THE EULER CHARACTERISTIC OF THE GENERALIZED KUMMER SCHEME OF AN ABELIAN THREEFOLD

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Abstract

Let \( X \) be an Abelian threefold. We prove a formula, conjectured by the first author, expressing the Euler characteristic of the generalized Kummer schemes \( K^n X \) of \( X \) in terms of the number of plane partitions. This computes the Donaldson-Thomas invariant of the moduli stack \([K^n X/X_\alpha]\).

1 Introduction

Let \( n > 0 \) be an integer. The \( n \)-th generalized Kummer scheme \( K^n X \) of an Abelian variety \( X \) is the fibre over \( 0_X \) of the composite map

\[
\text{Hilb}^n X \to \text{Sym}^n X \to X,
\]

where the first arrow is the Hilbert-Chow morphism and the second arrow takes a cycle to the weighted sum of its supporting points. The purpose of this note is to prove the following formula, which is the three-dimensional case of a conjecture from [7]:

\[\chi(K^n X) = n^5 \sum_{d | n} d^2.\]

Simultaneously with and independent of our work, Shen [8] has proven the conjecture in [7] for \( X \) an Abelian variety of arbitrary dimension \( g \), stating that

\[
\sum_{n \geq 0} P_{g-1}(n)q^n = \exp\left( \sum_{n \geq 1} \frac{\chi(K^n X)}{n^5} q^n \right),
\]

where \( P_d(n) \) denotes the number of \( d \)-dimensional partitions of \( n \). In fact, Shen proves a further generalization of this to the case of a product \( X \times Y \), where one factor \( X \) is an Abelian variety, and the other factor \( Y \) is an arbitrary quasi-projective variety. For \( g = 3 \), the formula in Theorem 1 is recovered from (1) by applying MacMahon’s product formula for plane partitions (cf. [9, Cor. 7.20.3]).

One motivation for the computation of \( \chi(K^n X) \) is as a test case for Donaldson–Thomas invariants for Abelian threefolds, as developed in [7]. In particular (see loc. cit.), the Donaldson-Thomas invariant of the moduli stack \([K^n X/X_\alpha]\) is the rational number

\[
\frac{(-1)^{n+1}}{n^6} \chi(K^n X) = \frac{(-1)^{n+1}}{n} \sum_{d | n} d^2.
\]
The formula (1) could be motivated by formally expanding Cheah’s formula, for the Euler characteristic of Hilbert schemes of points (cf. [2], and also [5] for a motivic refinement) up to first order in \( \chi(X) \), as follows:

\[
1 + \sum_{n \geq 1} \chi(\text{Hilb}^n X) q^n = 1 + \chi(X) \sum_{n \geq 1} \frac{\chi(K^n X)}{n^2} q^n \\
\exp\left(\chi(X) \log \sum_{n \geq 1} P_{g-1}(n) q^n\right) = 1 + \chi(X) \log \sum_{n \geq 0} P_{g-1}(n) q^n.
\]

The top equality comes from the étale cover \( X \times K^n X \to \text{Hilb}^n X \) of degree \( n^6 \), given by the translation action of \( X \) on the Hilbert scheme. The vertical equality is Cheah’s formula (cf. [2], and also [5] for a motivic refinement). For the bottom equality, we treat \( \chi(X)^2 \) as zero when expanding exp.

### 1.1 Conventions

We work over \( \mathbb{C} \). The symbol \( \chi \) denotes the topological Euler characteristic. We denote by \( \alpha \vdash n \) (one-dimensional) partitions of \( n = \sum_i i \alpha_i \), corresponding to classical Young tableaux. The number of \( d \)-dimensional partitions of \( n \) is denoted \( P_d(n) \). A higher dimensional partition can be seen as a generalized Young tableau, with \( (d+1) \)-dimensional boxes taking the rôle of squares. The convention is to set \( P_d(0) = 1 \).

