On Grothendieck groups and rings with an exact sequence for the Picard and class groups

Abolfazl Tarizadeh

Abstract. The main goal of this article is to investigate the Grothendieck groups, especially the Grothendieck ring $K_0(R)$, the Picard group $\text{Pic}(R)$ and the class group $\text{Cl}(R)$ of a given commutative ring $R$. Among the main results, we obtain a general theorem which asserts that for any commutative ring $R$ we have the following exact sequence of groups:

$$0 \rightarrow \text{Cl}(R) \rightarrow \text{Pic}(R) \rightarrow \text{Pic}(T(R))$$

where $T(R)$ denotes the total ring of fractions of $R$. As an application of this result, we obtain a canonical isomorphism of groups $\text{Cl}(R) \simeq \text{Pic}(R)$ whenever $T(R)$ has finitely many maximal ideals (e.g. $R$ is reduced with finitely many minimal primes). The latter result generalizes several classical theorems in the literature. It is also proved that the ring $K_0(R)$ modulo its nil-radical is canonically isomorphic to $H_0(R)$, the ring of all continuous functions $\text{Spec}(R) \rightarrow \mathbb{Z}$. Then as an application, the idempotents of the ring $K_0(R)$ are precisely determined in terms of the idempotents of $R$. We also give a new proof to the fact that the Picard group $\text{Pic}(R)$ can be canonically imbedded in the group of units of $K_0(R)$. Finally, we show that the localization of a monoid-ring $R[M]$ with respect to its multiplicative set of the unit vectors (monomials) is canonically isomorphic to the group-ring $R[G]$ when $G$ is the Grothendieck group of the commutative monoid $M$.

1. Introduction

The aim of this article is to investigate the Grothendieck groups, Grothendieck rings as well as the Picard groups and class groups of a commutative ring. There are various versions of Grothendieck groups and rings in the literature. In this article we are interested in studying the version which is constructed in the most standard and canonical way.

The following is a brief outline of the article. In §2, we recall some basic notions in order to facilitate easier reading.

In §3, we provide some improvements in two classical results on finitely generated projective modules (see Lemmas 3.1 and 3.3). These key results together with Lemma 3.4 and Corollary 3.5 allow us to reconstruct the Picard group of a commutative ring by a short and natural method. Theorem 3.7 is another main result of §3 which gives us a characterization of finitely generated projective modules in terms of the orthogonal idempotents.

---

2010 Mathematics Subject Classification. 13D15, 14A05, 16S34.

Key words and phrases. Grothendieck group; Grothendieck ring; Picard Group; Class group; Commutative monoid; Monoid-ring.
In §4, inspired by the classical class group in algebraic number theory (which is defined for integral domains), we first study this structure for arbitrary rings and several interesting results, including Lemma 4.1 and Theorems 4.2 and 4.4, are obtained. Then this investigation culminates in Theorem 4.9 which establishes the following exact sequence of groups:

\[ 0 \rightarrow \text{Cl}(R) \rightarrow \text{Pic}(R) \rightarrow \text{Pic}(T(R)). \]

This result has several nice consequences (see Corollaries 4.10 and 4.11) which gives us the canonical isomorphism of groups Cl(R) \(\cong\) Pic(R) under some natural assumptions which guarantees the triviality of the Picard group of T(R). Theorem 4.9 also shows that the triviality of the Picard group leads to the triviality of the class group. Theorem 4.12 is the further main result which is obtained in this regard. In Theorem 4.14, the relation of the Picard group Pic(R) with the group of units of the Grothendieck ring \(K_0(R)\) is established. Next in Theorem 4.16, it is proved that the ring \(K_0(R)\) modulo its nil-radical is canonically isomorphic to \(H_0(R)\), the ring of all continuous functions \(\text{Spec}(R) \rightarrow \mathbb{Z}\). Then as an application, the idempotents of the ring \(K_0(R)\) are precisely identified in terms of the idempotents of \(R\) (see Corollary 4.17).

Theorem 5.1 is one of the main results of §5 which establishes the following isomorphism of rings:

\[ S^{-1}(R[M]) \cong R[G] \]

where \(G\) is the Grothendieck group of the commutative monoid \(M\) and \(S\) denotes the multiplicative set of the unit vectors (monomials). Then two applications are given (see Corollaries 5.2 and 5.3). Theorem 5.4 is the next main result of this section which connects the Grothendieck group with the group of units. As an application of this result, Corollary 5.6 is deduced. In a classical article [7, §3], Levi proved a remarkable result which asserts that every torsion-free abelian group is a totally ordered group (see also [5, Theorem 6.31]). In Theorem 5.8 we generalize this result to the setting of monoids. Levi’s theorem is then deduced as an immediate consequence of our general result (see Corollary 5.9). Theorem 5.11 is another main result of this section.

2. Preliminaries

In what follows we briefly recall some basic background for the reader’s convenience. In this article, all monoids, groups, semirings and rings are assumed to be commutative. The group of units (invertible elements) of a ring \(R\) is denoted by \(R^*\).

2.1. Monoid-ring. Let \(R\) be a ring and \(M\) a monoid. Then the direct sum \(R[M] := \bigoplus_{a \in M} R\) can be made into a ring by the usual convolution product formula: \((r_a) \cdot (r'_b) = \sum_{(a,b) \in M^2} r_a r'_b \delta_{a+b}\). For each \(m \in M\) we denote the corresponding unit vector \((\delta_{a,m})_{a \in M}\) of \(R[M]\) by \(\epsilon_m\) or simply by \(m\) where \(\delta_{a,m}\) is the Kronecker delta. Then each \((r_m) \in R[M]\) can be written uniquely as \((r_m) = \sum_{m \in M} r_m \epsilon_m = \sum_{m \in M} r_m m\). The ring \(R[M]\) is called the monoid-ring of \(M\) over \(R\), or the monoid \(R\)-algebra of \(M\). If \(G\) is an abelian group, then \(R[G]\)
is also called the group-ring of $G$ over $R$. The sequence $\epsilon_0$ is the multiplicative identity of $R[M]$ where $0$ is the identity element of $M$. Clearly $\epsilon_m \cdot \epsilon_n = \epsilon_{m+n}$ for all $m, n \in M$. Hence, the set \( \{\epsilon_m : m \in M\} \) is a multiplicative set of $R[M]$. The map $\epsilon : M \to R[M]$ given by $m \mapsto \epsilon_m$ is a morphism of monoids from $M$ into the multiplicative monoid of the ring $R[M]$. Moreover the map $\eta : R \to R[M]$ given by $r \mapsto r\epsilon_0$ is a morphism of rings. The triple $(R[M], \epsilon, \eta)$ satisfies in the following universal property (see e.g.\[ Refer\ to\ a\ reference\ for\ details\]):\: For any such triple $(S, \varphi, \psi)$, i.e. $\varphi : M \to S$ is a morphism of monoids from $M$ into the multiplicative monoid of the ring $S$ and $\psi : R \to S$ is a morphism of rings, then there exists a unique morphism of rings $\theta : R[M] \to S$ such that $\varphi = \theta \circ \epsilon$ and $\psi = \theta \circ \eta$.

We call this, the universal property of the monoid-rings. In particular, if $f : M \to N$ is a morphism of monoids and $g : R \to S$ is a morphism of rings, then there exists a unique morphism of rings $h : R[M] \to S[N]$ such that the following diagrams are commutative:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{\epsilon_M} & & \downarrow{\epsilon_N} \\
R[M] & \xrightarrow{h} & S[N],
\end{array} \quad \begin{array}{ccc}
R & \xrightarrow{g} & S \\
\downarrow{\eta_R} & & \downarrow{\eta_S} \\
R[M] & \xrightarrow{h} & S[N].
\end{array}
\]

2.2. Grothendieck group. For a given monoid $M$, we may define an equivalence relation over the set $M^2 := M \times M$ as $(a, b) \sim (c, d)$ if there exists some $m \in M$ such that $m + (a + d) = m + (b + c)$. The set of all equivalence classes of this relation is denoted by $G(M)$ or simply by $G$. We also denote the equivalence class of a pair $(a, b) \in M^2$ simply by $[a, b]$. The set $G$ by the operation $[a, b] + [c, d] = [a + c, b + d]$ is an abelian group. Indeed, $[0, 0]$ is the identity element of $G$ where $0$ is the identity of $M$, and for each $[a, b] \in G$ its inverse is $[b, a]$. The abelian group $G$ is called the Grothendieck group of $M$. Note that the Grothendieck group $G$ is trivial if and only if for each $m \in M$ there exists some $m' \in M$ such that $m + m' = m'$. The canonical map $\gamma : M \to G$ given by $m \mapsto [m, 0]$ is a morphism of monoids and the pair $(G, \gamma)$ satisfies in the following universal property: For each such pair $(H, \psi)$, i.e. $H$ is an abelian group and $\psi : M \to H$ is a morphism of monoids, then there exists a unique morphism of groups $\varphi : G \to H$ such that $\psi = \varphi \circ \gamma$. Indeed, $\varphi([a, b]) = \psi(a) - \psi(b)$.

Grothendieck group is the fundamental construction of mathematics. For instance, the Grothendieck group of the additive monoid $\mathbb{N} = \{0, 1, 2, \ldots\}$ is called the additive group of integers and is denoted by $\mathbb{Z}$. This enables us to define the integers quite formally. That is, $n := [n, 0]$. Then $-n = [0, n]$ and $[m, n] = m - n$ for all $m, n \in \mathbb{N}$.

