Application of the canonical quantization of systems with curved phase space to the EMDA theory

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Abstract

The canonical quantization of dynamical systems with curved phase space introduced by I.A. Batalin, E.S. Fradkin and T.E. Fradkina is applied to the four-dimensional Einstein–Maxwell Dilaton–Axion theory. The spherically symmetric case with radial fields is considered. The Lagrangian density of the theory in the Einstein frame is written as an expression with first order in time derivatives of the fields. The phase space is curved due to the nontrivial interaction of the dilaton with the axion and the electromagnetic fields.

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1 Introduction

The main idea of I.A. Batalin, E.S. Fradkin and T.E. Fradkina [1]–[2] for the canonical quantization of systems with curved phase space consists of performing a dimensionality doubling of the original phase space by introducing a set of new variables, equal in number to the variables of the original phase space, so that each one of them is defined as the conjugate canonical momentum to each original phase variable. Further, the complete set of variables is subjected to special second class constraints in such a way that the formal exclusion of the new canonical momenta reduces the system back to the original phase space. It turns out that the new phase space is flat and its quantization proceeds along the lines of [3]–[4].

In this paper we shall apply this method to the bosonic sector of the truncated four–dimensional effective field theory of the heterotic string at tree level, better known as Einstein–Maxwell Dilaton–Axion (EMDA) theory. Since this truncated effective theory contains just massless bosonic modes, we shall consider a suitable purely bosonic model.

The paper is organized as follows: in Sec. 2 we present a brief outline of the generalized canonical quantization method for dynamical systems. We keep the notation and terminology of the authors. In Sec. 3 we consider the action of the four–dimensional EMDA theory, perform the ADM decomposition of the metric and write the Lagrangian density of the matter sector as an expression with first order in time derivatives of the fields. We further consider the spherically symmetric anzats and obtain a Lagrangian density which defines a curved phase space and possesses two irreducible first class constraints. We continue by canonically quantizing the resulting EMDA system along the lines of [3]–[4] in Sec. 4. In order to achieve this aim, a suitable generalization of the method has been performed. We sketch our conclusions in Sec. 5 and, finally, we present some useful mathematical identities and relationships in the Appendix A.

2 Outline of the method

Let there be a dynamical system described by the original Hamiltonian $H_0 = H_0(\Gamma)$ given as a function of $2N$ bosonic phase variables

$$\Gamma^A, \quad A = 1, 2, ..., 2N, \quad (1)$$

of certain manifold $\mathcal{M}$.

We assume that the dynamical system is unconstrained. The extension of the formalism to systems with constraints is straightforward. The Lagrangian of the system can be presented as an expression with first order in time derivatives (denoted by a dot over the variable) of the phase space variables $\Gamma^A$ [5]

$$L = a_A(\Gamma)\dot{\Gamma}^A - H_0(\Gamma). \quad (2)$$
The Euler–Lagrange equations for the Lagrangian (2) are given by the following relations

\[ \omega_{AB}(\Gamma) \dot{\Gamma}^B = \frac{\partial}{\partial \Gamma^A} H_0(\Gamma), \]  

where

\[ \omega_{AB}(\Gamma) = \frac{\partial}{\partial \Gamma^A} a_B(\Gamma) - \frac{\partial}{\partial \Gamma^A} a_A(\Gamma). \]  

The nondegenerate tensor \( \omega_{AB}(\Gamma) \) defines on the phase space manifold the covariant components of the symplectic metric. Also it is antisymmetric and satisfies the Jacobi identity. The Poisson bracket for any two functions \( X(\Gamma) \) and \( Y(\Gamma) \) of the phase space variables (1) is defined as follows:

\[ \{X, Y\} = \partial_A X \omega^{AB} \partial_B Y, \]

where \( \partial_A \equiv \partial/\partial \Gamma^A \).

Next, the reper field with contravariant components \( h^A_{\ a}(\Gamma) \) and its inverse \( h^a_{\ A}(\Gamma) \) are introduced as follows

\[ \omega^{AB}(\Gamma) = h^A_{\ a} \omega_{(0)}^{ab} h^B_{\ b}, \quad \omega_{AB}(\Gamma) = h^a_{\ A} \omega^{(0)}_{ab} h^b_{\ B}, \quad a, b = 1, 2, ..., 2N; \]

where \( \omega_{(0)}^{ab} \) and \( \omega^{(0)}_{ab} \) define a constant symplectic metric which we will assume that it has the form

\[ \omega_{ab}^{(0)} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \]

where \( I_N \) stands for the unit matrix of dimension \( N \).

The covariant derivatives in the phase space are defined as follows

\[ \nabla_C V_A \equiv \partial_C V_A - \Delta_{CB}^D V_D, \quad \nabla_C V^A \equiv \partial_C V^A + \Delta_{CD}^A V^D, \]

where \( \Delta_{CB}^D \equiv h^D_{\ a} \partial_C h^a_{\ B} \).

The commutator of the covariant derivatives is given by the following relations

\[ [\nabla_A, \nabla_B] = -\Lambda_{AB}^C \nabla_C, \]

where

\[ \Lambda_{AB}^C \equiv \Delta_{AB}^C - \Delta_{BA}^C. \]

The covariant derivatives of the reper fields and the symplectic metric are zero. By looking at the description of the classical dynamics of the system under consideration, it can
be verified that the equations of motion written in terms of the Poisson brackets (5) can be derived from the action

\[ S = \int \left[ \Gamma^A \mathcal{\omega}_{AB}(\Gamma) \dot{\Gamma}^B - H_0(\Gamma) \right] dt, \tag{11} \]

where

\[ \mathcal{\omega}_{AB}(\Gamma) \equiv \int \omega_A(\alpha \Gamma) \alpha d\alpha. \tag{12} \]

Now we introduce new bosonic variables \( \Pi_A \) equal in number with the initial phase variables \( \Gamma^A \). The new variables are taken as the conjugate canonical momenta to the initial variables. For any two functions of the \( \Gamma^A \) and \( \Pi_A \), \( X(\Gamma^A, \Pi_A), Y(\Gamma^A, \Pi_A) \), we define the following flat Poisson bracket

\[ \{X(\Gamma^A, \Pi_A), Y(\Gamma^A, \Pi_A)\}' \equiv \partial_A X \partial^A Y - \partial_A Y \partial^A X, \tag{13} \]

where \( \partial^A \equiv \partial / \partial \Pi_A \).

Let us consider the action

\[ S' = \int \left[ \Pi_A \dot{\Gamma}^A - H_0(\Gamma) - \Theta_A(\Gamma, \Pi) \lambda^A \right] dt, \tag{14} \]

where

\[ \Theta_A(\Gamma, \Pi) \equiv \Pi_A + \mathcal{\omega}_{AB}(\Gamma) \Gamma^B \tag{15} \]

and \( \lambda^A \) are Lagrange multipliers.

The flat Poisson bracket for any two of them is given by the following relation

\[ \{\Theta_A, \Theta_B\}' = \omega_{AB}(\Gamma). \tag{16} \]

The action (14) represents a system with Hamiltonian \( H_0(\Gamma) \) independent of the momenta \( \Pi_A \) subjected to special second class constraints \( \Theta_A(\Gamma, \Pi) \). It turns out that the equations of motion that follow from the action (14) are equivalent to those derived from the initial action (11) after the formal exclusion of the canonical momenta \( \Pi_A \).

In the case when the dynamical system possesses original irreducible first class constraints \( T'_a(\Gamma), a = 1, 2, ..., m' \) (this is indeed the case for the model we will consider later on), the expressions (11) and (14) are correspondingly modified as follows [2]:

\[ S = \int \left[ \Gamma^A \mathcal{\omega}_{AB}(\Gamma) \dot{\Gamma}^B - H_0(\Gamma) - T'_a(\Gamma) \lambda'^a \right] dt, \tag{17} \]

\[ S' = \int \left[ \Pi_A \dot{\Gamma}^A - H_0(\Gamma) - \Theta_A(\Gamma, \Pi) \lambda^A - T'_a(\Gamma) \lambda'^a \right] dt. \tag{18} \]
The next step is to perform the canonical quantization of the system. The variables \( \Gamma^A \) and \( \Pi_A \) are promoted to operators and are required to satisfy the following equal time commutation relations
\[
[\Gamma^A, \Pi_B] = i\hbar \delta^A_B. \tag{19}
\]

The second class constraints \( \Theta_A(\Gamma, \Pi) \) are converted into Abelian first class by introducing the new bosonic operators \( \Phi_a, a = 1, 2, ..., 2N \) [3]–[4] which satisfy the following equal time commutation relations
\[
[\Phi_a, \Phi_b] = -i\hbar \omega^{(0)}_{ab}, \tag{20}
\]
where \( \omega^{(0)}_{ab} \) is the same as in (7). The effective Abelian constraints are defined by the commutations relations
\[
[\mathcal{T}_A(\Gamma, \Pi, \Phi), \mathcal{T}_B(\Gamma, \Pi, \Phi)] = 0 \tag{21}
\]
and the boundary conditions \( \mathcal{T}_A(\Gamma, \Pi, 0) = \Theta_A(\Gamma, \Pi) \).

