Lattice formulation of 2D $\mathcal{N} = (2, 2)$ SQCD and twisted supercharges

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Abstract. We discuss on two kinds of lattice formulation of two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric QCD preserving one of its supercharges. The preserved supercharge is of the same form obtained by topological twist. One formulation is based on A model twist, and the other is based on B model twist. For the A model twist case, by employing the Ginsparg-Wilson operator, the lattice model can be constructed for general number of flavors of fundamental and anti-fundamental matter multiplets under the gauge group $G = U(N)$ or $SU(N)$. It preserves both of one supercharge ($Q$) and chiral flavor symmetry of matter multiplets. Furthermore, superpotential can be introduced with exact holomorphic or anti-holomorphic structure on lattice. For the B model twist case, the lattice action takes a simpler form compared with the A model twist case, because the admissibility condition is not required to the formulation. It can be applied to a restricted cases of $G = U(N)$ and of fundamental matters or anti-fundamental matters only.

1. Introduction

Lattice formulation of quantum field theory has been the most solid and conventional method to give its constructive definition as well as to explore its nonperturbative aspects, since Wilson’s formulation of lattice gauge theory. Lattice formulation of supersymmetric (gauge) theory has notorious difficulty in realizing supersymmetry (SUSY) on lattice. Namely, SUSY induces infinitesimal translations, although they are not symmetries on the lattice which is a grid with a finite spacing $a$. Such a mismatch vanishes in the continuum limit $a \to 0$ and SUSY is restored at the classical level. However, loop effects at the quantum level cause UV divergent radiative correction terms having inverse powers of $a$, which possibly prevent from restoring the SUSY in the continuum limit. If so, it will be necessary to introduce counter terms and fine-tune their parameters to be cancelled with the harmful radiative corrections.

Recently, for supersymmetric gauge theory with extended SUSY, various lattice models in which a part of supercharges is preserved on the lattice, have been constructed. The exact SUSY on the lattice helps reducing the number of parameters needed to be fine-tuned in the continuum limit. In dimensions lower than four it effectively works, and in particular for two dimensions there are lattice models where the target continuum SUSY theory can be obtained with no need of fine-tuning.

The supercharges preserved on the lattice are nilpotent up to internal symmetry, not inducing translations. It reminds us of scalar supercharges obtained from topological twist [6, 7].

1 This talk is based on the work [1, 2, 3, 4, 5].
For example, two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills (SYM) theory or supersymmetric QCD (SQCD), there are two kinds of lattice formulations. One preserves the supercharge $Q = -\frac{1}{\sqrt{2}}(Q_L + Q_R)$ which is the same as a scalar supercharge obtained by A-model twist, the other preserves $Q' = -\frac{1}{\sqrt{2}}(Q_L + Q_R)$ of the same form as the one obtained by B-model twist. The two-dimensional $\mathcal{N} = (2, 2)$ theory has two $U(1)_R$-symmetries, $U(1)_A$ and $U(1)_V$ at the classical level. The $Q$ invariant formulation respects $U(1)_A$ on the lattice, while the $Q'$ invariant one $U(1)_V$.

Let us note that we are considering the theory on flat space-time, where topological twists just amount to renaming fields variables and the continuum theory does not change by the twists. However, lattice models can become different depending on which of $Q$ and $Q'$ is exactly preserved.

In the following sections, we discuss two kinds of lattice models for two-dimensional $\mathcal{N} = (2, 2)$ SQCD, which is SYM theory coupled to $n_+$ fundamental and $n_-$ anti-fundamental matter multiplets, with $G = U(N)$ or $SU(N)$ gauge group. Lattice is two-dimensional square lattice with the spacing $a$, and compact link variables $U_{\mu} = e^{iaA_{\mu}}$ are used as gauge fields. Remaining field contents are defined on lattice sites. Each of the lattice models preserves either $Q$ or $Q'$.

