An Ising-type formulation of the six-vertex model

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Abstract. We show that the celebrated six-vertex model of statistical mechanics (along with its multistate generalizations) can be reformulated as an Ising-type model with only a two-spin interaction. Such a reformulation unravels remarkable factorization properties for row to row transfer matrices, allowing one to uniformly derive all functional relations for their eigenvalues and present the coordinate Bethe ansatz for the eigenvectors for all higher spin generalizations of the six-vertex model. The possibility of the Ising-type formulation of these models raises questions about the precedence of the traditional quantum group description of the vertex models. Indeed, the role of a primary integrability condition is now played by the star-triangle relation, which is not entirely natural in the standard quantum group setting, but implies the vertex-type Yang-Baxter equation and commutativity of transfer matrices as simple corollaries. As a mathematical identity the emerging star-triangle relation is equivalent to the Pfaff-Saalschütz-Jackson summation formula, originally discovered by J. F. Pfaff in 1797. Plausibly, all vertex models associated with quantized affine Lie algebras and superalgebras can be reformulated as Ising-type models with only two-spin interactions.
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1 Introduction

There are various types of integrable models of statistical mechanics [1] where interacting “spins” are assigned to different parts of the square lattice. These include the Ising-type (edge-interaction), vertex and iteration-round-a-face (IRF) models, where the Boltzmann weights of local spin configurations are attributed to the edges, vertices or faces of the lattice, respectively. The integrability of these models requires the local Boltzmann weights to satisfy a relevant type of the Yang-Baxter equation [2–4], which for the Ising-type models usually takes the form of Onsager’s “star-triangle relation” [5].

The vertex type solutions of the Yang-Baxter equation (YBE), commonly called $R$-matrices, are well understood. Their classification is based on the theory of Quantum Groups [6–8]. In particular, there are infinite series of the so-called trigonometric solutions [9–23], connected with the evaluation representations of quantized affine Lie algebras. Similar considerations also apply to the corresponding IRF models, which are closely related to the vertex models through the IRF-vertex correspondence [24–26]. Here we consider the case of the $U_q(\hat{sl}(2))$ algebra, where $q$ is a deformation (or “anisotropy”) parameter. The standard trigonometric $R$-matrix $R(\lambda|s_1,s_2)$ in this case depends on a spectral variable $\lambda$ and parameters $s_1, s_2$, which define two highest weight representations of the algebra $U_q(sl(2))$. The associated lattice model is usually referred to as “a higher spin generalization of the 6-vertex model”.

Contrary to this holistic “quantum group” picture, the algebraic nature of the star-triangle relation and associated Ising-type models is much less clear. Of course, these models are connected to the vertex and IRF type models. For instance, there is a “star-square” transformation (see (1.3) below) between the Ising-type and IRF models, which has been applied to some specific models [27–29]. However, these examples in no way suggest how to construct an Ising-type formulation for an arbitrary integrable IRF model, in particular, for the IRF version of the higher spin generalization of the 6-vertex model, mentioned above.

Fortunately, some missing pieces in understanding of relevant algebraic structures are coming from the 3-dimensional interpretation [30,31] of the Yang-Baxter equation based on Zamolodchikov’s tetrahedron equation [32–34] and properties of its solutions [35–40]. In fact, the present paper is motivated by a remarkable result of Mangazeev [41], who used the above 3D interpretation to construct a very simple representation for the $U_q(\hat{sl}(2))$ trigonometric $R$-matrix $R(\lambda|s_1,s_2)$ expressing it via a terminating basic hypergeometric series $4\varphi_3$. We found that this result allows one to construct the desired Ising-type formulation for an arbitrary integrable IRF model, in particular, for the IRF version of the higher spin generalization of the 6-vertex model, mentioned above.

To illustrate the last point consider the standard vertex configurations of the usual 6-vertex model, which are shown below in two ways: (i) by the edge arrangements (with bold and thin edges) and (ii) by the corresponding integer “heights” arrangements, $a \in \mathbb{Z}$, as in the unrestricted solid-on-solid (SOS) model,

$$
\begin{align*}
&\begin{array}{c|c|c|c|c|c|c|c}
& a & a + 2 & a + 1 & a + 1 & a & a + 1 & a & a + 1 & a & a + 1 & a & a \\
\hline
a & a & a + 1 & a & a + 1 & a & a & a + 1 & a & a & a + 1 & a & a \\
\hline
\omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5 & \omega_6 & \omega_7 & \omega_8 & \omega_9 & \omega_{10} & \omega_{11} & \omega_{12} & \omega_{13}
\end{array}
\end{align*}
$$

where $\omega_1, \omega_2, \ldots, \omega_{13}$ denote the corresponding Boltzmann weights. For all other configurations the

---

1 The rules are: (a) two faces separated by the thin edge have the same height, (b) the face height increases by $+1$ when crossing a bold edge from right to left or from bottom to top. Note, that this trivial vertex-SOS correspondence is different from that used in [24,25].
weights are assumed to vanish identically. Below we assume that $\omega_5 = \omega_6$, noting that this does not really reduce generality, since for periodic boundary conditions the weights $\omega_5, \omega_6$ always appear in pairs. Now we replace the lattice vertices by the faces of the dual square lattice and go over from the SOS to the IRF formulation of the model, as shown below,

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
 a \\
 b \\
 c \\
 d \\
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
 a \\
 b \\
 c \\
 d \\
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
\end{array}
\end{array}
\end{align*}
\]

It is not difficult to check that the resulting IRF weights can be represented in two equivalent forms

\[
W(a, b, c, d) = \omega_1 \frac{v(b - d)}{v(a - c)} \sum_{n = \max(b, c)}^{a} w_1(a - n) w_2(n - b) w_3(n - c) w_4(n - d)
\]

\[
= \omega_1 \frac{v(a - b)}{v(c - d)} \sum_{n = d}^{\min(b, c)} w_4(a - n) w_3(b - n) w_2(c - n) w_1(n - d),
\]

as Boltzmann weights of four-edge stars of an Ising-type model with four different types of oriented diagonal edges. The corresponding edge weights $w_i(a - b)$, where $i = 1, 2, 3, 4$, depend on the difference of spins $a$ and $b$ at the ends of the edge (with the arrow pointing from $b$ to $a$). These weights vanish for negative arguments

\[
w_i(n) = 0, \quad n < 0, \quad i = 1, 2, 3, 4,
\]

therefore both summations over the central spins in (1.2) are restricted to finite intervals. Moreover, there are weights $v(a - b)$ and $v(a - b)^{-1}$ associated with horizontal and vertical edges in (1.2). These weights only enter into the equivalence transformation factors in (1.3), which cancel out in the partition function for periodic boundary conditions.

The only edge weights that are required to reproduce (1.1) are given by

\[
w_1(0) = w_2(0) = w_3(0) = w_4(0) = 1,
\]

\[
w_1(1) = \epsilon (\omega_5^2 - \omega_3 \omega_4)/ (\omega_1 \omega_5), \quad w_2(1) = \epsilon \omega_3 / \omega_5 \quad w_3(1) = \epsilon \omega_4 / \omega_5, \quad w_4(1) = \omega_5 / (\epsilon \omega_1),
\]

\[
w_4(2) = \omega_5^2 (\omega_1 \omega_2 + \omega_3 \omega_4 - \omega_5^2) / (\epsilon^2 \omega_1^2 \omega_3 \omega_4), \quad \epsilon = v(1)/v(0).
\]

Evidently, the weights $W(a, b, c, d)$ only depend on spin differences. For instance, if one sets $d = 0$ then the central spin $n$ in (1.3) could only take three values 0, 1, 2. Finally note, that Eq.(1.3) relating the Ising-type and IRF models is usually called the “star-square transformation”, while the second line in (1.3) relating the Boltzmann weights of two different four-edge stars is called the “star-star relation”.

Let us now return to the general higher spin 6-vertex model, discussed above. Its transformation to an Ising-type model follows exactly the same steps. First one replaces the vertex model by a SOS and then by an IRF model, as explained above. Note that the edge indices in the vertex model (as well as the face spin differences across the edge) can now take arbitrary non-negative integer values. Next, the resulting IRF weights are represented as Boltzmann weights of the first four-edge star in (1.3) with certain edge weights $w_1(a - b)$ and $v(a - b)$ defined for all non-negative spin differences $a - b \in \mathbb{Z}_{\geq 0}$ but having the same vanishing conditions (1.4). As a result the range of summation over the central spin is finite and determined by the same inequalities as in (1.3). Essentially, this representation is equivalent to the Mangazeev’s formula for the $R$-matrix $\mathcal{R}(\lambda | s_1, s_2)$ for the higher spin 6-vertex model.
The existence of the Ising-type formulation for this case raises questions about the primacy of
the quantum group structures in the description of vertex models. It suggests, that the Yang-Baxter
equations for $R(\lambda | s_1, s_2)$ may now be derived from simpler relations imposed on the edge weights.
Indeed, we show that it is a corollary of the star-star relation, which has the same form (1.3) as in
the two-state case. It is presented in Sect.3.3, see Fig. 3 therein. The proof of this relation is based
on the second Sears’s transformation formula [42] for the basic hypergeometric series $4\varphi_3$.

The star-star relation as a primary integrability condition has appeared previously [30] in the
context of the $sl(n)$-generalized chiral Potts model [43, 44]. For all other known integrable Ising-
type models the star-star relation could also be derived (see e.g., [28]), but it is a corollary of a much
simpler star-triangle relation. Therefore, even though we already have a proof of the star-star relation,
mentioned above, it is important to understand whether there are simpler integrability conditions in
the 6-vertex model and its generalizations, in particular, is there a star-triangle relation?

Starting with Mangazeev’s $4\varphi_3$ formula for the $R$-matrix it was natural to first conduct the search
in the area of the basic hypergeometric series, especially because it is extremely well documented,
see, e.g., [45–47]. A review of this literature has eventually revealed that the required star-triangle
relation can be derived from 1910 Jackson’s $q$-analog [48] of the Pfaff-Saalschütz summation formula,
originally discovered by Johann Friedrich Pfaff in 1797 [49] and rediscovered by Saalschütz in 1890 [50].
An interesting feature of this star-triangle relation is that it does not have an apparent “difference
property”, i.e., some of its edge weights depend on two spectral variables rather than their ratio
(we use multiplicative spectral variables). Nevertheless, we show that this relation implies the star-
star relation and the Yang-Baxter equation and therefore does, indeed, play the role of a primary
integrability condition in the higher spin generalization of the 6-vertex model.

The organization of the paper is as follows. In Sect. 2 we formulate a new inhomogeneous Ising-type
model on the square lattice. In Sect. 3, we present an equivalent IRF model and discuss its integra-
bility conditions, including the Yang-Baxter equation, the star-star and the star-triangle relations.
Connection to the 6-vertex model and its higher spin generalization is discussed in Sect. 4, where the
most important finite-dimensional reductions of the model are also studied. Factorized $R$-matrices
are considered in Sect. 5. Next we derive all functional relations for transfer matrices and Baxter’s
Q-operators (Sect. 6) and present the Bethe ansatz solution (Sect. 7) for a general inhomogeneous
model. The column homogeneous case is considered in Sect. 8. Some generalizations and extensions
of the results to other models are discussed in Sect. 9. In the Conclusion we briefly summarize a few
important aspects of our work and mention some possible applications.

2 An Ising-type model

Here we introduce a new two-dimensional solvable edge-interaction model. The model can be formu-
lated on rather general planar graphs, however, for the purposes of this presentation it is convenient
to take a regular square lattice. Consider an oriented square lattice drawn diagonally as in Fig. 1. The
edges of the lattice are shown by thick lines and the sites are shown with either open or filled circles in
a checkerboard order. We will refer to the latter as to the “white” or “black” sites, respectively. The
edges of the lattice are oriented as indicated by arrows, namely, all the SE-NW edges have the same
(SE-NW) direction, while the SW-NE edges are oriented in a checkerboard order, always pointing
towards the black sites.

The thin vertical and horizontal lines in Fig. 1 represent the medial graph whose vertices lie on
the edges of the original square lattice. The lines are directed as shown by arrows. Each of these lines
carries its own parameter (in general complex) which we call the “rapidity” or “spectral variable”. In

\footnote{The paper by Pfaff was part of a 1793 collection of the “Analytical Observations to L.Euler’s Institute of Integral
Calculus” which was published under the heading “History 1793” in “New Transactions of Imperial Academy of Sciences
in St.Petersburg”, volume 11. This volume is listed as published in 1793, but Pfaff’s paper is marked as “Presented to
Academy 14 January 1797”. This is probably the reason of a frequent volume-year mismatch in referencing this paper,
though after two hundred years the exact publication date might not be that important.}
Figure 1: An oriented square lattice and its medial lattice shown by thick and thin lines, respectively. All edges and lines are oriented as indicated by arrows. There are two types of edges distinguished by relative orientation of the edge and the associated thin lines, passing through the edge, shown in (2.1).

In general, these variables may all be different for different lines as illustrated in Fig. 1. The minimal level of generality corresponds to the homogeneous case when one assigns the same variable $x$ to all lines directed upwards (i.e., $x_1 = x_2 = \ldots = x_N \equiv x$) and the same variable $x'$ to all lines directed downwards (i.e., $x'_1 = x'_2 = \ldots = x'_N \equiv x'$). Similarly, one assigns the same variables $y$ and $y'$ to all thin horizontal lines directed right and left, respectively.

At each lattice site place an integer spin variable $a \in \mathbb{Z}$. Two spins interact only if they are connected with an edge. There are two types of edges distinguished by relative orientation of the edge and the associated thin lines, passing through the edge,

\[(i) : \quad V_{x/y}(a-b), \quad (ii) : \quad \frac{1}{V_{x/y}(a-b)}.\]

The corresponding Boltzmann weights $V_{x/y}(a-b)$ and $(V_{x/y}(a-b))^{-1}$ depend on the difference of spins $a$ and $b$ at the ends of the edge and on the ratio of two thin line variables $x$ and $y$. The edge arrow pointing from $b$ to $a$ shows that the argument of the weight function is $(a-b)$ rather than $(b-a)$. For an easier perception the edges are also distinguished by their color: the blue color for the edges of type (i) and the red color for those of the type (ii).

The function $V_x(n)$ is defined as

\[V_x(n) = \left( \frac{q}{x} \right)^n \frac{(x^2;q^2)_n}{(q^2;q^2)_n}, \quad n \in \mathbb{Z},\]
where \( n \) is an integer, \( n \in \mathbb{Z} \),

\[
(x; q^2)_n = \prod_{j=0}^{n-1} (1 - x q^{2j}), \quad (x; q^2)_{-n} = \frac{1}{(xq^{-2n}; q^2)_n},
\]

and \( q \) is an arbitrary parameter of the model. Note that

\[
V_x(n) \equiv 0, \quad \text{for } n < 0,
\]

and

\[
\sum_{n=-\infty}^{\infty} V_{a-n}(x^{-1})V_{n-b}(x) = \delta_{a,b},
\]

where, due to (2.4), the summation is effectively restricted to a finite interval \( a \geq n \geq b \). Obviously, the edge weights are “chiral”, since

\[
V_x(n) \neq V_x(-n), \quad n \neq 0.
\]

Moreover, the vanishing property (2.4) is potentially dangerous for the type (ii) edges with negative spin difference arguments in (2.1), since their Boltzmann weights then diverge. However, this does not create any real problems, since such spin configurations will be excluded, see below.

Now we need to describe the boundary conditions, which play rather important role in the definition of the model. First, we assume that the values of white spins at the bottom row of the lattice (see Fig. 1) are ordered as

\[
a_1 \leq a_2 \leq a_3 \leq \ldots
\]

and so on, starting from right to left and, similarly, the values of white spins at the right vertical boundary of the lattice are ordered as

\[
a_1 \leq b_1 \leq c_1 \leq \ldots
\]

starting from bottom to top.

The Boltzmann weight of a particular spin configuration of the whole lattice is equal to the product of all the edge weights. The lattice spin configuration will be called admissible if all the spin differences appearing in the edge weight functions (2.1) are non-negative (i.e., for admissible configurations the values of spins cannot decrease along the edge directions). Remarkably, for the boundary conditions (2.7) the admissible configurations exhaust all lattice spin configurations with non-zero Boltzmann weight. Of course, the exclusion of non-admissible configurations containing only type (i) edges with negative spin arguments is obvious, since their weights vanish identically due to (2.4). However, this is not obvious for configurations containing type (ii) edges with \( a < b \) in (2.1), since their weights then become infinite. Our statement follows from the limiting procedure based on the replacement of (2.4) by

\[
V_x(n) = O(\varepsilon), \quad \varepsilon \to 0, \quad n < 0.
\]

Then a careful analysis shows, that as a consequence of (2.7), all divergent \( 1/\varepsilon \) pole factors coming from the type (ii) edges are always compensated by zeros in \( \varepsilon \) coming the type (i) edges. Actually, in all cases the degree of zero is always higher than the degree of pole, so that the Boltzmann weights of all non-admissible spin configurations with the boundary conditions (2.7) do indeed vanish. Some additional explanations of this point are given after (3.2) in the next section.

Next, we impose (quasi) periodic boundary conditions in the horizontal direction. The lattice in Fig. 1 has \( N \) sites per row. Clearly, for consistency with spin difference dependence of the Boltzmann weights, one needs to formulate the periodic boundary condition in terms of differences of spins on the left and right vertical boundaries. Here we choose the simplest option

\[
b_{N+1} - a_{N+1} = b_1 - a_1, \quad c_{N+1} - b_{N+1} = c_1 - b_1, \quad \text{etc.}
\]
by equating the jumps of boundary spins between successive rows on both sides of the lattice. If required, the periodic boundary conditions in the vertical direction should also be defined in terms of differences of boundary spins between successive columns of the lattice. Finally, introduce the “boundary fields” by assigning additional weights to the spin jumps on the horizontal and vertical boundaries of the lattice

$$w_{\text{fields}} = \left( \omega_{a}^{2-a_1} \omega_{a}^{3-a_2} \cdots \omega_{a}^{N+1-a_N} \right) \times \left( \omega_{b}^{h_1-a_1} \omega_{b}^{c_1-b_1} \cdots \right),$$

(2.10)

where \( \omega_{h} \) and \( \omega_{v} \) are arbitrary parameters.

The partition function is defined as a sum of the products of edge weights (2.1) and the field factors (2.10) over all admissible spin configurations,

$$Z = \sum_{(\text{admissible spins})} w_{\text{fields}} \prod_{(\text{edges})} \left( \text{edge weights (2.1)} \right).$$

(2.11)

Obviously, the weights remain unchanged if all lattice spins are simultaneously incremented by the same integer. Therefore, one particular spin, say the spin \( a_1 \) at the bottom right corner of the lattice in Fig. 1, should be set to some fixed value (we chose \( a_1 \equiv 0 \)) to avoid a repeated counting of equivalent spin configurations.

To summarize, we have defined an Ising-type model on the square lattice, see Fig. 1, with integer-valued spins at the lattice sites. The spins interact with their nearest neighbours (two spins interact only if they are connected by an edge). The edge Boltzmann weights depend on the difference of spins at the ends of the edge and the ratio of two spectral variables assigned to the medial graph lines passing through the edge. There are two types of edges, which are assigned with different Boltzmann weights defined in (2.1), (2.2). The admissible spin configurations are controlled by the boundary conditions (2.7), which play an essential role in the definition of the model. In the homogeneous lattice case the model has six arbitrary, in general complex, parameters: the “anisotropy” parameter \( q \), the boundary field parameters \( \omega_{h}, \omega_{v} \) and three independent ratios of the four spectral variables \( x, x', y, y' \). Note, that this counting does not include the trivial overall normalization of the Boltzmann weights. For a fully inhomogeneous lattice there are additional spectral variables: there are two such variables for each column and two variables for each row (these are the variables \( x_1, x'_1, x_2, x'_2, \ldots \) and \( y_1, y'_1, y_2, y'_2, \ldots \), respectively, shown in Fig. 1). Recall, the weights only depend on the ratios of these variables.

