Non-equilibrium statistical field theory for classical particles: Initially correlated grand canonical ensembles

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It was recently shown in Bartelmann et al. [1] how correlated initial conditions can be introduced into the statistical field theory for classical particles pioneered by Das and Mazenko [2]. In this paper we extend this development from the canonical to the grand canonical ensemble for a system satisfying statistical homogeneity and isotropy. We do this by translating the probability distribution for the initial phase space coordinates of the particles into an easy diagrammatic representation and then using a variant of the Mayer cluster expansion to sum over particle numbers. The grand canonical generating functional is then used in a structured approach to the derivation of the non-interacting cumulants of the two core collective fields, the density \( \rho \) and the response field \( B \). As a side-product we find several theorems pertaining to these cumulants which will be useful when investigating the interacting regime of the theory in future work.

I. INTRODUCTION

This work is part of a series aiming at finding a new approach to the problem of non-equilibrium particle kinetics using a statistical field theory for classical particles. While this work was begun with an application to cosmological structure formation in mind, we feel that our developments may be useful in other fields of statistical physics which is why we will try to keep our discussions as general as possible and only make the connection with cosmology when we deem it necessary.

The statistical field theory which forms the basis for our work was developed by Mazenko [3, 4] and Das and Mazenko [2, 5]. The basic premise of their work was to use the path integral approach for classical mechanics (cf. Martin et al. [6], Gozzi et al. [7], Penco and Mauro [8]) in order to describe the microscopic degrees of freedom of individual particles. Any macroscopic field is then collectively constructed from the microscopic information. This has several advantages over the standard approach of applying the path integral formalism directly to an effective theory for the macroscopic fields which were already discussed in Bartelmann et al. [1].

Two prominent examples are the relative structural simplicity of the equations of motion and the fact that multi-streaming does not pose a problem because the macroscopic fields are only assembled from the microscopic degrees of freedom at the time of interest by applying collective field operators.

While the work of Das and Mazenko on the theory was extensive they mostly concentrated on treating fluctuations around an equilibrium state with the help of fluctuation-dissipation relations. However, the theory gives a lot of freedom in choosing the initial state of the system and is thus applicable to a very wide range of problems. In Bartelmann et al. [1] we thus explored how to develop the non-equilibrium statistics of a system with correlations between the initial positions and momenta of the particles. This was done in the framework of a canonical ensemble with a fixed particle number. Two-particle interactions were implemented as a straightforward perturbation series in the interaction potential. We could then show in Bartelmann et al. [9, 10] that by describing the ‘free motion’ partly with Zel’dovich trajectories, the non-linear growth of the CDM power spectrum known from N-body simulations could be mimicked over a remarkable range of scales.

This work will extend the treatment of initially correlated systems with statistical homogeneity and isotropy to the grand canonical ensemble. We will see that this automatically leads to a formulation of the theory in terms of the connected n-point functions or cumulants of the collective fields and allows for a structured approach to their calculation. By contrast, in the canonical ensemble the n-point correlators seem to be the more natural quantities and a formulation in terms of cumulants must be obtained by hand which becomes cumbersome for higher orders of perturbation theory. In our next paper [11] we will then use the findings of this work to extend the self-consistent perturbation theory developed by Das and Mazenko to include initial correlations. It will allow us to obtain the linear growth of the CDM powerspectrum familiar from standard Eulerian perturbation theory directly from the theory itself without resorting to using Zel’dovich trajectories, but rather using the actual free Hamiltonian trajectories and the unmodified Newtonian potential. This is possible due to the fact that the grand canonical perturbation series sums up infinite classes of diagrams from the canonical one.

The outline of this paper is as follows. We begin in Sect. 2 by giving a clear definition of the grand canonical generating functional for a system satisfying statistical homogeneity and isotropy. In Sect. 3 we implement the initial phase space probability distribution of Bartelmann et al. [1] and express it in terms of a simple diagrammatic language. A technique known as the Mayer cluster expansion is then used to factorize the generating functional such that the summation over particle numbers can be performed exactly. In Sect. 4 we develop a systematic approach for managing the remaining combinatorics in deriving the non-interacting cumulants which is however reduced when compared to the canonical approach. Along the way we derive some general theorems for these cumulants which will prove helpful for dealing with perturbation theory in Fabis et al. [11].

While many of the quantities we will encounter have already been described in [2, 5] we will try to stay as close as possible to the notation introduced in [1] throughout this work. We refer the reader to the latter whenever we use some quantity without giving an explicit definition.
II. GRAND CANONICAL GENERATING FUNCTIONAL

A. Definition

For the following discussion it will be helpful to think of a generating functional as the normalization factor of some probability density $\mathcal{P}$ for the $6N$ phase space coordinates $x$ of a collection of $N$ particles in some volume $V$. The basic concept behind the theory is to generalize the canonical ensemble from the equilibrium Boltzmann distribution to any initial distribution $P_i$ and then to fix its evolution up to some arbitrary final time $t_f$ by requiring it to follow the classical trajectories $x^{cl}(t; x(t_i))$, i.e. the solution to some equations of motion which is determined by choosing of an initial state $x(t_i)$. This results in a new phase space probability density

$$\mathcal{P}_{P_i,N,V} [x(t_i), x(t_f)] = P_i(x(t_i)) \int Dx \delta_D \left[ x(t) - x^{cl}(t; x(t_i)) \right].$$

(1)

The canonical generating functional is obtained by integrating over both the initial and final states. In this framework defining the grand canonical ensemble is conceptually an easy step. We give up the notion of a fixed number of particles $N$ and replace it with an arbitrary probability distribution for the number of particles $P_N$. Using conditional probabilities we may write

$$\mathcal{P}_{P_i,N,V} [x(t_i), x(t_f), N] = \mathcal{P}_{P_i,N,V} [x(t_i), x(t_f)|N] \ P_N(N).$$

(2)

However, we know that the first factor must be the probability density in the canonical case where the particle number $N$ is fixed

$$\mathcal{P}_{P_i,N,V} [x(t_i), x(t_f), N] = \mathcal{P}_{P_i,N,V} [x(t_i), x(t_f)] \ P_N(N).$$

(3)

With this we may define the grand canonical partition functional simply as

$$Z_{gc} = \sum_{N=0}^{\infty} \int Dx(t) P_i(x(t_i)) P_N(N) \delta_D \left[ x(t) - x^{cl}(t; x(t_i)) \right],$$

(4)

where we absorbed the integration over initial and final states into the path integral.

B. Particle number probability distribution

We now need to specify the probability density $P_N$ for the number of particles. In order to do so we use the familiar textbook approach of embedding our system $S_{gc}$ into a much larger canonical system $S_c$. The grand canonical system $S_{gc}$ may exchange particles with its complement in $S_c$, and particles may interact across the boundary enclosing $S_{gc}$.

The standard approach for obtaining $P_N$ in equilibrium statistical physics would be to introduce a chemical potential defining the amount of energy needed for a single particle exchange between the two systems. Since we do not require equilibrium it would be in general quite hard to define this quantity since it might in principle depend on the entire instantaneous phase space configuration $x(t_i)$. Out of equilibrium we also lack a temperature defining an energy scale with which we can compare the chemical potential.

We can circumvent this problem by restricting ourselves to the case where the canonical system $S_c$ is statistically homogeneous and isotropic, i.e. its statistical properties on all scales of interest are invariant under translation and rotation at all times. This then means that the probability for finding an individual particle somewhere inside $S_c$ must be equal everywhere regardless of any kind of interactions or correlations. From this we can then immediately conclude that the probability $p$ for finding a particle inside the subsystem $S_{gc}$ is given by

$$p = \frac{V_{gc}}{V_c}. \quad (5)$$

The probability for $N_{gc}$ particles to be in $S_{gc}$ is thus given by a binomial distribution

$$N_{gc} \sim B \left( N_c, p = \frac{V_{gc}}{V_c} \right). \quad (6)$$

We now take the ‘thermodynamic’ limit of $N_c, V_c \to \infty$ while keeping the mean particle density $\bar{\rho} = N_c/V_c$ and thus the mean mass density constant. In this limit $p \to 0$ and we may approximate (6) by a Poisson distribution as

$$N_{gc} \sim B \left( N_c, p = \frac{V_{gc}}{V_c} \right) \to \mathcal{P} \left( N_c p = \bar{\rho} V_{gc} = \langle N_{gc} \rangle \right). \quad (7)$$

We now ignore the surrounding canonical system and drop the suffix ‘gc’. Pending normalization we may thus state

$$P_N(N) = \frac{\langle N \rangle^N}{N!} = \frac{\bar{\rho}^N V^N}{N!}. \quad (8)$$
III. INITIAL CORRELATIONS

A. Preliminaries

We now insert (8) into (4) and perform all the steps described in Bartelmann et al. 11. We rewrite the functional Dirac delta distribution as a Fourier transform and separate the

time dependence density-density correlations and density-momentum correlated sources are defined as

\[ Z_{\phi}[H, J, K] = e^{iS_{\Phi}} \sum_{N=0}^{\infty} e^{iH \cdot \phi} \int dp^{(i)} \frac{\hat{p}^{N} V^{N}}{N!} \frac{V^{-N}}{\sqrt{(2\pi)^{3N} \det C_{pp}}} \left( \hat{C} \left( \frac{\partial}{\partial \phi^{(i)}} \right) \exp \left( -\frac{1}{2} \bar{p}^{(i)T} C_{pp}^{-1} p^{(i)} \right) \right) e^{i\langle J_i, q^{(i)} \rangle} e^{i\langle \hat{J}_r, \rho^{(i)} \rangle} e^{-iS^{(i)}_{\Phi}[J,K]} . \]  

(9)

Since the interaction operator can be pulled out in front of the entire expression we only need to concern ourselves with the free grand canonical generating functional for the remainder of this work. We directly see that the power of the free volume \( V \) cancels nicely. The quantity \( \hat{C} \) is a polynomial operator containing density-density correlations and density-momentum correlations. Its explicit form will be given later in (23). Throughout the paper we will make implicit use of the fact that both these types of correlations as well as \( C_{pp} \) depend on the initial positions of particles, but not on their momenta, which is due to the initial momenta being fixed to the value of an initial random momentum field. The time averaged sources are defined as

\[ \hat{J}_{p,p} = \int_{t_i}^{t_f} dt \hat{J}(t)^{T} \hat{G}(t_i, t) \hat{P}_{p,p} \]  

