GEOMETRIC MEAN FLOWS AND THE CARTAN BARYCENTER ON THE WASSERSTEIN SPACE OVER POSITIVE DEFINITE MATRICES

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ABSTRACT. We introduce a class of flows on the Wasserstein space of probability measures with finite first moment on the Cartan-Hadamard Riemannian manifold of positive definite matrices, and consider the problem of differentiability of the corresponding Cartan barycentric trajectory. As a consequence we have a version of Lie-Trotter formula and a related unitarily invariant norm inequality. Furthermore, a fixed point theorem related to the Karcher equation and the Cartan barycentric trajectory is also presented as an application.

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1. Introduction and main theorem

Let $\mathbb{P}_m$ be the set of $m \times m$ positive definite matrices, which is a smooth Riemannian manifold with the Riemannian trace metric $\langle X, Y \rangle_A = \text{tr} \ A^{-1}XA^{-1}Y$, where $A \in \mathbb{P}_m$ and $X, Y \in \mathbb{H}_m$, the Euclidean space of $m \times m$ Hermitian matrices equipped with the inner product $\langle X, Y \rangle = \text{tr} \ XY$. Then $\mathbb{P}_m$ is a Cartan-Hadamard Riemannian manifold, a simply connected complete Riemannian manifold with non-positive sectional curvature (the canonical 2-tensor is non-negative). The Riemannian distance between $A, B \in \mathbb{P}_m$ with respect to the above metric is given by $d(A, B) = \| \log A^{-1/2}BA^{-1/2} \|_2$, where $\| X \|_2 = (\text{tr} X^2)^{1/2}$ for $X \in \mathbb{H}_m$, and the unique (up to parametrization) geodesic joining $A$ and $B$ is given as the curve of weighted geometric means

$$ t \in [0, 1] \mapsto A \#_t B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}. \quad (1.1) $$
Let $\mathcal{P}(\mathbb{P}_m)$ denote the set of all probability measures on the Borel sets of $\mathbb{P}_m$, and $\mathcal{P}^1(\mathbb{P}_m)$ be the set of $\mu \in \mathcal{P}(\mathbb{P}_m)$ with finite first moment, i.e., for some (equivalently for all) $Y \in \mathbb{P}_m$, $\int_{\mathbb{P}_m} d(X,Y) \, d\mu(X) < \infty$. For $\mu \in \mathcal{P}^1(\mathbb{P}_m)$, the Cartan barycenter $G(\mu) \in \mathbb{P}_m$ is uniquely defined as

$$G(\mu) = \arg\min_{Z \in \mathbb{P}_m} \int_{\mathbb{P}_m} \left[ d^2(Z,X) - d^2(Y,X) \right] \, d\mu(X)$$

independently of the choice of a fixed $Y \in \mathbb{P}_m$ (see [21]). For every $\mu \in \mathcal{P}^1(\mathbb{P}_m)$, $X = G(\mu)$ is characterized by the Karcher equation

$$\int_{\mathbb{P}_m} \log X^{-1/2} A X^{-1/2} \, d\mu(A) = 0,$$

which is equivalent to the gradient zero equation for the function $Z \mapsto \int_{\mathbb{P}_m} \left[ d^2(Z,X) - d^2(Y,X) \right] \, d\mu(X)$ on $\mathbb{P}_m$. See [9, Theorem 3.1].

When $A_1, \ldots, A_n \in \mathbb{P}_m$ and $w = (w_1, \ldots, w_n)$ is a weight vector (i.e., $w_j \geq 0$, $\sum_{j=1}^n w_j = 1$), we denote by $G_w(A_1, \ldots, A_n)$ the Cartan mean $G(\mu)$ of a finitely supported measure $\mu = \sum_{j=1}^n w_j \delta_{A_j}$, where $\delta_A$ is the point measure of mass 1 (or the Dirac mass) at $A \in \mathbb{P}_m$. In particular, $G_w(A,B)$ with $w = (1-t,t)$ for $0 \leq t \leq 1$ coincides with the weighted geometric mean in (1.1). For $n > 2$ we have no such formula, and properties of $G_w(A_1, \ldots, A_n)$ have to be established by indirect arguments. The multivariate mean $G_w(A_1, \ldots, A_n)$ has been the subject of intensive study in the past ten years, e.g., [20, 4, 14, 5, 18, 23].

We now introduce a class of flows induced by the weighted geometric mean map on the probability measure space $\mathcal{P}^1(\mathbb{P}_m)$.

