Numerical approach for bifurcation and orbital stability analysis of periodic motions of a 2-DOF autonomous Hamiltonian system

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Abstract. In Spaceflight Dynamics it is often necessary to obtain periodic motions of conservative mechanical systems and analyze their stability and bifurcation. These conservative systems can be described using Hamiltonian equations. We consider bifurcation and orbital stability problem for periodic motions of a 2-DOF autonomous Hamiltonian system. Since it is not possible to obtain analytical solutions to the aforementioned problem for all admissible values of its parameters a two-step numerical approach is proposed. On the first step the so-called base solutions are obtained analytically for particular values of problem's parameters. The base solutions are then continued to the borders of their existence domains using a numerical algorithm. In course of computation bifurcation points are identified and orbital stability is studied. On the second step new base solutions are identified in the neighborhood of bifurcation points and the continuation process is repeated. Finally, orbital stability and bifurcation diagrams of the resulting families of periodic motions are constructed. Poincare sections are also computed in the neighborhoods of bifurcation points to verify the results. To illustrate this approach, we computed the bifurcation and orbital stability diagrams for families of short-periodic motions originating from Regular precessions of a dynamically-symmetric satellite.

1. Introduction
Conservative mechanical systems appear in many problems of Spaceflight Dynamics such as the problem of attitude and orbital motion of a satellite. Analyzing behavior of a conservative system often requires computing its periodic motions and studying their properties such as stability and bifurcation. Stable periodic motions also play an important role in applications since they allow to design spacecraft orbits and maneuvers for increased propellant and energy efficiency. A common way to acquire equations of motion of a conservative system is to write them out in Hamiltonian form. Periodic motions of Hamiltonian systems have been previously studied rigorously in many works [1-5]. The analytical approach used in these works involves normalizing the initial Hamiltonian system in the neighborhood of known equilibria or stationary motions which makes it is possible to obtain the periodic motions as small parameter power series and apply KAM theory for nonlinear orbital stability analysis. However, these results only remain valid in small neighborhood of the aforementioned stationary motions and equilibria. To obtain periodic motions for all admissible values of the problem’s parameters as well as to perform an orbital stability and bifurcation analysis it is necessary to use a numerical approach.
Numerical algorithms for computing periodic motions of Hamiltonian systems have been discussed and developed in works [6-12]. The common way to obtain periodic motions relies on solving the boundary problem using shooting method-based algorithms. However, a predictor-corrector numerical continuation method proposed by A Deprit and J Henrard in [9] and developed in works [10-12] offers a more effective approach to computation of periodic motions.

In this paper we consider autonomous Hamiltonian system with two degrees of freedom:

\[
\begin{align*}
\dot{q}_1 &= \frac{\partial H}{\partial p_1}, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \quad (i = 1, 2) \\
q_1(t, \bar{A}), \quad p_1(t, \bar{A}) \quad (i = 1, 2)
\end{align*}
\]

where \(H(q_1, q_2, p_1, p_2, \bar{a})\) is its Hamiltonian, \(h\) is energy constant, \(\bar{a} = (a_1, ..., a_k, h)\) is parameter vector. We then assume that

\[
q_i(t, \bar{A}), \quad p_i(t, \bar{A}) \quad (i = 1, 2)
\]

is a known \(T\)-periodic motion of the system (1) with initial conditions

\[
q_{i0} = q_i(0, \bar{A}), \quad p_{i0} = p_i(0, \bar{A}), \quad (i = 1, 2)
\]

corresponding to fixed set of parameter values \(\bar{A}\), including energy constant value \(h_0\). Following [9,10] we use a term ‘base solutions’ for referring to the periodic motions (2).

