Anti-Gravitating Brane-World Solutions for a de Sitter Brane in Scalar-Tensor Gravity

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Abstract

In the context of a five-dimensional theory with a scalar field non-minimally-coupled to gravity, we look for solutions that describe novel black-string or maximally-symmetric solutions in the bulk. The brane line-element is found to describe a Schwarzschild-(Anti)-de Sitter spacetime, and, here, we choose to study solutions with a positive four-dimensional cosmological constant. We consider two different forms of the coupling function of the scalar field to the bulk scalar curvature, a linear and a quadratic one. In the linear case, we find solutions where the theory, close to our brane, mimics an ordinary gravitational theory with a minimally-coupled scalar field giving rise to an exponentially decreasing warp factor in the absence of a negative bulk cosmological constant. The solution is characterised by the presence of a normal gravity regime around our brane and an anti-gravitating regime away from it. In the quadratic case, there is no normal-gravity regime at all, however, scalar field and energy-momentum tensor components are well-defined and an exponentially decreasing warp factor emerges again. We demonstrate that, in the context of this theory, the emergence of a positive cosmological constant on our brane is always accompanied by an anti-gravitating regime in the five-dimensional bulk.

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1 Introduction

Black holes are among the most fundamental and, at the same time, most fascinating solutions of the General Theory of Relativity (GR). The different types of black holes predicted by GR have all been determined and classified according to their physical properties (mass, charge, angular momentum), and uniqueness theorems have been formulated (see, for example, [1, 2]). The emergence of theories [3, 4] based on the early concept of brane [5, 6] and postulating the existence of extra spacelike dimensions in nature has completely changed the landscape. Now, the higher-dimensional analogues of black holes cannot be easily classified or proven to be unique - moreover, they are supplemented by a large number of black objects such as black strings, black branes, black rings or black satellites [7].

In the limit where the self-energy of the brane is much smaller than the black-hole mass and the symmetries of the four-dimensional solutions may be extended to the higher-dimensional spacetime [3], analytical forms of higher-dimensional black holes, either spherically-symmetric or rotating, are easy to derive – in fact, they were derived long ago in [8, 9]. If, however, the brane self-energy is not negligible and contributes towards a particular profile of gravity along the extra dimensions [4], the analytic derivation of higher-dimensional black holes is extremely difficult. A first attempt [10], employing a straight-forward ansatz of a Schwarzschild line-element embedded into a warped extra dimension, has failed to lead to a five-dimensional black hole and has instead led to a black string, a five-dimensional solution with a horizon and a singularity at every point of the extra dimension. It was subsequently shown that these black strings are unstable under linear gravitational perturbations [11, 12], therefore these unphysical objects are unlikely to survive in the context of a fundamental gravitational theory.

Despite decades of efforts, the quest for the more physically-acceptable solutions of regular, localised close to our brane black holes has failed to lead to an analytical, non approximate form (see, Refs. [13] and [14]–[27] for an impartial list of works) – however, such solutions were successfully derived in lower-dimensional gravitational models [28–30]. Numerical solutions also emerged that described either small [31–33] or large black holes [34–37] in brane-world models. In an effort to derive the long-sought analytical black-hole solutions, in [38,39] the previously proposed idea [16], of adding a non-trivial profile along the extra dimension to the black-hole mass function in the original line-element used in [10], was extended to include also a dependence on the time and radial coordinate; in this way, the rather restricted Schwarzschild-type of brane background was extended to include additional terms [of an (Anti)-de Sitter or Reissner-Nordstrom type] and to allow also for non-static configurations. A large number of bulk scalar field theories were then investigated, however, no viable solutions that could sustain the line-element of a five-dimensional, regular, localised-close-to-our-brane black hole was found.

In contrast, the analysis performed in [38,39] have hinted towards the existence of solutions that were not characterized by the desired non-trivial profile of the mass-function in terms of the extra coordinate – these solutions could not therefore be localised black holes but rather novel black-string solutions. As a result, in this work, we focus on the careful investigation of the existence of these latter type of solutions in the context
of a theory with a scalar field non-minimally-coupled to gravity, and on the study of their physical properties. We demonstrate that, for very natural, simple choices of the coupling function between the scalar field and the five-dimensional scalar curvature, novel black-string solutions may indeed be found with rather interesting and provocative characteristics. Given the fact that the same theory has resisted in giving legitimate black-hole solutions, even for a wider number of choices of the coupling function [38,39], our present results add new ‘fuel’ to the long dispute around the question of why brane-world models lead quite easily to black-string solutions but not to localised black holes [40]–[50]. Indeed, higher-dimensional gravitational theories often allow for the emergence of uniform or non-uniform black-string solutions [51]–[59].

In our analysis, we will retain the ‘Vaidya form’ of the brane line-element used also in [17,38,39], since this form was shown not to lead to additional spacetime singularities in the bulk. As we are interested in finding static black-string solutions, here we abandon the dependence of the mass function on the time and extra-dimension coordinates, and allow for a general, radially-dependent form \( m(r) \). Our field equations will be straightforwardly integrated to determine the form of the mass function that is found to correspond to a Schwarzschild-Anti-de Sitter background. We will consider two simple forms of the coupling function, namely a linear and a quadratic one in terms of the scalar field. In both cases, we solve the set of field equations, and derive the scalar-field configurations and the physical properties of the model.

A common characteristic of the solutions derived in the two cases is the negative sign of the coupling function in front of the five-dimensional scalar curvature, either over the entire bulk (for a quadratic dependence) or at distances larger than a specific value (for a linear dependence). This clearly leads to the ‘wrong sign’ for gravity, however, as we will see, it is this negative sign that effectively creates an Anti-de Sitter spacetime and supports a Randall-Sundrum warp factor even in the absence of a negative bulk cosmological constant. In the case of the linear coupling function, the anti-gravitating regime arises away from our brane; this regime is pushed farther away the larger the warping coefficient and the smaller the cosmological constant on our brane are. In fact, for particular values of the parameters of the model, the theory resembles an ordinary, minimally-coupled scalar-tensor theory with normal gravity and a Randall-Sundrum warp factor.

Although the original objective of our analysis was to investigate the existence of novel black-string solutions in the context of a non-minimally coupled scalar-tensor theory, our solutions, in the limit of vanishing black-hole mass on the brane, reduce to maximally-symmetric brane-world solutions that are regular over the entire bulk apart from the AdS boundary. In this limit, the gravitational background on our brane is a pure (Anti)-de Sitter spacetime. In fact, we demonstrate that for the physically-motivated case of a positive-cosmological constant on our brane, the emergence of an anti-gravitating regime in the bulk is unavoidable.

Our paper has the following outline: in Section 2, we present the field equations and spacetime background. In Section 3, we study in detail the case of a linear coupling function, and determine the complete bulk solution, its physical properties as well as the
effective theory on the brane. A similar analysis is performed in Section 4 for the case of a quadratic coupling function. In Section 5, we present a mathematical argument that underlines the connection between the emergence of an anti-gravitating regime in the bulk and the positive sign of the cosmological constant on our brane. We finally present our conclusions in Section 6.

2 The Theoretical Framework

In this work, we focus on the following class of 5-dimensional gravitational theories with action functional

\[ S_B = \int d^4x \int dy \sqrt{-g^{(5)}} \left[ f(\Phi) R - \Lambda_5 - \frac{1}{2} \partial_L \Phi \partial^L \Phi - V_B(\Phi) \right] . \]  

The above theory contains the 5-dimensional scalar curvature \( R \), a bulk cosmological constant \( \Lambda_5 \), and a 5-dimensional scalar field \( \Phi \) with a self-interacting potential \( V_B(\Phi) \) and a non-minimal coupling to \( R \) via the general coupling function \( f(\Phi) \). Also, \( g^{(5)}_{MN} \) is the metric tensor of the 5-dimensional spacetime, and \( \kappa_5^2 = 8\pi G_5 \) is defined in terms of the 5-dimensional gravitational constant \( G_5 \). At a particular point along the fifth spatial dimension, whose coordinate we denote by \( y \), a 3-brane is introduced - without loss of generality, we assume that this takes place at \( y = 0 \). Then, the above bulk action must be supplemented by the brane one

\[ S_{br} = \int d^4x \sqrt{-g^{(br)}} (\mathcal{L}_{br} - \sigma) = - \int d^4x \int dy \sqrt{-g^{(br)}} \left[ V_b(\Phi) + \sigma \right] \delta(y) . \]  

Here, \( \mathcal{L}_{br} \) is related to the matter/field content of the brane and has been chosen to consist of an interaction term \( V_b(\Phi) \) of the bulk scalar field with the brane. Also, \( \sigma \) is the brane self-energy, and \( g^{(br)}_{\mu\nu} = g^{(5)}_{\mu\nu}(x^\lambda, y = 0) \) is the induced-on-the-brane metric tensor.

Note that, throughout our work, capital letters \( M, N, L, \ldots \) will denote 5-dimensional indices while lower-case letters \( \mu, \nu, \lambda, \ldots \) will be used for 4-dimensional indices.

