Nondeterministic Linear Logic

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Abstract

In this paper, we introduce Linear Logic with a nondeterministic facility, which has a self-dual additive connective. In the system the proofnet technology is available in a natural way. The important point is that nondeterminism in the system is expressed by the process of normalization, not by proof search. Moreover we can incorporate the system into Light Linear Logic and Elementary Linear Logic developed by J.-Y. Girard recently: Nondeterministic Light Linear Logic and Nondeterministic Elementary Linear Logic are defined in a very natural way.

1 Introduction

So far (untyped or typed) lambda calculi with the facility of nondeterminism have been studied: recently e.g., in [Aba94, DCLP93]. For example, in [DCLP93] nondeterminism is represented by using union type, while parallelism by using intersection type: this means that nondeterminism corresponds to the logical connective “or” and parallelism to “and”. Further this means that nondeterminism and parallelism are dual notions each other. Basically other researchers similarly classify nondeterminism and parallelism. In this paper, we advocate that nondeterminism and parallelism are not dual notions. For this we use the framework of Linear Logic [Gir87]. In Linear Logic, usual logical connectives are classified into two: multiplicative and additive connectives. Our advocacy is that nondeterminism and parallelism are classified in “computation as normalization” paradigm as follows:

- Nondeterminism = Additive.
- Parallelism = Multiplicative.

Already it has been pointed out that the multiplicative connectives are deeply related to parallelism since the appearance of [Gir87]. Currently V. Pratt studies the relationship intensively in the context of Chu space [Pra95, Pra97]. Here we point out that the additive connectives are deeply related to nondeterminism. We incorporate nondeterminism facility into the framework of Linear Logic by introducing new additive connective △ (nondeterministic with), which is self-dual. In the framework, nondeterminism is represented by reduction of cut between two △: by the reduction of △ from one proof

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net two proof nets are obtained. Note that standard Linear Logic is deterministic: this means proof nets have Church-Rosser property in the syntactical level and there are some “deterministic” denotational semantics of Linear Logic in the semantical level. So in order to incorporate nondeterminism into the framework of Linear Logic, we must introduce the new connective. Our advocacy has not been advocated before as far as we know in the context of “computation as normalization” paradigm. Also I believe that such a classification contributes to studies w.r.t. relationship between Linear Logic and Process Calculus.

On the other hand, our Nondeterministic Linear Logic also contributes to the study of the logical aspect of Complexity Theory in the context of “computation as normalization” paradigm. We can encode any nondeterministic polynomial-time Turing Machine into a proof (or a proof net) of Nondeterministic Light Linear Logic. The encoding is a nondeterministic version of Girard’s encoding deterministic polynomial-time Turing Machines into proofs of Light Linear Logic. But our polymorphic encoding of nondeterministic computations is original. Also by using the same method, we can formulate Nondeterministic Elementary Linear Logic and prove the similar statement. This is the main technical contribution of this paper.

2 The System

The system NDMALL is usual MALL (the multiplicative additive fragment of Linear Logic) with △ (nondeterministic with). The connective has arity 2 (hence in NDMALL A △ B is accepted as a formula if A and B are NDMALL formulas). The negation of A △ B is defined as follows:

\[(A \triangle B)^\perp \equiv A^\perp \triangle B^\perp\]

The inference rules for NDMALL are the same as MALL except for the following rule:

\[
\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \triangle B} \quad \text{(NDWITH)}
\]

The notion of proofs (in sequent calculus) of NDMALL is defined in usual manner. Obviously the connective △ belongs to additives. The problematic point of NDMALL is that any sequent of the form \[\vdash A \triangle B, A^\perp \triangle B^\perp\] does not have the proof of \(\eta\)-long normal form, i.e., consisting of just atomic formulas. But for example, many modal logics also do not have such proofs, even cut-free systems. We believe Nondeterministic Linear Logic can be accepted as a logical system. Though even you does not agree with the belief, you should accept our system as a type system for nondeterministic computations.