### 2 Proving the conjecture

#### 2.1 Stratification

The Hilbert scheme of points of any quasi-projective variety \( X \) admits a natural stratification by partitions,

\[
\text{Hilb}^n X = \bigsqcup_{\alpha \vdash n} \text{Hilb}^\alpha_n X
\]

where \( \text{Hilb}^\alpha_n X \) denotes the (locally closed) locus of subschemes of \( X \) having exactly \( \alpha_i \) components of length \( i \). Let \( X \) be an Abelian variety. Letting \( K^n X = K^n X \cap \text{Hilb}^\alpha_n X \), we get an induced stratification of the Kummer scheme:

\[
K^n X = \bigsqcup_{\alpha \vdash n} K^n_\alpha X.
\]

For each partition \( \alpha \vdash n \), let us define the subscheme

\[
V_\alpha = \{ \xi \in \text{Sym}^n_\alpha X \mid \Sigma \xi = 0 \} \subset \text{Sym}^n_\alpha X
\]

where \( \Sigma \) denotes addition of zero cycles under the group law on \( X \). The Hilbert-Chow morphism \( \text{Hilb}^n X \to \text{Sym}^n X \) restricts to morphisms

\[
\pi_\alpha : K^n_\alpha X \to V_\alpha.
\]
Fixing a point in $V_\alpha$ amounts to fixing the supporting points of the corresponding cycle and their multiplicities. Thus, each fibre of $\pi_\alpha$ is isomorphic to a product of punctual Hilbert schemes:

$$F_\alpha \cong \prod_i \text{Hilb}^i(A^3; 0)^{a_i}.$$ 

Hence, using (2), we find

$$\chi(K^n X) = \sum_{\alpha \vdash n} \chi(V_\alpha) \prod_i P_2(i)^{a_i},$$

where we have used $P_d(n) = \chi(\text{Hilb}^n(A^d; 0))$ (see [4] for $d = 2$ and [2], [5] for the general case).

### 2.2 Strategy of proof

Let $\sigma_2(n) = \sum_{d \mid n} d^2$ denote the square sum of divisors of an integer $n$. As is well known [1], these are related to the number of plane partitions by

$$nP_2(n) = \sum_{k=1}^n \sigma_2(k)P_2(n-k).$$

Let us define, for $\alpha \vdash n$, integers $c(\alpha) \in \mathbb{Z}$ by the recursion

$$c(\alpha) = \begin{cases} 
  n & \text{if } \alpha = (n^1), \\
  -\sum_{\ell,d,\ell_i \neq 0} c(\hat{\alpha}^i) & \text{otherwise},
\end{cases}$$

where, for a partition $\alpha = (1^{a_1} \ldots i^{a_i} \ldots \ell^{a_\ell}) \vdash n$, with $a_i \neq 0$, we let

$$\hat{\alpha}^i = (1^{a_1} \ldots i^{a_i-1} \ldots \ell^{a_\ell}) \vdash n-i.$$ 

We shall prove Theorem [1] in two steps, given by the two Lemmas that follow.

**Lemma 1.** The square sum of divisors $\sigma_2$ can be expressed in terms of the number of plane partitions $P_2$ as follows:

$$\sigma_2(n) = \sum_{\alpha \vdash n} c(\alpha) \prod_i P_2(i)^{a_i}. $$

**Lemma 2.** The Euler characteristics $\chi(V_\alpha) / n^3$ equal the numbers $c(\alpha)$ defined by recursion (5).

Assuming the two Lemmas, the main theorem follows:

**Proof of Theorem [1]** Equation (3) gives

$$\frac{\chi(K^n X)}{n^3} = \sum_{\alpha \vdash n} \frac{\chi(V_\alpha)}{n^3} \prod_i P_2(i)^{a_i} = \sum_{\alpha \vdash n} c(\alpha) \prod_i P_2(i)^{a_i} = \sigma_2(n).$$

We have applied Lemma [2] in the second equality, and Lemma [1] in the last equality. 

