Bootstrapping LCF Declarative Proofs

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Abstract

Suppose we have been sold on the idea that formalised proofs in an LCF system should resemble their written counterparts, and so consist of formulas that only provide signposts for a fully verified proof. To be practical, most of the fully elaborated verification must then be done by way of general purpose proof procedures. Now if these are the only procedures we implement outside the kernel of logical rules, what does the theorem prover look like? We give our account, working from scratch in the ProofPeer theorem prover [12], making observations about this new setting along the way.

1 Introduction

LCF [10] style theorem provers are based on small kernels of inference/typing rules. Ultimately, all theorems are proven by applying these rules, though in practice, users will call derived inference rules and higher-level proof procedures which call yet more derived inference rules and procedures until they ultimately hit the kernel rules. This leads to a zoo of often composable proof tools culminating in full decision procedures, which together afford the user a variety of ways to verify their theorems balancing elegance and simplicity against performance.

There is a more consolidated approach to verification with a less programmatic aesthetic. One can use a declarative language, such as those offered by Isabelle, HOL Light and Coq [4] [11] [17]. A user of such a language writes structured proofs by stating assumptions, variable declarations and intermediate conclusions, and indicating how these are connected by logical consequence. The languages often approximate written mathematical proofs, and could be viewed as the ideal implementation of a formal language for written mathematics. We might then expect them to be more accessible to working mathematicians, a core demographic for ProofPeer. Thus, the language with which users interface with our system, ProofScript [1], takes structured proof as a core part of its syntax, with the traditional panoply of proof procedures intended only to support these readable declarative proofs.

In both Isar and HOL Light’s Mizar Light, the heavy duty automation tends to be handed to clausal form automated theorem provers, and we have not sought to make ProofPeer an exception. In the following sections, we describe our streamlined approach to implementing just the scaffolding necessary...
to make such provers work in the novel setting of ProofScript, a dynamically typed, purely functional language with primitives for working with the terms and theorems of the safely encapsulated and secure logical kernel of ProofPeer.

2 Organisation

In the next section, we will give some motivation for the use of general purpose, external automated theorem provers in interactive theorem proving. In §4, we shall describe the minimal infrastructure we used to talk the basic language of these provers and in §5 we shall discuss what we needed to understand the response. We conclude in §6.

3 Accessing external tools

The Sledgehammer [9] tool in Isabelle began as an impressive effort to integrate external theorem provers, initially Vampire [8] and SPASS [16], into declarative proofs by automatically processing their certificates as inference rules in the Isar language. Since then, many more theorem provers have been integrated [3], but it has turned out that it is sufficient, and much easier besides, to just use the provers to minimise the number of lemmas needed for a first-order proof and then let it be found by Hurd’s METIS [6]. METIS is a generic resolution prover for first-order logic with equality, based on a small LCF style kernel with full proof-recording. As with other resolution provers, its job is to refute the negation of a posed problem, and because of its easily trusted kernel, when METIS claims to find a refutation, we have a high guarantee that the proof record really is a proof of the refutation.

Both Isabelle and METIS are written in Standard ML, so the automated theorem prover is easily usable as a library of the other interactive theorem prover. ProofPeer, on the other hand, runs on the JVM and Javascript, and is implemented entirely in Scala. One of our first efforts, then, was to write a clean port of METIS to Scala, where it can now be used as a library for other JVM or Javascript based-projects. Entirely under our control, this prover is in place to become the standard automated theorem prover which ProofPeer uses to reconstitute first-order proofs, just as its Standard ML variant does for Isabelle’s Sledgehammer.

To access METIS from ProofScript, we equip the language with a built-in function callmetis which can send and receive ProofScript data structures to and from a wrapper around the METIS library. This is currently a somewhat ad-hoc approach, but ProofScript’s data structures are easy to read and write and can serve as a general data exchange format in the manner of S-expressions and JSON, and we certainly intend to provide a general mechanism to integrate more external tools with ProofScript, communicating with wrappers in the same way.

A cautionary note is due because functions which get their data from third
parties may not behave nicely and return the same output for the same input, thereby breaking ProofScript’s promises about functional purity. For now, we can at least be reassured that our implementation of METIS is guaranteed to be fully deterministic, and that callmetis will always return the same output for the same input.

