Average subentropy, coherence and entanglement of random mixed quantum states

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Abstract

Compact expressions for the average subentropy and coherence are obtained for random mixed states that are generated via various probability measures. Surprisingly, our results show that the average subentropy of random mixed states approaches to the maximum value of the subentropy which is attained for the maximally mixed state as we increase the dimension. In the special case of the random mixed states sampled from the induced measure via partial tracing of random bipartite pure states, we establish the typicality of the relative entropy of coherence for random mixed states invoking the concentration of measure phenomenon. Our results also indicate that mixed quantum states are less useful compared to pure quantum states in higher dimension when we extract quantum coherence as a resource. This is because of the fact that average coherence of random mixed states is bounded uniformly, however, the average coherence of random pure states increases with the increasing dimension. As an important application, we establish the typicality of relative entropy of entanglement and distillable entanglement for a specific class of random bipartite mixed states. In particular, most of the random states in this specific class have relative entropy of entanglement and distillable entanglement equal to some fixed number (to within an arbitrary small error), thereby hugely reducing the complexity of computation of these entanglement measures for this specific class of mixed states.

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1 Introduction

Miniaturization [23] and technological advancements to handle and control systems at smaller and smaller scales necessitate the deeper understanding of concepts such as quantum coherence, entanglement and correlations [39, 16, 45, 30, 4, 25, 26, 33, 37, 24, 17, 22]. Two inequivalent resource theories of coherence has

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been proposed [11, 27, 2] realizing the importance of the coherence as a resource in various physical situations. Recently, it has been proved that the coherence of a random pure state sampled from the uniform Haar measure is generic for higher dimensional systems, i.e., most of the random pure states have almost the same amount of coherence [44]. The importance of this result and the similar results for entanglement of random bipartite pure states cannot be overemphasized. The average entanglement of random bipartite pure states, which is facilitated by the calculation of average entropy of the marginals of the random bipartite pure states [31, 8, 40, 42], is proved typical [14]. This has resulted in various interesting consequences in quantum information theory [14, 12, 6, 49, 29], in the context of black holes [32] and in particular, in explaining the equal a priori probability postulate of statistical physics [34, 9]. But as we approach towards the realistic implementations of quantum technology, mixed states are encountered naturally due to the interaction between the system of interest and the external world. Therefore, consideration of average entanglement and coherence content of random mixed states is of great importance in realistic scenarios. However, to the best of our knowledge, there is no known result on the average coherence of random mixed states.

Here, we aim at finding the average relative entropy of coherence of random mixed states sampled from various induced measures including the one obtained via the partial tracing of the Haar distributed random bipartite pure states. We first find the exact expression for the average subentropy of random mixed states sampled from induced probability measures and use it to find the average relative entropy of coherence of random mixed states. We note that the subentropy is a nonlinear function of state and therefore, it is expected that the average subentropy of a random mixed state should not be equal to the subentropy of the average state (the maximally mixed state). Surprisingly, we find that the average subentropy of a random mixed state approaches exponentially fast towards the maximum value of the subentropy, which is achieved for the maximally mixed state [20]. As one of the applications of our results, we note that the average subentropy may also serve as the state independent quality factor for ensembles of states to be used for estimating accessible information. Interestingly, we find that the average coherence of random mixed states, just like the average coherence of random pure states, shows the concentration phenomenon. This means that the relative entropies of coherence of most of the random mixed states are equal to some fixed number (within an arbitrarily small error) for larger Hilbert space dimensions. It is well known that the exact computation of the most of the entanglement measures for bipartite mixed states in higher dimensions is almost impossible [15]. However, using our results, we compute the average relative entropy of entanglement and distillable entanglement for a specific class of random bipartite mixed states and show their typicality for larger Hilbert space dimensions. It means that for almost all random states of this specific class, both the measures of entanglement are equal to a fixed number (that we calculate) within an arbitrarily small error, reducing hugely the computational complexity of both the measures for this specific class of bipartite mixed states. This is a very important practical application of the results obtained in this paper.
2 Quantum coherence and induced measures on the space of mixed states

2.1 Quantum coherence

Various coherence monotones, that serve as the faithful measures of coherence [2, 46, 48, 43], are proposed based on the resource theory of coherence [2]. These monotones include the $l_1$ norm of coherence, relative entropy of coherence [2] and the geometric measure of coherence based on entanglement [46]. In this work, unless stated otherwise, by coherence we mean the relative entropy of coherence throughout the paper. The relative entropy of coherence of a quantum state $\rho$, acting on an $m$-dimensional Hilbert space, is defined as

$$C_r(\rho) = S(\Pi(\rho)) - S(\rho),$$

where $\Pi(\rho) = \sum_{j=1}^{m} |j\rangle \langle j| \rho |j\rangle \langle j|$ for a fixed basis $\{|j\rangle : j = 1, \ldots, m\}$. $S(\rho) = -\text{Tr}(\rho \ln \rho)$ is the von Neumann entropy of $\rho$. All the logarithms that appear in the paper are with respect to natural base.

2.2 Induced measures on the space of mixed states

Unlike on the set of pure states, it is known that there exist several inequivalent measures on the set of density matrices, $D(C^m)$ (the set of trace one nonnegative $m \times m$ matrices). By the spectral decomposition theorem for Hermitian matrices, any density matrix $\rho$ can be diagonalized by a unitary $U$. It seems natural to assume that the distributions of eigenvalues and eigenvectors of $\rho$ are independent, implying $\mu$ to be product measure $\nu \times \mu_{\text{Haar}}$, where the measure $\mu_{\text{Haar}}$ is the unique Haar measure on the unitary group and measure $\nu$ defines the distribution of eigenvalues but there is no unique choice for it [50, 51].

The induced measures on the $(m^2 - 1)$-dimensional space $D(C^m)$ can be obtained by partial tracing the purifications $|\Psi\rangle$ in the larger composite Hilbert space of dimension $mn$ and choosing the purified states according to the unique measure on it. Following Ref. [50], the joint density $P_{m(n)}(\Lambda)$ of eigenvalues $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$ of $\rho$, obtained via partial tracing, is given by

$$P_{m(n)}(\Lambda) = C_{m(n)} K_1(\Lambda) \prod_{j=1}^{m} \lambda_j^{n-m} \theta(\lambda_j), \quad (2.1)$$

where the theta function $\theta(\lambda_j)$ ensures that $\rho$ is positive definite, $C_{m(n)}$ is the normalization constant and $K_1(\Lambda)$ is given by

$$K_1(\Lambda) = \delta \left( 1 - \sum_{j=1}^{m} \lambda_j \right) |\Delta(\Lambda)|^{2\gamma}, \quad (2.2)$$

for $\gamma = 1$ with $\Delta(\lambda) = \prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j)$. See Refs. [50, 51] for a good exposition of induced measures on the set of density matrices.

Now we introduce the family of integrals $I_m(a, \gamma)$ that will be a key to our work, where

$$I_m(a, \gamma) := \int_0^\infty \cdots \int_0^\infty K_\gamma(\Lambda) \prod_{j=1}^{m} \lambda_j^{a-1} d\lambda_j$$

$$= b_m(a, \gamma) \prod_{j=1}^{m} \frac{\Gamma(a + \gamma(j - 1)) \Gamma(1 + \gamma j)}{\Gamma(1 + \gamma)}, \quad (2.3)$$
with $\alpha, \gamma > 0$, $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt$ is the Gamma function, defined for $\text{Re}(z) > 0$ and $b_m(\alpha, \gamma) = \{\Gamma(\alpha m + \gamma m(m-1))\}^{-1}$. The value of above family of integrals can be obtained using Selberg’s integrals [50, 51, 1] (see Appendix 7.2 for a quick review of Selberg’s integrals). Let us define $C_m^{(\alpha, \gamma)} = 1/\mathcal{I}_m(\alpha, \gamma)$, which are called as normalization constants. A family of probability measures over $\mathbb{R}^m_+$ can be defined as:

$$d\nu_{\alpha, \gamma}(\Lambda) := C_m^{(\alpha, \gamma)} K_\gamma(\Lambda) \prod_{j=1}^m \lambda_j^{\alpha-1} d\lambda_j. \quad (2.4)$$

Also, $\nu_{\alpha, \gamma}$ is a family of normalized probability measures over the probability simplex

$$\Delta_{m-1} := \left\{ \Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ : \sum_{j=1}^m \lambda_j = 1 \right\},$$

i.e.,

$$\nu_{\alpha, \gamma}(\Delta_{m-1}) = \int d\nu_{\alpha, \gamma}(\Lambda) = 1.$$
are Schur concave, they are indeed entanglement monotones and can be used as alternative measures of entanglement. For the convenience, we list the properties of subentropy in the Appendix 7.1.

