The $B$-model connection and mirror symmetry for Grassmannians

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Abstract. We consider the Grassmannian $X = Gr_{n-k}(\mathbb{C}^n)$ and describe a ‘mirror dual’ Landau-Ginzburg model $(\tilde{X}, W_q : \tilde{X} \to \mathbb{C})$, where $\tilde{X}$ is the complement of a particular anti-canonical divisor in a Langlands dual Grassmannian $\tilde{X}$, and we express $W$ succinctly in terms of Plücker coordinates. First of all, we show this Landau-Ginzburg model to be isomorphic to the one proposed by the second author in [54]. Secondly we show it to be a partial compactification of the Landau-Ginzburg model defined earlier by Eguchi, Hori, and Xiong [15]. Finally we construct inside the Gauss-Manin system associated to $W$ a free submodule which recovers the trivial vector bundle with small Dubrovin connection defined out of Gromov-Witten invariants of $X$. We also give a $\mathbb{T}$-equivariant version of this isomorphism of connections. We conjecture that the free submodule above is isomorphic to the entire Gauss-Manin system (a tameness property for $W$). Our results imply a special case of [54, Conjecture 8.1]. They also imply [2, Conjecture 5.2.3] and resolve [48, Problem 13].

1. Introduction

Grassmannians are examples of smooth projective Fano varieties $X$ over $\mathbb{C}$, including also projective homogeneous spaces $G/P$, del Pezzo surfaces, etc. The (genus 0) Gromov-Witten invariants of such varieties, which relate to enumerative questions about rational curves in $X$, can be put together in various ways to define rich mathematical structures, such as the quantum cohomology rings, flat pencils of connections, Frobenius manifolds and other structures [10]. We consider these to make up the $A$-model of $X$. Mirror symmetry in one sense then seeks to describe these structures in terms of for example singularity theory or oscillating integrals of a regular function $W$ on a ‘mirror’ affine Calabi-Yau variety $\tilde{X}$. We call this mirror datum the $B$-model or Landau-Ginzburg (LG) model associated to $X$. The regular function $W$ is also called the ‘superpotential’.

One such structure on the $A$-model side is the small quantum cohomology ring. For general projective homogeneous spaces $G/P$ the quantum cohomology rings have already been reconstructed via Jacobi rings of associated superpotentials in previous work [54], recovering the remarkable presentations of these quantum cohomology rings due to Dale Peterson [46]. This paper is concerned with another structure, the small Dubrovin or Givental connection, which can be viewed as involving a kind of deformation (quantization) of the multiplication by a hyperplane class in quantum cohomology. See Section 2 for the definition in our setting. An equivariant analogue will appear in a future Appendix.

The main goal of this paper is to construct this $A$-model connection in terms of a Gauss-Manin system defined by the mirror LG model, in the case where $X$ is a Grassmannian. The LG model we work with is a reformulation of the one defined for general $G/P$ in [54], which applies to Grassmannians as a special case. Namely we give a new formula for this LG model in the Grassmannian case written out in terms of Plücker coordinates on a Langlands dual Grassmannian.

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Earlier (Laurent polynomial) LG models for Grassmannians were proposed by Eguchi, Hori and Xiong [15] and studied by Batyrev, Ciocan-Fontanine, Kim, and van Straten [2] in the 1990’s. We show that our LG model is isomorphic to a partial compactification of these.

Our main theorem implies a version of the mirror conjecture concerning solutions to the quantum differential equations, [54] Conjecture 8.1, for $X$ the Grassmannian of codimension $k$-subspaces in $\mathbb{C}^n$, which we denote by $Gr_{n-k}(\mathbb{C}^n)$ or $Gr_{n-k}(\mathbb{C}^n)$. Thanks to the comparison result with the superpotential of Eguchi, Hori and Xiong it also implies the ‘A-series conjecture’ of Batyrev, Ciocan-Fontanine, Kim and van Straten [2] Conjecture 5.2.3 about a series expansions for a coefficient of Givental’s $J$-function. This conjecture was restated also in [18] Problem 13 and [3] Section 3. In the latter reference [3] the A-series conjecture was proved in the special case of Grassmannians of 2-planes, by a different method.

We begin with a concrete introduction of the $A$-model structures. Then we give definitions of the $B$-model structures, as well as a careful statement of the main result. In Section 5.4 we show how our Plücker formula for the superpotential relates to the formulations in [15] [2] [54]. The proof of the main theorem begins in Section 7 and takes up the remainder of the paper. It makes heavy use of the cluster algebra structure of the homogeneous coordinate ring of a Grassmannian. In a future Appendix, we will prove a version of our result in the $T$-equivariant setting.

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2. The A-model introduction

We focus on an example to illustrate our results in the introduction. Let us suppose $X$ is the Grassmannian $Gr_{n-k}(\mathbb{C}^n) = Gr_2(\mathbb{C}^5)$, and denote the Schubert basis of $H^*(X) = H^*(X, \mathbb{C})$ by

$$\sigma^\emptyset, \sigma^\bullet, \sigma^\circ, \sigma^{\bullet\circ}, \sigma^{\bullet\bullet}, \sigma^{\circ\circ}, \sigma^{\bullet\bullet\circ}, \sigma^{\bullet\circ\circ}, \sigma^{\circ\bullet\circ}, \sigma^{\circ\circ\bullet}, \sigma^{\bullet\bullet\circ\circ}, \sigma^{\bullet\circ\bullet\circ}, \sigma^{\circ\bullet\bullet\circ}, \sigma^{\circ\circ\bullet\bullet}.$$

Here $\sigma^\lambda$ is in $H^{2|\lambda|}(X)$ and $|\lambda|$ denotes the number of boxes in the Young diagram $\lambda$. Furthermore $X^\lambda$ will be a Schubert variety associated to $\lambda$, representing the Poincaré dual homology class. For example $X^\emptyset$ is a hyperplane section for the Plücker embedding. The set of partitions fitting in an $(n-k) \times k$-rectangle indexing the Schubert basis for $Gr_{n-k}(\mathbb{C}^n)$ is denoted $P_{k,n}$.

In classical Schubert calculus, Monk’s rule says that the cup product with $\sigma^\emptyset$ takes any Schubert class $\sigma^\lambda$ to the sum of all the Schubert classes $\sigma^\mu$ corresponding to the $\mu$’s in $P_{k,n}$ made up of $\lambda$ and precisely one extra box. In the cohomology of $Gr_2(5)$,

$$\sigma^\emptyset \cup \sigma^{\bullet\circ} = \sigma^{\bullet\circ}, \quad \sigma^{\bullet\circ} \cup \sigma^{\bullet\bullet} = \sigma^{\bullet\bullet\circ}, \quad \sigma^{\bullet\circ\circ} \cup \sigma^{\bullet\bullet\circ\circ} = \sigma^{\bullet\bullet\circ\circ} \quad \text{and} \quad \sigma^{\emptyset} \cup \sigma^{\bullet\bullet\circ\circ} = 0.$$

This combinatorics of adding a box, in this context, just encodes what happens to a Schubert variety $X^\lambda$ homologically, when it is intersected with a hyperplane section in general position.

In the quantum cohomology ring [54] (at fixed parameter $q$), quantum Monk’s rule says that the quantum cup product with $\sigma^\emptyset$ takes any Schubert class $\sigma^\lambda$ to $\sigma^\emptyset \cup q \sigma^\mu$ plus, if it exists, a term $q \sigma^\nu$, where $\nu$ is obtained by removing $n-1 (= 4)$ boxes from $\lambda$, at least one in each row [7]. For example,

$$\sigma^\emptyset \star q \sigma^{\bullet\circ} = \sigma^{\bullet\circ} + q \sigma^{\bullet\bullet}, \quad \sigma^{\bullet\circ} \star q \sigma^{\bullet\bullet} = \sigma^{\bullet\bullet\circ} + q \sigma^{\bullet\circ\circ}, \quad \text{and} \quad \sigma^{\emptyset} \star q \sigma^{\bullet\bullet\circ\circ} = q \sigma^{\bullet\bullet\circ\circ}.$$

Very roughly speaking, the extra term $q \sigma^\nu$ in $\sigma^\emptyset \star q \sigma^\lambda$ corresponds intuitively to $\phi(\infty)$ sweeping out an element of the homology class $[X^\nu]$ as $\phi$ varies in the space of degree one maps $\phi : \mathbb{C}P^1 \hookrightarrow X$ satisfying that $\phi(0)$ lies in $X^\lambda$ and $\phi(1)$ lies in a fixed general position hyperplane.

The quantum products by degree two classes were used by Dubrovin and Givental [24] [14] to define a connection on the trivial bundle

$$H^2(X, \mathbb{C}) \times H^*(X, \mathbb{C}) \rightarrow H^2(X, \mathbb{C}).$$

In our setting we have $H^2(X, \mathbb{C}) = \mathbb{C}$ and we denote by $\tau$ the coordinate dual to the class $\sigma^\emptyset$. Let us recall the usual definition of the connection in the conventions of Dubrovin [10], depending on an additional
parameter $z$:

$$\nabla_{\partial_z} := \frac{d}{dt} + \frac{1}{z} \sigma \star e_z$$

We think of this connection as being dual to the one whose flat sections are constructed by Givental [26] in terms of descendent 2-point Gromov-Witten invariants, compare also Section 1.1.

Our main result is a $B$-model construction of the above connection. However we make some small adjustments first. Instead of $\tau$ we prefer to consider $q = e^\tau$, the coordinate on the torus $H^2(X, \mathbb{C})/H^2(X, 2\pi i \mathbb{Z}) \cong \mathbb{C}_q$. We write $\mathbb{C}_q$ to mean $\mathbb{C}$ with coordinate $q$, and similarly $\mathbb{C}_q^*$ for $\mathbb{C} \setminus \{0\}$ with (invertible) coordinate $q$. Also, following Dubrovin [14] we will extended the connection (2.1) in the $z$-direction, to give a flat meromorphic connection over a larger base. Namely, let $\mathcal{H}_A$ denote the (sheaf of regular sections of) the trivial vector bundle with fiber $H^*(X, \mathbb{C})$ over the extended base $\mathbf{P} = \mathbb{C}_z \times \mathbb{C}_q$, where the $z$ and $q$ are coordinates. We denote by $\mathcal{H}_{A, an}$ the sheaf of analytic sections of the vector bundle $\mathcal{H}_A$. Using the conventions of Iritani [32, Definition 3.1] we set:

$$A\nabla_{\partial_q} := q \frac{\partial}{\partial q} + \frac{1}{z} e_q \star -,$nabla_{\partial_z} := z \frac{\partial}{\partial z} + Gr - \frac{1}{z} e_{Tz}$$

where $Gr$ is a diagonal operator on $H^*(X)$ given by $Gr(\sigma) = k\sigma$ whenever $\sigma \in H^{2k}(X)$. These formulas define a flat meromorphic connection on $\mathcal{H}_A$.

The vector bundle $\mathcal{H}_A$ also comes equipped with a flat pairing which is non-degenerate over $\mathbb{C}_q^* \times \mathbb{C}_q$. Namely, let $\langle \cdot, \cdot \rangle_{H^*(X)}$ denote the Poincaré duality pairing on $H^*(X)$ and define $j: \mathbf{P} \to \mathbf{P}$ by $(z, q) \mapsto (-z, q)$. Then the pairing $S_A : j^* \mathcal{H}_A \otimes \mathcal{H}_A \to \mathcal{O}_{\mathbf{P}} A$ defined by

$$S_A(\cdot, \cdot) = (2\pi i z)^N \langle \cdot, \cdot \rangle_{H^*(X)} \rangle$$

where $N = \dim_{\mathbb{C}}(X)$, satisfies $dS_A(\sigma, \sigma') = S_A(A\nabla \sigma, \sigma') + S_A(\sigma, A\nabla \sigma')$, compare [32].

**Definition 2.1.** We define

$$H_A = \Gamma(\mathbb{C}_q^* \times \mathbb{C}_q^*, \mathcal{H}_A) = H^*(X, \mathbb{C}[z^{\pm 1}, q^{\pm 1}])$$

to be the module for $D_{\mathbf{P}} = \mathbb{C}[z^{\pm 1}, q^{\pm 1}] \langle \partial_z, \partial_q \rangle$ where $\partial_z, \partial_q$ act by $A\nabla_{\partial_z}$ and $A\nabla_{\partial_q}$, respectively. We also define the $\mathbb{C}[z, q]$-submodule $H_{A, 0} := H^*(X, \mathbb{C}[z, q])$, which is acted on by the subalgebra $D_{\mathbf{P}, 0}$ of $D_{\mathbf{P}}$ generated by $z, q, \tau(z\partial_z)$ and $z(q\partial_q)$. The pairing $H_A \otimes_{\mathbb{C}[z^{\pm 1}, q^{\pm 1}]} H_A \to \mathbb{C}[z^{\pm 1}, q^{\pm 1}]$ defined by $S_A$ is non-degenerate and denoted again by $S_A$.

The main goal of this paper is to construct the $D_{\mathbf{P}}$-module $H_A$ above, and with it the data, $\mathcal{H}_A, A\nabla$ and $S_A$, in terms of a Gauss-Manin system defined by a mirror LG model. The precise definitions of the LG model and Gauss-Manin system follow in Section 3. The main results are stated in Section 4.1

3. B-model introduction

3.1. The mirror LG model. To give our presentation of the mirror LG model of the Grassmannian $X = Gr_n-k(n)$ we need to introduce a new Grassmannian $\bar{X} := Gr_k(n)$. Both $X$ and $\bar{X}$ have dimension $N = k(n-k)$. To be more precise, if $X = Gr_n-k(C^n)$ then we think of $\bar{X}$ as $Gr_k((C^n)^*)$, which is embedded by a Plücker embedding in $\mathbb{P}(A^k(C^n)^*)$. The Plücker coordinates $\rho_i$ for $\bar{X}$ correspond in a natural way to the Schubert classes $\sigma^\lambda$ in $H^*(X)$ and are indexed by $\lambda \in \mathcal{P}_{k,n}$.

