ESTIMATE FOR THE GRADIENT IN $W^{1,A(x,\cdot)}$

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Abstract. Under appropriate assumptions on the $N(\Omega)$-function, the $L^{\Phi(A(x,\cdot))}$ estimate for the gradient of the minimizers of a class of energy functional in Musielak-Orlicz-Sobolev space $W^{1,A(x,\cdot)}$ is presented by using Calderón-Zygmund decomposition.

1. Introduction

Vast mathematical literature describes various aspects of partial differential equations related to the elliptic type operators including variable exponent, weighted, convex and double phase cases. Musielak-Orlicz-Sobolev spaces give an abstract framework of functional analysis to cover all of the above mentioned cases.

Some basic references of special Musielak-Orlicz spaces should be noticed. Before the key role in the modular spaces of functional analysis was played by the comprehensive book by Musielak [31], Nakano [32] provided the first framework approach to non-homogeneous setting with general growth conditions, which was followed by Skaff [34, 35] and Hudzik [22, 23]. In much earlier time variable exponent spaces were introduced by the pioneering work by Orlicz [33].

1.1. Linear and nonlinear Calderón-Zygmund theory. In this paper we mainly concern with the gradient integrability properties of the minimizers for some energy functional in Musielak-Orlicz-Sobolev space $W^{1,A(x,\cdot)}$ under some reasonable assumptions on the $N(\Omega)$-function $A$. Our results can be basically considered as the generalization of the classical Calderón-Zygmund theory for linear case started by the work of Calderón and Zygmund [7, 8]. From the viewpoint of contemporary mathematical research, these two foundation papers [7, 8] by Calderón and Zygmund yield a new research field in mathematics - the theory of harmonic analysis. Since the linear Calderón-Zygmund theory heavily depends on the existence of the fundamental solution, the nonlinear case of the theory does not exist before De Giorgi’s famous iteration method firstly appeared in [10].

The nonlinear case of the Calderón-Zygmund theory was started in the fundamental work of Iwaniec [24, 25]. Then important contributions by DiBenedetto and Manfredi [11] and Caffarelli and Peral [6] followed the time line of the investigation. After those papers, a large number of literature on the Nonlinear Calderón-Zygmund theory appears in recent years, see for instance [26, 27, 4, 5, 29, 30, 9].

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1.2. **Recent regularity results in Musielak-Orlicz-Sobolev spaces.** A highly important part of the mathematical literature in general Musielak-Orlicz-Sobolev spaces gives structural conditions on regularity analysis of the space in recent years.

In a recent work [2], Ahmida and collaborators prove the density of smooth functions in the modular topology in Musielak-Orlicz-Sobolev spaces, which extends the results of Gossez [20] obtained in the Orlicz-Sobolev settings. The authors impose new systematic regularity assumption on the modular function. And this allows to study the problem of density unifying and improving the known results in Orlicz-Sobolev spaces, as well as variable exponent Sobolev spaces.

In the paper [27], under some reasonable assumptions on the $N(\Omega)$-function, the De Giorgi process is presented by the authors in the framework of Musielak-Orlicz-Sobolev spaces. And as the applications, the local bounded property of the minimizer for a class of the energy functional in Musielak-Orlicz-Sobolev spaces is proved. Under similar assumptions as in [27], the authors in [36] prove the Hölder continuity of the minimizers for a class of the energy functionals in Musielak-Orlicz-Sobolev spaces.

1.3. **Motivation.** The priori estimate of $L^{\Phi(A(x, \cdot))}$ for the gradient in $W^{1, A(x, \cdot)}$ relates to many aspect of regularity results in the Musielak-Orlicz-Sobolev spaces, for example $C^{1, \alpha}$ estimate. Similar priori estimate results in the classical Sobolev spaces can be found in [3, 17, 18, 19]. In the paper [16], Fan presents the $L^{q(\cdot)}$ and $L^{\infty}$ estimate for the gradient of the minimizers of some kind of functionals in $W^{1, p(\cdot)}$. In this paper we generalized the results of Fan. We emphasize that the assumptions to derive the regularity results in this paper cover the variable exponent case, but can not cover the ones in double phase case in [9]. In order to get the estimate of $L^{\Phi(A(x, \cdot))}$ for the gradient, it is still interesting to find out the uniform assumptions including both variable exponent case and double phase case in Musielak-Orlicz-Sobolev spaces in order to derive these regularity results.

To find a more reasonable method to describe increasing conditions for the modular functions is always hard to overcome. Moreover, we wish the method we have developed are convenient to make the calculus for $N(\Omega)$ or $N$ functions. In this paper, based on the tools developed in [27, 36] (Lemma 2.1), we find some additional more accurate ways to describe increasing conditions on the $N(\Omega)$ functions in order to make calculus of “functions” accurately, see $\Delta_{R^+}$ and $\nabla_{R^+}$ conditions in Definition 2.4 and their basic properties in Lemma 2.2 and Lemma 2.3. For these basic assumptions, our fundamental viewpoint is that the more accurate estimate needs more detailed assumptions.

In most of existing research papers, the mathematicians are used to describing increasing condition by power functions. But it is natural to do this by the modular functions in Musielak-Orlicz spaces instead of the power functions. Furthermore, it is natural to describe modular functions by the convexity of the function in Musielak-Orlicz spaces. To see this, for example, it is well known that the Hölder inequality holds for convex modular functions (see Proposition 2.3(5)). We will use the uniform convexity to describe the increasing condition on modular functions in this paper.

To overcome the linearity dependence in the linear Calderón-Zygmund theory, we establish the Caccioppoli type inequality by De Giorgi's iteration, see Lemma 4.2. And by a new type of Gehring type inequality developed in Lemma 3.3 and an
abstract theorem developed by Calderón-Zygmund decomposition in Theorem 3.1, we rebuild the nonlinear $L^\Phi(A(x, \cdot))$ estimate for the gradient in $W^{1,A(x, \cdot)}$.

The main results of this paper are Theorem 3.1 in Section 3 and Theorem 4.1 in Section 4. These results extend part of the results in [17, 19, 16]. We also generalize the log-Hölder continuity of the variable exponent case in Musielak-Orlicz-Sobolev spaces in Section 4, see Lemma 4.9.

2. The Musielak-Orlicz-Sobolev Spaces

In this section, we list some definitions and propositions related to Musielak-Orlicz-Sobolev spaces. Firstly, we give the definition of $N$-function and generalized $N$-function as following.

Definition 2.1. A function $A : \mathbb{R} \rightarrow [0, +\infty)$ is called an $N$-modular function (or $N$-function), denoted by $A \in N$, if $A$ is even and convex, $A(0) = 0$, $0 < A(t) \in C^0$ for $t \neq 0$, and the following conditions hold

$$\lim_{t \to 0^+} \frac{A(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to +\infty} \frac{A(t)}{t} = +\infty.$$ 

Let $\Omega$ be a smooth domain in $\mathbb{R}^n$. A function $A : \Omega \times \mathbb{R} \rightarrow [0, +\infty)$ is called a generalized $N$-modular function (or generalized $N$-function), denoted by $A \in N(\Omega)$, if for each $t \in [0, +\infty)$, the function $A(\cdot, t)$ is measurable, and for a.e. $x \in \Omega$, we have $A(x, \cdot) \in N$.

Let $A \in N(\Omega)$, the Musielak-Orlicz space $L^A(\Omega)$ is defined by

$$\|u\|_{L^A(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} A\left(x, \frac{|u(x)|}{\lambda}\right) \, dx \leq 1 \right\},$$

with the (Luxemburg) norm

$$\|u\|_{L^A(\Omega)} = \left\| \frac{u}{A} \right\|_{L^1(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} A\left(x, \frac{|u(x)|}{\lambda}\right) \, dx \leq 1 \right\}.$$

The Musielak-Sobolev space $W^{1,A}(\Omega)$ can be defined by

$$W^{1,A}(\Omega) := \{ u \in L^A(\Omega) : |\nabla u| \in L^A(\Omega) \}$$

with the norm

$$\|u\|_{W^{1,A}(\Omega)} = \left\| \frac{u}{A} \right\|_{L^1(\Omega)} := \|u\|_A + \|\nabla u\|_A,$$

where $\|\nabla u\|_A := \|\nabla u\|_A$.

$A$ is called locally integrable if $A(\cdot, t_0) \in L^1_{\text{loc}}(\Omega)$ for every $t_0 > 0$.

Definition 2.2. We say that $a(x, t)$ is the Musielak derivative of $A(x, t) \in N(\Omega)$ at $t$ if for $x \in \Omega$ and $t \geq 0$, $a(x, t)$ is the right-hand derivative of $A(x, \cdot)$ at $t$; and for $x \in \Omega$ and $t \leq 0$, $a(x, t) := -a(x, -t)$. 

Define \( \tilde{A} : \Omega \times \mathbb{R} \to [0, +\infty) \) by
\[
\tilde{A}(x, s) = \sup_{t \in \mathbb{R}} (st - A(x, t)) \quad \text{for } x \in \Omega \text{ and } s \in \mathbb{R}.
\]
\( \tilde{A} \) is called the complementary function to \( A \) in the sense of Young. It is well known that if \( A \in N(\Omega) \), then \( \tilde{A} \in N(\Omega) \) and \( \tilde{A} \) is also the complementary function to \( A \).

For \( x \in \Omega \) and \( s \geq 0 \), we denote by \( a_+^{-1}(x, s) \) the right-hand derivative of \( \tilde{A}(x, \cdot) \) at \( s \) the same time define \( a_+^{-1}(x, s) = -a_+^{-1}(x, -s) \) for \( x \in \Omega \) and \( s \leq 0 \). Then for \( x \in \Omega \) and \( s \geq 0 \), we have
\[
a_+^{-1}(x, s) = \sup\{t \geq 0 : a(x, t) \leq s\} = \inf\{t > 0 : a(x, t) > s\}.
\]

**Proposition 2.1.** (See [13, 31].) Let \( A \in N(\Omega) \). Then the following assertions hold:

1. \( A(x, t) \leq a(x, t)t \leq A(x, 2t) \) for \( x \in \Omega \) and \( t \in \mathbb{R} \);
2. \( A \) and \( \tilde{A} \) satisfy the Young inequality
   \[
   st \leq A(x, t) + \tilde{A}(x, s) \quad \text{for } x \in \Omega \text{ and } s, t \in \mathbb{R}
   \]
   and the equality holds if \( s = a(x, t) \) or \( t = a_+^{-1}(x, s) \).

Let \( A, B \in N(\Omega) \). We say that \( A \) is weaker than \( B \), denoted by \( A \preceq B \), if there exist positive constants \( K_1, K_2 \) and \( h \in L^1(\Omega) \cap L^\infty(\Omega) \) such that
\[
A(x, t) \leq K_1B(x, K_2t) + h(x) \quad \text{for } x \in \Omega \text{ and } t \in [0, +\infty).
\]

**Proposition 2.2.** (See [13, 31].) Let \( A, B \in N(\Omega) \) and \( A \preceq B \). Then \( \tilde{B} \preceq \tilde{A} \), \( L^B(\Omega) \hookrightarrow L^A(\Omega) \) and \( L^A(\Omega) \hookrightarrow L^{\tilde{B}}(\Omega) \).

**Definition 2.3.** We say that a function \( A : \Omega \times [0, +\infty) \to [0, +\infty) \) satisfies the \( \Delta_2(\Omega) \) condition, denoted by \( A \in \Delta_2(\Omega) \), if there exists a positive constant \( K > 0 \) and a nonnegative function \( h \in L^1(\Omega) \) such that
\[
A(x, 2t) \leq KA(x, t) + h(x) \quad \text{for } x \in \Omega \text{ and } t \in [0, +\infty).
\]

If \( A(x, t) = A(t) \) is an \( N \)-function and \( h(x) \equiv 0 \) in \( \Omega \) in Definition 2.3, then \( A \in \Delta_2(\Omega) \) if and only if \( A \) satisfies the well-known \( \Delta_2 \) condition defined in [11, 12].

**Proposition 2.3.** (See [13].) Let \( A \in N(\Omega) \) satisfy \( \Delta_2(\Omega) \). Then the following assertions hold:

1. \( L^A(\Omega) = \{u : u \text{ is a measurable function, and } \int_{\Omega} A(x, |u(x)|) \, dx < +\infty\} \);
2. \( \int_{\Omega} A(x, |u|) \, dx < 1 \) (resp. \( = 1; > 1 \)) \( \iff \|u\|_A < 1 \) (resp. \( = 1; > 1 \)), where \( u \in L^A(\Omega) \);
3. \( \int_{\Omega} A(x, |u_n|) \, dx \to 0 \) (resp. \( 1; +\infty \)) \( \iff \|u_n\|_A \to 0 \) (resp. \( 1; +\infty \)), where \( \{u_n\} \subset L^A(\Omega) \);
4. \( u_n \to u \) in \( L^A(\Omega) \) \( \implies \int_{\Omega} \left| A(x, |u_n|) - A(x, |u|) \right| \, dx \to 0 \) as \( n \to \infty \);
5. If \( \tilde{A} \) also satisfies \( \Delta_2 \), then
   \[
   \left| \int_{\Omega} u(x)v(x) \, dx \right| \leq 2\|u\|_A\|v\|_{\tilde{A}}, \quad \forall u \in L^A(\Omega), v \in L^{\tilde{A}}(\Omega);
   \]
\[ a(\cdot, |u(\cdot)|) \in L^\tilde{A}(\Omega) \] for every \( u \in L^A(\Omega) \).

