Comparison of an Apocalypse-Free and an Apocalypse-Prone First-Order Low-Rank Optimization Algorithm

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Abstract: We compare two first-order low-rank optimization algorithms, namely P^2GD (Schneider and Uschmajew, 2015), which has been proven to be apocalypse-prone (Levin et al., 2021), and its apocalypse-free version P^2GDR obtained by equipping P^2GD with a suitable rank reduction mechanism (Olikier et al., 2022). Here an apocalypse refers to the situation where the stationarity measure goes to zero along a convergent sequence whereas it is nonzero at the limit. The comparison is conducted on two simple examples of apocalypses, the original one (Levin et al., 2021) and a new one. We also present a potential side effect of the rank reduction mechanism of P^2GDR and discuss the choice of the rank reduction parameter.

Keywords: Stationarity · Low-rank optimization · Determinantal variety · Steepest descent · Tangent cones.

AMS subject classifications: 14M12, 65K10, 90C30.

1. INTRODUCTION

As in Olikier et al. (2022), we consider the problem

$$\min_{X \in \mathbb{R}^{m \times n}} f(X)$$

of minimizing a differentiable function $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ with locally Lipschitz continuous gradient on the determinantal variety (Harris, 1992, Lecture 9)

$$\mathbb{R}^{m \times n}_{\leq r} := \{ X \in \mathbb{R}^{m \times n} \mid \text{rank } X \leq r \},$$

$m$, $n$, and $r$ being positive integers such that $r < \min\{m, n\}$. This problem appears in several applications such as matrix equations, model reduction, matrix sensing, and matrix completion; see, e.g., Schneider and Uschmajew (2015), Ha et al. (2020), and the references therein. As problem (1) is in general intractable—see Gillis and Glineur (2011)—, our goal is to find a stationary point of this problem, i.e., a zero of the stationarity measure

$$s_f : \mathbb{R}^{m \times n}_{\leq r} \rightarrow \mathbb{R} : X \mapsto \| P_{\mathbb{R}^{m \times n}_{\leq r}}(X)(-\nabla f(X)) \|,$$

denoted by $g^-$ in Schneider and Uschmajew (2015), that returns the norm of any projection of $-\nabla f(X)$ onto the tangent cone to $\mathbb{R}^{m \times n}_{\leq r}$ at $X$; the notation is introduced in Section 2.

To the best of our knowledge, the second-order method given in Levin et al. (2021, Algorithm 3.1) and the first-order method given in Olikier et al. (2022, Algorithm 2) are the only two algorithms in the literature that provably converge to stationary points. Other algorithms, such as Schneider and Uschmajew (2013, Algorithm 3), can fail in the sense that they can produce a feasible sequence $(X_i)_{i \in \mathbb{N}}$ that converges to some point $X$ with the property that $\lim_{i \rightarrow \infty} s_f(X_i) = 0 < s_f(X)$. Such a triplet $(X, (X_i)_{i \in \mathbb{N}}, f)$ is called an apocalypse and the point $X$, which necessarily satisfies rank $X < r$, is said to be apocalypse according to Levin et al. (2021, Definition 2.7).

In this paper, using synthetic instances of (1), we compare the behavior of the first-order algorithms given in Schneider and Uschmajew (2013, Algorithm 3) and Olikier et al. (2022, Algorithm 2), respectively dubbed P^2GD and P^2GDR, the latter consisting of the former equipped with a suitable rank reduction mechanism. We observe that the experiments corroborate the theory but also reveal that the choice of the rank reduction parameter can be significant in practice; see Section 5 for details.

This paper is organized as follows. After recalling some notation and preliminaries in Section 2, we compare in Section 3 the two algorithms on two simple examples of apocalypses, one on $\mathbb{R}^{3 \times 3}_{\leq 2}$ proposed in Levin et al. (2021, §2.2) and one on $\mathbb{R}^{2 \times 2}_{\leq 1}$, to illustrate how P^2GDR avoids following the apocalypses due to its rank reduction mechanism. In Section 4, we present a potential side effect
of that mechanism. We discuss the choice of the rank reduction parameter and draw conclusions in Section 5.

