WATER ARTIFICIAL CIRCULATION FOR EUTROPHICATION CONTROL

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Abstract. This work analyzes, from a mathematical point of view, the artificial mixing of water - by means of several pairs collector/injector that set up a circulation pattern in the waterbody - in order to prevent the undesired effects of eutrophication. The environmental problem is formulated as a constrained optimal control problem of partial differential equations, where the state system is related to the velocity of water and to the concentrations of the different species involved in the eutrophication processes, and the cost function to be minimized represents the volume of recirculated water. In the main part of the work, the wellposedness of the problem and the existence of an optimal control is demonstrated. Finally, a complete numerical algorithm for its computation is presented, and some numerical results for a realistic problem are also given.

1. Introduction. Artificial circulation and aeration are management techniques for oxygenating eutrophic waterbodies subject to quality problems, such as loss of oxygen, sediment accumulation and algal blooms. These techniques are typically applied in stratified shallow waters to mitigate problems associated with oxygen depletion. Artificial circulation disrupts stratification and minimizes the development of stagnant zones that may be subject to water quality problems. The movement of water and/or air is accomplished by use of bottom diffusers and water pumps to create a circulation pattern in shallow waters [11].

The general purpose of artificial mixing and oxygenation is to increase the dissolved oxygen content of the water. Various systems are available to help do this by mechanically mixing or agitating the water. Artificial circulation and aeration can increase fish and other aquatic animal habitat, prevent fishkills, reduce algal blooms, and improve the quality of domestic and industrial water supplies, decreasing treatment costs.

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This work deals with artificial circulation as a shallow water aeration technique. Large waterbodies (for instance, lakes or reservoirs) get much of their oxygen from the atmosphere through diffusion processes. Artificial circulation increases water’s oxygen by forcefully circulating the water to expose more of it to the atmosphere. Proper choice and design of an artificial circulation system depends on the management goals and the geophysical characteristics. Two techniques are the most common: air injection and mechanical mixing. The former has been analyzed, from an optimal control viewpoint in a few works (see, for instance, [1] and the references therein). However, in this paper we will focus our attention on the latter that, as far as we know, has remained unaddressed in the mathematical literature.

In air injection systems, a compressor delivers air through lines connected to perforated pipes or diffusers placed near the bottom. The rising air bubbles cause water in the bottom layer to also rise, pulling this water into the surface. This aeration technique is sometimes referred to as the air-lift method of circulation, since bottom waters are lifted to the lake surface through the action of the injected air [13]. On the contrary, mechanical mixing systems use a top-down approach to set up a circulation pattern. A flow pump takes water from the epilimnion (well aerated upper layer of water) by means of a collector, injecting it into the hypolimnion (poorly oxygenated bottom layer), setting up a circulation pattern that prevents stratification. In this way, oxygen-poor water from the bottom is circulated to the surface, where oxygenation from the atmosphere can occur. Essentially, this technique can be understood as an artificial promoter of water movement, recirculating bottom water to the surface [17].

Most of the effects of destratification and circulation are beneficial for the environment, and include two that turn out to be fundamental for our purposes of eutrophication remediation: (a) an increase in dissolved oxygen through the water column, and (b) a reduction of algal growth by reducing its exposure to light, by changing water chemistry (pH, CO$_2$ and so on), or by mixing of algae-grazing zooplankton.

So, in next section we will introduce a mathematical formulation of the environmental problem as a control/state constrained optimal control problem of partial differential equations, analyzing the corresponding state system, in order to set its wellposedness. Then, we will prove the existence of solution for the optimal control problem. Finally, we will deal with the numerical resolution of the problem, presenting a complete numerical algorithm and a realistic computational example.

2. Mathematical formulation of the control problem.

2.1. The physical domain. We consider a domain $\Omega \subset \mathbb{R}^3$ corresponding, for instance, to a lake. We suppose the existence of a set of $N_{CT}$ pairs collector-injector, assuming that each water collector is attached to a unique injector by a pipe with a pumping/turbine group.

We assume a smooth enough boundary $\partial \Omega$, such that it can be split into four disjoint subsets $\partial \Omega = \Gamma_S \cup \Gamma_B \cup \Gamma_C \cup \Gamma_T$, where:

- $\Gamma_S$ corresponds to the surface of the lake,
- $\Gamma_B$ corresponds to the bottom of the lake,
- $\Gamma_C$ corresponds to the part of the boundary where the water collectors $C^k$, $k = 1, \ldots, N_{CT}$, can be located,
- $\Gamma_T$ corresponds to the part of the boundary where the water injectors $T^k$, $k = 1, \ldots, N_{CT}$, can be located.
2.2. The state equations. We present first the hydrodynamic model, and then the model for the evolution of the concentrations of the different species involved in the eutrophication processes.

2.2.1. The hydrodynamic model: Modified Navier-Stokes equations. In order to simulate the dynamics of water we consider the modified Navier-Stokes equations following the Smagorinsky model of turbulence:

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v} - \nabla \cdot \mathbf{\Xi} + \nabla p &= \mathbf{F} \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot \mathbf{v} &= 0 \quad \text{in } \Omega \times (0, T), \\
\mathbf{v} &= \mathbf{v}_S \quad \text{on } \Gamma_S \times (0, T), \\
\mathbf{v} &= \mathbf{0} \quad \text{on } \Gamma_B \times (0, T), \\
\mathbf{v} &= \mathbf{0} \quad \text{on } (\Gamma_T \setminus \cup_{k=1}^{N_{CT}} T^k) \times (0, T), \\
\mathbf{v} &= \mathbf{0} \quad \text{on } (\Gamma_C \setminus \cup_{k=1}^{N_{CT}} C^k) \times (0, T), \\
\mathbf{v} &= \frac{g^k(t)}{\mu(C^k)} \mathbf{n} \quad \text{on } C^k \times (0, T), \quad \text{for } k = 1, \ldots, N_{CT}, \\
\mathbf{v} &= -\frac{g^k(t)}{\mu(T^k)} \mathbf{n} \quad \text{on } T^k \times (0, T), \quad \text{for } k = 1, \ldots, N_{CT}, \\
\mathbf{v}(0) &= \mathbf{v}_0 \quad \text{in } \Omega,
\end{align*}
\]

where \( \mathbf{v}(x, t) \) is the velocity of water, \( \mathbf{F}(x, t) \) if the source term, \( \mathbf{n} \) represents the unit outward normal vector to the boundary \( \partial \Omega \), \( \mu(S) \) denotes the usual measure of a generic set \( S \), the water velocity on the surface \( \mathbf{v}_S \) satifies that \( \mathbf{v}_S \cdot \mathbf{n} = 0 \), \( \mathbf{v}_0 \) is the initial velocity, and, for each \( k = 1, \ldots, N_{CT} \), \( g^k(t) \in H^s(0, T) \), with \( s \geq 1 \) a suitable index, represents the volume of water displaced by pump \( k \) at each time \( t \). This volume of water can be positive \( g^k(t) > 0 \) (we say that we are turbinating: water enters by the collector and is turbinate by the pipeline to the injector), or negative \( g^k(t) < 0 \) (in this case, we say that we are pumping: water enters by the injector and is pumped to the collector).

Finally, the term \( \mathbf{\Xi}(\mathbf{v}) \) is given by:

\[
\mathbf{\Xi}(\mathbf{v}) = \frac{\partial D(\mathbf{\epsilon})}{\partial \mathbf{\epsilon}} \bigg|_{\mathbf{\epsilon} = \mathbf{\epsilon}(\mathbf{v})}, \quad \text{with } \mathbf{\epsilon}(\mathbf{v}) = \frac{1}{2} (\nabla \mathbf{v} + \nabla \mathbf{v}^t),
\]

where \( D \) is a potential function (for instance, in the particular case of the classical Navier-Stokes equations, \( D(\mathbf{\epsilon}) = \nu [\mathbf{\epsilon} : \mathbf{\epsilon}] \) and, consequently, \( \mathbf{\Xi}(\mathbf{v}) = 2\nu \mathbf{\epsilon}(\mathbf{v}) \)). In our case, the Smagorinsky turbulence model, the potential function is [14]:

\[
D(\mathbf{\epsilon}) = \nu [\mathbf{\epsilon} : \mathbf{\epsilon}] + \frac{2}{3} \nu_{\text{tur}} [\mathbf{\epsilon} : \mathbf{\epsilon}]^{3/2},
\]

so,

\[
\mathbf{\Xi}(\mathbf{v}) = \left( \frac{\partial D(\mathbf{\epsilon})}{\partial \mathbf{\epsilon}} \bigg|_{\mathbf{\epsilon} = \mathbf{\epsilon}(\mathbf{v})} \right) \mathbf{\epsilon}(\mathbf{v}) = 2\nu \mathbf{\epsilon}(\mathbf{v}) + 2\nu_{\text{tur}} [\mathbf{\epsilon}(\mathbf{v}) : \mathbf{\epsilon}(\mathbf{v})]^{1/2} \mathbf{\epsilon}(\mathbf{v}) = \beta(\mathbf{\epsilon}(\mathbf{v})) \mathbf{\epsilon}(\mathbf{v}),
\]

where \( \beta(\mathbf{\epsilon}(\mathbf{v})) = 2\nu + 2\nu_{\text{tur}} [\mathbf{\epsilon}(\mathbf{v}) : \mathbf{\epsilon}(\mathbf{v})]^{1/2} \).

From a mathematical point of view, the advantage of considering the modified Navier-Stokes equations, besides being more appropriate for turbulent flows, lies in the fact that, if the potential function fulfills certain properties (see, for instance, [10, 9] and the references therein), it is possible to demonstrate the uniqueness of solution, in addition to gain in regularity with respect to that obtained for the
Interactions between five above species can be modelled by the following system of partial differential equations [4]:

\[
\begin{align*}
\frac{\partial u^i}{\partial t} + \mathbf{v} \cdot \nabla u^i - \nabla \cdot (\mu u^i \nabla u^i) &= A^i(u) + f^i \quad \text{in } \Omega \times (0, T), \\
\nabla u^i \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_S \times (0, T), \\
\nabla u^i \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_B \times (0, T), \\
\nabla u^i \cdot \mathbf{n} &= 0 \quad \text{on } (\Gamma_T \setminus \bigcup_{k=1}^{N_CT} T^k) \times (0, T), \\
\n\nabla u^i \cdot \mathbf{n} &= 0 \quad \text{on } (\Gamma_C \setminus \bigcup_{k=1}^{N_CT} C^k) \times (0, T), \\
\frac{1}{2} (1 + \text{sign}(g^k(t))) |\text{sign}(g^k(t))| u^i + \frac{1}{2} (1 - \text{sign}(g^k(t))) \nabla u^i \cdot \mathbf{n} &= \frac{1}{2} (1 + \text{sign}(g^k(t))) |\text{sign}(g^k(t))| \int_{C^k} u^i \, d\gamma \quad \text{on } T^k \times (0, T), \quad k = 1, \ldots, N_CT, \\
\frac{1}{2} (1 - \text{sign}(g^k(t))) |\text{sign}(g^k(t))| u^i + \frac{1}{2} (1 + \text{sign}(g^k(t))) \nabla u^i \cdot \mathbf{n} &= \frac{1}{2} (1 - \text{sign}(g^k(t))) |\text{sign}(g^k(t))| \int_{T^k} u^i \, d\gamma \quad \text{on } C^k \times (0, T), \quad k = 1, \ldots, N_CT, \\
u^i(0) &= u^i_0 \quad \text{in } \Omega,
\end{align*}
\]

for \( i = 1, \ldots, 5 \), where \( \text{sign}(y) \) denotes the sign function:

\[
\text{sign}(y) = \begin{cases} 
1 & \text{if } y > 0, \\
-1 & \text{if } y < 0, \\
0 & \text{if } y = 0,
\end{cases}
\]

and the reaction term \( A = (A^i) : \Omega \times (0, T) \times [\mathbb{R}_+]^5 \rightarrow \mathbb{R}^5 \) is defined by the following nonlinear expression:

\[
A(x, t, u) = \begin{bmatrix} 
-C_{nc} \left( L(x, t) \frac{u^1}{K_N + u^1 u^2 - K_r u^2} \right) + C_{nc} K_{rd} \Theta^{\theta(x, t) - 20} u^4 \\
C_f \frac{u^2}{K_F + u^2} u^3 - K_{mf} u^2 - K_z \frac{u^2}{K_F + u^2} u^3 \\
C_f \frac{K_z}{u^2} u^3 - K_{mz} u^3 \\
K_{mf} u^2 + K_{mz} u^3 - K_{rd} \Theta^{\theta(x, t) - 20} u^4 \\
C_{oc} \left( L(x, t) \frac{u^1}{K_N + u^1 u^2 - K_r u^2} \right) - C_{oc} K_{rd} \Theta^{\theta(x, t) - 20} u^4
\end{bmatrix}
\]

where:
• $C_{oc}$ is the oxygen-carbon stoichiometric relation,
• $C_{nc}$ is the nitrogen-carbon stoichiometric relation,
• $C_{zf}$ is the zooplankton grazing efficiency factor,
• $K_{rd}$ is the detritus regeneration rate,
• $K_r$ is the phytoplankton endogenous respiration rate,
• $K_{mf}$ is the phytoplankton death rate,
• $K_{mz}$ is the zooplankton death rate (including predation),
• $K_z$ is the zooplankton predation (grazing),
• $K_F$ is the phytoplankton half-saturation constant,
• $K_N$ is the nitrogen half-saturation constant,
• $\Theta$ is the detritus regeneration thermic constant,
• $\mu_i, i = 1, \ldots, 5,$ are the diffusion coefficients of each species,
• $f^i(x, t), i = 1, \ldots, 5,$ are the source terms of each species,
• $\theta(x, t)$ is the water temperature,
• $L$ is the luminosity function, given by:
\[
L(x, t, u^2) = \mu C_{t}^{\theta(x,t) - 20} \frac{I_0}{T_s} e^{-(\varphi_1 + \varphi_2 u^2(x,t))x_3},
\]
with $I_0$ the incident light intensity, $I_s$ the light saturation, $C_t$ the phytoplankton growth thermic constant, $\varphi_1$ the light attenuation due to depth, $\varphi_2$ the light attenuation due to phytoplankton, and $\mu$ the maximum phytoplankton growth rate.

All above coefficients are supposed to be nonnegative, except for the half-saturation constants that are strictly positive, Moreover, for the sake of simplicity in the theoretical part, we assume without loss of generality that the attenuation constant $\varphi_2 = 0$.

**Remark 1.** It is worthwhile noting here that the boundary conditions of problem (5) associated to collectors and injectors change depending on the pump operation mode (turbination, pumping or rest):

- **Turbination mode** ($g^k(t) > 0$): in this case, $\text{sign}(g^k(t)) = 1$, thus the corresponding boundary conditions read, for each $i = 1, \ldots, 5$:
  \[
  u^i(t) = \frac{1}{\mu(C^k)} \int_{C^k} u^i(t) d\gamma \quad \text{on} \quad T^k \times (0, T), \quad k = 1, \ldots, N_{CT},
  \]
  \[
  \nabla u^i(t) \cdot n = 0 \quad \text{on} \quad C^k \times (0, T), \quad k = 1, \ldots, N_{CT}.
  \]
  That is, concentration of species $i$ in the injector is the mean concentration in the collector, where water enters the pipeline to be turbinated.

