Topological Insulators at Strong Disorder

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Topological insulators are newly discovered materials with the defining property that any boundary cut into such crystal supports spectrum which is immune to the Anderson localization. The present paper summarizes our efforts on the rigorous characterization of these materials in the regime of weak and strong disorder. In particular, the defining property is rigorously proven under certain relevant conditions, for more than half of the classification table of topological insulators.

Keywords: Topological insulators; Index theory; Disordered materials.

1. Introduction

A topological insulator can be defined as a homogeneous crystal with the following physical characteristics: 1) It is insulating in the bulk, that is, the diagonal components of the bulk conductivity tensor vanish in the limit of zero temperature. 2) The crystal with a boundary, however, continues to conduct electricity even in this limit. 3) These features are robust at least in the regime of small disorder. The physics community put forward a set of conjectures which were confirmed by a large body of theoretical and computational work. They say that there are only three fundamental symmetries which can stabilize topological insulating phases. The combinations of these symmetries lead to precisely 10 classes of topological insulators and superconductors, as summarized in Table 1. Furthermore, it is conjectured that topological phases exists only in certain dimensions, depending on the class, and in such dimensions the phases can be labeled uniquely by either an integer number or by an integer modulo 2. When the classification is by \( \mathbb{Z} \) (or \( 2\mathbb{Z} \)), it is further conjectured that the integer label is provided by the Chern number of the Fermi projection in the even dimensions, and by the Chern number of the Fermi unitary operator in the odd dimensions.

In this paper I summarize recent results which confirm these conjectures for the topological phases highlighted in Table 1. More precisely: 1) The bulk invariants defining these phases are robust against strong disorder. As a result, to pass from one phase to another, one necessarily has to cross a localization-delocalization phase transition. 3) When a boundary is cut into a topological crystals, delocalized boundary spectrum emerges at the Fermi level.

This work was in collaboration with Jean Bellissard and Hermann Schulz-Baldes and it was recently published as a monograph in 8. The readers interested in the proofs of the results, are directed to this reference.
positive and finite constants, which can depend parametrically on $s$.

Malchanov criterium applies, for all $z$ has a mobility gap $\Delta$ if $\Delta$ is located in the essential spectrum of $H$. The probability measure on $\Omega$, compact topological space. We assume the existence of an ergodic and invariant $U$ denote the magnetic translations: $d$, $dure for disorder. Above, $H$ of the electrons is governed by the covariant family of finite range $H$ amiltonians classical dynamical system ($\Omega$, $\tau$, $\mathbb{Z}^d$), Let us start with the settings for the bulk. The disorder will be encoded in the $H$ as usual the presence of a uniform magnetic field. The dyn amics $x, y$ encodes as usual the presence of a uniform magnetic field. The dyn amics $d$ dimension $d = 0, \ldots, 8$. These phases are in one-to-one relation with the elements of the empty, $\mathbb{Z}$, $Z$ or $2\mathbb{Z}$ groups. The table is further divided into the complex classes $A$ and $AIII$ (top two rows), which are classified by the complex $K$-theory, and the real classes $AI$, $\ldots$, $CI$ (the remaining 8 rows), which are classified by the real $K$-theory. The phases highlighted in red are the ones cover by the results reported in this paper.

| $j$ | TRS | PHS | CHS | CAZ | 0, 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|-----|-----|-----|-----|-----|---|---|---|---|---|---|---|
| 0   | 0   | 0   | 0   | A   | $\mathbb{C}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{C}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 1   | +1  | +1  | 0   | AI  | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 2   | 0   | +1  | 0   | BDI | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 3   | +1  | 1   | 1   | DHI | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 4   | 0   | 0   | 0   | AIH | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 5   | +1  | 0   | 1   | CHI | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 6   | 0   | 1   | 0   | C   | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 7   | +1  | 0   | 1   | CI  | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |

