Entropic and information inequality for nonlinearly transformed two-qubit $X$-states

V. I. Man’ko$^{1,2}$ and R. S. Puzko$^2$

$^1$ P.N. Lebedev Physical Institute, Russian Academy of Science - Moscow 119991, Russia
$^2$ Moscow Institute of Physics and Technology (State University) - Dolgoprudny 117303, Russia

received 18 October 2014; accepted in final form 23 February 2015
published online 13 March 2015

PACS 03.67.-a – Quantum information
PACS 03.67.Bg – Entanglement production and manipulation
PACS 03.67.Mn – Entanglement measures, witnesses, and other characterizations

Abstract – The entropic and information inequalities for two-qubit $X$-states transformed by the nonlinear channels are given in explicit form. The subadditivity condition and non-negativity of the von Neumann quantum information are studied for both the initial $X$-state and the state after the action of the nonlinear channel. The partial case of the Werner state is considered in detail. We show that the von Neumann information increases due to the action of the nonlinear channel.

We generalize the results obtained for the Werner state given in our earlier article (J. Russ. Laser Res., 35 (2014) 362) to the case of a $X$-state of a two-qubit system. We study the influence of a nonlinear channel acting on the $X$-state of a two-qubit system onto the von Neumann mutual information.

Copyright © EPLA, 2015

Introduction. – The states of quantum systems are associated with wave functions, satisfying the linear Schrödinger equation [1]. The linearity of quantum evolution as well as the superposition principle are basic properties of quantum mechanics. The possible nonlinearities of quantum mechanics are discussed in the literature, e.g. the nonlinear quantum equations were presented in [2–7]. The linear Schrödinger equation is considered in these works as a linear approximation of nonlinear theories [6]. The nonlinear equation known as Gross–Pitaevskii equation [2,3] is used to describe boson gas. The Peres-Horodecki criterion of entanglement [8,9] is based on using a transposition map of the density matrix which cannot be presented in the Kraus form [10] and cannot be realized in quantum information experiments. Analogously to this example, although the nonlinear maps are not completely positive ones, and cannot be performed experimentally, they could be useful in quantum information.

The pure quantum states are described by the vectors in the Hilbert space [11]. The mixed quantum states are described by density operators or density matrices [12,13]. The density matrices $\rho$ have specific properties. They are Hermitian non-negative matrices with $\text{Tr}\rho = 1$. The eigenvalues of the density matrices are non-negative numbers. The linear transforms of the density matrices $\rho \to \Phi(\rho)$ which preserve the properties of the density matrices, i.e. hermiticity, non-negativity and trace of the matrices, are called positive maps [14].

The positive maps were studied in [10,14,15] and the maps of the special kind $\Phi(\rho) = \sum_n K_n \rho K_n^+$, where $\sum_n K_n^+ K_n = 1$, are called completely positive maps or quantum channels [16,17]. On the other hand, there exist the maps of density matrices $\rho \to \Phi(\rho)$ for which the matrix $\Phi(\rho)$ has the properties of density matrix and the maps are nonlinear [18]. In the context of the possibility of taking into account the influence of nonlinearities discussed in [2–7], it is interesting to study a model of the nonlinear maps of the density matrices, though, in the quantum information context, the quantum channels (completely positive maps) are considered as the only way to realize the transformation of density matrices which can be performed experimentally [19]. The nonlinear maps of density matrices were mentioned in [20,21]. The nonlinear map of the Werner state of two qubits with rational function $\Phi(\rho)$ was studied in [22]. This positive map was called “nonlinear channel”. The Werner state of two qubits is a partial case of the $X$-state studied in [23]. The experimental study of quantum discord in the $X$-states was done in [24].

The aim of our work is to extend the study of quantum transformations of density matrices having in mind the possibilities discussed in [2–7] and to study the model
of action of nonlinear quantum channels onto \(X\)-states of two qubits. We generalize our results obtained in [22] for Werner states of two qubits to the case of a generic \(X\)-state of two qubits. Also we study the properties of the von Neumann entropy for the \(X\)-state and states obtained by means of the action of the nonlinear channel on this state. We evaluate the entanglement of the nonlinearly transformed \(X\)-state of two qubits by using concurrence [25,26].

