On a general Syracuse problem with conjectures

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Abstract

In this paper, we study a general Syracuse problem. We give some necessary conditions concerning the existence of eventual non trivial cycles. Some properties based on linear logarithmic forms are established. New general conjectures are given. To illustrate the behavior of such a problem, some particular examples are presented.

1 Introduction

Let \( N = \{1, 2, 3, \ldots\} \) be the set of integers \( \geq 1 \). The Syracuse problem also known as the Collatz, Kakutani or 3x+1 problem, concerns the sequence of positive integers generated by the iterations associated to the following map \( S : N \to N \) defined by

\[
S(n) = \begin{cases} 
    n/2 & \text{if } n \text{ is even} \\
    3n + 1 & \text{if } n \text{ is odd}
\end{cases}
\]  

(1)

For every \( k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), the notation \( S^{(k)} \) stands for the \( k \)th iterate of the map \( S \). The well-known conjecture of the 3x+1 problem (or Collatz’s conjecture) states that, for any initial value \( n \in \mathbb{N} \), there is a positive integer \( k \in \mathbb{N}_0 \) such that \( S^{(k)}(n) = 1 \). A second conjecture in the Syracuse problem states that the trivial cycle \( (1 \to 2 \to 1) \) is the unique cycle in the graph associated to the map \( S \). The 3x+1 problem has been explored for more than 80 years by many authors. An extensive overviews on the literature concerning the Syracuse problem has been given by Lagarias [5, 6, 7]. The conjecture has been verified experimentally with a computer [8], for \( n \leq 5 \times 2^{60} \approx 5 \times 10^{26} \). The latest calculations based on the computation of \( \min(\Omega) > 87 \times 2^{60} \) increases minimum cycle length to 1043960591. Recently, Tao in [13] proved that Collatz conjecture is almost true for almost all numbers. Many generalizations of the Syracuse problem have been given in the literature, see [7] for more details.

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In this paper, we study a generalization to the Syracuse problem. The goal is to shed light on the sequence related to the $3x+1$ problem for a better understanding of the phenomenon. The general problem \"$qn+r$\" was first briefly introduced by Crandall in [3]. In this paper [3], Crandall mentioned some experimental results and gave some discussions on such a general problem. In our present paper, we will study such a problem in more details and we will give some new results.

The outline of this paper is as follows. In Section 2, we introduce the unified generalization of the Syracuse problem. We will give some notations, definitions and results relative to the general operator. In Section 3, we will give some results on the constraints on eventual non trivial cycle. This section is in fact a generalization of the work of Eliahou [4]. Section 4, we study a few examples with some results and conjectures.

2 The Syracuse generalized problem

We consider the map $T : \mathbb{N} \rightarrow \mathbb{N}$ given by

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ is even}, \\ (an+b)/2 & \text{if } n \text{ is odd}, \end{cases}$$

where $a$ and $b$ are both odd integers such that $a > -b$. Then, the number $an+b$ is an even positive integer for any odd number $n$ and one can iterate the $T$ function any number of times.

The Syracuse problem may be viewed as a direct graph whose vertices are the positive integers and whose edges are the connection from $n$ to $T(n)$. Following the terminology of Lagarias [5], such a graph will be called the Collatz graph. Given $n \in \mathbb{N}$, the trajectory of $n$ is the set $\Gamma(n) = \{n, T(n), T^2(n), \ldots\}$ of iterates. A cycle of the graph associated to the map $T$ having $k$ vertices is a trajectory $\Omega$ for which $T^k(x) = x$ for all $x \in \Omega$. The cardinal $\#\Omega = \text{Card}(\Omega) = k$, also called the length, of $\Omega$ is in fact the smallest integer such that $T^k(x) = x$ for all $x \in \Omega$. We will denote by $\Omega(\omega) = \{\omega, T(\omega), T^2(\omega), \ldots, T^{k-1}(\omega)\}$ the cycle associated to $T$ of length $k$ where $\omega$ is the smallest element of the cycle. We will also use the notation

$$\Omega(\omega) = (\omega \rightarrow T(\omega) \rightarrow T^2(\omega) \rightarrow \ldots \rightarrow T^{k-1}(\omega) \rightarrow \omega),$$

to denote the cycle $\Omega(\omega)$ having $k$ elements. For each choice of the parameters $a$ and $b$ the notation $C(a,b)$ stands for the set of all possible cycles associated to $T$.

There are two possible situations for a trajectory $\{T^{(k)}(n)\}_{k \in \mathbb{N}_0}$, for $n \in \mathbb{N}$:

(i) Convergent trajectory to one or other specific cycle $\Omega(\omega)$, assuming that such a cycle exists.

(ii) Divergent trajectory: $\lim_{k \to \infty} T^{(k)}(n) = \infty$. 

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Assuming that the cycle $\Omega(\omega)$ exists, we denote by $G(\omega)$ and $G(\infty)$ the subsets of $\mathbb{N}$ given by
\[ G(\omega) = \{ n \in \mathbb{N} : \exists k \in \mathbb{N}, \ T^{(k)}(n) = \omega \}, \]
and
\[ G(\infty) = \{ n \in \mathbb{N} : \lim_{k \to \infty} T^{(k)}(n) = \infty \}. \]

We observe that for all $\omega, \omega' \in \mathbb{N}$ such that the cycles $\Omega(\omega)$ and $\Omega(\omega')$ exist, if $\omega \neq \omega'$, then $G(\omega) \cap G(\omega') = \emptyset$ and $G(\infty) \cap G(\omega) = \emptyset$. The sets are pairwise disjoint and the set $G(\infty)$ may be empty. We have $\Omega(\omega) \subset G(\omega)$.

**Conjecture 2.1.** For any parameters $a$ and $b$ such that $a + b = 2^{\nu_0}$, where $\nu_0 \geq 1$, the set $\mathcal{C}(a, b)$ has finite cardinality: $1 \leq q^* = \# \mathcal{C}(a, b) < \infty$. Then, it follows that there is $q^*$ distinct integers $\omega_1, \ldots, \omega_{q^*}$, such that: $\mathcal{C}(a, b) = \{ \Omega(\omega_1^*), \ldots, \Omega(\omega_{q^*}^*) \}$, and
\[ \mathbb{N} = G(\omega_1^*) \cup \cdots \cup G(\omega_{q^*}^*) \cup G(\infty). \]

The set $G(\infty)$ may be empty or not empty.

**Questions:** Suppose that the condition $a + b = 2^{\nu_0}$ holds, then one can ask the following questions:

1. Under which conditions on the parameters $a$ and $b$ the set $G(\infty)$ is empty or not empty? In this case where the set $G(\infty)$ is empty, there is no divergent trajectory.

2. Is it possible to find an estimate for the cardinality $\# \mathcal{C}(a, b)$? Then, it will be possible to confirm that the set $\mathcal{C}(a, b)$ is finite.