The average subentropy over the set of mixed states is given by

\[ \mathcal{I}_m^Q(\alpha, \gamma) = \int d\mu_{\alpha,\gamma}(\rho) Q(\rho) = \int d\nu_{\alpha,\gamma}(\Lambda) Q(\Lambda). \] (3.2)

Apparently \( 0 \leq \mathcal{I}_m^Q(\alpha, \gamma) \leq 1 - \gamma_{\text{Euler}} \) since the subentropy is uniformly bounded, i.e., \( 0 \leq Q(\Lambda) \leq 1 - \gamma_{\text{Euler}} \), where \( \gamma_{\text{Euler}} \approx 0.57722 \) is Euler’s constant.

**Proposition 3.1.** For \( \gamma = 1 \) and arbitrary \( \alpha \), the average subentropy \( \mathcal{I}_m^Q(\alpha, 1) \) is given by

\[ \mathcal{I}_m^Q(\alpha, 1) = \frac{1}{m(m+\alpha-1)} \sum_{k=0}^{m-1} g_{mk}(\alpha) u_{mk}(\alpha), \] (3.3)

where

\[ g_{mk}(\alpha) = \psi(m(m+\alpha-1)+1) - \psi(2(m-1)+\alpha+1-k), \] (3.4)

\[ u_{mk}(\alpha) = \frac{(-1)^k \Gamma(2(m-1)+\alpha+1-k)}{\Gamma(k+1)\Gamma(m-k)\Gamma(m+\alpha-1-k)}, \] (3.5)

with \( \psi(z) = d \ln \Gamma(z) / dz \) being the digamma function.

**Proof.** See Appendix 7.3. \( \square \)

In the remaining, we consider the induced measure \( \mu_{m(n)}(m \leq n) \) over all the \( m \times m \) density matrices of the \( m \)-dimensional quantum system via partial tracing over the \( n \)-dimensional ancilla of uniformly Haar-distributed random bipartite pure states of system and ancilla, which is as follows: for \( \rho = U\Lambda U^\dagger \) with \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m) \) and \( U \in U(m) \),

\[ d\mu_{m(n)}(\rho) = d\nu_{m(n)}(\Lambda) \times d\mu_{\text{Haar}}(U), \] (3.6)

where \( d\nu_{m(n)}(\Lambda) = C_{m(n)} K_1(\Lambda) \prod_{j=1}^m \lambda_j^{n-m} d\lambda_j \) [50] is the joint distribution of eigenvalues \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m) \) of the density matrix \( \rho \), and \( d\mu_{\text{Haar}}(U) \) is the uniform Haar measure over unitary group \( U(m) \). Apparently Eq. (3.6) is a special case of Eq. (2.5) when \( (\alpha, \gamma) = (n-m+1, 1) \). That is, \( d\mu_{m(n)}(\rho) = d\nu_{n-m+1,1}(\rho) \) and \( d\nu_{m(n)}(\Lambda) = d\nu_{n-m+1,1}(\Lambda) \). From this, we see that

\[ \mathcal{I}_m^Q(n-m+1, 1) = \frac{1}{mn} \sum_{k=0}^{n-1} g_{mk}(n-m+1) u_{mk}(n-m+1). \] (3.7)

In fact, we find a closed-formula for the average subentropy:

**Lemma 3.2** (Closed-form of the average subentropy). The average subentropy of random mixed quantum states, induced by partial tracing the Haar-distributed random pure bipartite states in the Hilbert space of dimension \( m \otimes n \) with \( m \leq n \), is given by the following compact formula:

\[ \mathcal{I}_m^Q(n-m+1, 1) = 1 + H_{mn} - H_m - H_n. \] (3.8)

**Proof.** See Appendix 7.4. \( \square \)
The above expression can be rewritten as \( \mathcal{I}_m^Q(n - m + 1, 1) = (1 - \gamma_{\text{Euler}}) - (a_m + a_n - a_{mn}) \), where \( a_k = H_k - \ln k - \gamma_{\text{Euler}} \) for \( k = m, n \). Since the number series \( \{a_k\} \) is monotone decreasing and approaches to zero, it follows that \( a_m + a_n - a_{mn} \geq 0 \) and \( \lim_{m \to \infty}(a_m + a_n - a_{mn}) = 0 \) (note that \( m \leq n \)). Based on this fact, we get that

\[
\lim_{m \to \infty} \mathcal{I}_m^Q(n - m + 1, 1) = 1 - \gamma_{\text{Euler}} \approx 0.42278. \tag{3.9}
\]

If \( m = n \), this situation corresponds to the probability measure induced by the Hilbert-Schmidt distance [50], then

\[
\mathcal{I}_m^Q(1, 1) = \frac{1}{m^2} \sum_{k=0}^{m-1} g_{mk}(1)u_{mk}(1) = 1 + H_{m^2} - 2H_m. \tag{3.10}
\]

We find that it approaches exponentially fast towards the maximum value of the subentropy, which is achieved for the maximally mixed state [20]. The maximum value of \( Q(\rho) \) is approximately equal to 0.42278 [20]. This is surprising, since \( Q(\rho) \) is a nonlinear function of \( \rho \) and it is not expected that the average subentropy should match with the subentropy of the average state, which is the maximally mixed state.

### 4 The average coherence of random mixed states and typicality

Now, we are in a position to calculate the average coherence of random mixed states and establish its typicality. Let \( \rho = U \Lambda U^\dagger \) be a mixed full-ranked quantum state on \( \mathbb{C}^m \) with non-degenerate positive spectra \( \lambda_j \in \mathbb{R}^+ (j = 1, \ldots, m) \), where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m) \). Then coherence of the state \( \rho \) is given by \( \mathcal{C}_r(U \Lambda U^\dagger) = S(\Pi(U \Lambda U^\dagger)) - S(\Lambda) \). The average coherence of the isospectral density matrices can be expressed in terms of the quantum subentropy, von Neumann entropy, and \( m \)-Harmonic number as follows [5]:

\[
\mathcal{C}_r^{\text{iso}}(\Lambda) \ := \int d\mu_{\text{Haar}}(U)\mathcal{C}_r(U \Lambda U^\dagger) = H_m - 1 + Q(\Lambda) - S(\Lambda). \tag{4.1}
\]

Here \( Q(\Lambda) \) is the subentropy, given by Eq. (3.1), \( S(\Lambda) \) is the von Neumann entropy of \( \Lambda \) and \( H_m = \sum_{k=1}^m 1/k \) is the \( m \)-Harmonic number. From this, we see that the average coherence of isospectral full-ranked density matrices depends completely on the spectrum. Also, it is known that \( 0 \leq Q(\Lambda) \leq 1 - \gamma_{\text{Euler}} \). Now, using the product probability measures \( d\mu_{U, \gamma} = d\nu_{U, \gamma} \times d\mu_{\text{Haar}}(U) \), the average coherence of random mixed states is given by

\[
\mathcal{C}_r^{\text{iso}}(\alpha, \gamma) \ := \int d\mu_{U, \gamma}(\rho)\mathcal{C}_r(\rho) = \int d\mu_{U, \gamma}(U \Lambda U^\dagger)\mathcal{C}_r(U \Lambda U^\dagger)
\]

\[
= H_m - 1 + \mathcal{I}_m^Q(\alpha, \gamma) - \mathcal{I}_m^S(\alpha, \gamma), \tag{4.2}
\]

where \( \mathcal{I}_m^Q(\alpha, \gamma) = \int d\nu_{U, \gamma}(\Lambda)Q(\Lambda) \) and \( \mathcal{I}_m^S(\alpha, \gamma) = \int d\nu_{U, \gamma}(\Lambda)S(\Lambda) \). In the remaining, we again consider the induced measure \( \mu_{m(n)}(m \leq n) \) over all the \( m \times m \) density matrices of the \( m \)-dimensional quantum system via partial tracing over the \( n \)-dimensional ancilla of uniformly Haar-distributed random pure bipartite states of system and ancilla.
Theorem 4.1 (Closed-form of the average coherence). The average coherence of random mixed states of dimension \( m \) sampled from induced measures obtained via partial tracing of Haar distributed bipartite pure states of dimension \( mn \), for \( (\alpha, \gamma) = (n - m + 1, 1) \), is given by

\[
\bar{C}_r(n - m + 1, 1) = \frac{m - 1}{2n}. \tag{4.3}
\]

Proof. See Appendix 7.5.

For \( m = n \), which corresponds to the probability measure induced by the Hilbert-Schmidt distance, the average coherence of random mixed states is given by

\[
\bar{C}_r(1, 1) = \frac{m - 1}{2m} \to \frac{1}{2} \quad (m \to \infty). \tag{4.4}
\]

The asymptotic value \( \frac{1}{2} \) of the average coherence is also obtained by Puchała et al. \([35]\) using free probabilistic tools. However, free probabilistic theory can only be used to deal with asymptotic situation. Therefore, the case where \( m \neq n \) cannot be treated by such a method. Thus their result about average coherence is just an asymptotic value of a special case of our formula (4.3). Clearly this formula is completely exact, not asymptotically, and very simple.