Note that by the geometric Satake correspondence [23, 33, 39, 43] the A-model Grassmannian $X = Gr_{n-k}(C^n)$ may be embedded into the affine Grassmannian $Gr_G$ of $G = GL_n(C)$ in such a way that its cohomology $H^*(X)$ becomes identified with the fundamental representation $V_{\mathbf{G}_k}$ of a Langlands dual $GL_n(C)$. We think of the vector space $\wedge^k C^n$, whose dual appears in the definition the B-model Grassmannian $\bar{X}$ as precisely this representation $V_{\mathbf{G}_k}$ of the Langlands dual $GL_n(C)$, so that $\bar{X}$ is in fact canonically embedded in $\mathbb{P}(H^*(X^*))$. This means that $\Gamma(\mathcal{O}_{\bar{X}}(1)) = H^*(X)$, and that the Schubert classes of $X$ are canonically homogeneous coordinates on $\bar{X}$. Also, $\bar{X}$ should be viewed as a homogeneous space for the Langlands
We continue with the explicit example of $X = Gr_2(5)$ and $\tilde{X} = Gr_3(5)$, where $k = 3$ and $n = 5$. Define the rational function $W$ on $\tilde{X} \times \mathbb{C}_q$ by the formula

$$W = \frac{p_3}{p_0} + \frac{p_6}{p_4} + q \frac{p_5}{p_3} + \frac{p_7}{p_5} + \frac{p_8}{p_7},$$

in terms of Plücker coordinates $p_\lambda$. To obtain a regular function, remove from $\tilde{X}$ the 5 hyperplanes defined by the Plücker coordinates which appear in the denominators. The resulting affine variety is

$$\tilde{X} := \tilde{X} \setminus \{p_0 = 0\} \cup \{p_4 = 0\} \cup \{p_3 = 0\} \cup \{p_5 = 0\} \cup \{p_7 = 0\}.$$ 

Note that the anti-canonical class of $\tilde{X} = Gr_3(5)$ is $5p_\emptyset$, therefore $\tilde{X}$ is the complement of an anti-canonical divisor in $\tilde{X}$. Let $\omega$ be a choice of non-vanishing holomorphic volume form on $\tilde{X}$. We denote the regular function again by $\tilde{X}$, we may set $p_0 = 1$ henceforth so that the remaining $p_\lambda$ are actual coordinates on $\tilde{X}$.

For a general Grassmannian $X = Gr_{n-k}(n)$ we have a completely analogously defined affine subvariety $\tilde{X}$ of $\tilde{X} = Gr_k(n)$, along with a non-vanishing holomorphic volume form $\omega$ on $\tilde{X}$, and a superpotential $W : \tilde{X} \times \mathbb{C}_q \to \mathbb{C}$, see Section 3. We will show in Section 5.4 that this LG model is isomorphic to the one introduced by the second author in [54], and after restriction to an open subtorus becomes isomorphic to one introduced earlier by Eguchi, Hori and Xiong [17] and studied further in [2].

### 3.2. The Gauss-Manin system.

Denote by $\Omega^N(X)$ the space of holomorphic $N$-forms on $\tilde{X}$. We write $W_q : \tilde{X} \to \mathbb{C}$ for the superpotential where the coordinate $q \neq 0$ is fixed. There is a Gauss-Manin system associated to $W_q$, see [13, 15, 20], which should be thought of as describing holomorphic $N$-forms $\eta$ measured by ‘period integrals’ of the form $\int e^{zW_q} \eta$, see the definition below. Here we let both $z$ and $q$ vary to get a 2-parameter Gauss-Manin connection.

**Definition 3.1.** Consider the $\mathbb{C}[z^{\pm 1}, q^{\pm 1}]$-module $G^{W_q}$ defined by

$$G^{W_q} = \Omega^N(\tilde{X})[z^{\pm 1}, q^{\pm 1}] / ((d + z^{-1}dW_q \wedge -)(\Omega^{N-1}(\tilde{X})[z^{\pm 1}, q^{\pm 1}])).$$

Note that for fixed $(z_0, q_0) \in \mathbb{P}$ the ‘fiber’

$$F_{B,(z_0,q_0)} := \Omega^N(\tilde{X}) / ((d + z_0^{-1}dW_{q_0} \wedge -)(\Omega^{N-1}(\tilde{X}))),$$

is a twisted de Rham cohomology group in the sense going back to Witten [62]. There is a Gauss-Manin connection on $G^{W_q}$ defined on $\eta \in \Omega^N(\tilde{X})$ by

$$B_{\nabla_{q \partial_z}}[\eta] := \frac{1}{z} \left[ q \frac{\partial W}{\partial q} \eta \right],$$

$$B_{\nabla_{z \partial_q}}[\eta] := -\frac{1}{z} [W \eta],$$

and extended using Leibniz rule. By the flatness of $B_{\nabla}$ we get a $D_{\mathbb{P}}$-module structure on $G^{W_q}$ by letting $\partial_z$ and $\partial_q$ in $D_{\mathbb{P}} = \mathbb{C}[z^{\pm 1}, q^{\pm 1}] (\partial_z, \partial_q)$ act by the operators $B_{\nabla \partial_z}$ and $B_{\nabla \partial_q}$, respectively.

In order to state our main theorem we will define a $\mathbb{C}[z,q]$-submodule of $G^{W_q}$ which is to play the part of regular global sections of a trivial vector bundle $\mathcal{H}_{B}$ on $\mathbb{C}_q \times \mathbb{C}_z$.

**Definition 3.2.** Recall that $\tilde{X}$ has on it a non-vanishing holomorphic volume form $\omega$. Let $G^{W_q}_{\tilde{X}}$ be the $\mathbb{C}[z^{\pm 1}, q^{\pm 1}]$-submodule of $G^{W_q}$ spanned by the classes $[p_\lambda \omega]$ where $\lambda$ runs through $P_{k,n}$. Furthermore let $G^{W_q}_{0}$ be the $\mathbb{C}[z,q]$-submodule inside $G^{W_q}$ generated by the $[p_\lambda \omega]$, and let $F_{B,(z_0,q_0)}$ be the corresponding $\mathbb{C}$-linear subspace of $F_{B,(z_0,q_0)}$. [44, 45]
We will prove the following lemma in Section 5.5.

**Lemma 3.3.** $G^W_0$ is a free $\mathbb{C}[z,q]$-module with basis $\{[p_\lambda \omega], \lambda \in \mathcal{P}_{k,n}\}$.

By this lemma $G^W_0$ is indeed the space of regular sections of a trivial vector bundle on $P = \mathbb{C}_z \times \mathbb{C}_q$. We let $\mathcal{H}_B$ denote the sheaf of regular sections of this trivial bundle on $P$, and $\mathcal{H}_{B,an}$ the sheaf of analytic sections. The fiber of the vector bundle $\mathcal{H}_B$ at $(z_0, q_0)$ is $\mathcal{P}_{B,(z_0, q_0)}$ and has a basis given independently of $(z_0, q_0)$ by the classes $[p_\lambda \omega]$.

We conjecture that $G^W = G^W_0$. In order to prove this equality it would suffice to show that $W_q$ is tame in some form, for example cohomologically tame [57, 56]. Note that in the special case of projective space the $[p_\lambda \omega]$ do form a free basis of $G^W_0$, see [29]. Another related example is the orthogonal Grassmannian of lines in $\mathbb{C}^{2n+1}$. In this case the mirror LG-model of [57] was expressed in Plücker coordinates in [48], and proved to agree with one given by Gorbounov and Smirnov in [28], which they showed with Sabbah and Nemethi to be cohomologically tame.

4. Main results

4.1. Isomorphism of $D_P$-modules. The first main theorem recovers the $A$-model datum of $H_{A,0} = H^*(X, \mathbb{C}[z,q])$ with its small Dubrovin connection inside the Gauss-Manin system on $G^W_0$ of the $B$-model side.

**Theorem 4.1.** The $\mathbb{C}[z^{\pm 1}, q^{\pm 1}]$-module $G^W_0$ is a $D_P$-submodule of $G^W$, and the map

$$\Phi : H_A \rightarrow G^W_0, \quad \sigma^{\lambda} \mapsto [p_\lambda \omega]$$

is an isomorphism of $D_P$-modules. Under this isomorphism $H_{A,0} = H^*(X, \mathbb{C}[z,q])$ is identified with $G^W_0$, and $H_A$ is identified with $\mathcal{H}_B$. The pairing $S_A$ may be identified with a residue pairing $S_B$ on $G^W_0$.

The proof of this theorem hinges on verifying the following formulas for the action of $B\nabla$.

**Key result:** With the definitions as above we have

$$B\nabla_{q\partial_q}[p_\lambda \omega] = \frac{1}{z} \left( \sum_{\mu} [p_\mu \omega] + \sum_{\nu} q [p_\nu \omega] \right),$$

$$B\nabla_{z\partial_z}[p_\lambda \omega] = |\lambda||p_\lambda \omega| - \frac{1}{z} n \left( \sum_{\mu} [p_\mu \omega] + \sum_{\nu} q [p_\nu \omega] \right),$$

where $\mu$ and $\nu$ are exactly as in the quantum Monk’s rule for $s^a s_q \sigma^\lambda$.

In the concrete case of $X = Gr_2(5)$, we will see for example

$$B\nabla_{q\partial_q} \left( [p^{\mu\nu} \omega] \right) = \frac{1}{z} \left( [p^{\mu\nu} \omega] + [p^{\mu\nu} \omega] \right),$$

$$B\nabla_{q\partial_q} \left( [p^{\mu\nu} \omega] \right) = \frac{1}{z} \left( [p^{\mu\nu} \omega] + q [p^{\mu\nu} \omega] \right),$$

$$B\nabla_{q\partial_q} \left( [p^{\mu\nu} \omega] \right) = \frac{1}{z} \left( q [p^{\mu\nu} \omega] \right).$$

Similarly in the $z\partial_z$ direction:

$$B\nabla_{z\partial_z} \left( [p^{\mu\nu} \omega] \right) = 3 [p^{\mu\nu} \omega] - \frac{5}{z} \left( [p^{\mu\nu} \omega] + [p^{\mu\nu} \omega] \right),$$

$$B\nabla_{z\partial_z} \left( [p^{\mu\nu} \omega] \right) = 5 [p^{\mu\nu} \omega] - \frac{5}{z} \left( [p^{\mu\nu} \omega] + q [p^{\mu\nu} \omega] \right),$$

$$B\nabla_{z\partial_z} \left( [p^{\mu\nu} \omega] \right) = 6 [p^{\mu\nu} \omega] - \frac{5}{z} \left( q [p^{\mu\nu} \omega] \right).$$
4.2. Oscillating integrals. Recall the flat pairing $S_A$ in the $A$-model. We may think of this pairing as identifying the bundle $H^*(X) \times \mathbb{C}^*_z \times \mathbb{C}_q \to \mathbb{C}_z^* \times \mathbb{C}_q$ with its dual. The flatness condition can then be interpreted as saying that the dual connection to $A$ is given by the identical formulas (2.2) and (2.3) with the one change of replacing $z$ by $-z$. In other words the dual connection to $A$ takes the form

$$A\nabla^\vee = d + \text{Gr} \frac{dz}{z} + \frac{1}{z}(c_1(TX) \ast q) \frac{dz}{z} - \frac{1}{z} (\sigma^\mu, \sigma^\lambda) \frac{dq}{q}.$$ 

To study flat sections of this connection we need to consider analytic sections $H_{A,an}$. In [26, Corollary 6.3], Givental wrote down a basis of the space of all solutions $s \in H^*(X, \mathbb{C}[z^{-1}, \ln(q)][[q]])$ to the equation

$$(4.3)$$

$$A\nabla_{\mu}^\vee S = 0$$

using coefficients given in terms of two-point descendent Gromov-Witten invariants, see [10] for relevant definitions. Namely for $\mu \in \mathbb{P}_{k,n}$ there is a solution

$$(4.4)$$

$$s_\mu = \sum_{\lambda} \sum_d q^d \left( \frac{1}{z - \psi} e^{\ln(q) \cdot \sigma^\mu, \sigma^\lambda} \right)_{2,d}^{PD(\lambda)}$$

to (4.3). For the maximal $\mu$, which we denote $\mu_{n-k}$, the formula simplifies to

$$(4.5)$$

$$s_{\mu_{n-k}} = \sum_{\lambda} \sum_d q^d \left( \frac{1}{z - \psi} \sigma_{\mu_{n-k}, \sigma^\lambda} \right)_{2,d}^{PD(\lambda)}.$$ 

The equations (4.3) are also referred to as the small quantum differential equations. The following theorem implies an integral formula for the solution $s_{\mu_{n-k}} \in H^*(X, \mathbb{C}[z^{-1}][[q]])$ to (4.3).

**Theorem 4.2.** Let $\Gamma_0$ be a compact oriented real $N$-dimensional submanifold of $\tilde{X}$ representing a generator in $H^N(\tilde{X}, \mathbb{Z})$, and $\omega$ normalised so that $\frac{1}{(2\pi i)^N} \int_{\Gamma_0} \omega = 1$. Then the formula

$$S_{\Gamma_0}(z,q) := \frac{1}{(2\pi i)^N} \sum_{\lambda \in \mathbb{P}_{k,n}} \left( \int_{\Gamma_0} e^{\frac{1}{2}W_{\lambda}} \omega \right)^{PD(\lambda)}$$

defines a flat section for $A\nabla^\vee$ inside $H^*(X, \mathbb{C}[z^{-1}][[q]])$. In particular $S_{\Gamma_0}$ satisfies the quantum differential equations (4.3).

**Proof.** This statement follows in a standard way from Theorem 4.1 and the constructions. For any $(z, q) \in \mathbb{P}$ with $z \neq 0$, consider the linear form on the fiber $F_{B,(z,q)}$ of $H_B$ at the point $(z, q)$ defined by the formula

$$Osc_{\Gamma_0}(z,q) : [\eta] \mapsto \int_{\Gamma_0} e^{\frac{1}{2}W_{\lambda}} \ast \eta.$$ 

This formula defines a section $Osc_{\Gamma_0}$ of $H_{B,an}$ over $\mathbb{C}_z^* \times \mathbb{C}_q$, where $H_{B,an}$ denotes the sheaf of analytic sections of the dual bundle to $H_B$. The definition of the Gauss-Manin connection (4.1), (4.2) on $H_B$ is engineered so that $Osc_{\Gamma_0}$ is a flat section of $H_{B,an}$. We now use the symmetric pairing

$$S_B([p_{\lambda} \omega], [p_{\mu} \omega]) = (2\pi i)^N \delta_{\lambda,PD(\mu)}$$

which is flat by Theorem 4.1 and non-degenerate over $\mathbb{C}_z^* \times \mathbb{C}_q$, to identify the bundle $H_B^*$ with $H_B$ where $z \neq 0$. The dual basis to $\{ [p_{\lambda} \omega] \}$ is $\{ \frac{1}{(2\pi i)^N} [p_{PD(\lambda)} \omega] \}$. If we express $Osc_{\Gamma_0}$ in this basis,

$$Osc_{\Gamma_0} = \sum_{\lambda} m_{\lambda} \frac{1}{(2\pi i)^N} [p_{PD(\lambda)} \omega],$$

the coefficients $m_{\lambda}$ are naturally determined by

$$m_{\lambda} = S_B(Osc_{\Gamma_0}, [p_{\lambda} \omega]) = \int_{\Gamma_0} e^{\frac{1}{2}W_{\lambda}} \ast p_{\lambda} \omega.$$ 

Now the isomorphism between $H_A$ and $H_B$ from Theorem 4.1 identifies the flat section $Osc_{\Gamma_0}$ of $H_{B,an}$ with the section $S_{\Gamma_0}$ of $H_{A,an}$, which proves the theorem. \qed
**Remark 4.3.** Further local solutions $S = S_\Gamma$ to $A^\nabla_{q\partial q} S = 0$ can be obtained by replacing $\Gamma_0$ by some other, possibly non-compact integration cycle $\Gamma$. In this case it would be necessary to have conditions on the decay of $R(\frac{1}{2}W_q)$ in unbounded directions of $\Gamma$, and let $\Gamma$ vary with $z$ and $q$, to ensure convergence. Then the above proof of flatness for $S_{\Gamma_0}$ applies also to the more general local sections $S_\Gamma$. We note that the property $A^\nabla_{q\partial q} S_\Gamma = 0$ implies that the integrals
\[
\int_\Gamma e^{\frac{1}{2}W_q}\omega
\]
are solutions to the ‘quantum cohomology $D$-module’ [25]. It is an interesting problem to determine specific cycles $\Gamma_\mu$ such that the flat sections $S_{\Gamma_\mu}$ recover Givental’s basis of flat sections $s_\mu$ defined in the $A$-model. In particular this would give integral formulas for the coefficients of Givental’s $J$-function, compare Section 4 in [27]. See also [29] where explicit integration cycles are described for the case of $\mathbb{CP}^2$. Different integral expressions for the coefficients of the $J$-function of a Grassmannian were obtained by Bertram, Ciocan-Fontanine and Kim [3].

