T-STRUCTURES ON THE DERIVED CATEGORIES OF HOLONOMIC D-MODULES AND COHERENT O-MODULES

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Dedicated to Boris Feigin on his fiftieth birthday

Abstract. We give the description of the $t$-structure on the derived category of regular holonomic $\mathcal{D}$-modules corresponding to the trivial $t$-structure on the derived category of constructible sheaves via Riemann-Hilbert correspondence. We give also the condition for a decreasing sequence of families of supports to give a $t$-structure on the derived category of coherent $\mathcal{O}$-modules.

1. Introduction

It was one of the motivations of the introduction of the notions of $t$-structures and perverse sheaves by A. Beilinson, J. Bernstein, P. Deligne and O. Gabber ([1]) to ask what are the objects corresponding to regular holonomic $\mathcal{D}_X$-modules by the Riemann-Hilbert correspondence ([4]) $R\text{Hom}_{\mathcal{D}_X}(-, \mathcal{O}_X) : D^b_{\text{rh}}(\mathcal{D}_X) \to D^b_c(C^0_X)^\text{op}$. Here $X$ is a complex manifold, $D^b_c(C^0_X)^\text{op}$ is the opposite category of the derived category $D^b_c(C^0_X)$ of bounded complexes of sheaves on $X$ with constructible cohomologies, and $D^b_{\text{rh}}(\mathcal{D}_X)$ is the derived category of bounded complexes of $\mathcal{D}_X$-modules with regular holonomic cohomologies.

With the notion of $t$-structures, the answer is: the $t$-structure of the middle perversity on $D^b_c(C^0_X)$ corresponds to the trivial $t$-structure on $D^b_{\text{rh}}(\mathcal{D}_X)$.

The purpose of this paper is to answer the converse question: what is the $t$-structure on $D^b_{\text{rh}}(\mathcal{D}_X)$ corresponding to the trivial $t$-structure on $D^b_c(C^0_X)$. In fact, this $t$-structure can be extended to a $t$-structure on the derived category $D^b_c(\mathcal{D}_X)$ of bounded complexes of $\mathcal{D}_X$-modules with quasi-coherent cohomologies. In this paper, we treat an algebraic case, i.e. when $X$ is a smooth algebraic variety. In this case, the corresponding $t$-structure $(\mathcal{D}^{\leq 0}_{\text{qc}}(\mathcal{D}_X), \mathcal{D}^{> 0}_{\text{qc}}(\mathcal{D}_X))$ is given as follows.

$$\mathcal{D}^{\leq 0}_{\text{qc}}(\mathcal{D}_X) = \{ \mathcal{M} \in D^b_{\text{qc}}(\mathcal{D}_X) ; \text{codim Supp}(H^n(\mathcal{M})) \geq n \text{ for every } n \geq 0 \} ,$$

$$\mathcal{D}^{> 0}_{\text{qc}}(\mathcal{D}_X) = \{ \mathcal{M} \in D^b_{\text{qc}}(\mathcal{D}_X) ; H^k(\mathcal{M}) = 0 \text{ for any closed subset } Z \text{ and } k < \text{codim } Z \} .$$

Similar results hold for a complex analytic manifold $X$.

More general results are given in this paper. For a left Noetherian $\mathcal{O}_X$-ring $\mathcal{A}$ quasi-coherent over $\mathcal{O}_X$ and a decreasing sequence of families of supports $\Phi = \{ \Phi^n \}_{n \in \mathbb{Z}}$, the pair

$$\mathcal{D}^{\leq 0}_{\text{qc}}(\mathcal{A}) : = \{ M \in D^b_{\text{qc}}(\mathcal{A}) ; \text{Supp}(H^k(M)) \subset \Phi^k \text{ for any } k \} ,$$

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\[ \Phi D_{qc}^{>0}(\mathcal{A}) := \{ M \in D_{qc}^{b}(\mathcal{A}) \mid \mathcal{H}^{k}_{qc}(M) = 0 \text{ if } k < n \} \]
gives a \( t \)-structure on \( D_{qc}^{b}(\mathcal{A}) \) (Theorem 3.5). This construction is similar to the one of the perverse sheaves. The \( t \)-structure \( (\Phi D_{qc}^{<0}(\mathcal{A}), \Phi D_{qc}^{>0}(\mathcal{A})) \) corresponds to the case where \( \mathcal{A} = \mathcal{D}_X \) and \( \Phi = \mathcal{G} \), where \( \mathcal{G}^k \) is the family of supports consisting of closed subsets with codimension \( \geq k \).

However, this \( t \)-structure does not always induce a \( t \)-structure on \( D_{coh}^{b}(\mathcal{O}_X) \), the derived category of complexes of \( \mathcal{O}_X \)-modules with coherent cohomologies. We give the necessary and sufficient condition for \( \Phi \) to give a \( t \)-structure on \( D_{coh}^{b}(\mathcal{O}_X) \) (Theorem 5.9). 1 This condition resembles the one for perversity.

The paper is organized as follows. In §2, we briefly review the notion of \( t \)-structures.

In §3, we introduce the \( t \)-structure \( (\Phi D_{qc}^{<0}(\mathcal{A}), \Phi D_{qc}^{>0}(\mathcal{A})) \) on the triangulated category \( D_{qc}^{b}(\mathcal{A}) \), and studies its properties.

In §4, we study the \( t \)-structure on the derived category \( D_{coh}^{b}(\mathcal{O}_X) \) which corresponds to the standard \( t \)-structure on \( D_{coh}^{b}(\mathcal{O}_X) \) by the duality functor \( R \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X) \). We also give the relation between this \( t \)-structure and flatness (Proposition 4.6).

In §5, we give the condition for a decreasing sequence of families of supports to give a \( t \)-structure on \( D_{coh}^{b}(\mathcal{O}_X) \) (Theorem 5.9). This is a generalization of [6, Exercise X.2].

In §6, we study the \( t \)-structure on \( D_{qc}^{b}(\mathcal{O}_X) \).

In the last section, we give a proof of the stability of injectivity under filtrant inductive limits (Lemma 3.1).

While writing this paper, the author received the preprint [8] by A. Yekutieli and James J. Zhang, where they proposed similar \( t \)-structures.

2. \( T \)-STRUCTURE

In this section, we recall the notion of \( t \)-structures. For details, see [1, 6]. Let \( D \) be a triangulated category. Let \( D^{<0} \) and \( D^{>0} \) be full subcategories of \( D \). We set \( D^{\leq n} := D^{<0}[-n] \) and \( D^{\geq n} := D^{>0}[-n] \). We also use the notations: \( D^{<n} := D^{\leq n-1} \) and \( D^{>n} := D^{\geq n+1} \). The pair \( (D^{<0}, D^{>0}) \) is called a \( t \)-structure on \( D \) if it satisfies the following conditions:

\begin{align*}
\tag{2.1a} D^{<0} & \subset D^{<0} \text{ and } D^{>0} \subset D^{>0}, \\
\tag{2.1b} Hom_{D}(X, Y) = 0 & \text{ for } X \in D^{<0} \text{ and } Y \in D^{>0}, \\
\tag{2.1c} \text{For any } X \in D, \text{ there exists a distinguished triangle } X' \to X \to X'' & \xrightarrow{+1} \\
\text{with } X' \in D^{<0} \text{ and } X'' \in D^{>0}. \\
\end{align*}

Then one has

\begin{align*}
\tag{2.1d} \text{For a distinguished triangle } X' \to X \to X'' & \xrightarrow{+1}, \text{ if } X' \text{ and } X'' \text{ belong to } D^{<0} \text{ (resp. } D^{>0}), \text{ then so does } X. 
\end{align*}

Note that the \( t \)-structure is a self-dual notion: if \( (D^{\leq0}, D^{\geq0}) \) is a \( t \)-structure on a triangulated category \( D \), then \( (D^{\geq0})^{op}, (D^{\leq0})^{op} \) is a \( t \)-structure on the opposite triangulated category \( D^{op} \).

1After writing this paper, the author was informed the existence of the paper “Perverse coherent sheaves (after Deligne)”, math.AG/0005152 by Roman Bezrukavnikov, in which it is proved in a more general setting that the condition for \( \Phi \) to give a \( t \)-structure on \( D_{coh}^{b}(\mathcal{O}_X) \) is sufficient.
In the paper, we use the following results.

**Lemma 2.1.** Let $D$ be a triangulated category, and assume that $(D^<, D^>)$ satisfies (2.1d). Let $M_1 \to M \to M_2 \overset{+1}{\to}$ be a distinguished triangle. If one of the following conditions (i) and (ii) is satisfied, then there exists a distinguished triangle $M' \to M \to M'' \overset{+1}{\to}$ with $M' \in D^<$ and $M'' \in D^>.$

(i) $M_1 \in D^<$ and there exists a distinguished triangle $M'_2 \to M_2 \to M''_2 \overset{+1}{\to}$ with $M'_2 \in D^<$ and $M''_2 \in D^>.$

(ii) $M_2 \in D^>$ and there exists a distinguished triangle $M'_1 \to M_1 \to M''_1 \overset{+1}{\to}$ with $M'_1 \in D^<$ and $M''_1 \in D^>.$

**Proof.** Assume the condition (i). By the octahedral axiom,

one has distinguished triangles $M_1 \to M' \to M'_2 \overset{+1}{\to}$ and $M' \to M \to M''_2 \overset{+1}{\to}.$ Hence $M' \in D^<$ by (2.1d).

The condition (ii) is the dual form of (i). \qed

The following lemma is almost obvious.

**Lemma 2.2.** Let $F: D \to D'$ be an equivalence of triangulated categories. Let $(D^<, D^>)$ be a $t$-structure on $D,$ and let $(D'^<, D'^>)$ be a pair of full subcategories of $D'$ satisfying (2.1a) and (2.1b). If $F$ sends $D^<$ to $D'^<$ and $D^>$ to $D'^>,$ then $(D'^<, D'^>)$ is a $t$-structure on $D'.$

Let $D'$ be another triangulated category, and let $(D^<, D^>)$ and $(D'^<, D'^>)$ be a $t$-structure on $D$ and $D',$ respectively. A functor $F: D \to D'$ of triangulated categories is called left exact if $F$ sends $D^>$ to $D'^>.$ It is called right exact if it sends $D^<$ to $D'^<.$ It is called exact if it is left exact and right exact.

### 3. $t$-structures on a Noetherian scheme

Let $X$ be a finite-dimensional Noetherian separated scheme. Let $\mathcal{A}$ be an $\mathcal{O}_X$-ring, i.e., a (not necessarily commutative) ring on $X$ with a ring morphism $\mathcal{O}_X \to \mathcal{A}.$ We assume that $\mathcal{A}$ is quasi-coherent as a left $\mathcal{O}_X$-module, and that $\mathcal{A}$ is left Noetherian (e.g. see [5, Definition A.7]). Under the first assumption, the second is equivalent to saying that $\mathcal{A}(U)$ is a left Noetherian ring for any affine open subset $U.$
Let $\text{Mod}(\mathcal{A})$ be the category of (left) $\mathcal{A}$-modules, and $\text{Mod}_{qc}(\mathcal{A})$ (resp. $\text{Mod}_{coh}(\mathcal{A})$) its full subcategory of quasi-coherent (resp. coherent) $\mathcal{A}$-modules. Note that an $\mathcal{A}$-module is quasi-coherent over $\mathcal{A}$ if and only if it is quasi-coherent over $\mathcal{O}_X$.

One has the following lemma whose proof is given in the last section 7.

**Lemma 3.1.** Any filtrant inductive limit of injective objects of $\text{Mod}(\mathcal{A})$ is injective.

Let $D(\mathcal{A})$ be the derived category of $\text{Mod}(\mathcal{A})$, and let $D^b(\mathcal{A})$ be the full subcategory of $D(\mathcal{A})$ consisting of objects with bounded cohomologies. Let $D^b_{qc}(\mathcal{A})$ (resp. $D^b_{coh}(\mathcal{A})$) be the full subcategory of $D^b(\mathcal{A})$ consisting of objects with quasi-coherent (resp. coherent) cohomologies. We define

$$D^{\leq n}(\mathcal{A}) := \{ M \in D^b(\mathcal{A}) ; H^k(M) = 0 \text{ for } k > n \},$$

and similarly $D^{< n}(\mathcal{A})$, $D^{\leq n}(\mathcal{A})$, etc. We call the $t$-structure $(D^{< 0}(\mathcal{A}), D^{\geq 0}(\mathcal{A}))$ the *standard $t$-structure*. Let us denote by $\tau^{\leq n} : D^b(\mathcal{A}) \to D^{\leq n}(\mathcal{A})$ and $\tau^{> n} : D^b(\mathcal{A}) \to D^{> n}(\mathcal{A})$ the truncation functors with respect to the standard $t$-structure.

In this section, we shall give $t$-structures on $D^b(\mathcal{A})$ and $D^b_{qc}(\mathcal{A})$.

In this paper, a *family of supports* means a set $\Phi$ of closed subsets of $X$ satisfying the following conditions:

(i) If $Z \in \Phi$ and $Z'$ is a closed subset of $Z$, then $Z' \in \Phi$.

(ii) For $Z, Z' \in \Phi$, the union $Z \cup Z'$ belongs to $\Phi$.

