sigma model expansion parameter is $\alpha' R$. In the case at hand, this becomes $\alpha'/Q = 1/n$, which is obviously not small for the elementary fivebrane, which has $n = 1$. Since our solution has been constructed by solving the leading-order-in-$\alpha'$ beta function equations, ignoring all higher-order corrections, one can legitimately worry whether it makes any sense. In the next two sections we will present evidence that all higher-order corrections to this particular solution actually vanish, and the leading-order solution is exact. We will briefly discuss other solutions of interest for which higher-order corrections don’t vanish.

There is another perturbation theory issue to bring up here. String theory has two expansion parameters: the string tension $\alpha'$ and the string loop coupling constant $g_{\text{str}} \sim e^{-2\phi_0}$. The latter is the quantum expansion parameter of string theory and, in this paper, we are working to zeroth-order in an expansion in $g_{\text{str}}$. In effect, we are producing an exact solution, to all orders in $\alpha'$, of classical string field theory. However, as we have already pointed out, our solution has the unusual feature that $g_{\text{str}}$ grows without limit down the throat of a wormhole so that there is, strictly speaking, no reliable classical limit! Since virtually nothing is known about non-perturbative-in-$g_{\text{str}}$ physics, we don’t know what this means for the ultimate validity of this sort of solution. Similar issues arise in the matrix model/Liouville theory approach to two-dimensional quantum gravity, and
we hope eventually to gain some insight from that source (for a review, see the contribution of David Gross to this School).

Before proceeding to show that our solution is exact (in the sense just described), we want to briefly describe some inexact, but instructive, solutions of the basic heterotic field equations. A particularly interesting possibility (and this was Strominger’s original approach to this problem [7]) is to proceed along the lines of Sect. 2, but to determine the dilaton by solving the curl equation for $H$ perturbatively in $apm$. To do this, combine (2.11) and (2.3) to give

$$e^{-2\phi} \sim \alpha'(trR \wedge R - \frac{1}{30} TrF \wedge F).$$ \hspace{1cm} (3.10)

In a perturbative solution, this equation implies that $\partial \phi \sim O(\alpha')$ which, via (2.12), implies that $R \sim O(\alpha')$. Therefore, to leading order in $\alpha'$, one is entitled to drop the $R \wedge R$ term in (3.10). Substituting the explicit gauge field strength for an instanton of scale size $\rho$, one obtains the following dilaton solution:

$$e^{-2\phi} = e^{-2\phi_0} + 8\alpha' \left(\frac{x^2 + 2\rho^2}{(x^2 + \rho^2)^2}\right) + O(\alpha'^2).$$ \hspace{1cm} (3.11)

The metric and antisymmetric tensor fields are built out of this dilaton field according to the spacetime-supersymmetric ansatz of (2.7) and (2.8). This solution is very different from the previous one, obtained by setting $d \wedge H = 0$: The dilaton field and the metric are everywhere finite and the topology of the solution is $R^4$ rather than $R^4$ with semi-wormholes glued in. One can examine higher-order in $\alpha'$ corrections to the beta functions and verify that the solution must receive corrections. At the same time, one can examine higher-order corrections to the supersymmetry transformations [14] and verify that it is possible to maintain spacetime supersymmetry in the $\alpha'$-corrected solution. These solutions are very interesting in their own right and certainly have a soliton interpretation. On the other hand, since we do not know how to deal with the higher-order in $\alpha'$ corrections in any general manner, we will not pursue this line of development here.

Another interesting point concerns what happens when we lift the requirement of spacetime supersymmetry and look for solutions of the beta function equations rather than the condition that some supersymmetry charges annihilate the solution. Our solutions have the property that the mass (the ADM mass, to be precise) per unit fivebrane area is proportional to the axion charge: $M_5 = 2\pi \alpha'^{-3} Q$. This equality can be understood via a Bogomolny bound: any solution of the leading-order field equations with the fivebrane topology must satisfy the inequality $M_5 \geq 2\pi \alpha'^{-3} Q$ and our solution saturates the inequality. One can easily imagine a process in which mass, but not axion charge, is increased by sending a dilaton wave down one of the wormhole throats. Since the wormhole throat is semi-infinite, this wave need not be reflected back: It can continue to propagate down the throat forever, leaving an exterior solution for which $M_5 > 2\pi \alpha'^{-3} Q$. Such solutions of the leading-order beta function equations have indeed been found [15] and they resemble the familiar Reissner-Nordstrom sequence of charged black holes: They have an event horizon and a singularity, but the singularity retreats to infinity as the mass is decreased to the
extremal value that saturates the Bogomolny bound. Perhaps not surprisingly, the non-
extremal solutions are not annihilated by any spacetime supersymmetries, and we do not
expect to be able to find the corresponding exact conformal field theories. Nonetheless,
the fact that the extremal black hole is under exact string theory control should eventually
allow us to make progress on understanding the string physics of black holes, Hawking
radiation and the like.

In the rest of these lectures, we will pursue the much more limited goal of showing
that our special solution is an exact solution of string theory.

