A classical picture of lepton neutral current forces

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Abstract

When charged current weak interactions are excluded, the neutral current weak interaction is formally similar to ordinary electromagnetism with a massive photon. In this spirit, the Maxwell equations for the fields of the Z-boson are derived from the standard model. These describe the Z-boson scalar and vector potentials, and the Z-boson electric and magnetic fields whose sources are electron and neutrino distributions and currents. The Z-boson Maxwell equations are solved for point sources representing classical point-like electrons and neutrinos. The parity violation of the weak interaction is manifest in the structure of these solutions. As an application of this model, the neutral current contribution to the muonium hyperfine structure is computed using nonrelativistic perturbation theory.

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I. INTRODUCTION

The electromagnetic force is known to be the result of the exchange of photons between charged particles. Outside of the context of quantum field theory, this interaction is described in terms of classical scalar and vector potentials \( A^0 \) and \( \mathbf{A} \) with their electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \). This description of the electromagnetic interaction is a very useful model and forms the foundation of a large body of macroscopic, atomic, and condensed matter physics.

The electron also participates in weak interactions, both charged current (W-boson exchange) and neutral current (Z-boson exchange). For neutral current events, electrons (or neutrinos) remain as electrons (or neutrinos). We may imagine that one particle is the source of neutral current electric and magnetic fields and that the other particle feels the forces produced by these fields, in complete analogy with ordinary electromagnetism. In the charged current case, an initial electron state emerges as a final neutrino state or vice-versa. For this reason it is difficult to consider such a picture for the charged current interaction. Charged current interactions will not be considered in this paper, and hence, we will develop an incomplete picture of the weak force. This picture will, however, be useful when an interaction cannot proceed by W-boson exchange.

This paper begins with an analysis of the standard model Lagrangian for the first generation of leptons (the electron and its neutrino). A nonrelativistic weak-field Hamiltonian for the electron is developed which allows us to compute the interaction energy of an electron in the presence of a classical Z-boson field. The Maxwell equations for the Z-boson are then developed. In the absence of sources, the Maxwell equations are identical to those of ordinary electromagnetism but with a massive photon. The Maxwell equation source terms are derived from the interaction energies for both electron and neutrino sources.

The Maxwell equations derived here can be used to describe the (albeit small!) Z-boson field generated by macroscopic or atomic-scale distributions of electrons. They may also be used to visualize the Z-boson fields surrounding classical point-like electrons and neutrinos. The classical point particle solutions provide an interesting visualization of the parity violation in the standard model in terms of a vortex-like magnetic field structure oriented with the electron’s spin.
As an application of the formalism developed in this paper, a calculation of the neutral current contribution to the hyperfine structure of muonium ($\mu^+e^-\text{-bound state}$) is made using nonrelativistic perturbation theory. Other applications might be found in nuclear physics.

II. THE STANDARD MODEL LAGRANGIAN

We begin by considering the matter terms of the Lagrangian density for the standard model (with massless neutrinos) after spontaneous symmetry breaking (following the notation of Quigg):

\[
\mathcal{L}_{\text{electron}} = \bar{e}_R(i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu - g'\sin\theta_W \gamma^\mu Z_\mu)e_R \\
+\bar{e}_L(i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu + \frac{1}{2}(g\cos\theta_W - g'\sin\theta_W)\gamma^\mu Z_\mu)e_L \\
-\bar{m}_R e_L - m_Le_R ,
\]

(1)

\[
\mathcal{L}_{\text{neutrino}} = \bar{\nu}(i\gamma^\mu \partial_\mu - \frac{1}{2}(g\cos\theta_W + g'\sin\theta_W)\gamma^\mu Z_\mu)\nu .
\]

Note that the $W^\pm$ boson fields have been set to zero, decoupling electron and neutrino in the matter Lagrangian density. We do not wish to consider charged current interactions. In addition, the neutrino chirality is constrained such that $\frac{1}{2}(1 + \gamma^5)\nu = 0$.

The matter bispinors naturally decompose into left and right handed chiralities when using the chiral representation of the Dirac matrices:

\[
\gamma^\mu = \left( \begin{array}{cc}
0 & -1 \\
1 & 0
\end{array} \right) \left( \begin{array}{cc}
0 & \sigma \\
-\sigma & 0
\end{array} \right) .
\]

(2)

In the chiral representation, the matter equations of motion are

\[
i\partial_0 \begin{pmatrix}
e_R \\
e_L
\end{pmatrix} = \mathcal{H}_{\text{electron}} \begin{pmatrix}
e_R \\
e_L
\end{pmatrix}
\]

(3)

\[
\mathcal{H}_{\text{electron}} = \begin{pmatrix}
\sigma \cdot \pi_R - \Phi_R & -m \\
-m & -\sigma \cdot \pi_L - \Phi_L
\end{pmatrix}
\]

\[
i\partial_0 \nu = \mathcal{H}_{\text{neutrino}} \nu = (-\sigma \cdot \pi_n - \Phi_n)\nu ,
\]

(4)
where

\[ \pi_L = -i \nabla + eA + \frac{1}{2}(g \cos \theta_W - g' \sin \theta_W)Z, \]
\[ \pi_R = -i \nabla + eA - g' \sin \theta_W Z, \]
\[ \Phi_L = eA^0 + \frac{1}{2}(g \cos \theta_W - g' \sin \theta_W)Z^0, \]
\[ \Phi_R = eA^0 - g' \sin \theta_W Z^0, \]
\[ \pi_n = -i \nabla - \frac{1}{2}(g \cos \theta_W + g' \sin \theta_W)Z, \]
\[ \Phi_n = -\frac{1}{2}(g \cos \theta_W + g' \sin \theta_W)Z^0, \]
\[ A^\mu = \begin{pmatrix} A^0 & A \end{pmatrix}, \quad Z^\mu = \begin{pmatrix} Z^0 & Z \end{pmatrix}. \]