- **Pumping mode** ($g^k(t) < 0$): in this case, $\text{sign}(g^k(t)) = -1$, thus:
  \[
  \nabla u^i(t) \cdot n = 0 \quad \text{on} \quad T^k \times (0, T), \quad k = 1, \ldots, N_{CT},
  \]
  \[
  u^i(t) = \frac{1}{\mu(T^k)} \int_{T^k} u^i(t) d\gamma \quad \text{on} \quad C^k \times (0, T), \quad k = 1, \ldots, N_{CT}.
  \]
  That is, concentration of species $i$ in the collector is the mean concentration in the injector, where water enters the pipeline to be pumped.

- **Rest mode** ($g^k(t) = 0$): in this case, $\text{sign}(g^k(t)) = 0$, thus:
  \[
  \nabla u^i(t) \cdot n = 0 \quad \text{on} \quad T^k \times (0, T), \quad k = 1, \ldots, N_{CT},
  \]
  \[
  \nabla u^i(t) \cdot n = 0 \quad \text{on} \quad C^k \times (0, T), \quad k = 1, \ldots, N_{CT}.
  \]
  That is, the system is isolated.
2.3. The optimal control problem. Our main aim is related to mitigating the adverse effects of eutrophication. In order to do this, many strategies are possible, for example, optimizing the locations and/or sizes of the collectors and injectors, minimizing the pumped flow, etc. To fix ideas, we will focus in this last problem: the optimization of the volume of pumped water in order, for instance, to minimize the energy cost. The effects of eutrophication can be measured, for example, on the basis of the dissolved oxygen level in the deeper areas (which is usually low due to the decomposition of organic detritus). The idea is to get some circulation of water from the upper layers (rich in oxygen due to photosynthesis realized by phytoplankton and to the contact with the atmosphere) to lower areas (poor in oxygen). In addition, with this artificial circulation, we intend to break with the stratification of the species, which results in a better quality of water.

So, to address the mitigation of the harmful effects of eutrophication by controlling the flow pumped by the injectors/collectors, we assume that their geometry and position are fixed beforehand, and that we can only act on the pumped flow rate. Thus, the following optimal control problem needs to be solved:

\[
(P) \quad \min \{ J(g) : g \in U_{ad} \text{ and } \frac{1}{\mu(\Omega_C)} \int_{\Omega_C} u(t) \, dx \in K_C, \forall t \in [0,T] \},
\]

where

\[
U_{ad} = \{ g \in U : -c_1 \leq g^k(t) \leq c_2, \forall t \in [0,T], \forall k = 1, \ldots, N_C \},
\]

with \( U \) a suitable functional space (that will be detailed in next section) and the positive constants \( c_1, c_2 \) related to technological constraints on the pumps, where \( J(g) \) is the cost functional given by:

\[
J(g) = \frac{1}{2} \sum_{k=1}^{N_C} \int_0^T g^k(t)^2 \, dt,
\]

representing the energy saving related to pumps’ operation, and where \( K_C \subset \mathbb{R}^5 \) is the rectangle:

\[
K_C = [\lambda_1^m, \lambda_1^M] \times \cdots \times [\lambda_5^m, \lambda_5^M],
\]

with \( \lambda_k^m \) and \( \lambda_k^M \) the minimal and maximal mean concentrations allowed for the species \( k \), for \( k = 1, \ldots, 5 \), in the control domain \( \Omega_C \subset \Omega \). Finally, \((v,u)\) are the solutions of the state system (1) and (5) associated to control \( g \).

3. Mathematical analysis of the state system. In this section we proceed to analyze the existence and uniqueness of solution for the state system (1) and (5).

3.1. The hydrodynamic model: Modified Navier-Stokes equations. Next, we will analyze the existence of solution of the hydrodynamic model (1). For this, we begin analyzing the existence of solution in the homogeneous Dirichlet case, then we analyze the existence for the case in which the recirculated flow is of constant sign, to finally give a result of existence for the general case.

3.1.1. Existence and uniqueness of solution for the homogeneous Dirichlet case. Let us begin by analyzing the existence and uniqueness of the solution for the
homogeneous Dirichlet problem:

\[
\begin{aligned}
\frac{\partial \nu}{\partial t} + \nabla \nu \nu - \nabla : \mathcal{E}(\nu) + \nabla p &= F \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot \nu &= 0 \quad \text{in } \Omega \times (0, T), \\
\nu &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
\nu(0) &= \nu_0 \quad \text{in } \Omega,
\end{aligned}
\]  

\tag{10}

where \( \mathcal{E}(\nu) = \beta(\epsilon(\nu))\epsilon(\nu) \), with

\[\beta(\epsilon(\nu)) = 2\nu + 2\mu_{\text{tur}} [\epsilon(\nu) : \epsilon(\nu)]^{1/2},\]

\tag{11}

\( \mu, \mu_{\text{tur}} > 0, \ F \in L^2((0, T) \times \Omega), \) and \( \nu_0 \in V \) with

\[V = \{ \nu \in [W^{1,3}(\Omega)]^3 : \nabla \cdot \nu = 0 \text{ and } \nu|_{\partial \Omega} = 0 \}.\]

\tag{12}

Definition 3.1. Within the framework introduced at the beginning of this section, we will say that an element \( \nu \in W^{1,\infty,2}(0, T; V; [L^2(\Omega)]^3) \) is a solution of the problem (10) if it satisfies the following conditions:

1. \( \nu(0) = \nu_0 \) a.e. in \( \Omega \).
2. \( \nu \) verifies the variational formulation:

\[
\int_{\Omega} \frac{\partial \nu}{\partial t} \cdot \mathbf{z} \, d\mathbf{x} + \int_{\Omega} \nabla \nu \cdot \nabla \nu \cdot \mathbf{z} \, d\mathbf{x} + \int_{\Omega} \beta(\epsilon(\nu))\epsilon(\nu) : \epsilon(\nu) \, d\mathbf{x} \\
= \int_{\Omega} F \cdot \mathbf{z} \, d\mathbf{x}, \quad \text{a.e. } t \in (0, T), \ \forall \mathbf{z} \in V,
\]

\tag{13}

where we denote (see the monograph of Roubíček [16] for more details on the Sobolev-Bochner spaces):

\[W^{1,p,q}(0, T; X, Y) = \left\{ u \in L^p(0, T; X) : \frac{\partial u}{\partial t} \in L^q(0, T; Y) \right\}.
\]

Next, we will obtain a set of estimates that are not only valid for the solutions of (10), but also for Galerkin approximation of the problem (technique that we will use to prove the existence of solution).

Lemma 3.2. Let \( \nu \) be a solution of (10) in the sense of definition 3.1, then there exist continuous functions \( \Phi_1 \) and \( \Phi_2 \) such that

\[
\begin{aligned}
\| \nu \|_{L^\infty(0, T; [L^2(\Omega)]^3)} + \| \nabla \nu \|_{L^2(0, T; [L^2(\Omega)]^3 \times 3)} + \| \nabla \nu \|_{L^3(0, T; [L^3(\Omega)]^3 \times 3)} & \\
& \leq \Phi_1(\| \nu_0 \|_{[L^2(\Omega)]^3}, \| F \|_{L^2(0, T; [L^2(\Omega)]^3)}),
\end{aligned}
\]

\tag{14}

\[
\begin{aligned}
\left\| \frac{\partial \nu}{\partial t} \right\|_{L^2(0, T; [L^2(\Omega)]^3)} + \| \nabla \nu \|_{L^\infty(0, T; [L^2(\Omega)]^3 \times 3)} + \| \nabla \nu \|_{L^\infty(0, T; [L^3(\Omega)]^3 \times 3)} & \\
& \leq \Phi_2(\| \nabla \nu_0 \|_{[L^2(\Omega)]^3 \times 3}, \| \nabla \nu_0 \|_{[L^3(\Omega)]^3 \times 3}, \| F \|_{L^2(0, T; [L^2(\Omega)]^3)}).
\end{aligned}
\]

\tag{15}

Proof. First, we consider \( \mathbf{z} = \nu(t) \) as a test function. Thus, a.e. \( t \in (0, T) \):

\[
\begin{aligned}
\frac{1}{2} \| \nu(t) \|^2_{[L^2(\Omega)]^3} + \int_0^t \int_{\Omega} \beta(\epsilon(\nu(t)))\epsilon(\nu(t)) : \epsilon(\nu(t)) \, d\mathbf{x} \, ds \\
& \leq \frac{1}{2} \int_0^t \int_{\Omega} F(s) \, d\mathbf{x} \, ds + \frac{1}{2} \int_0^t \| \nu(s) \|^2_{[L^2(\Omega)]^3} \, ds + \frac{1}{2} \| \nu_0 \|^2_{[L^2(\Omega)]^3}.
\end{aligned}
\]

\tag{16}
Now, from the definition (11) of $\beta$ and Korn’s inequality, we have:

$$\int_\Omega \beta(\epsilon(\mathbf{v}(t))) \epsilon(\mathbf{v}(t)) : \epsilon(\mathbf{v}(t)) \, d\mathbf{x} \geq C_1 \left( \|\epsilon(\mathbf{v}(t))\|_{L^2(\Omega)^{3\times 3}}^2 + \|\epsilon(\mathbf{v}(t))\|_{L^3(\Omega)^{3\times 3}}^3 \right)$$

(17)

and

$$\geq C_2 \left( \|\nabla \mathbf{v}(t)\|_{L^2(\Omega)^{3\times 3}}^2 + \|\nabla \mathbf{v}(t)\|_{L^3(\Omega)^{3\times 3}}^3 \right),$$

where $C_1$ and $C_2$ are positive constants. Finally, taking into account estimates (16), (17) and Gronwall’s Lemma, we can derive the following estimate:

$$\|\mathbf{v}\|_{L^\infty(0,T;L^2(\Omega)^3)} + \|
abla \mathbf{v}\|_{L^2(0,T;L^2(\Omega)^{3\times 3})} + \|\nabla \mathbf{v}\|_{L^3(0,T;L^3(\Omega)^{3\times 3})} \leq \Phi_1(\|\mathbf{v}_0\|_{L^2(\Omega)^3} + \|\mathbf{F}\|_{L^2(0,T;L^2(\Omega)^{3\times 3})}),$$

(18)

where $\Phi_1$ is a continuous function. (It should be noted here that the $L^3$ boundedness of the velocity gradient is achieved by the presence of the term associated with turbulent viscosity. We will see, later on, that this boundedness is essential when obtaining additional regularity.)

Let us take now $\mathbf{z} = \partial \mathbf{v} / \partial t(t)$ as a test function (this is possible since $\nabla \cdot \mathbf{z} = 0$ and $\mathbf{z}|_{\partial \Omega} = 0$). Arguing as in [9], we obtain:

$$\left\| \frac{\partial \mathbf{v}}{\partial t}(t) \right\|_{L^2(\Omega)^3}^2 + \frac{d}{dt} \int_\Omega D(\epsilon(\mathbf{v}(t))) \, d\mathbf{x} = \int_\Omega \mathbf{F}(t) \cdot \frac{\partial \mathbf{v}}{\partial t}(t) \, d\mathbf{x} - \int_\Omega \nabla \mathbf{v}(t) \mathbf{v}(t) \cdot \frac{\partial \mathbf{v}}{\partial t}(t) \, d\mathbf{x}.$$  

Now, if we use the following inequalities:

$$\int_\Omega \mathbf{F}(t) \cdot \frac{\partial \mathbf{v}}{\partial t}(t) \, d\mathbf{x} \leq \|\mathbf{F}(t)\|_{L^2(\Omega)^3}^2 + \frac{1}{4} \left\| \frac{\partial \mathbf{v}}{\partial t}(t) \right\|_{L^2(\Omega)^3}^2,$$

$$\int_\Omega \nabla \mathbf{v}(t) \mathbf{v}(t) \cdot \frac{\partial \mathbf{v}}{\partial t}(t) \, d\mathbf{x} \leq \|\nabla \mathbf{v}(t)\|_{L^2(\Omega)^3}^2 + \frac{1}{4} \left\| \frac{\partial \mathbf{v}}{\partial t}(t) \right\|_{L^2(\Omega)^3}^2,$$

(19)

we obtain:

$$\frac{1}{2} \left\| \frac{\partial \mathbf{v}}{\partial t}(t) \right\|_{L^2(\Omega)^3}^2 + \frac{d}{dt} \int_\Omega D(\epsilon(\mathbf{v}(t))) \, d\mathbf{x} \leq \|\mathbf{F}(t)\|_{L^2(\Omega)^3}^2 + \|\nabla \mathbf{v}(t)\|_{L^2(\Omega)^3}^2.$$  

Integrating in the time interval $(0,t)$:

$$\frac{1}{2} \left\| \frac{\partial \mathbf{v}}{\partial t}(t) \right\|_{L^2(0,t;L^2(\Omega)^3)}^2 + \int_0^t D(\epsilon(\mathbf{v}(t))) \, d\mathbf{x} - \int_0^t D(\epsilon(\mathbf{v}(0))) \, d\mathbf{x} \leq \|\mathbf{F}\|_{L^2(0,t;L^2(\Omega)^3)}^2 + \|\nabla \mathbf{v}\|_{L^2(0,t;L^2(\Omega)^3)}^2,$$

from where, using the definition (3) of operator $D$, we have:

$$\frac{1}{2} \left\| \frac{\partial \mathbf{v}}{\partial t}(t) \right\|_{L^2(0,t;L^2(\Omega)^3)}^2 + C_1 \left( \|\epsilon(\mathbf{v}(t))\|_{L^2(\Omega)^{3\times 3}}^2 + \|\epsilon(\mathbf{v}(t))\|_{L^3(\Omega)^{3\times 3}}^3 \right)$$

$$\leq C_2 \left( \|\epsilon(\mathbf{v}(0))\|_{L^2(\Omega)^{3\times 3}}^2 + \|\epsilon(\mathbf{v}(0))\|_{L^3(\Omega)^{3\times 3}}^3 \right) + \|\mathbf{F}\|_{L^2(0,t;L^2(\Omega)^3)}^2 + \|\nabla \mathbf{v}\|_{L^2(0,t;L^2(\Omega)^3)}^2.$$
Using now Korn’s inequality:
\[
\frac{1}{2} \left\| \frac{\partial v}{\partial t} \right\|^2_{L^2(0,t;L^2(\Omega))^3} + C_3 \left( \left\| \nabla v(t) \right\|^2_{L^2(\Omega)^{3x3}} + \left\| \nabla v(t) \right\|^3_{L^2(\Omega)^{3x3}} \right) 
\leq C_4 \left( \left\| \nabla v_0 \right\|^2_{L^2(\Omega)^{3x3}} + \left\| \nabla v_0 \right\|^3_{L^2(\Omega)^{3x3}} \right) 
\]
where \( C_1, C_2, C_3 \) and \( C_4 \) are positive constants. On the other hand,
\[
\int_0^t \left\| \nabla v(s) \right\|^2_{L^2(\Omega)^{3x3}} \, ds \leq \int_0^t \left\| v(s) \right\|^2_{L^3(\Omega)} \left\| \nabla v(s) \right\|^2_{L^3(\Omega)^{3x3}} \, ds 
\leq C \int_0^t \left\| v(s) \right\|^2_{L^3(\Omega)^{3x3}} \, ds 
\leq \int_0^t \left\| \nabla v(s) \right\|^3_{L^3(\Omega)^{3x3}} \left( \left\| \nabla v(s) \right\|^2_{L^2(\Omega)^{3x3}} + \left\| \nabla v(s) \right\|^3_{L^2(\Omega)^{3x3}} \right) \, ds.
\]
Therefore, if we denote by \( z(t) = \left\| \nabla v(t) \right\|^2_{L^2(\Omega)^{3x3}} + \left\| \nabla v(t) \right\|^3_{L^3(\Omega)^{3x3}} \), we have:
\[
\frac{1}{2} \left\| \frac{\partial v}{\partial t} \right\|^2_{L^2(0,t;L^2(\Omega))^3} + C_3 z(t) \leq \int_0^t \psi_1(s)z(s) \, ds + \psi_2(t),
\]
where:
\[
\psi_1(t) = \left\| \nabla v(t) \right\|^2_{L^3(\Omega)^{3x3}}, \\
\psi_2(t) = \left\| F \right\|^2_{L^2(0,t;L^2(\Omega))^3} + C_4 \left( \left\| \nabla v_0 \right\|^2_{L^2(\Omega)^{3x3}} + \left\| \nabla v_0 \right\|^3_{L^2(\Omega)^{3x3}} \right).
\]
It is clear now that (and here turbulent viscosity has a crucial effect on increasing the regularity of the solution: without turbulent viscosity the estimate could not be done):
\[
\int_0^T \psi_1(s) \, ds = \int_0^T \left\| \nabla v(s) \right\|^2_{L^3(\Omega)^{3x3}} \, ds \leq C \left\| \nabla v \right\|^3_{L^2(0,T;L^3(\Omega)^{3x3})} < \infty
\]
and, using Gronwall’s Lemma, we obtain that:
\[
\left\| \frac{\partial v}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))^3} + \left\| \nabla v \right\|_{L^\infty(0,T;L^2(\Omega)^{3x3})} + \left\| \nabla v \right\|_{L^\infty(0,T;L^3(\Omega)^{3x3})} 
\leq \Phi_2 \left( \left\| \nabla v_0 \right\|_{L^2(\Omega)^{3x3}}, \left\| \nabla v_0 \right\|_{L^3(\Omega)^{3x3}}, \left\| F \right\|_{L^2(0,T;L^2(\Omega))^3} \right),
\]
where \( \Phi_2 \) is also a continuous function. That is, we are able to obtain continuous estimates with respect to data in the functional space
\[
v \in W^{1,\infty,2}(0,T; V, [L^2(\Omega)]^3),
\]
which ends the proof.