### 2. Continuum $\mathcal{N} = (2, 2)$ SQCD in two dimensions

The continuum theory of two-dimensional $\mathcal{N} = (2, 2)$ SQCD is obtained from four-dimensional $\mathcal{N} = 1$ SQCD by dimensional reduction. The multiplets $V = (A_\mu, \phi, \bar{\phi}; \lambda; D)$ are from four-dimensional $\mathcal{N} = 1$ vector multiplets. $A_\mu$ is a two-dimensional vector field, $\phi$, $\bar{\phi}$ are complex scalars, $\lambda$ is a gaugino, and $D$ is an auxiliary field. The fundamental matter multiplets $\Phi^+_{+I} = (\phi^+_I; \psi^+_{+IR}; \psi^+_{+IL}; F^+_{+I})$ and the anti-fundamental matter multiplets $\Phi^-_{-I'} = (\phi^-_{-I'}; \psi^-_{-IR}; \psi^-_{-IL}; F^-_{-I'})$ are from four-dimensional $\mathcal{N} = 1$ chiral multiplets. $I = 1, \ldots, n_+$ and $I' = 1, \ldots, n_-$ are flavor indices of fundamentals and anti-fundamentals, respectively. $\phi^+_I$ is a scalar, $\psi^+_{+IR}, \psi^+_{+IL}$ are fermions, and $F^+_{+I}$ is an auxiliary field. Similar notations are used for the anti-fundamentals. There are also anti-chiral multiplets belonging to the fundamental representation $\Phi^+_{+I} = \Phi^+_{+I} = (\phi^+_I; \bar{\psi}^+_{-IR}; \bar{\psi}^+_{-IL}; F^+_{+I})$ and to the anti-fundamental representation $\Phi^-_{-I'} = (\phi^-_{-I'}; \psi^-_{-IR}; \psi^-_{-IL}; F^-_{-I'})$.

The classical theory has two kinds of R-symmetries. $U(1)_V$ originates from the chiral symmetry of four-dimensional $\mathcal{N} = 1$ theory, and $U(1)_A$ is from a rotational symmetry on a dimensionally reduced two-dimensional plane.

The continuum action is expressed in terms of dimensionally reduced four-dimensional $\mathcal{N} = 1$ superfields as

\begin{align}
S &= S_{\text{SYM}} + S_{\text{mat}} + S_{\text{pot}} + S_{\text{FI,} \bar{\theta}}, \\
S_{\text{SYM}} &= \frac{1}{8g^2} \int d^2x \, \text{tr} \left( W^\alpha W_\alpha |_{\theta\theta} + \bar{W} \bar{W} |_{\bar{\theta}\bar{\theta}} \right), \\
S_{\text{mat}} &= \int d^2x \left[ \sum_{I=1}^{n_+} \Phi^+_{+I} e^V \Phi^+_I + \sum_{I'=1}^{n_-} \Phi^-_{-I'} e^{-V} \Phi^-_{I'} \right] |_{\theta\theta\bar{\theta}\bar{\theta}}, \\
S_{\text{pot}} &= \int d^2x \ W(\Phi^+_+, \Phi^-_-) |_{\theta\theta} + \int d^2x \ \bar{W}(\Phi^+_+, \Phi^-_-) |_{\bar{\theta}\bar{\theta}}, \\
S_{\text{FI,} \bar{\theta}} &= \int d^2x \, \text{tr} \left( -\kappa D + \frac{\bar{\theta}}{2\pi} F_{01} \right),
\end{align}

$S_{\text{pot}}$ gives superpotential terms, and $S_{\text{FI,} \bar{\theta}}$ represents the Fayet-Iliopoulos (FI) and $\bar{\theta}$-terms.
3. SYM part (A model twist)
After taking the Wess-Zumino gauge, the Euclidean action of the continuum SYM theory becomes

\[ S^{(E)}_{\text{SYM}} = \frac{1}{g^2} \int d^2 x \, \text{tr} \left( \frac{1}{2} F_{\mu \nu} F_{\mu \nu} + D_\mu \phi D_\mu \bar{\phi} + \frac{1}{4} [\phi, \bar{\phi}]^2 - D^2 \right. 
\]
\[ \left. + 4 \lambda_R D_2 \lambda_R + 4 \lambda_L D_2 \lambda_L + 2 \lambda_R [\phi, \lambda_L] + 2 \lambda_L [\phi, \lambda_R] \right). \]  

(6)

