The Constraint Algebra of Quantum Gravity in the Loop Representation

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We study the algebra of constraints of quantum gravity in the loop representation based on Ashtekar’s new variables. We show by direct computation that the quantum commutator algebra reproduces the classical Poisson bracket one, in the limit in which regulators are removed. The calculation illustrates the use of several computational techniques for the loop representation.

I. INTRODUCTION

An important element in the classical formulation of a canonical theory which has constraints is their algebra. If taking Poisson brackets of the constraints of the theory leads to quantities that can be expressed as combinations of the constraints, these are said to be first class. If the constraints are first class, quantization can proceed by requiring that the quantum operator version of the constraints annihilate the wavefunctions. Upon quantization, the Poisson bracket algebra of constraints should translate into an algebra of commutators where the commutator of any constraints should again be expressible as a combination of the constraints. The combination may involve coefficients that are functions of the canonical variables (which become operators at a quantum level). It is therefore crucial that such operators appear to the left. If this were not the case, imposing that the wavefunctions be annihilated by the constraints could imply —via commutation relations— extra conditions on the wavefunctions. The quantum theory found would therefore not necessarily correspond to the classical theory one started from.

From a physical point of view, constraints generate via canonical transformations the symmetries of the theory. Therefore respecting the constraint algebra at a quantum level is respecting the symmetries of the theory under quantization. It should be noticed that imposing the correct commutation relations between constraints at a quantum level without anomalies is far from a mere technicality. For instance in the case of string theory it is what determines that the theory of a bosonic string only is consistent in 26 dimensions.

The canonical formulation of general relativity has constraints. The issue of the commutation relations of the constraints therefore appears for any quantized version of the theory that one may consider. In this paper we will concentrate on the attempt to quantize General Relativity based on the Loop Representation constructed from Ashtekar’s new variables. The issue that we want to address is the consistency of the constraints that have been proposed. Although solutions to the proposed quantum version of the constraints have been found, this does not imply consistency of the constraint algebra.

We will perform the calculation directly in the Loop Representation. This will serve two purposes: firstly it shows that the expressions presented are well defined in the sense that commutators of the constraints can be computed. Secondly it provides an excellent example of the use of several techniques to perform computations in loop space. We have made this paper deliberately detailed and explicit, at the risk of being somewhat long, with the purpose of presenting in a clear fashion the techniques used. These techniques can be applied to other problems concerning the loop representation.
The problem of the constraint algebra in quantum gravity has received a fair share of attention, both in terms of traditional geometrodynamics and in terms of Ashtekar's new variables. In particular, it is quite clear that the issue cannot be completely analyzed on formal grounds: a suitable regularization is needed to avoid ambiguities that can arise in the constraints. A thorough discussion of these and other issues, as well as a historical introduction to the subject, can be found in the papers by Tsamis and Woodard [1] and Friedman and Jack [2].

In spite of what we have just said, in this paper we will only discuss the issue of the formal unregulated commutation relations. The motivation is twofold. First of all, even this calculation has ever been performed in terms of loops. Second, our main purpose is more to gain insight on how to operate with the recently introduced expressions for the constraints in the loop representation rather than to settle the issue of their consistency. The constraints in the loop representation have been used to find solutions to the equations [3], but operating twice with a constraint requires considerably more care than originally expected. Several lessons about how to operate with the diffeomorphism and Hamiltonian constraints in terms of loops will be learned during the discussions of the different commutators. A sharper understanding of how these operators act is gained through this analysis.

This paper is organized as follows. In section II we will briefly recall the definition of the constraints of canonical general relativity and the constraints of canonical quantum gravity in the Loop Representation. In section III we compute the commutator of two diffeomorphism constraints, in section IV the commutator of a diffeomorphism with a Hamiltonian, and in section V the commutator of two Hamiltonians. Several technical aspects of the computations are discussed in the appendices.

II. CONSTRAINTS IN THE LOOP REPRESENTATION

There exist well-established canonical formulations of general relativity. The more traditional one is based on the use of a three-dimensional metric $g_{ab}$ and its conjugate momentum density $\pi^{ab}$. Recently, a new formulation due to Ashtekar has exhibited interesting properties. It is based on the use of an $SU(2)$ connection $A^i_a$ as fundamental variable and its conjugate momentum density is a set of densitized triads $\tilde{E}^a_i$. In both formulations the variables are not free but are subject to constraints. These constraints group naturally into a vector and a scalar. In terms of Ashtekar’s new variables,

$$C(N) = \int d^3 x N^a(x) \tilde{E}^b_i(x) F^i_{ab}(x)$$

$$H(M) = \int d^3 x M^a_i(x) \tilde{E}^b_i(x) F^i_{ab}(x) e^{ijk}$$

where $F^i_{ab}$ is the curvature of the Ashtekar connection. It is important to remark that the Gauss constraint is not mentioned because in the Loop representation it is satisfied automatically for all loop functionals. We refer the reader to [1] for notation and conventions.

In the loop representation of quantum gravity [1], wavefunctions $\Psi(\gamma)$ are functions of loops $\gamma$. The word loop has in this context a very precise meaning: it is an equivalence class of unparametrized curves. The equivalence relation is that two curves are equivalent if they give rise to the same holonomy for all connections. This means that two loops are equivalent if they only differ by “tails” that immediately retrace themselves.

Because the theory started with an $SU(2)$ connection (the Ashtekar connection), wavefunctions inherit certain $SU(2)$ identities (Mandelstam identities),

$$\Psi(\gamma) = \Psi(\gamma^{-1})$$

$$\Psi(\gamma_1 \circ \gamma_2) = \Psi(\gamma_2 \circ \gamma_1)$$

$$\Psi(\gamma_1 \circ \gamma_2 \circ \gamma_3) + \Psi(\gamma_1 \circ \gamma_2 \circ \gamma_3^{-1}) = \Psi(\gamma_2 \circ \gamma_1 \circ \gamma_3) + \Psi(\gamma_2 \circ \gamma_1 \circ \gamma_3^{-1})$$

In this representation, the diffeomorphism and Hamiltonian constraints can be written by making use of the loop derivative. The loop derivative is the derivative that arises in loop space when two loops that differ by an infinitesimal element of area are considered close. The definition of the loop derivative of a function of a loop is,

$$\Delta_{ab}(\pi^0_i)\Psi(\gamma) = \lim_{\delta \gamma \to 0} \frac{\Psi(\pi^0_i \circ \delta \gamma \circ \pi^0_i \circ \gamma) - \Psi(\gamma)}{\sigma^{ab}}$$

where $\delta \gamma$ is a loop of infinitesimal area $\sigma^{ab}$ basepointed at $x$ and connected to the loop $\gamma$, through a path $\pi^0_i$. We will usually consider for practical computations infinitesimal loops formed by a parallelogram of infinitesimal sides $\bar{u}$ and $\bar{v}$.
Notice that the loop derivative is a path dependent object. Loop derivatives act on any loop or path dependence. In particular if one computes the commutator of two loop derivatives there will be a nonvanishing action of the second loop derivative on the path dependence of the first loop derivative, leading to a commutation relation of the form,

\[ [\Delta_{ab}(\pi^a_0), \Delta_{cd}(\chi^c_0)] = \Delta_{ab}(\pi^a_0)\Delta_{cd}(\chi^c_0) - \Delta_{cd}(\chi^c_0)\Delta_{ab}(\pi^a_0) \]

where the brackets denote that the second loop derivative only acts on the loop dependence of whatever is in the brackets.

Another relation of importance is to notice that the loop derivative satisfies a Bianchi identity. In order to define it, we need to introduce another derivative in loop space, the covariant derivative. The covariant derivative \( \nabla_a \) acts on functions by appending an infinitesimal element at the end of the path along the coordinate direction \( \tilde{a} \),

\[ (1 + \epsilon u^a \nabla_a) \Psi(\pi^a_0) = \Psi(\pi^a_0 + \epsilon u). \]

With this definition, the Bianchi identity reads,

\[ \nabla_b \Delta_{bc}(\pi^c_0) + \nabla_c \Delta_{ca}(\pi^a_0) + \nabla_a \Delta_{ab}(\pi^a_0) = 0. \]

The constraints of quantum gravity in the loop representation can be obtained using the loop transform. They have been discussed in refs [3,4] so we only discuss them briefly here to fix notation.

The diffeomorphism constraint is given by,

\[ C(N)\Psi(\gamma) = \int d^3xN^a(x)\oint_\gamma dy^b\delta^3(x - y)\Delta_{ab}(\gamma^b_0)\Psi(\gamma) \]

where \( N^a \) is a vector field on the three manifold along which the diffeomorphism is taken. This operator has been known for quite some time to generate deformations of the loop in the argument of the wavefunction corresponding to the diffeomorphism performed. As an intuitive picture of this expression one should remember that in the Ashtekar formulation of canonical gravity the diffeomorphism constraint is given by \( \int d^3xN^a(x)\text{Tr}[E^a(x)F_{ab}(x)] \). The expression in the loop representation can be heuristically thought of as a replacement \( E^a \rightarrow \oint dy^a \) and \( F_{ab} \rightarrow \Delta_{ab} \). In this paper we will only consider diffeomorphisms that leave unchanged the basepoint of the loops we are using, that is, \( \tilde{N} \) vanishes at the basepoint.

