Guiding Center and Gyrokinetic Theory for Large Electric Field Gradients and Strong Shear Flows

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The guiding center and gyrokinetic theory of magnetized particle motion is extended to the regime of large electric field gradients perpendicular to the magnetic field. A gradient in the electric field directly modifies the oscillation frequency and causes the Larmor orbits to deform from circular to elliptical trajectories. In order to retain a good adiabatic invariant, there can only be strong dependence on a single coordinate at lowest order, so that resonances do not generate chaotic motion that destroys the invariant. When the gradient across magnetic flux surfaces is dominant, the guiding center drift velocity becomes anisotropic in response to external forces and additional curvature drifts must be included. The electric polarization density remains gyrotropic, but both the polarization and magnetization are modified by the change in gyrofrequency. The theory can be applied to strong shear flows, such as are commonly observed in the edge transport barrier of a high-performance tokamak (H-mode) pedestal, even if the toroidal/guide field is small. Yet, the theory retains a mathematical form that is similar to the standard case and can readily be implemented within existing simulation tools.

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I. INTRODUCTION

Understanding the motion of charged particles in electromagnetic fields is an important subject in many fields of physics. For magnetized particles, the guiding center (GC) and gyrokinetic (GK) approaches, which exploit the adiabatic invariance of the magnetic moment, have been developed to a high degree of sophistication (recently reviewed in Refs. 13 and 14). Yet, for a number of applications, one must understand the effects of both strong magnetic fields, \( B \), and strong electric fields, \( E \). Thus, extending GC/GK theory to the smallest possible guide field would allow application to the widest variety of physical scenarios encountered in both nature and the laboratory.

The goal of this work is to derive a simple and useful extension of GC/GK theory to the largest electric field gradient that still maintains a good adiabatic invariant and to clearly explain the limits of these approximations with regard to shear flows. In fact, there is a simple extension to GC/GK theory that retains the essential modifications caused by an electric field gradient and unites the theory with the low-frequency limit of oscillation center theory and the gyration center drift velocity becomes anisotropic in response to external forces and additional curvature drifts must be included. The electric polarization density remains gyrotropic, but both the polarization and magnetization are modified by the change in gyrofrequency. The theory can be applied to strong shear flows, such as are commonly observed in the edge transport barrier of a high-performance tokamak (H-mode) pedestal, even if the toroidal/guide field is small. Yet, the theory retains a mathematical form that is similar to the standard case and can readily be implemented within existing simulation tools.

For magnetized plasmas, strong electric field gradients can occur at the interface between regions where the dominant nonambipolar transport processes change and, hence, require a change in the ambipolar electric field. This typically occurs across the last closed flux surface of a magnetic confinement device where field lines transition from being closed to openly contacting material surfaces, and clearly occurs within the plasma sheath itself. This can also potentially occur across a separatrix in the magnetic flux function that generates magnetic islands or at the interface between regions of chaotic and integrable magnetic field lines.

When large radial electric field gradients occur in a closed field line region, the strongly sheared flow can suppress turbulence and form a “transport barrier” where gradients in density and temperature also become large. The smallest observed scale lengths occur for edge transport barriers in the pedestal and scrape-off layer (SOL) of a high-performance (H-mode) tokamak, where shear flows are strong. In both the pedestal and the SOL, observations and predictions of the edge scale lengths can be similar in size to the poloidal gyroradius.

Although the poloidal gyroradius is larger than the toroidal gyroradius by the ratio of toroidal to poloidal field strength, the formal separation of scales needed for standard GC theory no longer exists. While a subsidiary expansion could be performed, such approximations would not apply to configurations where the poloidal and toroidal fields are comparable, such as spherical tokamaks, spheromaks, and reversed field pinches (RFP’s). Moreover, this suggests that in configurations without toroidal/guide field, such as field-reversed configurations (FRC’s) and Z-pinch’s, edge scale lengths could potentially reach the order of the gyroradius itself. If this is the case, then standard GC/GK theory would no longer be valid in the edge region.

Extending GC/GK theory to smaller guide field would also be useful in diverse applications such as accelerator and beam...
physics, astrophysical and space plasmas, sheath physics, laser-plasma interactions, and high energy density science. For example, GC/GK theory has been applied to plasma turbulence in astrophysical scenarios that rely on a static mean-field approximation for \( B \). However, astrophysical magnetic fields are often topologically nontrivial and have regions where the magnetic field must vanish. Again, it would be desirable to extend GC/GK theory to the smallest possible guide field in order to treat the largest possible domain.

The equations of motion for a nonrelativistic particle of mass \( m \) and charge \( q \) are

\[
\mathbf{r} = \mathbf{v} \quad \quad m\mathbf{v} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}),
\]

(1)

(For simplicity, only the non-relativistic case will be discussed.) It is well known that a perpendicular electric field leads to a perpendicular \( \mathbf{E} \times \mathbf{B} \) drift velocity, but a constant drift velocity can always be removed by a change of reference frame. Thus, it is only the variation of the fields that causes an obstruction to integrability. The electric field corrections become important when the electric oscillation frequency

\[
\Omega_E = \left(-q\mathbf{v} \cdot \mathbf{E} / m\right)^{1/2}
\]

becomes as strong as the cyclotron frequency \( \Omega_B = qB / m \); i.e. when the velocity gradient satisfies \( \nabla' \sim \Omega_B \). Note that, for a quasineutral system, \( \Omega_E \) is always much smaller than the plasma frequency, \( \omega_p \), since \( (\Omega_E / \omega_p)^2 = -\rho / qn \) is the ratio of charge density \( \rho \) to particle density \( n \).

