EVOLUTION OF THE EARLY UNIVERSE IN THE SCALE IN Variant THEORY

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(Received; Revised; Accepted)
Submitted to ApJ

ABSTRACT
Analytical solutions are obtained for the early cosmological phases in the scale invariant models with curvature $k = 0$. They complete the analytical solutions already found in the matter-dominated era by Jesus (2018). The physical properties in the radiative era are derived from the conservation laws and compared to those of current standard models. The critical runs of the temperature $T$(MeV) and of the expansion rate $H$ of the scale invariant models with low densities, e.g. $\Omega_m = 0.04$, are quite similar at the time of nucleosynthesis to those of standard models with $\Omega_m = 0.30$, leading to the same freezing number ratio of neutrons to protons. These results are consistent with the fact that the scale invariant models appear to not require the presence of dark matter.

Keywords: Cosmology: theory – primordial nucleosynthesis – dark matter
It is often ignored that when the cosmological constant $\Lambda$ is assumed to be equal to zero, the equations of General Relativity are scale invariant, which is also a property present in Maxwell equations of electrodynamics. However, for a non-zero $\Lambda$, the field equations of the gravitation no longer show the property of scale invariance. A fact, which as discussed by Bondi (1990), was one of the reasons of Einstein’s disenchantment for the cosmological constant. It is thus of interest to examine at what conditions the scale invariant properties of General Relativity may be restored, since current cosmological observations appear to support a positive cosmological constant.

A theoretical framework has been developed by Dirac (1973) and Canuto et al. (1977), in the so-called co-tensorial calculus based on the Integrable Weyl’s geometry. It offers a consistent basis to account for the properties of scale invariance of gravitation to a scale factor $\lambda$, as also illustrated by several properties studied by Bouvier & Maeder (1978). Scale invariant derivatives, modified affine connexions, modified Ricci tensor and curvatures can be obtained leading to a general scale invariant field equation. Dirac and Canuto et al. have expressed an action principle in the scale invariant framework, with a matter Lagrangian, as a coscalar of power -4 (varying like $\lambda^{-4}$). By considering the variations of this action, they also obtain the generalization of the Einstein field equation. This is Equation (7) in Maeder (2017a) from which the scale invariant cosmological equations derived.

In the integrable Weyls space, the scale factor remains undetermined without any other constraints as shown by Dirac and Canuto et al. Thus, these authors were fixing the scale factor by some external considerations based on the so-called Large Number Hypothesis, an hypothesis often disputed (Carter 1979). Thus, it seems appropriate to explore other conditions for setting the gauge. The proposition was thus made to fix the gauge factor by simply assuming the scale invariance of the empty space (Maeder 2017a). This means that the properties of the empty space should not change under a contraction or dilatation of the space-time. Indeed, we note, as shown by Carroll et al. (1992), that the current equation of the vacuum $P_{\text{vac}} = -\rho_{\text{vac}} c^2$ already implies that $\rho_{\text{vac}}$ should remain constant "if a volume of vacuum is adiabatically compressed or expanded". On this basis, the cosmological equations derived by Canuto et al. (1977) were greatly simplified (Maeder 2017a). A number of cosmological tests were performed, with positive results. Interestingly enough, these equations were then shown to have rather simple analytical solutions (Jesus 2018) for models of a matter dominated Universe with a zero curvature.

In order to express the motions of free particles, a geodesic equation was obtained (Bouvier & Maeder 1978) from a minimum action in the integrable Weyls space. In the weak field approximation, the geodesic equation leads to a modification of the Newton equation (Maeder & Bouvier 1979): it then contains a (currently very small) additional acceleration term proportional to the velocity of the particles. This equation was applied to study the internal motions of clusters of galaxies, the flat rotation curves of spiral galaxies and the age increase of the ”vertical” velocity dispersion of stars in galaxies (Maeder 2017c). The interesting result was that the observational properties of these various systems could be accounted without requiring to the current hypothesis of dark matter, and the same for the radial acceleration relation (RAR) of galaxies (Maeder 2018). The growth of the density fluctuations in the early Universe was also studied by Maeder & Gueorguiev (2018). The usual theory is that at recombination the baryons settle in the potential wells formed by dark matter previously assembly during the radiative era. In the scale invariant framework, the growth of the density fluctuation, predicted from the corresponding Euler, Poisson and continuity equations, is fast enough so that dark matter is not needed to account for the formation of galaxies.