The Hamiltonian constraint is,

\[ H(M)\Psi(\gamma) = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{2} \int d^3xM(x)\oint_\gamma dy^a\oint_\gamma dz^b\delta^3(x - y)f_s(y, y')[O_{y, y'}\Delta_{ab}(\gamma^b_0) + O_{y, y'}\Delta_{ab}(\gamma^b_0)]\Psi(\gamma). \]

In this paper we will omit the factor \( \frac{1}{2} \). This expression requires some discussion. First of all notice that we have introduced a regulator \( f_s(y, y') \) such that \( \lim_{\epsilon \rightarrow 0} f_s(y, y') \rightarrow \delta^3(y, y') \). The need for this regulator can be seen directly from the translation of the expression of the Hamiltonian constraint in the Ashtekar formulation \( \int d^3xM(x)\int d^3yf_s(y, y')\text{Tr}[E^a(y)E^b(y)F_{ab}(x)] \) due to the fact that the constraint is quadratic in momenta. The expression is naively zero if the loop is smooth (no kinks or intersections) due to the fact that \( \Delta_{ab} \) is antisymmetric. One can check that when the regulator is taken into account the expression also vanishes provided it acts on diffeomorphism invariant functions. Since wavefunctions in the loop representation take values on all possible loops we will generically consider a loop with a multiple intersection at a given point (the case with more than one intersection is trivially included since the action of the Hamiltonian is local).

A second point to notice is the appearance of the rerouting operator, \( O_{y, y'} \). Acting on a function of a loop, this operator has the following effect: it takes the loop formed by taking the original loop between \( y \) and \( y' \) no matter the original orientation and then it adds the portion between \( y \) and \( y' \) passing through the origin \( o \), no matter the original orientation. Of course, the points \( y \) and \( y' \) must lie on the loop, as they do in the expression of the Hamiltonian. The reader may be puzzled at this point: if one cuts and pastes a portion of a loop in the opposite orientation, one does not have in general a loop any more. This is true. To complete the procedure for arbitrary \( y \) and \( y' \) one must re-close the resulting loops (see appendix A). However, in the limit where regulators are removed, the points \( y \) and \( y' \) coincide with the intersection points and therefore the action of the rerouting simply amounts to a rerouting of certain “petals” of the loop. The “re-tracings” introduced to close the loop are of higher order in powers of the regulator in the limit where the regulators are removed.

The reader can realize that although expressing these expressions to a function of a loop seems reasonably straightforward, applying them for a second time can introduce complications. For instance: what is the action of two successive reroutings? What is the action of the loop derivative on the rerouting? These are the kinds of questions we would like to address in this paper.
III. DIFFEOMORPHISM ALGEBRA

We now proceed to compute the commutator of two diffeomorphism constraints. The reader may feel this an unneeded computation. After all, if the constraint is the infinitesimal generator of diffeomorphisms, it should, by definition, satisfy the correct algebra. Although this is a valid viewpoint, we feel that an explicit calculation is in order. Furthermore, the calculation can be viewed as a confirmation that the proposed constraint is equivalent to the generator of diffeomorphisms in loop space. The calculation is quite instructive in the sense that it makes a nontrivial use of several properties of loop space. This calculation is already present in the literature [9] but we feel a more careful and detailed derivation is in order. We find it convenient to explain the techniques used in some detail with the example of this calculation in order to be able to be more succinct in the following ones.

We will start with an auxiliary calculation that will prove useful in the sequel. We will evaluate the action of the loop derivative on a diffeomorphism constraint. To this aim we construct the following expression, which holds due to the very definition of loop derivative,

\[
(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z)) \int_\gamma dy^a \delta(y-x)\Delta_{ab}(\gamma_o^y)\Psi(\gamma) := \int_{\delta\gamma_o \circ \gamma} dy^a \delta(y-x)\Delta_{ab}((\delta\gamma_o \circ \gamma)^o_\gamma)\Psi(\delta\gamma_o \circ \gamma).
\]

In this expression, \(\delta\gamma_o \circ \gamma\) is the infinitesimal loop added to \(\gamma\) and connected through a path from the origin up to the point \(z\). That is, we evaluate the action of an infinitesimal deformation of area \(\sigma^{cd}\) acting on the infinitesimal generator of diffeomorphisms. We now evaluate the right member of this expression and this will enable us to read off the action of the loop derivative on the infinitesimal generator of diffeomorphisms.

With this aim we expand the right member of (12), partitioning the domain of integration, and using the definition of the loop derivative to expand \(\Psi(\delta\gamma \circ \gamma)\). We get,

\[
\int_{\gamma} dy^a\Delta_{ab}(\gamma_o^z)\delta(y-x)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a\delta(z+u-x)\Delta_{ab}(\gamma_o^{z+u}) - u^a\delta(z+u+v-x)\Delta_{ab}(\gamma_o^{z+u+v})
- v^a\delta(z+u-x)\Delta_{ab}(\gamma_o^{z+u+v+u})\}
(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+ \int_{\gamma} dy^a\delta(y-x)\Delta_{ab}((\delta\gamma_o \circ \gamma)^o_\gamma)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma)
\]

where the terms in curly braces correspond to the integration along the infinitesimal loop \(\delta\gamma\) which we take as being given by a parallelogram of infinitesimal sides given by \(u^a\) and \(v^b\).

The last term in this expression can be rewritten as

\[
\int_{\gamma} dy^a\delta(z-y)\Theta(z,y)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))[(\Delta_{ab}(\gamma_o^y))(1 + \sigma^{ef}\Delta_{ef}(\gamma_o^z))]\Psi(\gamma)
\]

where \(\Theta(z,y)\) is a Heaviside function that orders the points along the loop. We will be able to combine the zeroth order contribution (in terms of the infinitesimal loop) of this term with the first term in (13). It should be noticed that the first loop derivative does not act on everything to its right but only on the path inside the second loop derivative \(\gamma_o^y\), a fact we denoted by enclosing it in brackets, and that was discussed in detail in the previous section. We now consider the expansion of the terms containing infinitesimally shifted loop. They can be expressed with the use of the Mandelstam derivative,

\[
\Delta_{ab}(\gamma_o^{z+u}) = (1 + u^e D_e)\Delta_{ab}(\gamma_o^z)
\]

\[
\Delta_{ab}(\gamma_o^{z+u+v}) = (1 + v^d D_d)(1 + u^e D_e)\Delta_{ab}(\gamma_o^z)
\]

\[
\delta(z+u-x) = (1 + u^a \partial_a)\delta(z - x)
\]

We now expand again (13)

\[
\int_{\gamma} dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a(1 + u^e \partial_e)\delta(z-x)\Delta_{ab}(\gamma_o^z) -
\]

\[
\int_{\gamma} dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a(1 + u^e \partial_e)\delta(z-x)\Delta_{ab}(\gamma_o^z) -
\]

\[
\int_{\gamma} dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a(1 + u^e \partial_e)\delta(z-x)\Delta_{ab}(\gamma_o^z) -
\]

\[
\int_{\gamma} dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a(1 + u^e \partial_e)\delta(z-x)\Delta_{ab}(\gamma_o^z) -
\]

\[
\int_{\gamma} dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a(1 + u^e \partial_e)\delta(z-x)\Delta_{ab}(\gamma_o^z) -
\]

\[
\int_{\gamma} dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a(1 + u^e \partial_e)\delta(z-x)\Delta_{ab}(\gamma_o^z) -
\]

\[
\int_{\gamma} dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a(1 + u^e \partial_e)\delta(z-x)\Delta_{ab}(\gamma_o^z) -
\]

\[
\int_{\gamma} dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a(1 + u^e \partial_e)\delta(z-x)\Delta_{ab}(\gamma_o^z) -
\]

\[
\int_{\gamma} dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a(1 + u^e \partial_e)\delta(z-x)\Delta_{ab}(\gamma_o^z) -
\]

\[
\int_{\gamma} dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a(1 + u^e \partial_e)\delta(z-x)\Delta_{ab}(\gamma_o^z) -
\]

\[
\int_{\gamma} dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)(1 + \sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) +
+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a(1 + u^e \partial_e)\delta(z-x)\Delta_{ab}(\gamma_o^z) -
\]
\[-u^a(1 + v^cD_c)\delta(z - x)(1 + \sigma^dD_d)\Delta_{ab}(\gamma^z_o) - \\
v^a\delta(z - x)\Delta_{ab}(\gamma^z_o)(1 + \sigma^{cd}\Delta_{cd}(\gamma^z_o))\Psi(\gamma) + \\
+ \int dy^a\delta(y - x)\Theta(z, y)\sigma^{cd}\Delta_{cd}(\gamma^z_o)|\Delta_{ab}(\gamma^y_o)|(1 + \sigma^{ef}\Delta_{ef}(\gamma^z_o))\Psi(\gamma).\]

(18)

Of the terms in braces, it can be readily seen that only contributions proportional to \(u^a v^b\) are present, neglecting terms of higher order. The other terms combine to give the original expression. We can finally read off the contribution of the loop derivative,

\[\Delta_{cd}(\gamma^z_o) \int \gamma dy^a\delta(x - y)\Delta_{ab}(\gamma^y_o)\Psi(\gamma) =
2[\partial_c\delta(z - x)\Delta_{db}(\gamma^z_o) + \delta(z - x)D^c_d\Delta_{db}(\gamma^z_o)]\Psi(\gamma) + \\
\int \gamma dy^a\Theta(z, y)\delta(y - x)\Delta_{cd}(\gamma^z_o)|\Delta_{ab}(\gamma^y_o)|\Psi(\gamma) + \\
\int \gamma dy^a\delta(y - x)\Delta_{ab}(\gamma^y_o)\Delta_{cd}(\gamma^z_o)\Psi(\gamma).
\]

