SMALL AMPLITUDE WEAK ALMOST PERIODIC SOLUTIONS FOR THE 1D NLS

LUCA BIASCO, JESSICA ELISA MASSETTI, AND MICHELA PROCESI

Abstract. All the almost periodic solutions for non integrable PDEs found in the literature are very regular (at least $C^\infty$) and, hence, very close to quasi periodic ones. This fact is deeply exploited in the existing proofs. Proving the existence of almost periodic solutions with finite regularity is a main open problem in KAM theory for PDEs. Here we consider the one dimensional NLS with external parameters and construct almost periodic solutions which have only Sobolev regularity both in time and space. Moreover many of our solutions are so only in a weak sense. This is the first result on existence of weak, i.e. non classical, solutions for non integrable PDEs in KAM theory.

Contents

1. Introduction 1
2. Main results 6
3. Proof of Theorem 1 17
4. Functional setting 19
5. Degree decompositions, projections and normal forms 24
6. Proof of Theorems 3 and 4 27
7. Small divisors and Homological equation 29
8. Iterative Lemma and Proof of Theorem 5 39
9. Measure estimates 46
Appendix A. Technicalities 52
Appendix B. Topology, measure and continuous functions on infinite product spaces 58
References 62

1. Introduction

We present here the first result on existence of weak, i.e. non classical, solutions for non integrable PDEs in KAM theory. More precisely we study NLS equations on the circle finding almost periodic solutions, i.e. solutions which are limit (in the uniform topology in time) of time-quasi-periodic functions. We work on models with external parameters of the form:

$$(\text{NLS}_V) \quad iu_t + u_{xx} - V \ast u + f(|u|^2)u = 0, \quad u(t,x) = u(t,x + 2\pi),$$
where \( i = \sqrt{-1} \) and \( f(y) \) is real analytic in \( y \) in a neighborhood of \( y = 0 \) with \( f(0) = 0 \). Given \( V = (V_j)_{j \in \mathbb{Z}} \in [-1/4, 1/4]^{\mathbb{Z}} \subset \ell^\infty(\mathbb{R}) \), the Fourier multiplier \( V* \) is defined as the bounded\(^2\) linear operator \( V* : \mathcal{F}(\ell^1) \to \mathcal{F}(\ell^1) \) by

\[
u(x) = \sum_{j \in \mathbb{Z}} u_j e^{ijx} \mapsto (V* \nu)(x) = \sum_{j \in \mathbb{Z}} V_j u_j e^{ijx},
\]

where \( \mathcal{F}(\ell^1) \) is the Wiener algebra of \( 2\pi \)-periodic functions\(^3\) having Fourier coefficients in \( \ell^1 = \ell^1(\mathbb{C}) := \{u := (u_j)_{j \in \mathbb{Z}} \in \mathbb{C}^\mathbb{Z} : |u|_\ell^1 := \sum_{j \in \mathbb{Z}} |u_j| < \infty \} \), endowed with the corresponding norm \( |u(\cdot)|_{\mathcal{F}(\ell^1)} := |u|_{\ell^1} \). Note that the choice of the ball \([-1/4, 1/4]^{\mathbb{Z}} \) is rather arbitrary and any other ball is acceptable.

Besides classical solutions, namely functions \( u \) admitting continuous derivatives \( u_t \) and \( u_{xx} \) solving \((\text{NLS}_V)\), we are particularly interested in weak solutions according to the following

**Definition 1.1** (Global in time weak solutions). A function \( u : \mathbb{R}^2 \to \mathbb{C} \) which is \( 2\pi \)-periodic in \( x \) and such that the map \( t \mapsto u(t, \cdot) \in \mathcal{F}(\ell^1) \) is continuous is a weak solution of \((\text{NLS}_V)\) if for any smooth compactly supported function \( \chi : \mathbb{R}^2 \to \mathbb{R} \) one has

\[
\int_{\mathbb{R}^2} (-i\chi_t + \chi_{xx}) u - (V*u - f(|u|^2)u) \chi \, dx \, dt = 0.
\]

Note that according to our definition a weak solution is a continuous\(^4\) function on \( \mathbb{R}^2 \).

Local well posedness for equations as \((\text{NLS}_V)\) is well known even in lower regularity (see, e.g. [Bou93] and [Bou94]). In the context of integrable PDEs there are various results on weak almost-periodic solutions, we mention [KM18, KM17] for the KdV and mKdV, [GH17] for the Szego equation and [GKT] for the Benjamin-Ono. In the present paper we work close to the elliptic fixed point \( u = 0 \) and consider \((\text{NLS}_V)\) as a small (non integrable) perturbation of the (integrable) linear Schrödinger equation and prove the persistence of almost-periodic solutions.

**Theorem 1.** For almost every Fourier multiplier \( V \) there exist infinitely many small-amplitude weak almost-periodic solutions of \((\text{NLS}_V)\). Infinitely many of such solutions are not classical and infinitely many are classical.

Here by almost every we mean a full measure set with respect to the product probability measure on \([-1/4, 1/4]^{\mathbb{Z}} \). The Borel sets on such measure are respect to the product topology on \([-1/4, 1/4]^{\mathbb{Z}} \) (see Appendix B). This theorem will be derived in Section 3 from **Theorem 2** and **Theorem 3** stated below.

---

1. Clearly the operator norm is \( |V|_{\mathcal{F}(\ell^1), \mathcal{F}(\ell^1)} = |V|_\infty := \sup_j |V_j| \).
2. More precisely \( (V*u)(t, x) := (V*u(t, \cdot))(x) \) for every \( t \in \mathbb{R} \).
3. Note that all such functions are continuous.
4. Indeed \( |u(t, x) - u(t_0, x_0)| \leq |u(t, \cdot) - u(t_0, \cdot)|_{\mathcal{F}(\ell^1)} + |u(t_0, x) - u(t_0, x_0)| \).
Before describing in more detail our results, we briefly motivate why one is interested in almost-periodic solutions and particularly in those of low regularity.

In the study of finite dimensional nearly-integrable Hamiltonian systems KAM Theorems play a pivotal role, casting a light that illuminates the picture quite clearly. Indeed under some (generic) non-degeneracy assumptions most of the phase space of such systems is foliated by maximal invariant tori, whose dimension is half of the one of the whole space. In particular the system is not ergodic and the majority of initial data give rise to quasi-periodic solutions that densely fill some invariant torus and are, therefore, perpetually stable. Possible chaotic behavior is restricted to a set of small measure. On the other hand, in the infinite dimensional setting, for example in the PDEs case, the general picture is so far rather obscure and the main questions still remain unanswered. The typical solutions of an infinite dimensional integrable system are the almost-periodic ones that lie on maximal infinite dimensional invariant tori; what is their fate under perturbation? Is it still true that the majority of initial data produce perpetually stable solutions? The meaning and possible answers to these questions are deeply related to the regularity of the phase space in which one looks for solutions. These questions are completely open and only very partial answers are available. In particular the known results are on spaces of very high regularity (contained in $C^\infty$). On the other hand, finite regularity solutions appear naturally and are widely studied in the PDE-context. For instance the persistence of Sobolev almost periodic solutions is the first open problem on KAM for PDEs mentioned by S. Kuksin in [Kuk04] (see e.g. Problem 7.1). Even though our solutions are quite special, being mainly supported on sparse set of Fourier’s modes, the present work can be seen as a first step forward in this direction.

As is usual in KAM Theory, a key point in the study of the dynamics in the neighborhood of these invariant tori, consists in controlling the spectral properties of appropriate linear operators and dealing with the connected problem of small-divisor. The main difficulty is then to guarantee that suitable arithmetic (Diophantine) conditions on the frequencies are fulfilled all along the scheme, so that small-divisors can be bounded accordingly and the almost-periodic dynamics controlled. Extending Diophantine (or similar) estimates, which strongly depend on the dimension, to the infinite dimension is not straightforward. In fact all the results on almost-periodic solution for PDEs only deal with special model cases and consider parameter dependent equations.

Parameter vs. regularity issues. Quoting Bourgain [Bou05]: “the role of this parameter is essential to ensure appropriate non-resonance properties of the (modulated) frequencies along the iteration.”

As we shall see below, in all the existent literature, the external parameters have rather low regularity (giving a good frequency modulation) while
the almost periodic solutions have an extremely fast decay in their Fourier coefficients, which approach zero super-exponentially, exponentially or sub-exponentially (Gevrey). This means that those solutions are “very close” to quasi-periodic ones. For example Pöschel in [Pö02] studies an NLS with a multiplicative potential in $L^2$ (producing an infinite set of free parameters) and smoothing non-linearity and constructs almost-periodic solutions iteratively, through successive small perturbations of finite (but at each step higher) dimensional invariant tori. This leads to a very strong compactness property: in order to overcome the dependence of the KAM estimates on the dimension, the distances of these tori have to shrink super-exponentially, this leading to very regular solutions. See also [GX13] for a generalization of Pöschel’s approach to the analytic category, by using Toeplitz-Lipschitz function techniques. In his pioneering work [Bou05] on the quintic NLS with Fourier multipliers (providing external parameters in $\ell^\infty$), Bourgain proposed a different approach which does not rely on approximations by quasi-periodic functions by working directly in Fourier space, and relying on a Diophantine condition which is tailored for the infinite dimension. For most choices of the parameters, this leads to the construction of almost periodic invariant tori which support Gevrey solutions (see also [BMP21], [CY20]). In any case, all the above results are valid for most choices of the (infinitely many) external parameters. An open question is whether one can achieve a similar result by modulating finitely many parameters. This would lead to applications to more natural PDEs.

Of course, the most challenging scenario is represented by a fixed PDE, where the parameters should be “extracted” from the initial data.

**Almost-periodic solutions for a fixed PDE.** Again, quoting Bourgain “In the absence of external parameters, these (non resonance, ed.) conditions need to be realized from amplitude-frequency modulation and suitable restriction of the action-variables. This problem is harder. Indeed, a fast decay of the action-variables (enhancing convergence of the process) allows less frequency modulation and worse small divisors”.

Summarizing, the possibility of constructing almost-periodic solutions for a fixed PDE, i.e. “eliminating” the external parameters through amplitude-frequency modulation, appears to be intimately related to the regularity issues. Moreover, in the context of completely integrable PDEs, the invertibility of the amplitude-frequency map is known only in spaces of very low regularity (see [KMMT16]). It then becomes fundamental to look for almost-periodic solutions in lower regularity spaces if we want to bypass the introduction of external parameters. While it is possible to lower the regularity beyond the Gevrey class (see [Con]) up to now all the known solutions are at least $C^\infty$. On the other hand finding finite regularity solutions appears to be a very difficult question, due to extremely small divisors.

Comparable difficulties in tackling the Sobolev case appear in Birkhoff Normal Form theory for PDEs leading to two different types of results. In the analytic or Gevrey case one has sub-exponential stability times (see [FG13]
and \[ \text{CMW20} \]), whereas in the Sobolev case the stability times appear to be controlled by the Sobolev exponent (see \[ \text{BG06, FI, BD18, BMP20a, FM22} \). The counterpart of total and long time stability results is the construction of unstable trajectories, which undergo growth of the Sobolev norms, see \[ \text{Bou96, CKS}_10, \text{GK15, GHH}_18, \text{GGMP21} \].

**Comparison with the quasi-periodic case.** In the context of quasi-periodic solutions there is a wide literature regarding the finite regularity case (starting from the seminal paper \[ \text{Kuk88} \). A good strategy, which works also for fully-nonlinear PDEs, is to apply a Nash-Moser scheme and prove tame estimates on the inverse of the linearized equation at an approximate solution. This method was proposed in \[ \text{BB13} \] (generalizing the seminal works \[ \text{CW93, Bou99} \) concerning the analytic case) via multi-scale analysis, see also \[ \text{BB15} \) or \[ \text{BBM16} \) for a reducibility approach.

Note that in the existing KAM literature finite regularity solutions are always classical and due to finite regularity nonlinearities. Typically one looks for an invariant embedded finite dimensional torus in a fixed phase space of \( x \)-dependent functions. Then the regularity of the embedding is clearly related to the one of the non-linearity, which, in the Nash-Moser schemes, is required to be high enough increasing with the dimension of the torus. This is not surprising since in \( n \)-dimensional Hamiltonian systems the minimal regularity of the Hamiltonian vector field in order to construct a maximal Diophantine torus is essentially \( C^n \) (see \[ \text{Her86, Kou20, Pöts} \). For analytic nonlinearities one obtains analytic (or, at least, \( C^\infty \) embedded tori and hence smooth in time solutions. By bootstrap arguments, smoothness in space follows. We finally stress that in the integrable case one can construct even periodic solutions with very low regularity, see for instance \[ \text{GKT} \).

In comparison, most of the almost-periodic literature concentrates on the construction of the solutions rather than the invariant objects. Inspired by \[ \text{Bou05} \), a conceptual novelty of \[ \text{BMP21} \) was to look directly for an infinite dimensional invariant torus. The strategy developed in \[ \text{BMP21} \) applies also to non maximal tori both of infinite and finite dimension. The persistence result was achieved through an abstract normal form theorem “à la Herman” (in finite dimensional systems see \[ \text{Mas19, Mas18} \), whose estimates are uniform in the dimension of the considered torus (see \[ \text{BMP21 Theorems 3 and 7.1} \).

Note that in looking for infinite dimensional tori the topology of the phase space plays a fundamental role. In particular the analyticity of the embedded torus does not necessarily imply the analyticity in time of the solution, at least if the frequencies are unbounded as it is typical in PDEs.

**Small divisors for infinite frequencies in finite regularity.** As discussed above, lowering the regularity of the phase space requires dealing with very

---

\[ ^5 \text{In the infinite-dimensional case, whether this is an embedding depends strongly on the chosen topology. See discussion in Subsection 2.1} \]
small divisors. In order to apply the general approach of [BMP21] in a phase space with finite regularity one needs to look at the problem from a novel perspective and introduce new ideas. The starting point is to look for special tori which are approximately supported, in Fourier space, on a sparse subset of \( \mathbb{Z} \) called “tangential sites” (see Definition 2.2), then prove that our solutions are supported on such sites, see (3.2), up to a close to identity change of variables. The crucial fact is that the choice of tangential sites provides an extra set of parameters which can be used, for instance, in order to avoid resonances or simplify small divisor estimates (see for instance [PP15], [HP17]). In fact, we prove that an appropriate choice of the tangential sites, see definition 2.2, allows us to impose very strong Diophantine conditions, see Definition 2.3, so that our small divisors can be controlled similarly to the Gevrey case of [Bou05,BMP21].

Having constructed an invariant torus contained in the phase space, we show that it is the support of weak almost periodic solutions (according to Definition 1.1). Finally we prove that many tori support almost periodic solutions which are not classical.

Acknowledgments. We wish to thank D. Bambusi, M. Berti, L. Corsi, R. Feola, E. Haus, T. Kappeler, J.-P. Marco, and A. Maspero for useful discussions and suggestions. We are also grateful to the anonymous referees for their careful reading of the paper as well as for pointing out some bugs in the first version; their suggestions greatly helped us to improve the exposition.

The authors are partially supported, by PRIN 2020XB3EFL, Hamiltonian and Dispersive PDEs”. J.E. Massetti and M. Procesi acknowledge the support of the INdAM-GNAMPA grant “Spectral and dynamical properties of Hamiltonian systems”. J.E. Massetti acknowledges the support of the INdAM-GNAMPA grant “Chaotic and unstable behaviors of infinite-dimensional dynamical systems”.

2. Main results

It is convenient to exploit the Hamiltonian nature of (\( \text{NLS}_V \)). As was shown by Bourgain in [Bou05] it is most convenient to use a product space as phase space. To this purpose we introduce the scale of Banach spaces

\[
\|u\|_p := \left\{ u := (u_j)_{j \in \mathbb{Z}} \in \ell^1(\mathbb{C}) : \|u\|_p := \sup_{j \in \mathbb{Z}} |u_j|^{\lfloor j \rfloor^p} < \infty \right\}, \quad p > 1.
\]

where \( \lfloor j \rfloor := \max\{2, |j|\} \). We look for solutions in the Fourier-Lebesgue space \( \mathcal{F}(\mathbb{w}_p) \) of \( 2\pi \)-periodic functions whose Fourier coefficients belong to \( \mathbb{w}_p \). In the following we will identify functions and sequences writing, e.g.

\[\]
\[ \ell^1 \text{ instead of } \mathcal{F}(\ell^1) \text{ or } w_p \text{ instead of } \mathcal{F}(w_p) \] and so on. Moreover, we have the following standard immersion properties\footnote{For \( u \in w_p \), we have \( \|u\|_{L^\infty} \), \( \|u^{(k)}\|_{L^\infty} \leq \sum_j |j|^k |u_j| \leq \sum_j |j|^{k-p} |u_j| \). Moreover if \( u \in H^p \) then, by Parseval equality, \( \sum_j |j|^{2p} |u_j|^2 = \|u^{(p)}\|_{L^2}^2 \) and \( \sum_j |u_j|^2 = \|u\|_{L^2}^2 \), so that \( \sup_j |j|^{p} |u_j| \leq \|u^{(p)}\|_{L^2} \) and \( |u_0| \leq \|u\|_{L^2} \), proving the second inequality in (2.2).} For \( 0 \leq k < p - 1 \) we have \( C^p \subset H^p \subset w_p \subset C^k \) with estimate
\[ c \max\{\|u^{(k)}\|_{L^\infty}, \|u\|_{L^\infty}\} \leq |u|_{w_p} \leq 2 \max\{\|u^{(p)}\|_{L^2}, \|u\|_{L^2}\}, \]
where the first inequality holds for \( u \in w_p \) and the second one for \( u \in C^p \) and where \( c^{-1} := \sum_j |j|^{k-p} \).

We endow \( w_p \subset \ell^2 \) with the symplectic structure \( i \sum_j du_j \wedge d\bar{u}_j \) inherited from \( \ell^2 \). We define the NLS Hamiltonian
\[ H_V(u) := \sum_{j \in \mathbb{Z}} (j^2 + V_j)|u_j|^2 + P, \quad \text{with} \]
(2.3)
\[ P := -\int_T F(|\sum_{j} u_j e^{ijx}|^2)dx, \quad F(y) := \int_0^y f(s)ds. \]

Note that \( w_p \) is a Banach algebra w.r.t convolution and moreover \( V^* : w_p \to w_p \) is a linear bounded operator with norm \( |V|_{\infty} \). This implies that (2.3) is an analytic function on \( w_p \cap H^1 \), see Proposition 4.2.

By (2.2), if \( p > 3/2 \) then \( w_p \subset H^1 \). Otherwise, for \( 1 < p < 3/2 \) we might have infinite energy. Anyway, for our purposes, we only need that the Hamiltonian vector field
\[ X^{(j)}_{H_V} := i\partial_{\bar{u}_j} H_V = i(j^2 + V_j)u_j + i\partial_{u_j} P \]
is well defined component-wise. For completeness we remark that the flow of \( H_V \) is locally well-posed on \( w_p \), for \( p > 1 \). Indeed in Proposition 4.2 (see also (4.4)) we will show that the Hamiltonian vector field \( X_P \) is a uniformly bounded map from a suitable ball around the origin of \( w_p \) to \( w_p \); therefore, by standard variation of constants, we obtain the local well-posedness on \( w_p \).

The proof of Theorem 1 is based on the construction of an analytic change of variables on the phase space \( w_p \), which conjugates the nonlinear dynamics to a linear one, supported on an invariant flat torus. The regularity of the solution in the original variables is then deduced from the dynamics on the flat torus. Since these issues require some care, we find it instructive to first discuss the linear case, where the key difficulties become transparent.

2.1. The linear case. When \( f = 0 \) the Hamiltonian reduces to its quadratic part \( \sum_{j \in \mathbb{Z}} (j^2 + V_j)|u_j|^2 \), so that the linear actions \( |u_j|^2 \) are constants of motions and the dynamics is
\[ u_j(t) = u_j(0)e^{\nu j t}, \quad j \in \mathbb{Z}, \quad \nu = (\nu_j)_{j \in \mathbb{Z}}, \quad \nu_j := j^2 + V_j. \]
Let us call \( S_e := \{ j \in \mathbb{Z} \text{ s.t. } u_j(0) \neq 0 \} \). If \( S_e \) is a finite set, the corresponding solution \( u(t, x) := \sum_{j \in \mathbb{Z}} u_j(t)e^{ijx} \) is quasi-periodic and analytic both in time
and space. If $S_\ast$ is infinite, the regularity of $u(t,x)$ obviously depends on the one of the initial datum. If $u(0) := (u_j(0))_{j \in \mathbb{Z}} \in \ell^1$ then $u(t,x)$ is a weak solution of \(\text{NLS}_V\) in the sense of (1.2). Moreover, such solution is a time almost-periodic function, being limit in $\mathcal{F}(\ell^1)$ of the quasi periodic truncations $\sum_{|j| \leq n} u_j(0)e^{i\omega_j t + ijx}$ as $n \to \infty$. Furthermore if $(u_j(0))_{j \in \mathbb{Z}} \in \mathcal{W}_p$ then $t \mapsto u(t,\cdot) \in \mathcal{F}(\mathcal{W}_p)$ and its truncations converge\(^8\) in $\mathcal{W}_p'$ with $1 < p' < p$. Note that if $p > 3$ we have a classical solutions (recall (2.2)). Otherwise if $p \leq 2$ one can easily produce non classical solutions. Indeed take any infinite subset $S_\ast$ of $\mathbb{Z}$ and set $u_j(0) = (j)^{-p}$ for $j \in S_\ast$ and $u_j(0) = 0$ otherwise. Then, if $p \leq 2$

$$\limsup_{|j| \to \infty} j^2 |u_j(t)| = \limsup_{|j| \to \infty} j^2 |u_j(0)| > 0,$$

so $u(t,\cdot) \notin H^2$. Finally, if $S_\ast = \mathbb{Z}$ and $p \leq 3/2$ then $u(t,\cdot) \notin H^1$.

The support of each solution is an invariant torus in the following sense. Given $I := (I_j)_{j \in \mathbb{Z}} \in \mathcal{W}_{2p}$ with $I_j \geq 0$. We define the flat torus

$$T_I := \{ u \in \mathcal{W}_p : |u_j|^2 = I_j \ \forall j \in \mathbb{Z} \}$$

and the set

$$S_I := \{ j \in \mathbb{Z} \text{ s.t. } I_j > 0 \}.$$

Then the map $i: \mathbb{T}^{S_I} \to T_I \subset \mathcal{W}_p$, $\varphi = (\varphi_j)_{j \in S_I} \mapsto i(\varphi)$, with

$$i_j(\varphi) := \sqrt{I_j} e^{i\varphi_j} \text{ for } j \in S_I, \quad i_j(\varphi) := 0 \text{ otherwise},$$

is an analytic immersion\(^9\) provided that we endow $\mathbb{T}^{S_I}$ with the $\ell^\infty$-topology (see Lemma B.2 below for details). By construction the linear dynamics on the torus $T_I$ is $\varphi \mapsto \varphi + \nu t$.

Since the map $t \mapsto \nu t \in \mathbb{T}^{S_I}$ is not even continuous (endowing $\mathbb{T}^{S_I}$ with the $\ell^\infty$-topology), the regularity of $t \mapsto i(\nu t)$ depends on the choice of the actions $I_j$. In the examples discussed above it is not continuous w.r.t. the strong\(^10\) topology, see also [KM18]. In contrast with the finite dimensional case, even if $\nu$ has rationally independent entries, it is not straightforward to understand whether this invariant torus is densely filled\(^11\) by the solution’s orbit or not. In fact, this issue is related to the asymptotic behavior of $\nu$. We will discuss this in Lemma B.3.

---

\(^8\)Note that the convergence does not necessarily hold in $\mathcal{W}_p$. See the examples below.

\(^9\)Assuming also that $\inf_j \sqrt{I_j} (j)^p > 0$ the map $i$ is an embedded torus. Otherwise $i$ is a homeomorphism on the image only if we endow both source and target spaces with the product topology.

\(^10\)Note that the map is continuous endowing $\mathcal{W}_p$ with the product topology, which coincides with the weak * topology on bounded sets.

\(^11\)In the product topology such solutions are always dense.
Definition 2.1 (Invariant flat torus). Let $I \in \mathcal{W}_p$ and $\mathcal{T}_I$ defined in \eqref{eq:2.5}. Consider a vector field $X = (X^j)_{j \in \mathbb{Z}} : \mathcal{T}_I \to \mathbb{C}^\mathbb{Z}$. We say that $\mathcal{T}_I$ is an invariant flat torus for $X$ with frequency $\nu \in \mathbb{R}^\mathbb{Z}$ if 

$$
\dot{u}_j(t) = X^j(u(t)), \quad u_j(t) := u_j(0)e^{i\nu_j t}, \quad \forall t \in \mathbb{R}, \quad |u_j(0)|^2 = I_j, \quad j \in \mathbb{Z}.
$$

Note that the frequency $\nu_j$ is uniquely defined only for $j \in \mathcal{S}_I$.

2.2. The nonlinear case. In the nonlinear setting, as explained in the introduction, our invariant tori are approximately supported on $\mathcal{S}_I$ contained in an appropriately sparse set $\mathcal{S} \subset \mathbb{Z}$.

Definition 2.2 (Admissible tangential sites). a) An infinite subset $\mathcal{S}$ of $\mathbb{N}$, referred to as the set of tangential sites in $\mathbb{N}$, is said to be admissible if there exists a a smooth strictly increasing function $s : [0, +\infty) \to [0, +\infty)$ with the following properties:

- $\mathcal{S} = s(\mathbb{N})$
- there exists $i_* \geq 2$ such that

$$
(2.8a) \quad s(i) \geq e^{(\log i)^{1+\eta}}, \quad \forall i \geq i_*, \quad \text{for some given } 1 < \eta \leq 2;
$$

$$
(2.8b) \quad s(i + i') \geq s(i) + s(i'), \quad s(hi) \geq hs(i), \quad \forall h \geq 1, \quad i, i' \geq i_*;
$$

$$
(2.8c) \quad s(i^2) \geq s^2(i), \quad \forall i \geq i_*.
$$

We denote by $i(s)$ the inverse function of $s(i)$.

b) An infinite subset $\mathcal{S}$ of $\mathbb{Z}$, referred to as the set of tangential sites, is said to be admissible if there exist a finite subset $\mathcal{S}_0$ of $\mathbb{Z}$ and two admissible sets of tangential sites $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{N}$ so that $\mathcal{S}$ is one of the following subsets

$$
\mathcal{S}_1, \quad -\mathcal{S}_1, \quad (-\mathcal{S}_1) \cup \mathcal{S}_0, \quad \mathcal{S}_1 \cup \mathcal{S}_0 \quad (-\mathcal{S}_1) \cup \mathcal{S}_2 \cup \mathcal{S}_0.
$$

Remark 2.1. Definition 2.2 above gives a quantitative control on how "sparse" the set $\mathcal{S}$ should be. In particular by \eqref{eq:2.8a} the function $s(i)$ must grow faster than any polynomial.

Examples of admissible tangential sites in $\mathbb{N}$. $\mathcal{S} := \{2^i : i \in \mathbb{N}\}$; choose\footnote{The function $s(i)$ is obviously not unique but its restriction to $\mathcal{S}$ is unique. The same holds for its inverse $i(s)$.} $s(x) = 2^x$. Or, given $\eta > 0$ as above, set $\mathcal{S} := \{ \lfloor e^{(\log i)^{1+\eta}} \rfloor : i \in \mathbb{N}\}$ ([$\cdot$] being the integer part), and choose $s(x) = \lfloor e^{(\log x)^{1+\eta}} \rfloor$, for any $x > 0$.

In what follows, given a sequence indexed over $\mathbb{Z}$ we systematically decompose it over $\mathcal{S}$ and $\mathcal{S}^c := \mathbb{Z} \setminus \mathcal{S}$; for example, for the potential $V$ we write

$$
V = (V_{\mathcal{S}}, V_{\mathcal{S}^c}), \quad V_{\mathcal{S}} := (V_j)_{j \in \mathcal{S}}, \quad V_{\mathcal{S}^c} := (V_j)_{j \in \mathcal{S}^c}.
$$
Analogously, we define the set of tangential frequencies as
\[
Q_S := \left\{ \nu = (\nu_j)_{j \in S} \in \mathbb{R}^S : |\nu_j - j^2| < \frac{1}{2} \right\}.
\]

The cube $Q_S$ inherits the product topology\[14\] and the product probability measure $\text{meas}_{Q_S}$ from $[-1/2, 1/2]^S$ through the map
\[
\mathcal{V}_S^* : Q_S \to [-1/2, 1/2]^S, \quad \text{where} \quad \mathcal{V}_S^*(\nu) := \nu_j - j^2, \quad j \in S.
\]

Moreover the above map also endows $Q_S$ with the $\ell^\infty$-metric, which induces a finer topology.

