On Mach’s principle: Inertia as gravitation

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Abstract

In order to test the validity of Mach’s principle, we calculate the action of the entire universe on a test mass in its rest frame, which is an acceleration $\mathbf{g}^*$. We show the dependence of the inertia principle on the lapse and the shift. Using the formalism of linearized gravitation, we obtain the non-relativistic limit of $\mathbf{g}^*$ in terms of two integrals. We follow then two approaches. In the first one, these integrals are calculated in the actual time section $t = t_0$ up to the distance $R_U = ct_0$. In the more exact and satisfactory second approach, they are calculated over the past light cone using the formalism of the retarded potentials. The aim is to find whether the acceleration $\dot{\mathbf{v}}$ in the LHS of Newton’s second law can be interpreted as a reactive acceleration, in other words, as minus the acceleration of gravity $\mathbf{g}^*$ in the rest frame of the accelerated particle (i.e. to know whether or not $\mathbf{g}^* = -\dot{\mathbf{v}}$). The results strongly support Mach’s idea since the reactive acceleration for $\Omega_\Lambda = 0.7$ turns out to be about $\mathbf{g}^* = -1.1\dot{\mathbf{v}}$, in the first approach, and about $\mathbf{g}^* = -0.7\dot{\mathbf{v}}$, in the second. These results depend little on $\Omega_\Lambda$ if $\Omega_\Lambda < 0.9$. Even considering the approximations and idealizations made during the calculations, we deem these results as interesting and encouraging.

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1 Introduction

1.1 Mach’s critique of Newton’s laws

Elaborating further the ideas put forth by the physicist Christiaan Huygens and the philosophers Gottfried Leibniz and Bishop Berkeley, Mach proposed a radical criticism of Newton’s absolute space, more than thirty years before Einstein’s first paper on relativity [1, 2]. He said in 1872 “For me only relative motion exists ... When a body rotates relative to the fixed stars, centrifugal forces are produced; when it rotates [but] not relative to the fixed stars, no centrifugal forces are produced ... Obviously it does not matter if we think of the earth as turning round on its axis, or at rest while the fixed stars revolve round it”. From this premise, he concluded that the inertia is not a primary property of the bodies, as assumed by Newton in his Principia, but, quite on the contrary, an effect of its motion with respect to the fixed stars (or the distant galaxies, in current parlance). In other words, inertia would be an interaction that requires other bodies to manifest itself, so that it would have no sense in a universe consisting of only one mass.
Mach’s criticism had an influence on Einstein, who coined the expression “Mach principle” to denote the idea that inertia is an interaction with all the mass in the universe, to be expressed eventually by mathematical relations still to be discovered. Indeed Einstein himself acknowledged Mach’s influence in several occasions. In his student days, he had read with attention his main work [2], “a book which, with its critical attitudes toward basic concepts and basic laws, made a deep and lasting impression on me” [3]. When Mach died in 1916, he wrote in an obituary “... It is not improbable that Mach would have discovered the theory of relativity, if, at the time when his mind was still young and susceptible, the problem of the constancy of the speed of light had been discussed among physicists” [4] and in a letter to A. Weiner in 1930, “it is justified to consider Mach as the precursor of the general theory of relativity” [5]. Later, however, Einstein’s interest in Mach’s work waned, when he came to think that “Mach was a good scientist but a wretched philosopher”, a comment certainly due to his negative attitude towards the atoms. Nevertheless as Pais says, “After Einstein, the Mach principle faded but never died”.

Indeed, it is usually accepted that there must be something important in Mach’s principle, in particular that it has some relation with general relativity although in a subtle and not yet understood way. Indeed it is very vague: although Mach was probably thinking in gravitation, he suggested no explicit mechanism that could transmit any interaction from a fixed star to a mass. Nobody was able later to formulate the idea in a working way, so that, in Berry’s words, “the principle is half-baked” [6]. Sciama wrote an important and intriguing paper in 1953, suggesting that inertia and gravitation are the same phenomenon [7]-[10]. His proposal was certainly appealing, if only qualitative. The aim of this work is to find a way to understand Mach principle, baking it further in the frame of linearized general relativity.

