Abstract

We prove the embedding of $ISO_q(3) \hookrightarrow ISU^c_{\sqrt{q}}(2)$ and $ISO_q(2, 1) \hookrightarrow ISL^c_{q}(2, R)$ as $^*$-algebras and give a Hilbert space representation of $ISU^c_{\sqrt{q}}(2)$.

1 Introduction

The inhomogenized extensions of a large list of standard quantized Lie groups [1] have been given in [2, 3, 4, 5, 7, 8]. They form quantized versions of the classical inhomogeneous groups. For a real deformation parameter $q$ the representation theory of the homogeneous parts (e.g. corepresentations of the function algebra) is basically the same as for the classical groups, whereas for $q$ root of unity it is completely different. The representation theory of the noncommutative function algebra however differs for any $q \neq 1$ from the classical situation. Its relevance stems from the question whether a deformation exists on the $C^*$-algebra level. [6]

In part 2 we recall the properties of inhomogeneous quantum groups. In the 3rd. part we examine the algebraic embedding of the $ISO_q(3)$ into $ISU^c_{\sqrt{q}}(2)$ and
ISO_q(2, 1) into ISL_q^{ex}(2, R). Here the “extended inhomogeneous” quantum algebra \( IG^{ex} \) designates inhomogeneous quantum algebras containing two sets of coordinate functions.

In the last chapter we examine the representation theory of the \( ISU_q^{ex}(2) \).

2 The Hopf algebra structure of inhomogeneous quantum groups

Quantum groups may be considered to be deformations of the function algebra over the corresponding Lie groups. The deformation is given by a parameter \( q \in \mathbb{C} \) which has to be further restricted in order to get special cases of deformations. Quantum groups exhibit a Hopf algebra structure. The noncommutative algebra structure is controlled by an \( \hat{R} \)-matrix fulfilling the Quantum Yang–Baxter equation \( \hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \). In this paper we refer to \( \hat{R} \)-matrices in their standard form given in [1]. They are defined by their projector decomposition making use of the antisymmetrizer \( \hat{A}_{ijkl}^q \), the symmetrizer \( \hat{S}_{ijkl}^q \) and the trace projector \( \hat{T}_{ijkl}^q \propto C_{ijkl} \) with the metric \( C_{ijkl} \), existing for the \( q \)-orthogonal groups only.

\[
\hat{R}_{ijkl}^q = \begin{cases} 
q\hat{S}_{ijkl}^q - q^{-1}\hat{A}_{ijkl}^q & \text{for } SL_q(N) \\
q\hat{S}_{ijkl}^q - q^{-1}\hat{A}_{ijkl}^q + q^{1-N}\hat{T}_{ijkl}^q & \text{for } SO_q(N)
\end{cases}
\] (1)

The algebra relations for the generators \( M_{ij} \) of the unital \( C \)-algebra \( A \) are:

\[
\hat{R}_{ijkl}^q M_{ij}^{k'} M_{j'i'}^{l'} = M_{ii'}^{ij} M_{j'i'}^{j'k}
\] (2)

and

\[
\begin{align*}
\det M &= (-1)^{N-1}k_1...k_N M^{i_1...i_N}k_1...M^{i_N...i_1} \epsilon_{i_1...i_N} = 1 \quad (SL_q(N)) \\
C_{ijkl} M_{ij}^{i'} M_{j'i'}^{j'} &= C_{ij'}^{i'} C_{j'i'}^{j'} 1 \quad (SO_q(N))
\end{align*}
\] (3)

For the unimodularity condition we use the \( q \)-antisymmetric tensors \( \epsilon_q \) defined in [10].

In order to obtain inhomogeneous quantum groups the set of generators has to be enlarged not only by the coordinate functions \( x_i \) but by an invertible scaling operator \( \tilde{w} \) as well. Its existence is required by consistency of the comultiplication. The additional algebra relations of the extended Hopf algebra \( A^I \) are:

\[
\begin{align*}
(i) & \quad x^i M^j_k = \gamma \hat{R}_{ijkl}^q M^l_k x^m \\
(ii) & \quad \tilde{w} w = 1 \\
(iii) & \quad \tilde{w} M^i_j = M^i_j \tilde{w} \\
(iv) & \quad w M^i_j = M^i_j w \\
(v) & \quad \tilde{w} x^i = \frac{q}{\gamma} x^i \tilde{w} \\
(vi) & \quad w x^i = \frac{q}{\gamma} x^i w
\end{align*}
\] (4)

with

\[
\gamma = \begin{cases} 
q^{-1/N} & \text{for } SL_q(N) \\
1 & \text{for } SO_q(N)
\end{cases}
\] (5)