\( \square \)

**Theorem 3.3.** There exists a solution \( v \in W^{1,\infty,2}(0,T; V, [L^2(\Omega)]^3) \cap C([0,T]; V) \) of the problem (10), in the sense of definition 3.1, continuous with respect to data, that verifies the estimates (14) and (15).

**Proof.** The solvability of system (10) is proved, using the estimates obtained in Lemma 3.2, with the classical Galerkin method. That is, a sequence of functions \( \{v_n\} \subset W^{1,\infty,2}(0,T; V, [L^2(\Omega)]^3) \) is constructed, which solve finite-dimensional approximations of (10). Then, it is shown that a subsequence of \( \{v_n\} \) converges to an element \( v \) that is a solution of the system (10).

\( \square \)
**Theorem 3.4.** The solution of the problem (10), in the sense of definition 3.1, is unique.

**Proof.** Let us assume that \( v_1, v_2 \in W^{1,\infty, 2}(0, T; V, [L^2(\Omega)]^3) \) are two solutions of system (10). Let us denote \( v = v_1 - v_2 \). Then, subtracting the variational formulations corresponding to \( v_1 \) and \( v_2 \), we have that:

\[
\int_\Omega \frac{\partial v}{\partial t} \cdot z \, dx + \int_\Omega (\nabla v_1 v_1 - \nabla v_2 v_2) \cdot z \, dx
+ \int_\Omega (\beta(\epsilon(v_1)) \epsilon(v_1) - \beta(\epsilon(v_2)) \epsilon(v_2)) : \epsilon(z) \, dx = 0, \text{ a.e. } t \in (0, T), \forall z \in V. \tag{23}
\]

However, on the one hand,

\[
\nabla v_1 v_1 - \nabla v_2 v_2 = \nabla v_1 v + \nabla vv_2,
\]

and, on the other hand, by classical integration results,

\[
\beta(\epsilon(v_1)) \epsilon(v_1) - \beta(\epsilon(v_2)) \epsilon(v_2) = \frac{\partial D(\epsilon)}{\partial \epsilon} \bigg|_{\epsilon=\epsilon(v_1)} - \frac{\partial D(\epsilon)}{\partial \epsilon} \bigg|_{\epsilon=\epsilon(v_2)} = \int_0^1 \frac{d}{d\tau} \frac{\partial D(\epsilon)}{\partial \epsilon} \bigg|_{\epsilon=\epsilon(\tau)} \, d\tau = \int_0^1 \frac{\partial^2 D(\epsilon)}{\partial \epsilon^2} \bigg|_{\epsilon=\epsilon(\tau)} \, d\tau \epsilon(v),
\]

where \( \epsilon(\tau) = \tau \epsilon(v_1) + (1 - \tau) \epsilon(v_2) \). So, in particular, \( \frac{d}{d\tau} \epsilon(\tau) = \epsilon(v_1) - \epsilon(v_2) = \epsilon(v) \). Moreover, if we consider the definition (3) of \( D \), it is clear that

\[
\left( \int_0^1 \frac{\partial^2 D(\epsilon)}{\partial \epsilon^2} \bigg|_{\epsilon=\epsilon(\tau)} \, d\tau \right) \epsilon(v), \epsilon(v) \begin{bmatrix} [L^2(\Omega)]^3 \times 3 \\ [L^2(\Omega)]^3 \times 3 \end{bmatrix}
\]

\[
\geq C_1 \| \epsilon(v) \|^2_{[L^2(\Omega)]^3 \times 3} \int_0^1 \left( 1 + \| \epsilon \|_{[L^1(\Omega)]^3 \times 3} \right) \, d\tau
\]

\[
\geq C_2 \| \epsilon(v) \|^2_{[L^2(\Omega)]^3 \times 3},
\]

where \( C_1 \) and \( C_2 \) are two positive constants.

Therefore, if we choose as a test function \( z = v(t) \), take into account the previous estimates, and integrate in \([0, t] \) the equality (23), we obtain that:

\[
\frac{1}{2} \| v(t) \|^2_{[L^2(\Omega)]^3} + \int_0^t (\nabla v_1 v + \nabla vv_2, v)_{[L^2(\Omega)]^3} \, ds + C_2 \int_0^t \| \epsilon(v) \|^2_{[L^2(\Omega)]^3 \times 3} \, ds \leq 0,
\]

from where, using that

\[
(\nabla vv_2, v)_{[L^2(\Omega)]^3} = 0,
(\nabla v_1 v, v)_{[L^2(\Omega)]^3} = - (\nabla vv, v_1)_{[L^2(\Omega)]^3},
\]

we obtain, by Korn’s inequality, that

\[
\| v \|^2_{L^\infty(0, T; [L^2(\Omega)]^3)} + \| \nabla v \|^2_{L^2(0, T; [L^2(\Omega)]^3 \times 3)} \leq C_3 \int_0^t (\nabla vv, v_1)_{[L^2(\Omega)]^3} \, ds. \tag{24}
\]

However, since \( v_1 \in W^{1,\infty, 2}(0, T; V; [L^2(\Omega)]^3) \), and \( V \subset L^q(\Omega) \) for any \( 1 < q < \infty \) (cf. Corollary 1.22 in [16]), in particular, \( v_1 \in L^p(0, T; [L^q(\Omega)]^3) \) for any \( 1 < p, q < \infty \).
Thus, bearing in mind that, for domains $\Omega \subset \mathbb{R}^3$, we have (cf. [16]) that $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \subset L^{10/3}(0,T;L^{10/3}(\Omega))$, we obtain

\[
\int_0^t (\nabla\mathbf{v}, \mathbf{v}_1)_{L^2(\Omega)^3} \, ds \\
\leq C_4 \|\nabla\mathbf{v}\|_{L^2(0,t;L^2(\Omega))^{3\times 3}} \|\mathbf{v}_1\|_{L^2(0,t;L^2(\Omega))} \\
\leq C_4 \|\nabla\mathbf{v}\|_{L^2(0,t;L^2(\Omega))^{3\times 3}} \|\mathbf{v}\|_{L^{10/3}(0,t;L^{10/3}(\Omega))^{3\times 3}} \|\mathbf{v}_1\|_{L^5(0,t;L^5(\Omega)^3)} \\
\leq C_5 \|\mathbf{v}_1\|_{L^5(0,t;L^5(\Omega)^3)} \left( \|\mathbf{v}\|_{L^2(0,t;L^2(\Omega))^{3\times 3}} + \|\nabla\mathbf{v}\|_{L^2(0,t;L^2(\Omega))^{3\times 3}} \right),
\]

where $C_4$ and $C_5$ are positive constants.

Finally, if we denote $y(t) = \|\mathbf{v}\|_{L^2(0,t;L^2(\Omega))^{3\times 3}} + \|\nabla\mathbf{v}\|_{L^2(0,t;L^2(\Omega))^{3\times 3}}$ and $F(t) = \|\mathbf{v}_1\|_{L^5(0,t;L^5(\Omega)^3)}$, there will be a positive constant $C_y$ such that:

\[
y(t) \leq C_y F(t) y(t), \quad \forall t \in [0,T],
\]

with $\lim_{t \to 0} F(t) = 0$. From this, we can deduce that $y(t) = 0$, for any $t \in [0,T]$. Consequently, we have that $\mathbf{v} = \mathbf{0}$, i.e., $\mathbf{v}_1 = \mathbf{v}_2$.

3.1.2. Existence and uniqueness of solution for the non-homogeneous case. Let us assume that the flow rates $g^k(t)$ are, for all values of $k = 1, \ldots, N_{CT}$, continuous functions of constant sign over the entire interval $[0,T]$. We then analyze the following problem:

\[
\begin{cases}
\frac{\partial \mathbf{v}}{\partial t} + \nabla\mathbf{v}\mathbf{v} - \nabla \cdot \mathbf{\Xi}(\mathbf{v}) + \nabla p = \mathbf{F} & \text{in } \Omega \times (0,T), \\
\nabla \cdot \mathbf{v} = 0 & \text{in } \Omega \times (0,T), \\
\mathbf{v} = \mathbf{v}_S & \text{on } \Gamma_S \times (0,T), \\
\mathbf{v} = \mathbf{0} & \text{on } \Gamma_B \times (0,T), \\
\mathbf{v} = \mathbf{0} & \text{on } (\Gamma_T \setminus \cup_{k=1}^{N_{CT}} T^k) \times (0,T), \\
\mathbf{v} = \mathbf{0} & \text{on } (\Gamma_C \setminus \cup_{k=1}^{N_{CT}} C^k) \times (0,T), \\
\mathbf{v} = \frac{g^k(t)}{\mu(C^k)} \mathbf{n} & \text{on } C^k \times (0,T), \quad \text{for } k = 1, \ldots, N_{CT}, \\
\mathbf{v} = -\frac{g^k(t)}{\mu(T^k)} \mathbf{n} & \text{on } T^k \times (0,T), \quad \text{for } k = 1, \ldots, N_{CT}, \\
\mathbf{v}(0) = \mathbf{v}_0 & \text{in } \Omega,
\end{cases}
\tag{25}
\]

where $g^k(0) = 0$, for $k = 1, \ldots, N_{CT}$, $\mathbf{v}_0 \in \mathbf{V}$, and the surface velocity $\mathbf{v}_S$ is assumed null for the sake of simplicity. In order to establish the concept of solution, we need to introduce the following space:

\[
\mathbf{W} = \{ \mathbf{v} \in [W^{1,3}(\Omega)]^3 : \nabla \cdot \mathbf{v} = 0 \quad \text{and} \quad \mathbf{v}_{\partial\Omega \cup \{k=1 \cup N_{CT}(T^k \cup C^k)\}} = 0 \}. 
\tag{26}
\]

Remark 2. We must note that $\mathbf{V} \subset \mathbf{W}$. Thus, considering the initial condition in $\mathbf{V}$ makes sense. In this way, it is also evident the need for the compatibility condition between the initial and the boundary conditions.

Now, to prove the existence of solution of the problem with non-homogeneous Dirichlet conditions, we will use the techniques developed in [7]. That is, we will reconstruct from the values associated with the Dirichlet condition a function - in a suitable space - whose trace coincides with the given values. Once the above reconstruction is recovered, it will be used to reduce the search for a solution of the problem (25) to a homogeneous problem of the type (10).
Lemma 3.5. establishes the following result, that we state in the form of Lemma: the construction is sufficient to reach that of the homogeneous solution. This reconstruction coincides with the previous function \( g \)

\[ g(0) = 0, \text{ a.e. } x \in \partial \Omega \setminus \bigcup_{k=1}^{N_{CT}} (T^k \cup C^k). \]

Then, the vector satisfying that its normal component is equal to the vector \( g \) will be denoted by \( g \), that is, \( g = g_n n \in \mathcal{U} \).

If we take into account the previous notation, we can reformulate the problem (25) in the following terms:

\[
\begin{aligned}
\frac{\partial v}{\partial t} + \nabla v \cdot v - \nabla \cdot \mathbf{V}(v) + \nabla p &= \mathbf{F} \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot v &= 0 \quad \text{in } \Omega \times (0, T), \\
v &= g \quad \text{on } \partial \Omega \times (0, T), \\
v(0) &= v_0 \quad \text{in } \Omega.
\end{aligned}
\] (27)

The idea, as we commented before, is to reconstruct a function so that its trace coincides with the previous function \( g \), and such that the regularity of the reconstruction is sufficient to reach that of the homogeneous solution. This reconstruction can be done on the basis of Theorem 2.2 of [7] which, in our particular case, establishes the following result, that we state in the form of Lemma:

**Lemma 3.5.** There exists a linear, continuous mapping

\[ R : H^{s-1/2}(\partial \Omega \times (0, T)) \cap L^2(0, T; \widetilde{H}^1(\partial \Omega)) \rightarrow W^{1,2,2}(0, T; [H_s^*(\Omega)]^3, [H^{s-2}(\Omega)]^3) \]

\[ g_n \rightarrow R(g_n) = \xi_g, \]

such that \( \xi_g|_{\partial \Omega} = g_n n \) a.e. \( (x, t) \in \partial \Omega \times (0, T) \), where \( s > 3/2, \widetilde{H}^1(\partial \Omega) = \{ v \in H^1(\partial \Omega) : \int_{\partial \Omega} v \, d\gamma = 0 \} \), and \( [H_s^*(\Omega)]^3 = \{ u \in [H^s(\Omega)]^3 : \nabla \cdot u = 0 \} \).

The problem now is that establishing a normal component constant in space may be incompatible with previous reconstruction theorem for the trace. To overcome this problem, we will consider the following regularization of the normal component:

**Definition 3.6.** For the function \( g_n \) associated to the Dirichlet condition, introduced in Remark 3, and for any value \( s \geq 3/2 \), we will define its regularization \( \tilde{g}_n \in H^{s-1/2}(\partial \Omega \times (0, T)) \cap L^2(0, T; \widetilde{H}^1(\partial \Omega)) \), such that it satisfies:

\[ \text{sign}(\tilde{g}_n_{|C^k}(t)) \geq 0, \text{ a.e. } t \in (0, T), k = 1, \ldots, N_{CT}, \]

\[ \int_{C^k} \tilde{g}_n \, d\gamma = g^k(t), \text{ a.e. } t \in (0, T), k = 1, \ldots, N_{CT}, \]

\[ \text{sign}(\tilde{g}_n_{|T^k}(t)) \leq 0, \text{ a.e. } t \in (0, T), k = 1, \ldots, N_{CT}, \]

\[ \int_{T^k} \tilde{g}_n(t) \, d\gamma = -g^k(t), \text{ a.e. } t \in (0, T), k = 1, \ldots, N_{CT}, \]

\[ \tilde{g}_n(t) = 0, \text{ a.e. } (x, t) \in \left( \partial \Omega \setminus \bigcup_{k=1}^{N_{CT}} (T^k \cup C^k) \right) \times [0, T]. \]

Moreover, we will assume that this element \( \tilde{g}_n \) is consistent with the initial conditions of the problem, that is, \( \tilde{g}_n(0) = 0 \) a.e. \( x \in \partial \Omega \).
From now on, we will assume that the normal component of the velocity associated with the Dirichlet condition satisfies the regularity established in Lemma 3.5. For the sake of simplicity in the notation, we will denote by \( g \) that regularized normal component, and, given a value of \( s \), by:

\[
\mathcal{U} = \mathcal{H}^{r-1/2}(\partial \Omega \times (0, T)) \cap L^2(0, T; \tilde{H}^1(\partial \Omega))
\]  

(28)

the space associated to the Dirichlet conditions (clearly depending on \( s \), although this is not shown in order to simplify the notation).