4. Worldsheet Sigma Model Approach

To show conclusively that a given spacetime configuration is a solution of string theory,
we must show that it derives from an appropriate superconformal worldsheet sigma model.
In this section we will show that the worldsheet sigma models corresponding to the five-
branes constructed in section 2 possess extended worldsheet supersymmetry of type (4,4)
The notation derives from the fact that in a conformal field theory, the left-moving fields
(functions of $z$) and the right-moving fields (functions of $\bar{z}$) are dynamically independent.
It is therefore possible to have different numbers of right- and left-moving supercharges
$Q^I_{\pm}$. The general case, referred to as $(p,q)$ supersymmetry, is described by the algebra

$$
\{Q^I_+, Q^J_+\} = \delta^{IJ} \partial_z \quad I, J = 1 \ldots p
$$

$$
\{Q^I_-, Q^J_-\} = \delta^{IJ} \partial_{\bar{z}} \quad I, J = 1 \ldots q
$$

$$
\{Q^I_+, Q^J_-\} = 0.
$$

(4.1)

The minimal possibility, corresponding to a generic solution of the heterotic string, has
$(1,0)$ supersymmetry. Any left-right-symmetric, and therefore non-anomalous theory, will
have $(p,p)$ supersymmetry (this is sometimes referred to as $N=p$ supersymmetry). The
maximal possibility is $(4,4)$ which, it turns out, is what is realized in our fivebrane solution.
We will argue that, in the $(4,4)$ case, there is a nonrenormalization theorem which makes
the lowest-order in $\alpha'$ solution for the spacetime fields exact. The latter issue is closely
related to the question of finiteness of sigma models with torsion and with extended su-
persymmetry [10,17] and the results we find are slightly at variance with the conventional
wisdom, at least as we understand it. We will comment upon this at the appropriate point.

First we digress to explain why we expect four-fold extended supersymmetry in this
problem. The models of interest to us are structurally equivalent to a compactification of
ten-dimensional spacetime down to six dimensions: there are six flat dimensions (along the
crane) described by a free field theory and four ‘compactified’ dimensions (transverse
to the fivebrane) described by a nontrivial field theory. The fact that the ‘compactified’
space is not really compact has no bearing on the supersymmetry issue. The defining
property of all the fivebranes of section 2 is that they are annihilated by the generators of
a six-dimensional $N=1$ spacetime supersymmetry. That is, they provide a compactification
to six dimensions which maintains $N=1$ spacetime supersymmetry. Now, it is well-known
that in compactifications to four dimensions, the sigma model describing the six compact-
ified dimensions must possess $(2,0)$ worldsheet supersymmetry in order for the theory to
possess N=1 four-dimensional spacetime supersymmetry \[18\]. Roughly speaking, the conserved U(1) current of the (2,0) superconformal algebra defines a free boson which is used to construct the spacetime supersymmetry charges. It is also known that, if one wants to impose N=2 four-dimensional spacetime supersymmetry, the compactification sigma model must have (4,0) supersymmetry \[19\]. The conserved SU(2) currents of the (4,0) superconformal algebra are precisely what are needed to construct the larger set of N=2 spacetime supersymmetry charges. Since, by dimensional reduction, N=1 supersymmetry in six dimensions is equivalent to N=2 in four dimensions, the above line of argument implies that spacetime supersymmetric compactifications to six dimensions (including our fivebrane) require a compactification sigma model with at least (4,0) worldsheet supersymmetry. Since our solution is constructed to cancel the anomaly, it will be left-right symmetric and therefore automatically of type (4,4).

Now we turn to a study of string sigma models. The generic sigma model underlying the heterotic string describes the dynamics of D worldsheet bosons \(X^M\) and D right-moving worldsheet fermions \(\psi^M_R\) (where D, typically ten, is the dimension of spacetime) plus left-moving worldsheet fermions \(\lambda^a_L\) which lie in a representation of the gauge group \(G\) (typically \(SO(32)\) or \(E_8 \otimes E_8\)). The generic Lagrangian for this sigma model is written in terms of coupling functions \(G_{MN}, B_{MN}\) and \(A_M\) which eventually get interpreted as spacetime metric, antisymmetric tensor and Yang-Mills gauge fields. This Lagrangian has the explicit form \[20\]

\[
\frac{1}{4\pi\alpha'} \int d^2\sigma \{ G_{MN}(X) \partial_+ X^M \partial_- X^N + 2B_{MN}(X) \partial_+ X^M \partial_- X^N \\
+ iG_{MN}\psi^M_R \mathcal{D}_- \psi^N_R + i\delta_{ab} \lambda^a_L \mathcal{D}_+ \lambda^b_L + \frac{i}{2} (F_{MN})_{ab} \psi^M_R \psi^N_R \lambda^a_L \lambda^b_L \}
\]

where \(H = dB\). In this expression, the covariant derivatives on the left-moving fermions are defined in terms of the Yang-Mills connection, while the covariant derivatives on the right-moving fermions are defined in terms of a non-riemannian connection involving the torsion (which already appeared in section 2):

\[
\mathcal{D}_- \psi^A_R = \partial_- \psi^A_R + \Omega_-^A_B \partial_- X^N \psi^B_R, \\
\mathcal{D}_+ \lambda^a_L = \partial_+ \lambda^a_L + A^a_N \partial_+ X^N \lambda^b_L.
\]

As in section 2 we use indices of type \(M\) for coordinate space indices, type \(A\) for the tangent space and type \(a\) for the gauge group. An absolutely crucial feature of this action is that the connection appearing in the covariant derivative of the right-moving fermions is the generalized connection \(\Omega_-\), not the Christoffel connection. This action has a naive (1,0) worldsheet supersymmetry and can be written in terms of (1,0) superfields. Superconformal invariance is broken by anomalies of various kinds unless the coupling functions satisfy certain ‘beta function’ conditions \[1\] which are equivalent to the spacetime field equations discussed in section 2. The dilaton enters these equations in a rather roundabout, but by now well-understood, way \[20\].