For a 1 T magnetic field, the large shear condition, \( E' / B \sim \Delta_B \), occurs at 176 GV/m\(^2\) for electrons, 95.8 MV/m\(^2\) for protons, and \( \sim 1 \) MV/m\(^2\) for a singly-charged heavy (\( A \sim 100 \)) impurity ion. Tokamak H-mode pedestals and scrape-off layers typically have an electric shear on the order of \( \sim 0.1 - 10 \) MV/m\(^2\) in [23,25]. Thus, the corrections can be important for the main ion species and can be significant for impurity ions.

In the next section, the constraints of adiabatic invariance will be discussed. These constraints imply that the fields must only depend on a single coordinate, e.g. the magnetic flux function, to lowest order. The guiding center ordering assumptions are relaxed to the largest possible perpendicular electric field gradient in Sec. \( \ref{adiabatic} \). A natural and straightforward approach to the derivation of the guiding center drift and Hamiltonian is outlined in Sec. \( \ref{adiabatic} \). In order to clearly explain the issues involved when more than one oscillation frequency is large, the case of strong 2D variation near a fixed point in the electric potential is considered in Sec. \( \ref{adiabatic} \). Then, Sec. \( \ref{adiabatic} \) discusses the useful approximation that only the 1D variation across magnetic flux surfaces is dominant. In this case, the Larmor orbits become elliptical (Sec. \( \ref{adiabatic} \)) and the drifts (Sec. \( \ref{adiabatic} \)) become anisotropic in response to external forces. Section \( \ref{adiabatic} \) describes the modification to the linear GC polarization and magnetization response. Extension to gyrokinetic orbit theory for strong 1D variation is briefly described in Sec. \( \ref{adiabatic} \). The conclusions are summarized in the final section.

II. ADIABATIC INVARIANCE

When the electric field is as strong as the magnetic field, then no adiabatic invariants will exist unless there are approximte symmetries that are valid at lowest order. If \( \mathbf{E} \) depends on all of the coordinates, then the motion for any particular trajectory will only be cyclic if there is an effective potential well in all directions and the particle drift velocity will be determined by the location of the center of the well. However, if there is no scale separation, the three frequencies will be similar in magnitude. In this case, resonances will generate chaotic motion at the first order of perturbation theory and will generically induce Arnold diffusion [34,35].

For \( d + 1 \) space-time dimensions, there must be \( d \) symmetries in order for the trajectories to be integrable [33]. When the Hamiltonian only depends strongly on a single coordinate \( \psi \), there are \( d - 1 \) conserved momenta and time-independence implies conservation of the Hamiltonian. These constraints only allow a strong dependence of the fields and metric tensor on a single coordinate, \( \psi(r,t) \), at lowest order. This allows one to solve for the velocity \( v^\psi \) and canonical momentum \( p^\psi \). Hence one can determine the oscillation period \( 2\pi / \Omega = \int d\psi / 2\pi v^\psi \) and the adiabatic invariant \( J = \int p^\psi d\psi / 2\pi \). The frequency and shape of the trajectory must be determined numerically for each initial condition and averages over the oscillation period must be computed numerically for each orbit. While computationally expensive relative to GC theory, it is necessary for arbitrarily strong field variation.

For example, strong electric fields develop within the plasma sheath that forms near material surfaces. The geometry commonly employed to mitigate heat fluxes uses material surfaces that have a shallow angle of incidence with respect to magnetic field lines. In this case, the sheath is nearly perpendicular to the magnetic field lines and has a much weaker parallel component. As first shown by R. H. Cohen and D. D. Ryutov in Ref. [37] the adiabatic assumption applies here and can be used to define GC orbit theory in the sheath. However, if the Debye length is smaller than the gyroradius, then this requires the approach described in the preceding paragraph rather than the quadratic approximation described in the following.

The symmetry constraints only allow a strong dependence of the fields and metric tensor on a single coordinate, \( \psi(r,t) \), at lowest order. If the electric scalar potential \( \Phi \) and vector potential \( \mathbf{A} \) only depend strongly on space-like coordinate \( \psi \) to lowest order, then the fields must satisfy

\[
\mathbf{E} = -\nabla \psi \partial_\psi \Phi - \partial_\psi \partial_\psi \mathbf{A} \quad \quad \mathbf{B} = \nabla \psi \times \partial_\psi \mathbf{A}.
\]

(3)

Hence, to lowest order, \( \psi \) must be a constant of the magnetic field line trajectories, \( \mathbf{B} \cdot \nabla \psi = 0 \), and the field evolution must be ideal due to the fact that \( \mathbf{E} \cdot \mathbf{B} = 0 \). This implies that there is reference frame in which \( \mathbf{E} = -\nabla \Phi \) and that the topology of the field lines is fixed in time. Any magnetic field can be locally written in the Clebsch form \( \mathbf{B} = \nabla \Psi \times \nabla \alpha \), and, in a topologically toroidal region, \( \Psi \) can be taken to be a single-valued magnetic flux coordinate indicating the “radial” direction. However, \( \alpha \) is generically a multi-valued function, even for integrable field line trajectories. Hence, only
\( \psi(\Psi) \) will be a globally well-defined coordinate suitable for describing particle motion over a global region; this choice implies \( \nabla \alpha = \partial \psi A \).

### III. GUIDING CENTER ORDERING ASSUMPTIONS

#### A. Perturbation Theory

It is desirable to develop a perturbation theory that is as accurate as possible at lowest order. This is because the theory of adiabatic invariants only provides an asymptotic approximation to the exact particle trajectories.\textsuperscript{25,30} Hence, the asymptotic series will eventually diverge and is not carried to high order in practice. In fact, to this day, few codes, if any, have implemented a complete treatment of 2nd order GC effects. Methods for achieving superconvergent asymptotic expansions\textsuperscript{2,30} are based on determination of the best approximation to the frequency for each iteration. This is equivalent to the resummation of an infinite number of terms in the perturbation series. Thus, it is especially important to employ an accurate approximation to the frequency of cyclic orbits at lowest order.