2. NOTATION AND PRELIMINARIES

In this section, we recall some notation and preliminaries from Olikier et al. (2022) to which we refer for a more complete review of the background material. In what follows, \( \mathbb{R}^{m \times n} \) is endowed with the Frobenius inner product, and \( \| \cdot \| \) denotes the Frobenius norm. A nonempty subset \( \mathcal{C} \) of \( \mathbb{R}^{m \times n} \) is said to be a cone if, for every \( X \in \mathcal{C} \) and every \( \lambda \in [0, \infty) \), it holds that \( \lambda X \in \mathcal{C} \). For every nonempty subset \( \mathcal{S} \) of \( \mathbb{R}^{m \times n} \) and every \( X \in \mathcal{S} \), the set \( T_{S}(X) \) of all \( V \in \mathbb{R}^{m \times n} \) such that \( \{1\} \subseteq \mathbb{C} \) in \((0, \infty)\) converging to 0 and \((V_{i})_{i \in \mathbb{N}} \) in \( \mathbb{R}^{m \times n} \) converging to \( V \) such that \( X + t_{i}V_{i} \in \mathcal{S} \) for every \( i \in \mathbb{N} \) is a closed cone, not necessarily convex however, called the tangent cone to \( \mathcal{S} \) at \( X \). For every closed cone \( \mathcal{C} \) in \( \mathbb{R}^{m \times n} \) and every \( X \in \mathbb{R}^{m \times n} \), the set \( P_{\mathcal{C}}(X) := \text{argmin}_{X \in \mathcal{C}} \| X - Y \| \), called the projection of \( X \) onto \( \mathcal{C} \), is nonempty, compact, and all its elements have the same norm. If \( \alpha \) is a singleton, we identify it with its element.

The iteration map of P\(^2\)GD (Olikier et al. 2022, Algorithm 1) is given as Algorithm 1, the only difference with the one of Schneider and Uschmajew (2013, Algorithm 3) is that the initial step size for the backtracking procedure is chosen in a given bounded interval and not in \([1, \infty)\). The acronym "P\(^2\)GD" follows from the fact that the iteration map of this algorithm consists of a step along a projection of the negative gradient onto the tangent cone to \( \mathbb{R}^{m \times r}_{\leq r} \), followed by a projection onto \( \mathbb{R}^{m \times r}_{\leq r} \). The first projection can be computed by Schneider and Uschmajew (2013, Algorithm 2) and the second can be obtained by truncating an SVD in view of the Eckart–Young theorem. Because of the “Choose” statements, the P\(^2\)GD map is set-valued in general. In what follows, P\(^2\)GD(\(X; f, \alpha, \beta, \epsilon, \Delta\)) denotes the set of all possible outputs of Algorithm 1.

Algorithm 1 P\(^2\)GD map

Require: \( f, \alpha, \beta, \epsilon, \Delta \) where \( f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \) is differentiable with locally Lipschitz continuous gradient, \( 0 < \alpha \leq \beta < \infty \), and \( \beta \in (0,1) \).

Input: \( X \in \mathbb{R}^{m \times n} \) such that \( s_{f}(X) > 0 \).

Output: \( Y \in P\(^2\)GD(\(X; f, \alpha, \beta, \epsilon, \Delta\)) \).

1. Choose \( G \in P_{\mathbb{R}^{m \times n}_{\leq r}}(f(-\nabla f(X)), \alpha \in [\alpha, \bar{\alpha}], \) and \( Y \in P_{\mathbb{R}^{m \times n}_{\leq r}}(X + \alpha G) \); \n2. while \( \| f(Y) \| > \epsilon \) do \n3. \( \alpha \leftarrow 2\beta \); \n4. Choose \( Y \in P_{\mathbb{R}^{m \times n}_{\leq r}}(X + \alpha G) \); \n5. end while \n6. Return \( Y \).

As mentioned above, P\(^2\)GDR consists of P\(^2\)GD equipped with a rank reduction mechanism, hence the “R” in the acronym. This mechanism uses the numerical rank given \( \Delta \in [0, \infty) \) and \( X \in \mathbb{R}^{m \times n} \setminus \{0 \in \mathbb{R}^{m \times n}\} \), the \( \Delta \)-rank of \( X \) is defined as

\[
\text{rank}_{\Delta} X := \max \{ j \in \{1, \ldots, \text{rank} X \} \mid \sigma_{j}(X) > \Delta \},
\]

where \( \sigma(X) \geq \cdots \geq \sigma_{\text{min}(m,n)}(X) \) denote the singular values of \( X \), and the definition is completed by setting \( \text{rank}_{\Delta} 0_{m \times n} := 0 \). Based on this definition, the iteration map of P\(^2\)GDR (Olikier et al. 2022, Algorithm 3) is given as Algorithm 2. In particular, P\(^2\)GDR corresponds to P\(^2\)GD with \( \Delta := 0 \) and, more generally, the smaller \( \Delta \) is, the more P\(^2\)GDR tends to behave as P\(^2\)GD. Furthermore, by the Eckart–Young theorem, for every \( X \in \mathbb{R}^{m \times n} \) and every \( \epsilon \in \{0, \ldots, \text{rank} X\} \),

\[
P_{\text{rank}_{\epsilon} X}(X) = P_{\text{rank}_{\epsilon} X}(X).
\]

As the P\(^2\)GD map, the P\(^2\)GDR map is set-valued in general. In what follows, P\(^2\)GDR(\(X; f, \alpha, \Delta\)) denotes the set of all possible outputs of Algorithm 2.