2.3 Proof of Lemma 1: a recursion

Let us introduce the shorthand
\[ f(\alpha) = \prod_i P_2(i)^{\alpha_i}. \]
Expand the right hand side of (7), using the definition of \( c(\alpha) \):
\[
(8) \quad \sum_{\alpha \vdash n} c(\alpha) f(\alpha) = n P_2(n) - \sum_{\alpha \vdash n \ j \geq 1 \ \alpha_i \neq 0} c(\hat{\alpha}_j) f(\hat{\alpha}_j)
\]
On the other hand, by induction on \( n \), the identity (4) gives
\[
(9) \quad \sigma_2(n) = n P_2(n) - n - 1 \sum_{k=1}^{n-1} \sigma_2(k) P_2(n-k) = n P_2(n) - \sum_{k=1}^{n-1} \frac{n}{k} \sum_{\beta \vdash k} c(\beta) f(\beta) P_2(n-k).
\]
The sets over which the double sums in (8) and (9) run are clearly identified via \((k, \beta) = (n-j, \hat{\alpha}_j)\). Since \( f(\alpha) = \prod_i P_2(i)^{\alpha_i} \), it follows that the two expressions (8) and (9) are identical. Lemma 1 is established.

2.4 Proof of Lemma 2: an incidence correspondence

In this section we prove Lemma 2. The technique used is very similar to the one adopted in [4].

Later on, we will need the following:

Remark 2.1. Let \( \alpha = (n^1) \). Then \( V_\alpha \) is in bijection with the subgroup \( X_n \subset X \) of \( n \)-torsion points in \( X \). This implies that \( \chi(V_\alpha) = \chi(X_n) = n^6 \). In other words, \( \chi(V_\alpha)/n^5 = n = c(\alpha) \).

Now we fix a partition \( \alpha \vdash n \) different from \((n^1)\), and an index \( i \) such that \( \alpha_i \neq 0 \). We will compute \( \chi(V_\alpha) \) in terms of the partition \( \hat{\alpha}_i \vdash n - i \), thanks to an incidence correspondence between the spaces \( V_\alpha \subset \text{Sym}_n X \) and \( V_{\hat{\alpha}_i} \subset \text{Sym}_{n-i} X \).

Let us define the subscheme
\[
I = \{ (a, b; \xi) \in X^2 \times V_\alpha \mid \text{mult}_a \xi = i, (n-i)b = ia \text{ in } X \} \subset X^2 \times V_\alpha.
\]
We use the incidence correspondence
\[
\begin{array}{ccc}
I & \xrightarrow{\phi} & V_\alpha \\
\downarrow & & \downarrow \\
V_{\hat{\alpha}_i} & \xrightarrow{\psi} & V_\alpha
\end{array}
\]
where the map \( \phi \) is the one induced by the second projection, and \( \psi \) sends \((a, b; \xi)\) to the cycle \( T_b(\xi - ia) \), where \( T_b \) is translation by \( b \in X \).

The strategy is to compute \( \chi(I) \) twice: by means of the fibres of \( \phi \) and \( \psi \) respectively. This will enable us to compare \( \chi(V_\alpha) \) and \( \chi(V_{\hat{\alpha}_i}) \).
Fibres of $\phi$. Let $\zeta \in V_a$. This means $\zeta \in \text{Sym}^n X$ and $\sum \zeta = 0$ in $X$. We have

$$\phi^{-1}(\zeta) = \{ (a, b) \in X^2 \mid \text{mult}_a \zeta = i, (n-i)b = ia \} \subset X^2.$$ 