#### B. The small parameter: \( \varepsilon \ll 1 \)

The derivation of the GC drift equations is based on the assumption that the temporal and spatial scales for variation of the magnetic field \( B \) and electric field \( E \) are longer than the oscillation period \( 2\pi/\Omega \) and oscillation radius \( \rho \). Hence, the restrictions for any characteristic frequency \( \omega \) and wavenumber \( k \)

\[
\omega/\Omega \sim k\rho \lesssim \mathcal{O}(\varepsilon) \ll 1
\]

where \( \varepsilon \) is a small ordering parameter. For example, the magnetic field variations must be assumed to be small \( \rho \nabla B/B \sim \partial B/\Omega B \lesssim \mathcal{O}(\varepsilon) \) in order to develop a valid GC perturbation theory. Because the metric tensor and connection appear in the equations of motion, they must satisfy similar restrictions in order to develop a valid GC perturbation theory.

#### C. Standard Ordering: \( \Omega_E/\Omega_B \sim \mathcal{O}(\varepsilon^2) \)

The standard GC drift ordering \textsuperscript{11} assumes that the electric force is smaller than the magnetic force: \( E/v_\perp B \lesssim \mathcal{O}(\varepsilon) \), and, hence, that the electric drift velocity is smaller than the gyration velocity. Thus, the electric field shear is restricted to satisfy \( \nabla E/\Omega_B B \lesssim \mathcal{O}(\varepsilon^2) \). Unless a change of reference frame is employed, trajectories with \( v_\perp < E_\perp/B \) are excluded.

#### D. Large Flow Ordering: \( \Omega_E/\Omega_B \sim \mathcal{O}(\varepsilon) \)

The “large flow ordering”\textsuperscript{2,35} and, the recent approach of A. M. Dimits\textsuperscript{2,35}, are based on choosing the reference frame with the appropriate drift velocity. These theories have separate conditions for the components perpendicular \( E_\perp/v_\perp B \sim \mathcal{O}(1) \) and parallel \( E_\parallel/v_\perp B \sim \mathcal{O}(\varepsilon) \) to the magnetic field. The stronger restriction on the parallel electric field is essential for ensuring that the change in the parallel velocity per cycle remains small \( \Delta v_\parallel/v_\perp \lesssim \mathcal{O}(\varepsilon) \). For this ordering, the perpendicular electric field shear can be somewhat larger, \( \nabla E_\perp/\Omega_B B \lesssim \mathcal{O}(\varepsilon) \), but it must still be small.

### E. Maximal Ordering: \( \Omega_E/\Omega_B \sim \mathcal{O}(1) \)

Assume that the electric force variations in the plane perpendicular to the magnetic field can be large, \( \nabla \perp E_\perp/\Omega_B B \sim \mathcal{O}(1) \), but that the variations are weaker in the parallel direction: \( E_\parallel/v_\perp B \sim k_\parallel/k_\perp \lesssim \mathcal{O}(\varepsilon) \). GC theory still retains a simple form if one assumes that

\[
\nabla \perp E_\perp/\Omega_B B \sim \nabla \perp \nabla \perp \Phi/\Omega_B B \sim \mathcal{O}(1)
\]

but that the higher order gradients are small; i.e. for \( n \geq 2 \)

\[
\varepsilon^n \nabla^n E/v_\perp B \sim \varepsilon^n \nabla^n \Phi/v_\perp B \lesssim \mathcal{O}(\varepsilon).
\]

One natural way in which this ordering can occur is if all perpendicular gradient scale lengths, \( k_\perp \), are small in the sense of satisfying \( k_\parallel \rho \sim \mathcal{O}(\varepsilon) \). This allows the perpendicular electric field and flow velocity to satisfy \( E_\perp/v_\perp B \sim V_\perp/v_\perp \sim \mathcal{O}(\varepsilon^{-1}) \). Because this implies that \( \phi/\rho v_\perp B \sim \mathcal{O}(\varepsilon^{-2}) \), this requires the rather weak parallel scale variation \( k_\parallel \rho \sim \mathcal{O}(\varepsilon^3) \).

### IV. GUIDING CENTER DERIVATION

#### A. Lagrangian Approach

While there are many approaches to deriving the equations of motion\textsuperscript{14–17}, the route that appears to require the least amount of algebra is the approach of Ref.\textsuperscript{45} Consider modifying the charged particle action principle

\[
\mathcal{A} \left[ r, t \right] = \int \left[ \frac{1}{2} m \dot{r}^2 + q \mathbf{A} \cdot \dot{r} - q \Phi \right] dt
\]

by assuming that

\[
\mathbf{r} = \mathbf{R} + \rho' \hat{e}_i \quad \dot{r} = \dot{\mathbf{R}} + \dot{\rho}
\]

where the basis unit vectors \( \hat{e}_i(r, t) \) and the guiding center velocity \( \mathbf{R} = \mathbf{V}(r, t) \) are functions of \( r \) and \( t \). In the equations that follow, the time derivative in the GC reference frame is denoted by \( \dot{s} := ds/dt = (\partial_t + \mathbf{V} \cdot \nabla) s \) for any quantity \( s \).

