Perturbative calculation of quasi-normal modes\textsuperscript{1}

GEORGE SIOPSIS\textsuperscript{2}

\textit{Department of Physics and Astronomy,}

\textit{The University of Tennessee, Knoxville,}

\textit{TN 37996 - 1200, USA.}

E-mail: siopsis@tennessee.edu

Abstract

I discuss a systematic method of analytically calculating the asymptotic form of quasi-normal frequencies. In the case of a four-dimensional Schwarzschild black hole, I expand around the zeroth-order approximation to the wave equation proposed by Motl and Neitzke. In the case of a five-dimensional AdS black hole, I discuss a perturbative solution of the Heun equation. The analytical results are in agreement with the results from numerical analysis.

\textsuperscript{1}Presented at PASCOS 2004 / Nath Fest.

\textsuperscript{2}Research supported in part by the US Department of Energy under grant DE-FG05-91ER40627.
1 Introduction

This report summarizes recent work I did largely in collaboration with S. Musiri [1, 2, 3].

Quasi-normal modes (QNMs) describe small perturbations of a black hole. They are obtained by solving a wave equation for small fluctuations subject to the conditions that the flux be ingoing at the horizon and outgoing at asymptotic infinity. In general, one obtains a discrete spectrum of complex frequencies. The imaginary part determines the decay time of the small fluctuations ($\Im \omega = \frac{1}{\tau}$).

2 Schwarzschild black holes

For a Schwarzschild black hole,\

$$\frac{\omega_n}{T_H} = (2n + 1)\pi i + \ln 3$$

(1)

This was first derived numerically [4, 5, 6, 7, 8] and subsequently confirmed analytically [9]. $\Im \omega_n$ is large and easy to understand because the spacing of frequencies is $2\pi i T_H$, same as the spacing of poles of a thermal Green function. On the other hand, $\Re \omega_n$ is small. Its analytical value was conjectured by Hod [10] and is related to the Barbero-Immirzi parameter. It is intriguing from the loop quantum gravity point of view suggesting that the gauge group should be $SO(3)$ rather than $SU(2)$. Thus the study of QNMs may lead to a deeper understanding of black holes and quantum gravity.

The analytical derivation of the asymptotic form of QNMs [9] offered a new surprise as it heavily relied on the black hole singularity. It is intriguing that the unobservable region beyond the horizon influences the behavior of physical quantities. In the Schwarzschild background metric the wave equation for a spin-$j$ perturbation of frequency $\omega$ is\

$$-\frac{d^2 \Psi}{dz^2} + \frac{1}{\omega^2} V[r(z)] \Psi = \Psi, \quad V(r) = \left(1 - \frac{r_0}{r}\right) \left(\frac{\ell(\ell+1)}{r^2} + \frac{(1-j^2)r_0}{r^3}\right)$$

(2)

where $z = \omega(r + r_0 \ln(r/r_0 - 1) - i\pi r_0)$ and $r_0$ is the radius of the horizon. For QNMs we demand (assuming $\Re \omega > 0$) $e^{iz} \Psi \sim 1$ as $z \to +\infty$ and near the horizon ($z \to -\infty$), $e^{iz} \Psi \sim e^{2iz}$. The latter boundary condition is implemented by demanding that the monodromy around the
singular point \( r = r_0 \) in the complex \( r \)-plane be \( M(r_0) = e^{-4\pi\omega r_0} \) along a contour running counterclockwise. We may deform the contour so that the only contribution to the monodromy comes from the black hole singularity \( (r = 0) \)\(^9\). The potential can be written as a series in \( \sqrt{z} \), which is also a formal expansion in \( 1/\sqrt{\omega} \). We may then solve the wave equation perturbatively. To zeroth order, we obtain

\[
\Psi_0'' + \left\{ \frac{1-j^2}{4z^2} + 1 \right\}\Psi_0 = 0 \tag{3}
\]

whose solutions can be written in terms of Bessel functions. They lead to a zeroth-order monodromy \( M(r_0) = -(1+2\cos(\pi j)) \) yielding a discrete spectrum of complex frequencies (QNMs)

\[
\omega_n/T_H = (2n + 1)\pi i + \ln(1 + 2\cos(\pi j)) + o(1/\sqrt{n}) \tag{4}
\]

Expanding the wavefunction, \( \Psi = \Psi_0 + \frac{1}{\sqrt{-\omega r_0}} \Psi_1 + o(1/\omega) \), the first-order correction obeys

\[
\Psi_1'' + \left\{ \frac{1-j^2}{4z^2} + 1 \right\}\Psi_1 = \delta V \Psi_0 , \quad \delta V = -\frac{3\ell(\ell+1)-j^2}{6\sqrt{2}} z^{-3/2} \tag{5}
\]

