Walk/Zeta Correspondence
for quantum and correlated random walks

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Abstract

In this paper, following the recent paper on Walk/Zeta Correspondence by the first author and his coworkers, we compute the zeta function for the three- and four-state quantum walk and correlated random walk, and the multi-state random walk on the one-dimensional torus by using the Fourier analysis. We deal with also the four-state quantum walk and correlated random walk on the two-dimensional torus. In addition, we introduce a new class of models determined by the generalized Grover matrix bridging the gap between the Grover matrix and the positive-support of the Grover matrix. Finally, we give a generalized version of the Konno-Sato theorem for the new class. As a corollary, we calculate the zeta function for the generalized Grover matrix on the $d$-dimensional torus.

Keywords: Zeta function, Random walk, Correlated random walk, Quantum walk

1 Introduction

The quantum walk (QW) is a quantum counterpart of the correlated random walk (CRW). Note that the random walk (RW) is a special model of the CRW. The Grover walk is one of the most well-investigated models in the study of the QW. The zeta function for the Grover walk can be obtained by the so-called Konno-Sato theorem given in [3]. Recently, the first author and his coworkers gave explicit formulas for the generalized zeta function and the generalized Ihara zeta function corresponding to the flip-flop type Grover walk and its positive-support version respectively, on a class of graphs by using the Konno-Sato theorem in [4]. This relation is called “Grover/Zeta Correspondence”. In the subsequent paper [5], they obtained the zeta function for the wide class of walks including RW, CRW, QW, and open quantum random walk (OQRW) on the torus. The relation is called “Walk/Zeta Correspondence”. Moreover, their paper [7] presented the characteristic polynomial of the transition matrix related to the vertex-face walk on the two-dimensional torus. In these papers [4, 5, 7], they dealt with one-particle models including RW, CRW, QW, OQRW. On the other hand, they investigated multi-particle models with probabilistic or quantum interactions, called the interacting particle systems (IPS) in [6]. The relation between this model and the zeta function is called “IPS/Zeta Correspondence”. In addition, the model studied in [4, 5, 6, 7] is the discrete-time model, while they studied corresponding continuous-time model (CTM) in [8], called “CTM/Zeta Correspondence”.

In this paper, we extend the models studied in [5] and calculate the zeta function for the extended classes of models on the one- and two-dimensional torus via Walk/Zeta Correspondence. Furthermore, we introduce a new class of models determined by the generalized Grover matrix bridging the gap between the Grover matrix and the positive-support of the Grover matrix. Finally, we present a generalized version of the Konno-Sato theorem for the new class. As a corollary, we compute the zeta function for the generalized Grover matrix on the $d$-dimensional torus.

The rest of this paper is organized as follows. In Section 2, we review Walk/Zeta Correspondence on the torus investigated in [5]. Sections 3 and 4 are devoted to the three- and four-state QW and CRW on the one-dimensional torus. In Section 5, we treat the multi-state RW on the one-dimensional torus. Section 6 deals with the four-state QW and CRW on the two-dimensional torus. Section 7 introduces a new class of models determined by the generalized Grover matrix. In Section 8, a generalized version of the Konno-Sato theorem for the new class is given. By using this theorem, we calculate the zeta function
for the generalized Grover matrix on the \(d\)-dimensional torus. Furthermore, we mention a relation between Grover/Zeta and Walk/Zeta Correspondences. Section 9 summarizes our results.

2 Walk/Zeta Correspondence

First we introduce the following notation: \(\mathbb{Z}\) is the set of integers, \(\mathbb{Z}_{\geq}\) is the set of non-negative integers, \(\mathbb{Z}_{\geq}\) is the set of positive integers, \(\mathbb{R}\) is the set of real numbers, and \(\mathbb{C}\) is the set of complex numbers. Moreover, \(T^d_N\) denotes the \(d\)-dimensional torus with \(N^d\) vertices, where \(d, N \in \mathbb{Z}_{\geq}\). Remark that \(T^d_N = (\mathbb{Z} \text{ mod } N)^d\).

Following [5] in which Walk/Zeta Correspondence on \(\mathbb{T}\) was investigated, we will review our setting for \(2d\)-state discrete time walk with a nearest-neighbor jump on \(T^d_N\). The present paper will deal with the general \(d_c\)-state walk on \(T^d_N\) with \(d_c \in \mathbb{Z}_{\geq}\). However, it is easy to understand our setting for the case of \(2d\)-state walk with the nearest-neighbor jump, so we treat this type of model.

The discrete time walk is defined by using a shift operator and a coin matrix which will be mentioned below. Let \(f : T^d_N \rightarrow \mathbb{C}^{2d}\). For \(j = 1, 2, \ldots, d\) and \(\mathbf{x} \in T^d_N\), the shift operator \(\tau_j\) is defined by

\[
(\tau_j f)(\mathbf{x}) = f(\mathbf{x} - \mathbf{e}_j),
\]

where \(\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\}\) denotes the standard basis of \(T^d_N\). The coin matrix \(A = [a_{ij}]_{i,j=1,2,\ldots,2d}\) is a \(2d \times 2d\) matrix with \(a_{ij} \in \mathbb{C}\) for \(i, j = 1, 2, \ldots, 2d\). If \(a_{ij} \in [0,1]\) and \(\sum_{j=1}^{2d}a_{ij} = 1\) for any \(j = 1, 2, \ldots, 2d\), then the walk is a CRW. We should remark that, in particular, when \(a_{11} = a_{2} = \cdots = a_{2d}\) for any \(i = 1, 2, \ldots, 2d\), this CRW becomes a RW. If \(A\) is unitary, then the walk is a QW. So our class of walks contains RW, CRW, and QW as special models.

To describe the evolution of the walk, we decompose the \(2d \times 2d\) coin matrix \(A\) as

\[
A = \sum_{j=1}^{2d} P_j A,
\]

where \(P_j\) denotes the orthogonal projection onto the one-dimensional subspace \(\mathbb{C} \eta_j\) in \(\mathbb{C}^{2d}\). Here \(\{\eta_1, \eta_2, \ldots, \eta_{2d}\}\) denotes a standard basis on \(\mathbb{C}^{2d}\).

The discrete time walk associated with the coin matrix \(A\) on \(T^d_N\) is determined by the \(2dN^d \times 2dN^d\) matrix

\[
M_A = \sum_{j=1}^{d} \left(P_{2j-1} A \tau_j^{-1} + P_{2j} A \tau_j\right).
\]

(1)

The state at time \(n \in \mathbb{Z}_{\geq}\) and location \(\mathbf{x} \in T^d_N\) can be expressed by a \(2d\)-dimensional vector:

\[
\Psi_n(\mathbf{x}) = \begin{bmatrix}
\Psi^1_n(\mathbf{x}) \\
\Psi^2_n(\mathbf{x}) \\
\vdots \\
\Psi^{2d}_n(\mathbf{x})
\end{bmatrix} \in \mathbb{C}^{2d}.
\]

For \(\Psi_n : T^d_N \rightarrow \mathbb{C}^{2d} (n \in \mathbb{Z}_{\geq})\), Eq. 1 gives the evolution of the walk as follows.

\[
\Psi_{n+1}(\mathbf{x}) \equiv (M_A \Psi_n)(\mathbf{x}) = \sum_{j=1}^{d} \left(P_{2j-1} A \Psi_n(\mathbf{x} + \mathbf{e}_j) + P_{2j} A \Psi_n(\mathbf{x} - \mathbf{e}_j)\right).
\]

(2)
This equation means that the walker moves at each step one unit to the \(-x_j\)-axis direction with matrix \(P_{2j-1}A\) or one unit to the \(x_j\)-axis direction with matrix \(P_{2j}A\) for \(j = 1, 2, \ldots, d\). Moreover, for \(n \in \mathbb{Z}_+\) and \(x = (x_1, x_2, \ldots, x_d) \in T^d_N\), the \(2d \times 2d\) matrix \(\Phi_n(x_1, x_2, \ldots, x_d)\) is given by

\[
\Phi_n(x_1, x_2, \ldots, x_d) = \sum_* \Xi_n (l_1, l_2, \ldots, l_{2d-1}, l_{2d}),
\]

where the \(2d \times 2d\) matrix \(\Xi_n (l_1, l_2, \ldots, l_{2d-1}, l_{2d})\) is the sum of all possible paths in the trajectory of \(l_{2j-1}\) steps \(-x_j\)-axis direction and \(l_{2j}\) steps \(x_j\)-axis direction and \(\sum_*\) is the summation over \((l_1, l_2, \ldots, l_{2d-1}, l_{2d}) \in (\mathbb{Z}_+)^{2d}\) satisfying

\[
l_1 + l_2 + \cdots + l_{2d-1} + l_{2d} = n, \quad x_j = -l_{2j-1} + l_{2j} \quad (j = 1, 2, \ldots, d).
\]

Here we put

\[
\Phi_0(x_1, x_2, \ldots, x_d) = \begin{cases} I_{2d} & \text{if } (x_1, x_2, \ldots, x_d) = (0, 0, \ldots, 0), \\ O_{2d} & \text{if } (x_1, x_2, \ldots, x_d) \neq (0, 0, \ldots, 0), \end{cases}
\]

where \(I_n\) is the \(n \times n\) identity matrix and \(O_n\) is the \(n \times n\) zero matrix. Then, for the walk starting from \((0, 0, \ldots, 0)\), we obtain

\[
\Psi_n(x_1, x_2, \ldots, x_d) = \Phi_n(x_1, x_2, \ldots, x_d)\Phi_0(0, 0, \ldots, 0) \quad (n \in \mathbb{Z}_+).
\]

We call \(\Phi_n(x) = \Phi_n(x_1, x_2, \ldots, x_d)\) matrix weight at time \(n \in \mathbb{Z}_+\) and location \(x \in T^d_N\) starting from \(0 = (0, 0, \ldots, 0)\). When we consider the walk on not \(T^d_N\) but \(\mathbb{Z}^d\), we add the superscript "\((\infty)\)" to the notation like \(\Psi(\infty)\) and \(\Xi(\infty)\).