We seek for the solution in the following form
\[
\mathcal{T}_A(\Gamma, \Pi, \Phi) = \Theta_A(\Gamma, \Pi) + K_A(\Gamma, \Phi), \quad K_A(\Gamma, 0) = 0. \tag{22}
\]
Substituting this expression into (21) we get the following equation for \( K_A(\Gamma, \Phi) \):
\[
\nabla_A K_B - \nabla_B K_A + \Lambda^{C AB} K_C - (i\hbar)^{-1} [K_A, K_B] = \omega_{AB}(\Gamma). \tag{23}
\]

It is important here to note that the solution of (23) in the zero curvature case \( \Lambda^{C AB} = 0 \) is given by
\[
K_A = \exp (\Phi_a \partial^a) \tilde{K}_A(\Gamma, \varphi)|_{\varphi=0}, \quad \tilde{K}_A = \varphi_a h^a_A(\Gamma), \tag{24}
\]
where \( \varphi_a \) are classical variables and \( \partial^a \equiv \partial/\partial \varphi_a \).

We further construct from the initial first class constraints \( T'_i(\Gamma), i = 1, 2, ..., m' \) of our system, the quantities \( \tilde{T}'_i(\Gamma, \Phi) \) that commute with the effective Abelian constraints \( \mathcal{T}_A(\Gamma, \Pi, \Phi) \) [2]:
\[
[\tilde{T}'_i(\Gamma, \Phi), \mathcal{T}_A(\Gamma, \Pi, \Phi)] = 0, \quad A = 1, 2, ..., 2N; \quad i = 1, 2, ..., m'; \tag{25}
\]
with boundary conditions \( \tilde{T}'_i(\Gamma, 0) = T'_i(\Gamma) \).

After substituting (22) into (25) we obtain the following equation for \( \tilde{T}'_i(\Gamma, \Phi) \):
\[
\partial_A \tilde{T}'_i(\Gamma, \Phi) = (i\hbar)^{-1} [K_A(\Gamma, \Phi), \tilde{T}'_i(\Gamma, \Phi)]. \tag{26}
\]

It is worth noticing that the solution of the equation (26) in the special case \( \Lambda^{C AB} = 0 \) is given by the following relation [1]:
\[
\tilde{T}'_i(\Gamma, \Phi) = \exp (\Phi_a \partial^a) \tilde{T}_i(\Gamma, \varphi)|_{\varphi=0}, \tag{27}
\]
where $\tilde{T}_i(\Gamma, \varphi) = T_i'(\Gamma(x = 0))$.

The functions $\Gamma^A(x)$ are solutions of the differential equation

$$\frac{d\Gamma^A(x)}{dx} = \varphi_o \omega^{ab}_{(0)} h^A b(\Gamma), \quad \Gamma^A(x = 1) = \Gamma^A. \quad (28)$$

Now we can construct the fermion generating operator $\Omega$. In order to do that we introduce a pair (coordinate, conjugate momentum) $(\lambda, \pi)$ together with a pair of ghosts $(C, \overline{C})$ and antighosts $(\overline{P}, C)$ for every irreducible first class constraint, where $\lambda$ is an active Lagrange multiplier. The new introduced variables possess the following statistics $(\varepsilon)$ and ghost number $(gh)$ [2]:

$$\varepsilon(\lambda) = \varepsilon(\pi) = 1 + \varepsilon(C) = 1 + \varepsilon(\overline{C}) = 1 + \varepsilon(\overline{P}) = 1 + \varepsilon(P)$$

$$gh(C) = -gh(\overline{P}) = gh(\overline{C}) = -gh(C) = 1. \quad (29)$$

Only the following supercommutators of the above introduced operators are different from zero:

$$[\lambda, \pi] = [C, \overline{P}] = [P, \overline{C}] = i\hbar I, \quad (30)$$

where $I$ is the unit matrix with the subindices labelling the complete set of irreducible first class constraints and the supercommutator is defined as follows:

$$[A, B] = AB - BA(-1)^{\varepsilon(A)\varepsilon(B)}. \quad (31)$$

In the case when initial second class constraints are absent, the fermion generating operator $\Omega$ is given in the form

$$\Omega = \Omega'(\Gamma, \Phi, C', \overline{P}) + T_A(\Gamma, \Pi, \Phi) C^A + \Pi'_A P'^A + \pi_A P^A, \quad (32)$$

where the Fermi operator $\Omega'$ obeys the following equations

$$[\Omega', \Omega'] = [\Omega', T_A] = 0, \quad gh(\Omega') = 1, \quad (33)$$

with the boundary condition

$$\Omega'(\Gamma, \Phi, C', 0) = \tilde{T}'_i(\Gamma, \Phi) C''^i.$$

By virtue of equations (33), the operator (32) turns out to be nilpotent:

$$[\Omega, \Omega] = 0. \quad (34)$$

This is a very brief and incomplete exposition of the canonical quantization of systems with curved phase space introduced by I.A. Batalin, E.S. Fradkin and T.E. Fradkina. Along this line we are going to proceed in treating the four–dimensional Einstein–Maxwell Dilaton–Axion theory.

6
3 The four–dimensional EMDA system

The EMDA system arises as one of the simplest low–energy string gravity models. It arises as the corresponding truncation of the critical heterotic string theory (D=10, with 16 $U(1)$ vector fields) reduced to four dimensions with no moduli fields excited and just one non–vanishing $U(1)$ vector field (see, for instance, [6]–[7] and references therein). In the Einstein frame it is described by the action

$$S = \frac{1}{16\pi} \int d^4x |(4)g|^\frac{1}{2} \left\{ -R + 2\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} e^{4\phi} \partial_\mu \kappa \partial^\mu \kappa - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} - \kappa F_{\mu\nu} \tilde{F}^{\mu\nu} \right\},$$  

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \tilde{F}^{\mu\nu} = \frac{1}{2} E^{\mu\nu\lambda\sigma} F_{\lambda\sigma},$$

are the strength of the $U(1)$ vector field $A_\mu$ and its dual tensor, respectively, $R$ is the scalar curvature of the gravitational field, $\phi$ is the dilaton field, $\kappa$ is the pseudoscalar axion field, and $E^{\mu\nu\lambda\sigma} = \sqrt{|g|} e^{\mu\nu\lambda\sigma}$. Formally, the EMDA theory can be considered as an extension of the Einstein–Maxwell system to the case when one takes into account the (pseudo)scalar dilaton and axion fields.

Using the ADM decomposition of the four–dimensional metric tensor [8]–[10] we have

$$^{(4)}g_{\mu\nu} = \begin{pmatrix} N^i N_i - N^2 & N_j \\ N_i & g_{ij} \end{pmatrix}, \quad ^{(4)}g^{\mu\nu} = \begin{pmatrix} -N^2 & N^j/N^2 \\ N^i/N^2 & g^{ij} - N^i N^j/N^2 \end{pmatrix}.$$  

Thus, the ADM action is given by [8]–[9]

$$I_{Gr} = \int dt d^3x \left( \pi^{ij} \dot{g}_{ij} - N^i N^j \mathcal{H}_i^\perp - N^i \mathcal{H}_i \right),$$  

where

$$\mathcal{H}_i = g^{1/2} \left( \pi_{ij} \pi^{ij} - \frac{1}{2} \Pi^2 \right) - g^{1/2} (3)R,$$

$$\mathcal{H}_i = -2g_ik \pi_{kj} - 2(g_{ik,j} - g_{kj,i}) \pi^{kj},$$

$g_{ij}$ is the three–dimensional spatial metric, $g \equiv \det g_{ij}$ and $\Pi = g_{ij} \pi^{ij}$.

