EFFICIENCY AXIOMS FOR SIMPLICIAL COMPLEXES

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Abstract. We study the notion of efficiency for cooperative games on simplicial complexes. In such games, the grand coalition \([n]\) may be forbidden, and, thus, it is a non-trivial problem to study the total number of payoff \(v_\Delta\) of a cooperative game \((\Delta, v)\).

We address this question in the more general setting, by characterizing the individual values that satisfy the general efficient requirement, that is \(v_{\Delta}^{gen} = \sum_{T \in \Delta} a_T v(T)\) for a generic assignment of real coefficients \(a_T\). The traditional and the probabilistic efficiency are treated as a special case of this general efficiency.

Finally, we introduce a new notion of efficiency arising from the combinatorial and topological property of the simplicial complex \(\Delta\). The efficiency in this scenario is called simplicial and we characterize the individual values fulfilling this constraint.

In the traditional \(n\)-person game the characteristic function \(v : 2^{[n]} \to \mathbb{R}\) determines the worth of each coalition, where \([n] \overset{\text{def}}{=} \{1, \ldots, n\}\). The individual value \(\phi_i\) associated to such cooperative game \((n, v)\) measures the contribution of the player \(i\) in the game. We collect such values all together in the group value \(\phi = (\phi_1, \phi_2, \ldots, \phi_n)\). The assessment is optimistic (w. r. to \(v\)) if the sum of the payoff vector \(\sum_i \phi_i(v)\) is greater than the \(v([n])\), the worth of the grand coalition. If the contrary happens, then \(\phi\) is pessimistic (w. r. to \(v\)).

Consider the vector space of cooperative games \(\mathbb{R}^{2^n-1}\), that is the set of all characteristic function under the constrain \(v(\emptyset) = 0\). Provided a game \((n, v)\) in \(\mathbb{R}^{2^n-1}\), we consider the scaled game \((n, c \cdot v)\) given by the characteristic function \((c \cdot v)(T) = cv(T)\) where \(c\) is a real constant. It seems natural to assume that in this case the individual value is also scaled by \(c\), \(\phi_i(c \cdot v) = c\phi_i(v)\). Therefore consider a cone of games \(\mathfrak{Z}\) in \(\mathbb{R}^{2^n-1}\).

We are interesting in group values that are nor optimistic or pessimistic, despite this might be an artificial condition. This consideration leads to the constraint known as called Efficiency Axiom:

Efficiency Axiom.
For every cooperative game \((2^{[n]}, v)\) in \(\mathfrak{Z}\), one has \(\sum_{i=1}^n \phi_i(v) = v([n])\).

If certain coalitions are forbidden (take for instance the grand coalition), it is necessary to study what could take the place of the the total number of payoff \(v_\Delta\), that in the traditional case reduces to \(v_\Delta = v([n])\). In this work we focus on the specific instance of this problem for cooperative game on a simplicial complex [Mar20a, Mar20b]. We are going to shortly present our new results after introducing this new generalization for cooperative games.

Cooperative game on simplicial complex. Inspired by several articles [BDJLL01, BDJLL02, MTMZ19, MZ11, FV11, NZKI97, Zha99], the author has defined cooperative games on simplicial complexes [Mar20a]. In fact, a simplicial complex is a family \(\Delta\) of subsets of \([n]\) under the
constrain that every subset of \( X \in \Delta \), also belongs to the family \( \Delta \). A cooperative game on \( \Delta \) is defined by a characteristic function \( v: \Delta \to \mathbb{R} \) with \( v(\emptyset) = 0 \). In such game, a player \( i \) in \([n]\) may join a coalition \( T \) only if \( T \cup i \in \Delta \). In such case, the coalition is feasible. If every subset of \([n]\) is feasible, then \( \Delta = 2^{[n]} \) and we recover the literature case.

As in the classical case, the individual value function \( \phi_i(v) \) for the player \( i \) determines the worth of the participation of \( i \) in a feasible coalition during the cooperative game \((\Delta, v)\). As before, we may consider a cone \( \mathcal{Z} \) of cooperative games defined in \( \mathbb{R}^\Delta \), the vector space of all cooperative games on \( \Delta \).

