On generalized Clifford algebras and their physical applications¹ ²

R. Jagannathan³
Chennai Mathematical Institute
Plot H1, SIPCOT IT Park, Padur P.O.
Siruseri 603103, Tamilnadu, India

Dedicated to the memory of Professor Alladi Ramakrishnan

Abstract

Generalized Clifford algebras (GCAs) and their physical applications were extensively studied for about a decade from 1967 by Alladi Ramakrishnan and his collaborators under the name of \( L \)-matrix theory. Some aspects of GCAs and their physical applications are outlined here. The topics dealt with include: GCAs and projective representations of finite abelian groups, Alladi Ramakrishnan’s \( \sigma \)-operation approach to the representation theory of Clifford algebra and GCAs, Dirac’s positive energy relativistic wave equation, Weyl-Schwinger unitary basis for matrix algebra and Alladi Ramakrishnan’s matrix decomposition theorem, finite-dimensional Wigner function, finite-dimensional canonical transformations, magnetic Bloch functions, finite-dimensional quantum mechanics, and the relation between GCAs and quantum groups.

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³Formerly of MATSCIENCE, The Institute of Mathematical Sciences, Chennai.

Email: jagan@cmi.ac.in, jagan@imsc.res.in
1 Introduction

Extensive studies on Clifford algebra, its generalizations, and their physical applications were made for about a decade starting 1967, under the name of $L$-Matrix Theory, by Alladi Ramakrishnan and his collaborators at The Institute of Mathematical Sciences (MATSCIENCE) including me, his Ph.D student during 1971-76. When I joined MATSCIENCE in August 1971, as a student, the book [1], containing all the results of their papers on the subject, up to then, was getting ready to be released; I had participated in the final stage of proof reading of the book. Chandrasekaran had just completed his Ph.D. thesis on the topic [2]. Subsequently, I started my thesis work on the same topic under the guidance of Alladi Ramakrishnan. I had also the guidance of Ranganathan, Santhanam and Vasudevan, senior faculty members of the institute, who had also started their scientific careers under the guidance of Alladi Ramakrishnan and had contributed largely to the development of $L$-matrix theory. In my Ph.D. thesis [3] I had studied certain group theoretical aspects of generalized Clifford algebras (GCAs) and their physical applications. After my Ph.D. work also, I have applied the elements of GCAs in studies of certain problems in quantum mechanics, and quantum groups. Here, I would like to outline some aspects of GCAs and their applications essentially based on my work.

A generalized Clifford algebra (GCA) can be presented, in general, as an algebra having a basis with generators \( \{ e_j | j = 1, 2, \ldots, n \} \) satisfying the relations

\[
\begin{align*}
\omega_{jk} & = e_{jk} e_k e_j, \quad \omega_{jk} e_l = e_l \omega_{jk}, \quad \omega_{jk} \omega_{lm} = \omega_{lm} \omega_{jk}, \\
N_j & = 1, \quad \omega_j^N_j = \omega_j^N = 1, \quad \forall \ j, k, l, m = 1, 2, \ldots, n. \quad (1.1)
\end{align*}
\]

In any irreducible matrix representation, relevant for physical applications, one will have

\[
\omega_{jk} = \omega_{kj}^{-1} = e^{2\pi i \nu_{jk}/N_{jk}}, \quad N_{jk} = \gcd (N_j, N_k), \quad j, k = 1, 2, \ldots, n \quad (1.2)
\]

where \( \nu_{jk} \) are integers. Consequently, one can write

\[
\omega_{jk} = e^{2\pi i t_{jk}/\hat{N}}, \quad t_{kj} = -t_{jk}, \quad \hat{N} = \text{l.c.m} [N_{jk}], \quad j, k = 1, 2, \ldots, n. \quad (1.3)
\]
Thus, any GCA can be characterized by an integer \( \hat{N} \) and an antisymmetric integer matrix
\[
T = \begin{pmatrix}
0 & t_{12} & t_{13} & \ldots & t_{1n} \\
-t_{12} & 0 & t_{23} & \ldots & t_{2n} \\
-t_{13} & -t_{23} & 0 & \ldots & t_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-t_{1n} & -t_{2n} & -t_{3n} & \ldots & 0
\end{pmatrix}.
\]
(1.4)

In the following we shall study the representation theory of GCAs and physical applications of some special cases of these algebras.

2 Projective representations of finite abelian groups and GCAs

GCAs arise in the study of projective, or ray, representations of finite abelian groups. Let us consider the finite abelian group \( G \cong \mathbb{Z}_{N_1} \otimes \mathbb{Z}_{N_2} \otimes \ldots \otimes \mathbb{Z}_{N_n} \) with \( \{e_1^{m_1}e_2^{m_2}\ldots e_n^{m_n}\} \) as its generic element where the generators \( \{c_j\} \) satisfy the relations
\[
c_jc_k = c_kc_j, \quad c_j^{N_j} = 1, \quad j = 1, 2, \ldots, n.
\]
(2.1)

A projective representation \( D(G) \) of a group \( G \) is defined as
\[
D(g_j)D(g_k) = \varphi(g_j, g_k)D(g_jg_k), \quad \varphi(g_j, g_k) \in \mathbb{C}, \quad \forall g_j, g_k \in G,
\]
(2.2)

where the given factor set \( \{\varphi(g_j, g_k)\} \) is such that
\[
\varphi(g_j, g_k)\varphi(g_jg_k, g_l) = \varphi(g_j, g_kg_l)\varphi(g_k, g_l), \quad \forall g_j, g_k, g_l \in G,
\]
(2.3)

and
\[
\varphi(E, g_j) = \varphi(g_j, E) = 1, \quad \forall g_j \in G,
\]
(2.4)

with \( E \) as the identity element of \( G \). For an abelian group, equation (2.2) implies
\[
D(g_j)D(g_k) = \varphi(g_j, g_k)D(g_jg_k) = \varphi(g_j, g_k)D(g_kg_j) = \frac{\varphi(g_j, g_k)}{\varphi(g_k, g_j)}D(g_k)D(g_j), \quad \forall g_j, g_k \in G,
\]
(2.5)
or,
\[
D(g_j)D(g_k) = \Omega_\phi(g_j, g_k)D(g_k)D(g_j),
\]
with
\[
\Omega_\phi(g_j, g_k) = \frac{\varphi(g_j, g_k)}{\varphi(g_k, g_j)}, \quad \forall \ g_j, g_k \in G.
\]

Using (2.2) it is easy to see that we can write
\[
D \left( \prod_{j=1}^n c_{mj}^{e_j} \right) = \phi \left( \prod_{j=1}^n c_{mj}^{e_j} \right) \left\{ \prod_{j=1}^n D(c_j)^{m_j} \right\},
\]
with
\[
\phi \left( \prod_{j=1}^n c_{mj}^{e_j} \right) = \prod_{j=1}^n \prod_{p_j=1}^{m_{mj}} \varphi \left( c_j, \prod_{i=0}^{N_j-1} c_{j+i}^{m_{j-1}} \right)^{-1}.
\]

From this it follows that
\[
D \left( c_j^{N_j} \right) = \phi \left( c_j^{N_j} \right) D(c_j)^{N_j} = I, \quad \forall \ j = 1, 2, \ldots, n.,
\]
where
\[
\phi \left( c_j^{N_j} \right) = \prod_{p_j=1}^{N_j} \varphi \left( c_j, c_{j+p_j}^{N_j} \right)^{-1}.
\]

Let us now define
\[
e_j = \phi \left( c_j^{N_j} \right)^{1/N_j} D(c_j), \quad \forall \ j = 1, 2, \ldots, n.
\]

Then, it is found that the required representations satisfying (2.2-7), for the given factor set, are immediately obtained from (2.8-2.9) once the ordinary representations of \(\{e_j| j = 1, 2, \ldots, n\}\) are found such that
\[
e_j e_k = \omega^{(\varphi)}_{jk} e_k e_j, \quad \omega^{(\varphi)}_{jk} e_l = e_l \omega^{(\varphi)}_{jk}, \quad \omega^{(\varphi)}_{jk} \omega^{(\varphi)}_{lm} = \omega^{(\varphi)}_{lm} \omega^{(\varphi)}_{jk},
\]
\[
e_j^{N_j} = 1, \quad \left( \omega^{(\varphi)}_{jk} \right)^{N_j} = \left( \omega^{(\varphi)}_{jk} \right)^{N_k} = 1, \quad \text{with} \ \omega^{(\varphi)}_{jk} = \Omega_\phi(c_j, c_k),
\]
\[
\forall \ j, k, l, m = 1, 2, \ldots, n.
\]