3. Under which conditions on the parameters $a$ and $b$, the set $\mathcal{C}(a, b)$ is of cardinality $\# \mathcal{C}(a, b) = 1$? In this case, we will see that the unique cycle in $\mathcal{C}(a, b)$ is the trivial one $\Omega(1)$. If in addition $G(\infty) = \emptyset$, then for any integer $n$ there exists an integer $k$ such that $T^{(k)}(n) = 1$ and we will have $\mathbb{N} = G(1)$.

To find answers to the previous questions in a general situation, is a difficult task. We will study in the following sections the problem in some particular situations. The classical conjecture of Collatz may be reformulated as following.

**Conjecture 2.2.** (Collatz conjecture) For the parameters $a = 3$ and $b = 1$ the set $\mathcal{C}(3, 1)$ contains only one cycle (the trivial one): $\mathcal{C}(3, 1) = \{ \Omega(1) \}$, with $\Omega(1) = (1 \to 2 \to 1)$ and we have: $\mathbb{N} = G(1)$ and $G(\infty) = \emptyset$.

**Remark 2.1.** If we assume that the set $\mathcal{C}(a, b)$ is of finite cardinality. Then, the sets $G(\omega_1^*), \ldots, G(\omega_{q^*}^*), G(\infty)$ are the classes for the binary equivalence relation $\sim_T$ defined on $\mathbb{N}$ as following: For $n, n' \in \mathbb{N}$
\[ n \sim_T n' \iff \exists \omega \in \mathbb{N} \cup \{ \infty \} : \ n, n' \in G(\omega). \]
The quotient of \( \mathbb{N} \) by the relation \( \sim_T \) is

\[
\mathbb{N}/ \sim_T = \left\{ G(\omega_1^n), \ldots, G(\omega_q^n), G(\infty) \right\},
\]

if \( G(\infty) \neq \emptyset \) and

\[
\mathbb{N}/ \sim_T = \left\{ G(\omega_1^n), \ldots, G(\omega_q^n) \right\},
\]

if \( G(\infty) = \emptyset \). Furthermore,

\[
\mathbb{N} = G(\omega_1^n) \cup \cdots \cup G(\omega_q^n) \cup G(\infty).
\]

In the following theorem, we will see that if \( a \) and \( b \) satisfy the condition of Conjecture 2.1 then the set \( \mathcal{C}(a, b) \) is not empty.

**Theorem 2.1.** For any parameters \( a \) and \( b \) such that \( a + b = 2^{\nu_0} \), where \( \nu_0 \geq 1 \), the set \( \mathcal{C}(a, b) \) is not empty. It contains at least the trivial cycle of length \( \nu_0 \):

\[
\Omega(1) = (1 \rightarrow 2^{\nu_0-1} \rightarrow 2^{\nu_0-2} \rightarrow \ldots \rightarrow 2 \rightarrow 1).
\]

Furthermore, if in addition \( a = 2^{\nu_1} - \delta \), where \( \nu_1 \geq 1 \) for \( \delta = \pm 1 \) with \( \delta b > 0 \), then the set \( \mathcal{C}(a, b) \) contains in addition the following second trivial cycle of length \( \nu_1 \):

\[
\Omega(\delta b) = (\delta b \rightarrow \delta 2^{\nu_1-1} b \rightarrow \delta 2^{\nu_1-2} b \rightarrow \ldots \rightarrow 2\delta b \rightarrow \delta b).
\]

**Proof.** It is trivial to check that for all odd integers \( a \) and \( b \) such that \( a + b = 2^{\nu_0} \), we have \( T^{(\nu_0)}(1) = 1 \). Indeed, we have \( T(1) = (a + b)/2 = 2^{\nu_0-1} \) and then \( T^{(\nu_0)}(1) = 1 \). Furthermore, if in addition \( a = 2^{\nu_1} - \delta \) for \( \delta = \pm 1 \) with \( \delta b > 0 \), then we have \( T^{(\nu_1)}(\delta b) = \delta b \). Indeed, \( T(\delta b) = (\delta ab + b)/2 = b(\delta a + 1)/2 \). But as \( \delta a = \delta 2^{\nu_1} - 1 \), it follows that \( T(\delta b) = (\delta b)2^{\nu_1-1} \) and then \( T^{(\nu_1)}(\delta b) = \delta b \). \( \square \)

**Remark 2.2.** According to the previous Theorem, we may set \( \omega_1^* = 1 \). Furthermore, in the case where \( \delta b > 0 \), we have \( \omega_2^* = \delta b \) and \( \# \mathcal{C}(a, b) \geq 2 \).

**Examples 2.1.**

1. For the classical Syracuse problem, we have \( a = 3 = 2^{\nu_1} + \delta \) and \( b = 1 = 2^{\nu_0} - a \), where \( (\nu_0, \nu_1, \delta) = (1, 2, 1) \). In this case, \( \mathcal{C}(3, 1) \) has at least the trivial cycle of length \( \nu_1 = 2 \): \( \Omega(1) = (1 \rightarrow 2 \rightarrow 1) \).

   We observe that \( (a, b) = (3, 1) \) may be also written in the form \( a = 3 = 2^{\nu_1} - \delta \) and \( b = 2^{\nu_0} - a = 1 \) with \( (\nu_0, \nu_1, \delta) = (2, 2, 1) \) and we have \( \delta b > 0 \). So according to the previous theorem, \( \mathcal{C}(3, 1) \) must contains two trivial cycles \( \Omega(1) \) of length \( \nu_1 \) and \( \Omega(\delta b) \) of length \( \nu_0 \). Here of course the two trivial cycles coincide (\( \delta b = 1 \)).

2. For \( a = 7 \) and \( b = 9 \), the corresponding map \( T \) is given by

\[
T(n) = \begin{cases} 
  n/2 & \text{if } n \text{ is even}, \\
  (7n + 9)/2 & \text{if } n \text{ is odd}.
\end{cases}
\]

We have \( a = 7 = 2^{\nu_1} - \delta \) and \( b = 9 = 2^{\nu_0} - a \) with \( (\nu_0, \nu_1, \delta) = (4, 3, 1) \) and \( \delta b > 0 \). According to the previous theorem, \( \mathcal{C}(7, 9) \) contains at least the two trivial cycles the first one \( \Omega(1) \) of length \( \nu_0 = 4 \) and the second one \( \Omega(\delta b) \) of length \( \nu_1 = 3 \). We have: \( \Omega(1) = (1 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1) \) and \( \Omega(\delta b) = (9 \rightarrow 36 \rightarrow 18 \rightarrow 9) \).
3. For $a = 9$ and $b = -7$, the corresponding map $T$ is given by

$$T(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even,} \\
(9n - 7)/2 & \text{if } n \text{ is odd.}
\end{cases}$$

We have $a = 9 = 2^3 - \delta$ and $b = -7 = 2^0 - a$ with $(\nu_0, \nu_1, \delta) = (3, 1, -1)$ and $\delta b > 0$. According to the previous theorem, $\mathcal{C}(9, -7)$ contains at least the two trivial cycles the first one $\Omega(1)$ of length $\nu_1 = 1$ and the second one $\Omega(\delta b)$ of length $\nu_0 = 3$. We have: $\Omega(1) = (1 \rightarrow 1)$ and $\Omega(\delta b) = (7 \rightarrow 28 \rightarrow 14 \rightarrow 7)$.