(iii) $\Phi$ contains the empty set.

Then the set of families of supports has the structure of an ordered set by inclusion. The join of families of supports $\Phi_1$ and $\Phi_2$ is as follows:

$$\Phi_1 \cup \Phi_2 = \left\{ Z ; Z \text{ is a closed subset such that } Z \subset Z_1 \cup Z_2 \right\}$$

for some $Z_1 \in \Phi_1$ and some $Z_2 \in \Phi_2$$

$$= \left\{ Z ; Z \text{ is a closed subset such that } Z = Z_1 \cup Z_2 \right\}$$

for some $Z_1 \in \Phi_1$ and some $Z_2 \in \Phi_2$.

Note that an irreducible closed subset $Z$ is a member of $\Phi_1 \cup \Phi_2$ if and only if one has either $Z \in \Phi_1$ or $Z \in \Phi_2$. Their meet is given by

$$\Phi_1 \cap \Phi_2 = \{ Z ; Z \in \Phi_1 \text{ and } Z \in \Phi_2 \}.$$

We sometimes regard a closed subset $S$ of $X$ as the family of supports consisting of closed subsets of $S$. Hence for a closed subset $S$ and a family of supports $\Phi$, one has $\Phi \cap S = \{ Z \in \Phi ; Z \subset S \}$, and $\Phi \cup S$ is the the family of supports consisting of closed subsets $Z$ such that $Z \setminus S \in \Phi$.

For a sheaf $F$, one sets $\Gamma_\Phi(F) = \lim_{Z \in \Phi} \Gamma_Z(F)$. Then one has

$$\Gamma(U; \Gamma_\Phi(F)) = \left\{ s \in F(U) ; \text{supp}(s) \in \Phi \right\}$$

for any open subset $U$, because $X$ is Noetherian. Then $\Gamma_\Phi$ is a left exact functor from $\text{Mod}(\mathcal{A})$ to itself, and also it sends $\text{Mod}_{qc}(\mathcal{A})$ to itself. It commutes with filtrant inductive limits. It is well-known that $\Gamma_Z$ sends injective objects of $\text{Mod}(\mathcal{A})$ to injective objects. Hence, by Lemma 3.1, the functor $\Gamma_\Phi$ also sends injective objects of $\text{Mod}(\mathcal{A})$ to injective objects. Let
Lemma 3.4. Let $(i)$ follows immediately from $\Gamma_\Phi$ and $(ii)$ $\Gamma_\Phi$ commutes with the forgetful functor $\text{D}^b(\mathcal{A}) \to \text{D}^b(\mathcal{Z}_X)$. For $M \in \text{D}^b(\mathcal{A})$, let $\text{Supp}(M)$ denote the family of supports consisting of closed subsets $Z$ such that $Z \subset \bigcup_{i=1}^n \text{supp}(s_i)$ for open subsets $U_i$, integers $n_i$ and $s_i \in H^{n_i}(M)(U_i)$. Hence, $\text{Supp}(M) \subset \Phi$ is equivalent to saying that $\Gamma_\Phi(M) \sim \sim M$.

One sets $\mathcal{H}_\Phi^n(M) = H^n(\Gamma_\Phi(M))$. One has $\mathcal{H}_\Phi^n(M) \simeq \lim_{Z \in \Phi} \mathcal{H}_Z^n(M)$ for any $M \in \text{D}^b(\mathcal{A})$ and any integer $n$.

For any $M \in \text{D}^b_{qc}(\mathcal{A})$, one has an isomorphism (3.1)

$$\mathcal{H}_\Phi^n(M) \simeq \lim_{i} \mathcal{E}xt^n_{\mathcal{A}}(\mathcal{O}_X/I, M)$$

as $\mathcal{O}_X$-modules. Here the inductive limit is taken over coherent ideals $I$ of $\mathcal{O}_X$ such that $\text{Supp}(\mathcal{O}_X/I) \in \Phi$. Since an $\mathcal{A}$-module is quasi-coherent if and only if it is quasi-coherent over $\mathcal{O}_X$, the functor $\Gamma_{\Phi}$ induces a functor

$$\Gamma_{\Phi} : \text{D}^b_{qc}(\mathcal{A}) \to \text{D}^b(\mathcal{A}).$$

Remark 3.2. Although one neither gives a proof nor uses it in this paper, one has $\mathcal{H}_\Phi^n(M) \simeq \lim_{i} \mathcal{E}xt^n_{\mathcal{A}}(\mathcal{A}/\mathcal{A}I, M)$. Here the inductive limit is taken over $I$ as above. Hence an injective object of $\text{Mod}_{qc}(\mathcal{A})$ is a flabby sheaf (see Remark 7.4).

The following lemma is obvious.

Lemma 3.3. (i) For families of supports $\Phi$ and $\Phi'$, one has $\Gamma_{\Phi} \circ \Gamma_{\Phi'} = \Gamma_{\Phi \cap \Phi'}$.

(ii) For $M$, $K \in \text{D}^b(\mathcal{A})$ with $\text{Supp}(M) \subset \Phi$, one has

$$\text{Hom}_{\text{D}^b(\mathcal{A})}(M, \Gamma_{\Phi}K) \sim \sim \text{Hom}_{\text{D}^b(\mathcal{A})}(M, K)$$

and $\mathcal{E}xt^n_{\mathcal{A}}(M, \Gamma_{\Phi}K) \sim \sim \mathcal{E}xt^n_{\mathcal{A}}(M, K)$ for every $n$.

(iii) If $M \in \text{D}^{\geq n}(\mathcal{A})$, then $\Gamma_{\Phi}(M) \in \text{D}^{\geq n}(\mathcal{A})$ and $H^n(\Gamma_{\Phi}(M)) = \Gamma_{\Phi}(H^n(M))$.

(iv) For a locally closed subset $Z$ of $X$ and a family $\Phi$ of supports, one has $\Gamma_{\Phi} \circ \Gamma_{\Phi} \sim \sim \Gamma_{\Phi} \circ \Gamma_{\Phi} \circ \Gamma_{\Phi}$.

(v) For an open embedding $j : U \hookrightarrow X$, one has $\text{RJ}_* \circ \Gamma_{\Phi|U} \sim \sim \Gamma_{\Phi} \circ \text{RJ}_*$. Here $\Phi|U$ is the family of supports on $U$ given by

$$\Phi|U := \{ Z ; Z \text{ is a closed subset of } U \text{ such that } \overline{Z} \in \Phi \} = \{ Z \cap U ; Z \in \Phi \}.$$

Lemma 3.4. Let $M \in \text{D}^b(\mathcal{A})$ and $n$ an integer, and let $\Phi$ and $\Psi$ be families of supports.

(i) If $\Psi \subset \Phi$ and $\Gamma_{\Psi}(M) \in \text{D}^{\geq n}(\mathcal{A})$, then $\Gamma_{\Psi}(M) \in \text{D}^{\geq n}(\mathcal{A})$.

(ii) $\Gamma_{\Phi}(M) \in \text{D}^{\geq n}(\mathcal{A})$ if and only if $\Gamma_{\Phi}(M) \in \text{D}^{\geq n}(\mathcal{A})$ for every $Z \in \Phi$.

(iii) If $\Gamma_{\Phi}(M)$, $\Gamma_{\Psi}(M) \in \text{D}^{\geq n}(\mathcal{A})$, then $\Gamma_{\Phi \cup \Psi}(M) \in \text{D}^{\geq n}(\mathcal{A})$.

(iv) If $\mathcal{H}_\Phi^k(M)_x = 0$ for any $k < n$ and any $x$ such that $\{ x \} \in \Phi$, then $\Gamma_{\Phi}(M) \in \text{D}^{\geq n}(\mathcal{A})$.

Proof. (i) follows immediately from $\Gamma_{\Psi}(M) \simeq \Gamma_{\Psi} \Gamma_{\Phi}(M)$, and (ii) follows from (i) and $\mathcal{H}_\Phi^k(M) \simeq \lim_{Z \in \Phi} \mathcal{H}_Z^k(M)$. 
Let us show (iii). By (i) and (ii), it is enough to show that $\Gamma\Gamma_{Z_1 \cup Z_2}(M) \in D^{n\geq}(\mathcal{A})$ for closed subsets $Z_1$ and $Z_2$ such that $\Gamma\Gamma_{Z_1}(M), \Gamma\Gamma_{Z_2}(M) \in D^{n\geq}(\mathcal{A})$. Since one has

$$\tau^{<n}\Gamma\Gamma_{Z_1 \cup Z_2}(M)|_{X \setminus Z_1} \simeq \tau^{<n}\Gamma\Gamma_{Z_2}(M)|_{X \setminus Z_1} = 0,$$

the support of $\tau^{<n}\Gamma\Gamma_{Z_1 \cup Z_2}(M)$ is contained in $Z_1$. In the distinguished triangle

$$\Gamma\Gamma_{Z_1}(\tau^{<n}\Gamma\Gamma_{Z_1 \cup Z_2}(M)) \to \Gamma\Gamma_{Z_1}\Gamma\Gamma_{Z_1 \cup Z_2}(M) \to \Gamma\Gamma_{Z_1}(\tau^{<n}\Gamma\Gamma_{Z_1 \cup Z_2}(M)),$$

the second term is isomorphic to $\Gamma\Gamma_{Z_1}(M)$ which belongs to $D^{n\geq}(\mathcal{A})$, and the last term also belongs to $D^{n\geq}(\mathcal{A})$. Hence the first term, isomorphic to $\tau^{<n}\Gamma\Gamma_{Z_1 \cup Z_2}(M)$, belongs to $D^{n\geq}(\mathcal{A})$, and therefore it vanishes.

(iv) By the induction on $n$, one may assume that $\Gamma\Gamma_{\Phi}(M) \in D^{n\geq-1}(\mathcal{A})$. For an open set $U$ and $s \in \Gamma(U; \mathcal{H}_\Phi^{n-1}(M))$, set $S := \text{supp}(s)$. Let us show that $S = \emptyset$. Otherwise, let us take the generic point $x$ of an irreducible component of $S$. Then one has $\{x\} \in \Phi$. Since $\Gamma_S(\mathcal{H}_\Phi^{n-1}(M)) = \mathcal{H}_\Phi^{n-1}(M)$ and $\mathcal{H}_\Phi^{n-1}(M)_x = \mathcal{H}_\Phi^{n-1}(M)_x$, the germ of $s$ at $x$ must vanish, which is a contradiction.

A support datum is a decreasing sequence $\Phi = \{\Phi^n\}_{n \in \mathbb{Z}}$ of families of supports satisfying the following conditions:

$$\text{(3.2a)} \quad \text{for } n \ll 0, \Phi^n \text{ is the set of all closed subsets of } X,$$

$$\text{(3.2b)} \quad \text{for } n \gg 0, \Phi^n \text{ is } \{\emptyset\}.$$

For a support datum $\Phi$ and an integer $n$, let $\sigma^{\leq n}\Phi$ denote the support datum given by

$$(\sigma^{\leq n}\Phi)^k = \begin{cases} \Phi^k & \text{for } k \leq n, \\ \{\emptyset\} & \text{for } k > n. \end{cases}$$

Let $\mathbb{T}$ denote the support datum given by

$$\mathbb{T}^n = \begin{cases} \{\text{all closed subsets of } X\} & \text{for } n \leq 0, \\ \{\emptyset\} & \text{for } n > 0. \end{cases}$$

For a support datum $\Phi = \{\Phi^n\}_{n \in \mathbb{Z}}$, we set

$$\Phi D^{\leq n}(\mathcal{A}) := \{M \in D^b(\mathcal{A}); \text{Supp}(H^k(M)) \subset \Phi^{k-n} \text{ for any } k\},$$

$$\Phi D^{\geq n}(\mathcal{A}) := \{M \in D^b_{qc}(\mathcal{A}); \Gamma\Gamma_{\Phi^k}(M) \in D^{\geq k+n}(\mathcal{A}) \text{ for any } k\},$$

and $\Phi D^{\leq n}_{qc}(\mathcal{A}) := \Phi D^{\leq n}(\mathcal{A}) \cap D^b_{qc}(\mathcal{A}), \Phi D^{\geq n}_{qc}(\mathcal{A}) := \Phi D^{\geq n}(\mathcal{A}) \cap D^b_{qc}(\mathcal{A})$.

**Theorem 3.5.** $\Phi D^b(\mathcal{A}) := (\Phi D^{\leq 0}(\mathcal{A}), \Phi D^{\geq 0}(\mathcal{A}))$ and $\Phi D^b_{qc}(\mathcal{A}) := (\Phi D^{\leq 0}_{qc}(\mathcal{A}), \Phi D^{\geq 0}_{qc}(\mathcal{A}))$ are a t-structure on $D^b(\mathcal{A})$ and $D^b_{qc}(\mathcal{A})$, respectively.

Note that $\mathbb{T} D^b(\mathcal{A})$ is the standard t-structure.

Since the proof for $\Phi D^b_{qc}(\mathcal{A})$ is similar, we only give a proof for $\Phi D^b(\mathcal{A})$. We divide the proof of the theorem into several steps. It is evident that $\Phi D^b(\mathcal{A})$ satisfies the conditions (2.1a) and (2.1d). Let us show that it satisfies (2.1b).

