Sklyanin-like algebras for \((q\text{-})\)linear grids and 
\((q\text{-})\)para-Krawtchouk polynomials

Geoffroy Bergeron\textsuperscript{1} *, Julien Gaboriaud\textsuperscript{1} †, Luc Vinet\textsuperscript{1} ‡, Alexei Zhedanov\textsuperscript{2} §

\textsuperscript{1} Centre de Recherches Mathématiques, Université de Montréal, P.O. Box 6128, Centre-ville Station, Montréal (Québec), H3C 3J7, Canada.
\textsuperscript{2} School of Mathematics, Renmin University of China, Beijing, 100872, China.

August 10, 2020

Abstract

S-Heun operators on linear and \(q\)-linear grids are introduced. These operators are special cases of Heun operators and are related to Sklyanin-like algebras. The Continuous Hahn and Big \(q\)-Jacobi polynomials are functions on which these S-Heun operators have natural actions. We show that the S-Heun operators encompass both the bispectral operators and Kalnins and Miller’s structure operators. These four structure operators realize special limit cases of the trigonometric degeneration of the original Sklyanin algebra. Finite-dimensional representations of these algebras are obtained from a truncation condition. The corresponding representation bases are finite families of polynomials: the para-Krawtchouk and \(q\)-para-Krawtchouk ones. A natural algebraic interpretation of these polynomials that had been missing is thus obtained. We also recover the Heun operators attached to the corresponding bispectral problems as quadratic combinations of the S-Heun operators.

Keywords: Sklyanin algebras, bispectral orthogonal polynomials, \((q\text{-})\)para-Krawtchouk polynomials, Heun operators.

Introduction

In the study of orthogonal polynomials (OPs), many of their properties are expressed as structure relations between family members with different parameters, arguments or degrees, examples are the three term recurrence relation, the differential/difference equation, the backward/forward relation, etc. As it turns out, the operators involved in these formulas realize algebras that synthesize much of the characterization of these polynomial ensembles. The present paper relates to this framework.

One such instance that has proven very fruitful is the (algebraic) study of the two bispectral operators associated to hypergeometric OPs. These operators are the recurrence and the differential/difference operators. Let us focus on the developments related to the Askey–Wilson polynomials; since these polynomials sit at the top of the Askey scheme, the gist of their description descends onto all the lower families in the scheme. The two bispectral operators for the Askey–Wilson polynomials do not commute: they form an algebra whose relations have been found by Zhedanov in [1] and it is usually referred to as the Askey–Wilson algebra.

This algebra has appeared in a great variety of contexts, such as knot theory [2], double affine Hecke algebras and representation theory [3, 4, 5], Howe duality [6, 7], integrable models [8, 9, 10, 11], algebraic combinatorics [12, 13, 14, 15], the Racah problem for \(U_q(\mathfrak{sl}_2)\) [16, 17], etc. The abovementioned connections have some specializations for all entries of the Askey tableau.

*E-mail: geoffroy.bergeron@umontreal.ca
†E-mail: julien.gaboriaud@umontreal.ca
‡E-mail: vinet@CRM.UMontreal.CA
§E-mail: zhedanov@yahoo.com
The work of Kalnins and Miller based on the use of four structure or contiguity operators is another approach that illustrates the use of symmetry techniques in the study of OPs. These operators that shall be referred to as structure operators in the following correspond to the backward and forward operators, as well as to another pair of operators that “factorize” the differential/difference operator. It was recently observed that for the Askey–Wilson polynomials, these operators realize the relations of the trigonometric degeneration of the Sklyanin algebra. To our knowledge, the Sklyanin-like algebras similarly connected to other families of OPs have not been described so far and will be the center of attention here.

The differential/difference operator of which the OPs are eigenfunctions belongs to the intersection of the sets of operators involved in the two pictures. A natural question is the following: what is the most elementary set of operators that encompasses all operators in both of the approaches above? In the case of the Askey–Wilson polynomials, this answer was given in: it is the set of so-called S-Heun operators on the Askey–Wilson grid (these are special types of Heun operators that will be defined in the next section). Operators of the Heun type are related to the tridiagonalization procedure and have been given an algebraic formulation. They have been identified as Hamiltonians of quantum Euler–Arnold tops, they have been connected to band-time limiting and to the study of entanglement in spin chains and they have been studied quite a lot recently. As will be shown below, the S-Heun operators allow a factorization of these Heun operators. Let us note that in addition to the unification of the two approaches described above, the S-Heun framework has also led to a novel algebraic interpretation of the $q$-para-Racah polynomials. The goal of the present paper is to look at the grids of linear type from the S-Heun operators point of view. As a byproduct, an algebraic interpretation of the para-Krawtchouk and $q$-para-Krawtchouk polynomials will be obtained. These polynomials were first identified in the context of perfect state transfer and fractional revival on quantum spin chains and their algebraic interpretation was still lacking.

We will introduce the S-Heun operators on linear grids in Section 1. The simplest example of operators of this type will be worked out in Section 2 (this will involve differential operators, the Jacobi polynomials and the ordinary Heun operator). Section 3 will focus on the S-Heun operators on the discrete linear grid. A new degeneration of the Sklyanin algebra will be presented. Of relevance in this case, the Continuous Hahn polynomials will be seen to truncate to the para-Krawtchouk polynomials under a special condition and an algebraic interpretation of such a truncation will be given. The Heun operator on the uniform grid will also be recovered. The $q$-linear grid will be examined in Section 1 and the previous analysis will be repeated. The degeneration of the Sklyanin algebra that arises will be identified as $U_q(\mathfrak{sl}_2)$. The Big $q$-Jacobi polynomials will be involved, and they will be observed to reduce to the $q$-para-Krawtchouk polynomials under a certain condition. The Big $q$-Jacobi Heun operator will also be recovered as well. Connections between the three grids and the associated S-Heun operators and Sklyanin-type algebras will be presented in Section 5, followed by concluding remarks. The quadratic relations between the S-Heun operators for the three different types of grids are listed in Appendix A.

1 S-Heun operators on linear-type grids

S-Heun operators are defined as the most general second order differential/difference operators without diagonal term that obey a degree raising condition. Like Heun operators, they can be defined on different grids. We now introduce the three linear grids that we will use and obtain the S-Heun operators associated to each.

1.1 The discrete linear grid

Consider the operator $S$

$$S = A_1T_+ + A_2T_-$$

where

$$T_+ f(x) = f(x + 1), \quad T_- f(x) = f(x - 1)$$

are shift operators, and $A_{1,2}$ are functions in the real variable $x$. Impose that $S$ maps polynomials of degree $n$ onto polynomials of degree no higher than $n + 1$, namely,

$$SP_n(x) = \tilde{P}_{n+1}(x)$$

1

2
for all \( n = 0, 1, 2, \ldots \). This defines the S-Heun operators on the discrete linear grid.

It is sufficient to enforce this raising condition on monomials \( x^n \); for \( n = 0 \) and \( n = 1 \), it reads

\[
A_1 + A_2 = a_{00} + a_{01}x, \quad A_1(x + 1) + A_2(x - 1) = a_{10} + a_{11}x + a_{12}x^2,
\]

for some arbitrary parameters \( a_{ij} \). This can be rewritten as

\[
A_1 + A_2 = a_{00} + a_{01}x, \quad A_1 - A_2 = a_{10} + (a_{11} - a_{00})x + (a_{12} - a_{01})x^2.
\]

Straightforward induction shows that in general one has

\[
Sx^n = A_1(x + 1)^n + A_2(x - 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k [A_1 + (-1)^{n-k}A_2],
\]

which is a polynomial of degree \( n + 1 \). Thus, the functions \( A_1, A_2 \)

\[
A_1 = \frac{1}{2} \left[ (-a_{01} + a_{12})x^2 + (-a_{00} + a_{01} + a_{11})x + (a_{00} + a_{10}) \right], \quad (1.7a)
\]

\[
A_2 = \frac{1}{2} \left[ (+a_{01} - a_{12})x^2 + (+a_{00} + a_{01} - a_{11})x + (a_{00} - a_{10}) \right] \quad (1.7b)
\]

satisfy \( (1.5) \) and the operator \( (1.1) \) meets the degree raising condition.

**Proposition 1.1** With the functions \( A_1, A_2 \) given by \( (1.7) \), the operator \( S \) in \( (1.1) \) is the most general S-Heun operator on the linear grid. \( S \) depends on 5 free parameters and spans a 5-dimensional linear space. The elements

\[
L = \frac{1}{2} [T_+ - T_-], \quad (1.8a)
\]

\[
M_1 = \frac{1}{2} [T_+ + T_-], \quad (1.8b)
\]

\[
M_2 = \frac{1}{2} x [T_+ - T_-], \quad (1.8c)
\]

\[
R_1 = \frac{1}{2} x [(1 - 2x)T_+ + (1 + 2x)T_-], \quad (1.8d)
\]

\[
R_2 = \frac{1}{2} x [T_+ + T_-]. \quad (1.8e)
\]

form a basis for this space.

Using \( (1.6) \), one sees that the operator \( L \) is a lowering operator (it lowers by one the degree of polynomials in \( x \)), the operators \( M_1, M_2 \) are stabilizing operators (they do not change the degree) and the operators \( R_1, R_2 \) are raising operators (they raise it by one).