From this formula, we see that for fixed \( m \), the dimension of quantum system under consideration, when \( n \), the dimension of ambient environment, is larger, the average coherence is approximately vanishing. In fact, we have calculated the average coherence of random pure quantum states and also established its typicality \([44]\). The average coherence of random pure quantum states in \( m \)-dimensional Hilbert space is given by \( H_m - 1 \). Thus

\[
\frac{m - 1}{2m} < \frac{1}{2} \ll H_m - 1 \to \infty.
\]

In view of this, statistically, in \( m \)-dimensional Hilbert space, mixed quantum states is less useful than pure quantum states in higher dimension when we extract quantum coherence, as a resource, from quantum states.

Now, just like in the case of random pure states where the average coherence is a generic property of all random pure states \([44]\), one may ask if the average coherence of random mixed states is also a generic property of all random mixed states. The following theorem (Theorem 4.2) establishes that the average coherence is indeed a generic property of all random mixed states, i.e., as we increase the dimension of the density matrix, almost all the density matrices generated randomly have coherence approximately equal to the average relative entropy of coherence, given by Theorem 4.1. Thus, the average coherence of a random mixed state can be viewed as the typical coherence content of random mixed states.

Theorem 4.2. Let \( \rho_A \) be a random mixed state on an \( m \) dimensional Hilbert space \( \mathcal{H} \) with \( m \geq 3 \) generated via partial tracing of the Haar distributed bipartite pure states on \( mn \) dimensional Hilbert space. Then, for all \( \epsilon > 0 \), the relative entropy of coherence \( C_r(\rho_A) \) of \( \rho_A \) satisfies the following inequality:

\[
\Pr \left\{ \left| C_r(\rho_A) - \frac{m - 1}{2n} \right| > \epsilon \right\} \leq 2 \exp \left( -\frac{m\epsilon^2}{144\pi^3 \ln 2 (\ln m)^2} \right). \tag{4.5}
\]

Proof. See Appendix 7.6.
Corollary 4.3. Let $\rho_A$ be a random mixed state on an $m$-dimensional Hilbert space $\mathcal{H}$ with $m \geq 3$ generated via partial tracing of the Haar distributed bipartite pure states on $m \otimes m$ dimensional Hilbert space. Then, for all $\epsilon > 0$ and sufficiently large $m$, when $|\mathcal{C}_r(1,1) - \frac{1}{2}| < \frac{\epsilon}{2}$ (this is equivalent to $m > \frac{1}{4}$), its coherence is close to the number $\frac{1}{2}$, as the deviations become exponentially rare, i.e., the relative entropy of coherence $\mathcal{C}_r(\rho_A)$ of $\rho_A$ satisfies the following inequality:

$$\Pr \left\{ \left| \mathcal{C}_r(\rho_A) - \frac{1}{2} \right| > \epsilon \right\} \leq 2 \exp \left( -\frac{m^2 \epsilon^2}{576 \pi^3 \ln 2 (\ln m)^2} \right).$$

(4.6)

Proof. Since

$$\lim_{m \to \infty} \frac{1}{2} \mathcal{C}_r(1,1) = \frac{1}{2},$$

it follows from the triangular inequality that

$$\left\{ \rho : \left| \mathcal{C}_r(\rho) - \frac{1}{2} \right| < \epsilon \right\} \subseteq \left\{ \rho : \left| \mathcal{C}_r(\rho) - \frac{1}{2} \right| < \epsilon \right\}.$$

For dimension $m$ so large that $|\mathcal{C}_r(1,1) - \frac{1}{2}| < \frac{\epsilon}{2}$, this implies the following bounds,

$$\Pr \left\{ \left| \mathcal{C}_r(\rho_A) - \frac{1}{2} \right| < \epsilon \right\} \geq \Pr \left\{ \left| \mathcal{C}_r(\rho_A) - \mathcal{C}_r(1,1) \right| + \left| \mathcal{C}_r(1,1) - \frac{1}{2} \right| < \epsilon \right\} \geq \Pr \left\{ \left| \mathcal{C}_r(\rho_A) - \mathcal{C}_r(1,1) \right| < \epsilon - \left| \mathcal{C}_r(1,1) - \frac{1}{2} \right| \right\} \geq 1 - 2 \exp \left( -\frac{m^2 \epsilon^2}{144 \pi^3 \ln 2 (\ln m)^2} \right),$$

implying that

$$\Pr \left\{ \left| \mathcal{C}_r(\rho_A) - \frac{1}{2} \right| > \epsilon \right\} \leq 2 \exp \left( -\frac{m^2 \epsilon^2}{576 \pi^3 \ln 2 (\ln m)^2} \right).$$

This completes the proof. \qed

Next, we present an important consequence of Theorem 4.2 showing a reduction in computational complexity of certain entanglement measures for a specific class of mixed states.

5 Entanglement properties of a specific class of random bipartite mixed states

Consider a specific class $\mathcal{X}$ of random bipartite mixed states $\chi_{AB}$ of dimension $m \otimes m$ that are generated as follows. First generate random mixed states for a single quantum system $A$ in an $m$ dimensional Hilbert space via partial tracing the Haar distributed bipartite pure states on an $mn$ dimensional Hilbert space. Now bring in an ancilla $B$ in a fixed state $|0\rangle_B$ on a $d_B$-dimensional Hilbert space and apply the generalized CNOT gate, defined as

$$\text{CNOT} = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} |i\rangle_A \otimes |\text{mod}(i+j, m)\rangle_B \langle j| + \sum_{i=0}^{m-1} \sum_{j=0}^{d_B-1} |i\rangle_A \otimes |j\rangle_B.$$
on the composite system $AB$. The random bipartite mixed states, thus obtained, are given by

$$\chi_{AB} := \text{CNOT}[\rho_A \otimes |0\rangle\langle 0|_B] = \sum_{i,j=0}^{m-1} \rho_{ij} |ii\rangle\langle jj|_{AB},$$

where $\rho_A := \text{Tr}_{A_0}(|\psi\rangle\langle \psi|_{AA_0}) = \sum_{i,j=0}^{m-1} \rho_{ij} |i\rangle\langle j|_A$ is a random mixed state generated according to an induced measure via partial tracing as mentioned above. Now, using the results on convertibility of coherence into entanglement [46], we can estimate exactly the relative entropy of entanglement $E_r$ [47] and distillable entanglement $E_d$ [3, 36] of random mixed states in the class $\mathcal{X}$. In particular,

$$E^A_{r|B}(\chi_{AB}) = E^A_{d|B}(\chi_{AB}). \tag{5.1}$$

We can now use our exact results on the average relative entropy of coherence of random mixed states to find the average entanglement for the specific class of bipartite random mixed states in the class $\mathcal{X}$ as follows:

$$\mathbb{E} E^A_{r|B}(\chi_{AB}) = \int d\mu_{n-m+1,1}(\rho) E^A_{r|B}(\text{CNOT}[\rho_A \otimes |0\rangle\langle 0|_B])$$

$$= \int d\mu_{n-m+1,1}(\rho) \mathcal{E}_r(\rho_A) = \mathcal{E}_r(n - m + 1, 1). \tag{5.2}$$

Here $\mathcal{E}_r(n - m + 1, 1)$ is given by Theorem 4.1. Similarly, $\mathbb{E} E^A_{d|B}(\chi_{AB}) = \frac{m-1}{2n}$. The following corollary follows immediately from Theorem 4.2.

**Corollary 5.1.** Let $\chi_{AB} \in \mathcal{X}$ be a random mixed state on $m \otimes m$ dimensional Hilbert space with $m \geq 3$ generated as mentioned above. Then, for all $\epsilon > 0$

$$\Pr \left\{ \left| E^A_{r|B}(\chi_{AB}) - \frac{m-1}{2n} \right| > \epsilon \right\} \leq 2 \exp \left( -\frac{mne^2}{144\pi^3 \ln 2(\ln m)^2} \right) \tag{5.3}$$

and

$$\Pr \left\{ \left| E^A_{d|B}(\chi_{AB}) - \frac{m-1}{2n} \right| > \epsilon \right\} \leq 2 \exp \left( -\frac{mne^2}{144\pi^3 \ln 2(\ln m)^2} \right). \tag{5.4}$$

Corollary 5.1 establishes that most of the random states in the class $\mathcal{X}$ have almost the same fixed amount of distillable entanglement and relative entropy of entanglement in the large $m$ limit. Thus, our results help in estimating the entanglement content of most of the random states in the class $\mathcal{X}$ (which is an extremely hard task), asymptotically and show the typicality of entanglement for class $\mathcal{X}$ of mixed states.