**Lemma 3.6.** For $M \in \Phi D^{< 0}(\mathcal{A})$ and $K \in \Phi D^{\geq 0}(\mathcal{A})$, one has

$$\text{Hom}_{D^b(\mathcal{A})}(M, K) = 0 \text{ and } \mathcal{E}xt^n_{D^b(\mathcal{A})}(M, K) = 0 \text{ for } n \leq 0.$$
Proof. We shall show

\[(3.4) \quad \text{Hom}_{\mathcal{D}^b(\mathcal{A})}(M, K) = 0 \quad \text{for} \ M \in \Phi \mathcal{D}^{<0}(\mathcal{A}) \cap \mathcal{D}^{\leq n}(\mathcal{A}) \quad \text{and} \quad K \in \Phi \mathcal{D}^{\geq 0}(\mathcal{A})\]

by the induction on \(n\). Since \(\Phi \mathcal{D}^{\geq 0}(\mathcal{A}) \subset \mathcal{D}^{\geq n}(\mathcal{A})\) for \(n \ll 0\), (3.4) is satisfied for \(n \ll 0\). Assuming (3.4) \(n-1\), let us show (3.4) \(n\). One has \(\text{Supp}(H^n(M)[-n]) \subset \Phi^{n+1}\), \(H^n(M)[-n] \in \mathcal{D}^{\leq n}(\mathcal{A})\) and \(R\Gamma_{n+1} K \in \mathcal{D}^{\geq n}(\mathcal{A})\). Hence one has

\[
\text{Hom}_{\mathcal{D}^b(\mathcal{A})}(H^n(M)[-n], K) \simeq \text{Hom}_{\mathcal{D}^b(\mathcal{A})}(H^n(M)[-n], R\Gamma_{n+1}(K)) = 0.
\]

The distinguished triangle \(\tau^{\leq n} M \to M \to H^n(M)[-n] \overset{+1}{\longrightarrow}\) induces an exact sequence

\[
\text{Hom}_{\mathcal{D}^b(\mathcal{A})}(H^n(M)[-n], K) \to \text{Hom}_{\mathcal{D}^b(\mathcal{A})}(M, K) \to \text{Hom}_{\mathcal{D}^b(\mathcal{A})}(\tau^{< n} M, K).
\]

Since \(\text{Hom}_{\mathcal{D}^b(\mathcal{A})}(\tau^{< n} M, K) = 0\) by (3.4) \(n-1\), we obtain \(\text{Hom}_{\mathcal{D}^b(\mathcal{A})}(M, K) = 0\).

The proof of the second statement is similar. \(\Box\)

Hence, it remains to prove the following statement for any \(M \in \mathcal{D}^b(\mathcal{A})\):

\[(3.4) \quad \text{There exists a distinguished triangle} M' \to M \to M'' \overset{+1}{\longrightarrow} \text{with} \ M' \in \Phi \mathcal{D}^{<0}(\mathcal{A}) \text{and} \ M'' \in \Phi \mathcal{D}^{\geq 0}(\mathcal{A}).\]

Proof of (3.4). For \(n \in \mathbb{Z}\), let us consider the following statement:

\[(3.4) \quad \text{The property} \ (3.4) \ \text{holds if} \ M \in \mathcal{D}^b(\mathcal{A}) \ \text{satisfies} \ R\Gamma_{\Phi^i}(M) \in \mathcal{D}^{\geq i}(\mathcal{A}) \ \text{for any} \ i \leq n, \ i.e. \ if \ M \in \sigma^{\leq n} \Phi \mathcal{D}^{\geq 0}(\mathcal{A}).\]

Since every \(M \in \mathcal{D}^b(\mathcal{A})\) satisfies \(R\Gamma_{\Phi^i}(M) \in \mathcal{D}^{\geq i}(\mathcal{A})\) for \(i \ll 0\), it is enough to show (3.4) \(n\) for every \(n\). We shall show (3.4) \(n\) by the descending induction on \(n\). Note that (3.4) holds for sufficiently large \(n\) such that \(\Phi^{n+1} = \{0\}\), because such an \(M\) satisfies \(M \in \Phi \mathcal{D}^{\geq 0}(\mathcal{A})\).

We shall prove (3.4) \(n\) by assuming (3.4) \(n+1\). Suppose that \(M \in \mathcal{D}^b(\mathcal{A})\) satisfies \(R\Gamma_{\Phi^i}(M) \in \mathcal{D}^{\geq i}(\mathcal{A})\) for any \(i \leq n\). Let us consider a distinguished triangle

\[
\tau^{\leq n} R\Gamma_{\Phi^{n+1}}(M) \to M \to M'' \overset{+1}{\longrightarrow}.
\]

Since \(R\Gamma_{\Phi^{n+1}}(M) = R\Gamma_{\Phi^{n+1}} R\Gamma_{\Phi^n}(M)\), one has \(R\Gamma_{\Phi^{n+1}}(M) \in \mathcal{D}^{\geq n}(\mathcal{A})\) and

\[
\mathcal{H}^n_{\Phi^{n+1}}(M) \simeq \Gamma_{\Phi^{n+1}}(\mathcal{H}^n_{\Phi^n}(M)).
\]

Moreover one has

\[
\text{Supp}(H^i(\tau^{\leq n} R\Gamma_{\Phi^{n+1}}(M))) = \begin{cases} \emptyset & \text{for} \ i \neq n, \\ \Phi^{n+1} & \text{for} \ i = n, \end{cases}
\]

One concludes therefore that \(\tau^{\leq n} R\Gamma_{\Phi^{n+1}}(M) \in \Phi \mathcal{D}^{<0}(\mathcal{A})\). If one shows

\[(3.6) \quad R\Gamma_{\Phi^i}(M'') \in \mathcal{D}^{\geq i}(\mathcal{A}) \quad \text{for} \ i \leq n + 1,
\]

then we can apply (3.4) \(n+1\) to \(M''\) and (3.4) is satisfied for \(M''\). Hence Lemma 2.1 implies that (3.4) is satisfied for \(M\).

It remains to prove (3.6), i.e. \(H^k(R\Gamma_{\Phi^i}(M'')) = 0\) for \(k < i \leq n + 1\). For \(i \leq n + 1\) one has \(R\Gamma_{\Phi^i}(\tau^{\leq n} R\Gamma_{\Phi^{n+1}}(M)) = \tau^{\leq n} R\Gamma_{\Phi^{n+1}}(M)\). Hence we have a long exact sequence

\[
H^k(\tau^{\leq n} R\Gamma_{\Phi^{n+1}}(M)) \overset{\xi}{\longrightarrow} H^k(R\Gamma_{\Phi^i}(M)) \overset{\eta}{\longrightarrow} H^k(R\Gamma_{\Phi^i}(M'')) \overset{\nu}{\longrightarrow} H^{k+1}(R\Gamma_{\Phi^i}(M)).
\]

Hence it is enough to show that \(\xi\) is surjective and \(\eta\) is injective for \(k < i \leq n + 1\).

For \(k = n + 1\), \(\xi\) is an isomorphism, and for \(k < i \leq n\) one has \(H^k(R\Gamma_{\Phi^i}(M)) = 0\).
The morphism $\eta$ is injective for $i = n + 1$, and also for $k = n - 1 < i = n$ by (3.5). In the remaining case $k < n - 1$, one has $H^{k+1}(\tau_{\leq n} R\Gamma_{\Phi_{n+1}}(M)) = 0$. \hfill \Box

This $t$-structure behaves as follows under the local cohomology functors and the direct image functors.

**Lemma 3.7.**

(i) For a locally closed subset $Z$ of $X$, the functor $R\Gamma_Z : D^b_{qc}(\mathcal{A}) \to D^b_{qc}(\mathcal{A})$ is left exact, i.e. it sends $\Phi^* D_{qc}^{\leq 0}(\mathcal{A})$ to itself.

(ii) Let $j : U \hookrightarrow X$ be an open set of $X$. Then $j^{-1} : D^b_{qc}(\mathcal{A}) \to D^b_{qc}(\mathcal{A}|_U)$ is exact, and $Rj_* : D^b_{qc}(\mathcal{A}|_U) \to D^b_{qc}(\mathcal{A})$ is left exact.

**Proof.** (i) follows immediately from $R\Gamma_{\Phi^n} R\Gamma_Z(M) = R\Gamma_Z R\Gamma_{\Phi^n}(M)$, which is a consequence of Lemma 3.3 (iii), and (ii) follows from $R\Gamma_{\Phi^n} Rj_*(M) = Rj_* R\Gamma_{\Phi^n}(M)$. \hfill \Box

**Proposition 3.8.** For an open subset $U$ of $X$, let $\mathcal{C}_{qc}(U)$ be the heart $\Phi^* D_{qc}^{\leq 0}(\mathcal{A}|_U) \cap \Phi^* D_{qc}^{\geq 0}(\mathcal{A}|_U)$ of the $t$-structure $\Phi^* D^b_{qc}(\mathcal{A}|_U)$. Then, $U \mapsto \mathcal{C}_{qc}(U)$ is a stack on $X$.

**Proof.** Since $X$ is Noetherian, it is enough to show that, for open sets $U_1, U_2$ with $X = U_1 \cup U_2$, $M_{12} \in \mathcal{C}_{qc}(U_1 \cap U_2)$ and $M_i \in \mathcal{C}_{qc}(U_i)$ with an isomorphism $\varphi_i : M_{1|U_1 \cap U_2} \sim M_{12|U_1 \cap U_2}$ ($i = 1, 2$), there exist $M \in \mathcal{C}_{qc}(X)$ and isomorphisms $\psi_i : M|_{U_i} \sim M_i$ ($i = 1, 2$) such that the following diagram commutes.

\[
\begin{array}{ccc}
M|_{U_1 \cap U_2} & \xrightarrow{\psi_1} & M_1|_{U_1 \cap U_2} \\
\downarrow{\psi_2} & & \downarrow{\varphi_1} \\
M_2|_{U_1 \cap U_2} & \xrightarrow{\varphi_2} & M_{12}
\end{array}
\]

Let $j_i : U_i \hookrightarrow X$ ($i = 1, 2$) and $j_{12} : U_1 \cap U_2 \hookrightarrow X$ be the open embeddings, and let

\[
M \xrightarrow{\psi_1 \oplus \psi_2} Rj_1* M_1 \oplus Rj_2* M_2 \xrightarrow{-\varphi_1 \oplus \varphi_2} Rj_{12}* M_{12} \xrightarrow{+1}
\]

be a distinguished triangle. Since $Rj_{12*} M_{2|U_1} \to Rj_{12*} M_{12|U_1}$ is an isomorphism, $M|_{U_1} \to Rj_{1*} M_1|_{U_1} \sim M_1$ is an isomorphism. The commutativity of the diagram above is obvious by the construction. \hfill \Box

Let us remark that $(\Phi^* D_{qc}^{\leq 0}(\mathcal{A}) \cap D^b_{qc}(\mathcal{A}), \Phi^* D_{qc}^{\geq 0}(\mathcal{A}) \cap D^b_{qc}(\mathcal{A}))$ is not necessarily a $t$-structure on $D^b_{coh}(\mathcal{A})$ (see Example 5.1 and Example 6.1). In the coherent case, the proof of Theorem 3.5 fails, because $R\Gamma_{\Phi}$ does not respect coherency.

**4. THE DUAL STANDARD T-STRUCTURE ON $D^b_{coh}(\mathcal{O}_X)$**

In the sequel, we assume that $X$ is a finite-dimensional regular Noetherian separated scheme. Let us denote by $\mathcal{G}$ the support datum given by

\[
(4.1) \quad \mathcal{G}^d := \{ \mathcal{Z} ; \mathcal{Z} \text{ is a closed subset of } X \text{ such that codimZ} \geq d \}
= \{ \mathcal{Z} ; \mathcal{Z} \text{ is a closed subset of } X \text{ such that dim } \mathcal{O}_{X,x} \geq d \text{ for any } x \in Z \}.
\]

One has
\[
\begin{align*}
\mathcal{G}^d_{qc}(\mathcal{A}) &:= \{ M \in D^b_{qc}(\mathcal{A}) ; \text{Supp}(H^n(M)) \subset \mathcal{G}^n \text{ for every } n \geq 0 \}, \\
\mathcal{G}^d_{qc}(\mathcal{A}) &:= \{ M \in D^b_{qc}(\mathcal{A}) ; \mathcal{H}^n_Z(M) = 0 \text{ for any closed subset } Z \text{ and } n < \text{codimZ} \}.
\end{align*}
\]
Proposition 4.1. Let $Z$ be a locally closed subset of $X$. Then $\Gamma_Z: \mathcal{D}^b_{\text{qc}}(\mathcal{A}) \to \mathcal{D}^b_{\text{qc}}(\mathcal{A})$ is an exact functor with respect to the $t$-structure $\mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A})$.

Proof. It is already proved that $\Gamma_Z$ is left exact. Let us prove that it is right exact, i.e. it sends $\mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A})$ into itself.

