RELATIVISTIC SU(4) AND QUATERNIONS

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Abstract. A classification of hadrons and their interactions at low energies according to SU(4) allows to identify combinations of the fifteen mesons π, ω and ρ within the spin-isospin decomposition of the regular representation $\mathbf{15}$. Chirally symmetric SU(2)$\times$SU(2) hadron interactions are then associated with transformations of a subgroup of SU(4). Nucleon and Delta resonance states are represented by a symmetric third rank tensor $\mathbf{20}$ whose spin-isospin decomposition leads to $4 \oplus 16$ ‘tower states’ also known from the large-$N_c$ limit of QCD. Towards a relativistic hadron theory, we consider possible generalizations of the stereographic projection $\mathbb{S}^2 \to \mathbb{C}$ and the related complex spinorial calculus on the basis of the division algebras with unit element. Such a geometrical framework leads directly to transformations in a quaternionic projective ‘plane’ and the related symmetry group SL(2, $\mathbb{H}$). In exploiting the Lie algebra isomorphism sl(2, $\mathbb{H}$) $\cong$ su*(4) $\cong$ so(5,1), we focus on the Lie algebra su*(4) to construct quaternionic Dirac-like spinors, the associated Clifford algebra and the relation to SU(4) by Weyl’s unitary trick. The algebra so(5,1) contains the de Sitter-algebra so(4,1) which can be contracted to the algebra of the Poincaré group.

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1. Introduction

Nowadays, QCD is assumed to be the theory of hadronic interactions at high energies, however, there is no complete and consistent theory for hadrons at low and intermediate energies yet. Instead of trying to ‘reduce’ QCD towards a low energy theory of hadrons or dealing with ‘QCD inspired’ effective hadron models, we shall start with symmetry properties and quantum numbers known from hadron dynamics at low energies, especially from the \(\pi N\Delta\) system. On this basis, we can generalize the known symmetry properties of hadrons towards a relativistic quantum field theory.

2. Classification of hadrons

The simplest hadronic classification scheme is based on the group SU(2) of isospin symmetry which is realized in the Wigner-Weyl mode. However, when calculating pion-nucleon scattering processes [Adler and Dashen, 1968], it became apparent that this ‘static’ classification scheme is not sufficient to yield a complete description of pion dynamics but that it has to be extended by the more sophisticated concept of a spontaneously broken symmetry.

2.1. Chiral Dynamics

This more sophisticated approach to hadron dynamics is mainly based on an underlying ‘chiral symmetry’, described by the group SU(2)×SU(2). Spontaneous breakdown of this symmetry can be treated in terms of projections with respect to the diagonal subgroup SU\(_V\)(2) which has been identified with the isospin symmetry group [Adler and Dashen, 1968], [Weinberg, 1968]. Mathematically, this approach towards SU\(_V\)(2) isospin quantum numbers is well established in the framework of the coset decomposition SU(2)×SU(2)/SU\(_V\)(2) [Coleman et al., 1969], [Callan et al., 1969]. However, physically these so called ‘nonlinear sigma models’ raise a lot of serious problems. The general inherent problem results from the identification of physical particles with representations of the subgroup SU\(_V\)(2) so that the action of ‘chiral group transformations’ on irreducible (static) isospin representations of the field algebra generates inequivalent representations [Joos and Weimar, 1976] and thus changes the properties and the quantum numbers of the particle. Furthermore, not all automorphisms can be realized by unitary operators on linear representations of the subgroup SU\(_V\)(2) [Haag and Kastler, 1964], [Fabri et al., 1967]. Thus, due to the identification of SU\(_V\)(2) representations with physical particles, this concept leads necessarily to highly nonlinear models with enormous mathematical complications and to a loss of renormalizability. A typical SU(2)×SU(2) nonlinear sigma model [Leutwyler, 1991] describes the mesons by means of

\[
U = \exp(-i\gamma_5 \vec{\tau} \cdot \vec{\varphi}/f_\pi) .
\]
The related ‘quantum field theory’ is based on an effective (nonlinear) Lagrangian in terms of ‘fields’ $U$ and (covariant) derivatives $\nabla^\mu U$ where the ‘field’ $U$ is usually expanded into a power series (for details, see [Leutwyler, 1991] and references therein). Furthermore, nucleon resonances like the Delta isobar excitation are usually not considered in these models although it is known that both nucleon and Delta states are needed to saturate the Adler-Weisberger sum rule [Oehme, 1965], [Kirchbach and Riska, 1991].

2.2. Dynamic classification of the nucleon

In almost all effective classification schemes and models, the nucleon transforms relativistically according to the same Dirac representation as the electron. However, with respect to its interactions and its dynamic properties this description of the nucleon doesn’t work. Therefore, one has to include electromagnetic corrections from the very beginning like the Pauli (spin) term to correct the nucleon’s magnetic moments. Furthermore, the idea of a ‘composite particle’ is necessary to explain obvious deviations from the simple description using a fundamental Dirac spinor and to justify the use of (effective) formfactors in the description of photon interactions with the nucleon.

To avoid these phenomenological corrections to the (covariant) Dirac representation and the related problems, we want to revive and extend an investigation concerning the properties of dynamic nucleons in the framework of a Goldstone realization of pions [Sudarshan, 1965]. Trying to explain renormalization effects of the value of the axial coupling constant $g_A$, Sudarshan parametrized the dynamic nucleon according to

$$|N\rangle_{\text{dyn}} = a |N\rangle_{\text{stat}} + \sqrt{1-a^2} \int d\omega \pi(\omega) |N(\omega)\rangle_{\text{stat}}.$$  

(2)

The second part describes dynamic deviations of the static nucleon classification in terms of an integral over the nucleon/Goldstone pion continuum. The requirement $g_A = 1.26$ fixes the value of $a^2$ in the parametrization (2) to $a^2 \approx 2/3$.

Technically, this approach can be equally well understood in a quasiparticle picture [Dahm et al., 1994] if we reinterpret the quantum numbers of the coupled $\pi N$-system in eq. (2) in terms of the (static) Delta resonance. Appropriately, pure SU$_V$(2) fermion representations can be used as basic constituents of ‘mixed’ dynamic fermion representations which naturally occur in the quasiparticle picture. In order to fix the (chiral) dynamic eigenvectors of the $N\Delta$-system, the saturation of the Adler-Weisberger sum rule [Oehme, 1965], [Kirchbach and Riska, 1991] with nucleon and Delta states suggests to diagonalize the nucleon/Delta axial charge matrix,

$$Q_5 = \begin{pmatrix}
<N^+_\frac{1}{2} | A_3 | N^+_\frac{1}{2}> & <\Delta^+_\frac{3}{2} | A_3 | N^+_\frac{1}{2}>\\
<N^+_\frac{1}{2} | A_3 | \Delta^+_\frac{1}{2}> & <\Delta^+_\frac{3}{2} | A_3 | \Delta^+_\frac{1}{2}>
\end{pmatrix}.$$  

(3)

This diagonalization leads to the same structure of dynamic states as in Sudarshan’s picture [Dahm et al., 1994], and the states (2) are included in a more general set of dynamic fermion representations (‘Chirons’).
Due to the quasiparticle picture, it is obvious that starting with irreducible $SU_V(2)$ fermion representations the dynamics of chiral transformations will mix these states so that isospin will no longer be conserved. Furthermore, this ansatz suggests with respect to spin degrees of freedom that dynamic nucleons should not be associated with a fundamental Dirac spinor but with a higher multiplet structure which yields nucleonic 1/2 components as well as 3/2 (Delta) components. Such a description avoids the inconsistency that operators of the chiral algebra generating the compact symmetry group $SU(2) \times SU(2)$ need to connect different irreducible fermion representations $N$ and $\Delta$ of the isospin as well as the chiral group.