\( Q \)-SUSY transformation acts component fields as

\[ QA_\mu = \psi_\mu, \quad Q \psi_\mu = i D_\mu \phi, \]
\[ Q \phi = 0, \quad Q \bar{\phi} = \eta, \quad Q \eta = [\phi, \bar{\phi}], \]
\[ Q \chi = i D + i F_{01}, \quad Q D = -Q F_{01} - i [\phi, \chi], \]

(7)

where \( Q \) is nilpotent in the sense that

\[ Q^2 = (\text{infinitesimal gauge transformation with the parameter } \phi). \]  

(8)

Here, the gaugino fields were renamed as

\[ \psi_0 = \frac{1}{\sqrt{2}} (\lambda_L + \bar{\lambda}_R), \quad \psi_1 = \frac{i}{\sqrt{2}} (\lambda_L - \bar{\lambda}_R), \]
\[ \chi = \frac{1}{\sqrt{2}} (\lambda_R - \bar{\lambda}_L), \quad \eta = -i \sqrt{2} (\lambda_R + \bar{\lambda}_L). \]

(9)

Under the \( U(1)_A \) rotation, the fields change as

\[ \phi \to e^{i2\alpha} \phi, \quad \bar{\phi} \to e^{-i2\alpha} \bar{\phi}, \]
\[ \psi_\mu \to e^{i\alpha} \psi_\mu, \quad \chi \to e^{-i\alpha} \chi, \quad \eta \to e^{-i\alpha} \eta. \]

(10)

Note that the action (6) is expressed as \( Q \)-exact form:

\[ S^{(E)}_{\text{SYM}} = Q \frac{1}{g^2} \int d^2 x \, \text{tr} \left[ -i \chi (F_{01} - D) + \frac{1}{4} \eta [\phi, \bar{\phi}] - i \psi_\mu D_\mu \bar{\phi} \right]. \]

(11)

The action is manifestly symmetric under \( \lambda_R \leftrightarrow \bar{\lambda}_R, \lambda_L \leftrightarrow \bar{\lambda}_L \), but not \( \lambda_R \leftrightarrow \lambda_L \) and \( \bar{\lambda}_R \leftrightarrow \bar{\lambda}_L \).

3.1. \( Q \)-SUSY on lattice (A model twist)
As gauge fields on lattice, we consider compact link variables \( U_\mu(x) = e^{iA_\mu(x)}. \) \( x \) represents a lattice site, and all the other fields are put on sites. Then, the \( Q \)-SUSY satisfying (8) can be realized on the lattice as

\[ QU_\mu(x) = i \psi_\mu(x) U_\mu(x) \]
\[ Q\psi_\mu(x) = i \psi_\mu(x) \psi_\mu(x) + i a \nabla_\mu \phi(x) \]
\[ Q \phi(x) = 0 \]
\[ Q \bar{\phi}(x) = \eta(x), \quad Q \eta(x) = [\phi(x), \bar{\phi}(x)] \]
\[ Q \chi(x) = i D(x) + \frac{i}{2} \hat{\phi}(x), \quad Q D(x) = -\frac{1}{2} Q \hat{\phi}(x) - i [\phi(x), \chi(x)], \]

(12)
where \(a \nabla_\mu \phi(x) \equiv U_\mu(x)\phi(x + \mu)U_\mu(x)^{-1} - \phi(x)\) and
\[
\hat{\Phi}(x) \equiv -i(\langle U_{01}(x) - U_{10}(x) \rangle) / (1 - 1/|1 - U_{01}(x)|^2)
\]
corresponds to \(2F_{01}\) in the naive continuum limit. \(U_{\mu\nu}(x)\) are plaquette variables defined by
\[
U_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x + \mu)U_\mu(x + \nu)^{-1}U_\nu(x)^{-1},
\]
and \(\epsilon\) is a positive number chosen as \(0 < \epsilon < \frac{2}{\sqrt{N}}\) for \(G = U(N)\), \(0 < \epsilon < \frac{2\sqrt{2}}{\sqrt{N}}\) for \(G = SU(N)\) with \(N = 2, 3, 4\), and \(0 < \epsilon < 2\sin \left(\frac{\pi}{N}\right)\) for \(G = SU(N)\) with \(N \geq 5\), which is introduced to impose the admissibility condition for the plaquette variable
\[
||1 - U_{01}(x)|| < \epsilon
\]
at each site \(x\). Here, the norm of an arbitrary \(M \times M\) complex matrix \(A\) is defined as
\[
||A|| \equiv \sqrt{\frac{1}{M} \text{tr}(AA^\dagger)}.
\]
Note that, since \(\epsilon\) is independent of the lattice spacing \(a\), gauge fields become almost unconstrained as approaching the continuum limit. As is seen below, the admissibility condition plays a crucial role to construct a \(Q\) invariant lattice action exhibiting correct weak coupling behavior.