The variation of the complete action \( S = S_B + S_{br} \) with respect to the metric-tensor components \( g^{(5)}_{MN} \) yields the gravitational field equations that have the form

\[ f(\Phi) G_{MN} \sqrt{-g^{(5)}} = \kappa_5^2 \left[ (T_{MN}^{(\Phi)} - g_{MN} \Lambda_5) \sqrt{-g^{(5)}} - \left[ V_b(\Phi) + \sigma \right] g^{(br)}_{\mu\nu} \delta^\mu_M \delta^\nu_N \delta(y) \sqrt{-g^{(br)}} \right] , \]  

where

\[ T_{MN}^{(\Phi)} = \partial_M \Phi \partial_N \Phi + g_{MN} \left[ - \frac{\partial_L \Phi \partial^L \Phi}{2} - V_B(\Phi) \right] + \frac{1}{\kappa_5^2} [\nabla_M \nabla_N f(\Phi) - g_{MN} \Box f(\Phi)] . \]  

On the other hand, the variation of the action with respect to \( \Phi \) leads to the scalar-field equation

\[ - \frac{1}{\sqrt{-g^{(5)}}} \partial_M \left( \sqrt{-g^{(5)}} g^{MN} \partial_N \Phi \right) = \frac{\partial_b f}{2\kappa_5^2} R - \partial_b V_B - \frac{\sqrt{-g^{(br)}}}{\sqrt{-g^{(5)}}} \partial_b V_b \delta(y) . \]
We will also assume that the 5-dimensional gravitational background is given by the expression

\[ ds^2 = e^{2A(y)} \left\{ -\left[ 1 - \frac{2m(r)}{r} \right] dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right\} + dy^2. \]  

The above line-element is characterized by the presence of the warp factor \( e^{A(y)} \) that multiplies a 4-dimensional background. For \( m(r) = M \), this 4-dimensional line-element is just the Vaidya transformation of the Schwarzschild solution describing a black-hole with mass \( M \), and leads to the same black-string solutions found in [10]. A generalised Vaidya form, where \( m \) is not a constant but a function of the coordinates, was used in a number of works [17,38,39] in an effort to find regular, localised black-hole solutions. The motivation for the use of the Vaidya form of the 4-dimensional line-element, instead of the usual Schwarzschild one, was provided in [16,17]; in these, it was demonstrated that 4-dimensional line-elements with horizons, such as the Schwarzschild one, when embedded in 5-dimensional spacetimes, transform their coordinate singularities at the horizons to true spacetime ones [16]. In order to avoid this, in [17] the 4-dimensional Schwarzschild line-element was first transformed to its Vaidya form and then embedded in the warped fifth dimension; in that case, no new bulk singularities emerged.

Although the desired black-hole solutions have not yet been analytically found in brane-world models, the emergence of black-string solutions is more easily realised [10] [51]–[59]. Indeed, in the context of the theory (1), hints for the existence of novel black-string solutions described by the line-element (6) were given in [39]. Therefore, here we turn our attention to this question; we will keep the general \( r \)-dependence of the mass function, i.e. \( m = m(r) \), as shown in Eq. (6), in order to allow our brane metric background to deviate from the Schwarzschild form. Such a modification may allow for terms proportional to an effective cosmological constant or for terms of various forms associated with tidal charges to emerge. As the explicit form of the curvature invariant quantities for the line-element (6) (given in Appendix A) show, such a solution, if indeed supported by the theory (1), would describe a black-string solution with only the black-hole singularity extended over the fifth dimension and no other singularity present.

For the line-element (6), one may easily see that the relation \( \sqrt{-g^{(5)}} = \sqrt{-g^{(4)}} \) holds, and the gravitational equations are then simplified to

\[ f(\Phi) G^M_N = T^{(\Phi)}_N - \delta^M_N \Lambda_5 - [V_b(\Phi) + \sigma] g_{\mu\nu} g^{ML} \delta^\mu_L \delta^N_N \delta(y), \]  

with

\[ T^{(\Phi)}_N = \partial^M \Phi \partial_N \Phi + \nabla^M \nabla_N f + \delta^M_N (\mathcal{L}_\Phi - \Box f). \]  

In the above, we have defined

\[ \mathcal{L}_\Phi = -\frac{1}{2} \partial_L \Phi \partial^L \Phi - V_B(\Phi). \]  

In addition, for simplicity, we have absorbed the gravitational constant \( \kappa_5^2 \) in the expression of the general coupling function \( f(\Phi) \), and omitted the superscripts \( (5) \) and \( (4) \) from the bulk and brane metric tensors \( g_{MN} \) and \( g_{\mu\nu} \), respectively. In fact, we will now focus
on the gravitational equations in the bulk and thus altogether remove the brane-term proportional to $\delta(y)$ from Eq. (7) - when the junction conditions are studied, this term will be re-instated.

The non-vanishing components of the Einstein tensor $G^M_N$ for the background (6) are listed below:

\begin{align}
G^0_0 &= G^1_1 = 6A'^2 + 3A'' - \frac{2e^{-2A}\partial_r m}{r^2}, \\
G^2_2 &= G^3_3 = 6A'^2 + 3A'' - \frac{e^{-2A}\partial_r^2 m}{r}, \\
G^4_4 &= 6A'^2 - \frac{e^{-2A}(2\partial_r m + r\partial_r^2 m)}{r^2},
\end{align}

(10)

where a prime ('') denotes the derivative with respect to the $y$-coordinate. We will also assume that the bulk scalar field depends only on the coordinate along the fifth dimension, i.e. $\Phi = \Phi(y)$. Then, the non-vanishing mixed components of the energy-momentum tensor $T^{(\Phi)}_{MN}$ take in turn the form

\begin{align}
T^{(\Phi)}_{00} &= T^{(\Phi)}_{11} = T^{(\Phi)}_{22} = A'\Phi' \partial_\Phi f + \mathcal{L}_\Phi - \Box f, \\
T^{(\Phi)}_{44} &= (1 + \partial^2 \Phi f) \Phi'^2 + \Phi'' \partial_\Phi f + \mathcal{L}_\Phi - \Box f,
\end{align}

(11)

where, under the aforementioned assumptions, the quantities $\mathcal{L}_\Phi$ and $\Box f$ have the explicit forms

\begin{align}
\mathcal{L}_\Phi &= -\frac{1}{2} \Phi'^2 - V_B(\Phi), \\
\Box f &= 4A'\Phi' \partial_\Phi f + \Phi'^2 \partial_\Phi^2 f + \Phi'' \partial_\Phi f.
\end{align}

(12)

(13)

The gravitational field equations may now easily follow by substituting the components of $G^M_N$ and $T^M_N$, listed in Eqs. (10) and (11), respectively, in Eq. (7) evaluated in the bulk. We thus obtain three equations from the $(0_0)$, $(2_2)$ and $(4_4)$ components. Subtracting the $(0_0)$ and $(2_2)$ equations as well as the $(0_0)$ and $(4_4)$ equations, we arrive at two simpler ones that, together with the $(0_0)$ component, form the following system

\begin{align}
r \partial_r^2 m - 2\partial_r m &= 0, \\
f \left(3A'' + e^{-2A}\frac{\partial_r^2 m}{r} \right) &= \partial_\Phi f (A'\Phi' - \Phi'') - (1 + \partial_\Phi^2 f) \Phi'^2, \\
f \left(6A'^2 + 3A'' - \frac{2e^{-2A}\partial_r m}{r^2} \right) &= A'\Phi' \partial_\Phi f + \mathcal{L}_\Phi - \Box f - \Lambda_5.
\end{align}

(14)

(15)

(16)

The above gravitational equations are supplemented by the scalar-field equation of motion (5), that has the explicit form

\begin{align}
\Phi'' + 4A'\Phi' &= \partial_\Phi f \left(10A'^2 + 4A'' - e^{-2A}\frac{2\partial_r m + r\partial_r^2 m}{r^2} \right) + \partial_\Phi V_B.
\end{align}

(17)
Equation (14) can be easily integrated to yield the general form of the allowed mass function, and this is:

\[ m(r) = M + \Lambda r^3/6, \]  

(18)

where \( M \) and \( \Lambda \) are arbitrary integration constants the physical interpretation of which will be studied later (the coefficient 6 has been introduced for later convenience). The above solution may now be used in order to simplify the form of the remaining three equations (15)-(17). However, not all of them are independent: as we explicitly demonstrate in Appendix B, an appropriate manipulation and rearrangement of the gravitational equations (15)-(16) leads to the same result following also from a similar manipulation of the scalar-field equation (17). Indeed, in a fully determined theory, i.e. with given \( f(\Phi) \) and \( V_B(\Phi) \), we would only need three independent equations out of the existing four to find the two unknown metric functions \( m(r) \) and \( A(y) \) and the scalar field \( \Phi(y) \). Therefore, henceforth, we will altogether ignore Eq. (17) in our analysis, and retain Eqs. (15) and (16). We will then adopt the following approach: we will assume the well-known form [4] \( A(y) = -k|y| \), with \( k \) a positive constant, for the warp factor of the 5-dimensional line-element in order to ensure the localisation of gravity near the brane; for a chosen coupling function \( f(\Phi) \), we will then determine the scalar-field configuration by solving Eq. (15); finally, Eq. (16) will determine the form of the potential \( V_B(\Phi) \) that needs to be introduced to support the solution.

In the following sections, we present two simple choices for the coupling function \( f(\Phi) \), a linear and a quadratic one; for each one, we determine the corresponding solution for the scalar field and form of the potential, and discuss their physical characteristics.

