Distinguishability of quantum states under restricted families of measurements
with an application to quantum data hiding

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The starting point of the present investigation is the well-known result by Helstrom,
identifying the best achievable bias in distinguishing two quantum states under all mea-
measurements as (half) the trace norm of their difference. We turn this around, noticing that
every sufficiently rich set \(M\) of measurements on a fixed quantum system defines a statisti-
cal norm \(\|\cdot\|_M\) on the states of that system, via the optimal bias achievable when restricted
to \(M\). These norms are all upper bounded by the usual trace norm, and in finite dimen-
sional Hilbert spaces they are all equivalent to the trace norm in the sense that there exist
“constants of domination” \(\lambda\) and \(\mu\), such that

\[
\lambda \|\cdot\|_1 \leq \|\cdot\|_M \leq \mu \|\cdot\|_1
\]

which are optimal in the sense that there exist states such that the bounds are tight. In other
words, if we rate the performance of a set of measurements in distinguishing a given pair of
states (of equal prior probability) as the ratio of the largest bias that can be obtained by such
measurements to the best bias achievable when allowing all measurements, then \(\lambda\) and \(\mu\)
determine the worst and best case performance respectively for any pair of states. Here we
set ourselves the task of computing, or at least bounding such constants for various sets of
measurements \(M\).

Specifically, we look at the case that \(M\) consists only of a single measurement, namely
the uniformly random POVM, 2-designs, and 4-designs where we find asymptotically tight
bounds for \(\lambda\) and \(\mu\). Furthermore, we analyse the multipartite setting where the set of
measurements consists of all POVMs implementable by local operations and classical com-
unication (among other, related classes).

In the case of two parties, we show that the lower domination constant \(\lambda\) is the same as
that of a tensor product of local uniformly random POVMs up to a constant. This answers
in the affirmative an open question about the (near-)optimality of bipartite data hiding:
The bias that can be achieved by LOCC in discriminating two orthogonal states of a \(d \times d\)
bipartite system is \(\Omega(1/d)\), which is known to be tight. Finally, we use our analysis to derive
certainty relations (in the sense of Sanchez-Ruiz) for any such measurements and to lower
bound the locally accessible information for bipartite systems.

INTRODUCTION

Quantum measurements, as described by positive operator valued measures (POVMs) \((M_k) := \sum_{k=1}^{n} |k\rangle \langle k|\) on some Hilbert space \(\mathcal{H}\), can be viewed as completely positive maps from the set of
density operators to probability vectors

\[
\mathcal{M} : \xi \mapsto \sum_{k=1}^{n} |k\rangle \langle k| \operatorname{Tr}(\xi M_k),
\]
where we represent the probability vector as an operator, diagonal in the “label” basis \( \{ |k\rangle \} \) of the \( n \)-dimensional space \( \mathbb{C}^n \). As such, they are non-increasing for the statistical distance given by the trace norm \( \|X\|_1 = \text{Tr} |X| = \text{Tr} \sqrt{X^\dagger X} \): since \( \mathcal{M} \) is a completely positive and trace preserving (CPTP) map, we have that for any operator \( X \)

\[
\|\mathcal{M}(X)\|_1 \leq \|X\|_1. \tag{1}
\]

If \( X \geq 0 \), then equality holds as both the left, and the right hand side equal \( \text{Tr}(X) \). However, for traceless Hermitian operators \( X \), which are all – up to a scalar factor – of the form \( \rho - \sigma \) with states \( \rho \) and \( \sigma \) (i.e. positive semidefinite operators of trace 1), the inequality

\[
\|\mathcal{M}(\rho - \sigma)\|_1 \leq \|\rho - \sigma\|_1 \tag{2}
\]

is typically strict. On the other hand, there always exists a measurement that saturates this inequality – for instance the spectral measure of \( \rho - \sigma \). In fact, the two-outcome projective measurement \( (M_0, \mathbb{1} - M_0) \) with \( M_0 \) a projector onto the positive eigenspace of \( \rho - \sigma \) makes eq. [2] an equality. This is essentially the content of Helstrom’s Theorem: consider a situation where we want to discriminate between two (a priori equiprobable) states \( \rho \) and \( \sigma \). Then, for a given measurement POVM \( (M_k) \), we base our decision on the probability vectors \( \vec{p} = (\text{Tr}(\rho M_k)) \) and \( \vec{q} = (\text{Tr}(\sigma M_k)) \). The optimal decision rule is easily seen to be the maximum likelihood rule: observing \( k \), we decide on \( \rho \) if \( p_k > q_k \), otherwise on \( \sigma \). This gives us error probability

\[
P_E = \frac{1}{2} - \frac{1}{4} \sum_k |p_k - q_k| = \frac{1}{2} - \frac{1}{4} \|\vec{p} - \vec{q}\|_1 = \frac{1}{2} - \frac{1}{4} \|\mathcal{M}(\rho - \sigma)\|_1,
\]

where for the probability vectors we refer to the \( \ell^1 \)-norm and for the image of \( \mathcal{M} \) to the trace norm, since they coincide for diagonal matrices. The quantity

\[
\beta((M_k); \rho - \sigma) := \frac{1}{2} \|\vec{p} - \vec{q}\|_1 = \frac{1}{2} \|\mathcal{M}(\rho - \sigma)\|_1,
\]

ranging between 0 (for identical probability vectors) and 1 (for orthogonal probability vectors) is known as the bias of the POVM on the state pair \( (\rho, \sigma) \). Helstrom’s Theorem is the statement that the minimum error probability over all POVMs is \( \frac{1}{2} - \frac{1}{4} \|\rho - \sigma\|_1 \), corresponding to the maximum bias being \( \frac{1}{2} \|\rho - \sigma\|_1 \).

Thus we are motivated to look in generality at the situation where we are restricted in our choice of measurement. Formally, let \( \mathcal{M} \) be a set of POVMs. For convenience, take the elements of \( \mathcal{M} \) to be discrete POVMs, since we will show below that we lose nothing by restricting to two-outcome POVMs. The optimal bias achievable by \( \mathcal{M} \) for \( \xi = \rho - \sigma \) is given by \( \frac{1}{2} \|\xi\|_\mathcal{M} \) where the norm is defined by

\[
\|\xi\|_\mathcal{M} := \sup_{(M_k) \in \mathcal{M}} \|\mathcal{M}(\xi)\|_1. \tag{3}
\]

In Section 1, we start by recording a few simple implications and properties of this definition. First of all, we will see that it really is a norm. We make a connection to general norms in vector spaces, showing in particular that any norm on trace class operators can be interpreted as a norm of the above type. We then turn to a number of particular examples, highlighting especially the problem of determining the constants of domination of \( \| \cdot \|_\mathcal{M} \) with respect to \( \| \cdot \|_1 \). In Section 2, we investigate the particular case where \( \mathcal{M} \) consists of only one (necessarily informationally complete) POVM, finding the best constants of domination. These constants are attained for the isotropic (unitary invariant) POVM. We also show how to analyse the situation for POVMs
originating from 2- and 4-designs. In Section [3] we look at the situation that the system under consideration is bi- or multipartite, and that the POVMs are restricted to classes respecting the partition: local measurements, with or without classical communication between the parties, and extensions of this class. The existence of data hiding [10, 16, 25] states yields bounds on the constants of domination in one direction. Most notably, we show here that in the bipartite case, these bounds are optimal up to a constant factor by analysing the tensor product of two isotropic local POVMs: it turns out that the resulting measurement attains almost the same bias. Hence, the hiding states of [25] are already (near) optimal in the sense that we cannot hope to construct states which are less well distinguishable under LOCC operations. In Section [4] we make a connection to Sanchez-Ruiz’ “certainty relations” for mutually unbiased bases [23], which we show holds more generally for any 2-design POVM, and – even in a stronger form – for 4-designs. We also show how our results for bipartite systems imply a universal lower bound on the information accessible by LOCC from any pure state ensemble. Several appendices contain the proofs of more technical results in the main text.

1. FIRST OBSERVATIONS ON NORMS AND DUAL NORMS

Before turning to the essential observations that we will need later on, we first explain some basic concepts. At the heart of the Helstrom-Holevo Theorem on optimal state discrimination lies the duality between the operator norm \( \| \cdot \| \) and the trace norm \( \| \cdot \|_1 \): For operators \( \alpha, A \) on a Hilbert space \( \mathcal{H} \), these are dual to each other with

\[
\| \alpha \|_1 = \sup_{\| B \| \leq 1} | \text{Tr}(\alpha^\dagger B) |,
\]

\[
\| A \| = \sup_{\| \beta \|_1 \leq 1} | \text{Tr}(\beta \dagger A) |.
\]

In finite dimension, which we shall assume throughout this paper, the suprema are easily seen to be maxima. The duality persists when we restrict to Hermitian (self-adjoint) operators \( \alpha = \alpha^\dagger, A = A^\dagger \):

\[
\| \alpha \|_1 = \max_{B = B^\dagger, \| B \| \leq 1} \text{Tr}(\alpha B),
\]

\[
\| A \| = \max_{\beta = \beta^\dagger, \| \beta \|_1 \leq 1} \text{Tr}(\beta A).
\]

These equations are direct consequences of the singular value decomposition in the general, and of the spectral theorem in the Hermitian case.

The role of the Hilbert-Schmidt inner product, which makes the real vector space of Hermitian operators, \( B_{sa}(\mathcal{H}) \), a Euclidean space, becomes more evident in geometrical language by saying that the unit balls

\[
B_1(\| \cdot \|_1) = \{ \alpha = \alpha^\dagger : \| \alpha \|_1 \leq 1 \},
\]

\[
B_1(\| \cdot \|) = \{ A = A^\dagger : \| A \| \leq 1 \},
\]

are polar to each other. To explain this notion, note that the unit ball of any norm \( N \) on a finite dimensional real vector space,

\[
K := B_1(N) = \{ x : N(x) \leq 1 \},
\]
is a topologically closed, convex and symmetric set (i.e. \( K = -K \)), containing the origin 0 in its interior. Any such body \( K \) conversely determines a norm

\[
\|x\|_K = \inf \left\{ \frac{1}{t} : t > 0 \text{ and } tx \in K \right\},
\]

and it is immediately verified that \( K = B_1(\| \cdot \|_K) \) and \( N = \| \cdot \|_K \). That is, norms and convex closed symmetric bodies of full dimension are equivalent descriptions. Now, the polar of \( K \) in a Euclidean vector space with inner product \( \langle \cdot, \cdot \rangle \) is defined to be

\[
\tilde{K} := \{ y : \forall x \in K \langle x, y \rangle \leq 1 \}.
\]

It is easy to verify that if \( K \) is symmetric, convex and closed, and contains the origin in its interior, then \( \tilde{K} \) has the same properties, and \( \tilde{\tilde{K}} = K \).

By the above discussion, \( K \) is the unit ball of \( \| \cdot \|_K \), \( \tilde{K} \) is the unit ball of \( \| \cdot \|_{\tilde{K}} \) and one has the important, but elementary, formulas

\[
\|y\|_K = \max_{x \in K} \langle x, y \rangle,
\]
\[
\|x\|_{\tilde{K}} = \max_{y \in \tilde{K}} \langle x, y \rangle.
\]

which are the abstract versions of the equations above.

We are now ready to make a series of observations. First, we need to show that eq. (3) really does constitute a norm for trace class operators, i.e. for operators with a finite, well-defined, trace. We thereby call a set of POVMs \( M \) separating, or “informationally complete” if for any nonzero operator \( \xi \neq 0 \), there exists a POVM \( (M_k) \in M \) and an index \( k_0 \) such that \( \text{Tr}(\xi M_k) \neq 0 \).

**Lemma 1** Eq. (3) above defines a norm on the set of trace class operators if and only if the set of POVMs is separating. Furthermore, \( \| \cdot \|_M \leq \| \cdot \|_1 \).

