CONSTRUCTING THE HYPERDEFINABLE GROUP FROM THE GROUP CONFIGURATION

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Abstract. Under $\mathcal{P}(4)^\sim$-amalgamation, we obtain the canonical hyperdefinable group from the group configuration.

The group configuration theorem for stable theories given by Hrushovski [5], which extends Zilber’s result for $\omega$-categorical theories [17], plays a central role in producing deep results in geometric stability theory (For a complete exposition of it, see [14]). For example, it is pivotal in the proof of the dichotomy theorem for Zariski’ structures (See [9]). It is fair to say the group configuration theorem is one of the foundational theorems in geometric stability theory and its applications to algebraic geometry.

The theorem roughly says that one can get the canonical non-trivial type-definable group from the group configuration, a certain geometrical configuration, in stable theories. The complete generalization of the theorem into the context of simple theories seemed unreachable. In their topical paper [1], Ben-Yaacov, Tomasic and Wagner generalize the group configuration theorem by obtaining an invariant group from the group configuration in simple theories. However the group they produce does not completely fit into the first-order context.

On the other hand, Kolesnikov in his important thesis [12], categorizes simple theories by strengthening the type-amalgamation property (the independence theorem [10]), along the lines of early suppositions by Shelah [15] and Hrushovski [6]. These works suggest to us the possibility of using higher amalgamation for the group configuration problem. This approach proves successful, and in this paper we succeed in getting the canonical hyperdefinable group from the group configuration under stronger type-amalgamation in simple theories. The element of the group is a hyperimaginary, an equivalence class of a type-definable equivalence relation, and the group operation is type-definable, hence the group belongs to the domain of the standard first-order logic.

We assume that the reader is familiar with basics of simplicity theory [16]. Throughout the paper, $T$ is a complete simple theory. We work in a saturated model $\mathcal{M}$ of $T$ with hyperimaginaries, and $a, b, ...$ are (possibly infinitary) hyperimaginaries, $M, N$ are small elementary submodels. (Note that tuples from $\mathcal{M}^{eq}$ are also hyperimaginaries). As usual, $a \equiv_{A} b$ $(a \equiv_{A}^{t} b)$ means $a, b$ have the same type (Lascar strong type, resp.) over $A$. We point out that usually $\text{bdd}(a)$ denotes the set of all countable hyperimaginaries definable over $a$ [16, 3.1.7]. Here, depending on the context, it can be either a specific sequence which linearly orders the set $\text{bdd}(a)$; or, since a sequence of hyperimaginaries is again a hyperimaginary (of a large arity), a fixed hyperimaginary interdefinable with the sequence.

2000 Mathematics Subject Classification. Primary: 03C45.

The authors were supported by the Seggie Brown research fellowship, an NSF grant and an NSF grant DMS-0102502, in order.
We thank Frank O. Wagner for valuable correspondence on our earlier note which improves the presentation.

1. Around the Generalized Amalgamation Property

The usual amalgamation property (or the independence theorem) for Lascar strong types in simple theories is stated as follows: For $B \subseteq A, C$ with $A \subseteq B, C$, if $p$ is a Lascar strong type over $A$ and $p_B, p_C$ are nonforking Lascar strong type extensions of $p$ over $B, C$, respectively, then there is $d \models p_A \cup p_B$ such that $d \subseteq B \cup C$. We call it ‘3-amalgamation’ [8] (rather than 2-amalgamation [13]) which shall be compatible with Definition 1.3. Note that we can think of $B, C$ (after naming $A$) as two vertices of a base edge of a triangle and $d$ a top vertex, and $p_B = Lstp(d/B), p_C = Lstp(d/C)$ are the 2 types to be amalgamated. One would expect higher amalgamation to be a natural generalization of 3-amalgamation, using a tetrahedron and higher dimensional simplices instead of a triangle. Indeed, this is the case, but the following example draws attention to why we need extra care in defining the general $n$-amalgamation property.

Example 1.1. In the random graph $M$ in $\mathcal{L} = \{R\}$, choose distinct $a_i, b_i, c_i \in M$ and imaginary elements $d_i = \{a_i, b_i\} \ (i = 0, 1, 2)$. We can additionally assume that $R(a_0, c_0) \land R(b_0, c_1) \land \neg R(a_0, c_1) \land \neg R(b_0, c_0)$ and $tp(a_0b_0c_0c_1) = tp(a_1b_1c_1c_2) = tp(a_2b_2c_2c_0)$. Now it follows that $Lstp(d_0/c_0) = Lstp(d_0/c_0), Lstp(d_0/c_1) = Lstp(d_1/c_1)$ and $Lstp(d_1/c_2) = Lstp(d_2/c_2)$. However it is easy to see that $Lstp(d_0/c_0c_1), Lstp(d_1/c_1c_2), Lstp(d_2/c_2c_0)$ have no common realization.

In above example, $\{c_0, c_1, c_2\}$ can be considered as a base triangle, and $Lstp(d_0/c_0c_1), Lstp(d_1/c_1c_2), Lstp(d_2/c_2c_0)$ form other 3 triangles attached to the base triangle. The example shows that even if the edges of the 3 triangles are compatible over the base vertices, there is no common vertex joining the 3 triangles. On the other hand, due to the nature of the random graph if we only work in the home-sort, then any desired 3 types attached on a base triangle with compatible edges will be realized. As we want the notion of higher amalgamation to be preserved in interpreted theories, Kolesnikov suggests, in his revised works [12] [13], the following as higher amalgamation which we call here $K(n)$-amalgamation. We briefly explain the notation. In this paper, strong type indeed means Lascar strong type. Likewise, $p \in S_L(A)$ means $p$ is a Lascar strong type over $A$, and for $B \subseteq A, p|_L B$ (or simply $p|_B$) denotes $Lstp(a/B)$ for any (some) $a \models p$. Note that for $q \in S_L(B), q \subseteq p$ means $p|_B = q$ or equivalently $p \models q$.

Definition 1.2.  
- We say strong types $p_i \in S_L(A_i)$ are compatible over $A(\subseteq A_i)$ if each $p_i$ does not fork over $A$ and for $i, j, p_i|_L A_i \cap A_j = p_j|_L A_i \cap A_j$. (Hence $p_i|_L A = p_j|_L A$.) We say these $A$-compatible strong types $p_i$ are (generically) amalgamated if there is $q \in S_L(\bigcup_i A_i)$ nonforking over $A$ such that $\cup_i p_i \subseteq q$ (i.e. $q \models p_i$).
- We say $T$ has $K(n)$-amalgamation over $B$ if for $B$-independent $A = \{a_1, ..., a_n\}$ and any $B$-compatible $p_i \in S_L(BA_i)$ where $A_i = A \setminus \{a_i\} \ (i = 1, ..., n)$, whenever for $a \models p_i|_L B (= p_i|_L A)$ bdd$(aB) \subseteq \text{cl}(aB)$, then $p_1 \cup \ldots \cup p_n$ is generically amalgamated. We say $T$ has $K(n)$-amalgamation if it has $K(n)$-amalgamation over an arbitrary set.

The mend is that the realizations of strong types need be boundedly closed over the parameter set, i.e. in above $\text{bdd}(aB) \subseteq \text{cl}(aB)$. Note that $K(2)$-amalgamation is equivalent
to 3-amalgamation (usual amalgamation), and due to weak elimination of imaginaries, it can now be seen that the random graph has $K(n)$-amalgamation for all $n$. Each stable theory has $K(n)$-amalgamation as well, by stationarity.

However when we use inductive arguments for example, often we need to mind not only bounded closures of vertices of amalgamated types but also those of higher dimensional surfaces as well, since after naming parameters the surface dimension is increasing. Indeed, there exists in the literature another notion of amalgamation, called $\mathcal{P}(n)^{-}$-amalgamation, which was introduced by Hrushovski [6] prior to Kolesnikov’s work. In the notion, above concern is already taken care of. Moreover differently from $K(n)$-amalgamation (or the statement of the independence theorem), the base simplex is not regarded as an embedded parameter, but another type to be amalgamated. We think this is conceptually more correct and we shall take it to be $n$-amalgamation.

**Definition 1.3.** Let $I = \mathcal{P}(n)^{-} (= \mathcal{P}(n) \setminus \{n\})$, ordered by inclusion. Let $\{\{A_i\}_{i \in I}, \{\pi^j_i\}_{i \leq j \in I}\}$ be a directed family. Namely, each $\pi^j_i : A_i \rightarrow A_j$ is an elementary map between the two sets, and for $i \leq j \leq k \in I$, $\pi^j_k \circ \pi^i_j = \pi^k_j$ and $\pi^i_i = id$. We say $T$ has $\mathcal{P}(n)^{-}$-amalgamation, or simply $n$-amalgamation if whenever for any $u \in I$,

1. $\{\pi^j_i(A_{(i)}) : i \in u\}$ is $\pi^0_u(A_0)$-independent,
2. $A_u = bdd(\cup_{i \in u} \pi^j_u(A_{(i)}))$,

then we can extend the direct family to the one indexed by $\mathcal{P}(n)$ (by finding $A_n$ and $\pi^j_n$) so that (1),(2) hold for $n$ too. We say $T$ has $\mathcal{P}(n)^{-}$-amalgamation ($n$-amalgamation) over $A$, if $A_0 = bdd(A)$.

