THE ARTINIAN CONJECTURE
(FOLLOWING DJAMENT, PUTMAN, SAM, AND SNOWDEN)

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Abstract. This note provides a self-contained exposition of the proof of the
artinian conjecture, following closely Djament’s Bourbaki lecture. The original
proof is due to Putman, Sam, and Snowden.

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1. Introduction

This note provides a complete proof of the celebrated artinian conjecture. The
proof is due to Putman, Sam, and Snowden [6, 7]. Here, we follow closely the elegant
exposition of Djament in [3]. For the origin of the conjecture and its consequences,
we refer to those papers and Djament’s Bourbaki lecture [4]. In addition, the
expository articles by Kuhn, Powell and Schwartz in [5] are recommended.

There are two main result. Fix a locally noetherian Grothendieck abelian cate-
gory $A$, for instance, the category of modules over a noetherian ring.

Theorem 1.1. Let $A$ be a ring whose underlying set is finite. For the category
$\mathcal{P}(A)$ of free $A$-modules of finite rank, the functor category $\text{Fun}(\mathcal{P}(A)^{\text{op}}, A)$ is locally
noetherian.

This result amounts to the assertion of the artinian conjecture when $A$ is a finite
field and $A$ is the category of $A$-modules.

The first theorem is a direct consequence of the following.

Theorem 1.2. For the category $\Gamma$ of finite sets, the functor category $\text{Fun}(\Gamma^{\text{op}}, A)$
is locally noetherian.
The basic idea for the proof is to formulate finiteness conditions on an essentially small category \( C \) such that \( \text{Fun}(C^{\text{op}}, A) \) is locally noetherian. This leads to the notion of a Gröbner category. Such finiteness conditions have a ‘direction’. For that reason we consider contravariant functors \( C \to A \), because then the direction is preserved (via Yoneda’s lemma) when one passes from \( C \) to \( \text{Fun}(C^{\text{op}}, A) \).

2. Noetherian posets

Let \( C \) be a poset. A subset \( \mathcal{D} \subseteq C \) is a sieve if the conditions \( x \leq y \) in \( C \) and \( y \in \mathcal{D} \) imply \( x \in \mathcal{D} \). The sieves in \( C \) are partially ordered by inclusion.

**Definition 2.1.** A poset \( C \) is called

1. noetherian if every ascending chain of elements in \( C \) stabilises, and
2. strongly noetherian if every ascending chain of sieves in \( C \) stabilises.

For a poset \( C \) and \( x \in C \), set \( C(x) = \{ t \in C \mid t \leq x \} \). The assignment \( x \mapsto C(x) \) yields an embedding of \( C \) into the poset of sieves in \( C \).

**Lemma 2.2.** For a poset \( C \) the following are equivalent:

1. The poset \( C \) is strongly noetherian.
2. For every infinite sequence \( (x_i)_{i \in \mathbb{N}} \) of elements in \( C \) there exists \( i \in \mathbb{N} \) such that \( x_j \leq x_i \) for infinitely many \( j \in \mathbb{N} \).
3. For every infinite sequence \( (x_i)_{i \in \mathbb{N}} \) of elements in \( C \) there is a map \( \alpha : \mathbb{N} \to \mathbb{N} \) such that \( i < j \) implies \( \alpha(i) < \alpha(j) \) and \( x_{\alpha(j)} \leq x_{\alpha(i)} \).
4. For every infinite sequence \( (x_i)_{i \in \mathbb{N}} \) of elements in \( C \) there are \( i < j \) in \( \mathbb{N} \) such that \( x_j \leq x_i \).

**Proof.** (1) \( \Rightarrow \) (2): Suppose that \( C \) is strongly noetherian and let \( (x_i)_{i \in \mathbb{N}} \) be elements in \( C \). For \( n \in \mathbb{N} \) set \( C_n = \bigcup_{i \leq n} C(x_i) \). The chain \( (C_n)_{n \in \mathbb{N}} \) stabilises, say \( C_n = C_N \) for all \( n \geq N \). Thus there exists \( i \leq N \) such that \( x_j \leq x_i \) for infinitely many \( i \in \mathbb{N} \).