#### B. Drift Motion

In the limit of vanishing gyroradius, the GC equations of motion are the usual charged particle equations drifting in the non-inertial reference frame with velocity \( \mathbf{V} \):

\[
\mathbf{V} = \partial_t \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} = q (\mathbf{E} + \mathbf{V} \times \mathbf{B})/m.
\]
The zeroth order GC equation of motion can also be written as
\[
\frac{d}{dt} \mathbf{P}_{0, \text{gc}} + \nabla H_{0, \text{gc}} = \mathbf{V} \times (\nabla \times \mathbf{P}_{0, \text{gc}}) \tag{10}
\]
where, to zeroth order in gyroradius, the Hamiltonian and canonical momentum are
\[
H_{0, \text{gc}} = \frac{1}{2} m \mathbf{V}^2 + q \Phi \quad \mathbf{P}_{0, \text{gc}} = m \mathbf{V} + q \mathbf{A}. \tag{11}
\]
The assumption that \( \partial / \Omega \sim \theta(\varepsilon) \) implies that \( H_{0, \text{gc}} \) is conserved along a streamline, \( \mathbf{V} \cdot \nabla H_{0, \text{gc}} = 0 \), to \( \theta(\varepsilon) \). Note that, in certain works, such as Ref. \[13\] the term \( m \mathbf{V} \) is not retained in the momentum, and, in that case, the zeroth order Hamiltonian would have the opposite sign for the kinetic term.

These equations are just as difficult to solve as Eq. \[1\] if not more so, because this equation is to be interpreted as a PDE and the initial conditions for \( \mathbf{V} \) need to be chosen to eliminate fast oscillatory motions. For cases with symmetry, it is potentially possible to determine useful ansätze for solving this equations. However this is challenging for the general case, because the solution will typically have fast oscillatory motion unless it is pinned to a fixed point where \( \mathbf{E} = 0 \). One natural simplifying assumption is that the convective derivative is also small, so that the entire LHS, \( d\mathbf{V}/dt \sim \theta(\varepsilon) \), can be neglected to lowest order. In this case, the solution can be determined via an asymptotic expansion that begins with the usual lowest order approximation
\[
\mathbf{V}_0 = v_{\parallel} \hat{\mathbf{b}} + \mathbf{E} \times \hat{\mathbf{b}} / B. \tag{12}
\]

C. Oscillatory Motion

Due to the fact that \( \mathbf{V} \) satisfies the Euler-Lagrange equations of motion, the first variation of the action with respect to \( \rho \) is a total derivative \( d(\rho \cdot \mathbf{P}_0) / dt \).

The second variation of the action determines the oscillatory part of the motion and is derived to be
\[
\delta_{0, \text{osc}}[\rho, t] = \int \left\{ \frac{1}{2} m \dot{\rho}^2 + \rho \cdot \nabla q \mathbf{A} \cdot \dot{\rho} + \frac{1}{2} \rho \cdot \left[ (\nabla \nabla q \mathbf{A}) \cdot \mathbf{V} - \nabla \nabla \Phi \right] \right\} dt. \tag{13}
\]
The Euler-Lagrange equations of motion are
\[
\mathbf{p} = m \dot{\rho} + \rho \cdot \nabla q \mathbf{A} \tag{14}
\]
\[
\mathbf{p} = \nabla q \mathbf{A} \cdot \dot{\rho} + \rho \cdot \left[ (\nabla \nabla q \mathbf{A}) \cdot \mathbf{V} - \nabla \nabla \Phi. \right. \tag{15}
\]
The Euler-Lagrange equations are equivalent to
\[
m \ddot{\rho} = \dot{\rho} \times \mathbf{qB} + \rho \cdot \left[ (\nabla \nabla q \mathbf{A}) \cdot \mathbf{V} - \nabla \nabla \Phi \right] - \rho \cdot \frac{\partial}{\partial t} \nabla q \mathbf{A}. \tag{16}
\]
\[
m \ddot{\rho} = \dot{\rho} \times \mathbf{qB} + \rho \cdot [\nabla \mathbf{qE} - \nabla \mathbf{qB} \times \mathbf{V}]. \tag{17}
\]
Inserting the GC drift velocity from Eq. \[9\] yields the equivalent expression
\[
m \ddot{\rho} = \dot{\rho} \times \mathbf{qB} + \rho \cdot [\nabla m \mathbf{V} - \nabla \mathbf{V} \times \mathbf{qB}]. \tag{18}
\]
The assumptions imply that the \( \partial \mathbf{V} \) and \( \partial \mathbf{A} \) terms can be neglected at lowest order. However, the convective derivatives can still make a contribution to \( d\mathbf{V}/dt \) and \( d\mathbf{A}/dt \).

Equations \[14\]15 and hence, Eq. \[16\] are manifestly symplectic. However, for the completely equivalent Eqs. \[17\]18 the symplectic symmetry is not manifestly apparent because the term \( \rho \cdot \nabla \mathbf{A} \) has been split and absorbed into the \( \rho \cdot \mathbf{E} \) and \( \rho \cdot \nabla \mathbf{B} \times \mathbf{V} \) terms.

The canonical Hamiltonian to second order in gyroradius can be expressed in three completely equivalent ways as
\[
H_{0, \text{osc}} = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} \rho \cdot \left[ (\nabla \nabla q \mathbf{A}) \cdot \mathbf{V} - \nabla \nabla \Phi \right] \tag{19}
\]
\[
= \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} \rho \cdot \left[ (\nabla \nabla q \mathbf{A}) \cdot \mathbf{V} - \nabla \nabla \Phi \right] \tag{20}
\]
\[
= \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} \rho \cdot \left[ (\nabla \nabla q \mathbf{A}) \cdot \mathbf{V} - \nabla \nabla \Phi \right] \tag{21}
\]
This last form is equivalent to the perturbative analysis of Ref. \[4\]7 after neglecting the terms with time derivatives. Yet another useful expression for the Hamiltonian can be obtained by taking the inner product of Eq. \[17\] with \( \rho \) which yields
\[
H_{0, \text{osc}} = m \dot{\rho}^2 + \frac{1}{2} \left[ \rho \cdot \mathbf{E} - \rho \cdot \mathbf{B} \right] \cdot \left[ \rho \cdot \mathbf{E} - \rho \cdot \mathbf{B} \right]. \tag{22}
\]
The final term is a total derivative that vanishes for simple harmonic motion and vanishes when averaged over the gyroperiod.

