ON THE NON-INNER AUTOMORPHISM CONJECTURE OF FINITE $p$-GROUPS

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Abstract. A long-standing conjecture asserts that every finite non-abelian $p$-group has a non-inner automorphism of order $p$. In this paper, we settle the conjecture for a finite $p$-group ($p > 2$) of nilpotency class $n$ with certain conditions.

1. Introduction

There is a famous conjecture known as the Non-inner Automorphism Conjecture, listed in the renowned book “Unsolved Problems in Group Theory: The Kourovka Notebook”, which states that Every finite non-abelian $p$-group admits an automorphism of order $p$ which is not an inner. (see [10, Problem 4.13])

Some researchers showed interest in proving the sharpened version of the conjecture. They were interested in proving that every finite non-abelian $p$-group $G$ has a non-inner automorphism of order $p$ which fixes $\Phi(G)$ element-wise (for instance, see [4–8], and [9] for other references). The conjecture was first attacked by Liebeck [11]. He proved that for an odd prime $p$, every finite $p$-group $G$ of nilpotency class 2 has a non-inner automorphism of order $p$ fixing $\Phi(G)$ element-wise. In 2013, Abdollahi et al. [3] proved the validity of the conjecture for finite $p$-groups of nilpotency class 3. In particular, in Theorem 4.4, they proved that every finite $p$-group $G$ of odd order and of nilpotency class 3 has a non-inner automorphism of order $p$ that fixes $\Phi(G)$ element-wise. In 2014, Abdollahi et al. [2] showed that every finite $p$-group $G$ of co-class 2 has a non-inner automorphism of order $p$ leaving $Z(G)$ element-wise fixed. In 2017, Ruscitti et al. [13] confirmed the conjecture for finite $p$-groups of co-class 3, with $p \neq 3$.

If there is a maximal subgroup $M$ of a finite $p$-group $G$ with $|G| > p$ and $Z(M) \subseteq Z(G)$, then there exists a non-inner automorphism of $G$ of order $p$ (see, Rotman [12] Lemma 9.108). In 2002, Deaconescu and Silberberg [4] proved that if the conjecture is false for a finite $p$-group $G$, then $Z(G) < Z(M)$ for all maximal subgroups $M$ of $G$. This raises the following natural question:

Question. Given a finite $p$-group $G$ with $Z(G) < Z(M)$ for all maximal subgroups $M$ of $G$, does the conjecture hold?

In Theorem 2.1, we prove that every finite $p$-group $G$, $(p > 2)$ of nilpotency class $n$ such that $\exp(\gamma_n(G)) = p$, $|\gamma_n(G)| = p$ and $Z(C_G(x)) \leq \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$, has a non-inner automorphism of order $p$ which fixes $\Phi(G)$ element-wise. As a consequence, in Corollaries

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2.2 and 2.3, we give an affirmative answer to the above question under some conditions. In [13],
the authors proved the conjecture for all non-abelian finite $p$-groups of co-class 3, where $p$ is
a prime number such that $p \neq 3$. We also validate the conjecture for some non-abelian finite
3-groups of co-class 3 in Corollary 2.4.

Throughout $p$ denotes an odd prime number. For a group $G$, by $Z_m(G)$, $\gamma_m(G)$, $d(G)$ and
$\Phi(G)$, we denote, the $m$th term of the upper central series of $G$, the $m$th term of the lower
central series of $G$, the minimum number of generators of $G$ and the Frattini subgroup of $G$,
respectively. The nilpotency class and the exponent of a finite group $G$ are denoted by $\text{cl}(G)$
and $\exp(G)$, respectively. A finite $p$-group $G$ of order $p^n$ with $\text{cl}(G) = n - c$ is said to be of
class $c$. All other unexplained notations, if any, are standard.

2. Main results

Since the conjecture is true for all finite $p$-groups $G$ having nilpotency class 2 and 3, we consider
only finite $p$-groups $G$ with $\text{cl}(G) \geq 4$.

Theorem 2.1. Let $G$ be a finite $p$-group ($p > 2$) of class $n$ such that $|\gamma_n(G)| = \exp(\gamma_{n-1}(G)) = p$
and $Z(C_G(x)) \leq \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$. Then $G$ has a non-inner automorphism
of order $p$ that fixes $\Phi(G)$ element-wise.

Proof. Since $n = \text{cl}(G) \geq 4$ and $\exp(\gamma_{n-1}(G)) = p$, there exists an element $x \in \gamma_{n-1}(G) \setminus Z(G)$
of order $p$. Thus $[x, G] \subseteq \gamma_n(G)$, and therefore the order of conjugacy class of $x$ in $G$ is $p$. It
follows that $M = C_G(x)$ is a maximal subgroup of $G$. Let $g \in G \setminus M$. Then
\[(gx)^p = g^p x^p [x, g]^{p(p-1)/2} = g^p.\]

Consider the map $\beta$ of $G$ defined as $\beta(g) = gx$ and $\beta(m) = m$ for all $m \in M$. The map $\beta$
can be extended to an automorphism of $G$ fixing $\Phi(G)$ element-wise and of order $p$. We claim
that $\beta$ is a non-inner automorphism of $G$. For a contradiction, assume that $\beta = \theta_y$, the inner
automorphism of $G$ induced by some $y \in G$, which implies that $y \in C_G(M)$. If $y \notin M$, then
$G = M \langle y \rangle$. It follows that $y \in Z(G)$, which is a contradiction. Therefore $y \in Z(M)$. Since
$\beta = \theta_y$, we have $g^{-1} \theta_y(g) = [g, y] = x$. Now, by the given hypothesis $Z(C_G(x)) \leq \gamma_{n-1}(G)$
for all $x \in \gamma_{n-1}(G) \setminus Z(G)$, we have $y \in \gamma_{n-1}(G)$. Therefore
\[x = [g, y] \in \gamma_n(G) \leq Z(G),\]
which contradicts our choice of $x$ in $G$. Hence $G$ has a non-inner automorphism of order $p$ that
fixes $\Phi(G)$ element-wise. \qed