(5)

III. NONRELATIVISTIC-WEAK FIELD APPROXIMATION

To assess the nonrelativistic limit it will be more convenient to use the Dirac representation for the Dirac Matrices:

\[ \gamma^\mu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}. \]

(6)

In the chiral representation, the electron Hamiltonian in (3) becomes

\[ \mathcal{H}_{\text{electron}} = \begin{pmatrix} \frac{1}{2}\sigma \cdot (\pi_R - \pi_L) & \frac{1}{2}\sigma \cdot (\pi_R + \pi_L) \\ -\frac{1}{2}(\Phi_R + \Phi_L) + m & -\frac{1}{2}(\Phi_R - \Phi_L) \end{pmatrix} \begin{pmatrix} \frac{1}{2}\sigma \cdot (\pi_R + \pi_L) & \frac{1}{2}\sigma \cdot (\pi_R - \pi_L) \\ -\frac{1}{2}(\Phi_R - \Phi_L) & -\frac{1}{2}(\Phi_R + \Phi_L) - m \end{pmatrix}. \]

(7)

In the nonrelativistic limit the electron bispinor may be written in the form

\[ e = \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} e^{-i(m+E_{NR})t}, \]

(8)
where the nonrelativistic energy $E_{NR} \ll m$. We may solve for the “small” component spinor $\chi$ in terms of the “large” component spinor $\phi$; whence, to lowest order in the ratio $E_{NR}/m$:

$$\chi = \frac{1}{2m} \left( \frac{1}{2} \sigma \cdot (\pi_R + \pi_L) - \frac{1}{2} (\Phi_R - \Phi_L) \right) \phi .$$ (9)

Substituting for $\chi$ in the equations of motion allows the identification of the nonrelativistic Hamiltonian

$$\mathcal{H}_{NR} = \frac{1}{2m} \left( \frac{1}{2} \sigma \cdot (\pi_R + \pi_L) - \frac{1}{2} (\Phi_R - \Phi_L) \right)^2$$

$$+ \frac{1}{2} \sigma \cdot (\pi_R - \pi_L) - \frac{1}{2} (\Phi_R + \Phi_L) .$$ (10)

Use the identity $(\sigma \cdot A)(\sigma \cdot B) = A \cdot B + i \sigma \cdot (A \times B)$ and neglect terms quadratic in the fields (weak field approximation) to derive

$$\mathcal{H}_{NR} = -\frac{1}{2m} \nabla^2 + \mathcal{V} ,$$ (11)

$$\mathcal{V} = -\frac{i}{2m} \nabla \cdot \left( eA + \frac{1}{4} (g \cos \theta_W - 3g' \sin \theta_W)Z \right)$$

$$- \frac{i}{2m} \left( eA + \frac{1}{4} (g \cos \theta_W - 3g' \sin \theta_W)Z \right) \cdot \nabla$$

$$+ \frac{1}{2m} \sigma \cdot \left( \nabla \times \left( eA + \frac{1}{4} (g \cos \theta_W - 3g' \sin \theta_W)Z \right) \right)$$

$$- \frac{g \cos \theta_W + g' \sin \theta_W}{8m} \sigma \cdot \nabla Z^0 + \frac{1}{4} (g \cos \theta_W + g' \sin \theta_W) \sigma \cdot \nabla Z.$$ (12)

Note that it is understood that the differential operators operate only on the object immediately to its right.

**IV. INTERACTION ENERGY**

The interaction energy (12) is responsible for the Maxwell equation source terms. In the nonrelativistic-weak field approximation the interaction energy
for an electron source is just the expectation value of $V$ using the nonrelativistic electron wavefunction $\phi$. After integration by parts we have

$$V = -\frac{i}{2m} \int \left( eA + \frac{1}{4}(g \cos \theta_W - 3g' \sin \theta_W)Z \right) \cdot (\phi^\dagger \nabla \phi - (\nabla \phi^\dagger) \phi) d^3r$$

$$+ \frac{1}{2m} \int \left( \nabla \times \left( eA + \frac{1}{4}(g \cos \theta_W - 3g' \sin \theta_W)Z \right) \right) \cdot (\phi^\dagger \sigma \phi) d^3r$$

$$- \frac{i g \cos \theta_W + g' \sin \theta_W}{8m} \int Z^0 (\phi^\dagger \sigma \cdot \nabla \phi - (\nabla \phi^\dagger) \cdot \sigma \phi) d^3r$$

$$- \int \left( eA^0 + \frac{1}{4}(g \cos \theta_W - 3g' \sin \theta_W)Z^0 \right) \phi^\dagger \phi d^3r$$

$$- \frac{1}{4} (g \cos \theta_W + g' \sin \theta_W) \int Z \cdot (\phi^\dagger \sigma \phi) d^3r.$$

(13)

The nonrelativistic limit makes no sense for a massless neutrino. The neutrino source interaction energy may be found directly from $H_{\text{neutrino}}$ given by (4):

$$V = \frac{1}{2} (g \cos \theta_W + g' \sin \theta_W) \int Z^0 \nu^\dagger \nu d^3r$$

$$+ \frac{1}{2} (g \cos \theta_W + g' \sin \theta_W) \int Z \cdot (\nu^\dagger \sigma \nu) d^3r.$$  

(14)

V. MAXWELL EQUATIONS

The first order Lagrangian for the photon is

$$L = \int \left( -E \cdot (\nabla A^0 + \dot{A}) - B \cdot (\nabla \times A) + \frac{1}{2} (|B|^2 - |E|^2) \right) d^3r - V. $$  