### 1.2 The \( q \)-linear grid

Consider now the \( q \)-linear grid \( z = q^x \) (or exponential grid). The S-Heun operators \( \hat{S} \) on that grid are of the form

\[
\hat{S} = \hat{A}_1(z, q)\hat{T}_+ + \hat{A}_2(z, q)\hat{T}_-, \quad (1.9)
\]

with

\[
\hat{T}_\pm f(z) = f(q^{\pm 1}z), \quad (1.10)
\]

and are taken to map polynomials in \( z \) onto polynomials of at most one degree higher: \( \hat{S}P_n(z) = \hat{P}_{n+1}(z) \). Imposing this degree raising condition on the first monomials 1 and \( z \) yields

\[
\hat{A}_1(z, q) + \hat{A}_2(z, q) = a_{00} + a_{01}z, \quad (1.11a)
\]

\[
\hat{A}_1(z, q)q + \hat{A}_2(z, q)q^{-1} = a_{10}z^{-1} + a_{11} + a_{12}z. \quad (1.11b)
\]
Straightforward induction shows that in general one has

$$\hat{S}z^n = (\hat{A}_1q^n + \hat{A}_2q^{-n})z^n = z^n(\hat{A}_1q + \hat{A}_2q^{-1})\frac{q^n - q^{-n}}{q - q^{-1}} - z^n[\hat{A}_1 + \hat{A}_2]\frac{q^{n-1} - q^{1-n}}{q - q^{-1}}, \quad (1.12)$$

which is a polynomial of degree \(n+1\) in \(z\). Thus, an operator \(\hat{S}\) with \(\hat{A}_1(z, q)\) and \(\hat{A}_2(z, q)\) that satisfies \((1.11)\) will obey the degree raising condition on any monomial. We hence obtain:

$$\hat{A}_1(z, q) = \hat{A}_2(z, q^{-1}) = \frac{1}{(q - q^{-1})z}[a_{10} + (a_{11} - a_{00}q^{-1})z + (a_{12} - a_{01}q^{-1})z^2]. \quad (1.13)$$

**Proposition 1.2** With the functions \(\hat{A}_1(z, q), \hat{A}_2(z, q)\) given by \((1.13)\), the operator \(\hat{S}\) in \((1.9)\) is the most general S-Heun operator on the \(q\)-linear grid. \(\hat{S}\) depends on 5 free parameters and spans a 5-dimensional linear space. The elements

\[
\begin{align*}
\hat{L} &= \frac{1}{(q - q^{-1})}z^{-1}(\hat{T}_+ - \hat{T}_-), \quad \text{(1.14a)} \\
\hat{M}_1 &= \frac{1}{(q - q^{-1})}(-q^{-1}\hat{T}_+ + q\hat{T}_-), \quad \text{(1.14b)} \\
\hat{M}_2 &= \frac{1}{(q - q^{-1})}(\hat{T}_+ - \hat{T}_-), \quad \text{(1.14c)} \\
\hat{R}_1 &= \frac{1}{(q - q^{-1})}z(-q^{-1}\hat{T}_+ + q\hat{T}_-), \quad \text{(1.14d)} \\
\hat{R}_2 &= \frac{1}{(q - q^{-1})}z(\hat{T}_+ - \hat{T}_-). \quad \text{(1.14e)}
\end{align*}
\]

*can be chosen as a basis for this space.*

Looking at \((1.12)\) and \((1.13)\), one sees that the operator \(\hat{L}\) lowers the degrees, and that the \(\hat{M}_i\)'s and the \(\hat{R}_i\)'s are respectively stabilizing and raising operators.

### 1.3 The simplest case: differential S-Heun operators

The definition of the S-Heun operators on the real line goes as follows. Consider the first-order differential operator

$$\hat{S} = \hat{A}_1(x)\frac{d}{dx} + \hat{A}_2(x) \quad (1.15)$$

and impose the raising condition \(\hat{S}\rho_n(x) = \hat{p}_{n+1}(x)\) which demands that \(\hat{S}\) sends polynomials into polynomials of one degree higher. The general solution is given by

$$\hat{A}_1(x) = a_{10} + a_{11}x + a_{12}x^2, \quad \hat{A}_2(x) = a_{20} + a_{21}x. \quad (1.16)$$

This leads to the following set of five linearly independent S-Heun operators \([30]\)

\[
\begin{align*}
\hat{L} &= \frac{d}{dx}, \quad \hat{M}_1 = 1, \quad \hat{M}_2 = x\frac{d}{dx}, \quad \hat{R}_1 = x, \quad \hat{R}_2 = x^2\frac{d}{dx}, \quad (1.17)
\end{align*}
\]

which are once again labelled according to their property of lowering (\(\hat{L}\)), stabilizing (\(\hat{M}\)) or raising (\(\hat{R}\)) the degree of polynomials in the variable \(x\).

These S-Heun operators can also be obtained as a \(q \to 1\) limit of the ones defined on the \(q\)-linear grid. More precisely, writing \(q = e^h\) and letting \(h \to 0\), one obtains

$$\lim_{q \to 1} \hat{L} = \hat{L}, \quad \lim_{q \to 1} \hat{M}_1 = \hat{M}_1 - \hat{M}_2, \quad \lim_{q \to 1} \hat{M}_2 = \hat{M}_2, \quad \lim_{q \to 1} \hat{R}_1 = \hat{R}_1 - \hat{R}_2, \quad \lim_{q \to 1} \hat{R}_2 = \hat{R}_2. \quad (1.18)$$

This connects with the definition of the continuous S-Heun operators. These S-Heun operators will also be related to the ordinary Heun operator introduced in the next section.
2 The continuous case

The goal of this section is to revisit (mostly known) results with a point of view that will be adopted in the following sections. Here, we are interested in studying the OPs and algebras related to the set of the five S-Heun operators defined in Section 1.3.

2.1 The stabilizing subalgebra

We first study the subset $\{L, M_1, M_2\}$ of S-Heun operators that stabilize the set of polynomials of a given degree. Let us denote by $\tilde{Q}$ the most general quadratic combination of these operators. Using the relations of Appendix A, it is always possible to reduce $\tilde{Q}$ to an expression of the form

$$\tilde{Q} = \alpha_1L^2 + \alpha_2LM_1 + \alpha_3LM_2 + \alpha_4M_1^2 + \alpha_5M_1M_2 + \alpha_6M_2^2. \quad (2.1)$$

Using the realizations (1.17), the eigenvalue equation for the second-order differential operator $\tilde{Q}$ can be brought in the form

$$\tilde{D}P_n^{(\alpha,\beta)}(x) = n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}(x),$$

$$\tilde{D} = (x^2 - 1) \frac{d^2}{dx^2} + [(\alpha - \beta) + (\alpha + \beta + 2)x] \frac{d}{dx}, \quad (2.2)$$

which is recognized as the differential equation satisfied by the Jacobi polynomials [49].

We have thus identified the family of OPs related to these (ordinary) S-Heun operators, and as will be seen in the next subsection, certain combinations of these S-Heun operators provide the structure relations of these polynomials.

2.2 Jacobi polynomials and their structure relations

Consider the forward and backward operators for the Jacobi polynomials

$$\tau = L, \quad \tau^{(\alpha,\beta)^*} = -L + (\alpha - \beta)M_1 + (\alpha + \beta)R_1 + R_2. \quad (2.3a)$$

and the contiguity operators

$$\tilde{\mu}^{(\alpha)} = -L + \alpha M_1 + \tilde{M}_2, \quad \tilde{\mu}^{(\beta)^*} = \tilde{L} + \beta M_1 + \tilde{M}_2. \quad (2.3b)$$

These four operators act very simply on the Jacobi polynomials:

$$\tau P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n + \alpha + \beta + 1)P_n^{(\alpha+1,\beta+1)}(x), \quad (2.4a)$$

$$\tau^{(\alpha,\beta)^*} P_n^{(\alpha,\beta)}(x) = 2(n + 1)P_n^{(\alpha-1,\beta-1)}(x), \quad (2.4b)$$

$$\tilde{\mu}^{(\alpha)} P_n^{(\alpha,\beta)}(x) = (n + \alpha)P_n^{(\alpha+1,\beta+1)}(x), \quad (2.4c)$$

$$\tilde{\mu}^{(\beta)^*} P_n^{(\alpha,\beta)}(x) = (n + \beta)P_n^{(\alpha+1,\beta-1)}(x). \quad (2.4d)$$

The operators $\tilde{\mu}^{(\alpha)}$, $\tilde{\mu}^{(\beta)^*}$, $\tilde{\tau}$, $\tau^{(\alpha,\beta)^*}$ built from linear combinations of S-Heun operators are of the type studied by Kalnins and Miller [19].

We have mentioned in the introduction that S-Heun operators encompass both the structure operators of Kalnins and Miller and the bispectral operators. Let us indicate how the latter operators appear in this context. First, as mentioned above, the Jacobi differential operator appears as a quadratic combination of the stabilizing generators. We can actually provide a factorization of this operator either as a product of two contiguous operators or as the product of the forward and backward operator:

$$\tilde{D} = \tilde{\mu}^{(\alpha+1)^*}\tilde{\mu}^{(\beta)} - (\alpha + 1)\beta$$

$$= \tilde{\mu}^{(\beta+1)^*}\tilde{\mu}^{(\alpha)} - \alpha(\beta + 1)$$

$$= \tilde{\tau}^{(\alpha+1,\beta+1)^*}\tilde{\tau}$$

$$= \tau^{(\alpha,\beta)^*} - (\alpha + \beta). \quad (2.5)$$
The other bispectral operator $\bar{X}$ is the multiplication by the variable $x$. It can be directly expressed as $\bar{R}_1$, but since it will appear as a quadratic combination of the S-Heun operators for other grids, we shall write it here as

$$\bar{X} = \bar{R}_1\bar{M}_1. \quad (2.6)$$

We have thus recovered the two bispectral operators as quadratic combinations in the S-Heun operators. This completes the observation that the S-Heun operators are the elementary blocks behind the two factorizations.

### 2.3 The Sklyanin-like algebra realized by the structure operators

We now focus on the algebras that are realized by these sets of operators. On the one hand the pair of bispectral Jacobi operators is known \[50\] to generate the Jacobi algebra that has been well studied \[51\]. On the other hand, the algebra formed by the 4 linear operators $\bar{\mu}^{(\alpha)}$, $\bar{\mu}^{(\beta)^*}$, $\bar{\tau}$, $\bar{\tau}^{(\alpha,\beta)^*}$ can be seen to be a degeneration of the Sklyanin algebra \[24\].

We now give a presentation of this algebra. Denote $\nu = -\frac{1}{2}(\alpha + \beta)$ and set

$$\bar{A} = \bar{M}_2 - \nu\bar{M}_1, \quad \bar{B} = \bar{R}_2 - 2\nu\bar{R}_1, \quad \bar{C} = \bar{L}, \quad \bar{D} = \bar{M}_1. \quad (2.7)$$

These linear combinations of $\bar{\mu}^{(\alpha)}$, $\bar{\mu}^{(\beta)^*}$, $\bar{\tau}$, $\bar{\tau}^{(\alpha,\beta)^*}$ have been chosen in order to simplify the relations.

**Proposition 2.1** The operators $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ obey the homogeneous quadratic relations

$$[\bar{C}, \bar{D}] = 0, \quad [\bar{A}, \bar{C}] = -\bar{C}\bar{D}, \quad [\bar{A}, \bar{D}] = 0,$$

$$[\bar{B}, \bar{C}] = -2\bar{A}\bar{D}, \quad [\bar{A}, \bar{B}] = \bar{B}\bar{D}, \quad [\bar{B}, \bar{D}] = 0. \quad (2.8)$$

**Remark 2.1** One will notice that these relations are actually the relations of the $\mathfrak{sl}_2$ Lie algebra supplemented with a central element $\bar{D}$. The reason why we wrote these in a quadratic fashion is to make easier the comparison with the other Sklyanin algebras that will be obtained later.

