QUANTUM DIFFERENTIAL OPERATORS ON THE QUANTUM PLANE

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Abstract. The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ acts on its representation ring $R$ through $D(R)$, the ring of differential operators on $R$. A quantised universal enveloping algebra (or quantum group) is a deformation of a universal enveloping algebra and acts not through the differential operators of its representation ring but through the quantised differential operators of its representation ring. We present this situation for the quantum group of $sl_2$.

0. Introduction

Let $q$ be a transcendental element over $\mathbb{Q}$, and let $k$ be a field extension of $\mathbb{Q}(q)$ containing $\sqrt{q}$. Let $U_q$ denote the quantum group corresponding to the Lie algebra $sl_2(k)$. Let $R = k\langle x, y \rangle / xy - qyx$. We will call $R$ the coordinate ring of the quantum plane or sometimes just the quantum plane. This ring $R$ is a representation ring of $U_q$; that is, every type-1, irreducible, finite dimensional representation of $U_q$ appears in $R$ exactly once. Hence, $U_q$ acts on the quantum plane. This action is through the quantum- (or $q$-) differential operators (Section 3.3 of [LR1]). The weight space of $U_q$ is $\mathbb{Z}$, the group of integers. Thus $R$ is $\Gamma = \mathbb{Z} \times \mathbb{Z}$-graded as $\deg(x) = (1, 1)$ and $\deg(y) = (-1, 1)$. This corresponds to the fact that $x$ (resp. $y$) can be seen as the highest (resp. lowest) weight vector of weight 1 (resp. -1) of the unique, type-1, simple, 2-dimensional module.

The ring $R$ is also graded by the subgroup $\Lambda$ generated by $(1, 1)$ and $(-1, 1)$. In this paper we compute the ring (denoted by $D^\Lambda_q(R)$) of $q$-differential operators of $R$ viewing $R$ as $\Lambda$-graded, and study its properties. Note that the ring $D^\Lambda_q(R)$ differs

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from the ring $D^\Gamma_q(R)$ of $q$-differential operators on $\Gamma$-graded $R$. We shall address this in Section 1.

In Section 1, we give the requisite preliminaries. Section 2 deals with the description of first order $q$-differential operators on $\Gamma$-graded $R$. In Section 3 we find the generators of $D^\Lambda_q(R)$ explicitly. In Section 4 we describe some basic properties of $D^\Lambda_q(R)$ as a ring. In particular, we show that $D^\Lambda_q(R)$ is isomorphic to $D_q(k[t]) \otimes_k D_q(k[t])$ as rings (the ring $D_q(k[t])$ has been studied in [IM]) which is simple by Proposition 4.0.1. We also show that $D^\Lambda_q(R)$ is a domain. In Section 5 we explain the relevance of $U_q$ with $D^\Lambda_q(R)$. As an application, we show that any irreducible, finite dimensional $U(sl_2(k))$-module can be given a type-1, simple $U_q$-module structure where $U(sl_2(k))$ denotes the enveloping algebra of $sl_2(k)$ (Corollary 5.9.1).

We would like to acknowledge the formulae derived in the Mathematical Physics literature (see [DP], [D]), which have been rederived here.

1. Some definitions

In this section we recall some basic definitions from Section 3 of [LR1]. Let $G$ be an abelian group, and $S$ be a $G$-graded $k$-algebra. We fix a bicharacter $\beta : G \times G \to k^*$. The ring of $G$-graded and $k$-linear homomorphisms, $\text{grHom}_k(S, S)$ is an $S$-bimodule given by $(\varphi \cdot r)(s) = \varphi(rs)$ and $(r \cdot \varphi)(s) = r \varphi(s)$. For homogeneous $r$ of degree $a$, we let $[\varphi, r]_g = \varphi \cdot r - \beta(g, a) r \cdot \varphi$. For $g = 0$ the identity element, we write $[\varphi, r]_0$ simply as $[\varphi, r]$. This notation can be immediately generalised for a homogeneous $\psi$ of degree $a$ (and hence for any graded $\psi$) as $[\varphi, \psi]_g = \varphi \psi - \beta(g, a) \psi \varphi$. Again we let $[\varphi, \psi]$ denote $[\varphi, \psi]_0$. The graded $S$-bimodule $D^{n,G}_q$ or simply $D^n$ (where there is no confusion of the underlying group) is inductively defined as follows: We let $D^n = 0$ for $n < 0$. For each $n \geq 0$, we let $D^n$ be the $S$-bimodule generated by the set

$$Z_{n,q} = \{ \text{homogeneous } \varphi \in \text{grHom}_k(S, S) \mid \text{there is some } g \in G \text{ such that } [\varphi, r]_g \in D^{n-1} \text{ for all homogeneous } r \in S \}.$$
The $R$-bimodule $D_q^G(S) = \bigcup D^n$ is a ring since $D^i D^j \subset D^{i+j}$. This implies that $D^0$ is a ring, and each $D^n$ is a $D^0$-module.

For our ring $R$, we consider the bicharacter

$$\beta : \Gamma \times \Gamma \to \mathbb{k}^*, \quad ((a, b), (m, n)) \mapsto q^{\frac{am + bn}{2}}.$$ 

The grading action of $\Gamma$ on $R$ is the map which assigns to $(a, b) \in \Gamma$ the automorphism of $R$

$$\sigma_{(a,b)}(x^n y^m) = \beta((a, b), \deg(x^n y^m)) x^n y^m = q^{\frac{a(n-m) + b(n+m)}{2}} x^n y^m.$$ 

The set of all such grading homomorphisms will be denoted $\sigma_\Gamma$. These are always $q$-differential operators of order 0. In order to make this exposition more readable, we will sometimes write $\sigma_x$ for $\sigma_{(1,1)}$ and $\sigma_y$ for $\sigma_{(-1,1)}$.