Finally, we denote by $B_r(w_p)$ the open ball of radius $r$ centered at the origin of $w_p$, and for $r > 0$ we define the tangential actions
\[
I(p, r) := \{ I \in B_r(w_p) : I_j = 0 \text{ for } j \in S^c, \ I_j \geq 0 \text{ for } j \in S \}.
\]

We are now ready to state our main dynamical result regarding the existence of invariant KAM tori, parameterized by $I, V_S$, and by the frequency $\nu$.

**Theorem 2.** Assume that $S \subset \mathbb{Z}$ is an admissible set of tangential sites, $p > 1$ and $r > 0$ is sufficiently small. Moreover assume that
\[
I \in \mathcal{I}(p, r) \quad \text{and} \quad V_S = (V_j)_{j \in S^c} \in [-1/4, 1/4]^{|S^c|}.
\]

Then there exists a Cantor-like set $C \subset Q_S$ (recall (2.9)) of positive measure such that for any frequency $\nu \in C$, there exists
\[
V_S = (V_j)_{j \in S} \in [-1, 1]^{|S|}
\]

and a symplectic diffeomorphism $\Phi$ analytic on a small ball in $w_p$ such that, for all $\nu \in C$,
\[
\mathcal{T}_I = \{ u \in w_p : |u_j|^2 = I_j \ \forall j \in \mathbb{Z} \}
\]

is an invariant flat torus of frequency $\nu$ (recall Definition 2.4) for the Hamiltonian vector field of $H_\nu \circ \Phi$, with $V = (V_S, V_{S^c})$.

Theorem 2 above contains cubes of three different sidelengths, $[-1/4, 1/4]^{|S^c|}$, $Q_S$ and $[-1, 1]^{|S|}$. While the choice of the lengths $1/2, 1,$ and $2$ is rather arbitrary, the fact that we need three different scales is unavoidable, see Remark 2.2.

By Theorem 2, equation (NLS) has invariant tori on which the dynamics is the linear translation by $\nu t$. The statement above is a typical KAM Theorem, regarding the existence of an invariant torus. The fact that we are looking for an infinite torus however introduces various new difficulties, in particular related to the regularity of the dependence on $\nu, I, V_{S^c}$. This appears immediately when one wishes to prove that the Cantor set $C$ of “good frequencies” is measurable (and of positive measure) with respect to the product probability measure $\text{meas}_{Q_S}$ on $Q_S$. The $\sigma$-algebra of such measure, which is the natural one in this context, is given by the Borel sets of the

---

\[14\] Recall that if $X_i, i \in I,$ are topological metrizable spaces, the product topology on $X := \prod_{i \in I} X_i$ is the topology of the point-wise convergence, meaning that a sequence $x^{(k)} = (x_i^{(k)})_{i \in I}$ converges if $x_i^{(k)}$ converges for all $i \in I$. 
product topology, which is coarser than the one induced by the $\ell^\infty$-metric. Then a crucial point is that a function $f : Q_S \to \mathbb{R}$ which is Lipschitz (with respect to the $\ell^\infty$-metric) might be non continuous with respect to the product topology and, hence, non measurable. As typical in KAM schemes $C$ is defined as the intersection of sets of the form \{ $|f(\nu)| > \alpha > 0$ \} (see (9.1) and (9.2)) and $f : Q_S \to \mathbb{R}$ is a Lipschitz function. As explained above this does not assure measurability.\footnote{This problem does not appear if $f$ actually depends only on a finite number of variables as in the case of maximal tori, where $f(\nu) = \nu \cdot \ell$, with $|\ell| < \infty$.}

So we reformulate our theorem in a more technical way, carefully keeping track of the regularity w.r.t. all the parameters. To avoid working with functions defined only on a Cantor set, we suitably (see Lemma 4.1) extend all the functions so that they are defined for $\nu \in Q_S$. To summarize, the parameter dependences we need to control are:

- continuity w.r.t. the product topology for measure estimates both in $\nu$ and in $V$;
- Lipschitz dependence (w.r.t. $\ell^\infty$-metric) for implicit function theorems /contractions/extensions (see Lemma B.1).

The smallness condition (i.e. how close are our solutions to zero) is tied to the size of the non-linearity and to the regularity of the solution we are looking for. Recall that, since $f$ is analytic, for some $R > 0$ we have

\begin{equation}
\label{2.12}
f(y) = \sum_{d=1}^{\infty} f^{(d)} y^d, \quad |f|_R := \sum_{d=1}^{\infty} |f^{(d)}|_R < \infty.
\end{equation}

**Theorem 3.** Let $p_* > 1$ and $0 < \gamma \leq \min\{\frac{1}{4}, |f|_R\}$. There exists $\varepsilon_* = \varepsilon_*(p_*) > 0$ and $C = C(p_*) > 1$ such that, for all $r > 0$ satisfying

\begin{equation}
\label{2.13}
\varepsilon := \frac{|f|_R}{\gamma R} r \leq \varepsilon_*
\end{equation}

and for every

\begin{equation}
\label{2.14}
\frac{p_* + 1}{2} \leq p \leq p_*
\end{equation}

the following holds. There exist:

i) a map

\begin{equation}
\label{2.15}
\mathcal{V}_S : Q_S \times [-1/4,1/4]^S_c \times I(p,r) \to [-1,1]^S, \\
(\nu, V_{S_c}, I) \mapsto \mathcal{V}_S(\nu, V_{S_c}, I)
\end{equation}

which is continuous in $\nu, V_{S_c}$ w.r.t. the product topology and Lipschitz in all its variables w.r.t. the $\ell^\infty$ metric. In particular $\mathcal{V}_S$ is Lipschitz $C \varepsilon \gamma$-close w.r.t. $\nu$ to the map $\mathcal{V}_S^*$;

ii) a map

\[ \Phi : B_{3r}(\mu_p) \times Q_S \times [-1/4,1/4]^S_c \times I(p,r) \to B_{4r}(\mu_p), \]

\[ (u; \nu, V_{S_c}, I) \mapsto \Phi(u; \nu, V_{S_c}, I) \]
which is Lipschitz in all its variables and $r/16$-close to the identity w.r.t. $u$;
iii) a Cantor-like Borel set $C = C(V_{S^c}, I, \gamma) \subset Q_S$, with
\[ \text{meas}_{Q_S}(Q_S \setminus C(V_{S^c}, I, \gamma)) \leq C_0 \gamma, \]
\[ \text{meas}_{[-1/4,1/4]^S}(\mathcal{V}_S(Q_S \setminus C(V_{S^c}, I, \gamma), V_{S^c}, I)) \leq C_0 \gamma, \]
for a suitable absolute constant $C_0 > 0$.
Moreover for any $I \in \mathcal{I}(p, r)$, $V_{S^c} \in [-1/4,1/4]^{S^c}$ and $\nu \in C(V_{S^c}, I, \gamma)$, the map $\Phi(\cdot; \nu, V_{S^c}, I)$ is an analytic symplectic change of variables.
Finally, for $\nu \in C(V_{S^c}, I, \gamma)$, $\mathcal{T}_I$ defined in (2.5) is a KAM torus of frequency $\nu$ for $H_V \circ \Phi$ with $V = (\mathcal{V}_S(\nu, V_{S^c}, I), V_{S^c})$.

**Remark 2.2.** In Theorem 3 above the map $\mathcal{V}_S$ has oscillations of order $C_1 \epsilon \gamma$ so the target tube could be $[-1/2 - C_1 \epsilon \gamma, 1/2 + C_1 \epsilon \gamma]$, but in any case must have side-length greater than 1. On the other hand, in order to achieve the second estimate in (2.16), we need that the sidelength of $Q_S$ is greater than 1/2. This is the reason for choosing three distinct sidelengths (see comments after Theorem 2).

Note that the parameter $\gamma$ comes from a Diophantine condition, see Definition 2.3, and consequently controls the measure estimates (2.16). Of course, this estimate is meaningful only for small $\gamma$.

We now reformulate our result in terms of the Fourier multiplier $V$ (for the proof see Section 9).

**Corollary 2.1.** For $I \in \mathcal{I}(p, r)$, with $p, r$ as in the previous Theorem, let
\[ \mathcal{G}(I, \gamma) := \{ V \equiv (V_{S}, V_{S^c}) \in [-1/4,1/4]^Z : V_S \in \mathcal{V}_S(C(V_{S^c}, I, \gamma), V_{S^c}, I) \}. \]
$\mathcal{G}(I, \gamma)$ is a Borel set in $[-1/4,1/4]^Z$ with measure greater than $1 - C_0 \gamma$ ($C_0$ is the constant in (2.16)).

Therefore for all $I \in \mathcal{I}(p, r)$ and all $V \in \mathcal{G}(I, \gamma)$ the equation [NLS$_V$] has an invariant torus.

**Invariant tori and regularity of our almost periodic solutions.** The flat torus $\mathcal{T}_I$ in the original variables is realized by the analytic immersion $\varphi \mapsto \Phi(i(\varphi); \nu, V_{S^c}, I)$ with $i$ defined in (2.7). By construction the NLS dynamics on the torus $\mathcal{T}_I$ is $\varphi \mapsto \varphi + vt$ so our candidate for an almost periodic solution is $u(t, \cdot) := \Phi(i(\nu t); \nu, V_{S^c}, I)$ as discussed in Section 3.

We wish to stress that analyticity of $\Phi(i(\varphi))$ in the angles does not imply analyticity in time, since the map $t \mapsto vt \in \mathbb{T}^S$ is not even continuous (endowing $\mathbb{T}^S$ with the $\ell^\infty$-topology). On the other hand, since $\Phi$ is analytic,
the regularity issues are the same as in the linear case, recall Subsection 2.1.
In particular, for \( p > 3 \) our solutions are classical, while for \( p \leq 2 \) we can construct also merely weak solutions, see the proof of Theorem 1.

**The Cantor set \( \mathcal{C} \).** We can be rather explicit in our description of the set \( \mathcal{C} \) of Theorem 3. We start by fixing the hypercube

\[
Q := \left\{ \omega = (\omega_j)_j \in \mathbb{R}^\mathbb{Z} : |\omega_j - j^2| < \frac{1}{2} \right\}, \quad Q = Q_S \times Q_{S^c},
\]

endowed with the product topology, and by introducing the following closed set

**Definition 2.3 (Diophantine condition).** Let \( \gamma > 0 \). We say that a vector \( \omega \in Q \) belongs to \( D_{\gamma,S} \) if it satisfies

\[
|\omega \cdot \ell| \geq \gamma \prod_{s \in S} \frac{1}{(1 + |\ell_s|^2 (i(s))^2)^{3/2}} \quad \forall \ell : 0 < |\ell| < \infty, \quad \sum_{j \in S^c} |\ell_j| \leq 2, \quad \pi(\ell) = m(\ell) = 0,
\]

where \( i(s) \) is the inverse function of \( s(i) \), \( \pi(\ell) := \sum_{j \in \mathbb{Z}} j \ell_j \) is the “momentum” and \( m(\ell) := \sum_{j \in \mathbb{Z}} \ell_j \) is the “mass”.

**Theorem 4.** Under the hypotheses of Theorem 3, there exists a Lipschitz map \( \Omega : Q_S \times [-1/4, 1/4]^{S^c} \times I(p, r) \rightarrow Q_{S^c} \) which is continuous with respect to the product topology on \( Q_S \times [-1/4, 1/4]^{S^c} \) and for every \( j \in S^c \) satisfies

\[
|\Omega_j(\nu, V_{S^c}, I) - j^2 - V_j| \leq C_\gamma \varepsilon
\]

and the Lipschitz estimates

\[
\sup_{\nu' \neq \nu} \frac{|\Omega_j(\nu, V_{S^c}, I) - \Omega_j(\nu', V_{S^c}, I)|}{|\nu - \nu'|}_\infty \leq C\varepsilon,
\]

\[
\sup_{I' \neq I} \frac{|\Omega_j(\nu, V_{S^c}, I) - \Omega_j(\nu, V_{S^c}, I')|}{|I - I'|_2} \leq C\gamma r^{-2} \varepsilon.
\]

Moreover we can choose in Theorem 3

\[
\mathcal{C}(V_{S^c}, I, \gamma) := \{ \nu \in Q_S : \omega(\nu, V_{S^c}, I) \in D_{\gamma,S} \},
\]

where \( \omega(\nu, V_{S^c}, I) := (\nu, \Omega(\nu, V_{S^c}, I)) \).

In this way, the torus \( T_I \) defined in (2.5) is an elliptic invariant torus in the sense that its linearized dynamics in the “normal” directions is \( \dot{u}_j = i\Omega_j u_j \) for \( j \in S^c \).

---

19 Instead of \( 3/2 \) one can put every exponent \( \tau > 1 \).
20 As usual for integer vector \( \ell \in \mathbb{Z}^2 \) we set \( |\ell| = \sum_{j \in \mathbb{Z}} |\ell_j| \).
21 The constant \( C \) is the one of Theorem 3.
Remark 2.3. Note that (2.18) is a much stronger Diophantine condition than the one proposed in [Bou05] (or [BMP21]), where the denominators were of the form $1 + |\ell_j|^2j^2$. Of course the reasons why we can impose such strong diophantine conditions, still obtaining a positive measure set, are the structure of the set $S$ and the fact that we only need to consider denominators with $\sum_{j \in \mathcal{S}} |\ell_j| \leq 2$.

2.3. Plan of the paper, strategy and main novelties. Deducing Theorem 1 from Theorems 3 and 4 is a self contained argument, which we present in section 3. The proof is developed as follows: $V_{\mathcal{S}c}$ and $I \in I(p, r)$ being fixed, we first construct a set $\mathcal{C}'(V_{\mathcal{S}c}, I, \gamma) \subseteq \mathcal{Q}_S$ of large relative measure such that for all $\nu \in \mathcal{C}'(V_{\mathcal{S}c}, I, \gamma)$ the map $t \mapsto u(t, \cdot) := \Phi(i(\nu t); \nu, V_{\mathcal{S}c}, I) \in \mathcal{W}_p$ is almost-periodic in $\mathcal{W}_p'$ with $1 < p' < p$ since it is uniform limit of quasi periodic functions just as in the linear case (see Subsection 2.1). Such approximating functions are in fact classical solutions of the approximate equation $(\text{NLS}_{V_{\mathcal{S}c}})$ with $V_{\mathcal{S}c} = V_{\mathcal{S}c}(\nu) \rightarrow V(\nu)$. Secondly we show that $u$ is a weak solution in the sense of (1.2), for such $V = V(\nu)$. We next reformulate our result in terms of the external parameters $V$ instead of the frequencies $\nu$, namely we prove that for every $V$ in a large measure set $\mathcal{G}(I, \gamma) \subseteq [-1/4, 1/4]^\mathbb{Z}$ we can solve (1.2). A main point is to show that $\mathcal{G}(I, \gamma)$ is measurable. Next, taking the union over $\gamma$ we obtain a full measure set, thus showing that for a.e. $V$ there exists at least one solution. Finally moving $\mathcal{S}$ and suitably choosing $I$ we produce, for a.e. $V$, countably many different solutions.

In order to prove Theorems 3 and 4 (see end of Section 6) we follow the general strategy of [BMP21]. As recalled in Section 6, this consists in three main steps: the introduction of a suitable functional setting and degree decomposition, a formulation in terms of a counter-term theorem, solving a Homological equation, and proving the fast convergence of a KAM scheme. Regarding the first issue, the main additional difficulty in the present setting is to keep track of the regularity w.r.t. all the parameters. To this purpose in section 4 we start by defining the Poisson algebra of “regular Hamiltonians” (basically a space of normally analytic functions $H$ such that the corresponding Hamiltonian vector field $X_H$ is a normally analytic map $B_r(\mathcal{W}_p) \rightarrow \mathcal{W}_p$ and further introduce the space of parameter−depending regular Hamiltonians (see definitions 4.3 and 4.4). In particular, we set our ambient space as the closure, w.r.t. a suitable norm (see (4.9) together with (4.12)-(4.13)), of functions depending only on a finite number of frequencies. A Lipschitz extension result is given in Lemma 4.1.

Theorems 3 and 4 will be derived in two steps, through a technique known as “elimination of parameters”, firstly introduced by Rüssmann and Herman in the 80’s. This strategy consists in introducing some extra parameters (i.e. the counter-terms $\lambda_j$) in the perturbed Hamiltonian, in order to compensate its degeneracies (absence of twist properties for instance) and conjugate the modified Hamiltonian to one that admits the desired invariant KAM torus. This first normal form step contains the hard analysis. Then, we prove that
we can eliminate the counter-terms using the potential, by means of the classical implicit function theorem in Banach spaces. 

Hence, in our case, we first put $H_V$ in (2.3) in a suitable normal form with counter-terms in the spirit of Herman through Theorem 5, which contains the main KAM difficulties in dealing with Sobolev regularity.

Roughly speaking, this normal form theorem states that, under appropriate smallness conditions, for all $\omega \in D_\gamma, S, V_S, V_{S_c} \in [-1/4, 1/4]$ and $I \in I(p, r)$, there exist $\lambda \in \ell^\infty$ and a symplectic change of variables $\Psi$ (see (6.3)) such that

$$
\left( \sum_{j \in \mathbb{Z}} (\omega_j + \lambda_j)|u_j|^2 + P \right) \circ \Psi = \sum_{j \in \mathbb{Z}} \omega_j |u_j|^2 + O(||u||^2 - I)^2,
$$

where $R = O(||u||^2 - I)^2$ means that $R$ is a regular Hamiltonian which has a zero of order at least 2 at the torus $T_I$ defined in (2.5) (we formalize this definition in Section 5, introducing an appropriate degree decomposition). In order to deduce Theorems 3-4 we need to eliminate the counter terms (see equations in (6.9)): the main point is that we can solve with respect only to the tangential variables $V_{S \leftrightarrow \nu}$.

Coming back to the counter-term Theorem 5, let us explain the main new issues related to Sobolev regularity. As is habitual, the map $\Psi$ is constructed as the composition of a sequence of changes of coordinates whose generating function $S$ at every step is determined by solving a Homological equation

$$(2.22) \quad L_\omega S := \{ \sum_j \omega_j |u_j|^2, S \} = F,$$

where $F$ is a given analytic function on the phase space $w_p$ (see definition 4.4) which is at most quadratic in the normal variables ($j \notin S$). At a formal level, a solution $S = L^{-1}_\omega F$ of (2.22) is readily determined. In order to prove that $S$ is in fact analytic, one has to control the contribution given by small divisors (i.e. the eigenvalues of $L_\omega$). This is possible by imposing the arithmetic Melnikov conditions on the frequencies (2.18), where the constraint $\sum_{j \in S_c} |\ell_j| \leq 2$ comes from the fact that $F$ is at most quadratic in the normal variables, while the zero mass and momentum conditions come from the presence of the corresponding quadratic constants of motion. Also in this Sobolev context, due to the presence of small divisors, one bounds the solution $S$ at the cost of some “loss of information”. More explicitly, if the Hamiltonian vector field $X_F$ maps $B_r(w_p) \to w_p$, then $X_S$ maps $B_r(w_{p+\delta}) \to w_{p+\delta}$ (with $\delta > 0$); this means that $S$ is analytic in a smaller domain, since $w_{p+\delta} \subset w_p$. Then, at each iteration, one is able to define the solutions $S$ only on a (ball of a) smaller phase space of the Banach scale $(w_p)_{p>1}$, the target space shrinking accordingly. Resembling the finite dimensional case, we call $\delta$ the “loss of regularity”.

Of course, the convergence of a KAM scheme is achieved only if one loses a summable, amount of regularity $\delta_n$ at each step $n$. Nonetheless the typical characteristic of the Sobolev case is the presence of a lower bound on the
loss of regularity, so that a KAM scheme based only on such bound cannot converge. In KAM schemes for quasi-periodic Sobolev solutions a similar problem arises but it is bypassed by using tame estimates, or approximation by analytic functions. In the present context it is not clear if one can exploit similar ideas, since we are already working with analytic nonlinearities. Our point of view is to bypass this problem by taking full advantage of the fact that the tangential sites set $S$ is sparse (recall Definition 2.2). In order to get an intuition of our strategy take for simplicity $S = \{ s(i) = 2^i, \ i \in \mathbb{N} \}$ and start by considering a toy model with a non-linearity for which the set

\[(2.23) \quad U_S := \{ u \in \mathcal{W}_p : \ u_j = 0 \ \forall j \not\in S \} \simeq \{ v \in \ell^\infty \text{ s.t. } \sup_{i \in \mathbb{N}} |v_i|^{2ip} < \infty \}\]

is invariant for the dynamics; so that we may study the equation restricted to $S$. Then we are essentially in the analytic case (or Gevrey or slightly less if we take a slower growth for $s(i)$) and the Bourgain strategy in [Bou05] (or [BMP21], [Con]) applies\footnote{Note that Bourgain’s Diophantine condition reads $|\omega \cdot \ell| \geq \gamma \prod_{i \in \mathbb{N}} (1 + \ell^2 (i)^2)^{-1}$. This coincides with (2.18) when $\ell$ is supported only on $S$, modulo the renaming of the indexes in 2.23.} Of course for the NLS equation (NLS$_V$) the set $U_S$ is not invariant and the main difficulties arise from interaction between tangential and normal modes.

In the Diophantine estimates, to deal with the terms $\ell$ not supported only on the tangential sites, we use the constants of motion and the dispersive nature of the equation ($\omega_k \sim k^3$). Once one has guessed the correct Diophantine conditions (2.18), the proof of Proposition 7.1 (i.e. controlling the solution of the Homological equation 2.22) is the real core of our result. Again the proof is simple if $F$ is supported only on $S$ or $S^c$, on the other hand dealing with the interaction between tangential and normal sites requires a careful case analysis.

The final goal is to control the norm of $X_S$ as a map $B_r(\mathcal{W}_{p+\delta}) \to \mathcal{W}_{p+\delta}$, for arbitrarily small $\delta$. This should be compared with the corresponding estimate on $X_S$ in [BMP20a] Proposition 7.1 item (M). In the latter paper we take $S = \mathbb{Z}$ and then, in order to control $L^{-1}_\omega F$ we cannot take any $\delta > 0$ but instead must require $\delta \geq \delta_*$, where $\delta_* > 0$ is some fixed quantity. As one can expect the less sparse is $S$ the worst bounds one gets. The quantitative condition in Definition 2.2 is needed in order to ensure convergence of the iterative KAM scheme. We suitably choose the values of the parameter at each iterative step $n \in \mathbb{N}$, in particular the loss of regularity $\delta_n$ must be summable, e.g. $\delta_n \sim n^{-c}$, for some $c > 1$. Then the divergence due to small divisors, which is of order $\exp(\exp(n^{c/\eta}))$ by (7.3) and with $\eta$ defined in (2.8a), must be compensated by the super-exponential convergence $\exp(-\exp(Cn))$ given by the KAM quadratic scheme. This forces $c < \eta$ and $\eta > 1$.

The super-linearity assumptions (2.8b) and (2.8c) are essential for our estimate on the Homological equation to work. The asymptotic growth in (2.8a)
is only needed in the KAM step. A slower growth would give rise to a too large estimate on the solution of the Homological equation which would not be compensated anymore by the quadratic convergence of the KAM scheme.

3. Proof of Theorem 1

The solutions of Theorem 1 are constructed as the limit of sequences of smooth quasi-periodic functions. Fix \( p_0 \) and an admissible set of tangential sites \( \mathcal{S} \). Fix \( \gamma > 0 \) and take \( r \) such that (2.13) holds. For any potential \( \phi \in \{-1/4, 1/4\}^{\mathcal{S}} \) and \( I \in \mathcal{I}(p_0, r) \), we apply Theorem 3 and obtain, for all frequencies \( \nu \in \mathcal{C}(\phi, I, \gamma) \), an invariant torus.

Now define \( I^{(n)} = (I_j^{(n)}) \) by setting \( I_j^{(n)} = I_j \) if \( |j| \leq n \) and \( I_j^{(n)} = 0 \) otherwise. We apply Theorem 3 with \( I \sim \nu^{(n)} \), \( \gamma \sim \gamma/2 \) and set \( \mathcal{V}^{(n)}(\cdot) := \mathcal{V}(\cdot, \phi, I^{(n)}), \mathcal{I}^{(n)} \) a \( \mathcal{F}(\cdot, \cdot, \nu, \phi, I^{(n)}) \). We have obtained a sequence of NLS equations with potentials \( \mathcal{V}^{(n)}(\nu) \) for \( \nu \in \{-1/4, 1/4\}^{\mathcal{S}} \); each equation admits a finite dimensional invariant torus with frequency \( \nu \), for all \( \nu \in \mathcal{C}(\phi, I^{(n)}, \gamma/2) \). It is not hard to see that each torus supports a smooth quasi-periodic solution.

The idea is to show that (at least up to a sub-sequence) the limit over \( n \) is the desired almost-periodic solution.

First step (construction of the Cantor set). In order to apply Theorem 3 for each \( n \) (with the same frequency) we take the “good frequencies” in the (countable) intersection of the Cantor-like sets where all the tori are defined. Correspondingly, we define the set of “good potentials”. This is the content of the following result, which is proved in Section 9.

Lemma 3.1. There exists a sub-sequence \( n_k \to \infty \) (independent of \( \phi \)) such that the following holds. Defining the Borel sets

\[
\mathcal{C}'(\phi, I, \gamma) := \mathcal{C}(\phi, I, \gamma) \cap \bigcap_{k \in \mathbb{N}} \mathcal{C}(\phi, I^{(n_k)}, \gamma/2),
\]

\[
\mathcal{G}(\phi, I, \gamma) := \bigcap_{k \in \mathbb{N}} \mathcal{C}'(\phi^{(n_k)}(\mathbb{Z}), V, I),
\]

we have the estimates

\[
\text{meas}_{[-1/4, 1/4]} \left( \mathcal{G}(\phi, I, \gamma) \right) \geq 1 - C_0 \gamma,
\]

(3.1)

where \( C_0 \) is defined in Theorem 3.

Second step (construction of the convergent sub-sequence). We now prove that for every \( V \in \mathcal{G}(I, \gamma) \) we can solve (1.2) with \( V = (\phi, V_0) \). Indeed, for every \( \phi \in \mathcal{G}(\phi, I, \gamma) \) there exists \( \nu \in \mathcal{C}'(\phi, I, \gamma) \) such that
\[ \mathcal{V}_S(\nu, V_{\nu}, I) = V_S. \] We claim that
\[ u(t, x) := \Phi(v(t, x); \nu, V_{\nu}, I), \quad \text{where} \quad v(t, x) := \sum_{j \in S} \sqrt{T_j} e^{i(jx + \nu_j t)}, \]
satisfies (1.2) and, moreover, it is the uniform limit of the \( C^\infty \)-quasi-periodic functions
\[ u_k(t, \cdot) := \Phi_{n_k}(v_k(t, \cdot); \nu), \quad \text{where} \quad v_k(t, x) := \sum_{j \in S, |j| \leq n_k} \sqrt{T_j} e^{i(jx + \nu_j t)}. \]
Note that by construction \( v, u \in \mathbf{X}_{p_s} \) while \( t \mapsto \Phi_{n_k}(v_k(t, \cdot); \nu) =: u_k(t, \cdot) \) is a classical (actually \( C^\infty \)) quasi-periodic solution of (NLS\(_{(n_k)}\)) with \( V^{(n_k)} = (\mathcal{V}^{(n_k)})(\nu, V_{\nu}). \) Thus each \( u_k \) satisfies (1.2) with \( V^{(n_k)} \) in place of \( V. \)
Taking \( p < p_s \) satisfying (2.14) we get
\[ c_k := \sup_{|j| > n_k} \sqrt{T_j} |j|^p \to 0, \quad \text{as} \quad k \to \infty \]
and, for every \( t \in \mathbb{R}, \)
\[ |v(t, \cdot) - v_k(t, \cdot)|_p = c_k, \quad |I - I^{(n_k)}|_2 = c_k^2. \]
Since the map \( I \to \mathcal{V}_S(\nu, V_{\nu}, I) \) is Lipschitz then \( V^{(n_k)} \to V \) in \( \ell^\infty. \) Moreover since \( \Phi \) is also Lipschitz (w.r.t. \( u \) and \( I \)), for every \( t \in \mathbb{R} \)
\[ |u(t, \cdot) - u_k(t, \cdot)|_p = \left| \Phi(v(t, \cdot), \nu, V_{\nu}, I) - \Phi(v_k(t, \cdot), \nu, V_{\nu}, I^{(n_k)}) \right|_p \leq L (|v(t, \cdot) - v_k(t, \cdot)|_p + |I - I^{(n_k)}|_2) \leq L(c_k + c_k^2) \]
for a suitable \( L \geq 0. \) Then \( u_k \to u \) uniformly in \( \mathbb{R}^2. \) In order to prove that \( u \) satisfies (1.2) we have to show that
\[ \int_{\mathbb{R}^2} (-i\chi_t + \chi_{xx})(u - u_k) - (V * u - V^{(n_k)} * u_k) \chi - (f(|u|^2)u - f(|u_k|^2)u_k) \chi \, dx \, dt \]
tends to zero as \( k \to \infty. \) This follows since \( u_k \to u \) uniformly and observing that
\[ \|V^{(n_k)} * u_k - V * u\|_{L^\infty(\mathbb{R}^2)} \leq |V^{(n_k)} * u_k - V * u|_p, \]
\[ \leq |V^{(n_k)} - V|_{L^\infty}|u_k|_p + |V|_{L^\infty}|u - u_k|_p \xrightarrow{k \to \infty} 0. \]
This proves the claim in (3.2).