The following assumptions will be used.

\((C_1)\) \( \inf_{x \in \Omega} A(x, 1) = c_1 > 0; \)

**Proposition 2.4.** (See [13].) If \( A \in N(\Omega) \) satisfies \((C_1)\), then \( L^A(\Omega) \hookrightarrow L^1(\Omega) \) and \( W^{1,A}(\Omega) \hookrightarrow W^{1,1}(\Omega) \).

Let \( A \in N(\Omega) \) be locally integrable. We will denote

\[ W^{1,A}_0(\Omega) := C_0^\infty(\Omega) \] \[ D^{1,A}_0(\Omega) := C_0^\infty(\Omega) \| \nabla \cdot \|_{L^A(\Omega)}. \]

In the case that \( \| \nabla u \|_A \) is an equivalent norm in \( W^{1,A}_0(\Omega) \), \( W^{1,A}_0(\Omega) = D^{1,A}_0(\Omega) \).

**Proposition 2.5.** (See [13].) Let \( A \in N(\Omega) \) be locally integrable and satisfy \((C_1)\). Then

1. the spaces \( W^{1,A}(\Omega), W^{1,A}_0(\Omega) \) and \( D^{1,A}_0(\Omega) \) are separable Banach spaces, and
   \[ W^{1,A}_0(\Omega) \hookrightarrow W^{1,A}(\Omega) \hookrightarrow W^{1,1}(\Omega); \]
   \[ D^{1,A}_0(\Omega) \hookrightarrow D^{1,1}_0(\Omega) = W^{1,1}_0(\Omega); \]
2. the spaces \( W^{1,A}(\Omega), W^{1,A}_0(\Omega) \) and \( D^{1,A}_0(\Omega) \) are reflexive provided \( L^A(\Omega) \) is reflexive.

We always assume that the following condition holds.

**Proposition 2.6.** (See [13].) Let \( A, B \in N(\Omega) \) and \( A \) be locally integrable. If there is a compact imbedding \( W^{1,A}(\Omega) \hookrightarrow L^B(\Omega) \) and \( A \precsim B \), then there holds the following Poincaré inequality

\[ \| u \|_A \leq c \| \nabla u \|_A, \quad \forall u \in W^{1,A}_0(\Omega), \]

which implies that \( \| \nabla \cdot \|_A \) is an equivalent norm in \( W^{1,A}_0(\Omega) \) and \( W^{1,A}_0(\Omega) = D^{1,A}_0(\Omega) \).

The following assumptions will be used.

\((P_1)\) \( \Omega \subseteq \mathbb{R}^n (n \geq 2) \) is a bounded domain with the cone property, and \( A \in N(\Omega); \)
\((P_2)\) \( A : \overline{\Omega} \times \mathbb{R} \to [0, +\infty) \) is continuous and \( A(x, t) \in (0, +\infty) \) for \( x \in \overline{\Omega} \) and \( t \in (0, +\infty) \).

Let \( A \) satisfy \((P_1)\) and \((P_2)\). Denote by \( A^{-1}(x, \cdot) \) the inverse function of \( A(x, \cdot) \). We always assume that the following condition holds.
(P$_3$) $A \in N(\Omega)$ satisfies
\begin{equation}
\int_{0}^{1} \frac{A^{-1}(x,t)}{t^{\frac{1}{n+1}}} \, dt < +\infty, \; \forall \, x \in \overline{\Omega}.
\end{equation}

Under assumptions (P$_1$), (P$_2$) and (P$_3$), for each $x \in \overline{\Omega}$, the function $A(x, \cdot) : [0, +\infty) \to [0, +\infty)$ is a strictly increasing homeomorphism. Define a function $A^{-1} : \overline{\Omega} \times [0, +\infty) \to [0, +\infty)$ by
\begin{equation}
A^{-1}_s(x) = \int_{0}^{s} \frac{A^{-1}(x,\tau)}{\tau^{\frac{1}{n+1}}} \, d\tau \; \text{for} \; x \in \overline{\Omega} \; \text{and} \; s \in [0, +\infty).
\end{equation}
Then under the assumption (P$_3$), $A^{-1}_s$ is well defined, and for each $x \in \overline{\Omega}$, $A^{-1}_s(x, \cdot)$ is strictly increasing, $A^{-1}_s(x, \cdot) \in C^{1}((0, +\infty))$ and the function $A^{-1}_s(x, \cdot)$ is concave.

Let $X$ be a metric space and $f : X \rightarrow (-\infty, +\infty]$ be an extended real-valued function. For $x \in X$ with $f(x) \in \mathbb{R}$, the continuity of $f$ at $x$ is well defined. For $x \in X$ with $f(x) = +\infty$, we say that $f$ is continuous at $x$ if given any $M > 0$, there exists a neighborhood $U$ of $x$ such that $f(y) > M$ for all $y \in U$. We say that $f : X \rightarrow (-\infty, +\infty]$ is continuous on $X$ if $f$ is continuous at every $x \in X$. Define $\text{Dom}(f) = \{ x \in X : f(x) \in \mathbb{R} \}$ and denote by $C^{1-0}(X)$ the set of all locally Lipschitz continuous real-valued functions defined on $X$.

**Remark 2.1.** Suppose that $A \in N(\Omega)$ satisfy (P$_2$). Then for each $t_0 \geq 0$, $\overline{A}(x, t_0)$, $A_*(x, t_0)$ are bounded.

The following assumptions will also be used.

(P$_4$) $T : \overline{\Omega} \rightarrow [0, +\infty]$ is continuous on $\overline{\Omega}$ and $T \in C^{1-0}(\text{Dom}(T))$;

(P$_5$) $A_* \in C^{1-0}(\text{Dom}(A_*))$ and there exist three positive constants $\delta_0, C_0$ and $t_0$ with $\delta_0 < \frac{1}{2}$, $0 < t_0 < \min_{x \in \overline{\Omega}} T(x)$ such that
\begin{equation}
|\nabla_x A_*(x,t)| \leq C_0(A_*(x,t))^{1+\delta_0},
\end{equation}
for $x \in \Omega$ and $|t| \in [t_0, T(x))$ provided $\nabla_x A_*(x,t)$ exists.

Let $A, B \in N(\Omega)$. We say that $A \ll B$ if, for any $k > 0$,
\begin{equation}
\lim_{t \rightarrow +\infty} \frac{A(x, kt)}{B(x,t)} = 0 \; \text{uniformly for} \; x \in \Omega.
\end{equation}

**Remark 2.2.** Suppose that $A, B \in N(\Omega)$. Then $A \ll B \Rightarrow A \ll B$. 
Next we present two embedding theorems for Musielak-Sobolev spaces developed by Fan in [14].

**Theorem 2.7.** (See [14], [28].) Let \((P_1) - (P_5)\) hold. Then

(i) There is a continuous imbedding \(W^{1,A}(\Omega) \hookrightarrow L^{A^*}(\Omega)\);

(ii) Suppose that \(B \in N(\Omega), \ B : \overline{\Omega} \times [0, +\infty) \to [0, +\infty)\) is continuous, and \(B(x,t) \in (0, +\infty)\) for \(x \in \Omega\) and \(t \in (0, +\infty)\). If \(B \ll A_*\), then there is a compact imbedding \(W^{1,A}(\Omega) \hookrightarrow L^B(\Omega)\).

By Theorem 2.7, Remark 2.2 and Proposition 2.6, we have the following:

**Theorem 2.8.** (See [14], [28].) Let \((P_1) - (P_5)\) hold and furthermore, \(A, A_* \in N(\Omega)\). Then

(i) \(A \ll A_*\), and there is a compact imbedding \(W^{1,A}(\Omega) \hookrightarrow L^{A}(\Omega)\);

(ii) there holds the poincaré-type inequality

\[ \|u\|_A \leq C\|\nabla u\|_A \quad \text{for} \quad u \in W^{1,A}_0(\Omega), \]

i.e. \(\|\nabla u\|_A\) is an equivalent norm on \(W^{1,A}_0(\Omega)\).

In the following of this section, we suppose \(A\) satisfies Condition \((\mathcal{A})\), denoted by \(A \in \mathcal{A}\):

\((\mathcal{A})\) \(A\) satisfies assumptions \((P_1) - (P_5)\), \((P_6)\) in Section 2 and the following \((\overline{P}_4)\) \(T(x)\) defined in [27] satisfies \(T(x) = +\infty\) for all \(x \in \Omega\).

We recall some useful conclusions in [27] for the reader’s convenience.

**Lemma 2.1.** (See [27].) Suppose that \(A \in N(\Omega)\), and there exists a continuous and strictly increasing function \(\mathfrak{A} : [0, +\infty) \to [0, +\infty)\) such that

\[ A(x, \alpha t) \geq \mathfrak{A}(\alpha)A(x, t), \quad \forall \alpha \geq 0, t \in \mathbb{R}, x \in \Omega. \]

(i) Then there exists a continuous and strictly increasing function \(\widehat{\mathfrak{A}} : [0, +\infty) \to [0, +\infty)\), defined by

\[ \widehat{\mathfrak{A}}(\beta) = \begin{cases} \frac{1}{\mathfrak{A}^{-1}(\beta)}, & \text{for } \beta > 0, \\ 0, & \text{for } \beta = 0, \end{cases} \]

such that

\[ A(x, \beta t) \leq \widehat{\mathfrak{A}}(\beta)A(x, t), \quad \forall \beta > 0, t \in \mathbb{R}, x \in \Omega, \]

and furthermore \(\widehat{\mathfrak{A}} = \mathfrak{A}\);

(ii) If \(\mathfrak{A}\) satisfies

\[ n\mathfrak{A}(\alpha) > \alpha\mathfrak{A}'(\alpha) \]

for a.e. \(\alpha > 0\), then \(A_* \in N(\Omega)\), and there exists a continuous, strictly increasing and a.e. differentiable function \(\mathfrak{A}_* : [0, +\infty) \to [0, +\infty)\), defined by

\[ \mathfrak{A}_*^{-1}(\sigma) = \begin{cases} \frac{1}{\sigma + \mathfrak{A}(\sigma^{-1})}, & \text{for } \sigma > 0, \\ 0, & \text{for } \sigma = 0, \end{cases} \]
such that
\[(2.10)\quad A_*(x, \beta t) \leq A_*(\beta)A_*(x, t), \forall \beta > 0, t \in \mathbb{R}, x \in \Omega;\]

(iii) If \(A\) satisfies
\[(2.11)\quad \alpha A'(\alpha) > A(\alpha)
\]
for a.e. \(\alpha > 0\), then \(\vec{A} \in N(\Omega)\), and there exists a continuous, strictly increasing and a.e. differentiable function \(\vec{A} : [0, +\infty) \to [0, +\infty)\), defined by
\[(2.12)\quad \vec{A}^{-1}(\sigma) = \begin{cases} \frac{\sigma}{\alpha(\sigma)}, & \text{for } \sigma > 0, \\ 0, & \text{for } \sigma = 0, \end{cases}\]
such that
\[(2.13)\quad \vec{A}(x, \beta t) \leq \vec{A}(\beta)\vec{A}(x, t), \forall \beta > 0, t \in \mathbb{R}, x \in \Omega.\]

Remark 2.3. It is clear that \(A\) defined in (2.5) depends on \(\Omega\). To emphasize the situation, we denote \(A_\Omega(t) = A(t)\) for any \(t \geq 0\). To abbreviate the symbols, we write \(A(t)\) for \(A_\Omega(t)\) if we consider problems in the domain \(\Omega\) in this paper. And it is natural to assume that for \(\Omega_0 \subset \Omega\)
\[
A_\Omega(t) \leq A_{\Omega_0}(t) \leq A_{\Omega_\hat{0}}(t) \leq \hat{A}_\Omega(t), \forall t \geq 0.
\]

To give more precise increasing assumptions on the function \(A\) in Lemma 2.1, we need the following definition.

Definition 2.4. (1) Define the operators \(\hat{\cdot}, \star,\) and \(\hat{\cdot}\) for any function \(E : [0, +\infty) \to [0, +\infty)\) by (2.6), (2.9) and (2.12) provided \(E, E_*,\) and \(\hat{E}\) exist.
(2) We say that \(E : [0, +\infty) \to [0, +\infty)\) satisfies Condition \(\Delta_{R^+}\), denoted by \(E \in \Delta_{R^+}\), if there exist positive constants \(M_0, M_1,\) and \(M_2\) such that
\[(2.14)\quad M_1 E(\alpha)E(\beta) \leq E(M_0\alpha\beta) \leq M_2 \hat{E}(\alpha)\hat{E}(\beta), \forall \alpha, \beta > 0,
\]
provided \(\hat{E}\) exists.
(3) We say that \(E : [0, +\infty) \to [0, +\infty)\) satisfies Condition \(\nabla_{R^+}\), denoted by \(E \in \nabla_{R^+}\), if there exist positive constants \(M_0, M_1,\) and \(M_2\) such that
\[(2.15)\quad M_1 \hat{E}(\alpha)\hat{E}(\beta) \leq E(M_0\alpha\beta) \leq M_2 E(\alpha)E(\beta), \forall \alpha, \beta > 0,
\]
provided \(\hat{E}\) exists.

