Secondary bifurcations in semilinear ordinary differential equations

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Abstract

We consider the Neumann problem for the equation $u_{xx} + \lambda f(u) = 0$ in the punctured interval $(-1, 1) \setminus \{0\}$, where $\lambda > 0$ is a bifurcation parameter and $f(u) = u - u^3$. At $x = 0$, we impose the conditions $u(-0) + au_x(-0) = u(+0) - au_x(+0)$ and $u_x(-0) = u_x(+0)$ for a constant $a > 0$ (the symbols $+0$ and $-0$ stand for one-sided limits). The problem appears as a limiting equation for a semilinear elliptic equation in a higher dimensional domain shrinking to the interval $(-1, 1)$. First we prove that odd solutions and even solutions form families of branches $\{C^o_k\}_{k \in \mathbb{N}}$ and $\{C^e_k\}_{k \in \mathbb{N}}$, respectively. Both $C^o_k$ and $C^e_k$ bifurcate from the trivial solution $u = 0$. We then show that $C^e_k$ contains no other bifurcation point, while $C^o_k$ contains two points where secondary bifurcations occur. Finally we determine the Morse index of solutions on the branches. General conditions on $f(u)$ for the same assertions to hold are also given.

Keywords  Secondary bifurcation · Semilinear elliptic equation · Boundary value problem · Chafee–Infante problem · Matching condition

Mathematics Subject Classification 34B08 · 34B15 · 34B45 · 35J25
1 Introduction

This paper is concerned with the problem

\[
\begin{align*}
  u_{xx} + \lambda f(u) &= 0, & x &\in (-1, 1) \setminus \{0\}, \\
  u_x(-1) &= u_x(1) = 0, \\
  u(-0) + au_x(-0) &= u(+0) - au_x(+0), \\
  u_x(-0) &= u_x(+0),
\end{align*}
\]

(1.1)

where \( \lambda > 0 \) is a bifurcation parameter, \( a > 0 \) is a fixed constant and \(-0\) (resp. \(+0\)) stands for the left-hand limit (resp. the right-hand limit) as \( x \) approaches \( 0 \). Throughout the paper, we assume that \( f \) satisfies the following conditions:

\[
\begin{align*}
  f \in C^2(\mathbb{R}), \\
  f(-1) = f(0) = f(1) = 0, & f'(-1) < 0, & f'(0) > 0, & f'(1) < 0, \\
  \text{sgn}(u)f(u) > 0 & \text{ for } u \in (-1, 1) \setminus \{0\}, \\
  f(u) = -f(-u) & \text{ for } u \in (-1, 1).
\end{align*}
\]

(1.2)

Here \( \text{sgn} \) is the sign function. Our interest is the structure of solutions of (1.1) in the bifurcation diagram.

1.1 Background and known results

The problem (1.1) is related to the stationary problem of the scalar reaction-diffusion equation

\[
\begin{align*}
  u_t &= \Delta u + \lambda f(u), & (x, t) &\in \Omega \times (0, \infty), \\
  \partial_\nu u &= 0, & (x, t) &\in \Omega \times (0, \infty),
\end{align*}
\]

(1.3)

where \( \Omega \) is a bounded domain in a Euclidean space and \( \partial_\nu \) stands for the outward normal derivative. The existence of stable nonconstant stationary solutions is one of main interests in the study of (1.3). In [6, 15], it was shown that (1.3) has no stable nonconstant stationary solutions if \( \Omega \) is convex, while in [15], the existence of such solutions was proved when \( \Omega \) is a dumbbell-shaped domain. Here, by dumbbell-shaped domain, we mean a domain given by the union of two regions and a thin tubular channel connecting them. The structure of stable nonconstant stationary solutions was studied by deriving a finite-dimensional limiting equation as the thickness of the channel approaches zero. The limiting equation was obtained and examined by [25], and it was shown that a stable nonconstant stationary solution appears through a secondary bifurcation if \( \Omega \) is symmetric (see also [11]). In [17], the stability of nonconstant stationary solutions was investigated by constructing an invariant manifold (see also [8, 16, 18]).

The problem (1.1) is derived as another type of limiting equation. For small \( \varepsilon > 0 \), let \( \Omega_\varepsilon \) be a thin dumbbell-shaped domain shown in Fig. 1. Then we can expect that solutions of (1.3) are approximated by those of some equation in one space dimension, since \( \Omega_\varepsilon \) shrinks to the interval \((-1, 1)\) as \( \varepsilon \to 0 \). It is indeed shown in [13] that stationary solutions of (1.3) are approximated by solutions of (1.1) in the following sense: for any nondegenerate solution \( u \) of (1.1), there exists a corresponding stationary solution \( u_\varepsilon \) of (1.3) with \( \Omega = \Omega_\varepsilon \) such that \( u_\varepsilon \) converges to \( u \) in an appropriate sense as \( \varepsilon \to 0 \). Therefore the analysis of (1.1) provides information on the structure of solutions of (1.3).
Not only equations in higher dimensions, but also other equations relate to (1.1). Let \( u \) be a solution of (1.1) and let \( \bar{u} \) be defined by

\[
\bar{u}(x) :=
\begin{cases}
  u(x + a) & (x \in [-1 - a, -a]), \\
  \frac{u_x(+0) + u_x(-0)}{2} x + \frac{u(+0) + u(-0)}{2} & (x \in [-a, a]), \\
  u(x - a) & (x \in (a, 1 + a)).
\end{cases}
\]

Then, in a suitable sense, \( \bar{u} \) satisfies the problem

\[
\begin{cases}
  (\bar{u})_{xx} + \lambda (1 - \chi_{(-a,a)}(x)) f(\bar{u}) = 0, & x \in (-1 - a, 1 + a), \\
  (\bar{u})_x(-1 - a) = (\bar{u})_x(1 + a) = 0,
\end{cases}
\]

where \( \chi_A \) denotes the indicator function of a set \( A \). Therefore (1.1) can be regarded as one of boundary value problems for equations with variable coefficients. In many such problems, secondary bifurcations are observed. For instance, the Neumann problem for the equation

\[
(c(x)u_x)_x + \lambda (u - u^3) = 0
\]

with a piecewise constant function \( c(x) \) was studied by [10], and the Dirichlet problem for the equation

\[
u_{xx} + d(\lambda, x, u) = 0 \quad \text{with} \quad d(\lambda, x, u) = |x|^\lambda u^p, |x|^\lambda e^u, \text{or} \quad (1 - \chi_{(-\lambda, \lambda)}(x)) u^p
\]

was investigated by [12, 21, 23, 24].

We remark that the last two conditions of (1.1) also appear when we consider the Schrödinger operators with \( \delta' \)-interactions (for recent studies see [1–4, 9]).

The aim of this paper is to find bifurcation points on branches of solutions bifurcating from \( u = 0 \), and to determine the Morse index of solutions on the branches. We first observe that each branch consists of odd solutions or even solutions. Then, under additional assumptions on \( f \), we show that a secondary bifurcation occurs only on the branch of odd solutions. Finally, we prove that the Morse index of an odd solution with \( m \) zeros changes from \( m + 1 \) to \( m \) through a secondary bifurcation. In particular, we conclude that the Morse index of a monotone odd solution becomes zero after the secondary bifurcation. This fact is consistent with the result in [25].

1.2 Main result

To state our main result, we set up notation to be used throughout the paper. Set

\[
X_0 := \{ u \in C^2([-1, 1] \setminus \{0\}); u_{xx}|_{[-1,0]} \quad \text{and} \quad u_{xx}|_{(0,1]} \quad \text{are uniformly continuous}\}.
\]

If \( u \in X_0 \), then \( u|_{[-1,0]} \) (resp. \( u|_{(0,1]} \)) can be extended to a \( C^2 \) function on \([-1, 0]\) (resp. \([0, 1]\)). Hence we see that limits \( u(\pm 0) \) and \( u_x(\pm 0) \) exist for any \( u \in X_0 \) and that \( X_0 \) is a Banach space endowed with the norm

\[
\|u\|_{X_0} = \sup_{x \in [-1,1] \setminus \{0\}} |u(x)| + \sum_{k=1}^2 \sup_{x \in [-1,1] \setminus \{0\}} \left| \frac{d^k u}{dx^k}(x) \right|.
\]
We focus on solutions of (1.1) in the set \( X \subset X_0 \) defined by
\[
X := \{ u \in X_0; |u(x)| < 1 \text{ for } x \in [-1, 1] \setminus \{0\}. \]

Let \( S \) denote the set of all pairs \((\lambda, u) \in (0, \infty) \times X\) satisfying (1.1):
\[
S := \bigcup_{\lambda \in (0, \infty)} \{ \lambda \} \times S_{\lambda}, \quad S_{\lambda} := \{ u \in X; u \text{ satisfies (1.1)} \}. \]

Note that \( S \) contains a trivial branch \( \mathcal{L} := \{(\lambda, 0)\}_{\lambda \in (0, \infty)} \). We define \( S^0 \subset S \) and \( S^e \subset S \) by
\[
S^0 := \bigcup_{\lambda \in (0, \infty)} \{ \lambda \} \times S_{\lambda}^0, \quad S^e := \bigcup_{\lambda \in (0, \infty)} \{ \lambda \} \times S_{\lambda}^e, \]
\[
S_{\lambda}^0 := \{ u \in S_{\lambda}; u(-x) = -u(x) \text{ for } x \in [-1, 1] \setminus \{0\} \}, \quad S_{\lambda}^e := \{ u \in S_{\lambda}; u(-x) = u(x) \text{ for } x \in [-1, 1] \setminus \{0\} \}. \]

For a solution \( u \in S_{\lambda} \), we also discuss the linearized eigenvalue problem
\[
\begin{align*}
\varphi_{xx} + \lambda f'(u)\varphi &= \mu \varphi, \quad x \in (-1, 1) \setminus \{0\}, \\
\varphi_x(-1) &= \varphi_x(1) = 0, \\
\varphi(-0) + a\varphi_x(-0) &= \varphi(+0) - a\varphi_x(+0), \\
\varphi_x(-0) &= \varphi_x(+0).
\end{align*}
\]
(1.4)

It is shown that the set of eigenvalues of (1.4) consists of a sequence of real numbers which diverges to \(-\infty\) (see Lemma 2.7 in Sect. 2). We say that \( u \) is nondegenerate if all the eigenvalues are nonzero. The number of positive eigenvalues is called the Morse index and is denoted by \( i(u) \).

For \( n \in \mathbb{N} \), we define \( \lambda_n \in (0, \infty) \) by
\[
\lambda_n := \begin{cases} 
\frac{z_k^2}{f'(0)} & \text{if } n = 2k - 1, k \in \mathbb{N}, \\
\frac{(k\pi)^2}{f'(0)} & \text{if } n = 2k, k \in \mathbb{N}, 
\end{cases}
\]
where \( z_k \) denotes the unique root of the equation \( az \tan z = 1 \) in \(( (k-1)\pi, (k-1/2)\pi )\).

Note that \( \lambda_n < \lambda_{n+1} \). We set
\[
F(u) := 2 \int_0^u f(s) \, ds,
\]
and fix \( u_0 \in (0, 1) \).

The main result of this paper is stated as follows.

**Theorem 1.1** In addition to (1.2), assume that the following conditions are satisfied:

\[
f'(u) \begin{cases} > 0 & \text{if } u \in (-u_0, u_0), \\
< 0 & \text{if } u \in (-1, -u_0) \cup (u_0, 1), \end{cases} \quad (1.5a)
\]

\[
\text{sgn}(u) \frac{f'(u)F(u)}{f(u)^2} \text{ is strictly decreasing in } (-u_0, u_0) \setminus \{0\}, \quad (1.5b)
\]

\[
\text{sgn}(u) \frac{d}{du} \left( \frac{f'(u)F(u)^3}{f(u)^3} \right) \leq 0 \text{ for } u \in (-1, -u_0) \cup (u_0, 1). \quad (1.5c)
\]
Then for every $n \in \mathbb{N}$, there exists $C_n \subset S$ with the following properties.

(i) $C_n$ is a $C^1$ curve in $(0, \infty) \times X_0$ parametrized by $\lambda \in (\lambda_n, \infty)$ and bifurcates from $(\lambda_n, 0)$. Moreover,

$$S^o = \bigcup_{k=1}^{\infty} C_{2k-1} \cup C_{2k-1}^*, \quad S^e = \bigcup_{k=1}^{\infty} C_{2k} \cup C_{2k}^*,$$

where $C_n^*$ is defined by $C_n^* := \{(\lambda, -u); (\lambda, u) \in C_n\}$.

(ii) There is no bifurcation point on $L \setminus \{(\lambda_n, 0)\}_{n \in \mathbb{N}}$ and $S^e$.

(iii) For every odd number $n \in \mathbb{N}$, the curve $C_n$ (resp. $C_n^*$) has a unique bifurcation point $(\lambda_n^*, u_n^*)$ (resp. $(\lambda_n^*, -u_n^*)$). Furthermore, bifurcating solutions form a $C^1$ curve in a neighborhood of each bifurcation point.

(iv) Let $(\lambda, u) \in C_n \cup C_n^*$. If $n$ is even, then $u$ is nondegenerate and $i(u) = n$; if $n$ is odd, then $u$ is nondegenerate unless $\lambda = \lambda_n^*$ and

$$i(u) = \begin{cases} n & (\lambda < \lambda_n^*), \\ n - 1 & (\lambda \geq \lambda_n^*). \end{cases}$$

Here $\lambda_n^*$ is the number given in (iii).

The bifurcation diagram of (1.1) is shown in Fig. 2.

**Remark 1.2** $f(u) = u - u^3$ and $f(u) = \sin \pi u$ are typical examples satisfying (1.5), namely all of the conditions (1.5a), (1.5b) and (1.5c).

The assumption (1.5) is used to show the uniqueness of bifurcation points on $C_{2k-1}$. The assertions (i) and (ii) are verified under the weaker assumption

$$\frac{f'(u)u}{f(u)} < 1 \text{ for } u \in (-1, 1) \setminus \{0\}. \quad (1.6)$$

The fact that (1.5) implies (1.6) is not difficult to check. For the convenience of the reader, we give the proof of this fact in Appendix A.

Let $n$ be even and let $(\lambda, u) \in C_n$. Then we have $u_x(+0) = u_x(-0) = 0$, since $u$ is even. Combining this with (1.1) shows that $u$ can be extended smoothly up to $x = 0$. Therefore $u$ coincides with a solution $\tilde{u}$ of the usual Neumann problem for the equation $\tilde{u}_{xx} + \lambda f(\tilde{u}) = 0$ in $(-1, 1)$. It is, however, not obvious that $u$ and $\tilde{u}$ have the same Morse index, since the corresponding linearized eigenvalue problems are different.

The main task in the proof of Theorem 1.1 is to investigate the behavior of eigenvalues of (1.4). For $(\lambda, u) \in C_{2k-1}$ and $n \in \mathbb{N} \cup \{0\}$, let $\mu_n(\lambda)$ denote the $(n+1)$-th largest eigenvalue of (1.4). Then the condition $\mu_n(\lambda_0) = 0$ is necessary in order for a bifurcation to occur at $(\lambda_0, u)$. A theory of local bifurcations shows that $(\lambda_0, u)$ is indeed a bifurcation point if the additional condition $d\mu_n(\lambda_0)/d\lambda \neq 0$ is satisfied. The main difficulty arises from the verification of this condition. We will show in Sects. 4 and 5 that the assumption (1.5) gives $d\mu_{2k-2}(\lambda^*)/d\lambda < 0$ for any $\lambda^*$ satisfying $\mu_{2k-2}(\lambda^*) = 0$. This enables us to conclude that a secondary bifurcation occurs on $C_{2k-1}$ just once.

This paper is organized as follows. Section 2 provides some preliminaries. A main task is to reduce the problem (1.1) to a finite dimensional problem. In Sect. 3, we study primary branches bifurcating from the trivial branch $L$. In Sect. 4, we discuss the number of bifurcation points on the primary branches. Section 5 is devoted to the evaluation of the Morse index and the proof of Theorem 1.1.
2 Preliminaries

In this section, we first introduce a function $G$ to be used throughout the study, next convert (1.1) into a finite dimensional problem by the shooting method, then study general properties of eigenpairs of (1.4), and finally give sufficient conditions for a bifurcation by applying a bifurcation theorem developed in [7].

2.1 Definition and properties of $G$

Set
\[ \beta_0 := \sqrt{2 \int_0^1 f(s)ds} \quad (= \sqrt{F(1)} = \sqrt{F(-1)}) \quad , \quad I := (-\beta_0, \beta_0). \]

From (1.2), we see that the function
\[ (-1, 1) \ni u \mapsto \text{sgn}(u)\sqrt{F(u)} \in I \]

is strictly increasing. We then define a function $G : I \rightarrow (-1, 1)$ to be the inverse of this function, that is, $G(v)$ is determined by the relation
\[ F(G(v)) = v^2, \quad |G(v)| < 1, \quad G(v) \begin{cases} < 0 & \text{if } v \in (-\beta_0, 0), \\ > 0 & \text{if } v \in (0, \beta_0). \end{cases} \]

Furthermore, we set
\[ h(v) := 1 - \frac{G''(v)v}{G'(v)}, \quad H(v) := v^2 - \frac{G(v)v}{G'(v)}. \]

In the following lemma, we collect properties and formulas for $G$. Although part of the lemma is shown in [20], we prove all the assertions in Appendix B for readers’ convenience.
Lemma 2.1 The following hold.

(i) $G(v) \in C^2(I) \cap C^3(I \setminus \{0\})$ and $G''(v)v \in C^1(I)$.