**Proof.** An immediate consequence of the fact that \( \| \cdot \|_1 \) is a norm, and eq. (2). \( \square \)

We now show that we can restrict ourselves to POVMs with 2 outcomes. Intuitively, since we decide between two options (e.g. \( \rho \) and \( \sigma \) above), we can group the outcomes of each POVM in two. It is then not difficult to verify that

**Lemma 2** For any separating set \( M \) of POVMs \( (M_k)_{k=1}^n \) the set of 2-outcome POVMs,

\[
M_2 := \{ (M, \mathbb{1} - M) : \exists (M_k)_{k=1}^n \in M, \, \exists I \subseteq [n] \, \| \cdot \| = \sum_{k \in I} M_k \},
\]

satisfies \( \| \cdot \|_M = \| \cdot \|_{M_2} \). Furthermore, the set

\[
M := \text{conv} \{ 2M - \mathbb{1} : (M, \mathbb{1} - M) \in M_2 \}
\]

is a (closed) symmetric convex body, contained in the operator interval \([\mathbb{1}; \mathbb{1}] = \{ X : -\mathbb{1} \leq X \leq \mathbb{1} \}\) and containing \( \pm \mathbb{1} \), and is of full dimension, such that

\[
\|\xi\|_M = \max_{M \in M} |\text{Tr}(\xi M)| =: \|\xi\|_M.
\]
Proof. For a given $\xi = \rho - \sigma$ and POVM $(M_k)_{k=1}^n \in \M$, the bias achieved is simply $\sum_{k=1}^n |\Tr(M_k \xi)|$.

The same bias is achieved by the 2-outcome POVM $(M_+, M_-) \in \M_2$, where

$$M_+ = \sum_{k \in P} M_k, \quad P = \{ k \in [n] : \Tr(M_k \xi) \geq 0 \},$$

$$M_- = \sum_{k \in N} M_k = \mathbb{1} - M_+, \quad N = \{ k \in [n] : \Tr(M_k \xi) < 0 \},$$

and clearly no other grouping of the elements of this POVM can result in a larger bias.

\[\Box\]

Note that $\M$ has a non-empty interior (and then contains the origin in its interior) if and only if the collection $\M$ is informationally complete, which the case if and only if $\M_2$ is informationally complete. Mathematically the information-completeness is expressed by $\M$, spanning the whole operator space. Furthermore, note that from our discussion above we have that

Remark 3 The symmetric convex body $\M$ defines two norms, one on the observables and effects, the other on the trace class operators, via

$$\| M \|_\M = \inf \left\{ \frac{1}{t} : t > 0 \text{ and } tM \in \M \right\},$$

$$\| \xi \|_\M = \max_{M \in \M} \Tr(\xi M).$$

The first has exactly $\M$ as its unit ball, the second has as its unit ball the polar of $\M$, i.e.

$$\M^* = \{ \xi : \forall M \in \M \ \Tr(\xi M) \leq 1 \}.$$

The norm $\| \cdot \|_\M = \| \cdot \|_\M^*$ is dual to $\| \cdot \|_\M^*$:

$$\| \xi \|_\M = \max \{ \Tr(\xi M) : \| M \|_\M \leq 1 \},$$

$$\| M \|_\M^* = \max \{ \Tr(\xi M) : \| \xi \|_\M \leq 1 \}.$$

Putting everything together, we can now see that

Theorem 4 The norms $\| \cdot \|_\M$ associated to sets of POVMs are in one-to-one correspondence with full-dimensional symmetric closed convex bodies $\pm \mathbb{1} \in \M \subseteq \mathbb{1} \mathbb{1} \mathbb{1} \mathbb{1} \mathbb{1} \mathbb{1}$. As a consequence, any norm $\| \cdot \|_1 \leq \| \cdot \|_1$ can be written as $\| \cdot \|_1 = \| \cdot \|_\M$ for some set of POVMs.

Proof. First, starting with a set of POVMs $\M$ defining norms $\| \cdot \|_\M$, Lemma 2 describes how to construct $\M$, such that $\| \cdot \|_\M = \| \cdot \|_\M$.

Conversely, starting with a full-dimensional symmetric closed convex body $\M \subseteq \mathbb{1} \mathbb{1} \mathbb{1} \mathbb{1} \mathbb{1} \mathbb{1}$, we can construct a set of POVMs $\M = \{ (M, \mathbb{1} - M) : M \in \M \text{ and } M \geq 0 \}$ for which $\| \cdot \|_\M = \| \cdot \|_\M$.

We formalise the connection with the state discrimination problem in the following theorem.

Theorem 5 Let $\M$ be a set of POVMs on a given Hilbert space, and let $\M_2$ and $\M$ be defined as above. For any two states $\rho$ and $\sigma$, consider the minimum error probability $P_E^\M$ of discriminating between these (a priori equiprobable states). Then,

$$P_E^\M = \inf_{(M, \mathbb{1} - M) \in \M_2} \frac{1}{2} - \frac{1}{2} |\Tr((\rho - \sigma)M)| = \frac{1}{2} - \frac{1}{4} \| \rho - \sigma \|_\M.$$

That is, $\frac{1}{4} \| \rho - \sigma \|_\M$ is the bias achievable in discriminating $\rho$ from $\sigma$ when only measurements in $\M$ are allowed. \[\Box\]
In finite dimension, which is the case we stick to in this paper, the operators also form a finite-dimensional space, and all these norms are “equivalent” in the sense that there are \( \lambda', \mu' > 0 \) such that

\[
\lambda' \cdot \| \cdot \|_1 \leq \| \cdot \|_M \leq \mu' \cdot \| \cdot \|_1. \tag{6}
\]

By using the above correspondences and dualities, we see that this is equivalent to

\[
\lambda'\{-1; 1\} \subseteq M \subseteq \mu'\{-1; 1\}. \tag{7}
\]

We will use \( \lambda_1(M) (\mu_1(M)) \) to denote the largest \( \lambda' \) (smallest \( \mu' \)) in these equations. The numbers \( \lambda_1 \) and \( \mu_1 \) are called the constants of domination of the norm \( \| \cdot \|_M \) (with respect to \( \| \cdot \|_1 \)). In the following, our goal is to bound these constants of domination for various interesting classes of POVMs. These constants are especially interesting, since we know from Theorem 5 that they allow us to bound the bias that we can achieve when trying to distinguish two states \( \rho \) and \( \sigma \) with a restricted set of measurements.

Note that \( \mu_1(M) \) is trivially 1 since for \( \rho \geq 0, \| \rho \|_M = \| \rho \|_1 = \text{Tr}(\rho) \). Thus, we are motivated to restrict to traceless operators in eq. (6). This is also the setting for which bounds on the constants of domination give us a bound on the bias of distinguishing two a priori equiprobable states \( \rho \) and \( \sigma \). Let \( \lambda(M) \) and \( \mu(M) \) be the largest and smallest numbers \( \lambda' \) and \( \mu' \), respectively, such that

\[
\forall \xi \text{ with } \text{Tr}(\xi) = 0 \quad \lambda \| \xi \|_1 \leq \| \xi \|_M \leq \mu \| \xi \|_1. \tag{8}
\]

Equivalently, in the dual picture we have to go to the quotient modulo multiples of the identity, \( \mathbb{R}/1 \):

\[
\lambda\{-1; 1\}/\mathbb{R} \subseteq M/\mathbb{R} \subseteq \mu\{-1; 1\}/\mathbb{R}. \tag{9}
\]

The following lemma characterizes \( \lambda_1 (\mu_1) \) and \( \lambda (\mu) \), and their respective relations.

**Lemma 6** For a set \( M \) of POVMs with associated convex body \( M \), the constants of domination can be expressed as the solutions of the following optimisation problems:

\[
\frac{1}{2} \lambda(M) \leq \lambda_1(M) = \inf_{\| \xi \|_1 = 1} \sup_{M_k \in M} \| M(\xi) \|_1 \leq \inf_{\| \xi \|_1 = 1} \sup_{\text{Tr}(\xi) = 0} \| M(\xi) \|_1 = \lambda(M),
\]

\[
1 = \mu_1(M) = \sup_{\| \xi \|_1 = 1} \sup_{M_k \in M} \| M(\xi) \|_1 \geq \sup_{\| \xi \|_1 = 1} \sup_{\text{Tr}(\xi) = 0} \| M(\xi) \|_1 = \mu(M).
\]

Here, for the purpose of \( \lambda \) and \( \mu \), \( \xi \) may be thought of as \( \xi = \frac{1}{2} (\rho - \sigma) \) for orthogonal states \( \rho, \sigma \).

**Proof.** The optimisation problems are an immediate consequence of the definitions, and we already argued that \( \mu_1(M) = 1 \). To lower bound \( \lambda_1(M) \) we proceed as follows: Given any \( \xi \) of trace norm 1, we can write it as

\[
\xi = (1 - p) \rho - p \sigma = (1 - p) (\rho - \sigma) + (1 - 2p) \sigma = 2(1 - p) \xi_0 + (1 - 2p) \sigma,
\]

with orthogonal states \( \rho \) and \( \sigma \), and \( \xi_0 = \frac{1}{2} (\rho - \sigma) \). W.l.o.g. \( 0 \leq p \leq 1/2 \), otherwise use \( -\xi \). Now let \( X_0 \in M \) be optimal for \( \xi_0 \), i.e. \( \| \xi_0 \|_M = \text{Tr}(\xi_0 X_0) \), and test \( \xi \) with \( X = (1 + X_0)/2 \in M \). Note \( X \geq 0 \), so

\[
\| \xi \|_M = \text{Tr}(\xi X) = (1 - p) \text{Tr}(\xi_0 X) + \frac{1 - 2p}{2} \text{Tr}(\sigma X)
\]

\[
= (1 - p) \text{Tr}(\xi_0 X_0) + \frac{1 - 2p}{2} \text{Tr}(\sigma X)
\]

\[
\geq \frac{1}{2} \text{Tr}(\xi X_0) = \frac{1}{2} \| \xi_0 \|_M \geq \frac{1}{2} \lambda(M),
\]

since for any state \( \sigma \),

\[
\lambda = \sup_{\| \xi \|_1 = 1} \sup_{\text{Tr}(\xi) = 0} \| M(\xi) \|_1 = \sup_{\| \xi \|_1 = 1} \sup_{\text{Tr}(\xi) = 0} \frac{1}{2} \text{Tr}(\xi X_0).
\]
concluding the proof. \[\square\]

What is the relation of the constants of domination for different sets \(M\) and \(M'\)? Clearly, if \(M \subseteq M'\), then \(\lambda(M) \leq \lambda(M')\) and \(\mu(M) \leq \mu(M')\). More interesting relations are obtained by using the convex structure. For this purpose we look at convex combinations of POVMs in the sense of direct sums as follows. For POVMs \((M_k)\) and \((N_\ell)\), and a real \(0 \leq p \leq 1\), we denote by \(p(M_k) \oplus (1-p)(N_\ell)\) the POVM consisting of the \(m + n\) elements

\[pM_1, \ldots, pM_m, (1-p)N_1, \ldots (1-p)N_n.\]

If the associated CPTP maps of the two original POVMs are \(\mathcal{M}\) and \(\mathcal{N}\), then the direct sum convex combination has associated CPTP map

\[p\mathcal{M} \oplus (1-p)\mathcal{N}: \xi \mapsto \sum_{k=1}^m |k\rangle\langle k| p \text{Tr}(\xi M_k) + \sum_{\ell=1}^n |m+\ell\rangle\langle m+\ell|(1-p) \text{Tr}(\xi N_\ell).\]

