Inversion of the Weighted Spherical Mean

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Abstract. The paper contains the inversion formula for the weighted spherical mean. The interest to reconstruction a function by its integral by sphere grows tremendously in the last six decades, stimulated by the spectrum of new problems and methods of image reconstruction. We consider a generalization of the classical spherical mean and its inverse in the case when generalized translation acts to function instead of regular. As a particular case this problem includes action of spherical means on radially symmetric functions.

1 Introduction

Reconstruction of a function from a known subset of its spherical means is widely developed in pure and applied mathematics. Its connection with photoacoustic images is as follows. Let the speed of sound propagation in the medium be a constant value. Then the pressure at a certain point in time is expressed in terms of the spherical mean pressure and its time derivative at some previous point in time [1]. Therefore, this imaging technique requires the inversion of spherical means.

The problem of reconstruction a function \( f \) supported in a ball \( B \in \mathbb{R}^n \), if the spherical means of \( f \) are known over all geodesic spheres centered on the boundary \( \partial B \) was solved using different approach in [1–8]. It is remarkable that reconstruction formulas in [1–8] are different for even and odd dimension of Euclidean space \( n \).

Classical spherical mean has the form

\[
M(x, r, u) = \frac{1}{|S_n(1)|} \int_{S_n(1)} u(x + \beta r) dS, \quad x = (x_1, \ldots, x_n),
\]

where \( S_n(1) \) is unit sphere centered at the origin, \( \beta \) is a coordinate of the sphere \( S_n(1) \).

Operator (1) intertwines Laplace operator and one-dimensional Bessel operator with index \( n - 1 \). Great interest among various researchers is a generalization of the spherical mean (1). So, in paper [9] was considered a spherical mean in space with negative curvature, in [10] and [11] was studied a generalization of the spherical mean generated by the Dunkl transmutation operator.

In this paper we consider the spherical weighted mean (see (7)), which is the transmutation operator intertwining the multidimensional operator

\[
(\Delta_{\gamma})_x = \sum_{i=1}^{n} (B_{\gamma_i})_{x_i}, \quad (B_{\gamma_i})_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \quad \gamma_i \geq 0, \quad i = 1, \ldots, n
\]
and one-dimensional Bessel operator with index \( n + |\gamma| - 1 \) of the form
\[
(B_{n+|\gamma|-1})_t = \frac{d^2}{dt^2} + \frac{n + |\gamma| - 1}{t} \frac{d}{dt}, \quad t > 0, \quad |\gamma| = \gamma_1 + \ldots + \gamma_n.
\]  
(3)

Such spherical mean is closely related to B-ultra-hyperbolic equation of the form (see [12, 13])
\[
\sum_{j=1}^{n} (B_{\gamma_j})_x j u = \sum_{j=1}^{n} (B_{\gamma_j})_y j u, \quad u = u(x_1, \ldots, x_n, y_1, \ldots, y_n).
\]  
(4)

Not so long ago, various new methods for solving inverse problems with the Bessel operator appeared. The inverse problem involving recovery of initial temperature from the information of final temperature profile in the case of heat equation with Bessel operator was studied in [14].

V. V. Kravchenko with coauthors introduced a new method to solution of the inverse Sturm-Liouville problem (see [16–19]) which can be adapted to inverse problem with Bessel operator. Their idea is based on the observation that the potential can be recovered from the very first coefficient of the Fourier–Legendre series, and to find this coefficient a system of linear algebraic equations can be obtained directly from the Gel’fand-Levitan equation.

The paper contains an inversion formula for the weighed spherical mean (7) using the properties of the mixed Riesz hyperbolic B-potential (13).

2 Basic Definitions

In this section we give a summary of the basic notations, terminology and results which will be used in this article.

Suppose that \( \mathbb{R}^{n+1} \) is the \( n + 1 \)-dimensional Euclidean space,
\[
\mathbb{R}^{n+1}_+ = \{ (t, x) = (t, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}, \quad x_1 > 0, \ldots, x_n > 0 \},
\]
\( \gamma = (\gamma_1, \ldots, \gamma_n) \) is a multiindex consisting of fixed real numbers \( \gamma_i \geq 0, \ i = 1, \ldots, n \), and \( |\gamma| = \gamma_1 + \ldots + \gamma_n \). Let \( \Omega \) be a finite or infinite open set in \( \mathbb{R}^{n+1} \) symmetric with respect to each hyperplane \( x_i = 0, \ i = 1, \ldots, n, \) \( \Omega_+ = \Omega \cap \mathbb{R}^{n+1}_+ \) and \( \tilde{\Omega}_+ = \Omega \cap \mathbb{R}^{n+1}_+ \) where
\[
\mathbb{R}^{n+1}_+ = \{ (t, x) = (t, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}, x_1 \geq 0, \ldots, x_n \geq 0 \}.
\]