One observes that if $\nu$ is an integer or half-integer, the realization \[2.7\] is associated to a finite dimensional representation of dimension $2\nu + 1$.

### 2.4 Recovering the Heun operator

We now show how to recover the ordinary (differential) Heun operator from the knowledge of the S-Heun operators.

The generic Heun operator $\bar{W}$ can be expressed as the most general tridiagonalization of the hypergeometric operator \[27\]. It has been known to be

$$\bar{W} = Q_3(x)\frac{d^2}{dx^2} + Q_2(x)\frac{d}{dx} + Q_1(x), \quad (2.9)$$

where $Q_3(x)$, $Q_2(x)$ and $Q_1(x)$ are general polynomials of degree 3, 2 and 1 respectively.

Let us consider the most general quadratic combination of S-Heun operators that does not raise the degree of polynomials by more than one. Using the quadratic homogeneous relations of Appendix A, it is always possible to simplify such an expression to

$$\bar{W} = \alpha_1\bar{L}^2 + \alpha_2\bar{L}\bar{M}_1 + \alpha_3\bar{L}\bar{M}_2 + \alpha_4\bar{M}_1^2 + \alpha_5\bar{M}_1\bar{M}_2 + \alpha_6\bar{M}_2^2 + \beta_1\bar{M}_1\bar{R}_2 + \beta_2\bar{M}_2\bar{R}_1 + \beta_3\bar{M}_2\bar{R}_2. \quad (2.10)$$

From the differential expressions of the generators we obtain

$$\bar{W} = Q_3(x)\frac{d^2}{dx^2} + Q_2(x)\frac{d}{dx} + Q_1(x)\mathcal{I},$$

where $\mathcal{I}$ is the identity operator: $\mathcal{I}f(x) = f(x)$.
Proposition 2.2  The generic Heun operator \((2.9)\) can be obtained as the most general quadratic combination in the S-Heun generators \((1.17)\) that does not raise the degree of polynomials by more than one.

Calling upon the reordering relations of Appendix A, it is seen that the Heun operator generically factorizes as the product of a general S-Heun operator with a stabilizing S-Heun operator:

\[
\bar{W} = (\xi_1 \bar{L} + \xi_2 \bar{M}_1 + \xi_3 \bar{M}_2)(\eta_1 \bar{L} + \eta_2 \bar{M}_1 + \eta_3 \bar{M}_2 + \eta_4 \bar{R}_1 + \eta_5 \bar{R}_2) + \kappa. \tag{2.12}
\]

3 S-Heun operators on the linear grid

We now come to one of the main topics of the paper, namely the S-Heun operators defined on the linear grid.

3.1 The stabilizing subset

The subset of S-Heun operators that stabilizes the polynomials of a given degree is \(\{ L, M_1, M_2 \} \). The most general quadratic combination of these operators can always be reduced to an expression of the form

\[
Q = \alpha_1 L^2 + \alpha_2 LM_1 + \alpha_3 LM_2 + \alpha_4 M_1^2 + \alpha_5 M_1 M_2 + \alpha_6 M_2^2 \tag{3.1}
\]

using the relations of Appendix A. Substituting the expressions \((1.8)\), one sees that \(Q\) is a second-order difference operator. By straightforward manipulations, the eigenvalue equation for \(Q\) can be transformed into the difference equation of the Continuous Hahn polynomials \[49\]

\[
D P_n(\tilde{x}; a, b, c, d) = n(n + a + b + c + d - 1)P_n(\tilde{x}; a, b, c, d),
\]

\[
D = B(\tilde{x})T_2 - [B(\tilde{x}) + D(\tilde{x})]I + D(\tilde{x})T_2,
\]

\[
B(x) = (c - ix)(d - ix), \quad D(x) = (a + ix)(b + ix), \tag{3.2}
\]

with \(\tilde{x} = i\frac{x}{2}\) and where \(a, b, c, d\) are given in terms of the \(\alpha_i\). From this, we recognize that the key family of OPs related to these S-Heun operators is the Continuous Hahn family.

3.2 Continuous Hahn polynomials and their structure relations

The following combinations of S-Heun operators

\[
\tau = 2L, \quad \tau^{(a,b,c,d)*} = \mu_1 L + \mu_2 M_1 + \mu_3 M_2 + \mu_4 R_1 + \mu_5 R_2, \tag{3.3a, 3.3b}
\]

with

\[
\mu_1 = \frac{1}{2}(1 - (a + b + c + d)) + (ab + cd),
\]

\[
\mu_2 = \frac{1}{2}(a + b - c - d) - (ab - cd),
\]

\[
\mu_3 = \frac{1}{2}(c + d - a - b),
\]

\[
\mu_4 = -\frac{1}{4},
\]

\[
\mu_5 = \frac{1}{2}(a + b + c + d) - \frac{3}{4}
\]

turn out to be the forward and backward operators, while

\[
\mu^{(a,b,c,d)} = (d - a)L + (a + d - 1)M_1 + M_2, \quad \tag{3.3d}
\]

\[
\mu^{(a,b,c,d)*} = (c - b)L + (b + c - 1)M_1 + M_2, \quad \tag{3.3e}
\]
The 4 operators \( \mu \) Continuous Hahn polynomials: will act on polynomials as the contiguity relations. Indeed, these operators have the following actions on the product of the backward and forward operators:

\[
\tau \ P_n (i \frac{\tau}{2}, a, b, c, d) = i(n + a + b + c + d - 1) P_{n-1} \left( i \frac{\tau}{2}, a + \frac{1}{2} b + \frac{1}{2} c + \frac{1}{2} d + \frac{1}{2} \right), \tag{3.4a}
\]

\[
\tau^{(a,b,c,d)} \ P_n (i \frac{\tau}{2}, a, b, c, d) = -i(n + 1) P_{n+1} \left( i \frac{\tau}{2}, a - \frac{1}{2} b - \frac{1}{2} c - \frac{1}{2} d - \frac{1}{2} \right), \tag{3.4b}
\]

\[
\mu^{(a,b,c,d)} \ P_n (i \frac{\tau}{2}, a, b, c, d) = (n + a + d - 1) P_n \left( i \frac{\tau}{2}, a - \frac{1}{2} b + \frac{1}{2} c + \frac{1}{2} d - \frac{1}{2} \right), \tag{3.4c}
\]

\[
\mu^{(a,b,c,d)} \ P_n (i \frac{\tau}{2}, a, b, c, d) = (n + b + c - 1) P_n \left( i \frac{\tau}{2}, a + \frac{1}{2} b - \frac{1}{2} c - \frac{1}{2} d + \frac{1}{2} \right). \tag{3.4d}
\]

The 4 operators \( \mu^{(a,b,c,d)}, \mu^{(a,b,c,d)} \), \( \tau \), \( \tau^{(a,b,c,d)} \) have been studied by Kalnins and Miller in [19].

We now indicate how the two bispectral operators are formed from the S-Heun operators. As mentioned above, the Continuous Hahn difference operator can be formed by a quadratic combination of the stabilizing generators. Moreover, we can provide factorizations of this operator, either as a product of two contiguous operators or as the product of the backward and forward operators:

\[
D = \mu^{(a+\frac{1}{2},b-\frac{1}{2},c-\frac{1}{2},d+\frac{1}{2})} \mu^{(a,b,c,d)} - (a + d)(b + c - 1)
\]

\[
= \mu^{(a-\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d-\frac{1}{2})} \mu^{(a,b,c,d)} - (a + d - 1)(b + c)
\]

\[
= \tau^{(a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2})} \tau
\]

\[
= \tau \ \tau^{(a,b,c,d)} + 2 - (a + b + c + d).
\]

The remaining bispectral operator \( X \) is the multiplication by the variable \( x \) in this basis: \( X f(x) = x f(x) \). It appears as a quadratic combination in the S-Heun operators

\[
X = [M_2, R_2]. \tag{3.6}
\]

The framework of S-Heun operators presented here is thus seen to unite the symmetry techniques of Kalnins and Miller and the approach based on the bispectral operators (see [52] for more general context).

### 3.3 The Sklyanin-like algebra realized by the structure operators

Let us now look at the algebraic relations obeyed by these operators. On the one hand, the pair of bispectral Continuous Hahn operators realizes the Hahn algebra [39]. On the other hand, the algebra formed by the 4 linear operators \( \mu^{(a,b,c,d)}, \mu^{(a,b,c,d)} \), \( \tau \), \( \tau^{(a,b,c,d)} \) can be seen as a degeneration of the Sklyanin algebra.

This algebra can be presented as follows. Write \( \nu = -\frac{1}{2} (a + b + c + d) \) and take

\[
A = 2(\nu + 1) M_1 - 2 M_2,
\]

\[
B = \frac{1}{2} (2 \nu + 1)(2 \nu + 3) L - R_1 - (4 \nu + 3) R_2,
\]

\[
C = L,
\]

\[
D = M_1.
\]

These are linear combinations of \( \mu^{(a,b,c,d)}, \mu^{(a,b,c,d)} \), \( \tau \), \( \tau^{(a,b,c,d)} \) that have been chosen in order to simplify the relations.

**Proposition 3.1** The elements \( A, B, C, D \) obey the quadratic relations

\[
[C, D] = 0, \quad [A, C] = \{C, D\}, \quad [A, D] = \{C, C\}, \tag{3.8a}
\]

\[
[B, C] = \{D, A\}, \quad [B, D] = \{C, A\}, \quad [B, A] = \{B, D\}. \tag{3.8b}
\]

We shall refer to these relations as those of the Skl\(_4\) algebra.

The two quadratic Casimir elements are

\[
\Omega_1 = D^2 - C^2, \quad \Omega_2 = A^2 + D^2 - \{B, C\}. \tag{3.9}
\]
and they take the following values in the realization:
\[ \Omega_1 = 1, \quad \Omega_2 = (2\nu + 3)^2. \] (3.10)

**Remark 3.1** The stabilizing subalgebra of Skl\(_4\) (3.8a), which we shall denote by Skl\(_3\), has been identified in [53] as the algebra \(T_7\) \(((a, b) = (0, 0))\) whose relations are isomorphic to
\[ [x, y] = x^2, \quad [y, z] = 0, \quad [x, z] = zy. \] (3.11)

It enjoys nice properties such as being Koszul, PBW, and being derived from a twisted potential. That the above algebra is Skl\(_3\) is seen by setting \(x = \frac{1}{2}A, y = D, z = C\).

