Agency and the physics of numbers

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ABSTRACT

Analogous to Gödel’s incompleteness theorems is a theorem in physics to the effect that the set of explanations of given evidence is uncountably infinite. An implication of this theorem is that contact between theory and experiment depends on activity beyond computation and measurement—physical activity of some agent making a guess. Standing on the need for guesswork, we develop a representation of a symbol-handling agent that both computes and, on occasion, receives a guess from interaction with an oracle. We show: (1) how physics depends on such an agent to bridge a logical gap between theory and experiment; (2) how to represent the capacity of agents to communicate numerals and other symbols, and (3) how that communication is a foundation on which to develop both theory and implementation of spacetime and related competing schemes for the management of motion.

1. INTRODUCTION

Schrödinger, in his 1954 book “Nature and the Greeks,” laments:\footnote{Further author information: (Send correspondence to J.M.M.)\newline
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\ldots science in its attempt to describe and understand Nature simplifies this very difficult problem. The scientist subconsciously, almost inadvertently, simplifies his problem of understanding Nature by disregarding or cutting out of the picture to be constructed, himself, his own personality, the subject of cognizance. Inadvertently the thinker steps back into the role of an external observer. This facilitates the task very much. But it leaves gaps, enormous lacunae, leads to paradoxes and antimonies whenever, unaware of this initial renunciation, one tries to find oneself in the picture or to put oneself, one’s own thinking and sensing mind, back into the picture. This momentous step \ldots has consequences. So, in brief, we do not belong to this material world that science constructs for us. We are not in it, we are outside. We are only spectators.

For four decades we have been working to answer Schrödinger’s lament, and this is still work in progress. Currently we are trying to develop a notion of agency suitable for introduction into physics. In this effort we overlap efforts made prominent by Mermin, Fuchs, Schack, and others in QBism\footnote{Further author information: (Send correspondence to J.M.M.)\newline
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For us the key to this “suitable for physics” is to posit agents that write and read symbols, especially numerals, leading to a whole new arena to explore.

It is a commonplace that contact between experiment and theory in physics is numerical, but only recently has it been noticed that the numerals by which one writes and reads numbers are written and read by some sort of agents, and that this noticing has profound consequences. These consequences have been overlooked because physics is addicted to a notion of objectivity, long in need of an overhaul, that traces back at least to Descartes. This notion of objectivity wrongly denies the need to recognize agency in physics. Recently, this denial shows cracks. Notions of an agent are entering physics, for example in the characters “Alice” and “Bob” featured in quantum cryptography, in the Bayesian interpretation of quantum-theoretic probabilities, and in the interpretation of quantum mechanics that ascribes wave functions to the state of knowledge of an agent. We want to sharpen the notion of agency by focusing on the capacity of agents to deal with symbols, e.g. numerals.

In making use of numerals and also symbols of an alphabet, people, mathematicians in particular, act as symbol-handling agents. Attention to symbol-handling in connection with agency can be seen in the proof of Gödel’s incompleteness theorems. Those theorems imply that the system of symbols by which one deals with arithmetic allows endless logically undetermined questions, implying endless opportunities for appending axioms to arithmetic, which implicitly shows a role for an agent, namely an agent that asserts an axiom, thereby extending arithmetic. Gödel’s proof, while made in a fixed countable system of symbols, yet implies that the set of extended arithmetical systems is uncountable (meaning it cannot be mapped injectively to the set of natural numbers 0, 1, 2, …).

Gödel’s theorem hints at how to sharpen a notion of agency appropriate to physics. Aiming to show logical indeterminism in physics, we proved a theorem in quantum theory, to do with the set of explanations of given evidence. Although evidence excludes many candidates for explanations, we proved that evidence can never logically determine its explanation, for the set of explanations of given evidence is uncountable. The uncountable set of explanations implies a logical gap between theory and experiment. Bridging this gap with a particular explanation requires a guess beyond the reach of computation and measurement, and making this guess calls for physical activity that we characterize as agency. (Previously we had shown that the set of explanations of given evidence was infinite, but only countably infinite; what is new was the demonstration that the set is uncountable.)