(15)

From this Lagrangian we derive the relationship between the potentials and field strengths and two of the Maxwell equations:

$$E = -\nabla A^0 - \dot{A}, \quad B = \nabla \times A,$$

$$\nabla \cdot E = -e \dot{\phi},$$

$$\nabla \cdot B = e \phi^\dagger.$$  

(16)  

(17)
\[ \nabla \times \left( B + \frac{e}{m} \left( \phi^\dagger \sigma \phi \right) \right) - \dot{E} = -e \left( \frac{-i}{2m} \right) \left( \phi^\dagger \nabla \phi - (\nabla \phi^\dagger) \phi \right). \quad (18) \]

The first order Lagrangian for the Z-boson field neglecting three and four
boson couplings (see Appendix A for further discussion) is

\[ L = \int \left( -E \cdot (\nabla Z^0 + \dot{Z}) - B \cdot (\nabla \times Z) + \frac{1}{2} \left( |B|^2 - |E|^2 \right) \right) d^3r \]
\[ + \frac{1}{2} M^2 \int \left( (Z^0)^2 - |Z|^2 \right) d^3r - V. \quad (19) \]

This differs from the photon's Lagrangian by the addition of a mass
term \( (M = M_Z) \). Note that the symbols “\( E \)” and “\( B \)” are now used as the field
strengths for the Z-boson instead of the photon. The electron source terms
are modified for the Z-boson:

\[ \nabla \cdot E + M^2 Z^0 = -\frac{1}{4} (g \cos \theta_W - 3g' \sin \theta_W) \phi^\dagger \phi \]
\[ + \frac{1}{2} (g \cos \theta_W + g' \sin \theta_W) \left( \frac{-i}{4m} \right) \left( \phi^\dagger \sigma \cdot \nabla \phi - (\nabla \phi^\dagger) \cdot \sigma \phi \right), \quad (20) \]

\[ \nabla \times \left( B + \frac{1}{4m} (g \cos \theta_W - 3g' \sin \theta_W) \frac{1}{2} \left( \phi^\dagger \sigma \phi \right) \right) + M^2 Z - \dot{E} \]
\[ = -\frac{1}{4} (g \cos \theta_W - 3g' \sin \theta_W) \left( \frac{-i}{2m} \right) \left( \phi^\dagger \nabla \phi - (\nabla \phi^\dagger) \phi \right) \]
\[ + \frac{1}{2} (g \cos \theta_W + g' \sin \theta_W) \frac{1}{2} \left( \phi^\dagger \sigma \phi \right). \quad (21) \]

Similar Maxwell equations for a massless neutrino are

\[ \nabla \cdot E + M^2 Z^0 = \frac{1}{2} (g \cos \theta_W + g' \sin \theta_W) \nu^\dagger \nu, \quad (22) \]

\[ \nabla \times B + M^2 Z - \dot{E} = -(g \cos \theta_W + g' \sin \theta_W) \frac{1}{2} (\nu^\dagger \sigma \nu). \quad (23) \]

The remaining two Maxwell equations are just Bianchi identities and
remain unchanged from ordinary electromagnetism (Appendix A):

\[ \nabla \cdot B = 0, \quad \nabla \times E + \dot{B} = 0. \quad (24) \]
VI. ELECTRON FIELDS

It is interesting to consider the fields of a classical point-like electron. First consider the photon fields. The wavefunction bilinear combinations appearing as sources for the photon electric and magnetic fields have the following interpretations:

\begin{align*}
-e\phi^{\dagger}\phi & \quad \text{charge density } \rho , \\
-e\left(\frac{-i}{2m}\right) \left(\phi^{\dagger}\nabla\phi - (\nabla\phi^{\dagger})\phi\right) & \quad \text{current density } j , \\
\frac{1}{2} \left(\phi^{\dagger}\sigma\phi\right) & \quad \text{spin angular momentum density } s .
\end{align*}

The term \(-\frac{e}{m}\frac{1}{2}\phi^{\dagger}\sigma\phi\) in (18) has the physical interpretation of a magnetization (or a dipole moment density). From these considerations we find that the point-like classical electron has a charge \(-e\) and a magnetic moment of \(-\frac{e}{m}\times\text{its spin}\) (a gyromagnetic ratio of 2).

Now consider the Z-boson fields. There is the addition bilinear form 
\begin{align*}
\left(\frac{-i}{4m}\right) \left(\phi^{\dagger}\sigma \cdot \nabla\phi - (\nabla\phi^{\dagger}) \cdot \sigma\phi\right)
\end{align*}

which may be interpreted as a density for the helicity-velocity product but can be considered vanishing for the electron at rest. The Z-boson Maxwell equations for the point-like electron at rest with spin polarization \(S = \pm \frac{1}{2}\) along the z-axis derived from (20) and (21) are then

\begin{align*}
\nabla \cdot \mathbf{E} + M^2 \mathbf{Z}^0 &= -\frac{1}{4} \left(g \cos\theta_W - 3g' \sin\theta_W\right) \delta (\mathbf{r}) , \\
\nabla \times \left(\mathbf{B} + \frac{1}{4m} \left(g \cos\theta_W - 3g' \sin\theta_W\right) S\mathbf{Z} \delta (\mathbf{r})\right) + M^2 \mathbf{Z} &= \frac{1}{2} \left(g \cos\theta_W + g' \sin\theta_W\right) S\mathbf{Z} \delta (\mathbf{r}) .
\end{align*}

Unlike the electromagnetic Maxwell equations for the electron at rest, there is a nonvanishing magnetic source current density in (27) as well as the familiar electric charge density in (26).