The object of the present work is to search for the solutions in the radiative era of the universe in the scale invariant context. In Section 2, we derive the analytical solutions for cosmological models dominated by relativistic particles, and examine the values at the equilibrium of matter and radiation. In Section 3, the physical properties in the radiative era are studied. The conditions at the time of the cosmological nucleosynthesis are compared to the results of standard models.

2. THE COSMOLOGICAL SOLUTIONS IN THE RADIATIVE ERA

We study the physical conditions in the radiative era, in particular when the cosmological nucleosynthesis occurs. At this stage, the age of the Universe, of 1 second and more, is already $10^{43}$ times larger than the Planck time and thus the Universe dynamics is no longer dominated by quantum effects. Thus, if it occurs that scale invariance applies in the present day Universe, it should likely also be the case at the time of nucleosynthesis. For the matter era, analytical solutions were found by Jesus (2018) in the case of a flat Universe model with $k = 0$. Now, we also need the solutions of the cosmological models in the radiative era.
2.1. The equations and their solutions

The hypothesis of scale invariance assumes that the equations should be invariant to a transformation of the form $ds' = \lambda(x^\mu)ds$ (Weyl 1923; Dirac 1973; Canuto et al. 1977). The term $\lambda$ is the scale factor expressing a relation of the form $ds' = \lambda(x^\mu)ds$, where $ds'$ is the line element in general relativity and $ds$ the line element in the supposed more general scale invariant space. For reasons of homogeneity and isotropy the scale factor depends only on time $t$, and the condition of the invariance of the empty space imposes that $\lambda \sim 1/t$ (Maeder 2017a). Interestingly enough, this form is also one of those considered by Dirac (1973). The equations of the scale invariant cosmology under the above mentioned hypothesis are

$$\frac{8 \pi G \varrho}{3} = \frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + 2 \frac{\dot{R} \lambda}{R \lambda}, \quad (1)$$

$$-8 \pi G \rho = \frac{k}{R^2} + 2 \frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R} + 4 \frac{\dot{R} \lambda}{R \lambda}. \quad (2)$$

The combination of these two equations leads to

$$-\frac{4 \pi G}{3} (3p + \rho) = \frac{\dot{R}}{R} + \frac{\dot{R} \lambda}{R \lambda}. \quad (3)$$

The various terms have their usual significations, with dots representing time derivatives. These equations only differ from the usual ones by the presence in each case of an additional term containing the product $\frac{\dot{R} \lambda}{R \lambda}$. We see that if $\lambda(t)$ is a constant, the usual equations of cosmologies are recovered. This means that, at any fixed time, the effects that do not depend on time evolution are just those predicted by GR.

For the flat space with $k = 0$ in the matter dominated era, this equation has an analytical solution of the following form (Jesus 2018),

$$R(t) = \left( \frac{t^3 - \Omega_m}{1 - \Omega_m} \right)^{2/3}, \quad (4)$$

where $\Omega_m = \rho_{m,0}/\rho_{c,0}$ with $\rho_{c,0} = 3 H_0^2/(8 \pi G)$, the standard expression of the critical density at present. In the above equation, it is assumed that at the present time $t_0 = 1$, one has $R_0 = 1$. Interestingly enough, it is also possible in this context to have for a flat space different values of $\Omega_m$. The expression of the Hubble expansion term $H(t)$ resulting from (4) is

$$H(t) = \frac{2 \dot{R}^2}{t^3 - \Omega_m}, \quad \text{so that} \quad H_0 = \frac{2}{1 - \Omega_m}, \quad (5)$$

expressed here in the above scales of time and space. In Sect. 3, these quantities will be used with their appropriate physical units. We see from these two expressions that there is no solutions for $\Omega_m \geq 1$, since the presence of a significant matter content in the Universe is killing scale invariance (Maeder 2017a).