2 Inertia

According to Mach, this seems clear, the inertia of a test mass $m$ with velocity $\mathbf{v}$ and acceleration $\mathbf{a} = \ddot{\mathbf{v}}$ (overdot means time derivative) adopts the form of a reactive force $\mathbf{F}_I = -ma$ due to the fixed stars. Today we must speak instead not only of the distant galaxies, but of all the mass-energy in the entire visible universe. Since he insisted that only the relative situation with respect to other bodies could be a cause of the forces, the principle can be
given an operative status as follows. The reactive force $F_1$ can be calculated in the rest frame of the test mass as the gravitational pull of all the galaxies moving with acceleration $-\dot{v}$ and velocity $-v$. If this pull were shown to produce the reactive acceleration

$$g^* = -\dot{v},$$

in the limit $v \ll c$, the principle would be fully baked, since it would have an operative character (we will write $g^* = -\xi \dot{v}$ in the following). This work proposes a way to do that in the frame of linearized general relativity.

This reactive acceleration describes probably one of the most fundamental properties of the universe. The spacetime being a dynamical structure, it seems logic from the assumptions of Newtonian physics to consider inertia as a property of the matter itself. Nevertheless, on the grounds of General Relativity the problem adopts a new feature as long as spacetime play the role of a natural ingredient of that dynamics. Even more, the absence of an explanation of inertia can be considered as a test for the completeness of a dynamical theory.

In a general way, the motion of a particle depends entirely on geometry through the equations of geodesics.

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0,$$

where $\tau$ is the proper time and $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols, which in the weak field approximation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ are given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} \eta^{\alpha\gamma} (h_{\mu\gamma,\nu} + h_{\nu\gamma,\mu} - h_{\mu\nu,\gamma}).$$

(3)

In the Newtonian limit, the terms with $\mu = \nu = 0$ give the dominant part of the second term in the LHS of [2], which can be approximated as $\ddot{x}^i = -\Gamma^i_{00} \dot{x}^0 \dot{x}^0$, so that the acceleration of the test mass is

$$g^* = -\Gamma^i_{00} c^2.$$  

(4)

In the previous equations and in the rest of this work, the Newtonian three-velocities and three-accelerations, such as $v_i, g^*_j, \ldots$ with Latin subindexes are the non-relativistic limit of the relativistic quantities with the same superindexes and that $\beta_k = v_k/c$. The corresponding Christoffel symbols and acceleration of the point mass $g^*_i$ are therefore

$$\Gamma^i_{00} = -\frac{1}{2} (2h_{0i,0} - h_{00,i}), \quad g^*_i = c^2 (h_{0i,0} - \frac{1}{2} h_{00,i}),$$

(5)
in the Newtonian approximation where $\gamma = 1$. Strictly speaking, formula (5) is a prediction for the value of the reactive acceleration of a particle in geodesic motion, the inertia in other words.

Even being an unrealistic point of view, the most obvious approach to get a first insight is to consider the case of pure gravity or, equivalently, to investigate what geometry itself could suggest us on the origin of inertia. At this point, a better understanding is reached by using the formalism of Hamiltonian gravity. For this purpose, we pass from the usual four-dimensional description to the classical ADM(3+1) splitting of spacetime, with dynamical variables $N$, lapse, $N_i$, shift, and $q_{ij}$, three dimensional metric tensor\cite{11, 12}. Therefore, the reactive acceleration $g^*$ becomes a function of $N$ and $N_i$.

Dropping out total divergences and time derivatives, the (3+1) Einstein-Hilbert Lagrangian becomes

$$L = \int N \sqrt{q} (k_{ij}k^{ij} - k^2 + R^{(3)}) \, d^3x.$$ \hfill (6)

Here, the latin indexes run over the three dimensional space, $R^{(3)}$ is the three-dimensional scalar curvature and

$$k_{ij} = \frac{1}{2N} \left( D_i N_j - \frac{\partial q_{ij}}{\partial t} \right),$$

stands for the extrinsic curvature. There are no time derivatives of lapse and shift in the Lagrangian density, so that their canonical momenta vanish supplying two primary constraints. After a brief calculation, we get the corresponding Hamiltonian in the form

$$H = \int d^3x \left( NS + N_i V^i \right),$$

where $S$ and $V$ are the scalar and vector constraints, respectively, depending on the metric $q_{ij}$ and its canonical conjugate momentum $\Pi^{ij}$. The stability of the primary constraints leads us to the secondary ones $S = 0$ and $V^i = 0$. It is easy to verify that all of them are first class (symmetries), in such a way that the values of $\dot{N}$ and $\dot{N}_i$, and consequently of $g^*_i$, can be chosen arbitrarily (gauge fixing, remind that we are in pure gravity).