(19)

With this calculation in hand, it is straightforward to compute the successive action of two diffeomorphisms,

\[C(\vec{N})C(\vec{M})\Psi(\gamma) = \int d^3w^d N^d(w) \int \gamma dz^c\delta(w - z)\Delta_{cd}(\gamma^z_o) \int d^3x M^k(x) \int \gamma dy^a\delta(y - x)\Delta_{ab}(\gamma^y_o)\Psi(\gamma).
\]

(20)

Expanding this expression, we get six terms,

\[\int d^3w \int d^3x N^d(w)M^b(x) \int \gamma dz^c\delta(w - z)\partial_c\delta(z - x)\Delta_{ab}(\gamma^z_o)\Psi(\gamma) - \\
\int d^3w \int d^3x N^d(w)M^b(x) \int \gamma dz^c\delta(w - z)\partial_d\delta(z - x)\Delta_{cb}(\gamma^z_o)\Psi(\gamma) + \\
\int d^3w \int d^3x N^d(w)M^b(x) \int \gamma dz^c\delta(w - z)\delta(z - x)D^c_e\Delta_{eb}(\gamma^z_o)\Psi(\gamma) - \\
\int d^3w \int d^3x N^d(w)M^b(x) \int \gamma dz^c\delta(w - z)\delta(z - x)D^d_e\Delta_{eb}(\gamma^z_o)\Psi(\gamma) + \\
\int d^3w \int d^3x N^d(w)M^b(x) \int \gamma dz^c\delta(w - z)\delta(z - x)D^e_d\Delta_{eb}(\gamma^z_o)\Psi(\gamma) + \\
\int d^3w \int d^3x N^d(w)M^b(x) \int \gamma dz^c\delta(w - z)\Theta(z, y)\delta(y - x)\Delta_{cd}(\gamma^z_o)|\Delta_{ab}(\gamma^y_o)|\Psi(\gamma) + \\
\int d^3w \int d^3x N^d(w)M^b(x) \int \gamma dz^c\delta(w - z)\delta(y - x)\Delta_{ab}(\gamma^y_o)\Delta_{cd}(\gamma^z_o)\Psi(\gamma).
\]

(21)

We should now subtract the same terms with the replacement \(\vec{N} \leftrightarrow \vec{M}\). Since the calculation is tedious but straightforward we describe in words how the terms combine. The fifth and sixth terms, when combined with the similar terms coming from the substitution \(\vec{N} \leftrightarrow \vec{M}\) cancel taking into account the commutation relations for the loop derivatives \(\vec{N} \leftrightarrow \vec{M}\). The first and third term, combined with the first of the substitution \(\vec{N} \leftrightarrow \vec{M}\) form a total derivative. The fourth term, combined with the third and fourth of the substitution \(\vec{N} \leftrightarrow \vec{M}\) cancel due to the Bianchi identities of the loop derivatives. Finally, the second terms combine to produce exactly \(C(\mathcal{L}_\vec{N}\vec{M})\), which is the correct result of the calculation.

IV. COMMUTATOR OF A DIFFEOMORPHISM WITH A HAMILTONIAN

This calculation will teach us about two important new ingredients that were not present in the previous calculation: the action of reroutings in the Hamiltonian and how to deal with regularization. There has always been concerns about the use of a background dependent regularization in the Hamiltonian constraint, since it had potential to interfere with the action of diffeomorphisms. With the help of this calculation it is possible to detect the regularization problems explicitly, by showing where it is needed to remove the regulators for the computation to work. The commutator is,
\[ [\mathcal{C}(\tilde{M})H(\tilde{N}) - H(\tilde{N})\mathcal{C}(\tilde{M})] = \int d^3w M^b(w) \int d^3x \tilde{N}(x) \times \]

\[ \oint_L dz^a \delta(z - w) \Delta_{ab}(\gamma_o) \oint_\gamma dy^c \oint_\gamma dy'^d \delta(x - y') f_\epsilon(y - y')[O_{y y'} + O_{y_o y'}] \Delta_{cd}(\gamma_o)^D \Psi(\gamma) - \]

\[ \oint_\gamma dy^c \oint_\gamma dy'^d \delta(x - y') f_\epsilon(y - y')[O_{y y'} + O_{y_o y'}] \Delta_{cd}(\gamma_o)^D \oint_\gamma dz^a \delta(z - w) \Delta_{ab}(\gamma_o)^D \Psi(\gamma). \] (22)

We will now proceed as in the last section; first, we evaluate the action of the loop derivative of the diffeomorphism constraint on the loop derivative and the rerouting operators corresponding to the Hamiltonian constraint. We will prove that these terms minus the analogous ones coming from the action of the Hamiltonian on the diffeomorphism cancel each other. The remaining terms, which are the consequence of the action of the loop derivative on the loop dependence of integrals will give rise to the correct commutator. Since the terms involving \(O_{y o y'}\) behave in an analogous fashion to the ones that depend on \(O_{y y'}\), we will only concentrate on these.

The first contribution of the action of the diffeomorphism on the Hamiltonian is,

\[ \oint_\gamma dy^c \oint_\gamma dy'^d [\oint_\gamma dz^a + \oint_{\gamma_o y'} dz^a] \delta(z - w) \delta(x - y') f_\epsilon(y - y') \times \]

\[ \times \Delta_{ab}(\gamma_o) [O_{y y'} \Delta_{cd}(\gamma_o)^D \Psi(\gamma)] + \]

\[ + \oint_\gamma dy^c \oint_{\gamma_o} dy'^d [\oint_\gamma dz^a + \oint_{\gamma_o y'} dz^a] \delta(z - w) \delta(x - y') f_\epsilon(y - y') \times \]

\[ \times \Delta_{ab}(\gamma_o) [O_{y y'} \Delta_{cd}(\gamma_o)^D \Psi(\gamma)]. \] (23)

After commuting the rerouting and the derivative (see appendix A), and taking into account the sign introduced in this process we get,

\[ \oint_\gamma dy^c \oint_{\gamma_o y} dy'^d [- \oint_{\gamma_o y'} dz^a + \oint_{\gamma_o y'} dz^a] \delta(z - w) \delta(x - y') f_\epsilon(y - y') \times \]

\[ \times O_{y y'} \Delta_{ab}(\gamma_o) [\Delta_{cd}(\gamma_o)^D \Psi(\gamma)] + \]

\[ + \oint_\gamma dy^c \oint_{\gamma_o} dy'^d [\oint_\gamma dz^a - \oint_{\gamma_o y'} dz^a] \delta(z - w) \delta(x - y') f_\epsilon(y - y') \times \]

\[ \times O_{y y'} \Delta_{ab}(\gamma_o) [\Delta_{cd}(\gamma_o)^D \Psi(\gamma)]. \] (24)

The following step consists in noticing that the minus sign in front of the integral can be absorbed by integrating along the loop in the opposite direction, which we denote with an over-bar,

\[ \oint_\gamma dy^c \oint_{\gamma_o y} dy'^d [\oint_{\gamma_o y'} dz^a + \oint_{\gamma_o y'} dz^a] \delta(z - w) \delta(x - y') f_\epsilon(y - y') \times \]

\[ \times O_{y y'} \Delta_{ab}(\gamma_o) [\Delta_{cd}(\gamma_o)^D \Psi(\gamma)] + \]

\[ + \oint_\gamma dy^c \oint_{\gamma_o} dy'^d [\oint_\gamma dz^a + \oint_{\gamma_o y'} dz^a] \delta(z - w) \delta(x - y') f_\epsilon(y - y') \times \]

\[ \times O_{y y'} \Delta_{ab}(\gamma_o) [\Delta_{cd}(\gamma_o)^D \Psi(\gamma)], \] (25)

which in turns allows us to recombine the terms again into a single loop integral, making use of the definition of the rerouting operator,
\[ \oint dy^c \oint_{\gamma_y} dy^{rd} O_{y' y'} \oint_{\gamma} d\gamma^a \delta(z-w) \delta(x-y') f_c(y-y') \times \Delta_{ab}(\gamma_o^c) \left[ \Delta_{cd}(\gamma_o^r) \Psi(\gamma) \right] + \]
\[ + \oint dy^c \oint_{\gamma_y} dy^{rd} O_{y' y'} \oint_{\gamma} d\gamma^a \delta(z-w) \delta(x-y') f_c(y-y') \times \Delta_{ab}(\gamma_o^c) \left[ \Delta_{cd}(\gamma_o^r) \Psi(\gamma) \right]. \]  

\[ (26) \]