## 3 The Case of Linear Coupling Function

### 3.1 The Bulk Solution

We will first consider the case where the coupling function is of the general linear form, \( f(\Phi) = a\Phi + b \), where \( a \) and \( b \) are constants. Employing this together with the form of the mass function (18) and the exponentially-decreasing warp factor \( e^{A(y)} = e^{-ky} \) (assuming the usual \( \mathbb{Z}_2 \) symmetry in the bulk under the change \( y \to -y \), we henceforth focus on the positive \( y \)-regime), Eq. (15) takes the form

\[(a\Phi + b) \Lambda e^{2ky} = -a (k \Phi' + \Phi'') - \Phi'^2.\]  

(19)

In order to solve the above, we set: \( \Phi(y) = \Phi_0 e^{g(y)} \). Substituting in Eq. (19) and rearranging, we obtain

\[ a\Lambda \Phi_0 e^{2ky+g(y)} + b\Lambda e^{2ky} = -a(kg' + g'' + g'^2)\Phi_0 e^{g(y)} - g'^2\Phi_0^2 e^{2g(y)}, \]  

(20)

where a prime in \( g \) denotes, as before, the derivative with respect to \( y \). The above leads to a non-trivial solution only if \( g(y) = 2ky \). In that case, the following constraints should also hold

\[ a = -\frac{4k^2}{\Lambda} \Phi_0, \quad b = \frac{24k^4}{\Lambda^2} \Phi_0^2. \]  

(21)
The coefficient $b$ is clearly positive-definite however the sign of the coefficient $a$ depends on those of $\Phi_0$ and $\Lambda$.

Let us examine the type of solution that we have derived. Employing the form of the mass function (18) and the general expressions for the 5-dimensional curvature invariants given in Appendix A, the latter quantities are found to have the form:

\[
\begin{align*}
R &= -20k^2 + 4\Lambda e^{2ky} \\
R_{MN}R^{MN} &= 80k^4 - 32k^2\Lambda e^{2ky} + 4\Lambda^2 e^{4ky}, \\
R_{MNKL}R^{MNKL} &= 40k^4 - 16k^2\Lambda e^{2ky} + 8\Lambda^2 e^{4ky} + 48M^2 e^{4ky}.
\end{align*}
\]

For $M = \Lambda = 0$, we recover the curvature invariants of the 5-dimensional AdS spacetime. For $\Lambda = 0$ but $M \neq 0$, we obtain the black-string solution of [10], with the black-hole singularity at $r = 0$ extending over the entire fifth dimension up to the AdS boundary at $y \to \infty$. For $M = 0$ but $\Lambda \neq 0$, we find a solution that is everywhere regular apart from the AdS boundary. Finally, for $M \neq 0$ and $\Lambda \neq 0$, we obtain again a black-string solution with singular terms from both the black-hole and AdS boundary appearing in the expressions of the curvature invariants.

At this point, we should investigate the physical interpretation of the integration constants $M$ and $\Lambda$ appearing in the expression (18) of the mass function $m(r)$. To this end, we set $y = 0$ in the higher-dimensional line-element (6), and the projected-on-the-brane 4-dimensional background then reads

\[
ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).
\]

The above looks like a generalization of the Vaidya form of the Schwarzschild line-element in the presence of a cosmological constant. In order to convince ourselves, we apply a general coordinate transformation $v = h(t, r)$, where $h(t, r)$ will be defined shortly. Then, Eq. (23) assumes the standard, diagonal form

\[
ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)
\]

provided that $h(t, r) = t + g(r)$ and $g(r)$ satisfies the following condition

\[
\frac{dg(r)}{dr} = \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1}.
\]

The details of the transformation as well as the explicit form of the function $g(r)$, which is not of importance in the present analysis, can be found in the Appendix C. According to Eq. (24), the gravitational background on the brane is Schwarzschild-(Anti)-de Sitter with $M$ being the mass of the black hole and $\Lambda = \kappa_4^2\Lambda_4$, where $\Lambda_4$ is the cosmological constant on the brane.

It is of particular interest to study the profile of the non-minimal coupling function $f(\Phi)$ along the extra dimension: using the solution for the scalar field found above, we obtain

\[
f(y) = a\Phi(y) + b = \frac{4k^2\Phi_0^2}{\Lambda^2}\left(-\Lambda e^{2ky} + 6k^2\right).
\]
Figure 1: (a) The warp factor $e^{-k|y|}$ and scalar field $\Phi(y)$, and (b) the coupling function $f(y) = a\Phi(y) + b$, in terms of the coordinate $y$, for $k = 1$, $\Phi_0 = 0.01$, and $6k^2/\Lambda = 2/3, 50, 80, 95, 110, 125$ (from bottom to top).

For $\Lambda < 0$, i.e. for a negative cosmological constant on the brane, the above expression is everywhere positive-definite and gravity remains attractive over the whole bulk. However, for $\Lambda > 0$, we find that

$$f(y) > 0, \quad y < \frac{\ln(6k^2/\Lambda)}{2k}$$

$$f(y) = 0, \quad y = y_0 \equiv \frac{\ln(6k^2/\Lambda)}{2k}$$

$$f(y) < 0, \quad y > \frac{\ln(6k^2/\Lambda)}{2k}$$

That is, close to the brane and up to a maximum distance of $y = y_0$ the function $f(y)$ is positive and gravity acts as normal. However, at $y = y_0$, $f(y)$ vanishes, and gravity locally disappears, whereas, for $y > y_0$, $f(y)$ turns negative, and gravity acquires the “wrong sign”. We may therefore conclude that, for a positive cosmological constant on the brane, gravity becomes repulsive in the bulk at some finite distance away from the brane.

In Fig. 1a, we depict the form of the warp factor $e^{-k|y|}$ and the scalar field $\Phi(y)$ in terms of the coordinate $y$ along the fifth dimension, for $k = 1$ and $\Phi_0 = 0.01$. Although the former quantity exhibits the anticipated localisation close to our brane, the latter quantity increases away from the brane diverging at the boundary of spacetime. The displayed, qualitative behaviour of these two quantities is independent of the particular values of the parameters. In contrast to this, the profile of the coupling function $f(y)$, given in Eq. (26), depends strongly on the value of the dimensionless parameter $6k^2/\Lambda$: assuming that $\Lambda > 0$ on our brane, in Fig. 1b we display the form of $f(y)$, for $k = 1$, $\Phi_0 = 0.01$ and the values $6k^2/\Lambda = 2/3, 50, 80, 95, 110, 125$. For $6k^2/\Lambda < 1$, the function $f(y)$ does not have a vanishing point and is always negative; for $6k^2/\Lambda > 1$, a regime of positive values for $f(y)$ appears close to our brane that tends to become larger as $k^2/\Lambda$ gradually increases. In other words, the smaller the cosmological constant is on our brane, the farther away from our brane the anti-gravitating regime is located. It is
also interesting to note that the regime of positive values for the function \( f(y) \) around our brane is always characterized by a plateau, an area where the value of the coupling function is almost constant; therefore, close to our brane, gravity would not only act as normal but it would look as if the scalar curvature \( R \) does not have a coupling to the scalar field. In fact, for the particular value of \( 6k^2/\Lambda = 125 \), the coupling function \( f(\phi) \) around the brane is constant and approximately equal to unity, thus the model mimics ordinary, 5-dimensional gravity - with the difference that the bulk energy, that as we will see supports the complete bulk-brane solution, originates in fact from the scalar field.

In order to complete the analysis, we need to determine the potential of the scalar field \( V_B(\Phi) \). Substituting the forms of the functions \( m(r) \), \( A(y) \) and \( \Phi(y) \) in Eq. (16), we readily obtain

\[
V_B(\Phi) = -\Lambda_5 - 2k^2 \left( \frac{72k^4\Phi_0^2}{\Lambda^2} \right) - \frac{20k^2\Phi_0}{\Lambda} \Phi + 3\Phi^2.
\]  

(28)

Combining the above expression with the profile of the scalar field along the extra dimension, \( \Phi(y) = \Phi_0 e^{2ky} \), we notice the following: at the location of the brane, at \( y = 0 \), the scalar potential reduces to a constant value, namely

\[
V_B(y = 0) = -\Lambda_5 - 2k^2\Phi_0^2 \left( \frac{72k^4}{\Lambda^2} - \frac{20k^2}{\Lambda} + 3 \right).
\]  

(29)

The quantity inside the brackets has no real roots and is thus always positive-definite; that makes the second term a negative-definite quantity for all values of the parameters of the model. This means that, close to the brane, the scalar potential can mimic the role of the negative cosmological constant – thus making \( \Lambda_5 \) redundant – and support by itself an AdS spacetime in the bulk regime close to the brane.

In Fig. 2a, we depict the form of the scalar potential found above, for the choice of parameters \( 6k^2/\Lambda = 100 \), \( \Phi_0 = 0.01 \), \( k = 1 \) and for \( \Lambda_5 = 0 \). The regime close to our brane where \( V_B \) mimicks the negative cosmological constant is clearly present. As we move away from the brane, the scalar field starts increasing: this leads first to the formation of a small barrier (i.e. a local extremum), as a result of the competing roles of the linear and quadratic in \( \Phi \) terms in Eq. (28), and eventually to the divergence of \( V \) towards minus infinity at the boundary of spacetime. In the same plot, we depict the form of the coupling function \( f(y) \), for the same parameter values: this ensures us of the fact that the regime of the mimicking of “negative cosmological constant” and the location of the barrier lies well inside the normal gravitating regime. At the point where \( f(y) \) vanishes and gravity disappears, \( V(y) \) retains a moderate, finite value; allowing, however, to enter the anti-gravitating regime leads to arbitrary large negative values of the scalar potential.

