IS PURITY ETERNAL?

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ABSTRACT: Phenomenological and formal restrictions on the evolution of pure into mixed states are discussed. In particular, it is argued that, if energy is conserved, loss of purity is incompatible with the weakest possible form of Lorentz covariance.

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1. Introduction

In 1983, Hawking proposed that, due to quantum gravitational effects, a scattering process might result in an initial pure state becoming a final mixed state [1]. This idea (which had been suggested earlier as a phenomenological possibility worthy of investigation [2]) is clearly one of great intellectual power. It would represent a truly fundamental modification of the laws of physics.

Nevertheless, Hawking’s proposal quickly received criticism. It was noted [3,4] that conservation laws become decoupled from symmetry principles, so that (for example) rotation invariance would no longer imply conservation of angular momentum. Furthermore, actual calculations in models which admit systematic approximations showed no hint of the decay of purity [3] (see, however, ref. [5]). Also, since quantum mechanics is well tested in a variety of systems, corrections to the generalized hamiltonian would have to be quite small, with eigenvalues of order $10^{-21}$ GeV or less [4].

By far the most severe attack was leveled by Banks, Peskin, and Susskind (BPS) [6], who concluded that conservation of energy and momentum could not be preserved without losing locality. Since then, however, the prevailing notion of the meaning of locality has weakened somewhat, due to the discovery of the wormhole phenomenon [7]. Wormholes violate locality, but the phenomenological results turn out to be remarkably benign [8–10]. It therefore seems appropriate to reexamine Hawking’s proposal, to see whether or not the breakdown of locality noted by BPS is truly objectionable, and to see what other constraints, if any, may be placed on the decay of purity.

Let us briefly review some of the basic formalism for a theory in which pure states can evolve to mixed states; for more details, see refs. [2,4,6]. Such a theory must be formulated in terms of a density matrix $\rho$ rather than a state $|\psi\rangle$. The eigenvalues of $\rho$ represent probabilities, and so must be real, positive or zero, and sum to one (assuming normalizable states). Thus $\rho$ must be hermitian and satisfy $\text{Tr} \rho = 1$. The state is said to be pure if $\rho = |\psi\rangle\langle\psi|$ for some Hilbert space vector $|\psi\rangle$; in this case $\text{Tr} \rho^2 = \text{Tr} \rho = 1$. If $\text{Tr} \rho^2 < 1$, then $\rho$ cannot be written as $|\psi\rangle\langle\psi|$ for any $|\psi\rangle$, and the state is said to be mixed. The usual evolution equation for $\rho$ is $\dot{\rho} = -i[H,\rho]$, and this leaves both $\text{Tr} \rho$ and $\text{Tr} \rho^2$ constant in time. To allow loss of purity, we must modify this equation. Following refs. [2,4], we assume that the new evolution equation for $\rho$ is still linear, and still first order in time derivatives:

$$\dot{\rho}_{ab} = H_{ab}^{\cd} \rho_{dc} \cdot \quad (1)$$

The generalized hamiltonian $H$ must be constrained to preserve the hermiticity, positivity, and trace of $\rho$. As shown by BPS, the most general equation preserving $\rho^\dagger = \rho$ and $\text{Tr} \rho = 1$ is

$$\dot{\rho} = -i[H,\rho] - \frac{1}{2}g_{\alpha\beta}(Q^\alpha Q^\beta \rho + \rho Q^\alpha Q^\beta - 2Q^\beta \rho Q^\alpha) , \quad (2)$$
where the $Q$’s are any hermitian operators other than the identity, and $g_{\alpha\beta}$ is a hermitian matrix of coupling constants. The hamiltonian $H$ is unambiguously defined as the operator appearing in the commutator term. BPS also showed that a sufficient condition for $\rho$ to remain positive is that the eigenvalues of $g_{\alpha\beta}$ be positive or zero.