We consider the spherically symmetric case [11]–[12]. In the basis $(dr, d\theta, d\varphi)$ we have

$$g_{ij} = \text{diag} \left(e^{2\mu(r,t)}, e^{2\lambda(r,t)}, e^{2\lambda(r,t)} \sin^2 \theta \right),$$

so that the spatial line element has the form

$$g_{ij} dx^i dx^j = e^{2\mu(r,t)} dr^2 + e^{2\lambda(r,t)} \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right).$$
The conjugate momenta are given by the expressions

\[ \pi^{ij} = \text{diag} \left( \frac{1}{2} \pi_\mu(r, t)e^{-2\mu(r,t)} \sin \theta, \frac{1}{4} \pi_\lambda(r, t)e^{-2\lambda(r,t)} \sin \theta, \frac{1}{4} \pi_\lambda(r, t)e^{-2\lambda(r,t)}(\sin \theta)^{-1} \right) \]. \quad (42)

After integrating over the angles, the action (37) becomes [11]:

\[ I_{Gr} = 4\pi \int dt dr \left( \pi_\mu \ddot{\pi}_\mu + \pi_\lambda \ddot{\pi}_\lambda - N^\perp \mathcal{H}_\perp - N^r \mathcal{H}_r \right), \quad (43) \]

where

\[ \mathcal{H}_\perp = e^{-\mu-2\lambda} \left[ \frac{1}{8} \left( \pi_\mu^2 - 2\pi_\mu \pi_\lambda \right) + 2 e^{4\lambda} \left( 2\lambda'' + 3\lambda'^2 - 2\lambda' \mu' - e^{2(\mu-\lambda)} \right) \right], \quad (44) \]

\[ \mathcal{H}_r = \pi_\lambda \lambda' + \pi_\mu \mu' - \pi'_\mu, \quad \mathcal{H}_\theta = \mathcal{H}_\varphi = 0, \quad (45) \]

and the primes denote spatial derivatives.

Apart from an overall factor \( \frac{1}{16} \pi \) in the action (35), the part of the Lagrangian density concerning the dilaton can be written as an expression first order in time derivatives of the dilaton field

\[ L_d \equiv 2|^{(4)}g|^{\frac{1}{2}} \partial_\mu \phi \partial^\mu \phi = \pi_\phi \dot{\phi} + \frac{N^\perp}{8|^{(4)}g|^{1/2}} \pi_\phi^2 - \pi_\phi N^i \partial_i \phi + 2N^\perp g^{1/2} \pi^{ij} \partial_i \phi \partial_j \phi, \quad (46) \]

where, as it was pointed out above, \( g^{ij} \) is the three–dimensional spatial metric in the ADM decomposition.

In the same way, for the axion field we obtain the following relation

\[ L_k \equiv 2|^{(4)}g|^{\frac{1}{2}} e^{4\phi} \partial_\mu \kappa \partial^\mu \kappa = \pi_\kappa \dot{\kappa} + \frac{N^\perp}{2|^{(4)}g|^{1/2}} e^{-4\phi} \pi_\kappa^2 - \pi_\kappa N^i \partial_i \kappa + \frac{N^\perp}{4|^{(4)}g|^{1/2}} e^{4\phi} g^{1/2} \pi^{ij} \partial_i \kappa \partial_j \kappa, \quad (47) \]

and for the \( U(1) \) vector field we get

\[ L_{U(1)} \equiv -|^{(4)}g|^{\frac{1}{2}} \left( e^{-2\phi} F_{\mu\nu} F^{\mu\nu} + \kappa F_{\mu\nu} \tilde{F}^{\mu\nu} \right) = \pi^i \dot{A}_i - \left[ \frac{(N^\perp)^2}{8} e^{2\phi} g^{i0} \right] + 2N^\perp g^{1/2} g^{jk} N^j F_{jl} + \]

\[ 2(N^\perp)^3 g^{1/2} \kappa e^{2\phi} \tilde{F}^{0kl} \left[ \frac{1}{4N^\perp g^{1/2}} \pi U(1)_k + e^{-2\phi} \frac{N^j}{(N^\perp)^2} F_{jk} + \kappa g_{ik} \tilde{F}^{0i} \right] - \]

\[ N^\perp g^{1/2} e^{-2\phi} \left( g^{ij} g^{kl} - 2g^{ij} \frac{N^k N^l}{(N^\perp)^2} \right) F_{jl} F_{ik} - A_0 \partial_i \pi^i. \]

Now we restrict ourselves to the spherically symmetric case and, moreover, we assume manifest spherical symmetry for the fields [12] which means that the scalar fields depend
only on \( r, t \) and that the vector fields have only the radial component that depend on \( r, t \). We define as well \( p_\phi(r, t), p_\kappa(r, t), E^r(r, t) \) as follows

\[
\pi_\phi = p_\phi(r, t) \sin \theta, \quad \pi_\kappa = e^{4\phi} p_\kappa(r, t) \sin \theta, \quad \pi^r = e^{-2\phi} E^r(r, t) \sin \theta.
\]  

(49)

Finally, after integrating over the angles, the EMDA action \((35)\) takes the form

\[
I = \frac{1}{4} \int L_{\text{tot}} dt dr,
\]

(50)

where

\[
L_{\text{tot}} = -\pi_\mu \dot{\mu} - \pi_\lambda \dot{\lambda} + \pi_\phi \dot{\phi} + e^{4\phi} p_\kappa \dot{\kappa} + e^{-2\phi} E^r \dot{A}_r +
\]

\[
N^1 \left \{ e^{-\mu - 2\lambda} \left [ \frac{1}{8} \left ( \pi_\mu^2 - 2 \pi_\mu \pi_\lambda \right ) + 2 e^{4\lambda} \left ( 2 \lambda'' + 3 \lambda'^2 - 2 \lambda' \mu' - e^{2(\mu - \lambda)} \right ) \right ] +
\]

\[
\frac{p_\phi^2}{8} e^{-(\mu + 2\lambda)} + 2 \phi'^2 e^{2\lambda - \mu} + \frac{p_\kappa^2}{2} e^{4\phi - \mu - 2\lambda} + \frac{\kappa'^2}{2} e^{-2\phi - \mu + 2\lambda} - \frac{(E^r)^2}{8} e^{-2\phi + \mu - 2\lambda}
\]

\[
N^1 \left ( \pi'_\mu - \pi_\mu \mu' - \pi_\lambda \lambda' + p_\phi \phi' + e^{4\phi} p_\kappa \kappa' \right ) - A_0 e^{-2\phi} \left ( E^r' - 2 \phi' E^r \right ).
\]

Now we proceed as in [11] by imposing the coordinate condition

\[
r = e^\lambda
\]

(52)

and solving for \( \pi_\lambda \) the equation which results after putting the second constraint (the one multiplied by \( N^1 \)) equal to zero. Then \((51)\) is modified as follows

\[
L_{\text{tot}} = -\pi_\mu \dot{\mu} - \pi_\phi \dot{\phi} + e^{4\phi} p_\kappa \dot{\kappa} + e^{-2\phi} E^r \dot{A}_r +
\]

\[
N^1 e^{-\mu} \left [ \frac{1}{r^2} \left [ \frac{1}{8} \pi_\mu^2 - \frac{r}{4} \pi_\mu \left ( \pi'_\mu - \pi_\mu \mu' + p_\phi \phi' + e^{4\phi} p_\kappa \kappa' \right ) \right ] +
\]

\[
2 - 4 r \mu' - 2 e^{2\mu} + \frac{p_\phi^2}{8 r^2} + 2 r^2 \phi'^2 + e^{4\phi} \frac{p_\kappa^2}{2 r^2} + e^{4\phi} r^2 \frac{\kappa'^2}{2} - e^{2(\mu - \phi)} \frac{(E^r)^2}{8 r^2} \right ] - A_0 e^{-2\phi} \left ( E^r'' - 2 \phi' E^r \right ).
\]

We see that \((53)\) has eight variables spanning a curved phase space and two constraints. Now we can apply the I.A. Batalin, E.S. Fradkin and T.E. Fradkina canonical quantization to this model.
4 Canonical quantization of the EMDA system