4. For $a = 3$ and $b = 5$, the corresponding map $T$ is given by

$$T(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even,} \\
(3n + 5)/2 & \text{if } n \text{ is odd.}
\end{cases}$$

We have $a = 3 = 2^{
u_1} - \delta$ and $b = 5 = 2^0 - a$ with $(\nu_0, \nu_1, \delta) = (3, 2, 1)$ and $\delta b > 0$. According to the previous theorem, $\mathcal{C}(3, 5)$ contains at least the two trivial cycles. The first one $\Omega(1)$ of length $\nu_0 = 3$ and the second one $\Omega(\delta b)$ of length $\nu_1 = 2$. We have: $\Omega(1) = (1 \rightarrow 4 \rightarrow 2 \rightarrow 1)$ and $\Omega(\delta b) = (5 \rightarrow 10 \rightarrow 5)$. But, in this case, $\mathcal{C}(3, 5)$ has in addition two non trivial cycles of lengths 5, which are:

$\Omega(19) = (19 \rightarrow 31 \rightarrow 49 \rightarrow 76 \rightarrow 38 \rightarrow 19)$,
$\Omega(23) = (23 \rightarrow 37 \rightarrow 58 \rightarrow 29 \rightarrow 46 \rightarrow 23)$, and two other non trivial cycles of lengths 27, which are:

$\Omega(187) = (187 \rightarrow 283 \rightarrow 427 \rightarrow 643 \rightarrow 967 \rightarrow 1453 \rightarrow 2182 \rightarrow 1091 \rightarrow 1639 \rightarrow 2461 \rightarrow 3694 \rightarrow 1847 \rightarrow 2773 \rightarrow 4162 \rightarrow 2081 \rightarrow 3124 \rightarrow 1562 \rightarrow 781 \rightarrow 1174 \rightarrow 587 \rightarrow 883 \rightarrow 1327 \rightarrow 1993 \rightarrow 2992 \rightarrow 1496 \rightarrow 748 \rightarrow 374 \rightarrow 187)$ and

$\Omega(347) = (347 \rightarrow 523 \rightarrow 787 \rightarrow 1183 \rightarrow 1777 \rightarrow 2668 \rightarrow 1334 \rightarrow 667 \rightarrow 1003 \rightarrow 1507 \rightarrow 2263 \rightarrow 3397 \rightarrow 5098 \rightarrow 2549 \rightarrow 3826 \rightarrow 1913 \rightarrow 2872 \rightarrow 1436 \rightarrow 718 \rightarrow 359 \rightarrow 541 \rightarrow 814 \rightarrow 407 \rightarrow 613 \rightarrow 922 \rightarrow 461 \rightarrow 694 \rightarrow 347)$.

The arguments for the following theorem are similar to those from 3.

**Theorem 2.2.** For $b > 1$, $\exists n \in \mathbb{N}$ such that $\forall k \in \mathbb{N}_0$, we have $T^{(k)}(n) \neq 1$.

**Proof.** Let $b > 1$ be an odd integer and choose any integer $n$ such that $n \equiv 0 \pmod{b}$. Then $2T(n) = b(aq + 1) \equiv 0 \pmod{b}$, where $n = qb$. It follows that $2^\ell T(n) \equiv 0 \pmod{b}$ for any power $\ell \geq 1$. As $b$ is odd, by Gauss Theorem, we have also $T(n) \equiv 0 \pmod{b}$. It follows, that all iterates $T^{(k)}(n)$ of $n$ are multiple of $b$. Hence, for $n \equiv 0 \pmod{b}$ and $b > 1$, there is no integer $k$ such that $T^{(k)}(n) = 1$. $\square$

**Corollary 2.1.** If $b > 1$ and $\#\mathcal{C}(a, b) = 1$, then $G(\infty) \neq \emptyset$, i.e., the map $T$ has a divergent trajectory.

**Proof.** Its an immediate consequence of the previous theorem. $\square$
3 Bounds for cycle lengths

Let \( \xi \) be an irrational real number. We say that \( \mu = \mu(\xi) \) is an effective irrationality measure of \( \xi \) if for all \( \varepsilon > 0 \), there exists an integer \( q_0(\varepsilon) > 0 \) (effectively computable) such that

\[
\forall (p, q) \in \mathbb{Z} \times \mathbb{N}, \quad q > q_0(\varepsilon) \implies \left| \frac{p}{q} - \xi \right| > \frac{1}{q^{\mu + \varepsilon}},
\]

see [9] for more details. The following theorem is a consequence of Baker’s theory of linear forms in logarithms (see e.g. [1, 15] and the references therein). By definition, two positive rational numbers are multiplicatively independent if the quotient of their logarithms is irrational. In the following, we denote by \( \xi \) the real number \( \xi = \log(a) / \log(2) \).

We note that \( \xi \) is a transcendental real number. Indeed, suppose that \( \xi \) is a rational number, then there exist a pair of integers \( (p, q) \in \mathbb{N}^2 \) such that \( a^q = 2^p \), which is impossible since \( a \) is an odd integer number. Now, as \( \xi \) is irrational real number and \( 2^\xi = a \) is algebraic number, then according to the Gelfond-Schneider theorem, we get that \( \xi \) is transcendental real number.

**Theorem 3.1.** Let \( a_1, a_2, b_1, b_2 \) be positive integers with \( a_1 > a_2 \) and \( b_1 > b_2 \). Assume that \( a_1/a_2 \) and \( b_1/b_2 \) are multiplicatively independent. There exists an absolute, effectively computable, constant \( C \) such that

\[
\mu_{\text{eff}}\left( \frac{\log(a_1/a_2)}{\log(b_1/b_2)} \right) \leq C(\log a_1)(\log b_1).
\]

In our situation, we set \( a_1 = a > a_2 = 1 \) and \( b_1 = 2 > b_2 = 1 \). According to the previous theorem, we have \( \mu_{\text{eff}}(\xi) \leq C(\log a)(\log 2) \). The convergents of the continued fraction of the irrational number \( \xi \) imply that \( \mu_{\text{eff}}(\xi) \geq 2 \). Thus, the effective irrationality measure of \( \xi \) is a finite number:

\[
2 \leq \mu_{\text{eff}}(\xi) \leq C(\log a)(\log 2).
\]

In the following, we set \( \mu = \mu_{\text{eff}}(\xi) \).