We deal with the class \( C^m(\Omega_+) \) consisting of \( m \) times differentiable on \( \Omega_+ \) functions and denote by \( C^m(\tilde{\Omega}_+) \) the subset of functions from \( C^m(\Omega_+) \) such that all existing derivatives of these functions with respect to \( x_i \) for any \( i = 1, \ldots, n \) are continuous up to \( x_i = 0 \) and all existing derivative with respect to \( t \) are continuous for \( t \in \mathbb{R} \). Class \( C^m_{ev}(\tilde{\Omega}_+) \) consists of all functions from \( C^m(\tilde{\Omega}_+) \) such that
\[
\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}} \bigg|_{x=0} = 0
\]
for all nonnegative integer \( k \leq \frac{m-1}{2} \) and for \( i = 1, \ldots, n \) (see [20] and [21], p. 21). In the following we will denote \( C^m_{ev}(\mathbb{R}^{n+1}_+) \) by \( C^m_{ev} \). We set
\[
C^\infty_{ev}(\tilde{\Omega}_+) = \bigcap C^m_{ev}(\tilde{\Omega}_+)
\]
with intersection taken for all finite \( m \). Let \( C_{ev}^\infty(\mathbb{R}^{n+1}_+) = C_{ev}^\infty \). Assuming that \( \tilde{C}_{ev}^\infty(\tilde{\Omega}_+) \) is the space of all functions \( f \in C_{ev}^\infty(\tilde{\Omega}_+) \) with a compact support. We will use the notation \( \tilde{C}_{ev}^\infty(\tilde{\Omega}_+) = D_+(\tilde{\Omega}_+) \).

Let \( \mathcal{L}_p^\gamma(\Omega_+), 1 \leq p < \infty \) be the space of all measurable in \( \Omega_+ \) functions such that
\[
\int_{\Omega_+} |f(t, x)|^p x^\gamma dtdx < \infty,
\]
where and further
\[
x^\gamma = \prod_{i=1}^n x_i^{\gamma_i}.
\]

For a real number \( p \geq 1 \), the \( \mathcal{L}_p^\gamma(\Omega_+) \)-norm of \( f \) is defined by
\[
||f||_{\mathcal{L}_p^\gamma(\Omega_+)} = \left( \int_{\Omega_+} |f(t, x)|^p x^\gamma dtdx \right)^{1/p}.
\]

Let \( \mathcal{L}_p^\gamma = \mathcal{L}_p^\gamma(\mathbb{R}^{n+1}_+) \).

We will use the generalized convolution product defined by the formula
\[
(f \ast g)_{\gamma}(x, t) = \int_{\mathbb{R}^{n+1}} f(\tau, y)(\gamma \mathbb{T}_x y g)(t - \tau, x)y^\gamma d\tau dy,
\]
where \( \gamma \mathbb{T}_x y \) is multidimensional generalized translation
\[
(\gamma \mathbb{T}_x y f)(t, x) = (\gamma_{T_{x_1}}^{y_1} \ldots \gamma_{T_{x_n}}^{y_n} f)(t, x).
\]

Each of one–dimensional generalized translations \( \gamma_{T_{x_i}}^{y_i} \) is defined for \( i = 1, \ldots, n \) by the next formula (see [22], p. 122, formula (5.19))
\[
(\gamma_{T_{x_i}}^{y_i} f)(t, x) = \frac{\Gamma\left(\frac{\gamma_{T_{x_i}}^{y_i}}{2}\right)}{\Gamma\left(\frac{\gamma_{T_{x_i}}^{y_i}}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\pi} \sin^{\gamma_i - 1} \varphi_i \times
\]
\[
\times f(t, x_1, \ldots, x_{i-1}, \sqrt{x_i^2 + y_i^2 - 2x_i y_i \cos \varphi_i, x_{i+1}, \ldots, x_n}) d\varphi_i,
\]
\( \gamma_i > 0, i = 1, \ldots, n \) and for \( \gamma_i = 0 \) generalized translation \( \gamma_{T_{x_i}}^{y_i} \) is
\[
0_{T_{x_i}}^{y_i} = \frac{f(x + y) - f(x - y)}{2}.
\]

As the space of basic functions we will use the subspace of rapidly decreasing functions:
\[
S_{ev}(\mathbb{R}^{n+1}_+) = \left\{ f \in C_{ev}^\infty : \sup_{(t, x) \in \mathbb{R}^{n+1}_+} |t^\alpha x^\beta D^\gamma f(t, x)| < \infty \right\},
\]
where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_0, \beta_1, \ldots, \beta_n)$, $\alpha_0, \alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n$ are arbitrary integer non-negative numbers, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, $D^\beta = D_1^{\beta_0} D_2^{\beta_1} \ldots D_n^{\beta_n}$, $D_t = \frac{\partial}{\partial t}$, $D_{x_j} = \frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$.

