About quadratic residues in a class of rings

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Abstract

Let $R$ be a commutative ring with a collection of ideals $\{N_1, N_2, \ldots, N_{k-1}\}$ satisfying certain conditions, properties of the set of invertible quadratic residues of the ring $R$ are described in terms of properties of the set of invertible quadratic residues of the quotient ring $R/N_1$.

Keywords Lifting method, units, quadratic residues.

1 Introduction

Quadratic residues ([4]) have been studied since the 17th and 18th century by P. de Fermat, L. Euler, J.L. Lagrange A.M. Legendre, among other mathematicians. Nowadays quadratic residues are still a topic of study ([2],[6],[1],[10],[12]) and they have applications in areas which include acoustical engineering ([11]), cryptography ([5],[3]), in the study of Paley (conference) graphs, primality testing (Solovay-Strassen, Miller-Rabin), and integer factorization (quadratic sieve, number field sieve).

In the present note properties of invertible quadratic residues over a class of rings are given which are determined by properties of quadratic residues over a quotient ring, i.e., the properties are “lifted” from those of a quotient ring. An example of this class of rings includes the integers modulo $p^k$, $\mathbb{Z}_{p^k}$, where $p$ is a prime and $k$ a positive integer.

The manuscript is divided in four sections. In Section 2 notation and facts needed in the rest of the manuscript are presented. In section 3 the main results are given and in Section 4 applications of the results previously discussed are considered. Examples are given to illustrate the main results.

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2 Notations and basic facts

Given $R$ an associative ring with identity and $N$ a nil ideal of $R$, we begin our discussion recalling that units of the quotient ring $R/N$ can be lifted to the ring $R$. More precisely, if $R^*$ and $(R/N)^*$ denote the group of units of the ring $R$ and $R/N$ respectively, the following result holds.

**Proposition 2.1.** Let $R$ be an associative ring with identity, $N$ a nil ideal of $R$ and $\bar{\theta}: R \rightarrow R/N$ the canonical homomorphism from $R$ to the quotient ring $R/N$. Then,

1. $\bar{\theta}(f) = f + N \in (R/N)^*$ if and only if $f + N \subset R^*$.
2. If $R$ is finite the cardinality of $R^*$ and the cardinality of $(R/N)^*$ are related by the relation
   \[ |R^*| = |N||(R/N)^*|. \]  

**Proof:** The proof of this proposition can be found in [7], Proposition 2.1 and Remark 2.2.

Recall that an element $a$ of a ring $R$ is a **quadratic residue**, if there exists $x \in R$ such that $x^2 = a$ ([3],[9],[13]). Given $N$ an ideal of $R$, it is clear that if $a$ is a quadratic residue in the ring $R$, then $a + N$ is a quadratic residue in the quotient ring $R/N$. The following proposition provides sufficient conditions to prove the converse of this statement, as it will be seen, proposition 2.1 will be essential in the proof that will be presented.

**Proposition 2.2.** Let $R$ be a commutative ring with identity and $N$ a nil ideal of $R$. If $(g + N)^2 = a + N$ and $2g + N$ is an invertible element in $R/N$, then the function

\[ \eta: g + N \rightarrow a + N \]  
given by $\eta(g + m) = (g + m)^2$,

is bijective. In other words, if $a + N$ is a quadratic residue in $R/N$, then every element $b \in a + N$ is a quadratic residue in the ring $R$ and, for all $b \in a + N$ the quadratic equation

\[ y^2 = b \]  
has a unique solution in the set $g + N \subset R$.

**Proof:** Since $(g + N)^2 = a + N$, it is clear that the function $\eta$ is well defined. Since $N$ is a nil ideal of $R$ and $2g + N$ is an invertible element in $R/N$, from proposition 2.1 it follows that for all $p \in N$, the element $2g + p$ is an invertible element in $R$. Thus, if $\eta(g + m_1) = \eta(g + m_2)$ then,

\[ (2g + m_1 + m_2)(m_1 - m_2) = 0. \]  
Since $2g + m_1 + m_2$ is an invertible element in the ring $R$, it is concluded that $m_1 = m_2$. Therefore $\eta$ is an injective function. Now, it is proved that $\eta$ is surjective. First of all, note
that since $(g + N)^2 = a + N$, there exists $n_0 \in N$, such that $g^2 = a + n_0$. Now, given $n \in N$, it is easy to see that

$$\eta(g + (2g)^{-1}(n - n_0)) = a + n,$$

which proves the claim, finishing the proof of the proposition.

Now, definitions and notation that will be useful in the rest of the manuscript are introduced. The set $q(R^*)$ will denote the quadratic residues in the ring $R$ that are also units in $R$, that is

$$q(R^*) = \{a \in R|a \text{ is a quadratic residue in } R \text{ and } a \in R^*\}.$$

For $a$ a quadratic residue in the ring $R$, $s(a)$ will denote the set of solutions of the equation $x^2 = a$ in the ring $R$. In other words,

$$s(a) = \{x \in R|x^2 = a\}.$$

Finally, if $N$ is an ideal of the ring $R$ and $a \in R$, $T(a + N)$ will be denote the set of solutions of the equation $x^2 = b$, when $b$ varies in the equivalence class $a + N \in R/N$, in other words

$$T(a + N) = \{y \in R|y^2 \in a + N\}.$$

Based on propositions 2.1 and 2.2, sufficient conditions to lift quadratic residues from the quotient ring $R/N$ to the ring $R$, where $N$ is a nil ideal of the ring $R$ are established. In addition if $R$ is finite, formulas relating the cardinality of the sets $N, s(b), s(b + N), R^*, (R/N)^*, q(R^*)$ and $q((R/N)^*)$ are given.

**Proposition 2.3.** Let $R$ be a commutative ring with identity and $N$ a nil ideal of $R$ such that $2 + N$ is an invertible element in $R/N$. The following statements hold,

1. $a + N \in q((R/N)^*)$ if and only if $a + N \subset q(R^*)$.

2. The cardinality of the set $q(R^*)$ satisfies

$$|q(R^*)| = |N||q((R/N)^*)|. \quad (2)$$

3. If $a + N \in q((R/N)^*)$, then for all $b \in a + N$ the number of solutions of the quadratic equation

$$y^2 = b$$

in the ring $R$ is equal to the number of solutions of the quadratic equation

$$y^2 = b + N$$

in the ring $R/N$. In other words

$$|s(b)| = |s(a + N)|$$

for all $b \in a + N$. 