**Lemma 3.7.** We have the following continuous injections for \( s \geq 7/2 \):

\[
\begin{align*}
W^{1,2,2}(0, T; [H^s_\sigma(\Omega)]^3, [H^{s-2}(\Omega)]^3) & \\
\subset W^{1,\infty,2}(0, T; [H^{5/2}_\sigma(\Omega)]^3, [H^{3/2}(\Omega)]^3) & \cap C([0, T]; [H^{5/2}_\sigma(\Omega)]^3) \\
\subset W^{1,\infty,2}(0, T; [W^{2,3}_\sigma(\Omega)]^3, [W^{1,3}(\Omega)]^3) & \cap C([0, T]; [W^{2,3}_\sigma(\Omega)]^3).
\end{align*}
\]

**Proof.** For \( s = 7/2 \), we only need to consider the embedding:

\[
W^{1,2,2}(0, T; [H^{7/2}_\sigma(\Omega)]^3, [H^{3/2}(\Omega)]^3) \subset C([0, T]; [H^{5/2}_\sigma(\Omega)]^3)
\]

and the fact that \( H^{5/2}_\sigma(\Omega) \subset W^{2,3}(\Omega) \). \( \square \)

As remarked at the beginning of the section, we look for a solution of problem (25) in the way

\[
v = \zeta_g + z \in W^{1,\infty,2}(0, T; \mathbf{W}, [L^2(\Omega)]^3) \cap C([0, T]; \mathbf{W}),
\]

(29)

where \( \zeta_g \in W^{1,2,2}(0, T; [H^s_\sigma(\Omega)]^3, [H^{s-2}(\Omega)]^3) \) is the reconstruction of the trace \( g \), as given in Lemma 3.5, and \( z \in W^{1,\infty,2}(0, T; \mathbf{V}, [L^2(\Omega)]^3) \cap C([0, T]; \mathbf{V}) \) is a function to be determined, solution of a homogeneous problem similar to (10).

**Definition 3.8.** An element \( v \in W^{1,\infty,2}(0, T; \mathbf{W}, [L^2(\Omega)]^3) \cap C([0, T]; \mathbf{W}) \) is a solution of problem (25) if there exists an element \( z \in W^{1,\infty,2}(0, T; \mathbf{V}, [L^2(\Omega)]^3) \cap C([0, T]; \mathbf{V}) \) such that:

1. \( v = \zeta_g + z \), with \( \zeta_g \in W^{1,2,2}(0, T; [H^s_\sigma(\Omega)]^3, [H^{s-2}(\Omega)]^3) \), the reconstruction of the trace given in Lemma 3.5 for values of \( s \geq 7/2 \).
2. \( z(0) = v_0 - \zeta_g(0) \) a.e. in \( \Omega \).
3. \( z \) verifies the following variational formulation:

\[
\begin{align*}
\int_\Omega \frac{\partial z}{\partial t} \cdot \eta \, dx + \int_\Omega \nabla (\zeta_g + z) \cdot \eta \, dx + \int_\Omega \nabla \zeta_g \cdot \eta \, dx + 2 \nu \int_\Omega \epsilon (z) : \epsilon (\eta) \, dx \\
+ 2 \nu \int_\Omega \int \epsilon (\zeta_g + z) : \epsilon (\zeta_g + z) + \epsilon (z) : \epsilon (\eta) \, dx \\
= \int_\Omega f \cdot \eta \, dx - \int_\Omega \frac{\partial \zeta_g}{\partial t} \cdot \eta \, dx - 2 \nu \int_\Omega \epsilon (\zeta_g) : \epsilon (\eta) \, dx \\
- \int_\Omega \nabla \zeta_g \cdot \eta \, dx, \quad \text{a.e. } t \in (0, T), \quad \forall \eta \in \mathbf{V}.
\end{align*}
\]

**Remark 4.** It is worthwhile noting here that, thanks to the regularity required for the reconstruction of the trace, the second member associated to the variational formulation (30) lies in \( L^2(0, T; [L^2(\Omega)]^3) \), that is:

\[
H = F - \frac{\partial \zeta_g}{\partial t} + 2 \nu \nabla \cdot \epsilon (\zeta_g) - \nabla \zeta_g \zeta_g \in L^2(0, T; [L^2(\Omega)]^3).
\]

Moreover, it is straightforward that:

\[
\|H\|_{L^2(0, T; [L^2(\Omega)]^3)} \leq C \left( \|F\|_{L^2(0, T; [L^2(\Omega)]^3)} + \|\zeta_g\|_{W^{1,2,2}(0, T; [H^s_\sigma(\Omega)]^3, [H^{s-2}(\Omega)]^3)} \right).
\]
Lemma 3.9. Let $\mathbf{z}$ be a solution of (30), then for $s \geq 7/2$, there exist continuous functions $\Phi_1$ and $\Phi_2$ such that:

$$
\|\mathbf{z}\|_{L^\infty(0,T;L^2(\Omega)^3)} + \|\nabla \mathbf{z}\|_{L^2(0,T;[L^2(\Omega)]^3 \times 3)} + \|\nabla \mathbf{z}\|_{L^3(0,T;[L^3(\Omega)]^3 \times 3)} \\
\leq \Phi_1 \left( \|\mathbf{v}_0\|_{L^2(\Omega)^3}, \|\mathbf{F}\|_{L^2(0,T;[L^2(\Omega)]^3)}, \|\mathbf{c}_g\|_{W^{1,2,2}(0,T;H^2(\Omega)^3, [H^{2-2}\Omega]^3)} \right) \tag{31}
$$

$$
\left\| \frac{\partial \mathbf{z}}{\partial t} \right\|_{L^2(0,T;[L^2(\Omega)]^3)} + \|\nabla \mathbf{z}\|_{L^\infty(0,T;[L^3(\Omega)]^3 \times 3)} + \|\nabla \mathbf{z}\|_{L^\infty(0,T;[L^3(\Omega)]^3 \times 3)} \\
\leq \Phi_2 \left( \|\nabla \mathbf{v}_0\|_{L^2(\Omega)^3 \times 3}, \|\nabla \mathbf{v}_0\|_{L^2(\Omega)^3 \times 3}, \|\mathbf{F}\|_{L^2(0,T;[L^2(\Omega)]^3)}, \right. \\
\left. \|\mathbf{c}_g\|_{W^{1,2,2}(0,T;H^2(\Omega)^3, [H^{2-2}\Omega]^3)} \right). \tag{32}
$$

Proof. We divide the demonstration into two parts: one for each estimate.

a) First, in order to obtain the energy estimate, we consider $\eta = \mathbf{z}(t)$ as a test function. So,

$$
\begin{align*}
\frac{1}{2} \left\| \mathbf{z}(t) \right\|^2_{L^2(\Omega)^3} + \int_0^t \int_\Omega \nabla \mathbf{c}_g \cdot \mathbf{z} \, dx \, ds \\
+C_1 \left( \int_0^t \|\mathbf{e}(\mathbf{z})\|^2_{L^2(\Omega)^3 \times 3} \, ds + \int_0^t \|\mathbf{e}(\mathbf{c}_g + \mathbf{z})\|^3_{L^3(\Omega)^3 \times 3} \right) \\
\leq \frac{1}{2} \|\mathbf{v}_0 - \mathbf{c}_g(0)\|^2_{L^2(\Omega)^3} + \int_0^t \int_\Omega \mathbf{H} \cdot \mathbf{z} \, dx \, ds \\
+2\nu_{tur} \int_0^t \int_\Omega \left[ \mathbf{e}(\mathbf{c}_g + \mathbf{z}) : \mathbf{e}(\mathbf{c}_g + \mathbf{z}) \right]^{1/2} \mathbf{e}(\mathbf{c}_g + \mathbf{z}) : \mathbf{e}(\mathbf{c}_g) \, dx \, ds,
\end{align*}
$$

where $C_1$ is a positive constant. Now, in one hand,

$$
\int_0^t \int_\Omega \left[ \mathbf{e}(\mathbf{c}_g + \mathbf{z}) : \mathbf{e}(\mathbf{c}_g + \mathbf{z}) \right]^{1/2} \mathbf{e}(\mathbf{c}_g + \mathbf{z}) : \mathbf{e}(\mathbf{c}_g) \, dx \, ds \\
\leq \int_0^t \|\mathbf{e}(\mathbf{c}_g + \mathbf{z})\|^2_{[L^3(\Omega)]^3 \times 3} \|\mathbf{e}(\mathbf{c}_g)\|_{[L^3(\Omega)]^3 \times 3} \, ds \\
\leq \frac{2}{3} \int_0^t \|\mathbf{e}(\mathbf{c}_g + \mathbf{z})\|^3_{[L^3(\Omega)]^3 \times 3} \, ds + \frac{1}{3} \int_0^t \|\mathbf{e}(\mathbf{c}_g)\|^3_{[L^3(\Omega)]^3 \times 3} \, ds,
$$

but, in the other hand,

$$
\int_0^t \int_\Omega \nabla \mathbf{c}_g \cdot \mathbf{z} \, dx \, ds \leq \int_0^t \|\nabla \mathbf{c}_g\|_{[L^3(\Omega)]^3 \times 3} \|\mathbf{z}\|^2_{[L^3(\Omega)]^3} \, ds \\
+ \frac{1}{3\sqrt{\epsilon}} \int_0^t \|\mathbf{c}_g\|^3_{W^{1,3}(\Omega)^3} \, ds + \frac{2}{3}\sqrt{\epsilon} \int_0^t \|\mathbf{z}\|^3_{W^{1,3}(\Omega)^3} \, ds, \quad \forall \epsilon > 0.
$$

Thus, fitting the values of $\epsilon$ in previous expression, and using triangle inequality for the norms and Korn’s Lemma, we obtain the existence of a positive constant $C_2$ such that:

$$
\|\mathbf{z}(t)\|^2_{L^2(\Omega)^3} + \int_0^t \|\nabla \mathbf{z}(s)\|^2_{L^2(\Omega)^3 \times 3} \, ds + \int_0^t \|\nabla \mathbf{z}(s)\|^3_{L^3(\Omega)^3 \times 3} \, ds \\
\leq C_2 \left( \|\mathbf{v}_0 - \mathbf{c}_g(0)\|^2_{L^2(\Omega)^3} + \int_0^t \|\mathbf{c}_g\|^3_{W^{1,3}(\Omega)^3} \, ds \\
+ \int_0^t \|\mathbf{H}(s)\|^2_{L^2(\Omega)^3} \, ds + \int_0^t \|\mathbf{z}(s)\|^2_{L^2(\Omega)^3} \, ds \right)
$$

and, consequently, from Gronwall’s Lemma, we derive the existence of a continuous function $\Phi_1$ verifying:

$$
\|\mathbf{z}\|_{L^\infty(0,T;[L^2(\Omega)]^3)} + \|\nabla \mathbf{z}\|_{L^2(0,T;[L^2(\Omega)]^3 \times 3)} + \|\nabla \mathbf{z}\|_{L^3(0,T;[L^3(\Omega)]^3 \times 3)} \\
\leq \Phi_1 \left( \|\mathbf{v}_0\|_{L^2(\Omega)^3}, \|\mathbf{F}\|_{L^2(0,T;[L^2(\Omega)]^3)}, \|\mathbf{c}_g\|_{W^{1,2,2}(0,T;H^2(\Omega)^3, [H^{2-2}\Omega]^3)} \right). \tag{34}
$$
b) Second, for deriving the estimate on the time derivative, we will consider \( \eta = \partial z/\partial t(t) \) as a test function. Then, arguing as in the homogeneous case we have:

\[
\begin{align*}
\left\| \frac{\partial z}{\partial t} (t) \right\|_{L^2(\Omega)^3}^2 &+ \frac{\partial}{\partial t} \int_{\Omega} D(\epsilon(\zeta_0(t) + z(t))) \, dx \\
&= \int_{\Omega} F(t) \cdot \frac{\partial z}{\partial t} (t) \, dx - \int_{\Omega} \frac{\partial \zeta_0}{\partial t} (t) \cdot \frac{\partial z}{\partial t} (t) \, dx \\
&\quad - \int_{\Omega} \nabla(\zeta_0(t) + z(t))(\zeta_0(t) + z(t)) \cdot \frac{\partial z}{\partial t} (t) \, dx \\
&\quad + \int_{\Omega} \beta(\zeta_0(t) + z(t)) \epsilon(\zeta_0(t) + z(t)) : \epsilon \left( \frac{\partial \zeta_0}{\partial t} (t) \right) \, dx. \\
\end{align*}
\]

(35)

Denoting by \( H = F - \partial \zeta_0/\partial t \in L^2(0,T; [L^2(\Omega)]^3) \), we have the following estimates:

\[
\begin{align*}
\int_{\Omega} H(t) : \frac{\partial z}{\partial t} (t) \, dx &\leq \|H(t)\|_{L^2(\Omega)^3}^2 + \frac{1}{4} \left\| \frac{\partial v}{\partial t} (t) \right\|_{L^2(\Omega)^3}^2, \\
\int_{\Omega} \nabla(\zeta_0(t) + z(t))(\zeta_0(t) + z(t)) \cdot \frac{\partial z}{\partial t} (t) \, dx \\
&\leq \|\nabla(\zeta_0(t) + z(t))(\zeta_0(t) + z(t))\|_{L^2(\Omega)^3}^2 + \frac{1}{4} \left\| \frac{\partial v}{\partial t} (t) \right\|_{L^2(\Omega)^3}^2. \\
\end{align*}
\]

(36)