(2) \( \Rightarrow \) (3): Define \( \alpha : \mathbb{N} \to \mathbb{N} \) recursively by taking for \( \alpha(0) \) the smallest \( i \in \mathbb{N} \) such that \( x_j \leq x_i \) for infinitely many \( j \in \mathbb{N} \). For \( n > 0 \) set

\[
\alpha(n) = \min\{ i > \alpha(n-1) \mid x_j \leq x_i \leq x_{\alpha(n-1)} \} \text{ for infinitely many } j \in \mathbb{N} \}.
\]

(3) \( \Rightarrow \) (4): Clear.

(4) \( \Rightarrow \) (1): Suppose there is a properly ascending chain \( (C_n)_{n \in \mathbb{N}} \) of sieves in \( C \). Choose \( x_n \in C_{n+1} \setminus C_n \) for each \( n \in \mathbb{N} \). There are \( i < j \in \mathbb{N} \) such that \( x_j \leq x_i \). This implies \( x_j \in C_{i+1} \subseteq C_j \) which is a contradiction. \( \square \)

3. Functor categories

Let \( C \) be an essentially small category and \( A \) a Grothendieck abelian category. We denote by \( \text{Fun}(C^{\text{op}}, A) \) the category of functors \( C^{\text{op}} \to A \). The morphisms between two functors are the natural transformations. Note that \( \text{Fun}(C^{\text{op}}, A) \) is a Grothendieck abelian category.

Given an object \( x \in C \), the evaluation functor

\[
\text{Fun}(C^{\text{op}}, A) \to A, \quad F \mapsto F(x)
\]

admits a left adjoint

\[
A \to \text{Fun}(C^{\text{op}}, A), \quad M \mapsto M[C(-, x)]
\]

where for any set \( X \) we denote by \( M[X] \) a coproduct of copies of \( M \) indexed by the elements of \( X \). Thus we have a natural isomorphism

\[
\text{Fun}(C^{\text{op}}, A)(M[C(-, x)], F) \cong A(M, F(x)).
\]
Lemma 3.1. If \((M_i)_{i \in I}\) is a set of generators of \(\mathcal{A}\), then the functors \(M_i[\mathcal{C}(\mathcal{-}, x)]\) with \(i \in I\) and \(x \in \mathcal{C}\) generate \(\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{A})\).

Proof. Use the adjointness isomorphism (3.1). \(\square\)

A Grothendieck abelian category \(\mathcal{A}\) is \textit{locally noetherian} if \(\mathcal{A}\) has a generating set of noetherian objects. In that case an object \(M \in \mathcal{A}\) is noetherian iff \(M\) is \textit{finitely presented} (that is, the representable functor \(\mathcal{A}(M, -)\) preserves filtered colimits); see [8] Chap. V for details.

Lemma 3.2. Let \(\mathcal{A}\) be locally noetherian. Then \(\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{A})\) is locally noetherian iff \(M[\mathcal{C}(\mathcal{-}, x)]\) is noetherian for every noetherian \(M \in \mathcal{A}\) and \(x \in \mathcal{C}\).

Proof. First observe that \(M[\mathcal{C}(\mathcal{-}, x)]\) is finitely presented if \(M\) is finitely presented. This follows from the isomorphism (3.1) since evaluation at \(x \in \mathcal{C}\) preserves colimits. Now the assertion of the lemma is an immediate consequence of Lemma 3.1. \(\square\)

4. Noetherian Functors

Let \(\mathcal{C}\) be an essentially small category and fix an object \(x \in \mathcal{C}\). Set

\[
\mathcal{C}(x) = \bigsqcup_{t \in \mathcal{C}} \mathcal{C}(t, x).
\]

Given \(f, g \in \mathcal{C}(x)\), let \((f)\) denote the set of morphisms in \(\mathcal{C}(x)\) that factor through \(f\), and set \(f \leq g\) if \((f) \subseteq (g)\). We identify \(f\) and \(g\) when \((f) = (g)\). This yields a poset which we denote by \(\bar{\mathcal{C}}(x)\).