If we have two sets of POVMs, \(\mathcal{M}\) and \(\mathcal{N}\), then their direct sum convex combination is defined naturally as

\[p\mathcal{M} \oplus (1-p)\mathcal{N} = \{p(M_k) \oplus (1-p)(N_\ell) : (M_k) \in \mathcal{M}, (N_\ell) \in \mathcal{N}\}.\]

More generally, we can look at convex combinations of any finite or even countable number of POVMs and sets of POVMs. These constructions have a straightforward operational interpretation: implementing \(p(M_k) \oplus (1-p)(N_\ell)\) means tossing a biased coin, with \(p\) being the probability of heads, then measuring \((M_k)\) if heads showed, and \((N_\ell)\) for tails. The coin toss is part of the measurement result.

\[\text{Lemma 7} \quad \text{Let } \mathcal{M}_i \text{ be sets of POVMs and } p_i \geq 0 \text{ probabilities, and } \mathcal{N} = \bigoplus_i p_i \mathcal{M}_i. \text{ Denote the corresponding sets of operators } \mathcal{M}_i \text{ and } \mathcal{N}. \text{ Then,}\]

\[\mathcal{N} = \sum_i p_i \mathcal{M}_i,\]

\[\text{and consequently}\]

\[\lambda(\mathcal{N}) \geq \sum_i p_i \lambda(\mathcal{M}_i), \quad \mu(\mathcal{N}) \leq \sum_i p_i \mu(\mathcal{M}_i).\]

\[\text{Proof.}\] The first relation is by inspection. For the inequalities, note that since we have

\[\lambda(\mathcal{M}_i)[-1;1]_{/\mathbb{R}^1} \subseteq \mathcal{M}_i_{/\mathbb{R}^1} \subseteq \mu(\mathcal{M}_i)[-1;1]_{/\mathbb{R}^1},\]

we clearly get

\[\sum_i p_i \lambda(\mathcal{M}_i)[-1;1]_{/\mathbb{R}^1} \subseteq \sum_i p_i \mathcal{M}_i_{/\mathbb{R}^1} \subseteq \sum_i p_i \mu(\mathcal{M}_i)[-1;1]_{/\mathbb{R}^1}.\]

\[\square\]

In particular, since \([-1;1]\) is invariant under unitary conjugation, i.e. \(U[-1;1]U^\dagger = [-1;1]\), the constants of domination also have this invariance and so we obtain immediately...
Proposition 8 For a probability measure \( dp(U) \) on the unitary group on \( \mathcal{H} \), and any symmetric, full dimensional convex body \( \pm \mathbb{I} \subseteq \mathbb{M} \subseteq [-\mathbb{I}; \mathbb{I}] \),

\[
\lambda \left( \int dp(U) U M U^\dagger \right) \geq \lambda(\mathbb{M}),
\]
\[
\mu \left( \int dp(U) U M U^\dagger \right) \leq \mu(\mathbb{M}).
\]

In other words: symmetrisation makes \( \mathbb{M} \) “look more like \([-\mathbb{I}; \mathbb{I}]\)”. \( \square \)

2. SINGLE POVMs

Let us look now at the constants of domination \( \lambda \) and \( \mu \) in the case that \( \mathbb{M} \) consists of a single, informationally-complete, POVM. Let \( \mathcal{M} \) be the CPTP map associated with the POVM. With slight abuse of notation we denote the constants of domination \( \lambda(\mathcal{M}) \) and \( \mu(\mathcal{M}) \), and the associated norm \( \| \cdot \|_{\mathcal{M}} \).

A. Uniform POVM

As a consequence of Proposition 8, we arrive at the following theorem.

Theorem 9 The supremum of \( \lambda(\mathcal{M}) \), as well as the infimum of \( \mu(\mathcal{M}) \), over all POVMs in dimension \( d \) is attained by the uniform (unitary invariant POVM),

\[
(d|\psi\rangle\langle\psi|d\psi), \text{ with the normalised uniform distribution } d\psi \text{ on unit vectors.}
\]

Furthermore, denoting the CPTP map associated with it by \( \mathcal{U} \),

\[
\lambda(\mathcal{U}) = \min_{1 \leq a \leq d/2} \left( \frac{1}{d} \sum_{k=0}^{a-1} \left( \frac{a}{d} \right)^k \left( \frac{b}{d} \right)^{k+\ell} \right) = \frac{1}{d} \left( \sqrt{\frac{2}{\pi}} \pm o(1) \right) \tag{10}
\]
\[
\mu(\mathcal{U}) = \frac{1}{2}. \tag{11}
\]

Proof. We have already argued the supremum and infimum, so we are left to prove eqs. (10) and (11). Note that for any operator \( \xi \),

\[
\|\mathcal{U}(\xi)\|_1 = d \int d\psi |\text{Tr}(\psi \xi)|. \tag{12}
\]

Note that since \( \xi \) is Hermitian, we may again take \( \xi = (1 - p)\rho - p\sigma \) for orthogonal operators \( \rho \) and \( \sigma \). For eq. (10), we then have by the unitary invariance of the uniform POVM and the triangle inequality, \( \lambda(\mathcal{U}) \) is attained as \( \|\mathcal{U}(\xi)\|_1 \) for an operator of the form

\[
\xi = \frac{1}{2a} P - \frac{1}{2b} Q, \quad \text{with a projector } P \text{ of rank } a, \text{ and } Q = \mathbb{I} - P, \ b = d - a,
\]

where we may even take \( P \) to be the projector onto the subspace spanned by the first \( a \) computational basis vectors (again invoking unitary invariance). For this choice of operator, according to eq. (12), and letting \( p := a/d \),

\[
\|\mathcal{U}(\xi)\|_1 = d \int d\psi \left| \frac{1}{2a} \sum_{j=1}^{a} |\psi_j|^2 - \frac{1}{2b} \sum_{j=a+1}^{d} |\psi_j|^2 \right| = 1 - \frac{1}{d} \sum_{k=0}^{a-1} \sum_{\ell=0}^{b-1} p^k (1 - p)^\ell \binom{k+\ell}{k},
\]
by Lemma 23 in Appendix B. It is quite natural to conjecture that the minimal choice of ranks is \(a = \lfloor d/2 \rfloor\) and \(b = \lceil d/2 \rceil\). In this case we have

\[
\|U(\xi)\|_1 = 1 - \frac{1}{d} \sum_{k=0,\ldots,\lceil d/2 \rceil-1 \atop \ell=0,\ldots,\lfloor d/2 \rfloor-1} \left( \frac{\lfloor d/2 \rfloor}{d} \right)^k \left( \frac{\lceil d/2 \rceil}{d} \right)^\ell \binom{k + \ell}{k} = \frac{2}{\pi d} + O\left( \frac{1}{d} \right),
\]

for large \(d\). The analysis of the asymptotics is elementary but lengthy, and is here restricted to a few hints: We lose only terms of order \(O(1/d)\) by focusing on even \(d\), for which the formula evaluates to

\[
\lambda(U) = 1 - \frac{1}{d} \sum_{k, \ell=0}^{d/2 - 1} 2^{-k - \ell} \binom{k + \ell}{k} = \frac{1}{d} \sum_{k=0}^{d/2 - 1} 2^{-2k} \binom{2k}{k},
\]

where we have used the following identity from Lemma 24 proved by induction on \(k\):

\[
\sum_{\ell=0}^{k} 2^{-k - \ell} \binom{k + \ell}{\ell} = 1.
\]

Then a simple application of Stirling’s formula (with explicit error bounds) yields eq. (13).

However, we have not been able to prove that this is indeed the minimum. Instead, we follow a different route: From the proof of Lemma 23 in Appendix B we observe that for general \(a\) and \(b\),

\[
\|U(\xi)\|_1 = \mathbb{E} \left| \frac{1}{2 \pi} \int_{-\pi}^{\pi} \prod_{j=0}^{a-1} \left| X_j - \frac{a}{2b} \right| \prod_{j=a+1}^{d} X_j \right|,
\]

with independent \(X_j \geq 0\), each distributed according to a rescaled \(\chi^2\) law. By definition, their expectation and variance are \(\mathbb{E} X_j = 1\) and \(\text{Var} X_j = 1\), respectively (also, all higher moments are finite). Thus, by the central limit theorem

\[
\frac{1}{2a} \sum_{j=1}^{a} X_j \approx N\left( 1, \frac{1}{4a} \right), \quad \frac{1}{2b} \sum_{j=a+1}^{d} X_j \approx N\left( 1, \frac{1}{4b} \right),
\]

where \(Y_0\) and \(Y_1\) are normal distributed with means \(\mu\) and variance \(\nu\) as indicated by \(N(\mu, \nu)\), and the approximation signs indicate convergence in probability as \(a, b \to \infty\). (Note that since the third moment of the \(X_j\) is finite, this convergence is uniform in \(a\) and \(b\), thanks to the Berry-Esséen theorem which bounds the rate of convergence in the central limit theorem – see e.g. [9].)

Since \(Y_0 - Y_1 =: Z \sim N\left( 0, \frac{1}{4a} + \frac{1}{4b} \right)\), we obtain asymptotically

\[
\|U(\xi)\|_1 \sim \mathbb{E} |Z| = \sqrt{\frac{1}{4a} + \frac{1}{4b}} \sqrt{2\pi} \int_{-\infty}^{\infty} dx |x| e^{-x^2/2} = \sqrt{\frac{2}{\pi}} \sqrt{\frac{1}{4a} + \frac{1}{4b}},
\]

which is minimized for \(a = b = d/2\), yielding \(\lambda(U) \sim \sqrt{\frac{2}{\pi d}}\), as advertised.

For eq. (11), note that by the triangle inequality, \(\mu(M)\) of any POVM \(M\) is attained as \(\|M(\xi)\|_1\) for an extremal traceless \(\xi\) such that \(\|\xi\|_1 = 1\). These are easily seen to be of the form \(\xi = \frac{1}{2} |\phi_1\rangle\langle \phi_1| - \frac{1}{2} |\phi_2\rangle\langle \phi_2|\) for orthogonal pure state vectors \(|\phi_1\rangle, |\phi_2\rangle\). By unitary invariance of the
uniform POVM, any such $\xi$ will in fact yield the same value, so we may take $\xi = \frac{1}{2}|1\rangle\langle 1| - \frac{1}{2}|2\rangle\langle 2|$, so that by eq. (12),

$$\mu(U) = \|U(\xi)\|_1 = \frac{d}{2} \int d\psi |\psi_1|^2 - |\psi_2|^2 = \frac{1}{2},$$

once more by Lemma $\text{[23]}$ in Appendix $\text{[9]}$ applied with $a = b = 1$.

Note that in terms of the bias the above translates to

$$\frac{1}{2} \|\rho - \sigma\|_1 \geq \|\rho - \sigma\|_M \geq \frac{1}{\sqrt{d}} \left( \sqrt{\frac{2}{\pi}} - o(1) \right) \|\rho - \sigma\|_1.$$

**B. Almost optimal performance of 4-designs**

The results of the previous section provide the motivation to look at POVMs made from $t$-designs, as these are structures approximating the full random POVM better and better as $t \to \infty$. We thus intuitively expect to obtain a similar value for $\lambda$ as we obtained for the random POVM for larger $t$. Recall the following

**Definition 10** A (weighted) spherical $t$-design is an ensemble $(p_k, P_k)_{k=1}^n$ of 1-dimensional projectors $P_k$ and probabilities $p_k$ such that

$$\sum_k p_k P_k^{\otimes t} = \frac{1}{(d+t-1)} P^{(t)}_{\text{sym}} = \frac{1}{N(d,t)} \sum_{\pi \in S_t} U_\pi,$$

with the projector $P^{(t)}_{\text{sym}}$ onto the completely symmetric subspace of $(\mathbb{C}^d)^{\otimes t}$, which has dimension $(d+t-1)$. It can be expressed, by Schur duality, via the representation of the symmetric group (as this subspace is an irrep of multiplicity 1), as the sum of the permutation representations $U_\pi$, which is the permutation of the $t$ tensor factors by the permutation $\pi \in S_t$. Since $\text{Tr}(U_\pi) = d^{c(\pi)}$ with the number $c(\pi)$ of cycles of the permutation $\pi$, we get the normalisation factor $N(d,t) = \sum_{\pi \in S_t} d^{c(\pi)} = (d + t - 1) \cdots (d + 1) d$.