This type is moving shift model called M-type here. Another type is flip-flop shift model called F-type whose coin matrix is given by

\[
A^{(f)} = (I_d \otimes \sigma) A, \tag{3}
\]

where \(\otimes\) is the tensor product and

\[
\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

For example, when \(d = 2\) (two-dimensional case), we have

\[
I_2 \otimes \sigma = \begin{bmatrix} \sigma & O_2 \\ O_2 & \sigma \end{bmatrix}.
\]

The F-type model is also important, since it has a central role in the Konno-Sato theorem \[\text{[3]}\]. When we distinguish \(A\) (M-type) from \(A^{(f)}\) (F-type), we write \(A\) by \(A^{(m)}\).

The measure \(\mu_n(x)\) at time \(n \in \mathbb{Z}_+\) and location \(x \in T^d_N\) is defined by

\[
\mu_n(x) = \|\Psi_n(x)\|^p_{C_{2d}} = \sum_{j=1}^{2d} |\Psi^j_n(x)|^p,
\]

where \(\| \cdot \|_{C_{2d}}\) denotes the standard \(p\)-norm on \(\mathbb{C}^{2d}\). As for CRW and QW, we take \(p = 1\) and \(p = 2\), respectively. Then CRW and QW satisfy

\[
\sum_{x \in T^d_N} \mu_n(x) = \sum_{x \in T^d_N} \mu_0(x),
\]

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for any time $n \in \mathbb{Z}_>$. However, we do not necessarily impose such a condition for the walk we consider here. For example, the two-dimensional positive-support version of the Grover walk (introduced in Section 7) does not satisfy the condition. In this meaning, our walk is a generalized version for the usual walk.

To consider the zeta function, we use the Fourier analysis. To do so, we introduce the following notation: $K_N = \{0, 1, \ldots, N-1\}$ and $\overline{K}_N = \{0, 2\pi/N, \ldots, 2\pi(N-1)/N\}$.

For $f : \mathbb{R}_N^d \rightarrow \mathbb{C}^d$, the Fourier transform of the function $f$, denoted by $\hat{f}$, is defined by the sum

$$
\hat{f}(k) = \frac{1}{N^{d/2}} \sum_{x \in \mathbb{R}_N^d} e^{-2\pi i \langle x, k \rangle / N} f(x),
$$

where $k = (k_1, k_2, \ldots, k_d) \in \mathbb{R}_N^d$. Here $\langle x, k \rangle$ is the canonical inner product of $\mathbb{R}^d$, i.e., $\langle x, k \rangle = \sum_{j=1}^d x_j k_j$. Then we see that $\hat{f} : \mathbb{R}_N^d \rightarrow \mathbb{C}^d$. Moreover, we should remark that

$$
f(x) = \frac{1}{N^{d/2}} \sum_{k \in \mathbb{R}_N^d} e^{2\pi i \langle x, k \rangle / N} \hat{f}(k),
$$

where $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}_N^d$. By using

$$
\tilde{k}_j = \frac{2\pi k_j}{N} \in \overline{K}_N, \quad \tilde{k} = (\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_d) \in \overline{K}_N^d,
$$

we can rewrite Eqs. (4) and (5) in the following way:

$$
\tilde{g}(\tilde{k}) = \frac{1}{N^{d/2}} \sum_{x \in \mathbb{R}_N^d} e^{-i \langle x, \tilde{k} \rangle} g(x),
$$

$$
g(x) = \frac{1}{N^{d/2}} \sum_{\tilde{k} \in \mathbb{R}_N^d} e^{i \langle x, \tilde{k} \rangle} \tilde{g}(\tilde{k}),
$$

for $g : \mathbb{R}_N^d \rightarrow \mathbb{C}^d$ and $\tilde{g} : \overline{K}_N^d \rightarrow \mathbb{C}^d$. In order to take a limit $N \rightarrow \infty$, we introduced the notation given in Eq. (6). We should note that as for the summation, we sometimes write “$k \in \mathbb{R}_N^d$” instead of “$\tilde{k} \in \overline{K}_N^d$”. From the Fourier transform and Eq. (7), we have

$$
\hat{\Psi}_{n+1}(k) = \hat{M}_A(k) \hat{\Psi}_n(k),
$$

where $\hat{\Psi}_n : T_N^d \rightarrow \mathbb{C}^d$ and $2d \times 2d$ matrix $\hat{M}_A(k)$ is determined by

$$
\hat{M}_A(k) = \sum_{j=1}^d \left( e^{2\pi i k_j/N} P_{2j-1} + e^{-2\pi i k_j/N} P_{2j} \right).
$$

By using notations in Eq. (6), we get

$$
\hat{M}_A(\tilde{k}) = \sum_{j=1}^d \left( e^{i \tilde{k}_j} P_{2j-1} + e^{-i \tilde{k}_j} P_{2j} \right).
$$

Next we will consider the following eigenvalue problem for $2dN^d \times 2dN^d$ matrix $M_A$:

$$
\lambda \Psi = M_A \Psi,
$$
where \( \lambda \in \mathbb{C} \) is an eigenvalue and \( \Psi(\in \mathbb{C}^{2dN^d}) \) is the corresponding eigenvector. Noting that Eq. (8) is closely related to Eq. (2), we see that Eq. (8) is rewritten as

\[
\lambda \Psi(x) = (M_A \Psi)(x) = \sum_{j=1}^{d} \left( P_{2j-1}A \Psi(x + e_j) + P_{2j}A \Psi(x - e_j) \right),
\]

for any \( x \in K_N^d \). From the Fourier transform and Eq. (9), we obtain

\[
\hat{\lambda} \hat{\Psi}(k) = \hat{M_A}(k) \hat{\Psi}(k),
\]

for any \( k \in K_N^d \). Then the characteristic polynomials of \( 2d \times 2d \) matrix \( \hat{M_A}(k) \) for fixed \( k(\in K_N^d) \) is

\[
\det \left( \lambda I_{2d} - \hat{M_A}(k) \right) = \prod_{j=1}^{2d} \left( \lambda - \lambda_j(k) \right),
\]

where \( \lambda_j(k) \) are eigenvalues of \( \hat{M_A}(k) \). Similarly, the characteristic polynomials of \( 2dN^d \times 2dN^d \) matrix \( \hat{M_A} \) is

\[
\det \left( \lambda I_{2dN^d} - \hat{M_A} \right) = \prod_{j=1}^{2d} \prod_{k \in K_N^d} \left( \lambda - \lambda_j(k) \right).
\]

Thus we have

\[
\det \left( \lambda I_{2dN^d} - M_A \right) = \det \left( \lambda I_{2dN^d} - \hat{M_A} \right) = \prod_{j=1}^{2d} \prod_{k \in K_N^d} \left( \lambda - \lambda_j(k) \right).
\]

Therefore, by taking \( \lambda = 1/u \), we get the following key result.

\[
\det \left( I_{2dN^d} - uM_A \right) = \det \left( I_{2dN^d} - u\hat{M_A} \right) = \prod_{j=1}^{2d} \prod_{k \in K_N^d} \left( 1 - u\lambda_j(k) \right). \tag{11}
\]

We should note that for fixed \( k(\in K_N^d) \), eigenvalues of \( 2d \times 2d \) matrix \( \hat{M_A}(k) \) are expressed as

\[
\text{Spec}(\hat{M_A}(k)) = \{ \lambda_j(k) \mid j = 1, 2, \ldots, 2d \}.
\]

Moreover, eigenvalues of \( 2dN^d \times 2dN^d \) matrix not only \( \hat{M_A} \) but also \( M_A \) are expressed as

\[
\text{Spec}(\hat{M_A}) = \text{Spec}(M_A) = \{ \lambda_j(k) \mid j = 1, 2, \ldots, 2d, \ k \in K_N^d \}.
\]

By using notations in Eq. (6) and Eq. (10), we see that for fixed \( k(\in K_N^d) \),

\[
\det \left( I_{2d} - u\hat{M_A}(k) \right) = \prod_{j=1}^{2d} \left( 1 - u\lambda_j(k) \right). \tag{12}
\]

Furthermore, Eq. (7) gives the following important formula.

\[
\det \left( I_{2d} - u\hat{M_A}(k) \right) = \det \left( I_{2d} - u \sum_{j=1}^{d} \left( e^{i\tilde{k}_j} P_{2j-1}A + e^{-i\tilde{k}_j} P_{2j}A \right) \right).
\]
In this setting, we define the walk-type zeta function by

\[ \zeta(A, T^d_N, u) = \det \left( I_{2dN^d} - u\hat{M}_A \right)^{-1/N^d}. \]  

(13)

In general, for a \( d_c \times d_c \) coin matrix \( A \), we put

\[ \zeta(A, T^d_N, u) = \det \left( I_{d_cN^d} - u\hat{M}_A \right)^{-1/N^d}. \]

We should remark that the walk-type zeta function becomes the generalized zeta function \( \zeta(T^d_N, u) \) in [4] for the Grover walk (F-type). So we write the walk-type zeta function with a coin matrix \( A \) as \( \zeta(A, T^d_N, u) \). Moreover, our walk is defined on the “site” \( x(\in T^d_N) \). On the other hand, the walk in [4] is defined on the “arc” (i.e., oriented edge). However, both of the walks are the same for the torus case.