2. The Settings

Let us start with the settings for the bulk. The disorder will be encoded in the classical dynamical system ($\Omega$, $\tau$, $\mathbb{Z}^d$), where $\Omega = \Omega_0^\mathbb{Z}^d$, $(\tau_\omega)_\omega = \omega_{x+y}$ and $\Omega_0$ is a compact topological space. We assume the existence of an ergodic and invariant probability measure on $\Omega$, $d\mathbb{P}(\omega) = \prod_{\omega \in \mathbb{Z}^d} d\mathbb{P}_0(\omega_x)$, providing the averaging procedure for disorder. Above, $d\mathbb{P}_0$ is a probability measure on $\Omega_0$. Throughout, $U_y$ will denote the magnetic translations: $U_xU_y = e^{i\pi x \wedge y}U_{x+y}$, where the skew product $x \wedge y$ encodes as usual the presence of a uniform magnetic field. The dynamics of the electrons is governed by the covariant family of finite range Hamiltonians $H = \{H_\omega\}_{\omega \in \Omega}$, defined over $\mathbb{C}^N \otimes \mathbb{L}(\mathbb{Z}^d)$ and taking the generic form

$$H_\omega = \sum_{x,y \in \mathbb{Z}^d} W_y(\tau_\omega) \otimes |x\rangle\langle x|U_y, \quad U_yH_\omega U_y^{-1} = H_{\tau_\omega}, \quad |R| < \infty,$$

where $W_y$ are continuous functions over $\Omega$. It is useful to view $W_y$ as elements of the C*-algebra $\mathcal{M}_N(\mathbb{C}) \otimes \mathcal{C}(\Omega)$ endowed with the sup norm. We will say that $H$ has a spectral gap $\Delta$ if $\sigma(H_\omega) \cap \Delta = \emptyset$ for all $\omega \in \Omega$. We will also say that $H$ has a mobility gap $\Delta$ if $\Delta$ is located in the essential spectrum of $H$ and Aizenman-Malchanov criterium applies, namely, for any $s \in (0, 1)$ and $\delta > 0$,

$$\int_{\Omega} d\mathbb{P}(\omega) |\langle x, \alpha| (H_\omega - z)^{-1} |y, \beta\rangle|^s \leq A_s(\delta)e^{-\beta_s(\delta)|x-y|}, \quad (1)$$

for all $z \in \mathbb{C} \setminus \sigma(h)$ with $\text{dist}(z, \sigma(h) \setminus \Delta) \geq \delta$. Above, $A_s(\delta)$ and $\beta_s(\delta)$ are strictly positive and finite constants, which can depend parametrically on $s$ and $\delta$ but are
Let us start with the index theorems for the bulk case. It will be useful to form a matrix

\[ \langle x, n \rangle = \sum_{n,m < R} \sum_{y,x} \sum_{R \in \mathbb{Z}^d} W^\nu_{nm}(\tau_{x,n} \omega) \langle x, n \rangle \langle x, n \rangle U_{y,n-m}, \quad R, |R| < \infty, \]

which redefines the boundary condition. The family \( \tilde{H} = \{ \tilde{H}_\omega \}_{\omega \in \Omega} \) remains covariant w.r.t. the lattice shifts parallel to the boundary, but the dynamical system defined by these shifts is no longer ergodic w.r.t. \( d\mathbb{P} \). For this reason, our results apply only for boundary disorder, encoded in an ergodic dynamical system of the type \( (\Omega, \tau_{|\mathbb{Z}^d-1, \mathbb{Z}^d-1, d\mathbb{P}_L}) \), where \( d\mathbb{P}_L \) is the push forward measure induced by the following continuous map:

\[ p_L : \Omega_{d-1, L} \rightarrow \Omega_{d, L}, (p_L(\omega_L))_x = \begin{cases} \omega_x & \text{if } x_d \in I_L = \{-L, \ldots, L\} \\ \omega_x = 0, & \text{otherwise,} \end{cases} \]

where \( \Omega_{d-1, L} \) is considered with the product measure \( \prod_{x \in \mathbb{Z}^d-1, I_L} d\mathbb{P}_0(\omega_x) \). Under this probability measure, the disorder configurations where any of \( \omega_x \) are different from zero for \( x_d > L \) occur with zero probability.