### 3.2. \(Q\) invariant lattice action (A model twist)

The lattice action can be constructed as \(Q\)-exact form:
\[
S_{\text{SYM}}^{(\text{lat A})} = \frac{Q}{g_0^2} \sum_x \text{tr} \left[ \chi(x) \left\{ -i \hat{\Phi}(x) + iD(x) \right\} + \frac{1}{4} \eta(x)[\phi(x), \bar{\phi}(x)] - i \sum_\mu \psi_\mu(x) a \nabla_\mu \bar{\phi}(x) \right]
\]
for admissible gauge fields satisfying (15) for \(\forall x\), and
\[
S_{\text{SYM}}^{(\text{lat A})} = \infty
\]
(meaning that the Boltzmann weight vanishes) otherwise. Since the action (16) has the form \(Q\) acting to gauge invariant terms, it can be seen to be invariant under \(Q\) transformations from the nilpotency (8).

After the \(Q\) action in RHS of (16), we have
\[
S_{\text{SYM}}^{(\text{lat A})} = \frac{1}{g_0^2} \sum_x \text{tr} \left[ \frac{1}{4} \hat{\Phi}(x)^2 + a^2 \sum_\mu \nabla_\mu \phi(x) \nabla_\mu \bar{\phi}(x) + i \chi(x)Q\hat{\Phi}(x) 
\right.
\]
\[
+ i \sum_\mu \psi_\mu(x) a \nabla_\mu \eta(x) - D(x)^2
\]
\[
+ \frac{1}{4}[\phi(x), \bar{\phi}(x)]^2 - \chi(x)[\phi(x), \chi(x)] - \frac{1}{4} \eta(x)[\phi(x), \eta(x)]
\]
\[
- \sum_\mu \psi_\mu(x) \psi_\mu(x) \left( \bar{\phi}(x) + U_\mu(x)\bar{\phi}(x + \mu)U_\mu(x)^{-1} \right) \right].
\]
Without the admissibility (15) and the denominator of \(\hat{\Phi}\), gauge kinetic terms would be proportional to
\[
- \text{tr} (U_{01}(x) - U_{10}(x))^2 = \text{tr} (2 - U_{01}(x)^2 - U_{10}(x)^2),
\]
leading to the classical minima

\[ U_{01}(x) = \begin{pmatrix} \pm 1 & \cdots & \pm 1 \end{pmatrix} \] (up to gauge transformations) \hspace{1cm} (20)

for \( \forall x \). The minima have a huge degeneracy, and we should take into account fluctuations around all the minima, although the weakly coupled continuum theory is reproduced as fluctuations around a single minimum \( U_{01}(x) = 1 \). Thus, the connection of the lattice model to the continuum theory would become unclear. To avoid such situation and to single out the vacuum \( U_{01}(x) = 1 \), the admissibility and \( \hat{\Phi} \) are employed here [2].

In the lattice perturbation theory, we can show that the lattice model does not need any fine-tuning to obtain the target continuum theory with the full \( N = (2, 2) \) SUSY. Moreover, by computer simulation, ref. [8] performed a nonperturbative check that the full \( N = (2, 2) \) SUSY is restored in the continuum limit of the lattice theory with \( G = SU(2) \).

The FI and \( \vartheta \)-terms (5) can be latticized maintaining the \( Q \) invariance [3, 4].