2.3. SU(4) classification scheme

The ideas summarized in the last sections can be realized in a group theoretic framework if we find linear representations to classify quantum numbers of fermions and mesons in the physical spectrum which allow to represent all actions of the complete algebra of Chiral Dynamics. As mentioned in section 2.2, the saturation of the Adler-Weisberger sum rule leads to the qualitative suggestion to describe $SU_V(2)$ nucleon and Delta representations in the same representation so that the ‘chiral’ algebra can ‘connect’ these states when acting on a fermion multiplet. Sudarshan’s work resp. the quasiparticle picture allows for quantitative statements about the ‘mixture’ of nucleon and Delta fermions by fixing the parameter $a$ introduced in (2).

An ansatz on the basis of the Lie group SU(4) realizes these suggestions if we interpret SU(4) in terms of spin-flavour transformations and respect the SU(4) reduction according to the chain $SU(4) \supset USp(4) (\cong Sp(2)) \supset SU(2) \times SU(2) \supset SU(2)$. In this framework, dynamic nucleon and Delta states of Chiral Dynamics are represented by the third rank symmetric spinor representation $20$ of spin-flavour SU(4), $\Psi^{\alpha\beta\gamma}, 1 \leq \alpha, \beta, \gamma \leq 4$, which reduces with respect to spin-isospin quantum numbers [Dahm et al., 1994] according to

$$
\begin{align*}
\Psi^{111} &= \Delta_+^{1+}, & \Psi^{114} &= \sqrt{\frac{2}{3}} N_0^{1+} + \sqrt{\frac{1}{3}} \Delta_0^{1+}, \\
\Psi^{113} &= \Delta_+^{1+}, & \Psi^{144} &= \sqrt{\frac{2}{3}} N_0^{0+} + \sqrt{\frac{1}{3}} \Delta_0^{0+}, \\
\Psi^{133} &= \Delta_+^{0+}, & \Psi^{134} &= \sqrt{\frac{2}{3}} N_0^{1+} + \sqrt{\frac{1}{3}} \Delta_0^{1+}, \\
\Psi^{333} &= \Delta_+^{0+}, & \Psi^{124} &= \sqrt{\frac{2}{3}} N_0^{0+} + \sqrt{\frac{1}{3}} \Delta_0^{0+}, \\
\Psi^{112} &= \Delta_+^{1+}, & \Psi^{123} &= -\sqrt{\frac{2}{3}} N_0^{1+} + \sqrt{\frac{1}{3}} \Delta_0^{1+}, \\
\Psi^{334} &= \Delta_+^{0+}, & \Psi^{234} &= -\sqrt{\frac{2}{3}} N_0^{0+} + \sqrt{\frac{1}{3}} \Delta_0^{0+}, \\
\Psi^{221} &= \Delta_0^{1+}, & \Psi^{332} &= -\sqrt{\frac{2}{3}} N_0^{1+} + \sqrt{\frac{1}{3}} \Delta_0^{1+}, \\
\Psi^{443} &= \Delta_0^{0+}, & \Psi^{223} &= -\sqrt{\frac{2}{3}} N_0^{0+} + \sqrt{\frac{1}{3}} \Delta_0^{0+}, \\
\Psi^{222} &= \Delta_0^{1+}, & \Psi^{224} &= \Delta_0^{1+}, \\
\Psi^{244} &= \Delta_0^{0+}, & \Psi^{444} &= \Delta_0^{0+}.
\end{align*}
$$

(4)

The upper index denotes the charge and the lower one the spin projection of the
state. In symbolic notation this reduction reads as

\[ 20 \rightarrow \left( \frac{1}{2}; \frac{3}{2} \right) \oplus \left( \frac{3}{2}; \frac{3}{2} \right) \]

which yields exactly the tower states also obtained in the large-\( N_c \) limit of QCD.

The massive mesons \( \vec{\pi}, \omega \) and \( \vec{\rho} \) are identified within the regular representation \( 15 \) of SU(4), \( M^{\alpha \beta} \), which can be decomposed according to

\[ 15 \rightarrow (0, 1) \oplus (1, 0) \oplus (1, 1) \]

with respect to spin-isospin degrees of freedom. The explicit representation of the meson vector is given by

\[
M^{11} = \frac{1}{2}(\pi^0 + \omega^0 + \rho^0_0), \quad M^{12} = \sqrt{\frac{1}{2}}(\pi^+ + \rho^+_0), \\
M^{13} = \sqrt{\frac{3}{2}}(\omega_+ + \rho^0_+), \quad M^{14} = \rho^+_+, \\
M^{21} = \sqrt{\frac{3}{2}}(\pi^- - \rho^-_0), \quad M^{22} = \frac{1}{2}(-\pi^0 + \omega_0 - \rho^0_0), \\
M^{23} = \rho^-_-, \quad M^{24} = \sqrt{\frac{1}{2}}(\omega_+ - \rho^-_+), \\
M^{31} = \sqrt{\frac{3}{2}}(\omega_- + \rho^-_0), \quad M^{32} = \rho^+_-, \\
M^{33} = \frac{1}{2}(\pi^0 - \omega_0 - \rho^0_0), \quad M^{34} = \sqrt{\frac{1}{2}}(\pi^+ + \rho^-_0), \\
M^{41} = \rho^-_-, \quad M^{42} = \sqrt{\frac{1}{2}}(\omega_- - \rho^-_0), \\
M^{43} = \sqrt{\frac{1}{2}}(\pi^- - \rho^-_0), \quad M^{44} = \frac{1}{2}(-\pi^0 - \omega_0 + \rho^0_0),
\]

All these states have good spin and isospin projections \( S_3 \) and \( T_3 \) (electromagnetic charge), however, \( S^2 \) and \( T^2 \) are no Casimir operators of the rank 3 group SU(4), and neither total spin nor isospin are conserved in an SU(4) symmetric theory. It is well-known that spin is not conserved in relativistic dynamics, i.e. \( S^2 \) is not an appropriate Casimir operator with respect to a relativistic particle classification scheme. Nonconservation of isospin is already known from Chiral Dynamics where axial transformations 'connect' irreducible mesonic and fermionic isospin representations. Besides the axial transformations acting on the fermion space spanned by the \( N\Delta \) system, it is well-known that in the linear sigma model [Gell-Mann and Levy, 1960] the axial generators \( X_j \) connect an isospin singlet state \( \sigma \) \((T^2 \sigma = 0)\) with isospin triplet states \( \vec{\pi} \) \((T^2 \vec{\pi} = t(t+1)\vec{\pi} = 2\vec{\pi})\) states according to the commutation relations

\[
[X_j, \sigma] = i \pi_j, \quad [X_j, \pi_k] = -i \delta_{jk} \sigma.
\]

The generators \( T_j \) of the isospin subgroup SU\(_V\)(2) do not interchange the meson representations,

\[
[T_j, \sigma] = 0, \quad [T_j, \pi_k] = i \epsilon_{jkl} \pi_l.
\]

In a SU(4) hadron theory, the nonconservation of isospin has no influence on the definition of electromagnetic charges due to a well defined projection \( T_3 \), however,
isospin symmetry is slightly broken already by pure hadronic interactions. For example, the coupling of charged and neutral pions to the nucleon differs by \( \approx 10\% \) \cite{Dahm1994a}, \cite{Dahm1994b} if we identify physical hadronic states (in analogy to Chiral Dynamics) by the quantum numbers of the isospin reduction \( (7) \).