In order to be compatible with the notation of the Introduction we rename the field variables of (53) in the following way

\[ p_\phi = \Gamma^1, \quad E^r = \Gamma^2, \quad p_\kappa = \Gamma^3, \quad \pi_\mu = \Gamma^4, \]
\[ \phi = \Gamma^5, \quad A_r = \Gamma^6, \quad \kappa = \Gamma^7, \quad \mu = \Gamma^8. \] (54)

Then, the “canonical one–form” of (53) can be written as follows up to a total time derivative

\[ \pi_\phi \dot{\phi} + e^{4\phi} p_\kappa \dot{\kappa} + e^{-2\phi} E^r \dot{A}_r - \pi_\mu \dot{\mu} = a_A(\Gamma) \dot{\Gamma}^A, \quad A = 1, 2, \ldots, 8; \] (55)

where

\[ a_1 = -\frac{1}{2} \Gamma^5, \quad a_2 = -\frac{1}{2} e^{-2\Gamma^5} \Gamma^6, \quad a_3 = \frac{1}{2} e^{4\Gamma^5} \Gamma^7, \quad a_4 = \frac{1}{2} \Gamma^8, \]
\[ a_5 = \frac{1}{2} \left( \Gamma^2 - 4 e^{4\Gamma^3} \Gamma^3 \Gamma^7 + 2 e^{-2\Gamma^5} \Gamma^2 \Gamma^6 \right), \quad a_6 = \frac{1}{2} e^{-2\Gamma^5} \Gamma^2, \quad a_7 = \frac{1}{2} e^{4\Gamma^5} \Gamma^3, \quad a_8 = -\frac{1}{2} \Gamma^4. \] (56)

The symplectic metric is given by the relation

\[ \omega_{AB} = \frac{\partial a_B}{\partial \Gamma^A} - \frac{\partial a_A}{\partial \Gamma^B}. \] (57)

This is an invertible matrix which possesses the following form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-2\Gamma^5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e^{4\Gamma^5} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & -2 e^{-2\Gamma^5} \Gamma^2 & 4 e^{4\Gamma^5} \Gamma^3 & 0 \\
0 & -e^{-2\Gamma^5} & 0 & 0 & 2 e^{-2\Gamma^5} \Gamma^2 & 0 & 0 & 0 \\
0 & 0 & -e^{4\Gamma^5} & 0 & -4 e^{4\Gamma^5} \Gamma^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\] (58)

The expression for the inverse matrix reads

\[
\omega^{AB} = \begin{pmatrix}
0 & -2\Gamma^2 & 4\Gamma^3 & 0 & -1 & 0 & 0 & 0 \\
2\Gamma^2 & 0 & 0 & 0 & 0 & -e^{2\Gamma^5} & 0 & 0 \\
-4\Gamma^3 & 0 & 0 & 0 & 0 & 0 & -e^{-4\Gamma^5} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{2\Gamma^5} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{-4\Gamma^5} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}
\] (59)
We define as well the equal time Poisson bracket for any two functions \( X (\Gamma(r,t), \Gamma'(r,t)) \) and \( Y (\Gamma(r,t), \Gamma'(r,t)) \) as follows

\[
\{ X (\Gamma(r,t), \Gamma'(r,t)) , Y (\Gamma'(r,t), \Gamma'(r,t)) \} = \\
\int dr'' \left[ \frac{\partial}{\partial \Gamma^A(r'', t)} X (\Gamma, \Gamma') \right] \omega^{AB} \left( \Gamma(r'', t) \right) \frac{\partial}{\partial \Gamma^B(r'', t)} Y (\Gamma, \Gamma') ,
\]

where the prime over the \( \Gamma \) denotes derivative with respect to the spatial variable.

There are two irreducible constraints in the action (53), namely

\[
T_1'(\Gamma, \Gamma') = e^{-T^8} \left\{ \frac{1}{8r^2} \left[ (\Gamma^4)^2 - 2r \Gamma^4 \left( \Gamma^4' - \Gamma^4 \Gamma^8' + \Gamma^1 \Gamma^5' + e^{4r^5} \Gamma^3 \Gamma^7' \right) \right] + 2 - 4r \Gamma^8' - \\
2e^{2r^8} + \frac{(\Gamma^1)^2}{8r^2} + 2r^2 \left( \Gamma^5' \right)^2 + e^{4r^5} \left( \frac{\Gamma^3}{2r^2} + e^{4r^5} \frac{r^2 (\Gamma^7')^2}{2} - e^{2(r^8 - r^5)} \frac{(\Gamma^2)^2}{8r^2} \right) \right\}
\]

and

\[
T_2'(\Gamma, \Gamma') = e^{-2r^5} \left( 2\Gamma^2 \Gamma^5' - \Gamma^2' \right).
\]

These are first class constraints as they should be \([13]–[17]\) (see \([9]\) as well) and their Poisson brackets satisfy the following relations:

\[
\{ T_1'(\Gamma(r,t), \Gamma'(r,t)) , T_1'(\Gamma'(r',t), \Gamma'(r',t)) \} = \\
\frac{\Gamma^4(r,t)}{4r} T_1'(\Gamma(r,t), \Gamma'(r,t)) \frac{\partial}{\partial r'} \delta(r' - r) - (r \leftrightarrow r'),
\]

\[
\{ T_1'(\Gamma(r,t), \Gamma'(r,t)) , T_2'(\Gamma'(r',t), \Gamma'(r',t)) \} = 0,
\]

\[
\{ T_2'(\Gamma(r,t), \Gamma'(r,t)) , T_2'(\Gamma'(r',t), \Gamma'(r',t)) \} = 0.
\]

The next step is to determine the reper field \( h^a_A \) from the relations (6) in such a way that the curvature \( \Lambda_{AB}^C \) defined in the relation (10) is zero. This is a very crucial point because it simplifies enormously the forthcoming calculations. One such solution is given by the following matrix

\[
h^a_A(\Gamma) = \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-2r^5} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{4r^5} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -2e^{-2r^5} \Gamma^2 & 4e^{4r^5} \Gamma^3 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
Its inverse matrix has the following form

\[
 h^{A \, a}(\Gamma) = \begin{pmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & e^{2\Gamma^5} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & e^{-4\Gamma^5} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 0 & 2\Gamma^2 & -4\Gamma^3 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 \end{pmatrix}, \quad (67)
\]

and

\[
 \omega^{(0)}_{ab} = \begin{pmatrix}
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 \end{pmatrix}. \quad (68)
\]

It is worth noticing that with this choice of the reper field components $h^{a \, A} = 0$ we have $\Lambda^{C}_{AB} = 0$, a fact which will greatly simplify the computations when deriving the explicit expressions for some quantities. We have checked as well that the covariant derivative of the symplectic metric (58) and the reper field components (66) vanish, i.e., $\nabla_{C} \omega_{AB} = 0$ and $\nabla_{C} h^{a \, A} = 0$.

Now we introduce new bosonic fields $\Pi_{A}(r, t)$, which we promote into operators along with the variables $\Gamma^{A}(r, t)$. We also define the nonzero commutators as follows:

\[
 [\Gamma^{A}(r, t), \Pi_{B}(r', t)] = i\hbar \delta^{A}_{\ B} \delta(r - r'). \quad (69)
\]

This is a slightly modified version of the relation (19) since it involves the Dirac $\delta$–function which is not present in its original definition.