4. SYM part (B model twist)

In B-model twist case, we consider \( G = U(N) \) gauge group. It is convenient to take linear combinations of field variables as

\[
\begin{align*}
\psi_0' &= \frac{1}{\sqrt{2}}(\lambda_L - \lambda_R), \\
\psi_1' &= \frac{i}{\sqrt{2}}(\lambda_L + \lambda_R), \\
\chi' &= \frac{1}{\sqrt{2}}(\bar{\lambda}_L + \bar{\lambda}_R), \\
\eta' &= \frac{i}{\sqrt{2}}(\bar{\lambda}_L - \bar{\lambda}_R), \\
X_0 &= -\frac{1}{2}(\phi + \bar{\phi}), \\
X_1 &= -\frac{i}{2}(\phi - \bar{\phi}), \\
D' &= D - D_\mu X_\mu, \\
A_\mu &= A_\mu - iX_\mu, \\
A_\mu^\dagger &= A_\mu + iX_\mu.
\end{align*}
\] \hspace{1cm} (21)

Then, \( Q' \)-SUSY transforms the fields as

\[
\begin{align*}
Q' A_\mu &= 0, \\
Q' A_\mu^\dagger &= 2\psi_\mu', \\
Q' \psi_\mu' &= 0, \\
Q' \chi' &= -iF_{01}, \\
Q' \eta' &= D', \\
Q' D' &= 0,
\end{align*}
\] \hspace{1cm} (22)

where \( F_{01} \) is the field strength of the complexified gauge potentials \( A_\mu \). Note that \( Q' \) is exactly nilpotent: \( (Q')^2 = 0 \).

The action (6) can be recast into \( Q' \)-exact form:

\[
S_{SYM}^{(E)} = Q' \frac{1}{g^2} \int d^2x \tr \left[ -\eta'(D' + 2D_\mu X_\mu) + i\chi' F_{01}^\dagger \right].
\] \hspace{1cm} (23)

It is manifestly symmetric under \( \lambda_R \leftrightarrow \lambda_L \) and \( \bar{\lambda}_R \leftrightarrow \bar{\lambda}_L \), but not \( \lambda_R \leftrightarrow \bar{\lambda}_L \) and \( \lambda_L \leftrightarrow \bar{\lambda}_R \).

4.1. \( Q' \)-SUSY on lattice (B model twist)

In the lattice formulation based on B twist, we introduce positive definite hermitian variables \( V_\mu(x) = e^{-aX_\mu(x)} \) defined on sites for scalar fields. Then, \( V_\mu(x)U_\mu(x) \sim e^{iaA_\mu(x)} \) near the continuum limit.
The nilpotent $Q'$-SUSY can be realized on the lattice as
\begin{align}
Q'U_\mu(x) &= i\psi'_\mu(x)U_\mu(x), & Q'\psi_\mu(x) &= i\psi_\mu(x)\psi_\mu(x), \\
Q'V_\mu(x) &= i\psi'_\mu(x)V_\mu(x), & Q'\chi'(x) &= 1 - U_{01}(V^{-1}U)(x), \\
Q'\eta'(x) &= D'(x), & Q'D'(x) &= 0,
\end{align}
where $U_{01}(V^{-1}U)(x)$ means a plaquette variable constructed by the links $V_\mu^{-1}U_\mu \in GL(N, C)$.

4.2. $Q'$ invariant lattice action

On the analogy of the continuum case (23), we define the lattice action by
\begin{equation}
S^{(\text{lat B})}_{\text{SYM}} = \frac{1}{g_0^2} \sum_x \text{tr} \left[ -\eta'(x)(D'(x) + W(x)) + \chi'(x) \left( 1 - U_{01}(V^{-1}U)(x) \right)^2 \right], \tag{25}
\end{equation}
where $W(x)$ is a lattice counterpart of $2D_\mu X_\mu$ chosen as
\begin{equation}
W(x) = 2 \sum_x \left[ V_\mu(x)^{-1} + U_\mu(x - \hat{\mu})^{-1}V_\mu(x - \hat{\mu})U_\mu(x - \hat{\mu}) - 2 \right]. \tag{26}
\end{equation}

As a result of the $Q'$ action in RHS of (25), we have
\begin{equation}
S^{(\text{lat B})}_{\text{SYM}} = \frac{1}{g_0^2} \sum_x \text{tr} \left[ -D'(x)^2 - D'(x)W(x) + \left| 1 - U_{01}(V^{-1}U)(x) \right|^2 \right. \\
&\quad \left. + \chi'(x)Q'U_{01}(VU)(x) + \eta'(x)Q'W(x) \right]. \tag{27}
\end{equation}