**Third step (A set of good potentials).** We now show that for almost every \( V \in [-1/4, 1/4]^2 \) there exists at least one solution of (1.2). For every integer \( h \geq 1 \) take \( \gamma_h = |f|_{h^2}/h \) and \( r_h \) such that (2.13) holds as an equality, namely

\[ \text{We can take } L = 2 \text{ since } \Phi \text{ is } C\varepsilon \text{-close to the identity.} \]

\[ \text{Given } f : \mathbb{R}^2 \to \mathbb{C} \text{ we have } \|f\|_{L^\infty(\mathbb{R}^2)} = \sup_{t \in \mathbb{R}} \|f(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \sup_{t \in \mathbb{R}} |f(t, \cdot)|_p \text{ for every } p \geq 1. \]
\( r_h := \sqrt{\varepsilon \gamma R/|f|} = \sqrt{\varepsilon R/h} \). For every given sequence \( I_h \in \mathcal{I}(p_*, r_h) \) we set
\[
(3.3) \quad \mathcal{G} := \bigcup_{h \geq 1} \mathcal{G}(I_h, \gamma_h).
\]
By (3.1) \( \mathcal{G} \) has full measure in \([-1/4, 1/4]^Z\). This implies that for almost every \( V \in [-1/4, 1/4]^Z \) there exists an integer \( h \geq 1 \) such that \( V \in \mathcal{G}(I_h, \gamma_h) \). Then (3.2) (with \( I = I_h \)) gives a solution of (1.2).

**Fourth step (Abundance of solutions).** In order to find infinitely many solutions for almost every \( V \in [-1/4, 1/4]^Z \) we proceed as follows. First we choose in (3.3) \( \sqrt{|I_h|} := \frac{1}{2} r_h |j|^{-p_*} \). All the above construction depends on the choice of the set \( S \) of admissible tangential sites; in particular this holds for the set \( \mathcal{G} \) above. Let us now consider two distinct admissible sets \( S, S' \), and let us denote by \( \mathcal{G}, \mathcal{G}' \) the corresponding sets of potentials. We claim that for each \( V \in \mathcal{G} \cap \mathcal{G}' \) there exists (at least) two distinct almost-periodic solutions. Indeed, there exist \( h, h' \) such that \( V \in \mathcal{G}(I_h, \gamma_h) \cap \mathcal{G}'(I_{h'}, \gamma_{h'}) \), where \( \mathcal{G}(I, \gamma), \mathcal{G}'(I, \gamma) \) are the sets defined in Lemma (3.1) corresponding to \( S, S' \). Let us call \( u, v, \Phi \) and \( u', v', \Phi' \) the functions defined in (3.2), respectively for \( S \) and \( S' \). Since by Theorem 3 (point ii)) the maps \( \Phi \) and \( \Phi' \) are \( \bar{r}/16 \)-close to the identity where \( \bar{r} := \max\{r_h, r_{h'} \} \), then
\[
|u(t, \cdot) - u'(t, \cdot)|_{p_*} \geq |v(t, \cdot) - v'(t, \cdot)|_{p_*} - \bar{r}/8 \geq \bar{r}/2 - \bar{r}/8 > 0.
\]
The same holds for any translations \( u(t + t_0, x + x_0) \) and \( u'(t, x) \). Let us now consider a countable infinity of distinct admissible sets \( S \), and call \( \mathcal{G}_s \) the countable intersection of the corresponding sets \( \mathcal{G} \). The set \( \mathcal{G}_s \) has full measure in \([-1/4, 1/4]^Z\) as well. Then for every \( V \in \mathcal{G}_s \) we construct infinitely many solutions corresponding to different \( S \)'s.

**Regularity.** By (3.2) and (2.2) we know that if \( p_* > 3 \) then \( v_t \) and \( v_{xx} \) are continuous functions on \( \mathbb{R}^2 \). Moreover, by analyticity of \( v \rightarrow \Phi(v; \nu, I) \), also \( u_t \) and \( u_{xx} \) are continuous function on \( \mathbb{R}^2 \). Therefore \( u \) is a classical solution. On the other hand, when \( p_* \leq 2 \) and \( \sqrt{T_j} = \langle r/2 \rangle |j|^{-p_*} \) for all \( j \in S \), for every \( t \) the function \( u_{xx}(t, \cdot) \) is not in \( L^2 \). Otherwise its Fourier coefficients \( (u_{xx}(t, \cdot))_j = -j^2(u(t, \cdot))_j \) would belong to \( \ell_2 \). As in the linear case (recall (2.4)) we claim that
\[
\limsup_{|j| \to \infty} j^2 |(u(t, \cdot))_j| > 0.
\]
Indeed, since \( \Phi \) is close to the identity (in \( \mathcal{W}_{p_*} \) \( \Phi = Id + \Phi_1 \) with \( \|\Phi_1\| \) uniformly \( r/16 \)-small, we have that \( u(t, \cdot) := \Phi(v(t, \cdot)) = v(t, \cdot) + \Phi_1(v(t, \cdot)) \) and its Fourier coefficients satisfies \( |(u(t, \cdot))_j| \geq (r/4)|j|^{-p_*} \) for all \( j \in S \); since \( S \) is an infinite set this is a contradiction. \( \Box \)

4. **Functional setting**

Let us introduce the spaces of Hamiltonians used in the paper.
**Definition 4.1** (Multi-index notation). In the following we denote, with abuse of notation, by $\mathbb{N}^Z$ the set of multi-indexes $\alpha, \beta$ etc. such that $|\alpha| := \sum_{j \in \mathbb{Z}} \alpha_j$ is finite. As usual $\alpha! := \prod_{j \in \mathbb{Z}, \alpha_j \neq 0} \alpha_j$. Moreover $\alpha \preceq \beta$ means $\alpha_j \leq \beta_j$ for every $j \in \mathbb{Z}$, then $(\beta)^{\alpha} := \frac{\beta!}{(\beta - \alpha)!}$. Finally take $j_1 < j_2 < \ldots < j_n$ such that $\alpha_j \neq 0$ if and only if $j = j_i$ for some $1 \leq i \leq n$, as usual we set $\partial^{\alpha} f := \partial^{\alpha_{j_1}} \ldots \partial^{\alpha_{j_n}} f$; analogously for $\partial_{\bar{u}}^{\alpha} f$.

**Definition 4.2** (regular Hamiltonians). Consider a formal power series expansion
\[
H(u) = \sum_{(\alpha, \beta) \in \mathcal{M}} H_{\alpha, \beta} u^{\alpha} \bar{u}^{\beta}, \quad u^\alpha := \prod_{j \in \mathbb{Z}} u_j^{\alpha_j},
\]
where
\[
\mathcal{M} := \left\{ (\alpha, \beta) \in \mathbb{N}^Z \times \mathbb{N}^Z, \text{s.t. } |\alpha| = |\beta| < +\infty, \sum_{j \in \mathbb{Z}} j(\alpha_j - \beta_j) = 0 \right\},
\]
satisfying the reality condition
\[
H_{\alpha, \beta} = \bar{H}_{\beta, \alpha}, \quad \forall (\alpha, \beta) \in \mathcal{M}.
\]
We say that $H \in \mathcal{H}_{r,p}$ for $p > 1$, $r > 0$ if
\[
|H|_{r,p} := \frac{1}{r} \left( \sup_{|u| \leq r} \left| X_H \right|_p \right) < \infty,
\]
where $X_H$ denotes the Hamiltonian vector field, intrinsically defined through the standard symplectic form $\varpi = i \sum_j du_j \wedge d\bar{u}_j$ as
\[
i_{X_H} \varpi(\cdot) := \varpi(\cdot, X_H) = dH(\cdot),
\]
$H$ being the associated majorant Hamiltonian $H(u) := \sum_{(\alpha, \beta) \in \mathcal{M}} |H_{\alpha, \beta}| u^{\alpha} \bar{u}^{\beta}$ of $H(u)$, and $i_{X_H}$ being the usual contraction of a differential form with a vector field.

We denote by $\mathcal{H}_{r,p}(\mathbb{C})$ the space of $H$ satisfying (4.1) and (4.4) but not necessarily the reality condition (4.3).

Finally, given any $F, G \in \mathcal{H}_{r,p}(\mathbb{C})$, we define in the usual manner the Poisson brackets as $\{F, G\} := \varpi(X_F, X_G)$.

For details on the symplectic structure and Hamiltonian vector field see for instance [BMP20a].

Note that by Lemma 2.1 of [BMP21]
\[
|H|_{r,p} = \frac{1}{2} \sup_{(\alpha, \beta) \in \mathcal{M}} \sum_j |H_{\alpha, \beta}| (\alpha_j + \beta_j) u^{\alpha + \beta - 2\varepsilon_j},
\]
where $u_p = u_p(r)$ is defined as
\[
u p,j(r) := r^{|j|^{-p}}.
\]
Remark 4.1. Regarding $\mathcal{M}$ in (4.2) we note that the condition $|\alpha| = |\beta|$, i.e. $m(\alpha - \beta) = 0$, corresponds to mass conservation, namely the $H$ Poisson commutes with the mass $\sum_{j \in \mathbb{Z}} |u_j|^2$; moreover $\sum_{j \in \mathbb{Z}} j(\alpha_j - \beta_j) = 0$, i.e. $\pi(\alpha - \beta) = 0$, corresponds to momentum conservation, namely $H$ Poisson commutes with the momentum $\sum_{j \in \mathbb{Z}} j|u_j|^2$.

Note that $|·|_{r,p}$ is a semi-norm on $\mathcal{H}_{r,p}$ and a norm on its subspace

$$(4.7) \quad \mathcal{H}_{r,p}^0 := \{ H \in \mathcal{H}_{r,p} \text{ with } H(0) = 0 \},$$

endowing $\mathcal{H}_{r,p}$ with a Banach space structure. Moreover the space $\mathcal{H}_{r,p}$ enjoys the following algebra property with respect to Poisson brackets of Hamiltonians.

**Proposition 4.1** (Poisson structure). For any $F, G \in \mathcal{H}_{r+\rho,p}$, with $\rho > 0$, one has $\{F, G\} \in \mathcal{H}_{r,p}^0$ with the bound

$$(4.8) \quad |\{F, G\}|_{r,p} \leq 8 \max\left\{ 1, \frac{r}{\rho} \right\} |F|_{r+\rho,p} |G|_{r+\rho,p}.$$  

The proof is given in Appendix A. Of course the same estimates hold in $\mathcal{H}_{r,p}(\mathbb{C})$.

The next result controls the norm of the NLS non-linearity $P$ defined in (2.3) and it is based on the algebra property of $u_p$, $p > 1$, with respect to convolution.

**Proposition 4.2** (Proposition 2.1 of [BMP21]). There exist $c_1, c_2 > 1$ continuously depending on $p > 1$ such that if $c_1 r^2 \leq R$ then, recalling (2.12),

$$|P|_{r,p} \leq c_2 \frac{|f|_R}{R^{r^2}}.$$  

4.1. **Parameter families of regular Hamiltonians.** Throughout the paper our Hamiltonians will depend on two parameters, the frequency $\omega \in \mathbb{D}_{\gamma,S} \subseteq \mathcal{Q}$ and the action $I \in \mathcal{I}(p,r)$. In order to control the regularity w.r.t these parameters throughout the iterative scheme, we will introduce an appropriate weighted norm as follows.

Given $\gamma > 0$, a closed (w.r.t. the product topology) subset $\mathcal{O} \subseteq \mathcal{Q}$ and an open subset $\mathcal{I}$ of some Banach space, for $f : \mathcal{O} \times \mathcal{I} \rightarrow \mathbb{C}$, $(\omega, I) \mapsto f(\omega, I)$ we set

$$(4.9) \quad |f|_r = |f|_{r,\mathcal{O} \times \mathcal{I}} := \sup_{\omega \in \mathcal{O}} |f(\omega, I)| + \gamma \sup_{I \in \mathcal{I}} \sup_{\omega \neq \omega' \in \mathcal{O}} |\Delta_{\omega, \omega'} f|,$$

where as usual

$$(4.10) \quad \Delta_{\omega, \omega'} f := \frac{f(\omega, I) - f(\omega', I)}{|\omega - \omega'|_\infty}.$$  

\footnote{We put $\gamma$ in front of the Lipschitz semi-norm because of dimensional reason. See Definition 2.3.}
Definition 4.3. Given $\mathcal{O}$ and $\mathcal{I}$ as above let $C_{\text{Lip}}(\mathcal{O} \times \mathcal{I})$ be the Banach space of functions $f : \mathcal{O} \times \mathcal{I} \to \mathbb{C}$, which have finite norm $\|f\|_{\mathcal{O} \times \mathcal{I}}$ and which are:

- continuous w.r.t the product topology in $\mathcal{O}$;
- analytic in $\mathcal{I}$.

In $C_{\text{Lip}}(\mathcal{O} \times \mathcal{I})$ we consider the subalgebra $F(\mathcal{O} \times \mathcal{I})$ of functions which depend only on a finite number of $\omega_j$'s. The subalgebra $F(\mathcal{O} \times \mathcal{I})$ can be described as follows. Given $f \in F(\mathcal{O} \times \mathcal{I})$, which depends only on the variables $(\omega_k, \ldots, \omega_k)$, there exists a function $\tilde{f} : P_k(\mathcal{O}) \times \mathcal{I} \to \mathbb{C}$, where $P_k$ is the projection $P_k : \mathbb{R}^2 \to \mathbb{R}^{2k+1}$ defined as $P_k(\omega) := (\omega_k, \ldots, \omega_k)$, such that $f(\omega, I) = \tilde{f}(\omega_k, \ldots, \omega_k, I)$ for every $(\omega, I) \in \mathcal{O} \times \mathcal{I}$.

Finally denote the closure of $F(\mathcal{O} \times \mathcal{I})$ in $C_{\text{Lip}}(\mathcal{O} \times \mathcal{I})$ by $\mathcal{F}(\mathcal{O} \times \mathcal{I})$.

The spaces $C_{\text{Lip}}(\mathcal{O} \times \mathcal{I})$ and $\mathcal{F}(\mathcal{O} \times \mathcal{I})$ are Banach algebras (i.e. multiplicative algebras with constant equal to 1) w.r.t. the norm $\| \cdot \|_{\mathcal{O} \times \mathcal{I}}$.

The following extension result will be proved in Appendix A.

Lemma 4.1 (Lipschitz extension). Given $\mathcal{O} \subset Q$ and a ball $B_\rho$ in some complex Banach space $E$ and $f \in \mathcal{F}(\mathcal{O} \times B_\rho)$ there exists an extension $\tilde{f} : Q \times B_{\rho/2} \to \mathbb{C}$ such that $|\tilde{f}|_{\mathcal{O} \times B_{\rho/2}} \leq 2|f|_{\mathcal{O} \times B_\rho}$, $\tilde{f}$ is continuous w.r.t the product topology in $Q$, $\tilde{f}$ is Lipschitz on $B_{\rho/2}$ with estimate

$$|\tilde{f}(\omega, I) - \tilde{f}(\omega, I')| \leq 4\rho^{-1}|f|_{\mathcal{O} \times B_\rho} |I - I'|_E, \quad \forall \omega \in Q, \ I, I' \in B_{\rho/2}.$$ 

In the following we mainly consider $\mathcal{I} = \mathcal{I}(p, r)$ (see (2.11)), however in order to use Lemma 4.1 we need to pass to the complex. Then we define

$(4.11)\quad \mathcal{I}(\mathbb{C}) = \mathcal{I}(p, r, \mathbb{C}) := \{ I \in B_{r/4}(w_{2p}) \mid I_j = 0 \ \text{for} \ j \in S^c \}$,

which we systematically identify with the open ball of radius $r^2$ centered at the origin of the Banach space $\{ w = (w_j)_{j \in S} : \sup_{j \in S} |w_j| |j|^{2p} < \infty \}$.

Definition 4.4 (Real and complex Hamiltonians). Let $0 < r_0 \leq r$, $p_0 \geq p > 1$. Let $\mathcal{H}_{r,p} = \mathcal{H}_{r,p}^{\mathcal{O} \times \mathcal{I}}$ with $\mathcal{I} = \mathcal{I}(p_0, r_0)$ be the space of parameter depending real regular Hamiltonians $H : \mathcal{O} \times \mathcal{I} \ni (\omega, I) \mapsto H(\omega, I) \in \mathcal{H}_{r,p}$ such that

$(4.12)\quad H_{\alpha, \beta} \in \mathcal{F}(\mathcal{O} \times \mathcal{I}), \ \forall \alpha, \beta \in \mathcal{M}$ and $H_{\gamma} := \sum_{\alpha, \beta \in \mathcal{M}} |H_{\alpha, \beta}|^\gamma u^\alpha \bar{u}^\beta \in \mathcal{H}_{r,p}.$

We set

$(4.13)\quad \|H\|_{r,p} = \|H\|_{r,p}^{\mathcal{O} \times \mathcal{I}} := \|H_{\gamma}\|_{r,p} \frac{1}{2} \sup_{(\alpha, \beta) \in \mathcal{M}} \sum_j |H_{\alpha, \beta}|^\gamma (\alpha_j + \beta_j) u_{\alpha + \beta - 2e_j}.$

Respectively for $\mathcal{I}(\mathbb{C}) = \mathcal{I}(p_0, r_0, \mathbb{C})$ we define the space $\mathcal{H}_{r,p}(\mathbb{C}) = \mathcal{H}_{r,p}^{\mathcal{O} \times \mathcal{I}(\mathbb{C})}$ of parameter depending complex Hamiltonians $H : \mathcal{O} \times \mathcal{I}(\mathbb{C}) \ni (\omega, I) \mapsto H(\omega, I) \in \mathcal{H}_{r,p}(\mathbb{C})$ satisfying $(4.12)$ with $\mathcal{I}(\mathbb{C})$ instead of $\mathcal{I}$ and verifying the reality condition $H(\omega, I) \in \mathcal{H}_{r,p}(\mathbb{C})$ when $I \in \mathcal{I}$. 

Finally we denote by $H^0_{r,p}$, resp. $H^0_{r,p}(\mathbb{C})$ the subspace of $H_{r,p}$, resp. of $H_{r,p}(\mathbb{C})$, such that $H_{|u=0} = 0$.

The following result is proved in Appendix A.

**Lemma 4.2.** $(H^0_{r,p}, \| \cdot \|_{r,p})$ is a Banach-Poisson algebra in the following sense:
1. $(H^0_{r,p}, \| \cdot \|_{r,p})$ is a Banach space
2. for any $F, G \in H_{r+p}$ the following bound holds

\[
\|\{F, G\}\|_{r,p} \leq 8 \max\{1, \frac{r}{p}\} \|F\|_{r+p} \|G\|_{r+p}.
\]

**Proposition 4.3** (Monotonicity). The norm $\| \cdot \|_{r,p}$ is monotone decreasing in $p$ and monotone increasing in $r$:

\[
\| \cdot \|_{r,p+\delta} \leq \| \cdot \|_{r+p} \quad \forall \, \rho, \delta \geq 0.
\]

The fact that this norm is increasing in $r$ follows directly from mass conservation and the fact that $H(0) = 0$. Concerning the monotonicity in $p$, we refer the reader to [BMP20a, Proposition 6.3], where the proof is contained, written in the case of $| \cdot |_{r,p}$. The fact that it holds also in the Lipschitz frame, follows trivially.

**Proposition 4.4** (Hamiltonian flow). Let $S \in H_{r+p} = H_0^{\infty \times \mathcal{I}}$ with

\[
\|S\|_{r+p} \leq \delta := \frac{\rho}{16e(r+p)}.
\]

Then, for all $(\omega, I) \in \mathcal{O} \times \mathcal{I}$ the time 1-Hamiltonian flow of $S = S(\cdot, \omega, I)$ is well defined, analytic, symplectic; more precisely $\Phi^1_S : B_r(w_p) \to B_{r+p}(w_p)$ with

\[
\sup_{u \in B_r(w_p)} \|\Phi^1_S(u) - u\|_{r,p} \leq (r+p) \|S\|_{r+p} \leq \frac{\rho}{16e}.
\]

For any $H \in H_{r+p}$ we have that\(^{26}\) $H \circ \Phi^1_S = e^{(S, \cdot)}H \in H_{r,p}$, $e^{(S, \cdot)}H - H \in H^0_{r,p}$ and

\[
\|e^{(S, \cdot)}H\|_{r,p} \leq 2 \|H\|_{r+p},
\]

\[
\|(e^{(S, \cdot)} - \text{Id})H\|_{r,p} \leq \delta^{-1} \|S\|_{r+p} \|H\|_{r+p},
\]

\[
\|(e^{(S, \cdot)} - \text{Id} - \{S, \cdot\})H\|_{r,p} \leq \frac{1}{2} \delta^{-2} \|S\|_{r+p}^2 \|H\|_{r+p}.
\]

More generally for any $h \in \mathbb{N}$ and any sequence $(c_k)_{k \in \mathbb{N}}$ with $|c_k| \leq 1/k!$, we have

\[
\left\| \sum_{k \geq h} c_k \text{ad}^k_S (H) \right\|_{r,p} \leq 2 \|H\|_{r+p} \left( \|S\|_{r+p} / 2\delta \right)^h,
\]

\(^{26}\) $e^{(S, \cdot)}H := \sum_{k \in \mathbb{N}} \text{ad}^k_S (H)/k!$, where $\text{ad}_S = \{S, \cdot\}$.
where \(\text{ad}_S(\cdot) := \{S, \cdot\}\).

The proof is based on (4.14) and on the Lie series expansion for \(e^{\{S, \cdot\}}\), see [BMP20a] Proposition 2.1 and Lemma 2.1 for details.

**Remark 4.2.** If we are working in \(\mathcal{H}_{r,p}(\mathbb{C})\) all the estimates in Lemma 4.2, Proposition 4.3 and Proposition 4.4 still hold except for (4.17), which holds only for real \(I\). Indeed without the reality condition (4.3) the generating function \(S\) does not define anymore a Hamiltonian flow satisfying the reality condition \(u(t) = \overline{u}(t)\).

5. Degree decompositions, projections and normal forms

We want to prove that, in suitable variables, \(\mathcal{T}_I\) introduced in (2.5) is an invariant torus on which the flow is linear. To this purpose we introduce a suitable degree decomposition, whose main idea is to make a “power series expansion” in the variables \(|v|^2 - I\) and \(z\), which control the distance to the torus \(\mathcal{T}_I\), in order to highlight the terms which prevent \(\mathcal{T}_I\) from being a KAM torus. In the case of finite dimensional tori, one typically introduces action-angle variables but it is well known that this produces a singularity at \(I = 0\). In the infinite dimensional case this should be avoided and requires some care. An effective strategy is the expansion described below, first introduced in [Bou05] and further formalized in [BMP21, Section 4]. For convenience of the reader, we sketch here the main features of this decomposition. The main novelty here is to control the regularity with respect to the parameters. Fix \(S\) as in (2.2). Consider a Hamiltonian \(H(u)\) expanded in Taylor series at \(u = 0\) and tautologically rewrite \(H\) as

\[
H = \sum_{\substack{m, \alpha, \beta, a, b \in \mathbb{N}^S \\ \alpha \cap \beta = \emptyset}} H_{m, \alpha, \beta, a, b} |v|^{2m} \overline{v}^{\alpha} \overline{v}^{\beta} z^{a} \overline{z}^{b}
\]

where, by slight abuse of notation, \(u = (v, z)\) with \(v = (v_j)_{j \in S} := (u_j)_{j \in S}\) and \(z = (z_j)_{j \in S^c} := (u_j)_{j \in S^c}\). Moreover by \(\alpha \cap \beta = \emptyset\), here and in the following, we mean that \(\alpha_j \neq 0\) implies \(\beta_j = 0\).

Then introduce the auxiliary “action” variables \(w = (w_j)_{j \in S}\) substituting \(|v|^{2m} \overline{v}^{\alpha} \overline{v}^{\beta} z^{a} \overline{z}^{b} \rightarrow w^{m} \overline{v}^{\alpha} \overline{v}^{\beta} z^{a} \overline{z}^{b}\) in (5.1). Now we Taylor expand the Hamiltonian with respect to \(w\) and \(z\) at the point \(w_j = I_j\) for \(j \in S\) and \(z = 0\) respectively.

**Definition 5.1** (Degree decomposition). Let \(I \in \mathcal{I}(p, r)\) (recall (2.11)). For every integer \(d \geq -2\) and any regular Hamiltonian \(H \in \mathcal{H}_{r,p}\) we define the projection \((\Pi^d H)(u) = (\Pi^d H)(u) = H^{(d)}(u)\) defined as

\footnote{Consisting in a reordering of the indexes \(j\).}
Moreover we define, e.g.,
\[
H_{m,\alpha,\beta,a,b}(\frac{m}{\delta}) I^{m-\delta} (|v|^2 - 1) \delta^\alpha \zeta^\beta z^a z^b ,
\]
where \( \delta \leq m \) means that \( \delta_s \leq m_s \) for any \( s \in S \), \(|v|^2 = (|v_s|^2)_s \in S\); while the multi-index notations are introduced in Definition 4.1. We also set \( \Pi^{\leq d} := \sum_{d'=-2}^{d} \Pi^{d'} \) and \( \Pi^{>d} := \text{Id} - \Pi^{<d} \), analogously for \( \Pi^{<d} \) and \( \Pi^{>d} \).

Moreover we define, e.g., \( H(\leq d) := \Pi(\leq d) H \) and also \( H_{r,p} := \Pi^d H_{r,p} \) and, e.g., \( H_{r,p}^{\leq d} := \Pi^{\leq d} H_{r,p} \).

Note that, if \( S = \mathbb{Z} \), projections coincide with the ones of Section 4 of [BMP21], while if \( S = \emptyset \), \( H(\leq d) \) represents the usual homogeneous degree at \( z = 0 \).

In this way, given \( H \in H_{r,p} \), then
\[
(5.3) \quad H = H(\leq 0) + H(\geq 1) \equiv H(\leq -2) + H(\leq -1) + H(0) + H(\geq 1)
\]
where \( H(\leq -2) \) consists of terms which are constant w.r.t. both \( z \) and the "auxiliary action" \( w = |v|^2 \), \( H(\leq -1) \) is independent of the action but linear in the \( z_j \), while \( H(0) \) contributes with two terms: the one linear in the action and independent of \( z \), the second one quadratic in \( z \) and independent of the action. Finally, \( H(\geq 1) \) is what is left and \( X_{H(\geq 1)} \) vanishes on \( T_f \).

The operators \( \Pi^{d} \) define continuous projections (see Section 4 of [BMP21]) satisfying \( \Pi^{d} \Pi^{d'} = \Pi^{d} \Pi^{d'} = 0 \) for every \( d' \neq d, \, d' \geq -2 \). Moreover, this decomposition enjoys all the crucial properties required for a KAM scheme to converge, in particular they behave well with respect to Poisson brackets, that is:
\[
\forall F, G \in H_{r,p} \quad \{ F, G^{\geq 1} \}^{(-2)} = 0
\]
and
\[
F^{(-2)} = 0 \implies \{ F, G^{\geq 1} \}^{(\leq -1)} = 0,
\]
\[
F^{(-1)} = 0 = F^{(-2)} \implies \{ F, G^{\geq 1} \}^{(\leq 0)} = 0.
\]
Note also that if \( \Pi^{<d_1} F = \Pi^{<d_2} G = 0 \), then \( \Pi^{<d_1 + d_2} \{ F, G \} = 0 \). For all the properties of the projections see [BMP21] Proposition 4.1 and 4.2.

As is standard, on \( H_{r,p} \) we define the projections
\[
(5.4) \quad \Pi^K H := \sum_{\alpha \in \mathbb{N}^2} H_{\alpha,\alpha}|u|^{2\alpha}, \quad \Pi^K H := H - \Pi^K H,
\]
which are continuous on \( H_{r,p} \).

Correspondingly, we define the following subspaces of \( H_{r,p} \):
\[
(5.5) \quad H_{r,p}^{K} := \{ H \in H_{r,p} : \Pi^K H = H \}, \quad H_{r,p}^{R} := \{ H \in H_{r,p} : \Pi^K H = H \}.
\]
Moreover, e.g. \( \mathcal{H}_{r,p}^{0,R} := \mathcal{H}_{r,p}^{<0} \cap \mathcal{H}_{r,p}^{R} \). Note that \( \mathcal{H}_{r,p}^{K} \subseteq \text{ker } L_{\omega} \) and if \( \omega \in D_{r,s} \) then the two spaces coincide.