3 Equivalent models and integrability conditions

3.1 Interaction-round-a-face formulation

The edge interaction model introduced above could be reformulated either as a vertex model or as an interaction-round-a-face (IRF) model. In fact, this can be done in a variety of ways. Let us first describe an equivalent IRF model on the square lattice formed by the white sites in Fig. 1. The lattice in Fig. 1 can be divided into four-edge “stars” with white corner sites, such as the star shown Fig. 2. Applying the rules (2.1) one can write the IRF-type Boltzmann weights for this star as

$$S(a, b, c, d \mid x, x', y, y') = \sum_{n=\max(b,c)}^{a} \frac{V_{x/y}(a-n)V_{y'/x'}(n-b)V_{y/x}(n-c)}{V_{y'/x'}(n-d)}.$$  

(3.1)

The weights depend on the four corner spins \( a, b, c, d \in \mathbb{Z} \), which are supposed to obey the relations

$$a \geq b \geq d, \quad a \geq c \geq d.$$  

(3.2)

Outside this domain the weights (3.1) are assumed to vanish identically. It is important to understand the origin of these restrictions as well as the range of summation over the central spin in (3.1). Consider the four-edge star at the bottom right corner of Fig. 1. The boundary conditions (2.7) require the
ordering of the values of spins $a_2 \geq a_1$ at the bottom and $b_1 \geq a_1$ at the right side of that star. In the notation of Fig. 2 this corresponds to

$$c \geq d, \quad b \geq d.$$  

(3.3)

Let us now examine the sum in (3.1). To regularize the summand for the case when its denominator vanishes we replace the strict vanishing condition (2.4) by the regularized version (2.8). The function $V_{y/x'}(n-d)$ in the denominator of (3.1) vanishes if its argument is negative, i.e., when $n < d$. However, due to (3.3) this means that in this case $n < b$ as well as $n < c$, therefore, at least two factors $V_{y'/x'(n-b)}$ and $V_{y/x}(n-c)$ in the numerator of (3.1) must vanish, i.e., the degree of zero in $\epsilon$ is higher than the degree of the pole. Therefore the configurations with a vanishing denominator from the type (ii) edges do not contribute. Thus, the non-vanishing contributions to the sum (3.1) could only arise when $n \geq \max(b, c)$. Next, note that the first factor $V_{x'/y'}(a-n)$ in the numerator is non-zero only when $a \geq n$. This explains the range of summation in (3.1) as well as the domain (3.2) where the star weights (3.1) do not vanish.

Let us now return to the bottom right star in Fig. 1. The conditions (3.2) imply an ordering of spins at the top and left sides of that star,

$$b_2 \geq b_1, \quad b_2 \geq a_2.$$  

(3.4)

Combining these with (2.7) we can now apply the same reasonings to the stars which are either directly above or to the left of the bottom right corner star, that we started with. In this way the ordering of white spins in the horizontal and vertical directions, which is initially only implied by (2.7) to the boundary, will propagate to the whole lattice.\(^3\) Thus, we have reformulated our model as an IRF model on the square lattice formed by the white sites, with Boltzmann weights given by (3.1), (3.2). The partition function (2.11) of the edge-interaction model from the previous Section can now be equivalently rewritten

$$Z = \sum_{\text{(white spins)}} w_{\text{fields}} \prod_{\text{(black-center stars)}} S(a, b, c, d \mid x, x', y, y').$$  

(3.5)

3.2 Commuting transfer matrices

Below it will be convenient to use the IRF weights

$$W(a, b, c, d \mid x, x', y, y') = \frac{V_{y/y'}(b-d)}{V_{y'/x'}(a-c)} S(a, b, c, d \mid x, x', y, y').$$  

(3.6)

\(^3\)It is not difficult to see that this result is essentially equivalent to the statement about the admissible spin configurations with non-zero Boltzmann weights for the edge-interaction formulation presented in the previous section.
which only differ from (3.1) by simple equivalence transformation factors. Consider the transfer matrix of the edge-interaction model between the two bottom rows of white spins in Fig. 1, with the quasi-periodic boundary conditions (2.9),

$$b_{N+1} - a_{N+1} = b_1 - a_1. \quad (3.7)$$

Its matrix elements can be written using the IRF weights (3.6),

$$\left( \mathbb{T}(y, y' | x_N, x'_N, \ldots, x_1, x'_1) \right)_{b_{N+1}, b_N, \ldots, b_1}^{a_{N+1}, a_N, \ldots, a_1} = \omega_k^{b_1-a_1} \omega_v^{a_{N+1}-a_1} \prod_{i=1}^N W(b_{i+1}, b_i, a_{i+1}, a_i | x_i, x'_i, y, y') \quad (3.8)$$

with the a’s and b’s labeling matrix rows and columns, respectively. The parameters $y_1, y'_1$ from Fig. 1 have been replaced by $y, y'$. Note, that the equivalence transformation factors, introduced in (3.6), cancel out in the definition of the transfer matrix (i.e., the expression (3.8) remains unchanged, if the weights $W$ therein are replaced by the weights $S$). Recall, that the weights $W(a, b, c, d)$ take non-zero values only if the corner spins $a, b, c, d$ lie in the domain (3.2). As a result the matrix elements (3.8) are non-zero

$$\left( \mathbb{T}(y, y' | x_N, x'_N, \ldots, x_1, x'_1) \right)_{b_{N+1}, b_N, \ldots, b_1}^{a_{N+1}, a_N, \ldots, a_1} \neq 0, \quad \text{iff} \quad a_i \leq b_i, \quad i = 1, 2, \ldots, N + 1, \quad (3.9)$$

only if $a_i \leq b_i$ for all $i = 1, 2, \ldots, N + 1$.

The transfer matrix acts in the infinite-dimensional vector space $\mathcal{H}_N^{(IRF)}$ spanned by the vectors $\psi_a$ labeled by ordered spin sequences

$$\psi_a \in \mathcal{H}_N^{(IRF)}, \quad a = (a_{N+1}, a_N, \ldots, a_1) \quad a_i \in \mathbb{Z}, \quad a_{N+1} \geq a_N \geq \cdots \geq a_1. \quad (3.10)$$

As a consequence of (3.7), the transfer matrix preserves an integer quantity

$$M = \sum_{i=1}^N (a_{i+1} - a_i) = a_{N+1} - a_1 = b_{N+1} - b_1 = \sum_{i=1}^N (b_{i+1} - b_i), \quad (3.11)$$

which is similar to the conserved number of “down arrows” in the six-vertex model. For this reason the dependence on the vertical field parameter $\omega_v$ in (3.8) becomes trivial. Therefore, in what follows we set $\omega_v = 1$.

The lattice model defined above is integrable. Namely, the transfer matrices (3.8) with different values of $y$ and $y'$ (but the same sets of $\{x\}$ and $\{x'\}$) form a two-parameter commuting family

$$\left[ \mathbb{T}(y, y' | x_N, x'_N, \ldots, x_1, x'_1), \mathbb{T}(z, z' | x_N, x'_N, \ldots, x_1, x'_1) \right] = 0, \quad \forall y, y', z, z'. \quad (3.12)$$

This follows in a standard way from the IRF-type Yang-Baxter equation for the weights (3.6),

$$\sum_{a \in \mathbb{Z}} W(g, a, b, f | x, y) W(c, e, g, a | x, z) W(c, d, a, f | y, z) = \sum_{h \in \mathbb{Z}} W(c, h, g, b | y, z) W(h, d, b, f | x, z) W(c, e, h, d | x, y) \quad (3.13)$$

where we have introduced compact notations for pairs of spectral variables

$$x = (x, x'), \quad y = (y, y'), \quad z = (z, z'), \quad \text{etc.}, \quad (3.14)$$

since they always appear in pairs. Note that due to the corner spin constrains (3.2) for the weights (3.6) both summations in (3.13) are restricted to finite intervals.

For future references we mention here some elementary symmetry transformations of the YBE (3.13). In particular, it is not affected by the re-scaling of the weights

$$W(a, b, c, d | x, y) \rightarrow \frac{g_x(c, d)}{g_x(a, b)} \frac{g_y(a, c)}{g_y(b, d)} W(a, b, c, d | x, y), \quad (3.15)$$

with arbitrary non-vanishing factors $g_x(a, b)$. 

3.3 The star-star relation

The proof of Yang-Baxter equation (3.13) is based on the so-called “star-star” relation for the edge weights (2.1), which we present below. The Boltzmann weights (3.6), (3.1) are associated with the black-centered four-edge stars of the type shown in Fig. 2. Writing them in full, one gets

\[ W(a,b,c,d \mid x,x',y,y') = \frac{V_{y/y'}(b-d)}{V_{y/y'}(a-c)} \sum_{n=\max(b,c)}^{a} \frac{V_{x/x'}(a-n)V_{y/x'}(n-b)V_{y/x}(n-c)}{V_{y/x'}(n-d)}. \] (3.16)

Similarly, one can define IRF-type weights associated with the white-centered stars in Fig. 1 having black corner spins,

\[ W(a,b,c,d \mid x,x',y,y') = \frac{V_{x/x'}(a-b)}{V_{x/x'}(c-d)} \sum_{n=d}^{\min(b,c)} V_{x/x'}(n-d)V_{y/x'}(c-n)V_{y/x}(b-n). \] (3.17)

It turns out that the two definitions (with suitably chosen pre-factors in (3.16) and (3.17)) lead to the same result,

\[ W(a,b,c,d \mid x,x',y,y') = \overline{W}(a,b,c,d \mid x,x',y,y'). \] (3.18)

This relation is a statement of equality of the Boltzmann weights of the two “stars” shown graphically in Fig. 3, so it can naturally be called the star-star relation. A verification of this identity is based on

\[ \begin{array}{c}
\text{Figure 3: A graphical representation of the star-star relation (3.18).} \\
\end{array} \]

a repeated application of the second Sears’s transformation formula for basic hypergeometric series (see [42]). Details of these calculations together with a subsequent proof of the Yang-Baxter equation (3.13) are presented in the Appendix A.

3.4 The star-triangle relation

The “star-star” relation as a primary integrability condition has previously appeared in [30, 35] for 3-dimensional solvable models [30, 32], which also cover the $sl(n)$-generalized chiral Potts model in 2D [43,44]. For all other known integrable 2D edge interaction models the star-star relation could also be derived (see e.g., [28]), but it is a corollary of a much simpler star-triangle relation. Therefore, even though we already have a proof of the star-star relation (3.18), mentioned above (see Appendix A), it is important to understand whether there are simpler integrability conditions in our model, in particular, the star-triangle relation. Taking into account that the topic of the hypergeometric series has been intensively studied by many mathematician for more than two centuries, it appears that the relevant star-triangle relation — if it exists at all — should have already been discovered. Indeed, a purposeful study of the literature shows that the desired relation is the 1910 Jackson’s $q$-analog [48] of
the Pfaff-Saalschütz formula, originally discovered by Pfaff in 1797 [49] and rediscovered by Saalschütz in 1890 [50]. For a brief introduction see the review article [47].

To present the star-triangle relation we need to introduce yet another edge weight function \( W_{x,y}(a-b) \), which depends on the difference of edge spins \( a \) and \( b \) and on two spectral variables \( x \) and \( y \) (but not just on their ratio),

\[
W_{x,y}(n) = \left( \frac{y}{x} \right)^n \frac{(x^2;q^2)_n}{(y^2;q^2)_n}, \quad n \in \mathbb{Z}.
\] (3.19)

It possesses the following symmetries

\[
W_{x,y}(n) = W_{q/y,q/x}(-n) = 1
\] (3.20)

Note, in particular, that

\[
W_{x,q}(n) = V_x(n), \quad W_{q,x}(n) = \frac{1}{V_x(n)}.
\] (3.21)

The weights \( V \) and \( W \) satisfy the following two star-triangle relations,

\[
\sum_d V_{y/z}(d-a) W_{z,x}(b-d) V_{x/y}(c-d) = W_{z,y}(b-c) V_{x/z}(c-a) W_{y,x}(b-a), \quad (3.22a)
\]

and

\[
\sum_d V_{y/z}(a-d) W_{z,x}(d-b) V_{x/y}(d-c) = W_{z,y}(c-b) V_{x/z}(a-c) W_{y,x}(a-b), \quad (3.22b)
\]

where due to (2.4) the summations are restricted to finite intervals. The two relations are corollaries of each other under the substitution

\[
b \leftrightarrow c, \quad x \rightarrow q/z, \quad y \rightarrow q/y, \quad z \rightarrow q/x.
\] (3.23)

As mathematical identities, they are equivalent to the most general form of the Pfaff-Saalschütz-Jackson summation formula for the terminating balanced series \( _3\varphi_2 \), see eq.(3.5.1) in [47]. The details of the correspondence are given in Appendix B, where we also present a graphical representation of the star-triangle relation (3.22) and define yet another integrable edge interaction model involving the edge weights (3.19). Moreover, in the same Appendix we show that this relation implies the star-star relation (3.18) and, consequently, the Yang-Baxter equation (3.13) and commutativity of the transfer matrices (3.12). Therefore the star-triangle relation (3.22) does, indeed, play the role of a primary integrability condition in the model.

4 Connection to the six-vertex model

The reader might have already become perplexed on how exactly the above constructions are related to the six-vertex model mentioned in the title of the paper. These connections are discussed below.

4.1 Higher spin generalizations of the six-vertex model

In the previous section we have reformulated our edge interaction (Ising-type) model as an IRF model on the square lattice. Its face weights depend on differences of corner spins, therefore one can easily convert this IRF model into a vertex model by replacing the face weights by an equivalent vertex \( R \)-matrix,

\[
\mathcal{R}(x',y,y')_{ij}^{j_1,j_2} = \delta_{i_1+i_2,j_1+j_2} W(i_1+i_2,j_2,i_1,0|x,y,y').
\] (4.1)

where the weights \( W \) are defined by (3.16) (or, equivalently, by (3.6) and (3.2)). The matrix indices

\[
i_1 = c - d, \quad i_2 = a - c, \quad j_1 = a - b, \quad j_2 = b - d.
\] (4.2)
are defined as differences of corner spins, arranged as in Fig. 4. Due to (3.2) these indices take non-negative integer values
\[ i_1, i_2, j_1, j_2 \geq 0, \quad i_1, i_2, j_1, j_2 \in \mathbb{Z}. \]

The emergence of the conservation law
\[ i_1 + i_2 = j_1 + j_2, \tag{4.4} \]
appearing in the delta-function in (4.1) is a trivial consequence of the definitions (4.2). Using the two representations for the IRF weights (3.16) and (3.17) one can write (4.1) in the form
\[
\mathcal{R}(x, x', y, y')^{j_1, j_2}_{i_1, i_2} = \delta_{i_1+i_2,j_1+j_2} \sum_{n=0}^{\min(i_2,j_1)} \frac{V_{y/y'}(j_2) V_{y/y'}(i_2)}{V_{y/y'}(i_1+i_2-n)} V_{x/y'}(i_1-n) V_{y/x}(j_2-n).
\]

With these notations the Yang-Baxter equation (3.13) takes the form
\[
\sum_{j_1, j_2, j_3} \mathcal{R}(x, x', y, y')^{j_1, j_2}_{i_1, i_2} \mathcal{R}(x, x', z, z')^{j_3}_{j_1, j_2} \mathcal{R}(y, y', z, z')^{k_2, k_3}_{j_2, j_3} = \sum_{j_1, j_2, j_3} \mathcal{R}(y, y', z, z')^{j_2, j_3}_{i_2, i_3} \mathcal{R}(x, x', z, z')^{j_1, k_3}_{j_1, j_3} \mathcal{R}(x, x', y, y')^{k_1, k_2}_{j_1, j_2}.
\tag{4.6}
\]

Note that due to (4.3) and (4.4) all summations here are restricted to finite intervals. It is worth mentioning some simple symmetries of this equation. Obviously, it is invariant under the simultaneous transposition in all three vector spaces. Below we will use this symmetry in combination with the rescaling of the weights as in (3.15). It is easy to check that (4.6) is invariant under the transformation
\[
\mathcal{R}(x, x', y, y')^{j_1, j_2}_{i_1, i_2} \rightarrow \mathcal{R}^{T}(x, x', y, y')^{j_1, j_2}_{i_1, i_2} = \left( \frac{y_{j_1} y_{j_2}}{y_{i_1} y_{i_2}} \right)^{-1} \mathcal{R}^{T}(x, x', y, y')^{j_1, j_2}_{i_1, i_2},
\tag{4.7}
\]
for all three $R$-matrices (with corresponding substitutions of their arguments). Here the superscript $T$ denotes the matrix transposition of $\mathcal{R}$ in both spaces
\[
\mathcal{R}^{T}(x, x', y, y')^{j_1, j_2}_{i_1, i_2} = \mathcal{R}(x, x', y, y')^{i_1, i_2}_{j_1, j_2}.
\tag{4.8}
\]

The scaling factor in (4.7) exactly coincides to that in (3.15) with $g_{ab}(a, b) = x^b/x^a$ (taking into account (4.1), (4.2)).
Note, that the $R$-matrix (4.5) only depends on three independent ratios of the four spectral parameters $x, x', y, y'$. Therefore, it is useful to introduce a new set of parameters

$$x = \lambda_1 q^{-s_1}, \quad x' = \lambda_1 q^{s_1}, \quad y = \lambda_2 q^{-s_2}, \quad y' = \lambda_2 q^{s_2}, \quad z = \lambda_3 q^{-s_3}, \quad z' = \lambda_3 q^{s_3}. \quad (4.9)$$

Then it is easy to see that (4.5) only depends on the variables $\lambda_1/\lambda_2, s_1$ and $s_2$. Therefore we will write it as

$$\mathcal{R}(\lambda_1/\lambda_2 \mid s_1, s_2) = \mathcal{R}(x, x', y, y'), \quad (4.10)$$

provided the parameters are related as in (4.9). The Yang-Baxter equation (4.6) then takes the form

$$\sum_{j_1,j_2,j_3} \mathcal{R}(\lambda_1/\lambda_2 \mid s_1, s_2)^{j_1,j_2}_{i_1,i_2} \mathcal{R}(\lambda_1/\lambda_3 \mid s_1, s_3)^{k_1,j_3}_{j_1,i_3} \mathcal{R}(\lambda_2/\lambda_3 \mid s_2, s_3)^{k_2,k_3}_{j_2,j_3} = \sum_{j_1,j_2,j_3} \mathcal{R}(\lambda_2/\lambda_3 \mid s_2, s_3)^{j_2,j_3}_{i_2,i_3} \mathcal{R}(\lambda_1/\lambda_3 \mid s_1, s_3)^{j_1,k_3}_{j_1,j_3} \mathcal{R}(\lambda_1/\lambda_2 \mid s_1, s_2)^{k_1,k_2}_{j_1,j_2}. \quad (4.11)$$

Let us mention also two symmetry relations,

$$\mathcal{R}(\lambda \mid s_1, s_2)^{i_1,i_2}_{j_1,j_2} = \frac{V q^{-s_1} (i_1)V q^{-s_2} (i_2)}{V q^{-s_1} (j_1)V q^{-s_2} (j_2)} \mathcal{R}(\lambda \mid s_1, s_2)^{j_1,j_2}_{i_1,i_2}, \quad (4.12)$$

$$\mathcal{R}(\lambda \mid s_2, s_1)^{j_2,j_1}_{i_2,i_1} = \mathcal{R}(\lambda \mid s_1, s_2)^{j_1,j_2}_{i_1,i_2}, \quad (4.13)$$

and the normalization conditions

$$\lim_{\lambda \to 0} \mathcal{R}(\lambda \mid s_1, s_2)^{j_1,j_2}_{i_1,i_2} = \delta_{i_1,j_1} \delta_{i_2,j_2} q^{-2(s_1-j_1)(s_2-j_2)+2s_1s_2},$$

$$\lim_{\lambda \to \infty} \mathcal{R}(\lambda \mid s_1, s_2)^{j_1,j_2}_{i_1,i_2} = \delta_{i_1,j_1} \delta_{i_2,j_2} q^{2(s_1-j_1)(s_2-j_2)-2s_1s_2}, \quad (4.14)$$

which simply follow from (4.5) for fixed values of $s_1$ and $s_2$.

We state that the $R$-matrix (4.5) coincides with Mangazeev’s $R$-matrix [41] for the higher spin generalizations of the 6-vertex model. It is the trigonometric $R$-matrix associated with two infinite-dimensional evaluation representations of $U_q(\hat{sl}(2))$ with the highest weights $2s_1$ and $2s_2$. The details of the correspondence are presented in Appendix C.

