MINIMAL FOURIER MAJORANTS IN $L^p$

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Abstract. Denote the coefficients in the complex form of the Fourier series of a function $f$ on the interval $[-\pi, \pi)$ by $\hat{f}(n)$. It is known that if $p = 2j/(2j - 1)$ for some integer $j > 0$, then for each function $f$ in $L^p$ there exists another function $F$ in $L^p$ that majorizes $f$ in the sense that $|\hat{F}(n)| \geq |\hat{f}(n)|$ for all $n$, and for which $\|F\|_p \leq \|f\|_p$. When $j > 1$, the existence proofs for such small majorants do not provide constructions of them, but there is a unique majorant of minimal $L^p$ norm. We modify previous existence proofs to say more about the form of that majorant.

1. Introduction

Thus $\hat{f}(n) = (1/2\pi) \int_{-\pi}^{\pi} f(\theta)e^{-in\theta} \, d\theta$. Call $F$ a majorant of $f$, and $f$ a minorant of $F$, when $|\hat{f}(n)| \leq |\hat{F}(n)|$ for all integers $n$. In that case, $\|f\|_2 \leq \|F\|_2$; also, if $j$ is an integer greater than 1, then $F^j$ majorizes $f^j$, and hence

$$\left(\|f\|_2\right)^{2j} = \left(\|f^j\|_2\right)^2 \leq \left(\|F^j\|_2\right)^2 = \left(\|F\|_2\right)^{2j}.$$  

Finally, $\|f\|_\infty \leq \|F\|_\infty$ when $F$ majorizes $f$.

This pattern does not persist for other exponents. Hardy and Littlewood [6] considered the upper majorant property, asserting that there is a constant $U(p)$ for which

$$\|f\|_p \leq U(p)\|F\|_p$$  \hspace{1cm} (1.1)$$

whenever $F$ majorizes $f$. They gave an example showing that if this property holds in $L^3$ then the constant $U(3)$ must be strictly larger than 1. Later work [2, 4] revealed that the property fails completely for each exponent $p$ in the interval $(0, \infty)$ that is not even.
Here we consider the lower majorant property, also introduced in [6]. It holds when there is a constant \( L(p) \) so that each function \( f \) in \( L^p \) has a majorant \( F \) for which
\[
\|F\|_p \leq L(p)\|f\|_p.
\]
This is clearly true when \( p = 2 \) with \( L(2) = 1 \), letting \( F = |\hat{f}| \). It also holds when \( p = 1 \), since one can factor a given function \( f \) as \( f_1 f_2 \), where \( \|f_1\|_2 \|f_2\|_2 = \|f\|_1 \), form the exact majorants \( F_1 \) and \( F_2 \) of \( f_1 \) and \( f_2 \), and then take their product \( F = F_1 F_2 \), which majorizes \( f \), and for which
\[
\|F\|_1 \leq (\|F_1\|_2\|F_2\|_2 = (\|f_1\|_2\|f_2\|_2 = \|f\|_1).
\]
When \( p \in (1, \infty) \), a simple duality argument [2, Section 3] shows that if \( L^p \) has the lower majorant property, then its dual space \( L^{p'} \) has the upper majorant property, and \( U(p') \leq L(p) \). By the work cited above, this can only happen when \( p' \) is an even integer, thus ruling out all exponents \( p \) in the interval \((1, \infty)\) except for \( p = 2 \) and the special exponents for which \( p = 2j/(2j - 1) \), where \( j \) is an integer strictly greater than 1. When \( 1 < p < \infty \), a less simple duality argument [6] shows that the upper majorant property for \( L^{p'} \) implies the lower majorant property for \( L^p \), and that \( L(p) \leq U(p') \). So the latter property holds for the special exponents, and then \( L(p) = 1 \).

The duality proof in [6] and alternatives [11, 3, 10] to it do not specify a suitable majorant of a given function for these values of \( p \). For various good reasons, those arguments covered general exponents \( p \), but the duality mainly has impact for the special values of \( p \).

We exploit this hindsight by analysing those special cases and getting new information about majorants with minimal \( L^{2j/(2j-1)} \) norm. For an exponent \( p \) in the interval \((1, \infty)\), the majorants of a given function in \( L^p \) form a closed, convex subset. By the uniform convexity of the \( L^p \) norm, if that subset is nonempty, then it has a unique element of least norm.

**Theorem 1.1.** Let \( p = 2j/(2j - 1) \) for some integer \( j > 1 \), and let \( f \) be a function in \( L^p \). Then \( f \) has a majorant with no larger \( L^p \) norm. Its minimal majorant \( F \) in \( L^p \) factors as \((\hat{G})^{j-1} \hat{G}^j\), where \( \hat{G} \in L^{2j} \) and \( \hat{G} \geq 0 \). Moreover \( \hat{G} \) vanishes on the set where \( \hat{F} > |\hat{f}| \), and \( \hat{G} \) vanishes off the support of \( \hat{f} \).

**Corollary 1.2.** If \( S \) is the set of frequencies of \( f \), then the set of frequencies of the minimal majorant of \( f \) in \( L^{2j/(2j-1)} \) is included in the algebraic sum of \( j \) copies of \( S \) and \( j - 1 \) copies of \(-S \). If \( f \) is a trigonometric polynomial, so is its majorant of minimal norm in \( L^{2j/(2j-1)} \).
The corollary is new, as is the inclusion of the support of $\hat{G}$ in that of $\hat{f}$. Related conclusions appear in [4], which presents work that began later but ended earlier.

Parts of the following notion arose in earlier duality proofs.