By solving eq. (5), one obtains an \( o(1/\sqrt{n}) \) correction to the monodromy which yields the QNM frequencies \( \| \)

\[
\omega_n/T_H = (2n + 1)\pi i + \ln(1 + 2\cos(\pi j)) + \frac{e^{\pi j/2}}{\sqrt{n+1/2}} A + o(1/n) , \quad A = (1 - i) \frac{3\ell(\ell+1)-j^2}{24\sqrt{2\pi}^{3/2}} \frac{\sin(2\pi j)}{\sin(3\pi j/2)} \Gamma^2\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{4} + \frac{j}{2}\right) \Gamma\left(\frac{1}{4} - \frac{j}{2}\right) \tag{6}
\]

This analytical result is in agreement with numerical results for scalar \( (j = 0) \) and gravitational waves \( (j = 2) \).\(^6\)\(^11\) In the latter case, it also agrees with the result from a WKB analysis.\(^12\)

### 3 Kerr black holes

Extending the above discussion to rotating (Kerr) black holes is not straightforward. Numerical results paint a complicated picture \( \|\) and analytical calculations have yet to produce results. We have obtained explicit results \( \|2 \) in the case \( a = J/M \ll 1 \), where \( J \) is the angular momentum of the black hole and \( M \) is its mass. This regime includes the Schwarzschild black hole \( (a = 0) \). For the asymptotic range of frequencies

\[
1 \lesssim |\omega| \lesssim 1/a \tag{7}
\]
working as in the Schwarzschild case, we obtain [2]
\[ \Re \omega = \frac{1}{4\pi} \ln(1 + 2 \cos \pi j) + m\Omega + o(a^2) \]  
(8)
where \( m \) is the azimuthal eigenvalue of the wave and \( \Omega \approx a \) is the angular velocity of the horizon.

In the Schwarzschild limit, the range of frequencies (7) extends to infinity (1/\( a \to \infty \)) and we reproduce our earlier result (4). It would be interesting to extend this result to asymptotic frequencies for arbitrary values of \( a \) (0 < \( a \) < 1).

4 AdS Black Holes

According to the AdS/CFT correspondence, QNMs for an AdS black hole are expected to correspond to perturbations of the dual CFT. The establishment of such a correspondence is hindered by difficulties in solving the wave equation. In three dimensions, it reduces to a hypergeometric equation which is analytically solvable [14]. Numerical results have been obtained in four, five and seven dimensions [15].

In five dimensions, the wave equation reduces to a Heun equation which cannot be solved analytically. We shall build a perturbative expansion in the case of a large black hole based on an approximation to the wave equation which is valid in the high frequency regime [3]. The singularities are located at \( r^2 = \pm r_0^2 \), where \( r_0 \) is the radius of the horizon. In higher dimensions, there are more singularities, all lying on the circle \( |r| = r_0 \) in the complex \( r \)-plane. Even though these are unphysical singularities (with the exception of \( r = r_0 \)), they seem to play an important role in determining the QNMs, as in the Schwarzschild case [9].

Setting the AdS radius \( R = 1 \), the wave equation for a massive scalar of mass \( m \), frequency \( \omega \) and transverse momentum \( \vec{p} \) reads
\[ (y(y^2 - 1)\Psi')' + \left\{ \frac{\dot{\omega}^2}{4} \frac{y^2}{y^2 - 1} - \frac{\dot{\vec{p}}^2}{4} - m^2 y \right\} \Psi = 0 \]  
(9)
where \( y = \frac{r^2}{r_0^2} \), \( \dot{\omega} = \frac{\omega}{\pi T_H} \), \( \dot{\vec{p}} = \frac{|\vec{p}|}{\pi T_H} \). After isolating the behavior at the two singularities, \( y = \pm 1 \),
\[ \Psi(y) = (y - 1)^{-i\dot{\omega}/4}(y + 1)^{-i\dot{\omega}/4} F(y) \]  
(10)
it reduces to the Heun equation
\[ y(y^2 - 1)F'' + \left\{ \left(3 - \frac{i\dot{\omega}}{2}\right)y^2 + \frac{i\dot{\omega}}{2}y - 1 \right\} F' \]
+ \left\{ \hat{\omega} \left( \frac{i\hat{\omega}}{4} - 1 - i \right) y - m^2 y + (1 - i) \hat{\omega} - \frac{\hat{p}^2}{4} \right\} F = 0 \quad (11)

For large \( \hat{\omega} \) and in the physical regime \( r > r_0 \), this may be approximated by the hypergeometric equation

\[
(y^2 - 1) F''_0 + \left\{ \left( 3 - \frac{i + 1}{2} \hat{\omega} \right) y + \frac{1 - i}{2} \hat{\omega} \right\} F'_0 + \left\{ \frac{\hat{\omega}}{2} \left( \frac{i\hat{\omega}}{4} - 1 - i \right) - m^2 \right\} F_0 = 0 \quad (12)
\]

