Let $\mathcal{G}$ be the planar Galilean conformal algebra and $\tilde{\mathcal{G}}$ be its universal central extension. Then $\mathcal{G}$ (resp. $\tilde{\mathcal{G}}$) admits a triangular decomposition: $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{G}^0 \oplus \mathcal{G}^-$ (resp. $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}^+ \oplus \tilde{\mathcal{G}}^0 \oplus \tilde{\mathcal{G}}^-$). In this paper, we study universal and generic Whittaker $\mathcal{G}$-modules (resp. $\tilde{\mathcal{G}}$-modules) of type $\phi$, where $\phi : \mathcal{G}^+ \cong \tilde{\mathcal{G}}^+ \to \mathbb{C}$ is a Lie algebra homomorphism. We classify the isomorphism classes of universal and generic Whittaker modules. Moreover, we show that a generic Whittaker module of type $\phi$ is simple if and only if $\phi$ is nonsingular. For the nonsingular case, we completely determine the Whittaker vectors in universal and generic Whittaker modules. For the singular case, we concretely construct some proper submodules of generic Whittaker modules.

1. Introduction

Throughout the paper, we denote by $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$ the sets of complex numbers, integers, positive integers and nonnegative integers, respectively. All vector spaces are assumed to be over $\mathbb{C}$. For a Lie algebra $\mathcal{L}$, we use $U(\mathcal{L})$ to denote the universal enveloping algebra of $\mathcal{L}$. More generally, for a subset $X$ of $\mathcal{L}$, we use $U(X)$ to denote the universal enveloping algebra of the subalgebra of $\mathcal{L}$ generated by $X$. For a finite set $S$, we let $\#S$ denote the number of elements in $S$.

The non-relativistic limit of the AdS/CFT conjecture [20] has received a lot of attention. The main motivation is to study real life systems in condensed matter physics via the gauge-gravity duality. The study of a different non-relativistic limit was initiated in [4], where the authors proposed the Galilean conformal algebra as a different non-relativistic limit of the AdS/CFT conjecture and studied a non-relativistic conformal symmetry obtained by a parametric contraction of the relativistic conformal group. The finite-dimensional Galilean conformal algebra is associated with a certain non-semisimple Lie algebra which is regarded as a nonrelativistic analogue of conformal algebras. It was found that the finite-dimensional Galilean conformal algebra could be given an infinite-dimensional lift for all space-time dimensions (cf. [4, 5, 14, 21]). These infinite-dimensional extensions contain a subalgebra isomorphic to the (centerless) Virasoro algebra, which would suggest that they are important in physics. The planar Galilean conformal algebra $\mathcal{G}$, which was first introduced by Bagchi and Gopakumar in [4] and named by Aizawa in [2], is a Lie algebra with a basis $\{\mathbb{L}_m, \mathbb{H}_m, \mathbb{I}_m, \mathbb{J}_m | m \in \mathbb{Z}\}$ and the nontrivial Lie brackets defined by
The planar Galilean conformal algebra $G$ has a $\mathbb{Z}$-grading by the eigenvalues of the adjoint action of $L_0$. It follows that $G$ possesses the following triangular decomposition:

$$G = G^+ \oplus G^0 \oplus G^-,$$

where

$$G^\pm = \bigoplus_{m \in \mathbb{N}} CL_{\pm m} \oplus \bigoplus_{m \in \mathbb{N}} CH_{\pm m} \oplus \bigoplus_{m \in \mathbb{N}} CI_{\pm m} \oplus \bigoplus_{m \in \mathbb{N}} CJ_{\pm m}$$

and

$$G^0 = CL_0 \oplus CH_0 \oplus CI_0 \oplus CJ_0.$$

Set $B^- = G^0 \oplus G^-$. Biderivations, linear commuting maps and left-symmetric algebra structures of $G$ were studied in [9, 10], respectively. It is easy to see that $G$ is perfect and the universal central extension of $G$, denoted by $\tilde{G}$, was determined in [13].

Whittaker vectors and Whittaker modules play a critical role in the representation theory of finite-dimensional simple Lie algebras (cf. [3, 15]). Whittaker modules have been intensively studied for many important infinite dimensional Lie algebras such as the Virasoro algebra [12, 18, 23, 24], the super-Virasoro algebras [16], Heisenberg algebras [11], affine Kac–Moody algebras [1], the twisted affine Nappi–Witten Lie algebra [8], and so on. Analogous results in a similar setting have been worked out for many Lie algebras with triangular decompositions (cf. [7, 17, 19, 25–27]). Inspired by some works mentioned above, Batra and Mazorchuk generalized the ideas of both Whittaker modules and the underlying categories to a broad class of Lie algebras in [6]. Their framework allowed for a unified explanation of some important results (such as Lemma 2.1 in this paper). Meanwhile, they formulated some conjectures on the form of Whittaker vectors and Whittaker modules for Lie algebras with triangular decompositions. The aim of the present paper is to study Whittaker modules for the planar Galilean conformal algebra $G$ and its central extension $\tilde{G}$. Some ideas we use come from [1, 6, 23].

The present paper is organized as follows. In Section 2, we recall some notations and collect known facts about the planar Galilean conformal algebra $G$ and its central extension $\tilde{G}$. Also, two special Whittaker modules, i.e., the universal and generic Whittaker modules are constructed. In Section 3, we precisely determine all the Whittaker vectors in the universal and generic Whittaker $G$-modules (resp. $\tilde{G}$-modules) of nonsingular type. It turns out that the proof of Proposition 3.4 is quite non-trivial. Section 4 is devoted to studying generic Whittaker modules. We provide a sufficient and necessary condition for a generic Whittaker module to be simple, and classify the isomorphism classes of simple generic Whittaker modules of nonsingular type. We also concretely construct some proper submodules of generic Whittaker modules of singular type.
2. Preliminaries

2.1. Whittaker modules and their properties

We recall some results on Whittaker modules of complex Lie algebras with a triangular decomposition and establish related results (see a general Whittaker setup in [6]). Let $\mathfrak{g}$ be a complex Lie algebra with a triangular decomposition $\mathfrak{g} = \mathfrak{g}^{+} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{-}$ in the sense of [22]. Let $\phi : \mathfrak{g}^{+} \to \mathbb{C}$ be a Lie algebra homomorphism which will be called a Whittaker function, and $V$ be a $\mathfrak{g}$-module. A nonzero vector $v \in V$ is called a Whittaker vector of type $\phi$ if $xv = \phi(x)v$ for all $x \in \mathfrak{g}^{+}$. The module $V$ is said to be a Whittaker module of type $\phi$ if it is generated by a Whittaker vector of type $\phi$. We say that $\mathfrak{g}^{+}$ acts on $V$ locally nilpotently if for any $v \in V$ there is $s \in \mathbb{N}$ depending on $v$ such that $x_{1}x_{2} \cdots x_{s}v = 0$ for any $x_{1}, x_{2}, \ldots, x_{s} \in \mathfrak{g}^{+}$. Let $\mathfrak{g}^{+}_{(\phi)} = \{x - \phi(x) | x \in \mathfrak{g}^{+}\}$.

The following result comes from Lemma 3.1 in [1], Lemma 2.2 in [8], and Proposition 32 in [6].

**Lemma 2.1.** Let $V$ be a Whittaker $\mathfrak{g}$-module of type $\phi$. Suppose that $\mathfrak{g}^{+}$ acts locally nilpotently on $\mathfrak{g}^{+}/\mathfrak{g}^{-}$. Then the following statements hold.

(i) $\mathfrak{g}^{+}_{(\phi)}$ acts locally nilpotently on $V$. In particular, $x - \phi(x)$ acts locally nilpotently on $V$ for any $x \in \mathfrak{g}^{+}$.

(ii) All Whittaker vectors in $V$ are of type $\phi$.

(iii) Any nonzero submodule of $V$ contains a Whittaker vector of type $\phi$.

(iv) If the vector space of Whittaker vectors of $V$ is one-dimensional, then $V$ is simple.