Algorithm 3 P\(^2\)GDR(R)

Require: \( (X_{0}, f, \alpha, \bar{\alpha}, \Delta, \epsilon) \) where \( X_{0} \in \mathbb{R}^{m \times n} \) : \( f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \) is differentiable with locally Lipschitz continuous gradient, \( 0 < \alpha \leq \bar{\alpha} < \infty \), \( \beta, \epsilon \in (0,1) \), and \( \Delta, \epsilon \in [0, \infty) \).

1. \( i \leftarrow 0 \);
2. while \( \epsilon > 0 \) do \n3. Choose \( X_{i+1} \in \text{P}^{2}\text{GDR}(X_{i}; f, \alpha, \bar{\alpha}, \beta, \epsilon, \Delta) \); \n4. \( i \leftarrow i + 1 \); \n5. end while

The iterative process is summarized in Algorithm 3, where \( \Delta := 0 \) corresponds to the P\(^2\)GD algorithm and \( \Delta > 0 \) to the P\(^2\)GDR algorithm. Although the convergence analysis in Olikier et al. (2022) is conducted for \( \epsilon := 0 \), it is necessary to choose \( \epsilon > 0 \) in a practical implementation to guarantee that the algorithm terminates after a finite number of iterations.

By Olikier et al. (2022, Theorem 5.2 and Corollary 5.3), if \( \Delta > 0 \) and \( \epsilon = 0 \), then Algorithm 3 produces either a finite sequence the last term of which is stationary or an infinite sequence with the following two properties: its accumulation points are stationary and the stationarity measure \( s_{f} \) goes to zero along each convergent subsequence. Thus, in the second case, except if the sequence diverges to infinity, choosing \( \epsilon > 0 \) makes Algorithm 3 terminate after finitely many iterations. We do not have this guarantee if \( \Delta := 0 \). Indeed, if \( \epsilon := \Delta := 0 \) and Algorithm 3 produces a sequence \( (X_{i})_{i \in \mathbb{N}} \) that does not diverge to infinity, then, to the best of our knowledge, it is not known whether

\[
\lim_{i \rightarrow \infty} s_{f}(X_{i}) = 0.
\]
3. TWO EXAMPLES OF APOCALYPSES

In this section, we compare the behavior of $P^2$GD and $P^2$GDR on two examples of apocalypses. In Section 3.1 we compare the two algorithms empirically on the example of Levin et al. (2021). In Section 3.2 we compare them analytically on a simple example of an apocalypse on $\mathbb{R}^{2 \times 2}$.

3.1 The example of Levin et al.

In Levin et al. (2021), the following instance of (1) is considered: minimizing

$$f : \mathbb{R}^{3 \times 3} \to \mathbb{R} : X \mapsto Q(X_{1:2,1:2}) + \phi(X_{3,3})$$

on $\mathbb{R}^{3 \times 3}$, where $X_{1:2,1:2}$ is the upper-left $2 \times 2$ submatrix of $X$, $X_{3,3}$ its bottom-right entry, $\phi : \mathbb{R} \to \mathbb{R} : x \mapsto \frac{4}{(x+1)^2}$, $Q : \mathbb{R}^{2 \times 2} \to \mathbb{R} : Y \mapsto \frac{1}{2} \|D(Y - Y^*)\|^2$, $D := \text{diag}(1, \frac{1}{2})$, and $Y^* := \text{diag}(1, 0)$. First, it is observed that $\text{argmin} f = \text{diag}(1, 0, x_0) = X^*$, where $x_0 := \text{argmin} \phi \approx 1.32471795724475$, and $f^* := \min f = f(X^*) = \phi(x_0) \approx -1.932257884495233$. Second, it is proven analytically that $P^2$GD follows an apocalypse if used on this problem with $X_0 := \text{diag}(2, 1, 0)$, $\alpha := \frac{\alpha}{5}$, any $\beta \in (0, 1)$, and $c := \frac{1}{5}$.

In this subsection, we verify empirically that, on the same problem with the same input parameters, $P^2$GD and $P^2$GDR with $\Delta := 0.1$ respectively behave as predicted in Levin et al. (2021) and in agreement with the following theoretical guarantees: by Oliker et al. (2023, Theorem 5.2 and Corollary 5.3), since all sublevel sets of $f$ are bounded, for every initial iterate, $P^2$GDR produces a bounded sequence in $\mathbb{R}^{2 \times 3}$, the accumulation points of which are stationary and along which the stationarity measure $s_f$ goes to zero. We run our Matlab implementation of Algorithm 3 with $\beta := \frac{1}{2}$ and $\varepsilon := 10^{-8}$. This gives us the sequence $(X_i)_{i=0}^{36}$ for $P^2$GD, where

$$X_i := \text{diag}(1 + (-3/5)^i, (3/5)^i, 0)$$

for every $i \in \{0, \ldots, 37\}$. The sequence $(X_i)_{i=0}^{38}$ produced by $P^2$GDR obeys (3) for every $i \in \{0, \ldots, 38\}$. The only iteration of $P^2$GDR that differs from a $P^2$GD iteration is the fifth one, where rank$_2 X_5 = 1$, $\tilde{X}_5 := \text{diag}(1, 0, 0)$, and $X_6$ is selected in $P^2$GD($\tilde{X}_5$) but $P^2$GDR does not.