Let $G$ be a finite $p$-group such that $|Z(G)| = p$. Let $M$ be any maximal subgroup of $G$. Since
$Z(M)$ is a characteristic subgroup of $M$ and $M$ is a normal subgroup of $G$, we have $Z(M)$ is
a normal subgroup of $G$. Thus $Z(G) \leq Z(M)$ for all maximal subgroups $M$ of $G$. Hence, we
obtain the following Corollary from Theorem 2.1:

Corollary 2.2. Let $G$ be a finite $p$-group ($p > 2$) of class $n$ such that $|Z(G)| = \exp(\gamma_{n-1}(G)) = p$
and $Z(C_G(x)) \leq \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$. Then $G$ has a non-inner automorphism
of order $p$ that fixes $\Phi(G)$ element-wise.
Corollary 2.3. Let $G$ be a finite $p$-group ($p > 2$) of class $n$ such that $|Z(G)| = p$ and $Z(M) = \gamma_{n-1}(G)$ is of exponent $p$ for all maximal subgroups $M$ of $G$. Then $G$ has a non-inner automorphism of order $p$ that fixes $\Phi(G)$ element-wise.

Proof. Given that $d(G) = n$. It follows that $\gamma_n(G) \leq Z(G)$. Consequently, $|\gamma_n(G)| = p$.

Considering the provided hypothesis $Z(M) = \gamma_{n-1}(G)$ is of exponent $p$ for all maximal subgroups $M$ of $G$ and the proof of Theorem 2.1, we deduce that $Z(C_G(x)) \leq \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$. Hence, by Theorem 2.1, $G$ possesses a non-inner automorphism of order $p$ that fixes $\Phi(G)$ element-wise. \hfill \Box

Corollary 2.4. Let $G$ be a finite 3-group of order $3^n$ and of co-class 3 such that $Z(M) = \gamma_{n-4}(G)$ is of exponent 3 for all maximal subgroups $M$ of $G$. Then $G$ has a non-inner automorphism of order 3.

Proof. Assume that $G$ does not possess any non-inner automorphism of order 3. Then, it follows from [11 Corollary 2.3] that

$$d(Z_2(G)/Z(G)) = d(Z(G)) d(G).$$

(1)

Since $G$ is of co-class 3, we have $p^i \leq |Z_i(G)| \leq p^{i+2}$ and $\gamma_{n-i-2}(G) \leq Z_i(G)$ for all $1 \leq i \leq n-4$. Thus, by equation (1), $d(Z(G)) = 1$. Now, if $|Z(G)| = p^3$, then $|Z_2(G)| = p^4$, which contradicts equation (1). Furthermore, $|Z(G)|$ cannot be $p^2$ according to [13 Theorem 4.3]. Finally, assume that $|Z(G)| = p$. In this case, the conclusion follows from Corollary 2.3. \hfill \Box

We conclude the paper by giving an example of a 3-group of order $3^7$ which supports Theorem 2.1.

Example 2.5. Consider the following group:

$$G = \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7 \rangle,$$

with relations: $f_5 = [f_2, f_1], f_4 = f_1^3, f_3 = [f_3, f_1], f_6 = [f_3, f_2], f_7 = [f_5, f_1], f_5^3 = [f_4, f_2], f_3^3 = f_5^3 = f_6^3 = f_5^3 = [f_4, f_1] = [f_6, f_1] = [f_7, f_1] = [f_5, f_2] = [f_6, f_2] = [f_7, f_2] = [f_4, f_3] = [f_5, f_3] = [f_6, f_3] = [f_7, f_3] = [f_5, f_4] = [f_6, f_4] = [f_7, f_4] = [f_5, f_6] = [f_7, f_5] = [f_7, f_6] = 1$. Then

- $|G| = 3^7$.
- The nilpotency class of $G$ is 4.
- $Z(G) = \langle f_6, f_7 \rangle$.
- $\Phi(G) = \langle f_3, f_4, f_5, f_6, f_7 \rangle$.
- $\gamma_3(G) = \langle f_5, f_6, f_7 \rangle$.
- $\gamma_4(G) = \langle f_7 \rangle$.

Let $x = f_5$ and $M = C_G(x)$. Then $x \in \gamma_3(G) \setminus Z(G)$ is of order 3 and $Z(M) = \langle f_5, f_6, f_7 \rangle = \gamma_3(G)$. Consider the following automorphism:

$$\alpha(f_1f_2f_3f_4f_5f_6f_7) = f_1f_2f_3f_4f_5f_6f_7, \quad \alpha(f_1f_3f_5f_6) = f_1f_3f_5f_6, \quad \alpha(f_1f_2f_3f_4f_5f_6) = f_1f_2f_3f_4f_5f_6f_7.$$
Now, by using the relators of $G$, we have $\alpha(f_i) = f_i$ for all $i \in \{2, 3, 4, 5, 6, 7\}$ and $\alpha(f_1) = f_1f_5$. It is easy to verify that $\alpha$ is a non-inner automorphism of order 3 which fixes $\Phi(G)$ element-wise.

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4. Conflict of Interest

The authors declare that they have no conflict of interest.

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