WAVE-FRONT SETS OF BANACH FUNCTION TYPES

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Abstract. Let $\omega, \omega_0$ be appropriate weight functions and $\mathcal{B}$ be an invariant BF-space. We introduce the wave-front set, $WF_{\mathcal{B}(\omega)}(f)$ of the distribution $f$ with respect to weighted Fourier Banach space $\mathcal{B}(\omega)$. We prove that usual mapping properties for pseudo-differential operators $Op_t(a)$ with symbols $a$ in $S(\omega_0)$ hold for such wave-front sets. In particular we prove $WF_{\mathcal{B}(\omega/\omega_0)}(Op_t(a)f) \subseteq WF_{\mathcal{B}(\omega)}(f)$ and $WF_{\mathcal{B}(\omega)}(f) \subseteq WF_{\mathcal{B}(\omega/\omega_0)}(Op_t(a)f) \cup \text{Char}(a)$. Here $\text{Char}(a)$ is the set of characteristic points of $a$.

0. Introduction

In this paper we introduce wave-front sets with respect to Fourier images of translation invariant BF-spaces. The family of such wave-front sets contains the wave-front sets of Sobolev type, introduced by Hörmander in [23], the classical wave-front sets (cf. Sections 8.1 and 8.2 in [22]), and wave-front sets of Fourier Lebesgue types, introduced in [27]. Roughly speaking, for any given distribution $f$ and for appropriate Banach (or Frechét) space $\mathcal{B}$ of tempered distributions, the wave-front set $WF_{\mathcal{B}}(f)$ of $f$ consists of all pairs $(x_0, \xi_0)$ in $\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ such that no localizations of the distribution at $x_0$ belongs to $\mathcal{B}$ in the direction $\xi_0$.

We also establish mapping properties for a quite general class of pseudo-differential operators on such wave-front sets, and show that the micro-local analysis in [27] in background of Fourier Lebesgue spaces can be further generalized. It follows that our approach gives rise to flexible micro-local analysis tools which fit well to the most common approach developed in e.g. [22, 23]. In particular, we prove that usual mapping properties, which are valid for classical wave-front sets (cf. Chapters VIII and XVIII in [22]), also hold for wave-front sets of Fourier Banach types. For example, we show

$$WF_{\mathcal{B}(\omega/\omega_0)}(Op_t(a)f) \subseteq WF_{\mathcal{B}(\omega)}(f) \subseteq WF_{\mathcal{B}(\omega/\omega_0)}(Op_t(a)f) \cup \text{Char}(a).$$

(0.1)

That is, any operator $Op(a)$ shrinks the wave-front sets and opposite embeddings can be obtained by including $\text{Char}(a)$, the set of characteristic points of the operator symbol $a$.

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The symbol classes for the pseudo-differential operators are denoted by $S^{(\omega_0)}_{\rho,\delta}(\mathbb{R}^{2d})$, the set of all smooth functions $a$ on $\mathbb{R}^{2d}$ such that $a/\omega_0 \in S^0_{\rho,\delta}(\mathbb{R}^{2d})$. Here $\rho, \delta \in \mathbb{R}$ and $\omega_0$ is an appropriate smooth function on $\mathbb{R}^{2d}$. We note that $S^{(\omega_0)}_{\rho,\delta}(\mathbb{R}^{2d})$ agrees with the Hörmander class $S^r_{\rho,\delta}(\mathbb{R}^{2d})$ when $\omega_0(x,\xi) = \langle \xi \rangle^r$, where $r \in \mathbb{R}$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

The set of characteristic points $\text{Char}(a)$ of $a \in S^{(\omega)}_{\rho,\delta}$ is the same as in [27], and depends on the choices of $\rho, \delta$ and $\omega$ (see Definition 1.8 and Proposition 2.3). We recall that this set is smaller than the set of characteristic points given by [22]. It is empty when $a$ satisfies a local ellipticity condition with respect to $\omega$, which is fulfilled for any hypoelliptic partial differential operator with constant coefficients (cf. [27]). As a consequence of (0.1), it follows that such hypoelliptic operators preserve the wave-front sets, as expected (cf. Example 3.9 in [27]).

Information on regularity in background of wave-front sets of Fourier Banach types might be more detailed compared to classical wave-front sets, because of our choices of different weight functions $\omega$ and Banach spaces when defining our Fourier Banach space $\mathcal{FB}(\omega)(\mathbb{R}^d)$. For example, the space $\mathcal{FB}(\omega) = \mathcal{FL}^1_{(\omega)}(\mathbb{R}^d)$, with $\omega(x, \xi) = \langle \xi \rangle^N$ for some integer $N \geq 0$, is locally close to $C^N(\mathbb{R}^d)$ (cf. the Introduction of [27]). Consequently, the wave-front set with respect to $\mathcal{FL}^1_{(\omega)}$ can be used to investigate a sort of regularity which is close to smoothness of order $N$.

Furthermore, we are able to apply our results on pseudo-differential operators in context of modulation space theory, when discussing mapping properties of pseudo-differential operators with respect to wave-front sets. The modulation spaces were introduced by Feichtinger in [5], and the theory was developed in [7–9, 13]. The modulation space $M(\omega, \mathcal{B})$, where $\omega$ is a weight function (or time-frequency shift) on phase space $\mathbb{R}^{2d}$, appears as the set of tempered (ultra-)distributions whose short-time Fourier transform belong to the weighted Banach space $\mathcal{B}(\omega)$. These types of modulation spaces contains the (classical) modulation spaces $M^{p,q}_\omega(\mathbb{R}^{2d})$ as well as the space $W^{p,q}_\omega(\mathbb{R}^{2d})$ related to the Wiener amalgam spaces, by choosing $\mathcal{B} = L^p_{(\omega)}(\mathbb{R}^{2d})$ and $\mathcal{B} = L^p_{2d}(\mathbb{R}^{2d})$ respectively (see Remark 6.1). In the last part of the paper we define wave-front sets with respect to weighted modulation spaces, and prove that they coincide with the wave-front sets of Fourier Banach types.

Parallel to this development, modulation spaces have been incorporated into the calculus of pseudo-differential operators, in the sense of the study of continuity of (classical) pseudo-differential operators acting on modulation spaces (cf. [4, 25, 26, 33–35]), and the study of operators of non-classical type, where modulation spaces are used as symbol classes. We refer to [14–18, 20, 21, 25, 30, 31, 36–38, 40] for more...
facts about pseudo-differential operators in background of modulation space theory.

The paper is organized as follows. In Section 1 we recall the definition and basic properties for pseudo-differential operators, translation invariant Banach function spaces (BF-spaces) and (weighted) Fourier Banach spaces. Here we also define sets of characteristic points for a broad class of pseudo-differential operators. In Section 2 we prove some properties for the sets of characteristic points, which shows that our definition coincide with the sets of characteristic points defined in [27]. These sets might be smaller than characteristic sets in [22] (cf. [27, Example 3.11]).

In Section 3 we define wave-front sets with respect to (weighted) Fourier Banach spaces, and prove some of their main properties. Thereafter, in Section 4 we show how these wave-front sets are propagated under the action of pseudo-differential operators. In particular, we prove (0.1), when \( \omega_0 \) and \( \omega \) are appropriate weights and \( a \) belongs to \( S_{\rho,0}^{(\omega_0)} \) with \( \rho > 0 \).

In Section 5 we consider wave-front sets obtained from sequences of Fourier Banach spaces. These types of wave-front sets contain the classical ones (with respect to smoothness), and the mapping properties for pseudo-differential operators also hold in this context (cf. Section 18.1 in [22]).

Finally, Section 6 is devoted to study the definition and basic properties of wave-front sets with respect to modulation spaces. We prove that they can be identified with certain wave-front sets of Fourier Banach types.

1. Preliminaries

In this section we recall some notation and basic results. The proofs are in general omitted. In what follows we let \( \Gamma \) denote an open cone in \( \mathbb{R}^d \setminus 0 \). If \( \xi \in \mathbb{R}^d \setminus 0 \) is fixed, then an open cone which contains \( \xi \) is sometimes denoted by \( \Gamma_\xi \).

Assume that \( \omega, v \in L^{\infty}_{loc}(\mathbb{R}^d) \) are positive functions. Then \( \omega \) is called \( v \)-moderate if

\[
\omega(x + y) \leq C \omega(x)v(y)
\]

for some constant \( C \) which is independent of \( x, y \in \mathbb{R}^d \). If \( v \) in (1.1) can be chosen as a polynomial, then \( \omega \) is called polynomially moderate. We let \( \mathcal{P}(\mathbb{R}^d) \) be the set of all polynomially moderated functions on \( \mathbb{R}^d \). We say that \( v \) is submultiplicative when (1.1) holds with \( \omega = v \). Throughout we assume that the submultiplicative weights are even. If \( \omega(x, \xi) \in \mathcal{P}(\mathbb{R}^{2d}) \) is constant with respect to the \( x \)-variable (\( \xi \)-variable), then we sometimes write \( \omega(\xi) \) (\( \omega(x) \)) instead of \( \omega(x, \xi) \). In this case we consider \( \omega \) as an element in \( \mathcal{P}(\mathbb{R}^{2d}) \) or in \( \mathcal{P}(\mathbb{R}^d) \) depending on the situation.
We also need to consider classes of weight functions, related to $\mathcal{P}$. More precisely, we let $\mathcal{P}_0(\mathbb{R}^d)$ be the set of all $\omega \in \mathcal{P}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ such that $\partial^\alpha \omega / \omega \in L^\infty$ for all multi-indices $\alpha$. For each $\omega \in \mathcal{P}(\mathbb{R}^d)$, there is an equivalent weight $\omega_0 \in \mathcal{P}_0(\mathbb{R}^d)$, that is, $C^{-1} \omega_0 \leq \omega \leq C \omega_0$ holds for some constant $C$ (cf. [38, Lemma 1.2]).

Assume that $\rho, \delta \in \mathbb{R}$. Then we let $\mathcal{P}_{\rho,\delta}(\mathbb{R}^{2d})$ be the set of all $\omega(x, \xi)$ in $\mathcal{P}(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d})$ such that

$$\frac{\langle \xi \rangle^{\rho|\beta| - \delta|\alpha|}(\partial_x^\alpha \partial_\xi^\beta \omega)(x, \xi)}{\omega(x, \xi)} \in L^\infty(\mathbb{R}^{2d}),$$

for every multi-indices $\alpha$ and $\beta$. Note that in contrast to $\mathcal{P}_0$, we do not have an equivalence between $\mathcal{P}_{\rho,\delta}$ and $\mathcal{P}$ when $\rho > 0$. On the other hand, if $s \in \mathbb{R}$ and $\rho \in [0, 1]$, then $\mathcal{P}_{\rho,\delta}(\mathbb{R}^{2d})$ contains $\omega(x, \xi) = \langle \xi \rangle^s$, which are one of the most important classes in the applications.

For any weight $\omega$ in $\mathcal{P}(\mathbb{R}^d)$ or in $\mathcal{P}_{\rho,\delta}(\mathbb{R}^d)$, we let $L^p_{\omega}(\mathbb{R}^d)$ be the set of all $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that $f \cdot \omega \in L^p(\mathbb{R}^d)$.

The Fourier transform $\mathcal{F}$ is the linear and continuous mapping on $\mathcal{S}'(\mathbb{R}^d)$ which takes the form

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)e^{-i(x,\xi)} \, dx$$
when $f \in L^1(\mathbb{R}^d)$. We recall that $\mathcal{F}$ is a homeomorphism on $\mathcal{S}'(\mathbb{R}^d)$ which restricts to a homeomorphism on $\mathcal{S}(\mathbb{R}^d)$ and to a unitary operator on $L^2(\mathbb{R}^d)$.

Next we recall the definition of Banach function spaces.

**Definition 1.1.** Assume that $\mathcal{B}$ is a Banach space of complex-valued measurable functions on $\mathbb{R}^d$ and that $v \in \mathcal{P}(\mathbb{R}^d)$ is submultiplicative. Then $\mathcal{B}$ is called a (translation) invariant BF-space on $\mathbb{R}^d$ (with respect to $v$), if there is a constant $C$ such that the following conditions are fulfilled:

1. $\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{B} \subseteq \mathcal{S}'(\mathbb{R}^d)$ (continuous embeddings);
2. if $x \in \mathbb{R}^d$ and $f \in \mathcal{B}$, then $f(\cdot - x) \in \mathcal{B}$, and
\[
\|f(\cdot - x)\|_\mathcal{B} \leq C v(x) \|f\|_\mathcal{B};
\]  
(1.2)
3. if $f, g \in L^1_{\text{loc}}(\mathbb{R}^d)$ satisfy $g \in \mathcal{B}$ and $|f| \leq |g|$ almost everywhere, then $f \in \mathcal{B}$ and
\[
\|f\|_\mathcal{B} \leq C \|g\|_\mathcal{B}.
\]

Assume that $\mathcal{B}$ is a translation invariant BF-space. If $f \in \mathcal{B}$ and $h \in L^\infty$, then it follows from (3) in Definition 1.1 that $f \cdot h \in \mathcal{B}$ and
\[
\|f \cdot h\|_\mathcal{B} \leq C \|f\|_\mathcal{B} \|h\|_{L^\infty},
\]  
(1.3)

**Remark 1.2.** Assume that $\omega_0, v, v_0 \in \mathcal{P}(\mathbb{R}^d)$ are such $v$ and $v_0$ are submultiplicative, $\omega_0$ is $v_0$-moderate, and assume that $\mathcal{B}$ is a translation-invariant BF-space on $\mathbb{R}^d$ with respect to $v$. Also let $\mathcal{B}_0$ be the Banach
Remark 1.3. Let \( \mathcal{B} \) be an invariant BF-space. Then it is easy to find Sobolev type spaces which are continuously embedded in \( \mathcal{B} \) space which consists of all \( f \in L^1_{loc}(\mathbb{R}^d) \) such that \( \|f\|_{\mathcal{B}_0} = \|f \omega_0\|_{\mathcal{B}} \) is finite. Then \( \mathcal{B}_0 \) is a translation invariant BF-space with respect to \( \omega_0 \).