The paper is organized as follows. In the next section properties of specific nonlinear channels of 4 \(\times\) 4 density matrix are introduced. In the third section quantum tograms [27–31] of the state \(\Phi(\rho)\) are discussed. In the fourth section entropic and information properties of the nonlinearly transformed \(X\)-state are studied. In the fifth section concurrence and negativity characteristics of entanglement created by the nonlinear channels are evaluated. Conclusions and prospective are given in the last section.

Properties of nonlinear channels. – The nonlinear channel studied in this paper raises the density matrix to power \(n\),

\[
\rho_n = \frac{1}{\text{Tr}\rho^n} \rho^n.
\]

The transformed density matrix has to be normalized to satisfy the condition \(\text{Tr}\rho_n = 1\). We consider the specific state of a quantum two-qubit system called \(X\)-state. The density matrix of the state has the following form:

\[
\rho_X = \begin{pmatrix}
a & 0 & 0 & d \\
0 & b & c & 0 \\
c^* & b & 0 & 0 \\
d^* & 0 & 0 & a
\end{pmatrix},
\]

with parameters \(a, b, c, d\). By varying parameters of the density matrix \(\rho_X\) one can obtain states with different properties of entanglement. In the particular case \(a = \frac{1+|p|}{2}, b = \frac{1-|p|}{2}, c = 0, d = \frac{1}{2}\) this density matrix describes the Werner state [32] determined by the parameter \(p\).

The density matrix of the form (2) has the following eigenvalues:

\[
\lambda_1 = a + |d|, \quad \lambda_2 = b + |c|, \quad \lambda_3 = b - |c|, \quad \lambda_4 = a - |d|
\]

which are positive if \(a \geq |d|\) and \(b \geq |c|\). To examine the properties of entanglement we use the Peres-Horodecki criterion [8,9]. The criterion is based on the positive partial transposition of the density matrix \(\rho \rightarrow \rho^{\text{ppt}}\). The positive partial transposed matrix (ppt-matrix) for \(\rho_X\) takes the form

\[
\rho_X^{\text{ppt}} = \begin{pmatrix}
a & 0 & 0 & c \\
0 & b & d^* & 0 \\
c^* & d & 0 & 0 \\
0 & 0 & 0 & a
\end{pmatrix}.
\]

According to this criterion the state is separable if all of these eigenvalues are non-negative. Therefore, the state is separable if the conditions

\[
a > |c|, |d|, \\
b > |c|, |d|
\]

are satisfied. In the domain \(|c| > a > |d|\) \((b > |c|)\) and \((a > |d|) \cup (|d| > b > |c|)\) the density matrix \(\rho_X\) corresponds to the entangled state (see fig. 1). As can be seen from these formulas, the domains do not depend on the phases of complex parameters \(c, d\).

To find the form of the density matrix \(\rho_{X,n}\) produced by the channel in case of arbitrary value of the integer \(n\), we decompose the matrix \(\rho_X\),

\[
\rho_X = SDS^{-1},
\]

where \(D\) is the diagonal matrix with eigenvalues of the matrix \(\rho_X\) placed on the diagonal, and \(S\) is the unitary matrix containing the eigenvectors corresponding to these eigenvalues. The matrix \(\rho_X\) has the following eigenvalues and eigenvectors:

\[
\lambda_1 = a + |d| \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix}^T, \\
\lambda_2 = b + |c| \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix}^T, \\
\lambda_3 = b - |c| \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T, \\
\lambda_4 = a - |d| \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix}^T.
\]
Therefore, the decomposition for the matrix \( \rho_X \) can be written as
\[
\rho_{w,1} = SDS^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & -\frac{d^*}{|a|} \\
0 & 1 & -\frac{c}{|c|} & 0 \\
0 & \frac{c}{|c|} & 1 & 0 \\
\frac{d^*}{|a|} & 0 & 0 & 1
\end{pmatrix}.
\]
(9)

