Krein Regularization of $\lambda \phi^4$

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Abstract

We calculate the four-point function in $\lambda \phi^4$ theory by using Krein regularization and compare our result, which is finite, with the usual result in $\lambda \phi^4$ theory. The effective coupling constant ($\lambda_\mu$) is also calculated in this method.

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1 Introduction

Due to the appearance of infrared divergence in the two point function for the minimally coupled scalar field in de Sitter space, a new method of field quantization has been presented [1, 2]. A covariant quantization of minimally coupled scalar field has been constructed by positive and negative norm states [1, 3]. In the same manner as in the Gupta - Blueler quantization of the electrodynamic equations in Minkowski space, we have performed the field quantization in Krein space [4, 5].

It was conjectured that quantum metric fluctuations might smear out the singularities of Green functions on the light cone, but they do not remove other ultraviolet divergences of quantum field theory [6]. However, it has been shown that quantization in Krein space removes all ultraviolet divergences of QFT except the light cone singularity [1, 7, 8, 9, 10].

It has been shown that the Krein propagators which obey the Krein quantization and quantum metric fluctuation are finite [10, 11, 12, 13]. The most interesting result of this construction is a new method of regularization, which we have called ”Krein regularization”. A natural regularization of the one-loop approximation for an interacting quantum scalar field in Minkowski space ($\lambda \phi^4$) has been achieved through the application of Krein space quantization [14]. The Casimir effect and one-loop approximation of Moller scattering have been calculated in Krein space [15, 16]. In QED, the value of the Lamb shift was calculated in Krein space quantization including quantum metric fluctuation and the magnetic anomaly was computed in Krein space quantization [13].

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In this paper, we calculate the four-point function in Krein space quantization including quantum metric fluctuation through the application of Krein regularization. In this method, the effective coupling constant is also calculated.

2 Scalar Green Function in Krein space

In terms of positive and negative norms, the field operator in Krein space quantization can be written as [1]:

\[ \phi(x) = \frac{1}{\sqrt{2}}[\phi_p(x) + \phi_n(x)], \]  

(2.1)

where \( \phi(x) \) satisfies the Klein - Gordon equation and

\[ \phi_p(x) = \int d^3k[a(k)U_p(k, x) + a^\dagger(k)U_p^*(k, x)], \]  
\[ \phi_n(x) = \int d^3k[b(k)U_n(k, x) + b^\dagger(k)U_n^*(k, x)]. \]  

(2.2)

\( a(k) \) and \( b(k) \) are two independent operators. Two sets of solutions are given by \( U_p(k, x) = \frac{e^{-ik_x}}{\sqrt{(2\pi)^2k_0}} \) and \( U_n(k, x) = \frac{e^{ik_x}}{\sqrt{(2\pi)^2k_0}} \), (with the sign of the metric \((+,-,-,-)\)) where \( k_0 = \sqrt{k_\cdot k + m^2} \geq 0 \). Note that \( U_n \) has the negative norm.

The time-ordered product propagator for the scalar field is defined as [1]:

\[ iG_T(x, x') = \langle 0 | T\phi(x)\phi(x') | 0 \rangle = \theta(t - t')\mathcal{W}(x, x') + \theta(t' - t)\mathcal{W}(x', x). \]  

(2.3)

In this case we obtain:

\[ G_T(x, x') = \frac{1}{2}[G_F(x, x') + (G_F(x, x'))^*] = \Re G_F(x, x'), \]  

(2.4)

where the Feynman Green function is defined by [17]:

\[ G_F(x, x') = \int \frac{d^4k}{(2\pi)^4}e^{-ik(x-x')}\tilde{G}_F(k) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 - m^2 + i\varepsilon} = \]  

\[ -\frac{1}{8\pi}\delta(\sigma_0) + \frac{m^2}{8\pi}\theta(\sigma_0)\frac{J_1(\sqrt{2m^2}\sigma_0)}{\sqrt{2m^2}\sigma_0} - \frac{im^2}{4\pi^2}\theta(-\sigma_0)\frac{J_1(\sqrt{-2m^2}\sigma_0)}{\sqrt{-2m^2}\sigma_0}. \]  

(2.5)

in which \( 2\sigma_0 = (x - x')^2 = \eta_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu) \). Therefore:

\[ G_T(x, x') = \int \frac{d^4k}{(2\pi)^4}e^{-ik(x-x')}PP\frac{1}{k^2 - m^2} = -\frac{1}{8\pi}\delta(\sigma_0) + \frac{m^2}{8\pi}\theta(\sigma_0)\frac{J_1(\sqrt{2m^2}\sigma_0)}{\sqrt{2m^2}\sigma_0}, \]  

(2.6)

The contribution of the coincident point singularity \((x = x')\) merely appears in the imaginary part of \( G_F(x, x') \) ([7] and equation (9.52) in [17])

\[ G_F(x, x') = -\frac{2i}{(4\pi)^2 d - 4} + G_f^{\text{finite}}(x, x'), \]
where \( d \) is the space-time dimension and \( G_{\text{finite}}(x, x') \) becomes finite as \( d \to 4 \).