When the electron is placed in motion along the z-axis in either a state of positive or negative helicity, additional source terms are required to maintain
the covariance of the Maxwell equations. Specifically, a magnetic source current is generated due to the motion of the electric charge density just as with ordinary electromagnetism; however, unlike ordinary electromagnetism, an additional electric charge density coupled to the nonvanishing at-rest magnetic source current is required for covariance. Thus, in the limit of a classical point-like electron, we must make the identification

\[
\left( \frac{-i}{4m} \right) (\phi^\dagger \mathbf{\sigma} \cdot \nabla \phi - (\nabla \phi^\dagger) \cdot \mathbf{\sigma} \phi) \longrightarrow S \mathbf{\tilde{z}} \cdot \mathbf{v} \delta (\mathbf{r} - \mathbf{v}t),
\]

and the Maxwell equations (26) and (27) become more generally:

\[
\nabla \cdot \mathbf{E} + M^2 Z^0 = -\frac{1}{4} (g \cos \theta_W - 3g' \sin \theta_W) \delta (\mathbf{r} - \mathbf{v}t) + \frac{1}{2} (g \cos \theta_W + g' \sin \theta_W) S \mathbf{\tilde{z}} \cdot \mathbf{v} \delta (\mathbf{r} - \mathbf{v}t),
\]

\[
\nabla \times \left( \mathbf{B} + \frac{1}{4m} (g \cos \theta_W - 3g' \sin \theta_W) S \mathbf{\tilde{z}} \delta (\mathbf{r} - \mathbf{v}t) \right) + M^2 Z = \frac{1}{2} (g \cos \theta_W + g' \sin \theta_W) S \mathbf{\tilde{z}} \delta (\mathbf{r} - \mathbf{v}t) - \frac{1}{4} (g \cos \theta_W - 3g' \sin \theta_W) \mathbf{v} \delta (\mathbf{r} - \mathbf{v}t).
\]

The Maxwell equations for the point-like electron at rest (26) and (27) are solved in Appendix B. Physically there are three contributions to the Z-boson fields. The first component is in the form of a Yukawa potential due to a neutral current electric charge \( q = -\frac{1}{4} (g \cos \theta_W - 3g' \sin \theta_W) \):

\[
Z^0_{\text{Yukawa}} = \frac{q}{4\pi} \frac{e^{-Mr}}{r},
\]

\[
E_{\text{Yukawa}} = \frac{q}{4\pi} \left( \frac{M}{r} + \frac{1}{r^2} \right) e^{-Mr} \mathbf{\tilde{r}},
\]

The second component is due to a neutral current magnetic dipole moment \( \mu = \frac{m}{\mu} S \); in spherical coordinates:
\[ Z_{Dipole} = \frac{\mu}{4\pi} \left( \frac{M}{r} + \frac{1}{r^2} \right) e^{-Mr} \sin \theta \hat{\phi}, \quad (32) \]

\[ B_{Dipole} = \frac{\mu}{2\pi} \left( \frac{M}{r^2} + \frac{1}{r^3} \right) e^{-Mr} \cos \theta \hat{r} \]
\[ \quad + \frac{\mu}{4\pi} \left( \frac{M^2}{r} + \frac{M}{r^2} + \frac{1}{r^3} \right) e^{-Mr} \sin \theta \hat{\theta} + \frac{2}{3} \mu \hat{z} \delta(r). \quad (33) \]

Note that (33) implies that the electron has a neutral current gyromagnetic ratio identical (when quantum field theory radiative corrections are ignored) to the electromagnetic.

The third component is a vortex-like magnetic field near the equatorial plane due to a magnetic vortex charge \( \kappa S = \frac{1}{2} (g \cos \theta_W + g' \sin \theta_W) S \):

\[ Z_{PV} = -\frac{\kappa S}{2\pi} \left( \frac{1}{Mr^2} + \frac{1}{M^2r^3} \right) e^{-Mr} \cos \theta \hat{r} \]
\[ \quad - \frac{\kappa S}{4\pi} \left( \frac{1}{r} + \frac{1}{Mr^2} + \frac{1}{M^2r^3} \right) e^{-Mr} \sin \theta \hat{\theta} \]
\[ \quad + \frac{1}{3} \frac{\kappa S}{3 M^2} \hat{z} \delta(r), \quad (34) \]

\[ B_{PV} = \frac{\kappa S}{4\pi} \left( \frac{M}{r} + \frac{1}{r^2} \right) e^{-Mr} \sin \theta \hat{\phi}. \quad (35) \]

Note that (34) is not finite in the limit \( M \to 0 \). This is because there are no nonvanishing solutions to \( \mathbf{B} = \nabla \times \mathbf{Z} \) and \( \nabla \times \mathbf{B} = 0 \) with the same symmetries as (34) and (35). The fields (34) and (35) have no direct analogy in ordinary electromagnetism, although the axial symmetry of the magnetic field (35) is similar to that of a charge in motion.

The presence of the \( \delta \)-functions in (33) and (34) (discussed further in Appendix B) are required topologically to preserve Stokes integral identities.

The parameter \( \kappa \) is a direct measure of the parity violation in the standard model since it is the difference between the left-handed and right-handed electron coupling constants, while the charge \( q \) is their average. Under parity inversions \( \phi \to \pi + \phi, \theta \to \pi - \theta, \) and \( \hat{r} \to \hat{r}, \hat{\theta} \to -\hat{\theta}, \hat{\phi} \to \hat{\phi}. \) For the Yukawa part (31) both \( Z^0 \) and \( E \) are even under parity inversion. For the
dipole part (32) and (33) $Z$ is even and $B$ is odd; however, for (34) and (35) $Z$ is odd and $B$ is even, signaling parity violation.