In practice, the connective does not occur in conclusions of NDMALL proofs: if it occurs in them, then in one sided sequent calculus (or in the formulation of proof nets) it behaves like & in completely the same manner. Hence we can assume that △ does not occur in cut free NDMALL proofs. We omit cut elimination procedure for △ in NDMALL sequent calculus. But we will introduce the procedure using NDMALL proof nets in Section 4.
3 NDMALL proof nets

First we shall define NDMALL proof structures, which are basically the same as them in [Gir95b] except for connective $\triangle$. Simply by formulas we mean NDMALL formulas. Note that to each $\triangle$-link $L$ an eigenweight $p_L$ is assigned.

**Definition 1** A link $L$ is an $n + m$-tuple of formulas with a type: $P_1, \ldots, P_n$ $Q_1, \ldots, Q_m$. The type of a link is either ID, Cut, generalized axiom, $\otimes$, $\&$, $\oplus_1$, $\oplus_2$, or $\triangle$. To each type, $n$, a number of its premises and $m$, a number of its conclusions are assigned ($m, n \geq 0, m + n \neq 0$). The links with ID, Cut, generalized axiom, $\otimes$, $\&$, $\oplus_1$, $\oplus_2$, and $\triangle$ as types have the following forms:

- **ID-links** $A A^\perp$
- **Cut links** $A A^\perp$ with $A$ $B$
- **generalized axiom-links** $A_1 \cdots A_n$
- **$\otimes$-links** $A \otimes B$
- **$\&$-links** $A \& B$
- **$\oplus_1$-links** $A \oplus B$
- **$\oplus_2$-links** $B \oplus A$
- **$\triangle$-links** $A \triangle B$

We must distinguish a left premise (A) and a right premise (B) in $\otimes$, $\&$, and $\oplus$-links. For example, in a $\triangle$-link with $A \triangle A$ as the conclusion, the two premises $A$ and $A$ must be distinguished in an obvious way.

**Definition 2** To any $\&$-link or $\triangle$-link $L$ with $A \& B$ or $A \triangle B$ as its conclusion, we associate an eigenweight $p_L$, which is a boolean variable. The intuitive meaning of $p_L$ is the choice $\{l/r\}$ between the premises $A$ and $B$: $+p_L$ stands for the selection “left”, i.e., $A$ and $-p_L$ stands for the selection “right”, i.e., $B$. We use $\varepsilon p_L$ to speak of $+p_L$ or $-p_L$.

**Definition 3** A triple $\Theta = (V, E, w)$ is a proof structure if

- $(V, E)$ is a pair such that $V$ is a multiset of formulas and $E$ is a multiset of links between formulas occurring in $V$.
- $w$ is a function such that
  - (i) For each formula $A$ in $V$, a weight $w(A)$, i.e., a non-zero element of the boolean algebra generated by the eigenweights $p_1, \ldots, p_n$ of the $\&$-links or $\triangle$-links of $\Theta$;
  - (ii) For each link $L$ in $E$, a weight $w(L)$, i.e., a non-zero element of the boolean algebra generated by the eigenweights $p_1, \ldots, p_n$ of the $\&$-links or $\triangle$-links of $\Theta$.

Moreover, the following conditions must be satisfied:

- (a) Each formula in $V$ is the premise of at most one link and the conclusion of at least one link. The formulas which are not premises of some link are called the conclusions of $\Theta$;
(b) \( w(A) = \sum_{L \text{has } A \text{ as the conclusion}} w(L); \)

(c) If \( A \) is a conclusion of \( \Theta \), then \( w(A) = 1; \)

(d) If \( u \) is any weight occurring in \( \Theta \), then \( w(u) = 1 \);

(e) If \( u \) is a weight occurring in \( \Theta \) and containing \( \varepsilon \cdot p_L \), then \( u \leq w(L); \)

(f) If \( L \) is any non ID-link, with premises \( A \) and/or \( B \) then

- if \( L \) is any of \( \otimes, \wp \) and Cut, then \( w(L) = w(A) = w(B); \)
- if \( L \) is a \( \oplus_1 \)-link, then \( w(L) = w(A); \)
- if \( L \) is a \( \oplus_2 \)-link, then \( w(L) = w(B); \)
- if \( L \) is a \&-link, then \( w(A) = w(L) \cdot p_L \) and \( w(B) = w(L) \cdot \neg p_L; \)
- if \( L \) is a \( \triangle \)-link, then \( w(A) = w(L) \cdot p_L \) and \( w(B) = w(L) \cdot \neg p_L; \)