We then look for initial conditions

\[
q_{i0} = q_i^*(0, \bar{a}), \quad p_{i0} = p_i^*(0, \bar{a}), \quad (i = 1, 2)
\]

of periodic motions

\[
q_i^*(t, \bar{a}), \quad p_i^*(t, \bar{a}) \quad (i = 1, 2)
\]

which satisfy adherence conditions

\[
\lim_{\bar{a} \to \bar{A}} q_i(0, \bar{a}) = q_i(0, \bar{A}), \quad \lim_{\bar{a} \to \bar{A}} p_i(0, \bar{a}) = p_i(0, \bar{A}), \quad \lim_{\bar{a} \to \bar{A}} T^*(\bar{a}) = T(\bar{A}). \quad (i = 1, 2)
\]

Periodic motions (5) and (2) constitute a so-called natural family [9,13] emanating from base solution (2) and defined by adherence conditions (6).

The aim of the paper is to formulate and test a numerical approach which allows to perform bifurcation an orbital stability analysis simultaneously with computing the natural families constituted by periodic motions (5) and (2).

2. Materials and research methods

We base our approach on a predictor-corrector algorithm proposed by A Sokolskiy and S Karimov [11]. As an application we consider attitude motion of a dynamically-symmetric satellite. This problem is well-studied [3,4,14-18] analytically for both elliptical and circular orbits which allows for good verification of the proposed approach. In this paper we investigate bifurcation and orbital stability problems for families of short-periodic motions originating from the satellite’s Regular precessions on a circular orbit.

2.1 Numerical continuation of natural families

Since analytical computation of natural families is only possible in particular cases [9] we use the following numerical method. In accordance with [11], we introduce local coordinates \(\bar{w} = (\bar{n}_u, \bar{m}_u, \bar{\nu}_v, \bar{m}_v)^T\) by applying an univalent canonical transformations

\[
\xi_i = q_i^* - q_i, \quad \eta_i = p_i^* - p_i, \quad (i = 1, 2)
\]

and

\[
(\xi_1, \xi_2, \eta_1, \eta_2) = S \cdot \bar{w},
\]

where \(S\) is a symplectic orthogonal matrix [19]

\[
S = \frac{1}{V} \begin{pmatrix}
q_2 & q_1 & \bar{q}_2 & \bar{q}_1 \\
q_1 & q_2 & \bar{q}_1 & \bar{q}_2 \\
-p_2 & -p_1 & \bar{p}_2 & \bar{p}_1 \\
-p_1 & -p_2 & \bar{p}_1 & \bar{p}_2
\end{pmatrix}, \quad S^T S = I, \quad S^T S = E, \quad V = \sqrt{q_1^2 + q_2^2 + p_1^2 + p_2^2}.
\]

The derivatives \(q_1, q_2, p_1, p_2\) in (9) are calculated on the base solution (2). On applying (7) and (8) the initial canonical system takes on the following form
\[ n_u = \frac{\partial H^n}{\partial v}, \quad n_v = -\frac{\partial H^n}{\partial u}, \]
\[ m_u = \frac{\nu}{v} m_u + h_{14} n_u + h_{34} n_v + \sum_{j=1}^{k} h_j^2 a_j, \]
\[ m_v = -\frac{1}{v} H_u^T \cdot \vec{a}, \]

where
\[ H^n = \frac{1}{2} (h_{11} n_u^2 + h_{33} n_v^2 + 2n_u n_v h_{13}) + n_u \sum_{j=1}^{k} h_{11}^j a_j + n_v \sum_{j=1}^{k} h_{12}^j a_j \]
is its Hamiltonian and \( h_{11}, h_{13}, h_{14}, h_{33}, h_{34}, H_u^T, h_{11}^j, h_{12}^j, h_2^j, (j = 1 ... k) \) are time-dependent coefficients calculated on the base solution (2). Here
\[ \vec{a} = \vec{a} - \vec{A} \]
are parameter variations or ‘steps’ which are determined depending on desired rate and precision of the continuation. To determine these variations, we use a method proposed in [20].