The sign of the charge $q$ may be freely chosen, using $\sin^2 \theta_W = 0.23$, we have $|e| = 0.30$, $|q| = 0.014$, and $\kappa = -0.36 \text{ sgn}(q)$.

### VII. NEUTRAL CURRENT CONTRIBUTIONS TO THE MUONIUM HYPERFINE STRUCTURE

The $\mu^+e^-$-bound state (muonium) is an important system for high precision studies of quantum electrodynamics (QED). Its spectrum is qualitatively similar to that of Hydrogen. Using the formalism in this paper, it is a simple exercise in nonrelativistic perturbation theory to compute the energy level shifts of muonium due to the neutral current interaction. In particular, we calculate the neutral current perturbation to the ground state energy of muonium, and the hyperfine splitting between the $J = 0$ (singlet) and $J = 1$ (triplet) ground states.

Equation (13) is the energy perturbation when $\phi (= \psi \times \text{spinor})$ is the electron’s unperturbed nonrelativistic wave function, and $Z^0$ and $Z$ are the potentials generated by a point-like antimuon. The unperturbed ground state electron scalar wavefunction is

$$\psi = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \quad (36)$$

where $a = 4\pi/(m'_e e^2)$ is the muonium Bohr radius with $m'_e$ being the reduced mass of the electron in muonium. The potentials generated by the antimuon are given by equations (31) through (35) with $q$ replaced by $-q$, $\kappa$ replaced by $-\kappa$, and the electron’s mass replaced by the muon’s. Since both particles’ spins may be freely oriented, it is necessary to replace the fixed choices for the polarization axes with Pauli matrices, and in the end, take the expectation value using a spin state describing both particles. Following this procedure, there are three nonvanishing terms in (13):

$$V_C = q \int Z^0 \psi^\dagger \psi d^3r = -\frac{q^2}{\pi a^3} \frac{1}{(M + 2/a)^2}, \quad (37)$$

$$V_{NPV} = -\frac{q}{2m} \int <B \cdot \sigma> \psi^\dagger \psi d^3r \quad (38)$$
\[
= \frac{q^2}{m_em_\mu} <S_e \cdot S_\mu> \frac{2}{3\pi a^3} \left(1 - \frac{M^2}{(M + 2/a)^2}\right),
\]

\[
V_{PV} = -\frac{1}{2}\kappa \int <Z \cdot \sigma > \psi^\dagger \psi d^3r
\]

\[
= \frac{\kappa^2}{3\pi a^3} <S_e \cdot S_\mu> \left(\frac{2}{(M + 2/a)^2} + \frac{1}{M^2}\right).
\]

The “cross-terms” in (13) (proportional to \(\kappa q\)) involve the expectation value \(<S_e \times S_\mu>\) which can be shown to be vanishing.

The expectation value of the electron and muon spin operator dot product \(<S_e \cdot S_\mu>\) is evaluated by considering the expectation of the total spin-squared \(S^2 = (S_e + S_\mu)^2\), using the decomposition of the symmetric triplet singlet spin state into individual particle spinors: \(\uparrow\uparrow, (\uparrow\downarrow + \downarrow\uparrow)/\sqrt{2}, \downarrow\downarrow\) (triplet), and \((\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2}\) (singlet). In the triplet state \(<S_e \cdot S_\mu> = \frac{1}{4}\), and in the singlet state \(<S_e \cdot S_\mu> = -\frac{3}{4}\); hence, we may compute the neutral current contributions to the hyperfine splitting of the ground state of 1.7 Hz from (38), and 65 Hz from (39), for a total of 67 Hz. This compares to a 4.5 Ghz electromagnetic contribution which may be estimated from (38) with \(q\) replaced by \(e\) and \(M = 0\). The contribution of (37) shifts both singlet and triplet ground states by -0.10 Hz.

The neutral current contributions to the spectrum of hydrogen could be analyzed in similar fashion with suitable generalization to include the neutral current coupling of the proton. This coupling may be identified by summing the contributions from the constituent quarks with inclusion of radiative corrections. Extension to include heavier elements is limited by the complexity of atomic wavefunctions. However, for certain cases, such as Cesium, accurate calculations can be performed. Also of interest are muonic atoms because the muon’s Bohr radius is much smaller than the electron’s. In all these systems, the neutral current interaction introduces parity violation into the atomic structure due to the mixing of parity-even and parity-odd pure electronic states.

The analysis of positronium (\(e^+e^-\)-bound state) is however, more problematic because of the annihilation process \(e^+e^- \rightarrow\) virtual photon \(\rightarrow e^+e^-\). A precise treatment of positronium requires using the Bethe-Salpeter equation for relativistic bound states.
VIII. NEUTRINO FIELDS

The Maxwell equations for a point-like massless neutrino travelling through the origin at \( t = 0 \) towards the positive z-axis, after making the identifications \( \nu^\dagger \nu \rightarrow \delta(r - \hat{z} t) \) and \( \frac{1}{2} (\nu^\dagger \sigma \nu) \rightarrow S z \delta(r - \hat{z} t) \), are

\[
\nabla \cdot \mathbf{E} + M^2 Z^0 = \kappa \delta(r - \hat{z} t) , \tag{40}
\]

\[
\nabla \times \mathbf{B} + M^2 \mathbf{Z} - \dot{\mathbf{E}} = -2\kappa S z \delta(r - \hat{z} t) . \tag{41}
\]

