On a relative version of the Krichever correspondence

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February 18, 1997

Preface

This PhD thesis is the result of my work in the Graduiertenkolleg "Geometrie und Nichtlineare Analysis" at Humboldt University Berlin and in the DFG project KU 770/1-3.

It is published in the Bayreuther Mathematische Schriften 52 (1997), p.1-74.

At this point, I would like to express my thanks to all of the people who supported my mathematical development.

My special thanks go to Doz. Dr. sc. W. Kleinert, my thesis advisor and my professor since the very beginning of my studies. He also introduced me to the fascinating area of algebraic geometry, turned my attention to the theory of evolution equations, and kindly supported my work on this thesis.

The subject of the present work has been suggested to me by Prof. M. Mulase, whom I would like to express my gratitude for his interest in my work and the inspiration and the encouragement he gave me.

I gratefully thank Prof. H. Kurke for his keen interest in my work and lots of valuable hints. The discussions with him have been a wonderful help during the completion of this paper.

My special thanks also go to G. Hein, A. Matuschke, Dr. M. Pflaum and D. Roßberg for numerous inspiring discussions.

Berlin, October 1996

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0 Introduction

The aim of this paper is to construct a link ranging from a class of sheaves on curves over some base scheme via infinite Grassmannians to commutative algebras of differential operators and evolution equations.

The idea of studying relationships between algebraic curves, algebras of differential operators and partial differential equations is not new. This connection has been studied already at the beginning of the 20th century by G. Wallenberg, I. Schur, J. L. Burchnall and T. W. Chaundy. G. Wallenberg tried to find all commuting pairs of ordinary differential operators. During his classification of commuting operators $P$ and $Q$ of order 2 and 3, respectively, he found a certain relation to a plane cubic curve. However, Wallenberg did not continue to explore this relation.

The motivation to study commutative algebras instead of commuting pairs of differential operators was given by I. Schur in 1905, when he proved the following remarkable fact:

Let $P$ be an ordinary differential operator of order greater than zero, and let $B_P$ be the set of all differential operators which commute with $P$. Then $B_P$ is a commutative algebra.

The work of J. L. Burchnall and T. W. Chaundy about the relations between commuting differential operators and affine algebraic curves is extensive. For example, they proved that for commuting $P$ and $Q$ of positive order, the ring $\mathbb{C}[P, Q]$ has dimension 1. Furthermore, they analyzed at length certain examples of affine curves and the related differential operators. It is remarkable that a large part of the methods which have been systematically developed more than 50 years later, virtually already exist in these early papers by Burchnall and Chaundy, although mainly in examples.

After a long period of stagnation, another break-through came with the work of P. Lax about isospectral deformations of differential operators in the late 60’s.

It has been realized, already in the early stage of the theory, that a commutative algebra of differential operators carries a lot more information than only its algebraic-geometric spectrum.

In the 70’s, I. M. Krichever analyzed the behaviour of an operator at infinity, i.e., he constructed the line bundle on a complete curve corresponding to a given algebra by special extensions of the trivial bundle to the point at infinity (see K1, K2, K3). His approach may be considered as the source of the algebraic-geometric correspondences established later on.

Almost simultaneously, and inspired by the work of Krichever, D. Mumford established a correspondence between pointed curves equipped with a fixed line bundle and certain commutative algebras of differential operators. This article and the one of J.-L. Verdier also contain the very first constructions in the case of higher rank vector bundles.

Later on, infinite Grassmannians emerged in the study of evolution equations and their relations to vector bundles on curves. Without claiming to be complete, let us just mention the work of M. Sato, E. Previato, G. Segal and G. Wilson (PW, SW) and M. Mulase.
Among one of the culmination points of this newly established theory was the complete classification of elliptic commutative algebras of ordinary differential operators, and one of the affirmative solutions of the Schottky problem by M. Mulase in [M2].

In the approach of M. Mulase, some questions arise quite naturally: Can one generalize the correspondence between vector bundles on curves and elements of infinite Grassmannians to the case where the curve is not defined over a field, but for example over a $k$-algebra? In this case, does one also get a correspondence with commutative algebras of differential operators with coefficients in more general rings? The present paper gives an affirmative answer to both questions.

These questions are interesting from two points of view. First, the established correspondence enables us to construct certain classes of commutative algebras of partial differential operators. On the other hand, it gives us a powerful tool for the study of degenerations of curves and vector bundles in terms of differential operators and differential equations.

In the recently published paper [AMP] the authors also give a generalization of infinite Grassmannians to the relative case, which overlaps with ours in the case where the base scheme is defined over a field, and they investigate this Grassmannian from the point of view of representation theory.

The method presented in our paper serves the purpose of generalizing the techniques developed in [M2] to the relative case. Therefore, this article can be considered as a general reference and often will not be quoted in the sequel.

A preliminary version of this paper appeared as a preprint [Q]. In addition to some slight corrections, a great extension has been made: In the preprint only integral noetherian base schemes were allowed, whereas now we only need to assume the base scheme to be locally noetherian.

The paper is organized as follows: The first chapter is devoted to the generalization of the notion of infinite Grassmannians. In the second and third, the link with sheaves on relative curves is established. The fourth chapter aims at illustrating this correspondence. In chapter 5 we give a complete characterization of commutative elliptic algebras of differential operators with coefficients in the ring of formal power series over a $k$-algebra, $k$ being a field of characteristic zero. The appendix is included in order to help those readers who are interested in the details of the computations.
1 Relative infinite Grassmannians

To begin with, we want to generalize the notion of infinite Grassmannians. This can be done for arbitrary base schemes. To this end, let $S$ be any algebraic scheme. We denote by $\mathcal{O}_S[[z]]$ the sheaf defined by

$$\mathcal{O}_S[[z]](U) := \mathcal{O}_S(U)[[z]],$$

and $\mathcal{O}_S((z))$ is defined as the sheaf

$$\mathcal{O}_S((z))(U) := \mathcal{O}_S(U)((z)),$$

where $\mathcal{O}_S(U)((z))$ stands for the ring of formal Laurent series in $z$ with coefficients in $\mathcal{O}_S(U)$. $\mathcal{O}_S((z))$ has a natural filtration by subsheaves of the form $\mathcal{O}_S[[z]] \cdot z^n$.

**Remark** Neither $\mathcal{O}_S[[z]]$ nor $\mathcal{O}_S((z))$ are quasicoherent sheaves of $\mathcal{O}_S$-modules. To see this, take an open affine subset $\text{Spec}(R)$ of $S$ and choose an element $f \in R$ which is not a zero divisor. Then there is a natural inclusion

$$R[[z]]_f \hookrightarrow R[[z]]$$

which, in general, is not an isomorphism. Therefore, $\mathcal{O}_S[[z]]$ is not quasicoherent. The same holds true for $\mathcal{O}_S((z))$.

However, for each integer $n \in \mathbb{Z}$, the quotient sheaf

$$\mathcal{O}_S((z))/\mathcal{O}_S[[z]] \cdot z^n \cong \bigoplus_{m<n} \mathcal{O}_S \cdot z^m$$

is quasicoherent.

**Definition 1.1** Let $U$ be an open subset of $S$ and $v$ a local section of $\mathcal{O}_S((z))^\oplus r$ over $U$, for some $r$. Then the order of $v$ is defined to be the minimum integer $n$ such that $v \in \mathcal{O}_S(U)[[z]]^\oplus r \cdot z^{-n}$.

If $\mathcal{V}$ is a subsheaf of $\mathcal{O}_S((z))^\oplus r$, then we define

$$\mathcal{V}^{(n)} := \mathcal{V} \cap \mathcal{O}_S[[z]]^\oplus r \cdot z^{-n}.$$

**Remark** $\mathcal{O}_S((z))$ acts on $\mathcal{O}_S((z))^\oplus r$ by the natural assignment

$$\mathcal{O}_S(U)((z)) \times \mathcal{O}_S(U)((z))^\oplus r \rightarrow \mathcal{O}_S(U)((z))^\oplus r \quad (f, g = (g_1, \ldots, g_r)) \mapsto (f \cdot g_1, \ldots, f \cdot g_r).$$

For the multiplication defined this way we have the estimate:

$$\text{ord}(f \cdot g) \leq \text{ord}(f) + \text{ord}(g).$$

If $S$ is an integral scheme, then both sides are equal.

For more properties of power series and Laurent series with coefficients in arbitrary rings, the reader is referred to the appendix.
Definition 1.2 The Grothendieck group $K(S)$ is defined to be the quotient of the free abelian group generated by all coherent sheaves on $S$, by the subgroup generated by all expressions $F - F' - F''$ whenever there is an exact sequence $0 \to F' \to F \to F'' \to 0$ of coherent sheaves on $S$. If $F$ is a coherent sheaf on $S$ then we denote by $\gamma(F)$ its image in $K(S)$.

Definition 1.3 For any natural number $r$, integer $\alpha$, and element $F \in K(S)$, we define the infinite Grassmannian of rank $r$, index $F$, and level $\alpha$ over $S$ to be the set $G_{r,F,\alpha}(S)$ consisting of all quasicoherent subsheaves of $\mathcal{O}_S$-modules $W \subseteq \mathcal{O}_S((z))^{\oplus r}$ such that $W \cap \mathcal{O}_S[[z]]^{\oplus r} \cdot z^{-\alpha}$ and the quotient $\mathcal{O}_S((z))^{\oplus r}/(W + \mathcal{O}_S[[z]]^{\oplus r} \cdot z^{-\alpha})$ are coherent and furthermore:

$$F = \gamma(W \cap \mathcal{O}_S[[z]]^{\oplus r} \cdot z^{-\alpha}) - \gamma(\mathcal{O}_S((z))^{\oplus r}/(W + \mathcal{O}_S[[z]]^{\oplus r} \cdot z^{-\alpha})).$$

Remark The introduction of the level has a merely technical meaning. It will be used only in Chapter 3.

Now we introduce the concept of a Schur pair.

Definition 1.4 By a Schur pair of rank $r$ and index $F$ over $S$ we mean a pair $(\mathcal{A}, W)$ consisting of elements $\mathcal{A} \in \mathfrak{G}_G(S)$, for some $G \in K(S)$, and $W \in \mathfrak{G}_F(S)$ such that

- $\mathcal{A}$ is a sheaf of $\mathcal{O}_S$-subalgebras of $\mathcal{O}_S((z))$,
- The natural action of $\mathcal{O}_S((z))$ on $\mathcal{O}_S((z))^{\oplus r}$ induces an action of $\mathcal{A}$ on $W$, i.e., $\mathcal{A} \cdot W \subseteq W$.

We denote by $\mathfrak{G}_F(S)$ the set of Schur pairs of rank $r$ and index $F$ over $S$.

Remark 1 Let us include here a remark on Grothendieck groups. First assume that $S$ is integral. Then there is a surjective group homomorphism

$$rk : K(S) \longrightarrow \mathbb{Z}$$

induced by the map

$$\gamma(F) \mapsto rk(F),$$

where $rk(F)$ denotes the rank of $F$ at the generic point of $S$. If $S$ equals $Spec(k)$, for some field $k$, then the homomorphism $rk$ is an isomorphism.

If $S$ is reduced we still can define a “multirank” by taking the rank at every irreducible component. However, if $S$ is not reduced, the rank is no longer well-defined. That is why we use the Grothendieck group to define Grassmannians.
Remark 2  How are the notions of infinite Grassmannians and Schur pairs related to those introduced by M. Mulase? To answer this question, consider the embedding 

\[ O_S((z)) \rightarrow O_S((y)) \]

\[ z \mapsto y. \]

This leads to the natural identification:

\[ O_S((z))^{\oplus r} = \bigoplus_{i=0}^{r-1} O_S((y^r)) \cdot y^i = O_S((y)). \]

In particular, let \( S = \text{Spec}(k) \) for some field \( k \). Then \( GF(S) \) consists of all subspaces \( W \subseteq k((z))^{\oplus r} \) such that the composition of morphisms

\[ W \rightarrow k((z))^{\oplus r} \rightarrow k((z))^{\oplus r} / k[[z]]^{\oplus r} \]

is a Fredholm operator of index \( rk(F) \). In view of the previous identifications, this redefines the notion of an infinite Grassmannian as used by M. Mulase (cf. [M2]).

However, comparing the notions of Schur pairs, we see that our definition is a restricted version of the one given by Mulase. Later we will see that in the case where the ground field is of characteristic zero, this restriction is not substantial. For more details see Section 5.

Definition 1.5  Let \( \alpha : S \rightarrow S' \) be a morphism, and \((A', W')\) a Schur pair of rank \( r \) over \( S' \). We denote by \( \alpha(\ast)A' \) (resp. \( \alpha(\ast)W' \)) the image of \((A', W')\) under the map

\[ \alpha(\ast) : O_{S'}((z))^{\oplus r} \rightarrow O_S((z))^{\oplus r} \]

which is given by the pull-back of the coefficients.

Remark  \( (\alpha(\ast)A', \alpha(\ast)W) \) is a Schur pair of rank \( r \) over \( S \).

Now we can define homomorphisms of Schur pairs.

Definition 1.6  Let \((A, W)\) and \((A', W')\) be Schur pairs over \( S \) (resp. \( S' \)) of rank \( r \) (resp. \( r' \)). Then a homomorphism \( (\alpha, \xi) : (A', W') \rightarrow (A, W) \) consists of

1. A morphism \( \alpha : S \rightarrow S' \) such that \( \alpha(\ast)A' \subseteq A; \)

2. A homomorphism \( \xi \in \text{Hom}_{O_S[[z]]}(O_S[[z]]^{\oplus r'}, O_S[[z]]^{\oplus r}) \) such that for the induced homomorphism \( \xi \in \text{Hom}_{O_S((z))}(O_S((z))^{\oplus r'}, O_S((z))^{\oplus r}) \) the inclusion

\[ \xi(\alpha(\ast)W') \subseteq W \]

holds.

In this way, we get the category \( \mathcal{S} \) of Schur pairs.

Definition 1.7  We define a full subcategory \( \mathcal{S}' \) of \( \mathcal{S} \) as follows: \( (A, W) \in \mathcal{S}'(S) \) if and only if \( (A, W) \in \mathcal{S}'_F(S) \) and \( A \cap O_S[[z]] = O_S \).

The sense of this definition will become clear later on. For the time being, take it simply as a notation.
2 Families of curves and sheaves

In this section, we fix the geometric objects that we want to investigate, and we prove some basic properties of them. The definitions we are going to make might seem a little technical. That is why much space is given to illustrations and examples. Even more examples may be found in Chapter 4.

Our first aim is to study sheaves over families of curves. As to that, we need to fix three objects, namely: the base scheme, the total space and a sheaf on the total space.

2.1 Families of curves

As base schemes \( S \) we allow all locally noetherian schemes.

**Definition 2.1** By a pointed relative curve over \( S \) we understand a scheme \( C \) together with a locally projective morphism \( \pi : C \to S \) and a section \( P \subset C \) of \( \pi \) such that the following holds:

1. \( P \) is a relatively ample Cartier divisor in \( C \).
2. For the sheaf \( \mathcal{I} := I_P \) defining \( P \) in \( C \), \( \mathcal{I}/\mathcal{I}^2 \) is a free \( \mathcal{O}_P \)-module of rank 1.
3. Let \( \hat{\mathcal{O}}_C \) denote the formal completion of \( \mathcal{O}_C \) with respect to the ideal \( \mathcal{I} \). Then \( \hat{\mathcal{O}}_C \) is isomorphic to \( \mathcal{O}_P[[z]] \) as a formal \( \mathcal{O}_P \)-algebra.
4. \( \bigcap_{n \geq 0} \pi_*(\mathcal{O}_C(-nP)) = (0) \).

Let us include here a couple of remarks and examples.

**Remark**

- Since \( P \) is a section, \( \pi|_P : P \to S \) is an isomorphism. The sheaves \( \mathcal{I}/\mathcal{I}^2 \) and \( \hat{\mathcal{O}}_C \) have their support in \( P \). Consequently, Condition 2 is equivalent to \( \pi_*(\mathcal{I}/\mathcal{I}^2) \cong \mathcal{O}_S \), while Condition 3 translates into: \( \pi_*\hat{\mathcal{O}}_C \) is isomorphic to \( \mathcal{O}_S[[z]] \) as a formal \( \mathcal{O}_S \)-algebra.
- Condition 4 is equivalent to the fact that, for every integer \( n \in \mathbb{Z} \), the natural map

\[
\pi_*\mathcal{O}_C(nP) \to \pi_*\hat{\mathcal{O}}_C(nP)
\]

is injective.

**Example 1** Let \( S = \text{Spec}(k) \) for some field \( k \). Then \( C \) is a complete curve and \( P \) corresponds to some smooth, \( k \)-rational point of \( C \). The \( k \)-rationality is a consequence of the fact that \( P \) is a section. Since \( \hat{\mathcal{O}}_{C,P} \cong k[[z]] \), the ring \( \mathcal{O}_{C,P} \) is regular.

The Condition 4 is satisfied if and only if the curve \( C \) is reduced and irreducible.

**Example 2** The motivation for studying curves over base schemes which are different from one point comes mainly from the desire to investigate families of curves as considered in Example 1. Let us take, as an example, such a family over an integral \( k \)-scheme \( S \). Since \( S \) and the fibres of
\(\pi\) are irreducible, \(C\) is automatically irreducible. Let us assume, in addition, that \(C\) is reduced. Then Condition 4 of the previous definition is satisfied. Conditions 2 and 3 amount to saying that our family is constant locally around the section \(P\).

Notice that the condition we impose on \(P\) by assuming the triviality of \(I/I^2\) is very restrictive in the case where our base scheme \(S\) is complete. This is expressed in the following proposition, which can be found in \([ACGH]\):

**Proposition 2.2** Let \(f : X \to S\) be a family of nodal curves of genus \(g > 0\) over a reduced and irreducible complete curve, and let \(\Gamma \subset X\) be a section of \(f\) not passing through any of the singular points of the fibres. Suppose that the general fibre of \(f\) is smooth. Then

\[(\Gamma \cdot \Gamma) \leq 0.\]

Moreover, if \((\Gamma \cdot \Gamma) = 0\), then the family \(f : X \to S\), together with the section \(\Gamma\), is an isotrivial family of 1-pointed nodal curves.

It is not hard to generalize the whole set-up to the case where \(I/I^2\) is locally free, but not free. One simply has to use \(\prod_{n \geq 0} (I/I^2)^n\) instead of \(O_P[[z]]\). However, since we are mainly interested in local considerations, this generalization does not play such an important role that it would justify the technical effort.

Now, the first condition needs to be examined. The following lemma shows that it is almost automatically satisfied:

**Lemma 2.3**

- If \(S\) is irreducible, then \(P\) is a Cartier divisor if and only if the conormal sheaf \(I/I^2\) is a line bundle on \(P\).

- If \(P\) is a Cartier divisor on \(C\), and if the morphism \(\pi\) has irreducible fibres, then \(P\) is relatively ample.

**Remark** The assumption on \(P\) translates as follows: \(I = I_P\) is locally generated by one element which is not a zero divisor.

**Proof of the lemma** The first statement is easy. Let us prove the second one. The question is local on \(S\). So we are in the following situation:

\(R\) is a noetherian ring, \(\pi : C \to Spec(R)\) is a projective morphism, and \(P \subset C\) is a section of \(\pi\) and an effective Cartier divisor.