Comparing (2.13) with (1.1) it is clear that the problem of finding the projective representations of any finite abelian group for any given factor set reduces to the problem of finding the ordinary representations of a generalized Clifford algebra defined by (1.1).
3 Representations of GCAs

Let us now consider a GCA associated with a specific antisymmetric integer matrix $T$ as in (1.4) and an integer $\hat{N}$. The $T$-matrix can be related to its skew-normal form $\tilde{T}$ by a transformation as follows:

$$\mathcal{T} = \begin{pmatrix} 0 & t_1 \\ -t_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & t_s \\ -t_s & 0 \end{pmatrix} \oplus O_{n-2s},$$

$$T = U \tilde{T} \bar{U} (\pm \text{mod} \hat{N}),$$

where $O_{n-2s}$ is an $(n-2s) \times (n-2s)$ null matrix, $U = [u_{jk}]$ is a unimodular integer matrix with $|u_{jk}| \leq \hat{N}$, and $\bar{U}$ is the transpose of $U$. For any given antisymmetric integer matrix $T$ it is possible to get the skew normal form $\mathcal{T}$ and the corresponding $U$-matrix explicitly by a systematic procedure (see, e.g., [4]). Now, let $\{\epsilon_j|j = 1, 2, \ldots, n\}$ be a set of elements satisfying the commutation relations

$$\epsilon_{2j-1}\epsilon_{2j} = e^{2\pi i t_j/\hat{N}} \epsilon_{2j}\epsilon_{2j-1}, \quad j = 1, 2, \ldots, s,$$

$$\epsilon_k\epsilon_l = \epsilon_l\epsilon_k \quad \text{otherwise.}$$

(3.2)

It is clear that this set of relations generate a GCA corresponding to $\mathcal{T}$ as its $T$-matrix. It is straightforward to verify that if we construct $\{\epsilon_j|j = 1, 2, \ldots, n\}$ from $\{\epsilon_j|j = 1, 2, \ldots, n\}$ through a product transformation [3, 5]

$$\epsilon_j = \mu_j \epsilon_1^{u_{j1}} \epsilon_2^{u_{j2}} \cdots \epsilon_n^{u_{jn}}, \quad \forall j = 1, 2, \ldots, n,$$

(3.3)

where $[u_{jk}] = U$ and $\{\mu_j|j = 1, 2, \ldots, n\}$ are complex numbers, then, in view of (3.1),

$$\epsilon_j\epsilon_k = e^{2\pi i t_{jk}/\hat{N}} \epsilon_k\epsilon_j, \quad \forall j, k = 1, 2, \ldots, n.,$$

(3.4)

as required in (1.1)-(1.3); the complex numbers $\{\mu_j\}$ are normalization factors which are to be chosen such that

$$\epsilon_j^{\hat{N}_j} = 1, \quad \forall j = 1, 2, \ldots, n.$$

(3.5)

Now, let the matrix representations of $\{\epsilon_j|j = 1, 2, \ldots, 2s\}$ be given by

$$\epsilon_1 = I \otimes I \otimes I \otimes \cdots \otimes I \otimes A_1,$$

$$\epsilon_2 = I \otimes I \otimes I \otimes \cdots \otimes I \otimes B_1,$$
\[ \epsilon_3 = I \otimes I \otimes I \otimes \cdots \otimes A_2 \otimes I, \]
\[ \epsilon_4 = I \otimes I \otimes I \otimes \cdots \otimes B_2 \otimes I, \]
\[ \vdots \]
\[ \epsilon_{2s-1} = A_s \otimes I \otimes I \otimes \cdots \otimes I \otimes I, \]
\[ \epsilon_{2s} = B_s \otimes I \otimes I \otimes \cdots \otimes I \otimes I, \]

where

\[ A_j B_j = \omega_j \tau_j A_j, \]

with \( \omega_j = e^{2\pi i/N_j}, \) \( N_j = \hat{N}/(\text{g.c.d.}(t_j, \hat{N})), \)

\[ \tau_j = t_j/(\text{g.c.d}(t_j, \hat{N})), \quad j = 1, 2, \ldots, s, \quad (3.7) \]

and \( I_s \) are identity matrices of appropriate dimensions. Since \( \{\epsilon_k|k = 2s + 1, 2s + 2, \ldots, n\} \) commute among themselves and also with all other \( \{\epsilon_j|j = 1, 2, \ldots, 2s\} \) they are represented by unimodular complex numbers which can be absorbed in the normalization factors \( \{\mu_j\} \) in (3.3). This shows that if the matrix representations of all \( A_s \) and \( B_s \) satisfying (3.7) are known, then the problem of representation of the given GCA is solved. Explicitly, one has, apart from multiplicative normalizing phase factors,

\[ e_j \sim A_s^{u_j(2s-1)} B_s^{u_j(2s)} \otimes A_s^{u_j(2s-3)} B_s^{u_j(2s-2)} \otimes \cdots \otimes A_1^{u_j} B_1^{u_j}, \quad \forall \ j = 1, 2, \ldots, n. \quad (3.8) \]

Note that \( \omega_j \) in (3.7) are primitive roots of unity. Thus the representation theory of any GCA depends essentially on the central relation

\[ AB = \omega BA, \quad (3.9) \]

where \( \omega \) is a nontrivial primitive root of unity. If \( \omega \) is a primitive \( N \)th root of unity then the normalization relations for \( A \) and \( B \) can be

\[ A^{jN} = I, \quad B^{kN} = I, \quad \text{where} \ j, k = 1, 2, \ldots. \quad (3.10) \]

The central relation (3.9) determines the representation of \( A \) and \( B \) uniquely up to multiplicative phase factors and the normalization relation (3.10) fixes these phase factors. For more details on projective representations of finite abelian groups and their relation to GCAs, and other different approaches to GCAs, see [6]-[13].
4 The Clifford algebra

Hamilton’s quaternion, generalizing the complex number, is given by
\[ q = q_0 + q_1 i + q_2 j + q_3 k, \] (4.1)
where \( \{q_0, q_1, q_2, q_3\} \) are real numbers, 1 is the identity unit, and \( \{i, j, k\} \) are imaginary units such that
\[
\begin{align*}
ij &= -ji, \quad jk = -kj, \quad ki = -ik, \\
i^2 &= j^2 = k^2 = -1,
\end{align*}
\] (4.2)
and
\[
\begin{align*}
ij &= k, \quad jk = i, \quad ki = j.
\end{align*}
\] (4.3)

It should be noted that the relations in (4.3) are not independent of the commutation and normalization relations (4.2); to see this, observe that \( ijk \) commutes with each one of the imaginary units \( \{i, j, k\} \) and hence \( ijk \sim 1 \).

