Sobolev inequalities on product Sierpinski spaces

Xuan Liu* and Zhongmin Qian†

Abstract

On fractals, different measures (mutually singular in general) are involved to measure volumes of sets and energies of functions. Singularity of measures brings difficulties in (especially non-linear) analysis on fractals. In this paper, we prove a type of Sobolev inequalities, which involve different and possibly mutually singular measures, on product Sierpinski spaces. Sufficient and necessary conditions for the validity of these Sobolev inequalities are given. Furthermore, we compute the sharp exponents which appears in the sufficient and necessary conditions for the product Kusuoka measure, i.e. the reference energy measure on Sierpinski spaces.

Keywords. Product Sierpinski spaces, Sobolev inequalities

Mathematics Subject Classification. 28A80

1 Introduction

Fractals are spaces which in general possess singularities on one hand, and satisfy certain self-similar properties on the other. A large amount of research on fractal spaces was motivated by the study of disordered media in statistical physics. (See [34, 14].) Early literatures on analysis on fractals mainly addressed the following problems:

(i) constructions of analytic structures (Dirichlet forms in particular) on fractals and spectral properties of Laplacian operators defined as the associated self-adjoint operators (see, for example, [22, 8, 23, 27, 3, 24, 33] and references therein);
(ii) constructions of diffusion processes on fractals and heat kernel estimates for these processes (see, for example, [4, 1, 11, 28, 7, 2, 12, 26] and literature quoted in these works).

Properties of the Laplacian operators on fractals are relatively well understood. Recent researches on analysis on fractals have been concentrated on the following topics:

(iii) function spaces (such as Lipschitz spaces, Sobolev spaces, Besov spaces and etc.) on fractals ([10, 21, 35, 17, 6, 32] and etc.);
(iv) gradients on fractals and their applications on (non-linear) partial differential equations on fractals (see, for example, [37, 15, 16, 18, 19, 20, 31] and references listed in these works);
(v) stochastic partial differential equations on fractals ([18, 13, 38] and etc.).
Most of the existing literature addresses only one-fold fractal spaces, and research on product spaces of fractals however remains limited (but see \[36, 5\] and related references therein). As demonstrated by Euclidean spaces, analysis on multi-dimensional spaces are more complicated than that on one-dimensional space. For example, on \(\mathbb{R}\) or \([0, 1]\), functions with finite energy are automatically Hölder continuous, however, such functions on \(\mathbb{R}^n\) or \([0, 1]^n\) for \(n \geq 2\) are only Borel measurable. On the other hand, higher dimensionality introduces more interesting geometric and analytic features, which allows geometric objects (for example curves and surfaces) and analytic objects (for example functions and their derivatives) to have much richer and more profound properties. Similar situation also happens on fractal spaces. In fact, many fractals share the feature that the spectral dimension is strictly less than 2, and this is crucial to many results on fractals. (Some fractals, depending on the dimension of the ambient spaces, could have spectral dimensions larger than 2. For example, Sierpinski carpets in \(\mathbb{R}^3\) has spectral dimension greater than 2; see [2, p. 706].)

There is a remarkable difference between analysis on Euclidean spaces and that on fractals: different measures are involved to measure volume of sets and energy of functions, and these measures are singular to each other in general. The singularity between these measures introduces substantial difficulties in non-linear analysis on fractal spaces. Sobolev inequalities involving singular measures were established in [32] for (one-fold) Sierpinski spaces, where these inequalities were applied to semi-linear partial differential equations on fractals. In this paper, we consider products of Sierpinski spaces and study Sobolev inequalities involving singular measures on these spaces. The main difficulty in establishing these inequalities is that so far there is no appropriate analogue of the following Newton-Leibniz formula

\[
f(x) - f(y) = \int_0^1 \langle \dot{\gamma}(s), \nabla f(\gamma(s)) \rangle \, ds, \quad x, y \in \mathbb{R}^n,
\]

where \(\gamma : [0, 1] \to \mathbb{R}^n\) is the geodesic (parametrised by arc length) connecting \(x\) and \(y\). To overcome this, our main idea is to exploit the self-similar property of Sierpinski spaces and derive the Sobolev inequalities by an iteration argument.

The present paper is organized as follows. In Section 2, we set up the notations and briefly review some related results. In Section 3, we formulate and give the proof of Sobolev inequalities involving singular measures on product Sierpinski spaces. Sufficient and necessary conditions for any Sobolev inequality to hold are also given in this section. Section 4 is devoted to the sharp values of the exponents in the sufficient and necessary conditions introduced in Section 3. The main difficulty in computing these sharp values lies in the non-commutativity of the matrices involved in the harmonic structure on Sierpinski spaces. Though results in this paper are formulated and proved specifically for product Sierpinski spaces, however we believe that our methods should be easily adapted and most of the results (except those in Section 4) should remain valid for products (with possibly different components) of more general fractals.

### 2 Preliminaries

In this section, we shall introduce notations that will be in force throughout this paper, and give a brief review of analysis on Sierpinski spaces.

**Sierpinski spaces.**

Let \(V_{0,0} = \{p_1, p_2, p_3\} \subseteq \mathbb{R}^2\) with \(p_1 = (0, 0), \ p_2 = (1, 0), \ p_3 = (1/2, \sqrt{3}/2)\). Let \(F_i : \mathbb{R}^2 \to \mathbb{R}^2, \ i = 1, 2, 3\) be the contractions defined by

\[
F_i(x) = 2^{-1}(x + p_i), x \in \mathbb{R}^2.
\]
The \( m \)-lattices \( V_{0,m} \), \( m \in \mathbb{N} \) are the sets defined inductively by
\[
V_{0,m} = \bigcup_{i=1,2,3} F_i(V_{0,m-1}), \quad m \in \mathbb{N}_+.
\]

The one-fold compact Sierpinski space \( S_0 \) is defined to be the closure \( S_0 = \text{closure}(\bigcup_{m=0}^{\infty} V_{0,m}) \), and the one-fold infinite Sierpinski space \( S \) is defined to be
\[
S = \bigcup_{m=0}^{\infty} F_1^{-m} [S_0 \cup (-S_0)].
\]

Clearly, \( S \) can be written as \( S = \bigcup_{i \in \mathbb{Z}} S_i \), where \( S_i, i \in \mathbb{Z} \) are non-overlapping translations of \( S_0 \). One may order the two-ended sequence \( S_i, i \in \mathbb{Z} \) according to their distances to the origin \( p_1 \). However, the ordering of \( S_i, i \in \mathbb{Z} \) is not important in this paper.

We denote by \( V_{i,m} \subseteq S_i \) the \( m \)-lattice on \( S_i \), which is a translation of the lattice \( V_{0,m} \) on \( S_0 \), and define the \( m \)-lattice \( V_m \) on \( S \) to be the union
\[
V_m = \bigcup_{i \in \mathbb{Z}} V_{i,m}, \quad m \in \mathbb{N}.
\]

Regions and points in \( S_0 \) can be labelled systematically in the following way. Let
\[
W_* = \{ \omega = \omega_1 \omega_2 \omega_3 \ldots : \omega_i \in \{1, 2, 3\}, i \in \mathbb{N}_+ \}
\]
be the family of infinite sequences \( \omega = \omega_1 \omega_2 \omega_3 \ldots \) of symbols in \( \{1, 2, 3\} \). For each \( \omega \in W_* \), denote by \( [\omega]_m = \omega_1 \omega_2 \ldots \omega_m, m \in \mathbb{N} \) the truncation of \( \omega \) of length \( m \), and define the map
\[
F_{[\omega]_m} = F_{\omega_1} \cdots F_{\omega_m}.
\]

**Definition 2.1.** A subset \( S \subseteq S_0 \) of the form \( F_{[\omega]_m}(S_0) \), \( \omega \in W_*, m \in \mathbb{N} \) is called a dyadic simplex in \( S_0 \). Similarly, a subset \( S \subseteq S \) is called a dyadic simplex in \( S \), if \( S = F_1^{-k} \circ F_{[\omega]_m}(S_0) \) for some \( \omega \in W_* \) and some \( m, k \in \mathbb{N} \).

For any \( \omega \in W_* \), since \( F_i, i = 1, 2, 3 \) are contractions, the simplexes \( F_{[\omega]_m}(S_0) \) shrinks to a point \( \pi(\omega) \) in \( S_0 \). In other words, the set \( \cap_{m=0}^{\infty} F_{[\omega]_m}(S_0) \) contains the point \( \pi(\omega) \in S_0 \) as its unique element. The map \( \pi : W_* \to S_0, \omega \mapsto \pi(\omega) \) is surjective and gives a systematic labelling to points in \( S_0 \). This scheme applies similarly to the translations \( S_i \) of \( S_0 \).

**Definition 2.2.** The Hausdorff measure \( \nu \) on \( S \), normalized so that \( \nu(S_0) = 1 \), is the unique Borel measure on \( S \) such that \( \nu(F_{[\omega]_m}(S_i)) = 3^{-m} \) for all \( m \in \mathbb{N}, i \in \mathbb{Z}, \omega \in W_* \).

Notations on product spaces are similar. Specifically, for \( n \in \mathbb{N}_+ \), the \( n \)-fold compact Sierpinski space \( S^n_0 \) is defined to be the \( n \)-fold Cartesian product \( S^n_0 = S_0 \times \cdots \times S_0 \) with its product topology, and the \( n \)-fold infinite Sierpinski space \( S^n \) is defined to be the \( n \)-fold Cartesian product of \( S \). We shall denote a generic point in \( S^n \) by \( x = (x_1, \ldots, x_n), x_i \in S, 1 \leq i \leq n \).

For any \( n \)-tuple \( i = (i_1, \ldots, i_n) \in \mathbb{Z}^n \), define
\[
S^n_i = S^n_{(i_1, \ldots, i_n)} = S_{i_1} \times \cdots \times S_{i_n},
\]
where \( S_j, j \in \mathbb{Z} \) are the translations of \( S_0 \) defined above. Then \( S^n_i, i \in \mathbb{Z}^n \) are non-overlapping translations of \( S^n_0 \), and \( S^n = \bigcup_{i \in \mathbb{Z}^n} S^n_i \).
Definition 2.3. For any \( n \)-tuple \( \tau = (\tau_1, \ldots, \tau_n) \in \{1, 2, 3\}^n \), we define the map \( F_\tau : \mathbb{S}^n \to \mathbb{S}^n \) by

\[
F_\tau(x) = F_{(\tau_1, \ldots, \tau_n)}(x) = F_{\tau_1} \otimes \cdots \otimes F_{\tau_n}(x_1, \ldots, x_n) = (F_{\tau_1}(x_1), \ldots, F_{\tau_n}(x_n)), \quad x = (x_1, \ldots, x_n) \in \mathbb{S}^n.
\]

Note the difference between the notations \( F_{i_1 \ldots i_n} \) and \( F_{(i_1 \ldots i_n)} \). The former denotes the composition of maps \( F_{i_k}, 1 \leq k \leq n \) on \( \mathbb{S} \), while the latter denotes the product of these maps.