To proceed further, we must construct the specific sigma models corresponding to the fivebrane solutions. For the generic fivebrane, \[4.2\] undergoes a split into a nontrivial
four-dimensional theory and a free six-dimensional theory: the sigma model metric (as opposed to the canonical general relativity metric) then describes a flat six-dimensional spacetime times four curved dimensions. The right-moving fermions couple via the kinetic term to the generalized connection $\Omega^-$, which acts only on the four right-movers lying in the tangent space orthogonal to the fivebrane. The other six right-movers are free (we momentarily ignore the four-fermi coupling) so there is a six-four split of the right-movers as well. The left-moving fermions couple to an instanton gauge field which may or may not be identified with the other generalized connection, $\Omega_+$. In all the cases of interest to us, the gauge connection is an instanton connection and acts only in some $SU(2)$ subgroup of the full gauge group, so that four of the left-movers couple nontrivially, while the other 28 are free. Finally, the four-fermion interaction term couples together precisely those left- and right-movers which couple to the nontrivial gauge and $\Omega^-$ connections and is therefore consistent with the six-four split defined by the kinetic terms. The remaining variables can be regarded as defining a heterotic, but free, theory ($6X_6\psi_R$ and $28\lambda_L$) living in the six ‘uncompactified’ dimensions along the fivebrane. From now on, we focus our attention on the nontrivial piece of (4.2) referring to the four-dimensional part of the split. For string theory consistency, it must have a central charge of 6, which would be trivially true if the connections were all flat, but is far from obvious for a fivebrane.

Now let us further specialize to the sigma model underlying the left-right symmetric (and therefore non-anomalous) fivebrane solution of section 2. It is constructed by identifying the gauge connection with the ‘other’ generalized connection $\Omega_+$ and making that connection self-dual by imposing the condition $e^{-2\phi} = 0$ on the metric conformal factor. The result of this is that the four bosonic coordinates transverse to the fivebrane and the four nontrivially-coupled left- and right-moving fermions are governed by the worldsheet action

$$\frac{1}{4\pi\alpha'} \int d^2\sigma \{ G_{\mu\nu}(X) \partial_+ X^\mu \partial_- X^\nu + 2B_{\mu\nu}(X) \partial_+ X^\mu \partial_- X^\nu + iG_{\mu\nu}\psi^\mu_R D_-\psi^\nu_R + iG_{\mu\nu}\lambda^\mu_L D_+\lambda^\nu_L + \frac{1}{4} R(\Omega_+)_{\mu\nu}\lambda_\rho\psi^\mu_R\psi^\nu_R\lambda^\rho_L\lambda^\nu_L \}$$

(4.4)

where $D_\pm$ are the covariant derivatives built out of the generalized connections $\Omega_\pm$. In fact, as long as the $H$ appearing in $\Omega_\pm$ is given by $d\wedge B$, (4.4) is identical to the basic left-right symmetric, (1, 1) supersymmetric nonlinear sigma model with torsion [21]. Despite the apparent asymmetry of the coupling of $\lambda_L$ to $\Omega_+$ and $\psi_R$ to $\Omega_-$, the theory nonetheless has an overall left-right symmetry (under which $B \rightarrow -B$) and is non-anomalous. To exchange the roles of $\psi_R$ and $\lambda_L$ one has to replace the curvature of $\Omega_-$ by that of $\Omega_+$. This exchange symmetry property relies on the non-riemannian relation

$$R(\Omega_+)_{\mu\nu}\lambda_\rho = R(\Omega_-)_{\lambda\mu\nu}$$

(4.5)

which indeed holds for the generalized connection (2.4) when $d\wedge H = 0$. To summarize, we have shown that the heterotic sigma model describing the nontrivial four-dimensional geometry of the fivebrane is actually an example of a left-right symmetric sigma model with at least (1, 1) supersymmetry. As we will now show, it actually has (4, 4) worldsheet supersymmetry.
We now turn to the question of extended supersymmetry. The basic worldsheet supersymmetry of a \((1,1)\) model like (4.4) is

\[
\delta X^M = \epsilon_L \psi^M_R + \epsilon_R \psi^M_L
\]
\[
\delta \psi^A_L + [\Omega_{+M}]^{AB} \delta X^M \psi^B_L = \partial X^A \epsilon_R + \ldots
\]
\[
\delta \psi^A_R + [\Omega_{-M}]^{AB} \delta X^M \psi^B_R = \partial X^A \epsilon_L + \ldots
\] (4.6)

The worldsheet supersymmetry of the \((1,0)\) model is obtained by dropping the contributions of \(\epsilon_R\) and \(\psi_L\). The general structure of a possible second supersymmetry transformation is

\[
\hat{\delta} X^M = \epsilon_L f_R(X)^N \psi^N_R + \epsilon_R f_L(X)^N \psi^N_L
\]
\[
\hat{\delta} \psi^A_L + [\Omega_{+M}]^{AB} \delta X^M \psi^B_L = -f_L(X)^A_B \partial X^A \epsilon_R + \ldots
\]
\[
\hat{\delta} \psi^A_R + [\Omega_{-M}]^{AB} \delta X^M \psi^B_R = -f_R(X)^A_B \partial X^A \epsilon_L + \ldots
\] (4.7)

The function \(f\) is normalized and fully defined by the requirements that \(\{\hat{\delta}, \delta\} = 0\) and that \(\hat{\delta}\) anticommute with itself to give ordinary translations as in (4.1). The question is, what conditions must \(f\) satisfy in order for \(\hat{\delta}\) to be a symmetry and how many of them can there be?