4. The cardinality of the set $R^*$ satisfies the following relation

$$|R^*| = |N| \sum_{a+N \in q((R/N)^*)} |s(a + N)|. \quad (3)$$

5. If in addition, there exists $\alpha$ such that $|s(a + N)| = \alpha$ for all $a + N \in q((R/N)^*)$, then

$$a) \ |q((R/N)^*)| = \frac{|(R/N)^*|}{\alpha}, \quad b) \ |q(R^*)| = \frac{|N||(R/N)^*|}{\alpha}, \quad c) \ |q(R^*)| = \frac{|R^*|}{\alpha}. \quad (4)$$

**Proof:** It is easy to see that if $a + N \subset q(R^*)$, then $a + N \in q((R/N)^*)$. Now, it is proved the other implication. Assuming $a + N \in q((R/N)^*)$, there exists $g + N \in R/N$ such that $(g + N)^2 = a + N$. Since $a + N$ and $2 + N$ are invertible elements in the ring $R/N$, it follows that

$$(2 + N)(g + N) = 2g + N$$

is an invertible element in $R/N$, thus proposition 2.2 implies that every element $b \in a + N$ is a quadratic residue in the ring $R$. In addition, since $a + N \in (R/N)^*$ and $N$ is a nil ideal of the ring $R$, proposition 2.1 implies that for all $b \in a + N$, $b \in R^*$. This proofs that $a + N \subset q(R^*)$.

From claim 7, it follows that

$$q(R^*) = \bigcup_{a+N \in q((R/N)^*)} (a + N). \quad (5)$$

Then,

$$|q(R^*)| = \sum_{a+N \in q((R/N)^*)} |a + N| = |N| \sum_{a+N \in q((R/N)^*)} 1 = |N||q((R/N)^*)|,$$

which proves relation 2.

Note that since $a + N \in q((R/N)^*)$, for all $b \in a + N$, $s(b) \neq \emptyset$. In order to prove that $|s(b)| = |s(a + N)|$, it will be shown that the canonical homomorphism $\phi : R \to R/N$ restricted to the set $s(b)$, namely

$$x \in s(b) \to \phi(x) = x + N$$

determines a bijection between $s(b)$ and $s(b + N)$. In fact, if $x \in s(b)$ then $x + N \in s(b + N)$, thus the function $\phi$ is well defined. Now, if $\phi(x) = \phi(y) = x + N$ with $x, y \in s(b)$, then $(x + N)^2 = b + N = a + N$ with $2x + N$ an invertible element in the ring $R/N$. So, proposition 2.2 implies that the function

$$z \in x + N \to \eta_1(z) = z^2 \in b + N$$

is a bijective function, hence, since $x, y \in x + N$ and $x^2 = y^2 = b$, the injectivity of the function $\eta_1$ implies that $x = y$. Now, if $t + N \in s(b + N)$, then $(t + N)^2 = b + N = a + N$. 

4
Since \(2t + N\) is an invertible element in the ring \(R/N\), proposition 2.2 implies that the function
\[
z \in t + N \rightarrow \eta_2(z) = z^2 \in b + N
\]
is bijective. Thus there exists \(n_0 \in N\), such that \(\eta_2(t + n_0) = (t + n_0)^2 = b\). Hence, \(t + n_0 \in s(b)\) and \(\phi(t + n_0) = t + N\), hence \(\phi\) is a surjective function.

Note that the set \(R^*\) is a disjoint union of the sets \(T(a + N)\) with \(a + N \in q((R/N)^*)\), that is
\[
R^* = \bigcup_{a+N \in q((R/N)^*)} T(a+N).
\]

From this fact, it follows that
\[
|R^*| = \sum_{a+N \in q((R/N)^*)} |T(a+N)|.  \tag{6}
\]

In order to compute \(|T(a+N)|\), observe that \(T(a+N)\) can be written as a disjoint union of the sets \(s(b)\) with \(b \in a+N\),
\[
T(a+N) = \bigcup_{b\in a+N} s(b).  \tag{7}
\]

Thus, since for all \(b \in a+N\), \(|s(b)| = |s(a+N)|\),
\[
|T(a+N)| = \sum_{b\in a+N} |s(b)| = |s(a+N)| \sum_{b\in a+N} 1 = |s(a+N)||N|.  \tag{8}
\]

Finally, combining (6) and (8), relation in (3) follows easily.

Since \(|s(a+N)| = \alpha\) for all \(a+N \in q((R/N)^*)\), from relation (3), it follows that
\[
|R^*| = \alpha |N||q((R/N)^*)|,  \tag{9}
\]
and proposition 2.1 implies that,
\[
|R^*| = |N||(R/N)^*|.  \tag{10}
\]

Thus, by combining (9) and (10), relation in (5)-a is obtained. The relation in (5)-b is obtained from (2) and (5)-a). Finally, relation in (5)-c is obtained from (10) and (5)-b). ■

In the next lines the results in proposition 2.3 are illustrated with an example. Let \(p\) be a prime number different from 2, \(k\) a natural number and let \(R = \mathbb{Z}_{p^k}\) be the ring of integers modulo \(p^k\). It is clear that the ideal generated by \(p\) in \(R\), namely \(N = \langle p \rangle\), is a nilpotent ideal of index \(k\) of the ring \(R\). In addition,

\[
\frac{R}{N} = \frac{\mathbb{Z}_{p^k}}{\langle p \rangle} \cong \mathbb{Z}_p,
\]

thus \(|(R/N)^*| = p - 1\) and from Lagrange’s theorem it follows that \(|N| = p^{k-1}\). From the identity in (11), we have that \(|R^*| = p^{k-1}(p - 1)\). In addition, since, \(R/N \cong \mathbb{Z}_p\) is a field of characteristic different from 2, it follows that the number \(\alpha\) appearing in claim 5 of proposition 2.3 is \(\alpha = 2\). Thus, from proposition 2.4, it is concluded that:
• If \( a \in \mathbb{Z}^*_p \), and \( a \equiv b \mod (p) \), then \( a \) is a quadratic residue in \( \mathbb{Z}_p^k \) if and only if \( b \) is a quadratic residue in the ring \( \mathbb{Z}_p \).