The components of the energy-momentum tensor of the theory may be computed employing Eqs. (7) and (11). Using also the relations \( \rho = -T_0^0 \), \( p^i = T_i^i \) and \( p^y = T_y^y \), we find the explicit expressions:

\[
\rho = -p^i = \Lambda_5 - 2ak^2\Phi + 2k^2\Phi^2 + V_B(\Phi) = -4k^2\Phi_0^2 \left( \frac{6k^2}{\Lambda} - e^{2ky} \right)^2,
\]  

(30)
Figure 2: (a) The scalar potential $V_B$ and energy density $\rho$ of the system, and (b) the pressure components $p^y$ and $p^i$ in terms of the coordinate $y$ (from bottom to top in both plots), for $6k^2/\Lambda = 100$, $\Phi_0 = 0.01$, $k = 1$ and $\Lambda_5 = 0$. We also display the coupling function $f$ with its characteristic plateau, for easy comparison.

\begin{equation}
\rho^y = -\Lambda_5 + 8\alpha k^2 \Phi + 2k^2 \Phi^2 - V_B(\Phi) = 8k^2 \Phi_0^2 \left( \frac{18k^4}{\Lambda^2} - \frac{9k^2}{\Lambda} e^{2ky} + e^{4ky} \right) . \tag{31}
\end{equation}

The behaviour of the above quantities is also depicted in Fig. 2, for the same values of parameters as in Fig. 1 to allow for an easy comparison. As expected, close to the brane the profile of all components resembles that of a negative cosmological constant. At an intermediate distance, the energy density $\rho$ reaches a local, maximum value, and, far away from the brane – inside the anti-gravitating regime – it diverges to negative infinity. The pressure components $p^y$ and $p^i$ exhibit the exact opposite behaviour: starting from a constant value near the brane, they dive towards a local minimum and, inside the anti-gravitating regime, diverge to positive infinity. We readily observe that the total energy density $\rho$ of the system remains negative throughout the bulk – this is also obvious from its expression in Eq. (30); however, this is due to a physical, scalar field with a potential that turns out to be negative in order to create the local AdS spacetime and support the decreasing warp factor. Close to the brane, that potential is analytic and finite – should one wish to ban the diverging, anti-gravitating regime from the bulk spacetime, a second brane could easily be introduced at a distance $y = L < y_0$. The necessity of introducing a second brane in the model will be discussed shortly.

### 3.2 Junction Conditions and Effective Theory

Let us now address the issue of the junction conditions introduced in the model due to the presence of the brane at $y = 0$. The energy content of the brane is given by the combination $\sigma + V_b(\Phi)$, and it creates a discontinuity in the second derivatives of the warp factor and scalar field at the location of the brane. We write $A'' = \hat{A}'' + [A'] \delta(y)$ and $\Phi'' = \hat{\Phi}'' + [\hat{\Phi}'] \delta(y)$, where the hat quantities denote the distributional (i.e. regular)
parts of the second derivatives and [...] gives the discontinuities of the corresponding first
derivatives across the brane [64]. Then, if we re-introduce the delta-function terms both
in the Einstein equation (15) and scalar-field equation (17), and match the coefficients
of the delta-function terms, we obtain the conditions

\[ 3f(\Phi) [A'] = -[\Phi'] \partial_\Phi f - (\sigma + V_b), \quad (32) \]
\[ [\Phi'] = 4[A'] \partial_\Phi f + \partial_\Phi V_b, \quad (33) \]

where all quantities are evaluated at \( y = 0 \). Using the expressions for the warp factor
and the scalar field, as well as the symmetry in the bulk under the change \( y \to -y \), we
find their explicit forms

\[ \frac{8k^2}{\Lambda} k \Phi_0^2 \left( 1 - \frac{18k^2}{\Lambda} \right) = -\sigma - V_b(\Phi_0), \quad (34) \]
\[ 4k \Phi_0 \left( 1 - \frac{8k^2}{\Lambda} \right) = \partial_\Phi V_b \bigg|_{y=0}. \quad (35) \]

According to the second junction condition, in the absence of an interaction term \( V_b \) of
the scalar field with the brane, we should have \( k^2 = \Lambda/8 \). This result determines the sign
of the 4-dimensional cosmological constant, that must necessarily be positive, and relates
its magnitude to the scale of warping in the bulk. Moreover, the dimensionless quantity
\( k^2/\Lambda \), that determines the range of the gravitating regime, should be exactly \( 1/8 \): this
value, being smaller than \( 1/6 \), does not allow for a normal gravity regime anywhere
in the bulk, according to the discussion above. The first of the conditions also leads to the
result: \( \Phi_0^2 = 4\sigma/5k \); for the case \( k > 0 \), that ensures the decrease of the warp factor away
from our brane, the brane self-energy \( \sigma \) comes out to be positive, too and thus physically
acceptable.

As we showed above, the existence of a normal-gravity regime close to our brane
demands the presence of an interaction term \( V_b \) of the scalar field with the brane. Al-
though the number of choices for \( V_b \) is in this case infinite, one may draw some general
conclusions: if we assume again that \( k > 0 \) and that \( k^2/\Lambda > 1/6 \), so that a positive
\( f(\Phi) \)-regime exists around our brane, then Eq. (34) still ensures that the total energy
content of our brane \( \sigma + V_b(\Phi_0) \) is always positive. Assuming now, as an indicative case,
a linear form for the interaction term, too, i.e. \( V_b(\Phi) = \lambda_0 \Phi \), where \( \lambda_0 \) is a coupling
constant, we obtain the conditions

\[ \frac{8k^2}{\Lambda} k \Phi_0^2 \left( 1 - \frac{18k^2}{\Lambda} \right) = -\sigma - \lambda_0 \Phi_0, \quad 4k \Phi_0 \left( 1 - \frac{8k^2}{\Lambda} \right) = \lambda_0. \quad (36) \]

The above two conditions determine two out of the five parameters of the model:
\( (k, \Lambda, \Phi_0, \lambda_0, \sigma) \). Considering the bulk scalar field and the self-energy of the brane as
the constituents of the model that support the complete bulk-brane solution, the parameters
related to them, namely the value of the field on the brane \( \Phi_0 \), its coupling constant with the brane \( \lambda_0 \) and the brane self-energy \( \sigma \), may be naturally chosen as the
true independent quantities of the theory. On the other hand, the scale of the warping
\( k \) and the effective cosmological constant on the brane \( \Lambda \) are determined through the
junction conditions by the aforementioned three fundamental parameters. In this case, one may easily see that, for \( \lambda_0 \Phi_0 > 0 \), we obtain \( k^2/\Lambda < 1/8 \), which allows for a bulk that is everywhere anti-gravitating, while, for \( \lambda_0 \Phi_0 < 0 \), solutions with large values of \( k^2/\Lambda \) may be obtained that have their anti-gravitating regime pushed away from our brane.

We should finally address the issue of the effective theory on the brane. The negative sign of the coupling function \( f(\Phi) \) emerging at some distance from the brane as well as the diverging behaviour of the field \( \Phi \) in the same region raise concerns about the type of the effective theory that a 4-dimensional observer would witness. In order to answer this question, we need to derive the 4-dimensional effective action by integrating the 5-dimensional one, given in Eq. (1), over the fifth coordinate \( y \). Employing the first of Eqs. (22), we write:

\[
R = -20 k^2 + R^{(4)}e^{2ky},
\]

where \( R^{(4)} = 4\Lambda \) is the scalar curvature of the projected-on-the-brane line-element (23). Then, the action takes the form

\[
S = \int d^4x \, dy \, \sqrt{-g^{(5)}} \left[ \frac{f(\Phi)}{2} \left( e^{2ky} R^{(4)} - 20k^2 \right) - \Lambda_5 - \frac{1}{2} \Phi'^2 - V_B(\Phi) \right].
\] (37)

Using also that \( \sqrt{-g^{(5)}} = e^{-4k|y|/\sqrt{-g^{(4)}}} \), the 4-dimensional, effective gravitational constant would be given by the integral

\[
\frac{1}{\kappa^2_4} \equiv 2 \int_0^\infty dy \, e^{-2ky} f(\Phi) = \frac{8k^2 \Phi_0^2}{\Lambda^2} \int_0^\infty dy \left( -\Lambda + 6k^2 e^{-2ky} \right).
\] (38)

In the above, we have substituted the form of the coupling function \( f(\Phi) \) given in Eq. (26). We observe that, although the second term inside the brackets will lead to a finite result even for a non-compact fifth dimension – similarly to the Randall-Sundrum model, the first term will give a divergent contribution. As a result, the presence of a second brane at a distance \( y = L \) is imperative for a well-defined effective theory. In that case, the upper limit of the \( y \)-integral in Eq. (38) is replaced by \( L \), and we obtain

\[
\frac{M_{Pl}^2}{8\pi} = \frac{\Phi_0^2}{k} \frac{8k^2}{\Lambda} \left[ \frac{3k^2}{\Lambda} \left( 1 - e^{-2kL} \right) - kL \right].
\] (39)