The special second class constraints $\Theta_{A}(\Gamma, \Pi)$, defined in (15), are given by the following expression

\[
 \Theta_{1}(r, t) = \Pi_{1}(r, t) + \frac{1}{2} \Gamma^{5}(r, t), \quad \Theta_{2}(r, t) = \Pi_{2}(r, t) + \frac{1 - e^{-2\Gamma^{5}(r, t)} (1 + 2\Gamma^{5}(r, t))}{4 (\Gamma^{5}(r, t))^{2}} \Gamma^{6}(r, t), \\
 \Theta_{3}(r, t) = \Pi_{3}(r, t) + \frac{1 + e^{4\Gamma^{5}(r, t)} (-1 + 4\Gamma^{5}(r, t))}{16 (\Gamma^{5}(r, t))^{2}} \Gamma^{7}(r, t), \quad \Theta_{4}(r, t) = \Pi_{4}(r, t) - \frac{1}{2} \Gamma^{8}(r, t),
\]

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\[
\Theta_5(r, t) = \Pi_5(r, t) - \frac{1}{2} \Gamma^1(r, t) - \frac{1 - e^{-2\Gamma^5(r, t)} \left(1 + 2\Gamma^5(r, t) + 2(\Gamma^5(r, t))^2\right)}{2(\Gamma^5(r, t))^3} \Gamma^2(r, t)\Gamma^6(r, t) - \\
\frac{1 - e^{4\Gamma^5(r, t)} \left(1 - 4\Gamma^5(r, t) + 8(\Gamma^5(r, t))^2\right)}{8(\Gamma^5(r, t))^3} \Gamma^3(r, t)\Gamma^7(r, t),
\]
\[
\Theta_6(r, t) = \Pi_6(r, t) + \frac{1 - e^{-2\Gamma^5(r, t)} \left(1 + 2\Gamma^5(r, t) + 4(\Gamma^5(r, t))^2\right)}{4(\Gamma^5(r, t))^2} \Gamma^2(r, t),
\]
\[
\Theta_7(r, t) = \Pi_7(r, t) + \frac{1 - e^{4\Gamma^5(r, t)} \left(1 - 4\Gamma^5(r, t) + 16(\Gamma^5(r, t))^2\right)}{16(\Gamma^5(r, t))^2} \Gamma^3(r, t),
\]
\[
\Theta_8(r, t) = \Pi_8(r, t) + \frac{1}{2} \Gamma^4(r, t).
\]

These quantities satisfy the following commutation relations
\[
[\Theta_A(r, t), \Theta_B(r', t)] = i\hbar \omega_{AB}(r, t)\delta(r - r'),
\]
\[
\omega_{AB}(r, t) = \omega_{AB}(0),
\]
as well as the following flat Poisson bracket relations
\[
\{\Theta_A(r, t), \Theta_B(r', t)\}' = \int dr'' \left[ \partial_C \Theta_A(r, t) \partial^C \Theta_B(r', t) - \partial_C \Theta_B(r', t) \partial^C \Theta_A(r, t) \right] = \\
\omega_{AB}(r, t)\delta(r - r'),
\]
where
\[
\partial_C \equiv \frac{\partial}{\partial \Gamma^C(r'', t)}, \quad \partial^C \equiv \frac{\partial}{\partial \Pi_C(r'', t)}.
\]

Next, as prescribed in the Introduction, the second class constraints \(\Theta_A\) are converted into Abelian first class constraints by the introduction of the new bosonic fields \(\Phi_a, a = 1, 2, ..., 8\) which satisfy the following equal time commutation relations
\[
[\Phi_a(r, t), \Phi_b(r', t)] = -i\hbar \omega^{(0)}_{ab}\delta(r - r'),
\]
where \(\omega^{(0)}_{ab}\) is defined as in (68). We are considering the case \(\Lambda^C_{AB} = 0\), thus, from (22) and (23) the first class constraints \(T_A(\Gamma, \Pi, \Phi)\) are given by
\[
T_A(\Gamma, \Pi, \Phi) = \Theta_A(\Gamma, \Pi) + K_A(\Gamma, \Phi) = \Theta_A(\Gamma, \Pi) + \Phi_a \hbar^a A(\Gamma).
\]
Indeed, we can show that these first class constraints $\mathcal{T}_A(\Gamma, \Pi, \Phi)$ are Abelian since
\[
[\mathcal{T}_A(r, t), \mathcal{T}_B(r', t)] = \left[ \Theta_A(r, t) + \Phi_a(r, t)h^aA(r, t), \Theta_B(r', t) + \Phi_b(r', t)h^bB(r', t) \right] =
\]
\[
[\Theta_A(r, t), \Theta_B(r', t)] + \left[ \Phi_a(r, t)h^aA(r, t), \Phi_b(r', t)h^bB(r', t) \right] =
\]
\[\text{(75)}\]
\[
i\hbar \omega_{AB}\delta(r - r') - i\hbar h^aA(\Gamma)\omega^{(0)}_a h^bB(\Gamma)\delta(r - r') = 0.
\]
Now we proceed to construct the operators $\tilde{T}_i'(\Gamma, \Gamma', \Phi, \Phi')$ and $\tilde{\mathcal{T}}_2'(\Gamma, \Gamma', \Phi, \Phi')$ so that their commutator with $\mathcal{T}_A$ vanishes
\[
\left[ \tilde{T}_i'(\Gamma, \Gamma', \Phi, \Phi'), \mathcal{T}_A(\Gamma, \Pi, \Phi) \right] = 0
\]
and
\[
\tilde{T}_i'(\Gamma, \Gamma', 0, 0) = T_i'(\Gamma, \Gamma'), \quad i = 1, 2;
\]
where $T_i'(\Gamma, \Gamma')$ are the first class constraints given in (61) and (62). Here we have to note that the quantities $\tilde{T}_i'$ depend not only on the $\Phi_a$ fields, but also on their space derivatives and this is because of the existence of space derivatives of the phase space variables $\Gamma^A$ in the initial first class constraints $T_1'$ and $T_2'$. By substituting the relation (74) into (76) we obtain
\[
\left[ \tilde{T}_i'(\Gamma(r, t), \Gamma'(r, t), \Phi(r, t), \Phi'(r, t)), \ \Pi_A(r', t) \right] +
\]
\[
\left[ \tilde{T}_i'(\Gamma(r, t), \Gamma'(r, t), \Phi(r, t), \Phi'(r, t)), \ \Phi_a(r', t)h^aA(r', t) \right] = 0.
\]
Due to the relation (69) this equation adopts the following form
\[
\frac{\delta}{\delta \Gamma^A(r', t)} \tilde{T}_i'(\Gamma(r, t), \Gamma'(r, t), \Phi(r, t), \Phi'(r, t)) =
\]
\[
(i\hbar)^{-1} \left[ \Phi_a(r', t)h^aA(r', t), \ \tilde{T}_i'(\Gamma(r, t), \Gamma'(r, t), \Phi(r, t), \Phi'(r, t)) \right].
\]

Let us try to find a solution for equation (78) in the case when the constraint $T_2'(\Gamma, \Gamma')$ is given by the relation (62)
\[
T_2'(\Gamma, \Gamma') = e^{-2\Gamma^5} \left( 2\Gamma^2\Gamma^{5'} - \Gamma^{2} \right).
\]
We shall consider the function $T_2'(\Gamma) = -e^{-2\Gamma^5} (\Gamma^2 - 2\Gamma^2\Gamma^5)$. This relation comes from the expression of $T_2'(\Gamma, \Gamma')$ after ignoring the primes. Now we turn to solve the equation
\[
\frac{\delta}{\delta \Gamma^A(r', t)} \tilde{T}_2'(\Gamma(r, t), \Phi(r, t)) = (i\hbar)^{-1} \left[ \Phi_a(r', t)h^aA(r', t), \ \tilde{T}_2'(\Gamma(r, t), \Phi(r, t)) \right]
\]
\[\text{(79)}\]
The solution of equation (79) is given in the form

\[ \tilde{T}_2(\Gamma(r, t), 0) = T_2(\Gamma(r, t)). \]

The solution of equation (79) is given in the form

\[ \tilde{T}_2(\Gamma(r, t), \Phi(r, t)) = \exp \left( \int dr' \Phi_a(r', t) \frac{\partial}{\partial \Phi_a(r', t)} \right) T_2(\Gamma(x = 0)) |_{\varphi = 0}, \quad (80) \]

where the functions \( \Gamma^4(x) \) are solutions of the equations (28) and are given by the following relations

\[ \begin{align*}
\Gamma^1(x) &= \varphi_5(x - 1) + \Gamma^1, \quad \Gamma^2(x) = e^{-2\varphi_1(x-1)} \left[ \Gamma^2 + \varphi_6 e^{2\Gamma^5}(x - 1) \right], \\
\Gamma^3(x) &= e^{4\varphi_1(x-1)} \left[ \Gamma^3 + \varphi_7 e^{-4\Gamma^5}(x - 1) \right], \quad \Gamma^4(x) = \varphi_8(1 - x) + \Gamma^4, \\
\Gamma^5(x) &= \varphi_1(1 - x) + \Gamma^5, \quad \Gamma^6(x) = \varphi_2(1 - x) + \Gamma^6, \\
\Gamma^7(x) &= \varphi_3(1 - x) + \Gamma^7, \quad \Gamma^8(x) = \varphi_4(1 - x) + \Gamma^8. \end{align*} \quad (81) \]