### 4.2 Finite-dimensional reductions

As usual we consider the $R$-matrix (4.5) as a matrix acting in the tensor product of two vector spaces, assigning the matrix indices $i_1, j_1$ and $i_2, j_2$ to the first and second spaces, respectively. Following the terminology of the QISM we will call them as the “quantum” and “auxiliary” spaces. Having in mind further applications we will discuss various reductions associated with the second (auxiliary) space, however, in view of the symmetry (4.13), the same considerations also apply to the first space. When $2s_2$ is a non-negative integer the corresponding evaluation representation of $U_q(\hat{sl}(2))$ becomes reducible and splits into a semi-direct sum of the $(2s_2 + 1)$-dimensional and an infinite-dimensional representations. The $R$-matrix then takes the block-triangular form in the second space\(^4\),

$$2s_2 \in \mathbb{Z}_{\geq 0} : \quad \mathcal{R}(\lambda \mid s_1, s_2)^{j_1,j_2}_{i_1,i_2} = 0, \quad 0 \leq i_2 \leq 2s_2, \quad j_2 > 2s_2, \quad \forall i_1, j_1. \quad (4.15)$$

To select its finite-dimensional block one just needs to restrict the indices to a finite set

$$i_2, j_2 \in \mathcal{I}_{s_2} = \{0, 1, 2, \ldots , 2s_2\}. \quad (4.16)$$

Similarly, for the infinite-dimensional block one needs to select the complimentary set of non-negative integers,

$$i_2, j_2 \in \mathcal{I}_{\infty} = \{2s_2 + 1, 2s_2 + 2, \ldots , \infty\}. \quad (4.17)$$

The Yang-Baxter equation (4.11) turns into two equations of the same form, where the indices $i_2, j_2, k_2$ related to the second space are restricted to either the finite $\mathcal{I}_{s_2}$ or the infinite set $\mathcal{I}_{\infty}$. Below we consider a few examples of this reduction.

\(^4\)Note that in the considered case the formula (4.5) requires a limiting procedure where a non-integer $2s_2$ is continuously approaching the required integer value. In this way all the matrix elements (4.5) remain finite and well defined.
4.2.1 Inversion relation, $s_2 = 0$.

Setting $s_2 = 0$ in (4.9), using the definition (4.5) and the inversion relation (2.5) one obtains for the finite-dimensional part (4.16),

$$s_2 = 0 : \quad \mathcal{R}(\lambda \mid s_1, 0)^{j_1, 0}_{i_1, 0} = \delta_{i_1, j_1},$$

(4.18)

where $\lambda = \lambda_1/\lambda_2$. For the infinite-dimensional part (4.17) one get an important identity

$$s_2 = 0 : \quad \mathcal{R}(\lambda \mid s_1, 0)^{j_1, j_2}_{i_1, i_2} = \frac{[\lambda q^{-s_1}] [q^{j_2}]}{[\lambda q^{s_1}] [q^{j_2}]} \mathcal{R}(\lambda \mid s_1, -1)^{j_1, j_2-1}_{i_1, i_2-1}, \quad i_2, j_2 \geq 1,$$

(4.19)

relating the $R$-matrices with $s_2 = 0$ and $s_2 = -1$. Here and below we use the notation

$$[x] = x - x^{-1}.$$

(4.20)

4.2.2 The $L$-operator, $s_2 = \frac{1}{2}$.

The next important case to consider is when $s_2 = \frac{1}{2}$ and $s_1$ is arbitrary. With the relations (4.9) this corresponds to $y' = q^y$. Introducing the notation

$$\mathcal{R}(x, x', y, qy)^{j_1, j_2}_{i_1, i_2} = \frac{1}{[x'/y]} \mathcal{L}(x, x', y)^{j_1, j_2}_{i_1, i_2},$$

(4.21)

and selecting the 2-dimensional subspace $T_2$ of the second space one obtains from (4.5),

$$\mathcal{L}(x, x', y)^{j_1}_{i_1} = \left( \begin{array}{cc} L^{j_1, 0}_{i_1, 0} & L^{j_1, 1}_{i_1, 0} \\ L^{j_1, 0}_{i_1, 1} & L^{j_1, 1}_{i_1, 1} \end{array} \right) = \left( \begin{array}{cc} \delta_{i_1, j_1} [q^{-i_1}x'/y] & \delta_{i_1, j_1 + 1} [q^{i_1}] \\ \delta_{i_1 + 1, j_1} [q^{-i_1}x'/x] & \delta_{i_1, j_1} [q^{i_1}x/y] \end{array} \right),$$

(4.22)

where $[x] = x - 1/x$. Notice, that here for the arguments of $\mathcal{L}(x, x', y)$ we use the same (original) variables as in $\mathcal{R}(x, x', y, qy)$. As usual we consider the above expressions as matrix elements of the $L$-operator, which is regarded as a two by two matrix in the second space, whose elements are operators acting in the first space,

$$\mathcal{L}(x, x', y) = \left( \begin{array}{cc} \mu q^{\frac{H}{2}} - \mu^{-1} q^{-\frac{H}{2}} & F \\ E & \mu q^{\frac{H}{2}} - \mu^{-1} q^{-\frac{H}{2}} \end{array} \right), \quad q^{2s_1} = x'/x, \quad \mu = (xx')^{\frac{1}{2}}/y,$$

(4.23)

where the introduction of the dependent variable $\mu$ was used to bring the above formula to the standard form. Here $E, F, H$ stand for the generators of the quantum universal enveloping algebra $U_q(sl(2))$

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = (q - q^{-1})(q^H - q^{-H}).$$

(4.24)

The matrix elements (4.22) correspond to the infinite-dimensional highest weight representation $\pi^+_{s_1}$ of this algebra (the parameter $s_1$ is defined by $x'/x = q^{2s_1}$),

$$\pi^+_{s_1} [H] \mid j \rangle = (2s_1 - 2j) \mid j \rangle, \quad \pi^+_{s_1} [E] \mid j \rangle = [q^{2s_1 + 1 - j}] \mid j - 1 \rangle, \quad \pi^+_{s_1} [F] \mid j \rangle = [q^{j + 1}] \mid j + 1 \rangle,$$

(4.25)

spanned on the basis vectors $\mid j \rangle \in \mathbb{C}(\infty)$ with $j = 0, 1, 2, \ldots, \infty$.

For the infinite-dimensional part of the $R$-matrix with the indices $i_2, j_2 \geq 2$ one gets the relation,

$$s_2 = \frac{1}{2} : \quad \mathcal{R}(\lambda \mid s_1, \frac{1}{2})^{j_1, j_2}_{i_1, i_2} = \frac{[\lambda q^{-s_1 - \frac{1}{2}}] [\lambda q^{s_1 + \frac{1}{2}}] [q^{i_2}] [q^{j_2 - 1}]}{[\lambda q^{s_1 + \frac{1}{2}}] [\lambda q^{s_1 - \frac{1}{2}}] [q^{i_2}] [q^{j_2 - 1}]} \mathcal{R}(\lambda \mid s_1, -\frac{3}{2})^{j_1, j_2 - 2}_{i_1, i_2 - 2}, \quad i_2, j_2 \geq 2,$$

(4.26)

which connect the $R$-matrices with $s_2 = \frac{1}{2}$ and $s_2 = -\frac{3}{2}$. Note, that here we use the notations for $\mathcal{R}(\lambda \mid s_1, s_2) = \mathcal{R}(x, x', y, y')$, as it is defined in (4.10) and (4.9).
4.2.3 Six-vertex $R$-matrix, $s_1 = s_2 = \frac{1}{2}$.

Consider the case when $s_1 = s_2 = \frac{1}{2}$. Setting

$$R(\lambda \mid \frac{1}{2}, \frac{1}{2}) = \frac{1}{[\lambda q]} R^{(6v)}(\lambda),$$

(4.27)

and using (4.5), (4.9), one gets the standard $R$-matrix of the 6-vertex model,

$$R^{(6v)}(\lambda)_{00}^{00} = R^{(6v)}(\lambda)_{11}^{11} = \rho(\lambda, \lambda q - \lambda^{-1} q^{-1}), \quad R^{(6v)}(\lambda)_{10}^{01} = R^{(6v)}(\lambda)_{10}^{10} = \rho(\lambda, q - q^{-1}),$$

(4.28)

where $\lambda = \lambda_1/\lambda_2$ and $\rho(\lambda) \equiv 1$. Alternatively, one could get the same expression from (4.23) by choosing the 2-dimensional representation of (4.24) with $s_1 = \frac{1}{2}$.

4.2.4 Infinite-dimensional $R$-matrices, $2s_2 \in \mathbb{Z}_{\geq 0}$.

Here we present the generalization of the relations (4.19), (4.26),

$$R(\lambda \mid s_1, s_2)_{i_1,i_2}^{j_1,j_2} = \frac{V_y}{V_{y/x}(2s_2 + 1)} V_{y/x}(2s_2 + 1) R(\lambda \mid s_1, -s_2 - 1)_{i_1,i_2}^{j_1,j_2 - 2s_2 - 1}, \quad i_2, j_2 \geq 2s_2 + 1,$$

(4.29)

valid for $2s_2 \in \mathbb{Z}_{\geq 0}$ and arbitrary $s_1$. The factor in the RHS is expressed via the weight function (2.2), the parameter relations (4.9) are assumed and $\lambda = \lambda_1/\lambda_2$.

4.3 Symmetry transformation

Below we will also use an alternative variant of the $L$-operator originating from the $R$-matrix (4.7),

$$\overline{R}(x, x', y, q y)_{i_1,i_2}^{j_1,j_2} = \frac{1}{[x'/y]} \overline{L}(x, x', y)_{i_1,i_2}^{j_1,j_2}, \quad i_2, j_2 = 0, 1.$$

(4.30)

It is simply related to (4.21) (note the matrix transposition in both spaces)

$$\overline{L}(x, x', y)_{i_1,i_2}^{j_1,j_2} = (q^{j_1} y^{j_1} x^{j_2})^{-1} L(x, x', y)_{j_1,j_2}^{i_1,i_2}.$$

(4.31)

Writing $\overline{L}$ as a two by two matrix in the second space, but keeping the matrix indices of the first space, one gets

$$\overline{L}(x, x', y)_{i_1}^{j_1} = \begin{pmatrix} q^{-i_1} [\mu q^{s_1-j_1}] \delta_{i_1,j_1} & \mu^{-1} q^{s_1-j_1} [q^{2s_1-j_1}] \delta_{i_1,j_1+1} \\ \mu q^{s_1-j_1} [q^{j_1}] \delta_{i_1,j_1-1} & q^{2s_1-i_1} [\mu q^{-s_1+i_1}] \delta_{i_1,j_1} \end{pmatrix}$$

(4.32)

Here we use the same notations as in (4.9) and $\mu = \lambda_1 q^{\frac{1}{2}}/\lambda_2$. Similar to (4.23) we can rewrite (4.32) in the operator form

$$\overline{L} = \begin{pmatrix} q^{-s_1} (\mu q^H - \mu^{-1}) & \mu^{-1} F q^H \\ \mu E q^H & q^{s_1} (\mu - \mu^{-1} q^H) \end{pmatrix}$$

(4.33)

where $E, F, H$ denote the generators of the quantum universal enveloping algebra $U_q(sl(2))$, defined by (4.24). The matrix elements (4.32) correspond to the infinite-dimensional representation

$$\pi^+_s [H] |j\rangle = (2s_1 - 2j) |j\rangle, \quad \pi^+_s [E] |j\rangle = [q^j] |j - 1\rangle, \quad \pi^+_s [F] |j\rangle = [q^{2s_1-j}] |j + 1\rangle,$$

(4.34)
spanned on the basis vectors $|j\rangle$ with $j = 0, 1, 2, \ldots, \infty$. This representation is equivalent to (4.25).

It is instructive to rewrite (4.32) in terms of generators of the Weyl algebra

$$uv = q^2 v u.$$  \hspace{1cm} (4.35)

Choosing the representation

$$u |j\rangle = |j\rangle q^{2s_1 - 2j}, \quad v |j\rangle = |j - 1\rangle,$$  \hspace{1cm} (4.36)

one gets

$$\sim \mathcal{L} = \begin{pmatrix} \mu q^{-s_1} u - \mu^{-1} q^{-s_1} & \mu^{-1} q^{s_1} v^{-1} (u - q^{-2s_1}) \\ \mu q^{s_1} v (1 - q^{-2s_1} u) & \mu q^{s_1} - \mu^{-1} q^{s_1} u \end{pmatrix}.$$  \hspace{1cm} (4.37)

4.4 New solutions of the Yang-Baxter equation

The existence of the star-triangle relation (3.22) allows one in the standard way to construct new solutions of the Yang-Baxter equation. For completeness we present them in the Appendix B.4, though they are not relevant to the main topic of this paper. From the quantum group point of view the $R$-matrix (4.5) intertwines two infinite-dimensional evaluation highest weight representations of the $\mathcal{U}_q(\hat{sl}(2))$ algebra. The new $R$-matrices presented in Appendix B.4 are related to infinite-dimensional representations without highest and lowest weights.

5 Six-vertex model as a descendant of the six-vertex model

The Ising-type model presented above is inspired by a remarkable structure of Mangazeev’s $R$-matrix [41] for the higher spin generalizations of the 6-vertex model, defined in (4.5) (see Appendix C for connections with the notations of [41]). In this section we demonstrate how the observed Ising-type structure can be discovered by elementary calculations, based on the idea of factorized $R$-matrices and Baxter’s “propagation through the vertex” techniques [1]. Previously these methods were used in [51] for the chiral Potts model [52–54] which is related to cyclic representations of $\mathcal{U}_q(\hat{sl}(2))$ with $q$ being a root of unity. In this way the chiral Potts model (which is an Ising-type model) has turned out to be a “descendant” of the six-vertex model. Here we show that the same methods are equally powerful for arbitrary values of $q$ in the context of the six-vertex model itself.

5.1 Factorized $R$-matrices

In Sect. 3.1 and 4.1, we have reformulated our Ising-type model first as an IRF model and then as a vertex model. It is easy to see that to within boundary effects the same model can also be reformulated as yet another vertex model. Indeed, the square lattice in Fig. 1 can be formed by periodic translations of the “box diagrams” shown in Fig. 5. One can choose any of these two diagrams, for definiteness we take the left diagram in Fig. 5. The Boltzmann weight of this box can be conveniently associated with an $R$-matrix

$$\mathbb{R}(x, y)^{a', b'}_{a, b} = \frac{V_{y'/x'}(b - a') V_{x/y'}(a' - b') V_{y/x}(b' - a)}{V_{y'/x'}(b - a)},$$  \hspace{1cm} (5.1)

where $x = (x, x')$ and $y = (y, y')$ stand for the pairs of spectral variables in the vertical and horizontal directions. The spins $a, b, a', b' \in \mathbb{Z}$ are supposed to satisfy the relations

$$b \geq a' \geq b' \geq a.$$  \hspace{1cm} (5.2)

Outside this domain the RHS of (5.1) is assumed to vanish identically. The above inequalities ensure that all the spin differences appearing in (5.1) are non-negative, as it is required for admissible lattice spin configurations defined in Sect. 2.
Figure 5: Graphical representation of the factorized $R$-matrices. Eq. (5.1) corresponds to the left diagram.

Naturally, as the reader might have expected, the $R$-matrix (5.1) satisfies the vertex form Yang-Baxter equation

$$
\sum_{a',b',c'} R(x,y)_{a',b'} R(x,z)_{a',c'} R(y,z)_{b',c'} = \sum_{a',b',c'} R(y,z)_{b',c'} R(x,y)_{a',b'} R(x,z)_{a',c'} R(x,z)_{a',c'} R(x,z)_{a',c'}.
$$

A sketch of the proof, based on the star-star relation (3.18), is presented in the Appendix A.3. Note also, that apparently related factorized $R$-matrices in an operator form were previously constructed in [55,56].

At first sight the above equation (5.3) looks identical to (4.6) (apart from the relabelling of spin indices). The difference is in the range of allowed values of these indices. For (4.6) the indices can take arbitrary non-negative integer values and the restrictions on summations are determined by the conservation laws of the type (4.4) for every vertex. Contrary to this the external indices in (5.3) take arbitrary integer values, satisfying the inequalities

$$
a \leq c'' \leq b'' \leq a'' \leq c, \quad a \leq b \leq c,
$$

resulting from the the conditions (5.2) for each $R$-matrix. By the same reason the summation indices in (5.3) are restricted by

$$
a \leq b' \leq c'', \quad b'' \leq c' \leq a'', \quad b' \leq a' \leq \min(c',b)
$$

for the LHS and

$$
c'' \leq a' \leq b'', \quad a'' \leq b' \leq c, \quad \max(a',b) \leq c' \leq b'
$$

for the RHS. Note, that all the summations in (5.3) go over finite intervals.

A remarkable feature of the $R$-matrix (5.1) is that its matrix elements factorize into a product of factors depending only on two spins connected by an edge. On the other hand the $R$-matrices (5.1) and (4.5) describe the same model and, it is natural to expect that they should be related to each other by a linear transformation.\(^5\) The corresponding relation is given by a discrete Fourier transform

$$
\sum_{a',b' \in \mathbb{Z}} R(x,y)_{a',b'} \left( \frac{x}{x'} q^{2j_1} \right)^{\varepsilon a'} \left( \frac{y}{y'} q^{2j_2} \right)^{\varepsilon b'} = \sum_{i_1,i_2 \geq 0} \left( \frac{x}{x'} q^{2j_1} \right)^{\varepsilon a} \left( \frac{y}{y'} q^{2j_2} \right)^{\varepsilon b} \left( \frac{y^{j_1,j_2}}{y^{i_1,i_2}} \right)^{\varepsilon} \mathcal{R}^T(x,y)_{i_1,i_2}
$$

where $x = (x,x')$, $y = (y,y')$, $\varepsilon = \pm 1$, and the superscript $T$ denotes the matrix transposition in both spaces

$$
\mathcal{R}^T(x,y)_{i_1,i_2} = \mathcal{R}(x,y)_{j_1,j_2}^{i_1,i_2}.
$$

\(^5\)In the case when $q$ is a root of unity relations of this type were obtained in [28,29].
Note, that Eq. (5.7) contains two different (but equivalent) relations with \(\varepsilon = +1\) and \(\varepsilon = -1\). A proof of (5.7) is presented below. Separate considerations of the second diagram (on the right side of Fig. 5) are really not required, since its Boltzmann weight is expressed through (5.1) as \(\mathbb{R}(x, y)^{-a, -b}_{-a', -b'}\) and all relevant formulae can be obtained by simple symmetry transformations.

### 5.2 The RLL-relations.

As a preparation to the proof of (5.7) consider a few important cases of the Yang-Baxter equation (4.6), which are sufficient for an \textit{ab initio} calculation of the \textit{R}-matrix (4.5). First, quote the defining relation for the \textit{L}-operator,

\[
\sum_{j_1, j_2, j_3} \mathcal{L}(x, x', y)^{j_1, j_2}_{i_1, i_2} \mathcal{L}(x, x', z)^{k_1, k_3}_{j_1, j_3} \mathcal{R}^{(6v)}(y/z)^{j_2, j_3}_{k_2, k_3} = \sum_{j_1, j_2, j_3} \mathcal{R}^{(6v)}(y/z)^{j_2, j_3}_{i_2, i_3} \mathcal{L}(x, x', z)^{j_1, k_3}_{i_1, j_3} \mathcal{L}(x, x', y)^{k_1, k_2}_{j_1, j_2}
\]

(5.9)

where \(\mathcal{R}^{(6v)}\) is given by (4.28). This equation is obtained from (4.6) via the 2-dimensional reduction (see (4.21) in Sect. 4.2.2) in the second and third spaces when \(y' = qy\) and \(z' = qz\). The indices \(i_2, j_2, k_2, i_3, j_3, k_3 = 0, 1\) take two values, while the indices \(i_1, j_1, k_1 \geq 0\) take arbitrary non-negative integer values. It is well known that with the substitution (4.23) the above equation just reduces to the commutation relations (4.24) of the algebra \(U_q(sl(2))\). Thus, any representation of this algebra gives a solution of (5.9). Here we choose the representation (4.25), which leads to the expression (4.22) for the matrix elements of \(\mathcal{L}(x, x', y)^{j_1, j_2}_{i_1, i_2}\).

Next, consider the 2-dimensional reduction of (4.6) in the third space, when \(z' = qz\),

\[
\sum_{j_1, j_2, z \geq 0} \mathcal{L}(y, y', z)^{j_2}_{i_2} \mathcal{L}(x, x', z)^{j_1}_{i_1} \mathcal{R}(x, x', y)^{k_1, k_2}_{j_1, j_2} = \sum_{j_1, j_2} \mathcal{R}(x, x', y)^{j_2}_{i_2} \mathcal{L}(x, x', z)^{j_1}_{i_1} \mathcal{L}(y, y', z)^{k_2}_{j_2}.
\]

(5.10)

Note, that the boldface \(\mathcal{L}(x, x', z)^{j}_{i}\) and \(\mathcal{L}(y, y', z)^{j}_{i}\) are defined by (4.22). They are considered as two-by-two matrices and the above equation involves their matrix product. As usual we call Eq.(5.10) the \textit{“RLL-relation”}. Once the \textit{L}-operators \(\mathcal{L}(x, x', z)\) and \(\mathcal{L}(y, y', z)\) are fixed by (4.22) this relation determines \(\mathcal{R}(x, x', y, y')\) to within an overall normalization (see, e.g., Appendix C of ref. [41]). Furthermore, by using (5.7) this relation can be transformed to another \textit{RLL}-relation, which uniquely determines the factorized \textit{R}-matrix (5.1).