Taking, however, into account the commonly accepted universal validity of the inertia principle, one is naturally lead to assume the existence of a
cosmological universal source able to fix, universally as well, the value of $N$ and $N_i$. These arguments indicate that something more than geometry is needed, a point of view very close to Mach’s ideas, in which inertia plays more the role of a footprint of the contents of the universe rather than a test of General Relativity.

The next step for an understanding of inertia is, obviously, to add particles to the geometry. One can verify, using Hamiltonian methods, that this addition suffices to fix the values of $N$ and $N_i$. In particular, the standard model of the universe with dust, radiation and dark energy acquires therefore naturally inertia as a property.

3 The cosmological model and the calculation method

This paper accepts the standard cosmological model with flat space sections, i.e., with critical density $\rho_{\text{cr}}(t)$ and $k = 0$ in Friedmann’s equation. It assumes also that the universe consists of matter and dark energy, the latter being equivalent to a positive cosmological constant $\Lambda$ and the former, either ordinary or dark, with zero pressure (it can be treated as dust). The present densities of matter and dark energy are, respectively, $\rho_M = \Omega_M \rho_{\text{cr}}$ and $\rho_\Lambda = \Lambda / 8\pi G = \rho_{\text{cr}} \Omega_\Lambda$, with $\Omega_M + \Omega_\Lambda = 1$.

To show the validity of Mach’s idea that inertia is not an intrinsic property of particles and bodies but an interaction with the entire universe and, furthermore, that this interaction is just gravity, we follow several steps. (i) We take a point test mass moving with velocity $v$ and acceleration $\dot{v}$. (ii) We compute the force of a particular galaxy on this particle in the particle rest frame, in which the galaxy has opposite velocity $-v$ and acceleration $-\dot{v}$. This is done by solving the particle equation of motion in the gravitational field of the galaxy, using the formalism of linearized general relativity and taking the Newtonian limit with small velocity $v \ll c$. (iii) The sum of the forces of all the galaxies (plus the intergalactic mass-energy) is approximated by the integral over the universe of the effect of a uniform distribution of matter and energy with the critical density $\rho = \rho_{\text{cr}}$. This is possible because nearby bodies as the Milky Way have a negligible effect as compared with the distant galaxies.

Note that the linear approximation is used here to calculate the gravita-
tional field of a galaxy at a point, a standard problem in celestial mechanics in order to determine the reactive force. This, of course, does not imply that the metric of the standard model can be approximated as a Minkowskian one perturbed with a term proportional to \( G \).

The calculations are made for several values of \( \Omega_\Lambda \) following two approaches. In the first, the necessary integrals are evaluated in the actual time section \( t = t_0 \) up to the distance \( ct_0 \), with constant critical density \( \rho_{cr} \) and \( \Omega_\Lambda \); in other words, neglecting the time retard. The result will be that the reactive acceleration is in the interval \((-1.1\dot{v}, -1.2\dot{v})\) if \( 0 \leq \Omega_\Lambda \leq 0.9 \).

The second approach is more exact and satisfactory since the integrals are calculated over our (onion shaped) past light cone, taking into account the time retard, for which a singularity has to be eliminated. We will obtain thus reactive accelerations in the interval \((-0.7\dot{v}, -0.6\dot{v})\) for \( 0 \leq \Omega_\Lambda \leq 0.9 \). Being close to \( (1) \), these are certainly remarkable results.