**Definition.** For each $r \in R$, we define two multiplication homomorphisms $\lambda_r$ and $\rho_r$ in $\text{grHom}_\mathbb{k}(R, R)$ by

$$\lambda_r(s) = rs,$$

$$\rho_r(s) = sr.$$ 

When a ring is commutative, these two homomorphisms coincide. The ring $R$ is very close to being commutative in the following sense:

$$\lambda_x = \rho_x \sigma_y,$$

$$\lambda_y = \rho_y \sigma_x^{-1}.$$ 

Since $\sigma_\Gamma$ and these multiplication maps generate all $q$-differential operators of order 0, we have the following lemma.

**Lemma 1.0.1.** The $q$-differential operators of order 0 on $\Gamma$-graded $R$, are generated by $\sigma_\Gamma$, $\lambda_x$, and $\rho_y$. 

We have a corresponding Lemma for the $\Lambda$-graded ring: the ring $D^0_\sigma$ is generated by $\sigma$, $\lambda_x$, and $\rho_y$. This leads us to the general statement that for each $n \geq 0$, we have

$$D^n_\Lambda \subset D^n_\Gamma.$$ 

Before proceeding, we need some notation. For any integers $n$ and $a$ where $a \neq 0$, we put

$$[n]_a = \frac{q^n - 1}{q - 1} = 1 + q^a + q^{2a} + \cdots + q^{(n-1)a}.$$ 

Denote $[n]_1$ simply by $[n]$, and define $[n]_0 = n$.

**2. The first order $q$-differential operators**

We will define a family of homomorphisms which fill the role of the derivations on $R$ in the context of ordinary differential operators.

**Definition.** Let $a$ and $b$ be integers. Define $\partial^{\sigma_a}_x$ and $\partial^{\sigma_b}_y$ by

$$\partial^{\sigma_a}_x(x^i y^j) = [i]_a x^{i-1} y^j,$$
$$\partial^{\sigma_b}_y(x^i y^j) = [j]_b x^i y^{j-1}.$$ 

When either $a$ or $b$ is 1, we will omit it as a superscript. We will simply write $\partial_x$ and $\partial_y$ for $\partial^{\sigma_0}_x$ and $\partial^{\sigma_0}_y$ respectively.

We note the following

$$[\partial^{\sigma_a}_x, \rho_y] = 0,$$ 
$$[\partial^{\sigma_a}_x, \lambda_x] = \sigma^{a}_x,$$ 
$$[\partial^{\sigma_b}_y, \lambda_x] = 0,$$ 
$$[\partial^{\sigma_b}_y, \rho_y] = \sigma^{b}_y.$$ 

Here, $\sigma^{a}_x, \sigma^{b}_y$ denote $(\sigma_x)^a, (\sigma_y)^b$ respectively. This shows that $\partial^{\sigma_a}_x$ and $\partial^{\sigma_b}_y$ are $q$-differential operators of order 1. Furthermore, for $a, b \geq 2$

$$\partial^{\sigma_a}_x = \left(\frac{1 - q}{1 - q^{a}}\right) \partial_x^\sigma (1 + \sigma_x + \cdots + \sigma^{a-1}_x),$$
$$\partial^{\sigma_b}_y = \left(\frac{1 - q}{1 - q^{b}}\right) \partial_y^\sigma (1 + \sigma_y + \cdots + \sigma^{b-1}_y).$$
For $a, b \leq -1$, we have $\partial_x^{\alpha_a} = \sigma_x^{\alpha_a} \partial_x^{\alpha_a}$ and $\partial_y^{\beta_b} = \sigma_y^{\beta_b} \partial_x^{\beta_b}$. So with only $k, \sigma, \Lambda$ and the four homomorphisms $\partial_x, \partial_y, \partial_x^{\beta},$ and $\partial_y^{\beta}$, we may generate all other $\partial_x^{\alpha_a}$'s and $\partial_y^{\beta_b}$'s. We now prove the main theorem of this section.

**Theorem 2.0.1.** The $D^{0, \Gamma}$-module $D^{1, \Gamma}$, of first order $q$-differential operators is generated by $\{1, \partial_x^{\beta}, \partial_y^{\beta}, \partial_x, \partial_y\}$.

**Proof.** In this proof, we let $D^0 = D^{0, \Gamma}$ and $D^1 = D^{1, \Gamma}$. The $R$-bimodule $D^1$ of first order $q$-differential operators is generated as a $D^0$-module by homogeneous $\varphi \in \text{grHom}_k(R, R)$ such that $[\varphi, \lambda_r] \in D^0$ for any $r \in R$ (see Corollary 1.2.1 of [IM]). Since $R$ is generated by the two elements $x$ and $y$, it is enough to consider those homogeneous $\varphi$ such that $[\varphi, \lambda_x]$ and $[\varphi, \lambda_y]$ are in $D^0$. We need to show that $\varphi$ is in the $D^0$ span of the $\partial_x^{\alpha_a}$'s and $\partial_y^{\beta_b}$'s.