Definition 2.4. Let \( W^n_s = \{ \omega = \omega_1 \omega_2 \omega_3 \ldots : \omega_i \in \{1, 2, 3\}^n, i \in \mathbb{N}_+ \} \) be the family of infinite sequences \( \omega = \omega_1 \omega_2 \omega_3 \ldots \) of \( n \)-tuples \( \omega_i = (\omega_{i1}, \ldots, \omega_{in}) \in \{1, 2, 3\}^n \). For each \( \omega = \omega_1 \omega_2 \ldots \in W^n_s \), similar to the one-fold case, we denote by

\[
[\omega]_m = \omega_1 \ldots \omega_m = (\omega_{11}, \ldots, \omega_{1m}) \ldots (\omega_{nm}, \ldots, \omega_{mn})
\]

the truncation of \( \omega \) of length \( m \), and define the map \( F_{[\omega]_m} : \mathbb{S}^n \to \mathbb{S}^n \) by

\[
F_{[\omega]_m} = F_{\omega_1 \ldots \omega_m} = F_{(\omega_{11} \ldots \omega_{1m})} \circ \cdots \circ F_{(\omega_{nm} \ldots \omega_{mn})}.
\]

Though the same character \( \omega \) is used to denote elements of \( W_s \) and \( W^n_s \), it would be clear from the context that which of the families \( W_s \) and \( W^n_s \) is referred to.

Definition 2.5. The Hausdorff measure \( \nu_n \), normalized so that \( \nu_n(\mathbb{S}^n_0) = 1 \), on \( \mathbb{S}^n \) is defined to be the product \( \nu_n = \nu \times \cdots \times \nu \).

Standard Dirichlet forms.

Dirichlet forms on \( \mathbb{S}_0 \) and \( \mathbb{S} \) can be introduced by means of finite difference schemes. (Equivalent definitions of Dirichlet forms using sequence of random walks are also available. See, for example, [9, 29, 4] for more details.) For \( m \in \mathbb{N} \) and any function \( u \) on the lattice \( V_{0,m} \), define

\[
\mathcal{E}^{(m)}_0(u, u) = \sum_{x,y \in V_{0,m} : |x-y|=2^m} 1 \cdot \left( \frac{5}{3} \right)^m |u(x) - u(y)|^2.
\]

The scaling factor \( \frac{5}{3} \) is chosen so that the sequence \( \{\mathcal{E}^{(m)}_0\} \) of forms is consistent; that is, for any function \( u \) on \( V_{0,m} \),

\[
\mathcal{E}^{(m)}_0(u, u) = \min \{\mathcal{E}^{(m+1)}_0(w, w) : w \text{ is a function on } V_{0,m+1} \text{ and } w|_{V_{0,m}} = u\}. \quad (2.1)
\]

Clearly, \( \mathcal{E}^{(m+1)}_0(u, u) = \sum_{i=1,2,3} \frac{5}{3} \mathcal{E}^{(m)}_0(u \circ F_i, u \circ F_i) \) for all functions \( u \) on \( V_{0,m+1} \). For convenience, we denote

\[
\delta_s = \frac{1}{2/d_s - 1}, \quad (2.2)
\]

where \( d_s = \frac{2 \log 3}{\log 5} \in (1, 2) \) is the spectral dimension of \( \mathbb{S}_0 \). Then factor \( \frac{5}{3} \) can be written as \( \frac{5}{3} = 3^{1/\delta_s} \).

Let

\[
P = \begin{bmatrix}
\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}
\end{bmatrix}.
\]
Then \( \mathcal{E}^{(0)}(u,u) \) can be written as
\[
\mathcal{E}^{(0)}(u,u) = \frac{3}{2} u' P u. \tag{2.3}
\]

In view of the monotonicity (2.1) of the sequence \( \{\mathcal{E}^{(m)}_n\} \), the limit (possibly infinite)
\[
\mathcal{E}_0(u,u) = \lim_{m \to \infty} \mathcal{E}_0^{(m)}(u,u)
\]
exists for any function \( u \) on \( \bigcup_{m=0}^{\infty} V_{0,m} \). Moreover, the following self-similar property holds
\[
\mathcal{E}_0(u,u) = \sum_{i=1,2,3} \frac{5}{3} \mathcal{E}_0(u \circ F_i, u \circ F_i). \tag{2.4}
\]

Let
\[
\mathcal{F}(S_0) = \left\{ u : u \text{ is a function on } \bigcup_{m=0}^{\infty} V_{0,m} \text{ and } \mathcal{E}_0(u,u) < \infty \right\}.
\]

It is well known that every function \( u \in \mathcal{F}(S_0) \) is continuous on \( \bigcup_{m=0}^{\infty} V_{0,m} \), and hence admits a unique continuous extension onto \( S_0 \). (See, for example, [25, Theorem 2.2.6 and Theroem 3.3.4].) In other words, we have \( \mathcal{F}(S_0) \subseteq C(S_0) \). Moreover, the following Poincaré inequality holds
\[
\int_{S_0} |u - [u]_{S_0}|^2 \, d\nu \leq C_* \mathcal{E}_0(u,u) \text{ for all } u \in \mathcal{F}(S_0), \tag{2.5}
\]
where \([u]_{S_0} = \int_{S_0} u \, d\nu\), and \( C_* > 0 \) is a universal constant. (See, for example, [25, Lemma 2.3.9 and Theorem 3.3.4] or [32, Section 2].)

The form \( (\mathcal{E}_0, \mathcal{F}(S_0)) \), called the standard Dirichlet form on \( S_0 \), is a local Dirichlet from on \( L^2(S_0; \nu) \).

For any given function \( u \) on \( V_{0,0} \), there exists a unique \( h \in \mathcal{F}(S_0) \) such that \( h|_{V_{0,0}} = u \) and
\[
\mathcal{E}(h,h) = \min \{ \mathcal{E}(w,w) : w \in \mathcal{F}(S_0) \text{ and } w|_{V_{0,0}} = u \}.
\]

The function \( h \in \mathcal{F}(S_0) \) is called the harmonic function in \( S_0 \) with boundary value \( u \), and satisfies
\[
\mathcal{E}(h,h) = \mathcal{E}^{(m)}(h,h) = \mathcal{E}^{(0)}(u,u) \text{ for all } m \in \mathbb{N}.
\]

By the above and (2.1), the value of a harmonic function \( h \) on \( V_{0,1}\setminus V_{0,0} \) is the extreme point of a quadratic form, and is given by
\[
(h \circ F_i)|_{V_{0,0}} = A_i(h|_{V_{0,0}}), \quad i = 1, 2, 3, \tag{2.6}
\]
where \( A_i : \mathbb{R}^3 \to \mathbb{R}^3 \), \( i = 1, 2, 3 \) are the linear operators with matrix representations
\[
A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 0 & 1 & 0 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.7}
\]

Remark 2.6. The matrices \( A_i, i = 1, 2, 3 \) share the same eigenvalues \( \{1, \frac{2}{3}, \frac{1}{3}\} \), and are not mutually commutative.
For any $m \in \mathbb{N}$, the value of $h$ on $V_{0,m}$ can be given by iterations of (2.6)

\[
(h \circ F_{[\omega]_m})|_{V_{0,0}} = A_{[\omega]_m}(h|_{V_{0,0}}), \quad i = 1, 2, 3 \text{ for all } \omega \in W_*,
\]

where we have used the convention that

\[
A_{[\omega]_m} = A\omega_1\omega_2\ldots\omega_m = A\omega_m \cdots A\omega_2A\omega_1.
\]

Notice that, in the above, the order of subscripts in the product is reversed. The same notation will be used for all matrices that appear in this paper.

**Definition 2.7.** Let $m \in \mathbb{N}$. A function $h \in \mathcal{F}(S_0)$ is called an $m$-harmonic function in $S_0$, if $h \circ F_{[\omega]_m}$ is a harmonic function in $S_0$ for all $\omega \in W_*$. A function $h$ is called piecewise harmonic if it is $m$-harmonic for some $m \in \mathbb{N}$.

The standard Dirichlet form on $S$ can be defined similarly. For any function $u$ on $\bigcup_{m=0}^{\infty} V_m$, define

\[
E^{(m)}(u, u) = \sum_{x,y \in V_m; |x-y|=2^{-m}} \frac{1}{2} \left(\frac{5}{3}\right)^m |u(x) - u(y)|^2,
\]

\[
E(u, u) = \lim_{m \to \infty} E^{(m)}(u, u).
\]

The form $E$ satisfies the following self-similar property

\[
E(u, u) = \frac{5}{3} E(u \circ F_1).
\]

Every function $u$ on $\bigcup_{m=0}^{\infty} V_m$ with $E(u, u) < \infty$ admits a unique continuous extension onto $S$. Let

\[
\mathcal{F}(S) = L^2(S; \nu) \cap \left\{ u : u \text{ is a function on } \bigcup_{m=0}^{\infty} V_m \text{ and } E(u, u) < \infty \right\}.
\]

Then $\mathcal{F}(S) \subseteq L^2(S; \nu) \cap C_0(S)$, where $C_0(S)$ is the space of continuous functions on $S$ vanishing at infinity.

The form $(E, \mathcal{F}(S))$, called the **standard Dirichlet form** on $S$, is a local Dirichlet form on $L^2(S; \nu)$.

**Kusuoka measure and gradients.**

**Definition 2.8.** For any $u \in \mathcal{F}(S_0)$, the energy measure $\mu_{[u]}$ of $u$ is the unique Borel measure on $S_0$ such that

\[
\int_{S_0} \phi \, d\mu_{[u]} = 2\varepsilon_0(\phi u, u) - \varepsilon_0(\phi, u^2) \text{ for all } \phi \in \mathcal{F}(S_0).
\]

For any $u, w \in \mathcal{F}(S_0)$, the **mutual energy measure** $\mu_{[u, w]}$ is defined by the polarisation $\mu_{[u, w]} = \frac{1}{2}(\mu_{[u+w]} - \mu_{[u-w]})$.

By definition, $\mu_{[u]}(S) = E_0(u, u)$, which, together with the self-similar property (2.4), implies that

\[
\mu_{[u]}(F_{[\omega]_m}(S_0)) = \left(\frac{5}{3}\right)^m E_0(u \circ F_{[\omega]_m}, u \circ F_{[\omega]_m}) \text{ for all } \omega \in W_*, m \in \mathbb{N}.
\]

In particular, for a harmonic function $h$ with boundary value $h|_{V_{0,0}} = u$, by (2.3) and (2.8),

\[
\mu_h(F_{[\omega]_m}(S_0)) = \frac{3}{2} \left(\frac{5}{3}\right)^m u^t Y_{[\omega]_m}^t Y_{[\omega]_m} u,
\]
where $Y_i = P^i A_i P$, $i = 1, 2, 3$, and we used the fact $P A_i = P^i A_i P$.

For later use, we write down these matrices explicitly
\[
Y_1 = \begin{bmatrix}
\frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \\
-\frac{1}{5} & \frac{1}{5} & 0 \\
-\frac{1}{5} & 0 & \frac{1}{5}
\end{bmatrix}, \quad Y_2 = \begin{bmatrix}
\frac{1}{5} & -\frac{1}{5} & 0 \\
-\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\
0 & -\frac{1}{5} & \frac{1}{5}
\end{bmatrix}, \quad Y_3 = \begin{bmatrix}
\frac{1}{5} & 0 & -\frac{1}{5} \\
-\frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \\
\frac{1}{5} & \frac{2}{5} & -\frac{1}{5}
\end{bmatrix}.
\]

**Remark 2.9.** The matrices $Y_i$, $i = 1, 2, 3$ share the same eigenvalues $\{0, \frac{1}{5}, \frac{3}{5}\}$, and are not mutually commutative.