**4.3. $A$-series conjecture and the superpotential of Eguchi, Hori and Xiong.** In Section 5.3 we will recall the definition of the conjectural Laurent polynomial superpotential of Eguchi, Hori and Xiong [15] (the EHX superpotential). In that section we will also state in more detail and show the following comparison result.

**Proposition 4.4.** The Laurent polynomial associated to the Grassmannian $X$ by Eguchi, Hori and Xiong in [15] is isomorphic to the restriction of $W_q$ to a certain open torus inside $\tilde{X}$.

This result has the following consequence. Consider again (4.5), Givental’s special solution $s_{\mu_n-k}$ to the quantum differential equations. The coefficient of $\sigma^{\mu_n-k}$ in $s_{\mu_n-k}$ with $z$ specialised to 1 is an element of $\mathbb{C}[\![q]\!]$ and called the $A$-series in [2].

\[
A_X(q) = \sum_{d} q^d \left\langle \frac{1}{1 - \psi^{\mu_n-k}(q)} \mathfrak{s}, \sigma^0 \right\rangle_{2,d} = \sum_{d} q^d \left( \psi^{dn(\mu_n-k)}(q) \mathfrak{s}, \sigma^0 \right)_{2,d}.
\]

In [2] Batyrev, Ciocan-Fontanine, Kim and van Straten studied the Laurent polynomial superpotential of [15] and conjectured an explicit combinatorial formula for the $A$-series (4.6) in the case of a Grassmannian.

**Corollary 4.5.** [2] Conjecture 5.2.3] The $A$-series of the Grassmannian $X = Gr_{n-k}(\mathbb{C}^n)$ is given by
\[
A_X(q) = \sum_{m \geq 0} \frac{1}{(m!)^n} \sum_{(s_{i,j}) \in S_m} \left( \prod_{(i,j) \in \mathcal{I}_{k,n}} \frac{(s_{i+1,j})_{s_{i,j}}}{(s_{i,j})_{s_{i,j}}} \right) q^m.
\]

Here the indexing sets are
\[
\mathcal{I}_{k,n} = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid 0 < i < n - k, 0 < j < k\},
\]
\[
S_m = \{(s_{i,j}) \in (\mathbb{Z} \geq 0)^{\mathcal{I}_{k,n}} \mid s_{i+1,j} \geq s_{i,j}, s_{i,j+1} \geq s_{i,j}, s_{n-k,j} = s_{i,k} = m\}.
\]

The combinatorial formula (4.7) is obtained in [2] as a residue integral for the EHX superpotential. This corollary therefore follows from Theorem 4.2 together with Proposition 4.4 and the residue calculation of [2].

**4.4. The LG model on a Richardson variety.** The first LG-model for a Grassmannian which accurately recovers the quantum cohomology ring is one defined in [54]. This LG-model is a regular function $\mathcal{F}_q$ on an intersection of opposite Bruhat cells $\mathcal{R} = \mathcal{R}_{w,w_0}$ in a full flag variety. In Section 5.2 we explain how the formula of our LG model from Section 3 is obtained from the one given in [53], by mapping $\mathcal{R}$ into a Grassmannian and expressing $\mathcal{F}_q$ in terms of Plücker coordinates. The definition of the original LG model is given in detail in Section 5.2. We then prove the following comparison theorem in Section 5.4.

**Theorem 4.6.** The LG-models $(\tilde{X}, W_q)$ and $(\mathcal{R}_{w,w_0}, \mathcal{F}_q)$ are isomorphic.
As a consequence of Theorem 4.6 together with the results described in Section 4.1 and Section 4.2, we obtain a version of [54] Conjecture 8.1 for the case of Grassmannians.

**Corollary 4.7** (Special case of Conjecture 8.1 from [54]). For suitable middle dimensional (possibly non-compact) submanifolds $\Gamma \subset R_{w,P,w_0}$ oscillating integrals of the form

$$I_\Gamma (z, q) = \int_\Gamma e^{zF} \omega$$

define local solutions for the quantum cohomology $D$-module of the Grassmannian.

4.5. **Equivariant results.** The paper [54] also introduced a torus-equivariant version of the $G/P$ superpotential. It takes the form $F_T$ with $A$ diagonal maximal torus in our $A$-model $GL_n$. Note that $H^*_T(pt) = \mathfrak{h}^*$, which can be canonically identified with $\mathfrak{h}^*$, the Lie algebra of diagonal matrices in our $B$-model $\mathfrak{gl}_n(\mathbb{C})$. We also have an expression of this $T$-equivariant LG-model from [54] in terms of Plücker coordinates. In our running example of $X = Gr_2(\mathbb{C}^5)$ this looks as follows,

$$W_{q,x} = W_q + (x_1 + x_2)\ln(q) + (x_1 - x_2)\ln(p_{12}) + (x_2 - x_3)\ln(p_{23}) + (x_3 - x_4)\ln(p_{34}) + (x_4 - x_5)\ln(p_{45}).$$

Here the $x_i$ are the standard coordinates on $\mathfrak{h}$, which play the role of the equivariant parameters, that is the generators of $H^*_T(pt)$, in the equivariant quantum Monk’s rule [42]. In a future appendix will give a proof of the following equivariant version of our key result.

Let $*_q,x$ denote the equivariant quantum cohomology cup product. Then the equivariant quantum Monk’s rule reads

$$c^T_1(\mathcal{O}(1)) *_{q,x} \sigma^\lambda = \sum_\mu \sigma^\mu + q \sum_\nu \sigma^\nu + x_\lambda \sigma^\lambda,$$

where the $\mu$ and $\nu$ are as in the non-equivariant quantum Monk’s rule, and $x_\lambda$ is a particular linear combination of the $x_i$. In other words the small equivariant quantum cohomology ring $qH^*_T(X)$ is put together from the quantum cohomology ring and the equivariant cohomology ring without any mixing of quantum and equivariant parameters. This rule now reappears on the $B$-model side when replacing the superpotential $W_q$ with its equivariant version $W_{q,x}$.

**Theorem 4.8.** Consider the $\mathbb{C}[z, q, x]$-module

$$M = \Omega^N(\tilde{X})[z, q, x]/(d + \frac{1}{z}dW_{q,x})\Omega^{N-1}[z, q, x].$$

The Gauss-Manin connection in the $q$-direction defined for $\eta \in \Omega^N(\tilde{X})$ by

$$\nabla_{q\partial_q}([\eta]) = \left[q \frac{\partial}{\partial q} W_{q,x} \eta\right]$$

satisfies

$$\nabla_{q\partial_q}(p_{\lambda}\omega) = \sum_\mu p_{\mu}\omega + q \sum_\nu p_{\nu}\omega + x_\lambda p_{\lambda}\omega,$$

where $\mu, \nu$ and $x_\lambda$ are as in the equivariant quantum Monk’s rule for multiplication by $c^T_1(\mathcal{O}(1))$.

In analogy with Section 4.2 we obtain by this theorem solutions to the equivariant quantum differential equations

$$q \frac{\partial}{\partial q} S = \frac{1}{z} c^T_1(\mathcal{O}(1)) *_{q,x} S$$

which are of the form

$$S_{T,x}(z, q, x) := \frac{1}{(2\pi iz)^N} \sum_{\lambda \in P_{2n}} \left(\int_{\Gamma_x} e^{zW_{q,x} p_{\lambda}\omega}\right) \sigma^{P,D(\lambda)}.$$
In particular returning to the Richardson variety by the equivariant version of Theorem 1.0 we have that, under suitable convergence assumptions,

\[ \hat{S}_{\Gamma}(z, q, x) := \frac{1}{(2\pi i)^n} \sum_{\lambda \in \mathcal{P}_{k,n}} \left( \int_{\Gamma} e^{\frac{i}{2\pi i}(F_{\chi} + \ln(\phi(x, \omega)))} \sigma^{P_{D}(\lambda)} \right) \]

solves the equivariant quantum differential equations. Therefore

\[ S_A(\hat{S}_{\Gamma}(z, q, x), \sigma^0) = \int_{\Gamma} e^{\frac{1}{2\pi i}(F_{\chi} + \ln(\phi(x, \omega)))} \sigma \]

is a solution to the equivariant quantum cohomology D-module. This latter result proves part of Conjecture 8.2.

This concludes the summary of results.

5. The three versions of the B-model

There are three different definitions of a Landau-Ginzburg model dual to the A-model Grassmannian \( X = Gr_{n-k}(\mathbb{C}^n) \). We define them all in reverse chronological order, beginning with the one introduced in Section 3 and introduce relevant notation in each case. We also show how they are related to one another.

5.1. The Langlands dual Grassmannian B-model. The A-model Grassmannian, \( X = Gr_{n-k}(\mathbb{C}^n) \), is a homogeneous space for \( GL_n(\mathbb{C}) \) acting from the left. In the first instance we want to introduce the Landau-Ginzburg model taking place on the B-model Grassmannian. This Grassmannian is a Grassmannian of row vectors, \( \tilde{X} := Gr_{k}(\mathbb{C}^n)^* \), and we view it as homogeneous space for the Langlands dual -model Grassmannian, \( X = Gr_{n-k}(\mathbb{C}^n) \), acting from the right.

Elements of \( \tilde{X} \) may be represented by maximal rank \((k \times n)\)-matrices \( M \) in the usual way, with \( M \) representing its row-span. We think of \( \tilde{X} \) as embedded in \( \text{Proj}(\Lambda^k(\mathbb{C}^n)^*) \) by its Plücker embedding. The Plücker coordinates are all the maximal minors of \( M \), and are determined by a choice of \( k \) columns. We index the Plücker coordinates by partitions \( \lambda \in \mathcal{P}_{k,n} \) as follows. Associate to any partition \( \lambda \in \mathcal{P}_{k,n} \) a \( k \)-tuple in \( \{1, \ldots, n\} \) by interpreting \( \lambda \) as a path from the top right hand corner of its bounding \((n-k) \times k\) rectangle down to the bottom left hand corner, consisting of \( k \) horizontal and \( n-k \) vertical steps. The positions of the horizontal steps define a subset of \( k \) elements in \( \{1, \ldots, n\} \). We denote this subset, associated to \( \lambda \) by \( J_\lambda \). Suppose \( J_\lambda = \{j_1, \ldots, j_k\} \) with \( 1 \leq j_1 < \cdots < j_k \leq n \), then the Plücker coordinate associated to \( \lambda \) is defined to be the determinant of a \( k \times k \) submatrix of \( M \),

\[ p_\lambda(M) = \det((M_{i,j_k})_{i,k}) \].

A special role will be played by the \( n \) Plücker coordinates corresponding to the \( k \)-tuples which are (cyclic) intervals. These are \( J_i = L_{i+k} = (i+1, \ldots, i+k) \), where we denote by \( i \) the reduction of an integer \( i \) mod \( n \), and \( i \in \{1, \ldots, n\} \). The partition corresponding to \( J_i \) is denoted by \( \mu_i \) or \( \mu_{i,i+1} \). For example since \( J_1 = \{2, \ldots, k+1\} \) we have \( \mu_1 = (k) \), the maximal partition with one part. If \( n-k \geq 2 \), then \( \mu_2 \) is the maximal two row partition, \((k, k)\). For \( k = 3 \) and \( n = 7 \) we have for example,

\[ \mu_1 = \boxtimes, \quad \mu_2 = \boxed{\cdot}, \quad \mu_3 = \boxed{\cdot}, \quad \mu_4 = \boxed{\cdot}, \quad \mu_5 = \boxed{\cdot}, \quad \mu_6 = \boxed{\cdot}, \quad \mu_7 = \emptyset. \]

Always \( \mu_{n-k} \) is the maximal rectangle, and \( \mu_n \) is the empty partition. The Plücker coordinates corresponding to other rectangular Young diagrams will also play a special role. We denote the Plücker coordinate corresponding to an \( i \times j \) rectangle by \( p_{i \times j} \). So in the example where \((k, n) = (3, 7)\) we have \( p_{\mu_1} = p_{1 \times 3}, p_{\mu_2} = p_{2 \times 3} \) and so forth.

Each \( p_{\mu_i} \) is a section of \( O(1) \) for the Plücker embedding. Since the index of \( \tilde{X} \) is \( n \), the union of the hyperplane sections is an anticanonical divisor

\[ D = \{p_{\mu_1} = 0\} \cup \{p_{\mu_2} = 0\} \cup \ldots \cup \{p_{\mu_n} = 0\}. \]

Let \( \tilde{X} \) be the Zariski-open subset of \( \tilde{X} \), obtained by removing the anti-canonical divisor \((5.1)\). So

\[ \tilde{X} := \tilde{X} \setminus D = \{M \in \tilde{X} | p_{\mu_i}(M) \neq 0, \text{ all } i = 1, \ldots, n \}. \]
Note that the anticanonical divisor $D$ and its complement are invariant under the $\mathbb{Z}/n\mathbb{Z}$-action on $\hat{X}$ given by the cyclic shift

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1,n} \\
m_{21} & m_{22} & m_{23} & \cdots & m_{2,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_{k1} & m_{k2} & m_{k3} & \cdots & m_{k,n} \end{pmatrix} \mapsto M[1] = \begin{pmatrix} m_{12} & m_{13} & \cdots & m_{1,n} & m_{11} \\
m_{22} & m_{23} & \cdots & m_{2,n} & m_{21} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m_{k2} & m_{k3} & \cdots & m_{k,n} & m_{k1} \end{pmatrix}.$$  

The coordinate ring of $\hat{X}$ is described in detail in Section 7.

