ABSTRACT. We study the collapse of a many-body system which is used to model two-component Bose-Einstein condensates with attractive intra-species interactions and either attractive or repulsive inter-species interactions. Such a system consists of two different species for $N$ identical bosons in $\mathbb{R}^2$, interacting with potentials rescaled in the mean-field manner $-N^{2\beta-1}w(\sigma)(N^\beta x)$ with $\int_{\mathbb{R}^2} w(\sigma)(x)dx = 1$. Assuming that $0 < \beta < 1/2$, we first show that the leading order of the quantum energy is captured correctly by the Gross–Pitaevskii energy. For the totally attractive system, we investigate the blow-up behavior of the quantum energy as well as the ground states when $N \to \infty$ and the total interaction strength of intra-species and inter-species approaches sufficiently slowly a critical value $a_*$, which is the critical strength for the focusing Gross–Pitaevskii functional. We prove that the many-body ground states fully condensate on the (unique) Gagliardo–Nirenberg solution. A similar result is also obtained in the case of repulsive inter-species interactions, when the strengths of intra-species interactions of each component approach $a_*$ sufficiently slowly.

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1. INTRODUCTION

After the first realization of Bose-Einstein condensates (BEC) in the laboratory in 1995 [1, 10, 12, 15, 13, 29], theoretical studies have been developed for the one-component BEC [5, 6, 39, 51, 35, 62]. In that case, the Gross–Pitaevskii energy functional with attractive interaction is commonly used to predict a collapse of the system when the product of number of particles and the scattering length is too negative [9, 14, 58, 49, 4]. This effect has been observed in some experiments [20, 17]. Recently, BEC with multiple species have been realized in experiments
and some interesting phenomena absent in one-component BEC were observed and studied in theory \cite{2, 40, 30, 8, 11, 7, 18, 45, 55, 47, 46, 16}. The simplest multi-component BEC is the binary mixture, which can be used as a model for producing coherent atomic beams (also called atomic laser).

In this paper, we establish some results about 2D focusing mixture condensate in the critical regime of collapse. To be precise, we consider a Bose gas trapped into a quasi 2D layer by means of trapping potentials and we look at a non-linear Schrödinger many-body system arising in a regime of collapse. To be precise, we consider a Bose gas trapped into a quasi 2D layer by means of trapping potentials and we look at a non-linear Schrödinger many-body system arising in a regime of collapse. To be precise, we consider a Bose gas trapped into a quasi 2D layer by means of trapping potentials and we look at a non-linear Schrödinger many-body system arising in a regime of collapse. To be precise, we consider a Bose gas trapped into a quasi 2D layer by means of trapping potentials and we look at a non-linear Schrödinger many-body system arising in a regime of collapse. To be precise, we consider a Bose gas trapped into a quasi 2D layer by means of trapping potentials and we look at a non-linear Schrödinger many-body system arising in a regime of collapse. To be precise, we consider a Bose gas trapped into a quasi 2D layer by means of trapping potentials and we look at a non-linear Schrödinger many-body system arising in a regime of collapse.

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The parameters $a_1 > 0$ and $a_2 > 0$, which are of order 1, describe the strength of attractive intra-species inside each component. The inter-species interactions between two components of the system can be attractive ($a_{12} > 0$) or repulsive ($a_{12} < 0$). The choice of coupling constants proportional to $1/(N_j - 1)$ and $1/N$ ensures that the kinetic and the potential energies are comparable in the limit $N \to \infty$. In which limit regime, we assume that

$$\lim_{N \to \infty} \frac{N_1}{N} = c_1 \in (0, 1) \quad \text{and} \quad \lim_{N \to \infty} \frac{N_2}{N} = c_2 = 1 - c_1.$$  \hfill (1.5)

It is not restrictive to assume that the ratios $N_1/N$ and $N_2/N$ themselves are fixed, and so shall we henceforth.

We are interested in the large-$N$ behavior of the quantum energy per particle of $H_N$ in \eqref{1.1},

given by

$$E_N^Q := N^{-1} \inf_{\sigma_{H_N}} \langle \Psi_N | H_N | \Psi_N \rangle = N^{-1} \inf_{\Psi_N \in H_N, \|\Psi_N\|_{L^2} = 1} \langle \Psi_N | H_N | \Psi_N \rangle$$  \hfill (1.6)
and the corresponding ground state. Note that the energy per particle of the fully condensed trial function $u^{\otimes N}_{1} \otimes u^{\otimes N}_{2}$ is given by the $N$-dependent Hartree energy functional

$$
E^{H}(u_{1}, u_{2}) = c_{1} \int_{\mathbb{R}^{2}} \left[ \nabla u_{1}(x)^{2} + V_{1}(x)|u_{1}(x)|^{2} - \frac{a_{1}}{2}|u_{1}(x)|^{4} \right] dx
$$

$$
+ c_{2} \int_{\mathbb{R}^{2}} \left[ \nabla u_{2}(x)^{2} + V_{2}(x)|u_{2}(x)|^{2} - \frac{a_{2}}{2}|u_{2}(x)|^{4} \right] dx
$$

$$
- c_{1}c_{2}a_{12} \int_{\mathbb{R}^{2}} |u_{1}(x)|^{2}|u_{2}(x)|^{2} dx,
$$

(1.7)

where $c_{1}$ and $c_{2}$ are the ratios defined in (1.5). It turns out that the leading order of the quantum energy is captured by the effective Hartree energy in the mean-field regime. In fact, the Hartree energy, which is obtained by taking the infimum of the Hartree energy functional in (1.7) under the constraint $(u_{1}, u_{2}) \in H^{1}(\mathbb{R}^{2}) \times H^{1}(\mathbb{R}^{2})$ and $\|u_{1}\|_{L^{2}} = 1 = \|u_{2}\|_{L^{2}}$, is an upper bound to the quantum energy

$$
E^{Q}_{N} \leq \inf_{u_{1}, u_{2} \in H^{1}(\mathbb{R}^{2}) \atop \|u_{1}\|_{L^{2}} = 1 = \|u_{2}\|_{L^{2}}} E^{H}(u_{1}, u_{2}) =: E^{H}.
$$

(1.8)

When $N \to \infty$, since $u^{(\sigma)}_{\star} \approx \delta_{0}$ for $\sigma \in \{1, 2, 12\}$, the Hartree energy functional formally boils down to the non-linear Gross–Pitaevskii energy functional

$$
E^{GP}(u_{1}, u_{2}) = c_{1} \int_{\mathbb{R}^{2}} \left[ \nabla u_{1}(x)^{2} + V_{1}(x)|u_{1}(x)|^{2} - \frac{a_{1}}{2}|u_{1}(x)|^{4} \right] dx
$$

$$
+ c_{2} \int_{\mathbb{R}^{2}} \left[ \nabla u_{2}(x)^{2} + V_{2}(x)|u_{2}(x)|^{2} - \frac{a_{2}}{2}|u_{2}(x)|^{4} \right] dx
$$

$$
- c_{1}c_{2}a_{12} \int_{\mathbb{R}^{2}} |u_{1}(x)|^{2}|u_{2}(x)|^{2} dx.
$$

(1.9)

We can therefore expect that the Gross–Pitaevskii energy $E^{GP}$ and the quantum energy $E^{Q}_{N}$ are close. Here $E^{GP}$ is given by

$$
E^{GP} := \inf_{u_{1}, u_{2} \in H^{1}(\mathbb{R}^{2}) \atop \|u_{1}\|_{L^{2}} = 1 = \|u_{2}\|_{L^{2}}} E^{GP}(u_{1}, u_{2}).
$$

(1.10)

Note that $\|\nabla u_{i}\|_{L^{2}} \geq \|\nabla u_{i}\|_{L^{2}}$, for any $u_{i} \in H^{1}(\mathbb{R}^{2})$ and for $i \in \{1, 2\}$ (see [37, Theorem 7.8]). Therefore, $E^{GP}(u_{1}, u_{2}) \geq E^{GP}(|u_{1}|, |u_{2}|)$ and we can restrict the minimization problem (1.10) to non-negative functions. In particular, the ground states for $E^{GP}$ in (1.10), when it exists, can be chosen to be non-negative.

From now on, we always assume that $0 < a_{1}, a_{2} < a_{*}$ where $a_{*} > 0$ is the optimal constant of the Gagliardo–Nirenberg inequality

$$
\left( \int_{\mathbb{R}^{2}} |\nabla u(x)|^{2} dx \right) \left( \int_{\mathbb{R}^{2}} |u(x)|^{2} dx \right) \geq \frac{a_{*}}{2} \int_{\mathbb{R}^{2}} |u(x)|^{4} dx, \quad \forall u \in H^{1}(\mathbb{R}^{2}).
$$

(1.11)

Equivalently, $a_{*} = \|Q\|_{L^{2}}^{2}$ where $Q$ is the unique (up to translations) symmetric radial decreasing positive solution of the equation

$$
- \Delta Q + Q - Q^{3} = 0 \quad \text{in } \mathbb{R}^{2}.
$$

(1.12)

It is well-known (see [23, 44, 59, 60]) that $Q$ is the unique (up to dilations and translations) optimizer for the inequality (1.11). One can easily see from (1.11) and (1.12) that

$$
\|\nabla Q\|_{L^{2}}^{2} = \frac{1}{2} \|Q\|_{L^{4}}^{4} = \|Q\|_{L^{2}}^{2} = a_{*}.
$$
Actually, it was proved in [24, 25] that (1.10) admits a minimizer if \(0 < a_1, a_2 < a_s\) and either \(0 < a_{12} < \sqrt{c_1^{-1}c_2^{-1}(a_s - a_1)(a_s - a_2)}\) or \(a_{12} < 0\). Furthermore, \(E^{GP} = -\infty\) if either \(a_1 > a_s\) or \(a_2 > a_s\) or \(a_{12} > 2^{-1}c_1^{-1}c_2^{-1}(a_s - c_1a_1 - c_2a_2)\). Therefore, \(a_s\) is the critical interaction strength for the stability of the focusing two-component Gross–Pitaevskii ground state functional (1.9). The blow-up profile of the Gross–Pitaevskii energy (1.10) as well as its ground states were established by Guo, Zeng and Zhou in [24, 25] (see also Section 4 for a review). The purpose of the present paper is to investigate the blow-up behavior of the full many-body system (1.11), which is more difficult.

Our work and method are inspired by Lewin, Nam and Rougerie [35]. In the mentioned paper, the authors studied the collapse of the many-body system arising in a one-component BEC with an attractive interaction (see also [23] for the study in the one-body theory). In that one-component setting, we remark that \(a_s\) is also the critical interaction strength for the existence of a ground state for the focusing one-component Gross–Pitaevskii ground state functional. In addition, the convergence of the many-body ground states was proved for the single one-particle reduced density matrices. The two-component BEC presents more complicated phenomena than a single-component BEC since there are inter-species interactions between two components. In our two-component case, the convergence of the many-body ground states will be formulated using the double reduced density matrices. Depending on the inter-species interactions, we will discuss the blow-up behavior of the ground state energy (1.6) and its ground states as well. The precise statements of our results are represented in the next section. The remainder of the paper is then devoted to their proofs.

2. Main Results

2.1. The Case of Attractive Inter-Species Interactions. In the first part of this paper, we consider the totally attractive system, i.e. \(a_{12} > 0\). In that case, the existence of ground states for (1.10), under the assumptions that \(0 < a_1, a_2 < a_s\) and \(0 < a_{12} < \sqrt{c_1^{-1}c_2^{-1}(a_s - a_1)(a_s - a_2)}\), follows the standard direct method in the calculus of variations. Furthermore, there are no ground states when either \(a_1 \geq a_s\) or \(a_2 \geq a_s\) or \(a_{12} \geq 2^{-1}c_1^{-1}c_2^{-1}(a_s - c_1a_1 - c_2a_2)\). In addition, it was pointed out in [24] Theorems 1.2 and 1.3] that if \(0 < a_1, a_2 < a_s\) and

\[
\sqrt{c_1^{-1}c_2^{-1}(a_s - a_1)(a_s - a_2)} \leq a_{12} \leq 2^{-1}c_1^{-1}c_2^{-1}(a_s - c_1a_1 - c_2a_2)
\]

then there may exist a ground state, under some additional assumptions on \((a_1, a_2, a_{12})\). Especially, if \(c_1(a_s - a_1) = c_2(a_s - a_2)\) and \(\inf_{x \in \mathbb{R}^2}(V_1(x) + V_2(x)) \neq 0\) then a ground state for (1.10) exists at the threshold point. Thus, it is reasonable to study the behavior of the Gross–Pitaevskii ground states when \(V_1\) and \(V_2\) have a common minimum point. In that case, we will fix \(0 < a_{12} < a_s \min\{c_1^{-1}, c_2^{-1}\}\) and we take \((a_1, a_2) := (a_{1,N}, a_{2,N}) \wedge (a_s - c_2a_{12}, a_s - c_1a_{12})\) as \(N \to \infty\). The two components of the Gross–Pitaevskii ground states must blow up at the center of the trap and with the same rate. For the detail analysis, we refer the reader to the paper [24] (see also Subsection 4.1 for a review).

In this paper, we study the collapse of the full many-body system (1.11). When the inter-species interactions is attractive, we study its ground states in the regime where the total interaction strength of intra-species and inter-species tending to the critical value \(a_s\) sufficiently slowly. We prove that the many-body system (1.11) is fully condensed on the (unique) Gagliardo–Nirenberg solution (1.12). In our two-component setting, the convergence of ground states will be formulated using the double \((k, \ell)\)-particle reduced density matrices. It is defined, for any
\[ \Psi_N \in \mathcal{H}_N = \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}, \]  
by a partial trace

\[ \gamma_{\Psi_N}^{(k, \ell)} := \text{Tr}_{k+1 \to N_1} \otimes \text{Tr}_{\ell+1 \to N_2} |\Psi_N \rangle \langle \Psi_N|, \quad \forall k, \ell \in \mathbb{N}. \]

Equivalently, \( \gamma_{\Psi_N}^{(k, \ell)} \) is the trace class operator on \( \mathcal{H}_k \otimes \mathcal{H}_\ell \) with kernel

\[
\gamma_{\Psi_N}^{(k, \ell)}(x_1, \ldots, x_k, x'_1, \ldots, x'_k; y_1, \ldots, y_\ell, y'_1, \ldots, y'_\ell) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i z_k y_\ell} e^{-i z'_k y'_\ell} \Psi_N(x_1, \ldots, x_k, z; y_1, \ldots, y_\ell, z') \Psi_N(x'_1, \ldots, x'_k, z'; y'_1, \ldots, y'_\ell, z').
\]

To make the analysis precise, let us assume that the external potentials \( V_1 \) and \( V_2 \) are of the typical forms

\[ V_i(x) = |x - z_i|^{p_i}, \quad i \in \{1, 2\}, \quad (2.1) \]

where \( z_i \in \mathbb{R}^2 \) and \( p_i > 0 \). These anharmonic trapping potentials are most often used in laboratory experiments.

Let us introduce the following notation

\[ Q_0 = (a_*)^{-\frac{1}{2}} Q \quad (2.2) \]

which is the \( L^2(\mathbb{R}) \)-normalized function of the (unique) Gagliardo–Nirenberg solution of \( H_1 \).

In the case \( a_{12} > 0 \) and \( z_1 \equiv z_2 \), our first main result is the following.