(ii) (1.5a), (1.5b) and (1.5c) are equivalent to the following conditions (2.3a), (2.3b) and (2.3c), respectively:

$$h(v) \begin{cases} > 0 & \text{if } v \in (-v_0, v_0), \\ < 0 & \text{if } v \in (-\beta_0, -v_0) \cup (v_0, \beta_0), \end{cases}$$

$$\text{sgn}(v)h(v) \text{ is strictly decreasing in } (-v_0, v_0) \setminus \{0\},$$

$$\text{sgn}(v)\frac{d}{dv}(G'(v)h(v)) \leq 0 \text{ for } v \in (-\beta_0, -v_0) \cup (v_0, \beta_0).$$

Here $v_0 = G^{-1}(u_0)$. Moreover, (1.6) holds if and only if

$$\text{sgn}(v)H'(v) > 0 \text{ for } v \in I \setminus \{0\}.$$  

(iii) There are constants $c > 0$ and $C > 0$ such that for all $v \in I$,

$$c \leq G'(v) \leq C \sqrt{\beta_0 - |v|},$$

$$\text{sgn}(v)G''(v) \geq \frac{c}{(\beta_0 - |v|)^{3/2}} - C.$$  

The function $G$ is introduced in [20, 26, 27] to obtain a simple expression of solutions of the equation $w_{xx} + f(w) = 0$. A solution $w$ satisfying $|w| < 1$ corresponds to the closed orbit $w_0^2 + F(w) = c_0$ in the $ww_x$-plane, where $c_0$ is a nonnegative constant. By the change of variables $w = G(\tilde{w})$, the orbit is transformed into the circle $\tilde{w}^2 + \tilde{w}^2 = c_0$ in the $\tilde{w}w_x$-plane. Hence $w$ is written as $w(x) = G(\sqrt{c_0} \cos \rho(x))$ for some suitable function $\rho(x)$. The details of this argument are given in the next subsection.

2.2 Reduction to a finite dimensional problem

For $\beta_1, \beta_2 \in I$ and $\lambda \in (0, \infty)$, let $u_1$ and $u_2$ be solutions of the initial value problems

$$\begin{cases} (u_1)_{xx} + \lambda f(u_1) = 0, & x \in \mathbb{R}, \\ u_1(-1) = G(\beta_1), & (u_1)_x(-1) = 0, \end{cases} \quad \begin{cases} (u_2)_{xx} + \lambda f(u_2) = 0, & x \in \mathbb{R}, \\ u_2(1) = G(\beta_2), & (u_2)_x(1) = 0. \end{cases}$$

Then the function $u$ defined by

$$u(x) = \begin{cases} u_1(x) & (x \in [-1, 0)), \\ u_2(x) & (x \in (0, 1]) \end{cases}$$

satisfies (1.1) if and only if

$$\begin{cases} u_1(0) + a(u_1)_x(0) = u_2(0) - a(u_2)_x(0), \\ (u_1)_x(0) = (u_2)_x(0). \end{cases}$$

Now we introduce a solution $(U, V) = (U(y, \beta), V(y, \beta))$ of the initial value problem

$$\begin{cases} U_y = V, & y \in \mathbb{R}, \\ V_y = -f(U), & y \in \mathbb{R}, \\ (U(0, \beta), V(0, \beta)) = (G(\beta), 0). \end{cases}$$
Then we have
\[ u_1(x) = U\left(\sqrt{\lambda}(x+1), \beta_1\right), \quad u_2(x) = U\left(\sqrt{\lambda}(1-x), \beta_2\right), \]
\[ (u_1)_x(x) = \sqrt{\lambda}V\left(\sqrt{\lambda}(x+1), \beta_1\right), \quad (u_2)_x(x) = -\sqrt{\lambda}V\left(\sqrt{\lambda}(1-x), \beta_2\right). \quad (2.10) \]
Hence (2.8) is equivalent to the equation
\[ (P(\lambda, \beta_1), Q(\lambda, \beta_1)) = (P(\lambda, \beta_2), -Q(\lambda, \beta_2)). \quad (2.11) \]
where
\[ P(\lambda, \beta) := U(\sqrt{\lambda}, \beta) + a\sqrt{\lambda}V(\sqrt{\lambda}, \beta), \quad Q(\lambda, \beta) := V(\sqrt{\lambda}, \beta). \]
We remark that
\[ U(x, -\beta) = -U(x, \beta), \quad V(x, -\beta) = -V(x, \beta), \quad (2.12) \]
\[ P(\lambda, -\beta) = -P(\lambda, \beta), \quad Q(\lambda, -\beta) = -Q(\lambda, \beta), \quad (2.13) \]
which follow from the fact that \( f \) and \( G \) are odd functions.
Let us investigate the relation between (1.1) and (2.11). The set of solutions of (2.11) is denoted by
\[ T := \bigcup_{\lambda \in (0, \infty)} \{\lambda\} \times T_\lambda, \]
\[ T_\lambda := \{(\beta_1, \beta_2) \in I \times I; (P(\lambda, \beta_1), Q(\lambda, \beta_1)) = (P(\lambda, \beta_2), -Q(\lambda, \beta_2))\}. \]
We define \( C^2 \) mappings \( \Phi : X \to I \times I \) and \( \Psi_\lambda : I \times I \to X \) by
\[ \Phi(u) := \left(G^{-1}(u(-1)), G^{-1}(u(1))\right), \]
\[ \Psi_\lambda(\beta_1, \beta_2)(x) := \begin{cases} U\left(\sqrt{\lambda}(x+1), \beta_1\right) (= u_1(x)) & \text{for } x \in [-1, 0), \\ U\left(\sqrt{\lambda}(1-x), \beta_2\right) (= u_2(x)) & \text{for } x \in (0, 1]. \end{cases} \]
We note that
\[ \Psi_\lambda(-\beta_1, -\beta_2)(x) = -\Psi_\lambda(\beta_1, \beta_2)(x), \quad (2.14) \]
which is obtained immediately by (2.12).
By the following lemma, we see that (1.1) is reduced to the finite dimensional problem (2.11).

**Lemma 2.2** \( \Phi|_{S_\lambda} \) and \( \Psi_\lambda|_{T_\lambda} \) are one-to-one correspondences between \( S_\lambda \) and \( T_\lambda \). More precisely, \( \Phi(S_\lambda) = T_\lambda, \Psi_\lambda(T_\lambda) = S_\lambda, \Psi_\lambda \circ \Phi|_{S_\lambda} = \text{id}_{S_\lambda} \) and \( \Phi \circ \Psi_\lambda|_{T_\lambda} = \text{id}_{T_\lambda} \).

**Proof** That \( \Psi_\lambda(T_\lambda) \subset S_\lambda \) and \( \Phi \circ \Psi_\lambda|_{T_\lambda} = \text{id}_{T_\lambda} \) follow from the definitions of \( \Phi \) and \( \Psi_\lambda \). To see \( \Phi(S_\lambda) \subset T_\lambda \) and \( \Psi_\lambda \circ \Phi|_{S_\lambda} = \text{id}_{S_\lambda} \), let \( u \in S_\lambda \) and \((\beta_1, \beta_2) = \Phi(u)\). Then, by the first two conditions of (1.1), we see that \( u_1 := u|_{[-1,0]} \) and \( u_2 := u|_{[0,1]} \) satisfy (2.7). Hence \( u_1 \) and \( u_2 \) are given by (2.10), which yields \((\Psi_\lambda \circ \Phi)(u) = u\). Since the last two conditions of (1.1) lead to (2.8), we deduce that \( \Phi(u) = (\beta_1, \beta_2) \in T_\lambda \). We have thus proved \( \Phi(S_\lambda) \subset T_\lambda \) and \( \Psi_\lambda \circ \Phi|_{S_\lambda} = \text{id}_{S_\lambda} \). This completes the proof. \( \square \)
The nondegeneracy of a solution of (1.1) can also be determined by the corresponding solution of (2.11). We set

\[ D(\lambda, \beta_1, \beta_2) := \det \begin{pmatrix} P_\beta(\lambda, \beta_1) - P_\beta(\lambda, \beta_2) \\ Q_\beta(\lambda, \beta_1) - Q_\beta(\lambda, \beta_2) \end{pmatrix} = P_\beta(\lambda, \beta_1)Q_\beta(\lambda, \beta_2) + Q_\beta(\lambda, \beta_1)P_\beta(\lambda, \beta_2), \]

where \( P_\beta \) and \( Q_\beta \) stand for derivatives with respect to \( \beta \).

**Lemma 2.3** Let \( u \in S_\lambda \) and put \( (\beta_1, \beta_2) = \Phi(u) \in T_\lambda \). Then \( u \) is nondegenerate if and only if \( D(\lambda, \beta_1, \beta_2) \neq 0 \).

**Proof** Let \( u_1 \) and \( u_2 \) be given by (2.10) and define

\[ \varphi_j := \frac{1}{G'(\beta_j)} \frac{\partial u_j}{\partial \beta_j}, \quad j = 1, 2. \] (2.15)

Then, by the definitions of \( P \) and \( Q \), we have

\[ P_\beta(\lambda, \beta_j) = G'(\beta_j) \left[ \varphi_j + (-1)^{j-1}a(\varphi_j)_x \right] \bigg|_{x=0}, \quad Q_\beta(\lambda, \beta_j) = \frac{(-1)^{j-1}G'(\beta_j)}{\sqrt{\lambda}}(\varphi_j)_x \big|_{x=0}. \] (2.16)

Hence the condition \( D(\lambda, \beta_1, \beta_2) = 0 \) holds if and only if there is \((\alpha_1, \alpha_2) \neq (0, 0)\) such that

\[ \left\{ \begin{array}{l} \alpha_1 \varphi_1 + a(\varphi_1)_x \big|_{x=0} = \alpha_2 \varphi_2 - a(\varphi_2)_x \big|_{x=0}, \\
\alpha_1(\varphi_1)_x \big|_{x=0} = \alpha_2(\varphi_2)_x \big|_{x=0}, \end{array} \right. \] (2.17)

which means that \( (P_\beta(\lambda, \beta_1), Q_\beta(\lambda, \beta_1)) \) and \( (-P_\beta(\lambda, \beta_2), Q_\beta(\lambda, \beta_2)) \) are linearly dependent.

Suppose that \( D(\lambda, \beta_1, \beta_2) = 0 \), that is, the condition (2.17) holds for some \((\alpha_1, \alpha_2) \neq (0, 0)\). By definition, we see that \( \varphi_j \) satisfies

\[ (\varphi_j)_{xx} + \lambda f'(u_j)\varphi_j = 0, \quad (\varphi_j)_x \big|_{x=1} = 1, \quad (\varphi_j)_x \big|_{x=-1} = 0. \] (2.18)

We define \( \varphi \in X_0 \setminus \{0\} \) by

\[ \varphi(x) = \left\{ \begin{array}{ll} \alpha_1 \varphi_1(x) & (x \in [-1, 0)), \\
\alpha_2 \varphi_2(x) & (x \in (0, 1]). \end{array} \right. \] (2.19)

Then we see from (2.17) and (2.18) that \( \varphi \) satisfies (1.4) for \( \mu = 0 \). This means that \( u \) is not nondegenerate.

Suppose conversely that \( u \) is not nondegenerate, that is, there is \( \varphi \in X_0 \setminus \{0\} \) satisfying (1.4) for \( \mu = 0 \). If \( \varphi \) vanished at \( x = -1 \) or \( x = 1 \), we would have \( \varphi \equiv 0 \) due to the uniqueness of solutions of initial value problems. Hence both \( \varphi(-1) \) and \( \varphi(1) \) are nonzero. We put \((\alpha_1, \alpha_2) = (\varphi(-1), \varphi(1)) \neq (0, 0)\). From (1.4) and (2.18), we see that \( \varphi \) and \( \alpha_j \varphi_j \) satisfy the same initial value problem \( \varphi_{xx} + \lambda f'(u_j)\varphi = 0, \varphi \big|_{x=1} = \alpha_j, \varphi \big|_{x=-1} = 0 \). Therefore (2.19) holds. Substituting (2.19) into the last two conditions of (1.4), we have (2.17). Thus \( D(\lambda, \beta_1, \beta_2) = 0 \), and the lemma follows.

Let us derive explicit formulas for \( P \) and \( Q \) by solving (2.9). For \((y, \beta) \in \mathbb{R} \times I\), let \( \Theta \in \mathbb{R} \) be given implicitly by the relation

\[ \int_0^\Theta G'(\beta \cos \tau) \tau = y. \] (2.20)
Since $G'$ is positive, the left-hand side is strictly increasing with respect to $\Theta$ and diverges to $\pm \infty$ as $\Theta \to \pm \infty$. Hence $\Theta = \Theta(y, \beta)$ is uniquely determined for every $(y, \beta) \in \mathbb{R} \times I$. Moreover, by the implicit function theorem, we infer that $\Theta$ is of class $C^1$. We put $\theta(\lambda, \beta) := \Theta(\sqrt{\lambda}, \beta)$, that is, $\theta$ is determined by
\[ \int_0^\theta G'(\beta \cos \tau) d\tau = \sqrt{\lambda}. \] (2.21)
We note that $\theta > 0$.

**Lemma 2.4** $P$ and $Q$ are written as
\[ P = G(\beta \cos \theta) - a \sqrt{\lambda} \beta \sin \theta, \quad Q = -\beta \sin \theta. \] (2.22)

Furthermore,
\[
\begin{cases}
  P_\beta = G'(\beta \cos \theta) \cos \theta - a \sin \theta \int_0^\theta G'(\beta \cos \tau) d\tau \\
  + \left( \beta \sin \theta + a \frac{\beta \cos \theta}{G'(\beta \cos \theta)} \int_0^\theta G'(\beta \cos \tau) d\tau \right) \int_0^\theta G''(\beta \cos \tau) \cos \tau d\tau, \\
  Q_\beta = -\sin \theta + \frac{\beta \cos \theta}{G'(\beta \cos \theta)} \int_0^\theta G''(\beta \cos \tau) \cos \tau d\tau.
\end{cases}
\] (2.23)

**Proof** Define
\[ (U, V) := (G(\beta \cos \Theta), -\beta \sin \Theta). \]

Differentiating both sides of (2.20) yields $G'(\beta \cos \Theta) \Theta_y = 1$, and hence
\[ U_y = G'(\beta \cos \Theta) \cdot (-\beta \sin \Theta) \cdot \Theta_y = V, \]
\[ V_y = (-\beta \cos \Theta) \cdot \Theta_y = -\frac{\beta \cos \Theta}{G'(\beta \cos \Theta)} = -f(U), \]
where we have used (B.1) in deriving the last equality. Since $\Theta|_{y=0} = 0$, we have $(U, V)|_{y=0} = (G(\beta), 0)$. This shows that $(U, V)$ satisfies (2.9), and therefore we obtain (2.22).

By (2.22), we have
\[
\begin{cases}
  P_\beta = G'(\beta \cos \theta)(\cos \theta - \beta \theta \beta \sin \theta) - a \sqrt{\lambda}(\sin \theta + \beta \theta \beta \cos \theta), \\
  Q_\beta = -\sin \theta - \beta \theta \beta \cos \theta.
\end{cases}
\] (2.24)

Differentiating (2.21) gives
\[ \theta_\beta = -\frac{1}{G'(\beta \cos \theta)} \int_0^\theta G''(\beta \cos \tau) \cos \tau d\tau. \] (2.25)

(2.23) is then derived easily by plugging (2.21) and (2.25) into (2.24). \qed

**2.3 Properties of eigenpairs of (1.4)**

We recall known facts for the eigenvalue problem
\[
\begin{cases}
  \psi_{xx} + q(x) \psi = \nu w(x) \psi, & x \in (b, c), \\
  \psi_x(b) = \psi_x(c) = 0,
\end{cases}
\] (2.26)
where \( q \) and \( w \) are given functions and \( b, c \in \mathbb{R}, b < c \). Since we deal with the case where \( q \) and \( w \) are not necessarily continuous, we consider eigenfunctions in a generalized sense: by an eigenfunction of (2.26) we mean a function \( \psi \in W^{2,1}(b, c) \setminus \{0\} \) satisfying the differential equation in (2.26) almost everywhere and the boundary condition in the usual sense (note that \( W^{2,1}(b, c) \subset C^1([b, c]) \)).

**Theorem 2.5** ([5]) Suppose that \( q \) and \( w \) satisfy
\[
\begin{align*}
q, w & \in L^1(b, c), \\
w & \geq 0 \text{ in } (b, c), \\
w & > 0 \text{ in } (b, b + \delta) \cup (c - \delta, c) \text{ for some } \delta > 0, \\
\{w = 0\} & \subset \{q = 0\}. 
\end{align*}
\]
(2.27)

Then the following hold.

(i) The set of eigenvalues of (2.26) consists of real numbers \( \{\nu_n\}_{n=0}^\infty \) satisfying
\[
\nu_n > \nu_{n+1}, \quad \nu_n \to -\infty \quad (n \to \infty).
\]
Furthermore, each eigenvalue is simple, that is, the eigenspace associated with \( \nu_n \) is one-dimensional.

(ii) Any eigenfunction corresponding to \( \nu_n \) has exactly \( n \) zeros in \( (b, c) \).

For the proof of this theorem, see [5, Theorems 8.4.5 and 8.4.6].

In what follows, we denote by \( \mathcal{E} \) the set of all pairs \((q, w)\) satisfying (2.27), and write \( \nu_n(q) \) instead of \( \nu_n \) if we emphasize the dependence on \( q \).

**Lemma 2.6** For \( n \in \mathbb{N} \cup \{0\} \), the following hold.

(i) Let \((q, w), (\tilde{q}, \tilde{w}) \in \mathcal{E}\). Suppose that \( q \geq \tilde{q} \) in \((b, c)\) and \( q > \tilde{q} \) in some nonempty open subinterval of \((b, c)\). Then \( \nu_n(q) > \nu_n(\tilde{q}) \).

(ii) For any \( \varepsilon > 0 \) and \((q, w) \in \mathcal{E}\), there exists \( \delta > 0 \) such that \( |\nu_n(q) - \nu_n(\tilde{q})| < \varepsilon \) whenever \((\tilde{q}, \tilde{w}) \in \mathcal{E}\) and \( \|q - \tilde{q}\|_{L^1(b,c)} < \delta \). In other words, the mapping \( q \mapsto \nu_n(q) \) is continuous.

In the case where \( w \) is positive, the above lemma is well-known and can be shown by an argument based on Prüfer’s transformation. It is not difficult to check that the same argument works under the conditions stated in the lemma. We omit the detailed proof.