Integrating in the time interval \((0,t)\):

\[
\frac{1}{2} \left\| \frac{\partial z}{\partial t} \right\|_{L^2(0,t;[L^2(\Omega)]^3)}^2 + \int_{\Omega} D(\epsilon(\zeta_0(t) + z(t))) \, dx - \int_{\Omega} D(\epsilon(\zeta_0(0) + z(0))) \, dx \\
&\leq \|H\|_{L^2(0,t;L^2(\Omega)^3)}^2 + \|\nabla(\zeta_0 - z)(\zeta_0 - z)\|_{L^2(0,t;L^2(\Omega)^3)}^2 \\
&\quad + \int_0^t \int_{\Omega} \beta(\zeta_0(s) + z(s)) \epsilon(\zeta_0(s) + z(s)) : \epsilon \left( \frac{\partial \zeta_0}{\partial t} (s) \right) \, dx \, ds,
\]

from where:

\[
\frac{1}{2} \left\| \frac{\partial z}{\partial t} \right\|_{L^2(0,t;[L^2(\Omega)]^3)}^2 + C_1 \left( \|\epsilon(\zeta_0(t) + z(t))\|_{L^2(\Omega)^3}^2 + \|\epsilon(\zeta_0(t) + z(t))\|_{L^2(\Omega)^3}^2 \right) \\
&\leq C_2 \left( \|\epsilon(\nu_0)\|_{L^2(\Omega)^3}^3 + \|\epsilon(\nu_0)\|_{L^2(\Omega)^3}^3 \right) + \|H\|_{L^2(0,t;L^2(\Omega)^3)}^2 \\
&\quad + \int_0^t \int_{\Omega} \beta(\zeta_0(s) + z(s)) \epsilon(\zeta_0(s) + z(s)) : \epsilon \left( \frac{\partial \zeta_0}{\partial t} (s) \right) \, dx \, ds,
\]

where \( C_1 \) and \( C_2 \) are positive constants. We only have to estimate the last two terms of the previous inequality. So, for the first one it is very similar to the homogeneous case, that is,

\[
\int_0^t \|\nabla(\zeta_0(s) + z(s))(\zeta_0(s) + z(s))\|_{L^2(\Omega)^3}^3 \, ds \leq \int_0^t \|\nabla(\zeta_0(s) + z(s))\|_{L^2(\Omega)^3}^3 \times \left( \|\nabla(\zeta_0(s) + z(s))\|_{L^2(\Omega)^3}^2 + \|\nabla(\zeta_0(s) + z(s))\|_{L^2(\Omega)^3}^3 \right) \, ds.
\]

For the second one,

\[
\begin{align*}
\int_0^t \int_{\Omega} \beta(\zeta_0(s) + z(s)) \epsilon(\zeta_0(s) + z(s)) : \epsilon \left( \frac{\partial \zeta_0}{\partial t} (s) \right) \, dx \, ds \\
&= C_3 \left( \int_0^t \int_{\Omega} \epsilon(\zeta_0(s) + z(s)) : \epsilon \left( \frac{\partial \zeta_0}{\partial t} (s) \right) \, dx \, ds \\
&\quad + \int_0^t \int_{\Omega} \epsilon(\zeta_0(s) + z(s)) : \epsilon(\zeta_0(s) + z(s)) \right)^{1/2} \epsilon(\zeta_0(s) + z(s)) : \epsilon \left( \frac{\partial \zeta_0}{\partial t} (s) \right) \, dx \, ds \right).
\end{align*}
\]
with $C_3$ a positive constant. However,

$$
\int_0^t \int_\Omega [\epsilon(\zeta_g(s) + z(s)) : \epsilon \left( \frac{\partial \zeta_g}{\partial t}(s) \right)] \, dx \, ds \leq \int_0^t \|\epsilon(\zeta_g(s) + z(s))\|_{L^2(\Omega)}^2 \left\| \epsilon \left( \frac{\partial \zeta_g}{\partial t}(s) \right) \right\|_{L^2(\Omega)}^2 \, ds
$$

$$
\leq C_4 + \int_0^t \left( \|\epsilon(\zeta_g(s) + z(s))\|_{L^2(\Omega)}^2 + \|\epsilon(\zeta_g(s) + z(s))\|_{L^2(\Omega)}^3 \right)
\times \left\| \epsilon \left( \frac{\partial \zeta_g}{\partial t}(s) \right) \right\|_{L^2(\Omega)}^2 \, ds,
$$

where $C_4$ is a positive constant. Moreover,

$$
\int_0^t \int_\Omega [\epsilon(\zeta_g(s) + z(s)) : \epsilon(\zeta_g(s) + z(s))]^{1/2} \epsilon(\zeta_g(s) + z(s)) : \epsilon \left( \frac{\partial \zeta_g}{\partial t}(s) \right) \, dx \, ds
\leq C_5 + \int_0^t \left( \|\epsilon(\zeta_g(s) + z(s))\|_{L^2(\Omega)}^2 + \|\epsilon(\zeta_g(s) + z(s))\|_{L^2(\Omega)}^3 \right)
\times \left\| \epsilon \left( \frac{\partial \zeta_g}{\partial t}(s) \right) \right\|_{L^2(\Omega)}^3 \, ds,
$$

with $C_5$ a positive constant. Taking into account all above estimates, we can establish the following:

$$
\frac{1}{2} \left\| \frac{\partial z}{\partial t} \right\|_{L^2(0,T;[L^2(\Omega)]^3)}^2 + C_1 \left( \|\epsilon(\zeta_g(t) + z(t))\|_{L^2(\Omega)}^2 + \|\epsilon(\zeta_g(t) + z(t))\|_{L^2(\Omega)}^3 \right)
\leq C_2 \left( \|\epsilon(v_0)\|_{L^2(\Omega)}^2 + \|\epsilon(v_0)\|_{L^2(\Omega)}^3 + \|H\|_{L^2(0,T;[L^2(\Omega)]^3)} \right)
\cdot \int_0^t \left( \|\nabla(\zeta_g(s) + z(s))\|_{L^2(\Omega)}^2 + \|\nabla(\zeta_g(s) + z(s))\|_{L^2(\Omega)}^3 \right)
\times \left\| \nabla(\zeta_g(s) + z(s)) \right\|_{L^2(\Omega)}^3 \, ds
+ C_6 \int_0^t \left( \|\epsilon(\zeta_g(s) + z(s))\|_{L^2(\Omega)}^2 + \|\epsilon(\zeta_g(s) + z(s))\|_{L^2(\Omega)}^3 \right)
\times \left\| \epsilon \left( \frac{\partial \zeta_g}{\partial t}(s) \right) \right\|_{L^2(\Omega)}^3 \, ds
+ C_7 \int_0^t \left( \|\epsilon(\zeta_g(s) + z(s))\|_{L^2(\Omega)}^2 + \|\epsilon(\zeta_g(s) + z(s))\|_{L^2(\Omega)}^3 \right)
\times \left\| \epsilon \left( \frac{\partial \zeta_g}{\partial t}(s) \right) \right\|_{L^2(\Omega)}^{3/2} \, ds + C_8,
$$

where $C_6$, $C_7$ and $C_8$ are positive constants. Finally, using triangular inequality, Korn’s Lemma and Gronwall’s Lemma, we can conclude the existence of a continuous function $\Phi_2$, verifying the desired estimate:

$$
\left\| \frac{\partial z}{\partial t} \right\|_{L^2(0,T;[L^2(\Omega)]^3)} + \|\nabla z\|_{L^\infty(0,T;[L^2(\Omega)]^3)} + \|\nabla z\|_{L^\infty(0,T;[L^2(\Omega)]^3)}
$$
Theorem 3.10. Let us assume that $s \geq 7/2$. Then, there exists a solution $v \in W^{1,\infty;2}(0,T; W,[L^2(\Omega))^3]) \cap C([0,T]; W)$ of the problem (25), continuous with respect to data, that verifies the following estimates:

$$
\|v\|_{L^\infty(0,T;[L^2(\Omega)])} + \|\nabla v\|_{L^2(0,T;[L^2(\Omega)])^3 \times 3} + \|\nabla^2 v\|_{L^2(0,T;[H^1(\Omega)])^3 \times 3} \\
\|\frac{\partial v}{\partial t}\|_{L^2(0,T;[L^2(\Omega)])^3} + \|\nabla v\|_{L^\infty(0,T;[L^2(\Omega)])^3 \times 3} + \|\nabla v\|_{L^\infty(0,T;[L^2(\Omega)])^3 \times 3}
$$

$$\leq \Phi_3(\|v_0\|_{L^2(\Omega)}^3, \|v_0\|_{L^2(0,T;[L^2(\Omega)])^3}, \|F\|_{L^2(0,T;[L^2(\Omega)])^3}, \|\zeta\|_{W^{1,\infty;2}(0,T;[H^1(\Omega)])^3 \times [H^{s-2}(\Omega)])^3}, \|\zeta\|_{W^{1,\infty;2}(0,T;[H^1(\Omega)])^3 \times [H^{s-2}(\Omega)])^3}.
$$

with $\Phi_3$ and $\Phi_4$ continuous functions.

Proof. Using the estimates of Lemma 3.9 combined with the Galerkin method, we can obtain the existence of $z \in W^{1,\infty;2}(0,T; V,[L^2(\Omega)])^3]) \cap C([0,T]; W)$, solution of problem (30) and, consequently, from Lemma 3.8, the existence of $v = \zeta + z \in W^{1,\infty;2}(0,T; W,[L^2(\Omega)])^3]) \cap C([0,T]; W)$, solution of problem (25).

Theorem 3.11. The solution of the problem (25) is unique.

Proof. The proof is similar to the homogeneous case, whenever we employ the definition of non-homogeneous solution established in Definition 3.8.

3.1.3. Existence and uniqueness of solution for the general case. We are now ready to analyze the existence of solution of the general problem (1). Logically, the existence of solution of the general problem can only be achieved for certain pumping/turbination actuation regimes. That is, we will start from the base on which only a finite number of changes in the regime of operation of the pumps takes place. In order to simplify the exposure of the results, and without losing generality, we will assume that all pumps change their operating mode in unison. It is always possible to consider such a situation by introducing an appropriate refinement of time intervals in which it is possible for all groups to keep unchanged during the time period.

Let us assume then that $g = (g_1, \ldots, g_{NCT}) \in [C([0,T])]^{NCT}$ are functions verifying that there exist a finite number of time instants $t_1, \ldots, t_R \in (0,T)$ with $0 = t_0 < t_1 < \ldots < t_R < t_{R+1} = T$, such that:

$$\text{sign}(g^k(t)) = c^k, \quad \forall t \in (t_k,t_{k+1}), \quad k = 0, \ldots, R,
$$

where $c^k \in \{0,1,-1\}$ is a constant depending on the pumping operation mode of the group. The problem is reduced, therefore, to solve in a chained way the hydrodynamic model in each one of the intervals:

- Let $v_1 \in W^{1,\infty;2}((t_0,t_1); W,[L^2(\Omega)])^3]) \cap C([t_0,t_1]; W)$ be the solution of the hydrodynamic model in the interval $[t_0,t_1]$, taking as initial condition $v_0$.
- For $j = 2, \ldots, R+1$, let $v_j \in W^{1,\infty;2}((t_{j-1},t_j); W,[L^2(\Omega)])^3]) \cap C([t_{j-1},t_j]; W)$ be the solution of the hydrodynamic model in the interval $[t_{j-1},t_j]$, taking as initial condition $v_{j-1}(t_{j-1})$. 
Based on the results obtained in the previous subsection, we can guarantee the existence and uniqueness of the solution in each one of the time intervals, verifying the estimates (38)-(39). It should be noted that the time continuity of the solution in space $W$ is a key element to demonstrate the existence of solution. We recall again that the only restriction we ask the set of pumps is that the operation mode changes occur at the same time: this restriction is met naturally if we require that the groups can only change state a finite number of times in the whole time $[0, T]$. Then, we define the function $\mathbf{v} \in W^{1,\infty,2}(0, T; W, [L^2(\Omega)]^3) \cap C([0, T]; W)$, given by:

$$
\mathbf{v}_{[t_{j-1}, t_j]} = \mathbf{v}_j, \quad \forall j = 1, 2, \ldots, R + 1.
$$

It is evident that the solution constructed in this way is the only solution of the hydrodynamic model (1). Moreover, it is continuous with respect to the data in the sense that it verifies estimates analogous to those obtained in (38)-(39). So, we have demonstrated the following existence/uniqueness result:

**Theorem 3.12.** Let us assume that $s \geq 7/2$. Then, there exists a unique solution $\mathbf{v} \in W^{1,\infty,2}(0, T; W, [L^2(\Omega)]^3) \cap C([0, T]; W)$ of the problem (1), continuous with respect to data, that verifies the estimates (38) and (39).

### 3.2. The eutrophication model: Michaelis-Menten kinetics

In this section we will analyze the existence and uniqueness of the solution to the problem (5), for this, we will start by analyzing a certain type of parabolic equation that will later be used as the basis for the study of the existence of solution of the above problem.

#### 3.2.1. Analysis of a particular type of parabolic equation

Let us start by analyzing the existence and uniqueness of solution of the following particular equation that will later be used as the basis for the study of the existence of solution of the above problem.

Let $u \in W^{1,\infty,2}(0, T; \mathbb{R})$, $u \in C([0, T]; \mathbb{R})$, and $v \in C([0, T]; \mathbb{R})$. Then, there exists a unique solution $u \in W^{1,\infty,2}(0, T; \mathbb{R})$, continuous with respect to data, that verifies the estimates (38) and (39).
To look for solutions of the problem (42) in space $W^{1.2,2}(0, T; H^1(\Omega), H^1(\Omega)^*)$, we take

$$X = \{ u \in H^1(\Omega) : u|_{\Omega \cap T_{NCTk}} = 0 \},$$

(43)

as the space of test functions for the variational formulation associated with the problem (42).

We have the following technical result:

**Lemma 3.13.** The following injections are compact:

- $W^{1.2,2}(0, T; H^1(\Omega), H^1(\Omega)^*) \subset \subset L^6(\Omega)$,
- $W^{1.2,2}(0, T; H^1(\Omega), H^1(\Omega)^*) \subset \subset L^2(0, T; L^6(\Omega))$,
- $W^{1.2,2}(0, T; H^1(\Omega), H^1(\Omega)^*) \subset \subset L^2(0, T; L^4(\Omega))$.

**Proof.** The demonstration for the first inclusion is an immediate consequence of the compactness of injection of $H^1(\Omega)$ into $L^6(\Omega)$ and of the interpolation Lemma 7.8 of [16].

In order to prove the second and third compact inclusions, we can use the Aubin-Lions Lemma 7.7 of [16], along with the compact inclusions of $H^1(\Omega)$ into $L^6(\Omega)$ and $L^4(\Omega)$.

**Definition 3.14.** We will say that an element $u \in W^{1.2,2}(0, T; H^1(\Omega), H^1(\Omega)^*)$ is a solution of problem (42) if it satisfies the following conditions:

1. $u(0) = u_0$ a.e. $x \in \Omega$.
2. $u(t^+ \tau_k(0, T), \Omega)^* \subset \subset L^6(\Omega)$.
3. $u$ verifies the variational formulation:

$$\int_\Omega \frac{\partial u}{\partial t} \eta \, dx + \int_\Omega v \cdot \nabla u \eta \, dx + \mu \int_\Omega \nabla u \cdot \nabla \eta \, dx + \int_\Omega k_1 u \eta \, dx$$

$$= \int_\Omega b_2 u \eta \, dx + \int_\Omega f \eta \, dx, \quad \forall \eta \in X, \text{ a.e. } t \in (0, T).$$

(44)

**Remark 5.** It is worthwhile noting here that we also have the continuous injection

$$W^{1.2,2}(0, T; H^1(\Omega), H^1(\Omega)^*) \subset C([0, T]; L^2(\Omega)).$$

**Theorem 3.15.** There exists a unique solution $u$ of the problem (42), that satisfies $u(x, t) \geq 0$ a.e. $(x, t) \in (0, T) \times \Omega$, and verifies the estimate:

$$||u||_{W^{2.2}(0, T; H^1(\Omega), H^1(\Omega)^*)} \leq \phi_0 \left( ||u_0||_{L^2(\Omega)} + \sum_{k=1}^{NCT} ||k_1||_{L^6(0, T; L^6(\Omega))} \right)$$

$$+ ||v||_{L^\infty(0, T; W)} + ||f||_{L^2(0, T; L^2(\Omega))} + \sum_{k=1}^{NCT} ||h_k||_{L^2(0, T)},$$

(45)

where $\phi_0$ is a positive constant, possibly depending on upper bound $k_{\text{max}}$.