**Theorem 1.** Assume that \( 0 < a_{12} < a_* \min \{1, 2\} \) is fixed and \( V_1, V_2 \) are defined as in \( (2.1) \) with \( z_1 = 0 = z_2 \). Let \( \beta < 1/2 \) and let \( (a_1, a_2) := (a_{1,N}, a_{2,N}) \) satisfy \( (a_* - c_{2a_{12}}, a_* - c_{1a_{12}}) \) such that \( a_N := c_{1a_{1,N}} + c_{2a_{2,N}} + 2c_1c_2a_{12} = a_* - N^{-\gamma} \) with

\[
0 < \gamma < \min \left\{ \frac{p_0 + 2}{p_0}, \frac{p_0 + 2}{p_0} \big(1 - 2\beta\big) \right\}, \quad p_0 = \min\{p_1, p_2\}.
\]

Let \( \Psi_N \) be a ground state for \( H_N \). Let \( \Phi_N = \ell_{N}^{-1} \Psi_N \) where \( \ell_N = (a_* - a_N)^{-\frac{1}{2}} \)

\[
\Lambda = \left( \frac{p_0}{2} \right) \int_{\mathbb{R}^2} \left| x \right|^{p_0} |Q(x)|^2 \, dx \right)^{\frac{1}{p_0+2}} \quad \text{and} \quad \nu = \lim_{x \to 0} \frac{c_1V_1(x) + c_2V_2(x)}{|x|^{p_0}}. \quad (2.3)
\]

Then, up to extraction of a subsequence, we have

\[
\lim_{N \to \infty} \text{Tr} \left| \gamma_{\Phi_N}^{(k, \ell)} - |Q_0^{\otimes k} \otimes Q_0^{\otimes \ell}| \langle Q_0^{\otimes k} \otimes Q_0^{\otimes \ell}| \right| = 0, \quad \forall k, \ell \in \mathbb{N}, \quad (2.4)
\]

where \( Q_0 \) is given by \( (2.2) \). In addition, we have

\[
E_N^Q = E_{GP}^* + o(E_{GP})_{N \to \infty} = (a_* - a_N)^{\frac{p_0}{p_0+2}} \left( \frac{p_0 + 2}{p_0} \frac{\Lambda^2}{a_*} + o(1)_{N \to \infty} \right). \]

There exists another setting for which it is reasonable to study the blow-up behavior of ground states in the case \( a_{12} > 0 \): fix \( 0 < a_1, a_2 < a_* \) such that \( c_1(a_* - a_1) = c_1c_2a_* = c_2(a_* - a_2) \) and take \( a_{12} := \alpha_N \) as \( N \to \infty \). We have the following.

**Theorem 2.** Assume that \( 0 < a_1, a_2 < a_* \) are fixed such that \( c_1(a_* - a_1) = c_1c_2a_* = c_2(a_* - a_2) \) and \( V_1, V_2 \) are defined as in \( (2.1) \) with \( z_1 = 0 = z_2 \). Let \( 0 < \beta < 1/2 \) and let \( 0 < a_{12} := \alpha_N = a_* - N^{-\gamma} \) with

\[
0 < \gamma < \min \left\{ \frac{p_0 + 2}{p_0 + 3}, \frac{p_0 + 2}{p_0} \left(1 - 2\beta\right) \right\}, \quad p_0 = \min\{p_1, p_2\}. \]
Let $\Psi_N$ be a ground state for $H_N$. Let $\Phi = \ell^N_N\Psi_N(\ell^{-1}_N \cdot)$ where $\ell_N = \Theta(\alpha_s - \alpha_N)^{-\frac{1}{p_0+2}}$ with

$$\Theta = \left(\frac{p_0}{4c_1c_2}\int_{\mathbb{R}^2} |x|^{p_0}|Q(x)|^2dx\right)^{\frac{1}{p_0+2}} \quad \text{and} \quad \nu = \lim_{x \to 0} \frac{c_1V_1(x) + c_2V_2(x)}{|x|^{p_0}}. \quad (2.5)$$

Then, up to extraction of a subsequence, we have

$$\lim_{N \to \infty} \text{Tr} \left[ \gamma^{(k,\ell)}_{\Phi_N} - \langle Q_0^{\otimes k} \otimes Q_0^{\otimes \ell} \rangle \langle Q_0^{\otimes k} \otimes Q_0^{\otimes \ell} \rangle \right] = 0, \quad \forall k, \ell \in \mathbb{N}, \quad (2.6)$$

where $Q_0$ is given by $\Phi_N$. In addition, we have

$$E_N^Q = E_{GP} + o(E_{GP})_{N \to \infty} = \left(\alpha_s - \alpha_N\right)^{\frac{p_0}{p_0+2}} \left(2c_1c_2 \frac{p_0 + 2 \Theta^2}{p_0} \frac{a_s}{a_s} + o(1)_{N \to \infty}\right).$$

Remark:

• Theorem 1 covered the case $c_1(a_s - a_1) = c_2(a_s - a_2)$. Note that, with this assumption, Theorem 1 in [24] gave a complete classification of the existence and non-existence of ground states for (1.10).

• To obtain a similar result to Theorem 2 in the case $c_1(a_s - a_1) \neq c_2(a_s - a_2)$, a more evolved Gross–Pitaevskii model arises, where the constraint condition in (1.10) is replaced by $\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2 = 1$ (see [3, 21, 22]). However, the many-body theory behind this is still an open problem. We hope to come back this issue in the future.

2.2. The Case of Repulsive Inter-Species Interactions. In the second part, we consider the system with attractive intra-species interactions and repulsive inter-species interactions, i.e. $a_{12} < 0$. In that case, the existence of ground states for (1.10), under the assumptions that $0 < a_1, a_2 < a_s$ and $a_{12} < 0$ is fixed, follows the standard direct method in the calculus of variations. Furthermore, when either $a_1 \geq a_s$ or $a_2 \geq a_s$ or $a_1 = a_s = a_2$, there are no ground states for (1.10). The limit behavior of the Gross–Pitaevskii energy as well as its ground states, when $a_{12} < 0$ is fixed and $(a_1, a_2) := (a_{1,N}, a_{2,N}) \nearrow (a_s, a_s)$ as $N \to \infty$, have been analyzed in [25] (see Subsection 4.2 for a review). The two components of the Gross–Pitaevskii ground states prefer to blow up at the different center of the trap with different rates. This is somehow similar to the one-component setting where the minimization problems read

$$E_{i,GP} = \inf_{u_i \in H^1(\mathbb{R}^2), \|u_i\|_{L^2} = 1} E_{i,GP}(u_i), \quad i \in \{1, 2\}. \quad (2.7)$$

Here the Gross–Pitaevskii energy functionals are given by

$$E_{i,GP}(u_i) = \int_{\mathbb{R}^2} \left[|\nabla u_i(x)|^2 + V_i(x)|u_i(x)|^2 - \frac{ai}{2}|u_i(x)|^4\right] \, dx, \quad i \in \{1, 2\}. \quad (2.8)$$

For reader’s convenience, let us briefly recall the known results concerning the blow-up behavior of $E_{i,GP}$ in (2.7) as well as its ground states. From [23] we have, for $i \in \{1, 2\}$,

$$\lim_{\alpha_i \nearrow \alpha_s} \frac{E_{i,GP}}{(\alpha_s - \alpha_i)^{\frac{p_i}{p_i+2}}} = \frac{p_i + 2 \Lambda_i^2}{\alpha_i} \text{ where } \Lambda_i = \left(\frac{p_i}{2} \int_{\mathbb{R}^2} |x|^{p_i}|Q(x)|^2 \, dx\right)^{\frac{2}{p_i+2}}. \quad (2.9)$$

Here, we made use of potentials $V_i$, for $i \in \{1, 2\}$, which are given by (2.1). In addition, assume that $u_i$ is a positive ground state for $E_{i,GP}$ in (2.7) for each $0 < a_i < a_s$. Then, up to extraction of a subsequence, we have

$$\lim_{\alpha_i \nearrow \alpha_s} \left|\frac{1}{\alpha_i} \Lambda_i^{-1}(\alpha_s - \alpha_i)^{\frac{1}{p_i+2}} u_i(\Lambda_i^{-1}(\alpha_s - \alpha_i)^{\frac{1}{p_i+2}} x + z_i)\right| = Q_0(x).$$

strongly in $H^1(\mathbb{R}^2)$, where $z_i$ are minimum points of $V_i$ and $Q_0$ is given by (2.2).

In this paper, we study the collapse of the full many-body system (1.1). When the inter-species interactions is repulsive, we study its ground states in the regime where the interaction strength
of intra-species among particles in each component tending to the critical value $a_*$ sufficiently slowly. We prove that the many-body system (1.1) is fully condensed on the (unique) Gagliardo–Nirenberg solution (1.2). Our last main result is the following.

**Theorem 3.** Assume that $a_{12} < 0$ is fixed and $V_1, V_2$ are defined as in (2.1) with $z_1 \neq z_2$. Let $0 < \beta < 1/2$ and let $a_i := a_{i,N} = a_* - N^{-\gamma_i}$, for $i \in \{1, 2\}$, with

$$\frac{\gamma_1}{\gamma_2} = \frac{p_1 + 2}{p_2 + 2} \quad \text{and} \quad 0 < \gamma_i < \min\left\{\frac{p_i + 2}{p_i + 3}, \frac{1}{1 - 2\beta}\right\}.$$  

Let $\Psi_N$ be a ground state for $H_N$ in (1.1). Let

$$\Phi_N(x_1, \ldots, x_{N_1}; y_1, \ldots, y_{N_2}) = \Psi_N\left(\frac{x_1}{\ell_{1,N}} + z_1, \ldots, \frac{x_{N_1}}{\ell_{1,N}} + z_1; \frac{y_1}{\ell_{2,N}} + z_2, \ldots, \frac{y_{N_2}}{\ell_{2,N}} + z_2\right)$$

where $\ell_{i,N} = \Lambda_i(a_* - a_{i,N})^{-\frac{1}{p_i + 2}}$, for $i \in \{1, 2\}$, with $\Lambda_i$ are given by (2.9). Then, up to extraction of a subsequence, we have

$$\lim_{N \to \infty} \text{Tr} \left| \phi_{N}^{(k, \ell)} - |Q_0^{\otimes k} \otimes Q_0^{\otimes \ell}/\langle Q_0^{\otimes k} \otimes Q_0^{\otimes \ell}\rangle\right| = 0, \quad \forall k, \ell \in \mathbb{N}, \quad (2.10)$$

where $Q_0$ is given by (2.2). In addition, with $E_{1G}^\text{GP}$ and $E_{2G}^\text{GP}$ defined in (2.7), we have

$$E_N^Q = c_1 E_{1G}^\text{GP} + o(E_{1G}^\text{GP}) + c_2 E_{2G}^\text{GP} + o(E_{2G}^\text{GP})$$

$$= \sum_{i=1}^2 c_i (a_* - a_{i,N})^{p_i + 2}\left(\frac{p_i + 2}{p_i} a_* + o(1)_{N \to \infty}\right).$$

**Remark.** The condition $\frac{\gamma_1}{\gamma_2} = \frac{p_1 + 2}{p_2 + 2}$ is a technical assumption which yields that $\ell_{1,N}$ and $\ell_{2,N}$ have the same asymptotic behavior when $N \to \infty$. This will be only used to prove the convergence of ground states in (2.10), but not the asymptotic behavior of the quantum energy.

Note that, for $i \in \{1, 2\}$, the positive ground states $u_{a_i}$ of (2.7) decays exponentially (see [25 Proposition A]). More precisely, for any $R > 0$, there exists $C(R) > 0$ such that

$$u_{a_i}(x) \leq C(R) e^{-\delta|x-z_i|/(a_* - a_i)}^{\frac{1}{p_i + 2}} \quad \text{in } \mathbb{R}^2 \setminus B_R(x_i) \quad (2.11)$$

where $\delta > 0$ is independent of $R$, $z_i$ and $a_i$. The decay property (2.11) is used to study the blow-up profile of ground states for (1.10). In fact, as pointed out in [25], we are not able to give the optimal energy estimate for the Gross–Pitaevskii energy because of the presence of the cross term $-a_{12} \int_{\mathbb{R}^2} |u_1(x)|^2 |u_2(x)|^2 dx$ in (1.9). In the energy estimate, if $z_1 \neq z_2$ then the cross term can be made arbitrary small in the limit regime described above, by (2.11). Thus, we can give the refined estimate for the limit behavior of ground states as $(a_1, a_2) \nearrow (a_*, a_*)$. However, such a refined calculation for the energy is not known when $z_1 \equiv z_2$. In that case, we can not determine the accurate blow-up rate of ground states.

### 2.3. Methodology of the Proofs.

Similarly to what was done in [33] for the one-component setting, our proofs of BEC in this paper are based on a Feynman–Hellman-type argument. It relies strongly on the uniqueness of the limiting profile for Gross–Pitaevskii ground states, i.e. the unique positive solution of (1.12). The main difficulty in this paper, as well as in [33], is the energy estimate between the quantum energy and the Gross–Pitaevskii energy via the Hartree energy. In the next section, we will show that, under the intra-species interactions and inter-species interaction given by (1.12), we have

$$E^H \geq E_N^Q \geq E^H - CN^{2\beta-1}. \quad (2.12)$$
While the upper bound is trivial, by the variational principle, it is more complicated to obtain the lower bound in (2.12). Our strategy is to adapt the arguments in [32, Section 3] to our two-component setting. Next, we will compare the Hartree and Gross–Pitaevskii energies and show that the error term is $N^{-\beta}$, thanks to the assumption (1.4). We arrive at the final estimate

$$E_{\text{GP}} - CN^{-\beta} \geq E_N^Q \geq E_{\text{GP}} - CN^{-\beta} - CN^{2\beta - 1}. \quad (2.13)$$

The Gross–Pitaevskii energy functional is stable, i.e. $E_{\text{GP}} \geq 0$, under the assumptions that $0 < a_1, a_2 < a_*$ and either $0 < a_{12} < \sqrt{c_1^{-1}c_2^{-1}(a_* - a_1)(a_* - a_2)}$ or $a_{12} < 0$ (see [24, 25]). When $0 < \beta < 1/2$, (2.13) implies the convergence of the quantum energy to the Gross–Pitaevskii energy and the stability of second kind [38] for the many-body Hamiltonian system as well, i.e.,

$$H_N \geq CN. \quad (2.14)$$

In fact, if the inter-species interactions is repulsive ($a_{12} < 0$) then (2.14) is obtained directly from [32]. But it is much more complicated in the case $a_{12} > 0$. The novelty of our work in the present paper is to prove that the above condition between $a_1$, $a_2$ and $a_{12}$, which yields the stability of the Gross–Pitaevskii energy functional (1.9), is sufficient for the stability of the many-body system (1.1) as well. Finally, we note that the existence of the ground states for (1.6) follows easily from (2.14) and a standard compactness argument.

It is expected that the convergence of the quantum energy to the Gross–Pitaevskii energy holds true for any $0 < \beta < 1$. However, a proof for $1/2 \leq \beta < 1$ is much more involved (see [33, 34, 36, 51] for discussions in the one-component case). We hope that our study in this paper can serve as a first step for understanding the 2D focusing mixture boson gases.

**Organization of the paper.** We now describe the structure of this paper. In Section 3 we prove the convergence of the quantum energy to the Hartree energy for a more general system. In Section 4, we give proofs of Theorem 1, 2 and 3 after revisiting the blow-up phenomenon in the Gross–Pitaevskii theory and establishing energy estimates for the quantum energy.