2.3. Semirings and ring completion. A semiring is a triple $(S, +, \cdot)$ such that the pairs $(S, +)$ and $(S, \cdot)$ are monoids with the identity elements $0$ and $1$ (respectively) such that the multiplication distributes over the addition: $a \cdot (b + c) = a \cdot b + a \cdot c$, and that $a \cdot 0 = 0$ for all $a, b, c \in S$. A morphism of semirings is a function between semirings such that it is a monoid morphism of both the additive and multiplicative monoids.
Let $S := (S, +, \cdot)$ be a semiring. The Grothendieck group $G(S)$ of the additive monoid $(S, +)$, can be made into a ring by defining the multiplication on it as $[a, b] \cdot [c, d] = [ac + bd, ad + bc]$. The multiplicative identity of this ring is $[1, 0]$. The ring $G(S)$ is called the Grothendieck ring, or the ring completion of the semiring $S$ (see [3, p. 115-116]). The canonical map $\gamma : S \to G(S)$ given by $s \mapsto [s, 0]$ is a morphism of semirings and the pair $(G(S), \gamma)$ satisfies in the following universal property: For each such pair $(R, \varphi)$, i.e. $\varphi : S \to R$ is a morphism of semirings into a ring $R$, then there exists a unique morphism of rings $\theta : G(S) \to R$ such that $\varphi = \theta \circ \gamma$.

Indeed, $G(\_\_\_)$ is a covariant functor from the category of semirings to the category of rings. In particular, the Grothendieck ring of the semiring $(\mathbb{N}, +, \cdot)$ gives us the ring of integers $\mathbb{Z}$. In particular, the basic formula $(-1) \cdot (-1) = [0, 1] \cdot [0, 1] = [1, 0] = 1$ is easily obtained.

2.4. Definitions using Grothendieck groups. Let $R$ be a ring, and let $S(R)$ be the collection of isomorphism classes of finitely generated $R$-modules (the isomorphism class of each finitely generated $R$-module $M$ will be denoted by $[M]$, as is customary). The class $S(R)$ is actually a “set”. Indeed, every finitely generated $R$-module is isomorphic to a quotient of the free $R$-module $R^n$ for some $n \geq 0$. We consider $S(R)$ as a semiring with addition induced by direct sum of $R$-modules, and multiplication by tensor product of $R$-modules. More precisely, if $M$ and $N$ are finitely generated $R$-modules, then $[M] + [N] = [M \oplus N]$ and $[M] \cdot [N] = [M \otimes_R N]$.

Note that the isomorphism class of the zero module is the additive identity, and the isomorphism class of $R$ is the multiplicative identity of this semiring. Sometimes we will denote the isomorphism class $[M]$ simply by $M$ if there is no confusion. Let $S_0(R)$ be the set of isomorphism classes of finitely generated projective $R$-modules. If $M$ and $N$ are finitely generated projective $R$-modules, then $M \oplus N$ and $M \otimes_R N$ are as well. Hence, $S_0(R)$ is a sub-semiring of $S(R)$ which is of particular interest. We denote the Grothendieck rings of the semirings $S(R)$ and $S_0(R)$ respectively by $K(R)$ and $K_0(R)$. It is obvious that $S_0(R) \subseteq S(R)$. Note that, in general, none of these additive monoids have the cancellation property. Hence, the canonical ring map $K_0(R) \to K(R)$, induced by the above inclusion, is not necessarily injective. If $R$ is either a local ring or a PID, then the semiring $S_0(R)$ is isomorphic to the semiring $\mathbb{N}$ and hence the ring $K_0(R)$ is isomorphic to $\mathbb{Z}$.

Remember that if $(M_i)$ is an inductive (direct) system of $R$-modules over a directed poset $I$, then the family of the $R$-modules $\text{Hom}_R(M_i, N)$ is a projective (inverse) system over the same poset, and for any $R$-module $N$ we have the canonical isomorphism of $R$-modules $\text{Hom}_R(\lim_{i \in I} M_i, N) \simeq \lim_{i \in I} \text{Hom}_R(M_i, N)$. Its proof can be found in every homological algebra book. In particular, if $(M_i)_{i \in I}$ is a family of modules over a ring $R$, then we have the canonical isomorphism of $R$-modules $\text{Hom}_R(\bigoplus_{i \in I} M_i, N) \simeq \prod_{i \in I} \text{Hom}_R(M_i, N)$. Also note that direct sums and direct summands of projective modules are projective. More precisely, $\bigoplus_{i \in I} M_i$ is $R$-projective if and only if each $M_i$ is $R$-projective. Finally, every direct summand of a finitely generated projective $R$-module is a finitely generated projective $R$-module.
3. Finitely generated projective modules and Picard group

There is a minor gap in the last lines of the proof of [1] Chap. III, Proposition 7.4. In the following result, we fill it in.

Lemma 3.1. Let $M$ and $N$ be modules over a ring $R$. If $M \otimes_R N \simeq R^n$ as $R$-modules for some $n \geq 1$, then $M$ and $N$ are finitely generated projective $R$-modules.

Proof. Clearly $M \otimes_R N = (x_k \otimes y_k : k = 1, \ldots, d)$ is a finitely generated $R$-module where $d \geq n$. By the universal property of free modules, there is a (unique) morphism of $R$-modules $h : R^d \to M$ such that $h(\epsilon_k) = x_k$ for all $k$. Since each $x_k \otimes y_k$ is in the image of the induced morphism $h \otimes 1_N : R^d \otimes_R N \to M \otimes_R N$, thus it is surjective. In fact, $h \otimes 1_N$ is a split epimorphism. That is, we have the following short split exact sequence

$0 \to K \xrightarrow{inc} R^d \otimes_R N \xrightarrow{h \otimes 1_N} M \otimes_R N \to 0$

where $K$ is the kernel of $h \otimes 1_N$. Note that the short split exact sequences are left short split exact by additive functors. Hence, by applying the additive functor $M \otimes_R -$ to the above sequence, we obtain the following short split exact sequence

$0 \to M \otimes_R K \xrightarrow{\cdot y} R^{nd} \to M^n \xrightarrow{\cdot 1} 0$. Since $n \geq 1$, so $M$ is a direct summand of the free $R$-module $R^{nd}$. Hence, $M$ is a finitely generated projective $R$-module. Symmetrically, $N$ is also a finitely generated projective $R$-module. \hfill \Box

Note that Lemma 3.1 does not hold for $n = 0$. For instance, let $I$ and $J$ be coprime ideals of a ring $R$, then $R/I \otimes_R R/J = 0$ but $R/I$ and $R/J$ are not necessarily $R$-projective (or $R$-flat). As a specific example, in the ring of integers $\mathbb{Z}$, take $I = 2\mathbb{Z}$ and $J = 3\mathbb{Z}$.

Recall that if $M$ is a finitely generated flat module over a ring $R$, then for each $p \in \text{Spec}(R)$, there exists a (unique) natural number $\text{rank}_{R_p}(M_p) := n_p \geq 0$ such that $M_p \simeq (R_p)^{n_p}$ as $R_p$-modules, because it is well known that every finitely generated flat module over a local ring is a free module (see [8] Theorem 7.10)). In fact, this number $n_p$ is the dimension of $\kappa(p)$-vector space $M \otimes_R \kappa(p)$ where $\kappa(p) = R_p/pR_p$ is the residue field of $R$ at $p$. Hence, we obtain a function $r_M : \text{Spec}(R) \to \mathbb{Z}$ given by $p \mapsto \text{rank}_{R_p}(M_p)$. This function is called the rank map of $M$. The rank maps of the zero module and $R$ (as a module over itself) are the constant functions with value zero and 1, respectively. One can also observe that if $M$ and $N$ are finitely generated flat $R$-modules, then $M \otimes N$ and $M \otimes_R N$ are finitely generated flat $R$-modules and we have $r_{M \otimes N} = r_M + r_N$ and $r_{M \otimes_R N} = r_M \cdot r_N$.

Lemma 3.2. Let $M$ and $N$ be finitely generated projective modules over a ring $R$ whose rank maps are the same. Then every surjective morphism of $R$-modules $M \to N$ is an isomorphism.

Proof. It is an interesting exercise. See also [10] p. 39]. \hfill \Box

Let $M$ be a module over a ring $R$. The $R$-module $M^* := \text{Hom}_R(M, R)$ is called the dual of $M$. We have a canonical morphism of $R$-modules $M \otimes_R M^* \to R$ given by $x \otimes \varphi \mapsto \varphi(x)$. The image of this canonical morphism is an ideal of $R$ which is called the trace ideal of $M$ and is denoted by $\text{tr}_R(M)$ or simply by $\text{tr}(M)$ if there is no confusion.

The following result improves [10] Chap 3, Prop. 20].
Lemma 3.3. Let $M$ be a finitely generated projective module over a ring $R$. Then $\text{Supp}(M) = \text{Spec}(R)$ if and only if $\text{tr}(M) = R$.

Proof. First assume $\text{Supp}(M) = \text{Spec}(R)$. If $J := \text{tr}(M)$ is a proper ideal of $R$, then $J \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \text{Spec}(R)$. It is well known that $JM = M$. For its proof see e.g. [13, Theorem 3.1]. It follows that $(JR_\mathfrak{p})M_\mathfrak{p} = M_\mathfrak{p}$. Thus by the Nakayama lemma, $M_\mathfrak{p} = 0$ which is a contradiction. Conversely, assume $\text{tr}(M) = R$. It is well known (see e.g. [13, Corollary 3.2] and its proof) that there exists an idempotent $e \in R$ such that $\text{tr}(M) = Re$ and $I := \text{Ann}(M) = R(1-e)$. Thus $e = 1$ and so $\text{Supp}(M) = V(I) = V(0) = \text{Spec}(R)$.