2. RECENT PROOF OF UNCOUNTABLE EXPLANATIONS

Quantum theory expresses (an abstraction of) evidence in terms of probabilities of outcomes, and it offers mathematical language to express explanations of these probabilities of outcomes by a density operator $\rho$ and a Positive Operator-Valued Measure $M$. Any explanation implies a probability of an outcome $\omega$ of the form

$$\Pr(\omega) = \text{tr}[\rho M(\omega)].$$

(1)

The issue is that going the other way, from probabilities to explanations, is anything but unique. To show this, we let $\Omega$ be the space of outcomes, so that $M(\Omega) = 1$. As shown in more detail in [5], there is an uncountable set of inequivalent explanations of $\Pr(\omega)$, one explanation for each of the infinite
sequences $j_1, j_2, \ldots$:

$$\left(\rho^{(0)} \otimes \bigotimes_{n=1}^{\infty} \rho_{j_n}^{(n)}(\omega^{(n)}) \otimes \bigotimes_{n=1}^{\infty} M^{(n)}(\Omega^{(n)})\right),$$  \hspace{1cm} (2)$$

where $\rho$ and $M(\omega)$ in (1) are written in (2) respectively as $\rho^{(0)}$, and $M^{(0)}(\omega^{(0)})$. The explanations corresponding to different sequences $j_1, j_2, \ldots$ are inequivalent, because they extend to an expanded experiment which provides for attending to two or more distinct outcomes in each tensor product factor. To extend the explanation one attends to two or more outcomes as proper subsets of $\Omega^{(n)}$, for which two extended explanations imply different probabilities. Thus one gets to the following.

**Theorem**: The set of inequivalent explanations that exactly fit given probabilities is uncountably infinite.

**Corollary**: There is no logical ground to exclude any of the uncountable set of potential explanations of given evidence prior to obtaining additional evidence not yet on hand.

**Remarks** (restated from [5]):

1. The multiplicity of explanations has nothing to do with imperfections in the fit between evidence and its explanation; as proved above, it holds even in the ideal case in which one demands an exact fit. Requiring only an approximate fit, as is common practice, makes room for even more explanations of any given evidence.

2. Quantum-state tomography claims to determine a quantum state from evidence. To this claim we respond that quantum state tomography assumes that the measurement operators are known, and, by the Theorem, this knowledge cannot be obtained from evidence alone. In addition, quantum state tomography assumes some finite dimension of the Hilbert space, a dimension underivable from evidence.

3. It can also be noted that evidence is expressed by probabilities that are functions of parameters; we think of parameter values as settings of knobs, such as a knob by which to vary a magnetic field strength or a knob to vary the angle of a polarizer. Then not only the evidence but also the density operator and the POVM depend on knob settings. The proof of the Theorem goes through the same way for each knob setting.

4. For the physicist struggling to come up with some explanation of unexpected evidence, worrying about the possibility of other explanations may seem superfluous; yet being aware that something outside of logic is required to make an explanation liberates one from futile efforts to derive an explanation by logic alone.
3. AN AGENT’S OPEN CYCLE OF GUESSING AND TESTING

Out of a potential for uncountable inequivalent explanations, physicists write down particular explanations. By the Corollary, this takes something beyond logic and evidence. This ‘something’ can reasonably be called a **guess**. The guess, neither derived mathematically on the blackboard nor generated from an experiment on the work bench, comes from “somewhere else.” Guessed explanations, expressed in mathematical symbols, feed into the development of experimental devices and into the design of future experiments. Different guessed explanations lead to different experimental designs, leading to different bodies of evidence that call for more explanations and hence more guesses. Any particular explanation is essentially certain to require revision when tested over enough of its extensions. We thus arrive at

**Proposition:** The regularities that physicists find in the material world defy any final explanation.

With this Proposition, we see that quantum theory implies an endless evolution of physics as an activity of symbol-handling agents “guessing and testing” in an open cycle with no possibility of completion.