Note that by (5.2) and (5.4) we have

\[
(5.6) \quad d \text{ odd } \implies \mathcal{H}_{r,p}^{d,K} = \{0\}.
\]

In Lemma 4.3 of [BMP21] we proved that the map \( \lambda \rightarrow \Lambda \) defined by

\[
(5.7) \quad \Lambda = \sum_{j \in S} \lambda_{j}(|v_{j}|^{2} - I_{j}) + \sum_{j \in S^{c}} \lambda_{j}|z_{j}|^{2}
\]

is a linear isometry from \( \ell^{\infty} \) to \( \mathcal{K}_{r,p}^{0,K} \) for every \( r, p \). Note that, taking \( \Lambda \) as above

\[
(5.8) \quad H = \{\Lambda, G^{(d)}\} \implies H = H^{(d)}.
\]

The projections defined in (5.2) naturally extend to \( H \in \mathcal{H}_{r,p} \) or \( H \in \mathcal{H}_{r,p}(\mathbb{C}) \) setting

\[
(5.9) \quad (\Pi^{d}H)(u, \omega, I) := (\Pi^{d}_{I}H(\omega, I))(u).
\]

Similarly for (5.4).

The following result is proved in Appendix A.

**Proposition 5.1.** For every \( d \in \mathbb{N} \cup \{-2, -1\} \) and \( H \in \mathcal{H}_{r',p} = \mathcal{H}_{r',p}^{O \times I} \) where \( I = I(p, r) \) with \( r' \geq \sqrt{2r} \) the following holds.

(i) The projection operators \( \Pi^{d} : \mathcal{H}_{r',p} \rightarrow \mathcal{H}_{r',p} \) are continuous with bound

\[
(5.10) \quad \|\Pi^{d}H\|_{r',p} \leq 3^{d+1}\|H\|_{r,p}.
\]

(ii) Moreover the following representation formula holds

\[
(5.11) \quad \Pi^{d}H(u) = \sum_{\delta \in \mathbb{N}^{d}, a, b \in \mathbb{N}^{S^{c}}} (|v_{\delta}|^{2} - I)_{\delta} z_{a} \bar{z}^{b} \bar{H}_{\delta}(u) \quad \text{for } |u|_{p} < r',
\]

where \( \bar{H}_{\delta}(u) \) are analytic in \( B_{r}(w_{p}) \) and can be written in totally convergent power series in every ball \( |u|_{p} \leq \kappa_{s} r' \) with \( \kappa_{s} < 1 \).

Analogous statements hold for the complex case \( H \in \mathcal{H}_{r',p}(\mathbb{C}) = \mathcal{H}_{r',p}^{O \times I(\mathbb{C})} \) where \( I(\mathbb{C}) = I(p, r, \mathbb{C}) \) (recall formula (4.11)).

As above for \( \mathcal{H}_{r,p} \) we define the corresponding subspaces \( \mathcal{H}_{r,p}^{d}, \mathcal{H}_{r,p}^{K}, \mathcal{H}_{r,p}^{R}, \) etc. of \( \mathcal{H}_{r,p} \). Analogously for the complex case \( \mathcal{H}_{r,p}(\mathbb{C}) \). In particular we discuss the subspace \( \mathcal{H}_{r,p}^{0,K} \).

**Definition 5.2.** We denote by \( \mathcal{H}^{0,K} = \mathcal{H}^{0,K}(O \times I) \) the space of maps

\[
\mathcal{O} \times I \ni (\omega, I) \rightarrow \lambda(\omega, I) \in \ell^{\infty}(\mathbb{R}) \quad \text{with } \lambda_{j} \in \mathcal{F}(\mathcal{O} \times I) \quad \forall j \in \mathbb{Z},
\]

endowed with the norm (recall (4.9))

\[
(5.12) \quad \|\lambda\|_{\infty} := \sup_{j \in \mathbb{Z}} |\lambda_{j}|^{\gamma}.
\]
Proposition 5.1) that for $H$ since we mainly interested in the decomposition (5.3), we remark (see (5.13) one has $H$ in the complex case then by (5.7) $H$ there exists analogous estimates hold in the complex case $H \in \mathcal{H}_{r',p'}^{\mathcal{O} \times \mathcal{I}(\mathbb{C})}$.

Definition 6.1 (Normal Forms). For $\omega \in \mathcal{O}$ let

$$ D : \omega \mapsto D(\omega) := \sum_{j \in \mathbb{Z}} \omega_j |u_j|^2. $$

Let $0 < r < r_0$, $1 < p_0 \leq p$ and $\mathcal{I} = \mathcal{I}(p, r)$, as in (2.11). We will say that $N$ is an analytic family of normal forms if $N - D \in \mathcal{H}_{r_0,p_0}^{1,\mathcal{O} \times \mathcal{I}}$. We denote such affine subspace by $\mathcal{N}_{r_0,p_0} = \mathcal{N}_{r_0,p_0}^{\mathcal{O} \times \mathcal{I}}$. The same definition holds in the corresponding complex spaces.

Remark 6.1. Given $N \in \mathcal{N}_{r_0,p_0}^{\mathcal{O} \times \mathcal{I}}$ for every $\omega \in \mathcal{O}$ and $I \in \mathcal{I}(p, r)$, the flat torus $\mathcal{T}_I$ is invariant for the dynamics of $N(\omega, I)$.

Let us state our counter-term KAM result.

Theorem 5. Fix $0 < \gamma < 1$, $S$ as in (2.2), $r > 0$ and $p > 1$. Set $\mathcal{O} = \mathbb{D}_{\gamma,S}$ and $\mathcal{I} = \mathcal{I}(p, r)$. Consider $r_0, \rho, \delta > 0$ and $p_0 > 1$ with

$$ \rho \leq \frac{r_0 - r}{4}, \quad r \leq \frac{r_0}{2}, \quad \delta \leq \frac{p - p_0}{4}. $$

There exists $\tilde{\epsilon}, 1/\tilde{C} > 0$, decreasing functions of $\rho/r_0$ and $\delta$ such that the following holds. Consider $N_0 \in \mathcal{N}_{r_0,p_0}^{\mathcal{O} \times \mathcal{I}}$, $H - D \in \mathcal{H}_{r_0,p_0}^{1,\mathcal{O} \times \mathcal{I}}$ and assume that

$$ (1 + \Theta)^5 \epsilon \leq \tilde{\epsilon} $$

where

$$ \epsilon := \gamma^{-1} \sup_{I \in \mathcal{I}(p,r)} \|H - N_0\|_{r_0,p_0}, \quad \Theta := \gamma^{-1} \sup_{I \in \mathcal{I}(p,r)} \|D - N_0\|_{r_0,p_0}. $$

Then there exist a $\rho/4$-close to the identity symplectic diffeomorphism in $u$, Lipschitz in $\omega$ and analytic in $I$

$$ \Psi : B_{r_0}(w_p) \times \mathcal{O} \times \mathcal{I} \rightarrow B_{r_0}(w_p), \quad (u; \omega, I) \mapsto \Psi(u; \omega, I) $$

and a counter-term $\Lambda \in \mathcal{H}_{r_0,p_0}^{0,K}(\mathcal{O} \times \mathcal{I})$, where

$$ \Lambda = \sum_j \lambda_j \left( |u_j|^2 - I_j \right), \quad \|\lambda\|_\infty \leq \bar{C} \gamma (1 + \Theta)^2 \epsilon, $$

\[28\]Note that $D$ is a linear map into the space of formal quadratic polynomials.
such that \( \Psi(T;\omega,I) \) is an invariant torus for the dynamics of \( \Lambda(\omega,I)+H(\omega) \) for every \( \omega \in D_{\gamma,S} \) and \( I \in I(p,r) \).

More precisely there exists an analytic family of normal forms \( N \in N_{r_0-\rho,p_0+\delta} \), such that

\[
(\Lambda+H) \circ \Psi - N = 0.
\]

Finally if \( N_0 \) and \( H \) admit a complex extension on \( I(p,r,\mathbb{C}) \) satisfying \( \|P\|_{r_0-\rho,p_0} = |P|_{r_0,p_0} \), then also the counter-term \( \lambda \) extends to an holomorphic map in \( H^{0,K}(\mathcal{O} \times I(p,r,\mathbb{C})) \) satisfying \( (6.4) \).

Remark 6.2. i) The map \( \Psi \) is \( \rho/4 \)-close to the identity namely

\[
|\Psi(u;\omega,I) - u|_p \leq \rho/4, \quad \forall (u;\omega,I) \in B_{r_0-\rho}(w_p) \times \mathcal{O} \times I.
\]

ii) By \( (6.4) \) and Cauchy estimates we get that \( \lambda \) is uniformly Lipschitz in \( I(p,r/2) \)

\[
\frac{|\lambda(\omega,I) - \lambda(\omega,I')|_\infty}{|I-I'|_{2p}} \leq 4\bar{C}(1+\Theta)^2\gamma r^{-2}\epsilon,
\]

\( \forall \omega \in D_{\gamma,S}, \ I,I' \in I(p,r/2), \ I \neq I' \).

Proof of Theorem 3 and 4. Theorem 3 follows from Theorem 5 in a straightforward way. Fix \( p_0 = p - (p-1)/2, r_0 = 2r = 4r, \rho = r/8 \) and \( \delta = (p_0 - 1)/16 \). Note that by these choices

\[
(6.7)
\]

the constants \( \bar{\epsilon} \) and \( \bar{C} \) depend only on \( p_0 \).

One first rewrites \( H_V \) in \( (2.3) \) as \( D + \Lambda + P' \) where \( P' = P + \sum_j \lambda_j J_j \), by setting \( \lambda_j = j^2 - \omega_j + V_j \). Set \( H = D + P' \) and \( N_0 = D \) so that by definition \( \Theta = 0 \). Since the Hamiltonian \( P \in \mathcal{C}_{r_0,p_0}(\mathbb{C}) \) does not depend on \( \omega, I \), trivially one has \( P' \in \mathcal{C}_{r_0,p_0}(\mathbb{C}) \) and \( \|P'\|_{r_0,p_0} = |P|_{r_0,p_0} = |P|_{r_0,p_0} \).

Since by hypothesis \( \gamma \leq |f|_R \) we have by \( (2.13) \) that \( r^2/R \leq \epsilon_s \); then the hypothesis \( c_1 r^2 \leq \epsilon \) of Proposition 4.2 holds for all \( p \) satisfying \( (2.14) \) taking \( \epsilon_s \) small enough. Then, by Proposition 4.2

\[
(6.8)
\]

for some \( c(p_0) > 1 \) by \( (2.13) \). Again taking \( \epsilon_s = \epsilon_s(p_0) \) in \( (2.13) \) small enough, condition \( (6.2) \) is satisfied and Theorem 5 gives us the desired change of variables provided that \( \Lambda \in H^{0,K}(\mathbb{C}) \) is fixed accordingly.

Now we denote \( \omega_j = \nu_j \) if \( j \in S \) and \( \omega_j = \Omega_j \) otherwise. We get the equations

\[
(6.9)
\]

for \( \omega \in D_{\gamma,S} \) and \( I \in I(p,r,\mathbb{C}) \). By Lemma 4.1 we Lipschitz extend the map \( \lambda \) to the whole \( Q \times I(p,r,\mathbb{C}) \) (recall that \( r = r/2 \) in such a way

---

Note that the constants \( c_1, c_2 \) in Proposition 4.2 continuously depend on \( p \), which belongs to the compact \( ((p_0 + 1)/2, p_1] \).
that (6.4) and (6.6) hold for \( \omega, \omega' \in \mathcal{Q} \) (with \( \bar{C} \sim 2\bar{C} \)). From now on we can work only on the real \(^{30}\) case \( I \in \mathcal{I}(p, r) \). By (6.4) and taking \( \varepsilon_\ast(p_\ast) \) small enough such that \( 2\bar{C}\varepsilon \leq 2\bar{C}c(p_\ast)\varepsilon_\ast(p_\ast) \leq 1/2 \) (recall (6.8)) we use the Contraction Lemma (recall Lemma \(^{31}\) B.1), we solve the first equation finding \( \Omega : \mathcal{Q}_S \times [-1/4, 1/4]^{\mathcal{S}_c} \times \mathcal{I}(p, r) \rightarrow \mathcal{Q}_S \) which is continuous in the product topology of \( \mathcal{Q}_S \times [-1/4, 1/4]^{\mathcal{S}_c} \) and satisfies

\[
\tag{6.10}
|\Omega_j(\nu, V_{S^c}, I) - j^2 - V_{S^c}| \leq 2\bar{C}\gamma\varepsilon,
\]

\[
\sup_{\nu' \neq \nu} \frac{|\Omega_j(\nu, V_{S^c}, I) - \Omega_j(\nu', V_{S^c}, I)|}{|\nu - \nu'|_\infty} \leq 4\bar{C}\varepsilon,
\]

\[
\sup_{{V_{S^c}', \neq V_{S^c}}} \frac{|\Omega_j(\nu, V_{S^c}', I) - \Omega_j(\nu, V_{S^c}, I)|}{|V_{S^c'} - V_{S^c}|_\infty} \leq 4\bar{C}
\]

\[
\sup_{{I' \neq I}} \frac{|\Omega_j(\nu, V_{S^c}, I) - \Omega_j(\nu, V_{S^c}, I')|}{|I - I'|^{2p}_2} \leq 2\bar{C}\gamma^{-2}\varepsilon,
\]

for every \( j \in \mathcal{S}^c \) and where the first three estimates follows from (6.4) (recalling (5.12) and (4.9)) and the last one from (6.6). By (6.10) and (6.8) we prove (2.19) and (2.20) taking \( C \geq 4\bar{C}c(p_\ast) \). Note that this condition depends only on \( p_\ast \).

Finally the second equation in (6.9) uniquely defines

\[
\tag{6.11}
\nu_{S_j}(\nu, V_{S^c}, I) = \nu_j + \lambda_j(\nu, \Omega(\nu, V_{S^c}, I), I) - j^2
\]

for \( j \in \mathcal{S} \).

By Lemma \(^{32}\) A.1 iii) we can Lipschitz extend the map \( \Psi : B_{r_0 - \rho}(\nu_p) \times D_{\gamma, S} \times \mathcal{I}(p, r) \rightarrow B_{r_0}(\nu_p) \times D_{\gamma, S} \times \mathcal{I}(p, r) \) and set \( \Phi(u; \nu, V_{S^c}, I) := \Psi(u; \nu, V_{S^c}, I), I, \) where \( \nu(\nu, V_{S^c}, I) \) was defined in (2.21). The map \( \Phi \) conjugates the NLS Hamiltonian \( H_\nu \) to normal form for all \( \nu \) such that \( \omega(\nu, V_{S^c}, I) \in D_{\gamma, S}, \) namely for \( \nu \in \mathcal{C} \) (recalling (2.21)). Then the proofs of Theorems \(^{33}\) 3 and \(^{33}\) 4 are concluded provided that one shows that the set \( \mathcal{C} \) defined in (2.21) satisfies (2.16). This is the content of Lemma \(^{34}\) 9.1 proved in Section Section 9 where all the measure estimates are discussed.

\[\square\]

7. Small divisors and Homological equation

The proof of Theorem \(^{35}\) 5 is based on an iterative scheme that kills out the obstructing terms, namely terms belonging to \( \mathcal{H}_{r, p}^{-2}, \mathcal{H}_{r, p}^{-1} \) and \( \mathcal{H}_{r, p}^0, \) by solving Homological equations of the form

\[
\tag{7.1}
LF(d) = G(d), \quad G(d) \in \mathcal{H}_{r, p}^d, \quad d = -2, -1, 0.
\]

\(^{30}\)Obviously the same statements hold also in the complex case

\(^{31}\)We use it with \( F_j(\nu, V_{S^c}, w) := -\lambda_j(\nu, q + V_{S^c} + w), j \in \mathcal{S}^c, \) where \( q = (j^2)_{j \in \mathcal{S}^c} \) and \( r := \bar{C}\gamma\varepsilon.\)
where, for all \( \omega \in \mathbb{Q} \), \( L_\omega \) is an operator acting on formal Hamiltonians as
\[
L_\omega[\cdot] := \{D(\omega), \cdot\}, \quad L_\omega H(\omega, I) := i \sum_{\alpha, \beta \in \mathbb{N}^2} \omega \cdot (\alpha - \beta) H_{\alpha, \beta}(\omega, I) u^{\alpha} \bar{w}^{\beta}.
\]
The convergence of the iterative KAM scheme comes from a good control of \( L^{-1} \mathcal{G}(d) \) over the set \( D_{\gamma,S} \).

For \( \omega \in D_{\gamma,S} \) the Lie derivative operator \( L_\omega \) is formally invertible on the subspace \( \mathcal{H}_{r,p}^\mathbb{R} \) with inverse
\[
(L^{-1}_\omega H(\omega, I))_{\alpha, \beta} := -i H_{\alpha, \beta}(\omega, I) \omega \cdot (\alpha - \beta).
\]
We now show that the inverse is well defined also at a non formal level.

**Proposition 7.1.** Let \( \mathcal{O} = D_{\gamma,S} \) and \( \mathcal{I} = \mathcal{I}(p, r) \) and set \( \mathcal{H}_{r,p}^{\leq 0, \mathbb{R}} = \mathcal{H}_{r,p}^{\leq 0, \mathbb{R}, \mathcal{O} \times \mathcal{I}} \).

For every \( 0 < \delta < 1 \), (7.2) defines a bounded linear operator \( L^{-1} : \mathcal{H}_{r,p}^{\leq 0, \mathbb{R}} \rightarrow \mathcal{H}_{r,p}^{\leq \delta, \mathbb{R}} \) with estimate
\[
\|L^{-1} H\|_{r,p+\delta} \leq \frac{1}{\gamma} \exp \left( \exp \left( \frac{c}{\delta/\eta} \right) \right) \|H\|_{r,p},
\]
with \( i_s, \eta \) introduced in Definition in 2.2 and \( c = c(i_s) \). The same estimate holds in the corresponding complex spaces.

**Remark 7.1.** In this section by \( c \) we denote possibly different constants depending only on \( i_s \) introduced in Definition in 2.2.

**Proof.** Let us first show that if \( H_{\alpha, \beta} \in \mathcal{F} (\mathcal{O} \times \mathcal{I}) \) then the same holds for \( (L^{-1} H)_{\alpha, \beta} \). Indeed, since \( |\alpha| = |\beta| < \infty \), the expression \( \omega \cdot (\alpha - \beta) \) depends only on a finite number of frequencies \( \omega_j \) and hence is continuous w.r.t. the product topology. As for the Lipschitz dependence, \( \omega \in D_{\gamma,S} \) implies that \( 1/\omega \cdot (\alpha - \beta) \) is \( C^\infty \) and the result follows, since the product of Lipschitz functions is Lipschitz.

By (7.2) and (4.13) we get
\[
\|L^{-1} H\|_{r,p+\delta} \leq \frac{1}{\gamma} K \|H\|_{r,p},
\]
where
\[
K = \gamma \sup_{j \in \mathbb{Z}} \sup_{(\alpha, \beta) \in \mathcal{M}_j} \left( \frac{|j|^2}{\prod_{s \in S} |\alpha_s + \beta_s|} \right)^{\delta} \frac{1}{\omega \cdot (\alpha - \beta)},
\]
and (recall that \( H \in \mathcal{H}_{r,p}^{\leq 0, \mathbb{R}} \))
\[
\mathcal{M}_j := \left\{ (\alpha, \beta) \in \mathcal{M} \ : \ \alpha \neq \beta, \ \sum_{s \in S} \alpha_s + \beta_s \leq 2, \ \alpha_j + \beta_j \neq 0 \right\}.
\]

The constant \( 1 < \eta \leq 2 \) was introduced in (2.8a).
where $M$ was defined in (4.2). We define
\begin{equation}
\mathcal{M}_j := \left\{ (\alpha, \beta) \in \mathcal{M}_j, \quad \text{such that} \quad \left| \sum_{s \in \mathbb{Z}} (\alpha_s - \beta_s)s^2 \right| < 2 \sum_{s \in \mathbb{Z}} |\alpha_s - \beta_s| \right\},
\end{equation}
and consider
\begin{equation}
K_1 = \gamma \sup_{j \in \mathbb{Z}} \sup_{(\alpha, \beta) \in \mathcal{M}_j \setminus \mathcal{M}_j'} \left( \frac{|j|^2}{\prod_{s \in S} |\alpha_s + \beta_s|} \right) \frac{1}{|\omega \cdot (\alpha - \beta)|} \gamma^n,
\end{equation}
and
\begin{equation}
K_2 = \gamma \sup_{j \in \mathbb{Z}} \sup_{(\alpha, \beta) \in \mathcal{M}_j \setminus \mathcal{M}_j'} \left( \frac{|j|^2}{\prod_{s \in S} |\alpha_s + \beta_s|} \right) \frac{1}{|\omega \cdot (\alpha - \beta)|} \gamma^n.
\end{equation}
Then
\begin{equation}
K = \max\{K_1, K_2\}.
\end{equation}
We now have to give an upper bound on $K_1$ and $K_2$. First note that, by (2.17),
\begin{equation}
|\omega \cdot (\alpha - \beta)| \geq |\sum_{s \in \mathbb{Z}} s^2(\alpha_s - \beta_s)| - \frac{1}{2} \sum_{s \in \mathbb{Z}} |\alpha_s - \beta_s|
\end{equation}
so
\begin{equation}
\left| \sum_{s \in \mathbb{Z}} (\alpha_s - \beta_s)s^2 \right| \geq 2 \sum_{s \in \mathbb{Z}} |\alpha_s - \beta_s| \quad \Rightarrow \quad |\omega \cdot (\alpha - \beta)| \geq |\alpha - \beta| \geq 1.
\end{equation}

**Remark 7.2.** In the following we will strongly use the fact that the involved functions depend only on a finite number of $\omega_j$. In this case the Lipschitz semi-norm is bounded by the $\ell^1$-norm of the gradient.

The estimate of $K_2$ is trivial. Indeed, since $(\alpha, \beta) \in \mathcal{M}_j \setminus \mathcal{M}_j'$ then $|\omega \cdot (\alpha - \beta)| \geq |\alpha - \beta| \geq 1$. Therefore
\begin{equation}
\left| \frac{1}{\omega \cdot (\alpha - \beta)} \right| \leq \sup_{\omega \in \mathbb{D}_{\gamma, S}} \frac{1}{|\omega \cdot (\alpha - \beta)|} + \gamma \sup_{\omega \in \mathbb{D}_{\gamma, S}} \frac{|\alpha - \beta|}{(\omega \cdot (\alpha - \beta))^2} \leq 2,
\end{equation}
so that in conclusion
\begin{equation}
K_2 \leq 2.
\end{equation}
Let us now study $K_1$. Since $(\alpha, \beta) \in \mathcal{M}_j'$, then (recall Definition 2.3)
\begin{equation}
\sup_{\omega \in \mathbb{D}_{\gamma, S}} \left| \frac{1}{\omega \cdot (\alpha - \beta)} \right| \leq \frac{2}{\gamma d(\alpha - \beta)}, \quad \text{with} \quad d(\ell) := \prod_{s \in S} \frac{1}{(1 + |\ell_s|^2(i(s))^2)^{3/2}};
\end{equation}
as for the Lipschitz variation we estimate it as
\begin{equation}
\sup_{\omega \in \mathbb{D}_{\gamma, S}} \sum_j \left| \frac{\partial \omega_j}{\omega \cdot (\alpha - \beta)} \right| \leq \sup_{\omega \in \mathbb{D}_{\gamma, S}} \frac{|\alpha - \beta|}{(\omega \cdot (\alpha - \beta))^2} \leq \frac{4}{\gamma^2 (d(\alpha - \beta))^3},
\end{equation}
where the last inequality comes form \(|\alpha - \beta| \leq (d(\alpha - \beta))^{-1}\). Note that the sum in the left hand side above is over a finite number of indexes j’s. In conclusion we have proved that for \((\alpha, \beta) \in M'_{j}\)

\[
\left| \frac{1}{\omega \cdot (\alpha - \beta)} \right|^{\gamma} \leq \frac{14}{\gamma (d(\alpha - \beta))^3},
\]

and hence, recalling the definition of \(\eta\) in (2.8a), we have

\[
(7.13) \quad K_1 \leq 14K'_1,
\]

where

\[
K'_1 \equiv \sup_{j \in \mathbb{Z}} \sup_{(\alpha, \beta) \in M'_j} \left( \frac{|j|^2}{\prod_s (|\alpha_s + \beta_s|)^{\delta}} \prod_{s \in \mathcal{S}} \left(1 + |i(s)|^2 |\alpha_s - \beta_s|^2 \right)^{\epsilon_s} \right)^{1/2}.
\]

By (7.4), (7.10), (7.12), (7.13) and the following crucial estimate

\[
(7.14) \quad \log K'_1 \leq c \exp(135\delta^{-1/\eta}),
\]

Proposition 7.1 follows. \(\square\)

The rest of this section is devoted to the proof of estimate (7.14). We first need some preliminaries discussed in the following subsection.

### 7.1. Preliminaries

We first state an elementary result that will be useful below:

**Lemma 7.1.** If \(\ell \in \mathbb{Z}^2\) with \(|\ell| < \infty\), satisfies \(m(\ell) = \pi(\ell) = 0\), then \(|\ell|\) is even and \(|\ell| \neq 2\).

**Proof.** As usual we write in a unique way \(\ell = \ell^+ - \ell^-\) where \(\ell^+ \geq 0\) and \(\ell_s^+ \ell_s^- = 0\) for every \(s \in \mathbb{Z}\). Since \(m(\ell) = 0\) we get \(|\ell^+| = |\ell^-|\), therefore \(|\ell| = |\ell^+| + |\ell^-|\) is even.

Now assume by contradiction that \(|\ell^+| = |\ell^-| = 1\). Then \(\ell^+ = e_i, \ell^- = e_j\) for some \(i \neq j\) and \(\pi(\ell) = i - j\), which contradicts \(\pi(\ell) = 0\). \(\square\)

We now need some basic results and notations coming from [BMP20a].

**Definition 7.1.** Given a vector \(v \in \mathbb{N}^2\), \(0 < |v| < \infty\), we denote by \(\hat{n} = \hat{n}(v)\) the vector \((\hat{n}_i)_{i \in I}\) (where \(I \subset \mathbb{N}\) is finite) which is the decreasing rearrangement of

\[\{N \ni h > 1 \text{ repeated } v_h + v_{-h} \text{ times} \} \cup \{1 \text{ repeated } v_1 + v_{-1} + v_0 \text{ times} \}.\]

**Remark 7.3.** A good way of envisioning this list is as follows. Given the set of (commutative) variables \((x_j)_{j \in \mathbb{Z}}\), we consider a monomial (recall (4.1))

\[x^v = \prod_{i} x_i^{v_i} = x_{j_1} x_{j_2} \cdots x_{|v|},\]

then \(\hat{n}(v)\) is the decreasing rearrangement of the list \((\langle j_1 \rangle, \ldots, \langle j_{|v|} \rangle)\). As an example consider \(v = (v_j)_{j \in \mathbb{Z}}\) with \(v_{-6} = 1, v_{-3} = 4, v_{-1} = v_0 = 1, v_1 = v_6 = 2\), and \(v_j = 0\) otherwise, then \(\hat{n}(v) = (6, 6, 6, 3, 3, 3, 3, 1, 1, 1, 1, 1, 1).\)
Given \((\alpha, \beta) \in \mathcal{M}\) with \(|\alpha| = |\beta| > 1\), from now on we define
\[\hat{n} := \hat{n}(\alpha + \beta) .\]

We observe that there exists a choice of \(\sigma_i = \pm 1, 0\) such that by momentum conservation
\[(7.15) \sum_{l} \sigma_l \hat{n}_l = 0 .\]
with \(\sigma_l \neq 0\) if \(\hat{n}_l \neq 1\). Indeed we recall that
\[
\sum_{j \in \mathbb{Z}} j(\alpha_j - \beta_j) = \sum_{h \in \mathbb{N}} h(\alpha_h + \beta_{-h} - (\alpha_{-h} + \beta_h))
\]
and that each \(h > 1\) appears exactly \(\alpha_h + \alpha_{-h} + \beta_h + \beta_{-h}\) times in the sequence \(\hat{n}\). Hence we assign the value \(\sigma = 1, \alpha_h + \beta_{-h}\) times to the \(\hat{n}_l = h\) (and \(\sigma = -1\) the remaining times). The value \(h = 1\) instead appears in the sequence \(\hat{n}\), \(\alpha_1 + \alpha_{-1} + \beta_1 + \beta_{-1} + \alpha_0 + \beta_0\) times. Hence we assign the value \(\sigma = 1, \alpha_1 + \beta_{-1}\) times, the value \(\sigma = -1, \alpha_{-1} + \beta_1\) times and \(\sigma = 0\) all the remaining times.

From \((7.15)\) we deduce
\[(7.16) \hat{n}_1 \leq \sum_{l \geq 2} \hat{n}_l .\]

Indeed, if \(\sigma_1 = \pm 1\), the inequality follows directly from \((7.15)\); if \(\sigma_1 = 0\), then \(\hat{n}_1 = 1\) and consequently \(\hat{n}_l = 1 \forall l\). Since the mass is conserved, the list \(\hat{n}\) has at least two elements, and the inequality is achieved.

We finally need the following elementary result proved at the end of Appendix A.