**Definition 2.10.** The Kusuoka measure $\mu$ on $S$ is the unique Borel measure on $S$ such that
\[
\mu(F_{[\omega]_m}(S_i)) = \frac{1}{2} \left( \frac{5}{3} \right)^m \text{trace}(Y_{[\omega]_m}^i Y_{[\omega]_m}) \quad \text{for all } m \in \mathbb{N}, i \in \mathbb{Z}, \omega \in W_*.
\]

**Remark 2.11.** (i) The factor $\frac{1}{2}$ in (2.13) normalizes $\mu$ so that $\mu(S_0) = 1$.

(ii) Let $h_i$, $i = 1, 2, 3$ be the harmonic functions with boundary values $h_i|_{V_{0,0}} = 1_{\{p_i\}}$. It is easily seen from (2.11) and the definition of $\mu$ that
\[
\mu = \frac{1}{3} \left( \mu_{(h_1)} + \mu_{(h_2)} + \mu_{(h_3)} \right).
\]

(iii) The Kusuoka measure $\mu$ and the Hausdorff measure $\nu$ are mutually singular. (See [30, Example 1, Section 6].)

By (2.14), for any harmonic function $h$, its energy measure $\mu_{(h)}$ is absolutely continuous with respect to $\mu$. The same is true for any piecewise harmonic function in view of the self-similar property (2.4). By the denseness of piecewise harmonic functions in $\mathcal{F}(S_0)$, we see that $\mu_{(u)} \ll \mu$ for all $u \in \mathcal{F}(S_0)$.

Let $h_1$ be the harmonic function in $S_0$ with boundary value $h_1|_{V_{0,0}} = 1_{\{p_1\}}$, and define
\[
\nabla h_1 = -\sqrt{\frac{d\mu_{(h_1)}}{d\mu}},
\]
where the minus sign appears in the above definition as a convention. Note that $\nabla h_1 < 0$ $\mu$-a.e. as $\mu_{(h_1)}(F_{[\omega]_m}(S_0)) > 0$ for all $\omega \in W_*, m \in \mathbb{N}$.

**Definition 2.12.** For any $u \in \mathcal{F}(S_0)$, the gradient $\nabla u$ of $u$ is defined to be
\[
\nabla u = (\nabla h_1)^{-1} \frac{d\mu_{(u|h_1)}}{d\mu}.
\]

**Definition 2.13.** For any $u \in \mathcal{F}(S)$, the gradient $\nabla u$ of $u$ is defined to be
\[
\nabla u = \sum_{i \in \mathbb{Z}} \nabla (u|_{S_i}),
\]
where the gradients $\nabla (u|_{S_i})$ are taken with $u|_{S_i}$ regarded as functions in $\mathcal{F}(S_0)$.

By the definition of gradients, we have the representations
\[
\mathcal{E}_0(u,u) = \int_{S_0} |\nabla u|^2 \, d\mu, \quad \mathcal{E}(u,u) = \int_{S} |\nabla u|^2 \, d\mu.
\]

Moreover, for $G \in C^1(\mathbb{R}^k)$ and $u_1, \ldots, u_k \in \mathcal{F}(S_0)$, the following chain rule holds
\[
\nabla G(u_1, \ldots, u_k) = \sum_{i=1}^k \partial_i G(u_1, \ldots, u_k) \nabla u_i.
\]
3 Sobolev inequalities

We start with two lemmas regarding elementary properties of gradients and harmonic functions in $S_0$ respectively.

**Lemma 3.1.** Let $u \in \mathcal{F}(S_0)$. Then for any $\omega \in W_s$ and any $m \in \mathbb{N}_+$,

(a) \[
\mathcal{E}(u \circ F_{[\omega]m}) = \left( \frac{3}{5} \right)^m \int_{F_{[\omega]m}(S_0)} |\nabla u|^2 \, d\mu.
\] (3.1)

(b) \[
\nabla (u \circ F_{[\omega]m}) = \left( \frac{3}{5} \right)^{m/2} (\nabla u \circ F_{[\omega]m}) \cdot \left[ \frac{d(\mu \circ F_{[\omega]m})}{d\mu} \right]^{1/2}.
\] (3.2)

(c) For $\omega \in W_s$ and $m \in \mathbb{N}$,

\[
\left( \frac{1}{15} \right)^m \leq \frac{d(\mu \circ F_{[\omega]m})}{d\mu} \leq \left( \frac{3}{5} \right)^m, \; \mu\text{-a.e.}
\] (3.3)

Consequently, for any $r \geq 2$,

\[
\left( \frac{3^r}{5} \right)^{m/r} |\nabla u \circ F_{[\omega]m}| \cdot \left[ \frac{d(\mu \circ F_{[\omega]m})}{d\mu} \right]^{1/r} \leq |\nabla (u \circ F_{[\omega]m})| \leq \left( \frac{3}{5} \right)^{m/r} |\nabla u \circ F_{[\omega]m}| \cdot \left[ \frac{d(\mu \circ F_{[\omega]m})}{d\mu} \right]^{1/r}.
\] (3.4)

**Proof.** (a) This is an immediate corollary of (2.10) and the definition of gradients.

(b) For any $\omega, \omega' \in W_s$ and any $m, l \in \mathbb{N}_+$, by (a),

\[
\int_{F_{[\omega']l}(S_0)} |\nabla (u \circ F_{[\omega]m})|^2 \, d\mu = \left( \frac{5}{3} \right)^l \mathcal{E}(u \circ F_{[\omega]m} \circ F_{[\omega']l}) = \left( \frac{3}{5} \right)^m \int_{F_{[\omega]m \circ F_{[\omega']l}(S_0)}} |\nabla u|^2 \, d\mu = \left( \frac{3}{5} \right)^m \int_{F_{[\omega']l}(S_0)} |\nabla u \circ F_{[\omega]m}|^2 \, d(\mu \circ F_{[\omega]m}),
\]

which implies that

\[
|\nabla (u \circ F_{[\omega]m})|^2 = \left( \frac{3}{5} \right)^m |\nabla u \circ F_{[\omega]m}|^2 \cdot \frac{d(\mu \circ F_{[\omega]m})}{d\mu}.
\] (3.5)

In particular,

\[
|\nabla (h_1 \circ F_i)| = \sqrt{\frac{3}{5}} |\nabla h_1 \circ F_i| \cdot \frac{d(\mu \circ F_i)}{d\mu}, \; i = 1, 2, 3.
\] (3.6)

where $h_1$ is the harmonic function with boundary value $h_1|_{V_{0,0}} = 1_{(p_1)}$. Note that, by (2.6), $h_1 \circ F_1 = \frac{2}{5} + \frac{3}{5} h_1$ and $h_1 \circ F_2 = h_1 \circ F_3 = \frac{3}{5} h_1$. Therefore,

\[
\nabla (h_1 \circ F_1) = \frac{3}{5} \nabla h_1, \; \nabla (h_1 \circ F_2) = \nabla (h_1 \circ F_3) = \frac{2}{5} \nabla h_1.
\]

In view of (2.15), we see that $\nabla (h_1 \circ F_i) < 0 \; \mu\text{-a.e.}, \; i = 1, 2, 3$. Since $\nabla h_1 < 0 \; \mu\text{-a.e.}$, it follows from (3.6) that

\[
\nabla (h_1 \circ F_i) = \sqrt{\frac{3}{5}} \nabla h_1 \circ F_i \cdot \frac{d(\mu \circ F_i)}{d\mu}, \; i = 1, 2, 3.
\]
By the above and induction, it is easily seen that (3.2) holds for \( h_1 \). Moreover, (3.2) for general \( u \in \mathcal{F}(\mathbb{S}_0) \) follows from (3.2) for \( h_1 \) and polarisation of (3.5).

(c) We only need to prove (3.3). The inequality (3.4) follows from (3.3) immediately. For any \( \omega' \in \mathcal{W}_* \) and any \( l \in \mathbb{N} \), since spectrum \( (Y_i) = \{0, \frac{1}{3}, \frac{2}{3}, \frac{3}{3} \} \), we have

\[
\int_{F_{[\omega']}(\mathbb{S}_0)} d(\mu \circ F_{[\omega]}) = \mu(F_{[\omega]}) = \left( \frac{5}{3} \right)^{m+l} \text{ trace}(Y_{[\omega']_i} Y_{[\omega']_m} Y_{[\omega']_i}) \leq \left( \frac{5}{3} \right)^{m+l} \left( \frac{3}{5} \right)^{2m} \text{ trace}(Y_{[\omega']_i} Y_{[\omega']_i}) = \left( \frac{3}{5} \right)^m \mu(F_{[\omega']}(\mathbb{S}_0)).
\]

and similarly,

\[
\int_{F_{[\omega']}(\mathbb{S}_0)} d(\mu \circ F_{[\omega]}) \geq \left( \frac{5}{3} \right)^{m+l} \left( \frac{1}{5} \right)^{2m} \text{ trace}(Y_{[\omega']_i} Y_{[\omega']_i}) = \left( \frac{1}{15} \right)^m \mu(F_{[\omega']}(\mathbb{S}_0)).
\]

Now (3.3) follows readily from the above and the Lebesgue differentiation theorem.

\[ \square \]

**Lemma 3.2.** Let \( h_i, i = 1, 2, 3 \) be the harmonic functions in \( \mathbb{S}_0 \) with boundary values \( h_i|_{V_{0,0}} = 1_{(p_i)}, i = 1, 2, 3 \). Then

(a) \( h_1 + h_2 + h_3 = 1, |\nabla h_1|^2 + |\nabla h_2|^2 + |\nabla h_3|^2 = 3 \) \( \mu \)-a.e., and

\[
|\nabla h_i| \leq \sqrt{2} \text{ on } \mathbb{S}_0 \text{ } \mu \text{-a.e., } i = 1, 2, 3.
\]

(b) \( |\nabla h_i| \) has no strictly positive lower bound in any dyadic simplex \( S = F_{[\omega]_m}(\mathbb{S}_0) \subseteq \mathbb{S}_0: \)

\[
\text{ess inf}_{S} |\nabla h_i| = 0, \text{ } i = 1, 2, 3,
\]

where the essential infimum is taken with respect to the Kusuoka measure \( \mu \).

**Proof.** (a) The two identities in the statement are corollaries of the uniqueness of harmonic functions and the fact \( \mu = \frac{1}{3}(\mu(h_1) + \mu(h_2) + \mu(h_3)). \)

Since \( h_1 + h_2 + h_3 = 1 \), we have \( \sum_{i=1,2,3} \nabla h_i = 0 \), which together with \( \sum_{i=1,2,3} |\nabla h_i|^2 = 3 \) gives

\[
3 = \sum_{i=1,2,3} |\nabla h_i|^2 = 2(|\nabla h_1|^2 + |\nabla h_2|^2 + |\nabla h_3|^2) \geq \frac{3}{2} |\nabla h_1|^2.
\]

Therefore, \( |\nabla h_i| \leq \sqrt{2} \) \( \mu \)-a.e., \( i = 1, 2, 3. \)

(b) It suffices to prove ess inf_{\mathbb{S}_0} |\nabla h_1| = 0. We first show that ess inf_{\mathbb{S}_0} |\nabla h_1| = 0. Let \( \omega = 2333\ldots \in \mathcal{W}_* \). Then

\[
Y_{[\omega]_m+1} = Y_{\frac{2}{3}} Y_2 = \begin{bmatrix}
(\frac{1}{3})^{m+1} - \frac{1}{30} (\frac{3}{5})^m - \frac{3}{10} (\frac{1}{5})^m & -\frac{1}{30} (\frac{3}{5})^m + \frac{1}{10} (\frac{1}{5})^m \\
-\frac{1}{30} (\frac{3}{5})^m + \frac{3}{10} (\frac{1}{5})^m & (\frac{1}{5})^m - \frac{1}{30} (\frac{3}{5})^m - \frac{1}{10} (\frac{1}{5})^m \\
0 & (\frac{1}{5})^m - \frac{1}{30} (\frac{3}{5})^m - \frac{1}{10} (\frac{1}{5})^m
\end{bmatrix}.
\]

Therefore,

\[
\frac{1}{\mu(F_{[\omega]_m}(\mathbb{S}_0))} \int_{F_{[\omega]_m+1}} |\nabla h_1|^2 d\mu = \frac{3}{2} \text{ trace}(Y_{[\omega]_m+1}^{-1} Y_{[\omega]_m+1} e_1) \leq 9^{-m+1}.
\]
Definition 3.3. Let \(| \nabla h_1 | \leq 3^{m+1} \rangle > 0.