**Definition 1.3.** Let $p = 2j/(2j - 1)$ for some integer $j > 1$. Given a function $f$ in $L^p$, call a function $H$ in $L^{2j}$ a $p$-conjugate of $f$ if it has the following properties:

1. $\hat{H} \geq 0$.
2. The product $J$ given by $(\overline{H})^{j-1}H^j$ majorizes $f$.
3. $\hat{H}(n) = 0$ at all integers $n$ where $\hat{J}(n) > |\hat{f}(n)|$.
4. $\hat{H} = 0$ off the support of $\hat{f}$.

It will be shown later that property (4) follows from the others, and that the only $p$-conjugate of such a function $f$ is the function $G$ arising in Theorem 1.1. Property (2) can be restated as the requirement that the discrete convolution of $j$ copies of the nonnegative function $\hat{H}$ with $j - 1$ copies of the reflected transform $\hat{H}$ mapping $m$ to $\hat{H}(-m)$, which is $\hat{H}(-m)$ here, should be no smaller than $|\hat{f}|$.

When $f$ is a trigonometric polynomial, finding a $p$-conjugate becomes a finite-dimensional nonlinear-programming problem. The objective function is the $2j$-th power of the $L^{2j}$ norm of $H$; this is also the square of the $\ell^2$ norm of the convolution of $j$ copies of $\hat{H}$. The goal is to minimize that homogeneous polynomial of degree $2j$ in the coefficients of $H$ subject to the constraints (1) and (2). The slackness condition (3) is then necessary, as is condition (4).

The duality argument in [3] led us to another description of $H$ or $G$ when $f$ is a trigonometric polynomial. Minimize the $2j$-th power of the $L^{2j}$ norm of a function $g$ subject to having nonnegative coefficients that vanish off the support of $\hat{f}$, and for which

$$(1.3) \quad \sum_n |\hat{f}(n)||\hat{g}(n)| = 1.$$ 

Then rescale $g$ suitably to get $G$.

We elaborate on this in the next section, where we study the related notion of partial majorant for all exponents in the interval $(1, \infty)$. Theorem 1.1 then follows easily in Section 3 via a certain estimate in the special cases where $p'$ is even. We explain in Section 4 how the restriction on the support of $\hat{G}$ also follows by earlier methods.

The classical method shows that if $H$ is a $p$-conjugate of $f$, then the $L^p$ norm of $H^j(\overline{H})^{j-1}$ is at most $\|f\|_p$. In Section 5 we extend
this slightly to prove the following statements when \( p = 2j/(2j - 1) \) for some integer \( j \) strictly greater than one.

**Theorem 1.4.** Given a function \( G \) in \( L^{2j} \) with nonnegative coefficients, let \( F = (\overline{G})^{j-1}G^j \). Let \( f \) be any function in \( L^p \) with the property that \( |\hat{f}| \geq \hat{F} \) on the support of \( \hat{G} \). Then \( \|f\|_p \geq \|F\|_p \).

**Corollary 1.5.** Let \( H \) be a \( p \)-conjugate of a function \( f \) in \( L^p \). Then the minimal majorant of \( f \) in \( L^p \) is \( (\overline{H})^{j-1}H^j \), and \( H \) is the only \( p \)-conjugate of \( f \).

**Corollary 1.6.** If \( H \in L^{2j} \) and \( \hat{H} \geq 0 \), then \( (\overline{H})^{j-1}H^j \) is its own minimal majorant in \( L^p \).

Comparing the supports of the coefficients of \( H \) and \( (\overline{H})^{j-1}H^j \) shows that many functions in \( L^p \) with nonnegative coefficients are not minimal majorants in \( L^p \) of anything. Similar reasoning leads to many cases where \( G \) is not a multiple of the exact majorant of \( f \) because \( \hat{G} \) vanishes on part of the support of \( \hat{f} \). Earlier numerical work had also led us to such examples.

The functions, \( f \), say, that belong to \( L^{2j/(2j-1)} \) are those that factor as \( (k)^{j-1}k^j \) where \( k \in L^{2j} \). It turns out that the quantities \( \|F\|_p \) and \( \|f\|_p \) in Theorem 1.4 are equal if and only if the exact majorant \( E_k \) of \( k \) also belongs to \( L^{2j} \), with the same norm in that space as \( k \). Moreover, \( F = (E_k)^{j-1}(E_k)^j \) in those cases. A standard thinness condition applied to supports of coefficients yields many such examples.

**Remark 1.7.** As \( p \to 1^+ \) through the set of special exponents, Corollary 1.2 imposes progressively weaker restrictions on the frequencies of the minimal majorants of a trigonometric polynomial \( f \).

**Remark 1.8.** Instead of imposing condition (1.3) on suitable functions \( g \), we could, as in [1], maximize the left-hand side of equation (1.3) over suitable functions \( g \) in a closed sphere in \( L^{2j} \).

**Remark 1.9.** One way to formulate the upper majorant property in \( L^{2j} \) is that if two functions \( f \) and \( g \) in \( L^1 \) have the property that \( |\hat{f}| \leq |\hat{g}| \), then \( \|f\|_{2j} \leq \|g\|_{2j} \) in the cases where \( \hat{g} \geq 0 \). Theorem 1.4 yields a dual version of this, namely that if \( |\hat{f}| \leq |\hat{g}| \), then \( \|f\|_{2j/(2j-1)} \leq \|g\|_{2j/(2j-1)} \) when \( f \) factors as \( (\overline{H})^{j-1}H^j \) where \( H \in L^{2j} \) and \( \hat{H} \geq 0 \).