The multidimensional Fourier–Bessel transform of a function $f \in L^1_\gamma(\mathbb{R}_+^{n+1})$ is

$$ F_\gamma[f](\tau, \xi) = \hat{f}(\tau, \xi) = \int_{\mathbb{R}_+^{n+1}} f(t, x) e^{-i\tau \cdot \gamma(x)} dt dx, $$

where

$$ j_\gamma(x; \xi) = \prod_{i=1}^n j_{\gamma_i}(x_i, \xi_i), \quad \gamma_1 \geq 0, \ldots, \gamma_n \geq 0. $$

Now we introduce a weighted spherical mean. When constructing a weighted spherical mean, instead of the usual shift, a multidimensional generalized translation (6) is used.

Weighted spherical mean (see [12, 13, 23]) of function $f(x)$, $x \in \mathbb{R}^n_+$ for $n \geq 2$ is

$$ (M^\gamma_t f)(x) = (M^\gamma_t)_x[f(x)] = \frac{1}{|S^+_1(n)|_\gamma} \int_{S^+_1(n)} \gamma T^\theta_d f(x) \theta^\gamma dS, $$

(7)

where $\theta^\gamma = \prod_{i=1}^n \theta_i^{\gamma_i}$, $S^+_1(n) = \{ \theta : |\theta| = 1, \theta \in \mathbb{R}^n_+ \}$ is a part of a sphere in $\mathbb{R}^n_+$, and $|S^+_1(n)|_\gamma$ is given by

$$ |S^+_1(n)|_\gamma = \int_{S^+_1(n)} x^\gamma dS = \frac{\prod_{i=1}^n \Gamma \left( \frac{\gamma_i+1}{2} \right)}{2^{n-1} \Gamma \left( \frac{n+|\gamma|}{2} \right)}. $$

(8)

For $n = 1$ let $M^\gamma_t[f(x)] = \gamma T^\theta_d f(x)$.

Let $\nu > 0$. One-dimensional Poisson operator is defined for integrable function $f$ by the equality

$$ P^\nu_x f(x) = \frac{2C(\nu)}{x^{\nu-1}} \int_0^x (x^2 - t^2)^{\nu-1} f(t) dt, \quad C(\nu) = \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{\nu}{2} \right)}, $$

(9)

or

$$ P^\nu_x f(x) = C(\nu) \int_0^\pi f(x \cos \varphi) \sin^{\nu-1} \varphi d\varphi. $$

(10)

The constant $C(\nu)$ is chosen so that $P^\nu_x[1] = 1$. For $\nu = 0$ one-dimensional Poisson operator turns into an identical operator: $P^0_x = I$. 

4
3 Mixed Hyperbolic Riesz B–potential and Its Inversion

The given supplementary information in this section will be used in the 4th section where the main results are presented.

Mixed hyperbolic Riesz B–potential was studied in [23–25]. Here we provide a definition of the mixed hyperbolic Riesz B–potential and give some separation results that we will use in the next section.

Let $|x| = \sqrt{x_1^2 + ... + x_n^2}$. First for $(t, x) \in \mathbb{R}_{+}^{n+1}$, $\lambda \in C$ we define function $s^\lambda$ by the formula

$$s^\lambda(t, x) = \begin{cases} \frac{(t^2 - |x|^2)^\lambda}{N(\alpha, \gamma, n)}, & \text{when } t^2 \geq |x|^2 \text{ and } t \geq 0; \\ 0, & \text{when } t^2 < |x|^2 \text{ or } t < 0, \end{cases}$$

(11)

where

$$N(\alpha, \gamma, n) = \frac{2^{\alpha-n-1}}{\sqrt{\pi}} \prod_{i=1}^{n} \Gamma \left( \frac{\gamma_i + 1}{2} \right) \Gamma \left( \frac{\alpha - n - |\gamma| + 1}{2} \right) \Gamma \left( \frac{\alpha}{2} \right).$$