Acceptable phenomenology leads us to demand that the familiar conservation laws be satisfied. Energy conservation, for example, requires that $\text{Tr} f(H) \dot{\rho} = 0$, where $f(H)$ is any smooth function of the hamiltonian $H$. BPS have shown that, if $g_{\alpha\beta}$ has nonnegative eigenvalues and is, in addition, real and symmetric, then conservation of energy requires that $H$ commute with each of the $Q$’s. (If $g_{\alpha\beta}$ does not satisfy all these conditions, there may be other possibilities; this will be discussed later.) In a quantum field theory, there are very few operators available which commute with the hamiltonian, and all of them are global: that is, integrals over all space of a local density.* We always have at our disposal the hamiltonian $H$ and the total momentum operator $\vec{P}$ to press into service as $Q$’s. We may also have global charges corresponding to conserved quantities like baryon number. Is eq. (2), with the $Q$’s chosen from this list, a viable possibility?

BPS concluded that the answer is no, due to a breakdown of locality. In particular, they state that cluster decomposition will no longer hold. Given the density matrix $\rho$ at some fixed time, and localized operators $A(\vec{x})$ and $B(\vec{y})$, cluster decomposition states that

$$\text{Tr}[\rho A(\vec{x}) B(\vec{y})] \simeq \text{Tr}[\rho A(\vec{x})] \text{Tr}[\rho B(\vec{y})]$$

(3)

when $|\vec{x} - \vec{y}|$ becomes large. The precise meaning of “large” depends on the theory, and on $\rho$. In ordinary field theory, eq. (3) is usually quoted as a property of the vacuum density matrix $\rho_0 = |0\rangle\langle 0|$. If the lightest particle in the theory has mass $m$, and if we take $\rho = \rho_0$, then “large” means much bigger than $m^{-1}$. But if, for example, $\rho$ represents a pure state consisting of one or more particles whose wave functions are localized near the origin but coherent over a region of linear size $a \gg m^{-1}$, then “large” means much bigger than $a$. The question is, if eq. (3) is valid for some initial density matrix $\rho_{\text{init}}$, and for $|\vec{x} - \vec{y}|$ greater than some length $L$, will this still be true at later times?

If we assume that energy is conserved, then pure energy eigenstates like $\rho_0$ do not evolve in time, and so there is no modification of the cluster property for $\rho_{\text{init}} = \rho_0$. If we consider instead a $\rho_{\text{init}}$ corresponding to a localized excited state, we must ask how quickly its wave packet spreads out as time evolves. Even in ordinary field theory, wave packets spread out, and the minimum value of $|\vec{x} - \vec{y}|$ for which eq. (3) holds will grow with time. It turns out that eq. (2) does cause wave packets to spread out a bit faster than they otherwise would, but the effect is not dramatic; certainly it does not qualify as a violation of cluster decomposition. However, if we put some particles at both $\vec{x}$ and $\vec{y}$, then extra correlations do develop with time. For free nonrelativistic particles, these remain small at

* In a gauge theory, we have the local generators of the gauge symmetry; these annihilate all the physical states, however, and so would vanish in eq. (2).
all times, and actually disappear at late times. This is discussed in detail in sect. 2. We conclude that the nonlocality noted by BPS, while present, is in fact difficult to detect, and that eq. (2) cannot be dismissed because of it.

One may still question whether or not eq. (2), with global $Q$’s, has any reasonable chance to arise as the low energy limit of a more fundamental theory. I know of no such theory, but there is a strong passing resemblance between eq. (2) and the effective theories that arise due to wormholes [7–10] (which were originally thought to cause purity to decay). In wormhole theory, the usual field theory action $S_0 = \int d^4 x \mathcal{L}$ is modified to [10]

$$S = S_0 + g_{\alpha\beta} Q^\alpha Q^\beta + \ldots$$

(4)

where $Q^\alpha = \int d^4 x q^\alpha(x)$, $q^\alpha(x)$ is some local operator, and the ellipses stand for higher powers of $Q$’s. It seems not totally unreasonable to suppose that there could exist a variant of wormhole theory which ultimately results in eq. (2) rather than eq. (4). Of course, there are some important differences between eq. (2) and eq. (4): in eq. (4), the $Q$’s are four dimensional integrals, and they modify the action itself. They therefore do not lead to violations of quantum mechanics, but do look like they would produce severe violations of locality. As is now well known, this is not the case. Similarly, the apparent violations of locality in eq. (2) are not as drastic as they appear to be at first glance.