We denote \( n_u, n_v, m_u \) and \( \tau \) in form of linear combinations of parameter variations \( \vec{a} \) and the variation of energy constant \( \Delta h \):
\[ n_u = \sum_{j=1}^{k} n_u^j a_j + n_u^{k+1} \Delta h, \]
\[ n_v = \sum_{j=1}^{k} n_v^j a_j + n_v^{k+1} \Delta h, \]
\[ m_u = \sum_{j=1}^{k} m_u^j a_j + m_u^{k+1} \Delta h, \quad \tau = \sum_{j=1}^{k} \tau^j a_j + \tau^{k+1} \Delta h. \]

Substituting (13) into (10), we obtain the Predictor equations [5,8]:
\[ \dot{n}_{11} = h_{11} n_{11} + h_{33} n_{12}, \]
\[ \dot{n}_{12} = -h_{11} n_{11} - h_{13} n_{12}, \]
\[ \dot{n}_{21} = h_{11} n_{21} + h_{33} n_{22}, \]
\[ \dot{n}_{22} = -h_{11} n_{21} - h_{13} n_{22}, \]
\[ \dot{n}_p^j = n_p^j h_{13} + n_p^j h_{33} + h_{12}^j, \]
\[ \dot{n}_p^j = -n_p^j h_{13} - n_p^j h_{33} - h_{12}^j, \]
\[ \dot{m}_1 = \frac{\nu}{v} m_1 + h_{14} n_{11} + h_{34} n_{12}, \]
\[ \dot{m}_2 = \frac{\nu}{v} m_2 + h_{14} n_{21} + h_{34} n_{22}, \]
\[ \dot{m}_p^j = \frac{\nu}{v} m_p^j + h_{14} n_p^j + h_{34} n_p^j + h_{2}^j, \]
\[ j = 1 ... k + 1, \]
\[ h_{11}^{k+1} = h_{14}^{k+1}, h_{12}^{k+1} = h_{34}^{k+1}, h_2^{k+1} = 0. \]

Solving (8) and (9) with initial conditions \( n_{11} = 1, n_{12} = 0, n_{21} = 0, n_{22} = 1, m_1 = 0, m_2 = 0, n_p^1 = 0, n_p^2 = 0, m_p^j = 0 \), we find (12). After returning to the initial variables \( q_i, p_i (i = 1,2) \) we obtain approximate values of initial conditions (4) of a new periodic motion (5) belonging to a natural family emanating from (2). These approximate initial conditions can be corrected to a required level of precision using the corrector part of the algorithm presented in [11,19].

### 2.2 Bifurcation and linear orbital stability analysis

A base solution (2) is called critical [11, 21] if it is not possible to find a solution to the equations of normal deviations
\[ \dot{n}_{11} = h_{11} n_{11} + h_{33} n_{12}, \]
\[ \dot{n}_{12} = -h_{11} n_{11} - h_{13} n_{12}, \]
\[ \dot{n}_{21} = h_{11} n_{21} + h_{33} n_{22}, \]
\[ \dot{n}_{22} = -h_{11} n_{21} - h_{13} n_{22}, \]

which are part of both predictor and corrector steps of the numerical algorithm used in this work. This condition is called *termination* of a natural family according to [11,21] and can be also formulized by a criterion
\[ \Delta = \det(N(T) - E) = 0 \]
where $N(t)$ is the fundamental matrix of (15). However, meeting condition (16) can be associated with bifurcation of natural families and does not always mean principal impossibility to continue a natural family. While this work does not give an analytical solution to this problem, we propose a numerical approach to identify and analyze bifurcation of natural families.