For \( \varepsilon = 1 \), there exists an integer \( \hat{q}_0 > 0 \) such that

\[
\forall (p, q) \in \mathbb{Z} \times \mathbb{N}, \quad q > \hat{q}_0 \implies \left| \frac{p}{q} - \xi \right| > \frac{1}{q^{\mu+1}} \implies \left| p - q\xi \right| > \frac{1}{q^\mu}.
\]

(3)

Let us now recall some properties from the continued fractions which may be found in the literature of the theory of rational approximation. There is a unique representation of \( \xi \) as an infinite simple continued fraction:

\[
\xi = [a_0, a_1, a_2, a_3, \ldots]
\]

The integer numbers \( a_n \) are the partial quotients, the rational numbers

\[
\frac{P_n}{q_n} = [a_0, a_1, \ldots, a_n]
\]
are the convergents, where \( \gcd(p_n, q_n) = 1 \). The numbers \( \xi = [a_0, a_1, \ldots, a_n, x_{n+1}] = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}} \) are the complete quotients. From these definitions we deduce, for \( n \geq 0 \),

\[
\xi = [a_0, a_1, \ldots, a_n, x_{n+1}] = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}}.
\]

The integers \( p_n, q_n \) are obtained recursively as follows:

\[
p_n = a_np_{n-1} + p_{n-2},
q_n = a_nq_{n-1} + q_{n-2},
\]

with the initial values \( p_{-2} = 0 \), \( p_{-1} = 1 \) and \( q_{-2} = 1 \), \( 1_{-1} = 0 \). The sequences \( (p_n) \) and \( (q_n) \) are strictly increasing unbounded sequences. It is well known that the fractions \( p_n/q_n \) satisfy the following properties:

**Lemma 3.1.** For any pair of integers \( (p, q) \in \mathbb{N}^2 \) with \( 1 < q < q_n \),

\[
|p_n - q_n\xi| < |p - q\xi|.
\]

**Lemma 3.2.** For \( n \geq 0 \),

\[
\frac{1}{q_n + q_{n+1}} < |p_n - q_n\xi|.
\]

Let \( \Omega \) be a cycle of \( T \) and let \( \Omega_0 \) and \( \Omega_1 \) denote the subsets of \( \Omega \) of its even and odd elements, respectively. Let \( L = \#\Omega_0 \) and \( K = \#\Omega_1 \) be the cardinals of \( \Omega_0 \) and \( \Omega_1 \), respectively. We have the following theorem which may be seen as an extension to the one obtained by Eliahou [4].

**Lemma 3.3.** We assume \( b \geq 1 \). Then, \( K \) and \( L \) satisfy the following inequality

\[
0 < (K + L) - K\xi \leq \frac{bK}{a\log(2) \min(\Omega_1)}.
\]

which implies that

\[
0 < (K + L) - K\xi \leq \frac{b\#\Omega}{a\log(2) \min(\Omega)}.
\]

**Proof.** We have

\[
\left( \prod_{n \in \Omega} n \right) = \left( \prod_{n \in \Omega} T(n) \right) = \left( \prod_{n \in \Omega_0} T(n) \right) \times \left( \prod_{n \in \Omega_1} T(n) \right).
\]

Dividing the last equality by the member in the left hand side, we get

\[
1 < \frac{2^{K+L}}{a^K} = \prod_{n \in \Omega_1} \left( 1 + \frac{b}{an} \right) \leq \left( 1 + \frac{b}{a\min(\Omega_1)} \right)^K.
\]

Applying the log and the fact that \( \log(1 + x) < x \) for \( x > 0 \). We obtain the required result.
The following theorem is an extension to a result obtained by Crandall [3].

**Theorem 3.2.** Let us set $c_0 = \frac{a \log(2)}{b}$ with $b \geq 1$. Then, for all $n \geq 1$,

$$K \geq \min \left( q_n, \frac{c_0 \min(\Omega_1)}{q_n + q_{n+1}} \right),$$

which implies that

$$\#\Omega \geq \min \left( q_n, \frac{c_0 \min(\Omega)}{q_n + q_{n+1}} \right),$$

**Proof.** Let $n > 1$, if $K > q_n$, then the required inequality is obvious. Assume that $K \leq q_n$, it follows from Lemma 3.1 that $|(K + L) - K\xi| > |p_n - q_n\xi|$. By Lemma 3.2, we get

$$|(K + L) - K\xi| > \frac{1}{q_n + q_{n+1}}.$$

And now, from Lemma 3.3 we get

$$\frac{bK}{a \log(2) \min(\Omega_1)} > \frac{1}{q_n + q_{n+1}}.$$

Which prove the required inequalities.

**Remark 3.1.**

1. We observe that for the classical case with $a = 3$ and $b = 1$, we have $c_0 \simeq 2.07944\ldots \geq 2$ which the coefficient appearing in [3].

2. As an application of the previous theorem. For $a = 3$ and $b = 1$ we know that the Collatz conjecture has been verified experimentally with a computer [8], until $n \leq N_0 = 5 \times 2^{60} \simeq 5.764.607.523.034.234.880$. With $q_{19} = 397.573.379$ and $q_{20} = 6189.245.291$, the cardinal of an hypothetical nontrivial cycle $\Omega$ must satisfy

$$\#\Omega \geq 3.63974 \times 10^8 \simeq 363.974.000.$$

Similar result was obtained by Eliahou in [4] by using Farey fractions and also continued fractions.

**Theorem 3.3.** Let us set $c_0 = \frac{a \log(2)}{b}$ with $b \geq 1$ and let $\Omega$ be a cycle of $T$, then there exists an integer $n_0 > 0$ such that for all $n \geq n_0$, we have

$$K \geq \min \left( q_n, \frac{c_0 \min(\Omega_1)}{q_n} \right),$$

which implies,

$$\#\Omega \geq \min \left( q_n, \frac{c_0 \min(\Omega)}{q_n} \right).$$
Proof. Let $p_n/q_n$ be the convergents to $\xi$. Then, the sequence $(q_n)$ is strictly increasing unbounded sequences. It follows that there exists an integer $n_0 > 0$ such that $q_{n_0} > \hat{q}_0$ and $\forall n > n_0$, we have $q_n > q_{n_0} > \hat{q}_0$. So, according to (5), we have $|p_n - q_n\xi| > 1/q_n^\mu$. If $K \geq q_n$, the required inequality is trivial. Then, suppose $K < q_n$, it follows from Lemma 3.1 that $|(K + L) - K\xi| \geq |p_n - q_n\xi|$. According to (5), we get

$$\frac{bK}{a \min(\Omega_1) \log(2)} > \frac{1}{q_n^\mu}.$$ 

Which gives the required result. \qed

4 Bounds for $m$-circuits

Following the terminology of Davidson, see [2, 5] a cycle \n
$$\Omega(x_0) = (x_0 \longrightarrow T(x_0) \longrightarrow T^{(2)}(x_0) \longrightarrow \ldots \longrightarrow T^{(k)}(x_0) = x_0),$$