So the result of this manipulation can be written as,
\[ \oint dy^c \oint_{\gamma_y} dy^{rd} O_{y' y'} \oint_{\gamma} d\gamma^a \delta(z-w) \delta(x-y') f_c(y-y') \times \Delta_{ab}(\gamma_o^c) \left[ \Delta_{cd}(\gamma_o^r) \Psi(\gamma) \right] = \]
\[ = \oint dy^c \oint_{\gamma_y} dy^{rd} \delta(x-y') f_c(y-y') \times O_{y' y'} \]
\[ \oint_{\gamma} d\gamma^a \delta(z-w) \left[ \Delta_{cd}(\gamma_o^r) \Delta_{ab}(\gamma_o^c) + \Theta(z, y) \Delta_{ab}(\gamma_o^c) \right] \left[ \Delta_{cd}(\gamma_o^r) \right] \Psi(\gamma). \]  

\[ (27) \]

The corresponding terms arising from the action of the Hamiltonian constraint on the diffeomorphism are,
\[ \oint dy^c \oint_{\gamma_y} dy^{rd} O_{y' y'} \oint_{\gamma} d\gamma^a \delta(z-w) \delta(x-y') f_c(y-y') \times \Delta_{cd}(\gamma_o^r) \left[ \Delta_{ab}(\gamma_o^c) \Psi(\gamma) \right] = \]
\[ = \oint dy^c \oint_{\gamma_y} dy^{rd} \delta(x-y') f_c(y-y') \times O_{y' y'} \]
\[ \oint_{\gamma} d\gamma^a \delta(z-w) \left[ \Delta_{ab}(\gamma_o^c) \Delta_{cd}(\gamma_o^r) + \Theta(y, z) \Delta_{ab}(\gamma_o^c) \right] \left[ \Delta_{cd}(\gamma_o^r) \right] \Psi(\gamma). \]  

\[ (28) \]

When we subtract between (28) from (27) one immediately notices that the net result is zero if one uses the commutation relation for the loop derivatives \[ \oint \].

The loop derivative acts on the loop dependence of the integrals, exactly as in the commutator we considered in the previous section. We now concentrate on the terms that will give rise to the result of the commutator. These terms appear when the loop derivative acts on the loop dependence of the integrals, exactly as in the commutator we considered in the previous section. The contribution from \( C(\bar{M}) H(\bar{N}) \Psi(\gamma) \) is
\[ \int d^3 w M^b(w) \oint_{\gamma} d\gamma^a \delta(z-w) \int d^3 x \bar{N}(x) \times \]
\[ \left( 2 \oint_{\gamma} dy^{rd} \delta(x-y') D_i^{\gamma^r} [f_c(y-y') O_{y' y'} \Delta_{dj}^c(\gamma_o^c)] \Psi(\gamma) + \right. \]
\[ \left. + 2 \oint_{\gamma} dy^{rd} D_{\gamma}^i [\delta(x-z) f_c(y-z) O_{y' y'} \Delta_{dj}^c(\gamma_o^c)] \Psi(\gamma) \right). \]  

\[ (29) \]

And the contribution from \( H(\bar{N}) C(\bar{M}) \Psi(\gamma) \) is
\[ \int d^3 x \bar{N}(x) \int d^3 w M^b(w) \oint_{\gamma} dy^c \oint_{\gamma} dy^{rd} \delta(x-y') f_c(y-y') \times \]
\[ \times \left( 2 \oint_{\gamma} dy^{rd} D_{\gamma}^i [\delta(y-w) \Delta_{dj}^b(\gamma_o^r)] \Psi(\gamma) \right). \]  

\[ (30) \]

The result of computing the commutator is,
\[ 2 \oint_{\gamma} dy^c \oint_{\gamma} dy^{rd} \delta(x-y') D_i^{\gamma^r} [f_c(z-y') O_{z y'} \Delta_{dj}^c(\gamma_o^c)] \Psi(\gamma) + \]
\[ + 2 \oint_{\gamma} dy^c \oint_{\gamma} dy^{rd} D_{\gamma}^i [\delta(x-z) f_c(y-z) O_{y' y'} \Delta_{dj}^c(\gamma_o^c)] \Psi(\gamma) \]
\[ - \oint_{\gamma} dy^c \oint_{\gamma} dy^{rd} [\delta(x-y') f_c(y-y') 2 \oint_{\gamma} dy^{rd} D_{\gamma}^i [\delta(y-w) \Delta_{dj}^b(\gamma_o^r)] \Psi(\gamma). \]

We now explicitly write the antisymmetrizations of the first two terms, and use Leibnitz’ rule in the last one we get,
\[
\int d^3 w M^b(w) \int d^3 x N(x) \times \\
\times \left( - \oint dy^c \delta(y - w) \oint dy^d \delta(x - y') D^y_b [f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y)] \Psi(\gamma) + \\
+ \oint dy^c \oint dy^d \delta(x - y') f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y) \Psi(\gamma) + \\
+ \oint dy^c \oint dy^d \delta(x - y') f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y) \Psi(\gamma) \right). 
\]

The first and fourth terms can be integrated by parts on z. Taking into account that the diffeomorphisms we are considering have trivial action at the basepoint of the loops, the resulting terms cancel with the fifth term. In order for this cancelation to occur we have to remove the regulator so that \( \partial_\alpha^y \delta(y' - w) f_c(y - y') = \partial_\alpha^y \delta(y - w) f_c(y - y') \). The remaining terms are the second and third. By using the Bianchi identity in the last term of (32), this equation may be rewritten as

\[
\int d^3 w d^3 x M^b(w) N(x) \times \\
\times \left( - \oint dy^c \delta(y - w) \oint dy^d \delta(x - y') D^y_b [f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y)] \Psi(\gamma) + \\
+ \oint dy^c \oint dy^d \delta(x - y') f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y) \Psi(\gamma) + \\
+ \oint dy^c \oint dy^d \delta(x - y') f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y) \Psi(\gamma) \right). 
\]

By using Leibniz rule in the first term we get,

\[
\int d^3 w d^3 x M^b(w) N(x) \times \\
\times \left( - \oint dy^c \delta(y - w) \oint dy^d \delta(x - y') D^y_b [f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y)] \Psi(\gamma) + \\
+ \oint dy^c \oint dy^d \delta(x - y') f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y) \Psi(\gamma) + \\
+ \oint dy^c \oint dy^d \delta(x - y') f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y) \Psi(\gamma) \right). 
\]

Taking into account the definition of the rerouting operator, we have,

\[
D^y_b [f_c(y - y') O_{y'y'}] = D^y_b f_c(y - y') O_{y'y'} = -D^y_b f_c(y - y') O_{y'y'} 
\]

and using the Leibnitz rule again in (34) we get

\[
\int d^3 w d^3 x M^b(w) N(x) \times \\
\times \left( + \oint dy^c \oint dy^d D^y_b [\delta(y - w) \delta(x - y') f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y)] \Psi(\gamma) + \\
+ \oint dy^c \oint dy^d D^y_b [\delta(y - w) \delta(x - y') f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y)] \Psi(\gamma) + \\
- \oint dy^c \oint dy^d \delta(x - y') f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y) \Psi(\gamma) \\
- \oint dy^c \oint dy^d \delta(x - y') f_c(y - y') O_{y'y'} \Delta_{cd}(\gamma_0^y) \Psi(\gamma) \right). 
\]
The second term cancels with the first after interchanging $y$ and $y'$ (and again removing the regulator so that 
\( \delta(y - w) f_{x}(y) = \delta(y' - w) f_{x}(y') \)). If we interchange $y$ and $y'$ in the last term and integrate by parts in $x$ and $w$ we get the final result,

\[
\int d^{3}w d^{3}x (N(x) \partial_{x} M_{y}(w) - M_{y}(w) \partial_{x} N(x)) \int_{\gamma} d\gamma \int_{\gamma} d\gamma \int_{\gamma} d\gamma \int_{\gamma} d\gamma \int_{\gamma} d\gamma \int_{\gamma} d\gamma \times \\
\quad \delta(z - w) \delta(x - y') f_{x}(y) \Delta_{x x}(\gamma_{a}^{x}) \Psi_{y}(\gamma) = H(\mathcal{L}_{M} N)
\]  

(37)

which reproduces the classical commutator.

V. COMMUTATOR OF TWO HAMILTONIANS

The commutator of two Hamiltonians introduces another new computational requirement. While computing the commutator, one gets the product of two rerouting operators: one per each Hamiltonian. The result one expects involves only one rerouting, the one due to the presence of the metric in the function smear the resulting diffeomorphism. We will give greater details of this aspect of the calculation in appendix B.

The commutator of two Hamiltonians can be written as,

\[
[H(N)H(M) - H(M)H(N)]\Psi(L) = \int d^{3}x N(x) \int d^{3}w M(w) \times \\
\quad \int_{\gamma} dy_{c} \int_{\gamma} dy_{d} \delta(x - y') f_{x}(y) \delta(y - y') [O_{y} y' + O_{y} y'] \Delta_{x d}(\gamma_{a}^{x}) \times \\
\quad \int_{\gamma} dz_{p} \int_{\gamma} dz'_{d} \delta(w - z') f_{p}(z) \delta(z - z') [O_{z} z' + O_{z} z'] \Delta_{x p h}(\gamma_{a}^{x}) \Psi(\gamma) \times \\
\quad - \int d^{3}x N(x) \int d^{3}w M(w) \times \\
\quad \int_{\gamma} dz_{p} \int_{\gamma} dz'_{d} \delta(w - z') f_{p}(z) \delta(z - z') [O_{z} z' + O_{z} z'] \Delta_{p h}(\gamma_{a}^{x}) \Psi(\gamma).
\]  

(38)

We now proceed as in the previous case. Since several terms in the commutator cancel in similar fashion, we will show the cancellation explicitly for one case and will indicate how to proceed with the others verbally.