Since equations (15) are solved as part of predictor system (14) it is possible to calculate (16) on each computational step $n$ and check if it approaches zero by applying a halting condition:

$$\Delta_n \leq \epsilon$$  \hspace{1cm} (17)

where $\epsilon$ is a small user-defined value. Analyzing equations (15) also allows us to investigate the periodic motions’ orbital stability, i. e. their stability in respect to normal perturbations of their orbits. For this purpose, we write out the characteristic equation of (15):

$$\rho^2 - 2A\rho + 1 = 0,$$  \hspace{1cm} (18)

where $A = \frac{1}{2} Sp[N(T)]$ ($Sp[N]$ – is trace of the matrix $N$), and $T$ is period of the periodic solution in question. If the roots $\rho_1$, $\rho_2$ of equation (18) satisfy condition $|A| < 1$ then the periodic motion is linear orbital stable.

While in a rigorous sense it is impossible to continue a family beyond a critical solution using the method proposed in this work in practice if a bifurcation occurs it is sometimes possible to ‘overstep’ the termination by choosing the parameter steps (12) so that the critical solution falls between the start and end point of the step in the problem’s parameter space. To handle this situation we check for the following conditions:

$$\text{sign}(\Delta_n) = -\text{sign}(\Delta_{n+1})$$ \hspace{1cm} (19)

$$\text{sign} \left( \frac{\partial \Delta_n}{\partial a_j} \right) = -\text{sign} \left( \frac{\partial \Delta_n}{\partial a_j} \right), (j = 1..k + 1)$$ \hspace{1cm} (20)

If (19) or at least one of the conditions (20) apply, we use the bisection method to choose the parameter step values (12) and approach the critical solution until the condition (17) is fulfilled.

When a critical solution is identified, we construct Poincare sections in its neighborhood to investigate the bifurcation pattern and obtain initial conditions for the periodic motions branching off the initial natural family. These new periodic motions are then taken as base solutions for numerical continuation. Finally, to illustrate the bifurcations we present the resulting data in form of period versus parameter graphs.

### 2.3 Motion of a dynamically symmetric satellite about its center of mass

To illustrate the proposed approach we consider motion of a rigid-body satellite about its center of mass on a circular orbit in central Newtonian gravitational field. We introduce an orbital reference frame $OXYZ$ and a body-fixed frame $Oxyz$ (figure 1). The axes $OX$, $OY$ and $OZ$ of the orbital frame are aligned with transversal and normal vectors to the satellite’s orbit and with its center of mass’ radius-vector $\vec{R}$, respectively. The axes of the body-fixed frame $Oxyz$ are pointed along the satellite’s principal axes with corresponding moments of inertia being $J_1$, $J_2$ and $J_3$. We describe the relative position of these frames by Euler angles $\psi$, $\theta$, $\varphi$.

![Figure 1. The satellite’s reference frames.](image-url)
Taking $\psi, \theta, \varphi$ as generalized coordinates and assuming that the satellite is dynamically symmetric, i. e. $I_1 = I_2$, we obtain canonical equations describing the satellite’s motion about its center of mass

$$\frac{d\psi}{dv} = \frac{\delta H}{\delta p_\psi}, \quad \frac{d\theta}{dv} = \frac{\delta H}{\delta p_\theta}, \quad \frac{d\varphi}{dv} = -\frac{\delta H}{\delta \theta}, \quad \frac{dp_\theta}{dv} = -\frac{\delta H}{\delta \varphi}$$  \tag{21}

with Hamiltonian \cite{4}:

$$H = \frac{p_\psi^2}{2 \sin^2 \theta} + \frac{p_\theta^2}{2} - \left(\frac{\gamma \cos \theta}{\sin \theta} + \frac{\cos \psi}{\tan \theta}\right) p_\psi - p_\psi \sin \psi + \frac{\gamma^2}{2 \tan^2 \theta} + \frac{\gamma \cos \psi}{\sin \theta} + \frac{1}{2} \delta \cos^2 \theta.$$  \tag{22}

where $\nu = \omega_0 t$ and $\nu$ is a cyclic coordinate so its respective impulse retains constant value $p_\psi = \frac{l_3}{l_1} r_0 = \gamma$

where $r_0 = \dot{\psi} \cos \theta - \omega_0 \cos \psi \sin \theta$ and $\omega_0$ is the angular velocity of the radius-vector $\vec{R}$. $\delta = 3 \left(\frac{l_3}{l_1} - 1\right)$ is an inertial parameter ($-3 < \delta \leq 3$).