Using \( (4) \) and \( (7) \), a linear meson-fermion vertex describing hadronic interactions at low energies may be constructed according to the standard rules of SU(4) tensor algebra as \cite{Dahm1994a}, \cite{Dahm1994b}

\[
\mathcal{L}^\text{int} = G J^{\alpha \beta} M^{\alpha \beta} = G \Psi^{\tilde{\gamma} \tilde{\delta}} \Psi^{\beta \gamma \delta} M^{\alpha \beta}.
\] (10)

The algebra \( \text{su}(2) \oplus \text{su}(2) \) of Chiral Dynamics can be identified as a subalgebra of the Lie algebra \( \text{su}(4) \) and acts on the irreducible \( \text{su}(4) \) representations \( \Psi^{\alpha \beta \gamma} \) and \( M^{\alpha \beta} \) \cite{Dahm1994b}. Some results of the SU(4) coupling scheme and their comparison with experiments are given in \cite{Dahm1994a}, \cite{Dahm1994b}. However, as in the case of Chiral Dynamics it should be noted that the compact symmetry group SU(4) yields a good description of hadronic properties only at very low energies. With respect to a relativistic quantum field theory, it is at least necessary to find an appropriate noncompact symmetry group.

### 2.4. Towards a relativistic hadron theory

Before looking for such a noncompact symmetry group, it is noteworthy to mention some results obtained long ago from a completely different dynamic physical system, namely the classification scheme of ground states of nuclei.

In this context, Wigner \cite{Wigner1937} has already shown in 1937 that the Lie group SU(4) allows a reasonable classification of ground states. During the early 60ies, it has been shown on the basis of more complete experimental data that SU(4) indeed yields a good description of ground states of nuclei in the range of \( A=1 \) to \( A=140 \) \cite{Franzini1963}, \cite{Pais1964}. However, as soon as energy raises and dynamic effects become more important, SU(4) symmetry becomes worse.

This ‘SU(4) behaviour’ of the dynamically completely different system of nuclear ground states at low energies suggests to look for noncompact groups related to SU(4) by Weyl’s unitary trick so that at very low energies the various experiments cannot distinguish between SU(4) transformations and its related noncompact ‘counterpart(s)’. We are thus led to the group SU*(4) which results from embedding quaternions into complex vector spaces \cite{Helgason1962} by

\[
q_0 = 1_{2 \times 2}, \quad q_j = -i\sigma_j, \quad 1 \leq j \leq 3,
\] (11)

where \( \sigma_j \) denote the Pauli matrices.

However, besides these phenomenological considerations, based on the close relation of SU(4) and SU*(4) at low energies, there exists a straightforward mathematical approach. This direct approach towards a relativistic hadron theory is mainly inspired by eq. \( (4) \) which can be rewritten as

\[
U = \exp (\tilde{q} \cdot \tilde{\varphi}) = \cos \varphi + \tilde{q} \cdot \hat{\varphi} \sin \varphi, \quad \varphi = |\tilde{\varphi}|, \quad \hat{\varphi} = \frac{\tilde{\varphi}}{\varphi}
\] (12)
using the $2 \times 2$ complex matrix representations of quaternions as given by eq. (1) and their multiplication law $q_j q_k = -\delta_{jk} q_0 + \epsilon_{jkl} q_l$. For the sake of simplicity, we omitted in eq. (12) the rescaling $f_\pi$ of the parameters $\varphi$ as a redefinition of the ‘fields’ $\varphi_j$. Furthermore, with respect to the discussion of bosonic properties, we omitted the Dirac matrix $\gamma_5$ because of $\gamma_5^2 = 1$ and because $\gamma_5$ acts only on fermion spinors by exchanging upper and lower components. This exchange of the spinor components will be absorbed by the geometrical theory of the more general quaternionic transformations in section 3.2 so that the correspondence of $\gamma_5$ with the ‘fields’ $\varphi_j$ is related to reflections of quaternions at the quaternionic unit circle, $q \rightarrow q^{-1}$.

With the identity given in eq. (12), nonlinear realizations $U$ like eq. (1) which transform according to the representation $(\frac{1}{2}, \frac{1}{2})$ of SU(2)×SU(2) are nothing else but real quaternions normalized to unity,

$$||U||^2 = \frac{1}{2} \text{Tr} \left( U^+ U \right) = \cos^2 \varphi + \sin^2 \varphi = 1. \quad (13)$$

In analogy to the ‘polar’ and the ‘linear’ (cartesian) representation of complex numbers normalized to unity, $z = \exp(i\alpha) = x + iy$, $x = \cos \alpha$, $y = \sin \alpha$, the two identical representation schemes of the unit quaternion $U$ in eq. (12) can be denoted as ‘polar’ (‘nonlinear’) and ‘linear’ (cartesian) representations. The relevance of SU(2)×SU(2) and SO(4) transformations investigated in the framework of nonlinear [Weinberg, 1968], [Coleman et al., 1969], [Callan et al., 1969] resp. linear sigma models [Gell-Mann and Levy, 1960] then becomes obvious (section 3.3; Dahm and Kirchbach, 1995) due to the properties of the four dimensional projection plane and the isomorphism $U(1, \mathbb{H}) \cong SU(2, \mathbb{C})$. Thus, investigations of SU(2)×SU(2) nonlinear sigma models suggest strongly to identify ‘chiral’ transformations of hadron fields in the more general framework of quaternionic transformations.

After a short review of the wellknown complex case which leads to SU(2) spin in quantum mechanics and SL(2, C) ‘relativistic’ spinor theory, the necessary generalization is given for the quaternionic case.

3. Noncompact groups and spinors

3.1. $S^2 \rightarrow C$, the complex case

The treatment of spinors in quantum mechanics is closely related to the stereographic projection $S^2 \rightarrow \mathbb{C}$ [Gel’fand et al., 1963], [Penrose and Rindler, 1990]. Each point of the sphere $S^2$ can be described by two real parameters (angles). If we use the geometry given in [Penrose and Rindler, 1990] where the equator of the sphere lies in a plane parametrized by two real parameters (cartesian coordinates), the stereographic projection associates a point $P$ on the sphere with the two real coordinates of a point $P'$ of the equatorial plane which denotes the intersection of a line passing through the north pole $N$ and the point $P \in S^2$. If the point $P$ on the sphere moves on a continuous curve through the north pole $N$, the projection $P'$ has to move through one infinite point in the plane. This closure including one infinite point is possible by a relative complexification $i$ of the two real planar cartesian coordinates, and transformations of $P'$ can be investigated by appropriate transformations of
complex numbers in the complex ‘plane’\(^1\) \(\mathbb{C}\). In this context it is noteworthy, that
the projection \(S^2 \rightarrow \mathbb{C}\) *automatically* leads to a commutative theory independent
of the character and the further interpretation of the hypercomplex unit \(i\) since we
use only *one* hypercomplex unit as a relative complexification of the two real planar
coordinates. Therefore, this ‘minimal’ commutative approach to a projective theory
as used in quantum mechanics doesn’t justify a priori the assumption of \(i\) to be com-
mutative when we embed this complex projective theory into a higher hypercomplex
number system.