This question was first addressed in [22] for the case of left-right symmetric theories without torsion (i.e. without an antisymmetric tensor coupling term). The more complex case of left-right symmetry with torsion was subsequently dealt with in [23,16,17]. The basic result is that the pair of tensors \(f_{R,L}\) must be complex structures, covariantly constant with respect to the appropriate connection:

\[
f^2 = -1
\]
\[
D^+_A f^B_C = \partial_A f^B_C + \Omega^{(\pm)}_{AD} f^D_C - \Omega^{(\pm)}_{AC} f^D_D = 0
\] (4.8)

where the \(\pm\) notation is equivalent to the \(L,R\) notation. The tensors in (4.8) are written in tangent space indices which is why the generalized spin connections \(\Omega^{(\pm)}\) appear in the covariant derivative. The equation could, of course, also have been written in coordinate indices. In general, it is not obvious that such a pair of complex structures can be found, but, if one can, we know that the sigma model actually possesses \((2,2)\) worldsheet supersymmetry. A further question is whether multiple pairs \(f_{\pm}^{(r)}\) of such complex structures can be found. If we can find \(p - 1\) of them, then the sigma model has \((p,p)\) supersymmetry. It turns out that the only consistent possibility for multiple complex structures is that there be three of them [23] and that they satisfy the Clifford algebra

\[
f_{\pm}^{(r)} f_{\pm}^{(s)} = -\delta_{rs} + \epsilon_{rst} f_{\pm}^{(t)}
\] (4.9)

This corresponds to the case of \((4,4)\) supersymmetry. It is worth noting that each complex structure leads to a conserved (chiral) current:

\[
J_{\pm}^{(r)} = \psi^A_{\pm} (f_{\pm}^{(r)})_{AB} \psi^B_{\pm}
\] (4.10)
This yields a $U(1)$ symmetry in the $(2, 2)$ case and an $SU(2)$ symmetry in the $(4, 4)$ case.

The question of left-right asymmetric theories, such as those which underlie the ‘non-exact’ fivebranes discussed in section 3, is more delicate. According to [16], a heterotic sigma model will have $(p, 0)$ supersymmetry if there are $p - 1$ complex structures $f^{(r)}_+$ which are covariantly constant under the connection which couples to the right-moving fermions (those which do not couple to the gauge field) and if the gauge field (which affects the left-moving fermions) satisfies a condition which reduces, for a four-dimensional base space, to self-duality. The latter condition is met for all of the fivebranes of interest to us since they are all built on instanton gauge fields. Thus, in all cases, the essential issue is the existence of complex structures.

To count complex structures, we will use the theorem that a complex structure can be constructed from any covariantly constant spinor [24]. We start with a spinor $\eta$ (in our case four-dimensional) of definite chirality ($\gamma_5 \eta = \pm \eta$, say) and unit normalized ($\eta^\dagger \eta = 1$). Then we define a tensor

$$J_{AB} = -i\eta^\dagger \gamma_{AB} \eta$$

which we will identify as a complex structure tensor (in tangent space indices and with indices raised and lowered by the identity metric). It is then automatic that if the spinor is covariantly constant with respect to some connection, so is $J_{AB}$. A simple Fierz identity argument, quite similar to that found on p.52 of [24], then shows that $J$ squares to $-1$ ($J_{AB} J_{BC} = -\delta_{AC}$) and is indeed a complex structure.

We are now ready to construct the explicit complex structures. As was explained in the discussion after (2.10), on the fivebrane, constant spinors of definite four-dimensional chirality are covariantly constant. Using the Weyl representation for the four-dimensional gamma matrices, one has the following solutions of the two covariant constancy conditions:

$$\mathcal{D}_\mu (\Omega_+) \epsilon_+ = 0 \Rightarrow \epsilon_+ = \left( \begin{array}{c} \chi \\ 0 \end{array} \right)$$

$$\mathcal{D}_\mu (\Omega_-) \epsilon_- = 0 \Rightarrow \epsilon_- = \left( \begin{array}{c} 0 \\ \chi \end{array} \right),$$

where $\chi$ is any constant two-spinor (which we might as well unit normalize). Since there are three parameters needed to specify the general normalized two-spinor, there should be three independent choices for the two-spinor $\chi$ and therefore three choices for both $\epsilon_+$ and $\epsilon_-$. We will define the independent $\chi_r$ ($r = 1, 2, 3$) as those which give expectation values of the spin operator along the three coordinate axes:

$$\chi_r^\dagger \sigma^i \chi_r = \delta_{ir}. \quad (4.13)$$

This finally leads, with the help of (4.11), to the following set of three right- and left-handed complex structures:

$$J^+_1 = \left( \begin{array}{cc} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{array} \right) \quad J^-_1 = \left( \begin{array}{cc} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{array} \right)$$

$$J^+_2 = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \quad J^-_2 = \left( \begin{array}{cc} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{array} \right) \quad (4.14)$$

$$J^+_3 = \left( \begin{array}{cc} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{array} \right) \quad J^-_3 = \left( \begin{array}{cc} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{array} \right).$$
It is trivial to show that the $J^+$ commute with all the $J^-$ and that they satisfy the Clifford algebra (4.9). These are precisely the conditions needed to generate (4,4) supersymmetry in a left-right symmetric theory (or (4,0) supersymmetry in a heterotic theory). The complex structures are thus extremely simple indeed.