Let us examine the properties of eigenvalues of (1.4) using Theorem 2.5 and Lemma 2.6. For \( u \in X_0 \), we define a function \( \overline{u} \) by
\[
\overline{u}(x) := \begin{cases}
  u(x + a) & (x \in [-1 - a, -a)), \\
  \frac{u_x(+0) + u_x(-0)}{2} x + \frac{u(+0) + u(-0)}{2} & (x \in [-a, a]), \\
  u(x - a) & (x \in (a, 1 + a)).
\end{cases}
\]

Then it is straightforward to show that if \( u(-0) + a u_x(-0) = u(+0) - a u_x(+0) \) and \( u_x(-0) = u_x(+0) \), then \( \overline{u} \in W^{2,\infty}(-1 - a, 1 + a) \). We set
\[
w_0(x) := \begin{cases}
  1 & (x \in [-1 - a, -a) \cup (a, 1 + a]), \\
  0 & (x \in [-a, a]).
\end{cases}
\]
Lemma 2.7 Let $u \in X_0$. Then the set of eigenvalues of (1.4) consists of an infinite sequence of real numbers which diverges to $-\infty$. Furthermore, each eigenvalue is simple and continuous with respect to $(\lambda, u) \in (0, \infty) \times X_0$.

Proof We put

$$b = -1 - a, \quad c = 1 + a, \quad q = \lambda w_0 f'(\overline{u}), \quad w = w_0,$$

and prove the lemma by showing that there is a one-to-one correspondence between the set of eigenpairs of (1.4) and that of (2.26).

We assume that $(\mu, \varphi)$ is an eigenpair of (1.4). Then, by definition, we see that $\overline{\varphi}$ satisfies $\overline{\varphi} \in W^{2, \infty}(-1 - a, 1 + a)$, $(\overline{\varphi})_x(-1 - a) = (\overline{\varphi})_x(1 + a) = 0$ and

$$(\overline{\varphi})_{xx} + \lambda f'(\overline{u})\overline{\varphi} = \mu\overline{\varphi} \quad \text{in} \quad (-1 - a, -a) \cup (a, 1 + a), \quad (\overline{\varphi})_{xx} = 0 \quad \text{in} \quad (-a, a).$$

This means that $(\mu, \overline{\varphi})$ is an eigenpair of (2.26).

Conversely, we assume that $(\nu, \psi)$ is an eigenpair of (2.26). We define $\underline{\psi}$ by

$$\underline{\psi}(x) = \begin{cases} \psi(x - a) & (x \in [-1, 0)), \\ \psi(x + a) & (x \in (0, 1]). \end{cases}$$

Then we have $\underline{\psi} \in X_0$, because $q = \lambda w_0 f'(\overline{u})$ and $w = w_0$ are uniformly continuous on $[-1 - a, a) \cup (a, 1 + a]$, which shows that $\psi_x(= -q\psi + vw\psi)$ is also uniformly continuous on the same region. From (2.26), we see that the first two of (1.4) are satisfied for $(\mu, \varphi) = (\nu, \underline{\psi})$. Since $q = \lambda w_0 f'(\overline{u})$ and $w = w_0$ vanish in $(-a, a)$, we have $\psi_{xx} = 0$ in $(-a, a)$. Hence $\psi$ is a linear function in $(-a, a)$. In particular,

$$\psi_x(-a) = \psi_x(a) = \frac{\psi(a) - \psi(-a)}{2a} \quad (\text{the slope of the graph of } \psi \text{ in } (-a, a)).$$

This shows that the last two of (1.4) are satisfied for $(\mu, \varphi) = (\nu, \underline{\psi})$. We thus conclude that $(\nu, \underline{\psi})$ is an eigenpair of (1.4).

From the above argument, we deduce that the set of eigenpairs of (1.4) is given by

$$\{(\nu, \underline{\psi}) ; (\nu, \underline{\psi}) \text{ is an eigenpair of (2.26)}\}.$$  

Therefore we obtain the desired conclusion by (i) of Theorem 2.5 and (ii) of Lemma 2.6. □

From now on, for $n \in \mathbb{N} \cup \{0\}$, let $\mu_n(u)$ stand for the $(n + 1)$-th largest eigenvalue of (1.4).

Lemma 2.8 Let $\varphi$ be an eigenfunction of (1.4) corresponding to $\mu_n(u)$. Then

$$\varphi(-1) \varphi(1) \begin{cases} > 0 & \text{if } n \text{ is even}, \\ < 0 & \text{if } n \text{ is odd}. \end{cases}$$

Proof As shown in the proofs of Lemmas 2.3 and 2.7, we know that $\varphi(-1) \varphi(1)$ is nonzero and that $(\mu_n(u), \overline{\varphi})$ is the $(n + 1)$-th eigenpair of (2.26) with $b, c, q, w$ given by (2.28). By (ii) of Theorem 2.5, we see that $\overline{\varphi}$ has exactly $n$ zeros in $(-1 - a, 1 + a)$. Hence $\overline{\varphi}(-1 - a) = \varphi(-1)$ and $\overline{\varphi}(-1 - a) = \varphi(1)$ have the same sign if $n$ is even and have opposite signs if $n$ is odd. □

We conclude this subsection with a lemma which provides basic estimates of the Morse index.
Then the following hold.

Lemma 2.9 Suppose that (1.6) holds and that $u \in S_\lambda \setminus \{0\}$ vanishes at exactly $n$ points in $(-1, 1) \setminus \{0\}$. Then $\mu_{n+1}(u) < 0$ if $u(-0)u(+0) < 0$ and $\mu_n(u) < 0$ if $u(-0)u(+0) > 0$.

Proof Let $b = -1 - a$, $c = 1 + a$, $w = w_0$ and set

$$q := \begin{cases} \lambda w_0 \frac{f'(\overline{u})}{\overline{u}} & \text{if } \overline{u} \neq 0, \\ \lambda w_0 f'(0) & \text{if } \overline{u} = 0, \end{cases} \quad \tilde{q} := \lambda w_0 f'(\overline{u}).$$

Then one can easily check that $(q, w), (\tilde{q}, w) \in \mathcal{E}$. From the assumptions $u \neq 0$ and (1.6), we see that $q \geq \tilde{q}$ in $(b, c)$ and $q > \tilde{q}$ in a nonempty open subinterval of $(b, c)$. Hence (i) of Lemma 2.6 shows that $v_j(q) > v_j(\tilde{q})$ for all $j$. As shown in the proof Lemma 2.7, we know that the set of eigenpairs of (1.4) is given by (2.29). In particular, we have $\mu_j(u) = v_j(\tilde{q})$. Therefore the lemma is proved if we show that

$$v_{n+1}(q) = 0 \text{ if } u(-0)u(+0) < 0, \quad v_n(q) = 0 \text{ if } u(-0)u(+0) > 0.$$ \hspace{1cm} (2.30)

The assumption $u \in S_\lambda \setminus \{0\}$ shows that $\overline{u} \in W^{2, \infty}(-1 - a, 1 + a) \setminus \{0\}$ and

$$\begin{cases} (\overline{u})_x x + \lambda w_0(x) f'(\overline{u}) = 0, \\ (\overline{u})_x(-1-a) = (\overline{u})_x(1+a) = 0. \end{cases} \quad x \in (-1 - a, 1 + a),$$

Note that the above equation is written as $(\overline{u})_x x + q(x)\overline{u} = 0$. Hence we infer that $v_m(q) = 0$ for some $m$ and $\overline{u}$ is an eigenfunction corresponding to $v_m(q) = 0$. Since $u$ is assumed to vanish at exactly $n$ points in $(-1, 1) \setminus \{0\}$, we deduce that

$$(\text{the number of zeros of } \overline{u} \text{ in } (-1 - a, 1 + a)) = \begin{cases} n + 1 & \text{if } u(-0)u(+0) < 0, \\ n & \text{if } u(-0)u(+0) > 0. \end{cases}$$

By (ii) of Theorem 2.5, we conclude that $m = n + 1$ if $u(-0)u(+0) < 0$ and $m = n$ if $u(-0)u(+0) > 0$. Thus (2.30) is verified, and the proof is complete. \hfill \square

2.4 Conditions for a bifurcation

In this subsection, we observe that sufficient conditions for a solution to be a bifurcation point are described by means of $D(\lambda, \beta_1, \beta_2)$. The precise statement is given as follows.

Proposition 2.10 Let $J \subset (0, \infty)$ be an open interval containing a point $\lambda_0$ and suppose that $C = \{(\lambda, u(\cdot, \lambda))| \lambda \in J\} \subset S$ is a $C^1$ curve in $(0, \infty) \times X_0$. Set $(\beta_1(\lambda), \beta_2(\lambda)) := \Phi(u(\cdot, \lambda))$. Then the following hold.

(i) If $D(\lambda_0, \beta_1(\lambda_0), \beta_2(\lambda_0)) \neq 0$, then there is a neighborhood $\mathcal{N}$ of $(\lambda_0, u(\cdot, \lambda_0))$ in $(0, \infty) \times X_0$ such that $S \cap \mathcal{N} = C \cap \mathcal{N}$.

(ii) Suppose that

$$D(\lambda_0, \beta_1(\lambda_0), \beta_2(\lambda_0)) = 0, \quad \frac{d}{d\lambda} D(\lambda, \beta_1(\lambda), \beta_2(\lambda)) \bigg|_{\lambda = \lambda_0} \neq 0.$$ \hspace{1cm} (2.31)

Then there exists $\tilde{C} \subset S$ such that

$\tilde{C}$ is a $C^1$ curve in $(0, \infty) \times X_0$ intersecting $C$ transversally at $(\lambda_0, u(\cdot, \lambda_0))$, $S \cap \mathcal{N} = (C \cup \tilde{C}) \cap \mathcal{N}$ for some neighborhood $\mathcal{N}$ of $(\lambda_0, u(\cdot, \lambda_0))$ in $(0, \infty) \times X_0$. \hfill \square
Let $C$ and $Y$ be Banach spaces, $J$ an open interval and $F = F(\lambda, w)$ a $C^1$ mapping from $J \times X$ to $Y$. Assume that
\[ D^2_{ww}F \text{ and } D^2_{www}F \text{ exist and continuous in } J \times X, \]
\[ \langle F(\lambda, 0), 0 \rangle = 0 \text{ for all } \lambda \in J, \]
\[ \dim K(D_wF(\lambda_0, 0)) = \text{codim } R(D_wF(\lambda_0, 0)) = 1 \text{ for some } \lambda_0 \in J, \]
\[ D^2_{ww}F(\lambda_0, 0)\varphi_0 \notin R(D_wF(\lambda_0, 0)), \text{ where } K(D_wF(\lambda_0, 0)) = \text{span}\{\varphi_0\}. \]

Then there exist an open interval $\tilde{J}$ containing 0, a $C^1$ curve $\{(\lambda(s), W(s))\}_{s \in \tilde{J}} \subset J \times X$ and a neighborhood $N \subset J \times X$ of $(\lambda_0, 0)$ such that
\[ \{(\lambda(s), W(s))\}_{s \in \tilde{J}} \cap N = \{(\lambda_0, 0)\}_{\lambda \in J} \cup \{(\lambda(s), W(s))\}_{s \in \tilde{J}} \cap N, \]
where $W_s$ stands for the derivative of $W$ with respect to $s$.

We will use the following lemma to ensure the differentiability of an eigenpair of (1.4) with respect to $\lambda$. For the proof, we refer the reader to [14, Proposition I.7.2].

**Lemma 2.13** ([14]) In addition to the assumptions of Theorem 2.12, suppose that $X$ is continuously embedded in $Y$ and that $R(D_wF(\lambda_0, 0))$ is a complement of $K(D_wF(\lambda_0, 0)) = \text{span}\{\varphi_0\}$ in $Y$:
\[ Y = R(D_wF(\lambda_0, 0)) \oplus \text{span}\{\varphi_0\}. \]

Then there exist an open interval $\tilde{J}$ containing $\lambda_0$ and $C^1$ mappings $\tilde{J} \ni \lambda \mapsto \mu(\lambda) \in \mathbb{R}$ and $\tilde{J} \ni \lambda \mapsto \varphi(\lambda) \in X$ such that
\[ \mu(\lambda_0) = 0, \quad \varphi(\lambda_0) = \varphi_0, \quad D_wF(\lambda, 0)\varphi(\lambda) = \mu(\lambda)\varphi(\lambda). \]

**Remark 2.14** We will apply Lemma 2.13 in the special case where $Y$ is also continuously embedded in some Hilbert space $Z$ and
\[ R(D_wF(\lambda_0, 0)) = \text{span}\{\varphi_0\}^\perp \cap Y = \{\varphi \in Y; \langle \varphi, \varphi_0 \rangle = 0\}. \]

Here $^\perp$ and $\langle \cdot, \cdot \rangle$ stand for the orthogonal complement and the inner product in $Z$, respectively. Then by differentiating the equality $\langle D_wF(\lambda, 0)\varphi(\lambda), \varphi_0 \rangle = \mu(\lambda)\langle \varphi(\lambda), \varphi_0 \rangle$, we obtain the well-known formula
\[ \frac{d\mu}{d\lambda}(\lambda_0) = \frac{\langle D^2_{ww}F(\lambda_0, 0)\varphi_0, \varphi_0 \rangle}{\langle \varphi_0, \varphi_0 \rangle}. \]
To apply Theorem 2.12 to our problem, we set up function spaces \( \mathcal{X}, \mathcal{Y} \) and \( \mathcal{Z} \). We choose
\[
\mathcal{X} := \left\{ u \in X_0; \begin{array}{l}
\vspace{0.5cm}
\left\{ \begin{array}{l}
u_x(-1) = u_x(1) = 0 \\
u_x(-0) + au_x(-0) = u(+0) - au_x(+0)
\end{array} \right. \\
\end{array} \right\},
\]
\[
\mathcal{Y} := \{ u \in C([-1, 1] \setminus \{0\}); u|_{[-1,0)} \text{ and } u|_{(0,1]} \text{ are uniformly continuous}\},
\]
\[
\mathcal{Z} := L^2(-1, 1).
\]

Then \( \mathcal{X} \) is a closed linear subspace of \( X_0 \) and \( \mathcal{Y} \) is a Banach space endowed with the uniform norm. Let \( \langle \cdot, \cdot \rangle \) denote the inner product on \( \mathcal{Z} \):
\[
\langle u, v \rangle := \int_{-1}^{1} u(x)v(x)dx, \quad u, v \in \mathcal{Z}.
\]

We note that integrating by parts yields
\[
\langle \varphi_{xx} + q \varphi, \psi \rangle - \langle \varphi, \psi_{xx} + q \psi \rangle = \left[ \varphi_x \psi - \varphi \psi_x \right]_{x=-1}^{x=1} + \left[ \varphi_x \psi - \varphi \psi_x \right]_{x=0}^{x=+1}
\]
\[
= \left[ \varphi_x \psi - \varphi \psi_x \right]_{x=-1}^{x=1} + \left[ \varphi_x (\psi + a \varphi_x) - (q + a \varphi_x) \psi_x \right]_{x=-1}^{x=1}
\]
\[
- \{ \varphi_x (\psi - a \varphi_x) - (q - a \varphi_x) \psi_x \} = 0
\]
for all \( \varphi, \psi \in X_0 \) and \( q \in \mathcal{Y} \). We define a \( C^2 \) mapping \( \mathcal{G} : (0, \infty) \times \mathcal{X} \to \mathcal{Y} \) by
\[
\mathcal{G}(\lambda, u) := u_{xx} + \lambda f(u).
\]

By definition, we have \( \mathcal{S} = \{ (\lambda, u) \in (0, \infty) \times \mathcal{X}; \mathcal{G}(\lambda, u) = 0, |u| < 1 \} \).

The following lemma can be shown in a standard way. We give a proof in Appendix B for readers’ convenience.

Lemma 2.15 \( \) Let \( q \in \mathcal{Y} \) and define a linear operator \( T : \mathcal{X} \to \mathcal{Y} \) by \( T := d^2/dx^2 + q \).

Then
\[
R(T) = K(T)^\bot \cap \mathcal{Y} = \{ \varphi \in \mathcal{Y}; \langle \varphi, \psi \rangle = 0 \text{ for all } \psi \in K(T) \}.
\]

Let us prove Proposition 2.10.

Proof of Proposition 2.10 \( \) Set
\[
T_\lambda := D_u \mathcal{G}(\lambda, u(\cdot, \lambda)) = \frac{d^2}{dx^2} + \lambda f'(u(\cdot, \lambda)).
\]

From Lemma 2.3, we see that the condition \( D(\lambda_0, \beta_1(\lambda_0), \beta_2(\lambda_0)) \neq 0 \) gives \( K(T_{\lambda_0}) = \{0\} \).

This with Lemma 2.15 shows that the linear operator \( T_{\lambda_0} : \mathcal{X} \to \mathcal{Y} \) is an isomorphism if \( D(\lambda_0, \beta_1(\lambda_0), \beta_2(\lambda_0)) \neq 0 \). Hence the assertion (i) follows from the implicit function theorem.

We prove (ii) and (iii). In what follows, let (2.31) hold. We put
\[
\mathcal{F}(\lambda, w) := \mathcal{G}(\lambda, u(\cdot, \lambda) + w), \quad (\lambda, w) \in J \times \mathcal{X},
\]
and check that the conditions of Theorem 2.12 and Lemma 2.13 hold. We infer that \( \mathcal{F} : J \times \mathcal{X} \to \mathcal{Y} \) is continuously differentiable and has continuous derivatives \( D^2_w \mathcal{F} \) and \( D^2_{ww} \mathcal{F} \), since \( \mathcal{G} \) is of class \( C^2 \) and the mapping \( J \ni \lambda \mapsto u(\cdot, \lambda) \in \mathcal{X} \) is of class \( C^1 \). Moreover, by assumption, we have \( \mathcal{F}(\lambda, 0) = 0 \) for \( \lambda \in J \). Since the condition \( D(\lambda_0, \beta_1(\lambda_0), \beta_2(\lambda_0)) = 0 \) is assumed, we see from Lemmas 2.3 and 2.7 that for some \( \varphi_0 \in \mathcal{X} \setminus \{0\} \),
\[
K(T_{\lambda_0}) = \text{span}\{\varphi_0\}.
\]
For $j = 1, 2$, let $\varphi_j = \varphi_j(x, \lambda)$ be given by (2.15) with $\beta_j = \beta_j(\lambda)$. We put $\alpha_1 := \varphi_0(-1)$, $\alpha_2 := \varphi_0(1)$ and

$$
\varphi(x, \lambda) := \begin{cases} 
\alpha_1 \varphi_1(x, \lambda) & (x \in [-1, 0]), \\
\alpha_2 \varphi_2(x, \lambda) & (x \in (0, 1]).
\end{cases}
$$

By the same argument as in the proof of Lemma 2.3, we see that both $\alpha_1$ and $\alpha_2$ are nonzero and the assumption $D(\lambda_0, \beta_1(\lambda_0), \beta_2(\lambda_0)) = 0$ gives

$$
\varphi(\cdot, \lambda_0) = \varphi_0.
$$

By (2.16), we have

$$
-\frac{\alpha_1 \alpha_2 \sqrt{\lambda}}{G'(\beta_1(\lambda))G'(\beta_2(\lambda))} D(\lambda, \beta_1(\lambda), \beta_2(\lambda)) = \det \begin{pmatrix} (\varphi + a \varphi_x)|_{x=-0} & (\varphi - a \varphi_x)|_{x=+0} \\
\varphi_x|_{x=-0} & \varphi_x|_{x=+0} \end{pmatrix}.
$$

Differentiating this with respect to $\lambda$, we find that

$$
-\frac{\alpha_1 \alpha_2 \sqrt{\lambda}}{G'(\beta_1(\lambda))G'(\beta_2(\lambda))} \frac{d}{d\lambda} D(\lambda, \beta_1(\lambda), \beta_2(\lambda)) \bigg|_{\lambda=\lambda_0} = \det \begin{pmatrix} (\varphi_x + a \varphi_{xx})|_{x=-0} & (\varphi_x - a \varphi_{xx})|_{x=+0} \\
\varphi_x|_{x=-0} & \varphi_x|_{x=+0} \end{pmatrix} \bigg|_{\lambda=\lambda_0} + \det \begin{pmatrix} [\varphi_0 + a(\varphi_0)_x]|_{x=-0} & [\varphi_0 - a(\varphi_0)_x]|_{x=+0} \\
\varphi_x|_{x=-0} & \varphi_x|_{x=+0} \end{pmatrix} \bigg|_{\lambda=\lambda_0},
$$

where we have used the assumption $D(\lambda_0, \beta_1(\lambda_0), \beta_2(\lambda_0)) = 0$ and (2.38).