Therefore, \( \text{ess inf}_{S_0} | \nabla h_1 | = 0. \)

Next, we show that \( \text{ess inf}_{F_i(S_0)} | \nabla h_1 | = 0, i = 1, 2, 3. \) As seen in the proof of Lemma 3.1-(b), we have

\[ \nabla (h_1 \circ F_1) = 3 \nabla h_1, \quad \nabla (h_1 \circ F_2) = \nabla (h_1 \circ F_2) = \frac{2}{5} \nabla h_1. \]

This implies \( \text{ess inf}_{S_0} | \nabla (h_1 \circ F_i) | = 0. \) Since \( \mu \circ F_i \) and \( \mu \) are equivalent measures by virtue of (3.3), it follows from (3.2) that

\[ \text{ess inf}_{S_0} | \nabla h_1 | = \text{ess inf}_{S_0} | \nabla h_1 \circ F_i | = 0. \]

By induction, we see that \( \text{ess inf}_{S} | \nabla h_1 | = 0 \) holds for all dyadic simplexes. This completes the proof of (b).

We now give the definition of Sobolev spaces \( W^{1,r} \) on \( S_0^n \) and \( S^n \).

**Definition 3.3.** Let \( r \geq 2 \) and \( u \in F(S_0) \). Define

\[ \| u \|_{W^{1,r}(S_0)} = \| u \|_{L^r(S_0; \mu)}, \]

\[ \| u \|_{W^{1,r}(S_0)} = \left( \| u \|_{L^r(S_0; u)}^r + \| u \|_{W^{1,r}(S_0)}^r \right)^{1/r}. \]

The Sobolev space \( W^{1,r}(S_0) \) is defined to be the completion of

\[ \{ u \in F(S_0) : \| u \|_{W^{1,r}(S_0)} < \infty \} \]

with respect to the norm \( \| \cdot \|_{W^{1,r}(S_0)} \).

For a generic point \( x = (x_1, \ldots, x_i, \ldots, x_n) \in S_0^n \), we denote \( \tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \). To simplify notations, we abuse notations and denote

\( (x_i, \tilde{x}_i) = (x_1, \ldots, x_i, \ldots, x_n), \)

\( (\mu \times v_{n-1})(dx_i, dx_{\tilde{x}_i}) = (\mu \times v_{n-1})(dx_1, \ldots, dx_i, \ldots, dx_n). \)

**Definition 3.4.** Let \( r \geq 2 \), and let \( C(S_0^n) \) be the space of functions \( u \in C(S_0^n) \) such that \( u(\cdot, \tilde{x}_i) \in F(S_0) \) for all \( 1 \leq i \leq n \) and \( \tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \). For any \( u \in C(S_0^n) \), define

\[ \| u \|_{W^{1,r}(S_0^n)} = \left( \sum_{i=1}^n \int_{S_0^{n-1}} | \nabla_i u(x_i, \tilde{x}_i) |^r (\mu \times v_{n-1})(dx_i, dx_{\tilde{x}_i}) \right)^{1/r}, \]

\[ \| u \|_{W^{1,r}(S_0^n)} = \left( \| u \|_{L^r(S_0^n; v_{n-1})}^r + \| u \|_{W^{1,r}(S_0^n)}^r \right)^{1/r}, \]

where \( \nabla_i \) refers to the operator \( \nabla \) applied to the \( i \)-th variable \( x_i \). The Sobolev space \( W^{1,r}(S_0^n) \) is defined to be the completion of

\[ \{ u \in C(S_0^n) : \| u \|_{W^{1,r}(S_0^n)} < \infty \} \]

with respect to the norm \( \| \cdot \|_{W^{1,r}(S_0^n)} \).

The proposition below states that the space \( W^{1,r}(S_0) \) is sufficiently large.
Proposition 3.5. Let \( n \geq 1, r \geq 2 \). Then the space \( W^{1,r}(S_0) \) contains all piecewise harmonic functions in \( S_0 \), and \( C(S_0^n) \cap W^{1,r}(S_0^n) \) is dense in the space \( C(S_0^n) \) with respect to the supremum norm.

Proof. We first show that \( W^{1,r}(S_0) \) contains all harmonic functions in \( S_0 \). Let \( h_i, i = 1, 2, 3 \) be the harmonic functions with boundary values

\[ h_i|_{S_0,0} = 1_{\{p_i\}}, \ i = 1, 2, 3. \]

By Lemma 3.2, \( \forall h_i \in L^\infty(S_0; \mu) \) and therefore \( h_i \in W^{1,r}(S_0) \), \( i = 1, 2, 3 \). Furthermore, \( W^{1,r}(S_0) \) contains all harmonic functions in \( S_0 \) as any harmonic function is a linear combination of \( h_i, i = 1, 2, 3 \).

By (3.4) and the above, we see that \( W^{1,r}(S_0) \) contains all piecewise harmonic functions in \( S_0 \). Since the linear space generated by functions of the form

\[ u(x) = u_1(x_1) \cdots u_n(x_n), \ u_1, \ldots, u_n \in C(S_0) \]

is dense in \( C(S_0^n) \), so is the linear space generated by functions of the above form with each \( u_i \) being piecewise harmonic. Clearly, \( u_1(x_1) \cdots u_n(x_n) \in W^{1,r}(S_0^n) \) when \( u_i, i = 1, \ldots, n \) are piecewise harmonic. This completes the proof. \( \square \)

The following Poincaré inequality on \( S^n \), which is available on most fractal spaces, is the cornerstone of our arguments.

Lemma 3.6 (Poincaré inequality). There exists a universal constant \( C_* > 0 \) (also independent of \( n \)) such that

\[
\int_{S_0^n} |u - [u]|_{S_0^n}^2 \, d\nu_n \leq C_* \|u\|_{W^{1,2}(S_0^n)}^2, \ u \in W^{1,2}(S_0^n),
\]

where \([u]_{S_0^n} = \int_{S_0^n} u \, d\nu_n\).

Proof. Let \( u_k \) be the function on \( S_0^k \) defined by

\[ u_k(x_1, \ldots, x_k) = \int_{S_0^{n-k}} u(x_1, \ldots, x_n) \nu_{n-k}(dx_{k+1}, \ldots, dx_n), \ k = 0, 1, \ldots, n. \]

In particular, \( u_0 = [u]_{S_0^n} \) and \( u_n = u \). Clearly,

\[ u_{k-1}(x_1, \ldots, x_{k-1}) = \int_{S_0^n} u_k(x_1, \ldots, x_k) \nu(dx_k), \ k = 1, \ldots, n. \]

Moreover,

\[
\int_{S_0^n} |u - [u]|_{S_0^n}^2 \, d\nu_n = \int_{S_0^n} |u_n - u_0|^2 \, d\nu_n = \sum_{k=1}^n \int_{S_0^n} |u_k - u_{k-1}|^2 \, d\nu_k
\]

(3.9)

Now the Poincaré inequality (2.5) on \( S_0 \) gives

\[
\int_{S_0^n} |u_k(x_1, \ldots, x_k) - u_{k-1}(x_1, \ldots, x_{k-1})|^2 \, d\nu_k \leq C_* \int_{S_0^n} |\nabla_k u_k(x_1, \ldots, x_k)|^2 \mu(dx_k)
\]

\[ = C_* \int_{S_0^{n-k+1}} |\nabla_k u_k(x_k, \tilde{x}_k)|^2 (\mu \times \nu_{n-k+1})(dx_k, \ldots, dx_n), \ k = 1, \ldots, n. \]

Therefore,

\[
\int_{S_0^n} |u_k - u_{k-1}|^2 \, d\nu_k \leq C_* \int_{S_0^n} |\nabla_k u_k(x_k, \tilde{x}_k)|^2 (\mu \times \nu_{n-1})(dx_k, d\tilde{x}_k), \ k = 1, \ldots, n.
\]

This, together with (3.9), completes the proof. \( \square \)
We can now prove the first inequality for Sobolev functions, which is the key technical ingredient for the derivation of Sobolev inequalities on product Sierpinski spaces.

**Lemma 3.7.** Let \( u \in C(S_0^n) \cap W^{1,r}(S_0^n) \), \( n \geq 1 \). If \( r > 1 + (n-1)\delta \) and \( r \geq 2 \), then

\[
\text{osc}_{S_0^n}(u) \leq C_n \|u\|_{W^{1,r}(S_0^n)}.
\]  

(3.10)

**Remark 3.8.** The assumption \( r \geq 2 \) is only to guarantee that the gradient \( \nabla u \) has a proper definition. This is also the reason for having \( r \geq 2 \) in most of the results in this paper.

**Proof.** For any \( \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{W}_+^n \) with \( \omega_i = \omega_{i1} \omega_{i2} \ldots \in \mathbb{W}_+ \), \( 1 \leq i \leq n \), by the Poincaré inequality (3.8) and (3.4),

\[
\int_{S_0^n} \left| u \circ F(\omega_m) - [u \circ F(\omega_m)]_{S_0^n} \right|^2 \, d\nu_n
\]

\[
\leq C_n \sum_{i=1}^n \int_{S_0^n} \left| \nabla_i (u \circ F(\omega_m))(x_i, \hat{x}_i) \right|^2 \left( \mu \times \nu_{n-1} \right)(dx_i, d\hat{x}_i)
\]

\[
= C_n \left( \frac{3}{5} \right)^m \sum_{i=1}^n \int_{S_0^n} \left| \nabla_i (u \circ F(\omega_m))(x_i, \hat{x}_i) \right|^2 \left( \mu \times \nu_{n-1} \right)(dx_i, d\hat{x}_i)
\]

\[
= C_n \left( \frac{3}{5} \right)^m \sum_{i=1}^n \int_{F(\omega_m)(S_0^n)} \left| \nabla_i u(x_i, \hat{x}_i) \right|^2 \left( \mu \times \nu_{n-1} \right)(dx_i, d\hat{x}_i)
\]

\[
\leq C_n \left[ \int_{S_0^n} \left| u \circ F(\omega_m) - [u \circ F(\omega_m)]_{S_0^n} \right|^2 \, d\nu_n \right]^{1/2} \left\| u \right\|_{W^{1,r}(S_0^n)}^{1/2}
\]