To proceed, first note the Yang-Baxter equations (5.9) and (5.10) are invariant under the substitution

\[
\mathcal{R} \rightarrow \tilde{\mathcal{R}}, \quad \mathcal{L} \rightarrow \tilde{\mathcal{L}}, \quad \mathcal{R}^{(6v)} \rightarrow \tilde{\mathcal{R}}^{(6v)},
\]

(5.11)

where the modified \textit{R}-matrices are defined by (4.7), (4.31) and

\[
\tilde{\mathcal{R}}^{(6v)}(\lambda)^{j_1, j_2}_{i_1, i_2} = q^{i_2 - i_1} \lambda^{i_2 - j_2} \mathcal{R}^{(6v)}(\lambda)^{j_1, j_2}_{i_1, i_2}.
\]

(5.12)

As before, we will consider \(\tilde{\mathcal{L}}^j_i\), defined by (4.32), as a two-by-two matrix in the second space, with operator entries acting in the first space (the indices \(i, j\) label their elements). Changing basis in this space via the formula

\[
\sum_{\tilde{a}'} \mathcal{L}(x, x', y)^{\tilde{a}'}_a (x' x) (q^{2j})^{-\tilde{a}'} = \sum_i \left( \frac{x}{x'} q^{2i} \right)^{-\tilde{a}'} \tilde{\mathcal{L}}(x, x', y)^{\tilde{a}'}_i
\]

(5.13)

and using (4.32), one obtains the matrix elements of the transformed \textit{L}-operator

\[
\mathcal{L}(x, x', y)^{a'}_a = \begin{pmatrix}
\frac{x}{y} \delta_{a, a'} - \frac{y}{x} \delta_{a, a'} & \frac{q}{x} \delta_{a, a'} - \frac{x}{y} \delta_{a, a'} \\
\frac{q}{x} \delta_{a, a'} - \frac{x}{y} \delta_{a, a'} & \frac{x}{y} \delta_{a, a'} - \frac{y}{x} \delta_{a, a'}
\end{pmatrix}.
\]

(5.14)
Evidently, it still has the form (4.37), but the Weyl algebra (4.35) is now realized as
\[
\mathbf{u} | a \rangle = | a - 1 \rangle, \quad \mathbf{v} | a \rangle = | a \rangle q^{2a}.
\] (5.15)

Next, taking the “tilded” version of (5.10), obtained by the symmetry transformation (5.11), and using (5.13) and the variant of (5.7) with \( \varepsilon = -1 \) one obtains
\[
\sum_{a', b'} \mathbb{R}(x, x', y, y')^{a', b'}_{a, b} \mathbb{L}(x, x', z)^{a''}_{a} \mathbb{L}(y, y', z)^{b''}_{b} z
\]
\[
= \sum_{a', b'} \mathbb{L}(y, y', z)^{b'}_{b} \mathbb{L}(x, x', z)^{a'}_{a} \mathbb{R}(x, x', y, y')^{a''}_{a'}.
\] (5.16)

It should be stressed that this relation is equivalent to (5.10). It is obtained from the latter by the symmetry transformation (5.11), followed by a linear transformation, changing basis in the first and second vector spaces. At this stage we regard (5.7) as the definition of \( \mathbb{R}(x, x', y, y') \). However, remembering that the new RLL-relation (5.16) (with the \( \mathbb{L} \)-operators specified by (5.14)) determines \( \mathbb{R}(x, x', y, y') \) up to an overall normalization, one concludes that in order to prove (5.7) it suffices to show that the factorized expression (5.1) satisfies (5.16).

### 5.3 Propagation through the vertex techniques

The matrix elements of the \( \mathbb{L} \)-operator (5.14) possess important factorization properties, which we describe below. Let
\[
\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_i = \mathbf{v}_i^t, \quad i = 0, 1,
\] (5.17)
be the two-dimensional basis vectors. Introduce the following set of vectors labelled by \( a, a' \in \mathbb{Z} \)
\[
\Phi(x, y)_a^{a'} = \sum_{i=0,1} \Phi(x, y)^{a'}_{a,i} \mathbf{v}_i, \quad \Phi(x, y)_a^{-} = \sum_{i=0,1} \Phi(x, y)^{-}_{a,i} \mathbf{v}_i,
\] (5.18)
with the components
\[
\Phi(x, y)^{a'}_{a,i} = \sigma_i q^{-\sigma_i a} \left( (x/y)^{-\sigma_i} \delta_{a,a'} + \delta_{a,a'-1} \right),
\]
\[
\Phi(x, y)^{-}_{a,i} = q^{\sigma_i a} \left( \delta_{a,a'} - (x/y)^{\sigma_i} \delta_{a,a'-1} \right),
\] (5.19)
where \( i = 0, 1 \) and \( \sigma_0 = +1, \sigma_1 = -1 \). The above vectors satisfy important orthogonality relations
\[
\sum_{a'} \Phi(x, y)^{a'}_{a,i} \Phi(x, y)_a^{a''} = [y/x] \delta_{i,j}, \quad \sum_{i} \Phi(x, y)^{a'}_{a,i} \Phi(x, y)_a^{a''} = [y/x] \delta_{a',a''}.
\] (5.20)

Using the explicit form of the two-by-two \( \mathbb{L} \)-operator (5.14) it is easy to check that
\[
(\mathbb{L}(x, x', z)^{a'}_{a})_{i,i'} = \Phi(x', z)^{a'}_{a,i} \Phi(x, z)^{a'}_{a,i'}
\] (5.21)
Recall, that the indices \((i, i') = (0, 0)\) refer to the top left element in (5.14). Here we have used the following graphical notations

\[
\Phi(x, z)^{a'}_{a, i} = \begin{array}{c}
\Phi \\
\downarrow \\
i \end{array}
\]

\[
\Phi(x, z)^{a'}_{a, i} = \begin{array}{c}
\Phi \\
\downarrow \\
i \end{array}
\]

\[
\Phi(x, z)^{a'}_{a, i} = \begin{array}{c}
\Phi \\
\downarrow \\
\Phi \\
\downarrow \\
i \\
\end{array}
\]

\[
\Phi(x, z)^{a'}_{a, i} = \begin{array}{c}
\Phi \\
\downarrow \\
\Phi \\
\downarrow \\
i \\
\end{array}
\]

The are two types of spin variables in these pictures. The variables \(a, a' \in \mathbb{Z}\) are the “face spins” assigned to the shaded faces, while \(i, i' = \pm 1\) are the edge spins (like in vertex models) assigned to the directed solid lines in the unshaded areas. Moreover these lines (as well as their dashed continuations into shaded areas) carry a spectral variable \(z\). Another spectral parameter \(x\) is assigned to the directed vertical lines, which separate the shaded and unshaded areas.

The vectors (5.18) enter the so-called “SOS-vertex” correspondence and “propagation through the vertex techniques” invented by Baxter [4,25] and further developed by many authors including [57–59]. The approach we use here is most similar to that of [43], based on the factorized form of the \(L\)-operator (5.21). The key relation of the SOS-vertex correspondence in our case reads

\[
\sum_{j_2,j_3=0,1} \tilde{R}^{(6v)}(y/z)^{j_2,j_3}_{j_2,j_3} \Phi(x', z)^{b}_{c,j_3} \Phi(x', y)^{a}_{b,j_2} = \sum_{c=d} \Phi(x', y)^{a}_{d,j_2} \Phi(x', z)^{a}_{c,j_3} \mathcal{R}^{(6v)}(y/z)^{a-b,b-d}_{c-d,a-c} \tag{5.23}
\]

where \(\mathcal{R}^{(6v)}\) and \(\tilde{\mathcal{R}}^{(6v)}\) are given by (4.28) and (5.12), respectively. This relation is presented graphically in Fig. 6. Using the orthogonality (5.20) one can convert (5.23) into a similar relation containing the \(\Phi^{a'}\) vectors instead of \(\Phi^a\),

\[
\sum_{i_2,i_3=0,1} \Phi(x, y)^{a}_{a,i} \Phi(x, z)^{a'}_{b,j_3} \tilde{R}^{(6v)}(y/z)^{j_2,j_3}_{i_2,i_3} = \sum_{b=d} \mathcal{R}^{(6v)}(y/z)^{a-b,b-d}_{c-d,a-c} \Phi(x, z)^{b}_{d,j_3} \Phi(x, y)^{a}_{b,j_2} \tag{5.24}
\]

Figure 6: Graphical representation of the SOS-vertex duality (5.23) for the 6-vertex model.

Similarly to (5.18) we need to define additional vectors

\[
\Omega(x, y)^{a'}_{a,i} = \sigma^a q^{-\sigma^a} \left( \delta_{a,a'} + (yq/x)^{-\sigma^a} \delta_{a,a'-1} \right),
\]

\[
\overline{\Omega}(x, y)^{a'}_{a,i} = \sigma^a q^{\sigma^a} \left( (y/x)^{\sigma^a} \delta_{a,a'} - q^{\sigma^a} \delta_{a,a'-1} \right),
\]

satisfying the following orthogonality relations

\[
\sum_{a} \Omega(x, y)^{a'}_{a,i} \overline{\Omega}(x, y)^{a'}_{a',i'} = \left[ y/x \right] \delta_{i,i'}, \quad \sum_{i} \overline{\Omega}(x, y)^{a'}_{a,i} \Omega(x, y)^{a'}_{a',i} = \left[ y/x \right] \delta_{a,a'}. \tag{5.26}
\]
These vectors are represented graphically as

\[
\Omega(x, z)_{a,i} = \begin{array}{c}
i \\
z \\
a \end{array}
\]

\[
\overline{\Omega}(x, z)_{a,i} = \begin{array}{c}
\Omega \\
\overline{\Omega} \\
ad \\
\end{array}
\]

Note that here the directions of the vertical lines, carrying the spectral variable \(x\), are reversed in comparison with (5.22).

Now, return to Fig. 1 on page 6 and shade alternatively all faces of the medial lattice to cover those containing the white and black sites (the medial lattice is a square lattice formed by the thin rapidity lines). Then we can regard the spins in our Ising-type model as face spins and update the graphical notations (2.1) for the edge Boltzmann weights

\[
(i) : \Phi(x, z)_{a,i} \Phi(y, z)_{b,i} = V_{x/y}(a - b), \quad (ii) : \overline{\Phi}(y, z)_{b,i} \overline{\Phi}(z, x)_{a,i} = \frac{1}{V_{x/y}(a - b)}. \tag{5.28}
\]

where \(V_x(n)\) is defined in (2.2).

Now we can present three easily verifiable Yang-Baxter relations involving the SOS-vertex vectors (5.22), (5.27) and the edge weights (5.28). The first equation is

\[
V_{x/y}(a - b) \sum_i \Phi(x, z)_{a,i} \Phi(y, z)_{b,i} = \sum_i \overline{\Omega}(y, z)_{a,i} \Omega(x, z)_{b,i} V_{x/y}(a' - b'). \tag{5.29}
\]

It has the following graphical representation

Note, that after a substitution of the definitions (5.22), (5.27) this relation uniquely determines the function \(V_x(n)\), defined by (2.2)-(2.4), up to an overall normalization. The next relation is

\[
\sum_b V_{y/x}(a - b) \Phi(x, z)_{b,i} \overline{\Phi}(y, z)_{b,i} = \sum_b \Phi(y, z)_{a,i} \overline{\Phi}(x, z)_{a,i} V_{y/x}(b - c). \tag{5.31}
\]
It is represented graphically as

\[
\begin{align*}
\text{(5.32)}
\end{align*}
\]

Note the summation over the spin \( b \) corresponding to the internal face. Finally, the third relation reads

\[
\sum_b V_{y/x}(b-a) \Omega(x,z)^c_{b,i} \Phi(y,z)^c_{b,i'} = \sum_b \Omega(y,z)^b_{a,i} \Phi(x,z)^b_{a,i'} V_{y/x}(c-b).
\]

(5.33)

It is depicted below

\[
\begin{align*}
\text{(5.34)}
\end{align*}
\]

With the vectors (5.19) and (5.25) regarded as an input, each of the equations (5.31) and (5.33) can be considered as an equation for the weight function \( V_x(n) \). Similarly to (5.29), these equations also lead to the function \( V_x(n) \), defined by (2.2)-(2.4) (up to an overall normalization). All these equations can be viewed as (Yang-Baxter type) exchange relations, suitable for defining \( Z \)-invariant lattices models [60]. The models of this type are formulated on irregular lattices formed by an intersection of straight lines (with at most two lines intersecting at one point). Their partition functions remain unchanged upon parallel transport of individual lattice lines. In our case the lattices shown in (5.30), (5.32) and (5.34) are the simplest examples of such lattices. For instance, the lattice on the left side of (5.30) is formed by three straight lines associated with the spectral variables \( x, y \) and \( z \). There are four spins \( a, a', b, b' \), assigned to external faces and one spin \( i \) assigned to the internal edge on the \( z \)-line. The model contains the three-spin interactions, with the Boltzmann weights given by (5.19), (5.25), as well as the two-spin interaction with the weights (5.28). The LHS of (5.29) is the partition function of this model for the left lattice of (5.30) with fixed boundary spins on the external faces. Eq. (5.29) states that this partition function remains unchanged by moving the \( z \)-line down through the intersection point of the other two lines and transforming the lattice to that shown in the right side of (5.30). A similar interpretation exists for the other two relations (5.31) and (5.33).

We are now ready to give a simple visual proof of the Yang-Baxter equation (5.16). Substituting the factorized expressions (5.1) and (5.21) and using the above graphical notations one can represent this equation as shown in Fig. 7. Consider the diagram on the left side on this figure. Let us move the horizontal \( z \)-line down through the intersection points of the other lines and then consecutively use: (i) the relation (5.30), (ii) the relations (5.32) and (5.34) (in any order) and, finally, (iii) the relation (5.30) again. In this way the left diagram in Fig. 7 is transformed to the right one, thus proving (5.16). The above calculation completes our proof of the linear relation (5.7) between \( R \)-matrices arising from the “star” and “box” diagrams. Relations of this type commonly appear as (self-) duality transformations in the Ising-type lattice model [61,62].
Figure 7: A graphical representation of the LLR-relation (5.16).

6 Functional relations for the transfer matrices

6.1 Definition of the T- and Q-matrices.

To discuss the functional relations for the transfer matrices it is convenient to also use the equivalent vertex model, introduced in Sect. 4.1. Fig. 8 shows the correspondence between the corner spins of the IRF model and the edge indices of the vertex model,

\[
i_n = a_{n+1} - a_n, \quad j_n = b_{n+1} - b_n, \quad k_n = b_n - a_n, \quad n = 1, 2, \ldots N, \tag{6.1}
\]

which take arbitrary non-negative integer values. For the quasi-periodic boundary condition (involving the horizontal field factor \(\omega_{h_1}^{k_1}\) at the first column of the lattice) the elements of the transfer matrix read

\[
(T^{(\pm)}(y, y' | x_N, x'_N, \ldots, x_1, x'_1))_{i_N, j_N-1, \ldots, i_1}^{j_N, j_{N-1}, \ldots, j_1} = \rho(y, y') \sum_{k_1, \ldots, k_N = 0}^{\infty} \omega_{h_1}^{k_1} \mathcal{R}(x_N, x'_N, y, y')_{i_N, k_1}^{j_N, k_N} \times
\]

\[
\times \mathcal{R}(x_{N-1}, x'_{N-1}, y, y')_{i_N-1, k_{N-1}}^{j_{N-1}, k_{N-1}} \cdots \mathcal{R}(x_1, x'_1, y, y')_{i_1, k_1}^{j_1, k_1}, \tag{6.2}
\]

where the sum is taken over all nonnegative integer values of \(k_1, \ldots, k_N\). Actually, there is only a one-dimensional summation with \(k_1 = 0, 1, \ldots, \infty\), while the rest of the \(k\)'s are uniquely determined...
by the values of $i$'s, $j$'s and $k_1$ due to the conservation laws (4.4) at each vertex. The superscript (+) in the notation for the transfer matrix indicates that the sum over $k_1$ goes over an infinite interval. Using the correspondence (6.1) and the relation (4.1) it is easy to see that the expression inside the sum in (6.2) exactly coincides with the IRF transfer matrix (3.8) (with $\omega_v = 1$).

The transfer matrix (6.2) is an operator acting in the “quantum” space, formed by the direct product

$$\mathcal{H}^{(\text{vertex})} = \mathbb{C}^{(\infty)}_N \otimes \mathbb{C}^{(\infty)}_{N-1} \otimes \cdots \otimes \mathbb{C}^{(\infty)}_1,$$  \hspace{1cm} (6.3)

of $N$ identical infinite-dimensional vector spaces $\mathbb{C}^{(\infty)}$, each of which is spanned by the basis vectors $|j\rangle$ with $j = 0, 1, 2, \ldots, \infty$. The lower indices in (6.3) enumerate the columns of the lattice, as in Fig. 8. Below we will regard the variables $(x_N, x'_N, \ldots, x_1, x'_1)$ as fixed and will write the transfer matrix simply as $T^{(+)}(y, y')$ assuming the dependence on the fixed variables implicitly. The Yang-Baxter equation (4.6) implies that the transfer matrices $T^{(+)}(y, y')$ with different values of $y, y'$ form a commuting family,

$$[T^{(+)}(y, y'), T^{(+)}(z, z')] = 0, \quad \forall \quad y, y', z, z'. \hspace{1cm} (6.4)$$

Moreover, they preserve the number of particles $M$, previously introduced in (3.11). Indeed, the matrix elements (6.2) vanish, unless

$$i_1 + i_2 + \cdots + i_N = j_1 + j_2 + \cdots + j_N \equiv M, \quad M = 0, 1, 2, \ldots. \hspace{1cm} (6.5)$$

So, the transfer matrices act invariantly in each subspace of (6.3) with a fixed particle number $M = 0, 1, 2, \ldots, \infty$. It is convenient then to define diagonal matrices

$$w = q^{2M} \prod_{i=1}^{N} \frac{x_i}{x'_i}, \quad Z_0 = \sum_{k=0}^{\infty} (w \omega_k)^k = (1 - \omega_h w)^{-1}, \hspace{1cm} (6.6)$$

which also belong to the commuting family (6.4). The sum in $Z_0$ converges provided the horizontal field parameter $\omega_h$ is such that

$$|\omega_h w| < 1. \hspace{1cm} (6.7)$$

At this point it is worth noting that the definition of the $R$-matrix (4.5) only involves a finite summation, so it is well defined for arbitrary values of its parameters, in particular, for an arbitrary value of the “anisotropy” parameter $q$. However the convergence of the infinite sum in the definition of the transfer matrix (6.2) requires a special investigation. It is plausible that for $|q| < 1$ or $|q| = 1$ the restriction (6.7) for the field factor is sufficient for convergence of the sum (6.2). On the other hand, the considered model is extremely general, so that a more detailed analysis is, of course, needed for considerations of its particular cases. For simplicity, in what follows, we assume $|q| < 1$.

The normalization factor in (6.2) is chosen as

$$\rho(y, y') = (y'/y)^M \prod_{n=1}^{N} \frac{(y/x'_n)^2; q^2}{{(y/x'_n)^2; q^2}_\infty}, \hspace{1cm} (6.8)$$

where the $q$-products are defined in (2.3). Then, taking into account (4.5) one concludes \(^6\) that the transfer matrix (6.2) is a meromorphic function in the variables $y$ and $y'$. The reason for taking the normalization (6.8) will be clarified below. Next, define the factor

$$\varphi(y, y') = \prod_{n=1}^{N} \frac{(y'/x'_n)^2; q^2}{{(y/x'_n)^2; q^2}_\infty}, \hspace{1cm} (6.9)$$

and introduce two new transfer matrices

$$Q(y') = Z_0^{-1} \lim_{y' \to 0} \varphi(y, y') T^{(+)}(y, y'), \quad \bar{Q}(q^{-1} y) = Z_0^{-1} \lim_{y' \to 0} \varphi(y, y') T^{(+)}(y, y'), \hspace{1cm} (6.10)$$

\(^6\)The $R$-matrix (4.5) is a Laurent polynomial in the spectral variable $y'$ and a meromorphic function of $y$, having poles at $y^2 = x'^2 q^{-2n}, \; n = 0, 1, 2, \ldots$. 