Note that the random POVM $(d|\psi\rangle\langle \psi|d\psi)$ is an $\infty$-design. We call a $t$-design proper if all the probabilities are equal, $p_k = 1/n$. Note that any $t$-design is automatically also a $t'$-design for all $t' < t$. In particular, $\sum_k p_k P_k = \frac{1}{d} \mathbb{1}$, so it makes sense to associate a POVM with every $t$-design of the form

$$(M_k)_{k=1}^n, \quad \text{with} \; M_k = dp_k P_k,$$

which, as before, we also call a (weighted or proper) $t$-design. We shall consider the CPTP map $\mathcal{M}$ associated with such a $t$-design POVM.

It turns out that 4-designs already achieve essentially the same bias as the isotropic POVM. This was discovered by Ambainis and Emerson $\text{[1]}$, who showed, invoking a beautiful moment inequality by Berger, that

$$\|\rho - \sigma\|_M \geq \frac{1}{3\sqrt{d}} \|\rho - \sigma\|_2 \geq \frac{1}{3\sqrt{d}} \|\rho - \sigma\|_1.$$

We briefly review their argument, including the Berger inequality, as we need to return to this later on in Section $\text{[3]}$. 

\[\text{(15)}\]
Lemma 11 (Berger [8]) For a real random variable $S$,

$$E|S| \geq \frac{(ES^2)^{3/2}}{(ES^4)^{1/2}}.$$ 

Proof. That is just Hölder’s inequality, which states that for real random variables $f$ and $g$, and $\frac{1}{p} + \frac{1}{q} = 1$,

$$E(fg) \leq (E|f|^p)^{1/p} (E|g|^q)^{1/q}.$$ 

Here it is applied with $f = |S|^{2/3}$, $g = |S|^{4/3}$ and $p = 3/2$, $q = 3$.

Proof of eq. (15) – see [1]. For traceless $\xi$, consider the random variable $S$ which takes value $d \text{Tr}(\xi P_k)$ with probability $p_k$. Then clearly $E|S| = \|M(\xi)\|_1$, and Berger’s inequality can be used. The moments are easy calculations, using the fact that the POVM is a 4-design. First, the second moment,

$$E S^2 = \sum_k p_k d^2 \left(\text{Tr}(\xi P_k)\right)^2$$
$$= \sum_k p_k d^2 \text{Tr}((\xi \otimes \xi)(P_k \otimes P_k))$$
$$= d^2 \text{Tr} \left( (\xi \otimes \xi) \frac{2}{d(d + 1)} P^{(2)}_{\text{sym}} \right)$$
$$= \frac{d^2}{d(d + 1)} \text{Tr}((\xi \otimes \xi)(1 + F)) = \frac{d}{d + 1} \text{Tr}(\xi^2),$$

where $F$ is the swap operator, corresponding to the transposition (12), and we have made use of $\text{Tr}(\xi) = 0$. Similarly,

$$E S^4 = \sum_k p_k d^4 \left(\text{Tr}(\xi P_k)\right)^4$$
$$= \sum_k p_k d^4 \text{Tr}((\xi \otimes \xi \otimes \xi \otimes \xi)(P_k \otimes P_k \otimes P_k \otimes P_k))$$
$$= d^4 \text{Tr} \left( (\xi \otimes \xi \otimes \xi \otimes \xi) \frac{24}{d(d + 1)(d + 2)(d + 3)} P^{(4)}_{\text{sym}} \right)$$
$$= \frac{d^4}{d(d + 1)(d + 2)(d + 3)} \sum_{\pi \in S_4} \text{Tr}(\xi^{\otimes 4} U_{\pi})$$
$$= \frac{d^3}{(d + 1)(d + 2)(d + 3)} \left(3(\text{Tr}(\xi^2))^2 + 6 \text{Tr}(\xi^4)\right) \leq \left(\frac{d}{d + 1}\right)^3 9(\text{Tr}(\xi^2))^2,$$

where in the last line we have made use of $\text{Tr}(\xi) = 0$ to take care of all permutations with a fixed point. Thus,

$$\|M(\xi)\|_1 = E|S| \geq \frac{1}{3} \sqrt{\text{Tr}(\xi^2)} = \frac{1}{3} \|\xi\|_2 \geq \frac{1}{3\sqrt{d}} \|\xi\|_1.$$

In other words: $\lambda(M) \geq 1/(3\sqrt{d})$.

It is not known how to construct spherical 4-designs efficiently in general though Carathéodory’s Theorem tells us that there must exist a weighted 4-design of cardinality at most $1 + \left(\frac{d + 3}{4}\right)^2$. Constructions are known for a real vector space of small dimensions [15]. Ambainis and Emerson [1] construct approximate 4-designs which perform almost as good as eq. (15).
C. Performance of 2-designs

Unfortunately, we cannot give any bounds for the bias for 3-design POVMs, but here we show how to bound it for 2-designs. Consider first a proper 2-design with associated POVM \( M_k = \frac{2}{d} P_k \) \( k = 1 \). I.e.,

\[
\frac{1}{n} \sum_k P_k \otimes P_k = \frac{1}{d(d+1)} (\mathbb{1} + F) = \frac{2}{d(d+1)} P^{(2)}_{\text{sym}},
\]

with the projector \( P^{(2)}_{\text{sym}} \) onto the symmetric subspace of \( \mathbb{C}^d \otimes \mathbb{C}^d \) and the swap operator \( F \). Such POVMs are always tomographically complete – this will also follow from the theorem below.

An example of a 2-design is a complete set of \( d+1 \) mutually unbiased bases, which are known to exist if the dimension \( d \) is a prime power \([6, 26]\). Let \( \{ |\psi^b_s\rangle \}_{s=1,...,d} \), \( b = 0, \ldots, d \), be the basis vectors of the \( d+1 \) mutually unbiased bases, where \( |\psi^b_s\rangle \) is the \( s \)-th basis vector of the \( b \)-th basis. Then the set of basis state projectors \( P^b_s = |\psi^b_s\rangle \langle \psi^b_s| \) forms a proper spherical 2-design \([19]\). It is conjectured that in all dimensions there exist spherical 2-designs with the minimum number \( n = d^2 \) of elements \([22]\), giving rise to so-called symmetric informationally complete (SIC) POVMs. These are only known to exist up to dimension \( d = 45 \) \([22]\) by numerical results, and for even fewer dimensions up to \( d = 19 \) by mathematical construction. Zauner’s conjecture states that in every dimension there exists a SIC-POVM of a particularly beautiful group symmetric form \([27]\). We refer to \([3, 11]\) for more information.

As before, we look at the associated CPTP map,

\[
\mathcal{M} : \xi \mapsto \sum_{k=1}^n |k\rangle\langle k| \text{Tr}(\xi M_k) = \frac{d}{n} \sum_k |k\rangle\langle k| \text{Tr}(\xi P_k).
\]

Our objective is to prove the relation.

**Theorem 12** For any traceless Hermitian operator \( \xi \),

\[
\| \mathcal{M}(\xi) \|_1 \geq \frac{1}{2} \frac{1}{d+1} \| \xi \|_1.
\]

In other words, for any proper 2-design POVM as above, \( \lambda(\mathcal{M}) \geq \frac{1}{2} \frac{1}{d+1} \).

**Proof.** Since this is a homogeneous relation, we may w.l.o.g. assume that \( \| \xi \|_1 = 2 \), meaning that we can write \( \xi = \rho - \sigma \) with two orthogonal density operators \( \rho \) and \( \sigma \). Thus, what we need to show is \( \| \mathcal{M}(\rho) - \mathcal{M}(\sigma) \|_1 \geq \frac{1}{d+1} \).

For this, we use Proposition \([20]\) in Appendix A, ineq. \((A1)\), for the vectors \( \vec{p} \) and \( \vec{q} \) defined as

\[
p_k = \text{Tr}(\rho M_k) = \frac{d}{n} \text{Tr}(\rho P_k), \quad q_k = \text{Tr}(\sigma M_k) = \frac{d}{n} \text{Tr}(\sigma P_k).
\]

Namely,

\[
\| \mathcal{M}(\rho) - \mathcal{M}(\sigma) \|_1 = \| \vec{p} - \vec{q} \|_1 \\
\geq 1 - n \sum_k \frac{d^2}{n^2} (\text{Tr}(\rho P_k))(\text{Tr}(\sigma P_k)) \\
= 1 - d^2 \frac{1}{d+1} \sum_k \text{Tr}((\rho \otimes \sigma)(P_k \otimes P_k)).
\]
Now, the last sum can be evaluated as follows, using the property of spherical 2-design:

\[
\frac{1}{n} \sum_k \text{Tr}(\rho P_k \sigma P_k) = \frac{1}{d(d + 1)} \text{Tr}((\mathbb{1} + F)(\rho \otimes \sigma)) = \frac{1}{d(d + 1)}. 
\]

Inserting this above, we conclude

\[
\|\mathcal{M}(\rho) - \mathcal{M}(\sigma)\|_1 \geq 1 - d^2 \frac{1}{d(d + 1)} = \frac{1}{d + 1},
\]

as advertised. \(\square\)

**Corollary 13** For a POVM which is a weighted 2-design, and associated map \(\mathcal{M}\), the conclusion of Theorem 12 still holds: \(\lambda(\mathcal{M}) \geq \frac{1}{2} \frac{1}{d + 1}\).

**Proof.** The idea is to break down the probabilities \(p_k\) into smaller but approximately equal values. This increases the number of outcomes of the POVM, but makes it be approximated better and better by a proper 2-design, to which we can apply Theorem 12.

In detail, assume that our weighted 2-design is discrete, with \(n\) elements; choose an integer \(N \gg 1\), and for each \(k\) let \(N_k = \lfloor N p_k \rfloor\) and \(\epsilon_k = N p_k - N_k\). Define a new weighted 2-design with the same projectors \(P_k\) and “uniformized” weights

\[
\beta_{k\ell} = \begin{cases} 
\epsilon_k/N & \text{for } \ell = 0, \\
1/N & \text{for } \ell = 1, \ldots, N_k.
\end{cases}
\]

Then, applying the same proof as in Theorem 12 to this refined 2-design (which has \(N + n\) outcomes), we get

\[
\|\mathcal{M}(\rho) - \mathcal{M}(\sigma)\|_1 = \|\bar{p} - \bar{q}\|_1 \\
\geq 1 - (N + n) \sum_{k\ell} \beta_{k\ell} \left( \frac{1}{N} \text{Tr}(\rho P_k)(\text{Tr}(\sigma P_k)) \right) \\
\geq 1 - d^2 \frac{N + n}{N} \sum_{k\ell} \beta_{k\ell} \text{Tr}((\rho \otimes \sigma)(P_k \otimes P_k)) \\
= 1 - \frac{d^2}{d(d + 1)} \frac{N + n}{N} \text{Tr}([\mathbb{1} + F](\rho \otimes \sigma)) \\
= 1 - \frac{d}{d + 1} \left( 1 + \frac{n}{N} \right) \rightarrow \frac{1}{d + 1},
\]

where we have used \(\beta_{k\ell} \leq 1/N\) in the third line. \(\square\)

Note that the factor of \(1/(d + 1)\) in the bound (16) is essentially best possible (up to a constant independent of \(d\), as the example of \(d + 1\) mutually unbiased bases shows. Indeed, if the two states \(\rho\) and \(\sigma\) are distinct elements of one of the bases, then the measured output distributions for all the \(d\) other bases are the same, namely uniform, while in their proper basis the trace distance remains 2, so \(\|\mathcal{M}(\rho) - \mathcal{M}(\sigma)\|_1 = \frac{2}{d + 1}\), and hence \(\lambda(\mathcal{M}) \leq \frac{1}{d + 1}\).