### 3. Technical statements

Let us start with the index theorems for the bulk case. It will be useful to form a matrix \( \langle x|A|x \rangle \) out of the matrix elements \( (x, \alpha|A|x, \beta) \) of an operator over \( C^N \otimes \ell^2(\mathbb{Z}^d) \). We will denote by \( \text{tr} \) the trace of these matrices and will employ the Schatten norms \( \| \cdot \|_{(s)} \) for these matrices.

**Theorem 3.1 (Index Theorem for Bulk Projections (3, 6)).** Let \( d \) be even and let \( \{ P_\omega \}_{\omega \in \Omega} \) be a family of covariant projections over \( C^N \otimes \ell^2(\mathbb{Z}^d) \) such that

\[ \sum_{x \in \mathbb{Z}^d} (1 + |x|)^{d+1} \int_{\Omega} d\mathbb{P}(\omega) \| \langle x|P_\omega|x \rangle \|_{(d+1)}^{d+1} < \infty. \]

Consider the operator

\[ F_{\omega, x_0} = P_\omega \left( \frac{X + x_0\cdot \Gamma}{|X + x_0|} \right) P_\omega = \begin{pmatrix} 0 & G_{\omega, x_0} \\\ G_{\omega, x_0}^* & 0 \end{pmatrix}, \quad x_0 \in (0,1)^d, \]

where \( \Gamma = (\Gamma_1, \ldots, \Gamma_d) \) is an irreducible representation of the d-dimensional complex Clifford algebra \( \mathbb{C}_d \) and \( X \) is the position operator over \( \ell^2(\mathbb{Z}^d) \). The second equality gives the decomposition of \( F_{\omega, x_0} \) w.r.t. the natural grading of \( \mathbb{C}_d \). Then \( G_{\omega, x_0} \) is \( \mathbb{P} \)-almost surely a Fredholm operator on the range of \( P_\omega \). Its almost sure Fredholm
index is independent of \( x_0 \in \mathbb{Z}^d \) and \( \mathbb{P} \)-almost surely independent of \( \omega \in \Omega \), and is given by the formula

\[
\text{Ind } G_{\omega,x_0} = \Lambda_d \sum_{\rho \in S_d} (-1)^{\rho} \int_{\Omega} d\mathbb{P}(\omega) \text{ tr} \left\langle 0 \left| P_\omega \prod_{i=1}^d i [P_\omega, X_{\rho_i}] \right| 0 \right\rangle, \quad \Lambda_d = \frac{(2\pi)^d}{d!}.
\]

**Theorem 3.2 (Index Theorem for Bulk Unitaries (4, 7)).** Let \( d \) be odd and let \( \{U_\omega\}_{\omega \in \Omega} \) be a family of covariant unitary operators over \( \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d) \) such that

\[
\sum_{x \in \mathbb{Z}^d} (1 + |x|)^{d+1} \int_{\Omega} d\mathbb{P}(\omega) \left\| \langle x \left| U_\omega \right| x \right\|^{d+1}_{(d+1)} < \infty.
\]

Let \( E_{x_0} \) be the spectral projection onto the positive spectrum of \( (X + x_0) \cdot \Gamma \), \( x_0 \in (0,1)^d \). Then, \( \mathbb{P} \)-almost surely, the operator \( F_{\omega,x_0} = E_{x_0} U_\omega E_{x_0} \) is a Fredholm operator on the range of \( E_{x_0} \). Its almost sure Fredholm index is independent of \( x_0 \) and \( \mathbb{P} \)-almost surely independent of \( \omega \in \Omega \), and is given by the formula

\[
\text{Ind } F_{\omega,x_0} = \Lambda_d \sum_{\rho \in S_d} (-1)^{\rho} \int_{\Omega} d\mathbb{P}(\omega) \text{ tr} \left\langle 0 \left| P_\omega \prod_{i=1}^d i U_\omega^+ [U_\omega, X_{\rho_i}] \right| 0 \right\rangle, \quad \Lambda_d = \frac{(\pi)^{d-1}}{d!!}.
\]

In the absence of disorder and magnetic fields, the righthand sides of these index theorems are the classical even and odd Chern numbers over the non-commutative even and odd Chern numbers.