Our version of the Landau-Ginzburg model mirror dual to $X$ is a regular function $W : \hat{X} \times \mathbb{C}_q \to \mathbb{C}$ defined as follows. Denote by $\mu_i^{\mu}$ or $\mu_i$ or $\hat{\mu}_i$ the partition corresponding to $\hat{J}_i = \hat{L}_{i+1} := (i+1, i+2, \ldots, i+k-1, i+k+1)$. Unless $i \neq n-k$, the Young diagram of $\mu_i^{\mu}$ is obtained from the Young diagram of $\mu_i$ by adding a box. The particular shape of $\mu_i$ guarantees that there is only one way to do this. The partition $\mu_{n-k}$ is obtained by removing the entire rim from $\mu_{n-k}$ to give an $(n-k-1) \times (k-1)$ rectangle. We define

$$W := \sum_{i=1}^n \frac{p_{\mu_i^{\mu}}}{p_{\mu_i}} q^{\delta_i, n-k} = \sum_{\mu_i \neq n-k} \frac{p_{\mu_i^{\mu}}}{p_{\mu_i}} + q \frac{p_{\mu_{n-k}^{\mu}}}{p_{\mu_{n-k}}}$$  

Remark 5.1. Notice that in the quantum Schubert calculus of $X$ and for $i \neq n-k$,

$$\sigma_{\mathfrak{g}_{-1}}^{\mu_i^{\mu}} = \sigma_{\mu_i^{\mu}}, \quad \sigma_{\mathfrak{g}_{-1}}^{\mu_{n-k}^{\mu}} = q \sigma_{\mu_{n-k}^{\mu}}.$$  

We will now recall the definitions of the two earlier conjectured Landau-Ginzburg models for Grassmannians, starting with the LG-model of [54] followed by that of Eguchi, Hori and Xiong [15] [2], and explain how these previous definitions relate to this new one.

5.2. Richardson variety $B$-model. Let the (B-model) group $GL_n^\vee(\mathbb{C})$ act (now from the left) on a full flag variety. We fix some notation regarding this group $GL_n^\vee$. We let $B_+, B_-$ denote its upper-triangular and lower-triangular Borel subgroups, respectively, and $T$ denote the maximal torus of diagonal matrices. The unipotent radicals of $B_+$ and $B_-$ are denoted $U_+$ and $U_-$. Let $\mathfrak{h}$ denote the Lie algebra of $T$. We let $E_{i,j}$ denote the matrix with entry 1 in row $i$ and column $j$ and zeros elsewhere. Let $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$ be the usual Chevalley elements of $\mathfrak{g}_{\mathbb{C}}$, and let $\alpha_i \in \mathfrak{h}^*$ be the simple root corresponding to $e_i$. We define the usual 1-parameter subgroups $x_i : \mathbb{C} \to U_+$ and $y_i : \mathbb{C} \to U_-$,

$$x_i(t) = \exp(te_i), \quad y_i(t) = \exp(tf_i)$$  

for $i = 1, \ldots, n-1$. Let $W = N_{GL_n}(T)/T \cong S_n$ be the Weyl group, and choose the following representatives,

$$\hat{s}_i = x_i(1)y_i(-1)x_i(1).$$  

The $s_i = \hat{s}_iT$ generate $W$. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function. If $\ell(w) = m$ and $s_{i_1} \ldots s_{i_m}$ is a reduced expression for $w$, then the product $w = \hat{s}_{i_1} \ldots \hat{s}_{i_m}$ is a well defined element of $GL_n^\vee$ and independent of the reduced expression chosen. The longest element of $W$ is denoted $w_0$.

We will also require the root subgroups

$$x_{\alpha_j + \alpha_{j+1} + \cdots + \alpha_k}(a) := \hat{s}_{j+1}^{-1}\hat{s}_{j+2}^{-1} \cdots \hat{s}_{k-1}^{-1}x_k(a)\hat{s}_{k-1} \cdots \hat{s}_{j+2}\hat{s}_{j+1} = id + aE_{j,k+1}.$$  

We define $e_i^* : U_+ \to \mathbb{C}$ to be the map sending $u \in U_+$ to its $(i, i+1)$-entry, so $e_i^*(u) = u_{i,i+1}$. This entry is also the coefficient of $e_i$ in after embedding $U_+$ into the completed universal enveloping algebra of its Lie algebra, hence the notation.

Let $P \supset B_+$ be the maximal parabolic subgroup in $G$ generated by $B_+$ and the elements $\hat{s}_i$ for $i \neq n-k$. We also have $W_P = \langle s_i \mid i \neq n-k \rangle$, the corresponding parabolic subgroup of the Weyl group $W$. The longest element in $W_P$ is denoted $w_P$. Let $W^P$ denote the set of minimal length coset representatives in $W/W_P$. The longest element in $W^P$ is denoted $w^P$. Clearly $w^Pw_P = w_0$, the longest element of $W$.

We consider the closed Richardson variety, which is actually a Schubert variety inside $GL_n^\vee/B_-$, defined by,

$$\hat{X}_{w_P} = \overline{B_+w_PB_-/B_-}.$$
We think of $\tilde{X}_{w_p} \subset GL_n^G/B_-$ as Richardson variety, namely as closure of the intersection,

$$\tilde{X} = \mathcal{R}_{w_p, w_0} = B_+ \dot{w}_P B_- \cap B_- \dot{w}_0 B_- / B_-,$$

of opposite Bruhat cells. This intersection, $\tilde{X}$, is a smooth irreducible variety of dimension $\ell(w^P)$, namely $k(n - k)$ for our choice of $P$.

Let $C = C_{GL^n}$ denote the center of $GL_n^G$ and note that we have an isomorphism $T^{w_P} / C \rightarrow C_q^*$ via $q = \alpha_{n-k}$. Then we also have an isomorphism

$$\psi : (B_- \cap U_+ T^{w_P}(\dot{w}_P)^{-1} U_+) / C \xrightarrow{\sim} \tilde{X} \times C_q^*,$$

$$bC = u_1 t(\dot{w}_P)^{-1} u_2 C \mapsto (b\dot{w}_0 B_-, \alpha_{n-k}(t)).$$

**Definition 5.2** ([54]). For the A-model Grassmannian $X = Gr_{n-k}(\mathbb{C}^n)$ viewed as homogeneous space $GL_n / P$, the mirror superpotential is the map

$$\mathcal{F} : \tilde{X} \times C_q^* \rightarrow \mathbb{C}$$

defined as

$$\mathcal{F}(b\dot{w}_0 B_-, q) = \sum e_i^*(u_1) + \sum e_i^*(u_2),$$

if $b \in B_-$ factorizes as $b = u_1 t(\dot{w}_P)^{-1} u_2$ with $\alpha_{n-k}(t) = q$ and $u_1, u_2 \in U_+$. Note that this $\mathcal{F}$ is well-defined even though $u_1$ and $u_2$ are not uniquely determined by $b\dot{w}_0 B_-$ and $q$, see [54] Lemma 5.2.

Another way to express this function $\mathcal{F}$ is using the map

$$\mu : U_+ \cap B_- \dot{w}_0 w_P B_- \rightarrow U_+, \quad u_1 \mapsto u_{2,0}$$

defined by the condition that $u_{2,0} B_- = \dot{w}_0 w_P^{-1} u_1^{-1} B_-$. Note that then for $t \in T^{w_P}$ we have

$$u_1 t \dot{w}_P \dot{w}_0^{-1} (w_0 t)^{-1} u_{2,0} (w_0 t) = u_1 \dot{w}_P \dot{w}_0^{-1} u_{2,0} (w_0 t)^{-1} \in B_-.$$ 

If $u_1 \in U_+ \cap B_- \dot{w}_0 w_P B_-$, and $t \in T^{w_P}$ has $\alpha_{n-k}(t) = q$, then this means that the $u_2$ in the original definition of $\mathcal{F}$ is given by $u_2 = (w_0 t)^{-1} u_{2,0} (w_0 t)$. Therefore

$$\mathcal{F}(u_1 \dot{w}_P B_-, q) = \left( \sum_i e_i^*(u_1) \right) + \left( \sum_i e_i^*((w_0 t)^{-1} u_{2,0} (w_0 t)) \right) = \left( \sum_i e_i^*(u_1) \right) + \left( \sum_{i \neq k} e_i^*(u_{2,0}) \right) + q e_k^*(u_{2,0}),$$

where $u_{2,0} = \mu(u_1)$.

We recall the Peterson presentation, [46] of the quantum cohomology ring of $X = Gr_{n-k}(n)$, compare [51].

**Theorem 5.3.** [46] Let $Y^*_p$ be the Peterson variety associated to $X$ defined as follows, using the coadjoint action of $G$ on elements $e_i^*$ dual to the Chevalley generators $e_i$,

$$Y^*_p := \{ gB_- \in \tilde{X} \mid g^{-1} \cdot (\sum_i e_i^*) \in [n_-, n_-]^{-1} \}.$$ 

Then $qH^*(X, \mathbb{C})[q^{-1}] \cong \mathbb{C}[Y^*_p]$.

We refer to [51] for a detailed description of the isomorphism including the image of the Schubert classes and of $q$. We think $Y^*_p$ as coming with a finite morphism $Y^*_p \rightarrow C^*$ given by $q \in \mathbb{C}[Y^*_p]$. A result from [54] relates $\mathcal{F}$ to the Peterson variety.

**Theorem 5.4.** [54] The critical points inside $\tilde{X}$ of $\mathcal{F}$ lie in the Peterson variety and precisely recover the fibers of $q : Y^*_p \rightarrow C^*$. Moreover the restriction of the first projection $\tilde{X} \times C_q^* \rightarrow \tilde{X}$ to the subvariety of $\tilde{X} \times C_q^*$ defined by the partial derivatives $(\partial_{\tilde{R}} \mathcal{F})$ defines an isomorphism with $Y^*_p$ and we obtain

$$\mathbb{C}[\tilde{X} \times C_q^*] / (\partial_{\tilde{R}} \mathcal{F}) \cong \mathbb{C}[Y^*_p].$$
We want to define maps

\[ \hat{X} \xrightarrow{\pi_L} B_- \cap U_+ \hat{w}_P \hat{w}_0^{-1} U_+ \xrightarrow{\pi_R} \mathcal{R} \]

by setting \( \pi_L(b) := Pb \) and \( \pi_R(b) = b \hat{w}_0 B_- \). It is straightforward that \( \pi_R \) is well-defined and an isomorphism. The map \( \pi_L \) is a priori a map to \( \hat{X} \). However it lands in \( \hat{X} \) as a special case of a result by Knutson, Lam and Speyer, who studied projection maps from Richardson varieties to their ‘positroid varieties’ [37]. Their results imply that \( \pi_L \) is an isomorphism as well, so that we have \( \hat{X} \cong \mathcal{R} \).

**Proposition 5.5** ([37]). The map \( \pi_L \) is a well-defined isomorphism from \( B_- \cap U_+ \hat{w}_P \hat{w}_0^{-1} U_+ \) to \( \hat{X} \).

In Section 5.4 we will prove the following proposition.

**Proposition 5.6.** With the definitions from Section 5.1 and 5.2, the following diagram commutes:

\[ \begin{array}{ccc}
\hat{X} \times C_q^* & \xrightarrow{\pi_L \times \text{id}} & B_- \cap U_+ \hat{w}_P \hat{w}_0^{-1} U_+ \times C_q^* \\
W & \downarrow \cong & \mathcal{R} \times C_q^*
\end{array} \]

The Propositions 5.5 and 5.6 imply Theorem 4.6, that the Landau-Ginzburg model from Section 5.1 is isomorphic to the one from [54], recalled Definition 5.2. In the case of Lagrangian Grassmannians and for odd-dimensional quadrics, analogous formulas to Definition 5.2 and comparison results to the above Propositions were found by Pech and the second author in [14] [45].

5.3. **Laurent polynomial B-model.** The earliest construction of Landau-Ginzburg models for Grassmannians is due to Eguchi, Hori and Xiong [15] and associates to \( X = Gr_{n-k}(\mathbb{C}^n) \) a Laurent polynomial \( \mathcal{W} \) in \( k(n-k) \) variables (and parameter \( q \)). We let \( \mathcal{T} = (\mathbb{C}^*)^{k(n-k)} \) and define \( \mathcal{W} : \mathcal{T} \times C_q^* \to \mathbb{C} \) as follows (compare [24] [4]).

Let \( (V, A) \) be a quiver with vertices given by

\[ V = \{(i,j) \in \{1, \ldots, n-k\} \times \{1, \ldots, k\} \} \sqcup \{(0,1), (n-k,k+1)\} \]

and with two types of arrows \( a \in A \), namely

\[ (i,j) \to (i,j + 1) \quad \text{and} \quad (i,j) \to (i+1,j), \]

defined whenever \((i,j), (i,j + 1)\), and \((i,j), (i+1,j)\), respectively, are in \( V \). We write \( h(a) \) for the head of the arrow \( a \), and \( t(a) \) for the tail. To every vertex in the quiver associate a coordinate \( d_{i,j} \). We set \( d_{0,1} = 1 \) and \( d_{n-k,k+1} = q \), and let the remaining \( (d_{i,j})_{i=1,\ldots,n-k} \) be the coordinates on the big torus \( \mathcal{T} = (\mathbb{C}^*)^{k(n-k)} \).

**Definition 5.7.** [24] [15] Associate to every arrow \( a \in A \) a Laurent monomial by dividing the coordinate at the head by the coordinate at the tail. The regular function \( \mathcal{W} : \mathcal{T} \times C_q^* \to \mathbb{C} \) is defined to be the sum of the resulting Laurent monomials,

\[ \mathcal{W} = \sum_{a \in A} \frac{d_{h(a)}}{d_{t(a)}}, \]

keeping in mind that \( d_{0,1} = 1 \) and \( d_{n-k,k+1} = q \).

**Example 5.8.** Consider \( k = 3 \) and \( n = 5 \). So \( X = Gr_2(\mathbb{C}^5) \) and \( \hat{X} = Gr_3((\mathbb{C}^5)^*) \), the Grassmannian of 3-planes in the vector space of row vectors. The big torus is \( \mathcal{T} \cong (\mathbb{C}^*)^6 \) with coordinates \((d_{11}, d_{12}, d_{13}, d_{21}, d_{22}, d_{23})\). The superpotential is

\[ \mathcal{W} = d_{11} + \frac{d_{21}}{d_{11}} + \frac{d_{22}}{d_{12}} + \frac{d_{13}}{d_{23}} + \frac{d_{12}}{d_{11}} + \frac{d_{13}}{d_{12}} + \frac{d_{22}}{d_{21}} + \frac{d_{23}}{d_{22}} + q, \]

and is encoded in the quiver shown below.
The relationship between this superpotential $W$ and the Plücker coordinate superpotential $\tilde{W}$ defined by (5.2) is given in the following proposition.

**Proposition 5.9.** There is a (unique) embedding $\iota: T \rightarrow \tilde{X}$ for which the Plücker coordinates corresponding to rectangular Young diagrams are related to the $d_{ij}$ coordinates as follows,

$$
  d_{ij} = \frac{p_{i \times j}}{p(i-1) \times (j-1)}.
$$

The pullback of $W$ to $T \times \mathbb{C}^*_q$ under $\iota \times id$ is $\tilde{W}$.