The ‘geometric algebra’ of Clifford [14] has the generating relations
\[
\begin{align*}
\iota_j \iota_k &= -\iota_k \iota_j, \quad \text{for } j \neq k \\
\iota_j^2 &= -1, \quad \forall j, k = 1, 2, \ldots, n,
\end{align*}
\] (4.4)
obtained by generalizing (4.2). This is what has become the Clifford algebra defined by the generating relations
\[
\begin{align*}
e_j e_k &= -e_k e_j, \quad \text{for } j \neq k, \\
e_j^2 &= 1, \quad \forall j, k = 1, 2, \ldots, n,
\end{align*}
\] (4.5)
which differ from (4.4) only in the normalization conditions, and evolved into the GCA (1.1). Thus, the Clifford algebra (4.5) corresponds to (1.1) with the choice
\[
\omega_{jk} = -1, \quad N_j = 2, \quad \forall j, k = 1, 2, \ldots, n,
\] (4.6)
associated with the \( T \)-matrix
\[
T = \begin{pmatrix}
0 & 1 & 1 & \ldots & 1 \\
-1 & 0 & 1 & \ldots & 1 \\
-1 & -1 & 0 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \ldots & 0
\end{pmatrix}.
\] (4.7)
and

\[ \hat{N} = 2. \]  
(4.8)

The corresponding skew normal form is

\[ \mathbf{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]  
(4.9)

when \( n = 2m \). When \( n = 2m + 1 \),

\[ \mathbf{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus 0. \]  
(4.10)

In this case the \( U \)-matrices are

\[
U = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 \\
0 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1
\end{pmatrix},
\]

for \( n = 2m \),  
(4.11)

and

\[
U = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & 0 \\
0 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & 0 \\
-1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & 1
\end{pmatrix},
\]

for \( n = 2m + 1 \),  
(4.12)
such that

\[ T = U \tilde{T} U \left( \pm \text{mod}.2 \right). \]  

(4.13)

Now, equation (3.2) becomes in this case, for both \( n = 2m \) and \( n = 2m + 1 \),

\[ \epsilon_{2j-1} \epsilon_{2j} = -\epsilon_{2j} \epsilon_{2j-1}, \quad j = 1, 2, \ldots, m, \]
\[ \epsilon_k \epsilon_l = \epsilon_l \epsilon_k, \quad \text{otherwise}, \]  

(4.14)

with the matrix representations

\[ \epsilon_1 = I \otimes I \otimes I \otimes \cdots \otimes I \otimes A_1, \]
\[ \epsilon_2 = I \otimes I \otimes I \otimes \cdots \otimes I \otimes B_1, \]
\[ \epsilon_3 = I \otimes I \otimes I \otimes \cdots \otimes A_2 \otimes I, \]
\[ \epsilon_4 = I \otimes I \otimes I \otimes \cdots \otimes B_2 \otimes I, \]
\[ \vdots \]
\[ \epsilon_{2m-1} = A_m \otimes I \otimes I \otimes \cdots \otimes I \otimes I, \]
\[ \epsilon_{2m} = B_m \otimes I \otimes I \otimes \cdots \otimes I \otimes I, \]  

(4.15)

where

\[ A_j = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_j = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \forall \ j = 1, 2, \ldots, m. \]  

(4.16)

In the case of \( n = 2m + 1 \), since \( \epsilon_{2m+1} \) commutes with all other \( \epsilon_j \)'s it can be just taken to be 1. The matrices \( \sigma_1 \) and \( \sigma_3 \) are the well known first and the third Pauli matrices, respectively, and the second Pauli matrix is given by

\[ \sigma_2 = i \sigma_1 \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \]  

(4.17)

Then, in view of (3.8) and (4.11,4.12), the required representations of (4.5) are given in terms of the Pauli matrices by

\[ \epsilon_1 = \sigma_1 \otimes I \otimes \cdots \otimes I \otimes I, \]
\[ \epsilon_2 = \sigma_3 \otimes I \otimes \cdots \otimes I \otimes I, \]
\[ e_3 = \sigma_2 \otimes \sigma_1 \otimes I \otimes \cdots \otimes I \otimes I, \]
\[ e_4 = \sigma_2 \otimes \sigma_3 \otimes I \otimes \cdots \otimes I \otimes I \]
\[ \vdots \]
\[ e_{2m-1} = \sigma_2 \otimes \sigma_2 \otimes \cdots \otimes \sigma_2 \otimes \sigma_1, \]
\[ e_{2m} = \sigma_2 \otimes \sigma_2 \otimes \cdots \otimes \sigma_2 \otimes \sigma_3, \]
\[ e_{2m+1} = \sigma_2 \otimes \sigma_2 \otimes \cdots \sigma_2 \otimes \sigma_2. \] (4.18)

Note that this representation is Hermitian and unitary. One can show that this is an irreducible representation. Also it should be noted that the above representation matrices are defined only up to multiplication by \( \pm 1 \) since \( e_j^2 = 1 \) for all \( j \).

Let us now write down the generators of the first four Clifford algebras:

\[
C^{(2)} : e_1^{(2)} = \sigma_1, \quad e_2^{(2)} = \sigma_3, \\
C^{(3)} : e_1^{(3)} = \sigma_1, \quad e_2^{(3)} = \sigma_3, \quad e_3^{(3)} = \sigma_2, \\
C^{(4)} : e_1^{(4)} = \sigma_1 \otimes I, \quad e_2^{(4)} = \sigma_3 \otimes I, \quad e_3^{(4)} = \sigma_2 \otimes \sigma_1, \quad e_4^{(4)} = \sigma_2 \otimes \sigma_3, \\
C^{(5)} : e_1^{(5)} = \sigma_1 \otimes I, \quad e_2^{(5)} = \sigma_3 \otimes I, \quad e_3^{(5)} = \sigma_2 \otimes \sigma_1, \quad e_4^{(5)} = \sigma_2 \otimes \sigma_3, \quad e_5^{(5)} = \sigma_2 \otimes \sigma_2. \tag{4.19}
\]

where the superscript indicates the number of generators in the corresponding algebra. The dimension of the irreducible representation of the Clifford algebra with \( 2m \), or \( 2m + 1 \), generators is \( 2^m \). One can show that for the algebra with an even number of generators there is only one unique irreducible representation up to equivalence. In the case of the algebra with an odd number of generators there are two inequivalent irreducible representations where the other representation is given by multiplying all the matrices of the first representation by \(-1\). These statements form Pauli’s theorem on Clifford algebra.

An obvious irreducible representation of the identity and the three imaginary units of Hamilton’s quaternion algebra (4.2,4.3) is given by

\[ 1 = I, \quad i = -i\sigma_1, \quad j = -i\sigma_3, \quad k = i\sigma_2. \tag{4.20} \]

From the above it is clear that, as Clifford remarked [14], the geometric algebra, or the Clifford algebra, is a compound of quaternion algebras the units
of which are commuting with one another. Actually, the equations (3.2) and (3.3) correspond precisely to Clifford’s original construction of geometric algebra starting with commuting quaternion algebras; matrix representations and realization of commuting quaternion algebras in terms of direct products did not exist at that time. Later, obviously unaware of Clifford’s work, Dirac [15] used the same procedure to construct his four matrices \( \{\alpha_x, \alpha_y, \alpha_z, \beta\} \), building blocks of his relativistic theory of electron and other spin-1/2 particles, starting with the three Pauli matrices \( \{\sigma_1, \sigma_2, \sigma_3\} \). The Dirac matrices are given by

\[
\alpha_x = \sigma_1 \otimes \sigma_1, \quad \alpha_y = \sigma_1 \otimes \sigma_2, \quad \alpha_z = \sigma_1 \otimes \sigma_3, \quad \beta = \sigma_3 \otimes I, \quad (4.21)
\]

which can be shown to be equivalent to the representation of \( C(4) \) given above; as already mentioned, \( C(4) \) has only one inequivalent irreducible representation. Clifford algebra is basic to the theory of spinors, theory of fermion fields, Onsager’s solution of the two dimensional Ising model, etc. For detailed accounts of Clifford algebra and its various physical applications see, e.g., [16]-[18].