In fact as we have pointed out, the existence of inertia means, as a first requisite, the need to guess the correct dynamics able to fix the value of \( N \) and \( N_i \). Even more, the reactive acceleration must acquire, at least in the dominant terms, an expression of the form \(-\xi \dot{v}\), \( \xi \) being a constant with a value near to one. None of these conditions is, by any means, obvious and, curiously, a universe with particles has precisely these properties.

4 Linearized general relativity

4.1 Field equations in linearized general relativity

In this work, we will use the linearized equations of general relativity \([13, 14]\). Assuming the linear approximation to gravity with \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \eta_{\mu\nu} \equiv \text{diag}(1, -1, -1, -1), |h_{\mu\nu}| \ll 1 \), Einstein’s equations take the approximate form

\[
\Box h_{\mu\nu} = \frac{16\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T \right),
\]

\( \partial_\mu (h_{\nu}^{\ \mu} - \frac{1}{2} \delta_\nu^{\mu} h) = 0, \tag{7} \)

where \( h = \eta^{\alpha\beta} h_{\alpha\beta}, T_{\mu\nu} \) is the energy-momentum tensor and \( T \) its trace.
The solution of (7) can be found by means of the retarded Green function as
\[
h_{\mu\nu}(x) = \frac{16\pi G}{c^4} \int D_r(x - x') \left( T_{\mu\nu}(x') - \frac{1}{2} \eta_{\mu\nu} T(x') \right) \, d^4x'
\] (8)

where
\[
D_r(x - x') = \frac{1}{2\pi} \theta(x_0 - x'_0) \delta[(x - x')^2],
\] (9)

\(\theta\) being the step or Heaviside function (see, for instance, [15]).

In the cases of a perfect fluid without pressure (as in a dust universe) and of a point mass \(M\), the energy-momentum tensor takes the forms, respectively,
\[
T_{\mu\nu} = \rho u_\mu u_\nu, \quad T_{\mu\nu}(x) = Mc \int d\tau \, u_\mu(\tau) u_\nu(\tau) \delta^{(4)}[x - r(\tau)],
\] (10)

where \(\rho\) is the mass density, \(u_\mu = dx_\mu/d\tau\) the four-velocity field, \(\tau\) the proper time and \(r(\tau)\) the four-trajectory of the mass.

4.2 The effect of a galaxy

Equation (7) for the gravitational field created by a point galaxy of mass \(M\) at the position of the test particle in its rest frame can be solved with well-known standard methods [15]. The galaxy has three-velocity \(\mathbf{v}\) and three-acceleration \(\dot{\mathbf{v}}\) and its energy-momentum tensor \(T^{\mu\nu}\) is given by the second equation (10). We assume that the space-time positions of the galaxy and the test particle are \(r(\tau)\) and \(x\), \(\tau\) being the proper time of the galaxy, so that the four-vector from the galaxy to the particle is \(x - r(\tau)\). Note that \(dR/d\tau = -\mathbf{v}\) and \(dR/d\tau = -\mathbf{n} \cdot \mathbf{v}\).

The solution of (7) is
\[
h_{\mu\nu}(x) = \frac{-4GM}{c^4} \frac{c \left[ u_\mu(\tau) u_\nu(\tau) - \eta_{\mu\nu} c^2 / 2 \right]}{u \cdot [x - r(\tau)]} \bigg|_{\text{ret}},
\] (11)

where the subindex means that the RHS must be evaluated at the retarded time \(\tau'\). A comment is in order here. Eq. (11) does not verify “exactly” the second eq. (7); in fact this divergence turns out to be proportional to the product of the gravitational constant \(G\) times the four-acceleration of the galaxy at the retarded point. Now, the linear approximation that we are carrying out is the first order part of a formal expansion in series of
powers of $G$, which is tantamount to say in powers of non-dimensional parameters depending on the masses of the problem. Consequently, both $h_{\mu\nu}$ and the four-acceleration are proportional to $G$. In other words, this method is correct: the first equation (7) shows that the acceleration is of order $G$, so that the second is verified at linear order.