For all \(n \in \mathbb{N}\), we have the following exact sequence:

\[0 \to O_C((n-1)P) \to O_C(nP) \to O_P(nP) \to 0.\]

Since \(P\) is affine, this induces a long exact sequence of cohomology groups:

\[\begin{align*}
0 & \to H^0(O_C((n-1)P)) \to H^0(O_C(nP)) \to H^0(O_P(nP)) \\
& \to H^1(O_C((n-1)P)) \to H^1(O_C(nP)) \to 0.
\end{align*}\]
Hence we have surjections $H^1(\mathcal{O}_C((n-1)P)) \to H^1(\mathcal{O}_C(nP))$. Composing these surjections, we obtain, for each $n \in \mathbb{N}$, an epimorphism

$$\alpha_n : H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_C(nP)).$$

Let $M_n$ denote the kernel of the mapping $\alpha_n$, which is an $R$-submodule of $H^1(\mathcal{O}_C)$. From the very definition we get: $M_n \subseteq M_{n+1}$. But [H1], Thm. III.5.2., tells us that $H^1(\mathcal{O}_C)$ is a finitely generated $R$-module, hence noetherian. Therefore, there is an integer $N$ such that $M_N = M_{N+1} = \ldots$. This implies that, for all $n > N$, the following sequence is exact:

$$0 \to H^0(\mathcal{O}_C((n-1)P)) \to H^0(\mathcal{O}_C(nP)) \to H^0(\mathcal{O}_{PM}(nP)) \to 0. \quad (2)$$

But $P$ is affine. Therefore $\mathcal{O}_{PM}(nP)$ is globally generated. This implies, using Nakayama’s lemma and (2), that the global sections of $\mathcal{O}_C(nP)$ are contained in $P$. Thus $\mathcal{O}_C(nP)$ itself has no base-points, i.e., it is globally generated. The sections of $\mathcal{O}_C(nP)$ define an $R$-morphism

$$\beta : \mathcal{C} \to \mathbb{P}_R^M.$$

As $\beta$ is an $R$-morphism, it is compatible with $\pi$, i.e., $\pi = pr \circ \beta$, where $pr$ denote the natural projection from $\mathbb{P}_R^M$ onto $Spec(R)$. Since $P$ is a section, $\beta$ restricts to a closed embedding on $P$. In addition, we know that $\beta^{-1}(\beta(P)) = P$. Now we want to prove that $\beta$ has finite fibres. Assume, on the contrary, that there is a point $q \in \beta(\mathcal{C})$ such that $dim\beta^{-1}(q) \geq 1$. Let $X$ be an irreducible component of $\beta^{-1}(q)$ of dimension greater than 0. It is clear that $X$ does not intersect the divisor $P$. In particular, the intersection of $X$ with each fibre of $\pi$ is a closed subset of codimension at least 1. But the fibres of $\pi$ are assumed to be irreducible of dimension 1. Therefore $X$ cannot be contained in any fibre of $\pi$. But this is a contradiction, since $\pi(\beta^{-1}(q)) = pr(q)$. So $\beta$ is a quasi-finite morphism.

By [H1], Cor. II.4.8., $\beta$ is proper. From the Stein-factorization theorem (cf. [3], Cor. 4.3.3.) one knows that in this case $\beta$ is also finite.

By construction, $\mathcal{O}_{C}(nP) = \beta^*(\mathcal{O}_{PM}(1))$. $\mathcal{O}_{PM}(1)$ induces a very ample line bundle on $\beta(\mathcal{C})$. Since $\beta$ is a finite morphism, this implies that $\mathcal{O}_C(nP)$ is ample, too (cf. [12], Prop.I.4.4.).

**Remark** We have seen that a family of curves whose fibres satisfy the conditions of Definition 2.1 almost matches the definition already.

However, the converse is definitely not true. Namely, if $C$ is as in Definition 2.1, then it may occur that some fibres of $\pi$ are not even integral (see Section 4.2.4 for examples).

After this illustration we return to our general definition.

**Lemma 2.4** Let $C$ be as in Definition 2.1. Then:

1. $C$ is locally noetherian.

2. For each open affine subset $U$ of $S$, $\pi^{-1}(U) \setminus P$ is affine.
3. $S$ can be covered by open affine sets, $U_i = \text{Spec}(R_i)$, such that for each $i$ there is an open affine subset $V_i = \text{Spec}(B_i)$ of $\pi^{-1}(U_i)$ containing $P \cap \pi^{-1}(U_i)$.

Proof

1. $\pi$ is locally projective, hence it is locally of finite type. Therefore, together with $S$, $C$ is also locally noetherian.

2. This is a consequence of the relative ampleness of $P$.

3. $\pi$ is locally of finite type, i.e., we can choose an open covering of $S$ by affine sets $U_k' = \text{Spec}(R_k')$ such that, for all $k$, $\pi : \pi^{-1}(U_k') \to U_k'$ is proper, and there are finitely many open affine sets $V_{k,l} = \text{Spec}(B_{k,l})$ satisfying:

- $\pi^{-1}(U_k') = \bigcup_i V_{k,l, i}$,
- $B_{k,l}$ is a finitely generated $R_k'$ - algebra.

Let $U' := U_k'$ for some $k$. Choose a point $Q \in P \cap \pi^{-1}(U')$. $Q$ is contained in one of the $V_{k,l}$ which we denote by $V'$ for short. If $V'$ contains $P \cap \pi^{-1}(U')$, then we are done. Now let us assume that $V'$ does not contain $P \cap \pi^{-1}(U')$. $P \cap \pi^{-1}(U')$ is a closed subset of $\pi^{-1}(U')$. From the closedness of $\pi$ we conclude that $\pi((P \cap \pi^{-1}(U')) \setminus V')$ is closed in $U'$, and that $V'$ contains $P \cap \pi^{-1}(U' \setminus \pi(P \setminus V'))$. The open set $U' \setminus \pi(P \setminus V')$ can be covered by open affine sets of the kind $U'(f) := \text{Spec}((R_k)_{f})$, for certain elements $f \in R_k$, and we see that over each of the $U'(f)$’s the affine set $V'(f) = V_{k,l}(f) := \text{Spec}((B_{k,l})_{f})$ has exactly the required property. \(\square\)

Now let us illustrate Condition 3 of Definition 2.1.

Lemma 2.5 Assume that $I/I^2$ is trivial. Then $\hat{O}_C$ is isomorphic to $\mathcal{O}_P[[z]]$ if and only if each section of $I/I^2 \cong \mathcal{O}_P$ lifts to a section of $\hat{O}_C$. In particular, one may interpret $z$ as the lift of a generating section of $I/I^2$.

Remark Consequently, $\hat{O}_C$ is isomorphic to $\mathcal{O}_P[[z]]$ if and only if, for any integer $n \in \mathbb{N}$, $n \geq 2$, the map

$$H^0(I/I^n) \to H^0(I/I^2)$$

is surjective.

Proof of the lemma First assume that $\hat{O}_C \cong \mathcal{O}_P[[z]]$. Then $I/I^2 \cong \mathcal{O}_P \cdot z$ and of course

$$H^0(\mathcal{O}_P[[z]]) = H^0(\mathcal{O}_P)[[z]] \to H^0(\mathcal{O}_P) \cdot z.$$

Now let $z$ be a global section of $\hat{O}_C$ such that $z(\text{mod}(I^2))$ generates $I/I^2$. The sheaf $\hat{O}_C$ is defined to be the limit taken over the projective system $\mathcal{O}_C/I^n$. We restrict our consideration to an open affine set $U = \text{Spec}(R) \subseteq S$ such that there is an open subset $V = \text{Spec}(B)$ of $\pi^{-1}(U)$ which contains $p(U)$ and which is so that $I := I(V)$ is free. Then we construct the required isomorphism of sheaves locally on $U$. 11
Choose an element \( b \in I \) such that \( I = bB \) and \([b] = [z] \in I/I^2\). Now take \( f \in B/I^m \) for some \( m \). Identifying \( R \) with \( \pi^*(R) \subset B \), there is a uniquely determined element \( f_1 \in R \) such that \( f - f_1 \in I/I^m \). By assumption, \( I/I^2 \) equals \([b] \cdot R\), and thus there is a \( f_2 \in R \) such that \( f - f_1 - f_2b(\mod I^m) \in I^2/I^m \). Since \( I^n/I^{n+1} \) is generated by \( b^n(\mod I^{n+1}) \), we can continue this process and get well-defined maps

\[
B/I^m \to R \oplus R \cdot b \ldots \oplus R \cdot b^{m-1}.
\]

These maps give rise to a homomorphism of formal \( R \) - algebras:

\[
\hat{B} \to R[[b]],
\]

where \( \hat{B} \) denotes the completion of \( B \) with respect to the ideal \( I \). By [Mat], Thm.8.12., this is an isomorphism.

The fixed global section \( z \) of \( \hat{O}_C \) restricts on \( U \) to an element of \( \hat{B} \). Since \([z], [b] \in I/I^2\) coincide, the homomorphism constructed above maps \( z \) to an element \( b \cdot (1 + \alpha) \) for some \( \alpha \in R[[b]] : b \). Remark that all those elements \( 1 + \alpha \) are invertible in \( R[[b]] \) (cf. the appendix). Therefore the formal power series rings \( R[[b]] \) and \( R[[z]] \) are naturally isomorphic. So we finally get a well-defined isomorphism of \( R \) - algebras:

\[
\rho : \hat{B} \to R[[b]] \simeq R[[z]].
\]

Here, \( \rho \) does not depend on the choice of the local lift \( b \) of \([z]\). Therefore, these locally defined isomorphisms glue together. It follows from Lemma 2.4 that \( S \) has a covering by sets \( U \) as considered above. Thus the construction gives a well-defined isomorphism of sheaves

\[
\rho : \hat{O}_C \to \mathcal{O}_P[[z]].
\]

\[\square\]

### 2.2 Families of sheaves

**Definition 2.6** As sheaves \( \mathcal{F} \) on \( C \) we admit all coherent sheaves such that

1. The formal completion \( \hat{\mathcal{F}} \) of \( \mathcal{F} \) along \( P \) is a free \( \hat{O}_C \)-module.
2. \( \bigcap_{n \geq 0} \pi_* \mathcal{F}(-nP) = (0) \).

**Remark 1** Again, the second condition is equivalent to:

\[
\pi_* \mathcal{F}(nP) \hookrightarrow \pi_* \hat{\mathcal{F}}(nP), \quad \text{for all} \ n \in \mathbb{Z}.
\]

**Examples**

- If \( C \) is an integral scheme, then the second condition is satisfied if and only if \( \mathcal{F} \) is torsion free.

---

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• If $C$ is a complete curve, then the first condition is also satisfied for torsion free sheaves, since these are free in smooth points.

• For general $S$ and $C$ as in Definition 2.1 and for any vector bundle $F$ on $C$, Condition 2 of Definition 2.6 is satisfied.

**Remark 2**  The isomorphism
$$\rho : \hat{O}_C \xrightarrow{\sim} O_P[[z]]$$

makes $\hat{F}$ into an $O_P[[z]]$-module. If $F$ satisfies Condition 1, then there is an isomorphism of $\hat{O}_C$-modules
$$\Phi : \hat{F} \xrightarrow{\sim} \hat{O}_C^{\oplus s}$$
or, equivalently, an isomorphism of $O_P[[z]]$-modules
$$\rho \circ \Phi : \hat{F} \xrightarrow{\sim} O_P[[z]]^{\oplus s}.$$ 

A natural question is: What do the properties of $F$ imply in general? In order to get a flavor of what is happening, we prove the following lemma.

**Lemma 2.7** Let $F$ be a coherent sheaf of rank $r$, as in Definition 2.6. Then the following holds:

(i) The rank of $\hat{F}$ as an $\hat{O}_C$-module equals to the rank of $F$. In particular, there is an open, dense subset of $C$ on which $F$ has constant rank.

(ii) $F|_P$ is free of rank $r$.

(iii) $F$ is locally free in a neighborhood of $P$.

**Proof** (i) is a consequence of (iii). Let us turn to the proof of (ii). We assume that $\hat{F} \cong \hat{O}_C^{\oplus s}$, for some $s \in \mathbb{N}$. Thus
$$F|_P = F \otimes_{O_C} O_C/I = \hat{F}/I\hat{F} = (\hat{O}_C/I\hat{O}_C)^{\oplus s} = (O_C/I)^{\oplus s} = O_P^{\oplus s},$$
i.e., $F|_P$ is free of rank $s$.

Now let $x$ be a point of $P$. By Nakayama’s lemma we get a surjection
$$O_{C,x}^{\oplus s} \twoheadrightarrow F_x.$$ 

Denote by $K$ the kernel of this morphism. Taking the formal completion along $P$ we get an exact sequence
$$0 \rightarrow \hat{K} \rightarrow \hat{O}_{C,x}^{\oplus s} \rightarrow \hat{F}_x \rightarrow 0.$$  

By [Mat], Thm.2.4., $\hat{K}$ vanishes. On the other hand, we know that $\hat{O}_{C,x}$ is faithfully flat over $O_{C,x}$ (cf. [Mat], Thm.8.14.). This implies that $K = 0$ and we conclude: $O_{C,x}^{\oplus s} \cong F_x$. This completes the proof of the lemma. $\square$
Remark. In Section 4.2 we will see that, under certain conditions, the properties (ii) and (iii) of Lemma 2.7 are equivalent to Condition 1 in Definition 2.6. This will give us an explicit method to construct examples. A first step in this direction is the following lemma.

Lemma 2.8 Assume that \( C \) is as in Definition 2.1. Assume furthermore that \( F \) is locally free of rank \( r \) in some neighborhood of the section \( P \) and that the restriction of \( F \) to \( P \) is free. Then \( \hat{F} \) is a free \( \hat{O}_C \)-module if and only if, for all \( n \in \mathbb{N} \), the maps

\[
H^0(F/I^n \otimes F) \to H^0(F/I \otimes F)
\]

are surjective. \( \square \)

2.3 Definition of geometric data

To end this chapter, let us precisely define the geometric objects which we want to relate to Schur pairs. Let \( S \) be a locally noetherian scheme.

Definition 2.9 By a geometric datum of rank \( r \) and index \( F \) over \( S \), we mean a tuple

\[
(C, \pi, S, P, \rho, \Phi)
\]

such that

1. \( C \) is a scheme.
2. \( \pi : C \to S \) is a locally projective morphism.
3. \( P \subset C \) is a section of \( \pi \) such that
   - \( P \) is a relatively ample Cartier divisor in \( C \).
   - For the sheaf \( \mathcal{I} := \mathcal{I}_P \) defining \( P \) in \( C \), \( \mathcal{I}/\mathcal{I}^2 \) is a trivial line bundle on \( P \).
   - Let \( \hat{O}_C \) denote the formal completion of \( \mathcal{O}_S \) with respect to the ideal \( \mathcal{I} \). Then \( \hat{O}_C \) is isomorphic to \( \mathcal{O}_P[[z]] \) as a formal \( \mathcal{O}_P \)-algebra.
   - \( \bigcap_{n \geq 0} \pi_* \mathcal{O}_C(-nP) = (0). \)
4. \( \rho : \hat{O}_C \xrightarrow{\sim} \mathcal{O}_P[[z]] \) is an isomorphism of formal \( \mathcal{O}_P \)-algebras.
5. \( F \) is a coherent sheaf of rank \( r \) on \( C \) such that
   - The formal completion \( \hat{F} \) of \( F \) along \( P \) is a free \( \hat{O}_C \)-module of rank \( r \).
   - \( \bigcap_{n \geq 0} \pi_* F(-nP) = (0). \)
   - \( F = \gamma(\pi_* F) - \gamma(R^1 \pi_* F) \in K(S). \)
6. \( \Phi : \hat{F} \xrightarrow{\sim} \hat{O}_C^{\oplus r} \) is an isomorphism of sheaves of \( \hat{O}_C \)-modules.
Definition 2.10 Two geometric data

\[ (C, \pi, S, P, \rho, \Phi) \text{ and } (C', \pi', S', P', \rho', \Phi') \]

are identified if and only if

- \( S = S' \);
- There is an isomorphism \( \beta : C \to C' \) such that
  - The diagram
    \[
    \begin{array}{ccc}
    P & \subset & C \\
    \sim \downarrow & \beta & \rightarrow \\
    S & \to & C' \supset P'
    \end{array}
    \]
    is commutative;
  - \( \rho = \hat{\beta}^*(\rho') \);
- There is an isomorphism \( \Psi : \beta^*F' \to F \) such that \( \hat{\beta}^*(\Phi') = \Phi \circ \Psi \).

In the sequel, we denote by \( \mathfrak{D}_F^r(S) \) the set of equivalence classes of geometric data of rank \( r \) and index \( F \) over \( S \).

Definition 2.11 A homomorphism of geometric data is a collection

\[ (\alpha, \beta, \Psi) : (C, \pi, S, P, \rho, \Phi) \to (C', \pi', S', P', \rho', \Phi') \]

consisting of

1. A morphism \( \alpha : S \to S' \);
2. A morphism \( \beta : C \to C' \) such that
   (a) The following diagram is commutative:
   \[
   \begin{array}{ccc}
   P & \subset & C \\
   \sim \downarrow & \beta & \rightarrow \\
   S & \to & C' \supset P'
   \end{array}
   \]
   \( \beta^*(P') = P \) as Cartier divisors;
   (b) \( \rho = \hat{\beta}^*(\rho') \);
3. A homomorphism of sheaves \( \Psi : \beta^*F' \to F \).

Two homomorphisms are identified iff they differ only by an identification isomorphism as defined in the previous definition.
This establishes the category \( \mathcal{D} \) of geometric data.

Definition 2.11 requires some justification.

**Lemma 2.12** Let \((C, \pi, S, P, \rho, \Phi)\) and \((C', \pi', S', P', \rho', \Phi')\) be geometric data and \(\alpha : S \to S'\) and \(\beta : C \to C'\) morphisms such that Conditions 2(a) and 2(b) of Definition 2.11 are satisfied. Then

- \(\hat{\beta}^* \hat{\mathcal{O}}_{C'} \cong \hat{\mathcal{O}}_C\)
- \(\hat{\beta}^* \hat{\mathcal{F}}' \cong \hat{\mathcal{F}}\),

i.e., Condition 2(c) is well-formulated and for a homomorphism of sheaves \(\Psi : \beta^* \mathcal{F}' \to \mathcal{F}\), the composition \(\pi_* ((\rho \circ \Phi) \circ \Psi \circ (\hat{\beta}^* (\rho' \circ \Phi'))^{-1})\) belongs to \(\mathcal{H}om_{\mathcal{O}_S[[z]]}(\mathcal{O}_S[[z]]^{\oplus r'}, \mathcal{O}_S[[z]]^{\oplus r})\).

**Proof** We consider the exact sequence of \(\mathcal{O}_{C'}\)-modules

\[
0 \to \mathcal{I}^n \to \mathcal{O}_{C'} \to \mathcal{O}_{C'}/\mathcal{I}^n \to 0.
\]

The support of the sheaf \(\mathcal{O}_{C'}/\mathcal{I}^n\) is contained in \(P\), and, as a sheaf on \(P\), \(\mathcal{O}_{C'}/\mathcal{I}^n\) is known to be free. Therefore we can conclude, using [Mat], ch.18, Lemma 2, that

\[
\mathcal{U}or_1^{\mathcal{O}_{C'}}(\mathcal{O}_{C'}/\mathcal{I}^n, \mathcal{O}_C) \cong \mathcal{U}or_1^{\mathcal{O}_{C'}}(\mathcal{O}_{C'}/\mathcal{I}^n, \mathcal{O}_P) = (0).
\]

Consequently, the sequence

\[
0 \to \beta^* \mathcal{I}^n \to \beta^* \mathcal{O}_{C'} \to \beta^* (\mathcal{O}_{C'}/\mathcal{I}^n) \to 0
\]

is exact, as well. Using this we start our calculation:

\[
\hat{\beta}^* \hat{\mathcal{O}}_{C'} = \lim_{n \to \infty} \beta^* (\mathcal{O}_{C'}/\mathcal{I}^n)
\]

\[
\cong \lim_{n \to \infty} \beta^* \mathcal{O}_{C'}/\beta^* \mathcal{I}^n
\]

\[
\cong \lim_{n \to \infty} \mathcal{O}_{C}/\mathcal{I}^n
\]

\[
= \hat{\mathcal{O}}_C.
\]

We proceed analogously for the sheaf \(\mathcal{F}'\). The sequence \(\mathcal{B}\) stays exact after tensoring with \(\mathcal{F}'\), since we assumed \(\mathcal{F}'\) to be locally free near \(P'\). With the same conclusions as above we finally get an exact sequence

\[
0 \to \beta^* (\mathcal{F}' \otimes \mathcal{I}^n) \to \beta^* (\mathcal{F}') \to \beta^* (\mathcal{F}' \otimes (\mathcal{O}_{C'}/\mathcal{I}^n)) \to 0
\]

and, of course, \(\mathcal{F}' \otimes (\mathcal{O}_{C'}/\mathcal{I}^n) \cong \mathcal{F}'/(\mathcal{F}' \otimes \mathcal{I}^n)\). Now we can calculate again

\[
\hat{\beta}^* \hat{\mathcal{F}} = \lim_{n \to \infty} \beta^* (\mathcal{F}'/\mathcal{F}' \otimes \mathcal{I}^n)
\]

\[
\cong \lim_{n \to \infty} \beta^* (\mathcal{F}')/\beta^* (\mathcal{F}' \otimes \mathcal{I}^n)
\]

\[
\cong \lim_{n \to \infty} \beta^* (\mathcal{F}')/(\beta^* (\mathcal{F}') \otimes \mathcal{I}^n)
\]

\[
= \beta^* (\mathcal{F}).
\]

\[\Box\]

Let us include one more definition.
Definition 2.13 We define a full subcategory $\mathcal{D}'$ of $\mathcal{D}$ as follows: The objects of this category are the equivalence classes of geometric data $(C, \pi, S, P, \rho, F, \Phi)$ such that
\[ \pi_* \mathcal{O}_C = \mathcal{O}_S. \]
This subcategory will play an important role in Chapter 5.