A functor is \textit{noetherian} if every ascending chain of subfunctors stabilises.

Lemma 4.1. The functor \(\mathcal{C}(\mathcal{-}, x) : \mathcal{C}^{\text{op}} \to \text{Set}\) is noetherian iff the poset \(\bar{\mathcal{C}}(x)\) is strongly noetherian.

Proof. Sending \(F \subseteq \mathcal{C}(\mathcal{-}, x)\) to \(\bigsqcup_{t \in \mathcal{C}} F(t)\) induces an inclusion preserving bijection between the subfunctors of \(\mathcal{C}(\mathcal{-}, x)\) and the sieves in \(\bar{\mathcal{C}}(x)\). \(\square\)

For a poset \(\mathcal{T}\) let \(\text{Set}^{\downarrow} \mathcal{T}\) denote the category consisting of pairs \((X, \xi)\) such that \(X\) is a set and \(\xi : X \to \mathcal{T}\) is a map. A morphism \((X, \xi) \to (X', \xi')\) is a map \(f : X \to X'\) such that \(\xi(a) \leq \xi' f(a)\) for all \(a \in X\).

A functor \(\mathcal{C}^{\text{op}} \to \text{Set}^{\downarrow} \mathcal{T}\) is given by a pair \((F, \phi)\) consisting of a functor \(F : \mathcal{C}^{\text{op}} \to \text{Set}\) and a map \(\phi : \bigsqcup_{t \in \mathcal{C}} F(t) \to \mathcal{T}\) such that \(\phi(a) \leq \phi(F(f)(a))\) for every \(a \in F(t)\) and \(f : t' \to t\) in \(\mathcal{C}\).

Lemma 4.2. Let \(\mathcal{T}\) be a noetherian poset. If \(\mathcal{C}(\mathcal{-}, x)\) is noetherian, then any functor \((\mathcal{C}(\mathcal{-}, x), \phi) : \mathcal{C}^{\text{op}} \to \text{Set}^{\downarrow} \mathcal{T}\) is noetherian.

Proof. Let \((F_n, \phi_n)_{n \in \mathbb{N}}\) be a strictly ascending chain of subfunctors of \((F, \phi)\). The chain \((F_n)_{n \in \mathbb{N}}\) stabilises since \(\mathcal{C}(\mathcal{-}, x)\) is noetherian. Thus we may assume that \(F_n = F\) for all \(n \in \mathbb{N}\), and we find \(f_n \in \bigsqcup_{t \in \mathcal{C}} F(t)\) such that \(\phi_n(f_n) < \phi_{n+1}(f_n)\).

The poset \(\bar{\mathcal{C}}(x)\) is strongly noetherian by Lemma 4.1. It follows from Lemma 2.2 that there is a map \(\alpha : \mathbb{N} \to \mathbb{N}\) such that \(i < j\) implies \(\alpha(i) < \alpha(j)\) and \(f_{\alpha(j)} \leq x f_{\alpha(i)}\).

Thus

\[
\phi_{\alpha(n)}(f_{\alpha(n)}) < \phi_{\alpha(n)+1}(f_{\alpha(n)}) \leq \phi_{\alpha(n+1)}(f_{\alpha(n)}) \leq \phi_{\alpha(n+1)}(f_{\alpha(n+1)}).
\]

This yields a strictly ascending chain in \(\mathcal{T}\), contradicting the assumption on \(\mathcal{T}\). \(\square\)

Definition 4.3. A partial order \(\leq\) on \(\mathcal{C}(x)\) is \textit{admissible} if the following holds:

1. The order \(\leq\) restricted to \(\mathcal{C}(t, x)\) is total and noetherian for every \(t \in \mathcal{C}\).
2. For \(f, f' \in \mathcal{C}(t, x)\) and \(e \in \mathcal{C}(s, t)\), the condition \(f < f'\) implies \(fe < f'e\).
Fix an admissible partial order \( \leq \) on \( \mathcal{C}(x) \) and an object \( M \) in a Grothendieck abelian category \( A \). Let \( \text{Sub}(M) \) denote the poset of subobjects of \( M \) and consider the functor

\[
\mathcal{C}(\cdot, x) \downarrow: M: \mathcal{C}^{\text{op}} \longrightarrow \text{Set Sub}(M), \quad t \mapsto \{\mathcal{C}(t, x), (M)_{f \in \mathcal{C}(t,x)}\}.
\]