Finally, we come to the questions of finiteness and need for higher-order in $\alpha'$ corrections to our solutions. It is rather firmly established that two-dimensional nonlinear sigma models with (4,4) supersymmetry are in fact finite. The general proof was given quite some time ago by Alvarez-Gaume and Freedman and it consists in showing that no (4,4)-invariant counterterms of the needed dimension can be constructed. If the theory is finite, the beta-functions get no higher-order corrections and the choice of background fields which made the beta functions vanish at leading order must continue to make them vanish at all orders in $\alpha'$. Confirmation of this comes from a construction by Gates et.al., using (2,2) superfields, of the most general (4,4)-invariant action. The functional form of the action must satisfy certain conditions in order to have (4,4) supersymmetry and, with hindsight, one can see that the most general solution of these conditions corresponds precisely to our special multi-fivebrane solution.

As an aside, we mention that it has been argued that one really only needs (4,0) supersymmetry to achieve finiteness. This would apply to variations on the solution described in Sect. 2 in which, for example, the gauge instanton scale size did not match the wormhole throat transverse scale size. In the discussion given earlier in this section, we recall that the existence and properties of the right-moving complex structures $f^{(+)}_i$ have nothing to do with the properties of the gauge field (which governs the left-moving complex structures). So, if we keep the same metric then we should have the same $f^{(+)}_i$ and thus at least a (4,0) supersymmetry. While the solution may well exist, the anomalies probably mean that there will be corrections to the beta functions so that the theory is not finite, but constructible order by order. This subject has yet to be explored in any detail.

5. Algebraic CFT Approach

It is one thing to show that a sigma model is a superconformal field theory, as we have done in the previous section, and quite another to be able to classify its primary field content and calculate n-point functions of its vertex operators. Indeed, in order to answer all the interesting questions about string solitons, it would be desirable to have as detailed an algebraic understanding of the underlying conformal field theory as we already have for, say, the minimal models. We are far from having such an understanding, but in this section we will see that useful insight can be gained by studying a special limit which emphasizes the semi-wormhole throat.

Recall from section 2 that the (four-dimensional part of the) metric of the symmetric solution has the form

$$ds^2 = e^{-2\phi} dx^2$$  \hspace{1cm} (5.1)

where $dx^2$ is the flat metric on $R^4$ and

$$e^{-2\phi(x)} = e^{-2\phi_0} + \sum_{1}^{n} \frac{Q_i}{(x-x_i)^2}.$$  \hspace{1cm} (5.2)
The singularities in $e^{-2\phi}$ are associated with the semi-wormholes. Taking $n = 1$ and the limit $e^{-2\phi_0} \to 0$ gives

$$e^{-2\phi} = \frac{Q}{x^2}, \quad (5.3)$$

which is the solution corresponding to the wormhole throat itself. Using spherical coordinates centered on the singularity, and defining a logarithmic radial coordinate by $t = \sqrt{Q} \ln \sqrt{x^2/Q}$, the metric, dilaton and axion field strength of the throat may be written in the form

$$ds^2 = dt^2 + Qd\Omega_3^2,$$

$$\phi = -t/\sqrt{Q},$$

$$H = -Q\epsilon, \quad (5.4)$$

where $d\Omega_3^2$ is the line element and $\epsilon$ the volume form of the unit 3-sphere obeying $\int \epsilon = 2\pi^2$. The geometry of the wormhole is thus a 3-sphere of radius $\sqrt{Q}$ times the open line $R^1$ and the dilaton is linear in the coordinate of the $R^1$. Remarkably, these metric and antisymmetric tensor fields are such that the curvatures constructed from the generalized connections, defined in (2.4), are identically zero, reflecting the parallelizability of $S^3$. The axion charge $Q$ is integrally quantized. So, since $Q$ appears in the metric, the radius of the $S^3$ is quantized as well.

The sigma model defined by these background fields is an interesting variant of the Wess-Zumino-Witten model and the underlying conformal field theory can, it turns out, be analyzed in complete detail. The basic observation along these lines was made in [25] in the lorentzian context and euclideanized in [26,27]: the $S^3$ and the antisymmetric tensor field are equivalent to the $O(3)$ Wess-Zumino-Witten model of level

$$k = \frac{Q}{\alpha'}, \quad (5.5)$$

while the $R^1$ and the linear dilaton define a Feigin-Fuks-like free field theory with a background charge induced by the linear dilaton. Both systems are conformal field theories of known central charges:

$$c_{wzw} = \frac{3k}{k+2}, \quad c_{ff} = 1 + \frac{6}{k}. \quad (5.6)$$

The shift of the $R^1$ central charge away from unity is a familiar background charge effect which has been exploited in constructions of the minimal models [28] and in cosmological solutions [23].

For the combined theory to make sense, the net central charge must be four. Let us for the moment consider the bosonic string. If we expand $c_{wzw}$ in powers of $k^{-1}$ (this corresponds to the usual perturbative expansion in powers of $\alpha'$), we see instead that

$$c_{tot} = c_{wzw} + c_{ff} = 4 + O(k^{-2} \sim \alpha'^2). \quad (5.7)$$

But, we should not have expected to do any better: the field equations we solved in section 2 to get this solution are only the leading order in $\alpha'$ approximation to the full bosonic
string theory field equations and we must expect higher-order corrections to the fields and central charges. In fact, this issue can be studied in detail and it can be shown\cite{29} that the metric and antisymmetric tensor fields are not modified and that the only modification of the dilaton is to adjust the background charge of the $R^1$ (i.e. the coefficient of the linear term in $\phi$) so as to maintain $c_{tot}$ exactly equal to four.