The radiative era of the Universe is dominated by relativisitic particles. The density is $\varrho_{\text{rel}}$,

$$\varrho_{\text{rel}} = K_0 \varrho_\gamma, \quad \text{with} \quad K_0 = 1 + \frac{7}{8} \left( \frac{4}{11} \right)^{4/3} N_\nu. \quad (6)$$

There, $\varrho_\gamma$ is the density of photons. The term $K_0$ accounts for the neutrino contribution to the density of relativisitic particles (Coles & Lucchin 2002), where $N_\nu$ is the number of neutrino types. For a number $N_\nu = 3$, the value is $K_0 = 1.6813$. The appropriate conservation law for relativisitic particles has to be accounted for. According to Maeder (2017a), it is $\varrho_{\text{rel}} R^4 \lambda^2 = \text{const.}$, thus Equation (1) writes,

$$\lambda^2 \ddot{R}^2 R^2 + 2 \dot{R} R^3 \lambda \lambda = -C_{\text{rel}}, \quad (7)$$

with

$$C_{\text{rel}} = \frac{8 \pi G \varrho_{\text{rel}} R^4 \lambda^2}{3}. \quad (8)$$

The differential equation to be solved for $R(t)$ is thus,

$$\ddot{R}^2 R^2 t - 2 \dot{R} R^3 - C_{\text{rel}} t^3 = 0. \quad (9)$$
We now examine whether the above differential equation also possesses analytical solutions. We try to develop $R(t)$ as a product,

$$ R(t) = t \cdot v(t), \quad \text{thus} \quad \dot{R} = v + t \cdot \dot{v}. \quad (10) $$

Thus, Equation (9) leads to

$$ \dot{v} = \pm \sqrt{\frac{v^4 + C_{\text{rel}}}{v^2 t^2}}, \quad \text{and} \quad \frac{v \cdot dv}{\sqrt{v^4 + C_{\text{rel}}}} = \pm \frac{dt}{t}. \quad (11) $$

The equation is separable and from Bronstein & Semendiaev (1974) we obtain as an integral

$$ \ln \left( \sqrt{v^4 + C_{\text{rel}}} + v^2 \right) = \pm 2 \ln t + \text{const.} \quad (12) $$

Considering solutions of $R(t)$ growing with time, we take the sign “+” in the above equation and obtain

$$ \sqrt{v^4 + C_{\text{rel}}} + v^2 = c_2 t^2, \quad (13) $$

where $c_2$ is a constant, which will be fixed below from the relevant boundary conditions. The present solution does not apply at time $t_0$, but only in the radiative era, thus it is not possible to directly express the constants $C_{\text{rel}}$ and $c_2$ as functions of cosmological parameters (like $H_0$) at the present time. We get after some simple manipulations

$$ v = \left( \frac{c_2^2 t^4 - C_{\text{rel}}}{2 c_2 t^2} \right)^{1/2} \quad (14) $$

and for the expansion factor

$$ R(t) = t \left( \frac{c_2^2 t^4 - C_{\text{rel}}}{2 c_2 t^2} \right)^{1/2} = \frac{1}{\sqrt{2}} \left( \frac{c_2}{t} \right)^{1/2} \left( \frac{c_2 t^4}{2} - C_{\text{rel}} \right)^{1/2}. \quad (15) $$

The derivative of the expansion factor is

$$ \dot{R} = \left( \frac{c_2^2 t^4 - C_{\text{rel}}}{2 c_2 t^2} \right)^{1/2} \frac{1}{2} c_2^2 t^4 + C_{\text{rel}} \left( \frac{2 c_2}{c_2^2 t^4 - C_{\text{rel}}} \right)^{1/2}, \quad (16) $$

so that the Hubble expansion rate at a given time in the radiative era is,

$$ H = \frac{\dot{R}}{R} = \frac{1}{t} + \frac{1}{2 \cdot R^2} \frac{2 c_2^2 t^4 + C_{\text{rel}}}{c_2^2 t^3}. \quad (17) $$

There is thus an analytical solution of the scale invariant equation in the radiative era, which may greatly simplify all studies. The solution depends on the constants $C_{\text{rel}}$ and $c_2$, which we shall relate below to the densities of matter and of relativistic particles, respectively $\varrho_m$ and $\varrho_{\text{rel}}$ at the time when these two densities become equal.