### 3. From the Quantum Energy to the Gross–Pitaevskii Energy

#### 3.1. Convergence of the Quantum Energy to the Hartree Energy

In this subsection, we prove the convergence of the quantum energy to the Hartree energy under some assumptions on the kinetic and the potentials energies. We consider the general Hamiltonian

$$H_N = \sum_{i=1}^{N_1} h_{x_i} + \frac{1}{N_1 - 1} \sum_{1 \leq i < j \leq N_1} W^{(1)}(x_i - x_j) + \sum_{r=1}^{N_2} h_{y_r} + \frac{1}{N_2 - 1} \sum_{1 \leq r < s \leq N_2} W^{(2)}(y_r - y_s) + \frac{1}{N} \sum_{i=1}^{N_1} \sum_{r=1}^{N_2} W^{(12)}(x_i - y_r)\]$$

which is the many-body system for $N_1$ and $N_2$ identical bosons of different types in $\mathbb{R}^2$, acting on the Hilbert space $\mathcal{H}_N$ given by (1.2). Here we denote by $N = N_1 + N_2$ the total number of particles. The kinetic energy $h$, which is the non-interacting one-body Hamiltonian, is assumed to be real and positive preserving, that is $\langle u, hu \rangle \geq \langle |u|, h|u| \rangle$. The two-body interactions among particles of the same species $W^{(1)}$, $W^{(2)}$ and of different species $W^{(12)}$ satisfy $\hat{W}(\sigma) \in L^1(\mathbb{R}^2)$, for
$\sigma \in \{1, 2, 12\}$. For the sake of simplicity, we assume that the ratios $c_1 = N_1/N$ and $c_2 = N_1/N$ are fixed. The quantum energy per particle and the Hartree energy are given by

$$E^Q_N = N^{-1} \inf \sigma_{\mathcal{H}_N} H_N \quad \text{and} \quad E^H = \inf_{u_1, u_2 \in H^1(\mathbb{R}^2)} E^H(u_1, u_2).$$

Here the Hartree energy functional, which is obtain by considering the ansatz $u_1^{\otimes N_1} \otimes u_2^{\otimes N_2}$ as a trial wave function for the many-body Hamiltonian system, is given by

$$E^H(u_1, u_2) = c_1 \left[ \langle u_1, hu_1 \rangle + \frac{1}{2} \int_{\mathbb{R}^2} |u_1(x)|^2 (W^{(1)} \ast |u_1|^2)(x) dx \right]
+ c_2 \left[ \langle u_2, hu_2 \rangle + \frac{1}{2} \int_{\mathbb{R}^2} |u_2(x)|^2 (W^{(2)} \ast |u_2|^2)(x) dx \right]
+ c_1 c_2 \int_{\mathbb{R}^2} |u_1(x)|^2 (W^{(12)} \ast |u_2|^2)(x) dx.

**Theorem 4.** Under previous assumptions, we have

$$\lim_{N \to \infty} E^Q_N = E^H. \quad (3.1)$$

We remark that, based on quantum de Finetti theorem, the convergence (3.1) have been proven in [40, Theorem 4.1] for confined systems without convergence rate. Note that, in Theorem 4, we made only the assumption on the positivity preserving of the kinetic energy. Furthermore, we did not make any assumption on the sign of $W^\sigma$, for $\sigma \in \{1, 2, 12\}$, nor on its Fourier transform $\hat{W}^\sigma$. The intra-species and inter-species interactions can be either attractive or repulsive. If $\hat{W}^\sigma$ are not integrable (e.g. for Coulomb potentials), the proof can be done by an approximation argument. To prove Theorem 4, we will need the following lemma which relies on the estimate of the two-body interaction by a one-body term. We have the following

**Lemma 5.** Given the functions $w^{(1)}$, $w^{(2)}$ and $w^{(12)}$ in $\mathbb{R}^2$. Assume that the Fourier transforms $\hat{w}^{(1)}$, $\hat{w}^{(2)}$, $\hat{w}^{(12)}$ are positives and belong to $L^1(\mathbb{R}^2)$. Then for any integrable functions $\chi$ and $\zeta$ we have

$$\sum_{1 \leq i < j \leq N_1} w^{(1)}(x_i - x_j) \geq - \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi(x) \chi(y) w^{(1)}(x - y) dx dy
+ \sum_{i=1}^{N_1} (\chi \ast w^{(1)})(x_i) - \frac{N_1}{2} w^{(1)}(0), \quad (3.2)$$

$$\sum_{1 \leq r < s \leq N_2} w^{(2)}(y_r - y_s) \geq - \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \zeta(x) \zeta(y) w^{(2)}(x - y) dx dy
+ \sum_{r=1}^{N_2} (\zeta \ast w^{(2)})(y_r) - \frac{N_2}{2} w^{(2)}(0), \quad (3.3)$$

$$\sum_{i=1}^{N_1} \sum_{r=1}^{N_2} w^{(12)}(x_i - y_r) \geq \sum_{1 \leq i < j \leq N_1} w^{(12)}(x_i - x_j) + \sum_{i=1}^{N_1} [(\chi + \zeta) \ast w^{(12)}](x_i)
- \sum_{1 \leq r < s \leq N_2} w^{(12)}(y_r - y_s) + \sum_{r=1}^{N_2} [(\chi + \zeta) \ast w^{(12)}](y_r)$$
For \( N \) we define 

\[
- \frac{1}{2} \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi(x) \chi(y) w^{(12)}(x - y) dx dy \\
- \frac{1}{2} \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \zeta(x) \zeta(y) w^{(12)}(x - y) dx dy \\
- \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi(x) \zeta(y) w^{(12)}(x - y) dx dy - \frac{N}{2} w^{(12)}(0) .
\]

(3.4)

Proof. \([3.2], [3.3] \) and \([3.4] \) are obtained by expanding

\[
\int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f^{(\sigma)}(x) f^{(\sigma)}(y) (x - y) dx dy = 2\pi \int \int_{\mathbb{R}^2} \hat{w}^{(\sigma)}(k) |f^{(\sigma)}(k)|^2 dk \geq 0,
\]

for \( \sigma \in \{1, 2, 12\} \), where

\[
f^{(1)} = \sum_{i=1}^{N_1} \delta_{x_i} - \chi, \quad f^{(2)} = \sum_{r=1}^{N_2} \delta_{y_r} - \zeta \quad \text{and} \quad f^{(12)} = f^{(1)} + f^{(2)} .
\]

Now we follow the method described in \([3.2] \) Section 3 to prove Theorem \([4] \) for arbitrary potential \( W^{(\sigma)} \) satisfying \( \hat{W}^{(\sigma)} \in L^1(\mathbb{R}^2) \), for \( \sigma \in \{1, 2, 12\} \). The idea was in turn inspired by arguments of Lévy–Leblond \([31] \) and Dyson–Lenard \([19] \).

Proof of Theorem \([4] \). We first consider the case with an even number \( 2N_1 \) and \( 2N_2 \) of particles of different types. We denote by \( 2N = 2N_1 + 2N_2 \) the total number of particles that we split in two groups of \( N = N_1 + N_2 \). The position of the \( N \) first will be denoted by \( x_1, \ldots, x_{N_1} \) and \( y_1, \ldots, y_{N_2} \) whereas those of the others will be denoted by \( p_1 = x_{N_1+1}, \ldots, p_{N_2} = x_{2N_1} \) and \( q_1 = y_{N_2+1}, \ldots, q_{N_2} = y_{2N_2} \). Next we pick a \( 2N \)-particles state \( \Psi_{2N} \) and use its bosonic symmetry in two groups of \( 2N_1 \) and \( 2N_2 \) variables to write

\[
\frac{1}{2N} \langle \Psi_{2N} \bigg| \frac{1}{2N_1 - 1} \sum_{1 \leq i < j \leq 2N_1} W^{(1)}_{+}(x_i - x_j) + \frac{1}{2N_2 - 1} \sum_{1 \leq r < s \leq 2N_2} W^{(2)}_{+}(y_r - y_s) \\
+ \frac{1}{2N} \sum_{i=1}^{2N_1} \sum_{r=1}^{2N_2} W^{(12)}_{+}(x_i - y_r) \bigg| \Psi_{2N} \rangle
\]

Now, we define \( W^{(12)}_{N} = \frac{1}{N} W^{(12)} \), \( W^{(1)}_{N} = \frac{1}{N_1 - 1} W^{(1)} - W^{(12)}_{N} \) and \( W^{(2)}_{N} = \frac{1}{N_2 - 1} W^{(2)} - W^{(12)}_{N} \). For \( \sigma \in \{1, 2, 12\} \), we decompose \( W^{(\sigma)}_{N} = W^{(\sigma)}_{N,+} - W^{(\sigma)}_{N,-} \), where \( W^{(\sigma)}_{N,+} = (W^{(\sigma)}_{N})_{+} \geq 0 \) and \( W^{(\sigma)}_{N,-} = (W^{(\sigma)}_{N})_{-} \geq 0 \). We write the repulsive part using only the \( x_i \)'s and \( y_r \)'s as follow

\[
\frac{1}{2N} \langle \Psi_{2N} \bigg| \frac{1}{N_1 - 1} \sum_{1 \leq i < j \leq N_1} W^{(1)}_{+}(x_i - x_j) + \frac{1}{N_2 - 1} \sum_{1 \leq r < s \leq N_2} W^{(2)}_{+}(y_r - y_s) \\
+ \frac{1}{N} \sum_{i=1}^{N_1} \sum_{r=1}^{N_2} W^{(12)}_{+}(x_i - y_r) \bigg| \Psi_{2N} \rangle
\]

\[
= \frac{1}{N} \langle \Psi_{2N} \bigg| \frac{1}{N_1 - 1} \sum_{1 \leq i < j \leq N_1} W^{(1)}_{+}(x_i - x_j) + \frac{1}{N_2 - 1} \sum_{1 \leq r < s \leq N_2} W^{(2)}_{+}(y_r - y_s) \\
+ \frac{1}{N} \sum_{i=1}^{N_1} \sum_{r=1}^{N_2} W^{(12)}_{+}(x_i - y_r) \bigg| \Psi_{2N} \rangle
\]

\[
= \frac{1}{N} \langle \Psi_{2N} \bigg| \sum_{1 \leq i < j \leq N_1} W^{(1)}_{N,+}(x_i - x_j) + \sum_{1 \leq i < j \leq N_1} W^{(12)}_{N,-}(x_i - x_j)
\]
respectively only the $p_k$'s and $q_m$'s and both groups

$$
- \frac{1}{2N} \langle \Psi_{2N} \mid \frac{1}{2N_1 - 1} \sum_{1 \leq i < j \leq 2N_1} W_-^{(1)}(x_i - x_j) + \frac{1}{2N_2 - 1} \sum_{1 \leq r < s \leq 2N_2} W_-^{(2)}(y_r - y_s)
\]

$$

On the other hand, we express the attractive part as the difference of two terms, involving respectively only the $p_k$'s and $q_m$'s and both groups

$$
- \frac{1}{2N} \langle \Psi_{2N} \mid \sum_{1 \leq i < j \leq N_1} W_-^{(1)}(x_i - x_j) + \sum_{1 \leq k < \ell \leq N_1} W_-^{(2)}(p_k - p_\ell)
\]

$$

$$
+ \sum_{1 \leq m < n \leq N_2} W_-^{(2)}(q_m - q_n) + \sum_{1 \leq m < n \leq N_2} W_-^{(2)}(p_k - q_\ell)
\]

$$

$$
+ \sum_{k=1}^{N_1} \sum_{m=1}^{N_2} W_-^{(2)}(p_k - q_m) \mid \Psi_{2N} \rangle
\]

$$

$$
- \frac{1}{N} \langle \Psi_{2N} \mid \sum_{i=1}^{N_1} \sum_{k=1}^{N_1} W_-^{(1)}(x_i - p_k) + \sum_{r=1}^{N_2} \sum_{m=1}^{N_2} W_-^{(2)}(y_r - q_m)
\]

$$

$$
+ \sum_{i=1}^{N_1} \sum_{m=1}^{N_2} W_-^{(2)}(x_i - q_m) + \sum_{k=1}^{N_1} \sum_{r=1}^{N_2} W_-^{(2)}(p_k - y_r) \mid \Psi_{2N} \rangle
\]

$$

This means that $\langle \Psi_{2N} \mid H_{2N} \mid \Psi_{2N} \rangle / 2N = \langle \Psi_{2N} \mid \tilde{H}_N \mid \Psi_{2N} \rangle / N$ where

$$
\tilde{H}_N = \sum_{i=1}^{N_1} h_{x_i} + \sum_{r=1}^{N_2} h_{y_r}
$$

$$
+ \sum_{1 \leq i < j \leq N_1} W_-^{(1)}(x_i - x_j) + \sum_{1 \leq k < \ell \leq N_1} W_-^{(2)}(p_k - p_\ell)
\]

$$
+ \sum_{1 \leq r < s \leq N_2} W_-^{(2)}(y_r - y_s) + \sum_{1 \leq m < n \leq N_2} W_-^{(2)}(q_m - q_n)
\]

$$
- \frac{1}{N_1} \sum_{i=1}^{N_1} \sum_{k=1}^{N_1} W_-^{(1)}(x_i - p_k) - \frac{1}{N_2} \sum_{r=1}^{N_2} \sum_{m=1}^{N_2} W_-^{(2)}(y_r - q_m)
\]

$$
+ \sum_{1 \leq i < j \leq N_1} W_-^{(2)}(x_i - x_j) + \sum_{1 \leq r < s \leq N_2} W_-^{(2)}(y_r - y_s)
\]

$$
+ \sum_{i=1}^{N_1} \sum_{r=1}^{N_2} W_-^{(2)}(x_i - y_r) + \sum_{k=1}^{N_1} \sum_{m=1}^{N_2} W_-^{(2)}(p_k - q_m)
\]

$$
+ \sum_{1 \leq k < \ell \leq N_1} W_-^{(2)}(p_k - p_\ell) + \sum_{1 \leq m < n \leq N_2} W_-^{(2)}(q_m - q_n)
\]

$$
The Hamiltonian $\tilde{H}_N$ describes a system of $N = N_1 + N_2$ quantum particles that repel through potentials $W_{(\sigma)}^\sigma$, for $\sigma \in \{1, 2, 12\}$, and $N = N_1 + N_2$ classical particles that repel through potentials $W_{N,+}^{(1)}$, with repulsion potentials $W_{N,-}^{(2)}$ and attraction potentials $\frac{1}{N_1} W_{-}^{(1)}, \frac{1}{N_2} W_{-}^{(2)}$, $W_{N,-}$ between two groups. In order to bound $\tilde{H}_N$ from below, we first fix the positions $p_1, \ldots, p_{N_1}$, $q_1, \ldots, q_{N_2}$ of the particles in the second group and consider $\tilde{H}_N$ as an operator acting only over the $x_i$'s and $y_r$'s. Let $\Phi_N$ be any bosonic $N$-particles state in the $N = N_1 + N_2$ first variables. Applying Lemma 5 for the repulsive potential $W_{N,+}^{(1)}$ with $\chi = N_1 \rho_{\Phi_N}^{(1,0)}$, for $W_{N,+}^{(2)}$ with $\zeta = N_2 \rho_{\Phi_N}^{(0,1)}$, and for $W_{N,-}^{(1)}$ with $\chi = N_1 \rho_{\Phi_N}^{(1,0)}$, $\zeta = N_2 \rho_{\Phi_N}^{(0,1)}$ and using the Hoffmann-Ostenhof inequality [28] (see also [32] Lemma 3.2) we obtain

$$
\langle \Phi_N | \hat{\tilde{H}}_N | \Phi_N \rangle \geq N_1 \left\langle (h^2 r_{\Phi_N}^{(1,0)}), h^2 r_{\Phi_N}^{(1,0)} \right\rangle + N_2 \left\langle h^2 r_{\Phi_N}^{(0,1)}, h^2 r_{\Phi_N}^{(0,1)} \right\rangle
$$

$$
+ \frac{N_1^2}{2} \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \rho_{\Phi_N}^{(1,0)}(x) \rho_{\Phi_N}^{(1,0)}(y) \left( W_{N,+}^{(1)}(x) - W_{N,+}^{(2)}(y) \right) dx dy \tag{3.5}
$$

$$
+ \frac{N_2^2}{2} \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \rho_{\Phi_N}^{(0,1)}(x) \rho_{\Phi_N}^{(0,1)}(y) \left( W_{N,+}^{(2)}(x) - W_{N,+}^{(1)}(y) \right) dx dy \tag{3.6}
$$

$$
+ \sum_{1 \leq k < \ell \leq N_1} W_{N,-}^{(1)}(p_k - p_\ell) + \sum_{1 \leq m < n \leq N_2} W_{N,-}^{(2)}(q_m - q_n) \tag{3.7}
$$

$$
- \sum_{k=1}^{N_1} (\rho_{\Phi_N}^{(1,0)} \ast W_{-}^{(1)})(p_k) - \sum_{m=1}^{N_2} (\rho_{\Phi_N}^{(0,1)} \ast W_{-}^{(2)})(q_m) \tag{3.8}
$$

$$
+ \sum_{1 \leq k < \ell \leq N_1} W_{N,+}^{(1)}(p_k - p_\ell) + \sum_{1 \leq m < n \leq N_2} W_{N,+}^{(2)}(q_m - q_n) \tag{3.9}
$$

$$
\sum_{k=1}^{N_1} \sum_{m=1}^{N_2} W_{N,-}^{(12)}(p_k - q_m) \tag{3.10}
$$

$$
- N_1 \sum_{m=1}^{N_2} (\rho_{\Phi_N}^{(1,0)} \ast W_{N,-}^{(12)})(q_m) - N_2 \sum_{k=1}^{N_1} (\rho_{\Phi_N}^{(0,1)} \ast W_{N,-}^{(12)})(p_k) \tag{3.11}
$$

$$
+ N_1 N_2 \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \rho_{\Phi_N}^{(1,0)}(x) \rho_{\Phi_N}^{(0,1)}(y) W_{N,+}^{(12)}(x - y) dx dy
$$

$$
- \frac{N_1}{2} W_{N,+}^{(1)}(0) - \frac{N_2}{2} W_{N,+}^{(2)}(0) - \frac{N_1}{2} W_{N,+}^{(12)}(0).
$$