Since the motion is linear and the Hamiltonian constant to lowest order, it is also equal to an average over the gyroperiods \( H_{0, \text{osc}} = \langle H_{0, \text{osc}} \rangle \) and can be expressed as a sum over adiabatic invariants, \( H_{0, \text{osc}} = \sum I_j \Omega_j \), for each independent action invariant \( I_j \). For example, if there is only a single adiabatic invariant, then \( H_{0, \text{osc}} = J_0 \).

Note that the \( m \dot{\rho}^2 / 2 \) term contains the gyrogauge “space-time” scalar
\[
\mathcal{R} = \Omega \rho^j \partial_0 \rho \cdot \frac{\partial}{\partial \mathbf{e}_j} = \Omega \rho^j \partial_0 \rho \cdot (\partial_0 + \mathbf{V} \cdot \nabla) \mathbf{e}_j. \tag{23}
\]
In a number of works, the gyrogauge vector term \( \Omega \rho^j \nabla \mathbf{e}_j \cdot \partial_0 \rho \) is included in the momentum, but the gyrogauge scalar term \( \Omega \rho^j \partial_0 \mathbf{e}_j \cdot \partial_0 \rho \) is neglected in the Hamiltonian.

V. STRONG 2D VARIATION NEAR A FIXED POINT

If the gradients are strong in \( n \) directions, where \( n > 1 \), then there are generically \( n \) pairs of independent eigenvalues \( \pm \Omega_i \) and associated eigenvectors. In this case, a non-oscillatory solution for the drift velocity requires \( \Omega^{-1} \mathbf{V} \sim \theta(\varepsilon) \). This implies that the center of the orbital motion is locked to the fixed point location, \( \mathbf{R} \sim -\nabla \mathbf{E}^{-1} \mathbf{E} \). Thus, the GC velocity \( \mathbf{V} = d\mathbf{R}/dt \) is controlled by the motion of the fixed point and does not have the same freedom that exists in standard GC theory. There is also no guarantee that all frequencies, \( \pm \Omega_i \), are real. The \( \Omega_i \)’s generically come in complex conjugate pairs and, since both signs, \( \pm \Omega_i \), must appear, they generically come in quadruplets \( \{ \pm \Omega_i, \pm \Omega_i \} \). Moreover, since the frequencies are similar in magnitude, resonances can generate
chaotic motion at the first order of perturbation theory and the adiabatic approximation will no longer be valid.

Let us examine the orbits close to a fixed point of the electric potential. Assuming that the potential, \( \Phi \), is a function of magnetic flux \( \psi \), then this must also correspond to an O-point or X-point in the magnetic field. Hence, strong gradients can exist in both directions perpendicular to \( B \) point or \( X \)-point in the magnetic field. Hence, strong gradients can exist in both directions perpendicular to \( B \), but not the direction parallel to \( B \).Locally, the \( 2 \times 2 \) symmetric Hessian matrix of partial derivatives of the potential in the directions perpendicular to the field,

\[
\hat{\psi}_{ij} = \partial_i \partial_j \Phi_0 - (\partial_i \partial_j A_0) \cdot \mathbf{V}_0, \tag{24}
\]
can be diagonalized with an orthogonal transformation. The oscillatory motion in the perpendicular plane is a linear combination of two ellipses centered at the location of the fixed point, where \( E = 0 \). There are two pairs of oscillation frequencies \( \pm \Omega_+ \) and \( \pm \Omega_- \), given by

\[
\begin{align*}
\Omega_+^2 &= \frac{1}{4} \Omega_0^2 \pm \frac{1}{4} (\Omega_0^2 - 4 \Omega_D^2)^{1/2} \tag{25} \\
\Omega_-^2 &= \Omega_B^2 + \Omega_E^2 \tag{26} \\
\Omega_D^2 &= \text{tr} \hat{\psi} / m = \left[ \nabla^2 \Phi_0 - (\nabla^2 A_0) \cdot \mathbf{V} \right] q / m \tag{27} \\
\Omega_B^2 &= \text{det} \hat{\psi} / m. \tag{28}
\end{align*}
\]

If \( \Omega_D^2 > 0 \), then the motion is stable for \( \Omega_0^2 > \max (2 \Omega_D^2, 0) \), which leads to a theory with two adiabatic invariants. Figure 1 illustrates the case of two trajectories that are stable for both directions. However, if \( \Omega_D^2 < 0 \), then the motion corresponding to \( \Omega_-^2 < 0 \) is unstable. Figure 2 illustrates the case of two trajectories that are have unstable \( \Omega_-^2 < 0 \). While the theory of adiabatic invariants applies qualitatively, as illustrated by Fig. 3, the real difficulty lies in the fact that the quadratic approximation will break down because the orbit rapidly moves to a region that is far from the fixed point where the potentials may have a rather different dependence on space and time.

From the point of view of standard GC theory, the motion corresponding to \( \Omega_-^2 \) simply represents a very fast drift motion that has been promoted to a second adiabatic invariant. Hence, there is no additional freedom for the guiding center to wander from the fixed point. The lowest order momentum is \( \mathbf{P} = q \mathbf{A} + m \mathbf{V} \) and the lowest order Hamiltonian is

\[
H_0 = J_+ \Omega_+ + J_- \Omega_- + m \mathbf{V}^2 / 2 + q \langle \Phi \rangle (J_+, J_-). \tag{29}
\]

Here, the potential \( \langle \Phi \rangle (J_+, J_-) \) is the orbit-average of \( \Phi \) and its only spatial dependence is on the field line length \( \ell \).