If eq. (2) does not lead to serious violations of locality, and cannot be immediately rejected on grounds of implausibility, it becomes necessary to reexamine its viability. That is the purpose of this paper. Phenomenological constraints, including the anomalous spreading of wave packets, are examined in sect. 2. Formal restrictions are discussed in sect. 3, where it is found that eq. (2) is incompatible with the weakest possible form of Lorentz covariance (if energy is conserved). This is the main conclusion of this paper. However, a possible loophole in the argument is presented in sect. 4, and conclusions in sect. 5.

2. Phenomenological constraints

As noted in the introduction, the question of the locality of eq. (2) requires us to study the spreading of wave packets with time. Rather than immediately attacking this problem within quantum field theory, we will warm up with some examples from one-dimensional, nonrelativistic quantum mechanics that contain the essential physics. Let us first consider a harmonic oscillator of unit mass and frequency (so that the hamiltonian is $H = a^\dagger a + \frac{1}{2}$) which is in a coherent state. At time $t = 0$, the state is specified by an arbitrary complex parameter $\beta = |\beta| e^{i\phi}$:

$$|\beta, 0\rangle = e^{-|\beta|^2/2} e^{\beta a^\dagger} |0\rangle .$$

(5)

It is well known that at later times the state is still coherent, but with a different value of $\beta$:

$$|\beta, t\rangle = e^{-it/2} |\beta e^{-it}, 0\rangle .$$

(6)
Furthermore, the probability to find the oscillator at position \( x \) at time \( t \) is given by

\[
\langle x | \rho(\beta, t) | x \rangle = \frac{1}{\sqrt{\pi}} e^{-\left|x - \sqrt{2} |\beta| \cos(t - \phi)\right|^2}
\]  

(7)

where \( \rho(\beta, t) = |\beta, t\rangle \langle \beta, t| \) is the density matrix at time \( t \). Thus, the oscillator’s wave packet just moves back and forth without changing shape, its center following a classical trajectory with maximum amplitude \( \sqrt{2} |\beta| \).

Now consider modifying the quantum equation of motion to the general form of eq. (2). In order to conserve energy, the \( Q \)’s must commute with \( H \). For simplicity, we will take a single \( Q \) and set it equal to \( H \), and let the corresponding value of \( g \) be real and positive.

To compute the density matrix at time \( t \), we first expand the initial state \( |\beta, 0\rangle \) into energy eigenstates:

\[
|\beta, 0\rangle = e^{-|\beta|^2/2} \sum_{n=0}^{\infty} \frac{\beta^n}{(n!)^{1/2}} |n\rangle,
\]

(8)

where \( |n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle \) is a normalized energy eigenstate with energy \( n + \frac{1}{2} \). The time evolution of a density matrix element \( |n\rangle \langle m| \) in this basis is simple, and results in

\[
\rho(\beta, t) = e^{-|\beta|^2} \sum_{n,m=0}^{\infty} \frac{\beta^n \beta^m}{(n!m!)^{1/2}} e^{-i(n-m)t - g(n-m)^2t/2} |n\rangle \langle m| .
\]

(9)

The probability of finding the oscillator at position \( x \) at time \( t \) is again \( \langle x | \rho(\beta, t) | x \rangle \); let us study the behavior of this function as \( t \to \infty \). Looking at \( \rho(\beta, t) \), we see that its off-diagonal components decay away at late times, and it approaches

\[
\rho(\beta, \infty) = e^{-|\beta|^2} \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{n!} |n\rangle \langle n| .
\]