Let us now formulate the index theorems for the boundary. Again, it will be useful to form a matrix \( \langle x | \tilde{A} | x \rangle \) out of the matrix elements \( \langle x, x_d, \alpha | \tilde{A} | x, x_d, \beta \rangle \), for an operator \( \tilde{A} \) over \( \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^{d-1} \times \mathbb{N}) \). Note that this time the matrices are infinite. Recall that \( \left\| \cdot \right\|_{(s)} \) represents the \( s \)-Schatten norms for these matrices.

**Theorem 3.3 (Index Theorem for Boundary Projections (8)).** Let \( d \) be odd and let \( \{\tilde{P}_\omega\}_{\omega \in \Omega} \) be a family of covariant projections over \( \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d \otimes \mathbb{N}) \) such that

\[
\sum_{x \in \mathbb{Z}^{d-1}} (1 + |x|)^d \int_{\Omega} d\mathbb{P}_L(\omega) \left\| \langle x | \tilde{P}_\omega \right| x \right\|^{d}_{(d)} < \infty.
\]

Consider the operator

\[
\tilde{F}_{\omega,x_0} = \tilde{P}_\omega \left( \frac{\tilde{X} - \tilde{x}_0}{|\tilde{X} - \tilde{x}_0|} \cdot \tilde{\Gamma} \right) \tilde{P}_\omega = \begin{pmatrix} 0 & \tilde{G}_{\omega,\tilde{x}_0} \tilde{x}_0 \\ \tilde{G}_{\omega,\tilde{x}_0}^* & 0 \end{pmatrix}, \quad \tilde{x}_0 \in (0,1)^{d-1},
\]

where \( \tilde{X} \) is the position operator over \( \ell^2(\mathbb{Z}^{d-1}) \) and \( \tilde{\Gamma} = (\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_{d-1}) \) is an irreducible representation of the complex Clifford algebra \( \mathbb{C}_d \). Then, \( \mathbb{P}_L \)-almost surely, the operator \( \tilde{G}_{\omega,\tilde{x}_0} \) is a Fredholm operator on the range of \( \tilde{P}_\omega \). Its almost sure Fredholm index is independent of \( \tilde{x}_0 \) and \( \mathbb{P}_L \)-almost surely independent of \( \omega \in \Omega \), and is given by the formula

\[
\text{Ind } \tilde{G}_{\omega,\tilde{x}_0} = \Lambda_{d-1} \sum_{\rho \in S_{d-1}} (-1)^{\rho} \int_{\Omega} d\mathbb{P}_L(\omega) \text{ tr} \left\langle 0 \left| \tilde{P}_\omega \prod_{i=1}^{d-1} i [\tilde{P}_\omega, \tilde{X}_{\rho_i}] \right| 0 \right\rangle.
\]
Theorem 3.4 (Index Theorem for Boundary Unitaries (8)). Let \( d \) be odd and let \( \{ \tilde{U}_\omega \}_{\omega \in \Omega} \) be a family of covariant unitaries over \( \mathbb{C}^N \otimes \ell^2(\mathbb{Z} \otimes \mathbb{N}) \) such that
\[
\sum_{x \in \mathbb{Z}^{d-1}} (1 + |x|)^d \int_{\Omega} d\mathbb{P}_L(\omega) \| \langle x | \tilde{U}_\omega | x \rangle \|_d^d < \infty.
\]
Let \( \tilde{E}_{\tilde{x}_0} \) be the spectral projection onto the positive spectrum of \( (\tilde{X} - \tilde{x}_0) \cdot \tilde{\Gamma}, \tilde{x}_0 \in (0,1)^{d-1} \). Then, \( \mathbb{P}_L \)-almost surely, the operator \( \tilde{F}_{\omega, \tilde{x}_0} = \tilde{E}_{\tilde{x}_0} \tilde{U}_\omega \tilde{E}_{\tilde{x}_0} \) is a Fredholm operator on the range of \( \tilde{E}_{\tilde{x}_0} \). Its almost sure Fredholm index is independent of \( \omega \in \Omega \), and is given by the formula
\[
\text{Ind} \tilde{F}_{\omega, \tilde{x}_0} = \Lambda_{d-1} \sum_{\rho \in S_{d-1}} (-1)\rho \int_{\Omega} d\mathbb{P}_L(\omega) \text{tr} \left( 0 \left| \prod_{i=1}^{d-1} \tilde{U}_\omega \right| \tilde{X}_{\rho_i} \right) 0 .
\]