For a subfunctor \( F \subseteq M[\mathcal{C}(\cdot, x)] \) define a subfunctor \( \tilde{F} \subseteq \mathcal{C}(\cdot, x) \downarrow M \) as follows:

\[
\tilde{F}: \mathcal{C}^{\text{op}} \longrightarrow \text{Set Sub}(M), \quad t \mapsto \left(\mathcal{C}(t, x), \left(\pi_f(M[\mathcal{C}(t, x)]_f \cap F(t))\right)_{f \in \mathcal{C}(t,x)}\right)
\]

where \( \mathcal{C}(t, x)_f = \{g \in \mathcal{C}(t, x) \mid f \leq g\} \) and \( \pi_f: M[\mathcal{C}(t, x)]_f \to M \) is the projection onto the factor corresponding to \( f \). For a morphism \( e: t' \to t \) in \( \mathcal{C} \), the morphism \( \tilde{F}(e) \) is induced by precomposition with \( e \). Note that

\[
\pi_f(M[\mathcal{C}(t, x)]_f \cap F(t)) \subseteq \pi_{f_e}(M[\mathcal{C}(t', x)]_{f_e} \cap F(t'))
\]

since \( \leq \) is compatible with the composition in \( \mathcal{C} \).

**Lemma 4.4.** Suppose there is an admissible partial order on \( \mathcal{C}(x) \). Then the assignment which sends a subfunctor \( F \subseteq M[\mathcal{C}(\cdot, x)] \) to \( \tilde{F} \) preserves proper inclusions. Therefore \( M[\mathcal{C}(\cdot, x)] \) is noetherian provided that \( \mathcal{C}(\cdot, x) \downarrow M \) is noetherian.

**Proof.** Let \( F \subseteq G \subseteq M[\mathcal{C}(\cdot, x)] \). Then \( \tilde{F} \subseteq \tilde{G} \). Now suppose that \( F \neq G \). Thus there exists \( t \in \mathcal{C} \) such that \( F(t) \neq G(t) \). We have \( \mathcal{C}(t, x) = \bigcup_{f \in \mathcal{C}(t,x)} \mathcal{C}(t, x)_f \), and this union is directed since \( \leq \) is total. Thus

\[
F(t) = \sum_{f \in \mathcal{C}(t,x)} (M[\mathcal{C}(t, x)]_f \cap F(t))
\]

since filtered colimits in \( A \) are exact. This yields \( f \) such that

\[
M[\mathcal{C}(t, x)]_f \cap F(t) \neq M[\mathcal{C}(t, x)]_f \cap G(t).
\]

Choose \( f \in \mathcal{C}(t, x) \) maximal with respect to this property, using that \( \leq \) is noetherian. Now observe that the projection \( \pi_f \) induces an exact sequence

\[
0 \longrightarrow \sum_{f < g} (M[\mathcal{C}(t, x)]_g \cap F(t)) \longrightarrow F(t) \longrightarrow \pi_f(M[\mathcal{C}(t, x)]_f \cap F(t)) \longrightarrow 0
\]

since the kernel of \( \pi_f \) equals the directed union \( \sum_{f < g} M[\mathcal{C}(t, x)]_g \). For the directedness one uses again that \( \leq \) is total. Thus

\[
\pi_f(M[\mathcal{C}(t, x)]_f \cap F(t)) \neq \pi_f(M[\mathcal{C}(t, x)]_f \cap G(t))
\]

and therefore \( \tilde{F} \neq \tilde{G} \).

**Proposition 4.5.** Let \( x \in \mathcal{C} \). Suppose that \( \mathcal{C}(\cdot, x) \) is noetherian and that \( \mathcal{C}(x) \) has an admissible partial order. If \( M \in A \) is noetherian, then \( M[\mathcal{C}(\cdot, x)] \) is noetherian.