### 6 Conclusion

To conclude, we have provided analytical expressions for the average subentropy and the average relative entropy of coherence over the whole set of density matrices distributed according to the family of probability measures obtained via the spectral decomposition. We also have obtained the closed-form of the average subentropy (Lemma 3.2). Based on the compact form of the average subentropy, we find that as
we increase the dimension of the quantum system, the average subentropy approaches towards the maximum value of subentropy (attained for the maximally mixed state) exponentially fast, which is surprising as the subentropy is a nonlinear function of density matrix. We also have obtained the compact form of the average coherence $C_{m,n} = \frac{m+1}{2n}$ from combining the average entropy formula $S_{m,n} = H_{mn} - H_n - \frac{m-1}{2n}$ [31, 8, 40, 42] and the average subentropy formula $Q_{m,n} = 1 + H_{mn} - H_m - H_n$. Interestingly, using Lévy’s lemma, we prove that the coherence of random mixed states sampled from induced measures via partial tracing show the concentration phenomenon, establishing the generic nature of coherence content of random mixed states. As a very important application of our results, we show a huge reduction in the computational complexity of entanglement measures such as relative entropy of entanglement and distillable entanglement. We find the entanglement properties of a specific class random bipartite mixed states, thanks to Theorem 4.2. Since quantum coherence and entanglement are deemed as useful resources for implementations of various quantum technologies, our results will serve as a benchmark to gauge the resourcefulness of a generic mixed state for a certain task at hand. Furthermore, our results may have some applications in black hole physics as to how much coherence can be there in the Hawking radiation for non-thermal states [32], in thermalization of closed quantum systems and in catalytic coherence transformations.

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7 Appendix

Here we give a very brief review of the subentropy and the Selberg’s integrals. Also we provide the proofs of the Proposition 3.1, Lemma 3.2, and the two Theorems 4.1 and 4.2 of the main text.

7.1 Quantum Subentropy

The von Neumann entropy of a quantum system is of paramount importance in physics starting from thermodynamics [18, 19] to the quantum information theory, e.g., in studies of the classical capacity of a quantum channel and the compressibility of a quantum source [41], and serves as the least upper bound on the accessible information. The von Neumann entropy of an m dimensional density matrix $\rho$, is defined as $S(\rho) = -\sum_{j=1}^{m} \lambda_j \ln \lambda_j$, where $\lambda = \{\lambda_1, \cdots, \lambda_m\}$ are eigenvalues of $\rho$. An analogous lower bound on the accessible information, obtained in Ref. [20] and called as the subentropy $Q(\rho)$, is defined as $Q(\rho) = -\sum_{j=1}^{m} \lambda_j^m \left(\prod_{j\neq i}^{n} (\lambda_i - \lambda_j)\right)^{-1} \ln \lambda_i$. Also, when two or more of the eigenvalues $\lambda_j$ are equal, the value of $Q$ is determined by taking a limit starting with unequal eigenvalues, unambiguously. The upper bound $S(\rho)$ and the lower bound $Q(\rho)$ on the accessible information are achieved for the ensemble of eigenstates of $\rho$ and the Scrooge ensemble [20], respectively. Thus, the von Neumann entropy and the subentropy together define the range of the accessible information for a given density matrix. Main properties of subentropy are summarized here as the following proposition for the reader’s convenience, details can be found in [20, 13, 28, 7].

Proposition 7.1. The subentropy $Q(\rho)$ of a quantum state $\rho$ satisfies the following properties:

(I) $0 \leq Q(\rho) \leq 1 - \gamma_{\text{Euler}}$, where $\gamma_{\text{Euler}} \approx 0.57722$ is Euler’s constant. In particular, the lower bound is achieved only at all pure states; and the maximum value of subentropy is achieved at the maximally mixed state, that is, $Q(1_m / m) = \ln m - H_m + 1$, where $H_m$ is the m-th harmonic number. Moreover, $Q(\rho) \leq -\ln \lambda_{\text{max}}(\rho)$, where $\lambda_{\text{max}}(\rho)$ is the maximum eigenvalue of $\rho$. 

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(2) $Q(\rho)$ is a concave function in $\rho$ (which is also true for the von Neumann entropy). That is, if $\{\rho_j\}_{j=1}^n$ are density matrices and $\{q_j\}_{j=1}^n$ is a probability distribution, then
\[
Q\left(\sum_{j=1}^n q_j \rho_j\right) \geq \sum_{j=1}^n q_j Q(\rho_j).
\] (7.1)

In particular, $Q\left(\sum_i p_i U_i \rho U_i^\dagger\right) \geq Q(\rho)$, where $\{p_i\}$ is a probability distribution and $\{U_i\}$ is a set of unitary matrices.

(3) $Q(\rho)$ is Schur-concave in $\rho$. Note that a real-valued function $f(\rho)$ is called Schur concave in $\rho$ if $f(\rho) \geq f(\sigma)$ whenever $\rho$ is majorized by $\sigma$.

(4) $Q(\rho)$ is a continuous function in $\rho$. Assume $\rho, \rho' \in D(\mathbb{C}^m)$. If $\|\rho - \rho'\|_1 \leq \varepsilon^{-1}$, then
\[
|Q(\rho) - Q(\rho')| \leq (\ln m) \|\rho - \rho'\|_1 + \eta(\|\rho - \rho'\|_1),
\] (7.2)

where $\eta(x) = -x \ln x$.

(5) $Q(\rho_A \otimes \rho_B) \leq Q(\rho_A) + Q(\rho_B)$. In general, $Q(\rho_A \otimes \rho_B) \neq Q(\rho_A) + Q(\rho_B)$.

At present, we do not know whether the subadditivity inequality of subentropy is true or not: $Q(\rho_{AB}) \leq Q(\rho_A) + Q(\rho_B)$. A similarly question can be asked for the strong subadditivity of subentropy: $Q(\rho_{ABC}) + Q(\rho_B) \leq Q(\rho_{AB}) + Q(\rho_{BC})$. For a comparison between the von Neumann entropy and the subentropy, we rewrite both $S$ and $Q$ as contour integrals. $S$ can be represented as
\[
S(\rho) = -\frac{1}{2\pi i} \oint (\ln z) \text{Tr} \left( (1_m - \rho/z)^{-1} \right) dz,
\] (7.3)

where the contour encloses all the nonzero eigenvalues of $\rho$. $Q$ can be also represented as
\[
Q(\rho) = -\frac{1}{2\pi i} \oint (\ln z) \det \left( (1_m - \rho/z)^{-1} \right) dz,
\] (7.4)

$S(\rho)$ and $Q(\rho)$ are strikingly similar in above forms and where the trace appears in the formula for the von Neumann entropy, the determinant appears in the formula for the subentropy. Other comparison can also be seen in Refs. [20, 13, 28, 7]. Now, we present Selberg’s integrals and the calculation of the average subentropy of random mixed states.

### 7.2 Selberg’s Integrals and its consequences

**Proposition 7.2 (Selberg’s Integrals,[1]).** If $m$ is a positive integer and $\alpha, \beta, \gamma$ are complex numbers such that
\[
\text{Re}(\alpha) > 0, \quad \text{Re}(\beta) > 0, \quad \text{Re}(\gamma) > -\min \left\{ \frac{1}{m}, \frac{\text{Re}(\alpha)}{m-1}, \frac{\text{Re}(\beta)}{m-1} \right\},
\]

then
\[
S_m(\alpha, \beta, \gamma) = \int_0^1 \cdots \int_0^1 \left( \prod_{j=1}^m x_j^{\alpha-1}(1-x_j)^{\beta-1} \right) |\Delta(x)|^{2\gamma} [dx]
\]
\[
= \frac{m!}{\prod_{j=1}^m \Gamma(\alpha + \gamma(j-1)) \Gamma(\beta + \gamma(j-1)) \Gamma(1 + \gamma j)} \Gamma(\alpha + \beta + \gamma(m+j-2)) \Gamma(1+\gamma),
\] (7.5)
where $\Delta(x) = \prod_{1 \leq i < j \leq m} (x_i - x_j)$ and $|dx| = \prod_{j=1}^{m} dx_j$. Furthermore, if $1 \leq k \leq m$, then
\[
\int_{0}^{1} \cdots \int_{0}^{1} \left( \prod_{j=1}^{k} x_j \right) \left( \prod_{j=1}^{m} x_j^{\alpha-1} (1-x_j)^{\beta-1} \right) |\Delta(x)|^{2\gamma} |dx| = S_m(\alpha, \beta, \gamma) \prod_{j=1}^{k} \frac{\alpha + \gamma(m-j)}{\alpha + \beta + \gamma(2m-j-1)}. \tag{7.6}
\]

The following two integrals (Propositions 7.3 and 7.4) are direct consequences of Proposition 7.2.