Since $\rho_{\varphi(1)}$ is in $D^0$, we can replace $\varphi$ with $\varphi - \rho_{\varphi(1)}$ and so assume that $\varphi(1) = 0$. We consider the following cases separately:

1. If $\varphi$ is a first order $q$-differential operator such that

   
   \begin{align*}
   [\varphi, \lambda_x] &= 0, \\
   [\varphi, \lambda_y] &= \sigma_{(c,d)},
   \end{align*}

then $\varphi(xy) = q \varphi(yx)$ implies that $d = -c - 2$. Call such pairs $(c, -c - 2) \in \Gamma$ type C1. Since $\varphi(1) = 0$, we can conclude

$$\varphi(x^n y^m) = [m]_b x^n y^{m-1},$$

where $b = \frac{-c+d}{2} = -(c+1)$. Hence $\varphi = \partial_y^{\beta_b}$.

Similarly, if $\varphi$ is a first order $q$-differential operator such that

\begin{align*}
[\varphi, \lambda_x] &= \sigma_{(c,d)}, \\
[\varphi, \lambda_y] &= 0,
\end{align*}

then $\varphi(xy) = q \varphi(yx)$ implies that $d = c + 2$. Call such pairs $(c, c + 2) \in \Gamma$ type C2. Here again we can conclude

$$\varphi(y^n x^m) = [m]_a y^n x^{m-1}.$$
where $a = c + 1$. Thus we have $\varphi = \partial_x^{\gamma a}$.

2. Let $\varphi$ be homogeneous such that

$$[\varphi, \lambda_x] = 0,$$

$$[\varphi, \lambda_y] = \rho_s \sum_i \alpha_i \sigma(c_i, d_i),$$

for $\alpha_i \in k$. As $[\rho_s, \lambda_r] = 0$ for $r \in R$, we have $[\rho_s \varphi', \lambda_r] = \rho_s[\varphi', \lambda_r]$ for any $r \in R$ and $\varphi' \in \text{grHom}_k(R, R)$. Hence, we can assume without loss of generality that $s = 1$. This implies that

$$\varphi(x^n y^m) = \left( \sum_i \alpha_i [m]_{b_i} \right) x^n y^{m-1}$$

where $b_i = \frac{c_i + d_i}{2}$. It follows that $[\varphi, \lambda_y](x^n y^m) = \frac{1}{q^n}(\sum_i \alpha_i q^{mb_i}) x^n y^m$. Now we can use the fact that $[\varphi, \lambda_y] = \sum_i \alpha_i \sigma(c_i, d_i)$ to obtain

$$\frac{1}{q^n} \sum_i \alpha_i q^{mb_i} = \sum_i \alpha_i q^{(\frac{c_i+d_i}{2})} q^{mb_i}.$$ 

Let $A_a = \{ i \mid \frac{c_i + d_i}{2} = a \text{ and } \alpha_i \neq 0 \}$. We have,

$$\sum_a \left( \sum_{i \in A_a} \alpha_i q^{mb_i} \right) = \sum_a q^{n(a+1)} \left( \sum_{i \in A_a} \alpha_i q^{mb_i} \right)$$

for all nonnegative integers $m$ and $n$. This further implies that

$$\sum_{a \neq -1} \left( \sum_{i \in A_a} \alpha_i q^{mb_i} \right) = \sum_{a \neq -1} q^{n(a+1)} \left( \sum_{i \in A_a} \alpha_i q^{mb_i} \right)$$

for all $m, n \in \mathbb{N}$. If $A_a$ is not empty for $a \neq -1$, then we can fix an $m$ such that $\gamma_a = \sum_{i \in A_a} \alpha_i q^{mb_i} \neq 0$ for some $a \neq -1$. Note that if $c_i + d_i = c_j + d_j$ and $(c_i, d_i) \neq (c_j, d_j)$, then $-c_i + d_i \neq -c_j + d_j$. This implies that either $a = -1$ or $\alpha_i = 0$ for $i \in A_a$ which contradicts our assumption. Hence,

$$[\varphi, \lambda_x] = 0,$$

$$[\varphi, \lambda_y] = \sum_i \alpha_i \sigma(c_i, d_i),$$

where $c_i + d_i = -2$ (that is, type C1). Such a $\varphi$ is sum of first order differential operators covered in the first case.
The case where \( \varphi \) is a homogeneous homomorphism such that

\[
[\varphi, \lambda_y] = 0,
\]

\[
[\varphi, \lambda_x] = \sum_i \alpha_i \sigma(a_i, b_i),
\]

for \( \alpha_i \in k \), is dealt very similarly to the case just discussed.

3. Let \( \varphi \) be homogeneous such that

\[
[\varphi, \lambda_x] = \rho_r \sum_i \alpha_i \sigma(a_i, b_i)
\]

\[
[\varphi, \lambda_y] = \rho_s \sum_j \gamma_j \sigma(c_j, d_j),
\]

for \( \alpha_i, \gamma_j \in k \). Since \( \varphi \) is homogeneous, we have \( r = x^{n+1} y^m \) and \( s = x^n y^{m+1} \) for some integers \( m \) and \( n \). This implies that \( x \) is a factor of \( r \). Whenever \( a_i + b_j \neq 0 \), we can rewrite \( \rho_r \sigma(a_i, b_i) \) as \( \rho_r' x \sigma(a_i+1, b_i-1) \) for some \( r' \in R \) (because \( \lambda_x = \rho_x \sigma_y \)). Since

\[
[\sigma(a_i, b_i), \lambda_x] = (q^{a_i+b_i} - 1)x \sigma(a_i, b_i),
\]

we have

\[
(q^{a_i+b_i} - 1)^{-1}[\rho_r \sigma(a_i+1, b_i-1), \lambda_x] = \rho_r \sigma(a_i, b_i).
\]