Thus, from equation (80) we obtain for the solution \( \tilde{T}_2(\Gamma(r, t), \Phi(r, t)) \) of equation (79) the following expression

\[ \tilde{T}_2(\Gamma(r, t), \Phi(r, t)) = -e^{-2(\Phi_1 + \Gamma^5)} \left[ e^{2\Phi_1} \left( \Gamma^2 - \Phi_6 e^{2\Gamma^5} \right) - 2 \left( \Phi_1 + \Gamma^5 \right) e^{2\Phi_1} \left( \Gamma^2 - \Phi_6 e^{2\Gamma^5} \right) \right]. \quad (82) \]

From the relation (82) we can write down a solution of (78) empirically in the form

\[ \tilde{T}_2'(\Gamma(r, t), \Gamma'(r, t), \Phi(r, t), \Phi'(r, t)) = -e^{-2(\Phi_1 + \Gamma^5)} \left\{ \frac{d}{dr} \left[ e^{2\Phi_1} \left( \Gamma^2 - \Phi_6 e^{2\Gamma^5} \right) \right] - 2 \left( \Phi_1 + \Gamma^5 \right) e^{2\Phi_1} \left( \Gamma^2 - \Phi_6 e^{2\Gamma^5} \right) \right\}. \quad (83) \]

It is a remarkable fact that we can establish an analogy between each term of the constraint \( \tilde{T}_2'(\Gamma(r, t), \Gamma'(r, t), \Phi(r, t), \Phi'(r, t)) \) of the equation (83) and each term of the constraint \( T_2'(\Gamma, \Gamma') \) of the equation (62). The simplified expression for the equation (83) reads

\[ \tilde{T}_2'(\Gamma, \Gamma', \Phi, \Phi') = -e^{-2\Gamma^5} \left( \Gamma^2 - \Phi_6 e^{2\Gamma^5} - 2\Gamma^5 \Gamma^2 \right). \quad (84) \]

Let us check now that the expression (84) satisfies the equation (78). For instance we can compute the following relationship

\[ \frac{\delta}{\delta \Gamma^2(r', t)} \tilde{T}_2'(\Gamma(r, t), \Gamma'(r, t), \Phi(r, t), \Phi'(r, t)) = -e^{-2\Gamma^5(r, t)} \left[ \frac{\partial}{\partial r} \delta(r - r') - 2\Gamma^5(r, t) \delta(r - r') \right]. \quad (85) \]
On the other hand we have
\[ [\Phi_2(r', t)h^2_2(r', t), \bar{T}_2'(\Gamma(r, t), \Gamma'(r, t), \Phi(r, t), \Phi'(r, t))] = e^{-2t^3(r', t)} [\Phi_2(r', t), \Phi'_2(r, t)] = \]
\[ -\hbar e^{-2t^3(r', t)} \frac{\partial}{\partial r} \delta(r - r') = -\hbar e^{-2t^3(r', t)} \left[ \frac{\partial}{\partial r} \delta(r - r') - 2t^5'(r, t) \delta(r - r') \right], \quad (86) \]

where we have made use of the relations (102) and (103) of the Appendix A. It is easy to see that (84) satisfies the equation (78) in all cases and also that the boundary condition \( \bar{T}_2'(\Gamma, \Gamma', 0, 0) = T_2'(\Gamma, \Gamma') \) is satisfied.

We turn now to solve the equation (78) for the case when the constraint \( T_1'(\Gamma, \Gamma') \) defined in (61). In order to do that we shall follow the same procedure as we did with the constraint \( T_2'(\Gamma, \Gamma') \). Let us consider the function
\[ T_1(\Gamma) = e^{-t^8} \left\{ \frac{1}{8r^2} \left[ (\Gamma^4)^2 - 2r \Gamma^4 (\Gamma^4 - \Gamma^8 + \Gamma^4 \Gamma^5 + e^{4t^5} \Gamma^3 \Gamma^7) \right] + 2 - 4r \Gamma^8 - 2e^{2t^8} + \frac{(\Gamma^1)^2}{8r^2} + \frac{2r^2 (\Gamma^5)^2}{2r^2} + \frac{r^2}{2} e^{4t^5} (\Gamma^7)^2 - e^{2(t^8 - t^5)} \frac{(\Gamma^2)^2}{8r^2} \right\}. \quad (87) \]

As in the previous case, the expression for \( T_1(\Gamma) \) does come from the constraint \( T_1'(\Gamma, \Gamma') \) when ignoring the primes. Thus, in order to solve the following equation
\[ \frac{\delta}{\delta A_A(r', t)} \bar{T}_1(\Gamma(r, t), \Phi(r, t)) = (ih)^{-1} \left[ \Phi_A(r', t)h_{A}(r', t), \bar{T}_1(\Gamma(r, t), \Phi(r, t)) \right] \quad (88) \]

with
\[ \bar{T}_1(\Gamma(r, t), 0) = T_1(\Gamma(r, t)), \]

we need the expression for \( T_1(\Gamma(x = 0)) \) which is given by
\[ T_1(\Gamma(x = 0)) = e^{-t^7(\varphi + i\Gamma^8)} \left\{ \frac{1}{8r^2} \left[ (\varphi_8 + \Gamma^4)^2 - 2r (\varphi_8 + \Gamma^4) [(\varphi_8 + \Gamma^4) - \right] \]
\[ (\varphi_8 + \Gamma^4) (\varphi_4 + \Gamma^8) + (\varphi_5 + \Gamma^1) (\varphi_1 + \Gamma^5) + e^{4t^5} (\Gamma^3 - \Gamma^7 e^{-4t^5}) (\varphi_3 + \Gamma^7) \right] + \]
\[ 2 - 4r (\varphi_4 + \Gamma^8) - 2e^2(\varphi + i\Gamma^8) + \frac{(-\varphi_5 + \Gamma^1)^2}{8r^2} + 2r^2 (\varphi_1 + \Gamma^5)^2 + \]
\[ \frac{e^4(-\varphi + i\Gamma^5)}{2r^2} \left( \Gamma^3 - \varphi_7 e^{-4t^5} \right)^2 + e^4(\varphi + i\Gamma^5) \frac{r^2}{2} (\varphi_3 + \Gamma^7)^2 - e^2(\varphi + \varphi + i\Gamma^8 - \Gamma^5) \frac{(\Gamma^2 - \varphi_6 e^{4t^5})^2}{8r^2} \right\}. \quad (89) \]
We are looking for a solution of the equation (88) in the following form
\[
\tilde{T}_1(\Gamma(r,t), \Phi(r,t)) = \exp \left( \int dr' \Phi_a(r', t) \frac{\partial}{\partial \Phi_a(r', t)} \right) T_1(\Gamma(x = 0)) |_{\varphi = 0} .
\]  

(90)

There is one technical problem here: not all of the \( \Phi_a \) fields commute. We shall avoid this difficulty by making use of the M. Suzuki’s formula [18], that we quote in (104), in order to put these fields in a symmetric ordering. Thus, we obtain the following result:
\[
\tilde{T}_1(\Gamma(r,t), \Phi(r,t)) = \sum_{i=1}^{8} \tilde{T}_{1i}(\Gamma(r,t), \Phi(r,t)),
\]