This decomposition allows to find easily the result of raising matrix \( \rho_X \) to power \( n \),
\[
\rho_X^n = (SDS^{-1}) \cdot SDS^{-1} \cdot \ldots \cdot SDS^{-1} = SD^n S^{-1}. 
\]
(10)

To get the form of the matrix \( \rho_{X,n} \), one has to normalize the matrix \( \rho_X^n \). Therefore, the considered nonlinear channel gives the following matrix:
\[
\rho_{X,n} = \begin{pmatrix}
A_n & 0 & 0 & D_n \\
0 & B_n & C_n & 0 \\
0 & C^*_n & B_n & 0 \\
D^*_n & 0 & 0 & A_n
\end{pmatrix},
\]
(11)

where we use the following notations:
\[
A_n = (a + |d|)^n + (a - |d|)^n \\
B_n = (b + |c|)^n + (b - |c|)^n \\
C_n = 2(\lambda_1^0 + \lambda_2^0 + \lambda_3^0 + \lambda_4^0)^n \\
D_n = \frac{[(a + |d|)^n - (a - |d|)^n]}{|d|^n}.
\]
(12-15)

From (11) it can be seen that the nonlinear channel gives the matrix of the form similar to the initial matrix \( \rho_X \): the only non-zero elements are placed on the diagonal and anti-diagonal. Thus, this task becomes equivalent to the question about the entanglement of the state corresponding to the density matrix \( \rho_X \). The conditions of the positivity of the matrix \( \rho_{X,n} \) are \( A_n \geq |C_n| \) and \( B_n \geq |D_n| \). The two cases of odd and even integers \( n \) have to be separated. If the integer \( n \) is even, the matrix \( \rho_{X,n} \) is the density matrix for an arbitrary set of parameters \( a, b, c, d \). In the case of odd integer \( n \), there is a domain of parameters where the matrix \( \rho_{X,n} \) is negative. The condition of positivity for odd integers \( n \) can be rewritten in the following form:
\[
\begin{cases}
a \geq |d|, & b \geq |c|, & (\lambda_1^0 + \lambda_2^0 + \lambda_3^0 + \lambda_4^0) > 0, \\
a \leq |d|, & b \leq |c|, & (\lambda_1^0 + \lambda_2^0 + \lambda_3^0 + \lambda_4^0) < 0.
\end{cases}
\]
(16)

According to the Peres-Horodecki criterion the new state is entangled when at least one of the conditions \( A_n \geq |C_n| \) or \( B_n \geq |D_n| \) is violated. Figure 2 displays the domains of absolute values of \( c \) and \( d \) where matrix \( \rho_{X,n} \) describes the separable (white color) and entangled (gray color) state, respectively. The black domains correspond to set of parameters, where the matrix \( \rho_{X,n} \) is not the density matrix. Parameters \( a = 0.33 \) and \( b = 0.17 \). Figures displays cases of different power \( n \). (a) \( n = 3 \), (b) \( n = 5 \), (c) \( n = 2 \), (d) \( n = 4 \).

**Quantum tomography for nonlinear channels.**

Quantum tomography for a spin state shows the probability distribution of spin projections on selected directions. If the direction \( \vec{m} \) corresponds to the unitary matrix
\[
\begin{pmatrix}
\cos \left( \frac{\theta}{2} \right) e^{i\left(\varphi - \psi\right)} & \sin \left( \frac{\theta}{2} \right) e^{i\left(\varphi + \psi\right)} \\
-\sin \left( \frac{\theta}{2} \right) e^{i\left(\varphi - \psi\right)} & \cos \left( \frac{\theta}{2} \right) e^{i\left(\varphi + \psi\right)}
\end{pmatrix},
\]
(17)