(10)

Furthermore, this final state is independent of the specific choice of the \( Q \)’s as long as they all commute with \( H \). To compute \( \langle x | \rho(\beta, \infty) | x \rangle \) explicitly, note that the off-diagonal components of \( \rho(\beta, t) \) can also be removed by averaging over the phase of \( \beta \):

\[
\rho(\beta, \infty) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \rho(\beta, t) .
\]

(11)

Since the result is independent of time, we may as well use \( \rho(\beta, 0) = |\beta, 0\rangle \langle \beta, 0| \) on the right hand side. Then, taking the expectation value in a position eigenstate and using eq. (7) at \( t = 0 \), we find

\[
\langle x | \rho(\beta, \infty) | x \rangle = \frac{1}{2\pi^{3/2}} \int_0^{2\pi} d\phi e^{-\left|x - \sqrt{2} |\beta| \cos \phi\right|^2}.
\]

(12)
We thus see that the oscillator forgets its starting point, but conservation of energy forces it to remember its maximum amplitude $\sqrt{2} |\beta|$. If we make a change of integration variable in eq. (12), we get

$$
\langle x | \rho(\beta, \infty) | x \rangle = \int_{-\sqrt{2}\beta}^{+\sqrt{2}\beta} dy \left[ \frac{1}{\sqrt{\pi \hbar}} e^{-(x-y)^2/\hbar} \right] \frac{1}{\pi \sqrt{2} |\beta|^2 - y^2} \tag{13}
$$

where appropriate factors of $\hbar$ have been restored. Here $\pi^{-1}(2|\beta|^2 - y^2)^{-1/2}$ represents the probability for a classical oscillator with a specified maximum amplitude $\sqrt{2} |\beta|$ to be found at position $y$ at a randomly selected time. Eq. (13) tells us that this classical probability distribution is then smeared out over the width of the quantum wave packet. In the $\hbar \to 0$ limit, the wave packet becomes a delta function, and we recover the classical probability.

This is a rather innocuous result. In order to discover this sort of violation of quantum mechanics experimentally, one would have to let the oscillator run for a long time (since the new parameter $g$ is presumably small) while keeping precise track of the time. On the other hand, systems such as the earth revolving around the sun can be treated in an essentially similar way, have been around for billions of years, and have not been seen to exhibit any mysterious timing discrepancies. But it is hard to translate this into a limit on $g$ for elementary particles in any convincing way.

As another example, let us consider a free, nonrelativistic particle with unit mass in one dimension (so that the hamiltonian is $H = \frac{1}{2}p^2$). (For a related discussion, see ref. [11].) Take the initial state to be a gaussian wave packet of width $a$ and and mean velocity $p_0$:

$$
\psi(x, 0) = \frac{1}{\pi^{1/4} a^{1/2}} \exp \left[ ip_0 x - \frac{x^2}{2a^2} \right]. \tag{14}
$$

Then, at later times, we find (according to ordinary quantum mechanics) that the probability to find the particle at position $x$ is

$$
|\psi(x, t)|^2 = \frac{1}{\pi^{1/2}(a^2 + t^2/a^2)^{1/2}} \exp \left[ -\frac{(x - p_0 t)^2}{a^2 + t^2/a^2} \right]. \tag{15}
$$

This is easy to understand. The initial uncertainty in position $\Delta x \sim a$ implies an uncertainty in momentum $\Delta p \sim 1/a$. Then, at later times, the momentum uncertainty implies an additional uncertainty in position $(\Delta p) t \sim t/a$. These are then compounded quadratically to give the time dependent position uncertainty $\Delta x(t) \sim (a^2 + t^2/a^2)^{1/2}$ implied by eq. (15). Meanwhile, the center of the wave packet moves with constant velocity $p_0$, the only possibility consistent with conservation of momentum.