We shall first prove it when $Z$ is closed. We may assume that $X$ is affine. Writing $Z = \cap_{i=1}^n f_i^{-1}(0)$ with $f_i \in \Gamma(X; \mathcal{O}_X)$, and using $\Gamma_Z = \Gamma \circ \cap_{i=1}^n f_i^{-1}(0)$, the induction on $n$ reduces to the case $Z = f^{-1}(0)$ for some $f \in \Gamma(X; \mathcal{O}_X)$ by descent. It is enough to prove that $\text{Supp}(\mathcal{H}^k_Z(F[-d])) \subset \mathcal{S}$ for any $k$ and $F \in \mathcal{M}od_{\text{qc}}(\mathcal{A})$ satisfying $\text{Supp}(F) \subset \mathcal{S}$. Since $\mathcal{H}^k_Z(F) = 0$ for $k \neq 0, 1$. Since $\mathcal{H}^0_Z(F) \subset \mathcal{F}$, one has $\text{Supp}(\mathcal{H}^0_Z(F)) \subset \text{Supp}(F) \subset \mathcal{S}$. We can divide $S = S_1 \cup S_2$ with closed subsets $S_1$ and $S_2$ such that $S_1 \subset f^{-1}(0)$ and $\text{codim}(S_2 \cap f^{-1}(0)) > d$. Then $\text{Supp}(\mathcal{H}^0_Z(F)) \subset S_2 \subset \mathcal{S}^{d+1}$.

For a locally closed $Z$, let us show $\Gamma_Z(M) \in \mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A})$ for any $M \in \mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A})$. Writing $Z = Z_1 \setminus Z_2$ with closed $Z_2 \subset Z_1 \subset X$, by the distinguished triangle

$$\Gamma_Z(Z_1) \to \Gamma_Z(Z_2) \to \Gamma_Z(M) \xrightarrow{+1},$$

the desired result $\Gamma_Z(M) \in \mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A})$ follows from $\Gamma_Z(Z_1)[1], \Gamma_Z(Z_2) \in \mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A})$.

Proposition 4.2. Let $j: U \hookrightarrow X$ be an open embedding. Then $Rj_*: \mathcal{D}^b_{\text{qc}}(\mathcal{A}|_U) \to \mathcal{D}^b_{\text{qc}}(\mathcal{A})$ is exact with respect to the $t$-structures $\mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A})$ and $\mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A}|_U)$.

Proof. Since the left exactitude has been proved, let us show that $Rj_*$ sends $\mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A}|_U)$ to $\mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A})$. Let us denote by $\mathcal{E}^\tau \mathcal{D}^b_{\text{qc}}(\mathcal{A})$ the truncation functor with respect to the $t$-structure $\mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A})$. For $N \in \mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A}|_U)$, one has $j^{-1}(\mathcal{E}^\tau \mathcal{D}^b_{\text{qc}}(\mathcal{A}|_U) \simeq N$, and hence $Rj_* N \simeq \Gamma_U(\mathcal{E}^\tau \mathcal{D}^b_{\text{qc}}(\mathcal{A}|_U) \simeq N)$ belongs to $\mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{A})$ by Proposition 4.1.

In the rest of this section, we treat the case where $\mathcal{A} = \mathcal{O}_X$. Set

$$\mathcal{E}^0\mathcal{D}_{\text{coh}}(\mathcal{O}_X) := \mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{O}_X) \cap \mathcal{D}^b_{\text{coh}}(\mathcal{O}_X) \text{ and } \mathcal{E}^0\mathcal{D}_{\text{coh}}(\mathcal{O}_X) := \mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{O}_X) \cap \mathcal{D}^b_{\text{coh}}(\mathcal{O}_X),$$

and $\mathcal{E}^0\mathcal{D}_{\text{coh}}(\mathcal{O}_X) := (\mathcal{E}^0\mathcal{D}^b_{\text{qc}}(\mathcal{O}_X), \mathcal{E}^0\mathcal{D}^b_{\text{coh}}(\mathcal{O}_X))$.

Proposition 4.3. (i) The pair $\mathcal{E}^0\mathcal{D}^b_{\text{coh}}(\mathcal{O}_X)$ is a $t$-structure on $\mathcal{D}^b_{\text{coh}}(\mathcal{O}_X)$.

(ii) The equivalence of triangulated categories

$$R\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X): \mathcal{D}^b_{\text{coh}}(\mathcal{O}_X) \to \mathcal{D}^b_{\text{coh}}(\mathcal{O}_X)^{\text{op}}$$

sends $(\mathcal{E}^0\mathcal{D}_{\text{coh}}(\mathcal{O}_X), \mathcal{E}^0\mathcal{D}_{\text{coh}}(\mathcal{O}_X))$ to $(\mathcal{E}^0\mathcal{D}^b_{\text{coh}}(\mathcal{O}_X)^{\text{op}}, \mathcal{E}^0\mathcal{D}^b_{\text{coh}}(\mathcal{O}_X)^{\text{op}})$.

Remark 4.4. In [6, Exercise X.2] and Theorem 5.9, more general results are given.

Proof. By Lemma 2.2, it is enough to show that the functor $R\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$ sends $\mathcal{E}^0\mathcal{D}_{\text{coh}}(\mathcal{O}_X)$ and $\mathcal{E}^0\mathcal{D}_{\text{coh}}(\mathcal{O}_X)$ to $\mathcal{E}^0\mathcal{D}^b_{\text{coh}}(\mathcal{O}_X)$ and $\mathcal{E}^0\mathcal{D}^b_{\text{coh}}(\mathcal{O}_X)$, respectively.

First assume that $M \in \mathcal{E}^0\mathcal{D}_{\text{coh}}(\mathcal{O}_X)$. Then one has

$$\Gamma_{\mathcal{O}_X} R\mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X) \simeq \mathcal{H}om_{\mathcal{O}_X}(M, R\mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X)).$$

Since $\Gamma_{\mathcal{O}_X}(\mathcal{O}_X) \in \mathcal{D}^{d}_{\text{eq}}(\mathcal{O}_X)$, one has $\Gamma_{\mathcal{O}_X} R\mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X) \in \mathcal{D}^{d}_{\text{eq}}(\mathcal{O}_X)$.

Next assume $M \in \mathcal{E}^0\mathcal{D}_{\text{coh}}(\mathcal{O}_X)$, and set $S := \text{Supp}(H^d(R\mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X)))$. For $x \in S$, one has $H^d(R\mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X)) \simeq \text{Ext}^d_{\mathcal{O}_{X,x}}(M_x, \mathcal{O}_{X,x}) \neq 0$. Since the homological
dimension of $\mathcal{O}_{X,x}$ is equal to $\dim \mathcal{O}_{X,x}$, one has $d \leq \dim \mathcal{O}_{X,x}$. Hence one concludes that $S \in \mathcal{S}^d$.

The category $\mathcal{S}D_{\text{qc}}^{\geq 0}(\mathcal{O})$ is described by a more familiar notion: flatness.

**Definition 4.5.** An object of $D^b(\mathcal{O})$ is called with flat dimension $\leq 0$ if it is isomorphic to a bounded complex $M^\bullet$ of flat $\mathcal{O}_X$-modules such that $M^n = 0$ for $n < 0$.

**Proposition 4.6.** For $M \in D^b_{\text{qc}}(\mathcal{O})$, the following conditions are equivalent.

1. $M \in \mathcal{S}D_{\text{qc}}^{\geq 0}(\mathcal{O})$.
2. $M$ is of flat dimension $\leq 0$.
3. For any coherent $\mathcal{O}_X$-module $F$, one has $F \otimes_{\mathcal{O}_X} M \in D^b_{\text{qc}}(\mathcal{O})$.
4. For any $N \in D^b_{\text{qc}}(\mathcal{O})$, one has $N \otimes_{\mathcal{O}_X} M \in D^b_{\text{qc}}(\mathcal{O})$.

**Proof.** The equivalence (ii)---(iv) is more or less well-known.

(i) $\Rightarrow$ (ii) For every $n \geq 0$, one has $R\Gamma_{\mathcal{E}^n}(\mathcal{O}_X) \in D_{\text{qc}}^{\geq n}(\mathcal{O}_X)$. Hence (iv) implies that $R\Gamma_{\mathcal{E}^n}(M) \simeq R\Gamma_{\mathcal{E}^n}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} M$ belongs to $D_{\text{qc}}^{\geq n}(\mathcal{O}_X)$.

(ii) $\Rightarrow$ (iii) For a coherent $\mathcal{O}_X$-module $F$, set $N = R\mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X) \in \mathcal{S}D_{\text{qc}}^{\leq 0}(\mathcal{O}_X)$. Then one has $F \otimes_{\mathcal{O}_X} M \simeq R\mathcal{H}om_{\mathcal{O}_X}(N, M) \in D_{\text{qc}}^{\geq 0}(\mathcal{O}_X)$.\hfill \qed

**Remark 4.7.** Let $\mathcal{C}_{\text{qc}} := \mathcal{S}D_{\text{qc}}^{\leq 0}(\mathcal{O}_X) \cap \mathcal{S}D_{\text{qc}}^{\geq 0}(\mathcal{O}_X)$ be the heart of the $t$-structure $\mathcal{S}D_{\text{qc}}^{b}(\mathcal{O}_X)$, and let $\mathcal{C}_{\text{coh}} := \mathcal{C}_{\text{qc}} \cap D_{\text{coh}}^{b}(\mathcal{O}_X)$ be the heart of the $t$-structure $\mathcal{S}D_{\text{coh}}^{b}(\mathcal{O}_X)$. The abelian category $\mathcal{C}_{\text{coh}}$ is equivalent to $\text{Mod}_{\text{coh}}(\mathcal{O}_X)^{\text{op}}$. It is well-known that the category $\text{Mod}_{\text{qc}}(\mathcal{O}_X)$ is equivalent to the category $\text{Ind}(\text{Mod}_{\text{coh}}(\mathcal{O}_X))$ of ind-objects of the category of coherent modules. I conjecture that $\mathcal{C}_{\text{qc}}$ is equivalent to $\text{Ind}(\mathcal{C}_{\text{coh}}) \simeq \left(\text{Pro}(\text{Mod}_{\text{coh}}(\mathcal{O}_X))\right)^{\text{op}}$. Here Pro means the category of pro-objects.

## 5. $t$-Structures on $D_{\text{coh}}^{b}(\mathcal{O})$

For a support datum $\Phi$, one sets

$$\Phi D_{\text{coh}}^{\leq 0}(\mathcal{O}_X) = \Phi D_{\text{qc}}^{\leq 0}(\mathcal{O}_X) \cap D_{\text{coh}}^{b}(\mathcal{O}_X) \quad \text{and} \quad \Phi D_{\text{coh}}^{\geq 0}(\mathcal{O}_X) = \Phi D_{\text{qc}}^{\geq 0}(\mathcal{O}_X) \cap D_{\text{coh}}^{b}(\mathcal{O}_X),$$

and $\Phi D_{\text{coh}}^{b}(\mathcal{O}_X) := (\Phi D_{\text{coh}}^{\leq 0}(\mathcal{O}_X), \Phi D_{\text{coh}}^{\geq 0}(\mathcal{O}_X))$. As seen in the example below, $\Phi D_{\text{coh}}^{b}(\mathcal{O}_X)$ is not necessarily a $t$-structure on $D_{\text{coh}}^{b}(\mathcal{O}_X)$. In this section, we give a criterian for $\Phi D_{\text{coh}}^{b}(\mathcal{O}_X)$ to be a $t$-structure on $D_{\text{coh}}^{b}(\mathcal{O}_X)$.

For an integer $n$ and $M \in D_{\text{qc}}^{b}(\mathcal{O})$, we denote by $\Phi H^n(M) \in \Phi D_{\text{qc}}^{\leq 0}(\mathcal{O}) \cap \Phi D_{\text{qc}}^{\geq 0}(\mathcal{O})$ the $n$-th cohomology with respect to the $t$-structure $\Phi D_{\text{qc}}^{b}(\mathcal{O})$.

**Example 5.1.** Let $k$ be a field, and $A = k[x]$ with an indeterminate $x$. Let $X = \text{Spec}(A)$ be the line. Set

$$\Phi^i = \begin{cases} \mathbb{S}^0 & \text{for } i \leq 0, \\ \mathbb{S}^1 & \text{for } i = 1, 2, \\ \{\emptyset\} & \text{for } i \geq 3. \end{cases}$$
Then, the corresponding t-structure on $D^b_{qc}(\mathcal{O}_X)$ is given by:

(5.1a) $\Phi D^\leq_{qc}(\mathcal{O}_X) = \left\{ M \in D^\leq_{qc}(\mathcal{O}_X) : \Gamma(X; H^1(M)) \text{ and } \Gamma(X; H^2(M)) \right\}$

(5.1b) $\Phi D^\geq_{qc}(\mathcal{O}_X) = \left\{ M \in D^\geq_{qc}(\mathcal{O}_X) : \text{Hom}(M) \in D^\geq_{qc}(\mathcal{O}_X) \right\}$.