Remark

• Assume that $S = \text{Spec}(k)$, for some field $k$. Then the geometric datum $(C, \pi, S, P, \rho, F, \Phi)$ reduces to $(C, P, \rho, F, \Phi)$. Using the same method as in the remark on page \[\text{[3]}\] we see that this datum corresponds to a \textit{quintet} defined by M. Mulase \[\text{[M2]}\], where
\[ \rho : \hat{\mathcal{O}}_C \hookrightarrow k[[y]] \]
decomposes into
\[ \hat{\mathcal{O}}_C \sim k[[y]] \hookrightarrow k[[y]]. \]

• Now let $S$ be any scheme, $(C, \pi, S, P, \rho, F, \Phi)$ a geometric datum and $s \in S$ a closed point. Then we can restrict everything to the fibre $C_s$ of $C$ over $s$ and get a collection
\[ (C_s, \pi|_{C_s}, \{s\}, P|_{\{s\}}, \rho|_{C_s}, F|_{C_s}, \Phi|_{C_s}). \]
Assume that this is also a geometric datum. Then the restriction defines a morphism of geometric data
\[ (C_s, \pi|_{C_s}, \{s\}, P|_{\{s\}}, \rho|_{C_s}, F|_{C_s}, \Phi|_{C_s}) \rightarrow (C, \pi, S, P, \rho, F, \Phi). \]

• Another example of morphisms of geometric data is the following: Let $(C, \pi, S, P, \rho, F, \Phi)$ be a geometric datum and $\alpha : S' \rightarrow S$ a flat morphism. Then the fibre product gives rise to another geometric datum. This fact will be shown in Section \[\text{[1,2,4]}\].

3 The relative Krichever functor

After having defined both sides we want to relate, let us start with the construction of a bijective contravariant functor between the category of Schur pairs and the category of geometric data. Throughout the chapter let us assume that $S$ is a locally noetherian scheme.

3.1 Constructing Schur pairs

Assume we are given a geometric datum $(C, \pi, S, P, \rho, F, \Phi)$ of rank $r$ and index $F$. Let us start with

Lemma 3.1 For each integer $n$, the maps $\rho$ and $\Phi$ induce isomorphisms:
\[ \rho : \hat{\mathcal{O}}(n \cdot P) \sim \mathcal{O}_P[[z]] \cdot z^{-n}, \]
\[ \Phi : \hat{\mathcal{F}}(n \cdot P) \sim \mathcal{O}(n \cdot P)^{\oplus r}. \]
Proposition 3.4

The corresponding formal power series:

\[ I \cong \mathcal{O}_C(-P). \]

\[ \square \]

**Lemma 3.2** For any open affine subset \( U \) of \( S \), the natural maps

\[
\begin{align*}
H^0(\pi^{-1}(U), \mathcal{O}_C(n \cdot P)) & \to H^0(\pi^{-1}(U) \setminus P, \mathcal{O}_C) \\
H^0(\pi^{-1}(U), \mathcal{F}(n \cdot P)) & \to H^0(\pi^{-1}(U) \setminus P, \mathcal{F})
\end{align*}
\]

induce isomorphisms

\[
\begin{align*}
\lim_{n \to \infty} H^0(\pi^{-1}(U), \mathcal{O}_C(n \cdot P)) & \cong H^0(\pi^{-1}(U) \setminus P, \mathcal{O}_C) \\
\lim_{n \to \infty} H^0(\pi^{-1}(U), \mathcal{F}(n \cdot P)) & \cong H^0(\pi^{-1}(U) \setminus P, \mathcal{F})
\end{align*}
\]

**Proof** The maps in (4) are inclusions, because \( P \) is locally given by an element which is neither a zero divisor in \( \mathcal{O}_C \) nor in \( \mathcal{F} \). We know that \( P \) is ample relative to \( S \). Therefore, for sufficiently large \( n \in \mathbb{N} \), \( \mathcal{O}_{\pi^{-1}(U)}(n \cdot P) \) has a nonconstant global section. Now the assertion is a direct consequence of \([H1], \text{Lemma II.5.14} \). \( \square \)

**Definition 3.3** For a given geometric datum \( (C, \pi, S, \rho, \mathcal{F}, \Phi) \) of rank \( r \) and index \( F \), we define

\[(A, W) := \chi_{r, F}(C, \pi, S, \rho, \mathcal{F}, \Phi)\]

as follows:

\[
\begin{align*}
A(U) & := \pi_*(\rho)(H^0(\pi^{-1}(U) \setminus P, \mathcal{O}_C)) \\
& = \pi_*(\rho)(\lim_{n \to \infty} H^0(\pi^{-1}(U), \mathcal{O}_C(n \cdot P))) \\
& = \pi_*(\rho)(\lim_{n \to \infty} H^0(U, \pi_* \mathcal{O}_C(n \cdot P))) \\
& \subset \mathcal{O}_S(U)((z)),
\end{align*}
\]

\[
\begin{align*}
W(U) & := \pi_*(\rho \circ \Phi)(H^0(\pi^{-1}(U) \setminus P, \mathcal{F})) \\
& = \pi_*(\rho \circ \Phi)(\lim_{n \to \infty} H^0(\pi^{-1}(U), \mathcal{F}(n \cdot P))) \\
& = \pi_*(\rho \circ \Phi)(\lim_{n \to \infty} H^0(U, \pi_* \mathcal{F}(n \cdot P))) \\
& \subset \mathcal{O}_S(U)((z))^{\oplus r}.
\end{align*}
\]

**Remark** Assume that \( S = \text{Spec}(k) \) for some field \( k \). Then a geometric datum \( (C, \pi, S, \rho, \mathcal{F}, \Phi) \) gives us a quintet \( (C, P, \rho, \mathcal{F}, \Phi) \) as defined in \([M2] \), and \((A, W)\) is the corresponding Schur pair defined there.

In analogy to the case of a curve over a field we now want to identify the (relative) cohomology of \( \mathcal{O}_C \) and \( \mathcal{F} \) via the sheaves \( A \) and \( W \).

At first, we will see that the pole order of a local section along \( P \) is exactly the order of the corresponding formal power series:

**Proposition 3.4** For all integers \( n \in \mathbb{Z} \):

\[
\begin{align*}
A \cap \mathcal{O}_S[[z]] \cdot z^{-n} & = \pi_*(\rho)(\mathcal{O}_C(n \cdot P)), \\
W \cap \mathcal{O}_S[[z]] \cdot z^{-n} & = \pi_*(\rho \circ \Phi)(\mathcal{F}(n \cdot P)).
\end{align*}
\]
Proof  This is obviously a local property. Therefore let $S = Spec(R)$ be affine, $R$ a local ring, $P \subset V = Spec(B) \subset C$, $B$ another local ring, $I := I_P(V) = b \cdot B$, $b$ a non-zero divisor in $B$, and $M := H^0(V, \mathcal{F})$. The composition of the maps

$$H^0(C, \mathcal{F}(nP)) \to H^0(V, \mathcal{F}(nP)) \to H^0(\mathcal{F}(nP))$$

is assumed to be injective. Therefore the first map must be injective. The second one is injective, a priori, since $B$ is a local ring. So we only have to show that

$$M_b \cap \widehat{M} = M.$$ 

Of course, we are done if we can show:

$$M \cap b\widehat{M} = bM,$$

i.e.,

$$M \cap I\widehat{M} = IM.$$

But this clear: Take $m \in M$ and consider its image in $\widehat{M}$. It can be identified with the sequence $(m \mod I^nM)_{n \in \mathbb{N}}$. This sequence belongs to $I\widehat{M}$ if and only if, for all $n$, there are elements $i_n \in IM$ such that $m - i_n \in I^nM$. This implies $m \in IM$ immediately.

Of course, the statement concerning $A$ is proved in the same way. $\square$

Now we want to prove:

**Proposition 3.5** There are isomorphisms of sheaves of $\mathcal{O}_S$ - modules

$$\frac{\mathcal{O}_S((z))}{\mathcal{A} \cdot \mathcal{O}_S[[z]]} \cong R^1\pi_*\mathcal{O}_C$$

and

$$\frac{\mathcal{O}_S((z))^{\omega}}{\mathcal{W} \cdot \mathcal{O}_S[[z]]^{\omega}} \cong R^1\pi_*\mathcal{F}.$$

For the proof we need the following lemma.

**Lemma 3.6** Assume that $U = Spec(R)$ is an open affine subset of $S$, and $V = Spec(B) \subset \pi^{-1}(U)$ is an open affine set containing $P \cap \pi^{-1}(U)$ such that, on $V$, the divisor $P$ is given by a single element $b \in B$ and the restriction of $\mathcal{F}$ to $V$ is free. Then the natural maps

$$H^0(V, \mathcal{O}_C) \to H^0(V, \widehat{\mathcal{O}_C}),$$

$$H^0(V, \mathcal{F}) \to H^0(V, \widehat{\mathcal{F}})$$

induce isomorphisms of $R$ - modules:

$$\frac{H^0(V \setminus P, \mathcal{O}_C)}{H^0(V, \mathcal{O}_C)} \sim \frac{H^0(V \setminus P, \widehat{\mathcal{O}_C})}{H^0(V, \widehat{\mathcal{O}_C})}$$

and

$$\frac{H^0(V \setminus P, \mathcal{F})}{H^0(V, \mathcal{F})} \sim \frac{H^0(V \setminus P, \widehat{\mathcal{F}})}{H^0(V, \widehat{\mathcal{F}})}.$$
Proof  Let us start with the investigation of the structure sheaf. The map

\[ B \to \hat{B} \]

must be injective, since all elements of \( B \) are locally given by quotients of elements of \( \bigoplus_{n \geq 0} H^0(\pi^{-1}(U), \mathcal{O}_C(nP)) \). Let us fix a natural number \( n \). We obtain a diagram of embeddings:

\[
\begin{align*}
B & = H^0(V, \mathcal{O}_C) & \to & H^0(V, \hat{\mathcal{O}}_C) \quad = \quad \hat{B} \\
\frac{1}{b^n}B & = H^0(V, \mathcal{O}_C(n \cdot P)) & \to & H^0(V, \hat{\mathcal{O}}_C(n \cdot P)) \quad = \quad \frac{1}{b^n} \hat{B}.
\end{align*}
\] (5)

Now the proof of Proposition 3.4 implies that

\[ \hat{B} \cap \frac{1}{b^n}B = B. \]

From this we obtain a natural inclusion:

\[ \frac{1}{b^n}B \hookrightarrow \frac{1}{b^n} \hat{B}. \]

Taking the limit over \( n \), we get a monomorphism of \( R \)-modules:

\[
\frac{B_b}{B} = \frac{H^0(V \setminus P, \mathcal{O}_C)}{H^0(V, \mathcal{O}_C)} \hookrightarrow \lim_{n \to \infty} \frac{H^0(V, \mathcal{O}_C(n \cdot P))}{H^0(V, \hat{\mathcal{O}}_C)} = \frac{(\hat{B})_b}{\hat{B}}.
\]

We claim that this is, in fact, an isomorphism. Of course, we have \( R[b] \subseteq B \). So we obtain a diagram of inclusions:

\[
\begin{align*}
R[b] & \to B \\
\downarrow & \\
R(b) & \to B_b,
\end{align*}
\]

where \( R(b) \cap B = R[b] \), since \( b \) is a non-zero divisor in \( B \). Therefore we end up with a chain of inclusions:

\[
\frac{R(b)}{R[b]} \hookrightarrow \frac{H^0(V \setminus P, \mathcal{O}_C)}{H^0(V, \mathcal{O}_C)} \hookrightarrow \frac{H^0(V \setminus P, \hat{\mathcal{O}}_C)}{H^0(V, \hat{\mathcal{O}}_C)} = \frac{R((b))}{R[b]}.
\]

But the first and the last term are canonically isomorphic \( R \)-modules. Therefore all the monomorphisms appearing above are really isomorphisms.

Now we turn our attention to \( \mathcal{F} \). The isomorphism \( \mathcal{F}|_V \cong \mathcal{O}_V^{\oplus r} \) extends to an isomorphism \( \hat{\mathcal{F}} \cong \hat{\mathcal{O}}_V^{\oplus r} \), which also respects the localization by \( b \). So we get the claim concerning \( \mathcal{F} \) by applying the according statement on \( \mathcal{O}_C \). \( \Box \)

Let us proceed to the

Proof of the Proposition 3.5  As the isomorphism constructed in Lemma 3.6 is natural, it
is compatible with intersections of affine sets. Now, once again, it is sufficient to consider an open affine set $U$ of $S$, as in Lemma [3.4]. Then $\pi^{-1}(U)$ is covered by the affine sets $V$ and $\pi^{-1}(U) \setminus P$. Since $\pi|\pi^{-1}(U)$ is a separated morphism over an affine scheme, we can apply [H1], Thm. III.4.5., and get, with the aid of Lemmas [3.2] and [3.6],

$$R^1\pi_*\mathcal{O}_C(U) = H^1(\pi^{-1}(U), \mathcal{O}_C)$$

$$= \frac{H^0(V \setminus P, \mathcal{O}_C)}{H^0(V, \mathcal{O}_C) + H^0(\pi^{-1}(U) \setminus P, \mathcal{O}_C)}$$

$$= \frac{H^0(V \setminus P, \mathcal{O}_C)}{H^0(V, \mathcal{O}_C) + H^0(\pi^{-1}(U) \setminus P, \mathcal{O}_C)}$$

$$\cong \frac{\mathcal{O}_S((z))(U)}{\mathcal{O}_S[[z]](U) + \mathcal{A}(U)}.$$ 

The proof of the statement concerning $\mathcal{F}$ can be given in the same way. □

**Corollary 3.7** If $(C, \pi, S, P, \rho, \mathcal{F}, \Phi)$ is a geometric datum of rank $r$ and index $F$, then $(\mathcal{A}, \mathcal{W}) = \chi_{r,F}(C, \pi, S, P, \rho, \mathcal{F}, \Phi)$ is a Schur pair of rank $r$ and index $F$ over $S$.

**Proof** By construction, $\mathcal{A}$ is a quasicoherent sheaf of $\mathcal{O}_S$ - subalgebras of $\mathcal{O}_S((z))$, $\mathcal{W}$ is a quasicoherent sheaf of $\mathcal{O}_S$ - modules, and $\mathcal{A} \cdot \mathcal{W} \subseteq \mathcal{W}$. The fact that $\mathcal{A}$ and $\mathcal{W}$ are elements of the infinite Grassmannians of rank 1 and $r$, respectively, follows from Propositions [3.4] and [3.5], and the fact that $R^i\pi_*\mathcal{O}_C$ and $R^i\pi_*\mathcal{F}$ are coherent sheaves for locally projective morphisms $\pi$ and for all $i$ (cf. [H1], Thm. III.8.8). □

**Corollary 3.8** If $(C, \pi, S, P, \rho, \mathcal{F}, \Phi)$ and $(C', \pi', S', P', \rho', \mathcal{F}', \Phi')$ are identified as geometric data, then we obtain identical corresponding Schur pairs $(\mathcal{A}, \mathcal{W})$ and $(\mathcal{A}', \mathcal{W}')$.

**Proof** This follows from an easy calculation. We use the isomorphisms $\beta$ and $\Psi$ and get:

$$\mathcal{A} := \pi_*(\rho)(\lim_{n \to \infty} \pi_*\mathcal{O}_C(n \cdot P))$$

$$= \pi_*(\beta^* (\rho'))(\lim_{n \to \infty} \pi_* \beta^*\mathcal{O}_{C'}(n \cdot P'))$$

$$= \pi_*(\rho')(\lim_{n \to \infty} \pi_* \mathcal{O}_{C'}(n \cdot P'))$$

$$= \mathcal{A}' = \pi_*(\rho \circ \Phi')(\lim_{n \to \infty} \pi_* \beta^*\mathcal{F}'(n \cdot P'))$$

Analogously:

$$\mathcal{W}' := \pi'_*(\rho' \circ \Phi') (\lim_{n \to \infty} \pi'_* \mathcal{F}'(n \cdot P'))$$

$$= \pi'_*(\rho' \circ \Phi')(\lim_{n \to \infty} \pi'_* \beta^*\mathcal{F}'(n \cdot P'))$$

$$= \pi'_*(\rho' \circ \Phi')(\lim_{n \to \infty} \pi_* (\beta^*\mathcal{F}'(n \cdot P')))$$

$$= \pi'_*(\rho\circ \Phi)(\lim_{n \to \infty} \pi_* (\mathcal{F}(n \cdot P)))$$

$$= \pi'_*(\rho \circ \Phi)(\lim_{n \to \infty} \pi_*(\mathcal{F}(n \cdot P)))$$

$$= \mathcal{W}.$$
Proposition 3.9 Homomorphisms of geometric data induce homomorphisms of the corresponding Schur pairs.

Proof Let \((\alpha, \beta, \Psi) : (C, \pi, S, P, \rho, \Phi) \to (C', \pi', S', P', \rho', \Phi')\) be a homomorphism of geometric data. Let \((A, W)\) and \((A', W')\) be the Schur pairs associated to the given geometric data. We want to construct a homomorphism of Schur pairs \((\alpha, \xi) : (A', W') \to (A, W)\).

Of course, the morphism \(\alpha : S \to S'\) is taken directly from \((\alpha, \beta, \Psi)\), whereas \(\xi\) is defined as 
\[
\pi_*((\rho \circ \Phi) \circ \hat{\Psi} \circ (\beta^*(\rho' \circ \Phi'))^{-1}).
\]

Now we apply the properties of the given morphism \(\beta : C \to C'\) and derive:
\[
\alpha^{(*)} A' = \alpha^{(*)}(\pi'_*(\rho' \circ \Phi')((\lim_{n \to \infty} \pi'_* \mathcal{O}_{C'}(n \cdot P')))) \subseteq \pi_*(\beta^*(\rho' \circ \Phi'))((\lim_{n \to \infty} \pi_* \mathcal{O}_{C}(n \cdot P))) = A.
\]

As for \(W\) and \(W'\), we obtain:
\[
\xi(\alpha^{(*)} W') = \xi(\alpha^{(*)}(\pi'_*(\rho' \circ \Phi'))((\lim_{n \to \infty} \pi'_* \mathcal{F}(n \cdot P')))) \subseteq (\xi \circ \pi_*(\beta^*(\rho' \circ \Phi')))((\lim_{n \to \infty} \pi_* \mathcal{F}(n \cdot P'))) = \pi_*(\rho \circ \Phi \circ \hat{\Psi})((\lim_{n \to \infty} \pi_* \mathcal{F}(n \cdot P'))) \subseteq \pi_*(\rho \circ \Phi)((\lim_{n \to \infty} \pi_* \mathcal{F}(n \cdot P'))) = W.
\]

So we really have constructed a homomorphism of Schur pairs. □

Corollary 3.10 \(\chi\) is a contravariant functor from the category of geometric data to the category of Schur pairs. □

Definition 3.11 The functor \(\chi\) is called the Krichever functor.

3.2 Constructing geometric data

Now assume that we are given a Schur pair \((A, W)\) of rank \(r\) and index \(F\) over the scheme \(S\).

We start with some general observations.

Lemma 3.12 1. For all \(n \in \mathbb{Z}\), \(A^{(n)}\) and \(W^{(n)}\) are coherent sheaves. In particular, \(A^{(0)}\) is a coherent sheaf of \(\mathcal{O}_S\)-algebras.

2. If \(U = \text{Spec}(R)\) is an open affine subset of \(S\), then there is an integer \(M \in \mathbb{Z}\) (possibly depending on \(U\)) such that
\[
A^{(-M)}(U) = W^{(-M)}(U) = (0).
\]

3. All local sections of \(A^{(-1)}\) are nilpotent.
Proof

1. Let $U = \text{Spec}(R)$ be an affine open subset of $S$. We have to show that $\mathcal{A}^{(n)}(U)$ and $\mathcal{W}^{(n)}(U)$ are finitely generated. $\mathcal{A}^{(0)}(U)$ and $\mathcal{W}^{(0)}(U)$ are finitely generated by the definition of Schur pairs. Since $R$ is a noetherian ring, this immediately proves the statement for all $n \leq 0$. Now assume that $n > 0$. Then $\mathcal{A}^{(n)}(U)/\mathcal{A}^{(0)}(U)$ is isomorphic to a submodule of $R[[z]]/z^{-n}/R[[z]]$, hence finitely generated. But this already implies that $\mathcal{A}^{(n)}(U)$ itself is finitely generated. The same method of proof may be used for $\mathcal{W}$.

2. Without loss of generality we prove the statement for $\mathcal{A}$. $\mathcal{A}^{(0)}$ is generated by finitely many elements $f_1, \ldots, f_m$. Let us write $f_i = \sum_{j \geq 0} \lambda_{i,j} z^j$.

The statement $\mathcal{A}^{(-M)}(U) \neq (0)$ is equivalent to: There are elements $\mu_1, \ldots, \mu_m \in R$ such that $\sum_{i=1}^m \mu_i \lambda_{i,j} = 0$ for $j = 0, \ldots, M - 1$, but $\sum_{i=1}^m \mu_i f_i \neq 0$.