Similarly, for a SIC-POVM with \(d^2\) operators \((\frac{1}{d^2} P_k)\) it is easily verified that two states from the POVM, i.e. for instance \(\rho = P_1\) and \(\sigma = P_2\), have trace norm difference \(\|\rho - \sigma\|_1 = \frac{2d}{d + 1}\), while \(\|\mathcal{M}(\rho) - \mathcal{M}(\sigma)\|_1 = \frac{2}{d + 1}\), so \(\lambda(\mathcal{M}) \leq \frac{1}{d}\).
3. LOCAL POVMS

Consider now a multipartite system \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \) of local Hilbert spaces \( \mathcal{H}_j \) of dimension \( d_j \). (The total space’s dimension is denoted \( D = d_1 d_2 \cdots d_n \) in this section.) This partition suggests various classes of POVMs due to restrictions of locality. For instance, let \( \text{LO} \) be the class of all local operations, i.e. tensor product measurements:

\[
\text{LO} = \left\{ (M^{(1)}_{k_1} \otimes \cdots \otimes M^{(n)}_{k_n}) : (M^{(j)}_{k_j}) \text{ POVM on } \mathcal{H}_j \right\}.
\]

More generally, \( \text{LOCC} \) is the class of measurements that can be implemented by local operations and classical communication between the parties. \( \text{SEP} \) are the separable POVMs, i.e.

\[
\text{SEP} = \left\{ (M^{(1)}_{k_1} \otimes \cdots \otimes M^{(n)}_{k_n}) : M^{(j)}_{k_j} \geq 0, \sum_k M^{(1)}_{k_1} \otimes \cdots \otimes M^{(n)}_{k_n} = 1 \right\}.
\]

Finally, there is the class of all measurements positive partial transpose (PPT) operators: denoting the transpose operation (with respect to any basis) by \( T \), it is

\[
\text{PPT} = \left\{ (M_k) \text{ POVM} : \forall k \forall I \subset [n] \left( \bigotimes_{i \in I} T \otimes \bigotimes_{i \notin I} \text{id} \right) M_k \geq 0 \right\},
\]

i.e. all POVM elements have to be PPT with respect to every bipartition of the \( n \)-party system. Quite evidently,

\[
\text{LO} \subset \text{LOCC} \subset \text{SEP} \subset \text{PPT},
\]

and all inclusions are well-known to be strict, at least if the dimension is large enough. The corresponding symmetric convex bodies of operators are denoted

\[
\text{LO} \subset \text{LOCC} \subset \text{SEP} \subset \text{PPT}.
\]

These are interesting examples of POVM classes since we know due to so-called quantum data hiding \([10, 20, 25]\) that \( \|\xi\|_n \) for them can be much smaller than \( \|\cdot\|_1 \). Indeed, it was shown in these references that in a bipartite system \( \mathbb{C}^d \otimes \mathbb{C}^d \), the states \( \sigma = \frac{1+F}{d+1} \) and \( \alpha = \frac{1-F}{d-1} \), i.e. the (normalised) projectors onto the symmetric and antisymmetric subspace, respectively, obey

\[
\left\| \frac{1}{2} \rho - \frac{1}{2} \sigma \right\|_{\text{PPT}} = \frac{2}{d+1}.
\]

(In \([10]\) more general statements of this type for \( n \)-partite systems can be found.) Consequently, \( \lambda(\text{PPT}) \leq \frac{2}{d+1} \). The next result shows that this bound is not very far from the truth:

**Lemma 14** For any operator \( \xi \) on an \( n \)-partite system,

\[
\|\xi\|_{\text{SEP}} \geq \frac{2}{2^{n/2}} \|\xi\|_2 \geq \frac{2}{2^{n/2}} \frac{1}{\sqrt{D}} \|\xi\|_1.
\]

In particular, \( \lambda(\text{SEP}) \geq \frac{2}{2^{n/2}} \frac{1}{\sqrt{D}} \); for a bipartite system, we find \( \lambda(\text{SEP}) \geq \frac{1}{\sqrt{D}} \).
Proof. Gurvits and Barnum [14] have shown that for a bipartite system, within the set of Hermitian operators, the unit ball of the Hilbert-Schmidt norm centred on the identity operator contains only separable operators. More generally they proved in an \( n \)-partite system, that the ball of radius \( 2^{1-n/2} \) around the identity is fully separable [14].

It follows immediately that all the POVMs in the set \( \{ (M, \mathbb{1} - M) : \|2M - \mathbb{1}\|_2 \leq 2^{1-n/2} \} \) are separable. It is easy to see that the corresponding symmetric convex body (see Lemma 2) is the ball of radius \( 2^{1-n/2} \) in the Hilbert-Schmidt norm around the origin and so this is a subset of \( \text{SEP} \).

From this inclusion, and the fact that the Hilbert-Schmidt norm is self-dual,

\[
\|\xi\|_{\text{SEP}} = \max_{M \in \text{SEP}} \text{Tr}(M\xi) \geq \max_{\|M\|_2 \leq 2^{1-n/2}} \text{Tr}(M\xi) = \frac{2}{2^{n/2}} \|\xi\|_2,
\]

concluding the proof, if we recall \( \|\xi\|_1 \leq \sqrt{D}\|\xi\|_2 \).

We now come to the main technical result of the present section, showing that this order of magnitude goes through all the way to \( \mathcal{O}(\xi) \), indeed, a particular tensor product POVM on a bipartite system is already almost as good as the class of all separable POVMs, in terms of the constant of domination. Note that Proposition 8 gives us the local POVM with the largest \( \lambda \).

**Theorem 15** For any two states \( \rho \) and \( \sigma \) on a bipartite Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \), let \( \xi = \rho - \sigma \). Then,

\[
\|\xi\|_{\mathcal{U}_A \otimes \mathcal{U}_B} \geq \frac{1}{\sqrt{153}} \|\xi\|_2 \geq \frac{1}{\sqrt{153D}} \|\xi\|_1,
\]

where \( D = d_Ad_B \) is the Hilbert space dimension, and \( \mathcal{U}_A \) and \( \mathcal{U}_B \) are the CPTP maps of the isotropic POVMs on \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively. Consequently, \( \lambda(\mathcal{U}_A \otimes \mathcal{U}_B) \geq 1/\sqrt{153D} \).

**Proof.** We do exactly the same as in Subsection 2B, only that we have now a POVM on \( \mathcal{H}_A \otimes \mathcal{H}_B \) of the form

\[
(Dd\varphi d\psi|\varphi\rangle\langle\varphi| \otimes |\psi\rangle\langle\psi|),
\]

so \( S \) is the variable

\[
S = D\text{Tr}((|\varphi\rangle\langle\varphi| \otimes |\psi\rangle\langle\psi|)\xi),
\]

and the bias of the estimation based on the outcome is \( \mathbb{E}|S| \), as before in Subsection 2B.

We use Berger’s inequality, Lemma 11 again, for which we need the second and fourth moment. Because now we randomise independently over \( \mathcal{H}_A \) and \( \mathcal{H}_B \), we get

\[
\mathbb{E}S^2 = \frac{2^2d_A^2d_B^2}{d_A(d_B + 1)d_B(d_B + 1)} \text{Tr}((\Pi^A_{\text{sym}} \otimes \Pi^B_{\text{sym}})(\xi^{AB} \otimes \xi^{AB})),
\]

\[
\mathbb{E}S^4 = \frac{2^4d_A^4d_B^4}{d_A(d_A + 1)(d_A + 2)(d_A + 3)d_B(d_B + 1)(d_B + 2)(d_B + 3)} \times \text{Tr}((\Pi^A_{\text{sym}} \otimes \Pi^B_{\text{sym}})(\xi^{AB} \otimes \xi^{AB} \otimes \xi^{AB} \otimes \xi^{AB})),
\]

where the superscripts remind of the systems these operators act on.

Expanding the projectors into the permutations of two, respectively four, elements, we get

\[
\mathbb{E}S^2 = \frac{d_Ad_B}{(d_A + 1)(d_B + 1)} \left( \text{Tr}(\xi^2_A) + \text{Tr}(\xi^2_B) + \text{Tr}(\xi^2) \right),
\]

(17)
where \( \xi_A = \text{Tr}_B(\xi) \) and \( \xi_B = \text{Tr}_A(\xi) \), because we get terms with \( \mathbb{1}^{AA} \otimes \mathbb{1}^{BB}, \mathbb{1}^{AA} \otimes F^{BB}, F^{AA} \otimes \mathbb{1}^{BB} \) and \( F^{AA} \otimes F^{BB} \).

The fourth moment is considerably more complex: looking at
\[
\mathbb{E}S^4 = \frac{d_A^3 d_B^3}{(d_A + 1)(d_A + 2)(d_A + 3)(d_B + 1)(d_B + 2)(d_B + 3)} \sum_{\pi, \sigma \in S_4} \text{Tr}((U^{AAAA}_\pi \otimes U^{BBBB}_\sigma)\xi^4),
\]
we see that we need to calculate – or at least reasonably upper bound – the trace terms \( \text{Tr}((U^{AAAA}_\pi \otimes U^{BBBB}_\sigma)\xi^4) \). In Appendix C, Lemma 25 we show that
\[
\sum_{\pi, \sigma \in S_4} \text{Tr}((U^{AAAA}_\pi \otimes U^{BBBB}_\sigma)\xi^4) \leq 153(\text{Tr}(\xi^2))^2 + 126(\text{Tr}(\xi^2))(\text{Tr}(\xi_A^2)) + 126(\text{Tr}(\xi^2))(\text{Tr}(\xi_B^2))
+ 9(\text{Tr}(\xi_A^2))^2 + 9(\text{Tr}(\xi_B^2))^2 + 30(\text{Tr}(\xi_A^2))(\text{Tr}(\xi_B^2))
\leq 153(\text{Tr}(\xi^2) + \text{Tr}(\xi_A^2) + \text{Tr}(\xi_B^2))^2.
\]

Plugging this into eq. (18), we find
\[
\mathbb{E}S^4 \leq \left( \frac{d_A d_B}{(d_A + 1)(d_B + 1)} \right)^3 153(\text{Tr}(\xi^2) + \text{Tr}(\xi_A^2) + \text{Tr}(\xi_B^2))^2.
\]
Now we conclude as in the single-system case: by virtue of eqs. (17) and (19),
\[
\| (U_A \otimes U_B)\xi \|_1 = \mathbb{E}|S|
\geq \sqrt{\frac{\mathbb{E}S^2}{\mathbb{E}S^4}}
\geq \frac{1}{\sqrt{153}} \sqrt{\text{Tr}(\xi^2) + \text{Tr}(\xi_A^2) + \text{Tr}(\xi_B^2)}
\geq \frac{1}{\sqrt{153}} \|\xi\|_2 \geq \frac{1}{\sqrt{153D}} \|\xi\|_1,
\]
and we are done. \(\square\)

**Remark 16** From the proof we see that, just as in the single-system case of Subsection 2B, it is enough for the local measurements to be 4-designs.