(91)

where
\[
\tilde{T}_{11}(\Gamma, \Phi) = \frac{1 - 2r}{8r^2} \left[ (\Gamma^4 \Gamma^4) e^{-(\Phi_a + \Gamma^8)} + \frac{1}{2} \Phi_8 e^{-(\Phi_a + \Gamma^8)} \Phi_8 + \Gamma^4 \Phi_8 e^{-(\Phi_a + \Gamma^8)} + e^{-(\Phi_a + \Gamma^8)} \Phi_8 \right],
\]
\[
\tilde{T}_{12}(\Gamma, \Phi) = \frac{1}{4r} \left[ (\Gamma^4)^2 C + \Gamma^4 (\Phi_8 C + C \Phi_8) + \frac{1}{2} (\Phi_8 C \Phi_8) + (\Phi_8 C \Phi_8) + C \Phi_8^2 \right],
\]

where \( C = (\Phi_4 + \Gamma^8) e^{-(\Phi_4 + \Gamma^8)} \);
\[
\tilde{T}_{13}(\Gamma, \Phi) = -\frac{1}{4r} \left[ \Gamma^4 e^{-(\Phi_4 + \Gamma^8)} + \frac{1}{2} \left( e^{-(\Phi_4 + \Gamma^8)} \Phi_8 + \Phi_8 e^{-(\Phi_4 + \Gamma^8)} \right) \right] \times \left[ \Gamma^1 \Gamma^5 + (\Gamma^4 \Phi_1 - \Gamma^5 \Phi_5) - \frac{1}{2} (\Phi_1 \Phi_5 + \Phi_5 \Phi_1) \right],
\]
\[
\tilde{T}_{14}(\Gamma, \Phi) = -\frac{1}{4r} e^{4\Gamma^5} \left[ \Gamma^4 e^{-(\Phi_4 + \Gamma^8)} + \frac{1}{2} \left( e^{-(\Phi_4 + \Gamma^8)} \Phi_8 + \Phi_8 e^{-(\Phi_4 + \Gamma^8)} \right) \right] \times \left[ \Gamma^3 \Gamma^7 + (\Gamma^3 \Phi_3 - \Gamma^7 \Phi_3) - \frac{1}{2} (\Phi_7 \Phi_3 + \Phi_3 \Phi_7) \right],
\]
\[
\tilde{T}_{15}(\Gamma, \Phi) = e^{-(\Phi_4 + \Gamma^8)} \left[ 2 - 2 \Phi_a - (\Phi_a)^2 \right] + \frac{1}{2 \Gamma^2} \left( \Phi_5 - \Gamma^1 \right)^2 - \frac{1}{8r^2} e^{2(\Phi_4 + \Gamma^8)} \Phi_8 e^{2\Gamma^5} + \frac{1}{8r^2} e^{-(\Phi_4 + \Gamma^8)} \left( \Gamma^2 - \Phi_6 e^{2\Gamma^5} \right)^2 + \frac{1}{2 \Gamma^2} e^{-(\Phi_4 + \Gamma^8)} \left( \Gamma^3 - \Phi_7 e^{2\Gamma^5} \right)^2,
\]

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\[
\tilde{T}_{16}(\Gamma, \Phi) = -4r (\Phi_4 + \Gamma^8) e^{-(\Phi_4 + \Gamma^8)},
\]
\[
\tilde{T}_{17}(\Gamma, \Phi) = 2r^2 e^{-(\Phi_4 + \Gamma^8)} (\Phi_1 + \Gamma^5)^2,
\]
\[
\tilde{T}_{18}(\Gamma, \Phi) = \frac{r^2}{2} e^{-(\Phi_4 + \Gamma^8)} e^{4(\Phi_4 + \Gamma^5)} (\Phi_3 + \Gamma^7)^2.
\]

Each one of the expressions \(\tilde{T}_{1i}(\Gamma, \Phi)\) satisfies the equation (88). Now as in the previous case, we make use of the expression for \(\tilde{T}_{1i}(\Gamma, \Phi)\) in order to write down a solution for the equation (78). Thus, the solution is given by the following relations

\[
\tilde{T}_1'(\Gamma'(r, t), \Gamma'(r, t), \Phi'(r, t), \Phi'(r, t)) = \sum_{i=1}^{8} \tilde{T}_{1i}'(\Gamma(r, t), \Gamma'(r, t), \Phi(r, t), \Phi'(r, t)) ,
\]

where

\[
\tilde{T}_{11}'(\Gamma, \Gamma', \Phi, \Phi') = \frac{1}{8r^2} \left[ (\Gamma')^2 e^{-(\Phi_4 + \Gamma^8)} + \frac{1}{2} \Phi_8 e^{-(\Phi_4 + \Gamma^8)} \Phi_8 + 
\Gamma^4 \left( \Phi_8 e^{-(\Phi_4 + \Gamma^8)} + e^{-(\Phi_4 + \Gamma^8)} \Phi_8 \right) + \frac{1}{4} \left( \Phi_8^2 e^{-(\Phi_4 + \Gamma^8)} + e^{-(\Phi_4 + \Gamma^8)} \Phi_8^2 \right) \right] - 
\frac{1}{4r} \left[ \Gamma^4 \Gamma' e^{-(\Phi_4 + \Gamma^8)} + \frac{1}{4} \left( \Phi_8' e^{-(\Phi_4 + \Gamma^8)} \Phi_8 + \Phi_8 e^{-(\Phi_4 + \Gamma^8)} \Phi_8' \right) + 
\frac{\Gamma'}{2} \left( \Phi_8 e^{-(\Phi_4 + \Gamma^8)} + e^{-(\Phi_4 + \Gamma^8)} \Phi_8 \right) + \frac{\Gamma^4}{2} \left( \Phi_8 e^{-(\Phi_4 + \Gamma^8)} + e^{-(\Phi_4 + \Gamma^8)} \Phi_8' \right) + 
\frac{1}{4} \left( \Phi_8' \Phi_8 e^{-(\Phi_4 + \Gamma^8)} + e^{-(\Phi_4 + \Gamma^8)} \Phi_8' \Phi_8 \right) \right],
\]

\[
\tilde{T}_{12}'(\Gamma, \Gamma', \Phi, \Phi') = \frac{1}{4r} \left[ (\Gamma')^2 C_1 + \Gamma^4 (\Phi_8 C_1 + C_1 \Phi_8) + \frac{1}{2} \left( \Phi_8 C_1 \Phi_8 + \Phi_8^2 C_1 + C_1 \Phi_8^2 \right) \right],
\]

where \(C_1 = (\Phi_4 + \Gamma^8) e^{-(\Phi_4 + \Gamma^8)}\);

\[
\tilde{T}_{13}'(\Gamma, \Gamma', \Phi, \Phi') = -\frac{1}{4r} \left[ \Gamma^4 e^{-(\Phi_4 + \Gamma^8)} + \frac{1}{2} \left( e^{-(\Phi_4 + \Gamma^8)} \Phi_8 + \Phi_8 e^{-(\Phi_4 + \Gamma^8)} \right) \right].
\]
\[
\left[ \Gamma^1 \Gamma^5' + \left( \Gamma^1 \Phi' - \Gamma^5' \Phi_5 \right) - \frac{1}{2} (\Phi'_5 \Phi_5 + \Phi_5 \Phi'_5) \right],
\]

\[
\overline{T}'_{14} (\Gamma, \Gamma', \Phi, \Phi') = -\frac{1}{4r} e^{4r^5} \left[ \Gamma^4 e^{-(\Phi_4 + \Gamma^8)} + \frac{1}{2} (e^{-(\Phi_4 + \Gamma^8)} \Phi_8 + \Phi_8 e^{-(\Phi_4 + \Gamma^8)}) \right] \times
\]

\[
\left[ \Gamma^3 \Gamma^7' + \left( \Gamma^3 \Phi_3' - e^{-4r^5} \Gamma^7' \Phi_7 \right) - \frac{1}{2} e^{-4r^5} (\Phi_7 \Phi'_3 + \Phi'_3 \Phi_7) \right],
\]

\[
\overline{T}'_{15} (\Gamma, \Gamma', \Phi, \Phi') = \overline{T}_{15} (\Gamma, \Phi),
\]

\[
\overline{T}'_{16} (\Gamma, \Gamma', \Phi, \Phi') = -4r (\Phi_4' + \Gamma^8') e^{-(\Phi_4 + \Gamma^8)},
\]

\[
\overline{T}'_{17} (\Gamma, \Gamma', \Phi, \Phi') = 2r^2 e^{-(\Phi_4 + \Gamma^8)} (\Phi'_1 + \Gamma^5')^2,
\]

\[
\overline{T}'_{18} (\Gamma, \Gamma', \Phi, \Phi') = \frac{r^2}{2} e^{-(\Phi_4 + \Gamma^8)} e^{4(\Phi_4 + \Gamma^8)} (\Phi'_3 + \Gamma^7')^2.
\]