The comultiplication \( \Phi : A^I \rightarrow A^I \otimes A^I \), counit \( e : A^I \rightarrow \mathbb{C} \) and the antipode \( \kappa : A^I \rightarrow A^I \) are very easily given in matrix notation.
With
\[ M^I = \begin{pmatrix} \bar{w}Mx & 1 \\ 0 & 1 \end{pmatrix} \] (6)
we get
\[ \Phi(M^I) = M^I \otimes M^I \] (7)
and
\[ e(M^I) = \begin{pmatrix} E & 0 \\ 0 & 1 \end{pmatrix} \] (8)
with the unity matrix \( E \).

The antipode is given as
\[ \kappa(M^I) = \begin{pmatrix} \kappa(M)\bar{w} & -\kappa(M)\bar{w}\bar{x} \\ 0 & 1 \end{pmatrix}. \] (9)

### 2.1 Complex conjugation

The \(^*\)-operations on \( A \) are defined quite differently in the cases \( q \in \mathbb{R}^+ \) and \( |q| = 1 \).

**a)** \( q \in \mathbb{R}^+ \)

With the unitarity condition \((M^i_j)^* = \kappa(M^j_i)\), the quantum group \( SL_q(N) \) becomes a \( SU_q(N) \) quantum group. The same \(^*\)-structure holds for the orthogonal quantum groups \( SO_q(N) \).

**b)** \( |q| = 1 \)

For such \( q \) the \( R_q \) matrix has the property \( R^*_q = R_q^{-1} \). With the reality condition \((M^i_j)^* = M^i_j\) one finds the real representation of the quantum group \( SL_q(N) \) called \( SL_q(N,R) \) and the orthogonal quantum groups in this case have a metric which is indefinite, i.e. for \( N \) even we get \( SO_q(n,n) \) and for \( N \) odd \( SO_q(n,n+1) \), with \( N = 2n \) or \( N = 2n+1 \) respectively.

The complex conjugation for the inhomogeneous extensions of these function algebras have to be treated separately as well.

**a)** In the case \( q \in \mathbb{R}^+ \) we have to enlarge the generating set of the \(^*\)-Hopf algebra \( A^I \) by the conjugate coordinate functions \( \bar{x}_i \). The additional algebra relations are:

\[\begin{align*}
(i) \quad M^I_{s} \bar{x}_j &= \frac{1}{\gamma} \hat{R}^{ai}_{q ij} \bar{x}_a M^i_s \\
(ii) \quad w \bar{x}_i &= \frac{\gamma}{q} \bar{x}_i w \\
(iii) \quad \bar{w} \bar{x}_i &= \frac{q}{\gamma} \bar{x}_i \bar{w} \\
(iv) \quad x^i \bar{x}_j &= \frac{1}{q} \bar{x}_a x^b \hat{R}^{ai}_{q bj}.
\end{align*}\] (10)

The comultiplication of \( \bar{x}_i \) follows from being an \(^*\)-homomorphism and the antipode from the fact that \( \kappa \circ \ast \circ \kappa \circ \ast = \text{id} \).

**b)** When \( |q| = 1 \) the coordinate functions may be chosen to be real \((x^*_i = x_i)\), since applying the \(*\) to (3(i)) and taking into account that \( R^{ij*}_{q kl} = R^{-1*}_{q ij kl} \), we get:

\[ M^j_k x^i = \gamma^{-1} R^{-1*}_{q ji lm} x^m M^l_k \] (11)

or

\[ \gamma \hat{R}^{rs}_{q ji} M^j_k x^i = \delta^r_m \delta^s_x x^m M^l_k. \] (12)
3 Algebraic embedding

The \( \hat{R} \)-matrix of \( SU_{\sqrt{q}}(2) \) is decomposed using the \( q \)-antisymmetric \( \varepsilon \)-tensor:

\[
\hat{R}^{\mu\nu}_{\sqrt{q}\rho\sigma} = \sqrt{q} \delta_{\mu}^{\rho} \delta_{\nu}^{\sigma} + \varepsilon^{\mu\nu} \varepsilon_{\rho\sigma},
\]

with

\[
\varepsilon_{\mu\nu} := \begin{pmatrix} 0 & q^{-1/4} \\ -q^{1/4} & 0 \end{pmatrix}_{\mu\nu} = -\varepsilon_{\mu\nu}
\]

For the homogeneous parts the embedding \( SO_q(3) \hookrightarrow SU_{\sqrt{q}} \) is well known [9].