Let $\mathcal{C} := \Phi D^\leq_{qc}(\mathcal{O}_X) \cap \Phi D^\geq_{qc}(\mathcal{O}_X)$ be the heart of this t-structure. Since any object of $D^b(\mathcal{O}_X)$ has a form $\oplus M_n[-n]$ with $M_n \in \text{Mod}_{qc}(\mathcal{O}_X)$, one has

$$\mathcal{C} = \{ L \oplus M[-2] ; L \text{ is a vector space over } k(x) \text{ and } M \text{ is a torsion } k[x]-module \}.$$  

Here and in the rest of this example, we confuse $A$-modules and quasi-coherent $\mathcal{O}_X$-modules. Since $\text{Hom}_{qc}(L, M[-2]) = \text{Hom}_{qc}(M[-2], L) = 0$ for such an $L$ and $M$, the abelian category $\mathcal{C}$ is the direct sum of $\text{Mod}(k(x))$ and the abelian category $\text{Mod}_{tor}(k[x])$ of torsion $k[x]$-modules.

Let $K \in \text{Mod}(\mathcal{O}_X)$ be the sheaf of rational functions. Then $K$ and $(K/\mathcal{O}_X)[-2]$ belong to $\mathcal{C}$. Hence the distinguished triangle

$$(K/\mathcal{O}_X)[-2][1] \to \mathcal{O}_X \to K \xrightarrow{\eta}$$

implies

$$\Phi H^n(\mathcal{O}_X) = \begin{cases} (K/\mathcal{O}_X)[-2] & \text{for } n = -1, \\ K & \text{for } n = 0, \\ 0 & \text{for } n \neq -1, 0. \end{cases}$$

Hence, $(\Phi D^\leq_{qc}(\mathcal{O}_X) \cap D^b_{coh}(\mathcal{O}_X), \Phi D^\geq_{qc}(\mathcal{O}_X) \cap D^b_{coh}(\mathcal{O}_X))$ is not a t-structure on $D^b_{coh}(\mathcal{O}_X)$.

The categories $D^b(\mathcal{C})$ and $D^b_{qc}(\mathcal{O}_X)$ are not equivalent as categories. Indeed, the object $K \in \mathcal{C} \subset D^b(\mathcal{C})$ is a non-zero object of $D^b(\mathcal{C})$ such that any non-zero morphism $W \to K$ in $D^b(\mathcal{C})$ has a section. However there is no object of $D^b_{qc}(\mathcal{O}_X)$ with such properties. If $M \in D^b_{qc}(\mathcal{O}_X)$ has such properties, then, since $\text{Hom}_{D^b_{qc}(\mathcal{O}_X)}(\mathcal{O}_X[n], M) \neq 0$ for some $n$, $M$ must be a direct summand of $\mathcal{O}_X[n]$. Since $\text{End}_{D^b_{qc}(\mathcal{O}_X)}(\mathcal{O}_X[n]) \simeq k[x]$ does not have a non trivial idempotent, $M$ must be isomorphic to $\mathcal{O}_X[n]$. However, $x : \mathcal{O}_X[n] \to \mathcal{O}_X[n]$ is a non-zero morphism but not an epimorphism.

In [7], Y. T. Siu and G. Trautmann studied the vanishing and the coherency of local cohomologies. Although they discussed in the analytic framework, their main results, in our context, may be stated as follows. Let $M \in \text{Mod}_{coh}(\mathcal{O}_X)$, $Z$ a closed subset of $X$ and $n$ an integer. Then one has

(i) $\mathcal{H}^k_Z(M) = 0$ for any $k < n$ 

(ii) $\mathcal{H}^k_Z(M)$ is coherent for any $k < n$ 

We shall generalize these statements to the derived category case. We keep the notation: $X$ is a finite-dimensional regular Noetherian separated scheme. For $M \in D^b_{coh}(\mathcal{O}_X)$, let us denote by $M^*$ its dual: $M^* := \text{RHom}_{\mathcal{O}_X}(M, \mathcal{O}_X)$.

**Proposition 5.2.** Let $M \in D^b_{coh}(\mathcal{O}_X)$.

(i) For an integer $n$ and a closed subset $Z$ of $X$, $\text{RHom}_Z(M) \in D^\geq_{qc}(\mathcal{O}_X)$ if and only if $\text{codim}(Z \cap \text{Supp}(H^k(M^*))) \geq k + n$ for any $k$. 
(ii) For an integer \( n \) and a family of supports \( \Phi \), \( R\Gamma_\Phi(M) \in D^b_{qc}(\mathcal{O}_X) \) if and only if \( \Phi \cap \text{Supp}(H^k(M^*)) \subset \mathcal{G}^{k+n} \) for every \( k \).

Proof. (ii) is a consequence of (i). Let us show (i). \( R\Gamma_Z(M) \in D^b_{qc}(\mathcal{O}_X) \) if and only if

\[
R\mathcal{H}\text{om}(F, M) \in D^b_{coh}(\mathcal{O}_X)
\]

for any \( F \in \text{Mod}_{coh}(\mathcal{O}_X) \) with \( \text{Supp}(F) \subset Z \). Since \( (R\mathcal{H}\text{om}(F, M))^* = F \otimes_{\mathcal{O}_X} M^* \), (5.2) is equivalent to \( \text{Supp}(H^k(F \otimes_{\mathcal{O}_X} M^*)) \subset \mathcal{G}^{k+n} \) for every \( k \) by Proposition 4.3 (ii). The last condition is equivalent to \( Z \cap \text{Supp}(H^k(M^*)) \subset \mathcal{G}^{k+n} \) by the lemma below. \( \square \)

Lemma 5.3. Let \( M \in D^b_{coh}(\mathcal{O}_X) \) and \( Z, S \) closed subsets. Then the following conditions are equivalent.

(i) \( Z \cap \text{Supp}(\tau^{>0}M) \subset S \).

(ii) \( \text{Supp}(\tau^{>0}(F \otimes_{\mathcal{O}_X} M)) \subset S \) for any \( F \in \text{Mod}_{coh}(\mathcal{O}_X) \) with \( \text{Supp}(F) \subset Z \).

Proof. (i) \( \Rightarrow \) (ii) For \( k \geq 0 \), an exact sequence \( H^k(F \otimes_{\mathcal{O}_X} \tau^{<k}M) \to H^k(F \otimes_{\mathcal{O}_X} M) \to H^k(F \otimes_{\mathcal{O}_X} \tau^{\geq k}M) \) and \( H^k(F \otimes_{\mathcal{O}_X} \tau^{<k}M) = 0 \) implies

\[
\text{Supp}(H^k(F \otimes_{\mathcal{O}_X} M)) \subset \text{Supp}(H^k(F \otimes_{\mathcal{O}_X} \tau^{\geq k}M)) \subset \text{Supp}(F) \cap \text{Supp}(\tau^{\geq k}M) \subset S.
\]

(ii) \( \Rightarrow \) (i) Let \( x \in Z \cap \text{Supp}(\tau^{>0}M) \). Take the largest \( k \geq 0 \) such that \( x \in \text{Supp}(H^k(M)) \). If one chooses a coherent \( \mathcal{O}_X \)-module \( F \) with \( \text{Supp}(F) = Z \), then \( F_x \neq 0 \) and \( H^k(M)_x \neq 0 \), which implies that \( F_x \otimes_{\mathcal{O}_{X,x}} H^k(M)_x \neq 0 \). On the other hand, one has

\[
H^k(F \otimes_{\mathcal{O}_X} M)_x \simeq H^k(F_x \otimes_{\mathcal{O}_{X,x}} M_x) \simeq F_x \otimes_{\mathcal{O}_{X,x}} H^k(M)_x.
\]

Hence one obtains \( x \in \text{Supp}(H^k(F \otimes_{\mathcal{O}_X} M)) \subset S \). \( \square \)

Proposition 5.4. Let \( M \in D^b_{coh}(\mathcal{O}_X), \Phi \) a family of supports and \( n \) an integer. Then, \( \tau^{<n}(R\Gamma_\Phi(M)) \in D^b_{coh}(\mathcal{O}_X) \) if and only if \( \text{Supp}(H^k(M^*)) \subset \Phi \cup \Psi^{k+n} \) for any \( k \). Here \( \Psi^k \) is the family of supports consisting of closed subsets \( Z \) such that \( \Phi \cap Z \subset \mathcal{G}^k \).

Proof. Assume first \( M' := \tau^{<n}(R\Gamma_\Phi(M)) \in D^b_{coh}(\mathcal{O}_X) \). Let us complete the morphism \( M' \to M \) to a distinguished triangle \( M' \to M \to M'' \xrightarrow{+1} \). Since one has \( R\Gamma_\Phi(M'') \in D^{\geq n}_{qc}(\mathcal{O}_X) \), Proposition 5.2 implies that \( \text{Supp}(H^k(M'')) \subset \Psi^{k+n} \) for every \( k \). The distinguished triangle \( M'' \to M \to M'' \xrightarrow{+1} \) implies that \( \text{Supp}(H^k(M^*)) \subset \text{Supp}(H^k(M'')) \cup \text{Supp}(H^k(M'')) \). Since \( \text{Supp}(H^k(M'')) \subset \Phi \) and \( \text{Supp}(H^k(M'')) \subset \Phi \cup \Psi^{k+n} \), we obtain \( \text{Supp}(H^k(M'')) \subset \Phi \cup \Psi^{k+n} \).

Conversely, assuming that \( \text{Supp}(H^k(M^*)) \subset \Phi \cup \Psi^{k+n} \) for every \( k \), we shall prove \( \tau^{<n}(R\Gamma_\Phi(M)) \in D^b_{coh}(\mathcal{O}_X) \). By devisse, one may assume that \( M^* = F[-k] \) for \( F \in \text{Mod}_{coh}(\mathcal{O}_X) \) with \( \text{Supp}(F) \subset \Phi \cup \Psi^{k+n} \). Then there is an exact sequence \( 0 \to F' \to F \to F'' \to 0 \) with \( F', F'' \in \text{Mod}_{coh}(\mathcal{O}_X) \) such that \( \text{Supp}(F') \subset \Psi^{k+n} \) and \( \text{Supp}(F'') \subset \Phi \). Set \( M' := (F'[-k])^* \) and \( M'' := (F''[-k])^* \). Then \( R\Gamma_\Phi(M') \in D^{\geq n}_{qc}(\mathcal{O}_X) \) by Proposition 5.2. Since \( R\Gamma_\Phi(M'') \simeq M'' \), one has a distinguished triangle \( M'' \to R\Gamma_\Phi(M) \to R\Gamma_\Phi(M') \xrightarrow{+1} \). Since \( \tau^{<n}R\Gamma_\Phi(M') = 0 \), one has \( \tau^{<n}R\Gamma_\Phi(M) \simeq \tau^{<n}M'' \in D^b_{coh}(\mathcal{O}_X) \). \( \square \)
It is sometimes convenient to use $\mathbb{Z}$-valued functions on $X$ instead of support data. We say that a bounded $\mathbb{Z}$-valued function $p$ on $X$ is a supporting function if $p(y) \geq p(x)$ whenever $y \in \{x\}$.

**Lemma 5.5.** By the following correspondence, the set of support data is isomorphic to the set of supporting functions. To a support datum $\Phi$, one associates the supporting function $p_\Phi$ given by $p_\Phi(x) := \max\left\{ n \in \mathbb{Z} ; \frac{x}{n} \in \Phi^n \right\}$.

Conversely, to a supporting function $p$ on $X$, one associates the support datum $\Phi_p$ given by $\Phi_p^n = \{ Z ; Z \text{ is a closed subset such that } p(z) \geq n \text{ for any } z \in Z \}$.

This lemma immediately follows from the fact that any closed subset is a union of finitely many irreducible closed subsets and that any irreducible subset has a generic point.

One has
\[
\begin{align*}
p_{\mathbb{T}}(x) &= 0, & p_{\Phi \cap \Psi}(x) &= \min(p_\Phi(x), p_\Psi(x)), \\
p_{\mathbb{E}}(x) &= \text{codim}(\{x\}) = \dim(\mathcal{O}_x, x), & p_{\Phi \cup \Psi}(x) &= \max(p_\Phi(x), p_\Psi(x)).
\end{align*}
\]

For two support data $\Phi$ and $\Psi$, we define the support datum $\Phi \circ \Psi$ by
\[
(\Phi \circ \Psi)^n = \bigcup_{n=i+j} (\Phi^i \cap \Psi^j)
\]

Note that $\circ$ is commutative and associative, and $\mathbb{T}$ is the unit with respect to $\circ$: $\mathbb{T} \circ \Phi = \Phi$ for every $\Phi$.

The following lemma is obvious.

**Lemma 5.6.** Let $\Phi$ and $\Psi$ be support data.

(i) one has
\[
\bigcup_{n=i+j} (\Phi^i \cap \Psi^j) = \bigcap_{n+1=i+j} (\Phi^i \cup \Psi^j).
\]

(ii) Let $Z$ be an irreducible closed subset of $X$ such that $Z \in \Phi^a$ and $Z \notin \Phi^{a+1}$. Then $Z \in \Psi^b$ if and only if $Z \in (\Phi \circ \Psi)^{a+b}$.

(iii) $p_{\Phi \circ \Psi}(x) = p_\Phi(x) + p_\Psi(x)$ for any $x \in X$.

**Proof.** (iii) is obvious. Let us show (i) and (ii). The inclusion $\bigcup_{n=i+j} (\Phi^i \cap \Psi^j) \subset \bigcap_{n+1=i+j} (\Phi^i \cup \Psi^j)$ is obvious. Hence it is enough to show that for any irreducible closed subset $Z$, the conditions $Z \in \Phi^a$, $Z \notin \Phi^{a+1}$ and $Z \in \bigcap_{a+b+1=i+j} (\Phi^i \cup \Psi^j)$ imply $Z \in \Psi^b$.