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which, as we shall see below, play the role of the Baxter $Q$-matrices [4]. Evidently, they belong to the same commuting family (6.4),

$$
[T^{(+)}(y,y'), Q(z)] = [T^{(+)}(y,y'), \overline{Q}(z)] = [Q(z), Q(z')] = [Q(z), \overline{Q}(z')] = 0,
$$

(6.11)

where $y, y', z, z'$ take arbitrary values.

Recall, that $T^{(+)}(y,y')$ as well as $Q(y')$ and $\overline{Q}(y)$ implicitly depend on the set of fixed variables $(x_N, x'_N, \ldots, x_1, x'_1)$. Note also, that if a pair of these variables is constrained as

$$
x'_{n} = x_n q^{2s_n}, \quad 2s_n \in \mathbb{Z}_{\geq 0}
$$

(6.12)

for some particular $n$, then the above transfer matrices admit a $(2s_n + 1)$-dimensional reduction in the $n$-th factor of the product (6.3).

## 6.2 Functional relations

The elements of the $Q$-matrices (6.10) can be written explicitly by taking required limits of (6.2). The details are presented in Appendix D. It is worth noting that in the case of $U_q(\hat{sl}(2))$ considered here the algebraic origin of the $Q$-matrices (or the $Q$-operators) is very well understood [63,64]. Their main algebraic properties can be concisely expressed by a single factorization relation, which in our case takes the form

$$
\varphi(y, y') T^{(+)}(y, y') = Z_0 Q(y') \overline{Q}(q^{-1} y),
$$

(6.13)

derived in Appendix D. Let us also mention the normalization conditions

$$
Q(0) = \overline{Q}(0) = 1,
$$

(6.14)

where the $Q$-operators reduce to the identity operator.

Consider now the case when the spectral variables $y$ and $y'$ are related as

$$
y' = y q^{2s}, \quad 2s \in \mathbb{Z}_{\geq 0}.
$$

(6.15)

This is precisely the finite-dimensional reduction case (4.15)-(4.17), considered in Sect. 4.2 (see also (4.10)). This reduction implies that the sum in (6.2) splits into two parts:

(i) the first part, where all the indices $k_1, \ldots, k_N \in I_s = \{0, 1, \ldots, 2s\}$ take the finite set of values. Assuming the relation (6.15) we denote this part as $T_s(y)$ without the superscript $(+)$. Explicitly, for $2s \in \mathbb{Z}_{\geq 0}$ one obtains

$$
(T_s(y))_{i_N, j_{N-1}, \ldots, i_1}^{j_N, j_{N-1}, \ldots, j_1} = \rho(y, q^{2s} y) \sum_{k_1, \ldots, k_N \in I_s} \omega_{h}^{k_1} \mathcal{R}(x_N, x'_N, y, q^{2s} y)_{i_N, k_1}^{j_N, k_N} \times \mathcal{R}(x_{N-1}, x'_{N-1}, y, q^{2s} y)_{i_{N-1}, k_{N-1}}^{j_{N-1}, k_{N-1}} \ldots \mathcal{R}(x_1, x'_1, y, q^{2s} y)_{i_1, k_1}^{j_1, k_1}
$$

(6.16)

(ii) the second part where all the indices $k_1, \ldots, k_N \in I_s = \{2s + 1, 2s + 2, \ldots, \infty\}$ take the infinite set of values. Using (4.29) one can reduce this part to $T^{(+)}(y' q, y q^{-1})$.

In this way one obtains

$$
\varphi(y, y') T^{(+)}(y, y') = \varphi(y, y') T_s(y) + (\omega_h w)^{2s + 1} \varphi(y' q, y q^{-1}) T^{(+)}(y' q, y q^{-1}), \quad y' = y q^{2s},
$$

(6.17)

where the relation (6.15) is assumed. The above reduction for the transfer matrices is well known, see [64]. Combining the last equation with (6.13) one gets

$$
Z_0^{-1} \varphi(y, y q^{2s}) T_s(y) = Q(y') \overline{Q}(y q^{-1}) - (\omega_h w)^{2s + 1} Q(y q^{-1}) \overline{Q}(y').
$$

(6.18)
From (4.18) it follows that $T_0(y) \equiv 1$. Thus, setting $s = 0$ in (6.18) one obtains the quantum Wronskian relation for the $Q$-operators

$$Z_0^{-1} \varphi(y, y) = Q(y) Q^{-1}(y q^{-1}) - (\omega_h w) \ Q(y q^{-1}) Q^{-1}(y) .$$

(6.19)

Next, setting $s = 1/2$ in (6.18) one gets

$$Z_0^{-1} \varphi(y, y q) T_{1/2}(y) = Q(y q) Q^{-1}(y q^{-1}) - (\omega_h w)^2 \ Q(y q^{-1}) Q^{-1}(y q) .$$

(6.20)

Combining the last relation with (6.19) one obtains the famous $TQ$-equations [4]

$$T_{1/2}(y) Q(y) = f(y) \ Q(y q) + (\omega_h w) \ g(y) \ Q(y q^{-1}) ,$$

$$T_{1/2}(y) Q^{-1}(y q) = (\omega_h w) \ f(y) \ Q^{-1}(y q) + g(y) \ Q^{-1}(y q^{-1}) ,$$

(6.21)

where

$$f(y) = \frac{\varphi(y, y)}{\varphi(y, y q)} = \prod_{n=1}^{N} \left( 1 - (y/x_n)^2 \right) , \quad g(y) = \frac{\varphi(y q, y)}{\varphi(y, y q)} = \prod_{n=1}^{N} \left( 1 - (y/x_n)^2 \right) .$$

(6.22)

To conclude our derivation of the functional relations note, that the seemingly artificial normalization factors (6.8) and (6.9), entering the definitions of the transfer matrices and $Q$-operators were specially chosen to keep the functional relations in the standard form, with a minimal number of scalar factors. If necessary (for instance, when $|q| = 1$) the above normalization factors could be removed and the functional relations then be easily modified. Finally, note that essentially the same $Q$-operators were defined by Mangazeev in [41,65] (see Appendix C for more details).

### 6.3 Six-vertex model

By construction the eigenvalues of $\varphi(y, y') T^{(\pm)}(y, y')$, $Q(y')$ and $Q^{-1}(y)$ are polynomials in the variable $(y')^2$, and, in general, meromorphic functions of $y^2$ with poles in the finite part of the complex plane. An important exception is the finite-dimensional reduction case when the constraint (6.12) holds for all columns of the lattice $m = 1, 2, \ldots, N$. For an illustration consider the case of the six-vertex model with two-dimensional quantum space representations for all sites of the chain; that is possible if the spectral parameters obey the relations

$$x_n' = q x_n , \quad s_n = \frac{1}{2} , \quad m = 1, \ldots, N .$$

(6.23)

Using the definitions (6.2), (6.6), (6.8) and (6.16) one obtains

$$T_{1/2}(y) = z^{-1} T^{(6v)}(q^{-1/2} y) , \quad z = (\omega_h w)^{-1/2} ,$$

(6.24)

where

$$\left( T^{(6v)}(q^{-1/2} y) \right)_{i_1}^{j_1,i_2,i_3} = \cdots \cdots \cdots = (\omega_h)^{-1/2} \sum_{k_1, \ldots, k_N = 0,1} \omega_h^{k_1} R(x_N/y)^{j_N,k_N}_{i_N,i_{N-1},i_{N-2}} R(x_{N-1}/y)^{j_{N-2},k_{N-2}}_{i_{N-2},i_{N-3}} \cdots R(x_1/y)^{j_1,k_1}_{i_1,k_2} ,$$

(6.25)

where $R(\lambda)$ is given by (4.28) with $\rho(\lambda) = q^{-1/2} \lambda^{-1}$.

To present the expressions for the $Q$-operators (6.10) for this case consider the $q$-oscillator algebra

$$[H, \mathcal{E}_\pm] = \pm 2 \mathcal{E}_\pm , \quad q \mathcal{E}_+ \mathcal{E}_- - q^{-1} \mathcal{E}_- \mathcal{E}_+ = q - q^{-1}$$

(6.27)

and its two representations $\rho_{\pm}$, defined by the action on the vectors $|j\rangle$, $j = 0, 1, 2, \ldots, \infty$,

$$\rho_{\pm}(H)|j\rangle = \mp 2 j |j\rangle , \quad \rho_{\pm}(\mathcal{E}_\pm)|j\rangle = q^{\mp 1/2} (q^{-j} - q^j) |j+1\rangle , \quad \rho_{\pm}(\mathcal{E}_\mp)|j\rangle = \mp q^{\mp j+1/2} |j+1\rangle .$$

(6.28)
Define operators
\[ A_{\pm}(y) = Z_{0}^{-1} \text{Tr}_{\rho_{\pm}} \left\{ \omega_{h}^{\mp H/2} L_{\pm}^{(N)}(y/x_N) \cdots L_{\pm}^{(1)}(y/x_1) \right\}, \quad A_{\pm}(y) = 1 + O(y^2), \]
where
\[ L_{+}(y) = \begin{pmatrix} q_{2}^{y} & y \mathcal{E}_{-} \\ y \mathcal{E}_{+} & q_{-2}^{y} - q^{-1} y^2 \frac{q_{2}}{q_{2}} \end{pmatrix}, \quad L_{-}(y) = \begin{pmatrix} q_{-2}^{y} - q y^2 \frac{q_{2}}{q_{2}} & y \mathcal{E}_{+} \\ y \mathcal{E}_{-} & q_{2}^{y} \end{pmatrix} \]
are two by two matrices acting in the quantum space, whose entries are elements of the \( q \)-oscillator algebra (6.27).

Using (D.1), (D.4) and (D.5) from Appendix D it is not difficult to show that
\[ Q(y) = A_{+} \left( q^{-\frac{1}{2}} y \right), \quad \overline{Q}(y) = A_{-} \left( q^{-\frac{1}{2}} y \right). \]
The equations (6.21) can now be rewritten as
\[ T^{(6v)}(y) A_{\pm}(y) = z^{\pm 1} g(q^{-\frac{1}{2}} y) A_{\pm}(q y) + z^{\mp 1} g(q^{\frac{1}{2}} y) A_{\pm}(q^{-1} y), \quad z = (\omega_{h} w)^{-\frac{1}{2}}. \]
Note, that apart from a trivial change \( y \to y^2 = \zeta \) of the arguments of the transfer matrices this relation exactly coincides with Eq.(3.3) of ref. [66].

### 7 The homogeneous case

Consider the column homogeneous case
\[ x_{1} = x_{2} = \ldots = x_{N} \equiv x, \quad x'_{1} = x'_{2} = \ldots = x'_{N} \equiv x', \quad x' = q^{2s} x, \]
so that we can abbreviate the arguments of the transfer matrix (6.2) simply as \( T^{(+)}(y, y' \mid x, x') \). Obviously, the homogeneous model possesses the translational invariance
\[ [T^{(+)}(y, y' \mid x, x'), K] = 0, \]
where \( K \) is the one-site translation operator, whose matrix elements are given by
\[ (K)_{j_{N} j_{N-1} \ldots j_{1}}^{i_{N} i_{N-1} \ldots i_{1}}(\omega_{h})^{i_{1}} \delta_{i_{1}}^{j_{N}} \ldots \delta_{i_{2}}^{j_{1}} \delta_{i_{1}}^{j_{N}}. \]
In writing this formula we have taken into account the quasi-periodic boundary conditions, involving the horizontal field factor at the first column of the lattice.

#### 7.1 Hamiltonians

Apart from the translation operator the transfer matrix commuting family now contains spin chain Hamiltonians that are given by a sum of terms, each of which acts non-trivially only in two consecutive factors of the product (6.3). First consider a generic case, when \( 2s \) is not taking non-negative integer values. Using the definition (6.2) and (4.5) it is not difficult to check that
\[ \left\{ \rho(y, y')^{-1} T^{(+)}(y, y' \mid x, x') \right\}_{y = x \varepsilon, y' = x' \varepsilon} = K \left( 1 + \varepsilon H^{(1)} + \varepsilon' H^{(2)} \right) + O(\varepsilon^2, \varepsilon'^2, \varepsilon \varepsilon') \]
where the Hamiltonians \( H^{(1)} \) and \( H^{(2)} \) are defined as
\[ (H^{(1)})_{j_{N} j_{N-1} \ldots j_{1}}^{i_{N} i_{N-1} \ldots i_{1}} = \sum_{m=1}^{N} \omega_{h}^{(j_{N} - i_{N}) \delta_{m,N}^{i_{N}} j_{N}} \delta_{i_{N}}^{j_{N}} \ldots \delta_{i_{m+2}}^{j_{m+2}} (H^{(1)})_{i_{m+1} i_{m+1}}^{j_{m+1} j_{m+1}} \delta_{i_{m-1}}^{j_{m-1}} \ldots \delta_{i_{1}}^{j_{1}}, \]
and...
with $\ell = 1, 2$ and
\[
(H^{(1)})^{j,j'}_{i,i'} = \delta^j_i \delta^{j'}_{i'} F_i(s) + 2\delta_{i+j} \frac{\theta(i > j)}{[q^{-s}]} V_{q^{-2s}}(j) V_{q^{-2s}}(i),
\]
\[
(H^{(2)})^{j,j'}_{i,i'} = (H^{(1)})^{j',j}.
\] (7.6)

Here we use the notations
\[
\theta(a > b) = \begin{cases} 
1, & \text{if } a > b, \\
0, & \text{otherwise},
\end{cases}
\] (7.7)

and
\[
F_i(s) = -\sum_{k=0}^{i-1} q^{k-2s} + q^{-k+2s} [q^n] = q^n - q^{-n},
\] (7.8)

where $i = 0, 1, 2, \ldots, \infty$. The weight function $V_x(n)$, appearing in (7.6), is defined in (2.2). Moreover, the boundary conditions imply that $i_{N+1} = i_1$ and $j_{N+1} = j_1$.

It is worth remembering that the transfer matrix $T^{(+)}(y, y' \mid x, x')$ has two independent commuting Hamiltonians in the homogeneous case.

### 7.2 Q-operators

For the homogeneous case it is convenient to define different $Q$-operators
\[
Q^{(h)}(y') = \rho(y, y')^{-1} T^{(+)}(y, y')|_{y=x}, \quad \overline{Q}^{(h)}(q^{-1}y) = \rho(y, y')^{-1} T^{(+)}(y, y')|_{y'={x}},
\] (7.9)

with matrix elements,
\[
Q^{(h)}(y')_{i_{N\mathbf{N}}^{j_{N\mathbf{N}}-j_{\mathbf{N}}}1^{j_{\mathbf{N}}}1} = \sum_{k_1, \ldots, k_N} \omega_k^{k_1} \prod_{n=1}^{N} \delta_{i_n+k_{n+1}+j_n+k_n} \frac{V_{x/y'}(k_n) V_{y'/x'}(i_n-k_n)}{V_{x/x'}(i_n)}.
\] (7.10)

\[
\overline{Q}^{(h)}(q^{-1}y)_{i_{N\mathbf{N}}^{j_{N\mathbf{N}}-j_{\mathbf{N}}}1^{j_{\mathbf{N}}}1} = \sum_{k_1, \ldots, k_N} \omega_k^{k_1} \prod_{n=1}^{N} \delta_{i_n+k_{n+1}+j_n+k_n} \frac{V_{x/x'}(j_n) V_{y/x}(k_n)}{V_{y/x'}(j_n+k_n)}.
\] (7.11)

They satisfy the factorization relation (similar to (6.13))
\[
\rho(y, y') \overline{Q}^{(h)}(q^{-1}y) Q^{(h)}(y') = T^{(+)}(y, y').
\] (7.12)

and normalization conditions
\[
Q^{(h)}(x e^{x\varepsilon}) = K(1 + \varepsilon H^{(2)}) + O(\varepsilon^2), \quad Q^{(h)}(x e^{-x\varepsilon}) = 1 + \varepsilon H^{(1)} + O(\varepsilon^2),
\] (7.13)

\[
\overline{Q}^{(h)}(q^{-1}x e^{x\varepsilon}) = 1 + \varepsilon H^{(1)} + O(\varepsilon^2),
\] (7.14)

where the Hamiltonians are defined in (7.5), (7.6) and “1” stands for the identity operator. We have added the superscript $(h)$ to the notations for the new $Q$-operators (7.9) to distinguish them from the previously defined operators (6.10), normalized as (6.14). The latter can, of course, be specialized to the homogeneous case (7.1) and connected with (7.9),
\[
Q^{(h)}(y') = (\frac{x}{y'})^M Q(y') Q(x)^{-1} = (\frac{x}{y'})^M K Q(y') Q(x)^{-1},
\] (7.15)

and
\[
\overline{Q}^{(h)}(q^{-1}y) = (\frac{y}{x})^M \varphi(y, y)^{-1} \overline{Q}(q^{-1}) \left( \lim_{y \rightarrow x} \varphi(y, y)^{-1} \overline{Q}(q^{-1}) \right)^{-1}.
\] (7.16)
For completeness let us mention also the IRF version of the new operators (7.9)

$$Q^{(h)}(y)_{a_N, \ldots, a_1}^{b_N, \ldots, b_1} = \prod_{i=1}^{N} V_{y'/x'}(a_{i+1} - b_i)V_{x'/x}(b_i - a_i) / V_{x'/x}(a_{i+1} - a_i),$$  \hspace{1cm} (7.17)

$$Q^{(h)} (q^{-1}y)_{a_N, \ldots, a_1}^{b_N, \ldots, b_1} = \prod_{i=1}^{N} V_{y/x}(b_i - a_i)V_{x/x'}(b_{i+1} - b_i) / V_{y/x}(b_{i+1} - b_i),$$  \hspace{1cm} (7.18)

which satisfy the factorization relation

$$T(y, y') = Q^{(h)}(y')Q^{(h)}(q^{-1}y),$$  \hspace{1cm} (7.19)

with the IRF transfer matrix (3.8) (where the field factors are set as $\omega_h = \omega_v = 1$). Note, that the operator (7.17) possesses quite remarkable locality properties. It is a rather sparse matrix (similar to the Hamiltonians (7.6)), which is very convenient for numerical calculations.

### 7.3 Finite-dimensional reduction

Now consider the case when the parameter $2s$ in (7.1) is an integer, $2s \in \mathbb{Z}_{\geq 0}$. The corresponding Hamiltonians cannot be obtained from (7.5), (7.6) just by setting $2s$ to such a value, because this operation does not commute with taking the limit (7.4). Actually, for $2s \in \mathbb{Z}_{\geq 0}$ the limit (7.4) exists only when $\varepsilon = \varepsilon'$. So, there is only one Hamiltonian in this case. The quantum space of states (6.3) becomes reducible. Each factor of the product (6.3) splits into two components

$$\mathbb{C}^{(\infty)} = \mathbb{C}(2s+1) \oplus \mathbb{C}^{(\infty)}$$  \hspace{1cm} (7.20)

spanned by the vectors $|j\rangle$, $j \in \mathcal{I}_s = \{0,1,2,\ldots, 2s\}$ and $j \in \mathcal{I}_s = \{2s+1, 2s+2, \ldots, \infty\}$, respectively, which according to Sect. 4.2 support the $(2s+1)$-dimensional representations $\pi_s$ and the infinite-finite dimensional representation $\pi_{s-1}^+$ of the $U_q(sl(2))$ algebra (4.24). The Hamiltonian takes a block triangular form, acting invariantly in two blocks where one takes either the same $(2s+1)$-dimensional or the same infinite-finite dimensional components in all factors of the product (6.3).