(12)

Regular weighted distribution corresponding to (11) we will denote by $s^\lambda_+$. We introduce the mixed hyperbolic Riesz B–potential $I^\alpha_{s, \gamma}$ of order $\alpha$ as a generalized convolution product (5) with a weighted distribution $s_+^{\alpha-n-|\gamma|-1}$ and $f \in S_{ev}$ (see [25]):

$$(I^\alpha_{s, \gamma}f)(t, x) = \left( s_+^{\alpha-n-|\gamma|-1} * f \right)_{\gamma}(t, x).$$

(13)

The precise definition of the constant $N(\alpha, \gamma, n)$ allows to obtain the semigroup property or index law of the potential (13).

We can rewrite formula (13) as

$$(I^\alpha_{s, \gamma}f)(t, x) = \int_{\mathbb{R}^{n+1}}^{n+1} \frac{\alpha-n-|\gamma|-1}{s_+^{\alpha-n-|\gamma|-1}}(\tau, y)(\gamma T^y_x)f(t-\tau, x)y^\gamma d\tau dy.$$  

(14)

Integral (14) converges absolutely for $n+|\gamma|-1<\alpha$ for integrable with weight $y^\gamma$ on the part of the cone $\{|y| < \tau\} = \{y \in \mathbb{R}^n_+; |y| < \tau\}$, $0 < \tau < t$ function $f(\tau, y)$.

For $0 \leq \alpha \leq n + |\gamma| - 1$

$$(I^\alpha_{s, \gamma}f)(t, x) = \left( \frac{\partial^2}{\partial t^2} - \Delta_{\gamma} \right)^q (I^{\alpha+2q}_{s, \gamma}f)(t, x)$$

where $q = \left[ \frac{n+|\gamma|-\alpha+1}{2} \right]$.

In the case when $f(t, x) = h(t)F(x)$ we get

$$(I^\alpha_{s, \gamma}hF)(t, x) =$$

$$= \frac{1}{N(\alpha, \gamma, n)} \int_{0}^{\infty} h(t-\tau) d\tau \int_{|y|<\tau} (\tau^2 - |y|^2)^{\alpha-n-|\gamma|-1} \frac{\alpha-n-|\gamma|-1}{2} (\gamma T^y_x)F(x)y^\gamma dy.$$  

(15)
Theorem 1 \[24] \ Let n + |γ| - 1 < α < n + |γ| + 1, 1 ≤ p < \frac{n+|\gamma|+1}{\alpha}. \ For \ the \ next \ estimate

\[ ||I_{s,\gamma}^{\alpha}f||_{q,\gamma} \leq M||f||_{p,\gamma}, \quad f \in S_{ev} \]

(16)
to be valid it is necessary and sufficient that \( q = \frac{(n+|\gamma|+1)p}{n+|\gamma|+1-\alpha p} \). Constant \( M \) does not depend on \( f \).

**Remark.** By virtue of (16) there is unique extension of \( I_{s,\gamma}^{\alpha} \) to all \( \mathcal{L}_{p}^{\gamma} \), \( 1 < p < \frac{n+|\gamma|+1}{\alpha} \) preserving boundedness when \( n + |\gamma| - 1 < \alpha < n + |\gamma| \). It follows that this extension is introduced by the integral (14) from its absolute convergence.

Theorem 2 \[24] \ For \( f \in S_{ev} \) the Fourier–Hankel transform of mixed hyperbolic Riesz potential \( I_{s,\gamma}^{\alpha}f \) is

\[ \mathcal{F}_{\gamma}[I_{s,\gamma}^{\alpha}f](\tau, \xi) = \mathcal{D} \left| \tau^{2} - |\xi|^{2} \right|^{-\frac{\alpha}{2}} \cdot \mathcal{F}_{\gamma}[f(t, x)](\tau, \xi), \]

where

\[ \mathcal{D} = \begin{cases} 1, & |\xi|^{2} \geq \tau^{2}; \\ e^{-\frac{\alpha}{2} \tau^{2}}, & |\xi|^{2} < \tau^{2}, \tau \geq 0; \\ e^{\frac{\alpha}{2} \tau^{2}}, & |\xi|^{2} < \tau^{2}, \tau < 0. \end{cases} \]

For the inversion of the potential (13) approach based on the idea of approximative inverse operators (see [26]) was used. This method gives an inverse operator as a limit of regularized operators. Namely, taking into account the formula (17) we will construct inverse operator for the potential (13) in the form

\( (I_{s,\gamma}^{\alpha})^{-1}f = \lim_{\varepsilon \to 0} \left( \mathcal{F}_{\gamma}^{-1}(\mathcal{D}|\tau^{2} - |\xi|^{2}|^{-\frac{\alpha}{2}}e^{-\varepsilon|\tau| - |\xi|}) \ast f \right)_{\gamma} \),

where the limit is understood in the norm \( \mathcal{L}_{p}^{\gamma} \) or almost everywhere.