Appendix C considers the solution of the Maxwell equations (40) and (41). A physical \( (S = -\frac{1}{2}) \) point-like massless neutrino travelling along the z-axis at the speed of light generates scalar and vector potentials nonvanishing only in the plane perpendicular to \( \hat{z} \) at the neutrino’s instantaneous position. An observer could then only detect the neutrino’s presence at its moment of closest approach. This is not a property unique to the neutral current force; rather, it is a property related to Lorentz transformations properties of potentials and field strengths. The same phenomenon may be found in the ultra-relativistic limit of the Lienard-Wiechart potential for the electron. \(^{10}\)

The neutral current fields are given in cylindrical coordinates below:

\[
Z^0 = \frac{\kappa}{2\pi} K_0(M\rho)\delta(z - t) , \quad Z = \frac{\kappa}{2\pi} K_0(M\rho)\delta(z - t)\hat{z} ,
\]

\[
E = \frac{\kappa M}{2\pi} K_1(M\rho)\delta(z - t)\hat{\rho} , \quad B = \frac{\kappa M}{2\pi} K_1(M\rho)\delta(z - t)\hat{\phi} . \tag{42}
\]

where \( \kappa \) is the neutrino’s neutral current charge, and \( K_0 \) and \( K_1 \) are modified Bessel functions.

Under parity inversion the fields (42) represent a solution propagating backwards in time (antiparticle). Under both parity and time-reversal, \( E \) and \( B \) are even, while \( Z^0 \) and \( Z \) are odd.

If we consider solutions to the equations (40) and (41), for neutrinos of the wrong helicity \( (S = +\frac{1}{2}) \), the formal solution to (40) and (41) would then contain delta function derivatives (Appendix C).
IX. CONCLUSIONS

A nonrelativistic weak-field Hamiltonian suitable for describing neutral current interactions has been derived from the standard model. Maxwell equations for the Z-boson fields have also been derived. They are suitable for a description of the small Z-boson fields at an atomic or macroscopic scale.

The fields surrounding classical point-like leptons due to the neutral current interaction have been derived from the Z-boson Maxwell equations. The electron possesses Z-boson electric and magnetic fields analogous to its electromagnetic Coulomb and magnetic dipole fields. It also exhibits a Z-boson magnetic field forming a vortex-like structure oriented with the spin. This field is an explicit manifestation of the parity violation of the standard model. The fields of a point-like neutrino have also been shown to be nonvanishing only in the plane normal to its spin and containing the particle.

Using the nonrelativistic Hamiltonian and the solution of the Maxwell equations, we have calculated the 67 Hz contribution to the hyperfine splitting of the muonium ground state using simple nonrelativistic perturbation theory.

X. ACKNOWLEDGMENT

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XI. APPENDIX A: FIELD EQUATIONS FOR THE STANDARD MODEL

The gauge terms in the standard model Lagrangian are

\[ \mathcal{L}_{\text{gauge}} = -\frac{1}{4} F_{\alpha \mu \nu} F^{\alpha \mu \nu} - \frac{1}{4} f_{\mu \nu} f^{\mu \nu}, \]

\[ F_{\alpha \mu \nu} = \partial_{\nu} b_{\alpha \mu} - \partial_{\mu} b_{\alpha \nu} + g \varepsilon_{\alpha \beta \gamma \delta} b_{\gamma \mu} b_{\delta \nu}, \]

\[ f_{\mu \nu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu}. \]

where \( A_{\mu} \) and \( b_{\alpha \mu} \) are respectively the U(1) and SU(2) gauge fields, linear combinations of which form the physical boson fields:
\[ W^\pm = (b_1 \mp ib_2)/\sqrt{2}, \]
\[ Z = -A \sin \theta_W + b_3 \cos \theta_W, \quad (A2) \]
\[ A = A \cos \theta_W + b_3 \sin \theta_W. \]

Using the definitions (A2), field strengths for the physical fields may be defined as
\[ F_{\mu \nu}^\pm = \partial_\nu W_\mu^\pm - \partial_\mu W_\nu^\pm \pm ig \cos \theta_W (W_\mu^\pm Z_\nu - Z_\mu W_\nu^\pm) \pm ie (W_\mu^\pm A_\nu - A_\mu W_\nu^\pm), \]
\[ F_{\mu \nu}^Z = \partial_\nu Z_\mu - \partial_\mu Z_\nu - ig \cos \theta_W (W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+), \quad (A3) \]
\[ F_{\mu \nu}^A = \partial_\nu A_\mu - \partial_\mu A_\nu - ie (W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+). \]

where \( e = g \sin \theta_W = g' \cos \theta_W \). The gauge Lagrangian density, now including the boson mass terms, becomes
\[ \mathcal{L}_{\text{gauge}} = -\frac{1}{4} F_{\mu \nu}^A F^{\mu \nu} - \frac{1}{4} F_{\mu \nu}^Z F^{\mu \nu} - \frac{1}{2} F_{\mu \nu}^+ F^{-\mu \nu} + M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2} M_Z^2 Z_\mu Z^\mu. \quad (A4) \]

The equations of motion are
\[ \partial^\nu F_{\mu \nu}^\pm \pm ig \cos \theta_W F_{\mu \nu}^\pm Z^\nu \pm ie F_{\mu \nu}^\pm A^\nu \mp ie F_{\mu \nu}^A W_{\pm \nu} \pm ig \cos \theta_W F_{\mu \nu}^Z W^{\pm \nu} + M_W^2 W_\mu^+ = \text{weak charged current source (e.g., } e \rightarrow \nu \text{ and } \nu \rightarrow e), \]
\[ \partial^\nu F_{\mu \nu}^Z + ig \cos \theta_W F_{\mu \nu}^Z W^{+ \nu} - ig \cos \theta_W F_{\mu \nu}^Z W^{- \nu} + M_Z^2 Z_\mu = \text{weak neutral current source (e.g., } e \rightarrow e \text{ and } \nu \rightarrow \nu), \quad (A5) \]
\[ \partial^\nu F_{\mu \nu}^A + ie F_{\mu \nu}^A W^{+ \nu} - ie F_{\mu \nu}^A W^{- \nu} = \text{electromagnetic source (e.g., } e \rightarrow e). \]

where the matter terms in the full Lagrangian produce the sources for the boson fields in equations (A5).