(10)

with the projection operators

\[ \hat{P}_q = \begin{pmatrix} I_3 \\ 0 \end{pmatrix} \otimes I_N , \quad \hat{P}_q = \begin{pmatrix} 0 \\ I_3 \end{pmatrix} \otimes I_N , \]  

(11)

which take care of selecting either the position or momentum part from the right hand side of the free \( N \)-particle propagator \( \hat{G} = G \otimes I_N \), where \( G \) is the single particle free propagator, i.e. the Green’s function of the free equations of motion of a single particle. The purely source dependent action term \( S^{(N)}_{K} \) is given by

\[ S^{(N)}_{K}[J, K] = \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \hat{J}(t)^{T} \hat{G}(t, t') \hat{K}(t) . \]  

(12)

The angular brackets in (9) are not averages but define the scalar product for the tensors bundling properties of all \( N \) particles such that

\[ \langle A, B \rangle = \sum_{j=1}^{N} \vec{A}_j \cdot \vec{B}_j . \]  

(13)

theory into a free and an interacting part. We then introduce collective fields and their respective source fields \( \hat{S}_l \), which allows us to express the interaction by an operator \( \hat{S}_l \), which is independent of particle number. Then we also express the collective fields by an operator \( \Phi \), which enables us to execute the path integrals using the solution to the free equations of motion. The complete generating functional then reads

We may now apply the appropriate number of partial integrations to change all partial derivatives w.r.t. initial momenta present in the \( \hat{C} \) operator from acting on the Gaussian exponential to act on the phase factor \( e^{i\langle J_r, \rho^{(i)} \rangle} \) instead, picking up a minus sign for every partial integration. The fact that \( \hat{C} \) is polynomial and \( C_{pp} \) is positive-definite ensures that all boundary terms vanish. Then we execute all these derivatives to obtain

\[ \int dp^{(i)} \left( \hat{C} \left( \frac{\partial}{\partial \rho^{(i)}} \right) \exp \left( -\frac{1}{2} p^{(i)T} C_{pp}^{-1} p^{(i)} \right) \right) e^{i\langle J_r, \rho^{(i)} \rangle} \]  

\[ = \int dp^{(i)} \exp \left( -\frac{1}{2} p^{(i)T} C_{pp}^{-1} p^{(i)} \right) \hat{C}(-\hat{J}_p) e^{i\langle J_r, \rho^{(i)} \rangle} . \]  

(14)

The remaining integration over initial momenta is now a 3\( N \)-dimensional Fourier transform from \( p^{(i)} \) to \( \hat{J}_p \) which gives

\[ \int dp^{(i)} \frac{1}{\sqrt{(2\pi)^{3N} \det C_{pp}}} \exp \left( -\frac{1}{2} p^{(i)T} C_{pp}^{-1} p^{(i)} \right) e^{i\langle J_r, \rho^{(i)} \rangle} \]  

\[ = \exp \left( -\frac{1}{2} \hat{J}_p C_{pp} \hat{J}_p \right) . \]  

(15)

In this form we can easily split up \( C_{pp} \) into its diagonal part containing the auto-correlations of the momenta of individual particles and the remaining trace-free part which contains only cross-correlations between momenta of different particles and has entries of \( 3 \times 3 \) dimensional zero matrices on the diagonal. The statistical homogeneity of our system dictates that all entries on the diagonal must be independent of particle position and thus spatially constant. We may thus split \( C_{pp} \) as

\[ C_{pp} = \sigma_0^2 I_3 \otimes I_N + \sum_{j \neq k} \left( \rho_{j}^{(i)} \otimes \rho_{k}^{(i)} \right) \otimes E_{jk} \]  

(16)

where \( E_{jk} = e_i \otimes e_k \). When we later specialize to a curl-free initial velocity field we find \( \sigma_0^2 = \alpha \sigma_3^2 / 3 \) where \( \alpha \) is the constant conversion factor between velocity and momentum. For later use we define the second term as the cross-correlation matrix

\[ C_{pp}^{\ast} := \sum_{j \neq k} \left( \rho_{j}^{(i)} \otimes \rho_{k}^{(i)} \right) \otimes E_{jk} = \sum_{j \neq k} C_{p,p_0} \otimes E_{jk} \]  

(17)
Observe that in the notation of Bartelmann et al. \cite{Bartelmann} \( C_{p,p} = (B_{pp})_{jk} \). The diagonal part can be used to reintroduce the integral over initial momenta by reversing the Fourier transform. This leads to

\[
\exp \left\{ -\frac{1}{2} \hat{J}_p \left( \sigma_p^2 I_{3N} \right) \hat{J}_p \right\} = \int \mathcal{D}p^{(i)} \exp \left\{ -\frac{1}{2} \frac{\delta^2}{\hbar^2} \right\} e^{i(\hat{J}_p \cdot p^{(i)})} \exp \left\{ -\frac{1}{2} \frac{\delta^2}{\hbar^2} \right\} e^{i(\hat{J}_p \cdot p^{(i)})},
\]

where the \( p^{MB}_{\sigma_p} \) is now the Maxwell-Boltzmann distribution with momentum dispersion \( \sigma_p \). This is of course due to the fact that we have chosen a Gaussian random velocity field in the first place and just separated off the cross-correlations. While reintroducing an already performed operation may seem as a step backwards it will have the very desirable effect that we again have the complete free solution of the path integrals in the generating functional such that functional derivatives w.r.t. the sources \( J, K \) can be replaced by the phase-space quantities \( \chi, \chi \) conjugate to these sources. This will be important for performing the Mayer cluster expansion. Finally, we note that we may replace the time averaged source \( \bar{J}_p \) that appear in both \( \hat{C} \) and the momentum cross-correlation Gaussian as a functional derivative by using the relation

\[
-\bar{J}_p e^{-iS^{(2)}_p[J,K]} = \frac{\delta}{\hbar |K_p(t_i)|} e^{-iS^{(2)}_p[J,K]}.
\]

After these manipulations the free grand canonical generating functional has the form shown in \cite{Bartelmann}. Interestingly this expression for the initially correlated set of particles shows that we can obtain its generating functional by first finding the functional of an ideal gas where particles are initially uncorrelated in configuration space and have a Maxwell-Boltzmann distribution in momentum space. The initial correlations can then be induced by applying suitable operators.

\[
Z_{gc,n}[H, \mathbf{J}, \mathbf{K}] = \sum_{N=0}^{\infty} e^{iH\phi} \hat{C} \left( \frac{\delta}{\hbar |K_p(t_i)|} \right) \exp \left\{ -\frac{1}{2} \left( \frac{\delta}{\hbar |K_p(t_i)|} \right)^T C^{*}_{pp} \left( \frac{\delta}{\hbar |K_p(t_i)|} \right) \right\}
\]

\[
= \hbar N! \int dq^{(i)} \int \mathcal{D}p^{(i)} p^{MB}_{\sigma_p} \left( \mathbf{p}^{(i)} \right) e^{i(\hat{J}_p \cdot p^{(i)})} e^{-iS^{(2)}_p[J,K]}.
\]

\section{Diagrammatic representation of the initial correlations}

In the above form \cite{Bartelmann} the sum over particle numbers can not be performed straightforwardly because the initial correlations prevent the factorization of the free generating functional into single particle contributions. We thus organise the partition sum, representing the correlations in diagrammatic form. We begin with the momentum cross-correlations and rewrite the Gaussian factor in \cite{Bartelmann} in the following way, which may be recognized as the first step of a Mayer cluster expansion:

\[
\exp \left\{ -\frac{1}{2} C^{*}_{pp} \left( \frac{\delta}{\hbar |K_p(t_i)|} \right) \right\} = \prod_{j=1}^{N} \left\{ 1 + \exp \left\{ -\frac{\delta^2}{\hbar^2 |K_p(t_i)|} C_{p,p} \left( \frac{\delta}{\hbar |K_p(t_i)|} \right) \right\} - 1 \right\}
\]

\[
= \prod_{\{j,k\}} \left\{ 1 + \hat{C}_{p,p} \right\}.
\]

In the second line we expanded the quadratic form explicitly into a sum over all the different particles using the definition of \( C_{pp} \) from \cite{Bartelmann}. In the third line, we used the symmetry \( C_{p,p} = C_{p,p} \) to express the double sum as a sum over all different pairs \( \{j,k\} \). Each pair only appears once in the sum, e.g. \{1,2\} and \{2,1\} are considered equivalent and only one of them is summed over. In the fourth line, we defined a new scalar operator \( \hat{C}_{p,p} \), describing the momentum correlation between two particles. In Fig. \cite{Bartelmann} we now represent this new operator as a dashed line connecting two dots which represent particles \( i \) and \( j \). We write out the product \cite{Bartelmann} in terms of sums over different \( n \)-tuples of different particle pairs as

\[
\prod_{\{i,j\}} \left\{ 1 + C_{p,p} \right\} = 1 + \sum_{\{i,j\}} C_{p,p} + \sum_{\{i,j,l\}} C_{p,p} C_{p,p} + \cdots \quad (22)
\]

The tuples are different in the same sense as the particle pairs, for example \{1,2,3,4\} and \{3,4,2,1\} are considered equivalent, while \{1,3,2,4\} is different. The star indicates that the pairs \( \{i,j\} \) and \( \{k,l\} \) may not be the same. This means that \{1,2,2,1\} is excluded, while \{1,2,2,3\} is included. This scheme extends to all \( n \)-tuples, leading to the only restriction on the topology of diagrams:

\textbf{Rule 1:} Any pair of particles \( i \) and \( j \) may only be connected by at most one \( C_{p,p} \)-line.

We can now easily express the above sums in a diagrammatic form by going through all particle numbers \( 1 < n < N \) and
summed over, e.g., \( \{2,3,1\} \) is equivalent to \( \{1,2,3\} \) and not counted extra. The same holds for all other permutations of \( \{1,2,3\} \). Because of this, we had to draw three diagrams in Fig. 3 that are topologically identical which appears to be cumbersome. However, this has the very desirable effect of making the above expression invariant under particle exchange. This property will become important later on.