For each \( p \) and \( f \), then for each \( \mathcal{S} \) Sobolev type spaces which are continuously embedded in \( \mathcal{B} \) space which consists of all \( \mathcal{B} \) semi-norms \( L^p \) follows from (3) in Definition 1.1 that \( \mathcal{B} \) space with respect to the submultiplicative weight \( v \) large enough. This proves the assertion.

Then for each \( p \) fixed, the topology for \( \mathcal{S}(\mathbb{R}^d) \) can be defined by the semi-norms \( f \mapsto \|f\|_{Q^p_N} \), for \( N = 0, 1, \ldots \).

A combination of this fact and (1) and (3) in Definition 1.1 now shows that for each \( p \in [1, \infty] \) and integer \( N \geq 0 \), let \( Q^p_N(\mathbb{R}^d) \) be the set of all \( f \) such that \( \|f\|_{Q^p_N} < \infty \), where

\[
\|f\|_{Q^p_N} = \sum_{|\alpha + \beta| \leq N} \|x^\alpha D^\beta f\|_{L^p}.
\]

Then for each \( p \) fixed, the topology for \( \mathcal{S}(\mathbb{R}^d) \) can be defined by the semi-norms \( f \mapsto \|f\|_{Q^p_N} \), for \( N = 0, 1, \ldots \).

For future references we note that if \( \mathcal{B} \) is a translation invariant BF-space with respect to the submultiplicative weight \( v \) on \( \mathbb{R}^d \), then the convolution map \( * \) on \( \mathcal{S}(\mathbb{R}^d) \) extends uniquely to a continuous mapping from \( \mathcal{B} \times L^1_{(v)}(\mathbb{R}^d) \), and for some constant \( C \) it holds

\[
\|\varphi \ast f\|_{\mathcal{B}} \leq C \|\varphi\|_{L^1_{(v)}} \|f\|_{\mathcal{B}}, \quad \varphi \in L^1_{(v)}(\mathbb{R}^d), \quad f \in \mathcal{B}.
\] (1.4)

In fact, if \( f \in \mathcal{B} \) and \( g \) is a step function, then \( f \ast g \) is well-defined and belongs to \( \mathcal{B} \) in view of the definitions, and Minkowski’s inequality gives

\[
\|f \ast g\|_{\mathcal{B}} = \left\| \int f(\cdot - y)g(y) \, dy \right\|_{\mathcal{B}} \leq \int \|f(\cdot - y)\|_{\mathcal{B}} |g(y)| \, dy \leq C \int \|f\|_{\mathcal{B}} |g(\cdot)\| v(\cdot) \, dy = C \|f\|_{\mathcal{B}} \|g\|_{L^1_{(v)}}.
\]

Now assume that \( g \in C^\infty_0 \). Then \( f \ast g \) is well-defined as an element in \( \mathcal{S}' \cap C^\infty \), and by approximating \( g \) with step functions and using (1.4) it follows that \( f \ast g \in \mathcal{B} \) and that (1.4) holds also in this case. The assertion now follows from this fact and a simple argument of approximations, using the fact that \( C^\infty_0 \) is dense in \( L^1_{(v)} \).

For each translation invariant BF-space \( \mathcal{B} \) on \( \mathbb{R}^d \), and each pair of vector spaces \( (V_1, V_2) \) such that \( V_1 \oplus V_2 = \mathbb{R}^d \), we define the projection spaces \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) of \( \mathcal{B} \) by the formulae

\[
\mathcal{B}_1 \equiv \{ f \in \mathcal{S}'(V_1); f \otimes \varphi \in \mathcal{B} \text{ for every } \varphi \in \mathcal{S}(V_2) \} \quad (1.5)
\]
Proposition 1.4. Let $\mathcal{B}$ be a translation invariant BF-space on $\mathbb{R}^d$, and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the same as in (1.5) and (1.6). Then

$$\mathcal{B}_1 = \{ f \in \mathcal{S}'(V_2) ; f \otimes \varphi \in \mathcal{B} \text{ for every } \varphi \in \mathcal{S}(V_1) \}, \quad (1.5)'$$

and

$$\mathcal{B}_2 = \{ f \in \mathcal{S}'(V_2) ; \varphi \otimes f \in \mathcal{B} \text{ for some } \varphi \in \mathcal{S}(V_1) \} \quad (1.6)'$$

In particular, if $\varphi_j \in \mathcal{S}(V_j) \setminus 0$ for $j = 1, 2$ are fixed and $f_1 \in \mathcal{S}'(V_1)$ and $f \in \mathcal{S}'(V_2)$, then $\mathcal{B}_1$ and $\mathcal{B}_2$ are translation invariant BF-spaces under the norms

$$\| f \|_{\mathcal{B}_1} \equiv \| f \otimes \varphi_1 \|_{\mathcal{B}} \quad \text{and} \quad \| f \|_{\mathcal{B}_2} \equiv \| \varphi_2 \otimes f \|_{\mathcal{B}}$$

respectively.

Proof. We only prove $(1.6)'$. The other equality follows by similar arguments and is left for the reader. We may assume that $V_j = \mathbb{R}^{d_j}$ with $d_1 + d_2 = d$.

Let $\mathcal{B}_0$ be the right-hand side of $(1.6)'$. Then it is obvious that $\mathcal{B}_2 \subseteq \mathcal{B}_0$. We have to prove the opposite inclusion.

Therefore, assume that $f \in \mathcal{B}_0$, and choose $\varphi_0 \in \mathcal{S}(\mathbb{R}^{d_1}) \setminus 0$ such that $\varphi_0 \otimes f \in \mathcal{B}$. Also let $\varphi \in \mathcal{S}(\mathbb{R}^{d_1})$ be arbitrary. We shall prove that $\varphi \otimes f \in \mathcal{B}$.

Let $Q \subseteq \mathbb{R}^{d_1}$ be an open ball and $c > 0$ be chosen such that $|\varphi_0(x)| > c$ when $x \in Q$. Also let the lattice $\Lambda \subseteq \mathbb{R}^{d_1}$ and $\varphi_1 \in C_0^\infty(Q)$ be such that $0 \leq \varphi_1 \leq 1$ and

$$\sum_{\{x_j\} \in \Lambda} \varphi_1(\cdot - x_j) = 1.$$ 

Then $\varphi_1 \leq C|\varphi_0|$, for some constant $C > 0$, which gives

$$\| \varphi_1 \otimes f \|_{\mathcal{B}} \leq C\| \varphi_0 \otimes f \|_{\mathcal{B}} < \infty.$$ 

This in turn gives

$$\| \varphi \otimes f \|_{\mathcal{B}} \leq \sum \| (\varphi_1(\cdot - x_j) \varphi) \otimes f \|_{\mathcal{B}}$$

$$\leq \sum v(x_j, 0)\| (\varphi_1 \varphi(\cdot + x_j)) \otimes f \|_{\mathcal{B}}$$

$$\leq C \left( \sum v(x_j, 0)\| \varphi(\cdot + x_j) \|_{L^\infty(Q)} \right) \| \varphi_1 \otimes f \|_{\mathcal{B}}. \quad (1.7)$$

Since $v \in \mathcal{P}$ and $\varphi \in \mathcal{P}$, it follows that the sum in the right-hand side of (1.7) is finite. Hence $f \in \mathcal{B}_2$, and the proof is complete. \qed
Remark 1.5. We note that the last sum in \((1.7)\) is the norm
\[
\|\varphi\|_{W_\omega} \equiv \sum v(x_j,0)\|\varphi(\cdot + x_j)\|_{L^\infty(Q)}
\]
for the weighted Wiener space
\[
W_\omega(R^d) = \{ f \in L^\infty_{loc}(R^d) : \|f\|_{W_\omega} < \infty \}
\]
(cf. \([14]\)). The results in Proposition 1.4 can therefore be improved in such way that we may replace \(S\) by \(W_\omega\) in \((1.5), (1.6), (1.5)'\) and \((1.6)'.\)

Assume that \(B\) is a translation invariant BF-space on \(R^d\), and that \(\omega \in \mathcal{P}(R^d)\). Then we let \(\mathcal{FB}^\omega(\omega)\) be the set of all \(f \in \mathcal{P}(R^d)\) such that \(\xi \mapsto \hat{f}(\xi)\omega(x,\xi)\) belongs to \(B\). It follows that \(\mathcal{FB}^\omega(\omega)\) is a Banach space under the norm
\[
\|f\|_{\mathcal{FB}^\omega(\omega)} \equiv \|\hat{f}\|_B. \quad (1.8)
\]

Remark 1.6. In many situations it is convenient to permit an \(x\) dependency for the weight \(\omega\) in the definition of Fourier Banach spaces. More precisely, for each \(\omega \in \mathcal{P}(R^{2d})\) and each translation invariant BF-space \(B\) on \(R^d\), we let \(\mathcal{FB}^\omega(\omega)\) be the set of all \(f \in \mathcal{P}(R^d)\) such that
\[
\|f\|_{\mathcal{FB}^\omega(\omega)} = \|f\|_{\mathcal{FB}^\omega(\omega),x} \equiv \|\hat{f}\omega(x,\cdot)\|_B
\]
is finite. Since \(\omega\) is \(v\)-moderate for some \(v \in \mathcal{P}(R^{2d})\) it follows that different choices of \(x\) give rise to equivalent norms. Therefore the condition \(\|f\|_{\mathcal{FB}^\omega(\omega)} < \infty\) is independent of \(x\), and it follows that \(\mathcal{FB}^\omega(\omega)(R^d)\) is independent of \(x\) although \(\|\cdot\|_{\mathcal{FB}^\omega(\omega)}\) might depend on \(x\).

Recall that a topological vector space \(V \subseteq D'(X)\) is called local if \(V \subseteq V_{loc}\). Here \(X \subseteq R^d\) is open, and \(V_{loc}\) consists of all \(f \in D'(X)\) such that \(\varphi f \in V\) for every \(\varphi \in C_0^\infty(X)\). For future references we note that if \(B\) is a translation invariant BF-space on \(R^d\) and \(\omega \in \mathcal{P}(R^{2d})\), then it follows from \((1.4)\) that \(\mathcal{FB}^\omega(\omega)\) is a local space, i.e.
\[
\mathcal{FB}^\omega(\omega) \subseteq \mathcal{FB}^\omega(\omega)_{loc} \equiv (\mathcal{FB}^\omega(\omega))_{loc}. \quad (1.9)
\]

We need to recall some facts from Chapter XVIII in \([22]\) concerning pseudo-differential operators. Let \(a \in \mathcal{S}(R^{2d})\), and \(t \in R\) be fixed. Then the pseudo-differential operator \(\text{Op}_t(a)\) is the linear and continuous operator on \(\mathcal{S}(R^d)\), defined by the formula
\[
(\text{Op}_t(a)f)(x) = (2\pi)^{-d} \int \int a((1-t)x + ty,\xi)f(y)e^{i(x-y,\xi)}\, dyd\xi. \quad (1.10)
\]
For general \(a \in \mathcal{S}'(R^{2d})\), the pseudo-differential operator \(\text{Op}_t(a)\) is defined as the continuous operator from \(\mathcal{S}(R^d)\) to \(\mathcal{S}'(R^d)\) with distribution kernel
\[
K_{t,a}(x,y) = (2\pi)^{-d/2}(\mathcal{S}^{-1}a)((1-t)x + ty, x-y). \quad (1.11)
\]
Here $\mathcal{F}_2 F$ is the partial Fourier transform of $F(x, y) \in \mathcal{S}'(\mathbb{R}^{2d})$ with respect to the $y$-variable. This definition makes sense, since the mappings $\mathcal{F}_2$ and
\[
F(x, y) \mapsto F((1 - t)x + ty, x - y)
\]
are homeomorphisms on $\mathcal{S}'(\mathbb{R}^{2d})$. We also note that the latter definition of $\mathrm{Op}_t(a)$ agrees with the operator in (1.10) when $a \in \mathcal{S}(\mathbb{R}^{2d})$. If $t = 0$, then $\mathrm{Op}_t(a)$ agrees with the Kohn-Nirenberg representation $\mathrm{Op}(a) = a(x, D)$.