• If \( a \in q(\mathbb{Z}_p^*) \) and \( a \equiv b \mod (p) \), then the number of solutions of the equation \( x^2 = a \) in the ring \( \mathbb{Z}_p^k \) is equal to the number of solutions of the equation \( x^2 = b \) in the field \( \mathbb{Z}_p \), which is equal to 2, in other words

\[
s(a) = s(b) = 2.
\]

• The cardinality of the sets \( q(\mathbb{Z}_p^*) \), \( q(\mathbb{Z}_p^k) \) are given by

\[
|q(\mathbb{Z}_p^*)| = \frac{p-1}{2} \quad \text{and} \quad |q(\mathbb{Z}_p^k)| = \frac{p^{k-1}(p-1)}{2},
\]

respectively.

Next, the previous proposition is extended to a direct product of a finite collection of rings.

**Proposition 2.4.** Let \( R_1, R_2, \ldots, R_m \), be commutative rings with identity and let \( N_i \) be a nil ideal of the ring \( R_i \), such that \( 2 + N_i \in (R_i/N_i)^* \) for each \( i = 1, 2, \ldots, m \). The following statements hold:

1. \((a_1, \ldots, a_m) \in q((R_1 \times \cdots \times R_m)^*) \) if and only if \( a_i + N_i \in q((R_i/N_i)^*) \) for every \( i=1,2,\ldots,m \).

2. If \((a_1, \ldots, a_m) \in q((R_1 \times \cdots \times R_m)^*) \) then

\[
|s((a_1, \ldots, a_m))| = |s(a_1 + N_1)| \cdots |s(a_m + N_m)|. \tag{11}
\]

3. The cardinality of \( q((R_1 \times \cdots \times R_m)^*) \) satisfies the following relation

\[
|q((R_1 \times \cdots \times R_m)^*)| = |N_1||q((R_1/N_1)^*)| \cdots |N_m||q((R_m/N_m)^*)| \tag{12}
\]

4. If \( |s(a + N_i)| = \alpha_i \) for all \( a + N_i \in q((R_i/N_i)^*) \), then

\[
|q((R_1 \times \cdots \times R_m)^*)| = \frac{|N_1||q((R_1/N_1)^*)| \cdots |N_m||q((R_m/N_m)^*)|}{\alpha_1 \alpha_2 \cdots \alpha_m}, \tag{13}
\]

and

\[
|q((R_1 \times \cdots \times R_m)^*)| = \frac{|(R_1 \times \cdots \times R_m)^*|}{\alpha_1 \alpha_2 \cdots \alpha_m}. \tag{14}
\]
Proof: 1. The proof of this claim follows from claim 1 of proposition 2.3 and the fact that
\[ q((R_1 \times \cdots \times R_m)^*) = q(R_1^*) \times \cdots \times q(R_m^*). \] (15)

2. Since \( s((a_1, \ldots, a_m)) = s(a_1) \times \cdots \times s(a_m) \),
\[ |s((a_1, \ldots, a_m))| = |s(a_1)| \cdots |s(a_m)|. \]

Thus, relation (14) follows from 1) of proposition 2.4 and claim 3) of proposition 2.3.

3. The proof of relation in (12) is a consequence of the equality given in (15) and the identity in (2).

4. Relation (13) follows from (12) and claim (5)-a) of proposition 2.3. Finally, since \( (R_1 \times \cdots \times R_m)^* = R_1^* \times \cdots \times R_m^* \), relation (14) follows from the fact that \( |R_i^*| = |N_i|(R_i/N_i)^* \) for every \( i = 1, 2, \ldots, m \).

In the following lines the results of the previous proposition are illustrated with an example. Given \( n \) an odd natural number, if \( n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m} \) denotes the prime factorization of \( n \). The Chinese Remainder Theorem implies that
\[ \mathbb{Z}_n \cong \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_m^{k_m}}. \]

By setting \( R_i = \mathbb{Z}_{p_i^{k_i}} \) and \( N_i = \langle p_i \rangle \) for \( i = 1, 2, \ldots, m \), it is clear that \( R_i/N_i \cong \mathbb{Z}_{p_i} \) and that \( 2 + N_i \in (R_i/N_i)^* \). Thus, from proposition 2.4, it is concluded that:

- If \( a \in \mathbb{Z}_n^* \) and \( a \equiv a_i \mod (p_i) \) for \( i = 1, 2, \ldots, m \), \( a \) is a quadratic residue in \( \mathbb{Z}_n \) if and only if \( a_i \) is a quadratic residue in the ring \( \mathbb{Z}_{p_i} \) for all \( i = 1, 2, \ldots, m \).

- If \( a \in q(\mathbb{Z}_n^*) \) and \( a \equiv b_i \mod (p_i^{k_i}) \) for \( i = 1, 2, \ldots, m \), since \( s(b_i + N_i) = 2 \) for all \( i = 1, 2, \ldots, m \), the number of solutions of the equation \( x^2 = a \) in the ring \( \mathbb{Z}_n \) is equal to \( 2^m \), in other words
\[ s(a) = 2^m. \]

- The cardinality of the set \( q(\mathbb{Z}_n^*) \) is given by
\[ |q(\mathbb{Z}_n^*)| = \frac{p_1^{k_1-1}(p_1 - 1) \cdots p_m^{k_m-1}(p_m - 1)}{2^m}. \]

3 Main results

In this section the main results of the manuscript are presented. For \( R \) a commutative ring containing a collection of ideals \( \{N_1, N_2, \ldots, N_{k-1}\} \) satisfying a certain condition (the CNC condition, Definition 3.2), properties of the set of invertible quadratic residues of the ring \( R \) are described in terms of properties of the set of invertible quadratic residues of the quotient ring \( R/N_1 \).
Proposition 3.1. Let $R$ be a commutative ring and $N$ a nilpotent ideal of index $t \geq 2$ in $R$. Then the following statements hold:

1. For any prime number $p$ such that $p \geq t$, for all $n \in N$ and $a \in R$,
   \[(a + n)^p = a^p + pnr,
   \]
   for some $r \in R$.