By Eqs. (11), (12) and (13), we get

\[ \zeta(A, T^d_N, u)^{-1} = \exp \left[ \frac{1}{N^d} \sum_{\tilde{k} \in \tilde{K}^d_N} \log \left\{ \det \left( I_{2d} - u\hat{M}_A(\tilde{k}) \right) \right\} \right]. \]

Sometimes we write \( \sum_{k \in K^d_N} \) instead of \( \sum_{\tilde{k} \in \tilde{K}^d_N} \). Noting \( \tilde{k}_j = 2\pi k_j / N \) \( (j = 1, 2, \ldots, d) \) and taking a limit as \( N \to \infty \), we show

\[ \lim_{N \to \infty} \zeta(A, T^d_N, u)^{-1} = \exp \left[ \int_{[0,2\pi)^d} \log \left\{ \det \left( I_{2d} - u\hat{M}_A(\Theta^{(d)}) \right) \right\} \, d\Theta^{(d)}_{\text{unif}} \right], \]

if the limit exists. We should note that when we take a limit as \( N \to \infty \), we assume that the limit exists throughout this paper. Here \( \Theta^{(d)} = (\theta_1, \theta_2, \ldots, \theta_d) (\in [0,2\pi)^d) \) and \( d\Theta^{(d)}_{\text{unif}} \) denotes the uniform measure on \( [0,2\pi)^d \), that is,

\[ d\Theta^{(d)}_{\text{unif}} = \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi}. \]

Then the following result was obtained in [5].

**Theorem 1** (Komatsu, Konno and Sato [5]).

\[ \zeta(A, T^d_N, u)^{-1} = \exp \left[ \frac{1}{N^d} \sum_{\tilde{k} \in \tilde{K}^d_N} \log \left\{ \det \left( F(\tilde{k}, u) \right) \right\} \right], \]

\[ \lim_{N \to \infty} \zeta(A, T^d_N, u)^{-1} = \exp \left[ \int_{[0,2\pi)^d} \log \left\{ \det \left( F(\Theta^{(d)}, u) \right) \right\} \, d\Theta^{(d)}_{\text{unif}} \right], \]

where

\[ F(w, u) = I_{2d} - u\hat{M}_A(w), \]

with \( w = (w_1, w_2, \ldots, w_d) \in \mathbb{R}^d \).

Furthermore, we define \( C_r(A, T^d_N) \) by

\[ \overline{\zeta}(A, T^d_N, u) = \exp \left( \sum_{r=1}^{\infty} \frac{C_r(A, T^d_N)}{r} u^r \right). \]  

(14)
Sometimes we write $C_r(A, T^d_N)$ by $C_r$ for short. Combining Eq. (13) with Eq. (14) gives
\[
\det \left( I_{2dN^d} - uMA \right)^{-1/N^d} = \exp \left( \sum_{r=1}^{\infty} \frac{C_r}{r} u^r \right).
\]
Thus we get
\[
- \frac{1}{N^d} \log \left\{ \det \left( I_{2dN^d} - uMA \right) \right\} = \sum_{r=1}^{\infty} \frac{C_r}{r} u^r.
\]
Equation (15) follows from Eq. (11) that the left-hand of Eq. (15) becomes
\[
- \frac{1}{N^d} \log \left\{ \det \left( I_{2dN^d} - u\tilde{M}A \right) \right\} = - \frac{1}{N^d} \log \left\{ \det \left( I_{2dN^d} - u\hat{M}A \right) \right\} = \sum_{j=1}^{2d} \sum_{k \in \mathbb{Z}^d_N} \left( \lambda_j(k) \right)^r = \frac{1}{N^d} \sum_{j=1}^{2d} \sum_{k \in \mathbb{Z}^d_N} \sum_{r=1}^{\infty} \left( \lambda_j(k) \right)^r u^r.
\]
By this and the right-hand of Eq. (15), we have
\[
C_r(A, T^d_N) = \frac{1}{N^d} \sum_{j=1}^{2d} \sum_{k \in \mathbb{Z}^d_N} \left( \lambda_j(k) \right)^r = \frac{1}{N^d} \sum_{j=1}^{2d} \int_{0,2\pi} \lambda_j \left( \Theta^{(d)} \right)^r d\Theta_{\text{unif}}^{(d)}.
\]
Noting $\tilde{k}_j = 2\pi k_j / N$ ($j = 1, 2, \ldots, d$) and taking a limit as $N \to \infty$, we get
\[
\lim_{N \to \infty} C_r(A, T^d_N) = \frac{1}{N^d} \sum_{j=1}^{2d} \int_{0,2\pi} \lambda_j \left( \Theta^{(d)} \right)^r d\Theta_{\text{unif}}^{(d)} = \operatorname{Tr} \left( \hat{M}(\Theta^{(d)})^r \right) d\Theta_{\text{unif}}^{(d)} = \operatorname{Tr} \left( \Phi_r(\infty)(0) \right).\]

An interesting point is that $\Phi_r(\infty)(0)$ is the return “matrix weight” at time $r$ for the walk on not $T^d_N$ but $Z^d$. We should remark that in general $\operatorname{Tr}(\Phi_r(\infty)(0))$ is not the same as the return probability at time $r$ for QW and CRW.

From now on, we will present the result on only “$\lim_{N \to \infty}$” for $\tilde{C}(A, T^d_N)$ and $C_r(A, T^d_N)$, since the corresponding expression for “without $\lim_{N \to \infty}$” is the essentially same (see Theorems 1 and 2, for example).

## 3 One-Dimensional Three-State QW and CRW

This section is devoted to the three-state QW (case (i)) and CRW (case (ii)) on the one-dimensional torus $T^d_N$. Remark that a detailed study on the two-state QW and CRW was given in [5].
(i) QW case.

We consider the following $3 \times 3$ coin matrix $A^{(m)}_{QW}$ (M-type) and $A^{(f)}_{QW}$ (F-type) introduced by Machida [10].

$$A^{(m)}_{QW} = \begin{pmatrix} 
\frac{1 + \cos \eta}{2} & \frac{1 - \cos \eta}{2} & \frac{1 + \cos \eta}{2} \\
\frac{\sin \eta}{\sqrt{2}} & \frac{\sin \eta}{\sqrt{2}} & \frac{\sin \eta}{\sqrt{2}} \\
\frac{1 - \cos \eta}{2} & \frac{1 - \cos \eta}{2} & \frac{1 + \cos \eta}{2}
\end{pmatrix}, \quad A^{(f)}_{QW} = \begin{pmatrix} 
\frac{1 - \cos \eta}{2} & \frac{\sin \eta}{\sqrt{2}} & \frac{\sin \eta}{\sqrt{2}} \\
\frac{\sin \eta}{\sqrt{2}} & \frac{\cos \eta}{2} & \frac{\sin \eta}{\sqrt{2}} \\
\frac{1 + \cos \eta}{2} & \frac{\sin \eta}{\sqrt{2}} & \frac{\sin \eta}{\sqrt{2}}
\end{pmatrix},$$

for $\eta \in [0, 2\pi)$. If $\cos \eta = -1/3$, then the QW becomes the so-called Grover walk which is one of the most well-investigated model in the study of QWs. In this model, we take the projections

$$P_{-1} = \begin{pmatrix} 
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad P_0 = \begin{pmatrix} 
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad P_1 = \begin{pmatrix} 
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.$$

Then $3 \times 3$ matrix $\tilde{M}_{A^{(s)}_{QW}}(\tilde{k})$ for $s \in \{m, f\}$ is defined by

$$\tilde{M}_{A^{(s)}_{QW}}(\tilde{k}) = e^{i\frac{\tilde{k}}{2}}P_{-1}A^{(s)}_{QW} + P_0A^{(s)}_{QW} + e^{-i\frac{\tilde{k}}{2}}P_1A^{(s)}_{QW}.$$ 

We see that the walker moves at each step one unit to the left with $P_{-1}A^{(s)}_{QW}$ or one unit to the right with $P_1A^{(s)}_{QW}$ or stays with $P_0A^{(s)}_{QW}$. Thus we get

**Proposition 1.**

$$\lim_{N\to\infty} T_n^{(s)}(A^{(s)}_{QW}, T_N^{1}, u)^{-1} = (1 + (-1)^{\delta(s)}u) \exp \left[ \int_0^{2\pi} \log \left\{ F^{(s)}_{QW}(\theta, u) \right\} \frac{d\theta}{2\pi} \right],$$

(18)

for $s \in \{m, f\}$, where $\delta(s) = 1$ for $s = m$, $\delta(s) = 0$ for $s = f$ and

$$F^{(s)}_{QW}(\theta, u) = 1 - \left[ (-1)^{\delta(s)} \cos \eta + \left\{ (-1)^{\delta(s)} - \cos \eta \right\} \cos \theta \right] u + u^2.$$

Remark that the leading factor $(1 + (-1)^{\delta(s)}u)$ of the right-hand side of Eq. (18) corresponds to localization of the three-state Grover walk on $\mathbb{Z}$ (see Machida [10], for example). Localization means that there exists an initial state such that limsup for time $n$ of the probability that the walker returns to the starting location at time $n$ is positive. Specially, in the case of $\cos \eta = -1/3$ (Grover walk), we have

**Corollary 1.**

$$F^{(s)}_{QW}(\theta, u) = 1 - \frac{(-1)^{\delta(s)} \cdot 2}{3} \left( 1 + 2 \cos \theta + \delta(s)(1 - \cos \theta) \right) u + u^2.$$

This result corresponds to Corollary 8 in [5]. Moreover, we will compute $C_r(A^{(s)}_{QW}, T_N^{1})$ for this QW. Then we see that the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $\tilde{M}_{A^{(s)}_{QW}}(\tilde{k})$ can be written as follows.

$$\lambda_1 = -(-1)^{\delta(s)}, \quad \lambda_2 = \frac{t + i\sqrt{4 - t^2}}{2}, \quad \lambda_3 = \frac{t - i\sqrt{4 - t^2}}{2},$$

where

$$t = t(\tilde{k}) = (-1)^{\delta(s)} \cos \eta + \left\{ (-1)^{\delta(s)} - \cos \eta \right\} \cos \tilde{k}.$$

RemaRK that unitarity of $\tilde{M}_{A^{(s)}_{QW}}(\tilde{k})$ implies $|\lambda_1| = |\lambda_2| = |\lambda_3| = 1$. From Theorem 2 we get
Proposition 2.

\[ \lim_{N \to \infty} C_r(A^{(s)}_{QW}, T^1_N) = \int_0^{2\pi} G^{(s)}(\theta) \frac{d\theta}{2\pi}, \]

for \( s \in \{m, f\} \), where

\[ G^{(s)}(\theta) = \left\{ (-1)^{\delta(s)} \right\}^r + \left( \frac{t + i\sqrt{4 - t^2}}{2} \right)^r + \left( \frac{t - i\sqrt{4 - t^2}}{2} \right)^r, \]

\( t = t(\theta) = (-1)^{\delta(s)} + \cos \eta \left\{ (-1)^{\delta(s)} - \cos \eta \right\} \cos \theta. \)

(ii) CRW case.