2.2. Values of the constants and properties at the density equilibrium

In the radiation era, according to Equation (8) which expresses the appropriate conservation law, the density of relativistic particles at a given time $t$ is,

$$ \varrho_{\text{rel}} = \frac{3 C_{\text{rel}}}{8 \pi G R^3 \lambda^2}. \quad (18) $$

In the matter era, the corresponding expression obtained from the matter conservation law is (Maeder 2017a),

$$ \varrho_m = \frac{3 C_m}{8 \pi G R^3 \lambda}. \quad (19) $$

At the time $t_{eq}$ when both densities become equal, we have the following relation between the constants,

$$ C_{\text{rel}} = C_m \frac{R_{eq}}{t_{eq}} = \frac{4 \Omega_m}{(1 - \Omega_m)^2} \frac{R_{eq}}{t_{eq}}. \quad (20) $$
where $R_{\text{eq}}$ and $t_{\text{eq}}$ are the values at equilibrium and the constant $C_m = \frac{4\Omega_m}{(1-\Omega_m)^2}$ in the matter dominated era, as given by Equation (66) in Maeder (2017a). The term $\Omega_m$ is the usual parameter of matter density as discussed in Section (2.1). The constant $c_2$ is fixed by expression (13) applied to the time of equilibrium,

$$c_2 = \frac{v^2_{\text{eq}} + \sqrt{v^4_{\text{eq}} + C_{\text{rel}}}}{t^2_{\text{eq}}} , \quad \text{with} \quad v_{\text{eq}} = \frac{R_{\text{eq}}}{t_{\text{eq}}} .$$

The expression of $v_{\text{eq}}$ in terms of basic quantities is given below in Equation (23). The values of $R_{\text{eq}}$ and $t_{\text{eq}}$ are fixed thanks to the conservation laws. In the matter and radiative eras, we respectively have at a time $t$,

$$\varrho_m R^3 \lambda = \Omega_m \varrho_c, \quad \text{and} \quad \varrho_{\text{rel}} R^4 \lambda^2 = \varrho_{\gamma,0} R_{\text{eq}} \varrho_{c,0} ,$$

where the index $0$ indicates values at the present time $t_0$. The equality of the two densities defines the equilibrium point, we get

$$R_{\text{eq}} \lambda_{\text{eq}} = \frac{R_{\text{eq}}}{t_{\text{eq}}} = \frac{K_0 \varrho_{\gamma,0}}{\Omega_m \varrho_{c,0}} .$$

The numerical value of $v_{\text{eq}}$ is given by the above expression, then $t_{\text{eq}}$ is obtained by using the expression for $R(t)$ in the matter dominated era, as given by Equation (4). We first write,

$$v_{\text{eq}} = \frac{R_{\text{eq}}}{t_{\text{eq}}} = 1 - \frac{\Omega_m}{1 - \Omega_m} \left( \frac{1}{2} \left( \frac{2}{3} \right)^{2/3} \right) ,$$

and then search the solution of this equation, which is

$$t_{\text{eq}} = \left( \frac{2}{3 \varrho_{\text{eq}} (1 - \Omega_m)} \right) ^{3/2} + \left( \frac{2}{4 \varrho_{\text{eq}} (1 - \Omega_m)^2 + 4 \Omega_m} \right) ^{2/3} .$$

Since, $v_{\text{eq}}$ is known and $\Omega_m$ is chosen, the above equation gives the value of $t_{\text{eq}}$ and then of $R_{\text{eq}}$ by (24). The redshift at the equality of the two densities is obtained from expression (23) by,

$$(1 + z_{\text{eq}}) t_{\text{eq}} = \frac{\Omega_m \varrho_{c,0} K_0}{\varrho_{\gamma,0}} = 1/v_{\text{eq}} .$$