![Agent’s open cycle of guessing and testing](image)

With the recognition of an open cycle of guessing and testing, the blackboard of theory and the workbench of experiment can no longer encompass all of physics: something else comes into play, in order for agents (people or perhaps other organisms) to bridge the logical gap between evidence and its explanations. How, then, are we to think about agents?

I think of myself acting as an agent, sometimes at the blackboard of theory, other times at the workbench of experiment. In [5] we wrote about what might be called a “lone agent” without considering agents paired by a communications channel.
We think of an *agent* as a role that exercises agency. *Agents* and *agency* are terms of description available at widely varying levels of detail. I may see myself as an agent, and I may inquire into evidence of agency on the part of mitochondria within my cells. We propose that symbol-handling agents enter physical explanations.

... 

What capabilities are to be ascribed to an ‘agent’? We think of an agent as taking steps, one after another, and as equipped with a memory. Each “next step” of an agent is influenced both by the contents of its memory and by an inflow of symbols from an environment that includes other agents, and also by a logically undetermined “oracle” external to the agent.\(^8\) Guesses come from an agent interacting with an oracle, the workings of which we refrain from trying to penetrate. How to elaborate this proposal remains open; presumably investigators will conceive of a variety of expressions of ‘agents’ for applications varying from descriptions of viruses to descriptions of human mentality.

In order to distinguish between what can be computed and what must come from beyond computation (as guesses from interaction with an oracle), we imagine the extreme case of an agent that, while open to guessing, possesses maximal computational capacity. Thus we are unconcerned with practical limits on computing imposed by limits on memory or by limits on the rate at which an agent computes, leading us to assume that the agent has the ultimate computational capability of a Turing machine. The Turing machine, however, requires modification to offer a place for guesses from interaction with an oracle and for communication with other such machines.

Fortunately for our purposes, in a side remark in his 1936 paper, Turing briefly introduced an alternative machine called a *choice machine*, contrasted with the usual Turing machine that Turing called an a-machine:

> If at each stage the motion of a machine … is completely determined by the [memory] configuration, we shall call the machine an “automatic machine” (or a-machine). For some purposes we might use machines (choice machines or c-machines) whose motion is only partially determined by the configuration ….
> When such a machine reaches one of these ambiguous configurations, it cannot go on until some arbitrary choice has been made by an external operator. This would be the case if we were using machines to deal with axiomatic systems.\(^9\)

We picture a symbol-handling agent equipped with a c-machine modified to take part in a communications network by transmitting symbols to other such machines. We call the modified c-machine a Choice Machine. We posit that on occasion an “oracle” writes a symbol onto the scanned square of the Turing tape of the agent’s Choice Machine *privately*, in the sense that the symbol remains unknown to other agents unless and until the symbol-handling agent that receives the chosen symbol reports it to others.\(^10\) There is no limit to the number of symbols that the oracle can write.
4. AGENTS PAIRED BY COMMUNICATIONS CHANNELS

The notion of a lone agent goes back to Descartes “Cogito ergo sum”, with the (to us curious) notion that what I am most grounded in is myself, a notion compatible with the concept of an agent in QBism. What affects one agent as a symbol need not be a symbol to another agent, but there are special situations in which a pair of agents can communicate by sending symbols back and forth from one agent to the other. In this report we posit the capacity of agents, on occasion, to communicate, and we shift our focus from the “lone agent” in [5] to emphasize the pairing of one agent with another agent by means of a two-way communications channel over which symbols that mean something to one agent arrive at another agent that coherently responds.

Characterizations of flow of symbols, including their necessary synchronization, were discussed earlier. Here we remark on the mathematical forms—one might say data structures—that underpin concepts of space, time, and spacetime as used by physicists, and that also underpin arrangements of devices that implement these concepts. More generally, these forms would seem to be aids to thinking about how organisms, human and non-human, manage motion.

4.1 Forms of records used by agents to manage their motion

To think about agents managing their motions, we postulate that for some agents the Choice Machine expressing their symbol-handling capability is augmented by a second tape that we call a clock tape. Like the Turing tape, the clock tape is marked into squares, each of which can hold a single symbol, and at any moment a Choice Machine scans a single square of the clock tape, but the motion of the clock tape never reverses: at each move the scanned square of the clock tape shifts one place to the right. Consider symbols arriving one after another at an agent A from an agent B. Agent A can copy each symbol in the moment it arrives onto the scanned square of A’s clock tape. In this way A maps the dynamic, temporal order of arrival of the symbols to a static, spatial order along the clock tape.