Since the definition is not transparent to conceptualize with the above notation, we give a rewritten definition as in [2] or [7]. Recall that when we say a hyperimaginary $b = \bar{a}/E$ realizes a type $r$ over $d = \bar{c}/F$, we mean $r = r(\bar{x})$ is a (real) type such that i) $r(\bar{a})$; ii) whenever $E(e, e')$, then $r(\bar{e})$ iff $r(\bar{e'})$; iii) $r(\bar{a}')$ if $\bar{a}'/E = f(b)$ for some $d$-automorphism $f$. If additionally the converse of iii) holds, we call $r$ a complete type of $b$ over $d$.

**Definition 1.4.** We say $T$ has $n$-complete amalgamation over a set $B$ if the following holds: Let $W$ be a collection of subsets of $\{1, \ldots, n\} = u_n$, closed under subsets. For each $w \in W$, complete type $r_w(x_w)$ over $B$ is given where $x_w$ is possibly an infinite set of variables. Suppose that

1. for $w \subseteq w'$, $x_w \subseteq x_{w'}$ and $r_w \subseteq r_{w'}$.

Moreover for any $a_w \models r_w$,

2. $\{a_{(i)} : i \in w\}$ is $B$-independent,
3. $a_w$ is as a set $bdd(\cup_{i \in w} a_{(i)}B)$ (and the map $a_w \rightarrow x_w$ is a bijection).

Then there is a complete type $r_{u_n}(x_{u_n})$ over $B$ such that (1),(2),(3) hold for all $w \in W \cup \{u_n\}$. We say $T$ has $n$-complete amalgamation ($n$-CA) if it has $n$-complete amalgamation over any set.

We leave the reader to show that $T$ has $n$-CA over $B$ iff $T$ has $m$-amalgamation over $B$ for all $m \leq n$. The following can be freely used: For $B$-independent $A = \{a_1, \ldots, a_n\}$, $\{A_w \mid w \in \mathcal{P}(u_n)\}$ is a partition of $bdd(BA)$, where $A_w = bdd(\cup_{i \in w} A_i) \setminus \bigcup_{v \in \mathcal{P}(w)^{-}} bdd(\cup_{i \in v} A_i)$ for all $w \in \mathcal{P}(u_n)$.
is an elementary map \( g \) follows.

We can safely assume now that in particular have the same type over the set \( D \) for all \( n \). Many important simple structures also have \( n \)-CA for all \( n \) such as the random graph \( (1.6) \), every PAC-structure (over some parameter) \( [7] \), and ACFA \( [2] \).

In the recent work \([11]\), corrections of terminologies in \([12][13]\) in regard to \( n \)-CA are made. For instance, the definition of \( K(n) \)-simplicity is presented in terms of an infinite Morley sequence. Kolesnikov’s ideas in \([12]\) go through to show the equivalence of \( K(2) \)-simplicity and 4-amalgamation. (The equivalence of \( K(1) \)-simplicity and 3-amalgamation is the way of proving the independence theorem \([10]\).) Hence it is naturally conjectured that \( T \) being \( K(n) \)-simple and \( T \) having \( (n + 2) \)-CA are equivalent, for \( n > 2 \). However surprisingly, counterexamples are constructed. Then, the revised concept of \( n \)-simplicity (implying \( K(n) \)-simplicity) defined via a finite Morley sequence is shown to be equivalent to \( (n + 2) \)-CA for every \( n \).

The lemma 1.5 and 1.6.1.2 below essentially come from the proof of the generalized independence theorem \([2]\). We thank Zoe Chatzidakis for her explanation.

**Lemma 1.5.** Let \( T \) be stable.

1. Suppose that for a set \( C \), whenever \( a \upharpoonright C \) \( b \) with \( b = b_1 \cup \ldots \cup b_n \), then \( dcl(acl(ab_1C)\ldots acl(ab_nC)) \cap acl(bC) = dcl(acl(b_1C)\ldots acl(b_nC)) \) (\( \sharp \)). Then the following are satisfied.
   (a) \( tp(acl(ab_1C)\ldots acl(ab_nC)/acl(b_1C)\ldots acl(b_nC)) \) is stationary.
   (b) Let \( A = \{a_1, \ldots, a_n\} \), \( B = \{c_1, \ldots, c_n\} \) be \( C \)-independent, respectively. For \( 1 \leq i \leq n \), let \( v_i = \{1, \ldots, n\} \setminus \{i\} \). Now given \( k \leq n \), assume there is a bijective map \( h : \cup_i 1 \leq k acl(a_i, C) \rightarrow \cup_i 1 \leq k acl(c_i, C) \) where \( a_{v_i} = \{a_j \mid j \in v_i\} \) such that \( h(a_i) = c_i \), \( h[C] = id \) and, for each \( v_i \), \( h[acl(a_{v_i}, C)] \) is elementary. Then \( h \) is \( C \)-elementary.

2. In fact, the condition (\( \sharp \)) holds when the set \( C \) is a universe of a model \( M \). (Hence \((1)(a)\), \( (b) \) also are true over a model.)

**Proof.** We can safely assume \( a, a_i, b_i, c_i \) are finite tuples from \( M^{eq} = M \).

(1)(a) is immediate from (\( \sharp \)). (Recall that \( cb(c/d) \subseteq dcl(cd) \cap acl(d) \).) (1)(b) For \( 1 \leq i \leq k \), let \( h_i = h[acl(a_i, C)] \). Then let \( h = h_1 \cup \ldots \cup h_j \) \( (h^k = h) \) and \( D^a_i = dom(h^i) = \cup_{i=1, \ldots, j} acl(a_i, C) \) and \( D^c_i = ran(h^i) = \cup_{i=1, \ldots, j} acl(c_i, C) \). For induction, assume \( h_{j-1} \) is elementary \((1 < j) \). We shall show \( h_j \) is elementary too. Now for each \( i < j \) let \( w_i = v_i \cap w_i \). Then \( a_{v_i} = \{a_j \cup w_i\} \). Now since \( h_j \) is elementary, there is an automorphism \( \hat{h}_j \) extending \( h_j \). Then by induction, \( \hat{h}(D^a_{j-1}) \) and \( D^c_{j-1} \) have the same type via \( h_j^{-1} \circ \hat{h}^{-1} \), in particular have the same type over the set \( \cup_{i=1, \ldots, j-1} acl(w_i, C) \) fixed by \( h_j^{-1} \circ \hat{h}^{-1} \). Note now that \( c_j \upharpoonright C \) \( c_1 \cup c_2 \cup \ldots \cup c_{j-1} \) and \( c_v = \{c_j \cup w\} \). Hence we can apply (1)(a) to conclude that \( \hat{h}(D^a_{j-1}) \) and \( D^c_{j-1} = \cup_{i=1, \ldots, j-1} acl(c_i, C) \) also have the same type over \( acl(c_v, C) \), i.e. there is an elementary map \( g \) sending \( \hat{h}(D^a_{j-1}) \) to \( D^c_{j-1} \) fixing \( acl(c_{v_j}, C) = ran(h_j) \). Therefore it follows \( h \subseteq g \circ \hat{h} \) is elementary.
(2) It suffices to show for \( e \in acl(ab_1M)\ldots acl(ab_nM) \cap acl(bM) \), \( e \in acl(ab_1M)\ldots acl(b_nM) \). Since \( e \in acl(ab_1M)\ldots acl(ab_nM) \), there are \( e_1\ldots e_n \) and \( L(M) \)-formulas \( \varphi(x; y_1\ldots y_n) \), \( \psi_i(y_i, zw_i) \) with \( \varphi(e; e_1\ldots e_n) \), \( \psi_i(e, ab_i) \) such that \( \models \varphi(u; v) \) implies \( u \) is definable over \( vM \), and \( \psi_i(u', v') \) implies \( u' \) is algebraic over \( v'M \). Therefore
\[
\models \exists y_1\ldots y_n (\varphi(e, y_1\ldots y_n) \land \bigwedge_i \psi_i(y_i, ab_i)).
\]
Now since \( e \in acl(bM) \), \( a \perp_M eb \) and so \( tp(a/Meb) \) is a coheir extension of \( tp(a/M) \). Thus we have \( m \in M \) such that
\[
\models \exists y_1\ldots y_n (\varphi(e, y_1\ldots y_n) \land \bigwedge_i \psi_i(y_i, mb_i)).
\]
Hence \( e \in acl(ab_1M)\ldots acl(b_nM) \). \qed

**Proposition 1.6.**

(1) Let \( T \) be stable. If a set \( C \) satisfies \((\ddagger)\) in 1.5.1, then for each \( n \), \( T \) has \( n \)-CA over \( C \).

(2) All stable theories have \( n \)-CA over a model.

(3) The random graph has \( n \)-CA over any set.