Lemma 2.2. Suppose \(E, D \in \Delta_{R^+}\) (\(\nabla_{R^+}\) respectively) and \(\hat{E}, \hat{D}\) exist. Then
(1) \(\hat{E}, \hat{D} \in \nabla_{R^+}\) (\(\Delta_{R^+}\) respectively);
(2) for any \(C > 0\), there exit constants \(M = M(C, E) > 0\) and \(\hat{M} = \hat{M}(C, E) > 0\) such that
\[
M E(\alpha)E(\beta) \leq E(C\alpha\beta) \leq \hat{M} \hat{E}(\alpha)\hat{E}(\beta), \forall \alpha, \beta > 0,
\]
\[
(\hat{M} \hat{E}(\alpha)\hat{E}(\beta) \leq E(C\alpha\beta) \leq M E(\alpha)E(\beta), \forall \alpha, \beta > 0 \text{ respectively).}
\]
And especially there exit constants \( M = M(\mathcal{C}) > 0 \) and \( \hat{M} = \hat{M}(\mathcal{C}) > 0 \) such that

\[
M \mathcal{C}(\alpha) \mathcal{C}(\beta) \leq \mathcal{C}(\alpha\beta) \leq \hat{M} \mathcal{C}(\alpha) \mathcal{C}(\beta), \forall \alpha, \beta > 0,
\]

\[
(\hat{M} \mathcal{C}(\alpha) \mathcal{C}(\beta)) \leq \mathcal{C}(\alpha\beta) \leq M \mathcal{C}(\alpha) \mathcal{C}(\beta), \forall \alpha, \beta > 0 \text{ respectively};
\]

(3) \( \hat{\mathcal{D}} := \hat{\mathcal{C}}(\hat{\mathcal{D}}(\cdot)) = \hat{\mathcal{C}} \mathcal{D} \); 
(4) \( \mathcal{C} \mathcal{D} \in \Delta_{R^+} (\nabla_{R^+} \text{ respectively}) \).

**Proof.** (1) Since \( \mathcal{C} \in \Delta_{R^+} \), there exist constants \( M_0 > 0 \), \( M_1 > 0 \) and \( M_2 > 0 \) such that for any \( \alpha, \beta > 0 \)

\[
M_1 \mathcal{C}(\alpha) \mathcal{C}(\beta) \leq \mathcal{C}(M_0 \alpha \beta) \leq M_2 \hat{\mathcal{C}}(\alpha) \hat{\mathcal{C}}(\beta), \forall \alpha, \beta > 0.
\]

Then by (2.6) we get

\[
\frac{1}{M_2} \mathcal{C}(\alpha) \mathcal{C}(\beta) = \frac{1}{M_2 \hat{\mathcal{C}}(\alpha^{-1}) \mathcal{C}(\beta^{-1})} = \frac{1}{\mathcal{C}(M_0 \alpha^{-1} \beta^{-1})} = \frac{1}{M_0} \mathcal{C}(\alpha) \mathcal{C}(\beta),
\]

which implies that \( \hat{\mathcal{C}} \in \nabla_{R^+} \).

(2) Since \( \mathcal{C} \in \Delta_{R^+} \), there exist constants \( M_0 > 0 \), \( M_1 > 0 \) and \( M_2 > 0 \) such that for any \( \alpha, \beta > 0 \)

\[
M_1 \mathcal{C}(\alpha) \mathcal{C}(\beta) \leq \mathcal{C}(M_0 \alpha \beta) \leq M_2 \hat{\mathcal{C}}(\alpha) \hat{\mathcal{C}}(\beta), \forall \alpha, \beta > 0.
\]

Then by step (1) we get

\[
M_1^2 \mathcal{C}(\frac{C}{M_0}) \mathcal{C}(\alpha) \mathcal{C}(\beta) \leq M_1 \mathcal{C}(\frac{C}{M_0} \alpha) \mathcal{C}(\beta) = \mathcal{C}(C \alpha) \leq \mathcal{C}(M_0 \alpha)
\]

\[
= \mathcal{C}(M_0 \alpha) \leq M_2 \hat{\mathcal{C}}(\alpha) \hat{\mathcal{C}}(\beta) = M_2 \hat{\mathcal{C}}(\frac{C M_0}{M_1} \alpha) \mathcal{C}(\beta) \leq \frac{M_2}{M_1} \hat{\mathcal{C}}(\alpha) \mathcal{C}(\beta).
\]

Taking \( M := M_1^2 \mathcal{C}(\frac{C}{M_0}) \) and \( \hat{M} := \frac{M_2}{M_1} \hat{\mathcal{C}}(\alpha) \mathcal{C}(\beta) \) we get the conclusion of (2).

(3) It is easy to check that for any \( \sigma > 0 \)

\[
(\hat{\mathcal{D}}(\sigma)) \mathcal{D}(\hat{\mathcal{D}}(\sigma)) = \frac{1}{\mathcal{C}(\sigma^{-1})} = \frac{1}{\mathcal{C}(\sigma)} \mathcal{D}(\sigma).
\]

(4) It is easy to see that (4) holds. \( \square \)

It is easy to see that the following lemma holds.

**Lemma 2.3.** Suppose \( \mathcal{C} \in \Delta_{R^+} (\nabla_{R^+} \text{ respectively}) \) and \( \mathcal{C}^{-1}, \hat{\mathcal{C}}, \mathcal{C}_*, \tilde{\mathcal{C}} \) exist. Then

(1) \( \hat{\mathcal{C}} \) is commutative with \( \mathcal{C}^{-1}, \mathcal{C}_*, \tilde{\mathcal{C}} \);
(2) $\mathbf{e}^{-1}, \hat{\mathbf{e}}, \mathbf{e}_*, \tilde{\mathbf{e}} \in \nabla_{R^+}$ ($\Delta_{R^+}$ respectively) and $\hat{\mathbf{e}}^{-1}, \mathbf{e}_*^{-1}, \tilde{\mathbf{e}}^{-1} \in \Delta_{R^+}$ ($\nabla_{R^+}$ respectively).

Proof. (1) (a) We will prove $\hat{\sim}$ is commutative with $^{-1}$. In fact, it is easy to see by (2.13) that the following equation holds for any $\sigma > 0$,

$$\hat{\mathbf{e}}^{-1}(\sigma) = \frac{1}{\mathbf{e}^{-1}(\sigma^{-1})} = \hat{\mathbf{e}}^{-1}(\sigma).$$

(b) We will prove $\hat{\sim}$ is commutative with $\hat{\sim}$. In fact, by (2.9) and step (a), for any $\sigma > 0$, we get

$$\hat{\mathbf{e}}^{-1}(\sigma) = \frac{1}{\mathbf{e}^{-1}(\sigma^{-1})} = \frac{\mathbf{e}^{-1}(\sigma)}{\sigma^{\pi}}.$$ On the other hand,

$$\hat{\mathbf{e}}_*^{-1}(\sigma) = \hat{\mathbf{e}}^{-1}(\sigma) = \frac{\mathbf{e}^{-1}(\sigma)}{\sigma^{\pi}}.$$ Then we conclude $(\hat{\mathbf{e}})^{-1}(\sigma) = \hat{\mathbf{e}}_*^{-1}(\sigma).

(c) We will prove $\hat{\sim}$ is commutative with $\hat{\sim}$. In fact, by (2.12) and step (a), for any $\sigma > 0$, we get

$$\hat{\mathbf{e}}^{-1}(\sigma) = \hat{\mathbf{e}}^{-1}(\sigma) = \frac{1}{\mathbf{e}^{-1}(\sigma^{-1})} = \frac{\mathbf{e}^{-1}(\sigma)}{\sigma^{\pi}} = \hat{\mathbf{e}}^{-1}(\sigma).$$ On the other hand

$$(\hat{\mathbf{e}})^{-1}(\sigma) = \frac{\mathbf{e}^{-1}(\sigma)}{\sigma^{\pi}} = \sigma^{\pi} \mathbf{e}^{-1}(\sigma) = \sigma^{\pi}.$$ Then we conclude $(\hat{\mathbf{e}})^{-1}(\sigma) = (\hat{\mathbf{e}})^{-1}(\sigma).

(2) Taking $\mathbf{e}$, for example, if $\mathbf{e} \in \Delta_{R^+}$, we will prove $\mathbf{e}_*^{-1} \in \Delta_{R^+}$ and $\mathbf{e}_* \in \nabla_{R^+}$.

Since $\mathbf{e} \in \Delta_{R^+}$, there exist constants $M_0 > 0, M_1 > 0$ and $M_2 > 0$ such that for any $\alpha, \beta > 0$

$$(2.16) \quad M_1 \mathbf{e}(\alpha) \mathbf{e}(\beta) \leq \mathbf{e}(M_0 \alpha \beta) \leq M_2 \mathbf{e}(\alpha) \mathbf{e}(\beta), \forall \alpha, \beta > 0.$$ (d) Setting $a = \mathbf{e}(\alpha)$ and $b = \mathbf{e}(\beta)$ in (2.16), we get

$$\mathbf{e}^{-1}(M_1 ab) \leq M_0 \mathbf{e}^{-1}(a) \mathbf{e}^{-1}(b), \forall a, b > 0.$$ Then by (2.9) we conclude that for any $\alpha, \beta > 0$

$$\mathbf{e}_*^{-1}(\frac{1}{M_1 \alpha \beta}) = \frac{M_1^\pi}{\alpha^{\pi} \beta^{\pi} \mathbf{e}^{-1}(M_1 \frac{1}{\alpha \beta})} \geq \frac{M_1^\pi}{M_0 \alpha^{\pi} \beta^{\pi} \mathbf{e}^{-1}(\frac{1}{\alpha}) \mathbf{e}^{-1}(\frac{1}{\beta})} = \frac{M_1^\pi}{M_0} \mathbf{e}_*^{-1}(\alpha) \mathbf{e}_*^{-1}(\beta).$$

(e) On the other hand, setting $a = \hat{\mathbf{e}}(\alpha)$ and $b = \hat{\mathbf{e}}(\beta)$ in (2.14), we get

$$\mathbf{e}^{-1}(M_2 ab) \geq M_0 \hat{\mathbf{e}}^{-1}(\alpha) \mathbf{e}^{-1}(b), \forall a, b > 0.$$
Then by (2.18) and step (1) we conclude that for any $\alpha, \beta > 0$

\[
\mathbb{C}^{-1}(\frac{1}{M_2^{\alpha\beta}}) = \frac{\hat{M}_2}{M_2^{\alpha\beta}} \mathbb{C}^{-1}(\frac{1}{M_2^{\alpha\beta}}) \leq \frac{\hat{M}_2}{M_2^{\alpha\beta}} \mathbb{C}^{-1}(\frac{1}{M_2^{\alpha\beta}})
\]

\[
= \frac{\hat{M}_2}{M_0} \mathbb{C}^{-1}(\alpha) \mathbb{C}^{-1}(\beta) = \frac{\hat{M}_2}{M_0} \mathbb{C}^{-1}(\alpha) \mathbb{C}^{-1}(\beta)
\]

\[
= \frac{\hat{M}_2}{M_0} \mathbb{C}^{-1}(\alpha) \mathbb{C}^{-1}(\beta).
\]

(d) and (e) imply that $\mathbb{C}^{-1} \in \Delta_{\mathbb{R}^+}$ and for $a = \mathbb{C}^{-1}(\alpha)$ and $b = \mathbb{C}^{-1}(\beta)$

\[
\mathbb{C}_*(\frac{\hat{M}_2}{M_0}ab) \leq \frac{1}{M_1} \mathbb{C}_*(a) \mathbb{C}_*(b),
\]

and for $a = \mathbb{C}^{-1}_*(\alpha)$ and $b = \mathbb{C}^{-1}_*(\beta)$

\[
\mathbb{C}_*(\frac{\hat{M}_2}{M_0}ab) \geq \frac{1}{M_2} \mathbb{C}_*(a) \mathbb{C}_*(b),
\]

which implies $\mathbb{C}_* \in \nabla_{\mathbb{R}^+}$. 

\[
\mathbb{C}, \mathbb{C}^{-1} \in \nabla_{\mathbb{R}^+}, \mathbb{C}^{-1}, \mathbb{C}^{-1} \in \Delta_{\mathbb{R}^+}
\]

can be obtained by the equation (2.18) and equation (2.12). \hfill \Box

Denote the $n$-cube centered at $x_0$ with the edge length of $2R$ by $Q_R(x_0) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \|x_i - (x_0)_i\| \leq R, i = 1, \ldots, n\}$ and $u(U) = \int_U u(x) \, dx := |U|^{-1} \int_U u(x) \, dx$ for any open set $U \subset \mathbb{R}^n$ and any $u \in L^1(U)$.