Let us start by considering the terms that arise from the action of the derivative operator of the first Hamiltonian on the integrand of the second operator. We will discuss later on the terms that arise from the action of the derivative operator on the loop dependence of the integrals of the second Hamiltonian, which will give rise to the nonvanishing part of the commutator, which reproduces the classical result.

The calculation will be performed as follows. We will consider each of the above mentioned terms in $H(N)H(M)$ and perform operations that show that they are actually term in $H(N)H(M)$ and therefore cancel in the commutator. This will involve proving identities among integrals along petals of intersecting loops. For this it is convenient to introduce loop diagrams. In these diagrams we represent each loop as an oriented segment (from left to right) and we mark the ordering of the different variables in the integrands along the loops. We will see that in several cases the order in which variables appear implies that the integrals are actually zero. Several operations we will perform involving the rerouting operator will resemble the ones we performed in the previous section.

We now consider one of the contributions indicated above, extract the rerouting operator outside the loop integral as we did in the previous section and we get,

\[
\int_{\gamma} dy_{c} \int_{\gamma} dy_{d} \delta(x - y') f_{x}(y) \times \\
\quad \int_{[O_{y} y']_{a} y'} \int_{[O_{y} y']_{y} y} dz'_{d} \delta(w - z') f_{p}(z) \delta(z - z') [O_{z} z' + O_{z} z'] \Delta_{x d}(\gamma_{a}^{x}) \Psi(\gamma) + \\
\quad \int_{\gamma} dy_{c} \int_{\gamma} dy_{d} \delta(x - y') f_{x}(y) \times
\]
\[ \oint_{[O_y y']_{v'}} dz^{[p]} \oint_{[O_y y']_{v'}} dz^{[h]} \delta(w - z') f_\rho(z - z') O_{y y'} \Delta_{c d}(\gamma^y_0) \Delta_{p h}(\gamma^z_0) \Psi(\gamma) = \]

\[ = \oint_{[O_y y']_{v'}} dy'^{[c]} \oint_{[O_y y']_{v'}} dy'^{[d]} \delta(x - y') f_\epsilon(y - y') \times \]

\[ \oint_{[O_y y']_{v'}} dz^{[p]} \oint_{[O_y y']_{v'}} dz^{[h]} \delta(w - z') f_\rho(z - z') O_{y y'} \Delta_{c d}(\gamma^y_0) \Delta_{p h}(\gamma^z_0) \Psi(\gamma) - \]

\[ - \oint_{[O_y y']_{v'}} dy'^{[c]} \oint_{[O_y y']_{v'}} dy'^{[d]} \delta(x - y') f_\epsilon(y - y') \times \]

\[ \oint_{[O_y y']_{v'}} dz^{[p]} \oint_{[O_y y']_{v'}} dz^{[h]} \delta(w - z') f_\rho(z - z') \]

We now use the commutation relation for loop derivatives (7) to obtain the identity,

\[ \Delta_{c d}(\gamma^y_0) \Delta_{p h}(\gamma^z_0) \Psi(\gamma) = \]

\[ = [\Delta_{p h}(\gamma^z_0) \Delta_{c d}(\gamma^y_0) + \theta(z - y) \Delta_{c d}(\gamma^y_0)] \Delta_{p h}(\gamma^z_0) \Psi(\gamma) = \]

\[ = [\Delta_{p h}(\gamma^z_0) \Delta_{c d}(\gamma^y_0)] + \Delta_{c d}(\gamma^z_0) \Delta_{p h}(\gamma^z_0) - \Theta(y, z) \Delta_{p h}(\gamma^z_0) \Delta_{c d}(\gamma^y_0)] \Psi(\gamma) = \]

\[ = [\Theta(z, y) \Delta_{p h}(\gamma^z_0) \Delta_{c d}(\gamma^y_0)] + \Delta_{c d}(\gamma^y_0) \Delta_{p h}(\gamma^z_0) \Psi(\gamma) = \]

\[ = \Delta_{p h}(\gamma^z_0) \Delta_{c d}(\gamma^y_0) \Psi(\gamma). \]

(40)

It is now useful to draw the loop diagrams corresponding to the integrals present in (39). In the diagrams we indicate the relative domain of integration of the different variables in the first term of the expression.

\[ o \quad z \quad y' \quad y' \quad o \quad z' \quad y \quad y \]

By looking at the diagram, we see we can rewrite the integrals in the following way

\[ \oint_{[O_z z']_{y z}} dz^{[p]} \oint_{[O_z z']_{y z}} dz^{[h]} \delta(w - z') f_\rho(z - z') \]

\[ \oint_{[O_z z']_{y z}} dy'^{[c]} \oint_{[O_z z']_{y z}} dy'^{[d]} \delta(x - y') f_\epsilon(y - y') O_{y y'} \Delta_{c d}(\gamma^y_0) \Delta_{p h}(\gamma^z_0) \Psi(\gamma) - \]

\[ - \oint_{[O_z z']_{y z}} dy'^{[c]} \oint_{[O_z z']_{y z}} dy'^{[d]} \delta(x - y') f_\epsilon(y - y') O_{y y'} \Delta_{c d}(\gamma^y_0) \Delta_{p h}(\gamma^z_0) \Psi(\gamma). \]

(41)

We now use the fact that \( O_{y y'} O_{z z'} = O_{z' z} O_{y y'} \) and commute rerouting and derivative operators again.

\[ \oint_{[O_z z']_{y z}} dz^{[p]} \oint_{[O_z z']_{y z}} dz^{[h]} \delta(w - z') f_\rho(z - z') \]

\[ \oint_{[O_z z']_{y z}} dy'^{[c]} \oint_{[O_z z']_{y z}} dy'^{[d]} \delta(x - y') f_\epsilon(y - y') O_{z z'} \Delta_{c d}(\gamma^z_0) \Delta_{p h}(\gamma^z_0) \Psi(\gamma) - \]

\[ - \oint_{[O_z z']_{y z}} dy'^{[c]} \oint_{[O_z z']_{y z}} dy'^{[d]} \delta(x - y') f_\epsilon(y - y') O_{z z'} \Delta_{c d}(\gamma^z_0) \Delta_{p h}(\gamma^z_0) \Psi(\gamma) = \]

\[ = \oint_{[O_z z']_{y z}} dz^{[p]} \oint_{[O_z z']_{y z}} dz^{[h]} \delta(w - z') f_\rho(z - z') \]

\[ \oint_{[O_z z']_{y z}} dy'^{[c]} \oint_{[O_z z']_{y z}} dy'^{[d]} \delta(x - y') f_\epsilon(y - y') O_{z z'} \Delta_{c d}(\gamma^z_0) \Delta_{p h}(\gamma^z_0) \Psi(\gamma) + \]
by the second, which is similar in form. As before, we need only consider the proof for $z$ as it can be seen through the results of appendix A, we get, 

$$\oint z z \gamma \delta(x - y')f_x(y - y')O_{z z'}\Delta_{ph}(\gamma_0^z)\Omega_{y y'}\Delta_{cd}(\gamma_0^y)\Psi(\gamma) =$$

$$= \oint z z' \delta(x - y')f_x(y - y')O_{z z'}\Delta_{ph}(\gamma_0^z)\Omega_{y y'}\Delta_{cd}(\gamma_0^y)\Psi(\gamma).$$

(42)

This proves that all terms with four loop integrals that appear in the commutator cancel among themselves (each order of the operators in the commutator produces an equal contribution).

We now need to address the terms with three loop integrals. In order to do this we begin with the following observation. If in the second term of expression (42) we rename pairwise the names of the following variables, $z$ and $y$, $z'$ and $y'$, $p$ and $c$ and finally $h$ and $d$. The only real change in the last term of the commutator is that $x$ gets replaced by $w$ in the delta's arguments. Then, we may work with the first term and only at the end add the contribution of the second, which is similar in form. As before, we need only consider the proof for $O_{y y'}O_{z z'}$. Since in the terms we are considering the loop derivative acts only on the loop dependence of the integrals, terms with different reroutings can be treated in a similar fashion.

