Bernoulli-Taylor formula of $\psi$-umbral difference calculus

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Abstract

We shall present here the $\psi$-Bernoulli-Taylor* formula of a new sort with the rest term of the Cauchy type recently derived by the author in the case of $\psi$-difference calculus. The central importance of such a type formulas is beyond any doubt.

* see: historical remark at the beginning of this note

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1 One Historical Remark

Here are the famous examples of expansion

$$\partial_0 = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \frac{d^n}{dx^n}$$

or

$$\epsilon_0 = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \frac{d^n}{dx^n}$$

where $\partial_0$ is the divided difference operator while $\epsilon_0$ is at the zero point evaluation functional. If one compares these with "series universalissima"
of J.Bernoulli from *Acta Erudicorum* (1694) (see commentaries in [12]) and with
\[
exp\{yD\} = \sum_{k=0}^{\infty} \frac{y^k D^k}{k!}, \quad D = \frac{d}{dx},
\]
then confrontation with B.Taylor’s "Methodus incrementorum directa et inversa" (1715), London; entitles one to call the expansion formulas considered in this note "Bernoulli - Taylor formulas" or (for \(n \to \infty\)) "Bernoulli - Taylor series" [4].

2 Introduction

While deriving the Bernoulli-Taylor \(\psi\)-formula one is tempted to adapt the ingenious Viskov’s method [2] of arriving to formulas of such type for various pairs of operations. In our case these would be \(\psi\)-differentiation and \(\psi\)-integration (see: Appendix). However straightforward application of Viscov methods in \(\psi\)-extensions of umbral calculus leads to sequences which are not normal (Ward) hence a new invention is needed. This expected and verified here invention is the new specific \(\psi\) product of analytic functions or formal series. This note is based on [3] where the derivation of this new form of Bernoulli-Taylor \(\psi\) - formula was delivered due to the use of a specific \(\psi\) product of formal series.

3 Classical Bernoulli-Taylor formulas with the rest term of the Cauchy type by Viskov method

Let us consider the obvious identity
\[
\sum_{k=0}^{n} (\alpha_k - \alpha_{k+1}) = \alpha_0 - \alpha_{n+1}
\]
(1)
in which (1) we now put \(\alpha_k = a^k b^k; a, b \in A\). \(A\) is an associative algebra with unity over the field \(F=R,C\). Then we get
\[
\sum_{k=0}^{n} a^k (1-ab)b^k = 1 - a^{n+1} b^{n+1}; a, b \in A
\]
(2)
Numerous choices of \(a, b \in A\) result in many important specifications of (2).
Example 1. Let $\mathcal{F}$ denotes the linear space of sufficiently smooth functions $f : F \to F$. Let

$$a : \mathcal{F} \to \mathcal{F}; \quad (af)(x) = \int_a^b f(t)dt,$$

$$b : \mathcal{F} \to \mathcal{F}; \quad (bf)(x) = \left(\frac{d}{dx}\right)f(x); \quad (3)$$

$$l : \mathcal{F} \to \mathcal{F}; \quad (lf)(x) = f(x).$$

Then $[b,a]=1-ab=\varepsilon_\alpha$ where $\varepsilon_\alpha$ is evaluation functional on $\mathcal{F}$ i.e.

$$\varepsilon_\alpha(f) = f(\alpha) \quad (4)$$

Using now the text-book integral Cauchy formula ($k > 0$)

$$(a^k f)(x) = \int_a^x \frac{(x-t)^{k-1}}{(k-1)!} f(t)dt, \quad (5)$$

and under the choice $b\alpha=\Delta$ one gets from (2) the well-known Bernoulli-Taylor formula

$$f(x) = \sum_{k=0}^n \frac{(x-\alpha)^k}{k!} f^{(k)}(\alpha) + R_{n+1}(x) \quad (6)$$

with the rest term $R_{n+1}(x)$ in the Cauchy form

$$R_{n+1}(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \quad (7)$$

Example 2. Let $\mathcal{F}$ denotes the linear space of functions $f : Z_+ \to F; Z_+ = N \cup \{0\}$. Let