If $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and $s, t \in \mathbb{R}$, then there is a unique $b \in \mathcal{S}'(\mathbb{R}^{2d})$ such that $\mathrm{Op}_s(a) = \mathrm{Op}_t(b)$. By straight-forward applications of Fourier’s inversion formula, it follows that
\[
\mathrm{Op}_s(a) = \mathrm{Op}_t(b) \iff b(x, \xi) = e^{i(t-s)\langle D_x, D_\xi \rangle} a(x, \xi). \quad (1.12)
\]
(Cf. Section 18.5 in [22].)

Next we discuss symbol classes which we use. Let $r, \rho, \delta \in \mathbb{R}$ be fixed. Then we recall from [22] that $S^r_{\rho, \delta}(\mathbb{R}^{2d})$ is the set of all $a \in C^\infty(\mathbb{R}^{2d})$ such that for each pairs of multi-indices $\alpha$ and $\beta$, there is a constant $C_{\alpha, \beta}$ such that
\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} |\xi|^{-r - \rho|\beta| + \delta|\alpha|}.
\]
Usually we assume that $0 \leq \delta \leq \rho \leq 1$, $0 < \rho$ and $\delta < 1$.

More generally, assume that $\omega \in \mathcal{D}_{\rho, \delta}(\mathbb{R}^{2d})$. Then we recall from the introduction that $S^r_{\rho, \delta}(\mathbb{R}^{2d})$ consists of all $a \in C^\infty(\mathbb{R}^{2d})$ such that
\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} \omega(x, \xi) |\xi|^{-r - \rho|\beta| + \delta|\alpha|}. \quad (1.13)
\]
We note that $S^r_{\rho, \delta}(\mathbb{R}^{2d}) = S(\omega, g_{\rho, \delta})$, when $g = g_{\rho, \delta}$ is the Riemannian metric on $\mathbb{R}^{2d}$, defined by the formula
\[
(g_{\rho, \delta})_{(y, \eta)}(x, \xi) = \langle \eta \rangle^{2\delta} |x|^2 + \langle \eta \rangle^{-2\rho} |\xi|^2
\]
(cf. Section 18.4–18.6 in [22]). Furthermore, $S^r_{\rho, \delta} = S^r_{\rho, \delta}$ when $\omega(x, \xi) = \langle \xi \rangle^r$, as remarked in the introduction.

The following result shows that pseudo-differential operators with symbols in $S^{r}_{\rho, \delta}$ behave well. We refer to [22] or [27] for the proof.

**Proposition 1.7.** Let $\rho, \delta \in [0, 1]$ be such that $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$, and let $\omega \in \mathcal{D}_{\rho, \delta}(\mathbb{R}^{2d})$. If $a \in S^r_{\rho, \delta}(\mathbb{R}^{2d})$, then $\mathrm{Op}_t(a)$ is continuous on $\mathcal{S}(\mathbb{R}^{d})$ and extends uniquely to a continuous operator on $\mathcal{S}'(\mathbb{R}^{d})$.

We also need to define the set of characteristic points of a symbol $a \in S^r_{\rho, \delta}(\mathbb{R}^{2d})$, when $\omega \in \mathcal{D}_{\rho, \delta}(\mathbb{R}^{2d})$ and $0 \leq \delta < \rho \leq 1$. In Section 2 we show that this definition is equivalent to Definition 1.3 in [27]. We remark that our sets of characteristic points are smaller than the corresponding sets in [22]. (Cf. [22, Definition 18.1.5] and Remark 2.4 in Section 2.)
Definition 1.8. Assume that $0 \leq \delta < \rho \leq 1$, $\omega_0 \in \mathcal{P}_{\rho,\delta}(\mathbb{R}^{2d})$ and $a \in S_{\rho,\delta}^{(\omega)}(\mathbb{R}^{2d})$. Then $a$ is called $\psi$-invertible with respect to $\omega_0$ at the point $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$, if there exist a neighbourhood $X$ of $x_0$, an open conical neighbourhood $\Gamma$ of $\xi_0$ and positive constants $R$ and $C$ such that

$$|a(x, \xi)| \geq C\omega_0(x, \xi),$$

(1.14)

for $x \in X$, $\xi \in \Gamma$ and $|\xi| \geq R$.

The point $(x_0, \xi_0)$ is called characteristic for $a$ with respect to $\omega_0$ if $a$ is not $\psi$-invertible with respect to $\omega_0$ at $(x_0, \xi_0)$. The set of characteristic points (the characteristic set), for $a$ with respect to $\omega_0$ is denoted $\text{Char}(a) = \text{Char}_{(\omega_0)}(a)$.

We note that $(x_0, \xi_0) \notin \text{Char}_{(\omega_0)}(a)$ means that $a$ is elliptic near $x_0$ in the direction $\xi_0$. Since the case $\omega_0 = 1$ in Definition 1.8 is especially important we also make the following definition. We say that $c \in S^{(\omega)}_{\rho,\delta}(\mathbb{R}^{2d})$ is $\psi$-invertible at $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$, if $(x_0, \xi_0) \notin \text{Char}_{(\omega_0)}(c)$ with $\omega_0 = 1$. That is, there exist a neighbourhood $X$ of $x_0$, an open conical neighbourhood $\Gamma$ of $\xi_0$ and $R > 0$ such that (1.14) holds for $a = c$ and $\omega_0 = 1$, for some constant $C > 0$ which is independent of $x \in X$ and $\xi \in \Gamma$ such that $|\xi| \geq R$.

It will also be convenient to have the following definition of different types of cutoff functions.

Definition 1.9. Let $X \subseteq \mathbb{R}^d$ be open, $\Gamma \subseteq \mathbb{R}^d \setminus 0$ be an open cone, $x_0 \in X$ and let $\xi_0 \in \Gamma$.

(1) A smooth function $\varphi$ on $\mathbb{R}^d$ is called a cutoff function with respect to $x_0$ and $X$, if $0 \leq \varphi \leq 1$, $\varphi \in C^\infty_0(X)$ and $\varphi = 1$ in an open neighbourhood of $x_0$. The set of cutoff functions with respect to $x_0$ and $X$ is denoted by $\mathcal{C}_{x_0}(X)$;

(2) A smooth function $\psi$ on $\mathbb{R}^d$ is called a directional cutoff function with respect to $\xi_0$ and $\Gamma$, if there is a constant $R > 0$ and open conical neighbourhood $\Gamma_1$ of $\xi_0$ such that the following is true:

- $0 \leq \psi \leq 1$ and supp $\psi \subseteq \Gamma$;
- $\psi(t\xi) = \psi(\xi)$ when $t \geq 1$ and $|\xi| \geq R$;
- $\psi(\xi) = 1$ when $\xi \in \Gamma_1$ and $|\xi| \geq R$.

The set of directional cutoff functions with respect to $\xi_0$ and $\Gamma$ is denoted by $\mathcal{C}_{\xi_0}^{\text{dir}}(\Gamma)$.

Remark 1.10. We note that if $\varphi \in \mathcal{C}_{x_0}(X)$ and $\psi \in \mathcal{C}_{\xi_0}^{\text{dir}}(\Gamma)$ for some $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$, then $c \equiv \varphi \otimes \psi$ belongs to $S_{1,0}^{(\omega)}(\mathbb{R}^{2d})$ and is $\psi$-invertible at $(x_0, \xi_0)$.

2. Pseudo-differential calculus with symbols in $S_{\rho,\delta}^{(\omega)}$

In this section we make a review of basic results for pseudo-differential operators with symbols in classes of the form $S_{\rho,\delta}^{(\omega)}(\mathbb{R}^{2d})$, when $0 \leq \delta <
\[ \rho \leq 1 \text{ and } \omega \in \mathcal{P}_{\rho,\delta}(\mathbb{R}^{2d}). \]

For the standard properties in the pseudo-differential calculus we only state the results and refer to [22] for the proofs. Though there are similar stated and proved properties concerning sets of characteristic points, we include proofs of these properties in order to being more self-contained.

We start with the following result concerning compositions and invariance properties for pseudo-differential operators. Here we let

\[ \sigma_s(x, \xi) = \langle \xi \rangle^s, \]

where \( \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \) as usual. We also recall that \( S_{\rho,\delta}^{-\infty} = S_{1,0}^{-\infty} \) consists of all \( a \in C^\infty(\mathbb{R}^{2d}) \) such that for each \( N \in \mathbb{R} \) and multi-index \( \alpha \), there is a constant \( C_{N,\alpha} \) such that

\[ |\partial^\alpha a(x, \xi)| \leq C_{N,\alpha} \langle \xi \rangle^{-N}. \]

**Proposition 2.1.** Let \( 0 < \delta < \rho \leq 1, \mu = \rho - \delta > 0 \) and \( \omega, \omega_1, \omega_2 \in \mathcal{P}_{\rho,\delta}(\mathbb{R}^{2d}). \) Also let \( \{m_j\}_{j=0}^\infty \) be a sequence of real numbers such that \( m_j \to -\infty \) as \( j \to \infty \). Then the following is true:

1. if \( a_1 \in S_{\rho,\delta}^{(\omega_1)}(\mathbb{R}^{2d}) \) and \( a_2 \in S_{\rho,\delta}^{(\omega_2)}(\mathbb{R}^{2d}) \), then \( \text{Op}(a_1) \circ \text{Op}(a_2) = \text{Op}(c) \), for some \( c \in S_{\rho,\delta}^{(\omega_1\omega_2)}(\mathbb{R}^{2d}) \). Furthermore,

\[ c(x, \xi) = \sum_{|\alpha|<N} \frac{i^{|\alpha|}(D_\xi^\alpha a_1)(x, \xi)(D_x^\alpha a_2)(x, \xi)}{\alpha!} \in S_{\rho,\delta}^{(\omega_1\omega_2 - \omega_{\mu})}(\mathbb{R}^{2d}) \quad (2.1) \]

for every \( N \geq 0 \);

2. if \( M = \sup_{k \geq 0}(m_k), M_j = \sup_{k \geq j}(m_k) \) and \( a_j \in S_{\rho,\delta}^{(\omega_{\mu})}(\mathbb{R}^{2d}) \), then it exists \( a \in S_{\rho,\delta}^{(\omega_{\mu})}(\mathbb{R}^{2d}) \) such that

\[ a(x, \xi) = \sum_{|\alpha|<N} a_j(x, \xi) \in S_{\rho,\delta}^{(\omega_{\mu})}(\mathbb{R}^{2d}); \quad (2.2) \]

for every \( N \geq 0 \);

3. if \( a, b \in \mathcal{P}^{(\omega)}(\mathbb{R}^{2d}) \) and \( s, t \in \mathbb{R} \) are such that \( \text{Op}_s(a) = \text{Op}_t(b) \), then \( a \in S_{\rho,\delta}^{(\omega)}(\mathbb{R}^{2d}) \), if and only if \( b \in S_{\rho,\delta}^{(\omega)}(\mathbb{R}^{2d}) \), and

\[ b(x, \xi) = \sum_{k<N} \frac{(i(t-s)(D_x^s D_\xi^t))^k a(x, \xi)}{k!} \in S_{\rho,\delta}^{(\omega_{\mu})}(\mathbb{R}^{2d}) \quad (2.3) \]

for every \( N \geq 0 \).

As usual we write

\[ a \sim \sum a_j \quad \text{[2.2]} \]

when \( (2.2) \) is fulfilled for every \( N \geq 0 \). In particular it follows from \( (2.1) \) and \( (2.3) \) that

\[ c \sim \sum \frac{i^{|\alpha|}(D_\xi^\alpha a_1)(D_x^\alpha a_2)}{\alpha!} \quad \text{[2.1']}. \]
when \( \text{Op}(a_1) \circ \text{Op}(a_2) = \text{Op}(c) \), and
\[
b \sim \sum \frac{(i(t - s))\langle D_x, D_\xi \rangle^k a}{k!} \tag{2.3}
\]
when \( \text{Op}_s(a) = \text{Op}_t(b) \).

In the following proposition we show that the set of characteristic points for a pseudo-differential operator is independent of the choice of pseudo-differential calculus.

**Proposition 2.2.** Assume that \( s, t \in \mathbb{R}, 0 \leq \delta < \rho \leq 1, \omega_0 \in \mathcal{P}_{\rho, \delta} \) and that \( a, b \in S_{\omega_0}^{(\omega_0)}(\mathbb{R}^{2d}) \) satisfy \( \text{Op}_s(a) = \text{Op}_t(b) \). Then
\[
\text{Char}_{(\omega_0)}(a) = \text{Char}_{(\omega_0)}(b). \tag{2.4}
\]

**Proof.** Let \( \mu \) and \( \sigma \) be the same as in the proof of Proposition 2.1. By Proposition 2.1 (3) we have
\[
b = a + h,
\]
for some \( h \in S_{\omega_0}^{(\omega_0)}(\mathbb{R}^{2d}) \).