We begin studying the action of the loop derivative of the first Hamiltonian on the third and fourth loop integrals,

$$\oint \delta(y) \delta(x - y')f_x(y - y')\times$$

$$\oint \delta(y) \delta(x - y')f_x(y - y')O_{z z'}\Delta_{ph}(\gamma_0^z)\Omega_{y y'}\Delta_{cd}(\gamma_0^y)\Psi(\gamma).$$

(43)

In the first integral we need to do the following substitution (valid only in the limit when the regulator is removed),

$$f_{\rho}(z - z')O_{z z'}\Delta_{ph}(\gamma_0^z)\Omega_{y y'}\Delta_{cd}(\gamma_0^y).$$

(44)

As an aid to understand this expression one should remember that the corresponding expression in the connection representation would be,

$$f_{\rho}(z - z')Tr[F_{ph}(z)U_{z z'}U_{z y}U_{z y'}] = -f_{\rho}(z - z')Tr[F_{ph}(z')U_{z y}U_{z y'}]$$

(45)

Finally, working in an analogous way as in the previous section the result is,

$$\oint \delta(y) \delta(x - y')f_x(y - y') \times$$

$$O_{y y'} \left( 2 \oint \delta(w - z)D_{[c]y\rho}(y - z)O_{z y}\Delta_{ph}(\gamma_0^z)\Psi(\gamma) + \right.$$  

$$\left. + 2 \oint \delta(w - z)D_{[c]y\rho}(y - z)O_{z y}\Delta_{ph}(\gamma_0^z)\Psi(\gamma) \right).$$

(46)

Considering the fact that the Mandelstam derivative commutes with the loop integral and the rerouting operator, as it can be seen through the results of appendix A, we get,

$$\oint \delta(y) \delta(x - y')f_x(y - y') \times$$

$$\left( D_{[c]y\rho}(y - z)O_{z y}\Delta_{ph}(\gamma_0^z)\Psi(\gamma) - \right.$$  

$$\left. - D_{[c]y\rho}(y - z)O_{z y}\Delta_{ph}(\gamma_0^z)\Psi(\gamma) \right),$$

(47)
and removing the regulator we get,
\[
\frac{\partial \rho}{\partial y} \delta(x - y) f_{\rho}(y - y') 	imes
\left( \partial \rho \delta(w - y) O_{y' y} \int_{\gamma} \frac{dz}{\gamma} [f_{\rho}(y - z) O_{z o y} + f_{\rho}(z - y) O_{z y}] \Delta_{d p}(\gamma^o_o) \right) \Psi(\gamma) +
\frac{\partial \rho}{\partial y} \delta(w - y) [O_{y' y} \int_{\gamma} \frac{dz}{\gamma} [f_{\rho}(y - z) O_{z o y} + f_{\rho}(z - y) O_{z y}] \Delta_{d p}(\gamma^o_o) \right] \Psi(\gamma) -
\left(-\partial^w_{y} \delta(w - y) O_{y' y} \int_{\gamma} \frac{dz}{\gamma} [f_{\rho}(y - z) O_{z o y} + f_{\rho}(z - y) O_{z y}] \Delta_{d p}(\gamma^o_o) \right) \Psi(\gamma) -
\left(-\partial^w_{y} \delta(w - y) O_{y' y} \int_{\gamma} \frac{dz}{\gamma} [f_{\rho}(y - z) O_{z o y} + f_{\rho}(z - y) O_{z y}] \Delta_{c p}(\gamma^o_o) \right) \Psi(\gamma)\right)
\]
(48)

In the limit in which the regulator is removed the product \(\delta(x - y) \times \delta(w - y)\) is symmetric in \(x\) and \(w\) and therefore some of the terms in the above expression cancel with those coming from the other term in the commutator. The remaining terms are
\[
\frac{\partial \rho}{\partial y} \delta(x - y) f_{\rho}(y - y') 	imes
\left(-\partial^w_{y} \delta(w - y) O_{y' y} \int_{\gamma} \frac{dz}{\gamma} [f_{\rho}(y - z) O_{z o y} + f_{\rho}(z - y) O_{z y}] \Delta_{d p}(\gamma^o_o) \right) \Psi(\gamma) -
\left(-\partial^w_{y} \delta(w - y) O_{y' y} \int_{\gamma} \frac{dz}{\gamma} [f_{\rho}(y - z) O_{z o y} + f_{\rho}(z - y) O_{z y}] \Delta_{c p}(\gamma^o_o) \right) \Psi(\gamma)\right)
\]
(49)

The following step is to integrate by parts
\[
\int d^3 x N(x) \int d^3 w M(w) \times
\left( \int_{\gamma} \frac{dy}{\gamma} \int_{\gamma} \frac{dy^d}{\gamma} \delta(x - y') f_{\rho}(y - y') \times
\left(-\partial^w_{y} \delta(w - y) O_{y' y} \int_{\gamma} \frac{dz}{\gamma} [f_{\rho}(y - z) O_{z o y} + f_{\rho}(z - y) O_{z y}] \Delta_{d p}(\gamma^o_o) \right) \Psi(\gamma) -
\left(-\partial^w_{y} \delta(w - y) O_{y' y} \int_{\gamma} \frac{dz}{\gamma} [f_{\rho}(y - z) O_{z o y} + f_{\rho}(z - y) O_{z y}] \Delta_{c p}(\gamma^o_o) \right) \Psi(\gamma)\right)
\]
(50)

\[
= - \int d^3 x N(x) \int d^3 w D^w (M(w) \int_{\gamma} \frac{dy}{\gamma} \int_{\gamma} \frac{dy^d}{\gamma} \delta(x - y') f_{\rho}(y - y') \delta(w - y) \times
O_{y' y} \int_{\gamma} \frac{dz}{\gamma} [f_{\rho}(y - z) O_{z o y} + O_{z y}] \Delta_{d p}(\gamma^o_o) \right) \Psi(\gamma) +
\int d^3 x N(x) \int d^3 w \partial^w_{y} M(w) \int_{\gamma} \frac{dy}{\gamma} \int_{\gamma} \frac{dy^d}{\gamma} \delta(x - y') f_{\rho}(y - y') \delta(w - y) \times
O_{y' y} \int_{\gamma} \frac{dz}{\gamma} [f_{\rho}(y - z) O_{z o y} + O_{z y}] \Delta_{d p}(\gamma^o_o) \right) \Psi(\gamma) +
\int d^3 x N(x) \int d^3 w D^w (M(w) \int_{\gamma} \frac{dy}{\gamma} \int_{\gamma} \frac{dy^d}{\gamma} \delta(x - y') f_{\rho}(y - y') \delta(w - y) \times
O_{y' y} \int_{\gamma} \frac{dz}{\gamma} [f_{\rho}(y - z) O_{z o y} + O_{z y}] \Delta_{c p}(\gamma^o_o) \right) \Psi(\gamma)
\]

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- \int d^3x \mathcal{N}(x) \int d^3w \partial_\gamma^\mu M(w) \times \oint_\gamma dy^c \oint_\gamma dy^d \delta(x - y') f_\epsilon(y - y') \delta(w - y) \times

O_{y'y'} \oint_\gamma dz^p f_\rho(y - z) [O_{z o} + O_{z y}] \Delta_{d p}(\gamma_0^z)] \Psi(\gamma), \tag{51}

and to notice that since we are considering only the case of compact manifolds, the total derivatives that appear in the first and third terms do not contribute. The resulting terms can be rewritten as,

\int d^3x \int d^3w \mathcal{N}(x) \partial_\epsilon^w M(w) \times 2 \oint_\gamma dy^c \oint_\gamma dy^d \delta(x - y') f_\epsilon(y - y') \delta(w - y) \times

O_{y'y'} \oint_\gamma dz^p f_\rho(y - z) [O_{z o} + O_{z y}] \Delta_{d p}(\gamma_0^z)] \Psi(\gamma). \tag{52}

Taking into account the discussion at the beginning of this section, it is very easy to see (due to the symmetry in changing x and w in the deltas) that the other term of the commutator corresponding to the last one obtained is,

- \int d^3x \int d^3w M(w) \partial_\epsilon^x \mathcal{N}(x) \times 2 \oint_\gamma dy^c \oint_\gamma dy^d \delta(x - y') f_\epsilon(y - y') \delta(w - y) \times

O_{y'y'} \oint_\gamma dz^p f_\rho(y - z) [O_{z o} + O_{z y}] \Delta_{d p}(\gamma_0^z)] \Psi(\gamma).

The result then is,

\int d^3x \int d^3w (\mathcal{N}(x) \partial_\epsilon^w M(w) - M(w) \partial_\epsilon^x \mathcal{N}(x)) f_\epsilon(w - x) \times

O_{y'y'} \oint_\gamma dz^p f_\rho(y - z) [O_{z o} + O_{z y}] \Delta_{d p}(\gamma_0^z)] \Psi(\gamma) -

- \oint_\gamma dy^d \oint_\gamma dy^c \delta(x - y') f_\epsilon(y - y') \delta(w - y) \times

O_{y'y'} \oint_\gamma dz^p f_\rho(y - z) [O_{z o} + O_{z y}] \Delta_{d p}(\gamma_0^z)] \Psi(\gamma). \tag{54}

We now make some changes in this last expression so that we can use the equality of appendix B. Noticing that,

\delta(x - y') f_\epsilon(y - y') \delta(w - y) = \delta(x - y') f_\epsilon(w - x) \delta(w - y) \tag{55}

we can rewrite the expression of interest as,

\int d^3x \int d^3w (\mathcal{N}(x) \partial_\epsilon^w M(w) - M(w) \partial_\epsilon^x \mathcal{N}(x)) f_\epsilon(w - x) \times

\int d^3y \delta(w - \bar{y}) \int d^3z f_\rho(w - \bar{z}) \oint_\gamma dy^c \oint_\gamma dy^d \delta(y - y') \times

\left[ O_{y'y'} \oint_\gamma dz^p z - \bar{z} [O_{z o} + O_{z y}] \Delta_{d p}(\gamma_0^z)] \Psi(\gamma) -

- O_{y'y'} \oint_\gamma dz^p z - \bar{z} [O_{z o} + O_{z y}] \Delta_{d p}(\gamma_0^z)] \Psi(\gamma) \right], \tag{56}

and then as,

\int d^3x \int d^3w (\mathcal{N}(x) \partial_\epsilon^w M(w) - M(w) \partial_\epsilon^x \mathcal{N}(x)) f_\epsilon(w - x) \times