The correlation operator \( \hat{C} \) was derived in the appendix of Bartelmann et al. \[1\]. We first need to expand it explicitly with all the contributions from the individual particles appearing.

\[
\hat{C} = \prod_{i,j}(1 + \hat{C}_{\delta_{\delta}}) = 1 + \delta_{\delta}
\]

\[
+ \Sigma_{\{i,j\}} \delta_{\delta} + \Sigma_{\{i,j,k\}} \delta_{\delta} + \Sigma_{\{i,j,k,l\}} \delta_{\delta} + \cdots
\]

Keep in mind that while \( C_{p,p} \) is a matrix, \( C_{\delta_{\delta}} \) is a scalar quantity. The restriction \( n \neq l \) on the inner sums follows from \( \hat{C}_{\delta_{\delta}} = 0 \), which is a consequence of the statistical homogeneity and isotropy of the system. Primes on the sums mean that any particle index may appear only in one pair of the \( n \)-tuple of pairs. For example, \( \{1,2,\{2,3\}\} \) is forbidden. Primes on the product index mean that all particle indices present in the term of the preceding sum are excluded. Now define the new operator

\[
\hat{C}_{\delta_{\delta}} = -i \hat{C}_{\delta_{\delta}} \cdot \frac{\delta}{i \delta K_{\delta}}.
\]  

For this operator and \( C_{\delta_{\delta}} \), we introduce line diagram representations as basic building blocks in Fig. 4. We again expand the products into multiple sums and then reorder the entire expression in terms of the number of particles being correlated.

This means that we can express \( \hat{C} \) by going through all particle numbers \( \ell \) up to \( N \) and drawing for each \( \ell \) all diagrams that conform with a set of rules that we read off (23) and then summing these diagrams over all \( \ell \)-tupels of particles.
drawn diagrams with two particles here which must of course be connected. However, as soon as we arrive at four particles disconnected diagrams appear, the simplest example being two solid lines representing for example \( C_{\phi_1\phi_2} C_{\delta_3\delta_4} \).

\[
C_{\text{tot}} = 1 + \sum_{\{i,j\},\{k,l\}} C_{ij}^{(4)} C_{kl}^{(4)} + \sum_{\{i,j,k\}} C_{ijk}^{(3)} + \sum_{\{i,j,k,l\}} C_{ijkl}^{(2)} + \ldots
\]

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**FIG. 7.** Diagrammatic expansion of the total correlation operator up to two particles.

### C. Mayer cluster expansion

The Mayer cluster expansion is a technique first introduced in Mayer and Montroll [12]. Since we already have introduced a diagrammatic form for our generating functional we will use a suitable formulation found in Becker [13], adopting most of its notation and reasoning. The cluster expansion is normally used when one wants to compute the partition sum of an interacting gas in equilibrium statistical physics. The configuration integral cannot be performed exactly due to the interactions. The Mayer cluster expansion is then employed to expand the configuration integral into a perturbative series ordered by the number of particles taking part in the interaction allowing to execute the spatial integral at least numerically. A common application is then to derive expressions for the coefficients in the virial expansion of the equation of state, which corrects the non-interacting equation of state by terms of higher than quadratic order in the mean density.

As mentioned before, in our case the interaction has been separated into an operator and can be ignored for now. The Mayer cluster expansion is however more general in nature and may be applied to our problem of managing the initial correlations. As a first step we remind ourselves of the following result derived Eq. (35) of Bartelmann et al. [11] for the free N-particle generating functional, where we choose to leave out the averaging over the initial phase-space state \( x^{(0)} \):

\[
Z_0^{(N)}[J,K] := \int D\bar{x}(t) \int D\chi(t) e^{i S_0 + i \frac{1}{\hbar} \int dt \left[ J(0) x(t) + K(0) \chi(t) \right]} = e^{i \langle J, q^{(0)} \rangle} e^{i \langle J, p^{(0)} \rangle} e^{-i S_0^{(2)}[J,K]},
\]

where \( S_0 \) is the action containing only the free equations of motion. Inserting the first line into (27), we may pull both the collective field operator \( \Phi \) and our total correlation operator \( C_{\text{tot}} \) back under the path integrals and execute the functional derivatives contained in them. This replaces the collective field operator \( \Phi \) with the actual collective field \( \Phi \) and the argument of \( C_{\text{tot}} \) with

\[
\frac{\delta}{\delta \Phi_p(t_i)} \to \chi_p(t_i) .
\]

For now, we set the source terms \( J, K \) to zero because they are irrelevant for the following steps. They will be reintroduced later. With the exception of \( C_{\text{tot}} \) all other parts of the generating functional can now be factorized into contributions from single particles. For the differentials in the integral over the initial state and in the path integrals this is trivial and for the Maxwell-Boltzmann distribution we have

\[
P_{\sigma_p}^{\text{MB}}(p^{(i)}) = \prod_{j=1}^{N} \frac{e^{\frac{\sigma_p^2}{2} p^{(i,j)2}}}{\sqrt{(2 \pi \sigma_p^2)^3}} = \prod_{j=1}^{N} P_{\sigma_p}^{\text{MB}}(p^{(i)\sigma_p} \cdot \langle p^{(i)\sigma_p} \rangle) .
\]

By definition (cf. Bartelmann et al. [11]) the collective field \( \Phi \) can be separated into one particle contributions. This must also hold for the free action,

\[
S_0 = \sum_{j=1}^{N} S_{0,j} , \quad \Phi(1) = \sum_{j=1}^{N} \phi_j(1) .
\]

We can now gather all one-particle contributions into a trace operator defined as

\[
\text{Tr}_j := \bar{\rho} \int d^3 \bar{q}_j^{(i)} \int d^3 p_j^{(i)} P_{\sigma_p}^{\text{MB}}(p_j^{(i)}) \int D\nabla \chi_j^{(i)} \int D\nabla \phi_j^{(i)} e^{i S_{0,j} + \phi_j^{(i)} + H \phi_j} .
\]

and subsequently use this to write the non-interacting grand canonical generating functional in the very compact form

\[
Z_{\text{gc},0}[H] = \sum_{N=0}^{\infty} \frac{1}{N!} \left( \prod_{j=1}^{N} \text{Tr}_j^{(N)} \right) C^{(N)}_{\text{tot}}(\chi_p(t_i)) .
\]

Now consider a set of \( N \) particles. We want to separate the particles into a collection \( \{c_\ell\} \) of subsets containing \( \ell \) particles each. One such subset \( c_\ell \) is called a ‘cluster of size \( \ell \)’. Each such collection \( \{c_\ell\} \) also has a set of numbers \( \{m_\ell\} \) with \( m_\ell \) being the number of clusters of size \( \ell \) in \( \{c_\ell\} \). A set \( \{m_\ell\} \) is called a ‘cluster configuration’, where we only care about how many clusters of size \( \ell \) there are and not about which actual particles are in which clusters. Clearly, every cluster configuration \( \{m_\ell\} \) has different ‘realisations’ \( \{c_\ell\} \). Any configuration must of course obey the constraint

\[
\sum_{\ell=0}^{N} m_\ell \cdot \ell = N .
\]

In our diagrammatic language such a configuration corresponds to a **clustering pattern** of particle dots. This means that we group dots into clusters by drawing the dots with a clear separation between clusters. A realisation \( \{c_\ell\} \) of such a configuration is then given by specifying which actual particle is
assigned to which dot. As an example we pick a set of \( N = 6 \) particles and the realisation \( (1, 2, 3)(4, 5)(6) \) of the cluster configuration \( \{ m_1 = 1, m_2 = 1, m_3 = 1, m_4 = m_5 = m_6 = 0 \} \). This realisation corresponds to the fixed clustering pattern shown in Fig. 8. The notion of ‘fixed’ means that one needs to map the particle indices of the sequence \( (1, 2, 3)(4, 5)(6) \) to the dots by some bijective mapping. Once this is done, the pattern may not be changed in any way since this would correspond to another clustering realisation like e.g. \( (2, 1, 3)(4, 5)(6) \), where particles 1 and 2 have switched places. The actual clustering is now achieved by drawing the lines of Figs. 2, 4 between the particles of the individual clusters such that each cluster is the sum of all connected diagrams invariant. Each particle exchange would thus give a contribution \( \hat{\chi}_{p^3} \delta_1 \delta_2 \delta_3 \) \( \hat{\chi}_{p^1 p^2} \delta_4 \delta_5 \) \( \hat{\chi}_{p^4} \delta_6 \delta_7 \). For any realisation of a given cluster configuration as a fixed pattern we may thus go through all clusters of the realisation individually while keeping the correlation functions in the other clusters fixed, sum all connected diagrams compatible with the rules from the previous section for each cluster and then apply the trace operators \( \text{Tr}_i \). For the example of Figs. 9, 10 we would start with the 3-cluster. While holding the \( \hat{\chi}_{\text{total}} \) in the 2-cluster fixed, we first sum the two diagrams shown and then continue to sum all remaining connected 3-particle diagrams shown in Figs. 3, 6. This sum is defined as a single factor and then held constant while we sum the nine possible 2-particle diagrams of Fig. 7 in the 2-cluster. The 1-particle dot is trivially connected. In general, if we define

\[
\Sigma_\ell = \frac{1}{\ell!} \text{Tr}_{j_1} \cdots \text{Tr}_{j_\ell} \hat{\chi}_{p^j_{\ell}} \left( \chi_{p^j_{\ell}}(t_i) \right),
\]

where \( \hat{\chi}_{p^j_{\ell}} \) is the sum of all connected \( \ell \)-point diagrams, then the contribution from all diagrams in \( \hat{\chi}_{\text{total}} \) belonging to a specific realisation of an \( N \)-particle cluster configuration \( \{ m_\ell \} \) can be factorized as

\[
\prod_{\ell=0}^{N} \left( \ell! \Sigma_\ell \right)^{m_\ell}.
\]

For example, since one cannot have a disconnected correlation of two particles, \( \Sigma_0 \) can be seen as the 2-particle sum in Fig. 7. The \( (1, 2, 3)(4, 5)(6) \) realisation would give the contribution \( (1 \cdot 1 \cdot 2 \cdot \Sigma_3) (6 \cdot \Sigma_1) \). However, we now have to think about about all other possible realisations of a given cluster configuration \( \{ m_\ell \} \). While they all give the same contribution \( \Sigma_3 \) due to the trace operators and the summing of all connected diagrams, we still need to know how many of them are actually present in the expansion of \( \hat{\chi}_{\text{total}} \). In principle there are \( N! = 6! = 720 \) such realisations since this is just the question of arranging the particles in a certain order once we have drawn the clustering pattern and fixed the bijective mapping. However, there are equivalent realisations which lead to overcounting. We need to take two things into account:

- Exchanging the order of particles inside a cluster leaves the sum over all connected diagrams of the cluster invariant. Each particle exchange would thus give a new term that has already been accounted for, so we need to divide out all possible orderings within clusters. For a given \( N \)-particle cluster configuration these are \( \prod_{\ell=0}^{N} \ell! \Sigma_\ell \). For two clusters of equal size exchanging all particles between them also leaves their respective sums over connected diagrams invariant. For an arbitrary number of equally sized clusters this amounts to reordering the sequence of these clusters in the clustering pattern and thus to \( m_\ell \) possibilities.