Quasi-probabilistic values. The issue of studying \( v_{\Delta} \) already arises in the work of Bilbao, Driessen, Jiménez Losada and Lebrón [BDJLL01], where they introduce the notion of probabilistic efficiency for games over a matroid. One of the perks of matroids is that they are pure simplicial complex; in other words the maximal facets of a matroid have the same cardinality, see for instance [Sta12, Sta96, Sta91, Sta84, Ox111, Mar18, BM19]. Here we present the natural generalization to simplicial complexes that already appears in Section 6 of [Mar20a]:

Probabilistic Efficiency Axiom.
For every cooperative game \((\Delta, v)\) in \( \mathcal{Z} \), \( \sum_{i=1}^{n} \phi_i(v) = \sum_{F \in F_{s_{\Delta}}} c_F v(F) \),
with \( \sum_{F \in F_{s_{\Delta}}} c_F = 1 \) and \( c_F \geq 0 \) for every facet \( F \) of \( \Delta \).

In the above equation, \( F_{s_{\Delta}} \) is the set of facets of \( \Delta \), that are maximal elements by inclusion.

It is worth to recall that the Efficient axiom, and respectively the Probabilistic efficient are crucial to characterizes the Shapley values [Sha53, Sha72, Web88], and respectively the quasi-probabilistic values that can be written as sum of Shapley values [BDJLL01, Mar20a].

Our Approach. We adopt a differ point of view originated by the next consideration. The first axiom in the theory of probabilistic values is the Linearity Axiom for individual values, see Section 3 of [Web88]. We are also going to assume that the total number of payoff \( v_{\Delta} \) is a linear function:

\[ v_{\Delta}: \mathbb{R}^\Delta \to \mathbb{R}. \]

Therefore, the total number of payoff can be written as

\[ v_{\Delta} = \sum_{T \in \Delta} a_T v(T). \]

(Note that we allow \( T = \emptyset \), because \( v(\emptyset) = 0 \).) The choice of the coefficients \( a_T \) describes a diverse efficiency scenario: for instance, the efficiency axiom for \( (2^{[n]}, v) \) is given by setting \( a_T = 0 \) for every \( T \neq [n] \) and \( a_{[n]} = 1 \). Similarly, the probabilistic efficiency of Bilbao, Driessen, Jiménez Losada and Lebrón [BDJLL01] is obtained from the choice of \( a_F = c_F \) for every facet \( F \) of \( \Delta \) and \( a_T = 0 \), otherwise. Furthermore, a more flexible efficiency condition is needed in Theorem E of [Mar20b] to properly encode Shapley values on simplicial complexes.

Our first result characterize individual values that satisfies the general efficient condition.

**Theorem 2.1.** Let \( \Delta \) be a simplicial complex and let \( \mathcal{Z} \) be a cone of cooperative games \( v \) defined on \( \Delta \) containing the carrier games \( \mathcal{C} \) and \( \hat{\mathcal{C}} \).
Let $\phi$ be a group value on $I$ such that for each $i \in [n]$ and assume that for each $v \in \mathcal{I}$, we can write:

$$\phi_i(v) = \sum_{T \in \text{Link}_\Delta i} p_T^i (v(T \cup i) - v(T)).$$

The group value $\phi$ satisfies the simplicial efficiency axiom if and only if for all non-facet $T$ in $\Delta$

$$\sum_{i \in T} p_T^i - \sum_{j, T \in \text{Link}_\Delta j} p_T^j = a_T,$$

and for every facet $F$ of $\Delta$

$$\sum_{i \in F} p_F^i = a_F.$$

In Theorem 2.3, we specialize the previous result in the traditional setting and we reproduce Theorem 11 of [Web88].

Similarly, applying Theorem 2.1, we characterize the individual values that satisfy the probabilistic efficiency, see Theorem 2.5.

Moreover in Section 3, we introduce and study a new efficiency conditions: the simplicial efficiency.