Let

\[ g_{\alpha,\gamma,\varepsilon}(t, x) = \mathcal{F}_{\gamma}^{-1}(\mathcal{D}|\tau^{2} - |\xi|^{2}|^{-\frac{\alpha}{2}}e^{-\varepsilon|\tau| - |\xi|})(t, x), \]

then

\[ (I_{s,\gamma}^{\alpha})^{-1}f = \lim_{\varepsilon \to 0} (g_{\alpha,\gamma,\varepsilon} \ast f)_{\gamma}. \]

(18)

Theorem 3 \[24] \ The function \( g_{\alpha,\gamma,\varepsilon}(t, x) \) belongs to the space \( \mathcal{L}_{p}^{\gamma} \), \( 1 < p < \infty \) with additional restriction \( \frac{2(n+|\gamma|)-1}{2(n+|\gamma|)-2} < p \) for \( n + |\gamma| - 1 < \alpha < n + |\gamma| \) when \( n + |\gamma| + 1 \) is odd.

Theorem 4 \[24] \ Let \( n + |\gamma| - 1 < \alpha < n + |\gamma| + 1, 1 < p < \frac{n+|\gamma|}{\alpha} \) with the additional restriction \( p < \frac{2(n+1+|\gamma|)(n+|\gamma|)}{n+1+|\gamma|+2\alpha(n+|\gamma|)} \) when \( n + |\gamma| + 1 < \alpha < n + |\gamma| + n \) is odd. Then

\[ ((I_{s,\gamma}^{\alpha})^{-1}I_{s,\gamma}^{\alpha})^{-1}f)(t, x) = f(t, x), \quad f(t, x) \in \mathcal{L}_{p}, \]

where \( (I_{s,\gamma}^{\alpha})^{-1}f = \lim_{\varepsilon \to 0} (I_{s,\gamma}^{\alpha})^{-1}f \).
It is easy to see that if $\alpha = 2m$, $m \in \mathbb{N}$ the inverse to $I_{s,\gamma}^{2m}$ is $\left(\frac{\partial^2}{\partial t^2} - \Delta_\gamma\right)^m$:

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_\gamma\right)^m (I_{s,\gamma}^{2m} f)(t, x) = f(t, x), \quad f(t, x) \in L^r_\rho.$$  

The inverse Fourier–Hankel transform of $\mathcal{Q}|r^2 - |\xi|^2|^2 e^{-|\xi| - |\xi|}$ can be represented in the form

$$g_{\alpha, \gamma, \varepsilon}(t, x) = \mathcal{C}(n, \gamma, \alpha) \int_0^\infty |1 - r^2|^2 r^{n+|\gamma|-1} \times$$

$$\times \left[ e^{\frac{2\pi i}{2}(1-r)} \mathcal{F}^+_{n,\gamma,\alpha,\varepsilon}(r, x, t) + e^{\frac{2\pi i}{2}(1-r)} \mathcal{F}^-_{n,\gamma,\alpha,\varepsilon}(r, x, t) \right] dr,$$

where

$$\mathcal{C}(n, \gamma, \alpha) = \frac{\prod_{i=1}^n \Gamma\left(\frac{n+i}{2}\right) \Gamma(n + |\gamma| + 1 + \alpha)}{2^{n-1} \Gamma\left(\frac{n+|\gamma|}{2}\right)}$$

$$\mathcal{F}^+_{n,\gamma,\alpha,\varepsilon}(r, x, t) = \frac{2F_1\left(\frac{n+|\gamma|+1+\alpha}{2}, \frac{n+|\gamma|+2+\alpha}{2}, \frac{n+|\gamma|}{2}; -\frac{|x|^2r^2}{(\varepsilon+r+it)^{n+|\gamma|+1+\alpha}}\right)}{(\varepsilon+r+it)^{n+|\gamma|+1+\alpha}}$$

$$\mathcal{F}^-_{n,\gamma,\alpha,\varepsilon}(r, x, t) = \frac{2F_1\left(\frac{n+|\gamma|+1+\alpha}{2}, \frac{n+|\gamma|+2+\alpha}{2}, \frac{n+|\gamma|}{2}; -\frac{|x|^2r^2}{(\varepsilon+r-it)^{n+|\gamma|+1+\alpha}}\right)}{(\varepsilon+r-it)^{n+|\gamma|+1+\alpha}}.$$  

## 4 Inverse Problem

In this section we consider the recovery of a function $f$ from a knowledge of its weighted spherical mean $M^\gamma_\rho f$.