5 Alladi Ramakrishnan’s L-matrix theory and \( \sigma \)-operation

Representation theory of Clifford algebra has been expressed by Alladi Ramakrishnan [1] in a very nice framework called the L-matrix theory. Let

\[
L^{(2m+1)}(\lambda) = \sum_{j=1}^{2m+1} \lambda_j e_j^{(2m+1)},
\]

called an L-matrix, be associated with a \((2m+1)\)-dimensional vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2m+1}) \). It follows that

\[
(L^{(2m+1)}(\lambda))^2 = \left( \sum_{j=1}^{2m+1} \lambda_j^2 \right) I = ||\lambda||^2 I,
\]

where \( I \) is the \( 2^m \times 2^m \) identity matrix. Thus, \( L^2 \) represents the square of the norm, or the length, of the vector \( \lambda \). In other words, \( L \) is a square root of \( \sum \lambda_j^2 \) linear in \( \{\lambda_j\} \).
From (4.19) observe that
\[
e^{(5)}_1 = e^{(3)}_1 \otimes I, \quad e^{(5)}_2 = e^{(3)}_2 \otimes I, \quad e^{(5)}_3 = e^{(3)}_3 \otimes e^{(3)}_1, \quad e^{(5)}_4 = e^{(3)}_3 \otimes e^{(3)}_2, \quad e^{(5)}_5 = e^{(3)}_3 \otimes e^{(3)}_3.
\]
(5.3)

Thus, one can write
\[
L^{(5)}(\lambda) = \sum_{j=1}^{5} \lambda_j e^{(5)}_j = e^{(3)}_1 \otimes \lambda_1 I + e^{(3)}_2 \otimes \lambda_2 I + e^{(3)}_3 \otimes (\lambda_3 e^{(3)}_1 + \lambda_4 e^{(3)}_2 + \lambda_5 e^{(3)}_3),
\]
(5.4)
i.e., \(L^{(5)}\) can be obtained from \(L^{(3)}\) by replacing \(\lambda_1, \lambda_2, \) and \(\lambda_3\) by \(\lambda_1 I, \lambda_2 I,\) and \(L^{(3)}(\lambda_3, \lambda_4, \lambda_5)\), respectively. From (4.18) it is straightforward to see that this procedure generalizes: an \(L^{(2m+3)}\) can be obtained from an \(L^{(2m+1)}\) by replacing \((\lambda_1, \lambda_2, \ldots, \lambda_{2m})\), respectively, by \((\lambda_1 I, \lambda_2 I, \ldots, \lambda_{2m} I)\), and \(\lambda_{2m+1}\) by \(L^{(3)}(\lambda_{2m+1}, \lambda_{2m+2}, \lambda_{2m+3})\). This procedure is called \(\sigma\)-operation by Alladi Ramakrishnan. It can be shown that the induced representation technique of group theory takes this form in the context of Clifford algebra [19]. Actually, in this procedure any one of the parameters of \(L^{(2m+1)}\) can be replaced by an \(L^{(3)}\) and the remaining parameters \(\{\lambda_j\}\) can be replaced, respectively, by \(\{\lambda_j I\}\) with suitable relabelling. As we shall see below this \(\sigma\)-operation generalizes to the case of GCAs with ordered \(\omega\)-commutation relations.

Another interesting result of Alladi Ramakrishnan is on the diagonalization of an \(L\)-matrix. An \(L^{(2m+1)}\)-matrix of dimension \(2^m\) obeys
\[
(L^{(2m+1)})^2 = \sum_{j=1}^{2m+1} \lambda_j^2 I = \Lambda^2 I,
\]
(5.5)
and hence has \((\Lambda, -\Lambda)\) as its eigenvalues each being \(2^{m-1}\)-fold degenerate. In general, let us call the matrix \(e^{(2m+1)}_2\), or \(e^{(2m)}_2\), as \(\beta\):
\[
\beta = \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix},
\]
(5.6)
where \(I\) is the \(2^{m-1}\)-dimensional identity matrix. Thus, the diagonal form of \(L\) is \(\Lambda \beta\). Then, from the relation
\[
L(L + \Lambda \beta) = \Lambda^2 I + L \Lambda \beta = (L + \Lambda \beta) \Lambda \beta,
\]
(5.7)
it follows that \((L + \Lambda \beta)\) is the matrix diagonalizing \(L\) and the columns of \((L + \Lambda \beta)\) are the eigenvectors of \(L\). Note that an \(L^{(2m)}\)-matrix, of dimension \(2^m\), can be treated as an \(L^{(2m+1)}\)-matrix with one of the \(\lambda\)s as zero.

Let us now take a Hermitian \(L(\Lambda)\)-matrix where all the \(\lambda\)-parameters are real. Since \(\beta = e_2\) anticommutes with all the other \(e_j\)s we get

\[
(L + \Lambda \beta)^2 = 2\Lambda^2 I + \Lambda (L\beta + \beta L) = 2\Lambda (\Lambda + \lambda_2) I. \tag{5.8}
\]

Hence

\[
U = \frac{L + \Lambda \beta}{\sqrt{2\Lambda (\Lambda + \lambda_2)}} \tag{5.9}
\]

is Hermitian and unitary \((U = U^\dagger = U^{-1})\) and is such that

\[
U^{-1}LU = \Lambda \beta. \tag{5.10}
\]

Thus, the columns of \(U\) are normalized eigenvectors of the Hermitian \(L\). This result has been applied [1] to solve in a very simple manner Dirac’s relativistic wave equation [15],

\[
i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = \left[-i\hbar c \left(\alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} + \alpha_z \frac{\partial}{\partial z}\right) + mc^2 \beta\right] \psi(\vec{r},t), \tag{5.11}
\]

where \(\psi(\vec{r},t)\) is the 4-component spinor associated with the free spin-1/2 particle.

6 Dirac’s positive-energy relativistic wave equation

Students of Alladi Ramakrishnan got excellent training as professional scientists. He emphasized that the students should master any topic of research by studying the works of the leaders in the field and should communicate with their peers whenever necessary. In this connection, I would like to recall proudly an incident.
Following a suggestion of Santhanam, my fellow junior student Dutt and I started studying a paper of Dirac [20] in which he had proposed a positive-energy relativistic wave equation:

\[
\frac{i\hbar}{\partial t} \hat{\psi}(\vec{r}, t; q_1, q_2) = \left[-i\hbar c \left( \alpha_x' \frac{\partial}{\partial x} + \alpha_y' \frac{\partial}{\partial y} + \alpha_z' \frac{\partial}{\partial z} \right) + mc^2\beta' \right] \hat{\psi}(\vec{r}, t; q_1, q_2),
\]

(6.1)

\[\hat{\psi}\] being a 4-component column matrix with elements \((\hat{q}_1\psi, \hat{q}_2\psi, \hat{q}_3\psi, \hat{q}_4\psi)\) where

\[\begin{align*}
[\hat{q}_j, \hat{q}_k] &= -\beta'_{jk}, \quad j, k = 1, 2, 3, 4, \\
\beta' &= \sigma_2 \otimes I, \quad \alpha_x' = -\sigma_1 \otimes \sigma_3, \quad \alpha_y' = \sigma_1 \otimes \sigma_1, \quad \alpha_z' = \sigma_3 \otimes I.
\end{align*}\]

(6.2)

Unlike the standard relativistic wave equation for the electron (5.11) which has both positive and negative (antiparticle) energy solutions, the new Dirac equation (6.1) has only positive energy solutions. Further, more interestingly, this positive-energy particle would not interact with an electromagnetic field. Around November 1974, Dutt and I stumbled upon an equation which had only negative-energy solutions. Our negative-energy relativistic wave equation was exactly the same as Dirac’s positive-energy equation (6.1) except only for a slight change in the commutation relations of the internal variables \((\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_4)\) in the equation; instead of (6.2), we took

\[\begin{align*}
[\hat{q}_j, \hat{q}_k] &= \beta'_{jk}, \quad j, k = 1, 2, 3, 4.
\end{align*}\]

(6.4)

When I told Alladi Ramakrishnan about this he told us that we could not meddle with Dirac’s work and keep quiet. He suggested that I should write to Dirac and get his opinion on our work. I wrote to Dirac who was in The Florida State University at that time. I received a letter from him within a month! His reply was: “Dear Jagannathan, The equation you propose would correctly describe a particle with only negative-energy states. It would be the correct counterpart of the positive-energy equation, but of course it would not...
have any physical application. Yours sincerely, P. A. M. Dirac.” Immediately, Alladi Ramakrishnan forwarded our paper for rapid publication [21].

So far, no one has found any application for Dirac’s positive-energy equation. Attempts to modify it so that these positive-energy particles could interact with electromagnetic field have not succeeded. May be, these positive-energy Dirac particles and their negative-energy antiparticles constitute the dark matter of our universe.