**Definition 1.1.** For $X \in \mathbb{P}_m$, $\mu \in \mathcal{P}^1(\mathbb{P}_m)$ and $t \in \mathbb{R}$, define $X \#_t \mu \in \mathcal{P}^1(\mathbb{P}_m)$ by

$$X \#_t \mu := (f_t)_* \mu,$$

i.e., the push-forward of $\mu$ by the homeomorphic map $f_t : \mathbb{P}_m \to \mathbb{P}_m$ defined by $f_t(A) := X \#_t A$, where we use the notation $\#_t$ given in (1.1) without restricting to $0 \leq t \leq 1$ (indeed, the expression in (1.1) is meaningful for all $t \in \mathbb{R}$). We also define the Cartan barycentric trajectory of (1.3) by

$$\beta(t) = \beta_X^\mu(t) := G(X \#_t \mu), \quad t \in \mathbb{R}. \quad (1.4)$$
The one-parameter family $X \# t \mu$ provides a flow on $P^1(P_m)$ and is also considered as a $P_m$-valued Markov process (see Theorem 2.4 and Remark 2.5 for more details).

The main result of the paper is the following:

**Theorem 1.2.** Let $X \in P_m$ and $\mu \in P^1(P_m)$. Then the map $\beta : \mathbb{R} \to P_m$ defined by

$$
\beta'(0) = X^{1/2} \left( \int_{P_m} \log X^{-1/2} AX^{-1/2} \, d\mu(A) \right) X^{1/2}.
$$

The proof of the theorem will be presented in Section 3. The theorem has an important consequence on the Lie-Trotter formula for the Cartan barycenter, as shown in the rest of this introductory section.

For general square matrices $X$ and $Y$, the well-known **Lie-Trotter formula** expresses

$$
\lim_{n \to \infty} (e^{X/n} e^{Y/n})^n = e^{X+Y}.
$$

The symmetric form with a continuous parameter is also well-known as

$$
\lim_{t \to 0} (A^{t/2} B^t A^{t/2})^{1/t} = \exp(\log A + \log B)
$$

for $A, B \in P_m$. This formula has also been known in many other situations; for example, see [10, 8, 7, 1, 3] for $A \#_\alpha B$ and other means. The Lie-Trotter formula for the Cartan mean $G_w(A_1, \ldots, A_n)$ of a finite number of $A_j \in P_m$ (or a finitely supported measure $\mu = \sum_{j=1}^n w_j \delta_{A_j}$) is

$$
\lim_{t \to 0} G_w(A_1^t, \ldots, A_n^t)^{1/t} = \exp \left( \sum_{j=1}^n w_j \log A_j \right),
$$

as given in [6, 11]. In [9], the authors have extended this Lie-Trotter formula for a certain sub-class of $P^1(P_m)$ in such a way that

$$
\lim_{t \to 0} G(\mu^t)^{1/t} = \exp \int_{P_m} \log A \, d\mu(A)
$$

for any $\mu \in P(P_m)$ satisfying $\int_{P_m} (\|A\| + \|A^{-1}\|)^r \, d\mu(A) < \infty$ for some $r > 0$. Here, $\|A\|$ denotes the operator norm of $A$, while any two norms on $\mathbb{H}_m$ are equivalent due to finite dimensionality.
The action of $t$-th power $\mu^t$ on $\mathcal{P}(\mathbb{P}_m)$ is defined by the push-forward measure of $\mu$ by the matrix $t$-th power $A \mapsto A^t$ on $\mathbb{P}_m$, that is,

$$\mu^t(\mathcal{O}) = \mu\left(\{A^t : A \in \mathcal{O}\}\right)$$

(1.8)

for any Borel set $\mathcal{O} \subset \mathbb{P}_m$, which is indeed comparable to the case in (1.6) since $\mu^t = \sum_{j=1}^n w_j \delta_{A_j^t}$ for $\mu = \sum_{j=1}^n w_j \delta_{A_j}$. When $X$ is the identity matrix $I = I_m$, we have $\beta(t) = G(\mu^t)$ with $\beta(0) = I$. Theorem 1.2 implies that $\beta(t) = I + t\beta'(0) + o(t)$ so that

$$\frac{1}{t} \log \beta(t) = \beta'(0) + \frac{o(t)}{t} \longrightarrow \beta'(0) \text{ as } t \to 0.$$

Therefore,

$$\lim_{t \to 0} G(\mu^t)^{1/t} = \lim_{t \to 0} \beta(t)^{1/t} = \exp \beta'(0) = \exp \int_{\mathbb{P}_m} \log A \, d\mu(A).$$

This provides the following extension of the above Lie-Trotter formula to the most general case of $\mu \in \mathcal{P}^1(\mathbb{P}_m)$.