First, consider the $(2s+1)^N$-dimensional block. The Hamiltonian is given by the above formula (7.5), but with the superscript ($\ell$) omitted and

$$H_{i,i'}^{j,j'} = \delta_{ij}^{i'} \delta_{jj'}^{j'} (F_i(s) + F_i(s))$$

\begin{align*}
+ 2\delta_{i+i'+j+j'} (\theta(i > j) V_{q^{-2s}}(j) / [q^{i-j}] V_{q^{-2s}}(i) + \theta(i' > j') V_{q^{-2s}}(j') / [q^{i'-j'}] V_{q^{-2s}}(i')) .
\end{align*}  \hspace{1cm} (7.21)

for $0 \leq i + i' \leq 2s$ and

$$H_{i,i'}^{j,j'} = \delta_{ij}^{i'} \delta_{jj'}^{j'} (F_{2s-i}(s) + F_{2s-i}(s))$$

\begin{align*}
+ 2\delta_{i+i'+j+j'} (\theta(j > i) V_{q^{-2s}}(j) / [q^{j-i}] V_{q^{-2s}}(i) + \theta(j' > i') V_{q^{-2s}}(j') / [q^{j'-i'}] V_{q^{-2s}}(i')) .
\end{align*}  \hspace{1cm} (7.22)

for $2s \leq i,j \leq 4s$. Note, that (7.21) and (7.22) display the reflection symmetry

$$H_{i,i'}^{j,j'} = H_{2s-i,2s-i'}^{j,j'}.$$  \hspace{1cm} (7.23)

For the infinite dimensional block the local Hamiltonian is given by the same formula (7.22), but with $i, i', j, j' > 2s$. It is worth noting that for $i > 2s$

$$F_{2s-i}(s) = - \sum_{k=2s+1}^{i} q^k + q^{-k} / q^k - q^{-k} = F_{i-2s-1}(-s-1) - \frac{q^{2s+1} + q^{-2s-1}}{q^{2s+1} - q^{-2s-1}} .$$  \hspace{1cm} (7.24)
Moreover, with the l’Hôpital rule one obtains
\[
\frac{V_{q^{-2s}}(j)}{V_{q^{-2s}}(i)} = \frac{V_{q^{2s+2}}(i - 2s - 1)}{V_{q^{2s+2}}(j - 2s - 1)}.
\] (7.25)

Taking this into account it is easy to see that upon the shift of all indices as \(i \rightarrow i - 2s - 1\) the resulting Hamiltonian up to a constant coincides with \(H^{(1)} + H^{(2)}\) defined by (7.5), (7.6) with \(s\) replaced by \(-s - 1\).

8 The Bethe Ansatz

8.1 Coordinate Bethe ansatz

The commuting transfer matrices and Q-operators (6.4), (6.11) act in the infinite-dimensional quantum space \(H^{(\text{vertex})}\) defined by (6.3). As noted before (see (6.5)) all these matrices act invariantly in all finite dimensional subspaces of \(H^{(\text{vertex})}\) with a fixed number of particles \(M = 0, 1, 2, \ldots, \infty\). Their diagonalization problem can be solved via the coordinate Bethe ansatz by generalizing the results of [67–72]. Let \(\hat{\sigma}^+\) and \(\hat{h}\) be the “particle creation” and “particle number” operators acting on the basis vectors \(|j\rangle \in \mathbb{C}^\infty\) as
\[
\hat{\sigma}^+ |j\rangle = |j + 1\rangle, \quad \hat{h} |j\rangle = j |j\rangle, \quad j = 0, 1, 2, \ldots, \infty.
\] (8.1)
The eigenvectors describing \(M\)-particle states have the form
\[
\Psi = \sum_{1 \leq r_1 \leq r_2 \leq \cdots \leq r_M \leq N} \Psi(r_1, r_2, \ldots, r_M) \hat{\sigma}^+_{r_M} \cdots \hat{\sigma}^+_{r_1} \Psi_0, \quad M \geq 0,
\] (8.2)
where the (unnormalized) Bethe ansatz wave function reads as
\[
\Psi(r_1, \ldots, r_M) = \sum_{\hat{P}} A_{\hat{P}} \prod_{m=1}^{M} \phi_{\hat{P}m}(r_m)
\] (8.3)
with suitably chosen coefficients \(A_{\hat{P}}\) and functions \(\phi_{m}(r)\). The vector \(\Psi_0\) is the bare vacuum state
\[
\Psi_0 = |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle
\] (8.4)
which is the \(M = 0\) eigenstate of all operators in the commuting family, and \(\hat{\sigma}^+_r\) denotes the particle creation operator acting in the \(r\)-th factor of the tensor product (6.3). The summation is taken over all \(M!\) permutations \(\hat{P}\) of the integers \((1, 2, \ldots, M)\). Recall that the number of particles \(M\) is a conserved quantity, preserved by all operators in the commuting family. The coefficients \(A_{\hat{P}}\) read
\[
A_{\hat{P}} = \prod_{1 \leq j < m \leq M} \frac{q y^2_{\hat{P}j} - q^{-1} y^2_{\hat{P}m}}{y^2_{\hat{P}j} - y^2_{\hat{P}m}},
\] (8.5)
where the set of complex numbers \(\{y_m\}_{m=1}^{M}\) satisfies the system of algebraic equations (called the Bethe ansatz equations)
\[
\prod_{i=1}^{N} \frac{1 - y_m^2 / x_i^2}{1 - y_m^2 / x_i^2} = -(\omega w)^{-1} \prod_{j=1}^{M} \frac{y_j^2 - q^{-2} y_m^2}{y_j^2 - q^2 y_m^2}, \quad m = 1, 2, \ldots, M,
\] (8.6)
and the functions \(\phi_{m}(r)\) in (8.3) have the form
\[
\phi_{m}(r) = [q] \left( \prod_{i=1}^{r-1} [y_m / x_i] \right) \left( \prod_{i=r+1}^{N} [y_m / x_i] \right).
\] (8.7)
Note, that for $M = 1$ the coefficient $A_{\tilde{p}} \equiv 1$.

The corresponding eigenvalues of the $Q$-operators defined by (6.10)

$$Q(y) \Psi = Q(y) \Psi, \quad \overline{Q}(y) \Psi = \overline{Q}(y) \Psi,$$

(see also (D.1), (D.2) and (D.3) for explicit expressions of their matrix elements) can be expressed via

the functional relations should then be modified accordingly.

Note, that for $M = 1$ the so-called monodromy matrix

$$Q(y) = \prod_{m=1}^{M} \left(1 - y^2/y_m^2\right),$$

while the eigenvalues of $\overline{Q}(y)$ are, in general, meromorphic functions of this variable. They can be expressed through $Q(y)$ by solving the quantum Wronskian relation (6.19),

$$\overline{Q}(y) = Z_0^{-1} Q(y) \sum_{j=1}^{\infty} (\omega_n w)^{j-1} \frac{\varphi(q^j y, q^j y)}{Q(q^{j-1} y) Q(q^j y)}, \quad |q| < 1.$$  (8.10)

Note, that separate terms in the above sum contain poles arising from the zeroes of $Q(q^j y)$, $j = 1, 2, \ldots, \infty$. However, it is easy to see that all such poles cancel out between two successive terms of the sum by virtue of the Bethe ansatz equations (8.6). Thus, the poles in (8.10) could only originate from the function $\varphi(y, y)$, defined in (6.9). For arbitrary values of $\{s_n\}_{n=1}^{N}$ in (6.12) is has the poles located at

$$y^2 = q^{-2j} x_n^2, \quad n = 1, \ldots, N, \quad j = 1, 2, \ldots, \infty.$$  (8.11)

However, if $2s_n \in \mathbb{Z}_{\geq 0}$ for all $n = 1, 2, \ldots, N$ the function $\varphi(y, y)$ simplifies to

$$\varphi(y, y) = \prod_{n=1}^{N} \prod_{j=1}^{2s_n} \left(1 - (y/q^j x_n)^2\right).$$  (8.12)

and the eigenvalue $\overline{Q}(y)$ also becomes a polynomial in $y^2$.

The above construction for the eigenvectors (8.2)-(8.7) is very similar to Baxter’s results [72] for the eigenvectors of the inhomogeneous six-vertex model. The difference is that the single particle wave function (8.7) and the Bethe ansatz equations (8.6) in our case are slightly more general than those for the spin $s = 1/2$ six-vertex model where $x_n = q x_n$ ($n = 1, 2, \ldots, N$). Moreover, the sum in (8.2) allows any number of the particle positions $\{r_1, r_2, \ldots, r_N\}$ to coincide, whereas for the six-vertex model the inequalities in (8.2) are strict, due to the “exclusion principle”, that no more than one “down arrow” can be at the same site.

### 8.2 Algebraic Bethe ansatz

The Bethe state (8.2) may also be constructed within the framework of the QISM [73, 74]. Introduce the so-called monodromy matrix

$$\mathcal{M}(y) = \mathcal{M}(y | x_N, x'_N, \ldots, x_1, x'_1) = \mathcal{L}_N(x_N, x'_N, y) \mathcal{L}_{N-1}(x_{N-1}, x'_{N-1}, y) \cdots \mathcal{L}_1(x_1, x'_1, y),$$  (8.13)

where $\mathcal{L}_n(x_n, x'_n, y)$ is the two-by-two matrix (4.23), whose entries are elements of the $U_q(sl(2))$ algebra (4.24), acting in the $n$-th component of the tensor product $\mathcal{H}^{(\text{vertex})}$, defined by (6.3). Each of these components realizes an infinite-dimensional highest weight representation of this algebra given by (4.25) (where $s_1$ for the $n$-th component is replaced by $s_n$, such that $q^{2s_n} = x'_n/x_n$). It is convenient to denote the entries of the monodromy matrix as

$$\mathcal{M}(y) = \begin{pmatrix} A(y) & \hat{B}(y) \\ \hat{C}(y) & D(y) \end{pmatrix},$$  (8.14)

Of course, these fixed “kinematic” poles could easily be removed by multiplying $\overline{Q}(y)$ by a suitable factor. The functional relations should then be modified accordingly.
where \( \hat{A}, \hat{B}, \hat{C}, \hat{D} \) are operators acting in the quantum space (6.3). With the above notations, the transfer matrix \( T_{\frac{1}{2}}(y) \), defined by (6.16) with \( s = \frac{1}{2} \), is given by

\[
T_{\frac{1}{2}}(y) = \left( q^M \prod_{n=1}^{N} \left( \frac{y}{x_n} \right) \right) \left( \hat{A}(y) + \omega_h \hat{D}(y) \right),
\]

(8.15)

while the Bethe state (8.2)-(8.7), with the set \( \{ y_m \}_{m=1}^{M} \) solving the Bethe ansatz equations, is expressed as

\[
\Psi = \hat{B}(y_M) \cdots \hat{B}(y_2) \hat{B}(y_1) \Psi_0.
\]

(8.16)

The corresponding eigenvalue of the transfer matrix (8.15),

\[
T_{\frac{1}{2}}(y) \Psi = T_{\frac{1}{2}}(y) \Psi,
\]

(8.17)

reads [74, 75]

\[
T_{\frac{1}{2}}(y) = f(y) \prod_{j=1}^{M} \frac{1 - q^{-2}y^2/y_j^2}{1 - q^{-2}y^2/y_j^2} + (\omega_h w) g(y) \prod_{j=1}^{M} \frac{1 - q^{-2}y^2/y_j^2}{1 - q^{-2}y^2/y_j^2}.
\]

(8.18)

where \( f(y) \) and \( g(y) \) are defined in (6.22). Combining the last formula with (6.14) and (6.21) one immediately deduces the expression (8.9) for the eigenvalue of the operator \( Q(y) \).

Finally, from the definitions (8.13), (8.14) it follows that

\[
\hat{B}(y) = - \sum_{r=1}^{N} \left( \prod_{i=1}^{r-1} \left[ q^{-\hat{h}_i} y_m/x_i \right] \right) \left[ q^{\hat{h}_i} \right] \hat{\sigma}^+_r \left( \prod_{j=r+1}^{N} \left[ q^{\hat{h}_i} y_m/x_i \right] \right),
\]

(8.19)

where \( \hat{h}_i \) is the particle number operator (8.1) acting at the \( i \)-th component of the product (6.3). Substituting this into (8.16) and rearranging similar terms in a special way one obtains the coordinate Bethe ansatz expression for the eigenvectors given by (8.2)-(8.5). In the context of the XXX related models this statement first appeared in [76] (see eq.(4.8) therein). For generalizations to the XXZ case and derivations see [77, 78]. Note that this equivalence between the two forms of the “ansatz” for the eigenvectors holds for arbitrary values of \( \{ y_m \}_{m=1}^{M} \), in particular, it does not require them to satisfy the Bethe ansatz equations (8.6).

9 Generalizations to other models

Apparently, all vertex models associated with quantized affine Lie algebras [14, 15] and superalgebras [17] can be reformulated as Ising-type models. Indeed, the most important algebraic structure required for such a reformulation — the 3-dimensional interpretation based on the tetrahedron equation — is known for a majority of these models, [31, 79–83]. In support of our statement we present here a new class of Ising-type models related to the \( U_q(\widehat{sl}(n)) \) algebra, extending the results of this paper which was devoted to the \( n = 2 \) case. The formulation of these new models follows exactly the same description as in Sect. 2, except that each site of the lattice now carries an \( (n-1) \)-component integer spin variable \( a = (a_1, a_2, \ldots, a_{n-1}) \in \mathbb{Z}^{(n-1)} \) and the Boltzmann weight function \( V \), entering the definition of the edge weights (2.1), is now replaced with

\[
V_v(a, b) = q^{2Q(a, b) - Q(a, a) - Q(b, b)} \left( \frac{q}{x} \right)^{a_0 - b_0} \prod_{i=1}^{n-1} \left( q^{2}; q^2 \right)_{a_i - b_i}.
\]

(9.1)

Note, that this function depends on both spin variables \( a \) and \( b \) at the ends of the edge, rather than their difference, and

\[
a_0 = \sum_{i=1}^{n-1} a_i, \quad Q(a, b) = \sum_{1 \leq i < j \leq n-1} a_i b_j.
\]

(9.2)
Similar to (3.16), (3.17) one can define the corresponding IRF weight

\[ \mathcal{W}^n(a, b, c, d | x, x', y, x') = \frac{V_{y/y'}(b, d)}{V_{y/y'}(a, c)} \sum_{n = \max(b, c)}^{a} \frac{V_{x/x'}(a, n)V_{y/y'}(n, b)V_{y/y'}(n, c)}{V_{y/y'}(n, d)} \],

(9.3a)

\[ = \frac{V_{x/x'}(a, b)}{V_{x/x'}(c, d)} \sum_{n = \min(b, c)}^{d} \frac{V_{x/x'}(n, d)V_{y/y'}(c, n)V_{y/y'}(b, n)}{V_{y/y'}(a, n)} \].

(9.3b)

where the summation is taken over \((n - 1)\) independent integer variables \(n_i\), such that \(\max(b_1, c_i) \leq n_i \leq a_i\) in (9.3a) and, similarly, in (9.3b). The corner spins should obey the relations \(a_i \geq b_i, c_i \geq d_i\) \((i = 1, 2, \ldots, n - 1)\), otherwise the weights \(\mathcal{W}^n(a, b, c, d)\) are assumed to vanish identically. The equality of two different expressions in (9.3) is the star-star relation, which ensures the integrability of the model. It has exactly the same graphical representation as in Fig. 3 and reduces to (3.18) for \(n = 2\). For \(n > 3\) the star-star relation (9.3) is a new identity which we currently claim as a conjecture (though we have thoroughly verified it for many particular cases).

It is not difficult to check that the IRF weights \(\mathcal{W}^n(a, b, c, d | x, x', y, x')\) only depend on the differences between the corner spins, so one can trivially convert these weights into a vertex \(R\)-matrix (similar to (4.1)). The resulting solutions of the Yang-Baxter equation are associated with infinite-dimensional highest weight evaluations representations of the \(U_q(sl(n))\) algebra, namely, the symmetric tensor representations [79].

The vertex \(R\)-matrix, corresponding to (9.3) reads

\[ \mathcal{R}^n(x, x', y, y')^{j_1, j_2}_{i_1, i_2} = \delta_{i_1 + i_2, j_1 + j_2} \mathcal{W}^n(i_1 + i_2, j_1, i_1, 0 | x, x', y, y') \],

(9.4)

where \(j_1, j_2, i_1, i_2 \in \mathbb{Z}_{\geq 0}^{n-1}\). Note, if \(x' = q^{\mu_1}x\) and \(y' = q^{\mu_2}y\), with \(\mu_1, \mu_2 \in \mathbb{Z}_{\geq 0}\), then the above \(R\)-matrix admits a finite-dimensional reduction. For instance, the Cherednik-Perk-Shultz \(R\)-matrix [9,10] (vector representations of \(U_q(sl(n))\)) appears in the case \(x' = qx\), \(y' = qy\). Let

\[ e_0 = (0, \ldots, 0), \quad e_i = (0, \ldots, 1\text{ at }i^{th}\text{ place}, \ldots, 0), \]

(9.5)

and \(\varepsilon_{i,j} = -\varepsilon_{j,i} = 1\) for \(i < j\), except the case \(i = 0\) or \(j = 0\) where \(\varepsilon_{i,j} = 0\). Then,

\[ \mathcal{R}_{e_i, e_i} = 1, \quad \mathcal{R}_{e_i, e_j} = q^{\varepsilon_{i,j}} \frac{x/y}{[qx/y]}, \quad \mathcal{R}_{e_j, e_i} = \left[ \frac{y}{x} \right]^{\varepsilon_{i,j}} \frac{[q]}{[qx/y]}, \quad i \neq j, \]

(9.6)

where we have omitted the superscript “\(sl(n)\)”.

10 Conclusion

In this paper we have presented a new approach to the six-vertex model and its higher spin generalizations. We have reformulated these models as Ising-type models with only a two-spin interaction across edges of the square lattice. This reformulation leads to a significant simplification of the algebraic theory of the higher spin six-vertex model, since the edge Boltzmann weights are given by a very simple formula (2.2). As a result explicit expressions for associated \(R\)-matrices, as well as their properties, become much more transparent.

An Ising-type model can be equivalently converted into a vertex model in (at least) two different ways. In the first way, the \(R\)-matrix defining the vertex weights is deduced from the Bolzmann weights of the four-edge star (like the one shown in Fig. 2) by using the star-square transformation followed by the IRF-vertex correspondence. In our case this procedure leads to Eq.(4.5). In the second way the vertex-type \(R\)-matrix is identified with the “box diagram”, shown in Fig. 5, that leads to Eq.(5.1). In Sect. 5 we have shown these two \(R\)-matrices are related by a linear relation (5.7), which is not unnatural to expect, since they both describe exactly the same model.
From the quantum group point of view these $R$-matrices intertwine equivalent representations of the $U_q(\hat{sl}(2))$ algebra, connected by a similarity transformation in the representation space. For the first of these representation (corresponding to (4.32)) the Cartan element is diagonal, while for the other one (corresponding to (5.14)) it is realised as a shift operator. It should be stressed, that it is the shift operator realization of the Cartan elements that leads to the factorized $R$-matrices. This fact has been previously observed [51] for the chiral Potts model and its generalizations [43], which are related to the $U_q(\hat{sl}(n))$ algebra. We expect a similar phenomenon to take place for all other quantized affine algebras. One disadvantage of the factorized $R$-matrices is the necessity to work with infinite-dimensional representations, whereas the models with a finite number of discrete spin states only appear via a reduction. In practice, however, this does not create real problems. Actually, it would be fair to say that certain infinite-dimensional representations (in particular, the $q$-oscillator representations) cannot be avoided, since they are required for the transfer matrix construction of the Baxter $Q$-operators [63, 64]. Moreover, they play the important role of “prefundamental” representations in the theory of quantized affine algebras [84,85].

As is well known, the six-vertex model has a vast number of important applications to various problems of physics and mathematics (for reviews see [86,87]). We mention, in particular, the enumeration of the alternating sign matrices [88,89], asymmetric stochastic exclusion processes and stochastic lattice models [90, 91]. Most recently the (inhomogeneous) six-vertex model was involved in fascinating connections to black hole sigma models in quantum field theory [92–99]. It would be interesting to revisit these problems and connections in view of the Ising-type structures studied in our work.

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Appendix A. The star-star and Yang-Baxter relations

A.1 Proof of the star-star relation.

In this Appendix we will prove the star-star (3.18) and the Yang-Baxter (3.13) relations. For the readers’ convenience, let us reproduce here the definitions of the IRF-type weights given by (3.6) and (3.17),

$$W(a,b,c,d \mid x,x',y,y') = \frac{V_{y'/y}(b-d)}{V_{y'/y}(a-c)} \sum_{n=\max(b,c)}^{a} \frac{V_{x/y}(a-n)V_{y'/x}(n-b)V_{y'/x}(n-c)}{V_{y'/x}(n-d)}, \quad (A.1)$$

$$\overline{W}(a,b,c,d \mid x,x',y,y') = \frac{V_{x'/x}(a-b)}{V_{x'/x}(c-d)} \sum_{n=\min(b,c)}^{\min(b,c)} \frac{V_{x/y}(n-d)V_{y'/x}(c-n)V_{y/x}(b-n)}{V_{y/x}(a-n)}. \quad (A.2)$$

We need to prove that the two definitions lead to the same result

$$W(a,b,c,d \mid x,x',y,y') = \overline{W}(a,b,c,d \mid x,x',y,y'). \quad (A.3)$$

This is the star-star relation, which states the equality of the Boltzmann weights of the two “stars” shown in Fig. 3. Below it will be more convenient to modify the graphical representations (2.1) for the edge weights. We will abolish the rapidity lines, but instead indicate the rapidity ratio arguments of
the weights for all edges of the lattice. These arguments are shown near the edges, as in the pictures below,

\[(i): \quad (x) = V_x(a - b); \quad (ii): \quad (x) = \frac{1}{V_x(a - b)}.\]  

(A.4)

Otherwise, the rules remain unchanged: the edge arrow pointing from \( b \) to \( a \) means that the spin difference argument of the weight function is \((a - b)\) rather than \((b - a)\), moreover, the colour distinguishes the type (i) (blue) and the type (ii) (red) edges with the Boltzmann weights \( V_x(a - b) \) and \( (V_x(a - b))^{-1} \), respectively. Recall, that the function \( V_x(n) \) is defined in (2.2). With the new conventions the star-star relation (A.3) is presented in Fig. 9.