Accounting for these restrictions, we may now rewrite the traces over \( \hat{\chi}_{\text{total}} \) from (31) as a sum over cluster configurations

\[
\prod_{\ell=0}^{N} \left( \ell! \Sigma_\ell \right)^{m_\ell}.
\]

\[
= \left( \text{Tr}_1 \text{Tr}_2 \text{Tr}_3 \hat{\chi}_{p^1 \delta_1} \hat{\chi}_{p^2 \delta_2} \right) \left( \text{Tr}_4 \text{Tr}_5 \text{Tr}_6 \hat{\chi}_{p^4 \delta_4} \hat{\chi}_{p^5 \delta_5} \right) \left[ \text{Tr}_6 \right].
\]
\( \{ m_\ell \} \) subject to the constraint (32). This allows writing the free grand canonical partition functional into

\[
Z_{gc,0}[H] = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\{m_\ell\}} \frac{N!}{m!} \prod_{\ell=0}^{N} (\ell! \Sigma_\ell) \prod_{\ell=0}^{N} (\ell! \Sigma_\ell)^{m_\ell} \\
= \sum_{N=0}^{\infty} \sum_{\{m_\ell\}} \prod_{\ell=0}^{\infty} (\Sigma_\ell)^{m_\ell} m_\ell!
\]

\[
= \prod_{\ell=0}^{\infty} \exp \{ \sum_{\ell=0}^{\infty} \Sigma_\ell \}.
\]

\[ (37) \]

Going from the second to the third line we use that first summing over all cluster configurations with the \( N \)-particle constraint \( \{m_\ell\} \) and then summing over all particle numbers is identical to summing over all cluster configurations without this constraint. In the third line \( \ell' \) runs over all cluster sizes present in the preceding cluster configuration. We then reorder the sum not in terms of cluster configurations but in terms of cluster size.

Readers familiar with QFT will recognize the above equation (37) as what is often called exponentiation of disconnected diagrams. The reason that we obtain this familiar structure for the generating functional is that the topological principles behind the Mayer cluster expansion and the Feynman diagrams of QFT are the same. Showing how these principles can be implemented for the initial correlations between particles sampling a Gaussian random field, rather than for the particle interactions, should be seen as one of the main achievements of this paper. We also mention that it should in principle be possible to include slightly non-Gaussian fields into this formalism, since higher order cumulants of the random field would be the equivalent of vertices in our diagrammatic language.

Since it does not matter which actual \( \ell \) particles are used in calculating \( \Sigma_\ell \) we can always think of some representative set of \( \ell \)-particles for which the trace operators in (35) are defined. We include the sources \( J, K \) defined for this representative set by modifying the trace operators as

\[
\text{Tr}_J \rightarrow \text{Tr}_J \exp \left\{ i \int_{t_i}^{t_f} dt \left( J_f(t) \cdot \dot{X}_f(t) + K_f(t) \cdot \dot{X}_f(t) \right) \right\}.
\]

\[ (38) \]

The complete grand canonical generating functional is now easily obtained by applying the interaction operator

\[
Z_{gc}[H, J, K] = e^{\hat{S}_I} \exp \left\{ \sum_{\ell=0}^{\infty} \Sigma_\ell[H, J, K] \right\}.
\]

\[ (39) \]

### IV. Non-Interacting Cumulants

#### A. General form

Having found the general form of \( Z_{gc,0} \), we will now present a scheme how the connected non-interacting correlators of the collective field \( \Phi \) can be derived. For the remainder of this paper we will call these \( n \)-point cumulants. Their general definition is given by

\[
G_{\Phi_{n_1} \cdots \Phi_{n_\ell}}(1, \ldots, n) = \frac{\delta}{\delta \hat{H}_{\alpha_1}(1)} \cdots \frac{\delta}{\delta \hat{H}_{\alpha_\ell}(n)}
\ln Z_{gc}[H, J, K] \bigg|_{H=J=K=0},
\]

\[ (40) \]

where we use the notation \( 1 = (t_1, \vec{k}_1) \) to bundle time and Fourier space coordinates into an ‘external label’. We further shorten the notation by understanding that \( \Phi_{n_\ell} \) means the collective field of type \( \alpha \) evaluated at the external label 1. In the non-interacting case of \( \delta_1 = 0 \) the exponential form of \( Z_{gc,0} \) cancels against the logarithm. Due to the ordering by the number \( \ell \) of representative particles it will be advantageous to also separate \( G_{\Phi_{n_1} \cdots \Phi_{n_\ell}}^{(0)} \) into its contributions from \( \ell \) particles as

\[
G_{\Phi_{n_1} \cdots \Phi_{n_\ell}}^{(0)} = \frac{\delta}{\delta \hat{H}_{\alpha_1}(1)} \cdots \frac{\delta}{\delta \hat{H}_{\alpha_\ell}(n)} \ln Z_0[H, J, K] \bigg|_{H=J=K=0}
\]

\[ (41) \]

Observe that all collective quantities are now intrinsically defined for the \( \ell \) representative particles that appear in \( \Sigma_\ell \). In the second line we returned to expressing the collective fields as operators whose single-particle contributions in Fourier space read (cf. Eqs. (54),(58) in [1])

\[
\hat{\Phi}_\ell(\vec{k}_1, t_1) = \left\{ \begin{array}{ll}
\frac{\delta}{\delta \hat{H}_{\alpha_1}(1)} & \\
\frac{\delta}{\delta \hat{H}_{\alpha_\ell}(n)} & \\
\left. \exp \left\{ -i \vec{k}_1 \cdot \delta_\ell \right\} \right| \end{array} \right.
\]

\[ (42) \]

acting on \( Z_0[H, J, K] \) which is the deterministic canonical generating functional [26], only defined for the \( \ell \) representative particles. Furthermore, we again used [27] to express the factors of \( \delta_\ell \) in the initial correlations as functional derivatives leading to the operator \( \delta_\ell \) which absorbed the factor \( \frac{1}{\sqrt{2}} \) in (35). The integration over initial conditions is only over the ideal gas part

\[
\int d\vec{r}_i^{(f)} = \rho \int d\vec{q}^{(i)} \int d\vec{p}^{(i)} \exp \left( \frac{i \vec{p} \cdot \vec{q}}{\sqrt{2} \sigma_F} \right)
\]

\[ (43) \]
to notice that by specifying a maximum order of correlations to be taken into account, one can effectively truncate the series in \( C^{(\ell)}_{\text{con}} \). This is due to the fact that a maximum order of correlations translates into a maximum number of lines in the diagrams making up \( C^{(\ell)}_{\text{con}} \). Since these diagrams are connected one needs at least \( \ell - 1 \) correlation lines to connect \( \ell \) particles. A maximum number \( n \) of correlation lines thus means a truncation at \( \ell = n + 1 \) particles.

B. Effects of collective field and initial correlation operators

The effects of collective field operators were derived in Bartelmann et al. \( \Pi \). One particular advantage of the operator approach is that one may freely choose the order in which the various functional derivatives are to be applied. Since the \( B \) field operator factorizes as \( \hat{b}_j(1) \hat{\phi}_j(1) \) it is advantageous to first calculate density-only cumulants since mixed cumulants between density and response fields can be obtained from them by multiplying with appropriate \( b(1) \) prefactors.

We adopt the notion of a particle \( j \) ‘carrying’ an external label 1 if a single particle operator \( \hat{\phi}_j \) is applied at label 1. Physically this represents the contribution of the particle \( j \) to the Fourier mode \( \vec{k}_j \) of the collective field vector \( \Phi \) at the time \( t_1 \). The effect of applying multiple operators \( \hat{\phi}_j(1) \) all belonging to the same particle \( j \) but with different external labels will result in a shift of the \( J \) source,

\[
\hat{\phi}_j(1) \ldots \hat{\phi}_j(s_j) Z_0^{(\ell)}[J, K] = Z_0^{(\ell)}[J + L_j, K],
\]

with the shift tensor defined as

\[
L_j(t) = -\sum_{\{s_j\}} \delta_0 \left( t - t_{s_j} \right) \left( \vec{k}_j / 0 \right) \mathcal{O} \vec{e}_j
\]

and \( \{s_j\} \) is the set of external labels carried by the particle \( j \). For more than one particle, the shift tensors are added,

\[
L(t) := \sum_{j=1}^{\ell} L_j(t).
\]

Having applied all \( \hat{\phi}_j \), for all particles present, we need to account for the effects of applying \( \hat{b}_j(1) \) and the functional derivatives in \( C^{(\ell)}_{\text{con}} \) to \( Z_0^{(\ell)} \). Application of \( \hat{b}_j(1) \) will result in a \( b_j(1) \) factor:

\[
b_j(1) Z_0^{(\ell)}[J + L, K] = b_j(1) Z_0^{(\ell)}[J + L, K],
\]

\[
b_j(1) = -i \int_t^{t_{1}} dt \left( J(t) + \sum_{\alpha=1}^{\ell} L_{\alpha}(t), G(t, t_1) \right) \left( 0_3 / \vec{k}_1 \right) \mathcal{O} \vec{e}_j.
\]

Once all derivatives have been applied we turn off the sources \( J, K \) and the factor reduces substantially to

\[
b_j(1) = i \vec{k}_1 \cdot \sum_{\{s_j\}} \vec{k}_{s_j} g_{q_{s_j}} (t_{s_j}, t_1).
\]

In complete analogy we can define factors for the result of the application of the initial correlation operators.