**The simplicial efficiency.** In this scenario is built on the following point of view. If the grand coalition $[n]$ is forbidden, the largest possible coalitions are precisely the elements of $\text{Fs} \Delta$. However, the facets may intersect. Thus, in Section 3, with an inclusion-exclusion argument we study the following number of total payoff:

$$v_{\Delta}^{\text{simp}l} = \sum_{l=1}^k \sum_{(F_j) \in (\text{Fs}_\Delta)} (-1)^{j+1} v(F_i_1 \cap \cdots \cap F_i_l).$$

We classify the values fulfilling this requirement in Theorem 3.5. In the specific case of matroids, we show that $v_{\Delta}^{\text{simp}l}$ only depends by the dimension-zero and codimension-one skeleton of $\Delta$, see Theorem 3.8. Moreover in Remarks 3.3 and 3.4 we argue that the probabilistic efficiency is a first-hand approximation of the simplicial efficiency.

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1. Preliminaries

In this manuscript $n$ is a positive integer and we denote by $[n] \overset{\text{def}}{=} \{1, \ldots, n\}$. A (finite) simplicial complex $\Delta$ over $n$ vertices is a family of subsets of $[n]$ with the simplicial condition:

$$T \in \Delta, S \subseteq T \Rightarrow S \in \Delta.$$

The set of elements of the simplicial complex is denoted by Set($\Delta$). A facet of $\Delta$ is a set $F$ in $\Delta$ that is maximal by inclusion. The set of facets is $\text{Fs} \Delta$. If every facets have the same cardinality then the simplicial complex is said to be pure.

In this paper, we refer to results of [Mar20a] and [Mar20b] that use the notions of star and link of a vertex $i$ in $\Delta$. We recall those for completeness, even if they will not be central in our study.
If $S$ is an element in $\Delta$, then $\bar{S} \overset{\text{def}}{=} 2^S$ is the $(|S| - 1)$-dimensional simplex defined on the vertices of $\bar{S}$.

**Definition 1.1.** The star of an element $S$ in $\Delta$ is the simplicial complex defined to be the collection of all subset in $\bar{T}$ with $T$ being in $\Delta$ and containing $S$,

$$\text{Star}_\Delta S = \{A : A \in \bar{T}, T \in \Delta, S \subseteq T\}.$$  

We highlight when $S = \{i\}$ is a vertex, then $\text{Star}_\Delta i$ is the set of simplex $\bar{T}$ containing $i$, that is

$$\text{Star}_\Delta i = \{A : A \subseteq T, i \in T \in \Delta\}.$$  

**Definition 1.2.** The link of an element $S$ in a simplicial complex $\Delta$ is made by the subsets $A$ of $T \in \Delta$, such that $T$ is disjoint by $S$ and can be completed by $S$, $S \cup T$, to an element in $\Delta$:

$$\text{Link}_\Delta S = \{A : A \in \bar{T} \text{ with } T \in \Delta \text{ such that } S \cap T = \emptyset, S \cup T \in \Delta\}.$$  

The case when $S$ is the singleton $\{i\}$ will be extremely relevant in our work: $\text{Link}_\Delta i$ is the set of simplex $T$ in $\Delta$ with $i \notin T$ such that $T \cup i \in \Delta$:

$$\text{Link}_\Delta i = \{T \in \Delta : i \notin T \text{ and } T \cup i \in \Delta\}.$$  

1.3. **Matroids.** Since they were introduced by Whitney [Whi35] in 1935, matroids are at the crossroads of Algebra, Combinatorics, Geometry, and Topology. New variations have appeared in the traditional one and we refer for a detailed description to [Sta12, Sta96, Sta91, Sta84, Oxl11].

A matroid on the ground set $[n]$ is a collection $I$ of subsets of $[n]$ (called independent sets), such that (I1) $\emptyset \in I$, (I2) $A \subseteq B \in I \Rightarrow A \in I$, and (I3) $A, B \in I$, $|A| < |B| \Rightarrow \exists b \in B \setminus A : A \cup \{b\} \in I$.

The first two axioms make $I$ into a (non-empty) simplicial complex. Axiom (I3) is sometimes referred as independent set exchange property (or independence augmentation axiom.). Let $I$ be a matroid on the ground set $[n]$, and let $A \subseteq [n]$. All maximal independent subsets of $A$ have the same cardinality, called the rank $\text{rk}(A)$ of $A$, whereas the corank of $A$ is $\text{cork}(A) = \text{rk}([n]) - \text{rk}(A)$. Hence a matroid is a pure simplicial complex, see for instance Figure 1a. Not every pure simplicial complex is a matroid; in fact Figure 1b shows a rank three simplicial complex that is not a matroid. Indeed, the independent set $\{5\}$, cannot be extended to a base by any element in the independent set $\{2, 3\}$. Simplicial complexes that are not matroids are extremely important in Mathematics; few example can be found in [Mar15, GM16, GM18].