The projective transformations can be described by Möbius transformations in
the complex plane,

\[
f(z) \rightarrow f'(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C},
\]

so that curves on \(S^2\) are mapped onto curves in \(\mathbb{C}\). In \(\mathbb{C}\) one benefits from complex
analysis and well defined contour integrals which are related to finite paths on \(S^2\), a
nice feature which is for example used when applying dispersion relations in physics.
The possibility to ‘close’ these paths by adding *one* infinite point is intimately related
to the fact that eq. (14) has exactly *one* singularity, i.e. that the equation

\[
\gamma z + \delta = 0, \quad \gamma, \delta \in \mathbb{C},
\]

derived from the denominator of (14) has an *unique* solution. However, the geometry
described by Möbius transformations (14) may be equally well treated in terms of
matrix algebras by the identification

\[
f(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \longleftrightarrow A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C},
\]

which is formally motivated by the introduction of homogeneous coordinates \(z \rightarrow
z_1/z_2\) in eq. (14). Thus, two identical formalisms are available to treat the relevant
transformations:

- The use of coordinates \(z\) to describe points in the complex ‘plane’ allows to
define an involution \(\overline{z}\) (complex conjugation). This involution in \(\mathbb{C}\), when re-
stricted to the unit circle \(|z| = 1\), is equivalent to the replacement \(z \rightarrow z^{-1}\).
Transformations are described by Möbius transformations (14) and the appro-
priate complex analysis. Furthermore, Möbius transformations constitute a
group with respect to composition.

- Equivalently, one may define two dimensional complex spinors

\[
\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\]

when using homogeneous coordinates \(z_1\) and \(z_2\). The appropriate spinorial
transformations can be identified according to eq. (16) with matrices \(A, A \in
\]

\(^1\) We’ll denote the projective one dimensional complex and quaternionic ‘lines’ by ‘planes’ al-
though the nomenclature ‘plane’ is justified only in the complex case with respect to the involved
two real parameters.
\( C_{2 \times 2} \), so that the composition of Möbius transformations is equivalent to simple matrix multiplication. The related matrix groups in the complex case \( S^2 \rightarrow \mathbb{C} \) are the compact group \( SU(2) \) with respect to rotations of the sphere and the noncompact group \( SL(2, \mathbb{C}) \) with respect to general (noneuclidean) transformations. Restricting rotations of the sphere \( S^2 \) to rotations with the fixed cartesian axis \( z \) (choosing the ‘quantization axis’ \( \sim \hat{z} \)) and complexifying \( y \) relative to \( x \), this geometry leads to the groups \( SO(2) \) and \( U(1, \mathbb{C}) \). On the representation space of square-integrable functions this spontaneous symmetry breaking leads to the decomposition of spherical harmonics \( Y_{lm}(\theta, \phi) \) in terms of (nonlinear) Legendre polynomials and exponentials as representations of \( U(1, \mathbb{C}) \).

If we want to generalize this concept on the basis of a projective ‘plane’ and a noneuclidean geometry, it is not straightforward to generalize nothing but the matrix formalism on the basis of the related group theory to \( SU(n) \) or \( SL(n, \mathbb{C}) \), \( n > 2 \). However, with respect to eq. (15) derived from the denominator of the Möbius transformation, it is obvious that for the ‘numbers’ \( \gamma, z \) and \( \delta \) multiplication as well as addition has to be defined. To avoid problems with zero divisors of the necessary algebra and in order to add only one infinite point, the simplest possible generalization is a generalization on the basis of division algebras. This leads directly to the use of quaternions in eq. (14) and the related projection \( S^4 \rightarrow \mathbb{H} \).

### 3.2. \( S^4 \rightarrow \mathbb{H} \), the quaternionic case

Projections from the sphere \( S^4 \) can be understood on the same geometrical footing as in the projection \( S^2 \rightarrow \mathbb{R}^2 \) and the additional complexification to \( \mathbb{C} \). In the case \( S^4 \rightarrow \mathbb{H} \), however, it is the symmetry group \( SO(5) \) which acts transitively on \( S^4 \). Furthermore, fixing the projection point at the intersection of \( S^4 \) with the fifth cartesian (real) axis, the remaining compact symmetry group in the projection plane with respect to rotations restricted to the fifth axis is \( SO(4) \). The corresponding spinors can be defined by a complexification of all four real variables relative to each other which leads now to noncommutative hypercomplex units constituting the division algebra of quaternions. The spinors related to restricted rotations, i.e. to circles in the projection plane and to the orthogonal symmetry group \( SO(4) \), correspond to \( U(1, \mathbb{H}) \times U(1, \mathbb{H}) \) transformations of the quaternions (see section 4.3). However, since \( U(1, \mathbb{H}) \) is isomorphic to the group \( SU(2, \mathbb{C}) \), the spinor representations of the \( SO(4) \) linear sigma model can also be defined in terms of the complex covering group \( SU(2) \times SU(2) \). A similar geometry holds in the complex case (in quantum mechanics), where \( SO(3) \) acts transitively on \( S^2 \) and appropriate spinors can be defined using the covering group \( SU(2) \) or the noncompact group \( SL(2, \mathbb{C}) \) whereas in the case of restricted rotations of \( S^2 \) with a fixed \( z \)-axis the compact groups \( SO(2) \) resp. \( U(1, \mathbb{C}) \) describe the symmetry transformations in the complex plane.

Now, if we generalize eq. (14) to the division algebra of quaternions, a fraction of quaternions has to be defined carefully due to their noncommutativity. A suitable definition can be introduced by

\[
f(q) = \frac{aq + b}{cq + d} := \frac{aq + b}{cq + d} \equiv (aq + b) (cq + d)^{-1}
\]  

(17)
with \( a, b, c, d, q \in H \). Using this definition, it is possible to handle the relevant quaternionic Möbius transformations in analogy to the complex case in two equivalent formalisms:

- The use of coordinates \( q \) to describe points in the quaternionic projective ‘plane’ allows to define an involution \( \overline{q} \) (quaternionic conjugation). Quaternionic transformations in the plane are described by generalized Möbius transformations (17) and an appropriate quaternionic analysis. The transformations (17) constitute a group with respect to composition, too. Like in the case of complex numbers, the equation \( cq + d = 0 \) derived from the denominator in eq. (17) has an \emph{unique} solution \( q \) so that it is only necessary to take care of one singularity in eq. (17).

- As a second description, one may define two dimensional quaternionic spinors

\[
\Psi = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}
\]

when using homogeneous coordinates \( q_1 \) and \( q_2 \) to relate the spinor \( \Psi \) to points \( q = q_1/q_2 \) in the quaternionic projective ‘plane’. The appropriate matrix transformations can be identified with matrices \( \mathcal{A} \) according to

\[
f(q) = \frac{aq + b}{cq + d} \mapsto \mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in H,
\]

so that the composition of generalized Möbius transformations (17) is equivalent to simple matrix multiplication. The matrix groups in the quaternionic case \( S^4 \rightarrow H \) are the compact groups Sp(2) and its subgroup SU(2) \( \times \) SU(2) related to rotations of the sphere and the noncompact group SL(2,\( H \)) describing general (noneuclidean) transformations.

However, we are faced with the problem that we don’t have an appropriate quaternionic analysis yet. Thus, instead of investigating infinitesimal properties of the quaternionic transformations \( f(q) \) on the basis of an analysis, we want to benefit from the theory of Lie groups which allows to represent and investigate local transformations as well. Due to the Lie algebra isomorphism \( \text{sl}(2,\( H \)) \cong \text{su}^*(4) \cong \text{so}(5,1) \), all infinitesimal (local) properties of \( f(q) \) resp. \( \text{SL}(2,\( H \)) \) may be equally well discussed in terms of \( \text{so}(5,1) \) on real representation spaces or in terms of \( \text{su}^*(4) \) on complex representation spaces. This Lie algebra isomorphism is the basis of the algebraic hadron theory presented in section 4.