2. In addition, assuming there exists a natural number $s > 1$ such that $sN = \{0\}$ and such that all the prime factors of $s$ are greater than or equal to the nilpotency index $t$ of the ideal $N$. The function $H : R/N \rightarrow R$ given by
   \[H(x + N) = x^s\]
   is well defined and it is multiplicative, i.e., it satisfies $H((x + N)(y + N)) = H(x + N)H(y + N)$, for all $x, y \in R$.

3. Under the assumptions of claim 2, if $a + N$ is a quadratic residue in the quotient ring $R/N$, then $H(a + N) = a^s$ is a quadratic residue in $R$. More precisely, if $g \in R$ is such that \((g + N)^2 = a + N\), then
   \[(g^s)^2 = a^s.\]

Proof:

1. Since $n^t = 0$, 
   \[(a + n)^p = \sum_{j=0}^{p} \binom{p}{j} a^{p-j}n^j = a^p + \sum_{j=1}^{t-1} \binom{p}{j} a^{p-j}n^j.
   \]
   Since $p$ is a prime number, $p$ divides $\binom{p}{j}$ for all $1 \leq j \leq p - 1$. Also, since $t \leq p$, 
   \[(a + n)^p = a^p + pn \left( k_1a^{p-1} + k_2a^{p-2}n + \cdots + k_{t-1}a^{p-t+1}n^{t-2} \right)
   \]
   where $k_i = \binom{p}{i}/p$. Therefore,
   \[(a + n)^p = a^p + pnr,
   \]
   with $r = k_1a^{p-1} + k_2a^{p-2}n + \cdots + k_{t-1}a^{p-t+1}n^{t-2} \in R$.

2. Let $p_1, p_2, p_3, \ldots, p_m$ be the prime numbers, not necessarily different, appearing in the prime factor decomposition of the integer $s$, with $p_i \geq t$, for $i = 1, 2, 3, \ldots, m$. Since $y + N = x + N$, there exists $n \in N$ such that $y = x + n$. Since $p_1 \geq t$, from claim 2
   \[y^{p_1} = (x + n)^{p_1} = x^{p_1} + p_1nr_1,
   \]
for some \( r_1 \in \mathbb{R} \). Similarly, since \( p_2 \geq t \) and \( p_1nr_1 \in N \), it follows from claim 1 and the previous relation that
\[
y^{p_1p_2} = (x^{p_1} + p_1nr_1)^{p_2} = x^{p_1p_2} + p_2(p_1nr_1)r_2,
\]
for some \( r_2 \in \mathbb{R} \). In the same way, it is possible to verify that
\[
y^{p_1p_2 \cdots p_m} = x^{p_1p_2 \cdots p_m} + (p_1p_2 \cdots p_m)n(r_1r_2 \cdots r_m),
\]
with \( r_1, r_2, \ldots, r_m \in \mathbb{R} \). In other words,
\[
y^s = x^s + sh,
\]
where \( h = nr_1r_2 \cdots r_m \in N \). Finally, since \( h \in N \) and \( sN = 0 \), it follows that \( y^s = x^s \).
Hence, the function \( H \) is well defined. It is easily verified that the function \( H \) is multiplicative.

3. Since \((g + N)^2 = a + N\), it follows that \((H(g + N))^2 = H(a + N)\), thus \((g^s)^2 = a^s\), i.e., \( a^s \) is a quadratic residue in the ring \( R \).

An example illustrating the previous proposition is presented which it allows to discuss additional properties of the function \( H \). Consider \( R = \mathbb{Z}_{25} \) and \( N = \langle 5 \rangle = \{0, 5, 10, 15, 20\} \). It is clear that \( N \) has nilpotency index \( t = 2 \), \( sN = 0 \) for \( s = 5 \) and
\[
\frac{\mathbb{Z}_{25}}{(5)} \cong \mathbb{Z}_5.
\]

- By setting \( x + N = \overline{x} \), it is not difficult to see that \( H(\overline{0}) = 0, H(\overline{1}) = 1, H(\overline{2}) = 7, H(\overline{3}) = 18 \) and \( H(\overline{4}) = 24 \). Since \( \overline{0}, \overline{1}, \overline{4} \) are the quadratic residues in the ring \( \mathbb{Z}_{25}/(5) \), then \( 0, 1 \) and \( 24 \) are quadratic residues in the ring \( \mathbb{Z}_{25} \).

- Since \((3 + N)^2 = 4 + N \) in \( \mathbb{Z}_{25}/(5) \). From proposition 2.2, it follows that the function \( \eta : \{3, 8, 13, 18, 23\} \to \{4, 9, 14, 19, 24\} \) given by \( \eta(x) = x^2 \mod (25) \) is a bijective function, in particular, all the elements in the equivalence class \( 4 + N \) are quadratic residues in the ring \( \mathbb{Z}_{25} \). Thus, the only quadratic residue in the equivalence class \( 4 + N \) that is mapping by the function \( H \) is \( 24 \).

- Since \( H(1 + N + 1 + N) = 7 \) and \( H(1 + N) + H(1 + N) = 1 + 1 = 2 \), then the function \( H \) is not in general a ring homomorphism.

It is to be noticed that the hypothesis in claim 2 of Proposition 3.1, which requires that all prime factors of \( s \) be greater or equal than the nilpotency index \( t \) of the ideal \( N \), restricts enormously the number of applications of that proposition. For instance, if we consider \( R = \mathbb{Z}_{2^t} \) with \( 2 \leq t \) and \( N = (2) \), it is clear that \( N^t = \{0\} \) and \( sN = \{0\} \) for \( s = 2^{t-1} \). Thus,
according to claim 2 of Proposition 3.1 in order to get quadratic residues in the ring $R$ by computing the quadratic residues in the ring $R/N \cong \mathbb{Z}_2$, it is necessary that $t \leq 2$, therefore $t = 2$. Hence, we can only obtain quadratic residues in the ring $\mathbb{Z}_4$, which is easily done by hand. In the following lines, we show how to overcome such restrictions.