Assume that \( (x_0, \xi_0) \notin \text{Char}_{(\omega_0)}(a) \). By the definitions, there is a neighbourhood \( X \) of \( x_0 \), an open conical neighbourhood \( \Gamma \) of \( \xi_0 \), \( C > 0 \) and \( R > 0 \) such that
\[
|a(x, \xi)| \geq C\omega_0(x, \xi) \quad \text{and} \quad |h(x, \xi)| \leq C\omega_0(x, \xi)/2,
\]
as \( x \in X, \xi \in \Gamma \) and \( |\xi| \geq R \). This gives
\[
|h(x, \xi)| \geq C\omega_0(x, \xi)/2, \quad \text{when} \quad x \in X, \xi \in \Gamma, |\xi| \geq R,
\]
and it follows that \( (x_0, \xi_0) \notin \text{Char}_{(\omega_0)}(b) \). Hence \( \text{Char}_{(\omega_0)}(b) \subset \text{Char}_{(\omega_0)}(a) \). By symmetry, the opposite inclusion also holds. Hence \( \text{Char}_{(\omega_0)}(a) = \text{Char}_{(\omega_0)}(b) \), and the proof is complete. \( \square \)

The following proposition shows different aspects of set of characteristic points, and is important when investigating wave-front properties for pseudo-differential operators. In particular it shows that \( \text{Op}(a) \) satisfy certain invertibility properties outside the set of characteristic points for \( a \). More precisely, outside \( \text{Char}_{(\omega_0)}(a) \), we prove that
\[
\text{Op}(b) \text{Op}(a) = \text{Op}(c) + \text{Op}(h), \tag{2.5}
\]
for some convenient \( b, c \) and \( h \) which take the role of inverse, identity symbol and smoothing remainder respectively.

**Proposition 2.3.** Let \( 0 \leq \delta < \rho \leq 1, \omega_0 \in \mathcal{P}_{\rho, \delta}(\mathbb{R}^{2d}), a \in S_{\omega_0}^{(\omega_0)}(\mathbb{R}^{2d}), (x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0), \) and let \( \mu = \rho - \delta \). Then the following conditions are equivalent:

1. \( (x_0, \xi_0) \notin \text{Char}_{(\omega_0)}(a) \);
2. there is an element \( c \in S_{\omega_0}^{(0/\omega_0)}(\mathbb{R}^{2d}) \) which is \( \psi \)-invertible at \( (x_0, \xi_0) \), and an element \( b \in S_{\omega_0}^{(1/\omega_0)}(\mathbb{R}^{2d}) \) such that \( ab = c \);
(3) there is an element \( c \in S^0_{\rho,\delta} \) which is \( \psi \)-invertible at \((x_0, \xi_0)\), and elements \( h \in S^{-\mu}_{\rho,\delta} \) and \( b \in S^{(1/\omega_0)}_{\rho,\delta} \) such that (2.5) holds;

(4) for each neighbourhood \( X \) of \( x_0 \) and conical neighbourhood \( \Gamma \) of \( \xi_0 \), there is an element \( c = \varphi \otimes \psi \) where \( \varphi \in \mathscr{C} x_0(X) \) and \( \psi_{\xi_0}^{\text{dir}}(\Gamma) \), and elements \( h \in \mathscr{I} \) and \( b \in S^{(1/\omega_0)}_{\rho,\delta} \) such that (2.5) holds. Furthermore, the supports of \( b \) and \( h \) are contained in \( X \times \mathbb{R}^d \).

For the proof we note that \( \mu \) in Proposition 2.3 is positive, which in turn implies that \( \cap_{j \geq 0} S^{(\omega_0\sigma - j\rho)}(\mathbb{R}^{2d}) \) agrees with \( S^{-\infty}(\mathbb{R}^{2d}) \).

Proof. The equivalence between (1) and (2) follows by letting \( b(x, \xi) = \varphi(x)\psi(\xi)/a(x, \xi) \) for some appropriate \( \varphi \in \mathscr{C} x_0(\mathbb{R}^d) \) and \( \psi \in \mathscr{C}^{\text{dir}}_{\xi_0}(\mathbb{R}^d \setminus 0) \).

(4) \( \Rightarrow \) (3) is obvious in view of Remark 1.10. Assume that (3) holds. We shall prove that (1) holds, and since \(|b| \leq C/\omega_0\), it suffices to prove that

\[
|a(x, \xi)b(x, \xi)| \geq 1/2 \tag{2.6}
\]

when

\[
(x, \xi) \in X \times \Gamma, \ |\xi| \geq R \tag{2.7}
\]

holds for some conical neighbourhood \( \Gamma \) of \( \xi_0 \), some open neighbourhood \( X \) of \( x_0 \) and some \( R > 0 \).

By Proposition 2.1 (1) it follows that \( ab = c+h \) for some \( h \in S^{-\mu}_{\rho,\delta} \). By choosing \( R \) large enough and \( \Gamma \) sufficiently small conical neighbourhood of \( \xi_0 \), it follows that \( c(x, \xi) = 1 \) and \(|h(x, \xi)| \leq 1/2 \) when (2.7) holds. This gives (2.6), and (1) follows.

It remains to prove that (1) implies (4). Therefore assume that (1) holds, and choose an open neighbourhood \( X \) of \( x_0 \), an open conical neighbourhood \( \Gamma \) of \( \xi_0 \) and \( R > 0 \) such that (1.14) holds when \((x, \xi) \in X \times \Gamma \) and \(|\xi| > R \). Also let \( \varphi_j \in \mathscr{C} x_0(X) \) and \( \psi_j \in \mathscr{C}^{\text{dir}}_{\xi_0}(\Gamma) \) for \( j = 1, 2, 3 \) be such that \( \varphi_j = 1 \) on \( \text{supp} \varphi_{j+1} \), \( \psi_j = 1 \) on \( \text{supp} \psi_{j+1} \) when \( j = 1, 2 \), and \( \psi_j(\xi) = 0 \) when \(|\xi| \leq R \). We also set \( c_j = \varphi_j \otimes \psi_j \) when \( j \leq 2 \) and \( c_j = c_2 \) when \( j \geq 3 \).

If \( b_1(x, \xi) = \varphi_1(x)\psi_1(\xi)/a(x, \xi) \in S^{(1/\omega_0)}_{\rho,\delta} \), then the symbol of \( \text{Op}(b_1) \text{Op}(a) \) is equal to \( c_1 \mod (S^{-\mu}_{\rho,\delta}) \). Hence

\[
\text{Op}(b_j) \text{Op}(a) = \text{Op}(c_j) + \text{Op}(h_j) \tag{2.8}
\]

holds for \( j = 1 \) and some \( h_1 \in S^{-\mu}_{\rho,\delta} \).

For \( j \geq 2 \) we now define \( \tilde{b}_j \in S^{(1/\omega_0)}_{\rho,\delta} \) by the Neumann serie

\[
\text{Op}(\tilde{b}_j) = \sum_{k=0}^{j-1} (-1)^k \text{Op}(\tilde{r}_k),
\]
where $\text{Op}(\tilde{r}_k) = \text{Op}(h_1)^k \text{Op}(b_1) \in \text{Op}(S_{\rho,\delta}^{(\sigma-k\mu/\omega_0)})$. Then (2.8) gives

$$\text{Op}(\tilde{b}_j) \text{Op}(a) = \sum_{k=0}^{j-1} (-1)^k \text{Op}(h_1)^k \text{Op}(b_1) \text{Op}(a)$$

$$= \sum_{k=0}^{j-1} (-1)^k \text{Op}(h_1)^k (\text{Op}(c_1) + \text{Op}(h_1)).$$

That is

$$\text{Op}(\tilde{b}_j) \text{Op}(a) = \text{Op}(c_1) + \text{Op}(\tilde{h}_{1,j}) + \text{Op}(\tilde{h}_{2,j}), \quad (2.9)$$

where

$$\text{Op}(\tilde{h}_{1,j}) = (-1)^{j-1} \text{Op}(h_1)^j \in \text{Op}(S_{\rho,\delta}^{-j\mu}) \quad (2.10)$$

and

$$\text{Op}(\tilde{h}_{2,j}) = -\sum_{k=1}^{j-1} (-1)^k \text{Op}(h_1)^k \text{Op}(1-c_1) \in \text{Op}(S_{\rho,\delta}^{-\mu}).$$

By Proposition 2.1 (1) and asymptotic expansions it follows that

$$\text{Op}(\tilde{h}_{2,j}) = -\sum_{k=1}^{j-1} (-1)^k \text{Op}(h_1)^k \text{Op}(1-c_1) \text{Op}(h_1)^k$$

$$+ \text{Op}(\tilde{h}_{3,j}) + \text{Op}(\tilde{h}_{4,j}), \quad (2.11)$$

for some $\tilde{h}_{3,j} \in S_{\rho,\delta}^{-\mu}$ which is equal to zero in $\text{supp } c_1$ and $\tilde{h}_{4,j} \in S_{\rho,\delta}^{-\mu}$. Now let $b_j$ and $r_k$ be defined by the formulae

$$\text{Op}(b_j) = \text{Op}(c_2) \text{Op}(\tilde{b}_j) \in \text{Op}(S_{\rho,\delta}^{(1/\omega_0)})$$

$$\text{Op}(r_k) = \text{Op}(c_2) \text{Op}(\tilde{r}_k) \in \text{Op}(S_{\rho,\delta}^{(\sigma-k\mu/\omega_0)}).$$

Then

$$\text{Op}(b_j) = \sum_{k=0}^{j-1} (-1)^k \text{Op}(r_k)$$

and (2.9)–(2.11) give

$$\text{Op}(b_j) \text{Op}(a) = \text{Op}(c_2) \text{Op}(c_1) + \text{Op}(c_2) \text{Op}(\tilde{h}_{1,j})$$

$$- \sum_{k=1}^{j-1} (-1)^k \text{Op}(c_2) \text{Op}(1-c_1) \text{Op}(h_1)^k + \text{Op}(c_2) \text{Op}(\tilde{h}_{3,j}) + \text{Op}(c_2) \text{Op}(\tilde{h}_{4,j}).$$
Since $c_1 = 1$ and $\tilde{h}_{3,j} = 0$ on $\text{supp} \ c_2$, it follows that
\[
\text{Op}(c_2) \text{ Op}(c_1) = \text{ Op}(c_2) \mod \text{ Op}(S^{-\infty}),
\]
\[
\text{Op}(c_2) \text{ Op}(\tilde{h}_{1,j}) \in \text{ Op}(S_{\rho,\delta}^{-j\mu}),
\]
\[
\sum_{k=1}^{j-1} (-1)^k \text{ Op}(c_2) \text{ Op}(1-c_1) \text{ Op}(h_1)^k \in \text{ Op}(S^{-\infty}),
\]
\[
\text{Op}(c_2) \text{ Op}(\tilde{h}_{3,j}) \in \text{ Op}(S^{-\infty})
\]
and
\[
\text{Op}(c_2) \text{ Op}(\tilde{h}_{4,j}) \in \text{ Op}(S_{\rho,\delta}^{-j\mu}).
\]
Hence, (2.8) follows for some $h_j \in S_{\rho,\delta}^{-j\mu}$.

By choosing $b_0 \in S^{(1/\omega)}_{\rho,\delta}$ such that
\[
b_0 \sim \sum r_k,
\]
it follows that $\text{Op}(b_0) \text{ Op}(a) = \text{ Op}(c_2) + \text{ Op}(h_0)$, with
\[
h_0 \in S^{-\infty}.
\]
The assertion (4) now follows by letting
\[
b(x, \xi) = \varphi_3(x) b_0(x, \xi), \quad c(x, \xi) = \varphi_3(x) c_2(x, \xi),
\]
and
\[
h(x, \xi) = \varphi_3(x) h_0(x, \xi),
\]
and using the fact that if $\varphi_3 \in C_0^\infty(\mathbb{R}^d)$ and $h_0 \in S^{-\infty}(\mathbb{R}^{2d})$, then $\varphi_3(x) h_0(x, \xi) \in \mathcal{S}(\mathbb{R}^{2d})$. The proof is complete. \qed

**Remark 2.4.** By Proposition 2.3 it follows that Definition 1.3 in [27] is equivalent to Definition 1.8. We also remark that if $a$ is an appropriate symbol, and $\text{Char}'(a)$ the set of characteristic points for $a$ in the sense of [22, Definition 18.1.5], then $\text{Char}_{(\omega_0)}(a) \subseteq \text{Char}'(a)$. Furthermore, strict embedding might occur, especially for symbols to hypoelliptic partial operators with constant coefficients, which are not elliptic (cf. Example 3.11 in [27]).

### 3. Wave Front Sets with Respect to Fourier Banach Spaces

In this section we define wave-front sets with respect to Fourier Banach spaces, and show some basic properties.