depending on Euler’s angles \( \theta, \varphi, \psi \), the quantum tomogram for the spin state with density matrix \( \rho \) is produced by the formula
\[
W(m, \vec{m}) = \langle m | \rho | m \rangle,
\]
(18)

where \( m \) is the spin projection on the direction \( \vec{m} \). The quantum tomogram for a two-qubit system and directions...
$\overrightarrow{a}$, $\overrightarrow{b}$ is given by the formula

$$W(m_1, \overrightarrow{a}, m_2, \overrightarrow{b}) = (j_1, m_1, j_2, m_2) U_\rho U^+ |j_1, m_1, j_2, m_2\rangle,$$

where $U = u_1 \otimes u_2$, the product being a tensor product of the unitary matrices $u_1, u_2$ corresponding to directions $\overrightarrow{a}$, $\overrightarrow{b}$, and $m_1, m_2$ are spin projections on these directions.

The general form of the matrices, produced by the considered nonlinear channel, is the $X$-matrix with specific matrix elements for each power $n$ of the channel. Thus, it is possible to write the formula for the quantum tomogram in the following form:

$$W(\overrightarrow{a}, \overrightarrow{b}) = W(\overrightarrow{a}, \overrightarrow{b}) = A_n \cdot f_+ (\theta_a, \theta_b) + B_n \cdot f_- (\theta_a, \theta_b) + \psi(C_n, D_n, \theta_a, \theta_b, \psi_a, \psi_b),$$

where $A_n, B_n, C_n, D_n$ are given by eqs. (12)–(15), $f_+ (\theta_a, \theta_b) = \cos^2 \left(\frac{\theta_a}{2}\right) \cos^2 \left(\frac{\theta_b}{2}\right) + \sin^2 \left(\frac{\theta_a}{2}\right) \sin^2 \left(\frac{\theta_b}{2}\right)$, $f_- (\theta_a, \theta_b) = \cos^2 \left(\frac{\theta_a}{2}\right) \sin^2 \left(\frac{\theta_b}{2}\right) + \sin^2 \left(\frac{\theta_a}{2}\right) \cos^2 \left(\frac{\theta_b}{2}\right)$ and $\psi(C_n, D_n, \theta_a, \theta_b, \psi_a, \psi_b) = \frac{1}{2} \sin \theta_a \sin \theta_b \cdot \text{Re}(ce^{i(\psi_a-\psi_b)})$. The formulas are written for the arbitrary integer $n$ determining the nonlinear channel. The only difference between cases of different $n$ is due to the matrix elements $A_n, B_n, C_n, D_n$. In particular, the quantum tomogram for directions $\overrightarrow{a}$ and $\overrightarrow{b}$ in the case of the initial Werner state takes the form

$$W(\overrightarrow{a}, \overrightarrow{b}) = W(\overrightarrow{a}, \overrightarrow{b}) = A(p) f_+ (\theta_a, \theta_b) + B(p) f_- (\theta_a, \theta_b) + \frac{C(p)}{2} \sin (\theta_a) \sin (\theta_b) \cos (\psi_a + \psi_b),$$

where $A(p) = \frac{1}{2} \sqrt{(1-p)^n + (1-3p)^n}$, $B(p) = \frac{(1-p)^n}{2(1-p)^n + (1+3p)^n}$ and $C(p) = A(p) - B(p)$.

Entropic inequalities for nonlinear channels $\rho_X \rightarrow \Phi(\rho_X)$ – The von Neumann entropy of an arbitrary system with density matrix $\rho$ is defined as follows [16]:

$$S = -\text{Tr} (\rho \ln \rho).$$

Let us consider the case of a two-qubit system with qubits 1, 2. In the case of a two-qubit system state $\rho(1,2)$, there are entropy $S(1,2)$ for the system and entropies $S(1), S(2)$ for subsystems corresponding to the density matrices $\rho(1) = \text{Tr}_2 \rho(1,2)$ and $\rho(2) = \text{Tr}_1 \rho(1,2)$,