Since $\varphi_j$ satisfies (2.18), we see that $\varphi_{xx} + \lambda f'(u(\cdot, \lambda))\varphi = 0$ in $(-1, 1) \setminus \{0\}$ and $\varphi_x = 0$ at $x = \pm 1$. Hence, using (2.35) for $\varphi = \varphi_0$, $\psi = \varphi(\cdot, \lambda)$ and $q = \lambda f'(u(\cdot, \lambda))$, we have

$$
\langle T_\lambda \varphi_0, \varphi \rangle = \left[ (\varphi_0)_x (\varphi + a \varphi_x) - [\varphi_0 + a(\varphi_0)_x] \varphi_x \right]|_{x=-0} - \left[ (\varphi_0)_x (\varphi - a \varphi_x) - [\varphi_0 - a(\varphi_0)_x] \varphi_x \right]|_{x=+0}.
$$

We differentiate this equality with respect to $\lambda$. By (2.37), the derivative of the left-hand side at $\lambda = \lambda_0$ is computed as

$$
\frac{d}{d\lambda} \langle T_\lambda \varphi_0, \varphi \rangle \bigg|_{\lambda=\lambda_0} = \frac{d}{d\lambda} \langle T_\lambda \varphi_0, \varphi_0 \rangle \bigg|_{\lambda=\lambda_0} + \langle T_\lambda \varphi_0, \varphi_\lambda \rangle \bigg|_{\lambda=\lambda_0} = \langle D^2_{\lambda w} \mathcal{F}(\lambda_0, 0) \varphi_0, \varphi_0 \rangle.
$$

Therefore

$$
\langle D^2_{\lambda w} \mathcal{F}(\lambda_0, 0) \varphi_0, \varphi_0 \rangle = \left[ (\varphi_0)_x (\varphi_\lambda + a \varphi_{x\lambda}) - [\varphi_0 + a(\varphi_0)_x] \varphi_{x\lambda} \right]|_{x=-0, \lambda=\lambda_0} - \left[ (\varphi_0)_x (\varphi_\lambda - a \varphi_{x\lambda}) - [\varphi_0 - a(\varphi_0)_x] \varphi_{x\lambda} \right]|_{x=+0, \lambda=\lambda_0}.
$$

Notice that the right-hand side of this equality coincides with that of (2.39), since $\varphi_0$ satisfies

$$
[\varphi_0 + a(\varphi_0)_x]|_{x=-0} = [\varphi_0 - a(\varphi_0)_x]|_{x=+0}, \quad (\varphi_0)_x|_{x=-0} = (\varphi_0)_x|_{x=+0}.
$$

Thus

$$
\langle D^2_{\lambda w} \mathcal{F}(\lambda_0, 0) \varphi_0, \varphi_0 \rangle = -\frac{\alpha_1 \alpha_2 \sqrt{\lambda}}{G'(\beta_1(\lambda_0))G'(\beta_2(\lambda_0))} \frac{d}{d\lambda} D(\lambda, \beta_1(\lambda), \beta_2(\lambda)) \bigg|_{\lambda=\lambda_0}.
$$
From (2.40) and the second condition of (2.31), we see that \( \langle D_{\lambda,w}^2 F(\lambda_0,0) \varphi_0, \varphi_0 \rangle \neq 0 \). Combining this with Lemma 2.15 and (2.37) gives

\[
D_{\lambda,w}^2 F(\lambda_0,0) \varphi_0 \notin \text{span}\{\varphi_0\}^\perp \cap \mathcal{V} = R(T_{\lambda_0}) = R(D_w F(\lambda_0,0)).
\]

Hence all the conditions of Theorem 2.12 and Lemma 2.13 are satisfied. Applying Theorem 2.12, we obtain a \( C^1 \) curve \( \{(\Lambda(s), W(\cdot, s))\}_{s \in J} \subset J \times \mathcal{X} \) satisfying (2.33). Then one can directly check that \( \tilde{C} : = \{(\Lambda(s), u(\cdot, \Lambda(s)) + W(\cdot, s))\}_{s \in J} \) is the one having the desired properties stated in (ii).

It remains to derive (2.32). By Lemma 2.13, we have an eigenvalue \( \mu(\lambda) \) of \( D_w F(\lambda,0) = T_\lambda \) which is of class \( C^1 \) and satisfies \( \mu(\lambda_0) = 0 \). It is easily seen that \( \mu(\lambda) \) is an eigenvalue of (1.4) for \( u = u(\cdot, \lambda) \). Hence \( \mu(\lambda) \) coincides with \( \mu_n(u(\cdot, \lambda)) \), since Lemma 2.7 shows that each eigenvalue \( \mu_m(u(\cdot, \lambda)) \), \( m \in \mathbb{N} \cup \{0\} \), is isolated and continuous with respect to \( \lambda \).

In particular, \( \mu_n(u(\cdot, \lambda)) \) is continuously differentiable in a neighborhood of \( \lambda_0 \). As noted in Remark 2.14, we can compute the derivative of \( \mu(\lambda) = \mu_n(u(\cdot, \lambda)) \) by the formula (2.34). Therefore we see from (2.40) and Lemma 2.8 that

\[
\text{sgn} \left( \frac{d}{d\lambda} \mu_n(u(\cdot, \lambda)) \bigg|_{\lambda=\lambda_0} \right) = \text{sgn} \left( -\varphi_0(-1) \varphi_0(1) \frac{d}{d\lambda} D(\lambda, \beta_1(\lambda), \beta_2(\lambda)) \bigg|_{\lambda=\lambda_0} \right) = (-1)^{n-1} \text{sgn} \left( \frac{d}{d\lambda} D(\lambda, \beta_1(\lambda), \beta_2(\lambda)) \bigg|_{\lambda=\lambda_0} \right).
\]

We thus obtain (2.32), and the proof is complete. \( \square \)

### 3 Primary branches

In this section, we examine primary branches of solutions of (1.1) bifurcating from the trivial branch \( \mathcal{L} = \{(\lambda, 0)\}_{\lambda \in (0, \infty)} \). To set up notation, we introduce the function

\[
g(\beta, \phi) := \begin{cases} 
G(\beta \cos \phi) - a \sin \phi \int_0^\phi G'(\beta \cos \tau) d\tau & \text{if } \beta \neq 0, \\
G'(0) \cos \phi - a G'(0) \phi \sin \phi & \text{if } \beta = 0.
\end{cases}
\]

In Lemma 3.2 below, we will show that for fixed \( k \in \mathbb{N} \) and \( \beta \in I \), the equation \( g(\beta, \phi) = 0 \) has a unique root \( \phi \) in \((k - 1)\pi, (k - 1/2)\pi\). We denote such \( \phi \) by \( \phi_k(\beta) \). In other words, \( \phi_k(\beta) \) is defined implicitly by the relations

\[
\phi_k(\beta) \in \left((k - 1)\pi, \left(k - \frac{1}{2}\right)\pi\right), \quad g(\beta, \phi_k(\beta)) = 0.
\]

We will also show that \( \phi_k(\beta) \) is of class \( C^1 \) as a function of \( \beta \).

Put

\[
\lambda_k^\circ(\beta) := \left( \int_0^{\phi_k(\beta)} G'(\beta \cos \tau) d\tau \right)^2, \quad \lambda_k^c(\beta) := \left( \int_0^{k\pi} G'(\beta \cos \tau) d\tau \right)^2,
\]

\[
u_k^\circ(x, \beta) := \Psi_{\lambda_k^\circ(\beta)}(\beta, -\beta)(x), \quad \nu_k^c(x, \beta) := \Psi_{\lambda_k^c(\beta)}(\beta, \beta)(x),
\]

\[
C_k^\circ := \{(\lambda_k^\circ(\beta), \nu_k^\circ(\cdot, \beta))\}_{\beta \in I}, \quad C_k^c := \{(\lambda_k^c(\beta), \nu_k^c(\cdot, \beta))\}_{\beta \in I}.
\]
Let $z_k$ be defined by the unique root of the equation $az \tan z = 1$ in $((k - 1)\pi, (k - 1/2)\pi)$. Then, by the definitions of $g$ and $\phi_k$, we have

$$\phi_k(0) = z_k. \tag{3.1}$$

This together with the fact that $G'(0) = 1/\sqrt{f'(0)}$ (see (B.3) below) gives

$$\lambda_k^0(0) = (\phi_k(0)G'(0))^2 = \lambda_{2k-1}, \quad \lambda_k^e(0) = (k\pi G'(0))^2 = \lambda_{2k}. \tag{3.2}$$

Therefore $C_{k, +}^0$ (resp. $C_{k, -}^0$) is a $C^1$ curve in $(0, \infty) \times X$ which intersects $L$ at $(\lambda_k^0(0), u_k^0(\cdot, 0)) = (\lambda_{2k-1}, 0)$ (resp. $(\lambda_k^e(0), u_k^e(\cdot, 0)) = (\lambda_{2k}, 0)$). We define $C_{k, +}^e, C_{k, -}^e, C_{k, +}^0$ and $C_{k, -}^0$ by

\begin{align*}
C_{k, +}^o & := \{ (\lambda_k^0(\beta), u_k^0(\cdot, \beta)) \}_{\beta \in (0, \beta_0)}, \quad C_{k, -}^o := \{ (\lambda_k^0(\beta), u_k^0(\cdot, \beta)) \}_{\beta \in (-\beta_0, 0)}, \\
C_{k, +}^e & := \{ (\lambda_k^e(\beta), u_k^e(\cdot, \beta)) \}_{\beta \in (0, \beta_0)}, \quad C_{k, -}^e := \{ (\lambda_k^e(\beta), u_k^e(\cdot, \beta)) \}_{\beta \in (-\beta_0, 0)}.
\end{align*}

We prove the following proposition.

**Proposition 3.1** The following hold.

(i) There hold

\begin{align*}
&\bigcup_{k=1}^{\infty} C_{k, +}^0 \cup C_{k, -}^0 = \mathcal{S}^o, \quad \bigcup_{k=1}^{\infty} C_{k, +}^e \cup C_{k, -}^e = \mathcal{S}^e. \tag{3.3}
\end{align*}

Furthermore, for every $\lambda \in (0, \infty)$, there exists a neighborhood $\mathcal{N}$ of $(\lambda, 0)$ in $(0, \infty) \times X_0$ such that

$$\mathcal{S} \cap \mathcal{N} = \begin{cases} 
(C_{k, +}^o \cup L) \cap \mathcal{N} & \text{if } \lambda = \lambda_{2k-1}, \\
(C_{k, -}^o \cup L) \cap \mathcal{N} & \text{if } \lambda = \lambda_{2k}, \\
L \cap \mathcal{N} & \text{if } \lambda \neq \lambda_{2k-1}, \lambda_{2k}. \tag{3.4}
\end{cases}$$

(ii) Assume (1.6). Then

$$\sgn(\beta) \frac{d\lambda_k^0}{d\beta}(\beta) > 0, \quad \sgn(\beta) \frac{d\lambda_k^e}{d\beta}(\beta) > 0 \quad \text{for } \beta \in I \setminus \{0\}. \tag{3.5}$$

In particular, $C_{k, +}^o$ and $C_{k, -}^o$ (resp. $C_{k, +}^e$ and $C_{k, -}^e$) are parametrized by $\lambda \in (\lambda_{2k-1}, \infty)$ (resp. $\lambda \in (\lambda_{2k}, \infty)$).

We begin with solving the equation $g(\beta, \phi) = 0$.

**Lemma 3.2** For $k \in \mathbb{N}$ and $\beta \in I$, there exists $\phi_k(\beta) \in ((k - 1)\pi, (k - 1/2)\pi)$ with the following properties:

$$\{ \phi \in [0, \infty); g(\beta, \phi) = 0 \} = \{ \phi_k(\beta) \}_{k=1}^{\infty}, \tag{3.6}$$

$$\phi_k(-\beta) = \phi_k(\beta), \tag{3.7}$$

$$\lim_{\beta \to \pm \beta_0} \phi_k(\beta) = (k - 1)\pi. \tag{3.8}$$

Furthermore, $\phi_k \in C^1(I)$ and

$$\frac{d\phi_k}{d\beta}(\beta) = \frac{J(\beta, \phi_k(\beta))}{I(\beta, \phi_k(\beta))}. \tag{3.9}$$
where
\[
I(\beta, \phi) := \beta \left\{ \left( \frac{G'(\beta \cos \phi) \beta \sin \phi}{G(\beta \cos \phi)} + \frac{\cos \phi}{\sin \phi} \right) \int_0^\phi G'(\beta \cos \tau) d\tau + G'(\beta \cos \phi) \right\},
\]
\[
J(\beta, \phi) := \left( \frac{G'(\beta \cos \phi) \beta \cos \phi}{G(\beta \cos \phi)} - 1 \right) \int_0^\phi G'(\beta \cos \tau) d\tau - \beta \int_0^\phi G''(\beta \cos \tau) \cos \tau d\tau.
\]

**Proof** For the moment suppose that \( k \) is odd. We note that \( g \in C^1(I \times \mathbb{R}) \), since \( g \) is written as
\[
g(\beta, \phi) = \cos \phi \int_0^1 G'(t\beta \cos \phi) dt - a \sin \phi \int_0^\phi G'(\beta \cos \tau) d\tau.
\]
From the fact that \( G' \) is positive, we have
\[
g(\beta, (k - 1)\pi) > 0, \quad g(\beta, \phi) < 0 \quad \text{for} \quad \phi \in \left( \left( k - \frac{1}{2} \right) \pi, k\pi \right).
\]
Furthermore,
\[
g_\phi(\beta, \phi) = -(1 + a)G'(\beta \cos \phi) \sin \phi - a \cos \phi \int_0^\phi G'(\beta \cos \tau) d\tau
\]
\[
< 0 \quad \text{for} \quad \phi \in \left( (k - 1)\pi, \left( k - \frac{1}{2} \right) \pi \right).
\]
Therefore there exists \( \phi_k(\beta) \in ((k - 1)\pi, (k - 1/2)\pi) \) such that
\[
\{ \phi \in [(k - 1)\pi, k\pi]; g(\beta, \phi) = 0 \} = \{ \phi_k(\beta) \},
\]
and the implicit function theorem shows that \( \phi_k \in C^1(I) \). The case where \( k \) is even can be dealt with in the same way. We have thus shown the existence of \( \phi_k(\beta) \) and (3.6).

The fact that \( G \) is odd yields \( g(-\beta, \phi) = g(\beta, \phi) \), and hence \( g(-\beta, \phi_k(\beta)) = g(\beta, \phi_k(\beta)) = 0 \). From (3.6), we obtain (3.7).

We prove (3.8). To obtain a contradiction, suppose that \( \phi_k(\beta) \) does not converge to \((k - 1)\pi\) as \( \beta \to \beta_0 \) or \( \beta \to -\beta_0 \). Then we can take \( \{ \beta_j \}_{j=1}^\infty \subset I \) and \( \delta \in (0, \pi/2) \) such that
\[
|\beta_j| < |\beta_{j+1}|, \quad |\beta_j| \to \beta_0 \quad (j \to \infty), \quad \phi_k(\beta_j) \geq (k - 1)\pi + \delta.
\]
By (2.5) and the monotone convergence theorem,
\[
\int_0^{\phi_k(\beta_j)} G'(\beta_j \cos \tau) d\tau \geq c \int_0^{(k-1)\pi+\delta} \frac{d\tau}{\sqrt{\beta_0 - |\beta_j| \cos \tau}}
\]
\[
\quad \to c \int_0^{(k-1)\pi+\delta} \frac{d\tau}{\sqrt{\beta_0 - \beta_0 \cos \tau}} = \infty \quad (j \to \infty).
\]
From this and the fact that \( |\sin \phi_k(\beta_j)| \geq |\sin \delta| \), we have \( |g(\beta_j, \phi_k(\beta_j))| \to \infty \) as \( j \to \infty \). This contradicts the equality \( g(\beta_j, \phi_k(\beta_j)) = 0 \), and therefore (3.8) holds.