**Corollary 1.3.** The formula (1.7) holds true for every $\mu \in \mathcal{P}^1(\mathbb{P}_m)$.

It turns out [9, Corollary 4.5] that $|||G(\mu^t)^{1/t}|||$ is increasing as $t \downarrow 0$ for any unitarily invariant norm $||| \cdot |||$. As a byproduct of Corollary 1.3 we have:

**Corollary 1.4.** Let $\mu \in \mathcal{P}^1(\mathbb{P}_m)$. Then for every unitarily invariant norm $||| \cdot |||$ and for every $t > 0$,

$$|||G(\mu^{-t})^{1/t}||| = |||G(\mu^t)^{1/t}||| \leq \left|\left|\exp \int_{\mathbb{P}_m} \log X \, d\mu(X)\right|\right|,$$

and $|||G(\mu^t)^{1/t}|||$ increases to $|||\exp \int_{\mathbb{P}_m} \log X \, d\mu(X)|||$ as $t \downarrow 0$.

2. **Geometric mean flows on the probability measure space**

Let $X \in \mathbb{P}_m$ and $\mu \in \mathcal{P}^1(\mathbb{P}_m)$. For every $t \in \mathbb{R}$, define $X \#_t \mu$ as in Definition 1.1 that is, $X \#_t \mu = (f_t)_* \mu$ is the push-forward of $\mu$ by $f_t : \mathbb{P}_m \to \mathbb{P}_m, f_t(A) = X \#_t A$.

**Lemma 2.1.** We have $X \#_t \mu \in \mathcal{P}^1(\mathbb{P}_m)$ for every $t \in \mathbb{R}$.

**Proof.** It is immediate to see that

$$\| \log A \| = \log \max\{\|A\|, \|A^{-1}\|\}, \quad A \in \mathbb{P}_m.$$ 

(2.1)
When $t > 0$, we have
\[
\|X^\#_t A\| \leq \|X\| \|X^{-1/2}AX^{-1/2}\|^t \leq \|X\| \|X^{-1}\|^t \|A\|^t,
\]
\[
\|(X^\#_t A)^{-1}\| = \|X^{-1/2}(X^{1/2}A^{-1}X^{1/2})^tX^{-1/2}\| \leq \|X\| \|X^{-1}\|^t \|A^{-1}\|^t.
\]
Therefore, by (2.1) we have
\[
\|\log(X^\#_t A)\| \leq (1 + t) \|\log X\| + t \|\log A\|, \quad A \in \mathbb{P}_m,
\]
which implies that $X^\#_t \mu \in \mathcal{P}^1(\mathbb{P}_m)$ since $d(X, I) = \|\log X\| \leq m \|\log X\|$ for all $X \in \mathbb{P}_m$ and
\[
\int_{\mathbb{P}_m} \|\log A\| d(X^\#_t \mu)(A) = \int_{\mathbb{P}_m} \|\log(X^\#_t A)\| d\mu(A) < \infty.
\]
When $t < 0$, the argument is similar since $X^\#_t A = X^{1/2}(X^{1/2}A^{-1}X^{1/2})^{-t}X^{1/2}$. □

Note that $I^\#_t \mu = \mu^t$, where $\mu^t$ is defined in (1.8), $X^\#_0 \mu = \delta_X$ and $X^\#_1 \mu = \mu$. When $t \neq 0$, since $X^\#_t Z = A$ if and only if $Z = X^\#_{1/t} A$, i.e., $f_{-1}^t = f_{1/t}$, we see that
\[
(X^\#_t \mu)(O) = \mu(\{X^\#_{1/t} A : A \in O\})
\]
for any Borel set $O \subset \mathbb{P}_m$. Moreover, note that if $\mu = \frac{1}{n} \sum_{j=1}^n \delta_{A_j}$, then $X^\#_t \mu = \frac{1}{n} \sum_{j=1}^n X^\#_{1/t} A_j$.

The 1-Wasserstein distance $d_1^W$ on $\mathcal{P}^1(\mathbb{P}_m)$ is defined by
\[
d_1^W(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{P}_m \times \mathbb{P}_m} d(X, Y) d\pi(X, Y), \quad \mu, \nu \in \mathcal{P}^1(\mathbb{P}_m),
\]
where $\Pi(\mu, \nu)$ is the set of all couplings for $\mu, \nu$, i.e., $\pi \in \mathcal{P}(\mathbb{P}_m \times \mathbb{P}_m)$ whose marginals are $\mu$ and $\nu$. Recall (see [21]) that $\mathcal{P}^1(\mathbb{P}_m)$ is a complete metric space with the metric $d_1^W$ and that the set $\mathcal{P}_0(\mathbb{P}_m)$ of uniform probability measures with finite support (i.e., the measures of the form $\frac{1}{n} \sum_{j=1}^n \delta_{A_j}$) is dense in $\mathcal{P}^1(\mathbb{P}_m)$. An important fact called the fundamental contraction property in [21] (also [9, Theorem 2.3]) is that the Cartan barycenter $G : \mathcal{P}^1(\mathbb{P}_m) \to \mathbb{P}_m$ is a Lipschitz map with Lipschitz constant 1; namely, for every $\mu, \nu \in \mathcal{P}^1(\mathbb{P}_m)$,
\[
d(G(\mu), G(\nu)) \leq d_1^W(\mu, \nu). \quad (2.2)
\]