\int d^3y \delta(w - \bar{y}) \int d^3z f_\rho(w - \bar{z}) \oint_\gamma dy^c \oint_\gamma dy^d \delta(y - y') \times

\oint_\gamma dz^p z - \bar{z} [O_{y'y'} [O_{z o} + O_{z y}] + O_{y'y'} [O_{z o} + O_{z y}] \Delta_{d p}(\gamma_0^z)] \Psi(\gamma). \tag{57}
Finally, using the result of appendix B we can rewrite the above expression in terms that only involve one rerouting operator, as is required by the desired final result (which only involves one rerouting coming from the metric that appears smearing the diffeomorphism constraint). As a consequence of this the commutator is,

$$\int d^3x \int d^3w \left( \mathcal{N}(x) \partial_c^w \mathcal{M}(w) - \mathcal{M}(w) \partial_c^w \mathcal{N}(x) \right) f_\epsilon(w - x)^\times$$

$$\int d\pi^3 \delta(w - \pi) \int d\pi^3 f_\epsilon(w - \pi)^\times$$

$$2 \left[ \oint_\gamma d^3y^\delta(y - \pi) \oint_\gamma d^3y^d(x - y') \oint_{O_y y'} d^3z^p \delta(z - \pi) \left[ O_{yz} \frac{1}{2} \Delta_{dp} - O_{yz} \frac{1}{2} \Delta_{dp} \right] \Psi(\gamma) \right].$$

We now prove that in the above expression two of the terms cancel each other and the other two give rise to the desired result. Consider the following two terms,

$$\oint_\gamma d^3y^\delta(y - \pi) \oint_\gamma d^3y^d(x - y') \oint_{O_y y'} d^3z^p \delta(z - \pi) O_{yz} \frac{1}{2} \Delta_{dp} \Psi(\gamma)$$

$$+ \oint_\gamma d^3y^\delta(y - \pi) \oint_\gamma d^3y^d(x - y') \oint_{O_y y'} d^3z^p \delta(z - \pi) \frac{1}{2} \Delta_{dp} \Psi(\gamma).$$

We can rewrite them as,

$$\int d^3\bar{y}^\delta(x - \bar{y}) \oint_\gamma d^3y^\delta(y - \pi) \oint_\gamma d^3y^d(y' - \bar{y}) \oint_{O_y y'} d^3z^p \delta(z - \pi) O_{yz} \frac{1}{2} \Delta_{dp} \Psi(\gamma)$$

$$+ \int d^3\bar{y}^\delta(x - \bar{y}) \oint_\gamma d^3y^\delta(y - \pi) \oint_\gamma d^3y^d(y' - \bar{y}) \oint_{O_y y'} d^3z^p \delta(z - \pi) \frac{1}{2} \Delta_{dp} \Psi(\gamma).$$

It is convenient to write the product of the last two loop integrals symbolically and study them with diagrams, assuming for example that $y'$ precedes $y$ in the loop $\gamma$. The alternative case can be studied in a similar way. So,

$$\int_{\gamma_{yo}} d^3y^d \left[ \int_{\gamma_{yo'}} d^3z^p + \int_{\gamma_{yo'}} d^3z^p + \int_{\gamma_{yo'}} d^3z^p \right] +$$

$$\int_{\gamma_{yo'}} d^3y^d \left[ \int_{\gamma_{yo'}} d^3z^p + \int_{\gamma_{yo'}} d^3z^p + \int_{\gamma_{yo'}} d^3z^p \right] =$$

$$\int_{\gamma_{oy}} d^3z^p \left[ \int_{\gamma_{oy}} d^3y^d + \int_{\gamma_{oz}} d^3y^d + \int_{\gamma_{ox}} d^3y^d \right] +$$

$$\int_{\gamma_{oy}} d^3z^p \left[ \int_{\gamma_{oy}} d^3y^d + \int_{\gamma_{oz}} d^3y^d + \int_{\gamma_{ox}} d^3y^d \right].$$

We can then rewrite expression (60) as,

$$\int d^3\bar{y}^\delta(x - \bar{y}) \oint_\gamma d^3y^\delta(y - \pi) \oint_{O_y y'} d^3z^p \delta(\bar{y} - z) \oint_{O_y y'} d^3y^d(y' - \bar{y}) O_{yz} \frac{1}{2} \Delta_{dp} \Psi(\gamma)$$

$$+ \int d^3\bar{y}^\delta(x - \bar{y}) \oint_\gamma d^3y^\delta(y - \pi) \oint_{O_y y'} d^3z^p \delta(\bar{y} - z) \oint_{O_y y'} d^3y^d(y' - \bar{y}) \frac{1}{2} \Delta_{dp} \Psi(\gamma).$$
and making the change \( y' \) by \( z \) (and removing both regulators so that one can change \( y' \) by \( z \) only in the last two deltas) we get,

\[
\int d^3 y \delta (x - y') \oint_\gamma dy' \delta (y - y') \oint_\gamma dy'' \delta (y' - y') \oint_{O_{x} y \gamma} dz' \delta (z - z') O_{y} \Delta_{dp}(\gamma_{y}^y) \Psi(\gamma)
\]

\[
+ \int d^3 y \delta (x - y') \oint_\gamma dy' \delta (y - y') \oint_\gamma dy'' \delta (y' - y') \oint_\gamma dz' \delta (z - z') \frac{1}{2} \Delta_{dp}(\gamma_{y}^y) \Psi(\gamma).
\]  

Finally, taking into account that \( O_{y} \Delta_{dp}(\gamma_{y}^y) = -O_{x} \Delta_{dp}(\gamma_{y}^y) \) , and changing \( d \) by \( p \) we get,

\[
- \int d^3 y \delta (x - y') \oint_\gamma dy' \delta (y - y') \oint_\gamma dy'' \delta (y' - y') \oint_{O_{x} y \gamma} dz' \delta (z - z') O_{y'} \Delta_{dp}(\gamma_{y}^y) \Psi(\gamma)
\]

\[
- \int d^3 y \delta (x - y') \oint_\gamma dy' \delta (y - y') \oint_\gamma dy'' \delta (y' - y') \oint_\gamma dz' \delta (z - z') \frac{1}{2} \Delta_{dp}(\gamma_{y}^y) \Psi(\gamma).
\]  

We have therefore proved that expression \( 60 \) is equal to minus itself. Using the same tools one can demonstrate that the two remaining terms in \( 58 \) are equivalent to,

\[
\int d^3 x \int d^3 w \left( N(x) \partial^w M(w) - M(w) \partial^x N(x) \right) f_x(w - x) \times
\]

\[
\int dy \delta (w - y) \int dz \delta (w - z) \times
\]

\[
2 \left[ \oint_\gamma dy' \delta (y - y') \oint_\gamma dy'' \delta (y' - y') \oint_{O_{x} y \gamma} dz' \delta (z - z') O_{y} \Delta_{dp}(\gamma_{y}^y) \Psi(\gamma)
\]

\[
+ \oint_\gamma dy' \delta (y - y') \oint_\gamma dy'' \delta (y' - y') \times \oint_\gamma dz' \delta (z - z') \frac{1}{2} \Delta_{dp}(\gamma_{y}^y) \Psi(\gamma) \right].
\]  

Inserting the factor \( \frac{\xi}{2} \) that was omitted, this last expression is the regularized form of the product of the metric operator times the diffeomorphism constraint, which is consistent with the classical result (see appendix C).

VI. CONCLUSIONS

In this paper we have studied the formal commutation relations of the constraints of quantum gravity in the loop representation. We have shown that the expressions available for the constraint are operationally useful to compute the commutators. We use a background dependent regulator and only recover the correct commutation relations in the formal limit in which the regulator is removed. Our computations set the stage for a future computation keeping the regulators at the lowest order.

VII. ACKNOWLEDGEMENTS

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APPENDIX A:

We need a gauge invariant regularization prescription. If one performs a point-splitting at intersections, one needs to close the loops to preserve gauge invariance after the rerouting process. In the following example we denote the intersecting point as \( I \). We will use the following notation in this appendix: \( \gamma_{y'} \) denotes the portion of \( \gamma \) from \( y \) to \( y' \) without passing through the origin no matter the original orientation of that portion. The other cases that appear have an analogous meaning. Explicitly, the gauge invariance-preserving action of the rerouting operator can be defined as,
\[ f_r(y - y') \, O_{y'y'} \, \Psi(\gamma) = f_r(y - y') \Psi(O_{y'y'} \gamma) = f_r(y - y') \, \Psi(\gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'}) = \]

Let us now consider the action of the rerouting operator on the loop derivative. Consider that the point \( z \) is in the piece \( \gamma_{y'y'} \) (\( y \) before \( y' \), for example) \([14]\). Then,

\[ O_{y'y'} \, \Delta_d \, (\gamma_z) \, \Psi(\gamma) = - \Delta_d \, (\gamma_z) \, O_{y'y'} \, \Psi(\gamma) = \Delta_d \, (\gamma_{y'y'} \, (\gamma_z)) \, \Psi(O_{y'y'} \gamma). \]  

The minus sign appears because the rerouting affects the loop dependence of the loop derivative. It is a situation analogous to action of the parity operator in quantum mechanics on the partial derivative,

\[ P \, \partial_y \, \Psi(x) = - \partial_y \, P \, \Psi(x) = \partial_y \, \Psi(x'). \]

If the point \( z \) of the loop derivative is in \( \gamma_{y'y'} \) there is no sign change.