Suppose that at some but not all of its moments, an agent A receives a symbol from B and writes the received symbol on A’s clock tape. If the symbols arrive only occasionally, A records these arriving symbols sparsely, so that successive symbols occupy not successive squares of A’s clock tape but on squares separated by stretches of A’s tape, stretches on which A may have written symbols for its own use or symbols arriving from agents other than B. Consider a stretch of A’s clock tape after such a performance. The ratio of the number of symbols recorded from B to the number of squares on A’s clock tape gives the average frequency ratio of symbol arrivals from B to steps of A. Such frequency ratios are the recorded evidence of the speed ratio of A’s receptions from B relative the speed of A’s stepping along its clock tape. These ratios are the basic form of evidence of motion.

The example in Fig. 2 shows a relative frequency

\[
\frac{\text{number of } B\text{-symbols received}}{\text{number of } A\text{-moves}} \approx \frac{1}{4}
\]

Now we turn from frequency to distance. The basic form for distance in terms of records on a tape is in terms of an echo count. An echo count is the number of squares from a symbol sent to the return of an echo. That is, agent A, at the moment of scanning square n of its clock tape, transmits a symbol
to $B$, $B$ echoes the symbol back to $A$, and the echoed symbol arrives as $A$ scans square $n + k$; then $k$ is an *echo count* from $A$ to $B$ and back.

The notion of *echo count* is a basis for distance. As discussed in more detail in [12][13], in special cases echo counts it can be aligned with distances as defined by Einstein in special relativity, but echo count gives a property of records that is more basic than relativistic distance, in that the concepts needs no assumption of any spacetime metric.

In discussing a lone agent, one might want to ask “what belongs to the agent?”, for example a record of odds that the agent assigns to a bet could be said to “belong to the agent.” But when agents communicate, some records are shared, so that ownership can be shared; communicating agents differ drastically from lone agents. We take as fundamental the capacity of an agent, on occasion, to pair with another agent in communicating symbols. Then the question “what belongs to an agent?” becomes embedded in a context of asking “what strings of symbols flow from one agent to another?”. Strings of symbols bind the agents without belonging totally to one or the other agent. It is a potentially interesting discipline to explore this and related questions, not just in engineered systems but also in organisms, human and non-human, to explore situations in which one can ask: “which record, what agents have access to the record, where and when (along their own tapes) do they have this access?”

### 4.2 Evolution of forms of records

The analysis of the form of records of frequency and distance has the following implication. Given that agents, by definition, can be seen as in interaction with oracles that are logically undetermined, and given that records of symbols in the memories of agents are the basic form to frequency and distance, we arrive at the insight that agents managing speed and distance operate not as a single system of a logic of symbols landing on and departing from the squares of tapes, but like extended arithmetics, they form an uncountable set, involving the evolution through the logically undetermined entrance of symbols, e.g. from oracles.

We are interested in various possible paths of the evolution of ways in which agents manage motion, especially in cases in which agents leave measurable tracks in the form of their clock tapes. For
example, human agents develop evolving national and international time standards. One such thread of evolution consists in the concepts and the implementation of these concepts in International Atomic Time (TAI). Recall that TAI rests on its base reference frequency of the electromagnetic radiation resonant with the transition between hyperfine energy levels of cesium 133. This reference frequency is employed, either directly or indirectly, in almost all physical investigation. But Cesium (or its contemplated replacement by an optical frequency) is by no means the only type of reference imaginable or potentially fruitful. Other such threads of evolution of symbolic communication among agents are candidates for exploration, and some are in use. For example, the cutting-edge optical clocks now operating in National Metrology Laboratories have an instability of about 1/100 of the cesium clocks that define the second. Therefore it makes no sense to compare two such optical clocks by comparing each to the second: one compares them directly. There is an interesting potential for alternative schemes of measuring motion based on symbol exchange in the LIGO experiment. Still other schemes are candidates for explorations of the management of motion by biological organisms, both human and non-human. “The frog starves to death in the presence of dead flies, not because he is fastidious, but because he does not see them”. What alternative to spacetime is it in which the frog lives?