2.2. The planar Galilean conformal algebra and its universal central extension, their Whittaker modules

Recall from [13] that the universal central extension $\tilde{G}$ of the planar Galilean conformal algebra $G$ is a Lie algebra with a basis $\{L_{m}, H_{m}, I_{m}, J_{m}, C_{1}, C_{2}, C_{3} | m \in \mathbb{Z}\}$ and the nontrivial Lie brackets are given by

$$
[L_{m}, L_{n}] = (m - n)L_{m+n} + n^{2}\delta_{m+n, 0}C_{1},$$

$$[L_{m}, H_{n}] = mH_{m+n} + n^{2}\delta_{m+n, 0}C_{2}, \quad [H_{n}, H_{m}] = n\delta_{m+n, 0}C_{3},$$

$$[L_{m}, I_{n}] = (m - n)I_{m+n}, \quad [L_{m}, J_{n}] = (m - n)J_{m+n},$$

$$[H_{m}, I_{n}] = I_{m+n}, \quad [H_{m}, J_{n}] = -J_{m+n}, \quad \forall m, n \in \mathbb{Z}.$$

By definition, it is easy to see the following facts.

(1) Let $\mathbb{C}[C_{1}, C_{2}, C_{3}]$ be the polynomial algebra generated by $C_{1}, C_{2}, C_{3}$. Then the center of $U(\tilde{G})$ is $\mathbb{C}[C_{1}, C_{2}, C_{3}]$.

(2) $\tilde{G}$ is a semi-direct product of the Heisenberg–Virasoro algebra

$$\tilde{HV} := \bigoplus_{m \in \mathbb{Z}} \mathbb{C}L_{m} \oplus \bigoplus_{m \in \mathbb{Z}} \mathbb{C}H_{m} \oplus \mathbb{C}C_{1} \oplus \mathbb{C}C_{2} \oplus \mathbb{C}C_{3}$$

and the commutative ideal

$$I \oplus J = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}I_{m} \oplus \bigoplus_{m \in \mathbb{Z}} \mathbb{C}J_{m},$$

where

$$I := \bigoplus_{m \in \mathbb{Z}} \mathbb{C}I_{m} \quad \text{and} \quad J := \bigoplus_{m \in \mathbb{Z}} \mathbb{C}J_{m}.$$

(3) The Cartan subalgebra (modulo center) of $\tilde{G}$ is spanned by $L_{0}$ and $H_{0}$.
Denote by $\tilde{G}$ has a $\mathbb{Z}$-grading by the eigenvalues of the adjoint action of $L_0$. It follows that $\tilde{G}$ possesses the following triangular decomposition:

$$\tilde{G} = \tilde{G}^+ \oplus \tilde{G}^0 \oplus \tilde{G}^-,$$

where

$$\tilde{G}^\pm = \bigoplus_{m \in \mathbb{N}} CL_{\pm m} \oplus \bigoplus_{m \in \mathbb{N}} CH_{\pm m} \oplus \bigoplus_{m \in \mathbb{N}} CL_{\pm m} \oplus \bigoplus_{m \in \mathbb{N}} CJ_{\pm m},$$

and

$$\tilde{G}^0 = CL_0 \oplus CH_0 \oplus CI_0 \oplus CC_1 \oplus CC_2 \oplus CC_3.$$

As a Lie algebra, $\tilde{G}^+$ (resp. $\tilde{G}^-$) is generated by $L_1, L_2, H_1, I_1$ and $J_1$ (resp. $L_{-1}, L_{-2}, H_{-1}, I_{-1}$ and $J_{-1}$). Set $\tilde{B}^- = \tilde{G}^0 \oplus \tilde{G}^-.$

Now we define a partition $i$ to be a sequence of non-negative integers $i := (i_r, ..., i_2, i_1)$ with $i_r \geq ... \geq i_2 \geq i_1 \geq 0$. Note that $r$ depends on $i$. For $k \in \mathbb{Z}_+$, set $i(k) := \#\{1 \leq s \leq r | i_s = k\}$. Denote by $P$ the set of all partitions. For $j, i, h, l \in P$, we define

$$|i| = i_1 + i_2 + \cdots + i_r,$$

$$|(j, i, h, l)| = |j| + |i| + |h| + |l|,$$

$$\Delta(j) = j(0) + j(1) + \cdots,$$

$$L^1 = L_{-1} \cdots L_{-i_1} L_{j_1} = \cdots L^{[1(2)]}_{-1} L_{0}^{[1(0)]},$$

$$H^h = H_{-i_2} \cdots H_{-i_1} H_{j_1} = \cdots H^{[h(2)]}_{-1} H_{0}^{[h(0)]},$$

$$I^l = I_{-i_2} \cdots I_{-i_1} I_{j_1} = \cdots I^{[l(2)]}_{-1} I_{0}^{[l(0)]},$$

$$J^j = J_{-j_2} \cdots J_{-j_1} J_{j_1} = \cdots J^{[j(2)]}_{-1} J_{0}^{[j(0)]}.$$

Define $0 = (\cdots, 2^0, 1^0, 0^0)$, and write $L^0 = H^0 = I^0 = J^0 = 1 \in U(\tilde{G}).$

For any $(j, i, h, l) \in P^4$ and $f(C) \in \mathbb{C}[C_1, C_2, C_3]$. It is obvious that

$$f(C)j^l H^h L^1 \in U(\tilde{G})_{-(|j, i, h, l|)}.$$
According to the PBW Theorem, $U(B^-)$ (resp. $U(B^-)$) has a basis \{${}^tI^HL^L$ \mid $(j, i, h, l) \in P^4$\} (resp. \{${}^tC^aI^HL^L$ \mid $(j, i, h, l) \in P^4, \alpha \in \mathbb{Z}_+^3$\}), where $C^a = C^a_1 C^a_2 C^a_3, \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_+^3$. Thus, $M(\phi)$ (resp. $\tilde{M}(\phi)$) has a basis
\[
\{{}^tI^HL^Lw_\phi \mid (j, i, h, l) \in P^4\} \quad \text{(resp.} \quad \{{}^tC^aI^HL^Lw_\phi \mid (j, i, h, l) \in P^4, \alpha \in \mathbb{Z}_+^3\}\) (2.1)
and $uw_\phi \neq 0$ whenever $0 \neq u \in U(B^-)$ (resp. $U(B^-)$).

For any nonzero element $v = \sum_{(j, i, h, l) \in P^4} f_{j, i, h, l} {}^tI^HL^Lw_\phi \in M(\phi)$ (resp. $v = \sum_{(j, i, h, l) \in P^4} f_{j, i, h, l}(C) {}^tI^HL^Lw_\phi \in \tilde{M}(\phi)$) with $f_{j, i, h, l} \in C, f_{j, i, h, l}(C) \in \mathbb{C}[C_1, C_2, C_3]$, we define
\[
\max \deg(v) := \max \{\|(j, i, h, l)\| \mid f_{j, i, h, l} \neq 0 \text{ (resp.} \quad f_{j, i, h, l}(C) \neq 0\}\},
\]
\[
\max_{L_{\phi, H_{\phi}}}(v) := \max \{\|l(s) + h(s)\| \mid f_{l(s), h(s)} \neq 0 \text{ (resp.} \quad f_{l(s), h(s)}(C) \neq 0\}\}, \quad s \in \mathbb{Z}_+.
\]
We set $\max \deg(0) = -\infty$.

**Definition 2.5.** Let $\phi : G^+ \cong \tilde{G}^+ \rightarrow \mathbb{C}$ be a Lie algebra homomorphism and $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3$. Set
\[
\tilde{L}_{\phi, \xi} = \tilde{M}(\phi) / \left(\sum_{i=1}^{3}(C_i - \xi_i)\tilde{M}(\phi)\right),
\]
and $L_{\phi} = M(\phi)$. We call $L_{\phi}$ (resp. $\tilde{L}_{\phi, \xi}$) a generic Whittaker $G$-module (resp. $\tilde{G}$-module) of type $\phi$.

Denote by $w_\phi$ (resp. $\tilde{w}_\phi$) the cyclic Whittaker vector for $L_{\phi}$ (resp. $\tilde{L}_{\phi, \xi}$).

**Remark 2.6.** For any $x \in U(G^+)$ (resp. $U(\tilde{G}^+)$), and $w' = uw_\phi$ for $u \in U(B^-)$ (resp. $U(\tilde{B}^-)$), we have
\[
(x - \phi(x))w' = [x, u]w_\phi, \quad \forall \ x \in G^+ \cong \tilde{G}^+.
\]

### 3. Whittaker vectors for Whittaker modules of nonsingular type

The aim of this section is to precisely determine all the Whittaker vectors in the universal and generic Whittaker $G$-modules (resp. $\tilde{G}$-modules) of type $\phi$ when $\phi$ is nonsingular. For this purpose, we first give a series of lemmas which will be used to prove our main results.