We now explain that Skl\(_4\) is a degeneration of the Sklyanin algebra. We rewrite the \(\tau^{(a, b, c, d)^*}\) in terms of \(A, B, C, D\), using \(e_1 = a + b + c + d\):
\[ \tau^{(a, b, c, d)^*} = \frac{1}{4}(a + b - c - d)A + \frac{1}{4}B + \left[\frac{1}{8}(1 - e_1)(1 + e_1) + ab + cd\right] C + \left[\frac{1}{4}e_1(a + b - c - d) - ab + cd\right] D. \] (3.12)

Two analogs of an identity due to Rains [54] can be obtained for \(\tau^{(a, b, c, d)^*}\). These are the quasi-commutation relations:
\[ \tau^{(a + e, b, c, d - e)^*} \tau^{(a - \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d - \frac{1}{2}e)^*} = \tau^{(a, b, c, d)^*} \tau^{(a - \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d - \frac{1}{2}e)^*}, \] (3.13)
\[ \tau^{(a, b - e, c, d)^*} \tau^{(a + \frac{1}{2}b - \frac{1}{2}c + \frac{1}{2}d + \frac{1}{2}e)^*} = \tau^{(a, b, c, d)^*} \tau^{(a + \frac{1}{2}b - \frac{1}{2}c + \frac{1}{2}d + \frac{1}{2}e)^*}. \] (3.14)

**Proposition 3.2** Either of the quasi-commutation relation (3.13), (3.14) repackages the relations (3.8) of the Skl\(_4\) algebra.

**Proof:** Substituting the relation (3.12) into (3.13) and bringing all terms to the rhs, one obtains (\(u = b - c, v = a - b - c + d\)):
\[
0 = \frac{1}{4} \left\{ \frac{1}{2}(AB - BA) + u(CB - BC) + \frac{1}{4} \left[ (2 - v)BD + vDB \right] + u \left[ (2 - v)AD + vDA - 2(1 - v)C^2 \right] \right.
  - \frac{1}{4} \left[ (v^2 + 4u^2 - 4v + 3)AC - (v^2 + 4u^2 - 1)CA \right],
  
  + \frac{1}{4} \left[ v^3 - 4u^2v + 8u^2 - 2v^2 - v + 2 \right] CD - \frac{1}{4} \left[ v^3 - 4u^2v - 4v^2 + 3v \right] DC \right\}. \] (3.15)

The dependence on the free parameter \(e\) factors out. Taking \(v \to \infty\), one obtains immediately that
\[ CD - DC = 0. \] (3.16)

Also, taking \(u \to 0\) and \(v \to 0\), one gets
\[ AB - BA = -2BD + \frac{3}{2} AC + \frac{1}{2} CA - CD. \] (3.17)

Substituting these relations back in (3.15) leads to
\[
0 = \frac{1}{4} \left\{ u(CB - BC) + \frac{v}{2} [DB - BD] + u \left[ (2 - v)AD + vDA - 2(1 - v)C^2 \right] \right.
  - \frac{1}{4} \left[ (v^2 + 4u^2 - 4v)AC - (v^2 + 4u^2)CA \right] + \frac{1}{4} \left[ 8u^2 + 2v^2 - 4v \right] CD \right\}. \] (3.18)

Repeating a similar process, the remaining relations of (3.8) are found. A similar derivation starting from (3.14) instead yields the same relations. \(\square\)
3.4 Finite-dimensional representations

It is known that finite-dimensional representations of the Hahn algebra relate to the Hahn polynomials [51]. We now wish to obtain finite-dimensional representations of the $Skl_4$ algebra; looking at (3.7), it is seen that one needs $\nu$ to be either an integer or half-integer. It will be shown that this corresponds in fact to a truncation of the Jacobi matrix of the Continuous Hahn polynomials.

Let us write the condition ($\nu$ is either an integer or half-integer) as

$$1 - (a + b + c + d) = N, \quad (3.19)$$

where $N$ is a positive integer that corresponds to the maximal degree of the truncated family of polynomials.

This truncation condition is known [46] to be the one that takes the Wilson polynomials to the para-Racah polynomials. In the present case, we start from the Continuous Hahn OPs so the result of the truncation leads to a different family of para-polynomials.

Proposition 3.3 The polynomials that arise from the truncation condition (3.19) form a basis that supports $(N+1)$-dimensional representations of the degenerate Sklyanin algebra $Skl_4$ and are identified as the para-Krawtchouk polynomials [45].

We indicate below how the recurrence relation of the para-Krawtchouk polynomials is obtained from that of the Continuous Hahn polynomials by imposing (3.19).

3.4.1 $N = 2j + 1$ odd

In the case where $N = 2j + 1$ is odd ($j$ is a non-negative integer), we parametrize the truncation condition as follows

$$c = -a - j + e_1 t, \quad b = -d - j + e_2 t \quad (3.20)$$

and then take the limit $t \to 0$. We shall choose $e_1 = e_2$: this will lead to simpler expressions. The more general solutions corresponding to $e_1 \neq e_2$ can be recovered from the simpler solutions by the procedure of isospectral deformations, see for instance [55]. Using the chosen parametrization, the recurrence coefficients $A_n, C_n$ appearing in the recurrence relation of the Continuous Hahn polynomials

$$(a + ix)P_n(x; a, b, c, d) = A_n P_{n+1}(x; a, b, c, d) + C_n P_{n-1}(x; a, b, c, d) - (A_n + C_n) P_n(x; a, b, c, d),$$

$$P_n(x; a, b, c, d) = \frac{n!}{i^n(a+c)_n(a+d)_n} p_n(x; a, b, c, d) \quad (3.21)$$

become in the limit $t \to 0$:

$$A_n = -\frac{(n - N)(n + a + d)}{2(2n - N)}, \quad (3.22a)$$

$$C_n = +\frac{n(n - N - a - d)}{2(2n - N)}. \quad (3.22b)$$

Now take $\gamma$ to be

$$\gamma = (b + c) - (a + d), \quad (3.23)$$

it follows that (3.22) can be rewritten in view of (3.19) as

$$A_n = -\frac{1}{2} \frac{(N - n)(N - 1 - 2n + \gamma)}{2(2n - N)}, \quad (3.24a)$$

$$C_n = -\frac{1}{2} \frac{n(N + 1 - 2n - \gamma)}{2(2n - N)}. \quad (3.24b)$$

These are recognized as the recurrence coefficients of the para-Krawtchouk polynomials in the variable $-\frac{x}{2}$ introduced in [49]. These polynomials are defined on the union of two linear lattices and the parameter $\gamma$ describes the displacement of one lattice with respect to the other.
3.4.2 \( N = 2j \) even

In the case where \( N = 2j \) is even, we use the parametrization
\[
c = -a - j + e_1 t, \quad b = -d - j + e_1 t + 1
\]
and then take the limit \( t \to 0 \). The recurrence coefficients in the recurrence relation of the Continuous Hahn polynomials become
\[
A_n = \frac{(n - N)(n + a + d)}{2(2n - N + 1)}, \quad \text{(3.26a)}
\]
\[
C_n = \frac{n(n - N - a - d)}{2(2n - N - 1)}, \quad \text{(3.26b)}
\]
and upon writing
\[
\gamma = 1 + (b + c) - (a + d),
\]
we obtain
\[
A_n = \frac{1}{2} \frac{(N - n)(N - 2 - 2n + \gamma)}{2(2n - N + 1)}, \quad \text{(3.28a)}
\]
\[
C_n = -\frac{1}{2} \frac{n(N + 2 - 2n - \gamma)}{2(2n - N - 1)}. \quad \text{(3.28b)}
\]
These are the recurrence coefficients of the para-Krawtchouk polynomials in the variable \(-\frac{x}{2}\). The expressions for the monic polynomials are given in [46].

3.4.3 A remark on the truncation condition

It can be checked that in the realization (3.7), applying the truncation condition (3.19) seems to suggest that the raising operator \( B \) annihilates the monomial \( x^{N+1} \) and not \( x^N \). A priori, this means that the truncation condition amounts to looking at \((N + 2)\)-dimensional representations of the algebra \( Skl_4 \), which would seem to contradict the fact that the para-Krawtchouk polynomials were truncated to have degrees at most \( N \) (and thus to span a space of dimension \( N + 1 \)).

Looking at the situation more closely, one observes that \( B \) indeed maps para-Krawtchouk polynomial of maximal degree \( N \) to a certain polynomial of degree \( N + 1 \). But this polynomial of degree \( N + 1 \) corresponds to the characteristic polynomial of the (upper block of the) truncated Jacobi matrix, hence it is null on the orthogonality grid points. Keeping in mind that the para-Krawtchouk polynomials are the basis vectors for the finite-dimensional representation of \( Skl_4 \), this characteristic polynomial thus corresponds to a null vector. Therefore the dimension of the space on which the representation of the \( Skl_4 \) algebra acts is indeed \( N + 1 \).

3.5 Recovering the associated Heun operator

The Heun operator associated to the Continuous Hahn polynomials was implicitly defined in [39]. This operator \( W_{CH} \) is the most general second order operator that acts on the discrete linear grid and maps polynomials of degree \( n \) into polynomials of degree \( n + 1 \). It can be expressed as
\[
W_{CH} = A_1 T_+ + A_0 I + A_2 T_-,
\]
where \( A_{1,2} \) are general polynomials of degree 3 with the same leading order coefficient, and \( A_0 + A_1 + A_2 = \pi_1(x) \), with \( \pi_1(x) \) a general polynomial of degree 1.

We now consider the most general quadratic combination of S-Heun operators that does not raise the degree of polynomials by more than one. Upon using the quadratic homogeneous relations of Appendix A this general combination can be brought into the form
\[
W = \alpha_1 L^2 + \alpha_2 LM_1 + \alpha_3 LM_2 + \alpha_4 M_1^2 + \alpha_5 M_1 M_2 + \alpha_6 M_2^2 + \beta_1 M_1 R_2 + \beta_2 M_2 R_1 + \beta_3 M_2 R_2.
\]

(3.30)
Substituting the expressions of the S-Heun basis operators (1.8), we obtain

\[ W = A_1 T_+^2 + A_0 I + A_2 T_-^2, \]

\[ A_1 = \frac{1}{4}[-2\beta_2 x^3 + (\alpha_6 - 3\beta_2 + \beta_3)x^2 + (\alpha_3 + \alpha_5 + \alpha_6 + \beta_1 - \beta_2 + \beta_3)x + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \beta_1)], \]

\[ A_2 = \frac{1}{4}[-2\beta_2 x^3 + (\alpha_6 + 3\beta_2 - \beta_3)x^2 + (\alpha_3 - \alpha_5 - \alpha_6 + \beta_1 - \beta_2 + \beta_3)x + (\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 + \alpha_5 - \beta_1)], \]

\[ A_0 = (\beta_1 + \beta_2 + \beta_3)x + \alpha_4 - (A_1 + A_2). \]

**Proposition 3.4** The generic Heun-Continuous Hahn operator (3.29) can be obtained as the most general quadratic combination in the S-Heun generators (1.8) that does not raise the degree of polynomials by more than one.