Hence, we can assume that \( \varphi \) satisfies

\[
[\varphi, \lambda_x] = \rho_r \sum_i \alpha_i \sigma(-a_i, a_i)
\]

\[
[\varphi, \lambda_y] = \rho_s \sum_j \gamma_j \sigma(c_j, d_j),
\]

where \( r = x^{n+1} y^m \) and \( s = x^n y^{m+1} \). The relation

\[
[\varphi, \lambda_x] = \rho_r \sum_i \alpha_i \sigma(-a_i, a_i)
\]

implies that

\[
[\varphi, \lambda_x^n] = n x^{n-1} \rho_r \sum_i \alpha_i \sigma(-a_i, a_i).
\]
Since \( \varphi(1) = 0 \), we obtain
\[
\varphi(x^t y) = \left( \sum_j \gamma_j + \frac{t}{q^{n+1}} \sum_i \alpha_i q^{a_i} \right) x^{t+n} y^{m+1}
\]
for all \( t \geq 0 \); whereas,
\[
\varphi(q^t x^t y) = \left( \sum_i \alpha_i + \sum_j \gamma_j q^{(\frac{c_j + d_j}{2} + 1)t} \right) x^{t+n} y^{m+1},
\]
for all \( t \geq 0 \). This implies that for all \( t \geq 0 \),
\[
\sum_j \gamma_j = \sum_j \gamma_j q^{(\frac{c_j + d_j}{2} + 1)t}
\]
and
\[
\frac{1}{q^n} \sum_i \alpha_i = \frac{1}{q^{n+1}} \sum_i \alpha_i q^{a_i}.
\]
Hence the pairs \((c_j, d_j)\) are of type \( C^2 \) or \( \gamma_j = 0 \) for all \( j \). Either conclusion takes us back to the second case.

This proves the theorem. \( \square \)

3. The generators of \( D^\Lambda_q(R) \)

Let us define two subalgebras of \( \text{grHom}_k(R, R) \),
\[
D_x = k\langle \lambda_x, \partial_x^a \mid a = -1, 0, 1 \rangle \quad \text{and} \quad D_y = k\langle \rho_y, \partial_y^b \mid b = -1, 0, 1 \rangle.
\]
Since \([\partial_x^a, \lambda_x] = \sigma_x^a\) and \([\partial_y^b, \rho_y] = \sigma_y^b\), we have \( D_x \) to be the \( k \)-subalgebra generated by \( \lambda_x, \partial_x^a, \partial_x, \) and \( \sigma_x^{-1} \). A similar statement can be made for \( D_y \). Each of \( D_x \) and \( D_y \) is isomorphic to the ring of quantum differential operators on a polynomial ring with one variable (see [IM]). Moreover, elements of \( D_x \) commute with those of \( D_y \).

The rings \( D_x \) and \( D_y \) are subrings of \( D^\Lambda_q(R) \). We will show that \( D_x \) and \( D_y \) generate all of \( D^\Lambda_q(R) \).

The following is a straight-forward generalisation of Lemma 2.0.2 in [IM].

**Lemma 3.0.2.** For any \( \varphi \in D^\Lambda_q(R) \) and any \( r \in R \), there are \( c_1, c_2, \ldots, c_k \) in \( \Lambda \) such that
\[
[\cdots [ [\varphi, \lambda_r]_{c_1}, \lambda_r]_{c_2} \cdots, \lambda_r]_{c_k} = 0.
\]
Corollary 3.0.1. For $r = x$, the $c_i$ may be taken to be multiples of $(1, 1)$.

Proof. For any $c \in \Lambda$ we have, $\beta(c, (1, 1)) = q^m$ for some integer $m$. Thus,

$$ [\varphi, \lambda_x]_c = \varphi \lambda_x - q^m \lambda_x \varphi = [\varphi, \lambda_x]_{(m, m)} $$

for any graded homogenous endomorphism $\varphi$.

Lemma 3.0.3. The operators $\{\lambda_x^i, \rho_y^j, \sigma_d\}_{i,j \geq 0, d \in \Lambda}$ are linearly independent over $k$.

Proof. Since the given collection of operators are graded, it is enough to check for linear independence of homogeneous operators. This reduces to check that the set $\{\sigma_d\}$ is linearly independent over $k$ which is clear.

Lemma 3.0.4. If $\varphi \in D^\Lambda_q(R)$ and $[\varphi, \rho_y] = 0$, then $\varphi \in D_x(\rho_y)$.

Proof. By the corollary above there are $c_1, c_2, \ldots, c_k$ which are multiples of $(1, 1)$ and such that

$$ [\cdots [[\varphi, \lambda_x]_{c_1}, \lambda_x]_{c_2} \cdots, \lambda_x]_{c_k} = 0. $$

Put $\varphi_1 = \varphi$ and $\varphi_{i+1} = [\varphi_i, \lambda_x]_{c_i}$. The operators $[, \lambda_x]_{c_i}$ and $[, \rho_y]$ commute, so we have $[\varphi_i, \rho_y] = 0$ for each $i$. We will prove the lemma by descending induction on $k$.

To start, $[\varphi_k, \lambda_x]_{c_k} = 0$, so $\varphi_k$ is in $D^{0, \Lambda}$. Hence

$$ \varphi_k = \sum_{i,j \in \mathbb{Z}, d \in \Lambda} \alpha_{ijd} \rho_y^j \lambda_x^i \sigma_d $$

where $\alpha_{ijd} \in k$. However, $[\varphi_k, \rho_y] = 0$, so for each $i$, $j$, and $d$, either $\alpha_{ijd} = 0$ or $\sigma_d$ commutes with $\rho_y$ (by Lemma 3.0.3). Hence, each $\alpha_{ijd} \sigma_d$ is in $D_x$, and thus $\varphi_k$ is in $D_x(\rho_y)$.