The following two Poincaré type inequalities in Musielak-Sobolev space are presented with a short proof.

**Lemma 2.4.** (Poincaré type inequality.) For any bounded, connected, open $U \subset \mathbb{R}^n$ with the cone property, there exists a constant $C = C(n, A, U)$, such that

\[
\|u - u_U\|_A \leq C\|\nabla u\|_A, \forall u \in W^{1,A}(U),
\]

in which $A \in \mathscr{A}$.

**Proof.** Suppose on the contrary that, for each integer $k \in \mathbb{N}$, there exists a function $u_k \in W^{1,A}(U)$ satisfying

\[
\|u_k - (u_k)_U\|_A \geq \frac{k}{\|\nabla u_k\|_A}.
\]

Define

\[
v_k := \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_A}
\]

for $k \in \mathbb{N}$.

Then

\[
(v_k)_U = 0, \|v_k\|_A = 1,
\]

and (2.17) implies

\[
\|\nabla v_k\|_A < \frac{1}{k} \text{ for } k \in \mathbb{N}.
\]
It is clear that the functions \( \{v_k : k \in \mathbb{N}\} \) are bounded in \( W^{1,A}(U) \). By Theorem 2.8 there exists a subsequence \( \{v_{k_j} : j \in \mathbb{N}\} \) of \( \{v_k : k \in \mathbb{N}\} \) and a \( v \in L^A(U) \) such that \( v_{k_j} \to v \) in \( L^A(U) \). Then by (2.18) it follows that

\[
(2.20) \quad v_U = 0, \quad \|v\|_A = 1.
\]

On the other hand (2.19) implies for each \( i = 1, \ldots, n \) and any \( \phi \in C_0^\infty(\Omega) \) that

\[
\hat{U} v \phi_x \ dx = \lim_{k_j \to +\infty} \int_U v_{k_j} \phi_x \ dx = -\lim_{k_j \to +\infty} \int_U (v_{k_j})_x \phi \ dx = 0.
\]

Then \( \nabla v = 0 \) a.e.. Since \( v \in W^{1,A}(U) \) and \( U \) is connected, \( v \) is a.e. a constant. And this contradicts to (2.20).

\[\square\]

**Lemma 2.5.** (Poincaré type inequality for a cube.) There exists a constant \( C = C(n, A) \), such that

\[
(2.21) \quad \|u - u_R\|_A \leq CR\|\nabla u\|_A, \quad \forall u \in W^{1,A}(Q_R(x)),
\]

in which \( u_R = u_{Q_R(x)} \) and \( A \in \mathcal{A} \).

**Proof.** The case \( U = Q_1(0) \) follows from Lemma 2.4. In general if \( u \in W^{1,A}(Q_R(x)) \), denote \( v(y) := u(x + ry) \) for any \( y \in Q_1(0) \), then \( v \in W^{1,A}(Q_1(0)) \), which implies

\[
\|v - v_1\|_{L^A(Q_1(0))} \leq C\|\nabla v\|_{L^A(Q_1(0))}.
\]

Changing variables, we get (2.21).

\[\square\]

3. Assumptions and Lemmas

In this section, in order to get the conclusions of this paper, we state the assumptions on \( A \). Firstly, the following assumption on \( A \) will be used.

\( (P_0^*) \quad A \in N(\Omega), \) and for a.e. \( x \in \Omega, A^{\frac{1}{n}}(x,t) \) is convex and differentiable on the variable \( t \) for \( \{t > 0\} \).

**Definition 3.1.** Under the assumption \( (P_0^*) \) we define an even function on the second variable \( A^* : \Omega \times \mathbb{R} \to [0, +\infty) \) as follows:

\[
(3.1) \quad (A^*)^{-1}(x,s) := s^{\frac{n+1}{n}}(A^{-1}(x,s))',
\]

for a.e. \( x \in \Omega \) and \( s > 0 \), where \( (A^{-1}(x,s))' \) is the derivative of \( A^{-1}(x,s) \) on the variable \( s \); and \( A^*(x,0) = 0 \) for a.e. \( x \in \Omega \).

**Remark 3.1.** Under the assumption \( (P_0^*) \), \( A^* \) in Definition 3.1 is well defined.
Proof. We only need to prove that, for a.e. $x \in \Omega$, the right hand side of (3.1) is monotone on $\{s > 0\}$. Indeed, setting $A(x,t) = s$, we get

$$s^{\frac{n+1}{n}}(A^{-1}(x,s))'_{s} = \frac{(A(x,t))^{1+\frac{1}{n}}}{A'_{t}(x,t)} = \frac{1}{-n(A^{-n}(x,t))_{t}'}.$$ 

Then by the assumption $(P_{*}^{0})$, for a.e. $x \in \Omega$, the right hand side of (3.1) is monotone on $\{s > 0\}$. □

We give the assumptions on $A^{*}$ induced by $A$ to proceed.

$(P_{1}^{*})$ $\Omega \subset \mathbb{R}^{n}(n \geq 2)$ is a bounded domain with the cone property, and $A^{*} \in N(\Omega);

$(P_{2}^{*})$ $A^{*}: \overline{\Omega} \times \mathbb{R} \to [0, +\infty)$ is continuous and $A^{*}(x,t) \in (0, +\infty)$ for $x \in \overline{\Omega}$ and $t \in (0, +\infty);

$(P_{5}^{*})$ There exist three positive constants $\delta_{0}, C_{0}$ and $t_{0}$ with $\delta_{0} < \frac{1}{n}$, $t_{0} > 0$ such that

$$|\nabla_{x} A^{*}(x,t)| \leq C_{0}(A^{*}(x,t))^{1+\delta_{0}}, \quad j = 1, \ldots, n,$$

for $x \in \Omega$ and $|t| \in [t_{0}, +\infty)$ provided $\nabla_{x} A^{*}(x,t)$ exists.

Remark 3.2. Under the assumption $(P_{0}^{*})$, the corresponding assumptions for $A^{*}$ analogous to $(P_{3})$ and $(P_{4})$ for $A$ in Section 2 are automatically satisfied.

Under the assumptions $(P_{0}^{*})$, $(P_{1}^{*})$, $(P_{2}^{*})$ and $(P_{5}^{*})$ it is clear that the following conclusions hold.

Lemma 3.1. Assume that $(P_{0}^{*})$, $(P_{1}^{*})$, $(P_{2}^{*})$ and $(P_{5}^{*})$ hold. Then

1. $(A^{*})_{*} = A$;
2. $A \in N(\Omega)$ satisfies the assumption $(P_{1})$ and $(P_{2})$;
3. There is a continuous embedding $W^{1,A^{*}}(\Omega) \hookrightarrow L^{A}(\Omega)$;
4. $A^{*} \ll A$, and there is a compact embedding $W^{1,A^{*}}(\Omega) \hookrightarrow \hookrightarrow L^{A^{*}}(\Omega)$.

The following assumption of increasing condition on $A^{*} \in N(\Omega)$ will be used.

$(P^{*})$ For any $\Omega_{0} \subseteq \Omega$ with cone property, there exists a strictly increasing differentiable function $A_{\Omega_{0}}^{*} \in \nabla_{R^{+}}$, with the uniform constants $M_{0}, M_{1}$ and $M_{2}$ for any $\Omega_{0} \subseteq \Omega$ in Definition 2.4 such that

$$A^{*}(x, \alpha t) \leq A_{\Omega_{0}}^{*}(\alpha)A^{*}(x,t), \quad \forall \alpha \geq 0, t \in \mathbb{R}, x \in \Omega_{0},$$

and

$$nA_{\Omega_{0}}^{*}(\alpha) > \alpha(A_{\Omega_{0}}^{*})'(\alpha), \quad \forall \alpha \geq 0.$$

We give a generalized Sobolev Poincaré type inequality in Musielak-Orlicz-Sobolev space with a short proof.
Lemma 3.2. (Sobolev Poincaré type inequality.) For any \( Q_R(x) \subset \mathbb{R}^n \), there exists a constant \( C = C(n, A^*) \), such that
\[
\mathcal{A}_R^{-1} \left( \frac{1}{Q_R(x)} \int_{Q_R(x)} A(y, \frac{|u - u_R|}{R}) \, dy \right) \leq C \mathcal{A}_R^{-1} \left( \frac{1}{Q_R(x)} \int_{Q_R(x)} A^*(y, |\nabla u|) \, dy \right)
\]
for any \( u \in W^{1,A^*}(Q_R(x)) \), where \( u_R = u|_{Q_R(x)} \) and \( \mathcal{A}_R = \mathcal{A}_{Q_R(x)} \).

Proof. By Lemma 2.5 for \( x = 0 \) and \( R = 1 \), there exists a constant \( C = C(n, A) \) such that
\[
\|v - v_1\|_{A^*} \leq C \|\nabla v\|_{A^*}, \quad \forall v \in W^{1,A^*}(Q_1(0)).
\]
Then by Lemma 3.1 (3), we get that, for any \( v \in W^{1,A^*}(Q_1(0)) \)
\[
\int_{Q_1(0)} A(y, |v - v_1|) \, dy = \int_{Q_1(0)} A(y, \frac{|v - v_1|}{\|v - v_1\|_A} \|v - v_1\|_A) \, dy
\]
\[
\leq \mathcal{A}_1(\|v - v_1\|_A) \int_{Q_1(0)} A(y, \frac{|v - v_1|}{\|v - v_1\|_A}) \, dy = \mathcal{A}_1(\|v - v_1\|_A)
\]
\[
\leq \mathcal{A}_1(\|v - v_1\|_{A^*} + \|\nabla v\|_{A^*})
\]
\[
\leq \mathcal{A}_1(C\mathcal{A}_1^{-1}(\int_{Q_1(0)} A^*(y, |\nabla v|) \, dy))
\]
\[
\leq \mathcal{A}_1(C\mathcal{A}_1^{-1}(\int_{Q_1(0)} A^*(y, |\nabla v|) \, dy)).
\]
Set \( u(x + R y) = v(y) \) for \( y \in Q_1(0) \). Then we get that, for any \( u \in W^{1,A^*}(Q_R(x)) \)
\[
\int_{Q_R(x)} A(y, |u - u_R|) \, dy \leq \mathcal{A}_R^{-1}(C\mathcal{A}_R^{-1}(\int_{Q_R(x)} A^*(y, |\nabla u|) \, dy))
\]
or equivalently, by setting \( u = \tilde{u} / \tilde{A}_R \), we conclude that, for all \( \tilde{u} \in W^{1,A^*}(Q_R(x)) \),
\[
\mathcal{A}_R^{-1}(\int_{Q_R(x)} A(y, \frac{\tilde{u} - \tilde{u}_R}{R}) \, dy) \leq C \mathcal{A}_R^{-1}(\int_{Q_R(x)} A^*(y, |\nabla \tilde{u}|) \, dy),
\]
where \( C = C(n, A^*) \).

The following lemma plays an important role in this paper, which is a generalization of Lemma 1 in [17].

Lemma 3.3. (Gehring type inequality.) Suppose that \( \mathcal{B} : [0, \infty) \to [0, \infty) \) is non-decreasing and differentiable, \( a \in (1, \infty) \), and \( h, H : [\mathcal{B}^{-1}(1), \infty) \to [0, \infty) \) are non-increasing with
\[
\lim_{t \to \infty} h(t) = 0, \quad \lim_{t \to \infty} H(t) = 0
\]
such that for \( t \in [\mathcal{B}^{-1}(1), \infty) \) and a constant \( C > 0 \),
\[
- \int_t^\infty \mathcal{B}(s) \, dh(s) \leq a\mathcal{B}(t) h(t) + CH(t).
\]
Then, for $\epsilon \in [0, \frac{1}{a-1})$,

$$-\int_{\mathbb{B}^{-1}(1)}^{\infty} \mathfrak{B}^{1+\epsilon}(t) \, dh(t) \leq \frac{1}{1 - \epsilon(a - 1)} \left( - \int_{\mathbb{B}^{-1}(1)}^{\infty} \mathfrak{B}(t) \, dh(t) \right) + \frac{C \epsilon}{1 - \epsilon(a - 1)} \left( - \int_{\mathbb{B}^{-1}(1)}^{\infty} \mathfrak{B}^{1+\epsilon}(t) \, dH(t) \right).$$

(3.6)

Proof. **Step 1.** Suppose that there exists a $j \in (\mathbb{B}^{-1}(1), \infty)$ such that $h(t) = 0$ and $H(t) = 0$ for $t \in [j, \infty)$, and for each $\epsilon \in (0, \infty)$ set

$$I(\epsilon) := -\int_{\mathbb{B}^{-1}(1)}^{\infty} \mathfrak{B}^{1+\epsilon}(t) \, dh(t) = -\int_{\mathbb{B}^{-1}(1)}^{j} \mathfrak{B}^{1+\epsilon}(t) \, dh(t).$$