**Proof.** The demonstration for the case of all groups in turbination mode can be performed using standard techniques for parabolic equations. An analogous result for the case of homogeneous Neumann boundary conditions has been proved - employing these techniques - by the authors in their recent work [4]. Interested readers can consult further details, for instance, in the monograph of Roubíček [16].

The demonstration for the other cases (with groups also pumping or at rest) is completely similar to the previous one, since the only changes are related to the particular location of the nonhomogeneous Dirichlet and the homogeneous Neumann conditions, but the other issues are exactly the same.
Theorem 3.17. Under some technical assumptions (see (58) below), there exists a unique solution \(u\) of the problem (46) such that:

1. \(u(0) = u_0^i\) a.e. \(x \in \Omega\).
2. \(u_{T^k} = \frac{1}{\mu(C_k)} \int_{C_k} u^i \, d\gamma\) a.e. \((x, t) \in T^k \times (0, T)\).
3. \(u^i\) verifies the variational formulation:

\[
\int_{\Omega} \frac{\partial u^i}{\partial t} \eta \, dx + \int_{\Omega} v \cdot \nabla u^i \eta \, dx + \mu^i \int_{\Omega} \nabla u^i \cdot \nabla \eta \, dx = \int_{\Omega} f^i \eta \, dx + \int_{\Omega} A^i(u) \eta \, dx, \quad \forall \eta \in X, \quad \text{a.e. } t \in (0, T).
\]

(47)

Definition 3.16. We will say that an element \(u \in W^{1,2,2}(0, T; [H^1(\Omega)]^5, [H^1(\Omega)]^{5'})\) is a solution of problem (46) if it satisfies the following conditions for \(i = 1, \ldots, 5\):

1. \(u^i(0) = u_0^i\) a.e. \(x \in \Omega\).
2. \(u_{T^k} = \frac{1}{\mu(C_k)} \int_{C_k} u^i \, d\gamma\) a.e. \((x, t) \in T^k \times (0, T)\).
3. \(u^i\) verifies the variational formulation:

\[
\int_{\Omega} \frac{\partial u^i}{\partial t} \eta \, dx + \int_{\Omega} v \cdot \nabla u^i \eta \, dx + \mu^i \int_{\Omega} \nabla u^i \cdot \nabla \eta \, dx = \int_{\Omega} f^i \eta \, dx + \int_{\Omega} A^i(u) \eta \, dx, \quad \forall \eta \in X, \quad \text{a.e. } t \in (0, T).
\]

Theorem 3.17. Under some technical assumptions (see (58) below), there exists a unique solution \(u \in W^{1,2,2}(0, T; [H^1(\Omega)]^5, [H^1(\Omega)]^{5'})\) of the problem (46) such that:

1. \(u^i(x, t) \geq 0\) a.e. \((x, t) \in \Omega \times (0, T), \text{ for } i = 1, \ldots, 4\).
2. \(\|u\|_{W^{1,2,2}(0, T; [H^1(\Omega)]^5, [H^1(\Omega)]^{5'})} \leq \phi_T(\|u_0\|_{L^2(\Omega)^5} + \|v\|_{L^\infty(0, T; W)} + \|f\|_{L^2(0, T; L^2(\Omega)^5)})\), with \(\phi_T\) a positive constant depending on the parameters of the problem.
3. \(u\) is locally Lipschitz continuous with respect to data.

Proof. In order to demonstrate the existence of solution, we consider the mapping:

\[
K: \Omega \times T \rightarrow \Omega \times T \quad \text{such that } \quad K(z_\Omega, z_T) = (u_\Omega, u_T),
\]

where \(\Omega\) is the set defined by:

\[
\Omega = \left\{(z_\Omega, z_T) \in [L^{10\epsilon}(0, T; L^{10\epsilon}(\Omega))]^5 \times [L^2((0, T))]^{5NCT}\right\}.
\]
\[ z^i_{1\Omega}(x, t) \geq 0, \ a.e. \ (x, t) \in \Omega \times (0, T), \ \forall i = 1, \ldots, 4, \]
\[ z^{2\Omega}_{T_k}(t) \geq 0, \ a.e. \ t \in (0, T), \ \forall i = 1, \ldots, 4, \ \forall k = 1, \ldots, N_{CT} \]

and the image of \( K \) is given by:
\[
\mathbf{u}_{\Omega} = \mathbf{u}, \quad \mathbf{u}_{T_k} = \frac{1}{\mu(C^k)} \int_{C^k} \mathbf{u} \, d\gamma, \quad \text{for} \ k = 1, \ldots, N_{CT},
\]

with \( \mathbf{u} \in W^{1,2,2}(0, T; [H^1(\Omega)]^5, [H^1(\Omega)]^5) \) the solution of the following uncoupled system of equations (solved in the order \( i = 2 \to i = 3 \to i = 4 \to i = 1 \to i = 5)\):
\[
\begin{align*}
\frac{\partial u^i}{\partial t} - \mathbf{v} \cdot \nabla u^i - \nabla \cdot (\mu^i \nabla u^i) + R_i^1(\mathbf{z}_\Omega, \mathbf{u})u^i &= R_i^2(\mathbf{z}_\Omega, \mathbf{u})u^i + F_i^i(\mathbf{u}) \quad \text{in} \ (0, T) \times \Omega, \\
\nabla u^i \cdot \mathbf{n} &= 0 \quad \text{on} \ (\partial \Omega \cup \cup_{k=1}^{N_{CT}} T^k) \times (0, T), \\
u^i &= z^i_{T_k} \quad \text{on} \ T^k \times (0, T), \quad \text{for} \ k = 1, \ldots, N_{CT}, \\
u^i(0) &= u_0^i \quad \text{in} \ \Omega,
\end{align*}
\]

where the reaction terms are given by:
\[
\mathbf{R}_1 = \begin{bmatrix}
C_{nc}L(x, t) \frac{u^2}{K_N + z^1_{\Omega}} \\
K_r + K_m f + K_z \frac{z^3_{\Omega}}{K_F + z^3_{\Omega}} \\
K_{mz} \\
K_{ra} \Theta^{L(x, t) - 20} \\
0
\end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix}
0 \\
L(x, t) \frac{z^1_{\Omega}}{K_N + z^1_{\Omega}} \\
C_{fz} K_z \frac{u^2}{K_F + u^2} \\
0 \\
0
\end{bmatrix},
\]

and the source term is given by:
\[
\mathbf{F} = \begin{bmatrix}
f^1(x, t) + C_{nc} K_r u^2 + C_{nc} K_{ra} \Theta^{L(x, t) - 20} u^4 \\
f^2(x, t) \\
f^3(x, t) \\
f^4(x, t) + K_m f u^2 + K_m z u^3 \\
f^5(x, t) + C_{nc} \left(L(x, t) \frac{u^1}{K_N + u^1} u^2 - K_r u^2 \right) - C_{nc} K_{ra} \Theta^{L(x, t) - 20} u^4
\end{bmatrix}.
\]

The non-negativity of \( \mathbf{u}_{\Omega} \) is a direct consequence of the Theorem 3.15, and for proving the non-negativity of \( \mathbf{u}_{T} \) we only have to regularize the equation (42) in order to obtain a continuous solution and then pass to the limit. Now we will show that the mapping \( K \) has a fixed point in a certain subset \( \hat{B} \) of \( B \). On the one hand, thanks to the Theorem 3.15 (for the equations corresponding to nutrients, phytoplankton, zooplankton and organic detritus), and to classic results for parabolic equations (for the equation associated with dissolved oxygen), we have that the application \( K \) is well defined.

In addition, following the order of resolution mentioned above, we can obtain the following estimates:

- For phytoplankton:
\[
\|u^2\|_{W^{1,2,2}(0, T; H^1(\Omega), H^1(\Omega)')} \leq C_2 \left( 1 + \sum_{k=1}^{N_{CT}} \|z^2_{T_k}\|_{L^2((0, T))} + \|z^3_{\Omega}\|_{L^{\infty}((0, T); L^{\infty}(\Omega))} \right), \quad (48)
\]
that they satisfy the compatibility conditions: 

\[ \sum_{k=1}^{N_{CT}} \frac{1}{\mu_i^k} \| u_i^2 \|_{L^2(0,T;L^1(C^k))} \leq C_T \| u_i^2 \|_{W^{1,2,1}(0,T;H^1(\Omega),H^1(\Omega'))}. \]

• For zooplankton:

\[ \| u_i^3 \|_{W^{1,2,1}(0,T;H^1(\Omega),H^1(\Omega'))} \leq C_3 \left( 1 + \sum_{k=1}^{N_{CT}} \| z_k^3 \|_{L^2((0,T))} \right), \]

\[ \sum_{k=1}^{N_{CT}} \frac{1}{\mu_i^k} \| u_i^4 \|_{L^2(0,T;L^1(C^k))} \leq C_9 \| u_i^4 \|_{W^{1,2,1}(0,T;H^1(\Omega),H^1(\Omega'))}. \]

• For organic detritus:

\[ \| u_i^1 \|_{W^{1,2,1}(0,T;H^1(\Omega),H^1(\Omega'))} \]

\[ \leq C_1 \left( 1 + \sum_{k=1}^{N_{CT}} \frac{1}{\mu_i^k} \| z_k^1 \|_{L^2((0,T))} + \sum_{k=1}^{N_{CT}} \| z_k^2 \|_{L^2((0,T))} \right), \]

\[ \sum_{k=1}^{N_{CT}} \frac{1}{\mu_i^k} \| u_i^1 \|_{L^2(0,T;L^1(C^k))} \leq C_6 \| u_i^1 \|_{W^{1,2,1}(0,T;H^1(\Omega),H^1(\Omega'))}. \]

• For dissolved oxygen:

\[ \| u_i^5 \|_{W^{1,2,1}(0,T;H^1(\Omega),H^1(\Omega'))} \]

\[ \leq C_5 \left( 1 + \sum_{k=1}^{N_{CT}} \| z_k^2 \|_{L^2((0,T))} + \sum_{k=1}^{N_{CT}} \| z_k^2 \|_{L^2((0,T))} \right), \]

\[ \sum_{k=1}^{N_{CT}} \frac{1}{\mu_i^k} \| u_i^5 \|_{L^2(0,T;L^1(C^k))} \leq C_{10} \| u_i^5 \|_{W^{1,2,1}(0,T;H^1(\Omega),H^1(\Omega'))}. \]

Above constants \( C_j \), for \( j = 1, \ldots, 10 \) can possibly depend on the initial conditions, on the water velocity, and on the source terms. In addition, if we assume that they satisfy the compatibility conditions:

\[ C_j C_{5+j} < 1, \quad \forall j = 1, \ldots, 5, \]
then, by simple arithmetic operations, we can assure the existence of new positive constants \(\hat{C}_j\), for \(j = 1, \ldots, 10\), such that, if we consider the subset \(\hat{B}\) of \(B\) defined by:

\[
\hat{B} = \left\{ (z_0, z_T) \in [L^{\frac{10}{5}}(0, T; L^{\frac{10}{5}}(\Omega))]^5 \times [L^2((0, T))]^{5N_{CT}} : \right. \\
\left. \|z_0^k\|_{L^{\frac{10}{5}}(0, T; L^{\frac{10}{5}}(\Omega))} \leq \hat{C}_i, \forall i = 1, \ldots, 5, \right. \\
\left. \sum_{k=1}^{N_{CT}} \|z_T^k\|_{L^2((0, T))} \leq \hat{C}_{5+i}, \forall i = 1, \ldots, 5, \right. \\
\left. z_0^k(x, t) \geq 0, \text{ a.e. } (x, t) \in (0, T) \times \Omega, \forall i = 1, \ldots, 4, \right. \\
\left. z_T^k(t) \geq 0, \text{ a.e. } t \in (0, T), \forall i = 1, \ldots, 4, \forall k = 1, \ldots, N_{CT} \right\},
\]

then, \(K(z_0, z_T) \in \hat{B}, \forall (z_0, z_T) \in \hat{B}\). To prove this, if we consider an element \((z_0, z_T) \in \hat{B}\), we have that \((u_{0i}, u_T) = K(z_0, z_T)\) verifies the following estimates:

\[
\|u_{01}^1\|_{W^{1,2}(0, T; H^1(\Omega))} \leq C_1(1 + \hat{C}_6 + \hat{C}_7 + \hat{C}_8 + \hat{C}_9 + \hat{C}_3), \\
\|u_{02}^2\|_{W^{1,2}(0, T; H^1(\Omega))} \leq C_2(1 + \hat{C}_7 + \hat{C}_3), \\
\|u_{03}^3\|_{W^{1,2}(0, T; H^1(\Omega))} \leq C_3(1 + \hat{C}_8), \\
\|u_{04}^4\|_{W^{1,2}(0, T; H^1(\Omega))} \leq C_4(1 + \hat{C}_7 + \hat{C}_8 + \hat{C}_9 + \hat{C}_3), \\
\|u_{05}^5\|_{W^{1,2}(0, T; H^1(\Omega))} \leq C_5(1 + \hat{C}_7 + \hat{C}_8 + \hat{C}_9 + \hat{C}_{10} + \hat{C}_3),
\]

and

\[
\sum_{k=1}^{N_{CT}} \|u_{01}^1\|_{L^2((0, T))} \leq C_6 C_1(1 + \hat{C}_6 + \hat{C}_7 + \hat{C}_8 + \hat{C}_9 + \hat{C}_3), \\
\sum_{k=1}^{N_{CT}} \|u_{02}^2\|_{L^2((0, T))} \leq C_7 C_2(1 + \hat{C}_7 + \hat{C}_3), \\
\sum_{k=1}^{N_{CT}} \|u_{03}^3\|_{L^2((0, T))} \leq C_8 C_3(1 + \hat{C}_8), \\
\sum_{k=1}^{N_{CT}} \|u_{04}^4\|_{L^2((0, T))} \leq C_9 C_4(1 + \hat{C}_7 + \hat{C}_8 + \hat{C}_9 + \hat{C}_3), \\
\sum_{k=1}^{N_{CT}} \|u_{05}^5\|_{L^2((0, T))} \leq C_{10} C_5(1 + \hat{C}_7 + \hat{C}_8 + \hat{C}_9 + \hat{C}_{10} + \hat{C}_3).
\]