In the limit where \( \Omega_B \gg \Omega_E \), the magnetic moment is \( J_+ \) and gyrofrequency is \( \Omega_+ \approx \Omega_0 \approx \Omega_B + \Omega_E^2 / 2 \Omega_B + \ldots \), which agrees with the prediction of the large flow GC ordering. In this case, the slower eigenvalue, \( \Omega_-^2 \approx -\Omega_D^2 / \Omega_0^2 \), is comparable to the frequency of the drift motion around the magnetic surface and the entire trajectory must be determined to obtain an accurate approximation to \( J_- \) and \( \Omega_- \). However, this case can be approximated well by considering 1D variation alone, as discussed in the following section.

VI. STRONG 1D VARIATION ACROSS MAGNETIC FLUX SURFACES

A. Oscillation Frequency

If the region far from a fixed point has strong variation along the magnetic flux coordinate, \( x := \psi \), then there is single adiabatic invariant. Expand the potentials via

\[
\Phi = \Phi_0 x + \Phi_0'' x^2 / 2 \tag{30}
\]
\[
\Lambda = \Lambda_0 x + \Lambda_0'' x^2 / 2 = B_0 \hat{x} - \mu_0 J_0 x^2 / 2. \tag{31}
\]

In this case, the effective potential is defined as

\[
\Psi''_0 := \Phi''_0 - q \mathbf{V}_0 \cdot \mathbf{V}_0 = \Phi''_0 + \mu_0 J_0 \cdot \mathbf{V}_0 \tag{32}
\]

and determines the electric oscillation frequency via

\[
\Omega_E^2 = q \Psi''_0 / m = \Omega_B \partial_x \mathbf{V}_0, \tag{33}
\]

FIG. 1. Two orbits near an O-point in the electric potential \( \Phi \) and magnetic flux \( \psi \).

FIG. 2. Two orbits near an X-point in the electric potential \( \Phi \) and magnetic flux \( \psi \).
There is only a single pair of oscillation frequencies $\pm \Omega_0$ where
\[
\Omega_0^2 = \Omega_B^2 + \Omega_E^2 = \Omega_B^2 + \Omega_B \hat{b} \cdot \nabla \times \mathbf{v}_0.
\] (34)

It is clear that near a potential minimum, $q \Psi_0^0 > 0$, the electric field gradient increases the oscillation frequency, whereas near a potential maximum, $q \Psi_0^0 < 0$, the electric field gradient decreases the oscillation frequency. The orbits are stable for
\[
\Omega_0^2 / \Omega_B^2 = \Psi_0^0 / B_0 \Omega_B > -1.
\] (35)

The Taylor series for $\Omega_0 / \Omega_B$, expressed as a function of $\Omega_E^2 / \Omega_B^2$, only converges in the region $|\Omega_E / \Omega_B|^2 < 1$. Hence, it is clear that standard GC and GK perturbation series do not converge outside of this region.

B. Oscillatory Motion

The Hamiltonian in the guiding center reference frame can be written to second order as
\[
H_{0, \text{osc}} = [p_x^2 + (p_y - xqB_{0,z})^2 + p_z^2]/2m + q \Psi_0^0 v_x^2/2.
\] (36)

The lack of dependence of the Hamiltonian on $y$ and $z$ to this order implies that the momenta $p_y$ and $p_z = mv_z$ are conserved. The equation of motion for $v_x$
\[
mv_x = q(v_y B_{0,z} - \Psi_0^0 v_x)
\] (37)
can be solved by inserting $mv_x = p_y - xqB_{0,z}$. This yields simple harmonic motion with frequency $\Omega_0$ (Eq. 26) and average position $X$
\[
\dot{v}_x = \Omega_0^2 (X - x) \quad X = p_y / \Omega_B / m \Omega_0^2.
\] (38)

The motion in $y$ is oscillatory, but with a different amplitude
\[
v_y = (X - x) \Omega_B \quad y = v_y / \Omega_B / \Omega_0^2.
\] (39)

The gyro-orbit is given by
\[
\rho = \hat{e}_x a_x \sin(\theta) - \hat{e}_y a_y \cos(\theta)
\] (40)
\[
w = \hat{e}_x u_x \cos(\theta) + \hat{e}_y u_y \sin(\theta).
\] (41)

The angle $\theta$ is right-handed, but the rotation is diamagnetic; i.e. left-handed for a positively charged particle. Thus, to lowest order, $w = -\Omega_0 \partial_\theta \rho$ and $u = \Omega_0 a$. However, due to the strong electric field, the oscillatory motion is elliptical rather than circular. The ratio of the principle axes of the ellipse is determined via the relation
\[
a_y / a_x = u_y / u_x = \Omega_B / \Omega_0.
\] (42)

As shown in Fig. 3, the ellipse is longer in the $x$ direction for a potential well, $q \Psi_0^0 > 0$, and shorter in the $x$ direction for a potential hill, $q \Psi_0^0 < 0$.

The adiabatic invariant associated with the motion is
\[
J = m(v_x^2 + \Omega_0^2 v_x^2)/2\Omega_0 = m v_x^2/2\Omega_0 = ma_x^2 \Omega_0 / 2.
\] (43)

The second order Hamiltonian is simply $H_{0, \text{osc}} = \Omega_0 \mathbf{v}$.