The equations (21) possess the following particular solutions known as Regular precessions \cite{22-24}:

$$g_0 = \frac{\pi}{2}, \quad \cos \psi_0 = -\gamma, \quad p_{\theta_0} = \sin \psi_0, \quad p_{\varphi_0} = 0$$  \tag{23}

$$g_0 = \frac{\pi}{2}, \quad \psi_0 = \pi, \quad p_{\theta_0} = 0, \quad p_{\varphi_0} = 0$$  \tag{24}

$$\sin g_0 = \frac{\gamma}{\delta - 1}, \quad \psi_0 = 0, \quad p_{\theta_0} = 0, \quad p_{\varphi_0} = \delta \sin g_0 \cos g_0$$  \tag{25}

Solutions (23), (24) and (25) presented in figure 2 describe stationary motions of the satellite in the orbital reference frame and are known as Hyperboloidal, Cylindrical and Conical precession, respectively.

![Figure 2. (a) Hyperboloidal, (b) cylindrical and (c) conical precessions of a symmetric satellite.](image)

If a Regular precession is Lyapunov-stable \cite{25} there exist two types of periodic motions in its neighborhood: short-periodic motions with period close to $\frac{2\pi}{\omega_2}$ and long-periodic motions with period close to $\frac{2\pi}{\omega_1}$ where $\omega_1$ and $\omega_2$ are the frequencies of the linearized system.

3. Results and discussion

The short-periodic motions emanating from Regular precessions of a dynamically-symmetric satellite are defined by the following analytical expressions \cite{26}:

$$\psi = \psi_0 + ckA_{12}^J \sin \Omega (v - v_0) + O(c^2),$$

$$\theta = \theta_0 + ckA_{22}^J \sin \Omega (v - v_0) + O(c^2)$$

$$p_\psi = p_{\psi_0} + ckA_{32}^J \sin \Omega (v - v_0) + O(c^2),$$

$$p_{\theta_0} = p_{\theta_0} + ckA_{42}^J \sin \Omega (v - v_0) + O(c^2)$$  \tag{26}

with period $T = \frac{2\pi}{\Omega}$, where
\[ \Omega = \omega_2 + 4c^2a + O(c^4), \]

\( c(h) \) is a small parameter and \( \kappa, \alpha, A_{12}^j, A_{32}^j, A_{32}^j, A_{42}^j (j = \Gamma, Z, K) \) are coefficients depending on \( \gamma \) and \( \delta \) obtained during normalization of the Hamiltonian system (1) in the neighborhood of Regular precessions.

We use the numerical method described in paragraph 1.1 to continue motions (26) to the borders of their existence domains. Following [26] we refer to the natural families of short-periodic motions emanating from Hyperboloidal, Cylindrical and Conical precession as \( \Gamma_s, Z_s \) and \( K_s \), respectively. Figure 3 shows traces of the satellite’s principal axis on a unit sphere for motions belonging to natural families \( \Gamma_s \) (a), \( Z_s \) (b) and \( K_s \) (c) for parameter values \( \gamma = 0.5, \delta = 1.0 \) and \( \Delta h = 0.01 \) (blue curve) and \( \Delta h = 0.1 \) (red curve). \( \Delta h \) is deviation of the energy constant from its value for the corresponding Regular precession.

\[ \text{(a)} \quad \text{(b)} \quad \text{(c)} \]

**Figure 3.** Traces of the satellite’s principal axis on a unit sphere for short-periodic motions arising from (a) Hyperboloidal, (b) Cylindrical and (c) Conical precessions for \( \gamma = 0.5, \delta = 1.0 \) and \( \Delta h = 0.01 \) (blue curve) and \( \Delta h = 0.1 \) (red curve). \( \Delta h \) is deviation of the energy constant from its value for the corresponding Regular precession.