By using the positivity of $W_{N,+}^{(1)}$, $W_{N,+}^{(2)}$, and $W_{N,+}^{(12)}$ we have

$$
\tag{3.5} \geq \frac{N_1}{2} \int \rho_{\Phi_N}^{(1,0)}(x) (W_{N,+}^{(1)} \ast \rho_{\Phi_N}^{(1,0)})(x) dx
$$

$$
\tag{3.6} \geq \frac{N_2}{2} \int \rho_{\Phi_N}^{(0,1)}(x) (W_{N,+}^{(2)} \ast \rho_{\Phi_N}^{(0,1)})(x) dx.$$
Next we apply again Lemma 2 for $W_{N_{1}}^{(1)}$ with $\chi = (N_{1} - 1)\rho_{\Phi_{N}}^{(1),(1)}$, for $W_{N_{1}}^{(2)}$ with $\zeta = (N_{2} - 1)\rho_{\Phi_{N}}^{(0),(1)}$, and for $W_{N_{1}}^{(12)}$ with $\chi = (N_{1} - 1)\rho_{\Phi_{N}}^{(1),(0)}$, $\zeta = (N_{2} - 1)\rho_{\Phi_{N}}^{(0),(0)}$, we obtain
\begin{align}
\mathbf{(3.12)} & \geq -\frac{(N_{1} - 1)^{2}}{2} \int_{R^{2} \times R^{2}} \rho_{\Phi_{N}}^{(1),(0)}(x)\rho_{\Phi_{N}}^{(1),(0)}(y)(W_{N_{1}}^{(1)} + W_{N_{1}}^{(12)})(x - y) dx dy \\
\mathbf{(3.13)} & \geq -\frac{(N_{2} - 1)^{2}}{2} \int_{R^{2} \times R^{2}} \rho_{\Phi_{N}}^{(0),(1)}(x)\rho_{\Phi_{N}}^{(0),(1)}(y)(W_{N_{1}}^{(2)} + W_{N_{1}}^{(12)})(x - y) dx dy \\
\mathbf{(3.14)} & \geq -\frac{(N_{1} - 1)}{2} W_{N_{1}}^{(1)}(0) - \frac{N_{2}}{2} W_{N_{1}}^{(2)}(0) - \frac{N_{1} N_{2}}{2} W_{N_{1}}^{(12)}(0).
\end{align}

Again, by using the positivity of $W_{N_{1}}^{(1)}$, $W_{N_{1}}^{(2)}$, and $W_{N_{1}}^{(12)}$ we have
\begin{align}
\mathbf{(3.12)} & \geq -\frac{N_{1}}{2} (N_{1} - 1) \int_{R^{2} \times R^{2}} \rho_{\Phi_{N}}^{(1),(0)}(x)\rho_{\Phi_{N}}^{(1),(0)}(y)(W_{N_{1}}^{(1)} + W_{N_{1}}^{(12)})(x - y) dx dy \\
& = -\frac{N_{1}}{2} \int_{R^{2} \times R^{2}} \rho_{\Phi_{N}}^{(1),(0)}(x)\rho_{\Phi_{N}}^{(1),(0)}(y)W_{N_{1}}^{(1)}(x - y) dx dy,
\end{align}

\begin{align}
\mathbf{(3.13)} & \geq -\frac{N_{2}}{2} (N_{2} - 1) \int_{R^{2} \times R^{2}} \rho_{\Phi_{N}}^{(0),(1)}(x)\rho_{\Phi_{N}}^{(0),(1)}(y)(W_{N_{1}}^{(2)} + W_{N_{1}}^{(12)})(x - y) dx dy \\
& = -\frac{N_{2}}{2} \int_{R^{2} \times R^{2}} \rho_{\Phi_{N}}^{(0),(1)}(x)\rho_{\Phi_{N}}^{(0),(1)}(y)W_{N_{1}}^{(2)}(x - y) dx dy.
\end{align}

Putting all of the above together and using $W_{\sigma}^{(+)} - W_{\sigma}^{(-)} = W_{\sigma}$, for $\sigma \in \{1, 2, 12\}$, we arrive at
\begin{align}
\langle \Phi_{N} | \hat{H}_{N} | \Phi_{N} \rangle & \geq N_{1} \langle \sqrt{\rho_{\Phi_{N}}^{(1),(0)}}, h \sqrt{\rho_{\Phi_{N}}^{(1),(0)}} \rangle + N_{2} \langle \sqrt{\rho_{\Phi_{N}}^{(0),(1)}}, h \sqrt{\rho_{\Phi_{N}}^{(0),(1)}} \rangle \\
& + N_{1} \int_{R^{2} \times R^{2}} \rho_{\Phi_{N}}^{(1),(0)}(x)\rho_{\Phi_{N}}^{(1),(0)}(y)W_{N_{1}}^{(1)}(x - y) dx dy \\
& + N_{2} \int_{R^{2} \times R^{2}} \rho_{\Phi_{N}}^{(0),(1)}(x)\rho_{\Phi_{N}}^{(0),(1)}(y)W_{N_{1}}^{(2)}(x - y) dx dy \\
& + \frac{N_{1} N_{2}}{N} \int_{R^{2} \times R^{2}} \rho_{\Phi_{N}}^{(1),(0)}(x)\rho_{\Phi_{N}}^{(0),(1)}(y)W_{N_{1}}^{(12)}(x - y) dx dy + \mathcal{R}_{1} \\
& = N \mathcal{E}_{\mathcal{H}}^{\Phi} \left( \sqrt{\rho_{\Phi_{N}}^{(1),(0)}}, \sqrt{\rho_{\Phi_{N}}^{(0),(1)}} \right) + \mathcal{R}_{1},
\end{align}

Here we abbreviated by $\mathcal{R}_{1}$ the error terms
\begin{align}
\mathcal{R}_{1} & = -\frac{N_{1}}{2(N_{1} - 1)} (W_{N_{1}}^{(1)}(0) + W_{N_{1}}^{(12)}(0)) - \frac{N_{2}}{2(N_{2} - 1)} (W_{N_{1}}^{(2)}(0) + W_{N_{1}}^{(12)}(0)) \\
& - \frac{1}{N} \sum_{k=1}^{N_{1}} (\rho_{\Phi_{N}}^{(0),(1)} W_{N_{1}}^{(12)}(0)) - \frac{1}{N} \sum_{m=1}^{N_{2}} (\rho_{\Phi_{N}}^{(0),(1)} W_{N_{1}}^{(12)}(0)) \\
& + \frac{N - 1}{N} \int_{R^{2} \times R^{2}} \rho_{\Phi_{N}}^{(1),(0)}(x)\rho_{\Phi_{N}}^{(0),(1)}(y)W_{N_{1}}^{(12)}(x - y) dx dy.
\end{align}
By Young’s inequality we have
\[
\tag{3.14}
\frac{N_1}{N} \norm{\rho_N (0,0) \|_{L^1_w}} \|W^{(12)}_w\|_{L^1_w} - \frac{N_2}{N} \norm{\rho_N (0,1) \|_{L^1_w}} \|W^{(12)}_w\|_{L^1_w}
\]
\[
= - \norm{W^{(12)}_w \|_{L^\infty}} \geq -(2\pi)^{-1} \norm{\hat{W}^{(12)}_w \|_{L^1}} \geq -(2\pi)^{-1} \norm{\hat{W}^{(12)}_w \|_{L^1}}.
\]
\[
\tag{3.15}
\geq - \norm{W^{(12)}_w \|_{L^\infty}} \rho_N (1,0) \|_{L^1_w} \rho_N (0,1) \|_{L^1_w} \geq -(2\pi)^{-1} \norm{\hat{W}^{(12)}_w \|_{L^1}}.
\]
Noting that \( W^{(i)}_w (0) + W^{(i)}_w (0) = (2\pi)^{-1} \norm{\hat{W}^{(i)}_w \|_{L^2}}, \) for \( i \in \{1, 2\}. \) Hence we conclude
\[
\frac{\langle \Phi_N | \tilde{H}_N | \Phi_N \rangle}{N} \geq E^H - \frac{(2\pi)^{-1}}{N} \left( \frac{N_1 \norm{\hat{W}^{(1)}_w \|_{L^1}}}{2(N_1 - 1)} + \frac{N_2 \norm{\hat{W}^{(2)}_w \|_{L^1}}}{2(N_2 - 1)} + 2 \norm{\hat{W}^{(12)}_w \|_{L^1}} \right).
\]
Since the right hand side is independent of the \( p_k \)‘s and \( q_m \)‘s, the bound
\[
\frac{\tilde{H}_N}{N} \geq E^H - \frac{C}{N} \left( \norm{\hat{W}^{(1)}_w \|_{L^1}} + \norm{\hat{W}^{(2)}_w \|_{L^1}} + \norm{\hat{W}^{(12)}_w \|_{L^1}} \right)
\]
holds in the sense of operators in the \( 2N \)-particles space, where \( 2N = 2N_1 + 2N_2 \) with \( N_1 \geq 2 \) and \( N_2 \geq 2 \). Minimizing over \( \Psi_{2N} \) and recalling the upper bound \( E_{2N}^Q \leq E^H \) gives the final estimate
\[
\tag{3.16}
E^H \geq E_{2N}^Q \geq E^H - \frac{C}{N} \left( \norm{\hat{W}^{(1)}_w \|_{L^1}} + \norm{\hat{W}^{(2)}_w \|_{L^1}} + \norm{\hat{W}^{(12)}_w \|_{L^1}} \right)
\]
for \( N = N_1 + N_2 \) and \( N_1 \geq 4 \) and \( N_2 \geq 4 \). \( \square \)

3.2. Convergence of the Hartree Energy to the Gross–Pitaevskii Energy. In this subsection, we compare the Hartree and the Gross–Pitaevskii energies. We first note that if \( W^{(1)}_w, W^{(2)}_w \) and \( W^{(12)}_w \) in \( (3.16) \) are replaced by \( w^{(1)}_N, w^{(2)}_N \) and \( w^{(12)}_N \) in \( (1.3) \) then we obtain
\[
\tag{3.17}
E^H \geq E_N^Q \geq E^H - CN^{2\beta - 1},
\]
where \( E_N^Q \) and \( E^H \) are defined as in \( (1.6) \) and \( (1.8) \). This follows from the fact that the Fourier transform of \( w^{(\sigma)}_N \) satisfies \( \norm{\hat{w}^{(\sigma)}_N \|_{L^1}} \leq CN^{2\beta}, \) for \( \sigma \in \{1, 2, 12\}. \)
Our next step is to estimate the Hartree energy by the Gross–Pitaevskii energy. We start with the upper bound. Recalling that \( \int_{\mathbb{R}^2} w^{(\sigma)}(x) dx = 1, \) for \( \sigma \in \{1, 2, 12\}. \) By introduction the variable \( z = N^\beta (x - y), \) we write
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u_i(x)|^2 N^{2\beta} w^{(\sigma)}(N^\beta (x - y)) |u_j(y)|^2 dx dy - \int_{\mathbb{R}^2} |u_i(x)|^2 |u_j(x)|^2 dx
\]
\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u_i(x)|^2 w^{(\sigma)}(z) \left( |u_j(x - N^{-\beta} z)|^2 - |u_j(x)|^2 \right) dx dz
\]
\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u_i(x)|^2 w^{(\sigma)}(z) \left( \int_0^1 (\nabla |u_j|^2 (x - tN^{-\beta} z) \cdot (N^{-\beta} z) dt \right) dx dz. \tag{3.18}
\]
Now we come to the lower bound. Note that, for any \( \kappa > 0 \), let \((u, R)\) where we abbreviated by \( |z| \) from (3.18), (3.20) and noting that (1 + \( |z| \)) \( u(\sigma)(z) \in L^1(\mathbb{R}^2) \), for \( \sigma \in \{1, 2, 12\} \), we obtain
\[
|E^H(u_1, u_2) - E^{GP}(u_1, u_2)| \leq N^{-\beta} \mathcal{R}_2
\]
where we abbreviated by \( \mathcal{R}_2 \) the error terms
\[
\mathcal{R}_2 = c_1 a_1 \|u_1\|_{L^6}^2 \|\nabla u_1\|_{L^6} + c_2 a_2 \|u_2\|_{L^6}^3 \|\nabla u_2\|_{L^6}^2 + 2c_1 c_2 a_{12} \|u_1\|_{L^6}^2 \|u_2\|_{L^6} \|\nabla u_2\|_{L^6}^2.
\]
Let \((u_1, u_2)\) be a ground state for \( E^{GP} \). It follows from (3.21) that
\[
E^Q_N \leq E^H \leq E^{GP} - N^{-\beta} \mathcal{R}_2.
\]
Now we come to the lower bound. Note that, for any \( \kappa > 0 \) and for \( \sigma \) given by (3.19), we have
\[
\int_{\mathbb{R}^2} |u_i, N(x)|^2 (w_N^{(\sigma)} \ast |u_{j,N}|^2)(x) dx \leq \frac{\kappa}{2} \int_{\mathbb{R}^2} |u_i(x)|^4 dx + \frac{1}{2 \kappa} \int_{\mathbb{R}^2} |u_j(x)|^4 dx.
\]
This follows from the Cauchy–Schwarz inequality and the fact that \( \int_{\mathbb{R}^2} w_N^{(\sigma)}(x) dx = 1 \). Applying (3.23) several times to \( \kappa = 1 \) with \( i = 1 = j \) and \( i = 2 = j \), we can estimate the intra-species interactions in (1.7) by those in (1.9) from below. However, it is not obvious how to estimate their inter-species interactions. Depending on its sign, we will estimate the Hartree energy by the Gross–Pitaevskii energy from below, by using (3.18), (3.20) and (3.23).

4. Collapse of the Many-Body System

Through this section, we assume that the potentials \( V_1 \) and \( V_2 \) are of the forms (2.1), i.e.,
\[
V_i(x) = |x - z_i|^p_i, \quad i \in \{1, 2\},
\]
where \( z_1, z_2 \in \mathbb{R}^2 \) and \( p_1, p_2 > 0 \).

4.1. Proofs of Theorems 1 and 2. The purpose of this subsection is to prove Theorems 1 and 2 which give the blow-up profile for the many body system (1.1) when the total interaction strength of intra-species and inter-species tends to a critical number. We first revisit the blow-up phenomenon for the Gross–Pitaevskii minimization problem (1.10). In the case \( a_{12} > 0 \), the existence of the Gross–Pitaevskii ground states follows the standard direct method in the calculus of variations. The following is taken from [24 Theorem 1.1] but the statement is adapted to our model (1.10).