Since $Z \in \Phi^{a+1} \cup \Psi^b$, one obtains $Z \in \Psi^b$. 

**Lemma 5.7.** Let $\Phi$, $\Psi_1$ and $\Psi_2$ be three support data.

(i) $\Phi \circ (\Phi \cap \Psi_2) = (\Phi \circ \Psi_1) \cap (\Phi \circ \Psi_2)$ and $\Phi \circ (\Psi_1 \cup \Psi_2) = (\Phi \circ \Psi_1) \cup (\Phi \circ \Psi_2)$.

(ii) If $\Phi \circ \Psi_1 \subset \Phi \circ \Psi_2$, then one has $\Psi_1 \subset \Psi_2$.

(iii) If $\Phi \circ \Psi_1 = \Phi \circ \Psi_2$, then one has $\Psi_1 = \Psi_2$.

**Proof.** (i) is obvious. (ii) follows from Lemma 5.6 (iii), since $\Psi_1 \subset \Psi_2$ if and only if $p_{\Psi_1}(x) \leq p_{\Psi_2}(x)$ for every $x \in X$. (iii) is an immediate consequence of (ii).
Lemma 5.8. Let $\Phi$ and $\Theta$ be a pair of support data. For an integer $a$, set
\[ \Psi^n := \{ Z ; Z \text{ is a closed subset such that } \Phi^k \cap Z \subset \Theta^{k+n} \text{ for every } k \} , \]
\[ (\Psi_a)^n := \{ Z ; Z \text{ is a closed subset such that } \Phi^k \cap Z \subset \Theta^{k+n} \text{ for every } k \leq a \} \]
\[ = \{ Z \in (\Psi_{a-1})^n ; \Phi^a \cap Z \subset \Theta^{a+n} \} . \]
and $\Psi := \{ \Psi^n \}_{n \in \mathbb{Z}}$, $\Psi_a := \{ \Psi_a^n \}_{n \in \mathbb{Z}}$.

(i) $\Psi$ and $\Psi_a$ are support data, and $p_{\Psi}(x) = \min \{ p_{\Theta}(y) - p_{\Phi}(y) ; y \in \overline{\{x\}} \}$.
(ii) If a support datum $\Psi'$ satisfies $\Phi \circ \Psi' = \Theta$, then $\Psi' = \Psi$.
(iii) There exists a support datum $\Psi'$ such that $\Phi \circ \Psi' = \Theta$ if and only if $0 \leq p_{\Phi}(y) - p_{\Phi}(x) \leq p_{\Theta}(y) - p_{\Theta}(x)$ whenever $y \in \overline{\{x\}}$.
(iv) If $\Phi \circ \Psi = \Theta$, then one has $(\sigma^{\leq a}) \circ \Psi_a = \Theta$ for every integer $a$.
(v) Assume that $\Phi \circ \Psi = \Theta$. Then one has $(\Psi_{a-1})^n \subset \Phi^a \cup \Psi^n$ for every $n$.
(vi) Assume that $(\Psi_{a-1})^n \subset \Phi^a \cup (\Psi_a)^n$ for every $a$ and $n$. Then $\Phi \circ \Psi = \Theta$.

Proof. (i) is obvious.
(ii) One has $\Psi' \subset \Psi$ and $\Phi \circ \Psi' \subset \Theta$. Hence
\[ \Theta = \Phi \circ \Psi' \subset \Phi \circ \Psi \subset \Theta . \]
Hence $\Phi \circ \Psi' = \Phi \circ \Psi$ and Lemma 5.7 (iii) implies $\Psi' = \Psi$.
(iii) is obvious.
(iv) We shall apply (iii) and (ii). If $y \in \overline{\{x\}}$, then one has
\[ p_{\sigma^{\leq a} \Phi}(y) - p_{\sigma^{\leq a} \Phi}(x) = \min(p_{\Phi}(y), a) - \min(p_{\Phi}(y), a) \leq p_{\Phi}(y) - p_{\Phi}(x) \leq p_{\Theta}(y) - p_{\Theta}(x) . \]
Here the first inequality follows from $p_{\Phi}(y) \geq p_{\Phi}(x)$.
(v) Let $Z \in (\Psi_{a-1})^n$ be an irreducible closed subset. We shall show that if $Z \not\subset \Phi^a$, then $Z \in \Psi^n$. Let us take an integer $i$ such that $Z \in \Phi^i$ and $Z \not\subset \Phi^{i+1}$. Then one has $i < a$.
Since $Z \in (\Psi_{a-1})^n$, one has $Z = \Phi^i \cap Z \subset \Theta^{i+n} = (\Phi \circ \Psi)^{i+n}$. Hence $Z \in \Psi^n$.
(vi) Let $Z \in \Theta^n$ be an irreducible closed subset. Let us show $Z \in (\Phi \circ \Psi)^n$. Take an integer $i$ such that $Z \in \Phi^i$ and $Z \not\subset \Phi^{i+1}$. Let us show $Z \in \Psi^{n-i}$. Since $\Psi_a = \Psi$ for $a \gg 0$, it is enough to show $Z \in (\Psi_a)^{n-i}$ for every $a$. We shall show it by the induction on $a$. It is obvious that $Z \in (\Psi_a)^{n-i}$ for $a \ll 0$. Assuming that $Z \in (\Psi_{a-1})^{n-i}$, let us show $Z \in (\Psi_a)^{n-i}$. Since $Z \in (\Psi_{a-1})^{n-i} \subset \Phi^a \cup (\Psi_a)^{n-i}$, we may assume that $Z \in \Phi^a$, which implies $a \leq i$. Therefore one has $\Phi^a \cap Z = Z \in \Theta^n \subset \Theta^{a-i+n}$. Together with $Z \in (\Psi_{a-1})^{n-i}$, one obtains $Z \in (\Psi_a)^{n-i}$.

For a support datum $\Phi$, let $\Phi_*$ denote the support datum given by
\[ \Phi_*^n := \{ Z ; Z \text{ is a closed subset such that } \Phi^k \cap Z \subset \Theta^{a+k} \text{ for every } k \} . \]

Now we are ready to give a criterian for $\Phi^{\text{D}_{\text{coh}}^b}(\mathcal{O}_X)$ to be a $t$-structure, which is a generalization of [6, Exercise X.2].

Theorem 5.9. Let $\Phi$ be a support datum. Then the following conditions are equivalent.
(i) $\Phi^{\text{D}_{\text{coh}}^b}(\mathcal{O}_X)$ is a $t$-structure.
(ii) For any irreducible closed subsets $Z$ and $S$ such that $S \subset Z$ and $S \subset \Phi^k$, one has $Z \in \Phi^{k+\text{codim}(Z) - \text{codim}(S)}$. In terms of supporting functions, one has
\[ p_{\Phi}(y) - p_{\Phi}(x) \leq \text{codim}(\{y\}) - \text{codim}(\{x\}) = \text{dim}(\mathcal{O}_{X,y}) - \text{dim}(\mathcal{O}_{X,x}) \text{ if } y \in \overline{\{x\}} . \]
(iii) $\Phi \circ \Phi_* = \mathcal{O}$. 
There exists a support datum \( \Psi \) such that \( \Phi \circ \Psi = \emptyset \).

Moreover if these conditions are satisfied, the equivalence

\[
R \mathcal{H}om \mathcal{O}_X(-, \mathcal{O}_X) : D_{\mathcal{O}_X}^b(\mathcal{O}_X) \stackrel{\sim}{\longrightarrow} D_{\mathcal{O}_X}^b(\mathcal{O}_X)^{\text{op}}
\]

sends the \( t \)-structure \( ^pD_{\mathcal{O}_X}(\mathcal{O}_X) \) to \( ^pD_{\mathcal{O}_X}(\mathcal{O}_X)^{\text{op}} \).

\textbf{Proof.} The last statement easily follows from Proposition 5.2. The equivalence of (ii)—(v) is already shown.

Let us show (v) \( \Rightarrow \) (i). The proof is similar to the one of Theorem 3.5. It is enough to show that any \( M \in D_{\mathcal{O}_X}^b(\mathcal{O}_X) \) satisfies the following property:

\[
\text{(5.5) } \exists \text{ a distinguished triangle } M' \to M \to M'' \xrightarrow{+1} \text{ with } M' \in ^pD_{\mathcal{O}_X}(\mathcal{O}_X) \text{ and } M'' \in ^pD_{\mathcal{O}_X}(\mathcal{O}_X).
\]

For \( n \in \mathbb{Z} \), let us consider the following statement:

\[
\text{(5.5)}_n \text{ The property (5.5) holds if } M \in \sigma^{\leq n}D_{\mathcal{O}_X}(\mathcal{O}_X).
\]

We shall prove \( (5.5)_n \) by assuming \( (5.5)_{n+1} \). Let us consider a distinguished triangle

\[
\tau^{\leq n}R\Gamma_{\phi_{n+1}}(M) \to M \to M'' \xrightarrow{+1}.
\]

In the course of the proof of Theorem 3.5, we have proved that \( \tau^{\leq n}R\Gamma_{\phi_{n+1}}(M) = \mathcal{H}_{\phi_{n+1}}^n(M)[-n] \in \Phi D_{\mathcal{O}_X}^{<0} \) and \( M'' \in \sigma^{\leq n+1}D_{\mathcal{O}_X}(\mathcal{O}_X) \). Hence it is enough to show that \( \mathcal{H}_{\phi_{n+1}}^n(M) \) is coherent. Since \( \mathcal{H}_{\phi_{n+1}}^i(M) = 0 \) for \( i > n+1 \), Proposition 5.2 implies that \( \text{Supp}(H^i(M^*)) \cap \Phi^k \subseteq \mathcal{G}^k \) for every \( k < n \) and \( i \). Hence one has \( \text{Supp}(H^i(M^*)) \subseteq (\sigma^{\leq n})^i \) for every \( i < n+1 \).

Let us show that (i) implies (ii). In order to see this, it is enough to show that the following situation cannot happen:

\( Z \) is an irreducible closed subset of \( X \) and \( S \) is an irreducible closed subset of \( Z \) with codimension 1 such that \( Z \in \Phi^a \), \( Z \not\subseteq \Phi^{a+1} \) and \( S \in \Phi^b \) with \( b > a + 1 \).

Set \( M = \mathcal{O}_Z[-a] \in \Phi D_{\mathcal{O}_X}^{\leq 0} \). Here \( \mathcal{O}_Z \) is the structure sheaf of \( Z \) endowed with the reduced scheme structure. Let

\[
M' \to M \to M'' \xrightarrow{+1}
\]

be a distinguished triangle with \( M' \in \Phi D_{\mathcal{O}_X}^{<0} \) and \( M'' \in \Phi D_{\mathcal{O}_X}^{\leq 0} \). Then one has \( \text{Supp}(M'') \subseteq Z \), and hence \( M'' = R\Gamma\Phi^a(M'') \in D_{\mathcal{O}_X}^{\leq a} \), which implies that \( M' \in D_{\mathcal{O}_X}^{<a} \). The exact sequence \( 0 \to H^a(M') \to \mathcal{O}_Z \), along with \( \text{Supp}(H^a(M')) \subseteq \Phi^{a+1} \), implies that \( H^a(M') = 0 \). Hence one has \( M' \in D_{\mathcal{O}_X}^{>a} \). On the other hand, one has the exact sequence

\[
\mathcal{H}_{\Phi_{a+1}}^a(M') \to \mathcal{H}_{\Phi_{a+1}}^{a+1}(M) \to \mathcal{H}_{\Phi_{a+1}}^{a+1}(M'').
\]

Since \( R\Gamma_S(M'') \in D^{>b}_{\mathcal{O}_X} \), one has \( \mathcal{H}_{\Phi_{a+1}}^{a+1}(M'') = 0 \). Since \( M' \in D_{\mathcal{O}_X}^{<a} \), one has \( \mathcal{H}_{\Phi_{a+1}}^{a+1}(M') = \Gamma_S(H^{a+1}(M')) \), which is a coherent \( \mathcal{O}_X \)-module. Hence \( \mathcal{H}_{\Phi_{a+1}}^{a+1}(M) = \mathcal{H}_{\Phi_{a+1}}^a(M_Z) \) is a coherent \( \mathcal{O}_X \)-module. This is a contradiction. The last step follows from either Proposition 5.4 or the following easy lemma, whose proof is omitted.