**Proof.** Combine Lemmas 4.2 and 4.3.

5. Gröbner categories

**Definition 5.1.** An essentially small category \( \mathcal{C} \) is a Gröbner category if the following holds:

1. The functor \( \mathcal{C}(\cdot, x) \) is noetherian for every \( x \in \mathcal{C} \).
2. There is an admissible partial order on \( \mathcal{C}(x) \) for every \( x \in \mathcal{C} \).

**Example 5.2.** A strongly noetherian poset (viewed as a category) is a Gröbner category.
Theorem 5.3. Let $\mathcal{C}$ be a Gröbner category and $\mathcal{A}$ a Grothendieck abelian category. If $\mathcal{A}$ is locally noetherian, then $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{A})$ is locally noetherian.

Proof. Combine Lemma 5.1 and Proposition 5.5. □

6. Base change

Given functors $F, G : \mathcal{C}^{\text{op}} \to \text{Set}$, we write $F \rightsquigarrow G$ if there is a finite chain

$$F = F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_{n-1} \hookrightarrow F_n = G$$

of epimorphisms and monomorphisms of functors $\mathcal{C}^{\text{op}} \to \text{Set}$.

Definition 6.1. A functor $\phi : \mathcal{C} \to \mathcal{D}$ is contravariantly finite if the following holds:

1. Every object $y \in \mathcal{D}$ is isomorphic to $\phi(x)$ for some $x \in \mathcal{C}$.
2. For every object $y \in \mathcal{D}$ there are objects $x_1, \ldots, x_n$ in $\mathcal{C}$ such that

$$\bigsqcup_{i=1}^n \mathcal{C}(-, x_i) \to \mathcal{D}(\phi-, y).$$

The functor $\phi$ is covariantly finite if $\phi^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ is contravariantly finite.

Note that a composite of contravariantly finite functors is contravariantly finite.

Lemma 6.2. Let $f : \mathcal{C} \to \mathcal{D}$ be a contravariantly finite functor and $\mathcal{A}$ a Grothendieck abelian category. Fix $M \in \mathcal{A}$ and suppose that $M[\mathcal{C}(-, x)]$ is noetherian for all $x \in \mathcal{C}$. Then $M[\mathcal{D}(-, y)]$ is noetherian for all $y \in \mathcal{D}$.

Proof. A finite chain

$$\bigsqcup_{i=1}^n M[\mathcal{C}(-, x_i)] = F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_{n-1} \hookrightarrow F_n = M[\mathcal{D}(\phi-, y)]$$

of epimorphisms and monomorphisms induces a chain

$$\bigsqcup_{i=1}^n \mathcal{A}[\mathcal{C}(-, x_i)] = F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_{n-1} \hookrightarrow F_n = \mathcal{A}[\mathcal{D}(\phi-, y)]$$

of epimorphisms and monomorphisms in $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{A})$. Thus $M[\mathcal{D}(\phi-, y)]$ is noetherian. It follows that $M[\mathcal{D}(-, y)]$ is noetherian, since precomposition with $\phi$ yields a faithful and exact functor $\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{A}) \to \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{A})$. □

Proposition 6.3. Let $f : \mathcal{C} \to \mathcal{D}$ be a contravariantly finite functor and $\mathcal{A}$ a locally noetherian Grothendieck abelian category. If $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{A})$ is locally noetherian, then $\text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{A})$ is locally noetherian.

Proof. Combine Lemmas 5.2 and 6.2. □

7. Categories of finite sets

Let $\Gamma$ denote the category of finite sets (a skeleton is given by the sets $\mathbf{n} = \{1, 2, \ldots, n\}$). The subcategory of finite sets with surjective morphisms is denoted by $\Gamma_{\text{sur}}$. A surjection $f : \mathbf{m} \to \mathbf{n}$ is ordered if $i < j$ implies $\min f^{-1}(i) < \min f^{-1}(j)$. We write $\Gamma_{\text{ord}}$ for the subcategory of finite sets whose morphisms are ordered surjections. Given a surjection $f : \mathbf{m} \to \mathbf{n}$, let $f^j : \mathbf{n} \to \mathbf{m}$ denote the map given by $f^j(i) = \min f^{-1}(i)$. Note that $ff^j = \text{id}$, and $gf = f^jg^j$ provided that $f$ and $g$ are ordered surjections.