We denote by $\mathcal{N}_l$ the submodule of $R^m$ generated by the vectors $(\lambda_0, \ldots, (\lambda, j))$. These modules form an ascending chain of submodules of $R^m$:

$$\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \ldots \subseteq R^m.$$  

Since $R^m$ is a noetherian module, there is an integer $T$ such that $\mathcal{N}_l = \mathcal{N}_T$, for all $l \geq T$. We claim that $\mathcal{A}^{(-T-1)}(U) = (0)$. If this were false, we could find elements $\mu_1, \ldots, \mu_m \in R$ such that $\sum_{i=1}^m \mu_i \lambda_{i,j} = 0$, for $j = 0, \ldots, T$. But by the definition of $\mathcal{N}_T$ this already implies $\sum_{i=1}^m \mu_i f_i = 0$. So we are done.

3. The last part of the lemma is easy. Assume that for some $U$, $\mathcal{A}^{(-1)}(U)$ would contain an element $f$ which is not nilpotent. Then, for all $n \in \mathbb{N}$,

$$0 \neq f^n \in \mathcal{A}^{(-n)}(U),$$

and this is a contradiction.

\[\square\]

Lemma 3.13 Let $U = \text{Spec}(R)$ be an open affine subset of $S$. Then $\mathcal{A}(U)$ is a finitely generated $R$-algebra of relative dimension 1.

Proof By assumption, $\frac{R((z))}{R[[z]] + \mathcal{A}(U)}$ is a finitely generated $R$-module. So we can choose finitely many elements $b_1, \ldots, b_m \in R((z))$ such that $[b_1], \ldots, [b_m]$ are generators of this module. Denote by $N$ the maximum of the orders of the $b_j$'s. We may assume that $N \geq 2$. Now it is straightforward that:

$$R((z)) = R[[z]] \cdot z^{-N} + \mathcal{A}(U).$$
Therefore, for all \( n > N \), there is a monic element of order \( n \) in \( \mathcal{A}(U) \). Let us choose monic elements \( a, b \in \mathcal{A}(U) \) of order \( 2N + 2 \) and \( 2N + 1 \), respectively. We claim that

\[
R((z)) = R[[z]] \cdot z^{-(2N+1)(2N+2)} + R[a, b].
\]  
(6)

In order to prove this, we choose an integer \( n \leq a \)

\[
A \subset R \{ n \geq N \text{ as an R} \text{u}\}
\]

for \( n \in \mathbb{N} \) written as a linear combination of the \( N \) is greater than \( \text{Lemma 3.14} \)

\[
\text{Obviously, } b \text{ does not lie in } R[a]. \text{ Let } \{u_1, \ldots, u_q\} \text{ be a set of generators of } R[a, b]^{((2N+1)(2N+2)-1)} \text{ as an } R- \text{module, and put }
\]

\[
v_m := a^{m+l} \cdot b^{2N+2-l}
\]

for \( n \geq (N + 1)(N + 2) \) and \( n = (2N + 1)(2N + 2) + m \cdot (2N + 2) + l \) (see above). Then the set \( \{u_1, \ldots, u_q\} \cup \{v_n\}_{n \geq (2N+1)(2N+2)} \) generates the \( R- \text{module } R[a, b] \). For all \( M, a^M \) can be written as a linear combination of the \( n \)'s and \( v_j \)'s. Since none of the \( v_j \)'s is a power of \( a \), and since there are only finitely many \( n \)'s, the representing linear combination for a sufficiently high power of \( a \) is exactly the required polynomial relation between \( a \) and \( b \). This completes the proof. □

**Lemma 3.14** Let \( U = \text{Spec}(R) \) be an open affine subset of \( S \) and let \( N \in \mathbb{N} \) be such that \( R((z)) = R[[z]] \cdot z^{-N} + \mathcal{A}(U) \) (cf. Lemma 3.13). Then, for all \( n, m \geq 2N + 1 \):

\[
\mathcal{A}(U)^{(m)} \cdot \mathcal{A}(U)^{(n)} = \mathcal{A}(U)^{(m+n)}.
\]

**Proof** Of course, \( \mathcal{A}(U)^{(m)} \cdot \mathcal{A}(U)^{(n)} \subseteq \mathcal{A}(U)^{(m+n)} \). To see the other inclusion it is sufficient to show that \( \mathcal{A}(U)^{(m)} \cdot \mathcal{A}(U)^{(n)} \) contains monic elements of the orders \( n + i, \ldots, m + n \).

First take \( 1 \leq i \leq n - N \). Then we can split \( m + i = (m - N) + (N + i) \). Since \( m - N \) is greater than \( N \), by assumption, \( \mathcal{A}(U)^{(m)} \) contains a monic element of order \( m - N \). On the other hand, \( N + 1 \leq N + i \leq n \). Therefore, \( \mathcal{A}(U)^{(n)} \) contains a monic element of order \( N + i \).

As a second case consider now the terms \( m + i \) for \( n - N < i \leq n \). In this case, \( i \) is greater than \( N \), so \( \mathcal{A}(U)^{(n)} \) contains a monic element of order \( i \) and, of course, \( \mathcal{A}(U)^{(m)} \) contains a monic element of order \( m \). This completes the proof. □

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Now we aim at defining geometric objects from a given Schur pair. First, we define a sheaf of graded $O_S$-algebras as follows:

$$grd(A) := O_S \oplus \bigoplus_{n \geq 1} A(n),$$

i.e., $grd(A)_0 = O_S$ and $grd(A)_n = A(n)$, for $n \geq 1$. Now, we define a scheme $C$ by

$$C := \text{Proj}(grd(A)).$$

This scheme comes equipped with a projection morphism $\pi$ to $S$, and Lemma 3.13 says that $C$ is a curve over $S$.

**Remark**

- The morphism $\pi$ factors over the scheme $\text{Spec}(A(0))$. By Lemma 3.12, this scheme is finite over $S$, and $\text{Spec}(A(0))_{\text{red}} = S_{\text{red}}$.

- It is well-known that, for any $m \in \mathbb{N}$, the scheme $C$ is naturally isomorphic to

$$C^{(m)} := \text{Proj}(O_S \oplus \bigoplus_{n \geq 1} A^{(mn)}).$$

On the other hand, Lemma 3.14 implies that for any affine open subset $U \subseteq S$ there is a number $m$ such that $O_S(U) \oplus \bigoplus_{n \geq 1} A^{(mn)}(U)$ is generated by $A^{(m)}(U)$ as an $O_S(U)$-algebra. Moreover, we know that, for all $m$, $A^{(m)}$ is coherent (see Lemma 3.12). Then we obtain from the general theory developed in [H1], II.7., that $C$ is locally projective over $S$.

- The localization of $grd(A)$ by the section $1 \in grd(A)_1$ can be identified with $A$, thus $C$ contains the (relatively) affine subset $\text{Spec}(A)$.

Analogously, we define

$$grd(W) := \bigoplus_{n \in \mathbb{Z}} W^{(n)}.$$

This gives us a locally finitely generated $grd(A)$-module. The sheaf $\mathcal{F}$ on $C$ is defined to be

$$\mathcal{F} := (grd(W))^{-}.$$

We would like to see that the curve $C$ just constructed is exactly a curve of the type we started with.

**Theorem 3.15** There is a section $P \subset C$ of $\pi$ such that $C \setminus P$ is precisely $\text{Spec}(A)$. $P$ is a relatively ample Cartier divisor, its conormal sheaf is free of rank 1 on $P$, and $\hat{O}_C$ is isomorphic to $O_P[[z]]$. Finally, $\bigcap_{n \geq 0} \pi_* O_C(-nP) = (0)$.  

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Proof Again, we restrict everything to an affine open subset $U = Spec(R)$ of $S$. Let $a$ and $b$ be the elements we constructed in Lemma 3.13. We can view $a$ as an element of $(\text{grd}(A))(U)_{2N+2}$. Let us localize $\text{grd}(A)$ by $a$:

$$B := (\text{grd}(A))(U)_{(a)} = \left\{ \frac{g}{n} / n \in \mathbb{N}, g \in (\text{grd}(A))(U)_{(2N+2) \cdot n} \right\} = \left\{ \frac{g}{n} / n \in \mathbb{N}, g \in A(U)_{(2N+2) \cdot n} \right\} \subseteq R[[z]].$$

Especially, $y := \frac{b}{a}$ gives us a monic element of $B$ of order -1. But of course, for this element $y$ we obtain:

$$R[[y]] \cong R[[z]],$$

since all elements of order zero with invertible leading term are invertible in $R[[z]]$. Thus, we have to consider the situation:

$$R[y] \subseteq B \subseteq R[[y]] = R[[z]]. \quad (8)$$

Let $I := B \cap R[[z]] \cdot z$. Obviously, this is an ideal of $B$. Let $P$ be the closed subscheme of $Spec(B)$ defined by $I$. One easily sees from (8) that $B/I$ is naturally isomorphic to $R$. Therefore, $P$ is a section of the projection morphism $\pi$. Observe that the definition of $P$ does not depend on the choice of $a$ and $b$. For the ideal sheaf $\mathcal{I}$ of $P$, we get immediately:

$$\mathcal{I}/\mathcal{I}^2 = [y] \cdot (\mathcal{O}_{C}/\mathcal{I}) \cong \mathcal{O}_P \cdot z,$$

i.e., $\mathcal{I}/\mathcal{I}^2$ is free of rank 1.

Now let us prove that $C \setminus P = Spec(A)$. We do this again on the affine open subset $U = Spec(R)$ of $S$. In a first step, we restrict our consideration to the open set $D(a) = Spec(B)$. $P$ is contained in this set, and $D(a)$ and $Spec(A(U))$ cover $\pi^{-1}(U)$. Assume, on the contrary, that there is a graded prime ideal $p \subset \text{grd}(A(U))$ such that $1 \in p$ and $a \in p_{2N+2}$. By the choice of $a$, this already implies that $p_n = \text{grd}(A(U))_n$ for all sufficiently large $n$ (cf. Lemma 3.14). This is a contradiction.

On $D(a) = Spec(B)$, $Spec(A(U))$ is given as $D(\frac{1}{a})$. Now the second step is the following: Let $p$ be a prime ideal of $B$ containing $\frac{1}{a}$. We want to show that $p$ belongs to $P$. This holds if and only if $p$ contains the ideal

$$\left\{ \frac{g}{a^n} / n \in \mathbb{N}, \text{ord}(g) < \text{ord}(a)n \right\}.$$ 

Assume that $\text{ord}(g) < \text{ord}(a)n$ and set $m := n \cdot \text{ord}(a) - \text{ord}(g)$. Then:

$$\frac{g^{\text{ord}(a)}}{a^n} = \frac{g^{\text{ord}(a)}}{a^{\text{ord}(g)+m}} \cdot \frac{1}{a^m} \in p.$$

But $p$ was assumed to be prime, hence $\frac{g}{a^m} \in p$. So we see that $\frac{1}{a} \cdot B$ defines the same closed subset of $D(a)$ as $I$.

As for the last step, we have to prove that $D(\frac{1}{a})$ does not intersect $P$. Take a prime ideal $p \in P$. It is sufficient to show that

$$B_{1/a} \cdot p = B_{1/a} = \mathcal{A}_a.$$
But this is obvious.

Our next aim is to show that the completion of $\mathcal{O}_C$ along $P$ is isomorphic to $\mathcal{O}_P[[z]]$. To do this, we use the inclusions (8). By [Mat], Thm. 8.1, we only need to prove that the $(y)$-adic topology on $R[[y]]$ induces the $I$-adic topology on $B$, and this one induces the $(y)$-adic topology on $R[y]$. Note that once we have shown the first fact, the second one follows immediately.

We saw that $I$ and $1/a$ define the same closed subset of $\text{Spec}(B)$. Therefore, both ideals define the same topology on $B$. Since, obviously, $(1/a B)^n \subseteq R[[y]] \cdot y^{n(2N+2)} \cap B,$ there only remains to show that for each $k \in \mathbb{N}$ there is an integer $N(k)$ such that $R[[y]] \cdot y^{N(k)} \cap B \subseteq (1/a B)^k.$

We claim that this is true for $N(k) = k \cdot (2N + 2) = k \cdot (\text{ord}(a))$.

From the definition of $B$ we get:

$$R[[y]] \cdot y^{k(\text{ord}(a))} \cap B = \{ \frac{g}{a^\alpha} / g \in A(U), \text{ord}(g) \leq \text{ord}(a)(\alpha - k) \}.$$ 

Let us consider such an element $\frac{g}{a^\alpha} \in R[[y]] \cdot y^{k(\text{ord}(a))} \cap B$. The inequality $\text{ord}(g) \leq \text{ord}(a)(\alpha - k)$ implies $\text{ord}(g \cdot a^k) \leq \text{ord}(a)\alpha$. Therefore, $\frac{g}{a^\alpha}$ is an element of $B$, i.e., $\frac{g}{a^\alpha} \in (1/a B)^k$.

So we have shown that the $I$-adic completion of $B$ is isomorphic to $R[[y]] = R[[z]]$. This isomorphism obviously does not depend on the choice of $a$ and $b$. Therefore, it extends to an isomorphism

$$\rho : \hat{\mathcal{O}}_C \xrightarrow{\sim} \mathcal{O}_P[[z]].$$

Furthermore, it is now an easy consequence of [Mat], Thm. 7.5., that the ideal $I$ is locally free of rank 1. In fact, along $P$, this ideal is generated by the element $y$. This implies that $P$ is a Cartier divisor.

The relative ampleness of $P$ is an easy consequence of the fact that for each open affine subset $U$ of $S$, $\pi^{-1}(U) \setminus P = \text{Spec}(A(U))$ is affine.

Finally, one easily sees that $\pi_* (\mathcal{O}_C(nP))$ can be identified with $A^{(n)}$. So, the fact that $\bigcap_{n \geq 0} \pi_*\mathcal{O}_C(-nP) = (0)$ is a consequence of Lemma 3.12. □

Analogously, one can prove:

**Theorem 3.16** \(\hat{F}\) is a free \(\hat{\mathcal{O}}_C\)-module, and the inclusion of \(\mathcal{W}\) in \(\mathcal{O}_S((z))^{\oplus r}\) induces an isomorphism of sheaves of \(\hat{\mathcal{O}}_C\)-modules:

$$\Phi : \hat{F} \xrightarrow{\sim} \hat{\mathcal{O}}_C^{\oplus r}.$$ 

The intersection $\bigcap_{n \geq 0} \pi_* F(-nP)$ vanishes. Furthermore, $F = \gamma(\pi_* F) - \gamma(R^1 \pi_* F)$.

The proofs of the first statements are pure analogies to Theorem 3.15. The very last assertion is a consequence of Theorem 3.5. □
Definition 3.17  For a given Schur pair \((A, W)\) of rank \(r\) and index \(F\), we define

\[
\eta_{r,F}(A, W) := (C, \pi, S, P, \rho, F, \Phi)
\]

with the objects described above. This defines a map

\[
\eta_{r,F} : \mathcal{S}_F(S) \to \mathcal{D}_F(S).
\]

Now we are ready to prove the converse of Theorem 3.9.

Theorem 3.18  Homomorphisms of Schur pairs induce homomorphisms of the corresponding geometric data.

Proof  Let \((\alpha, \xi) : (A', W') \to (A, W)\) be a homomorphism of Schur pairs. We want to construct a homomorphism \((\alpha, \beta, \Psi) : (C, \pi, S, P, \rho, F, \Phi) \to (C', \pi', S', P', \rho', F', \Phi')\) of the associated geometric data. We proceed in two steps:

First we assume that \((A, W) = (\alpha^* A', \alpha^* W')\). This gives us a morphism \(\beta : C = \text{Proj}(\text{grd}(\alpha^* A')) \to C'\) which makes the following diagram commute

\[
\begin{array}{ccc}
C & \xrightarrow{\beta} & C' \\
\pi \downarrow & & \downarrow \pi' \\
S & \xrightarrow{\alpha} & S'.
\end{array}
\]

Note that, in general, \(C\) is different from the fibre product \(C' \times_{S'} S\) (cf. Section 4.2.4).

Since \(C \setminus P = \text{Spec}(\alpha^* A')\) maps to \(\text{Spec}(A') = C' \setminus P'\), we get \(\beta^{-1}(P') = P\). Then, from the construction of \(P\) in Theorem 3.15, it is also clear that \(\beta^* P' = P\).

Recall that the local trivializations \(\rho\) and \(\rho'\) have been defined by the inclusions \(\alpha^* A' \subseteq \mathcal{O}_S((z))\) and \(A' \subseteq \mathcal{O}_S((z))\). So it is obvious that \(\rho = \beta^*(\rho')\).

Finally, we consider the sheaves \(\mathcal{F}\) and \(\mathcal{F'}\). As \(\mathcal{F} = (\text{grd}(\alpha^* W'))^\sim\) and \(\mathcal{F'} = (\text{grd}(W'))^\sim\), there is a natural map \(\beta^* \mathcal{F} \to \mathcal{F}\) which is an isomorphism near \(P\).

Now let us return to the general case. As \((A', W')\) is a Schur pair, the induced object \((\alpha^* A', \alpha^* W')\) is also a Schur pair, i.e., the given homomorphism decomposes as follows:

\[
(A', W') \xrightarrow{\alpha \times id} (\alpha^* A', \alpha^* W') \xrightarrow{id \times \xi} (A, W).
\]

So it just remains to consider the case where \(S = S', \alpha = id_S, A' \subseteq A\) and \(\xi(W') \subseteq W\).

The inclusion \(A' \subseteq A\) induces \(\text{grd}(A') \hookrightarrow \text{grd}(A)\) and, therefore, a morphism

\[
\beta : \text{Proj}(\text{grd}(A)) = C \to C' = \text{Proj}(\text{grd}(A'))
\]
which restricts to
\[ \beta : \text{Spec}(A) \to \text{Spec}(A') \]
and which, in addition, is an isomorphism near \( P \). Therefore, \( \beta \) fits into the diagram
\[ P \subset C \xrightarrow{\beta} C' \supset P' \]
\[ S = \sim \]
and \( \beta^*(P') = P \). The statement \( \rho = \tilde{\beta}^*(\rho') \) is obvious.

Now let us define the homomorphism of sheaves. We know that \( \xi(W') \subseteq W \). Since \((A, W)\) is a Schur pair, this implies:
\[ \xi(A \cdot W') = A \cdot \xi(W') \subseteq W. \]
\( \xi \) is determined by the images of the basis elements \( e_1, \ldots, e_{r'} \). By the definition of \( \xi \), either \( \text{ord}(\xi(e_j)) \leq 0 \) or \( \xi(e_j) = 0 \). Therefore, \( \xi((A \cdot W')^{(n)}) \subseteq W^{(n)} \), for all \( n \in \mathbb{Z} \), and \( \xi \) induces a homomorphism \( \xi : \text{grd}(A \cdot W') \to \text{grd}(W) \), i.e., a homomorphism of sheaves
\[ \Psi : \text{grd}(A \cdot W') \sim = \beta^* F' \to F = \text{grd}(W) \sim. \]
Obviously, for this homomorphism \( \Psi \), \( \xi \) is recovered by \( \xi = \pi_*((\rho \circ \Phi) \circ \tilde{\Psi} \circ \tilde{\beta}^*(\rho' \circ \Phi')^{-1}) \). \( \square \)

**Corollary 3.19** \( \eta \) is a contravariant functor from the category of Schur pairs to the category of geometric data. \( \square \)

**Theorem 3.20** The Krichever functor \( \chi \) and the functor \( \eta \) are equivalences of the categories \( D \) and \( S \) and inverse to each other. Under this categorical equivalence, the subcategory \( D' \) corresponds to \( S' \).

**Proof** This is an easy consequence of Theorem II.5.14 \[H1\]. \( \square \)

4 Applications

Once the correspondence between geometric data and Schur pairs is established, we are, of course, interested in seeing how this relation works practically. For example, assume that the given geometric objects have additional properties. How do these properties display in the corresponding Schur pair?

On the other hand, we had to impose some strong conditions on our family of curves and the sheaf on it (cf. Definitions 2.1 and 2.6). How substantial are these conditions? Are there still interesting and significant examples?
4.1 Translation of geometric properties

Let \((C, \pi, S, P, \rho, \mathcal{F}, \Phi)\) be a geometric datum of rank \(r\) and index \(F\), and \((\mathcal{A}, \mathcal{W})\) the associated Schur pair. A particular question is: What happens if \(C\) or \(\mathcal{F}\) is \(S\)-flat?

**Lemma 4.1** The sheaf \(\mathcal{F}\) is flat over \(S\) if and only if \(W \subset \mathcal{O}_S((z)^{\oplus r})\) is locally free. \(\pi\) is a flat morphism if and only if \(A \subset \mathcal{O}_S((z))\) is locally free.