![Figure 9: The star-star relation (A.3) with alternative graphical notations for the edge weight, as in (A.4).](image)

Below we will use the second Sears’s transformation formula for the basic hypergeometric series (see [42]),

\[
\Phi_3^4 \left( \frac{q^{-2n}}{a; \ldots; c; q^2} \right) = \left( \frac{\frac{a}{b}, \ldots; \frac{a}{c}; q^2}{\frac{b}{a}, \ldots; \frac{c}{a}; q^2} \right) \left( \frac{\cdots}{\cdots} \right),
\]

(A.5)

where the integer \( n \geq 0 \) and the other parameters satisfy the constraint

\[
a b c = q^{2n-2} d e f. \quad \text{(A.6)}
\]

Here we use the standard notations

\[
\Phi_{r+1}(q^{-2n}, a_1, a_2, \ldots, a_r; q^2) = \sum_{k=0}^{n} \frac{(q^{-2n}, a_1, a_2, \ldots, a_r; q^2)_k}{(q^2, b_1, b_2, \ldots, b_r; q^2)_k} q^{2k},
\]

(A.7)

and

\[
(a; q^2)_k = \prod_{\ell=0}^{k-1} (1 - a q^{2\ell}); \quad (a_1, a_2, \ldots, a_m; q^2)_k = (a_1; q^2)_k (a_2; q^2)_k \cdots (a_m; q^2)_k.
\]

(A.8)
We are now ready to prove the star-star relation (A.3). First, consider the case \( b \geq c \). From the definition (2.2) it is easy to verify, that

\[
V_x(a - n) = V_x(a) \left( \frac{q}{x} \right)^n \frac{(q^{-2a}; q^2)_n}{(q^{2-2a}x^{-2}; q^2)_n}.
\]  

(A.9)

Using this formula and the definition (A.7) one can rewrite (A.1) and (A.2) in the form

\[
\mathcal{W} = \frac{V_{y/y'}(b - d) V_{y'/x'}(a - b)V_{y/x}(a - c)}{V_{y/y'}(a - c) V_{y/x}(a - d)} 
\times 4\varphi_3 \begin{pmatrix}
q^{2(a-b)}, & q^{2(a-c)}, & (x/y)^2, & q^{2(1-a+d)}(x'/y)^2 \\
q^{2(a-b)}, & q^{-2(a-c)}, & (x/y')^2, & q^{2(1-a+d)}(x'/y^2) \\
q^{2(1-a+b)}(x'/y)^2, & q^{-2(a-d)}, & q^{2(1-a+c)}(x/y)^2 \\
q^{2(1-a+b)}(x'/y^2), & q^{-2(a-d)}, & q^{2(1-b+d)}(x/y)^2 
\end{pmatrix},
\]  

(A.10)

and

\[
\overline{\mathcal{W}} = \frac{V_{x/x'}(a - b) V_{y'/x'}(c - d)V_{y/x}(b - d)}{V_{x/x'}(c - d) V_{y/x}(a - d)} 
\times 4\varphi_3 \begin{pmatrix}
q^{2(c-d)}, & q^{-2(b-d)}, & (x/y')^2, & q^{2(1-a+d)}(x'/y)^2 \\
q^{2(c-d)}, & q^{2(b-d)}, & (x/y)^2, & q^{2(1-a+d)}(x'/y^2) \\
q^{2(1-c+d)}(x'/y)^2, & q^{2(1-a+b)}(y' y)^2, & q^{2(1-c+b)} \\
q^{2(1-c+d)}(x'/y^2), & q^{2(1-a+b)}(y'y)^2, & q^{2(1-c+b)} 
\end{pmatrix},
\]  

(A.11)

where we have omitted the arguments of \( \mathcal{W} \) and \( \overline{\mathcal{W}} \) since they do not change during these calculations. Applying the second Sears transformation (A.5) to both (A.10) and (A.11), one obtains

\[
\mathcal{W} = \frac{V_{x/x'}(a - b) V_{y/y'}(a - d)V_{y/x}(b - c)}{V_{y/y'}(a - c) V_{y/x}(a - d)} 
\times 4\varphi_3 \begin{pmatrix}
q^{2(a-b)}, & q^{2(c-d)}, & q^2(x/y)^2, & (y'/x')^2 \\
q^{2(1-a+d)}(y'/y)^2, & (x/x')^2, & q^{2(1-c+b)} 
\end{pmatrix},
\]  

(A.12)

and

\[
\overline{\mathcal{W}} = \frac{V_{x/x'}(a - b) V_{y/y'}(a - d)V_{y/x}(b - c)}{V_{y/y'}(a - c) V_{y/x}(a - d)} 
\times 4\varphi_3 \begin{pmatrix}
q^{2(c-d)}, & q^{2(a-b)}, & q^2(x/y)^2, & (y'/x')^2 \\
q^{2(1-a+d)}(y'/y^2), & (x/x')^2, & q^{2(1-c+b)} 
\end{pmatrix},
\]  

(A.13)

The resulting expressions coincide due to the obvious symmetry of \( 4\varphi_3 \) upon permutations of its upper line arguments. Evidently, this proves the star-star relation (A.3). The above computation is valid for \( b \geq c \). The case \( b < c \) can be obtained by the transformation \( b \leftrightarrow c \), \( x' \leftrightarrow y^{-1} \) and \( y' \leftrightarrow x^{-1} \), which maps the star-star relation into itself.

### A.2 Proof of the IRF-type Yang-Baxter equation

Let us now prove the Yang-Baxter relation (3.13). The star-star relation (A.3) could be used in two ways. Firstly, one could express the sum in (A.1) via the sum in (A.2) (of course, taking into account the pre-factors in front of the sums). We will call this operation as the “direct star-star map centered at \( n \)”, where \( n \) denotes the summation index in the first sum (A.1), which is the central spin in the left star depicted in Fig. 9. Similarly, one could express the second sum (A.2) via the sum in (A.1). We will call this operation as the “reverse star-star map centered at \( n \)”, where \( n \) now refers to the central spin in the right star in Fig. 9.
Return now to the Yang-Baxter equation (3.13). For convenience we reproduce it here

\[ \sum_{a \in \mathbb{Z}} W(g, a, b, f \mid x, y) \ W(c, e, g, a \mid x, z) \ W(e, d, a, f \mid y, z) = \]
\[ = \sum_{h \in \mathbb{Z}} W(c, h, g, b \mid y, z) \ W(h, d, b, f \mid x, z) \ W(c, e, h, d \mid x, y) \]  

(A.14)

where the bold symbols denote the pairs of spectral variables \( x = (x, x') \), \( y = (y, y') \) and \( z = (z, z') \). Consider the LHS of this equation and use the expression (A.1) for each of the weights \( W(g, a, b, f \mid x, y) \), \( W(c, e, g, a \mid x, z) \) and \( W(e, d, a, f \mid y, z) \) denoting the central spins by \( n_1 \), \( n_2 \) and \( n_3 \), respectively. Then, with the graphical representation of the weight (A.1) shown on the left side of Fig. 9, the complete LHS of (A.14) will be represented as shown below

(A.15)

Note, that the external sites, corresponding to fixed spins in (A.14), are represented by open circles, while the internal sites, corresponding to the summation spin indices, are shown by filled circles.

Now, make the reverse star map centered at \( a \). This transformation involves six edges, the four edges surrounding the site \( a \) and two boundary edges \((f, n_1)\) and \((f, n_3)\) (see the shaded area on the figure (A.15) above). The result is shown in the left side of (A.16).

(A.16)

The shading on this figure now indicates the area of the next transformation, which is the reverse star map centered at \( n_3 \). The result is shown on the right side of (A.16). Next, make the reverse star map centered at \( n_1 \) followed by the direct star map centred at \( n_2 \). The results are shown on the left and right sides of (A.17), respectively.
Finally, redenoting the summation indices
\[ n_3 \to n_1, \quad a \to n_2, \quad n_1 \to n_3, \quad n_2 \to h, \]
one converts the LHS of (A.14) to its RHS. This proves the Yang-Baxter relation (3.13) and commutativity of transfer matrices (3.12) in the main text.

A.3 Proof of the vertex-type Yang-Baxter equation

Now consider the Yang-Baxter equation (5.3) for the \( R \)-matrix (5.1). For convenience let us reproduce it here,
\[
\sum_{a',b',c'} R_{a',b'}(x,y) R_{a',c'}(x,z) R_{b',c'}(y,z) = \sum_{b',c'} R_{b',c'}(y,z) R_{a',c'}(x,z) R_{a,b'}(x,y). \tag{A.19}
\]

With the graphical notations (A.4) the LHS of this equation can be represented as
\[
\begin{align*}
\sum_{a',b',c'} R_{a',b'}(x,y) & R_{a',c'}(x,z) R_{b',c'}(y,z) = \sum_{b',c'} R_{b',c'}(y,z) R_{a',c'}(x,z) R_{a,b'}(x,y).
\end{align*}
\tag{A.20}
\]

The proof of this equation consists of the repeated application of the star-star relation (A.3), represented graphically in Fig. 9. The working is very similar to the proof of (A.14) in the previous subsection. In particular we will use the notions of the “direct” and “reverse” star-star maps introduced in the first paragraph of Sect. A.2.

Start from the left hand side of (A.19) and then

1. make the reverse star map centred at \( a' \),
2. make the reverse star map centred at $b'$,
3. make the reverse star map centred at $c'$,
4. make the direct star map centred at $a'$ again,

and then making the change

\[ a' \to a' \, , \, b' \to c' \, , \, c' \to b' \, , \]  \hspace{1cm} (A.21)

one converts the left hand side of (A.19) to its right hand side.

**Appendix B. Star-triangle relation as a primary integrability condition**

**B.1 Star-triangle relation.**

Here we prove the star-triangle relation (3.22) by reducing it to the Pfaff-Saalshcütz-Jackson summation formula (see eq.(3.5.1) in [47]),

\[
\Phi_2 \left( \begin{array}{cc}
q^{-2n}, & a \\
q^{-2n}a b/c & \end{array} \right| q^2, q^2 \right) = \frac{(c/a, c/b; q^2)^n}{(q, c/a b; q^2)^n} (B.1)
\]

where the notations are defined in (A.7) and (A.8). Here $n$ is a non-negative integer $n \geq 0$, and $q$, $a$, $b$ and $c$ stand for arbitrary complex parameters.

For the readers’ convenience we reproduce the relations (3.22) here

\[
\sum_d V_{y/z}(d - a) W_{x,y}(b - d) V_{x/y}(c - d) = W_{z,y}(b - c) W_{x/z}(c - a) W_{y,x}(b - a) \, , \hspace{1cm} (B.2a)
\]

and

\[
\sum_d V_{y/z}(a - d) W_{x,z}(d - b) V_{x/y}(d - c) = W_{z,y}(c - b) W_{x/z}(a - c) W_{y,x}(a - b) \, . \hspace{1cm} (B.2b)
\]

The above two relations are equivalent (see (3.23)), so it is enough to consider one of them. For definiteness we choose the second one (B.2b). Due to (2.4) both sides of this relation vanish identically unless $a \geq c$. So in the following we assume $a \geq c$. Next, we change the summation variable there from $d$ to $n = a - d$, which takes values in the interval $0 \leq n \leq a - c$ (since otherwise the summand in (B.2b) vanishes). Using the above definitions one can transform the LHS of (B.2b) as follows

LHS of (B.2b) =

\[
V_{x/y}(a - c) W_{x,z}(a - b) \Phi_2 \left( \begin{array}{cc}
q^{-2(a-c)}, & (y/z)^2 \\
q^{2(1-a+c)}(y/x)^2, & q^2 \\
q^{2(1-a+b)}z^{-2} & q^2 \end{array} \right) \hspace{1cm} (B.3a)
\]

\[
V_{x/z}(a - c) W_{x,y}(a - b) \frac{(q^{2(1-a+c)})(z/x)^2, q^2 y^2, q^2)}{(q^{2(1-a+c)})(y/x)^2, q^2 b^2, q^2 a-c} \hspace{1cm} (B.3b)
\]

\[
W_{z,y}(c - b) V_{x/z}(a - c) W_{y,x}(a - b) \, , \hspace{1cm} (B.3c)
\]

where in the first line we used (2.2), (3.19) and (A.9) as well as the formula

\[
W_{x,y}(a - n) = W_{x,y}(a) \left( \frac{y}{x} \right)^n \frac{(q^{2-2a}y^{-2}; q^2)^n}{(q^{2-2a}x^{-2}; q^2)^n} \, . \hspace{1cm} (B.4)
\]
In the next line (B.3b) we simply substituted the summation formula (B.1). Finally Eqs. (A.9) and (B.4) were used again to rewrite the result in the form (B.3c), which is identical to the RHS of (B.2b), thus completing the proof of this relation.

### B.2 Star-star relation.

Next, we will show how to derive the star-star relation (3.18) from the star-triangle relation. Setting \( y = q \) in (B.2) one obtains

\[
\frac{V_x(b - c) V_{x/z}(c - a)}{V_x(b - a)} = \sum_k V_{q/z}(k - a) W_{z,x}(b - k) V_{x/q}(c - k),
\]

\[
= \sum_k W_{x/z,x}(k - a) V_{qz/x}(b - k) V_{x/q}(k - c), \tag{B.5b}
\]

where the first equality follows from (B.2a) while the second one follows from (B.2b) after an additional substitution \( a \leftrightarrow b \) and \( z \rightarrow x/z \).

The LHS of (3.18) can be transformed as

\[
\text{LHS of (3.18)} = \sum_n \left( \frac{V_{y/y'}(n - b) V_{y/y'}(b - d)}{V_{y/y'}(n - d)} \right) \frac{V_{x/y'}(a - n) V_{y/x}(n - c)}{V_{y/y'}(a - c)} \tag{B.6}
\]

\[
= \sum_k V_{y/y'}(k - d) V_{y/qy'}(b - k) W_{x,y'/y',x'}(a - k) W_{y'/x',x'}(c - k),
\]

where at first (B.5a) is applied to the product of three \( V \)'s enclosed by the parenthesis and then the summation over \( n \) is performed with help of (B.2b). Consider now the RHS of (3.18), given by (3.17),

\[
\text{RHS of (3.18)} = \sum_n \left( \frac{V_{x/x'}(a - b) V_{y/x}(b - n)}{V_{y/x'}(a - n)} \right) \frac{V_{y/y'}(c - n) V_{x/y'}(n - d)}{V_{x/x'}(c - d)} \tag{B.7}
\]

\[
= \sum_{n,k,k'} V_{y/y'}(k - n) V_{y/qy'}(b - k) W_{x,x',y,y'}(a - k) V_{qy'/y'}(k' - d) V_{x/y'}(n - k') W_{y'/x',x'}(c - k') \tag{B.7}
\]

\[
= \sum_k V_{y/qy'}(b - k) W_{x,x',y,y'}(a - k) V_{qy'/y'}(k - d) W_{y'/x',x'}(c - k),
\]

where at first (B.5a) is applied to both products of three \( V \)'s grouped in the parentheses and then the summation over \( n \) is removed by using the inversion relation (2.5). The resulting expression coincides with (B.6), thereby proving (3.18). Thus the star-triangle relation (3.22) implies the star-star relation (3.18) and, consequently, the Yang-Baxter equation (3.13) and the commutativity of the transfer matrices (3.12). Therefore, the star-triangle relation (3.22) does, indeed, plays the role of a primary integrability condition in the model.

### B.3 Alternative graphical notations.

To present the star-triangle relation graphically we need to introduce a new variant of graphical notations for the edge Boltzmann weights. They involve the edges of the type \( (i) \) and type \( (iii) \)

\[
(i) : \quad y \Rightarrow a \Rightarrow x \Rightarrow y = V_{x/y}(a - b), \quad (iii) : \quad a \Rightarrow b = W_{x,y}(a - b). \tag{B.8}
\]
which are distinguished by relative orientation of the edge and associated spectral parameter (dashed) lines. The corresponding Boltzmann weights $V_{x/y}(a - b)$ and $W_{x,y}(a - b)$ are defined by (2.2) and (3.19), respectively. Note that for the type $(iii)$ edges the directed rapidity lines cross the edge from the same side, unlike the type $(i)$ edges where they come from different sides. For better visibility the type $(iii)$ edges will be shown by double lines.

As in (2.1) the weights in (B.8) depend on the difference of spins at the ends of the edge, but, in general, they do not have the spectral parameter “difference property”. The function $W$ depends non-trivially on both rapidities $x$ and $y$, not just on their ratio. With the graphical rules (B.8) the first relation in (3.22) is represented graphically as in Fig. 10. For the second relation one needs to reverse all arrows (both edge arrows and the dashed line arrows).

**B.4 New solutions of the Yang-Baxter equations**

The existence of the star-triangle relation (3.22) allows one to define new integrable Ising-type models in the standard way. First, let us present them in the IRF form. To make the notations uniform it is convenient to redenote weights (3.16) as

$$W^{(1,1)}(a, b, c, d | x, y) = W(a, b, c, d | x, y)$$

(B.9)

where $W$ is given by (3.16) and $x = (x, x'), y = (y, y')$ denote pairs of spectral variables. Define new weight functions,

$$W^{(2,1)}(a, b, c, d | x, y) = \theta(a \geq c) \sum_n W_{x,y}(a - n)V_{y/x'}(b - n)W_{y',x}(c - n)V_{x'/y'}(n - d),$$

(B.10)

$$W^{(1,2)}(a, b, c, d | x, y) = \theta(a \geq b) \sum_n W_{x,y}(a - n)W_{y,x'}(b - n)V_{y/x'}(n - c)V_{x'/y'}(d - n),$$

(B.11)

and

$$W^{(2,2)}(a, b, c, d | x, y) = \sum_n W_{x,y}(a - n)V_{y/x'}(b - n)V_{y'/x}(n - c)V_{x'/y'}(n - d).$$

(B.12)

Here $\theta(a \geq b) = 1$ if $a \geq b$ and vanishes otherwise. Using the notations (B.8) we can represent these

![Figure 10: Graphical representation of the star-triangle relation (3.22a).]
There are eight different Yang-Baxter equations for $W^{(i,j)}$:

$$
\sum_a W^{(i,j)}(g,a,b,f | x,y) W^{(i,k)}(c,e,g,a | x,z) W^{(j,k)}(e,d,a,f | y,z) = \\
\sum_h W^{(j,k)}(c,h,g,b | y,z) W^{(i,k)}(h,d,b,f | x,z) W^{(i,j)}(c,e,h,d | x,y)
$$

for all possible choices of the indices $i,j,k = 1, 2$.