The minimum of the bosonic part should satisfy
\begin{equation}
W(x) = 0, \quad U_{01}(V^{-1}U)(x) = 1, \tag{28}
\end{equation}
from which we obtain \(^2\)
\begin{equation}
0 = \sum_x \text{tr} W(x) = \sum_x \text{tr} \left[ (V_\mu(x)^{-1/2} - V_\mu(x)^{1/2})^2 \right], \tag{29}
\end{equation}
leading to $V_\mu(x) = 1$, and thus $U_{01}(x) = 1$. In this lattice formulation, a single desirable minimum $U_{01}(1) = 1$ is obtained for gauge fields, and flat directions for scalar fields are lifted.

5. Matter part

As seen in the SYM part, $Q$-SUSY respects the symmetry under $\lambda_R \leftrightarrow \bar{\lambda}_L$ and $\lambda_L \leftrightarrow \bar{\lambda}_R$, while $Q'$-SUSY respects the symmetry under $\lambda_R \leftrightarrow \lambda_L$ and $\bar{\lambda}_R \leftrightarrow \bar{\lambda}_L$. Correspondingly, for the matter part, $Q$-SUSY respects the symmetry under
\begin{equation}
\psi_{\pm IR} \leftrightarrow \bar{\psi}_{\mp IL}, \quad \psi_{\pm IL} \leftrightarrow \bar{\psi}_{\mp IR}, \tag{30}
\end{equation}
while $Q'$-SUSY respects the symmetry under
\begin{equation}
\psi_{\pm IR} \leftrightarrow \psi_{\pm IL}, \quad \bar{\psi}_{\pm IR} \leftrightarrow \bar{\psi}_{\pm IL}. \tag{31}
\end{equation}
\(^2\) Note that $V_\mu(x)^{1/2}$ is unambiguously defined, since $V_\mu(x)$ is positive definite.
5.1. $Q$ invariant lattice formulation (A model twist)
In ref. [3], the Wilson terms were introduced to suppress the bosonic and fermionic doublers. Note that the Wilson terms play a role of $\mathcal{O}(1/a)$ mass terms coupling $\Phi_{+\hat{I}}$ with $\Phi_{-\hat{I}}$ and coupling $\Phi_{+\hat{I}}^\dagger$ with $\Phi_{-\hat{I}}^\dagger$. So, it is possible to maintain the $Q$ invariance only when $n_+ = n_- (\equiv n)$. The Wilson terms breaks chiral flavor symmetry $U(n_+) \times U(n_-)$ to its diagonal subgroup $U(n)$.

Ref. [4] applied Ginsparg-Wilson formulation [9] to realize both of $Q$-SUSY and the chiral flavor symmetry for general $n_\pm$. As an interesting feature of this construction, holomorphic structure of superpotential is not ruined by lattice artifacts.

When $n_+ \neq n_-$, $U(1)_A$ R-symmetry is anomalous. The Ginsparg-Wilson formulation reproduces the correct anomaly in the form of the trace of the Ginsparg-Wilson Dirac operator $\hat{D}$:

$$\text{Tr} \left( \gamma_3 a \hat{D} \right) \simeq \frac{1}{\pi} \int d^2 x \text{tr} F_{01}, \quad (32)$$

which arises from the $U(1)_A$ transformation to path-integral measure of the lattice theory.

When using the Ginsparg-Wilson formulation, $\epsilon$ appearing in the admissibility condition is chosen as follows. For $G = U(N)$ without $\vartheta$-term, $0 < \epsilon < \frac{1}{5}$ if $N = 1, \ldots, 100$, and $0 < \epsilon < \frac{2}{\sqrt{N}}$ if $N \geq 101$. For $G = U(N)$ with $\vartheta$-term, $0 < \epsilon < \frac{1}{5}$ if $N = 1, \ldots, 25$, and $0 < \epsilon < \frac{1}{\sqrt{N}}$ if $N \geq 26$. For $G = SU(N)$, $0 < \epsilon < \frac{1}{5}$ if $N = 2, 3, \ldots, 31$, and $0 < \epsilon < 2 \sin \left( \frac{\pi}{N} \right)$ if $N \geq 32$.