Let $a_1, \ldots, a_n$ be the $a_i$ points, in the support of $\zeta$, having multiplicity $i$ (recall that $i$ is fixed). Then

$$\phi^{-1}(\zeta) = \bigcup_{1 \leq j \leq a_i} H_j,$$

where $H_j = \{ b \in X \mid (n-i)b = ia_j \}$. Each $H_j$ is the kernel of the translated isogeny $b \mapsto (n-i)b - ia_j$, which has degree $(n-i)^6$, so $\chi(H_j) = (n-i)^6$. This yields $\chi(\phi^{-1}(\zeta)) = a_i(n-i)^6$. Hence,

$$\chi(I) = \chi(V_a)a_i(n-i)^6.$$ 

Fibres of $\psi$. Let $C \in V_a$. A point $(a, b; \zeta) \in \psi^{-1}(C)$ determines $\zeta$ as

$$\zeta = T_b^{-1}(C) + ia,$$

and the condition $\text{mult}_a \zeta = i$ translates into $\text{mult}_a(T_b^{-1}(C) + ia) = i$, which means $a \notin \text{Supp}(T_b^{-1}(C))$, i.e. $a + b \notin \text{Supp}(C)$.

Let us define the subscheme

$$B = \{ (a, b) \mid (n-i)b = ia \} \subset X^2.$$ 

Then we note that

$$\psi^{-1}(C) = \{ (a, b) \in B \mid a + b \notin \text{Supp}(C) \} = B \setminus \bigcup_{c \in \text{Supp}(C)} Y_c,$$

where

$$Y_c = \{ (a, b) \in B \mid a + b = c \} \cong \{ b \in X \mid nb = ic \} \cong X_n.$$

Now, if we map $B \to X$ through the second projection, we see that the fibres are all isomorphic (to $X_n$ the group of $i$-torsion points in $X$). Hence, as $\chi(X) = 0$, we find that $\chi(B) = 0$. Thus, remembering that $\text{Supp}(C)$ consists of $(\sum a_i) - 1$ distinct points, we find

$$\chi(\psi^{-1}(C)) = - \sum_{c \in \text{Supp}(C)} \chi(Y_c) = -n^6 \cdot \left( \sum a_i - 1 \right).$$

Finally,

$$\chi(I) = -\chi(V_a)n^6 \cdot \left( \sum a_i - 1 \right).$$

Compare (10) and (11) to get

$$\chi(V_a) = -\frac{a_i(n-i)^6}{n^6(\sum a_i - 1)} \chi(V_a).$$
We now conclude by showing that the numbers \( \chi(V_\alpha)/n^5 \) satisfy the same recursion (5) fulfilled by the \( c(\alpha)'s \). If \( \alpha = (n^1) \), we know by Remark 2.1 that
\[
\frac{1}{n^5}\chi(V_\alpha) = n.
\]
For \( \alpha \neq (n^1) \), we can use the above computations to find (the sums run over all indices \( i \) for which \( \alpha_i \neq 0 \)):
\[
-\sum_i \frac{1}{(n-i)^5}\chi(V_\hat{\alpha}_i) = \sum_i \frac{1}{(n-i)^5} \frac{\alpha_i(n-i)^6}{n^6} \cdot (\sum_i \alpha_i - 1) \chi(V_\alpha)
= \frac{1}{n^5} \sum_i \alpha_i(n-i) \chi(V_\alpha)
= \frac{1}{n^5} \sum_i \alpha_i - \sum_i i\alpha_i \chi(V_\alpha)
= \frac{1}{n^5} \chi(V_\alpha).
\]
Lemma 2 is proved. As noted in Section 2.2, this completes the proof of Theorem 1.

REMARK 2.2. For an Abelian variety \( X \) of arbitrary dimension \( g \), Shen [8] observes that from an equality of formal power series in \( q \),
\[
\sum_{n \geq 0} P_{g-1}(n)q^n = \exp\left(\sum_{n \geq 1} s_n q^n\right),
\]
defining the sequence \( \{s_n\}_{n \geq 1} \), one obtains by application of the operator \( q \frac{d}{dq} \) the identity
\[
nP_{g-1}(n) = \sum_{k=1}^n ks_k P_{g-1}(n-k).
\]
Starting with this equality, our proofs of Lemmas 1 and 2 with \( \chi(V_\alpha)/n^5 \) replaced by \( \chi(V_\alpha)/n^{2g-1} \), go through without change, and we recover the identity (1).

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