\[
c_{b(p_j)} = -i \int \mathcal{O} \vec{e}_j \sum_{\{s_j\}} \vec{k}_{s_j} g_{q_{s_j}} (t_{s_j}, t_1), \tag{49}
\]

\[
c_{b(p_j)} = \exp \left\{ -\left( \sum_{\{s_j\}} \vec{k}_{s_j} g_{q_{s_j}} (t_{s_j}, t_1) \right)^\top \right\} \exp \left\{ -\left( \sum_{\{s_j\}} \vec{k}_{s_j} g_{q_{s_j}} (t_{s_j}, t_1) \right) \right\} - 1. \tag{50}
\]

Finally, the free non-averaged generating functional reduces to

\[
Z_0^{(\ell)}[J, K]_{J=K=0} = \exp \left\{ i \int_t^{t_{1}} dt \left\{ L(t), G(t, t_1) \mathcal{O} \right\} \right\}
\]

\[
= \prod_{j=1}^{\ell} \left\{ -i \left( \sum_{\{s_j\}} g_{q_{s_j}} (t_{s_j}, t_1) \vec{k}_{s_j} \right) \cdot \vec{q}_j \right\}
\]

\[
\exp \left\{ -i \left( \sum_{\{s_j\}} g_{q_{s_j}} (t_{s_j}, t_1) \vec{k}_{s_j} \right) \cdot \vec{q}_j \right\}. \tag{51}
\]

Only two components of the free single particle propagator \( G(t, t') \) are needed namely \( g_{q_1} \) and \( g_{q_2} \). In most systems we have \( g_{q_{s_j}} (t, t_1) = \Theta(t - t_1) = 1 \) since we are only interested in times \( t \geq t_1 \). Thus only \( g_{q_{s_j}} \) remains. For convenience of notation we now define

\[
g_{12} := g_{q_{s_j}} (t_{1}, t_{2}) \quad \text{and} \quad g_1 := g_{q_{s_j}} (t_1, t_1) \tag{52}
\]

Concerning the \( b_j \) factors we can now derive a straightforward theorem that will help us reduce the number of terms that we need to calculate later.

Theorem 1. If in a term contributing to some \( C^{(\ell)}_{\Phi_{s_1} \ldots \Phi_{s_n}} \) a particle ‘\( j \)’ carries only external labels belonging to \( B \)-fields, then the term vanishes.

Proof. This is best done iteratively, beginning with a particle carrying only a single \( B \)-field label 1. Then immediately \( b_j(1) = i k_1^2 \mathcal{O} \mathcal{G}_{s_1} = 0 \) due to \( \mathcal{G}_{s_1} = g_{q_{s_1}} (t_1, t_1) = 0 \). For two \( B \)-fields \( B(1), B(2) \) we have

\[
b_j(1) b_j(2) = \left( i \vec{k}_1 \cdot \vec{k}_2 \right) g_{12} \mathcal{O} \Theta(t_2 - t_1) \Theta(t_1 - t_2).
\]

The only possibility for both Heaviside functions not to vanish would be \( t_1 = t_2 \). But this leads to factors \( g_{q_{s_j}} (t_1, t_1) = g_{q_{s_j}} (t_2, t_2) = 0 \).

In the general case of \( n \) \( B \)-fields we use a diagrammatic argument. Picture every time coordinate included in the external labels as a point. Due to \( \{\mathcal{O} \mathcal{G}_{s_j} \} \) a factor \( b_j(1) \) can be seen as a sum of lines between \( t_1 \) and all other \( n - 1 \) time coordinates \( t_i \), representing the \( g_{13} \). The product \( b_j(1) b_j(2) \ldots b_j(n) \) then
results in a sum of \((n-1)^n\) terms each containing \(n\) lines connecting all \(n\) instances in time. As in the case \(n = 2\), any pair of points connected by two lines, i.e. a closed loop contributes zero. The argument for a 2-point loop is easily extended to a general \(m\)-point loop as
\[
\Theta(t_{i_1} - t_{i_2}) \Theta(t_{i_2} - t_{i_3}) \ldots \Theta(t_{i_{m-1}} - t_{i_m}) \Theta(t_{i_m} - t_{i_1})
\]
\[
\Rightarrow t_{i_1} \geq t_{i_2} \geq t_{i_3} \ldots \geq t_{i_m}
\]
Again this can only be satisfied if all time coordinates are identical which leads to vanishing propagators. But as we need to connect \(n\) instances in time with \(n\) lines in each term of the product \(b_j(1) b_j(2) \ldots b_j(n)\) there is no possibility of doing so without creating a closed loop and thus all terms vanish.

A straightforward corollary that we can immediately derive from this theorem is the following:

**Corollary 2.** Any non-interacting B-field-only cumulant vanishes: \(G^{(0,0)}_{\tilde{B}_1, \ldots, \tilde{B}_n} = 0\).

The above Theorem 1 is merely a consequence of the theory respecting the causality of interactions. We can derive yet another theorem.

**Theorem 3.** If in a term contributing to some \(G^{(0,0)}_{\Phi_{\nu_1} \ldots \Phi_{\nu_n}}\) one of the particles involved, say particle ‘a’, carries no external labels, i.e. no \(\hat{\phi}_{\nu_a}\) has been applied, then the term vanishes.

**Proof.** Since the particle \(a\) is involved in the cumulant it must be connected in a diagrammatic sense with one of the three line types of our diagrammatic representation.

If the particle is ‘inside the diagram’, i.e. connected to more than one line, then the diagram rules demand that at least one of the lines connecting to \(a\) is of the dashed \(p\)-type. Thus either a factor of \(c_{\nu_a \nu_p}\) or \(c_{\nu_p \nu_a}\) is present. But since \(a\) carries no external labels the set \(\{s_a\}\) in (49), (50) is empty and thus \(c_{\nu_a \nu_p} = 0 = c_{\nu_p \nu_a}\).

If the particle is ‘at the boundary of the diagram’, i.e. connected to only one line, then we need to distinguish three different cases.

- The particle is connected to a dashed \(p\)-type line. The same argument as above applies.
- It is connected by a \(C_{\delta, \delta_j}\) line. Since \(\{s_a\}\) is empty we have for the factor from the free generating functional
  \[
  \exp\left\{-i \sum_{\nu_a} \tilde{k}_{\nu_a} \cdot \vec{q}_{\nu_a}^{(0)}\right\} = \exp(0) = 1
  \]
  and thus the only quantity left that depends on \(\vec{q}_{\nu_a}^{(0)}\) is \(C_{\delta, \delta_j}\). This leaves us with
  \[
  \int d^3 \vec{q}_{\nu_a}^{(0)} C_{\delta, \delta_j} = \int d^3 \vec{q}_{\nu_a}^{(0)} \xi(\vec{q}_{\nu_a}^{(0)}, \vec{q}_{\nu_j}^{(0)}) = 0
  \]
  since we define our 2-point correlation function \(\xi\) as the Fourier transform of a power spectrum \(P_{\delta}(k)\) which vanishes at \(k = 0\).

- It is connected to the solid \(\delta\)-side of a \(C_{\delta, \delta_j}\). With the arguments from the previous case we have
  \[
  \int d^3 \vec{q}_{\nu_a}^{(0)} C_{\delta, \delta_j} = \left(\int d^3 \vec{q}_{\nu_a}^{(0)} \delta(\vec{q}_{\nu_a}^{(0)})\right) \bar{\rho}(\vec{q}_{\nu_a}^{(0)}) = 0
  \]
  since the density contrast must obey particle conservation. Notice that \(\bar{\rho}(\vec{q}_{\nu_a}^{(0)})\) is the initial momentum field evaluated at the initial position of the particle \(i\).

\(\square\)

We can now combine Theorems 1 and 3 to derive another corollary.

**Corollary 4.** For all \(\ell > n\), the cumulant \(G^{(0,\ell)}_{\tilde{p}_1 \ldots \tilde{p}_n, \tilde{\rho}_{e1} \ldots \tilde{\rho}_{en}} = 0\).

**Proof.** We begin by considering density-only cumulants first. These are given by

\[
G^{(0,\ell)}_{\tilde{p}_1 \ldots \tilde{p}_n} = \int d\Gamma_i C^{(i)}_{\rho, \bar{J}} \left(\sum_{j=1}^{\ell} \hat{\phi}_{\tilde{p}_j}(1)\right) \ldots \left(\sum_{j=1}^{\ell} \hat{\phi}_{\tilde{p}_j}(n)\right) Z^{(0)}_{\alpha}[\mathbf{J}, \mathbf{K}] |_{\mathbf{J}=\mathbf{K}=0}.
\]

(53)

Expanding the product of operators we get a sum where in each term every external label appears exactly once, i.e. each term consists of \(n\) factors. For \(\ell > n\), there will be at least one particle in each term that will not carry an external label and thus all terms vanish according to Theorem 3. Next we can add any number \(m\) of \(B\)-fields to the correlator. This leads to a sum of terms with \(n + m\) factors. But as \(\ell > n\) for every term we still have at least one particle which either carries no external label which causes the term to vanish or it only carries \(B\)-field indices which also makes the term vanish according to Theorem 1.

\(\square\)

It is also interesting to note that the scaling of any cumulant \(G^{(0,\ell)}_{\Phi_{\nu_1} \ldots \Phi_{\nu_n}}\) with the mean particle density \(\bar{\rho}\) is only controlled by the number of representative particles due to \(\Sigma_{\nu_a} \propto \bar{\rho}\). This relates to the fact that in a continuous fluid picture of the density field any density-only \(n\)-point cumulant \(G^{(0,\ell)}_{\tilde{p}_1 \ldots \tilde{p}_n, \tilde{\rho}_{e1} \ldots \tilde{\rho}_{en}}\) has only terms scaling with \(\bar{\rho}\). In our particle picture we have terms of all possible scalings \(p\) for \(1 \leq \ell \leq n\) due to (41). All terms with \(\ell < n\) may consequently be interpreted as shot noise terms where some of the \(n\) particles have been identified with each other, thus reducing the amount of possible initial correlation.