7 GCAs with ordered ω-commutation relations

We shall now consider a GCA (1.1) with ordered ω-commutation relations, i.e.,

\[ e_j e_k = \omega e_k e_j, \quad \omega = e^{2\pi i/N}, \quad \forall \ j < k, \]
\[ e_j^N = 1, \quad j, k = 1, 2, \ldots, n. \quad (7.1) \]

The associated \( T \)-matrix has elements

\[ t_{jk} = \begin{cases} 
1, & \text{for } j < k, \\
0, & \text{for } j = k, \\
-1, & \text{for } j > k,
\end{cases} \quad (7.2) \]

and \( \tilde{N} = N \). This is exactly same as for the Clifford algebra except for the value of \( \tilde{N} \). So, the treatment of representation theory of this GCA is along the same lines as for the Clifford algebra: \( T \) matrix is the same as in (4.7) for any \( n \) and \( T \) and \( U \) matrices are the same as in (4.9) and (4.11) for \( n = 2m \) and (4.10) and (4.12) for \( n = 2m + 1 \), respectively. The only difference is that in the case of Clifford algebra \( A_j^{-1} = A_j \) and \( B_j^{-1} = B_j \) for any \( j \) where as now \( A_j^{-1} = A_j^{N-1} \) and \( B_j^{-1} = B_j^{N-1} \) for any \( j \). Thus, in view of (3.8) and (4.11,4.12), the required representations of (7.1) are given by

\[ e_1 = A \otimes I \otimes \cdots \otimes I \otimes I, \]
\[ e_2 = B \otimes I \otimes \cdots \otimes I \otimes I, \]
\[ e_3 = \mu A^{-1}B \otimes A \otimes I \otimes \cdots \otimes I \otimes I, \]
\[ e_4 = \mu A^{-1}B \otimes B \otimes I \otimes \cdots \otimes I \otimes I, \]
\[ e_5 = \mu^2 A^{-1}B \otimes A^{-1}B \otimes A \otimes I \otimes \cdots \otimes I, \]
\[ e_6 = \mu^2 A^{-1} B \otimes A^{-1} B \otimes B \otimes I \otimes \cdots \otimes I, \]
\[
\vdots \]
\[ e_{2m-1} = \mu^{m-1} A^{-1} B \otimes A^{-1} B \otimes \cdots \otimes A^{-1} B \otimes A, \]
\[ e_{2m} = \mu^{m-1} A^{-1} B \otimes A^{-1} B \otimes \cdots \otimes A^{-1} B \otimes B, \]
\[ e_{2m+1} = \mu^m A^{-1} B \otimes A^{-1} B \otimes \cdots A^{-1} B \otimes A^{-1} B, \]
(7.3)
where \( \mu = \omega^{(N+1)/2} \) and
\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]
(7.4)
\[
B = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \omega & 0 & \cdots & 0 \\
0 & 0 & \omega^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega^{N-1}
\end{pmatrix},
\]
(7.5)
\( N \times N \) unitary matrices, obeying
\[
AB = \omega BA, \quad A^N = B^N = I.
\]
(7.6)
The matrices \( A \) and \( B \) in (7.4) and (7.5), respectively, provide the only irreducible representation for the relation (7.6) [22]. It can also be shown that the GCA \( C_N^{(n)} \) defined by (7.1) has only one \( N^m \)-dimensional irreducible representation, as given by (7.3) without \( e_{2m+1} \), when \( n = 2m \) and there are \( N \) inequivalent irreducible representations of dimension \( N^m \) (differing from (7.3) only by multiplications by powers of \( \omega \)) when \( n = 2m + 1 \) (see, e.g., [23, 24]). This is the generalization of Pauli’s theorem for the GCA (7.1). When \( N = 2 \) it is seen that \( A = \sigma_1, B = \sigma_3 \), and the representation (7.3) becomes the representation (4.18) of the Clifford algebra.

From the structure of the representation (7.3) it is clear that the \( \sigma \)-operation procedure should work in this case also. Let the \( n \)-dimensional vector \( \mathbf{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be associated with an \( \mathcal{L} \)-matrix defined by
\[
\mathcal{L}^{(n)} = \sum_{j=1}^{n} \lambda_j e_j^{(n)}
\]
(7.7)
Then, from the commutation relations (7.1) it follows that

\[(L^{(n)})^N = \left(\sum_{j=1}^{n} \lambda_j^N\right)I.\]  

Thus, the \(N\)-th root of \(\sum_{j=1}^{n} \lambda_j^N\) is given by \(L^{(n)}\) which is linear in \(\lambda_j\)s. This fact helps linearize certain \(N\)-th order partial differential operators using the GCA [6] similar to the way Clifford algebra helps linearize certain second order partial differential operators (e.g., Dirac’s linearization of \(\hat{H}^2 = -\hbar^2 c^2 \nabla^2 + m^2 c^4\) to get his relativistic Hamiltonian \(\hat{H} = -i\hbar c (\alpha_x \partial/\partial x + \alpha_y \partial/\partial y + \alpha_z \partial/\partial z) + mc^2 \beta\)). Now, it can be easily seen [1] that \(L^{(2m+3)}\) is obtained from \(L^{(2m+1)}\) by the \(\sigma\)-operation: replace \((\lambda_1, \lambda_2, \ldots, \lambda_{2m})\) in \(L^{(2m+1)}\) by \((\lambda_1 I, \lambda_2 I, \ldots, \lambda_{2m} I)\), respectively, where \(I\) is the \(N\)-dimensional identity matrix, and \(\lambda_{2m+1}\) by \(L^{(3)}(\lambda_{2m+1}, \lambda_{2m+2}, \lambda_{2m+3})\).

From the above it is clear that the matrices \(A\) and \(B\) in (7.4) and (7.5), respectively, obeying the relation (7.6), play a central role in the study of GCAs. If we want to have two matrices \(A_j\) and \(B_j\) obeying

\[A_j B_j = e^{2\pi ij/N} B_j A_j, \quad \text{g.c.d}(j, N) = 1, \]  

then, \(A_j\) is same as \(A\) in (7.4) and \(B_j\) is given by \(B\) in (7.5) with \(\omega\) replaced by \(\omega^j\), up to multiplicative factors which are to be determined by the required normalization relations like (3.10). In the following we shall outline some of the physical applications of the matrices \(A\) and \(B\).

One approach to study the representation theory of the GCA with ordered \(\omega\)-commutation relations (7.1) is to study the vector, or the ordinary, representations of the group

\[G : \{\omega^j e_1^{j_1} e_2^{j_2} \ldots e_n^{j_n} | j_0, j_1, j_2, \ldots j_n = 0, 1, 2, \ldots N - 1\}.\]  

This group has been called a generalized Clifford group (GCG) and the study of its representation theory involves interesting number theoretical aspects ([23],[24]). Particularly, by studying the representations of the lowest order GCG generated by \(A\), \(B\) and \(\omega\) one can show that \(A\) and \(B\) have only one irreducible representation as given by (7.4) and (7.5). Study of spin systems defined on a GCG also involves very interesting number theoretical problems [25]. Alladi Ramakrishnan and collaborators used the \(L\)-matrix theory for studying several topics like idempotent matrices, special unitary
groups arising in particle physics, algebras derived from polynomial conditions, Duffin-Kemmer-Petiau algebra, and para-Fermi algebra (for details see [1]). They studied essentially the GCA with ordered \( \omega \)-commutation relations (7.1). The more general GCAs (1.1) were studied later in ([3], [5], [13], [23], [24]). In gauge field theories Wilson operators and 't Hooft operators satisfy commutation relations of the form in (7.6) and the corresponding algebra is often called the 't Hooft-Weyl algebra (see, e.g., [26]). For the various other physical applications of GCAs see, e.g., ([27], [28]).