Lemma 7.2. Let \(x_1 \geq x_2 \geq \ldots \geq x_N \geq 2\). Then
\[(7.17) \frac{\sum_{1 \leq \ell \leq N} x_{\ell}}{\prod_{1 \leq \ell \leq N} \sqrt{x_{\ell}}} \leq \sqrt{x_1} + \frac{4}{\sqrt{x_1}} .\]

7.2. Proof of estimate \((7.14)\). For \(j \in \mathbb{Z}\) and \((\alpha, \beta) \in \mathcal{M}_j\) by Lemma 7.2 we can write
\[(7.18) K' \leq K'' := \sup_{j \in \mathbb{Z}} \sup_{(\alpha, \beta) \in \mathcal{M}_j} \left( \frac{3}{\prod_{l \geq 3} |\hat{n}_l|^{1/2}} \right) \prod_{s \in S} \left( 1 + |i(s)|^2 |\alpha_s - \beta_s|^2 \right)^{9/2} .\]

Then \((7.14)\) follows by
\[(7.19) \log K'' \leq c \exp(135\delta^{1/\eta}) .\]
It remains to prove (7.19).

In the following we call \( j_1, j_2 \in S^c \) with \( |j_1| \geq |j_2| \) the possible normal sites. Set

\[
\alpha_i = \alpha_{s(i)}, \quad \beta_i = \beta_{s(i)},
\]

the quantity in the right hand side of (7.18) reads

\[
\frac{3}{\Pi_{i \geq 3}[\hat{n}_i]^{1/2}} \prod_{i \in \mathbb{N}} \left( 1 + \langle i \rangle^2 |\alpha_i - \beta_i|^2 \right)^{9/2}.
\]

By Lemma 7.1, the constraint \( \sum_{s \in S^c} \alpha_s + \beta_s \leq 2 \) implies that there exists at least one \( s \in S \) such that \( \alpha_s + \beta_s \neq 0 \). We denote the largest in absolute value \( s \in S \) with this property as \( s_M \) and \( i_M := i(s_M) \).

In proving (7.19) we often meet the quantity

\[
A_k(\delta) := \sum_{i \leq i_M} -\frac{\delta}{9} k_i \log |s(i)| + \log (1 + \langle i \rangle^2 k_i^2),
\]

defined for \( k \in \mathbb{N} \), which is estimated by the following result, whose proof is postponed in Appendix A.

**Lemma 7.3.** For every \( k \in \mathbb{N} \)

\[
A_k(\delta) \leq c \exp(45 \delta^{1/\eta}).
\]

We divide the proof of (7.19) in six cases according to how many and in which position the normal sites appear in the list \( \hat{n} \).

**Case 1.** \( \hat{n}_2 > s_M \). Here, since there are at most two normal sites we get

\[
\Pi_{i \geq 3}[\hat{n}_i] = \prod_{i \leq i_M} |s(i)|^{\alpha_i + \beta_i}.
\]

Recalling that

\[
\langle i(s) \rangle |\alpha_s - \beta_s| = \langle i \rangle |\alpha_i - \beta_i| \leq \langle i \rangle (\alpha_i + \beta_i), \quad \text{if} \quad s = s(i), \quad i \in \mathbb{N},
\]

we get

\[
\log K'' \leq \log(3^\delta)
\]

\[
+ \sup_{\alpha, \beta} \left( -\frac{\delta}{2} \sum_{i \leq i_M} (\alpha_i + \beta_i) \log |s(i)| + 9/2 \sum_{i \leq i_M} \log \left( 1 + \langle i \rangle^2 (\alpha_i + \beta_i)^2 \right) \right)
\]

\[
= \log(3^\delta) + 9/2 \sup_{k \in \mathbb{N}} A_k(\delta) \leq c \exp(45 \delta^{1/\eta}),
\]

where \( A_k \) was defined in (7.21), estimated in Lemma 7.3.

---

\[^{33}\text{Recall Definition 4.1. Note that the sum over } i \text{ is restricted to the indexes such that } k_i \geq 1.\]
Case 2. \( \hat{n}_1 > s_k = \hat{n}_2 \) and only one normal site. We have \( \sum_{i \in S^c} \alpha_i + \beta_i = 1 \) and the normal site must be \( \hat{n}_1 \). Moreover
\[
(7.23) \quad \Pi_{i \geq 3} [s(i)] = [s(i)]^{\alpha_i + \beta_i} \prod_{i < n} [s(i)]^{\alpha_i + \beta_i},
\]
so
\[
K''_1 \leq 3^\delta \sup_{\alpha, \beta} [s(i)]^{-(\alpha_i + \beta_i - 1)\delta/2} \left( 1 + \langle i \rangle^2 |\alpha_i - \beta_i|^2 \right)^{9/2}
\]
(7.24)
\[
\times \prod_{i < s} [s(i)]^{-(\alpha_i + \beta_i)\delta/2} \left( 1 + \langle i \rangle^2 (\alpha_i + \beta_i)^2 \right)^{9/2}.
\]

If \( \alpha_{i_k} = \beta_{i_k} \) and hence \( i_k \) does not appear in the small divisors, then we proceed as in case 1.

(a) If \( \alpha_{i_k} + \beta_{i_k} \geq 2 \) then we claim that
\[
(7.25) \quad \log \left( [s(i_k)]^{-(\alpha_{i_k} + \beta_{i_k} - 1)\delta/2} \left( 1 + \langle i \rangle^2 |\alpha_i - \beta_i|^2 \right)^{9/2} \right) \leq \frac{c}{\delta}
\]
In order to prove our claim we consider two cases \( i_k \leq i_* \) and \( i_k > i_* \). In the first case, letting \( x = \alpha_{i_k} + \beta_{i_k} - 1 \geq 1 \), the left hand side of (7.25) is bounded by
\[
\frac{9}{2} \left( -\delta x \log 2 + \log (1 + i_k^2 (x + 1)^2) \right) \quad \text{with} \quad \delta := \frac{\delta}{9}
\]
which is negative for \( x \geq c/\delta^2 \) and, therefore, bounded by \( c \log \delta \).

Consider now the case \( i_k > i_* \). By (2.8a) the left hand side of (7.25) is bounded by
\[
\frac{9}{2} \left( -\delta x \log (1 + \langle i \rangle (x + 1)^2) \right) \leq \frac{9}{2} f(i_k, x),
\]
with \( f \) defined in (A.6). Reasoning as above we can estimate \( f(i_k, x) \) by \( c/\delta \) obtaining (7.25). By (7.24) and (7.25) we have
\[
\log K''_1 \leq \log (3^\delta) + \frac{c}{\delta}
\]
\[
+ \sup_{\alpha, \beta} \left( \frac{\delta}{2} \sum_{i < i_k} (\alpha_i + \beta_i) \log [s(i)] + \frac{9}{2} \sum_{i < i_k} \log (1 + \langle i \rangle^2 (\alpha_i + \beta_i)^2) \right)
\]
\[
\leq \log (3^\delta) + \frac{c}{\delta} + \sup_{k \in \mathbb{N}^c} \left| A_k(\delta) \right| \leq c \exp(45\delta^{-1/\eta}),
\]
by (7.21) and Lemma 7.3.

(b) If \( \alpha_{i_k} + \beta_{i_k} = 1 \) (then \( |\alpha_{i_k} - \beta_{i_k}| = 1 \)), here the second factor in (7.24) is equal to one. Thus in order to bound the third factor (i.e. \( (1 + \langle i \rangle)^2 |\alpha_i - \beta_i|^2 \leq 2^{9/2} |i_k|^2 \)) we distinguish two cases: \( i_k \leq i_* \) and \( i_k > i_* \). In the first case \( 2^{9/2} |i_k|^2 \leq 2^{9/2} |i_*|^2 \) and the estimate of \( K''_1 \) in (7.24) proceeds as in case 1 above. On the other hand when \( i_k > i_* \) we need a different argument.

---

Recalling that \( i_k > i_* \geq 21 \) and \( \log 21 \geq 3 \).
Given $u \in \mathbb{Z}^\mathbb{Z}$, with $|u| < \infty$, consider the set

$$\{ j \neq 0, \text{ repeated } |u_j| \text{ times} \},$$

where $D < \infty$ is its cardinality. Define the vector $m = m(u)$ as the reordering of the elements of the set above such that $|m_1| \geq |m_2| \geq \cdots \geq |m_D| \geq 1$. Given $\alpha \neq \beta \in \mathbb{N}^\mathbb{Z}$, with $|\alpha| = |\beta| < \infty$ we consider $m = m(\alpha - \beta)$ and $\hat{n} = \hat{n}(\alpha + \beta)$.

**Lemma 7.4** (Lemma C.4 of [BMP20a]). Given $\alpha \neq \beta \in \mathbb{N}^\mathbb{Z}$, with $1 \leq |\alpha| = |\beta| < \infty$ and satisfying (7.7), we have

$$|m_1| \leq \sum_{l \geq 3} \hat{n}_l^2.$$

Note that

$$\alpha_{i_M} \neq \beta_{i_M} \implies s_M \leq |m_1|.$$

In the present case $|\alpha_{i_M} - \beta_{i_M}| = 1$, by (7.27)

$$s(i_M) = s_M \leq |m_1| \leq \sum_{l \geq 3} \hat{n}_l^2$$

$$= \sum_{i < i_M} |s(i)|^2 (\alpha_i + \beta_i) \leq \sum_{i \leq i_M} |s(i)|^2 (\alpha_i + \beta_i)$$

$$+ \sum_{i_M < i < i_{M'}} s^2(i) (\alpha_i + \beta_i) \leq \sum_{i \leq i_M} s^2_i (\alpha_i + \beta_i) + 31 \sum_{i_M < i < i_{M'}} s^2(i) (\alpha_i + \beta_i)$$

using that $s(i)$ is increasing. By (2.8a)-(2.8c), for the inverse function $i(s)$ we have for integer $j \geq 1$ and $s, s' \geq s_* := s(i_*)$

$$i(s + s') \leq i(s) + i(s'), \quad i(js) \leq ji(s), \quad i(s^2) \leq i^2(s).$$

Applying the inverse function $i(s)$ to the inequalities in (7.28) and using (7.29) we get

$$i_M \leq \sum_{i \leq i_M} \star_i (\alpha_i + \beta_i) + \sum_{i_M < i < i_{M'}} \star_i^2 (\alpha_i + \beta_i) \leq c \sum_{i \leq i_M} \star_i^2 (\alpha_i + \beta_i)$$

using that $|s(i)|$ is increasing.

$$\hat{n} = \hat{n}(\alpha + \beta).$$

$\text{Lemma C.4 of [BMP20a]}$.

Given $\alpha \neq \beta \in \mathbb{N}^\mathbb{Z}$, with $1 \leq |\alpha| = |\beta| < \infty$ and satisfying (7.7), we have

$$|m_1| \leq 31 \sum_{l \geq 3} \hat{n}_l^2.$$

Note that

$$\alpha_{i_M} \neq \beta_{i_M} \implies s_M \leq |m_1|.$$

In the present case $|\alpha_{i_M} - \beta_{i_M}| = 1$, by (7.27)

$$s(i_M) = s_M \leq |m_1| \leq \sum_{l \geq 3} \hat{n}_l^2$$

$$= \sum_{i < i_M} |s(i)|^2 (\alpha_i + \beta_i) \leq \sum_{i \leq i_M} |s(i)|^2 (\alpha_i + \beta_i)$$

$$+ \sum_{i_M < i < i_{M'}} s^2(i) (\alpha_i + \beta_i) \leq \sum_{i \leq i_M} s^2_i (\alpha_i + \beta_i) + 31 \sum_{i_M < i < i_{M'}} s^2(i) (\alpha_i + \beta_i)$$

using that $s(i)$ is increasing. By (2.8a)-(2.8c), for the inverse function $i(s)$ we have for integer $j \geq 1$ and $s, s' \geq s_* := s(i_*)$

$$i(s + s') \leq i(s) + i(s'), \quad i(js) \leq ji(s), \quad i(s^2) \leq i^2(s).$$

Applying the inverse function $i(s)$ to the inequalities in (7.28) and using (7.29) we get

$$i_M \leq \sum_{i \leq i_M} \star_i (\alpha_i + \beta_i) + \sum_{i_M < i < i_{M'}} \star_i^2 (\alpha_i + \beta_i) \leq c \sum_{i \leq i_M} \star_i^2 (\alpha_i + \beta_i)$$

$$\leq c \prod_{i < i_M} \left(1 + \langle i \rangle^2 (\alpha_i + \beta_i)^2 \right)$$

$\text{Lemma 7.4 (Lemma C.4 of [BMP20a]). Given } \alpha \neq \beta \in \mathbb{N}^\mathbb{Z}, \text{ with } 1 \leq |\alpha| = |\beta| < \infty \text{ and satisfying (7.7), we have}$

$$|m_1| \leq 31 \sum_{l \geq 3} \hat{n}_l^2.$$
where by \(\sum^*\) we mean that the sum is only over the indexes \(i\) such that \(\alpha_i + \beta_i \geq 1\). By (7.24) we get

\[
\log K_1'' \leq c + \log(3^\delta) + \sup_{\alpha, \beta} \sum_{i < i_M} \left( -\frac{\delta}{2} (\alpha_i + \beta_i) \log |s(i)| + \frac{27}{2} \log \left( 1 + (i)^2 (\alpha_i + \beta_i)^2 \right) \right)
\]

(7.31) \[
\leq c + \log(3^\delta) + \frac{27}{2} \sup_{k \in \mathbb{N}^2} A_k(\delta/3) \leq c \exp(135\delta^{-1/\eta}),
\]

again by (7.21) and Lemma 7.3.

**Case 3.** \(\hat{n}_1 > s_M = \hat{n}_2\) and two normal sites.

Now \(\Pi_{l \geq 3}|\hat{n}_l| = |j_2| s(i_M)|^{\alpha_{i_M} + \beta_{i_M} - 1} \prod_{i < i_M} |s(i)|^{\alpha_i + \beta_i}

(7.32)

where \(j_2\) is the smallest\(^{36}\) normal site. We have

\[
K_1'' \leq 3^\delta |j_2|^{-\delta/2} \sup_{\alpha, \beta} |s(i_M)|^{-\left(\alpha_{i_M} + \beta_{i_M} - 1\right)\delta/2} \left( 1 + (i_M)^2 |\alpha_{i_M} - \beta_{i_M}|^2 \right)^{9/2}
\]

(7.33)

\[
\times \prod_{i < i_M} |s(i)|^{-\left(\alpha_i + \beta_i\right)\delta/2} \left( 1 + (i)^2 (\alpha_i + \beta_i)^2 \right)^{9/2}.
\]

(a) If \(\alpha_{i_M} = \beta_{i_M}\), or if \(\alpha_{i_M} + \beta_{i_M} \geq 2\) then we proceed as in case 2-(a), since \(|j_2|^{-\delta/2} \leq 1\) and does not bother.

(b) Let now \(\alpha_{i_M} + \beta_{i_M} = 1\) (so that (7.27) holds). The analogous of (7.30) is

\[
i_M \leq c \left( (|j_2|) \right)^2 \prod_{i < i_M} \left( 1 + (i)^2 (\alpha_i + \beta_i)^2 \right).
\]

(7.34)

Then by (7.33) we get

\[
\log K_1'' \leq c + \log(3^\delta) + 18 \log i(|j_2|) - \frac{\delta}{2} \log |j_2|
\]

(7.35)

\[
+ \sup_{\alpha, \beta} \sum_{i < i_M} \left( -\frac{\delta}{2} \log |s(i)| + \frac{27}{2} \log \left( 1 + (i)^2 (\alpha_i + \beta_i)^2 \right) \right).
\]

Since

\[
\sup_{x \geq 2} \left( 18 \log i(x) - \frac{\delta}{2} \log x \right) = \sup_{y \geq i(2)} \left( 18 \log y - \frac{\delta}{2} \log y \right)
\]

\[
\leq c + \sup_{y \geq 4} \left( 18 \log y - \frac{\delta}{2} \log^2 y \right) \leq c + \frac{162}{\delta},
\]

we obtain

\[
\log K_1'' \leq c + \log(3^\delta) + \frac{162}{\delta} + \frac{27}{2} \sup_{k \in \mathbb{N}^2} A_k(\delta/3) \leq c \exp(135\delta^{-1/\eta}),
\]

---

\(^{36}\)In absolute value.
again by (7.21) and Lemma 7.3.

**Case 4** \( \hat{n}_1 = s_M \) and the (eventual) normal sites are \( < \hat{n}_2 \). Recall that we have to estimate from above the quantity in (7.20).

Let us start with the case of two normal sites \( j_1, j_2 \).

We start considering the case \( \hat{n}_1 = \hat{n}_2 \), which implies \( \alpha_{i_M} + \beta_{i_M} \geq 2 \). We get

\[
(7.36) \quad \Pi_{l \geq 3} [\hat{n}_l] = |j_1| |j_2| [s(i_M)]^{\alpha_{i_M} + \beta_{i_M} - 2} \prod_{i < i_M} [s(i)]^{\alpha_i + \beta_i}.
\]

(a) If \( \alpha_{i_M} = \beta_{i_M} \), or if \( \alpha_{i_M} + \beta_{i_M} \geq 2 \) then we are reduced to case 3-(a).

(b) If \( \alpha_{i_M} = 2, \beta_{i_M} = 0 \) (or vice versa), we proceed as in case 3-(b) and apply Lemma C.4. Since, again, (7.27) holds, the analogous of (7.34) is

\[
(7.37) \quad i_M \leq c (i(|j_1|))^2 (i(|j_2|))^2 \prod_{i < i_M} \left( 1 + i^2 (\alpha_i + \beta_i)^2 \right).
\]

Then the analogous of (7.35) is

\[
\log K''_1 \leq c + \log(3^5) + 36 \log i(|j_1|) - \frac{\delta}{2} \log |j_1| + \sup_{\alpha_i \beta_i} \sum_{i < i_M} \left( -\frac{\delta}{2} \log [s(i)] + \frac{27}{2} \log \left( 1 + i^2 (\alpha_i + \beta_i)^2 \right) \right).
\]

We conclude as in case 3-(b).

We now consider the case \( \hat{n}_1 > \hat{n}_2 \).

Note that \( \alpha_{i_M} + \beta_{i_M} = 1 \) (so that (7.27) holds). Let us denote by \( s'_{i_M} \) the second largest tangential site and set \( i_M' := i(s'_{i_M}) \). By construction \( s'_{i_M} = \hat{n}_2 \) and \( \alpha_{i_M'} + \beta_{i_M'} \geq 1 \). Then

\[
\Pi_{l \geq 3} [\hat{n}_l] = |j_1| |j_2| [s(s'_{i_M})]^{\alpha_{i_M'} + \beta_{i_M'} - 1} \prod_{i < i_M'} [s(i)]^{\alpha_i + \beta_i}.
\]

In this case the analogous of (7.37) is

\[
i_M' \leq c (i(|j_1|))^4 \prod_{i < i_M'} \left( 1 + i^2 (\alpha_i + \beta_i)^2 \right),
\]

which follows by (7.28) where the second line holds with \( i_M' \) instead of \( i_M \). Then we proceed as in the case \( \hat{n}_1 = \hat{n}_2 \).

Note If there is only one normal site or if there is none, then the same arguments apply word by word with the only "advantage" that there is only one \( |j_1| \) or none in (7.36).

\(^{37}\) In the notation of BMP20a \( \hat{n}_1 = |m_1| \)

\(^{38}\) Recall that the tangential sites we are considering are positive.
Case 5 \( \hat{n}_1 = s_M \) and only one normal site \( j_1 = \hat{n}_2 \). Here (7.23) holds and we proceed as in case 2.

Case 6 \( \hat{n}_1 = s_M, j_1 = \hat{n}_2 \) and two normal sites. The proof follows word by word the one of case 3. This concludes the proof of (7.19), which, by (7.18), implies (7.14).

8. Iterative Lemma and Proof of Theorem 5

Let \( r, r_0, p, p_0, \rho, \delta \) be as in (6.1) and \( 1 < \eta \leq 2 \) as in Definition 2.2. Let \( \{ \rho_n \}_{n \in \mathbb{N}}, \{ \delta_n \}_{n \in \mathbb{N}} \) be the summable sequences:

\[
\rho_n = \frac{\rho_2^{-n}}{6}, \quad \delta_n = c_\eta \delta_n^{-1 + \frac{\eta}{2}} \quad \forall n \geq 1, \quad c_\eta^{-1} := \frac{24}{5} \sum_{n \geq 1} n^{-1 + \frac{\eta}{2}} > 12, \quad \delta_0 = \frac{\delta}{8};
\]

Let us define recursively

\[
\begin{align*}
 r_{n+1} &= r_n - 3\rho_n \to r_\infty := r_0 - \rho \geq 7r/4 \quad \text{(decreasing)}, \\
 p_{n+1} &= p_n + 3\delta_n \to p_\infty := p_0 + \delta < p \quad \text{(increasing)},
\end{align*}
\]

recalling (6.1).

Set \( O = D_{r, S}, I = I(p, r) \) and \( I(\mathbb{C}) = I(p, r, \mathbb{C}) \) (recall (2.18), (2.11) and (4.11)). Since these sets are fixed, we omit to write them explicitly in the notations of this section. For instance we denote

\[
H_{r_n, p_n} = H_{r_n, p_n}^{O \times I(\mathbb{C})}.
\]

By (5.13) and (8.2), we can use projections \( \Pi^0, \Pi^{-1}, \Pi^\geq 1 \), on these spaces for all \( n \).

Remark 8.1. A crucial point for the convergence of the algorithm is that, thanks to (5.13), no small divisor appears in the estimate of the counter-terms, see (8.22) and (8.24) below.

Let

\[
H_0 := D + G_0 + \Lambda_0, \quad G_0 \in H_{r_0, p_0}(\mathbb{C}), \quad \Lambda_0 \in H_{r, p}^{0, \mathcal{K}},
\]

where \( D \) is defined in (6.1) and the counter-terms \( \Lambda_0 = \sum_{j \in \mathbb{Z}} \lambda_{0,j} (|u_j|^2 - I_j) \) with \( \lambda_0 = (\lambda_{0,j})_{j \in \mathbb{Z}} \) free parameters in \( \ell^\infty \). We define

\[
\varepsilon_0 := \gamma^{-1} \left( \left\| G_0^{(0, \mathcal{K})} \right\|_\infty + \left\| G_0^{(0, \mathcal{R})} \right\|_{r_0, p_0} + \left\| G_0^{(-2)} \right\|_{r_0, p_0} + \left\| G_0^{(-1)} \right\|_{r_0, p_0} \right),
\]

\[
\Theta_0 := \gamma^{-1} \left\| G_0^{(2)} \right\|_{r_0, p_0} + \varepsilon_0.
\]

\[\text{Note that } \frac{1 + \eta}{2} > 1.\]
Lemma 8.1 (Iterative step). There exists a constant $C > 1$ large enough\footnote{Depending only on $i_*, \eta$ defined in (2.8a).} such that if

$$

\varepsilon_0 \leq (1 + \Theta_0)^{-5}K^{-3}, \quad K := \left(\frac{r_0}{\rho}\right)^6 \sup_{n \geq 1} 2^{6n} \exp \left(\frac{\zeta n^2 - \chi n(1 - \chi/2)}{\xi_{n^2}}\right),

(8.5) \quad \text{where } \zeta := \exp \left(\frac{\xi_{n^2}}{\xi_{n^2}}\right), \quad \xi := \frac{1 + n}{2\eta} < 1, \quad \chi := 3/2,

$$

with $\eta$ as in (2.8a), then we can iteratively construct a sequence of generating functions $S_i = S_i^{(-2)} + S_i^{(-1)} + S_i^{(0)} \in H_{r_i - \rho_i, r_{i+1}}(\mathbb{C})$ and a sequence of counter-terms $\Lambda_i \in H^{0, K}(\mathbb{C})$ such that the following holds, for $n \geq 0$.

(1) For all $\omega \in D_{\gamma, S}, I \in I(p, r)$, for all $i = 0, \ldots, n - 1$ and all $p' \geq p_{i+1}$ the time-1 Hamiltonian flow $\Phi_{S_i}$ generated by $S_i = S_i(\omega, I)$ satisfies

$$

\sup_{u \in B_{r_{i+1}}(u_{p'})} |\Phi_{S_i}(u)|_{p'} \leq \rho 2^{-2i - 7}.

(8.6) \quad \text{Moreover}
$$

$$

\Psi_n := \Phi_{S_0} \circ \cdots \circ \Phi_{S_{n-1}}

(8.7) \quad \text{is a well defined, analytic map } B_{r_n}(u_{p'}) \to B_{r_0}(u_{p'}) \text{ for all } p' \geq p_n \text{ with the bound}
$$

$$

\sup_{u \in B_{r_n}(u_{p'})} |\Psi_n(u) - \Psi_{n-1}(u)|_{p'} \leq \rho 2^{-2n - 3}.

(8.8) \quad \text{Moreover}
$$

(2) We set $L_0 := 0$ and for $i = 1, \ldots, n$

$$

L_i := e^{(S_{i-1})^{-1}}(L_{i-1} + \text{Id}), \quad \Lambda_i := \Lambda_{i-1} - \tilde{\Lambda}_{i-1}, \quad H_i = e^{(S_{i-1})^{-1}} H_{i-1}

(8.9) \quad \text{where } \Lambda_i \in H^{0, K}_{r_{i-1}, p_i} \text{ are free parameters and } L_i : H^{0, K}(\mathbb{C}) \to H_{r_{i-1}, p_i}(\mathbb{C}) \text{ are bounded linear operators. Note that } \Lambda_i \text{ are free parameters, while } \tilde{\Lambda}_i \text{ are given functions of } (\omega, I). \text{ We have}
$$

$$

H_i = D + G_i + (\text{Id} + L_i)\Lambda_i, \quad G_i, \in H_{r_{i-1}, p_i}(\mathbb{C}).

(8.10) \quad \text{Setting for } i = 0, \ldots, n
$$

$$

\varepsilon_i := \gamma^{-1} \left(\left\|G_i^{(0, K)}\right\|_{\infty} + \left\|G_i^{(0, \mathcal{R})}\right\|_{r_i, p_i} + \left\|G_i^{(-2)}\right\|_{r_i, p_i} + \left\|G_i^{(-1)}\right\|_{r_i, p_i}\right),

\Theta_i := \gamma^{-1} \left\|G_i^{(1)}\right\|_{r_i, p_i} + \varepsilon_i,

(8.11) \quad \text{we have}
$$

$$

\varepsilon_i \leq \varepsilon_0 e^{-\chi^{-1}}, \quad \Theta_i \leq \Theta_0 \sum_{j=0}^{i} 2^{-j},

(8.12) \quad \left\|(L_i - L_{i-1})h\right\|_{r_i, p_i} \leq K\varepsilon_0 (1 + \Theta_0)^2 2^{-i} \left\|h\right\|_{\infty}

(8.13) \quad \left\|L_i h\right\|_{r_i, p_i} \leq K(1 + \Theta_0)^2 \varepsilon_0 \sum_{j=1}^{i} 2^{-j} \left\|h\right\|_{\infty}.
for all $h \in \mathcal{H}^{0,K}(\mathbb{C})$. Finally the counter-terms satisfy the bound
\begin{equation}
\| \bar{A}_{i-1} \|_\infty \leq \gamma K \varepsilon_{i-1} (1 + \Theta_0)^2, \quad i = 1, \ldots, n.
\end{equation}

**Proof of Theorem 3.** Starting from the Hamiltonian $H$ satisfying (6.2), we set $G_0 = H - D$ in (8.3). The smallness conditions (8.5) are met, provided that we choose $\bar{\varepsilon}$ and $C$ appropriately.

Using (8.8) we define $\Psi$ as the limit of the $\Psi_n$ (which define a Cauchy sequence) and $\Lambda = \Lambda_0 = \sum_j \lambda_j < \infty$. Note that the series is summable by (8.14). For more details see [BMP21, Section 6].

**Proof of the iterative Lemma.** We start with a Hamiltonian $H_0 = D + \Lambda_0 + G_0$ with $\Lambda_0 \in \mathcal{H}^{0,K}$ and $D$.