Then, we can take the constants \(\hat{C}_j\), for \(j = 1, \ldots, 10\), defined as the solution of the following \(10 \times 10\) linear system:

\[
C_1(1 + \hat{C}_6 + \hat{C}_7 + \hat{C}_8 + \hat{C}_9 + \hat{C}_3) = \hat{C}_1, \\
C_2(1 + \hat{C}_7 + \hat{C}_3) = \hat{C}_2, \\
C_3(1 + \hat{C}_8) = \hat{C}_3, \\
C_4(1 + \hat{C}_7 + \hat{C}_8 + \hat{C}_9 + \hat{C}_3) = \hat{C}_4, \\
C_5(1 + \hat{C}_7 + \hat{C}_8 + \hat{C}_9 + \hat{C}_{10} + \hat{C}_3) = \hat{C}_5, \\
C_6 C_1(1 + \hat{C}_6 + \hat{C}_7 + \hat{C}_8 + \hat{C}_9 + \hat{C}_3) = \hat{C}_6, \\
C_7 C_2(1 + \hat{C}_7 + \hat{C}_3) = \hat{C}_7, \\
C_8 C_3(1 + \hat{C}_8) = \hat{C}_8 C_9 C_4(1 + \hat{C}_7 + \hat{C}_8 + \hat{C}_9 + \hat{C}_3) = \hat{C}_9, \\
C_{10} C_5(1 + \hat{C}_7 + \hat{C}_8 + \hat{C}_9 + \hat{C}_{10} + \hat{C}_3) = \hat{C}_{10}.
\]
and \(b = (C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10})^T\). By solving explicitly the system, we obtain the following solution:

\[
\hat{C}_1 = C_1 - \frac{C_1^2 C_6}{C_1 C_6 - 1} + \frac{C_1 C_4 C_9}{C_1 C_3} - \frac{(C_1 C_6 - 1) (C_2 C_7 - 1) (C_4 C_9 - 1)}{C_1 C_2 C_7} + \frac{(C_1 C_6 - 1) (C_2 C_7 - 1) (C_4 C_9 - 1)}{C_1 C_3 C_8 (C_3 - C_2 C_7 + 1)}
\]

\[
\hat{C}_2 = C_2 - \frac{C_2^2 C_7}{C_2 C_7 - 1} - \frac{C_2^2 C_7}{C_2 C_7 - 1} + \frac{C_2 C_3^2 C_8}{C_2 C_7 - 1} (C_3 C_8 - 1)
\]

\[
\hat{C}_3 = C_3 - \frac{C_3^2 C_8}{C_3 C_8 - 1},
\]

\[
\hat{C}_4 = C_4 - \frac{C_4^2 C_8}{C_3 C_8 (C_3 - C_2 C_7 + 1)} - \frac{C_3 C_4}{C_3 C_4 (C_3 C_8 - 1) (C_3 C_8 - 1)} + \frac{(C_2 C_7 - 1) (C_3 C_8 - 1) (C_4 C_9 - 1)}{(C_2 C_7 - 1) (C_3 C_8 - 1) (C_4 C_9 - 1)}
\]

\[
\hat{C}_5 = C_5 - \frac{C_5^2 C_{10}}{C_5 C_{10} - 1} + \frac{C_4 C_5 C_9}{C_4 C_9 (C_5 C_{10} - 1)} - \frac{(C_2 C_7 - 1) (C_4 C_9 - 1) (C_5 C_{10} - 1)}{C_2 C_5 C_7 (C_3 - C_2 C_7 + 1)} + \frac{(C_2 C_7 - 1) (C_3 C_8 - 1) (C_4 C_9 - 1) (C_5 C_{10} - 1)}{(C_2 C_7 - 1) (C_3 C_8 - 1) (C_4 C_9 - 1) (C_5 C_{10} - 1)}
\]

\[
\hat{C}_6 = \frac{C_1 C_4 C_6 C_9}{(C_1 C_6 - 1) (C_4 C_9 - 1)} - \frac{C_1 C_3 C_6}{C_1 C_6 - 1} - \frac{C_1 C_4 C_9}{C_1 C_6 C_8 (C_3 - C_2 C_7 + 1)} + \frac{(C_1 C_6 - 1) (C_2 C_7 - 1) (C_4 C_9 - 1)}{(C_1 C_6 - 1) (C_2 C_7 - 1) (C_4 C_9 - 1)}
\]

\[
\hat{C}_7 = \frac{C_2 C_3^2 C_7 C_8}{(C_2 C_7 - 1) (C_3 C_8 - 1)} - \frac{C_2 C_3 C_7}{C_2 C_7 - 1} - \frac{C_2 C_7}{C_2 C_7 - 1}
\]

\[
\hat{C}_8 = -\frac{C_3 C_8}{C_3 C_8 - 1}
\]
As we said at the beginning of these work, we will act on the pumped/turbinated flow for each of the pairs collector-injector. Since we are assuming that we are in a situation in which all groups are turbinating, we will consider the following set of admissible controls $\hat{u}$ normal component of velocity on the boundary):

$$
\hat{C}_9 = \frac{C_3C_4C_9}{(C_2C_7 - 1)(C_4C_9 - 1)} - \frac{C_4C_9}{C_1C_9 - 1} \frac{C_2C_4C_7C_9}{(C_2C_7 - 1)(C_4C_9 - 1)}.
$$

$$
\hat{C}_{10} = \frac{C_4C_5C_9C_{10}}{(C_4C_9 - 1)(C_5C_{10} - 1)} - \frac{C_4C_5C_{10}}{C_2C_5C_7C_{10}} - \frac{C_5C_{10} - 1}{C_3C_5C_8C_{10}(C_3 - C_2C_7 + 1)}.
$$

We can observe that, since $C_jC_{5+j} - 1 < 0$, $\forall j = 1, \ldots, 5$, then $\hat{C}_i > 0$, $\forall i = 1, \ldots, 10$, and then its clear that $K(z_{i1}, z_T) \in \hat{B}$, $\forall (z_{i1}, z_T) \in \hat{B}$.

Moreover, the continuity of the application $K$ can be demonstrated by techniques analogous to those described in [4], and its relative compactness is a straightforward consequence of the compact inclusions demonstrated in the Lemma 3.13. Thus, we can apply the Schauder fixed-point Theorem, which guarantees the existence of a fixed point for the application $K$ that is, by its definition, a solution of the state system (46).

Finally, for the uniqueness of solution and its continuity with respect to data, we refer the reader to the reference [4], since their demonstration can performed using analogous techniques.

3.2.3. Analysis of the model for the general case. The existence and uniqueness of solution for the general model (5) - that is, the case in which some pairs collector-injector are pumping, other ones turbinating, and other ones at rest - can be proved following the same idea as that developed in the section 3.1.3, mainly due to the fact that $W^{1,2,2}(0, T; H^1(\Omega), H^1(\Omega)) \subset C([0, T]; L^2(\Omega))$, and that the initial conditions $u_0^i$ are taken in $L^2(\Omega)$, for $i = 1, \ldots, 5$.

4. Mathematical analysis of the optimal control problem. Let us assume, without loss of generality, that all the groups are in turbination mode (the generalization to the rest of the cases does not present problems). We begin by establishing the precise formulation of the control problem $(P)$ in the following terms:

$$
(P) \quad \min \{ J(g) : g \in \mathcal{U}_{ad}^* \},
$$

where:

- **Control constraints:** As we said at the beginning of these work, we will act on the pumped/turbinated flow for each of the pairs collector-injector. Since we are assuming that we are in a situation in which all groups are turbinating, we will consider the following set of admissible controls $g$ (normal component of velocity on the boundary):

$$
\mathcal{U}_{ad} = \{ g \in H^{k-1/2}(\partial\Omega \times (0, T)) \cap L^2(0, T; \tilde{H}^1(\partial\Omega)) : 
\begin{align*}
&g(x, t) = 0, \ a.e. \ (x, t) \in (\partial\Omega \setminus \bigcup_{k=1}^{N_{CT}} (T^k \cup C^k)) \times (0, T), \\
&-c_1 \leq g(x, t) \leq 0, \ a.e. \ (x, t) \in T^k \times (0, T), \ k = 1, \ldots, N_{CT}, \\
&0 \leq g(x, t) \leq c_2, \ a.e. \ (x, t) \in C^k \times (0, T), \ k = 1, \ldots, N_{CT}, \\
&\left\| g \right\|_{L^2} \leq c_3,
\end{align*}
\}.
$$

\[ (60) \]
where \( c_1, c_2, c_3 > 0 \) are technological constants, and the parameter \( s \geq 7/2 \).

Given a normal velocity on the boundary \( g \in U_{ad} \), we denote by \( g = gn + 0\tau \) the associated velocity, and by \( \zeta_g \in W^{1,2,2}(0,T;[H^s/2(\Omega)]^3,[H^{s-2}/2(\Omega)]^3) \) its corresponding reconstruction, as given in Lemma 3.5.

- **State equations:** Given an admissible control \( g \in U_{ad} \), we denote by
  \[
  v \in W^{1,\infty,2}(0,T;W,[L^2(\Omega)]^3) \cap C([0,T];W),
  \]
  \[
  u \in W^{1,2,2}(0,T;[H^1(\Omega)]^5,[H^1(\Omega)]^5),
  \]
  the solutions of compositions:

  \[
  W^{1,\infty,2}(0,T;[H^s(\Omega)]^3,[H^{s-2}/2(\Omega)]^3) \rightarrow W^{1,2,2}(0,T;[H^s(\Omega)]^3,[H^{s-2}/2(\Omega)]^3)
  \]

  \[
  \zeta_g \rightarrow v \quad \text{(Theorem 3.10)}
  \]

  \[
  W^{1,\infty,2}(0,T;[H^s(\Omega)]^3,[H^{s-2}/2(\Omega)]^3) \rightarrow W^{1,2,2}(0,T;[H^1(\Omega)]^5,[H^1(\Omega)]^5)
  \]

  \[
  \zeta_g \rightarrow v \quad \text{(Theorem 3.10)}
  \]

- **State constraints:** We define the mapping:
  \[
  G : U_{ad} \rightarrow [C(0,T)]^5
  \]
  \[
  g \rightarrow G(g) = \frac{1}{\mu(\Omega_C)} \int_{\Omega_C} u \, dx,
  \]
  that is, in order to evaluate \( G \), it is necessary to consider the following chain of compositions:

  \[
  U_{ad} \rightarrow W^{1,2,2}(0,T;[H^s(\Omega)]^3,[H^{s-2}/2(\Omega)]^3)
  \]

  \[
  g \rightarrow \zeta_g \quad \text{(Lemma 3.5)}
  \]

  \[
  W^{1,2,2}(0,T;[H^s(\Omega)]^3,[H^{s-2}/2(\Omega)]^3) \rightarrow W^{1,\infty,2}(0,T;W,[L^2(\Omega)]^3) \cap C([0,T];W)
  \]

  \[
  \zeta_g \rightarrow v \quad \text{(Theorem 3.10)}
  \]

  \[
  W^{1,\infty,2}(0,T;[L^2(\Omega)]^3) \cap C([0,T];W) \rightarrow W^{1,2,2}(0,T;[H^1(\Omega)]^5,[H^1(\Omega)]^5)
  \]

  \[
  \zeta_g \rightarrow v \quad \text{(Theorem 3.17)}
  \]

  \[
  W^{1,2,2}(0,T;[H^1(\Omega)]^5,[H^1(\Omega)]^5) \rightarrow [C(0,T)]^5
  \]

  \[
  \frac{1}{\mu(\Omega_C)} \int_{\Omega_C} u \, dx.
  \]

Then, we consider the following state constraints:

\[
G(g)(t) \in K_C, \quad \forall t \in [0,T],
\]

where \( K_C \subset \mathbb{R}^5 \) is the rectangle given by \( K_C = [\lambda_1^n,\lambda_1^M] \times \cdots \times [\lambda_5^n,\lambda_5^M] \), and assume that the set \( U^*_{ad} = \{ g \in U_{ad} : G(g)(t) \in K_C, \forall t \in [0,T] \} \neq \emptyset \).

So, we can demonstrate the following existence result for the control problem:

**Theorem 4.1.** Assuming that the coefficients and data of the problem satisfy the necessary conditions for the existence of a solution of the state systems, as established in the Theorems 3.10 and 3.17, there exists, at least, a solution of the optimal control problem \( (P) \).

**Proof.** Let us consider a minimizing sequence \( \{ g_n \}_{n \in \mathbb{N}} \subset U^*_{ad} \) of the control problem, that is, satisfying:

\[
\lim_{n \to \infty} J(g_n) = \inf \{ J(g) : g \in U^*_{ad} \}.
\]

Thus, thanks to Lemma 3.5, the sequence of reconstructions of the trace \( \{ \zeta_{g_n} \}_{n \in \mathbb{N}} \subset W^{1,2,2}(0,T;[H^s(\Omega)]^3,[H^{s-2}/2(\Omega)]^3) \) is bounded.
Now, by Theorems 3.10 and 3.17, we have that the sequences corresponding to the associated states \( \{v_n\}_{n \in \mathbb{N}} \subset W^{1,\infty;2}(0, T; \mathbf{W}, [L^2(\Omega)]^3) \cap C([0, T]; \mathbf{W}) \) and \( \{u_n\}_{n \in \mathbb{N}} \subset W^{1,2;2}(0, T; [H^1(\Omega)]^5, [H^1(\Omega)]^5) \) are also bounded in their respective functional spaces, where, for each \( n \in \mathbb{N} \):

- \( v_n \in W^{1,\infty;2}(0, T; \mathbf{W}, [L^2(\Omega)]^3) \cap C([0, T]; \mathbf{W}) \) is given by \( v_n = \zeta_{g_n} + z_n \), with \( z_n(0) = v_0 - \zeta_{g_0}(0) \) a.e. \( x \in \Omega \), and \( z_n \in W^{1,\infty;2}(0, T; \mathbf{V}, [L^2(\Omega)]^3) \cap C([0, T]; \mathbf{V}) \) is the solution of the variational formulation:

\[
\int_\Omega \frac{\partial z_n}{\partial t} \cdot \eta \, dx + \int_\Omega \nabla (\zeta_{g_n} + z_n) \cdot \eta \, dx
\]

+ \int_\Omega \nabla z_n \zeta_{g_n} \cdot \eta \, dx + 2\nu \int_\Omega \epsilon(\zeta_{g_n} + z_n) : \epsilon(\eta) \, dx
\]

+ 2\nu_{tur} \int_\Omega [\epsilon(\zeta_{g_n} + z_n) : \epsilon(\zeta_{g_n} + z_n)]^{1/2} \epsilon(\zeta_{g_n} + z_n) : \epsilon(\eta) \, dx
\]

\[= \int_\Omega F \cdot \eta \, dx - \int_\Omega \frac{\partial \zeta_{g_n}}{\partial t} \cdot \eta \, dx - \int_\Omega \nabla \zeta_{g_n} \zeta_{g_n} \cdot \eta \, dx, \text{ a.e. } t \in (0, T), \forall \eta \in \mathbf{V}. \]  

- \( u_n \in W^{1,2;2}(0, T; [H^1(\Omega)]^5, [H^1(\Omega)]^5) \) is such that \( u_n(0) = u_0 \), a.e. \( x \in \Omega \),

\[
\int_\Omega \frac{\partial u_n}{\partial t} \cdot \eta \, dx + \int_\Omega \nabla u_n \cdot \eta \, dx + \int_\Omega \Lambda_p \nabla u_n : \nabla \eta \, dx
\]

\[= \int_\Omega \mathbf{f} \cdot \eta \, dx + \int_\Omega \mathbf{A}(u_n) \cdot \eta \, dx, \text{ a.e. } t \in (0, T), \forall \eta \in \mathbf{X}. \]

So, there exist subsequences - still denoted in the same way for simplicity - such that verify the following convergences:

- \( g_n \rightharpoonup \bar{g} \) in \( H^{s - 1/2}(\partial \Omega \times (0, T)) \cap L^2(0, T; H^1(\partial \Omega)) \).
- \( \eta_n \rightharpoonup \bar{\eta} \) in \( C([0, T] \times \Omega) \).
- \( \mathcal{J}_{g_n} \rightharpoonup \bar{\mathcal{J}}_{\bar{g}} \) in \( L^2(0, T; [H^s_0(\Omega)]^5) \).
- \( \frac{\partial \mathcal{J}_{g_n}}{\partial t} \rightharpoonup \frac{\partial \bar{\mathcal{J}}_{\bar{g}}}{\partial t} \) in \( L^2(0, T; [H^{s - 2}(\Omega)]^5) \).
- \( \mathcal{J}_{g_n} \rightharpoonup \bar{\mathcal{J}}_{\bar{g}} \) in \( L^p(0, T; [H^s_0(\Omega)]^5) \), for any \( 1 < p < \infty \), (this convergence, the only non-immediate, can be obtained thanks to the Aubin-Lions Theorem (Lemma 7.7 of [16] with pivot space \( H^{s - 1/4}(\Omega) \)), and to the compact inclusion of \( H^{s - 1/4}(\Omega) \) into \( H^s(\Omega) \) for any \( s \geq 7/2 \) (Theorem 16.1 of [15]).
- \( z_n \rightharpoonup \bar{z} \) in \( L^\infty(0, T; \mathbf{V}) \).
- \( \frac{\partial z_n}{\partial t} \rightharpoonup \frac{\partial \bar{z}}{\partial t} \) in \( L^2(0, T; [L^2(\Omega)]^3) \).
- \( z_n \rightharpoonup \bar{z} \) in \( L^p(0, T; L^q(\Omega)) \), for any \( 1 < p < \infty \) and \( 1 \leq q < \infty \).
- \( v_n \rightharpoonup \bar{v} \) in \( L^\infty(0, T; \mathbf{W}) \).
- \( \frac{\partial v_n}{\partial t} \rightharpoonup \frac{\partial \bar{v}}{\partial t} \) in \( L^2(0, T; [L^2(\Omega)]^3) \).
- \( \bar{v} \) in \( L^p(0, T; L^q(\Omega)) \), for any \( 1 < p < \infty \) and \( 1 \leq q < \infty \).
- \( u_n \rightharpoonup \bar{u} \) in \( L^2(0, T; [H^1(\Omega)]^5) \).
- \( \frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial \bar{u}}{\partial t} \) in \( L^2(0, T; [H^1(\Omega)]^5) \).
- \( u_n \rightharpoonup \bar{u} \) in \( [L^{10/3 - \epsilon}(0, T; L^{10/3 - \epsilon}(\Omega))]^5 \), for any \( 0 < \epsilon \leq 7/3 \).
- \( u_n \rightharpoonup \bar{u} \) in \( [L^2(0, T; L^{4 - \epsilon}(\Omega))]^5 \), for any \( 0 < \epsilon \leq 3 \).
- \( u_n \rightharpoonup \bar{u} \) in \( C_c([0, T]; L^2(\Omega)) \).
Now, on the one hand, based on the strong convergence of \( g_n \) to \( \tilde{g} \), on its weak convergence in \( U \), and on the weak lower semicontinuity of the norm, we have that \( \tilde{g} \in U_{ad} \).

