Center conditions for polynomial differential equations: discussion of some problems

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Abstract
Classifications of irreducible components of the set of polynomial differential equations with a fixed degree and with at least one center singularity lead to some other new problems on Picard-Lefschetz theory and Brieskorn modules of polynomials. In this article we explain these problems and their connections to such classifications.

0 Introduction

The set of polynomial 1-forms \( \omega = P(x, y)dy - Q(x, y)dx \), \( \deg P, \deg Q \leq d \), \( d \geq 2 \) is a vector space of finite dimension and we denote by \( F(d) \) its projectivization. Its subset \( F(d) \) containing all \( \omega \)'s with \( P \) and \( Q \) relatively prime and \( \deg(\omega) := \max\{\deg P, \deg Q\} = d \) is Zariski open in \( \overline{F(d)} \). We denote the elements of \( F(d) \) by \( F(\omega) \) or \( F \) if there is no confusion about the underlying 1-form \( \omega \) in the text. Any \( F(\omega) \) induces a holomorphic foliation \( F \) in \( \mathbb{C}^2 \) i.e., the restrictions of \( \omega \) to the leaves of \( F \) are identically zero. Therefore, we name an element of \( F(d) \) a (holomorphic) foliation of degree \( d \).

The points in \( \text{sing}(F(\omega)) = \{P = 0, Q = 0\} \) are called the singularities of \( F(\omega) \). A singularity \( p \in \mathbb{C}^2 \) of \( F(\omega) \) is called reduced if \( (P_x Q_y - P_y Q_x)(p) \neq 0 \). A reduced singularity \( p \) is called a center singularity or center for simplicity if there is a holomorphic coordinates system \( (\tilde{x}, \tilde{y}) \) around \( p \) with \( \tilde{x}(p) = 0, \tilde{y}(p) = 0 \) such that in this coordinates system \( \omega \wedge d(\tilde{x}^2 + \tilde{y}^2) = 0 \). One can call \( f := \tilde{x}^2 + \tilde{y}^2 \) a local first integral around \( p \). The leaves of \( F \) around the center \( p \) are given by \( \tilde{x}^2 + \tilde{y}^2 = c \). Therefore, the leaf associated to the constant \( c \) contains the one dimensional cycle \( \{ (\sqrt{\tilde{x}}, \sqrt{\tilde{y}}) | (\tilde{x}, \tilde{y}) \in \mathbb{R}^2, \tilde{x}^2 + \tilde{y}^2 = 1 \} \) which is called the vanishing cycle. We consider the subset of \( F(d) \) containing \( F(\omega) \)'s with at least one center and we denote its closure in \( \overline{F(d)} \) by \( M(d) \). It turns out that \( M(d) \) is an algebraic subset of \( F(d) \) (see for instance [Mo1]). Now the problem of identifying the irreducible components of \( M(d) \) arises. This problem is also known by the name ”Center conditions” in the context of real polynomial differential equations. Let us introduce some of irreducible components of \( M(d) \).

For \( n \in \mathbb{N} \cup \{0\} \), let \( P_n \) denote the set of polynomials of degree at most \( n \) in \( x \) and \( y \) variables. Let also \( d_i \in \mathbb{N} \), \( i = 1, 2, \ldots, s \) with \( \sum_{i=1}^{s} d_i = d - 1 \) and \( \mathcal{L}(d_1, \ldots, d_s) \) be the set of logarithmic foliations

\[ F(f_1 \cdots f_s \sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i}), \quad f_i \in P_{d_i}, \quad \lambda_i \in \mathbb{C} \]

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For practical purposes, one assumes that \( \deg f_i = d_i, \lambda_i \in \mathbb{C}^s, 1 \leq i \leq s \) and that \( f_i \)'s intersect each other transversally, and one obtains an element in \( \mathcal{F}(d) \). Such a foliation has the logarithmic first integral \( f_1^{\lambda_1} \cdots f_s^{\lambda_s} \). Since \( \mathcal{L}(d_1, \ldots, d_s) \) is parameterized by \( \lambda_i \) and \( f_i \)'s it is irreducible.