Lemma 3.4. Let $M$ be a finitely generated projective module over a ring $R$. Then $M^\ast$ and $\text{End}(M) \cong \text{Hom}_R(M, M)$ are finitely generated projective $R$-modules with $r_{M^\ast} = r_M$ and $r_{\text{End}(M)} = r_M^2$.

Proof. There exist an $R$-module $N$ and some natural number $n \geq 0$ such that $M \oplus N \cong R^n$ as $R$-modules. It follows that $M^\ast \oplus N^\ast \cong R^n$ and $\text{End}(M) \oplus \text{Hom}_R(N, M) \cong M^\ast$. Hence, $M^\ast$ and $\text{End}(M)$ are finitely generated projective $R$-modules. Suppose $r_M(p) = d$. By [10, $\S$3.4, Proposition 18], we have $(M^\ast)p \cong M^\ast \otimes_R R_\mathfrak{p} \cong \text{Hom}_R(M_\mathfrak{p}, R_\mathfrak{p}) \cong (R_\mathfrak{p})^d$. So $r_{M^\ast}(p) = d$. Hence, $r_{M^\ast} = r_M$. Similarly, we have $\text{End}(M) \otimes_R R_\mathfrak{p} \cong \text{Hom}_R(M_\mathfrak{p}, M_\mathfrak{p}) \cong (R_\mathfrak{p})^d \cong (R_\mathfrak{p})^{d^2}$. Thus $r_{\text{End}(M)}(p) = d^2$.

The above results enable us to provide an alternative proof to the following well known result.

Corollary 3.5. Let $M$ be a module over a ring $R$. Then $M$ is a finitely generated projective $R$-module of constant rank 1 if and only if the canonical morphism of $R$-modules $M \otimes_R M^\ast \to R$ is an isomorphism.

Proof. Assume $M$ is a finitely generated projective $R$-module of constant rank 1. Then by Lemma 3.3, $\text{tr}(M) = R$ and so the canonical map $M \otimes_R M^\ast \to R$ is surjective. By Lemma 3.4, $M^\ast$ is a finitely generated projective $R$-module of constant rank 1. Thus $M \otimes_R M^\ast$ is a finitely generated projective $R$-module of constant rank 1. Then by Lemma 3.2, the above map is an isomorphism. The reverse implication is an immediate consequence of Lemma 3.1.

Corollary 3.6. Let $M$ and $N$ be modules over a ring $R$ such that $M \otimes_R N \cong R$ as $R$-modules. Then $N \cong M^\ast$.

Proof. By Lemma 3.1, $M$ is a finitely generated projective $R$-module of constant rank 1. Then using Corollary 3.5, we have $N \cong N \otimes_R R \cong N \otimes_R (M \otimes_R M^\ast) \cong (N \otimes_R M) \otimes_R M^\ast \cong R \otimes_R M^\ast \cong M^\ast$.

By an invertible module over a ring $R$ we mean an $R$-module which satisfies one of the equivalent conditions of Corollary 3.5. Let $\text{Pic}(R)$ be the set of isomorphism classes of invertible $R$-modules. By Corollary 3.5, the set $\text{Pic}(R)$ by the binary operation $[M] \cdot [N] = [M \otimes_R N]$ induced by the tensor product of $R$-modules is an abelian group whose identity element is the isomorphism class of $R$ and the inverse of each $[M] \in \text{Pic}(R)$ is the isomorphism class of its dual $[M^\ast]$. The group $\text{Pic}(R)$ is called the Picard group of $R$. If there is no confusion, we shall denote each element $[M]$ of $\text{Pic}(R)$ simply by $M$. If $\varphi : R \to S$ is a morphism of rings, then the map $\text{Pic}(R) \to \text{Pic}(S)$ given by $M \mapsto M \otimes_R S$ is well-defined and a morphism
of groups. In fact, Pic(−) is a covariant and essentially surjective functor from the category of commutative rings to the category of abelian groups. Essentially surjectiveness means that for any abelian group $G$ there exists a Dedekind domain $R$ such that $\text{Pic}(R) \simeq \text{Cl}(R) \simeq G$ where $\text{Cl}(R)$ denotes the class group of $R$ (see §4).

It is well known that the rank map of a finitely generated flat $R$-module $M$ is continuous (where $\mathbb{Z}$ is equipped with the discrete topology), or equivalently locally constant, if and only if $M$ is a projective $R$-module. For its proof see e.g. [2] Tag 00NX. Remember that “locally constant” means that the rank map of $M$ is constant in an open neighborhood of each prime ideal of $R$ (the quasi-compactness of $\text{Spec}(R)$ yields that every locally constant rank map takes finitely many values. But it is important to notice that its converse does not hold. Indeed, it is easy to construct a finitely generated flat module whose rank map takes finitely many values, but it is not a projective module). This fact allows us to provide a characterization of finitely generated projective modules in terms of the orthogonal idempotents:

**Theorem 3.7.** Let $M$ be a finitely generated module over a ring $R$. Then $M$ is a projective $R$-module if and only if there exists a finite sequence $e_0, \ldots, e_n$ of orthogonal idempotents of $R$ such that $\sum_{k=0}^{n} e_k = 1$ and $M_{p} \simeq (R_{p})^{k}$ for all $p \in D(e_k)$.

In this case, the annihilator of the $R$-module $\Lambda^k(M)$ is generated by the idempotent $\sum_{i=0}^{k-1} e_i$ for all $k \in \{1, \ldots, n, n+1\}$.

**Proof.** If $M$ is $R$-projective, then its rank map $r_M$ is continuous. Using the quasi-compactness of the prime spectrum, then there exists a natural number $n \geq 0$ such that $\text{Spec}(R) = \bigcup_{k=0}^{n} r_M^{-1}(\{k\})$. Clearly each $r_M^{-1}(\{k\})$ is a clopen subset of $\text{Spec}(R)$. Thus by the canonical correspondence between the idempotents and clopens (see e.g. [4] Theorem 1.1), there exists an idempotent $e_k \in R$ such that $r_M^{-1}(\{k\}) = D(e_k)$. Now the desired assertions are easily deduced. Conversely, using the hypothesis, we first obtain that $M$ is a flat $R$-module, because flatness is a local property. Again by the hypothesis, the rank map of $M$ is continuous. Hence, $M$ is $R$-projective. Now we show that the annihilator of $\Lambda^k(M)$ is generated by $\sum_{i=0}^{k-1} e_i$. Indeed, we have the canonical isomorphism of $R_p$-modules $\Lambda^k_{R_p}(M) \otimes_R R_p \simeq \Lambda^k_{R_p}(M_p)$. Also remember that if $F$ is a free $R$-module of rank $d \geq 0$, then $\Lambda^k(F)$ is a free $R$-module of rank $\binom{d}{k}$ and hence $\Lambda^k(F) \neq 0$ for all $0 \leq k \leq d$ and $\Lambda^k(F) = 0$ for all $k > d$. Therefore $\text{Supp}(\Lambda^k(M)) = \bigcup_{i=k}^{n} D(e_i) = D(\sum_{i=k}^{n} e_i)$. Since $\Lambda^k(M)$ is a finitely generated projective $R$-module, so its annihilator is generated by an idempotent element $e \in R$, because the annihilator of every finitely generated projective module is generated by an idempotent element (see e.g. [3] Corollary 3.2). Thus $\text{Supp}(\Lambda^k_{R}(M)) = V(e) = D(1-e)$. It follows that $1-e = \sum_{i=k}^{n} e_i$, hence $e = \sum_{i=0}^{k-1} e_i$. \qed
Remark 3.8. Let $M$ be a finitely generated projective module over a ring $R$. Using the notations of Theorem 3.7 then in particular $\text{Ann}(M) = Re_0$. Note that some of the $e_i$’s may be zero. In fact, $e_k \neq 0$ if and only if $\text{rank}_{R_p}(M_p) = k$ for some $p \in \text{Spec}(R)$. If $e_k \neq 0$ then $\Lambda^i(M) \neq 0$ for all $i \leq k$. Clearly the nonzero $e_k$’s are pairwise distinct. Also note that $\Lambda^k(M) = 0$ for all $k \geq n + 1$. In summary, if $I_k$ denotes the annihilator of $\Lambda^k(M)$, then we have $I_0 = 0 \subseteq I_1 = Re_0 \subseteq I_2 = R(e_0 + e_1) \subseteq \ldots \subseteq I_n = R(\sum_{i=0}^{n-1} e_i) \subseteq I_{n+1} = R$. Finally, if $R$ has no nontrivial idempotents, then there exists an integer $d \geq 0$ such that $M_p \simeq (R_p)^d$ for all $p \in \text{Spec}(R)$, that is to say, $M$ has constant rank $d$.

Example 3.9. We illustrate Theorem 3.7 with two examples. If $R$ is a ring then for $M := R^2$ we have the sequence $e_0 = e_1 = 0$ and $e_2 = 1$. As another example, if $e \in R$ is an idempotent then for the projective $R$-module $M := Re$ we have the sequence $e_0 = 1 - e$ and $e_1 = e$.

We conclude this section with the following auxiliary result.

Proposition 3.10. Let $M$ be a module over a ring $R$ with the property that for each $p \in \text{Spec}(R)$ there exists some $f \in R \setminus p$ such that $M_f$ is a finitely generated $R_f$-module. Then $M$ is a finitely generated $R$-module.