(g) For any \( A \in V \), if the links whose conclusion is \( A \) are \( L_1, \ldots, L_m \) then for each \( 1 \leq i, j \leq m \), whenever \( i \neq j \), then \( w(L_i) \neq w(L_j). \)

**Definition 4** Let \( \phi \) be a valuation for a proof structure \( \Theta = (V, E, w) \), i.e. a function from the set of eigenweights of \( \Theta \) to \{0, 1\}, which is extended to a function (still denoted \( \phi \)) from the weights of \( \Theta \) to \{0, 1\}. A pair \( \phi(\Theta) = (V_0, E_0) \) is the slice by \( \phi \) if \( V_0 \) is the restriction to the formulas \( A \) in \( V \) such that \( \phi(w(A)) = 1 \) and \( E_0 \) is the restriction of \( E \) by \( V_0 \) where the definition of \&-links and \( \triangle \)-links is changed such that they have exactly one premise and one conclusion.

The definition of the dependencies of the weights and the formulas in proof structures on an eigenweight is the same as that of [Gir95b].

**Definition 5** Let \( \phi \) be a valuation of \( \Theta \), let \( p_L \) be an eigenweight. A weight \( w \) (in \( \phi(\Theta) \)) depends on \( p_L \) (in \( \phi_L(\Theta) \)) if \( \phi(w) \neq \phi_L(w) \), where the valuation \( \phi_L \) is defined as follows:

- \( \phi_L(p_L) = \neg(\phi(p_L)); \)
- \( \phi_L(p_{L'}) = \phi(p_{L'}) \) if \( L' \neq L. \)

A formula \( A \) of \( \Theta \) is said to depend on \( p_L \) (in \( \phi(\Theta) \)), if \( A \) is the conclusion of a link \( L' \) such that \( \phi(w(L')) = 1 \) and \( \phi_L(w(L')) = 0. \)

**Definition 6** A switching \( S = (\phi_S, \text{select}_\wp, \text{select}_\&, \text{select}_\triangle) \) of a proof structure \( \Theta \) consists in:

- A choice of a valuation \( \phi_S \) for \( \Theta; \)
- A function \( \text{select}_\wp \) from the set of all \( \wp \)-links \( L \) of \( \phi_S(\Theta) \) to \{l, r\} whose element represents a choice for premises of a \( \wp \)-link.
• A selection $\text{select}_\&$ for each $\&$-link $L$ of $\phi_S(\Theta)$ a formula $\text{select}_\&(L)$, the jump of $L$, depending on $p_L$ in $\phi_S(\Theta)$. There is always a normal choice of jump for $L$, namely the premise $A$ of $L$ such that $\phi_S(w(A)) = 1$.

• A selection $\text{select}_\bigtriangleup$ for each $\bigtriangleup$-link $L$ of $\phi_S(\Theta)$ a formula $\text{select}_\bigtriangleup(L)$, the jump of $L$, depending on $p_L$ in $\phi_S(\Theta)$. There is always a normal choice of jump for $L$, namely the premise $A$ of $L$ such that $\phi_S(w(A)) = 1$.

**Definition 7** Let $S$ be a switching of a proof structure $\Theta$: the graph $\Theta_S = (V_S, E_S)$ corresponding to $S$ consists in:

- the vertices $V_S$ is $V_0$ of $\phi_S(\Theta) = (V_0, E_0)$;
- the edges $E_S$ are consists of:
  1. the edge between the conclusions for any ID-link of $\phi_S(\Theta)$;
  2. the edge between the premises for any Cut-link of $\phi_S(\Theta)$;
  3. the edge between the conclusion and the premise for any $\oplus$-links of $\phi_S(\Theta)$;
  4. the edges between the left premise and the conclusion, and between the right premise and the conclusion for any $\otimes$-link of $\phi_S(\Theta)$;
  5. the edge between the the premise (left or right) selected by $\text{select}_\&(L)$ and the conclusion of any $\&$-link $L$.
  6. the edge between the jump $\text{select}_\&(L)$ of $L$ and the conclusion for any $\&$-link $L$.
  7. the edge between the jump $\text{select}_\bigtriangleup(L)$ of $L$ and the conclusion for any $\bigtriangleup$-link $L$.

**Definition 8** A proof structure $\Theta$ is said to be a proof net if for any switching $S$, the graph $\Theta_S$ is connected and acyclic.