**Proof** Flatness is a local property. So we may assume that \(S = \text{Spec}(R)\), \(R\) a noetherian ring. We know that \(\mathcal{O}_C(P)\) is ample on \(C\) relative to \(S\). Let \(N\) be so that \(\mathcal{O}_C(N \cdot P)\) is very ample relative to \(S\). We know from the proof of [11], Thm. III.9.9., that \(\mathcal{F}\) is \(S\)-flat if and only if, for sufficiently large \(n\), \(\pi_* (\mathcal{F}(nN \cdot P))\) is a locally free sheaf of \(\mathcal{O}_S\) - modules of finite rank. Remember that \(\Phi\) and \(\rho\) induce an isomorphism of \(\pi_* (\mathcal{F}(nN \cdot P))\) with \(W \cap \mathcal{O}_S[[z]]^{\oplus r} \cdot z^{-nN}\) (cf. Corollary 3.4). By assumption, \(W \cap \mathcal{O}_S[[z]]^{\oplus r}\) is coherent, hence is of finite rank. As \(S\) is assumed to be noetherian, this implies that, for all \(m\), \(W \cap \mathcal{O}_S[[z]]^{\oplus r} \cdot z^{-m}\) is also of finite rank. This implies that the \(S\)-flatness of \(\mathcal{F}\) is equivalent to the local freeness of \(W \cap \mathcal{O}_S[[z]]^{\oplus r} \cdot z^{-nN}\) for sufficiently large \(n\). But on the other side, we also know that \(\mathcal{O}_S((z))^{\oplus r}/(W + \mathcal{O}_S[[z]]^{\oplus r})\) is coherent.

Set \(W := H^0(W)\). Then \(R((z))^{\oplus r}/(W + R[[z]]^{\oplus r})\) is a finitely generated \(R\)-module, hence, for sufficiently large \(n:\)

\[ W + R[[z]]^{\oplus r} \cdot z^{-nN} = R((z))^{\oplus r}. \]

Denote by \(\{e_1, \ldots, e_r\}\) the standard basis of the \(R[[z]]\) - module \(R[[z]]^{\oplus r}\). Then, for all \(i = 1, \ldots, r\) and \(j > nN\), there are elements

\[ w_{i,j} = e_i \cdot z^{-j} + \text{terms of lower order} \in W. \]

Let \(\tilde{W}\) be the free \(R\)-submodule of \(W\) generated by these elements. Then, of course,

\[ W = \tilde{W} \oplus (W \cap R[[z]]^{\oplus r} \cdot z^{-nN}), \]

and we see that \(W\) is locally free if and only if this holds true for \(W \cap R[[z]]^{\oplus r} \cdot z^{-nN}\).

The proof of the second statement may be completed in the same way. \(\square\)

At this time, let us outline a result which follows immediately from Lemma 3.12.

**Lemma 4.2** If \((C, \pi, S, P, \rho, \mathcal{F}, \Phi)\) is a geometric datum then \(\pi_* \mathcal{O}_C(-nP)\) and \(\pi_* \mathcal{F}(-nP)\) vanish for sufficiently large \(n \in \mathbb{N}\). \(\square\)

Now we turn our attention to the stability of sheaves.

**Definition 4.3** We call \(\mathcal{F}\) strongly semistable with respect to the section \(P\) iff there is an integer \(N\) such that

\[ \pi_* \mathcal{F}(N \cdot P) = R^i \pi_* \mathcal{F}(N \cdot P) = 0. \]

Later on we will see that this notion of semistability is the most convenient one for the examination of commutative algebras of differential operators corresponding to sheaves over relative curves.

Translating the last definition in terms of Schur pairs, we get immediately:
Lemma 4.4 \( F \) is strongly semistable with respect to \( P \) if and only if
\[
W \oplus \mathcal{O}_S[[z]]^{\mathfrak{m}} \cdot z^{-N} = \mathcal{O}_S((z))^{\mathfrak{m}},
\]
for some \( N \in \mathbb{Z} \). \( \Box \)

Corollary 4.5 If \( F \) is strongly semistable with respect to \( P \) then, in particular, \( F \) is flat over \( S \). \( \Box \)

Definition 4.6 A coherent sheaf \( F \) on \( C \) is called simple if
\[
\pi_*\text{End}_{\mathcal{O}_C}(F) = \mathcal{O}_S.
\]

In the set-up of Schur pairs it is easier to handle isomorphisms than homomorphisms. That is why we are interested in the following statement.

Lemma 4.7 Let \( S \) be reduced and assume that, for each point \( s \in S \), the residue field \( k(s) \) is infinite. Then \( F \) is simple if and only if \( \pi_*\text{Aut}_{\mathcal{O}_C}(F) = \mathcal{O}_S^* \).

Proof One implication is obvious. To prove the other one, we can assume, without loss of generality, that \( S = \text{Spec}(R) \) is affine. We want to prove
\[
\text{End}_{\mathcal{O}_C}(F) = R
\]
under the assumption that \( \text{Aut}_{\mathcal{O}_C}(F) = R^* \). Obviously, \( R \) is contained in \( \text{End}_{\mathcal{O}_C}(F) \). Now assume that \( \text{End}_{\mathcal{O}_C}(F) \) contains an element \( \phi \) which does not belong to \( R \). For \( r \in R \), we consider the endomorphism \( r + \phi \) and restrict it to the fibres of \( \pi \): \( (r + \phi)_s := (r + \phi)|_{C_s} \). Since \( S \) is reduced, \( r + \phi \) is an isomorphism if and only if, for all \( s \in S \), \( (r + \phi)_s \) is an isomorphism. We define
\[
S(r) := \{ s \in S / (r + \phi)_s \text{ is an isomorphism} \}.
\]
Obviously, these are open, possibly empty, subsets of \( S \). Now we show that, for each \( s \in S \), there is an element \( r \in R \) such that \( s \in S(r) \):

Let us fix \( s \in S \). We write
\[
\mathcal{G}_r := \ker((r + \phi)_s).
\]
Then \( \mathcal{G}_r \) is a subsheaf of \( \mathcal{F}|_{C_s} \). We show that the sheaves \( \mathcal{G}_r \) and \( \mathcal{G}_{r'} \) intersect only in the zero section whenever \( r - r' \) is not contained in the prime ideal defining \( s \) in \( \text{Spec}(R) \). This can be seen locally on \( C_s \). Let \( V \) be an open affine subset of \( C_s \), and \( F := \mathcal{F}(V) \).

Assume there is an element \( f \in F \) such that \( (r + \phi)_s(f) = (r' + \phi)_s(f) = 0 \). Then \( (r-r') \cdot f = 0 \), which implies that \( f = 0 \).

As \( k(s) \) is assumed to be infinite, there are infinitely many elements \( r \in R \) so that their pairwise differences are not contained in the ideal of \( s \).

Hence we obtain an infinite chain
\[
\mathcal{G}_1 \subseteq \ldots \subseteq \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \ldots \subseteq \mathcal{F}.
\]
Since $C_s$ is a noetherian scheme and $\mathcal{F}_{C_s}$ is coherent, this chain must become stationary, i.e., there are infinitely many $r \in R$ such that $(r + \phi)_s$ is an injective homomorphism between two sheaves with the same Hilbert polynomial, hence it must be an isomorphism, and this means $s \in S(r)$ for all those $r$.

This proves that the sets $S(r)$ form a covering of $S$. Now, $(r + \phi)|S(r)$ is an isomorphism, hence corresponds to an element of $H^0(S(r), \mathcal{O}_{S_r})^*$. Therefore, $\phi|S(r) \in H^0(S(r), \mathcal{O}_{S_r})$, i.e., $\phi \in H^0(S, \mathcal{O}_S) = R$. This is a contradiction. □

Now let us return to Schur pairs. The sheaf of groups $\text{Isom}_{\mathcal{O}_S[[z]]}(\mathcal{O}_S[[z]]^\oplus)$ acts on $\mathfrak{S}_r(S)$, for every $F \in K(S)$. Using Corollary 3.8 we draw the following two conclusions:

**Corollary 4.8** Let $S$ be as in Lemma 4.7. Then $W$ corresponds to a simple sheaf if and only if

$$\text{Stab}_W \text{Isom}_{\mathcal{O}_S[[z]]}(\mathcal{O}_S[[z]]^\oplus_r) = \mathcal{O}_S^*.$$ □

**Corollary 4.9** Again let $S$ be as above. Then a sheaf $F$ belonging to a geometric datum $(C, \pi, S, P, \rho, F, \Phi)$ is simple if and only if for all equivalent geometric data $(C, \pi, S, P, \rho, F, \Phi')$:

$$\Phi' = \lambda \Phi$$

for some $\lambda \in H^0(S, \mathcal{O}_S)^*$. □

Now let us see to what determinant line bundles correspond. Assume that $S$ is noetherian, regular and separated. $W$ is a quasi-coherent subsheaf of $\mathcal{O}_S((z))^{\oplus r}$. We know that $\mathcal{O}_S((z))^{\oplus r}/(W + \mathcal{O}_S[[z]]^{\oplus r})$ is coherent. Together with the fact that the base scheme $S$ is noetherian, this implies that there is an integer $N$ satisfying

$$\mathcal{O}_S((z))^{\oplus r} = W + \mathcal{O}_S[[z]]^{\oplus r} \cdot z^{-N}. \quad (9)$$

Therefore,

$$W/W^{(N)} = (W + \mathcal{O}_S[[z]]^{\oplus r} \cdot z^{-N})/\mathcal{O}_S[[z]]^{\oplus r} \cdot z^{-N} = \mathcal{O}_S((z))^{\oplus r}/\mathcal{O}_S[[z]]^{\oplus r} \cdot z^{-N}$$

which is a trivial sheaf of $\mathcal{O}_S$-modules. Subsequently, $\text{det}(W) = \text{det}(W^{(N)})$ is a well-defined line bundle on $S$. Note that this definition does not depend on the choice of the integer $N$ occurring in the condition (9). Furthermore, additivity holds for exact sequences.

On the other hand, for the given sheaf $F$ of $\mathcal{O}_C$-modules we may consider the so-called **determinant of the cohomology** (after P. Deligne)

$$\lambda(F) := \text{det}(\pi_* F) \otimes (\text{det}(R^1 \pi_* F))^{-1}.$$ We can prove the following result.
Proposition 4.10 \( \lambda(F) \cong \det(W) \).

Proof This is an easy consequence of Proposition 3.5. Using the fact proven there, we get:
\[
\lambda(F) \cong \det(W(0)) \otimes \det(O_S((z))^{\oplus r}/(W + O_S[[z]]^{\oplus r}))^{-1}.
\]
Now we consider the exact sequences of quasicoherent sheaves on \( S \):
\[
0 \to W/W(0) \to O_S((z))^{\oplus r}/O_S[[z]]^{\oplus r} \to O_S((z))^{\oplus r}/(W + O_S[[z]]^{\oplus r}) \to 0
\]
and
\[
0 \to W(0) \to W \to W/W(0) \to 0.
\]
The statement of the lemma follows then from the additivity of \( \det \).

Remark In the case that \( F \) is flat and \( S \) is separable, \( \det(W) \) is again well-defined, even if \( S \) is not regular. Consequently this determinant generalizes the determinant of the cohomology.

4.2 Examples of geometric data

First we prove a criterion which will be highly useful for the construction of examples.

Proposition 4.11 Assume that the base scheme \( S \) satisfies: \( H^1(S, O_S) = 0 \), and that, for the section \( P \), \( \mathcal{I}_P/I_P^2 \) is free of rank 1. Then

1. \( \hat{\mathcal{O}}_C \cong O_P[[z]] \).

2. If \( F \) is a coherent sheaf of \( \mathcal{O}_C \)-modules such that
   - \( F \) is locally free in a neighborhood of \( P \),
   - \( F|_P \cong O_P^{\oplus r} \)

   then \( \hat{F} \cong \hat{\mathcal{O}}_C^{\oplus r} \).

Remark For example, the cohomological condition is fulfilled for all affine schemes \( S \).

Proof of the proposition At first, observe that the condition that \( H^1(S, O_S) \) vanishes is, of course, equivalent to \( H^1(P, O_P) = 0 \).

By Lemma 2.3 and the remark following it the first claim is equivalent to:
\[
H^0(C, \mathcal{I}/\mathcal{I}^n) \to H^0(C, \mathcal{I}/\mathcal{I}^2) \quad \forall n \in \mathbb{N}, n \geq 2.
\]
One easily sees that this is the case if and only if for all \( n \geq 2 \):
\[
H^0(C, \mathcal{I}/\mathcal{I}^{n+1}) \to H^0(C, \mathcal{I}/\mathcal{I}^n).
\]
We have the exact sequence of sheaves of $\mathcal{O}_P$-modules
\[ 0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{I}/\mathcal{I}^n \to \mathcal{I}/\mathcal{I}^{n+1} \to 0 \] (10)
which induces a long exact sequence of cohomology groups
\[ 0 \to H^0(\mathcal{I}^n/\mathcal{I}^{n+1}) \to H^0(\mathcal{I}/\mathcal{I}^n) \to H^0(\mathcal{I}/\mathcal{I}^{n+1}) \to H^1(\mathcal{I}^n/\mathcal{I}^{n+1}) \to \ldots \]
By assumption, $\mathcal{I}^n/\mathcal{I}^{n+1} = (\mathcal{I}/\mathcal{I}^2)^n \cong \mathcal{O}_P$. So our assumption on $\mathcal{S}$ implies that $H^1(\mathcal{I}^n/\mathcal{I}^{n+1}) = 0$ for all $n \in \mathbb{N}$ and we are done.

Now we come to the second part. By Lemma S the claim is equivalent to the following fact:
\[ H^0(\mathcal{F}/(\mathcal{I} \otimes \mathcal{F})) \to H^0(\mathcal{F}/(\mathcal{I} \otimes \mathcal{F})), \forall n \in \mathbb{N}. \] (11)
We consider one more exact sequence of coherent sheaves of $\mathcal{O}_C$-modules:
\[ 0 \to \mathcal{I}^n \to \mathcal{O}_C \to \mathcal{O}_C/\mathcal{I}^n \to 0 \] (12)
Since $\mathcal{F}$ is locally free in some neighborhood of $P$, and $\mathcal{O}_C/\mathcal{I}^n = 0$ outside $P$, the sequences (10) and (12) stay exact when we tensor with $\mathcal{F}$. So we get
\[ 0 \to (\mathcal{I}^{n-1}/\mathcal{I}^n) \otimes \mathcal{F} \to (\mathcal{O}_C/\mathcal{I}^n) \otimes \mathcal{F} \to (\mathcal{O}_C/\mathcal{I}^{n-1}) \otimes \mathcal{F} \to 0 \] (13)
and (12) implies: $\mathcal{O}_C/\mathcal{I}^n \otimes \mathcal{F} \cong \mathcal{O}_C/\mathcal{I}^{n-1} \otimes \mathcal{F}$. Now we write down the long exact sequence of cohomology groups induced by (13):
\[ \begin{align*}
0 & \to H^0((\mathcal{I}^{n-1}/\mathcal{I}^n) \otimes \mathcal{F}) \to H^0(\mathcal{F}/(\mathcal{I} \otimes \mathcal{F})) \to H^1((\mathcal{I}^{n-1}/\mathcal{I}^n) \otimes \mathcal{F}) \to \ldots
\end{align*} \] (14)
Since the restriction of $\mathcal{F}$ to $P$ is free, and $\mathcal{I}^{n-1}/\mathcal{I}^n$ is isomorphic to $\mathcal{O}_P$, $(\mathcal{I}^{n-1}/\mathcal{I}^n) \otimes \mathcal{F}$ is a free $\mathcal{O}_P$-module. Therefore we finally get:
\[ H^1((\mathcal{I}^{n-1}/\mathcal{I}^n) \otimes \mathcal{F}) \cong H^1(P, \mathcal{O}_P)^{\oplus r} = 0, \]
which, together with the sequence (14), implies the surjectivity in (11). $\square$

Now we come to explicit examples.

### 4.2.1 Trivial families of curves

The easiest case, but which is not without interest, is the one of a trivial family of curves with some sheaf on it.

**Proposition 4.12** Let $K$ be a complete, integral curve over some field $k$ and $p \in K$ a smooth, $k$-rational point. Let $S$ be a locally noetherian $k$-scheme and set $C := K \times_{\text{Spec}(k)} S$. Denote by $\pi$ the projection from $C$ to $S$, and by $P$ the section $\{(p, s)/s \in S\}$. Then $\hat{\mathcal{O}}_C \cong \mathcal{O}_P[[z]]$.  

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Proof Let $\mathcal{J} = \mathcal{J}_p$ be the sheaf of ideals defining $p$ in $K$. Since $p$ is a smooth point, $\mathcal{J}$ is generated by one element $z$ near $p$, and we get:

- $\mathcal{J}/\mathcal{J}^2 = [z] \cdot (\mathcal{O}_K/\mathcal{J})$;
- $\hat{\mathcal{O}}_K \cong k[[z]]$.

Since $C = K \times \text{Spec}(k) S$ and $P = \{p\} \times \text{Spec}(k) S$, we obtain:

- $I_P = \mathcal{J} \otimes_k \mathcal{O}_S$, hence $I_P/I_P^2 = [z] \cdot (\mathcal{O}_C/I_P) = [z] \cdot \mathcal{O}_P$.
- $\hat{\mathcal{O}}_C = \hat{\mathcal{O}}_K \otimes_k \mathcal{O}_S$, i.e., $\hat{\mathcal{O}}_C \cong \mathcal{O}_P[[z]]$.

4.2.2 Elliptic curves

To describe a nontrivial family of integral curves which fits into our set-up, we define a family of elliptic curves over $A^2 = \mathbb{A}^2$ as follows:

$$C := \{(A, B, z_0 : z_1 : z_2) \in \mathbb{A}^2 \times \mathbb{P}^2/z_0 z_2 = z_1^3 + A z_0^2 z_1 + B z_0^3\} \quad \downarrow \pi \quad A^2$$

(15)

A section of $\pi$ can be defined by $P := \{(A, B, 0 : 0 : 1)/(A, B) \in \mathbb{A}^2\}$. One easily sees that $\pi$ is a flat, projective morphism with reduced, irreducible fibres of dimension 1 and that $C$ is reduced.

We want to study the conormal sheaf of $P$. Since $P$ does not intersect the hyperplane $\mathbb{A}^2 \times (z_2 = 0)$, we can restrict our consideration to its affine complement. Let us denote $y_i := z_i/z_2$ for $i = 0, 1$. Then $C \cap (z_2 \neq 0) \subset \mathbb{A}^2 \times \mathbb{A}^2$ is given by the equation

$$y_0 = y_1^3 + A y_0^2 y_1 + B y_0^3.$$  

(16)

Let $R$ be the affine coordinate ring of $C \cap (z_2 \neq 0)$. The ideal $I$ of $P$ in $R$ is generated by $y_0$ and $y_1$. But $y_0 \in I^2$, i.e., $I/I^2 = y_1 R/I$. This implies that $I/I^2$ is a trivial line bundle.

Note that $\hat{\mathcal{O}}_C \cong \mathcal{O}_P[[z]]$ since $H^1(\mathbb{A}^2) = 0$.

Now we want to find a suitable (formal) local parameter on $C$ along $P$. From the above calculation we know that $y_1 = z_1/z_2$ is such a local parameter, i.e., the formal completion $\hat{R}$ of $R$ with respect to $I$ equals $k[A, B][[y_1]]$. We claim that for $\alpha := \sqrt{z_0/z_1}$, $\hat{R} \cong k[A, B][[\alpha]]$. We use the equation (16) and calculate:

$$y_0 = y_1^3 + A y_0^2 y_1 + B y_0^3$$

$$y_0/y_1 = y_1^2 + A y_0^2/y_1$$

$$- y_1^2 - A y_0^2 = (B y_0^2 - 1)y_0/y_1.$$
(By_0^2 - 1) is an invertible element of \( k[A, B][[y_1]] \). Therefore

\[ z_0/z_1 = y_0/y_1 = (By_0^2 - 1)^{-1}(-y_1^2 - Ay_0^2) \in k[A, B][[y_1]] \]

is an element of order \(-2\) with leading coefficient 1. So, the square root of \( z_0/z_1 \) is a well-defined monic element \( \alpha \) of \( \hat{R} \) of order \(-1\). This implies that \( \hat{R} \cong k[A, B][[\alpha]] \).