**Proposition 7.3 ([1]).** With the same conditions on the parameters $\alpha, \gamma$,
\[
\int_{0}^{\infty} \cdots \int_{0}^{\infty} |\Delta(x)|^{2\gamma} \prod_{j=1}^{m} x_j^{\alpha-1} e^{-x_j} dx_j = \prod_{j=1}^{m} \frac{\Gamma(\alpha + \gamma(j-1)) \Gamma(1+\gamma j)}{\Gamma(1+\gamma)}. \tag{7.7}
\]

**Proposition 7.4 ([1]).** With the same conditions on the parameters $\alpha, \gamma$, and $1 \leq k \leq m$,
\[
\int_{0}^{\infty} \cdots \int_{0}^{\infty} \left( \prod_{j=1}^{k} x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^{m} x_j^{\alpha-1} e^{-x_j} dx_j = \left( \prod_{j=1}^{k} (\alpha + \gamma(m-j)) \right) \left( \prod_{j=1}^{m} \frac{\Gamma(\alpha + \gamma(j-1)) \Gamma(1+\gamma j)}{\Gamma(1+\gamma)} \right). \tag{7.8}
\]

In the following, we prove Propositions 7.5 and 7.6 from Propositions 7.3 and 7.4, respectively, using the Laplace transform.

**Proposition 7.5 ([50, 51]).** It holds that
\[
\frac{1}{C_m(\alpha, \gamma)} := \int_{0}^{\infty} \cdots \int_{0}^{\infty} \delta \left( 1 - \sum_{j=1}^{m} x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^{m} x_j^{\alpha-1} dx_j = \frac{1}{\Gamma(\alpha m + \gamma(m-1))} \prod_{j=1}^{m} \frac{\Gamma(\alpha + \gamma(j-1)) \Gamma(1+\gamma j)}{\Gamma(1+\gamma)}. \tag{7.9}
\]

**Proof.** Let
\[
F(t) := \int_{0}^{\infty} \cdots \int_{0}^{\infty} \delta \left( t - \sum_{j=1}^{m} x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^{m} x_j^{\alpha-1} dx_j.
\]

Applying the Laplace transform ($t \to s$) to $F(t)$ gives us
\[
\mathcal{L}(F)(s) = \int_{0}^{\infty} F(t) e^{-st} dt = \int_{0}^{\infty} \cdots \int_{0}^{\infty} \exp \left( -s \sum_{j=1}^{m} x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^{m} x_j^{\alpha-1} dx_j
\]
\[
= s^{-\alpha m - 2\gamma} \int_{0}^{\infty} \cdots \int_{0}^{\infty} |\Delta(y)|^{2\gamma} \prod_{j=1}^{m} y_j^{\alpha-1} e^{-y_j} dy_j,
\]

leading to the following via the inverse Laplace transform ($s \to t$) to $\mathcal{L}(F)(s)$:
\[
F(t) = \frac{t^{\alpha m + \gamma(m-1)-1}}{\Gamma(\alpha m + \gamma(m-1))} \int_{0}^{\infty} \cdots \int_{0}^{\infty} |\Delta(x)|^{2\gamma} \prod_{j=1}^{m} x_j^{\alpha-1} e^{-x_j} dx_j.
\]

Therefore, we have
\[
\frac{1}{C_m(\alpha, \gamma)} = F(1) = \frac{1}{\Gamma(\alpha m + \gamma(m-1))} \times \int_{0}^{\infty} \cdots \int_{0}^{\infty} |\Delta(x)|^{2\gamma} \prod_{j=1}^{m} x_j^{\alpha-1} e^{-x_j} dx_j.
\]

Hence the desired identity via Eq. (7.7).
Proposition 7.6. It holds that, for $1 \leq k \leq m$,
\[
\int_0^\infty \cdots \int_0^\infty \left( \prod_{j=1}^k x_j \right) \delta \left( 1 - \sum_{j=1}^m x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{-1} dx_j = \frac{1}{\Gamma(am + \gamma m(m-1) + k)} \int_0^\infty \cdots \int_0^\infty \left( \prod_{j=1}^k x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{-1} e^{-x_j} dx_j.
\] (7.10)

Proof. Similarly, let
\[
f(t) := \int_0^\infty \cdots \int_0^\infty \left( \prod_{j=1}^k x_j \right) \delta \left( t - \sum_{j=1}^m x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{-1} dx_j.
\]
Then, the Laplace transform of $f(t)$ is given by
\[
\bar{f}(s) = \int_0^\infty \cdots \int_0^\infty \left( \prod_{j=1}^k x_j \right) \exp \left( - \sum_{j=1}^m sx_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{-1} dx_j.
\]

Therefore, we have
\[
f(t) := \frac{t^{am + \gamma m(m-1) + k-1}}{\Gamma(am + \gamma m(m-1) + k)} \int_0^\infty \cdots \int_0^\infty \left( \prod_{j=1}^k y_j \right) |\Delta(y)|^{2\gamma} \prod_{j=1}^m y_j^{-1} e^{-y_j} dy_j.
\]

By setting $t = 1$ in the above equation, we derived the desired identity via Eq. (7.8).

Proposition 7.7. It holds that
\[
\frac{d}{dt} \left( \frac{\Gamma(t+a)}{\Gamma(t+b)} \right) = (\psi(t+a) - \psi(t+b)) \frac{\Gamma(t+a)}{\Gamma(t+b)},
\] (7.11)
where $\psi(t) = \frac{d}{dt} \ln \Gamma(t)$.

7.3 The proof of Proposition 3.1 of the main text

A family of probability measures over $\mathbb{R}_+^m$ can be defined as:
\[
d\nu_{a,\gamma}(\Lambda) := C_{m}^{(a,\gamma)} K_{\gamma}(\Lambda) \prod_{j=1}^m \lambda_j^{a-1} d\lambda_j,
\] (7.12)
where $K_1(\Lambda)$ is given by
\[
K_1(\Lambda) = \delta \left( 1 - \sum_{j=1}^m \lambda_j \right) |\Delta(\lambda)|^2,
\] (7.13)
with $\Delta(\lambda) = \prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j)$ and $C_{m}^{(a,\gamma)} = 1/\mathcal{I}_m(a, \gamma)$ with
The subentropy of a state \( \rho \) with the spectrum \( \Lambda = \{ \lambda_1, \cdots, \lambda_m \} \) can be written as \([20, 13, 28, 7]\)

\[
Q(\Lambda) = (-1) \frac{m(m-1)}{2} \sum_{i=1}^{m-1} \lambda_i \ln \lambda_i \prod_{j \in i} \phi'(\lambda_j),
\]

\[
|\Delta(\lambda)|^2 = \left| \prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j) \right|^2.
\]

The average subentropy over the set of mixed state is given by

\[
I_m^\varpi(\alpha, \gamma) = \int \mu_{\varpi, \gamma}(\rho) Q(\rho) = \int \nu_{\varpi, \gamma}(\Lambda) Q(\Lambda).
\]

Denote \( \phi(x) := \prod_{j=1}^{m}(x - x_j) \). Then \( \phi'(x) = \sum_{i=1}^{m} \prod_{j \in i} (x - x_j) \). Thus \( \phi'(x_i) = \prod_{j \in i} (x_i - x_j) \). Furthermore, we have

\[
\prod_{i=1}^{m} \phi'(x_i) = \prod_{i=1}^{m} \prod_{j \in i} (x_i - x_j) = (-1)^{m(m-1)/2} |\Delta(x)|^2.
\]

Here \( |\Delta(x)|^2 = |\Delta(x_1, \ldots, x_m)|^2 \) is called the discriminant of \( \phi \) \([1]\). We also have

\[
\phi'(\lambda_2) \cdots \phi'(\lambda_m) = (-1)^{m(m-1)/2} \phi'(\lambda_1) |\Delta(\lambda_2, \ldots, \lambda_m)|^2.
\]

If we expand the polynomial \( \phi(x) \), then we have:

\[
\phi(x) = x^m - \left( \sum_{j=1}^{m} x_j \right) x^{m-1} + \cdots + (-1)^m \prod_{j=1}^{m} x_j = \sum_{j=0}^{m} (-1)^j e_j x^{m-j},
\]

where \( e_j (j=1, \ldots, m) \) is the \( j \)-th elementary symmetric polynomial in \( x_1, \ldots, x_m \), with \( e_0 \equiv 1 \).

In what follows, we calculate the integral \( I_m^\varpi(\alpha, \gamma) \) for \( \gamma = 1 \). Propositions 7.5 and 7.6 will be used frequently for \( \gamma = 1 \).

\[
I_m^\varpi(\alpha, 1) = -mc_m^{(\alpha, 1)} \sum_{k=0}^{m-1} (-1)^k \int_0^1 d\lambda_1 \lambda_1^{2(m-1)+\alpha-k} \ln \lambda_1 \times \int_0^\infty \cdots \int_0^\infty e_k \delta \left( 1 - \lambda_1 - \sum_{j=2}^{m} \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^{m} \lambda_j^{\alpha-1} d\lambda_j.
\]

It suffices to calculate a family of integrals in terms of the following form: for \( k = 0, 1, \ldots, m-1 \),

\[
\int_0^\infty \cdots \int_0^\infty e_k \delta \left( 1 - \lambda_1 - \sum_{j=2}^{m} \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^{m} \lambda_j^{\alpha-1} d\lambda_j.
\]

If \( k = 0 \), then

\[
\int_0^\infty \cdots \int_0^\infty e_0 \delta \left( 1 - \lambda_1 - \sum_{j=2}^{m} \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^{m} \lambda_j^{\alpha-1} d\lambda_j = (1 - \lambda_1)^{(m-1)(m+\alpha-2) - 1} \int_0^\infty \delta \left( 1 - \sum_{j=1}^{m-1} x_j \right) |\Delta(x_1, \ldots, x_{m-1})|^2 \prod_{j=1}^{m-1} x_j^{\alpha-1} dx_j = (1 - \lambda_1)^{(m-1)(m+\alpha-2) - 1} \prod_{j=1}^{m-1} \frac{\Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma((m - 1)(m + \alpha - 2))},
\]

\[
(7.20)
\]
Here we used Proposition 7.5 in the last equality.