Then integration by parts yields

$$I(\epsilon) = -\int_{\mathbb{B}^{-1}(1)}^{j} \mathfrak{B}(t) \mathfrak{B}'(t) \, dh(t)$$

$$= -\int_{\mathbb{B}^{-1}(1)}^{j} \mathfrak{B}(t) \, dh(t) + \int_{\mathbb{B}^{-1}(1)}^{j} (1 - \mathfrak{B}'(t)) \mathfrak{B}(t) \, dh(t)$$

$$= I(0) + (1 - \mathfrak{B}'(t)) \left( - \int_{t}^{j} \mathfrak{B}(s) \, dh(s) \right) \bigg|_{t=\mathbb{B}^{-1}(1)}^{t=j}$$

$$+ \epsilon \int_{\mathbb{B}^{-1}(1)}^{j} \mathfrak{B}^{1+\epsilon}(t) \, dh(t)$$

$$= I(0) + \epsilon J,$$

(3.7)

where

$$J := \int_{\mathbb{B}^{-1}(1)}^{j} \mathfrak{B}^{1+\epsilon}(t) \left( - \int_{t}^{j} \mathfrak{B}(s) \, dh(s) \right) \, d\mathfrak{B}(t).$$
With (3.5) and once more integration by parts we obtain

\[
J \leq \left[ a - 1 \right] B^{-1}(1) B^{-1}(1) \] 

\[
\frac{1}{1 + \epsilon} \int_{B^{-1}(1)}^{j} \mathcal{B}^e(t) \text{d}h(t) + C \int_{B^{-1}(1)}^{j} \mathcal{B}^{e-1}(t) H(t) \text{d}B(t) 
\]

\[
= \frac{1}{1 + \epsilon} \left[ -a B^{-1}(1) B^{-1}(1) - CH(B^{-1}(1)) \right] 
\]

\[
- \frac{a}{1 + \epsilon} \int_{B^{-1}(1)}^{j} \mathcal{B}^{1+e}(t) \text{d}h(t) + C \int_{B^{-1}(1)}^{j} \mathcal{B}^{e-1}(t) H(t) \text{d}B(t) 
\]

\[
= \frac{1}{1 + \epsilon} \int_{B^{-1}(1)}^{j} \mathcal{B}(t) \text{d}h(t) - \frac{a}{1 + \epsilon} \int_{B^{-1}(1)}^{j} \mathcal{B}^{1+e}(t) \text{d}h(t) 
\]

\[
+ C \int_{B^{-1}(1)}^{j} \mathcal{B}(t) H(t) \text{d}B(t) + \frac{C}{1 + \epsilon} H(B^{-1}(1)) 
\]

\[
= \frac{1}{1 + \epsilon} \int_{B^{-1}(1)}^{j} \mathcal{B}(t) \text{d}h(t) - \frac{a}{1 + \epsilon} \int_{B^{-1}(1)}^{j} \mathcal{B}^{1+e}(t) \text{d}h(t) 
\]

\[
- \frac{C}{1 + \epsilon} \int_{B^{-1}(1)}^{j} \mathcal{B}^{1+e}(t) \text{d}H(t) 
\]

\[
= \frac{1}{1 + \epsilon} I(0) + \frac{a}{1 + \epsilon} I(\epsilon) - \frac{C}{1 + \epsilon} \int_{B^{-1}(1)}^{j} \mathcal{B}^{1+e}(t) \text{d}H(t). 
\]

(3.8)

Then (3.7) and (3.8) imply that

\[
I(\epsilon) \leq \frac{1}{1 - \epsilon(a - 1)} I(0) - \frac{C \epsilon}{1 - \epsilon(a - 1)} \int_{B^{-1}(1)}^{j} \mathcal{B}^{1+e}(t) \text{d}H(t), 
\]

whenever \( \epsilon \in (0, \frac{1}{a-1}) \). So (3.6) follows.

**STEP 2.** In the general case, for each \( j \in (B^{-1}(1), \infty) \) set

\[
h_j(t) = \begin{cases} 
        h(t) & \text{if } t \in [B^{-1}(1), j), \\
        0 & \text{if } t \in [j, \infty) 
\end{cases} 
\]

and

\[
H_j(t) = \begin{cases} 
        H(t) & \text{if } t \in [B^{-1}(1), j), \\
        0 & \text{if } t \in [j, \infty). 
\end{cases} 
\]

Then \( h_j, H_j : [B^{-1}(1), \infty) \to [0, \infty) \) is non-increasing and for \( t \in [B^{-1}(1), \infty) \),

\[
- \int_{t}^{\infty} \mathcal{B}(s) \text{d}h_j(s) \leq a \mathcal{B}(t) h_j(t) + C H_j(t). 
\]
Hence by STEP 1, for $\epsilon \in [0, \frac{1}{a-1})$,
\[
- \int_{\mathcal{B}^{-1}(1)}^{j} \mathcal{B}^{1+\epsilon}(t) \, dh(t) = - \int_{\mathcal{B}^{-1}(1)}^{j} \mathcal{B}^{1+\epsilon}(t) \, dh(t) \\
\leq \frac{1}{1 - \epsilon(a - 1)} \left( - \int_{\mathcal{B}^{-1}(1)}^{j} \mathcal{B}(t) \, dh(t) \right) \\
+ \frac{Ce}{1 - \epsilon(a - 1)} \left( - \int_{\mathcal{B}^{-1}(1)}^{j} \mathcal{B}^{1+\epsilon}(t) \, dH(t) \right) \\
\leq \frac{1}{1 - \epsilon(a - 1)} \left( - \int_{\mathcal{B}^{-1}(1)}^{\infty} \mathcal{B}(t) \, dh(t) \right) \\
+ \frac{Ce}{1 - \epsilon(a - 1)} \left( - \int_{\mathcal{B}^{-1}(1)}^{\infty} \mathcal{B}^{1+\epsilon}(t) \, dH(t) \right).
\]

Then we obtain \(3.10\) by letting \(j \to \infty\). \(\square\)

We get the following main theorem of this paper.

**Theorem 3.1.** Suppose that $\mathcal{C} \in \Delta_{\mathbb{R}^n \cap \mathbb{N}}$ and $0 \leq g \in L^1(Q_1(x_0))$ with $\int_{Q_1(x_0)} g(x) \, dx = 1$ and for any $x \in Q_1 = Q_1(x_0) := \{x \in \mathbb{R}^n : |x^i - x_0^i| \leq 1, i = 1, \ldots, n\}$, any $R < \text{dist}(x, \partial Q_1)$, the following estimate (reverse type inequality) holds

\[
\int_{Q_1} g(x) \, dx \leq b\mathcal{C} \left( \int_{Q_n(x)} \mathcal{C}^{-1}(g) \, dx \right) + b
\]

with $b > 0$ being a constant. Then there exists a constant $\epsilon' > 0$, depending only on $\mathcal{C}$ and $b$, such that for $\epsilon \in [0, \epsilon')$,

\[
\int_{Q_1} \mathcal{M}(g) \, dx \leq c\mathcal{M} \left( \int_{Q_1} g \, dx + 1 \right),
\]

where $\mathcal{M}(t) := t \cdot (\mathcal{C}^{-1}(t))^e$; and

\[
\text{STEP 1.} \quad \text{We will show that}
\]

\[
\int_{E(G,t)} G \, dx \leq a \left[ \frac{t}{\mathcal{C}^{-1}(t)} \int_{E(G,t)} \mathcal{C}^{-1}(G) \, dx + C \int_{E(F,t)} F \, dx \right]
\]
for \( t \in [t_1, \infty) \), where \( a \) is a positive constant depending only on \( C, b; C \) is a positive constant depending only on \( C \) and \( F(x) := |Q_1| = a \) a constant.

Fix \( t \in [t_1, \infty) \), and for \( \epsilon_0 > 0 \), define \( s = s(t, \epsilon_0) \) by

\[
s := bc_3\left(\frac{5 + (2c_0c_4 + 1)\epsilon_0}{\epsilon_0c_1}\right)t,
\]

where \( c_0 = c_0(C), c_1 = c_1(C) > 0, c_3 = c_3(C) > 0 \) and \( c_4 = c_4(C, g) \) are constants determined in later equations (3.19) and (3.21). And take \( \epsilon_0 \) small enough such that \( s > t \). It is clear that

\[
Q_1 = \cup_{k \geq 0} C_k.
\]

Since

\[
\int_{Q_1} G \, dx = \int_{Q_1} t_1 g(x)|Q_1| \, dx = t_1,
\]

we can employ the famous Calderón-Zygmund decomposition to obtain a sequence of parallel \( n \)-cubes \( Q^k_j \subset C_k, j \in \mathbb{N} \), such that

\[
G \leq s, \quad \text{a.e. in } C_k \setminus \cup_j Q^k_j
\]

and

\[
s < \int_{Q^k_j} G \, dx \leq 2^n s, \quad \forall j \in \mathbb{N}.
\]

Then \( |E(G, s) \setminus \cup_{k,j} Q^k_j| = 0 \), which implies from the above two inequalities that

\[
\int_{E(G, s)} G \, dx \leq \Sigma_{k,j} \int_{Q^k_j} G \, dx \leq 2^n s \Sigma_{k,j} |Q^k_j|.
\]

Denote by \( \overline{Q}^k_j \) the parallel cube to \( Q^k_j \) with the same center and double side, and \( D_k := \cup_j \overline{Q}^k_j \). It is clear that

\[
\overline{Q}^k_j \subset \cup_{j=k-1}^{k+1} C_i.
\]

Since \( D_k \) is bounded, by the well-known covering theorem, there exists a countable disjoint cubes \( \{\overline{Q}^k_{j_h}\}_{h=1}^{\infty} \) which is a subsequence of cubes \( \{\overline{Q}^k_j\} \) such that

\[
|D_k| \leq 5^n \Sigma_{i=1}^{\infty} |\overline{Q}^k_{j_h}|.
\]
Denote by $\overline{Q}_R$ as one of the cubes $\{\overline{Q}_h^k\}_{h=1}^\infty$. Multiplying (3.19) by $t_1|Q_1|$, by (3.15), we obtain

\[
(3.19) \quad s < \int_{Q} G \, dx \\
\leq bt_1|Q_1|\mathcal{C}\left(\int_{\overline{Q}_R} \mathcal{C}^{-1}(g) \, dx \right) + bt_1|Q_1| \\
\leq \frac{bt_1|Q_1|}{\mathcal{C}(c_1^{-1}\mathcal{C}^{-1}(t_1|Q_1|))} \mathcal{C}\left(\frac{\mathcal{C}^{-1}(t_1|Q_1|)}{C_1} \right) \mathcal{C}\left(\frac{1}{t_1|Q_1|} \int_{\overline{Q}_R}\mathcal{C}^{-1}(G) \, dx \right) \\
\leq \frac{bt_1|Q_1|}{\mathcal{C}(c_1^{-1}\mathcal{C}^{-1}(t_1|Q_1|))} \mathcal{C}\left(\int_{\overline{Q}_R}\mathcal{C}^{-1}(G) \, dx \right) + bt_1|Q_1| \\
= bc_3 \mathcal{C}\left(\int_{\overline{Q}_R}\mathcal{C}^{-1}(G) \, dx \right) + bt_1|Q_1|,
\]

where $c_1 = c_1(\mathcal{C}) > 0$ satisfies $\mathcal{C}^{-1}(\alpha\beta) \leq c_1\mathcal{C}^{-1}(\alpha)\mathcal{C}^{-1}(\beta)$, $\forall \alpha, \beta > 0$, $c_2 = c_2(\mathcal{C}) > 0$ satisfies $\mathcal{C}(\alpha)\mathcal{C}(\beta) \leq c_2\mathcal{C}(\alpha), \forall \alpha, \beta > 0$; and $c_3 := \frac{t_1|Q_1|}{\mathcal{C}(c_1^{-1}\mathcal{C}^{-1}(t_1|Q_1|))}$. Then by the definition of $s$ we can see

\[
t \leq \mathcal{C}\left(\frac{\epsilon_0 c_1^{-1}}{5 + (2c_0c_4 + 1)\epsilon_0} \right) \left(\mathcal{C}\left(\int_{\overline{Q}_R}\mathcal{C}^{-1}(G) \, dx \right) + c_3^{-1}t_1|Q_1| \right).
\]

Since $\mathcal{C} \in N$ and $\mathcal{C}^{-1} \in \nabla R^+$ is concave, it is easy to see that

\[
(3.20) \quad \mathcal{C}^{-1}(t) \leq c_1 \frac{\epsilon_0 c_1^{-1}}{5 + (2c_0c_4 + 1)\epsilon_0} \left(\int_{\overline{Q}_R}\mathcal{C}^{-1}(G) \, dx \right) + \mathcal{C}^{-1}(c_3^{-1}t_1|Q_1|),
\]

which implies that

\[
(3.21) \quad \frac{5 + (2c_0c_4 + 1)\epsilon_0}{\epsilon_0} \mathcal{C}^{-1}(t)|\overline{Q}_R| \leq \int_{\overline{Q}_R}\mathcal{C}^{-1}(G) \, dx + \mathcal{C}^{-1}(c_3^{-1}t_1|Q_1|)|\overline{Q}_R| \\
= \int_{\overline{Q}_R}\mathcal{C}^{-1}(G) \, dx + c_4|\overline{Q}_R|,
\]

where $c_4 := \mathcal{C}^{-1}(c_3^{-1}t_1|Q_1|)$. We will estimate the right-hand side of (3.21). It is easy to see that

\[
(3.22) \quad \int_{\overline{Q}_R}\mathcal{C}^{-1}(G) \, dx \leq \int_{E(G,t_2)|\overline{Q}_R}\mathcal{C}^{-1}(G) \, dx + \mathcal{C}^{-1}(t)|\overline{Q}_R|.
\]