3.2 An apocalypse on $\mathbb{R}^{2 \times 2}$

For the function

$$f : \mathbb{R}^{2 \times 2} \to \mathbb{R} : X \mapsto X_{11}^2 + (X_{22} - 1)^2 + (X_{12} - X_{21})^2$$

we have $\min_{\mathbb{R}^{2 \times 2}} f = 0$ and $\text{argmin}_{\mathbb{R}^{2 \times 2}} f = \text{diag}(0, 1)$. Proposition 1 states that $P^2$GD used with an initial step size for the backtracking procedure smaller than 1 can follow an apocalypse by trying to minimize $f$ on $\mathbb{R}^{2 \times 2}$. Before introducing that proposition, we give an intuitive explanation of the result. Given any point $x_0$ with $x_0 \in (0, \infty)$, $P^2$GDR produces a sequence converging to $0_{2 \times 2}$, thereby minimizing the first term of $f$. However, no iteration affects the second term because the search direction $\text{diag}(0, 1)$, which would enable the minimization of the second term, is not available until $0_{2 \times 2}$ is reached.

\[(\text{available at https://sites.uclouvain.be/abeul/2022.02)\]
which never happens. The third term of $f$ makes its global minimizer on $\mathbb{R}^{2}_{\leq 1}$ unique without affecting the iterations.

**Proposition 1.** Let $x_0 \in (0, \infty)$ and $\alpha \in (0, 1)$. With $f$ on $\mathbb{R}^{2x_0}_\leq 2$ as defined above, starting from $X_0 := \text{diag}(x_0, 0)$, and using $\alpha := \alpha$, $\beta \in (0, 1)$, and $c \in (0, \frac{1}{2})$, P2GD produces the sequence $(X_i)_{i \in \mathbb{N}}$ defined by

$$X_i := \text{diag}((1-\alpha)x_0, 0) \quad (4)$$

for every $i \in \mathbb{N}$. Moreover, $s_f(X_i) = (1-\alpha)^i x_0$ for every $i \in \mathbb{N}$. In particular, since $s_f(0_{x_0\times x_0}) = \|\nabla f(0_{x_0\times x_0})\| = 1$,

$$(0_{x_2 \times x_2}, (X_i)_{i \in \mathbb{N}}, f)$$

is an apocalypse.

**Proof.** The formula $(4)$ holds for $i = 0$. Furthermore, for every $X \in \mathbb{R}^{2x_2}$,

$$\nabla f(X) = X - \begin{bmatrix} 0 & X_{1,2} , \ 1 \end{bmatrix}.$$ 

Therefore, for every $i \in \mathbb{N}$,

$$-\nabla f(X_i) = \text{diag}(-(1-\alpha)^i x_0, 1),$$

and the formula for $s_f(X_i)$ is valid. Thus, for every $i \in \mathbb{N}$,

$$X_{i+1} = X_i + \alpha \text{diag}(-\nabla f(X_i)), \quad f(X_{i+1}) \leq f(X_i) - c \alpha s_f(X_i)^2,$$

which shows that the sequence defined by $(4)$ is indeed the one produced by P2GD. The expression for $s_f(0_{x_0 \times x_0})$ follows from the fact that $-\nabla f(0_{x_0 \times x_0}) = \text{diag}(0, 1) \in \mathbb{R}^{2x_2}_\leq 2 = T_{\mathbb{R}^{2x_2} \leq 2}(0_{x_0 \times x_0}).$

The next proposition shows that P2GDR escapes the apocalypse due to its rank reduction mechanism. During the first iterations, P2GDR produces the same iterates as P2GD. However, when the numerical rank of the iterate becomes smaller than its rank, i.e., when its smallest singular value becomes smaller than or equal to $\Delta$, P2GDR realizes that a stronger decrease of $f$ is obtained by first reducing the rank and then applying an iteration of P2GD. As a result, the first term of $f$ is minimized within a finite number of iterations, after which the minimization of the second term can start.

**Proposition 2.** Consider the same problem as in Proposition 1 with the same parameters and $\Delta \in (0, \infty)$. Then, P2GDR produces the sequence $(X_i)_{i \in \mathbb{N}}$ defined by

$$X_i := \begin{cases} \text{diag}((1-\alpha)^i x_0, 0) & \text{if } i \leq i_\Delta^1 \\
\text{diag}(0, 1 - (1-\alpha)^{i-i_\Delta}) & \text{if } i > i_\Delta \end{cases} \quad (5)$$

where $i_\Delta := \max \left\{ \left\lfloor \frac{\ln(\frac{\Delta}{\alpha})}{\ln(1-\alpha)} \right\rfloor, 0 \right\}$. In particular, $(X_i)_{i \in \mathbb{N}}$ converges to diag(0,1) and $\lim_{i \to \infty} s_f(X_i) = 0$.