\textbf{Lemma 5.10.} Let \( A \) be a 1-dimensional Noetherian local ring with a maximal ideal \( \mathfrak{m} \). Then \( H^1_{\mathfrak{m}}(A) \) is not a finitely generated \( A \)-module.
6. T-structure on $D^b_{\text{qc}}(\mathcal{D}_X)$ and $D^b_{\text{hol}}(\mathcal{D}_X)$

In the sequel, we shall treat the case $\mathcal{A} = \mathcal{D}$. Let $X$ be an algebraic manifold over a field $k$ of characteristic 0, i.e. a quasi-compact separated scheme smooth over $k$. Let $\mathcal{D}_X$ be the sheaf of differential operators on $X$. Let us denote by $D^b_{\text{qc}}(\mathcal{D}_X)$ the derived category of bounded complexes of $\mathcal{D}_X$-modules with quasi-coherent cohomologies. Let us denote by $D^b_{\text{coh}}(\mathcal{D}_X)$, $D^b_{\text{hol}}(\mathcal{D}_X)$, $D^b_{\text{rh}}(\mathcal{D}_X)$ the full subcategories of $D^b_{\text{qc}}(\mathcal{D}_X)$ consisting of bounded complexes with coherent, holonomic, regular holonomic $\mathcal{D}_X$-modules as cohomologies, respectively. For a morphism $f : X \to Y$ of algebraic manifolds, we denote by $\mathbb{D} f^* : D^b_{\text{qc}}(\mathcal{D}_Y) \to D^b_{\text{qc}}(\mathcal{D}_X)$ and $\mathbb{D} f_* : D^b_{\text{qc}}(\mathcal{D}_X) \to D^b_{\text{qc}}(\mathcal{D}_Y)$ the inverse image and the direct image functors (see e.g. [5]). Note that they respect $D^b_{\text{hol}}$ and $D^b_{\text{rh}}$.

Let us define a t-structure on $D^b_{\text{qc}}(\mathcal{D}_X)$ as follows.

\begin{align}
\mathcal{E}D^<_{\text{qc}}(\mathcal{D}_X) &= \{ M \in D^b_{\text{qc}}(\mathcal{D}_X) : \text{Supp}(H^n(M)) \subset \mathcal{E} \text{ for every } n \}, \\
\mathcal{E}D^>_{\text{qc}}(\mathcal{D}_X) &= \{ M \in D^b_{\text{qc}}(\mathcal{D}_X) : \text{RG}_{\text{qc}}(M) \in D^>_{\text{qc}}(\mathcal{D}_X) \text{ for every } n \}.
\end{align}

It is indeed a t-structure by Theorem 3.5. Let $\mathcal{C}_{\text{qc}} = \mathcal{E}D^<_{\text{qc}}(\mathcal{D}_X) \cap \mathcal{E}D^>_{\text{qc}}(\mathcal{D}_X)$ be its heart.

We note that $(\mathcal{E}D^<_{\text{qc}}(\mathcal{D}_X) \cap D^b_{\text{coh}}(\mathcal{D}_X)) = \mathcal{E}D^<_{\text{qc}}(\mathcal{D}_X) \cap D^b_{\text{coh}}(\mathcal{D}_X)$ is a t-structure on $D^b_{\text{coh}}(\mathcal{D}_X)$ (when $\dim X > 1$), as seen in the following example.

**Example 6.1.** Set $X = \text{Spec}(\mathbb{C}[x, y])$ and $Y = \text{Spec}(\mathbb{C}[t])$ with indeterminates $x$, $y$ and $t$. Set $t = f(x, y) = xy : X \to Y$. Set $\mathcal{M} := \mathcal{D}_X / \mathcal{D}_X(\partial_x - y\partial_y)$ and $\mathcal{N} := \mathcal{D}_Y$. Note that $\mathcal{N} \simeq \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y = \bigoplus_{n=0}^\infty \mathcal{O}_X v_n$ with $v_n = 1_{X \to Y} \otimes \partial_t^n$ as an $\mathcal{O}_X$-module. The defining relations of $\{v_n\}_{n \in \mathbb{Z}_{\geq 0}}$ as a $\mathcal{D}_X$-module are

$$
\partial_x v_n = yv_{n+1} \text{ and } \partial_y v_n = xv_{n+1}.
$$

Hence $\mathcal{N} \subset \mathcal{C}_{\text{qc}}$. One has a morphism $\mathcal{M} \to \mathcal{N}$ by 1 mod $\mathcal{D}_X(\partial_x - y\partial_y) \mapsto v_0$. Set $E = \mathbb{C}^{\oplus \mathbb{Z}_{\geq 0}} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{C} w_n$. Then one has $\mathcal{B}_{[0]|X} \otimes_{\mathbb{C}} E[-2] \in \mathcal{C}_{\text{qc}}$. There is an exact sequence

$$
0 \to \mathcal{M} \to \mathcal{N} \xrightarrow{g} \mathcal{B}_{[0]|X} \otimes_{\mathbb{C}} E \to 0
$$
in $\text{Mod}(\mathcal{D}_X)$. Here $g$ is given by

$$
g(v_n) = \sum_{1 \leq k \leq n} (-1)^{n-k} \frac{1}{(n-k)!} (\partial_x \partial_y)^{n-k} \delta \otimes w_k,
$$

where $\delta \in \mathcal{B}_{[0]|X}$ is the generator with the defining relations $x \delta = y \delta = 0$. Hence one has a distinguished triangle $\mathcal{B}_{[0]|X} \otimes_{\mathbb{C}} E[-2][1] \to \mathcal{M} \to \mathcal{N} \xrightarrow{+1}$. Thus we obtain

$$
\mathcal{E}H^n(\mathcal{M}) = \begin{cases} 
\mathcal{B}_{[0]|X} \otimes_{\mathbb{C}} E[-2] & \text{for } n = -1, \\
\mathcal{N} & \text{for } n = 0, \\
0 & \text{for } n \neq -1, 0.
\end{cases}
$$

Since $\mathcal{B}_{[0]|X} \otimes_{\mathbb{C}} E$ is not coherent, $(\mathcal{E}D^<_{\text{qc}}(\mathcal{D}_X) \cap D^b_{\text{coh}}(\mathcal{D}_X))$ is not a t-structure on $D^b_{\text{coh}}(\mathcal{D}_X)$. Note that one has $\mathcal{H}_{\text{qc}}(\mathcal{M}) = 0$ and $\mathcal{H}_{\text{qc}}(\mathcal{M}) \approx \mathbb{B}_{[0]|X} \otimes_{\mathbb{C}} E$.

Let us denote by $\mathcal{E}D^<_{\text{hol}}(\mathcal{D}_X) : = \mathcal{E}D^<_{\text{qc}}(\mathcal{D}_X) \cap D^b_{\text{hol}}(\mathcal{D}_X)$ and $\mathcal{E}D^>_{\text{hol}}(\mathcal{D}_X) : = \mathcal{E}D^>_{\text{qc}}(\mathcal{D}_X) \cap D^b_{\text{hol}}(\mathcal{D}_X)$. Similarly we define $\mathcal{E}D^<_{\text{rh}}(\mathcal{D}_X)$ and $\mathcal{E}D^>_{\text{rh}}(\mathcal{D}_X)$.

**Theorem 6.2.** $(\mathcal{E}D^<_{\text{hol}}(\mathcal{D}_X), \mathcal{E}D^>_{\text{hol}}(\mathcal{D}_X))$ and $(\mathcal{E}D^<_{\text{rh}}(\mathcal{D}_X), \mathcal{E}D^>_{\text{rh}}(\mathcal{D}_X))$ are a t-structure on $D^b_{\text{hol}}(\mathcal{D}_X)$ and $D^b_{\text{rh}}(\mathcal{D}_X)$, respectively.
Proof. In order to show that \((\mathcal{D}^{0}_{\text{hol}}(\mathcal{D}_X), \mathcal{D}^0_{\text{hol}}(\mathcal{D}_X))\) is a t-structure on \(\mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)\), it is enough to show that, for any any \(\mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)\), there exists a distinguished triangle

\[ \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \xrightarrow{+1} \]

with \(\mathcal{M}' \in \mathcal{D}^0_{\text{hol}}(\mathcal{D}_X)\) and \(\mathcal{M}'' \in \mathcal{D}^0_{\text{hol}}(\mathcal{D}_X)\).

Let us show this by the induction on the codimension \(d\) of \(S := \text{Supp}(.\mathcal{M})\). One has \(\tau^{<d}.\mathcal{M} \in \mathcal{D}^0_{\text{hol}}(\mathcal{D}_S)\). By the distinguished triangle \(\tau^{<d}.\mathcal{M} \to \mathcal{M} \to \tau^{\geq d}.\mathcal{M} \xrightarrow{+1}\), we may assume that \(\mathcal{M} \simeq \tau^{\geq d}.\mathcal{M}\) by Lemma 2.1.

Let \(S_0\) be a \(d\)-codimensional smooth open subset of \(S\) such that \(S_1 := S \setminus S_0\) is of codimension \(> d\). Set \(U := X \setminus S_1\). Then one has \(S_0 = U \cap S\). Let \(j : U \to X\) and \(i : S_0 \to U\) be the open embedding and the closed embedding, respectively. Then there exists \(\mathcal{N} \in \mathcal{D}_{\text{hol}}^d(\mathcal{D}_{S_0})\) such that \(\mathcal{M}_U\) is isomorphic to \(\mathbb{D}i_\ast.\mathcal{N}\). By shrinking \(S_0\) if necessary, we may assume that the cohomologies of \(\mathcal{N}\) are locally free \(\mathcal{O}_{S_0}\)-modules. Hence \(\mathbb{D}i_\ast.\mathcal{N}\) belongs to \(\mathcal{D}^0_{\text{hol}}(\mathcal{D}_U)\). Hence \(\mathcal{M}'' := Rj_\ast \mathbb{D}i_\ast.\mathcal{N}\) belongs to \(\mathcal{D}^0_{\text{hol}}(\mathcal{D}_X)\) by Lemma 3.7. Let us consider a distinguished triangle

\[ \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \xrightarrow{+1} \]

Since \(\text{Supp}(\mathcal{M}') \subset S_1\), the codimension of \(\text{Supp}(\mathcal{M}')\) is greater than \(d\). Then the induction proceeds by Lemma 2.1.

The regular holonomic case is proved similarly, because the regular holonicity is also preserved by the direct image functors. □

Remark 6.3. By Proposition 3.8, \(U \mapsto \mathcal{D}^{\leq 0}_s(\mathcal{D}_U) \cap \mathcal{D}^0_{\text{hol}}(\mathcal{D}_U)\) is a stack on \(X\) for \(\ast = \text{qc, hol, rh}\).

For a closed point \(x \in X\), let us denote by \(\mathcal{B}_x|_X\) the regular holonomic \(\mathcal{D}_X\)-module \(\mathcal{H}^\dim X(\mathcal{O}_X)|_x\).

Proposition 6.4. For \(\mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)\), the following conditions are equivalent.

\begin{enumerate}
  \item \(\mathcal{M} \in \mathcal{D}^0_{\text{hol}}(\mathcal{D}_X)\).
  \item \(\mathcal{M}\) is of flat dimension \(\leq 0\) as a complex of \(\mathcal{O}_X\)-modules.
  \item For any closed subset \(Z\) of \(X\), one has \(\mathcal{H}^i_Z(\mathcal{M}) = 0\) for any \(i < \text{codim}Z\).
  \item For any locally closed subset \(Z\) of \(X\), one has \(\mathcal{H}^i_Z(\mathcal{M}_Z) = 0\) for any \(i < \text{codim}Z\).
  \item For any closed point \(x \in X\) and \(i < 0\). Here \(k_x\) is the sheaf \(\mathcal{O}_X/m_x\) of functions vanishing at \(x\).
\end{enumerate}

Proof. The equivalence of (i)–(iv) are evident. Since \(\text{RHom}_{\mathcal{D}_X}(\mathcal{B}_x|_X, \mathcal{M}) \simeq k_x \otimes_{\mathcal{O}_X} \mathcal{M}[-\dim X]\), the implication (i) \(\Rightarrow\) (v) and the equivalence (v) \(\Leftrightarrow\) (vi) are evident.

It remains to prove (vi) \(\Rightarrow\) (i). Let us assume that \(\mathcal{M}\) satisfies (vi). Let us show \(\mathcal{M} \in \mathcal{D}^0_{\text{hol}}(\mathcal{D}_X)\) by the induction on the codimension \(d\) of \(S := \text{Supp}(\mathcal{M})\). Let \(S_0\) be a \(d\)-codimensional smooth open subset of \(S\) such that \(S_1 := S \setminus S_0\) is of codimension \(> d\). Let \(i : S_0 \to X\) be the inclusion. Then there exists \(\mathcal{N} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_{S_0})\) such that \(\mathcal{M}\) is isomorphic to \(\mathbb{D}i_\ast.\mathcal{N}\) on a neighborhood of \(S_0\). By shrinking \(S_0\) if necessary, we may assume that the cohomologies of \(\mathcal{N}\) are locally free \(\mathcal{O}_{S_0}\)-modules. Then \(k_x \otimes_{\mathcal{O}_{S_0}} \mathcal{N} \simeq k_x \otimes_{\mathcal{O}_X} \mathcal{M}[d]\), and hence one has \(H^k(\mathcal{N})_x = 0\) for any \(k < d\) and any closed point \(x\) of \(S_0\). Therefore, \(\mathcal{N} \in \mathcal{D}^{\leq d}(\mathcal{D}_{S_0})\) and \(\mathcal{M}'' := \mathbb{D}i_\ast.\mathcal{N} \simeq \text{Rf}_{S_0}.\mathcal{M}\) belongs to \(\mathcal{D}^0_{\text{hol}}(\mathcal{D}_X)\). Consider a distinguished triangle

\[ \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \xrightarrow{+1} \]
The exact sequence $H^{k-1}(k_x \otimes_{\mathcal{O}_X} \mathcal{M}^\prime) \to H^k(k_x \otimes_{\mathcal{O}_X} \mathcal{M}^\prime) \to H^k(k_x \otimes_{\mathcal{O}_X} \mathcal{M})$ implies that $H^k(k_x \otimes_{\mathcal{O}_X} \mathcal{M}^\prime) = 0$ for $k < 0$. Since codim $\text{Supp}(\mathcal{M}^\prime) > d$, the induction hypothesis implies that $\mathcal{M}^\prime \in \mathcal{D}^0_{\text{hol}}(\mathcal{D}_X)$. Hence one concludes that $\mathcal{M} \in \mathcal{D}^0_{\text{hol}}(\mathcal{D}_X)$.