Let $f = f(x) \in C^2(\mathbb{R}^n_+)$, such that $\frac{\partial f}{\partial x_i} \bigg|_{x_i=0} = 0$, $i = 1, \ldots, n$. The weighted spherical mean $M^\gamma_\rho f$ is the transmutation operator intertwining $(\Delta_\gamma)_x$ and $(B_{n+|\gamma|-1})_t$ (see [29]):

$$(B_{n+|\gamma|-1})_t(M^\gamma_\rho f)(x) = (M^\gamma_\rho(\Delta_\gamma)_x f)(x).$$

Let consider the integral operator

$$(\mathcal{M}^\gamma_\rho f)(x) = \frac{1}{|S^+(n)|_\gamma} \int_{\mathbb{R}^n_+} (\gamma T^y_{\rho} f(x))(t^2 - |y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma dy.$$  

Operator $t^{1-k}\mathcal{M}^\gamma_\rho$ intertwines $(\Delta_\gamma)_x$ and $(B_k)_t$ when $k > n + |\gamma| - 1$:

$$(B_k)_t(t^{1-k}\mathcal{M}^\gamma_\rho f)(x) = (t^{1-k}\mathcal{M}^\gamma_\rho(\Delta_\gamma)_x f)(x).$$
We’ll tend to use spherical coordinates in (21) when \( k > n + |\gamma| - 1 \), then, using (9) we can write

\[
(\mathcal{M}^{\gamma,k}_t f)(x) = \frac{1}{|S_1^+(n)|_\gamma} \int_{\{|y| < t\}^+} \left( \tau T_x^y f(x) \right) (t^2 - |y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma \, dy = \{y = \rho \theta\} =
\]

\[
= \frac{1}{|S_1^+(n)|_\gamma} \int_0^t (t^2 - \rho^2)^{\frac{k-n-|\gamma|-1}{2}} \rho^{|\gamma|-1} \int_{S_1^+(n)} (\tau T_x^y f(x)) \theta^\gamma dS =
\]

\[
= \int_0^t (t^2 - \rho^2)^{\frac{k-n-|\gamma|-1}{2}} \rho^{|\gamma|-1} (M^\gamma_\rho f)(x) d\rho =
\]

\[
= \frac{t^{k-n-|\gamma|}}{2C(k-n-|\gamma|+1)} (\mathcal{P}^{k-n-|\gamma|+1} t^{n+|\gamma|-1} (M^\gamma_\rho f)(x))(t).
\]

Now let find the inverse operator for \( \mathcal{M}^{\gamma,k}_t \). Let multiply (21) by \( h(t - \tau) \) and integrate by \( \tau \) from 0 to \( \infty \). The function \( h(t) \) should be chosen such that the function \( h(t - \tau)(\mathcal{M}^{\gamma,k}_t f)(x) \) is an integrable by \( \tau \) by the interval from 0 to \( \infty \).

We obtain

\[
\int_0^\infty h(t - \tau)(\mathcal{M}^{\gamma,k}_t f)(x) d\tau =
\]

\[
= \frac{1}{|S_1^+(n)|_\gamma} \int_0^\infty h(t - \tau) d\tau \int_{\{|y| < t\}^+} (\tau^2 - |y|^2)^{\frac{k-n-|\gamma|-1}{2}} (\tau T_x^y f(x)) y^\gamma dy.
\]