8 Weyl-Schwinger unitary basis for matrix algebra and Alladi Ramakrishnan’s matrix decomposition theorem

Heisenberg’s canonical commutation relation between position and momentum operators of a particle, the basis of quantum mechanics, is

\[
[\hat{q}, \hat{p}] = i\hbar. \tag{8.1}
\]

Weyl [22] wrote it in exponential form as

\[
e^{i\eta \hat{q}/\hbar} e^{i\xi \hat{p}/\hbar} = e^{i\xi \eta/\hbar} e^{i\xi \hat{p}/\hbar} e^{i\eta \hat{q}/\hbar}, \tag{8.2}
\]

where the parameters \( \xi \) and \( \eta \) are real numbers, and studied its representation as the large \( N \) limit of the relation:

\[
AB = \omega BA, \quad \omega = e^{2\pi i/N}. \tag{8.3}
\]

Note that the Heisenberg-Weyl commutation relation (8.2) takes the form (8.3) when \( \xi \eta/\hbar = 2\pi/N \). Weyl established that the relation (8.3), subject to the normalization condition

\[
A^N = B^N = I, \tag{8.4}
\]

has only one irreducible representation as given in (7.4) and (7.5). Analysing the large \( N \) limits of \( A \) and \( B \), he showed that the relation (8.2), or equivalently the Heisenberg commutation relation (8.1), has the unique (upto equivalence) irreducible representation given by the Schrödinger representation

\[
\hat{q}\psi(q) = q\psi(q), \quad \hat{p}\psi(q) = -i\hbar \frac{d}{dq}\psi(q), \quad \text{for any } \psi(q). \tag{8.5}
\]
This result, or the Stone-von Neumann theorem obtained later by a more rigorous approach, is of fundamental importance for physics since it establishes the uniqueness of quantum mechanics. Thus, Weyl viewed quantum kinematics as an irreducible Abelian group of unitary ray rotations in system space.

Following the above approach to quantum kinematics Weyl gave his correspondence rule for obtaining the quantum operator \( \hat{f}(\hat{q}, \hat{p}) \) for a classical observable \( f(q, p) \):

\[
\hat{f}(\hat{q}, \hat{p}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi d\eta \, g(\xi, \eta) e^{i(\xi \hat{q} + \eta \hat{p})},
\]

\[
g(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq dp \, f(q, p) e^{-i(\xi q + \eta p)}. \tag{8.6}
\]

The fact that the set of \( N^2 \) linearly independent unitary matrices \( \{A^k B^l | k, l = 0, 1, 2, \ldots, (N-1)\} \) forms a basis for the \( N \times N \)-matrix algebra is implicit in this suggestion that any quantum operator corresponding to a classical observable can be written as a linear combination of the unitary operators \( \{e^{i(\xi \hat{q} + \eta \hat{p})}\} \).

Schwinger [29] studied in detail the role of the matrices \( A \) and \( B \) in quantum mechanics and hence the set \( \{A^k B^l | k, l = 0, 1, 2, \ldots, (N-1)\} \) is often called Schwinger’s unitary basis for matrix algebra. Let us write an \( N \times N \) matrix \( M \) as

\[
M = \sum_{k,l=0}^{N-1} \mu_{kl} A^k B^l. \tag{8.7}
\]

From the structure of the matrices \( A \) and \( B \) it is easily found that

\[
\text{Tr} \left[ (A^k B^l)^\dagger (A^m B^n) \right] = N \delta_{km} \delta_{ln}. \tag{8.8}
\]

Hence,

\[
\mu_{kl} = \frac{1}{N} \text{Tr} \left[ (A^k B^l)^\dagger M \right] = \text{Tr} \left[ B^{-l} A^{-k} M \right]. \tag{8.9}
\]

Alladi Ramakrishnan wrote (8.7) equivalently as

\[
M = \sum_{k,l=0}^{N-1} c_{kl} B^k A^l, \tag{8.10}
\]
and expressed the coefficients \{c_{kl}\} in a very nice form [1]:

\[
C = S^{-1}R,
\]

(8.11)

\[
C = \begin{pmatrix}
  c_{00} & c_{01} & c_{02} & \cdots & c_{0,N-1} \\
  c_{10} & c_{11} & c_{12} & \cdots & c_{1,N-1} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_{N-2,0} & c_{N-2,1} & c_{N-2,2} & \cdots & c_{N-2,N-1} \\
  c_{N-1,0} & c_{N-1,1} & c_{N-1,2} & \cdots & c_{N-1,N-1}
\end{pmatrix}.
\]

(8.12)

\[
S^{-1} = \frac{1}{N} \begin{pmatrix}
  1 & 1 & 1 & \cdots & 1 \\
  1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\
  1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(N-1)} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & \omega^{-(N-2)} & \omega^{-(N-2)} & \cdots & \omega^{-(N-2)(N-1)} \\
  1 & \omega^{-(N-1)} & \omega^{-(N-1)} & \cdots & \omega^{-(N-1)(N-1)}
\end{pmatrix},
\]

(8.13)

\[
R = \begin{pmatrix}
  M_{00} & M_{01} & M_{02} & \cdots & M_{0,N-1} \\
  M_{11} & M_{12} & M_{13} & \cdots & M_{10} \\
  M_{22} & M_{23} & M_{24} & \cdots & M_{21} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  M_{N-2,N-2} & M_{N-2,N-1} & M_{N-2,0} & \cdots & M_{N-2,N-3} \\
  M_{N-1,N-1} & M_{N-1,1} & M_{N-1,2} & \cdots & M_{N-1,N-2}
\end{pmatrix}.
\]

(8.14)

Note that \(S^{-1}\) is the inverse of the Sylvester, or the finite Fourier transform, matrix. He called (8.10)-(8.14) as a matrix decomposition theorem.

Comparing (8.7) and (8.10) it is clear that \(\mu_{kl} = \omega^{-kl}c_{lk}\).

### 9 Finite-dimensional Wigner function

Let \(N = 2\nu + 1\) and choose

\[
A = \begin{pmatrix}
  0 & 1 & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1 \\
  1 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

(9.1)
and

\[ B = \begin{pmatrix}
\omega^{-\nu} & 0 & 0 & \ldots & 0 & 0 \\
0 & \omega^{-\nu+1} & 0 & \ldots & 0 & 0 \\
0 & 0 & \omega^{-\nu+2} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \omega^{\nu-1} & 0 \\
0 & 0 & 0 & \ldots & 0 & \omega^{\nu}
\end{pmatrix}. \quad (9.2) \]

where \( \omega = e^{2\pi i/(2\nu+1)} \). Note that \( AB = \omega BA \) and \( A^{2\nu+1} = B^{2\nu+1} = I \). Let us now write a \( 2\nu + 1 \)-dimensional matrix \( M \) as

\[ M = \sum_{k,l=\nu}^{\nu} v_{kl} \omega^{kl/2} B^k A^l, \quad (9.3) \]

where

\[ v_{kl} = \frac{1}{2\nu + 1} \text{Tr} \left[ \omega^{-kl/2} A^{-l} B^{-k} M \right]. \quad (9.4) \]

If the matrix \( M \) is to be Hermitian, i.e., \( M^\dagger = M \), then the condition to be satisfied is that \( v_{kl}^* = v_{-k,-l} \).

Let \( W = (w_{kl}) \), with \( k, l = -\nu, -\nu + 1, \ldots, \nu - 1, \nu \), be a real matrix and define the finite-dimensional Fourier transform

\[ v_{\xi\eta} = \frac{1}{2\nu + 1} \sum_{k,l=\nu}^{\nu} w_{kl} \omega^{-\xi k - \eta l}. \quad (9.5) \]

We have

\[ v_{\xi\eta}^* = v_{-\xi,-\eta}. \quad (9.6) \]

Hence, the matrix

\[ H = \sum_{\xi,\eta=-\nu}^{\nu} v_{\xi\eta} \omega^{\xi\eta/2} B^\xi A^\eta = \frac{1}{2\nu + 1} \sum_{\xi,\eta=-\nu}^{\nu} \sum_{k,l=-\nu}^{\nu} w_{kl} \omega^{-\xi k - \eta l + (\xi\eta/2)} B^\xi A^\eta \quad (9.7) \]
is Hermitian. This property, that to every real matrix \( W \) there is associated a unique Hermitian matrix \( H \), is the basis of the Weyl correspondence (8.6). For a given Hermitian matrix \( H \) the associated real matrix \( W \) is obtained from (9.7) as

\[
w_{kl} = \text{Tr} \left[ \omega^{\xi k + \eta l - (\xi \eta / 2)} A^{-\eta} B^{-\xi} H \right].
\] (9.8)

In the large \( \nu \) limit this provides the converse of the Weyl rule (8.6) for obtaining the classical observable corresponding to a quantum operator or the Wigner transform of a quantum operator; in particular, the Wigner phase-space quasiprobability distribution function can be obtained as the limiting case of (9.8) corresponding to the choice of \( H \) as the quantum density operator [3]. Thus, the formula (9.8) can be viewed as an expression of the finite-dimensional Wigner function corresponding to the case when \( H \) is a finite-dimensional density matrix. For more details on finite-dimensional, or discrete, Wigner functions, which are of current interest in quantum information theory, see, e.g., [30].