**Theorem 6.** We have the followings

(i) If \( 0 < a_1, a_2 < a_s \) and \( 0 < a_{12} < \sqrt{c_1^{-1} c_2^{-1} (a_s - a_1)(a_s - a_2)} \) then \( E^{GP} \geq 0 \) and it has at least one ground state.

(ii) If either \( a_1 > a_s \) or \( a_2 > a_s \) or \( a_{12} > 2^{-1} c_1^{-1} c_2^{-1} (a_s - c_1 a_1 - c_2 a_2) \) then \( E^{GP} = -\infty \).
As pointed out in [24, Theorem 1.2 and Theorem 1.3], when $0 < a_1, a_2 < a_*$ and
\[
\sqrt{c_1^{-1}c_2^{-1}(a_* - a_1)(a_* - a_2)} \leq a_1 \leq 2^{-1}c_1^{-1}c_2^{-1}(a_* - c_1a_1 - c_2a_2)
\]
then there may exist ground states for (1.10), under additional assumptions on $(a_1, a_2, a_{12})$, especially when $z_1 \neq z_2$. Therefore, it is reasonable to consider the case $z_1 \equiv z_2$ in order to study the limit behavior of ground states when they do not exist at the threshold. The following is taken from [24, Theorem 1.5].

**Theorem 7.** Assume that $0 < a_{12} < a_* \min\{c_1^{-1}, c_2^{-1}\}$ is fixed and $V_1, V_2$ are defined as in (2.1) with $z_1 = 0 = z_2$. Then for every sequence $(a_{1,N}, a_{2,N}) \nrightarrow (a_* - c_2a_{12}, a_* - c_1a_{12})$ as $N \to \infty$, we have
\[
E_{GP} = (a_* - a_N)\frac{p_0}{\rho_0} - \frac{p_0 + 2L^2}{a_*} + o(1)_{N \to \infty},
\]
(4.1)
where $a_N = c_1a_{1,N} + c_2a_{2,N} + 2c_1c_2a_{12}, p_0 = \min\{p_1, p_2\}$ and $\Lambda$ is given by (2.3).

In addition, assume that $(a_{1,N}, a_{2,N})$ is a positive ground state for $E_{GP}$ in (1.10) for each $0 < a_{1,N} < a_* - c_2a_{12}$ and $0 < a_{2,N} < a_* - c_1a_{12}$. Then, up to extraction of a subsequence, we have
\[
\lim_{N \to \infty} E_{GP}^{-1}u_{1,N}(E_{GP}^{-1}) = Q_0 = \lim_{N \to \infty} E_{GP}^{-1}u_{2,N}(E_{GP}^{-1})
\]
strongly in $H^1(\mathbb{R}^2)$, where $E_{GP} = \Lambda(a_* - a_N)\frac{p_0}{\rho_0}$ and $Q_0$ is given by (2.2).

Theorem 7 gave the blow-up profile for the Gross–Pitaevskii exact ground states when they exist. In order to establish the blow-up behavior of the many-body ground states via the Feynman–Hellman-type argument, we need to extend that blow-up result to the Gross–Pitaevskii approximate ground states. We have the following.

**Theorem 8.** Assume that $0 < a_{12} < a_* \min\{c_1^{-1}, c_2^{-1}\}$ is fixed and $V_1, V_2$ are defined as in (2.1) with $z_1 = 0 = z_2$. Let $(a_{1,N}, a_{2,N}) \nrightarrow (a_* - c_2a_{12}, a_* - c_1a_{12})$ as $N \to \infty$. Let $(u_{1,N}, u_{2,N}) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ be a sequence of a coupled positive functions such that $\|u_{1,N}\|_{L^2} = 1 = \|u_{2,N}\|_{L^2}$ and
\[
E_{GP}(u_{1,N}, u_{2,N}) = E_{GP} + o(E_{GP})_{N \to \infty} = (a_* - a_N)\frac{p_0}{\rho_0} - \frac{p_0 + 2L^2}{a_*} + o(1)_{N \to \infty},
\]
(4.2)
where $a_N = c_1a_{1,N} + c_2a_{2,N} + 2c_1c_2a_{12}, p_0 = \min\{p_1, p_2\}$ and $\Lambda$ is given by (2.3). Then we have
\[
\lim_{N \to \infty} E_{GP}^{-1}u_{1,N}(E_{GP}^{-1}) = Q_0 = \lim_{N \to \infty} E_{GP}^{-1}u_{2,N}(E_{GP}^{-1})
\]
strongly in $L^2(\mathbb{R}^2)$, where $Q_0$ is defined as in Theorem 7 and $Q_0$ is given by (2.2).

**Proof.** Denote $\tilde{u}_{1,N} = E_{GP}^{-1}u_{1,N}(E_{GP}^{-1})$ and $\tilde{u}_{2,N} = E_{GP}^{-1}u_{2,N}(E_{GP}^{-1})$, then $\|\tilde{u}_{1,N}\|_{L^2} = 1 = \|\tilde{u}_{2,N}\|_{L^2}$. We rewrite $E_{GP}$, with $\tilde{a}_{1,N} = a_{1,N} + c_2a_{12}$ and $\tilde{a}_{2,N} = a_{2,N} + c_1a_{12}$, as
\[
E_{GP}(u_{1,N}, u_{2,N}) = c_1\int_{\mathbb{R}^2} \left[|\nabla \tilde{u}_{1,N}(x)|^2 - \frac{\tilde{a}_{1,N}}{2} |\tilde{u}_{1,N}(x)|^4 \right] dx
\]
\[
+ c_2\int_{\mathbb{R}^2} \left[|\nabla \tilde{u}_{2,N}(x)|^2 - \frac{\tilde{a}_{2,N}}{2} |\tilde{u}_{2,N}(x)|^4 \right] dx
\]
\[
+ \frac{c_1}{\ell_N^p} \int_{\mathbb{R}^2} |x|^{p_1} |\tilde{u}_{1,N}(x)|^2 dx + \frac{c_2}{\ell_N^q} \int_{\mathbb{R}^2} |x|^{p_2} |\tilde{u}_{2,N}(x)|^2 dx
\]
\[
+ \frac{c_1c_2a_{12}}{2} \ell_N^2 \int_{\mathbb{R}^2} (|\tilde{u}_{1,N}(x)|^2 - |\tilde{u}_{2,N}(x)|^2)^2 dx
\]
(4.3)
For each $N$, we may assume without loss of generality that $\tilde{u}_{1,N} \leq \tilde{u}_{2,N}$. By (4.3) and (1.11) we have, with $a_N = c_1 \tilde{a}_{1,N} + c_2 \tilde{a}_{2,N}$,

$$C_{\text{GP}}(u_{1,N}, u_{2,N}) \geq c_1^2 \frac{a_s - \tilde{a}_{1,N}}{a_s} \int_{\mathbb{R}^2} |\nabla \tilde{u}_{1,N}(x)|^2 dx \geq c_1^2 \frac{a_s - a_N}{a_s} \int_{\mathbb{R}^2} |\nabla \tilde{u}_{1,N}(x)|^2 dx. \quad (4.4)$$

Using the assumption (4.2) and the asymptotic formula for $E_{\text{GP}}$ in (4.1) we then deduce from (4.4) that $\{\tilde{u}_{1,N}\}$ is bounded in $H^1(\mathbb{R}^2)$. Next we prove that $\tilde{u}_{2,N}$ is also bounded in $H^1(\mathbb{R}^2)$. We notice that, by the Cauchy–Schwarz inequality and Minkowski’s inequality we have

$$||\tilde{u}_{1,N}||_{L^4} - ||\tilde{u}_{2,N}||_{L^4}^4 \leq ||\tilde{u}_{1,N}||^2 + ||\tilde{u}_{2,N}||_{L^2}^2 ||\tilde{u}_{1,N}||^2 - ||\tilde{u}_{2,N}||_{L^2}^2 \leq (||\tilde{u}_{1,N}||_{L^4}^2 + ||\tilde{u}_{2,N}||_{L^4}^2) ||\tilde{u}_{1,N}||^2 - ||\tilde{u}_{2,N}||_{L^2}^2 ||\tilde{u}_{1,N}||^2$$

which implies that

$$||\tilde{u}_{1,N}||_{L^4}^2 - ||\tilde{u}_{2,N}||_{L^4}^2 \leq ||\tilde{u}_{1,N}||^2 - ||\tilde{u}_{2,N}||_{L^2}^2 \leq 2c_1^{-1}c_2^{-1}a_{12}^{-2}c_N^{-2}C_{\text{GP}}(u_{1,N}, u_{2,N}). \quad (4.5)$$

On the other hand, it follows from (4.3) that

$$||\tilde{u}_{1,N}||^2 - ||\tilde{u}_{2,N}||_{L^2}^2 \leq 2c_1^{-1}c_2^{-1}a_{12}^{-2}c_N^{-2}C_{\text{GP}}(u_{1,N}, u_{2,N}). \quad (4.6)$$

and

$$\int_{\mathbb{R}^2} |\nabla \tilde{u}_{2,N}(x)|^2 dx - \frac{\tilde{a}_2}{2} \int_{\mathbb{R}^2} |\tilde{u}_{2,N}(x)|^4 dx \leq c_2^{-1}c_1^{-2}a_{12}^{-2}c_N^{-2}C_{\text{GP}}(u_{1,N}, u_{2,N}). \quad (4.7)$$

We deduce from (4.5), (4.6), (4.7), (1.2) and (1.1) that $\{\tilde{u}_{2,N}\}$ is bounded in $H^1(\mathbb{R}^2)$. Since this holds for each $N$, we conclude that the boundedness of $\{\tilde{u}_{1,N}\}$ and $\{\tilde{u}_{2,N}\}$ hold for the whole sequence. Thus, $\tilde{u}_{1,N}$ (resp. $\tilde{u}_{2,N}$) converges to a function $W_1$ (resp. $W_2$) weakly in $H^1(\mathbb{R}^2)$ and pointwise almost everywhere in $\mathbb{R}^2$. But then by taking the limit $N \to \infty$ in (4.6) we conclude that $W_1 = W_0 = W_2$ almost everywhere in $\mathbb{R}^2$. Furthermore, since $p_0 = \min\{p_1, p_2\}$, we deduce from (4.3), (1.2) and (1.1) that either $\int_{\mathbb{R}^2} |x|^{p_1} |\tilde{u}_{1,N}(x)|^2 dx$ or $\int_{\mathbb{R}^2} |x|^{p_2} |\tilde{u}_{2,N}(x)|^2 dx$ is bounded. It then follows that either $\tilde{u}_{1,N}$ or $\tilde{u}_{2,N}$ converges to $W_0$ strongly in $L^r(\mathbb{R}^2)$, for $2 \leq r < \infty$. In particular, we have $||W_0||_{L^2} = 1$. Moreover, by taking the limit $N \to \infty$ in (4.5) and using (4.6), (1.2) and (1.1) we obtain that both $\tilde{u}_{1,N}$ and $\tilde{u}_{2,N}$ converge to $W_0$ strongly in $L^4(\mathbb{R}^2)$. In fact, those convergences hold in $L^r(\mathbb{R}^2)$, for $4 \leq r < \infty$, by the $H^1(\mathbb{R}^2)$-boundness of $\{\tilde{u}_{1,N}\}$ and $\{\tilde{u}_{2,N}\}$. Taking the limit $N \to \infty$ in (1.7) and using Fatou’s lemma and the Hardy–Littlewood–Sobolev inequality, we obtain

$$\int_{\mathbb{R}^2} |\nabla W_0(x)|^2 dx - \frac{a_s}{2} \int_{\mathbb{R}^2} |W_0(x)|^4 dx \leq 0.$$ 

Thus, $W_0$ is an optimizer for (1.11). Recall that (1.11) admits a unique optimizer, up to translation and dilations. Therefore, a simple scaling and the uniqueness (up to translation) of positive solutions of (1.12) allow us to conclude that

$$W_0(x) = (a_s)^{-\frac{1}{2}} b Q(bx + x_0)$$

for some constant $b \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}^2$. Here $Q$ is the unique (up to translation) solution of (1.12). We will show that $b = 1$ and $x_0 = 0$. Indeed, it follows from (1.1), (1.2), (1.3), (1.11) and Fatou’s lemma that

$$\frac{p_0 + 2}{p_0} \Lambda^2 \geq \frac{b^2 \Lambda^2}{a_s} \int_{\mathbb{R}^2} |\nabla Q(x)|^2 dx + \frac{\nu}{b^{p_0} \Lambda^{p_0}} \int_{\mathbb{R}^2} |x|^{p_0} |Q(x + b^{-1}x_0)|^2 dx. \quad (4.8)$$

Here we have used the fact that $c_1 + c_2 = 1$ and the assumption $p_0 = \min\{p_1, p_2\}$. Note that $||\nabla Q||_{L^2}^2 = ||Q||_{L^2}^2 = a_s$ and

$$\int_{\mathbb{R}^2} |x|^{p_0} |Q(x + b^{-1}x_0)|^2 dx \geq \int_{\mathbb{R}^2} |x|^{p_0} |Q(x)|^2 dx, \quad (4.9)$$
by the Hardy–Littlewood rearrangement inequality as \( Q \) is a radial symmetric decreasing function. Thus, (4.8) reduces to
\[
\frac{p_0 + 2}{p_0} \Lambda^2 \geq b^2 \Lambda^2 + \frac{\nu}{b^{p_0} \Lambda^{p_0}} \int_{\mathbb{R}^2} |x|^{p_0} |Q(x)|^2 \, dx.
\] (4.10)

It is elementary to check that
\[
\inf_{\lambda > 0} \left( \lambda^2 + \frac{\nu}{\lambda^{p_0}} \int_{\mathbb{R}^2} |x|^{p_0} |Q(x)|^2 \, dx \right) = \frac{p_0 + 2}{p_0} \Lambda^2
\]
with the unique optimal value \( \lambda = \Lambda \). Therefore, the equality in (4.10) must occurs, and hence \( b = 1 \). This also implies that the equality in (4.11) must occurs, and hence \( x_0 = 0 \). \( \square \)

Next, we settle an estimate for the quantum energy and ground states. Using (3.17) and the arguments in [35] we have the following asymptotic formula of the quantum energy.

**Lemma 9.** Assume that \( 0 < a_{12} < \min \{ c_{1}^{-1}, c_{2}^{-1} \} \) is fixed and \( V_1, V_2 \) are defined as in (2.1) with \( z_1 = 0 = z_2 \). Let \( 0 < \beta < 1/2 \) and let \( (a_1, a_2) := (a_N, a_N) \) \( \not\sim (a_* - c_2a_{12}, a_* - c_1a_{12}) \) such that \( a_N := c_1a_{1N} + c_2a_{2N} + 2c_1c_2a_{12} = a_* - N^{-\gamma} \) with
\[
0 < \gamma < \min \left\{ \frac{p_0 + 2}{p_0 + 3}, \frac{p_0 + 2}{p_0} (1 - 2\beta) \right\}, \quad p_0 = \min \{ p_1, p_2 \}.
\]
Then we have, with \( \Lambda \) given by (2.3),
\[
E_N^Q = E^{GP} + o(E^{GP})_{N \to \infty} = (a_* - a_N)^{\frac{p_0}{p_0 + 3}} \left( \frac{p_0 + 2 \Lambda^2}{p_0} - \frac{a_*}{a_*} + o(1)_{N \to \infty} \right).
\]

**Proof.** We start with the upper bound. We deduce from (3.22) and the asymptotic behavior of the Gross–Pitaevskii ground state (see e.g., [24] Proposition 3)) that
\[
E_N^Q \leq E_N^H \leq E^{GP} + CN^{-\beta} \tilde{\ell}_N^3
\]
\[
= (a_* - a_N)^{\frac{p_0}{p_0 + 3}} \left[ \frac{p_0 + 2 \Lambda^2}{p_0} - \frac{a_*}{a_*} + CN^{-\beta} (a_* - a_N)^{\frac{p_0 + 3}{p_0 + 2}} \right], \quad (4.11)
\]
The error term \( N^{-\beta} (a_* - a_N)^{\frac{p_0 + 3}{p_0 + 2}} \) is of order 1 when \( a_* - a_N = N^{-\gamma} \) with \( 0 < \gamma < \frac{p_0 + 2}{p_0 + 3} \).