**Remark 1.10.** Corollary 1.6 also follows easily from part (ii) of Lemma 2 in [1].
2. Partial majorants

In this section, we show that much of the pattern in Theorem 1.1 persists for other values of $p$ in the interval $(1, \infty)$ with a different notion of majorant. In the next section, we observe that the two notions coincide for minimal majorants when $p'$ is even, and we explain why a key estimate holds in those special cases.

Given a bounded sequence $c = (c_n)_{n=-\infty}^{\infty}$, regard it as giving the coefficients of some $2\pi$-periodic distribution, $\tilde{c}$ say, which may or may not belong to $L^p$.

Definition 2.1. An integrable function $F$ is a partial majorant of $\tilde{c}$ if $\hat{F}(n) \geq |c(n)|$ at all indices $n$ where $c(n) \neq 0$.

If $\hat{F} \geq 0$ off the support of $c$ too, call $F$ a full majorant of $\tilde{c}$. When $\tilde{c}$ is integrable, this coincides with the notion of $F$ being a majorant of $\tilde{c}$.

The functions in $L^p$ that partially majorize $\tilde{c}$ form a closed convex subset of $L^p$. If it is nonempty, and $p \in (1, \infty)$, then this subset has a unique element of minimal $L^p$ norm. As noted earlier, the same comments apply to the set of functions in $L^p$ that fully majorize $\tilde{c}$. Since that set is included in the set of partial majorants of $\tilde{c}$ in $L^p$, the minimal norm of full majorants cannot be smaller than the minimal norm of partial majorants.

Recall that complex function-valued functions $F$ factor as $|F| \operatorname{sgn}(F)$, where $\operatorname{sgn}(F)$ vanishes off the support of $F$. Also, when $F \in L^p$, where $1 < p < \infty$, letting

\[
G = |F|^{p/p'} \operatorname{sgn}(F) = |F|^{p-1} \operatorname{sgn}(F)
\]

puts $G$ in $L^{p'}$, and then

\[
F = |G|^{p'/p} \operatorname{sgn}(G) = |G|^{p'-1} \operatorname{sgn}(G).
\]

Definition 2.2. Given a nontrivial bounded sequence $c$, let $R(c)$ be the set of trigonometric polynomials $g$ with nonnegative coefficients that vanish off the support of $c$, and for which

\[
\sum_n |c(n)||\hat{g}(n)| = 1.
\]

Given an exponent $p$ in the interval $(1, \infty)$, let

\[
K_p(c) = \inf \{\|g\|_{p'} : g \in R(c)\}.
\]

Lemma 2.3. Let $1 < p < \infty$. A nontrivial distribution $\tilde{c}$ has a partial majorant in $L^p$ if and only if $K_p(c) > 0$; the minimal $L^p$ norm of partial majorants of $\tilde{c}$ is then equal to $1/K_p(c)$. The partial majorant of minimal $L^p$ norm is a rescaled copy of $|h|^{p'-1} \operatorname{sgn}(h)$ for the function $h$.
of minimal $L^{p'}$ norm in the closure of the set $R(c)$ in $L^p$. Finally, $\hat{h}$ vanishes off the support of $c$, and on the part of that support where the transform of the minimal majorant is strictly larger than $|c|$.

**Corollary 2.4.** Let $1 < p < \infty$. When $\tilde{c}$ has a partial majorant in $L^p$, let $F$ be its minimal partial majorant in $L^p$, and let $G = |F|^{p-1} \sgn(F)$. Then $G \geq 0$, and $G$ vanishes off the support of $c$. Moreover, $\hat{G}$ also vanishes on the part of that support where $\hat{F} > |c|$.

**Proofs.** Suppose that $\tilde{c}$ has a partial majorant, $E$ say, in $L^p$. For functions $g$ in $R(c)$, only finitely-many of the terms in the sum $\sum |\hat{g}(n)|c(n)|$ are nontrivial, and then

$$\hat{g}(n)|c(n)| \leq \hat{g}(n)\hat{E}(n) = \hat{g}(n)\bar{E}(n).$$

Therefore,

$$\sum_n \hat{g}(n)|c(n)| \leq \sum_n |g(n)|\bar{E}(n) = \frac{1}{2\pi} \int g(\theta)\overline{E(\theta)}\,d\theta \leq \|g\|_{p'}\|E\|_p$$

by Hölder’s inequality. The assumption that $\sum_n \hat{g}(n)|c(n)| = 1$ yields that $\|g\|_{p'} \geq 1/\|E\|_p$. Hence $K_p(c) \geq 1/\|E\|_p$; in particular, $K_p(c) > 0$, as asserted in the lemma.

For the converse, suppose that $K_p(c) > 0$. The set $R(c)$ is nonempty since it contains the function $t \mapsto e^{int}/|c(n)|$ for each index $n$ in the support of $c$. Because $R(c)$ is convex, and $1 < p' \leq \infty$, the closure of $R(c)$ in $L^{p'}$ then contains an element, $h$ say, of minimal $L^{p'}$ norm. That norm must be $K_p(c)$; moreover, $\hat{h}$ is nonnegative, and $\hat{h}$ vanishes off the support of $c$.

Let

$$J = |h|^{p'-1} \sgn(h) = |h|^{p'/p} \sgn(h).$$

Then $\|J\|_p = 1$ when $\|h\|_{p'} = 1$, that is when $K_p(c) = 1$. We claim that $J$ is a partial majorant of $\tilde{c}$ in this special case.