If $1 \leq k \leq m - 1$, it suffices to calculate the following:

$$
\int_0^\infty \cdots \int_0^\infty \left( \frac{k}{\prod_{j=1}^m \lambda_{j+1}} \right) \delta \left( (1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{a-1} d\lambda_j
$$

$$
= (1 - \lambda_1)^{(m-1)(m+\alpha-2)+k-1} \times \\
\int_0^\infty \cdots \int_0^\infty \left( \frac{k}{\prod_{j=1}^m x_j} \right) \delta \left( (1 - \sum_{j=1}^{m-1} x_j) - \sum_{j=2}^m \lambda_j \right) |\Delta(x_1, \ldots, x_{m-1})|^2 \prod_{j=1}^{m-1} x_j^{a-1} dx_j
$$

$$
= (1 - \lambda_1)^{(m-1)(m+\alpha-2)+k-1} \frac{\prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j) \Gamma(m + \alpha - 1)}{\Gamma((m - 1)(m + \alpha - 2) + k) \Gamma(m + \alpha - 1 - k)}.
$$ (7.21)

Here we used Proposition 7.6. Next, we calculate the integral

$$
\int_0^1 d\lambda_1 \lambda_1^t \int_0^\infty \cdots \int_0^\infty \delta \left( (1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{a-1} d\lambda_j.
$$

(1). If $k = 0$, then

$$
\int_0^1 d\lambda_1 \lambda_1^t \int_0^\infty \cdots \int_0^\infty \delta \left( (1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{a-1} d\lambda_j
$$

$$
= \frac{\prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma((m - 1)(m + \alpha - 2))} \times \int_0^1 \lambda_1^t \Gamma(1 - \lambda_1)^{(m-1)(m+\alpha-2)-1} d\lambda_1
$$

$$
= \frac{\prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma((m - 1)(m + \alpha - 2))} \times \frac{\Gamma(t+1) \Gamma((m - 1)(m + \alpha - 2))}{\Gamma(t+1+(m - 1)(m + \alpha - 2))}
$$

$$
= \frac{\Gamma(t+1) \prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma(t+1+(m - 1)(m + \alpha - 2))}.
$$

By taking the derivative with respect to $t$ on both sides, we get

$$
\int_0^1 d\lambda_1 \lambda_1^t \ln \lambda_1 \int_0^\infty \cdots \int_0^\infty \delta \left( (1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{a-1} d\lambda_j
$$

$$
= [\psi(t+1) - \psi(t+1+(m - 1)(m + \alpha - 2))] \frac{\Gamma(t+1) \prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma(t+1+(m - 1)(m + \alpha - 2))}.
$$

For $t = 2(m - 1) + \alpha$, we have

$$
\int_0^1 d\lambda_1 \lambda_1^{2(m-1)+\alpha} \ln \lambda_1 \int_0^\infty \cdots \int_0^\infty \delta \left( (1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{a-1} d\lambda_j
$$

$$
= [\psi(2(m - 1) + \alpha + 1) - \psi(m(m + \alpha - 1) + 1)] \frac{\Gamma(2(m - 1) + \alpha + 1) \prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma(m(m + \alpha - 1) + 1)}.
$$
(2). If \( 1 \leq k \leq m - 1 \), then

\[
\int_0^1 d\lambda_1 \lambda_1^j \int_0^\infty \cdots \int_0^\infty e_k \delta \left( 1 - \lambda_1 - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha_j - 1} d\lambda_j
\]

\[
= \binom{m-1}{k} \int_0^1 d\lambda_1 \lambda_1^j \int_0^\infty \cdots \int_0^\infty \left( \prod_{j=1}^k \lambda_{j+1} \right) \delta \left( 1 - \lambda_1 - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha_j - 1} d\lambda_j
\]

\[
= \binom{m-1}{k} \frac{\Gamma(m-j-1)\Gamma(1+j)}{\Gamma(m+1)(m+\alpha-2+k)} \prod_{j=1}^k (m+\alpha-j-1) \times \int_0^1 \lambda_1^j (1-\lambda_1)^{(m-1)(m+\alpha-2)+k-1} d\lambda_1
\]

\[
= \binom{m-1}{k} \frac{\Gamma(t+1)\prod_{j=1}^{m-1} \Gamma(\alpha-j-1)\Gamma(1+j)}{\Gamma(t+1+(m-1)(m+\alpha-2)+k)} \prod_{j=1}^k (m+\alpha-j-1) \frac{\Gamma(m+\alpha-j-1)}{\Gamma(m+\alpha-j-1)}.
\]

By taking the derivative with respect to \( t \), we get

\[
\int_0^1 d\lambda_1 \lambda_1^j \ln \lambda_1 \int_0^\infty \cdots \int_0^\infty e_k \delta \left( 1 - \lambda_1 - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha_j - 1} d\lambda_j
\]

\[
= \binom{m-1}{k} \left[ \psi(t+1) - \psi(t+1+(m-1)(m+\alpha-2)+k) \right]
\]

\[
\times \frac{\Gamma(t+1)\prod_{j=1}^{m-1} \Gamma(\alpha-j-1)\Gamma(1+j)}{\Gamma(t+1+(m-1)(m+\alpha-2)+k)} \frac{\Gamma(m+\alpha-j-1)}{\Gamma(m+\alpha-j-1)}.
\]

For \( t = 2(m-1) + \alpha - k \), we have

\[
\int_0^1 d\lambda_1 \lambda_1^j (2(m-1)+\alpha-k) \ln \lambda_1 \int_0^\infty \cdots \int_0^\infty e_k \delta \left( 1 - \lambda_1 - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha_j - 1} d\lambda_j
\]

\[
= \binom{m-1}{k} \left[ \psi(2(m-1)+\alpha-k+1) - \psi(m+\alpha-1+1) \right]
\]

\[
\times \frac{\Gamma(2(m-1)+\alpha-k+1)\prod_{j=1}^{m-1} \Gamma(\alpha-j-1)\Gamma(1+j)}{\Gamma(m+\alpha-j-1+1)} \frac{\Gamma(m+\alpha-1+1)}{\Gamma(m+\alpha-1)}.
\]
In summary, we get

\[ I_m^{Q}(\alpha,1) = -mc^{(a,1)}_m \left[ \int_0^1 d\lambda_1 \lambda_1^{2(m-1)+\alpha} \ln \lambda_1 e_0 \int_0^\infty \cdots \int_0^\infty \delta \left( 1 - \lambda_1 - \sum_{j=2}^m \lambda_j \right) \mid \Delta(\lambda_2,\ldots,\lambda_m) \right]^2 \prod_{j=2}^m \lambda_j^{a-1} d\lambda_j \]

\[ + \sum_{k=1}^{m-1} (-1)^k \int_0^1 d\lambda_1 \lambda_1^{2(m-1)+\alpha-k} \ln \lambda_1 \int_0^\infty \cdots \int_0^\infty e_k \delta \left( 1 - \lambda_1 - \sum_{j=2}^m \lambda_j \right) \mid \Delta(\lambda_2,\ldots,\lambda_m) \right]^2 \prod_{j=2}^m \lambda_j^{a-1} d\lambda_j \]

\[ = -mc^{(a,1)}_m \sum_{k=0}^{m-1} (-1)^k \left( \frac{m-1}{k} \right) \left[ \psi(2(m-1)+\alpha-k+1) - \psi(m+\alpha-1+1) \right] \times \frac{\Gamma(2(m-1)+\alpha-k+1) \prod_{j=1}^{m-1} \Gamma(\alpha+j-1) \Gamma(1+j)}{\Gamma(m+\alpha-1+1) \Gamma(m+\alpha-1-k)} \]

\[ = -\frac{1}{m(m+\alpha-1)} \sum_{k=0}^{m-1} (-1)^k \left[ \psi(2(m-1)+\alpha-k+1) - \psi(m+\alpha-1+1) \right] \times \frac{\Gamma(2(m-1)+\alpha-k+1) \Gamma(k+1) \Gamma(m-k) \Gamma(m+\alpha-1-k)}{\Gamma(1) \Gamma(m-k) \Gamma(m+\alpha-1-k)} \].