Since \( \mu(F(\omega_m)(S_0^n)) \leq (3/5)^m \), we obtain that

\[
\frac{1}{\nu_n(F(\omega_m)(S_0^n))} \int_{F(\omega_m)(S_0^n)} \left| u - [u]_{F(\omega_m)(S_0^n)} \right|^2 \, d\nu_n
\]

\[
\leq \left[ \int_{S_0^n} \left| u \circ F(\omega_m) - [u \circ F(\omega_m)]_{S_0^n} \right|^2 \, d\nu_n \right]^{1/2} \left\| u \right\|_{W^{1,r}(S_0^n)}^{1/2}
\]

For any \( x \in S_0^n \), choose \( \omega \in \mathbb{W}_+^n \) such that \( F(\omega_m)(S_0^n) \rightarrow x \) as \( m \rightarrow \infty \). Let \( [u]_m = [u]_{F(\omega_m)(S_0^n)} \) for \( m \in \mathbb{N} \). Then, by the above inequality, we have

\[
\left| [u]_m - [u]_{m+1} \right| \leq \frac{1}{\nu_n(F(\omega_m+1)(S_0^n))} \int_{F(\omega_m+1)(S_0^n)} \left| u - [u]_m \right| \, d\nu_n
\]

\[
\leq \frac{3^n}{\nu_n(F(\omega_m)(S_0^n))} \int_{F(\omega_m)(S_0^n)} \left| u - [u]_m \right| \, d\nu_n \leq C_n 3^{-m[1/(r\delta) - (n-1)/r]} \left\| u \right\|_{W^{1,r}(S_0^n)}
\]

Since \( 1/(r\delta) - (n-1)/r > 0 \), we may conclude that

\[
\left| [u]_k - [u]_0 \right| \leq \sum_{m=0}^{k-1} \left| [u]_m - [u]_{m+1} \right| \leq C_n \sum_{m=0}^{\infty} 3^{-m[1/(r\delta) - (n-1)/r]} \left\| u \right\|_{W^{1,r}(S_0^n)} = C_n \left\| u \right\|_{W^{1,r}(S_0^n)}
\]

Using the Lebesgue differentiation theorem and setting \( k \rightarrow \infty \) gives

\[
\left| u(x) - \int_{S_0^n} u \, d\nu_n \right| \leq C_n \left\| u \right\|_{W^{1,r}(S_0^n)}
\]

which implies (3.10). \( \square \)
Remark 3.9. When \( n = 1, p = 2 \), the inequality (3.10) reduces to the following well-known inequality on \( S_0 \) (see, for example, [22])

\[
\operatorname{osc}(u) \leq C \varepsilon_0(u, u)^{1/2}, \quad u \in F(S_0).
\]

Proposition 3.10. Suppose that \( u \in W^{1,r}(S^n_0), r > 1 + (n - 1)\delta_\varepsilon \). Then \( u \in C(S^n_0) \) and

\[
\operatorname{osc}(u \circ F_{[\omega]_m}) \leq C_n \left\| u \right\|_{W^{1,r}(F_{[\omega]_m}(S^n_0))}^{r},
\]

where

\[
\alpha_r = 1/(r'\delta_\varepsilon) - (n - 1)/r.
\]

Proof. For any \( \omega \in W^n_\varepsilon \) and any \( m \in \mathbb{N} \), by (3.10),

\[
\operatorname{osc}(u \circ F_{[\omega]_m}) \leq C_n \left\| u \circ F_{[\omega]_m} \right\|_{W^{1,r}(S^n_0)}.
\]

By (3.4), we deduce that

\[
\left\| u \circ F_{[\omega]_m} \right\|_{W^{1,r}(S^n_0)}^{r} = \sum_{i=1}^{n} \int_{S^n_0} |\nabla_i (u \circ F_{[\omega]_m})(x_i, \hat{x}_i)|^r (\mu \times \nu_{n-1})(dx_i, d\hat{x}_i)
\]

\[
\leq 3^{-m(r-1)/\delta_\varepsilon} \sum_{i=1}^{n} \int_{S^n_0} |\nabla_i (u \circ F_{[\omega]_m})(x_i, \hat{x}_i)|^r ([\mu \circ F_{[\omega]_m}] \times \nu_{n-1})(dx_i, d\hat{x}_i)
\]

\[
= 3^{-m(r-1)/\delta_\varepsilon} \sum_{i=1}^{n} \int_{S^n_0} |\nabla_i (u \circ F_{[\omega]_m})(x_i, \hat{x}_i)|^r (\mu \times \nu_{n-1})(dx_i, d\hat{x}_i)
\]

which completes the proof. \( \square \)

To proceed further, we shall need the result below on the Kusuoka measure.

Lemma 3.11. There exists an \( \omega \in W_\varepsilon \) such that

\[
\lim_{k \to \infty} \frac{\mu \circ F^m_{1}(F_{[\omega]_k}(S_0))}{\mu(F_{[\omega]_k}(S_0))} = \inf_{\omega \in W_\varepsilon, k \in \mathbb{N}^+} \frac{\mu \circ F^m_{1}(F_{[\omega]_k}(S_0))}{\mu(F_{[\omega]_k}(S_0))} = \left( \frac{1}{15} \right)^m.
\]

Proof. By definition,

\[
\frac{\mu \circ F^m_{1}(F_{[\omega]_k}(S_0))}{\mu(F_{[\omega]_k}(S_0))} = \frac{\operatorname{trace}(Y^m_{1} Y^t_{[\omega]_k} Y_{[\omega]_k})}{\operatorname{trace}(Y^t_{[\omega]_k} Y_{[\omega]_k})}, \quad \omega \in W_\varepsilon.
\]

Since spectrum(\( Y_1 \)) = \{ 0, \frac{1}{3}, \frac{3}{3} \}, we deduce that

\[
\operatorname{trace}(Y^m_{1} Y^t_{[\omega]_k} Y_{[\omega]_k}) \geq 5^{-2m} \operatorname{trace}(Y^t_{[\omega]_k} Y_{[\omega]_k}),
\]

which implies that

\[
\inf_{\omega \in W_\varepsilon, k \in \mathbb{N}^+} \frac{\mu \circ F^m_{1}(F_{[\omega]_k}(S_0))}{\mu(F_{[\omega]_k}(S_0))} \geq \left( \frac{1}{15} \right)^m.
\]
Corollary 3.12. Let \( m \in \mathbb{N} \). Then

\[
\left\| \frac{d(\mu \circ F_{1}^{-m})}{d\mu} \right\|_{L^{\infty}(\mathbb{S}; \mu)} = \left\| \frac{d\mu}{d(\mu \circ F_{1}^{m})} \right\|_{L^{\infty}(\mathbb{S}; \mu)} = 15^{m}.
\] (3.15)
The following proposition shows that a function \( u \in W^{1,r}(\mathbb{S}^n) \) with \( r > 1 + (n - 1)\delta \), \( r \geq 2 \) has at most polynomial growth.

**Proposition 3.13.** Suppose \( u \in W^{1,r}(\mathbb{S}^n) \), \( r > 1 + (n - 1)\delta \), \( r \geq 2 \). Let \( S_{0,m} = F_{(1,\ldots,1)}^{-m}(\mathbb{S}^n_0) \), \( m \in \mathbb{N} \). Then

\[
\text{osc}_{S_{0,m}}(u) \leq 3^m \beta_r \norm{u}_{W^{1,r}(S_{0,m})},
\]

where

\[
\beta_r = (1/\delta + 1)/r' - n/r.
\]

**Proof.** By (3.2),

\[
\nabla_i u = \nabla_i (u \circ F^{-m}_{(1,\ldots,1)} \circ F^m_{(1,\ldots,1)})
\]

\[
= \left(\frac{3}{5}\right)^{m/2} \nabla_i (u \circ F^{-m}_{(1,\ldots,1)} \circ F^m_{(1,\ldots,1)}) \cdot \left[\frac{d(\mu \circ F^{-m}_{(1,\ldots,1)})}{d\mu}\right]^{1/2}.
\]

Therefore,

\[
\nabla_i (u \circ F^{-m}_{(1,\ldots,1)}) = 3^{m/(2\delta)} \nabla_i u \circ F^{-m}_{(1,\ldots,1)} \cdot \left[\frac{d(\mu \circ F^{-m}_{(1,\ldots,1)})}{d\mu}\right]^{1/2}.
\]

It follows from the above and Corollary 3.12 that

\[
\left|\nabla_i (u \circ F^{-m}_{(1,\ldots,1)})\right| \leq 3^{m(1/\delta + 2)/r' - 1} \left|\nabla_i u \circ F^{-m}_{(1,\ldots,1)}\right| \cdot \left[\frac{d(\mu \circ F^{-m}_{(1,\ldots,1)})}{d\mu}\right]^{1/r}.
\]

Therefore,

\[
\int_{S^0_0} \left|\nabla_i (u \circ F^{-m}_{(1,\ldots,1)})(x_i, \hat{x}_i)\right|^r (\mu \times v_{n-1})(dx_i, d\hat{x}_i)
\]

\[
\leq 3^{m(1/\delta + 2)(r - 1) - r} \int_{S^0_0} \left|\nabla_i u \circ F^{-m}_{(1,\ldots,1)}(x_i, \hat{x}_i)\right|^r \left[(\mu \circ F^{-m}_{(1,\ldots,1)}) \times v_{n-1}\right](dx_i, d\hat{x}_i)
\]

\[
= 3^{m(1/\delta + 1)(r - 1) - n} \int_{S_{0,m}} \left|\nabla_i u(x_i, \hat{x}_i)\right|^r (\mu \times v_{n-1})(dx_i, d\hat{x}_i),
\]

which yields that

\[
\norm{u \circ F^{-m}_{(1,\ldots,1)}}_{W^{1,r}(S_{0,m})} \leq 3^m \beta_r \norm{u}_{W^{1,r}(S_{0,m})}.
\]

Now (3.16) follows immediately from (3.10) and the above inequality. \( \square \)

To formulate our first main result, let us first introduce the setting that we shall work on. Let \( \sigma \) be a positive Radon measure on \( \mathbb{S}^n \) satisfying the following condition: there exist constants \( 0 < \underline{\delta} < \bar{\delta} \leq \infty \) with \( \bar{\delta} \geq 1 \) and \( C_\sigma > 0 \) such that

\[
\begin{align*}
\sigma(S) &\leq C_\sigma v_n(S)^{1/\bar{\delta}}, \quad \text{if } 0 < \text{diam}(S) \leq 1, \\
\sigma(S) &\leq C_\sigma v_n(S)^{1/\underline{\delta}}, \quad \text{if } \text{diam}(S) > 1
\end{align*}
\]

for all dyadic simplexes \( S \subseteq \mathbb{S}^n \).

**Remark 3.14.** Note that the restriction \( \bar{\delta} \geq 1 \) in (M) is necessary in view of the countable additivity of \( \sigma \) and \( v_n \) and that \( \sigma \) is finite on compact subsets.

We list some examples of the Radon measure \( \sigma \).
Example 3.15. (i) The Hausdorff measure $\nu_n$, for which $\delta = \delta = 1$.

(ii) The product Kusuoka measure $\mu_n = \mu \times \cdots \times \mu$, for which the sharp constants $\delta$ and $\overline{\delta}$ will be given later. (See Corollary 4.2-(a).)

(iii) Dirac measures, for which $\delta = \overline{\delta} = \infty$.