Taking into account (15) we get

\[
\frac{|S_1^+(n)|_\gamma}{N(k, \gamma, n)} \int_0^\infty h(t - \tau)(\mathcal{M}^{\gamma,k}_t f)(x) d\tau = (I^{k,\gamma}_h f)(t, x),
\]

where \( N(k, \gamma, n) \) defined by (12) and \( I^k_{k,\gamma} \) is the mixed hyperbolic Riesz B–potential (14) of order \( k > 0 \) acting to function \( h(t) f(x) \). Therefore, using theorem 4 we obtain

\[
h(t) f(x) = \frac{|S_1^+(n)|_\gamma}{N(k, \gamma, n)} \left( (I^{k,\gamma}_h)^{-1} \int_0^\infty h(\phi - \tau)(\mathcal{M}^{\gamma,k}_\tau f)(y) d\tau \right)(t, x), \tag{23}
\]

where \( n + |\gamma| - 1 < k < n + 1 + |\gamma| \) and \((I^{k,\gamma}_h)^{-1}\) is given by (13). So, in the inverse formula (23) we have an arbitrary parameter \( k \in (n + |\gamma| - 1, n + 1 + |\gamma|) \) and an arbitrary non-zero function \( h \) (such that the function \( h(t - \tau)(\mathcal{M}^{\gamma,k}_\tau f)(x) \) is an integrable by \( \tau \) depends on one variable.)
In order to find the inverse to weighed spherical mean the formula (23) can be simplified. We can take \( k = 2m > n + |\gamma| - 1, \ m \in \mathbb{N} \). In this case

\[
(I_{\alpha,\gamma}^{2m})^{-1} = \left( \frac{\partial^2}{\partial t^2} - \Delta_\gamma \right)^m
\]

and

\[
h(t)f(x) = \frac{|S_t^+(n)|_\gamma}{N(2m, \gamma, n)} \left( \frac{\partial^2}{\partial t^2} - \Delta_\gamma \right)^m \int_0^\infty h(t - \tau)(\mathcal{M}_{\gamma,2m}f)(x)d\tau. \tag{24}
\]

So we obtain the main statement.

**Theorem 5** Let \( f = f(x) \in C^2(\mathbb{R}^n_+) \), such that \( \frac{\partial f}{\partial x_i} \bigg|_{x_i=0} = 0, i = 1, \ldots, n \) and

\[
(\mathcal{M}_{t}^\gamma f)(x) = \frac{t^{k-n-|\gamma|}}{2C(k-n-|\gamma|+1)} \left( P_t^{k-n-|\gamma|+1}i_{n+|\gamma|-1}(M_\rho^\gamma f)(x) \right)(t),
\]

where \( M_\rho^\gamma f \) is the weighted spherical mean [7] of the function \( f \), is \( P_t^\nu \) the one-dimensional Poisson operator [9], \( C(\nu) \) is the constant defined in [7]. Then the function \( f \) can be reconstructed by its weighted spherical mean by the formula

\[
h(t)f(x) = \frac{|S_t^+(n)|_\gamma}{N(2m, \gamma, n)} \left( \frac{\partial^2}{\partial t^2} - \Delta_\gamma \right)^m \int_0^\infty h(t - \tau)(\mathcal{M}_{\gamma,2m}f)(x)d\tau,
\]

where function \( h(t) \) is arbitrary such that the function \( h(t - \tau)(\mathcal{M}_{\gamma,2m}f)(x) \) is an integrable by \( \tau \) by the interval from 0 to \( \infty \), \( |S_t^+(n)|_\gamma \) is given by [8], \( N(2m, \gamma, n) \) is given by (12).

**Example.** Let \( h(t) = e^t \), \( (M_\rho^\gamma f)(x) = j_\gamma(x, \xi) j_{\frac{n+|\gamma|}{2}}(\rho|\xi|) \), where \( \xi = (\xi_1, \ldots, \xi_n) \) some vector.

\[
(\mathcal{M}_{\gamma}^\gamma f)(x) = \int_0^\tau (\tau^2 - \rho^2) \frac{2m-n-|\gamma|-1}{2} \rho^{n+|\gamma|-1}(M_\rho^\gamma f)(x)d\rho =
\]

\[
= j_\gamma(x, \xi) \int_0^\tau (\tau^2 - \rho^2) \frac{2m-n-|\gamma|-1}{2} \rho^{n+|\gamma|-1} j_{\frac{n+|\gamma|}{2}}(\rho|\xi|)d\rho =
\]

\[
= 2^{\frac{n+|\gamma|}{2}-1} \frac{\Gamma \left( \frac{n + |\gamma|}{2} \right)}{\Gamma \left( \frac{1-n+|\gamma|}{2} \right)} \int_0^\tau (\tau^2 - \rho^2) \frac{2m-n-|\gamma|-1}{2} \rho^{\frac{n+|\gamma|}{2}} j_{\frac{n+|\gamma|}{2}}(\rho|\xi|)d\rho.
\]
Using formula 2.12.4.6 from \[28\] we obtain