Let us now consider specifically the case that appears in the Hamiltonian constraint, where the loop derivative is evaluated at one of the points of the action of the rerouting operator,

\[ \sigma d \, f_r(y - y') \, O_{y'y'} \, \Delta_d \, (\gamma_z) \, \Psi(\gamma) \equiv f_r(y - y') \, \delta_y \, \Psi(\gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'}) \]

where the result of the operator \( \delta_y \)

\[ \delta_y \, \Psi(\gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'}) = \Psi(\delta_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'}) - \Psi(\gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'} \circ \gamma_{y'y'}) \]

Notice that the definition of the operator \( \delta_y \) implies that the action of the loop derivative is with respect to \( \gamma \) and the rerouting operator acts \textit{afterwards}.

\[ \textbf{APPENDIX B:} \]

In this appendix we will prove that expressions \([57]\) and \([58]\), are equal \([10]\). First we write the Mandelstam identities for wavefunctions. For loops \( \gamma, \gamma_1, \gamma_2, \) and \( \gamma_3 \),

\[ \Psi(\gamma) = \Psi(\gamma^{-1}) \]
\[ \Psi(\gamma_1 \circ \gamma_2) = \Psi(\gamma_2 \circ \gamma_1) \]
\[ \Psi(\gamma_1 \circ \gamma_2 \circ \gamma_3) + \Psi(\gamma_1 \circ \gamma_2 \circ \gamma_3^{-1}) = \Psi(\gamma_2 \circ \gamma_1 \circ \gamma_3) + \Psi(\gamma_2 \circ \gamma_1 \circ \gamma_3^{-1}). \]

Considering equation \([A3]\), (with \( p = z \)) we can write the action of the loop operators in expression \([57]\),

\[ \delta_3 \, \Psi(O_{y'y'} O_{z \circ y' \gamma}) + \delta_3 \, \Psi(O_{y'y'} O_{z \circ y' \gamma}) + \delta_z \, \Psi(O_{y'y'} O_{z \circ y' \gamma}) + \delta_z \, \Psi(O_{y'y'} O_{z \circ y' \gamma}) \]

Additionally, using the results of the appendixes A and the Mandelstam identities of Appendix B we can show \( (p \) is the intersecting point of \( \gamma_1 \circ \gamma_2 \circ \gamma_3) \)

\[ \delta_p \, \Psi(\gamma_1 \circ \gamma_2 \circ \gamma_3) + \delta_p \, \Psi(\gamma_3 \circ \gamma_1 \circ \gamma_2) + \delta_p \, \Psi(\gamma_2 \circ \gamma_1 \circ \gamma_3) + \delta_p \, \Psi(\gamma_1 \circ \gamma_2 \circ \gamma_3) = 2 \delta_p \, \Psi(\gamma_3 \circ \gamma_2 \circ \gamma_1) + \frac{1}{2} \delta_p \, \Psi(\gamma_3 \circ \gamma_2 \circ \gamma_1) - \delta_p \, \Psi(\gamma_1 \circ \gamma_3 \circ \gamma_2) - \frac{1}{2} \delta_p \, \Psi(\gamma_1 \circ \gamma_3 \circ \gamma_2). \]

Using this last identity, the integrand in \([57]\) can be written as,

\[ 2 \delta_p \, \Psi(O_{y'y'} \gamma) + \frac{1}{2} \delta_p \, \Psi(\gamma - y') \, \Psi(O_{y'y'} \gamma) - \frac{1}{2} \delta_p \, \Psi(\gamma). \]

On the other hand, we can make a partition in the loop integrals so that expression \([57]\) splits in 6 terms. Symbolically we can write,

\[ \int_{y} d\gamma \int_{y} d\gamma' \int_{O_{y'y'}} d\gamma'' \int_{O_{y'y'}} d\gamma''' \int_{O_{y'y'}} d\gamma'''' \int_{O_{y'y'}} d\gamma''''] \times \]

\[ \times \left[ \int_{O_{\gamma_y}} d\gamma''' + \int_{O_{\gamma_y}} d\gamma''' + \int_{O_{\gamma_y}} d\gamma''' + \int_{O_{\gamma_y}} d\gamma''' \right]. \]
Then, combining (B6) and (B7), we finally get expression (58), which in shorthand writing is,

$$
2 \oint d\gamma \oint \frac{d\gamma}{\gamma} \oint d\gamma \oint d\gamma \oint \frac{d\gamma}{\gamma} dz^p\left[O_{zy} \Delta_d p(\gamma y) - O_{zy} \Delta_d p(\gamma y)\right] +
$$

$$
+ 2 \oint d\gamma \oint \frac{d\gamma}{\gamma} \oint d\gamma \oint d\gamma \oint \frac{d\gamma}{\gamma} dz^p\left[\frac{1}{2} \Delta_d p(\gamma y) - \frac{1}{2} \Delta_d p(\gamma y)\right]
$$

(B8)

As an aid to visualize the meaning of the identity (B5), its counterpart in the connection representation is the following identity between SU(2) matrices,

$$
Tr[x^d B A^{-1} C] + Tr[x^d C B A^{-1}] + Tr[x^d A^{-1} B^{-1} C] + Tr[x^d B^{-1} C A^{-1}] =
$$

$$
2[Tr[B x^d C] Tr[A] - \frac{1}{2} Tr[AB x^d C] - Tr[A x^d B] Tr[C] + \frac{1}{2} Tr[A x^d BC]]
$$

$$
= 2[\frac{1}{2} Tr[AB x^d C] + Tr[B x^d CA^{-1}] - \frac{1}{2} Tr[A x^d BC] - Tr[A x^d BC^{-1}]]
$$

(B9)

Where $x^d = \frac{1}{1 + \sigma^d}$, and $\sigma^d$ for $1 \leq d \leq 3$ are the Pauli matrices and $A, B$ and $C$, are complex SU(2) matrices. This identity can be easily proved noting that,

$$
Tr[x^d PQ^{-1}] + Tr[x^d Q^{-1} P] = 2 Tr[x^d P] Tr[Q] - Tr[x^d PQ] - Tr[x^d Q P]
$$

(B10)

and,

$$
Tr[P^{-1} x^d Q] = -Tr[x^d P] Tr[Q] + Tr[x^d PQ]
$$

$$
Tr[Q x^d P^{-1}] = Tr[x^d Q P] - Tr[x^d P] Tr[Q]
$$

(B11)

(B12)

**APPENDIX C:**

To obtain the expression for the metric operator in the loop representation we first compute it in the connection representation and apply the loop version. The action of the metric operator is,

$$
f_\varepsilon(\overline{\omega} - \overline{\tau}) \frac{\delta}{\delta \overline{A(\overline{w})}} Tr[U(\gamma)] = f_\varepsilon(\overline{w} - \overline{\tau}) \oint \frac{dz^\alpha d(z - \overline{\tau})}{\gamma} \times
$$

$$
\frac{1}{2} \left[ \oint \frac{dw^b \delta(w - \overline{\omega}) Tr[U(\gamma z)] U(\overline{w}_z \omega_z) + \oint \frac{dw^b \delta(w - \overline{\omega}) Tr[U(\gamma z)] U(\overline{w}_z \omega_z)}{\gamma z} \right] +
$$

$$
+ f_\varepsilon(\overline{w} - \overline{\tau}) \frac{1}{4} \oint \frac{dz^\alpha d(z - \overline{\tau})}{\gamma} \oint \frac{dw^b \delta(w - \overline{w}) Tr[U(\gamma)]}{\gamma z w} =
$$

$$
= \frac{1}{2} f_\varepsilon(\overline{w} - \overline{\tau}) \oint \frac{dz^\alpha d(z - \overline{\tau})}{\gamma} \left[ \oint \frac{dw^b \delta(w - \overline{w}) O_{w z} Tr[U(\gamma)]}{\gamma z w} + \oint \frac{dw^b \delta(w - \overline{w}) O_{z w} Tr[U(\gamma)]}{\gamma z w} \right]
$$

$$
+ f_\varepsilon(\overline{w} - \overline{\tau}) \frac{1}{4} \oint \frac{dz^\alpha d(z - \overline{\tau})}{\gamma} \oint \frac{dw^b \delta(w - \overline{\omega}) Tr[U(\gamma)]}{\gamma z w} =
$$

$$
= \frac{1}{2} f_\varepsilon(\overline{w} - \overline{\tau}) \oint \frac{dz^\alpha d(z - \overline{\tau})}{\gamma} \oint \frac{dw^b \delta(w - \overline{\omega}) O_{w z} Tr[U(\gamma)]}{\gamma z w} +
$$

$$
+ f_\varepsilon(\overline{w} - \overline{\tau}) \frac{1}{4} \oint \frac{dz^\alpha d(z - \overline{\tau})}{\gamma} \oint \frac{dw^b \delta(w - \overline{\omega}) Tr[U(\gamma)]}{\gamma z w}
$$

(C1)

Where we have used that $O_{w z} Tr[U(\gamma)] = O_{z w} Tr[U(\gamma)]$. Recall, when comparing with section VI, that we have omitted a factor $\frac{1}{4}$ in the commutator of the Hamiltonians.

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