At the same time, (2.12) and Young inequality for $\mathcal{C} \in \Delta R^+$ yield that

\[
|\overline{Q}_R| = \mathcal{C}^{-1}(\frac{|Q_R|}{t}) |\overline{Q}_R| \\
\leq c_0 \mathcal{C}^{-1}(t) \mathcal{C}^{-1}(\frac{|Q_R|}{t}) |\overline{Q}_R| \\
\leq c_0 \mathcal{C}^{-1}(t) \left[ \mathcal{C}^{-1}(\frac{1}{t} |E(1,t) \cap Q_R| + |\overline{Q}_R|) \right] |\overline{Q}_R| \\
\leq c_0 \mathcal{C}^{-1}(t) \left[ E(1,t) \cap Q_R | + 2c_0 \mathcal{C}^{-1}(t)|\overline{Q}_R|.
\]
Therefore, (3.21), (3.22) and (3.23) imply that
\begin{equation}
(3.24) \quad \frac{5}{\epsilon_0} \int_{\partial R_n} \mathcal{C}^{-1}(G) \, dx + c_0 c_4 \frac{1}{t} \left| E(1, t) \cap \partial R_n \right|.
\end{equation}
Combining (3.17), (3.18) and (3.24) we get
\[|D_k| \leq \epsilon_0 5^{n-1} \sum_{i=k+1}^n \left[ \frac{1}{\epsilon^{-1}(t)} \int_{E(G, t) \cap C_i} \mathcal{C}^{-1}(G) \, dx + c_0 c_4 \frac{1}{t} \left| E(|Q_1|, t) \cap C_i \right| \right].\]
Taking sum with \( k \) we can see
\[
\sum_k |D_k| \leq \epsilon_0 5^n \left[ \frac{1}{\epsilon^{-1}(t)} \int_{E(G, t)} \mathcal{C}^{-1}(G) \, dx + c_0 c_4 \frac{1}{t} \left| E(|Q_1|, t) \cap Q_1 \right| \right],
\]
with (3.16), which implies
\begin{equation}
(3.25) \quad \int_{E(G, s)} G \, dx \leq a_1 b \left[ \frac{t}{\epsilon^{-1}(t)} \int_{E(G, t)} \mathcal{C}^{-1}(G) \, dx + c_0 c_4 \left| E(|Q_1|, t) \right| \right]
\end{equation}
with
\[a_1 := 10^n \epsilon_0 c_3 \mathcal{C} \left( \frac{5 + (2c_0 c_4 + 1)\epsilon_0}{\epsilon_0} \right) .\]
On the other hand, by the definition of \( s \) we get
\[
\int_{E(G, t) \setminus E(G, s)} G \, dx \leq \int_{E(G, t) \setminus E(G, s)} \frac{G}{\mathcal{C}^{-1}(G)} \mathcal{C}^{-1}(G) \, dx
\]
\[
\leq \frac{s}{\epsilon^{-1}(t)} \int_{E(G, t)} \mathcal{C}^{-1}(G) \, dx
\]
\[
\leq b c_3 \mathcal{C} \left( \frac{5 + (2c_0 c_4 + 1)\epsilon_0}{\epsilon_0} \right) \frac{t}{\epsilon^{-1}(t)} \int_{E(G, t)} \mathcal{C}^{-1}(G) \, dx
\]
\[
:= a_2 b \left[ \frac{t}{\epsilon^{-1}(t)} \int_{E(G, t)} \mathcal{C}^{-1}(G) \, dx \right]
\]
with \( a_2 := c_3 \mathcal{C} \left( \frac{5 + (2c_0 c_4 + 1)\epsilon_0}{\epsilon_0} \right). \) The above inequality and (3.25) imply
\[
\int_{E(G, t)} G \, dx \leq a \left[ \frac{t}{\epsilon^{-1}(t)} \int_{E(G, t)} \mathcal{C}^{-1}(G) \, dx + c_0 c_4 \frac{1}{|Q_1|} \int_{E(F, t)} F \, dx \right]
\]
with \( a = (a_1 + a_2) b \), where \( F(x) := |Q_1| = \text{a constant} \). The proof of (3.12) is completed.

**STEP 2.** We will prove the conclusion of the lemma. Now for \( t \in [t_1, \infty) \), set \( h(t) := \int_{E(G, t)} \mathcal{C}^{-1}(G) \, dx = \int_{E(\epsilon^{-1}(G), \epsilon^{-1}(t))} \mathcal{C}^{-1}(G) \, dx \) and \( H(t) := \int_{E(F, t)} F \, dx = \int_{E(\epsilon^{-1}(F), \epsilon^{-1}(t))} \mathcal{C}(\mathcal{C}^{-1}(F)) \, dx \). Then \( h, H : [t_1, \infty) \to [0, \infty) \) is non-increasing and
\[
\lim_{t \to \infty} h(t) = 0, \quad \lim_{t \to \infty} H(t) = 0,
\]
and it is easy to verify that
\begin{equation}
(3.26) \quad \int_{E(F, t)} \mathcal{D}(P(x)) \, dx = - \int_t^\infty \frac{\mathcal{D}(s)}{s} \, dh(s)
\end{equation}
for any $\mathfrak{D} \in N$, any $P \in L^D(\Omega)$ and any $t \in [t_1(\mathfrak{D}), \infty)$. Then (3.12) implies that $h$ satisfies (3.26) for $t' = \mathcal{C}^{-1}(t)$ by setting $\mathfrak{D}(s) = \frac{\epsilon(s)}{s}$ for $s \in [t_1, \infty)$ in Lemma 3.3. Then the conclusion of Lemma 3.3 implies, for $\epsilon \in [0, \frac{1}{a-1})$, that

$$
\int_{t_1}^{\infty} \mathcal{C}(t') \left( \frac{\mathcal{C}(t')}{t'} \right)^{\epsilon} \, dh(t') \leq \frac{1}{1 - \epsilon(a - 1)} \left( - \int_{t_1}^{\infty} \mathcal{C}(t') \frac{dh(t')}{t'} \right) + \frac{aC\epsilon}{1 - \epsilon(a - 1)} \left( - \int_{t_1}^{\infty} \mathcal{C}(t') \left( \frac{\mathcal{C}(t')}{t'} \right)^{\epsilon} \, dh(t') \right).
$$

Combining with (3.26) we get

$$
\int_{E(G, t_1)} G \cdot \left( \frac{G}{\mathcal{C}^{-1}(G)} \right)^{\epsilon} \, dx \leq \frac{1}{1 - \epsilon(a - 1)} \int_{E(G, t_1)} G \, dx + \frac{aC\epsilon}{1 - \epsilon(a - 1)} \int_{E(F, t_1)} F \cdot \left( \frac{F}{\mathcal{C}^{-1}(F)} \right)^{\epsilon} \, dx,
$$

where $C > 0$ is a constant depending on $\mathcal{C}$ and $b$. Denote $\mathfrak{U}(t) := \left( \frac{1}{\mathcal{C}(t)} \right)^{\epsilon}$. Since $G\mathfrak{U}(G) \leq G$ in $Q_1 \setminus E(G, t_1)$, we conclude for $\epsilon \in [0, \frac{1}{a-1})$, there exists a constant $C_3 > 0$ such that

$$
\int_{Q_1} G\mathfrak{U}(G) \, dx \leq \frac{C_3}{1 - \epsilon(a - 1)} \int_{Q_1} G \, dx + \frac{C_3\epsilon}{1 - \epsilon(a - 1)} \int_{Q_1} F \cdot \mathfrak{U}(F) \, dx.
$$

It is easy to verify $\mathfrak{U} \in \Delta_{R+}$. With $\mathfrak{U}(t_1) = 1$ and (3.13), we get

$$
\int_{Q_1} t_1 |Q_1| g \cdot \mathfrak{U}(t_1) \mathfrak{M}(g) \, dx \leq C_4 \int_{Q_1} t_1 |Q_1| g \, dx \cdot \mathfrak{U} \left( \int_{Q_1} t_1 |Q_1| g \, dx \right) + C_4 \int_{Q_1} F \cdot \mathfrak{U}(F) \, dx,
$$

which implies that

$$
\int_{Q_{1/2}} \mathfrak{M}(g) \, dx \leq c(n) \int_{Q_1} \mathfrak{M}(g) \, dx \leq c\mathfrak{M} \left( \int_{Q_1} g \, dx + 1 \right),
$$

where $c$ is a positive constant depending on $n$, $\mathcal{C}$, $b$ and $\epsilon$. $\square$

We give another short remark for Theorem 3.1 to end this section.

**Remark 3.3.** It is easy to verify that $\mathfrak{M} \in N$ and $\mathfrak{M} \gg \mathcal{C}$.

### 4. The regularity of the minimizers

In the this this section, we suppose $A$ satisfies Condition $\mathcal{A}^*$, denoted by $A \in \mathcal{A}^*$:

- $(\mathcal{A}^*)$ A satisfies assumptions $(P_0^*)$, $(P_1^*)$, $(P_2^*)$, $(P_3^*)$ and $(P^*)$ in Section 3.

Consider the integral functionals as follows

$$
E(v) = E(v, \Omega) = \int_{\Omega} f(x, \nabla v(x)) \, dx,
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$. The main goal of this section is to prove the following theorem:

**Theorem 4.1.**

- Suppose $A \in \mathcal{A}^*$ and $(\mathcal{A}^*)$ holds.
- Let $u \in W_{0}^{1, \mathcal{C}^{-1}(\Omega)}(\Omega)$ be a minimizer of $E$ in $W_{0}^{1, \mathcal{C}^{-1}(\Omega)}(\Omega)$.
- Then $u \in C^{1, \alpha}(\Omega)$ for some $0 < \alpha < 1$.

**Proof.**

...
where \( v \in W^{1,A}(\Omega) \) and \( f(x, z) \) is a Carathéodory function on \( \Omega \times \mathbb{R}^n \) satisfying
\[
(4.2) \quad b_2 A\left(x, \sum_{i=1}^n |z_i|\right) - b_1 \leq f(x, z) \leq b_3 A\left(x, \sum_{i=1}^n |z_i|\right) + b_1
\]
with \( b_1, b_2 \) and \( b_3 \) being non-negative constants, \( A \in N(\Omega) \cap \mathcal{S}'^* \) satisfying \( (C_1) \) (see in Section 2).

**Definition 4.1.** A function \( u \in W^{1,A}_{\text{loc}}(\Omega) \) is said to be a local minimizer of \( E \) if
\[
(4.3) \quad E(u; \text{supp}\varphi) \leq E(u + \varphi; \text{supp}\varphi) \quad \text{for any} \quad \varphi \in W^{1,A}_0(\Omega) \quad \text{with} \quad \text{supp}\varphi \subset \subset \Omega.
\]

**Lemma 4.1.** Suppose \( u \in W^{1,A}_{\text{loc}}(\Omega) \) is a local minimizer of the functional \( E \). Then there exist constants \( \theta = \theta(\eta_1, A) \in (0, 1) \), \( c = c(\eta_1, A) > 0 \) and \( c' = c'(\eta_1, A) > 0 \), such that, for any \( x_0 \in \Omega \), \( Q_R(x_0) \subset \Omega \), \( 0 < t < s < R \) and \( a \in (-\infty, +\infty) \), the following inequality holds:
\[
(4.4) \quad \int_{Q_s(x_0)} A(x, |\nabla u|) \, dx \leq \theta \int_{Q_s(x_0)} A(x, |\nabla u|) \, dx + c \int_{Q_s(x_0)} A(x, \left|\frac{u-a}{s-t}\right|) \, dx + c'|Q_s(x_0)|.
\]

**Proof.** For \( x_0 \in \Omega \), \( Q_R(x_0) \subset \Omega \), \( 0 < t < s < R \) and \( a \in (-\infty, +\infty) \), choose \( \eta \in C_0^\infty(\Omega) \) such that \( 0 \leq \eta \leq 1 \), \( \eta|_{Q_t} \equiv 1 \) and \( |\nabla \eta| \leq \frac{2}{s-t} \). Set \( v = u - \eta(u-a) \).