The next lemma will play a role, which was given in [17, Lemma 2.2] in a more general setting.
Lemma 2.2. Let $f : \mathbb{P}_m \to \mathbb{P}_m$ be a Lipschitz map with Lipschitz constant $C$. Then the push-forward map $f_* : \mathcal{P}^1(\mathbb{P}_m) \to \mathcal{P}^1(\mathbb{P}_m)$, $\mu \mapsto f_* \mu$, is Lipschitzian with respect to $d^W_1$ with Lipschitz constant $C$.

Lemma 2.3. For every $\mu, \nu \in \mathcal{P}^1(\mathbb{P}_m)$ and $t, s \in [0, 1]$,
\[
d^W_1(X \#_t \mu, Y \#_s \nu) \leq (1 - t)d(X, Y) + td^W_1(\mu, \nu) + |t - s|d^W_1(\delta_Y, \nu).
\]

Proof. It is known (see [2]) that
\[
d(A \#_t B, C \#_t D) \leq (1 - t)d(A, C) + td(B, D), \tag{2.3}
\]
\[
d(A \#_t B, A \#_s B) = |t - s|d(A, B), \quad t, s \in [0, 1].
\]
By the triangular inequality, for every $t, s \in [0, 1]$,
\[
d(A \#_t B, C \#_s D) \leq d(A \#_t B, C \#_t D) + d(C \#_t D, C \#_s D)
\]
\[
\leq (1 - t)d(A, C) + td(B, D) + |t - s|d(C, D).
\]
For $\mu = \frac{1}{n} \sum_{j=1}^n \delta_{A_j}$, $\nu = \frac{1}{n} \sum_{j=1}^n \delta_{B_j}$ in $\mathcal{P}_0(\mathbb{P}_m)$, it is known (see Introduction of [22]) that
\[
d^W_1(\mu, \nu) = \min_{\sigma \in S_n} \frac{1}{n} \sum_{j=1}^n d(A_j, B_{\sigma(j)}),
\]
where $S_n$ is the permutation group on $\{1, \ldots, n\}$. Therefore, for every $t, s \in [0, 1]$ we find a $\sigma \in S_n$ so that
\[
d^W_1(X \#_t \mu, Y \#_s \nu) = \frac{1}{n} \sum_{j=1}^n d(X \#_t A_j, Y \#_s B_{\sigma(j)})
\]
\[
\leq \frac{1}{n} \sum_{j=1}^n [(1 - t)d(X, Y) + td(A_j, B_{\sigma(j)}) + |t - s|d(Y, B_{\sigma(j)})]
\]
\[
\leq (1 - t)d(X, Y) + td^W_1(\mu, \nu) + |t - s|d^W_1(\delta_Y, \nu).
\]
Hence the required inequality holds for all $\mu, \nu \in \mathcal{P}_0(\mathbb{P}_m)$. Since $d(X \#_t A, X \#_t B) \leq td(A, B)$ by (2.3), we see by Lemma 2.2 that $\mu \mapsto X \#_t \mu$ is Lipschitzian with Lipschitz constant $t$. Since $\mathcal{P}_0(\mathbb{P}_m)$ is dense in $\mathcal{P}^1(\mathbb{P}_m)$, the result follows. \hfill $\Box$

Theorem 2.4. For each $X \in \mathbb{P}_m$, the map $\Phi_X : \mathbb{R} \times \mathcal{P}^1(\mathbb{P}_m) \to \mathcal{P}^1(\mathbb{P}_m)$ defined by
\[
\Phi_X(t, \mu) = X \#_t \mu
\]
is a continuous flow satisfying
\[ \Phi_X(ts, \mu) = \Phi_X(s, \Phi_X(t, \mu)), \quad t, s \in \mathbb{R}. \] (2.4)

Moreover, for a fixed \( \mu \in \mathcal{P}^1(\mathbb{P}_m) \), the map \( t \in \mathbb{R} \mapsto X#t\mu \in \mathcal{P}^1(\mathbb{P}_m) \) is locally Lipschitz continuous with respect to \( d^W_1 \), that is, for every \( T > 0 \) there exists a constant \( C_T > 0 \) such that
\[ d^W_1(X#t\mu, X#s\mu) \leq C_T|t - s|, \quad t, s \in [-T, T]. \]

**Proof.** It is immediate to see that \( X#s(X#tA) = X#stA \) for every \( t, s \in \mathbb{R} \), which yields
\[ X#st\mu = X#s(X#t\mu), \quad t, s \in \mathbb{R}. \] (2.5)
This is nothing but (2.4). Continuity follows from Lemma 2.3.