Proof. Using the quasi-compactness of $\text{Spec}(R)$, there exist finitely many elements $f_1, \ldots, f_n \in R$ such that $\text{Spec}(R) = \bigcup_{i=1}^n D(f_i)$ and $M_{f_i} = (x_{i,1}/1, \ldots, x_{i,d_i}/1)$ is a finitely generated $R_{f_i}$-module for all $i \in \{1, \ldots, n\}$. We show that $M$ as $R$-module is generated by the elements $x_{i,1}, \ldots, x_{i,d_i}$ with $i = 1, \ldots, n$. If $m \in M$ then for each $i \in \{1, \ldots, n\}$ we may write $m/1 = \sum_{k=1}^{d_i} (r_{i,k}f_{i}^{s_k})(x_{i,k}/1)$. Thus there exists a natural number $N \geq 1$ such that $f_i^Nm = \sum_{k=1}^{d_i} r_{i,k}x_{i,k}$ for all $i \in \{1, \ldots, n\}$. We have $\text{Spec}(R) = \bigcup_{i=1}^n D(f_i^N)$ and so $1 = \sum_{i=1}^n r_i f_i^N$. It follows that $m = \sum_{i,k} r_{i,k}x_{i,k}$. This completes the proof. \hfill \Box

4. Picard group versus class group and $K_0(R)^*$

Let $R$ be an arbitrary ring. Then $Z(R) = \{a \in R : \text{Ann}(a) \neq 0\}$ is called the set of zero divisors of $R$. The localization $T(R) := S^{-1}R$ with respect to the multiplicative set $S := R \setminus Z(R)$ is called the total ring of fractions of $R$. Identify the ring $R$ with its canonical image in $T(R)$, then remember that an $R$-submodule $I$ of $T(R)$ is called a fractional ideal of $R$ if $aI \subseteq R$ for some $a \in R \setminus Z(R)$. Clearly every finitely generated $R$-submodule of $T(R)$ is a fractional ideal. If $I$ and $J$ are fractional ideals of $R$, then by $IJ$ we mean the set of all finite sums $\sum_{k=1}^n x_ky_k$ with $x_k \in I$ and $y_k \in J$ for all $k$. Clearly $IJ$ is a fractional ideal of $R$.

In particular, if $I$ is a fractional ideal of $R$ then $Ix$ is also a fractional ideal of $R$ for all $x \in T(R)$. A fractional ideal $I$ of $R$ is called an invertible (fractional) ideal of $R$ if there exists another fractional ideal $J$ of $R$ such that $IJ = R$. In this case, $J = \{x \in T(R) : xI \subseteq R\}$ and $J$ is often denoted by $I^{-1}$. Also note that if $I$ and
J are $R$-submodules of $T(R)$ with $IJ = R$, then $I$ is an (invertible) fractional ideal of $R$.

**Lemma 4.1.** Let $I$ and $J$ be fractional ideals of a ring $R$. If one of them is $R$-flat, then $I \otimes_R J$ is canonically isomorphic to $IJ$ as $R$-modules.

**Proof.** We show that the canonical map $I \otimes_R J \to IJ$ which sends each pure tensor $x \otimes y$ of $I \otimes_R J$ into $xy \in IJ$ is an isomorphism of $R$-modules. This map is clearly a surjective morphism of $R$-modules. To see its injectivity, suppose $\sum_{i=1}^n x_i y_i = 0$ where $x_i \in I$ and $y_i \in J$ for all $i$. The canonical map $I \otimes_R J \to T(R) \otimes_R T(R)$ is injective, because $T(R)$ and by hypothesis, one of $I$ or $J$ are flat $R$-modules. Hence, it will be enough to show that in $T(R) \otimes_R T(R)$ the element $\sum_{i=1}^n x_i \otimes y_i$ is zero. We have $aI \subseteq R$ for some $a \in R \setminus J(R)$ and so $ax_i \in R$ for all $i$. Therefore in $T(R) \otimes_R T(R)$ we may write $\sum_{i=1}^n x_i \otimes y_i = \sum_{i=1}^n ax_i \otimes (1/a)y_i = \sum_{i=1}^n 1 \otimes ax_i (1/a)y_i = 1 \otimes (\sum_{i=1}^n x_i y_i) = 0$. This completes the proof. \qed

**Theorem 4.2.** If $I$ is an invertible ideal of a ring $R$, then $I$ is a finitely generated projective $R$-module of rank 1.

**Proof.** There exist some fractional ideal $J$ of $R$ such that $IJ = R$. Then we may write $1 = \sum_{k=1}^n x_k y_k$ where $x_k \in I$ and $y_k \in J$ for all $k$. Clearly $I = \sum_{k=1}^n Rx_k$, because if $x \in I$ then we have $x = \sum_{i=1}^n (xy_k)x_k$ where $xy_k \in IJ = R$ for all $k$. Hence, $I$ is a finitely generated $R$-module. Then we show that $I$ is a projective $R$-module. Consider the surjective morphism of $R$-modules $\varphi : R^n \to I$ which sends each unit vector $\epsilon_k \in R^n$ into $x_k$. The map $\psi : I \to R^n$ given by $\psi(x) = (xy_1, \ldots, xy_n)$ is a morphism of $R$-modules and $\varphi \circ \psi$ is the identity map of $I$. Thus the short exact sequence $0 \longrightarrow \ker \varphi \longrightarrow R^n \xrightarrow{\varphi} I \longrightarrow 0$ splits and so $I$ is a projective $R$-module. Finally, we show that for each $p \in \text{Spec}(R)$ then $I_p \simeq R_p$ as $R_p$-modules. By Lemma 4.1, $I \otimes_R J \cong R$ as $R$-modules. This yields that $J_p \simeq R_p$ and $I_p \simeq (R_p)^n$ for some natural numbers $m, n \geq 0$, because every finitely generated projective (even flat) module over a local ring is a free module. It follows that $mn = 1$ and so $m = n = 1$. \qed

**Proposition 4.3.** Every invertible ideal $I$ of a ring $R$ contains a non-zerodivisor of $R$. In particular, every invertible ideal is faithful.

**Proof.** We have $IJ = R$ for some fractional ideal $J$ of $R$. Then we have $1 = \sum_{k=1}^n x_k y_k$ where $x_k \in I$ and $y_k \in J$ for all $k$. Also $aJ \subseteq R$ for some $a \in R \setminus Z(R)$. Thus $a = \sum_{k=1}^n ay_k x_k \in I$, because $ay_k \in R$ for all $k$. \qed

The following result can be viewed as the converse of Theorem 4.2.

**Theorem 4.4.** Let $I$ be a fractional ideal of a ring $R$. If $I$ is a projective $R$-module and contains a non-zerodivisor of $R$, then it is an invertible ideal of $R$. 
Proof. Let $I = \sum_{k \in S} Rx_k$ with $x_k \in T(R)$ for all $k$. Then there exists a surjective morphism of $R$-modules $\varphi : F = \bigoplus_{k \in S} R \to I$ such that $\varphi(\epsilon_k) = x_k$ for all $k$. Since $I$ is $R$-projective, there exists a morphism of $R$-modules $\psi : I \to F$ such that $\varphi \psi : I \to I$ is the identity map. For each $i \in S$, by the universal property of free modules, there exists a (unique) morphism of $R$-modules $g_i : R \to R$ such that $g_i(\epsilon_k) = \delta_{i,k}$ for all $k$. By hypothesis, there is some $b \in R \setminus Z(R)$ such that $b \in I$. Then clearly $J := \sum_{i \in S} R y_i$ is a fractional ideal of $R$ where $y_i := (1/b) g_i(\psi(b)) \in T(R)$. To conclude the proof it suffices to show that $IJ = R$. There is a non-zerodivisor $a \in R$ such that $aI \subseteq R$. We have $abx_k y_i = (ax_k)(g_i(\psi))(b) = (g_i(\psi))(abx_k)(x_k) = ab(g_i(\psi))(x_k)$. It follows that $x_k y_i = (g_i(\psi))(x_k) \in R$, because $ab$ is invertible in $T(R)$. Hence, $IJ = \sum_{k,i \in S} Rx_k y_i \subseteq R$. To see the reverse inclusion, if $\psi(b) = (r_i)_{i \in S}$ then each $r_i = (g_i(\psi))(b)$. Remember that $r_i = 0$ for all but a finite number of indices $i$. So $1 = (1/b)(\psi(b))(1/b) \sum_{i \in S} r_i x_i = \sum_{i \in S} x_i y_i \in IJ$. \hfill \Box

Remark 4.5. The assumption of “containing a non-zerodivisor” in Theorem 4.4 is vital. In other words, a fractional ideal which is also a finitely generated projective module of rank 1 is not necessarily an invertible ideal.

Corollary 4.6. If $I$ is an invertible ideal of a ring $R$, then $I^{-1} \simeq I^*$ as $R$-modules.

Proof. By Theorem 4.2, $I$ is $R$-projective and so it is $R$-flat. Thus by Lemma 4.1, $I \otimes_R I^{-1} \simeq R$. Now the desired conclusion easily follows from Corollary 3.6. \hfill \Box

Let $R$ be a ring. Then $T(R)^* = \{a/b \in T(R) : a \in R \setminus Z(R)\}$. Clearly a principal fractional ideal $Rx$ of $R$ with $x \in T(R)$ is invertible if and only if $x$ is invertible in $T(R)$. The set of invertible ideals of $R$ under the operation of multiplication of fractional ideals is an abelian group. This group modulo its subgroup $H := \{Rx : x \in T(R)^*\}$ is called the ideal class group or simply the class group of $R$ and is denoted by $\text{Cl}(R)$. For given invertible ideals $I$ and $J$ of $R$, then in the group $\text{Cl}(R)$ we have $IH = JH$ if and only if $aI = bJ$ for some $a, b \in R \setminus Z(R)$.