\[
\begin{align*}
(M^{\gamma,2m f})(x) = & \quad 2^{n+|\gamma|} \Gamma \left( \frac{n + |\gamma|}{2} \right) \frac{|\xi|^{-n+|\gamma|/2}}{n!} j_\gamma(x, \xi) \frac{2^{m-n-|\gamma|-1/2}}{|\xi|^{2m-n-|\gamma|-1/2}} \\
& \times \Gamma \left( \frac{2m - n - |\gamma| + 1}{2} \right) J_{n-1/2}(|\xi|) = \\
= & \quad 2^{m-\frac{3}{2} \tau_{m-\frac{1}{2}} \Gamma \left( \frac{n + |\gamma|}{2} \right) \Gamma \left( \frac{2m - n - |\gamma| + 1}{2} \right) \frac{|\xi|^{-n+|\gamma|/2}}{n!} j_\gamma(x, \xi) J_{m-\frac{1}{2}}(|\xi|) .
\end{align*}
\]

Taking into account that \( h(t) = e^t \) we obtain

\[
\int_0^\infty h(t - \tau)(M^{\gamma,2m f})(x)d\tau = \\
= e^t 2^{m-\frac{3}{2}} \frac{\Gamma \left( \frac{n + |\gamma|}{2} \right) \Gamma \left( \frac{2m - n - |\gamma| + 1}{2} \right)}{|\xi|^{m-\frac{1}{2}} \Gamma \left( \frac{n + |\gamma|}{2} \right) \Gamma \left( \frac{2m - n - |\gamma| + 1}{2} \right) \frac{|\xi|^{-n+|\gamma|/2}}{n!} j_\gamma(x, \xi) J_{m-\frac{1}{2}}(|\xi|) d\tau} \\
= e^t 2^{m-\frac{3}{2}} \Gamma \left( \frac{n + |\gamma|}{2} \right) \Gamma \left( \frac{2m - n - |\gamma| + 1}{2} \right) \frac{|\xi|^{-n+|\gamma|/2} \Gamma \left( \frac{n + |\gamma|}{2} \right) \Gamma \left( \frac{2m - n - |\gamma| + 1}{2} \right)}{\Gamma \left( \frac{n + |\gamma|}{2} \right) \Gamma \left( \frac{2m - n - |\gamma| + 1}{2} \right) \frac{|\xi|^{-n+|\gamma|/2}}{n!} j_\gamma(x, \xi) J_{m-\frac{1}{2}}(|\xi|)} \\
= \Gamma(m) \frac{\Gamma \left( \frac{n + |\gamma|}{2} \right) \Gamma \left( \frac{2m - n - |\gamma| + 1}{2} \right)}{2^{2-2m} \sqrt{\pi} (1 + |\xi|^2)^m e^t j_\gamma(x, \xi)}.
\]

Let calculate the constant

\[
\frac{|S^+(n)|_\gamma}{N(2m, \gamma, n)} = \\
= \prod_{i=1}^n \Gamma \left( \frac{\gamma_i + 1}{2} \right) \frac{\sqrt{\pi}}{2^{n-1} \Gamma \left( \frac{n + |\gamma|}{2} \right) 2^{2m-n-1} \prod_{i=1}^n \Gamma \left( \frac{\gamma_i + 1}{2} \right) \Gamma \left( \frac{2m-n-|\gamma|+1}{2} \right) \Gamma(m)} \\
= \frac{2^{2-2m} \sqrt{\pi}}{\Gamma(m) \Gamma \left( \frac{n + |\gamma|}{2} \right) \Gamma \left( \frac{2m-n-|\gamma|+1}{2} \right)} .
\]
Therefore
\[
\frac{|S^+_1(n)|_\gamma}{N(2m, \gamma, n)} \left( \frac{\partial^2}{\partial t^2} - \Delta_\gamma \right)^m \int_0^\infty h(t - \tau)(\mathcal{M}_{\gamma, 2m} f)(x) d\tau = \frac{1}{(1 + |\xi|^2)^m} \left( \frac{\partial^2}{\partial t^2} - \Delta_\gamma \right)^m e^t j_\gamma(x, \xi) = e^t j_\gamma(x, \xi),
\]
that gives \( f(x) = j_\gamma(x, \xi) \). In the formula (25) we used the fact that \( \Delta_\gamma j_\gamma(x; \xi) = -|\xi| j_\gamma(x; \xi) \) [12].

This result confirmed by the formula (see [23])
\[
(M^x_{\rho})_\gamma j_\gamma(x, \xi) = j_\gamma(x, \xi) j_{\frac{n}{2} + \frac{1}{2}}(\rho|\xi|).
\]

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