1.4. **Cooperative games on simplicial complexes.** In [Mar20a], the author introduces the notion of cooperative game on the simplicial complex $\Delta$, inspired by the work of Bilbao, Driessen, Jiménez Losada and Lebrón [BDJLL01].

Here we shortly recall that a cooperative game on a simplicial complex $\Delta$ is the pair $(\Delta, v)$ where $v$ is a characteristic function $v : \text{Set}(\Delta) \to \mathbb{R}$ under the constrain $v(\emptyset) = 0$. The vertices of $\Delta$ are the players of the cooperative game and a coalition $T$ is feasible if $T \in \Delta$. The set $\mathbb{R}^\Delta$ of characteristic functions on $\Delta$ is naturally a real vector space.

Given a cooperative game $(\Delta, v)$, one can re-scale the characteristic function with a scalar $c$ and obtain a new cooperative game $(\Delta, cv)$, where $cv(T) = c(v(T))$ for every subset $T \in \Delta$.

**Definition 1.5.** An individual value for a player $i$ in $[n]$ is a function $\phi_i : \mathbb{R}^\Delta \to \mathbb{R}$.  


(a) This simplicial complex is a matroid.

(b) This simplicial complex is not a matroid.

Figure 1. Two examples of pure simplicial complexes.

The goal of each individual value \( \phi_i(v) \) is assessing the worth of the participation of the player \( i \) in to the game. Naturally, we are looking for values such that \( \phi_i(cv) = c\phi_i(v) \), because the worth of each player is just re-scaled. For this reason we often consider a cone \( \mathcal{S} \) of cooperative game in \( \mathbb{R}^\Delta \).

1.6. Efficiency axioms previously introduced in literature. In the traditional case the sum of such values is often compare with the total number of payoff \( v([n]) \) of the grand coalition \([n]\). If the sum \( \sum_i \phi_i(v) \) is greater than \( v([n]) \), then the assessment is going to distribute to all the player a larger amount than the one eventually obtained. This is the optimistic (w. r. to \( v \)) setting. Vice versa, the group value \( \phi(v) = \{\phi_1(v), \ldots, \phi_n(v)\} \) is pessimistic.

It may be artificial, but it is surely interesting to consider group values that are not optimistic or pessimistic. This leads to the so called Efficiency Axiom:

**Efficiency Axiom.**

For every cooperative game \((2^{[n]}, v)\) in \( \mathcal{S} \), \( \sum_{i=1}^n \phi_i(v) = v([n]) \).

This axiom is part of the requirement in the characterization of the Shapley values [Sha53, Sha72, Web88].

The effort of generalizing such requirement can be found already in the work [BDJLL01], where they deal with the notion of probabilistic efficiency for a cooperative game over a matroid. Here we present the natural generalization to simplicial complexes:

**Probabilistic Efficiency Axiom.**

For every cooperative game \((\Delta, v)\) in \( \mathcal{S} \), \( \sum_{i=1}^n \phi_i(v) = \sum_{F \in \mathcal{F}_\Delta} c_F v(F) \),

with \( \sum_{F \in \mathcal{F}_\Delta} c_F = 1 \) and \( c_F \geq 0 \) for every facet \( F \).

As the the previous case, also this axiom is used as a necessary and sufficient condition for writing the quasi-probabilistic values as sum of Shapley values [BDJLL01, Mar20a].

1.7. Carrier games. There are two set of games that have a crucial role in the theory of probabilistic values [Web88]. We are going to called both set carrier games, even if in literature this terminology often refer to the first one:

\[
C = \{v_T : \emptyset \neq T \subset [n]\}, \quad \hat{C} = \{\hat{v}_T : \emptyset \neq T \subset [n]\},
\]
where $v_T$ and $\hat{v}_T$ are so defined:

$$ v_T(S) = \begin{cases} 1 & T \subseteq S \\ 0 & \text{otherwise.} \end{cases}, \quad \hat{v}_T(S) = \begin{cases} 1 & T \subseteq S \\ 0 & \text{otherwise.} \end{cases} $$