Definition 3.2. [8, Definition 3.2] We say that a collection $\{N_1, \ldots, N_k\}$ of ideals of a ring $R$ satisfies the CNC-condition if the following properties hold:

1. **Chain condition:** $\{0\} = N_k \subset N_{k-1} \subset \cdots \subset N_1 \subset N_0 = R$.

2. **Nilpotency condition:** for $i = 1, 2, 3, \ldots, k-1$, there exists $t_i \geq 2$ such that $N_i^{t_i} \subset N_{i+1}$.

3. **Characteristic condition:** for $i = 1, 2, 3, \ldots, k-1$, there exists $s_i \geq 1$ such that $s_i N_i \subset N_{i+1}$. In addition, the prime factors of $s_i$ are greater than or equal to $t_i$.

The minimum number $t_i$ satisfying the nilpotency condition will be called the nilpotency index of the ideal $N_i$ in the ideal $N_{i+1}$. Similarly, the minimum number $s_i$ satisfying the characteristic condition will be called the characteristic of the ideal $N_i$ in the ideal $N_{i+1}$.

The nilpotency condition and the characteristic condition of the previous definition can be stated as follows:

a. The nilpotency condition is equivalent to the following condition: for $i = 1, 2, \ldots, k-1$, $N_i/N_{i+1}$ is a nilpotent ideal of index $t_i$ in the ring $R/N_{i+1}$, (for details see [8, Definition 3.2]).

b. The characteristic condition is equivalent to the following condition: for $i = 1, 2, \ldots, k-1$, there exists a natural number $s_i \geq 1$ such that $s_i(N_i/N_{i+1}) = 0$ in the ring $R/N_{i+1}$, (for details see [8, Definition 3.2]).

Theorem 3.3. Let $R$ be a commutative ring, $\{N_1, N_2, \ldots, N_k\}$ a collection of ideals of $R$ satisfying the CNC-condition and let $s_i$ be the characteristic of the ideal $N_i$ in the ideal $N_{i+1}$. If $a + N_1$ is a quadratic residue in $R/N_1$, then $a^{s_1 s_2 \cdots s_{k-1}}$ is a quadratic residue in $R$. More precisely, if $g \in R$ is such that $(g + N_1)^2 = a + N_1$, then

$$
(g^{s_1 s_2 \cdots s_{k-1}})^2 = a^{s_1 s_2 \cdots s_{k-1}}. 
$$

(16)

Proof: Note first that since the ideals $N_i$ satisfy the chain condition given in definition 3.2 for all $i = 1, 2, \ldots, k-1$, the following isomorphism holds

$$
R/N_i \cong \frac{R/N_{i+1}}{N_i/N_{i+1}}.
$$

(17)
In addition, since the ideals \( N_i \) satisfy the nilpotency condition and characteristic condition with characteristics \( s_i \) respectively, from claim \( 2 \) of proposition \( 3.1 \) it follows that for \( i = 1, 2, \ldots, k - 1 \), the functions
\[
H_i : R/N_i \to R/N_{i+1}, \quad H_i(x + N_i) = x^{s_i} + N_{i+1}
\]
are well defined and multiplicative. Hence, if \((g + N_1)^2 = a + N_1\),
\[
H_{k-1} \circ \cdots \circ H_1((g + N_1)^2) = H_{k-1} \circ \cdots \circ H_1(a + N_1),
\]
whence the identity in (16) is obtained.

**Remark 3.4.** It follows from the proof of the Theorem (3.3) that, if \( a \in R \) is such that \( a + N_1 \) is a quadratic residue in \( R/N_1 \), then \( H_1(a + N_1) = a^{s_1} + N_2 \) is a quadratic residue in \( R/N_2 \). In the same way, \( H_2(a^{s_1} + N_2) = a^{s_1s_2} + N_3 \) is a quadratic residue in \( R/N_3 \), and so on. At the end of this process, it is obtained that \( a^{s_1s_2\cdots s_{k-1}} \) is a quadratic residue in \( R \). The following chain of multiplicative functions,
\[
\frac{R}{N_1} \xrightarrow{H_1} \frac{R}{N_2} \xrightarrow{H_2} \cdots \xrightarrow{H_{k-2}} \frac{R}{N_{k-1}} \xrightarrow{H_{k-1}} \frac{R}{N_k} = R, \text{ with } H_i(x + N_i) = x^{s_i} + N_{i+1}
\]
appears naturally in that process.

**Theorem 3.5.** Let \( R \) be a commutative ring with identity, \( \{N_1, N_2, \ldots, N_k\} \) a collection of ideals of \( R \) satisfying both the Chain condition and the Nilpotency condition. Assuming that \( 2 + N_1 \in (R/N_1)^* \), the following claims hold

1. \( a + N_1 \in q((R/N_1)^*) \) if and only if \( a + N_1 \subset q(R^*) \).

2. The cardinality of the set \( q(R^*) \) is given by
\[
|q(R^*)| = |N_1||q((R/N_1)^*)| \tag{18}
\]

3. If \( a + N_1 \in q((R/N_1)^*) \), then
\[
s(a) = s(a + N_{k-1}) = \cdots = s(a + N_1). \tag{19}
\]

4. If for each \( i = 1, 2, 3, \ldots, k - 1 \), there exists \( \alpha_i \) such that, \( |s(a + N_i)| = \alpha_i \) for all \( a + N_i \in q((R/N_i)^*) \), then
\[
|(R/N_{i+1})^*| = \alpha_i|q((R/N_{i+1})^*)|. \tag{20}
\]