**Theorem 1.** ([Mo2]) The set \( \mathcal{L}(d_1, \ldots, d_s) \) is an irreducible component of \( \mathcal{M}(d) \), where \( d = \sum_{i=1}^s d_i - 1 \).

In the case \( s = 1 \) we can assume that \( \lambda_1 = 1 \) and so \( \mathcal{L}(d+1) \) is the space of foliations of the type \( \mathcal{F}(df) \), where \( f \) is a polynomial of degree \( d+1 \). This case is proved by Ilyashenko in [1].

In general the aim is to find \( d_i \in \mathbb{N} \cup \{0\}, i = 1, 2, \ldots, k \) and parameterize an irreducible component \( X = X(d_1, d_2, \ldots, d_k) \) of \( \mathcal{M}(d) \) by \( \mathcal{P}_{d_1} \times \mathcal{P}_{d_2} \times \cdots \times \mathcal{P}_{d_k} \). In the above example \( k = 2s \) and \( d_{s+1} = \cdots d_{2s} = 0 \). Once we have done this, we can reformulate the fact that \( X \) is an irreducible component of \( \mathcal{M}(d) \) in a meaningful way as follows:

**Theorem 2.** There exists an open dense subset \( U \) of \( X \) with the following property: for all \( \mathcal{F} \in U \) parameterized with \( f_i \in \mathcal{P}_{d_i}, i = 1, 2, \ldots, k \) and a center \( p = C^2 \) of \( \mathcal{F} \) let \( \mathcal{F}_\epsilon \) be a holomorphic deformation of \( \mathcal{F} \) in \( \mathcal{F}(d) \) such that its unique singularity \( p_\epsilon \) near \( p \) is still a center. Then there exist polynomials \( f_{i\epsilon} \in \mathcal{P}_{d_i} \) such that \( \mathcal{F}_\epsilon \) is parameterized by \( f_{i\epsilon} \)'s. Here \( f_{i\epsilon} \)'s are holomorphic in \( \epsilon \) and \( f_{i0} = f_i \).

The above theorem also says that the persistence of one center implies the persistence of all other type of singularities.

## 1 Usual method

To prove theorems like Theorem 2 usually one has to take \( U \) the complement of \( X \cap \text{sing}(\mathcal{M}(d)) \) in \( X \). But this is not an explicite description of \( U \). In practice one defines \( U \) by conditions like: \( f_i, i = 1, 2, \ldots, k \) is of degree \( d_i \), \( f_i \)'s have no common factors, \( \{f_i = 0\}'s \) intersect each other transversally and so on. To prove Theorem 2 after finding such an open set \( U \), it is enough to prove that for at least one \( \mathcal{F} \in U \)

\[
(1) \quad T_X X = T_X \mathcal{M}(d)
\]

where \( T_X \) means the tangent bundle at \( \mathcal{F} \). Note that for a foliation \( \mathcal{F} \in X \) the equality \( (1) \) does not imply that \( \mathcal{F} \in U \). There may be an irreducible component of \( \mathcal{M}(d) \) of dimension lower than the dimension of \( X \) such that it passes through \( \mathcal{F} \) and its tangent space at \( \mathcal{F} \) is a subset of \( T_X X \). For this reason after proving \( (1) \) for \( \mathcal{F} \) with some generic conditions on \( f_i \)'s, we may not be sure that \( U \) defined by such generic conditions on \( f_i \)'s is \( X - (X \cap \text{sing}(\mathcal{M}(d))) \). However, in the bellow \( U \) can mean \( X - (X \cap \text{sing}(\mathcal{M}(d))) \) or some open dense subset of \( X \).

An element \( \mathcal{F} \) of the irreducible component \( X \) may have more than one center. The deformation of \( \mathcal{F} \) within \( X \) may destroy some centers but it preserves at least one center. Therefore, we have the notion of stable and unstable center for elements of \( X \). A stable center of \( \mathcal{F} \) is a center which persists after any deformation of \( \mathcal{F} \) within \( X \). An unstable center is a center which is not stable. It is natural to ask

**P 1. Are all the centers of a foliation \( \mathcal{F} \in U \) stable?**
The answer is positive for \( X = \mathcal{L}(d_1, d_2, \ldots, d_s) \) in Theorem 1. Every element \( F \in U \) has \( d^2 - \sum_{i<j} d_id_j \) stable center. Here \( U \) means just an open dense subset of \( X \).