Two independent solutions are

\[
\mathcal{K}_{\pm} = (x + 1)^{-a_{\pm}} F(a_{\pm}, c - a_{\mp}; a_{\pm} - a_{\mp} + 1; 1/(x + 1)) \quad (13)
\]

where \( a_{\pm} = h_{\pm} - \frac{i + 1}{2} \hat{\omega}, \ c = \frac{3}{2} - i \hat{\omega}, \ h_{\pm} = 1 \pm \sqrt{1 + m^2/4} \) and \( x = \frac{y - 1}{2} \). The solution which is well-behaved at the boundary \( (r \to \infty) \) is \( F_0 = \mathcal{K}_+ \). Near the horizon \( (x \to 0) \), we have \( F_0 \sim \mathcal{A}_0 + \mathcal{B}_0 x^{1-c} \), where

\[
\mathcal{A}_0 = \frac{\Gamma(1-c)\Gamma(1-a_+ + a_-)}{\Gamma(1-a_-)\Gamma(1-c + a_+)} , \quad \mathcal{B}_0 = \frac{\Gamma(c-1)\Gamma(1+a_- - a_+)}{\Gamma(a_+)\Gamma(c-a_-)} \quad (14)
\]

For a QNM, we demand regularity at the horizon, so \( \mathcal{B}_0 = 0 \). This yields the spectrum

\[
\hat{\omega}_n = -2(1 + i)(n + h_+ - \frac{3}{2}) \ , \ n = 1, 2, \ldots \quad (15)
\]

which agrees with numerical results [16].

The first-order correction may be written as

\[
F_1(x) = \mathcal{K}_-(x) \int_x^\infty \frac{\mathcal{K}_+ H_1 F_0}{W} - \mathcal{K}_+(x) \int_x^\infty \frac{\mathcal{K}_- H_1 F_0}{W} \quad (16)
\]

where \( W \) is the Wronskian and

\[
H_1 = \frac{1}{2x+1} \left\{ \frac{1}{2} \frac{d}{dx} + (i - 1) \frac{\hat{\omega}}{4} + \frac{\hat{p}^2}{4} \right\} \quad (17)
\]

At the horizon, \( F_1(x) \sim \mathcal{A}_1 + \mathcal{B}_1 x^{1-c} \), where

\[
\mathcal{B}_1 = \frac{\Gamma(c-1)\Gamma(1+a_- - a_+)}{\Gamma(a_-)\Gamma(c-a_-)} \int_0^\infty \frac{\mathcal{K}_+ H_1 \mathcal{K}_+}{W} \quad (18)
\]

The QNM frequencies to first order are solutions of \( \mathcal{B}_0 + \mathcal{B}_1 = 0 \). To solve this equation, we need to calculate the integral in (18). A somewhat tedious calculation [3] leads to corrections to the zeroth-order expression for QNM frequencies (15) which are of order \( 1/h_+ \sim 1/m \).
5 Conclusions

Even though QNMs have long been known numerically, we have only recently made progress toward calculating them analytically. Their behavior appears to rely on unobservable, or even unphysical singularities. It is desirable to fully investigate the behavior of QNMs in general space-time backgrounds. In asymptotically AdS spaces, we would like to understand their relevance to the AdS/CFT correspondence. In asymptotically flat spacetimes, they should lead to the equivalent of Bohr’s correspondence principle for the gravitational force which seems to exist in the case of Schwarzschild black holes but remains elusive for Kerr black holes. This may shed some light on the quantum theory of gravity.

References

[1] S. Musiri and G. Siopsis, Class. Quant. Grav. 20 (2003) L285.

[2] S. Musiri and G. Siopsis, Phys. Lett. B579 (2004) 25.

[3] S. Musiri and G. Siopsis, Phys. Lett. B563 (2003) 102; B576 (2003) 309.
   G. Siopsis, Phys. Lett. B590 (2004) 105.

[4] S. Chandrasekhar and S. Detweiler, Proc. R. Soc. London, Ser. A 344 (1975) 441.

[5] E. W. Leaver, Proc. R. Soc. London, Ser. A 402 (1985) 285.

[6] H. P. Nollert, Phys. Rev. D47 (1993) 5253.

[7] N. Andersson, Class. Quant. Grav. 10 (1993) L61.

[8] A. Bachelot and A. Motet-Bachelot, Annales Poincaré Phys. Theor. 59 (1993) 3.

[9] L. Motl, Adv. Theor. Math. Phys. 6 (2003) 1135.
   L. Motl and A. Neitzke, Adv. Theor. Math. Phys. 7 (2003) 2.

[10] S. Hod, Phys. Rev. Lett. 81 (1998) 4293.

[11] E. Berti and K. D. Kokkotas, Phys. Rev. D68 (2003) 044027.
[12] A. Maassen van den Brink, *J. Math. Phys.* **45** (2004) 327.

[13] E. Berti, *et al.*, *Phys. Rev.* **D68** (2003) 124018.

[14] V. Cardoso and J. P. S. Lemos, *Phys. Rev.* **D63** (2001) 124015.

D. Birmingham, *et al.*, *Phys. Rev. Lett.* **88** (2002) 151301.

[15] G. T. Horowitz and V. E. Hubeny, *Phys. Rev.* **D62** (2000) 024027.

A. O. Starinets, *Phys. Rev.* **D66** (2002) 124013.

R. A. Konoplya, *Phys. Rev.* **D66** (2002) 044009; *ibid.* 084007.

[16] A. Núñez and A. O. Starinets, *Phys. Rev.* **D67** (2003) 124013.