We consider the following 3 \( \times \) 3 coin matrix \( A^{(s)}_{CRW} \) given by using the Hadamard product of coin matrix \( A^{(s)}_{QW} \) for \( s \in \{m, f\} \). That is,

\[ A^{(m)}_{CRW} = A^{(m)}_{QW} \odot A^{(m)}_{QW} = \begin{bmatrix} \frac{(1+\cos \eta)^2}{4} & \frac{\sin^2 \eta}{2} & \frac{(1-\cos \eta)^2}{4} \\ \frac{\sin^2 \eta}{2} & \cos^2 \eta & \frac{\sin^2 \eta}{2} \\ \frac{(1-\cos \eta)^2}{4} & \frac{\sin^2 \eta}{2} & \frac{(1+\cos \eta)^2}{4} \end{bmatrix}, \]

\[ A^{(f)}_{CRW} = A^{(f)}_{QW} \odot A^{(f)}_{QW} = \begin{bmatrix} \frac{(1-\cos \eta)^2}{4} & \frac{\sin^2 \eta}{2} & \frac{(1+\cos \eta)^2}{4} \\ \frac{\sin^2 \eta}{2} & \cos^2 \eta & \frac{\sin^2 \eta}{2} \\ \frac{(1+\cos \eta)^2}{4} & \frac{\sin^2 \eta}{2} & \frac{(1-\cos \eta)^2}{4} \end{bmatrix}, \]

where \( \odot \) is the Hadamard product. For this model, similarly we obtain

Proposition 3.

\[ \lim_{N \to \infty} \zeta \left( A^{(s)}_{CRW}, T^1_N, u \right)^{-1} = \exp \left[ \int_0^{2\pi} \log \left( F^{(s)}_{CRW}(\theta, u) \right) \frac{d\theta}{2\pi} \right], \]

for \( s \in \{m, f\} \), where

\[ F^{(s)}_{CRW}(\theta, u) = 1 - \frac{u}{2} \left[ \left\{ 1 - (-1)^{\delta(s)} \cos \eta \right\}^2 \cos \theta + 2 \cos^2 \eta \right] \]

\[ - \frac{(-1)^{\delta(s)} \cdot u^2}{2} \left[ \left\{ 1 - (-1)^{\delta(s)} \cos \eta \right\}^2 \left\{ (-1)^{\delta(s)} + 2 \cos \eta \right\} \cos \theta + \cos \eta(1 + \cos^2 \eta) \right] \]

\[ - \frac{(-1)^{\delta(s)} \cdot u^3}{2} \cos \eta(1 - 3 \cos^2 \eta). \]

Here \( \delta(s) = 1 \) for \( s = m \), \( \delta(s) = 0 \) for \( s = f \).

4 One-Dimensional Four-State QW and CRW

In this section, we deal with the four-state QW (case (i)) and CRW (case (ii)) on the one-dimensional torus \( T^1_N \).

(i) QW case.
We consider the following $4 \times 4$ coin matrix $A_{QW}^{(m)}$ (M-type) and $A_{QW}^{(f)}$ (F-type) introduced by Watabe et al. [1].

\begin{align}
A_{QW}^{(m)} &= \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}, \\
A_{QW}^{(f)} &= \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix},
\end{align}

where $p, q \in [0, 1]$ and $q = 1 - p$. If $p = q = 1/2$, then the QW becomes the Grover walk. In this model, we take the projections \{P_2, P_1, P_1, P_2\} by

\begin{align*}
P_{-2} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
P_{-1} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
P_1 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
P_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{align*}

For $s \in \{m, f\}$, we define $4 \times 4$ matrix $\tilde{M}_{A_{QW}^{(s)}} (\tilde{k})$ by

\[ \tilde{M}_{A_{QW}^{(s)}} (\tilde{k}) = e^{2ik} P_{-2} A_{QW}^{(s)} + e^{ik} P_{-1} A_{QW}^{(s)} + e^{-ik} P_1 A_{QW}^{(s)} + e^{-2ik} P_2 A_{QW}^{(s)}. \]

The walker moves at each step two unit to the left with $P_{-2} A_{QW}^{(s)}$ or one unit to the left with $P_{-1} A_{QW}^{(s)}$ or one unit to the right with $P_1 A_{QW}^{(s)}$ or two unit to the right with $P_2 A_{QW}^{(s)}$. Then we have

**Proposition 4.**

\[
\begin{align*}
\lim_{N \to \infty} \zeta\left( A_{QW}^{(m)}, T_{N}^{1}, u \right)^{-1} &= \exp \left[ \int_{0}^{2\pi} \log \left\{ F_{QW}^{(m)} (\theta, u) \right\} \frac{d\theta}{2\pi} \right], \\
\lim_{N \to \infty} \zeta\left( A_{QW}^{(f)}, T_{N}^{1}, u \right)^{-1} &= (1 - u^2) \exp \left[ \int_{0}^{2\pi} \log \left\{ F_{QW}^{(f)} (\theta, u) \right\} \frac{d\theta}{2\pi} \right].
\end{align*}
\]

Here

\[
\begin{align*}
F_{QW}^{(m)} (\theta, u) &= 1 + \{ \cos \theta + \cos(2\theta) - 2i p_* (\sin \theta + \sin(2\theta)) \} u - 4i p_* \sin(3\theta) u^2 \\
& \quad - \{ \cos \theta + \cos(2\theta) + 2i p_* (\sin \theta + \sin(2\theta)) \} u^3 + u^4, \\
F_{QW}^{(f)} (\theta, u) &= 1 - \sqrt{1 - 4p_*^2} (\cos \theta + \cos(2\theta)) u + u^2,
\end{align*}
\]

where $p_* = p - 1/2 \in [-1/2, 1/2])$.

As a special case, if $p_* = 0$ ($p = 1/2$), then the QW becomes the Grover walk. Then we obtain

**Corollary 2.**

\[
\begin{align*}
\lim_{N \to \infty} \zeta\left( A_{QW}^{(s)}, T_{N}^{1}, u \right)^{-1} &= (1 - u^2) \exp \left[ \int_{0}^{2\pi} \log \left\{ 1 - (-1)^{s} (\cos \theta + \cos(2\theta)) u + u^2 \right\} \frac{d\theta}{2\pi} \right],
\end{align*}
\]

where $\delta(s) = 1$ for $s = m$, $\delta(s) = 0$ for $s = f$.
We note that Inui and Konno [2] investigated our M-type model with \( p_* = 0 \) case and showed localization occurs. In fact, Proposition 4 implies that concerning the M-type, localization occurs for only \( p_* = 0 \). In contrast to M-type model, localization occurs for F-type model with any \( p_* \) showed localization occurs. In contrast, Proposition 4 implies that concerning the M-type, localization occurs for only \( p_* = 0 \). In contrast to M-type model, localization occurs for F-type model with any \( p_* \) showed localization occurs.

From now on, we focus on F-type model, since \( F_{QW} \) is a quadratic polynomial with respect to \( u \). Then the eigenvalues \( \lambda \) of \( M_{A^{(s)}_{QW}}(\tilde{k}) \) with \( |\lambda| = 1 \) can be written as follows.

\[
\lambda = \pm 1, \quad \alpha_q(\tilde{k}) \pm i \sqrt{1 - \alpha_q(\tilde{k})^2},
\]

where

\[
\alpha_q(\tilde{k}) = \frac{1}{2} \sqrt{1 - 4p_*^2} (\cos \tilde{k} + \cos(2\tilde{k})).
\]

Then we obtain

**Proposition 5.**

\[
\lim_{N \to \infty} C_r(A^{(f)}_{QW}, T_N^1) = \int_0^{2\pi} G^{(f)}_{QW}(\theta) \frac{d\theta}{2\pi},
\]

where

\[
G^{(f)}_{QW}(\theta) = 1 + (-1)^r + \left( \alpha_q(\theta) + i \sqrt{1 - \alpha_q(\theta)^2} \right)^r + \left( \alpha_q(\theta) - i \sqrt{1 - \alpha_q(\theta)^2} \right)^r.
\]

(ii) CRW case.