The inclusion \( \subset \) in the equality 1 is trivial. To prove the other side \( \supset \), we fix a stable center singularity \( p \) of \( F \) and make a deformation \( F_\epsilon(\omega + \epsilon \omega_1 + \cdots) \) of \( F = F(\omega) \). Here \( \omega_1 \) represents an element \([\omega_1]\) of \( T_F \mathcal{M}(d) \). Let \( f \) be a local first integral in a neighborhood \( U' \) of \( p \), \( s \) a holomorphic function in \( U' \) such that \( \omega = s \cdot df \), \( \delta \) a vanishing cycle in a leaf of \( F \) in \( U' \) and \( \Sigma \simeq (\mathbb{C}, 0) \) a transverse section to \( F \) in a point \( p \in \delta \). We assume that the transverse section \( \Sigma \) is parameterized by \( t = f |_\Sigma \). The holonomy of \( F \) along \( \delta \) is identity. Let \( h_\epsilon(t) \) be the holonomy of \( F_\epsilon \) along the path \( \delta \). It is a holomorphic function in \( \epsilon \) and \( t \) and by hypothesis \( h_0(t) = t \). We write the Taylor expansion of \( h_\epsilon(t) \) in \( \epsilon \)

\[
h_\epsilon(t) - t = M_1(t)\epsilon + M_2(t)\epsilon^2 + \cdots + M_i(t)\epsilon^i + \cdots, \quad i! \cdot M_i(t) = \frac{\partial^i h_\epsilon}{\partial \epsilon^i} |_{\epsilon=0}
\]

The function \( M_i \) is called the \( i \)-th Melnikov function of the deformation \( F_\epsilon \) along the path \( \delta \). It is well-known that the first Melnikov function is given by

\[
M_1(t) = -\int_{\delta_t} \frac{\omega_1}{s}
\]

where \( \delta_t \) is the lifting up of \( \delta \) in the leaf through \( t \in \Sigma \), and the multiplicity of \( M_1 \) at \( t = 0 \) is the number of limit cycles (more precisely the number of fixed points of the holonomy \( h_\epsilon \)) which appears around \( \delta \) after the deformation (see for instance [Mo1]). This fact shows the importance of these functions in the local study of Hilbert 16-th problem.

Now, if in the deformation \( F_\epsilon \) the deformed singularity \( p \), near \( p \) is center then \( h_\epsilon = id \) and in particular

\[
\int_{\delta_t} \frac{\omega_1}{s} = 0, \quad \forall t \in \Sigma
\]

Let \( T^*_F X \) be the set of \([\omega_1] \in T_F \mathcal{F}(d) \) with the above property. It is easy to check that the above definition does not depends on the choice of \( f \) (see [Mo1]). We have seen that \( T_F \mathcal{M}(d) \subset T^*_F X \). The following question arises:

**P 2.** Is \( T_F \mathcal{M}(d) = T^*_F X \)?

If the answer is positive then it means that form the vanishing of integrals 2 one must be able to prove that \( \omega_1 \in T_F X \). Otherwise, calculating more Melnikov functions to get more and more information on \( \omega_1 \) is necessary. The proof of Theorem 1 with \( s = 1 \) shows that the answer of P 2 is positive in this case. However, the answer of P 2 for \( X = \mathcal{L}(d_1, d_2, \ldots, d_s) \) is not known.

## 2 Some singularities of \( \mathcal{M}(d) \)

The method explained in the previous section has two difficulties: First, identifying \( U := X \cap \text{sing}(\mathcal{M}(d)) \) and second to know the dynamics and topology of the original foliation \( F \). A way to avoid these difficulties is to look for foliations \( \mathcal{F}(df) \), where \( f \) is a degree \( d+1 \) polynomial in \( \mathbb{C}^2 \). We already know that such foliations lie in the irreducible component \( \mathcal{L}(d+1) \). But if we take \( f \) a non-generic polynomial then \( \mathcal{F}(df) \) may lie in other irreducible components of \( \mathcal{M}(d) \) and even worse, \( \mathcal{F}(df) \) may not be a smooth point of such irreducible components.
P 3. Do all irreducible components of $\mathcal{M}(d)$ intersect $\mathcal{L}(d+1)$?