Let $\omega \in \mathcal{P}(\mathbb{R}^{2d})$, $\Gamma \subseteq \mathbb{R}^d \setminus 0$ be an open cone and let $\mathcal{B}$ be a translation invariant BF-space on $\mathbb{R}^d$. For any $f \in \mathcal{E}'(\mathbb{R}^d)$, let
\[
|f|_{\mathcal{B}(\omega, \Gamma)} = |f|_{\mathcal{B}(\omega, \Gamma)_\iota} \equiv \|\hat{f}\omega(x, \cdot)\chi_{\Gamma}\|_{\mathcal{B}}.
\]
We note that \( \hat{\omega}(x, \cdot) \chi_{\Gamma} \in \mathcal{B}_{\text{loc}} \) for every \( f \in \mathcal{E}' \). If \( \hat{\omega}(x, \cdot) \chi_{\Gamma} \notin \mathcal{B} \), then we set \( |f|_{\mathcal{F}(\mathcal{B}(\mathcal{B}))} = +\infty \). Hence \( | \cdot |_{\mathcal{F}(\mathcal{B}(\mathcal{B}))} \) defines a semi-norm on \( \mathcal{E}' \) which might attain the value +\( \infty \). Since \( \omega \) is \( v \)-moderate for some \( v \in \mathcal{P}(\mathbb{R}^{d}) \), it follows that different \( x \in \mathbb{R}^{d} \) gives rise to equivalent semi-norms. Furthermore, if \( \Gamma = \mathbb{R}^{d} \setminus 0 \) and \( f \in \mathcal{F}(\mathcal{B}) \cap \mathcal{E}' \), then \( |f|_{\mathcal{F}(\mathcal{B}(\mathcal{B}))} \) agrees with \( \| f \|_{\mathcal{F}(\mathcal{B}(\mathcal{B}))} \). For simplicity we write \( |f|_{\mathcal{F}(\mathcal{B}(\mathcal{B}))} \) instead of \( |f|_{\mathcal{F}(\mathcal{B}(\mathcal{B}))} \) when \( \omega = 1 \).

For the sake of notational convenience we set

\[
| \cdot |_{\mathcal{B}(\Gamma)} = | \cdot |_{\mathcal{F}(\mathcal{B}(\mathcal{B}))}, \quad \text{when } \mathcal{B} = \mathcal{F}(\mathcal{B}).
\]

We let \( \Theta_{\mathcal{B}}(f) = \Theta_{\mathcal{F}(\mathcal{B}(\mathcal{B}))}(f) \) be the set of all \( \xi \in \mathbb{R}^{d} \setminus 0 \) such that \( |f|_{\mathcal{B}(\Gamma)} < \infty \), for some \( \Gamma = \Gamma_{\xi} \). We also let \( \Sigma_{\mathcal{B}}(f) \) be the complement of \( \Theta_{\mathcal{B}}(f) \) in \( \mathbb{R}^{d} \setminus 0 \). Then \( \Theta_{\mathcal{B}}(f) \) and \( \Sigma_{\mathcal{B}}(f) \) are open respectively closed subsets in \( \mathbb{R}^{d} \setminus 0 \), which are independent of the choice of \( x \in \mathbb{R}^{d} \).

\textbf{Definition 3.1.} Let \( \mathcal{B} \) be a translation invariant BF-space on \( \mathbb{R}^{d} \), \( \omega \in \mathcal{P}(\mathbb{R}^{d}), \mathcal{B} \) be as in (3.2), and let \( X \) be an open subset of \( \mathbb{R}^{d} \). The wave-front set of \( f \in \mathcal{P}(X) \), \( \text{WF}_{\mathcal{B}}(f) \equiv \text{WF}_{\mathcal{F}(\mathcal{B})(\mathcal{B})}(f) \) with respect to \( \mathcal{B} \) consists of all pairs \((x_{0}, \xi_{0}) \) in \( X \times (\mathbb{R}^{d} \setminus 0) \) such that \( \xi_{0} \in \Sigma_{\mathcal{B}}(\varphi f) \) holds for each \( \varphi \in C_{0}^{\infty}(X) \) such that \( \varphi(x_{0}) \neq 0 \).

We note that \( \text{WF}_{\mathcal{B}}(f) \) in Definition 3.1 is a closed set in \( X \times (\mathbb{R}^{d} \setminus 0) \), since it is obvious that its complement is open. We also note that if \( x_{0} \in \mathbb{R}^{d} \) is fixed and \( \omega_{0}(\xi) = \omega(x_{0}, \xi) \), then \( \text{WF}_{\mathcal{F}(\mathcal{B})(\mathcal{B})}(f) = \text{WF}_{\mathcal{F}(\mathcal{B})(\mathcal{B})}(f) \), since \( \Sigma_{\mathcal{B}} \) is independent of \( x_{0} \).

The following theorem shows that wave-front sets with respect to \( \mathcal{F}(\mathcal{B})(\mathcal{B}) \) satisfy appropriate micro-local properties. It also shows that such wave-front sets decreases when the local Fourier BF-spaces increases, or when the weight \( \omega \) decreases.

\textbf{Theorem 3.2.} Let \( X \subseteq \mathbb{R}^{d} \) be open, \( \mathcal{B}_{1}, \mathcal{B}_{2} \) be translation invariant BF-spaces, \( \varphi \in C^{\infty}(\mathbb{R}^{d}) \), \( \omega_{1}, \omega_{2} \in \mathcal{P}(\mathbb{R}^{d}) \) and \( f \in \mathcal{P}(X) \). If \( \text{WF}_{\mathcal{B}_{1}(\omega_{1})_{\text{loc}}} \subseteq \text{WF}_{\mathcal{B}_{2}(\omega_{2})_{\text{loc}}} \), then 

\[
\text{WF}_{\mathcal{B}_{2}(\omega_{2})}(\varphi f) \subseteq \text{WF}_{\mathcal{B}_{1}(\omega_{1})}(f).
\]

\textit{Proof.} It suffices to prove

\[
\Sigma_{\mathcal{B}_{2}}(\varphi f) \subseteq \Sigma_{\mathcal{B}_{1}}(f).
\]

when \( \mathcal{B}_{j} = \mathcal{F}(\mathcal{B}_{j}(\omega_{j})) \), \( \varphi \in \mathcal{P}(\mathbb{R}^{d}) \) and \( f \in \mathcal{E}'(\mathbb{R}^{d}) \), since the statement only involve local assertions. The local properties and Remark 1.2 also imply that it is no restriction to assume that \( \omega_{1} = \omega_{2} = 1 \).

Let \( \xi_{0} \in \Theta_{\mathcal{B}_{2}}(f) \), and choose open cones \( \Gamma_{1} \) and \( \Gamma_{2} \) in \( \mathbb{R}^{d} \) such that \( \overline{\Gamma_{2}} \subseteq \Gamma_{1} \). Since \( f \) has compact support, it follows that \( |\hat{f}(\xi)| \leq C|\xi|^{N_{0}} \) for some positive constants \( C \) and \( N_{0} \). The result therefore follows if
we prove that for each $N$, there are constants $C_N$ such that

$$ |\varphi f|_{B_2(\Gamma_2)} \leq C_N \left( |f|_{B_2(\Gamma_1)} + \sup_{\xi \in \mathbb{R}^d} \left( |\hat{f}(\xi)| \langle \xi \rangle^{-N} \right) \right) $$

when $\Gamma_2 \subseteq \Gamma_1$ and $N = 1, 2, \ldots$. (3.4)

By using the fact that $\omega$ is $\nu$-moderate for some $\nu \in \mathcal{P}(\mathbb{R}^d)$, and letting $F(\xi) = |\hat{f}(\xi)|$ and $\psi(\xi) = |\hat{\varphi}(\xi)|$, it follows that $\psi$ turns rapidly to zero at infinity and

$$ |\varphi f|_{B_2(\Gamma_2)} = |\varphi f|_{\mathcal{B}_2(\Gamma_2)} = \|\mathcal{F}(\varphi f)\chi_{\Gamma_2}\|_{\mathcal{B}_2} $$

$$ \leq C \left\| \left( \int_{\mathbb{R}^d} \varphi(\cdot - \eta) \hat{f}(\eta) \, d\eta \right) \chi_{\Gamma_2} \right\|_{\mathcal{B}_2} \leq C(J_1 + J_2) $$

for some positive constant $C$, where

$$ J_1 = \left\| \left( \int_{\Gamma_1} \varphi(\cdot - \eta) \hat{f}(\eta) \, d\eta \right) \chi_{\Gamma_2} \right\|_{\mathcal{B}_2} \quad (3.5) $$

and

$$ J_2 = \left\| \left( \int_{\mathbb{R}^d \setminus \Gamma_1} \varphi(\cdot - \eta) \hat{f}(\eta) \, d\eta \right) \chi_{\Gamma_2} \right\|_{\mathcal{B}_2} \quad (3.6) $$

and $\chi_{\Gamma_2}$ is the characteristic function of $\Gamma_2$. First we estimate $J_1$. By (3) in Definition 1.1 and (1.4), it follows for some constants $C_1, \ldots, C_5$ that

$$ J_1 \leq C_1 \left\| \int_{\Gamma_1} \varphi(\cdot - \eta) \hat{f}(\eta) \, d\eta \right\|_{\mathcal{B}_2} = C_1 \| \varphi * (\chi_{\Gamma_1} \hat{f}) \|_{\mathcal{B}_2} $$

$$ = C_2 \| \varphi \mathcal{F}^{-1}(\chi_{\Gamma_1} \hat{f}) \|_{\mathcal{B}_2} \leq C_3 \| \varphi \mathcal{F}^{-1}(\chi_{\Gamma_1} \hat{f}) \|_{\mathcal{B}_1} $$

$$ = C_4 \| \varphi * (\chi_{\Gamma_1} \hat{f}) \|_{\mathcal{B}_1} \leq C_5 \| \hat{\varphi} \|_{L^1(\Gamma_1)} \| \chi_{\Gamma_1} \hat{f} \|_{\mathcal{B}_1} = C_\psi \| f \|_{\mathcal{B}_1(\Gamma_1)}, \quad (3.7) $$

where $C_\psi = C_5 \| \varphi \|_{L^1(\Gamma_1)} < \infty$, since $\varphi$ turns rapidly to zero at infinity. In the second inequality we have used the fact that $(\mathcal{F}\mathcal{B}_1)_{loc} \subseteq (\mathcal{F}\mathcal{B}_2)_{loc}$.

In order to estimate $J_2$, we note that the conditions $\xi \in \Gamma_2$, $\eta \notin \Gamma_1$ and the fact that $\Gamma_2 \subseteq \Gamma_1$ imply that $|\xi - \eta| > c \max(|\xi|, |\eta|)$ for some constant $c > 0$, since this is true when $1 = |\xi| \geq |\eta|$. We also note that if $N_1$ is large enough, then $\langle \cdot \rangle^{-N_1} \in \mathcal{B}_2$, because $\mathcal{F}$ is continuously embedded in $\mathcal{B}_2$. Since $\psi$ turns rapidly to zero at infinity, it follows
that for each \(N_0 > d + N_1\) and \(N \in \mathbb{N}\) such that \(N > N_0\), it holds

\[
J_2 \leq C_1 \left\| \left( \int_{\mathcal{G}_1} \langle \cdot - \eta \rangle^{-(2N_0+N)} F(\eta) \, d\eta \right) \chi_{\mathcal{F}_2} \right\|_{\mathcal{F}}
\]

\[
\leq C_2 \left\| \left( \int_{\mathcal{G}_1} \langle \cdot \rangle^{-N_0} \langle \eta \rangle^{-N_0} \langle \eta \rangle^{-N} F(\eta) \, d\eta \right) \chi_{\mathcal{F}_2} \right\|_{\mathcal{F}}
\]

\[
\leq C_2 \int_{\mathcal{G}_1} \left\| \langle \cdot \rangle^{-N_0} \chi_{\mathcal{F}_2} \langle \eta \rangle^{-N_0} \langle \eta \rangle^{-N} F(\eta) \right\| \, d\eta
\]

\[
\leq C \sup_{\eta \in \mathbb{R}^d} |\langle \eta \rangle^{-N} F(\eta)|, \quad (3.8)
\]

for some constants \(C_1, C_2 > 0\), where \(C = C_2 \left\| \langle \cdot \rangle^{-N_0} \right\|_{\mathcal{F}_2} \left\| \langle \cdot \rangle^{-N_0} \right\|_{L^1} < \infty\). This proves (3.4), and the result follows. \(\Box\)

4. Mapping properties for pseudo-differential operators on wave-front sets

In this section we establish mapping properties for pseudo-differential operators on wave-front sets of Fourier Banach types. More precisely, we prove the following result (cf. (0.1)):

**Theorem 4.1.** Let \(\rho > 0, \omega \in \mathcal{P}(\mathbb{R}^d), \omega_0 \in \mathcal{P}_{\rho,0}(\mathbb{R}^d), a \in S^{(\omega_0)}_{\rho,0}(\mathbb{R}^d), \) and \(f \in \mathcal{S}'(\mathbb{R}^d)\). Also let \(\mathcal{B}\) be a translation invariant BF-space on \(\mathbb{R}^d\). Then

\[
WF_{\mathcal{F}\mathcal{B}(\omega/\omega_0)}(Op(a)f) \subseteq WF_{\mathcal{F}\mathcal{B}(\omega)}(f)
\]

\[
\subseteq WF_{\mathcal{F}\mathcal{B}(\omega/\omega_0)}(Op(a)f) \cup \text{Char}_{\omega_0}(a). \quad (4.1)
\]

We shall mainly follow the proof of Theorem 3.1 in [27]. The following restatement of Proposition 3.2 in [27] shows that \((x_0, \xi) \notin WF_{\mathcal{F}\mathcal{B}(\omega/\omega_0)}(Op(a)f)\) when \(x_0 \notin \text{supp} \, f\).