It remains to prove (3.9). Differentiating \( g(\beta, \phi_k(\beta)) = 0 \) yields
\[
\frac{d\phi_k}{d\beta}(\beta) = -\frac{g_\beta(\beta, \phi_k(\beta))}{g_\phi(\beta, \phi_k(\beta))}
\]
\[
= \frac{G'(\beta \cos \phi) \cos \phi - G(\beta \cos \phi) / \beta - a \beta \sin \phi \int_0^\phi G''(\beta \cos \tau) \cos \tau d\tau}{(1 + a)G'(\beta \cos \phi) \beta \sin \phi + a \beta \cos \phi \int_0^\phi G'(\beta \cos \tau) d\tau} \bigg|_{\phi = \phi_k(\beta)}.
\]

(3.10)
We note that
\[
\frac{1}{a} = \frac{\beta \sin \phi}{G(\beta \cos \phi)} \int_0^\phi G'(\beta \cos \tau) d\tau \bigg|_{\phi=\phi_k(\beta)},
\] (3.11)
which follows from \( g(\beta, \phi_k(\beta)) = 0 \). Plugging this into (3.10), we obtain (3.9). Therefore the lemma follows. \( \square \)

We prove the further property of \( \phi_k \) to be used later.

**Lemma 3.3** For each \( k \in \mathbb{N} \),
\[
\lim \inf_{\beta \to \pm \beta_0} \frac{\vert \sin \phi_k(\beta) \vert}{\sqrt{\beta_0 - \vert \beta \vert}} > 0.
\]

**Proof** We use (2.5) to obtain
\[
G' (\beta \cos \tau) \leq \frac{C}{\sqrt{\beta_0 - \vert \beta \cos \tau \vert}} \leq \frac{C}{\sqrt{\beta_0 - \vert \beta \vert}}.
\]
Hence, by (3.11), we have
\[
\frac{1}{a} \leq \frac{C \vert \beta \phi_k(\beta) \vert}{\vert G(\beta \cos \phi_k(\beta)) \vert} \cdot \frac{\vert \sin \phi_k(\beta) \vert}{\sqrt{\beta_0 - \vert \beta \vert}}.
\]
The desired inequality is verified by combining this with (3.8). \( \square \)

It is known (see [19, 20, 22]) that the condition (1.6) implies the inequality
\[
\text{sgn}(\beta) \int_0^{l \pi} G''(\beta \cos \tau) \cos \tau d\tau > 0 \quad (\beta \in I \setminus \{0\}, l \in \mathbb{N}),
\]
which is equivalent to
\[
\text{sgn}(\alpha) \frac{d}{d\alpha} \int_0^\alpha \frac{ds}{\sqrt{F(\alpha) - F(s)}} > 0 \quad (\alpha \in (-1, 1) \setminus \{0\}).
\]
In order to show (3.5), we generalize this inequality.

**Lemma 3.4** Let (1.6) hold. Then for all \( \beta \in I \setminus \{0\} \) and \( \phi \in (0, \infty) \),
\[
\beta \int_0^\phi G''(\beta \cos \tau) \cos \tau d\tau > \begin{cases} - \left( G'(\beta \cos \phi) \cos \phi - \frac{G'(\beta)}{G(\beta)} G(\beta \cos \phi) \right) \frac{1}{\sin \phi} & \text{if } \sin \phi \neq 0, \\ 0 & \text{if } \sin \phi = 0. \end{cases}
\]

**Proof** We first consider the case \( \sin \phi \neq 0 \). Put
\[
\tilde{H}(v) := \begin{cases} \frac{G(v)}{v} & (v \neq 0), \\ G'(0) & (v = 0). \end{cases}
\]
From Lemma 2.1, one can check that $\tilde{H} \in C^1(I)$ and $\tilde{H}'(v) = G'(v)H(v)/v^3$. Then

$$\beta \int_0^\phi G''(\beta \cos \tau) \cos \tau d\tau = \beta \int_0^\phi \left( G''(\beta \cos \tau) - \frac{G'(\beta)H(\beta \cos \tau)}{G(\beta)} \right) \cos \tau d\tau$$

$$+ \frac{G'(\beta)\beta^2}{G(\beta)} \int_0^\phi \tilde{H}(\beta \cos \tau) \cos \tau d\tau$$

$$= - \int_0^\phi \frac{\partial}{\partial \tau} \left( G'(\beta \cos \tau) - \frac{G'(\beta)H(\beta \cos \tau)}{G(\beta)} \right) \cos \frac{\tau}{\sin \tau} d\tau$$

$$+ \frac{G'(\beta)}{G(\beta)\beta} \int_0^\phi \frac{H(\beta \cos \tau)G'(\beta \cos \tau)}{\cos^2 \tau} d\tau. \quad (3.12)$$

We apply integration by parts to obtain

$$\int_0^\phi \frac{\partial}{\partial \tau} \left( G'(\beta \cos \tau) - \frac{G'(\beta)H(\beta \cos \tau)}{G(\beta)} \right) \cos \frac{\tau}{\sin \tau} d\tau$$

$$= \left( G'(\beta \cos \phi) - \frac{G'(\beta)H(\beta \cos \phi)}{G(\beta)} \right) \cos \frac{\phi}{\sin \phi}$$

$$+ \int_0^\phi \left( G'(\beta \cos \tau) - \frac{G'(\beta)H(\beta \cos \tau)}{G(\beta)} \right) \frac{1}{\sin^2 \tau} d\tau. \quad (3.13)$$

This computation is valid, since

$$G'(\beta \cos \tau) - \frac{G'(\beta)H(\beta \cos \tau)}{G(\beta)} = O(1 - |\cos \tau|) = O(\sin^2 \tau) \quad \text{as} \quad \tau \to l\pi, \; l \in \mathbb{Z}. \quad (3.14)$$

Note that the integrand of the second term on the right of (3.13) is written as

$$G'(\beta \cos \tau) - \frac{G'(\beta)H(\beta \cos \tau)}{G(\beta)} \tilde{H}(\beta \cos \tau) = \frac{G'(\beta)G'(\beta \cos \tau)}{G(\beta)\beta} \left( \frac{G(\beta)H(\beta \cos \tau) - G'(\beta \cos \tau)}{G(\beta)\beta} \right)$$

$$= \frac{G'(\beta)G'(\beta \cos \tau)}{G(\beta)\beta} \left( \frac{H(\beta \cos \tau)}{\cos^2 \tau} - H(\beta) \right).$$

Therefore

$$\int_0^\phi \frac{\partial}{\partial \tau} \left( G'(\beta \cos \tau) - \frac{G'(\beta)H(\beta \cos \tau)}{G(\beta)} \right) \cos \frac{\tau}{\sin \tau} d\tau$$

$$= \left( G'(\beta \cos \phi) \cos \phi - \frac{G'(\beta)}{G(\beta)\beta} G(\beta \cos \phi) \right) \frac{1}{\sin \phi}$$

$$+ \frac{G'(\beta)}{G(\beta)\beta} \int_0^\phi \left( \frac{H(\beta \cos \tau)}{\cos^2 \tau} - H(\beta) \right) \frac{G'(\beta \cos \tau)}{\sin^2 \tau} d\tau.$$

Substituting this into (3.12), we find that

$$\beta \int_0^\phi G''(\beta \cos \tau) \cos \tau d\tau = \left( G'(\beta \cos \phi) \cos \phi - \frac{G'(\beta)}{G(\beta)\beta} G(\beta \cos \phi) \right) \frac{1}{\sin \phi}$$

$$+ \frac{G'(\beta)}{G(\beta)\beta} \int_0^\phi \frac{G'(\beta \cos \tau)(H(\beta) - H(\beta \cos \tau))}{\sin^2 \tau} d\tau.$$

Since the assumption (1.6) implies (2.4), we deduce that the second term on the right of this equality is positive. Thus we obtain the desired inequality.
In the case \( \sin \phi = 0 \), we see from (3.14) that the first term on the right of (3.13) vanishes, and hence the same argument works. \( \square \)

To obtain odd and even solutions of (1.1), we find solutions of (2.11) satisfying either \( \beta_1 = -\beta_2 \) or \( \beta_1 = \beta_2 \).

**Lemma 3.5** There hold

\[
\{(\lambda, \beta) \in (0, \infty) \times (I \setminus \{0\}); (\lambda, \beta, -\beta) \in T\} = \bigcup_{k=1}^{\infty} \{(\lambda_k^o(\beta), \beta)\}_{\beta \in I \setminus \{0\}},
\]

\[
\{(\lambda, \beta) \in (0, \infty) \times (I \setminus \{0\}); (\lambda, \beta, \beta) \in T\} = \bigcup_{k=1}^{\infty} \{(\lambda_k^e(\beta), \beta)\}_{\beta \in I \setminus \{0\}}.
\]

**Proof** We see from (2.13) that \( (\lambda, \beta, -\beta) \in T \) if and only if \( P(\lambda, \beta) = 0 \), and that \( (\lambda, \beta, \beta) \in T \) if and only if \( Q(\lambda, \beta) = 0 \). Hence

\[
\{(\lambda, \beta); (\lambda, \beta, -\beta) \in T, \beta \neq 0\} = \{(\lambda, \beta); P(\lambda, \beta) = 0, \beta \neq 0\},
\]

\[
\{(\lambda, \beta); (\lambda, \beta, \beta) \in T, \beta \neq 0\} = \{(\lambda, \beta); Q(\lambda, \beta) = 0, \beta \neq 0\}.
\]

We fix \( \beta \in I \setminus \{0\} \). By (2.21), (2.22) and the definition of \( g \), we have \( P(\lambda, \beta) = \beta g(\beta, \theta(\lambda, \beta)) \). This together with (3.6) and (2.21) yields

\[
\{\lambda; P(\lambda, \beta) = 0\} = \{\lambda; g(\beta, \theta(\lambda, \beta)) = 0\} = \bigcup_{k=1}^{\infty} \{\lambda; \theta(\lambda, \beta) = \phi_k(\beta)\} = \bigcup_{k=1}^{\infty} \{\lambda_k^o(\beta)\}.
\]

Moreover, by (2.22),

\[
\{\lambda; Q(\lambda, \beta) = 0\} = \{\lambda; \sin \theta(\lambda, \beta) = 0\} = \bigcup_{k=1}^{\infty} \{\lambda; \theta(\lambda, \beta) = k\pi\} = \bigcup_{k=1}^{\infty} \{\lambda_k^e(\beta)\}.
\]

Therefore we obtain the desired conclusion. \( \square \)

Put

\[
\tilde{S}_X^o := \{u \in S_X \setminus \{0\}; u(-1) = -u(1)\}, \quad \tilde{S}_X^e := \{u \in S_X \setminus \{0\}; u(-1) = u(1)\}.
\]

**Lemma 3.6** There hold \( S_X^o = \tilde{S}_X^o \) and \( S_X^e = \tilde{S}_X^e \).

**Proof** It is clear that \( S_X^o \subset \tilde{S}_X^o \). To show \( \tilde{S}_X^o \subset S_X^o \), we let \( u \in \tilde{S}_X^o \). Then \( u_1 := u|_{[-1,0]} \) and \( u_2 := u|_{[0,1]} \) satisfy (2.7) for \( \beta_1 = G^{-1}(u(-1)) \) and \( \beta_2 = G^{-1}(u(1)) \). Since the assumption \( u \in \tilde{S}_X^o \) yields \( \beta_1 = -\beta_2 \), we see that \( u_1(x) \) and \( -u_2(-x) \) satisfy the same initial value problem. Hence \( u_1(x) = -u_2(-x) \), which gives \( u \in S_X^o \). We have thus proved \( S_X^o = \tilde{S}_X^o \). The equality \( S_X^e = \tilde{S}_X^e \) can be shown in the same way. \( \square \)

We are now in a position to prove Proposition 3.1.

**Proof of Proposition 3.1** By (2.14), (3.7) and the fact that \( G \) is odd, we have

\[
(\lambda_k^o(-\beta), u_k^o(-\beta)) = (\lambda_k^o(\beta), -u_k^o(\beta)), \quad (\lambda_k^e(-\beta), u_k^e(-\beta)) = (\lambda_k^e(\beta), -u_k^e(\beta)). \quad (3.15)
\]

Hence (3.2) follows.
Lemmas 2.2 and 3.6 yield
\[ S^\circ = \tilde{S}^\circ = \{\lambda \psi_\lambda(\beta, -\beta) \mid (\beta, -\beta) \in T_\lambda, \beta \neq 0\}, \]
\[ S^e = \tilde{S}^e = \{\lambda \psi_\lambda(\beta, \beta) \mid (\beta, \beta) \in T_\lambda, \beta \neq 0\}. \]

Combining these with Lemma 3.5 shows that
\[ S^\circ = \{\lambda, \psi_\lambda(\beta, -\beta) \mid (\lambda, \beta, -\beta) \in T, \beta \neq 0\} = \bigcup_{k=1}^{\infty} \{\lambda \psi_k(\beta, \cdot, \beta) \} \}_{\beta \in I \setminus \{0\}}, \]
\[ S^e = \{\lambda, \psi_\lambda(\beta, \beta) \mid (\lambda, \beta, \beta) \in T, \beta \neq 0\} = \bigcup_{k=1}^{\infty} \{\lambda \psi_k(\beta, \cdot, \beta) \} \}_{\beta \in I \setminus \{0\}}. \]

Therefore we obtain (3.3).

By (2.21) and the fact that \( G'(0) = 1/\sqrt{f'(0)} \) (see (B.3) below), we deduce that \( \theta(\lambda, 0) = \sqrt{\lambda}/G'(0) = \sqrt{f'(0)}\lambda. \) Hence it follows from (2.23) that
\[ D(\lambda, 0, 0) = 2 P_\beta(\lambda, 0) Q_\beta(\lambda, 0) \]
\[ = -\frac{2}{\sqrt{f'(0)}} \left( \cos \sqrt{f'(0)}\lambda - a \sqrt{f'(0)} \sin \sqrt{f'(0)}\lambda \right) \sin \sqrt{f'(0)}\lambda. \]

This shows that \( D(\lambda, 0, 0) = 0 \) if and only if \( \sqrt{f'(0)}\lambda = z_k \) or \( \sqrt{f'(0)}\lambda = k\pi \) for some \( k \in \mathbb{N} \). Moreover,
\[ \left. \frac{d}{d\lambda} D(\lambda, 0, 0) \right|_{\lambda = z_k^2/ f'(0)} = \frac{\sqrt{f'(0)}}{z_k} \{(1 + a) \sin z_k + az_k \cos z_k \} \sin z_k > 0, \]
\[ \left. \frac{d}{d\lambda} D(\lambda, 0, 0) \right|_{\lambda = (k\pi)^2/ f'(0)} = -\frac{\sqrt{f'(0)}}{k\pi} < 0. \]

Thus, using (i) and (ii) of Proposition 2.10 for \( \mathcal{C} = \mathcal{L} \), we conclude that (3.4) holds.

What is left is to show (ii). Assume (1.6) and let \( \beta \in I \setminus \{0\} \). The estimate for \( \lambda^e_k(\beta) \) is directly derived from Lemma 3.4. Indeed,
\[ \beta \frac{d}{d\beta} \sqrt{\lambda^e_k(\beta)} = \beta \int_0^{k\pi} G''(\beta \cos \tau) \cos \tau d\tau > 0. \]

Let us consider the estimate for \( \lambda^e_k(\beta) \). We use (3.9) to obtain
\[ \beta \frac{d}{d\beta} \sqrt{\lambda^e_k(\beta)} = \beta \left( \int_0^{\phi_k(\beta)} G''(\beta \cos \tau) \cos \tau d\tau + \frac{d\phi_k(\beta)}{d\beta} G'(\beta \cos \phi_k(\beta)) \right) \]
\[ = \beta \frac{I(\beta, \phi)}{\phi_k(\beta)} \left( I(\beta, \phi) \int_0^{\phi} G''(\beta \cos \tau) \cos \tau d\tau + J(\beta, \phi) G'(\beta \cos \phi) \right) \bigg|_{\phi = \phi_k(\beta)}. \]

Let \( \phi \in ((k-1)\pi, (k-1/2)\pi) \). Then
\[ \frac{I(\beta, \phi)}{\beta} = \left( \frac{G'(\beta \cos \phi) \sin \phi}{G(\beta \cos \phi)} + \frac{\cos \phi}{\sin \phi} \right) \int_0^{\phi} G'(\beta \cos \tau) d\tau + G'(\beta \cos \phi) > 0. \]

(3.16)
By a direct computation, we have

\[
I(\beta, \phi) \int_0^\phi G''(\beta \cos \tau) \cos \tau d\tau + J(\beta, \phi) G'(\beta \cos \phi) \int_0^\phi G'(\beta \cos \tau) d\tau \\
= \left( \frac{G'(\beta \cos \phi) \beta \sin \phi + \cos \phi}{G(\beta \cos \phi)} + \frac{\cos \phi}{\sin \phi} \right) \beta \int_0^\phi G''(\beta \cos \tau) \cos \tau d\tau \\
+ \left( \frac{G'(\beta \cos \phi) \beta \cos \phi}{G(\beta \cos \phi)} - 1 \right) G'(\beta \cos \phi). \quad (3.17)
\]

Applying Lemma 3.4 shows that the right-hand side of this equality is bounded below by

\[
\left( \frac{G'(\beta \cos \phi) \beta \sin \phi}{G(\beta \cos \phi)} + \frac{\cos \phi}{\sin \phi} \right) \cdot \left\{ - \left( \frac{G'(\beta \cos \phi) \cos \phi - \frac{G'(\beta)}{G(\beta) \beta \cos \phi}}{G(\beta) \beta \cos \phi} \right) \frac{1}{\sin \phi} \right\} \\
+ \left( \frac{G'(\beta \cos \phi) \beta \cos \phi}{G(\beta \cos \phi)} - 1 \right) G'(\beta \cos \phi) \\
= \left( \beta^2 \sin^2 \phi + \frac{G(\beta \cos \phi) \beta \cos \phi}{G'(\beta \cos \phi)} - \frac{G(\beta) \beta}{G'(\beta)} \right) \frac{G'(\beta) G'(\beta \cos \phi)}{G(\beta) \beta \sin^2 \phi} \\
= \frac{G'(\beta) G'(\beta \cos \phi) (H(\beta) - H(\beta \cos \phi))}{G(\beta) \beta \sin^2 \phi}.
\]

This is positive, since the assumption (1.6) implies (2.4). Hence it follows that

\[
I(\beta, \phi) \int_0^\phi G''(\beta \cos \tau) \cos \tau d\tau + J(\beta, \phi) G'(\beta \cos \phi) > 0. \quad (3.18)
\]

Combining (3.16) and (3.18), we obtain

\[
\beta \frac{d}{d\beta} \sqrt{\lambda_0^\phi(\beta)} > 0.
\]

Thus (ii) is verified, and the proof is complete. \( \square \)

4 Secondary bifurcations

In this section, we consider bifurcation points on \( S^o \) and \( S^e \).