(7.22)

Let us define

\[ g_{mk}(\alpha) = \psi(m+\alpha-1+1) - \psi(2(m-1)+\alpha+1-k), \]

(7.23)

and

\[ u_{mk}(\alpha) = \frac{(-1)^k \Gamma(2(m-1)+\alpha+1-k)}{\Gamma(k+1) \Gamma(m-k) \Gamma(m+\alpha-1-k)}. \]

(7.24)

Then, from Eq. (7.22), we have

\[ I_m^{Q}(\alpha,1) = \frac{1}{m(m+\alpha-1)} \sum_{k=0}^{m-1} g_{mk}(\alpha) u_{mk}(\alpha). \]

(7.25)

This completes the proof of Proposition 3.1 of main text. For \((\alpha, \gamma) = (n-m+1,1)\), we have

\[ I_m^{Q}(n-m+1,1) = \frac{1}{mn} \sum_{k=0}^{m-1} g_{nk}(n-m+1) u_{mk}(n-m+1). \]

(7.26)

If \(m = n\), this situation corresponds to the measure induced by the Hilbert-Schmidt distance \([50]\), then we have

\[ I_m^{Q}(1,1) = \frac{1}{m^2} \sum_{k=0}^{m-1} g_{nk}(1) u_{mk}(1). \]

(7.27)

In Eqs. (7.26) and (7.27), the functions \(g_{nk}\) and \(u_{mk}\) are given by Eqs. (7.23) and (7.24).
7.4 The proof of Lemma 3.2 of the main text

Now that
\[ T^Q_m(n - m + 1, 1) = \frac{1}{mn} \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m + n - k)}{k! \Gamma(m - k) \Gamma(n - k)} \left[ \psi(mn + 1) - \psi(m + n - k) \right], \]  
(7.28)
by the fact that \( \psi(v + 1) = H_v - \gamma \) for any positive integer \( v \), it follows that
\[ T^Q_m(n - m + 1, 1) = \frac{1}{mn} \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m + n - k)}{k! \Gamma(m - k) \Gamma(n - k)} (H_{mn} - H_{m+n-1-k}). \]  
(7.29)
To obtain the closed formula: \( T^Q(n - m + 1, 1) = 1 + H_{mn} - H_m - H_n \), it suffices to prove the following identities:
\[ \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m + n - k)}{k! \Gamma(m - k) \Gamma(n - k)} = mn, \]  
(7.30)
\[ \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m + n - k)}{k! \Gamma(m - k) \Gamma(n - k)} H_{m+n-1-k} = mn(H_m + H_n - 1). \]  
(7.31)
To this end, we need to fix some notations.
\[ \binom{z}{n} := \frac{z(z-1) \cdots (z-n+1)}{n!} \quad \text{for } n \in \mathbb{N} \text{ and arbitrary } z. \]

Binomial coefficients can be generalized to multinomial coefficients which is defined to be the number:
\[ \binom{n}{m_1, m_2, \ldots, m_q} = \frac{n!}{m_1! m_2! \cdots m_q!}, \]
where \( n = \sum_{j=1}^{q} m_j \). For any real \( z \in \mathbb{R} \) and positive integer pairs \( (m, n) \) with \( m \leq n \), it holds that\(^1\)
\[ \binom{z}{m} \binom{z}{n} = \sum_{k=0}^{m} \binom{m+n-k}{k,m-k,n-k} \binom{z}{m+n-k}. \]  
(7.32)
This identity had appeared earlier in Gould’s book [10, Eq. (6.44), pp57], and it was due to Riordan [38].
We know that, for any given real \( z \) and positive integer \( k \),
\[ \binom{-1}{k} = (-1)^k, \]  
(7.33)
\[ \binom{z}{k} = \frac{z(z-1)}{k(k-1)}, \]  
(7.34)
\[ \frac{d}{dz} \binom{z}{k} = \binom{z}{k} \sum_{i=0}^{k-1} \frac{1}{z-i}. \]  
(7.35)
Then
\[ \frac{d}{dz} \binom{z-1}{m-1} = \binom{z-1}{m-1} \sum_{i=0}^{m-2} \frac{1}{(z-1) - i} = \binom{z-1}{m-1} \sum_{i=1}^{m-1} \frac{1}{z-i}. \]  
(7.36)
\(^1\)This identity can be referred to the link: https://en.wikipedia.org/wiki/Binomial_coefficient
implying that
\[
\frac{d}{dz} \bigg|_{z=-1} \binom{z-1}{m-1} = \binom{-2}{m-1} \sum_{i=1}^{m-1} \frac{1}{-1-i} = (-1)^m m (H_m - 1).
\] (7.37)

Similarly
\[
\frac{d}{dz} \bigg|_{z=-1} \binom{z-1}{n-1} = \binom{-2}{n-1} \sum_{i=1}^{n-1} \frac{1}{-1-i} = (-1)^n n (H_n - 1).
\] (7.38)

We also see that
\[
\frac{d}{dz} \bigg|_{z=-1} \binom{z}{k} = \binom{-1}{k} \sum_{i=0}^{k-1} \frac{1}{-1-i} = (-1)^{k+1} H_k.
\] (7.39)

Note that
\[
\binom{z}{m} \binom{z}{n} = \sum_{k=0}^{m} \frac{(m+n-k)!}{(m-k)!(n-k)!k!} \binom{z}{m-n-k}.
\]

Replacement of \((m, n)\) by \((m-1, n-1)\) gives rise to
\[
\binom{z}{m-1} \binom{z}{n-1} = \sum_{k=0}^{m-1} \frac{(m+n-2-k)!}{(m-1-k)!(n-1-k)!k!} \binom{z}{m-n-2-k} \\
= \sum_{k=0}^{m-1} \frac{(m+n-1-k)!}{(m-1-k)!(n-1-k)!k!} \cdot \frac{1}{n+m-1-k} \binom{z}{m-n-2-k} \\
= \sum_{k=0}^{m-1} \frac{\Gamma(m+n-k)}{\Gamma(m-k)\Gamma(n-k)!} \cdot \frac{1}{n+m-1-k} \binom{z}{m-n-2-k},
\]
that is,
\[
\binom{z}{m-1} \binom{z}{n-1} = \sum_{k=0}^{m-1} \frac{\Gamma(m+n-k)}{\Gamma(m-k)\Gamma(n-k)!} \cdot \frac{1}{n+m-1-k} \binom{z}{m-n-2-k},
\]

multiplying both sides by \((z+1)\), we get
\[
(z+1) \binom{z}{m-1} \binom{z}{n-1} = \sum_{k=0}^{m-1} \frac{\Gamma(m+n-k)}{\Gamma(m-k)\Gamma(n-k)!} \cdot \frac{z+1}{n+m-1-k} \binom{z+1}{m-n-1-k}.
\]

Therefore
\[
(z+1) \binom{z}{m-1} \binom{z}{n-1} = \sum_{k=0}^{m-1} \frac{\Gamma(m+n-k)}{\Gamma(m-k)\Gamma(n-k)!} \cdot \frac{z+1}{n+m-1-k} \binom{z+1}{m-n-1-k}.
\]

Again, replacement of \(z+1\) by \(z\) gives rise to
\[
z \binom{z-1}{m-1} \binom{z-1}{n-1} = \sum_{k=0}^{m-1} \frac{\Gamma(m+n-k)}{\Gamma(m-k)\Gamma(n-k)!} \cdot \frac{z}{m+n-1-k}.
\] (7.40)

(i) Letting \(z = -1\) in Eq. (7.40) gives rise to
\[
\sum_{k=0}^{m-1} \frac{\Gamma(m+n-k)}{\Gamma(m-k)\Gamma(n-k)!} (-1)^{m+n-1-k} = - \binom{-2}{m-1} \binom{-2}{n-1} = (-1)^{m+n-1} mn.
\] (7.41)
Hence,

\[ \frac{1}{k!} \sum_{k=0}^{m-1} (-1)^k \Gamma (m + n - k) = mn. \] (7.42)

(ii). Furthermore, differentiating both sides at \( z = -1 \) leads to

\[ \sum_{k=0}^{m-1} \frac{\Gamma (m + n - k)}{\Gamma (m - k) \Gamma (n - k) k!} (-1)^{m+n-k} H_{m+n-1-k} \]

\[ = \left[ \left( \frac{z}{m-1} \right) \left( \frac{z}{n-1} \right) \left( 1 + z \sum_{i=1}^{m-1} \frac{1}{z+i} + z \sum_{i=1}^{n-1} \frac{1}{z+i} \right) \right]_{z=-1} \]

\[ = (-1)^m n \ln (H_m + H_n - 1). \]

Finally, we divide by \((-1)^{m+n}\) on both sides and get the conclusion:

\[ \sum_{k=0}^{m-1} \frac{\Gamma (m + n - k)}{\Gamma (m - k) \Gamma (n - k) k!} \frac{(-1)^k}{k!} H_{m+n-1-k} = mn (H_m + H_n - 1). \] (7.43)

Hence the result.