We can also consider an \(n\)-point density-only cumulant with \(\ell = n\) and then change any number of density fields into response fields. According to Corollary 4 the resultant cumulant will vanish unless we also reduce the number of particles \(\ell\) by one for each \(B\)-field. Just as in the canonical ensemble we thus see that for fixed \(n\)-point order the response field leads to an identification of particles since it encodes how the effects of interactions are transported forward through time by single particles which then may contribute to other collective fields at a later time.
C. Hierarchy of label combinatorics

A general density-only \( \ell \)-particle \( n \)-point cumulant has the form given in (53). We first multiply out all the sums into individual terms. Next we organise them in a hierarchical fashion.

The first level of the hierarchy should tell us how many of the \( n \) external labels are carried by each of the \( \ell \) particles. We will call this category a ‘label distribution’ \( \#_1|\ldots|\#_\ell \). An example for \( \ell = 2 \) would be \( 1|2 \) where one particle carries one label and the other two labels. Note that we do not ask which particle carries the single label, both possibilities are in the same distribution and the same holds for higher \( \ell \).

The next lower level category is called a ‘label grouping’ \( \{s_1|\ldots|s_{\ell}\} \) specifying which specific labels are grouped due to being carried by the same particle. Considering an \( \ell = 3 \) particle \( n = 4 \)-point cumulant a possible label grouping would be \( (1; 2; 3, 4) \) as well as \( (1; 3; 2, 4) \) and both belong to the label distribution \( 1|1|2 \).

The lowest category is a ‘labeling’ representing a single term in (53). For the \( \ell = 3 \) particle \( n = 4 \)-point cumulant with particles \( a, b, c \) one could realise \( (1; 2; 3, 4) \) as e.g. \( a_1b_2c_3c_4, b_1c_2a_3a_4, a_1c_2b_3b_4 \) and so on. The hierarchy for this example can be seen in Fig. 11. The important point to realise is that

- Find all possible label distributions \( \#_1|\ldots|\#_\ell \) where all particles carry at least one label (distributions where one particle does not carry a label vanish due to Theorem 3).
- For each of these identify all possible label groupings and pick an arbitrary representative.
- For this representative label grouping go through all possible labelings, execute the integral over the initial particle positions taking into account diagrams of initial correlations up to a desired order. Gather the results into one function \( T^{(\ell)}_{\#_1|\ldots|\#_\ell} \). The complete contribution from a label distribution \( \#_1|\ldots|\#_\ell \) is obtained by summing \( T^{(\ell)}_{\#_1|\ldots|\#_\ell} \) evaluated with all groupings of external labels belonging to the distribution.

This has the advantage that for any distribution one only needs to calculate a single function. Another benefit of this approach is that we directly see how labels of the collective fields are grouped onto particles, allowing us to drop certain terms according to Theorem 3. We will see how this works in detail when we compute the general \( \ell = 2 \) particle cumulants.

D. Explicit form of initial correlations

Since the primary motivation for this work is cosmological structure formation and because we want to be able to give explicit expressions for the cumulants we now adopt the specific form of the 2-particle correlations given in [1]. The initial density field is characterized by the power spectrum of its contrast in the standard way as

\[
\left\langle \delta (k_1) \delta (k_2) \right\rangle = (2\pi)^3 \delta_D (k_1 + k_2) P_0 (k_1) .
\]

The initial momentum field is assumed to be irrotational so there exists a momentum potential \( \psi \) with

\[
\tilde{p}_j^{(i)} = \alpha \nabla \psi (\tilde{r})_{|\tilde{r}=q_j^{(i)}} .
\]

The continuity equation demands that this potential is linked to the initial density contrast as

\[
\delta_j = \delta (q_j^{(i)}) = - \left( \nabla^2 \psi (\tilde{r}) \right)_{|\tilde{r}=q_j^{(i)}} .
\]

This allows us to express all three types of correlations as Fourier transforms of the density power spectrum. Due to statistical homogeneity and isotropy they will only depend on the modulus of the separation vector of the two particles involved.

\[
C_{p,q} = \int \frac{d^3 h}{(2\pi)^3} e^{i \tilde{h} \cdot (q^{(i)} - q^{(j)})} \widetilde{P}_0 (h) .
\]

\[
C_{\delta,p} = -i \alpha \int \frac{d^3 h}{(2\pi)^3} e^{i \tilde{h} \cdot (q^{(i)} - q^{(j)})} \frac{\tilde{h}}{h} \widetilde{P}_0 (h) .
\]

\[
C_{\delta,q} = \alpha^2 \int \frac{d^3 h}{(2\pi)^3} e^{i \tilde{h} \cdot (q^{(i)} - q^{(j)})} \frac{\tilde{h} \otimes \tilde{h}}{h^3} \widetilde{P}_0 (h) .
\]
E. 1-particle cumulants

The 1-particle cumulants \( G_{\phi_a}^{(0,1)} \) can be written down directly to any desired order in \( n \) as was already shown in (53). Initial correlations have no effect as \( C_{\text{con}} = 1 \). Since there is only one particle, the only possible grouping of external labels is their entirety, i.e., \( \{s_a\} = \{1, \ldots, n\} \). This means that integrating the spatial part of (51) over the initial position of the single particle \( a \) gives

\[
\int d^3\vec{q}_a \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \tilde{q}_a \right\} = (2\pi)^3 \delta_D \left( \sum_{i=1}^{n} \tilde{q}_i \right).
\]  

(61)

Combining this with the Gaussian cutoff, the 1-particle contribution to the \( n \)-point density-only cumulant is given by

\[
G_{\rho, 1-p}^{(0,1)} = \tilde{\rho} (2\pi)^3 \delta_D \left( \sum_{i=1}^{n} \tilde{q}_i \right) \exp \left\{ -\frac{\sigma^2}{2} \sum_{i=1}^{n} \tilde{g}_i \tilde{q}_i' \right\}.
\]  

(62)

Any cross correlator between \( \rho \) and \( B \) is obtained by applying the appropriate \( h_{a}(s) = b(s) \) factor

\[
G_{\rho, 1-p}^{(0,1)} = \tilde{\rho} (2\pi)^3 \delta_D \left( \sum_{i=1}^{n} \tilde{q}_i \right) \exp \left\{ -\frac{\sigma^2}{2} \sum_{i=1}^{n} \tilde{g}_i \tilde{q}_i' \right\}.
\]  

(63)

The most interesting cases are the one and two point cumulants.

\[
G_{\rho, 1}^{(0,1)} = \tilde{\rho} (2\pi)^3 \delta_D \left( \tilde{q}_1 \right) \quad G_{B, 1}^{(0,1)} = 0
\]  

(64)

\[
G_{\rho, 2}^{(0,1)} = \tilde{\rho} (2\pi)^3 \delta_D \left( \tilde{q}_1 + \tilde{q}_2 \right) \exp \left\{ -\frac{\sigma^2}{2} \tilde{q}_1^2 (\tilde{g}_1 - \tilde{g}_2)^2 \right\}
\]  

\[
G_{B, 1}^{(0,1)} = -i k_1 \tilde{g}_1 B_2^{(0,1)}
\]  

(65)

As expected the 1-point correlator of the density field just gives the mean particle density. The interpretation of the 2-point density cumulant as an exponentially damped shot-noise contribution was motivated at the end of section 5.1. In our next work [11], we will show that \( G_{\rho B}^{(0,1)} \) and \( G_{B, 1}^{(0,1)} \) can be understood as propagators for the density field \( \rho \) in a statistical sense.

F. 2-particle cumulants

The two-particle correlation operator \( C_{\text{con}}^{(2)} \) is the argument of the two-particle sum seen in Fig. 7[11] missing only a prefactor of \( 2^{-1} \). It is advantageous to note that the general expression (53) for a density-only cumulant is by construction invariant under particle exchange or renumbering. We use this to combine diagrams that transform into one another under such renumbering, i.e. those diagrams that have the same non-invariant topology. For the present \( \ell = 2 \) case this is shown in Fig. 11[11]. Next we have to think about which of the diagrams in Fig. 2 we want to include. We aim to take into account all terms up to second order in the initial power spectrum \( P_0 \). This means that all but the last diagram must be considered. The \( C_{\rho, 2-p}^{(2)} \)-line represents an exponential function of \( C_{\rho, 2-p} \propto P_0 \), while the other two line types are linear in \( P_0 \). We thus expand the isolated \( C_{\rho, 2-p}^{(2)} \)-line up to second order

\[
\begin{align*}
\begin{array}{c}
\text{a} \quad \text{b} + \quad \text{a} \quad \text{b} \\
\begin{array}{c}
\text{a} \text{b} + \quad \text{a} \quad \text{b}
\end{array}
\end{array}
\end{align*}
\]  

(66)

which leads to an expansion of the \( C_{\rho, 2-p} \) factor of (50) as

\[
c_{\rho, 2-p} = - \left( \sum_{i=1}^{n} \tilde{q}_a \tilde{g}_a \right) \frac{C_{\rho, 2-p}}{\left( \sum_{i=1}^{n} \tilde{q}_a \tilde{g}_a \right)^2} + \frac{1}{2} \left( \sum_{i=1}^{n} \tilde{q}_a \tilde{g}_a \right)^2 \frac{C_{\rho, 2-p}^{(2)}}{\left( \sum_{i=1}^{n} \tilde{q}_a \tilde{g}_a \right)^2} + \ldots
\]  

(66)

All other dashed lines inside a larger diagram only represent the linear term of (66). Overall, this leads to the contributions shown in Figs. 13 and 14. The simple dashed \( C_{\rho, 2-p}^{(2)} \)-line in both orders only represents the respective first and second order term of its expansion.

\[
\begin{align*}
\begin{array}{c}
\text{a} \quad \text{b} + \quad 2 \quad \text{a} \quad \text{b} \\
\begin{array}{c}
\text{a} \quad \text{b} + \quad 2 \quad \text{a} \quad \text{b}
\end{array}
\end{array}
\end{align*}
\]  

(66)

FIG. 12. Combining diagrams by particle exchange/relabeling.