4 Conversion

Any time a user wishes to make an inference in a declarative proof, they will implicitly have in mind a conjecture that the next formula to be derived is a logical consequence of the assumptions currently in force. To a resolution prover, this is turned into a conjunctive normal form problem asserting that the contradiction of the next step together with the assumptions in force entails a contradiction. It therefore suffices to show that such a refutation entails the user’s conjecture.

Normally, however, we show something much stronger, namely that the user’s conjecture and the refutation of its conjunctive normal form are logically equivalent, by computing the latter from the former by successive term rewriting. From its earliest days, the original LCF theorem prover implemented its rewriters via conversions [14], which still represent an impressive demonstration of how higher-order functions can be used to design combinator languages [15].

ProofScript has excellent support for higher-order functions, and even though it eschews the static type system familiar to users of Standard ML, we found this gave us no real difficulty when it came to implementing and using conversions.

In the next few sections, we shall discuss how we go about building the basic infrastructure to compute the CNF form of the user’s conjecture. We shall then describe how we send the version of the problem to METIS via callmetis and how we interpret METIS’s resolution proof trace from its own kernel.

4.1 Primitive inference rules

Our kernel implements rules for a classical, monomorphic, simply typed lambda calculus with just two primitive types: the type of propositions \( \mathbb{P} \) and the type \( \mathcal{U} \) of ZFC sets. All theorems generated by this kernel are associated with rich context objects, which provide the hypotheses of the theorem, and information on how to resolve names to fully qualified identifiers with a logical type.

The contexts are used to support structured declarative proof. At any point in ProofScript code, there is an assumed context of constants that have been introduced and assumptions which have been made of these constants, which can be extended via primitive assume and let statements. ProofScript code is then structured in nested blocks, allowing us to escape back into a parent context as we leave each code block.

The use of these contexts means that there is no need to distinguish constants from free variables, as we do in traditional HOL based theorem provers.
In ProofScript, we do not instantiate free variables in terms, but only specialise quantifiers. We believe this makes ProofScript conceptually simpler for end-users, though as we shall discuss in the coming sections, it presents some complications as far as interfacing with our resolution prover, which does make such syntactic distinctions.

4.2 Conversions and conversionals

Historically, conversions were functions which take a term \( t \) and either produce an equational theorem \( t = t' \) or else throw an exception. These exceptions are standardly caught by derived conversions which may try an alternative conversion strategy. We do not support such an exception mechanism in ProofScript, so our conversions must signal any errors via appropriate return values.

Conversions are operated upon using higher-order functions called conversionals, which allow us to build conversion strategies out of existing conversions, targeting individual parts of a term and responding to failure by trying alternative strategies.

Most of the 1500 lines of ProofScript library code which integrates ProofScript with METIS does so by way of conversions. We define about 50 conversions and conversionals which are used around 500 times.

4.2.1 Basic conversions

Some basic conversions are mostly implemented directly in terms of the kernel’s inference rules or via simple derived rules. Other conversionals exist simply to deal with failure. The conversion \( c_1 \text{orElseConv} c_2 \) takes a term \( t \) and tries \( c_1 t \), checks if it fails, and if so, returns \( c_2 t \) instead. Another conversion, noConv, is the identity of \( \text{orElseConv} \), and is just the constant function to \( \text{nil} \), representing a conversion that always fails.

The conversional \( \text{thenConv} \) provides sequencing, taking two conversions \( t \rightsquigarrow t' \) and \( t' \rightsquigarrow t'' \), and producing the conversion \( t \rightsquigarrow t'' \) via a derived rule for transitivity of equality. This conversional also has an identity, allConv, which is just an alias for the kernel’s reflexivity inference rule.

As noticed by Paulson [14], both conversionals are associative, with \( \text{thenConv} \) distributing over \( \text{orElseConv} \) on the left. We also notice that, if we make conversions non-deterministic such that they return lists of equations with the empty list representing failure, we retain right-distributivity, while if we have them return sets, we get all the axioms of a ring.