Equations (20) and (21) provide the constants $C_{\text{rel}}$ and $c_2$. The numerical values of the various input parameters seen above are the following ones,

$$H_0 = 3.2408 \cdot 10^{-18} h \ [s^{-1}]$$

$$\varrho_{\gamma,0} = K_0 \frac{a T^4}{c^2} = 4.6485 \cdot 10^{-34} K_0 \ [g/cm^3]$$

$$\varrho_{m,0} = \Omega_m \varrho_{c,0} = 1.8788 \cdot 10^{-29} h^2 \Omega_m \ [g/cm^3]$$

$$v_{\text{eq}} = \frac{R_{\text{eq}}}{t_{\text{eq}}} = 2.4741 \cdot 10^{-5} \frac{K_0}{\Omega_m h^2} .$$

Table 1 gives for different values of $\Omega_m$ the values of the useful parameters for the model description in the early phases according to the above equations and constants.

3. PHYSICAL PROPERTIES IN THE RADIATIVE ERA

We now examine the evolution of the physical properties during the radiative era. The conservation laws of matter and radiation densities (22), together with the constants given by (27) lead to

$$\varrho_m(t) = 1.8788 \cdot 10^{-29} h^2 \Omega_m t \frac{R(t)}{R^4(t)} \ [g/cm^3] ,$$

$$\varrho_{\text{rel}}(t) = K_0 \varrho_{\gamma} = 4.6485 \cdot 10^{-34} K_0 \frac{t^2}{R^4(t)} \ [g/cm^3] ,$$

$$T(t) = 2.726 \frac{t^{1/2}}{R(t)} \ [K] .$$
Table 1. Values of parameters at the equilibrium point between matter and density in the scale invariant models with $k = 0$ and various $\Omega_m$, as well as the constants $C_{rel}$ and $c_2$. The value of $h = 0.70$ is chosen here together with a number of neutrinos $N_\nu = 3$. The time $t_{eq}$ is the equilibrium time and $t_0$ the initial time, both in a timescale where $t_0 = 1$. The time $\tau_{eq}$ is the equilibrium time in years, in the scale where the present age of the Universe is 13.8 Gyr.

| $\Omega_m$ | $\tau_{eq}$ | $t_{eq}$ | $R_{eq}$ | $(1+z)_{eq}$ | $\rho_{eq}$ | $T_{eq}[K]$ | $C_{rel}$ | $c_2$ | $\tau_{eq}$ | $t_{in}$ |
|-----------|-------------|----------|----------|-------------|------------|------------|-----------|------|-------------|---------|
| 0.50      | 1.697892E-04| 0.793709 | 1.347619E-04 | 742.5       | 1.492821E-18 | 1.802135E-04 | 1.358314E-03 | 5.850414E-02 | 2.76864E+04 | .7937007 |
| 0.30      | 2.829820E-04| 0.669434 | 1.844378E-04 | 5278.8      | 2.719636E-19 | 1.177370E-04 | 6.930173E-04 | 5.67737E+04 | 6.964333 |
| 0.20      | 4.244731E-04| 0.584806 | 2.483247E-04 | 4028.4      | 7.039421E-04 | 2.037876E-04 | 5.035313E-04 | 1.41636E+04 | 1.10360E+05 | .5841616 |
| 0.10      | 8.489461E-04| 0.464169 | 3.945551E-04 | 2537.7      | 6.983735E-04 | 4.713107E+03 | 4.192327E-04 | 2.80515E+05 | .4641616 |
| 0.04      | 2.122365E-03| 0.342048 | 7.259523E-04 | 1377.5      | 3.292346E-04 | 1.177362E+04 | 6.930173E-04 | 1.12216E+04 | .3420085 |

There, $R(t)$ is given by Equation (15). The initial $t_{in}$ of the Universe for a given model is obtained by setting $R(t) = 0$ in this equation,

$$t_{in} = \frac{C_{rel}^{1/4}}{c_2^{1/2}},$$

which applies in the scales where $t_0 = 1$ and $R_0 = 1$. We notice that the values obtained by this last expression are very close to those obtained by imposing $R(t) = 0$ in Equation (4) for the matter-dominated era, which gives simply $t_{in} = \Omega_m^{1/3}$. The more appropriate values (29) for the initial time in the radiative era are generally slightly larger than $\Omega_m^{1/3}$, with a minor difference concerning only the sixth decimal. (It may also be recalled (Maeder 2017a) that $t_{in} = 0$ in the above mentioned scale only when $\Omega_m = 0$.)