5 First approach to the reactive acceleration

Once known the expressions of $h_{\alpha\beta}$ from (11) and taking into account (2)-(4), we recover the expression (3) to calculate $g^*$ from (11). Following then a standard method, it is easy to find in the limit of small velocity $\beta \ll 1$, that

$$h_{0i,0} = (4GM/c^4) \dot{v}/R, \quad h_{00,i} = (2GM/c^3)[(\dot{\beta} \cdot \mathbf{n})n_i/R + n_i c/R^2],$$  

(12)

where $\mathbf{R}$ is the three-vector from the test mass to the galaxy, $\dot{v}$ is the galaxy acceleration and $\mathbf{n} = \mathbf{R}/R$. All the quantities are taken at the retarded time.

In this first approach we start with a dust universe of constant density $\rho_{cr}(t_0)$ consisting in all the galaxies plus the intergalactic matter, which produce a gravitational field like (12). The acceleration of the test mass is then obtained by integration in the time section until the distance $ct_0$.

To obtain the acceleration of the test mass, we insert (12) into (5), change the sign of $\dot{v}$ and integrate over the time section $t = t_0$ until the radius $R_U = ct_0$ with the critical density $\rho_{cr} = 3H_0^2/8\pi G$. This leads to

$$g^* = -\frac{16\pi G}{c^2} \dot{v} \int_0^{R_U} \rho_{cr} RdR + \frac{4\pi G}{3c^2} \dot{v} \int_0^{R_U} \rho_{cr} RdR,$$

(13)

where the two terms are the contribution to the reactive acceleration of the terms in $h_{0i,0}$ and $h_{00,i}$, respectively. Since we take constant $\rho_{cr}$, the integrals are immediate. It is easy to see that the effect of terms containing $h_{ij}$ or $h_{ii}$ can be neglected in the limit $\beta \rightarrow 0$.

Three points must be underscored: i) we are accepting the approximation that all the galaxies have the same acceleration; ii) the terms proportional to $v$ are not considered in (12) because we are taking the limit $\beta \ll 1$; and iii) it is easy to see that the terms in $h_{ij}$ or $h_{ii}$ can be neglected in the same limit.
The dust universe. If \((\Omega_M, \Omega_\Lambda) = (1, 0)\), it happens that \(H_0 t_0 = 2/3\), the total reactive acceleration on a test mass being therefore

\[
g^* = -\frac{11}{4} (H_0 t_0)^2 \dot{v} = -\frac{11}{9} \dot{v} = -1.22 \dot{v} .
\]

(14)

Being close to the correct value \(-\dot{v}\), this result is remarkable. However, we have not yet considered the dark energy.

5.1 Introducing the dark energy

In the case of a perfect fluid with mass density \(\rho\) and pressure \(p\), the expression of the energy-momentum tensor is

\[
T_{\mu\nu} = (\rho + p/c^2) u_\mu u_\nu - \eta_{\mu\nu} p .
\]

(15)

As is well known, both the dark energy and the cosmological constant have negative pressure. In fact \(\rho = \rho_{cr}(t_0) = \rho_M + \rho_\Lambda\) is the addition of the mass densities of matter and dark energy, and \(p = p_\Lambda = -\rho_\Lambda c^2\) is the pressure of the dark energy. Since \(\rho_\Lambda = \Omega_\Lambda \rho_{cr}\), \(T_{\mu\nu}\) and its trace can be written

\[
T_{\mu\nu} = \rho_{cr}(1-\Omega_\Lambda) u_\mu u_\nu + \rho_{cr} c^2 \Omega_\Lambda \eta_{\mu\nu} , \; \; T = \rho_{cr} u_\mu u^\mu + pu^\mu u^\nu/c^2 - 4p = \rho_{cr} c^2 - 3p .
\]

(16)

The standard model. The source of \(h_{\mu\nu}\) in (7) is \(T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T\), so that those of \(h_{00}\) and \(h_{0i}\) are, respectively,

\[
\rho_{cr}(u_0)^2 (1-\Omega_\Lambda) - \frac{1}{2} \rho_{cr} c^2 (1 + \Omega_\Lambda) \; \; \text{and} \; \; \rho_{cr} (1-\Omega_\Lambda) u_0 u_i ,
\]

(17)

which in the Newtonian limit take the form

\[
\frac{\rho_{cr} c^2}{2} (1 - 3\Omega_\Lambda) , \; \; \text{and} \; \; \rho_{cr} c v_i (1 - \Omega_\Lambda) ,
\]

(18)

where \(\rho_{cr} = 3H_0^2/8\pi G\); one term is multiplied by \((1 - 3\Omega_\Lambda)\), the other by \((1 - \Omega_\Lambda)\). Note that this neglects terms in derivatives of \(\Omega_\Lambda\), but this is acceptable since they contain the factor \(H_0 v\) which is always very small and, moreover, we take the limit \(v/c \to 0\).