**Corollary 17** The constants of domination, for locality-restricted measurements on a \( d \times d \)-system, are in the following relations:
\[
\frac{1}{\sqrt{153d}} \leq \lambda(U \otimes U) \leq \lambda(LO) \leq \lambda(LOCC) \leq \lambda(SEP) \leq \lambda(PPT) \leq \frac{2}{d + 1}.
\]
For separable measurements we have the even tighter bounds,
\[
\frac{1}{d} \leq \lambda(SEP) \leq \lambda(PPT) \leq \frac{2}{d + 1}.
\]

**Proof.** The first inequality in (20) is just Theorem 15, the chain is by inclusion of the sets of POVMs, with the last bound following from the data hiding states \( \alpha_d \) and \( \sigma_d \), the (appropriately normalised) projections onto the (anti-)symmetric subspace of \( \mathbb{C}^d \otimes \mathbb{C}^d \) – see [10, 25] and [20]. By Lemma 14 finally, \( \lambda(SEP) \geq \frac{1}{\sqrt{D}} = \frac{1}{d} \). \(\square\)
Remark 18 The first inequality \([20]\) in Corollary \([17]\) proves a conjecture about the optimal bias achievable with LOCC measurements \([20, \text{Conjecture 7}]\). Compare also with \([25]\), where a bias of order \(1/d^2\) was proven using a particular tomographically complete measurement, and it was suggested there that better POVMs might exist.

This result shows that in a very strong sense the original data hiding states, the symmetric and anti-symmetric subspace projections, are essentially optimal: up to a constant factor they achieve the best available bias, which is \(\Theta(1/d)\).

Remark 19 The \(\ell^2\)-bound in Theorem \([15]\) has another notable consequence for data hiding: observing that for orthogonal states \(\rho\) and \(\sigma\),

\[
\|\rho - \sigma\|_2 = \sqrt{\text{Tr}(\rho^2) + \text{Tr}(\sigma^2)} \geq \max \{\|\rho\|_2, \|\sigma\|_2\},
\]

we conclude that data hiding states have to be highly mixed. If one of them has rank bounded by \(r\), say, Theorem \([15]\) places a lower bound of \(1/13r\) on the bias achievable by LOCC measurements.

Indeed, all known constructions of data hiding states endow them with considerable entropy (comparable to or larger than the size of the “shares”), see \([10, 16, 25]\). Our bound tells us that this has to be so to guarantee security of the scheme. We intend to return to this issue on a separate occasion.

4. CERTAINTY RELATIONS

The results on \(\lambda(M)\) for the isotropic POVM, tensor products of isotropic POVMs, and 2-designs have nice interpretations as “certainty relations” in the sense of Sanchez-Ruiz \([23]\). Namely, for a complete set of \(d + 1\) mutually unbiased bases in \(\mathbb{C}^d\) with associated basis measurements \(B_k\), he shows that for any pure state \(\varphi = |\varphi\rangle\langle \varphi|\),

\[
(d + 1) \log \frac{d + 1}{2} \leq \sum_{k=0}^{d} S_2(B_k(\varphi)) \leq (d + 1) \log d - \log(d - 1), \tag{22}
\]

where \(S_2(B) = -\log \sum_x |\langle x|\varphi\rangle|^4\) is the Rényi entropy of order 2 for the orthonormal basis \(B = \{|1\rangle, \ldots, |d\rangle\}\). The right hand side of eq. (22) is referred to as a certainty relation, and intuitively states that for the chosen measurements there exists no pure state that will lead to maximum entropy for all measurements simultaneously. It quantifies the fact (quite natural, after a moment of thought) that not all the tomographic data from measuring those bases is equally informative in the sense of Shannon. The certainty relation of \([23]\) also holds for the Shannon entropy. Let \(M\) be the measurement formed by measuring in one of the \(d + 1\) bases at random. Using the concavity of the log, the certainty relation can then be rewritten as

\[
\log (d(d + 1)) - S_2(M(\varphi)) \geq \frac{1}{d + 1} \log(d - 1).
\]

From our results in the previous section, we can infer similar certainty relations. First of all, from Theorem \([12]\) we get the following more general but weaker bound for any proper 2-design POVM with \(n\) outcomes:

\[
\log n - S_2(M(\varphi)) \geq \log n - S(M(\varphi)) = D(M(\varphi)||M(\mathbb{I}/d)) \geq \frac{1}{2\ln 2} \|M(\varphi - \mathbb{I}/d)\|_1^2 \geq \frac{1}{4\ln 2} \frac{d - 1}{d(d + 1)^2} \geq \frac{1}{6 \ln 2} \frac{1}{(d + 1)^2},
\]

where \(\|\cdot\|_1^2\) is the sum of the squared entries of a matrix.
where the second inequality follows from the Pinsker inequality \( D(\rho||\sigma) \geq \frac{1}{2 \ln 2} ||\rho - \sigma||^2_1 \).

For uni- and bipartite 4-designs, in particular the isotropic POVMs, we get considerably better bounds, due to the appearance of the Hilbert-Schmidt norm. Consider any ensemble of quantum states, \( \rho = \sum_x p_x \rho_x \). For the Shannon mutual information between the preparation variable \( X \) (distributed according to \( \rho \)), the Pinsker inequality

\[
I(X : U) = \sum_x p_x D(U(\rho_x) || U(\rho))
\]

\[
\geq \sum_x p_x \frac{1}{2 \ln 2} ||U(\rho_x) - U(\rho)||^2_1
\]

\[
\geq \sum_x p_x \frac{1}{18 \ln 2} ||U(\rho_x) - U(\rho)||^2_2
\]

\[
= \frac{1}{18 \ln 2} \left( \sum_x p_x \text{Tr}(\rho_x^2) - \text{Tr}(\rho^2) \right) = \frac{1}{18 \ln 2} \left( S_L(\rho) - \sum_x p_x S_L(\rho_x) \right).
\]  

(23)

In other words, we get a lower bound on the accessible information of the ensemble in terms of so-called “linear entropies” \( S_L(\rho) = 1 - \text{Tr}(\rho^2) \). In the above derivation we have used the well-known relation between mutual information and relative entropy, the Pinsker inequality and eq. (15).

A particularly interesting case is that of a pure state ensemble \( \rho_x = |\varphi_x\rangle\langle \varphi_x| \); all the \( S_L(\rho_x) \) are zero, so we get a positive lower bound for the accessible information

\[
I_{\text{acc}}(\{p_x, \varphi_x\}) \geq I(X : U) \geq \frac{1}{18 \ln 2} (1 - \text{Tr}(\rho^2)),
\]

which is a small but positive constant, depending only on \( \rho \). It turns out that the best possible lower bound on the accessible information in terms solely of \( \rho \) is known: it is the so-called subentropy \( Q(\rho) \) of Jozsa, Robb and Wootters [18], attained on a particular ensemble decomposition of \( \rho \), named after Ebenezer Scrooge. Incidentally, for this ensemble all complete (i.e., rank-1) POVMs have the same information gain. It is largest on the maximally mixed state, and bounded by \( \frac{1-\gamma}{\ln 2} \approx 0.6099 \), where \( \gamma \) is Euler’s constant [18].

For bipartite systems we furthermore obtain a lower bound for \( f_{\text{acc}}^{\text{LOCC}}(\cdot) \), that is the accessible information when we are restricted to performing LOCC measurements. This bound is obtained by using Theorem [15] to lower bound \( I(X : U_A \otimes U_B) \) – the mutual information when the locally unitarily invariant continuous POVM is used. This quantity is studied as a lower bound on the locally accessible information in [24] (where it is denoted \( \Lambda_L(\{p_x, \varphi_x\}) \)). Unlike the subentropy, this quantity depends on the ensemble (rather than the ensemble average alone) even when it is a pure state ensemble. However, in [24] it is interpreted differently as the average of the mutual information over all complete product basis measurements. Since some measurements of this form cannot be performed by LOCC, the authors (unnecessarily) restrict their claim that it is a lower bound on the locally accessible information to bipartite systems of \( 2 \times n \) dimensions (where it is known that any complete product basis measurement can be performed by LOCC). This is unnecessary because, as described in Section [3], \( I(X : U_A \otimes U_B) \) is also the mutual information yielded by the protocol where Alice and Bob independently measure according to the unitarily invariant continuous POVM and share their results (which is clearly accomplished by LOCC). As noted in [24], this bound is saturated by Scrooge ensembles.

No general closed form is known for \( I(X : U_A \otimes U_B) \) (although some special cases are derived in [24]) so it is useful to note that by using the same derivation as in (23), but invoking Theorem [15], we get that for an arbitrary ensemble on a bipartite system,

\[
f_{\text{acc}}^{\text{LOCC}}(\{p_x, \rho_x\}) \geq I(X : U_A \otimes U_B) \geq \frac{1}{306 \ln 2} \left( S_L(\rho) - \sum_x p_x S_L(\rho_x) \right).
\]  

(24)
It is worth noting that in the case of an ensemble of pure states this lower bound, unlike $I(X : U_A \otimes U_B)$, depends only on the ensemble average. Hence we get a lower bound of

$$Q^{\text{LOCC}}(\rho) := \inf_{\rho = \sum_x p_x \varphi_x} I^{\text{LOCC}}_{\text{acc}}(\{p_x, \varphi_x\}) \geq \frac{1}{306 \ln 2} (1 - \text{Tr}(\rho^2))$$

on the LOCC-subentropy of $\rho$.

5. CONCLUSION

We have introduced a formalism of norms on states/density operators linked to their (pairwise) distinguishability by a given, restricted, class of measurements. This allows us to study the relation between these norms in convex geometric terms. We went on to investigate the constants of domination for the resulting norms with respect to the well-known trace norm: for a single measurement we looked at the isotropic POVM, 4- and 2-designs. Furthermore, we considered several classes of locally restricted measurements, such as LOCC or PPT POVMs. The results here have strong connection to data hiding: indeed, we proved that up to a constant factor the hiding states of [25] achieve already the best possible bias. We leave many questions open, such as the eventual determination of the locally accessible information and better bounds on the constants of domination. More importantly, one ought to be able to obtain more information on the geometry of the convex bodies $M$ and the unit balls of $\| \cdot \|_M$ – here we only compared them with the trace and the Hilbert-Schmidt norms, but it would be interesting to get more insight into their geometric shape. It is an intriguing open question regarding single measurements where to place 3-design POVMs relative to 2- and 4-designs.

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APPENDIX A: AN $\ell_1$-INEQUALITY FOR PROBABILITY VECTORS AND DENSITY OPERATORS

Proposition 20  For probability vectors $\vec{p}, \vec{q}$ in $\mathbb{R}^n$ (i.e. $p_i \geq 0$ and $\sum_{i=1}^n p_i = 1$, and likewise for $q_i$),

$$\|\vec{p} - \vec{q}\|_1 \geq 1 - n \vec{p} \cdot \vec{q},$$

(A1)

where on the left is the statistical distance between the distributions, namely the $\ell_1$-norm of their difference, and on the right we have the usual Euclidean inner product of vectors.

Corollary 21 (Quantum case) Ineq. (A1) has a straightforward quantum generalisation: for any two density operators $\rho$ and $\sigma$ on an $n$-dimensional Hilbert space,

$$\|\rho - \sigma\|_1 \geq 1 - n \text{Tr}(\rho\sigma),$$

(A2)

where now on the left is the trace norm, and on the right is the Hilbert-Schmidt inner product on operator space.

This actually follows from the classical case, as follows: $\rho$ is diagonalised in some basis, with a probability vector $\vec{p}$ along the diagonal. Denote the dephasing operation in this basis by $\mathcal{E}$ – it is a CPTP map with $\mathcal{E}(\rho) = \rho$. Denoting $\sigma' = \mathcal{E}(\sigma)$, which is now diagonalised in the same basis, with a probability vector $\vec{q}$ along the diagonal, we now have

$$\frac{1}{2} \|\rho - \sigma\|_1 \geq \frac{1}{2} \|\rho - \sigma'\|_1 \text{ and } \text{Tr}(\rho\sigma) = \text{Tr}(\rho\sigma'),$$

so all we need to prove is

$$\frac{1}{2} \|\rho - \sigma'\|_1 \geq 1 - n \text{Tr}(\rho\sigma').$$

But because of

$$\frac{1}{2} \|\rho - \sigma'\|_1 = \frac{1}{2} \|\vec{p} - \vec{q}\|_1 \text{ and } \text{Tr}(\rho\sigma') = \vec{p} \cdot \vec{q}$$

this is precisely (A1).