$$a : Z_+ \to \mathcal{F}; \quad (af)(x) = \sum_{k=0}^{x-1} f(k),$$

$$b : Z_+ \to \mathcal{F}; \quad (bf)(x) = f(x+1) - f(x), \quad (8)$$

$$l : Z_+ \to \mathcal{F}; \quad (lf)(x) = f(x).$$

It is easy to see that $[b,a]=1-ab=\varepsilon_0$ where $\varepsilon_0$ is evaluation functional i.e. $\varepsilon_0(f) = f(0)$. $b = \Delta$ is the standard difference operator with its left inverse
definite summation operator a. The corresponding \( \Delta \)-calculus Cauchy formula is also known (see formula (31 p.310 in [5]);

\[
(a^k f)(x) = \sum_{r=0}^{x-1} \frac{(x-r-1)^{k-1}}{(k-1)!} f(r); k > 0
\]  

(9)

where \( a^n = x(x-1)(x-2)...(x-n+1) \).

Under the choice (8) one gets from (2) the \( \Delta \)-calculus Bernoulli-Taylor formula [1]

\[
f(x) = \sum_{k=0}^{n} x^k \frac{(\Delta^k f)(0)}{k!} + R_{n+1}(x)
\]  

(10)

with the rest term \( R_{n+1}(x) \) in the Cauchy \( \Delta \) form

\[
R_{n+1}(x) = \sum_{r=0}^{x-1} \frac{(x-r-1)^n}{n!} (\Delta^{n+1} f)(r);
\]  

(11)

4 "\( \psi \) realization" of Bernoulli identity.

Now a specifically new form of the Bernoulli-Taylor formula with the rest term of the Cauchy type as well as Bernoulli-Taylor series is to be supplied in the case of \( \psi \)-difference umbral calculus (see [5-8] and [9,10] and references therein). For that to do we use natural \( \psi \)-umbral representation [13,14] of Graves-Heisenberg-Weyl (GHW) algebra [11,12] generators \( \hat{p} \) and \( \hat{q} \) and then we use Bernoulli identity (12)

\[
\hat{p} \sum_{k=0}^{n} \frac{(-\hat{q})^k \hat{p}^k}{k!} = \frac{(-\hat{q})^n \hat{p}^{n+1}}{n!}
\]  

(12)

derived by Viskov from (11) under the substitution (see (28) in [2])

\[
\alpha_0 = 0, \quad \alpha_k = (-1)^k (\hat{q})^{k-1} \hat{p}^k (k-1)!, \quad k = 1, 2, ... 
\]

due to \( \hat{p} \hat{q}^n = \hat{q}^n \hat{p} + n \hat{q}^{n-1} \) (n=1,2,...) resulting by induction from

\[
[\hat{p}, \hat{q}] = 1
\]  

(13)

Example 1. The choice \( \hat{p} = D \equiv \frac{d}{dx} \) and \( \hat{q} = \dot{x} - y, y \in F; \dot{x} f(x) = x f(x) \) after substitution into Bernoulli identity (12) and integration \( \int_{\alpha}^{x} \frac{dt}{t} \) gives the Bernoulli-Taylor formula (15)
Example 2. The choice $\hat{p} = \Delta$ and $\hat{q} = \hat{x} \circ E^{-1}$ where $E^\alpha f(x) = f(x+\alpha)$ after substitution into Bernoulli identity (12) and "$\Delta$ - integration" $\sum_r^{\alpha-1} g_r = 0$ and $\hat{x} \circ E^{-1}$ gives the Bernoulli - Mac laurin formula of the following form ($\alpha, x \in \mathbb{Z}, \nabla = 1 - E^{-1}$) with the rest term $R_{n+1}(x)$

$$f(0) = \sum_{k=0}^{n} \frac{\alpha^k}{k!} (-1)^{k+1} (\nabla^k f)(\alpha) + R_{n+1}(\alpha); \quad (14)$$

$$R_{n+1}(\alpha) = (-1)^n \sum_{r=0}^{\alpha-1} \frac{r^n}{n!} (\nabla^{n+1} f)(\alpha + 1). \quad (15)$$

Example 3. Here $f^{(k)} \equiv \partial^p f$ and $f(x)*\psi g(x) \equiv f(\hat{x}\psi) g(x)$ - see Appendix. The choice $\hat{p} = \partial^p$ and $\hat{q} = \hat{z} \psi (z = x - y)$ where $\hat{x}\psi x^n = \frac{n+1}{(n+1)!} x^{n+1}$ after substitution into Bernoulli identity (12) and "$\partial^p$ - integration" $\int_{\alpha}^{x} d\psi t$ (see: Appendix) gives another Bernoulli - Taylor $\psi$-formula of the form:

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} (x - \alpha)^k \psi^k f^{(k)}(\alpha) + R_{n+1}(x) \quad (16)$$

with the rest term $R_{n+1}(x)$ in the Cauchy-form

$$R_{n+1}(x) = \frac{1}{n!} \int_{\alpha}^{x} d\psi t(x - t)^n \psi^\alpha f(n+1)(t) dt \quad (17)$$

In the above notation $x^{0*\psi} = 1$, $x^\alpha = x^\alpha (x^{\alpha-1}) = x \psi ... \psi x$. Naturally $\partial^p x^{n*\psi} = n x^{(n-1)*\psi}$ and in general $f, g$ - may be formal series for which

$$\partial^p (f * \psi g) = (Df) * \psi g + f * (\partial^p \psi g) \quad (18)$$

i.e. Leibniz $*\psi$ rule holds \[13\] \[14\] \[15\].

Summary: These another forms of both the Bernoulli - Taylor formula with the rest term of the Cauchy type \[3\] as well as Bernoulli - Taylor series are quite easily handy due to the technique developed in \[13\] \[14\] where one may find more on $*\psi$ product devised perfectly suitable for the Ward's "calculus of sequences" \[12\] or more exactly $*\psi$ is devised perfectly suitable for the so-called $\psi$ - extension on Finite Operator Calculus of Rota (see \[9\] \[10\] \[14\] \[15\] and references therein)
5 Appendix

*ψ product

Let \( n - q \psi \equiv q_n; \quad q_n \neq 0; \quad n > 0 \). Let \( \partial_\psi \) be a linear operator acting on formal series and defined accordingly by \( \partial_\psi x^n = q_n x^{n-1} \).

We introduce now a intuition appealing \( \partial_\psi \)-difference-ization rules for a specific new \( *_\psi \) product of functions or formal series. This \( *_\psi \) product is what we call: the \( \psi \)-multiplication of functions or formal series as specified below.

**Notation A.1.**

For \( n \geq 0 \) hence \( x *_\psi 1 = (1)_\psi^{-1} x \neq x \) therefore \( x *_\psi \alpha 1 = \alpha 1 *_\psi x = x *_\psi \alpha = \alpha *_\psi x = \alpha (1)_\psi^{-1} x \) and \( \forall x, \alpha \in F; \)

\[
f(x) *_\psi x^n = f(\hat{x}\psi)x^n.
\]

For \( k \neq n \) \( x^n *_\psi x^k \neq x^k *_\psi x^n \) as well as \( x^n *_\psi x^k \neq x^{n+k} \) - in general.

In order to facilitate the formulation of observations accounted for on the basis of \( \psi \)-calculus representation of GHW algebra we shall use what follows.

**Definition A.1.** With Notation A.1. adopted define the \( *_\psi \) powers of \( x \) according to \( x^{n*\psi} \equiv x *_\psi x^{(n-1)*\psi} = \hat{x}\psi x^{(n-1)*\psi} = x *_\psi x *_\psi \ldots *_\psi x = \frac{n!}{n_q!} x^n; \quad n \geq 0 \). Note that \( x^{n*\psi} *_\psi x^{k*\psi} = \frac{k!}{k_q!} x^{n+k*\psi} \) for \( k \neq n \) and \( x^{0*\psi} = 1 \).

This noncommutative \( \psi \)-product \( *_\psi \) is devised so as to ensure the following observations.

**Observation A.1.**

\begin{itemize}
  \item[a)] \( \partial_\psi x^{n*\psi} = n x^{(n-1)*\psi}; \quad n \geq 0 \)
  \item[b)] \( \exp_\psi(\alpha x) \equiv \exp \{ \alpha \hat{x}\psi \} 1 \)
  \item[c)] \( \exp [\alpha x] *_\psi \exp_\psi(\beta \hat{x}\psi) 1 = (\exp_\psi(\alpha + \beta \hat{x}\psi) 1 \)
  \item[d)] \( \partial_\psi(x^k *_\psi x^{n*\psi}) = (Dx^k) *_\psi x^{n*\psi} + x^k *_\psi (\partial_\psi x^{n*\psi}) \)
  \item[e)] \( \partial_\psi(f *_\psi g) = (Df) *_\psi g + f *_\psi (\partial_\psi g); \quad f, g - \text{formal series} \)
  \item[f)] \( f(\hat{x}\psi)g(\hat{x}\psi) 1 = f(x) *_\psi \hat{g}(x); \quad \hat{g}(x) = g(\hat{x}\psi) 1. \)
\end{itemize}