If one considers very early epochs of the Universe, we may write the time as $t = t_{eq} + \Delta t$. This quantity $\Delta t$ represents the age counted from the origin of the Universe, in a scale where the interval $1 - t_{in}$ is the present age of the Universe. Thus, the age $\tau$, the cosmic time expressed in seconds, for a given event at the time $t$ may be written,

$$\tau = 4.355 \cdot 10^{17} \frac{\Delta t}{1 - t_{in}} [s],$$

(30)

where a present age of 13.8 Gyr has been adopted. This relates the ages $\tau$ and $\Delta t$ for a given cosmological model (since $t_{in}$ depends on the model). In the very early stages, the quantity $\Delta t$ is very small. For example, for the cosmological nucleosynthesis which occurs during, say, the first three minutes $\Delta t < 4.1 \cdot 10^{-16}$, whatever the cosmological model. Before the equilibrium time of matter and radiation, say of the order of 10^9 yr, one has $\Delta t < 7 \cdot 10^{-6}$. Thus, we may consider that during the whole radiative phase the value of $\Delta t$ is $\ll 1$ for any flat scale invariant model. This leads to an interesting simplification of the expression of $R(t)$ given by Equation (15). We may write,

$$t^4 = (t_{in} + \Delta t)^4 \approx t_{in}^4 + 4 t_{in}^3 \Delta t,$$

and

$$t^2 \approx t_{in}^2 + 2 t_{in} \Delta t.$$

(31)

The expansion factor becomes,

$$R(t) = \frac{1}{\sqrt{2}} \left( c_2 t_{in}^4 - \frac{C_{rel}}{c_2} \right)^{1/2} \approx \frac{1}{\sqrt{2}} \left( c_2 t_{in}^4 + 4 c_2 t_{in}^3 \Delta t - \frac{C_{rel}}{c_2} \right)^{1/2},$$

(32)

which may then be written with expression (29),

$$R(t) \approx \sqrt{2} c_2^{1/2} t_{in}^{3/2} \Delta t^{1/2},$$

(33)

and

$$R(t) \approx B \Delta t^{1/2},$$

(34)

with $B = \sqrt{2} c_2^{1/2} t_{in}^{3/2} = \sqrt{2} \frac{C_{rel}}{c_2^{1/4}}$.

(35)
The temperature expressed in K or in MeV as a function of the time in sec is

\[ T(t) \propto \frac{1.272 \cdot 10^9}{C_{\text{rel}}^{1/4}(1-t_{\text{in}})^{1/2}} \frac{1}{\tau^{1/2}} \ [K], \]

where \( t_{\text{in}} = 1.0 \times 10^3 \text{ sec} \) and \( \tau = \frac{t_{\text{in}}}{R(t)} \), and the 

\[ H(t) = \frac{1}{2} \frac{1}{\Delta t} \]  

which applies to the scale where \( t_0 = 1 \). In the early phases, we can approximate expressions (28) by,

\[ \varrho_\text{m}(t) \propto 1.8788 \cdot 10^{-29} \ h^2 \Omega_\text{m} \ \frac{t_{\text{in}}}{R^3(t)} \propto \frac{1.8788 \cdot 10^{-29} \ h^2 \Omega_\text{m}}{2^{3/2} \ h^2 \Omega_\text{m} \ c_2^{1/4} \ C_{\text{rel}}^{7/8} \ 1 \Delta t^{3/2}} \ [g/cm^3], \]

\[ \varrho_{\text{rel}}(t) \propto K_0 \varrho_\gamma \propto 4.653 \cdot 10^{-34} \ K_0 \ \frac{t_{\text{in}}^2}{R^3(t)} \propto \frac{4.653 \cdot 10^{-34} \ K_0}{4} \ C_{\text{rel}} \ 1 \Delta t^2 \ [g/cm^3], \]

\[ T(t) \propto 2.726 \ \frac{t_{\text{in}}^{1/2}}{R(t)} \propto \frac{2.726}{\sqrt{2}} \ C_{\text{rel}}^{1/4} \ 1 \Delta t^{1/2} \ [K]. \]  