**Proposition 4.2.** Let \(\omega \in \mathcal{P}(\mathbb{R}^d), \omega_0 \in \mathcal{P}_{\rho,\delta}(\mathbb{R}^d), 0 \leq \delta \leq \rho, 0 < \rho, \delta < 1,\) and let \(a \in S^{(\omega_0)}_{\rho,0}(\mathbb{R}^d)\). Also let \(\mathcal{B}\) be a translation invariant BF-space, and let the operator \(L_a\) on \(\mathcal{S}'(\mathbb{R}^d)\) be defined by the formula

\[
(L_a f)(x) \equiv \varphi_1(x)(Op(a)(\varphi_2 f))(x), \quad f \in \mathcal{S}'(\mathbb{R}^d), \quad (4.2)
\]

where \(\varphi_1 \in C^\infty(\mathbb{R}^d)\) and \(\varphi_2 \in S^{(\omega_0)}_{0,0}(\mathbb{R}^d)\) are such that

\[
\text{supp} \, \varphi_1 \cap \text{supp} \, \varphi_2 = \emptyset.
\]

Then the kernel of \(L_a\) belongs to \(\mathcal{S}(\mathbb{R}^{2d})\). In particular, the following is true:

1. \(L_a = Op(a_0)\) for some \(a_0 \in \mathcal{S}(\mathbb{R}^{2d})\);
2. \(WF_{\mathcal{F}\mathcal{B}(\omega/\omega_0)}(L_a f) = \emptyset\).
Next we consider properties of the wave-front set of \( \text{Op}(a)f \) at a fixed point when \( f \) is concentrated to that point.

**Proposition 4.3.** Let \( \rho, \omega, \omega_0, a \) and \( \mathcal{B} \) be as in Theorem 4.1. Also let \( f \in \mathcal{E}'(\mathbb{R}^d) \). Then the following is true:

1. if \( \Gamma_1, \Gamma_2 \subseteq \mathbb{R}^d \setminus 0 \) are open cones such that \( \Gamma_2 \subseteq \Gamma_1 \), and \( |f|_{\mathcal{B}(\omega, \Gamma_1)} < \infty \), then \( |\text{Op}(a)f|_{\mathcal{B}(\omega/\omega_0, \Gamma_2)} < \infty \);

2. \( \text{WF}_{\mathcal{B}(\omega/\omega_0)}(\text{Op}(a)f) \subseteq \text{WF}_{\mathcal{B}(\omega)}(f) \).

We note that \( \text{Op}(a)f \) in Proposition 4.3 makes sense as an element in \( \mathcal{S}'(\mathbb{R}^d) \), by Proposition 1.7.

**Proof.** We shall mainly follow the proof of Proposition 3.3 in [27]. We may assume that \( \omega(x, \xi) = \omega(\xi) \), \( \omega_0(x, \xi) = \omega_0(\xi) \), and that \( \text{supp } a \subseteq K \times \mathbb{R}^d \) for some compact set \( K \subseteq \mathbb{R}^d \), since the statements only involve local assertions.

Let \( F(\xi) = |\hat{f}(\xi)| \omega(\xi) \), and let \( \mathcal{F}_1 a \) denote the partial Fourier transform of \( a(x, \xi) \) with respect to the \( x \) variable. By straightforward computation, for arbitrary \( N \) we have

\[
|\mathcal{F}(\text{Op}(a)f)(\xi)\omega(\xi)/\omega_0(\xi)| \leq C \int_{\mathbb{R}^d} \langle \xi - \eta \rangle^{-N} F(\eta) \, d\eta,
\]

for some constant \( C \) (cf. (3.6) and (3.8) in [27]).

We have to estimate

\[
|(\text{Op}(a)f)|_{\mathcal{B}(\omega/\omega_0, \Gamma_2)} = \|\mathcal{F}(\text{Op}(a)f)\omega/\omega_0\chi_{\Gamma_2}\|_{\mathcal{B}}.
\]

By (4.3) we get

\[
\|\mathcal{F}(\text{Op}(a)f)\omega/\omega_0\chi_{\Gamma_2}\|_{\mathcal{B}} \leq C \left( \int_{\mathbb{R}^d} \langle \cdot - \eta \rangle^{-N} F(\eta) \, d\eta \right) \chi_{\Gamma_2} \|_{\mathcal{B}} \leq C (J_1 + J_2),
\]

where \( C \) is a constant and

\[
J_1 = \left\| \left( \int_{\Gamma_1} \langle \cdot - \eta \rangle^{-N} F(\eta) \, d\eta \right) \chi_{\Gamma_2} \right\|_{\mathcal{B}}
\]

and

\[
J_2 = \left\| \left( \int_{\mathbb{R}^d \setminus \Gamma_1} \langle \cdot - \eta \rangle^{-N} F(\eta) \, d\eta \right) \chi_{\Gamma_2} \right\|_{\mathcal{B}}.
\]

In order to estimate \( J_1 \) and \( J_2 \) we argue as in the proof of (3.3). More precisely, by (1.4) we get

\[
J_1 \leq \left\| \int_{\Gamma_1} \langle \cdot - \eta \rangle^{-N} F(\eta) \, d\eta \right\|_{\mathcal{B}} = \left\| \langle \cdot \rangle^{-N} \ast (\chi_{\Gamma_1} F) \right\|_{\mathcal{B}} \leq C \left\| \langle \cdot \rangle^{-N} \right\|_{L^1(\mathbb{R}^d)} \left\| \chi_{\Gamma_1} F \right\|_{\mathcal{B}} < \infty.
\]
Next we estimate $J_2$. Since $\Gamma_2 \subseteq \Gamma_1$, we get

$$|\xi - \eta| \geq c \max(|\xi|, |\eta|), \quad \text{when} \quad \xi \in \Gamma_2, \quad \text{and} \quad \eta \in \mathcal{C} \Gamma_1,$$

for some constant $c > 0$. (Cf. the proof of Proposition 3.3.)

Since $f$ has compact support, it follows that $F(\eta) \leq C\langle \eta \rangle^{t_1}$ for some constant $C$. By combining these estimates we obtain

$$J_2 \leq \left\| \left( \int_{\Gamma_1} F(\eta) \langle \cdot - \eta \rangle^{-N} d\eta \right) \chi_{\Gamma_2} \right\|_{B}$$

$$\leq C \left\| \left( \int_{\Gamma_1} \langle \eta \rangle^{t_1} \langle \cdot \rangle^{-N/2} \langle \eta \rangle^{-N/2} d\eta \right) \chi_{\Gamma_2} \right\|_{B}$$

$$\leq C \left\| \langle \cdot \rangle^{-N/2} \chi_{\Gamma_2} \right\|_{B} \int_{\Gamma_1} \langle \eta \rangle^{-N/2 + t_1} d\eta.$$

Hence, if we choose $N$ sufficiently large, it follows that the right-hand side is finite. This proves (1).

The assertion (2) follows immediately from (1) and the definitions. The proof is complete. \(\square\)

**Proof of Theorem 4.1** By Proposition 2.1 it is no restriction to assume that $t = 0$. We start to prove the first inclusion in (4.1). Assume that $$(x_0, \xi_0) \notin \WF_{\mathcal{F}(\omega)}(f),$$

let $\chi \in C_0^\infty(\mathbb{R}^d)$ be such that $\chi = 1$ in a neighborhood of $x_0$, and set $\chi_1 = 1 - \chi$ and $a_0(x, \xi) = \chi(x) a(x, \xi)$. Then it follows from Proposition 4.2 that

$$(x_0, \xi_0) \notin \WF_{\mathcal{F}(\omega/\omega_0)}(\Op(a)(\chi_1 f)).$$

Furthermore, by Proposition 4.3 we get

$$(x_0, \xi_0) \notin \WF_{\mathcal{F}(\omega/\omega_0)}(\Op(a_0)(\chi f)),$$

which implies that

$$(x_0, \xi_0) \notin \WF_{\mathcal{F}(\omega/\omega_0)}(\Op(a)(\chi f)), $$

since $\Op(a)(\chi f)$ is equal to $\Op(a_0)(\chi f)$ near $x_0$. The result is now a consequence of the inclusion

$$\WF_{\mathcal{F}(\omega/\omega_0)}(\Op(a)f)$$

$$\subseteq \WF_{\mathcal{F}(\omega/\omega_0)}(\Op(a)(\chi f)) \cup \WF_{\mathcal{F}(\omega/\omega_0)}(\Op(a)(\chi_1 f)).$$

It remains to prove the last inclusion in (4.1). By Proposition 4.2 it follows that it is no restriction to assume that $f$ has compact support. Assume that

$$\WF_{\mathcal{F}(\omega/\omega_0)}(\Op(a)f) \cup \WF_{\mathcal{F}(\omega/\omega_0)}(\Op(a)(\chi_1 f)),$$

and choose $b$, $c$ and $h$ as in Proposition 2.3 (4). We shall prove that $$(x_0, \xi_0) \notin \WF_{\mathcal{F}(\omega)}(f).$$ Since

$$f = \Op(1 - c)f + \Op(b) \Op(a)f - \Op(h)f,$$
the result follows if we prove

\[(x_0, \xi_0) \notin S_1 \cup S_2 \cup S_3,\]

where

\[S_1 = \WF_{\mathcal{F}B(\omega)}(\Op(1-c)f), \quad S_2 = \WF_{\mathcal{F}B(\omega)}(\Op(b)\Op(a)f)\]

and \[S_3 = \WF_{\mathcal{F}B(\omega)}(\Op(h)f).\]

We start to consider \(S_2\). By the first embedding in (1.1) it follows that

\[S_2 = \WF_{\mathcal{F}B(\omega)}(\Op(b)\Op(a)f) \subseteq \WF_{\mathcal{F}B(\omega/\omega)}(\Op(a)f).\]

Since we have assumed that \((x_0, \xi_0) \notin \WF_{\mathcal{F}B(\omega/\omega)}(\Op(a)f)\), it follows that \((x_0, \xi_0) \notin S_2\).

Next we consider \(S_3\). Since \(h \in \mathcal{S}\), it follows that \(\Op(h)f \in \mathcal{S}\). Hence \(S_3\) is empty.

Finally we consider \(S_1\). By the assumptions it follows that \(c_0 = 1 - c\) is zero in \(\Gamma\), and by replacing \(\Gamma\) with a smaller cone, if necessary, we may assume that \(c_0 = 0\) in a conical neighborhood of \(\Gamma\). Hence, if \(\Gamma \equiv \Gamma_1, \Gamma_2, J_1 \text{ and } J_2\) are the same as in the proof of Proposition 4.3, then it follows from that proof and the fact that \(c_0(x, \xi) \in S^0_{\rho,0}\) is compactly supported in the \(x\)-variable, that \(J_1 < +\infty\), and that for each \(N \geq 0\), there are constants \(C_N\) and \(C'_N\) such that

\[|\Op(c_0)f|_{\mathcal{F}B(\omega/\omega, \Gamma_2)} \leq C_N(J_1 + J_2)
\leq C'_N\left(J_1 + \left\| \int_{\Gamma_1} \langle \cdot \rangle^{-N} \langle \eta \rangle^{-N} d\eta \chi_{\Gamma_2} \right\|_{\mathcal{F}B} \right).\]  
(4.4)

By choosing \(N\) large enough, it follows that

\[|\Op(c_0)f|_{\mathcal{F}B(\omega/\omega, \Gamma_2)} < \infty.\]

This proves that \((x_0, \xi_0) \notin S_1\), and the proof is complete. \(\square\)

**Remark 4.4.** We note that the statements in Theorems 4.1 are not true if \(\omega_0 = 1\) and the assumption \(\rho > 0\) is replaced by \(\rho = 0\). (Cf. Remark 3.7 in [27].)

Next we apply Theorem 4.1 on operators which are elliptic with respect to \(S^\omega_{\rho,\delta}(\mathbb{R}^{2d})\), where \(\omega_0 \in \mathcal{P}_{\rho,\delta}(\mathbb{R}^{2d})\). More precisely, assume that \(0 \leq \delta < \rho \leq 1\) and \(a \in S^\omega_{\rho,\delta}(\mathbb{R}^{2d})\). Then \(a\) and \(\Op(a)\) are called (locally) elliptic with respect to \(S^\omega_{\rho,\delta}(\mathbb{R}^{2d})\) or \(\omega_0\), if for each compact set \(K \subseteq \mathbb{R}^d\), there are positive constants \(c\) and \(R\) such that

\[|a(x, \xi)| \geq c\omega_0(x, \xi), \quad x \in K, \ |\xi| \geq R.\]

Since \(|a(x, \xi)| \leq C\omega_0(x, \xi)|\xi|\), it follows from the definitions that for each multi-index \(\alpha\), there are constants \(C_{\alpha,\beta}\) such that

\[|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha,\beta}|a(x, \xi)||\xi|^{-\rho|\beta|+\delta|\alpha|}, \quad x \in K, \ |\xi| > R,\]
when $a$ is elliptic. (See e.g. [2, 22].)

It immediately follows from the definitions that $\text{Char}_{\omega_0}(a) = \emptyset$ when $a$ is elliptic with respect to $\omega_0$. The following result is now an immediate consequence of Theorem 4.1.