(iv) Examples (ii) and (iii) can be generalized to linear combinations of measures of the form $\sigma = \sigma_1 \times \sigma_2$, where $\sigma_1, \sigma_2$ are Radon measures on $\mathbb{S}^k$ and $\mathbb{S}^{n-k}$ satisfying conditions (M) on the corresponding spaces. Another particular case of such measures is

$$\sigma = \sum_{i=1}^{n} \sum_{j=1,2,3} \delta_{p_j} \times \nu_{n-i},$$

where $\delta_{p_j}$ is the Dirac measure concentrated at $p_j \in \mathbb{V}_{0,0}$. Applying Theorem 3.16 below to the above measure $\sigma$ gives the trace theorem for functions in $W^{1,r}(\mathbb{S}^n_0)$.

We are now in a position to formulate the Sobolev inequalities on the infinite product space $\mathbb{S}^n$.

Theorem 3.16. Suppose $\sigma$ is a positive Radon measure on $\mathbb{S}^n$ satisfying the condition (M). Let $r > 1 + (n-1)\delta$, $r \geq 2$, $p \geq 1$, $\min \{p, r\} \leq q \leq \infty$. Then

$$\left\| u \right\|_{L^q(\sigma)} \leq C \sum_{i=1,2} \left[ \left\| u \right\|_{W^{1,r}(\mathbb{S}^n_i)}^{a_1} \right]^{1-a_1} \left[ \left\| u \right\|_{L^p(\mathbb{S}^n_{i,k})}^{a_2} \right]^{1-a_2},$$

where

$$a_1 = \left[ \frac{1/p - 1/(q \delta)}{1/p - 1/r + [1/(r \delta)] + 1/r/\overline{\delta}} \right]^+, \quad a_2 = \left[ \frac{1/p - 1/(q \overline{\delta})}{1/p - 1/r + [1/(r \delta)] + 1/r/\delta} \right]^+,$$

and $C > 0$ is a constant depending only on the constant $C_\sigma$ in (M).

Proof. Let $\mathbb{S}^n_i = \mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_n}$, $i = (i_1, \ldots, i_n) \in \mathbb{Z}^n$ be non-overlapping translations of $\mathbb{S}^n_0$ such that $\mathbb{S}^n = \bigcup_i \mathbb{S}^n_i$ (the specific order of $\mathbb{S}^n_i$ does not matter). For any $m \in \mathbb{Z}$, let

$$S_{i,m} = F_{(1,\ldots,1)}^{-m}(\mathbb{S}^n_i) = (F_i^\mathbb{S}^n)^{-m}(\mathbb{S}^n_i).$$

Then $S_{i,m}$ are dyadic simplexes with $\text{diam}(S_{i,m}) = 2^m$, and $\mathbb{S}^n = \bigcup_i S_{i,m}$. Denote

$$\nu_n(m) = \nu_n(S_{0,m}) = 3^m, \quad |u|_{S_{i,m}} = \frac{1}{\nu_n(m)} \int_{S_{i,m}} u \ d\nu_n, \quad i, m \in \mathbb{Z}.$$

For $m \geq 0$, by Proposition 3.13 and (M),

$$\left\| u \right\|_{L^q(\sigma)}^q \leq 2^{q-1} \sum_i \left[ \int_{S_{i,m}} \left| u - [u]_{S_{i,m}} \right|^q \ d\sigma + \sigma(S_{i,m}) \left| [u]_{S_{i,m}} \right|^q \right]$$

$$\leq 2^{q-1} \sum_i \left[ \sigma(S_{i,m}) \nu_n(m)^{1/\delta + q/n} \left\| u \right\|_{W^{1,r}(S_{i,m})}^q + \sigma(S_{i,m}) \nu_n(m)^{1/\delta - q/p} \left( \int_{S_{i,m}} |u|^p \ d\nu_n \right)^{q/p} \right]$$

$$\leq C_\sigma 2^{q-1} \sum_i \left[ \nu_n(m)^{1/\delta + q/n} \left( \sum_i \left\| u \right\|_{W^{1,r}(S_{i,m})}^q \right)^{q/r} + \nu_n(m)^{1/\delta - q/p} \left( \sum_i \int_{S_{i,m}} |u|^p \ d\nu_n \right)^{q/p} \right]$$

$$\leq C_\sigma 2^{q-1} \left[ \nu_n(m)^{1/\delta + q/n} \left( \sum_i \left\| u \right\|_{W^{1,r}(S_{i,m})}^q \right)^{q/r} + \nu_n(m)^{1/\delta - q/p} \left( \sum_i \int_{S_{i,m}} |u|^p \ d\nu_n \right)^{q/p} \right]$$

$$\leq C_\sigma 2^{q-1} \left[ \nu_n(m)^{1/\delta + q/n} \left\| u \right\|_{W^{1,r}(\mathbb{S}^n)}^q + \nu_n(m)^{1/\delta - q/p} \left\| u \right\|_{L^p(\mathbb{S}^n_i)}^q \right].$$
where \( \alpha, \beta \) be the exponents given by (3.12) and (3.17) respectively.

Therefore,

\[
\|u\|_{L^q(\sigma)} \leq C \left[ v_n (m)^{1/(q\delta) + \beta/n} \|u\|_{W^{1,r}(\mathbb{S}^n)} + v_n (m)^{1/(q\delta) - 1/p} \|u\|_{L^p(\nu_n)} \right],
\]  

(3.20)

where \( C > 0 \) is a constant depending only on \( C_\sigma \) (the constant \( C \) can chosen to be independent of \( q \) as \( q > 1 \)).

Similarly, for \( m \leq 0 \), we have

\[
\|u\|_{L^q(\sigma)} \leq C \left[ v_n (m)^{1/(q\delta) + \alpha/n} \|u\|_{W^{1,r}(\mathbb{S}^n)} + v_n (m)^{1/(q\delta) - 1/p} \|u\|_{L^p(\nu_n)} \right].
\]

(3.21)

Without loss of generality, we may assume that \( \|u\|_{W^{1,r}(\mathbb{S}^n)} > 0 \).

For the case \( \|u\|_{W^{1,r}(\mathbb{S}^n)} \leq \|u\|_{L^p(\nu_n)} \), setting

\[
m = \inf \left\{ m \geq 0 : v_n (m)^{\beta/n + 1/p} \geq \|u\|_{L^p(\nu_n)}/\|u\|_{W^{1,r}(\mathbb{S}^n)} \right\}
\]

in (3.20) gives

\[
\|u\|_{L^q(\sigma)} \leq C \left[ v_n \right]^{a_1} \|u\|_{W^{1,r}(\mathbb{S}^n)}^{1-a_1} \|u\|_{L^p(\nu_n)}.
\]

For the case \( \|u\|_{W^{1,r}(\mathbb{S}^n)} > \|u\|_{L^p(\nu_n)} \), setting

\[
m = \sup \left\{ m \leq 0 : v_n (m)^{\beta/n + 1/p} \leq \|u\|_{L^p(\nu_n)}/\|u\|_{W^{1,r}(\mathbb{S}^n)} \right\}
\]

in (3.21) gives

\[
\|u\|_{L^q(\sigma)} \leq C \left[ v_n \right]^{a_2} \|u\|_{W^{1,r}(\mathbb{S}^n)}^{1-a_2} \|u\|_{L^p(\nu_n)}.
\]

This completes the proof.

\( \square \)

Remark 3.17. (i) Recall that the Sobolev inequality on \( \mathbb{R}^n \) takes the form

\[
\|u\|_{L^q(\mathbb{R}^n)} \leq \|\nabla u\|_{L^p(\mathbb{R}^n)}^{1/a} \|u\|_{L^p(\mathbb{R}^n)},
\]

(3.22)

where \( a \in [0, 1] \) is given by

\[
a = \frac{1/p - 1/q}{1/p - 1/r + 1/n}.
\]

(3.23)

Let us compare (3.18) and (3.22). That the exponent \( q \) in (3.23) is changed to the two exponents \( q\delta \) and \( q\delta' \). This is due to the different scaling rates of the measures \( v_n \) and \( \sigma \), and the inhomogeneity of the scaling rate of \( \sigma \) under shrinkage and expansion. The factor before \( 1/n \) in (3.23) is changed from 1 to the pair \( 1/(r' \delta) + 1/r' + 1/r' \delta' \) and \( 1/(r \delta) + 1/r' \). When \( r = 2 \), these numbers are both equal to \( 1/(2 \delta) + 1/2 \), which is the natural factor for \( \mathbb{S}^n \) as \( \delta = 1 \) if the spectral dimension were \( d_s = 1 \). When \( r > 2 \), these numbers depend on the exponent \( r \). This suggests that \( r - 2 \), the exceeding part of \( r \), has an distorting effect on the dimension \( n \). Such distorting effect can also be seen from (3.2) and (3.4).

(ii) Only the term corresponding to \( a_1 \) is needed on the right hand side of (3.18) if \( \mathbb{S}^n \) is replaced by \( \mathbb{S}^n_0 \), since only the first part of (M) is involved. The proof of this is similar to that of Theorem 3.16 and hence omitted.

We now show that the condition (M) is also necessary for the Sobolev inequality (3.18) to hold.
Theorem 3.18. Suppose that $\sigma$ is a positive Radon measure on $\mathbb{S}^n$, and there exist constants $p \geq 1$, $1 \leq q < \infty$, $r > 1 + (n-1)\delta$, $r \geq 2$, $a_i \in [0, 1]$, $1 \leq i \leq k$ and $C > 0$ such that

$$\|u\|_{L^q(\sigma)} \leq C \sum_{i=1}^k \|u\|_{W^{1,r}(\mathbb{S}^n)}^{a_i} \|u\|_{L^p(v_i)}^{1-a_i} \quad \text{for all } u \in W^{1,r}(\mathbb{S}^n).$$

(3.24)

Then $\sigma$ satisfies the condition (M) for some $0 < \delta < \bar{\delta} \leq \infty$ with $\bar{\delta} \geq 1$ and $C \sigma > 0$.

