Schatten–von Neumann Properties in the Weyl Calculus, and Calculus of Metrics on Symplectic Vector Spaces

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Abstract Let $s^w_p$ be the set of all $a \in \mathcal{A}'$ such that $a^w(x, D)$ is Schatten $p$-operator on $L^2$. Then we prove the following:

- $S(m, g) \subseteq s^w_p$ iff $m \in L^p$. Furthermore, $L^p \cap S(m, g) \subseteq s^w_p$ when $h^{N/2} m \in L^p$. Consequently, $S^p_{\rho,\delta} \cap L^\infty \subseteq s^w_\infty$ when $0 \leq \delta < \rho \leq 1$;
- if $g(z, \zeta) = \sum \lambda_j (z_j^2 + \zeta_j^2)$, then $G(z, \zeta) = \sum \lambda_j^{\alpha} (z_j^2 + \zeta_j^2)$ is symplectically invariantly defined. Moreover, if $0 \leq \alpha \leq 1$ and $g \leq g^\sigma$ is slowly varying (and $\sigma$-temperate), then the same is true for $G$;
- a generalization of sharp Gårding’s inequality.

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1. Introduction

In [17, 18], Hörmander present a new approach to the Weyl calculus of pseudo-differential operators. Here among others, the symplectic invariance is established, as well as a general family of symbol classes is introduced where each symbol class is parameterized by certain weight function and Riemannian metric on the phase space. Moreover, important results concerning continuity, half-continuity and compactness are established, e.g. an improvement of Melin’s inequality is presented.

Some further improvements have been done since [17, 18]. In [19], Hörmander present, among others, a proof of Fefferman and Phong’s inequality, and in [3, 4, 6], Bony and Lerner present important extensions and improvements in several directions of the calculus. For example, certain generalizations of Gårding’s and Fefferman and Phong’s inequalities were

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established by Bony in [4]. Some detailed study of compactness in terms of Schatten–von Neumann properties have also been done by Buzano and Nicola in [8].

The aim of the present paper is to investigate continuity, especially Schatten–von Neumann properties in the calculus and to discuss properties of Riemannian metrics defined on symplectic vector spaces. (cf. [35].) In the first part we establish necessary and sufficient conditions on the involved symbol classes in order to the corresponding Weyl operators should belong to Schatten–von Neumann classes of certain degrees. In particular, our investigations concern \( L^2 \)-continuity, compactness, trace-class and Hilbert–Schmidt properties in the calculus. (cf. e.g. [17, 19, 28, 29, 33].) Recall from the above that similar questions have been considered in [8]. We shall here give an alternative approach and prove that the Schatten–von Neumann results in [8] hold in more general situations. Later on, we present further improvements, where, in particular, Theorem 3.9 in [18] and Theorem 18.6.3 in [19] are generalized and put into a more general context. Roughly speaking, the results here are more focused on Schatten–von Neumann properties of single pseudo-differential operators instead of whole classes of such operators.

In these considerations as well as for the Weyl calculus in general, geometric properties for metrics defined on symplectic vector spaces are important, depending on the fact that the general definition of the symbol classes are formulated in terms of such metrics. (See [3, 5, 6, 10, 17, 19].) Some attentions are therefore paid to present some new properties for such metrics which are needed. However, we note that these investigations are independent of pseudo-differential calculus in general, and the results which are obtained might be valuable in other fields as well.

In order to describe our results more in details we recall the definition of the Weyl quantization. Let \( V \) be a vector space of dimension \( n < \infty \), \( V' \) its dual space and let \( W = T^*V = V \oplus V' \). Then the Weyl quantization \( a_w(x, D) \) of \( a \in \mathscr{S}(W) \) is the linear and continuous operator on \( \mathcal{F}(V) \), defined by the formula

\[
a_w(x, D) f(x) = \text{Op}^w(a) f(x) = (2\pi)^{-n} \int \int a((x + y)/2, \xi) f(y) e^{i(y - \xi)/2} dyd\xi
\]

when \( f \in \mathcal{F}(V) \). (We use the usual notation for function and distribution spaces, see e.g. [19].) The definition of \( a_w(x, D) \) extends to each \( a \in \mathscr{S}^\prime(W) \), giving that \( a_w(x, D) \) is continuous from \( \mathcal{F}(V) \) to \( \mathcal{F}^\prime(V) \). (See [19, 31–33].)

We are especially concerned with the family \( s^w_p(W) \), \( p \in [1, \infty) \), of Banach spaces of distributions \( a \in \mathscr{S}^\prime(W) \) such that \( a_w(x, D) \) is a Schatten–von Neumann operator to the order \( p \). In particular, \( a_w(x, D) \) is \( L^2 \)-continuous, Hilbert–Schmidt or trace-class operator, if and only if \( a \in s^w_\infty \), \( a \in s^w_2 \) and \( a \in s^w_p \) respectively. In general it is difficult to find simple characterizations for the \( s^w_p \)-spaces. A question in the calculus therefore deals with finding necessary and sufficient conditions for an element \( a \) to belong to \( s^w_p \) for certain \( p \).

Such questions have been carefully investigated from a point of view of harmonic analysis, when the symbol classes consist of Sobolev spaces, Besov spaces (see [33]), or modulation spaces (see [13–15, 34]). For example, in [33] sharp embeddings between Sobolev spaces, Besov spaces and \( s^w_p \)-spaces are presented. From these investigations it follows that

\[
H^p_{2\mu(2/p - 1)}(W) \subseteq s^w_p(W) \subseteq H^p_{-\mu(2/p - 1)}(W), \quad 1 \leq p \leq 2,
H^p_{\mu(1 - 2/p)}(W) \subseteq s^w_p(W) \subseteq H^p_{-\mu(1 - 2/p)}(W), \quad 2 \leq p \leq \infty,
\]

for any \( \mu > 1 \) (cf. Theorem 2.6 in [33]). Here \( H^p \) denotes the Sobolev space of distributions with \( s \) derivatives in \( L^p \). (See, e.g., [28] or Section 2.) In particular, \( s^w_2 = L^2 \). We also observe \( \Box \) Springer.