Now we again consider modifying the quantum equation of motion. Since the hamiltonian is so simple in this case, we can take the $Q$’s to be arbitrary functions of $p$ and
still conserve energy and momentum. Let us work out one example in detail: a single $Q$, this time (for later mathematical convenience) equal to the momentum $p$. Expanding in momentum eigenstates, we see that the probability to find the particle at position $x$ at time $t$ is

$$
\langle x|\rho(t)|x \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp \, dk \, \tilde{\psi}(p,0)\tilde{\psi}^*(k,0) \, e^{i(p-k)x-i(p^2-k^2)t/2-g(p-k)^2t/2} \, .
$$

where $\tilde{\psi}(p,0)$ is the Fourier transform of $\psi(x,0)$:

$$
\tilde{\psi}(p,0) = a^{1/2} \frac{1}{\pi^{1/4}} \exp \left[ -\frac{1}{2} a^2 (p-p_0)^2 \right] \, .
$$

Because the energies are continuous in this case, we cannot simply discard the off-diagonal terms as we did for the harmonic oscillator. Instead, we note that the choice of $Q=p$ has rendered $\langle x|\rho(t)|x \rangle$ as a gaussian integral which can be evaluated exactly at arbitrary times:

$$
\langle x|\rho(t)|x \rangle = \frac{1}{\pi^{1/2}(a^2+t^2/a^2+2gt)^{1/2}} \exp \left[ -\frac{(x-p_0 t)^2}{a^2+t^2/a^2+2gt} \right] \, .
$$

We see that there is a new contribution to the uncertainty in position, $\Delta x_{\text{new}} \sim (2gt)^{1/2}$, which is compounded with the ordinary quantum uncertainties. The $t^{1/2}$ dependence of $\Delta x_{\text{new}}$ is characteristic of a one-dimensional random walk. Note that, since momentum is still conserved, the center of the wave packet still moves at constant velocity $p_0$.

This change in the nature of how wave packets spread out is again an innocuous one. In fact, for $g \ll 1$, the effect of $\Delta x_{\text{new}}$ is negligible, since then $gt \ll a^2+t^2/a^2$ at all times.

Now let us consider two free particles, with hamiltonian $H = \frac{1}{2}(p_1^2 + p_2^2)$ and initial wave function

$$
\psi(x_1,x_2,0) \sim \exp \left[ ip_{10}x_1 + ip_{20}x_2 - \frac{(x_1-x_{10})^2}{2a^2} - \frac{(x_2-x_{20})^2}{2a^2} \right] \, .
$$

We can analyze this system in the same way as before; taking $Q$ to be equal to the total momentum $p_1 + p_2$ of the two particles, we eventually find

$$
\langle x_1,x_2|\rho(t)|x_1,x_2 \rangle \sim \exp \left[ -\frac{(a^2+t^2/a^2+2gt)(y_1^2+y_2^2)-4gt\,y_1y_2}{(a^2+t^2/a^2)(a^2+t^2/a^2+4gt)} \right] \, .
$$

where $y_i = x_i - (x_{i0} + p_{i0} \, t)$. The $y_1y_2$ term indicates that the presence of a second particle influences the first. Even though they are spatially uncorrelated initially, correlations develop, independent of how far apart they are. This represents a violation of cluster decomposition. However, for $g \ll 1$, the new correlations are small at all times, and actually decrease with time for $t \gtrsim a^2$. Furthermore, if we trace over the states of the
second particle (integrate eq. (20) over $x_2$), we reproduce eq. (18). Thus the second particle has no effect if we know nothing of its state.

A quantum field theory will exhibit similar behavior. Indeed, for free field theory this analysis can be taken over completely, except that we should use a relativistic dispersion relation, and choose $Q$’s which respect Lorentz covariance. The change in the dispersion relation prevents exact evaluation of the integrals, and even a steepest descent analysis is quite involved, but the behavior should be qualitatively similar. This means that loss of purity does not force us to rethink the LSZ reduction paradigm for scattering theory, since wave packets will remain sufficiently compact, and extra correlations will always be small. A more serious problem is the requirement of Lorentz covariance, which, it turns out, is impossible to satisfy. This is the subject of the next section.

