Information Theory and Quadrature Rules

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Abstract—Quadrature rules estimate $\int_a^b f(x) \, dx$ when $f$ is defined by a table of $n+1$ values. Every binary string of length $n$ defines a quadrature rule by choosing which endpoint of each interval represents the interval. The standard rules, such as Simpson’s Rule, correspond to strings of low Kolmogorov complexity, making it possible to define new quadrature rules with no smoothness assumptions, as well as in higher dimensions. Error results depend on concepts from compressed sensing. Good quadrature rules exist for “sparse” functions, which also satisfy an error–information duality principle.

I. INTRODUCTION

Researchers have been showing how information theory clarifies results about mathematics and computing ever since Shannon [5] defined the basic concepts. This work considers quadrature or numerical integration from an information theory perspective. The basic problem is to estimate $\int_a^b f(x) \, dx$ from a table of values $f_i = f(x_i)$, $i = 0, \ldots, n$. This kind of problem arises naturally in applications, where, for example, one may only be able to estimate the value of a function during a satellite pass, or at a discrete set of ambient conditions such as temperature, or, in the social sciences, on Tuesdays.

Standard works on numerical analysis (eg., [1], [4]) develop quadrature methods that require one of two conditions that are impossible to guarantee. Many methods (eg, Gaussian quadrature) require evaluation of $f$ at arbitrary points in its domain, which is impossible in the situation at hand. Other methods (eg Newton–Cotes integration; see below) impose smoothness conditions on $f$. This, too, is problematic: imagine the effect of earthquake, phase transition, or scandal on the functions whose measurement is described above.

Estimation without control over the error is unsatisfying. Integrating a function from a table is a kind of signal processing, and ideas from signal reconstruction lead to two error estimates, at least for functions that have a sparse (although perhaps unknown) representation. The first is a kind of error–information duality for integration; briefly, the information in the error is the error in the information. The second is the existence of good quadrature rules for sparse functions. Section V has details, including the definition of sparse.

Here is the outline. Section II defines a primitive quadrature rule for estimating $\int f(x) \, dx$ from any binary string of length $n$. A quadrature program for $\int f(x) \, dx$ is the mean of the estimates from several primitive quadrature rules.

Section III develops

Theorem 1: Each Newton–Cotes estimate for $\int f(x) \, dx$ corresponds to a quadrature program based on strings of low Kolmogorov complexity.

II. QUADRATURE PROGRAMS

The Riemann integral $\int_a^b f(x) \, dx$ depends on having full information about $f$. (By scaling and translation, restricting to integrals over the domain $[0,1]$ causes no loss of generality.) Briefly, the domain is subdivided, and $f$ is sampled in each subdomain. One then takes the limit, as the mesh goes to zero, of the sums $\sum f(x_i^*) \Delta x_i$, where $x_i^*$ is the sample point and $\Delta x_i$ is the size of the corresponding subinterval.

Sampling and, therefore, computing the limit is not feasible when $f$ is known by a table of values $f_i = f(x_i/n)$, $i = 0, \ldots, n$. In this case, one typically chooses one of the endpoints of each interval as the sample point.

For convenience, let $h = 1/n$ denote the size of each subinterval.

Definition 1: A primitive quadrature rule from the binary string $b$ for $f$ is the sum $\sum_{i=0}^n f_i h + \cdots + f_n h$, where

$$f_i = \begin{cases} f_{i-1} & \text{if } b_i = 0 \\ f_i & \text{if } b_i = 1. \end{cases}$$

In other words, the binary string $b$ is an input to the pseudocode program below.

```c
float Quadrature(float f, bool b[], int n, float h) {
    int i = 0;
    float q = 0.0;
    for (i = 0; i < n; i++) {
        if (b[i] == 0) {
            q += h*f[i]; // left endpoint
        } else {
            q += h*f[i+1]; // right endpoint
        }
    }
    return q;
}
```

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Definition 2: A *quadrature rule* is the estimate obtained from the mean of the estimates from a finite set of primitive quadrature rules.