Compared to the Randall-Sundrum model [4], the expression for the 4-dimensional gravity scale \( M_{Pl}^2 \) involves the quantity \( \Phi_0^2 \) – with units \( [M]^3 \) – and the dimensionless parameter \( k^2/\Lambda \) on its right-hand side. This signifies the fact that, in the context of the theory (1), the 5-dimensional gravity scale \( M_5^2 \) may be altogether replaced by the coupling function \( f(\Phi) \). If one chooses large values for the \( k^2/\Lambda \) parameter, then the value of the effective Planck scale may differ from that of \( \Phi_0 \) by orders of magnitude. In fact, the smaller the cosmological constant is on our brane, the more extended is the positive-value regime for \( f(\Phi) \), as we saw in the previous subsection, and the larger the difference between \( M_{Pl}^2 \) and \( \Phi_0^2 \). Equation (39) contains also a term linear in the inter-brane distance \( L \), which was absent in the Randall-Sundrum case. Therefore, one should take care that the inequality \( kL < 3k^2/\Lambda \) is always satisfied – however, for small values of the 4-dimensional cosmological constant on the brane, as argued above, this constraint should be easily satisfied.
The introduction of the second brane in order to ensure a finite effective theory on our brane is supplemented by a second set of junction conditions at the location \( y = L \). A brane source-term of the form 
\[-[\hat{\sigma} + \hat{V}_b(\Phi)] \delta(y - L)\] 
should be introduced in the action, where \( \hat{\sigma} \) and \( \hat{V}_b(\Phi) \) are the self-energy of the second brane and the interaction term of the scalar field with that brane, respectively. We follow a similar procedure as at \( y = 0 \), and arrive at a set of junction conditions similar to those in Eq. (33) but with all quantities evaluated at \( y = L \). Their explicit form reads

\[
8k^2 \left( \frac{18k^2 \Lambda}{\Lambda} - e^{2kL} \right) \Phi^2_0 = -\hat{\sigma} - \hat{V}_b(\Phi) \big|_{y=L}, \tag{40}
\]

\[
4k \Phi_0 \left( \frac{8k^2 \Lambda}{\Lambda} - e^{2kL} \right) = \partial_\Phi \hat{V}_b \big|_{y=L}. \tag{41}
\]

The above set of conditions may be used to determine two more parameters of the model: one may be the inter-brane distance \( L \) and the other a parameter associated with the interaction term \( \hat{V}_b \). The self-energy of the second brane \( \hat{\sigma} \) as well as the functional form of \( \hat{V}_b(\Phi) \) remain completely arbitrary.

To complete the derivation of the effective theory on the brane, we finally compute the effective cosmological constant - this may be used as a consistency check of our results. The cosmological constant on the brane is given by the integral of the remaining terms in Eq. (37) - since \( \Phi \) is only \( y \)-dependent, no dynamical field will survive in the effective theory. These terms will be supplemented by the source terms of the two branes as well as the Gibbons-Hawking terms at the boundaries of spacetime [65]. In total, we will have:

\[
-\Lambda_4 = \int_{-L}^{L} dy \ e^{-4k|y|} \left[ -10k^2 f(\Phi) - \Lambda_5 - \frac{1}{2} \Phi^2 - V_B(\Phi) + f(\Phi)(-4A'') \big|_{y=0} \right. \\
+ f(\Phi)(-4A'') \big|_{y=L} - [\sigma + \hat{V}_b(\Phi)] \delta(y) - [\hat{\sigma} + \hat{V}_b(\Phi)] \delta(y - L) \right]. \tag{42}
\]

Employing the expressions for the coupling function and scalar potential, Eqs. (26) and (28), respectively, as well as the junction conditions (34) and (40), and integrating over \( y \), we finally obtain the result

\[
\Lambda_4 = 8k^2 \Phi_0^2 \left[ \frac{3k}{\Lambda} (1 - e^{-2kL}) - L \right] = \frac{\Lambda}{\kappa_4^2}, \tag{43}
\]

where we have used the expression for the effective gravitational scale \( M_{Pl}^2/8\pi = 1/\kappa_4^2 \) given in Eq. (39). As expected, the derivation of the effective theory has confirmed the interpretation of the metric parameter \( \Lambda \) as the product \( \kappa_4^2 \Lambda_4 \), that followed also by comparing the projected-on-the-brane line-element (23) with the standard Schwarzschild-de Sitter background.

We would like finally to note that the presence of the mass parameter \( M \) has played no role either in the profile of the scalar field and the energy-momentum tensor components or in the derivation of the junctions conditions and the effective theory on the brane. Its presence creates a Schwarzschild-de Sitter background on the brane and an extended
singularity into the bulk leading to a 5-dimensional black-string stretching between the two branes. If we set this parameter equal to zero, then the 4-dimensional background on the brane reduces to a pure de Sitter spacetime while the 5-dimensional background is free of singularities as long as \( L < \infty \). For \( L > y_0 \), the bulk will also contain an anti-gravitating regime (unavoidable for \( \Lambda_4 > 0 \), as we will see in the next section).

4 The Quadratic Case

We now move to the case where the coupling function has a quadratic form, i.e. \( f(\Phi) = a \Phi^2 \), where \( a \) is again a constant. Employing, as in the previous subsection, the form of the mass function (18) and the warp factor \( e^{A(y)} = e^{-ky} \), Eq. (15) now takes the form

\[
a \Lambda e^{2ky} \Phi^2 = -2a \Phi (k \Phi' + \Phi'') - (1 + 2a)\Phi^2.
\]

Again, we set: \( \Phi(y) = \Phi_0 e^{g(y)} \), and the above equation is rewritten as

\[
a \Lambda e^{2ky} = -2a (kg' + g'') - (1 + 4a)g^2.
\]

The above calls for an exponential dependence for the function \( g(y) \) - we thus set \( g(y) = g_0 e^{\lambda y} \), with \( g_0 \) and \( \lambda \) constant coefficients, and write the above equation as

\[
a \Lambda e^{2ky} = -2ag_0 \lambda (k + \lambda) e^{\lambda y} - (1 + 4a)g_0^2 \lambda^2 e^{2\lambda y}.
\]

There is again only one non-trivial solution that satisfies the aforementioned equation, and this corresponds to the choice \( \lambda = 2k \). Then, the following constraints should hold

\[
a = -\frac{1}{4}, \quad g_0 = -\frac{\Lambda}{12k^2}.
\]

The coefficient \( a \) is negative-definite, therefore in this case gravity acts as a repulsive force over the entire bulk. We should note here that an attempt to generalise the form of the coupling function according to the ansatz \( f(\Phi) = a \Phi^2 + b \Phi + c \), where \( (a, b, c) \) are constant coefficients, failed to lead to a consistent solution. Had such a solution been possible, we could perhaps find regimes in the \( y \)-coordinate where gravity would act as normal, hopefully close to our brane. Unfortunately such a solution has not emerged, and therefore for a quadratic coupling function, the theory always lead to an anti-gravitating bulk. This feature is strongly connected to the presence of the cosmological constant on the brane – we will return to this point in the following section.

Let us, however, investigate the remaining aspects of the model. The warp factor assumes the standard Randall-Sundrum form, i.e. \( e^{A(y)} = e^{-ky} \), and is displayed in Fig. 3a. The profile of the scalar field depends on the sign of the parameter \( \Lambda \): using the second of the constraints (47), we find that

\[
\Phi(y) = \Phi_0 \exp \left( -\frac{\Lambda}{12k^2} e^{2ky} \right).
\]

The projected-on-the-brane line-element is still given by Eq. (23), thus, \( \Lambda \) is again proportional to the brane cosmological constant. Then, Eq. (48) tells us that, for a positive cosmological constant on the brane, the scalar field takes its maximum value \( \Phi = \Phi_0 e^{-\Lambda/12k^2} \).
Figure 3: (a) The warp factor $e^{-k|y|}$ and scalar field $\Phi(y)$, and (b) the coupling function $f(y) = a\Phi^2(y)$, in terms of the coordinate $y$, for $k = 1$, $\Phi_0 = 1$, and $6k^2/\Lambda = 3/2, 3, 10, 20, 100, 500$ (from top to bottom).

at the location of our brane, and decreases fast as we move away from the brane. Therefore, the scalar field exhibits a localisation around our brane similar to that of the warp factor; in particular, for small values of the parameter $k^2/\Lambda$, the profile of the scalar field exhibits a cusp at the location of the brane ($y = 0$) while, as $k^2/\Lambda$ increases, a plateau appears around the brane. The coupling function, $f(\Phi) = a\Phi^2$, assumes a similar profile by decreasing very fast, as $y$ increases; as a result, the anti-gravitating regime associated with $f(y)$ is rather small. The profiles of the scalar field and coupling function, for $\Lambda > 0$, are depicted in Figs. 3a and 3b, respectively. On the other hand, for a negative cosmological constant on the brane, the scalar field increases very fast away from the brane blowing-up at the boundary of the spacetime, and the same behaviour is exhibited by the coupling function $f(\Phi)$. In what follows, we ignore this unattractive solution and explore further the more interesting one with a positive cosmological constant on the brane.