4.1 Nonexistence of bifurcation points on \( S^e \)

The following lemma shows that no bifurcation point exists on \( S^e \).

**Lemma 4.1** Assume (1.6). Then for every \((\lambda, u) \in S^e\), there is a neighborhood \( N \) of \((\lambda, u) \) in \((0, \infty) \times X_0\) such that \( S \cap N = S^e \cap N \).

**Proof** By the assumption (1.6) and Proposition 3.1, we see that \( S^e \) is the union of \( C^1 \) curves parametrized by \( \lambda \). Therefore, according to (i) of Proposition 2.10, we only need to show that
\[ D(\lambda_k^\varepsilon(\beta), \beta, \beta) \neq 0 \text{ for } \beta \in I \setminus \{0\}. \]

Using (2.23) and the fact that \( \theta(\lambda_k^\varepsilon(\beta), \beta) = k\pi \) gives

\[ D(\lambda_k^\varepsilon(\beta), \beta, \beta) = 2 P_\beta(\lambda_k^\varepsilon(\beta), \beta) Q_\beta(\lambda_k^\varepsilon(\beta), \beta) \]

\[ = 2 \left\{ G'(\beta) + a \frac{\beta}{G'(\beta)} \left( \int_0^{k\pi} G'(\beta \cos \tau)d\tau \right) \left( \int_0^{k\pi} G''(\beta \cos \tau) \cos \tau d\tau \right) \right\} \]

\[ \times \frac{\beta}{G'(\beta)} \int_0^{k\pi} G''(\beta \cos \tau) \cos \tau d\tau. \]

Lemma 3.4 shows that this is positive, and hence the lemma follows. \( \square \)

### 4.2 The number of bifurcation points on \( S^0 \)

We show that \( C_{k,+}^0 \) and \( C_{k,-}^0 \) have a unique bifurcation point, provided that (1.5) holds.

**Proposition 4.2** Assume (1.5). Then for \( k \in \mathbb{N} \), there exists a \( C^1 \) curve \( \tilde{C}_{k,+}^0 \subset S \) such that \( \tilde{C}_{k,+}^0 \) intersects \( C_{k,+}^0 \) transversally at some point \( (\lambda_{k,+}^*, u_{k,+}^*) \). Moreover, for each \( (\lambda, u) \in C_{k,+}^0 \), there is a neighborhood \( N \) of \( (\lambda, u) \) such that

\[ S \cap N = \begin{cases} C_{k,+}^0 \cap N & \text{if } (\lambda, u) \neq (\lambda_{k,+}^*, u_{k,+}^*), \\ (C_{k,+}^0 \cup \tilde{C}_{k,+}^0) \cap N & \text{if } (\lambda, u) = (\lambda_{k,+}^*, u_{k,+}^*). \end{cases} \]

The same assertion holds for \( C_{k,-}^0 \) in place of \( C_{k,+}^0 \).

To prove this proposition, we examine the behavior of \( D(\lambda_k^0(\beta), \beta, -\beta) \). First we consider

\[ P_\beta(\lambda_k^0(\beta), \beta). \]

**Lemma 4.3** If (1.6) holds, then \((-1)^{k-1} P_\beta(\lambda_k^0(\beta), \beta) > 0 \) for all \( k \in \mathbb{N} \) and \( \beta \in I \setminus \{0\} \).

**Proof** By definition, we have

\[ \theta(\lambda_k^0(\beta), \beta) = \phi_k(\beta). \quad (4.1) \]

This together with (2.23) and (3.11) yields

\[ P_\beta(\lambda_k^0(\beta), \beta) = \left. \frac{G'(\beta \cos \phi) \beta \cos \phi - G(\beta \cos \phi)}{\beta} \right|_{\phi = \phi_k(\beta)} + \left. \left( \beta \sin \phi + \frac{G(\beta \cos \phi) \cos \phi}{G'(\beta \cos \phi) \sin \phi} \right) \int_0^{\phi} G''(\beta \cos \tau) \cos \tau d\tau \right|_{\phi = \phi_k(\beta)}. \]

It follows from (3.17) that

\[ \frac{G'(\beta \cos \phi) \beta}{G(\beta \cos \phi)} \int_0^{\phi} G'(\beta \cos \tau)d\tau \left|_{\phi = \phi_k(\beta)} \cdot P_\beta(\lambda_k^0(\beta), \beta) \right. \]

\[ = \left( I(\beta, \phi) \int_0^{\phi} G''(\beta \cos \tau) \cos \tau d\tau + J(\beta, \phi)G'(\beta \cos \phi) \right) \left|_{\phi = \phi_k(\beta)}. \right. \]

Therefore the desired inequality follows from (3.18). \( \square \)

Next we consider \( Q_\beta(\lambda_k^0(\beta), \beta) \). We note that (2.23) and (4.1) yield

\[ Q_\beta(\lambda_k^0(\beta), \beta) = R(\beta, \phi_k(\beta)), \quad R(\beta, \phi) := - \sin \phi + \frac{\beta \cos \phi}{G'(\beta \cos \phi)} \int_0^{\phi} G''(\beta \cos \tau) \cos \tau d\tau. \]
Lemma 4.4 For every $k \in \mathbb{N}$,

$$(-1)^{k-1} Q_\beta(\lambda_\kappa^\alpha(0), 0) < 0, \quad \lim_{\beta \to \pm \beta_0} (-1)^{k-1} Q_\beta(\lambda_\kappa^\alpha(\beta), \beta) = \infty. \quad (4.2)$$

Proof We see from (3.1) that $Q_\beta(\lambda_\kappa^\alpha(0), 0) = R(0, z_k) = -\sin z_k$. Hence the first inequality of (4.2) holds.

We examine the limit of $Q_\beta(\lambda_\kappa^\alpha(\beta), \beta)$ as $\beta \to \pm \beta_0$. By (2.6), we have

$$\beta \int_0^{\phi_k(\beta)} G''(\beta \cos \tau) \cos \tau d\tau \geq \int_0^{\phi_k(\beta)} |\beta \cos \tau| \left\{ \frac{c}{(\beta_0 - |\beta \cos \tau|)^{3/2}} - C \right\} d\tau \geq c \int_{(k-1)\pi}^{\phi_k(\beta)} \frac{|\beta \cos \tau|}{(\beta_0 - |\beta \cos \tau|)^{3/2}} d\tau - C|\beta|\phi_k(\beta).$$

Since

$$\beta_0 - |\beta \cos \tau| = \beta_0 - |\beta| + |\beta|(1 - |\cos \tau|) \leq \beta_0 - |\beta| + \beta_0 \sin^2 \tau,$$

we find that

$$\int_{(k-1)\pi}^{\phi_k(\beta)} \frac{|\cos \tau|}{(\beta_0 - |\beta \cos \tau|)^{3/2}} d\tau \geq \int_{(k-1)\pi}^{\phi_k(\beta)} \frac{|\cos \tau|}{(\beta_0 - |\beta| + \beta_0 \sin^2 \tau)^{3/2}} d\tau$$

$$= \int_0^{\sqrt{\beta_0}} \frac{1}{\sqrt{\beta_0} \beta_0 - |\beta|} \int_0^{\sqrt{\beta_0}} \frac{|\cos \tau|}{(1 + \eta^2)^{3/2}} d\eta d\tau,$$

where we have used the change of variables $\eta = \sqrt{\beta_0} \sin \tau / \sqrt{\beta_0 - |\beta|}$. Lemma 3.3 implies that the integral on the right is bounded below by some positive constant. Therefore

$$\beta \int_0^{\phi_k(\beta)} G''(\beta \cos \tau) \cos \tau d\tau \geq \frac{\tilde{c}}{\beta_0 - |\beta|} - C|\beta|\phi_k(\beta), \quad (4.3)$$

where $\tilde{c} > 0$ is a constant. From (2.5), we see that

$$G'(\beta \cos \phi_k(\beta)) \leq \frac{C}{\sqrt{\beta_0 - |\beta \cos \phi_k(\beta)|}} \leq \frac{C}{\sqrt{\beta_0 - |\beta|}}. \quad (4.4)$$

Combining (4.3), (4.4) and (3.8) gives

$$(-1)^{k-1} Q_\beta(\lambda_\kappa^\alpha(\beta), \beta) \geq (-1)^k \sin \phi_k(\beta) + (-1)^{k-1} \cos \phi_k(\beta) \left\{ \frac{\tilde{c}}{\sqrt{\beta_0 - |\beta|}} - C|\beta|\phi_k(\beta)\sqrt{\beta_0 - |\beta|} \right\} \to \infty \quad (\beta \to \pm \beta_0).$$

We thus obtain (4.2). \qed

Lemma 4.4 shows that

$$Q_\beta(\lambda_\kappa^\alpha(\beta^*), \beta^*) = 0 \quad \text{for some } \beta^* \in I \setminus \{0\}.$$

In what follows, we fix such $\beta^*$ and set

$$\phi^* := \phi_k(\beta^*).$$
Since \( Q_\beta(\lambda_k^*(\beta), \beta) = R(\beta, \phi_k(\beta)) \), we see that the condition \( Q_\beta(\lambda_k^*(\beta^*), \beta^*) = 0 \) is equivalent to
\[
\frac{\beta^* \cos \phi^*}{G'(\beta^* \cos \phi^*)} \int_0^{\phi^*} G''(\beta^* \cos \tau) \cos \tau d\tau = \sin \phi^*.
\] (4.5)

We investigate the properties of \((\beta^*, \phi^*)\).

**Lemma 4.5** There holds
\[
1 - \frac{\beta^* \sin \phi^*}{\cos \phi^*} \frac{d\phi_k}{d\beta}(\beta^*) > 0.
\]

**Proof** A direct computation yields
\[
\frac{1}{\beta^*} \beta^* I(\beta^*, \phi^*) - \frac{\sin \phi^*}{\cos \phi^*} J(\beta^*, \phi^*) = \frac{1}{\sin \phi^* \cos \phi^*} \int_0^{\phi^*} G'(\beta^* \cos \tau) d\tau + G'(\beta^* \cos \phi^*) \cos \phi^* \int_0^{\phi^*} G''(\beta^* \cos \tau) \cos \tau d\tau
\]
\[
= \frac{1}{\sin \phi^* \cos \phi^*} \int_0^{\phi^*} G'(\beta^* \cos \tau) d\tau + \frac{\beta^* \sin \phi^*}{\cos \phi^*} \int_0^{\phi^*} G''(\beta^* \cos \tau) \cos \tau d\tau > 0,
\]
where we have used (4.5) in deriving the second equality. This together with (3.9) and (3.16) shows that
\[
1 - \frac{\beta^* \sin \phi^*}{\cos \phi^*} \frac{d\phi_k}{d\beta}(\beta^*) = \frac{\beta^*}{\beta^* I(\beta^*, \phi^*)} \left( \frac{1}{\beta^*} \beta^* I(\beta^*, \phi^*) - \frac{\sin \phi^*}{\cos \phi^*} J(\beta^*, \phi^*) \right) > 0,
\]
as claimed. \(\square\)

We recall that the function \( h(v) \) is given by (2.2).

**Lemma 4.6** If (1.5a) and (1.6) hold, then \( h(\beta^* \cos \phi^*) > 0 \).

**Proof** To obtain a contradiction, we suppose that \( h(\beta^* \cos \phi^*) \leq 0 \). By Lemma 3.4, we have
\[
\beta^* \int_0^{(k-1)\pi} G''(\beta^* \cos \tau) \cos \tau d\tau \geq 0,
\]
which gives
\[
\beta^* \int_0^{\phi^*} G''(\beta^* \cos \tau) \cos \tau d\tau \geq \beta^* \int_{(k-1)\pi}^{\phi^*} G''(\beta^* \cos \tau) \cos \tau d\tau.
\] (4.6)

According to (ii) of Lemma 2.1, we can use (2.3a). Hence, from the assumption \( h(\beta^* \cos \phi^*) \leq 0 \), we find that
\[
h(\beta^* \cos \tau) < 0 \quad \text{for all } \tau \in [(k-1)\pi, \phi^*).
\]
This particularly implies that the function 
\[ G'(\beta \cos \tau)/|\cos \tau| \] 
is decreasing on \([k - 1, \phi^*]\), since 
\[ (G'(v)/v)' = -G'(v)h(v)/v^2. \] Therefore 
\[ G''(\beta \cos \tau) \beta \cos \tau = G'(\beta \cos \tau)(1 - h(\beta \cos \tau)) \]
\[ > G'(\beta \cos \phi^*) |\cos \tau| \quad \text{for all } \tau \in [(k - 1, \phi^*]. \]

Plugging this into (4.6), we obtain 
\[ \beta^* \int_0^{\phi^*} G''(\beta \cos \tau) \cos \tau d\tau > G'(\beta \cos \phi^*) \int_0^{\phi^*} |\cos \tau| d\tau = \frac{G'(\beta \cos \phi^*) \sin \phi^*}{\cos \phi^*}. \]

This contradicts (4.5), and therefore 
\[ h(\beta \cos \phi^*) > 0. \]

\[ \square \]

**Lemma 4.7** Let (1.5a) and (1.5b) hold and assume that 
\[ h(\beta^*) > 0. \] Then 
\[ R_{\beta}(\beta^*, \phi^*) \beta^* \sin \phi^* > 0. \]

**Proof** We note that the assumptions (1.5a) and (1.5b) give (2.3a) and (2.3b). By (2.3a) and the assumption 
\[ h(\beta^*) > 0, \] we have 
\[ |\beta^* \cos \tau| < v_0 \quad \text{for all } \tau \in [0, \phi^*. \]

A direct computation yields 
\[ R_{\beta}(\beta, \phi) = \frac{G''(\beta \cos \phi) \beta \cos^2 \phi}{G'(\beta \cos \phi)^2} \int_0^{\phi} G''(\beta \cos \tau) \cos \tau d\tau - \frac{\cos \phi}{G'(\beta \cos \phi) \beta} \frac{\partial}{\partial \beta} \left( \beta \int_0^{\phi} G''(\beta \cos \tau) \cos \tau d\tau \right). \]

Since 
\[ (G''(v)v)' = -(G'(v)h(v))' + G''(v), \] we have 
\[ \frac{\partial}{\partial \beta} \left( G''(\beta \cos \tau) \beta \cos \tau \right) = -\frac{d}{dv} (G'(v)h(v))|_{v=\beta \cos \tau} \cdot \cos \tau + G''(\beta \cos \tau) \cos \tau \]
\[ = \frac{\partial}{\partial \tau} \left( G'(\beta \cos \tau)h(\beta \cos \tau) \right) \cdot \cos \tau \cos \tau + G''(\beta \cos \tau) \cos \tau, \]

and hence 
\[ R_{\beta}(\beta, \phi) = \frac{h(\beta \cos \phi) \cos \phi}{G'(\beta \cos \phi)^2} \int_0^{\phi} G''(\beta \cos \tau) \cos \tau d\tau \]
\[ + \frac{\cos \phi}{G'(\beta \cos \phi) \beta} \frac{\partial}{\partial \tau} \left( G'(\beta \cos \tau)h(\beta \cos \tau) \right) \cdot \cos \tau \cos \tau \cdot \sin \tau. \]

Note that 
\[ h(\beta \cos \phi) \int_0^{\phi} G''(\beta \cos \tau) \cos \tau d\tau \]
\[ = (h(\beta \cos \phi) - h(\beta)) \int_0^{\phi} G''(\beta \cos \tau) \cos \tau d\tau - \frac{h(\beta)}{\beta} \int_0^{\phi} \left( \frac{\partial}{\partial \tau} G'(\beta \cos \tau) \right) \cdot \cos \tau \cos \tau \cdot \sin \tau. \]
Substituting this into (4.9), we obtain
\[
R(\beta, \phi) = \frac{(h(\beta \cos \phi) - h(\beta)) \cos \phi}{G'(\beta \cos \phi)} \int_0^\phi G''(\beta \cos \tau) \cos \tau d\tau \\
+ \frac{\cos \phi}{G'(\beta \cos \phi) \beta} \int_0^\phi \frac{\partial}{\partial \tau} \left\{ G'(\beta \cos \tau)(h(\beta \cos \tau) - h(\beta)) \right\} \cdot \frac{\cos \tau}{\sin \tau} d\tau.
\]

We now apply integration by parts to the second term on the right. Then, since
\[
h(\beta \cos \tau) - h(\beta) = O(1 - |\cos \tau|) = O(\sin^2 \tau)
\]
as \(\tau \to \pm \pi, l \in \mathbb{Z}\), we have
\[
\int_0^\phi \frac{\partial}{\partial \tau} \left\{ G'(\beta \cos \tau)(h(\beta \cos \tau) - h(\beta)) \right\} \cdot \frac{\cos \tau}{\sin \tau} d\tau
\]
\[
= \frac{G'(\beta \cos \phi)(h(\beta \cos \phi) - h(\beta)) \cos \phi}{\sin \phi} + \int_0^\phi \frac{G'(\beta \cos \tau)(h(\beta \cos \tau) - h(\beta))}{\sin^2 \tau} d\tau.
\]

This together with (4.5) shows that
\[
R(\beta^*, \phi^*) \beta^* \sin \phi^* = \left( \frac{\beta^* \sin \phi^* \cos \phi^*}{G'(\beta^* \cos \phi^*)} \int_0^\phi G''(\beta^* \cos \tau) \cos \tau d\tau + \cos^2 \phi^* \right) (h(\beta^* \cos \phi^*) - h(\beta^*)) \\
+ \frac{\sin \phi^* \cos \phi^*}{G'(\beta^* \cos \phi^*)} \int_0^\phi G'(\beta^* \cos \tau)(h(\beta^* \cos \tau) - h(\beta^*)) d\tau \\
= h(\beta^* \cos \phi^*) - h(\beta^*) + \frac{\sin \phi^* \cos \phi^*}{G'(\beta^* \cos \phi^*)} \int_0^\phi G'(\beta^* \cos \tau)(h(\beta^* \cos \tau) - h(\beta^*)) d\tau.
\]

We see from (2.3b) and (4.8) that the right-hand side is positive, and (4.7) is proved. \(\square\)

**Lemma 4.8** Let (1.5a), (1.5c) and (1.6) hold and assume that \(h(\beta^*) \leq 0\). Then
\[
R(\beta^*, \phi^*) \beta^* \sin \phi^* > h(\beta^* \cos \phi^*). \quad (4.10)
\]