A. Example
Suppose that \( n = 7 \), so \( f \) is defined by \( f_0, \ldots, f_7 \). The string \( b = 0011101 \) yields the estimate
\[
(f_0 + f_1 + f_3 + f_4 + f_5 + f_5 + f_7)h
\]
while \( b = 0001111 \) yields
\[
(f_0 + f_1 + f_2 + f_4 + f_5 + f_6 + f_7)h.
\]

B. Strings of Low Complexity
The binary strings of lowest complexity are 000\ldots0 and 111\ldots1. These correspond to using the left and the right endpoints of each interval, respectively. In the first case, though, the final value \( f_n \) has no effect on the estimate of the integral, while in the second the initial value \( f_0 \) is ignored. A remedy to this situation is to take the mean of the two estimates so obtained. A simple calculation shows that the estimate is then
\[
\frac{f_0 + 2f_1 + \cdots + 2f_{n-1} + f_n}{2n},
\]
which is the well-known trapezoid rule.

Proposition 1: The trapezoid rule is the mean of the quadrature rules 0\ldots0 and 1\ldots1.

Alternatively, the trapezoid rule is the mean of the quadrature rules 0101\ldots01 and 1010\ldots10.

The next most complex strings are 0101\ldots01 and 1010\ldots10. Simpson’s Rule (\( \Pi \)) estimates the integral as
\[
\frac{f_0 + 4f_1 + 2f_2 + \cdots + 4f_{n-1} + f_n}{3n}.
\]

Proposition 2: Simpson’s Rule is the mean of the quadrature rules 0\ldots0, 1\ldots1, and 1010\ldots10.

III. COMPARISON WITH NEWTON–COTES QUADRATURE
More generally, Newton–Cotes Integration uses the Lagrange interpolation polynomial of degree \( d \) to derive an approximation that is exact when \( f \) has degree \( \leq n \); see \( \Pi \). Here are some common Newton–Coles formulas, along with their interpretations as quadrature programs. Notice that in each case the complexity of the strings involved is quite low. Also notice that in each case the estimate has the form
\[
\sum_{i=0}^{n} a_i f_i
\]
with \( \sum a_i = 1 \).

A. \( n = 3 \), or Simpson’s Three–Eights Rule
In this case, the function is sampled at four equally-spaced points \((x_0, y_0), (x_1, y_1), (x_2, y_2), \) and \((x_3, y_3)\).
\[
\int_{x_0}^{x_3} f(x) \, dx \approx \frac{3h}{8} \left[ y_0 + 3y_1 + 3y_2 + y_3 \right].
\]
This estimate is the mean of eight primitive quadrature rules: three from 000, three from 111, one from 100, and one from 110. To confirm, the 000 rule yields \( y_0 + y_1 + y_2 \); the 111 rule yields \( y_1 + y_2 + y_3 \); the 100 rule yields \( y_1 + y_1 + y_2 \); and the 110 rule yields \( y_1 + y_2 + y_2 \). These add up to \( 3y_0 + 9y_1 + 9y_2 + 3y_3 \); divide by 8 to get the mean, and factor out the 3.

B. \( n = 4 \)
In this case, the function is sampled at five equally-spaced points \((x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), \) and \((x_4, y_4)\), and
\[
\int_{x_0}^{x_4} f(x) \, dx \approx \frac{2h}{45} \left[ 7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4 \right].
\]
This corresponds to the mean of 45 primitive quadrature rules: 12 each from 0000 and 1111, two from 0011, and 19 instances from 1010.

This is four of the samples of Simpson’s Rule, plus seven more 1010s and two more 0011. The latter choice of endpoints concentrate on the center of the table, while the former concentrates on the alternate endpoints.

At this point the proof of Theorem \( \Pi \) is clear.

IV. HIGHER DIMENSIONS
The one-dimensional Newton-Cotes methods use an interpolating polynomial of degree \( d \). One needs \( d + 1 \) distinct points to determine the coefficients of this polynomial. This is easy when the domain is an interval.

The situation is different in higher dimensions. The dimension of the vector space of polynomials of degree at most \( d \) in \( n \) variables is
\[
\binom{d+n}{n};
\]
this is the number of coefficients, or, since passing through a given point imposes one linear constraint on the polynomial, the number of points required to determine the coefficients uniquely.

The number of points in a cubic grid is \( 2^d \), but adjoining adjacent cubes leads to other grid point counts. The difficulty is matching the number of grid points to the number of coefficients. As a rule, this is impossible.