Proof. For the first part of (M), it suffices to show that $\sup_{i \in \mathbb{Z}^n} \sigma(S_i^0) < \infty$ and that $\bar{\delta} = \infty$. (Note that $\bar{\delta} = \infty$ is the only valid value when $\sigma$ is a Dirac measure.) Let $\varphi_0$ be the 1-harmonic function in $S_0$ with boundary value $\varphi_0|_{V_{0,1}} = 1_{F_1(p_1)}$. Clearly, $\varphi_0|_{V_{0,0}} = 0$. Therefore, setting $\varphi_0 = 0$ on $S \setminus S_0$ gives a function $\varphi_0 \in \mathcal{F}(S)$. It is easily seen that

$$\frac{2}{5} \leq \varphi_0 \leq 1 \quad \text{on } F_1 \circ F_2(S_0), \quad \text{supp}(\varphi_0) \subseteq S_0.$$

Note that $(\varphi_0 \circ F_1)|_{S_0}$ is the harmonic function in $S_0$ with boundary value $(\varphi_0 \circ F_1)|_{V_{0,0}} = 1_{(p_2)}$. By (3.7), (3.2) and (3.15),

$$\|\nabla \varphi_0\|_{L^\infty(S; \mu)} = \|\nabla \varphi_0 \circ F_1\|_{L^\infty(S; \mu)} \leq \sqrt{\frac{5}{3}} \left\|\nabla (\varphi_0 \circ F_1)\right\|_{L^\infty(S_0; \mu)} \leq \frac{d\mu}{d(\mu \circ F_1)} \frac{1}{L^\infty(S; \mu)} \leq 5 \sqrt{2}.$$

Therefore,

$$\|\nabla \varphi_0\|_{W^{1,r}(S)} \leq 5 \sqrt{2}.$$

For any $j \in \mathbb{Z}$, let $\varphi_j \in \mathcal{F}(S)$ be the translation of $\varphi_0$ such that $\text{supp}(\varphi_j) \subseteq S_j$. For each $i = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, let

$$u_i(x) = \varphi_{i_1}(x_1) \cdots \varphi_{i_n}(x_n), \quad x = (x_1, \ldots, x_n) \in \mathbb{S}^n.$$

Then $u_i \in W^{1,r}(\mathbb{S}^n)$, supp($u_i$) $\subseteq S_i$. Let $\tau_i, i \in \mathbb{Z}^n$ be the translations mapping $S_0$ onto $S_i$, and

$$S_i = \tau_i(F_{(1, \ldots, 1)} \circ F_{(2, \ldots, 2)} S_0^0).$$

Then

$$\left(\frac{2}{5}\right)^n \leq u_i \leq 1 \quad \text{on } S_i, \quad \|u_i\|_{W^{1,r}(\mathbb{S}^n)} \leq 5 \sqrt{2} \ n^{1/r}.$$

Setting $u = u_i$ in (3.24) gives $\sup_{i \in \mathbb{Z}^n} \sigma(S_i^{1/q}) < \infty$. Since $q < \infty$, it is seen that $\sup_{i \in \mathbb{Z}^n} \sigma(S_i) < \infty$. Clearly, this implies $\sup_{i \in \mathbb{Z}^n} \sigma(S_i^{1/q}) < \infty$, as we may change the boundary value of $\varphi_0$ to $\varphi_0|_{V_{0,1}} = 1_{F_1(p_2)}$ and $\varphi_0|_{V_{0,0}} = 1_{F_1(p_1)}$ and similar results hold.

We now prove the second part of (M). Let $u_i$ and $S_i$ be the functions and simplexes defined above. It suffices to consider simplexes $S$ of the form $S = F_{(1, \ldots, 1)}^{-k}(S_i), k \in \mathbb{N}_+, i \in \mathbb{Z}^n$. The desired conclusion follows readily by changing the boundary value of functions.

By (3.4),

$$\|u_i \circ F_{(1, \ldots, 1)}^k\|_{W^{1,r}(\mathbb{S}^n)} \leq 3 k^{\beta_r} \|u_i\|_{W^{1,r}(\mathbb{S}^n)} \leq 3 k^{\beta_r} \cdot 5 \sqrt{2} n^{1/2} \quad \text{for all } k \in \mathbb{N}_+, \quad (3.25)$$

where $\beta_r > 0$ is the constant given by (3.17).

Let $S \subseteq \mathbb{S}^n$ be a simplex of the form $S = F_{(1, \ldots, 1)}^{-k}(S_i), k \in \mathbb{N}_+, i \in \mathbb{Z}^n$. By (3.25), setting $u = u_i \circ F_{(1, \ldots, 1)}^k$ in (3.24) gives

$$\sigma(S)^{1/q} \leq c_n \cdot 3^{k(n/p + \beta_r/n)} = c_n \cdot V_n(S)^{1/p + \beta_r/n},$$

where $c_n > 0$ is a constant depending only on $n$ and the constant $C$ in (3.24). Therefore, the second part of (M) holds with $\delta = q(1/p + \beta_r/n) > 0$.

This completes the proof. \qed
We may exchange the positions of $\sigma$ and $v$ in (3.18). Let $\sigma$ be a Radon measure satisfying the following condition: there exist constants $0 < \underline{\delta} \leq \overline{\delta} \leq \infty$ with $\overline{\delta} \leq 1$ and $C_\sigma > 0$ such that

\[
\begin{cases}
C_\sigma^{-1} v_n(S)^{1/\overline{\delta}} \leq \sigma(S), & \text{if } 0 < \text{diam}(S) \leq 1, \\
C_\sigma^{-1} v_n(S)^{1/\underline{\delta}} \leq \sigma(S), & \text{if } \text{diam}(S) > 1
\end{cases}
\]  

(M')

for all dyadic simplexes $S \subseteq S^n$. Note that, as in (M), the restriction $\underline{\delta} \leq 1$ is necessary in view of the countable additivity of measures.

Remark 3.19. The sharp constants for $(M')$ when $\sigma$ is the product Kusuoka measure will be given later. (See Corollary 4.2-(b).)

For such Radon measures, we have the following two theorems, of which the proofs are similar to those of Theorem 3.16 and Theorem 3.18, and hence will be omitted.

**Theorem 3.20.** Suppose $\sigma$ is a positive Radon measure on $S^n$ satisfying the condition $(M')$. Let $r > 1 + (n - 1) \delta$, $r \geq 2$, $p \geq 1$, $\min\{p, r\} \leq q \leq \infty$. Then

\[\|u\|_{L^q(v_n)} \leq C \sum_{i=1,2} \|u\|_{W^{1,r}(S^n)}^a \|u\|_{L^p(\sigma)}^{1-a_i},\]

where

\[a_1 = \left[\frac{1/(p \underline{\delta}) - 1/q}{1/(p \underline{\delta}) - 1/r + \left[1/(r \overline{\delta}) + 1/r\right]/n}\right]^+, \quad a_2 = \left[\frac{1/(p \overline{\delta}) - 1/q}{1/(p \overline{\delta}) - 1/r + \left[1/(r \overline{\delta}) + 1/r\right]/n}\right]^+,\]

and $C > 0$ is a constant depending only on the constant $C_\sigma$ in $(M')$.

**Theorem 3.21.** Suppose $\sigma$ is a positive Radon measure on $S^n$, and there exist constants $p \geq 1$, $1 \leq q < \infty$, $r > 1 + (n - 1) \delta$, $r \geq 2$, $a_i \in [0,1]$, $1 \leq i \leq k$ and $C > 0$ such that

\[\|u\|_{L^q(v_n)} \leq C \sum_{i=1}^k \|u\|_{W^{1,r}(S^n)}^{a_i} \|u\|_{L^p(\sigma)}^{1-a_i} \text{ for all } u \in W^{1,r}(S^n).\]

Then $\sigma$ satisfies the condition $(M')$ for some $0 < \underline{\delta} \leq \overline{\delta} \leq \infty$ with $\overline{\delta} \leq 1$ and $C_\sigma > 0$.

### 4 Sharp exponents for product Kusuoka measures

We now give the sharp values of the exponents $\underline{\delta}$ and $\overline{\delta}$ in $(M)$ and $(M')$ for the product Kusuoka measure $\mu_n = \mu \times \cdots \times \mu$.

**Proposition 4.1.** The following holds

\[\inf_{\omega \in \mathcal{W}_+} \liminf_{m \to \infty} \left[\text{trace}(\mathbf{Y}^{i_1}_{\omega_1} \cdots \mathbf{Y}^{i_m}_{\omega_m})\right]^{1/m} = \frac{3}{25}.
\]

Moreover, the infimum (4.1) can be achieved by some $\omega \in \mathcal{W}_+$.

**Proof.** Let $\mathbf{M}_i$, $i = 1, 2, 3$ be the $2 \times 2$ matrices given by (3.14). As seen in the proof of Lemma 3.11, it suffices to prove the lemma for $\mathbf{M}_i$, $i = 1, 2, 3$.

Note that

\[\det(\mathbf{M}_1) = \det(\mathbf{M}_2) = \det(\mathbf{M}_3) = \frac{3}{25}.
\]
We see that
\[
\det \left( \mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_m}^t \mathbf{M}_{\omega_m} \cdots \mathbf{M}_{\omega_1} \right) = \prod_{i=1}^{m} \det \left( \mathbf{M}_{\omega_i} \right)^2 = \left( \frac{3}{25} \right)^{2m}.
\]
Since \( \mathbf{M}_{\omega_1} \cdots \mathbf{M}_{\omega_m} \mathbf{M}_{\omega_m} \cdots \mathbf{M}_{\omega_1} \) is non-negative definite, by the arithmetic-geometric mean inequality,
\[
2^{-1} \cdot \text{trace} \left( \mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_m}^t \mathbf{M}_{\omega_m} \cdots \mathbf{M}_{\omega_1} \right) \geq \left[ \det \left( \mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_m}^t \mathbf{M}_{\omega_m} \cdots \mathbf{M}_{\omega_1} \right) \right]^{1/2} = \left( \frac{3}{25} \right)^{m}.
\]
This implies that
\[
\inf_{\omega \in W_+} \liminf_{m \to \infty} \left[ \text{trace} \left( \mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_m}^t \mathbf{M}_{\omega_m} \cdots \mathbf{M}_{\omega_1} \right) \right]^{1/m} \geq \frac{3}{25}.
\]
To show the reverse, we construct a finite sequence \( (\omega_1 \ldots \omega_r) \in \{1, 2, 3\}^r \) such that \( \mathbf{M}_{\omega_1} \cdots \mathbf{M}_{\omega_1} \) has no real eigenvalues. Once such a finite sequence can be found, since \( \mathbf{M}_i, i = 1, 2, 3 \) are \( 2 \times 2 \) matrices, the spectrum of \( \mathbf{M}_{\omega_1} \cdots \mathbf{M}_{\omega_1} \) must consist of a pair of conjugate eigenvalues \( \lambda, \bar{\lambda} \in \mathbb{C} \). Therefore, \( \mathbf{M}_{\omega_1} \cdots \mathbf{M}_{\omega_1} \) is similar to the diagonal matrix
\[
\begin{bmatrix}
\lambda & 0 \\
0 & \bar{\lambda}
\end{bmatrix}.
\]
Moreover, we have
\[
|\lambda| = \left( \frac{3}{25} \right)^{r/2}
\]
in view of
\[
\det \left( \mathbf{M}_{\omega_1} \cdots \mathbf{M}_{\omega_1} \right) = \prod_{i=1}^{r} \det \left( \mathbf{M}_{\omega_i} \right) = \left( \frac{3}{25} \right)^{r}.
\]
Now set \( \omega \) to be the repetition
\[
\omega = (\omega_1 \ldots \omega_r)(\omega_1 \ldots \omega_r) \ldots \in W_+.
\]
Then, with \( m = rk, k \in \mathbb{N}_+ \),
\[
\text{trace} \left( \mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_1}^t \mathbf{M}_{\omega_1} \cdots \mathbf{M}_{\omega_1} \right) = |\lambda|^2k + |\bar{\lambda}|^2k = 2 \cdot \left( \frac{3}{25} \right)^{rk} = 2 \cdot \left( \frac{3}{25} \right)^{m}, k \in \mathbb{N}_+.
\]
This implies that
\[
\lim_{m \to \infty} \left[ \text{trace} \left( \mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_1}^t \mathbf{M}_{\omega_1} \cdots \mathbf{M}_{\omega_1} \right) \right]^{1/m} = \frac{3}{25}.
\]
By a direct search, it can be seen that the least possible value of \( r \) is \( r = 3 \), and accordingly, (312) is a finite sequence satisfying the desired property. In particular, the product
\[
\mathbf{M}_2 \mathbf{M}_1 \mathbf{M}_3 = \begin{bmatrix}
\frac{6}{125} & \frac{\sqrt{3}}{125} \\
\frac{125}{\sqrt{3}} & \frac{4}{125}
\end{bmatrix}
\]
is a such product having eigenvalues \( \frac{1}{25} \left( 1 \pm \frac{\sqrt{3}}{2}i \right) \).
We may now take \( \omega = (312)/(312) \ldots \in W_+ \) and complete the proof.