As in the case of the one-dimensional three-state CRW in the previous section, we treat the following \( 4 \times 4 \) coin matrix \( A^{(s)}_{CRW} \) given by using the Hadamard product of coin matrix \( A^{(s)}_{QW} \) for \( s \in \{ m, f \} \) as follows.

\[
A^{(m)}_{CRW} = A^{(m)}_{QW} \odot A^{(m)}_{QW} = \begin{pmatrix} (p-1)^2 & p^2 & pq & pq \\ p^2 & (p-1)^2 & pq & pq \\ pq & pq & (q-1)^2 & q^2 \\ pq & pq & q^2 & (q-1)^2 \end{pmatrix},
\]

\[
A^{(f)}_{CRW} = A^{(f)}_{QW} \odot A^{(f)}_{QW} = \begin{pmatrix} \text{\( p \times p \)} & \text{\( q \times q \)} \\ \text{\( p \times p \)} & \text{\( q \times q \)} \end{pmatrix},
\]

where \( p, q \in [0,1] \) and \( q = 1 - p \). If \( p = q = 1/2 \), the CRW becomes a RW. For \( s \in \{ m, f \} \), like QW, we define \( 4 \times 4 \) matrix \( \tilde{M}_{A^{(s)}_{CRW}}(\tilde{k}) \) by

\[
\tilde{M}_{A^{(s)}_{CRW}}(\tilde{k}) = e^{2i\tilde{k}} P_{-2} A^{(s)}_{CRW} + e^{i\tilde{k}} P_{-1} A^{(s)}_{CRW} + e^{-i\tilde{k}} P_i A^{(s)}_{CRW} + e^{-2i\tilde{k}} P_2 A^{(s)}_{CRW}.
\]

Thus we have

**Proposition 6.**

\[
\lim_{N \to \infty} \mathcal{T} \left( A^{(m)}_{CRW}, T_N^1, u \right)^{-1} = \exp \left[ \int_0^{2\pi} \log \left\{ F^{(m)}_{CRW}(\theta, u) \right\} \frac{d\theta}{2\pi} \right],
\]

\[
\lim_{N \to \infty} \mathcal{T} \left( A^{(f)}_{CRW}, T_N^1, u \right)^{-1} = (1 + 4p_*^2u^2) \exp \left[ \int_0^{2\pi} \log \left\{ F^{(f)}_{CRW}(\theta, u) \right\} \frac{d\theta}{2\pi} \right],
\]
where \( p_* = p - 1/2 \) and

\[
F_{\text{CRW}}^{(m)}(\theta, u) = 1 - \frac{1}{2} \left\{ (1 + 4p_*^2)(\cos \theta + \cos(2\theta)) - 4ip_*(\sin \theta + \sin(2\theta)) \right\} u \\
- 2ip_*(1 + 4p_*^2) \sin(3\theta) u^2 \\
+ 2p_*^2 \left\{ (1 + 4p_*^2)(\cos \theta + \cos(2\theta)) + 4ip_*(\sin \theta + \sin(2\theta)) \right\} u^3 - 16p_*^4 u^4,
\]

\[
F_{\text{CRW}}^{(f)}(\theta, u) = 1 - \frac{1}{2} (1 - 4p_*^2)(\cos \theta + \cos(2\theta)) u - 4p_*^2 u^2.
\]

When \( p_* = 0 \) (\( p = 1/2 \)), our CRW becomes a four-state RW. In this model, Proposition 6 implies

\[
\lim_{N \to \infty} \mathcal{C} \left( A_{\text{CRW}}^{(s)}(T_N^1) \right)^{-1} = \exp \left[ \int_0^{2\pi} \left\{ 1 - \frac{u}{2} (\cos \theta + \cos(2\theta)) \right\} \frac{d\theta}{2\pi} \right],
\]

for \( s \in \{m, f\} \). Note that this result is equivalent to Corollary 5 in Section 5.

From now on, we focus on F-type model, since \( F_{\text{CRW}}^{(f)}(\theta, u) \) is a quadratic polynomial with respect to \( u \). Then the eigenvalues \( \lambda \) of \( M_{\text{A}_{\text{CRW}}^{(f)}}(\tilde{k}) \) can be written as follows.

\[
\lambda = \pm 2ip_*, \quad \alpha_c(\tilde{k}) \pm \sqrt{\alpha_c(\tilde{k})^2 + 4p_*^2}
\]

where

\[
\alpha_c(\tilde{k}) = \frac{1}{4} (1 - 4p_*^2)^2 (\cos \tilde{k} + \cos(2\tilde{k})).
\]

Therefore we have

**Proposition 7.**

\[
\lim_{N \to \infty} C_r( A_{\text{CRW}}^{(f)}(T_N^1) ) = \int_0^{2\pi} G_{\text{CRW}}^{(f)}(\theta) \frac{d\theta}{2\pi},
\]

where

\[
G_{\text{CRW}}^{(f)}(\theta) = (2ip_*)^r + (-2ip_*)^r + \left( \alpha_c(\theta) + \sqrt{\alpha_c(\theta)^2 + 4p_*^2} \right)^r + \left( \alpha_c(\theta) - \sqrt{\alpha_c(\theta)^2 + 4p_*^2} \right)^r
\]

5 One-Dimensional Multi-State RW

In Sections 3 and 4, we dealt with the three- and four-state QW and CRW on the one-dimensional torus \( T_N^1 \). When the number of the state is larger than five, the similar study becomes complicated. Therefore, we treat the RW with multi-state on \( T_N^1 \) in this section. The random walker considered here jumps at each step to location \( x \in \{-L, -(L-1), \ldots, L-1, L\} \) with probability \( p_x \), where \( L \in \mathbb{Z}_+ \). Here \( \{p_x\} \) satisfies \( p_x \in [0, 1] \) and \( \sum_{x=-L}^{L} p_x = 1 \). We should note that RW is a special case of CRW and its coin matrix \( A_{\text{RW}} \) can be considered as real value 1, that is, \( A_{\text{RW}} = 1 \). So we get

\[
\widehat{M}_{A_{\text{RW}}}(\tilde{k}) = \sum_{x=-L}^{L} e^{-ix\tilde{k}} p_x.
\]

Then we obtain
Proposition 8.

\[
\lim_{N \to \infty} \zeta(A_{RW}, T_N^1, u)^{-1} = \exp \left\{ \int_0^{2\pi} \log \left( 1 - u \left( \sum_{x=-L}^{L} e^{-ix\theta} p_x \right) \right) \frac{d\theta}{2\pi} \right\},
\]

\[
\lim_{N \to \infty} C_r(A_{RW}, T_N^1) = \int_0^{2\pi} \left( \sum_{x=-L}^{L} e^{-ix\theta} p_x \right)^r \frac{d\theta}{2\pi}.
\]

From now on, we consider the following special case:

\[p_x = p_* \quad (x \in \{-L, -(L-1), \ldots, L-1, L\}\setminus\{0\}).\]

That is, \(p_x\) is the same as \(p_*\) except the origin. So we have

\[p_0 + 2L \cdot p_* = 1 \quad (p_0, p_* \in [0,1]).\]

Then we see that

\[\hat{M}_{A_{RW}}(\vec{k}) = p_0 + 2p_* \sum_{\ell=1}^{L} \cos(\ell \theta).\]

Therefore we obtain

Proposition 9.

\[
\lim_{N \to \infty} \zeta(A_{RW}, T_N^1, u)^{-1} = \exp \left\{ \int_0^{2\pi} \log \left( 1 - u \left( p_0 + 2p_* \sum_{\ell=1}^{L} \cos(\ell \theta) \right) \right) \frac{d\theta}{2\pi} \right\},
\]

\[
\lim_{N \to \infty} C_r(A_{RW}, T_N^1) = \int_0^{2\pi} \left( p_0 + 2p_* \sum_{\ell=1}^{L} \cos(\ell \theta) \right)^r \frac{d\theta}{2\pi}.
\]

Then the following result is given by Proposition 9 for \(L = 1, p_0 = 0, p_* = 1/2\) case.

Corollary 3.

\[
\lim_{N \to \infty} \zeta(A_{RW}, T_N^1, u)^{-1} = \exp \left\{ \int_0^{2\pi} \log(1 - u \cos \theta) \frac{d\theta}{2\pi} \right\}, \quad (23)
\]

\[
\lim_{N \to \infty} C_r(A_{RW}, T_N^1) = \int_0^{2\pi} (\cos \theta)^r \frac{d\theta}{2\pi}.
\]

We should note that this is equivalent to Corollary 7 in [5]. Moreover, for this case, in order to obtain an explicit form of the right-hand side of Eq. (23), we prepare the following lemma.

Lemma 1.

\[
\log \left( \frac{1 + \sqrt{1 - x^2}}{2} \right) = -\sum_{n=1}^{\infty} \frac{1}{2n} \binom{2n}{n} \left( \frac{x^2}{4} \right)^n.
\]

Proof. Let \(f(x)\) be the left-hand side of Eq. (24). Differentiating \(f(x)\) by \(x\) gives

\[
f'(x) = \frac{1}{x} \left( 1 - \left( 1 - x^2 \right)^{-\frac{1}{2}} \right).
\]
Here, the Taylor expansion of \((1 - x^2)^{-1/2}\) implies the following formula.

\[
(1 - x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{1}{2}\right)^n x^{2n}.
\]

(26)

Substituting Eq. (26) into Eq. (25) leads to

\[
f'(x) = \frac{1}{x} \left\{ 1 - \sum_{n=1}^{\infty} \binom{2n}{n} \left(\frac{1}{2}\right)^n x^{2n} \right\} = -\sum_{n=1}^{\infty} \binom{2n}{n} \left(\frac{1}{2}\right)^n x^{2n-1}.
\]

Therefore we have

\[
f(x) = \int_0^x f'(y) \, dy = -\sum_{n=1}^{\infty} \binom{2n}{n} \left(\frac{1}{2}\right)^n \int_0^x y^{2n-1} \, dy = -\sum_{n=1}^{\infty} \frac{1}{2n} \binom{2n}{n} \left(\frac{1}{2}\right)^n x^{2n}.
\]

Now we have the following explicit form of the right-hand side of Eq. (23).

Corollary 4.

\[
\lim_{N \to \infty} \zeta(A_{RW}, T_N, u)^{-1} = \frac{1 + \sqrt{1 - u^2}}{2}.
\]

Proof. In order to compute the right-hand side of Eq. (23), we see that

\[
\frac{1}{\pi} \int_0^{2\pi} \log(1 - u \cos \theta) \, d\theta = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{u^{2n}}{n} \int_0^{\pi} (\cos \theta)^{2n} \, d\theta
\]

\[
= -\sum_{n=1}^{\infty} \frac{1}{2n} \binom{2n}{n} \left(\frac{u^2}{4}\right)^n = \log \left(\frac{1 + \sqrt{1 - u^2}}{2}\right).
\]

The last equality comes from Lemma 1.