In order to write the extensive expressions for the contribution of the individual diagrams to the density-only cumulant in a more compact form we introduce the following shorthand notation for sums of wavevectors over label sets

\[
\tilde{K}_i := \sum_{x} \tilde{k}_i.
\]  

(67)

Since they will show up in the factors coming from initial correlations, e.g. (66), we also define sums of the combination of a free propagator and a wavevector in accordance with [10] as

\[
\tilde{K}_i := \sum_{x} \tilde{k}_i = \sum_{x} g_x \tilde{k}_x.
\]  

(68)

These quantities are not to be confused with the source terms \( \tilde{K}_i \) for individual particles which have been turned off by now.
We are now ready to implement the strategy for organising the different terms contributing to an \(n\)-point cumulant as discussed in section IV C. For \(\ell = 2\) particles with indices \(a\) and \(b\), all non-vanishing labeling distributions are given by \(m (n - m)\) with \(1 < m \leq [n/2]\) and all label groupings \((\{s_1\}; \{s_2\}) = \{i_1, \ldots, i_m; j_1, \ldots, j_{n-m}\}\) contain only two labelings \(a_{i_1} \ldots a_{i_m} b_{j_1} \ldots b_{j_{n-m}}\) and \(a_{j_1} \ldots a_{j_{n-m}} b_{i_1} \ldots b_{i_m}\). Let us pick such a general labeling group \((\{s_1\}; \{s_2\})\) and start with the simplest diagram, the \(C_{d,\rho}\)-line in Fig. [13]. The damping function \(\mathcal{D}_{(\{s_1\}; \{s_2\})}\) will be omitted for now. As the diagram is invariant under particle exchange both labelings contained in any label grouping will give the same result and thus a factor of 2. Using the spatial part of (51) and (58) we find

\[
2 \frac{1}{2} \rho^2 \int d^3 q_{\rho}^{(i)} \int d^3 q_{\rho}^{(j)} e^{-i q_{\rho}^{(i)} \cdot \vec{k}_{i}^{(1)}} e^{-i q_{\rho}^{(j)} \cdot \vec{k}_{i}^{(2)}} = \rho^2 (2\pi)^3 \delta_D \left( \vec{h} - \vec{k}_{i}^{(1)} \right) \delta_D \left( \vec{h} + \vec{k}_{i}^{(2)} \right) P_0(h) = \rho^2 (2\pi)^3 \delta_D \left( \sum_{i=1}^{n} \vec{k}_i \right) P_0 \left( \left| \vec{k}_{i}^{(1)} \right| \right). \tag{69}
\]

The next diagram is the bare \(C_{\delta,\rho}\)-line in Fig. [13]. The calculation proceeds in largely the same way as for the first diagram except that the two labelings contained in the label grouping now give different results since the diagram is not symmetric anymore. Using (49), (51), and (59) we find the result given in (70) for the \(a_{i_1} \ldots a_{i_m} b_{j_1} \ldots b_{j_{n-m}}\) realisation and the second realisation gives the same result just with the external label sets \(\{s_1\} = \{i_1, \ldots, i_m\}\) and \(\{s_2\} = \{j_1, \ldots, j_{n-m}\}\) exchanged:

\[
2 \frac{1}{2} \rho^2 \int d^3 q_{\rho}^{(i)} \int d^3 q_{\rho}^{(j)} e^{-i q_{\rho}^{(i)} \cdot \vec{k}_{i}^{(1)}} e^{-i q_{\rho}^{(j)} \cdot \vec{k}_{i}^{(2)}} \left( (-i)^2 \alpha \int \frac{d^3 h}{(2\pi)^3} e^{i \vec{h} \cdot (q_{\rho}^{(i)} - q_{\rho}^{(j)})} P_0(h) \frac{\vec{h}}{h^2} \right) \cdot \vec{k}_{i}^{(1)}
\]

\[
= -\alpha \rho^2 (2\pi)^3 \int \frac{d^3 h}{(2\pi)^3} \delta_D \left( \vec{h} - \vec{k}_{i}^{(1)} \right) \delta_D \left( \vec{h} + \vec{k}_{i}^{(2)} \right) P_0(h) \frac{\vec{h}}{h^2} = -\alpha \rho^2 (2\pi)^3 \delta_D \left( \sum_{i=1}^{n} \vec{k}_i \right) P_0 \left( \left| \vec{k}_{i}^{(1)} \right| \right) \frac{\vec{k}_{i}^{(1)} \cdot \vec{k}_{i}^{(2)}}{\left( \left| \vec{k}_{i}^{(1)} \right| \right)^2}. \tag{70}
\]

Observe that the argument of \(P_0\) in both expressions can be exchanged between the two label sets by using the Dirac delta distribution in front. At this point the pattern should become clear, hence the only other diagram we will compute explicitly is the first diagram of Fig. [14] in order to show the combined usage of (69) and the linear term in (66) which give the first term in parentheses in the first line of (71). Due to the symmetry of the diagram both labelings give the same result.

\[
2 \frac{1}{2} \rho^2 \int d^3 q_{\rho}^{(i)} \int d^3 q_{\rho}^{(j)} e^{-i q_{\rho}^{(i)} \cdot \vec{k}_{i}^{(1)}} e^{-i q_{\rho}^{(j)} \cdot \vec{k}_{i}^{(2)}} \left( -\alpha^2 \int \frac{d^3 h}{(2\pi)^3} e^{i \vec{h} \cdot (q_{\rho}^{(i)} - q_{\rho}^{(j)})} \frac{\vec{h}}{h^2} P_0(h) \right) \left( \int \frac{d^3 h}{(2\pi)^3} e^{i \vec{h} \cdot (q_{\rho}^{(i)} - q_{\rho}^{(j)})} P_0(h) \right) = -\alpha \rho^2 (2\pi)^3 \int \frac{d^3 h}{(2\pi)^3} \delta_D \left( \vec{h}_1 + \vec{h}_2 - \vec{k}_{i}^{(1)} \right) \delta_D \left( \vec{h}_1 + \vec{h}_2 + \vec{k}_{i}^{(2)} \right) P_0(h_1) P_0(h_2) \frac{\vec{h}_1}{h_1^2} \frac{\vec{h}_2}{h_2^2} \frac{\vec{k}_{i}^{(1)} \cdot \vec{h}_1}{h_1^2} \frac{\vec{k}_{i}^{(2)} \cdot \vec{h}_2}{h_2^2} \frac{\vec{k}_{i}^{(1)} \cdot \vec{k}_{i}^{(2)}}{h_1^2 h_2^2} \tag{71}
\]

In the last line we renamed \(\vec{h}_1 \rightarrow \vec{h}\). The damping function is easily calculated to be

\[
\mathcal{D}_{(\{s_1\}; \{s_2\})} = \exp \left\{ -\frac{\alpha^2}{2} \left( \frac{\vec{k}_{i}^{(1)} \cdot \vec{k}_{i}^{(2)}}{h_1^2} + \frac{\vec{k}_{i}^{(1)} \cdot \vec{k}_{i}^{(2)}}{h_2^2} \right) \right\}. \tag{72}
\]

Once all diagrams have been calculated we gather the results into two functions as described in our strategy of section IV C. The general 2-particle \(n\)-point density cumulant up to second order in the initial correlations then reads

\[
G^{(0,2)}_{\rho_{\ell-\rho}} = \rho^2 (2\pi)^3 \delta_D \left( \sum_{i=1}^{n} \vec{k}_i \right) \sum_{m=1}^{n/2} \sum_{\{s_1\}; \{s_2\}} \left( \mathcal{T}^{(2,1)}_{\rho_{\ell-\rho}} \left( \{s_1\}; \{s_2\} \right) + \mathcal{T}^{(2,2)}_{\rho_{\ell-\rho}} \left( \{s_1\}; \{s_2\} \right) \right). \tag{73}
\]

In this, \(\mathcal{T}^{(2,1)}_{\rho_{\ell-\rho}}\) contains all results from diagrams of first order in \(P_0\) found in Fig. [13] and \(\mathcal{T}^{(2,2)}_{\rho_{\ell-\rho}}\) contains the results from the second order diagrams of figure [14]. Both can be found in Appendix A. Their length might look daunting at first. However, they reduce considerably when we consider explicit small \(n\)-point cumulants. For \(n = 1\) we have two terms in (53) where in each of them one of the two particles does not carry an external label. Theorem 3 then tells us that we have

\[
G^{(0,2)}_{\rho_1} = 0 \quad \text{and} \quad G^{(0,2)}_{B_1} = 0. \tag{74}
\]

In the \(n = 2\) case there is trivially only one possible distribution, namely \(1|1\) and the only grouping is \((1;2)\). The overall Dirac delta allows us to set \(\vec{k}_2 = -\vec{k}_1\). The \(\mathcal{T}^{(2,2)}_{1|1}\) function thus
reduces to
\[ T_{111}^{(2,1)}(1; 2) = P_0(k_1)(1+\alpha g_1)(1+\alpha g_2) \exp \left\{ -\frac{\sigma_p^2}{2} k_1^2 (g_1^2 + g_2^2) \right\} \tag{75} \]
which is the general form of Eq. (44) in Bartelmann et al. [9]. This contribution corresponds to the linear power spectrum of the standard Euler-Poisson system and is equivalent to it if one neglects the damping and uses Zel’Dovich trajectories. The \( T_{111}^{(2,2)} \) term is thus a generalization of Eq. (48) in [9] which only contained the fourth diagram of Fig. 14.

\[ T_{111}^{(2,2)}(1; 2) = \exp \left\{ -\frac{\sigma_p^2}{2} k_1^2 (g_1^2 + g_2^2) \right\} \int \frac{d^3 h}{(2\pi)^3} P_0 (\vec{k}_1 - \vec{h}) \]
\[ P_0(h) \left\{ \alpha^2 g_1 g_2 \left( \frac{\vec{k}_1 \cdot \vec{h}}{h^2} \right)^2 \right\} \left( 1 + \alpha (g_1 + g_2) \frac{\vec{k}_1 \cdot (\vec{k}_1 - \vec{h})}{(\vec{k}_1 - \vec{h})^2} \right) \]
\[ + \frac{\alpha^2}{2} g_1 g_2 \left( \frac{\vec{k}_1 \cdot (\vec{k}_1 - \vec{h})}{\vec{k}_1 - \vec{h})^2 \right) \right\} + \alpha^2 g_1 g_2 \left( \frac{\vec{k}_1 \cdot \vec{h}}{h^2} \right) \left( \frac{\vec{k}_1 \cdot (\vec{k}_1 - \vec{h})}{(\vec{k}_1 - \vec{h})^2} \right) \right\} \tag{76} \]

This term represents the coupling of modes at the initial time which are then transported forward by the free propagator. An interesting question for future work would be whether comparable terms can be found in the non-interacting limit of the Euler-Poisson system or if (76) already contains effects of multi-streaming.