If so, then $J$ must be the partial majorant of minimal $L^p$ norm, since the discussion after the relations (2.4) makes $1/K_p(c)$ a lower bound for the $L^p$ norms of all partial majorants of $\tilde{c}$. Since $\hat{h}$ is real-valued, $\hat{h}(\theta) = \bar{\hat{h}(\theta)}$ for almost all $\theta$. It follows that $J(\theta) = \bar{J(\theta)}$ for almost all $\theta$, and hence that $\hat{J}$ is real-valued.

Fix an integer $n$ in the support of $c$. Let $z_n$ be the function mapping $\theta$ to $e^{inz_n}$, and let $\phi$ map real numbers $r$ to $(|h + rz_n|_{p'})^{p'}$; as in [6 Lemma 2], but with $p$ replaced by $p'$,

$$\phi'(0) = p' \text{Re} \hat{J}(n) = p'\hat{J}(n).$$  

(2.5)
If \( g \in R(c) \), and \( r > 0 \), then the function \( (g + rz_n)/(1 + r|c(n)|) \) also belongs to the set \( R(c) \). So the quotient

\[
\frac{h + rz_n}{1 + r|c(n)|}
\]

belongs to the closure of \( R(c) \) when \( r > 0 \). By the minimality of \( \|h\|_{p'} \) in \( R(c) \), the derivative at \( r = 0 \) of

\[
\frac{\|h + rz_n\|_{p'}}{(1 + r|c(n)|)^{p'}}
\]

must be nonnegative. That derivative is equal to

\[
\frac{p'\hat{J}(n)[1 + 0]^{p'}}{(1 + 0)^{p'}} - \frac{\|h + 0\|_{p'}^{p'}[1 + 0]^{p' - 1}|c(n)|}{[1 + 0]^{2p'}}.
\]

This is nonnegative for all \( n \) in the support of \( c \) if and only if

\[
\hat{J}(n) \geq |c(n)|
\]

in all those cases. So \( J \) is indeed a partial majorant of \( \hat{c} \).

When \( \hat{h}(n) > 0 \), the quotient (2.6) also belongs to the closure of the set \( R(c) \) when \( r \) is negative and close enough to 0. In those cases, the derivative (2.7) must be equal to 0, and then \( \hat{J}(n) = |c(n)| \). So the part of the support of \( c \) where \( \hat{J}(n) > |c(n)| \) is disjoint from the support of \( \hat{h} \), and \( \hat{h} \) must vanish on that part.

If \( K_p(c) \) is positive but differs from 1, let \( c' = c/K_p(c) \), and note that \( K_p(c') = 1 \). Let \( h' = K_p(c)h; \) this is the function of minimal \( L^{p'} \) norm in the set \( R(c') \). Then \( \|h'\|_{p'} = 1 \), and the function \( J' \) that factors as \( |h'|^{p' - 1}\text{sgn}(h') \) is the minimal partial majorant in \( L^p \) of the inverse transform of \( c' \). Moreover \( \hat{h}'(n) = 0 \) on the part of the support of \( c' \) where \( \hat{J}'(n) > |c'(n)| \). Let

\[
F = K_p(c)J' = K_p(c)^{p'}J;
\]

this is the minimal partial majorant of \( \hat{c} \) in \( L^p \), and the rest of the lemma follows. Corollary 2.4 also follows, and

\[
G = K_p(c)^{p'/p'}h' = K_p(c)^{p}h.
\]

Remark 2.5. The classical method was applied to some instances of partial majorants and related notions in [6, page 308]. It will be used in Section 4 to give another proof of Corollary 2.4.
3. Special exponents

The lower majorant property for $L^p$ holds in the special cases because of two extra things that are true in those cases.

Lemma 3.1. Let $p = 2j/(2j - 1)$ for some integer $j > 1$, and let $c$ be a bounded sequence. If $\hat{c}$ has a partial majorant in $L^p$, then its partial majorant of minimal $L^p$ norm is also a full majorant of minimal $L^p$ norm. If there is a nontrivial function $f$ in $L^p$ for which $|c| \leq |\hat{f}|$, then $K_p(c) \geq 1/\|f\|_p$.

Proof. For these special values of $p$, if $\hat{c}$ has a partial majorant in $L^p$, then the factorization of the minimal partial majorant $F$ in Lemma 2.4 takes the form

$$F = |G|^{p'-2}G \operatorname{sgn}(G) = \{|G|^2\}^{j-1}G = \{\hat{G}G\}^{j-1}G = \hat{G}^{j-1}G^j,$$

where $G \in L^{2j}$ and $\hat{G} \geq 0$. Choose trigonometric polynomials $G_m$ with nonnegative coefficients so that the sequence $\{G_m\}_{m=1}^\infty$ converges in $L^{p'}$ norm to $G$. Then the sequence $\{\{\hat{G}_m\}^{j-1}(G_m)^j\}_{m=1}^\infty$ converges in $L^p$ norm to $F$. Since the coefficients of each function $G_m$ are nonnegative, the same is true for the terms in the sequence $\{\{\hat{G}_m\}^{j-1}(G_m)^j\}_{m=1}^\infty$, and hence for the $L^p$-norm limit $F$ of that sequence. So the partial majorant $F$ is a full majorant of $\hat{c}$. Its $L^p$ norm is minimal, since $\|F\|_p$ is minimal among norms of partial majorants of $\hat{c}$.