$$S(1,2) = -\text{Tr} [\rho(1,2) \ln \rho(1,2)],$$

$$S(1) = -\text{Tr} [\rho(1) \ln \rho(1)],$$

$$S(2) = -\text{Tr} [\rho(2) \ln \rho(2)],$$

For the X-state of two qubits we have $\rho(1,2) = \rho_X$. In the case of matrices produced by the nonlinear channel from $\rho_X$

$$\rho(1) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \rho(2) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

and $S(1) = S(2) = \ln 2$. The entropy of system is

$$S(1,2) = -\sum_{k=1}^4 \lambda_k \ln \lambda_k,$$

where $\lambda_k$ are $A_n \pm |D_n|$, $B_n \pm |C_n|$ eigenvalues of $\rho_X$. The quantum mutual information of the system with subsystems 1, 2 is defined as

$$I_N = S(1) + S(2) - S(1,2).$$

In the case of the initial Werner state the information $I_N(p)$ is

$$I_N(p) = \ln 4 + \ln [(1+3p)^n + 3(1-p)^n] - (1+3p)^n \ln (1+3p)^n + 3(1-p)^n \ln (1-p)^n - (1+3p)^n + 3(1-p)^n.$$

It has a maximum at $p = 1$ where $I_N = \ln 4$. The quantum mutual information $I_N(p)$ increases after the action of the nonlinear channel on the initial state. With the growing integer $n$ determining the nonlinear channel, the information is increasing (see fig. 3).

The tomographic Shannon entropy [33] is associated with a tomographic probability distribution. For subsystems of the considered two-qubit systems the marginal distributions are

$$W_1(\overrightarrow{a}, \overrightarrow{b}) = W(\overrightarrow{a}, \overrightarrow{b}),$$

$$W_2(\overrightarrow{a}, \overrightarrow{b}) = W(\overrightarrow{a}, \overrightarrow{b}).$$

Tomographic entropies of qubit subsystems are

$$H(1) = -W_1(\overrightarrow{a}, \overrightarrow{a}) \ln W_1(\overrightarrow{a}, \overrightarrow{a}) - W_1(\overrightarrow{b}, \overrightarrow{b}) \ln W_1(\overrightarrow{b}, \overrightarrow{b}),$$

$$H(2) = -W_2(\overrightarrow{a}, \overrightarrow{a}) \ln W_2(\overrightarrow{a}, \overrightarrow{a}) - W_2(\overrightarrow{b}, \overrightarrow{b}) \ln W_2(\overrightarrow{b}, \overrightarrow{b}).$$

50005-p4
Entropy and information of nonlinearly transformed two-qubit $X$-states

The tomographic entropy for a two-qubit system state is given by the formula

$$ H(1, 2) = -W(\uparrow, \bar{a}, \uparrow, \bar{b}) \ln W(\uparrow, \bar{a}, \uparrow, \bar{b}) $$

$$ - W(\downarrow, \bar{a}, \downarrow, \bar{b}) \ln W(\downarrow, \bar{a}, \downarrow, \bar{b}) $$

$$ - W(\downarrow, \bar{a}, \downarrow, \bar{b}) \ln W(\downarrow, \bar{a}, \downarrow, \bar{b}) $$

$$ - W(\uparrow, \bar{a}, \uparrow, \bar{b}) \ln W(\uparrow, \bar{a}, \uparrow, \bar{b}). $$

For states $\rho_{X,n}$ the entropies of the subsystems are equal to $H(1) = H(2) = \ln 2$. The Shannon information for the two-qubit system is non-negative and it is given by the formula

$$ I_S = H(1) + H(2) - H(1, 2). $$

The relation between the two definitions of the information (26) and (31) is

$$ I_S \leq I_N. $$

For both mutual and Shannon information the non-negativity conditions exist: $I_N \geq 0$, $I_S \geq 0$.