**Sketch of proof.** We describe the embedding $\iota: T \rightarrow \tilde{X}$ defined in Proposition 5.9 concretely, using a construction similar to one defined earlier in [52], of a map $T \rightarrow \mathcal{R}$. To explain the construction in our setting we continue with Example 5.8. Let us decorate the above quiver by elements $\hat{s}_i$ as follows and remove the arrow with head labeled $q$.

```
Row 1
  \hat{s}_1
Row 2
  \hat{s}_2  \hat{s}_2
Row 3
  1  d_{11}  d_{12}  d_{13}
Row 4
  d_{21}  d_{22}  d_{23}  \hat{s}_4
```

We call a path in the quiver which has precisely one vertical step a 1-path. Notice that each 1-path traverses a well-defined row. Then we read off a sequence of $\hat{s}_i$'s and of 1-paths, going column by column from right to left. Namely in each column we list, starting from the top row going down row by row, any $\hat{s}_i$'s followed by the rightmost 1-paths which end in that column. In this example the sequence is:

```
  \hat{s}_4  \hat{s}_2  \hat{s}_2  \hat{s}_1
```

To a 1-path $\gamma$ in row $i$ we associate a factor $x_i \left( \frac{d_{t(\gamma)}}{d_{h(\gamma)}} \right)$, where $t(\gamma)$ is the starting point and $h(\gamma)$ is the ending point of $\gamma$. The other elements of the sequence correspond in the obvious way to factors $\hat{s}_i$. These factors multiplied together to give an element $g(d_{ij})$ of $GL_n^\vee$. In the above example we have the element of $GL_5^\vee$ given by

$$
  g(d_{ij}) := \hat{s}_4 x_3 (d_{13}) x_4 \left( \frac{d_{23}}{d_{13}} \right) \hat{s}_2 x_3 (d_{12}) x_4 \left( \frac{d_{22}}{d_{12}} \right) \hat{s}_1 \hat{s}_2 x_3 (d_{11}) x_4 \left( \frac{d_{21}}{d_{11}} \right).
$$
Finally the map $\iota : \mathcal{T} \rightarrow \bar{X}$ is given by

$$
\iota : (d_{ij}) \mapsto P \hat{w}_0 g(d_{ij}).
$$

To see that this map is the embedding alluded to in the Proposition it suffices to check that the $d_{ij}$ are related to Plücker coordinates of $P \hat{w}_0 g(d_{ij})$ as follows,

\begin{equation}
\begin{align*}
\hat{d}_{11} &= \frac{p_0}{\hat{p}_0}, \\
\hat{d}_{12} &= \frac{\hat{p}_0}{p_0}, \\
\hat{d}_{13} &= \frac{\hat{p}_0}{p_0}, \\
\hat{d}_{21} &= \frac{\hat{p}_0}{p_0}, \\
\hat{d}_{22} &= \frac{\hat{p}_0}{p_0}, \\
\hat{d}_{23} &= \frac{\hat{p}_0}{p_0}.
\end{align*}
\end{equation}

This holds in general as is straightforward to check. Similarly substituting $P \hat{w}_0 g(d_{ij})$ into the formula $[5.2]$ for $W$ can be checked to recover the Laurent polynomial $W$. \hfill \Box

5.4. Comparison of the Grassmannian $B$-model with the Richardson variety $B$-model. In this section we prove Proposition $5.3$ which says that the superpotential $W$ defined in $[5.2]$ coincides with the Richardson variety superpotential $\mathcal{F}$ from $[5.4]$.

Let $\bar{F}$ denote the pull back $\mathcal{F}$ to $\bar{X} \times \mathbb{C}^*_q$ via $(\pi_L \times id)^{-1} \circ (\pi_R \times id)$, where

$$
\bar{X} \times \mathbb{C}^*_q \xrightarrow{\pi_l \times id} B_\sim \cap U_+ \hat{w}_p \hat{w}_0^{-1} U_+ \times \mathbb{C}_q \xrightarrow{\pi_R \times id} \mathcal{R} \times \mathbb{C}^*_q,
$$

were defined in Section $5.2$ and we are using Proposition $5.5$. Then Proposition $5.6$ says that $\bar{F}$ agrees with $W$.

It suffices to show that $\bar{F}$ and $W$ agree on an open dense subset of $\bar{X}$. Therefore we may assume without loss of generality that $Pg \in \bar{X}$ is of the form

$$
P_g = P\hat{w}_0^{-1}g(d_{ij})
$$

for an element $(d_{ij}) \in \mathcal{T}$, compare Section $5.3$. We call this subset $\hat{X}^f act$. Any element, $\hat{w}_0^{-1}g(d_{ij})$ can also be written in the form, $\hat{w}_p \hat{w}_0^{-1}u_2$ for $u_2 \in U_+$ factorized as $u(1) \cdots u(k)$, where each $u(j)$ is in a product of root subgroups,

$$
u(j) = x_{\alpha_j + \cdots + \alpha_k} (m_{jk}) x_{k+1} (m_{j,k+1}) \cdots x_{n-1} (m_{j,n-1}).
$$

Let $U^f act_+ \subset U_+$ denote the subset of such factorized elements $u_2 = u(1) \cdots u(k)$, with nonzero $m_{j,t}$. Then we have $\hat{X}^f act = P\hat{w}_p \hat{w}_0^{-1}U^f act$, and we may assume

$$
P_g = P\hat{w}_p \hat{w}_0^{-1}u_2,
$$

where $u_2 \in U^f act_+$.

We can now define a map

$$
\hat{\mu} : U^f act_+ \rightarrow U_+
$$

by letting $\hat{\mu}(u_2) := u_{1,0}$ where $u_{1,0} \in U_+$ such that $u_{1,0}^{-1}B_\sim = \hat{w}_p \hat{w}_0^{-1}u_2 B_\sim$. Notice that $\hat{w}_p \hat{w}_0^{-1}u_2 B_\sim$ lies in $U_+ B_\sim / B_\sim$ since it equals $\hat{w}_p^{-1} g(d_{ij}) B_\sim \hat{w}_0^{-1} B_\sim \hat{w}_0 B_\sim = U_+ B_\sim$, and therefore $\hat{\mu}$ is well-defined.

Comparing with Definition $5.2$ we see that $\bar{F}$ on $P\hat{w}_p \hat{w}_0^{-1}u_2$ is given by the formula

$$
\bar{F}(P\hat{w}_p \hat{w}_0^{-1}u_2) = \sum_i e^*_i(u_2) + \sum_{i \neq n-k} e^*_i(u_{1,0}) + qe^*_{n-k}(u_{1,0}).
$$

Let $\Delta^f_J(g)$ denote the minor with row set $I$ and column set $J$. We then have the following lemma about minors.

**Lemma 5.10.** Let $u_2 \in U^f act_+$ and $u_{1,0} = \hat{\mu}(u_2)$ and $b = u_{1,0} \hat{w}_p \hat{w}_0^{-1}u_2 \in B_\sim$. Then we have

(a): \[ e^*_i(u_{1,0}) = \begin{cases} 
0 & 1 \leq i \leq n - k - 1; \\
\frac{\Delta^f_{n-k+1} \cdots \Delta^f_{i+1}}{\Delta^f_{n-k+1} \cdots \Delta^f_{n-k+1}(b)} & n - k \leq i \leq n - 1.
\end{cases} \]
We leave the details of the proof to the reader.

(Recall that $J_i = \{i+1, \ldots, i+k\}$ and $J_\infty = \{i+1, \ldots, i+k-1, i+k+1\}$.) The proof is a combination of the special shape of $u_2$ and the factorisation $b = u_{1,0}w_p\hat{w}_0^{-1}u_2 \in B_\infty$. For example the equality (b) for $k \leq i \leq n-1$ is the consequence of the vanishing of a $(k+1) \times (k+1)$-minor of $\hat{w}_0^{-1}u_2$ which takes the form

$$e_i^*(u_2)\Delta_{\sum_{k=b}^{n-k+1}}(\hat{w}_0^{-1}u_2) - \Delta_{\sum_{k=b}^{n-k+1}}(\hat{w}_0^{-1}u_2).$$

We leave the details of the proof to the reader.

Proposition 5.6 now follows immediately, since the $\Delta_{\sum_{k=b}^{n-k+1}}(\hat{w}_0^{-1}u_2)$ are nothing other than the summands of $W(Pb)$ as defined in (6.2). Therefore this concludes the proof of Theorem 4.6.

5.5. A consequence of the comparison result. Recall Definitions 3.1 and 3.2 of the Gauss-Manin system $G^W$ and its submodule $\overline{G}^W$. We can now quickly deduce Lemma 3.3. Namely we have:

**Lemma 5.11.** $\overline{G}^W_0$ is a free $\mathbb{C}[z,q]$-module with basis $\{[p_\lambda\omega], \lambda \in P_{k,n}\}$ and

$$\overline{G}^W = \overline{G}^W_0 \otimes_{\mathbb{C}[z,q]} \mathbb{C}[z^{\pm 1}, q^{\pm 1}].$$

In particular $\overline{G}^W$ is a free $\mathbb{C}[z^{\pm 1}, q^{\pm 1}]$-module with basis $\{[p_\lambda\omega], \lambda \in P_{k,n}\}$.

**Proof.** By the comparison result, Theorem 4.6 the Jacobi ring $\mathbb{C}[\hat{X} \times \mathbb{C}_q^*/(\partial Z W)$ is isomorphic to the Jacobi ring $\mathbb{C}[\hat{X} \times \mathbb{C}_q^*/(\partial R \mathcal{F})$. Therefore by Theorem 5.3 and Theorem 5.4 the Jacobi ring $\mathbb{C}[\hat{X} \times \mathbb{C}_q^*/(\partial Z W)$ is isomorphic to $qH^*(X, \mathbb{C})$. Tracing through these isomorphisms we see that the Schubert class $\sigma^\lambda$ goes to the class of the Plücker coordinate $p_\lambda$ in $\mathbb{C}[\hat{X} \times \mathbb{C}_q^*/(\partial Z W)$. Now since setting $z = 0$ in $\overline{G}^W_0$ recovers precisely this Jacobi ring of $W$, and therefore the $[p_\lambda\omega]$ are a free $\mathbb{C}[q]$-module basis at $z = 0$. The Lemma follows. \[\square\]

6. Begin of the proof of Theorem 4.1.

Our main aim in Sections 4 to 9 is to show that the following hold for all $\lambda \in P_{k,n}$ (Theorem 9.2):

\begin{equation}
\begin{aligned}
[q \frac{\partial W}{\partial q} p_\lambda \omega] &= \sum_\mu [p_\mu \omega] + q \sum_\nu [p_\nu \omega]; \\
\frac{1}{z}[W p_\lambda \omega] &= \sum_\mu [p_\mu \omega] + q \sum_\nu [p_\nu \omega] - |\lambda|[p_\lambda \omega],
\end{aligned}
\end{equation}

where $\mu, \nu$ are exactly as in the quantum Monk’s rule for $\sigma^{\Delta}_q \ast_q \sigma^\lambda$.

In Section 7 we recall the cluster structure on the Grassmannian, following Scott [58]. In Section 8 we use the cluster structure to define a vector field $X_\lambda$ on the Grassmannian, together with twisted versions $X^{(m)}_\lambda$, $m \in [1, n]$ (with $X^{(n)}_\lambda = X_\lambda$). We first show that:

\begin{equation}
\begin{aligned}
X_\lambda W_q &= \left( \sum_\mu p_\mu + q \sum_\nu p_\nu \right) - q \frac{\partial W_q}{\partial q} p_\lambda; \\
n \sum_{m=1}^n X^{(m)}_\lambda W_q &= n \left( \sum_\mu p_\mu + q \sum_\nu p_\nu \right) - W_q p_\lambda,
\end{aligned}
\end{equation}

where, in each case, $\mu, \nu$ are exactly as in the quantum Monk’s rule for $\sigma^{\Delta}_q \ast_q \sigma^\lambda$. This leads to the results (6.1), (6.2) (see Theorem 9.7) via an insertion argument.
7. The coordinate ring $\mathbb{C}[\bar{X}]$ as a cluster algebra

By [58], the coordinate ring of $\bar{X}$ has a cluster algebra structure (see also [20, 21]), which we now recall. A skew-symmetric cluster algebra with coefficients is defined as follows [17]. Let $r, m \in \mathbb{N}$ and consider the field $\mathbb{F} = \mathbb{C}(u_1, \ldots, u_{r+m})$ of rational functions in $r + m$ indeterminates. A seed in $\mathbb{F}$ is a pair $(\bar{x}, \bar{Q})$ where $\bar{x} = \{x_1, x_2, \ldots, x_{r+m}\}$ is a free generating set of $\mathbb{F}$ over $\mathbb{C}$ and $\bar{Q}$ is a quiver with vertices $1, 2, \ldots, r + m$. The vertices $r + 1, \ldots, r + m$ are said to be frozen. The subset $x = \{x_1, x_2, \ldots, x_r\}$ of $\bar{x}$ is known as a cluster while $\bar{x}$ is known as an extended cluster.

Given $1 \leq k \leq r$, the seed $(\bar{x}, \bar{Q})$ can be mutated at $k$ to produce a new seed $\mu_k(\bar{x}, \bar{Q}) = (\bar{x}', \mu_k \bar{Q})$ where $\bar{x}' = (\bar{x} \setminus \{x_k\}) \cup \{x'_k\}$, and

$$x_k x'_k = \prod_{i \to k} x_i + \prod_{k \to i} x_i.$$

The new quiver, $\mu_k \bar{Q}$, is obtained from $\bar{Q}$ as follows:

1. For every path $i \to k \to j$ in $\bar{Q}$, add an arrow $i \to j$ (with multiplicity).
2. Reverse all arrows incident with $k$.
3. Cancel a maximal collection of 2-cycles in the resulting quiver.

The cluster algebra associated to $(\bar{x}, \bar{B})$ is the $\mathbb{C}$-subalgebra of $\mathbb{F}$ generated by the elements of all of the extended clusters which can be obtained from $(\bar{x}, \bar{Q})$ by arbitrary finite sequences of mutations. Note that in the usual definition, the cluster algebra is defined as a subring or $\mathbb{Q}$-subalgebra, but in the context here this is a good definition since we want to work with coordinate rings over $\mathbb{C}$.

Recall that in the B-model we are working with $\bar{X} = P\setminus GL^*_n$, a Grassmannian of $k$-planes in $(\mathbb{C}^n)^*$, in its Plücker embedding. We have the following:

**Theorem 7.1.** [58] (see also [20, 21]). The homogeneous coordinate ring $\mathbb{C}[\bar{X}]$ is a cluster algebra.

We follow [58], which describes the cluster structure on $\mathbb{C}[P\setminus GL^*_n]$ in terms of Postnikov diagrams, i.e. alternating strand diagrams from [17]. We restrict here to the Postnikov diagrams arising in the cluster structure of the Grassmannian.