Next the following result can be obtained by Proposition 9 for \(L = 2, \, p_0 = 0, \, p_* = 1/4\) case.

Corollary 5.

\[
\lim_{N \to \infty} \zeta(A_{RW}, T_N, u)^{-1} = \exp \left\{ \int_0^{2\pi} \log \left(1 - \frac{u}{2} (\cos \theta + \cos(2\theta))\right) \, d\theta \right\},
\]

\[
\lim_{N \to \infty} C_r(A_{RW}, T_N) = \int_0^{2\pi} \left\{ \frac{1}{2} (\cos \theta + \cos(2\theta)) \right\}^r \, \frac{d\theta}{2\pi}.
\]

Finally, we consider the case in which the walker moves at each step to each location with the same probability \(p_0 = p_* = 1/(2L + 1)\). In a similar fashion, we have

Proposition 10.

\[
\lim_{N \to \infty} \zeta(A_{RW}, T_N, u)^{-1} = \exp \left\{ \int_0^{2\pi} \log \left(1 - \frac{u}{2L + 1} (\cos((L + 1)\theta) - \cos(L\theta))\right) \, d\theta \right\},
\]

\[
\lim_{N \to \infty} C_r(A_{RW}, T_N) = \int_0^{2\pi} \left\{ \frac{1}{2L + 1} (\cos((L + 1)\theta) - \cos(L\theta)) \right\}^r \, \frac{d\theta}{2\pi}.
\]
6 Two-Dimensional Four-State QW and CRW

In this section, we treat the four-state QW (case (i)) and CRW (case (ii)) on the two-dimensional torus $T^2_N$.

(i) QW case.

We consider the following $4 \times 4$ coin matrix $A^{(m)}_{QW}$ (M-type) and $A^{(f)}_{QW}$ (F-type).

$$A^{(m)}_{QW} = \begin{bmatrix}
p - 1 & p & \sqrt{pq} & \sqrt{pq} 
p & p - 1 & \sqrt{pq} & \sqrt{pq} 
\sqrt{pq} & \sqrt{pq} & q - 1 & q 
\sqrt{pq} & \sqrt{pq} & q & q - 1
\end{bmatrix}, \quad (27)$$

$$A^{(f)}_{QW} = \begin{bmatrix}
p - 1 & p & \sqrt{pq} & \sqrt{pq} 
p & p - 1 & \sqrt{pq} & \sqrt{pq} 
\sqrt{pq} & \sqrt{pq} & q - 1 & q 
\sqrt{pq} & \sqrt{pq} & q & q - 1
\end{bmatrix} = (I_2 \otimes \sigma) A^{(m)}_{QW}, \quad (28)$$

where $p, q \in [0, 1]$ and $q = 1 - p$. If $p = q = 1/2$, then the QW becomes the Grover walk. Note that Eq. (27) equals Eq. (19), however, Eq. (28) (given by Eq. (3)) does not equal Eq. (20). In this model, we take the projections $P_i$ for $s$.

For $s \in \{m, f\}$, we define $4 \times 4$ matrix $\tilde{M}_{A^{(s)}_{QW}}(\tilde{k}_1, \tilde{k}_2)$ as follows.

$$\tilde{M}_{A^{(s)}_{QW}}(\tilde{k}_1, \tilde{k}_2) = e^{ik_1} P_1 A^{(s)}_{QW} + e^{-ik_1} P_2 A^{(s)}_{QW} + e^{ik_2} P_3 A^{(s)}_{QW} + e^{-ik_2} P_4 A^{(s)}_{QW}.$$ 

The walker moves at each step one unit to the left with $P_1 A^{(s)}_{QW}$, or one unit to the right with $P_2 A^{(s)}_{QW}$, or one unit to the down with $P_3 A^{(s)}_{QW}$, or one unit to the up with $P_4 A^{(s)}_{QW}$. Thus we obtain

**Proposition 11.**

$$\lim_{N \to \infty} \zeta \left( A^{(s)}_{QW}, T^2_N, u \right)^{-1} = (1 - u^2) \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left\{ F^{(s)}(\theta_1, \theta_2, u) \right\} \frac{d\theta_1 d\theta_2}{2\pi^2} \right],$$

for $s \in \{m, f\}$. Here $p_\ast = p - 1/2$ and

$$F^{(s)}_{QW}(\theta_1, \theta_2, u) = 1 - (-1)^{\delta(s)} \left[ \left( 1 + (-1)^{\delta(s)} \cdot 2p_\ast \right) \cos \theta_1 + \left( 1 - (-1)^{\delta(s)} \cdot 2p_\ast \right) \cos \theta_2 \right] u + u^2,$$

where $\delta(s) = 1$ for $s = m$, $\delta(s) = 0$ for $s = f$.

Specially, Proposition 11 for $p = 1/2$ (Grover walk) gives

**Corollary 6.**

$$F^{(s)}_{QW}(\theta_1, \theta_2, u) = 1 - (-1)^{\delta(s)} (\cos \theta_1 + \cos \theta_2) u + u^2,$$

for $s \in \{m, f\}$. Here $\delta(s) = 1$ for $s = m$, $\delta(s) = 0$ for $s = f$.
Proposition 13.

Then we get

$$\lambda = \pm 1, \quad \beta_q^{(s)}(k_1, k_2) \pm i \sqrt{1 - \beta_q^{(s)}(k_1, k_2)^2},$$

where \( p_* = p - 1/2 \) and

$$\beta_q^{(s)}(k_1, k_2) = \left( \frac{1}{2} - \delta(s) + p_* \right) \cos k_1 + \left( \frac{1}{2} - \delta(s) - p_* \right) \cos k_2.$$  

Then we have

Proposition 12.

$$\lim_{N \to \infty} C_r(A^{(s)}_{QW}, T_N^2) = \int_0^{2\pi} \int_0^{2\pi} G_{QW}(\theta_1, \theta_2) \frac{d\theta_1 \, d\theta_2}{2\pi \, 2\pi},$$

for \( s \in \{m, f\} \), where

$$G_{QW}(\theta_1, \theta_2) = 1 + (-1)^r + \left( \beta_q^{(s)}(\theta_1, \theta_2) + i \sqrt{1 - \beta_q^{(s)}(\theta_1, \theta_2)^2} \right)^r$$

$$+ \left( \beta_q^{(s)}(\theta_1, \theta_2) - i \sqrt{1 - \beta_q^{(s)}(\theta_1, \theta_2)^2} \right)^r,$$

$$\beta_q^{(s)}(\theta_1, \theta_2) = \left( \frac{1}{2} - \delta(s) + p_* \right) \cos \theta_1 + \left( \frac{1}{2} - \delta(s) - p_* \right) \cos \theta_2.$$  

(ii) CRW case.

As in the case of the one-dimensional four-state CRW in Section 4, we treat the following 4 \times 4 coin matrix \( A_{CRW}^{(s)} \) given by using the Hadamard product of coin matrix \( A_{QW}^{(s)} \) for \( s \in \{m, f\} \) as follows.

$$A_{CRW}^{(m)} = A_{QW}^{(m)} \odot A_{QW}^{(m)} = \begin{bmatrix} (p - 1)^2 & p^2 & pq & pq \\ p^2 & (p - 1)^2 & pq & pq \\ pq & pq & (q - 1)^2 & q^2 \\ pq & pq & q^2 & (q - 1)^2 \end{bmatrix},$$

$$A_{CRW}^{(f)} = A_{QW}^{(f)} \odot A_{QW}^{(f)} = \begin{bmatrix} (p - 1)^2 & p^2 & pq & pq \\ p^2 & (p - 1)^2 & pq & pq \\ pq & pq & q^2 & (q - 1)^2 \\ pq & pq & (q - 1)^2 & q^2 \end{bmatrix} = (I_2 \otimes \sigma) A_{CRW}^{(m)},$$

(29)

(30)

where \( p, q \in [0, 1] \) and \( q = 1 - p \). If \( p = q = 1/2 \), the CRW becomes a RW. Note that Eq. (29) is equal to Eq. (21), however, Eq. (30) (given by Eq. (3)) is not equal to Eq. (22).

For \( s \in \{m, f\} \), like QW, we define 4 \times 4 matrix \( \widetilde{M}_{A_{CRW}^{(s)}}(k_1, k_2) \) by

$$\widetilde{M}_{A_{CRW}^{(s)}}(k_1, k_2) = e^{ik_1} P_1 A_{CRW}^{(s)} + e^{-ik_1} P_2 A_{CRW}^{(s)} + e^{ik_2} P_3 A_{CRW}^{(s)} + e^{-ik_2} P_4 A_{CRW}^{(s)}.$$  

Then we get

Proposition 13.

$$\lim_{N \to \infty} \zeta \left( A_{CRW}^{(s)}, T_N^2, u \right)^{-1} = (1 - 4p^2u^2) \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left\{ F_{CRW}^{(s)}(\theta_1, \theta_2, u) \right\} \frac{d\theta_1 \, d\theta_2}{2\pi \, 2\pi} \right],$$

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for \( s \in \{m, f\} \). Here
\[
F_{CRW}^{(s)}(\theta_1, \theta_2, u) = 1 - \frac{1}{2} \left[ (-1)^{\delta(s)} + 2p_* \right]^2 \cos \theta_1 + \left[ (-1)^{\delta(s)} - 2p_* \right]^2 \cos \theta_2 \right] u + 4p_*^2 u^2,
\]
where \( p_* = p - 1/2 \).