**Lemma 3.1.** The following statements hold.

(i) For $n \in \mathbb{N}$, we have
\[
\max \deg \left([F_n, {}^tI^HL^L]w_\phi\right) \leq \|(j, i, h, l)\| - n + 1, \quad \text{where} \quad F_n = I_n \text{ or } J_n.
\]

(ii) For any $k \in \mathbb{Z}_+$ and $a \in \mathbb{N}$, we have
\[
[I_{k+1}, L^a_{-k}] = v_1 - a(2k + 1)L^a_{-k-1}I_1,
(I_{k+1}, H^a_{-k}] = v_2 - aH^a_{-k-1}I_1,
[J_{k+1}, L^a_{-k}] = v_3 - a(2k + 1)L^a_{-k-1}j_1,
[J_{k+1}, H^a_{-k}] = v_4 + aH^a_{-k-1}j_1,
\]
where $\max \deg(v_2w_\phi) < (a - 1)k$ if $k > 0$, and $\max_{L_{\phi, H_{\phi}}}(v_2w_\phi) < a - 1$ if $k = 0$ for $1 \leq s \leq 4$. 

(iii) Suppose $\mathbf{l} := (\ldots, 2^{[2]}, 1^{[1]}, 0^{[0]}), \mathbf{h} := (\ldots, 2^{[h(2)]}, 1^{[h(1)]}, 0^{[h(0)]})$ and $k \in \mathbb{Z}_+$ is the minimal non-negative integer such that $\mathbf{l}(k) \neq 0$ or $\mathbf{h}(k) \neq 0$, then

$$[I_{k+1}, J^L H^L L^L] w_\phi = w_1 - (2k + 1) \mathbf{l}(k) \phi(I_1) J^L H^L L^L w_\phi \quad \text{(3.1)}$$

$$[I_{k+1}, J^L H^L L^L] w_\phi = w_2 - (2k + 1) \mathbf{l}(k) \phi(J_1) J^L H^L L^L w_\phi + \mathbf{h}(k) \phi(J_1) J^L H^L L^L w_\phi \quad \text{(3.2)}$$

where if $k > 0$, then $\max \deg (w_s) < |(\mathbf{j}, \mathbf{i}, \mathbf{h}, \mathbf{l})| - k$; if $k = 0$, then $w_s = w_s' + w_s''$ with $\max \deg (w_s') < |(\mathbf{j}, \mathbf{i}, \mathbf{h}, \mathbf{l})|$, max $L_s, L_h(w_s'') < l(0) + h(0) - 1$ for $s = 1, 2$. Meanwhile, $\bar{\mathbf{l}}$ and $\bar{\mathbf{h}}$ satisfy $l(\mathbf{m}) = l(\mathbf{m}), h(\mathbf{m}) = h(\mathbf{m})$ if $m \neq k$, and $l(k) = l(k) - 1, h(k) = h(k) - 1$.

**Proof.**

(i) Let $\mathbf{l} := (l_r, \ldots, l_2, l_1)$, $\mathbf{h} := (h_3, \ldots, h_2, h_1)$. Then

$$[F_n, J^L H^L L^L] = \sum_{p=1}^r J^L H^L L_{-p} \cdots [F_n, L_{-l_p}] \cdots L_{-l_1}$$

$$\quad + \sum_{q=1}^s J^L H_{-h_q} \cdots [F_n, H_{-h_q}] \cdots H_{-h_1} L^L \quad \text{(3.3)}$$

$$= \sum_{j=1}^a d_{j_1, i_1, h_1, l_1} J^L H_{h_1} L^L + \sum_{j=2}^b b_{j_2, i_2, h_2, l_2} J^L H_{h_2} L^L + \sum_{j=3}^c c_{j_3, i_3, h_3, l_3} J^L H_{h_3} L^L,$$

where $d_{j_1, i_1, h_1, l_1}, b_{j_2, i_2, h_2, l_2}, c_{j_3, i_3, h_3, l_3} \in \mathbb{C}$, $0 < n_2, n_3 \leq n$ and

$$|\mathbf{j}, \mathbf{i}, \mathbf{h}, \mathbf{l}| - n = |\mathbf{j}_1, \mathbf{i}_1, \mathbf{h}_1, \mathbf{l}_1|$$

$$= |\mathbf{j}_2, \mathbf{i}_2, \mathbf{h}_2, \mathbf{l}_2| - n_2$$

$$= |\mathbf{j}_3, \mathbf{i}_3, \mathbf{h}_3, \mathbf{l}_3| - n_3.$$

Since $I_{n_2} w_\phi = J_{n_2} w_\phi = 0$ for $n_2, n_3 > 1$, we have

$$\max \deg ([F_n, J^L H^L L^L] w_\phi) \leq |\mathbf{j}, \mathbf{i}, \mathbf{h}, \mathbf{l}| - n + 1.$$

(ii) Follows from the following formulae in $U(\tilde{G})$, which can be proved by induction on $a \in \mathbb{N}$.

$$[I_{k+1}, L^a_{-k}] = \sum_{s=1}^a \binom{a}{s} \prod_{t=0}^{s-1} (tk - 2k - 1) L^a_{s-k-1} L_{i_1+k-\cdots-k_1},$$

$$[J_{k+1}, L^a_{-k}] = \sum_{s=1}^a \binom{a}{s} \prod_{t=0}^{s-1} (tk - 2k - 1) L^a_{s-k-1} J_{i_1+k-\cdots-k_1},$$

$$[I_{k+1}, H^a_{-k}] = \sum_{s=1}^a (-1)^s \binom{a}{s} H^a_{s-k} I_{i_1+k-\cdots-k_1},$$

$$[J_{k+1}, H^a_{-k}] = \sum_{s=1}^a \binom{a}{s} H^a_{s-k} J_{i_1+k-\cdots-k_1}.$$

(iii) We only prove (3.1), since (3.2) can be managed by the same way. Denote

$$L^L = L^L L_{-k}^L$$

and $H^h = H^h H_{-k}^h$. 




where \( I'(s) = I(s), \ h'(s) = h(s) \) for all \( s \neq k \) and \( I'(k) = h'(k) = 0 \). Then
\[
\begin{align*}
[I_{k+1}, j_l l^H h^T l^I] w_\phi &= j_l l^H h^T [I_{k+1}, L_{-k}^I] w_\phi + j_l l^H [I_{k+1}, H^h H_{h,k}^h] L^I w_\phi \\
&= j_l l^H h^T [I_{k+1}, L_{-k}^I] w_\phi + j_l l^H h^T [I_{k+1}, L_{-k}^I] w_\phi \\
&\quad + j_l l^H [I_{k+1}, h^h L_{-k}^l] w_\phi + j_l l^H h^T [I_{k+1}, H_{h,k}^h] L^I w_\phi.
\end{align*}
\]
Considering the first and third terms in the right hand side of the above equality, we have \([I_{k+1}, L_{-k}^I], [I_{k+1}, H^h] \in U(B^-) \)
and
\[
\max \ \text{deg} \left( j_l l^H h^T [I_{k+1}, L_{-k}^I] w_\phi \right), \max \ \text{deg} \left( j_l l^H [I_{k+1}, H^h] H_{h,k}^h \right) w_\phi < |(j, i, h, l)| - k.
\]
By applying (ii) to the second and last terms, we see that
\[
\begin{align*}
j_l l^H h^T l^I [I_{k+1}, L_{-k}^I] w_\phi &= j_l l^H h^T l^I v_1 w_\phi - l(k)(2k + 1) \phi(I_l) j_l l^H l^I L_{-k}^I w_\phi, \\
j_l l^H h^T [I_{k+1}, H_{h,k}^h] L^I w_\phi &= j_l l^H h^T (v_2 - h(k) H_{h,k}^h - f_j l^I w_\phi \\
&= j_l l^H l^I v_2 L^I w_\phi - h(k) j_l l^H H_{h,k}^h - [I_{j, l}] w_\phi \\
&\quad - h(k) \phi(I_j) j_l l^H H_{h,k}^h - [I_{j, l}] w_\phi.
\end{align*}
\]
If \( k > 0 \), then
\[
\max \ \text{deg} \left( j_l l^H h^T l^I v_1 w_\phi \right) < |(j, i, h, l')| + (l(k) - 1)k = |(j, i, h, l)| - k,
\]
\[
\max \ \text{deg} \left( j_l l^H h^T v_2 L^I w_\phi \right) < |(j, i, h', l)| + (h(k) - 1)k = |(j, i, h, l)| - k,
\]
\[
\max \ \text{deg} \left( h(k) j_l l^H H_{h,k}^h - [I_{j, l}] w_\phi \right) < |(j, i, h, l)| - k.
\]
If \( k = 0 \), one can observe that
\[
\max \ L_0, h_0 (j_l l^H h^T l^I v_1 w_\phi) < l(0) + h(0) - 1.
\]
Furthermore, both \( j_l l^H h^T v_2 L^I w_\phi \) and \( h(0) j_l l^H h^T H_{h,k}^h - [I_{j, l}] w_\phi \) can be divided into the sum of two parts \( w' \) and \( w'' \) with \( \max \ \text{deg} \ (w'') < |(j, i, h, l)| \) and \( \max \ L_0, h_0 (w'') < l(0) + h(0) - 1 \). Thus (3.1) holds, completing the proof.