**Proof.** (1) In a stable theory \( T \) we can work in \( M^{eq} \) and substitute algebraic closures for bounded closures. We use the notation in 1.4. It suffices to show the case \( W = \mathcal{P}(u_n)^- \) with the corresponding types \( r_w(x_w)(w \in W) \). Again for \( 1 \leq i < k \leq n \), let \( v_i = \{1, \ldots, n\} \setminus \{i\} \) and \( w_i = v_k \cap v_i \). We shall show that \( \cup_{1 \leq i \leq n} r_{v_i} \) is consistent and realized by \( a_{v_1}, \ldots, a_{v_n} \) such that \( \{a_{v_1}, \ldots, a_{v_n}\} \) is \( B \)-independent. (Then the type of its algebraic closure over \( B \) extending \( \cup_{1 \leq i \leq n} r_{v_i} \) is the desired \( r_{u_n}(x_{u_n}) \).) Now due to usual amalgamation there is \( a_{v_1}a_{v_2} \models r_{v_1} \cup r_{v_2} \) such that \( \{a_{v_1}, \ldots, a_{v_n}\} \) is \( B \)-independent. Then for induction, assume that there is \( a_{v_1}\ldots a_{v_{k-1}} \models r_{v_1} \cup \ldots \cup r_{v_{k-1}} (2 < k) \) such that \( a_{v_1}\ldots a_{v_{k-1}} \) extends \( \{a_{v_1}, \ldots, a_{v_n}\} \). Now let \( b_{v_k} \models r_{v_k} \). Then there is a map \( h : \cup_{1 \leq i < k} b_{v_i} \rightarrow \cup_{1 \leq i < k} a_{w_i} \) such that \( h \) sends \( b_{v_i} \) to \( a_{w_i} \). Hence by 1.5.1(b), \( h \) is elementary and hence extended to an automorphism \( \hat{h} \). Then we have \( a_{v_k} = \hat{h}(b_{v_k}) \models r_{v_k} \). Now then \( a_{v_1}\ldots a_{v_k} \) realizes \( r_{v_1} \cup \ldots \cup r_{v_k} \) if for \( y = x_{v_k} \cap (x_{v_1} \cup \ldots \cup x_{v_{k-1}}) \), \( a_{v_1}\ldots a_{v_{k-1}}[y = a_{v_k}]y \). But this clearly holds since from the remark after 1.4, \( y = x_{w_1} \cup \ldots \cup x_{w_{k-1}} \). This finishes the proof of (1).

(2) It follows from 1.5.2 and (1) above.

(3) Note that for the random graph \( M = (\bar{M}, R) \), we can work in \( M^{eq} \) and substitute algebraic closures for bounded closures. Now since the random graph has weak elimination of imaginaries, for any \( A \) there is \( A' \) in the home sort \( \bar{M} \) such that \( acl(A) = dcl(A') \). Hence when we check \( n \)-CA of 1.4, we can assume each \( r_w \) is a type of a set in \( \bar{M} \). Then in \( \bar{M} \), since \( tp(A/B) \) is determined by equality and \( R \) relations of pairs in \( A \cup B \), due to randomness of \( R \) we clearly have the desired unifying type of a set in \( \bar{M} \). \qed

However, there is a stable theory which does not even have 4-amalgamation over an algebraically closed set. We thank Ehud Hrushovski for supplying us with this example.

**Example 1.7.** Let \( A \) be an infinite set with \( |A|^2 = \{\{a, b\}|a, b \in A, a \neq b\} \), and let \( B = [A]^2 \times \{0, 1\} \) where \( \{0, 1\} = \mathbb{Z}/2\mathbb{Z} \). Also let \( E \subseteq A \times [A]^2 \) be a membership relation, and let \( P \) be a subset of \( \mathbb{B}^3 \) such that \( ((w_1, \delta_1)(w_2, \delta_2)(w_3, \delta_3)) \in P \) iff there are distinguish \( a_1, a_2, a_3 \in A \) such that for \( \{i, j, k\} = \{1, 2, 3\} \), \( w_i = \{a_j, a_k\} \), and \( \delta_1 + \delta_2 + \delta_3 = 0 \). Now let \( M \) be a model with the 3-sorted universe \( A, [A]^2, B \) equipped with relations \( E, P \) and the projection \( f : B \rightarrow [A]^2 \). Then since \( M \) is a reduct of \( (A, \mathbb{Z}/2\mathbb{Z})^{eq} \), \( M \) is stable. We work in \( M \) and show \( M \) does not have \( \mathcal{P}(A)^{-}\)amalgamation. Note first that \( acl(\emptyset) = acl(\emptyset) \), and for \( a \in A \), \( acl(a) = \)}
Therefore we have complete types \( r_{ij} = tp(a_i a_j) \), and let \( x_{ij} \) be the variable for \( b_{ij} \). Note that \( b_{ij} = (a_i, a_j, \delta) \) and \( b'_{ij} = (a_i, a_j, \delta + 1) \) have the same type over \( a_i a_j \). Hence there is \( (a_i a_j)' = (b'_{ij}, \ldots) \) also realizing \( r_{ij}(x_{ij}) \). Therefore we have complete types \( r_{ijk}(x_{ijk}) \), \( r'_{ijk}(x'_{ijk}) \) both extending \( r_{ij}(x_{ij}) \cup r_{ik}(x_{ik}) \cup r_{jk}(x_{jk}) \) realized by some enumerations of \( acl(a_i a_j a_k) \) such that, respectively, \( P(x_{ij} x_{ik} x_{jk}) \in r_{ijk} \) where \( \neg P(x_{ij} x_{ik} x_{jk}) \in r'_{ijk} \). Then it is easy to see that \( r_{123} \cup r_{124} \cup r_{134} \cup r'_{234} \) is inconsistent.

In the example, \( (\{a_2, a_3\}, 0) \in dcl(acl(a_1 a_2) \cup acl(a_1 a_3)) \), since \( (\{a_2, a_3\}, 0) \) is a unique solution to \( P((\{a_1, a_2\}, 0), (\{a_1, a_3\}, 0), x) \). But \( (\{a_2, a_3\}, 0) \notin acl(acl(a_2) \cup acl(a_3)), \) i.e. 1.5.1(\( \sharp \)) does not hold over an algebraically closed set. In [8], Hrushovski shows that if a stable \( T \) eliminates generalized finite imaginaries then \( T \) has 4-amalgamation.

**Proposition 1.8.** If \( T \) has 4-amalgamation over \( B \), then it has \( K(3) \)-amalgamation over \( B \).

**Proof.** Assume \( T \) has 4-amalgamation. Now suppose that \( B \)-independent \( A = \{a_1, a_2, a_3\} \) and \( B \)-compatible \( p_i \in S_L(BA) \) where \( A_i = A \setminus \{a_i\} \) \( (i = 1, 2, 3) \) are given. Also for \( d_i = p_i[eB], bdd(d_i B) \subseteq dcl(d_i B) \) (*). Now let \( r_{01}(x_0) = tp(bdd(B B)) \) and let \( r_{i1}(x_i) = tp(bdd(a_i B)/B), r_i(x_i) = tp(bdd(d_i B)/B) \) extending \( r_{01}(x_0) \). Now we have \( r_{14}(x_{14}) = tp(bdd(a_1 d_2 B)/B) \) extending \( r_{11} \cup r_{12} \) since due to independence \( x_1 \cap x_0 = x_6 \). Let \( f_1 : bdd(a_1 d_2 B) \rightarrow x_{14} \) be the realization map. Now note that due to compatibility of types \( p_i \), for \( i \in \mathbb{Z}/3\mathbb{Z} \), there is an automorphism \( h_i \) sending \( d_{i2} \) to \( d_{i1} \) fixing \( bdd(a_1 B) \). Then due to (*), \( h_{i+2} \circ h_i \circ bdd(d_{i+2} B) = h_{i+1}^{-1} \circ bdd(d_{i+2} B) \) (**). Now via \( f_1 \circ h_1 : bdd(a_1 d_3 B) \rightarrow x_{14}, bdd(a_1 d_3 B) \models r_{14}(x_{14}) \). Then there is \( r_{24}(x_{24}) = tp(bdd(a_2 d_3 B)/B) \) extending \( r_{2}(x_2) \cup r_4(x_4) \) since also \( x_{14} \cap x_2 = x_6 \). Thus by the map \( f_2 \circ h_2 : bdd(a_2 d_1 B) \rightarrow x_{24} \) where \( f_2 : bdd(a_2 d_3 B) \rightarrow x_{24}, bdd(a_2 d_1 B) \models r_{24}(x_{24}) \). Note that \( f_2 \circ h_1 : bdd(d_3 B) = f_1 \circ h_1 \circ bdd(d_3 B) \). We too have \( r_{34}(x_{34}) = tp(bdd(a_3 d_1 B)/B) \) extending \( r_3(x_3) \cup r_4(x_4) \). Let \( f_3 : bdd(a_3 d_1 B) \rightarrow x_{34} \). Note again that \( f_3 \circ h_3 : bdd(a_3 d_2 B) \rightarrow x_{34} \) extends \( f_1 \circ bdd(d_2 B) : bdd(d_2 B) \rightarrow x_4 \) since from (**), on \( bdd(d_2 B) \), \( f_3 \circ h_3 = (f_2 \circ h_2) \circ h_3 = (f_1 \circ h_1 \circ h_2) \circ h_3 = f_1 \). Therefore \( f_1 \cup f_3 \circ h_3 : bdd(a_1 d_3 B) \cup bdd(a_3 d_2 B) \rightarrow r_{14}(x_{14}) \cup r_{34}(x_{34}) \) is a well-defined realization map extending the realizations of \( r_j(x_j) \) \( (j = 1, 3, 4) \). Then now it is easy to find additional types \( r_w \) so that they satisfy (1),(2),(3) of 1.4 for \( n = 4 \). Therefore by 4-amalgamation we have \( d(\equiv_B d_i) \) such that \( \{a_1, a_2, a_3, d\} \) is \( B \)-independent and the type of \( bdd(a_1 a_2 a_3 d B)/B \) extends types \( r_w \). Obviously, \( d \) is the generic realization of \( p_1 \cup p_2 \cup p_3 \).}