C. Guiding Center Drifts

Now that the Hamiltonian has been computed, the guiding center drift equation can be refined to include the $\partial (\rho^2)$ contribution
\[
m \dot{\mathbf{V}} + m \mathbf{V} \cdot \nabla \mathbf{V} + J \nabla \Omega_0 = q \mathbf{E} + q \mathbf{V} \times \mathbf{B}.
\] (44)

The zeroth order electric field $\mathbf{E}_0$ drives the zeroth order drift $\mathbf{V}_0 = \mathbf{E}_0 \times \mathbf{b}_0 / B_0$. The first order drift is driven by the first order force
\[
\mathbf{F}_1 = q \mathbf{E}_1 + \mathbf{V}_0 \times q \mathbf{B}_1 - \partial_t m \mathbf{V}_0 - m \mathbf{V}_0 \mathbf{V}_0 \cdot \nabla \hat{e}_\perp - J \nabla \Omega_0.
\] (45)

This expression is quite familiar from the standard case. Now, there are additional curvature forces $-V^\perp \mathbf{V} \cdot \nabla \hat{e}_\perp$, because both $V^\perp$ and $V^\parallel$ can be large. In fact, the term $V^\perp \partial_t \hat{e}_\perp \sim \partial (\rho^2)$ is dominant.

If one expands the velocity in terms of $\varepsilon$ as $\mathbf{V} = \sum_n \varepsilon^n \mathbf{V}_n$, then $\mathbf{V}_{n, \perp}$ is determined by an equation of the form
\[
m \mathbf{V}_{n, \perp} \cdot \nabla \mathbf{V}_0 = \mathbf{F}_{n, \perp} + \varepsilon \mathbf{F}_{n, \parallel} \times \mathbf{b}_0
\] (46)

where $\mathbf{F}_n$ denotes the sum of all additional “force” terms at order $n$. The solution for the drift
\[
V_{n, \perp} = F_{n, \perp} / (qB_{0,z} + m \partial_t \mathbf{V}_{0,y}) = F_{n, \perp} / \Omega_B / m \Omega_0^2
\] (47)
\[
V_{n, \parallel} = -F_{n, \parallel} / qB_{0,z} = -F_{n, \parallel} / m \Omega_B
\] (48)
is anisotropic in its response to the forces due to the additional inertia provided by the strongly sheared flow. Thus the first order drifts are given by:
\[
V_1^\perp = (qE_{1,y} + V^\perp B_{1,z}) - J \partial_t \Omega_0 - \hat{e}_\perp \cdot \partial_t m \mathbf{V}_0 - m \mathbf{V}_0 \mathbf{V}_0 \cdot \nabla \hat{e}_\perp \Omega_B / \Omega_0^2
\] (49)

and
\[
V_1^\parallel = (-qE_{1,x} - V^\perp B_{1,z} + qV^\perp B_{1,y} + J \partial_t \Omega_0 + \hat{e}_\perp \cdot \partial_t m \mathbf{V}_0 + m \mathbf{V}_0 \mathbf{V}_0 \cdot \nabla \hat{e}_\perp) / \Omega_B.
\] (50)
The time derivative of the drift flow $dV_{n,\perp}/dt$ generates a polarization drift and polarization charge density at one higher order in $\varepsilon$:

$$V_{n+1,\text{pol}} = \frac{d}{dt} \left( \frac{F_{n,\perp}}{m\Omega_0^2} \right)$$

$$N_{n+1,\text{pol}} = -\nabla \cdot \frac{N_{gc} F_{n,\perp}}{m\Omega_0^2}.$$  

(51)

Interestingly enough, the polarization is gyrotropic, i.e. isotropic in the directions perpendicular to the field.

D. Polarization and Magnetization

The polarization and magnetization can be computed in a straightforward manner following the microscopic “bottom-up” approach [40]. The intrinsic microscopic polarization vector $\pi$ and macroscopic polarization density $P$ in guiding center space are defined by

$$\pi = q \int \left[ \rho - \frac{1}{2} \nabla \cdot \rho \mathbf{v} + \ldots \right] d\theta / 2\pi$$

(53)

$$P = \int \pi F dJ \Omega_d dV_\parallel.$$  

(54)

where $F$ is the guiding center distribution function.

For any force $F$, the time derivative of the flow introduces a polarization drift. The finite displacement for $\langle \rho \rangle = \oint dV_{2,\text{pol}}$ is derived from the polarization drift, Eq. \[51\] which leads to

$$\langle \rho \rangle = \frac{F_{\perp}}{m\Omega_0^2}.$$  

(55)

Thus, the result for the guiding center polarization is

$$\pi = q(qE_{\perp} - J\nabla \cdot \Omega_0)/m\Omega_0^2 - \frac{1}{2} \nabla \cdot q \left[ a_0^2 \hat{e}_x + a_0^2 \hat{e}_y \right]$$

(56)

$$P = q(qN_{gc} E_{\perp} - P_{gc,xx} \nabla \cdot \ln(\Omega_0))/m\Omega_0^2 - \nabla \cdot q \pi \Omega_{\perp}/2m\Omega_0^2.$$  

(57)

where $P_{gc,ij}$ is the zeroth order guiding center pressure tensor.

Note that when comparing to magnetized fluid theory, it is important to recognize that the guiding center density, $N_{gc}$, itself differs from the particle density, $n_0$, in the limit of vanishing gyroradius. As explained in Ref. [50] the difference $N_{gc} - n_0$ contributes an amount that is equal to the pressure term in Eq. [57]. Accounting for the contribution of both terms doubles the magnitude of the diamagnetic polarization in the adiabatic drift-reduced fluid theory.