We generalize these notation for any element $T$ of a simplicial complex. Indeed for every partially order set $(P, \leq_P)$ and every element $q$ in $P$ we consider the following function:

$$ u^p_{q}(s) \overset{\text{def}}{=} \begin{cases} 1 & q \leq_P s \\ 0 & \text{otherwise.} \end{cases}, \quad \hat{u}^p_{q}(s) \overset{\text{def}}{=} \begin{cases} 1 & q <_P s \\ 0 & \text{otherwise.} \end{cases} $$

Thus, we define

$$ v_T(S) \overset{\text{def}}{=} u^\Delta_T(S), \quad \hat{v}_T(S) \overset{\text{def}}{=} \hat{u}^\Delta_T(S). $$

It is easy to see that in the classical case (when $\Delta$ is a full simplex on $n$ vertices) these functions reproduce the carrier games.

**Definition 1.8.** Let $\Delta$ be a simplicial complex. The sets of carrier games are so defined:

$$ C = \{ v_T : \emptyset \neq T \in \Delta \}, \quad \hat{C} = \{ \hat{v}_T : \emptyset \neq T \in \Delta \}, $$

where $v_T(\overset{\text{def}}{=} u^\Delta_T(S)$ and $\hat{v}_T(\overset{\text{def}}{=} \hat{u}^\Delta_T(S)$; moreover, $\hat{v}_\emptyset = \hat{u}^\Delta_\emptyset$.

### 2. A unique approach to efficiency

Let us assume that the total number of payoff is prescribed as

$$ v^\text{gen}_\Delta = \sum_{T \in \Delta} a_T v(T). $$

and we require the following efficiency constrain:

**Generic Efficiency Axiom.**

For every cooperative game $(2^{[n]}, v)$ in $\mathcal{I}$, \( \sum_{i=1}^n \phi_i(v) = v^\text{gen}_\Delta. \)

Then, we characterize the individual values that can be written in the classical sum of marginal contributions and that satisfy the **Generic Efficiency Axiom**.

**Theorem 2.1.** Let $\Delta$ be a simplicial complex and let $\mathcal{I}$ be a cone of cooperative games $v$ defined on $\Delta$ containing the carrier games $C$ and $\hat{C}$.

Let $\phi$ be a group value on $\mathcal{I}$ such that for each $i \in [n]$ and assume that for each $v \in \mathcal{I}$, we can write:

$$ \phi_i(v) = \sum_{T \in \text{Link}_\Delta i} p^i_T (v(T \cup i) - v(T)). $$

The group value $\phi$ satisfies the simplicial efficiency axiom if and only if for all non-facet $T$ in $\Delta$

$$ \sum_{i \in T} p^i_{T \setminus i} - \sum_{j, T \in \text{Link}_\Delta j} p^j_T = a_T, \quad (2) $$

and for all facet $F$ of $\Delta$

$$ \sum_{i \in F} p^i_{F \setminus i} = a_F. \quad (3) $$
Proof. Let us show that the equations (2) and (3) are necessary. Using our assumption, this is:

\[
\sum_{i=1}^{n} \phi_i(v) = \sum_{i \in [n]} \sum_{T \in \text{Link}_\Delta i} p^i_T \left( v(T \cup i) - v(T) \right).
\]

We reorder the terms in the sum as

\[
\sum_{i=1}^{n} \phi_i(v) = \sum_{T \in \Delta} v(T) \left( \sum_{i \in T} p^i_T - \sum_{j, T \in \text{Link}_\Delta j} p^j_T \right).
\]

Since \( v^\text{gen}_\Delta = \sum_{T \in \Delta} a_T v(T) \), when \( T = F \) is a facet, we get (3). If \( T \) is not a facet, then it is clear that \( \phi \) satisfies (2).

To prove the opposite direction one needs to note that if \( T \) is not a facet, then

\[
\sum_{i=1}^{n} \phi_i(v_T) - \sum_{i=1}^{n} \phi_i(\hat{v}_T) = \sum_{i \in T} p^i_T - \sum_{j, T \in \text{Link}_\Delta j} p^j_T.
\]

By hypothesis the latter equals \( a_T \). Similarly if \( F \) is a facet then

\[
\sum_{i=1}^{n} \phi_i(v_F) = \sum_{i \in \overline{T}} p^i_T,
\]

and by hypothesis the latter is equal \( a_F \). □

2.2. The Traditional Efficiency. As a corollary of the previous theorem we can easily obtain Theorem 11 of [Web88]. In fact, in the traditional cooperative game on a full simplicial complex with \( n \) vertices, the efficiency axiom is the following:

Efficiency Axiom.