Let \( \mu \in \mathcal{P}^1(\mathbb{P}_m) \) be fixed. Lemma 2.3 shows in particular that \( d^W_1(X#t\mu, X#s\mu) \leq C_1|t - s| \) for every \( t, s \in [0, 1] \) with \( C_1 := d^W_1(\delta_X, \mu) \). When \( t, s \in [0, 1] \), since \( X#-tA = X(X#tA)^{-1}X \) and \( X#-sA = X(X#sA)^{-1}X \), we have
\[ d(X#-tA, X#-sB) = d(X#tA, X#sB), \quad A, B \in \mathbb{P}_m, \]
which immediately gives
\[ d^W_1(X#-t\mu, X#-s\mu) = d^W_1(X#t\mu, X#s\mu) \leq C_1|t - s|. \]
Moreover,
\[ d^W_1(X#t\mu, X#-s\mu) \leq d^W_1(X#t\mu, \delta_X) + d^W_1(\delta_X, X#-s\mu) \leq C_1t + C_1s = C_1|t - (-s)|. \]
Hence the result holds for \( T = 1 \).

For any \( T > 0 \) and \( t, s \in [-T, T] \) write \( t = t'T \) and \( s = s'T \) with \( t', s' \in [-1, 1] \). Then by (2.5) we can write \( X#t\mu = X#t'\mu' \) and \( X#s\mu = X#s'\mu' \) with \( \mu' := X#T\mu \). By the above case with \( \mu' \) in place of \( \mu \) we have
\[ d^W_1(X#t\mu, X#s\mu) \leq C_1'|t' - s'| = \frac{C_1'}{T}|t - s| \]
for some constant \( C_1' \). Hence the result follows with \( C_T := C_1'/T \). \( \square \)
Remark 2.5. Theorem 2.4 says that $\Phi_X(\mu, t) = X\#_t\mu$ ($t \in \mathbb{R}$) is a multiplicative $\mathbb{R}$-flow on $\mathcal{P}^1(\mathbb{P}_m)$. Modifying as $\Psi_X(\mu, t) := X\#e^{-t}\mu$ ($t \geq 0$), we have an additive $\mathbb{R}_+$-flow on $\mathcal{P}^1(\mathbb{P}_m)$ starting at $\mu$ ($t = 0$) and attracted to $\delta_X$ (as $t \to \infty$). This flow is also considered as a $\mathbb{P}_m$-valued Markov stochastic process $X_t(A) := X\#_t A$ (with smooth sample paths) on the probability space $(\mathbb{P}_m, \mu)$.

3. Proof of Theorem 1.2

In the following we fix $X \in \mathbb{P}_m$ and $\mu \in \mathcal{P}^1(\mathbb{P}_m)$. For notational simplicity we write $X_t \in \mathbb{P}_n$ for $\beta(t) = G(X\#_t\mu)$ (with $X_0 = X$), which is uniquely characterized by the Karcher equation (see (1.2))

$$
\int_{\mathbb{P}_n} \log X_t^{-1/2} AX_t^{-1/2} d(X\#_t\mu)(A) = 0,
$$

that is,

$$
\int_{\mathbb{P}_n} \log X_t^{-1/2}(X\#_t A)X_t^{-1/2} d\mu(A) = 0. \quad (3.1)
$$

We set $\mu_X := (g_X)_*\mu$, where $g_X : \mathbb{P}_m \to \mathbb{P}_m$ is defined by $g_X(A) := X^{-1/2}AX^{-1/2}$. (Note that $\mu_X$ is $M.\mu$ with $M = X^{-1/2}$ in the notation in [13].) Moreover, let

$$
\mu_X^t := (\mu_X)^t,
$$

where the action of $t$-th power $\mu^t$ on $\mathcal{P}(\mathbb{P}_m)$ is defined by the push-forward measure of $\mu$ by the matrix $t$-th power $A \mapsto A^t$ on $\mathbb{P}_m$, that is, $\mu^t = I\#_t\mu$.