We also need to derive the form of the potential $V_B(\Phi)$ of the scalar field in the bulk. This follows easily from Eq. (16) leading to the expression

$$V_B(\Phi) = -\Lambda_5 + k^2\Phi^2 \left[ \frac{3}{2} + 2 \ln \left( \frac{\Phi}{\Phi_0} \right) + 2 \ln^2 \left( \frac{\Phi}{\Phi_0} \right) \right], \quad (49)$$

or, in terms of the $y$-coordinate

$$V_B(y) = -\Lambda_5 + k^2 \left( \frac{3}{2} - \frac{\Lambda}{6k^2} e^{2ky} + \frac{\Lambda^2}{72k^4} e^{4ky} \right) \Phi_0^2 \exp \left( -\frac{\Lambda}{6k^2} e^{2ky} \right). \quad (50)$$

The bulk potential in principle consists of the negative cosmological-constant term and a term that is related to the scalar field. For $\Lambda > 0$, this second term decreases very fast exhibiting also a localisation around our brane – its profile is shown in Fig. 4a. Setting $z = \Lambda e^{2ky}/6k^2$, one may easily see that the 2nd-order polynomial inside the brackets has no real roots, and is thus always positive-definite. As a result, the second
The scalar potential $V_B$ and energy density $\rho$ of the system, and (b) the pressure components $p^y$ and $p^i$ in terms of the coordinate $y$ (from top to bottom in both plots), for $6k^2/\Lambda = 100$, $\Phi_0 = 1$, $k = 1$ and $\Lambda_5 = 0$. We also display the coupling function $f$, for comparison.

The components of the energy-momentum tensor may be also easily derived from Eqs. (7) and (11) using the solution for the scalar field and scalar potential. They have the form:

\[
\rho = -p^i = \Lambda_5 + 2a \Phi (3A' \Phi' + \Phi''') + V_B(\Phi) = \frac{3}{2} k^2 \Phi^2 \left(1 - \frac{\Lambda}{6k^2} e^{2k y}\right), \quad (51)
\]

\[
p^y = -\Lambda_5 + \frac{1}{2} \Phi'^2 - 8a \Phi A' \Phi' - V_B(\Phi) = -\frac{3}{2} k^2 \Phi^2 \left(1 - \frac{\Lambda}{3k^2} e^{2k y}\right). \quad (52)
\]

The form of the energy-momentum components are depicted in Figs. 4a and 4b. The energy density matches the value of the potential at the location of the brane and decreases slightly faster than the latter away from the brane; it remains predominantly positive apart from a small regime at large distances from the brane. The pressure components exhibit the exact opposite behaviour regarding their sign. Overall, the energy-momentum components resemble those of a positive cosmological constant close to our brane, then decrease fast and finally vanish at large distances exhibiting a nice localisation pattern. We should stress that, according to our analysis, no negative, bulk cosmological constant needs to be introduced by hand. It is in fact the negative value of the coupling function $f(\Phi)$ that turns the coupling term between the scalar field and the Ricci scalar to a form of a negative distribution of energy; it is this term then that manages to support the exponentially falling warp factor even in the absence of a typical AdS spacetime.

The presence of the brane, with its non-trivial energy content, introduces once again discontinuities in the derivatives of the warp factor and scalar field. The associated
junction conditions at $y = 0$ have the same form as in Eqs. (33). Their explicit forms, however, are bound to be different and are given by

$$\frac{k\Phi_0^2}{2} e^{-\Lambda/6k^2} \left(3 + \frac{\Lambda}{3k^2}\right) = -\sigma - V_b\big|_{y=0}, \quad (53)$$

$$2k\Phi_0 e^{-\Lambda/12k^2} \left(2 + \frac{\Lambda}{6k^2}\right) = -\partial_\Phi V_b\big|_{y=0}. \quad (54)$$

Since the left-hand-sides of the above equations are positive-definite, the interaction term $V_b$ of the scalar field with the brane must be, not only non-vanishing, but necessarily negative (with a negative first derivative, too) in order to avoid a negative brane self-energy $\sigma$. As before, the above conditions may fix the parameters $k$ and $\Lambda$ while the scalar-field parameters $\Phi_0$ and $V_b$, as well as $\sigma$, may remain arbitrary.

We should, however, stress that this particular solution, being either a black string or regular in the bulk, cannot constitute a realistic model due to the negative sign of the coupling function $f(\Phi)$. This sign will be carried over to the 4-dimensional effective theory leading to anti-gravity on the brane. Indeed, working as in the previous subsection and isolating the coefficient of $R^{(4)}$ in the action, we arrive at the relation

$$\frac{1}{\kappa_4^2} = -\frac{\Phi_0^2}{2} \int_0^{\infty} dy \, e^{-2k\nu} \exp\left(-\frac{\Lambda}{6k^2} e^{2k\nu}\right) = -\frac{\Phi_0^2}{4k} \left(e^{-\Lambda/6k^2} - \frac{\Lambda}{3k} I\right), \quad (55)$$

where

$$I \equiv \int_0^{\infty} dy \, \exp\left(-\frac{\Lambda}{12k^2} e^{2k\nu}\right). \quad (56)$$

The integral $I$ may be computed numerically and yields a finite result; therefore, there is no need for the introduction of a second brane in this model. Nevertheless, the value of the effective gravitational constant $\kappa_4^2$ turns out to be negative – this becomes clear if one looks at the middle part of Eq. (55), where a negative coefficient multiplies a positive-definite integral. This result is catastrophic, therefore, the model is not a viable one. Its emergence, however, reveals two facts: (i) that anti-gravitating solutions in the context of the theory (1) are somehow associated to the positive cosmological constant on the brane since two such solutions have emerged for two different choices of the coupling function, and (ii) that, when $M \neq 0$, the theory of a non-minimally-coupled scalar field to gravity gives rise to yet another undesired black-string solution rather than a physically motivated, and long sought-for, localised black-hole solution.

5 A Theoretical Argument

In the previous section, we have constructed explicit solutions that emerge from the 5-dimensional field equations, and describe a 4-dimensional Schwarzschild-de Sitter
background on the brane. From the bulk point of view, these solutions describe either black strings, if $M \neq 0$, or regular, maximally-symmetric solutions over the whole bulk apart from its boundary at $y \to \infty$, if $M = 0$ – a second brane could easily shield the boundary singularity creating two-brane models with a compact fifth dimension. In both cases, however, the bulk solution is characterized, either globally or over particular regimes, by a negative coupling function $f(\Phi)$ that leads to an anti-gravitating theory. In this section, we examine from the mathematical point-of-view why the emergence of such solutions is possible in the context of the given theory, and why they do so particularly for the physically-motivated case of a positive cosmological constant on the brane.

The analysis will focus on the gravitational equation (15). By employing the relations
\[
\partial_y f = \Phi' \partial_\Phi f , \quad \partial_y^2 f = \Phi'^2 \partial_\Phi^2 f + \Phi'' \partial_\Phi f ,
\]
as well as the expressions $A(y) = -k|y|$ and $m(r) = M + \Lambda r^3/6$, Eq. (15) is written as
\[
\Lambda e^{2k|y|} f = -\Phi'^2 + \partial_y^2 f - k \text{sgn}(y) \partial_y f .
\]
We assume once again the existence of the $Z_2$-symmetry in the bulk, and restrict our analysis to the positive $y$-regime for simplicity. Consequently, we write:
\[
\Phi'^2 = -\partial_y^2 f - k \partial_y f - \Lambda e^{2ky} f .
\]
The first derivative of the scalar field $\Phi'$ may vanish at particular values of the coordinate $y$, but is assumed to be in general non-vanishing to allow for a non-trivial scalar field in the bulk. Also, both functions $f = f(y)$ and $\Phi = \Phi(y)$ should be real in their whole domain. Therefore, both sides of Eq. (59) should be non-negative, which finally leads to the constraint
\[
\partial_y^2 f + k \partial_y f + \Lambda e^{2ky} f \leq 0 .
\]
The above constraint should be satisfied for every solution of the field equations (14)-(16), including the ones presented in sections 3.1 and 3.2. These were characterized by $\Lambda > 0$, in which case the combination $\Lambda e^{2ky}$, appearing in the last term of the above expression, diverges to $+\infty$ at the boundary of spacetime. But there, the coupling function $f(y)$ is negative for both solutions, and this renders the last, dominant term smaller than zero as the constraint demands. Also, for all other values of $y$, one may easily check that the profiles of the function $f(y)$ found in sections 3 and 4 always satisfy the constraint (60).

In what follows, we would like to investigate whether physically-acceptable solutions with $f(y) > 0$ may arise in the case where $\Lambda$ is also positive. To this, we will add the demand that the components of the energy-momentum tensor may be localised close to the brane, and certainly non-diverging at the boundary of spacetime. These may be written as
\[
\rho = -p^i = \frac{1}{2} \Phi'^2 + V_B(\Phi) + \Lambda_5 + 3A' \partial_y f + \partial_y^2 f ,
\]
\[
p^y = \frac{1}{2} \Phi'^2 - V_B(\Phi) - \Lambda_5 - 4A' \partial_y f .
\]
From Eq. (16), one may solve for the general form of the scalar potential to find
\[
V_B(\Phi) = -\Lambda_5 - \frac{1}{2} \Phi'^2 - 3A' \partial_y f - \partial_y^2 f - f (6k^2 - \Lambda e^{2ky}) .
\]
Employing the above into the expressions (61)-(62), together with Eq. (59), the energy-momentum tensor components simplify to

\[ \rho = -p^i = -6k^2 f(\Phi) + f(\Phi) \Lambda e^{2ky}, \]  
\[ p^y = 6k^2 f(\Phi) - 2f(\Phi) \Lambda e^{2ky}. \]  

We observe that all components contain the diverging combination \( \Lambda e^{2ky} \). Therefore, we should demand the vanishing of the coupling function \( f(\Phi) \) at the boundary of spacetime at least as fast as \( e^{-2ky} \).