Now we construct the corresponding subring \( A \) of \( k[A, B][[\alpha]] \). We see that \( P \) is exactly the intersection of \( C \) with the hyperplane \( (z_0 = 0) \). Therefore, the affine ring of coordinates of \( C \setminus P \) is

\[ k[A, B, z_1/z_0, z_2/z_0]/((z_2/z_0)^2 - (z_1/z_0)^3 - A(z_1/z_0)^2 - B) \]

We express the generating elements of this \( k[A, B] \) - algebra in terms of the above chosen formal parameter \( \alpha \):

\[
\begin{align*}
z_1/z_0 &= \alpha^{-2} \\
z_2/z_0 &= \alpha^{-2} \cdot y_1^{-1}.
\end{align*}
\]

\( y_1^{-1} \) is an element of \( k[A, B][[\alpha]] \). We want to find out its special form. Using the equation (16) we get

\[
1/y_1^2 = y_1/y_0 + A(y_0/y_1) + B(y_0/y_1)^2 = \alpha^{-2} + A\alpha^2 + B\alpha^4.
\]

So, finally:

\[
A = k[A, B, \alpha^{-2}, \alpha^{-2} \cdot \sqrt[2]{\alpha^{-2} + A\alpha^2 + B\alpha^4}].
\]

The term \( \alpha^{-2} \in A \) reflects the 2:1 covering

\[ C \rightarrow \mathbb{A}^2 \times \mathbb{P}^1. \]

### 4.2.3 Families of line bundles over a curve

Let \( C \) be a complete integral complex curve and \( p \in C \) a point. We choose a formal local trivialization \( \rho \) of \( C \) near \( p \) and construct the corresponding subring \( A \subset C((z)) \).

We take the Picard variety \( \text{Pic}^n(C) \) of \( C \), for some \( n \), as a base scheme. As described in Proposition 4.12, \( \rho \) extends to a local trivialization of \( C \times \text{Pic}^n(C) \) near \( \{L\} \times \text{Pic}^n(C) \) and we get for the corresponding sheaf of \( \mathcal{O}_{\text{Pic}^n(C)} \)- algebras:

\[
A = A \otimes C \mathcal{O}_{\text{Pic}^n(C)} \subset \mathcal{O}_{\text{Pic}^n(C)}((z)).
\]

Now we consider the Poincaré bundle \( \mathcal{P}_C^n \) of degree \( n \) on \( C \) (normalized with respect to the fixed point \( p \)). \( \mathcal{P}_C^n \) is a line bundle on \( C \times \text{Pic}^n(C) \) satisfying:

- \( \mathcal{P}_C^n \mid (C \times \{L\}) \cong L \) for every \( L \in \text{Pic}^n(C) \),
- \( \mathcal{P}_C^n \mid \{p\} \times \text{Pic}^n(C) \) is trivial,
- \( \mathcal{P}_C^n \) is flat over \( \text{Pic}^n(C) \).
For more details see [LB].

Let $U \subset \text{Pic}^n(C)$ be an open affine subset. We apply Proposition 4.11 and conclude that $\mathcal{P}_C^U(C \times U)$ satisfies Condition 1 of Definition 2.6. So we can construct (for some local trivialization) the corresponding sheaf $\mathcal{W}(U) \subseteq O_U((z))$ and we obtain a Schur pair $(\mathcal{A}(U), \mathcal{W}(U))$.

(Keep in mind that $\mathcal{A}$ is globally defined whereas $\mathcal{W}$ is not!)

$\mathcal{W}(U)$ defines a map from $U$ to $\mathfrak{S}_\mu(\text{Spec}(C))$, $\mu = n + 1 - g(C)$, and as the image we get $U$ as a subset of the Grassmannian.

M. Mulase proved in [M2] that every finite dimensional integral manifold of the KP-flows on some quotient of the Grassmannian $\mathfrak{S}_\mu(\text{Spec}(C))$ has the linear structure of a Jacobian of a curve. The quotient has been taken in order to eliminate different local trivializations. In particular, the differences arising from the local construction of $\mathcal{W}$ cancel out completely. So, the above result implies that the integral manifold also carries the algebraic-geometric structure of the Jacobian.

### 4.2.4 Base change

Here, we want to investigate the behaviour of geometric data under base changes. In general, the pull-back of a geometric datum over $S$ under a base change $\alpha : S' \to S$ does not give rise to another geometric datum. However, let us start with a positive example.

**Lemma 4.13** Let $(C, \pi, S, P, \rho, \mathcal{F}, \Phi)$ be a geometric datum of rank $r$ over $S$ and $\alpha : S' \to S$ a flat morphism. The fibre product construction then defines a collection $(C', \pi', S', P', \rho', \mathcal{F}', \Phi')$. We claim that this collection forms a geometric datum of rank $r$ over $S'$ and that $(\alpha, \alpha', \text{id})$ is a homomorphism of geometric data

$$(\alpha, \alpha', \text{id}) : (C', \pi', S', P', \rho', \mathcal{F}', \Phi') \to (C, \pi, S, P, \rho, \mathcal{F}, \Phi),$$

where $\alpha'$ is defined as the fibre product morphism,

$$
\begin{array}{ccc}
C' & \overset{\alpha'}{\longrightarrow} & C \\
\downarrow \pi' & & \downarrow \pi \\
S' & \overset{\alpha}{\longrightarrow} & S.
\end{array}
$$

**Proof** Let us check the properties listed in Definition 2.9. Some of them are easy to see. Of course, $C'$ is a scheme, $\pi' : C' \to S'$ is a locally projective morphism, $P'$ is a relatively ample Cartier divisor and $\mathcal{I}/\mathcal{I}^2 = \alpha'^* (\mathcal{I}/\mathcal{I}^2)$ is free of rank 1 on $P'$.

In Lemma 2.3 we saw that the condition $\mathcal{O}_C \cong \mathcal{O}_P[[z]]$ is equivalent to the fact that the section $1 \in H^0(C, \mathcal{I}/\mathcal{I}^2) = H^0(P, \mathcal{O}_P)$ lifts to a section of $\mathcal{I}/\mathcal{I}^n$ for all $n \geq 2$. Working through the diagram

$$
\begin{array}{ccc}
H^0(\mathcal{I}/\mathcal{I}^n) & \xrightarrow{\alpha^*} & H^0(\mathcal{I}/\mathcal{I}^2) \\
\downarrow & \alpha'^* & \downarrow \alpha^* \\
H^0(\mathcal{I}/\mathcal{I}^n) & \xrightarrow{\alpha'^*} & H^0(\mathcal{I}/\mathcal{I}^2)
\end{array}
$$

we obtain: $\mathcal{O}_{C'} \cong \mathcal{O}_{P'}[[z]]$. 

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Of course, $\rho$ and $\alpha$ induce an isomorphism
\[ \rho' : \hat{\mathcal{O}}_{C'} \sim \mathcal{O}_{P'}[[z]]. \]

Now we turn our attention to the sheaf $\mathcal{F}'$. It is easy to see that
\[ \mathcal{F}'/\mathcal{I}' \otimes \mathcal{F}' \sim \alpha'^* (\mathcal{F}/\mathcal{I} \otimes \mathcal{F}) \sim \mathcal{O}^\oplus_{P'} \]
and that $\mathcal{F}'$ is locally free near $P'$.

In order to prove that the completion of $\mathcal{F}'$ along $P'$ is free, we have to show that the generating sections $e_1, \ldots, e_r$ of $\mathbb{H}_0(\mathcal{O}^\oplus_{P'}) = \mathbb{H}_0(\mathcal{F}'/\mathcal{I}' \otimes \mathcal{F'})$ lift to sections of $\mathcal{F}'/\mathcal{I}^n \otimes \mathcal{F}'$ for all $n \in \mathbb{N}$. This is done as above.

Again, $\Phi$ and $\alpha$ induce an isomorphism of $\hat{\mathcal{O}}_{C'}$-modules
\[ \Phi' : \hat{\mathcal{F}}' \sim \hat{\mathcal{O}}^\oplus_{C'}. \]

Finally, note that for sufficiently large $n \in \mathbb{N}$,
\[ \pi_* \mathcal{O}_C(-nP) = 0 \text{ and } \pi_* \mathcal{F}(-nP) = 0 \]
(cf. Lemma [1.2]). Consequently, using [11], Thm.II.9.3.,
\[ \pi'_* \mathcal{O}_{C'}(-nP') = \alpha^* \pi_* \mathcal{O}_C(-nP) = 0 \text{ and } \pi'_* \mathcal{F}'(-nP') = \alpha^* \pi_* \mathcal{F}(-nP) = 0 \]
for sufficiently large $n$. This completes the list of the properties which we had to check and $(C', \pi', S', P', \rho', \mathcal{F}', \Phi')$ is really a geometric datum.

The fact that $(\alpha, \alpha', id)$ is a morphism of geometric data is straightforward. $\blacksquare$

As we mentioned at the beginning, the pull-back of a geometric datum over $S$ via a morphism $\alpha : S' \to S$ does not always define a new geometric datum. A typical situation for this is the restriction to one point of the base scheme $S$.

Let us give an example in terms of Schur pairs. As a base scheme we choose $S = \mathbb{A}^1_k = \text{Spec}(k[t])$, and our Schur pair is given by $(A, A)$ for
\[ A = k[t] \oplus tk[t]z^{-1} \oplus k[t][z^{-1}] \cdot z^{-2}. \]

Now we consider the fibre of the corresponding projective curve $C$ over the point $0 \in S$. Its affine part outside the section $P$ is given by the ring
\[ A_0 = k \oplus (tk[t]/t^2k[t])z^{-1} \oplus k[z^{-1}] \cdot z^{-2}. \]

This is a cuspidal curve with an embedded point. In particular, the fibre is not reduced and therefore cannot be a part of a geometric datum over $k$.

However, $(A, A)$ in fact induces a Schur pair over $S' = \text{Spec}(k)$, namely $(A', A')$ with
\[ A' = \text{Im}(A \subset k[t][(z)] \to (k[t]/t \cdot k[t])(((z)))) = k \oplus k[z^{-1}]z^{-2}. \]

This ring corresponds to the cuspidal curve, i.e., to the integral component of the fibre passing through the section $P$. 

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5 Families of commutative algebras of differential operators

In [M2] M. Mulase used the equivalence of the category of Schur pairs and the category of quintets for a complete classification of commutative algebras of ordinary differential operators with coefficients in \( k[[x]] \). This leads us to the natural question of whether it is possible to extend these results to the relative case, at least in some special situations. It is hard to do this in the set-up of sheaves. Therefore we will restrict our observations to the case of an affine base scheme \( S = \text{Spec}(R) \), where \( R \) is a commutative noetherian \( k \)-algebra for some field \( k \) of characteristic zero.

Before beginning let us fix a convention: Whenever in this chapter we speak about Schur pairs we mean elements of \( S' \) and all geometric data occuring here belong to \( D' \).

Let us start analyzing our objects in this special case.

5.1 Schur pairs over affine base schemes

Definition 5.1 Let \( A \) be an \( R \)-subalgebra of \( R((y)) \), and \( r \in \mathbb{N} \). \( A \) is said to be an algebra of pure rank \( r \), if

1. \( r = \gcd(\text{ord}(a)/a \in A) \) and
2. There are monic elements \( a \) and \( b \) of positive order in \( A \) such that \( \gcd(\text{ord}(a), \text{ord}(b)) = r \).

Let us see what these properties imply:

Lemma 5.2 Let \( A \subseteq R((y)) \) be an \( R \)-subalgebra of pure rank \( r \). Then there is a monic element \( z \in R[[y]] \) of order \(-r\) such that

- \( A \subseteq R((z)) \),
- \( R((z))/(A + R[[z]]) \) is a finitely generated \( R \)-module.

Proof Choose monic elements \( a \) and \( b \) of \( A \) of positive order such that \( \gcd(\text{ord}(a), \text{ord}(b)) = r \). Then there are natural numbers \( i \) and \( j \) such that

\[
r = i(\text{ord}(a)) - j(\text{ord}(b)).
\]

Define \( z := a^{-i}b^j \). Since the inverse of a monic element of \( R((y)) \) is again a well-defined element of \( R((y)) \), we have constructed a monic element \( z \in R[[y]] \) of order \(-r\). Now let us prove that the localization of \( A \) by \( a \) is contained in \( R((z)) \):

Of course, \( z \in A_a \). We choose an element \( v \) of \( A_a \) and denote its order by \( \alpha \). \( v \) has the form \( \frac{w}{z^m} \) for some elements \( w \in A \) and \( m \in \mathbb{N} \). \( r \) divides the orders of \( w \) and \( a \), therefore \( r \) divides \( \alpha \). Since \( z \) is monic of order \(-r\), there is some \( n \in \mathbb{Z} \) and \( v_0 \in R \), such that \( v - v_0 \cdot z^n \in A_a \) is an element of order less than \( \alpha \). Now the assertion is proved inductively.

In particular, this shows that \( A \) itself is contained in \( R((z)) \).

For the second part see the proof of Lemma 3.13. \( \square \)
Remark  The converse of this lemma is true, as well:
Let \( A \subset R((y)) \) be an \( R \)-subalgebra satisfying

- \( A \subset R((z)) \) for some monic element \( z \) of order \(-r\);
- \( R((z))/(A + R[[z]]) \) is a finitely generated \( R \)-module

then \( A \) is an \( R \)-algebra of pure rank \( r \).

Now let us define

**Definition 5.3** As an embedded Schur pair of rank \( r \), index \( F \) and level \( \alpha \) over \( \text{Spec} \( R \) \) we denote a pair \((A, W)\) consisting of

- \( A \subseteq R((y)) \) an \( R \)-subalgebra of pure rank \( r \) satisfying \( A \cap R[[y]] = R \);
- \( W \subseteq R((y)) \) with \( W \in \mathfrak{S}_{F,\alpha}^1(\text{Spec}(R)) \)

such that \( A \cdot W \subseteq W \).

We write \( \mathcal{E}_\alpha \mathfrak{S}_{F,\alpha}^r(R) \) for the set of embedded Schur pairs of rank \( r \), index \( F \) and level \( \alpha \) over \( \text{Spec}(R) \).

This is a natural generalization of the notion of Schur pairs introduced by M. Mulase \[M2\]. Now we want to see how these objects are related to the Schur pairs on \( \text{Spec}(R) \) we have defined earlier.

**Proposition 5.4** For all \( \alpha \in \mathbb{Z} \) there is a canonical one-to-one correspondence between Schur pairs of rank \( r \) and index \( F \) and embedded Schur pairs \((A, W)\) of rank \( r \), index \( F \) and level \( \alpha \) with the extra-condition that

\[ A \subset R((y^r)). \]

**Proof** The method of the proof has been outlined already in the remark on page \[\[\].

Let us start with an embedded Schur pair \((A, W)\) of rank \( r \), index \( F \) and level \( \alpha \) such that \( A \subset R((y^r)) \). Set \( z := y^r \). Then, by Lemma \[5.2\], \( A \subset R((z)) \) is an element of \( \mathfrak{S}_{\alpha}^1(\text{Spec}(R)) \) for some \( G \in K(\text{Spec}(R)) \). Now we identify:

\[
R[[y]] \cdot y^{-\alpha} = \bigoplus_{i=-\alpha+r-1}^{\alpha} R[[y^r]] \cdot y^i = R[[z]]^{\oplus r}.
\]

This identification extends to an isomorphism of \( R((y)) \) with \( R((z))^{\oplus r} \) and so we end up with \( W \subset R((z))^{\oplus r} \). Since \( R[[y]] \cdot y^{-\alpha} \) translates into \( R[[z]]^{\oplus r} \), \( W \) gives an element of \( \mathfrak{S}_{F,\alpha}^r(\text{Spec}(R)) \).

That also clarifies the inverse construction. We formally set \( z := y^r \) and translate the data back using Lemma \[5.2\] and its converse. \(\square\)
5.2 Formal pseudo-differential operators

We saw that embedded Schur pairs are closely related to Schur pairs, while Schur pairs themselves correspond to geometric data via the Krichever functor. Now a natural question is how to identify embedded Schur pairs which lead to “similar” geometric data, where similar means that they differ only by a very special change of the local trivializations. This is done with the help of formal pseudo-differential operators. Furthermore, formal pseudo-differential operators will be the main tool for the classification of commutative algebras of differential operators.

Consider the ring $R[[x]]$ of formal power series in one variable with coefficients in $R$, and write $\partial := \frac{d}{dx}$. $\partial$ acts on $R[[x]]$ by derivation:

$$\partial(\sum_{i \geq 0} a_i x^i) = \sum_{i \geq 1} i a_i x^{i-1},$$

while, for any $n \in \mathbb{N}$, $\partial^n$ acts by repeated derivation:

$$\partial^n(f) := \partial(\partial^{n-1}f)$$

for $f \in R[[x]]$. $\partial^0$ is defined to be the identity.

For given elements $f, g \in R[[x]]$ and $n \in \mathbb{N}$ we define:

$$(f \partial^n)(g) = f \partial^n(g),$$

$$(\partial^n f)(g) = \partial^n(fg).$$

In this way, the ring of ordinary differential operators with coefficients in $R[[x]]$, $D := R[[x]][\partial]$, turns out to be a subring of the endomorphism ring $\text{End}_R(R[[x]])$. The multiplication of elements of $D$ is determined by the Leibniz rule:

For $f, g \in R[[x]]$ and $n \in \mathbb{N}$:

$$\partial^n(fg) = \sum_{i=0}^{\infty} \binom{n}{i} f^{(i)} \partial^{n-i}(g).$$

(17)

It is our aim to make the operator $\partial$ invertible. In fact, we want to introduce $\partial^{-1}$ with $\partial \partial^{-1} = \partial^{-1} \partial = 1$ and define a multiplication on the set

$$E := \{ \sum_{n \in \mathbb{Z}} f_n \partial^n/f_n \in R[[x]], f_n = 0 \text{ for } n \gg 0 \} \supset D,$$

which is compatible with the multiplication on $D$. One can define $\partial^{-1}$ as an endomorphism on $R[[x]]$ by formal integration:

$$\partial^{-1}(\sum_{i \geq 0} a_i x^i) = \sum_{i \geq 0} \frac{a_i}{i+1} x^{i+1}.$$ 

But obviously, the so-defined $\partial^{-1}$ is not inverse to $\partial$ as an endomorphism of $R[[x]]$. That is why we consider the action of $\partial$ and $\partial^{-1}$ on the quotient $R\text{-module } R[[x]]/R[x]$. One easily sees that
one may interpret $R[x][\partial, \partial^{-1}]$ as a subring of $\text{End}_R(R[[x]]/R[[x]])$ and that for this embedding $\partial^{-1}$ is the inverse of $\partial$.

Observe that the Leibniz rule (17) also holds true for negative $n$, i.e., for formal integration, where, for arbitrary $n \in \mathbb{Z}$ and $i \in \mathbb{N}$, the binomial coefficient $\binom{n}{i}$ is defined as follows:

$$\binom{n}{i} := \frac{n \cdot (n-1) \cdot \ldots \cdot (n-i+1)}{i \cdot (i-1) \cdot \ldots \cdot 1} \in \mathbb{N}.$$ 

For negative $n \in \mathbb{Z}$, the summation in (17) is really an infinite one, while, for nonnegative $n$, it is finite.

The formula (17) defines a multiplication rule for elements of $E$ of the form

$$\sum_{n=-M}^{N} \left( \sum_{i=0}^{n} f_{i,n} x^i \right) \partial^n.$$

But now it is clear that the so-defined multiplication extends to a multiplication on all of $E$, which restricts to the usual one on $D$ and has the form:

$$(\sum_{m=0}^{\infty} a_m \partial^{M-m}) \cdot (\sum_{n=0}^{\infty} b_n \partial^{N-n}) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \binom{M-m}{i} a_m b_n^{(i)} \partial^{M+N-m-n-i}$$

$$(\sum_{m=0}^{l} \sum_{n=0}^{l-m} \binom{M-m}{i} a_m b_{l-m-i}^{(i)} \partial^{M+N-l}.$$ 

**Definition 5.5** $E$ is called the ring of formal pseudo-differential operators with coefficients in $R[[x]]$.

**Remark 1** In our notation we follow M. Mulase [M2]. Other authors use the name of micro-differential operators for the objects which we call formal pseudo-differential operators.

**Remark 2**

- From the above construction it is clear that $E$ is an associative, non-commutative ring, which has the additional structure of a left $R[[x]]$-module.
- $E$ has a filtration by left $R[[x]]$-submodules

$$E^{(m)} := \left\{ \sum_{n \in \mathbb{Z}} f_n \partial^n / f_n \in R[[x]], f_n = 0 \text{ for } n > m \right\}.$$ 

The order of an element $P \in E$ is defined to be the minimum $m \in \mathbb{Z}$ such that $P \in E^{(m)}$. In particular, the order of an element of $D$ coincides with its degree when we consider it as a polynomial in the variable $\partial$.