In particular,
\[
|R^*| = \alpha_{k-1}|q(R^*)|. \tag{21}
\]
Proof: 1. It is easy to see that if \( a + N_1 \subseteq q(R^*) \), then \( a + N_1 \in q((R/N_1)^*) \). Now, we proceed to prove the other implication of the statement. From the isomorphism given in (17), the fact that \( N_i/N_{i+1} \) is a nilpotent ideal of index \( t_i \) in the ring \( R/N_{i+1} \) and the fact that \( 2 + N_1 \in (R/N_1)^* \), from proposition (2.1), it follows that

\[
(2 + N_{i+1}) + N_i/N_{i+1} \subseteq ((R/N_{i+1})/(N_i/N_{i+1}))^* 
\]

for all \( i \in 1, 2, 3, \ldots, k \). Now, let \( b \in a + N_1 \), since \( b + N_1 = a + N_1 \in q((R/N_1)^*) \), it follows from the isomorphism

\[
R/N_1 \cong \frac{(R/N_2)}{(N_1/N_2)}. 
\]

that \( (b + N_2) + N_1/N_2 \subseteq q(((R/N_2)/(N_1/N_2))^*) \), thus, from claim 1 of proposition 2.3 it follows that

\[
(b + N_2) + N_1/N_2 = \{b + n + N_2 | n \in N_1 \} \subseteq q((R/N_2)^*),
\]

in particular, it is concluded that \( b + N_2 \in q((R/N_2)^*) \). Similarly, from the isomorphism

\[
R/N_2 \cong \frac{(R/N_3)}{(N_2/N_3)},
\]

it follows that \( (b + N_3) + N_2/N_3 \in q(((R/N_3)/(N_2/N_3))^*) \), thus, from item 1 of proposition 2.3 it follows that

\[
(b + N_3) + N_2/N_3 = \{b + n + N_3 | n \in N_2 \} \subseteq q((R/N_3)^*),
\]

in particular, it is concluded that \( b + N_3 \in q((R/N_3)^*) \). Continuing this process, it is finally shown that \( b + N_k \in q((R/N_k)^*) \), which immediately implies that \( b \in q(R^*) \). This shows that \( a + N_1 \subseteq q(R^*) \), as we wanted to prove.

2. From the isomorphism given in (17) and item 2 of proposition 2.3 it follows that

\[
|N_i/N_{i+1}|q((R/N_i)^*) = |q(((R/N_{i+1})/(N_i/N_{i+1}))^*)| = |q((R/N_{i+1})^*)|, 
\]

(22)

thus from Lagrange’s theorem,

\[
|q(R^*)| = |q((R/N_k)^*)| = \frac{|N_{k-1}|}{|N_k|}|q((R/N_{k-1})^*)| = \frac{|N_{k-1}|}{|N_k|}\frac{|N_{k-2}|}{|N_{k-1}|}\cdots\frac{|N_1|}{|N_2|}q((R/N_1)^*),
\]

whence the identity in (18) is obtained.

3. Again, from the isomorphism given in (17), it follows that

\[
s(a + N_i) = s(a + N_{i+1} + N_i/N_{i+1}) 
\]

for all \( i \in 1, 2, 3, \ldots, k - 1 \). On the other hand, since \( N_i/N_{i+1} \) is a nilpotent ideal of index \( t_i \) in the ring \( R/N_{i+1} \) and the fact that \( (2 + N_{i+1}) + N_i/N_{i+1} \subseteq ((R/N_{i+1})/(N_i/N_{i+1}))^* \), from claim 3 of proposition 2.3 it follows that

\[
s(a + N_{i+1} + N_i/N_{i+1}) = s(a + N_{i+1}).
\]
for $i = 1, 2, 3, \ldots, k - 1$. From the previous identities, it follows that $s(a + N_i) = s(a + N_{i+1})$ for $i = 1, 2, 3, \ldots, k - 1$, this of course implies the equalities appearing in (19).

4. It follows from the isomorphism given in (17) and claim 4 of proposition 2.3 that

$$|(R/N_{i+1})^*| = |N_i/N_{i+1}| \sum_{a+N_i \in q((R/N_i)^*)} |s(a + N_i)|.$$ 

Since, $|s(a + N_i)| = \alpha_i$, it is deduced from the former identity that

$$|(R/N_{i+1})^*| = \alpha_i |N_i/N_{i+1}| q((R/N_i)^*).$$

Finally, identity in (20) follows from (22).

4 Applications of the main results

In this section Theorems 3.3 and 3.5 will be used in order to describe properties of the set of invertible quadratic residues for several classes of rings which include: rings containing a nilpotent ideal; group rings $RG$ where $R$ is a commutative ring containing a collection of ideals satisfying the CNC-condition and $G$ is a commutative group; polynomial ring $R[x]$ where $R$ is a commutative ring containing a collection of ideals satisfying the CNC-condition. Examples are given illustrating the results.

4.1 Rings containing a nilpotent ideal

If $R$ is a commutative ring containing a nilpotent ideal $N$, by invoking Theorems 3.3 and 3.5 properties of the set of invertible quadratic residues of the ring $R$ are described.

**Proposition 4.1.** Let $R$ be a commutative ring and $N$ a nilpotent ideal of nilpotency index $k \geq 2$ in $R$. Then, the following statements hold,

1. Let $s > 1$ be the characteristic of the quotient ring $R/N$. If $a + N$ is a quadratic residue in $R/N$, then $a^{s^{k-1}}$ is a quadratic residue in $R$. More precisely, if $g \in R$ is such that $(g + N)^2 = a + N$, then

$$\left(g^{s^{k-1}}\right)^2 = a^{s^{k-1}}. \quad (23)$$

2. Assuming that $2 + N \in (R/N)^*$, the following claims hold,

   a). $a + N \in q((R/N)^*)$ if and only if $a + N \subset q(R^*)$.

   b). The cardinality of the set $q(R^*)$ is given by

$$|q(R^*)| = |N||q((R/N)^*)|$$. \quad (24)
c). If \( a+N \in q((R/N)^*) \), then
\[
s(a) = s(a+N^{k-1}) = \cdots = s(a+N).
\] (25)

d). If there exists \( \beta \) such that, for all \( a+N \in q((R/N)^*) \), then
\[
|(R/N^{i+1})^*| = \beta |q((R/N^{i+1})^*)|.
\] (26)

In particular,
\[
|R^*| = \beta |q(R^*)|.
\] (27)

**Proof:** In [7] (Proposition 4.1), it is proven that the collection
\[
B = \{N, N^2, \ldots, N^k\}
\]
of ideals of the ring \( R \) satisfies the CNC-condition with nilpotency index and characteristic of the ideal \( N^i \) in the ideal \( N^{i+1} \) being \( t_i = 2 \) and \( s_i = s \) for all \( i = 1, 2, 3, \ldots, k-1 \). Therefore, the proof of this proposition is now a clear consequence of Theorems 3.3 and 3.5.