3. Formal Constraints

Since we have a theory in which correlations can appear between spacelike separated systems, finding a Lorentz covariant version may seem to be a hopeless task. This is not necessarily the case: theories with tachyons, whatever their other faults, are Lorentz covariant, and have superluminal correlations. Thus a reasonable question to ask is, what is the weakest possible form of Lorentz covariance which can be demanded of eq. (2)?

Let us begin with a concrete example of eq. (2) which is Lorentz covariant. Consider a scalar field $\varphi$ with lagrangian $L = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - V(\varphi) + J \varphi$, where $J$ is a gaussian random variable with $\langle J(x)J(y) \rangle = g \delta^4(x - y)$. Here angle brackets denote an average over the probability distribution for $J$. This theory is manifestly Lorentz invariant, and, as shown by BPS, leads to the evolution equation

$$\dot{\rho} = -i[H, \rho] - \frac{1}{2} g \int d^3x \left[ \varphi(\vec{x}), [\varphi(\vec{x}), \rho] \right],$$

which is a specific form of eq. (2). This theory is not viable, since BPS demonstrated that it produces enormous violations of energy and momentum conservation. It does, however, serve as an example of the form eq. (2) could take in a Lorentz covariant framework. We can rewrite eq. (21) as

$$\partial^0 \rho = -i[P^0, \rho] - \frac{1}{2} g \int d\Sigma^0 \left[ \varphi(\vec{x}), [\varphi(\vec{x}), \rho] \right].$$

Here $\vec{x}$ specifies a point on the spacelike hypersurface $\Sigma$ whose unit normal defines the time direction, and $d\Sigma^\mu = \frac{1}{6} \epsilon^{\mu jk} dx_i dx_j dx_k$. Clearly, each term in eq. (22) carries a vector index in the time direction, and no other explicit indices. I wish to argue that this must be true of any version of eq. (2) which can arise from an underlying, Lorentz invariant framework. It is by no means clear that this condition is sufficient to ensure the existence
of a full set of Lorentz transformations with the correct algebra, but it would appear to be necessary. This condition on eq. (2) is what I mean by the “weakest possible” form of Lorentz covariance.

We do not need to know much more about Lorentz transformations to reach a strong conclusion. This is fortunate: since eq. (2) tells us that infinitesimal time translations are not generated by the Hamiltonian $H$, it is clearly unsafe to assume that space translations, rotations, and boosts will still be generated by the usual operators.* If Lorentz invariance is to have any meaning at all, however, we must demand that the Lorentz generators (whatever form they may take) satisfy certain basic properties, including the following:

(1) The total energy and momentum operators $H$ and $\vec{P}$ should transform as a four-vector under infinitesimal Lorentz transformations (so that, for example, the generator of boosts in the $z$-direction, acting on $H$, results in $P_z$).

(2) The infinitesimal time and space translations $dt$ and $d\vec{x}$ must also transform as a four-vector.

(3) The usual product rules should hold, so that, for example, $M^2 \equiv H^2 - \vec{P}^2$ is still invariant under Lorentz transformations.

This information alone is enough to allow us to conclude that $\dot{\rho}$ and $[H, \rho]$ both transform in the same way. I have argued that Lorentz covariance will be lost if the final term in eq. (2) does not also have this property. Therefore the product of two $Q$’s must transform like $H$. The simplest possibility is to let one $Q$ be a Lorentz scalar, perhaps $M^2 \equiv H^2 - \vec{P}^2$, and let the other $Q$ be $H$ itself. The problem with this is that it implies a matrix $g_{\alpha\beta}$ which is purely off-diagonal, and which therefore has a negative eigenvalue. One can see the problem explicitly in free field theory. For definiteness, take $Q_1 = H$ and $Q_2 = M^2$, with $g_{11} = g_{22} = 0$ and $g_{12} = g_{21} = g$. All one-particle states have the same value of $M^2$, and so the extra term in eq. (2) vanishes for all one-particle states, pure or mixed. If, however, we consider a two-particle state $|p_1 p_2\rangle = a^\dagger(p_1)a^\dagger(p_2)|0\rangle$, and evolve the initial density matrix $\rho(0) = |p_1 p_2\rangle\langle p_3 p_4|$, we find at later times