**Corollary 4.2.** (a) The measure \( \sigma = \mu_n \) satisfies the condition \( (M) \) with \( \delta = 1 \) and \( \overline{\delta} = \delta_s \). Conversely, if \( \mu_n \) satisfies \( (M) \) for some \( \delta \) and \( \overline{\delta} \), then \( \delta \leq 1, \overline{\delta} \geq \delta_s \).

(b) The measure \( \sigma = \mu_n \) satisfies the condition \( (M') \) with \( \delta = (1 + 1/\delta_s)^{-1} \) and \( \overline{\delta} = 1 \). Conversely, if \( \mu_n \) satisfies \( (M') \) for some \( \delta \) and \( \overline{\delta} \), then \( \delta \leq (1 + 1/\delta_s)^{-1}, \overline{\delta} \geq 1 \).
Proof. (a) We only need to prove the statements on $\bar{\delta}$, as $\mu_n(S) = \nu_n(S) = 3^k$ for dyadic simplexes $S \subseteq S^n$ with $\text{diam}(S) = 2^k$, $k \in \mathbb{N}$. Since spectrum($Y_i$) = \{0, \frac{1}{5}, \frac{2}{5}\}, $i = 1, 2, 3$, we see that
\[
\mu(\text{F}_{[\omega]_m}(S_0)) \leq \frac{1}{2} \left(\frac{5}{3}\right)^m \left[2 \cdot \left(\frac{3}{5}\right)^{2m} + \left(\frac{1}{5}\right)^{2m}\right] = 3^{m/\delta_s} = \nu(\text{F}_{[\omega]_m}(S_0))^{1/\delta_s},
\]
for all $\omega \in W_+, m \in \mathbb{N}$. Therefore,
\[
\mu_n(\text{F}_{[\omega]_m}(S_0)) \leq \nu_n(\text{F}_{[\omega]_m}(S_0))^{1/\delta_s},
\]
which shows that the first part of (M) with $\bar{\delta} = \delta_s$.
Conversely, suppose $\mu_n$ satisfies the first part of (M) for some $\bar{\delta}$. Let $S_m = \text{F}_1^m(S_0), m \in \mathbb{N}$. Since
\[
\mu(\text{F}_1^m(S_0)) = \frac{1}{2} \left(\frac{5}{3}\right)^m \left[\left(\frac{3}{5}\right)^{2m} + \left(\frac{1}{5}\right)^{2m}\right] \geq \frac{3^m}{2},
\]
we see that
\[
1/\bar{\delta} \leq \lim_{m \to \infty} \frac{\log \mu_n(S_m)}{\log \nu_n(S_m)} = 1/\delta_s.
\]
Therefore, $\bar{\delta} \geq \delta_s$.
(b) As in the proof of (a), we only need to prove the statements on $\bar{\delta}$. By Proposition 4.1,
\[
\mu(\text{F}_{[\omega]_m}(S_0)) \geq \frac{1}{2} \left(\frac{5}{3}\right)^m \left(\frac{3}{25}\right)^m = \frac{1}{2} 3^{m(1+1/\delta_s)} = \frac{1}{2} \nu(\text{F}_{[\omega]_m}(S_0))^{1+1/\delta_s},
\]
for all $\omega \in W_+, m \in \mathbb{N}$. Therefore,
\[
\mu_n(\text{F}_{[\omega]_m}(S_0)) \geq \frac{1}{2^n} \nu_n(\text{F}_{[\omega]_m}(S_0))^{1+1/\delta_s}.
\]
This shows the first part of (M') holds with $\bar{\delta} = (1 + 1/\delta_s)^{-1}$.
Conversely, suppose $\mu_n$ satisfies the first part of (M') for some $\bar{\delta}$. By Proposition 4.1, there exists an $\omega \in W_+$ such that
\[
\lim_{m \to \infty} \left[\text{trace}(Y_{[\omega]_m}^t Y_{[\omega]_m})\right]^{1/m} = \frac{3}{25}.
\]
Therefore,
\[
\lim_{m \to \infty} \mu(\text{F}_{[\omega]_m}(S_0))^{1/m} = \frac{5}{3} \lim_{m \to \infty} \left[\text{trace}(Y_{[\omega]_m}^t Y_{[\omega]_m})\right]^{1/m} = \frac{5}{3},
\]
Let $S_m = \text{F}_{[\omega]_m}(S_0) \times \cdots \times \text{F}_{[\omega]_m}(S_0)$. The above and (M') give
\[
1/\bar{\delta} \geq \lim_{m \to \infty} \frac{\log \mu_n(S_m)}{\log \nu_n(S_m)} = \frac{\log 5}{\log 3} = 1 + 1/\delta_s.
\]
This completes the proof. \[\square\]

Bibliography

[1] Barlow, M. T. and Bass, R. F. (1989). The construction of Brownian motion on the Sierpinski carpet. In Annales de l’IHP Probabilités et statistiques, volume 25, pages 225–257.

[2] Barlow, M. T. and Bass, R. F. (1999). Brownian motion and harmonic analysis on Sierpinski carpets. Canadian Journal of Mathematics, 51(4):673–744.
[3] Barlow, M. T. and Kigami, J. (1997). Localized eigenfunctions of the Laplacian on p.c.f self-similar sets. *Journal of the London Mathematical Society*, 56(02):320–332.

[4] Barlow, M. T. and Perkins, E. A. (1988). Brownian motion on the Sierpinski gasket. *Probability Theory and Related Fields*, 79(4):543–623.

[5] Bockelman, B. and Strichartz, R. S. (2007). Partial differential equations on products of Sierpinski gaskets. *Indiana University Mathematics Journal*, 56(3):1361–1375.

[6] Derfel, G., Grabner, P., and Vogl, F. (2008). The Zeta function of the Laplacian on certain fractals. *Transactions of the American Mathematical Society*, 360(2):881–897.

[7] Fitzsimmons, P. J., Hambly, B. M., and Kumagai, T. (1994). Transition density estimates for Brownian motion on affine nested fractals. *Communications in Mathematical Physics*, 165(3):595–620.

[8] Fukushima, M., and Shima, T. (1992). On a spectral analysis for the Sierpinski gasket. *Potential Analysis*, 1(1):1–35.

[9] Goldstein, S. (1987). Random walks and diffusions on fractals. In *Percolation Theory and Ergodic Theory of Infinite Particle Systems*, pages 121–129. Springer.

[10] Grigor’yan, A., Hu, J., and Lau, K. S. (2003). Heat kernels on metric measure spaces and an application to semilinear elliptic equations. *Transactions of the American Mathematical Society*, 355(5):2065–2095.

[11] Hambly, B. M. (1992). Brownian motion on a homogeneous random fractal. *Probability Theory and Related Fields*, 94(1):1–38.

[12] Hambly, B. M. and Kumagai, T. (1999). Transition density estimates for diffusion processes on post critically finite self-similar fractals. *Proceedings of the London Mathematical Society*, 78(2):431–458.

[13] Hambly, B. M. and Yang, W. (2016). Existence and space-time regularity for stochastic heat equations on p.c.f. fractals. *preprint, arXiv:1609.08960*.

[14] Havlin, S. and Ben-Avraham, D. (1987). Diffusion in disordered media. *Advances in Physics*, 36(3):695–798.

[15] Hino, M. (2008). Martingale dimensions for fractals. *Institute of Mathematical Statistics*, 36(3):971–991.

[16] Hino, M. (2010). Energy measures and indices of Dirichlet forms, with applications to derivatives on some fractals. *Proceedings of the London Mathematical Society*, 100(1):269–302.

[17] Hino, M. and Kumagai, T. (2006). A trace theorem for Dirichlet forms on fractals. *Journal of Functional Analysis*, 238(2).

[18] Hinz, M., Röckner, M., and Teplyaev, A. (2013). Vector analysis for Dirichlet forms and quasilinear PDE and SPDE on metric measure spaces. *Stochastic Processes and Their Applications*, 123(12):4373–4406.

[19] Hinz, M. and Teplyaev, A. (2013). Dirac and magnetic Schrödinger operators on fractals. *Journal of Functional Analysis*, 265(11):2830–2854.
[20] Hinz, M. and Teplyaev, A. (2015). Finite energy coordinates and vector analysis on fractals. *Fractal Geometry and Stochastics V*, 70:209–227.

[21] Hu, J. (2003). A note on Hajłasz–Sobolev spaces on fractals. *Journal of Mathematical Analysis and Applications*, 280(1):91–101.

[22] Kigami, J. (1989). A harmonic calculus on the Sierpinski spaces. *Japan Journal of Applied Mathematics*, 6(2):259–290.

[23] Kigami, J. (1993). Harmonic metric and Dirichlet form on the Sierpinski gasket. *Pitman Research Notes in Mathematics Series*, pages 201–218.

[24] Kigami, J. (1998). Distributions of localized eigenvalues of Laplacians on post critically finite self-similar sets. *Journal of Functional Analysis*, 156(1):170–198.

[25] Kigami, J. (2001). *Analysis on Fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press.

[26] Kigami, J. (2008). Measurable Riemannian geometry on the Sierpinski gasket: the Kusuoka measure and the Gaussian heat kernel estimate. *Mathematische Annalen*, 340(4):781–804.

[27] Kigami, J. and Lapidus, M. L. (1993). Weyl’s problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals. *Communications in Mathematical Physics*, 158(1):93–125.

[28] Kumagai, T. (1993). Estimates of transition densities for Brownian motion on nested fractals. *Probability Theory and Related Fields*, 96(2):205–224.

[29] Kusuoka, S. (1987). A diffusion process on a fractal. In Itô, K. and Ikeda, N., editors, *Probabilistic Methods in Mathematical Physics: Proceedings of the Taniguchi International Symposium (Katata and Kyoto, 1985)*, pages 251–274. Academic Press.

[30] Kusuoka, S. (1989). Dirichlet forms on fractals and products of random matrices. *Publications of the Research Institute for Mathematical Sciences*, 25(4):659–680.

[31] Liu, X. and Qian, Z. (2016). Brownian motion on the Sierpinski gasket and related stochastic differential equations. *preprint, arXiv:1612.01297*.

[32] Liu, X. and Qian, Z. (2017). Sobolev inequalities and semi-linear parabolic equations on Sierpinski gaskets. *preprint, arXiv:1706.01590*.

[33] Meyers, R., Strichartz, R. S., and Teplyaev, A. (2004). Dirichlet forms on the Sierpiński gasket. *Pacific Journal of Mathematics*, 217(1):149–174.

[34] Rammal, R. and Toulouse, G. (1983). Random walks on fractal structures and percolation clusters. *Journal de Physique Lettres*, 44(1):13–22.

[35] Strichartz, R. S. (2003). Function spaces on fractals. *Journal of Functional Analysis*, 198(1):43–83.

[36] Strichartz, R. S. (2005). Analysis on products of fractals. *Transactions of the American Mathematical Society*, 357(2):571–615.

[37] Teplyaev, A. (2000). Gradients on fractals. *Journal of Functional Analysis*, 174:128–154.

[38] Yang, W. (2016). Existence and space-time regularity for damped stochastic wave equations on p.c.f. fractals. *preprint, arXiv:1611.04874*.