The removal of a link of a proof structure $\Theta$ in NDMALL is defined in the same manner as [Gir95b] except for $\bigtriangleup$-links. Here the definition of the removal for $\bigtriangleup$-links is only added.

**Definition 9** The case where $L$ is a $\bigtriangleup$-link with premises $A$ and $B$ such that $w(L) = 1$ and $L$ is a conclusion of $\Theta$, and $\Gamma, A \bigtriangleup B$ is the set of conclusions of $\Theta$. The removal of $L$ is the operation which first removes the conclusion $A \bigtriangleup B$ and the link $L$, gets a proof structure $\Theta'$ and then forms two proof structures $\Theta_A$ and $\Theta_B$ from $\Theta'$:

* In $\Theta'$ make the substitution $p_L = 1$, and keep only those links $L'$ whose weight is still non-zero, together with the premises and conclusions of such links: the result is by definition $\Theta_A$, a proof structure with conclusions $\Gamma, A$.
* In $\Theta'$ make the substitution $p_L = 0$, and keep only those links $L'$ whose weight is still non-zero, together with the premises and conclusions of such links: the result is by definition $\Theta_B$, a proof structure with conclusions $\Gamma, B$. 
**Definition 10** A proof structure Θ is sequentializable if

1. Θ is an ID-link, or;

2. the proof structures which are obtained by the removal of a terminal link in Θ are sequentializable.

The proof of the following theorem is completely the same as that of [Gir95b] which uses the empire for each valuation and each formula, since in fixed proof nets △-links behave in the same manner as &-links. However the behavior of △-link in cut elimination is different from that of & which is defined in the next section.

**Theorem 1** ([Gir95b]) Θ is a proof net iff Θ is sequentializable.

### 4 Lazy Cut Elimination in NDMALL

**Definition 11** A cut-link L is ready if

- \( w(L) = 1 \) and;
- If the premises of L are A and A⊥ then both A and A⊥ are the conclusion of exactly one link.

**Definition 12 (lazy cut elimination)** Let \( L_0 \) be a ready cut in a proof net Θ, whose premises \( B \triangle C \) and \( B^\perp \triangle C^\perp \) are the respective conclusions of links \( L \) and \( L' \). Then we define the contractums \( \Theta' \) and \( \Theta'' \) of redex \( \Theta \) when reducing \( L_0 \) in \( \Theta \).

- If \( L \) is a △-link (with premises \( B \) and \( C \)) and \( L' \) is a △-link (with premises \( B^\perp \) and \( C^\perp \)), then \( \Theta' \) and \( \Theta'' \) are obtained in three steps (the reduction is called △-reduction):
  - how to get \( \Theta' \) (resp. \( \Theta'' \)):
    1. to show \( \Theta' \) is a proof net: Let \( \phi' \) be a valuation for \( \Theta' \). Then we define the valuation \( \phi \) for \( \Theta \) from \( \phi' \) as follows:
      \[
      \phi(p_M) = \begin{cases} 
      1 & \text{ if } M = L \text{ or } L' \\
      \phi'(p_M) & \text{ if otherwise.}
      \end{cases}
      \]

**Proposition 1** If Θ' is obtained from a proof net Θ by lazy cut elimination, then Θ' is a proof net and has the same conclusions as Θ.

**Proof.** We only consider △-reduction:

- Here we use the same meta symbols as the definition of lazy cut elimination.

  1. to show Θ' is a proof net: Let \( \phi' \) be a valuation for Θ'. Then we define the valuation φ for Θ from \( \phi' \) as follows:

\[
\phi(p_M) = \begin{cases} 
1 & \text{if } M = L \text{ or } L' \\
\phi'(p_M) & \text{otherwise.}
\end{cases}
\]
Let $S'$ be any switching with the valuation $\phi'$ for $\Theta'$. Then we define the switching $S$ for $\Theta$ from $S'$ as follows:

- the valuation of $S$ is $\phi$;
- select$^S_p$ and select$^S_\land$ are the same as $S'$;
- select$^S_\triangledown (M) = \begin{cases} B & \text{if } M = L; \\ B & \perp \text{ if } M = L'; \\ \text{select}^{S'}_{\triangledown} (M) & \text{if otherwise.} \end{cases}$

Since $\Theta$ is a proof net by assumption, $\Theta_S$ is acyclic and connected. Then from this it is immediate that $\Theta_{S'}$ is acyclic and connected. Hence $\Theta'$ is a proof net.