On the other hand, the pass to the limit in the terms associated with the variational formulations (65) and (66) can be performed based on above convergences, using standard techniques similar to which can be found in [10], [3] and [2]. So, we obtain that \( \tilde{v} = \zeta \tilde{g} + \tilde{z} \) and \( \tilde{u} \) are the states associated to \( \tilde{g} \).

Finally, passing to the limit in the cost functional \( J \) and in the state constraints given by \( G \) is straightforward, due to above convergences. Thus, we have demonstrated that the admissible control \( \tilde{g} \in U_{ad}^* \) is a (not necessarily unique) solution of the optimal control problem \((\mathcal{P})\), with associated states \( \tilde{v} \) and \( \tilde{u} \).

5. Numerical solution of the optimal control problem. In this section we will see how to solve numerically the control problem \((\mathcal{P})\). For this, we will consider a space-time discretization combining the method of the characteristics (for the time discretization) and the finite element method (for the space discretization). For the numerical resolution of the nonlinear coupling between the velocity components and those of the eutrophication model we will use a fixed-point algorithm. Finally, the numerical resolution of each of the Stokes-type problems derived from the fixed-point algorithm associated with the hydrodynamic model will be carried out using a penalty method [18, 5, 6].

For the discretization of the problem, let us consider a partition \( 0 = t_0 < t_1 < \ldots < t_N = T \) of the time interval \([0, T]\) such that \( t_{n+1} - t_n = \Delta t = 1/\alpha, \forall n = 0, \ldots, N - 1 \), and a family of meshes for the domain \( \Omega \) with characteristic size \( h \). Associated to this family of meshes, we also consider three compatible finite element spaces \( W_h, M_h \) and \( Z_h \) corresponding, respectively, to the velocity and the pressure of water, and to the concentrations of the species involved in the eutrophication model.

In order to simplify the presentation of the numerical problem, we will assume that all the groups collector-injector are in turbination mode, with the controls only depending on the time variable. Therefore, we try to solve the following constrained nonlinear minimization problem resulting from discretizing the control problem \((\mathcal{P})\):

\[
(\mathcal{Q}) \quad \min \{ J(g) : g \in U_{ad}^* \},
\]

where:

- **Discretized control constraints:** As already commented, we will assume that the inlet/outlet velocities for the collectors and injectors is constant at each step of time. We will then consider the following set of admissible discrete controls:

\[
U_{ad} = \{ g \in \mathbb{R}^{N \times N_{CT}} : 0 \leq g^{k,n} \leq c, \forall k = 1, \ldots, N_{CT}, \forall n = 1, \ldots, N \}, \quad (67)
\]

with \( c \) a bound related to pumps’ technical characteristics.

- **Discretized state equations:** Assumed known the initial velocity \( v_0 \in W_h \) and the initial concentrations for the species \( u_0 \in Z_h \), and given an admissible control

\[
g = (g_1^{1,1}, g_1^{2,1}, \ldots, g_1^{N_{CT},1}, \ldots, g_N^{1,N}, g_N^{2,N}, \ldots, g_N^{N_{CT},N}) \in U_{ad},
\]

we compute the associated state in the following way:
We consider the following discretized cost functional:

\[ \text{Discretized cost functional:} \]

\[ J(n) = \frac{1}{2} \sum_{n=1}^{N_{CT}} \sum_{k=1}^{N} (g^{k,n})^2. \]  

Remark 6. It should be noted that we employ a different number of time steps in both models, that is, in the eutrophication model one more step of time is solved
than in the hydrodynamic model. This shift is motivated by the following dependency scheme:

\[
\begin{array}{c c}

v^0 & u^0 \\
\downarrow & \downarrow \\
g^1 & u^1 \\
\downarrow & \downarrow \\
g^2 & v^2 \\
\downarrow & \downarrow \\
g^3 & v^3 \\
\vdots & \vdots \\
g^N & v^N \\
\downarrow & \downarrow \\
u^{N+1} & G^N(g) = G^N(g_1, g_2, \ldots, g_N)
\end{array}
\]

It is easy to see that, due to the time discretization used, the control associated to the first step of time, begins to have influence from the second step of time for the species of the eutrophication model. If we consider the same number of steps of time for the two models, the control associated with the last time step has no effect on the state constraints. To avoid this inconsistency, we propose to solve the eutrophication model an additional step of time, and to displace one time step the state constraints. Finally, as a direct consequence of above scheme, the Jacobian associated to state constraints is a block lower triangular matrix:

\[
\delta_g G(g) = \begin{pmatrix}
\delta_{g_1} G^1(g) & 0 & \ldots & 0 & 0 \\
\delta_{g_1} G^2(g) & \delta_{g_2} G^2(g) & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\delta_{g_1} G^{N-1}(g) & \delta_{g_2} G^{N-1}(g) & \ldots & \delta_{g_{N-1}} G^{N-1}(g) & 0 \\
\delta_{g_1} G^N(g) & \delta_{g_2} G^N(g) & \ldots & \delta_{g_{N-1}} G^N(g) & \delta_{g_N} G^N(g)
\end{pmatrix}
\]

From the computational viewpoint, for the generation of the mesh associated to the domain and for the numerical resolution of the state equations we have used FreeFem++ [12], and for the numerical resolution of the nonlinear minimization problem, the interior-point algorithm IPOPT [19]. In final section we give further details associated to the numerical test that we present there.

One of the necessary requirements to use the IPOPT algorithm in the numerical resolution of the nonlinear minimization problem (Q) is to have a function that evaluates the Jacobian associated to the state constraints (71) of our problem. For this, there are two possibilities, to use the linearized equations or to use the adjoint state equations. The choice of one method or another depends on the relationship between the size of the control and the constraints. That is, if we want to use the linearized equations to calculate the Jacobian matrix, we must solve $N_{C_T} \times N$ times the linearized equations, whereas if we employ the adjoint state to calculate the Jacobian matrix, we must solve $5N$ times the adjoint state system. If we take into account that the computational cost associated with the resolution of the linearized equations and the adjoint state is similar, we have that if $N_{C_T} < 5$ it is more convenient to use the linearized equations, whereas if $N_{C_T} > 5$, it is more advantageous to employ the adjoint state. In the example presented in the next section, we have considered four pairs collector-injector, therefore, we will use the linearized equations for computing the Jacobian of the discretized state constraints.
So, using the linearized state equations, the Jacobian associated to the state constraints can be computed by:

$$\delta G^n(g)(\delta g) = \frac{1}{\mu(\Omega_C)} \int_{\Omega_C} \delta u^{n+1} \, dx, \quad \forall n = 1, \ldots, N,$$

where $\delta v^0 = 0$, $\delta u^0 = 0$ and, given a direction $\delta g$, the linearized state is obtained by:

(i) The linearized hydrodynamic model: For each $n = 0, 1, \ldots, N - 1$, the pair $(\delta v_{h}^{n+1}, \delta p_{h}^{n+1}) \in W_{h} \times M_{h}$, with

$$\delta v_{h}^{n+1} = -\frac{\delta g_{h}^{n+1}}{\mu(T^h)} \mathbf{n}, \quad \delta p_{h}^{n+1} = \frac{\delta g_{h}^{n+1}}{\mu(C^h)} \mathbf{n}, \quad \forall k = 1, \ldots, N_{CT},$$

is the solution of:

$$\begin{align*}
\alpha \int_{\Omega} \delta v^{n+1} \cdot z \, dx &+ 2\nu \int_{\Omega} \epsilon(\delta v^{n+1}) : \epsilon(z) \, dx \\
+ 2\nu_{\text{tur}} \int_{\Omega} |\epsilon(\nu^{n+1}) : \epsilon(\nu^{n+1})|^{1/2} \epsilon(\delta v^{n+1}) : \epsilon(z) \, dx \\
- \int_{\Omega} \nabla \cdot \delta v^{n+1} q \, dx &- \lambda \int_{\Omega} \delta p^{n+1} q \, dx \\
= \alpha \int_{\Omega} (\delta v^n \circ X^n) \cdot z \, dx &- \int_{\Omega} (\nabla v^n \circ X^n) \delta v^n \cdot z \, dx, \quad \forall v \in V_h, \quad \forall q \in M_h.
\end{align*} \tag{72}$$

(ii) The linearized eutrophication model: For each $n = 0, \ldots, N$, the element $\delta u^{n+1} \in Z_h$, with

$$\delta u_{h}^{n+1} = \frac{1}{\mu(C^h)} \int_{C^h} \delta u^{n+1} \, d\gamma, \quad \forall k = 1, \ldots, N_{CT},$$

is the solution of:

$$\begin{align*}
\alpha \int_{\Omega} \delta u^{n+1} \cdot \eta \, dx &+ \int_{\Omega} \Lambda_{\mu} \nabla u^{n+1} : \nabla \eta \, dx = \alpha \int_{\Omega} (\delta u^n \circ X^n) \cdot \eta \, dx \\
- \int_{\Omega} (\nabla u^n \circ X^n) \delta v^n \cdot \eta \, dx &+ \int_{\Omega} \delta_{u} A^{n+1}(u^{n+1}) \delta u^{n+1} \cdot z \, dx, \quad \forall \eta \in X_h.
\end{align*} \tag{73}$$

6. **Numerical results.** In this final section we will present a few numerical results that we have obtained by solving our problem in a realistic scenario that, for the sake of simplicity, corresponds to a two-dimensional case.

For this purpose, we have considered a rectangular domain $\Omega = [0, 14] \times [0, 16]$ (measured in meters), corresponding to a reservoir, in which we have distributed $N_{CT} = 4$ pairs collector-injector with the geometric configuration shown in Figure 1. The control domain we have chosen is the lower rectangle (shaded in green in Figure 1) of dimensions $\Omega_C = [0, 14] \times [0, 2] \subset \Omega$. The parameters used for the numerical resolution of the state equations are given in Table 1.

For time discretization we have considered a time step of $\Delta t = 3600$ seconds (1 hour), and for space discretization we have used a regular mesh formed by triangles of characteristic size $h = 0.25$ meters (corresponding to 3075 vertices). The finite element spaces we have employed for space discretizations have been the mini-element ($P_1$-bubble/$P_1$) for the hydrodynamic model, and the Lagrange $P_1$ element for the eutrophication model. In Figure 2 we compare the evolution of the state
In view of the evolution of the state constraints, we observe that the mean concentration of dissolved oxygen undergoes a decrease in the lower layers of the domain.
Figure 2. Comparison of the constraints on the five species (nutrient, phytoplankton, zooplankton, organic detritus and dissolved oxygen) for an uncontrolled case (dashed line) and for a controlled one (solid line), corresponding to 48 steps of time.

This decay in concentration is mainly due to the decomposition of organic detritus which, by the effects of sedimentation, are deposited in the bottom. In order to mitigate this oxygen decrease, it is observed that the pumping of water from upper to lower layers relieves the effects discussed above. However, a side effect of this turbination is the increase in the rate of deposition of the organic detritus in the lower layers, which may aggravate the problem when we stop acting on the system. In Figures 3 and 4 we show the concentrations of dissolved oxygen and of organic detritus in the last step of time ($n = 49$), where previous behaviour can be observed (that is, with mechanical mixing of water, stratification is completely broken for both species, and aeration of the bottom layer is accomplished).

In this first approach to solving the problem, we are only interested in controlling the concentration of dissolved oxygen in the lower layer, for which we consider the state constraints rectangle given by $K_C = (-\infty, \infty) \times (-\infty, \infty) \times (-\infty, \infty) \times (-\infty, \infty) \times \left[\lambda_5^m, \infty\right)$, where $\lambda_5^m \in \mathbb{R}^N$ is such that $\lambda_5^{n,m} = \min\{4.6, G_5^{n,m}(\mathbf{g}_{ref})\}$, for $n = 1, \ldots, N$. (this is, we consider as a lower threshold for the dissolved oxygen that one obtained by turbinating water at medium power until reaching a concentration...
Figure 3. Concentrations of dissolved oxygen at final time with the four pumps at rest (up) and with medium power pumps (down).

of 4.6 mg/l). It is then expected that the optimal control $g_{opt}$ is lower, from the point at which the threshold of 4.6 mg/l is exceeded, than that used to generate previous threshold, with the consequent, desired energy saving.

Now, let us present the results we have obtained by optimizing the process during the first 12 steps of time. In Figure 5 we can see the optimal control $g_{opt}$ obtained for the four groups against the reference control $g_{ref} = 5.0 \times 10^4 m/s$, and the evolution of the constraint associated with the dissolved oxygen for the optimal case, for the reference case, and for the uncontrolled case. We observe that the control is optimal in the sense that it saturates the constraint associated with the dissolved oxygen. On the other hand, it can be also noted that, with this optimal configuration, groups
Figure 4. Concentrations of organic detritus at final time with the four pumps at rest (up) and with medium power pumps (down).

2 and 4 remain almost inactive, while groups 1 and 3 are turbinating to a greater capacity than the reference control. Finally, in Figure 6 we show the concentration of dissolved oxygen in the last step of time (n = 13) corresponding to the optimal control.

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Figure 5. Optimal control $g_{opt}$ for the four groups (up), and mean concentrations of dissolved oxygen for $g_{ref}$, $g = 0$ and $g_{opt}$ (down).

Figure 6. Concentration of dissolved oxygen at final time for the optimal control $g_{opt}$.

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