In the momentum space for this propagator we have [18]

\[
\tilde{G}_T(k) = \frac{1}{2} [\tilde{G}_F(k) + \tilde{G}_F(k)^*] = \frac{1}{2} \left[ \frac{1}{k^2 - m^2 + i\epsilon} + \frac{1}{k^2 - m^2 - i\epsilon} \right] = PP \frac{1}{k^2 - m^2}, \quad (2.7)
\]

The quantum field theory in Krein space, including the quantum metric fluctuation \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), removes all ultraviolet divergencies of the theory [6, 10]:

\[
\langle G_T(x - x') \rangle = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2\langle \sigma^2 \rangle}} \exp \left( -\frac{\sigma_0^2}{2\langle \sigma^2 \rangle} \right) + \frac{m^2}{8\pi} \theta(\sigma_0) J_1(\sqrt{2m^2\sigma_0}) \sqrt{2m^2\sigma_0}, \quad (2.8)
\]

where \( 2\sigma = g_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu) \) and \( \sigma = \sigma_0 + \sigma_1 + O(h^2) \). The average value is taken over the quantum metric fluctuations.

The Fourier transformation of the second part of the equation (2.8) is [11, 14]:

\[
PP \frac{m^2}{k^2(k^2 - m^2)}. \quad (2.9)
\]

This propagator has been used by some authors to improve the UV behavior in relativistic higher-derivative correction theories [9, 19] and also appears in the supersymmetry theory [20].

For diagrams with loops there are two possibilities: the diagram is convergent or divergent due to the singularity of the delta function. In the first case we chose the propagator as \( PP \frac{1}{k^2-m^2} \). For the second case we selected the propagator as (2.9) which would result in the answer being finite and with no need for any renormalization. This means there are no added counter terms in the Lagrangian.

### 3 \( \lambda \phi^4 \) in Krein Space Quantization

The S matrix elements are the most important quantities in the interacting QFT, which can be written in terms of the time order product of the two free field operator by applying the LSZ reduction formulas, Wick’s theorem and time evolution operator [18].

The un-physical states can not propagate in the physical system because they are not observed in nature, so the following conditions have been imposed on the physical states for eliminating these unphysical states:

\[
b(k)|\text{physical states}\rangle = 0.
\]

The tree order S-matrix elements do not change because the propagator in the two methods is the same \((k^2 \neq m^2) \) [12].

In the \( \lambda \phi^4 \) theory, for calculating the four-point function the negative norm states appear in the calculations of the loop expansion, where they cause the negative mode states to propagate in the loop.
3.1 Four-Point Green Function

By using the S-matrices elements [21]:

\[ s_{fi} = \langle in, p_3, p_4 | 1 + S^{(1)} + S^{(2)} + ... | p_1, p_2, in \rangle. \]

In studying the second-order Feynman graphs only \( S^{(2)} \) is important which is:

\[ S^{(2)} = \frac{1}{2!} \left( -\frac{i\lambda}{4!} \right)^2 \int d^4x_1 \int d^4x_2 T \left[ \phi^4(x_1)\phi^4(x_2) \right]. \] (3.1)

There are 3 different diagrams: the \( s, t \) and \( u \) channels. The variables \( s, t \) and \( u \) are known as manelstam variables. \( \Gamma^{(4)}(t) \) and \( \Gamma^{(4)}(u) \) are obtained from \( \Gamma^{(4)}(s) \) by interchanging \( p_3 \) and \( p_4 \), \( p_2 \) and \( -p_3 \) and the sum of these three graphs are:

\[ \langle p_3, p_4 | S^{(2)} | p_1, p_2 \rangle = \Gamma^{(4)}(s) + \Gamma^{(4)}(t) + \Gamma^{(4)}(u) = \]

\[ \frac{(-i\lambda)^2}{2} \left[ I(p_1, p_2, p_3, p_4) + I(p_1, p_2, p_4, p_3) + I(p_1, -p_3, -p_2, p_4) \right] K_{1234}, \] (3.2)

where \( K_{1234} = \frac{(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)}{\sqrt{2\omega_1}2\omega_2\omega_3\omega_4} \), \( \omega_i = \sqrt{p_i^2 + m_0^2} \) and

\[ I(p_1, p_2, p_3, p_4) = \int \frac{d^4k}{(2\pi)^4} \tilde{G}_T(k)\tilde{G}_T(k+p), \] (3.3)