If the answer of the above question is positive then the classification of irreducible components of $\mathcal{M}(d)$ leads to the classification of polynomials of degree $d + 1$ in $\mathbb{C}^2$ according to their Picard-Lefschetz theory and Brieskorn modules. If not, we may be interested to find an irreducible component $X$ which does not intersect $\mathcal{L}(d+1)$. In any case, the method which we are going to explain below is useful for those $X$ which intersect $\mathcal{L}(d+1)$.

The foliation $\mathcal{F} = \mathcal{F}(df)$ has a first integral $f$ and so it has no dynamics. The function $f$ induces a $(C^\infty)$ locally trivial fibration on $\mathbb{C} - C$, where $C$ is a finite subset of $\mathbb{C}$. The points of $C$ are called critical values of $f$ and the associated fibers are called the critical fibers. We have Picard-Lefschetz theory of $f$ and the action of monodromy

$$\pi_1(\mathbb{C} - C, b) \times H_1(f^{-1}(b), \mathbb{Q}) \to H_1(f^{-1}(b), \mathbb{Q})$$

where $b \in \mathbb{C} - C$ is a regular fiber. Let $\delta' \in H_1(f^{-1}(b), \mathbb{Q})$ be the monodromy of $\delta$ (the vanishing cycle around a center singularity of $\mathcal{F}(df)$) along an arbitrary path in $\mathbb{C} - C$ with the end point $b$. From analytic continuation of the integral \( \int_{\gamma} f \omega \) one concludes that $\int_{\gamma} f \omega = 0$.

P 4. Determine the subset $\pi_1(\mathbb{C} - C).\delta \subset H_1(f^{-1}(b), \mathbb{Q})$.

In the case of a generic polynomial $f$, Ilyashenko has proved that in $\mathbb{C}[t]$ the equality happens. To prove Theorem 5 I have used a polynomial $f$ which is a product of $d + 1$ lines in general position and I have proved that $\pi_1(\mathbb{C} - C).\delta$ together with the cycles at infinity generate $H_1(f^{-1}(b), \mathbb{Q})$. Cycles at infinity are cycles around the points of compactification of $f^{-1}(b)$.

Parallel to the above topological theory theory, we have another algebraic theory associated to each polynomial. The Brieskorn module $H = \bigoplus_{i=0}^{d} \Omega^1_{\mathbb{C}^2,d+1} df$, where $\Omega^i, i = 0, 1, 2$ is the set of polynomial differential $i$-forms in $\mathbb{C}^2$, is a $\mathbb{C}[t]$-module in a natural way and we have the action of Gauss-Manin connection

$$\nabla : H_C \to H_C$$

where $H_C$ is the localization of $H$ over the multiplicative subgroup of $\mathbb{C}[t]$ generated by $t - c, c \in C$ (see [Mo2]).

P 5. Find the torsions of $H$ and classify the kernel of the maps $\nabla^i = \nabla \circ \nabla \circ \cdots \circ \nabla$ $i$-times.

When $f$ is the product of lines in general position then $H$ has no torsions and the classification of the kernel of $\nabla^i$ is done in [Mo2] using a theorem of Cerveau-Mattei.

Solutions to the both problems P4 and P5 are closely related to the position of $\mathcal{F}(df)$ in $\mathcal{M}(d)$. Using solutions to P4 and P5 one calculates the Melnikov functions $M_i$'s by means of integrals of 1-forms (the data of the deformation) over vanishing cycles and one calculates the tangent cone $TC_{\mathcal{F}} \mathcal{M}(d)$ of $\mathcal{F} = \mathcal{F}(df)$ in $\mathcal{M}(d)$ and compare it with the tangent cone of suspicious irreducible components of $\mathcal{M}(d)$. For instance, to prove Theorem 1 we have taken $f$ the product of $d + 1$ lines in general position and we have proved that

\begin{equation}
(3) \quad \bigcup_{\sum_{i=1}^{s} d_i = d - 1} TC_{\mathcal{F}} \mathcal{L}(d_1, d_2, \ldots, d_s) = TC_{\mathcal{F}} \mathcal{M}(d)
\end{equation}

All the varieties $\mathcal{L}(d_1, \ldots, d_s), \sum_{i=1}^{s} d_i = d - 1$ pass through $\mathcal{F} = \mathcal{F}(df)$.
P 6. Are \( L(d_1, \ldots, d_s) \)'s all irreducible components of \( M(d) \) through \( F(df) \)?