The main theme of this paper is finding the canonical group from the group configuration, a generalization of the group configuration theorem of stable theories into the simple context. We succeed in obtaining the hyperdefinable group from the group configuration under 4-amalgamation. What we are going to use is 4-amalgamation over a parameter properly containing a model (See the proof of 2.6). But as indicated even a stable theory need not have such a property. Hence to make it work in more general context, we introduce the notion of model-\( n \)-CA, a little variation of \( n \)-CA.

**Definition 1.9.** We say \( T \) has model-\( n \)-complete amalgamation if the following holds: Let \( u_n = \{1, \ldots, n\} \), and \( W_n = \mathcal{P}(u_{n+1}) \setminus \{u_n\} \). Let \( W \) be a collection of subsets of \( W_n \), closed
under subsets. For each $w \in W$, complete type $r_w(x_w)$ over a model $M$ is given where $x_w$ is possibly an infinite set of variables. Suppose that

(1) for $w \subseteq w'$, $x_w \subseteq x_{w'}$ and $r_w \subseteq r_{w'}$.
Moreover for any $a_w \models r_w$,

(2) $\{a_{i}\mid i \in w\}$ is $M$-independent,

(3) $a_w$ is as a set $\text{bdd}(\cup_{i \in w} a_{i})$ (and the map $a_w \rightarrow x_w$ is a bijection).

Then there is a complete type $r_{u_{n+1}}(x_{u_{n+1}})$ over $M$ such that (1), (2), (3) hold for all $w \in W \cup \{u_{n+1}\}$.

Each of stability, $(n+1)$-CA over models, or $n$-CA implies model-$n$-CA for every $n$. Model-$n$-CA also holds in aforementioned algebraic examples such as ACFA and PAC-structures. Model-4-CA is the property we shall use, hence covers the case that $T$ is stable.

2. THE GROUP CONFIGURATION

**Definition 2.1.** By a group configuration we mean a 6-tuple of hyperimaginaries $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ over a hyperimaginary $e$ such that, for $\{i, j, k\} = \{1, 2, 3\}$,

(1) $f_i \in \text{bdd}(f_j, f_k; e)$,
(2) $x_i \in \text{bdd}(f_j, x_k; e)$,
(3) all other triples and all pairs from $C$ are independent over $e$.

If the group configuration $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ over $e$ has the property that $\text{bdd}(f_i; e) = \text{bdd}(c b(x_i, x_k; f_i; e); e)$, we call such $C$ a bounded quadrangle. If additionally $x_i, x_j$ are inter-definable over $f_k e$, then we call $C$ a definable quadrangle over $e$.

![Diagram of group configuration]

**Fact 2.2.** (1) If $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ is a group configuration/bounded quadrangle over $e$ and $\text{bdd}(f_i e) = \text{bdd}(f_i' e)$, $\text{bdd}(x_i e) = \text{bdd}(x_i' e)$, then $C' = (f'_1, f'_2, f'_3, x'_1, x'_2, x'_3)$ is also a group configuration/bounded quadrangle over $e$. In this case, we say $C$ and $C'$ are (boundedly) equivalent over $e$.

(2) For $C$ a group configuration/bounded quadrangle over $e$ and $e' \supseteq e$, if $C \models e e'$ then $C$ also is a group configuration/bounded quadrangle over $e'$.

(3) In a group configuration $(f_1, f_2, f_3, x_1, x_2, x_3)$ over $e$ even if we replace $f_i$ by $f'_i = c b(x_j, x_k; f_i; e_i)$ for $\{i, j, k\} = \{1, 2, 3\}$, $(f'_1, f'_2, f'_3, x_1, x_2, x_3)$ is still a group configuration (hence a bounded quadrangle) over $e$.

**Proof.** We sketch the proof. (1) Obvious for a group configuration. For a bounded quadrangle notice that in general $cb(a_i/a_2)$ and $cb(b_1/b_2)$ are interbounded as far as $a_i, b_i$ are interbounded. (2) Easy. (3) Since $x_i x_j \downarrow f_k f_k e$ and $x_i x_j f_k e$, $x_i, x_j$ are interbounded over $f_k e$. On the other hand, $x_j \downarrow f_k e f_i$ implies $x_i x_j \downarrow f_k f_i f_j$ (**), $x_i x_j \downarrow f_k f_i f_j$ and thus $x_i x_j x_k \downarrow f_k f_i f_j f_k e$ and $x_i x_j x_k \downarrow f_k f_i f_j f_k e$. Then from (**), it follows $x_i x_k \downarrow f_k f_i f_j f_k e$.
and from (*) \( x_i x_j \downarrow_{f_i f_j} f_i f_j e \). Therefore \( f'_k = cb(x_i x_j / f_k e) = cb(x_i x_j / f_i f_j e) \in bdd(f'_i f'_j) \). Other independences over \( e \) come easily. □

From now on, assume that a group configuration over \( \dot{e} = A / \bar{E} \) is given. We shall produce the non-trivial canonical hyperdefinable group from it. By above 2.2.3, we can replace it by a bounded quadrangle \( C = (f, g, \dot{h}, \dot{a}, \dot{b}, \dot{c}) \) over a model \( M \) containing \( A \). After naming \( M \), we freely assume that \( \emptyset = bdd(\emptyset) \). We further suppose \( f, g, \dot{h}, \dot{a}, \dot{b}, \dot{c} \) are all boundedly closed (by extending each to its bounded closure, if necessary.) Clearly \( C \) still is a bounded quadrangle over \( \emptyset \). Let \( p = tp(f) (= \text{Lstp}(f)), q = tp(g), r = tp(h) \) and let \( \Gamma_q(uv) = q(u) \land q(v) \land u \downarrow v \).

(Later we shall omit \( q \) in \( \Gamma_q \).) Now we can think of \( \dot{h} \) as a multi-valued function such that \( \text{dom}(\dot{h}) = \text{tp}(\dot{a} / \dot{h}) = \text{Lstp}(\dot{a} / \dot{h}) \) and \( r q(\dot{h}) = \text{Lstp}(\dot{b} / \dot{h}) \). More precisely \( b \in k_r(a) \) means \( k_r \models r, k_r a \equiv \dot{h} b \). Similarly we write \( a \in h_q(c), b \in g_p(c) \) for \( h_q ca \models \dot{g} \dot{a} c, g_p cb \models \dot{f} \dot{b} c \), respectively. In the same way, \( b \in \text{dom}(f_p) \equiv \exists c(f_p bc \models \text{tp}(\dot{f} \dot{b} c)), \) and so on.

We say a set \( A \) is \( n \)-independent if any subset of \( A \) having \( n \) elements is independent. Now we define \( R = R^q \) to be a symmetric type-definable relation over \( \emptyset \) on the set of independent realizations of \( q \) such that

\[
R(f g; f' g') \text{ iff } f, g, f', g' \models q, \{f, g, f', g'\} \text{ 3-independent, and there are } b \text{ and } a \downarrow f g f' g' \text{ such that } f(a) \cap g(b) \neq \emptyset, f'(a) \cap g'(b) \neq \emptyset.
\]

It is easy to see that \( a \downarrow f g f' g' \) above can be replaced by \( b \downarrow f g f' g' \). Similarly, one can define \( R^p, R^{pq} \) by replacing \( f, g, f', g' \models q \) by \( f, g, f', g' \models p \) or \( f, f' \models p, g, g' \models q \), respectively.

**Lemma 2.3.**

(1) If \( fg \models \Gamma_q \), and \( c \in f(a) \cap f(b) \) with \( c \downarrow fg \), then

\begin{enumerate}
  \item (a) \( f, g \) are interbounded over \( e := bc(ba / fg) \), and
  \item (b) \( e \downarrow f, e \downarrow g \).
\end{enumerate}

(2) (1) still holds when we replace \( f, g \models q \) by \( f, g \models p \), or \( f \models p, g \models q \).

(3) If \( \langle fg, f'g' \rangle \models R \) (or \( R^p, R^{pq} \)), then any element in \( \{f, g, f', g'\} \) is in the bounded closure of the other 3 elements.