Suppose that $|c| \leq |\hat{f}|$ for some $f$ in $L^p$; then there is sequence $(\varepsilon(n))$ of numbers of absolute-value at most 1 for which $|c(n)| = \varepsilon(n)|\hat{f}(n)|$ for all indices $n$. Let $g \in R(c)$, and let $k$ be the trigonometric polynomial for which $\hat{k}(n) = \varepsilon(n)\hat{g}(n)$ for all $n$. Much as in the relations (2.4),

$$\sum_n \hat{g}(n)|c(n)| = \sum_n \hat{g}(n)\varepsilon(n)\overline{f(n)}$$

$$= \sum_n \hat{k}(n)\overline{f(n)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\theta)\overline{f(\theta)} \, d\theta.$$  \hspace{1cm} (3.2)

By Hölder’s inequality and the upper majorant property with constant 1 in $L^{2j}$,

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\theta)\overline{f(\theta)} \, d\theta \right| \leq \|k\|_{2j}\|f\|_p \leq \|g\|_{2j}\|f\|_p.$$  \hspace{1cm} (3.3)

The assumption that $\sum_n \hat{g}(n)|c(n)| = 1$ yields that $\|g\|_{2j} \geq 1/\|f\|_p$ for all functions $g$ in $R(c)$, and that $K_p(c) \geq 1/\|f\|_p$. $\square$
Proof of Theorem 1.1. Let $p = 2j/(2j - 1)$, and let $f$ be a nontrivial function in $L^p$. By the second part of the lemma above,

$$K_p(\hat{f}) \geq \frac{1}{\|f\|_p} > 0.$$ 

By the first part of Lemma 2.3, the function $f$ has a partial majorant in $L^p$, and its partial majorant $F$ of minimal $L^p$ norm satisfies the condition that

$$\|F\|_p = \frac{1}{K_p(\hat{f})} \leq \|f\|_p.$$ 

By Lemma 3.1, the minimal partial majorant $F$ is also a full majorant, with minimal $L^p$ norm, of $f$. \hfill $\square$

Proof of Corollary 1.2. The conclusion is evident if $G$ is a trigonometric polynomial. In general, $G$ is again a limit in $L^{2j}$ norm of a sequence $(G_m)$ of polynomials whose coefficients vanish off the support of $\hat{f}$. The frequencies of $(\overline{G})^{j-1}G^j$ have the desired property, because the frequencies of the terms in the sequence $\{(\overline{G}_m)^{j-1}(G_m)^j\}_{m=1}^{\infty}$ do.

Remark 3.2. Since the products in the sums (3.2) all vanish off the support of $\hat{g}$, requiring that $|c(n)| = \varepsilon(n)|\overline{f}(n)|$ and $\hat{k}(n) = \varepsilon(n)\hat{g}(n)$, where $|\varepsilon(n)| \leq 1$, is only needed when $\hat{g}(n) \neq 0$. The conclusion that

$$\sum_n \hat{g}(n)|c(n)| \leq \|g\|_{2j}\|f\|_p$$

therefore follows from the weaker hypothesis that $|c| \leq |\hat{f}|$ on the support of $\hat{g}$. Moreover, the assumption that $\|g\|_{2j} = 1$ was not used for this. These observations will be useful in the first part of Section 5.

Remark 3.3. The cases where $p' = 2j$ for a positive integer $j$ are the only ones [14] where the function $\text{sgn}(G)|G|^{p'-1}$ must have nonnegative coefficients if $G$ does. Hence the part of Lemma 3.1 making $\hat{F}$ nonnegative fails when $p'$ is not even. So does the other part, using the upper majorant property for $L^{p'}$ to get a strictly positive lower bound for $K_p(\hat{f})$. This is no accident, since one way [16, 8, 13] to disprove the upper majorant property when $p'$ is not even is to use a function $g$ with nonnegative coefficients for which $\text{sgn}(g)|g|^{p'-1}$ has a negative coefficient.

Remark 3.4. Proving Lemma 3.1 only requires Lemma 2.3 in the cases where $p'$ is even. The proof of the latter lemma is more elementary in those cases, because the function $\phi$ in formula (2.5) is then a polynomial.
4. Previous proofs

The classical version [6] of Corollary 2.4 applies to full majorants of minimal norm. It does not include the statement that \( \hat{G} \equiv 0 \) off the support of \( c \), but it asserts when \( \hat{c} \in L^p \) that
\[
\hat{G}(n) = 0 \text{ if } \hat{F}(n) > |c(n)|
\]
whether or not \( n \) lies in the support of \( c \).

We now show how the conclusion in Theorem 1.1 that \( \hat{G} \equiv 0 \) off the support of \( c \) follows from condition (4.1) in the special cases where \( p' \) is even. In nontrivial cases, view the coefficients of \( F \) as being given by the convolution of the coefficients of \( G \) with those of \( (|G|^2)^{j-1} \).

Now \( \hat{G}(n) = \hat{G}(-n) = \hat{G}(-n) \) for all \( n \). So the coefficients of \( \overline{G} \) are all nonnegative, and the same is true for the coefficients of \( |G|^2 \), which come from convolving the coefficient sequences of \( G \) and \( \overline{G} \). Moreover,
\[
(|G|^2)^{\sim}(0) = \sum_n \hat{G}(n)\hat{\overline{G}}(-n) = \sum_n \hat{G}(n)^2 > 0.
\]

By induction on \( j \), the coefficients of \( |G|^{2j-2} \) are all nonnegative, and its 0-th coefficient is strictly positive. Therefore
\[
\hat{F}(n) = \sum_m \hat{G}(n-m)(|G|^{2j-2})^{\sim}(m) \geq \hat{G}(n)|G|^{2j-2}(0),
\]
and the last expression is strictly positive if \( \hat{G}(n) > 0 \). Hence the support of \( \hat{G} \) is included in the support of \( \hat{F} \).