Now suppose that $\varphi_i$ is in $D_x(\rho_y)$. Then so is $\sigma_{c_i-1}^{-1} \varphi_i$ by our assumptions on the $c_i$. Then we have

$$ \sigma_{c_i-1}^{-1} \varphi_i = [\sigma_{c_i-1}^{-1} \varphi_{i-1}, \lambda_x] = \sum_j f_j \rho_y^j $$
for some \( f_j \in D_x \). By [IM], we have \( F_j \) in \( D_x \) such that

\[
[F_j, \lambda_x] = f_j.
\]

Put

\[
\psi = \sum F_j \rho_y^j.
\]

Then \( \psi \in D_x (\rho_y) \), and we have

\[
[\psi - \sigma_{c_i-1}^{-1} \varphi_{i-1}, \lambda_x] = [\psi, \lambda_x] - [\sigma_{c_i-1}^{-1} \varphi_{i-1}, \lambda_x]
\]

\[
= \sum f_j \rho_y^j - \sigma_{c_i-1}^{-1} [\varphi_{i-1}, \lambda_x] c_{i-1}
\]

\[
= 0.
\]

As \( \sigma_{c_i-1}^{-1}, \varphi_{i-1} \), and \( \psi \) all commute with \( \rho_y \), \([\psi - \sigma_{c_i-1}^{-1} \varphi_{i-1}, \rho_y] = 0\). This implies that \( \psi - \sigma_{c_i-1}^{-1} \varphi_{i-1} \in D_0^{0, \Lambda} \). By applying the base case to \( \psi - \sigma_{c_i-1}^{-1} \varphi_{i-1} \), we conclude \( \psi - \sigma_{c_i-1}^{-1} \varphi_{i-1} \in D_x (\rho_y) \).

**Remark 3.0.1.** The lemma above is interesting in its own right. A similar proof can be used to show that if \( \varphi \in D_q^\Lambda (R) \) and \( [\varphi, \lambda_x] = 0 \), then \( \varphi \in D_y (\lambda_x) \).

**Theorem 3.0.2.** The ring \( D_q^\Lambda (R) \) is generated by \( D_x \) and \( D_y \).

**Proof.** Let \( \varphi \) be a homogeneous \( q \)-differential operator of order \( m \). Then \( \varphi = \sum \varphi_i \) where each \( \varphi_i \) has the property that for some \( c \in \Lambda, [\varphi_i, \lambda_r]_c \) is in \( D^{m-1, \Lambda} \) for any \( r \in R \). Then \( \varphi \) will be in the algebra generated by \( \partial_x, \partial_y, \partial_y^2, \partial_y^3 \), and \( D^{0, \Lambda} \) if each \( \varphi_i \) is so. Thus we will assume that there is a \( c \in \Lambda \) such that for any \( r \in R \), \([\varphi, \lambda_r]_c \in D^{m-1, \Lambda} \) and proceed by induction on \( m \).

The base case is immediate, so we will suppose already that all \( q \)-differential operators of order \( m - 1 \) or less can be expressed with the generators in the set above. In particular, \([\varphi, \lambda_y]_c \) is in the span of \( D_x \) and \( d_y \). Since \([\varphi, \lambda_y]_c = [\varphi, \lambda_y]_d \sigma_x \) for some \( d \in \Lambda \) which depends on \( c \) and \( \varphi \), we have \([\varphi, \lambda_y]_d \) is in the span of \( D_x \) and \( D_y \). Since \([\sigma_d^{-1} \varphi, \rho_y] = \sigma_d^{-1} [\varphi, \rho_y]_c \), and \( c \in \Lambda \), we can assume without loss of generality that \([\varphi, \rho_y] = \sum f_i g_i \) for some \( f_i \in D_x \) and \( g_i \in D_y \).
By Theorem 2.0.1 of [IM], there are $G_i \in D_y$ such that $[G_i, \rho_y] = g_i$. Put $\psi = \sum f_i G_i - \varphi$. Then $[\psi, \rho_y] = 0$. Hence $\psi \in D_x(\rho_y)$. It follows that $\varphi$ is in the ring generated by $D_x$ and $D_y$.

4. Properties of the ring $D^\Lambda_q(R)$

In this section, we consider the properties of $D^\Lambda_q(R)$ as a ring.

**Proposition 4.0.1.** The ring $D_x \otimes_k D_y$ is simple.

**Proof.** Let $I$ be an ideal of $D_x \otimes_k D_y$. Every element $\theta$ of $I$ can be written as $\sum_i \psi_i \otimes \eta_i$ where $\psi_i \in D_x$ and $\eta_i \in D_y$ for all $i$. Since $\theta(\lambda_x \otimes 1) - q^n(\lambda_x \otimes 1)\theta$ is in $I$ for every integer $n$, we conclude there is some $\theta \in I$ such that each $\psi_i$ is in $k[\lambda_x]$. Similarly, we can assume that each $\eta_i$ is in $k[\rho_y]$. When we consider the commutators of $\theta$ with $\partial_x \otimes 1$ and $1 \otimes \partial_y$, we conclude $I$ contains $1 \otimes 1$.

**Theorem 4.0.3.** The ring $D^\Lambda_q(R)$ is isomorphic to $D_x \otimes_k D_y$ as graded rings.

**Proof.** Note that the elements of $D_x$ and those of $D_y$ commute with each other and that the algebras $D_x$ and $D_y$ generate $D^\Lambda_q(R)$. Hence, we have a surjective map from $D_x \otimes_k D_y$ to $D^\Lambda_q(R)$. This map is injective by the proposition above. Hence the theorem.