At the $n$'th step we have an expression of the form
\begin{equation*}
H_n = D + (\text{Id} + \mathcal{L}_n) \Lambda_n + G_n,
\end{equation*}
with $G_n \in \mathcal{H}_{r_n,p_n}$. To proceed to the step $n + 1$ we apply the change of variables $e^{(S_n^i)}$. The generating function $S_n$ and the counter-term $\bar{A}_n$ are fixed as the unique solutions of the Homological equation
\begin{equation}
\Pi^{\leq 0}\left\{ S_n, D + G_n^{\geq 1} \right\} + (\text{Id} + \mathcal{L}_n) \bar{A}_n + G_n^{(-2,\mathcal{R})} = 0,
\end{equation}
recalling that $G_n^{(-1,\mathcal{R})} = 0$ by (5.6). This equation can be written component-wise as a triangular system and solved consequently. Indeed we have
\begin{align}
\left\{ S_n^{(-2)}, D \right\} + \Pi^{-2,\mathcal{R}} \mathcal{L}_n \bar{A}_n + G_n^{(-2,\mathcal{R})} &= 0, \\
\left\{ S_n^{(-1)}, D \right\} + \Pi^{-1} \left\{ S_n^{(-2)}, G_n^{\geq 1} \right\} + \Pi^{-1} \mathcal{L}_n \bar{A}_n + G_n^{(-1)} &= 0, \\
\Pi^{0,K} \left\{ S_n^{(-2)} + S_n^{(-1)}, G_n^{\geq 1} \right\} + \bar{A}_n + \Pi^{0,K} \mathcal{L}_n \bar{A}_n + G_n^{(0,K)} &= 0, \\
\left\{ S_n^{(0,\mathcal{R})}, D \right\} + \Pi^{0,\mathcal{R}} \left\{ S_n^{(-2)} + S_n^{(-1)}, G_n^{\geq 1} \right\} + \Pi^{0,\mathcal{R}} \mathcal{L}_n \bar{A}_n + G_n^{(0,\mathcal{R})} &= 0.
\end{align}

We start by solving the equations for $S_n$ "modulo $\bar{A}_n\”$, then we determine the counter-term by inversion of an appropriate linear operator resulting from inserting the equations for $S_n$ into equation (8.18).

We hence have by Proposition 7.1
\begin{align}
S_n^{(-2)} &= L^{-1} \left( \Pi^{-2} \mathcal{L}_n \bar{A}_n + G_n^{(-2)} \right), \\
S_n^{(-1)} &= L^{-1} \left( \Pi^{-1} \left\{ S_n^{(-2)}, G_n^{\geq 1} \right\} + \Pi^{-1} \mathcal{L}_n \bar{A}_n + G_n^{(-1)} \right), \\
S_n^{(0,\mathcal{R})} &= L^{-1} \left( \Pi^{0,\mathcal{R}} \left\{ S_n^{(-2)} + S_n^{(-1)}, G_n^{\geq 1} \right\} + \Pi^{0,\mathcal{R}} \mathcal{L}_n \bar{A}_n + G_n^{(0,\mathcal{R})} \right).
\end{align}
Plugging them into (8.18) we thus get

\[ \Pi^{0,K}\{L^{-1}(\Pi^{\leq-1}L_n\tilde{A}_n + \Pi^{-1}\{L^{-1}\Pi^{-2}L_n\tilde{A}_n, G_n^{\geq1}\}), G_n^{\geq1}\} + \tilde{A}_n + \Pi^{0,K}L_n\tilde{A}_n = -\Pi^{0,K}\{L^{-1}(G_n^{\leq-1} + \Pi^{-1}\{L^{-1}G_n^{(-2)}, G_n^{\geq1}\}), G_n^{\geq1}\} - G_n^{(0,K)}. \]

Note that the left hand side of the equation above can be written as \((\text{Id} + M_n)\tilde{A}_n\), where

\[ M_n : H^0,K \to H^0,K \] is

\[ (8.21) \]

\[ M_n h := \Pi^{0,K}\{L^{-1}(\Pi^{\leq-1}L_n h + \Pi^{-1}\{L^{-1}\Pi^{-2}L_n h, G_n^{\geq1}\}), G_n^{\geq1}\} + \Pi^{0,K}L_n h. \]

Similarly to Lemma 6.2 of [BMP21] one has

\[ (8.22) \]

\[ \|M_n h\|_\infty \leq \|h\|_\infty / 2. \]

In order to prove (8.22), we treat the three summands of \(M_n\) separately, we recall that by (5.13), (8.13) and by the smallness condition in (8.5)

\[ \|\Pi^{0,K}L_n h\|_\infty \leq 3\|L_n h\|_{r_n,p_n} \leq 3K(1 + \Theta)^2 \varepsilon_0 \sum_{j=1}^{2^{-j} \|h\|_\infty} < \frac{1}{6} \|h\|_\infty. \]

In the other summands we use the identification \(\|\Pi^{0,K}F\|_\infty \leq 3\|F\|_{r',p'}\) for any \(r', p'\) (see formulas (5.7) and (5.13)) such that \(r' \geq r\sqrt{2}\) and \(p' \leq p\).

We have by (4.15), (5.13), Propositions 4.1 and 7.1

\[ (8.23) \]

\[ \|\Pi^{0,K}\{\Pi^{\leq-1}L^{-1}_\omega L_n h, G_n^{\geq1}\}\|_\infty \leq 3\|\{\Pi^{\leq-1}L^{-1}_\omega L_n h, G_n^{\geq1}\}\|_{r_n,p} \]

\[ \leq 120 \|\Pi^{\leq-1}L^{-1}_\omega L_n h\|_{r_n,p} \|G_n^{\geq1}\|_{r_n,p} \leq 240\gamma \|L^{-1}_\omega L_n h\|_{r_n,p} \Theta_n \]

\[ \leq 240\exp\left(\exp\left(\frac{c}{(3\delta)^{1/\eta}}\right)\right) \|L_n h\|_{r_n,p} \Theta_n \]

\[ \leq \frac{1}{6}K^2\varepsilon_0(1 + \Theta_0)^3\|h\|_\infty \leq \frac{1}{6} \|h\|_\infty, \]

where, in order to control the exponential term, we used the definition of \(K\) given in (8.5).

Now we estimate the remaining term in (8.21). We have, again by (4.15),
\[\Pi^{0,K} \{ L^{-1} \Pi^{-1} \{ L^{-1} \Pi^{-2} \mathcal{L}_n h, G_n^{21} \} , G_n^{21} \} \|_\infty \leq 3 \| L^{-1} \Pi^{-1} \{ L^{-1} \Pi^{-2} \mathcal{L}_n h, G_n^{21} \} \|_{\sqrt{2r,p}} \]

\[\leq 400 \| L^{-1} \Pi^{-1} \{ L^{-1} \Pi^{-2} \mathcal{L}_n h, G_n^{21} \} \|_{3r/2,p} \| G_n^{21} \|_{3r/2,p} \]

\[\leq 400 \gamma \| L^{-1} \Pi^{-1} \{ L^{-1} \Pi^{-2} \mathcal{L}_n h, G_n^{21} \} \|_{3r/2,p} \Theta_n \]

\[\leq 400 \exp \left( c \exp \left( \frac{4}{3\delta} \right) \right) \| L^{-1} \Pi^{-2} \mathcal{L}_n h, G_n^{21} \|_{r_n, (p + p_n)/2} \Theta_n^2 \]

\[\leq 2^{14} \gamma \exp \left( 2c \exp \left( \frac{4}{3\delta} \right) \right) \| L^{-1} \Pi^{-2} \mathcal{L}_n h, G_n^{21} \|_{r_n, (p + p_n)/2} \Theta_n^2 \]

\[\leq \frac{1}{6} K^2 \varepsilon_0 (1 + \Theta_0)^4 \| h \|_\infty \leq \frac{1}{6} \| h \|_\infty ,\]

noting that by (6.1) and (8.2)

\[r_n - \frac{3}{2} r \geq r_\infty - \frac{3}{2} r \geq \frac{r}{4}, \quad \frac{p - p_n}{2} \geq \frac{p - p_\infty}{2} \geq \frac{3}{\delta},\]

and estimating the exponential term as in (8.23). This concludes the proof of (8.22).

Then we have that:

\[(8.24) \quad \bar{A}_n = - (\text{Id} + M_n)^{-1} \Pi^{0,K} \{ L^{-1} \left( G_n^{\leq -1} \Pi^{-1} \left( L^{-1} G_n^{(-2)}, G_n^{21} \right) \right) , G_n^{21} \} - (\text{Id} + M_n)^{-1} G_n^{(0,K)} \]

is well defined. In order to prove (8.14) we split (8.24) in three terms and note that by (8.22) the operator norm of (Id + M_n)^{-1} is bounded by 2.

Regarding the first one we obtain

\[\| (\text{Id} + M_n)^{-1} \Pi^{0,K} \{ L^{-1} G_n^{\leq -1} , G_n^{21} \} \|_\infty \leq 6 \| L^{-1} G_n^{\leq -1} , G_n^{21} \|_{\sqrt{2r,p}} \]

\[\leq 240 \| L^{-1} G_n^{\leq -1} \|_{r_n,p} \| G_n^{21} \|_{r_n,p} \Theta_n \]

\[\leq 240 \gamma \| L^{-1} G_n^{\leq -1} \|_{r_n,p} \Theta_n \]

\[\leq \frac{1}{4} K \varepsilon_0 (1 + \Theta_0) ,\]
taking \( C \) large enough here and below. Regarding the second term we get
\[
\| (\text{Id} + M_n)^{-1} \Pi^{0, \mathcal{K}} \left\{ L^{-1} \Pi^{-1} \left\{ L^{-1} G_n^{(-2)}, G_n^{(\geq 1)} \right\}, G_n^{(\geq 1)} \right\} \|_{\infty}
\]
\[
\leq 6 \| L^{-1} \Pi^{-1} \left\{ L^{-1} G_n^{(-2)}, G_n^{(\geq 1)} \right\} \|_{\sqrt{2r, p}}
\]
\[
\leq 800 \| L^{-1} \Pi^{-1} \left\{ L^{-1} G_n^{(-2)}, G_n^{(\geq 1)} \right\} \|_{3r/2, p} \| G_n^{(\geq 1)} \|_{3r/2, p}
\]
\[
\leq 800 \gamma \| L^{-1} \Pi^{-1} \left\{ L^{-1} G_n^{(-2)}, G_n^{(\geq 1)} \right\} \|_{3r/2, p} \Theta_n
\]
\[
\leq 800 \exp \left( \exp \left( c \left( \frac{2}{3\delta} \right)^{1/\eta} \right) \right) \| L^{-1} G_n^{(-2)} \|_{3r/2, p} \Theta_n
\]
\[
\leq 2^{15} \gamma \exp \left( \exp \left( c \left( \frac{2}{3\delta} \right)^{1/\eta} \right) \right) \| L^{-1} G_n^{(-2)} \|_{3r/2, p} \Theta_n^2
\]
\[
\leq 2^{15} \exp \left( 2 \exp \left( c \left( \frac{2}{3\delta} \right)^{1/\eta} \right) \right) \| G_n^{(-2)} \|_{r, p} \Theta_n
\]
\[
\leq 2 \gamma \varepsilon_n.
\]
so (8.14) follows.

By substituting in the equations (8.20) we get the final expressions for \( S_n^{(-2)} \) and \( S_n^{(-1)} \) and finally \( S_n^{(0, \mathcal{K})} \) which by (4.14), (7.3), (8.11), (8.13), (8.5) and (8.14) yield the estimates
\[
\| S_n^{(-2)} \|_{r_n, p_n + \delta_n} \leq (1 + K^2 (1 + \Theta_0)^4 \varepsilon_0) D_n \varepsilon_n \leq 2 D_n \varepsilon_n
\]
\[
\| S_n^{(-1)} \|_{r_n - \rho_n, p_n + 2\delta_n} \leq (1 + 16 r_n \rho_n D_n \Theta_0) 2 D_n \varepsilon_n
\]
\[
\| S_n^{(0)} \|_{r_n - 2\rho_n, p_n + 3\delta_n} \leq (1 + 16 r_n \rho_n D_n \Theta_0)^2 3 D_n \varepsilon_n
\]
where
\[
D_n := \exp \left( \exp \left( \frac{c}{\delta_n^{1/\eta}} \right) \right) \leq \exp \left( \frac{1}{3} \zeta^{n^{\xi}} \right)
\]
(\( \zeta \) and \( \xi \) were defined in (8.5)) systematically using the inductive hypothesis and the first bound in (8.5). The final bound thus reads (recall (8.1) and (8.2))
\[
(8.26) \quad \| S_n \|_{r_n - 2\rho_n, p_n + 1} \leq \frac{10^2}{\rho^2} 4^{n+8} \exp \left( \zeta^{n^{\xi}} \right) \varepsilon_n (1 + \Theta_0)^2 \leq \frac{\rho}{2^{2n + 10} r_0}.
\]

Then we can apply Proposition 4.4 since (4.16) is satisfied by \( S_n \) with \( \rho \to \rho_n \) and \( r \to r_{n+1} \). Then item (1) of Lemma 8.1 is easily proved. In particular
(8.6) follows by (4.17) and (8.26) (for complete details see the analogous proof of [BMP21, Lemma 6.1]).

Regarding item (2), by construction we have

\[ L_{n+1} - L_n = \left(e^{\left\{ S_n, \cdot \right\}} - \text{Id} \right) \circ (L_n + \text{Id}), \]

hence by (4.19), (4.16), (8.1) and (5.7)

\[ \| (L_{n+1} - L_n) h \|_{r_{n+1}, p_{n+1}} \leq \frac{\varepsilon_0 2^{n+9}}{\rho} \| S_n \|_{r_{n-2\rho_n, p_{n+1}}} \| (L_n + \text{Id}) h \|_{r_{n-2\rho_n, p_{n+1}}}, \]

which by (8.5) proves (8.13).

As for the expression of \( G_{n+1} \), by (8.10) and (8.9) we have

\[ G_{n+1} = e^{\left\{ S_n, \cdot \right\}} H_n - [D + (\text{Id} + L_{n+1}) \tilde{\Lambda}_{n+1}]. \]

Since \( S_n \) solves the Homological equation (8.15), we have that by (8.9)

\[ G_{n+1} = G_n^{(-2, \mathcal{K})} + G_n^{\geq 1} + \Pi^{\geq 1}(L_{n+1} \tilde{\Lambda}_n + \{S_n, G_n^{\geq 1}\}) + G_{n+1,*}, \]

\[ G_{n+1,*} = \{S_n, G_n^{< 0}\} + \Pi^{\leq 0}(L_{n+1} - L_n) \tilde{\Lambda}_n + \left(e^{\left\{ S_n, \cdot \right\}} - \text{Id} - \{S_n, \cdot \}\right) G_n \]

\[ - \sum_{h=2}^{\infty} \frac{(\text{ad} S_n)^{h-1}}{h!} \left( \Pi^{\leq 0}(\text{Id} + L_n) \tilde{\Lambda}_n + G_n^{\leq 0} + \Pi^{\leq 0}\{S_n^{(-1)} + S_n^{(-2)}, G_n^{\geq 1}\} \right), \]

using that \( \Pi^{\leq 0}(\{S_n, D\}) = \{S_n, D\} \) by (5.8) and that \( \{S_n, G_n^{(-2, \mathcal{K})}\} = 0 \). Note that \( G_{n+1,*} \) is quadratic in \( S_n \sim G_n^{\leq 0} \).
Recalling (8.1), (8.2), (4.14), (5.13), (8.5), (8.11), (8.13), (8.14) and Proposition 4.4, which can be applied by (8.26), we have

\[ \|G_{n+1,}\|_{r_{n+1},p_{n+1}} \leq \frac{2^{n+6}r_0}{\rho} \|S_n\|_{r_n-2\rho_n,p_{n+1}} \gamma \varepsilon_n \]

\[ + \frac{r_0^3}{\rho^2} 8^{n+9} \exp \left( \zeta_n^e \right) \varepsilon_n \left( 1 + \Theta_0 \right)^2 \|\Lambda_n\|_\infty + \frac{2^{2n+21}r_0^2}{\rho^2} \|S_n\|_{r_n-2\rho_n,p_{n+1}}^2 \gamma \Theta_0 \]

\[ + \frac{2^{n+10}r_0}{\rho} \|\Pi^{\leq 0} (Id + \Lambda_n) + G_n \| \]

\[ + \Pi^{\leq 0} \{ S_n^{(-1)} + S_n^{(-2)} \} \|r_n-2\rho_n,p_{n+1}\| \|S_n\|_{r_n-2\rho_n,p_{n+1}} \]

Taking \( C \) large enough. Recalling (8.11) and (8.28) this implies the first estimates in (8.12).

By the above estimate and recalling (8.11) and (8.28) we get

\[ \Theta_{n+1} \leq \varepsilon_{n+1} + \Theta + e^{-\chi^{n+1}} \varepsilon_0 + \gamma^{-1} \|\Pi^{\geq 1} (\Lambda_n + \{ S_n, G_n^{\leq 1} \}) \|_{r_{n+1},p_{n+1}} \]

\[ \leq \Theta + 4e^{-\chi^{n+1}} \varepsilon_0 + 2^4 \varepsilon_n + 2^{n+7}r_0 \gamma^{-1} \|S_n\|_{r_n-2\rho_n,p_{n+1}} \Theta_0 \leq \Theta + \Theta_0 2^{-n-1}, \]

again by (5.13), (8.5), (8.13), (8.14), (4.14), (8.26). This finally implies the second estimates in (8.12).

The analyticity of \( \Psi_n \) and \( \Lambda_n \) follows by Proposition 7.1 point (ii) of Proposition 5.1 and recalling Remark 4.2.

9. Measure estimates

Lemma 9.1 (Measure estimates). The set \( \mathcal{C} \) defined in (2.21) satisfies (2.16), namely

\[ \text{meas}_{Q_S} (Q_S \setminus \mathcal{C}(V_{S'^e}, I, \gamma)) \leq C_0 \gamma , \]

\[ \text{meas}_{[-\ell/4,\ell/4]} (\mathcal{C}(V_{S'^e}, I, \gamma), V_{S'^e}, I) \leq C_0 \gamma . \]

Proof. We start proving the first estimate in (2.16).

Take \( \gamma \leq 1/2 \). For \( \ell \neq 0 \) with \(|\ell| < \infty\), \( \sum_{s \in S^e} |\ell_s| \leq 2\), \( \pi(\ell) = 0 \) and

\[ \text{meas}_{[-\ell/4,\ell/4]} (\mathcal{C}(V_{S'^e}, I, \gamma), V_{S'^e}, I) \leq C_0 \gamma . \]
\( m(\ell) = 0 \), set for simplicity \( R_\ell = R_\ell(V_{S^c}, I, \gamma) \) and define the resonant set
\( R_\ell := \left\{ \nu \in Q_S : |\omega(\nu, V_{S^c}, I) \cdot \ell| \leq \gamma \prod_{s \in S} \frac{1}{(1 + |\ell_s|^2 \langle i(s) \rangle^2)^{3/2}} =: \gamma d(\ell) \right\} \).

Recalling the definition of \( \omega(\nu, V_{S^c}, I) \) in (2.21) we note that by the continuity with respect to the product topology of the functions \( \nu \mapsto \Omega_j(\nu, V_{S^c}, I) \) with \( j \in S^c \), \( I \in \mathcal{I}(p, r) \) (and since \( |\ell| < \infty \)) we have the crucial fact that the sets \( R_\ell \) are closed (w.r.t. the product topology) and, therefore, measurable with respect to the product probability measure on \( Q_S \). Moreover, since \( Q_S \) is compact all the \( R_\ell \) are compact too.

In the following we will often omit to write the immaterial dependence on \( V_{S^c} \) and \( I \).

Set \( q(\ell) := \sum_{s \in \mathbb{N}} s^2 \ell_s \). If \( |q(\ell)| \geq |\ell| \) then
\[ |\omega \cdot \ell| \geq \left| \sum_{s \in \mathbb{N}} s^2 \ell_s \right| - \frac{1}{2} |\ell| \geq \frac{1}{2} |\ell| \geq \frac{1}{2} \]
and \( R_\ell = \emptyset \). Recall the definition of the set \( C \) given in (2.21) and (2.18). Denoting by \( \text{meas} \) the product probability measure on \( Q_S \), we have
\[ Q_S \setminus C = \bigcup_{\ell \in A} R_\ell, \quad \text{which implies} \quad \text{meas}(Q_S \setminus C) \leq \sum_{\ell \in A} \text{meas}(R_\ell), \]
where
\[ A := \left\{ \ell \in \mathbb{Z}^\infty : 0 < |\ell| < \infty, \sum_{s \in S^c} |\ell_s| \leq 2, m(\ell) = \pi(\ell) = 0, |q(\ell)| < |\ell| \right\}. \]
Fix \( \ell \in A \). We note that
\[ \exists \bar{s} = \bar{s}(\ell) \in S \quad \text{(depending only on} \ \ell) \text{such that} \ \ell_{\bar{s}} \neq 0, \]
otherwise \( 0 < |\ell| = \sum_{s \in S^c} |\ell_s| \leq 2 \), which contradicts Lemma 7.1. Since \( \omega(\nu) = (\nu, \Omega(\nu)) \), we get
\[ |\ell|^{-1} |(\omega(\nu + t\bar{s}) \cdot \ell) - (\omega(\nu) \cdot \ell)| \geq |\ell_{\bar{s}}| - 2 \sup_{s \in S^c} |\Omega_s|_{\text{lip}} \geq 1 - 2C \varepsilon \geq 1/2, \]
taking \( \varepsilon_*(p_\nu) \) small enough in (2.13). Set \( \hat{\nu} = (\nu_\bar{s})_{s \neq \bar{s}} \). Then for every \( \hat{\nu} \) there exist \( a_\ell(\hat{\nu}) = a_\ell(\hat{\nu}, V_{S^c}, I, \gamma) < b_\ell(\hat{\nu}) = b_\ell(\hat{\nu}, V_{S^c}, I, \gamma) \) satisfying
\[ \{ \nu_\bar{s} \ s.t. \ (\nu_\bar{s}, \hat{\nu}) \in R_\ell \} \subseteq \{ a_\ell(\hat{\nu}), b_\ell(\hat{\nu}) \}, \quad \text{with} \ b_\ell(\hat{\nu}) - a_\ell(\hat{\nu}) \leq 4\gamma d(\ell), \]
which implies
\[ \text{meas}\{ \nu_\bar{s} \ s.t. \ (\nu_\bar{s}, \hat{\nu}) \in R_\ell \} \leq 4\gamma d(\ell). \]
Since \( R_\ell \) is measurable, by Fubini’s Theorem we have that
\[ \text{meas}(R_\ell) \leq 4\gamma d(\ell) = 4\gamma \prod_{s \in S} \frac{1}{(1 + |\ell_s|^2 \langle i(s) \rangle^2)^{3/2}} = 4\gamma \prod_{\ell \in \mathbb{N}} \frac{1}{(1 + |\ell_s|^2 \langle i(s) \rangle^2)^{3/2}}. \]
Therefore

\begin{equation}
\tag{9.6}
\text{meas}(\mathcal{Q}_S \setminus C) \leq 4\gamma \sum_{\ell \in A} \prod_{i \in \mathbb{N}} \frac{1}{1 + |\ell_s(i)|^2 2^{(i)^2}}^3/2.
\end{equation}

We claim that

\begin{equation}
\tag{9.7}
\sum_{\ell \in A} d(\ell) = \sum_{\ell \in A} \prod_{i \in \mathbb{N}} \frac{1}{1 + |\ell_s(i)|^2 2^{(i)^2}}^3/2 \leq C_0 \frac{17}{17},
\end{equation}

taking $C_0$ large enough.

Given $k \in \mathbb{Z}^N$ with $|k| < \infty$ we define $\ell \in \mathbb{Z}^\mathbb{N}$ supported on $S$ setting

$$\ell_s := k_{i(s)}, \text{ for } s \in S, \quad \ell_s = 0 \text{ for } s \notin S.$$

Now for each $\ell \in A$ there exist unique $k \in \mathbb{Z}^N$ with $|k| < \infty$ and $s_1, s_2 \in S^c$ and $\sigma_1, \sigma_2 = \pm 1, 0$ such that

\begin{equation}
\tag{9.8}
\ell = \ell^k + \sigma_1 e_{s_1} + \sigma_2 e_{s_2}.
\end{equation}

On the other hand, given $k \in \mathbb{Z}^N$ with $|k| < \infty$, there exist at most $36(|k| + 2)$ vectors $\ell \in A$ satisfying (9.8). Indeed we prove that, given $\sigma_1, \sigma_2 = \pm 1, 0$, there exist at most $4(|k| + 2)$ couples $(s_1, s_2) \in S^c \times S^c$ such that $\ell$ in (9.8) belongs to $A$. Indeed they have to satisfy

$$\begin{cases}
\sigma_1 s_1 + \sigma_2 s_2 = -\pi(\ell^k) \\
\sigma_1 s_1^2 + \sigma_2 s_2^2 = q(\ell^k) + h,
\end{cases}$$

for some $|h| < |\ell| \leq |k| + 2$.

Then by (9.6) we get

$$\sum_{\ell \in A} \prod_{i \in \mathbb{N}} \frac{1}{1 + |\ell_s(i)|^2 2^{(i)^2}}^3/2 \leq 36\gamma \sum_{k \in \mathbb{Z}^N, 0 < |k| < \infty} (|k| + 2) \prod_{i \in \mathbb{N}} \frac{1}{1 + |k_i|^2 2^{(i)^2}}^3/2.$$

Since

$$|k| + 2 \leq 2 \prod_{i \in \mathbb{N}} (1 + |k_i|^2 2^{(i)^2})^{1/2},$$

we get

$$\sum_{\ell \in A} \prod_{i \in \mathbb{N}} \frac{1}{1 + |\ell_s(i)|^2 2^{(i)^2}}^3/2 \leq 72\gamma \sum_{k \in \mathbb{Z}^N, 0 < |k| < \infty} \prod_{i \in \mathbb{N}} \frac{1}{1 + |k_i|^2 2^{(i)^2}}.$$

where the last sum converges (see [Bou05] or Lemma 4.1 of [BMP20a]). This concludes the proof of (9.7), taking $C_0$ large enough and, by (9.6), the proof of the first estimate in (2.16).

Let us now prove the second estimate in (2.16). Let us denote for brevity

\begin{equation}
\tag{9.9}
\text{meas}'(E) := \text{meas}_{[-1/4, 1/4]^S}(E \cap [-1/4, 1/4]^S)
\end{equation}

the product probability measure on $[-1/4, 1/4]^S$. Since $\mathcal{Y}_S$ (defined in (2.15)) is Lipschitz $C\varepsilon\gamma$-close to the map $\mathcal{Y}_S^*$ (defined in (2.10)) we have that

$$\mathcal{Y}_S(Q_S, V_{S'}, I') \supset [-1/4, 1/4]^S.$$
for every $V_{S_e} \in [-1/4, 1/4]^{S_e}$ and $I' \in \mathcal{I}(p, r)$. The function $\mathcal{Y}_S(\nu, V_{S_e}, I)$ is invertible (w.r.t. $\nu$). In particular there exists a function

$$g : [-3/8, 3/8]^S \times [-1/4, 1/4]^{S_e} \times \mathcal{I}(p, r) \to [-4C\gamma_\epsilon, 4C\gamma_\epsilon]^S$$

which is continuous w.r.t. the product topology in $[-3/8, 3/8]^S \times [-1/4, 1/4]^{S_e}$ and satisfies

$$(9.12) \quad |g(V'_S, V_{S_e}, I) - g(V_S, V_{S_e}, I)|_{\ell_e^S} \leq 6\overline{C}\varepsilon|V'_S - V_S|_{\ell_e^S}.$$ 

Indeed recalling (6.11) $g$ satisfies the fixed point equation (e.g. by a slightly modified version of Lemma [B.1])

$$g = -\lambda(q + V_S + g(V_S, V_{S_e}, I), I);$$

then (9.10) and (9.12) follow from (6.4) (recalling (5.12) and (4.9)). For the remaining of the proof we drop the dependence on $V_{S_e}$ and $\gamma$. Fix $\ell \in A$, first we note that for all $I, I' \in \mathcal{I}(p, r)$ the set $\mathcal{Y}_S(\mathcal{R}_\ell(I), I')$ is measurable since $\mathcal{R}_\ell(I)$ is closed and $\mathcal{Y}_S$ is continuously invertible. Recalling (9.11) we have

$$(9.11) \quad \mathcal{Y}_S(\mathcal{R}_\ell(I), I') \cap [-1/4, 1/4]^S = \{V_S \in [-1/4, 1/4]^S : \exists \nu \in \mathcal{R}_\ell(I), V_S = \mathcal{Y}_S(\nu, I')\}$$

As usual we set $\ell := (\ell_S, \ell_{S_e})$. Let $S := \{j \in S : \ell_j \neq 0\}$. We split $\ell_S := (\ell_S, 0)$.

Decomposing $V_S = (V'_S, V_{S_e})$ we set

$$E(V_{S_e}; S) := \{V_S \in [-1/4, 1/4]^S : (V'_S, V_{S_e}) \in E\}.$$ 

We claim that for every $V_{S_e} \in [-1/4, 1/4]^{S_e}$

$$(9.13) \quad \text{meas}_{[-1/4, 1/4]^{S_e}}(E(V_{S_e}; S)) \leq 16\gamma d(\ell).$$

Then by Fubini’s theorem we get

$$(9.14) \quad \text{meas}_{[-1/4, 1/4]^S}(E) = \text{meas}'(\mathcal{Y}_S(\mathcal{R}_\ell(I), I')) \leq 16\gamma d(\ell).$$

Then

$$\text{meas}_{[-1/4, 1/4]^S}(\mathcal{Y}_S(\mathcal{Q}_S \setminus \mathcal{C}(I), I')) \leq \sum_{\ell \in A} 16\gamma d(\ell)$$

and the second estimate in (2.16) follows by (9.7) and taking $I' = I$. 