The support of \( \hat{G} \) is also included in the support of \( c \) if \( \hat{G} \) vanishes on the part of the support of \( \hat{F} \) lying outside the support of \( c \). If \( n \) belongs to that part, then \( \hat{F}(n) = 0 = |c(n)| \). Since condition (4.1) makes \( \hat{G}(n) = 0 \) in that case, the support of \( \hat{G} \) is indeed included in the support of \( c \).

Another way to reach this conclusion is to prove Corollary 2.4 by the classical method, as was done in special cases in [6]. To that end, let \( p \in (1, \infty) \), let \( F \) be a minimal partial majorant of \( \hat{c} \) in \( L^p \), and let \( G = |F|^{p-1} \text{sgn}(F) \). The function \( \tilde{F} \) mapping \( \theta \) to \( \overline{F(-\theta)} \) has the same \( L^p \) norm as \( F \), and \( \tilde{F} = \overline{F} \). So \( \tilde{F} \) is also a minimal partial majorant of \( \hat{c} \) in \( L^p \), and must therefore coincide almost everywhere with \( F \) by the minimality of \( \|F\|_p \). It follows that \( \hat{G} \) coincides almost everywhere with \( G \), and hence that \( \hat{G} \) is real-valued too.

Fix an integer \( n \) outside the support of the sequence \( c \). When \( r \) is real, let \( \psi(r) = (\|F+rz_n\|_p)^p \), where \( z_n \) is again the function mapping \( \theta \) to \( e^{in\theta} \). As in [6 page 311], the derivative \( \psi'(0) \) exists and is equal to
the real part of $p\hat{G}(n)$, which is just $p\hat{G}(n)$ here. The functions $F + rz_n$ are also partial majorants of $\hat{c}$. By the minimality of $\|F\|_p$ among partial majorants, $\psi'(0) = 0$. Hence $\hat{G}(n) = 0$ for all $n$ outside the support of $c$.

Argue similarly when $n$ lies in the support of $c$, and $\hat{F}(n) > |c(n)|$. At the remaining points $n$ in that support, $\hat{F}(n) = |c(n)|$, and the functions $F + rz_n$ with $r > 0$ are partial majorants of $\hat{c}$. So $\psi'(0) \geq 0$, and $\hat{G}(n) \geq 0$ at these points.

The proof in [3] was also applied to full majorants rather than partial majorants. Given a nontrivial function $f$ in $L^p$, form the set, $\tilde{R}^j(\hat{f})$ say, of all functions $g$ in $L^{p'}$ where the coefficients of $g$ are nonnegative

$$\sum_n \hat{g}(n)|\hat{f}(n)| \geq 1.$$  

This set is larger than the closure of $R(\hat{f})$ in $L^{p'}$, since the sum above is allowed to exceed 1, and there is no restriction on the support of $\hat{g}$. When $p'$ is even, however, the upper majorant property with constant 1 makes the minimum value of $\|g\|_{p'}$ on $\tilde{R}^j(\hat{f})$ the same as its minimum value on the closure of $R(\hat{f})$ in $L^{p'}$.

The authors of [3] use continuous linear functionals whose real parts separate the set $\tilde{R}^j(\hat{f})$ from closed balls of radius less than $K_p(\hat{f})$ about 0. Representing those functionals by integration against functions in $L^p$, and rescaling suitably, yielded the desired majorant for $f$. It is then only a short step to consider the function $g$ that achieves the minimum norm in the set $\tilde{R}^j(\hat{f})$, form $(\tilde{g})^j - \tilde{g}^j$, and proceed as we do.

5. COMPLEMENTS

We now prove Theorem 1.4 and its corollaries. Then we apply them to related questions.

Because $\|F\|_p = (\|G\|_{2j})^{2j-1}$, the conclusion in the theorem, namely that $\|F\|_p \leq \|f\|_p$, follows if $(\|G\|_{2j})^{2j-1} \leq \|f\|_p$. Consider the set, $S_G$ say, of trigonometric polynomials that are majorized by $G$ and have only nonnegative coefficients. Since $G$ belongs to the closure in $L^{2j}$ of this set, it is enough to show that

$$\tag{5.1} (\|g\|_{2j})^{2j-1} \leq \|f\|_p \quad \text{for all } g \text{ in } S_G.$$ 

For any such function $g$, let $F' = (\tilde{g})^j - \tilde{g}^j$. Then $F'$ is majorized by $F$, so that $\tilde{F}^j \leq \hat{F} \leq |\hat{f}|$ on the support of $\hat{G}$, and hence on the
support of \( \hat{g} \). Let \( c = \hat{F} \); then Remark 3.2 applies, and yields that

\[
(5.2) \quad \sum_n \hat{F}'(n) \hat{g}(n) = \sum_n |c(n)| \hat{g}(n) \leq \|f\|_p \|g\|_{2j}.
\]

The sum on the left above is equal to

\[
(5.3) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} F'(\theta) \overline{g(\theta)} \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) \, d\theta \right]^{j-1} \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) \, d\theta} \right\} g(\theta) \, d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(\theta)|^{2j} \, d\theta = \left( \|g\|_{2j} \right)^{2j}.
\]

So \( \|g\|_{2j}^{2j} \leq \|f\|_p \|g\|_{2j} \), and inequality (5.1) holds.