10 Finite-dimensional quantum canonical transformations

As seen above, the relation (7.6) has a unique representation for \( A \) and \( B \) as given by (7.4) and (7.5). Let us take \( N \) to be even and make a transformation

\[
A \rightarrow A' = \omega^{-kl/2} A^k B^l, \quad B \rightarrow B' = \omega^{-mn/2} A^m B^n,
\] (10.1)

where \((k, l, m, n)\) can be in general taken to be nonnegative integers in \([0, N - 1]\), and require

\[
A' B' = \omega B' A', \quad A'^N = B'^N = I.
\] (10.2)

This implies that we should have

\[
kn - lm = 1(\text{mod}).N,
\] (10.3)

and the factors \( \omega^{-kl/2} \) and \( \omega^{-mn/2} \) ensure that \( A'^N = B'^N = I \). The uniqueness of the representation requires that there should be a definite solution to the equivalence relation

\[
SA = A'S, \quad SB = B'S.
\] (10.4)
Substituting the explicit matrices for $A$ and $B$ from (7.4) and (7.5) it is straightforward to solve for $S$. We get

$$S_{xy} = \omega^{-(nx^2 - 2xy + ky^2)/2m}, \quad x, y = 0, 1, 2, \ldots, N - 1. \quad (10.5)$$

From the association, following Weyl,

$$A \rightarrow e^{i\eta \hat{p}/\hbar}, \quad B \rightarrow e^{i\xi \hat{q}/\hbar}, \quad (10.6)$$

it follows that in the limit of $N \rightarrow \infty$ the finite-dimensional transformation (10.1) becomes the linear canonical transformation of the pair $(\hat{q}, \hat{p})$,

$$\hat{q}' = n\hat{q} + m\hat{p}, \quad \hat{p}' = l\hat{q} + k\hat{p}. \quad (10.7)$$

By taking the corresponding limit of the matrix $S$ in (10.5) one gets the unitary transformation corresponding to the quantum linear canonical transformation (10.7) ([3], [31]) (for details of the quantum canonical transformations see [32]).

11 Magnetic Bloch functions

For an electron of charge $-e$ and mass $m$ moving in a crystal lattice under the influence of an external constant homogeneous magnetic field the stationary state wavefunction corresponding to the energy eigenvalue $E$ satisfies the Schrödinger equation

$$\hat{H} \psi(\vec{r}) = E \psi(\vec{r}),$$

$$\hat{H} = \frac{1}{2m} (\vec{\hat{p}} + e\vec{A})^2 + V(\vec{r}), \quad (11.1)$$

where $\vec{p}$ is the momentum operator $-i\hbar \vec{\nabla}$, $V(\vec{r})$ is the periodic crystal potential, and $\vec{A} = \frac{1}{2}(\vec{B} \times \vec{r})$ is the vector potential of the magnetic field $\vec{B}$. In the absence of the magnetic field the Hamiltonian is invariant under the group of lattice translations and as a consequence the corresponding wavefunction takes the form of a Bloch function:

$$\psi_{\vec{B}=0}(\vec{r}) = \sum_{\vec{R}} e^{-i\vec{K} \cdot \vec{R}} u(\vec{r} + \vec{R}), \quad (11.2)$$

where $\{\vec{R}\}$ is the set of all lattice vectors and $\vec{K}$ is a reciprocal lattice vector within a Brillouin zone. This is the basis of the band theory of solids. In
the presence of a magnetic field the Hamiltonian $\hat{H}$ is not invariant under the lattice translation group. Now, the invariance group is the so-called magnetic translation group with its generators given by, apart from some phase factors, $\left\{ \tau_j = e^{i\vec{a}_j \cdot (\vec{p} - e\vec{A})} | j = 1, 2, 3 \right\}$ where $\vec{a}_j$s are the primitive lattice vectors. These generators obey the algebra:

$$\tau_j \tau_k = e^{-i\vec{e} \cdot \vec{a}_j \times \vec{a}_k / \hbar} \tau_k \tau_j, \quad j, k = 1, 2, 3,$$

(11.3)

a GCA! We can obtain the irreducible representations of this algebra in terms of $A$ and $B$ matrices. Once the inequivalent irreducible representations of the magnetic translation group are known, using the standard group theoretical techniques we can construct the symmetry-adapted basis functions for the Schrödinger equation (11.1). This leads to a generalization of the Bloch function (11.2), the magnetic Bloch function, given by

$$\psi(\vec{r}) = \sum_{\vec{R}} e^{-i\left[(\vec{K} + \frac{2}{\hbar} \vec{B} \times \vec{r}) \cdot \vec{R} + \phi(\vec{R})\right]} u(\vec{r} + \vec{R}),$$

(11.4)

where

$$\phi(n_1\vec{a}_1 + n_2\vec{a}_2 + n_3\vec{a}_3) = \frac{e}{2\hbar} \cdot (n_1n_2\vec{a}_1 \times \vec{a}_2 + n_1n_3\vec{a}_1 \times \vec{a}_3 + n_2n_3\vec{a}_2 \times \vec{a}_3).$$

(11.5)

If the term $\phi(\vec{R})$ is dropped from this expression then it reduces to the well known form proposed by Peierls (for more details see ([31], [33], [34]) and references therein). Understanding the dynamics of a Bloch electron in a magnetic field is an important problem of condensed matter physics with various practical applications.

### 12 Finite-dimensional quantum mechanics

Following are the prophetic words of Weyl [22]: The kinematical structure of a physical system is expressed by an irreducible Abelian group of unitary ray rotations in system space. .... If the group is continuous this procedure automatically leads to Heisenberg’s formulation. .... Our general principle allows for the possibility that the Abelian rotation group is entirely discontinuous, or that it may even be a finite group. .... But the field of discrete
groups offers many possibilities which we have not yet been able to realize in Nature; perhaps, these holes will be filled by applications to nuclear physics.

Keeping in mind the above statement of Weyl and the later work of Schwinger [29], a finite-dimensional quantum mechanics was developed by Santhanam and collaborators. Following Weyl, let us make the association

\[ A \to e^{i\hat{p}/\hbar}, \quad B \to e^{i\hat{q}/\hbar}. \]  \tag{12.1}

Now if we interpret the finite dimensional matrices \( A \) and \( B \) as corresponding to finite-dimensional momentum and position operators, say, \( P \) and \( Q \), respectively, with finite discrete spectra, then, the corresponding system will have confinement purely as a result of its kinematical structure. The matrices \( P \) and \( Q \) can be obtained by taking the logarithms of \( A \) and \( B \). The commutation relation between \( P \) and \( Q \) was first calculated by Santhanam and Tekumalla [35] (Tekumalla was my senior fellow student at our institute). Further work by Santhanam ([36]-[40]) along these lines resulted in the study of the Hermitian phase operator in finite dimensions as a precursor to the currently well known Pegg-Barnett formalism (see, e.g., [41]).

Later, we developed a formalism of finite-dimensional quantum mechanics (FDQM) ([42]-[44]) in which we studied the solutions of the Schrödinger equation with finite-dimensional matrix Hamiltonians obtained by replacing the position and momentum operators by finite-dimensional matrices \( Q \) and \( P \). In [44] I interpreted quark confinement as a kinematic confinement as a consequence of its Weylian finite-dimensional quantum mechanics. Recently, dynamics of wave packets has been studied within the formalism of FDQM [45].