\footnote{The terminology follows that introduced by Auslander and Smalø [1] for an inclusion functor.}
Lemma 7.1. (1) The inclusion $\Gamma_{\text{sur}} \to \Gamma$ is contravariantly finite.

(2) The inclusion $\Gamma_{\text{os}} \to \Gamma_{\text{sur}}$ is contravariantly finite.

Proof. (1) For each integer $n \geq 0$ there is an isomorphism

$$\bigsqcup_{m \to n} \Gamma_{\text{sur}}(-, m) \xrightarrow{\sim} \Gamma(-, n)$$

which is induced by the injective maps $m \to n$.

(2) For each integer $n \geq 0$ there is an isomorphism

$$\Gamma_{\text{os}}(-, n) \times \mathcal{S}_n \xrightarrow{\sim} \Gamma_{\text{sur}}(-, n)$$

which sends a pair $(f, \sigma)$ to $\sigma f$. The inverse sends a surjective map $g : m \to n$ to $(\tau^{-1}g, \tau)$ where $\tau \in \mathcal{S}_n$ is the unique permutation such that $g^{\tau}$ is increasing. □

Fix an integer $n \geq 0$. Given $f, g \in \Gamma(n)$ we set $f \leq g$ if there exists an ordered surjection $h$ such that $f = gh$.

Lemma 7.2. The poset $(\Gamma(n), \leq)$ is strongly noetherian.

Proof. We fix some notation for each $f \in \Gamma(m, n)$. Set $\lambda(f) = m$. If $f$ is not injective, set

$$\mu(f) = m - \max\{i \in m \mid \text{there exists } j < i \text{ such that } f(i) = f(j)\}$$

and $\pi(f) = f(m - \mu(f))$. Define $\tilde{f} \in \Gamma(m - 1, n)$ by setting $\tilde{f}(i) = f(i)$ for $i < m - \mu(f)$ and $\tilde{f}(i) = f(i + 1)$ otherwise.

Note that $f \leq \tilde{f}$. Moreover, $\mu(f) = \mu(g)$, $\pi(f) = \pi(g)$, and $\tilde{f} \leq \tilde{g}$ imply $f \leq g$.

Suppose that $(\Gamma(n), \leq)$ is not strongly noetherian. Then there exists an infinite sequence $(f_r)_{r \in \mathbb{N}}$ in $\Gamma(n)$ such that $i < j$ implies $f_j \nleq f_i$; see Lemma 7.2. Call such a sequence bad. Choose the sequence minimal in the sense that $\lambda(f_1)$ is minimal for all bad sequences $(g_r)_{r \in \mathbb{N}}$ with $g_j = f_j$ for all $j < i$. There is an infinite subsequence $(f_{\alpha(r)})_{r \in \mathbb{N}}$ (given by some increasing map $\alpha : \mathbb{N} \to \mathbb{N}$) such that $\mu$ and $\pi$ agree on all $f_{\alpha(r)}$, since the values of $\mu$ and $\pi$ are bounded by $n$. Now consider the sequence $f_0, f_1, \ldots, f_{\alpha(0) - 1}, f_{\alpha(0)}, f_{\alpha(1)}, \ldots$ and denote this by $(g_r)_{r \in \mathbb{N}}$. This sequence is not bad, since $(f_r)_{r \in \mathbb{N}}$ is minimal. Thus there are $i < j$ in $\mathbb{N}$ with $g_j \not\leq g_i$. Clearly, $j < \alpha(0)$ is impossible. If $i < \alpha(0)$, then

$$f_{\alpha(i - \alpha(0))} \leq f_{\alpha(j - \alpha(0))} = g_j \leq g_i = f_i,$$

which is a contradiction, since $i < \alpha(0) \leq \alpha(j - \alpha(0))$. If $i \geq \alpha(0)$, then $f_{\alpha(j - \alpha(0))} \leq f_{\alpha(i - \alpha(0))}$; this is a contradiction again. Thus $(\Gamma(n), \leq)$ is strongly noetherian. □

Proposition 7.3. The category $\Gamma_{\text{os}}$ is a Gröbner category.