To obtain the result for the standard model of the universe with \(\Omega_\Lambda \neq 0\), one must use instead of (13) the expression

\[
g^* = -\frac{16\pi G}{c^2} \dot{v} (1-\Omega_\Lambda) \int_0^{R_U} \rho_{cr} R \, dR + \frac{4\pi G}{3c^2} \dot{v} (1-3\Omega_\Lambda) \int_0^{R_U} \rho_{cr} R \, dR ,
\]

(19)
from which it follows easily that the prediction for the reactive acceleration of the test mass due to the rest of the universe is

$$g^* = \frac{-11 + 9\Omega \Lambda}{4} (H_0 t_0)^2 \dot{v}. \quad (20)$$

For $\Omega \Lambda = 0.7$, this is $-1.09 \dot{v}$. It is a simple matter to show that the reactive acceleration is between about $-1.2 \dot{v}$ and $-1.1 \dot{v}$, according to eq. (20), for all the values of $\Omega \Lambda$ in the interval $(0, 0.9)$. This is very close to eq. (1). A warning however is necessary. The reactive acceleration depends here on what we consider to be the radius of the universe. Nevertheless, even taking into account the simplifications in the physical model, this is intriguing (see figure 2).

6 Second approach to the reactive acceleration

In this second approach, the integrals that give $h_{00}$ and $h_{0i}$ are calculated along the past of our light cone, using therefore the retarded time. This solves the problem just mentioned of the definition of the radius of the universe, although at the price of a slight complication of the analysis. The equation of a light ray moving radially to or from Earth in the Robertson-Walker metric with $k = 0$ (with constant $\theta$ and $\varphi$ and $ds = 0$) is equal to

$$c \, dt = \mp \frac{S(t) \, dr}{(1 - kr^2)^{1/2}} \quad (21)$$

where $S$ is the scale factor.

The double sign in (21) is minus or plus according to whether the photon approaches to or recedes from Earth, respectively. Let us take $k = 0$. Denoting $f(t) = \int dt/S(t)$ and applying this equation to an incoming photon that passes the radial coordinate $r$ at $t$ and reaches the Earth at $r = 0$ and $t = t_0$, we have

$$r = c \int_t^{t_0} \frac{dt}{S} = c[f(t_0) - f(t)]. \quad (22)$$

In the Einstein-de Sitter model (with $k = 0$, $\Omega_M = 1$, $S = (t/t_0)^{2/3}$, $t_0 = 1/(6\pi G \rho_0)^{1/2}$ and $\rho(t) = 1/6\pi G t^2$), eq. (22) gives $f(t) = 3t_0^{2/3} t^{1/3}$ and
\[ r/c = f(t_0) - f(t) = 3t_0(1 - \tau^{1/3}) \] with \( \tau = t/t_0 \). The proper distance of the photon to us is therefore

\[ R = rS = 3ct_0(\tau^{2/3} - \tau). \] (23)

This is the equation of the light cone at the cosmological level, represented in figure 1 as the solid line [17]. The line gives the time as a function of the proper distance to the test mass along a light ray with trajectory (23). As we see, it has not shape of cone but of onion. It could be called therefore “the light onion”. Light from a galaxy reaching us at present time is emitted

![Figure 1: Light cones in the universe for the cosmological models with k = 0 and \( \Omega_\Lambda = 0 \) (solid line), \( \Omega_\Lambda = 0.7 \) (long dashes) and \( \Omega_\Lambda = 0.9 \) (short dashes). Each curve refers to the particular value of \( t_0 \) corresponding to its \( \Omega_\Lambda \). The maximum value of the distance for \( \Omega_\Lambda = 0 \) is \( R_{\text{max}} = 4ct_0/9 \) at time \( 8t_0/27 \) (explanation in the text.)

at the point of intersection of our past light-onion with the world line of the galaxy. The maximum proper distance to the galaxy is \( R_{\text{max}} = 4ct_0/9 \) at \( \tau = 8/27 \approx 0.296 \). Note that for \( \tau > 8/27 \) the photons approach Earth while for \( \tau < 8/27 \) they recede from it.