Note that the equality \( \mathfrak{M} \) does not give an answer to this problem. There may be an irreducible component of \( M(d) \) through \( F(df) \) and different form \( L(d_1, d_2, \ldots, d_s) \)'s such that its tangent cone at \( F(df) \) is a subset of \( \mathfrak{M} \). In this case the definition of other notions of tangent cone based on higher order 1-forms in the deformation of \( F(df) \) seems to be necessary.

The first case in which one may be interested to use the method of this section can be:

P 7. Let \( l_i = 0, \ i = 0, 1, \ldots, d \) be lines in the real plane and \( m_i, \ i = 0, 1, \ldots, d \) be integers numbers. Put \( f = l_0^{m_0} \cdots l_d^{m_d} \). Find all irreducible components of \( M(d) \) through \( F(df) \).

In this problem the line \( l_i \) has multiplicity \( m_i \) and it would be interesting to see how the classification of irreducible components through \( F(df) \) depends on the different arrangements of the lines \( l_i \) in the real plane and the associated multiplicities. In particular, we may allow several lines to pass through a point or to be parallel. When there are lines with negative multiplicities then we have a third kind of singularities \( \{ l_i = 0 \} \cap \{ l_j = 0 \} \) called dicritical singularities, where \( l_i \) (resp. \( l_j \)) has positive (resp. negative) multiplicity. They are indeterminacy points of \( f \) and are characterized by this property that there are infinitely many leaves of the foliation passing through the singularity. Also in this case there are saddle critical points of \( f \) which are not due to the intersection points of the lines with positive (resp. negative) multiplicity. The reader may analyze the situation by the example \( f = \frac{1}{l_2 l_3} \).

3 Looking for irreducible components of \( M(d) \)

To apply the methods of previous sections one must find some irreducible subsets of \( M(d) \) and then one conjectures that they must be irreducible components of \( M(d) \). The objective of this section is to do this.

Classification of codimension one foliations on complex manifolds of higher dimension is a subject related to center conditions. We state the problem in the case of \( \mathbb{C}^n, \ n > 2 \) which is compatible with this text. However, the literature on this subject is mainly for projective spaces of dimension greater than two (see [CL]).

The set of polynomial 1-forms \( \omega = \sum_{i=1}^n P_i(x) dx_i, \deg P_i \leq d \) is a vector space of finite dimension and we denote by \( \mathcal{F}(n, d) \) its projectivization. Its subset \( \mathcal{F}(n, d) \) containing all \( \omega \)'s with \( P_i \)'s relatively prime and \( \deg(\omega) := \max\{\deg P_i, i = 1, 2, \ldots, n\} = d \) is Zariski open in \( \mathcal{F}(n, d) \). An element \( [\omega] \in \mathcal{F}(n, d) \) induces a holomorphic foliation \( \mathcal{F} = \mathcal{F}(\omega) \) in \( \mathbb{C}^n \) if and only if \( \omega \) satisfies the integrability condition

\[ \omega \wedge d\omega = 0 \] (4)

This is an algebraic equation on the coefficients of \( \omega \). Therefore, the elements of \( \mathcal{F}(n, d) \) which induce a holomorphic foliation in \( \mathbb{C}^n \) form an algebraic subset, namely \( M(n, d) \), of \( \mathcal{F}(n, d) \). Now we have the problem of identifying the irreducible components of \( M(n, d) \). We define \( \mathcal{F}(2, d) := \mathcal{F}(d) \) and \( M(2, d) := M(d) \).