While this is quite interesting, we are really interested in the superstring case. The leading-order-in-$\alpha'$ metric, dilaton etc. fields are the same as in the bosonic case (and, because of the non-renormalization theorems, we expect no corrections to them) but various fermionic terms are added to the previous purely bosonic sigma model. The structure is that of the $(1,1)$ worldsheet supersymmetric sigma model\cite{4.4} discussed in section 3. There is still an $S^3 \times R^1$ split, but the component theories are supersymmetrized versions of Wess-Zumino-Witten and Feigin-Fuks. The Feigin-Fuks theory is still essentially free. In the supersymmetric WZW theory, the four-fermi terms vanish identically because, as pointed out above, the generalized curvature vanishes for this background. As a consequence, the generalized connections are locally pure gauge and can be eliminated from the fermion kinetic terms by a gauge rotation of the frame field. Since the fermions are effectively free, they make a trivial addition to the central charges of both the $S^3$ and the $R^1$ models:

$$c_{wzw} = \frac{3k}{k+2} + \frac{3}{2}, \quad c_{ff} = 1 + \frac{6}{k} + \frac{1}{2}. \quad (5.8)$$

There is, however, a small subtlety: the gauge rotation which decouples the fermions is chiral, and therefore anomalous, because the left- and right-moving fermions couple to two different pure gauge generalized connections, $\Omega_+$ and $\Omega_-$. The entire effect of this anomaly on the central charge turns out to be the replacement in $c_{wzw}$ of $k$ by $k - 2$ (the details can be found in\cite{30}) with the result that

$$c_{wzw} = \frac{3(k-2)}{k} + \frac{3}{2}, \quad c_{tot} = c_{wzw} + c_{ff} = 6. \quad (5.9)$$

Six is, of course, exactly the value we want for the central charge. The remarkable fact is that, in the supersymmetric theory, the expansion of $c_{wzw}$ in powers of $k^{-1}$ terminates at first non-trivial order and no modification of the dilaton field is needed to maintain the desired central charge of six. These results are consistent with the non-renormalization theorems discussed in section 4, but are not tied to perturbation theory, since they derive from exactly-solved conformal field theories. On the other hand, since the present discussion makes no reference to the $(4,4)$ supersymmetry which was crucial in proving the perturbative non-renormalization theorems of section 4, an important element is still missing.

This is a good point to remind the reader of the hierarchy of superconformal algebras. Much of what we know about conformal field theory comes from studying the representation theory of these algebras. The basic N=1 superconformal algebra contains an energy-momentum tensor $T(z)$ and its superpartner $G(z)$. The essential information is
contained in the singular terms in their operator product expansion:

\[ T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \ldots \]

\[ T(z)G(w) = \frac{\frac{3}{2} G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{(z-w)} + \ldots \]

\[ G(z)G(w) = \frac{c/6}{(z-w)^3} + \frac{\frac{7}{2} T(w)}{(z-w)} + \ldots \]

The central charge \( c \) is unconstrained. All superstring theories have at least this much worldsheet supersymmetry. The \( N=2 \) superconformal algebras differ from this by having a conserved current \( J(z) \) and two supercharges \( G^\pm(z) \) distinguished by the value \((\pm)\) of their charge with respect to the current \( J(z) \). This charge also plays a key role in the GSO projection which rids the theory of tachyons. The important new algebraic relations are contained in the operator products

\[ J(z)G^\pm(w) = \pm \frac{G^\pm(w)}{(z-w)} + \ldots \]

\[ G^\pm(z)G^\pm(w) \sim 1 \]

\[ G^+(z)G^-(w) = \frac{c/6}{(z-w)^3} + \frac{\frac{7}{2} J(w)}{(z-w)^2} + \ldots \]

There is an \( N=1 \) subalgebra generated by \( T(z) \) and \( G(z) = \frac{1}{\sqrt{2}}(G^+(z) + G^-(z)) \). The ‘practical’ utility of the \( N=2 \) algebra is that the conserved current defines a free field \( H \) by the relation \( J(z) = i\sqrt{\frac{c}{3}}\partial_z H(z) \) and this free field can be used to construct the \( N=1 \) spacetime supersymmetry charge in a compactification to four dimensions. Once again, the central charge, \( c \), is unconstrained. One further extension, to four supercharges, turns out to be possible (and it can be shown [22] that this is the maximal extension). There are now three conserved currents \( J^i \) which generate an \( SU(2) \) Kac-Moody algebra and the supercharges \( G^\alpha(z), G^\alpha(z) \) are in \( I = 1/2 \) representations of the conserved \( SU(2) \). The relevant operator product expansions are

\[ T(z)T(w) = \frac{3k}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \ldots \]

\[ G^\alpha(z)G^\beta(w) = \frac{k\delta_{\alpha\beta}}{(z-w)^2} + \frac{J^i(w)\sigma^i_{\alpha\beta}}{(z-w)^2} + \frac{T(w)\delta_{\alpha\beta}}{(z-w)} + \ldots \]

\[ J^i(z)J^j(w) = -\frac{i}{2}\frac{k\delta_{ij}}{(z-w)^2} + \epsilon_{ijk} \frac{J^k(w)}{(z-w)} + \ldots \]

\[ J^i(z)G^\alpha(w) = -\frac{i}{2}\frac{\sigma^i_{\alpha\beta}G^\beta(w)}{(z-w)} + \ldots \]
dimensions. The $SU(2)$ Kac-Moody algebra is of arbitrary level $k$, but we can see by comparison with (5.10) that the central charge is constrained to be $6k$. Since the level is constrained by unitarity to be integer, the only allowed values of the central charge are $6, 12, \ldots$. Fortunately, $c = 6$ is just what we need, and this suggests that the N=4 algebra will be important role for us.