Any sequences of digits modulo \( 2^n - 1 \) still determines a primitive quadrature rule. Look, for example, at an \( m \times m \) array in dimension 2, which is made up from \( m^2 \) *primitive* \((2 \times 2)\) squares. Arbitrarily label the corners of each square 0, 1, 2, and 3, for example starting at the northwest corner and proceeding clockwise.

Now, consider the mean of the four low-complexity sequences 000\ldots0, 111\ldots1, 222\ldots2, and 333\ldots3. (Each has length \( m^2 \).) In each primitive square, the corresponding entry in the sequence determines which grid point to choose.

The result is a quadrature rule that weights each of the corner points with weight 1, each of the non-corner edge points with weight 2, and each of the interior points with weight 4; the weighted sum is then divided by 4.

Theorem 2: The mean of the four low-complexity sequences 000\ldots0, 111\ldots1, 222\ldots2, and 333\ldots3 defines a quadrature rule with weights...
A. Error–Information Duality

Recently, Donoho (3) and others have investigated the problem of Compressed Sensing (CS), which is to reconstruct a signal represented as a vector from a sample of its entries. Donoho showed that knowing that a vector can be compressed is enough to reconstruct it, even without knowing what the compressed version might have been. When integrating the goal is to process the signal rather than to reconstruct it, but the same principle applies.

This section contains two results. The first, following Donoho (3), relates the information in the error in an integral estimate to the error in the information in the description of the integrand, a kind of Error–Information Duality. The second proves that for sparse functions (see below) there exists a quadrature program estimating the integral to arbitrary precision.

A. Error–Information Duality

Following Donoho, the functions of interest have the form $f(x) = \sum a_j \phi_j(x)$, where the functions $\phi_j$ form a basis for an appropriate space of functions. (The space for which they form a basis is intentionally left vague in order to be as general as possible.) The function is sparse if for some $R > 0$, $\|a\|_p < R$, where $0 < p < 2$ and $\|a\|_p$ is the $L^p$ norm of the series of coefficients $a_1, a_2, \ldots$. A function whose expansion has many small terms fails to be sparse by this definition, while a finite degree polynomial expansion is sparse.

Let $X_{p,n}(R)$ denote the space of functions given by a table of $n$ values which are $L^p$ sparse in the sense above. This is the space of functions of interest.

Begin with the functions $f$ with $d + 1$ nonzero coefficients, generalizing the space of polynomials if degree $\leq d$. Renumber if necessary so that the nonzero coefficients are $a_0, \ldots, a_d$.

The entries in the table of values $f = [f_0 \cdots f_n]^T$ are $\sum_{j=0}^d a_j \phi_j(\frac{i}{n})$. Let $\Phi$ denote the $(n + 1) \times d$ matrix with entries $\phi_j(\frac{i}{n})$. Let $a = [a_0 \cdots a_d]^T$. Then $f = \Phi a$. The matrix $\Phi$ only depends on the basis $\{\phi_j\}_i$.

Now, integrate $f$. First, let $q_j = \int_0^1 f_j(x) \, dx$, and let $Q = [q_0, \ldots, q_d]^T$; like $\Phi$, $Q$ only depend on the basis. Since $f(x) = \sum_{j=0}^d a_j \phi_j(x)$, $f_0 \int_0^1 f(x) \, dx = \sum_{j=0}^d a_j q_j = Q a$.

Next, suppose that $\Phi$ has a left inverse $\Phi^{-1}$, noting that this is never the case when $n + 1 < d$. Then $a = \Phi^{-1} f$, and

**Theorem 3:** When the expansion of $f$ has $d + 1$ coefficients and $n > d + 1$ then $\int_0^1 f(x) \, dx = Q \Phi^{-1} f$.

Compare this theorem with the exactness results for Newton–Cotes integrals of degree $d$ polynomials.

When there are more than $n$ nonzero coefficients, the integral can be estimated by truncating the series expansion to include the $n$ “most important” coefficients. The truncated function is integrated exactly, so the error in the estimate comes from the coefficients that were ignored. The truncated function contains $n$ floats worth of information, plus a little more to describe where these coefficients are in the series expansion. The table of values has $n$ floats worth of information as well. The information in the error in the integral estimate is exactly the information in the ignored coefficients. Hence

**Theorem 4 (Error–Information Duality):** The information content of the error is (a digest of) the error in the known information about the integrand.