2. to show $\Theta''$ is a proof net: Let $\phi''$ be a valuation for $\Theta''$. Then we define the valuation $\phi$ for $\Theta$ from $\phi''$ as follows:

$$\phi(p_M) = \begin{cases} 0 & \text{if } M = L \text{ or } L'; \\ \phi''(p_M) & \text{if otherwise.} \end{cases}$$

Let $S''$ be any switching for $\Theta''$ with the valuation $\phi''$. Then we define the switching $S$ for $\Theta$ from $S''$ as follows:

- the valuation of $S$ is $\phi$;
- select$^S_p$ and select$^S_\land$ are the same as $S''$;
- select$^S_\triangledown (M) = \begin{cases} C & \text{if } M = L; \\ C & \perp \text{ if } M = L'; \\ \text{select}^{S''}_{\triangledown} (M) & \text{if otherwise.} \end{cases}$

Since $\Theta$ is a proof net by assumption, $\Theta_S$ is acyclic and connected. Then from this it is immediate that $\Theta_{S''}$ is acyclic and connected. Hence $\Theta''$ is a proof net. □

Since $\triangledown$ is a variant of additive connectives, by the same method as that in [Gir95b], the following proposition is easily proved.

**Proposition 2** By lazy cut elimination, any MALL proof net is reduced to a unique normal form (which contains ready cuts) in linear time of its size.

## 5 Nondeterministic Light Linear Logic

In [Gir95c], it is shown that (1) any p-time Deterministic Turing Machine are representable in Light Linear Logic (for short LLL) and (2) under the condition of bounded depth any LLL proof net is reduced to a normal form in p-time of its size. In this section we show that (1') any p-time Nondeterministic Turing Machine are representable in Nondeterministic Light Linear Logic (for short NDLLL) and (2') under the condition of bounded depth any NDLLL proof net is reduced to a normal form by lazy cut elimination in p-time of its size. The system NDLLL is obtained from LLL by adding the inference rule (NDWITH) in Section 2. It is not difficult to show (2') if we follow Girard’s proof for LLL, since any NDMALL proof net is reduced a normal form by...
lazy cut elimination in linear time of its size (Proposition 2) and △ connective does not interact with any exponential connectives.

Let a Nondeterministic Turing Machine be \( M \). Let \( \Sigma \) be the set of the symbols used in \( M \) and \( \mathcal{Q} \) be the set of the states used in \( M \). Let \( p \) be the number of the symbols used in \( M \), i.e., the cardinal of \( \Sigma \) and \( q \) be the number of the states used in \( M \), i.e., the cardinal of \( \mathcal{Q} \). In order to prove (1'), we only show the move (transition) relation of the Nondeterministic Turing Machine \( M \) is representable in NDMALL, since from a representation in NDMALL of the move relation of \( M \) we can easily construct a proof net with \( \forall I \otimes \text{Tur}^{p,q} \to \text{Tur}^{p,q} \) as the conclusion that represents \( M \) completely, where

\[
\text{Tur}^{p,q} = \text{list}^p \otimes \text{list}^q \otimes \text{bool}^2, \text{list}^p = \forall X, (\exists (X \to oX) \to oX) \to oX, \text{and} \text{list}^q = \forall Y, (\exists (Y \to oY) \to oY) \to oY.
\]