**Definition 7.2.** A Postnikov diagram of type $(k,n)$ consists of a disk with $2n$ marked points $b_1, b'_1, b_2, b'_2, \ldots, b_n, b'_n$ marked clockwise on its boundary, together with $n$ oriented curves in the disk, known as strands. Strand $i$ starts at $b_i$ or $b'_i$ and ends at $b_{i+k}$ or $b'_{i+k}$. The arrangement must satisfy the following additional conditions:

(a) Only two strands can intersect at any given point and all such crossings must be transversal.
(b) There are finitely many crossing points.
(c) If strand $i$ starts at $b_i$ (respectively, $b'_i$), the first strand crossing it (if such a strand exists) comes from the right (respectively, left). Similarly, if strand $i$ ends at $b_{i+k}$ (respectively, $b'_{i+k}$), the last string crossing it (if such a strand exists) comes from the right (respectively, left). Following a strand from its starting point to its ending point, the crossings alternate in sign.
(d) A strand has no self-crossings.
(e) If two strands cross at successive points $X$ and $Y$, then one strand is oriented from $X$ to $Y$, while the other is oriented from $Y$ to $X$.

Postnikov diagrams are considered up to isotopy (noting that such an isotopy can neither create nor delete crossings), together with equivalence under the twisting/untwisting move and the boundary twist; see Figures 1 and 2. These moves are local in the sense that no other strands must cross the strands involved in the area where the rule is applied.

When actually drawing Postnikov diagrams, we usually drop the labels of the vertices $b_i$ and $b'_i$ and instead indicate the start of strand $i$ by writing an $i$ in a circle (i.e. at $b_i$ or $b'_i$) and the end of strand $i$ (i.e. at $b_{i+k}$ or $b'_{i+k}$) by writing $i$ in a rectangle. We draw a dotted line between $b_i$ and $b'_i$ to make it clearer where they are. Thus each $i$ in a rectangle should be linked by a dotted line to $i + k$ in a circle. For an example of a Postnikov diagram, of type $(3,6)$, see Figure 3.
Figure 1. The twisting/untwisting move

Figure 2. The boundary twist

Figure 3. A Postnikov diagram for $Gr_3(6)$.

The complement of a Postnikov diagram in the disk is a union of disks, called faces. A face whose boundary includes part of the boundary of the disk is called a boundary face. A face whose boundary (excluding the boundary of the disk) is oriented (respectively, alternating) is said to be an oriented (respectively, alternating) face; it is easy to check that all faces are of one of these types.

We label each alternating face $F$ with the subset $L(F)$ of $\{1, 2, \ldots, n\}$ which contains $i$ if and only if $F$ lies to the left of strand $i$. The corresponding minor is denoted $p_F = p_{L(F)}$.

The geometric exchange on a Postnikov diagram is the move given in Figure 4.

We recall the following (see [58, Props. 5 and 6]).

**Theorem 7.3. (Postnikov)**

(a) Each Postnikov diagram of type $(k, n)$ has exactly $k(n - k) + 1$ alternating faces.
(b) Each alternating face is labelled by a $k$-subset of $\{1, 2, \ldots, n\}$.
(c) Every $k$-subset of $\{1, 2, \ldots, n\}$ appears as the label of an alternating face in some Postnikov diagram of type $(k, n)$. 
(d) Any two Postnikov diagrams of type \((k,n)\) are connected by a sequence of geometric exchanges.

(e) The labels of the faces on the boundary of any Postnikov diagram are the \(L_i = \{i-k+1, \ldots, i\}\) for \(i = 1, 2, \ldots, n\). In fact, \(L_i\) labels the face between \(b_i'\) and \(b_i+1\).

Scott [58, Sect. 5] defines a quiver \(Q = Q(D)\) for any Postnikov diagram \(D\). The vertices of \(Q\) are the alternating faces of \(D\). If two faces \(X, Y\) of \(D\) are related as in Figure 5 then there is an arrow in \(Q\) from \(X\) to \(Y\). We call arrows between coefficient vertices boundary arrows, and all other arrows internal arrows.

We consider the field \(\mathbb{F}\) given by adjoining indeterminates \(u_I\) for \(I\) the label of an alternating face in \(D\) to \(\mathbb{C}\). Let \(\tilde{x}(D)\) be the free generating set for \(\mathbb{F}\) containing these indeterminates, where we regard the indeterminates corresponding to boundary faces (the \(L_i\)) as frozen variables. Note that there are \(k(n-k)-n+1\) nonfrozen variables and \(n\) frozen variables, making a total of \(k(n-k)+1\) variables. Each variable is naturally associated to an alternating face of \(D\) and thus to a vertex of \(Q\).

Let \(A(D)\) be the cluster algebra corresponding to the initial seed \((\tilde{x}(D), Q(D))\). The Postnikov diagram \(D\) can be thought of as a base diagram for the definition of \(A(D)\): it does not matter which Postnikov diagram of type \((k,n)\) that we take, but we choose one. Then we have:

**Theorem 7.4.** [58, Thm. 2]

(a) Let \(D\) be a Postnikov diagram of type \((k,n)\). Then there is an isomorphism \(\varphi_D\) from \(\mathbb{C}[P^\vee \setminus G^\vee]\) to \(A(D)\).

(b) Let \(D'\) be any Postnikov diagram of type \((k,n)\) and let \(M(D')\) be the collection of minors corresponding to the \(k\)-subsets labelling the alternating faces of \(D'\). Then \(S(D') := (\varphi(M(D')), Q(D'))\) is a seed of \(A(D)\) (via the isomorphisms \(\varphi_D\), \(\varphi_{D'}\) from part (a)).

(c) If \(D', D''\) are two Postnikov diagrams of type \((k,n)\) related by a geometric exchange corresponding to a quadrilateral face \(X\) of \(D'\) then \(S(D'')\) is the mutation at \(X\) of \(S(D')\).

Thus we see that, via the identification \(\varphi_D\), the cluster variable \(u_I\), for \(I\) a \(k\)-subset of \([1,n]\) (regarded as an element of \(A(D)\) for some \(D\)), corresponds to the minor \(p_I\).

Let \(A'(D)\) be the cluster algebra defined the same way as for \(A(D)\) except that the elements \(u_{L_i}^{-1}\), for \(i = 1, 2, \ldots, n\), of \(\mathbb{F}\) are added to the generating set. Thus \(A'(D)\) is the localisation of \(A(D)\) obtained by adjoining inverses to the elements \(u_{L_i}\) (see [21, Sect. 3.4]). Note that \(\tilde{X}\) is defined to be the subset of \(P^\vee \setminus G^\vee\) where the \(p_J\) do not vanish. Then we have the following:

**Proposition 7.5.**

(a) Let \(D\) be a Postnikov diagram of type \((k,n)\). Then there is an isomorphism \(\varphi'_D\) from \(\mathbb{C}[\tilde{X}]\), the coordinate ring of \(\tilde{X}\), to \(A'(D)\).
b) Let $D'$ be any Postnikov diagram of type $(k,n)$ and let $M(D')$ be the collection of minors corresponding to the $k$-subsets labelling the alternating faces of $D$. Then $S(D') := (\varphi(M(D')), Q(D'))$ is a seed of $\mathcal{A}'(D)$.

c) If $D', D''$ are two Postnikov diagrams of type $(k,n)$ related by a geometric exchange corresponding to a quadrilateral face $X$ of $D'$ then $S_D'' := (\varphi(M(D'')), Q(D''))$ is the mutation at $X$ of $S_D'$.

The proof of this result involves applying [21, Prop. 3.37] to get (a). See also [18, Prop. 11.1]. The proof of (b) and (c) is the same as that for Theorem 7.4 in [58].

Remark 7.6. If $k = 1$ or $n - 1$, there is a unique Postnikov diagram of type $(k,n)$ up to equivalence. It can be chosen to have no crossings at all. The case $k = 1, n = 6$ is shown in Figure 6.

8. The vector fields $X^{(m)}_{\lambda}$

In this section we define a family of vector fields $X^{(m)}_{\lambda}, \lambda \in \mathcal{P}_{k,n}, m \in [1,n]$, on $\tilde{X}$ and study some of their additivity properties, denoting $X^{(m)}_{\lambda}$ by $X_{\lambda}$. We start by defining coefficients $c^{(m)}_{\lambda}(\mu)$ which we use in the definition of the vector fields. We then prove that they satisfy an additivity property which is used in checking regularity.

Fix a Young diagram $\lambda \in \mathcal{P}_{k,n}$. For each $\mu \in \mathcal{S}_{n-k,k}$ we define a nonnegative integer $c^{(m)}_{\lambda}(\mu)$ as follows. Write $J_\mu \setminus J_\lambda = \{m_1 < m_2 < \cdots < m_r\}$ and $J_\lambda \setminus J_\mu = \{l_1 < l_2 < \cdots < l_r\}$ in increasing numerical order. Then set

$$c_{\lambda}(\mu) = |\{1 \leq j \leq r : m_j > l_j\}|.$$

Note that $c_{\lambda}(\lambda) = c_{\lambda}(\phi) = 0$.

Given a cluster $\mathcal{C}$ (for the cluster structure on $\mathbb{C}[\tilde{X}]$ discussed in Section 7), we define:

$$\tilde{X}_C = \{x \in \tilde{X} : f(x) = 0 \text{ for all } f \in \mathcal{C}\}.$$

Note that $\tilde{X}_C$ is isomorphic to $(\mathbb{C}^*)^{k(n-k)}$ by [58, Theorem 4]. If $\mathcal{C}$ is a cluster of $\mathbb{C}[\tilde{X}]$ coming from a Postnikov diagram containing a region labelled $J_\lambda$ (so that $p_\lambda \in \mathcal{C}$), we consider the regular vector field

$$(8.1) X_{\lambda,C} := p_\lambda \sum_{p_\mu \in \mathcal{C}} c_{\lambda}(\mu)p_\mu \frac{\partial}{\partial p_\mu}$$

on $\tilde{X}_C$. We shall see later that $X_{\lambda,C}$ can be extended to a regular vector field on the whole of $\tilde{X}$.

We also consider a family of ‘twisted’ versions of $X_{\lambda}$, defined as follows. For $m \in [1,n]$, let $\mu^{(m)}$ denote the partition corresponding to the subset $J_\mu - m \mod n$.

For $m \in [1,n]$ and Young diagrams $\lambda, \mu$ in $\mathcal{P}_{k,n}$, we set

$$c^{(m)}_{\lambda}(\mu) = c_{\lambda^{(m)}}(\mu^{(m)}) - c_{\lambda^{(m)}}(\phi^{(m)}).$$
Note that $\phi^{(m)}(m) = \mu_{n-m,n-m+1}$ for all $m$ and that, by definition, $c^{(m)}_\lambda(\phi) = 0$ for any $m, \lambda$. We also have that $c^{(n)}_\lambda(\mu) = c_\lambda(\mu)$ for any $\lambda, \mu$.

For a cluster $C$ as above, we have the regular vector field

$$X^{(m)}_{\lambda,C} := p_x \sum_{p_\mu \in C} c^{(m)}_\lambda(\mu) \frac{\partial}{\partial p_\mu}$$

on $\hat{X}_C$. Note that $X^{(n)}_{\lambda,C} = X_{\lambda,C}$.

Let $P_n$ be a regular polygon with vertices $1, 2, \ldots, n$ numbered clockwise. Then, in [38], two $k$-subsets $I, J$ of $\{1, 2, \ldots, n\}$ are said to be weakly separated provided none of the chords between the vertices of $P_n$ corresponding to $I \setminus J$ crosses any of the chords between the vertices corresponding to $J \setminus I$. Scott [58] uses the term non-crossing and we shall use this terminology.

We recall the following result of Scott:

**Theorem 8.1.** Let $D$ be a Postnikov diagram of type $(k, n)$. Then the collection of $k$-subsets labelling the alternating faces of $D$ is an inclusion-maximal collection of pairwise non-crossing $k$-subsets of $\{1, 2, \ldots, n\}$.

We will now show that the coefficients $c^{(m)}_\lambda(\mu)$ satisfy a certain additivity property on Postnikov diagrams.

**Proposition 8.2.** Fix a partition $\lambda \in \mathcal{P}_{k,n}$ such that $J_\lambda$ is not a coefficient, and let $D$ be a Postnikov diagram with a region labelled $J_\lambda$. Let $D \neq I_\lambda$ be a non-boundary alternating face of $D$, labelled by the $k$-subset $J_\mu$ corresponding to a partition $\mu \in \mathcal{P}_{k,n}$. Let $I_1, I_1', I_2, I_2', \ldots, I_r, I_r'$ be the $k$-subsets labelling the alternating faces adjacent to $Y$, in order clockwise around $Y$, with corresponding partitions $\mu(1), \mu'(1), \mu(2), \mu'(2), \ldots, \mu(r), \mu'(r)$. Then, for $m \in [1, n]$, we have

$$\sum_{i=1}^r c^{(m)}_\lambda(\mu(i)) = \sum_{i=1}^r c^{(m)}_\lambda(\mu'(i)).$$

In particular, taking $m = n$, we have:

$$\sum_{i=1}^r c_\lambda(\mu(i)) = \sum_{i=1}^r c_\lambda(\mu'(i)).$$

**Proof.** We consider first the case $m = n$. It is easy to check that there is some $r \geq 1$ and $a_1, \ldots, a_r$, $b_1, b_2, \ldots, b_r \in \{1, 2, \ldots, n\}$ such that $J_\mu = \{b_1, b_2, \ldots, b_r\} \cup J'$ and, for all $1 \leq i \leq r$, $I_i = (J_\lambda \setminus \{b_i\}) \cup \{a_i\}$ and $I'_i = (J_\mu \setminus \{b_i\}) \cup \{a_i-1\}$ where $i-1$ is interpreted modulo $r$ (with representatives in $1, 2, \ldots, r$). See Figure 7 for an illustration of the Postnikov diagram locally around $Y$. Note that, since $Y$ is on the right hand side of the corresponding strand, no $a_i$ lies in $J_\mu$, so that $a_i \neq b_j$ for all $i, j$. 
For subsets $I, I'$ of $\{1, 2, \ldots, n\}$ we write $I < I'$ to indicate that every element of $I$ is less than every element of $I'$ and for an element $a$ we write $a < I$ (respectively, $I < a$) to denote $\{a\} < I$ (respectively, $I < \{a\}$). Note that, since $X \neq Y$, $J_\lambda \neq J_\mu$. So, by Theorem 8.1 there are four possibilities for the subsets $J_\lambda$ and $J_\mu$, as listed below. In each case, we define integers $N_{a_i}$ and $N_{b_i}$ for $i = 1, 2, \ldots, r$, as follows.