Using the relations of Appendix A, one can see that the Heun operator generically factorizes as the product of a general S-Heun operator with a stabilizing S-Heun operator:

\[ W = (\xi_1 L + \xi_2 M_1 + \xi_3 M_2)(\eta_1 L + \eta_2 M_1 + \eta_3 M_2 + \eta_4 R_1 + \eta_5 R_2) + \kappa. \]  

(3.32)

### 4 The case of the \( q \)-linear grid

We consider now the S-Heun operators associated to the \( q \)-linear (or exponential) grid.

#### 4.1 The stabilizing subspace

The stabilizing subset of S-Heun operators is \( \{ \hat{L}, \hat{M}_1, \hat{M}_2 \} \). Using the relations of Appendix A, it is always possible to reduce the most general quadratic combination of these operators to

\[ \hat{Q} = \alpha_1 L^2 + \alpha_2 \hat{L} M_1 + \alpha_3 \hat{M}_2 + \alpha_4 \hat{M}_2^2 + \alpha_5 \hat{M}_1 \hat{M}_2 + \alpha_6 \hat{M}_2^2. \]  

(4.1)

Substituting the expressions (1.14), one recognizes \( \hat{Q} \) as a second-order \( q \)-difference operator whose eigenvalue problem can be cast as the difference equation

\[ \hat{D} P_n(z; \alpha, \beta, \gamma; \tilde{q}) = (\tilde{q}^{-n} - 1)(1 - \alpha \beta \tilde{q}^{n+1}) P_n(z; \alpha, \beta, \gamma; \tilde{q}), \]

\[ \hat{D} = B(z) \hat{T}_+^2 - [B(z) + D(z)] I + D(z) \hat{T}_-^2, \]

\[ B(z) = \frac{\alpha \tilde{q}(z-1)(\beta z - \gamma)}{z^2}, \]

\[ D(z) = \frac{(z - \alpha \tilde{q})(z - \gamma \tilde{q})}{z^2} \]

(4.2)

of the Big \( q \)-Jacobi polynomials [49] in base \( \tilde{q} = q^2 \), making those the OPs associated to S-Heun operators on the exponential lattice.

#### 4.2 Big \( q \)-Jacobi polynomials and their structure relations

Focusing on the structure and contiguity relations of the Big \( q \)-Jacobi polynomials, we shall show how the set of S-Heun operators spans a space that contains the relevant operators. Let

\[ \hat{\tau} = (q - q^{-1}) \hat{L}, \]

\[ \hat{\tau}^{(a,b,c,d)} = \mu_1 \hat{L} + \mu_2 \hat{M}_1 + \mu_3 \hat{M}_2 + \mu_4 \hat{R}_1 + \mu_5 \hat{R}_2, \]

(4.3a)

(4.3b)

with

\[ \mu_1 = -(q - q^{-1}), \]

\[ \mu_2 = (a + b)q^{-1} - q(c^{-1} + d^{-1}), \]

\[ \mu_3 = (a + b) - (c^{-1} + d^{-1}), \]

\[ \mu_4 = -abq^{-2} + q^2 c^{-1} d^{-1}, \]

\[ \mu_5 = -abq^{-1} + qc^{-1} d^{-1}, \]

(4.3c)
operators actually provide factorizations of this operator in terms of contiguity operators as well as backward and forward 
q studied by Kalnins and Miller in [19].

The 4 operators \( \hat{\mu}^{(a,b,c,d)} \) and \( \hat{\nu}^{(a,b,c,d)} \) relative to the S-Heun operators:

The S-Heun operators thus underscore much of the characterization of the Big q-Jacobi polynomials. Moreover, one can

\[ \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = P_n(z;\alpha,\beta,\gamma,\tilde{q}). \]

It is clear that the parameter \( a \) is redundant. One has \( \Phi_n^{(1,\beta/\gamma,\alpha,\tilde{q})}(z;\tilde{q}) = P_n(z;\alpha,\beta,\gamma,\tilde{q}). \) It is seen that

\[ \hat{\tau} \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = \frac{aq(1-q^{-2n})(1-acdq^{2n-2})}{(1-ad)} \Phi_{n-1}^{(aq,bq,cq,dq)}(z;\tilde{q}), \]

\[ \hat{\hat{\tau}}^{(a,b,c,d)} \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = \frac{(acq^2)(adq^2)}{acdq} \Phi_{n+1}^{(aq^{-1},bq^{-1},cq^{-1},dq^{-1})}(z;\tilde{q}), \]

\[ \hat{\mu}^{(a,b,c,d)} \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = \frac{q}{d(1-ad^{-2})} \Phi_n^{(aq^{-1},bq,cq,dq^{-1})}(z;\tilde{q}), \]

\[ \hat{\nu}^{(a,b,c,d)} \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = \frac{-q(adq^2-2n)(1-bcq^{2n-2})}{c(1-ad)} \Phi_n^{(aq,bq^{-1},cq^{-1},dq)}(z;\tilde{q}). \]

The 4 operators \( \hat{\mu}^{(a,b,c,d)}, \hat{\nu}^{(a,b,c,d)}, \hat{\tau}, \hat{\hat{\tau}}^{(a,b,c,d)} \) built from linear combinations of S-Heun operators have been studied by Kalnins and Miller in [19].

Let us further indicate how the bispectral operators show up in this context. As mentioned above, the Big q-Jacobi difference operator appears as a quadratic combination of the stabilizing generators. Moreover, one can actually provide factorizations of this operator in terms of contiguity operators as well as backward and forward operators:

\[ \hat{D} = \alpha\gamma q^3 \mu^{(q,\frac{a}{\gamma},\alpha q^2,\gamma q)} \mu^{(1,\frac{\alpha}{\gamma},q\gamma^2)} \mu^{(1,\frac{a}{\gamma},\alpha q^2,\gamma q^2)} - (1 - q^2)(1 - \frac{\alpha q^2}{\gamma}) \]

\[ = \alpha\gamma q^3 \mu^{(q^{-1},\frac{a}{\gamma},q\gamma^2)} \mu^{(1,\frac{\alpha}{\gamma},q\gamma^2)} \mu^{(1,\frac{a}{\gamma},q\gamma^2)} - (1 - \gamma)(1 - \frac{\alpha q^2}{\gamma}) \]

\[ = -\alpha\gamma q^3 \hat{\tau}^{(q,\frac{a}{\gamma},q\gamma^2)} \tau \hat{\tau}^{(1,\frac{\alpha}{\gamma},q\gamma^2)} - (1 - q^2)(1 - \alpha\beta). \]

The second bispectral operator \( \hat{X} \) is the multiplication by the variable \( z \): \( \hat{X}f(z) = zf(z) \). It also appears as the quadratic combination of S-Heun operators:

\[ \hat{X} = \hat{M}_2 \hat{R}_1 - \hat{M}_1 \hat{R}_2. \]

The S-Heun operators thus underscore much of the characterization of the Big q-Jacobi operators.

4.3 The Sklyanin-type algebra realized by the structure operators

The pair of bispectral Big q-Jacobi operators is known to realize the Big q-Jacobi algebra [56, 42]. The algebra generated by the 4 linear operators \( \hat{\mu}^{(a,b,c,d)}, \hat{\nu}^{(a,b,c,d)}, \hat{\tau}, \hat{\hat{\tau}}^{(a,b,c,d)} \) is a familiar degeneration of the Sklyanin algebra [24].

Denote \( q^{-\nu} = (abcd)^{\frac{1}{4}} \) and form

\[ \hat{A} = q^{-\nu}(M_1 + qM_2), \]

\[ \hat{B} = \frac{1}{2(q - q^{-1})}[q^{2\nu}(\hat{R}_1 + q^{-1}\hat{R}_2) - q^{-2\nu}(\hat{R}_1 + q\hat{R}_2)], \]

\[ \hat{C} = 2\hat{L}, \]

\[ \hat{D} = q^{\nu}(M_1 + q^{-1}M_2). \]
Proposition 4.1  The operators $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ obey the quadratic relations
\[ \hat{A}\hat{B} = q\hat{B}\hat{A}, \quad \hat{B}\hat{D} = q\hat{D}\hat{B}, \quad \hat{C}\hat{A} = q\hat{A}\hat{C}, \quad \hat{D}\hat{C} = q\hat{C}\hat{D}, \]
\[ [\hat{B}, \hat{C}] = \frac{\hat{A}^2 - \hat{D}^2}{q - q^{-1}}, \quad [\hat{A}, \hat{D}] = 0 \]  
which define $U_q(\mathfrak{sl}_2)$.

When $\nu$ is an integer or a half-integer, one obtains finite-dimensional representations of $U_q(\mathfrak{sl}_2)$ of dimension $2\nu + 1$. In that case, the maximal degree of the polynomials obtained from the action of the raising operator $\hat{B}$ is $N$.

Remark 4.1  The $q \to 1$ limit of this realization yields the $\mathfrak{sl}_2$ commutation relations. In fact $(4.8)$ tends to the differential Bargmann realization of $\mathfrak{sl}_2$. Under the limit, the $q$-linear grid becomes the continuum, and the above combinations of shift operators turn into differential operators.

4.4  Finite-dimensional representations

We now wish to obtain finite-dimensional representations of $U_q(\mathfrak{sl}_2)$ corresponding to a particular truncation of the Jacobi matrix of the Big $q$-Jacobi polynomials. As mentioned previously, this can be accomplished by taking $\nu$ to be either an integer or a half-integer. In order to do so, we are led to take
\[ \sqrt{abcd} = q^{1-N}, \]  
(4.10)

where $N$ is a positive integer that corresponds to the maximal degree of the truncated family of polynomials.