By including the quantum metric fluctuation, \( \tilde{G}_T(k) \) and \( \tilde{G}_T(k+p) \) in the above equation must be replaced by \( < \tilde{G}_T(k) > \) and \( < \tilde{G}_T(k+p) > \). Since \( < \tilde{G}_T(k) > \) and \( < \tilde{G}_T(k+p) > \) are finite, the four-point function in this formalism is automatically regularized and no divergent term is encountered

\[ \Gamma^{(4)}_{kr}(p) = \frac{3\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} < \tilde{G}_T(k) > < \tilde{G}_T(k+p) >. \] (3.4)

4 Krein Regularization for the Four-Point Function

Because of the delta function singularity in the propagators, the integral (3.3) is divergent at the ultraviolet limit, whereas the integral (3.4) is finite:

\[ \Gamma^{(4)}_{kr}(p) = \frac{3\lambda^2}{8} \int \frac{d^4k}{(2\pi)^4} PP \left( \frac{1}{k^2 - m^2} - \frac{1}{k^2} \right) PP \left( \frac{1}{(p-k)^2 - m^2} - \frac{1}{(p-k)^2} \right). \] (4.1)

In order to solve this integral, we use Feynman parameters:

\[ \Gamma^{(4)}_{kr}(p) = \frac{3\lambda^2}{2} \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \left[ \frac{1}{(l^2 + x(1-x)p^2 - m^2)^2} - \frac{1}{(l^2 + x(1-x)p^2 - m^2)^2} + \frac{1}{(l^2 + x(1-x)p^2 - m^2)^2} \right]. \] (4.2)
where \( k = l - x p \) \([22, 23]\).

The integral over \( l \) is no longer divergent. In order to solve the integral over \( l \), we apply the Wick rotation in a manner that if \( i \epsilon \) exists in the denominator, the substitution \( l_E^0 = -i l^0 \) is used, and if, conversely, \( -i \epsilon \) is present in the denominator, the change of variable \( l_E^0 = i l^0 \) is applied and we arrive at the following integral \([23]\):

\[
\Gamma_{kr}^{(4)}(p) = -\frac{3 \lambda^2}{32 \pi^2} \int_0^1 dx \left\{ \ln \left( 1 - x(1-x) \frac{p^2}{m^2} \right) + \ln \left( -\frac{p^2}{m^2} \right) - \ln \left( 1 - \frac{p^2}{m^2} \right) - \ln \left( 1 - (1-x) \frac{p^2}{m^2} \right) \right\}.
\]

(4.3)

In Hilbert space, we have \([22]\):

\[
\Gamma_{Hr}^{(4)}(p) = \frac{3 \lambda^2}{32 \pi^2} \int_0^1 dx \left\{ \frac{2}{4 - d} - \ln \left( 1 - x(1-x) \frac{p^2}{m^2} \right) \right\}.
\]

(4.4)

### 4.1 Effective Coupling Constant

The effective potential was calculated in Krein space quantization \([11]\). It was used to calculate the \( \beta \)-function in Krein space and the answer was:

\[
\beta = \frac{d \lambda_{\text{eff}}}{dt} = \frac{3 \lambda^2}{16 \pi^2},
\]

where \( \lambda_{\text{eff}} = \frac{d V_{\text{eff}}}{d \varphi^2} |_{\varphi^2 = \mu^2}, \mu = e^{-t} \) and the \( \beta \)-function was similar to it’s counterpart in Hilbert space.

In our method, by defining the scattering matrix elements and using equation (4.3) at the scale of energy \( p^2 = -\mu^2 \), the effective coupling constant would be as below:

\[
\lambda_{\mu} = \lambda + \frac{3 \lambda^2}{32 \pi^2} \int_0^1 dx \left\{ \ln \left( 1 + (x - x^2) \frac{\mu^2}{m^2} \right) + \ln \left( \frac{\mu^2}{m^2} \right) - 2 \ln \left( 1 + x \frac{\mu^2}{m^2} \right) \right\},
\]

(4.5)

which is finite. The \( \beta \)-function would be defined as follows:

\[
\beta = \mu \frac{d \lambda_{\mu}}{d \mu} = \frac{3 \lambda^2}{16 \pi^2},
\]

(4.6)

which is in agreement with the result from the previous methods.

### 5 Conclusion

In this paper we have explicitly calculated the four-point function in the one loop approximation. In Krein space quantization including quantum metric fluctuation, the Green function is finite and does not have any divergent term in the ultraviolet and infrared limit.

Using the four-point function, the coupling constant is written at the scale of energy \( \mu \) and the \( \beta \)-function is calculated. The result is similar to that gained from the effective potential and Hilbert space methods.
In this method $\lambda \phi^4$ is automatically regularized and renormalization is not applied. This method can be easily generalized to non-Abelian gauge theory and quantum gravity in the background field method. It can also be employed in the calculation of the $\beta$-function in QED.

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