We will now show that a closer examination of the algebraic structure of the wormhole conformal field theory reveals the existence of just the right extended supersymmetry. An important clue to understanding the structure of the (4,4) superconformal symmetry comes from the fact that there must be two $SU(2)$ Kac-Moody symmetries: The first is part of the standard N=4 superalgebra. This algebra contains the energy-momentum tensor $T(z)$, four supercurrents $G^a(z)$ and three currents $J_i(z)$ of conformal weight 1, which generate an $SU(2)$ Kac-Moody algebra of a level tied to the conformal anomaly (in our case, level one). The second is the $SU(2)$ Kac-Moody algebra of the Wess-Zumino-Witten part of the wormhole conformal field theory. It has a general level $n$, related to the area of the wormhole cross-section (or, equivalently, its axion charge) and is clearly distinct from the N=4 $SU(2)$ Kac-Moody. Since the superconformal algebra is quite tightly constrained, it is not a priori obvious that such an $SU(2) \otimes SU(2)$ Kac-Moody is compatible with N=4 supersymmetry and useful information, such as restrictions on allowed values of the central charge, might be obtained by explicitly constructing the algebra (assuming a consistent one to exist). Quite remarkably, precisely the algebra we need has already been constructed by Sevrin et. al., who discovered an alternate N=4 superalgebra, containing an $SU(2) \otimes SU(2) \otimes U(1)$ Kac-Moody algebra, which had been missed in previous attempts at a general classification of extended superalgebras. In what follows we briefly summarize enough of their work to explain its significance for the wormhole problem and, in particular, to verify the assertions made in section 4 about the role of (4,4) supersymmetry. In addition to establishing the presence of a (4,4) superconformal symmetry, this construction is a useful starting point for studying the correspondence between the instanton moduli space and perturbations of the superconformal field theory.

The construction discovered by Sevrin et.al. goes as follows: Start with the bosonic WZW model for an $SU(2) \otimes U(1)$ group manifold (this is the geometry of the wormhole if we let the radius of the $U(1)$ be infinite). The conformal model contains four dimension-one Kac-Moody currents $J^a$ satisfying the usual KM algebra:

\begin{align*}
J^0(z)J^0(w) &= -\frac{1}{2}(z-w)^{-2} \\
J^0(z)J^i(w) &= O((z-w)^0) \\
J^i(z)J^j(w) &= -\frac{1}{4}n\delta^{ij}(z-w)^{-2} + \epsilon^{ijk}(z-w)^{-1}J^k(w)
\end{align*}

(5.13)

where $i = 1, 2, 3$ indexes the currents of an $SU(2)$ algebra of level $n$ and $J^0$ is the current of the $U(1)$ algebra. This is supersymmetrized by adding a set of four dimension-1/2 fields $\psi^a$ satisfying the free fermion algebra

\begin{equation}
\psi^a(z)\psi^b(w) = -\frac{1}{4}\delta^{ab}(z-w)^{-2}
\end{equation}

(5.14)

* This discussion was developed in collaboration with E. Martinec
(this is motivated by the arguments given earlier in this section that the fermions in a
supersymmetric WZW model are, modulo anomalies, free).

As usual, the Sugawara construction provides an energy-momentum tensor

\[ T(z) = -J^0 J^0 - \frac{1}{n+2} J^i J^i - \partial \psi^a \psi^a \]  

(5.15)

with respect to which the fields \( J^a (\psi^a) \) are primaries of weight 1 (1/2) and which has the expected \( S_{WZW} \) conformal anomaly

\[ c_{SWZW} = \frac{3n}{n+2} + 3 = \frac{6(n+1)}{(n+2)}. \]  

(5.16)

There is also a Sugawara-like construction of four real supersymmetry charges \( G^a \), with \( a = 0, \ldots, 3 \):

\[
G^0 = 2[J^0 \psi^0 + (1/\sqrt{n+2}) J^i \psi^i + (2/\sqrt{n+2}) \psi^1 \psi^2 \psi^3] \\
G^1 = 2[J^0 \psi^1 + (1/\sqrt{n+2})(-J^1 \psi^0 + J^2 \psi^3 - J^3 \psi^2) - (2/\sqrt{n+2}) \psi^0 \psi^2 \psi^3]
\]  

(5.17)

(plus cyclic expressions for \( G^2 \) and \( G^3 \)). These supercharges could have been packaged as a complex \( I = 1/2 \) multiplet, as in (5.12). The operator product expansion of these supercharges with themselves reads

\[
G^a(z)G^b(w) = 4 (\frac{n+1}{n+2}) \delta^{ab} (z-w)^{-3} + 2 \delta^{ab} T(w)(z-w)^{-1} \\
-8[\frac{1}{n+2} \alpha_{ab}^+ \alpha_{ab}^-(w) + \frac{n+1}{n+2} \alpha_{ab}^+ \partial \alpha_{ab}^- (w)](z-w)^{-2} \\
-4[\frac{1}{n+2} \alpha_{ab}^+ \partial \alpha_{ab}^- (w) + \frac{n+1}{n+2} \alpha_{ab}^- \partial \alpha_{ab}^+ (w)](z-w)^{-1}
\]  

(5.18)

where \n\alpha_{ab}^+ = \pm \delta_i^a \delta_j^b + \frac{1}{2} \epsilon_{ijk}\n\alpha_{ab}^- = \frac{1}{2} \epsilon_{ijk}\n(5.19)

and

\[
A_i^- = \psi^0 \psi_i + \epsilon_{ijk} \psi^j \psi^k \\
A_i^+ = -\psi^0 \psi_i + \epsilon_{ijk} \psi^j \psi^k + J^i
\]  