Proposition 4.2  The polynomials that arise from the truncation condition $(4.10)$ form a basis that supports $(N + 1)$-dimensional representations of $U_q(\mathfrak{sl}_2)$ in the realization $(4.8)$. The $q$-para-Krawtchouk polynomials $(4.10)$ are the ones that arise from this truncation condition.

We show below how their recurrence relation is obtained from the one of the Big $q$-Jacobi polynomials.

4.4.1  $N = 2j + 1$ odd

In the case where $N = 2j + 1$ is odd, we write
\[ d = a^{-1}q^{-2j+\epsilon_1 t}, \quad b = c^{-1}q^{-2j+\epsilon_1 t} \]  
(4.11)

and then take the limit $t \to 0$. Using this parametrization, the recurrence relation of the Big $q$-Jacobi polynomials
\[ zP_n(z; a, b, c; \tilde{q}) = A_nP_{n+1}(z; a, b, c; \tilde{q}) + C_nP_{n-1}(z; a, b, c; \tilde{q}) + [1 - (A_n + C_n)]P_n(z; a, b, c; \tilde{q}) \]  
(4.12)

has for coefficients
\[ A_n = + \frac{(1 - acq^{2n})(1 - q^{2n-2N})}{(1 + q^{2n-N+1})(1 - q^{4n-2N})}, \]  
(4.13a)
\[ C_n = - \frac{q^{2n-N-1}(1 - q^{2n})(ac - q^{2n-2N})}{(1 + q^{2n-N-1})(1 - q^{4n-2N})} \]  
(4.13b)

after the use of (4.11) and the limit $t \to 0$. Now letting
\[ ac = c_3q^2 \]  
(4.14)

it follows that (4.13) can be rewritten as
\[ A_n = + \frac{(1 - c_3q^{2n+2})(1 - q^{2n-2N})}{(1 + q^{2n-N+1})(1 - q^{4n-2N})}, \]  
(4.15a)
\[ C_n = - \frac{q^{2n-N+1}(1 - q^{2n})(c_3 - q^{2n-2N-2})}{(1 + q^{2n-N-1})(1 - q^{4n-2N})}, \]  
(4.15b)

and one recognizes the recurrence coefficients of the $q$-para-Krawtchouk polynomials in the base $\tilde{q} = q^2$ introduced in $(56)$ when $N$ is odd. These polynomials are defined on the union of two $q$-linear lattices and the parameter $c_3$ describes the shift of one lattice with respect to the other.
4.4.2 $N = 2j$ even

In the case where $N = 2j$ is even, we take

$$d = a^{-1}q^{-2j+e_1t}, \quad b = c^{-1}q^{-2j+e_2t+2},$$

(4.16)

which ensures (4.10) in the limit $t \to 0$. Using this parametrization and after letting $t \to 0$, the recurrence coefficients of the Big $q$-Jacobi polynomials become

$$A_n = \frac{(1 - acq^{2n})(1 - q^{2n-2N})}{(1 + q^{2n-N})(1 - q^{4n-2N}+2)}; \quad (4.17a)$$

$$C_n = -\frac{q^{2n-N-2}(1 - q^{2n})(ac - q^{2n-2N})}{(1 + q^{2n-N})(1 - q^{4n-2N}-2)}, \quad (4.17b)$$

and upon letting

$$ac = c_3q^2 \quad (4.18)$$

$A_n$ and $C_n$ can be rewritten as

$$A_n = \frac{(1 - c_3q^{2n+2})(1 - q^{2n-2N})}{(1 + q^{2n-N})(1 - q^{4n-2N}+2)}; \quad (4.19a)$$

$$C_n = -\frac{q^{2n-N}(1 - q^{2n})(c_3 - q^{2n-2N-2})}{(1 + q^{2n-N})(1 - q^{4n-2N}-2)} \quad (4.19b)$$

These are the recurrence coefficients of the $q$-para-Krawtchouk polynomials in the base $\tilde{q} = q^2$ for $N$ even. For more detail, see [56].

4.4.3 A remark on the truncation condition

There is once again an apparent mismatch in the dimensions of the representations of the algebra and those of the representation basis. The same remark as the one made in the preceding section applies here. It can be checked that in the realization (4.8), applying the truncation condition (4.10) seems to suggest that the raising operator $\hat{B}$ annihilates the monomial $z^{N+1}$ and not $z^N$, which means that the truncation condition leads to representations of the algebra $U_q(\mathfrak{sl}_2)$ of dimension $N + 2$. This would contradict the fact that the $q$-para-Krawtchouk polynomials were truncated to a maximal degree $N$ (and thus span a space of dimension $N + 1$).

It can be observed that $\hat{B}$ maps the $q$-para-Krawtchouk polynomial of degree $N$ to a polynomial of degree $N + 1$. The resulting polynomial is the characteristic polynomial of the (upper block of the) truncated Jacobi matrix, hence it is again null on the orthogonality grid points. In the representation basis with which we are working (i.e. where the $q$-para-Krawtchouk polynomials are the basis elements), this characteristic polynomial corresponds to a null vector. Hence, the dimension of the space on which the realization of the $U_q(\mathfrak{sl}_2)$ algebra acts is indeed $N + 1$.

4.5 Recovering the related Heun operator

The Heun operator associated to the Big $q$-Jacobi polynomials is given in [42] and had also been introduced previously in [36]. This operator $W_{BJ}$ is the most general second order $q$-difference operator that acts on the $q$-linear grid and maps polynomials of degree $n$ into polynomials of degree $n + 1$. Its expression is

$$W_{BJ} = \mathcal{A}_1 \hat{T}_+ + \mathcal{A}_0 \hat{T} + \mathcal{A}_2 \hat{T}_-, \quad (4.20)$$

where

$$\mathcal{A}_1 = \frac{\pi_3(z)}{z^2}, \quad \mathcal{A}_2 = \frac{\tilde{q} \pi_3(z) + z \pi_2(z)}{z^2} \quad (4.21)$$

and $\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 = \pi_1(z)$, with $\pi_k(z)$ a generic polynomial of degree $k$ and $\tilde{q}$ the base.
Let us consider the most general quadratic combination of S-Heun operators that does not raise the degree of polynomials by more than one. Using the quadratic homogeneous relations of Appendix A we arrive at

\[ W = \alpha_1 \hat{L}^2 + \alpha_2 \hat{L} \hat{M}_1 + \alpha_3 \hat{L} \hat{M}_2 + \alpha_4 \hat{M}_1^2 + \alpha_5 \hat{M}_1 \hat{M}_2 + \alpha_6 \hat{M}_2^2 + \beta_1 \hat{M}_1 \hat{R}_2 + \beta_2 \hat{M}_2 \hat{R}_1 + \beta_3 \hat{M}_2 \hat{R}_2. \]  

(4.22)

Substituting the expressions (1.14) for the generators we obtain

\[ W = A_1 \hat{T}_z^2 + A_0 \hat{L} + A_2 \hat{T}_z, \]
\[ A_1 = \frac{1}{z^2(1-q^2)^2} [ (q^2 \alpha_1 + (q^2 \alpha_3 - q \alpha_2)z + (q^2 \alpha_6 - q \alpha_5 + \alpha_4)z^2 + (q^3 \beta_3 - q^2 \beta_1 - q^2 \beta_2)z^3] , \]
\[ A_2 = \frac{1}{z^2(1-q^2)^2} [ (q^3 \alpha_1 + (q^2 \alpha_3 - q^3 \alpha_2)z + (q^2 \alpha_6 - q^3 \alpha_5 + q^4 \alpha_4)z^2 + (q \beta_3 - q^2 \beta_1 - q^2 \beta_2)z^3] , \]
\[ A_0 = \beta_2 z + \alpha_4 - (A_1 + A_2). \]

(4.23)

**Proposition 4.3** The generic Heun-Big $q$-Jacobi operator (4.20) (with base $q^2$) can be obtained as the most general quadratic combination in the S-Heun generators (1.14) that does not raise the degree of polynomials by more than one.

Moreover, using the relations of Appendix A we see that the Heun operator typically factorizes as the product of a raising S-Heun operator with a stabilizing S-Heun operator:

\[ \hat{W} = (\xi_1 \hat{L} + \xi_2 \hat{M}_1 + \xi_3 \hat{M}_2)(\eta_1 \hat{L} + \eta_2 \hat{M}_1 + \eta_3 \hat{M}_2 + \eta_4 \hat{R}_1 + \eta_5 \hat{R}_2) + \kappa. \]  

(4.24)

## 5 Connections between the different cases

It is well known that the three grids on which we have defined S-Heun operators can be obtained as limiting cases or contractions of the Askey–Wilson grid. We now observe that this translates into limits/contractions of the associated Sklyanin algebras.

Let us denote the points of the Askey–Wilson grid by

\[ \lambda_s = z_s + z_s^{-1}, \quad z_s = q^s. \]  

(5.1)

The associated Sklyanin algebra was introduced in [23] as the trigonometric degeneration of the Sklyanin algebra [24] and was studied from the perspective of S-Heun operators in [25]. The defining relations read

\[ DC = qCD, \quad CA = qAC, \quad [A, D] = \frac{(q - q^{-1})^3}{4} C^2, \]
\[ [B, C] = \frac{A^2 - D^2}{q - q^{-1}}, \]
\[ AB - qBA = qDB - BD = -\frac{q^2 - q^{-2}}{4}(DC - CA). \]

(5.2)

The $q$-linear (or exponential) grid

\[ \lambda_s = z_s, \quad z_s = q^s \]

(5.3)

is obtained from the Askey–Wilson one in the asymptotic expansion $z_s \to \infty$ and the same limit takes the Askey–Wilson polynomials into the Big $q$-Jacobi OPs. At the level of the algebras, this corresponds to the following contraction. Writing

\[ A = \epsilon \hat{A}, \quad B = \hat{B}, \quad C = \epsilon^2 \hat{C}, \quad D = \epsilon \hat{D} \]

(5.4)
and taking $\epsilon \to 0$, one recovers $U_q(\mathfrak{sl}_2)$:
\[
\hat{A}\hat{B} = q\hat{B}\hat{A}, \quad \hat{B}\hat{D} = q\hat{D}\hat{B}, \quad \hat{C}\hat{A} = q\hat{A}\hat{C}, \quad \hat{D}\hat{C} = q\hat{C}\hat{D},
\]
\[
[\hat{B}, \hat{C}] = \frac{\hat{A}^2 - \hat{D}^2}{q - q^{-1}}, \quad [\hat{A}, \hat{D}] = 0.
\]

We now compare the discrete linear grid to the continuum. A rescaling similar to the one discussed above takes this grid to the real line. This also takes the Continuous Hahn polynomials into the Jacobi ones. From the perspective of the algebras, (5.4) will relate one algebra to the other. The Sklyanin algebra (3.8) associated to the discrete grid is
\[
[C, D] = 0, \quad [A, C] = \{C, D\}, \quad [A, D] = \{C, C\},
\]
\[
[B, C] = \{D, A\}, \quad [B, D] = \{C, A\}, \quad [B, A] = \{B, D\}
\]
and upon writing
\[
A = \epsilon \hat{A}, \quad B = \hat{B}, \quad C = \epsilon^2 \hat{C}, \quad D = \epsilon \hat{D}
\]
and taking $\epsilon \to 0$, we recover
\[
[C, D] = 0, \quad [A, C] = -CD, \quad [A, D] = 0,
\]
\[
[\hat{B}, \hat{C}] = -2\hat{A}\hat{D}, \quad [\hat{A}, \hat{B}] = \hat{B}\hat{D}, \quad [\hat{B}, \hat{D}] = 0.
\]

We recall that the latter algebra is essentially the $\mathfrak{sl}_2$ Lie algebra with a central element $D$.