With the \( q \)-deformed Clebsch-Gordan-coefficients \( c_{i \mu\nu} \) of the product decomposition of the \( SU_{\sqrt{q}}(2) \) [11]

\[
c_1^{\mu\nu} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\mu\nu}, \quad c_2^{\mu\nu} := \frac{1}{\sqrt{1 + q}} \begin{pmatrix} 0 & \sqrt{q} \\ 1 & 0 \end{pmatrix}_{\mu\nu}, \quad c_3^{\mu\nu} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\mu\nu}
\]

and \( c_{i \mu\nu} = c_{i \mu\nu}^{\mu\nu} \) the matrix elements \( M_{ij} \) of \( SO_q(3) \) are given in terms of \( SU_{\sqrt{q}}(2) \) elements \( m_{\mu\nu} \) by

\[
M_{ij} := c_{i \mu\nu} m_{\mu\rho} m_{\nu\sigma} c_{j \rho\sigma}^{\rho\sigma}.
\]

This is verified by checking the \( \hat{R} \)-matrix of the \( SO_q(3) \) group to be:

\[
\frac{1}{q} \eta^{i i'} \eta^{j j'} c_{i \mu\nu} c_{j \rho\sigma} \hat{r}_{\lambda\mu'\nu'} \hat{r}_{\lambda' \rho'\sigma'} c_{i' \rho'\sigma'} c_{j' \lambda' \lambda} c_{\lambda' \lambda},
\]

where \( \hat{r} \) denotes the \( \hat{R} \)-matrix of the \( SU_{\sqrt{q}}(2) \) group and \( \eta = \text{diag}(1, i, 1) \), \( i \) being the imaginary unit. It just produces a base change to more convenient coordinates. Note as well the decomposition of the symmetric projector \( \hat{S}_{\sqrt{q}\rho\sigma} = c_{i \mu\nu}^{\mu\nu} c_{i \rho\sigma}^{\rho\sigma} \).

In order to clear out nasty indices we want to make use of a graphical notation, which has been given in [10]. With

\[
c_{i \mu\nu} := \quad \varepsilon \quad \mu \quad \nu
\]

and the equality

\[
\begin{array}{c}
\varepsilon \\
\end{array} = \sqrt{q} \quad \begin{array}{c}
\varepsilon \\
\end{array} + \quad \begin{array}{c}
\varepsilon \\
\end{array}
\]

we can disentangle the matrix:
This is the $\hat{R}$-matrix of $SO_q(3)$ since obviously

$$C^{ij} = \eta^i_{i'} \eta^j_{j'}$$

and

$$\hat{A}_{q,kl}^{ij} = \eta^i_{k'} \eta^j_{l'} \eta^{k'}_{i'} \eta^{l'}_{j'}$$

Of course the same construction holds for the $SO_q(2,1)$ group. Then $\eta$ has to be chosen as identity matrix.

b) Since we know the $q$-antisymmetrizer we are able to find the $SO_q(3)$ covariant quantum plane in terms of spinor variables. To obtain sufficiently many degrees of freedom we have to take at least two copies of $q$-spinors $x$ and $y$ having the same commutation relations with $m$. This provides an extended inhomogeneous algebra called $ISU^{c,\sqrt{q}}(2)$. We want to mention that the extended algebra does not have a
correct coalgebra structure. This is not important for the algebraic embedding. The 3-dimensional quantum space has the form:

\[
\begin{align*}
    z^i &= \eta^j_{\ j} \\
    &= \left( \begin{array}{c}
        j \\
        \overline{x} \overline{x} \\
\end{array} + \begin{array}{c}
        j \\
        \overline{y} \overline{y} \\
\end{array} \right)
\end{align*}
\]  

(22)

We fix the \(x,y\)-relations such that the element \(\varepsilon_{\nu\mu} x^\nu y^\mu\) commutes with the coordinate functions and get

\[
\begin{align*}
    \begin{array}{c}
        \overline{y} \\
       \overline{x} \\
\end{array} \\
    &= \begin{array}{c}
        \overline{x} \\
       \overline{y} \\
\end{array}
\end{align*}
\]  