Now we turn to the lower bound. It follows from (3.17) that
\[
E_N^Q \geq E^H - CN^{2\beta - 1}, \quad (4.12)
\]
Next, we compare the Hartree and the Gross–Pitaevskii energies. Let \( (u_{1N}, u_{2N}) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \) be a ground state for \( E^H \) in (1.7). Applying (3.23) several times to \( \kappa = 1, i = 1, j = 1, i = 2, j = 2, \) we obtain that, with \( \tilde{\alpha}_{1N} = a_{1N} + c_2a_{12} \) and \( \tilde{\alpha}_{2N} = a_{2N} + c_1a_{12}, \)
\[
E^H \geq c_1 \int_{\mathbb{R}^2} \left( |\nabla u_{1N}(x)|^2 - \frac{\tilde{\alpha}_{1N}}{2} |u_{1N}(x)|^4 \right) \, dx + c_2 \int_{\mathbb{R}^2} \left( |\nabla u_{2N}(x)|^2 - \frac{\tilde{\alpha}_{2N}}{2} |u_{2N}(x)|^4 \right) \, dx.
\]
For each \( N \), we may assume without loss of generality that \( \tilde{\alpha}_{1N} \leq \tilde{\alpha}_{2N} \). By the above inequality and (1.11) we have, with \( a_N = c_1\tilde{\alpha}_{1N} + c_2\tilde{\alpha}_{2N}, \)
\[
E^H \geq c_1 \frac{a_* - \tilde{\alpha}_{1N}}{a_*} \int_{\mathbb{R}^2} |\nabla u_{1N}(x)|^2 \, dx \geq c_1 \frac{a_* - a_N}{a_*} \int_{\mathbb{R}^2} |\nabla u_{1N}(x)|^2 \, dx.
\] (4.13)
Now we denote \( \tilde{u}_{1N} = \ell_{N}^{-1} u_{1N}(\ell_{N}^{-1}) \) and \( \tilde{u}_{2N} = \ell_{N}^{-1} u_{2N}(\ell_{N}^{-1}) \) and \( \ell_{N} \) is defined as in Theorem 7. It follows from (4.13) and the upper bound of \( E^H \) in (4.11) that \( \{ \tilde{u}_{1N} \} \) is bounded in \( H^1(\mathbb{R}^2) \). In addition, we have
\[
\int_{\mathbb{R}^2} |\nabla \tilde{u}_{2N}(x)|^2 \, dx - \frac{\tilde{\alpha}_{2N}}{2} \int_{\mathbb{R}^2} |\tilde{u}_{2N}(x)|^4 \, dx \leq c_2^2 \ell_{N}^2 E^H. \quad (4.14)
\]
On the other hand, by applying (3.23) several times to \( \kappa = 1, i = 1 = j, i = 2 = j \) and to \( \kappa = 2, i = 1, j = 2 \), and using (1.11) we obtain

\[
E^H \geq \frac{c_1c_2a_{12}}{2} - \int_{\mathbb{R}^2} |u_{1,N}(x)|^4 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |u_{2,N}(x)|^4 \, dx
\]

which implies that

\[
- \int_{\mathbb{R}^2} \bar{u}_{1,N}(x)|^4 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} \bar{u}_{2,N}(x)|^4 \, dx \leq 2c_1^{-1}c_2^{-1}a_{12}^{-2}E^H. \tag{4.15}
\]

Hence we deduce from (4.14), (4.15) and the upper bound of \( E^H \) in (4.11) that \( \{\bar{u}_{2,N}\} \) is bounded in \( H^1(\mathbb{R}^2) \). Since this holds for each \( N \), we conclude that the boundedness of \( \{\bar{u}_{1,N}\} \) and \( \{\bar{u}_{2,N}\} \) hold for the whole sequence. We may apply (3.13) and (3.20) with \( i = 1, j = 2 \) to obtain

\[
E_N^Q \geq E^\text{GP} - CN^{2\beta - 1} - CN^{-\beta}\beta_N = (a_s - a_N)^{p_0/p_0+2} \left( \frac{p_0 + 2}{p_0} \frac{\Lambda^2}{a_s} + R_3 \right).
\]

Here we abbreviated by \( R_3 \) the error terms

\[
R_3 = -CN^{2\beta - 1}(a_s - a_N)^{-p_0/p_0+2} - CN^{-\beta}(a_s - a_N)^{-p_0+3/p_0+2}
\]

which is of order 1 when \( a_s - a_N = N^{-\gamma} \) with

\[
0 < \gamma < \min \left\{ \frac{p_0 + 2}{p_0 + 3}, \frac{p_0 + 2}{p_0} (1 - 2\beta) \right\}.
\]

This concludes the proof of Lemma 9. \( \square \)

In the case of the totally attractive system, we have seen in Theorems 7 and 8 that the Gross–Pitaevskii exact/approximate ground states prefer to have the same behavior when \( a_{12} > 0 \) is fixed and \( (a_1, a_2) \not\sim (a_s - c_2a_{12}, a_s - c_1a_{12}) \). It is the same situation when \( 0 < a_1, a_2 < a_s \) are fixed and \( a_{12} > 0 \) tends to a critical number. In which case, the limit behavior of the Gross–Pitaevskii exact ground states have not studied yet in [24]. But it is somehow similar to the previous case. As mentioned in the introduction, we will assume that \( c_1(a_s - a_1) = c_1c_2a_s = c_2(a_s - a_2) \). Note that, with this assumption, Theorem 6 gave a complete classification of the existence and non-existence of ground states for (1.10). In the following, we address the limiting profile of the general Gross–Pitaevskii approximate ground states in the limit regime that the inter-species interactions tend to the critical number \( a_s \). We first note that we have the following estimate

\[
\limsup_{a_{12} \to a_s} \frac{E^\text{GP}_{a_{12} \to a_s}}{(a_s - a_{12})^{p_0/p_0+2}} \leq 2c_1c_2\frac{p_0 + 2}{p_0} \Theta^2 \frac{\Lambda^2}{a_s}, \tag{4.16}
\]

where \( p_0 = \min \{p_1, p_2\} \) and \( \Theta \) is given by (2.20). To see this fact, we simply take

\[
u_1(x) = u_2(x) = (a_s)^{3/2} \lambda(a_s - a_{12})^{-1/p_0+2} Q(\lambda(a_s - a_{12})^{-1/p_0+2} x)
\]

as a trial function for \( E^\text{GP} \) in (1.14) and minimizes it over \( \lambda > 0 \).

**Theorem 10.** Assume that \( 0 < a_1, a_2 < a_s \) are fixed such that \( c_1(a_s - a_1) = c_1c_2a_s = c_2(a_s - a_2) \) and \( V_1, V_2 \) are defined as in (2.1) with \( z_1 = 0 = z_2 \). Let \( 0 < a_{12} := \alpha_N \not\sim \alpha_s \) as \( N \to \infty \). Let \( (u_{1,N}, u_{2,N}) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \) be a sequence of a coupled positive functions such that \( \|u_{1,N}\|_{L^2} = 1 = \|u_{2,N}\|_{L^2} \) and

\[
E^\text{GP}(u_{1,N}, u_{2,N}) = E^\text{GP} + o(E^\text{GP})_{N \to \infty} = (a_s - \alpha_N)^{p_0/p_0+2} \left( 2c_1c_2\frac{p_0 + 2}{p_0} \Theta^2 \frac{\Lambda^2}{a_s} + o(1)_{N \to \infty} \right) \tag{4.17}
\]
where \( p_0 = \min\{p_1, p_2\} \) and \( \Theta \) is given by (2.5). Then we have
\[
\lim_{N \to \infty} \ell_N^{-1} u_{1,N} (\ell_N^{-1}) = Q_0 = \lim_{N \to \infty} \ell_N^{-1} u_{2,N} (\ell_N^{-1})
\] (4.18)
strongly in \( L^2(\mathbb{R}^2) \), where \( \ell_N = \Theta (\alpha_* - \alpha_N)^{-\frac{1}{p_0 - 2}} \) and \( Q_0 \) is given by (2.2).

Proof. Denote \( \tilde{u}_{1,N} = \ell_N^{-1} u_{1,N} (\ell_N^{-1}) \) and \( \tilde{u}_{2,N} = \ell_N^{-1} u_{2,N} (\ell_N^{-1}) \), then \( \| \tilde{u}_{1,N} \|_{L^2} = 1 = \| \tilde{u}_{2,N} \|_{L^2} \). We rewrite \( \mathcal{E}^{GP} \) as
\[
\mathcal{E}^{GP} (u_{1,N}, u_{2,N}) = c_1 \ell_N^2 \int_{\mathbb{R}^2} \left[ |\nabla \tilde{u}_{1,N}(x)|^2 - \frac{a_1 + c_2 \alpha_N}{2} |\tilde{u}_{1,N}(x)|^4 \right] \, dx \\
+ c_2 \ell_N^2 \int_{\mathbb{R}^2} \left[ |\nabla \tilde{u}_{2,N}(x)|^2 - \frac{a_2 + c_1 \alpha_N}{2} |\tilde{u}_{2,N}(x)|^4 \right] \, dx \\
+ \frac{c_1}{\ell_N^2} \int_{\mathbb{R}^2} |x|^p_1 |\tilde{u}_{1,N}(x)|^2 \, dx + \frac{c_2}{\ell_N^2} \int_{\mathbb{R}^2} |x|^p_2 |\tilde{u}_{2,N}(x)|^2 \, dx \\
+ \frac{c_1 c_2 \alpha_N}{2} \ell_N^2 \int_{\mathbb{R}^2} (|\tilde{u}_{1,N}(x)|^2 - |\tilde{u}_{2,N}(x)|^2)^2 \, dx
\] (4.19)
By (4.19) and (1.11) we have
\[
\mathcal{E}^{GP} (u_{1,N}, u_{2,N}) \geq c_1 c_2 \ell_N^2 \left( \frac{\alpha_* - \alpha_N}{a_*} \right) \int_{\mathbb{R}^2} (|\nabla \tilde{u}_{1,N}(x)|^2 + |\nabla \tilde{u}_{2,N}(x)|^2) \, dx \\
+ \frac{c_1}{\ell_N^2} \int_{\mathbb{R}^2} |x|^p_1 |\tilde{u}_{1,N}(x)|^2 \, dx + \frac{c_2}{\ell_N^2} \int_{\mathbb{R}^2} |x|^p_2 |\tilde{u}_{2,N}(x)|^2 \, dx \\
+ \frac{c_1 c_2 \alpha_N}{2} \ell_N^2 \int_{\mathbb{R}^2} (|\tilde{u}_{1,N}(x)|^2 - |\tilde{u}_{2,N}(x)|^2)^2 \, dx
\] (4.20)
Using (4.17), (4.16) we then deduce from (4.20) that \( \{ \tilde{u}_{1,N} \} \) and \( \{ \tilde{u}_{2,N} \} \) are bounded in \( H^1(\mathbb{R}^2) \). Thus, \( \tilde{u}_{1,N} \) (resp. \( \tilde{u}_{2,N} \)) converges to a function \( W_1 \) (resp. \( W_2 \)) weakly in \( H^1(\mathbb{R}^2) \) and pointwise almost everywhere in \( \mathbb{R}^2 \). On the other hand, it follows from (4.19) that
\[
\| |\tilde{u}_{1,N}|^2 - |\tilde{u}_{2,N}|^2 \|_{L^2}^2 \leq 2c_1 c_2 \ell_N^{-1} \ell_N^{-2} \mathcal{E}^{GP} (u_{1,N}, u_{2,N}).
\] (4.21)
Taking the limit \( N \to \infty \) in (4.21) and using (4.17), (4.16) we conclude that \( W_1 = W_0 = W_2 \) almost everywhere in \( \mathbb{R}^2 \). Furthermore, since \( p_0 = \min\{p_1, p_2\} \), we deduce from (4.19), (4.17) and (4.16) that either \( \int_{\mathbb{R}^2} |x|^p_1 |\tilde{u}_{1,N}(x)|^2 \, dx \) or \( \int_{\mathbb{R}^2} |x|^p_2 |\tilde{u}_{2,N}(x)|^2 \, dx \) is bounded. It then follows that either \( \tilde{u}_{1,N} \) or \( \tilde{u}_{2,N} \) converges to \( W_0 \) strongly in \( L^r(\mathbb{R}^2) \), for \( 2 \leq r < \infty \). In particular, we have \( \| W_0 \|_{L^2} = 1 \). On the other hand, by the same arguments as in the proof of Theorem S we obtain that both \( \tilde{u}_{1,N} \) and \( \tilde{u}_{2,N} \) converge to \( W_0 \) strongly in \( L^r(\mathbb{R}^2) \), for \( 4 \leq r < \infty \). In addition, it follows from (4.19) and (1.11) that
\[
\int_{\mathbb{R}^2} |\nabla \tilde{u}_{1,N}(x)|^2 \, dx - \frac{a_1 + c_2 \alpha_N}{2} \int_{\mathbb{R}^2} |\tilde{u}_{1,N}(x)|^4 \, dx \leq c_1^{-1} \ell_N^{-2} \mathcal{E}^{GP} (u_{1,N}, u_{2,N}).
\] (4.22)
Taking the limit \( N \to \infty \) in (4.22) and using Fatou’s lemma and the Hardy–Littlewood–Sobolev inequality, the above implies that
\[
\int_{\mathbb{R}^2} |\nabla W_0(x)|^2 \, dx - \frac{a_*}{2} \int_{\mathbb{R}^2} |W_0(x)|^4 \, dx \leq 0.
\]
Thus \( W_0 \) is an optimizer for (1.11). Recall that (1.11) admits a unique optimizer \( Q \), up to translation and dilations. Therefore, a simple scaling and the uniqueness (up to translation) of positive solutions of (1.12) allow us to conclude that
\[
W_0(x) = (a_*)^{-\frac{1}{4}} b Q(bx + x_0)
\]
for some constant $b \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}^2$. Here $Q$ is the unique (up to translation) solution of (1.12). We will show that $b = 1$ and $x_0 = 0$. Indeed, it follows from (4.17), (4.16), (4.19), (1.11) and Fatou’s lemma that
\begin{equation}
2c_1 c_2 \frac{p_0}{p_0} + 2 \frac{\Theta^2}{\Theta} \geq 2c_1 c_2 \frac{b^2 \Theta^2}{a_*} \int_{\mathbb{R}^2} |\nabla Q(x)|^2 dx + \frac{\nu}{b \rho_0 \rho_0} \int_{\mathbb{R}^2} |x|^p_0 |Q(x + b^{1}x_0)|^2 dx.
\end{equation}
Here we have used the assumption $p_0 = \min\{p_1, p_2\}$. Note that $\|\nabla Q\|^2_{L^2} = \|Q\|^2_{L^2} = a_*$ and
\begin{equation}
\int_{\mathbb{R}^2} |x|^{p_0} |Q(x + b^{-1}x_0)|^2 dx \geq \int_{\mathbb{R}^2} |x|^{p_0} |Q(x)|^2 dx,
\end{equation}
by the Hardy–Littlewood rearrangement inequality as $Q$ is a radial symmetric decreasing function. Thus, (4.23) reduces to
\begin{equation}
2c_1 c_2 \frac{p_0}{p_0} + 2 \frac{\Theta^2}{\Theta} \geq 2c_1 c_2 b^2 \frac{\Theta^2}{\Theta} + \frac{\nu}{b \rho_0 \rho_0} \int_{\mathbb{R}^2} |x|^{p_0} |Q(x)|^2 dx.
\end{equation}
It is elementary to check that
\begin{equation}
\inf_{\theta > 0} \left( 2c_1 c_2 \theta^2 + \frac{\nu}{b \rho_0} \int_{\mathbb{R}^2} |x|^{p_0} |Q(x)|^2 dx \right) = 2c_1 c_2 \frac{p_0}{p_0} + 2 \frac{\Theta^2}{\Theta}
\end{equation}
with the unique optimal value $\theta = \Theta$. Therefore, the equality in (4.25) must occurs, and hence $b = 1$. This also implies that the equality in (4.24) must occurs, and hence $x_0 = 0$. \hfill \Box

Remark. If $(u_{1,N}, u_{2,N})$ is an ground state for $E_{GP}$ then one can easily seen from the proof of Theorem 10 that we have the following estimate
\begin{equation}
\liminf_{a_{12}/a_* \to \infty} \frac{E_{GP}}{(a_* - a_{12})^{\rho_0/p_0 + 2}} \geq 2c_1 c_2 \frac{p_0}{p_0} + 2 \frac{\Theta^2}{a_*}.
\end{equation}
Together with (4.16) we obtain the asymptotic behavior of the Gross–Pitaevskii energy. In addition, we also obtain the $H^1(\mathbb{R}^2)$ strong convergence (4.18) for the Gross–Pitaevskii exact ground states.