**Proof.** (1)(a) Note that from \( ab \downarrow e fg \) and that \( a, b \) interbounded over \( fg \), it follows that \( a, b \) are interbounded over \( e \), too (*). Now from \( c \downarrow f g, ca \downarrow g fe \). Moreover from \( c \downarrow fg, c \downarrow fe \) and then by (*), \( ca \downarrow e \). Hence \( g \models bdd(cb(ca / g)) \subseteq bdd(fe) \). By a similar argument, \( f \in bdd(ge) \) can be shown too.

(1)(b) There are \( h_1 u_1, h_2 u_2 \) such that \( h_1 f bu_1 c \), \( h_2 gau_2 c \models \dot{f} \dot{g} \dot{a} \dot{b} \dot{c} \). Then since \( c \) is boundedly closed, by amalgamation we have

\[
hu \models \text{tp}(h_1 u_1 / cbf) \cup \text{tp}(h_2 u_2 / cag).
\]

such that \( \{u, c, f, g\} \) independent. Then we have \( k, k' \) such that \( h g k a u c, h f k' b u c \models \dot{f} \dot{g} \dot{a} \dot{b} \dot{c} \).

From \( c \downarrow f g h \), we have \( ba \downarrow f g k k' \). From \( b \downarrow f g h \) and \( u, a \in bdd(k k' b) \), it follows \( ba \downarrow k k' f g \) and thus \( e \in bdd(k k') \) (†). On the other hand, \( f \downarrow h g \) implies \( k' \downarrow h g k \) and \( k' \downarrow g k \). Hence \( \{g, k, k'\} \) is independent. Similarly \( \{f, k, k'\} \) is independent. Then from (†), \( e \downarrow f, e \downarrow g \).

(2) Similar to (1).

(3) There are \( c, c', b \) and \( a \downarrow f g f' g' \) such that \( c \in f(a) \cap g(b), c' \in f'(a) \cap g'(b) \). Hence \( ab \downarrow f g f' g' \) and \( ab \downarrow f' g f g \). Therefore \( e = cb(ab / fg) \) and \( e' = cb(ab / f' g) \) are interbounded (**). From (1)(a), \( f, g \) are interbounded over \( e \), and so are \( f', g' \) over \( e' \). Hence it follows from (**), \( g' \in bdd(f' e') \subseteq bdd(f' f g) \) and similarly for the other relations. □
The proof of 2.3.1(b) above is essentially due to Frank O. Wagner.

**Lemma 2.4.** (1) For $fg, f'g' \models \Gamma_q$, $R(fg, f'g')$ iff there are $b$ and $a \downarrow fgf'g'$ such that $f(a) \cap g(b) \neq \emptyset$, $f'(a) \cap g'(b) \neq \emptyset$ and $fg \downarrow_e f'g'$ where $e = cb(ba/fg)$.

(2) Given independent $f, g \models q$, there are $f', g'$ such that $R(fg, f'g')$.

(3) Above (1)(2) hold for $R^a$, $R^{ap}$, as well.

**Proof.** (1) (⇒) Note that since $a \downarrow fgf'g'$, $ba \downarrow fg f'g'$, $ba \downarrow f'g' f g$. Hence $e, e' = cb(ba/f'g')$ are interbounded. Then due to 2.3.1(a), $f', g'$ are interbounded over $e$. Now from $fg \downarrow f'$, $fg \downarrow e f'$, thus $fg \downarrow e f'g'$.

(⇐) Again since $a \downarrow fgf'g'$, $e, e'$ are interbounded. Then from $fg \downarrow e f'g'$, equivalently $fg \downarrow e f'g'$, and 2.3.1(b), $\{f, g, f', g'\}$ is 3-independent.

(2) By amalgamation there is $c \in \text{ran}(f) \cap \text{ran}(g)$ such that $c \downarrow fg$. Choose $a \in g^{-1}(c), b \in f^{-1}(c)$. Then, by the extension axiom, we have $f'g'$ such that $\{ab, fg, f'g'\}$ is $e$-independent where $e = cb(ba/fg)$ and $f'g' \equiv_{ab e} fg$. Then the right hand side of (1) follows easily.

(3) Clear.

The following lemma is crucial to our argument.

**Lemma 2.5.** Let $R(fg, f'g')$. Namely, $\{f, g, f', g'\}$ 3-independent, and we can find $d$ and $a \downarrow fgf'g'$ such that $c \in f(a) \cap g(d), c' \in f'(a) \cap g'(d)$. Then there are $h, h' \models p$ and $b$ such that $c \in h(b), c' \in h'(b), b \downarrow hh'ffg'g'$, $\{f, h, f', h'\}$ is 3-independent, and $hh' \downarrow ff'gg'$. It follows $R^{ap}(hf, hh')$ and $R^{ap}(hg, hh')$.

**Proof.** Note since $fac, f'ac' \models \hat{g}a\hat{c}$, there are $k_0, b_0; k_1, b_1$ such that $f'k_0ab_0c', fk_1ab_1c \models \hat{g}\hat{h}a\hat{b}\hat{c}$. Then by amalgamation, we have $kb, hh'$ such that

$$kb \models tp(k_0b_0/af') \cup tp(k_1b_1/af), kb \downarrow_a ff' \text{ and } h'f'kabc', hfkabc \models \hat{f}\hat{g}\hat{h}a\hat{b}\hat{c}.$$  

We can further assume that $hh' \downarrow ff'gg'$. Then it follows from $ab \downarrow_k ff'$, $b \downarrow_k f'hh'$ and $ab \downarrow_{hh'ff} gg'$, $b \downarrow ff'gg'hh'$. Moreover from $f \downarrow_k ff'$, $hf \downarrow_k h'f'$, $\{f, h, f', h'\}$ is 3-independent. Then since $hh' \downarrow ff'gg'$, $\{g, h, g', h'\}$ is 3-independent as well. Therefore $R^{ap}(hf, hh')$ and $R^{ap}(hg, hh')$.

Now define $R' = (R')^a$ by
$R'(fg; f'g')$ iff $f, g, f'g' \models q, \{f, g, f', g'\}$ 3-independent, and for any $a, b$ such that $f(a) \cap g(b) \neq \emptyset$ and $a \uparrow fg$, there are $a'b' \equiv_f g$ ab such that $a' \uparrow ffg'g'$, $f'(a') \cap g'(b') \neq \emptyset$.

Again we also define $(R')^p, (R')^{pq}$ by substituting $f, g, f'g' \models p$ or $f, f \models p, gg' \models q$, respectively.

We shall prove that $R$ and $R'$ are equivalent under 4-amalgamation. For the rest of this paper, we assume that $T$ has 4-CA, or more weakly model-4-CA. Note that clearly $R'$ implies $R$.

**Notation** For bounded closed sequences $a, b, c$, we use $\overline{abc}$ to denote some sequence of $\text{bdd}(abc)$ extending the orderings of $a, b, c$.

**Theorem 2.6.** $R$ and $R'$ are equivalent.

**Proof.** It shall show $R$ implies $R'$. Let $R(fg, f'g')$. We use Lemma 2.5 with the same notation. We then have $c', ab, hh'$ such that $c \in f(a) \cap h(b) \cap g(d)$; $c' \in f'(a) \cap h'(b) \cap g'(d)$; $b \uparrow fgh'f'g'h'$; $f' \in \text{bdd}(fhh'), g' \in \text{bdd}(ghh')$; and $\{f, g, h, h'\}$ is independent. Now let $\overline{hj}, \overline{hg}, \overline{hbc} = \text{bdd}(hhb) = \text{bdd}(hcb), \overline{fac} = \text{bdd}(fc) = \text{bdd}(fa), \overline{gdc} = \text{bdd}(gc) = \text{bdd}(gd)$ be sequences of boundedly closed sets extending the boundedly closed sequences $f, g, h, c, a, b, d$ (See Notation above 2.6). Since $hbc \equiv h'bc'$ we also have $\overline{hbc'} \equiv \overline{hbc}$.