For an operator $P = \sum_{n=0}^{\infty} f_n \partial^{N-n}$ of order $N$, $f_0$ is called its leading coefficient.
An operator $P \in E$ can be written in the right normal form $P = \sum_{m=0}^{\infty} a_m \partial^{M-m}$ or in the left normal form $P = \sum_{n=0}^{\infty} \partial^{N-n} b_n$. It is an easy consequence of the Leibniz rule that the order of an operator is the same in the left and the right normal form and the leading coefficient does not change. So we see that $E$ is also a right $R[[x]]$-module and that $E^{(m)}$ gives rise to a filtration of $E$ by right $R[[x]]$-submodules.

For more properties of formal pseudo-differential operators see the appendix A.

$E$ contains the right ideal $xE$ generated by $x$. Denote by $\sigma : E \to E/xE$ the projection. So we have for formal pseudo-differential operators in the right normal form:

$$\sigma\left(\sum_{n=0}^{\infty} f_n \partial^{N-n}\right) = \sum_{n=0}^{\infty} f_n(0) \partial^{N-n} \in R((\partial^{-1})) = E/xE.$$ 

Let us set

$$y := \partial^{-1}.$$ 

Obviously we get, for all $n \in \mathbb{Z}$:

$$\sigma(E^{(n)}) = R[[y]] \cdot y^{-n}.$$ 

**Definition 5.6** The projection map $\sigma$ defines an action of $E$ on $R((y))$ as follows: Take $P \in E$ and $v \in R((y))$. Then there is an operator $Q \in E$ such that $v = \sigma(Q)$. Define

$$P(v) := \sigma(QP).$$

**Remark**

- This definition does not depend on the choice of $Q$. Note that for an invertible operator $P \in E$, $P^{-1} : R((y)) \to R((y))$ is inverse to the map $P : R((y)) \to R((y))$. If $P$ is an operator of order 0 with invertible leading coefficient, then $P$ is invertible and the induced map is an automorphism preserving orders, i.e., for all $n \in \mathbb{Z}$,

$$P : R[[y]] \cdot y^n \rightarrow R[[y]] \cdot y^n.$$ 

- If $P$ is a formal pseudo-differential operator with constant coefficients, $P \in R((\partial^{-1}))$, then we can regard $P$ as an element of $R((y))$ and it is easy to see that in this case the action of $P$ on $R((y))$ coincides with the usual multiplication in $R((y))$:

$$\sigma(QP) = \sigma(Q) \cdot P = P \cdot \sigma(Q).$$ 

- At this point, our way differs slightly from the one taken by M. Mulase [M2]. There, the quotient is taken by $Ex$ and operators $P \in E$ act on $R((y))$ from the left. Our approach will find its justification in Section 5.4.
Proposition 5.7 Let $P \in E$ be an operator of order 0 with invertible leading coefficient and $P : R((y)) \rightarrow R((y))$ the induced automorphism defined above. Then $P$ induces an automorphism

$$P : \mathfrak{g}_{F,a}^1(\text{Spec}(R)) \rightarrow \mathfrak{g}_{F,a}^1(\text{Spec}(R))$$

for all integers $\alpha \in \mathbb{Z}$ and all elements $F \in K(\text{Spec}(R))$.

Proof cf. [M2], Prop. 4.2.

Definition 5.8 A formal pseudo-differential operator $T \in E$ is called admissible if it is an operator of order 0 with invertible leading coefficient such that

$$T \partial T^{-1} \in R((\partial^{-1})).$$

The group of admissible operators is denoted by $\Gamma_a$.

Lemma 5.9 1. An operator $T$ is admissible if and only if it has the form

$$T = \exp(c_1 x) \cdot \left( \sum_{i=0}^{\infty} f_i \partial^{-i} \right),$$

where $c_1 \in R$, $f_i \in R[[x]]$ is a polynomial of degree at most $i$, and $f_0 \in R$ is invertible.

2. Let $v \in R((\partial^{-1}))$ be a monic element of order $-r$, $r \neq 0$. Then there is an admissible operator $T \in \Gamma_a$ such that $T \partial^{-r} T^{-1} = v$.

Proof This is a direct consequence of Lemma A.4 and its corollaries.

Definition 5.10 Two embedded Schur pairs $(A_1, W_1)$ and $(A_2, W_2)$ of rank $r$, index $F$ and level $\alpha$ are said to be equivalent if there is an admissible operator $T$ such that

$$T A_2 T^{-1} = A_1, \quad T W_2 = W_1,$$

where $A_1$ and $A_2$ are understood to be subalgebras of $R((\partial^{-1}))$, i.e., $T$ acts by conjugation, while $W_1$ and $W_2$ are understood to be subspaces of $R((y))$ and the action of $T$ on $W_2$ is defined by Definition 5.6. So we get $\mathfrak{g}_a \mathfrak{g}_{F,r}^1(\text{Spec}(R))/\Gamma_a$.

Now let $(A, W)$ be an arbitrary embedded Schur pair of rank $r$. By Lemma 5.2, $A \subset R((z))$ for some element $z \in R((y))$ of order $-r$. By Lemma 5.3 there is an operator $T \in \Gamma_a$ such that

$$T^{-1} z T = y^r.$$ 

Consequently, $A \subset R((z))$ implies that

$$T^{-1} A T \subset T^{-1} R((z)) T = R(T^{-1} z T) = R((y^r)).$$

This proves
Lemma 5.11  Every equivalence class of embedded Schur pairs contains a representative which corresponds to a Schur pair. ☐

Definition 5.12  Let \( \alpha \in \mathbb{Z} \) be an integer. Two Schur pairs of rank \( r \) and index \( F \) are said to be \( \alpha \)-equivalent if the associated embedded Schur pairs of level \( \alpha \) are equivalent.

We call two geometric data \( \alpha \)-equivalent if the corresponding Schur pairs are \( \alpha \)-equivalent.

Remark  Observe that the \( \alpha \)-equivalence depends on the congruence class of \( \alpha \) modulo \( r \).

Now let us study the \((-1)\)-equivalence in more detail.

Lemma 5.13  Let \( T \in \Gamma_a \) be such that for two algebras \( A_1 \) and \( A_2 \) of pure rank \( r \) which are both contained in \( R((y^r)) \):

\[
TA_1T^{-1} = A_2.
\]

Then \( TR[[y^r]]T^{-1} = R[[y^r]] \).

Conversely, for every algebra \( A_1 \) of pure rank \( r \) with \( A_1 \subseteq R((y^r)) \) and every \( T \in \Gamma_a \) satisfying \( TR[[y^r]]T^{-1} = R[[y^r]] \), \( TA_1T^{-1} \subseteq R((y^r)) \) is also an algebra of pure rank \( r \).

Proof  The last part is obvious. As for the first one, remember that there are elements \( a, b \in A_1 \) such that

\[
a^{-1}b = y^r + \sum_{i \geq 2} \alpha_i y^r
\]
(c.f. Lemma 5.2). Then \( TA_1T^{-1} = A_2 \subseteq R((y^r)) \) implies:

\[
Ta^{-1}bT^{-1} = Ta^{-1}T^{-1}TbT^{-1} = (TaT^{-1})^{-1}TbT^{-1} \subseteq R[[y^r]]
\]

hence \( TR[[a^{-1}b]]T^{-1} \subseteq R[[y^r]] \). But \( R[[y^r]] \) equals \( R[[a^{-1}b]] \), and so we obtain \( TR[[y^r]]T^{-1} \subseteq R[[y^r]] \). The second inclusion we get using the fact that \( A_1 = T^{-1}A_2T \). ☐

Now let us consider the action of \( T \in \Gamma_a \) on the second component of an embedded Schur pair \((A, W)\) of level \(-1\)

\[W \subseteq R((y)) = \bigoplus_{i=1}^r R((y^r)) \cdot y^i.\]

For \( c_i \in R((y^r)) \) we have

\[
T(\sum_{i=1}^r c_i y^i) = \sum_{i=1}^r (T^{-1}c_i T)(y^i).
\]

In particular, by Corollary A.4, \( T \) is determined by \( A \) and \( T^{-1}AT \) up to an operator with constant coefficients. So we obtain:
Proposition 5.14  1. If two geometric data \((C, \pi, \operatorname{Spec}(R), P, \rho_1, \Phi_1)\) and \((C, \pi, \operatorname{Spec}(R), P, \rho_2, \Phi_2)\) of rank \(r\) and index \(F\) are \((-1)\)-equivalent then there is an automorphism of \(R\)-algebras \(h : R[[z]] \to R[[z]]\) satisfying \(\rho_2 = h \circ \rho_1\).

2. Two geometric data
\[(C, \pi, \operatorname{Spec}(R), P, \rho, F, \Phi_1)\] and \[(C, \pi, \operatorname{Spec}(R), P, \rho, F, \Phi_2)\] of rank \(r\) and index \(F\) are \((-1)\)-equivalent if and only if there are elements \(d_1, \ldots, d_{r-1} \in R[[z]]\) and \(d_0 \in R[[z]]^*\) such that
\[
M = \begin{pmatrix}
  d_0 & d_1 & \ldots & d_{r-2} & d_{r-1} \\
  d_{r-1} y^r & d_0 & \ldots & d_{r-3} & d_{r-2} \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  d_1 y^r & d_2 y^r & \ldots & d_{r-1} y^r & d_0
\end{pmatrix}
\]
\[
\rho \Phi_2 = M \circ \rho \Phi_1.
\]
\[\square\]

Corollary 5.15  Note that in the case \(r = 1\), two geometric data \((C, \pi, \operatorname{Spec}(R), P, \rho_1, \Phi_1)\) and \((C, \pi, \operatorname{Spec}(R), P, \rho_2, \Phi_2)\) are always \((-1)\)-equivalent. \(\square\)

5.3 Classification of commutative algebras of differential operators

After these preliminaries we now come to the main object: the classification of commutative algebras of ordinary differential operators with coefficients in \(R[[x]]\), where \(R\) is a commutative noetherian \(k\)-algebra, for some field \(k\) of characteristic zero.

Definition 5.16  A commutative subalgebra \(B\) of \(D\) is said to be elliptic of pure rank \(r\) if

- \(r = \gcd(\operatorname{ord}(P)/P \in B)\);
- There are monic elements \(P, Q \in B\) such that \(\gcd(\operatorname{ord}(P), \operatorname{ord}(Q)) = r\).

The set of all such subalgebras of \(D\) is denoted by \(B_r(R)\).

Let us start the observations with the following

Lemma 5.17  If \(B\) is an elliptic subalgebra of rank \(r\) of \(D\) then there is a formal pseudo-differential operator \(X\) of order 0 with coefficients in \(R[[x]]\) and invertible leading coefficient such that
\[
A := X^{-1}BX \subseteq R((\partial^{-1}))
\]
and \(A\) is an algebra of pure rank \(r\).
Proof Choose a monic operator $P \in B$ of order $N$ greater than 0. Then by Lemma A.4 there exists a formal pseudo-differential operator $X$ of order 0 with invertible leading coefficient such that

$$X^{-1}P X = \partial^N.$$  

Let $Q \in B$ be an arbitrary element of $B$. Since $P$ and $Q$ commute, we get

$$0 = X^{-1}(PQ - QP)X = (X^{-1}PX)(X^{-1}QX) - (X^{-1}QX)(X^{-1}PX) = \partial^N(X^{-1}QX) - (X^{-1}QX)\partial^N.$$  

From the proof of Corollary A.6 one gets that then $X^{-1}QX$ must have constant coefficients, i.e., $X^{-1}QX \in R((\partial^{-1}))$. Finally, observe that the rank and the monicity of an operator are preserved under conjugation by $X$. So, $A$ is in fact an algebra of pure rank $r$. ✷

Let us look how far the name “elliptic” is justified for such an algebra $B$. Let $m$ be a maximal ideal of $R$. Then $B/m$ is a commutative algebra of differential operators with coefficients in $K[[x]]$ for some field $K$ containing $k$. A monic operator $P \in B$ of positive order, which exists by definition, gives us a monic operator $P \in B/m$ of positive order. From Lemma 5.17, applied to $B/m$, we know that every $Q \in B/m$ must have constant leading coefficient. In particular, since $K$ is a field, the leading coefficient of $Q$ is an element of $K^*$. This implies that $Q$ is really an elliptic ordinary differential operator, i.e., $B$ is a family of algebras of elliptic operators parametrized by $R$.

However, this is not the only way to interpret $B$. Let us take a commutative subalgebra $R \subset k[[t_1, \ldots, t_m]][\frac{d}{dt_1}, \ldots, \frac{d}{dt_m}]$. Then $Q \in B$ is a partial differential operator, which does not need to be elliptic.

Let us continue with our construction.

**Lemma 5.18 (Sato)** A formal pseudo-differential operator $P \in E$ is a differential operator if and only if it preserves $\sigma(D)$ in $R((y))$, i.e.,

$$P\sigma(D) \subseteq \sigma(D).$$  

In the proof one may follow [M2], Lemma 7.2. ✷

Now let us have a look at another definition of infinite Grassmannians.

**Definition 5.19** The Sato Grassmannian is defined to be

$$SG^+ := \{ J \subset E \text{closed subspace} \mid J \oplus E^{(-1)} = E, DJ \subseteq J \}$$  

This is a relative version of the Grassmannian originally used by Sato. There is the following connection to the Grassmannians we have considered until now:
Theorem 5.20 (Sato) 1. Let $\Gamma_m$ be the group of monic formal pseudo-differential operators of order zero and let $SG^+$ be the Sato Grassmannian defined as above. Then there is a natural bijection $\alpha : \Gamma_m \rightarrow SG^+$ given by

$$\Gamma_m \ni X \mapsto \alpha(X) = J = DX^{-1} \in SG^+.$$ 

2. Set $\mathfrak{S}^{+}(\text{Spec}(R)) := \{ W \in \mathfrak{S}_{0,-1}(\text{Spec}(R))/W \oplus R[[y]]y = R((y)) \}$. Then the projection $\sigma : E \rightarrow R((y))$ induces a bijection

$$\sigma : SG^+ \sim \mathfrak{S}^{+}(\text{Spec}(R)).$$

Proof Lemma A.3 implies that $\Gamma_m$ is a group. Now we can apply the proof of [M2], Thm. 7.4., and obtain our result. □

Remark $\mathfrak{S}^{+}(\text{Spec}(R))$ is the generalization of the big cell of the Grassmannian of rank 1, index 0 and level −1 (cf. [M2]).

Definition 5.21 We call two algebras $B_1, B_2 \in \mathcal{B}_r(R)$ equivalent if there is an invertible element $f \in R[[x]]$ such that

$$B_1 = fB_2f^{-1}.$$ 

We denote by $\mathcal{B}_r(R)$ the set of these equivalence classes.

Theorem 5.22 For all $r \geq 1$, there is a canonical bijection

$$\mu_r : \mathcal{B}_r(R) \rightarrow \mathcal{E}_{-1}\mathfrak{S}_{0}^{+}(\text{Spec}(R))/\Gamma_a,$$

where $\mathcal{E}_{-1}\mathfrak{S}_{0}^{+}(\text{Spec}(R))$ denotes the subset of $\mathcal{E}_{-1}\mathfrak{S}_{0}^{+}(\text{Spec}(R))$ consisting of embedded Schur pairs $(A, W)$ with $W \in \mathfrak{S}^{+}(\text{Spec}(R))$.

Proof Given $B \in \mathcal{B}_r(R)$. Using Lemma 5.17 and Theorem 5.20, we construct $A := X^{-1}BX \subseteq R((\partial^{-1}))$ and $W := X\sigma(D) \in \mathfrak{S}^{+}(\text{Spec}(R))$. $A$ is an algebra of pure rank $r$.

Since $B$ is contained in $D$, we get $B\sigma(D) \subseteq \sigma(D)$ (cf. Lemma 5.18). This implies

$$A \cdot W = X^{-1}BX(\sigma(DX)) = \sigma(DXX^{-1}BX) = \sigma(DBX) \subseteq X\sigma(D) = W.$$ 

So we actually constructed an embedded Schur pair of the required type.

If $X_1$ is another operator satisfying $X_1^{-1}BX_1 \subseteq R((\partial^{-1}))$ then $T := X^{-1}X_1$ is an admissible operator and we end up with an equivalent embedded Schur pair.

Now, what happens if we take $fBf^{-1}$ instead of $B$ for some $f \in R[[x]]^*$? Then: $A' = (fX)^{-1}(fBf^{-1})(fX) = A$ and $W' = (fX)(\sigma(D)) = f_0 \cdot W = W$, where $f_0$ denotes the (invertible) constant coefficient of $f$.

As for the inverse way, let us take $(A, W) \in \mathcal{E}_{-1}\mathfrak{S}_{0}^{+}(\text{Spec}(R))$. From Theorem 5.20 we get a unique operator $S \in \Gamma_m$ such that $W = S\sigma(D)$. Let us define $B := SAS^{-1} \subseteq E$. Using
Lemma 5.18, we obtain that $B \subseteq D$. We only have to check that $\sigma(DB)$ is contained in $\sigma(D)$, or, equivalently, that $\sigma(DB)$ is a subset of $\sigma(D)$:

$$S\sigma(DB) = \sigma(DBS) = \sigma(DSS^{-1}BS) = A \cdot W \subseteq W = S\sigma(D).$$

If we start with an equivalent embedded Schur pair $(A, W)$, for some $T \in \Gamma_a$, we get an operator $S_1 \in \Gamma_m$ such that

$$S_1(\sigma(D)) = TW = TS(\sigma(D)).$$

Again Lemma 5.18 implies $S_1^{-1}TS \in D$. But this operator is an invertible formal pseudodifferential operator of order 0. So $S_1^{-1}TS$ is in fact an invertible element $f$ of $R[[x]]$ and we conclude that

$$B' = S_1TAT^{-1}S_1^{-1} = fSAS^{-1}f^{-1} = fBf^{-1}.$$

So we in fact end up with an equivalent algebra. $\Box$

We have established a one-to-one correspondence of elliptic commutative algebras of differential operators with coefficients in $R[[x]]$ and certain equivalence classes of embedded Schur pairs. Using the results of the sections 5.1 and 5.2 we can now state:

**Corollary 5.23** There is a bijection between equivalence classes of commutative elliptic subalgebras of $R[[x]][\frac{d}{dx}]$ of pure rank $r$ and $(-1)$-equivalence classes of geometric data

$$(C, \pi, \text{Spec}(R), P, \rho, F, \Phi)$$

of rank $r$ and index 0 with the extra-condition that

$$H^0(F) = H^1(F) = 0.$$

In particular, every sheaf corresponding to a commutative algebra of differential operators is strongly semistable with respect to $P$ (cf. Section 4.1). $\Box$

### 5.4 Eigenvalue problems

As a motivation, let us approach the above constructed relation from another side. We take an elliptic commutative algebra $B$ of differential operators with coefficients in $R[[x]]$. It is an interesting problem to find out the common eigenfunctions of all operators belonging to $B$. Let $f \in R[[x]]$ be such a common eigenfunction, i.e., for each $P \in B$, there is some $\lambda(P) \in R$ such that

$$P(f) = \lambda(P) \cdot f.$$

One easily sees that, for a given $f$, the map

$$\lambda : B \to R$$

$$P \mapsto \lambda(P)$$

is a homomorphism of $R$-algebras.
On the other hand, given a homomorphism \( \lambda : B \to R \), what are the eigenfunctions of \( B \) with respect to \( \lambda \)?

Let \((A, W)\) be an embedded Schur pair corresponding to \( B \), \( W = \sigma(DS) \), \( A = S^{-1}BS \), for a formal pseudo-differential operator \( S \) of order zero with invertible leading coefficient. For a given function \( f \in R[[x]] \), we define an \( R \)-linear map

\[
 f : W \to R
\]

by

\[
 f(\sigma(QS)) := \sigma(Q(f)), \text{ for } Q \in D.
\]

This map is well-defined. For, if \( \sigma(QS) = \sigma(Q'S) \), then \( \sigma(Q) = \sigma(Q') \), i.e., \( Q - Q' \in xE \). Consequently, \( \sigma((Q - Q')(f)) = 0 \).

Now we claim

**Proposition 5.24** \( f \in R[[x]] \) is a common eigenfunction of the elements of \( B \) with the eigenvalue \( \lambda \) if and only if \( \lambda \) makes the map (18) \( A \)-linear, i.e., for \( a \in A \), \( a = S^{-1}PS \), \( P \in B \):

\[
 f(a \cdot w) = \lambda(P) \cdot f(w).
\]

**Proof** First assume that \( f \) is an eigenfunction as above. Then \( P(f) = \lambda(P) \cdot f \) for all \( P \in B \). Therefore, for \( w = \sigma(QS) \in W \) and \( a = S^{-1}PS \in A \)

\[
 f(a \cdot w) = f(\sigma(QSS^{-1}PS)) = f(\sigma(QPS)) = \sigma((QP)(f)) = \sigma(Q(\lambda(P) \cdot f)) = \lambda(P) \cdot \sigma(Q(f)) = \lambda(P) \cdot f(w).
\]

On the other hand, assume that the \( A \)-linearity holds. Then, for all ordinary differential operators \( Q \),

\[
 \sigma(Q(P(f))) = \sigma(Q(\lambda(P) \cdot f)).
\]

We claim that this implies \( P(f) = \lambda(P) \cdot f \). But this is clear, since, for every \( g = \sum_{n \geq 0} g_n x^n \in R[[x]] \), \( \sigma(\partial^n(g)) = n! \cdot g_n \), i.e., \( g \) is in fact determined by \( \sigma(Q(g)) \), \( Q \in D \). \( \square \)

### 5.5 Examples

To get a better idea of the formal correspondence constructed in this chapter, let us give some easy examples. Certainly, the results we are going to obtain are not new. They only serve to illustrate the construction.