Lemma 11. Assume that $0 < a_{12}, a_{2} < a_{*}$ are fixed such that $c_1(a_{*} - a_{1}) = c_1 c_2 a_{*} = c_2(a_{*} - a_{2})$ and $V_1, V_2$ are defined as in (2.11) with $z_1 = 0 = z_2$. Let $0 < \beta < 1/2$ and let $a_{12} := a_{N} = a_{*} - N^{-\gamma}$ with
\begin{equation}
0 < \gamma < \min \left\{ \frac{p_0 + 2}{p_0 + 3}, \frac{p_0 + 2}{p_0} (1 - 2\beta) \right\}, \quad p_0 = \min\{p_1, p_2\}.
\end{equation}
Then we have, with $\Theta$ given by (2.5),
\begin{equation}
E_{N}^{Q} = E_{GP} + o(E_{GP})_{N \to \infty} = (a_{*} - a_{N})^{\rho_0/p_0 + 2} \left( 2c_1 c_2 \frac{p_0}{p_0} + 2 \frac{\Theta^2}{a_*} + o(1)_{N \to \infty} \right).
\end{equation}
Proof. We start with the upper bound. We deduce from (3.22) and the asymptotic behavior of the Gross–Pitaevskii ground states that
\begin{equation}
E_{N}^{Q} \leq E_{N}^{H} \leq E_{GP}^H + C N^{-\beta} \frac{p_0}{p_0 + 2} \left[ 2c_1 c_2 \frac{p_0}{p_0} + 2 \frac{\Theta^2}{a_*} + C N^{-\beta} (a_{*} - a_{N})^{-\rho_0/p_0 + 3} \right].
\end{equation}
The error term $N^{-\beta}(a_{*} - a_{N})^{-\rho_0/p_0 + 3}$ is of order 1 when $a_{*} - a_{N} = N^{-\gamma}$ with $0 < \gamma < \frac{p_0 + 2}{p_0 + 3}$. Now we turn to the lower bound. We use again (3.17) to obtain
\begin{equation}
E_{N}^{Q} \geq E_{N}^{H} - C N^{2\beta - 1}.
\end{equation}
Next, we compare the Hartree and the Gross–Pitaevskii energies. Let \((u_{1,N}, u_{2,N}) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\) be a ground state for \(E^H\). Applying (3.23) several times to \(\kappa = 1\), \(i = 1 = j\), \(i = 2 = j\) and \(i = 1, j = 2\) and using (11.11) we obtain
\[
E^H \geq c_1 c_2 \frac{\alpha_* - \alpha_N}{a_*} \int_{\mathbb{R}^2} \left[ |\nabla u_{1,N}(x)|^2 + |\nabla u_{2,N}(x)|^2 \right] \, dx.
\]

Now we denote \(\tilde{u}_{2,N} = \ell_N^{-1} u_{1,N}(\ell_N^{-1} \cdot)\) and \(\tilde{u}_{1,N} = \ell_N^{-1} u_{2,N}(\ell_N^{-1} \cdot)\) where \(\ell_N = \Theta(\alpha_* - \alpha_N)^{-\frac{p_0 + \gamma}{2}}\). It follows from the above inequality and the upper bound of \(E^H\) in (4.27) that \(\{\tilde{u}_{1,N}\}\) and \(\{\tilde{u}_{2,N}\}\) are both bounded in \(H^1(\mathbb{R}^2)\). We may apply (3.15) and (3.20) with \(\gamma = 1\) and \(\ell_N^2 = 2\) to obtain
\[
E^Q_N \geq E^{GP}_N - C N^{2 \beta - 1} - C N^{- \beta} \ell_N^2 = (\alpha_* - \alpha_N)^{-\frac{p_0}{p_0 + \gamma}} \left( 2 c_1 c_2 \frac{p_0 + 2 \Theta^2}{p_0} a_* + R_4 \right).
\]

Here we abbreviated by \(R_4\) the error terms
\[
R_4 = - C N^{2 \beta - 1} (\alpha_* - \alpha_N)^{-\frac{p_0}{p_0 + \gamma}} - C N^{- \beta} (\alpha_* - \alpha_N)^{-\frac{p_0 + \gamma}{p_0 + \gamma}}
\]
which is of order 1 when \(\alpha_* - \alpha_N = N^{- \gamma}\) with
\[
0 < \gamma < \min \left\{ \frac{p_0 + 2}{p_0}, \frac{p_0 + 2}{p_0} (1 - 2 \beta) \right\}.
\]
This concludes the proof of Lemma 11. □

Now we are in the position to give the proofs of Theorems 1 and 2. We only prove Theorem 1 since the proof of Theorem 2 is analogously.

**Proof of Theorem 1.** Let \(\eta > 0\) be a small parameter and \(A\) be a bounded self–adjoint operator on \(L^2(\mathbb{R}^2)\). Consider the perturbed Hamiltonian in the group of \(N_1\) particles
\[
H_{N,\eta} = \sum_{i=1}^{N_1} \left( - \Delta x_i + V_1(x_i) + \eta A x_i \right) - \frac{1}{N_1 - 1} \sum_{1 \leq i < j \leq N_1} w_N^{(1)}(x_i - x_j)
+ \sum_{r=1}^{N_2} \left( - \Delta y_r + V_2(y_r) \right) - \frac{1}{N_2 - 1} \sum_{1 \leq r < s \leq N_2} w_N^{(2)}(y_r - y_s)
- \frac{1}{N} \sum_{i=1}^{N_1} \sum_{r=1}^{N_2} w_N^{(12)}(x_i - y_r)
\tag{4.29}
\]
with the quantum energy per particle denoted \(E^Q_\eta\) hereafter. The associated Gross–Pitaevskii ground state functional is
\[
E^{GP}_\eta(u_1, u_2) = c_1 \int_{\mathbb{R}^2} \left[ |\nabla u_1(x)|^2 + V_1(x)|u_1(x)|^2 - \frac{a_{1,N}}{2} |u_1(x)|^4 \right] \, dx + c_1 \eta \langle u_1, Au_1 \rangle
+ c_2 \int_{\mathbb{R}^2} \left[ |\nabla u_2(x)|^2 + V_2(x)|u_2(x)|^2 - \frac{a_{2,N}}{2} |u_2(x)|^4 \right] \, dx
- c_1 c_2 a_{12} \int_{\mathbb{R}^2} |u_1(x)|^2 |u_2(x)|^2 \, dx
\]
with the corresponding Gross–Pitaevskii energy \(E^{GP}_\eta\) (note that \(E^{GP} = E^{GP}_0\)). Let \((u_{1,N,\eta}, u_{2,N,\eta})\) be a ground state for \(E^{GP}_\eta\). Let \(\Psi_N\) be a ground state for \(H_N = H_{N,0}\). By the argument in the proof of Lemma 11 we have
\[
\eta c_1 \text{Tr}(A \gamma^{(1,0)}_N) = N^{-1} \langle \Psi_N, H_{N,\eta} \Psi_N \rangle - N^{-1} \langle \Psi_N, H_{N,0} \Psi_N \rangle \geq E^Q_\eta - E^Q_N
\]
The phenomenon for the Gross–Pitaevskii minimization problem \((1.9)\) is the case among particles in each component tending to a critical number. We first revisit the blow-up profile of the many-body system \((1.1)\) when the interaction strength of intra-species calculus of variations. In \([25, \text{Theorem 1.1}]\), the authors proved that if \(0 < \gamma < \min\{\frac{2}{3}, \frac{2}{3}\}\) one can pick \(\eta = \eta_N = o\left(\frac{1}{N^{\frac{3}{2}} r_{\gamma, \eta}}\right) \to 0\) as \(N \to \infty\) such that
\[
\lim_{N \to \infty} \eta_N^{-1} \left(N^{\frac{3}{2}} r_{\gamma, \eta}^{-\frac{1}{2}} + N^{2\beta - 1}\right) = 0. \tag{4.30}
\]
Then, it follows from the above estimate and repeating the argument with \(A\) changes to \(-A\) yields
\[
\langle u_{1, N, \eta}, Au_{1, N, \eta} \rangle + o(1)_{N \to \infty} \leq \text{Tr}(A\gamma_{\eta, N}^{(1, 0)}) \leq \langle u_{1, N, -\eta}, Au_{1, N, -\eta} \rangle + o(1)_{N \to \infty}. \tag{4.31}
\]
On the other hand, since \((u_{1, N, \eta}, u_{2, N, \eta})\) is a ground state for \(E_G^{GP}\) (recall that \(E_G = E_0^{GP}\), we have, with the choice of \(\eta\) in \([14.31]\),
\[
E_G \leq \mathcal{E}_G^{\eta}(u_{1, N, \eta}, u_{2, N, \eta}) \leq \mathcal{E}_G^{\eta} + \eta \|A\| \leq \mathcal{E}_G^{\eta}(u_{1, N, 0}, u_{2, N, 0}) + \eta \|A\| \leq E_G + 2\eta \|A\|.
\]
It follows from the above that \((u_{1, N, \eta}, u_{2, N, \eta})\) and \((u_{1, N, -\eta}, u_{2, N, -\eta})\) are sequences of quasi-ground states for \(E_G\). We may apply Theorem \([8\), together with \(4.31\)], we get the trace-class weak*-convergence of \(\gamma_{\eta, N}^{(1, 0)}\) to \(|\ell_N \Phi_N(\ell_N \cdot)|(\ell_N \Phi_N(\ell_N \cdot))\), where \(\ell_N\) is defined as in Theorem \([1\) and \(Q_0\) is given by \([2.2\). Since no mass is lost in the limit, the convergence must hold in trace-class norm. Equivalently, \(\gamma_{\Phi_N}^{(1, 0)}\) converges to \(|Q_0\rangle \langle Q_0|\) where \(\Phi_N = \ell_N^{-\frac{1}{2}} \Psi_N(\ell_N^{-\frac{1}{2}})\). This gives \([2.4\) for \((k, l) = (1, 0)\).

Now we consider the perturbed Hamiltonian in the group of \(N_2\) particles
\[
H_{N, \gamma} = \sum_{i=1}^{N_1} \left(-\frac{1}{N_1 - 1} \sum_{1 \leq i < j \leq N_1} w_N^{(1)}(x_i - x_j) - \frac{1}{N_2 - 1} \sum_{1 \leq r < s \leq N_2} w_N^{(2)}(y_r - y_s) \right) - \frac{1}{N} \sum_{i=1}^{N_1} \sum_{r=1}^{N_2} w_N^{(12)}(x_i - y_r).
\tag{4.32}
\]
At this stage, by repeating the above argument, we also obtain the convergence of \(\gamma_{\Phi_N}^{(0, 1)}\) to \(|Q_0\rangle \langle Q_0|\) in trace-class. This gives \([2.4\) for \((k, l) = (0, 1)\).

To obtain \([2.4\) for all \((k, l)\) with \(k, l \in \mathbb{N}\), we observe that \(\gamma_{\Phi_N}^{(1, 0)}\) and \(\gamma_{\Phi_N}^{(0, 1)}\) converge in trace-class norm to a rank-one operator. It is well known that this implies the convergence of higher order density matrices to tensor powers of the limiting operator (see, e.g., the discussion following \([17\) Section 3]).

### 4.2. Proof of Theorem \([8\)

The purpose of this subsection is to prove Theorem \([8\) which gives the blow-up profile of the many-body system \([1.1\) when the interaction strength of intra-species among particles in each component tending to a critical number. We first revisit the blow-up phenomenon for the Gross–Pitaevskii minimization problem \([1.9\). In the case \(a_{12} < 0\), the existence of the Gross–Pitaevskii ground states follows the standard direct method in the calculus of variations. In \([22\) Theorem 1.1\), the authors proved that if \(0 < a_1, a_2 < a_\ast\) then
$E^{GP} \geq 0$ and it has at least one ground state. On the other hand, $E^{GP} = -\infty$ when either $a_1 > a_*$ or $a_2 > a_*$. This is somehow similar to the one-component setting (see [23]).

The next result concerns the limit behavior of Gross–Pitaevskii ground state energy and its ground states. For the system of the attractive intra-species interactions and repulsive inter-species interactions, we are not able to determine the accurate blow-up rate of ground states when $z_1 = z_2$, due to the absence of a refined energy estimate. When $z_1 \neq z_2$, the decays property (2.11) allows us to control the cross term in (1.9). In [25], the authors prove that for any fixed $a_{12} < 0$ and $z_1 \neq z_2$ we have

$$c_1 E^{GP}_1 + c_2 E^{GP}_2 \leq E^{GP} \leq c_1 E^{GP}_1 + c_2 E^{GP}_2 + C \left( e^{-\delta_0(a_* - a_1)^{-\frac{1}{p_1 + 2}}} + e^{-\delta_0(a_* - a_2)^{-\frac{1}{p_2 + 2}}} \right).$$  \hspace{1cm} (4.33)

Here $E^{GP}_i$ is defined in (2.7) and $\delta_0 = \delta|z_1 - z_2| > 0$ with $\delta > 0$ given in (2.11). The limiting profile of the the Gross–Pitaevskii exact ground states follows the energy estimate (4.33) and the asymptotic behavior of the Gross–Pitaevskii ground state energy in the one-component setting, which is given by (2.9) (see [25, Theorem 1.2]).

In this subsection, we only study the blow-up profile of the many-body system (1.1) in the case $z_1 \neq z_2$. Similarly to what were done in Section 4.1, we need to extend the blow-up result in [25] to the Gross–Pitaevskii approximate ground states. The following is sufficient for our purpose.

**Theorem 12.** Assume that $V_i$, for $i \in \{1, 2\}$, are defined as in (2.11) with $z_i \in \mathbb{R}^2$. Let $a_{i,N} \rightarrow a_*$ as $N \rightarrow \infty$. Let $u_{i,N} \in H^1(\mathbb{R}^2)$ be a sequence of positive functions such that $\|u_{i,N}\|_{L^2} = 1$ and

$$E^{GP}_i(u_{i,N}) = E^{GP}_i + o(E^{GP}_i)_{N \rightarrow \infty} = (a_* - a_{i,N}) \frac{p_i}{p_i + 2} - \frac{2}{p_i} \Lambda_i \left( \frac{p_i + 2}{a_*} \right) + o(1)_{N \rightarrow \infty}$$  \hspace{1cm} (4.34)

where $E^{GP}_i$ are defined in (2.8) and $\Lambda_i$ are given by (2.9). Then we have

$$\lim_{N \rightarrow \infty} \ell_{1,N}^{-1} u_{i,N}(\ell_{1,N}^{-1} \cdot + z_i) = Q_0$$  \hspace{1cm} (4.35)

strongly in $L^2(\mathbb{R}^2)$, where $\ell_{i,N} = \Lambda_i(a_* - a_{i,N})^{-\frac{1}{p_i + 2}}$ and $Q_0$ is given by (2.2).