**Proof.** The formula $(5)$ is correct for $i = 0$. If $i_\Delta > 0$, then $(1-\alpha)^i x_0 > \Delta$ for every $i \in \{0, \ldots, i_\Delta - 1\}$, and $(5)$ holds for every $i \in \{0, \ldots, i_\Delta\}$ in view of Proposition 1. It remains to prove $(5)$ for every integer $i > i_\Delta$. Let us look at iteration $i_\Delta$. Since $X_{i_\Delta} = 0_{x_2 \times x_2}$, $-\nabla f(0_{x_2 \times x_2}) = \text{diag}(0, 1) \in \mathbb{R}^{2x_2}_\leq 2 = T_{\mathbb{R}^{2x_2} \leq 2}(0_{x_0 \times x_0})$, $s_f(0_{x_2 \times x_2}) = 1$, $\hat{X}_{i_\Delta} = -\alpha \nabla f(X_{i_\Delta}) = \text{diag}(0, \alpha)$ and

$$f(\hat{X}_{i_\Delta}) = f(\text{diag}(0, \alpha)) \geq c \alpha s_f(\hat{X}_{i_\Delta})^2,$$

we have $\hat{X}_{i_\Delta} = \text{diag}(0, \alpha)$. As $\hat{X}_0 = X_{i_\Delta}$, Proposition 1 yields $\hat{X}_0^0 = \text{diag}((1-\alpha)^{i_\Delta + 1} x_0, 0)$.

$$f(\hat{X}_{i_\Delta}) = \frac{(1-\alpha)^2}{2} < \frac{(1-\alpha)^2((i_\Delta + 1)^2 - 1)}{2} = f(\hat{X}_{i_\Delta})$$

we have $X_{i_\Delta + 1} = X_{i_\Delta}$, in agreement with $(5)$. Let us now assume that $(5)$ holds for some integer $i > i_\Delta$ and prove that it also holds for $i + 1$. As $\hat{X}_0 = X_i$, $-\nabla f(X_i) = \text{diag}(0, (1-\alpha)^{i+i_\Delta}) \in \mathbb{R}^{2x_2}_\leq 2$, and $f(X_i) = f(\text{diag}(0, 1 - (1-\alpha)^{i+i_\Delta})) \geq c \alpha s_f(X_i)^2$, we have $\hat{X}_i^0 = \text{diag}(0, 1 - (1-\alpha)^{i+i_\Delta})$. Since $\Delta > 0$, then $\alpha^2 f(X_i) = \text{diag}(0, 1 - (1-\alpha)^{i+i_\Delta})$, and from what precedes, $\hat{X}_i^1 = \text{diag}(0, \alpha)$. Since $f(\hat{X}_i^0) < f(\hat{X}_i^1)$, we have $X_{i+1} = X_0$, as wished. The other two claims follow.

The iterates of P2GDR computed in Proposition 2 are represented in Figure 2 which summarizes this subsection. As explained, P2GDR follows an apocalypse because, at any point diag(0, 1) with $x \in (0, \infty)$, the projection of $-\nabla f$ onto the tangent cone to $\mathbb{R}^{2x_2}_\leq 2$ is parallel to the $x$-axis, and can thus minimize only the first term of $f$. The descent direction diag(0, 1), which enables the minimization of the second term of $f$, becomes accessible only at diag(0, 0).

![Fig. 2. Iterates $X_i$ produced by P2GDR for the problem of Section 12 with $x_0 := 1$, $\alpha := \frac{1}{2}$, and $\Delta := \frac{1}{2}$ in the $xy$-plane of diag(x, y) matrices. The arrows represent $-\alpha \nabla f(X_i)$.

Although P2GDR avoids the apocalypse for every $\Delta > 0$, it should be noted that, if $\Delta \geq \alpha$, then its rank reduction mechanism makes it apply the P2GD map to 0_{x_2 \times x_2} in at least one iteration from iteration $i_\Delta + 1$, thereby constructing points that are not used, as shown in the proof of Proposition 2. For those iterations, P2GDR therefore produces the same iterates as P2GD at a higher computational cost.

We close this section by discussing how Algorithm 3 with $\varepsilon > 0$ behaves on this problem. If $\Delta := 0$, it returns the sequence $(X_i)_{i \in \mathbb{N}}$ defined by $(5)$, where $i_\varepsilon := \max \left\{ \left\lfloor \frac{\ln(\frac{\Delta}{\alpha})}{\ln(1-\alpha)} \right\rfloor, 0 \right\}$. Thus, in view of $(5)$, for Algorithm 3 to avoid stopping while it is heading towards the apocalyptic point, we must have $i_\Delta < i_\varepsilon$, i.e., $\Delta \geq (1-\alpha)^{i_\varepsilon + 1} x_0$.