B. Good Quadrature Rules

Now suppose that $f$ is a sparse function in the sense of the section above, so that there exists a good estimate

\[ (\dagger) \int_0^1 f(x) \, dx = \sum_{i=0}^n a_i f_i. \]

This section shows

**Theorem 5:** For any $\varepsilon > 0$ there exists a quadrature program that approximates $(\dagger)$ within $\varepsilon$.

**Proof.** Choose rational numbers $y_i/r$ such that $\sup\{|a_i - y_i/r|\} < \varepsilon/n$. Here $r$ is any convenient common denominator. Notice that $\sum y_i = r$, because of the weighted average nature of $(\dagger)$. The proof finds quadrature programs that reproduce the coefficients $y_i/r$.

Choose $r$ quadrature rules $b_1^{(l)}, b_2^{(l)}, \ldots, b_n^{(l)}$ where $l$ runs from 1 to $r$. Each $b_i^{(l)}$ leads to an estimate as in Section II, part A.

Now, consider the contribution of each $f_i$. The only contribution from $f_0$ occurs when $b_i^{(l)} = 0$, so

\[ y_0 = h \sum_{l=1}^r (1 - b_i^{(l)}). \]

The only contribution from $f_n$ occurs when $b_n^{(l)} = 1$, so

\[ y_n = h \sum_{l=1}^r b_i^{(l)}. \]

The contribution from $f_i$, where $i$ is neither 1 nor $n$ occurs when $b_{i-1}^{(l)} = 0$ (left endpoint) or when $b_i^{(l)} = 1$, so

\[ y_i = h \sum_{l=1}^r (1 - b_{i-1}^{(l)} + b_i^{(l)}). \]
Next, solve for the \( b_{i}^{(l)} \). From the \( f_{0} \) coefficient, \( h \sum b_{0}^{(l)} = r - y_{0} \). Plug this into the relation for the \( f_{1} \) coefficient, so \( y_{1} = h \sum_{l=1}^{r} (1 - b_{0}^{(l)} + b_{1}^{(l)}) \), implying that \( h \sum b_{1}^{(l)} = 2r - y_{0} - y_{1} \). Continuing in this way shows that \( h \sum b_{i}^{(l)} = ir - y_{0} - y_{1} - \cdots - y_{i} \).

Finally, \( y_{n} = h \sum b_{n}^{(l)} \), but this is redundant since the \( f_{n-1} \) coefficient satisfies \( h \sum b_{n-1}^{(l)} = nr - y_{0} - y_{1} - \cdots - y_{n-1} \).

Since \( \sum y_{i} = 1 \), the theorem is proved. ■

VI. FURTHER WORK

One foresees two kinds of further work. The first involves the concept of integration. Suppose that one makes a random choice of binary string(s) to define a quadrature program: what is the probability that this program is good? The sample space here is well-defined, namely, binary strings, but the concept of "good" needs refinement, especially with regard to the space of functions to be integrated. Integrating smooth functions allows one to compare the results with Newton–Cotes quadrature, but seems excessively restrictive in terms of the applications in the introduction. Perhaps it would be better to survey, say, \( L^{2} \) functions, by choosing random coefficient for a wavelet basis.

There is also further work possible from the perspectives of signal processing, compressed sensing, and cryptography. One way to think of \( \int_{0}^{1} f(x) \, dx \) is to think of \( f \) as a message and the integral as a message digest. From a cryptographic perspective, this is not a good message digest, because the information from the high order bits of the message has no effect on the low-order bits of the digest, while an ideal message digest should appear random. Can one characterize other message digests in terms the information content added by the algorithm? Is this a measure of security?

From the signal processing perspective, the function \( f \) represents some signal and the integral is a simple form of on-line processing. It is a simple matter to integrate against a kernel \( K(t) \), that is, to estimate \( \int K(t)f(t) \, dt \), as long as one has enough information about \( K \). But what of more complex processes like convolution? These problems are particularly interesting in the context of compressed sensing: what is the information-theoretic meaning of an integral transform when the function \( f \) is compressible?

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