---

43In the present proof we actually take $I' = I$. We have introduced $I'$ for estimate (9.14) that will be used in the proof of Lemma 3.1.
Let us finally prove the claim in (9.13). Fix \( V_{\mathcal{S}^4, \mathcal{S}} \in [-1/4, 1/4]^4 \). Set for brevity

\[
f(V_{\mathcal{S}}) := \omega(q + V_S + g(V_S, I')I) \cdot \ell
\]

(2.21) \( V_S \cdot \ell_S + (q + g(V_S, I')) \cdot \ell_S + \Omega(q + V_S + g(V_S, I'), I) \cdot \ell_{\mathcal{S}^c}. \)

The aim now is to prove that the increment of \( f \) in a suitable direction is bounded from below. Since \( \ell \in A \) by (9.4) we get \( |\ell_S| \geq 1 \) and \( |\ell_{\mathcal{S}^c}| \leq 2. \)

Define \( \sigma_{\mathcal{S}} \) by \( \sigma_{\mathcal{S}, j} = \text{sign} \ell_j \) for \( j \in \mathcal{S}; \) then \( \sigma_{\mathcal{S}} \cdot \ell_S = |\ell_S| = |\ell| \geq 1 \) and \( |\sigma|_\infty = 1. \) Then we get

\[
|\ell|^{-1}|f(V_{\mathcal{S}} + t\sigma_{\mathcal{S}}) - f(V_{\mathcal{S}})| \geq |\ell_S| - 16C\epsilon|\ell| \geq |\ell_S|/2;
\]

by (9.12) and (6.10) and taking \( \epsilon = p_* \) in (2.13) small enough such that

\[
16C\epsilon \leq 16C\epsilon(p_*)\epsilon \leq 1/2 \quad \text{recall (6.8), (6.7)}.
\]

Then we get that

\[
\text{meas}\{t \in \mathbb{R} : V_S + t\sigma_{\mathcal{S}} \in \mathbb{E}(V_{\mathcal{S}^4, \mathcal{S}})\} \leq 4\gamma d(\ell)/|\ell_S|.
\]

We need the following (finite dimensional) result proved in the Appendix.

**Lemma 9.2.** Let \( E \subset [-1/4, 1/4]^n \) be a measurable set. Let \( \xi = (\xi, \pm 1) \) with \( \hat{\xi} \in \mathbb{R}^{n-1}. \) Assume that for every \( x \in [-1/4, 1/4]^n \) we have \( \text{meas}_{\mathbb{E}}\{t \in \mathbb{R} : x + t\xi \in E\} \leq \delta. \) Then \( \text{meas}_{\mathbb{E}}(E) \leq 2^{1-n} \delta |\xi|_2^2, \) where \( |\cdot|_2 \) denotes the Euclidean norm.

Then (9.13) follows by the above lemma with

\[
E = \mathbb{E}(V_{\mathcal{S}^4, \mathcal{S}}), \quad n = 2^4, \quad \xi = \sigma_{\mathcal{S}}, \quad x = V_{\mathcal{S}}, \quad \delta = 4\gamma d(\ell)/|\ell_S|
\]

and noting that \( |\sigma|_\infty^2 = n \leq |\ell_S| \) and \( \text{meas}_{[-1/4, 1/4]^n}(A) = 2^n \text{meas}_{\mathbb{R}^n}(A) \) for every \( A \subset [-1/4, 1/4]^n. \)

**Proof of Corollary 2.1.** It is equivalent to prove that

\[
\mathcal{B}(I, \gamma) := \{V \equiv (V_S, V_{\mathcal{S}^c}) \in [-1/4, 1/4]^Z : V_S \in \mathcal{Y}_S(Q_{\mathcal{S}} \mathcal{C}(V_{\mathcal{S}^c}, I, \gamma), V_{\mathcal{S}^c}, I)\}
\]

is a Borel set in \([0, 1/4]^Z\) with measure bounded by \( C_0\gamma \) (\( C_0 \) is the constant in (2.16)).

The function \( h : [-1/4, 1/4]^Z \to \mathcal{Q}_{\mathcal{S}} \times [0, 1/4]^\mathcal{S}^c \) defined by

\[
h(V) := (q + V_S + g(V_S, V_{\mathcal{S}^c}, I, V_{\mathcal{S}^c})
\]

is continuous with respect to the product topology. Then, for every \( \ell \in A \) (recall (9.3)), the set

\[
\mathcal{B}_\ell(I, \gamma) := \{V \equiv (V_S, V_{\mathcal{S}^c}) \in [-1/4, 1/4]^Z : |\omega(h(V)) \cdot \ell| \leq \gamma d(\ell)\}
\]

(9.10), (9.11)

\[
\{V \equiv (V_S, V_{\mathcal{S}^c}) \in [-1/4, 1/4]^Z : V_S \in \mathcal{Y}_S(R(\ell(V_{\mathcal{S}^c}, I), V_{\mathcal{S}^c}, I))\}
\]

(9.15)

44 Recall (2.21) and (9.11).
45 \( \text{meas}_{\mathbb{E}}, \text{resp. meas}_{\mathbb{E}}\), being the standard measure in \( \mathbb{R}, \) resp \( \mathbb{R}^n. \)
46 Here we consider the case when \( \ell_S = -1 \) the +1 case is analogous.
47 \( g \) was defined in (9.10). By (9.11) \( h \) is the inverse of the function \( (\nu, V_{\mathcal{S}^c}) \mapsto (\mathcal{Y}_S(\nu, V_{\mathcal{S}^c}), V_{\mathcal{S}^c}). \)
is closed and, therefore, measurable. Then by Fubini’s theorem and (9.14) (with \(I = I'\)) we get \(\text{meas}_{[-1/4,1/4]}(B_\ell) \leq 16\gamma d(\ell)\). Since \(B\) defined in (9.15) can be written as \(B = \bigcup_{\ell \in A} B_\ell\) (recall (9.2)), by (9.7) we get \(\text{meas}_{[-1/4,1/4]}(B) \leq C_0\gamma\).

\[\square\]

**Proof of Lemma 3.1.** The first estimate in (3.1) is equivalent to

\[
\text{meas}_{[-1/4,1/4]} \left( \mathcal{Y}_S \left( Q_S \setminus C' \left( V_{S^c}, I, \gamma \right), V_{S^c}, I \right) \right) \leq C_0 \gamma ,
\]

where \(C_0\) was defined in Theorem 3.

In the following we drop the dependence on \(V_{S^c}\). We first note that recalling (9.1), (9.3) and setting \(L := C^{-2} \epsilon\), we get

\[
|I - I'|_{2p^*} \leq d(\ell)/4L \implies \mathcal{R}_\ell (I', \gamma/2) \subset \mathcal{R}_\ell (I, \gamma), \quad \forall \ell \in A.
\]

Indeed, since \(\ell \in A\),

\[
|\omega(\nu, I) \cdot \ell| \leq |\omega(\nu, I') \cdot \ell| + 2\gamma L |I - I'|_{2p^*} \leq \gamma d(\ell)/2 + 2\gamma L |I - I'|_{2p^*} \leq \gamma d(\ell).
\]

Therefore

\[
Q_S \setminus \left( C(I, \gamma) \cap C(I', \gamma/2) \right) = \left( Q_S \setminus C(I, \gamma) \right) \bigcup \left( Q_S \setminus C(I', \gamma/2) \right) \bigcup_{\ell \in A, \ d(\ell) < 4L |I - I'|_{2p^*}} \mathcal{R}_\ell (I, \gamma) \bigcup_{\ell \in A, \ d(\ell) < 4L |I - I'|_{2p^*}} \mathcal{R}_\ell (I', \gamma/2).
\]

Then

\[
Q_S \setminus C'(I, \gamma) := \bigcup_{\ell \in A} \mathcal{R}_\ell (I, \gamma) \bigcup_{k \in \mathbb{N}} \bigcup_{\ell \in A, \ d(\ell) < 4L |I - I'(n_k)|_{2p^*}} \mathcal{R}_\ell (I^{(n_k)}, \gamma/2)
\]

and (recalling the notation in (9.9))

\[
\text{meas}' \left( \mathcal{Y}_S \left( Q_S \setminus C'(I, \gamma); I \right) \right) \leq \sum_{\ell \in A} \text{meas}' \left( \mathcal{Y}_S \left( \mathcal{R}_\ell (I, \gamma); I \right) \right)
\]

\[
+ \sum_{k \in \mathbb{N}} \sum_{\ell \in A, \ d(\ell) < 4L |I - I^{(n_k)}|_{2p^*}} \text{meas}' \left( \mathcal{Y}_S \left( \mathcal{R}_\ell (I^{(n_k)}, \gamma); I \right) \right)
\]

\[
\leq 16\gamma \sum_{\ell \in A} d(\ell) + 16\gamma \sum_{k \in \mathbb{N}} \sum_{\ell \in A, \ d(\ell) < 4L |I - I^{(n_k)}|_{2p^*}} d(\ell) \leq 17\gamma \sum_{\ell \in A} d(\ell) \leq C_\gamma ,
\]

taking the sub-sequence \((n_k)_{k \in \mathbb{N}}\) growing fast enough\(^{48}\). This proves (9.16). The second estimate in (3.1) follows from the first one and Fubini Theorem.

\(^{48}\)Since the positive term series in (9.7) converges.
Consider now the case of a general \( \omega \)-continuity in \( X \). Then the extended function \( \tilde{f} \) is Lipschitz continuous, namely \( \tilde{f} \in L^d(\mathbb{R}^2, \gamma, I, I') \) for every \( (\omega, I) \in \mathcal{O} \times B_{\rho/2} \). By Cauchy estimates
\[
|\tilde{f}(\omega, I) - \tilde{f}(\omega, I')| \leq 2\rho^{-1}|f|^{\gamma, \mathbb{O} \times B_{\rho/2}}|I - I'|_E, \quad \forall \omega \in \mathcal{O}, \ I, I' \in B_{\rho/2}.
\]

We need the following

**Lemma A.1** (Lipschitz extension). Let \( X \) be a metric space endowed with the metric \( d(\cdot, \cdot) \) and \( \emptyset \neq U \subseteq X \).

i) Let \( f : U \to \mathbb{R} \) be a \( L \)-Lipschitz function, namely \( |f(u) - f(v)| \leq Ld(u, v) \) for every \( u, v \in U \). Then \( \tilde{f}(x) := \inf_{u \in U} f(u) + Ld(x,u), \ x \in X, \) is a \( L \)-Lipschitz extension of \( f \).

ii) Moreover if \( \sup_U |f| =: M < \infty, \) then \( \hat{f} := \max\{-M, \min\{\tilde{f}, M\}\} \) is a \( L \)-Lipschitz extension of \( f \) satisfying \( \sup_X |\hat{f}| = M \).

iii) Let \( g : U \to \ell^\infty(\mathbb{R}) \) a \( L \)-Lipschitz function, namely \( |g(u) - g(v)|_{\ell^\infty} \leq Ld(u,v) \) for every \( u,v \in U \). Then there exists a \( L \)-Lipschitz extension \( \tilde{g} : X \to \ell^\infty(\mathbb{R}) \) of \( g \). An analogous statement holds for every \( \ell^\infty \)-weighted space, in particular for \( \mathfrak{w}_p \).

Before proving Lemma A.1 we conclude the proof of Lemma 4.1. Then Lemma A.1 with \( X = P_k(\mathbb{Q}) \times B_{\rho/2} \), distance \( d((\omega, I), (\omega', I')) = |\omega - \omega'| + 2\gamma\rho^{-1}|I - I'|_E \), \( U = P_k(\mathbb{Q}) \times B_{\rho/2} \), \( f = \hat{f}, \ L = \gamma^{-1}|f|^{\gamma, \mathbb{Q} \times B_{\rho/2}} \) implies Lemma 4.1 in the case \( f \in F(\mathbb{Q} \times B_{\rho}) \). Note in particular that, since \( P_k(\mathbb{Q}) \) is finite dimensional the product topology on it coincides with one induced by the norm \( | \cdot |_\infty \). Then the extended function \( \tilde{f} \in F(\mathbb{Q} \times B_{\rho/2}) \) and its continuity in \( \omega \) follows by its Lipschitz-continuity in \( \omega \). Moreover, in this case, \( |\tilde{f}|^{\gamma, \mathbb{Q} \times B_{\rho/2}} = |f|^{\gamma, \mathbb{Q} \times B_{\rho}} \) and
\[
|\tilde{f}(\omega, I) - \tilde{f}(\omega, I')| \leq 2\rho^{-1}|f|^{\gamma, \mathbb{Q} \times B_{\rho/2}}|I - I'|_E, \quad \forall \omega \in \mathbb{Q}, \ I, I' \in B_{\rho/2}.
\]

Consider now the case of a general \( f \in F(\mathbb{Q} \times B_{\rho}) \). By definition there exists a sequence \( f_n \in F(\mathbb{Q} \times B_{\rho}) \) such that \( |f_n - f|^{\gamma, \mathbb{Q} \times B_{\rho}} \leq 4^{-n-1}|f|^{\gamma, \mathbb{Q} \times B_{\rho}} \) and \( g_0 := f_0, \ g_n := f_n - f_{n-1} \) for \( n \geq 1 \). Then \( f = \sum_{n \geq 0} g_n \). Moreover
\[ |g_0|_{\gamma_2 \times B_{\rho}} \leq 5/4 |f|_{\gamma_2 \times B_{\rho}}, |g_n|_{\gamma_2 \times B_{\rho}} = |f_n - f|_{\gamma_2 \times B_{\rho}} + |f - f_{n-1}|_{\gamma_2 \times B_{\rho}} \leq 5 \cdot 4^{-n-1} |f|_{\gamma_2 \times B_{\rho}} \] and \( g_n \in F(\mathcal{O} \times B_{\rho}) \). Then there exist extensions \( \tilde{g}_n \in F(\mathcal{O} \times B_{\rho/2}) \), \( n \geq 0 \) such that \( \tilde{g}_n = g_n \) on \( \mathcal{O} \times B_{\rho/2} \), \( |\tilde{g}_n|_{\gamma_2 \times B_{\rho/2}} \leq 5 \cdot 4^{-n-1} |f|_{\gamma_2 \times B_{\rho}}, \) \( \tilde{g}_n \) is continuous w.r.t the product topology in \( \mathcal{O} \), finally \( \tilde{g}_n \) is Lipschitz on \( B_{\rho/2} \) with estimate

\[ |\tilde{g}_n(\omega, I) - \tilde{g}_n(\omega, I')| \leq 10 \rho^{-1} 4^{-n-1} |f|_{\gamma_2 \times B_{\rho}} |I - I'|_{E}, \quad \forall \omega \in \mathcal{O}, I, I' \in B_{\rho/2}. \]

Finally set \( \tilde{f} := \sum_{n \geq 0} \tilde{g}_n. \)

**Proof of Lemma A.1**

i) It is McShane’s Theorem. Incidentally we note that \( \tilde{f} \) is the greatest possible extension while the smaller one is \( \underline{f}(x) := \sup_{u \in U} f(u) - L d(x, u). \)

ii) The fact that \( \tilde{f} \) is \( L \)-Lipschitz follows applying twice the following result: Given a \( L \)-Lipschitz \( g : X \to \mathbb{R} \) and \( c \in \mathbb{R} \) the functions \( \tilde{g} := \max\{g, c\} \) and \( \underline{g} := \min\{g, c\} \) are \( L \)-Lipschitz. We prove it only for \( \tilde{g} \), the other case being analogous. We first note that \( \tilde{g} = (g + c + |g - c|)/2. \) Then

\[ |\tilde{g}(x) - \tilde{g}(y)| \leq \frac{1}{2} \left( |g(x) - g(y)| + |g(x) - c| - |g(y) - c| \right) \leq |g(x) - g(y)| \leq L d(x, y). \]

iii) By the McShane’s Theorem in point i), every component \( g_j : \alpha \to \mathbb{R} \) of \( g \) can be extended to \( \tilde{g}_j : X \to \mathbb{R} \) with the same Lipschitz constant \( L \). Then, setting \( \tilde{g} = (\tilde{g}_j)_{j \in \mathbb{Z}} \), we have that \( \tilde{g} \in \ell^\infty \); in fact, for every fixed \( x \in X \), \( x_0 \in \alpha \) and \( j \in \mathbb{Z} \),

\[ |\tilde{g}_j(x)| \leq |\tilde{g}_j(x) - \tilde{g}_j(x_0)| + |g_j(x_0)| \leq L d(x, x_0) + \|g(x_0)\|_{\ell^\infty}. \]

Finally for every \( x_1, x_2 \in X \) and \( j \in \mathbb{Z} \), we have that \( |g_j(x_1) - g_j(x_2)| \leq L d(x, x_0) \) and, finally, \( |g(x_1) - g(x_2)|_{\ell^\infty} \leq L d(x, x_0) \). The extension to \( \ell^\infty(\mathbb{C}) \) and to every \( \ell^\infty \)-weighted space, in particular for \( u_\rho \), is straightforward. \( \square \)

**Proof of Proposition 4.1**

Writing \( F = \sum F_{\alpha', \beta'} u^{\alpha'} \bar{v}^{\beta'} \) and \( G = \sum G_{\alpha'', \beta''} u^{\alpha''} \bar{v}^{\beta''} \) we have

\[
\{F, G\} = i \sum_{\alpha', \beta', \alpha'', \beta''} F_{\alpha', \beta'} G_{\alpha'', \beta''} \sum_j (\alpha'_j \beta''_j - \beta'_j \alpha''_j) u^{\alpha' + \alpha'' - \epsilon_j} \bar{v}^{\beta' + \beta'' - \epsilon_j} \\
= : H = \sum_{\alpha, \beta} H_{\alpha, \beta} u^{\alpha} \bar{v}^{\beta},
\]

where

\[
H_{\alpha, \beta} := i \sum_j \sum_{\alpha' + \alpha'' - \epsilon_j = \alpha} F_{\alpha', \beta'} G_{\alpha'', \beta''} (\alpha'_j \beta''_j - \beta'_j \alpha''_j).
\]

Note that \( \{F, G\}(0) = 0 \), indeed \( H_{0, 0} = 0 \) since \( \alpha' + \alpha'' = \beta' + \beta'' = \epsilon_j \) (namely \( \alpha'_j + \alpha''_j = \beta'_j + \beta''_j = 1 \) and \( \alpha'_k + \alpha''_k = \beta'_k + \beta''_k = 0 \) for \( k \neq j \)).
implies that $\alpha_j'\beta_{j'}'' - \beta_j''\alpha_j' = 0$ by mass conservation $|\alpha| = |\beta|$. Recalling (4.5), we have

\[ |\{F, G\}|_{r, p} \leq \sup_{\ell} \sum_j \sum_{\alpha', \beta', \alpha'', \beta''} |F_{\alpha', \beta'} G_{\alpha'', \beta''} (\alpha_j'\beta_{j'}'' + \alpha_j''\beta_j') (\beta'_{\ell} + \beta''_{\ell}) u_p^{\alpha' + \beta'_{\ell} - 2e_\ell + \alpha'' + \beta''_{\ell} - 2j}, \]

where $u_p = u_p(r)$ was defined in (4.6). We split in four terms the right hand side of (A.2) according to the splitting

\[(\alpha_j'\beta_{j'}'' + \alpha_j''\beta_j') (\beta'_{\ell} + \beta''_{\ell}) = \alpha_j'\beta_{j'}''\beta_{\ell}' + \alpha_j'\beta_{j'}\beta''_{\ell} + \alpha_j''\beta_j\beta_{\ell}' + \alpha_j''\beta_j\beta_{\ell}' ; \]

we will consider only the term

\[ A := \sup_{\ell} \sum_j \sum_{\alpha', \beta', \alpha'', \beta''} |F_{\alpha', \beta'} G_{\alpha'', \beta''} |\beta_j'' u_p^{\alpha'' + \beta''_{\ell} - 2j} \times u_p^{\alpha' + \beta'_{\ell} - 2e_\ell + \alpha'' + \beta''_{\ell} - 2j}, \]

the others being analogous. Noting that $\forall j \in \mathbb{Z}$,

\[ \sum_{\alpha'', \beta''} |G_{\alpha'', \beta''}| |\beta_j'' u_p^{\alpha'' + \beta''_{\ell} - 2j} \leq |G|_{r, p}, \]

we have

\[ A \leq |G|_{r, p} \sup_{\ell} \sum_{\alpha', \beta'} |F_{\alpha', \beta'}| \sum_j |\alpha_j'\beta_{j'}\beta_{\ell}' u_p^{\alpha' + \beta_{\ell}' - 2e_\ell} \]

\[ = |G|_{r, p} \sup_{\ell} \sum_{\alpha', \beta'} |F_{\alpha', \beta'}||\alpha'| \beta_{\ell}' u_p^{\alpha' + \beta_{\ell}' - 2e_\ell} \]

\[ \leq |G|_{r, p} \sup_{\ell} \sum_{\alpha', \beta'} |F_{\alpha', \beta'}| \beta_{\ell}' \tilde{u}_p^{\alpha' + \beta_{\ell}' - 2e_\ell} |\alpha'| \beta_{\ell}' \left( \frac{r}{r + \rho} \right)^{|\alpha' + \beta'|_{r, p}^2}, \]

where $\tilde{u}_p$ is short for $u_p(r + \rho)$ (recall (4.6)). Since

\[ \sup_{|\alpha'| = |\beta'|} |\alpha' + \beta'| \left( \frac{r}{r + \rho} \right)^{|\alpha' + \beta'|_{r, p}^2} \leq \sup_{x \geq 2} \left( \frac{r}{r + \rho} \right)^{x-2} \leq 2 \max \left\{ 1, \frac{r}{\rho} \right\}, \]

we get

\[ A \leq 2 \max \left\{ 1, \frac{r}{\rho} \right\} |G|_{r, p} \sup_{\ell} \sum_{\alpha', \beta'} |F_{\alpha', \beta'}| \beta_{\ell}' \tilde{u}_p^{\alpha' + \beta_{\ell}' - 2e_\ell}, \]

and, recalling the definition in (4.5), this concludes the proof of (4.8). \( \square \)

\[ 49 \text{Indeed, setting } y := \rho/r, \text{ we have that } \sup_{x \geq 2} x \left( \frac{r}{r + \rho} \right)^{x-2} = \sup_{x \geq 2} x (1 + y)^{2-x} = 2 \]

if $y \geq \sqrt{\rho} - 1$. On the other hand, when $0 < y < \sqrt{\rho} - 1$ we have

\[ \sup_{x \geq 2} x (1 + y)^{2-x} = \frac{(1 + y)^2}{e \ln(1 + y)} \leq \frac{1}{y} \sup_{0 < y < \sqrt{\rho} - 1} \frac{(1 + y)^2}{e \ln(1 + y)} = \frac{2}{y}. \]
Remark A.1. Note that Proposition 4.1 and its proof hold for any $\ell^\infty$-weighted norm, namely with norm $\sup_{j \in \mathbb{Z}} w_j |u_j|$, with $w_j \to \infty$.

Proof of Lemma 4.2. Let us first prove the Banach structure. Consider a Cauchy sequence of Hamiltonians $H^{(n)} \in \mathcal{H}_{r,p}^{0}$. For all $\omega \in \mathcal{O}$, $I \in \mathcal{I}$, $H^{(n)}(\omega, I)$ is a Cauchy sequence in $\mathcal{H}_{r,p}^{0}$ then we define

$$H(\omega, I) = \sum_{\alpha, \beta \in \mathcal{M}} H_{\alpha, \beta}(\omega, I) u^\alpha \bar{u}^\beta \in \mathcal{H}_{r,p}^{0}$$

for all $\omega \in \mathcal{O}$ $I \in \mathcal{I}$, one has point-wise convergence $H^{(n)}(\omega, I) \to H(\omega, I) \in \mathcal{H}_{r,p}$. Moreover, for all $\alpha, \beta$ the sequence $H_{\alpha, \beta}^{(n)}$ is a Cauchy sequence w.r.t the norm $|\cdot|^\gamma$. Since $\mathcal{F}(\mathcal{O} \times \mathcal{I})$ is Banach, for all $\alpha, \beta \in \mathcal{M}$ $H_{\alpha, \beta}^{(n)} \to H_{\alpha, \beta} \in \mathcal{F}(\mathcal{O} \times \mathcal{I})$.

By hypothesis $\forall \varepsilon > 0$ there exist $N$ such that for all $n, m > N$ one has

$$\frac{1}{2} \sup_{j} \sum_{(\alpha, \beta) \in \mathcal{M}} |H_{\alpha, \beta}^{(n)} - H_{\alpha, \beta}^{(m)}|^{\gamma}(\alpha_j + \beta_j) u^\alpha \bar{u}^\beta < \varepsilon^2$$

so taking the liminf on $m$ we get for all $j$

$$\frac{1}{2} \liminf_{m} \sum_{(\alpha, \beta) \in \mathcal{M}} |H_{\alpha, \beta}^{(n)} - H_{\alpha, \beta}^{(m)}|^{\gamma}(\alpha_j + \beta_j) u^\alpha \bar{u}^\beta < \varepsilon.$$

Then, for all $\alpha, \beta$ one has

$$H_{\alpha, \beta}^{(n)} - H_{\alpha, \beta} \leq \liminf_{m} |H_{\alpha, \beta}^{(n)} - H_{\alpha, \beta}^{(m)}|^{\gamma}$$

so

$$\sum_{(\alpha, \beta) \in \mathcal{M}} |H_{\alpha, \beta}^{(n)} - H_{\alpha, \beta}|^{\gamma}(\alpha_j + \beta_j) u^\alpha \bar{u}^\beta$$

$$\leq \liminf_{m} \sum_{(\alpha, \beta) \in \mathcal{M}} |H_{\alpha, \beta}^{(n)} - H_{\alpha, \beta}^{(m)}|^{\gamma}(\alpha_j + \beta_j) u^\alpha \bar{u}^\beta$$

$$\leq \liminf_{m} \sum_{(\alpha, \beta) \in \mathcal{M}} |H_{\alpha, \beta}^{(n)} - H_{\alpha, \beta}^{(m)}|^{\gamma}(\alpha_j + \beta_j) u^\alpha \bar{u}^\beta < \varepsilon,$$

by Fatou’s lemma. Taking the supremum over $j$ we have proved that $H^{(n)} \to H$ in the $\mathcal{H}_{r,p}^{0}$ norm.

Concerning the Poisson algebra property, it suffices to use the fact that $|\cdot|^\gamma$ has the algebra property with respect to standard multiplication and from [A.1] we deduce

$$|H_{\alpha, \beta}|^{\gamma} \leq \sum_{j} \sum_{\alpha', \alpha'' \neq j = \alpha} |F_{\alpha', \beta'}^{\gamma}| |G_{\alpha'', \beta''}^{\gamma}|^{\gamma} (\alpha_j' \beta_j'' + \beta_j' \alpha_j'').$$

Then the proof follows verbatim the one of Proposition 4.1. □
Proof of Proposition 5.1. Items (i) and (ii) directly follow by Propositions 4.1 and 4.2 of [BMP21], respectively. Here we only discuss the analyticity with respect to \( I \). By formula (4.30) in and recalling the notations in (5.9) we get the representation formula for every \( |u|_p \leq r' \)

\[
H^{(d)}(u, \omega, I) = \sum_{\alpha, \beta, \zeta, \delta} \sum_{\delta \geq \zeta} \sum_{m \geq \delta} A
\]

(A.5)

\[
A = \left( \frac{m}{\delta} \right) \left( \frac{\delta}{\zeta} \right) (-1)^{\delta-\zeta} \frac{m^{\zeta-c}}{H^{(m-\zeta)}} \frac{m^{\zeta-c}}{H^{(m-\zeta)}} H_{m, \alpha, \beta, a, b}(\omega, I) |v|^2 v^a v^b z^a \bar{z}^b,
\]

where \( m, \alpha, \beta, \zeta, \delta \in \mathbb{N}^S, \alpha, b \in \mathbb{N}^S, \alpha \cap \beta = \emptyset \).