4. A POTENTIAL SIDE EFFECT OF THE RANK REDUCTION MECHANISM

In this section, we report our observation that, for some instances of P2GD and P2GDR may converge to different stationary points with different costs, especially if $\Delta$ is large. In Section 13, we present an example
where the stationary point to which $P^2$-GD converges has a lower cost than the one to which $P^2$-GDR converges. The converse situation happens in the example given in Section 4.2. In both examples, the initial iterate is $X_0 := \text{diag}(1, 0)$, $P^2$-GDR is used with $\Delta := 1$, and we have $0 = \text{rank}_A X_0 < \text{rank} X_0 = 1$. Moreover, for the two proposed cost functions, applying an iteration of $P^2$-GD to $X_i^1 := 0_{2 \times 2}$ decreases the cost more than to $X_i^0 := X_0$, and $P^2$-GDR therefore computes $X_1$ by applying a $P^2$-GDR iteration to $0_{2 \times 2}$. In this case, using $P^2$-GDR with $\Delta := 1$ is not surprising that another output is produced.

4.1 $P^2$GD may find a better stationary point than $P^2$GD

Define the function $f : \mathbb{R}^{2 \times 2} \to \mathbb{R}$ by

$$f(X) := \frac{(X_{1,1} - 4)^2 + 3(X_{2,2} - 2)^2 + (X_{1,2} - X_{2,1})^2}{2}.$$  

Then,

$$\min_{\mathbb{R}^{2 \times 2}} f = 0, \quad \text{argmin} f = \left\{ \begin{array}{c} 4 \\ \pm 2\sqrt{2} \\ 2 \end{array} \right\}.$$  

The next two propositions show that, for some input parameters, $P^2$-GD converges to a stationary point with lower cost than the one to which $P^2$-GDR converges. Starting from $\text{diag}(1, 0)$, neither of the two algorithms converges to one of the two global minimizers of $f$ on $\mathbb{R}^{2 \times 2}$. On the other hand, $P^2$-GD produces a sequence converging to $\text{diag}(0, 0)$, thereby minimizing the first term of $f$ and achieving a cost of $6$. On the other hand, at the first iteration, $P^2$-GDR prefers to apply the $P^2$-GD map to $0_{2 \times 2}$ because this yields a stronger decrease of $f$ thanks to the factor 3 in the second term. After that first iteration, $P^2$-GDR produces the same iterates as $P^2$-GD and constructs a sequence converging to $\text{diag}(0, 2)$, thus minimizing the second term and achieving a cost of $8$. The third term of $f$ makes its minimizers on $\mathbb{R}^{2 \times 2}$ finite without affecting the iterations of $P^2$-GD and $P^2$GDR.

Proposition 3. Let $\alpha := \frac{1}{4}$. With $f$ on $\mathbb{R}^{2 \times 2}$ as defined above, starting from $X_0 := \text{diag}(1, 0)$, and using $\bar{\alpha} := \alpha, \beta \in (0, 1)$, and $\epsilon \in (0, 2)$, $P^2$-GD produces the sequence $(X_i)_{i \in \mathbb{N}}$ defined by $X_i := \text{diag}(4 - 3(1 - \alpha)^i, 0)$ for every $i \in \mathbb{N}$. In particular, $(X_i)_{i \in \mathbb{N}}$ converges to $\text{diag}(0, 0)$, $\lim_{i \to \infty} s_f(X_i) = 0$, and $s_f(\text{diag}(4, 0)) = 0$.

Proof. The formula holds for $i = 0$. Let us prove that, if it holds for some $i \in \mathbb{N}$, it also holds for $i + 1$. For every $X \in \mathbb{R}^{2 \times 2}$,

$$\nabla f(X) = \begin{bmatrix} X_{1,1} - 4 & X_{1,2} - X_{2,1} \\ 2X_{2,1} - X_{1,2} & 2(X_{2,2} - 2) \end{bmatrix}.$$  

Thus,

$$-\nabla f(X_i) = \text{diag}(3(1 - \alpha)^i, 6), \quad P_{t \geq 2}(X_i)(-\nabla f(X_i)) = \text{diag}(3(1 - \alpha)^i, 0),$$  

and $s_f(X_i) = 3(1 - \alpha)^i$. Since

$$X_i + \alpha \text{diag}(3(1 - \alpha)^i, 0) = \text{diag}(4 - 3(1 - \alpha)^{i+1}, 0) \in \mathbb{R}^{2 \times 2},$$

and

$$f(X_i) - f(\text{diag}(4 - 3(1 - \alpha)^{i+1}, 0)) \geq \alpha s_f(X_i)^2,$$

$X_{i+1}$ has the required form. The other claims follow. $\Box$

Proposition 4. Consider the same problem as in Proposition 3 with the same parameters and $\Delta := 1$. Then, $P^2$-GDR produces the sequence $(X_i)_{i \in \mathbb{N}}$ defined by $X_i := \text{diag}(0, 2 - 2(1 - 3\alpha)^i)$ for every $i \in \mathbb{N} \setminus \{0\}$. In particular, $(X_i)_{i \in \mathbb{N}}$ converges to $\text{diag}(0, 2)$, $\lim_{i \to \infty} s_f(X_i) = 0$, and $s_f(\text{diag}(0, 2)) = 0$.