**Proof** We see from (ii) of Lemma 2.1 that (2.3a) and (2.3c) are satisfied. By (2.3a), Lemma 4.6 and the assumption \(h(\beta^*) \leq 0\), we can take \(\tau_0 \in [0, \phi^* - (k - 1)\pi)\) such that \(|\beta^* | \cos \tau_0 = v_0\). Put
\[
I := ([k - 1]\pi + \tau_0, \phi^*] \cup \bigcup_{j=1}^{k-1} ((j - 1)\pi + \tau_0, j\pi - \tau_0).
\]

Then, from (2.3a), we have
\[
|\beta^* \cos \tau| \begin{cases} < v_0 & \text{if } \tau \in I, \\ \geq v_0 & \text{if } \tau \in [0, \phi^*] \setminus I, \end{cases} \quad (4.11)
\]
and
\[
h(\beta^* \cos \tau) \begin{cases} > 0 & \text{if } \tau \in I, \\ = 0 & \text{if } \tau \in \partial I \setminus \{\phi^*\}. \end{cases} \quad (4.12)
\]
We estimate the second term on the right of (4.9). Using integration by parts and (4.12), we have
\[
\int_I \frac{\partial}{\partial \tau} \left( G'(\beta^* \cos \tau) h(\beta^* \cos \tau) \right) \cdot \frac{\cos \tau}{\sin \tau} \, d\tau = \frac{G'(\beta^* \cos \phi^*) h(\beta^* \cos \phi^*) \cos \phi^*}{\sin \phi^*} + \int_I \frac{G'(\beta^* \cos \tau) h(\beta^* \cos \tau)}{\sin^2 \tau} \, d\tau > \frac{G'(\beta^* \cos \phi^*) h(\beta^* \cos \phi^*) \cos \phi^*}{\sin \phi^*}.
\]
Furthermore, (2.3c) and (4.11) yield
\[
\int_{[0, \phi^*]} I \frac{\partial}{\partial \tau} \left( G'(\beta^* \cos \tau) h(\beta^* \cos \tau) \right) \cdot \cos \tau \sin \tau \, d\tau = - \left. \int_{[0, \phi^*]} \left\{ v \frac{d}{dv} (G'(v) h(v)) \right\} \right|_{v=\beta^* \cos \tau} d\tau \geq 0.
\]
Plugging these inequalities into (4.9) and using (4.5), we obtain
\[
R_{\beta} (\beta^* \cdot \phi^*) \beta^* \sin \phi^* > \frac{h(\beta^* \cos \phi^*) \beta^* \sin \phi^* \cos \phi^*}{G'(\beta^* \cos \phi^*)} \int_0^{\phi^*} G''(\beta^* \cos \tau) \cos \tau d\tau + h(\beta^* \cos \phi^*) \cos^2 \phi^* = h(\beta^* \cos \phi^*),
\]
which proves the lemma. \(\square\)

**Lemma 4.9** Assume (1.5). Then
\[
(-1)^{k-1} \text{sgn}(\beta^*) \frac{d}{d\beta} Q_{\beta}(\lambda_k^0(\beta), \beta) \bigg|_{\beta=\beta^*} > 0. \tag{4.13}
\]

**Proof** Since \(Q_{\beta}(\lambda_k^0(\beta), \beta) = R(\beta, \phi_k(\beta))\), we have
\[
\frac{d}{d\beta} Q_{\beta}(\lambda_k^0(\beta), \beta) \bigg|_{\beta=\beta^*} = R_{\beta}(\beta^*, \phi^*) + R_{\phi}(\beta^*, \phi^*) \frac{d\phi_k}{d\beta}(\beta^*).
\]
To estimate this, we compute \(R_{\phi}(\beta^*, \phi^*)\). A direct computation gives
\[
R_{\phi}(\beta, \phi) = - \cos \phi - \frac{(G'(\beta \cos \phi) - G''(\beta \cos \phi) \beta \sin \phi)}{G'(\beta \cos \phi)^2} \int_0^\phi G''(\beta \cos \tau) \cos \tau d\tau + \frac{\beta \cos \phi}{G'(\beta \cos \phi)} \cdot G''(\beta \cos \phi) \cos \phi
\]
\[
= - \left( \cos \phi + \frac{\beta \sin \phi}{G'(\beta \cos \phi)} \int_0^\phi G''(\beta \cos \tau) \cos \tau d\tau \right) h(\beta \cos \phi).
\]
From (4.5), we find that
\[
R_{\phi}(\beta^*, \phi^*) = \frac{h(\beta^* \cos \phi^*)}{\cos \phi^*}. \tag{4.14}
\]
We consider the two cases: $h(\beta^*) > 0$ and $h(\beta^*) \leq 0$. We first consider the latter case. Lemma A.1 shows that (1.6) is satisfied, and hence we can apply Lemmas 4.6 and 4.8. From (4.10) and (4.14), we have
\[
\beta^* \sin \phi^* \cdot \frac{d}{d\beta} Q_h(\lambda_k^*(\beta), \beta) \bigg|_{\beta = \beta^*} > h(\beta^* \cos \phi^*) \left( 1 - \frac{\beta^* \sin \phi^*}{\cos \phi^*} \frac{d\phi_k}{d\beta}(\beta^*) \right).
\]
Lemmas 4.5 and 4.6 show that the right-hand side is positive. Since the sign of $\beta^* \sin \phi^*$ coincides with that of $(-1)^{k-1} \mathrm{sgn}(\beta^*)$, we obtain (4.13).

Let us consider the other case $h(\beta^*) > 0$. Then we have $(-1)^{k-1} \mathrm{sgn}(\beta^*) R_\beta(\beta^*, \phi^*) > 0$ by Lemma 4.7. Moreover, we see from Lemma 4.6 and (4.14) that $(-1)^{k-1} R_\beta(\beta^*, \phi^*) < 0$. Therefore it is sufficient to show that
\[
\mathrm{sgn}(\beta^*) \frac{d\phi_k}{d\beta}(\beta^*) \leq 0.
\]
To prove this, we define
\[
\tilde{h}(v) := \left\{ \begin{array}{ll}
G'(v) v & (v \neq 0), \\
1 & (v = 0).
\end{array} \right.
\]
By (i) of Lemma 2.1, we see that $\tilde{h} \in C(I) \cap C^2(I \setminus \{0\})$ and $G \tilde{h} \in C^1(I)$. Notice that
\[
\frac{d}{dv} \left( \frac{G(v)^2 \tilde{h}'(v)}{G'(v)} \right) = -G(v) h'(v) \quad \text{for } v \in I \setminus \{0\}.
\]
(2.3b) shows that the right-hand side of this equality is nonnegative if $v \in (-v_0, v_0)$, and hence
\[
\mathrm{sgn}(v) \frac{G(v)^2 \tilde{h}'(v)}{G'(v)} \geq \lim_{v \to 0} \mathrm{sgn}(v) \frac{G(v)^2 \tilde{h}'(v)}{G'(v)} = 0 \quad \text{for } v \in (-v_0, v_0) \setminus \{0\}.
\]
From this we have
\[
\mathrm{sgn}(v) \tilde{h}(v) \text{ is nondecreasing in } (-v_0, v_0) \setminus \{0\}. \tag{4.15}
\]
Since $G'(v) + G''(v) v = (G(v) \tilde{h}(v))'$, we see that $J(\beta, \phi)$ is written as
\[
J(\beta, \phi) = \tilde{h}(\beta \cos \phi) \int_0^\phi G'(\beta \cos \tau) d\tau + \int_0^\phi \frac{\partial}{\partial \tau} \left(G(\beta \cos \tau) \tilde{h}(\beta \cos \tau)\right) \cdot \frac{1}{\beta \sin \tau} d\tau
\]
\[
= (\tilde{h}(\beta \cos \phi) - \tilde{h}(\beta)) \int_0^\phi G'(\beta \cos \tau) d\tau
\]
\[
+ \int_0^\phi \frac{\partial}{\partial \tau} \left(G(\beta \cos \tau) (\tilde{h}(\beta \cos \tau) - \tilde{h}(\beta))\right) \cdot \frac{1}{\beta \sin \tau} d\tau.
\]
Integrating by parts shows that the second term on the right is computed as
\[
\int_0^\phi \frac{\partial}{\partial \tau} \left(G(\beta \cos \tau)(\tilde{h}(\beta \cos \tau) - \tilde{h}(\beta))\right) \cdot \frac{1}{\beta \sin \tau} d\tau
\]
\[
= (\tilde{h}(\beta \cos \phi) - \tilde{h}(\beta)) \frac{G(\beta \cos \phi)}{\beta \sin \phi} + \int_0^\phi \frac{\tilde{h}(\beta \cos \tau) - \tilde{h}(\beta)}{\beta \sin \phi} G(\beta \cos \tau) \cos \frac{\beta \cos \tau}{\beta^2} \sin \frac{\beta \cos \tau}{\beta^2} d\tau.
\]
Therefore
\[
J(\beta^*, \phi^*) = (\tilde{h}(\beta^* \cos \phi^*) - \tilde{h}(\beta^*)) \left( \int_0^{\phi^*} G'(\beta^* \cos \tau) d\tau + \frac{G(\beta^* \cos \phi^*)}{\beta^* \sin \phi^*} \right) + \int_0^{\phi^*} \left( \tilde{h}(\beta^* \cos \tau) - \tilde{h}(\beta^*) \right) \frac{G(\beta^* \cos \tau) \cos \tau}{\beta^* \sin^2 \tau} d\tau.
\]
(4.16)

By (2.3a) and the assumption \( h(\beta^*) > 0 \), we have \( |\beta^* \cos \tau| < v_0 \) for all \( \tau \in [0, \phi^*] \). It follows from (4.15) that the right-hand side of (4.16) is nonpositive, and hence \( J(\beta^*, \phi^*) \leq 0 \). This together with (3.16) shows that
\[
\beta
\]

Therefore there exist
\[
\beta^*_+ \in (0, \beta_0) \quad \text{and} \quad \beta^*_- \in (-\beta_0, 0)
\]
such that
\[
D = \{ \beta^*_+, \beta^*_- \}.
\]
(4.18)

We can now prove Proposition 4.2.

**Proof of Proposition 4.2** By (2.13), we have
\[
D(\lambda_k^0(\beta), \beta, -\beta) = 2 P_\beta(\lambda_k^0(\beta), \beta) Q_\beta(\lambda_k^0(\beta), \beta).
\]
In particular, Lemma 4.3 yields
\[
D := \{ \beta \in I \setminus \{0\}; \ D(\lambda_k^0(\beta), \beta, -\beta) = 0 \} = \{ \beta \in I \setminus \{0\}; \ Q_\beta(\lambda_k^0(\beta), \beta) = 0 \}.
\]
We see from Lemmas 4.3 and 4.4 that
\[
D(\lambda_k^0(\beta), \beta, -\beta) \begin{cases} < 0 \quad \text{if } |\beta| \text{ is small}, \\ > 0 \quad \text{if } |\beta| \text{ is close to } \beta_0. \end{cases}
\]
Furthermore, Lemmas 4.3 and 4.9 show that for any \( \beta^* \in D \),
\[
\text{sgn}(\beta^*) \frac{d}{d\beta} D(\lambda_k^0(\beta), \beta, -\beta) \bigg|_{\beta=\beta^*} = 2 \text{sgn}(\beta^*) P_\beta(\lambda_k^0(\beta^*), \beta^*) \cdot \frac{d}{d\beta} Q_\beta(\lambda_k^0(\beta), \beta) \bigg|_{\beta=\beta^*} > 0.
\]
(4.17)

Therefore there exist \( \beta^*_+ \in (0, \beta_0) \) and \( \beta^*_- \in (-\beta_0, 0) \) such that
\[
D = \{ \beta^*_+, \beta^*_- \}.
\]
(4.18)

From Lemma A.1, we know that (1.6) is satisfied under the assumption (1.5). Hence (ii) of Proposition 3.1 shows that \( C_{k,+}^a \) and \( C_{k,-}^a \) are written as
\[
C_{k,+}^a = \{ (\lambda, u_k^a(\cdot, \beta^*_+(\lambda))) \}_{\lambda \in (\lambda_{2k-1}, \infty)}, \quad C_{k,-}^a = \{ (\lambda, u_k^a(\cdot, \beta^*_-(\lambda))) \}_{\lambda \in (\lambda_{2k-1}, \infty)},
\]
(4.19)

where \( \beta^*_+(\lambda) \) (resp. \( \beta^*_-(\lambda) \)) is the inverse of the function \( (0, \beta_0) \ni \beta \mapsto \lambda_k^0(\beta) \) (resp. \( (-\beta_0, 0) \ni \beta \mapsto \lambda_k^0(\beta) \)). Then, by (4.17), (4.18) and (3.5), we have
\[
\begin{cases} D(\lambda, \beta^*_+(\lambda), -\beta^*_+(\lambda)) = 0 \quad \text{if and only if } \lambda = \lambda_k^0(\beta^*_+), \\ \frac{d}{d\lambda} D(\lambda, \beta^*_+(\lambda), -\beta^*_+(\lambda)) \bigg|_{\lambda = \lambda_k^0(\beta^*_+)} = \left( \frac{d\lambda_k^0}{d\beta^*_+}(\beta^*_+) \right)^{-1} \cdot \frac{d}{d\beta} D(\lambda_k^0(\beta), \beta, -\beta) \bigg|_{\beta=\beta^*_+} > 0. \end{cases}
\]
(4.20)

Thus we obtain the desired conclusion by applying (i) and (ii) of Proposition 2.10. \( \square \)
Remark 4.10 Since (2.13) and (3.15) give $Q_\beta(\lambda^0_k(-\beta), -\beta) = -Q_\beta(\lambda^0_k(\beta), \beta)$, we infer that $\beta^- = -\beta^+$. Hence the bifurcation points $(\lambda^0_k(\beta^+), u^0_k(\cdot, \beta^+)) \in C^0_{k,+}$ and $(\lambda^0_k(\beta^-), u^0_k(\cdot, \beta^-)) \in C^0_{k,-}$ obtained in Proposition 4.2 satisfy

$$(\lambda^0_k(\beta^-), u^0_k(\cdot, \beta^-)) = (\lambda^0_k(\beta^+), -u^0_k(\cdot, \beta^+)).$$

4.3 Remark on the assumption (1.5)

At the end of this section, we observe that Proposition 4.2 is still true for $k = 1$ if we drop the assumption (1.5b).

Proposition 4.11 Under the assumptions (1.5a), (1.5c) and (1.6), Proposition 4.2 holds for $k = 1$.

Proof Let $\beta^* \in I \setminus \{0\}$ satisfy $Q(\lambda^0_1(\beta^*), \beta^*) = 0$. If (4.13) is satisfied for $k = 1$, then the proposition can be proved in the same way as Proposition 4.2. Therefore we only need to show that

$$h(\beta^*) \leq 0,$$

(4.21)

since this enables us to apply Lemma 4.8 to obtain (4.13). Contrary to (4.21), suppose that $h(\beta^*) > 0$. Then (2.3a) yields

$$h(\beta^* \cos \tau) > 0 \text{ for all } \tau \in [0, \phi^*],$$

where $\phi^* := \phi_1(\beta^*) \in (0, \pi/2)$. From this and the fact that $(G'(v)/v)' = -G'(v)h(v)/v^2$, we see that the function $G'(\beta^* \cos \tau)/\cos \tau$ is increasing on $[0, \phi^*]$. Hence

$$\beta^* \int_0^{\phi^*} G''(\beta^* \cos \tau) \cos \tau d\tau = \int_0^{\phi^*} G'(\beta^* \cos \tau)(1 - h(\beta^* \cos \tau))d\tau$$

$$< \int_0^{\phi^*} \frac{G'(\beta^* \cos \phi^*)}{\cos \phi^*} \cos \tau d\tau$$

$$= \frac{G'(\beta^* \cos \phi^*)}{\cos \phi^*} \sin \phi^*.$$

This gives $Q(\lambda^0_1(\beta^*), \beta^*) < 0$, a contradiction. Therefore we obtain (4.21), and the proof is complete. $\square$

5 Proof of Theorem 1.1

To prove Theorem 1.1, we compute the Morse index of solutions on $S^e$ and $S^o$. We write $\lambda^*$ for the number $\lambda^*_{k,+}$ obtained in Proposition 4.2.

Proposition 5.1 For $k \in \mathbb{N}$, the following hold.

(i) Let (1.6) hold and let $(\lambda, u) \in C^e_k$. Then $u$ is nondegenerate and $i(u) = 2k$.

(ii) Let (1.5) hold and let $(\lambda, u) \in C^o_k$. Then $u$ is nondegenerate unless $\lambda \neq \lambda^*$ and

$$i(u) = \begin{cases} 2k - 1 & (\lambda < \lambda^*), \\ 2k - 2 & (\lambda \geq \lambda^*). \end{cases}$$
In what follows, we fix \( k \in \mathbb{N} \). For \( n \in \mathbb{N} \cup \{0\} \), let \( \mu_n^\alpha(\beta) \) (resp. \( \mu_n^\varepsilon(\beta) \)) denote the \((n+1)\)-th largest eigenvalue of (1.4) for \((\lambda, u) = (\lambda_k^\varepsilon(\beta), u_k^\varepsilon(\cdot, \beta)) \in C_k^\varepsilon \) (resp. \((\lambda, u) = (\lambda_k^\varepsilon(\beta), u_k^\varepsilon(\cdot, \beta)) \in C_k^\varepsilon \)). We see from Lemma 2.7 that \( \mu_n^\alpha(\beta) \) and \( \mu_n^\varepsilon(\beta) \) are continuous with respect to \( \beta \). In the following two lemmas, we give basic estimates of \( \mu_n^\alpha(\beta) \) and \( \mu_n^\varepsilon(\beta) \).