Now, to show $R(fg, f'g')$, assume there are $a_1, d_1, c_1$ such that $c_1 \in f(a_1) \cap g(d_1)$ and $c_1 \uparrow fg$. We also have $f_a c_1 \equiv f_{ac} \equiv g_d c_1 \equiv g_{dc}$. Then by 4-amalgamation, there are $c_2, b_2, a_2, d_2$ such that

$$\overline{fa_2c_2 gdc_2 fg} \equiv \overline{fa_1c_1 gdc_1 f_g}; \overline{hf hbc_2} \equiv \overline{h c hbc}; \text{and} \overline{hg hbc_2} \equiv \overline{h g hbc} \; \text{(*)}.$$

What we are going to amalgamate next are the following strong types:

$$\text{Lstp}(\overline{hbc_2 fg}; h), \text{Lstp}(\overline{hbc'/h'} f'; h) \text{and} \text{Lstp}(\overline{hbc'h' g}; h).$$

Here the base parameter is $h$ (here is the point where we need model-4-CA, since indeed the parameter is $Mh$), and each realization is boundedly closed over the parameter. Each type does not fork over $h$. Additionally due to (*), it can be seen that the 3 strong types are $h$-compatible. Hence we have $\overline{hbc_2}$, a generic solution of the types. Moreover we have $\overline{hbc} = \text{bdd}(f, bc; h)$ extending $\overline{fac}, \overline{hbc}$. Also since $c' \in \text{bdd}(h', bc; h)$ we have $\overline{h h'bc'} = \text{bdd}(h', bc; h)$ extending $\overline{hbc, h'bc'}$. Similarly there is $\overline{gbdc} = \text{bdd}(g, bc; h)$ extending $\overline{hbc, gdc}$. Note that here 4-amalgamation indeed says that there exist elementary maps $\overline{h_1}, \overline{h_2}, \overline{h_3}$ with

$$\begin{align*}
\text{dom}(\overline{h_1}) &= \text{bdd}(f, bc; h) \text{bdd}(h', bc; h) \text{bdd}(f h'; h), \\
\text{dom}(\overline{h_2}) &= \text{bdd}(f, b_2c_2; h) \text{bdd}(g, b_2c_2; h) \text{bdd}(fg; h), \\
\text{dom}(\overline{h_3}) &= \text{bdd}(h', bc; h) \text{bdd}(g, bc; h) \text{bdd}(hh'; h),
\end{align*}$$

fixing $\text{bdd}(fh'; h), \text{bdd}(fg; h) \text{bdd}(gh'; h)$, respectively such that $\overline{h_1(hbc)} = \overline{h_2(hbc_2)} = \overline{h_3(hbc)} = \overline{hbc_3}$ the generic solution. Moreover they are compatible with elementary maps sending $\overline{hbc} \rightarrow \overline{ha_3b_3c_3}, \overline{gbdc} \rightarrow \overline{gb_3d_3c_3}$ and $\overline{h h'bc'} \stackrel{id}{\rightarrow} \overline{h h'bc'}$. (In particular, maps $\overline{h_1(hbd(h', bc; h)) = h_3(hbd(h', bc; h))}$. Hence there are $a_3, c_3, d_3$ such that $a_3 = h_1(a) = h_2(a_2), c_3 = h_1(c') = h_3(c'), d_3 = h_2(d_2) = h_3(d)$ and

1. $\overline{fha_3b_3c_3} \equiv_{\text{bdd}(f h'; h)} \overline{habc} \overline{hh'bc'}$;
2. $\overline{fha_3b_3c_3} \equiv_{\text{bdd}(fg; h)} \overline{fa_2b_2c_2 hgb_2d_2c_2}$;

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(3) \[ hh'bc_3c_3' hgbd_3c_3 \equiv_{bdd(gh';h)} hh'bc'd' hgbdc. \]

Then by (*) and (2), \( a_1d_1 \equiv_{f'g} a_2d_2 \equiv_{f'g} a_3d_3. \) Also since \( f' \in bdd(fh'h) \), from (1), \( c_3' \in f'(a_3) \cap h'(b_3) \) and \( c_3 \in f(a_3). \) Note that, since \( R(hg, h'g') \), \( g' \in bdd(gh'h) \). Then, similarly from (3), \( c_3' \in g'(d_3) \) and \( c_3 \in g(d_3). \) Therefore \( R'(fg, f'g') \). \( \square \)

### 3. Type-definability of the transitive closure of \( R \)

In this section, we use \( R \equiv R' \) (Theorem 2.6) to prove that, the transitive closure of \( R \) is type-definable. The proof is similar to the proof in [4] that the transitive closure of the relation \( \sim_1 \) forms a hyperimaginary canonical base, (or the improvement of this proof in [16, 3,3.1]).

Let \( \bar{R} \) be the transitive closure \( R \). We remark that if both \( \{a, b, c, d\}, \{a', b', c, d\} \) are 3-independent and \( ab \downarrow_{cd} a'b' \), then \( \{a, b, a', b'\} \) is also 3-independent.

**Lemma 3.1.** Suppose that \( R(fg, hk), R(hk, f'g') \) and \( fg \downarrow_{hk} f'g' \). Then \( R(fg, f'g') \) and \( fg \downarrow_{f'g'} hk \).

**Proof.** By the previous remark, \( \{f, g, f', g'\} \) is 3-independent (*). Now since \( R(fg, hk) \), there are \( b \) and \( a \) such that \( f(a) \cap g(b) \neq \emptyset, h(a) \cap k(b) \neq \emptyset \) (so \( ab \downarrow_{hk} fg \)). Then since \( R'(hk, f'g') \), there are \( a'b' \equiv_{hk} ab \) such that \( a' \downarrow_{hk} f'g' \), \( f'(a') \cap g'(b') \neq \emptyset. \) Hence by amalgamation, we have \( a''b'' \models \text{Lstp}(ab/hk, fg) \cup \text{Lstp}(a'b'/hk, f'g') \), and \( a''b'' \downarrow_{hk} fgf'g' \).

It follows then \( a'' \downarrow_{f} fgf'g' \) and \( f(a'') \cap g(b'') \neq \emptyset, f'(a'') \cap g'(b'') \neq \emptyset. \) This with (*) says \( R(fg, f'g') \). It remains to show \( fg \downarrow_{f'g'} hk \). Since \( g \downarrow_{hk} f'g' \), \( g \downarrow_{f'g'} hk \). Now by 2.3.3, \( f \in bdd(gf'g') \), and therefore \( fg \downarrow_{f'g'} hk \). The proof is finished. \( \square \)

**Theorem 3.2.** The following are equivalent.

1. \( \bar{R}(\bar{f}, \bar{g}) \).
2. For some \( \bar{h}, R(\bar{h}, \bar{f}) \) and \( R(\bar{h}, \bar{g}) \).
3. For some \( \bar{h} \) with \( \bar{h} \downarrow_{\bar{f}} \bar{g} \) and \( \bar{h} \downarrow_{\bar{g}} \bar{f}, R(\bar{h}, \bar{f}) \) and \( R(\bar{h}, \bar{g}) \).

**Proof.** It suffices to show (1) implies (3). We prove this by induction on the length of an \( R \)-chain. Note that 2.4.2 gives the induction step for length 0. Now assume that there are \( \bar{f}, \bar{f}_n, \bar{h} \) such that \( \bar{R}(\bar{f}, \bar{f}_n) \) with the \( R \)-chain length \( n \) and \( R(\bar{f}_n, \bar{h}) \). By the induction hypothesis for \( n \), there is \( \bar{h} \) such \( \bar{h} \downarrow_{\bar{f}} \bar{f}_n \) and \( \bar{h} \downarrow_{\bar{f}_n} \bar{f} \) (*). By extension, we can assume \( \bar{h} \downarrow_{\bar{f}_n} \bar{h} \) (**). Then by (*), \( \bar{h} \downarrow_{\bar{f}_n} \bar{h} \bar{f} \) (**). In particular, \( \bar{h} \downarrow_{\bar{f}_n} \bar{h} \). Hence from the lemma 3.1, \( R(\bar{h}, \bar{h}) \) and \( \bar{h} \downarrow_{\bar{h}} \bar{f}_n \). Then it follows from (**), \( \bar{h} \downarrow_{\bar{h}} \bar{f} \). Moreover again by (*)(**), we have \( \bar{h} \downarrow_{\bar{f}} \bar{h} \). Hence the induction step for \( n + 1 \) is shown. \( \square \)

### 4. The generic group operation on \( \Gamma/\bar{R} \)

Recall that in section 2, we define \( \Gamma(xy) = \Gamma_q(xy) = q(x) \land q(y) \land x \downarrow y \). Now since \( R \) is symmetric, clearly \( \bar{R} \) is an equivalence relation on \( \Gamma \). By putting \( (\bar{R}(\bar{x}, \bar{y}) \land \Gamma(\bar{x}) \land \Gamma(\bar{y})) \lor \bar{x} = \bar{y} \), we can extend \( \bar{R} \) to a type-definable equivalence relation on the whole universe. We shall find the canonical hyperdefinable group from the hyperdefinable generic group operation on \( \Gamma/\bar{R} \). First we state some more properties of \( R \) and \( \bar{R} \).
Lemma 4.1. Let \( \bar{f} = f_1 f_2, \bar{g} \models \Gamma \), and let \( e = \bar{f}/\bar{R} \).

1. \( \bar{R}(\bar{f}, \bar{g}) \) and \( \bar{f} \downarrow_\text{e} \bar{g} \) iff \( R(\bar{f}, \bar{g}) \).
2. For \( a, b \) such that \( f_1(a) \cap f_2(b) \neq \emptyset \) and \( a \downarrow \bar{f} \), \( e \) is interbounded with \( cb(ab/\bar{f}) \).

Proof. (1) \( (\Rightarrow) \) By 3.2, there is \( \bar{h} \) such that \( \bar{h} \downarrow \bar{f} \bar{g}, \bar{h} \downarrow \bar{g} \bar{f} \), \( R(\bar{h}, \bar{f}) \) and \( R(\bar{h}, \bar{g}) \). Then since \( \bar{f} \downarrow_\text{e} \bar{g}, \bar{f} \downarrow_\text{h} \bar{g} \), and then by 3.1, \( R(\bar{f}, \bar{g}) \).