4.2.2 Subterm conversionals

ProofScript terms compose only as combinations and abstractions, and so it suffices to define two basic conversionals which can be used to arbitrarily rewrite subterms. The conversional \( \text{combConv} \) is implemented in terms of the primitive kernel destructor \( \text{destcomb} \), which splits combinations, and the inference rule \( \text{combine} \) which says that combining equal terms yields equal combinations. The
conversional accepts conversions $f \rightsquigarrow g$ and $x \rightsquigarrow y$ and yields the conversion $f \ x \rightsquigarrow g \ y$.

Another conversional deals with abstractions via the primitive kernel destructor $\text{destabs}$, which tears apart abstractions, and the inference rule $\text{abstract}$ which sends $\forall x. \phi(x) = \psi(x)$ to $(x \mapsto \phi(x)) = (y \mapsto \psi(y))$. Using these, $\text{absConv}$ sends the conversion $\phi(x) \rightsquigarrow \psi(x)$ to the conversion $x \mapsto \phi(x) \rightsquigarrow x \mapsto \psi(x)$.

It should be noted that ProofScript’s abstraction destructor differs from those of traditional HOL systems, which would normally return the body of the abstraction as a term with a new free variable. As mentioned in §4.1 all ProofScript variables are bound, leaving only the constants. Thus, when passed an abstraction, $\text{destabs}$ returns a triple $(v, \phi(v), \text{ctx})$, where $\text{ctx}$ extends the current context with a new declaration for a new constant $c$, and such that $\phi(c)$ is a valid term in this context.

By locally entering the returned context $\text{ctx}$, we can apply our conversion $\phi(x) \rightsquigarrow \psi(x)$ which, if successful, yields the equational theorem $\phi(c) = \psi(c)$. When this is returned to the enclosing context, the kernel discharges the constant declaration and replaces the constant in the theorem with a universally quantified variable. We can then apply the abstraction inference rule to yield our desired result.

```plaintext
def absConv conv =
  tm =>
    match destabs tm
    case [ctx, _, body] =>
      match incontext <ctx> conv body
      case nil => nil
      case th => abstract (lift! th)
    case _ => nil
```

### 4.2.3 Other primitive conversions

As mentioned, ProofScript’s kernel inference rule $\text{reflexive}$ is already a conversion. There is one other kernel rule which is a conversion, namely $\text{normalize}$. This sends a term to its $\beta\eta$ long normal form.

The following are the only other primitive conversions we need:

- **subsConv** Given a theorem $\vdash t = t'$, sends $t$ to a normalized $t'$, and any other term to nil.

- **rewrConv1** Given a theorem

  $$\vdash \forall x_1 x_2 \ldots, x_n. \phi(x_1, x_2, \ldots, x_n) = \psi(x_1, x_2, \ldots, x_n)$$

  sends $t$ to $t'$ if it finds a substitution $\theta$ such that $t = \phi(x_1, x_2, \ldots, x_n)[\theta]$ and $t' = \psi(x_1, x_2, \ldots, x_n)[\theta]$. If no such substitution is found, returns nil\[^1\].

\[^1\]The matching algorithm which finds $\theta$ is quite rudimentary, but sufficient for our purposes.
Of these, only the primitive rewriting conversion \texttt{rewrConv1} has a non-trivial implementation in the form of a pattern matching algorithm \texttt{for} terms. The pattern is represented by the left hand side of the quantified equation in the theorem passed to \texttt{rewrConv1}.

Here, there is some minor awkwardness in not having free variables. In a regular HOL, a pattern is straightforwardly represented by a term, with the substitutable atoms the free variables, and the unsubstitutable atoms the constants. ProofScript does not support free variables, so we must instead represent the substitutable parts of a pattern with bound variables and keep track of substitutions by associating new constants with binding positions.

The conversion \texttt{rewrConv1} therefore takes a quantified equational theorem whose left-hand side is the pattern to match against. The quantified variables are the ones we can substitute for, and the quantifiers are stripped by repeated applications of \texttt{destabs}, giving us the equational body in a new context \texttt{ctx} where the quantified variables have been replaced with fresh variables $x_1, x_2, \ldots, x_n$ in the quantification order. These variables will have been declared in \texttt{ctx}.