Furthermore, for \( s \in \{m, f\} \), the eigenvalues \( \lambda \) of \( M_{A_{CRW}^{(s)}}(\vec{k}) \) can be written as follows.
\[
\lambda = \pm 2p_*, \quad \beta_c^{(s)}(\vec{k}_1, \vec{k}_2) \pm \sqrt{\beta_c^{(s)}(\vec{k}_1, \vec{k}_2)^2 - 4p_*^2},
\]
where
\[
\beta_c^{(s)}(\vec{k}_1, \vec{k}_2) = \left( \frac{1}{2} - \delta(s) + p_* \right)^2 \cos \vec{k}_1 + \left( \frac{1}{2} - \delta(s) - p_* \right)^2 \cos \vec{k}_2.
\]

Then we have
\[
\lim_{N \to \infty} C_r(A_{CRW}^{(s)}, T_N^2) = \int_0^{2\pi} \int_0^{2\pi} G_{CRW}^{(s)}(\theta_1, \theta_2) \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi},
\]
for \( s \in \{m, f\} \), where
\[
G_{CRW}^{(s)}(\theta_1, \theta_2) = (2p_*)^r + (-2p_*)^r + \left( \frac{\beta_c^{(s)}(\theta_1, \theta_2)}{\beta_c^{(s)}(\vec{k}_1, \vec{k}_2)^2 - 4p_*^2} \right)^r - \frac{\beta_c^{(s)}(\theta_1, \theta_2) - \sqrt{\beta_c^{(s)}(\vec{k}_1, \vec{k}_2)^2 - 4p_*^2}}{\beta_c^{(s)}(\vec{k}_1, \vec{k}_2)^2 - 4p_*^2}^r,
\]
\[
\beta_c^{(s)}(\theta_1, \theta_2) = \left( \frac{1}{2} - \delta(s) + p_* \right)^2 \cos \theta_1 + \left( \frac{1}{2} - \delta(s) - p_* \right)^2 \cos \theta_2.
\]

## 7 Generalized Grover Matrix

In this section, we introduce a new class of models determined by the general Grover matrix \( U(a) \) with parameter \( a \in [0, 1] \) which connects the positive-support of the Grover matrix \( (a = 0) \) and the Grover matrix \( (a = 1) \). In fact, \( d_c \times d_c \) general Grover matrix \( U(a) = [U(a)_{ij}] \) is defined by
\[
U(a)_{ij} = \left( \frac{2}{d_c} - 1 \right) a + 1 - \delta_{ij} = \begin{cases} \frac{2}{d_c} a - 1 & (i = j), \\ \frac{2}{d_c} - 1 & (i \neq j), \end{cases}
\]
for \( a \in [0, 1] \), where \( \delta_{ij} = 1 (i = j), 0 (i \neq j) \). We should remark that if \( d_c = 2 \), then \( U(a) \) does not depend on \( a \). That is,
\[
U(a) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (a \in [0, 1]).
\]

Thus \( U(a) \) is unitary for any \( a \in [0, 1] \) if \( d_c = 2 \). On the other hand, we see that “\( U(a) \) is unitary” if and only if “\( a = 1 \)” for \( d_c \geq 3 \).
From now on, we compute zeta functions for the models determined by $U(a)$. First, as in a similar fashion of Section 3, we deal with the three-state model on the one-dimensional torus $T_N^1$ defined by the following $3 \times 3$ coin matrix $U^{(m)}(a)$ (M-type) and $U^{(f)}(a)$ (F-type).

$$U^{(m)}(a) = \begin{bmatrix} -\frac{a}{3} & -\frac{a}{3} + 1 & -\frac{a}{3} + 1 \\ -\frac{a}{3} + 1 & -\frac{a}{3} & -\frac{a}{3} + 1 \\ -\frac{a}{3} + 1 & -\frac{a}{3} + 1 & -\frac{a}{3} \end{bmatrix}, \quad U^{(f)}(a) = \begin{bmatrix} -\frac{a}{3} + 1 & -\frac{a}{3} + 1 & -\frac{a}{3} \\ -\frac{a}{3} + 1 & -\frac{a}{3} & -\frac{a}{3} + 1 \\ -\frac{a}{3} + 1 & -\frac{a}{3} + 1 & -\frac{a}{3} + 1 \end{bmatrix}.$$ 

By a similar argument in Section 3 we have

**Proposition 15.**

$$\lim_{N \to \infty} \zeta \left( U^{(m)}(a), T_N^1, u \right)^{-1} = \exp \left[ \int_0^{2\pi} \log \left\{ F^{(m)}(\theta, u, a) \right\} \frac{d\theta}{2\pi} \right],$$

$$\lim_{N \to \infty} \zeta \left( U^{(f)}(a), T_N^1, u \right)^{-1} = (1 + u^2) \exp \left[ \int_0^{2\pi} \log \left\{ F^{(f)}(\theta, u, a) \right\} \frac{d\theta}{2\pi} \right],$$

where

$$F^{(m)}(\theta, u, a) = 1 + \frac{a}{3} (1 + 2 \cos \theta)u + \frac{2a - 3}{3} (1 + 2 \cos \theta)u^2 + (a - 2)u^3,$$

$$F^{(f)}(\theta, u, a) = 1 + \frac{a - 3}{3} (1 + 2 \cos \theta)u - (a - 2)u^2.$$  

Moreover, similarly in Section 4, we treat the four-state model on the one-dimensional torus $T_N^1$ defined by the following $4 \times 4$ coin matrix $U^{(m)}(a)$ (M-type) and $U^{(f)}(a)$ (F-type).

$$U^{(m)}(a) = \begin{bmatrix} -\frac{a}{3} & -\frac{a}{3} + 1 & -\frac{a}{3} + 1 & -\frac{a}{3} + 1 \\ -\frac{a}{3} + 1 & -\frac{a}{3} & -\frac{a}{3} + 1 & -\frac{a}{3} + 1 \\ -\frac{a}{3} + 1 & -\frac{a}{3} + 1 & -\frac{a}{3} & -\frac{a}{3} + 1 \\ -\frac{a}{3} + 1 & -\frac{a}{3} + 1 & -\frac{a}{3} + 1 & -\frac{a}{3} \end{bmatrix},$$

$$U^{(f)}(a) = \begin{bmatrix} -\frac{a}{3} + 1 & -\frac{a}{3} + 1 & -\frac{a}{3} + 1 & -\frac{a}{3} + 1 \\ -\frac{a}{3} + 1 & -\frac{a}{3} & -\frac{a}{3} + 1 & -\frac{a}{3} + 1 \\ -\frac{a}{3} + 1 & -\frac{a}{3} + 1 & -\frac{a}{3} & -\frac{a}{3} + 1 \\ -\frac{a}{3} + 1 & -\frac{a}{3} + 1 & -\frac{a}{3} + 1 & -\frac{a}{3} + 1 \end{bmatrix}.$$  

Then we get the following result.

**Proposition 16.**

$$\lim_{N \to \infty} \zeta \left( U^{(m)}(a), T_N^1, u \right)^{-1} = \exp \left[ \int_0^{2\pi} \log \left\{ F^{(m)}(\theta, u, a) \right\} \frac{d\theta}{2\pi} \right],$$

$$\lim_{N \to \infty} \zeta \left( U^{(f)}(a), T_N^1, u \right)^{-1} = (1 - u^2) \exp \left[ \int_0^{2\pi} \log \left\{ F^{(f)}(\theta, u, a) \right\} \frac{d\theta}{2\pi} \right],$$

where

$$F^{(m)}(\theta, u, a) = 1 + a(\cos \theta + \cos 2\theta)u + 2(a - 1)(1 + \cos \theta + \cos 3\theta)u^2 + (3a - 4)(\cos \theta + \cos 2\theta)u^3 + (2a - 3)u^4,$$

$$F^{(f)}(\theta, u, a) = 1 + (a - 2)(\cos \theta + \cos 2\theta)u - (2a - 3)u^2.$$  

Finally, as in a similar fashion of Section 5 we consider the four-state model on the two-dimensional torus $T_N^2$ defined by the following $4 \times 4$ coin matrix $U^{(m)}(a)$ (M-type) and
\[ U^{(f)}(a) \] (F-type).

\[
U^{(m)}(a) = \begin{bmatrix}
-a + \frac{1}{2} & -a + 1 & -a + 1 & -a + 1 \\
-a + 1 & -a + 1 & -a + 1 & -a + 1 \\
-a + 1 & -a + 1 & -a + 1 & -a + 1 \\
-a + 1 & -a + 1 & -a + 1 & -a + 1 \\
\end{bmatrix},
\]

(33)

\[
U^{(f)}(a) = \begin{bmatrix}
-a + \frac{1}{2} & -a + \frac{1}{2} & -a + 1 & -a + 1 \\
-a + \frac{1}{2} & -a + \frac{1}{2} & -a + 1 & -a + 1 \\
-a + \frac{1}{2} & -a + \frac{1}{2} & -a + 1 & -a + 1 \\
-a + \frac{1}{2} & -a + \frac{1}{2} & -a + 1 & -a + 1 \\
\end{bmatrix} = (I_2 \otimes \sigma) U^{(m)}(a).
\]

(34)

Remark that Eq. (33) is equal to Eq. (31), however, Eq. (34) (given by Eq. (3)) is not equal to Eq. (32). Then we show

**Proposition 17.**

\[
\lim_{N \to \infty} \zeta \left( \frac{U^{(m)}(a), T_N^2, u}{u} \right)^{-1} = \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left\{ F^{(m)}(\theta_1, \theta_2, u, a) \right\} \frac{d\theta_1 d\theta_2}{2\pi 2\pi} \right],
\]

\[
\lim_{N \to \infty} \zeta \left( \frac{U^{(f)}(a), T_N^2, u}{u} \right)^{-1} = (1 - u^2) \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left\{ F^{(f)}(\theta_1, \theta_2, u, a) \right\} \frac{d\theta_1 d\theta_2}{2\pi 2\pi} \right],
\]

where

\[
F^{(m)}(\theta_1, \theta_2, u, a) = 1 + a(\cos \theta_1 + \cos \theta_2)u + 2(a - 1)(1 + 2 \cos \theta_1 \cos \theta_2)u^2 \\
+ (3a - 4)(\cos \theta_1 + \cos \theta_2)u^3 + (2a - 3)u^4,
\]

\[
F^{(f)}(\theta_1, \theta_2, u, a) = 1 + (a - 2)(\cos \theta_1 + \cos \theta_2)u - (2a - 3)u^2.
\]

As for the M-type model, if \(a = 0\) and \(a = 1\), then we have the following result for the positive-support of Grover matrix and the Grover matrix, respectively.