of length $k$ is said to be a circuit if there is a value $i$ for which

$$x_0 > T(x_0) > \ldots > T^{(i)}(x_0)$$

and

$$T^{(i)}(x_0) < T^{(i+1)}(x_0) < \ldots < T^{(k)}(x_0) = x_0.$$ 

For the classical sequence of Collatz, Steiner [12] showed that the only cycle that is a circuit is the trivial cycle. Here, we have assumed that the sequence starts with an odd integer $x_0$, increases in $i + 1$ steps until an even number is encountered, then the sequence decreases in $k - i + 1$ steps until the odd number $x_0$ is again encountered. In such a situation, we will say that the cycle is of one oscillation. In [11], Simons and Weger derived lower and upper bounds for hypothetical cycle with $m$ oscillations. A cycle with $m$ oscillations contains $m$ local minimum odd integers $x_0, x_1, \ldots, x_{m-1}$ and $m$ local maximum even integers $y_0, y_1, \ldots, y_{m-1}$ such that the sequence starts with the odd integer $x_0$, increases in $k_0$ steps until the even number $y_0$ is encountered, then the sequence decreases in $\ell_0$ steps, until the odd integer $x_1$ is encountered, again increases in $k_1$ steps until the even number $y_1$ is encountered, and so on until the odd integer $x_{m-1}$ is encountered. Then, the sequence increases in $k_{m-1}$ steps until the even integer $y_{m-1}$ is encountered and finally the sequence increases in $\ell_m$ steps until the integer $x_0$ is again encountered. The number $K$ of odd integers and the number $L$ of even numbers in the cycle $\Omega$ are given by

$$K = \sum_{i=0}^{m-1} k_i, \quad \text{and} \quad L = \sum_{i=0}^{m-1} \ell_i.$$ 

The following Lemma, as we will see, is easy to obtain.
Lemma 4.1. For $i = 0, \ldots, m - 1$, we have
\[ y_i = T^{(k_i)}(x_i) = \left(\frac{a}{2}\right)^{k_i} x_i + \frac{b}{a - 2} \left(\frac{a}{2}\right)^{k_i} - 1 = 2^{\ell_i} x_{i+1}, \tag{6} \]
with the conditions $x_m = x_0$, $k_m = k_0$ and $\ell_m = \ell_0$.

Proof. For $i = 0, \ldots, m - 1$, we have $T^{(j)}(x_i)$ is odd for $j = 0, \ldots, k_i - 1$. Then
\[ T^{(j)}(x_i) = \left(\frac{a}{2}\right)^{j} x_i + \frac{b}{2} \sum_{\ell=0}^{j-1} \left(\frac{a}{2}\right)^{\ell}, \]
wish is also odd. Then,
\[ y_i = T^{(k_i)}(x_i) = \left(\frac{a}{2}\right)^{k_i} x_i + \frac{b}{2} \sum_{\ell=0}^{k_i-1} \left(\frac{a}{2}\right)^{\ell} + \frac{b}{2}. \]
It follows that
\[ y_i = T^{(k_i)}(x_i) = \left(\frac{a}{2}\right)^{k_i} x_i + \frac{b}{2} \sum_{\ell=0}^{k_i-1} \left(\frac{a}{2}\right)^{\ell} = \left(\frac{a}{2}\right)^{k_i} x_i + \frac{b}{2} \left(\frac{a}{2} - 1\right), \]
wish get the relation \(\Box\).

Lemma 4.2. Let $b \geq 1$. A necessary condition that a nontrivial cycle $\Omega$ having $m$ oscillations exists is
\[ 0 < (K + L) - K \frac{\log(a)}{\log(2)} < \frac{b}{(a - 2) \log(2)} \sum_{i=0}^{m-1} \frac{1}{x_i}, \tag{7} \]
which implies that
\[ 0 < (K + L) - K \frac{\log(a)}{\log(2)} < \frac{mb}{(a - 2) \log(2) \min(\Omega)}. \tag{8} \]

Proof. From the relation \(\Box\), we have
\[ \frac{2^{k_i+\ell_i}}{a^{k_i}} \frac{x_{i+1}}{x_i} = 1 + \frac{b}{(a - 2)x_i} \left(1 - \left(\frac{2}{a}\right)^{k_i}\right), \quad i = 0, \ldots, m - 1. \]
Taking the product from $i = 0$ to $i = m - 1$, and using the fact that $x_m = x_0$, we get
\[ 1 < \frac{2^{K + L}}{a^K} \prod_{i=0}^{m-1} \left(1 + \frac{b}{(a - 2)x_i} \left(1 - \left(\frac{2}{a}\right)^{k_i}\right)\right) < \prod_{i=0}^{m-1} \left(1 + \frac{b}{(a - 2)x_i}\right). \]
Now, applying the log and using the inequality $\log(1 + x) < x$ for $x > 0$ we obtain the relation \(\Box\) which implies obviously the relation \(\Box\).
Theorem 4.1. Let us set \( c_1 = \frac{(a - 2) \log(2)}{b} \) with \( b \geq 1 \). If a nontrivial cycle \( \Omega \) with \( m \) oscillations exists, then for all \( n \geq 1 \),

\[ m \geq \min \left( q_n, \frac{c_1 \min(\Omega)}{q_n + q_{n+1}} \right). \]

Proof. It is similar to the proof of Theorem 3.2 by using Lemma 4.2 \( \square \)

Theorem 4.2. Let us set \( c_1 = \frac{a \log(2)}{b} \) with \( b \geq 1 \). If a nontrivial cycle \( \Omega \) with \( m \) oscillations exists, then for all \( n \geq 1 \), there exists an integer \( n_0 > 0 \) such that for all \( n \geq n_0 \), we have

\[ m \geq \min \left( q_n, \frac{c_1 \min(\Omega)}{q_n} \right), \]

where \( \mu \) is a measure of irrationality of \( \frac{\log(a)}{\log(2)} \).

Proof. It is similar to the proof of Theorem 3.3 by using Lemma 4.2 \( \square \)

5 Particular Syracuse problems

5.1 The \((2^\nu + 1)n + (2^\nu - 1)\) problem

For the choice of the parameters \( a = 2^\nu + 1 \) and \( b = 2^\nu - 1 \) for \( \nu \geq 1 \). The corresponding map \( T \) is given by

\[ T(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{(2^\nu + 1)n + (2^\nu - 1)}{2} & \text{if } n \text{ is odd.} \end{cases} \]

We have \( a + b = 2^{\nu+1} \), then according to Theorem 2.1 the set \( C(2^\nu + 1, 2^\nu - 1) \) contains at least the trivial cycle \( \Omega(1) \) of length \( \nu + 1 \), where

\[ \Omega(1) = (1 \rightarrow 2^\nu \rightarrow 2^{\nu-1} \rightarrow \ldots \rightarrow 2 \rightarrow 1). \]

For \( \nu = 1 \) we recover the classical Syracuse problem with \( a = 3 \) and \( b = 1 \) and we have the classical Collatz conjecture, see Conjecture 2.2