We then prime our matching algorithm with a list of those constants, so that it knows it can treat these as substitutable while all other variables must be treated as fixed constants. The matching algorithm returns an instantiation as an association list $[[x_1, t_1], [x_2, t_2], \ldots, [x_n, t_n]]$. We can then specialise the quantifiers of the original equational theorem in turn with $t_1, t_2, \ldots, t_n$, apply the equational form of Modus Ponens\footnote{This rule sends $\vdash \phi = \psi$ and $\vdash \phi$ to $\vdash \psi$.} and thus obtain the desired right-hand side.

### 4.2.4 Proving theorems by conversion

By definition, a successful conversion proves a theorem given a term. But they can also be used to infer non-equational theorems via \texttt{convRule}, which takes a theorem, applies a conversion to its term, and if successful, uses the equational form of Modus Ponens to prove the converted form of the theorem. We can also prove non-equational theorems $\vdash \phi$ is if we have a conversion $\phi \leadsto \top$. Equivalences of a proposition to true can be eliminated to the proposition via a simple derived inference rule.

### 4.3 CNF Conversion

A full CNF conversion from ProofScript conjectures can be obtained through the following passes:

- convert the term to first-order by:
  - eliminating set builder notation;
  - lifting abstractions, by pulling them out to the front and making them the arguments to $\beta$ redexes (let-definitions);
lifting any terms of type \( P \) used as first-order function or predicate arguments, pulling them out to the front and making them the arguments to \( \beta \) redexes;

- converting the term to its negation normal form:
  - eliminating all propositional connectives other than \( \land, \lor \) and \( \neg \);
  - pushing all negations to the leaves of the term and eliminating double negations;

- converting to prenex normal form by pulling all quantifiers out to the front of the formula;

- converting the propositional matrix of the quantified formula to conjunctive normal form;

- eliminating all quantifiers by skolemisation.

### 4.4 Tautology checking

The parts of the CNF conversion which manipulate propositional formulas are easy enough to implement once we have proven a number of simple propositional equivalences. These are tedious to obtain by hand, so it is helpful to be able to decide and verify propositional tautologies automatically.

A simple but complete tautology checker can be implemented by what amounts to truth-table evaluation. First, we collect a list of equational rewrite theorems, that tell us the semantics of each propositional connective in terms of its valuations on \( \top \) and \( \bot \) (theorems such as \( \vdash \forall p. (p \lor \top) = \top \)). Then, given a propositional conjecture, we sweep through and extract each variable \( p \), performing a case-analysis \( \vdash p = \top \lor p = \bot \). Each complete case-analysis corresponds to a row of a truth-table, and gives us a set of equations which evaluate each variable. Next, using a derived conversional \( \text{upConv} \), we descend to the leaves of a term and then work upwards, applying our equations and rewrite rules, calculating the truth value of the whole proposition and hoping to reach the \( \top \).

We thus have a tautology checker which can be used to prove classical propositional theorems of a few variables (and at most three at a time suits our purposes). As a decision procedure, it can be easily integrated into declarative proof by the ProofScript keyword \text{by}, as in the following code:

```plaintext
theoremandDeMorgan: \forall p q. (\neg (p \land q)) = (\neg p \lor \neg q)
by taut
```

Case splitting on propositional variables, as we have done, requires that we have first proven the law of excluded middle, which can be done via Diaconescu’s [5] proof, suitably modified for Hilbert’s indefinite description operator, which assumes full classical choice. ProofScript has a novel implementation of choice, and, unlike the HOLs, does not take indefinite descriptions as primitive. The classical inference is instead achieved automatically by ProofScript’s rules for lifting theorems out of contexts:
choose choiceDef : 'ε : (P → P) → P'
let 'p : P → P'
assume ex : '∃x. p x'
choose 'chosen : P' ex

Here, we define an ε function based on a nested proof. This proof assumes that there exists some object satisfying p and then chooses a witness. When the assertion that the witness satisfies p is lifted out of the context, the assumptions are discharged and a quantifier introduced for the variable p, yielding the theorem

∀p. ∃chosen. (∃x. p x) → p chosen.

By applying choose to this universal theorem, ProofScript will first skolemize to

∃chosen. ∀p. (∃x. p x) → p (chosen p)

and then introduce ε as a new Skolem constant.