Figures 4-6 show existence and linear orbital stability domains of natural families \( \Gamma_s, Z_s \) and \( K_s \) obtained using the numerical method described in paragraph 2.2. Gray areas represent the existence domains and cross-hatched areas represent domains of orbital instability.

\[ \text{(a)} \quad \text{(b)} \quad \text{(c)} \]

**Figure 4.** Existence (grey) and orbital instability (hatched) domains of family of short-periodic motions arising from Cylindrical precession for (a) \( \delta = 0.5 \), (b) \( \delta = 1.0 \) and (c) \( \delta = 2.8 \).
The family $Z_{s}$ (figure 4) emanates from Cylindrical precession at the curve $S_{1}^{Z}$ and terminates at curve $S_{2}^{Z}$. With increase of parameter $\gamma$ it becomes orbitally unstable at curves $S_{1}^{K}$, $S_{2}^{Z}$ and $S_{2}^{Z}$ due to bifurcation either with natural family $K_{s}$ (curve $S_{1}^{K}$) or with long-periodic motions originating from Cylindrical precession. The family $Z_{s}$ then becomes linear orbitally stable at the curve $S_{1}^{K}$ due to bifurcation with natural family $\Gamma_{s}$ which emanates from Hyperboloidal precession.

The family $\Gamma_{s}$ (figure 5) emanates from Hyperboloidal precession at the curve $S_{0}^{\Gamma}$ and terminates at curve $S_{1}^{\Gamma}$. With increase of $\gamma$ it becomes orbitally unstable at curve $S_{2}^{\Gamma}$ and then becomes linear orbitally stable at curve $S_{1}^{\Gamma}$. This change in orbital stability takes place to bifurcation with long-periodic motions originating from Hyperboloidal precession. Further study of these long-periodic motions and their bifurcation can be found in \cite{27}.

The family $K_{s}$ (figure 6) is orbitally unstable for $0 < \delta < 1$. For $\delta > 1$ it exits between the curves $S_{1}^{K}$ and $S_{1}^{K}$ and is linear orbitally stable in a subdomain below the curve $S_{1}^{K}$.

![Figure 5](image1.png)

**Figure 5.** Existence (grey) and orbital instability (hatched) domains of family of short-periodic motions arising from Hyperboloidal precession for (a) $\delta = 0.5$, (b) $\delta = 1.0$ and (c) $\delta = 2.8$.

![Figure 6](image2.png)

**Figure 6.** Existence (grey) and orbital instability (hatched) domains of short-periodic motions arising from Conical precession for (a) $\delta = 0.5$ and (b) $\delta = 2.8$.

Figure 7 shows existence domains of families $\Gamma_{s}$ (inclined hatch), $Z_{s}$ (gray fill) and $K_{s}$ (horizontal hatch) plotted together for fixed value $\delta = 0.5$ and a period vs. parameter bifurcation diagram of these natural families for fixed value of the energy constant $h = 0.35$. 
In Figure 7b, $T_\gamma$, $T_\Gamma$, and $T_K$ are periods of motions belonging to families $\Gamma_s$, $Z_s$, and $K_s$, respectively. Points $P_1$ and $P_3$ belong to the curve $S_0^F$ corresponding to Cylindrical precession, point $P_2$ belongs to curve $S_2^K$ on which the family $K_s$ terminates and $B_{1,2}$ are bifurcation points belonging to curves $S_1^K$ and $S_1^F$. If we continue the natural family $Z_s$ from the point $P_3$ ($\gamma = 1.84$) where it emanates from Cylindrical precession we arrive at point $B_2$ ($\gamma = 1.11$) where a bifurcation occurs: motions belonging to family $Z_s$ become orbitally unstable and a linear orbitally stable family $\Gamma_s$ branches off from $Z_s$. Further continuation brings us to the point $B_1$ ($\gamma = 0.78$) at which an orbitally unstable family $K_s$ branches off from the family $Z_s$ with the latter becoming linear orbitally stable. Finally, natural family $Z_s$ terminates at point $P_2$ ($\gamma = 0.16$) by converging into Cylindrical precession. The family $\Gamma_s$ can be continued beyond $\gamma = 0$ and $K_s$ terminates at point $P_2$.