In the following, we only prove (4.35) for $i = 1$ since the proof of the case $i = 2$ is analogously.

**Proof.** Denote $\bar{u}_{1,N} = \ell_{1,N}^{-1} u_{1,N}(\ell_{1,N}^{-1} \cdot + z_1)$ then $\|\bar{u}_{1,N}\|_{L^2} = 1$. By (1.11) we have

$$E^{GP}_1(u_{1,N}) = \ell_{1,N}^2 \int_{\mathbb{R}^2} \left| \nabla \bar{u}_{1,N}(x) \right|^2 - \frac{a_{1,N}}{2} |\bar{u}_{1,N}(x)|^2 \right] \, dx + \frac{1}{\ell_{1,N}^{p_1}} \int_{\mathbb{R}^2} |x|^{p_1} |\bar{u}_{1,N}(x)|^2
\geq \ell_{1,N}^2 \frac{a_* - a_{1,N}}{a_*} \int_{\mathbb{R}^2} \left| \nabla \bar{u}_{1,N}(x) \right|^2 \, dx + \frac{1}{\ell_{1,N}^{p_1}} \int_{\mathbb{R}^2} |x|^{p_1} |\bar{u}_{1,N}(x)|^2 \, dx.$$  \hspace{1cm} (4.36)

Using (4.34) and the asymptotic formula for $E^{GP}_1$ in (2.9) we then deduce from (4.36) that \{\bar{u}_{1,N}\} is bounded in $H^1(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} |x|^{p_1} |\bar{u}_{1,N}(x)|^2 \, dx \leq C$. It then follows that $\bar{u}_{1,N}$ converges to a function $W_1$ weakly in $H^1(\mathbb{R}^2)$, strongly in $L^r(\mathbb{R}^2)$ for $2 \leq r < \infty$ and pointwise almost everywhere in $\mathbb{R}^2$. In particular, we have $\|W_1\|_{L^2} = 1$. In addition, it follows from (4.36) that

$$\int_{\mathbb{R}^2} \left| \nabla \bar{u}_{1,N}(x) \right|^2 \, dx - \frac{a_{1,N}}{2} \int_{\mathbb{R}^2} |\bar{u}_{1,N}(x)|^4 \, dx \leq \ell_{1,N}^{-2} E^{GP}_1(u_{1,N}).$$  \hspace{1cm} (4.37)

Taking the limit $N \rightarrow \infty$ in (4.37) and using the asymptotic formula of $E^{GP}_1$ in (2.9) together with Fatou’s lemma and the Hardy–Littlewood–Sobolev inequality, we obtain that

$$\int_{\mathbb{R}^2} \left| \nabla W_1(x) \right|^2 \, dx - \frac{a_*}{2} \int_{\mathbb{R}^2} |W_1(x)|^4 \, dx \leq 0.$$
Thus $W_1$ is an optimizer for (1.11). Recall that (1.11) admits a unique optimizer $Q$, up to translation and dilations. Therefore, a simple scaling and the uniqueness (up to translation) of positive solutions of (1.12) allow us to conclude that
\[
W_1(x) = (a_*)^{-\frac{1}{2}}b_1Q(b_1x + x_1)
\]
for some constant $b_1 \in \mathbb{R}^+$ and $x_1 \in \mathbb{R}^2$. Here $Q$ is the unique (up to translation) solution of (1.12). We will show that $b_1 = 1$ and $x_1 = 0$. Indeed, it follows from (4.36) and Fatou’s lemma that
\[
\frac{p_1 + 2}{p_1} \Lambda_1^2 \geq \frac{b_1^2}{a_*} \int_{\mathbb{R}^2} |\nabla Q(x)|^2 dx + \frac{1}{b_1^p \Lambda_1^p} \int_{\mathbb{R}^2} |x|^{p_1} |Q(x + b_1^{-1}x_1)|^2 dx.
\]  
(4.38)
Note that $\|\nabla Q\|^2_{L_2} = \|Q\|^2_{L_2} = a_*$ and
\[
\int_{\mathbb{R}^2} |x|^{p_1} |Q(x + b_1^{-1}x_1)|^2 dx \geq \int_{\mathbb{R}^2} |x|^{p_1} |Q(x)|^2 dx
\]  
(4.39)
by the Hardy–Littlewood rearrangement inequality as $Q$ is a radial symmetric decreasing function. Thus, (4.38) reduces to
\[
\frac{p_1 + 2}{p_1} \Lambda_1^2 \geq \frac{b_1^2}{a_*} \int_{\mathbb{R}^2} |\nabla Q(x)|^2 dx + \frac{1}{b_1^p \Lambda_1^p} \int_{\mathbb{R}^2} |x|^{p_1} |Q(x)|^2 dx.
\]  
(4.40)
On the other hand, it is elementary to check that
\[
\inf_{\lambda_1 > 0} \left( \lambda_1^2 + \frac{1}{\Lambda_1^p} \int_{\mathbb{R}^2} |x|^{p_1} |Q(x)|^2 dx \right) = \frac{p_1 + 2}{p_1} \Lambda_1^2
\]
with the unique optimal value $\lambda_1 = \Lambda_1$. Therefore, the equality in (4.40) must occur, and hence $b_1 = 1$. This also implies that the equality in (4.39) must occur, and hence $x_1 = 0$.

**Lemma 13.** Assume that $a_{i_2} < 0$ is fixed and $V_1, V_2$ are defined as in (2.1) with $z_1 \neq z_2$. Let $0 < \beta < 1/2$ and let $a_{i,N} = a_* - N^{-\gamma_i}$, for $i \in \{1, 2\}$, such that
\[
0 < \gamma_i < \min \left\{ \frac{p_i + 2}{p_i + 3}, \frac{p_i + 2}{p_i} (1 - 2\beta) \right\}.
\]

Then we have, with $E_1^{GP}$ and $E_2^{GP}$ defined in (2.7),
\[
E_N^Q = c_1 E_1^{GP} + o(E_1^{GP}) + c_2 E_2^{GP} + o(E_2^{GP})
\]
\[
= \sum_{i=1}^{2} c_i (a_* - a_{i,N})^{p_i + 2} \left( \frac{p_i + 2}{p_i} \Lambda_1^2 \right) + o(1)_{N \to \infty}
\]
where $\Lambda_1$ and $\Lambda_2$ are given by (2.9).

**Proof.** We start with the lower bound. For which we use again (3.17) to obtain
\[
E_N^Q \geq E^H - CN^{2\beta - 1}.
\]  
(4.41)
Next, we compare the Hartree and the Gross–Pitaevskii energies. Let $(u_{1,N}, u_{2,N}) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ be a ground state for $E^H$. Applying (3.23) several times to $i = 1, i = 1 = j, i = 2 = j$ and using the non-negativity of the inter-species interactions we obtain that
\[
E^H \geq c_1 E_1^{GP} (u_{1,N}) + c_2 E_2^{GP} (u_{2,N}) \geq c_1 E_1^{GP} + c_2 E_2^{GP}.
\]  
(4.42)
Putting together (4.41) and (4.42) we obtain
\[
E_N^Q \geq c_1 E_1^{GP} + c_2 E_2^{GP} - CN^{2\beta - 1}
\]
\[
\geq c_1 (a_* - a_{1,N})^{p_1 + 2} \left( \frac{p_1 + 2}{p_1} \Lambda_1^2 + R_5 \right) + c_2 (a_* - a_{2,N})^{p_2 + 2} \left( \frac{p_2 + 2}{p_2} \Lambda_2^2 + R_6 \right).
\]
Here we abbreviated by $\mathcal{R}_5$ and $\mathcal{R}_6$ the error terms

$$\mathcal{R}_5 = -CN^{2\beta-1}(a_s - a_{1,N})^{-\frac{p_i}{p_i+2}} \quad \text{and} \quad \mathcal{R}_6 = -CN^{2\beta-1}(a_s - a_{2,N})^{-\frac{p_i}{p_i+2}}$$

which are of order 1 when $a_s - a_{i,N} = N^{-\gamma_i}$ with $0 < \gamma_i < \frac{p_i+2}{p_i}(1 - 2\beta)$, for $i \in \{1, 2\}$.

Now we turn to the upper bound. We deduce from (4.22), (4.33), [25, Proof of Lemma 3.1] and the asymptotic behavior of the Gross–Pitaevskii ground states (see e.g., [23, Lemma 4]) that

$$E^Q_N \leq E^H \leq E^{GP}(u_{1,N}, u_{2,N}) + C N^{-\beta}(\ell^3_{1,N} + \ell^3_{2,N}) \leq c_1 E^1_{GP} + c_2 E^2_{GP} + \mathcal{R}_7.$$  \hspace{1cm} (4.43)

where $\ell_{i,N} = \Lambda_i(a_s - a_{i,N})^{-\frac{1}{p_i+2}}$, for $i \in \{1, 2\}$. Here we abbreviated by $\mathcal{R}_7$ the error terms

$$\mathcal{R}_7 := C N^{-\beta}(\ell^3_{1,N} + \ell^3_{2,N}) + C(\epsilon^{-\delta_0\ell_{1,N}} + e^{-\delta_0\ell_{2,N}}).$$  \hspace{1cm} (4.44)

When $a_{i,N} = a_s - N^{-\gamma_i}$, for $i \in \{1, 2\}$, with $0 < \gamma_i < \frac{p_i+2}{p_i+3}\beta$ we have $\mathcal{R}_7 = o(E^1_{GP}) + o(E^2_{GP}).$

The proof of Lemma 13 is completed.

**Proof of Theorem 5.** Let $\eta > 0$ be a small parameter and $A$ be a bounded self-adjoint operator on $L^2(\mathbb{R}^2)$. Consider the perturbed Hamiltonian in the group of $N_1$ particles as in [4, 29] with the quantum energy per particle denoted $E^Q_{\eta}$. The associated Gross–Pitaevskii ground state functional now is

$$\mathcal{E}^{GP}_{\eta}(u_1, u_2) = c_1 \mathcal{E}^{GP}_{1,N}(u_1) + c_2 \mathcal{E}^{GP}_{2,N}(u_2) - c_1 c_2 a_{12} \int_{\mathbb{R}^2} |u_1(x)||u_2(x)|^2 dx,$$

where we have introduced the Gross–Pitaevskii energy functionals

$$\mathcal{E}^{GP}_{\eta}(u) = \mathcal{E}^{GP}_1(u) + \eta (u, Au)$$

with the corresponding Gross–Pitaevskii energy $E^{GP}_{1,N}(u)$. Let $u_{1,N,\eta}$ be a ground states for $E^1_{1,N}$. Let $\Psi_N$ be a ground state for $H_N = H_{N,0}$. By the arguments in the proof of Lemma 13 we have

$$\eta c_1 \text{Tr}(A^{(1,0)}_\eta) = N^{-1}(\Psi_N, H_{N,\eta}\Psi_N) - N^{-1}(\Psi_N, H_N\Psi_N) \geq E^Q_{\eta} - E^Q_N \geq c_1 E^1_{1,\eta} + c_2 E^2_{1,\eta} - c_1 E^1_{1,\eta} - c_2 E^2_{1,\eta} - CN^{2\beta-1} - \mathcal{R}_7 \geq \eta c_1 (u_{1,N,\eta}, Au_{1,N,\eta}) - CN^{2\beta-1} - \mathcal{R}_7.$$

Here $\mathcal{R}_7$ is defined as in (4.44). Under the assumption that $a_s - a_{1,N} = N^{-\gamma_1}$ and $a_s - a_{2,N} = N^{-\gamma_2}$ with

$$\frac{\gamma_1}{\gamma_2} = \frac{p_1 + 2}{p_2 + 2} \quad \text{and} \quad 0 < \gamma_i \leq \min \left\{ \frac{p_i + 2}{p_i + 3} \beta, \frac{p_i + 2}{p_i} (1 - 2\beta) \right\}, \quad i \in \{1, 2\},$$

one can pick $\eta = \eta_{1,N} = o((a_s - a_{1,N})^{-\frac{p_i}{p_i+2}}) \rightarrow 0$ as $N \rightarrow \infty$ such that

$$\lim_{N \rightarrow \infty} \eta^{-1}(CN^{2\beta-1} - \mathcal{R}_7) = 0.$$

(4.45)

Then, it follows from the above estimate and repeating the argument with $A$ changes to $-A$ yields

$$\langle u_{1,N,\eta}, Au_{1,N,\eta} \rangle + o(1)_{N \rightarrow \infty} \leq \text{Tr}(A^{(1,0)}_\eta) \leq \langle u_{1,N,-\eta}, Au_{1,N,-\eta} \rangle + o(1)_{N \rightarrow \infty}. \hspace{1cm} (4.46)$$

On the other hand, since $u_{1,N,\eta}$ is a ground state for $E^1_{1,\eta}$ (recall that $\mathcal{E}^{GP}_1 = \mathcal{E}^{GP}_{1,0}$), we have, with the choice of $\eta$ in (4.45),

$$E^1_{1,\eta} \leq \mathcal{E}^{GP}_{1,N}(u_{1,N,\eta}) \leq E^1_{1,\eta} + \eta ||A|| \leq \mathcal{E}^{GP}_{1,N}(u_{1,N,0}) + \eta ||A|| \leq E^1_{1,\eta} + 2\eta ||A||.$$
It follows from the above that \( \{u_{1,N,\gamma}\} \) and \( \{u_{1,N,-\gamma}\} \) are sequences of quasi-ground states for \( E_1^{GP} \). We may apply Theorem 12 together with (4.46), we get the trace-class weak-* convergence of \( \gamma_{\Psi_N}^{(1,0)} \) to \( \|\ell_{1,N}Q_0(\ell_{1,N}(\cdot - z_1))\| \ell_{1,N}Q_0(\ell_{1,N}(\cdot - z_1)) \), where \( \ell_{1,N} \) is defined as in Theorem 3 \( z_1 \) is the minimum point of \( V_1 \) in (2.1) and \( Q_0 \) is given by (2.2). Since no mass is lost in the limit, the convergence must hold in trace-class norm. This gives (2.10) for \((k,l) = (1,0)\).

At this stage, by considering the perturbed Hamiltonian in the group of \( N_2 \) particles as in (4.32) and repeating the same argument as above, we also obtain the convergence of \( \gamma_{\Psi_N}^{(0,1)} \) to \( \|\ell_{2,N}Q_0(\ell_{2,N}(\cdot - z_2))\| \ell_{2,N}Q_0(\ell_{2,N}(\cdot - z_2)) \) in trace-class norm, where \( \ell_{2,N} \) is defined as in Theorem 3 and \( z_2 \) is the minimum point of \( V_2 \) in (2.1). Equivalently, both \( \gamma_{\Phi_N}^{(1,0)} \) and \( \gamma_{\Phi_N}^{(0,1)} \) converge to \( |Q_0\rangle\langle Q_0| \) in trace-class norm, where

\[
\Phi_N(x_1, \ldots, x_{N}; y_1, \ldots, y_{N_2}) = \frac{\Psi_N}{\ell_{1,N}^N,\ell_{2,N}^N} \left( \frac{x_1}{\ell_{1,N}^N} + \frac{x_2}{\ell_{2,N}^N} \right) \frac{y_1}{\ell_{1,N}^N} + \frac{z_1}{\ell_{1,N}^N} \frac{z_2}{\ell_{2,N}^N}. \]

It follows again from [37] Section 3] that (2.10) holds for all \((k,l)\) with \( k,\ell \in \mathbb{N} \).

ACKNOWLEDGEMENTS

This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy – EXC-2111 – 390814868.

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