**Negativity and concurrence.** We use such characteristics as negativity and concurrence [25,26] to analyze the action of the considered quantum channel on the entanglement of the $X$-state. The negativity for the system having density matrix $\rho$ is defined as follows:

$$ N = \text{Tr}(|\rho^{pp}\rangle\langle\rho^{pp}|), $$

$$ N = \text{Tr}(|\rho^{pp}\rangle\langle\rho^{pp}|). $$

Fig. 4: (Color online) Negativity for states produced by the quantum channel with (a) $n = 2$, (b) $n = 4$ for different values of $|c|$ and $|d|$. Parameters $a = 0.33$ and $b = 0.17$. Level $N = 1$ corresponds to a separable state. If the state is entangled, then $N > 1$.

Fig. 5: (Color online) Concurrence for states produced by the quantum channel with (a) $n = 2$, (b) $n = 4$ for different values of $|c|$ and $|d|$. Parameters $a = 0.33$ and $b = 0.17$. For both mutual and Shannon information the non-negativity conditions exist: $I_N \geq 0$, $I_S \geq 0$. **Negativity and concurrence.** We use such characteristics as negativity and concurrence [25,26] to analyze the action of the considered quantum channel on the entanglement of the $X$-state. The negativity for the system having density matrix $\rho$ is defined as follows:

$$ N = \text{Tr}(|\rho^{pp}\rangle\langle\rho^{pp}|), $$

$$ N = \text{Tr}(|\rho^{pp}\rangle\langle\rho^{pp}|). $$
where $\rho^{\text{pt}}$ is the ppt-matrix for $\rho$. The partial transposed density matrix for the separable state remains non-negative and saves the property $\text{Tr}\rho = 1$. The negativity $N$ of the entangled state deviates from this value. The more entangled the state, the greater the negativity $N$.

Figure 4 displays the negativity for the density matrices produced by the channel $\rho X \rightarrow \rho_{X,n}$ for different cases of power $n$. The surfaces show the dependences of the negativity on the absolute values of parameters $c$ and $d$ of the matrix $\rho X$. For odd $n$ the behaviors of the surfaces distinguish significantly from the cases of even $n$. However, one can see that for both cases the domains of the parameters $c$ and $d$, where the state is entangled, expand with power $n$. The value of negativity is growing with integer $n$.

The concurrence for the two-qubit state with the density matrix $\rho$ is defined by the formula

$$
C = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}),
$$

(34)

where $\lambda_k, k = 1, 2, 3, 4$ are eigenvalues of the matrix $R = \rho\rho_C$, and $\lambda_1$ has the maximum value. Matrix $\rho_C$ is the result of the spin flip operation on the matrix $\rho$,

$$
\rho_C = (\sigma_y \otimes \sigma_y)\rho^\ast (\sigma_y \otimes \sigma_y),
$$

(35)

where $\sigma_y$ is the Pauli matrix. In the case of the initial $X$-state, one can obtain the matrix $\rho_{X,C}$,

$$
\rho_{X,C} = (\sigma_y \otimes \sigma_y)\rho_X^\ast (\sigma_y \otimes \sigma_y) = \rho_X.
$$

(36)

Thus, the matrix $R = \rho_X^2$ and its eigenvalues are $(|A| \pm |D|)^2$, $(|B| \pm |C|)^2$. Utilizing the property of the considered nonlinear channel $\rho X \rightarrow \Phi(\rho X)$ for which new state density matrix has the structure of the $X$-state density matrix, one can find the concurrence for the arbitrary integer $n$. The eigenvalues of the matrix $R_n = \rho_{X,n}\rho_{X,n,C}$ are $(|A_n| \pm |D_n|)^2$, $(|B_n| \pm |C_n|)^2$. The concurrence takes value 0 for separable states and greater for entangled states. In fig. 5 the concurrence for the density matrices $\rho_{X,n}$ for different cases of power $n$ is shown.