For Corollary 1.5, let \( H \) be a \( p \)-conjugate of \( f \), and let \( J = (\overline{H})^{j-1} H^j \). Then \( \hat{J} \) and \( |f| \) agree on the support of \( \hat{H} \). It follows that if \( K \) is any majorant of \( f \), then \( K \geq \hat{J} \) on the support of \( \hat{H} \). Applying Theorem 1.4 with \( F \) replaced by \( J \) and \( f \) replaced temporarily by \( K \) yields that \( \|J\|_p \leq \|K\|_p \). Because \( J \) is a majorant of the original function \( f \) in the corollary, it must be the minimal one in \( L^p \). Since \( J \) is then unique, so is the function \( H \) in \( L^{2j} \) obtained by letting \( H = |J|^{1/(2j-1)} \text{sgn}(J) \).

Corollary 1.6 follows, because \( H \) is a \( p \)-conjugate of \( (\overline{H})^{j-1} H^j \).

Combining Corollary 1.5 with Theorem 1.1 yields that a function \( F \) in \( L^p \) is a minimal majorant of a function \( f \) in that space if and only if \( |F|^{1/(2j-1)} \text{sgn}(F) \) is a \( p \)-conjugate of \( f \). Similarly, the functions \( F \) in \( L^p \) that are minimal majorants of something in \( L^p \) are those for which the coefficients of \( |F|^{1/(2j-1)} \text{sgn}(F) \) are nonnegative.

There are several ways to see that some functions in \( L^p \) have nonnegative coefficients, but are not minimal majorants in \( L^p \) of anything. As in Remark 3.3 with the rôles of \( p \) and \( p' \) exchanged, one can appeal to the fact [14] that if \( p \) is not an even integer, then there exist functions, \( F \), say, in \( L^p \) with nonnegative coefficients for which some coefficient of \( |F|^{1/(2j-1)} \text{sgn}(F) \) fails to be nonnegative.

It is simpler, however, to consider the supports, \( S_j \) say, of the transforms of products \( \overline{G}^j G \) when \( G \) is a trigonometric polynomial for which \( \hat{G} \geq 0 \). If \( G \) has at most one nonzero coefficient, then the same is true for \( \overline{G}^j G \) when \( j > 1 \). In the remaining cases, since the support of the coefficients of \( \overline{G} \) contains all differences of members of \( S_1 \), that support contains 0 as well as some positive integer and its negative. Then \( S_2 \) includes \( S_1 \), and it also contains an integer to left of \( S_1 \) and an integer to the right of \( S_1 \). Similarly, \( S_{j+1} \) contains at least two more points than \( S_j \). In particular, \( S_j \) must contain at least 4 integers when \( j > 1 \). It follows that any trigonometric polynomial \( F \)
with exactly 2 or 3 nonzero coefficients cannot be factored in the form specified in Theorem 1.1.

This approach also provides many examples where the support of the coefficients of the function $G$ in Theorem 1.1 is strictly smaller than the support of $\hat{f}$. Just let $f$ be any exact minorant of $(\hat{G})^{j-1}G^j$, where $G$ is a trigonometric polynomial with nonnegative coefficients, with at least two of them different from 0. In particular, $G$ cannot be a multiple of the exact majorant $\hat{E}_f$ of $f$, because $\hat{G}$ vanishes on part of the support of $\hat{E}_f$.

We now discuss the cases where $\|F\|_p = \|f\|_p$ in Theorem 1.4. They are the ones where the $p$-conjugate $G$ has the property that the inequality

$$\|(G)_{2j}\|^{2j-1} \leq \|f\|_p$$

is not strict. When $f$ is a trigonometric polynomial, so is $G$, and the relations (5.2) and (5.3) apply when $g = G$ and $F' = F$. Matters then reduce to determining when the inequality

$$\sum_n |c(n)|\hat{g}(n) \leq \|f\|_p \|g\|_{2j}$$

is not strict.

In the proof of that inequality, we used Remark 3.2. It concerns the equations (3.2) and the inequalities (3.3), in which the sequence $c$ is factored on the support of $\hat{g}$ as $\varepsilon \hat{f}$, where $|\varepsilon| \leq 1$ on that support, and $k$ is the minorant of $g$ for which $\hat{k} = \varepsilon \hat{g}$ there. The inequalities in line (3.3) are the instance of Hölder’s inequality where

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\theta) f(\theta) d\theta \right| \leq \|k\|_{2j} \|f\|_{2j/(2j-1)},$$

and the fact that

$$\|k\|_{2j} \leq \|g\|_{2j}$$

when $|\varepsilon(n)| \leq 1$ for all $n$ in the support of $\hat{g}$. The norms of $f$ and $(\hat{g})^{j-1}g$ in $L^{2j/(2j-1)}$ are equal if and only if equality holds in both inequalities displayed just above.

Inequality (5.6) is strict if the inequality $|\varepsilon(n)| \leq 1$ is strict for some $n$ in the support of $\hat{g}$. It follows that if $g$ and $k$ have the same norms in $L^{2j}$, then $|\varepsilon| = 1$ on the support of $\hat{g}$. That is $|\hat{f}| = |c|$ on that set, and $k$ is an exact minorant of $g$. 
In nontrivial cases, inequality (5.4) is strict unless $f$ factors as $(\hat{k})^{j-1}k^j$ multiplied by some constant, $C$ say. Then $C > 0$ because
\[
\int_{-\pi}^{\pi} k(\theta)\mathcal{F}(\theta)\,d\theta = \sum \hat{g}(n)|c(n)| > 0.
\]
Now $\|f\|_p = C(\|k\|_{2j})^{2j-1} = C(\|g\|_{2j})^{2j-1}$, while $\|F\|_p = (\|g\|_{2j})^{2j-1}$. So $C = 1$ if $f$ and $F$ have the same norms in $L^p$. Then $f$ does indeed factor as $(\hat{k})^{j-1}k^j$ where $k$ has the same norm in $L^{2j}$ as its exact majorant.