\( (\Leftarrow) \) \( R(\bar{f}, \bar{g}) \) implies \( R(\bar{f}, \bar{g}) \). By the extension axiom, there is \( g' \models tp(g/e) \) such that \( g' \downarrow_\text{e} \bar{g} \) \( (\ast) \). Hence \( R(\bar{g}, g') \) and by the proof of \( (\Rightarrow) \), \( R(\bar{g}, g') \). Again, by extension, we can assume that \( f \downarrow_{e\bar{g}} g' \). Hence \( \bar{f} \downarrow g \bar{g}' \) and then by 3.1, \( \bar{f} \downarrow g \bar{g} \). Therefore by \( (\ast) \), \( \bar{f} \downarrow_\text{e} \bar{g} \).

(2) Let \( e_1 = cb(ab/f) \). By extension, there is \( \bar{f}' \models tp(f/abe_1) \) such that \( \bar{f} \downarrow_\text{abe_1} \bar{f}' \). Hence \( \{ab, \bar{f}, \bar{f}'\} \) is \( e_1 \)-independent and from 2.4.1, \( R(\bar{f}, \bar{f}') \) and \( e_1 = cb(ab/\bar{f}') \). Then by (1), \( \bar{f} \downarrow_\text{e} \bar{f}' \) \( (** \ast) \). Let \( e_2 = cb(\bar{f}/\bar{f}') \). Then due to \( (**) \), \( e_2 \in bdd(e) \). Moreover since \( \bar{f} \downarrow_\text{e} \bar{f}', e \downarrow_\text{e} \bar{f}', \) and \( e \in dcl(\bar{f}') \), \( e_2 \in bdd(\bar{f}') \), we have \( e \in bdd(e_2) \). Thus \( bdd(e) = bdd(e_2) \). Similarly since \( \bar{f} \downarrow_\text{e}\bar{f}' \), it can be too seen \( bdd(e_1) = bdd(e_2) \). Therefore \( bdd(e_1) = bdd(e) \). \( \Box \)

The proof of the following lemma uses 4-CA.

Lemma 4.2. Let \( R(gh, uw) \). Then for any \( c \in g(a), d \in v(a) \) with \( a \downarrow g v \), there are \( c'a'd' \equiv_{gv} \) \( cad \) and \( b' \) such that \( a' \downarrow gh uw, c' \in h(b'), d' \in w(b') \).

Proof. By 2.5, there are \( a_0, b_0, c_0, d_0, t_0 \) and \( f, u \) such that \( c_0 \in g(a_0) \cap f(t_0) \cap h(b_0), d_0 \in v(a_0) \cap u(t_0) \cap \{f, g, u, v\} \) is 3-independent, and \( f u \downarrow_{gv} \) \( hw \). It follows \( R^{m}(fh, uw), R^{m}(fg, uv) \) and \( g u \downarrow_{fu} hw \) \( (*) \). Let \( e = bdd(cb(t_0a_0/fg)) \). Then from 2.4.3, \( e = bdd(cb(t_0a_0/uv)) \) and \( fg \downarrow_\text{e} \) \( uw \). Similarly for \( k = bdd(cb(t_0b_0/fg)), \) \( fh \downarrow_\text{k} \) \( uw \). Now let \( vac_0a = bdd(ea_0b_0) = bdd(ea_0), vac_0 = bdd(egf), va_0d_0 = bdd(va_0), vbd_0 = bdd(vb_0), vbd_0 = bdd(vb_0), \) be sequences of bounded closed sets (See Notation above 2.6). Note that there are sequences \( gac, vad \) such that \( gac_0c_0 \equiv gac \) and \( vad_0d_0 \equiv vad \).

Then by 4-amalgamation, there are \( a_1, c_1, d_1, t_1 \) such that

\[
\begin{align*}
(1) \quad & gac_1c_1 = c_1c_1 = _{egf} cac_0c_0 = cac_0c_0; \\
(2) \quad & vad_1d_1 = vad_1d_1 = _{evu} va_0d_0 = va_0d_0; \\
(3) \quad & gac_1c_1 = gac_1c_1 = _{gac} va_1d_1 = va_1d_1.
\end{align*}
\]

and \( \{a_1, g, v, e\} \) is independent. Then from 2.3.1, it can be seen so is \( \{t_1, f, u, e\} \) \( (** \ast) \). Due to (1)(2), \( ft_0c_0 = ft_1c_1 \) and \( ut_0d_0 = ut_1d_1 \). Hence there are enumerations such that \( ft_0c_0 = ft_1c_1 \) and \( ut_0d_0 = ut_1d_1 \).

Again by 4-CA, we have \( c_2, d_2, t_2, b_2 \) such that

\[
\begin{align*}
(4) \quad & ft_2c_2 = kbf_t2 \equiv_{kbf} ft_0c_0 = kb_0t_0; \\
(5) \quad & ut_2d_2 = kbw_2 \equiv_{kbw} ut_0d_0 = kb_0t_0; \\
(6) \quad & ft_2c_2 = ut_2d_2 \equiv_{fu} ft_1c_1 = ut_1d_1,
\end{align*}
\]

and \( \{t_2, f, u, k\} \) is independent. Hence due to \( (\ast), (**), 2.3.1(\ast) \) and (6), we can apply amalgamation to have

\[ d'c't' = Lstp(d_1c_1t_1/fu; g v) \cup Lstp(d_2c_2t_2/fu; hw) \]
such that \( t' \downarrow_{f_u} ghvw \) (†). Then there are the desired \( a', b' \) such that
\[
(7) \quad d'c't'a' \equiv_{f_{gw}} d_1 c_1 t_1 a_1, \quad \text{and} \quad d'c't'b' \equiv_{f_{hw}} d_2 c_2 t_2 b_2.
\]
Hence it follows from (†), \( a' \downarrow ghvw \). Moreover by (3)(7), \( c'a'd' \equiv_L^{g_{fv}} \text{cad} \); by (4)(7), \( c' \in h(b') \); and by (5)(7), \( d' \in w(b') \). The proof is finished. \( \square \)

We are ready to define the promised generic operation on \( \Gamma/\bar{R} \). Let

\[
\{x_1 y_1, x_2 y_2; x_3 y_3 \} := \exists x y z (\bar{R}(x_1, y_1) \land \bar{R}(x_2, y_2) \land \bar{R}(x, z) \land \wedge_{i=1,2,3} \Gamma(x i y i) \land x_1 y_1 / \bar{R} \downarrow x_2 y_2 / \bar{R} \land \{ x, y, z \} \text{ is independent}).
\]

Note that for \( x_1 y_1, x_2 y_2 \models \Gamma, x_1 y_1 / \bar{R} \downarrow x_1 y_1 / \bar{R} \text{ iff } \exists x'_1 y'_1 x'_2 y'_2 \bar{R}(x_1 y_1; x'_1 y'_1) \land \bar{R}(x_2 y_2; x'_2 y'_2) \land \{ x'_1, y'_1, x'_2, y'_2 \} \text{ independent. Hence } \ast \text{ is a partial type over } \emptyset.

Claim 1. The relation \( \ast \) is a hyperdefinable partial type over \( \emptyset \) such that, for any independent \( e_1 = f_1 g_1 / \bar{R}, e_2 = f_2 g_2 / \bar{R} \in \Gamma / \bar{R} \), there is \( e_3 = f h / \bar{R} \in \Gamma / \bar{R} \) such that \( (e_1, e_2; e_3) \) realizes \( \{x_1 y_1, x_2 y_2; x_3 y_3 \} \). (See the explanation above 1.4): It suffices to show there exist \( f, g, h \) such that \( e_1 = f g / \bar{R}, e_2 = g h / \bar{R} \) and \( \{ f, g, h \} \) is independent. Now by 2.3.1, \( e_i \downarrow g_i, e_i \downarrow f_i \) and \( f_i, g_i \) are interbounded over \( e_i \). Then, by amalgamation, there exists \( g \models \text{tp}(f_2 / e_1) \cup \text{tp}(g_1 / e_2) \) and \( \{ g, e_1, e_2 \} \) independent. We also have \( f, h \) such that \( f g \equiv_{e_1} f_1 g_1, g h \equiv_{e_1} f_2 g_2 \). Then, \( \{ f, g, h \} \) is independent too.