Set for brevity \( X := X_{\Pi^d H} \) the Hamiltonian vector field of \( \Pi^d H \) (recall (5.9)). Then define \( Y \) component-wise for \( j \in \mathbb{Z} \) as

\[
Y_j := \frac{1}{2} \sum_{\alpha, \beta, \zeta, \delta} \sum_{m \geq \delta} \left( \frac{m}{\delta} \right) \left( \frac{\delta}{\zeta} \right) (-1)^{\delta-\zeta} \frac{m^{\zeta-c}}{H^{(m-\zeta)}} \frac{m^{\zeta-c}}{H^{(m-\zeta)}} H_{m, \alpha, \beta, a, b}(\omega, I) |v|^2 v^a v^b z^a \bar{z}^b,
\]

with \( \xi := 2\zeta + \alpha + \beta + a + b \) and again \( m, \alpha, \beta, \zeta, \delta \in \mathbb{N}^S, \alpha, b \in \mathbb{N}^S, \alpha \cap \beta = \emptyset \) and where \( u_p = u_p(r') \) was defined in (4.6) and \( I_p := u^2_p / 2 \). By the formula after (4.30) in [BMP21] we have \( Y \in \mathfrak{w}_p \) with \( |Y|_p \leq 3^{d+1} |H|_{r'} \) and \( |X_j(u, \omega, I)| \leq Y_j \) for every \( j \in \mathbb{Z} \), \( u \in B_{r'}(\mathfrak{w}_p) \), \( \omega \in \mathfrak{O} \), \( I \in \mathcal{I}(p, r) \) (resp. \( I \in \mathcal{I}(p, r, \mathbb{C}) \) in the complex case). Therefore \( H^{(d)}(u) \) is analytic in \( \mathcal{I}(p, r) \) (resp. \( I \in \mathcal{I}(p, r, \mathbb{C}) \) in the complex case) since can be written in a totally (a fortiori uniformly) convergent series (see e.g. Theorem 2, Appendix A of [PT87]).

Proof of Lemma 7.2. \(^{50}\) By induction over \( N \). The result is obvious for \( N = 1 \). Let assume it for \( N \) and show it for \( N + 1 \). We have

\[
\sum_{1 \leq \ell \leq N + 1} x_\ell \leq \frac{\sum_{1 \leq \ell \leq N} x_\ell + x_{N+1}}{\prod_{1 \leq \ell \leq N + 1} \sqrt{x_\ell}} \leq \sqrt{\frac{\sum_{1 \leq \ell \leq N} x_\ell + x_{N+1}}{\sum_{1 \leq \ell \leq N} \sqrt{x_\ell}}} \leq \sqrt{\frac{1}{\sqrt{x_{N+1}}}} + \frac{\sqrt{x_{N+1}}}{\sqrt{2}}.
\]

since \( x_1 \geq x_2 \geq \ldots \geq x_N \geq 2 \) implies \( \prod_{1 \leq \ell \leq N} \sqrt{x_\ell} \geq \sqrt{2} \). It remains to prove that

\[
\left( \frac{\sqrt{x_1} + \frac{4}{\sqrt{x_1}}} \right) \frac{1}{\sqrt{x_{N+1}}} + \frac{\sqrt{x_{N+1}}}{\sqrt{2}} \leq \sqrt{\frac{x_1}{\sqrt{2}} + \frac{4}{\sqrt{x_1}}}.
\]

\(^{50}\) Note that in Lemma C.4 of [BMP20a] was stated the following erroneous result: Let \( 0 < a < 1 \) and \( x_1 \geq x_2 \geq \ldots \geq x_N \geq 2 \). Then \( \sum_{1 \leq \ell \leq N} x_\ell^{1-a} \leq x_1^{1-a} + \frac{2}{a-1} \). Anyway in [BMP20a] we only had used the above result with \( a = 1 \), which is exactly the content of Lemma 7.2.
Denoting \( t := \sqrt{\frac{2}{1}} \) and \( s := \sqrt{\frac{2}{N+1}} \), the above inequality is equivalent to

\[
f(t, s) := 2t^2s - \sqrt{2}ts^2 + 8s - 2t^2 - 8 \geq 0
\]

for \( \sqrt{2} \leq s \leq t \). Since \( f \) is a concave function of \( s \) we have that

\[
f(t, s) \geq \min \{ f(t, \sqrt{2}), f(t, t) \}.
\]

It is immediate to see that

\[
\min_{t \geq \sqrt{2}} f(t, \sqrt{2}) = 7\sqrt{2} - 9 > 0, \quad \min_{t \geq \sqrt{2}} f(t, t) = 12\sqrt{2} - 16 > 0,
\]

showing that \( f(t, s) > 0 \) for \( \sqrt{2} \leq s \leq t \) and concluding the proof. \( \square \)

**Proof of Lemma 7.3.** Let \( \tilde{\delta} := \frac{\delta}{9} \). We split the sum in (7.21) into two terms \( A_k = A_k^+ + A_k^- \), with

\[
A_k^+ := \sum_{i < i_k, k_i \geq 1} -\delta k_i \log |i| + \log (1 + \langle i \rangle^2 k_i^2),
\]

\[
A_k^- := \sum_{i \leq i_k, k_i \geq 1} -\delta k_i \log |i| + \log (1 + \langle i \rangle^2 k_i^2).
\]

Regarding the first term we get, recalling that \( s(i) \geq i \),

\[
A_k^+ \leq \sum_{i < i_k, k_i \geq 1} -\delta k_i \log |i| + 1 + 2 \log(i) + 2 \log k_i
\]

\[
\leq 8i_* \log i_* - \sum_{i < i_k, k_i \geq 1} ((\tilde{\delta} \log 2)k_i - 2 \log k_i) \leq 8i_* (\log i_* + \log \frac{1}{\tilde{\delta}}),
\]

using that \( \max_{x \geq 1} -(\tilde{\delta} \log 2)x + 2 \log x \leq 2 \log 1/\tilde{\delta} \).

Consider now the second term. By (2.8a) we get

\[
A_k^- \leq \sum_{i_ \leq i_k, k_i \geq 1} -\delta k_i \log^{1+\eta} |i| + \log (1 + \langle i \rangle^2 k_i^2) \leq A_{k,1}^- + A_{k,2}^-,
\]

where

\[
A_{k,1}^- := \sum_{i \leq i_k, k_i \geq 1} 2f(i, k_i), \quad A_{k,2}^- := \sum_{i \leq i_k, 1 \leq k_i < 28/\tilde{\delta}} 2f(i, k_i),
\]

and \( f(i, k) := -\delta k \log^{1+\eta} i + 3 \log i + 2 \log(\delta k) + 2 \log(1/\tilde{\delta}). \)

When \( k_i \geq 28/\tilde{\delta} \) we have \( f(i, k) \):

\[
f(i, k_i) \leq -\frac{1}{2} \delta k_i \log^{1+\eta} i + 2 \log(\delta k_i) + 2 \log(1/\tilde{\delta})
\]

\[
\leq -\frac{1}{4} \delta k_i \log^{1+\eta} i + 2 \log(1/\tilde{\delta}) \leq -7 \log^{1+\eta} i + 2 \log(1/\tilde{\delta}),
\]

51 With that \( i_* \geq 3 \).

52 Using that \( -\frac{1}{4}x + 2 \log x \leq 0 \) for \( x \geq 28 \) and \( \log i \geq \log i_* \geq 1 \).
which is negative for \( i \geq 1/\bar{\delta} \). Then we get
\[
A_{k,1} \leq \frac{4}{\bar{\delta}} \log(1/\bar{\delta})
\]
We finally consider the term \( A_{k,2} \), namely when \( k_i < 28/\bar{\delta} \). In this case
\[
f(i, k_i) \leq \left( 3 - \bar{\delta} \log \eta I \right) \log i + 7 + 2 \log(1/\bar{\delta})
\]
We claim that the last quantity is negative for \( i \geq \exp(4\bar{\delta} - 1/\eta) \).

\[\square\]

Proof of Lemma 9.2. To fix ideas we consider the case \( \xi_n = -1 \). Set \( x =: (\hat{x}, x_n) \). Let us introduce the portion of hyperplane (which is a graph over \( \hat{x} \))
\[
P := \{ (\hat{x}, \hat{\xi} \cdot \hat{x}) : \hat{x} \in [-1/4, 1/4]^{n-1} \},
\]
orthogonal to \( \xi \). Note that for every \( y \in E \) there exist unique \( t \in \mathbb{R} \) and \( \hat{x} \in [-1/4, 1/4]^{n-1} \) such that
\[
y = (\hat{x}, \hat{\xi} \cdot \hat{x}) + t\xi.
\]
Then by Fubini’s theorem we have that
\[
\text{meas}(E) = |\xi|_2 \int_P \text{meas}\{ t \in \mathbb{R} : (\hat{x}, \hat{\xi} \cdot \hat{x}) + t\xi \in E \} \, d\sigma
\leq |\xi|_2 \delta \int_P d\sigma = |\xi|_2 \delta \int_{[-1/4, 1/4]^{n-1}} \sqrt{1 + |\xi|^2} d\hat{x}
\leq 2^{1-n} \delta |\xi|^2.
\]
\[\square\]

APPENDIX B. TOPOLOGY, MEASURE AND CONTINUOUS FUNCTIONS ON INFINITE PRODUCT SPACES

Product topology. Fix \( \varrho > 0 \). Let us consider the set \([-\varrho, \varrho]^\mathbb{Z} \) endowed with the product topology, namely the coarsest topology (i.e. the topology with the fewest open sets) for which all the projections \( \pi_j : [-\varrho, \varrho]^\mathbb{Z} \to [-\varrho, \varrho] \), with \( \pi_j(\omega) := \omega_j, j \in \mathbb{Z} \), are continuous.

We call cylinder a subset \( \bigotimes_{n \in \mathbb{N}} A_n \) of \([-\varrho, \varrho]^\mathbb{Z} \) such that \( A_n \subseteq [-\varrho, \varrho] \) with \( A_n \neq [-\varrho, \varrho] \) only for finitely many \( n \in \mathbb{N} \). Then a basis of the product topology is given by the open cylinders, namely cylinders \( \bigotimes_{n \in \mathbb{N}} A_n \), where \( A_n \) are open.
By the Tychonoff’s theorem $[-\varrho, \varrho]^\mathbb{Z}$ with the product topology is a compact Hausdorff space.

**Product measures.** The product $\sigma$-algebra (of the Borel sets) of $[-\varrho, \varrho]^\mathbb{Z}$ is generated by the set of cylinders $\bigotimes_{n \in \mathbb{N}} A_n$, where $A_n$ are Borel sets (w.r.t. the standard topology on $[-\varrho, \varrho]$).

The probability product measure $\mu$ on product $\sigma$-algebra of $[-\varrho, \varrho]^\mathbb{Z}$ is defined by

$$
\mu(\bigotimes_{n \in \mathbb{N}} A_n) := \prod_{n \in \mathbb{N}} \frac{1}{2\varrho} |A_n|,
$$

where $|A_n|$ denotes the usual Lebesgue measure of the Borel set $A_n$.

Through the bijective map $\mathcal{F}^* : Q \to [-1/2, 1/2]^\mathbb{Z}$, where $\mathcal{F}_j^*(\nu) := \nu_j - j^2$, $j \in \mathbb{Z}$, we induce the product topology and the probability product measure on the set $Q$. Analogously for $Q_S$ and $Q_{Sc}$.

**Product measures.** Given a compact Hausdorff space $X$ and a Banach space $E$ we denote by $C(X, E)$ the Banach space of continuous functions $f : X \to E$ endowed with the uniform norm

$$
|f|_{C(X, E)} := \sup_{x \in X} |f(x)|_E = \max_{x \in X} |f(x)|_E.
$$

**Lemma B.1** (Lipschitz fixed point). Let $\mathcal{C}$ be the closed subset of the Banach space $C(Q_S \times [-1/4, 1/4]^{Sc}, \ell^{\infty}_{Sc})$ (with the product topology on $Q_S \times [-1/4, 1/4]^{Sc}$) defined as

$$
\mathcal{C} := \{w \in C(Q_S \times [-1/4, 1/4]^{Sc}, \ell^{\infty}_{Sc}) \text{ s.t. } |w|_{C(Q_S \times [-1/4, 1/4]^{Sc}, \ell^{\infty}_{Sc})} \leq r\},
$$

for some $0 < r < 1/4$. Let $F \in C(Q_S \times [-1/4, 1/4]^{Sc} \times [-r, r]^{Sc}, \ell^{\infty}_{Sc})$ with $|F(\nu, V_{Sc}, w)|_{\ell^{\infty}_{Sc}} \leq r$, $|F(\nu, V_{Sc}, w) - F(\nu, V_{Sc}, w')|_{\ell^{\infty}_{Sc}} \leq 1/2|w - w'|_{\ell^{\infty}_{Sc}}$, for all $\nu \in Q_S, w, w' \in Q_{Sc}$. Then there exists a unique $w \in \mathcal{C}$ such that

$$
w(\nu, V_{Sc}) = F(\nu, V_{Sc}, w(\nu, V_{Sc})).
$$

Moreover if $F$ is $L$-Lipschitz for some $L > 0$ w.r.t. $\nu \in Q_S$ (endowed with the $\ell^{\infty}$-metric) then $w$ is $2L$-Lipschitz in $\nu$. Analogously if $F$ is $L'$-Lipschitz w.r.t. some other parameter $I$ in some Banach space then $w$ is $2L'$-Lipschitz in $I$.

**Proof.** First note that if $w \in \mathcal{C}$ then $F(\nu, V_{Sc}, w(\nu, V_{Sc})) \in \mathcal{C}$, since the product topology in $[-r, r]^{Sc}$ is weaker than the one induced by the $\ell^{\infty}$-norm. Set $\Phi : \mathcal{C} \to \mathcal{C}$ by

$$
(\Phi(w))(\nu, V_{Sc}) := F(\nu, V_{Sc}, w(\nu, V_{Sc})).
$$
Let us check that $\Phi$ is a contraction on $C$. Indeed
\[
|\Phi(w) - \Phi(w')|_C = \sup_{\nu \in Q_S, V_S \in [-1/4,1/4]} |F(\nu, V_S, w(\nu, V_S)) - F(\nu, V_S, w'(\nu, V_S))|_{\ell_S^C} \\
\leq \frac{1}{2} \sup_{\nu \in Q_S, V_S \in [-1/4,1/4]} |w(\nu, V_S) - w'(\nu, V_S)|_{\ell_S^C} = \frac{1}{2} |w - w'|_C.
\]

The existence of the fixed point follows from the Contraction Mapping Theorem on Banach spaces. Assume now that $F$ is $L$-Lipschitz for some $L > 0$ w.r.t. $\nu \in Q_S$ (endowed with the $\ell^\infty$-metric $d_\infty$). Then
\[
|w(\nu, V_S) - w'(\nu, V_S)|_{\ell_S^C} \leq |F(\nu, V_S, w(\nu, V_S)) - F(\nu, V_S, w'(\nu, V_S))|_{\ell_S^C} \\
+ |F(\nu, V_S, w'(\nu, V_S)) - F(\nu, V_S, w'(\nu, V_S))|_{\ell_S^C} \\
\leq Ld_\infty(\nu, \nu') + \frac{1}{2} |w(\nu, V_S) - w'(\nu, V_S)|_{\ell_S^C}
\]
implying
\[
|w(\nu, V_S) - w'(\nu, V_S)|_{\ell_S^C} \leq 2Ld_\infty(\nu, \nu').
\]

The next lemma regards the analyticity of the map $i$ in (2.7). Without loss of generality we consider here only the case $S = \mathbb{Z}$ since we are not assuming $I_j \neq 0$. Let us first introduce the notation $C_s^\infty := \{\varphi = (\varphi_j)_{j \in \mathbb{Z}} \text{ with } |\text{Im } \varphi_j| < s, \forall j \in \mathbb{Z}\}$ for some $s > 0$ and the equivalence relation $\varphi \sim \varphi'$ iff $\varphi - \varphi' \in 2\pi\mathbb{Z}$. We set $T_s^\infty := C_s^\infty / \sim$.

**Lemma B.2.** For $\sqrt{t} \in \tilde{B}_r(w_p)$ the map $i$ in (2.7) can be extended to an analytic map
\[
i : T_s^\infty \to w_p, \quad [\varphi] = [(\varphi_j)_{j \in \mathbb{Z}}] \mapsto (\sqrt{t} e^{i\varphi_j})_{j \in \mathbb{Z}},
\]
for any $s > 0$.

**Proof.** For $\varphi \in C_s^\infty$ we denote by $[\varphi] \in T_s^\infty$ the equivalence class of $\varphi$. $T_s^\infty$ is a metric space endowed with the distance
\[
d([\varphi], [\psi]) := \min_{\psi' \in [\psi]} |\varphi - \psi'|_\infty,
\]
where $| \cdot |_\infty$ is the norm on $\ell^\infty = \ell^\infty(\mathbb{C})$. Moreover it is a Banach manifold. Indeed given every point $[\varphi] \in T_s^\infty$, set $\rho := \min\{1, s - |\text{Im } \varphi|\}/2$ and consider the open ball $B_\rho([\varphi]) \subset T_s^\infty$ of radius $\rho$ centered in $[\varphi]$ and the open ball $U_\rho$ of radius $\rho$ centered at the origin of $\ell^\infty$; then a local chart is $\Phi : B_\rho([\varphi]) \to U_\rho$ defined so that $\Phi^{-1}(\psi) := [\varphi + \psi]$. We claim that
\[
g := i \circ \Phi^{-1} : U_\rho \to w_p, \quad g(\psi) := (\sqrt{t} e^{i[\varphi_j + \psi_j]})_{j \in \mathbb{Z}}
\]
is analytic since the Frechet derivative $Dg : U_p \to \mathcal{L}(\ell^\infty, w_p)$ is continuous. Indeed for every $\psi' \in \ell^\infty$

$$Dg(\psi)[\psi'] = (i\sqrt{T}\psi) e^{i(\psi_j + \psi_j')} j \in \mathbb{Z}$$

and for $\psi, \tilde{\psi} \in U_p$ the operator norm (recall $\sqrt{T} \in \mathcal{B}(w_p)$) satisfies

$$\|Dg(\psi) - Dg(\tilde{\psi})\|_{op} = \sup_{\psi' \in \ell^\infty, \|\psi'\| = 1} (i\sqrt{T}\psi) e^{i(\psi_j + \psi_j')} j \in \mathbb{Z}, \|w_p\| \leq r e^{|\text{Im} \varphi| + \rho \left((1 - e^{i(\psi_j - \psi_j')})\right) |j| \in \ell^\infty}$$

$$\leq r e^{|\text{Im} \varphi| + 2\rho |\tilde{\psi} - \psi| \in \ell^\infty}.$$  

□

On the density of linear flow $\nu t$ on the flat torus $\mathcal{T}_1$ (recall (2.5)). We note that if $|\nu|_{\ell^\infty} < \infty$ the linear flow $t \to \nu t$ is not dense on the torus $\mathcal{T}_1$, otherwise $\nu \in \mathcal{S}_1$ would be a countable dense subset but $\ell^\infty$-based topologies are not separable. On the other hand if the sequence $\nu_j$ increases very fast (e.g. super-exponentially) the flow is dense.

**Lemma B.3.** If for any $n \geq 1$ the vector $\nu^{(n)}$ with entries $\nu_j$, $|j| \leq n$, $j \in \mathcal{S}_1$ is rationally independent and

$$\lim_{n \in \mathcal{S}_1, |n| \to \infty} |\nu_n| \sum_{j \in \mathcal{S}_1, |j| > |n|} |\nu_j|^{-1} = 0$$

then the flow is dense.

**Proof.** Fix $\delta > 0$, $\bar{\varphi}$ on the torus and call $d$ the distance on the one dimensional torus $d_{\mathcal{T}_1}(\varphi, \psi) = \min_{k \in \mathbb{Z}} |\varphi - \psi - 2\pi k|$ for $\varphi, \psi \in \mathcal{T}_1$. We want to show that there exists a time $T \in \mathbb{R}$ such that $d_{\mathcal{T}_1}(\bar{\varphi}, \nu_0 T) \leq \delta$ for every $n \in \mathcal{S}_1$. Let $n_0 \in \mathcal{S}_1$ large enough such that $|\nu_n| \sum_{j \in \mathcal{S}_1, |j| > |n_0|} |\nu_j|^{-1} \leq \delta/2\pi$ for every $n \in \mathcal{S}_1$, $|n| \geq |n_0|$. Since $\nu^{(n_0)}$ is rationally independent there exists $T_0 > 0$ such that $d_{\mathcal{T}_1}(\bar{\varphi}, \nu_j T_0) \leq \delta/2$ for every $j \in \mathcal{S}_1, |j| \leq |n_0|$. For every $j \in \mathcal{S}_1, |j| > |n_0|$ there exists $t_j \in \mathbb{R}$ with $|t_j| \leq \pi/|\nu_j|$ such that $d_{\mathcal{T}_1}(\bar{\varphi}, \nu_j (T_0 + t_j)) = 0$. Set $T := T_0 + T'$ where $T' := \sum_{j \in \mathcal{S}_1, |j| > |n_0|} t_j$. Note that $|T| \leq \delta/2|\nu_{n_0}|$. Then for $n \in \mathcal{S}_1, |n| \leq |n_0|$ we have $d_{\mathcal{T}_1}(\bar{\varphi}, \nu_{n_0} T) \leq \delta/2 + |\nu_n T'| \leq \delta$. Finally for $n \in \mathcal{S}_1, |n| > |n_0|$ we have $d_{\mathcal{T}_1}(\bar{\varphi}, \nu_{n_0} T) \leq |\nu_n| \sum_{j \in \mathcal{S}_1, |j| > |n_0|, j \neq 0} |t_j| \leq \delta/2$, concluding the proof. □

Finally, since in our application to NLS $\nu_j \sim j^2$, if $\mathcal{S}_1$ is sparse enough the condition of Lemma [B.3] is satisfied and the flow is dense.

---

53Recall the definition of $\mathcal{S}_1$ in (2.6).
References

[BB13] M. Berti and Ph. Bolle. Quasi-periodic solutions for Schrödinger equations with Sobolev regularity of NLS on $T^d$ with a multiplicative potential. J. European Math. Society, 15:229–286, 2013.

[BB15] Massimiliano Berti and Philippe Bolle. A Nash-Moser approach to KAM theory. In Hamiltonian partial differential equations and applications, volume 75 of Fields Inst. Commun., pages 255–284. Fields Inst. Res. Math. Sci., Toronto, ON, 2015.

[BBM16] P. Baldi, M. Berti, and R. Montalto. KAM for autonomous quasilinear perturbations of KdV. Ann. Inst. H. Poincare Anal. Non Lineaire, 33:1589–1638, 2016.

[BD18] M. Berti and J.M. Delort. Almost global existence of solutions for capillary-gravity water waves equations with periodic spatial boundary conditions. Springer, 2018.

[BG06] D. Bambusi and B. Grébert. Birkhoff normal form for partial differential equations with tame modulus. Duke Math. J., 135(3):507–567, 2006.

[BMP20a] L. Biasco, J. E. Massetti, and M. Procesi. An Abstract Birkhoff Normal Form Theorem and Exponential Type Stability of the 1d NLS. Comm. Math. Phys., 375(3):2089–2153, 2020.

[BMP20b] L. Biasco, J. E. Massetti, and M. Procesi. A note on the construction of Sobolev almost periodic invariant tori for the 1d NLS. Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 31(4):981–993, 2020.

[BMP21] L. Biasco, J.E. Massetti, and M. Procesi. Almost-periodic invariant tori for the NLS on the circle. Ann. Inst. H. Poincaré Anal. Non Linéaire, 38(3):711–758, 2021.

[Bou93] J. Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. Geom. Funct. Anal., 3(2):107–156, 1993.

[Bou94] J. Bourgain. Periodic nonlinear Schrödinger equation and invariant measures. Comm. Math. Phys., 166(1):1–26, 1994.

[Bou96] J. Bourgain. On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE. Internat. Math. Res. Notices, (6):277–304, 1996.

[Bou99] J. Bourgain. Global solutions of nonlinear Schrödinger equations, volume 46 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 1999.

[Bou05] J. Bourgain. On invariant tori of full dimension for 1D periodic NLS. J. Funct. Anal., 229(1):82–94, 2005.

[CKS+10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. Invent. Math., 181(1):39–113, 2010.

[CMW20] H. Cong, L. Mi, and P. Wang. A Nekhoroshev type theorem for the derivative nonlinear Schrödinger equation. J. Differential Equations, 268(9):5207–5256, 2020.

[Con] H. Cong. The existence of full dimensional KAM tori for Nonlinear Schrödinger equation. arXiv:2103.14777.

[CW93] W. Craig and C. E. Wayne. Newton’s method and periodic solutions of nonlinear wave equations. Comm. Pure Appl. Math., 46(11):1409–1498, 1993.

[CY20] H. Cong and X. Yuan. The existence of full dimensional invariant tori for 1-dimensional nonlinear wave equation. Annales de l’Institut Henri Poincaré C, Analyse non linéaire, 38, 09 2020.

[FG13] E. Faou and B. Grébert. A Nekhoroshev-type theorem for the nonlinear Schrödinger equation on the torus. Anal. PDE, 6(6):1243–1262, 2013.
SMALL AMPLITUDE WEAK ALMOST PERIODIC SOLUTIONS FOR THE 1D NLS

[FI] R. Feola and F. Iandoli. Long time existence for fully nonlinear NLS with small Cauchy data on the circle. To appear on Annali della SNS, classe di Scienze. DOI:10.2422/2036 – 2145.201811003.

[FM22] R. Feola and J. E. Massetti. Sub-exponential stability for the beam equation. preprint arXiv:2207.09986, 2022.

[GGMP21] F. Giuliani, M. Guardia, P. Martin, and S. Pasquali. Chaotic-like transfers of energy in hamiltonian pdes. Communications in Mathematical Physics, 384(2):1227–1290, 2021.

[GH17] P. Gerard and Koch H. The cubic Szegö flow at low regularity., volume Exp. No. XIV of Séminaire Laurent Schwartz Équations aux dérivées partielles et applications. Année 2016/2017. Ed. Éc. Polytech., Palaiseau, 2017.

[GHH+18] M. Guardia, Z. Hani, E. Haus, A. Maspero, and M. Procesi. Strong nonlinear instability and growth of Sobolev norms near quasiperiodic finite-gap tori for the 2D cubic NLS equation. arXiv:1810.03694, to appear on JEMS, 2018.

[GK15] M. Guardia and V. Kaloshin. Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation. J. Eur. Math. Soc. (JEMS), 17(1):71–149, 2015.

[GKT] P. Gérard, T Kappeler, and P. Topalov. On the Benjamin-Ono equation on $\mathbb{T}$ and its periodic and quasiperiodic solutions. arXiv:2103.09291.

[GX13] J. Geng and X. Xu. Almost periodic solutions of one dimensional Schrödinger equation with the external parameters. J. Dynam. Differential Equations, 25(2):435–450, 2013.

[Her86] M. Herman. Sur les courbes invariantes par les difféomorphismes de l’anneau. Vol. 2. Astérisque, (144):248, 1986. With a correction to: it On the curves invariant under diffeomorphisms of the annulus, Vol. 1 (French) [Astérisque No. 103-104, Soc. Math. France, Paris, 1983; MR0728564 (85m:58062)].

[HP17] E. Haus and M. Procesi. KAM for beating solutions of the quintic NLS. Comm. Math. Phys., 354(3):1101–1132, 2017.

[KM17] Th. Kappeler and J.C. Molnar. On the well-posedness of the defocusing mKdV equation below $L^2$. SIAM J. Math. Anal., 49(3):2191–2219, 2017.

[KM18] T. Kappeler and J. Molnar. On the wellposedness of the kdv equation on the space of pseudomeasures. Sel. Math. New Ser., 24:1479–1526, 2018.

[KMKT16] Thomas Kappeler, Alberto Maspero, Jan Molnar, and Peter Topalov. On the convexity of the KdV Hamiltonian. Comm. Math. Phys., 346(1):191–236, 2016.

[Kou20] C. E. Koudjinan. A KAM theorem for finitely differentiable Hamiltonian systems. J. Differential Equations, 269(6):4720–4750, 2020.

[Kuk88] S. B. Kuksin. Perturbation of conditionally periodic solutions of infinite-dimensional Hamiltonian systems. Izv. Akad. Nauk SSSR Ser. Mat., 52(1):41–63, 240, 1988.

[Kuk04] Sergei B. Kuksin. Fifteen years of KAM for PDE. In Geometry, topology, and mathematical physics, volume 212 of Amer. Math. Soc. Transl. Ser. 2, pages 237–258. Amer. Math. Soc., Providence, RI, 2004.

[Mas18] J. E. Massetti. A normal form à la Moser for diffeomorphisms and a generalization of Rüssmann’s translated curve theorem to higher dimensions. Anal. PDE, 11(1):149–170, 2018.

[Mas19] J. E. Massetti. Normal forms for perturbations of systems possessing a Diophantine invariant torus. Ergodic Theory Dynam. Systems, 39(8):2176–2222, 2019.

[Pöș] J. Pöschel. KAM below $C^n$. arXiv:2104.01866.
[Pöschel02] J. Pöschel. On the construction of almost periodic solutions for a nonlinear Schrödinger equation. Ergodic Theory Dynam. Systems, 22(5):1537–1549, 2002.

[PP15] M. Procesi and C. Procesi. A KAM algorithm for the non–linear Schrödinger equation. Advances in Math., 272:399–470, 2015.

[PT87] Jürgen Pöschel and Eugene Trubowitz. Inverse spectral theory, volume 130 of Pure and Applied Mathematics. Academic Press, Inc., Boston, MA, 1987.

Università degli Studi Roma Tre
Email address: biasco@mat.uniroma3.it

Università degli Studi Roma Tre
Email address: jmassetti@mat.uniroma3.it

Università degli Studi Roma Tre
Email address: procesi@mat.uniroma3.it