4.5 Skolemisation

A higher-order theorem which captures the main rewrite rule we need to perform skolemisation is given by:

\[ \vdash ∀p. (\forall x. ∃y. p x y) = (∃f. ∀x. p x (f x)). \] (1)

Repeated rewriting with this theorem allows us to fully skolemize. To take an example:

\[ ∀x y : U. ∃z : U. P x y z = ∀x : U. ∃f : U → U. ∀y. P x y (f y) = ∃g : U → U → U. ∀x. ∀y. P x y (g y x). \]

But notice that the type of f in the second formula differs from the type of g in the third, meaning that we are applying variants of (1) with different types. This is no issue in traditional HOLs which support a simple form of type polymorphism and type instantiation. ProofScript, however, due to a deliberate decision to limit the expressivity of its higher-order metalogic, can only express the above as a monomorphic theorem.

This means that an arbitrary number of variants of (1) may be needed to skolemize theorems in ProofScript, each requiring its own proof, because there are infinitely many possible types that the Skolem functions may inhabit. The best we can hope to do is capture the theorems in a single definition, a metatheorem expressed in the language of ProofScript, as it were:

def skolemThm [a, b] =
  theorem '∀p. (∀x. ∃y. p x y) = (∃f : <a> → <b>. ∀x. p x (f x))'
...

8
Here, ProofScript inserts the type variables passed as function arguments \( a \) and \( b \) into the \(< >\) quoted positions of the theorem.

This is still an unpleasant state of affairs, since every time we call \texttt{skolemThm}, we may find ourselves redoing a proof, namely when the arguments \( a \) and \( b \) have been passed in before. Our solution is to introduce a primitive memoisation mechanism via the ProofScript keyword \texttt{table}. This can replace \texttt{def} in the above code and will cause ProofScript to save the result of calls to \texttt{skolemThm} and return the saved copy whenever the function is called with the same arguments.

In general, it is worth being wary of the consequences of one’s decisions to simplify a logic because one does not have need of the expressive power. While we have chosen to keep the logic monomorphic to encourage users to work as much as possible within the ZF object logic, we would not want to push this to the point of using a first-order logic, where we cannot properly express axioms such as the axiom of comprehension, and where we would require the system to waste time proving arbitrarily many instances of metatheorems for any generally useful consequence of such axioms.

With conversions, the ProofScript code which performs the CNF conversion is structurally very similar to the pseudocode algorithms which do the same, but with the benefit that any result is formally verified for correctness. Conversions give us a powerful way to drive \textit{computation}, and are well suited to this sort of algebraic normalisation.

## 5 Talking to METIS

\textsc{METIS}’s logical kernel deals only in CNF clauses, which are represented as sets of literals. There are no connectives or quantifiers in the logical syntax.

### 5.1 Representing \textsc{METIS} clauses and certificates

\textsc{METIS} proof certificates are returned to ProofScript as a tree, rooted at the empty clause which represents the refutation, and with nodes labelled with clauses derived and the inference rule used. The possible rules are:

- **Axiom** leaf nodes containing an arbitrary clause;
- **Assume** leaf nodes with clause of the form \( \{ \phi, \neg \phi \} \);  
- **Refl** leaf nodes with clause of the form \( \{ x = x \} \);  
- **Equality**(\( L(x), s, t \)) leaf nodes with a clause of the form \( \{ s \neq t, L', L[t/x] \} \) where \( L' \) is the negation of \( L[s/x] \);  
- **RemoveSym** a derived rule in Hurd’s implementation, this node has a clause which is the same as its child, minus duplicate equalities and inequalities (up to symmetry of equality);
Irreflexive a node which is the same as its child, but without literals of the form \(x \neq x\);

Subst(\(\theta\)) a node whose clause is a substitution \(\theta\) of its child clause;

Resolve(L) uses of the resolution rule mark nodes with two children, whose clauses are of the form \(C_1 \cup \{L\}\) and \(C_2 \cup \{L'\}\) where \(L'\) is the negation of \(L\). The resolution’s clause is \(C_1 \cup C_2 - \{L, L'\}\).