5. DISCUSSION

Descartes’s “Cogito ergo sum” says “I’m sure of me but a little doubtful about you.” In the perspective developed above “my friend can be a real as myself.” While our discussion of a “lone agent” was hardly avoidable in developing the notion of an agent as possessing the symbol-handling capability of a Choice Machine, a single agent is like “one hand clapping”: Agents can’t function as agents without being paired with other agents via back-and-forth communication of symbols. As discussed in [12], conflicts in synchronization place severe limitations the potential for a given agent to communicate with other agents, and the potential for conflicts generally limits the number channels that can operate concurrently to pair one agent with another. Agent A may have to step out of communication with B if it is to enter communication with C.

As we see it, the notion of objectivity has to do with expectations of agreement among records made by different agents, rather than a notion of behavior devoid of agent participation.

The clock tape along which an agent steps is the core image of “local time,” without encumbering it with the notion, which for 40 years we have found baffling, of any “global time”.

Guesses by an agent, here thought of as arising in the interaction of an agent with an oracle, influence not only how an agent resolves certain choices but also the agent’s organization; that is, the agent’s capacity to perceive a situation that calls for a choice.

For physics, the answer is clear: “An important task of the theoretical physicist lies in distinguishing between trivial and nontrivial discrepancies between theory and experiment.” [17, p. 3]

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APPENDIX A. ENTANGLEMENT

Here is an example of two explanations of given evidence, the first of which involves no entanglement, while the second involves an entangled state. Consider a simple case of evidence with two outcomes: \( \omega_1^{(0)} \) and \( \omega_2^{(0)} \), and with probabilities given, for some \( 0 \leq a \leq 1 \), by

\[
\text{Pr}(\omega_1^{(0)}) = a, \quad \text{and} \quad \text{Pr}(\omega_2^{(0)}) = 1 - a. \tag{3}
\]

These probabilities accord with an explanation \( \alpha \) involving a Hilbert space of just two real dimensions, \( \mathcal{H}^{(0)} = \mathbb{R}^2 \), with a basis \( \{ |x^{(0)}\rangle, |y^{(0)}\rangle \} \), so that

\[
|x^{(0)}\rangle\langle x^{(0)}| + |y^{(0)}\rangle\langle y^{(0)}| = 1^{(0)}, \tag{4}
\]

with

\[
\rho_\alpha = (\sqrt{a}|x^{(0)}\rangle + \sqrt{1-a}|y^{(0)}\rangle)(\sqrt{a}\langle x^{(0)}| + \sqrt{1-a}\langle y^{(0)}|); \tag{5}
\]

\[
M_\alpha^{(0)}(\omega_1^{(0)}) = |x^{(0)}\rangle\langle x^{(0)}| \quad \text{and} \quad M_\alpha^{(0)}(\omega_2^{(0)}) = |y^{(0)}\rangle\langle y^{(0)}| \tag{6}
\]

Here is an alternate explanation \( \beta \) of the probabilities given in (3), involving an additional tensor-product space \( \mathcal{H}^{(1)} = \mathbb{R}^2 \), again a real vector space of dimension 2, with basis vectors \( \{ |x^{(1)}\rangle, |y^{(1)}\rangle \} \). An alternate explanation \( \beta \), involving an entangle state, is then

\[
\rho_\beta = (\sqrt{a}|x^{(0)}\rangle|x^{(1)}\rangle + \sqrt{1-a}|y^{(0)}\rangle|y^{(1)}\rangle)(\sqrt{a}\langle x^{(0)}|\langle x^{(1)}| + \sqrt{1-a}\langle y^{(0)}|\langle y^{(1)}|) \tag{7}
\]

\[
M_\beta = M_\alpha^{(0)} \otimes 1^{(1)} \tag{8}
\]

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