To characterize the classical fields surrounding a lepton we consider only configurations for which the lepton remains invariant in its initial state.
There are no weak charged currents and hence \( W^\pm = 0 \) is a consistent solution to the fields equations and the remaining fields equations for \( A \) and \( Z \) decouple.

There are also Bianchi identities for the fields strengths which are satisfied identically:

\[
\partial^\nu * F^\pm_{\mu\nu} + ig \cos \theta_W * F^\pm_{\mu\nu} Z^\nu_\pm \mp ie * F^\pm_{\mu\nu} A^\nu_\pm \mp ig \cos \theta_W * F^Z_{\mu\nu} W^\pm_\nu = 0,
\]

\[
\partial^\nu * F^Z_{\mu\nu} + ig \cos \theta_W * F^Z_{\mu\nu} W^\nu_\mp \pm ie * F^A_{\mu\nu} W^\pm_\nu = 0, \tag{A6}
\]

where \( * F^\mu_\nu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F^\rho_\sigma \). When \( W^\pm = 0 \) the Bianchi identities decouple, and since there are no boson mass terms in the Bianchi identities, they reduce to the familiar form of classical electrodynamics for both the photon and the \( Z \)-boson.

The structure of the classical field theories for \( A \) and \( Z \) are now identical except that the \( Z \)-boson has a mass \( M = M_Z \) which appears in its equation of motion but not in its Bianchi identity.

**XII. APPENDIX B: SOLUTION FOR THE ELECTRON**

Without sources, \( B \) and \( Z \) are solutions to \( \nabla \times (\nabla \times U) + M^2 U = 0 \). When \( U \) is a solution then so is \( \nabla \times U \). Two solution pairs in spherical coordinates are

\[
U = z^{-\frac{3}{2}} \left\{ \begin{array}{c} I_{\frac{1}{2}}(z) \\ K_{\frac{1}{2}}(z) \end{array} \right\} \sin \theta \hat{\phi}, \tag{B1}
\]

\[
\nabla \times U = 2 M z^{-\frac{3}{2}} \left\{ \begin{array}{c} I_{\frac{3}{2}}(z) \\ K_{\frac{3}{2}}(z) \end{array} \right\} \cos \theta \hat{r} - M z^{-\frac{3}{2}} \left\{ \begin{array}{c} z I_{\frac{1}{2}}(z) - I_{\frac{3}{2}}(z) \\ -z K_{\frac{1}{2}}(z) - K_{\frac{3}{2}}(z) \end{array} \right\} \sin \theta \hat{\theta},
\]

where \( I_n(z) \) and \( K_n(z) \) are modified Bessel functions with \( z = Mr; \) explicitly
\[ I_\frac{3}{2}(z) = \sqrt{\frac{2}{\pi z}} \left( \cosh z - \frac{\sinh z}{z} \right) , \quad I_\frac{1}{2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z , \]

\[ K_\frac{3}{2}(z) = \sqrt{\frac{\pi}{2z}} \left( 1 + \frac{1}{z} \right) e^{-z} , \quad K_\frac{1}{2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} . \] (B2)

Consider a spherical volume \( V \) containing the uniform magnetic source density \( \vec{\kappa}S\hat{z} = \frac{\kappa}{V} S \left( \cos \theta \hat{r} - \sin \theta \hat{\theta} \right) \). Consider homogeneous solutions for \( Z \) and \( B \) inside (upper) and outside (lower) the sphere:

\[ Z = 2z^{-\frac{3}{2}} \left\{ \begin{array}{c} aI_\frac{1}{2}(z) \\ bK_\frac{1}{2}(z) \end{array} \right\} \cos \theta \hat{r} - z^{-\frac{3}{2}} \left\{ \begin{array}{c} azI_\frac{1}{2}(z) - aI_\frac{3}{2}(z) \\ -bzK_\frac{1}{2}(z) - bK_\frac{3}{2}(z) \end{array} \right\} \sin \theta \hat{\theta} , \]

\[ B = -Mz^{-\frac{3}{2}} \left\{ \begin{array}{c} aI_\frac{1}{2}(z) \\ bK_\frac{1}{2}(z) \end{array} \right\} \sin \theta \hat{\phi} . \] (B3)

A particular solution to \( \nabla \times B + M^2 Z = \vec{\kappa}S\hat{z} \) must be added inside the sphere, such as

\[ Z = M^{-\frac{3}{2}} \frac{\kappa}{V} S\hat{z} , \quad B = 0 . \] (B4)

The constants \( a \) and \( b \) are determined from the boundary conditions that the components of \( Z \) and \( B \) parallel to the interface at \( r = R \) are continuous across that interface:

\[ a = -\frac{3\kappa}{4\pi} S M (MR)^{-\frac{3}{2}} K_\frac{3}{2}(MR) \rightarrow R^{-3} \text{ as } R \rightarrow 0 , \] (B5)

\[ b = -\frac{3\kappa}{4\pi} S M (MR)^{-\frac{3}{2}} I_\frac{3}{2}(MR) \rightarrow -\frac{\kappa}{4\pi} SM \sqrt{\frac{2}{\pi}} \text{ as } R \rightarrow 0 . \]

The point-like electron solution corresponds to the \( R \rightarrow 0 \) limit; however, note that the constant \( a \) diverges like \( R^{-3} \) implying that \( Z \) may possess a \( \delta \)-function at the origin. To determine its weight, integrate \( \nabla \times B + M^2 Z \) over a spherical volume and apply Stokes’ theorem for volume integration:

\[ \kappa S\hat{z} = \int (\nabla \times B)d^3 r + M^2 \int Zd^3 r = \int (d^2 S \times B) + M^2 \int Zd^3 r . \] (B6)
This holds for any radius but in the limit as the radius tends to \( \infty \) the magnetic field vanishes. \( Z \) must have a term \( \frac{eS}{3M^2} \delta (r) \hat{z} \) for equality to hold.