Note, that the geometrical generalization given in this section is a direct mathematical generalization on the basis of the four division algebras \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) and \( \mathbb{O} \), denoting real numbers, complex numbers, quaternions and octonions, respectively. In this geometrical scheme, there is \emph{no need} to refer to the physical motivation already given for \( \text{SU}^*(4) \) in section 2 but all symmetry properties are derived from the projection onto quaternions.

4. An algebraic theory

If we use Lie theory to investigate infinitesimal and global properties of the quaternionic projective geometry, there exist two further possibilities to investigate quaternionic projective transformations (19). The Lie algebra isomorphism \( \text{sl}(2,H) \cong \text{sl}^*(4) \cong \text{so}(5,1) \).
su*(4) \cong so(5,1) suggests investigations on real and complex representation spaces by means of the Lie algebras su*(4) and so(5,1). Taking all the relevant quaternionic, complex and real algebras and groups into account, an appropriate algebraic theory leads to the scheme given in figure 1.

This algebraic theory comprises three possible reduction schemes related to the three division algebras \( \mathbb{H}, \mathbb{C}, \text{and} \mathbb{R} \) involved in its representation theory. Using quaternions, the chain \( SU(2,\mathbb{H}) \supset SP(2) \supset SP(1) \times SP(1) \supset SP(1) \) is possible where one should put special emphasis on the quaternionic projective space \( HP(1) \cong SP(2)/SP(1) \times SP(1) \). The same symmetry transformations can be investigated in terms of an isomorphic complex representation theory by use of the chain \( SU^*(4) \supset USp(4) \supset SU(2) \times SU(2) \supset SU(2) \). Besides the possibility to discuss the projection \( USp(4)/SU(2) \times SU(2) \) in terms of complex numbers, we can use Weyl’s unitary trick to relate the noncompact group \( SU^*(4) \) to the compact group \( SU(4) \) which we used in section 2 as a nonrelativistic classification scheme of hadrons.

The reduction scheme on real spaces begins with the group \( SO(5,1) \) which is covered twice by \( SU^*(4) \). In this chain, we have more possibilities to identify groups which are relevant in classical physics. If we look for the noncompact subgroups \( SO(4,1) \) and \( SO(3,1) \) of \( SO(5,1) \), these groups can be identified with the de Sitter and the Lorentz group. The de Sitter group allows an approach to the Poincaré and the Galilei group which is discussed in section 4.2. The groups \( SO(6), SO(5) \) and \( SO(4) \) emerge by Weyl’s unitary trick where \( SO(5) \) has been discussed in the context of linear relativistic wave equations (Bhabha equations), the Duffin-Kemmer-Pétiau ring [Fischbach et al., 1974], [Krajcik and Nieto, 1977], and in Pauli’s approach towards a unified theory [Pauli, 1933]. \( SO(4) \) is the symmetry group of meson...
transformations in the linear sigma model, whereas its covering group SU(2) × SU(2) is used to define the appropriate fermion spinors or 'nonlinear' meson representations. SO(6) has been used as a generalization of the SO(4) sigma model where the mesons have been associated with the regular representation \( \mathbf{15} \) of SO(6), and third rank spinor representations of the covering group SU(4) have been used to classify hadronic fermions [Dahm and Kirchbach, 1995].

Here, we cannot treat all the algebras and groups involved in the proposed scheme in detail. Instead, we focus on a more detailed discussion of the two Lie algebras \( \mathfrak{su}^*(4) \) and so(5,1) on complex and real representation spaces which are equivalent to \( \mathfrak{sl}(2,\mathbb{H}) \) and thus describe the same infinitesimal 'physics'. The Lie algebra \( \mathfrak{su}^*(4) \) allows to define a Clifford product and can be related to the Dirac algebra \( \Gamma \) on \( \mathbb{C}^4 \times \mathbb{C}^4 \) whereas so(5,1) allows to obtain the Poincaré algebra. As further examples for the powerful quaternionic projective geometry, we discuss in section 4.3 the geometrically simple relations of the linear sigma model [Gell-Mann and Levy, 1960] to typical nonlinear sigma models currently used in hadron physics and in section 4.4 the classification according to the proposed theory. Section 4.5 is devoted to the effective description of special relativity in terms of complex quaternions and their relation to the quaternionic projective theory as well as to the relation of space-time and internal degrees of freedom.

4.1. Complex representation and \( \mathfrak{su}^*(4) \)

A general basis of \( \mathbb{C}^4 \times \mathbb{C}^4 \) consists of 32 elements which we want to parametrize by tensor products of two quaternions \( (q_\alpha, q_\beta) \) (which we denote by 'qquaternions' [Dahm, 1994]) with complex coefficients. Using the embedding of quaternions, we obtain a 16-dimensional algebra where the first quaternion \( q_\alpha \) determines the block structure and the second one, \( q_\beta \), specifies the entry. Allowing further for qquaternions with complex coefficients, we obtain a 32-dimensional algebra in \( \mathbb{C}^4 \times \mathbb{C}^4 \) which is a complete matrix algebra.

However, heading towards a decomposition of this 32-dimensional matrix algebra with respect to Lie group theory it is very useful to subdivide it using hermitean conjugation, i.e. according to the eigenspaces \(+/-\) of the involution \( \tau \). In this context, it is of great use to define a new 15-dimensional skew-hermitean basis \( \mathcal{Y}_{\alpha\beta} \), \( \mathcal{Y}^+ = -\mathcal{Y} \), \( 0 \leq \alpha, \beta \leq 3 \), in \( \mathbb{C}^4 \times \mathbb{C}^4 \) by

\[
\mathcal{Y}_{j0} = (q_j, q_0), \quad \mathcal{Y}_{0k} = (q_0, q_k), \quad \mathcal{Y}_{jk} = i(q_j, q_k),
\]

(20)

1 \( \leq j, k \leq 3 \). The exponential mapping

\[
\exp : \mathcal{Y} \longrightarrow \{U\}_{4 \times 4}
\]

relates \( \mathcal{Y}_{j0}, \mathcal{Y}_{0k} \) and \( \mathcal{Y}_{jk} \) to unitary matrices \( U \) in \( \mathbb{C}^4 \times \mathbb{C}^4 \) and to the group SU(4).

In the complete 32-dimensional matrix algebra \( \mathbb{C}^4 \times \mathbb{C}^4 \), we can further identify the Lie algebra \( \mathfrak{su}^*(4) \) according to the definition [Helgason, 1962]

\[
\mathfrak{su}^*(4) = \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \middle| A_1, \ldots, A_4 \in \mathbb{C}^2 \right\}
\]

(22)
if \( A_3 = -\overline{A}_2, \ A_4 = \overline{A}_1 \) and \( \text{Tr}(A_1 + \overline{A}_1) = 0 \) where \( \overline{A} \) denotes complex conjugation. In the skew-hermitean basis (20), we find the generators of \( SU^*(4) \) given by

\[
su^*(4) = \{ \mathcal{H}_{02}, \mathcal{H}_{10}, \mathcal{H}_{20}, \mathcal{H}_{30}, \mathcal{H}_{11}, \mathcal{H}_{13}, \mathcal{H}_{21}, \mathcal{H}_{23}, \mathcal{H}_{31}, \mathcal{H}_{33} \}_{A_0, A_1, A_2, A_3, A_4}
\]

(23)

Introducing two sets \( G_1 \) and \( G_2 \) by

\[
G_1 = \{ \mathcal{H}_{02}, \mathcal{H}_{10}, \mathcal{H}_{20}, \mathcal{H}_{30}, \mathcal{H}_{11}, \mathcal{H}_{13}, \mathcal{H}_{21}, \mathcal{H}_{23}, \mathcal{H}_{31}, \mathcal{H}_{33} \}
\]