Generally, the more powerful METIS’s kernel rules, the more work we have cut out for us on the ProofScript side. For one example, METIS implements clauses as sets of literals, and so assumes a bunch of rules for working with disjunctions without telling us how they are applied. For another, we have taken the liberty of replacing some of METIS’s derived kernel rules with primitive ones, but each such liberty comes at a cost of extra work in certificate translation.

To deal with literals as sets, it makes sense to ensure that our ProofScript clause representation is in some normal form, and for us, that just means making sure we are always normalised with respect to associativity and idempotency of disjunction. In that way, we can treat our clauses as lists of literals with unique elements. Passes over the clause to remove duplicates are only needed to implement Subst and Resolve, and are handled with a conversion nubClauseConv. We need no pass for associativity, which is always preserved by our implementation of the METIS kernel rules.

In ProofScript, we represent proof certificates as a tree made by nesting lists, with clauses as sets and literals encoded in an S-expression like format according to \(\theta\):

\[
\theta(t) = \begin{cases} 
  t, & \text{if } t \text{ is a variable} \\
  [F, \theta(t_1), \theta(t_2), \ldots, \theta(t_n)], & \text{if } t \text{ is the function application} \\
  F(t_1, t_2, \ldots, t_n) \text{ with function symbol } F \\
  \text{true}, [P, \theta(t_1), \theta(t_2), \ldots, \theta(t_n)], & \text{if } t \text{ is the positive literal} \\
  P(t_1, t_2, \ldots, t_n) \text{ with predicate symbol } P \\
  \text{false}, [P, \theta(t_1), \theta(t_2), \ldots, \theta(t_n)], & \text{if } t \text{ is the negative literal} \\
  \neg P(t_1, t_2, \ldots, t_n) \text{ with predicate symbol } P.
\end{cases}
\]

Our port of METIS has literals which are polymorphic in variable, function symbol and predicate symbol alphabets, and it is up to the caller to decide what alphabets to use. So can get away with just using actual ProofScript function terms as METIS function symbols and Proofscript predicate terms as predicate symbols.

5.1.1 Free variables aren’t

METIS variables often need to be instantiated, so when represented as a ProofScript theorem, they must be bound in clauses by quantifiers. Since they must also be instantiated on a clause-by-clause basis, we need a set of bindings per
clause. However, METIS likes to be able to identify variables across clauses, even as it substitutes for them independently.

We cannot make this identification reliably in ProofScript, which only identifies bound variables between clauses up to $\alpha$-equivalence, and even if we have some control over the naming of variables, we cannot reliably control ProofScript’s naming of fresh ones.

The best we can do is single out a variable by the index of its binding site (the DeBruijn index in this case), but with METIS potentially generating hundreds of fresh variables during resolution, it would be disastrous to take such indices as a direct representation of METIS variables.

Instead, we introduce maps from bound variable indices to METIS variables and carry the mapping around in our clauses. A clause of $n$ variables and $m$ literals is therefore represented by a pair consisting of a list of METIS variables and a ProofScript theorem:

$$([v_1, v_2, v_3, \ldots, v_n], \vdash \forall x_1 x_2 \ldots x_n. \phi_1 \lor \phi_2 \lor \ldots \lor \phi_m).$$

where we assume that the $i^{th}$ bound variable corresponds to the METIS variable $v_i$. Plain integers make a convenient choice for the representation of these METIS variables, and at the call-site, we ask the resolution prover to use them both internally and in proof certificates. For a concrete example, the METIS clause $\{p(3,8), \neg q(2,5)\}$ could be represented by any of the ProofScript pairs:

$$([3,8,2,5], \vdash \forall x y z u. p x z \lor \neg q y w);$$
$$([8,5,3,2], \vdash \forall a b c d. \neg q d b \lor (p c a));$$
$$([5,2,8,3], \vdash \forall x_1 x_2 x_3 x_4, p x_4 x_3 \lor \neg q x_2 x_1).$$

Now suppose we want to perform the substitution $[h(1,2,9)/8]$. The resulting clause will be $\{p(3,h(1,2,9)), \neg q(2,5)\}$, represented by a ProofScript clause that is $\alpha$-equivalent to

$$\vdash \forall x y z u v. p x (h y z u) \lor \neg q z v.$$

To find this, we must first figure out how many bound variables will be needed after the substitution, generate fresh constants for them, and then recreate the mapping from indices to METIS free variables.