5.2. $Q'$ invariant lattice formulation (B model twist)
From the symmetry (31) for fermions, $Q'$ invariant formulation seems more natural to preserve the chiral flavor symmetry.

As discussed in ref. [5], in the case $n_- = 0$ (only fundamental matters) or $n_+ = 0$ (only anti-fundamental matters), the $Q'$ invariant lattice action can be shown to have a single minimum at $V_0(x) = 1, U_{01}(x) = 1$.

Since the $Q'$ invariant lattice action is not required to impose the admissibility condition, it takes simpler form compared with the $Q$ invariant action. However, it can be applied to the gauge group $G = U(N)$, not $G = SU(N)$ at present. It is difficult to introduce superpotential with preserving $Q'$-SUSY. It is because the anti-holomorphic part of superpotential $\overline{W}$ is written as $Q'$-exact while the holomorphic part $W$ is not as $Q'$-exact.

6. Summary and discussion
We have presented the two kinds of lattice formulations of two-dimensional $\mathcal{N} = (2, 2)$ SQCD preserving one supercharge on the lattice.

One is the $Q$ invariant formulation, which is based on A model twist. The lattice model can be constructed for $G = U(N)$ or $SU(N)$. Moreover, by employing the Ginsparg-Wilson operator which realizes exact chiral flavor symmetry on the lattice, it is possible to formulate the model for general $n_\pm$ and for general superpotential.

The other is the $Q'$ invariant formulation, based on B model twist. It overcomes difficulty of degenerate minima and flat directions without using the admissibility condition, and thus the lattice action has a simpler form compared with the case of the $Q$ invariant formulation. However, its applicability is restricted to $G = U(N)$ and to $(n_+, n_-) = (n_+, 0)$ or $(0, n_-)$ at present.

There are different lattice formulations of SYM theories based on orbifolding/deconstruction approach from matrix models [10, 11, 12, 13]. Ref. [14] pointed out that the supercharge preserved in such formulations corresponds to $Q'$ from B model twist. Since bifundamental fields play a key role in the formulation, it is applicable to $G = U(N)$, but difficult to $G = SU(N)$, because it is impossible to impose a traceless condition on fundamental fields in a gauge invariant way.
In ref. [15], two-dimensional $\mathcal{N} = (2, 2)$ SQCD has been constructed in the orbifolding-deconstruction approach. Let us start from matrix models with eight supercharges, corresponding to $\mathcal{N} = (4, 4)$ in two dimensions. By orbifolding and deconstruction, $U(N_c) \times U(N_f)$ quiver gauge theory in two dimensions is obtained. If $U(N_f)$ adjoint fields are killed by hand, $U(N_f)$ gauge symmetry reduces to global flavor symmetry with $Q'$-SUSY intact. Then, SQCD with $n_+ = n_- (= N_f)$ and without superpotential can be constructed on two-dimensional lattice. The lattice theory has somewhat different structure from our $Q'$ invariant formulation.

For future work, a number of interesting subjects are waiting for us. Firstly, the Ginsparg-Wilson formulation to two-dimensional $\mathcal{N} = (2, 2)$ SQCD [4] can be applied to the lattice study of gauged linear sigma models. Due to the exact chiral structure, we can separately couple matter multiplets belonging to different representations of $G$ to different chiral sectors of the model. Two-dimensional $\mathcal{N} = (2, 2)$ SQCD models with various superpotentials have been analytically investigated based on the effective twisted superpotentials [16, 17, 18]. The number of the vacua or the Witten index of the models has been computed for various $N, n_{\pm}$, and an analog of the Seiberg duality in four dimensions has been discussed based on the symmetry of Grassmannians. It will be worth confirming those properties and exploring new aspects, which are not yet investigated there, from the first principle computation using this lattice formulation.

Secondly, concerning nonperturbative aspects of superstring theory, 2d $\mathcal{N} = (8, 8)$ SYM with $G = U(N)$ is known as matrix string theory [19], which is conjectured to give type IIA superstring theory in low energy regime at large $N$. It will be important to construct its lattice formulation with no fine-tuning required.

Finally, we hope that the above lattice formulation will give some insights to explore cases of higher-dimensional theories.

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