Recall that by a semi-local ring we mean a ring with finitely many maximal ideals.

Lemma 4.7. Every finitely generated flat module of constant rank over a semi-local ring is a free module.

Proof. See [2] Tags 00NX, 00NZ, 02M9]. \hfill \Box

Note that in the above lemma, the “constant rank” assumption is crucial. For instance see [12, Remark 3.12].

Corollary 4.8. The Picard group of every semi-local ring is trivial.

Proof. It follows from Lemma 4.7. \hfill \Box

Our next goal is to prove one of the main results of this article:

Theorem 4.9. For any ring $R$ we have the following exact sequence of groups:

\[ 0 \longrightarrow \text{Cl}(R) \longrightarrow \text{Pic}(R) \longrightarrow \text{Pic}(T(R)) \]
First we realize the canonical embedding \( \text{Cl}(R) \to \text{Pic}(R) \). Let \( I \) be an invertible fractional ideal of \( R \). Then by Theorem \ref{thm:invertible_ideal} it is a finitely generated projective \( R \)-module of rank 1. Thus \([I]\) is a member of \( \text{Pic}(R) \). Hence, the assignment \( I \mapsto [I] \) from the group of invertible fractional ideals of \( R \) into \( \text{Pic}(R) \) is a well-defined map. By Lemma \ref{lem:group_morphism} it is also a group morphism. If \( Rx \) is an invertible principal (fractional) ideal of \( R \) with \( x \in T(R)^\star \), then \( \text{Ann}_R(x) = 0 \) and so \( Rx \cong R \) as \( R \)-modules. Conversely, if \( \varphi : R \to I \) is an isomorphism of \( R \)-modules, then \( I = Ry \) where \( y := \varphi(1) \in T(R)^\star \). Thus the above morphism induces an injective group map from \( \text{Cl}(R) \) into \( \text{Pic}(R) \). Then to conclude the assertion, it suffices to show that the image of this embedding is the kernel of the group morphism \( \text{Pic}(R) \to \text{Pic}(T(R)) \) which is given by \( M \mapsto M \otimes_R T(R) \). To accomplish this, let \( I \) be an invertible ideal of \( R \). Then \( IJ = R \) for some \( R \)-submodule \( J \) of \( T(R) \). Since \( T(R) \) is \( R \)-flat, so we obtain an injective morphism of \( R \)-modules \( I \otimes_R T(R) \xrightarrow{\sim} T(R) \otimes_R T(R) \xrightarrow{\sim} T(R) \) which is given by \( x \otimes y \mapsto xy \). Indeed, it is a morphism of \( T(R) \)-modules. This map is also surjective, because if we take \( x \in T(R) \) then the element \( \sum_{i=1}^n x_i \otimes x y_i \) of \( I \otimes_R T(R) \) is mapped into \( x \) where \( 1 = \sum_{i=1}^n x_i y_i \) with \( x_i \in I \) and \( y_i \in J \) for all \( i \). So \( I \otimes_R T(R) \cong T(R) \). To see the reverse inclusion, let \( M \) be an invertible \( R \)-module (i.e., a finitely generated projective \( R \)-module of rank 1) such that \( M \otimes_R T(R) \cong T(R) \) as \( T(R) \)-modules. To complete the proof, it will be enough to show that \( M \) as an \( R \)-module is isomorphic to an invertible ideal of \( R \). Since \( M \) is \( R \)-flat, so the canonical injective map \( R \to T(R) \) gives us an injective morphism of \( R \)-modules \( f : M \xrightarrow{\sim} M \otimes_R R \xrightarrow{\otimes_R T(R)} M \otimes_R T(R) \xrightarrow{\sim} T(R) \). Hence, \( M \cong \text{Im}(f) \) as \( R \)-modules. For any morphism of groups \( \varphi : G \to H \), if \( g \in \text{Ker}(\varphi) \) then clearly \( g^{-1} \in \text{Ker}(\varphi) \). This yields that \( M^* \otimes_R T(R) \cong T(R) \) as \( T(R) \)-modules. Then similarly above, we obtain an (injective) morphism of \( R \)-modules \( f' : M^* \to T(R) \).

So the map \( h : R \xrightarrow{\sim} M \otimes_R M^* \xrightarrow{\otimes_R T(R)} T(R) \otimes_R T(R) \xrightarrow{\sim} T(R) \) is also an injective morphism of \( R \)-modules. Clearly \( \text{Im}(h) = Rx \) where \( x = h(1) \). It is also obvious that \( \text{Im}(h) = IJ \) where \( I = \text{Im}(f) \) and \( J = \text{Im}(f') \). But \( x \) is an invertible element of \( T(R) \), because we may write \( x = a/b \) with \( a \in R \) and \( b \in R \setminus Z(R) \), if \( ab' = 0 \) for some \( a' \in R \), then \( h(a') = a'h(1) = 0 \) and so \( a' = 0 \), hence \( a \) is a non-zero divisor of \( R \). Thus \( I(x^{-1}J) = R \).

The above theorem, in particular, tells us that for any ring \( R \) if the Picard group of \( T(R) \) is trivial, then we have the canonical isomorphism of groups \( \text{Cl}(R) \cong \text{Pic}(R) \), or equivalently, every invertible \( R \)-module is isomorphic to an invertible fractional ideal of \( R \). In the next two results we will observe that this condition is fulfilled quite naturally:

**Corollary 4.10.** Let \( R \) be a ring such that \( T(R) \) has finitely many maximal ideals. Then we have the canonical isomorphism of groups \( \text{Cl}(R) \cong \text{Pic}(R) \).

**Proof.** By Corollary \ref{cor:finite_maximal} the Picard group of \( T(R) \) is trivial. Then apply Theorem \ref{thm:invertible_ideal}.

**Corollary 4.11.** If \( R \) is a reduced ring with finitely many minimal primes, then we have the canonical isomorphism of groups \( \text{Cl}(R) \cong \text{Pic}(R) \).
Proof. Since $R$ is reduced, thus $Z(R) = \bigcup_{p \in \text{Min}(R)} p$. Then using the prime avoidance lemma, we observe that $T(R)$ has finitely many maximal ideals (whose Krull dimension is also zero). Hence, the assertion is deduced from Corollary 4.10. □

Theorem 4.9 also tells us that if the Picard group of a ring is trivial, then its class group is also trivial. In particular, by [2, Tag 0BCH], the Picard group of each UFD and so its class group are trivial.

The following result is already well known for semi-local integral domains. We will observe that it can be considerably generalized to every semi-local ring (with fairly the same proof).

**Theorem 4.12.** Every invertible fractional ideal of a semi-local ring is principal.

**Proof.** Let $R$ be a semi-local ring and let $I \subseteq T(R)$ be an invertible fractional ideal of $R$. So there exists an $R$-submodule $J$ of $T(R)$ such that $IJ = R$. Let $M_1, \ldots, M_n$ be the maximal ideals of $R$. For each $k$ we may choose some $x_k \in I$ and $y_k \in J$ such that $x_ky_k \notin M_k$. Also, for each $k$ there exists some $a_k \in \bigcap_{1 \leq i \leq n, i \neq k} M_i$ such that $a_k \notin M_k$. Then clearly $y := \sum_{k=1}^{n} a_k y_k \in J$. Thus $Iy$ is an ideal of $R$ (i.e., $Iy \subseteq IJ = R$). We claim that $Iy$ is the unit ideal of $R$. If not, then $Iy \subseteq M_i$ for some $i$. It follows that $x_iy = (x_iy_i)a_i + \sum_{1 \leq k \leq n, k \neq i} (x_iy_k)a_k \in M_i$. Note that each $x_iy_k \in IJ = R$ and so $\sum_{1 \leq k \leq n, k \neq i} (x_iy_k)a_k \in M_i$. This yields that $(x_iy_i)a_i \in M_i$ which is a contradiction. Therefore $Iy = R$. Now to conclude the assertion, it suffices to show that $y$ is invertible in $T(R)$, because in this case we will have $I = Ry^{-1}$. Since $y = a/b$ for some $a \in R$ and $b \in R \setminus Z(R)$, it will be enough to show that $a \notin Z(R)$. Suppose $a \in Z(R)$. Since $Iy = R$, so $1 = xy$ for some $x \in I$. But we may write $x = c/d$ where $c \in R$ and $d \in R \setminus Z(R)$. It follows that $bd = ac \in Z(R)$ which is a contradiction, because $bd$ is a non-zerodivisor of $R$. This completes the proof. □

As an immediate consequence of the above result we obtain that: every Noetherian Prüfer domain, or equivalently every Dedekind domain, with finitely many maximal ideals is a PID.

**Corollary 4.13.** The class group of every semi-local ring is trivial.

**Proof.** It follows from Theorem 4.12. As a second proof, it follows from Corollary 4.8 and Theorem 4.9. □

**Theorem 4.14.** Let $R$ be a ring. Then the group $\text{Pic}(R)$ can be canonically embedded in the group $K_0(R)^*$.

**Proof.** We show that the map $\text{Pic}(R) \to K_0(R)^*$ given by $M \mapsto [M, 0]$ is an injective morphism of groups. Let $M \in \text{Pic}(R)$. Then by Corollary 4.5 $M \otimes_R M^* \simeq R$. Thus $[M, 0] \cdot [M^*, 0] = [R, 0]$. Hence, $[M, 0]$ is a member of $K_0(R)^*$ and so the above map is well-defined. This map is clearly a morphism of groups. For injectivity, suppose $[M, 0] = [R, 0]$. Then there exists some natural number $n \geq 0$ such that...
$M \oplus R^n \simeq R^{n+1}$ as $R$-modules. It suffices to show that $M \simeq R$. We will use the exterior powers to establish this isomorphism. It is well known that for any two modules $M$ and $N$ over a ring $R$, we have the canonical isomorphism of $R$-modules $\Lambda^k(M \oplus N) \simeq \bigoplus_{p+q=k} \Lambda^p(M) \otimes_R \Lambda^q(N)$. It is also well known that if $M$ is finitely generated projective $R$-module of rank $d \geq 0$, then $\Lambda^k(F)$ is a free $R$-module of rank $\binom{d}{k}$ for all $0 \leq k \leq d$ and $\Lambda^k(F) = 0$ for all $k > d$. Finally, since $M$ is a finitely generated projective $R$-module of rank 1, thus $\Lambda^k_R(M) = 0$ for all $k \geq 2$. Indeed, it is deduced from the fact that the exterior powers commute with the localization.