Lemma 3.1. For $X \in \mathbb{P}_m$, $\mu \in \mathcal{P}^1(\mathbb{P}_m)$ and $t \in \mathbb{R}$. Then $X\#_t\mu = (\mu_X^t)_{X^{-1}}$ and

$$
\beta(t) = X^{1/2}G(\mu_X^t)X^{1/2}, \quad t \in \mathbb{R}. \quad (3.2)
$$

Proof. Since $f_t(A) = X^{1/2}(X^{-1/2}AX^{-1/2})^tX^{1/2}$, we have $f_t = g_{X^{-1}} \circ h_t \circ g_X$, where $h_t(A) := A^t$. Therefore,

$$
X\#_t\mu = (g_{X^{-1}} \circ h_t \circ g_X)_*\mu
= (g_{X^{-1}} \circ h_t)_*\mu_X
= (g_{X^{-1}})_*\mu_X^t = (\mu_X^t)_{X^{-1}}.
$$

Now, we recall (see [13]) that the Cartan barycenter has the invariance property $G((g_X)_*\mu) = g_X(G(\mu))$, i.e., $G(\mu_X) = X^{-1/2}G(\mu)X^{-1/2}$. Hence (3.2) follows. \qed
To prove Theorem 1.2, we may and do assume that $X = I$ from (3.2). In this case, $X_t = G(I#_t\mu) = G(\mu_t^0)$ (with $X_0 = I$) and (1.3) is simply $\beta(0) = \int_{\mathbb{P}_m} \log X \, d\mu(X)$.

**Lemma 3.2.** For any $T > 0$ there exists a constant $K_T > 0$ such that for every $\alpha \in [-1, 1]$ and every $t, s \in [-T, T]$, 

$$\|X_t^\alpha - X_s^\alpha\| \leq K_T |s - t|.$$

**Proof.** For any $T > 0$, by Lemma 2.3 we have

$$d_W^1(I#_t\mu, I#_s\mu) \leq C_T |t - s|, \quad t, s \in [-T, T].$$

Applying this to the fundamental contraction property (2.2) and using the exponential metric increasing property (EMI) (see [2, Theorem 6.1.4])

$$\|\log A - \log B\|_2 \leq d(A, B), \quad A, B \in \mathbb{P}_m,$$

we have

$$\|\log X_t - \log X_s\| \leq C_T |t - s|, \quad T, S \in [-T, T].$$

In particular, $\|\log X_t\| \leq C_T T$ for all $t \in [-T, T]$. For any $\alpha \in [-1, 1]$ and $t, s \in [-T, T]$ we find that

$$\|X_t^\alpha - X_s^\alpha\| = \|\exp(\alpha \log X_t) - \exp(\alpha \log X_s)\|$$

$$\leq \sum_{k=1}^{\infty} \| (\alpha \log X_t)^k - (\alpha \log X_s)^k \| \frac{1}{k!}$$

$$\leq \sum_{k=1}^{\infty} \| (\log X_t)^k - (\log X_s)^k \| \frac{1}{k!}$$

$$\leq \sum_{k=1}^{\infty} kC_T(C_T T)^{k-1} |t - s| = K_T |t - s|,$$

where $K_T := C_T e^{C_T T}$. \qed

**Lemma 3.3.** There exists a constant $C > 0$ such that

$$\frac{1}{|t|} \| \log X_t^{-1/2} A^t X_t^{-1/2} \| \leq C + \| \log A \|$$

for every $t \in [-1, 1] \setminus \{0\}$ and every $A \in \mathbb{P}_n$. 

Proof. From $X_{-t} = G(\mu^{-t}) = G((\mu^{-1})^t)$, we may assume that $t \in (0, 1]$. Since
\[ \|X_t^{-1/2}A'X_t^{-1/2}\| \leq \|X_t^{-1}\|\|A\|, \quad \|X_t^{1/2}A^{-1}X_t^{1/2}\| \leq \|X_t\|\|A^{-1}\|, \]
we have
\[ \frac{1}{t} \log \|X_t^{-1/2}A'X_t^{-1/2}\| \leq \frac{1}{t} \log \|X_t^{-1}\| + \log \|A\|, \]
\[ \frac{1}{t} \log \|X_t^{1/2}A^{-1}X_t^{1/2}\| \leq \frac{1}{t} \log \|X_t\| + \log \|A^{-1}\|. \]
Let $M_t := X_t - I$ and $M'_t := X_t^{-1} - I$. Then we have
\[ \frac{1}{t} \log \|X_t^{-1}\| = \frac{1}{t} \log \|I - X_t^{-1/2}M_tX_t^{-1/2}\| \]
\[ \leq \frac{1}{t} \log (1 + \|X_t^{-1}\|\|M_t\|) \leq \|X_t^{-1}\|\frac{\|M_t\|}{t}, \]
\[ \frac{1}{t} \log \|X_t\| = \frac{1}{t} \log \|I - X_t^{1/2}M'_tX_t^{1/2}\| \]
\[ \leq \frac{1}{t} \log (1 + \|X_t\|\|M'_t\|) \leq \|X_t\|\frac{\|M'_t\|}{t}. \]
Note here that $\|X_t\|$, $\|X_t^{-1}\|$, $\|M_t\|/t$ and $\|M'_t\|/t$ are all uniformly bounded for $t \in (0, 1]$ by Lemma 3.2. Combining the above estimates together with (2.1), we find a constant $C > 0$ such that
\[ \frac{1}{t} \| \log X_t^{-1/2}A'X_t^{-1/2} \| \leq C + \| \log A \|, \quad t \in (0, 1]. \]