Proof. Let us first prove the formula for $i = 1$. By the proof of the preceding proposition, as $X_0 := X_0$, we have $X_0^0 = \text{diag}(1 + 3\alpha, 0)$. Since $X_1^1 = 0_{2 \times 2}$,

$$-\nabla f(0_{2 \times 2}) = \text{diag}(4, 6), \quad P_{t \geq 2}(0_{2 \times 2})(-\nabla f(0_{2 \times 2})) = P_{t \geq 2}(\text{diag}(0, 6), s_f(0_{2 \times 2}) = 6, \quad \bar{X}^1 + \alpha \text{diag}(0, 6) = \text{diag}(0, 6\alpha),$$

and

$$f(\bar{X}^1) - f(\text{diag}(0, 6\alpha)) \geq \alpha s_f(\bar{X}^0)^2.$$  

we have $X_1 = \bar{X}^0$, in agreement with the formula. Let us now assume that the formula holds for some $i \in \mathbb{N} \setminus \{0\}$ and prove that it also holds for $i + 1$. Observe that

$$\text{rank}_A X_i = 1. \quad \text{Since} \quad -\nabla f(X_i) = \text{diag}(4, 6(1 - 3\alpha)^i), \quad P_{t \geq 2}(X_i)(-\nabla f(X_i)) = \text{diag}(0, 6(1 - 3\alpha)^i), s_f(X_i) = 6(1 - 3\alpha)^i), \quad X_{i+1} + \alpha \text{diag}(0, 6(1 - 3\alpha)^i) = \text{diag}(0, 2 - 2(1 - 3\alpha)^{i+1}) \in \mathbb{R}^{2 \times 1},$$

and

$$f(X_{i+1}) - f(\text{diag}(0, 2 - 2(1 - 3\alpha)^{i+1})) \geq \alpha s_f(X_i)^2,$$

the formula is valid for $i + 1$. The other claims follow. $\Box$

4.2 $P^2$GDR may find a better stationary point than $P^2$GD

Define the function $f : \mathbb{R}^{2 \times 2} \to \mathbb{R}$ by

$$f(X) := \frac{(X_{1,1} - 2)^2 + (X_{2,2} - 3)^2 + (X_{1,2} - X_{2,1})^2}{2}.$$  

Then,

$$\min_{\mathbb{R}^{2 \times 2}} f = 0, \quad \text{argmin} f = \left\{ \begin{array}{c} 2 \\ \pm 6 \\ 3 \end{array} \right\}.$$  

The next two propositions show that, for some input parameters, $P^2$-GD and $P^2$GDR achieve the costs of $\frac{9}{2}$ and $2$, respectively.

Proposition 5. Let $\alpha \in (0, 1)$. With $f$ on $\mathbb{R}^{2 \times 2}$ as defined above, starting from $X_0 := \text{diag}(1, 0)$, and using $\bar{\alpha} := \alpha, \beta \in (0, 1)$, and $\epsilon \in (0, 2)$, $P^2$- GD produces the sequence $(X_i)_{i \in \mathbb{N}}$ defined by $X_i := \text{diag}(2 - (1 - \alpha)^i, 0)$ for every $i \in \mathbb{N}$. In particular, $(X_i)_{i \in \mathbb{N}}$ converges to $\text{diag}(2, 0)$, $\lim_{i \to \infty} s_f(X_i) = 0$, and $s_f(\text{diag}(2, 0)) = 0$.

Proof. The formula holds for $i = 0$. Let us prove that, if it holds for some $i \in \mathbb{N}$, it also holds for $i + 1$. For every $X \in \mathbb{R}^{2 \times 2}$,

$$\nabla f(X) = X - \begin{bmatrix} 2 & 0 \\ X_{1,2} & 3 \end{bmatrix}.$$  

Thus,

$$-\nabla f(X_i) = \text{diag}((1 - \alpha)^i, 3), \quad P_{t \geq 2}(X_i)(-\nabla f(X_i)) = \text{diag}((1 - \alpha)^i, 0),$$

and

$$f(X_i) - f(\text{diag}((1 - \alpha)^i, 0)) \geq \alpha s_f(X_i)^2,$$

$X_{i+1}$ has the required form. The other claims follow. $\Box$
and \( s_f(X_i) = (1 - \alpha)^i \). Since 
\[
X_i + \alpha \text{diag}((1 - \alpha)^i, 0) = \text{diag}(2 - (1 - \alpha)^{i+1}, 0) \in \mathbb{R}^{2 \times 2}
\]
and
\[
f(X_i) - f(\text{diag}(2 - (1 - \alpha)^{i+1}, 0)) \geq c \alpha s_f(X_i)^2,
\]
\( X_{i+1} \) has the required form. The other claims follow. \( \square \)

**Proposition 6.** Consider the same problem as in Proposition 5 with the same parameters and \( \Delta := 1 \). Then, P^2GD produces the sequence \((X_i)_{i \in \mathbb{N}}\) defined by \( X_i := \text{diag}(0, 3 - 3(1 - \alpha)^i) \) for every \( i \in \mathbb{N} \setminus \{0\} \). In particular, \((X_i)_{i \in \mathbb{N}}\) converges to \((0, 3)\), \( \lim_{i \to \infty} X_i = (0, 3) \), and \( s_f(\text{diag}(0, 3)) = 0 \).