(5.20)

are commuting \( SU(2) \) Kac-Moody algebras of levels 1 and \( n+1 \), respectively. The c-number term (the central charge) and the term involving \( T(z) \) are obligatory in any higher-N superalgebra, while the terms involving dimension 1 operators are what differentiate the various possible extended superalgebras. With further effort, one shows that the \( G \cdot A^\pm \) OPE generates combinations of \( G^a \) and \( \psi^a \) while the \( G \cdot \psi \) OPE yields \( A^\pm \) and \( J^0 \). No new operators appear in further iterations, so the complete algebra generated by the supercharges contains just \( T \) (dimension 2), \( G^a \) (dimension 3/2), \( A_i^\pm \) and \( J^0 \) (dimension 1) and \( \psi^a \) (dimension 1/2). The Kac-Moody algebra defined by the dimension 1 operators is evidently \( SU(2) \times SU(2) \times U(1) \), which accords with our expectations derived from the wormhole geometry.
The superalgebra whose construction we have outlined above is a particular example of a one-parameter family of N=4 algebras dubbed the $A_\gamma$ algebras. The only problem with it is that the sigma model analysis of extended supersymmetry (see for example [17]) makes quite clear that the canonically defined supercharges and energy-momentum tensor must satisfy the standard N=4 algebra, which closes on $T$, $G^a$ and a single level-one $SU(2)$ Kac-Moody algebra $J^i$. The supercharges defined above obviously do not have that property. However, if we ‘improve’ them as follows

$$\tilde{T} = T - \frac{1}{n + 2} \partial J^0 \quad \tilde{G}^a = G^a - \frac{2}{n + 2} \partial \psi^a,$$

we can show that $\tilde{T}$, $\tilde{G}^a$ and $A^i_\gamma$ (the level-one Kac-Moody current) close on themselves and enjoy precisely the standard N=4 superalgebra. This says that the full algebra has the standard algebra as a subalgebra, perhaps no great surprise.

This improvement has a simple physical interpretation: $J^0$ generates a $U(1)$ symmetry which can be regarded as a translation in a free coordinate $\rho$ (that is, we can write $J(z) \sim \partial z \rho(z)$ where $\rho$ is a free scalar field). The original algebra (5.18) makes no reference to the dilaton and corresponds physically to a constant dilaton field. It is well-known that, if one turns on a dilaton which is linear in a free coordinate $\rho$, this has the effect of adding a term proportional to $\partial^2 \phi \sim \partial z J^0(z)$ to $T(z)$ and shifting the central charge of the superconformal algebra by a constant. With a little care we can show that the linear dilaton implicit in (5.21) is precisely what we obtained earlier in this section in our discussion of the WZW-Feigin-Fuks conformal field theory of the wormhole. This is a further piece of evidence that the improved energy-momentum tensor $\tilde{T}$ is the physically relevant one. Now comes the miracle: $T$ is, in any event, not physically acceptable because it has a central charge of $6(n + 1)/(n + 2)$. The central charge of $\tilde{T}$, however, can easily be shown to be 6, precisely the required value!

This shows that there is an exact conformal field theory of just the right central charge associated with the wormhole geometry and verifies the key rôle of N=4 extended supersymmetry in establishing the physics of the model. There are many fivebrane applications of this exact wormhole conformal field theory which are just beginning to be worked out. Perhaps the most interesting concern the vertex operators of excitations about the wormhole, among which one must find the moduli of the exact solutions. In any event, this line of argument shows that the dramatic consequences of (4,4) superconformal symmetry, which we first extracted from perturbative considerations, seem to have nonperturbative status.

6. Conclusion

In these lectures, we have constructed a special set of conformal field theories which have the interpretation of soliton solutions of heterotic string theory. We first constructed them as solutions of the leading order in $\alpha'$ beta function conditions and then showed that, owing to an extended worldsheet supersymmetry, the associated nonlinear sigma model is an exact conformal field theory. It is the existence of an explicit and exact conformal field theory associated with the soliton solution which distinguishes the solution described
here from previous attempts to construct string theory solitons. There are several lines of inquiry which can be pursued now that "exact" string solitons exist. One issue concerns the mass of the soliton. In all previous discussions of string solitons, the mass has been computed using the lowest-order spacetime effective action (2.1) and is therefore known only to lowest order in $\alpha'$. It would obviously be desirable to know the mass exactly, but for that one needs to develop a conformal field theory method of computing soliton masses. Our exact soliton conformal field theory should provide a useful laboratory for developing such methods. A second issue is the question of stringy collective coordinates and their semiclassical quantization. It should be an instructive challenge to translate the well-known standard field theory physics of collective coordinates into the string theory context. This is a nontrivial matter because motion in collective coordinate space becomes motion in a space of conformal field theories and it is a nontrivial matter to find the action associated with such motions (and knowing the exact underlying conformal field theories should help). Yet another question to pursue is that of stringy black hole physics. We noted in Sect. 3 that our solitons were similar to the extreme Reissner-Nordstrom black holes in the sense that, while they have no singularity or event horizon, if one increases their mass by any finite amount (while keeping the axion charge fixed), an event horizon and a singularity (lying at a finite geodesic distance from any finite point) will appear. Such black hole solitons can easily be created by scattering some external particle on an extremal soliton and, by studying stringy scattering theory about the extremal soliton, one should be able to explore, in a controlled way, how a stringy black hole Hawking radiates and the nature of the final state it approaches. These are quite difficult questions, but having precise control of the underlying conformal field theory may allow us to make progress on them. Perhaps it will be possible to report on progress along these lines at the next Swieca Summer School.
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