All four IRF weights $W^{(i,j)}$ allow a corresponding vertex form $R^{(i,j)}$ by means of (4.1) (4.2). From the quantum group point of view these $R$-matrices are related to infinite-dimensional evaluation representations of the $U_q(\hat{sl}(2))$ algebra. The indices $i,j = 1$ in the notation $R^{(i,j)}$ indicate representations with highest weights, while $i,j = 2$ indicate representations without highest or lowest weights. As in Sect. 4.2 the $R$-matrices, associated with the highest weight representations admit finite-dimensional reductions. For instance, when $y' = qy$, the $R$-matrix $L = R^{(2,1)}(x,x';y,qy)$ has a block-triangular structure in the second space with a 2-dimensional block given by

$$
L_{i_1,0}^{j_1,0} = \delta_{i_1,j_2} \frac{[q^{i_1}y]}{[y]} , \quad L_{i_1,0}^{j_1,1} = -\delta_{i_1,j_1+1} \frac{[q^{j_1}x']}{[y]} , \\
L_{i_1,1}^{j_1,0} = \delta_{i_1+1,j_1} \frac{[q^{i_1}x]}{[y]} , \quad L_{i_1,1}^{j_1,1} = -\delta_{i_1,j_1} \frac{[q^{i_1-1}xx'/y]}{[y]} .
$$
It can be written in the canonical form (the same as (4.23))

\[
\mathcal{L} = \frac{1}{\mu q^{s+1}} \begin{pmatrix}
\mu q^{\frac{H}{2}} - \mu^{-1} q^{-\frac{H}{2}} & F \\
E & \mu q^{\frac{H}{2}} - \mu^{-1} q^{-\frac{H}{2}}
\end{pmatrix},
\]

where \(E, F, H\) stands for the generators of the quantum universal enveloping algebra \(U_q(sl_2)\), defined in (4.24), and the matrix elements (B.17) correspond to the infinite-dimensional representation \(\pi_{s,s'}^+\) of this algebra.

\[
\pi_{s,s'}^+[H] |j\rangle = (2s + 2s' - 2j) |j\rangle, \quad \pi_{s,s'}^+[E] |j\rangle = [q^{2s+1-j}] |j - 1\rangle, \quad \pi_{s,s'}^+[F] |j\rangle = [q^{j+1-2s'}] |j + 1\rangle,
\]

where \(j \in \mathbb{Z}\) and the parameters are defined as

\[
y = (\mu q^{s+s'})^{-1}, \quad x = q^{-2s}, \quad x' = q^{1-2s'}.
\]

Note, that for the generic case this representation is irreducible and does not have highest or lowest weights.

**Appendix C. Mangazeev’s \(R\)-matrix for higher spin 6-vertex model**

In ref. [41] Mangazeev obtained a remarkably simple formula for the most general higher spin \(R\)-matrix related to the 6-vertex model, expressing it via a \(4\varphi_3\) \(q\)-hypergeometric series. The result of [41] can be represented in various equivalent forms. Here we will use its variant from [79], eq.(3.22) therein, which we quote below preserving their notations

\[
[R_{I,J}(\lambda)]^{i,j'}_{i,j} = \delta_{i+j,i'+j'} + \begin{pmatrix} q^2, q^2 & (q^2,q^2)_{i+j} \\
(q^2,q^2)_{i+j} & (q^2,q^2)_{i'+j'}
\end{pmatrix}
\times \frac{(\lambda^2 q^{-I-J}; q^2)_{i'} (\lambda^2 q^{-I-J}; q^2)_{i} (q^{-2j}; q^2)_{j'}}{(\lambda^2 q^{-I-J}; q^2)_{i+j} (q^{-2j}; q^2)_{j'}}
\times \sum_{n \geq 0} \frac{(q^{-2i}, q^{-2j'}, \lambda^2 q^{-I-J}, \lambda^2 q^{2-I-J-2i-2j}; q^2)_n}{(q^2, q^{-2i-2j}, \lambda^2 q^{2-I-J-2i}, \lambda^2 q^{2-I-J-2j}; q^2)_n} q^{2n}.
\]

Apart from the parameter \(q\), this expression contains three independent variables \(q^I, q^J\) and \(\lambda^2\). They only appear in certain combinations, so it is convenient to introduce the variables \(x, x', y, y'\), such that

\[
\left(\frac{x}{y}\right)^2 = \lambda^2 q^{-I-J}, \quad \left(\frac{x'}{y}\right)^2 = \lambda^2 q^{I+J}, \quad \left(\frac{x}{y'}\right)^2 = \lambda^2 q^{-I}, \quad \left(\frac{x'}{y'}\right)^2 = \lambda^2 q^{I-J}.
\]

Clearly, there are only three independent ratios of these new variables, but the fourth equation above is a corollary of the other three. Next, let us make a substitution of the integer indices in (C.1),

\[
i = a - b, \quad j = b - d, \quad i' = c - d, \quad j' = a - c, \quad a, b, c, d \in \mathbb{Z},
\]

which automatically solves the \(\delta\)-function constraint. Further, let us combine all \(q\)-product factors in front of the sum in (C.1) into a product of the weight functions \(V_\lambda(n)\), using the definition (2.2), and then write the sum in (C.1) as a \(4\varphi_3\) series using (A.7). Then, after all these substitutions, the resulting expression for (C.1) almost literally coincides with the RHS of (A.10). More precisely, one obtains,

\[
\left[\text{RHS of (C.1)}\right] = q^{ij'-ij} \left(\lambda q^{(I-J)/2}\right)^{i'-i} \times \left[\text{RHS of (A.10)}\right].
\]

Taking now into account the definitions (4.1), (4.9), (4.10) and the fact that (A.10) coincides with (A.1) one concludes that the \(R\)-matrix (C.1) is simply related to that defined by (4.10), (4.5),

\[
[R_{I,J}(\lambda)]^{i_1,j_1}_{i_2,j_2} = q^{(j_1j_2-i_1i_2)} (q^{s_1-s_2} \lambda)^{i_1-j_1} \mathcal{R}(\lambda \mid s_1, s_2)^{j_1,j_2}_{i_1,i_2}
\]
where
\[ I = 2s_1, \quad J = 2s_2, \] (C.6)
and \( \lambda \) is the same in both sides. So the two \( R \)-matrices are related by the matrix transpostion in both spaces, followed by a multiplication by some gauge transformation factors. Note, that the latter cancel out in the Yang-Baxter equation. The reason we have dropped these factors is that it is impossible to consistently attribute them to the edge Boltzmann weights in the Ising type formulation of the model.

Let us now present the connection with Mangazeev’s definitions of the \( Q \)-operators. First compare two definitions, our Eq.(D.3), defining \( \overline{R}^{j_1,j_2}_{i_1,i_2} \), and Eq.(6.21) of [41], which defines \([A_-^{(I)}(\lambda)]_{n,i}^{n',i'}\). These quantities are related as follows,
\[ [A_-^{(I)}(\lambda)]_{n,i}^{n',i'} = (-\lambda)^{I-I'} \left((y/x')^2;q^2\right)_I \overline{R}(y|x,x')_{i',n'} q^{n'-n} \frac{c_{n'}}{c_n}, \] (C.7)
where
\[ \left(\frac{y}{x'}\right)^2 = \lambda^{-2}q^{1-I}, \quad \left(\frac{y}{x}\right)^2 = \lambda^{-2}q^{1+I}, \quad c_n = q^{-n(1+I)/2}(q^2;q^2)_n. \] (C.8)
The extra power \((-\lambda)^{I-I'}\) in (C.7) appears because in Mangazeev’ formulas \( Q_- \) is the Laurent polynomial.

Thus, Mangazeev’s \( Q_-^{(\lambda)} \) is related to \( \overline{Q}(q^{-1}y) \) by
\[ Q_-^{(\lambda)} = \lambda^{NI-M}\overline{Q}(q^{-1}y). \] (C.9)
for integer \( I = 2s_1 \).

Appendix D. Q-matrices and functional relations

The elements of the \( Q \)-matrices (6.10)
\[ (Q(y'))^{j_2,\ldots,j_1}_{i_N,\ldots,i_1} = Z_0^{-1} \sum_{k_1=0}^{\infty} \omega_h^{k_1} \prod_{m=1}^{N} \mathcal{R}(y'|x_m,x'_m)_{i_m,k_m}^{j_m,k_m} \] (D.1)
\[ \varphi(y,y)^{-1}(\overline{Q}(q^{-1}y))_{i_N,\ldots,i_1}^{j_N,\ldots,j_1} = Z_0^{-1} \sum_{k_1=0}^{\infty} \omega_h^{k_1} \prod_{m=1}^{N} \overline{R}(y|x_m,x'_m)_{i_m,k_m}^{j_m,k_m} \]
where \( k_{N+1} = k_1 \), are expressed through the \( R \)-matrices,
\[ \mathcal{R}(y'|x,x')_{i_1,i_2}^{j_1,j_2} = q^{i_2-j_2} \frac{(q^2;q^2)_{j_2}}{(q^2;q^2)_{i_2}} \lim_{y' \rightarrow 0} \left(\frac{y'}{y}\right)^{j_1} \mathcal{R}(x,x',y,y')_{i_1,i_2}^{j_1,j_2} \] (D.2)
\[ = \delta_{i_1+i_2,j_1+j_2} \left(\frac{x'}{y'}\right)^{-i_1} \sum_{n=0}^{\min(i_2,j_1)} \left(\frac{x'}{x}\right)^{n-i_2} \frac{(q^2;q^2)_{i_1+i_2-n}}{(q^2;q^2)_{i_2-n}} V_{x/y}(n)V_{y/x'}(j_1-n) \]
and
\[ \overline{R}(y|x,x')_{i_1,i_2}^{j_1,j_2} = (-)^{i_2-j_2} q^{i_2-j_2} \frac{(q^2;q^2)_{j_2}}{(q^2;q^2)_{i_2}} \lim_{y' \rightarrow 0} \left(\frac{y'}{y}\right)^{i_1} \overline{R}(x,x',y,y')_{i_1,i_2}^{j_1,j_2} \] (D.3)
\[ = \delta_{i_1+i_2,j_1+j_2} \left(\frac{x'}{y'}\right)^{j_1} \sum_{n=0}^{\min(i_2,j_1)} \left(-\frac{x'}{x}\right)^{-n} \frac{q^{n^2}}{(q^2;q^2)_n(q^2;q^2)_{j_1-n}} V_{y/x}(i_2-n) V_{y/x'}(i_1+i_2-n). \]

obtained by suitable limits of (4.5).
At \( x = q^{-1}x' \), which is \( s = 1/2 \), one has

\[
\mathfrak{R}^{0,j_2}_{i,j_2} = \delta_{i,j_2} q^{-j_2}, \quad \mathfrak{R}^{1,j_2}_{0,i_2} = -\delta_{i_2,j_2+1} \frac{y'}{x'} q^{-j_2},
\]

\[
\mathfrak{R}^{0,j_2}_{1,j_2} = -\delta_{i_2+1,j_2} \frac{y'}{x'} [q^{j_2}], \quad \mathfrak{R}^{1,j_2}_{1,j_2} = \delta_{i_2,j_2} \frac{y'}{x'} [q^{j_2}],
\]

and

\[
\mathfrak{R}^{0,j_2}_{0,j_2} = \delta_{i,j_2} \left[ \frac{y}{x} q^{j_2} \right], \quad \mathfrak{R}^{1,j_2}_{0,j_2} = -\delta_{i_2,j_2+1} \left[ \frac{y}{x'} q^{j_2} \right],
\]

\[
\mathfrak{R}^{0,j_2}_{1,j_2} = \delta_{i_2+1,j_2} \left[ \frac{y}{x'} q^{j_2} \right], \quad \mathfrak{R}^{1,j_2}_{1,j_2} = -\delta_{i_2,j_2} \left[ \frac{y}{x} q^{j_2} \right].
\]

In the short notations

\[
\mathfrak{R} = \begin{pmatrix} q^{H/2} & \mu \mathcal{E}_- \\ \mu \mathcal{E}_+ & q^{-H/2} - q^{-1} \mu^2 q^{H/2} \end{pmatrix}
\]

where

\[
\mu = q^{1/2} \frac{y'}{x'}, \quad \langle i | q^{H/2} | j \rangle = \delta_{i,j} q^{-j},
\]

and

\[
\langle i | \mathcal{E}^- | j \rangle = -\delta_{i,j+1} q^{-j-1/2}, \quad \langle i | \mathcal{E}^+ | j \rangle = \delta_{i+1,j} (q^{-j} - q^{j}) q^{-1/2}.
\]

Similarly,

\[
\overline{\mathfrak{R}} = \frac{1}{1 - \mu \mu^2} \begin{pmatrix} q^{-H/2} - \mu \mu^2 q^{H/2} & \mu \mathcal{E}_+ \\ \mu \mathcal{E}_- & q^{H/2} \end{pmatrix},
\]

where

\[
\mu = q^{-1/2} \frac{y}{x'}, \quad \langle i | q^{H/2} | j \rangle = \delta_{i,j} q^{j},
\]

and

\[
\langle i | \mathcal{E}^- | j \rangle = \delta_{i+1,j} (q^{-j} - q^{j}) q^{1/2}, \quad \langle i | \mathcal{E}^+ | j \rangle = -\delta_{i,j+1} q^{j+1/2}.
\]

In both cases

\[
[H, \mathcal{E}_\pm] = \pm 2 \mathcal{E}_\pm, \quad q \mathcal{E}_+ \mathcal{E}_- - q^{-1} \mathcal{E}_- \mathcal{E}_+ = q - q^{-1}.
\]

Using the correspondence (6.1) one can write the IRF version of the above operators (D.1) as

\[
(Q(y'))_{a_{N+1}, a_N, \ldots, a_1}^{b_N+1, b_N, \ldots, b_1} = \prod_{i=1}^{N} \left( \frac{q x'_i}{y'} \right)^{a_{i+1} - a_i} \left( \sum_{m_i} \left( \frac{x'_i}{x_i} \right)^{a_{i+1} - m_i} \frac{(q^2; q^2)_{m_i-a_i} V_{x_i/y'} (b_{i+1} - m_i) V_{y'/x'_i} (m_i - b_i)}{(q^2; q^2)_{m_i-a_i+1}} \right),
\]

\[
(Q^{-1}(y))_{a_{N+1}, a_N, \ldots, a_1}^{b_N+1, b_N, \ldots, b_1} = \prod_{i=1}^{N} \left( \frac{q x'_i}{y} \right)^{b_{i+1} - b_i} \left( \sum_{m_i} \left( -\frac{q x'_i}{x_i} \right)^{n_i - b_i} \frac{(q^{b_{i+1}-n_i})^2 V_{x_i/y} (n_i - a_{i+1})}{(q^2; q^2)_{b_{i+1}-n_i} (q^2; q^2)_{n_i-b_i} V_{y/x'_i} (n_i - a_i)} \right),
\]

where the summations run over the intervals

\[
\max(a_{i+1}, b_i) \leq n_i \leq b_{i+1}, \quad \max(a_{i+1}, b_i) \leq m_i \leq b_{i+1},
\]
\[ a_{N+1} = M + a_1, \quad b_{N+1} = M + b_1, \quad \text{(D.15)} \]

with \( M \) being the conserved number of up spins \((6.5)\). With the same correspondence between indices \((6.1)\) the matrix elements \((D.1)\) can now be written as

\[
(Q(y'))_{i^N \ldots j_1} = Z_0^{-1} \sum_{a_1 \leq b_1 \leq \infty} \omega_h^{b_1-a_1} (Q(y'))_{a_{N+1},a_{N+1},a_{N+1},a_{N+1}}^{b_{N+1},b_{N+1},b_{N+1},b_{N+1}}
\]

where the sum is taken over the spin \( b_1 \). Consider now the product

\[
\varphi(y,y)^{-1} (Q(q^{-1} y))_{i^N \ldots j_1} = Z_0^{-2} \sum_{a_1 \leq b_1 \leq \infty} \omega_h^{b_1-a_1} \sum_{c_1 \leq c_2 \leq c_3 \leq \infty} Q(q^{-1} y)_{a_{N+1},a_{N+1},a_{N+1}}^{b_{N+1},b_{N+1},b_{N+1}}.
\]

Note that the elements \((D.13)\) have the same vanishing conditions as those of the IRF transfer matrix \((3.9)\). Therefore, the summation over \( c_1, c_2, \ldots, c_N \) is restricted to a finite domain

\[
c_{i+1} \geq c_i, \quad a_i \leq c_i \leq b_i, \quad i = 1, 2, \ldots, N,
\]

\[
 c_{N+1} = M + c_1,
\]

where \( c_{N+1} = M + c_1 \). Using \((D.13)\) one obtains

\[
(D.17) = Z_0^{-2} \left( \frac{y'}{y} \right)^M \sum_{b_1 = a_1} \omega_h^{b_1-a_1} \sum_{\{n_i,m_i\}} \prod_{i=1}^{N} \frac{V_{x_i/y'} (b_{i+1} - m_i) V_{y'/x_i} (m_i - b_i) V_{y/x_i} (n_i - a_{i+1})}{V_{y/x_i} (m_i - a_i)} G_i,
\]

where

\[
G_i = \left( \frac{x_{i-1}'}{x_{i-1}} \right)^{(n_{i-1} - m_{i-1})} \sum_{c_i = n_{i-1}} q^{(n_{i-1} - c_i)}\frac{(-q)^{n_{i-1} - c_i}}{(q^2;q^2)^{c_i-n_{i-1}}(q^2;q^2)^{n_{i-1} - c_i}}.
\]

with the boundary conditions

\[
n_0 = n_N - M, \quad m_0 = m_N - M.
\]

One can show that for all the non-vanishing contributions to \((D.19)\) the spins variables \( \{n_i\} \) and \( \{m_i\} \) are ordered as

\[
m_{i+1} \geq n_i, \quad n_{i+1} \geq n_i, \quad m_i \geq n_i.
\]

The considerations are similar to those presented in Sect. 3.1, where the ordering of white spins in Fig. 1 was discussed. Note that spins \( \{n_i\} \) and \( \{m_i\} \) correspond to black sites in the same figure. With the substitution

\[
a = n_i - n_{i-1}, \quad b = m_i - n_{i-1}, \quad c = m_i - n_{i-1},
\]

the sum in \((D.20)\) can be written as

\[
F(a, b, c) = \sum_{n=0}^{\min(a,b)} \frac{(-q)^n q^{n(n-1)} (q^2;q^2)_{c-n}}{(q^2;q^2)_n(q^2;q^2)_{a-n}(q^2;q^2)_{b-n}} = q^{2ab} \frac{(q^2;q^2)_{c-a}(q^2;q^2)_{c-b}}{(q^2;q^2)_a(q^2;q^2)_b(q^2;q^2)_{c-a-b}}.
\]

It is evaluated exactly by using the Heine’s \( q \)-analog of the Gauss summation formula for the hypergeometric function \([46]\. It is not difficult to see that

\[
F(a, b, c) = \begin{cases} 0, & \text{if } \max(a, b) \leq c < a + b, \\ q^{2ab}, & \text{if } c = a + b, \\ \text{non-zero}, & \text{otherwise}. \end{cases}
\]

\[47\]
Note that with the substitution (D.23) the first inequality \( \max(a, b) \leq c \) in (D.25) is satisfied for all sums in (D.20) by virtue of (D.22). Let us now examine the difference \( \Delta = c - a - b \). In terms of the variables (D.23) it reads
\[
\Delta_i = (m_i - n_i) - (m_{i-1} - n_{i-1}).
\] (D.26)

Evidently, with the boundary condition (D.21) one gets
\[
\sum_{i=1}^{N} \Delta_i = 0.
\] (D.27)

Therefore, either \( \Delta_i = 0 \) for all \( i = 1, 2, \ldots, N \) or at least one \( \Delta_i < 0 \) (for some particular value of \( i \)). In the latter case the corresponding \( G_i \) vanishes and so does the whole product in (D.19). Thus the non-vanishing contributions to (D.19) could only arise when \( \Delta_i = 0 \) for all \( i = 1, 2, \ldots, N \). This gives \( N - 1 \) independent constraints between the spins \( \{n_i\} \) and \( \{m_i\} \) which allow one to easily perform the summation over all but one of the spins \( \{m_i\} \). Then, instead of \( m_1 \) we choose the integer variable
\[
\delta = m_1 - n_1, \quad 0 \leq \delta \leq b_1 - a_1,
\] (D.28)
whose allowed values are determined by the inequalities (D.14) and (D.22). Eq.(D.19) then takes the form
\[
(D.17) = Z_0^{-2} \left( \frac{y'}{y} \right)^M \sum_{b_1=a_1}^{\infty} \omega_h^{b_1-a_1} \prod_{\delta=0}^{b_1-a_1} w^{\delta} \sum_{i=1}^{N} \mathcal{W}(b_{i+1}, b_i, a_{i+1} + \delta, a_i + \delta \mid x_i, x'_i \mid y, y')
\] (D.29)
where we used (3.16) and
\[
Z_0 = \sum_{k=0}^{\infty} (w \omega_h)^k, \quad w = q^{2M} \prod_{i=1}^{N} \frac{x_i}{x'_i}.
\] (D.30)

Remembering the definitions (4.1), (6.2), (6.8), using a simple identity
\[
(y'/y)^M \varphi(y, y) = \varphi(y, y') \rho(y, y'),
\] (D.31)
and rearranging the summations one gets from (D.29),
\[
\overline{Q}(q^{-1} y) Q(y') = Z_0^{-1} \varphi(y, y') T^{(+)}(y, y')
\] (D.32)
as stated in (6.13) in the main text.

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