\[\square\]

**Lemma 3.2.** The following statements hold.

(i) \( \) For \( n \in \mathbb{N} \), we have
\[
\max \ \text{deg} \left( [G_n, j_l l^H h^T] w_\phi \right) \leq |(j, i, h)| - n + 1, \quad \text{where} \ G_n = L_n \text{ or } H_n.
\]

(ii) \( \) For any \( k \in \mathbb{Z}_+ \) and \( a \in \mathbb{N} \), we have
\[
[I_{k+1}, I_{-k}^a] = -a(2k + 1) I_{-k}^{-1} I_1,
\]
\[
[I_{k+1}, J_{-k}^a] = -a(2k + 1) J_{-k}^{-1} J_1,
\]
\[
[H_{k+1}, I_{-k}^a] = a I_{-k}^{-1} I_1,
\]
\[
[H_{k+1}, J_{-k}^a] = -a J_{-k}^{-1} J_1.
\]

(iii) \( \) Let \( i := (\cdots, 2^{(2)}, 1^{(1)}, 0^{(0)}), j := (\cdots, 2^{(2)}, 1^{(1)}, 0^{(0)}), h := (\cdots, 2^{(2)}, 1^{(1)}, 0^{(0)}), \) and \( k \in \mathbb{Z}_+ \) be the minimal non-negative integer such that \( i(k) \neq 0 \) or \( j(k) \neq 0 \). Suppose \( h(s) = 0 \) for any \( 0 \leq s \leq k \). Then
\[ [L_{k+1}, j^j H^h] w_\phi = w_1 - (2k + 1)i(k)\phi(I_1)j^j H^h w_\phi \]
\[ - (2k + 1)j(k)\phi(J_1)j^j H^h w_\phi, \]  
(3.4) 
\[ [H_{k+1}, j^j H^h] w_\phi = w_2 + i(k)\phi(I_1)j^j H^h w_\phi - j(k)\phi(J_1)j^j H^h w_\phi, \]  
(3.5) 
where \( \max \deg (w_k) < |(j, i, h)| - k \) for \( s = 1, 2 \). Meanwhile, \( \tilde{i} \) and \( \tilde{j} \) satisfy that \( \tilde{i}(m) = i(m), \tilde{j}(m) = j(m) \) if \( m \neq k \) and \( \tilde{i}(k) = i(k) - 1, \tilde{j}(k) = j(k) - 1 \).

Proof. (i) and (ii) follow from similar arguments to those in the proof of Lemma 3.1(i), (ii).

(iii) We only prove (3.4), since (3.5) can be managed similarly. Let 
\[ j^j = j^j I^j_{-} \quad \text{and} \quad j^j = j^j I^j_{-}, \]
where \( i'(s) = i(s), j'(s) = j(s) \) for all \( s \neq k \) and \( i'(k) = j'(k) = 0 \). Then 
\[ [L_{k+1}, j^j H^h] w_\phi = j^j [L_{k+1}, H^h] w_\phi + j^j [L_{k+1}, j^j I^j_{-k}] H^h w_\phi \]
\[ + [L_{k+1}, j^j I^j_{-k}] j^j H^h w_\phi \]
\[ = j^j [L_{k+1}, H^h] w_\phi + j^j [L_{k+1}, j^j I^j_{-k}] H^h w_\phi + j^j j^j [L_{k+1}, j^j I^j_{-k}] j^j H^h w_\phi \]
\[ + [L_{k+1}, j^j] j^j I^j_{-k} j^j H^h w_\phi \]
It follows from the assumption on \( k \) that \( [L_{k+1}, H^h], [L_{k+1}, j^j I^j_{-k}], [L_{k+1}, j^j I^j_{-k}] \in U(B^-) \) and \( \max \deg (j^j [L_{k+1}, H^h] w_\phi), \max \deg (j^j [L_{k+1}, j^j I^j_{-k}] H^h w_\phi), \max \deg ([L_{k+1}, j^j] j^j I^j_{-k} j^j H^h w_\phi) \) are all strictly smaller than \( |(j, i, h)| - k \). Now using (2) to the third term and last term, we see that 
\[ j^j [L_{k+1}, j^j I^j_{-k}] H^h w_\phi = u_1 - (2k + 1)i(k)\phi(I_1)j^j H^h w_\phi, \]
\[ j^j [L_{k+1}, j^j I^j_{-k}] j^j H^h w_\phi = u_2 - (2k + 1)j(k)\phi(J_1)j^j H^h w_\phi, \]
where \( \max \deg (u_s) < |(j, i, h)| - k \) for \( s = 1, 2 \). Meanwhile, \( i \) and \( j \) satisfy \( \tilde{i}(m) = i(m), \tilde{j}(m) = j(m) \) if \( m \neq k \) and \( \tilde{i}(k) = i(k) - 1, \tilde{j}(k) = j(k) - 1 \). The proof is complete. \( \square \)

We have the following classification of isomorphism classes of universal and generic Whittaker \( G \)-modules (resp. \( \tilde{G} \)-modules).

**Corollary 3.3.** Let \( \phi : G^+ \cong \tilde{G}^+ \to C \) and \( \phi' : G^+ \cong \tilde{G}^+ \to C \) be Lie algebra homomorphisms, and \( \xi, \zeta' \in C^5 \). Then the following statements hold.

(i) \( \tilde{M}(\phi) \cong M(\phi') \) as \( G \)-modules if and only if \( \phi = \phi' \).

(ii) \( \tilde{M}(\phi) \cong M(\phi') \) as \( \tilde{G} \)-modules if and only if \( \phi = \phi' \).

(iii) \( \tilde{L}_\phi, \xi \cong \tilde{L}_{\phi', \zeta'} \) as \( \tilde{G} \)-modules if and only if \( \phi = \phi' \) and \( \xi = \zeta' \).

Proof. 
(i) Suppose that \( \rho : M(\phi) \to M(\phi') \) is an isomorphism of \( G \)-modules. Then 
\[ E\rho(w_\phi) = \rho(Ew_\phi) = \phi(E)\rho(w_\phi) = \phi' \rho(w_\phi), \quad \forall E \in G^+ \]
Thus \( \rho(w_\phi) \) is a nonzero Whittaker vector of type \( \phi \), which implies \( \phi = \phi' \) by Lemma 2.1(ii).

(ii) Follows from similar arguments as (i).

(iii) Suppose that \( \varphi : \tilde{L}_\phi, \xi \to \tilde{L}_{\phi', \zeta'} \) is an isomorphism of \( \tilde{G} \)-modules. Similar arguments as (i) yield that \( \phi = \phi' \). Moreover,
Assume that \( f \) is a vector, where \( f \) is nonsingular.

\[ \xi_s \rho(\bar{w}_\phi) = \rho(C_i \bar{w}_\phi) = C_i \rho(\bar{w}_\phi) = \xi'_s \rho(\bar{w}_\phi), \quad 1 \leq s \leq 3. \]

Hence \( \xi = \xi' \).