For every cooperative game \((2^{[n]}, v)\) in \( \mathcal{I} \), one has \( \sum_{i=1}^{n} \phi_i(v) = v([n]) \).

Thus, in equation (1), \( a_T = 0 \) for every subset of \([n]\), but \( a_{[n]} = 1 \). Hence, Theorem 11 of [Web88] follows as a corollary of our result:

Theorem 2.3. Let \( \mathcal{I} \subset \mathbb{R}^{2^{n-1}} \) be the cone of cooperative games containing the (classical) carrier games \( \mathcal{C} \) and \( \hat{\mathcal{C}} \).

Let \( \phi \) be a group value on \( \mathcal{I} \) such that for each \( i \in [n] \) and assume that for each \( v \in \mathcal{I} \), we can write:

\[
\phi_i(v) = \sum_{T \in [n], i} p^i_T \left( v(T \cup i) - v(T) \right).
\]

The group value \( \phi \) satisfies the simplicial efficiency axiom if and only if for all non-facet \( T \) in \( \Delta \)

\[
\sum_{i \in T} p^i_T - \sum_{j \notin T} p^j_T = 0,
\]

and for all facet \( F \) of \( \Delta \)

\[
\sum_{i \in [n]} p^i_{[n], i} = 1.
\]

Proof. Observe that \( \text{Link}_{2^{[n]}} i = [n] \setminus i \) and there is only one facet, \( F = [n] \). Then, apply Theorem 2.1 with \( v_\Delta = v([n]) \). □
2.4. The Probabilistic Efficiency. We can also characterize the individual values that satisfy the probabilistic efficiency, introduced in [BDJLL01] for cooperative games on matroids and generalized for games on every simplicial complex in in Section 6 of [Mar20a]:

**Probabilistic Efficiency Axiom.**

For every cooperative game \((\Delta, v)\) in \(\mathcal{F}\), \(\sum_{i=1}^{n} \phi_i(v) = \sum_{F \in \mathcal{F} \Delta} c_F v(F)\), with \(\sum_{F \in \mathcal{F} \Delta} c_F = 1\) and \(c_F \geq 0\) for every facet \(F\).

Thus, the total number of payoff can be written as in equation (1) by setting \(a_T = 0\) for every subset of \([n]\), but the facets where \(a_F = c_F\). In next results we characterize the individual values that fulfill this efficiency request.

**Theorem 2.5.** Let \(\Delta\) be a simplicial complex and let \(\mathcal{F}\) be a cone of cooperative game defined on \(\Delta\) containing the carrier games \(C\) and \(\hat{C}\). Let \(\phi\) be a group value on \(\mathcal{F}\) such that for each \(i \in [n]\) and each \(v \in \mathcal{F}\), we can write:

\[
\phi_i(v) = \sum_{T \in \text{Link}_i \Delta} p^i_T (v(T \cup i) - v(T)).
\]

The group value \(\phi\) satisfies the probabilistic efficiency axiom if and only if for all non-facet \(T\) in \(\Delta\)

\[
\sum_{i \in T} p^i_{T \setminus j} - \sum_{j \in \text{Link}_i \Delta} p^j_T = 0,
\]

and for all facet \(F\) of \(\Delta\)

\[
\sum_{i \in F} p^i_{F \setminus j} = c_F
\]

**Proof.** Simply apply Theorem 2.1 with \(v_\Delta = \sum_{F \in \mathcal{F} \Delta} c_F v(F)\). □

**Remark 2.6.** We note that the conditions \(\sum_{F \in \mathcal{F} \Delta} c_F = 1\) and \(c_F \geq 0\) are irrelevant in the Theorem 2.5. Thus the statement holds also for the individual values that satisfy the condition \(\sum_{i=1}^{n} \phi_i(v) = \sum_{F \in \mathcal{F} \Delta} c_F v(F)\) without any requirement on \(c_F\). For instance, \(c_F\) may be a negative number.