**Example 4.2.** Let \( p \) be an odd prime number, \( i \in \mathbb{N} \) and let \( R = \{a+bu : a, b \in \mathbb{Z}_{p^i}, a^2 = 0\} \). It is readily seen that \( R \) with the (obvious) addition and multiplication operations is a commutative ring with cardinality \( |R| = p^{2i} \). It is also easily seen that \( R \) is isomorphic to the ring of polynomials with coefficients in \( \mathbb{Z}_{p^i} \) modulo the ideal generated by \( x^2 \), that is \( \mathbb{Z}_{p^i}[x]/(x^2) \). It is readily seen that
\[
R^* = \{a+bu : a \in (\mathbb{Z}_{p^i})^*, b \in \mathbb{Z}_{p^i}\},
\]
so the cardinality of \( R^* \) is \( |R^*| = \varphi(p^i)p^i = (p-1)p^{2i-1} \), where \( \varphi \) denotes the Euler totient function. On the other hand, it is verified that the ideal \( N = \langle p, u \rangle \) has nilpotency index \( k = i + 1 \) and that \( |N| = p^{2i-1} \), then it follows that \( N \) is a maximal ideal of \( R \) with
\[
\frac{R}{N} \cong \mathbb{Z}_p
\]
whence \( |(R/N)^*| = p-1 \) and the characteristic of the quotient ring \( R/N \) is \( s = p \). From the latter isomorphism and proposition 4.1, it is concluded that

- \( a + bu \in q(R^*) \) if and only if \( a \mod (p) \in q(\mathbb{Z}_p^*) \).

- Let \( a + bu \in R \) if \( a \mod (p) \in q(\mathbb{Z}_p^*) \) then for all \( b \in \mathbb{Z}_{p^i} \)
\[
(a + bu)^{p^i} = (a \mod (p))^{p^i}
\]
is an invertible quadratic residue in \( R \).
• The number of invertible quadratic residues of the ring $R$ is given by

$$|q(R^*)| = |N||q((R/N)^*)| = \frac{p^{2i-1}(p-1)}{2}$$

• Let $a + bu \in R$, if $a \mod (p) \in q(\mathbb{Z}_p^*)$ then for all $b \in \mathbb{Z}_p^*$ the number of solutions in $R$ of the equation $x^2 = a + bu$ is equal to 2, in other words

$$s(a + bu) = 2.$$

An easy application of the previous result is the following:

**Corollary 4.3.** Let $R$ be a commutative ring and $c$ a nilpotent element of index $k \geq 2$ in $R$. Then, the following statements hold:

1. Let $s > 1$ be the characteristic of the quotient ring $R/\langle c \rangle$. If $a + \langle c \rangle$ is a quadratic residue in $R/\langle c \rangle$, then $a^{k-1}$ is a quadratic residue in $R$. More precisely, if $g \in R$ is such that $(g + \langle c \rangle)^2 = a + \langle c \rangle$, then

$$\left(g^{k-1}\right)^2 = a^{k-1}. \quad (28)$$

2. Assuming that $2 + \langle c \rangle \in (R/\langle c \rangle)^*$, the following claims hold:

   a. $a + \langle c \rangle \in q((R/\langle c \rangle)^*)$ if and only if $a + \langle c \rangle \subset q(R^*)$.

   b. The cardinality of the set $q(R^*)$ is given by

$$|q(R^*)| = |\langle c \rangle||q((R/\langle c \rangle)^*)|. \quad (29)$$

   c. If $a + \langle c \rangle \in q((R/\langle c \rangle)^*)$, then

$$s(a) = s(a + \langle c^{k-1} \rangle) = \cdots = s(a + \langle c \rangle). \quad (30)$$

   d. If there exists $\beta$ such that $|s(a + \langle c \rangle)| = \beta$ for all $a + \langle c \rangle \in q((R/\langle c \rangle)^*)$, then

$$|(R/\langle c^{i+1} \rangle)^*)| = \beta|q((R/\langle c^{i+1} \rangle)^*)|. \quad (31)$$

In particular,

$$|R^*| = \beta|q(R^*)|. \quad (32)$$

**Proof:** Since $R$ is a commutative ring, $\langle c \rangle$ is a nilpotent ideal of nilpotency index $k$ in $R$, and the result follows immediately from Proposition 4.1.
4.2 Group rings

If $R$ is a commutative ring containing a collection of ideals satisfying the CNC-condition and $G$ is a commutative group, by invoking Theorems 3.3 and 3.5, properties of the set of invertible quadratic residues of the group ring $RG$ are described.

Proposition 4.4. Let $R$ be a commutative ring and $G$ a commutative group. Let $\{N_1, N_2, \ldots, N_k\}$ be a collection of ideals of $R$ satisfying the CNC-condition. Then, the following statements hold:

1. Let $s_i$ be the characteristic of the ideal $N_i$ in the ideal $N_{i+1}$. If $a + N_1G$ is a quadratic residue in $(R/N_1)G$, then $a^{s_1s_2 \cdots s_{k-1}}$ is a quadratic residue in $RG$. More precisely, if $g \in RG$ is such that $(g + N_1G)^2 = a + N_1G$, then

$$ (g^{s_1s_2 \cdots s_{k-1}})^2 = a^{s_1s_2 \cdots s_{k-1}}. \quad (33) $$

2. Assuming that $2 + N_1G \in ((R/N_1)G)^*$, then the following claims hold:
   
   a. $a + N_1G \in q(((R/N_1)G)^*)$ if and only if $a + N_1G \subset q((RG)^*)$.
   
   b. The cardinality of the set $q((RG)^*)$ is given by
   
   $$ |q((RG)^*)| = |N_1|^{[G]}|q(((R/N_1)G)^*)|. \quad (34) $$
   
   c. If $a + N_1G \in q(((R/N_1)G)^*)$ then,

   $$ s(a) = s(a + N_{k-1}G) = \cdots = s(a + N_1G). \quad (35) $$

   d. If there exists $\beta$ such that $|s(a + N_iG)| = \beta$ for all $a + N_1G \in q(((R/N_1)G)^*)$ then,

   $$ |((R/N_{i+1})G)^*| = \beta|q(((R/N_{i+1})G)^*)|, \quad (36) $$
   
   for $i = 1, 2, \ldots, k - 1$. In particular,

   $$ |(RG)^*| = \beta|q((RG)^*)|. \quad (37) $$

Proof: In [7] (Proposition 4.9), it is shown that the collection

$$ B = \{N_1G, N_2G, \ldots, N_kG\} $$

of ideals of the ring $RG$ satisfies the CNC-condition with nilpotency index and characteristic of the ideal $N_iG$ in the ideal $N_{i+1}G$ being exactly the same nilpotency index and characteristic of the ideal $N_i$ in the ideal $N_{i+1}$. Therefore, the proof of this proposition is a direct consequence of Theorems 3.3 and 3.5.