13 GCAs and quantum groups

Experience of working on GCAs helped me later in my work on quantum groups. An \( n \times n \) linear transformation matrix \( M \) acting on the noncommutative \( n \)-dimensional Manin vector space and its dual is a member of the quantum group \( GL_q(n) \) if its noncommuting elements \( m_{jk} \) satisfy certain commutation relations. For example, the elements of a \( 2 \times 2 \) quantum matrix belonging to \( GL_q(2) \),

\[ M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \]  \tag{13.1}
have to satisfy the commutation relations

\[
\begin{align*}
  m_{11}m_{12} &= q^{-1}m_{12}m_{11}, \\
  m_{12}m_{22} &= q^{-1}m_{22}m_{12}, \quad m_{21}m_{22} = q^{-1}m_{22}m_{21}, \\
  m_{12}m_{21} &= m_{21}m_{12}, \quad m_{11}m_{22} - m_{22}m_{11} = (q^{-1} - q)m_{12}m_{21}. \quad (13.2)
\end{align*}
\]

Some of these relations are already GCA-like, or Heisenberg-Weyl-like. It was shown in ([46],[47]) that, in general, all the commutation relations of \( GL_q(n) \) can be formulated in a similar form and hence the representations of these elements can be found utilising the representation theory of the Heisenberg-Weyl relations. Extending these ideas further, we developed in [48] a systematic scheme for constructing the finite and infinite dimensional representations of the elements of the quantum matrices of \( GL_q(n) \), where \( q \) is a primitive root of unity, and discussed the explicit results for \( GL_q(2), GL_q(3) \) and \( GL_q(4) \). In this work we essentially used the product transformation technique ([3],[5]) developed in the context of representation theory of GCAs. In [49] we extended this formalism to the two-parameter quantum group \( GL_{p,q}(2) \) and the two-parameter quantum supergroup \( GL_{p,q}(1|1) \).

14 Conclusion

To summarize, I have reviewed here some aspects of GCAs and their physical applications, mostly related to my own work. I learnt about it in the school of Alladi Ramakrishnan and it has been useful to me throughout my academic career so far. I would like to conclude with the following remark on GCAs by Alladi Ramakrishnan [50]:

\[\text{The structure is too fundamental to be unnoticed, too consistent to be ignored, and much too pretty to be without consequence.}\]

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References

[1] Alladi Ramakrishnan, *L-\textit{Matrix Theory or Grammar of Dirac Matrices*}, Tata-McGraw Hill, 1972.

[2] P. S. Chandrasekaran, *Clifford Algebra, its Generalization, and their Applications to Symmetries and Relativistic Wave Equations*, Ph.D. Thesis, University of Madras, 1971.

[3] R. Jagannathan, *Studies in Generalized Clifford Algebras, Generalized Clifford Groups, and their Physical Applications*, Ph.D. Thesis, University of Madras, 1976.

[4] M. Newman, *Integral Matrices*, Academic Press, 1972.

[5] Alladi Ramakrishnan and R. Jagannathan, *Topics in Numerical Analysis - II*, Ed. J. H. Miller, Academic Press, p.133, 1976.

[6] K. Morinaga and T. Nono, *J. Sci. - Hiroshima Univ.* A 16 (1952).

[7] K. Yamazaki, *J. Fac. Sci. - Univ. Tokyo*, Sec.1, 10 (1964) 147.

[8] I. Popovici and C. Gheorghe, *C. R. Acad. Sci.* (Paris), A 262 (1966) 682.

[9] A. O. Morris, *Quart. J. Math.*, 18 (1967) 7.

[10] A. O. Morris, *J. London Math. Soc.* (2), 7 (1973) 235.

[11] Alladi Ramakrishnan (Ed.), *Proc. MATSCIENCE Conf. Clifford algebra, its generalizations, and applications*, MATSCIENCE, 1971.

[12] N. B. Backhouse and C. J. Bradley, *Proc. Amer. Math. Soc.*, 36 (1972) 260.

[13] R. Jagannathan, *Springer Lecture Notes in Mathematics* 1122 (1984) (Proc. 4th MATSCIENCE Conf. on Number Theory), Ed. K. Alladi, p.130.

[14] W. K. Clifford, *Amer. J. Math. Pure and Appl.*, 1 (1878) 350; see *Mathematical Papers by W. K. Clifford*, Ed. H. Robert Tucker, Chelsea, 1968.

[15] P. A. M. Dirac, *Proc. Roy. Soc.*, A 177 (1928) 610.
[16] D. Hestenes, *Space-Time Algebra*, Gordon & Breach, 1966.

[17] J. S. R. Chisholm and A. K. Common (Eds.), *Clifford Algebras and their Applications in Mathematical Physics*, D. Reidel, 1986.

[18] C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, Camb. Univ. Press, 2003.

[19] Alladi Ramakrishnan and I. V. V. Raghavacharyulu, *Symposia in Theoretical Physics and Mathematics*, 8, Plenum Press, 1968, Alladi Ramakrishnan (Ed.), p.25.

[20] P. A. M. Dirac, *Proc. Roy. Soc.*, A 322 (1971) 435.

[21] R. Jagannathan and H. N. V. Dutt, *J. Math. Phys. Sci.* (IIT-Madras) IX (1975) 301.

[22] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover, (1950).

[23] R. Jagannathan and N. R. Ranganathan, *Rep. Math. Phys.* 5 (1974) 131.

[24] R. Jagannathan and N. R. Ranganathan, *Rep. Math. Phys.* 7 (1975) 229.

[25] R. Jagannathan and T. S. Santhanam, *Springer Lecture Notes in Mathematics* 938, (1982) (Proc. 3rd MATSCIENCE Conf. on Number Theory), Ed. K. Alladi, p.82.

[26] V. P. Nair, *Quantum Field Theory - A Modern Perspective*, Springer, (2005).

[27] A. K. Kwasniewski, *J. Phys. A: Math. Gen.* 19 (1986) 1469.

[28] A. K. Kwasniewski, W. Bajguz, and I. Jaroszewski, *Adv. Appl. Clifford Algebras* 8 (1998) 417.

[29] J. Schwinger, *Quantum Kinematics and Dynamics*, Benjamin, (1970).

[30] S. Chaturvedi, E. Ercolessi, G. Marmo, G. Morandi, N. Mukunda, and R. Simon, *J. Phys. A: Math. Gen.* 39 (2006) 1405.

[31] R. Jagannathan, *MATSCIENCE Report* 87 (1977) 6.
[32] M. Moshinsky and C. Quesne, *J. Math. Phys.* **12** (1971) 1772.

[33] N. R. Ranganathan and R. Jagannathan, *Proc. 2nd International Colloquium on Group Theoretical Methods in Physics*, Univ. Nijmegen, (1973) p.232.

[34] R. Jagannathan and N. R. Ranganathan, *Phys. Stat. Sol. B* **74** (1976) 74.

[35] T. S. Santhanam and A. R. Tekumalla, *Found. Phys.* **6** (1976) 583.

[36] T. S. Santhanam, *Phys. Lett. A* **56** (1976) 345.

[37] T. S. Santhanam, *Found. Phys.* **7** (1977) 121,

[38] T. S. Santhanam, *Nuovo Cim. Lett.* **20** (1977) 13.

[39] T. S. Santhanam, *Uncertainty Principle and Foundations of Quantum Mechanics*, Eds. W. Price and S. S. Chissick, John Wiley, (1977) p.227.

[40] T. S. Santhanam and K. B. Sinha, *Aust. J. Phys.* **31** (1978) 233.

[41] R. Tanas, A. Miranowicz, and Ts. Gantsog, *Progress in Optics XXXV* (1996) 355.

[42] R. Jagannathan, T. S. Santhanam, and R. Vasudevan, *Int. J. Theor. Phys.* **20** (1981) 755.

[43] R. Jagannathan and T. S. Santhanam, *Int. J. Theor. Phys.* **21** (1982) 351.

[44] R. Jagannathan, *Int. J. Theor. Phys.* **22** (1983) 1105.

[45] J. Y. Bang and M. S. Berger, *Phys. Rev. A* **80** (2009) 022105.

[46] E. G. Floratos, *Phys. Lett. B* **233** (1989) 395.

[47] J. Weyers, *Phys. Lett. B* **240** (1990) 396.

[48] R. Chakrabarti and R. Jagannathan, *J. Phys. A: Math. Gen.* **24** (1991) 1709.

[49] R. Chakrabarti and R. Jagannathan, *J. Phys. A: Math. Gen.* **24** (1991) 5683.
[50] Alladi Ramakrishnan, in The Structure of Matter - Rutherford Centennial Symposium, 1971, University of Canterbury, Christchurch, New Zealand, Ed. B. G. Wybourne, 1972.