We have used the fact that \( t = t_{\text{in}} + \Delta t \approx t_{\text{in}} \). The expansion factor is given by Equation (34), with Equations (29) and (35) for \( t_{\text{in}} \) and \( B \). The values of \( C_{\text{rel}} \) and \( c_2 \) for different models are given in Table 1. In the above expressions, the time \( \Delta t \) is in the scale where \( t_0 = 1 \). We need these equations in the current units. With Equation (30), we have with the time \( \tau \) in seconds,

\[ \varrho_\text{m}(\tau) \propto 1.9091 \cdot 10^{-3} \ h^2 \Omega_\text{m} \ C_{\text{rel}}^{7/8}(1-t_{\text{in}})^{3/2} \ \tau^{3/2} \ [g/cm^3], \]

\[ \varrho_{\text{rel}}(\tau) \propto 22.0409 \ K_0 \ C_{\text{rel}}(1-t_{\text{in}})^{3/2} \ \tau^{3/2} \ [g/cm^3]. \]  

The temperature expressed in K or in MeV as a function of the time in sec is

\[ T(\tau) \propto \frac{1.272 \cdot 10^9}{C_{\text{rel}}^{1/4}(1-t_{\text{in}})^{1/2}} \frac{1}{\tau^{1/2}} \ [K], \]

\[ T(\tau) \propto 0.1096 \ C_{\text{rel}}^{1/4}(1-t_{\text{in}})^{1/2} \ \tau^{1/2} \ [MeV]. \]  

Thus, the time dependence of the expansion factor \( R(t) \) for the early Universe in the scale invariant context is the same as in standard models. The Hubble term \( H \) becomes

\[ H(t) = \frac{1}{2} \frac{1}{\Delta t} \]  

Figure 1. Temperatures in Mev as a function of time in seconds for some models. The black line shows the values given by Steigman (2007) and Weinberg (2008), which are very similar. The brown line is the standard model based on the value of the Planck temperature and Planck time scale as given by Mukhanov (2004). The red and green lines correspond to the scale invariant models for \( \Omega_\text{m} = 0.30 \) and \( \Omega_\text{m} = 0.04 \) respectively, with in both cases a value \( h = 0.7 \) and a number of neutrino \( N_\nu = 3. \)
The Hubble rate expressed in seconds is, consistently with Equations (30) and (36),

$$H(\tau) \approx \frac{1}{2} \frac{1}{\tau} \text{ [s}^{-1}], \quad (40)$$

which is the same as in standard models. The above relations allow us to describe the physical conditions in the scale invariant models with $k = 0$ of the early Universe. Different values of the density parameter $\Omega_m$ lead to different models.

4. THE PHYSICAL CONDITIONS AT THE TIME OF THE COSMOLOGICAL NUCLEOSYNTHESIS

It is interesting to compare the physical conditions predicted by the scale invariant models at the time of the nucleosynthesis with the corresponding values in standard models. The key parameters are the temperature $T$ and the expansion rate $H$. The runs $T(\tau)$, with the time $\tau$ is in seconds, are illustrated for some standard and scale invariant models in Fig. 1. The results of the standard models by Steigman (2007) and Weinberg (2008) are very similar (black line), while the results by Mukhanov (2004) are slightly different. We see that the scale invariant models with $\Omega_m = 0.30$ show a significantly higher temperature at a given time than the standard models. For $\Omega_m = 0.04$, there is a nice agreement. This is interesting since in the scale invariant context, there is no need to call for an unknown dark matter, thus the matter content $\Omega_m$ is only that resulting from the baryons, currently of the order of 0.04 (Frieman et al. 2008).