(23)

We still have to prove that \(\hat{A}^{ij}_{q \ kl} z^i z^j\) vanishes. This follows from the equation

\[
\begin{align*}
    \alpha \left( \begin{array}{c}
        \overline{x} \overline{y} \\
       \overline{y} \overline{x} \\
    \end{array} + \begin{array}{c}
        \overline{y} \overline{x} \\
       \overline{x} \overline{y} \\
    \end{array} \right) &= 0,
\end{align*}
\]  

(24)

or

\[
\begin{align*}
    \begin{array}{c}
        \overline{x} \\
       \overline{y} \\
\end{array} \\
    &= \begin{array}{c}
        \overline{y} \\
       \overline{x} \\
\end{array}
\end{align*}
\]  

(25)

Next we observe that

\[
\begin{align*}
    \begin{array}{c}
        \overline{i} \\
       \overline{j} \\
\end{array} \\
    &= \begin{array}{c}
        \overline{i} \\
       \overline{j} \\
\end{array}
\end{align*}
\]  

(26)

Now we have to take a look to the *-operation. The behaviour is for \(q\) complex quite different from \(q\) real, since \((c_{i\mu})^* = c_{i\mu}^\dagger\) in the first and \((c_{i\mu})^* = c_{i\mu}^\dagger\) in the
second case. From this we see immediately that the algebraic embedding in the case of $ISO_q(2,1)$ respects the $\ast$-operation, whereas for the $ISO_q(3)$ quantum group we have to examine the $\ast$-structure more closely. At first for the homogeneous part we have:

$$\kappa(c^i_{\mu\nu}m^\rho_{\sigma\rho}c^j_{\sigma\rho}) = c^i_{\mu\nu}(m^\sigma_{\nu})^\ast(m^\rho_{\mu})^\ast c^j_{\rho\sigma} = (c^j_{\rho\sigma}m^\rho_{\mu}m^\sigma_{\nu}c^i_{\mu\nu})^\ast. \quad (27)$$

To use the graphical technique for the translations we have to introduce the “transposed $c$-matrix” $c^T$:

$$\begin{align*}
(c^T)^{\nu\mu}_i &= 
\begin{array}{c}
\mu
\end{array}
\begin{array}{c}
\nu
\end{array}

&= 
\begin{array}{c}
\nu
\end{array}
\begin{array}{c}
\mu
\end{array}

&= c^{\mu\nu}_i
\end{align*} \quad (28)$$

Of course $(z^i)^{\ast} =: \bar{z}_i = (c^T)^{\mu\nu}_i x^\mu x^\nu$. Calculating now the mixed commutation relations one has:

$$\begin{align*}
\bar{x} \bar{x} = & \frac{1}{\gamma^4} \quad \text{and} \\
q^2 &= \begin{array}{c}
\mu
\end{array}
\begin{array}{c}
\nu
\end{array}

&= 
\begin{array}{c}
\nu
\end{array}
\begin{array}{c}
\mu
\end{array}

&= c^{\mu\nu}_i
\end{align*} \quad (29)$$

To reproduce the correct $z - \bar{z}$ relations which use diagrams similar to that of (29) we have to redefine $z$ as $z \rightarrow z^i = x^\mu x^\nu c^j_{\mu\nu} \bar{\omega}^n$. Now the calculation is similar to the one above and we find $n = 2/3$. This relation finishes the prove of the algebraic embedding.

Remark:

a) The antipode of the coordinates $z$ just differs by a minus sign, when expressed in terms of $\kappa(x)$.

b) The Coproduct of the coordinates $z$ can’t be embedded by obvious reasons.
4 Hilbert space representation for $ISO_q(3)$

The problem to find a representation for $ISO_q(3)$ and $ISO_q(2,1)$ is now reduced to that of representing $ISU_{\sqrt{q}}^{ex}(2)$ and $ISL_{q}^{ex}(2,R)$.