Denote by \( M(\phi)_\phi \) (resp. \( \tilde{M}(\phi)_\phi \)) the set of all Whittaker vectors in \( M(\phi) \) (resp. \( \tilde{M}(\phi) \)). The following result precisely determines all Whittaker vectors in \( M(\phi) \) and \( \tilde{M}(\phi) \) when \( \phi \) is nonsingular.

**Proposition 3.4.** If \( \phi \) is nonsingular, then \( M(\phi)_\phi = \mathbb{C}w_{\phi} \) and \( \tilde{M}(\phi)_\phi = \mathbb{C}[C_1, C_2, C_3]w_{\phi} \).

**Proof.** We only prove the assertion for the case \( \tilde{M}(\phi) \). Similar arguments yield the assertion for the case \( M(\phi) \).

It is obvious that \( \mathbb{C}[C_1, C_2, C_3]w_{\phi} \subseteq \tilde{M}(\phi)_\phi \) as \( \mathbb{C}[C_1, C_2, C_3] \) is in the center of \( U(\tilde{G}) \). It suffices to prove that \( \tilde{M}(\phi)_\phi \subseteq \mathbb{C}[C_1, C_2, C_3]w_{\phi} \). For that, let \( w = uw_{\phi} \in \tilde{M}(\phi)_\phi \) be an arbitrary nonzero vector, where \( u \in U(\tilde{B}^-) \). We can write \( w \) as a linear combination of basis (2.1) of \( \tilde{M}(\phi) \):

\[ w = \sum_{(j, \mathbf{i}, \mathbf{h}, \mathbf{l}) \in P^1} f_{j, \mathbf{i}, \mathbf{h}, \mathbf{l}}(C)j^jH^hL^lw_{\phi}, \quad (3.6) \]

where \( f_{j, \mathbf{i}, \mathbf{h}, \mathbf{l}}(C) \in \mathbb{C}[C_1, C_2, C_3] \). We want to show that if there is \( (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \neq (j, \mathbf{i}, \mathbf{h}, \mathbf{l}) \in P^1 \), such that \( f_{j, \mathbf{i}, \mathbf{h}, \mathbf{l}}(C) \neq 0 \), then there is \( E_n \in \{L_n, H_n, I_n, J_n | n \in \mathbb{N}\} \), such that \( (E_n - \phi(E_n))w \neq 0 \), from which the assertion follows. Set

\[ N := \max\{|(j, \mathbf{i}, \mathbf{h}, \mathbf{l})| f_{j, \mathbf{i}, \mathbf{h}, \mathbf{l}}(C) \neq 0\}, \quad (3.7) \]

\[ A_N := \{(j, \mathbf{i}, \mathbf{h}, \mathbf{l})| f_{j, \mathbf{i}, \mathbf{h}, \mathbf{l}}(C) \neq 0, |(j, \mathbf{i}, \mathbf{h}, \mathbf{l})| = N\}. \quad (3.8) \]

Assume that \( f_{j, \mathbf{i}, \mathbf{h}, \mathbf{l}}(C) \neq 0 \) for some \( (j, \mathbf{i}, \mathbf{h}, \mathbf{l}) \neq (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \). By Remark 2.6,

\[ (E_n - \phi(E_n))w = \left[ E_n, \sum_{(j, \mathbf{i}, \mathbf{h}, \mathbf{l}) \in P^1} f_{j, \mathbf{i}, \mathbf{h}, \mathbf{l}}(C)J^jH^hL^lw_{\phi} \right]. \]

**Case 1.** There exists some \( \mathbf{l} \neq \mathbf{0} \) for which \( f_{j, \mathbf{i}, \mathbf{h}, \mathbf{l}}(C) \neq 0 \).

Denote

\[ N' := \max\{|(j, \mathbf{i}, \mathbf{h}, \mathbf{l})| \mathbf{l} \neq \mathbf{0}, f_{j, \mathbf{i}, \mathbf{h}, \mathbf{l}}(C) \neq 0\}. \]

Set

\[ A_{N'} := \{(j, \mathbf{i}, \mathbf{h}, \mathbf{l}) \in P^1 | (j, \mathbf{i}, \mathbf{h}, \mathbf{l}) = N', f_{j, \mathbf{i}, \mathbf{h}, \mathbf{l}}(C) \neq 0\}, \]

\[ k' := \min\{m \in \mathbb{Z}_+ | \mathbf{l}(m) \neq 0 \text{ or } \mathbf{h}(m) \neq 0 \text{ for some } (j, \mathbf{i}, \mathbf{h}, \mathbf{l}) \in A_{N'}\} \]

and

\[ K' := \max\{\mathbf{l}(k') + \mathbf{h}(k') | (j, \mathbf{i}, \mathbf{h}, \mathbf{l}) \in A_{N'}\}. \]

Note that \( N' \leq N \) and \( A_{N'} \) is a nonempty set.

**Subcase 1.** \( N' = N. \)
In this subcase, we compute

\[(I_{k+1}^\prime - \phi(I_{k+1}^\prime))w = \sum_{(j, i, h, l) \in \Lambda_N} f_{j, i, h, l}(C) [I_{k+1}^\prime, j^j H^h L^l] w_\phi \]

+ \sum_{(j, i, h, l) \in \Lambda_N} f_{j, i, h, l}(C) [I_{k+1}^\prime, j^j H^h L^l] w_\phi ,

+ \sum_{(j, i, h, l) \in \Lambda_N} f_{j, i, h, l}(C) [I_{k+1}^\prime, j^j H^h L^l] w_\phi .

By using Lemma 3.1(i) to the first summand, we know that its degree is strictly smaller than \(N - k\). As for the second summand, note that \(l(s) = h(s) = 0\) for \(0 \leq s \leq k\), we have

\[ [I_{k+1}^\prime, j^j H^h L^l] = j^j [I_{k+1}, H^h] L^l + j^j H^h [I_{k+1}, L^l] \in U(B^-) .\]

Thus, its degree is also strictly smaller than \(N - k\). Now applying Lemma 3.1(iii) to the third summand and last summand, respectively, we know that it is of the form

\[ v_1 - \phi(I_1) \sum_{(j, i, h, l) \in \Lambda_N} \frac{h(k') f_{j, i, h, l}(C) j^j H^h L^l w_\phi}{l(k') = 0, h(k') \neq 0} \]

\[ - \phi(I_1) \sum_{(j, i, h, l) \in \Lambda_N} (2k' + 1) l(k') f_{j, i, h, l}(C) j^j H^h L^l w_\phi , \]

\[ - \phi(I_1) \sum_{(j, i, h, l) \in \Lambda_N} h(k') f_{j, i, h, l}(C) j^j H^h L^l w_\phi , \]

\[ = v_1 - \phi(I_1) \sum_{(j, i, h, l) \in \Lambda_N} (2k' + 1) l(k') f_{j, i, h, l}(C) j^j H^h L^l w_\phi , \]

\[ - \phi(I_1) \sum_{(j, i, h, l) \in \Lambda_N} h(k') f_{j, i, h, l}(C) j^j H^h L^l w_\phi , \]

where \( v_1 = v'_1 + v''_1 \) with max deg \( (v'_1) < N - k' \) and max deg \( (v''_1) < k' - 1 \). Meanwhile, \( \tilde{I} \) and \( \tilde{h} \) satisfy \( \tilde{l}(m) = l(m), \tilde{h}(m) = h(m) \) if \( m \neq k' \) and \( \tilde{l}(k') = l(k') - 1, \tilde{h}(k') = h(k') - 1 \). The preceding discussion shows finally that
\[(I_{k+1}' - \phi(I_{k+1}'))w = w_1 - \phi(I_1) \sum_{(j, i, h, l) \in \Lambda_N} (2k' + 1)l I(k') f_{j, i, h, l}(C) J^H L^1 w_{l, h} \]

\[= -\phi(I_1) \sum_{(j, i, h, l) \in \Lambda_N} h(k') f_{j, i, h, l}(C) J^H L^1 w_{l, h}, \quad (3.9)\]