We have so far discussed the following contractions, denoted by full arrows:

```
AW grid      ......      discrete linear grid
            ↓                      ↓
q-linear grid ......      continuum
```

One could wonder if it is possible to complete the diagram with the dotted arrows. The bottom arrow is easy to add: this amounts to taking the limit $q \to 1$. This limit takes the $q$-linear grid to the continuum, the Big $q$-Jacobi polynomials to the Jacobi polynomials, and at the level of the algebra, it takes $U_q(\mathfrak{sl}_2)$ to $\mathfrak{sl}_2$.

The details corresponding to the upper arrow remain to be worked out. It is likely that an intermediary step related to the quadratic grid $\lambda_s = s^2$ should be required. Indeed, it is known that the $q \to 1$ limit of the Askey–Wilson grid leads to the quadratic grid. It should thus be possible to apply the S-Heun construction to the quadratic grid; the polynomial should be those of Wilson, and the related Sklyanin algebra would stand in between the one of Askey–Wilson type (5.2) and the one of the discrete linear type (3.8).

**6 Conclusion**

The results of this paper are summarized as follows. We have introduced S-Heun operators on linear and $q$-linear grids. These operators are special cases of second order Heun operators with no diagonal term. On the real line and the discrete and $q$-linear grids, the sets of five S-Heun operators were constructed and shown to be related to the Jacobi, Continuous Hahn and Big $q$-Jacobi polynomials respectively. These S-Heun operators were also shown to encompass the bispectral and structure operators for each family of orthogonal polynomials. A presentation of the relations for the four structure operators of Kalnins and Miller was given in each case and identified as realizing degenerations, contractions or limits of the Sklyanin algebra. For the discrete and $q$-linear grids, the finite-dimensional representations of the Sklyanin-type algebras were obtained from a truncation condition on the Jacobi matrix of the associated polynomials; this yielded the para-Krawtchouk and $q$-para-Krawtchouk polynomials.
as bases of the finite representations and provided algebraic interpretations of these sets of OPs that had so far been missing.

The Sklyanin-like algebra related to the discrete linear grid \([3.8]\) has a simple presentation and a detailed study of its representation theory would be interesting. It would also be instructive to examine the types of Sklyanin algebra that the S-Heun operators on the quadratic grid would lead to. We plan on undertaking this in the near future. Note that we have restricted ourselves to Heun operators defined by actions on polynomials. The exploration of the generalizations that result from the extension to spaces of rational functions have been initiated in \([11]\) and should be actively pursued in the S-Heun framework in particular.

Acknowledgments

The authors would like to thank Jean-Michel Lemay for useful discussions. JG holds an Alexander-Graham-Bell scholarship from the Natural Sciences and Engineering Research Council of Canada (NSERC). The research of LV is funded in part by a Discovery Grant from NSERC. AZ gratefully holds a CRM-Simons professorship and his work is supported by the National Science Foundation of China (Grant No.11771015).

A The homogeneous quadratic algebraic relations

The 14 quadratic homogeneous relations associated to all three sets of 5 S-Heun operators are collected here. One notes that all three sets of relations display a similar structure. These relations can be thought of as reordering relations and are especially useful when considering the most general quadratic combinations in the generators.

A.1 The continuum

The relations between the S-Heun operators \(\bar{L}, \bar{M}_1, \bar{M}_2, \bar{R}_1, \bar{R}_2\) defined in (1.17) can be presented as the fourteen following relations:

\[
\begin{align*}
\bar{M}_1\bar{L} = \bar{L}\bar{M}_1, \\
\bar{M}_2\bar{L} = \bar{L}\bar{M}_2 - \bar{M}_1\bar{L}, \\
\bar{M}_2\bar{M}_1 = \bar{M}_1\bar{M}_2, \\
\bar{M}_1^2 = 1,
\end{align*}
\][A.1]

\[
\begin{align*}
\bar{L}\bar{R}_1 = 1 + \bar{M}_1\bar{M}_2, \\
\bar{L}\bar{R}_2 = \bar{M}_2^2 + \bar{M}_1\bar{M}_2, \\
\bar{R}_1\bar{L} = \bar{M}_1\bar{M}_2, \\
\bar{R}_2\bar{L} = \bar{M}_2^2 - \bar{M}_1\bar{M}_2, \\
\bar{R}_2\bar{R}_1 = \bar{R}_1\bar{R}_2 + \bar{R}_1^2, \\
\bar{R}_1\bar{M}_1 = \bar{M}_2\bar{R}_1 - \bar{M}_1\bar{R}_2, \\
\bar{R}_2\bar{M}_1 = \bar{M}_2\bar{R}_2 - \bar{M}_1\bar{R}_2, \\
\bar{R}_2\bar{M}_2 = \bar{M}_2\bar{R}_2 - \bar{M}_1\bar{R}_2, \\
\bar{M}_1\bar{R}_1 = \bar{M}_2\bar{R}_1 - \bar{M}_1\bar{R}_2.
\end{align*}
\]

A.2 The discrete linear grid

Here are the relations between the S-Heun operators \(L, M_1, M_2, R_1, R_2\) that have been defined in (1.8):

\[
\begin{align*}
M_1L = LM_1, \\
M_2L = LM_2 - LM_1, \\
M_2M_1 = M_1M_2 - L^2, \\
M_1^2 = 1 + L^2, \\
LR_1 = 1 - 2M_2^2 - M_1M_2, \\
LR_2 = 1 + M_1M_2, \\
R_1L = 3M_1M_2 - 3L^2 - 2M_2^2, \\
R_2L = M_1M_2 - L^2, \\
R_2R_1 = 2R_2^2 + R_1R_2 - 4M_2^2, \\
R_1M_1 = 3M_1R_2 - 2M_2R_2 - 3LM_1, \\
R_1M_2 = 2M_2R_2 - 3M_1R_2 + 3LM_2 + M_2R_1, \\
R_2M_1 = M_1R_2 - LM_1, \\
R_2M_2 = M_2R_2 - M_1R_2 + LM_2, \\
M_1R_1 = 3M_1R_2 - 2M_2R_2 - 4LM_2.
\end{align*}
\][A.2]
A.3 The $q$-linear grid

We remind the reader that the $q$-number 2 is written as $[2]_q = q + q^{-1}$. The S-Heun operators $\hat{L}$, $\hat{M}_1$, $\hat{M}_2$, $\hat{R}_1$, $\hat{R}_2$ defined in (1.14) obey the fourteen quadratic relations:

\[
\begin{align*}
\hat{M}_1 \hat{L} &= [2]_q \hat{L} \hat{M}_1 + \hat{L} \hat{M}_2, \\
\hat{M}_2 \hat{L} &= -\hat{L} \hat{M}_1, \\
\hat{M}_2 \hat{M}_1 &= \hat{M}_1 \hat{M}_2, \\
[2]_q \hat{M}_1 \hat{M}_2 &= 1 - \hat{M}_1^2 - \hat{M}_2^2, \\
\hat{L} \hat{R}_1 &= 1 - \hat{M}_2^2, \\
\hat{L} \hat{R}_2 &= [2]_q \hat{M}_2^2 + \hat{M}_1 \hat{M}_2, \\
\hat{R}_1 \hat{L} &= 1 - \hat{M}_1^2, \\
\hat{R}_2 \hat{L} &= -\hat{M}_1 \hat{M}_2, \\
[2]_q \hat{R}_1 \hat{R}_2 &= -\hat{R}_1^2 - \hat{R}_2^2, \\
[2]_q \hat{R}_1 \hat{M}_1 &= -[2]_q^2 \hat{M}_1 \hat{R}_2 - [2]_q \hat{M}_2 \hat{R}_2 + \hat{M}_2 \hat{R}_1, \\
[2]_q \hat{R}_1 \hat{M}_2 &= [2]_q \hat{M}_1 \hat{R}_2 + \hat{M}_2 \hat{R}_2, \\
\hat{R}_2 \hat{M}_1 &= [2]_q \hat{M}_1 \hat{R}_2 + \hat{M}_2 \hat{R}_2, \\
\hat{R}_2 \hat{M}_2 &= -\hat{M}_1 \hat{R}_2, \\
\hat{M}_1 \hat{R}_1 &= -[2]_q \hat{M}_1 \hat{R}_2 - \hat{M}_2 \hat{R}_2. 
\end{align*}
\]

(A.3)

References

[1] A. S. Zhedanov, “Hidden symmetry” of Askey-Wilson polynomials, Theoretical and Mathematical Physics 89, 1146–1157 (1991).

[2] D. Bullock and J. H. Przytycki, Multiplicative structure of Kauffman bracket skein module quantizations, Proceedings of the American Mathematical Society 128, 923–931 (1999), arXiv:math/9902117.

[3] T. H. Koornwinder, The Relationship between Zhedanov’s Algebra AW(3) and the Double Affine Hecke Algebra in the Rank One Case, Symmetry, Integrability and Geometry: Methods and Applications 3, 063 (2007), arXiv:math/0612730.

[4] T. H. Koornwinder, Zhedanov’s Algebra AW(3) and the Double Affine Hecke Algebra in the Rank One Case. II. The Spherical Subalgebra, Symmetry, Integrability and Geometry: Methods and Applications 4, 052 (2008), arXiv:0711.2320v3.

[5] M. Mazzocco, Confluences of the Painlevé equations, Cherednik algebras and $q$-Askey scheme, Nonlinearity 29, 2565–2608 (2016), arXiv:1307.6140.