(24)

and

\[
G_2 = \{ \mathcal{H}_{01}, \mathcal{H}_{03}, \mathcal{H}_{12}, \mathcal{H}_{22}, \mathcal{H}_{32} \}
\]

(25)

the Lie algebras \( su^*(4) \) and \( su(4) \) can be represented as

\[
su^*(4) = G_1 \oplus iG_2
\]

\[
su(4) = G_1 \oplus G_2
\]

(26)

which reflects the relation of \( su^*(4) \) and \( su(4) \) by Weyl’s unitary trick. The subalgebra \( G_1 \) is isomorphic to the Lie algebra \( sp(2,\mathbb{C}) \) and generates the maximal compact subgroup \( Sp(2) \) of \( SU^*(4) \). The proof is straightforward by applying the definition of the Lie algebra \( sp(2,\mathbb{C}) \) [Helgason, 1962],

\[
sp(2,\mathbb{C}) = \left\{ \left( \begin{array}{cc} A_1 & A_2 \\ A_3 & A_4 \end{array} \right) \left| A_1, \ldots, A_4 \in \mathbb{C}_{2\times2} \right. \right\}
\]

(27)

where \( A_2^T = A_2, \ A_3^T = A_3 \) and \( A_4 = -A_4^T \). Thus, with respect to hermitean conjugation we find the Cartan decomposition of \( su^*(4) \) generators according to \( 10 \oplus 5 \) where the 10 skew-hermitean operators generate the compact subgroup \( Sp(2) \). Besides this decomposition, it is noteworthy to mention the nontrivial anticommutators of the operators \( \mathcal{H}_{\alpha\beta} \) in eq. (20). If we calculate their commutation and anticommutation properties, the elements \( \mathcal{H}_{\alpha\beta} \) of the above Lie algebras fulfil the commutation relations

\[
[A_{00}, A_{\alpha\beta}] = 0, \quad [A_{0j}, A_{0k}] = 2 \epsilon_{jkl} A_{0l}, \quad [A_{0j}, A_{\alpha k}] = 2 \epsilon_{jlm} A_{km}, \\
[A_{10}, A_{\alpha k}] = 2 \epsilon_{jkl} A_{10}, \quad [A_{1j}, A_{\alpha l}] = 2 \epsilon_{jkm} A_{ml},
\]

(28)

where the unit matrix in \( \mathbb{C}_{4\times4} \) is denoted by \( A_{00} \). From these commutation relations it is obvious that the two three dimensional sets \( A_{0j} \) and \( A_{k0} \) generate an \( SU(2) \times SU(2) \) subgroup of \( SU(4) \). The anticommutation relations read as

\[
\{ A_{00}, A_{\alpha\beta} \} = -2 \delta_{\alpha\beta} A_{00}, \quad \{ A_{0j}, A_{0k} \} = -2 \delta_{jk} A_{00}, \\
\{ A_{0j}, A_{k0} \} = 2i \delta_{jk} A_{00}, \quad \{ A_{0j}, A_{\alpha l} \} = -2i \delta_{j\alpha} A_{k0}, \\
\{ A_{10}, A_{\alpha k} \} = -2 \delta_{jk} A_{10}, \quad \{ A_{10}, A_{\alpha l} \} = -2i \delta_{j\alpha} A_{k0}, \\
\{ A_{jk}, A_{im} \} = -2 \delta_{jk} \epsilon_{knp} A_{00} + 2i \epsilon_{jlm} \epsilon_{knp} A_{n0}.
\]

(29)
Furthermore, using the definition
\[ \gamma_0 = i \mathcal{H}_0 \in i G_2, \quad \gamma^j = \mathcal{H}_j \in G_1, \] (30)
the matrices \( \gamma_0, \gamma^j \in \mathfrak{su}(4) \) fulfil the Clifford product
\[ \frac{1}{2} \{ \gamma_0, \gamma_0 \} = 1_{4\times4}, \quad \frac{1}{2} \{ \gamma_0, \gamma^j \} = 0, \quad \frac{1}{2} \{ \gamma^j, \gamma^k \} = -\delta_{jk} 1_{4\times4}, \] (31)
and the full Dirac algebra can be constructed according to
\[ \gamma_5 = \mathcal{H}_{02}, \quad \gamma_5 \gamma^0 = i \mathcal{H}_{01}, \quad \gamma_5 \gamma^j = -i \mathcal{H}_{j3}, \quad \sigma^{0j} = i \mathcal{H}_{j2}, \quad \sigma^{jk} = \epsilon_{jkl} \mathcal{H}_{l0}. \] (32)
Thus, adding the unit element \( 1_{4\times4} \) to the adjoint representation of \( \mathfrak{su}(4) \), the algebra is isomorphic to the Dirac algebra, and the sixteen coefficients (‘fields’) of the decomposition
\[ \Gamma = s1_{4\times4} + p \gamma_5 + v_\mu \gamma^\mu + a_\mu \gamma_5 \gamma^\mu + f_\mu\nu \sigma^{\mu\nu} \] (33)
can be associated with \textit{real} coefficients of the regular representation of the group \( \text{SU}(4) \). Besides the nice possibility to identify various ‘fields’ used in perturbative and nonperturbative/effective models, this theory allows to determine the appropriate global and infinitesimal transformation properties \textit{exactly}. Furthermore, the application of Lie group theory allows to handle \textit{finite} transformations as well as infinitesimal (local) transformations which is interesting with respect to summing up a complete perturbation series.

\subsection*{4.2. Real representation and \textit{so}(5,1)}

The relation of quaternionic projective theoretical to observables in classical physics can be established by representing the isomorphic Lie algebra \( \text{so}(5,1) \) on real spaces.

The relevant algebra of classical dynamics is the Poincaré algebra \( \mathcal{P} \) which generates finite (macroscopic) space-time transformations. However, it is wellknown [Levy-Nahas, 1967] that only two Lie algebras can be contracted [Gilmore, 1974] to the Poincaré algebra, namely the de Sitter algebra \( \text{so}(4,1) \) and the anti-de Sitter algebra \( \text{so}(3,2) \). Thus, as a direct possibility, we can identify the subalgebra \( \text{so}(4,1) \) of \( \text{so}(5,1) \) straightforward with the de Sitter algebra to obtain the Poincaré algebra by contraction, i.e. in the limit of vanishing curvature. This interpretation allows to identify the ten generators of Poincaré space-time transformations in a contracted (projective) limit of quaternionic generators in \( \mathfrak{sl}(2,\mathbb{H}) \). In this sense, classical dynamics on real representation spaces and Dirac theory on complex representation spaces can be understood as \textit{two facets} of one and the same quaternionic projective theory. It is noteworthy, that the Dirac algebra is \textit{isomorphic} to the quaternionic projective theory in terms of \( \mathfrak{su}(4) \) whereas classical dynamics, described by means of the Poincaré algebra, appears after an additional contraction, i.e. in a special (projective) limit. Because \( \mathfrak{su}(4) \) (resp. \( \mathfrak{su}(4) \) as given in section 3) already describes internal symmetry (flavour) degrees of freedom \( \mathcal{I} \), quaternionic theory suggests that one \textit{shouldn’t} handle dynamic and internal symmetry by simply assuming...
direct/semidirect products like $\mathcal{P} \times \mathcal{I}$. Moreover, space-time and internal symmetry transformations are connected by commutation/anticommutation relations of the generators of $\mathfrak{sl}(2, \mathbb{H}) \cong \mathfrak{su}^*(4) \cong \mathfrak{so}(5,1)$.