Conjecture 5.1. Let \( \nu = 2 \) and let the parameters \( a = 2^\nu + 1 = 5 \) and \( b = 2^\nu - 1 = 3 \), then the set \( C(5, 3) \) contains exactly 7 cycles, we have

\[ C(5, 3) = \{ \Omega(1), \Omega(3), \Omega(39), \Omega(43), \Omega(51), \Omega(53), \Omega(61) \}, \]

and

\[ \mathbb{N} = G(1) \cup G(3) \cup G(39) \cup G(43) \cup G(51) \cup G(53) \cup G(61) \cup G(\infty), \]

with \( G(\infty) \neq \emptyset \). The trivial cycle of length 3 is \( \Omega(1) = (1 \rightarrow 4 \rightarrow 2 \rightarrow 1) \), the cycle \( \Omega(3) \) of length 5 is \( \Omega(3) = (3 \rightarrow 9 \rightarrow 24 \rightarrow 12 \rightarrow 6 \rightarrow 3) \) and the other 5 nontrivial cycles of length 7 are:
\[ \Omega(39) = (39 \rightarrow 99 \rightarrow 249 \rightarrow 624 \rightarrow 312 \rightarrow 156 \rightarrow 78 \rightarrow 39), \]
\[ \Omega(43) = (43 \rightarrow 109 \rightarrow 274 \rightarrow 137 \rightarrow 344 \rightarrow 172 \rightarrow 86 \rightarrow 43), \]
\[ \Omega(51) = (51 \rightarrow 129 \rightarrow 324 \rightarrow 162 \rightarrow 81 \rightarrow 204 \rightarrow 102 \rightarrow 51), \]
\[ \Omega(53) = (53 \rightarrow 134 \rightarrow 67 \rightarrow 169 \rightarrow 424 \rightarrow 212 \rightarrow 106 \rightarrow 53) \text{ and } \]
\[ \Omega(61) = (61 \rightarrow 154 \rightarrow 77 \rightarrow 194 \rightarrow 97 \rightarrow 244 \rightarrow 122 \rightarrow 61). \]

**Conjecture 5.2.** Let \( \nu \geq 3 \) and let the parameters \( a = 2^{\nu} + 1 \) and \( b = 2^{\nu} - 1 \), then the set \( C(a, b) \) contains exactly one cycle. We have \( C(a, b) = \{ \Omega(1) \} \) and
\[ N = G(1) \cup G(\infty), \]
with \( G(\infty) \neq \emptyset \) and the trivial cycle, of length \( \nu + 1 \), is
\[ \Omega(1) = (1 \rightarrow 2^{\nu} \rightarrow 2^{\nu-1} \rightarrow \ldots \rightarrow 2 \rightarrow 1). \]

As in the particular case, the problem in general case lies in the irregular behavior of the successive iterates. First, we recall the definition of the expansion factor \( s(n) \) given in [5] by:
\[ s(n) = \sup_{k \geq 0} \frac{T^{(k)}(n)}{2}, \]
if \( n \) has a bounded trajectory \( \Gamma(n) \) and \( s(n) = +\infty \) if \( \Gamma(n) \) is a divergent trajectory. The following result gives an idea on the computational difficulties encountered also in the general problem. It is similar to the one given in [3].

**Theorem 5.1.** For \( \nu \geq 1 \) and the parameters \( a = 2^{\nu} + 1 \) and \( b = 2^{\nu} - 1 \), the sequence \( (s(n))_{n \in \mathbb{N}} \) is unbounded.

**Proof.** For \( k \in \mathbb{N}, \) let \( n_k = 2^k - 1. \) We have immediately
\[ T(n_k) = (a n_k + b) / 2 = a 2^{k-1} - 1, \text{ for } k > 1. \]

And
\[ T^{(2)}(n_k) = a^2 2^{k-2} - 1, \text{ for } k > 2. \]

Thus by induction
\[ T^{(j)}(n_k) = a^j 2^{k-j} - 1, \text{ for } k > j. \]

Then, for \( k > 1, \) the number \( T^{(k-1)}(n_k) = a^{k-1} 2 - 1 \) belongs to the trajectory \( \Gamma(n_k). \) It follows that for \( k > 1: \)
\[ s(n_k) = \sup_{k \geq 0} \frac{T^{(k)}(n_k)}{n_k} \geq \frac{a^{k-1} 2 - 1}{2^k - 1} > \left( \frac{a}{2} \right)^{k-1}, \]
as \( a = 2^{\nu} + 1 > 2 \) for \( \nu \geq 1, \) the right-hand side goes to \( \infty \) as \( k \rightarrow \infty. \)

The following result follows the ideas given by Rozier in [10] for the classical Syracuse problem.
Theorem 5.2. Let $\nu \geq 1$ and let the parameters $a = 2^{\nu} + 1$ and $b = 2^{\nu} - 1$, a necessary condition that a nontrivial cycle $\Omega$ for $T$ having one oscillation exists is that there exists an integer $n_0 > 0$ such that for $K > n_0$, we have

$$2^K - 1 \leq \frac{K^\mu}{\log(2)}, \quad L = 1 - K + \left\lfloor \frac{\log(a)}{\log(2)} K \right\rfloor \quad \text{and} \quad \frac{a^K - 2^K}{2^{K+L} - a^K} \in \mathbb{N},$$

where $\mu$ is a measure of irrationality of $\frac{\log(a)}{\log(2)}$, $K$ and $L$ are the number of odd and even integers in the cycle $\Omega$, respectively.

Proof. Suppose that such a nontrivial cycle $\Omega$ exists with one oscillation $m = 1$. From the relation (9), we have

$$y_0 = T^K(x_0) = \left(\frac{a}{2}\right)^K x_0 + \frac{b}{a - 2} \left(\left(\frac{a}{2}\right)^K - 1\right) = 2^L x_0. \quad (9)$$

From the last relation and as $b = a - 2$, we obtain that

$$x_0 = \frac{a^K - 2^K}{2^{K+L} - a^K} \in \mathbb{N},$$

and

$$1 < \frac{2^{K+L}}{a^K} = 1 + \frac{1}{x_0} \left(1 - \frac{2^K}{a^K}\right) < 1 + \frac{1}{x_0}.$$

Applying the $\log$ to the last inequality and taking in account into the inequality $\log(1 + x) < x$ for $x > 0$, we get

$$0 < \frac{K + L}{K} - \frac{\log(a)}{\log(2)} K < \frac{1}{x_0 \log(2)} \frac{1}{K}, \quad (10)$$

which may be also written as

$$\frac{\log(a)}{\log(2)} K < K + L < \frac{\log(a)}{\log(2)} K + \frac{1}{x_0 \log(2)}.$$