Conclusion. — To resume, we point out the main results of our work. We studied the influence of specific nonlinearities of quantum transformations of density matrices on quantum correlations and the entanglement phenomenon. We have shown that the nonlinear channels acting on the initial $X$-state of two qubits change the entanglement characteristics of the state. The entropy of the transformed state is shown to decrease with $n \rightarrow \infty$. The entanglement degree expressed by means of the concurrence and negativity parameters is shown to increase under the influence of the nonlinear channel. The possibility to apply the obtained results and study the influence of nonlinearities in quantum and quantum-like phenomena needs extra clarification. The action of nonlinear channels on the other quantum states will be studied in a future publication.

REFERENCES

[1] Schrödinger E., Ann. Phys., 79 (1926) 489.
[2] Gross E. P., Nuovo Cimento, 20 (1961) 454.
[3] Pitaevskii L. P., Sov. Phys. JETP, 13 (1961) 451.
[4] Doebner H.-D. and Goldin G. A., Phys. Rev. A, 54 (1996) 3764.
[5] Weinberg S., Ann. Phys., 194 (1989) 336.
[6] Bialynicki-Birula I. and Mycielski J., Ann. Phys., 100 (1976) 62.
[7] De Broglie L., Non-linear Wave Mechanics (Elsevier, Amsterdam) 1960.
[8] Peres A., Phys. Rev. Lett., 77 (1996) 1413.
[9] Horodecki M., Horodecki P. and Horodecki R., Phys. Lett. A, 223 (1996) 1.
[10] Kraus K., States, Effects, and Operations: Fundamental Notions in Quantum Theory (Springer-Verlag) 1983.
[11] Dirac P. A. M., The Principles of Quantum Mechanics (Oxford University Press, London) 1958.
[12] Landau L., Z. Phys., 45 (1927) 430.
[13] Neumann J., Nachr. Ges. Wiss. Götting., 11 (1927) 245.
[14] Sudarchan E. S. G., Mathews P. M. and Rau J., Proc. Phys. Rev., 121 (1961) 920.
[15] Stinespring W. F., Proc. Am. Math. Soc., 6 (1955) 211.
[16] Nielsen M. A. and Chuang I. L., Quantum Computation and Quantum Information (Cambridge University Press) 2000.
[17] Holevo A. S., Statistical Structure of Quantum Theory, Lect. Notes Phys. Monogr. Ser., Vol. 67 (Springer) 2001.
[18] Man’ko V. I., Marmo G., Simoni A. and Ventriglia F., Phys. Lett. A, 372 (2008) 6490.
[19] Peres A., Quantum Theory: Concepts and Methods (Kluwer Academic Publishers) 2002.
[20] Chernega V. N. and Man’ko O. V., J. Russ. Laser Res., 35 (2014) 27.
[21] Chernega V. N., Man’ko V. I. and Man’ko O. V., J. Russ. Laser Res., 35 (2014) 278.
[22] Man’ko V. I. and Puzko R. S., J. Russ. Laser Res., 35 (2014) 362.
[23] Yu T. and Eberly J. H., Quantum Inf. Comput., 7 (2007) 459.
[24] Feildman E. B. and Zenchuk A. I., Phys. Rev. A, 86 (2012) 012303.
[25] Hill S. and Wootters W. K., Phys. Rev. Lett., 78 (1997) 5022.
[26] Wootters W. K., Phys. Rev. Lett., 80 (1998) 2245.
[27] Dodonov V. V. and Man’ko V. I., Phys. Lett. A, 229 (1997) 335.
[28] Man’ko V. I. and Man’ko O. V., JETP, 85 (1997) 430.
[29] Ibranov A., Man’ko V. I., Marmo G., Simoni A. and Ventriglia F., Phys. Lett. A, 317 (2010) 2614.
[30] Ibranov A., Man’ko V. I., Marmo G., Simoni A. and Ventriglia F., Phys. Scr., 79 (2009) 065013.
[31] Man’ko O. V., Man’ko V. I. and Marmo G., Phys. Scr., 62 (2000) 446.
[32] Werner R. F., Phys. Rev. A, 40 (1989) 4277.
[33] Shannon C. E., Bell Syst. Tech. J., 27 (1948) 379.