Since \( u \) is the local minimizer of \( E \), we get
\[
(4.5) \quad E(u, Q_s) \leq E(v, Q_s).
\]

From \( (4.2) \) and \( (4.5) \) we conclude
\[
(4.6) \quad -b_1 |Q_s| + b_2 \int_{Q_s} A(x, |\nabla u|) \, dx \leq b_1 |Q_s| + b_3 \int_{Q_s} A(x, |\nabla v|) \, dx,
\]
which implies
\[
(4.7) \quad \int_{Q_s} A(x, |\nabla u|) \, dx \leq b_5 \int_{Q_s} A(x, |\nabla v|) \, dx + b_4 |Q_s|,
\]
where \( b_4 = \frac{b_5}{b_2} \) and \( b_5 = \frac{b_1}{b_2} \). By \( \nabla v = (1-\eta)\nabla u - (u-a)\nabla \eta \) we get
\[
\int_{Q_s} A(x, |\nabla v|) \, dx
\]
\[
\leq \int_{Q_s} A(x, |(1-\eta)\nabla u| + |(u-a)\nabla \eta|) \, dx
\]
\[
\leq \int_{Q_s} A(x, 2\max\{|(1-\eta)\nabla u|, |(u-a)\nabla \eta|\}) \, dx
\]
\[
\leq \hat{A}(2) \left( \int_{Q_s} A(x, |(1-\eta)\nabla u|) \, dx + \int_{Q_s} A(x, |(u-a)\nabla \eta|) \, dx \right)
\]
\[
\leq \hat{A}(2) \int_{Q_s \setminus Q_t} A(x, |\nabla u|) \, dx + \left( \hat{A}(2) \right)^2 \int_{Q_s} A(x, \left|\frac{u-a}{s-t}\right|) \, dx.
\]
Then (4.7) and (4.8) imply

\[
\int_{Q_{r}} A(x, |\nabla u|) \, dx \leq \int_{Q_{r}} A(x, |\nabla u|) \, dx
\]  

(4.9)

\[
\leq c_{1} \int_{Q_{s} \setminus Q_{r}} A(x, |\nabla u|) \, dx + c_{2} \int_{Q_{r}} A(x, \frac{|u - a|}{s - t}) \, dx + b_{4}|Q_{s}|,
\]

where \( c_{1} = b_{5} \mathfrak{A}(2) \) and \( c_{2} = b_{5} (\mathfrak{A}(2))^{2} \). Adding \( c_{1} \int_{Q_{s} \setminus Q_{r}} A(x, |\nabla u|) \, dx \) on both sides of (4.9), we conclude

\[
(1 + c_{1}) \int_{Q_{r}} A(x, |\nabla u|) \, dx \leq c_{1} \int_{Q_{r}} A(x, |\nabla u|) \, dx
\]  

(4.10)

\[
+ c_{2} \int_{Q_{r}} A(x, \frac{|u - a|}{s - t}) \, dx + b_{4}|Q_{s}|.
\]

Then we can get the conclusion of the lemma from (4.10) if we take \( \theta = \frac{c_{2}}{1+c_{1}} \) and \( c' = \frac{b_{4}}{1+c_{1}} \).

**Lemma 4.2.** (Caccioppoli type inequality) Suppose \( u \in W^{1,A}_{\text{loc}}(\Omega) \) is a local minimizer of the functional \( E \), and there exists a constant \( T_{0} \geq 1 \) such that \( \mathfrak{A} \) induced by \( \mathfrak{A}(T_{0}) = 1 \). Then there exist positive constants \( c_{3} = c_{3}(b_{1}, A) \) and \( c_{4} = c_{4}(b_{1}, A), \) such that, for any \( x_{0} \in \Omega, Q_{R}(x_{0}) \subset \Omega, 0 < \rho < R \) and \( a \in (-\infty, +\infty), \) the following inequality holds:

\[
\int_{Q_{\rho}(x_{0})} A(x, |\nabla u|) \, dx \leq c_{3} \int_{Q_{\rho}(x_{0})} A(x, \frac{|u - a|}{R - \rho}) \, dx + c_{4}|Q_{R}(x_{0})|.
\]

(4.11)

In particular, there exist positive constants \( c_{5} = c_{5}(b_{1}, n, A) \) and \( c_{6} = c_{6}(b_{1}, n, A), \) such that, for any \( x_{0} \in \Omega \) and \( Q_{R}(x_{0}) \subset \Omega, \) the following inequality holds:

\[
\int_{Q_{\rho}(x_{0})} A(x, |\nabla u|) \, dx \leq c_{5} \int_{Q_{R}(x_{0})} A(x, \frac{|u - u_{R}|}{R}) \, dx + c_{6}.
\]

(4.12)

**Proof.** Take any \( x_{0} \in \Omega, Q_{R}(x_{0}) \subset \Omega, 0 < \rho < R \) and \( a \in (-\infty, +\infty). \) For \( t \in (0, R], \) set \( f(t) = \int_{Q_{t}} A(x, |\nabla u|) \, dx. \) Choose \( 0 < r < 1 \leq T_{0} \) such that \( \theta \mathfrak{A}(T_{0}/r) < 1. \) Let

\[
t_{0} = \rho, \; t_{i+1} - t_{i} = \left(1 - \frac{r}{T_{0}}\right) \left(\frac{r}{T_{0}}\right)^{i}(R - \rho), \; i = 1, 2, \ldots,
\]

and denote \( L := \int_{Q_{R}} A(x, \frac{|u - a|}{R - \rho}) \, dx. \) Iterating (4.4) we get

\[
f(t) = f(t_{0}) \leq \theta f(t_{1}) + c \int_{Q_{t_{1}}} A(x, \frac{|u - a|}{(1 - \frac{r}{T_{0}})(R - \rho)}) \, dx + c'|Q_{t_{1}}|
\]

\[
\leq \theta f(t_{1}) + cL\mathfrak{A}(\frac{1}{1 - \frac{r}{T_{0}}}) + c'|Q_{R}| \leq \ldots
\]

\[
\leq \theta^{k} f(t_{k}) + cL\mathfrak{A}(\frac{1}{1 - \frac{r}{T_{0}}})\Sigma_{i=0}^{k-1} \left(\frac{\theta \mathfrak{A}(T_{0}/r)}{r}\right)^{i} + c'|Q_{R}|\Sigma_{i=0}^{k-1}\theta^{i}.
\]
Sending $k \to +\infty$ we can get (4.11). And (4.12) is obtained by taking $\rho = R/2$ and $a = u_R$ in (4.11).

\[\square\]

Lemma 4.3. For any $Q_R \subset \Omega$, there exists a positive constant $c_7 = c_7(n, A)$ such that, for any $u \in W^{1,A}(Q_R)$, the following inequality holds:

\[
\int_{Q_R} A(x, |\nabla u|) \, dx \leq c_7 \mathfrak{A}_R \mathfrak{A}_R^{-1} \left( \int_{Q_R} \mathfrak{A}_R \mathfrak{A}_R^{-1} (A(x, |\nabla u|)) \, dx \right) + c_7. \tag{4.13}
\]

Proof. Since $A \in N(\Omega)$ satisfies

\[A(x, \alpha t) \geq \mathfrak{A}_R(\alpha) A(x, t), \quad \forall \alpha \geq 0, t \in \mathbb{R}, x \in Q_R,\]

denoting $\mathfrak{A}_R(\alpha) = \beta$ and $A(x, t) = s$, we can get

\[A(x, \mathfrak{A}_R^{-1}(\beta) A^{-1}(x, s)) \geq \beta s, \quad \forall \beta \geq 0, s \geq 0, x \in \Omega,\]

or equivalently

\[
A^{-1}(x, \beta s) \leq \mathfrak{A}_R^{-1}(\beta) A^{-1}(x, s), \quad \forall \beta \geq 0, s \geq 0, x \in \Omega. \tag{4.14}
\]

By assumption ($P_2^*$) and Lemma 3.1 (2), there exists a positive constant $\tilde{C}$, such that

\[
A^*(x, A^{-1}(x, 1)) \leq \tilde{C}, \quad \forall x \in \overline{\Omega}. \tag{4.15}
\]

By $\widehat{\mathfrak{A}} \in \nabla_{R+}$, we get

\[
\widehat{\mathfrak{A}}(\alpha \beta) \leq C_1 \widehat{\mathfrak{A}}(\alpha) \widehat{\mathfrak{A}}(\beta), \quad \forall \alpha, \beta > 0. \tag{4.16}
\]
Lemma 4.4. Suppose \( u \in W^{1,A}(\Omega) \) and there exist two constants \( R_0 > 0 \) and \( L > 0 \), not depending on \( R \), such that

\[
\mathfrak{A}_R(t^{-1}) \leq L \mathfrak{A}_R(t^{-1})
\]

for any \( Q_R \subset Q_1 \) and any \( t \) with \( |Q_R| \leq t \leq |Q_{R_0}| \). Then there exists a constants \( c_8 = c_8(n, A, \Omega, L) \), not depending on \( R \), and a constant \( R_1 \), such that for any \( \mathfrak{A}_R(|Q_R|) \leq s \leq \mathfrak{A}_R(|Q_{R_1}|) \),

\[
\mathfrak{A}_R \mathfrak{A}_R^{-1}(s^{-1}) \leq c_8 \mathfrak{A}_R \mathfrak{A}_R^{-1}(s^{-1}).
\]

Proof. Setting \( s_{R, t} := \mathfrak{A}_R(t) \), by (2.8), there exists a constant \( c_0 > 0 \), not depending on \( R \), and a constant \( R_1 \leq R_0 \), such that

\[
\frac{1}{s_{R, t}} = (\mathfrak{A}_R(t))^\frac{1}{n} \geq c_0 t, \forall t \text{ with } |Q_R| \leq t \leq |Q_{R_1}|
\]

and

\[
s_{R_1, |Q_{R_1}|} = (\mathfrak{A}_{R_1}(|Q_{R_1}|))^\frac{1}{n} = c_0 |Q_{R_0}|.
\]
By (4.17), we get
\[(4.19) \quad \hat{A}_R(t) \leq L\hat{A}_R(t)\]
or equivalently,
\[(4.20) \quad A_R(t)A_R\left(\frac{1}{L}\right) \geq L^{-1}\]
for all \(t \) with \(|Q_R| \leq t \leq |Q_{R_{1}}|\). Then
\[
\hat{A}_R^{-1}(s_{R,t}) \leq \hat{A}_R^{-1}(Ls_{R,t}).
\]
By (4.9) it is clear that
\[
\hat{A}_R^{*^{-1}}(s_{R,t}) = \hat{A}_R^{-1}(s_{R,t})s_{R,t}^{\frac{1}{R}}.
\]
Then for any \(t \) with \(|Q_R| \leq t \leq |Q_{R_{1}}|\), we can see
\[
\hat{A}_R\hat{A}_R^{*^{-1}}(s_{R,t}) = \frac{1}{A_R\left(\frac{1}{A_R^{*^{-1}}(s_{R,t})s_{R,t}^{\frac{1}{R}}}\right)} = \hat{A}_R(\hat{A}_R^{-1}(s_{R,t})s_{R,t}^{\frac{1}{R}})
\]
\[
\leq \hat{A}_R(\hat{A}_R^{-1}(Ls_{R,t})s_{R,t}^{\frac{1}{R}}) \leq C_1 Ls_{R,t} \cdot \hat{A}_R(s_{R,t}^{\frac{1}{R}})
\]
\[
= C_1 Ls_{R,t} \cdot \hat{A}_R(c_0)^{\frac{1}{R}} \leq C_1^2 L^2 s_{R,t} \cdot \hat{A}_R(c_0)^{\frac{1}{R}}
\]
\[
\leq C_1^2 L^2 \cdot \hat{A}_R \cdot \hat{A}_R^{-1}(s_{R,t}) \cdot \hat{A}_R(c_0)^{\frac{1}{R}} \cdot \hat{A}_R(s_{R,t}^{\frac{1}{R}})
\]
\[
\leq C_1^2 L^2 \cdot \hat{A}_R \cdot \hat{A}_R^{-1}(s_{R,t}) \cdot \hat{A}_R(c_0)^{\frac{1}{R}} \cdot \hat{A}_R(s_{R,t}^{\frac{1}{R}})
\]
\[
\leq C_2 L^2 \cdot \hat{A}_R \cdot \hat{A}_R^{-1}(s_{R,t}) \cdot \hat{A}_R(c_0)^{\frac{1}{R}} \cdot \hat{A}_R(s_{R,t}^{\frac{1}{R}})
\]
\[
:= c_8 A_R A_R^{*^{-1}}(s_{R,t})
\]
or equivalently
\[
\hat{A}_R\hat{A}_R^{*^{-1}}(s_{R,t}) \leq c_8 A_R A_R^{*^{-1}}(s_{R,t})
\]
for any \(s \) with \(A_R(|Q_R|) \leq s \leq A_R(|Q_{R_{1}}|)\). \(\square\)

By Jensen’s inequality it is easy to get the following lemma.