From the last relation and as $\min(\Omega) > 2$, $\Omega$ is assumed to be nontrivial cycle) we get also that

$$K + L = 1 + \left\lfloor \frac{\log(a)}{\log(2)} K \right\rfloor.$$

The following relation, from the relation (9),

$$2^K (x_0 2^L + 1) = a^K (x_0 + 1).$$

shows that $2^K$ divide $x_0 + 1$, then

$$x_0 > 2^K - 1. \quad (11)$$
From the definition of the measure of rationality \( \mu \) of \( \frac{\log(a)}{\log(2)} \), there exists an integer \( n_0 > 0 \) such that for all pair \((p, q) \in \mathbb{Z} \times \mathbb{N} \) with \( q > n_0 \), we have
\[
\left| \frac{p}{q} - \frac{\log(a)}{\log(2)} \right| > \frac{1}{q^{\mu+1}}.
\]

Then for \( K > n_0 \), we have
\[
\frac{K + L}{K} - \frac{\log(a)}{\log(2)} > \frac{1}{K^{\mu+1}}.
\]

It follows from (10) that,
\[
\frac{1}{x_0 \log(2)} = \frac{1}{K} > \frac{K + L}{K} - \frac{\log(a)}{\log(2)} > \frac{1}{K^{\mu+1}}.
\]

which gives from (11) that
\[
\frac{K^\mu}{\log(2)} > 2^K - 1.
\]

This concludes the proof. \( \square \)

The following corollary gives the main result given in [10].

**Corollary 5.1.** For \( \nu = 1 \), we get the classical case of the Syracuse problem corresponding to \( a = 3 \) and \( b = 1 \). In this case, it follows that there is no cycle with only one oscillation other than the trivial cycle.

**Proof.** From the transcendental number theory, see [9, 14], we have
\[
\left| \frac{p}{q} \right| > \frac{1}{q^{\mu+1}}, \quad \forall (p, q) \in \mathbb{N}^2, \text{ with } q \geq 2.
\]

Then \( \mu + 1 = 15 \), which implies that \( \mu = 14 \). According to the previous Theorem, it follows that for the classical Syracuse problem, necessary condition is that for \( K \geq 2 \), we have
\[
2^K - 1 \leq \frac{K^{14}}{\log(2)}, \quad L = 1 - K + \left\lfloor \frac{\log(a)}{\log(2)} \right\rfloor \quad \text{and} \quad x_0 = \frac{3^K - 2^K}{2^K + L - 3^K} \in \mathbb{N},
\]

The condition \( 2^K - 1 \leq \frac{K^{14}}{\log(2)} \) holds if and only if \( K \leq 91 \) and there is no integer \( K \leq 91 \) for which \( x_0 = \frac{3^K - 2^K}{2^K + L - 3^K} \) is an integer for \( L = 1 - K + \left\lfloor \frac{\log(a)}{\log(2)} K \right\rfloor \). \( \square \)

The following Lemme is obviously obtained from the expression of the parameter \( A_\nu \) given in the Lemma.
Lemma 5.1. For all integer \( \nu \geq 1 \), we set \( A_\nu = \log \left( \frac{(2\nu + 1)\nu - 1}{2\nu - 1} \right) / \log(2) = \log \left( \frac{a^\nu - 1}{b} \right) / \log(2) \).
Then
- We have \( a^\nu - 1 = 2A_\nu b \).
- For \( \nu = 1 \), \( A_\nu = \nu \times (\nu - 1) + 1 = 1 \) and for \( \nu = 2 \), \( A_\nu = \nu \times (\nu - 1) + 1 = 3 \).
- For all integer \( \nu \geq 3 \), we have \( \lfloor A_\nu \rfloor = \nu(\nu - 1) \). More precisely, \( \nu(\nu - 1) < A_\nu < \nu(\nu - 1) + 1 \).
- As \( \nu \to \infty \), we have \( A_\nu \sim \nu(\nu - 1) \).

Theorem 5.3. The cycle \( \Omega(b) \) of length \( \nu + A_\nu \) where
\[
\Omega(b) = (b \to a2^{\nu-1} - 1 \to a^22^{\nu-2} - 1 \to \ldots \to a^\nu - 1 = 2A_\nu b \to 2A_{\nu-1}b \to 2A_{\nu-2}b \to \ldots \to 2b \to b),
\]
exists only for the both cases where \( A_\nu \) is an integer, namely for \( \nu = 1 \) or \( \nu = 2 \).

Proof. We have \( T^{(\nu)}(b) = a^\nu - 1 \). According to the previous Lemma \( a^\nu - 1 = 2A_\nu b \), which gives in the case where \( A_\nu \) is an integer the relation \( T^{(\nu + A_\nu)}(b) = b \). But \( A_\nu \) is an integer if and only if \( \nu = 1 \) or \( \nu = 2 \). For \( \nu = 1 \), we have \( b = 2^\nu - 1 = 1 \) and \( \Omega(b) = \Omega(1) \) is the trivial cycle. For \( \nu = 2 \), we have \( b = 3 \), \( A_\nu = 3 \) and the cycle \( \Omega(b) \) is of length \( \nu + A_\nu = 2 + 3 = 5 \), is exactly the one given in the theorem, we have
\[
\Omega(b) = (3 \to a2^{\nu-1} - 1 = 9 \to a^2 - 1 = 24 = 2^a b = 2^3 \times 3 \\
\to 2A_{\nu-1}b = 2^2 \times 3 \to 2b = 2 \times 3 \to 3),
\]
\[
\Omega(b) = (3 \to 9 \to 24 \to 12 \to 6 \to 3).
\]

\[ \Box \]

5.2 The \((2^\nu - 1)n + (2^\nu + 1)\) problem

For the choice of the parameters \( a = 2^\nu - 1 \) and \( b = 2^\nu - 1 \) for \( \nu \geq 1 \). The corresponding map \( T \) is given by
\[
T(n) = \begin{cases} 
 n/2 & \text{if } n \text{ is even,} \\
 (2^\nu - 1)n + (2^\nu + 1))/2 & \text{if } n \text{ is odd.}
\end{cases}
\]
We have \( a + b = 2^{\nu+1} \), then according to Theorem 2.1 the set \( C(2^\nu - 1, 2^\nu - 1) \) contains at least two trivial cycles. The first one \( \Omega(1) \) of length \( \nu + 1 \), where
\[
\Omega(1) = (1 \to 2^\nu \to 2^{\nu-1} \to 2^{\nu-2} \to \ldots \to 2 \to 1),
\]
and the second one \( \Omega(b) \) of length \( \nu \), where
\[
\Omega(b) = (b \to 2^{\nu-1}b \to 2^{\nu-2}b \to \ldots \to 2b \to b).
\]
Conjecture 5.3. For $\nu = 2$, $a = 2^\nu - 1 = 3$ and $b = 2^\nu + 1 = 5$, the set $C(a, b)$ contains exactly 6 cycles $C(a, b) = \{\Omega(1), \Omega(5), \Omega(19), \Omega(23), \Omega(187), \Omega(347)\}$, and

$$N = G(1) \cup G(5) \cup G(19) \cup G(23) \cup G(187) \cup G(347).$$

We have $G(\infty) = \emptyset$ with the cycles of length 3, 2, 5, 27 and 27, respectively, are