We start with the easiest case: Let \( B \in B_1(R) \) be an algebra containing a monic element of order 1. We prove

**Lemma 5.25** All algebras \( B \in B_1(R) \) containing a monic element of order 1 are equivalent.
**Proof** Take \( u = \sum_{i \geq 0} v_i x^i \in R[[x]] \). It suffices to prove that there is an invertible formal power series \( f = \sum_{i \geq 0} f_i x^i \in R[[x]] \) such that

\[
f^{-1} \partial f = \partial + u.
\]

Let us construct \( f \):

\[
f^{-1} \partial f = f^{-1} f \partial + f^{-1} f' = \partial + f^{-1} f';
\]

therefore we only need to construct \( f \) such that \( u = f^{-1} f' \), i.e., \( fu = f' \). We write this expression as a formal power series

\[
\sum (\sum_{j=0}^{i} f_j u_{i-j}) x^i = \sum_{i \geq 0} (i+1) f_{i+1} x^i;
\]

so that we get as a necessary and sufficient condition:

\[
f_{i+1} = \frac{1}{i+1} \sum_{j=0}^{i} f_j u_{i-j},
\]

which is solvable for an arbitrarily given \( u \). \( \square \)

What does this say in terms of curves and sheaves? The curve associated to such a \( B \) is obviously \( \mathbb{P}^1_R \) with some section \( P \). One possible line bundle with vanishing cohomologies is \( \mathcal{O}_{\mathbb{P}^1_R}(-P) \). The different local parametrizations of \( \mathbb{P}^1_R \) and local trivializations of \( \mathcal{O}_{\mathbb{P}^1_R}(-P) \) along \( P \) factor out under the equivalence relation. Now the above lemma reads as:

**Corollary 5.26** For any two sections of \( \mathbb{P}^1_R \) there is an isomorphism of \( \mathbb{P}^1_R \) mapping one section into the other. Furthermore, given a section \( P \), \( \mathcal{O}_{\mathbb{P}^1_R}(-P) \) is the only coherent sheaf on \( \mathbb{P}^1_R \) of rank 1 with vanishing cohomology groups which satisfies the conditions of Definition 2.9. \( \square \)

The next interesting case is that of algebras \( B \in \mathcal{B}_1(R) \) containing monic elements of order 2 and 3, but without any element of order 1. These correspond to families of reduced and irreducible curves with arithmetic genus 1.

First take \( R = k \). Then we get a single curve. One may ask how the singularity of the curve is displayed in the associated algebra of differential operators:

**Proposition 5.27** An algebra \( B \) as above corresponds to a singular plane cubic with its point at infinity as a section if and only if there is a formal pseudo-differential operator \( T \) of order 0 with invertible leading coefficient such that \( TBT^{-1} \) is an algebra of differential operators with constant coefficients.

**Remark** Keep in mind that this does not say that \( B \) is equivalent to an algebra of differential operators with constant coefficients. The above transformation changes the sheaf induced by \( B \).
Proof  Let $B$ be given as above. Then for the corresponding Schur pair $(A, W)$ we get:

$$A = k[y^{-2} + \alpha y^{-1} + \beta, y^{-3} + \gamma y^{-2} + \delta y^{-1} + \epsilon]$$

with coefficients $\alpha, \beta, \gamma, \delta, \epsilon \in k$. Note that of course $\beta$ and $\epsilon$ may be changed arbitrarily, and $\gamma$ may be set to 0.

Assume that $\alpha$ is different from 0. We show that such an $A$ (understood as an element of $B_1(k)$ via the identification $\partial^{-1} = y$) is in fact equivalent to an algebra of differential operators containing $\partial^2$, i.e., we could have chosen $T$ such that $T BT^{-1}$ already contains $\partial^2$.

Given $\alpha, \delta \in k$ we want to construct an invertible power series

$$f = \sum_{i \geq 0} f_i x^i \in k[[x]]$$

such that:

- $f^{-1} \partial^2 f = \partial^2 + \alpha \partial + \beta$ for some $\beta \in k$;
- There are numbers $\epsilon, a, b, c \in k$ such that

$$f^{-1}(\partial^3 + a \partial^2 + b \partial + c)f = \partial^3 + \delta \partial + \epsilon.$$

Let’s start the calculation:

$$f^{-1} \partial^2 f = f^{-1}(f \partial^2 + 2 f' \partial + f'') = \partial^2 + 2 f^{-1} f' \partial + f^{-1} f''.$$

We have already seen that the equation $f' = \frac{\alpha}{2} f$ is solvable. For $f$ chosen as such we get:

$$f^{-1} \partial^2 f = \partial^2 + \alpha \partial + \frac{\alpha^2}{4} \text{ and }$$

$$f^{-1}(\partial^3 + a \partial^2 + b \partial + c)f =$$

$$f^{-1}(f \partial^3 + 3 f' \partial^2 + 3 f'' \partial + f''') = \partial^3 + (3 f^{-1} f' + a) \partial^2 + (3 f^{-1} f'' + 2 a f^{-1} f' + b) \partial +$$

$$(f^{-1} f''' + a f^{-1} f'' + b f^{-1} f' + c)$$

$$= \partial^3 + (\frac{3 a}{2} + a) \partial^2 + (\frac{3 a^2}{4} + a a + b) \partial + (\frac{a^3}{8} + \frac{a a^2}{4} + b^2 + c).$$

From this we see that all conditions can be satisfied.

We still have to show that the rings $A = k[y^{-2}, y^{-3} + \delta y^{-1}]$ correspond to singular plane cubics and that all such cubics occur.

A singular plane cubic has the equation

$$z_0 z_2^2 = z_1 (z_1 + \delta z_0)^2$$

with $P = (0 : 0 : 1)$ being its point at infinity. Note that it has a cusp if $\lambda = 0$ and a node in the remaining cases. Using the methods of 4.2.2 we get

$$A = k[y^{-2}, y^{-3} + \delta y^{-1}]$$
as the associated ring, and the proof is complete. □

Now let us have a look at the partial differential equations which are produced by the algebras \( B \) as considered above. \( B \) is generated by two elements

\[
\begin{align*}
L &= \partial^2 + u\partial + v \\
P &= \partial^3 + \alpha \partial^2 + \beta \partial + \gamma.
\end{align*}
\]

We have seen before that we can assume that \( u = 0 \) if we are only interested in the equivalence class of \( B \).

What does it mean for \( B \) to be commutative? \( [P, L] = 0 \) can be interpreted as

\[
\begin{align*}
(I) & \quad 2\alpha' = 0 \\
(II) & \quad \alpha'' + 2\beta' - 3v' = 0 \\
(III) & \quad \beta'' + 2\gamma' - 3v'' - 2\alpha v' = 0 \\
(IV) & \quad \gamma'' - v''' - \alpha v'' - \beta v' = 0.
\end{align*}
\]

The first equation says that \( \alpha \) is a constant. So we can choose another normal form

\[
\begin{align*}
L &= \partial^2 + v \\
P &= \partial^3 + \beta \partial + \gamma,
\end{align*}
\]

\( v \) and \( \gamma \) without constant term. The class of the algebra \( B \) uniquely determines \( P \) and \( L \), and on the other side, these two elements, and by (II) and (III) in fact \( \beta \), uniquely determine the equivalence class of \( B \).

Substituting (I), (II) and (III) in (IV) we get

\[
\frac{1}{6} \beta'''' - \frac{2}{3} \beta \beta' = 0,
\]

which is nothing but the stationary Korteveg - de Vries equation.

We saw that \( B \) corresponds to a singular curve with its point at infinity if it may be transformed into \( \bar{B} = k[\partial^2, \partial^3 + \beta \partial] \), \( \beta \) being a constant. So we get:

**Corollary 5.28** A pointed integral curve \( (C, p) \) of arithmetic genus 1 is isomorphic to a singular plane cubic with its point at infinity if and only if there is a torsion free sheaf on \( C \) generating a constant solution of the KdV equation. The singularity is a node if this constant if different from zero; it is a cusp if the constant equals to zero. □

After this study of special cases we are now interested in the general structure of \( B_1(\mathbb{C}) \). From the correspondence established in the first part of the section \( \mathbb{B} \) it is clear that two algebras of differential operators lead to the same pointed curve if and only if they differ only by conjugation with a formal pseudo-differential operator of order 0 with invertible leading coefficient. So we obtain:
Proposition 5.29 For any \( B \in \mathcal{B}_1(\mathbb{C}) \) such that \( \text{Spec}(B) \) is smooth, the set
\[
\mathcal{B}_1(\mathbb{C})(B) := \left\{ B' \in \mathcal{B}_1(\mathbb{C}) \mid \exists \text{ pseudo-diff. op. } X \text{ of order } 0 \text{ with inv. leading coeff. such that } X^{-1}B'X = B \right\}
\]
has the structure of a reduced, irreducible, affine complex variety of dimension
\[
g = \text{card}\{n \in \mathbb{N} \mid \text{ there is no differential operator of order } n \text{ in } B\}.
\]

Proof Let \((C, p)\) be the pointed curve given by \(B\). We consider \(\text{Pic}^{g-1}(C)\). The theta divisor in it is given by the condition \(h^0 \neq 0\). This divisor is known to be ample, so its complement
\[
U = \{\mathcal{L} \text{ line bundle of degree } g-1 \text{ on } C/h^0(\mathcal{L}) = h^1(\mathcal{L}) = 0\}
\]
is affine. It is also reduced and irreducible. Now we take the Poincaré bundle \(\mathcal{P}^{g-1}_C\) on \(\text{Pic}^{g-1}(C) \times C\) (normalized with respect to \(p\)) and restrict it to \(U \times C\). We choose some local parametrization of \(U \times C\) near \(U \times \{p\}\) and some local trivialization of \(\mathcal{P}^{g-1}_C\) along \(U \times \{p\}\) (for more details see section 4.2.3). Different parametrizations and trivializations cancel out under the equivalence relation. Now we construct the associated algebra \(B\) of ordinary differential operators with coefficients in \(R[[x]]\) for \(U = \text{Spec}(R)\). From the universality of the Poincaré bundle we get that \(B\) uniquely parametrizes the equivalence classes of algebras \(B' \in \mathcal{B}_1(\mathbb{C})\) corresponding to the given pointed curve \((C, p)\), hence all of \(\mathcal{B}_1(\mathbb{C})(B)\). \(\square\)

Remark In the case that \(g = 1\) the so-defined \(B \in \mathcal{B}_1(R)\) carries a nontrivial family of solutions of the KdV equation.

A Basic facts on power series rings and formal pseudo-differential operators with coefficients in power series rings

Here we want to sum up some easy properties of power series (resp. Laurent series) with coefficients in rings and of formal pseudo-differential operators having coefficients in rings of power series.

Assume that \(R\) is a commutative ring.

Lemma A.1 The units of the ring \(R[[z]]\) are exactly the power series with invertible constant coefficient.

Proof Given \(s = \sum_{i=0}^{\infty} s_i z^i \in R[[z]]\), we want to determine its inverse. Set \(t = \sum_{j=0}^{\infty} t_j z^j\). Then
\[
s \cdot t = (\sum_{i=0}^{\infty} s_i z^i)(\sum_{j=0}^{\infty} t_j z^j)
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} s_i t_j z^{i+j}
= \sum_{l=0}^{\infty} (\sum_{j=0}^{l} s_{l-j} t_j) z^l.
\]

Hence, \(s \cdot t = 1\) if and only if
• $1 = s_0 t_0$;
• $\forall l \geq 1, s_0 t_l = -\sum_{j=0}^{l-1} s_{l-j} t_j$.

One can construct an element $t \in R[[z]]$ with these properties if and only if the constant term $s_0$ of $s$ is invertible.

Lemma A.2 Let $N$ be an integer and assume that the order $M$ of a monic element $s \in R((z))$ is divisible by $N$. Then there is a uniquely determined monic element $t \in R((z))$ of order $\frac{M}{N}$ such that $t^N = s$.

Proof Set $s = z^M (1 + \sum_{i=1}^{\infty} s_i z^i)$ and $t = z^{M/N} (1 + \sum_{j=1}^{\infty} t_j z^j)$. Then $t^N = s$ if and only if

$$1 + \sum_{i=1}^{\infty} s_i z^i - (1 + \sum_{j=1}^{\infty} t_j z^j)^N = 0.$$ 

For $n \geq 1$, the coefficient of $z^n$ of this term is

$$0 = s_n - t_n - p_n,$$

where $p_n$ is a polynomial in $t_1, \ldots, t_{n-1}$. Therefore, all $t_j$ are well-determined by the coefficients $s_j$. □

From now on we assume that $R$ is a $k$-algebra for some field $k$ of characteristic zero.

Lemma A.3 A formal pseudo-differential operator $P = \sum_{n=0}^{\infty} s_n \partial^{-n}$ of order 0 is invertible by another formal pseudo-differential operator of order 0 if and only if its leading coefficient $s_0 \in R[[x]]$ is invertible.

Proof Set $Q := \sum_{n=0}^{\infty} t_n \partial^{-n}$ and $S := \sum_{n=0}^{\infty} r_n \partial^{-n}$. Now let us write down $QP$.

$$QP = \sum_{l=0}^{\infty} \sum_{m=0}^{l-m} (-m \choose i) t_m s^{(i)}_{l-m-i} \partial^{-l}.$$ 

If $QP = 1$ then $t_0 s_0 = 1$, i.e., $s_0$ must be a unit in $R[[x]]$. Now let us assume that $s_0$ is invertible. From the equation above one sees that $QP = 1$ if and only if

• $t_0 = s_0^{-1}$;
• For $l \geq 1$: $t_l = s_0^{-1} \cdot f_l$, where $f_l$ is a polynomial in $t_0, \ldots, t_{l-1}$ and $s_0, \ldots, s_l$ and its derivatives.

These equations are solvable, i.e., such coefficients $t_i \in R[[x]]$ exist.

On the other hand,

$$PS = \sum_{l=0}^{\infty} \sum_{m=0}^{l-m} (-m \choose i) s_m r^{(i)}_{l-m-i} \partial^{-l}$$

equals 1 if and only if
• \( r_0 = s_0^{-1}; \)

• For \( l \geq 1: \ r_l = s_l^{-1} \cdot g_l, \) \( g_l \) being a polynomial in \( t_0, \ldots, t_{l-1} \) and its derivatives and \( s_0, \ldots, s_l. \)

So there are formal pseudo-differential operators \( Q \) and \( S \) such that \( QP = 1 = PS. \) From the associativity of \( E \) we get \( Q = R = P^{-1}. \) □

**Lemma A.4** Let \( L = \sum_{i=0}^{\infty} u_i \partial^{N-i} \) be a monic formal pseudo-differential operator of order \( N, \) \( N \neq 0, \) with coefficients in \( R[[x]]. \) Then there is a formal pseudo-differential operator \( X = \sum_{i=0}^{\infty} s_i \partial^{-i} \) of order 0 with invertible leading coefficient such that

\[ X^{-1}LX = \partial^N. \]

**Proof** \( X^{-1}LX = \partial^N \) is equivalent to \( LX = X\partial^N, \) i.e.,

\[ \sum_{i=0}^{\infty} s_i \partial^{N-i} = (\sum_{i=0}^{\infty} u_i \partial^{N-i}) (\sum_{i=0}^{\infty} s_i \partial^{-i}) = \sum_{i=0}^{\infty} (\sum_{m=0}^{l} \sum_{i=0}^{l-m} \binom{N-m}{i} u_m s_i^{(i)} ) \partial^{N-i}. \]

Comparing coefficients, this gives us:

\[ s_l = \sum_{m=0}^{l} \sum_{i=0}^{l-m} \binom{N-m}{i} u_m s_i^{(i)}, \]

and so

\[ 0 = \sum_{m=0}^{l} \sum_{i=0}^{l-m} \binom{N-m}{i} u_m s_i^{(i)}. \]

For \( (m, i) = (0, 1) \) or \( (1, 0), \) the term \( s_{l-1} \) is involved. In all other terms, \( s \) occurs with lower index. So we get

\[ Nu_0 s_{l-1} + u_1 s_{l-1} = P_{l-2}, \]

where \( P_{l-2} \) is a linear combination of the \( s_0, \ldots, s_{l-2} \) and its derivatives with coefficients consisting of integer multiples of \( u_0, \ldots, u_l. \) Using the fact that \( u_0 = 1 \) we get the equation

\[ s_{l-1} = -\frac{1}{N} u_1 s_{l-1} + \frac{1}{N} P_{l-2}. \]  \tag{19} \]

Let us write

\[ s_{l-1} = \sum_{i=0}^{\infty} s_{l-1,i} z^i, \quad u_1 = \sum_{i=0}^{\infty} u_{1,i} z^i, \quad P_{l-2} = \sum_{i=0}^{\infty} P_{l-2,i} z^i \]

with \( s_{l-1,i}, u_{1,i}, P_{l-2,i} \in R. \) Then the equation (19) can be written as

\[ \sum_{i=0}^{\infty} s_{l-1,i+1} (i + 1) z^i = -\frac{1}{N} \left( \sum_{i=0}^{\infty} \sum_{j=0}^{i} u_{1,i-j} s_{l-1,j} z^j \right) + \frac{1}{N} \sum_{i=0}^{\infty} P_{l-2,i} z^i. \]
Equivalently, for all $i = 0, 1, 2, \ldots$:

$$(i + 1)s_{i-1, i+1} = -\frac{1}{N} \sum_{j=0}^{i} u_{1, i-j} s_{i-1, j} + \frac{1}{N} P_{l-2, i}.$$ 

So we see that we can construct an operator $X$ with the required properties. 

**Corollary A.5** Assume that $L$ as above is an operator with constant coefficients. Then the operator $X$ constructed in the lemma A.4 has the form

$$X = \exp \left( -\frac{1}{N} u_{1} x \right) \sum_{i=0}^{\infty} f_{i} \partial^{-i},$$

where $f_{i}$ is a polynomial of degree at most $l$.

**Proof** We consider again the equation (19). In the given case, it has a solution of the form

$$s_{l-1} = \exp \left( -\frac{1}{N} u_{1} x \right) f_{l-1}$$

with $\exp \left( -\frac{1}{N} u_{1} x \right) f_{l-1} = \frac{1}{N} P_{l-2}$. Set $l = 1$:

$$s_{0} = \exp \left( -\frac{1}{N} u_{1} x \right) c, \ c \in R^*.$$ 

Inductively, we conclude that

$$P_{l-2} = \exp \left( -\frac{1}{N} u_{1} x \right) \cdot g_{l-2}$$

for some polynomial $g_{l-2}$ of degree at most $l - 2$. So $f_{l-1}$ is in fact a polynomial of degree $\leq l - 1$. 

**Corollary A.6** The operator $X$ in Lemma A.4 is determined up to an operator with constant coefficients; i.e., if two operators $X_{1}$ and $X_{2}$ satisfy:

$$X_{1}^{-1} L X_{1} = X_{2}^{-1} L X_{2} = \partial^{N}$$

then there is an operator $X$ with constant coefficients such that $X_{1} X = X_{2}$.

**Proof** Denote $X := X_{1}^{-1} X_{2}$. We show that $X$ has constant coefficients. By assumption, $\partial^{N} = X \partial^{N} X^{-1}$, i.e., $X \partial^{N} = \partial^{N} X$. So

$$\sum_{l \geq 0} s_{l} \partial^{N-l} = \partial^{N} \left( \sum_{l \geq 0} s_{l} \partial^{-l} \right) = \sum_{l \geq 0} \sum_{i=0}^{l} \binom{N}{i} s_{l-i} \partial^{N-l}.$$

This implies that $\sum_{i=1}^{l} \binom{N}{i} s_{l-i} = 0$. Now we proceed inductively to show: $s'_{m} = 0$ for all $m$:

- **m = 0:** Set $l = 1$: $\binom{N}{1} s'_{0} = 0$, hence $s'_{0} = 0$.

- Assume that $s'_{0} = \ldots = s'_{m-1} = 0$. Then of course also the higher derivatives of these terms vanish and for $l = m + 1$ we again get: $\binom{N}{l} s'_{m} = 0$.

$\square$
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