**Lemma 4.5.** Suppose there exists a constant \(R_0 > 0\), such that for any \(Q_R \subset \Omega\) with \(R \leq R_0\), \(A_R A_R^{*^{-1}} \in N(Q_R)\). Then for any \(0 \leq g \in L^1(\Omega)\), there holds
\[
\int_{Q_R} A_R A_R^{-1}(g) \, dx \leq c_0 A_R A_R^{*^{-1}} \left( \int_{Q_R} g \, dx \right).
\]

**Lemma 4.6.** Suppose \(A_R^{*^{-1}} \in N\) and there exist two constants \(R_0 > 0\) and \(T_0 > 0\) such that \(A_R^{*^{-1}}(t)\) is convex on \(\{t \geq T_0\}\) for any \(R \leq R_0\). Then there
exists a constant $c_{10} > 0$, not depending on $R$, such that for any $0 \leq g \in L^1(\Omega)$, there holds
\[ A_R A_R^{-1} \left( \int_{Q_R} A_R^* A_R^{-1}(g) \, dx \right) \leq c_{10} A R^{-1} \left( \int_{Q_R} A^* A^{-1}(g) \, dx + 1 \right). \]

Proof. It is clear that
\[ A_R A_R^{-1} \left( \int_{Q_R} A_R^* A_R^{-1}(g) \, dx \right) \leq A_R A_R^{-1} \left( \int_{Q_R} h(x) \, dx \right), \]
where
\[ h(x) = \begin{cases} A_R A_R^{-1}(g(x)), & \text{if } A_R A_R^{-1}(g(x)) \geq T_0, \\ T_0, & \text{if } A_R A_R^{-1}(g(x)) < T_0. \end{cases} \]

Since $A^* A^{-1} A_R A_R^{-1}(t)$ is convex on $\{ t \geq T_0 \}$ for any $R \leq R_0$, from Jensen’s inequality we get
\[
\begin{align*}
& A_R A_R^{-1} \left( \int_{Q_R} h(x) \, dx \right) = A R^{-1} A R^{-1} A R^{-1} A R^{-1} \left( \int_{Q_R} h(x) \, dx \right) \\
& \leq A R^{-1} \left( \int_{Q_R} A R^{-1} A R^{-1} (h(x)) \, dx \right) \\
& \leq A R^{-1} \left( \frac{1}{|Q_R|} \int_{\{ x \in Q_R : A_R A_R^{-1}(g) \geq T_0 \}} A_R A_R^{-1}(g) \, dx + \frac{1}{|Q_R|} \int_{\{ x \in Q_R : A_R A_R^{-1}(g) < T_0 \}} A R^{-1} A R^{-1} (T_0) \right) \\
& \leq A R^{-1} \left( \int_{Q_R} A R^{-1}(g) \, dx + A R^{-1} (T_0) \right) \\
& \leq c_9 (A R^{-1} \left( \int_{Q_R} A^* A^{-1}(g) \, dx \right) + A R^{-1} (T_0)) \\
& \leq c_9 (A R^{-1} \left( \int_{Q_R} A^* A^{-1}(g) \, dx \right) + A R^{-1} (T_0)) \\
& \leq c_{10} (A R^{-1} \left( \int_{Q_R} A^* A^{-1}(g) \, dx \right) + 1),
\end{align*}
\]
where $c_9$ and $c_{10} > 0$ are constants depending on $A$. \hfill \Box

Lemma 4.7. Under the assumptions of Lemma 4.4, there exists a constant $c_{14} > 0$, not depending on $R$, such that for any $0 \leq g \in L^1(\Omega)$ with $\int_{Q_R} g \, dx \leq 1$ and any $Q_R \subset \Omega$,
\begin{equation}
(4.21) \quad \overline{A_R A_R^{-1}} \left( \int_{Q_R} A_R^* A_R^{-1}(g) \, dx \right) \leq c_{14} A R^{-1} \left( \int_{Q_R} A^* A^{-1}(g) \, dx \right).
\end{equation}

Proof. By Lemma 4.3 (2.3) and (2.11) we get
\[ \int_{Q_R} A R^{-1} A_R^{-1}(g) \, dx \leq c_9 A R^{-1} \left( \frac{1}{|Q_R|} \right) \leq c_{12} \frac{1}{|Q_R|} \leq c_{13} A R \left( \frac{1}{|Q_R|} \right). \]
Then Lemma 4.4 and Lemma 4.6 imply that

\[
\mathcal{A}_R \mathcal{A}_R^{-1} \left( \int_{Q_R} \mathcal{A}_R^{-1}(g) \, dx \right) \leq c_{14} \mathcal{A}_R \mathcal{A}_R^{-1} \left( \int_{Q_R} \mathcal{A}_R^{-1}(g) \, dx + 1 \right) \\
\leq c_{15} \mathcal{A}^*-1 \left( \int_{Q_R} \mathcal{A}^{-1}(g) \, dx + 1 \right).
\]

□

From Lemma 4.3 and Lemma 4.7 we conclude the following lemma.

**Lemma 4.8.** Under the assumption of Lemma 4.4–4.6, for any \( R \leq R_0 \), there exists a positive constant \( c = c(n, A) \) such that, for any \( u \in W^{1,A}(Q_R) \), the following inequality holds:

\[
\int_{Q_R} A(x, |\nabla u|) \, dx \leq c \mathcal{A}^*-1 \left( \int_{Q_R} \mathcal{A}^{-1}(A(x, |\nabla u|)) \, dx \right) + c.
\]

**Lemma 4.9.** The following two statement are equivalent:

1. There exists a constant \( L > 0 \), not depending on \( R \), such that \( \mathcal{A}_R(|Q_R|^{-1}) \leq L \mathcal{A}_R(|Q_R|^{-1}) \) for any \( Q_R \subset \Omega \).
2. There exists a constant \( L > 0 \), not depending on \( R \), such that \( \mathcal{A}_R(t^{-1}) \leq L \mathcal{A}_R(t^{-1}) \) for any \( Q_R \subset \Omega \) and any \( t \geq |Q_R| \).

**Proof.** If (1) holds, then (2) holds. In fact, from (1), we get

\[
\mathcal{A}_R(t^{-1}) \leq \mathcal{A}_R(t^{-1}) \leq L \mathcal{A}_R(t^{-1}) \leq \mathcal{A}_R(t^{-1}),
\]

where \( |Q_{R_t}| = t \). □

**Remark 4.1.** For the variable exponent case \( A(x, |t|) = |t|^{p(x)} \), where \( 1 < \inf_{x \in \Omega} p(x) =: p_- \leq p(x) \leq p_+ := \sup_{x \in \Omega} p(x) < n \) and \( p \in C(\Omega) \), the log-Hölder continuity of \( p \) can imply that Condition (4.23) holds. Check the next section for details.

Changing variables, by Theorem 3.1, Lemma 4.8 and Lemma 4.9, we get the following main theorem of this paper.

**Theorem 4.1.** Suppose that \( (P^n_0) \), \( (P^n_1) \), \( (P^n_2) \), \( (P^n_3) \) and \( (P^n_4) \) hold; that for any \( Q_R \subset \Omega \), \( \mathcal{A}_R \mathcal{A}_R^{-1} \in N(Q_R) \); that there exists a constant \( L > 0 \), not depending on \( R \), such that

\[
\mathcal{A}_R(|Q_R|^{-1}) \leq L \mathcal{A}_R(|Q_R|^{-1});
\]
that there exist two constants $R_0 > 0$ and $T_0 > 0$ such that $A^* A^{-1} A^* R^{-1}(t)$ is convex on \( \{ t \geq T_0 \} \) for any $R \leq R_0$. If $u \in W^{1, A}_\text{loc}(\Omega)$ is a local minimizer of $E$, then there exists a positive constant $\epsilon = \epsilon(n, A)$ such that

$$A(x, |\nabla u|) \in L^p_{\text{loc}}(\Omega), \quad (\text{equivalently } |\nabla u| \in L^p_{\text{loc}}(\Omega),)$$

where $\Phi(t) := t \left( \frac{A(t) - A^{-1}(t)}{A^* A^{-1}(t)} \right)^p$ for any $t > 0$. And there exists a constant $c = c(n, A, \epsilon, L, |\nabla u|_{L^p(\Omega)})$ such that for every $x \in \Omega$, there exists an $n$-cube $Q_r \subset \Omega$ with the center $x$, such that

$$\int_{Q_r} \Phi(A(x, |\nabla u|)) \, dx \leq c\Phi \left( \int_{Q_r} A(x, |\nabla u|) \, dx + 1 \right).$$

**Remark 4.2.** It is clear that $\Phi \in N$ and $N(\Omega) \ni \Phi(A) \gg A$.

5. An Example

**Example.** Let $p \in C^{1,0}(\overline{\Omega})$ and $\frac{n}{n-1} < q \leq p(x) \leq p_+ := \sup_{x \in \overline{\Omega}} p(x) < n \ (q \in \mathbb{R})$ for $x \in \overline{\Omega}$. Define $A : \overline{\Omega} \times [0, +\infty) \to [0, +\infty)$ by

$$A(x, t) = t^{p(x)}.$$

Since $q > \frac{n}{n-1}$,

$$A^*(x, t) = (p(x))^{\frac{np(x)}{np(x) + np_+}} \in N(\Omega).$$

It is clear that

$$A_*(x, t) = \left( \frac{n - p(x)}{np(x)} \right)^{\frac{np(x)}{np(x) + np_+}} \in N(\Omega),$$

and

$$\mathfrak{A}^*(\alpha) := \begin{cases} \frac{np_-}{\alpha^{n+p_-}}, & \text{for } t \geq 1, \\ \frac{np_+}{\alpha^{n+p_+}}, & \text{for } t < 1, \end{cases} \quad \mathfrak{A}^*(\alpha) = \begin{cases} \frac{np_-}{\alpha^{n+p_-}}, & \text{for } t \geq 1, \\ \frac{np_+}{\alpha^{n+p_+}}, & \text{for } t < 1, \end{cases}$$

$$\mathfrak{A}(\alpha) := \begin{cases} \alpha^{p_-}, & \text{for } t \geq 1, \\ \alpha^{p_+}, & \text{for } t < 1, \end{cases} \quad \mathfrak{A}(\alpha) = \begin{cases} \alpha^{p_-}, & \text{for } t \geq 1, \\ \alpha^{p_+}, & \text{for } t < 1, \end{cases}$$

$$\mathfrak{A}_*(\alpha) := \begin{cases} \frac{np_-}{\alpha^{n-p_-}}, & \text{for } t \geq 1, \\ \frac{np_+}{\alpha^{n-p_+}}, & \text{for } t < 1, \end{cases} \quad \mathfrak{A}_*(\alpha) = \begin{cases} \frac{np_-}{\alpha^{n-p_-}}, & \text{for } t \geq 1, \\ \frac{np_+}{\alpha^{n-p_+}}, & \text{for } t < 1. \end{cases}$$

The above six control functions satisfy the following inequalities:

$$\mathfrak{A}^*(\alpha) A(x, t) \leq A^*(x, \alpha t) \leq \mathfrak{A}^*(\alpha) A(x, t),$$

$$\mathfrak{A}(\alpha) A(x, t) \leq A(x, \alpha t) \leq \mathfrak{A}(\alpha) A(x, t),$$

and

$$\mathfrak{A}_*(\alpha) A(x, t) \leq A_*(x, \alpha t) \leq \mathfrak{A}_*(\alpha) A(x, t),$$

for $x \in \Omega$ and $t \in \mathbb{R}^n$. Moreover, we can find that $\mathfrak{A}(\alpha) \in \Delta_{\mathbb{R}_+}$ and $\mathfrak{A}^*, \mathfrak{A}_* \in \nabla_{\mathbb{R}_+}$. In addition, for $x \in \Omega$,

$$\nabla_x A^*(x, t) = \left[ \frac{n^2}{(n + p(x))^2} \log (p(x) t) + \frac{n}{n + p(x)} \right] \left( p(x) t \right)^{\frac{np(x)}{np(x) + np_+}} \nabla_x p(x).$$
Since for any $\epsilon > 0$, $\log s \rightarrow 0$ as $s \rightarrow +\infty$, we conclude that there exist constants $\delta_1 < \frac{1}{n}$, $c_1$ and $t_1$ such that

$$\left| \frac{\partial A^*(x, t)}{\partial x_j} \right| \leq c_1 (A^*)^{1+\delta_1}(x, t),$$

for all $x \in \Omega$ and $t \geq t_1$. Then Condition $(P^*_5)$ is satisfied.

Denote $p_R := p|_{Q_R}$. If $p$ is log-Hölder continuous, i.e. for any $x, y \in Q_R \subset \Omega$ there exists a constant $L > 0$ such that

$$|p_R(x) - p_R(y)| \log |x - y|^{-1} \leq L,$$

then for small $R > 0$

$$R^-(p_R^n - p_R^{-n}) \leq |x - y|^{-1}|p_R^n(x) - p_R^n(y)| \leq e^L.$$

It is easy to see that

$$\widehat{A}_R(|Q_R|^{-1}) \leq e^{nL} A_R(|Q_R|^{-1}),$$

where $A_R$ is in the sense of Remark 2.3.

By the expression of $A^*_\delta$, $A^*$ and $A_\delta$, the other increasing conditions can be verified. The readers can also find the verification of the increasing conditions in [10] accordingly. This example generalized some parts of the conclusions in [10] and some of the corollaries therein.

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