Now we are in position to calculate the fermion generating operator \( \Omega \). In order to achieve this aim, let us recall that we have the following set of irreducible first class constraints

\[
\overline{T}_a^\prime \equiv (\tilde{T}'_a, T'_A), \quad a = 1, 2; \quad A = 1, 2, ..., 8; \quad (93)
\]

whose expressions are given in the equations (74), (84) and (91). We put into correspondence with the complete set of irreducible first class constraints (93) the following ordered operator pairs

\[
\begin{pmatrix}
\lambda^a(t, r), & \pi^a(t, r), \\
\lambda^A(t, r), & \pi_A(t, r)
\end{pmatrix} \longrightarrow
\begin{pmatrix}
\mathcal{C}^a(t, r), & \mathcal{P}_{a}'(t, r), \\
\mathcal{C}^A(t, r), & \mathcal{P}_{A}'(t, r)
\end{pmatrix}, \quad
\begin{pmatrix}
\mathcal{P}^a(t, r), & \mathcal{C}_{a}'(t, r), \\
\mathcal{P}^A(t, r), & \mathcal{C}_{A}'(t, r)
\end{pmatrix}; \quad (94)
\]

where the active Lagrange multipliers read \( \lambda^1' = N^1, \lambda^2' = A_0 \). The ghost numbers and the statistics of these magnitudes read

\[
gh(C) = gh(\mathcal{P}) = 1, \quad gh(C) = gh(\mathcal{P}) = -1,
\]

\[
\varepsilon(\lambda) = \varepsilon(\pi) = 0, \quad \varepsilon(C) = \varepsilon(C) = \varepsilon(\mathcal{P}) = \varepsilon(\mathcal{P}) = 1.
\]

The following supercommutators are nonzero

\[
[\lambda^j(t, r), \pi_j(t', r)] = i\hbar \delta(r - r'),
\]

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\[ [C^i(r, t), \overline{P}_j(r', t)] = i\hbar \delta(r - r'), \quad (95) \]

\[ [\mathcal{P}^i(r, t), \overline{C}_j(r', t)] = i\hbar \delta(r - r'), \quad (96) \]

while all remaining supercommutators of these quantities vanish, i.e., \( [C^i(r, t), C^j(r', t)] = 0, \]
\( [\mathcal{P}^i(r, t), \overline{P}_j(r', t)] = 0, \) etc., where \( i, j = a, A. \)

We have assumed as well that the following relations take place

\[ C^i(r, t)C^a(r', t)\delta(r - r') = \overline{C}_i(r, t)C_i(r', t)\delta(r - r') = 0, \]

\[ \mathcal{P}^i(r, t)\mathcal{P}^i(r', t)\delta(r - r') = \overline{\mathcal{P}}_i(r, t)\overline{\mathcal{P}}_i(r', t)\delta(r - r') = 0. \]

Thus, the fermion generating operator \( \Omega \) adopts the form given by the expression (32)

\[ \Omega = \Omega'(\Gamma, \Lambda, \Phi, \Phi', C', \overline{P}) + \mathcal{T}_A(\Gamma, \Pi, \Phi)C^A + \Pi^a_a\mathcal{P}^a + \pi_A C^A, \quad a = 1, 2; \quad A = 1, 2, \ldots 8; \quad (97) \]

where the the Fermi operator \( \Omega' \) obeys the following relations \([\Omega, \Omega'] = [\Omega', \mathcal{T}_A] = 0, \) possesses the ghost number \( g\hbar(\Omega') = 1, \) and satisfies the boundary condition \( \Omega'(\Gamma, \Lambda, \Phi, \Phi', C', 0) = \overline{T}_a'(\Gamma, \Lambda, \Phi, \Phi')C^a. \)

We try to find the solution for the equation (97) in the following form (here we use a Weyl basis)

\[ \Omega' = \overline{T}_a'(\Gamma, \Phi, \Phi')C^a + \overline{U}^a_a(\Gamma, \Phi, \Phi') \left[ \overline{P}_a C^{ac} \left( \frac{\partial}{\partial r} C^{cb} \right) + \left( \frac{\partial}{\partial r} C^{ab} \right) \overline{P}_a C^{cb} + C^{ac} \left( \frac{\partial}{\partial r} C^{cb} \right) \overline{P}_a \right]. \quad (98) \]

By keeping only terms with lowest number of ghost (2–ghost) we obtain

\[ [\Omega, \Omega'] = \left[ \overline{T}_a'(r, t), \overline{T}_b'(r', t) \right] C^{ab}(r, t) C^{bc}(r', t) + \]

\[ 6i\hbar U^a_{bc}(r, t) \overline{T}_a'(r, t) C^{ac}(r, t) \left( \frac{\partial}{\partial r} C^{bc}(r, t) \right) \delta(r - r') + \]

\[ 9\hbar^2 \delta(r - r') \left[ U^i_{bj}(r, t), U^j_{ij}(r', t) \right] \left( \frac{\partial}{\partial r} C^{bc}(r, t) \right) C^{ef}(r', t) + \]

\[ 4 - \text{ghost terms} + 6 - \text{ghost terms}. \]

In our case the following commutation relations for the constraints \( \overline{T}_a'(r, t) \) take place

\[ \left[ \overline{T}_1'(r, t), \overline{T}_2'(r', t) \right] = 0, \quad \text{while the commutator } \left[ \overline{T}_1'(r, t), \overline{T}_1'(r', t) \right] \quad \text{can be written in the form} \]

\[ \left[ \overline{T}_1'(r, t), \overline{T}_1'(r', t) \right] = i\hbar F_1(r) \frac{\partial}{\partial r'} \delta(r' - r) + i\hbar F_2(r)$ \delta(r - r'). \quad (100) \]
Thus, we have

\[
\left[ \tilde{T}_a^r(r, t), \tilde{T}_b^{r'}(r', t) \right] C^{\mu a}(r, t)C^{\nu b}(r', t) = i\hbar F_1(r)C^{\alpha 1}(r, t)C^{\alpha 1}(r', t) \frac{\partial}{\partial r'} \delta(r' - r) =
\]

\[
-\hbar F_1(r)C^{\mu a}(r, t) \left( \frac{\partial}{\partial r} C^{\alpha 1}(r, t) \right) \delta(r - r'). \tag{101}
\]

The exact form of the function \( F_1(r) \) and possible solutions of (97) are currently under investigation.

5 Conclusions

The canonical quantization of dynamical systems introduced by I.A. Batalin and E.S. Fradkin in [1] and I.A. Batalin, E.S. Fradkin and T.E. Fradkina in [2] is applied to the four-dimensional Einstein–Maxwell Dilaton–Axion theory.

By performing an ADM decomposition of the metric and considering the spherically symmetric anzats with radial fields, the total Lagrangian density of the theory (gravity coupled to matter fields) is written as an expression first order in time derivatives of the fields which defines a curved phase space with two irreducible first class constraints. Thus, the canonical quantization method mentioned above can be applied to this effective system. However, in order to achieve this goal, some generalizations of the method have been performed.

This paper actually constitutes the first part of our investigation because the explicit form of the fermion generating operator \( \Omega \) and the total unitarizing Hamiltonian of the theory \( H \) are not included. This is the subject of our current research activity.

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Appendix A

For any continuous differential function \( f(z) \) the following identity holds

\[
f(z)\delta'(z - z_0) = f(z_0)\delta'(z - z_0) - f'(z_0)\delta(z - z_0),
\]

where now the prime denotes derivatives with respect to the variable \( z \).

The space derivative of the commutator of two \( \Phi \) fields is given by the following relation

\[
[\Phi_a(r, t), \Phi'_b(r', t)] = -i\hbar \omega_{ab}^{(0)} \frac{\partial}{\partial r'} \delta(r' - r).
\]

One of the identities of Ref. [18] is useful for the case we are considering, namely, the following one

\[
e^{\lambda(A + B)} = e^{(\lambda/2)A} e^{\lambda B} e^{(\lambda/2)A} + \frac{1}{2} \int_0^\lambda dt \int_0^t ds \ e^{(t/2)A} e^{tB} G(s) e^{(t/2)A} e^{(\lambda-t)(A+B)},
\]

where

\[
G(s) = \int_0^s du \left\{ \frac{1}{2} e^{(u/2)A} [A, [A, B]] e^{-(u/2)A} + e^{-uB} [B, [A, B]] e^{uB} \right\}.
\]

In our case the function \( G(s) = 0 \). We also set \( \lambda = 1 \). Finally, we should point out that this formula is used just in the case of noncommuting \( \Phi_a \) fields.
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