Claim 2. \( e_1 \cdot e_2 = e_3 = f h / \bar{R} \) does not depend on the choice of \( f, g, h \), i.e. \( e_1 \cdot e_2 \) is unique: Suppose there are \( f', g', h' \) such that \( e_1 = f' g' / \bar{R}, e_2 = g' h' / \bar{R} \) and \( \{ f', g', h' \} \) is independent. Then we can also find independent \( \{ u, v, w \} \) such that \( u \downarrow_{e_1 e_2} f g h f' g' h' \) and \( e_1 = u v / \bar{R}, e_2 = v w / \bar{R} \). Hence, from 4.1.2, \( u v w \downarrow e_1 e_2 f g h f' g' h' \), and \( \{ f, u, e_1, e_2 \} \) is independent (†). We shall prove that \( \bar{R}(f h, u w) \). (Then the by the same proof, \( \bar{R}(f h', u w) \) and thus \( \bar{R}(f h, f' h') \)). Note from (†) and 4.1.1, \( \bar{R}(f g, u v) \) and \( \bar{R}(g v, u w) \). Hence there are \( b, a \downarrow f g w \) and \( c \in g(a) \cap f(b), d \in v(a) \cap u(b) \). Moreover, by 4.2, we have \( a'b'c'd' \) such that \( c'a'd' \equiv_L^{g_{fv}} \text{cad} \) and \( a' \downarrow ghvw, c' \in g(a') \cap h(b'), d' \in v(a') \cap w(b') \). Now again due to (†) and 2.3.1, we have \( e_1 \downarrow_{g w} e_2, f u \downarrow_{g w} hw \) (⋆), and \( a \downarrow_{g w} f u, cad \downarrow_{g w} f u, c'a'd' \downarrow_{g w} hw \). Hence, by amalgamation, we have

\[
c_1 a_1 d_1 \models Lstp(cad/gv; f u) \cup Lstp(c'a'd'/gv; hw),
\]
such that \( a_1 \downarrow_{g w} f h u w \) (⋆⋆). Then we have \( b_1, b_1' \) such that
\[
c_1 a_1 d_1 b_1 \equiv_{f_{gw}} cadb, c_1 a_1 d_1 b_1' \equiv_{g_{hw}} c'a'd'b'.
\]
Thus, \( c_1 \in f(b_1) \cap h(b_1') \), \( d_1 \in u(b_1) \cap w(b_1') \) and from (⋆⋆), \( b_1 \downarrow f h u w \). Moreover from (⋆) and the remark above 3.1, \( \{ f, h, u, w \} \) is 3-independent. Therefore \( \bar{R}(f h, u w) \), as desired.

Claim 3. This generically given group satisfies the genericity properties in [16, 4.7.1]: Note that since \( \{ f, g, h \} \) independent, it follows for \( i = 1, 2, e_i \downarrow e_i \cdot e_2 \). For generic associativity, let \( \{ k_1, k_2, k_3 \} \) be independent realizations of \( \Gamma / \bar{R} \).

Subclaim. There exists independent \( \{ h_1, h_2, h_3, h_4 \} \) such that \( h_1 h_2 / \bar{R} = k_1, h_2 h_3 / \bar{R} = k_2 \) and \( h_3 h_4 / \bar{R} = k_3 \): As in the proof of Claim 1, we can find \( h_2, h_3, h_4 \) independent such that \( h_2 h_3 / \bar{R} = k_2 \) and \( h_3 h_4 / \bar{R} = k_3 \). Now, for \( h_1 h_2 / \bar{R} = k_1 \), amalgamation of \( Lstp(h_2' / k_1) \) and \( Lstp(h_2 / k_2 k_3) \) gives the subclaim.
Now, by the subclaim, \( k_1 \cdot k_2 = h_1 h_3 / \bar{\rho} \) and \( k_2 \cdot k_3 = h_2 h_4 / \bar{\rho} \). Then \( k_1 \cdot (k_2 \cdot k_3) = h_1 h_4 / \bar{\rho} \) and \( (k_1 \cdot k_2) \cdot k_3 = h_1 h_4 / \bar{\rho} \) as well. Finally it can be easily seen that \( \cdot \) is generically surjective. Hence Claim 3 is verified.

Therefore we have the following:

**Theorem 4.3.** Given the group configuration, there exists a canonical hyperdefinable group and a definable bijection mapping \( \Gamma / \bar{\rho} \) to the generic types of the group such that \( \cdot \) is mapped to the group multiplication generically.

5. 1-based theories

One application of 4.3 is the following result. This extends the theorem [3, 3.23] that, in any 1-based non-trivial \( \omega \)-categorical simple \( T \), an infinite vector space over some finite field is definably recovered in \( \mathcal{M}^q \). Recall that \( T \) is non-trivial if there are hyperimaginaries \( a_1, a_2, a_3 \) and \( A \) such that for \( 1 \leq i < j \leq 3 \), \( a_i, a_j \) are independent over \( A \) whereas \( \{a_1, a_2, a_3\} \) is dependent over \( A \).

**Theorem 5.1.** Suppose that \( T \) is 1-based, non-trivial, having model-\( 4 \)-CA. Then there is a hyperdefinable infinite bounded-by-Abelian group \( V \) over a model \( M \) of \( SU \)-rank 1 generic types. Moreover for the bounded subgroup \( V_0 = V \cap bdd(M) \), \( V / V_0 \) forms a vector space over the division ring \( R \) of \( bdd(M) \)-endomorphisms of \( V \) such that for \( b, a_1, ..., a_n \in V \), \( b \in bdd(a_1...a_n) \) iff \( b + V_0 = \alpha_1(a_1 + V_0) + ... + \alpha_n(a_n + V_0) \) for some \( \alpha_i \in R \).

**Proof.** By the proof of Lemma 3.22 in [3], there exists a non-trivial rank-1 \( Lstp \) \( p \) over some model \( M \). For convenience, let \( M = \emptyset \) after naming the model. As \( p \) is non-trivial, there exists \( \{a, b, c\} \) realizing \( p \) such that \( b, c \) is independent and \( a \in bdd(b, c) \setminus bdd(b) \cup bdd(c) \). Let \( yx \) realize \( tp(ab/c) \) with \( yx \perp c ab \). Then \( \dim(ay/bx) = 1 \) as \( y \in bdd(abx) \) and \( a \perp bx \). Let \( z = cb(Lstp(ay/bx)) \), then by 1-basedness, \( z \in bdd(ay) \cap bdd(bx) \). Moreover, by a straightforward rank calculation, \( SU(z) = 1 \). This gives a bounded quadrangle \( (a, b, c, x, y, z) \). Now by Theorem 4.3, we obtain a hyperdefinable group \( G \) over \( \emptyset \) such that the generic types all have \( SU \)-rank 1. The group \( G \) is 1-based since the underlying theory is 1-based. Now we use the following fact [16, 4.8.4],

**Fact 5.2.** Suppose \( G \) is an 1-based group hyperdefinable over \( \emptyset \) in a simple theory. Then for the normal subgroup \( G^0_\emptyset \), the smallest \( \emptyset \)-hyperdefinable subgroup of bounded index, the commutator subgroup \( (G^0_\emptyset)' \) of \( G^0_\emptyset \) has boundedly many elements and contained in the center of \( G^0_\emptyset \).

Therefore, if we set \( G^0_\emptyset = V \), then \( V \) is the desired bounded-by-Abelian hyperdefinable group. Note that by above \( V' \) is contained in the normal subgroup \( V_0 = V \cap bdd(\emptyset) \). Indeed again from [16, 4.8.18], the Abelian group \( V / V_0 \) forms a vector space over a division ring \( R \) of \( bdd(\emptyset) \)-endomorphisms of \( V \), and dependence in \( V / V_0 \) is given by linear dependence of the vector space. \( \square \)

**References**

[1] I. Ben-Yaacov, I. Tomasic and F. Wagner, ‘Constructing an almost hyperdefinable group’ will appear in *Journal of Math. Logic.*
[2] Z. Chatzidakis and E. Hrushovski, ‘Model theory of difference fields’, Transactions of AMS 351 (1999) 2997-3071.
[3] T. de Piro, B.Kim, ‘The geometry of 1-based minimal types’, Transactions of American Math. Soc., 355 (2003) 4241-4263.
[4] B. Hart, B. Kim and A. Pillay, ‘Coordinatization and canonical bases in simple theories’, Journal of Symbolic Logic, 65 (2000) 293-309.
[5] E. Hrushovski, ‘Unimodular minimal theories’, Journal of London Math. Soc., 46 (1992) 385-396.
[6] E. Hrushovski, ‘Simplicity and the Lascar group’, preprint (1997).
[7] E. Hrushovski, ‘Pseudo-finite fields and related structures’, will appear in Quaderni di Matematica.
[8] E. Hrushovski, ‘Relative Et’, preprint (2004).
[9] E. Hrushovski and B. Zilber, ‘Zariski geometries’, Journal of AMS 9 (1996) 1-56.
[10] B. Kim and A. Pillay, ‘Simple theories’, Ann. Pure and Applied Logic 88 (1997) 149-164.
[11] B. Kim, A. Kolesnikov and A. Tsuboi, ‘Generalized type-amalgamation and n-simplicity’, in preparation.
[12] A. Kolesnikov, ‘Generalized amalgamation in simple theories and characterization of dependence in non-elementary classes’, Ph. D. Thesis, Carnegie Mellon University, 2004.
[13] A. Kolesnikov, ‘n-simple theories’, Ann. Pure and Applied Logic 131 (2005) 227-261.
[14] A. Pillay Geometric stability theory, Oxford University Press, Oxford (1996).
[15] S. Shelah, ‘Toward classifying unstable theories’, Ann. Pure and Applied Logic 80 (1996) 229-255.
[16] F. O. Wagner, Simple theories, Kluwer Academic Publishers, Dordrecht (2000).
[17] B. I. Zilber, Uncountably categorical theories, AMS Translations of Mathematical Monographs, Vol. 117.

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