- $\Omega(1) = (1 \rightarrow 4 \rightarrow 2 \rightarrow 1)$,
- $\Omega(5) = (5 \rightarrow 10 \rightarrow 5)$,
- $\Omega(19) = (19 \rightarrow 31 \rightarrow 49 \rightarrow 76 \rightarrow 38 \rightarrow 19)$,
- $\Omega(23) = (23 \rightarrow 37 \rightarrow 58 \rightarrow 29 \rightarrow 46 \rightarrow 23)$,
- $\Omega(187) = (187 \rightarrow 283 \rightarrow 427 \rightarrow 643 \rightarrow 967 \rightarrow 1453 \rightarrow 2182 \rightarrow 1091 \rightarrow 1639 \rightarrow 2461 \rightarrow 3694 \rightarrow 1847 \rightarrow 3124 \rightarrow 1562 \rightarrow 781 \rightarrow 1174 \rightarrow 883 \rightarrow 1327 \rightarrow 2992 \rightarrow 1496 \rightarrow 748 \rightarrow 374 \rightarrow 187)$,
- $\Omega(347) = (347 \rightarrow 523 \rightarrow 787 \rightarrow 1183 \rightarrow 1777 \rightarrow 2668 \rightarrow 1334 \rightarrow 667 \rightarrow 1003 \rightarrow 1507 \rightarrow 2263 \rightarrow 3397 \rightarrow 5098 \rightarrow 3826 \rightarrow 1913 \rightarrow 2872 \rightarrow 1436 \rightarrow 718 \rightarrow 359 \rightarrow 541 \rightarrow 814 \rightarrow 407 \rightarrow 613 \rightarrow 922 \rightarrow 461 \rightarrow 694 \rightarrow 347).$

Conjecture 5.4. For $\nu \geq 1$ with $\nu \neq 2$, $a = 2^\nu - 1$ and $b = 2^\nu + 1$, the set $C(a, b)$ contains exactly two cycles. We have $C(a, b) = \{\Omega(1), \Omega(b)\}$, and $N = G(1) \cup G(b) \cup G(\infty)$, with $G(\infty) = \emptyset$ for $\nu = 1$ and $G(\infty) \neq \emptyset$ for $\nu \geq 3$. The trivial cycle of length $\nu + 1$ is

$$\Omega(1) = (1 \rightarrow 2^\nu \rightarrow 2^{\nu-1} \rightarrow 2^{\nu-2} \rightarrow \ldots \rightarrow 2 \rightarrow 1),$$

and the second cycle of length $\nu$ is

$$\Omega(b) = (b \rightarrow 2^{\nu-1}b \rightarrow 2^{\nu-2}b \rightarrow \ldots \rightarrow 2b \rightarrow b).$$

5.3 The $(2^\nu + 1)n - (2^\nu - 1)$ problem

For the choice of the parameters $a = 2^\nu + 1$ and $b = -2^\nu + 1 < 0$ for $\nu \geq 1$. The corresponding map $T$ is given by

$$T(n) = \begin{cases} 
n/2 & \text{if } n \text{ is even,} \\
((2^\nu + 1)n - (2^\nu - 1))/2 & \text{if } n \text{ is odd.}
\end{cases}$$

We have $a + b = 2$, then according to Theorem 2.1, the set $C(2^\nu + 1, -2^\nu + 1)$ contains at least two trivial cycles. The first one $\Omega(1)$ of length 1, where

$$\Omega(1) = (1 \rightarrow 1),$$

and the second one $\Omega(-b)$ of length $\nu$, where

$$\Omega(-b) = (-b \rightarrow -b2^{\nu-1} \rightarrow -b2^{\nu-2} \rightarrow \ldots \rightarrow -2b \rightarrow -b).$$
Conjecture 5.5. Let $\nu = 1$ and let the parameters $a = 2^\nu + 1 = 3$ and $b = -2^\nu + 1 = -1$, then the set $C(a, b)$ contains exactly three cycles $C(a, b) = \{\Omega(1), \Omega(5), \Omega(17)\}$, and

$$\mathbb{N} = G(1) \cup G(5) \cup G(17).$$

We have $G(\infty) = \emptyset$ with the cycles of length 1, 3 and 11, respectively, are

$$\begin{align*}
\Omega(1) &= (1 \rightarrow 1), \\
\Omega(5) &= (5 \rightarrow 7 \rightarrow 10 \rightarrow 5), \\
\Omega(17) &= (17 \rightarrow 25 \rightarrow 37 \rightarrow 55 \rightarrow 82 \rightarrow 41 \rightarrow 91 \rightarrow 136 \rightarrow 68 \rightarrow 34 \rightarrow 17).
\end{align*}$$

Conjecture 5.6. Let $\nu \geq 2$ and let the parameters $a = 2^\nu + 1$ and $b = -2^\nu + 1$, then the set $C(a, b)$ contains exactly two cycles. We have

$$C(a, b) = \{\Omega(1), \Omega(-b)\}, \quad \text{and} \quad \mathbb{N} = G(1) \cup G(-b) \cup G(\infty),$$

with $G(\infty) \neq \emptyset$. The trivial cycle of length 1 is $\Omega(1) = (1 \rightarrow 1)$ and the second cycle of length $\nu$ is

$$\Omega(-b) = (-b \rightarrow -b2^{\nu-1} \rightarrow -b2^{\nu-2} \rightarrow \ldots \rightarrow -2b \rightarrow -b).$$

5.4 The $(2^\nu - 1)n + 1$ problem

For the choice of the parameters $a = 2^\nu - 1$ and $b = 1$ for $\nu \geq 2$. The corresponding map $T$ is given by

$$T(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even,} \\
\frac{(2^\nu - 1)n + 1}{2} & \text{if } n \text{ is odd.}
\end{cases}$$

We have $a + b = 2^\nu$, then according to Theorem 2.1, the set $C(2^\nu - 1, 1)$ contains at least the trivial cycle $\Omega(1)$, of length $\nu$, where

$$\Omega(1) = (1 \rightarrow 2^{\nu-1} \rightarrow 2^{\nu-2} \rightarrow \ldots \rightarrow 2 \rightarrow 1).$$

For $\nu = 2$ we recover the classical Syracuse problem with $a = 3$ and $b = 1$ and we have the classical Collatz conjecture, see Conjecture 2.2.

Conjecture 5.7. Let $\nu \geq 3$ and let the parameters $a = 2^\nu - 1$ and $b = 1$, then the set $C(a, b)$ contains exactly one cycle, $C(a, b) = \{\Omega(1)\}$. We have

$$\mathbb{N} = G(1) \cup G(\infty),$$

with $G(\infty) \neq \emptyset$ where the trivial cycle $\Omega(1)$, of length $\nu$ is

$$\Omega(1) = (1 \rightarrow 2^{\nu-1} \rightarrow 2^{\nu-2} \rightarrow \ldots \rightarrow 2 \rightarrow 1).$$
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