The move relation \( R \) of \( M \) is represented as a subset of \( (\Sigma \times \mathcal{Q}) \times (\Sigma \times \mathcal{Q} \times \{←, →\}) \). Then it is sufficient to represent the move relation \( R \) by a NDLLL proof net with \( \text{bool}^{p \times q} \to \text{bool}^{p \times q^2} \) as the conclusion, since we can easily see the set \( (\Sigma \times \mathcal{Q}) \) is represented by \( \text{bool}^{p \times q} \) and \( (\Sigma \times \mathcal{Q} \times \{←, →\}) \) by \( \text{bool}^{p \times q \times 2} \), we can easily construct any proof net with \( \text{bool}^p \otimes \text{bool}^q \to \text{bool}^{p \times q} \) as the conclusion and with \( \text{bool}^{p \times q \times 2} \to \text{bool}^p \otimes \text{bool}^q \otimes \text{bool}^2 \) as the conclusion by using a general version of \( D \) in Section 11.3 in [GL89], and given any proof net with \( \text{bool}^{p \times q} \to \text{bool}^{p \times q \times 2} \) as the conclusion, by composing these proof nets we can easily construct any proof net with \( \text{bool}^p \otimes \text{bool}^q \to \text{bool}^p \otimes \text{bool}^q \otimes \text{bool}^2 \) as the conclusion. Let \( m \) be \( \max \{ \{(y,t,d) : (x,s,(y,t,d)) \in R \} : x \in \Sigma, s \in \mathcal{Q} \} \). The following NDLL proof corresponds to the intended proof net:

\[
\begin{align*}
(1) & \\
\vdots & \\
\vdots & \\
\end{align*}
\]

\[
\begin{array}{c}
\frac{X \& \cdots \& X \vdash X \wedge \cdots \wedge X}{p \times q^2} \\
\frac{X \& \cdots \& X \vdash (X \wedge \cdots \wedge X) \& \cdots \& (X \wedge \cdots \wedge X)}{p \times q} \\
\frac{(X \wedge \cdots \wedge X) \& \cdots \& (X \wedge \cdots \wedge X) \to (X \wedge \cdots \wedge X) \vdash X \& \cdots \& X \to oX}{p \times q^2} \\
\frac{\frac{(X \wedge \cdots \wedge X) \& \cdots \& (X \wedge \cdots \wedge X) \to (X \wedge \cdots \wedge X)}{p \times q^2}}{\forall Y, (\exists (Y \to oY) \to oY) \to oY} (\forall) \\
\end{array}
\]

The programming of the move relation \( R \) corresponds to the proofs between (1) and
\(\langle p \times q \rangle\), the move relation \(R\). By the way there may be \(x \in \Sigma\) and \(s \in Q\) such that \(|\{(y, t, d) : (x, s, (y, t, d)) \in R\}| < m\). Then we introduce a new state “halt” and can construct a new proof net with \(\vdash \text{bool}^{p \times q} \rightsquigarrow \text{bool}^{p \times (q+1) \times 2}\) as the conclusion from the already obtained proof net by turning \(m - |\{(y, t, d) : (x, s, (y, t, d)) \in R\}|\) transitions into “halt” state.

From what precedes the following theorem is proved.

**Theorem 2** Any \(p\)-time Nondeterministic Turing Machine are representable in Nondeterministic Light Linear Logic.

It is obvious that in the context of Elementary Linear Logic, the same theorem is proved.

6 Concluding-remarks

As to the semantics of Nondeterministic Linear Logic, in this paper, we just presented a very primitive operational semantics: lazy cut elimination procedure (see Section 4). In order to justify Nondeterministic Linear Logic, we must develop denotational and operational semantics of Nondeterministic Linear Logic in more sophisticated ways:

- **Denotational Semantics**
  In the usual coherent semantics, self-dual connectives like \(\triangle\) connective are not allowed. However in [Gir96], J.-Y. Girard has developed a semantics not only accommodating usual connectives of Linear logic, but also self-dual additive connectives: in the semantics formulas are interpreted by coherent Banach spaces (which are named by Girard) and proofs by vectors in the spaces. Since in NDMALL the Church-Rosser property does not hold and the result of normalization of a proof leads many normal proofs, the NDMALL proofs in the semantics are interpreted by the sum of some vectors (i.e., a vector) in the coherent Banach spaces. The details will be left elsewhere.
  Also interpretations of NDLL into Chu spaces [Pra95, Pra97] and Game Semantics [AG94] are interesting. Such researches will lead some insights on the relationship between Linear Logic and Concurrency Theory.

- **Operational Semantics**
  For Linear Logic very elegant operational semantics have been developed: Geometry of Interaction (for short GOI). In [Gir95a], GOI has been extended to the system accommodating the additives. The study of GOI for Nondeterministic Linear Logic is interesting. In GOI for MALL a simple logic programming language is used. It is not difficult to incorporate nondeterminism with logic programming. Hence it seems that the development of GOI for NDMALL is not so difficult.

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