4. The Bulk-Boundary Correspondence Principle

Let us state first the results for the even dimensions. Recall that all topological phases classified by \( \mathbb{Z} \) (or \( \mathbb{Z}^2 \)) in even dimensions were conjectured to be uniquely defined by the top even Chern numbers of the Fermi projections \( P_\omega = \chi(H_\omega \leq \mu) \), where \( \mu \) represents the Fermi level.

Theorem 4.1 (Bulk-Boundary Principle for Even Dimensions (8)). (1)
If the Fermi level is located in a mobility gap, then the conditions of Theorem [3A] are satisfied and as a consequence the even Chern number
\[
\text{Ch}_d(P_\omega) := \Lambda_d \sum_{\rho \in S_d} (-1)\rho \int_{\Omega} d\mathbb{P}(\omega) \text{tr} \left( 0 \left| P_\omega \prod_{i=1}^{d} X_{\rho_i} P_\omega \right| 0 \right)
\]
is quantized and \( \mathbb{P} \)-almost surely does not fluctuate from one disorder configuration to another.

(2) The invariant remains constant under continuous deformations of the functions \( W_\mu \) in the definition of the Hamiltonians, as long as the Fermi level remains in a mobility gap. In other words, the only way \( \text{Ch}_d(P_\omega) \) can change its quantized value is through an Anderson localization-delocalization transition.

(3) Let \( \tilde{H}_\omega \) be a covariant family of half-space Hamiltonians for \( H_\omega \) and assume that \( H_\omega \)‘s have a spectral gap. Define \( \tilde{U}_\omega = \exp \left( 2\pi i f(\tilde{H}_\omega) \right) \), where \( f \) is a smooth function such that \( f = 1, 0 \) below/above an arbitrarily small interval around the Fermi level. Then \( \tilde{U}_\omega \) satisfies the conditions of Theorem [3A] and as a consequence the odd Chern number
\[
\text{Ch}_{d-1}(\tilde{U}_\omega) := \Lambda_{d-1} \sum_{\rho \in S_{d-1}} (-1)\rho \int_{\Omega} d\mathbb{P}_L(\omega) \text{tr} \left( 0 \left| \prod_{i=1}^{d-1} \tilde{U}_\omega \right| \tilde{X}_{\rho_i} \right) 0
\]
is quantized and \( \mathbb{P}_L \)-almost surely does not fluctuate from one disorder configuration to another. It is also independent of \( L \).
If the spectrum is localized at the Fermi level, then the function \( f \) defining \( \tilde{U}_\omega \) can be deformed into a step function without violating the conditions of Theorem 3.4. In this case \( \tilde{U}_\omega = I \) and its boundary invariant is zero. In other words, if \( \text{Ch}_{d-1}(\tilde{U}_\omega) \neq 0 \), then the boundary spectrum at the Fermi level is necessarily delocalized.

Furthermore, an equality between the bulk and boundary invariants holds

\[
\text{Ch}_d(P_\omega) = \tilde{\text{Ch}}_{d-1}(\tilde{U}_\omega),
\]

which is a direct consequence of Theorem A10 of 4. Combining with the previous point, we now can confirm that a non-trivial bulk invariant induces delocalized boundary spectrum at the Fermi level.