**Proof.** Let us first prove the formula for \( i = 1 \). By the proof of the preceding proposition, as \( \hat{X}^0_0 = X_0 \), we have \( \hat{X}^0_0 = \text{diag}(1 + \alpha, 0) \). Since \( \hat{X}^1_0 = 0_{2 \times 2} \), \(-\nabla f(0_{2 \times 2}) = \text{diag}(2, 3)\), \( P_{2 \times 2}(-\nabla f(0_{2 \times 2})) = P_{2 \times 2}(-\nabla f(0_{2 \times 2})) = \text{diag}(0, 3) \), and \( \alpha \text{diag}(0, 3) = \text{diag}(0, 3\alpha) \), and
\[
f(\hat{X}^1_0) - f(\text{diag}(0, 3\alpha)) \geq c \alpha s_f(\hat{X}^1_0)^2,
\]
we have \( \hat{X}^1_0 = \text{diag}(0, 3\alpha) \). As 
\[
f(\hat{X}^1_0) = \frac{2}{4 + 9(1 - \alpha)^2} < \frac{(1 - \alpha)^2 + 9}{2} = f(\hat{X}^0_0),
\]
we have \( X_1 = \hat{X}^1_0 \), in agreement with the formula. Let us now assume that the formula holds for some \( i \in \mathbb{N} \setminus \{0\} \) and prove that it also holds for \( i + 1 \). Observe that \( \text{rank} \Delta X_i = 1 \). Since \( -\nabla f(X_i) = \text{diag}(2, 3(1 - \alpha)^i)\), \( P_{2 \times 2}(-\nabla f(X_i)) = \text{diag}(0, 3(1 - \alpha)^i)\), \( s_f(X_i) = 3(1 - \alpha)^i \), \( X_i + \alpha \text{diag}(0, 3(1 - \alpha)^i) = \text{diag}(0, 3 - 3(1 - \alpha)^{i+1}) \in \mathbb{R}^{2 \times 2} \), and
\[
f(X_i) - f(\text{diag}(0, 3 - 3(1 - \alpha)^{i+1})) \geq c \alpha s_f(X_i)^2,
\]
the formula is valid for \( i + 1 \). The other claims follow. \( \square \)

5. CONCLUSION

This paper compares P^2GD and P^2GDR on synthetic instances of (1). The simplicity of those instances enables both analytical and empirical investigations. This allows us to observe two behaviors:

1. P^2GD and P^2GDR respectively following and escaping apocalypses (Section 4).
2. P^2GD and P^2GDR converging to different stationary points with different cost values (Section 4).

Concerning P^2GDR, i.e., Algorithm 3 with \( \Delta > 0 \), we also observe that, if \( \varepsilon > 0 \), which is always the case in a practical implementation, then the choice of \( \Delta \) can be significant. Indeed, when \( \varepsilon := 0 \), the convergence analysis of Algorithm 3 given in Olikier et al. (2022) holds for every \( \Delta > 0 \). However, as remarked in Section 3, if \( \varepsilon > 0 \), then \( \Delta \) must be chosen large enough to let the rank reduction mechanism prevent the algorithm from stopping while heading towards an apocalyptic point. On the other hand, choosing \( \Delta \) too large can make the rank reduction mechanism work inefficiently in the sense that, for some iterations, P^2GDR produces the same iterates as P^2GD at a higher computational cost. In any case, it is good practice, in order to avoid following an apocalyptic, to apply a rank reduction to the last iterate and to look at the effect of a P^2GD iteration on the obtained point. Besides these apocalypses-related considerations, the larger \( \Delta \) is, the more side effects of the rank reduction mechanism are likely to arise, as noticed in Section 4.

We close this paper with two open questions regarding P^2GD and P^2GDR.

1. As pointed out at the end of Section 2, it is not known whether there exists an instance of (1) for which P^2GD produces a sequence with the following two properties: it does not diverge to infinity and the stationarity measure \( s_f \) does not go to zero along any convergent subsequence. For such an instance, there would exist \( \varepsilon > 0 \) such that Algorithm 3 with \( \Delta := 0 \) does not terminate.
2. Is there an instance of (1) for which P^2GDR converges to a nonstationary point having a lower cost than the stationary point to which P^2GDR converges?

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