**Theorem 4.5.** Let $\omega \in \mathcal{D}R(\mathbb{R}^{2d})$, $\omega_0 \in \mathcal{D}R_{\rho,0}(\mathbb{R}^{2d})$, $\rho > 0$, and let $a \in S^\omega_{\rho,0}(\mathbb{R}^{2d})$ be elliptic with respect to $\omega_0$. Also let $B$ be a translation invariant $BF$-space. If $f \in \mathcal{D}R(\mathbb{R}^d)$, then

$$WF_{\mathcal{F}B(\omega/\omega_0)}(\text{Op}(a)f) = WF_{\mathcal{F}B(\omega)}(f).$$

5. WAVE-FRONT SETS OF SUP AND INF TYPES AND PSEUDO-DIFFERENTIAL OPERATORS

In this section we put the micro-local analysis in a more general context compared to previous sections, and define wave-front sets with respect to sequences of Fourier $BF$-spaces.

Let $\omega_j \in \mathcal{D}R(\mathbb{R}^{2d})$ and $B_j$ be translation invariant $BF$-space on $\mathbb{R}^d$ when $j$ belongs to some index set $J$, and consider the array of spaces, given by

$$(B_j) \equiv (B_j)_{j \in J}, \; \text{where} \; B_j = \mathcal{F}B_j(\omega_j), \; j \in J. \; \; \; (5.1)$$

If $f \in \mathcal{D}'(\mathbb{R}^d)$, and $(B_j)$ is given by (5.1), then we let $\Theta_{(B_j)}^{\text{sup}}(f)$ be the set of all $\xi \in \mathbb{R}^d \setminus 0$ such that for some $\Gamma = \Gamma_\xi$ and each $j \in J$ it holds $|f|_{B_j(\Gamma)} < \infty$. We also let $\Theta_{(B_j)}^{\text{inf}}(f)$ be the set of all $\xi \in \mathbb{R}^d \setminus 0$ such that for some $\Gamma = \Gamma_\xi$ and some $j \in J$ it holds $|f|_{B_j(\Gamma)} < \infty$. Finally we let $\Sigma_{(B_j)}^{\text{sup}}(f)$ and $\Sigma_{(B_j)}^{\text{inf}}(f)$ be the complements in $\mathbb{R}^d \setminus 0$ of $\Theta_{(B_j)}^{\text{sup}}(f)$ and $\Theta_{(B_j)}^{\text{inf}}(f)$ respectively.

**Definition 5.1.** Let $J$ be an index set, $B_j$ be translation invariant $BF$-space on $\mathbb{R}^d$, $\omega_j \in \mathcal{D}R(\mathbb{R}^{2d})$ when $j \in J$, $(B_j)$ be as in (5.1), and let $X$ be an open subset of $\mathbb{R}^d$.

1. The wave-front set of $f \in \mathcal{D}'(X)$, $WF_{(B_j)}^{\text{sup}}(f) = WF_{(\mathcal{F}B_j(\omega_j))}^{\text{sup}}(f)$, of sup-type with respect to $(B_j)$, consists of all pairs $(x_0, \xi_0)$ in $X \times (\mathbb{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{(B_j)}^{\text{sup}}(\varphi f)$ holds for each $\varphi \in C^\infty(X)$ such that $\varphi(x_0) \neq 0$;

2. The wave-front set of $f \in \mathcal{D}'(X)$, $WF_{(B_j)}^{\text{inf}}(f) = WF_{(\mathcal{F}B_j(\omega_j))}^{\text{inf}}(f)$, of inf-type with respect to $(B_j)$, consists of all pairs $(x_0, \xi_0)$ in $X \times (\mathbb{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{(B_j)}^{\text{inf}}(\varphi f)$ holds for each $\varphi \in C^\infty(X)$ such that $\varphi(x_0) \neq 0$.

**Remark 5.2.** Let $\omega_j(x, \xi) = (\xi)^{-j}$ for $j \in J = \mathbb{N}_0$ and $B_j = L^{q_j}$, where $q_j \in [1, \infty]$. Then it follows that $WF_{(B_j)}^{\text{sup}}(f)$ in Definition 5.1 is equal to the standard wave front set $WF(f)$ in Chapter VIII in [22].

The following result follows immediately from Theorem 4.1 and its proof. We omit the details.
Theorem 4.1. Let $\rho > 0$, $\omega_j \in \mathcal{P}(\mathbb{R}^{2d})$ for $j \in J$, $\omega_0 \in \mathcal{P}_{\rho,0}(\mathbb{R}^{2d})$, $a \in S^\omega_{\rho,0}(\mathbb{R}^{2d})$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. Also let $\mathcal{B}_j$ be a translation invariant $\mathcal{B}$-$\mathcal{F}$-space on $\mathbb{R}^d$ for every $j$. Then

$$WF^{\sup}_{(\mathcal{B}_j(\omega_j/\omega_0))}(\text{Op}(a)f) \subseteq WF^{\sup}_{(\mathcal{B}_j(\omega_j))}(f) \subseteq WF^{\sup}_{(\mathcal{B}_j(\omega_j/\omega_0))}(\text{Op}(a)f) \cup Char(\omega_0)(a), \quad (4.1)'$$

and

$$WF^{\inf}_{(\mathcal{B}_j(\omega_j/\omega_0))}(\text{Op}(a)f) \subseteq WF^{\inf}_{(\mathcal{B}_j(\omega_j))}(f) \subseteq WF^{\inf}_{(\mathcal{B}_j(\omega_j/\omega_0))}(\text{Op}(a)f) \cup Char(\omega_0)(a). \quad (4.1)''$$

The following generalization of Theorem 4.5 is an immediate consequence of Theorem 4.1, since $Char(\omega_0)(a) = \emptyset$, when $a$ is elliptic with respect to $\omega_0$.

Theorem 4.5. Let $\rho > 0$, $\omega_j \in \mathcal{P}(\mathbb{R}^{2d})$ for $j \in J$, $\omega_0 \in \mathcal{P}_{\rho,0}(\mathbb{R}^{2d})$ and let $a \in S^\omega_{\rho,0}(\mathbb{R}^{2d})$ be elliptic with respect to $\omega_0$. Also let $\mathcal{B}_j$ be a translation invariant $\mathcal{B}$-$\mathcal{F}$-space on $\mathbb{R}^d$ for every $j$. If $f \in \mathcal{S}'(\mathbb{R}^d)$, then

$$WF^{\sup}_{(\mathcal{B}_j(\omega_j/\omega_0))}(\text{Op}(a)f) = WF^{\sup}_{(\mathcal{B}_j(\omega_j))}(f)$$

and

$$WF^{\inf}_{(\mathcal{B}_j(\omega_j/\omega_0))}(\text{Op}(a)f) = WF^{\inf}_{(\mathcal{B}_j(\omega_j))}(f).$$

Remark 5.3. We note that many properties valid for the wave-front sets of Fourier Banach type also hold for wave-front sets in the present section. For example, the conclusions in Remark 4.4 hold for wave-front sets of sup- and inf-types.

Finally we remark that there are some technical generalizations of Theorem 4.1 which involve pseudo-differential operators with symbols in $S^\omega_{\rho,0}(\mathbb{R}^{2d})$ with $0 \leq \delta < \rho \leq 1$. From these generalizations it follows that

$$WF(\text{Op}(a)f) \subseteq WF(f) \subseteq WF(\text{Op}(a)f) \cup Char(\omega_0)(a),$$

when $0 \leq \delta < \rho \leq 1$, $\omega_0 \in \mathcal{P}_{\rho,0}(\mathbb{R}^{2d})$, $a \in S^\omega_{\rho,0}(\mathbb{R}^{2d})$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. (Cf. Theorem 5.3' and Theorem 5.5 in [27].)

6. Wave front sets with respect to modulation spaces

In this section we define wave-front sets with respect to modulation spaces, and show that they coincide with wave-front sets of Fourier Banach types. In particular, all micro-local properties for pseudo-differential operators in the previous sections carry over to wave-front sets of modulation space types.
We start with defining general types of modulation spaces. Let \( \phi \in \mathcal{S}(\mathbb{R}^d) \setminus 0 \) be fixed, and let \( f \in \mathcal{S}(\mathbb{R}^d) \). Then the short-time Fourier transform \( V_\phi f \) is the element in \( \mathcal{S}(\mathbb{R}^{2d}) \), defined by the formula

\[
(V_\phi f)(x, \xi) \equiv \mathcal{F}(f \cdot \overline{\phi(\cdot - x)})(\xi).
\]

We usually assume that \( \phi \in \mathcal{S}(\mathbb{R}^d) \), and in this case the short-time Fourier transform \( (V_\phi f) \) takes the form

\[
(V_\phi f)(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) \overline{\phi(y - x)} e^{-i(y, \xi)} dy,
\]

when \( f \in \mathcal{S}(\mathbb{R}^d) \).

Now let \( \mathcal{B} \) be a translation invariant BF-space on \( \mathbb{R}^{2d} \), with respect to \( v \in \mathcal{P}(\mathbb{R}^{2d}) \). Also let \( \phi \in \mathcal{S}(\mathbb{R}^d) \setminus 0 \) and \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \) be such that \( \omega \) is \( v \)-moderate. Then the modulation space \( M(\omega) = M(\omega, \mathcal{B}) \) consists of all \( f \in \mathcal{S}(\mathbb{R}^d) \) such that \( V_\phi f \cdot \omega \in \mathcal{B} \). We note that \( M(\omega, \mathcal{B}) \) is a Banach space with the norm

\[
\| f \|_{M(\omega, \mathcal{B})} \equiv \| V_\phi f \omega \|_\mathcal{B} \quad (6.1)
\]

(cf. [7]).

**Remark 6.1.** Assume that \( p, q \in [1, \infty] \), \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \) and let \( L_1^{p,q}(\mathbb{R}^{2d}) \) and \( L_2^{p,q}(\mathbb{R}^{2d}) \) be the sets of all \( F \in L_1^{p,q}(\mathbb{R}^{2d}) \) such that

\[
\| F \|_{L_1^{p,q}} \equiv \left( \int \left( \int |F(x, \xi)|^p \, dx \right)^{q/p} \, d\xi \right)^{1/q} < \infty
\]

and

\[
\| F \|_{L_2^{p,q}} \equiv \left( \int \left( \int |F(x, \xi)|^p \, dx \right)^{q/p} \, d\xi \right)^{1/q} < \infty,
\]

respectively (with obvious modifications when \( p = \infty \) or \( q = \infty \)). Then \( M(\omega, \mathcal{B}) \) is equal to the usual modulation space \( M(\omega, \mathcal{B}) \) when \( \mathcal{B} = L_1^{p,q}(\mathbb{R}^{2d}) \). If instead \( \mathcal{B} = L_2^{p,q}(\mathbb{R}^{2d}) \), then \( M(\omega, \mathcal{B}) \) is equal to the space \( W_{\omega}^{p,q}(\mathbb{R}^{2d}) \), related to Wiener-amalgam spaces.

In the following proposition we list some important properties for modulation spaces. We refer to [14] for the proof.

**Proposition 6.2.** Assume that \( \mathcal{B} \) is a translation invariant BF-space on \( \mathbb{R}^{3d} \) with respect to \( v \in \mathcal{P}(\mathbb{R}^{2d}) \), and that \( \omega_0, v_0 \in \mathcal{P}(\mathbb{R}^{2d}) \) are such that \( \omega \) is \( v \)-moderate. Then the following is true:

1. if \( \phi \in M_1^{\infty}(\mathbb{R}^d) \setminus 0 \), then \( f \in M(\omega, \mathcal{B}) \) if and only if \( V_\phi f \omega \in \mathcal{B} \). Furthermore, (6.1) defines a norm on \( M(\omega, \mathcal{B}) \), and different choices of \( \phi \) gives rise to equivalent norms;
2. \( M_{\omega_0,v_0}^p(\mathbb{R}^d) \subseteq M(\omega, \mathcal{B}) \subseteq M_{\omega_0,v_0}^\infty(\mathbb{R}^d) \).
The following generalization of Theorem 2.1 in [29] shows that modulation spaces are locally the same as translation invariant Fourier BF-spaces. We recall that if \( \varphi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\} \) and \( \mathcal{B} \) is a translation invariant BF-space on \( \mathbb{R}^{2d} \), then it follows from Proposition 6.3 that
\[
\mathcal{B}_0 \equiv \{ f \in \mathcal{S}(\mathbb{R}^d) ; \varphi \otimes f \in \mathcal{B} \}
\]
is a translation invariant BF-space on \( \mathbb{R}^d \) which is independent of the choice of \( \varphi \).

**Proposition 6.3.** Let \( \varphi \in C^\infty_0(\mathbb{R}^d) \setminus \{0\} \), \( \mathcal{B} \) be a translation invariant BF-space on \( \mathbb{R}^{2d} \), and let \( \mathcal{B}_0 \) be as in (6.2). Also let \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \), and \( \omega_0(\xi) = \omega(x_0, \xi) \), for some fixed \( x_0 \in \mathbb{R}^d \). Then
\[
M(\omega, \mathcal{B}) \cap \mathcal{E}'(\mathbb{R}^d) = \mathcal{B}_0(\omega_0) \cap \mathcal{E}'(\mathbb{R}^d).
\]
Furthermore, if \( K \subseteq \mathbb{R}^d \) is compact, then
\[
C^{-1}\|f\|_{M(\omega, \mathcal{B})} \leq \|f\|_{M(\omega, \mathcal{B}_0)} \leq C\|f\|_{M(\omega_0, \mathcal{B}_0)}, \quad f \in \mathcal{E}'(K),
\]
for some constant \( C \) which only depends on \( K \).