More precisely, if $p \in \text{Spec}(R)$ then we have the canonical isomorphisms of $R_p$-modules $(\Lambda^k_R(M))_p \simeq \Lambda^k_R(M) \otimes_R R_p \simeq \Lambda^k_R(M_p) \simeq \Lambda^k_R_R(R_p) = 0$ for all $k \geq 2$.

Now using these observations, we have $R \simeq \bigwedge^{n+1}(R^n) \simeq \bigwedge^{n+1}(M \otimes_R R^n) \simeq \bigoplus_{p+q=n+1} \Lambda^p(M) \otimes_R \Lambda^q(R^n) \simeq \Lambda^n(M) \otimes_R \bigwedge^n(R^n) \simeq M \otimes_R R \simeq M$. This completes the proof.

\[\square\]

**Remark 4.15.** It is very important to notice that the isomorphism which is obtained in Theorem 4.14 does not hold in general. More precisely, let $M$ be a module over a ring $R$ such that there exists natural numbers $d$, $n \geq 0$ for which $M \oplus R^d \simeq R^n$ as $R$-modules (in this case, $M$ is called a stably free module). Then clearly $d \leq n$ and $M$ is a finitely generated projective $R$-module of rank $n - d$. If $n - d \geq 2$ then $M$ is not necessarily a free module. That is, there are stably free modules which are not free (see e.g. [4, Chap. 3], [5, p. 301] or [6, Chap. XXI, §2]).

Remember that a ring $R$ modulo its nilradical is denoted by $R_{\text{red}}$. Also recall that by $H_0(R)$ we mean the ring of all continuous functions $\text{Spec}(R) \rightarrow \mathbb{Z}$ where $\mathbb{Z}$ is equipped with the discrete topology. For more information on this ring see e.g. [14, §5].

**Theorem 4.16.** Let $R$ be a ring. Then we have the canonical isomorphism of rings $K_0(R)_{\text{red}} \simeq H_0(R)$.

**Proof.** Any two isomorphic finitely generated flat $R$-modules have the same rank maps. Moreover if $M$ is a finitely generated projective $R$-module, the its rank map $r_M : \text{Spec}(R) \rightarrow \mathbb{Z}$ is a continuous function. Hence, the map $S_0(R) \rightarrow H_0(R)$ given by $[M] \mapsto r_M$ is well-defined. In fact, it is a morphism of semirings. Then using the universal property of Grothendieck rings, we obtain a morphism of rings $K_0(R) \rightarrow H_0(R)$ given by $[M,N] \mapsto r_M - r_N$. To conclude the assertion, it suffices to show that this ring map is surjective and its kernel is the nilradical of $K_0(R)$. Let $f : \text{Spec}(R) \rightarrow \mathbb{Z}$ be a continuous function. Thus for each $n \in \mathbb{Z}$, $f^{-1}\{n\}$ is a clopen (both open and closed) subset of $\text{Spec}(R)$. Then by [14, Theorem 1.1], there exists an idempotent $e_n \in R$ such that $f^{-1}\{n\} = D(e_n)$. The quasi-compactness of $\text{Spec}(R)$ yields that all of the $e_n = 0$ except for finite number of indices $n$. Hence, $\sum_{n \in \mathbb{Z}} n[Re_n,0]$ is a member of $K_0(R)$. Note that if $n > 0$ then $n[Re_n,0] = \left[ \bigoplus_{n=1}^{\infty} Re_n,0 \right]$. If $n < 0$ then $n[Re_n,0] = \left[ 0, \bigoplus_{n=-\infty}^{n} Re_n \right]$. If $n = 0$ then $n[Re_n,0] = 0$. Note that if $e \in R$ is an idempotent then $r_{Re}(p)$ is either 0 or 1, according as $e \in p$ or $e \notin p$. Using this, then we show that $f = \sum_{n \in \mathbb{Z}} n \cdot r_{Re_n}$. If $p \in \text{Spec}(R) = \bigcup_{n \in \mathbb{Z}} D(e_n)$ then there exists a unique $k \in \mathbb{Z}$
such that \( p \in D(\epsilon_k) \). This shows that \( r_{Re_k}(p) = 1 \) and \( r_{Re_k}(p) = 0 \) for all \( n \neq k \). Therefore \( (\sum_{n \in \mathbb{Z}} n \cdot r_{Re_n}(p)) = \sum_{n \in \mathbb{Z}} n \cdot r_{Re_n}(p) = k = f(p) \). Hence, the ring map rank is surjective. Thus \( K_0(R) \) modulo the kernel of this ring map is isomorphic to \( H_0(R) \). It is easy to see that \( H_0(R) \) is a reduced ring. It follows that the nilradical of \( K_0(R) \) is contained in the kernel of the map rank. By [10] Chap. II, §4, Theorem 4.6 or Corollary 4.6.1), the kernel of the map rank is also contained in the nilradical of \( K_0(R) \). This completes the proof.

If \( e \) and \( e' \) are idempotents of a ring \( R \) such that \( e - e' \) is contained in the Jacobson radical of \( R \), then \( e = e' \). Indeed, \( a(1 - e + e') = 1 \) for some \( a \in R \), thus \( e = aee' \) and so \( e(1 - e') = 0 \). This yields that \( e = ee' \). Similarly we get that \( e' = ee' \), because \( b(1 + e - e') = 1 \) for some \( b \in R \). Hence, \( e = e' \). Using this observation, then we have the following result.

**Corollary 4.17.** The map \( e \mapsto [Re, 0] \) is a bijection from the set of idempotents of a ring \( R \) onto the set of idempotents of the ring \( K_0(R) \).

**Proof.** If \( e \in R \) is an idempotent then \( Re \otimes_R Re \simeq Re \otimes_R R/(1 - e) \simeq Re/Re(1 - e) \simeq Re \) and so \( [Re, 0] \cdot [Re, 0] = [Re \otimes_R Re, 0] = [Re, 0] \). Hence, \( [Re, 0] \) is an idempotent. Thus the above map is well-defined. Suppose \( [Re, 0] = [Re', 0] \) for some idempotents \( e, e' \in R \). To prove \( e = e' \) it suffices to show that \( D(e) = D(e') \). There exists some \( n \geq 0 \) such that \( Re \oplus R^n \cong Re' \oplus R^n \). If \( p \in D(e) \) then \( (Re)_p \cong R_p \). It follows that \( (Re')_p \cong R_p \neq 0 \) and so \( p \in D(e') \). Finally, we show that the above map is surjective. If \( z \in K_0(R) \) is an idempotent then its image \( z' \) under the canonical ring map rank : \( K_0(R) \to H_0(R) \) is an idempotent. Thus there exists an idempotent \( e \in R \) such that \( r_{Re} = z' \). It follows that \( [Re, 0] - z \) is contained in the kernel of rank which is by Theorem 4.10 the nilradical of \( K_0(R) \). This yields that \( z = [Re, 0] \). \( \square \)

5. SOME NEW RESULTS ON MONOIDS AND MONOID-RINGS

We begin this section with the following identification:

**Theorem 5.1.** Let \( R \) be a ring and \( M \) a monoid. Then we have the canonical isomorphism of rings \( S^{-1}(R[M]) \cong R[G] \) where \( S := \{ \epsilon_m : m \in M \} \) and \( G \) is the Grothendieck group of \( M \).

**Proof.** By the universal property of monoid-rings, there exists a (unique) ring map \( \varphi : R[M] \to R[G] \) such that \( \eta' = \varphi \circ \eta \) and the following diagram is commutative:

\[
\begin{array}{ccc}
M & \xrightarrow{\gamma} & G \\
\downarrow{\epsilon_M} & & \downarrow{\epsilon_G} \\
R[M] & \xrightarrow{\varphi} & R[G]
\end{array}
\]

where \( \eta : R \to R[M] \) and \( \eta' : R \to R[G] \) are the canonical ring maps and \( \gamma : M \to G \) is the canonical morphism of monoids. Therefore, \( \varphi(\sum_{m \in M} r_m \epsilon_m) = \sum_{m \in M} r_m \epsilon_{[m, 0]} \).