Analogously, we have
\[(J_{k+1}' - \phi(J_{k+1}'))w = w_2 - \phi(J_1) \sum_{(j, i, h, l) \in \Lambda_N} (2k' + 1)l I(k') f_{j, i, h, l}(C) J^H L^1 w_{l, h} \]

\[+ \phi(J_1) \sum_{(j, i, h, l) \in \Lambda_N} h(k') f_{j, i, h, l}(C) J^H L^1 w_{l, h}, \quad (3.10)\]

Note that \(w_s(s = 1, 2)\) have the same property as that of \(v_1\). Suppose on the contrary that for any \(E_n \in \{I_{n1}, H_{n1}, I_{n2}, H_{n2}| n \in \mathbb{N}\}\), we have \((E_n - \phi(E_n))w = 0\). Especially, \((I_{k+1}' - \phi(I_{k+1}'))w = (J_{k+1}' - \phi(J_{k+1}'))w = 0\). Now (3.9) and (3.10) yield
\[
\phi(I_1) \sum_{(j, i, h, l) \in \Lambda_N} (2k' + 1)l I(k') f_{j, i, h, l}(C) J^H L^1 w_{l, h} = 0
\]

\[= -\phi(I_1) \sum_{(j, i, h, l) \in \Lambda_N} h(k') f_{j, i, h, l}(C) J^H L^1 w_{l, h}, \]

and
\[
\phi(J_1) \sum_{(j, i, h, l) \in \Lambda_N} (2k' + 1)l I(k') f_{j, i, h, l}(C) J^H L^1 w_{l, h} = 0
\]

\[= \phi(J_1) \sum_{(j, i, h, l) \in \Lambda_N} h(k') f_{j, i, h, l}(C) J^H L^1 w_{l, h}. \]

Combining the above two formulae with the assumption that \(\phi\) is nonsingular, we obtain
\[
\sum_{(j, i, h, l) \in \Lambda_N} (2k' + 1)l I(k') f_{j, i, h, l}(C) J^H L^1 w_{l, h} = 0
\]

and
\[
\sum_{(j, i, h, l) \in \Lambda_N} h(k') f_{j, i, h, l}(C) J^H L^1 w_{l, h} = 0,
\]

which are absurd, proving the result.

**Subcase 2.** \(N' < N\).

In this subcase, note that \(l = 0\) for those \((j, i, h, l)\) satisfying \(N' < |(j, i, h, l)| \leq N\). Thus, we have
\[
\begin{align*}
\sum_{|\langle j, i, h \rangle| < N} f_{j, i, h, 1}(C)^{H_{k, i}}L_{j, h}^1w_{\phi} 
+ \sum_{\langle j, i, h \rangle \in \Lambda_N} f_{j, i, h, 1}(C)^{H_{k, i}}L_{j, h}^1w_{\phi} 
+ \sum_{\langle j, i, h, 1 \rangle \in \Lambda_N} f_{j, i, h, 1}(C)^{H_{k, i}}L_{j, h}^1w_{\phi} 
+ \sum_{N'' < |\langle j, i, h \rangle| \leq N} f_{j, i, h}(C)^{H_{k, i}}L_{j, h}^1w_{\phi}. 
\end{align*}
\] (3.11)

If \( h = 0 \) for any \( \langle j, i, h \rangle \) with \( N'' < |\langle j, i, h \rangle| \leq N \) and \( f_{j, i, h, 1}(C) \neq 0 \). It follows from similar arguments as those for the Subcase 1 that the conclusion holds. If there exists some \( h \neq 0 \) for some \( \langle j, i, h \rangle \) with \( N'' < |\langle j, i, h \rangle| \leq N \) and \( f_{j, i, h, 1}(C) \neq 0 \). Denote
\[
N'' = \max \{|\langle j, i, h \rangle| \mid N'' < |\langle j, i, h \rangle| \leq N, \ h \neq 0, \ f_{j, i, h}(C) \neq 0\}.
\]

Also, set
\[
\Lambda_{N''} = \{\langle j, i, h \rangle \in P^3 \mid |\langle j, i, h \rangle| = N'', \ f_{j, i, h, 1}(C) \neq 0\},
\]
\[
k'' := \min \{m \in \mathbb{Z}^+ \mid (h(m) \neq 0 \mid |\langle j, i, h \rangle| = N'' \ and \ f_{j, i, h}(C) \neq 0\}
\]
and
\[
K'' := \max \{|\langle j, i, h \rangle| \mid \langle j, i, h \rangle \in \Lambda_{N''}\}.
\]

Also, note that \( N'' \leq N \) and \( \Lambda_{N''} \) is a nonempty set. If \( N'' = N \), we write \( w \) as
\[
\begin{align*}
w &= \sum_{\langle j, i, h \rangle \in \Lambda_N} f_{j, i, h}(C)^{H_{k, i}}L_{j, h}^1w_{\phi} 
+ \sum_{\langle j, i, h \rangle \in \Lambda_N} f_{j, i, h}(C)^{H_{k, i}}L_{j, h}^1w_{\phi} 
+ \sum_{\langle j, i, h \rangle \in \Lambda_N} f_{j, i, h}(C)^{H_{k, i}}L_{j, h}^1w_{\phi} 
+ \sum_{\langle j, i, h \rangle \in \Lambda_N} f_{j, i, h}(C)^{H_{k, i}}L_{j, h}^1w_{\phi}.
\end{align*}
\]

If \( N'' < N \), then \( h = 0 \) for those \( \langle j, i, h \rangle \) satisfying \( N'' < |\langle j, i, h \rangle| \leq N \). Now we rewrite \( w \) as
\[
\begin{align*}
w &= \sum_{\langle j, i, h \rangle \in \Lambda_{N''}} f_{j, i, h}(C)^{H_{k, i}}L_{j, h}^1w_{\phi} 
+ \sum_{\langle j, i, h \rangle \in \Lambda_{N''}} f_{j, i, h}(C)^{H_{k, i}}L_{j, h}^1w_{\phi} 
+ \sum_{\langle j, i, h \rangle \in \Lambda_{N''}} f_{j, i, h}(C)^{H_{k, i}}L_{j, h}^1w_{\phi} 
+ \sum_{\langle j, i, h \rangle \in \Lambda_{N''}} f_{j, i, h}(C)^{H_{k, i}}L_{j, h}^1w_{\phi}.
\end{align*}
\]

In either case, using the same arguments as those in Subcase 1, we obtain
\[
(I_{k'' + 1} - \phi(I_{k'' + 1}))w = w_3 + \phi(I_1) \sum_{\langle j, i, h \rangle \in \Lambda_{N''}} f_{j, i, h}(C)^{H_{k''}}L_{j, h}^1w_{\phi},
\]
where \( w_3 = w_3' + w_3'' \) with \( \deg (w_3') < N'' - k'' \) and \( \deg (w_3'') < K'' - 1 \). Meanwhile, \( \bar{h} \) satisfies \( \bar{h}(m) = h(m) \) if \( m \neq k'' \) and \( \bar{h}(k'') = h(k'') - 1 \). It follows from \( \phi(I_1) \neq 0 \) that \( (I_{k'' + 1} - \phi(I_{k'' + 1}))w_3' \neq 0 \).

**Case 2.** \( l = 0 \) for any \( \langle j, i, h, l \rangle \) with \( f_{j, i, h, l, 1}(C) \neq 0 \).

In this case, we simply write \( \langle j, i, h, 0 \rangle \) as \( \langle j, i, h \rangle \), that is,
\[
w = \sum_{\langle j, i, h \rangle \in P^3} f_{j, i, h}(C)^{H_{k, i}}L_{j, h}^1w_{\phi}.
\]

Set
\[ k := \min\{m \in \mathbb{Z}_+ | h(m) \neq 0 \text{ or } i(m) \neq 0 \text{ or } j(m) \neq 0 \text{ for some } (j, i, h) \in \Lambda_N \}. \]

**Subcase 1.** There exists some \((j, i, h) \in \Lambda_N\) with \(h(k) \neq 0\).

In this situation, set \(K := \max \{h(k) | (j, i, h) \in \Lambda_N\}\). By a similar discussion as that in Subcase 1 of Case 1, we can prove that
\[
(I_{k+1} - \phi(I_{k+1}))w = w_4 - \phi(I_1) \sum_{(j, i, h, l) \in \Lambda_N} h(k)f_{j, i, h, l}(C)j^lH^hI^1w_\phi,
\]
where \(w_4 = w'_4 + w''_4\) with max deg \((w'_4) < N - k\) and max \(L_{j, h, l}(w'_4) < K - 1\). Meanwhile, \(\tilde{h}\) satisfies that \(\tilde{h}(m) = h(m)\) if \(m \neq k\) and \(\tilde{h}(k) = h(k) - 1\). It follows that from \(\phi(I_1) \neq 0\) that \((I_{k+1} - \phi(I_{k+1}))w \neq 0\).