Proof of Proposition 20  We use the well-known relation between trace distance and fidelity [12]:

$$\frac{1}{2} \|\vec{p} - \vec{q}\|_1 \geq 1 - \sum_i \sqrt{p_i q_i},$$

hence we are done once we show

$$2 \left( 1 - \sum_i \sqrt{p_i q_i} \right) \geq 1 - n \sum_i p_i q_i,$$

which – introducing the shorthand $t_i = \sqrt{p_i q_i}$ – is equivalent to

$$\sum_i t_i \leq \frac{1}{2} + \frac{1}{2} n \sum_i t_i^2.$$

Now, for fixed $s = \sum_i t_i \leq 1$, the right hand side here is minimal for $t_1 = \ldots = t_n = \frac{s}{n}$, in which case it reduces to $\frac{1}{2} + \frac{1}{2} s^2$, which is indeed always $\geq s$. \hfill \Box

Remark 22  Ineq. (A1) becomes false when introducing a factor $c < 1$ on the left hand side, for sufficiently large $n$. Ashley Montanaro [personal communication] pointed out to us the following class of examples:

Consider $\vec{p} = (x, 0, \frac{1-x}{n-2}, \ldots, \frac{1-x}{n-2})$ and $\vec{q} = (0, x, \frac{1-x}{n-2}, \ldots, \frac{1-x}{n-2})$, which have $c \|\vec{p} - \vec{q}\|_1 = 2cx$, whereas $1 - n \vec{p} \cdot \vec{q} = 1 - \frac{n}{n-2} (1 - x)^2 \sim 2x + x^2$ for large $n$.
APPENDIX B: AN INTEGRAL OVER THE UNIT SPHERE

**Lemma 23** Let $P$ and $Q$ be mutually orthogonal projectors of rank $a$ and $b$, respectively, in $\mathbb{C}^d$. Then, for the uniform distribution on the unit vectors $|\psi\rangle = \sum_{j=1}^d |\psi_j\rangle \in \mathbb{C}^d$,

$$
\mathbb{E} \left[ \frac{1}{2a} \text{Tr}(\psi P) - \frac{1}{2b} \text{Tr}(\psi Q) \right] = d \int d\psi \left[ \frac{1}{2a} \sum_{j=1}^a |\psi_j|^2 - \frac{1}{2b} \sum_{j=a+1}^{a+b} |\psi_j|^2 \right]
= 1 - \frac{1}{a+b} \sum_{k=0}^{a-1} \sum_{\ell=0}^{b-1} p^k (1-p)^\ell \binom{k+\ell}{k},
$$

where $p = a/(a+b)$.

**Proof.** Introduce a random Gaussian vector $|\varphi\rangle \sim \mathcal{N}_{\mathbb{C}^d}(0,1)$ [Z], i.e. $|\varphi\rangle = \frac{1}{\sqrt{2d}} \sum_{j=1}^d (\alpha_j + i\beta_j) |j\rangle$ with independent Gaussian distributed real and imaginary parts $\alpha_j, \beta_j \sim \mathcal{N}(0,1)$ of zero mean and unit variance. In particular, $\mathbb{E} \langle \varphi | \varphi \rangle = 1$.

Now, using this and the unitary invariance of the distribution of $|\varphi\rangle$, we see

$$
\mathbb{E} \left[ \frac{1}{2a} \text{Tr}(\varphi P) - \frac{1}{2b} \text{Tr}(\varphi Q) \right] = \mathbb{E}_\varphi \left( \langle \varphi | \varphi \rangle \mathbb{E}_\psi \left[ \frac{1}{2a} \text{Tr}(\psi P) - \frac{1}{2b} \text{Tr}(\psi Q) \right] \right)
= \mathbb{E}_\varphi \left[ \frac{1}{2a} \text{Tr}(\varphi P) - \frac{1}{2b} \text{Tr}(\varphi Q) \right]
= \frac{1}{2} \mathbb{E}_{\alpha_j,\beta_j \sim \mathcal{N}(0,1)} \left[ \frac{1}{2a} \sum_{j=1}^a (\alpha_j^2 + \beta_j^2) - \frac{1}{2b} \sum_{j=a+1}^{a+b} (\alpha_j^2 + \beta_j^2) \right]
= \frac{1}{2} \mathbb{E}_{X,Y} \left[ \frac{1}{2a} X - \frac{1}{2b} Y \right].
$$

The sums of squares of Gaussian components occurring here are well-studied, and known under the name of $\chi^2$-distributions:

$$
\sum_{j=1}^a (\alpha_j^2 + \beta_j^2) =: X \sim \chi^2_{2a}, \quad \sum_{j=a+1}^{a+b} (\alpha_j^2 + \beta_j^2) =: Y \sim \chi^2_{2b},
$$

their probability density being given by

$$
\Pr\{X \in [x; x+dx]\} = \frac{1}{2(a-1)!} (x/2)^{a-1} e^{-x/2} dx, \quad \Pr\{Y \in [y; y+dy]\} = \frac{1}{2(b-1)!} (y/2)^{b-1} e^{-y/2} dy.
$$

This allows us to evaluate the latter expectation as follows, denoting the indicator function of a set $\{\ldots\}$ as $1\{\ldots\}$:

$$
\frac{1}{2} \mathbb{E}_{X,Y} \left[ \frac{1}{2a} X - \frac{1}{2b} Y \right] = \frac{1}{2} \mathbb{E}_{X,Y} \left( \int dr \, 1\{X/2a \leq r \leq Y/2b\} + \int dr \, 1\{Y/2b \leq r \leq X/2a\} \right)
= \frac{1}{2} \int_0^\infty dr \left( \mathbb{E} 1\{X \leq 2ar, Y \geq 2br\} + \mathbb{E} 1\{X \geq 2ar, Y \leq 2b\} \right)
= \frac{1}{2} \int_0^\infty dr \left( \Pr\{X \leq 2ar\} \Pr\{Y \geq 2br\} + \Pr\{X \geq 2ar\} \Pr\{Y \leq 2b\} \right)
= \frac{1}{2} \int_0^\infty dr \Pr\{X \geq 2ar\} + \frac{1}{2} \int_0^\infty dr \Pr\{Y \geq 2br\}
- \int_0^\infty dr \Pr\{X \geq 2ar\} \Pr\{Y \geq 2br\}.
$$
Using the $\chi^2$ densities, the probabilities under the integrals are easily evaluated:

$$\Pr\{X \geq 2ar\} = e^{-ar} \sum_{k=0}^{a-1} \frac{(ar)^k}{k!}, \quad \Pr\{Y \geq 2br\} = e^{-br} \sum_{\ell=0}^{b-1} \frac{(br)^\ell}{\ell!}.$$ 

This finally gives

$$E \left| \frac{1}{2a} \text{Tr}(\psi P) - \frac{1}{2b} \text{Tr}(\psi Q) \right| = \frac{1}{2} E_{X,Y} \left| \frac{1}{2a} X - \frac{1}{2b} Y \right|$$

$$= \frac{1}{2} + \frac{1}{2} - \int_0^\infty dr e^{-r(a+b)} \sum_{k=0, \ell=0}^{a-1, b-1} \frac{(ar)^k(br)^\ell}{k!\ell!}$$

$$= 1 - \frac{1}{a+b} \sum_{k=0, \ell=0}^{a-1, b-1} \left( \frac{a}{a+b} \right)^k \left( \frac{b}{a+b} \right)^\ell \left( k + \ell \right),$$

where we have used the integral for the Gamma function.

We will also need the following small lemma

**Lemma 24** Let $S_k$ denote $\sum_{l=0}^{k} 2^{-(k+l)}(k+l)$. We claim that for integers $k \geq 0$, $S_k = 1$.

**Proof.** Using the well known ‘addition formula’ $\binom{n}{m} + \binom{n-1}{m} = \binom{n}{m-1}$

$$S_{k+1} = \sum_{l=0}^{k+1} 2^{-(k+1+l)} \binom{k+1}{l} + \sum_{l=0}^{k-1} 2^{-(k+1+l)} \binom{k+1}{l}$$

$$= \frac{1}{2} S_k + 2^{-(2k+2)} \binom{2k+1}{k+1} + \sum_{l=0}^{k-1} 2^{-(2k+1+l)} \binom{k+l+1}{l}$$

$$= \frac{1}{2} S_k + 2^{-(2k+2)} \binom{2k+1}{k+1} + \frac{1}{2} S_{k+1} - \frac{1}{2} 2^{-(2k+2)} \binom{2k+2}{k+1}$$

so

$$S_{k+1} = S_k + 2^{-(2k+2)} \left( 2 \binom{2k+1}{k+1} - \binom{2k+2}{k+1} \right) = S_k$$

where the final equality is due to the addition formula and the symmetry $\binom{2k+1}{k+1} = \binom{2k+1}{k-1}$. To complete the proof we note that $S_0 = 1$. □

**APPENDIX C: UPPER BOUNDS ON CERTAIN TRACES**

**Lemma 25** Let $\xi$ be a traceless Hermitian operator on a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $P_{\text{sym}}^{(4)} A$ and $P_{\text{sym}}^{(4)} B$ denote the projector onto the completely symmetric subspace of $\mathcal{H}_A^{\otimes 4}$ and $\mathcal{H}_B^{\otimes 4}$, respectively.

Then, with the shorthands $t := \text{Tr}(\xi^2)$, $a := \text{Tr}(\xi_A^2)$ and $b := \text{Tr}(\xi_B^2)$, where $\xi_A = \text{Tr}_A(\xi)$ and $\xi_B = \text{Tr}_B(\xi)$,

$$\text{Tr}\left( (P_{\text{sym}}^{(4)} A \otimes P_{\text{sym}}^{(4)} B) \xi^{\otimes 4} \right) \leq \frac{1}{4!^2} (153t^2 + 126ta + 126tb + 9a^2 + 9b^2 + 30ab).$$  \hspace{1cm} (C1)
The proof is conceptually simple but a little long. We write the projection operators as averages over the unitary operators which permute the four subsystems. Defining, for permutations $\pi \in S_4$, the representation

$$U^A_\pi := \sum_{j \in \{1, \ldots, d\}^m} \bigotimes_{i=1}^4 |j_{i\pi(i)}^A \rangle \langle j_i^A|,$$

where $\{|j_i^A\rangle\}_{1 \leq j \leq d}$ is an orthonormal basis for the $i$-th copy of $H_A$ in $H_A^{\otimes 4}$, and defining $U^B_\pi$ similarly:

$$\text{Tr} \left( \left( D^{(4)}_{\text{sym} A} \otimes D^{(4)}_{\text{sym} B} \right) \xi^{\otimes 4} \right) = \frac{1}{24^2} \sum_{\pi \in S_4, \sigma \in S_4} \text{Tr} \left( U^A_\pi \otimes U^B_\sigma \xi^{\otimes 4} \right)$$

Clearly $(\pi, \sigma) \to U^A_\pi \otimes U^B_\sigma$ is a representation of $S_4 \times S_4$. $S_4 \times S_4$ has a subgroup consisting of all the elements of the form $(g, g)$, which we’ll denote by $R$.