**Lemma 5.2** There hold \( \mu_{2k-2}^\varepsilon(0) > 0 \) and \( \mu_{2k-1}^\alpha(0) > 0 \).

**Proof** It is elementary to show that the \((n+1)\)-th eigenvalue \( \mu_n \) of (1.4) for \( u = 0 \) is given by

\[
\mu_n = \lambda f'(0) - \{(j - 1)\pi\}^2, \quad \mu_n = \lambda f'(0) - z_j^2 \quad (j \in \mathbb{N}).
\]

Hence

\[
\mu_{2k-2}^\varepsilon(0) > \mu_{2k-1}^\alpha(0) = \lambda f'(0) - z_k^2 = 0,
\]

\[
\mu_{2k-1}^\alpha(0) > \mu_{2k}^\varepsilon(0) = \lambda f'(0) - (k\pi)^2 = 0,
\]

as desired. \( \square \)

**Lemma 5.3** Assume that (1.6) holds. Then \( \mu_{2k-1}^\alpha(\beta) < 0 \) and \( \mu_{2k}^\varepsilon(\beta) < 0 \) for all \( \beta \in I \setminus \{0\} \).

**Proof** Let \( Z(w) \) denote the number of zeros of a function \( w \) in \((-1, 1) \setminus \{0\} \). By Lemma 2.9, it suffices to show that

\[
Z(u_k^\varepsilon(\cdot, \beta)) = 2k - 2, \quad Z(u_k^\alpha(\cdot, \beta)) = 2k,
\]

\[
u_k^\varepsilon(-0, \beta)u_k^\varepsilon(+0, \beta) < 0, \quad u_k^\varepsilon(-0, \beta)u_k^\varepsilon(+0, \beta) > 0
\]

for \( \beta \in I \setminus \{0\} \). To derive these, we recall that any \( u \in S_\lambda \) is written as

\[
u(x) = \begin{cases}
U \left( \sqrt{\lambda}(x + 1), \beta_1 \right) = G \left( \beta_1 \cos \Theta \left( \sqrt{\lambda}(x + 1), \beta_1 \right) \right) & \text{for } x \in [-1, 0), \\
U \left( \sqrt{\lambda}(1 - x), \beta_2 \right) = G \left( \beta_2 \cos \Theta \left( \sqrt{\lambda}(1 - x), \beta_2 \right) \right) & \text{for } x \in (0, 1],
\end{cases}
\]

where \( \beta_1 = G(u(-1)) \) and \( \beta_2 = G(u(1)) \). This implies that if \( \beta_1 \beta_2 \neq 0 \) and

\[
\left( m_j - \frac{1}{2} \right) \pi < \theta(\lambda, \beta_j) = \Theta(\sqrt{\lambda}, \beta_j) \leq \left( m_j + \frac{1}{2} \right) \pi \quad \text{for some } m_j \in \mathbb{N} \cup \{0\},
\]

then \( Z(u) = m_1 + m_2 \) and \( \text{sgn}(u(-0)u(+0)) = \text{sgn}((-1)^{m_1+m_2} \beta_1 \beta_2) \). Since we know that \( \theta(\lambda_k^\varepsilon(\beta), \beta) = \phi_k(\beta) \in \{(k - 1)\pi, (k - 1/2)\pi\} \) and \( \theta(\lambda_k^\varepsilon(\beta), \beta) = k\pi \), we have

\[
Z(u_k^\varepsilon(\cdot, \beta)) = (k - 1) + (k - 1) = 2k - 2, \quad Z(u_k^\varepsilon(\cdot, \beta)) = k + k = 2k,
\]

\[
\text{sgn}(u_k^\varepsilon(-0, \beta)u_k^\varepsilon(+0, \beta)) = \text{sgn}((-1)^{(k-1)+(k-1)} \cdot (-\beta^2)) = \text{sgn}(-\beta^2) < 0,
\]

\[
\text{sgn}(u_k^\varepsilon(-0, \beta)u_k^\varepsilon(+0, \beta)) = \text{sgn}((-1)^{k+k} \cdot \beta^2) = \text{sgn}(\beta^2) > 0
\]

for \( \beta \in I \setminus \{0\} \). Therefore the lemma follows. \( \square \)

Let us show Proposition 5.1.

**Proof of Proposition 5.1** First we prove (i). Lemma 5.2 shows that \( \mu_{2k-1}^\varepsilon(\beta) \) is positive if \(|\beta|\) is small enough. As shown in the proof of Lemma 4.1, we know that \( D(\lambda_k^\varepsilon(\beta), \beta) \neq 0 \) for \( \beta \in I \setminus \{0\} \). From this and Lemma 2.3, we see that \( \mu_{2k-1}^\varepsilon(\beta) \) never vanishes. Therefore \( \mu_{2k-1}^\varepsilon(\beta) > 0 \) for all \( \beta \in I \). Thus (i) is verified by combining this with Lemma 5.3.
Next we prove (ii). We recall that (4.18) holds. Hence Lemma 2.3 gives
\[ \mu_n^o(\beta) \neq 0 \text{ for all } n \in \mathbb{N} \cup \{0\} \text{ and } \beta \in I \setminus \{0, \beta_+^*, \beta_-^*\}, \]
\[ \mu_{n_+}^o(\beta_+^*) = \mu_{n_-}^o(\beta_-^*) = 0 \text{ for some } n_+, n_- \in \mathbb{N} \cup \{0\}. \] (5.1)
Moreover, Lemma 5.2 shows that \( \mu_n^o(\beta) \) is positive if \( |\beta| \) is small and \( n \leq 2k - 2 \).

Combining these with Lemma 5.3, we deduce that
\[ \mu_{2k-2}^o(\beta_+^*) = \mu_{2k-2}^o(\beta_-^*) = 0, \] (5.2)
\[ \mu_{2k-3}^o(\beta) > 0 \text{ for all } \beta \in I, \text{ provided } k \geq 2. \] (5.3)

To investigate the behavior of \( \mu_{2k-2}^o(\beta) \), we apply (iii) of Proposition 2.10. For this purpose, we use the parametrization of \( C_{2k+}^o \) and \( C_{2k+}^o \) as in (4.19). Then \( \mu_n^o(\beta_\pm^o(\lambda)) = \mu_n(u_k^o(\cdot, \beta_\pm^o(\lambda))). \)

By (5.2), we can apply (2.32) for \( n = 2k - 2, u(\cdot, \lambda) = u_k^o(\cdot, \beta_\pm^o(\lambda)), \beta_1(\lambda) = \beta_\pm^o(\lambda) \) and \( \beta_2(\lambda) = -\beta_\pm^o(\lambda) \) to obtain
\[
\text{sgn} \left( \frac{d}{d\lambda} \mu_{2k-2}^o(\beta_\pm^o(\lambda)) \Bigg|_{\lambda=\lambda_k^o(\beta_\pm^o)} \right) = -\text{sgn} \left( \frac{d}{d\lambda} D(\lambda, \beta_\pm^o(\lambda), -\beta_\pm^o(\lambda)) \Bigg|_{\lambda=\lambda_k^o(\beta_\pm^o)} \right).
\]

According to (4.20), we know that the right-hand side is negative. Therefore it follows from (5.1) and (5.2) that
\[
\mu_{2k-2}^o(\beta_\pm^o(\lambda)) = \begin{cases} 
> 0 & \text{if } \lambda < \lambda_k^o(\beta_\pm^o), \\
< 0 & \text{if } \lambda > \lambda_k^o(\beta_\pm^o).
\end{cases}
\]

Combining this with Lemma 5.3 and (5.3), we conclude that
\[
i(u_k^o(\cdot, \beta_\pm^o(\lambda)) = \begin{cases} 
2k - 1 & \text{if } \lambda < \lambda_k^o(\beta_\pm^o), \\
2k - 2 & \text{if } \lambda \geq \lambda_k^o(\beta_\pm^o).
\end{cases}
\]

As noted in Remark 4.10, we know that \( \lambda_k^o(\beta_\pm^o) = \lambda_k^o(\beta_-^*). \) Consequently, (ii) is proved. \( \square \)

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1** We note that by the assumption (1.5), the condition (1.6) is satisfied. We put \( C_{2k-1} = C_{k+}^o, C_{2k} = C_{k+}^o. \) Then (i) and (ii) follow immediately from Proposition 3.1 and Lemma 4.1. Moreover, (iii) and (iv) are direct consequences of Propositions 4.2 and 5.1 and Remark 4.10. Therefore the proof is complete. \( \square \)

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**Appendix A: Relation between (1.5) and (1.6)**

In this section, we show that (1.5) implies (1.6).

**Lemma A.1** If (1.5) holds, then (1.6) is satisfied.
Proof Assume (1.5). First we show that
\[
\frac{f'(u)F(u)}{f(u)^2} < 1 \quad \text{for } u \in (-1, 1) \setminus \{0\}. \tag{A.1}
\]

We see from (1.5b) that for \( u \in (-u_0, u_0) \setminus \{0\}, \)
\[
\frac{f'(u)F(u)}{f(u)^2} < \lim_{s \to 0} \frac{f'(s)F(s)}{f(s)^2} = f'(0) \lim_{s \to 0} \frac{2f(s)}{2f(s)f'(s)} = 1,
\]
where we have used L'Hôpital's rule in deriving the first equality. Moreover, (1.5a) yields \( f'(u)F(u)/f(u)^2 \leq 0 \) for \( u \in (-1, -u_0] \cup [u_0, 1) \). Therefore (A.1) holds.

Now we derive (1.6). Notice that
\[
1 - \frac{f'(u)F(u)}{f(u)^2} = \frac{d}{du} \left( \frac{F(u)}{f(u)} - u \right).
\]
From this and the fact that \( F(u)/f(u) \to 0 \) as \( u \to 0 \), we have
\[
\int_0^u \left( 1 - \frac{f'(s)F(s)}{f(s)^2} \right) ds = \frac{F(u)}{f(u)} - u = F(u) \frac{1 - f'(u)u}{f(u)} - u \left( 1 - \frac{f'(u)F(u)}{f(u)^2} \right).
\]
Hence
\[
1 - \frac{f'(u)u}{f(u)} = \frac{f(u)u}{F(u)} \left( 1 - \frac{f'(u)F(u)}{f(u)^2} \right) + \frac{f(u)}{F(u)} \int_0^u \left( 1 - \frac{f'(s)F(s)}{f(s)^2} \right) ds.
\]
(1.6) then follows immediately from (A.1).

\[\square\]

Appendix B: Proofs of Lemmas 2.1 and 2.15

This section provides the proofs of Lemmas 2.1 and 2.15.

Proof of Lemma 2.1 By (1.2), we see that the function given by (2.1) belongs to \( C^3((-1, 1) \setminus \{0\}) \) and its derivative does not vanish in \((-1, 1) \setminus \{0\}\). Hence the inverse function theorem shows that \( G \in C^3(I \setminus \{0\}) \).

Let \( u = G(v) \). Differentiating the equality \( F(u) = v^2 \), we have
\[
G'(v) = \frac{v}{f(u)} = \frac{\text{sgn}(u)\sqrt{F(u)}}{f(u)} , \quad G''(v) = \frac{1}{f(u)} \left( 1 - \frac{f'(u)F(u)}{f(u)^2} \right) , \tag{B.1}
\]
\[
\frac{d}{dv} (G'(v)h(v)) = -G''(v)v = \frac{\sqrt{F(u)}}{f(u)} \frac{d}{du} \left( \frac{f'(u)F(u)}{f(u)^3} \right) , \tag{B.2}
\]
provided that \( v \neq 0 \). Since Taylor's theorem yields
\[
f'(s) = f'(0) + f''(0)s + o(s) , \quad f(s) = f'(0)s + \frac{1}{2} f''(0)s^2 + o(s^2) , \quad F(s) = f'(0)s^2 + \frac{1}{3} f''(0)s^3 + o(s^3) \quad \text{as } s \to 0 ,
\]
we deduce that
\[
\lim_{v \to 0} G'(v) = \frac{1}{\sqrt{f'(0)}} , \quad \lim_{v \to 0} G''(v) = -\frac{f''(0)}{3 f'(0)^2} . \tag{B.3}
\]
Therefore $G \in C^2(I)$. Note that $G''(v)v$ is written as $G''(v)v = G'(v)(1 - f'(u)G'(v)^2)$. We hence have $G''(v)v \in C^1(I)$, and (i) is verified.

From (B.1), we see that $G'(v) > 0$ and

$$h(v) = \frac{f'(u)F(u)}{f(u)^2}.$$

The equivalence of (1.5) and (2.3) then follows easily from this and (B.2). We also see from (B.5) in the classical sense. Hence it follows from (2.35) that for any $\eta$ suppose that

$$G(v) = \psi(v)\text{ is symmetric, since the right-hand side of (2.35) vanish if } \varphi, \psi \in \mathcal{X}.'$$

First we claim that the following holds:

if $\eta \in \mathcal{Y}$ and $\varphi \in \mathcal{Z}$ satisfy $\langle \varphi, T\psi \rangle = \langle \eta, \psi \rangle$ for any $\psi \in \mathcal{X}$, then $\varphi \in \mathcal{X}$ and $T\varphi = \eta$.

(B.4)

Suppose that $\eta \in \mathcal{Y}$, $\varphi \in \mathcal{Z}$ and $\langle \varphi, T\psi \rangle = \langle \eta, \psi \rangle$ for $\psi \in \mathcal{X}$. Then $\varphi$ satisfies the following equation in the distributional sense:

$$\varphi_{xx} + q(x)\varphi = \eta(x), \quad x \in (-1, 1) \setminus \{0\}.$$

(B.5)

From a regularity theory for differential equations, we see that $\varphi$ belongs to $X_0$ and satisfies (B.5) in the classical sense. Hence it follows from (2.35) that for any $\psi \in \mathcal{X}$,

$$0 = \psi(1)\varphi_x(1) - \psi(-1)\varphi_x(-1) - \varphi_x(0)\left\{(\varphi(-0) + a\varphi_x(-0)) - (\varphi(+0) + a\varphi_x(+0))\right\}$$

$$+ \left\{(\varphi(-0) + a\varphi_x(-0))(\varphi_x(0) - \varphi_x(+0))\right\}.$$

Since $\psi \in \mathcal{X}$ is arbitrary, we find that $\varphi_x(1) = \varphi_x(-1) = 0, \varphi(-0) + a\varphi_x(-0) = \varphi(+0) - a\varphi_x(+0)$ and $\varphi_x(-0) = \varphi_x(+0)$. Therefore $\varphi \in \mathcal{X}$ and $T\varphi = \eta$, as claimed.

Next we show that

$$\|\varphi\|_\mathcal{Z} \leq C\|T\varphi\|_\mathcal{Z} \quad \text{for all } \varphi \in K(T) \cap \mathcal{X},$$

(B.6)
where $C > 0$ is a constant. Suppose that this claim were false. Then we could find a sequence \( \{\phi_n\} \subset K(T) \cap \mathcal{X} \) such that \( \|\phi_n\|_\mathcal{Z} = 1 \) and \( T \phi_n \to 0 \) in \( \mathcal{Z} \) as \( n \to \infty \). Since \( \|(\phi_n)_{xx}\|_\mathcal{Z} = \|T \phi_n - q \phi_n\|_\mathcal{Z} \leq \|T \phi_n\|_\mathcal{Z} + \|q\| \|\phi_n\|_\mathcal{Z} \), we see that \( \{(\phi_n)_{xx}\} \) is bounded in \( \mathcal{Z} \). According to the Rellich–Kondrachov theorem, there are a subsequence \( \{\tilde{\phi}_n\} \) of \( \{\phi_n\} \) and \( \varphi \in \mathcal{Z} \) such that \( \tilde{\phi}_n \to \varphi \) in \( \mathcal{Z} \). Since \( T \) is symmetric, we have
\[
\langle \varphi, T \psi \rangle = \lim_{n \to \infty} \langle \tilde{\phi}_n, T \psi \rangle = \lim_{n \to \infty} \langle T \tilde{\phi}_n, \psi \rangle = 0 \quad \text{for all } \psi \in \mathcal{X}.
\]
By (B.4), we infer that \( \varphi \in \mathcal{X} \) and \( T \varphi = 0 \). Hence \( \varphi \in K(T) \). On the other hand, since \( \varphi_n \) satisfies \( \|\varphi_n\|_\mathcal{Z} = 1 \) and \( \varphi_n \in K(T) \perp \), we have \( \|\varphi\|_\mathcal{Z} = 1 \) and \( \varphi \in K(T) \perp \). This leads to a contradiction, because no nonzero function exists in \( K(T) \cap K(T) \perp \). Therefore (B.6) holds.

Finally let us prove (2.36). We have \( R(T) \perp = K(T) \), since using (B.4) for \( \eta = 0 \) yields \( R(T) \perp \subset K(T) \) and the fact that \( T \) is symmetric gives \( K(T) \subset R(T) \perp \). From this it follows that \( \overline{R(T)} = K(T) \perp \), where \( \overline{R(T)} \) is the closure of \( R(T) \) in \( \mathcal{Z} \). Therefore we only need to show that
\[
\overline{R(T)} \cap \mathcal{Y} = R(T).
\]
Since it is clear that \( R(T) \subset \overline{R(T)} \cap \mathcal{Y} \), what is left is to prove \( \overline{R(T)} \cap \mathcal{Y} \subset R(T) \). Let \( \eta \in \overline{R(T)} \cap \mathcal{Y} \). By definition, there exists a sequence \( \{\psi_n\} \subset \mathcal{X} \) such that \( T \psi_n \to \eta \) in \( \mathcal{Z} \) as \( n \to \infty \). Since \( \mathcal{X} = (K(T) \perp \cap \mathcal{X}) \oplus K(T) \), we can take \( \tilde{\psi}_n \in K(T) \perp \cap \mathcal{X} \) such that \( \varphi_n - \tilde{\psi}_n \in K(T) \). Then
\[
T \tilde{\psi}_n = T \varphi_n \to \eta \quad \text{in } \mathcal{Z} \text{ as } n \to \infty.
\]
This together with (B.6) shows that \( \{\tilde{\psi}_n\} \) is a Cauchy sequence in \( \mathcal{Z} \). Hence \( \tilde{\psi}_n \to \varphi \) in \( \mathcal{Z} \) for some \( \varphi \in \mathcal{Z} \). Combining this with (B.8) and the fact that \( T \) is symmetric, we deduce that
\[
\langle \varphi, T \psi \rangle = \lim_{n \to \infty} \langle \tilde{\psi}_n, T \psi \rangle = \lim_{n \to \infty} \langle T \tilde{\psi}_n, \psi \rangle = \langle \eta, \psi \rangle \quad \text{for all } \psi \in \mathcal{X}.
\]
We see from (B.4) that \( \varphi \in \mathcal{X} \) and \( T \varphi = \eta \), which means \( \eta \in R(T) \). Thus (B.7) holds, and the proof is complete.

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