[6] L. Frappat, J. Gaboriaud, E. Ragoucy, and L. Vinet, The dual pair $(U_q(\mathfrak{su}(1,1)), \mathfrak{o}_q(2n))$, $q$-oscillators, and Askey-Wilson algebras, Journal of Mathematical Physics 61, 041701 (2020), arXiv:1908.04277.

[7] J. Gaboriaud, L. Vinet, and S. Vinet, Howe duality and algebras of the Askey-Wilson type: an overview, (2019), arXiv:1911.08314.

[8] P. Baseilhac, Deformed Dolan–Grady relations in quantum integrable models, Nuclear Physics B 709, 491–521 (2005), arXiv:hep-th/0404149v3.

[9] P. Baseilhac, An integrable structure related with tridiagonal algebras, Nuclear Physics B 705, 605–619 (2005), arXiv:math-ph/0408025.

[10] P. Baseilhac and K. Koizumi, A new (in)finitive dimensional algebra for quantum integrable models, Nuclear Physics B 720, 325–347 (2005), arXiv:math-ph/0503036.

[11] L. Vinet and A. Zhedanov, Quasi-Linear Algebras and Integrability (the Heisenberg Picture), Symmetry, Integrability and Geometry: Methods and Applications 4, 015 (2008), arXiv:0802.0744.

[12] P. Terwilliger and R. Vidunas, Leonard pairs and the Askey-Wilson relations, Journal of Algebra and Its Applications 03, 411–426 (2004), arXiv:math-ph/0305356.

[13] P. Terwilliger, The Universal Askey–Wilson Algebra, Symmetry, Integrability and Geometry: Methods and Applications 7, 069 (2011), arXiv:1104.2813.

[14] P. Terwilliger, The Universal Askey-Wilson Algebra and DAHA of Type $(C_1^\vee, C_1)$, Symmetry, Integrability and Geometry: Methods and Applications 9, 047 (2013), arXiv:1202.4673.

[15] P. Terwilliger, The $q$-Onsager Algebra and the Universal Askey–Wilson Algebra, Symmetry, Integrability and Geometry: Methods and Applications 14, 044 (2018), arXiv:1801.06083.
[16] Y. A. Granovskii and A. Zhedanov, Hidden Symmetry of the Racah and Clebsch-Gordan Problems for the Quantum Algebra $\mathfrak{sl}_q(2)$, Journal of Group Theoretical Methods in Physics 1, 161–171 (1993), arXiv:hep-th/9304138.

[17] H.-W. Huang, An embedding of the universal Askey-Wilson algebra into $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, Nuclear Physics B 922, 401–434 (2017), arXiv:1611.02130v2.

[18] E. G. Kalnins and W. Miller Jr., Symmetry techniques for q-series: Askey-Wilson polynomials, Rocky Mountain Journal of Mathematics 19, 223–230 (1989).

[19] E. G. Kalnins and W. Miller Jr., q-Series and Orthogonal Polynomials Associated with Barnes’ First Lemma, IMA Preprint Series (1987).

[20] W. Miller Jr., A note on Wilson polynomials, SIAM Journal on Mathematical Analysis 18, 1221–1226 (1987).

[21] L. Infeld and T. E. Hull, The Factorization Method, Reviews of Modern Physics 23, 21–68 (1951).

[22] T. H. Koornwinder, The structure relation for Askey–Wilson polynomials, Journal of Computational and Applied Mathematics 207, 214–226 (2007), arXiv:math/0601303.

[23] A. S. Gorsky and A. V. Zabrodin, Degenerations of Sklyanin algebra and Askey–Wilson polynomials, Journal of Physics A: Mathematical and General 26, L635–L640 (1993), arXiv:hep-th/9303026.

[24] E. K. Sklyanin, Some algebraic structures connected with the Yang–Baxter equation. Representations of quantum algebras, Functional Analysis and Its Applications 17, 273–284 (1983).

[25] J. Gaboriaud, S. Tsujimoto, L. Vinet, and A. Zhedanov, Degenerate Sklyanin algebras, Askey-Wilson polynomials and Heun operators, (2020), arXiv:2005.06961.

[26] M. E. H. Ismail and E. Koelink, Spectral properties of operators using tridiagonalisation, Analysis and Applications 10, 327–343 (2012), arXiv:1108.5718.

[27] F. A. Grünbaum, L. Vinet, and A. Zhedanov, Tridiagonalization and the Heun equation, Journal of Mathematical Physics 58, 031703 (2017), arXiv:1602.04840.

[28] F. A. Grünbaum, The Bispectral Problem: An Overview, in Special Functions 2000: Current Perspective and Future Directions (Springer, Dordrecht, 2001), pp. 129–140.

[29] F. A. Grünbaum, L. Vinet, and A. Zhedanov, Algebraic Heun Operator and Band-Time Limiting, Communications in Mathematical Physics 364, 1041–1068 (2018), arXiv:1711.07862.

[30] A. V. Turbiner, The Heun operator as a Hamiltonian, Journal of Physics A: Mathematical and Theoretical 49, 26LT01 (2016), arXiv:1603.02053.

[31] D. Slepian, Some Comments on Fourier Analysis, Uncertainty and Modeling, SIAM Review 25, 379–393 (1983).

[32] H. J. Landau, An Overview of Time and Frequency Limiting, in Fourier Techniques and Applications (Springer, Boston, 1985), pp. 201–220.

[33] N. Crampé, R. I. Nepomechie, and L. Vinet, Free-Fermion entanglement and orthogonal polynomials, Journal of Statistical Mechanics: Theory and Experiment 9, 093101 (2019), arXiv:1907.00044.

[34] N. Crampé, R. I. Nepomechie, and L. Vinet, Entanglement in Fermionic Chains and Bispectrality, (2020), arXiv:2001.10576.

[35] K. Takemura, Degenerations of Ruijsenaars-van Diejen operator and q-Painlevé equations, Journal of Integrable Systems 2, 1–27 (2017), arXiv:1608.07265.

[36] K. Takemura, On q-Deformations of the Heun Equation, Symmetry, Integrability and Geometry: Methods and Applications 14, 061 (2018), arXiv:1712.09564.

[37] P. Baseilhac, S. Tsujimoto, L. Vinet, and A. Zhedanov, The Heun–Askey–Wilson Algebra and the Heun Operator of Askey–Wilson Type, Annales Henri Poincaré 20, 3091–3112 (2019), arXiv:1811.11407.

[38] P. Baseilhac and R. A. Pimenta, Diagonalization of the Heun–Askey–Wilson operator, Leonard pairs and the algebraic Bethe ansatz, Nuclear Physics B 949, 114824 (2019), arXiv:1909.02464.
L. Vinet and A. Zhedanov, *The Heun operator of Hahn-type*, Proceedings of the American Mathematical Society 147, 2987–2998 (2019), arXiv:1808.00153

N. Crampé, L. Vinet, and A. Zhedanov, *Heun algebras of Lie type*, Proceedings of the American Mathematical Society 148, 1079–1094 (2020), arXiv:1904.10643

S. Tsujimoto, L. Vinet, and A. Zhedanov, *The rational Heun operator and Wilson biorthogonal functions*, (2019), arXiv:1912.11571

P. Baseilhac, L. Vinet, and A. Zhedanov, *The q-Heun operator of big q-Jacobi type and the q-Heun algebra*, The Ramanujan Journal 52, 367–380 (2020), arXiv:1808.06695

G. Bergeron, L. Vinet, and A. Zhedanov, *Signal Processing, Orthogonal Polynomials, and Heun Equations*, in AIMS-SIAM Conference on Orthogonal Polynomials and Their Applications, (Birkhäuser, Cham, 2020), pp. 195–214, arXiv:1903.00144

G. Bergeron, N. Crampé, S. Tsujimoto, L. Vinet, and A. Zhedanov, *The Heun-Racah and Heun-Bannai-Ito algebras*, Journal of Mathematical Physics 61, 081701 (2020), arXiv:2003.09558

L. Vinet and A. Zhedanov, *Para-Krawtchouk polynomials on a bi-lattice and a quantum spin chain with perfect state transfer*, Journal of Physics A: Mathematical and Theoretical 45, 265304 (2012), arXiv:1110.6475

J.-M. Lemay, L. Vinet, and A. Zhedanov, *The para-Racah polynomials*, Journal of Mathematical Analysis and Applications 438, 565–577 (2016), arXiv:1511.05215

V. X. Genest, L. Vinet, and A. Zhedanov, *Quantum spin chains with fractional revival*, Annals of Physics 371, 348–367 (2016), arXiv:1507.05919

É.-O. Bossé and L. Vinet, *Coherent Transport in Photonic Lattices: A Survey of Recent Analytic Results*, Symmetry, Integrability and Geometry: Methods and Applications 13, 074 (2017), arXiv:1705.04841v2

R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer Monographs in Mathematics (Springer, 2010), 578 pp.

V. X. Genest, M. E. H. Ismail, L. Vinet, and A. Zhedanov, *Tridiagonalization of the hypergeometric operator and the Racah-Wilson algebra*, Proceedings of the American Mathematical Society 144, 4441–4454 (2016), arXiv:1506.07803

Y. I. Granovskii, I. M. Lutzenko, and A. S. Zhedanov, *Mutual Integrability, Quadratic Algebras, and Dynamical Symmetry*, Annals of Physics 217, 1–20 (1992)

V. X. Genest, L. Vinet, and A. Zhedanov, *The Racah algebra and superintegrable models*, Journal of Physics: Conference Series 512, 012011 (2014), arXiv:1312.3874

N. Iyudu and S. Shkarin, *Classification of quadratic and cubic PBW algebras on three generators*, (2018), arXiv:1806.06844

E. M. Rains, *BC_n-symmetric abelian functions*, Duke Mathematical Journal 135, 99–180 (2006), arXiv:math/0402113v2

J.-M. Lemay, L. Vinet, and A. Zhedanov, *An analytic spin chain model with fractional revival*, Journal of Physics A: Mathematical and Theoretical 49, 335302 (2016), arXiv:1509.08965

S. Tsujimoto, L. Vinet, and A. Zhedanov, *Tridiagonal representations of the q-oscillator algebra and Askey-Wilson polynomials*, Journal of Physics A: Mathematical and Theoretical 50, 235202 (2017), arXiv:1612.04038

J.-M. Lemay, L. Vinet, and A. Zhedanov, *A q-generalization of the para-Racah polynomials*, Journal of Mathematical Analysis and Applications 462, 323–336 (2018), arXiv:1708.03368