We now state the results for odd dimensions. By examining Table 1, one can see that all topological phases classified by \( \mathbb{Z} \) (or \( 2\mathbb{Z} \)) in odd dimensions possess the chiral symmetry. This means there exists a symmetry \( J, J^* = J, J^2 = I \), such that \( JH_\omega J^{-1} = -H_\omega \). Among other things, this symmetry constrains the Fermi level to be pinned at zero, in order to satisfy charge neutrality. To avoid states pinned at zero, hence to allow for insulating states, the fiber of Hilbert space must necessarily be even. Hence, we assume that the bulk Hamiltonians are defined over \( \mathbb{C}^{2N} \otimes \ell^2(\mathbb{Z}^d) \) and similarly for the half-space Hamiltonians. In this case, the ground state can be encoded in the unitary operator \( U_\omega \) defined over \( \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d) \) by the following implicit expression

\[
\text{sgn}(H_\omega) = \begin{pmatrix} 0 & U_\omega^* \\ U_\omega & 0 \end{pmatrix},
\]

where the decomposition above is w.r.t. the grading induced by \( J \).

**Theorem 4.2 (Bulk-Boundary Principle for Odd Dimensions).**

1. If the Fermi level is located in a mobility gap, then the conditions of Theorem 3.2 are satisfied and as a consequence the odd Chern number

\[
\text{Ch}_d(U_\omega) := \Lambda_d \sum_{p \in \mathcal{S}_d} (-1)^\rho \int_{\Omega} d\mathbb{P}(\omega) \text{ tr} \left< 0 \left| \prod_{i=1}^d dU_\omega^* \left[ U_\omega, X_{p_i} \right] \right| 0 \right>
\]

is quantized and \( \mathbb{P} \)-almost surely does not fluctuate from one disorder configuration to another.

2. The invariant remains constant under continuous deformations of the functions \( W_y \) in the definition of the Hamiltonians, as long as the Fermi level remains in a mobility gap. In other words, the only way \( \text{Ch}_d(U_\omega) \) can change its quantized value is through an Anderson localization-delocalization transition.

3. Let \( \tilde{H}_\omega \) be a covariant family of half-space Hamiltonians for \( H_\omega \) and assume that \( H_\omega \) have a spectral gap. Define

\[
\tilde{P}_\omega = e^{-i\tilde{f}(\tilde{H}_\omega)} \text{diag}(I_N, 0_N) e^{i\tilde{f}(\tilde{H}_\omega)}
\]
where \( f \) is a smooth function, odd under inversion and \( f = \pm 1 \) above/below a small interval around \( \mu \). Then \( \tilde{P}_\omega \) satisfies the conditions of Theorem 3.3 and as a consequence the even Chern number

\[
\tilde{\text{Ch}}_{d-1}(\tilde{P}_\omega) := \Lambda_{d-1} \sum_{\rho \in S_{d-1}} (-1)^\rho \int_\Omega d\nu L(\omega) \text{tr} \left\langle 0 \right| \tilde{P}_\omega \prod_{i=1}^{d-1} [\tilde{P}_\omega, \tilde{X}_\rho] \left| 0 \right\rangle
\]

is quantized and \( \mathbb{P}_L \)-almost surely does not fluctuate from one disorder configuration to another. It is also independent of \( L \).

(4) If the spectrum is localized at the Fermi level, then the function \( f \) defining \( \tilde{P}_\omega \) can be deformed into a step function without violating the conditions of Theorem 3.3. In this case \( \tilde{P}_\omega = \text{diag}(1_N, 0) \) and its boundary invariant is zero. In other words, if \( \tilde{\text{Ch}}_{d-1}(\tilde{P}_\omega) \neq 0 \), then the boundary spectrum at the Fermi level is necessarily delocalized.

(5) Furthermore, the following equality between the bulk and boundary invariants holds

\[
\text{Ch}_d(U_\omega) = \tilde{\text{Ch}}_{d-1}(\tilde{P}_\omega),
\]

which is a direct consequence of Theorem A10 of 4. Combining with the previous point, we now can confirm that a non-trivial bulk invariant induces delocalized boundary spectrum at the Fermi level.

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