Let us be given a polynomial map \( F : \mathbb{C}^2 \to \mathbb{C}^n, \ n \geq 2 \) and a codimension one foliation \( \mathcal{F} = \mathcal{F}(\omega) \) in \( \mathbb{C}^n \). In the case \( n > 2 \), let us suppose that \( F \) is regular in a point \( p \in \mathbb{C}^2 \). This implies that \( F \) around \( p \) is a smooth embedding. We assume that \( F(\mathbb{C}^2, p) \) has a tangency with the leaf of \( \mathcal{F} \) through \( F(p) \). In the case \( n = 2 \), we assume that \( F \) is singular.
at \( p \). In both cases, after choosing a generic \( F \) and \( \mathcal{F} \), the pullback of \( \mathcal{F} \) by \( F \) has a center singularity at \( p \).

**P 8.** Fix an irreducible component \( X \) of \( \mathcal{F}(n,d) \). Is

\[
\{ F^* \mathcal{F}, \mathcal{F} \in X, \ \deg f_i \leq d_i, \ i = 1, 2, \ldots, n \}
\]

where \( F = (f_1, f_2, \ldots, f_n) \), an irreducible component of \( \mathcal{M}(d'') \) for some \( d'' \in \mathbb{N} \)?

For instance in Theorem I the elements of \( \mathcal{L}(d_1, d_2, \ldots, d_s) \) are pull backs of holomorphic foliations \( \mathcal{F}(x_1 x_2 \cdots x_s \sum_{i=1}^s \lambda_i \frac{dx_i}{x_i}) \), \( \lambda_i \in \mathbb{C}^* \) in \( \mathbb{C}^s \) by the polynomial maps \( F = (f_1, f_2, \ldots, f_s), \deg f_i \leq d_i \).

Another way to find irreducible subsets of \( \mathcal{M}(d) \) is by looking for foliations of lower degree. Take a polynomial of degree \( d \) in \( \mathbb{C}^2 \) with the generic conditions considered by Ilyashenko, i.e. \( f \) has non degenerated singularities with distinct images. Now \( \mathcal{F}(df) \) has degree \( d - 1 \) which is less than the degree of a generic foliation in \( \mathcal{F}(d) \).

**P 9.** Classify all irreducible components of \( \mathcal{M}(d) \) through \( \mathcal{F}(df) \).

All \( \mathcal{L}(d_1, \ldots, d_s) \)'s pass through \( \mathcal{F}(df) \). There are other candidates as follows:

1. \( A_i = \{ \mathcal{F}(\frac{dp}{q} + d(p)) \mid \deg(p) = 1, \deg(q) = d \} i = 0, 1, 2, \ldots, d; \)

2. \( B_1 = \{ \mathcal{F}(\frac{dp}{q} + d(p)) \mid \deg(p) = 1, \deg(q) = d \}; \)

An element of \( A_i \) (resp. \( B_1 \)) has a first integral of the type \( pe^{q/p} \) (resp. \( qe^p \)). These candidates are supported by Dulac’s classification (see [Du] and [CL] p.601) in the case \( d = 2 \).

We can look at our problem in a more general context. Let \( M \) be a projective complex manifold of dimension two. We consider the space \( \mathcal{F}(L) \) of holomorphic foliations in \( M \) with the normal line bundle \( L \) (see for instance [Mo1]). Let also \( \mathcal{M}(L) \) be its subset containing holomorphic foliation with at least one center singularity. Again \( \mathcal{M}(L) \) is an algebraic subset of \( \mathcal{F}(L) \) and one can ask for the classification of irreducible components of \( \mathcal{M}(L) \). For \( M = \mathbb{C}P(2) \) some irreducible components of \( \mathcal{M}(L) \) are identified in [Mo1].

**P 10.** Prove a theorem similar to Theorem I for an arbitrary projective manifold of dimension two.

In this generality one must be careful about trivial centers which we explain now. Let \( \mathcal{F} \) be a holomorphic foliation in \( \mathbb{C}^2 \) and 0 a regular point of \( \mathcal{F} \). We make a blow up (see [CaSa]) at 0 and we obtain a divisor \( \mathbb{C}P(1) \) which contains exactly one singularity of the blow up foliation and this singularity is a center.

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