Proof. Fix an integer $n \geq 0$. The poset $\Gamma_{\text{os}}(n)$ is strongly noetherian by Lemma 7.2 and it follows from Lemma 4.1 that the functor $\Gamma_{\text{os}}(-, n)$ is noetherian.

The admissible partial order on $\Gamma_{\text{os}}(n)$ is given by the lexicographic order. Thus for $f, g \in \Gamma_{\text{os}}(m, n)$, we have $f < g$ if there exists $j \in m$ with $f(j) < g(j)$ and $f(i) = g(i)$ for all $i < j$. □

Theorem 7.4. Let $\mathcal{A}$ be a locally noetherian Grothendieck abelian category. Then the category $\text{Fun}(\Gamma_{\text{os}}, \mathcal{A})$ is locally noetherian.

Proof. The category $\Gamma_{\text{os}}$ is a Gröbner category by Proposition 7.3. It follows from Theorem 5.3 that $\text{Fun}(\Gamma_{\text{os}}^{\text{op}}, \mathcal{A})$ is locally noetherian. The inclusion $\Gamma_{\text{os}} \to \Gamma$ is contravariantly finite by Lemma 7.1. Thus $\text{Fun}(\Gamma^{\text{op}}, \mathcal{A})$ is locally noetherian by Proposition 6.3. □
THE ARTINIAN CONJECTURE

8. The artinian conjecture

Let \( A \) be a ring. We denote by \( \mathcal{P}(A) \) the category of free \( A \)-modules of finite rank. If \( A \) is finite, then the functor \( \Gamma \to \mathcal{P}(A) \) sending \( X \) to \( A[X] \) is a left adjoint of the forgetful functor \( \mathcal{P}(A) \to \Gamma \).

Lemma 8.1. Let \( A \) be finite. Then the functor \( \Gamma \to \mathcal{P}(A) \) is contravariantly finite.

Proof. The assertion follows from the adjointness isomorphism
\[
\mathcal{P}(A)(A[X], P) \cong \Gamma(X, P).
\]

Theorem 8.2. Let \( A \) be a finite ring and \( \mathcal{A} \) a locally noetherian Grothendieck abelian category. Then the category \( \text{Fun}(\mathcal{P}(A)^{op}, \mathcal{A}) \) is locally noetherian.

Proof. Combine Theorem 7.4 with Lemma 8.1 and Proposition 6.3.

9. FI-modules

The proof of the artinian conjecture yields an alternative proof of the following result due to Church, Ellenberg, Farb, and Nagpal.

Let \( \mathcal{G}_{\text{inj}} \) denote the category whose objects are finite sets and whose morphisms are injective maps.

Theorem 9.1 ([2, Theorem A]). Let \( \mathcal{A} \) be a locally noetherian Grothendieck abelian category. Then the category \( \text{Fun}(\mathcal{G}_{\text{inj}}, \mathcal{A}) \) is locally noetherian.

Proof. The following argument has been suggested by Kai-Uwe Bux. Consider the functor \( \phi: \mathcal{G}_{\text{os}} \to (\mathcal{G}_{\text{inj}})^{op} \) which is the identity on objects and takes a map \( f: m \to n \) to \( f': n \to m \) given by \( f'(i) = \min f^{-1}(i) \). This functor is contravariantly finite, since for each integer \( n \geq 0 \) the morphism
\[
\Gamma_{\text{os}}(\cdot, n) \times \mathfrak{S}_n \longrightarrow \Gamma_{\text{inj}}(n, \cdot)
\]
which sends a pair \((f, \sigma)\) to \( f'\sigma \) is an epimorphism.

It follows from Proposition 6.3 that \( \text{Fun}(\mathcal{G}_{\text{inj}}, \mathcal{A}) \) is locally noetherian, since \( \text{Fun}(\Gamma_{\text{os}}^{op}, \mathcal{A}) \) is locally noetherian by Proposition 7.3 and Theorem 5.3.

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