**Corollary 7.**

\[
\lim_{N \to \infty} \zeta \left( \frac{U^{(m)}(0), T_N^2, u}{u} \right)^{-1} = \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left\{ F^{(m)}(\theta_1, \theta_2, u, 0) \right\} \frac{d\theta_1 d\theta_2}{2\pi 2\pi} \right],
\]

\[
\lim_{N \to \infty} \zeta \left( \frac{U^{(m)}(1), T_N^2, u}{u} \right)^{-1} = (1 - u^2) \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left\{ F^{(m)}(\theta_1, \theta_2, u, 1) \right\} \frac{d\theta_1 d\theta_2}{2\pi 2\pi} \right],
\]

where

\[
F^{(m)}(\theta_1, \theta_2, u, 0) = 1 - 2(1 + 2 \cos \theta_1 \cos \theta_2)u^2 - 4(\cos \theta_1 + \cos \theta_2)u^3 - 3u^4,
\]

\[
F^{(m)}(\theta_1, \theta_2, u, 1) = 1 + (\cos \theta_1 + \cos \theta_2)u + u^2.
\]

We should remark that the results for \(a = 0\) and \(a = 1\) correspond to Corollaries 13 and 11 in [5], respectively.

As for the F-type model, if \(a = 0\) and \(a = 1\), then we have the following result for the positive-support of Grover matrix and the Grover matrix, respectively.

**Corollary 8.**

\[
\lim_{N \to \infty} \zeta \left( \frac{U^{(f)}(0), T_N^2, u}{u} \right)^{-1} = (1 - u^2) \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left\{ F^{(f)}(\theta_1, \theta_2, u, 0) \right\} \frac{d\theta_1 d\theta_2}{2\pi 2\pi} \right],
\]

\[
\lim_{N \to \infty} \zeta \left( \frac{U^{(f)}(1), T_N^2, u}{u} \right)^{-1} = (1 - u^2) \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left\{ F^{(f)}(\theta_1, \theta_2, u, 1) \right\} \frac{d\theta_1 d\theta_2}{2\pi 2\pi} \right],
\]
Note that the matrix $G$ is a connected graph with $n$ vertices and $m$ edges for each $v \in V(G)$. The adjacency matrix $A(G) = [a_{ij}]$ is the square matrix such that $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent, and $a_{ij} = 0$ otherwise. If $\deg_{G} v = k$ (constant) for each $v \in V(G)$, then $G$ is called $k$-regular. Moreover, the $n \times n$ matrix $P_n = \mathbf{P}(G) = [P_{uv}]_{u,v \in V(G)}$ is given as follows:

$$
P_{uv} = \begin{cases} 
\frac{1}{\deg_{G} u} & \text{if } (u, v) \in D(G), \\
0 & \text{otherwise.}
\end{cases}
$$

Note that the matrix $\mathbf{P}(G)$ is the transition probability matrix of the simple random walk on $G$. If $G$ is a $(q+1)$-regular graph, then we have $\mathbf{P}(G) = \frac{1}{q+1} A(G)$.

In this setting, Konno and Sato presented the following result which is called the Konno-Sato theorem here.

**Theorem 3 (Konno and Sato)**. Let $G$ be a simple connected $(q+1)$-regular graph with $n$ vertices and $m$ edges for $q \in \mathbb{Z}_\geq$. Then we have

$$
\det \left( I_{2m} - u \ U^{(j)}(0) \right) = (1 - u^2)^{m-n} \det \left( (1 + qu^2)I_n - (q + 1)uP_n \right),
$$

$$
\det \left( I_{2m} - u \ U^{(j)}(1) \right) = (1 - u^2)^{m-n} \det \left( (1 + u^2)I_n - 2uP_n \right).
$$

Moreover, for the generalized Grover matrix $U(a)$ with $a \in [0, 1]$ bridging the gap between the positive-support of the Grover matrix ($a = 0$) and the Grover matrix ($a = 1$), we can extend the Konno-Sato theorem (Theorem 3) as follows.

**Theorem 4**. Let $G$ be a simple connected $(q+1)$-regular graph with $n$ vertices and $m$ edges for $q \in \mathbb{Z}_\geq$. Then we have

$$
\det \left( I_{2m} - u \ U^{(j)}(a) \right) = (1 - u^2)^{m-n} \det \left[ \left\{ 1 + (q + (1 - q)a) u^2 \right\} I_n - \{1 + q + (1 - q)a\} u P_n \right],
$$

for $a \in [0, 1]$.
Proof. The proof is the almost same as that of Theorem 4.1 in [3], so we omit the details. The essential point is that we change $B_{ef}$, $w_{uv}$, $d_{uv}$ in the following way.

$$B_{ef} = \frac{2}{d_{o(f)}} \rightarrow \left( \frac{2}{d_{o(f)}} - 1 \right) a + 1 = \frac{(1-a)d_{o(f)} + 2a}{d_{o(f)}} \quad (e, f \in D(G)),$$

$$w_{uv} = \frac{2}{d_v} \rightarrow \left( \frac{2}{d_v} - 1 \right) a + 1 = \frac{(1-a)d_v + 2a}{d_v} \quad (u, v \in V(G)),$$

$$d_{uv} = 2 \rightarrow (1-a)d_v + 2a \quad (u, v \in V(G)).$$

We should remark that our walk with F-Type in Walk/Zeta Correspondence is defined on the “site” $x(\in T^d_N)$, on the other hand, the walk in Grover/Zeta Correspondence of [4] is defined on the “arc” (i.e., oriented edge). However, both of the walks are the same for the torus case. This relation holds for the model given by a generalized Grover matrix $U(a)$ with $a \in [0,1]$.

Finally, we focus on the model determined by the $2d \times 2d$ generalized Grover matrix $U(a)$ with $a \in [0,1]$ on the $d$-dimensional torus $T^d_N$ which is $2d$-regular with $n = N^d$ vertices and $m = dN^d$ edges. By using a similar method in “Section 6 Torus case” in [4], from Theorem 4 we obtain the following more general result. Indeed, $a = 0$ case corresponds to Corollary 2 in [4] (Grover/Zeta Correspondence) and Corollary 16 in [5] (Walk/Zeta Correspondence), and $a = 1$ case corresponds to Corollary 1 in [4] (Grover/Zeta Correspondence) and Corollary 14 in [5] (Walk/Zeta Correspondence).

Corollary 9.

$$\zeta \left(U^{(f)}(a), T^d_N, u\right)^{-1} = (1-u^2)^{d-1} \exp \left[ \frac{1}{N^d} \sum_{\tilde{k} \in \mathbb{Z}^d_N} \log \left\{ F^{(f)}(\tilde{k}, u, a) \right\} \right],$$

$$\lim_{N \to \infty} \zeta \left(U^{(f)}(a), T^d_N, u\right)^{-1} = (1-u^2)^{d-1} \exp \left[ \int_{[0,2\pi]^d} \log \left\{ F^{(f)}(\Theta^{(d)}, u, a) \right\} d\Theta^{(d)}_{\text{unif}} \right],$$

where

$$F^{(f)}(w, u, a) = 1 - \frac{2(d + (1-d)a)}{d} \left( \sum_{j=1}^d \cos w_j \right) \cdot u + \{2d - 1 + 2(1-d)a\} \cdot u^2,$$

for $a \in [0,1]$. Here $\tilde{k} = (\tilde{k}_1, \ldots, \tilde{k}_d) \in \mathbb{Z}^d_N$, $\Theta^{(d)} = (\theta_1, \theta_2, \ldots, \theta_d) \in [0,2\pi]^d$, $w = (w_1, w_2, \ldots, w_d) \in \mathbb{R}^d$, and $d\Theta^{(d)}_{\text{unif}}$ denotes the uniform measure on $[0,2\pi]^d$, that is,

$$d\Theta^{(d)}_{\text{unif}} = \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi}.$$

Note that this result comes from an approach based on Grover/Zeta Correspondence. In particular, Corollary 9 for $d = 2$ case is equivalent to Proposition 17 for F-type model, which is given by another approach based on Walk/Zeta Correspondence.

9 Summary

Following the previous paper on Walk/Zeta Correspondence by the first author and his coworkers [5], we calculated the zeta function for various extended classes of QWs and
CRWs on the torus determined by a more general coin matrix via the Fourier analysis. In Section 2, we reviewed Walk/Zeta Correspondence on the torus studied in [5]. In Sections 3 and 4, we investigated the three- and four-state QW and CRW on the one-dimensional torus. Section 5 treated the multi-state RW on the one-dimensional torus. Section 6 dealt with the four-state QW and CRW on the two-dimensional torus. In Section 7, we introduced a new class of models determined by the generalized Grover matrix, which connects the Grover matrix and the positive-support of the Grover matrix. Section 8 presented a generalized version of the Konno-Sato theorem for the new class. As a corollary, we computed the zeta function for the generalized Grover matrix on the \(d\)-dimensional torus. Moreover, we mentioned the relation between Grover/Zeta and Walk/Zeta Correspondences. To extend our class of models determined by the generalized Grover matrix \(U(a)\) to a more general class would be one of the interesting problems [9].

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