![Bifurcation diagram of short-periodic motion families $\Gamma_s$, $Z_s$, and $K_s$.](image)

**Figure 7.** Bifurcation diagram of short-periodic motion families $\Gamma_s$, $Z_s$, and $K_s$. Right graph shows the periods $T_\Gamma$, $T_\gamma$, $T_K$ of these motions plotted against parameter $\gamma$ for $\delta = 0.5$, $h = 0.35$. Dotted lines show periods of orbital instability.

To illustrate the bifurcation of families $\Gamma_s$, $Z_s$, and $K_s$ and identify other families of periodic motions we constructed Poincare maps in the neighborhood of points $B_1, B_2$ (Figures 8-9). On these maps $\Gamma_s$, $Z_s$, and $K_s$ refer to stationary points representing short-periodic motions which emanate from Hyperboloidal, Cylindrical and Conical precessions, respectively. Numerical analysis show that stationary points $Z_i$ ($i = 1.5$) present on these maps correspond to long-periodic motions emanating from Cylindrical precession and stationary point $K_i$ corresponds to long-periodic motion emanating from Conical precession. Stationary points $\Gamma_3$ and $\Gamma_4$ correspond to long-periodic motions emanating from Hyperboloidal precession in case of fourth-order resonance. Natural families of these motions were computed and studied in [27].

For small deviations of the energy constant $h$ from the Regular precessions the results of this numerical analysis correspond well with the analytical results presented in [2,3]. Aside from presenting interest for general research in the field of Hamiltonian mechanics, orbitally stable periodic motions obtained in this work may be useful for planning attitude motion and control for small-scale satellites in circular orbits in case it is acceptable to model the satellite with a rigid body. Since these motions are conditioned by the inherent properties of the mechanical system and do not require external energy inputs to maintain they can be potentially used for designing more propellant and energy efficient attitude maneuvers.
Figure 8. Poincare sections in the neighborhood of bifurcation points $B_1$ and $B_2$ for $\gamma = 0.77$ and $\gamma = 1.05$. $\Gamma_{3,4}$, $Z_i$ ($i = 1.5$) and $K_1$ are long-periodic motions arising from Hyperboloidal, Cylindrical and Conical precessions, respectively.

Figure 9. Poincare sections in the neighborhood of bifurcation points $B_1$ and $B_2$ for $\gamma = 0.77$ and $\gamma = 1.05$. $\Gamma_{3,4}$, $Z_i$ ($i = 1.5$) and $K_1$ are long-periodic motions arising from Hyperboloidal, Cylindrical and Conical precessions, respectively.

4. Conclusion
In this work we build upon the numerical continuation method proposed by A Sokolskiy and S Karimov and adapt it for investigating bifurcation and linear orbital stability of periodic motions of a Hamiltonian system with two degrees of freedom for all admissible values of problems’ parameters. The proposed approach involves run-time linear orbital stability analysis and identification of bifurcation points which is followed by additional bifurcation analysis using Poincare maps. The resulting bifurcation diagrams are presented as period versus parameter plots. As an application we investigated the linear orbital stability and bifurcation problems for short-periodic motions originating from Regular precessions of a symmetric satellite on circular orbit. The resulting orbital stability domains and bifurcation diagrams correspond well with the analytical conclusions obtained in earlier works. The families of orbitally-stable periodic motions obtained herein present both theoretical interest in the field of Hamiltonian mechanics and may be also used for deriving propellant efficient attitude control for small dynamically symmetric satellites on circular orbits.

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