**Case I:** If $J_\mu \setminus J_\lambda < J_\lambda \setminus J_\mu$, then set

$$
N_{a_i} = \begin{cases} 
0 & a_i \in J_\lambda; \\
1 & a_i < J_\lambda \setminus J_\mu, a_i \notin J_\lambda; \\
0 & a_i > J_\lambda \setminus J_\mu, a_i \notin J_\lambda;
\end{cases}
\quad \text{and} \quad
N_{b_i} = \begin{cases} 
-1 & \mu \notin J_\lambda; \\
-1 & \mu < J_\mu \setminus J_\lambda, b_i \in J_\lambda; \\
0 & \mu > J_\mu \setminus J_\lambda, b_i \in J_\lambda.
\end{cases}
$$

**Case II:** If $J_\lambda \setminus J_\mu = K_1 \sqcup K_2$ where $K_1$ and $K_2$ are nonempty and $K_1 < J_\mu \setminus J_\lambda < K_2$, then set

$$
N_{a_i} = \begin{cases} 
0 & a_i \notin J_\lambda; \\
-1 & a_i < J_\mu \setminus J_\lambda, a_i \in J_\lambda; \\
0 & a_i > J_\mu \setminus J_\lambda, a_i \in J_\lambda;
\end{cases}
\quad \text{and} \quad
N_{b_i} = \begin{cases} 
0 & \mu \notin J_\lambda; \\
1 & \mu < J_\lambda \setminus J_\mu, b_i \in J_\lambda; \\
0 & \mu > J_\lambda \setminus J_\mu, b_i \in J_\lambda.
\end{cases}
$$

**Case III:** If $J_\lambda \setminus J_\mu < J_\lambda \setminus J_\mu$, then set

$$
N_{a_i} = \begin{cases} 
0 & a_i \in J_\lambda; \\
0 & a_i < J_\lambda \setminus J_\mu, a_i \notin J_\lambda; \\
1 & a_i > J_\lambda \setminus J_\mu, a_i \notin J_\lambda;
\end{cases}
\quad \text{and} \quad
N_{b_i} = \begin{cases} 
-1 & \mu \notin J_\lambda; \\
0 & \mu < J_\lambda \setminus J_\mu, b_i \in J_\lambda; \\
-1 & \mu > J_\lambda \setminus J_\mu, b_i \in J_\lambda.
\end{cases}
$$

**Case IV:** If $J_\mu \setminus J_\lambda = K_1 \sqcup K_2$ where $K_1$ and $K_2$ are non-empty and $K_1 < J_\lambda \setminus J_\mu < K_2$, then set

$$
N_{a_i} = \begin{cases} 
0 & a_i \in J_\lambda; \\
0 & a_i < J_\lambda \setminus J_\mu, a_i \notin J_\lambda; \\
1 & a_i > J_\lambda \setminus J_\mu, a_i \notin J_\lambda;
\end{cases}
\quad \text{and} \quad
N_{b_i} = \begin{cases} 
0 & \mu \notin J_\lambda; \\
0 & \mu < J_\lambda \setminus J_\mu, b_i \notin J_\lambda; \\
-1 & \mu > J_\lambda \setminus J_\mu, b_i \notin J_\lambda.
\end{cases}
$$

**Claim:**
(a) For $1 \leq i \leq r$, we have $c_\lambda(\mu(i)) = c_\lambda(\mu) + N_{a_i} + N_{b_i}$.
(b) For $1 \leq i \leq r$, we have $c_\lambda(\mu'(i)) = c_\lambda(\mu) + N_{a_{i-1}} + N_{b_i}$.

**Proof of claim:** The proof is by case by case. Let $1 \leq j \leq r$ and set $i = j$ or $j - 1$. Let $L = I_j$ in the former case and $L = I_j'$ in the latter case so that $L = (J_\mu \setminus \{b_j\}) \cup \{a_j\}$. Let $\kappa$ be the partition corresponding to $L$. We set the proof out as a table, with conditions on $a_i, b_j$ in the first two columns and the values of $N_{a_i}, N_{b_j} \mu(c_\lambda(\kappa)) - c_\lambda(\mu)$ in the last three columns. The proof then follows from the observation that the sum of the entries in the last two columns equals the entry in the last column. (If these columns are empty, it indicates that this situation cannot occur). We also remark that, since $a_i \notin J_\mu$ and $b_j \notin J_\lambda$, we can carry out the following operations on $J_\mu \setminus J_\lambda$ and $J_\mu \setminus J_\mu$ in order to obtain $L \setminus J_\lambda$ and $J_\mu \setminus L$ respectively.

Below the table for each case is a diagram illustrating the arrangement of the subsets $J_\lambda \setminus J_\mu$ and $J_\mu \setminus J_\lambda$. The subset $\{1, 2, \ldots, n\}$ is drawn as a circle (with the numbering $1, 2, \ldots, n$ clockwise around the boundary). A dotted line cutting across the circle indicates the gap between 1 and $n$.

(a) If $a_i, b_j \notin J_\lambda$, remove $a_i$ from and add $b_j$ to $J_\lambda \setminus J_\mu$.
(b) If $a_i \in J_\lambda, b_j \notin J_\lambda$, remove $a_i$ from $J_\lambda \setminus J_\mu$ and remove $b_j$ from $J_\mu \setminus J_\lambda$.
(c) If $a_i \notin J_\lambda$ and $b_j \in J_\lambda$, add $a_i$ to $J_\mu \setminus J_\lambda$ and add $b_j$ to $J_\lambda \setminus J_\mu$.
(d) If $a_i, b_j \notin J_\lambda$, add $a_i$ to and remove $b_j$ from $J_\mu \setminus J_\lambda$.

**Case I:** $J_\mu \setminus J_\lambda < J_\lambda \setminus J_\mu$.
### Case II: \( J_\lambda \setminus J_\mu = K_1 \uplus K_2, K_1, K_2 \neq \emptyset \) and \( K_1 < J_\mu \setminus J_\lambda < K_2 \)

| Membership of \( J_\lambda \) | Condition(s) | \( N_{a_i} \) | \( N_{b_j} \) | \( c_\lambda(\kappa) - c_\lambda(\mu) \) |
|-------------------------------|--------------|-------------|-------------|----------------------------------|
| \( a_i, b_j \in J_\lambda \) | \( b_j < J_\mu \setminus J_\lambda \) | 0           | -1          | -1                                |
|                              | \( b_j > J_\mu \setminus J_\lambda \) | 0           | 0           | 0                                |
| \( a_i \in J_\lambda, b_j \notin J_\lambda \) |                             | -1          | 0           | -1                                |
| \( a_i \notin J_\lambda, b_j \in J_\lambda \) | \( a_i < J_\lambda \setminus J_\mu, b_j < J_\mu \setminus J_\lambda \) | 1           | -1          | 0                                |
|                              | \( a_i < J_\lambda \setminus J_\mu, b_j > J_\mu \setminus J_\lambda \) | 1           | 0           | 1                                |
|                              | \( a_i > J_\lambda \setminus J_\mu, b_j < J_\mu \setminus J_\lambda \) |                             |             |                                   |
|                              | \( a_i > J_\lambda \setminus J_\mu, b_j > J_\mu \setminus J_\lambda \) |                             |             |                                   |
| \( a_i, b_j \notin J_\lambda \) | \( a_i < J_\lambda \setminus J_\mu \) | 1           | -1          | 0                                |
|                              | \( a_i > J_\lambda \setminus J_\mu \) | 0           | -1          | -1                                |

### Case III: \( J_\lambda \setminus J_\mu < J_\mu \setminus J_\lambda \)

| Membership of \( J_\lambda \) | Condition(s) | \( N_{a_i} \) | \( N_{b_j} \) | \( c_\lambda(\kappa) - c_\lambda(\mu) \) |
|-------------------------------|--------------|-------------|-------------|----------------------------------|
| \( a_i, b_j \in J_\lambda \) | \( b_j < J_\mu \setminus J_\lambda \) | 0           | 0           | 0                                |
|                              | \( b_j > J_\mu \setminus J_\lambda \) | 0           | -1          | -1                                |
| \( a_i \in J_\lambda, b_j \notin J_\lambda \) |                             | 0           | -1          | -1                                |
| \( a_i \notin J_\lambda, b_j \in J_\lambda \) | \( a_i < J_\lambda \setminus J_\mu, b_j < J_\mu \setminus J_\lambda \) | 0           | 0           | 0                                |
|                              | \( a_i < J_\lambda \setminus J_\mu, b_j > J_\mu \setminus J_\lambda \) |                             |             |                                   |
|                              | \( a_i > J_\lambda \setminus J_\mu, b_j < J_\mu \setminus J_\lambda \) |                             |             |                                   |
|                              | \( a_i > J_\lambda \setminus J_\mu, b_j > J_\mu \setminus J_\lambda \) |                             |             |                                   |
| \( a_i, b_j \notin J_\lambda \) | \( a_i < J_\lambda \setminus J_\mu \) | 0           | -1          | -1                                |
|                              | \( a_i > J_\lambda \setminus J_\mu \) | 1           | -1          | 0                                |
Case IV: \( J_\mu \setminus J_\lambda = K_1 \sqcup K_2, K_1, K_2 \neq \phi \) and \( K_1 < J_\lambda \setminus J_\mu < K_2 \)

| Membership of \( J_\lambda \) | Condition(s) | \( N_{a_i} \) | \( N_{b_j} \) | \( c_\lambda(\kappa) - c_\lambda(\mu) \) |
|-----------------|-------------|-------------|-------------|-----------------|
| \( a_i, b_j \in J_\lambda \) | \( b_j < J_\lambda \setminus J_\mu \) | 0 | 1 | 1 |
| \( a_i \in J_\lambda, b_j \notin J_\lambda \) | \( b_j > J_\lambda \setminus J_\mu \) | 0 | -1 | -1 |
| \( a_i \notin J_\lambda, b_j \in J_\lambda \) | \( a_i < J_\lambda \setminus J_\mu \) | 0 | 0 | 0 |
| \( a_i > J_\lambda \setminus J_\mu \) | \( a_i > J_\lambda \setminus J_\mu \) | 1 | 0 | 1 |
| \( a_i, b_j \notin J_\lambda \) | \( a_i < J_\lambda \setminus J_\mu, b_j < J_\lambda \setminus J_\mu \) | 0 | 0 | 0 |
| \( a_i < J_\lambda \setminus J_\mu, b_j > J_\lambda \setminus J_\mu \) | 0 | -1 | -1 |
| \( a_i > J_\lambda \setminus J_\mu, b_j < J_\lambda \setminus J_\mu \) | 0 | 0 | 0 |
| \( a_i > J_\lambda \setminus J_\mu, b_j > J_\lambda \setminus J_\mu \) | 0 | -1 | -1 |

By the claim, and noting that \( N_{a_i} \) (respectively, \( N_{b_j} \)) depends only on \( \mu, \lambda \) and \( a_i \) (respectively, \( b_j \)), we have

\[
\sum_{i=1}^{r} c_\lambda(\mu(i)) = \sum_{i=1}^{r} c_\lambda(\mu'(i)) = r c_\lambda(\mu) + \sum_{i=1}^{r} (N_{a_i} + N_{b_i}).
\]

Note that relabelling strand \( i \) as \( i - m \) modulo \( n \) in \( D \) for each \( i \), we obtain a new Postnikov diagram \( D^{(m)} \) with a region labelled \( J_{\lambda^{(m)}} \). Furthermore, the partitions corresponding to the \( k \)-subsets labelling the regions surrounding this region are \( \mu(1)^{(m)}, \mu'(1)^{(m)}, \ldots, \mu(r)^{(m)}, \mu'(r)^{(m)} \). Hence, applying the above with \( \lambda \) replaced by \( \lambda^{(m)} \), we have, for \( m \in [1, n] \):

\[
\sum_{i=1}^{r} c_{\lambda^{(m)}}(\mu(i)) = \sum_{i=1}^{r} c_{\lambda^{(m)}}(\mu'(i)) = \sum_{i=1}^{r} c_{\lambda^{(m)}}(\mu(i)^{(m)}) = \sum_{i=1}^{r} c_{\lambda^{(m)}}(\mu'(i)^{(m)}) = \sum_{i=1}^{r} c_{\lambda^{(m)}}(\mu'(i)) = \sum_{i=1}^{r} c_{\lambda^{(m)}}(\mu(i))
\]

and we are done.

\( \square \)

**Lemma 8.3.** Fix a cluster \( C \). Then there is a regular vector field \( X_\lambda \) on \( \bar{X} \) such that \( X_\lambda \) and \( X_{\lambda,C} \) coincide on \( \bar{X}_C \). Similarly, for each \( m \in [1, n] \), there is a regular vector field \( X_{\lambda^{(m)}} \) on \( \bar{X} \) such that \( X_{\lambda^{(m)}} \) and \( X_{\lambda,C}^{(m)} \) coincide on \( \bar{X}_C \).
Proof. We consider the cluster mutation of \( \mathcal{C} \) at a cluster variable \( p_\mu \in \mathcal{C} \), giving a new cluster \( \mathcal{C}(\mu) \). Let \( \mu_i, \mu'_i \), for \( i = 1, 2, \ldots, r \) be as in Proposition 8.2. Then the mutation corresponds to the change of variables \( \tilde{\mu}_\kappa = p_\kappa \) for \( \kappa \in \mathcal{C}, \kappa \neq \mu \), and

\[
\tilde{p}_\mu = \frac{\prod_{i=1}^r p_\mu(i) + \prod_{i=1}^r p_{\mu'(i)}}{p_\mu}.
\]

We can thus compute:

\[
p_{\mu(i)} \frac{\partial}{\partial p_{\mu(i)}} = \tilde{p}_{\mu(i)} \frac{\partial}{\partial \tilde{p}_{\mu(i)}} = \tilde{p}_{\mu(i)} \frac{\partial}{\partial \tilde{p}_{\mu'_i(i)}} + \tilde{p}_{\mu(i)} \frac{\partial}{\partial \tilde{p}_{\mu'}} \frac{\partial}{\partial \tilde{p}_{\mu}}.
\]

Similarly,

\[
p_{\mu'(i)} \frac{\partial}{\partial p_{\mu'(i)}} = \tilde{p}_{\mu'(i)} \frac{\partial}{\partial \tilde{p}_{\mu'(i)}} + \prod_{j=1}^r \tilde{p}_{\mu'(j)} \tilde{p}_\mu \frac{\partial}{\partial \tilde{p}_\mu}.
\]

We also have:

\[
p_\mu \frac{\partial}{\partial p_\mu} = p_\mu \left( \frac{1}{p_\mu^2} \right) \frac{\partial}{\partial \tilde{p}_\mu} = -\frac{1}{p_\mu} \left( \prod_{j=1}^r p_{\mu(j)} + \prod_{j=1}^r p_{\mu'(j)} \right) \frac{\partial}{\partial \tilde{p}_\mu}.
\]

Hence, using Proposition 8.2, we have:

\[
p_\lambda \sum_{\kappa \in \mathcal{C}} c_\lambda(\kappa) p_\kappa \frac{\partial}{\partial p_\kappa} = \tilde{p}_\lambda \sum_{\kappa \in \mathcal{C}, \kappa \neq \mu} c_\lambda(\kappa) \tilde{p}_\kappa \frac{\partial}{\partial \tilde{p}_\kappa} + \left( \sum_{i=1}^r c_\lambda(\mu(i)) \prod_{j=1}^r \tilde{p}_{\mu(j)} + \sum_{i=1}^r c_\lambda(\mu'(i)) \prod_{j=1}^r \tilde{p}_{\mu'(j)} - c_\lambda(\mu) \right) \tilde{p}_\lambda \tilde{p}_\mu \frac{\partial}{\partial \tilde{p}_\mu}.
\]

It follows that there is an extension of \( X_{\lambda, \mathcal{C}} \) to \( \cup_\mu \tilde{X}_{\mathcal{C}(\mu)} \) where the union is over the clusters \( \mathcal{C}(\mu) \) obtained from \( \mathcal{C} \) by mutating at \( p_\mu \) for some cluster variable \( p_\mu \) in \( \mathcal{C} \) with \( \mu \neq \lambda \).