In real representation theory, we can use two mathematical approaches. Either the generators are represented by matrices acting on appropriate real vector spaces or we can choose representations of the generators in terms of differential operators acting on spaces of (square-integrable) functions. The second approach allows to discuss transformation laws in terms of differential equations and appropriate polynomial systems as solutions of these equations. Thus, if we represent the generators of the Lie algebra $\mathfrak{so}(5,1)$ and their subalgebras as differential operators [Helgason, 1962], [Gilmore, 1974], we can introduce appropriate coordinate systems and relate the differential equations to dynamic laws known from classical physics.

4.3. Chiral Dynamics revisited

After the identification of the $\text{SU}(2) \times \text{SU}(2)$ meson representation (12) as unit quaternion, there exists an elegant geometrical approach towards linear and nonlinear sigma models and their relations. Restricting the quaternions in eq. (17) to a special structure, $b = c = 0$, $a, d \in U(1, \mathbb{H})$ and $q = U \in U(1, \mathbb{H})$, the generalized M"obius transformation takes the form

$$f(U) = a Ud^{-1}. \tag{34}$$

The treatment of quaternions differs from the complex case in that the quaternions $U$ and $d^{-1} (= d^+ \text{ for } d \in U(1, \mathbb{H}))$ in general do not commute. Therefore, the appropriate symmetry group in the case of the quaternionic circle is $U(1, \mathbb{H}) \times U(1, \mathbb{H})$ respectively $\text{SU}(2) \times \text{SU}(2)$ if we use the isomorphism $\text{SU}(2) \cong U(1, \mathbb{H})$. Thus, invariance under the symmetry group $\text{SU}(2) \times \text{SU}(2)$ reflects on complex representation spaces the fact that $U(1, \mathbb{H})$ is closed under quaternionic multiplication, $f(q) \in U(1, \mathbb{H})$ in eq. (34), and the group transformations map unit quaternions onto unit quaternions (they act on the quaternionic ‘unit circle’).

The product structure of the ‘chiral’ group stems from the quaternionic projective transformation law (19) if we use the description of quaternions in the projective plane and from the fact that quaternions in general do not commute. The decoupling of the two $\text{SU}(2)$ groups in the ‘chiral’ group becomes even more obvious in terms of the matrix representation (15) if we choose as above $b = c = 0$ and $a, d \in \text{SU}(2, \mathbb{C})$. Note, that in this theory there is no necessity to introduce ‘handedness’ with respect to a chiral structure of objects or to think about mass of fermions and bosons. The ‘chiral’ structure originates directly from the higher hypercomplex and noncommutative geometry. In the complex case, this ‘chiral’ structure doesn’t exist because the symmetry group in the projective plane reduces to $U(1, \mathbb{C})$ due to the commutativity of complex numbers. Thus, we are left with the $U(1, \mathbb{C})$ symmetry which allows to choose a complex phase like in quantum mechanics.

Geometrically, $U(1, \mathbb{H}) \times U(1, \mathbb{H})$ symmetry stems from restricted rotations of the sphere $S^4$ with respect to a fixed axis through its north pole, i.e. if the symmetry group $\text{SO}(5)$ (resp. its covering $\text{Sp}(2)$) is restricted to $\text{SO}(4)$ (resp. its covering
SU(2) × SU(2). This geometry justifies the identification

\[ \sigma = \cos \varphi, \quad \vec{\pi} = \hat{\varphi} \sin \varphi \]  

already given in \((12)\) in the context of linear and nonlinear representations of unit quaternions.

As discussed at the beginning of this section, the norm \((13)\),

\[ \|U\|^2 = \frac{1}{2} \text{Tr} (U^+ U) = \sigma^2 + \vec{\pi}^2 = 1_{2 \times 2} \]

of a unit quaternion \(U\) is conserved under \(U(1, \mathbf{H}) \times U(1, \mathbf{H})\). The invariance of this norm, however, can be equally well interpreted in terms of \(\text{SO}(4)\) acting on \(\mathbb{R}^4\) (respectively the sphere \(S^3\)) by introducing a four dimensional real vector \(B = (\sigma, \vec{\pi})\) such that

\[ \|U\|^2 = \frac{1}{2} \text{Tr} (U^+ U) = B^2. \]  

Thus, the linear sigma model [Gell-Mann and Levy, 1960] covers a special aspect of quaternionic projective theory, and the three parameters \(\varphi_j\) of the nonlinear representation \(U\) in eq. \((12)\) can be related to an appropriate linear parametrization by using coordinates in \(\mathbb{R}^4\) which are related to angles of \(S^3\) (see for example the set of coordinates in [Boyer, 1971]).

However, it is obvious that expansions of \(U\) in terms of parameters \(\varphi\) as used in nonlinear effective hadron models [Leutwyler, 1991] spoil this geometrical concept. Since the transcendental functions in \((13)\) respectively the ‘fields’ \(\sigma\) and \(\vec{\pi}\) can be interpreted as complete sums of an even and an odd power series in \(\varphi\), the linear sigma model yields a complete description of mesonic properties as well as a nonlinear theory in terms of \(U\). This is not true for any expansions of \(U\) in terms of \(\varphi_j\) up to a certain finite order. The description of fermions with respect to general \(\text{SU}^*(4)\) or \(\text{Sp}(2)\) transformations has to use quaternions, or, in an appropriate symmetry reduction caused by a fixed rotation axis of the sphere \(S^4\), fermion spinors can be described by representations of the group \(\text{SU}(2) \times \text{SU}(2)\) like in Chiral Dynamics. Moreover, \((14)\) allows for a direct identification of flavour \(\text{SU}(2)\) quantum numbers in the reduction scheme \(\text{SU}(2) \times \text{SU}(2)/\text{SU}(2)\). Since the ‘chiral’ product structure of \(\text{SU}(2) \times \text{SU}(2)\) originates from the noncommutative character of quaternionic projective transformations \((17)\) there is no further geometrical meaning in a naive generalization to arbitrary \(\text{SU}(n) \times \text{SU}(n)\) groups for \(n > 2\).

### 4.4. Representation theory and the classification of matter fields

The relation of the Dirac algebra \(\Gamma\) and the Lie group \(\text{SU}^*(4)\) in section 4.1 suggests to identify massive matter fields with appropriate representations of \(\text{SU}^*(4)\).

Due to the description \((18)\) of the fundamental Dirac spinor in terms of quaternions, quantum electrodynamics suggests to associate the electron with the fundamental representation of \(\text{SU}^*(4)\). The quaternionic projective approach then allows investigations of gauge theories as well as investigation of classical theories by appropriate identifications of the potentials respectively the classical fields \(\vec{E}\) and \(\vec{B}\) in \((23)\).
Thus, tracing the coefficients in (33) back to the generalized Möbius transformation (17) one is to lead to a geometrical interpretation of QED in terms of quaternionic projective transformations.

As motivated in sections 2.2 and 2.3, higher spinorial representations of SU\(^{*}(4)\) should be associated with massive hadrons in the particle spectrum. Appropriately, the third rank symmetric spinor \(\Psi^{\alpha\beta\gamma}\) describes the 20 nucleon/Delta fermions whereas the regular representation describes the massive meson fields \(\vec{\pi}, \omega\) and \(\vec{\rho}\).

Due to this classification scheme of the matter fields, there is no need to use the concept of a spontaneously broken SU(2)\(\times\)SU(2) symmetry with necessarily massless Goldstone pions and an additional explicit chiral symmetry breaking to restore the pion mass in order to explain the pseudovector coupling of pions to fermions. Geometrically, it is interesting that the fundamental spinor corresponds to a point in the quaternionic projective plane whereas the higher spinorial representations correspond to extended objects with certain symmetry properties and thus have additional degrees of freedom.