We fix the following notations:

$$
\tau_x = \lambda_x \partial_x.
$$

$$
\tau_y = \rho_y \partial_y.
$$

**Theorem 4.0.4.** The ring $D^\Lambda_q(R)$ is a domain.

**Proof.** For each $c \in A$, let $(D^\Lambda_q(R))_c$ denote the set of all homogeneous operators in $D^\Lambda_q(R)$ of degree $c$. Then $(D^\Lambda_q(R))_{(0,0)} = (D_x)_0 \otimes_k (D_y)_0$ where $(D_x)_0$ and $(D_y)_0$ are respectively the ring of operators in $D_x$ and $D_y$ of degree 0. By Lemma 3.0.5 of [IM], $(D_x)_0 = k[\tau_x, \sigma_x^\pm]$ and $(D_y)_0 = k[\tau_y, \sigma_y^\pm]$ are localised polynomial rings. Hence, $(D^\Lambda_q(R))_{(0,0)} = k[\tau_x, \tau_y, \sigma_x^\pm, \sigma_y^\pm]$ is a localised polynomial ring and in particular a domain.
The operator $\lambda x$ is not a left zero-divisor in $D_q^A(R)$ because the ring $R$ is a domain. Similarly, $\rho y$ is not a right zero-divisor. Suppose that $\lambda x$ is a right zero-divisor in $D_q^A(R)$ and $\varphi\lambda x = 0$. Then for any $c \in \Lambda$, $[\varphi, \lambda x]_c = q^m \lambda x \varphi$ where $\sigma_c(x) = q^m x$. It follows that for any $c_1, c_2, \ldots, c_k$ in $\Lambda$ there is some integer $m$ such that

$$[\cdots [ [\varphi, \lambda x]_{c_1}, \lambda x]_{c_2} \cdots, \lambda x]_{c_k} = q^m \lambda x^k \varphi.$$ 

Since $\lambda x$ is not a left zero-divisor, this cannot be 0. Hence, by the Lemma 3.0.2, $\varphi$ is not in $D_q^A(R)$. A similar argument shows that $\rho y$ cannot be a zero-divisor on the left.

Finally, suppose $\varphi \psi = 0$. It is sufficient to consider the case when $\varphi, \psi$ are homogeneous of degrees $(a, b), (c, d)$ respectively, in the sense that $\varphi(x^a y^m) = \alpha x^{a+b} y^{m+b}$, and similarly for $\psi$. By suitably multiplying by (or factoring of) powers of $\lambda x$ and $\rho y$, we can assume that $\varphi$ and $\psi$ are in $(D_q^A(R))((0,0))$, which we know is a domain. Hence the theorem.

5. THE MOTIVATING DIAGRAM

The universal enveloping algebra $U(sl_2)$ of the Lie algebra $sl_2$ acts on its representation ring $k[u, v]$ through $D(k[u, v])$, the ring of differential operators on $k[u, v]$. Similarly, the quantised universal enveloping algebra (or quantum group) $U_q(sl_2)$, a deformation of $U(sl_2)$, acts through the quantum differential operators of its representation ring, $R$. Understanding these actions on the underlying representation rings is the goal of this section.

5.1. Quantum group on $sl_2$. Let $U_q$ denote the quantum group corresponding to the Lie algebra $sl_2(k)$. That is, $U_q$ is a $k$-algebra generated by $E, F, K, K^{-1}$,
with relations given by

\[ KK^{-1} = 1 = K^{-1}K, \]
\[ KEK^{-1} = q^2E, \]
\[ KFK^{-1} = q^{-2}F, \]
\[ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \]

Extensive treatments of \( U_q \) can be found in [CP] and [J], but we only need from these the fact that \( U_q \) is a Hopf-algebra with a co-multiplication map \( \Delta \) given by

\[ \Delta(E) = E \otimes 1 + K \otimes E, \]
\[ \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \]
\[ \Delta(K) = K \otimes K. \]

5.2. **The integral form of \( U_q \).** Let \( A = \mathbb{Q}[q, q^{-1}] \) be the Laurent polynomial ring over \( \mathbb{Q} \). Let \([m] = \frac{q^m - q^{-m}}{q - q^{-1}}\) and \([m]! = \prod_{1 \leq i \leq m} [i] \). By convention, \([0]! = 1\). The integral form \( U_{q,A} \) of \( U_q \) is the \( A \)-subalgebra of \( U_q \) generated by \( E(m), F(m), K, K^{-1} \) for positive integers \( m \) where

\[ E(m) = \frac{E^m}{[m]!}, \quad F(m) = \frac{F^m}{[m]!}. \]

5.3. **The ring \( D_A \).** Let \( D_A \) be the \( A \)-subalgebra of \( D^A_q(R) \) generated by

\[ \{ \lambda_x, \rho_y, \sigma_d, \partial_x, \partial_y, (\partial_x^\beta)^{(m)}, (\partial_y^\beta)^{(m)} \}_{m \in \mathbb{N}, d \in \Lambda} \]

where

\[ (\partial_x^\beta)^{(m)} = \frac{(\partial_x^\beta)^m}{[2]^m [m]!}, \quad (\partial_y^\beta)^{(m)} = \frac{(\partial_y^\beta)^m}{[2]^m [m]!}. \]

The completion of localisation of \( D_A \) at \((q - 1)\) will be denoted by \( \hat{D}_A \).