Here we restrict ourselves to the representation of $ISO_q(3)$. (We want to mention that $SL_q(2,R)$ does not exist on the Hilbert space level anyhow [12].) As well we confine the value of $q$ to $(0,1)$. The case $q > 1$ is however isomorphic. The relations of $SU_{\sqrt{q}}(2)$ are ($\mu = \sqrt{q}$):

$$\alpha \gamma = \mu \gamma \alpha, \quad \alpha \gamma^* = \mu \gamma^* \alpha, \quad \gamma \gamma^* = \gamma^* \gamma,$$

$$\alpha^* \alpha + \gamma^* \gamma = 1, \quad \alpha \alpha^* + \mu^2 \gamma^* \gamma = 1$$

(30)

The second coordinate has the following commutation relations: (Remember that in our convention the quantum plane coordinates are $x^{(\tau)}, \tau = 1,2$)

$$x^{(2)} \alpha = \mu^{-1} \alpha x^{(2)}, \quad x^{(2)} \gamma = \mu \gamma x^{(2)},$$

$$x^{(2)} \alpha^* = \mu \alpha^* x^{(2)}, \quad x^{(2)} \gamma^* = \mu^{-1} \gamma^* x^{(2)},$$

(31)

Applying the antipode to (31) gives

$$\alpha^* \kappa(x^{(2)}) = \mu^{-1} \kappa(x^{(2)}) \alpha^*, \quad \gamma \kappa(x^{(2)}) = \mu \kappa(x^{(2)}) \gamma$$

$$\alpha \kappa(x^{(2)}) = \mu \kappa(x^{(2)}) \alpha, \quad \gamma^* \kappa(x^{(2)}) = \mu^{-1} \kappa(x^{(2)}) \gamma^*.$$  

(32)

These relations hold for any $SU_{\sqrt{q}}(2)$-covariant plane.

We have other relations for the functions on the quantum planes $x^{(\tau)}$ and $y^{(\tau)}, \tau = 1,2$:

$$x^{(2)} y^{(2)} = \mu y^{(2)} x^{(2)}, \quad \kappa(x^{(2)}) \kappa(y^{(2)}) = \mu^{-1} \kappa(y^{(2)}) \kappa(x^{(2)}),$$

$$x^{(\tau)} \kappa(x^{(\delta)}) = \mu^{-2} \kappa(x^{(\delta)}) x^{(\tau)}, \quad x^{(\tau)} \kappa(y^{(\delta)}) = \mu^{-1} \kappa(y^{(\delta)}) x^{(\tau)},$$

$$\kappa(x^{(\delta)}) \kappa(y^{(2)}) = \mu^2 \kappa(y_2) \kappa(x_2), \quad \kappa(b^{(\delta)}) \kappa(b^{(\tau)}) = \kappa(b^{(\tau)}) \kappa(b^{(\delta)}), \quad a, b \in \{x, y\},$$

$$y^{(\tau)} \kappa(x^{(\delta)}) = \mu^{-1} \kappa(x^{(\delta)}) y^\tau + (\mu^{-2} - 1) \kappa(y^{(\delta)}) x^{(\tau)},$$

(33)

where $\delta, \tau \in \{1,2\}$.

These and the conjugated relations together with the obvious relations for $\omega$ contain the whole algebraic information of $ISU_{\sqrt{q}}(2)$ since:

$$a^{(1)} = (\mu \gamma)^{(-1)} \left( \omega u \left( a^{(2)} \right) + \alpha a^{(2)} \right), \quad a, b \in \{x, y\}.$$  

(34)

Remember that $\gamma$ is an invertible element.

The algebra may still be simplified by a nonlinear transformation in the functions of coordinates. With $Q^4 = \mu^2 = q$ and $v^3 = \omega$ we define:

$$\rho_1 = \bar{v}^2 \gamma^{-1} x^{(2)} \quad \Theta_1 = q^{-1} v \gamma^{-1} \kappa(x^{(2)})$$

$$\rho_2 = \bar{v}^2 \gamma^{-1} y^{(2)} \quad \Theta_2 = q^{-1} v \gamma^{-1} \kappa(y^{(2)})$$