If $(\pi', \sigma') = r^{-1}(\pi, \sigma)r$ for some $r \in R$, we write $(\pi', \sigma') \sim (\pi, \sigma)$ and note that the corresponding terms are equal since

$$\text{Tr} \left( U^A_{\pi'} \otimes U^B_{\sigma'} \xi^{\otimes 4} \right) = \text{Tr} \left( \left( U^A_\pi \otimes U^B_\sigma \right) \left( U^A_g \otimes U^B_g \right) \xi^{\otimes 4} \left( U^A_{g^{-1}} \otimes U^B_{g^{-1}} \right) \right) = \text{Tr} \left( U^A_\pi \otimes U^B_\sigma \xi^{\otimes 4} \right).$$

Essentially, conjugation by an element of $R$ corresponds to a permutation of the identical $\xi$ operators, and therefore leaves the term unchanged.

The set of all $24!^2$ terms is partitioned by the equivalence relation $\sim$ with the terms in each subset all equal to each other. We shall refer to these subsets as the $R$-conjugacy classes of $S_4 \times S_4$. Clearly, the $R$-conjugacy classes form a finer partition of $S_4 \times S_4$ than the normal conjugacy classes.

By demonstrating an appropriate upper-bound for the terms in each $R$-conjugacy class, and calculating the size of each class, we will prove the upper bound (C1).

Tensor Diagrams. Let establish an orthonormal basis $\{|i\rangle_A\}$ for $H_A (H_B)$. In this basis, we can write $\xi$ in component form thus $\xi_{i,j} = \langle k | A \otimes l | B \rangle \xi_{i,j} \langle k | A \otimes l | B \rangle$

We would like to demonstrate upper bounds for terms of the form

$$\text{Tr} \left( U^A_\pi \otimes U^B_\sigma \xi^{\otimes 4} \right) = \xi_{a_1,b_1}^{a(1),b(1)} \xi_{a_2,b_2}^{a(2),b(2)} \xi_{a_3,b_3}^{a(3),b(3)} \xi_{a_4,b_4}^{a(4),b(4)},$$

where the $ai$ and $bi$ ($i \in \{1, 2, 3, 4\}$) are dummy variables to be contracted over according to the Einstein summation convention. Using indices in our calculations would be rather messy and confusing. Instead we use the ingenious tensor diagrams of Penrose [21].

We denote our bipartite Hermitian operator $\xi$ by $\square$. The “terminals” of this diagram correspond to indices like so

$$\xi_{i,j}^{k,l} = \begin{array}{c} (\text{Alice}) \\ k \end{array} \square \begin{array}{c} i \\ l \end{array} \begin{array}{c} (\text{Bob}) \\ j \end{array}.$$ 

Joining the terminals with “wires” denotes contraction of the corresponding indices

$$\xi_{i,j}^{k,r,m,n} = \begin{array}{c} (\text{Alice}) \\ k \end{array} \begin{array}{c} r \\ j \end{array} \begin{array}{c} m \\ p \end{array} \begin{array}{c} (\text{Bob}) \\ q \end{array}.$$
\[
\xi_A := \text{Tr}_B(\xi) = \begin{array}{c}
\end{array}, \quad \xi_B := \text{Tr}_A(\xi) = \begin{array}{c}
\end{array},
\]

\[
\text{Tr}(\xi) = \begin{array}{c}
\end{array} = 0, \quad \text{Tr}(\xi^2) = \begin{array}{c}
\end{array} = t,
\]

\[
\text{Tr}(\xi_A^2) = \begin{array}{c}
\end{array} = a, \quad \text{Tr}(\xi_B^2) = \begin{array}{c}
\end{array} = b.
\]

In an effort to keep the diagrams tidy and compact, we sometimes use a pair of vertical grey lines, one with wires entering from the right and the other with a matching set of wires entering from the left. A diagram with this feature is to be read as equivalent to the diagram one obtains by identifying the grey lines in parallel to join the matching wires. It should not be confused with the bars drawn across wires (by Penrose and others) to denote (anti-)symmetrization.

Here is an example showing how a diagram corresponds to a particular term of the form (C2):

\[
\begin{array}{c}
\end{array} = \epsilon_{j,q} \epsilon_{l,p} \epsilon_{k,m} \epsilon_{i,m} \epsilon_{n,p} \epsilon_{i,n}
\]

In Fig. 1, we provide a table with a diagram representative of each of the \(R\)-conjugacy classes organised by the conjugacy class of \(S_4 \times S_4\) which contains it.

The size of each \(R\)-conjugacy class is written to the right of the corresponding diagram. An upper bound is given and diagrams which are identically 0 (by virtue of having a factor of \(\text{Tr}(\xi) = 0\)) are drawn in a lighter shade of grey.

**Proofs of upper bounds.** We give bounds for the terms shown in the upper-right triangle of Fig. 1. Bounds for those terms below the diagonal follow from these by exchanging the roles of the parties. We will make repeated use of the Cauchy-Schwarz inequality for the Hilbert-Schmidt inner product,

**Lemma 26** \[|\text{Tr}(A^\dagger B)|^2 \leq (\text{Tr}(A^\dagger A))(\text{Tr}(B^\dagger B))\].

Let \(P\) denote a positive semidefinite hermitian operator. We have the inequality \(\text{Tr}(P^2) \leq (\text{Tr}(P))^2\) (by the spectral decomposition of \(P\) for example). From this fact and the Cauchy-Schwarz inequality it follows that

**Lemma 27** If \(P\) and \(Q\) are both positive semidefinite, then \(\text{Tr}(PQ) \leq (\text{Tr}(P))(\text{Tr}(Q))\).

Third, since the partial transpose map is selfadjoint,

**Lemma 28** The quantities \(t, a\) and \(b\) are unchanged if we replace \(\xi\) with \(\xi^\Gamma\).

**Proof of Lemma 25.** We go through the types one by one.

\((2,2):(2,2)\)

\[
\begin{array}{c}
\end{array} = \text{Tr}(\xi^2) = t^2.
\]

To show that the same bound applies to

\[
\begin{array}{c}
\end{array},
\]

we note that it can be written as \(\text{Tr}((\text{Tr}_A(Z))^2)\), where

\[
Z = (\xi \otimes 1_C)(1_A \otimes |\Phi \rangle \langle \Phi|)(\xi \otimes 1_C)
\]

and \(|\Phi\rangle = \sum_{i=1}^d |i\rangle_B \otimes |i\rangle_C\). Since \(Z = ((\xi \otimes 1_C)(1_A \otimes |\Phi\rangle))( (\xi \otimes 1_C)(1_A \otimes |\Phi\rangle))^\dagger\), it is positive semidefinite and as such \(\text{Tr}((\text{Tr}_A(Z))^2) \leq (\text{Tr}(Z))^2\). The result follows by noting that \(\text{Tr}(Z) = t\).
(2,2):(1,1,1) \[\begin{array}{c|c|c}
\hline
\hline
\hline
(a) & (b) & (c) \\
\hline
\hline
\hline
\end{array}\] = (Tr(\(\xi_A^2\)))^2 = a^2. \\

(2,1,1):(2,1,1) \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] = ab. \\

(4):(4) Noting that \(\xi^2\) is positive semidefinite, and applying Lemma 27, we get \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] = Tr(\(\xi^4\)) \leq (Tr(\(\xi^2\)))^2 = t^2. The partial-transpose of \(\xi\), \(\xi^\Gamma\), has the diagrammatic representation \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] (we choose to take the transpose on Bob’s system). Substituting, this for \(\xi\) in \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] results in the diagram \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] = (Tr(\(\xi^\Gamma\)))^4, so Lemma 28 shows that the same bound applies here. The Cauchy-Schwarz inequality yields \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] = \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\], which can be seen to be \(\leq t^2\) because of the previous two bounds. \\

(4):(2,1,1) \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] = Tr\((\text{Tr}_B(\xi^2))\xi_A^2\) \leq (Tr(\(\xi^2\))(Tr(\(\xi_A^2\))) = ta, by Lemma 27. \\
(4):(3,1) \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] \leq \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] \[\begin{array}{c|c|c|c}
\hline
\hline
\hline
\hline
\end{array}\]. Using the results for these two diagrams and the arithmetic-geometric mean inequality we can bound this expression by \(t(t + a)/2\), as was claimed. \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] is given by substituting \(\xi^\Gamma\) into the previous diagram, so by Lemma 28 the previous bound applies. \\

(4):(2,2) \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] = Tr(Tr\(_B(\xi^2))^2 \leq t^2. For the other diagram we use the Cauchy-Schwarz inequality: \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] \leq \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] \[\begin{array}{c|c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] = \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] \leq t^2. \\
(2,2):(2,1,1) \[\begin{array}{c|c|c}
\hline
\hline
\hline
\hline
\end{array}\] = ta. For the other diagram in this class it is useful to define \(Y_B := \text{Tr}_A(\xi(\xi_A \otimes 1_B))\). We define \(Y_A\) similarly but with the roles of the parties reversed. \\
\[\begin{align*}
\text{Tr}(Y_B^2) &= \text{Tr}\left( \left( \text{Tr}_A(\xi(\xi_A \otimes 1_B)) \right) \cdot Y_B \right) \\
&= \text{Tr}(\xi(\xi_A \otimes 1_B)(1_A \otimes Y_B)) \\
&= \text{Tr}(\xi(\xi_A \otimes Y_B)) \\
&\leq \sqrt{\text{Tr}(\xi^2)(\text{Tr}(\xi_A^2)(\text{Tr}(Y_B^2)))},
\end{align*}\]
and therefore

\[ \text{Tr}(Y_B^2) \leq (\text{Tr}(\xi^2))(\text{Tr}(\xi_A^2)) = ta. \]

Similarly \( \text{Tr}(Y_A^2) \leq tb. \) Hence, \( \underbrace{\text{Tr}(\xi_A^4)}_{t} = \text{Tr}(Y_B^2) \leq ta. \)

\((4):(1,1,1,1)\) \( \underbrace{\text{Tr}(\xi_A^4)}_{t} \leq (\text{Tr}(\xi_A^2))^2 = a^2. \)

\((3,1):(3,1)\) \( \underbrace{\text{Tr}(\xi_A^4)}_{t} = \text{Tr}((\xi_A \otimes I_B)\xi^2(1_A \otimes \xi_B)) = \text{Tr}(\xi^2(\xi_A \otimes \xi_B)). \) Using the Cauchy-Schwarz inequality we upper bound this by \( \sqrt{\text{Tr}(\xi^4)\text{Tr}(\xi_A^2)(\text{Tr}(\xi_B^2))}, \) which in turn is bounded by \( (\text{Tr}(\xi^2))\sqrt{\text{Tr}(\xi_A^2)(\text{Tr}(\xi_B^2))} \leq (ta + tb)/2, \) using arithmetic-geometric mean inequality at the end. \( \underbrace{\text{Tr}(\xi_B^2)}_{t} \) is given by substituting \( \xi^2 \) into the previous diagram, so by Lemma 28 the same bound applies.

\((3,1):(2,2)\) \( \underbrace{\text{Tr}(\xi_B^2)}_{t} \leq \sqrt{\text{Tr}(\text{Tr}(\xi_B^2))(\text{Tr}(Y_A^2))^2} \leq t(t + b)/2. \)

\((3,1):(2,1,1)\) \( \underbrace{\text{Tr}(\xi_A^4)}_{t} \leq \sqrt{\text{Tr}(\xi_A^2)(\text{Tr}(Y_A^2))^2} \leq a(t + b)/2. \)

Now, collecting terms according to the multiplicities found in the table of Fig. 1 we conclude the proof. \( \square \)

**Remark 29** Note that for every pair of conjugacy classes of permutations, all the types falling into the corresponding box in Fig. 1 share the same upper bound.
FIG. 1: Sizes and upper-bounding expressions of the $R$-conjugacy classes. The faded diagrams are identically zero (because they contain a factor of $\text{Tr}(\xi)$).