Let us assume that the base field $k$ is the complex number field $\mathbb{C}$, and let us denote by $X_{\text{an}}$ the associated complex manifold. Let us denote by $D^b_c(\mathbb{C}_{X_{\text{an}}})$ the derived category of bounded complexes of $\mathbb{C}_{X_{\text{an}}}$-modules with constructible cohomologies.

**Theorem 6.5.** The equivalence of triangulated categories

$$\text{Sol}_x := R\mathcal{H}\text{om}_{\mathcal{D}_X}(-, \mathcal{O}_{X_{\text{an}}}) : D^b_{\text{rh}}(\mathcal{D}_X) \sim D^b_c(\mathbb{C}_{X_{\text{an}}})^{\text{op}}$$

sends the $t$-structure $((\mathcal{D}^0_{\text{rh}}(\mathcal{D}_X), \mathcal{D}^0_{\text{rh}}(\mathcal{D}_X)))$ to $(D^0_c(\mathbb{C}_{X_{\text{an}}})^{\text{op}}, D^0_c(\mathbb{C}_{X_{\text{an}}})^{\text{op}})$.

**Proof.** Since $R\mathcal{H}\text{om}_{\mathcal{D}_X}(-, \mathcal{O}_{X_{\text{an}}})$ is an equivalence of triangulated categories, it is enough to show that $\mathcal{M} \in D^b_c(\mathcal{D}_X)$ belongs to $\mathcal{D}^0_{\text{rh}}(\mathcal{D}_X)$ if and only if $R\mathcal{H}\text{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X_{\text{an}}})$ belongs to $D^0_c(\mathbb{C}_{X_{\text{an}}})$. This immediately follows from Proposition 6.4 and

$$R\mathcal{H}\text{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X_{\text{an}}}) \simeq \text{Hom}_C(\mathcal{C}_x \otimes_{\mathcal{O}_X} \mathcal{M}, C).$$

7. Proof of Lemma 3.1

In this section, we give a proof of Lemma 3.1.

**Lemma 7.1.** Let $\mathcal{M}$ be a coherent $\mathcal{A}$-module, and $\mathcal{N}$ a (not necessarily coherent) $\mathcal{A}$-submodule of $\mathcal{M}$, and $Z$ a closed subset of $X$. Then, for the generic point $\xi$ of any irreducible component of $Z$, there exist a coherent $\mathcal{A}$-submodule $\mathcal{L}$ of $\mathcal{M}$ and an open neighborhood $U$ of $\xi$ such that

$$\mathcal{N}|_U = (\mathcal{N}|_{X \setminus Z} + \mathcal{L})|_U.$$

**Proof.** Assume first that $X$ is affine. Set $R = \mathcal{O}_X(X)$, $A = \mathcal{A}(X)$, $M = \mathcal{M}(X)$. Let $L \subset M$ be the inverse image of $\mathcal{N}_\xi$ by the morphism $M \to \mathcal{M}_\xi$. Then $L$ is a finitely generated $A$-module. Let $\mathcal{L}$ be the coherent $\mathcal{A}$-module associated with $L$: $\mathcal{L} := \mathcal{A} \otimes_A L \simeq \mathcal{O}_X \otimes_R L$. Then one has $\mathcal{N}_\xi = \mathcal{L}_\xi$. Hence there exists an open neighborhood $U$ of $\xi$ such that $\mathcal{L}|_U \subset \mathcal{N}|_U$. We may assume further that $U \cap Z = U \cap \{\xi\}$. Let us show that $\mathcal{N}|_U = (\mathcal{N}|_{X \setminus Z} + \mathcal{L})|_U$. It is evident that the last member is contained in the first. We shall prove the opposite inclusion. Let $V$ be an affine open subset contained in $U$. Then $\mathcal{M}(V) = \mathcal{O}_X(V) \otimes_R M$. Let $N_V$ be the inverse image of $\mathcal{N}(V)$ by the morphism $M \to \mathcal{M}(V)$. Then one has $\mathcal{N}(V) = \mathcal{O}_X(V) \otimes_R N_V$.

Assume first $\xi \in V$. Then by the commutative diagram

$$\begin{array}{ccc}
N_V & \longrightarrow & \mathcal{N}(V) \\
\downarrow & & \downarrow \\
M & \longrightarrow & \mathcal{M}(V)
\end{array}$$

$N_V$ is contained in $L$. Hence $\mathcal{N}(V) \simeq \mathcal{O}_X(V) \otimes_R N_V \subset \mathcal{O}_X(V) \otimes_R L \simeq \mathcal{L}(V)$. If $\xi$ is not contained in $V$, then one has $V \subset X \setminus Z$, and hence $\mathcal{N}(V) \subset (\mathcal{N}|_{X \setminus Z})(V)$. Hence one obtains $\mathcal{N}|_U \subset (\mathcal{N}|_{X \setminus Z} + \mathcal{L})|_U$. 


In the general case, let us take an affine open subset $W$ of $X$ containing $\xi$. Then it is enough to remark that any coherent $\mathcal{A}|_W$-submodule of $\mathcal{M}|_W$ can be extended to a coherent $\mathcal{A}$-submodule of $\mathcal{M}$. Note that a quasi-coherent $\mathcal{A}$-submodule of a coherent $\mathcal{A}$-module is coherent.

The next lemma is not used in this paper.

**Lemma 7.2.** Let $\mathcal{M}$ be a coherent $\mathcal{A}$-module, and $\mathcal{N}$ a (not necessarily coherent) $\mathcal{A}$-submodule of $\mathcal{M}$. Then, there exist finite families of open subsets $U_i$ of $X$ and coherent $\mathcal{A}$-submodules $\mathcal{M}_i$ of $\mathcal{M}$ such that

$$\mathcal{N} = \sum_i (\mathcal{M}_i)_{U_i}.$$

**Proof.** Let $\mathcal{W}$ be the set of open subsets $W$ of $X$ such that there exist finite families of open subsets $U_i$ of $W$ and coherent $\mathcal{A}$-submodules $\mathcal{M}_i$ of $\mathcal{M}$ such that $\mathcal{N}|_W = (\sum_i (\mathcal{M}_i)_{U_i})|_W$. Since $X$ is Noetherian, $\mathcal{W}$ has a maximal element. Let $W$ be such a maximal element. Assuming that $W \neq X$, let us derive a contradiction. Let $\xi$ be the generic point of an irreducible component of $X \setminus W$. Then by Lemma 7.1, there exist an open neighborhood $U$ of $\xi$ and a coherent $\mathcal{A}$-submodule $\mathcal{L}$ such that $\mathcal{N}|_U = (\mathcal{M}_W + \mathcal{L})|_U$. Hence one has $\mathcal{N}|_{W \cup U} = (\sum_i (\mathcal{M}_i)_{U_i} + \mathcal{L}_U)|_{W \cup U}$. This contradicts the choice of $W$. □

The following lemma, an analogue of [2], is a corollary of Lemma 7.1,

**Lemma 7.3.** Let $\mathcal{M}$ be an $\mathcal{A}$-module. Then the following conditions are equivalent.

(i) $\mathcal{M}$ is an injective $\mathcal{A}$-module.

(ii) For any coherent $\mathcal{A}$-module $\mathcal{F}$, one has $\mathcal{E}xt^1_{\mathcal{A}}(\mathcal{F}, \mathcal{M}) = 0$ and $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{M})$ is a flabby sheaf.

**Proof.** (i)$\Rightarrow$(ii) is well-known. Let us show (ii)$\Rightarrow$(i). In order to see that $\mathcal{M}$ is injective, it is enough to show that for any left ideal $\mathcal{I}$ of $\mathcal{A}$ with $\mathcal{I} \neq \mathcal{A}$ and a morphism $\varphi: \mathcal{I} \to \mathcal{M}$, there exists a left ideal $\mathcal{I}'$ strictly containing $\mathcal{I}$ such that $\varphi$ extends to a morphism $\mathcal{I}' \to \mathcal{M}$. Let $Z = \text{Supp}(\mathcal{A}/\mathcal{I}) \neq \emptyset$, and let $\xi$ be the generic point of an irreducible component of $Z$. Then by Lemma 7.1, there exist a neighborhood $U$ of $\xi$ and a coherent left ideal $\mathcal{J}$ of $\mathcal{A}$ such that $\mathcal{I}^U = (\mathcal{A}_X \setminus Z + \mathcal{J})^U$. Set $\mathcal{J}' = \mathcal{I} + \mathcal{A}_U \neq \mathcal{I}$. The exact sequence $0 \to \mathcal{J} \to \mathcal{I'} \to (\mathcal{A}/\mathcal{J})_{Z \cap U} \to 0$ induces an exact sequence

$$\mathcal{H}om_\mathcal{A}(\mathcal{J}', \mathcal{M}) \to \mathcal{H}om_\mathcal{A}(\mathcal{I}, \mathcal{M}) \to \mathcal{E}xt^1_{\mathcal{A}}((\mathcal{A}/\mathcal{J})_{Z \cap U}, \mathcal{M}).$$

Here the last term vanishes by the assumptions:

$$\mathcal{E}xt^1_{\mathcal{A}}((\mathcal{A}/\mathcal{J})_{Z \cap U}, \mathcal{M}) = H^1_{Z \cap U}(U; \mathcal{R}\mathcal{H}om_\mathcal{A}(\mathcal{A}/\mathcal{J}, \mathcal{M}))$$

$$= H^1_{Z \cap U}(U; \mathcal{H}om_{\mathcal{A}}(\mathcal{A}/\mathcal{J}, \mathcal{M})) = 0.$$

Therefore, the morphism $\varphi: \mathcal{I} \to \mathcal{M}$ extends to a morphism $\mathcal{I}' \to \mathcal{M}$. □

**Proof of Lemma 3.1.** Let $\{\mathcal{M}_i\}_{i \in I}$ be a filtrant inductive family of injective $\mathcal{A}$-modules, and $\mathcal{M} = \lim_i \mathcal{M}_i$. For any coherent $\mathcal{A}$-module $\mathcal{F}$, one has

$$\mathcal{E}xt^k_{\mathcal{A}}(\mathcal{F}, \mathcal{M}) \simeq \lim_i \mathcal{E}xt^k_{\mathcal{A}}(\mathcal{F}, \mathcal{M}_i),$$

and the condition (ii) in the lemma above is satisfied. Note that any filtrant inductive limit of flabby sheaves on a Noetherian scheme is flabby. □
Remark 7.4. Let $\mathcal{M}$, $\mathcal{N}$ be coherent $\mathcal{A}$-modules, and $Z$ a closed subset of $X$. When $\mathcal{A}$ is commutative, an injective object of $\text{Mod}_{qc}(\mathcal{A})$ is an injective object of $\text{Mod}(\mathcal{A})$, and $\text{Hom}_{D_{qc}(\mathcal{A})}(\mathcal{M}, R\Gamma_Z(\mathcal{N})[n])$ is calculated by $H^n(\text{Hom}_{\mathcal{A}}(\mathcal{M}, \Gamma_Z(S^\bullet)))$ for an injective resolution $S^\bullet$ of $\mathcal{N}$ in $\text{Mod}_{qc}(\mathcal{A})$. But it is not true in general. Any injective object of $\text{Mod}_{qc}(\mathcal{A})$ is a flabby sheaf (cf. Remark 3.2), and $\Gamma_Z(S^\bullet)$ certainly calculates $R\Gamma_Z(\mathcal{N})$. But $S^\bullet$ is not necessarily a complex of injective objects of $\text{Mod}(\mathcal{A})$, and the functor $\Gamma_Z$ does not send injective objects of $\text{Mod}_{qc}(\mathcal{A})$ to injective objects of $\text{Mod}_{qc}(\mathcal{A})$.

For example, take $X = \text{Spec}(\mathbb{C}[x])$, $\mathcal{A} = \mathcal{D}_X$ and $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X \partial_x$, $Z = \{0\}$. Then one has by Hilbert’s Nullstellensatz

$$\text{Hom}_{\mathcal{A}}(\mathcal{M}, \Gamma_Z(\mathcal{N})) \simeq \lim_{n>0} \text{Hom}_{\mathcal{A}}(\mathcal{D}_X/(\mathcal{D}_X \partial_x + \mathcal{D}_X x^n), \mathcal{N}) = 0$$

for any $\mathcal{N} \in \text{Mod}_{qc}(\mathcal{A})$, but one has $\text{Hom}_{D_{qc}(\mathcal{A})}(\mathcal{M}, R\Gamma_Z(\mathcal{M})[2]) \simeq \mathbb{C}$.

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