For each \( m \in M \), then \( \varphi(\epsilon_m) = \epsilon_{[m, 0]} \) is invertible in \( R[G] \), because \( \epsilon_{[m, 0]} \cdot \epsilon_{[0, m]} = \epsilon_{[m, m]} = 1 \). Thus by the universal property of localizations, there exists a (unique) ring map \( \psi : S^{-1}(R[M]) \to R[G] \) such that \( \varphi = \psi \circ \pi \) where \( \pi : R[M] \to S^{-1}(R[M]) \) is the canonical ring map. Hence, \( \psi((\sum_{m \in M} r_m \epsilon_m)/\epsilon_a) = \sum_{m \in M} r_m \epsilon_{[m, a]} \). In order to find the inverse of \( \psi \) we act as follows. The map \( \mu : G \to S^{-1}(R[M]) \) given
by \([a, b] \mapsto \epsilon_a / \epsilon_b\) is well-defined and a morphism of monoids from \(G\) into the multiplicative monoid of the ring \(S^{-1}(R[M])\). Thus by the universal property of the monoid-rings, there exists a (unique) ring map \(\theta : R[G] \to S^{-1}(R[M])\) such that \(\mu = \theta \circ \epsilon_G\) and \(\pi \circ \eta = \theta \circ \eta'\). Therefore \(\theta\left(\sum_{[a, b] \in G} r_{a, b} \epsilon_{[a, b]}\right) = \sum_{[a, b] \in G} (r_{a, b} \epsilon_a)/\epsilon_b\). Now by the direct computations, we easily observe that \(\theta \circ \psi : S^{-1}(R[M]) \to S^{-1}(R[M])\) and \(\psi \circ \theta : R[G] \to R[G]\) are the identity maps. This completes the proof. \(\square\)

We improve the following result by adding (iii) and (iv) as new equivalents.

**Corollary 5.2.** For a monoid \(M\) with the Grothendieck group \(G\) the following statements are equivalent.

(i) \(M\) has the cancellation property.

(ii) The canonical map \(M \to G\) is injective.

(iii) For any ring \(R\), the unit vector \(\epsilon_m\) is a non-zerodivisor of \(R[M]\) for all \(m \in M\).

(iv) For any ring \(R\), the canonical ring map \(\varphi : R[M] \to R[G]\) is injective.

**Proof.** (i) \(\Leftrightarrow\) (ii) : It is an easy exercise and well-known.

(i) \(\Rightarrow\) (iii) : Suppose \((r_a) \cdot \epsilon_m = 0\) for some \((r_a) \in R[M]\). Then \(r'_n := \sum_{(a, k) \in M^2} r_n \delta_{k, m} =\)

\[\sum_{a \in M, b \in M} r_a = 0\] for all \(a \in M\). Since \(M\) has the cancellation property, thus for each \(b \in M\), we have \(r'_{b + m} = \sum_{a + b = m} r_a = r_b = 0\). Hence, \((r_a) = 0\).

(iii) \(\Rightarrow\) (i) : Suppose \(m + a = m + b\) for some \(a, b, m \in M\). Then in \(\mathbb{Z}[M]\) we have \(\epsilon_m (\epsilon_a - \epsilon_b) = 0\). By hypothesis, \(\epsilon_m\) is a non-zerodivisor, thus \(\epsilon_a = \epsilon_b\) and so \(a = b\).

(iii) \(\Leftrightarrow\) (iv) : By the proof of Theorem 5.1, \(\varphi = \psi \circ \pi\) where \(\psi : S^{-1}(R[M]) \to R[G]\) is an isomorphism, \(\pi : R[M] \to S^{-1}(R[M])\) is the canonical ring map and \(S := \{\epsilon_m : m \in M\}\). Therefore \(\varphi\) is injective if and only if \(\pi\) is as well, or equivalently, \(S\) is contained in the set of non-zerodivisors of \(R[M]\). \(\square\)

For the additive monoid \(\mathbb{N} = \{0, 1, 2, \ldots\}\) the monoid-ring \(R[\mathbb{N}]\) is called the ring of polynomials over \(R\) with the variable \(x := \epsilon_1 = (\delta_{1, n})_{n \in \mathbb{N}}\) and is denoted by \(R[x]\).

Similarly, for the additive monoid \(\mathbb{N}^d\) with \(d \geq 1\), by setting \(a_i := (\delta_{i, k})_{k=1}^d\) for \(i \in \{1, \ldots, d\}\) the monoid-ring \(R[\mathbb{N}^d]\) is called the ring of polynomials over \(R\) with the variables \(x_1 := \epsilon_{a_1}, \ldots, x_d := \epsilon_{a_d}\) and is denoted by \(R[x_1, \ldots, x_d]\). More generally, consider the additive monoid \(M := \bigoplus_{i \in I} \mathbb{N}\) with \(I\) a set, then the monoid-ring \(R[M]\) is denoted by \(R[x_i : i \in I]\) and is called the ring of polynomials over \(R\) with the variables \(x_i := \epsilon_{a_i}\) where \(a_i := (\delta_{i, k})_{k \in I} \in M\) for all \(i \in I\). The localization of the polynomial ring \(R[x_1, \ldots, x_d]\) with respect to the multiplicative set of monomials \(S = \{x_1^{c_1} \ldots x_d^{c_d} : c_1, \ldots, c_d \geq 0\}\) is called the ring of Laurent polynomials in several variables and denoted by \(R[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]\). The additive monoid \(\mathbb{N}^d\) has the cancellation property and hence by Corollary 5.2, the monomial \(x_1^{c_1} \ldots x_d^{c_d}\) is a non-zerodivisor of \(R[x_1, \ldots, x_d]\) for all \((c_1, \ldots, c_d) \in \mathbb{N}^d\). Therefore, \(R[x_1, \ldots, x_d]\) is a subring of \(R[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]\). In particular, the localization of the polynomial ring \(R[x]\) with respect to the multiplicative set \(\{1, x, x^2 \ldots\}\) is denoted by \(R[x, x^{-1}]\). Finally, the localization of the ring \(R[x_i : i \in I]\) with respect to the multiplicative set of monomials \(\{\epsilon_m : m \in M = \bigoplus_{i \in I} \mathbb{N}\}\) is denoted by \(R[x_i^{\pm 1} : i \in I]\). Similarly to the above, \(R[x_i : i \in I]\) is a subring of this ring. The Grothendieck group of
the additive monoid $\mathbb{N}^d$ is the additive group $\mathbb{Z}^d$. More generally, the Grothendieck group of the additive monoid $\bigoplus_{i \in I} \mathbb{N}$ is the additive group $\bigoplus_{i \in I} \mathbb{Z}$.

Using the above observations, then Theorem 5.1 easily yields the following result which its particular case is presumably well-known.

Corollary 5.3. For the additive group $G := \bigoplus_{i \in I} \mathbb{Z}$, the group-ring $R[G]$ is canonically isomorphic to the ring $R[x_1^{\pm 1} : i \in I]$. In particular, $R[\mathbb{Z}^d] \simeq R[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ for all $d \geq 1$.

Let $S$ be a multiplicative subset of a ring $R$. Then clearly $S$ is the multiplicative submonoid of $R$. Let $S'$ be the set of all $a \in R$ such that $aa' \in S$ for some $a' \in R$. Then clearly $S'$ is a multiplicative subset of $R$ containing $S$. In the next result, the group of units is identified in terms of the Grothendieck group.

Theorem 5.4. For a multiplicative subset $S$ of a ring $R$ the following assertions hold.

(i) The Grothendieck group of $S$ can be canonically embedded in $(S^{-1}R)^*$.
(ii) The group $(S^{-1}R)^*$ is canonically isomorphic to the Grothendieck group of $S'$.

Proof. (i) : Let $G$ be the Grothendieck group of the multiplicative monoid $S$. The map $\mu : S \to (S^{-1}R)^*$ given by $s \mapsto s/1$ is a morphism of monoids. Thus by the universal property of Grothendieck groups, there exists a (unique) morphism of groups $\varphi : G \to (S^{-1}R)^*$ such that $\mu = \varphi \circ \gamma$ where $\gamma : S \to G$ is the canonical map. Therefore, $\varphi([s,t]) = s/t$. If $[s,t] \in \text{Ker}(\varphi)$, then $ss' = ts'$ for some $s' \in S$. It follows that $[s,t] = [1,1]$ is the identity element of $G$. Hence, $\varphi$ is injective.

(ii) : Let $G'$ be the Grothendieck group of the multiplicative monoid $S'$. Clearly the map $S^{-1}R \to (S')^{-1}R$ given by $r/s \mapsto r/s$ is an isomorphism of rings and hence their groups of units are isomorphic. Using part (i), then the map $\psi : G' \to (S^{-1}R)^*$ given by $[a,b] \mapsto ab'/bb'$ is an injective morphism of groups where $bb' \in S$ for some $b' \in R$. If $a/s \in (S^{-1}R)^*$ then $a \in S'$. Thus $\psi([a,s]) = a/s$. Hence, $\psi$ is surjective.

The following result is an immediate consequence of the above theorem.

Corollary 5.5. Let $S$ be a multiplicative subset of a ring $R$ with the Grothendieck group $G$. If $S = S'$, then $G \simeq (S^{-1}R)^*$.

Corollary 5.6. For any ring $R$, the Grothendieck group of the multiplicative monoid $R \setminus \mathbb{Z}(R)$ is canonically isomorphic to $T(R)^*$.

Proof. Setting $S := R \setminus \mathbb{Z}(R)$. If $a \in S'$ then $ab \in S$ for some $b \in R$. If $ac = 0$ for some $c \in R$, then $abc = 0$ and so $c = 0$. Thus $a \in S$. The desired conclusion now follows from Corollary 5.3.