Proof of Theorem 1.2. For $t \in [-1, 1] \setminus \{0\}$ let $H_t := X_t^{-1/2} - I$. We will prove that $H_t/t$ converges as $t \to 0$. Since $\|H_t/t\|$ is bounded by Lemma 3.2, we may prove that a limit point of $H_t/t$ as $t \to 0$ is unique. Note that for each $A \in \mathbb{P}_n$
\[ X_t^{-1/2}A'X_t^{-1/2} = (I + H_t)(I + t \log A + o(t))(I + H_t) \]
\[ = I + 2H_t + t \log A + o(t). \]
Now, assume that $H_{t_k}/t_k \to L$ for a sequence $t_k \in [-1, 1] \setminus \{0\}$ with $t_k \to 0$, so that
\[ \frac{1}{t_k} \log X_{t_k}^{-1/2}A^kX_{t_k}^{-1/2} = 2\frac{H_{t_k}}{t_k} + \log A + \frac{o(t_k)}{t_k} \to 2L + \log A. \]
as \( k \to \infty \). By (3.1) we have

\[
\int_{\mathcal{P}_n} \frac{1}{t_k} \log X_t^{-1/2} A^t X_t^{-1/2} \, d\mu(A) = 0. \tag{3.3}
\]

Thanks to Lemma 3.3, the Lebesgue convergence theorem can be applied to (3.3) so that we obtain

\[
2L = -\int_{\mathcal{P}_n} \log A \, d\mu(A).
\]

Therefore, \( L \) is a unique limit point of \( H_t/t \) as \( t \to 0 \). This means that \( t \mapsto Y_t := X_t^{-1/2} \) is differentiable at \( t = 0 \) with the derivative \( L \). Since \( X_t = Y_t^{-2} \), we find that \( \beta(t) = X_t \) is differentiable at \( t = 0 \) and

\[
\beta'(0) = -2L = \int_{\mathcal{P}_n} \log A \, d\mu(A),
\]

which is the desired conclusion (as we assumed that \( X = I \)). \( \square \)

**Theorem 3.4.** Let \( X \in \mathcal{P}_m, \mu \in \mathcal{P}^1(\mathcal{P}_m) \) and let \( \beta(t) = G(X \# t\mu) \) be as in Theorem 1.2. Then the following are equivalent:

(i) \( \beta'(0) = 0 \);

(ii) \( X = G(\mu) \);

(iii) \( X = G(X \# t\mu) \) for all \( t \in \mathbb{R} \) (equivalently for some \( t \neq 0 \));

(iv) \( I = G(\mu_X) \) for all \( t \in \mathbb{R} \) (equivalently for some \( t \neq 0 \)).

**Proof.** For every \( \mu \in \mathcal{P}^1(\mathcal{P}_m) \), the Karcher equation (1.2) is equivalent to \( \beta'(0) = 0 \) thanks to (1.5). Hence we have (i) \( \iff \) (ii). Moreover, we note that

\[
\int_{\mathcal{P}_m} \log X^{-1/2} AX^{-1/2} \, d(X \# t\mu)(A) = \int_{\mathcal{P}_m} \log X^{-1/2}(X \# tA)X^{-1/2} \, d\mu(A)
\]

\[
= t \int_{\mathcal{P}_m} \log X^{-1/2} AX^{-1/2} \, d\mu(A)
\]

\[
= t \int_{\mathcal{P}_m} \log A \, d\mu_X(A)
\]

\[
= \int_{\mathcal{M}_m} \log A \, d\mu_X^t(A).
\]

Therefore, it immediately follows that (ii)–(iv) are equivalent. \( \square \)