**Subcase 2.** There exists some \((j, i, h) \in \Lambda_N\) with \(i(k) \neq 0\), and \(h(k) = 0\) for any \((j, i, h) \in \Lambda_N\).

We have
\[
(L_{k+1} - \phi(L_{k+1}))w = \sum_{(j, i, h) \in \Lambda_N} f_{j, i, h}(C)[L_{k+1}, j^1H^h]w_\phi
+ \sum_{(j, i, h) \in \Lambda_N} f_{j, i, h}(C)[L_{k+1}, j^1H^h]w_\phi
+ \sum_{(j, i, h) \in \Lambda_N} f_{j, i, h}(C)[L_{k+1}, j^1H^h]w_\phi
+ \sum_{(j, i, h) \in \Lambda_N} f_{j, i, h}(C)[L_{k+1}, j^1H^h]w_\phi.
\]

By using Lemma 3.2(1) to the first summand, we know that its degree is strictly smaller than \(N - k\). For the second summand, note that \(j(s) = i(s) = h(s) = 0\) for \(0 \leq s \leq k\), we have
\[
[L_{k+1}, j^1H^h] = [L_{k+1}, j^1]H^h + j^1[L_{k+1}, I^1]H^h + j^1[L_{k+1}, H^h] \in U(B^-).
\]
Thus, its degree is also strictly smaller than \(N - k\). Now using Lemma 3.2(iii) to the third summand and last summand, respectively, we know that it is of the form
\[
v_2 - \phi(I_1) \sum_{(j, i, h) \in \Lambda_N} (2k + 1)j(k)f_{j, i, h}(C)j^1H^h w_\phi
- \phi(I_1) \sum_{(j, i, h) \in \Lambda_N} (2k + 1)i(k)f_{j, i, h}(C)j^1H^h w_\phi
- \phi(I_1) \sum_{(j, i, h) \in \Lambda_N} (2k + 1)j(k)f_{j, i, h}(C)j^1H^h w_\phi
= v_2 - \phi(I_1) \sum_{(j, i, h) \in \Lambda_N} (2k + 1)i(k)f_{j, i, h, l}(C)j^1H^h w_\phi
- \phi(I_1) \sum_{(j, i, h) \in \Lambda_N} (2k + 1)j(k)f_{j, i, h}(C)j^1H^h w_\phi,
\]
where \( \text{max deg} (v_s) < N - k \). Meanwhile, \( \bar{i} \) and \( \bar{j} \) satisfy that \( \bar{i}(m) = i(m), \bar{j}(m) = j(m) \) if \( m \neq k \) and \( \bar{i}(k) = i(k) - 1, \bar{j}(k) = j(k) - 1 \). Putting our observation together gives

\[
(L_{k+1} - \phi(L_{k+1}))w = w_5 - \phi(I_1) \sum_{(j, i, h) \in \Lambda_N \atop i(k) \neq 0} (2k + 1)i(k)f_{j, i, h}(C)\bar{j}^i I^h w_\phi
\]

\[
- \phi(J_1) \sum_{(j, i, h) \in \Lambda_N \atop j(k) \neq 0} (2k + 1)j(k)f_{j, i, h}(C)\bar{j}^i I^h w_\phi.
\]

(3.12)

Similarly, we have

\[
(H_{k+1} - \phi(H_{k+1}))w = w_6 + \phi(I_1) \sum_{(j, i, h) \in \Lambda_N \atop i(k) \neq 0} i(k)f_{j, i, h}(C)\bar{j}^i I^h w_\phi
\]

\[
- \phi(J_1) \sum_{(j, i, h) \in \Lambda_N \atop j(k) \neq 0} j(k)f_{j, i, h}(C)\bar{j}^i I^h w_\phi.
\]

(3.13)

Note that \( \text{max deg} (w_s) < N - k \) for \( s = 5, 6 \). Also, suppose on the contrary that for any \( E_n \in \{L_n, H_n, I_n, J_n | n \in \mathbb{N} \} \), we have \( (E_n - \phi(E_n))w = 0 \). In particular,

\[
(L_{k+1} - \phi(L_{k+1}))w = (H_{k+1} - \phi(H_{k+1}))w = 0.
\]

Now (3.12) and (3.13) force

\[
\phi(I_1) \sum_{(j, i, h) \in \Lambda_N \atop i(k) \neq 0} i(k)f_{j, i, h}(C)\bar{j}^i I^h w_\phi = -\phi(J_1) \sum_{(j, i, h) \in \Lambda_N \atop j(k) \neq 0} j(k)f_{j, i, h}(C)\bar{j}^i I^h w_\phi
\]

and

\[
\phi(I_1) \sum_{(j, i, h) \in \Lambda_N \atop i(k) \neq 0} i(k)f_{j, i, h}(C)\bar{j}^i I^h w_\phi = \phi(J_1) \sum_{(j, i, h) \in \Lambda_N \atop j(k) \neq 0} j(k)f_{j, i, h}(C)\bar{j}^i I^h w_\phi.
\]

Using the above two formulae along with the assumption that \( \phi \) is nonsingular, we have

\[
\sum_{(j, i, h) \in \Lambda_N \atop i(k) \neq 0} i(k)f_{j, i, h}(C)\bar{j}^i I^h w_\phi = 0,
\]

which is impossible, proving the result.

**Subcase 3.** There exists some \( (j, i, h) \in \Lambda_N \) with \( j(k) \neq 0 \), and \( h(k) = i(k) = 0 \) for any \( (j, i, h) \in \Lambda_N \).

By a similar discussion as that in Subcase 2, we can show that the conclusion holds. We complete the proof. \( \square \)

**Proposition 3.5.** Suppose \( \phi \) is nonsingular and \( \xi \in \mathbb{C}^3 \). Then any Whittaker vector in \( \bar{L}_{\phi, \xi} \) is of the form \( cw_\phi \) for some \( c \in \mathbb{C} \).

**Proof.** It is easy to see that the set \( \{\bar{j}^i I^h \bar{l}w_\phi | (j, i, h, l) \in P^4 \} \) forms a basis of \( \bar{L}_{\phi, \xi} \). Then we can use the same arguments as those in **Proposition 3.4** to complete the proof by replacing the polynomials \( f_{j, i, h, l}(C) \) with scalars \( f_{j, i, h, l} \) whenever necessary. \( \square \)
4. Simplicity and classification of generic Whittaker modules

4.1. Generic Whittaker modules of nonsingular type

In this subsection, we study generic Whittaker modules of nonsingular type. We first have the following result on simplicities of generic Whittaker modules of nonsingular type.

**Proposition 4.1.** Suppose the Lie algebra homomorphism \( \phi : \mathcal{G}^+ \cong \tilde{\mathcal{G}}^+ \to \mathbb{C} \) is nonsingular, and \( \xi \in \mathbb{C}^3 \). Then the generic Whittaker \( \mathcal{G} \)-module \( L_\phi \) (resp. \( \tilde{\mathcal{G}} \)-module \( \tilde{L}_{\phi, \xi} \)) is simple.

**Proof.** It follows from Lemma 2.1(iv), and Propositions 3.4 and 3.5. \( \square \)

The following result presents a classification of simple \( \mathcal{G} \)-Whittaker (resp. \( \tilde{\mathcal{G}} \)-Whittaker) modules of nonsingular type.

**Theorem 4.2.** Let \( \phi : \mathcal{G}^+ \cong \tilde{\mathcal{G}}^+ \to \mathbb{C} \) be a Lie algebra homomorphism of nonsingular type. Then the following statements hold.

(i) Let \( S \) be a simple Whittaker \( \mathcal{G} \)-module of type \( \phi \). Then \( S \cong L_\phi \).