By a totally (linearly) ordered monoid we mean a monoid $M$ equipped with a total ordering $<$ such that its operation is compatible with its ordering, i.e. if $a < b$ for some $a, b \in M$, then $a + c < b + c$ for all $c \in M$. Note that in this definition $a + c < b + c$ if and only if $M$ has the cancellation property. Clearly, every totally ordered monoid is torsion-free (i.e. every nonidentity element is of infinite order). Recall that a monoid $M$ is called strongly torsion-free if whenever $na = nb$ for some $n \geq 2$ and $a, b \in M$, then $a = b$. Every strongly torsion-free monoid is torsion-free. The converse also holds for groups.
Remark 5.7. Let \( \{M_i : i \in I\} \) be a family of totally ordered monoids and let \( M = \prod_{i \in I} M_i \) be their direct product. Then \( M \) is a totally ordered monoid via the lexicographical ordering induced by the orderings on the \( M_i \)'s. In fact, using the well-ordering theorem, the index set \( I \) can be well-ordered. Take \( a = (a_i), b = (b_i) \in M \). If \( a \neq b \), then the set \( \{i \in I : a_i \neq b_i\} \) is nonempty. Let \( k \) be the least element of this set. Then the lexicographical ordering \( \prec_{\text{lex}} \) is defined on \( M \) as \( a \prec_{\text{lex}} b \) or \( b \prec_{\text{lex}} a \), depending on whether \( a_k < b_k \) or \( b_k < a_k \), where \( < \) is the ordering on \( M_k \). Hence, \( (M, \prec_{\text{lex}}) \) is a totally ordered monoid. In particular \( \bigoplus_{i \in I} M_i \), the direct sum of the \( M_i \)'s, is also a totally ordered monoid, because every submonoid of a totally ordered monoid is itself a totally ordered monoid.

Theorem 5.8. Let \( M \) be a cancellative monoid. Then \( M \) is a totally ordered monoid if and only if \( M \) is strongly torsion-free.

Proof. The implication \( \Rightarrow \) is clear. Conversely, let \( G \) be the Grothendieck group of \( M \). Consider \( G \) as a \( \mathbb{Z} \)-module and put \( S := \mathbb{Z} \setminus \{0\} \). Clearly, \( M \) is strongly torsion-free if and only if \( G \) is torsion-free. Hence, the canonical map \( G \to S^{-1}G \) is injective. Note that \( S^{-1}G \) is an \( S^{-1}\mathbb{Z} \)-module. We know that \( S^{-1}\mathbb{Z} = \mathbb{Q} \) is the field of rational numbers. Hence the \( \mathbb{Q} \)-vector space \( S^{-1}G \) is canonically isomorphic to a direct sum of copies of \( \mathbb{Q} \). Using that \( \mathbb{Q} \) is a totally ordered group, then by Remark 5.7, every direct sum of copies of \( \mathbb{Q} \) is also a totally ordered group. Therefore \( S^{-1}G \) and hence also \( G \) are totally ordered groups. The canonical map \( M \to G \) is also injective. Therefore \( M \) is also a totally ordered monoid, because every submonoid of a totally ordered monoid is itself a totally ordered monoid. \( \square \)

As an application, the following famous result of Levi \([7, \S 3]\) immediately follows from the above theorem.

Corollary 5.9. Every torsion-free abelian group is a totally ordered group.

Remark 5.10. After proving Theorem 5.8 we were informed that it is already proven in Northcott \([9, \S 2.12, \text{Theorem 22}]\). But we must point out that our method is quite different and also shorter than his approach.

Let \( S \) be a semiring and \( M \) a monoid. Then exactly like the monoid-ring construction, we may define a new semiring which is called the monoid-semiring of \( M \) over \( S \) and is denoted by \( S[M] \). More precisely, by \( S[M] \) we mean the set of all sequences \( (s_a)_{a \in M} \) such that each \( s_a \in S \) and \( s_a = 0 \) for all but a finite number of indices \( a \in M \). This set by the componentwise addition and a multiplication (which is defined exactly like the multiplication of monoid-rings) can be made into a semiring. The map \( \epsilon : M \to S[M] \) given by \( m \mapsto \epsilon_m \) is a morphism of monoids from \( M \) into the multiplicative monoid of the semiring \( S[M] \). Also, the map \( \eta : S \to S[M] \) is a morphism of semirings. In fact, the triple \((S[M], \epsilon, \eta)\) satisfies the following universal property. For any such triple \((T, \varphi, \psi)\), i.e. \( \varphi : M \to T \) is a morphism of monoids from \( M \) into the multiplicative monoid of a semiring \( T \) and \( \psi : S \to T \) is a morphism of semirings, then there exists a unique morphism of semirings \( \theta : S[M] \to T \) such that \( \varphi = \theta \circ \epsilon \) and \( \psi = \theta \circ \eta \). We have then the following result.

Theorem 5.11. Let \( S \) be a semiring and \( M \) a monoid. Then the Grothendieck ring of the semiring \( S[M] \) is canonically isomorphic to the monoid-ring \( R[M] \) where \( R \) denotes Grothendieck ring of the semiring \( S \).
Proof. We shall denote the Grothendieck ring of the semiring \( S[M] \) by \( R' \). We have to show that \( R' \simeq R[M] \). The map \( \gamma' \circ \eta : S \rightarrow R' \) is a morphism of semirings where \( \eta : S \rightarrow S[M] \) and \( \gamma' : S[M] \rightarrow R' \) are the canonical maps. Thus by the universal property of Grothendieck rings, there is a (unique) ring map \( \varphi : R \rightarrow R' \) such that \( \gamma' \circ \eta = \varphi \circ \gamma \) where \( \gamma : S \rightarrow R \) is the canonical map. Hence, \( \varphi([a,b]) = [ae_0,be_0] \).

The map \( \gamma' \circ \epsilon : M \rightarrow R' \) is a morphism of monoids from \( M \) into the multiplicative monoid of the ring \( R' \) where \( \epsilon : M \rightarrow S[M] \) is the canonical map. Thus by the universal property of monoid-rings, there is a (unique) ring map \( \psi : R[M] \rightarrow R' \) such that \( \gamma' \circ \epsilon = \psi \circ \epsilon' \) and \( \varphi = \psi \circ \eta' \) where \( \epsilon' : M \rightarrow R[M] \) and \( \eta' : R \rightarrow R[M] \) are the canonical maps. Therefore, \( \psi(\sum_{m \in M} [a_m,b_m]e'_m) = \sum_{m \in M} [a_m e_m, b_m e_m] \).

Then we will find the inverse of \( \psi \). The map \( \eta' \circ \gamma : S \rightarrow R[M] \) is a morphism of semirings. Thus by the universal property of monoid-semirings, there exists a (unique) morphism of semirings \( \varphi' : S[M] \rightarrow R[M] \) such that \( \epsilon' = \varphi' \circ \epsilon \) and \( \eta' \circ \gamma = \varphi' \circ \eta \). So \( \varphi'(\sum_{m \in M} a_m e_m) = \sum_{m \in M} [a_m,0]e'_m \). Then by the universal property of Grothendieck rings, there is a (unique) morphism of rings \( \theta : R' \rightarrow R[M] \) such that \( \varphi' = \theta \circ \gamma' \). Hence, \( \theta([\sum_{m \in M} a_m e_m, \sum_{m \in M} b_m e_m]) = \sum_{m \in M} [a_m, b_m]e'_m \). Now one can easily see that \( \theta \circ \psi \) and \( \psi \circ \theta \) are the identity maps. \( \square \)

We conclude this article by proposing the following problem.

**Conjecture 5.12.** Let \( I = \bigoplus_{i \in S} R_i \) be the direct sum ideal of a direct product ring \( R = \prod_{i \in S} R_i \). If \( \text{Pic}(R/I) = 0 \) then we have the canonical isomorphism of groups:

\[
\text{Pic}(R) \simeq \prod_{i \in S} \text{Pic}(R_i).
\]

**Acknowledgments.** We would like to give sincere thanks to Professor Pierre Deligne, who generously shared with us his very valuable and excellent ideas.

**References**

[1] H. Bass, Algebraic K-Theory, W.A. Benjamin, Inc. (1968).
[2] A.J. de Jong et al., The Stacks Project, see [http://stacks.math.columbia.edu](http://stacks.math.columbia.edu), (2022).
[3] D. Husmoller, Fibre Bundles, 3rd ed., Springer-Verlag New York, Inc. (1994).
[4] F. Ischebeck and R.A. Rao, Ideals and Reality: Projective Modules and Number of Generators of Ideals, Springer-Verlag, (2005).
[5] T.Y. Lam, A First Course in Noncommutative Rings, Springer-Verlag, (2001).
[6] S. Lang, Algebra, rev. 3rd ed., Springer-Verlag, (2002).
[7] F.W. Levi, Ordered groups, Proc. Indian Acad. Sci. A 16(4) (1942) 256-263.
[8] H. Matsumura, Commutative Ring Theory, Cambridge University Press, Cambridge, (1989).
[9] D.G. Northcott, Lessons on Rings, Modules and Multiplicities, Cambridge University Press (1968).
[10] J.R. Silvester, Introduction to Algebraic K-Theory, Chapman and Hall, (1981).
[11] A. Tarizadeh, A fresh look into monoid rings and formal power series rings, J. Algebra Appl. 19(1) (2020) 2050003.
[12] A. Tarizadeh, Notes on finitely generated flat modules, Bull. Korean Math. Soc. 57(2) (2020) 419-427.
[13] A. Tarizadeh, Some results on pure ideals and trace ideals of projective modules, Acta Math. Vietnam. 47 (2022) 475-481.
[14] A. Tarizadeh and P.K. Sharma, Structural results on lifting, orthogonality and finiteness of idempotents, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. (RACSAM), 110(1), 54 (2022).
[15] C.A. Weibel, The K-Book: An Introduction to Algebraic K-theory, Graduate studies in mathematics (volume 145), American Mathematical Society, Providence, Rhode Island, (2013).

Department of Mathematics, Faculty of Basic Sciences, University of Maragheh, P. O. Box 55136-553, Maragheh, Iran.
Email address: ebulfez1978@gmail.com