7.5 The proof of Theorem 4.1 of the main text

For \((\alpha, \gamma) = (n - m + 1, 1)\) the value of average subentropy \( T_m^Q (n - m + 1, 1) \) is given by Eq. (7.26). From the results of Page [31] and others [8, 40, 42] it is also known that

\[ T_m^Q (n - m + 1, 1) = H_m - H_n - \frac{m-1}{2n}. \] (7.44)

Let \( a_n = H_n - \ln n - \gamma_{\text{Euler}}. \) Clearly \( \lim_{n \to \infty} a_n = 0. \) Now

\[ T_m^Q (n - m + 1, 1) = \left( \ln n - \frac{m-1}{2n} \right) + (a_m - a_n). \]

We get \( T_m^S (1, 1) \propto \ln m - \frac{1}{2} \) when \( m \) becomes very large.

The average coherence of random mixed states is given by

\[ \overline{\gamma}_r (\alpha, \gamma) := \int d\mu_{\alpha, \gamma} (\rho) \overline{\gamma}_r (\rho) = \int d\mu_{\alpha, \gamma} (U \Lambda U^\dagger) \overline{\gamma}_r (U \Lambda U^\dagger) \]

\[ = \int d\nu_{\alpha, \gamma} (\Lambda) \left[ \int d\mu_{\text{Haar}} (U) S (U \Lambda U^\dagger) \right] - S (\Lambda) \]

\[ = H_m - 1 + \int d\nu_{\alpha, \gamma} (\Lambda) (Q (\Lambda) - S (\Lambda)) \]

\[ = H_m - 1 + T_m^Q (\alpha, \gamma) - T_m^S (\alpha, \gamma), \] (7.45)

where \( T_m^Q (\alpha, \gamma) = \int d\nu_{\alpha, \gamma} (\Lambda) Q (\Lambda) \) and \( T_m^S (\alpha, \gamma) = \int d\nu_{\alpha, \gamma} (\Lambda) S (\Lambda). \) Also, we have used the fact that the average coherence of the isospectral density matrices can be expressed in terms of the quantum subentropy, von Neumann entropy, and \( m\)-Harmonic number as follows [5]:

\[ \overline{\gamma}_r^{\text{iso}} (\Lambda) := \int d\mu_{\text{Haar}} (U) \overline{\gamma}_r (U \Lambda U^\dagger) \]

\[ = H_m - 1 + Q (\Lambda) - S (\Lambda). \]
Here $Q(\Lambda)$ is the subentropy, given by Eq. (7.15), $S(\Lambda)$ is the von Neumann entropy of $\Lambda$ and $H_m = \sum_{k=1}^{m} 1/k$ is the $m$-Harmonic number. Now using Eqs. (7.26) and (7.44), in Eq. (7.45) completes the proof of the theorem and $\overline{c}_r(n - m + 1, 1)$ is given by, via Lemma 3.2,

$$\overline{c}_r(n - m + 1, 1) = \frac{m - 1}{2n} + \left[ T_m^S(n - m + 1, 1) - (1 + H_m - H_n) \right] = \frac{m - 1}{2n}. \tag{7.46}$$

Similarly,

$$\overline{c}_r(1, 1) = \frac{m - 1}{2m}. \tag{7.47}$$

7.6 The proof of Theorem 4.2 of the main text

To prove Theorem 4.2 of the main text, we use the concentration of measure phenomenon and in particular, Lévy’s lemma [21, 14], which can be stated as follows:

**Lévy’s Lemma:** Let $\mathcal{F} : S^k \rightarrow \mathbb{R}$ be a Lipschitz function from $k$-sphere to real line with the Lipschitz constant $\eta$ (with respect to the Euclidean norm) and a point $X \in S^k$ be chosen uniformly at random. Then, for all $\epsilon > 0$,

$$\Pr \{ |\mathcal{F}(X) - \mathbb{E}\mathcal{F}| > \epsilon \} \leq 2 \exp \left( \frac{-(k + 1)\epsilon^2}{9\pi^3\eta^2 \ln 2} \right). \tag{7.48}$$

Here $\mathbb{E}(\mathcal{F})$ is the mean value of $\mathcal{F}$. But before we present the proof we need to find the Lipschitz constant for the relevant function on $S^k$ which is $G : S^{nm} \mapsto \mathbb{R}$, defined as $G(|\psi_{AB}|) = S \left( \rho_A^{(d)} \right) - S(\rho_A) = \overline{c}_r(\rho_A)$ where $\rho_A^{(d)}$ is the diagonal part of $\rho_A = \mathrm{Tr}_B(|\psi_{AB}\rangle \langle \psi_{AB}|)$.

**Lemma 7.8.** The function $\tilde{F} : S^{nm} \mapsto \mathbb{R}$, defined as $\tilde{F}(|\psi_{AB}|) = S(\rho_A^{(d)})$ where $\rho_A^{(d)} = \mathrm{Tr}_B(|\psi_{AB}\rangle \langle \psi_{AB}|)$ and $S$ is the von Neumann entropy, is a Lipschitz continuous function with Lipschitz constant $\sqrt{8} \ln m$.

**Proof.** The proof is given in Ref. [14].

**Lemma 7.9.** The function $F : S^{nm} \mapsto \mathbb{R}$, defined as $F(|\psi_{AB}|) = S(\rho_A^{(d)})$ where $\rho_A^{(d)}$ is the diagonal part of $\rho_A = \mathrm{Tr}_B(|\psi_{AB}\rangle \langle \psi_{AB}|)$ and $S$ is the von Neumann entropy, is a Lipschitz continuous function with Lipschitz constant $\sqrt{8} \ln m$.

**Proof.** We follow the proof strategy of Ref. [14]. Let $|\psi_{AB}| = \sum_{i=1}^{m} \sum_{j=1}^{m} \psi_{ij} |ij\rangle_A^B$ and therefore, $\rho_A^{(d)} = \sum_{i=1}^{d} p_i |i\rangle \langle i|$ with $p_i = \sum_{j} |\psi_{ij}|^2$. Now, $F(|\psi_{AB}|) = - \sum_{i=1}^{m} p_i \ln p_i$. The Lipschitz constant for $F$ can be bounded as follows:

$$\eta^2 := \sup_{|\psi\rangle \leq 1} \nabla F \cdot \nabla F = 4 \sum_{i=1}^{m} p_i [1 + \ln p_i]^2 \leq 4 \left( 1 + \sum_{i=1}^{m} p_i (\ln p_i)^2 \right) \leq 4 \left( 1 + (\ln m)^2 \right) \leq 8(\ln m)^2,$$

where the last inequality is true for $m \geq 3$. Therefore, $\eta \leq \sqrt{8} \ln m$ for $d \geq 3$. 

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Lemma 7.10. The function $G : S^{nm} \mapsto \mathbb{R}$, defined as $G(\langle \psi_{AB} \rangle) = S\left(\rho_A^{(d)}\right) - S(\rho_A)$ where $\rho_A^{(d)}$ is the diagonal part of $\rho_A = \text{Tr}_B(\langle \psi_{AB} \rangle \langle \psi_{AB} \rangle^\dagger)$ and $S$ is the von Neumann entropy, is a Lipschitz continuous function with the Lipschitz constant $2\sqrt{8\ln m}$.

Proof. Take $\sigma_A = \text{Tr}_B (\langle \phi_{AB} \rangle \langle \phi_{AB} \rangle^\dagger)$.

$$|G(\langle \psi_{AB} \rangle) - G(\langle \phi_{AB} \rangle)| := |S\left(\rho_A^{(d)}\right) - S\left(\sigma_A^{(d)}\right) - [S(\rho_A) - S(\sigma_A)]|$$
$$\leq |S\left(\rho_A^{(d)}\right) - S\left(\sigma_A^{(d)}\right)| + |S(\rho_A) - S(\sigma_A)|$$
$$\leq \sqrt{8\ln m} \|\psi_{AB}\rangle - \|\phi_{AB}\rangle_2 + \sqrt{8\ln m} \|\psi_{AB}\rangle - \|\phi_{AB}\rangle_2$$
$$\leq 2\sqrt{8\ln m} \|\psi_{AB}\rangle - \|\phi_{AB}\rangle_2.$$ 

Thus, $G$ is a Lipschitz continuous function with the Lipschitz constant $2\sqrt{8\ln m}$. 

Now applying Lévy’s lemma, Eq. (7.48), to the function $G(\langle \psi_{AB} \rangle) = \mathcal{C}_r(\rho_A)$, we have

$$\Pr\{|\mathcal{C}_r(\rho_A) - \mathcal{C}_r(n - m + 1,1)| > \epsilon\} \leq 2 \exp\left(-\frac{mn\epsilon^2}{144\pi^3 \ln 2(\ln m)^2}\right), \tag{7.49}$$

for all $\epsilon > 0$. This completes the proof of Theorem 4.2 of the main text.