$$\rho(t) = \exp\left[-i(E_{12} - E_{34})t - g(E_{12} - E_{34})(M_{12}^2 - M_{34}^2)t\right]|p_1 p_2\rangle\langle p_3 p_4|$$

(23)

where $E_{ij} = E_i + E_j$, $E_i = (p_i^2 + m^2)^{1/2}$, $M_{ij}^2 = (E_i + E_j)^2 - (\vec{p}_i + \vec{p}_j)^2$, and $m$ is the mass of one particle. The $g$ dependent term is not always negative; indeed its sign is frame dependent. (The same is true if we replace $M^2$ by some other Lorentz scalar, such as a charge.) An off-diagonal matrix element of $\rho$ which is monotonically increasing guarantees that $\rho$ will eventually have a negative eigenvalue. Therefore this Lorentz covariant scheme must be rejected.

* In section 2, we implicitly assumed that the total momentum operator $\vec{P}$ generates space translations; relaxing this may provide another way out of unpleasant conclusions regarding nonlocal effects.
The only other possibility arises in a supersymmetric theory: then we have the supercharges at our disposal. They commute with the hamiltonian, and their product has the correct transformation properties. The appropriate version of eq. (2) is

\[ \dot{\rho} = -i[H, \rho] - \frac{1}{2} g \left[ \rho H + \rho H - 2(\bar{\sigma}^0)^{\hat{\alpha}\hat{\alpha}}(Q^\dagger_{\hat{\alpha}} \rho Q_{\hat{\alpha}} + Q_{\hat{\alpha}} \rho Q^\dagger_{\hat{\alpha}}) \right]. \] (24)

\[ \text{Tr} \dot{\rho} = 0 \] follows from the supersymmetry algebra \( \{Q_{\hat{\alpha}}, Q^\dagger_{\hat{\alpha}}\} = (\sigma^\mu)_{\hat{\alpha}\hat{\beta}} P_\mu. \) (We may use conventions in which both \( (\sigma^0)_{\hat{\alpha}\hat{\beta}} \) and \( (\bar{\sigma}^0)^{\hat{\alpha}\hat{\beta}} \) are equal to \( \delta_{\hat{\alpha}\hat{\beta}}. \)) Eq. (24) also conserves energy and momentum, since \( Q_{\hat{\alpha}} \) and \( Q^\dagger_{\hat{\alpha}} \) commute with \( H \) and \( \bar{P}. \)

The problem with eq. (24) is that it does not conserve angular momentum. Although it is straightforward to show (using the transformation properties of \( Q_{\hat{\alpha}} \) and \( Q^\dagger_{\hat{\alpha}} \)) that while \( \text{Tr} \bar{J} \rho \) remains constant, \( \text{Tr} \bar{J}^2 \rho \) does not. It is easy to see this physically: \( Q \sim b^\dagger a, \) where \( b^\dagger \) creates a fermion and \( a \) destroys a boson. An initial state of one boson at rest will be converted to a state of one fermion at rest by eq. (24). This fermion is equally likely to have spin up or spin down, and so \( \text{Tr} \bar{J} \rho \) remains zero, but clearly \( \text{Tr} \bar{J}^2 \rho \) has changed. Thus this scheme must be rejected as well, and we are forced to conclude that eq. (2) is incompatible with the weakest possible form of Lorentz covariance.