(ii) Let \( \tilde{S} \) be a simple Whittaker \( \tilde{\mathcal{G}} \)-module of type \( \phi \). Then \( \tilde{S} \cong \tilde{L}_{\phi, \xi} \) for some \( \xi \in \mathbb{C}^3 \). Moreover, the set \( \{ \tilde{L}_{\phi, \eta} | \eta \in \mathbb{C}^3 \} \) exhausts all pairwise non-isomorphic simple \( \tilde{\mathcal{G}} \)-Whittaker modules of type \( \phi \).

**Proof.**

(i) Follows from Proposition 4.1 and Remark 2.4.

(ii) Let \( w \in \tilde{S} \) be a nonzero cyclic Whittaker vector of type \( \phi \). By Schur’s lemma, the center of \( U(\tilde{\mathcal{G}}) \) acts on \( \tilde{S} \) by scalars. This implies that there exists \( \tilde{\xi} = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3 \) such that \( C_s v = \xi_s v, \ \forall \ v \in \tilde{S}, 1 \leq s \leq 3 \). Meanwhile from the universal property of \( \tilde{M}(\phi) \), there exists a surjective homomorphism \( \pi : \tilde{M}(\phi) \to \tilde{S} \) such that \( \pi(w_\phi) = w \). Then

\[
\pi \left( \sum_{s=1}^{3} (C_s - \xi_1)\tilde{M}(\phi) \right) = \sum_{s=1}^{3} (C_s - \xi_1)\pi(\tilde{M}(\phi)) = \sum_{s=1}^{3} (C_s - \xi_1)\tilde{S} = 0.
\]

So,

\[
\sum_{s=1}^{3} (C_s - \xi_1)\tilde{M}(\phi) \subseteq \ker \pi \subseteq \tilde{M}(\phi).
\]

Since \( \tilde{L}_{\phi, \xi} \) is simple by Proposition 4.1 and \( \ker \pi \neq \tilde{M}(\phi) \), we have \( \ker \pi = \sum_{s=1}^{3} (C_s - \xi_1)\tilde{M}(\phi) \). Hence the first statement follows. The second statement follows from Corollary 3.3. We complete the proof. \( \square \)

4.2. Generic Whittaker modules of singular type

Suppose the Lie algebra homomorphism \( \phi : \mathcal{G}^+ \cong \tilde{\mathcal{G}}^+ \to \mathbb{C} \) is singular, and \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3 \). In this subsection, we study generic Whittaker modules of type \( \phi \). We will show that all generic Whittaker modules of singular type are reducible. For that, let

\[
\Omega_\phi := U(\mathcal{G})(\mathcal{I} \oplus \mathcal{J})L_\phi, \quad \Gamma_\phi := U(\mathcal{G})\mathcal{I}L_\phi, \quad \Upsilon_\phi := U(\mathcal{G})\mathcal{J}L_\phi
\]

and

\[
\tilde{\Omega}_{\phi, \xi, c} := U(\tilde{\mathcal{G}})(\mathcal{I} \oplus \mathcal{J})\tilde{L}_{\phi, \xi} + (H_0 - c)\tilde{L}_{\phi, \xi}, \quad \tilde{\Gamma}_{\phi, \xi} := U(\tilde{\mathcal{G}})\mathcal{I}\tilde{L}_{\phi, \xi}, \quad \tilde{\Upsilon}_{\phi, \xi} := U(\tilde{\mathcal{G}})\mathcal{J}\tilde{L}_{\phi, \xi}
\]

where \( c \in \mathbb{C} \).
Proposition 4.3. Keep notations as above. The following statements hold.

(i) If $\phi(I_1) = \phi(J_1) = 0$, then the generic Whittaker $G$-module $L_\phi$ has a proper submodule $\Omega_\phi$, and the corresponding quotient $L_\phi/\Omega_\phi$ is simple if and only if $\phi(H_1) \neq 0$. Moreover, the generic Whittaker $\tilde{G}$-module $\tilde{L}_\phi$ has proper submodules $\tilde{\Omega}_\phi, z, \phi$ for $c \in \mathbb{C}$. Let $\tilde{L}_\phi, z, \phi := \tilde{L}_\phi, z, \phi / \tilde{\Omega}_\phi, z, \phi$ be the corresponding quotient module. Then we have

(a) If $\xi_3 \neq 0$, then $\tilde{L}_\phi, z, \phi$ is simple if and only if $2\xi_3\phi(L_2) + \phi(H_1)^2 - 4\xi_3\phi(H_1) \neq 0$ or $\xi_3\phi(L_1) + (c - \xi_2)\phi(H_1) \neq 0$.

(b) If $\xi_3 = 0$, then $\tilde{L}_\phi, z, \phi$ is simple if and only if $\phi(H_1) \neq 0$.

(ii) If $\phi(I_1) = 0$, $\phi(J_1) \neq 0$, then the generic Whittaker $G$-module $L_\phi$ (resp. $\tilde{G}$-module $\tilde{L}_\phi$) has a proper submodule $\Gamma_\phi$ (resp. $\tilde{\Gamma}_\phi$).

(iii) If $\phi(I_1) \neq 0$, $\phi(J_1) = 0$, then the generic Whittaker $G$-module $L_\phi$ (resp. $\tilde{G}$-module $\tilde{L}_\phi$) has a proper submodule $\Upsilon_\phi$ (resp. $\tilde{\Upsilon}_\phi$).

Proof.

(i) It follows from direct computations that

$$\Omega_\phi = \text{span}_\mathbb{C}\{H^jL^jL^jw_\phi \mid (j, i, h, l) \in P^4, \ |i| + |j| > 0\},$$

and

$$\tilde{\Omega}_\phi, z, \phi, c = \text{span}_\mathbb{C}\{H^jL^jL^jw_\phi, (H_0 - c)H^jL^jw_\phi \mid (j, i, h, l) \in P^4, \ |i| + |j| > 0\}$$

are proper submodules of $L_\phi$ and $\tilde{L}_\phi, z, \phi$, respectively. Then $L_\phi/\Omega_\phi$ is a Whittaker $G$-module with trivial action by $I \oplus J$, so that it is a universal Whittaker module over the Heisenberg-Virasoro algebra $\mathcal{H}V$ with trivial action by $C_i$ for $1 \leq i \leq 3$. Hence, it follows from [19, Theorem 15(2)] that $L_\phi/\Omega_\phi$ is simple if and only if $\phi(H_1) \neq 0$. Similar arguments yield the second assertion by [19, Theorem 15].

(ii) It follows from direct computations that

$$\Gamma_\phi = \text{span}_\mathbb{C}\{H^jL^jL^jw_\phi \mid (j, i, h, l) \in P^4, \ |i| > 0\}$$

is a proper submodules of $L_\phi$. Similarly, $\tilde{\Gamma}_\phi, z, \phi$ is a a proper submodules of $\tilde{L}_\phi, z, \phi$.

(iii) It follows from direct computations that

$$\Upsilon_\phi = \text{span}_\mathbb{C}\{H^jL^jL^jw_\phi \mid (j, i, h, l) \in P^4, \ |j| > 0\}$$

is a proper submodules of $L_\phi$. Similarly, $\tilde{\Upsilon}_\phi, z, \phi$ is a a proper submodules of $\tilde{L}_\phi, z, \phi$.

We finally have the following result asserting that all generic Whittaker modules of singular type are reducible.

Theorem 4.4. Suppose the Lie algebra homomorphism $\phi : G^+ \cong \tilde{G}^+ \rightarrow \mathbb{C}$ is singular, and $\xi \in \mathbb{C}^3$. Then the generic Whittaker $G$-module $L_\phi$ and $\tilde{G}$-module $\tilde{L}_\phi, z, \phi$ are reducible.

As a direct consequence of Proposition 4.1 and Theorem 4.4, we have the following result which provides a sufficient and necessary condition for a generic Whittaker module to be simple.

Corollary 4.5. Suppose $\phi : G^+ \cong \tilde{G}^+ \rightarrow \mathbb{C}$ is a Lie algebra homomorphism, and $\xi \in \mathbb{C}^3$. Then the generic Whittaker $G$-module $L_\phi$ (resp. $\tilde{G}$-module $\tilde{L}_\phi, z, \phi$) is simple if and only if $\phi$ is nonsingular, i.e., $\phi(I_1)\phi(J_1) \neq 0$.

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