For the case \( \mu = \lambda \), we have

\[
p_\lambda p_\lambda(i) \frac{\partial}{\partial p_\lambda(i)} = p_\lambda \tilde{p}_\lambda(i) \frac{\partial}{\partial \tilde{p}_\lambda(i)} + p_\lambda \tilde{p}_\lambda(i) \frac{\partial}{\partial \tilde{p}_\lambda(i)} \frac{\partial}{\partial \tilde{p}_\lambda} = p_\lambda \tilde{p}_\lambda(i) \frac{\partial}{\partial \tilde{p}_\lambda(i)} + p_\lambda \tilde{p}_\lambda(i) \prod_{j=1, j \neq i}^r \tilde{p}_{\lambda(j)} \frac{\partial}{\partial \tilde{p}_\lambda} = p_\lambda \tilde{p}_\lambda(i) \frac{\partial}{\partial \tilde{p}_\lambda(i)} + \prod_{j=1}^r \tilde{p}_{\lambda(j)} \frac{\partial}{\partial \tilde{p}_\lambda}.
\]
and, similarly:

\[ p_\lambda p_{\lambda'}(i) \frac{\partial}{\partial p_{\lambda'}(i)} = p_\lambda \tilde{p}_{\lambda'}(i) \frac{\partial}{\partial \tilde{p}_{\lambda'}(i)} + \left( \prod_{j=1}^{r} \tilde{p}_{\lambda'(j)} \right) \frac{\partial}{\partial \tilde{p}_\lambda}. \]

As before, we have

\[ p_\lambda \frac{\partial}{\partial p_\lambda} = -\tilde{p}_\lambda \frac{\partial}{\partial \tilde{p}_\lambda}. \]

Hence, we have

\[ p_\lambda \sum_{p_\nu \in C} c_\lambda(\kappa)p_\nu \frac{\partial}{\partial p_\nu} = p_\lambda \sum_{\tilde{p}_\nu \in C(\lambda), \tilde{p}_\nu \neq \tilde{p}_\lambda} c_\lambda(\kappa)\tilde{p}_\nu \frac{\partial}{\partial \tilde{p}_\nu} + \left( \prod_{i=1}^{r} c_\lambda(\lambda(i)) \prod_{j=1}^{r} \tilde{p}_{\lambda(j)} + \prod_{i=1}^{r} c_\lambda(\lambda'(i)) \prod_{j=1}^{r} \tilde{p}_{\lambda'(j)} - \tilde{p}_\lambda \right) \frac{\partial}{\partial \tilde{p}_\lambda}. \]

Hence there is a regular extension of \( X_\lambda, C \) to \( \tilde{X}_C \cup \bigcup_{\mu \in C} \tilde{X}_{C(\mu)} \). By [22, Lemma 2.3], the complement of \( \tilde{X}_C \cup \bigcup_{\mu \in C} \tilde{X}_{C(\mu)} \) in \( \tilde{X} \) has codimension at least two. Hence, by Hartog’s Theorem, \( X_\lambda, C \) extends to a regular vector field on the whole of \( \tilde{X} \), as required. The same argument can be used for \( X_\lambda^{(m)} \), again using the additivity given by Proposition 9.2.

Note that we have not shown that \( X_\lambda \) is independent of the choice of initial cluster \( C \). We expect this to hold, but we do not need it here.

9. ACTION OF THE VECTOR FIELD

In this section we first we compute the action of \( X_\lambda^{(m)} \) on \( W_q \) for each \( m \in [1, n] \) (Proposition 9.1). We then use this to compute expressions for \( q \frac{\partial W_q}{\partial q} [p_\lambda \omega] \) and \( \frac{1}{z} [W_q p_\lambda \omega] \) in terms of quantum Monk’s rule (Theorem 9.7).

**Theorem 9.1.** Let \( \lambda \) be an arbitrary Young diagram in \( P_{k, n} \). Then we have:

\[ X_\lambda^{(m)} W_q = \left( \sum_\mu p_\mu + q \sum_\nu p_\nu \right) - q^{\delta_{mn}} \frac{p_{L_m}}{p_{L_m}} p_\lambda, \]

where \( \mu, \nu \) are exactly as in the quantum Monk’s rule for \( \sigma^\Theta \ast_q \sigma^\lambda \).

The proof of the theorem uses a result of the first author and J. Scott, and will appear in a subsequent version of this paper.

We then have:

**Theorem 9.2.** Let \( \lambda \) be an arbitrary Young diagram in \( P_{k, n} \). Then we have:

(a)

\[ X_\lambda W_q = \left( \sum_\mu p_\mu + q \sum_\nu p_\nu \right) - q \frac{\partial W_q}{\partial q} p_\lambda; \]
(b) \[ \sum_{m=1}^{n} X^{(m)}_{\lambda} W_q = n \left( \sum_{\mu} p_{\mu} + q \sum_{\nu} p_{\nu} \right) - W_q p_\lambda, \]

where \( \mu, \nu \) are exactly as in the quantum Monk’s rule for \( \sigma^D \ast_q \sigma^\lambda \).

**Proof.** Part (a) is the case \( m = n \) in Proposition 9.1. To see part (b) we add up the cases \( m = 1, 2, \ldots, n \). \( \square \)

**Lemma 9.3.** Let \( \xi \) be a regular vector field on \( \dot{\mathcal{X}} \). Then we obtain the following relation in \( G^W \):

\[ [d \xi \omega] + \frac{1}{z} [(\xi \cdot W_q) \omega] = 0, \]

where \( \xi \omega \) denotes the insertion of \( \xi \) into \( \omega \).

**Proof.** Let \( \xi \) be a regular vector field on \( \dot{\mathcal{X}} \). Then we have

\[ [(d + \frac{1}{z} d W_q \wedge - ) (\xi \omega)] = 0 \]

in \( G^W \). Note that \( d W_q \wedge \omega = 0 \), so we have

\[ 0 = d \xi (d W_q \wedge \omega) = (d \xi d W_q) \omega - d W_q \wedge d \xi \omega \]

from which the result follows. \( \square \)

**Lemma 9.4.** Let \( \lambda \in \mathcal{P}_{k,n} \). Then we have:

\[ d(i_{X^{(m)}_{\lambda}} \omega) = - c_{\lambda^{(m)}} (\phi^{(m)}) [p_{\lambda} \omega]. \]

**Proof.** We have:

\[ d(i_{X^{(m)}_{\lambda}} \omega) = \sum_{\mu \in \mathcal{C}} d \left( i_{c^{(m)}_{\lambda}(\mu) p_{\lambda} p_{\mu} \partial_{p_{\mu}}} \omega \right) \]

\[ = \sum_{\mu \in \mathcal{C}} d \left( c^{(m)}_{\lambda}(\mu) p_{\lambda} \wedge \bigwedge_{\varepsilon \in \mathcal{C}, \varepsilon \neq \mu} \frac{dp_{\varepsilon}}{p_{\varepsilon}} \right) \]

\[ = \sum_{\mu \in \mathcal{C}} c^{(m)}_{\lambda}(\mu) dp_{\lambda} \wedge \bigwedge_{\varepsilon \in \mathcal{C}, \varepsilon \neq \mu} \frac{dp_{\varepsilon}}{p_{\varepsilon}} \]

\[ = c^{(m)}_{\lambda}(\lambda) dp_{\lambda} \wedge \bigwedge_{\varepsilon \in \mathcal{C}, \varepsilon \neq \mu} \frac{dp_{\varepsilon}}{p_{\varepsilon}} \]

\[ = c^{(m)}_{\lambda}(\lambda)[p_{\lambda} \omega] \]

\[ = (c_{\lambda^{(m)}}(\lambda^{(m)}) - c_{\lambda^{(m)}}(\phi^{(m)}))[p_{\lambda} \omega] \]

\[ = - c_{\lambda^{(m)}}(\phi^{(m)})[p_{\lambda} \omega], \]

and we are done. \( \square \)

**Lemma 9.5.** Fix an arbitrary Young diagram \( \lambda \in \mathcal{P}_{k,n} \). Then for \( 1 \leq i \leq n \), we have

\[ c_{\lambda}(\mu_{i,i+1}) = \begin{cases} |J_{\lambda}| \cap [1, i], & 1 \leq i \leq n - k; \\ |J^c_{\lambda}| \cap [i + 1, n], & n - k + 1 \leq i \leq n, \end{cases} \]

where \( J^c_{\lambda} \) denotes the complement of \( J_{\lambda} \) in \([1, n]\).
Proof. Suppose first that $1 \leq i \leq n - k$. We can write

$$[1, n] = [1, i] \cup [i + 1, i + k] \cup [i + k + 1, n],$$

and

$$J_\lambda = (J_\lambda \cap [1, i]) \cup (J_\lambda \cap [i + 1, i + k]) \cup (J_\lambda \cap [i + k + 1, n]).$$

We have

$$J_\lambda \setminus J_i = (J_\lambda \cap [1, i]) \cup (J_\lambda \cap [i + k + 1, n]).$$

Since $J_i \setminus J_\lambda \subseteq J_i = [i + 1, i + k]$, we see from its definition that

$$c_{J_i} = |J_\lambda \cap [1, i]|.$$ 

Suppose next that $n - k + 1 \leq i \leq n$. We write

$$[1, n] = [1, i + k - n] \cup [i + k - n + 1, i] \cup [i + 1, n].$$

We have

$$J_\lambda = (J_\lambda \cap [1, i + k - n]) \cup (J_\lambda \cap [i + k - n + 1, i]) \cup (J_\lambda \cap [i + 1, n]).$$

Furthermore:

$$J_\lambda \setminus J_i = (J_\lambda \cap [i + k - n + 1, i]).$$

Since $J_i \setminus J_\lambda \subseteq J_i = [i + 1, n]$, we see from its definition that $c_{J_i}$ is the number of elements of $[i + 1, n]$ not in $J_\lambda$, i.e.

$$c_{J_i} = |J_\lambda^c \cap [i + 1, n]|,$$

and we are done. \(\square\)

Lemma 9.6. Let $\lambda$ be an arbitrary Young tableau in $\mathcal{P}_{k,n}$. Then

$$\sum_{m=1}^{n-1} c_{\lambda^{(m)}} (\phi^{(m)}) = |\lambda|.$$ 

Proof. If $1 \leq m \leq k - 1$, then $n - k + 1 \leq n - m \leq n - 1$, so, by Lemma 9.5,

$$c_{\lambda^{(m)}} (\phi^{(m)}) = |J_\lambda^{(m)} \cap [n - m + 1, n]| = |J_\lambda^{(m)} \cap [n - m + 1, n]| = |J_\lambda^{(m)} \cap [1, m]|.$$

An element $j \in J_\lambda^{(m)} \cap [1, k - 1]$ contributes 1 to the term $J_\lambda^{(m)} \cap [1, m]$ in the sum

$$\sum_{m=1}^{k-1} |J_\lambda^{(m)} \cap [1, m]|$$

for all $m \geq j$, and zero otherwise. It follows that

$$\sum_{m=1}^{k-1} c_{\lambda^{(m)}} (\phi^{(m)}) = \sum_{j \in J_\lambda^{(m)} \setminus [1, k - 1]} k - j,$$

which is the sum over $j \in J_\lambda^{(m)} \cap [1, k - 1]$ of the number of boxes in the row of $\lambda$ to the left of the vertical step numbered $j$ and strictly to the right of the leading diagonal, i.e. the total number of boxes in $\lambda$ strictly to the right of the leading diagonal.

If $k \leq m \leq n - 1$, then $1 \leq n - m \leq n - k$ and, by Lemma 9.5,

$$c_{\lambda^{(m)}} (\phi^{(m)}) = |J_\lambda^{(m)} \cap [1, n - m]| = |J_\lambda^{(m)} \cap [1, n - m]| = |J_\lambda \cap [m + 1, n]|.$$
An element $j$ in $J_\lambda \cap [k + 1, n]$ contributes 1 to the term $|J_\lambda \cap [m + 1, n]|$ in the sum
\[ \sum_{m=k}^{n-1} |J_\lambda \cap [m + 1, n]| \]
if $m \leq j - 1$, and zero otherwise. It follows that
\[ \sum_{m=k}^{n-1} c_{\lambda(m)}(\phi^{(m)}) = \sum_{j \in J_\lambda, j \geq k+1} j - k, \]
which is the sum over $j \in J_\lambda \cap [k + 1, n]$ of the number of boxes in the column of $\lambda$ above the horizontal step numbered $j$ which are on or below the leading diagonal, i.e. the total number of boxes in $\lambda$ on or below the leading diagonal. Combining this with the above gives the claimed result. \qed

We can now put all of the above results together, to obtain:

**Theorem 9.7.** Let $\lambda \in P_{k,n}$. Then

(a) \[ q \frac{\partial W_q[\lambda \omega]}{\partial q} = \sum_{\mu} [p_{\mu} \omega] + q \sum_{\nu} [p_{\nu} \omega], \]

and

(b) \[ \frac{1}{z} [W_q p_{\lambda} \omega] = \frac{n}{z} \left( \sum_{\mu} [p_{\mu} \omega] + q \sum_{\nu} [p_{\nu} \omega] \right) - |\lambda| [p_{\lambda} \omega], \]

where, in each case, $\mu, \nu$ are exactly as in the quantum Monk's rule for $\sigma^\omega \ast q \sigma^\lambda$.

**Proof.** By Lemma 9.3
\[ d(i_{X_{\lambda}} \omega) = d(i_{X_{\lambda}^{(n)}} \omega) = -c_{\lambda}(\phi) [p_{\lambda} \omega] = 0. \]

By Lemma 9.3 and Lemma 8.3 we have $[X_{\lambda} \cdot W_q \omega] = 0$, so part (a) follows from Theorem 9.2(a). Using Theorem 9.2(b) and Lemmas 9.4 and 9.6 we obtain:

(9.1)
\[ [di_{\sum_{m=1}^{n} X_{\lambda(m)}^{(m)}} \omega] + \frac{1}{z} \sum_{m=1}^{n} X_{\lambda(m)}^{(m)} W_q \omega] = -\sum_{m=1}^{n} c_{\lambda}(\phi^{(m)}) [p_{\lambda} \omega] + \frac{n}{z} \left( \sum_{\mu} [p_{\mu} \omega] + q \sum_{\nu} [p_{\nu} \omega] \right) - \frac{1}{z} [W_q p_{\lambda} \omega] \]

\[ = -|\lambda| [p_{\lambda} \omega] + \frac{n}{z} \left( \sum_{\mu} [p_{\mu} \omega] + q \sum_{\nu} [p_{\nu} \omega] \right) - \frac{1}{z} [W_q p_{\lambda} \omega], \]

and the result then follows from Lemma 9.3 and Lemma 8.3. \qed

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