Corollary 4.5. Let $G$ be a commutative group, $R$ be a commutative ring and $N$ a nilpotent ideal of index $k$ in $R$. Then, the following statements hold:
1. Let $s > 1$ be the characteristic of the quotient ring $R/N$. If $a + NG$ is a quadratic residue in $(R/N)G$, then $a^{s-1}$ is a quadratic residue in $RG$. More precisely, if $g \in RG$ is such that $(g + NG)^2 = a + NG$, then

$$
(g^{s-1})^2 = a^{s-1}.
$$

(38)

2. Assuming that $2 + NG \in ((R/N)G)^*$, the following claims hold

a. $a + NG \in q((R/N)G)^*$ if and only if $a + NG \subset q((RG)^*)$.

b. The cardinality of the set $q((RG)^*)$ is given by

$$
|q((RG)^*)| = |N|^{|G|} |q((R/N)G)^*|.
$$

(39)

c. If $a + NG \in q((R/N)G)^*$, then

$$
s(a) = s(a + N^{k-1}G) = \cdots = s(a + NG).
$$

(40)

d. If there exists $\beta$ such that $|s(a + NG)| = \beta$ for all $a + NG \in q((R/N)G)^*$ then,

$$
|(R/N^{i+1}G)^*| = \beta |q((R/N^{i+1}G)^*)|,
$$

for $i = 1, 2, \ldots, k-1$. In particular,

$$
|(RG)^*| = \beta |q((RG)^*)|.
$$

(42)

**Proof:** The proof of this corollary is a direct consequence of Proposition 4.4 and the fact that the collection $\{N, N^2, \ldots, N^k\}$ of ideals of the ring $R$ satisfies the CNC-condition with constant characteristic $s_i = s$ for all $i = 1, 2, 3, \ldots, k-1$.

**Example 4.6.** Let $p$ be an odd prime number, $i \in \mathbb{N}$ and let $R = \{a + bu : a, b \in \mathbb{Z}_p, \ u^2 = 1\}$ be the group ring $\mathbb{Z}_pG$ where $G = \{1, u\}$ is the cyclic group of order $n = 2$. It is readily seen that $R$ with the (obvious) addition and multiplication operations is a commutative ring with cardinality $|R| = p^{2i}$. It is also easily seen that $R$ is isomorphic to the ring of polynomials with coefficients in $\mathbb{Z}_p$ modulo the ideal generated by $x^2 - 1$ in $R$, that is $\mathbb{Z}_pG \cong \mathbb{Z}_p[x]/\langle x^2 - 1 \rangle$. It is readily seen that

$$
(\mathbb{Z}_pG)^* = \{a + bu : a \neq b, a \neq -b\},
$$

so the cardinality of $(\mathbb{Z}_pG)^*$ is $|((\mathbb{Z}_pG)^*)| = (p-1)^2$. In addition, since $N = \langle p \rangle$ has nilpotency index $k = i$ in $\mathbb{Z}_pG$, and

$$
\frac{\mathbb{Z}_pG}{\langle p \rangle G} \cong \mathbb{Z}_pG,
$$

then it is deduced that $|((\mathbb{Z}_pG)^*)| = |(\mathbb{Z}_pG)^*| |\langle p \rangle G| = (p-1)^2 p^{2(i-1)}$. From the latter isomorphism and the proposition 4.5, it is concluded that:
• \(a + bu \in q((\mathbb{Z}_p^iG)^*)\) if and only if \((a \mod (p)) + (b \mod (p))u \in q((\mathbb{Z}_p^iG)^*)\).

• Let \(a + bu \in \mathbb{Z}_p^iG\), if \((a \mod (p)) + (b \mod (p))u \in q((\mathbb{Z}_p^iG)^*)\), then
\[
(a + bu)^{p^i-1} = ((a \mod (p)) + (b \mod (p))u)^{p^i-1}
\]
is an invertible quadratic residue in \(R = \mathbb{Z}_p^iG\).

• The number of invertible quadratic residues of the ring \(R\) is given by
\[
|q((\mathbb{Z}_p^iG)^*)| = |N|^{[G]}|q((\mathbb{Z}_pG)^*)| = p^{2(i-1)}|q((\mathbb{Z}_pG)^*)|
\]

• Let \(a+bu \in \mathbb{Z}_p^iG\), if \((a \mod (p)) + (b \mod (p))u \in q((\mathbb{Z}_p^iG)^*)\) and \(|s((a \mod (p)) + (b \mod (p))u)| = \beta\), then
\[
|s(a + bu)| = \beta.
\]
If additionally, \(|s((a \mod (p)) + (b \mod (p))u)| = \beta\) for all \((a \mod (p)) + (b \mod (p))u \in q((\mathbb{Z}_p^iG)^*)\) then,
\[
|(\mathbb{Z}_p^iG)^*| = \beta|q((\mathbb{Z}_p^iG)^*)|.
\]

For instance, if \(p = 3\), it is easy to see that \(q((\mathbb{Z}_3^iG)^*) = \{1\}\) and the number of solutions in \(\mathbb{Z}_3^iG\) of the equation \(x^2 = 1\) is equal to 4, in other words \(|s(1)| = 4\). Thus, if \(a + bu \in \mathbb{Z}_3^iG\) is such that \(a \equiv 1 \mod (3)\) and \(b \equiv 0 \mod (3)\), then
\[
(a + bu)^{3^i-1} = 1,
\]
\(|s(a + bu)| = 4, |q((\mathbb{Z}_3^iG)^*)| = 3^{2(i-1)}\) and \(|(\mathbb{Z}_3^iG)^*| = (4)3^{2(i-1)}\).

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