In the standard model, the integrals must be calculated over the past light cone, i.e. with the distance \( R \) and the time \( t \) related by the curves in
figure 1, each one obtained by inserting in \(S(t) = ((1 - \Omega_\Lambda)/\Omega_\Lambda)^{1/3}\sinh^{2/3}([\sqrt{3}\Lambda t/2])\) with \(\Lambda = 3H_0^2\Omega_\Lambda\). For simplicity, we approximate the different light cones for \(0 \leq \Omega_\Lambda \leq 0.9\) by that of \(\Omega_\Lambda = 0\), which has a much simpler analytical expression. This is justified because of their closeness. In other words, we assume that the relation between \(R\) and \(t\) is given by (23), even if the expressions for the cosmological quantities as the density, the age of the universe or the Hubble constant are assumed to depend on time according to the standard model based on corresponding scale factor. As usual, the subindex \(\Lambda\) indicate here present value, while \(\lambda\) stands for the time dependent functions.

In order to proceed with the calculation, we must insert in the integrals \(\int_0^1\) the retarded values of all the time dependent quantities, i.e. of the density \(\rho_{cr} = 3H_0^2/8\pi G = (\Lambda/8\pi G)\coth^2[At]\) and \(\Omega_\lambda = \tanh^2[At]\) with \(A = \sqrt{3\Lambda/4}\). Since \(\Lambda = 3H_0^2\Omega_\Lambda\), then \(\rho_{cr} = (3H_0^2\Omega_\Lambda/8\pi G)\coth^2[At]\) and \(A = 3H_0\Omega_\Lambda^{1/2}/2\).

Putting inside the integral the time dependent critical density and \(\Omega_\Lambda\), we have instead of (19)
\[
g^* = -\frac{16\pi G\dot{v}}{c^2} \int_{lc} \rho_{cr}(t')(1-\Omega_\Lambda(t'))RdR + \frac{4\pi G\dot{v}}{3c^2} \int_{lc} \rho_{cr}(t')(1-3\Omega_\Lambda(t'))RdR, \tag{24}
\]
where \(t'\) is the retarded time along the light cone as given by eq. (23), from which it follows that \(RdR = 9(ct_0)^2f(\tau)d\tau\) with
\[
f(\tau) = \epsilon(\tau)[(2/3)\tau^{1/3} - (5/3)\tau^{2/3} + \tau] \tag{25}
\]
where \(\epsilon(\tau)\) is \(-1\) for \(\tau > 8/27\) and \(+1\) for \(\tau < 8/27\) in order to take into account the two possible signs in eq. (21), according whether the photons approach to or recede from Earth.

We define the dimensionless non negative integrals
\[
J = \int_0^1 \coth^2(a\tau)f(\tau)d\tau, \quad K = \int_0^1 \coth^2(a\tau)\Omega_\Lambda(a\tau)f(\tau)d\tau, \tag{26}
\]
where \(\Omega_\lambda = \tanh^2(a\tau)\) and \(a = At_0 = 3(H_0t_0)\Omega_\Lambda^{1/2}/2\). The second is immediate. In fact, it is easy to see that \(K = \int_0^1 f(\tau)d\tau\) and that \(K = (3ct_0)^{-2}[\int_0^{R_{max}} - f_0^{R_{max}}]RdR = (3ct_0)^{-2}R_{max}^2 = 16/729\), since \(R_{max} = 4ct_0/9\).