Next suppose we want to resolve our new clause with $\{q(2,5), r(4,6)\}$ which might happen to be represented by

$$([5,4,2,6], \vdash \forall x y z w. r y w \lor q z x).$$

We can strip away quantifiers from one of the clauses to get at the literals, introducing new constants into the current context along the way, but we will have to make sure that we instantiate rather introduce variables in the other clause whenever our bound variable mappings tell us there is a match. And after resolution, we will have lost the free METIS variable 5, and so will be deleting it from our index mapping.
This requires a lot of delicate work, and a clear understanding of how the introduction of universal quantifiers by lifting out of contexts can be exploited programmatically. It is not the sort of thing we expect the average ProofPeer user to get their hands dirty with, though if they feel the need to, the tight fit between ProofScript’s structured proof language and its programming language should make it seem natural. But the regular user, we hope, will mostly enjoy writing simple theorems such as the following, based on our working implementation of METIS:

```plaintext
let oneDef: 'one = \emptyset'
let twoDef: 'two = \sec{one}'

theorem one: '\forall x. x \in one = (x = \emptyset)'
  by metis [empty, oneDef, power, subset, ext]

theorem two: '\forall x. x \in two = (x = \emptyset \lor x = one)'
  by metis [empty, one, twoDef, power, subset, ext]

theorem oneNotZero: '-(\emptyset = one)'
  by metis [empty, one]
```

6 Conclusion

The scripting language for ProofPeer, aptly named ProofScript, is intended to be the programmable metalanguage for an LCF kernel, but what distinguishes it from other languages designed for this purpose, such as Standard ML, is that it aims primarily to be the natural vehicle for writing accessible, declarative proofs, where black-box automation is left to fill out the tedious gaps.

The route we have taken towards this goal has not required any stops at the tactic systems or contextual rewriters of traditional LCF, but we have made extensive use of conversions and higher-order conversions in order to connect to the METIS first-order theorem prover. If the success of the Isabelle Sledgehammer project is anything to go by, this could be the only integrated tool that we need when it comes to first-order automated proof.

ProofScript shares qualities of the ML dialects in its succinctness and support for higher-order functions, and we were pleased that we were able to reason easily with what would become quite complex types in ML. We did not really miss the static type system reminding us not to mix list parameterised conversions with term parameterised ones. This is not to say that there are not other benefits to having a type system, in terms of automatic documentation in type-signatures and in supporting refactoring of complex code-bases, but our needs are quite modest for now.

When designing the logic of ProofScript, we aimed at some conceptual simplifications so as not to distract users. Users are not left to choose between building mathematical structures out of user-defined types and exhibiting those
structures in ZFC sets as they are in HOL/ZFC \cite{agerholm1995}. Our type inference \cite{schwabe2013} algorithm guides users away from polymorphic universes to the ZFC one. And users writing declarative proofs are not left wondering what the difference is between a universally quantified theorem and one where all the variables are free to be instantiated. The distinction between free variables and constants is just a logical confusion to a user working in a declarative setting where constants can be freely introduced in local contexts.

But when it comes to the nuts-and-bolts of programming proofs, there is sometimes another story. Simple polymorphic types can improve the efficiency of proofs by replacing infinitely many metatheorems with a single instantiable one. And free variables, effectively being named holes in terms and theorems which can be identified across contexts, become a means by which a programmer can represent their own instantiation mechanisms. Not having such things meant that in sections \[4.2.3\] and \[5.1.1\] we were forced to shadow DeBruijn indices with our own metadata. This is just the usual reminder that there are balances to be struck between conceptual simplicity and practicalities. We feel justified that we have struck such a balance.

Future work on automation in ProofPeer will focus on replicating the impressive and ongoing development of Sledgehammer and its HOL Light counterpart HOL(y) Hammer \cite{schwabe2013}, thus allowing Proofscript to access a wide variety of powerful ATP tools which can then be used to advise the METIS prover. As we proceed, we expect that the necessary infrastructure of ProofScript will remain lean compared to those of its friends in the LCF tradition.

References

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