We will consider the most general such form, namely \( f(y) = Ae^{-\sum_{n=1}^{N} b_n y^n} \), where \( A \) and \( b_n \) are arbitrary constants, and \( N \) a positive integer. The first and second derivative of \( f(y) \) are found to be:

\[ \partial_y f = -Ae^{-\sum_{n=1}^{N} b_n y^n} \left( \sum_{n=1}^{N} b_n ny^{n-1} \right), \]
\[ \partial_y^2 f = Ae^{-\sum_{n=1}^{N} b_n y^n} \left[ \left( \sum_{n=1}^{N} b_n ny^{n-1} \right)^2 - \sum_{n=1}^{N} b_n n(n-1)y^{n-2} \right]. \]

Both quantities quickly tend to zero which ensures the finiteness of the scalar potential (63). Then, the inequality constraint of Eq. (60) reads

\[ f(\Phi) \left[ \left( \sum_{n=1}^{N} b_n ny^{n-1} \right)^2 - \sum_{n=1}^{N} b_n n(n-1)y^{n-2} - k \sum_{n=1}^{N} b_n ny^{n-1} + \Lambda e^{2ky} \right] \leq 0. \]  

Since \( f(\Phi) \) is demanded to be everywhere positive, it is the expression inside the square brackets that needs to be negative-definite. For \( N = 1 \), the latter reduces to \( b_1(b_1 - k) + \Lambda e^{2ky} \); but this, for \( \Lambda > 0 \), is always positive-definite since \( b_1 \geq 2k \) according to the argument below Eqs. (64)-(65). For \( N > 1 \), as \( y \) increases away from the brane, the first and last term are clearly the dominant ones in Eq. (67); but these are again positive-definite. Therefore, in all cases the constraint (60) is violated either over the entire \( y \)-regime (as in the case studied in section 4) or at a distance from the brane (as in the case studied in Section 3).

Conclusively, we have demonstrated that, for a function \( f(\Phi) \) positive and decreasing at large distances from our brane – assumptions that guarantee the correct sign of the gravitational force and the localisation of the energy-momentum tensor in the bulk – no viable solutions arise in the context of the theory (1) when \( \Lambda > 0 \) on our brane. On the other hand, for \( \Lambda \) either zero or negative, solutions with \( f > 0 \) are much easier to arise.\(^6\)  

E.g. for \( \Lambda = -|\Lambda| < 0 \), Eq. (60) is now written as

\[ \partial_y^2 f + k\partial_y f - f|\Lambda| e^{2ky} \leq 0. \]

\(^6\)In fact, a static brane-world solution with \( M = 0 \) and \( \Lambda = 0 \) on our brane was presented in [61] where a quadratic coupling function \( f(\Phi) = 1 - \xi \Phi^2 \) between the scalar field and the Ricci scalar was considered.
One may readily see that this constraint is much easier to satisfy: for $f(\Phi)$ positive and decreasing, the second and third term are already negative-definite. For instance, the choice considered above for $f(\Phi)$, namely $f(\phi) = Ae^{-\sum_{n=1}^{N} b_n \phi^n}$, satisfies the constraint (68) over the entire $\phi$-regime for appropriate choices of the parameters. A detailed analysis on the emergence of legitimate solutions in the context of the theory (1) with a Minkowski or Anti de Sitter background on our brane will be given elsewhere [66].

6 Conclusions

Motivated by the results of previous works [38, 39], where despite intensive efforts regular, localised-on-the-brane black-hole solutions were not found in the context of a theory with a scalar field non-minimally-coupled to gravity, in this work we have focused on the derivation and study of the properties of black-string solutions that, in contrast, seem to emerge quite naturally in the context of the same theory. To this end, we have retained the ‘Vaidya form’ of the spacetime line-element, that on the brane leads to a Schwarzschild black hole while in the bulk produces solutions with the minimum number of spacetime singularities. We have in addition allowed for an arbitrary mass function $m(r)$ in an effort to accommodate, if possible, solutions with a more general profile including an (Anti)-de Sitter or Reissner-Nordstrom type of background.

The integration of an appropriate re-arrangement of the equations of motion has allowed us to uniquely determine the form of the mass function, namely $m(r) = M + \Lambda r^3/6$. Performing an inverse coordinate transformation on the brane, we readily identified the parameter $M$ with the black-hole mass and the parameter $\Lambda$ as the product $\kappa^2 4 \Lambda_4$, where $\Lambda_4$ is the four-dimensional cosmological constant on the brane. As a result, the brane background assumes the form of a Schwarzschild-(Anti)-de Sitter spacetime. As the expressions of the five-dimensional curvature invariants reveal, these solutions may have a dual description from the bulk point of view: they may describe either black strings, if $M \neq 0$, or brane-world maximally-symmetric solutions, if $M = 0$.

The properties of these five-dimensional solutions strongly depend on the form of the non-minimal coupling function $f(\Phi)$ between the scalar field and the five-dimensional scalar curvature. We have considered two simple choices for $f(\Phi)$, a linear and a quadratic one in terms of the scalar field. For a linear coupling function, the scalar field is found to increase exponentially away from the brane, and to drive the coupling function to negative values at a distance from the brane. When $6k^2/\Lambda > 1$, there is always a positive-value regime for $f(\Phi)$ close to our brane while the anti-gravitating regime, with $f(\Phi) < 0$, is pushed away from our brane as the value of $6k^2/\Lambda$ gradually increases. For fairly large values of $6k^2/\Lambda$, i.e. for a large warping factor $k$ or a small cosmological constant on our brane, the profile of the coupling function exhibits a wide plateau around our brane. When $6k^2/\Lambda \simeq 125$, this plateau is centered around the value of unity, and, therefore, the theory mimics a five-dimensional scalar-tensor theory with a minimally-coupled scalar field and normal gravity around our brane – the anti-gravitating regime is however still lurking at the boundaries of the extra dimension. The latter may be cut short or altogether removed from the theory by adding a second brane; this is also necessary in order to obtain a finite four-dimensional gravitational scale, as we have
explicitly demonstrated. The anti-gravitating regime is also characterized by a diverging scalar field that results in the divergence of the energy-momentum tensor components, too. However, after the introduction of the second brane at a finite distance from the first, all energy-momentum tensor components are well-behaved. In fact, the energy density takes on an almost constant, negative value around our brane, thus mimicking a bulk cosmological constant (which, in this case, is redundant) and supporting a Randall-Sundrum warp factor.

For a quadratic coupling function $f(\Phi)$, the scalar field is found to be everywhere finite and, in fact, to exhibit a localisation around our brane - the same behaviour is exhibited by all the energy-momentum tensor components. The four-dimensional gravitational scale comes out to be finite, therefore, in this case there is no reason to introduce a second brane. The warp factor takes a form identical to the one in the Randall-Sundrum model even in the absence of a negative, bulk cosmological constant and for positive values of the energy-momentum tensor around our brane. What in fact creates the Anti-de Sitter spacetime in the bulk and supports the exponentially decreasing warp factor is the coupling itself between the scalar field and the bulk scalar curvature, that is everywhere negative. This, of course, leads to an anti-gravitating theory over the whole spacetime and eventually to an unphysical gravitational theory on our brane. This model, being far from a realistic theory, is nevertheless a characteristic example of the variety of solutions that may arise in brane-worlds; more specifically, it underlines the easiness with which unphysical black-string solutions (in the case where $M \neq 0$) emerge in contrast to the physically-motivated localised black-hole solutions.

The discussion of the second model with the quadratic coupling function served also another purpose: together with the first one with a linear $f(\Phi)$, they were both derived under the assumption of a positive cosmological constant on our brane. Also, both models were characterized, either globally or over particular regimes, by a negative coupling function $f(\Phi)$ that led to an anti-gravitating theory. In order to investigate the potential connection between a Schwarzschild-de Sitter spacetime on our brane and an anti-gravitating regime in the bulk, in section 5, we examined from the mathematical point-of-view why the field equations in the present theory seem to favor the emergence of these solutions. By turning a particular combination of the field equations into a constraint relating solely the coupling function, its derivatives and the effective cosmological constant, we demonstrated that, for $\Lambda > 0$, this constraint is impossible to satisfy for $f(\Phi)$ also positive for the entire extra dimension. Therefore, in this class of theories, with a non-minimally-coupled scalar field and a general coupling function, the emergence of an effective four-dimensional theory on our brane with a positive cosmological constant is always accompanied by a problematic anti-gravitating regime in the five-dimensional bulk.

The aforementioned conclusion opens the way for the derivation of solutions with normal gravity either in the case of a Minkowski or Anti-de Sitter spacetime on our brane. Although less physically-motivated, it would still be of interest to investigate whether a scalar-tensor theory in the bulk could support a solution (either a black string or a regular one) with a decaying warp factor but without the need for a constant distribution of a negative energy density in the higher-dimensional spacetime. Another
question, we also hope to come back to, is that of the stability of the solutions, the ones derived here and the ones to be soon presented, under perturbations. Note that, the non-minimal coupling of the scalar field to the gravitational field makes the stability analysis highly non-trivial compared to the pure gravitational solutions derived in the literature (see, for example [60]). Given the fact that usually black-string solutions suffer from the Gregory-Laflamme instability [11, 12] – which up to now has served as a method to get rid of unphysical solutions, it will be extremely interesting to see whether the presence of the scalar field and the sign of its coupling function $f(\Phi)$ affects in any way the stability of these solutions.