With all 2-particle density-only correlators described by (73) we can now address mixed correlators of \( \rho \) and \( B \). For the 2-point cumulant we directly infer from Theorem 1 that
\[ G_{\rho B}^{(0,2)} = 0 = G_{B \rho}^{(0,2)} . \tag{77} \]

Let us thus look at the 3-point cumulant. There the only label distribution is 121 with three possible label groupings so without specifying \( T_{112}^{(2)} \) we have
\[ G_{\rho \rho \rho}^{(0,2)} = \bar{\rho}^3 \left( 2\pi \right)^3 \delta \left( \vec{k}_1 + \vec{k}_2 + \vec{k}_3 \right) \]
\[ \left\{ T_{112}^{(2)}(1; 2; 3) + T_{121}^{(2)}(2; 1; 3) + T_{121}^{(2)}(3; 1; 2) \right\} . \tag{78} \]

Replacing the last density field with a \( B \)-field each of the \( T_{112}^{(2)} \) gets a \( b(3) \) factor in front of it which depends on where 3 is placed in the overall grouping. This leads to
\[ G_{\rho \rho \rho}^{(2)} = \left. 3 \bar{\rho}^3 \left( 2\pi \right)^3 \delta \left( \vec{k}_1 + \vec{k}_2 + \vec{k}_3 \right) \left( \vec{k}_3 \cdot \vec{k}_2 g_{23} T_{121}^{(2)}(1; 2; 3) + \vec{k}_3 \cdot \vec{k}_1 g_{13} T_{112}^{(2)}(2; 1; 3) \right) \right. \tag{79} \]

The third term \( T_{121}^{(2)}(3; 1; 2) \) vanished due to Theorem 1 as one particle only carried the \( B \)-field label 3. If we add one more \( B \)-field this holds for all three terms, i.e. we directly see Corollary 4 in action and thus \( G_{\rho \rho \rho}^{(0,2)} = 0 \). This implies a general strategy for how to calculate any mixed correlator.

- Calculate the \( T \) functions for the corresponding \( n \)-point density-only cumulant.
- Replace density by response fields as desired. For each label distribution identify then those label groupings where there are groups of only \( B \)-field labels and drop their corresponding \( T \)-functions.
- Place \( b(s) \)-factors appropriate to the number of \( B \)-fields in front of all remaining \( T \)-functions, where the \( b(s) \) can be directly read off from the grouping.

G. 3-particle cumulants

While technically possible, writing down general \( n \)-point cumulants becomes infeasible quite quickly for more than two particles. For the \( l = 3 \) case we will contend ourselves with the 3-point cumulants. 4-point cumulants are given in Appendix B. In terms of diagrams we will only consider the lowest possible order of \( O(P_0^2) \) which already has 36 diagrams. We can again reduce this number significantly by combining all non-invariant diagrams of the same topology leading to the form of \( C_{\text{con}}^{(3)} \) shown in figure 15 with only 7 diagrams left. For

\[ \text{FIG. 15. All diagrams of order } O(P_0^2) \text{ contributing to } C_{\text{con}}^{(3)} \text{ after combining those of equal topology.} \]

the 3-point cumulant the only possible index distribution is 1111 and the only group ordering subsequently (1; 2; 3). For each of the 7 diagrams one thus needs to evaluate \( 3! = 6 \) terms leading to a total of 42 individual contributions which make up
\[ T_{1111}^{(3,2)}(1; 2; 3) = \exp \left\{ -\frac{\sigma_p^2}{2} \left( K_1^2 + K_2^2 + K_3^2 \right) \right\} \]
\[ \left\{ P_0(k_1) P_0(k_2) (1 + \alpha g_1)(1 + \alpha g_2) \right\} \]
\[ \left\{ \alpha^2 K_3 \cdot \vec{k}_1 \frac{K_1 \cdot \vec{k}_2 K_3 \cdot \vec{k}_2}{k_1 k_2} - \vec{k}_3 \left( \frac{\vec{k}_1}{k_1} + \frac{\vec{k}_2}{k_2} \right) \right\} \]
\[ + \text{cyc. perm.} \right\} . \tag{80} \]

Due to Corollary 4 the density-only cumulant is the only 3-particle 3-point cumulant.
\[ G_{\rho \rho \rho}^{(0,3)}(1; 2; 3) = \bar{\rho}^3 \left( 2\pi \right)^3 \delta_0 \left( \vec{k}_1 + \vec{k}_2 + \vec{k}_3 \right) T_{1111}^{(3,2)}(1; 2; 3) \tag{81} \]
This is again a generalization of a result from [9], namely for the bispectrum given in Eq. (77).
V. SUMMARY

This paper extends the statistical field theory for classical particles pioneered by [Das and Mazenko] and adapted to correlated initial conditions in Bartelmann et al. [1] from the canonical to the grand canonical ensemble. We were able to do this in the case of statistically homogeneous and isotropic canonical to the grand canonical ensemble. We were able to have some general theorems for them will be advantageous in the study of the interacting grand canonical ensemble.

- Initial correlations between the degrees of freedom of the individual particles can be brought into the form of correlation operators acting on the free generating functional of an ideal gas. These operators can be expressed in a simple diagrammatic representation.

- Using a variant of the Mayer cluster expansion the sum over particle numbers can be transformed into an exponential of the sum over connected $\ell$-particle generating functionals. Having only connected cumulants as the intrinsic building blocks of perturbation theory and having some general theorems for them will be advantageous in the study of the interacting grand canonical ensemble.

- The ordering of the generating functional by $\ell$ representative connected particles together with the diagrammatic representation of initial correlations allows a structured scheme when calculating free cumulants, reducing the combinatorial efforts necessary when compared with the canonical approach. With the help of this scheme we derived generalizations of the cumulants found in Bartelmann et al. [9].

Building upon these, our next paper [11] will focus on generalizing the self-consistent grand canonical perturbation theory of [2–5] to include initial correlations.

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Appendix A: 2-particle $T$-functions

The two functions containing the contribution from a single label grouping $\{(s_1); \{s_2\}\}$ of a distribution $\#\{s_1\}\#\{s_2\}$ to some $n = \#\{s_1\} + \#\{s_2\}$ point cumulant in first and second order in the initial power spectrum $P_0$ are shown (A1) and (A2). In the first order function the first term represents the first diagram of figure [12] the second and third term the second diagram and the fourth term the third diagram. In the second order function the first term represents the third diagram of figure [13] the second term the first diagram, the third and fourth term the second diagram and the fifth term the fourth diagram.

\[ \mathcal{T}^{(2,1)}_{\#\{s_1\}\#\{s_2\}}(\{s_1\}; \{s_2\}) = \mathfrak{D}_{\{s_1\},\{s_2\}} P_0 \left( \left\{ \vec{k}_{s_1} \right\} \right) \left( 1 - \alpha \frac{\vec{k}_{s_1}}{k_{s_1}^2} \cdot \vec{K}_{s_2} - \alpha \frac{\vec{k}_{s_2}}{k_{s_2}^2} \cdot \vec{K}_{s_1} + \alpha^2 \frac{\vec{k}_{s_1}}{k_{s_1}^2} \cdot \vec{k}_{s_1} \cdot \vec{K}_{s_2} \cdot \vec{K}_{s_2} \right) \]  

(A1)

\[ \mathcal{T}^{(2,2)}_{\#\{s_1\}\#\{s_2\}}(\{s_1\}; \{s_2\}) = \mathfrak{D}_{\{s_1\},\{s_2\}} \int \frac{d^3h}{(2\pi)^3} P_0(h) P_0 \left( \left\{ \vec{k}_{s_1} - \vec{h} \right\} \right) \left( 1 + \alpha \frac{\vec{k}_{s_1}}{k_{s_1} - \vec{h}} \cdot \frac{\vec{h}}{h^2} - \alpha \frac{\vec{k}_{s_2}}{k_{s_2} - \vec{h}} \cdot \frac{\vec{h}}{h^2} + \frac{\alpha^2}{2} \frac{\vec{k}_{s_1}}{k_{s_1} - \vec{h}} \cdot \vec{k}_{s_1} \cdot \vec{K}_{s_2} \cdot \vec{K}_{s_2} \right) \]  

(A2)

Appendix B: 3-particle 4-point cumulants

The 4-point cumulant has only one index distribution $1|1|2$ but $\binom{4}{2} = 6$ different groupings. The $T$ function can in principle be read off from the 3-point case of (80) by replacing one of the single labels by a set of 2 labels and is given in

\[ G^{(0,3)}_{\rho_1\rho_2\rho_3} = \tilde{\rho}^3 (2\pi)^3 \delta_D \left( \sum_{j=1}^4 \vec{k}_j \right) \times \left[ \mathcal{T}^{(3,2)}_{1|1|2}(1; 2; 3, 4) + \mathcal{T}^{(3,2)}_{1|1|2}(1; 3; 2, 4) + \mathcal{T}^{(3,2)}_{1|1|2}(1; 4; 2, 3) \right] \]
Notice that this is again a shot-noise like contribution due to the \( \bar{\rho}^3 \) scaling. The only mixed 3-particle 4-point cumulant is

\[
\mathcal{T}^{(3,2)}_{1|1|2}(1; 2; 3, 4) = \left\{ P_0(k_1) P_0(k_2)(1 + \alpha g_1)(1 + \alpha g_2) \left[ a^2 \frac{(k_3 + k_4) \cdot (k_1 + k_2)}{k_1^2 k_2^2} - \alpha \frac{(k_3 + k_4) \cdot (k_1 + k_2)}{k_1^2 + k_2^2} \right] \right. \\
+ \left[ P_0(k_1)(1 + \alpha g_1) P_0(k_3 + k_4) \right] \left\{ 1 + \alpha \left( k_3 + k_4 \right) \cdot \frac{(k_3 + k_4)}{k_1^2 k_4^2} - \alpha \frac{k_3 \cdot k_4}{k_1^2} - \alpha \frac{k_2 \cdot (k_3 + k_4)}{(k_3 + k_4)^2} \right\} \left. \exp \left\{ -\frac{\sigma^2}{2} \left( k_3^2 + k_4^2 + (k_3 + k_4)^2 \right) \right\} \right\} \\
+ (1 \leftrightarrow 2) \right\}
\]

(B3)

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