4.5. A noncanonical decomposition of SU\(^{*}(4)\) and space-time

As a further interesting feature of SU\(^{*}(4)\) representation theory, we want to connect quaternionic projective theory to the wellknown description of space-time and relativistic dynamics in terms of quaternions with complex coefficients [Blaton, 1935], [Blaschke, 1959], so called biquaternions [Clifford, 1878].

Therefore, we map su\(^{*}(4)\) not by the canonical mapping \(\exp : G_1 \oplus iG_2 \to \text{SU}^\ast(4)\) onto the group but we focus on the properties of

\[
X = \exp (-\alpha_1 i\mathbf{1}_{12} - \alpha_2 i\mathbf{1}_{22} - \alpha_3 i\mathbf{1}_{32}) \tag{37}
\]

emerging in the noncanonical parametrization

\[
g \in \text{SU}^\ast(4) = \exp (\beta_1 i\mathbf{R}_{01}) \exp (\beta_2 i\mathbf{R}_{03}) \exp (\{G_1\}) * \exp (-\alpha_1 i\mathbf{1}_{12} - \alpha_2 i\mathbf{1}_{22} - \alpha_3 i\mathbf{1}_{32}) \tag{38}
\]

The elements \(P \in \{i\mathbf{R}_{01}, i\mathbf{R}_{03}\}\) parametrized by real coefficients \(\beta_1\) and \(\beta_2\) fulfill \(P^2 = 1\) and thus give rise to four projection operators. The (symbolic) exponential on the rhs of the first equation in (38) maps the Lie algebra sp(2, \(\mathbb{C}\)) onto the maximal compact subgroup Sp(2) of SU\(^{*}(4)\). The additional negative signs of the real parameters \(\alpha_j\) are introduced in the argument of the exponential (37) to simplify calculations when using the quaternionic basis defined in section 4.1 instead of the skew-hermitean basis (20).

Rewriting (37) in terms of quaternionic operators, we find

\[
X = \exp (\alpha_1 (q_1, q_2) + \alpha_2 (q_2, q_2) + \alpha_3 (q_3, q_2)) \rightarrow \bar{t}(q_0, q_0) + x_1(q_1, q_2) + x_2(q_2, q_2) + x_3(q_3, q_2). \tag{39}
\]

Furthermore, \((q_0, q_0)\) commutes with the \((q_j, q_2)\) and \((q_j, q_2)\) is isomorphic to \((q_2, q_2)\) which can be seen by changing the order of indices and using the symmetry of
quaternionic multiplication. Therefore, we can associate the basis elements \((q_2, q_j)\) related to the finite coordinates \(x_j\) with the matrix representations

\[
(q_2, q_j) = \begin{pmatrix} 0 & -q_j \\ q_j & 0 \end{pmatrix}
\]

(40)
due to the construction scheme of qquaternions. Using the \(2 \times 2\) matrix representation

\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad i^2 = -1
\]

(41)
of the imaginary (commutative) unit \(i\), it is obvious that the event \(\bar{X}\) in (37) emerging in the noncanonical decomposition (38) of \(SU^\ast(4)\) group elements is isomorphic to the wellknown description of (hermitean) space-time in terms of biquaternions,

\[
\bar{X} = x_0 q_0 + iq_j x_j.
\]

(42)

Thus, it is possible to express all transformations of the Lorentz group, i.e. rotations and boosts, in a common framework [Dahm, 1994] on the basis of noncanonical decompositions of qquaternionic mappings.

Furthermore, because the ‘norm’ \(|\bar{X}|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2\) of space-time events as given in (39) resp. (42) is conserved under Lorentz transformations, the Wick rotation as a complexification of the time component allows to factor out and omit the overall sign,

\[
|\bar{X}|^2 \rightarrow (ix_0)^2 - x_1^2 - x_2^2 - x_3^2 = - (x_0^2 + x_1^2 + x_2^2 + x_3^2).
\]

(43)

 Appropriately, the norm on the rhs of (43) is conserved under \(SO(4)\) transformations which reflects the relation of the Lorentz group \(SO(3,1)\) and the compact group \(SO(4)\) by Weyl’s unitary trick. Thus, we can express (Wick rotated) space-time events in terms of four dimensional vectors with real coordinates which are related phenomenologically to the division algebra of quaternions, to the symmetry group \(SO(4)\) and to Gegenbauer polynomials.

This approach has the further nice feature that it allows to explain some properties of quantum mechanics and the first quantization [Dahm, 1994] although we cannot explain Planck’s constant with respect to the simple, restricted algebraical considerations given above. Applying a Wick rotation to space-time, the conservation of the norm (43) of ‘space-time’ events leads to the symmetry group \(SO(4)\) acting on a four dimensional real euclidean space. However, fixing the time component in the representation (39), i.e. looking only for stationary problems at fixed time, one benefits from the direct product decomposition of \(SO(4)\) according to \(SO(4) \cong SO(3) \times SO(3)\). Instead of \(SO(4)\) transformations mixing (Wick-rotated) time with the three space components, the remaining transformations in coordinate space at fixed time respect automatically \(SO(3)\) rotation symmetry. But this is exactly the point where nonrelativistic quantum mechanics starts by solving differential equations for square-integrable functions on \(\mathbb{R}^3\).
5. Summary and Outlook

We have presented a generalization of ‘effective’ hadron models towards an algebraic theory based on the division algebra of quaternions. This theory allows from ‘first principles’ to embed Chiral Dynamics completely into quaternionic transformations, to identify particle representations and quantum numbers and to relate various effective hadron models on a geometrical basis. Furthermore, real and complex representations of quaternionic projective theory allow to identify symmetry transformations in classical physics as well as transformations of quantum field theory as two facets of one and the same quaternionic theory (section 4). In quaternionic theory, space-time events and internal degrees of freedom are treated in an unified framework of quaternionic transformations like in other approaches based on Clifford algebras [Keller, 1995].

Besides the topics covered in section 4, there is a large variety of other very interesting features comprised in the algebraic theory presented in figure 1. Especially with respect to the mass spectrum and a discussion of relativistic wave equations in terms of induced representations [Niederer and O’Raifeartaigh, 1974], the quaternionic projective theory allows to focus on coset decomposition like SU$^*$\( (4)/USp(4)\) or the projective space \(\mathbb{HP}(1)\). For the framework of Chiral Dynamics is completely embedded in this algebraic theory on the basis of quaternionic transformations and their representations, it seems possible to handle relativistic transformations of physical hadrons in a complete and exact framework. In this context, explicit calculations of cross sections and further observables can be achieved by using representation theory of the groups SU$^*$\( (4)\) and SO\( (5,1)\) so that local as well as global transformation properties of the representations are exactly determined.

Comparison of these calculations with experiments should be done in an energy range where it is possible to discriminate between the compact group SU\( (4)\) and the related noncompact group SU\(^* (4)\) but without having too much influence of higher resonances. Practically, this restricts investigations to the dynamics of the \(\pi N\Delta\)-system where the quaternionic projective theory has its roots. However, in this energy regime we have access to experimental data by high precision experiments which are possible at the electron accelerator MAMI [Walcher, 1994].

With respect to an algebraic construction \(\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}\) of the division algebras, one further extension of the above quaternionic geometry is possible on the basis of octonions. However, due to the nonassociative structure of the algebra of octonions [Dixon, 1994] such a model cannot be discussed completely in terms of groups and by simple matrix representations but one has to subdivide octonions to cover certain aspects with tools like group theory or standard representation theory.

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