Next consider a spherical volume \( V \) which contains the uniform magnetization \( M = \frac{\mu}{V} \hat{z} = \frac{\mu}{V} (\cos \theta \hat{r} - \sin \theta \hat{\theta}) \). Solutions for \( Z \) and \( B \) are

\[
\begin{align*}
Z &= -z^{-\frac{1}{2}} \left\{ aI_{\frac{3}{2}}(z) \right\} \sin \theta \hat{\phi}, \\
B &= -2Mz^{-\frac{3}{2}} \left\{ aI_{\frac{3}{2}}(z) \right\} \cos \theta \hat{r} + Mz^{-\frac{1}{2}} \left\{ azI_{\frac{3}{2}}(z) - aI_{\frac{1}{2}}(z) \right\} \sin \theta \hat{\theta}.
\end{align*}
\]

The vector potential \( Z \) must be continuous across the interface at \( r = R \). The other boundary condition involves \( H = B - M \). It is easy to demonstrate using Stokes’ theorem that, without surface current sources, the tangential component of \( H \) must be continuous across the interface, The solutions for the constants \( a \) and \( b \) are

\[
\begin{align*}
a &= -\frac{3}{4\pi} \mu M^2 (MR)^{-\frac{3}{2}} K_{\frac{3}{2}}(MR) \to R^{-3} \text{ as } R \to 0 \\
b &= -\frac{3}{4\pi} \mu M^2 (MR)^{-\frac{3}{2}} I_{\frac{3}{2}}(MR) \to -\frac{\mu}{4\pi} M^2 \sqrt{\frac{2}{\pi}} \text{ as } R \to 0
\end{align*}
\]

The divergence of the constant \( a \) signifies the presence of a \( \delta \)-function in the magnetic field. To determine its weight, compute the volume integral of \( \nabla \times Z = B \) and use the Stokes theorem for volume integration as before. There is a similar singularity in the point-like electron’s electromagnetic dipole field \([13]\).

The solution for the electric field is a straightforward application of Gauss’ law.

**XIII. APPENDIX C - SOLUTION FOR THE MASSLESS NEUTRINO**

Consider a massless neutrino whose trajectory takes it through the origin and then along the positive z-axis. Its potentials and fields will be functions of \( u = z - t \) and the cylindrical coordinate \( \rho \). The Maxwell equations (40) and (41) may be written as second order differential equations for the potentials using cylindrical coordinates:

\[
\text{(B6)}
\]
\[
\begin{align*}
- \frac{\partial^2 Z^+}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial Z^+}{\partial \rho} + M^2 Z^+ - 2 \frac{\partial^2 Z^-}{\partial u^2} + \frac{\partial}{\partial u} \left( \frac{\partial Z^\rho}{\partial \rho} + \frac{1}{\rho} Z^\rho \right) &= \frac{\kappa}{2} (1 - 2S) \delta(\mathbf{r} - \hat{\mathbf{z}}t) & (C1) \\
- \frac{\partial^2 Z^-}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial Z^-}{\partial \rho} + M^2 Z^- &= \frac{\kappa}{2} (1 + 2S) \delta(\mathbf{r} - \hat{\mathbf{z}}t) & (C2) \\
\frac{\partial^2 Z^\rho}{\partial \rho \partial u} - \frac{1}{2} Z^\rho &= 0 & (C3) \\
- \frac{\partial^2 Z^\phi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial Z^\phi}{\partial \rho} + \left( \frac{1}{\rho^2} + M^2 \right) Z^\phi &= 0 & (C4)
\end{align*}
\]

where \( Z^+ = \frac{1}{2} (Z^0 + \hat{\mathbf{z}} \cdot \mathbf{Z}) \), \( Z^- = \frac{1}{2} (Z^0 - \hat{\mathbf{z}} \cdot \mathbf{Z}) \), \( Z^\rho = \hat{\mathbf{\rho}} \cdot \mathbf{Z} \), and \( Z^\phi = \hat{\phi} \cdot \mathbf{Z} \).

Equation (C4) is the modified Bessel equation. With no singularity at the origin, equation (C4) has no bounded solution at infinity other than \( Z^\phi = 0 \). Similarly, for \( S = -\frac{1}{2} \) we must choose \( Z^- = Z^\rho = 0 \). Equation (C1) is then satisfied by the modified Bessel function \( Z^+(\rho, u) = a K_0(M\rho) \delta(u) \).

To determine the weight \( a \), Gauss’s law is applied to a tube of radius \( R \) concentric with the \( z \)-axis:

\[
\kappa = \int \mathbf{E} \cdot d^2 \mathbf{S} + M^2 \int Z^0 d^3 \mathbf{r} = 2\pi a M R K_1(MR) + 2\pi a M^2 \int_0^R K_0(M\rho) \rho d\rho
\]

\[
= 2\pi a \lim_{\rho \to 0} M \rho K_1(M\rho) = 2\pi a
\]

(C5)

When \( S = +\frac{1}{2} \), equation (C2) has a solution proportional to \( K_0(M\rho) \delta(u) \). The \( \delta \)-function is then differentiated in (C1) and (C3).
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