At energies above 1 MeV, when electrons and positrons are still in equilibrium with radiation, the equilibrium between neutrons and protons is maintained by the weak interactions, $\nu + n \leftrightarrow p + e^-$, $n + e^+ \leftrightarrow p + \bar{\nu}$ and $n \rightarrow p + e^- + \bar{\nu}$. The ratio of the numbers of neutrons to protons is then defined by a Boltzman distribution of the form $\frac{dn}{n} \propto e^{Q/(kT)}$, where $Q = 1.293$ MeV is the energy corresponding to the mass difference of neutrons and protons. At some energy below 1 MeV, the breakdown of the equilibrium makes the first two exchange rates to become rapidly negligible, only the third one is remaining, slowly reducing the number of free neutrons until deuterium is formed.

Fig. 2 shows as a a function of temperature the rates $\Gamma(n \rightarrow p)\tau_n$ and $\Gamma(p \rightarrow n)\tau_n$ (Durrer 2008), the $\Gamma$-rates are the products $\sigma v n$, expressed in sec$^{-1}$, of the corresponding cross-sections, velocities and concentrations. At energies above 1 MeV, we see the equality of the two rates. Below it, the rate $\Gamma(p \rightarrow n)$ rapidly vanishes, while the neutron slowly decays to protons at a timescale $\tau_n = 885.7$ sec, so that the product $\Gamma(n \rightarrow p)\tau_n$ tends towards unity. The expansion rate $\tau_n H$ is also illustrated. When the $\Gamma$-rates are higher than the expansion rate $H$, the nuclear process are
faster and thus tend towards equilibrium. At the opposite, when the \( \Gamma \)-rates become lower, the cosmological expansion and the associated cooling rapidly dominate, the nuclear interactions cease, except for the neutron disintegration. The crossing temperature of the \( \tau_n H \)- and \( \tau_n \Gamma \)-lines defines the critical temperature, fixing the fraction of neutrons, which after the disintegration of a small part of them, will finally be turned to helium-4, when deuterium formation starts at 0.085 MeV.

Fig. 2 also shows the expansion rates \( \tau_n H \) of the standard models together with the values of the scale invariant models already considered above. From Equations (39) and (40), we eliminate the time \( \tau \) and obtain the relation between \( H \) in \( s^{-1} \) and \( T \) in MeV,

\[
H(\tau) = 41.6245 \cdot C_{rel}^{1/2} (1 - t_{in}) T^2 .
\]  
(41)

We see as before that the scale invariant model with \( \Omega_m = 0.30 \) somehow deviates from the standard models and would suggest a lower value of the freezing temperature for the neutron-proton decoupling. The model with a lower matter density (baryonic) of \( \Omega_m = 0.04 \) shows an excellent agreement with the standard models, the minor differences being of the same order as the differences between the various standard models. The estimated freezing energy for neutrons is around 0.6 to 0.7 MeV. The agreement of the standard models and the low density scale invariant model is encouraging and makes worth a detailed study of the cosmological nucleosynthesis in the scale invariant framework.

5. CONCLUSIONS

The runs of \( T(\text{MeV}) \) and of the expansion rate \( H \) are essential parameters for the cosmological nucleosynthesis, which is used in standard models to fix the value of the density parameter \( \Omega_m \), typically found around 0.30. It is thus amazing that the scale invariant models, which usually do not need the presence of dark matter as shown by a number of tests (Maeder 2017a,b,c, 2018; Maeder & Gueorguiev 2018), find an excellent agreement with the conditions of the standard models for much lower values of \( \Omega_m \) of about 0.04. Such a density parameter leads to the same value of the freezing energy of the neutron to proton ratio as in standard models. These positive results are stimulating for the further study of the cosmological nucleosynthesis in the scale invariant context, they come after a series of other positives tests as mentioned in the introduction.

Acknowledgments: The author expresses his deep gratitude to his wife and to D. Gachet for their encouragements, as well to Dr. Vesselin G. Gueorguiev for his continuous support and constructive remarks.

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