This objection would not arise in two dimensions, where there is no angular momentum. Perhaps eq. (24) has some role to play on the superstring worldsheet.

4. A possible loophole

The conclusion just reached—that the evolution of pure to mixed states is incompatible with the weakest possible form of Lorentz covariance—was based on the assumption that the \( Q \)'s in eq. (2) must commute with \( H. \) BPS showed that this is the only possibility if \( g_{\alpha\beta} \) is a real symmetric matrix with nonnegative eigenvalues. We must still consider the possibility that \( g_{\alpha\beta} \) does not obey these conditions (since positivity alone is a sufficient condition to avoid development of negative probabilities, and even that may not be necessary). Is it possible to conserve energy, momentum, and angular momentum, even if the corresponding operators do not commute with the \( Q \)'s? Surprisingly, the answer is yes. The example I have found is not a viable theory because it allows \( \rho \) to develop negative eigenvalues; however, I have been unable to prove that a model without this fatal flaw does not exist. This leaves open a small chance to maintain the weak form of Lorentz covariance along with the familiar conservation laws.

The simplest theory which conserves energy even though \([H, Q] \neq 0\) is based on the hamiltonian \( H = b^\dagger b, \) where \( b \) is a fermion operator obeying the usual anticommutation relations \( \{b, b\} = 0 \) and \( \{b^\dagger, b\} = 1. \) We take \( Q_1 = b^\dagger + b \) and \( Q_2 = i(b^\dagger - b), \) with \( g_{11} = -g_{22} = g \) and \( g_{12} = g_{21} = 0. \) Obviously, \( g_{\alpha\beta} \) has a dangerous negative eigenvalue.
Eq. (2) becomes
\[ \dot{\rho} = -i[H, \rho] - 2g(b^\dagger \rho b^\dagger + b \rho b) . \] (25)

We can now see why energy is conserved; if we compute \( \text{Tr} H^n \dot{\rho} \), all the extra terms vanish because we always get a factor of either \( b^2 = 0 \) or \( b^\dagger 2 = 0 \). However, it is not hard to show that, in general, an initially pure \( \rho \) immediately evolves into one with a negative eigenvalue.*

The model may be made less trivial by extending it to a set of independent operators \( b_i \), with \( H = \sum_i b_i^\dagger b_i \):
\[ \dot{\rho} = -i[H, \rho] - 2g \sum_i (b_i^\dagger \rho b_i^\dagger + b_i \rho b_i) . \] (26)

It is easy to check that we still have \( \text{Tr} H^n \dot{\rho} = 0 \). This version can be promoted to a field theory of free (Majorana) fermions:
\[ \dot{\rho} = -i[H, \rho] - 2g \sum_s \int d^3p [b^\dagger(p, s) \rho b^\dagger(p, s) + \text{h.c.}] , \] (27)

where \( b^\dagger(p, s) \) creates a fermion with momentum \( p \) and spin \( s \). The extra term has the correct Lorentz transformation properties, but is nothing very simple (or local) in position space. And, as before, it leads to negative probabilities.

This existence of this example is quite annoying, since it prevents a general proof of the necessity of \([H, Q] = 0\) for energy conservation, yet is both untenable (because of the negative probabilities) and extremely contrived. It is hard to imagine a physical process (such as wormholes) leading to anything as complicated in position space as eq. (27).

5. Conclusions

A reexamination of the question of whether or not pure states can evolve into mixed states shows that phenomenological problems associated with loss of locality are less severe than previously believed. However, it seems impossible to maintain both Lorentz covariance of the modified evolution equation and the familiar conservation laws. A loophole in the general argument leading to this conclusion was noted, but the only example I could find which sneaks through it is rather special, and in any case results in negative probabilities. The fascinating possibility that purity may not be eternal is still out of reach.

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* After this paper was circulated as a preprint, Jun Liu found a modified version of eq. (25) which preserves the positivity of \( \rho \) [12].
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