For the converse, suppose that a trigonometric polynomial $k$ has the same $L^{2j}$ norm as its exact majorant $g$. Letting $f = (\hat{k})^{j-1}k^j$, and $F = (\hat{g})^{j-1}g^j$ then gives $F$ the same $L^p$ norm as $f$. Now $F$ clearly majorizes $f$, and it is minimal in $L^p$ if $g$ is a $p$-conjugate of $f$. It is easy to see that $g$ satisfies the first two conditions in Definition 1.3. Argue as in lines (5.2) and (5.3) to get that
\[
(\|g\|_{2j})^{2j} = \sum_n \hat{F}(n)\hat{g}(n), \quad \text{and} \quad (\|f\|_{2j})^{2j} = \sum_n \hat{f}(n)\hat{k}(n).
\]
Since these norms are equal, so are the sums. Since
\[
|\hat{f}(n)\hat{k}(n)| \leq \hat{F}(n)\hat{g}(n)
\]
for all $n$, the two sides above are equal. Since $|\hat{k}(n)| = \hat{g}(n)$, the quantities $|\hat{f}(n)|$ and $\hat{F}(n)$ agree on the support of $\hat{g}$. So $g$ satisfies the third condition in Definition 1.3. It was shown in Section 4 that the first three conditions in the definition imply the remaining one.

As in [6] and [1], both directions in the reasoning above extends to general functions $k$ and $g$ in $L^{2j}$, because formulas like
\[
\int_{-\pi}^{\pi} k(\theta)\mathcal{F}(\theta)\,d\theta = \sum_n \hat{k}(n)\hat{f}(n)
\]
do.

Finally, we ask when a function, $G$ say, in $L^{2j}$, with nonnegative coefficients has the same norm in $L^{2j}$ as all functions that it majorizes exactly. This is equivalent to requiring for all exact minorants $k$ of $G$, that $k^j$ have the same norm in $L^2$ as $\hat{G}^j$. Since $k^j$ is majorized by $G^j$, that happens if and only if the coefficients of $k^j$ all have the same absolute value as those of $G^j$.

The support, $T_j$ say, of $\hat{G}^j$ is the algebraic sum of $j$ copies of the support, $S$ say, of $\hat{G}$. An integer, $n$ say, belongs to $T_j$ if and only if there is a function $\alpha$ mapping $S$ into the nonnegative integers for which
\[
\sum_{k \in S} \alpha(k) = j, \quad \text{and} \quad n = \sum_{k \in S} \alpha(k)k.
\]
As in [7], call $S$ a $B_j$ set, or a Sidon set of order $j$, if for each integer $n$ in the set $T_j$ there is at most one function $\alpha$ with the properties specified above. It is easily to see that the range of a sequence $(\lambda_i)$ of positive integers for which $\lambda_{i+1} > (1+\varepsilon)\lambda_i$ for all $i$ has this property if $\varepsilon$ is large enough; more subtle examples were used in [15, §4]. As explained in the survey [12], some authors use the term $B_j$ set with slightly different meanings, but the one above is widely used, and it fits our situation.

A function $G$ in $L^{2j}$ with nonnegative coefficients has the same norm as all its exact minorants if and only if the support of $\hat{G}$ is a Sidon set of order $j$. This is likely well known for exponential sums. To confirm that it extends to other functions, suppose first that $\hat{G}$ vanishes off such set. Then, in the context of the formulas (5.7), the coefficient $\hat{G}^j(n)$ is a sum of a suitable number of copies of the same product

$$\prod_{k \in S} \hat{G}(k)^{\alpha(k)}, \quad (5.8)$$

and similarly for $\hat{k}^j$ for each exact minorant $k$ of $G$. Hence $\hat{k}^j$ and $\hat{G}^j$ have the same absolute values, and $\|k^j\|_2 = \|G^j\|_2$.

On the other hand, if the support of $G$ is not a Sidon set of order $j$, then there is an integer $n$ in the support of $\hat{G}^j$ for which the second of the equations (5.7) holds for more than one choice of $\alpha$. Then $\hat{G}^j(n)$ is again equal to a sum of copies of products of the form displayed in line (5.8), but with different $\alpha$’s being used in some products. As before, $\hat{k}^j(n)$ is equal to a sum of similar products, but now the arguments of the coefficient of $k$ can be chosen to create some cancelation in that sum. Then $|\hat{k}^j(n)| < |\hat{G}^j(n)|$, and $\|k\|_{2j} < \|G\|_{2j}$.

Remark 5.1. The term “Sidon set” is also widely used with a very different meaning, as in [9] and [5]. For clarity, those sets could be called “Sidon interpolation sets.” Among their many properties is the fact that for functions whose transforms vanish off such a set each $L^p$ norm, where $p \in [1, \infty)$, is equivalent to the $L^2$ norm. So the exact majorant of every such function in $L^p$ also belongs to $L^p$, and $\|E_f\|_p \leq C\|f\|_p$ in this case, where $C$ depends on $p$ and the choice of Sidon interpolation set, but not on $f$. Both notions of Sidon set are discussed in [15, §4], but in that context the term just meant Sidon interpolation set.

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