(35)
All algebraic relations with coordinate-functions are given by:

\[ \begin{align*}
\rho_i \alpha &= Q \rho_i \\
\rho_i \alpha^\ast &= Q^{-1} \alpha^\ast \rho_i \\
\rho_i \gamma &= Q \gamma \rho_i \\
\rho_i \gamma^\ast &= Q^{-1} \gamma^\ast \rho_i \\
\Theta_i \alpha &= Q \alpha \Theta_i \\
\Theta_i \alpha^\ast &= Q^{-1} \alpha^\ast \Theta_i \\
\rho_i \rho_i^\ast &= Q^2 \rho_i \rho_i^\ast \\
\Theta_i \gamma &= Q^{-1} \gamma \Theta_i \\
\rho_i \Theta_i^\ast &= Q^{-1} \Theta_i \rho_i^\ast \\
\Theta_i \Theta_i^\ast &= \Theta_i^\ast \Theta_i \\
\Theta_1 \Theta_2 &= Q^2 \Theta_2 \Theta_1 \\
\Theta_1 \rho_2 &= Q^2 \rho_2 \rho_1 \\
\rho_1 \Theta_2 &= Q^{-1} \Theta_2 \rho_1 \\
\Theta_1 \rho_2 &= (Q^2 - Q^{-2}) \Theta_2 \rho_1 + Q \rho_2 \Theta_1 \\
\Theta_1 \Theta_2 &= \Theta_2 \Theta_1 \\
\Theta_1 \Theta_1^\ast &= \Theta_1^\ast \Theta_1
\end{align*} \]

with \( i = 1, 2 \).

It is easy to find a maximal real subalgebra \( D \) of commuting elements. In order to rely the representation of \( ISO_q(3) \) to that of \( SU_q(2) \) we choose:

\[ D := \{ \alpha \alpha^\ast, \gamma \gamma^\ast, \rho_1 \rho_1^\ast, \Theta_1 \Theta_1^\ast, \Theta_2 \Theta_2^\ast, v \} \]

These operators are used to label the eigenvectors of the Hilbertspace. They are normalised so that the representation is given by:

\[ \begin{align*}
\pi(\alpha)|n, m, k, r, s, v\rangle &= \sqrt{1 - Q^{4n}} |n - 1, m, k, r, s, v\rangle \\
\pi(\alpha^\ast)|n, m, k, r, s, v\rangle &= \sqrt{1 - Q^{4(n+1)}} |n + 1, m, k, r, s, v\rangle \\
\pi(\gamma)|n, m, k, r, s, v\rangle &= Q^{2n} |n, m - 1, k, r, s, v\rangle \\
\pi(\gamma^\ast)|n, m, k, r, s, v\rangle &= Q^{2n+2} |n, m + 1, k, r, s, v\rangle \\
\pi(v)|n, m, k, r, s, v\rangle &= |n, m, k - 1, r, s, v\rangle \\
\pi(v^\ast)|n, m, k, r, s, v\rangle &= |n, m, k + 1, r, s, v\rangle \\
\pi(\rho_1)|n, m, k, r, s, v\rangle &= Q^{-(n+m+k)+r+2s-v} |n, m, k, r - 1, s, v\rangle \\
\pi(\rho_1^\ast)|n, m, k, r, s, v\rangle &= Q^{-(n+m+k)+r+1+2s-v} |n, m, k, r + 1, s, v\rangle \\
\pi(\rho_2)|n, m, k, r, s, v\rangle &= Q^{-(n+m+k)+2r+3s-v} |n, m, k, r - 1, s, v - 1\rangle \\
\pi(\rho_2^\ast)|n, m, k, r, s, v\rangle &= Q^{-(n+m+k)+2r+1+3s-v} |n, m, k, r + 1, s, v + 1\rangle \\
\pi(\Theta_1)|n, m, k, r, s, v\rangle &= Q^{-(n-m+k+r+v)} |n, m, k, r, s - 1, v\rangle \\
\pi(\Theta_1^\ast)|n, m, k, r, s, v\rangle &= Q^{-(n-m+k+r+v)} |n, m, k, r, s + 1, v\rangle \\
\pi(\Theta_2)|n, m, k, r, s, v\rangle &= Q^{-(n-m+k+r-s+v)} |n, m, k, r, s - 1, v - 1\rangle \\
\pi(\Theta_2^\ast)|n, m, k, r, s, v\rangle &= Q^{-(n-m+k+r-s+v)} |n, m, k, r, s + 1, v + 1\rangle
\end{align*} \]

5 Conclusion

We have given the algebraic embedding of two \( q \)-euclidean groups in three dimensions. We have given an irreducible Hilbert space representation for the function algebra.
of $ISO_q(3)$. With a reasoning along the lines of \[6\] it should be possible to prove that $ISO_q(3)$ exists on a $C^*$-algebra level.

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