We need the following lemma for the proof.

**Lemma 6.4.** Assume that \( f \in \mathcal{E}'(\mathbb{R}^d) \). Then the following is true:

1. if \( \phi \in C^\infty_0(\mathbb{R}^d) \), then there exists \( 0 \leq \varphi \in C^\infty_0(\mathbb{R}^d) \) such that
   \[
   (V_\phi f)(x, \xi) = \varphi(x)\hat{\phi}((\mathcal{F}(\phi(\cdot - x)))\xi);
   \]
2. if \( \varphi \in C^\infty_0(\mathbb{R}^d) \), then there exists \( \phi \in C^\infty_0(\mathbb{R}^d) \) such that
   \[
   (\phi \otimes \hat{f})(x, \xi) = \varphi(x)V_\phi f(x, \xi).
   \]

**Proof.** (1) Let \( \varphi \in C^\infty_0 \) be equal to \( (2\pi)^{d/2} \) in a compact set containing the support of the map \( x \mapsto V_\phi f(x, \xi) \). Then (1) is a straight-forward consequence of Fourier’s inversion formula.

The assertion (2) follows by choosing \( \phi \in C^\infty_0 \) such that \( \phi = 1 \) on \( \text{supp} \ f - \text{supp} \varphi \). \( \square \)

**Proof of Proposition 6.3.** We may assume that \( \omega = \omega_0 = 1 \) in view of Remark 1.2. Assume that \( f \in \mathcal{E}' \) and \( \varphi \in C^\infty_0(\mathbb{R}^d) \setminus \{0\} \). From (2) of Lemma 6.4 it follows that there exists \( \phi \in C^\infty_0 \) such that
\[
\|f\|_{M(\mathcal{B})} = \|V_\phi f\|_{\mathcal{B}} = \|\varphi \otimes \hat{f}\|_{\mathcal{B}} = \|\hat{f}\|_{\mathcal{B}_0},
\]
and (6.3) follows. The proof is complete. \( \square \)

Let \( \mathcal{B} \) be a translation invariant BF-space on \( \mathbb{R}^{2d} \), \( \phi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\} \) be fixed, \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \), \( \Gamma \subseteq \mathbb{R}^d \setminus \{0\} \) be an open cone, and let \( \chi_\Gamma(x, \xi) = \chi_\Gamma(\xi) \) be the characteristic function of \( \Gamma \). For any \( f \in \mathcal{E}'(\mathbb{R}^d) \) we set
\[
|f|_{\mathcal{B}(\Gamma)} = |f|_{M(\omega, \mathcal{B}, \Gamma)} = |f|_{M(\omega, \mathcal{B}, \Gamma)} = \|(V_\phi f)\omega \chi_\Gamma\|_{\mathcal{B}}
\]
when \( \mathcal{B} = M(\omega, \mathcal{B}) \). (6.6)
Proposition 6.5. Let $\omega$ be a translation invariant BF-space on $\mathbb{R}^d$, $\omega \in \mathcal{P}(\mathbb{R}^d)$, $f \in \mathcal{D}'(X)$, and let $\mathcal{B} = M(\omega, \mathcal{B})$. Then $\Theta_B(f)$, $\Sigma_B(f)$ and the wave-front set $WF_B(f)$ of $f$ with respect to the modulation space $\mathcal{B}$ are defined in the same way as in Section 3 after replacing the semi-norms of Fourier Banach types in (3.2) with the semi-norms in (6.6).

In Theorem 6.9 below we prove that wave-front sets of Fourier BF-spaces and modulation space types agree with each others. As a first step we prove that $WF_{M^\phi(\omega, \mathcal{B})}(f)$ is independent of $\phi$ in (6.6).

Proposition 6.5. Let $X \subseteq \mathbb{R}^d$ be open, $f \in \mathcal{D}'(X)$ and $\omega \in \mathcal{P}(\mathbb{R}^d)$. Then $WF_{M^\phi(\omega, \mathcal{B})}(f)$ is independent of the window function $\phi \in \mathcal{P}(\mathbb{R}^d) \setminus 0$.

We need some preparation for the proof, and start with the following lemma. We omit the proof (the result can be found in [3]).

Lemma 6.6. Let $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then for some constant $N_0$ and every $N \geq 0$, there are constants $C_N$ such that

$$|V_\phi f(x, \xi)| \leq C_N(x)^{-N}(\xi)^{N_0}.$$ 

The following result can be found in [14]. Here $\hat{\ast}$ is the twisted convolution, given by the formula

$$(F \hat{\ast} G)(x, \xi) = (2\pi)^{-d/2} \int \int F(x - y, \xi - \eta)G(y, \eta)e^{-i(x-y, \eta)}\,dyd\eta,$$

when $F, G \in \mathcal{S}(\mathbb{R}^d)$. The definition of $\hat{\ast}$ extends in such way that one may permit one of $F$ and $G$ to belong to $\mathcal{S}'(\mathbb{R}^d)$, and in this case it follows that $F \hat{\ast} G$ belongs to $\mathcal{S}' \cap C^\infty$.

Lemma 6.7. Let $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\phi_j \in \mathcal{S}(\mathbb{R}^d)$ for $j = 1, 2, 3$. Then

$$(V_{\phi_1} f) \hat{\ast} (V_{\phi_2} \phi_3) = (\phi_3, \phi_1)_{L^2} \cdot V_{\phi_2} f.$$ 

Proof of Proposition 6.5. We assume that $f \in \mathcal{S}'(\mathbb{R}^d)$ and that $\omega(x, \xi) = \omega(\xi)$, since the statements only involve local assertions. Assume that $\phi, \phi_1 \in \mathcal{S}(\mathbb{R}^d) \setminus 0$ and let $\Gamma_1$ and $\Gamma_2$ be open cones in $\mathbb{R}^d$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$. The assertion follows if we prove that

$$|f|_{M^\phi(\omega, \Gamma_2, \mathcal{B})} \leq C(|f|_{M^{\phi_1}(\omega, \Gamma_1, \mathcal{B})} + 1)$$

(6.7)

for some constant $C$.

When proving (6.7) we shall mainly follow the proof of (3.4). Let $v \in \mathcal{P}$ be chosen such that $\omega$ is $v$-moderate, and let

$$\Omega_1 = \{(x, \xi); \xi \in \Gamma_1\} \subseteq \mathbb{R}^{2d} \quad \text{and} \quad \Omega_2 = \Omega_1 \subseteq \mathbb{R}^{2d};$$

with characteristic functions $\chi_1$ and $\chi_2$ respectively. Also set

$$F_k(x, \xi) = |V_{\phi_1} f(x, \xi)\omega(\xi)\chi_k(x, \xi)| \quad \text{and} \quad G = |V_{\phi_1} f(x, \xi)v(\xi)|.$$
By Lemma [6.1] and the fact that $\omega$ is $v$-moderate we get
\[ |V_\phi f(x, \xi)\omega(x, \xi)| \leq C((F_1 + F_2) * G)(x, \xi), \]
for some constant $C$, which implies that
\[ |f|_{M^\phi(\omega, \Gamma_2, \mathcal{B})} \leq C(J_1 + J_2), \tag{6.8} \]
where
\[ J_k = \|(F_k * G)\chi_{\Gamma_2}\|_{\mathcal{B}} \]
and $\chi_{\Gamma_2}(x, \xi) = \chi_{\Gamma_2}(\xi)$ is the characteristic function of $\Gamma_2$. Since $G$ turns rapidly to zero at infinity, (6.4) gives
\[ J_1 \leq \|F_1 * G\|_{\mathcal{B}} \leq \|G\|_{L^1(\nu)} \|F_1\|_{\mathcal{B}} = C|f|_{M^{\phi_1}(\omega, \Gamma_1, \mathcal{B})}, \tag{6.9} \]
where $C = \|G\|_{L^1(\nu)}$.

Next we consider $J_2$. Since, for each $N \geq 0$, there are constants $C_N$ such that
\[ F_2(x, \xi) = 0, \quad \text{and} \quad \langle \xi - \eta \rangle^{-2N} \leq C_N \langle \xi \rangle^{-N} \langle \eta \rangle^{-N} \]
when $\xi \in \Gamma_2$ and $\eta \in \partial \Gamma_1$, it follows from Lemma [5.9] and the computations in (6.8) that
\[ (F_2 * G)(x, \xi) \leq C_N \langle \xi \rangle^{-N} \langle \xi \rangle^{-N}, \quad \xi \in \Gamma_2. \]
Consequently, $J_2 < \infty$. The estimate (6.7) is now a consequence of (6.8), (6.9) and the fact that $J_2 < \infty$. This completes the proof. \hfill $\square$

Since $WF_{M^\phi(\omega, \mathcal{B})}(f)$ is independent of $\phi$ we usually omit $\phi$ and write $WF_{M(\omega, \mathcal{B})}(f)$ instead. We are now able to prove the following.

**Proposition 6.8.** Assume that $\mathcal{B}$ is a translation invariant BF-space on $\mathbb{R}^d$, $\mathcal{B}_0$ is given by (6.2), $\phi \in \mathcal{P}(\mathbb{R}^d) \setminus 0$ and $\omega \in \mathcal{P}(\mathbb{R}^{2d})$. Also assume that $f \in \mathcal{E}'(\mathbb{R}^d)$. Then
\[ \Theta_{M^\phi(\omega, \mathcal{B})}(f) = \Theta_{\mathcal{F}\mathcal{B}_0}(\omega)(f) \quad \text{and} \quad \Sigma_{M^\phi(\omega, \mathcal{B})}(f) = \Sigma_{\mathcal{F}\mathcal{B}_0}(\omega)(f). \tag{6.10} \]

**Proof.** We may assume that $\omega = 1$ in view of Lemma [1.2]. Let $\Gamma_1, \Gamma_2$ be open cones in $\mathbb{R}^d \setminus 0$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$, let $\chi_{\Gamma_2}(x, \xi) = \chi_{\Gamma_2}(\xi)$ be the characteristic function of $\Gamma_2$, and let $\varphi$ and $\phi$ be chosen such that (1) in Lemma [6.3] is fulfilled.

By (6.4) it follows that
\[ |V_\phi f(x, \xi)| \leq \varphi(x)(|\hat{f}| * |\mathcal{F}\phi|)(\xi). \]
This gives
\[ |f|_{M^\phi(\Gamma_2, \mathcal{B})} = |V_\phi f\chi_{\Gamma_2}|_{\mathcal{B}} \leq C|\varphi \otimes ((|\hat{f}| * |\mathcal{F}\phi|)\chi_{\Gamma_2})|_{\mathcal{B}} \]
\[ = C(||\hat{f}| * |\mathcal{F}\phi|)|_{\mathcal{B}_0} \leq C(J_1 + J_2), \]
for some constant $C$, where $J_1$ and $J_2$ are the same as in (3.5) and (3.6) with $\mathcal{B}_2 = \mathcal{B}_0$, $\psi = |\mathcal{F}\phi|$ and $F = |\hat{f}|$. \hfill $\square$
A combination of the latter estimate, (3.7) and (3.8) now gives that
for each $N \geq 0$, there is a constant $C_N$ such that
$$|f|_{M^\phi(\Gamma_2, B)} \leq C_N \left( |f|_{\mathcal{F}B_0} + \sup_\xi |\hat{f}(\xi)|^{-N} \right).$$
Hence, by choosing $N$ large enough it follows that $|f|_{M^\phi(\Gamma_2, B)}$ is finite when $|f|_{\mathcal{F}B_0} < \infty$. Consequently,
$$\Theta_{\mathcal{F}B_0}(f) \subseteq \Theta_M(f). \quad (6.11)$$

In order to get a reversed inclusion we choose $\varphi$ and $\phi$ such that
Lemma 6.4 (2) is fulfilled. Then (6.5) gives
$$|f|_{\mathcal{F}B_0(\Gamma)} = \|\varphi \otimes (\hat{f} \chi_{\Gamma})\|_{\mathcal{B}} = \|(\varphi \otimes 1)(V_\phi f \chi_{\Gamma})\|_{\mathcal{B}} \leq C_1 \|\varphi\|_{L^\infty} \|V_\phi f \chi_{\Gamma}\|_{\mathcal{B}} = C_2 |f|_{M(\omega, B)},$$
for some constants $C_1, C_2 > 0$. This proves that (6.11) holds with reversed inclusion. The proof is complete. □

The following result is now an immediate consequence of Proposition 6.8.

**Theorem 6.9.** Assume that $\mathcal{B}$ is a translation invariant BF-space on $\mathbb{R}^{2d}$, $\mathcal{B}_0$ is given by (6.2), $\omega \in \mathcal{P}(\mathbb{R}^{2d})$, $X \subseteq \mathbb{R}^d$ is open and that $f \in \mathcal{D}'(X)$. Then
$$\text{WF}_{\mathcal{F}B_0(\omega)}(f) = \text{WF}_{M(\omega, \mathcal{B})}(f).$$

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