After a bit of algebra, the reactive acceleration (24) can be written then as
\[
g^* = -\Omega_\Lambda(H_0t_0)^2 \left[\frac{99}{2}J - \frac{8}{9}\right]\dot{v}, \tag{27}
\]
which is the expression for the reactive acceleration in our second approach and the main result of this work. The integral $J$ can be written as the sum of two terms

$$J = J_1 + J_2 = \int_{8/27}^{1} \coth^2(a\tau)f(\tau)d\tau + \int_{0}^{8/27} \coth^2(a\tau)f(\tau)d\tau \quad (28)$$

Unfortunately, $J_2$ is unbounded because of the divergence of the integrand at $\tau = 0$.

Changing the variable from $\tau$ to $u = a\tau$, the integrals $J_1$ and $J_2$ can be written as

$$J_1 = -\frac{2}{3a^{4/3}}I_{1/3} + \frac{5}{3a^{5/3}}I_{2/3} - \frac{1}{a^2}I_1,$$

$$J_2 = \frac{2}{3a^{4/3}}I'_{1/3} - \frac{5}{3a^{5/3}}I'_{2/3} + \frac{1}{a^2}I'_1,$$

where $I_\lambda = \int_{a8/27}^{a} \coth^2(u) u^\lambda du$ and $I'_\lambda = \int_{0}^{a8/27} \coth^2(u) u^\lambda du$. The three integrals $I'_\lambda$ are singular because the integrands diverge at $u = 0$. As is easy to show, however, that the singularities are eliminated by subtracting to the integrands $u^{-5/3}$, $u^{-4/3}$ and $u^{-1}$, respectively, what leaves regular series which vanish at $u = 0$. In fact, the three integrands are, respectively, equal to

$$u^{-5/3} + 2u^{1/3}/3 + u^{7/3}/15 + O(u)^{13/3},$$

$$u^{-4/3} + 2u^{2/3}/3 + u^{8/3}/15 + O(u)^{14/3},$$

$$u^{-1} + 2u/3 + u^3/15 + O(u)^5$$

After eliminating the singularities in this way, $J$ is easily calculated and can be inserted in eq. (27).

The result of this second approach for the coefficient $\xi$ (so that the reactive acceleration is $g^* = -\xi\dot{v}$) are plotted as the lower curve in Figure 2. As is seen, $\xi$ is in the interval $(0.7, 0.6)$, approximately, if $\Omega_\Lambda < 0.9$. For $\Omega_\Lambda = 0.7$, $\xi \simeq 0.65$. We estimate that the accuracy of these numbers is about 10%, but we have made no attempt to refine them any further.

Instead of eliminating the singularities this way, we could try to find a cut-off. It can then be shown easily that, if the integral $J_2$ is cut-off at $\tau \simeq 0.18$, corresponding to $t \simeq 2.5$ Gy, one obtains the right result $\xi = 1$ for $\Omega_\Lambda = 0.7$.

This is not a bad result; quite on the contrary it is very encouraging. The reactive acceleration is of the order of $-\dot{v}$ and varies little with $\Omega_\Lambda$. 

7 Summary and conclusions

In order to evaluate Mach’s principle, we have studied the gravitational effect of the entire universe on an accelerated body or test particle in its rest frame, using the formalism of linearized gravitation. The aim is to show that the LHS of Newton second law $\ddot{v} = F/m$ is minus the reactive acceleration due to the gravitational action of the all the mass-energy in the universe on the particle in its rest frame. The results are expressed in terms of two integrals depending on the critical density and the relative density of dark energy. We have followed two approaches. In the first one, the integrals are calculated in the actual time section $t = t_0$ up to the radius of the universe $R_U = ct_0$. In the second, the integration is done over the past light cone of the particle, using the formalism of the retarded fields. In both cases the results are enticing: the reactive acceleration turns out to be close to $-\dot{v}$: we have found that, if $\Omega_\Lambda = 0.7$, it is about $-1.1 \dot{v}$ in the first approach, and close to $-0.7 \dot{v}$ in the second.

All this means that the second Newton equation for a particle can be obtained either as an approximation to the equation of the geodesics or as the expression of the equilibrium between the forces due to nearby objects and those of background gravity in the rest frame of the particle. To summarize,
our results strongly support Mach’s idea that inertia is not an intrinsic properties of bodies or particles but an interaction, whose source fixes lapse \( N_i \) and shift \( N_i \), and, furthermore, that this interaction is gravity as suggested by Sciama.

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