\( \psi \)-Integration Let: \( \partial_\psi x^n = x^{n-1} \). The linear operator \( \partial_\theta \) is identical with divided difference operator. Let \( Q f(x) = f(qx) \). Recall also that to the "\( \partial_\psi \) difference-ization" there corresponds the \( q \)-integration which is a right inverse operation to "\( q \)-difference-ization". Namely

\[
F(z) := \left( \int q \varphi \right)(z) := (1 - q) z \sum_{k=0}^{\infty} \varphi(q^k z) q^k
\] (19)
i.e.

\[ F(z) \equiv \left( \int_q \varphi \right)(z) = (1 - q)z \sum_{k=0}^{\infty} q^k \hat{Q}^k \varphi(z) = (1 - q)z \frac{1}{1 - qQ} (z). \quad (20) \]

Of course

\[ \partial_q \circ \int_q = id \quad (21) \]

as

\[ \frac{1 - q\hat{Q}}{(1 - q)} \partial_0 \left( (1 - q) \frac{1}{1 - qQ} \right) = id. \quad (22) \]

Naturally (22) might serve to define a right inverse operation to ”q-difference-ization” \( (\partial_q \varphi)(x) = \frac{1 - qQ}{(1 - q)} \partial_0 \varphi(x) \) and consequently the ”q-integration” as represented by (19) and (20). As it is well known the definite q-integral is an numerical approximation of the definite integral obtained in the \( q \to 1 \) limit.

Finally we introduce the analogous representation for \( \partial_\psi \) difference-ization

\[ \partial_\psi = \hat{n}_\psi \partial_0; \quad \hat{n}_\psi x^{n-1} = n_\psi x^{n-1}; \quad n \geq 1 \quad (23) \]

Then

\[ \int_\psi x^n = \left( \hat{x} \frac{1}{\hat{n}_\psi} \right) x^n = \frac{1}{(n + 1)_\psi} x^{n+1}; \quad n \geq 0 \quad (24) \]

and of course \( \left( \int_\psi \equiv \int d_\psi \right) \)

\[ \partial_\psi \circ \int_\psi = id \quad (25) \]

Naturally

\[ \partial_\psi \circ \int_\psi^x f(t)d_\psi t = f(x) \]

The formula of ”per partes” \( \psi \)-integration is easily obtainable from (Observation A.1 e) and it reads:

\[ \int_a^b (f \ast_\psi \partial_\psi g)(t)d_\psi t = \left[(f \ast_\psi g)(t)\right]^b_a - \int_a^b (Df \ast_\psi g)(t)d_\psi t \quad (26) \]
Two Closing Remarks:
I. All these above may be quite easily extended \[15\] to the case of any \(Q \in \text{End}(P)\) linear operator that reduces by one the degree of each polynomial \[16\]. Namely one introduces \[15\]:

Definition A.2.

\[\hat{x}Q \in \text{End}(P), \hat{x}Q : F[x] \to F[x]\]

such that \((x^n) = \frac{(n+1)}{(n+1)} q_{n+1}; n \geq 0; \text{where } Qq_n = nq_{n-1} \).

Then \(\ast_Q\) product of formal series and \(Q\)-integration are defined analogously. (This has been accomplished by my student E. Krot).

II. In 1937 Jean Delsarte \[17\] had derived the general Bernoulli-Taylor formula for a class of linear operators \(\delta\) including linear operators that reduce by one the degree of each polynomial. The rest term of the Cauchy-like type in his Taylor formula (I) is given in terms of the unique solution of a first order partial differential equation in two real variables. This first order partial differential equation is determined by the choice of the linear operator \(\delta\) and the function \(f\) under expansion. In our Bernoulli-Taylor-formula (16)-(17) or in its straightforward \(\ast_Q\) product of formal series and \(Q\)-integration generalization - there is no need to solve any partial differential equation.

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