**Corollary 3.5.** Let \( \mu \in \mathcal{P}^1(\mathcal{P}_m) \) and let \( \beta(t) := G(\mu \# t\mu) \). Then \( \beta \) is differentiable at \( t = 0 \) with \( \beta'(0) = 0 \).
When \( \mu \in \mathcal{P}^2(\mathbb{P}_m) \), i.e., \( \mu \) has finite second moment, the equivalence of (ii) and (iii) of Theorem 3.4 was shown in [13, Theorem 3.1]. The Karcher equation or equivalently \( \beta'(0) = 0 \) has played a crucial role in the Riemannian geometric approach of multivariate geometric means as in [20, 18, 16], which has been extended to the Cartan barycenter in [12, 13, 9]. For a finitely supported measure \( \mu = \frac{1}{n} \sum_{j=1}^{n} w_j \delta_{A_j} \), the fixed point Cartan mean equation \( X = G(X \#_t \mu) = G_{w}(X \#_t A_1, \ldots, X \#_t A_n) \) appeared in [18] and [16]. The formula (1.5) is evidently new and deserves to receive its attention due to its relation to the Karcher equation.

4. Final remarks and open problems

(1) In the present paper, we first prove the differentiability of the Cartan barycentric trajectory \( \beta(t) \) at \( t = 0 \) and then use it to prove the Lie-Trotter formula for \( \lim_{t \to 0} G(\mu t)^{1/t} \). One can also proceed in the opposite way. Indeed, we have a direct proof of the Lie-Trotter formula in Corollary 1.3 which in turn shows Theorem 1.2 immediately. It is worth noting that the Lebesgue convergence theorem is essential in our direct proof of (1.7) for \( \mu \in \mathcal{P}^1(\mathbb{P}_m) \), as it is so in the proof of Theorem 1.2 in Section 3.

(2) We are also interested in the extension of Theorem 1.2 to any \( t \in \mathbb{R} \), that is, in the differentiability problem of \( \beta(t) \) and, in this case, in what is the form of derivative \( \beta'(t) \). It does not seem possible to generalize the above proof for \( \beta'(0) \) to the case for \( \beta'(t) \) at \( t \neq 0 \). But, under a stronger assumption that \( \int_{\mathbb{P}_n} (\|A\| + \|A^{-1}\|)^{2\alpha} d\mu(A) < \infty \) with some \( \alpha > 0 \), we can prove the differentiability of \( \beta(t) \) for \( t \in [-\alpha, \alpha] \), though the expression of \( \beta'(t) \) is much complicated.

(3) Given \( \mu \in \mathcal{P}^1(\mathbb{P}_m) \), it is well-known (see [15]) that the (Euclidean) gradient of the function \( \psi(X) := \frac{1}{2} \int_{\mathbb{P}_m} [d^2(X, A) - d^2(Y, A)] d\mu(A) \) at \( X \in \mathbb{P}_m \) is

\[
\nabla \psi(X) = X^{-1/2} \left( \int_{\mathbb{P}_n} \log X^{1/2} A^{-1/2} \right) X^{-1/2},
\]

and the Riemannian gradient of \( \psi \) at \( X \) is \( \nabla_{\text{Riem}} \psi(X) = X \nabla \psi(X) X \). Hence the Riemannian gradient flow on \( \mathbb{P}_m \) is introduced as the solution of the Cauchy problem

\[
\frac{dX_t}{dt} = -\nabla_{\text{Riem}} \psi(X_t) = X_t^{1/2} \left( \int_{\mathbb{P}_m} \log X_t^{-1/2} A X_t^{-1/2} d\mu(A) \right) X_t^{1/2}.
\]
with initial value $X_0 = X \in \mathbb{P}_m$. In [19], Lim and Pálfia have discussed this gradient flow (called an ODE flow there) and obtained its description by using the resolvent operator defined by

$$J^\mu_\lambda(X) := G\left(\frac{\lambda}{\lambda + 1} \mu + \frac{1}{\lambda + 1} \delta_X\right)$$

for $\lambda \geq 0$ and $X \in \mathbb{P}_m$. Note that $J^\mu_\lambda(X)$ is the Cartan barycentric trajectory of the arithmetic mean flow $\lambda \geq 0 \mapsto (\lambda \mu + \delta_X)/(\lambda + 1)$ on $\mathcal{P}^1(\mathbb{P}_m)$. When $t = \lambda/(\lambda + 1)$, from the arithmetic-geometric mean inequality $X_A \leq (X + \lambda A)/(\lambda + 1)$, we can see that $X_A \leq (\lambda \mu + \delta_X)/(\lambda + 1)$ in the partial order on $\mathcal{P}(\mathbb{P}_m)$ considered in [13] [9].

By the monotonicity property of the Cartan barycenter (see [9, Theorem 3.2]) we have $\beta^\mu_\lambda(t) \leq J^\mu_\lambda(X)$ for $t = $ $\lambda/(\lambda + 1)$. It might be interesting to find more relations of the trajectory $\beta(t) = \beta^\mu_\lambda(t)$ with $J^\mu_\lambda(X)$ and the gradient flow.

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