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A Curvature Invariant Quantities

Employing the expression of the line-element (6), one may compute the scalar curvature invariant quantities. These have the form:

$$R = -8A'' - 20A^2 + \frac{2e^{-2A} (r \partial_r^2 m + 2 \partial_r m)}{r^2},$$

(A.1)

$$R_{MN}R^{MN} = 2e^{-4A} \left[ e^{2A} (A'' + 4A^2) - \frac{\partial_r^2 m}{r} \right]^2 + 2 \frac{e^{-4A} \left[ r^2 e^{2A} (A'' + 4A^2) - 2 \partial_r m \right]^2}{r^4} + 16 \left( A'' + A^2 \right)^2,$$

(A.2)

$$R_{MNKL}R^{MNKL} = - \frac{8e^{-2A} A^2 (r \partial_r^2 m + 2 \partial_r m)}{r^2} + 40A^4 + 16A'' (A'' + 2A^2)$$

$$+ 4e^{-4A} \left[ \frac{(\partial_r m)^2}{r^2} + \frac{4[2(\partial_r m)^2 + (m - r \partial_r m) \partial_r^2 m]}{r^4} + \frac{4(3m^2 - 4rm \partial_r m)}{r^6} \right],$$

(A.3)

and may be used for the geometric characterization of the solutions derived from the field equations.

B Independent Field Equations

Here, we will demonstrate that the three field equations (15)-(17) are not all independent. To this end, we substitute the mass function $m(r) = M + \Lambda r^3 / 6$ into Eq. (15); as shown in Section 5, the latter may then be brought to the form

$$\Phi'' = -f(3A'' + \Lambda e^{-2A}) + A' \partial_y f - \partial_y^2 f.$$  

(B.1)
Taking the derivative of both sides with respect to \(y\), we obtain
\[
2\Phi'\Phi'' = -f(3A'' - 2\Lambda A'e^{-2A}) - \partial_y f(2A'' + \Lambda e^{-2A}) + A'\partial_y^2 f - \partial_y^3 f. \tag{B.2}
\]

Next, we consider Eq. (16) which we solve for the potential \(V\) to find
\[
V = -\Lambda_5 - \frac{1}{2} \Phi^2 - f(6A'^2 + 3A'' - \Lambda e^{-2A}) - 3A'\partial_y f - \partial_y^2 f. \tag{B.3}
\]
If we take again the derivative with respect to \(y\), we arrive at the result
\[
\partial_y V = -\Phi'\Phi'' - f(12A'A'' + 3A'' + 2\Lambda A'e^{-2A}) - \partial_y f(6A'^2 + 6A'' - \Lambda e^{-2A}) - 3A'\partial_y^2 f - \partial_y^3 f. \tag{B.4}
\]

We now use the above expression in the scalar-field equation (17) after multiplying first the latter by \(\Phi'\); we eventually obtain
\[
2\Phi'\Phi'' = -f(3A'' - 2\Lambda A'e^{-2A}) - \partial_y f(2A'' + \Lambda e^{-2A}) + A'\partial_y^2 f - \partial_y^3 f. \tag{B.5}
\]

We see that equations (B.2) and (B.5) are identical, which means that the three field equations from which these equations were derived are not independent. We are thus entitled to keep only two of them in our analysis and to ignore the third one.

### C Inverse Generalised Vaidya Transformation

Starting from the projected-on-the-brane line-element (23), that we will write it, for simplicity, as
\[
ds^2 = -\left(1 - \frac{2m(r)}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{C.1}
\]
where \(m(r) = M + \Lambda r^3/6\), we will seek to determine the coordinate transformation of the Vaidya time-variable \(v\), if existent, that will bring the aforementioned line-element to a diagonal, Schwarzschild-like form
\[
ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{C.2}
\]

We will consider the following general transformation:
\[
v = h(t,r) \Rightarrow dv = \partial_t h \ dt + \partial_r h \ dr. \tag{C.3}
\]
Substituting the above expression of \(dv\) into equation (C.1), we obtain
\[
ds^2 = -\left(1 - \frac{2m(r)}{r}\right) (\partial_t h)^2 dt^2 + \left[-\left(1 - \frac{2m(r)}{r}\right) (\partial_r h)^2 + 2\partial_r h\right] dr^2 + 2\partial_t h \left[-\left(1 - \frac{2m(r)}{r}\right) \partial_r h + 1\right] dt dr + r^2 d\Omega^2. \tag{C.4}
\]
We now demand the vanishing of the off-diagonal term in Eq. (C.4): for \( \partial_t h \neq 0 \), this leads to the constraint
\[
\partial_t h = \left( 1 - \frac{2m(r)}{r} \right)^{-1}.
\] (C.5)

Provided that the above holds, the coefficient of \( dr^2 \) in Eq. (C.4) reduces to \( \partial_r h \), therefore
\[
f(r) = \frac{1}{\partial_r h} = \left( 1 - \frac{2m(r)}{r} \right).
\] (C.6)

Comparing finally the coefficients of \( dt^2 \) in Eqs. (C.2) and (C.4), we conclude that \( \partial_t h \) must be equal to unity. Therefore, if the coordinate transformation \( v = t + g(r) \) is applied to the line-element (C.1), the latter takes indeed the diagonal form
\[
ds^2 = - \left( 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} \right) dt^2 + \left( 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta 
\]
\[
d\phi^2), \quad (C.7)
\]
that describes a 4-dimensional Schwarzschild-(Anti) de Sitter background depending on the sign of the parameter \( \Lambda \), which turns out to be proportional to the cosmological constant on the brane.

To complete the analysis, we need to determine the value of the function \( g(r) \) through the integral
\[
g(r) = \int \frac{dr}{f(r)} = \int \frac{dr}{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}}. \tag{C.8}
\]
Evaluating the above integral amounts to calculating the tortoise coordinate for the specific black-hole background. The steps of the evaluation depend on the sign of the parameter \( \Lambda \). Let us start with the case \( \Lambda > 0 \), where the 4-dimensional background is a Schwarzschild-de Sitter one. The function \( f(r) \) has, in the most general case, two real, positive roots \( r_h \) and \( r_c \) corresponding to the black-hole and cosmological horizon, respectively. Then, the aforementioned integral becomes
\[
g(r) = \frac{3}{\Lambda} \int \frac{r 
\]
\[
dr}{-r^3 + 3r\Lambda - 6M\Lambda} = \frac{3}{\Lambda} \int \frac{r dr}{(r - r_h)(r_c - r)(r + r_c + r_h)}, \tag{C.9}
\]
where the two horizons satisfy the relations
\[
(r_c + r_h)^2 - r_c r_h = \frac{3}{\Lambda}, \quad (r_c + r_h) r_c r_h = \frac{6M}{\Lambda}. \tag{C.10}
\]
Splitting the fraction in the integral (C.9) into three separate ones and performing the corresponding integrations, we arrive at the result [67]
\[
g(r) = r_h \ln(r - r_h) - \frac{r_c \ln(r_c - r)}{1 - \Lambda r^2_h} - \frac{(r_c + r_h) \ln(r + r_c + r_h)}{1 - \Lambda(r_c + r_h)^2} + C_1, \tag{C.11}
\]
where \( C_1 \) is an arbitrary integration constant.

If, on the other hand, \( \Lambda = -|\Lambda| < 0 \), then the background on the brane is of a Schwarzschild-Anti-de Sitter type. The function \( f(r) \) vanishes only at \( r = r_h \), i.e. at the location of the black-hole horizon. We then write:
\[
g(r) = \int \frac{r dr}{\frac{|\Lambda| r^3}{3} + r - 2M} = \frac{3}{|\Lambda|} \int \frac{r dr}{(r - r_h)(r^2 + r_h \beta + \beta)}, \tag{C.12}
\]
24
where $\beta = 6M/|\Lambda|r_h$. Note that the quadratic polynomial $r^2 + r_h r + \beta$ has no real, positive roots. We then split the fraction inside the integral into two separate ones of the form

$$\frac{1}{(r - r_h)(r^2 + r_h r + \beta)} = \frac{A}{r - r_h} + \frac{Br + D}{r^2 + r_h r + \beta},$$

where

$$A = \frac{1}{2r_h^2 + \beta}, \quad B = -A, \quad D = -2r_h A.$$  \hfill (C.14)

Substituting Eq. (C.13) into Eq. (C.12) and applying standard integration techniques, we finally arrive at the result

$$g(r) = \frac{3}{|\Lambda|(2r_h^2 + \beta)} \left[ r_h \ln \left( \frac{r - r_h}{\sqrt{r^2 + r_h r + \beta}} \right) + \frac{r_h^2 + 2\beta}{\sqrt{4\beta - r_h^2}} \arctan \left( \frac{2r + r_h}{\sqrt{4\beta - r_h^2}} \right) \right] + C_2,$$

where $C_2$ is again an arbitrary integration constant and the horizon radius may be expressed as

$$r_h = \frac{1}{(-3A^2 M + \sqrt{9A^4 M^2 + |\Lambda|^3})^{1/3}} - \frac{(-3A^2 M + \sqrt{9A^4 M^2 + |\Lambda|^3})^{1/3}}{|\Lambda|}.$$  \hfill (C.16)

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