The intrinsic microscopic magnetization vector $\mu$ and macroscopic magnetization density $M$ are defined by

$$\mu = q \int \left[ \rho \times \mathbf{V}_{\perp} + \frac{1}{2} \rho \times \frac{d}{dt} \rho + \ldots \right] d\theta / 2\pi$$

(58)

$$M = \int \mu F dJ \Omega_d dV_\parallel.$$  

(59)

where the first term in Eq. [58] is the moving dipole term due to the electric polarization. The results for the magnetization

$$\mu = -q(J\Omega_0 + mV_{\perp}^2)\Omega_0 \mathbf{b}/m\Omega_0^2$$

$$M = -q(P_{gc,xx} + N_{gc} mV_{\perp}^2)\Omega_0 \mathbf{b}/m\Omega_0^2.$$  

(60)

(61)

are modified by the change in frequency and by the addition of the zeroth order ram pressure.

VII. EXTENSION TO GYROKINETIC ORBIT THEORY

GK theory extends the limits of GC theory to arbitrary $k_\perp r$, where $k_\perp$ is the perpendicular wavenumber. However, this assumes that the amplitude of the variation is sufficiently small. Consideration of the particle motion near the potential minimum of a sinusoidal wave implies that the restriction requires

$$\delta_E = k_\perp |E_{\perp}| / \Omega_B B \lesssim 1.$$  

(62)

This restriction is satisfied by both the large flow $\delta_0 \sim O(\varepsilon)$ and standard $\delta_0 \sim O(\varepsilon^2)$ ordering.

The GC theory for strong variation across magnetic flux surfaces, presented in Sec. VII can readily be used to develop a more accurate GK theory. The main differences are due to the fact that the drift motion is anisotropic and that the orbits are elliptical. Given the convention adopted in Eq. [40], the quantity $k \cdot \rho = c_k \sin(\theta - \theta_k)$ has the magnitude and reference angle:

$$c_k^2 = (k \cdot a)^2 = (k_a x)^2 + (k_a y)^2$$

(63)

$$\tan(\theta_k) = k_a x/k_a y = k_\perp \Omega_B / k_\perp \Omega_0.$$  

(64)

Many of the formal expressions that appear in the GK theory are identical with the replacement of the frequency $\Omega_0 \rightarrow \Omega_0$ and the expressions above for $c_k$ and $\theta_k$. However, the drifts must be modified to the anisotropic form given in Eqs. [47-48].

The perturbed first-order electric potential $\Phi_1(r)$ in guiding center coordinates is defined via

$$\Phi_1(R, \mu, \theta) = \sum_k \int dr e^{i(k \cdot R - \theta)} \Phi_1(r) / (2\pi)^3$$

(65)

$$= \sum_k \int dr e^{-i k \cdot \rho} \Phi_{1k} / (2\pi)^3.$$  

(66)

Hence, the potential in real space can be expanded as

$$\Phi_1(r) = \sum_k \sum_n e^{i(n(\theta - \theta_k) + k R)} J_n(c_k) \Phi_{1k}$$

(67)

where $J_n(x)$ is the Bessel function of order $n$. The contributions to the Hamiltonian from the vector potential that are of the form $V \cdot A_1$ and $V \cdot A_{1\parallel}$ can be expressed in a similar fashion. The contribution of the perpendicular component $(v - V) \cdot A_{1\perp}$ can be written in terms of $B_{1\parallel} := \mathbf{b} \cdot \nabla \times A_1$ as

$$(v - V) \cdot A_{1\perp}(r) = \sum_k \sum_n e^{i(n(\theta - \theta_k) + k R)} [J_n(c_k) - J_{n-1}(c_k)] \mu B_{1\parallel} / 2c_k \Omega_0.$$  

(68)

Here, the magnetic moment, $\mu = J q / m$, is used to represent the adiabatic invariant in order to eliminate any possible confusion with the Bessel functions.
The first order GK Hamiltonian is simply the $n = 0$ orbit average of these quantities. The two required gyroaveraging operators are defined by $J_0(c_k)$ and

$$ G_1(c_k) = J_1(c_k)/c_k = [J_0(c_k) + J_2(c_k)]/2. \tag{69} $$

Each averaging operator, defined by its Fourier expansion, $\hat{O}(k)$, acts on a function, $f(R)$, as an integro-differential convolution operator, $\hat{O}$, via

$$ \hat{O}f := \int dR \sum_k e^{ik \cdot (R - R')} \hat{O}(k) f(R')/(2\pi)^3. \tag{70} $$

The first order GK Hamiltonian is

$$ H_{1,gk} = q(J_0\Phi_1 - V \cdot J_0 A_{1\perp} - \nabla \cdot J_0 A_{1\parallel}) + \mu \dot{G}_1 B_{1\parallel} \Omega_B/\Omega_0. \tag{71} $$

If one assumes that the perturbation amplitudes are of order $\delta$, where $\epsilon \ll \delta \ll 1$, then the total Hamiltonian is

$$ H = H_{0,gk} + H_{0,osc} + H_{1,gk}. \tag{72} $$

The gyrokinetic drifts are anisotropic and result from applying Eqs. [47],[48] to the first order Hamiltonian and momentum.

VIII. CONCLUSION

In conclusion, guiding center (GC) and gyrokinetic (GK) theory can be extended to the regime of large electric field gradient, which directly modifies the oscillation frequency and causes the orbits to become elliptical. In order for the trajectories to retain a robust adiabatic invariant and a slower drift motion, the spatial variations can only depend strongly on a single coordinate at lowest order. This situation naturally applies to the large shear flows observed in the H-mode pedestal even if the toroidal/guide magnetic field is as small or even smaller than the poloidal field. The resulting GC/GK theory displays anisotropy in the drift motion and modifies the polarization and magnetization, but otherwise retains a similar mathematical form to the standard case. Thus, the changes needed to improve the accuracy can readily be implemented in existing GC/GK simulation tools. It is of great interest to continue exploring the physical implications of the extended ordering and its convergence for strong field variations. It would also be valuable to further develop the connection to oscillation center theory and the theory of waves in magnetized plasmas.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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