5.4. **The action of \( U(\mathfrak{sl}_2) \) on \( \mathbb{Q}[u,v] \).** The ring of \( \mathbb{Q} \)-linear usual differential operators on the polynomial ring \( \mathbb{Q}[u,v] \) is the second Weyl algebra. That is, \( D(\mathbb{Q}[u,v]) \) is a \( \mathbb{Q} \) algebra with generators \( \lambda_u, \lambda_v, \partial_u, \partial_v \) and relations \([\partial_u, \lambda_u] = 1, [\partial_v, \lambda_v] = 1\), and all the other commutators equal 0.
The action of $U(sl_2)$ on $Q[u,v]$ gives a homomorphism $\psi : U(sl_2) \to D(Q[u,v])$ by

$$
\psi(e) = \lambda_u \partial_v,
\psi(h) = \lambda_u \partial_u - \lambda_v \partial_v,
\psi(f) = \lambda_v \partial_u.
$$

Let $A[\tilde{h}]$ denote a polynomial ring over $A$ in one variable $\tilde{h}$. We let

$$
\psi(\tilde{h}) = \lambda_u \partial_u + \lambda_v \partial_v.
$$

Consider the projection of the centre $Z \subset U(sl_2)$ to $Q[h] = U^0 \subset U(sl_2)$ corresponding to the triangular decomposition $U(sl_2) = U^- U^0 U^+$. Since $Q[h] \cong Q[\tilde{h}]$ as rings, we can consider $Q[\tilde{h}]$ as a $Z$-module via the projection. Then, the map $\psi$ gives rise to a map

$$
\psi : U(sl_2) \otimes_Z Q[\tilde{h}] \to D(Q[u,v]).
$$

This map extends trivially to a homomorphism of power series rings

$$
\psi : (U(sl_2) \otimes_Z Q[\tilde{h}])[t] \to D(Q[u,v])[t]
$$

which is $Q[t]$ linear.

5.5. The action of $U_q, A$ on $R$. The quantum group $U_q$ acts on the quantum plane as

$$
K(1) = 1, \quad K(x) = q x, \quad K(y) = \frac{1}{q} y,
E(1) = 0, \quad E(x) = 0, \quad E(y) = x,
F(1) = 0, \quad F(x) = y, \quad F(y) = 0.
$$
with this action extended to all of $R$ via $\Delta$. The resulting action can be listed as follows:

$$K(x^iy^j) = q^{i-j}x^iy^j,$$
$$E(x^iy^j) = q^j[x]_{-2}x^{i+1}y^{j-1},$$
$$F(x^iy^j) = q^i[y]_{-2}x^{i-1}y^{j+1}.$$ 

That is, $K$ acts on $R$ as an automorphism, $E$ acts as a left $K$-derivation, and $F$ acts as a right $K^{-1}$-derivation.

The action of $E$ on $R$ is the same as the $q$-differential operator $\lambda_x\sigma_x\partial_y^{2-1}$. This is an element of $D_A$ which can be expressed as $\lambda_x\sigma_x\sigma_y^{-2}(\partial_y^2)^{(1)}(1 + \sigma_y)$. Similarly, we can identify the action of $F$ on $R$ and with the action of an element of $D_A$. This leads us to define an $A$-linear homomorphism $\phi : U_{q,A} \to D_A$ by putting

$$\phi(E) = \sigma_{(2,0)}\lambda_x(\partial_y^2)^{(1)}(1 + \sigma_{(1,-1)}),$$
$$\phi(F) = \sigma_{(-2,0)}\rho_y(\partial_x^2)^{(1)}(1 + \sigma_{(-1,-1)}),$$
$$\phi(K) = \sigma_{(2,0)}.$$ 

If we define $d_y$ and $d_x \in D_A$ as

$$d_y = \sigma_{(2,0)}(\partial_y^2)^{(1)}(1 + \sigma_{(1,-1)}),$$
$$d_x = \sigma_{(-2,0)}(\partial_x^2)^{(1)}(1 + \sigma_{(-1,-1)}),$$

then $d_y^{(m)} = d_y^m/\lceil m \rceil!$ and $d_x^{(m)} = d_x^m/\lceil m \rceil!$ are elements of $D_A$. Since $d_y\lambda_x = q\lambda_x d_y$ and $d_x\rho_y = q\rho_y d_x$, we have

$$\phi(E^{(m)}) = q^{\frac{m(m-1)}{2}}(\lambda_x)^m d_y^{(m)},$$
$$\phi(F^{(m)}) = q^{\frac{m(m-1)}{2}}(\rho_y)^m d_x^{(m)}.$$ 

The action of $\sigma_{(0,2)}$ is compatible with the action of the centre $Z_q \subset U_{q,A}$.

That is, consider the projection $p : Z_q \to U^{o}$ corresponding to the triangular decomposition $U_q = U_q^{-}U_q^{0}U_q^{+}$. Now, think of the action of $K, K^{-1}$ as $\sigma_{(0,2)}$ and $\sigma_{(0,-2)}$ respectively. We can thus consider $A[\sigma_{(0,2)}]$ as a module over $Z_q$. Then the
ring homomorphism $\phi: U_{q,A} \otimes_A A[\sigma_{(0,2)}] \to D_A$ factors through $Z_q$ to give a ring homomorphism

$$\phi: U_{q,A} \otimes_{Z_q} A[\sigma_{(0,2)}] \to D_A.$$ 

Let $U_{q,A} \otimes_{Z_q} A[\sigma_{(0,2)}]$ denote the inverse limit of $U_{q,A} \otimes_{Z_q} A[\sigma_{(0,2)}]$ with respect to the ideal $(q-1)$. Then we have the required map (by abuse of notation, we call it $\phi$ again)

$$\phi: U_{q,A} \otimes_{Z_q} A[\sigma_{(0,2)}] \to \hat{D}_A.$$

5.6. The set-up. There is a commutative diagram

$$
\begin{array}{ccc}
U_{q,A} \otimes_{Z_q} A[\sigma_{(0,2)}] & \xrightarrow{\phi} & \hat{D}_A \\
\downarrow \phi & & \downarrow \nu \\
(U(sl_2) \otimes_{Z} A[\hat{h}])[t] & \xrightarrow{\psi} & D(Q[u,v])[t]
\end{array}
$$

which we shall now describe.

5.7. The map $\nu$. For any expression $a$ in $D(Q[u,v])$, let

$$q^{(a)} = \sum_{n=0}^{\infty} \binom{a}{n} t^n,$$

where

$$\binom{a}{n} = \frac{(a)(a-1) \cdots (a-n+1)}{n!} \text{ for } n \in \mathbb{Z}, n \geq 1,$$

$$\binom{a}{0} = 1.$$ 

We use parentheses to distinguish these power series in $t$ from mere powers of $q$. Then $q^{(1)} = 1 + t$ and $q^{(a+b)} = q^{(a)} q^{(b)}$.

For any $a \in D(Q[u,v])$, the expression $q^{(a)} - 1$ is divisible by $a$ and $t$. Hence

$$P(a) = \frac{(q^{(a)} - 1)}{at}$$

is a well-defined power series in $t$ with coefficients in $D(Q[u,v])$, and it is invertible since its constant coefficient is 1.
We define the $\mathbb{Q}$-linear homomorphism $\nu : \hat{D}_A \to \hat{D}(\mathbb{Q}[u,v])[t]$ by
\[
\begin{align*}
\nu(q) &= q^{(1)}, \\
\nu(\sigma_x) &= q^{(\lambda_u \partial_u)}, \\
\nu(\lambda_x) &= \lambda_u, \\
\nu(\partial_x) &= \partial_u, \\
\nu(\sigma_y) &= q^{(\lambda_v \partial_v)}, \\
\nu(\rho_y) &= \lambda_v, \\
\nu(\partial_y) &= \partial_v, \\
\nu(\partial^3_x) &= \partial_u P(\lambda_u \partial_u), \\
\nu(\partial^3_y) &= \partial_v P(\lambda_v \partial_v).
\end{align*}
\]

### 5.8. The map $\mu$

Here we shall use the same notation as in the previous subsection. For $a \in U_{q,A} \otimes_{\mathbb{Z}_q} A[\sigma_{(0,2)}]$, let $q^{(a)} = \sum_{n=0}^{\infty} \binom{a}{n} t^n$ where the binomial coefficients are defined formally. Again, define the invertible $P(a) = (q^{(a)} - 1) / at$ for any $a \in U_q \otimes_{\mathbb{Z}_q} A[\sigma_{(0,2)}]$. Now we can define
\[
\begin{align*}
\mu : U_q \otimes_{\mathbb{Z}_q} A[\sigma_{(0,2)}] &\to \left(U(sl_2) \otimes \mathbb{Q}[\bar{h}]\right)[t], \\
\mu(q) &\mapsto (1 + t), \\
\mu(K) &\mapsto q^{(h)}, \\
\mu(E) &\mapsto q^{(h)} eP\left(\frac{-h+\bar{h}}{2}\right) (1 + q^{(\frac{h-\bar{h}}{2})}) \over 1 + q^{(1)}, \\
\mu(F) &\mapsto q^{(-h)} fP\left(\frac{h+\bar{h}}{2}\right) (1 + q^{(\frac{-h+\bar{h}}{2})}) \over 1 + q^{(1)}, \\
\mu(\sigma_{(0,2n)}) &\mapsto q^{(nh)}.
\end{align*}
\]

This completes the commutative diagram.

### 5.9. An Application.

**Proposition 5.9.1.** Given an irreducible, finite dimensional $U(sl_2)$-module, there is a natural way to define a $U_q$-action to give a type-1 irreducible, finite dimensional module of $U_q$. 

Proof. Let $V$ be an $n + 1$-dimensional irreducible module of $U(sl_2)$, for $n \geq 0$. Without loss of generality, we can assume that a $k$-basis of $V$ is

$$\{y^n, y^{n-1}x, y^{n-2}x^2, \ldots, x^n\}$$

where $x^n$ is the highest weight vector, and that the action of $U(sl_2)$ is given by the map $\psi$. By using the map $\mu$, we get the required $U_q$ action on $V$. □

6. Conclusions and Acknowledgements

We suspect that $D^\Gamma_q(R) = D^\Lambda_q(R)\langle \sigma_{(1,0)} \rangle$. The evidence for this is that any left $\sigma_{(1,0)}$-derivation is a $q$-differential operator of order 0, not 1. This suggests that if $\varphi$ is a $q$-differential operator of order $m$, then so is $\varphi \cdot r - \sigma_{(1,0)}(r) \cdot \varphi$ for every $r$ in $R$. It follows that $(D^\Gamma_q(R))^m$ would be generated as a $(D^\Gamma_q(R))^0$-module by $Z_q^m$.

Furthermore, we expect results similar to those in Section 5. In particular, for a general semi-simple Lie algebra $G$ we expect a map analogous to $\mu$ could be constructed.

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q-DIFFERENTIAL OPERATORS

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