3. The Simplicial Efficiency

In this section we want to introduce a new concept of efficiency that differs from the previous approaches. Since the grand coalition \([n]\) is forbidden, the largest possible coalitions are precisely the elements of \(\mathcal{F} \Delta\), but the facets may intersect. If we work under the constrain that \(\Delta\) is a connected simplicial complex, if \(\Delta \neq 2^{[n]}\), then every facet intersect at least another facet. This reasoning leads to the inclusion exclusion problem for computing the total number of payoff \(v_\Delta\) for the cooperative game \((\Delta, v)\). Let us set up a proper arithmetic for doing this.

Let \(F_1, F_2, ..., F_k\) be a random order of the facets. We begin by considering \(v(F_1) + v(F_2)\). If they intersect, then the worth of intersection is double counted and we subtract this: \(v(F_1) + v(F_2) - v(F_1 \cap F_2)\). Next, we add \(v(F_3)\) and we might need to correct again our computation subtracting \(v(F_3 \cap (F_1 \cup F_2))\). Now, \(v(F_3 \cap (F_1 \cup F_2))\) may not be a simplex, but it is the union of simplicies. Therefore, we use the following trick: let \(K\) be a sub-complex of \(\Delta\), then

\[
v(K) \overset{\text{def}}{=} \sum_{F \in \mathcal{F} \Delta K} v(F).
\]
With this in mind, we keep going in our computation by rewriting \( v(F_3 \cap (F_1 \cup F_2)) \) and adding \( v(F_4) \) and so on. Thus, we define \( v^{\text{simp}}_\Delta \) as the following real number:

\[
(7) \quad v^{\text{simp}}_\Delta \overset{\text{def}}{=} v(F_1) + v(F_2) - v(F_1 \cap F_2) + v(F_3) - v(F_3 \cap (F_1 \cup F_2)) + \ldots
\]

Let us now prove that such number is well defined, by showing that is is independent by the ordering of the facets.

**Remark 3.1.** The arithmetic trick we are using is not so far from the idea proposed in [BDJLL01]. Indeed their efficiency axiom can be seen as the probabilistic (weighted) version of \( \sum_{i=1}^n \phi_i(v) = v(\Delta) \) by (6) \( \Delta \subseteq \sum_{F \in Fs_\Delta} v(F) \).

We recall that if \( A \) is a finite set, \( \binom{A}{l} \) is the set of subsets of \( A \) of cardinality \( l \).

**Theorem 3.2.** Let \( k \) be the number of facets of a simplicial complex \( \Delta \), \( k \overset{\text{def}}{=} \# Fs_\Delta \). The total number of payoff \( v_\Delta \) for the cooperative game \( (\Delta, v) \) defined in equation (7) is precisely

\[
(8) \quad v_\Delta = \sum_{i=1}^k \sum_{(F_{i_1} \cap \cdots \cap F_{i_l})} (-1)^{l+1} v(F_{i_1} \cap \cdots \cap F_{i_l})
\]

where \( F_{i_j} \) is the \( i_j \)-th elements of the set \( \{F_{i_1}, \ldots, F_{i_l}\} \).

Moreover, if \( \Delta \) is the full simplex \( 2^{[n]} \), then \( v_\Delta = v([n]) \).

**Proof.** Let us consider the sum of the characteristic function evaluated over all facets \( \sum_{F \in Fs_\Delta} v(F) \). Of course, we are double counting (at least!) everything that is in the intersection of two facets, and to correct our computation we take away this quantity, leading to:

\[
\sum_{F \in Fs_\Delta} v(F) - \sum_{(F_{i_1}, F_{i_2}) \subseteq Fs_\Delta} v(F_{i_1} \cap F_{i_2}).
\]

If a certain set \( T \) appears in more than in one intersection \( F_{i_1} \cap F_{i_2} \), we are subtracting \( v(T) \) too many times. So we should again correct our partial step by adding triple intersections:

\[
\sum_{F \in Fs_\Delta} v(F) - \sum_{(F_{i_1}, F_{i_2}) \subseteq Fs_\Delta} v(F_{i_1} \cap F_{i_2}) + \sum_{(F_{i_1}, F_{i_2}, F_{i_3}) \subseteq Fs_\Delta} v(F_{i_1} \cap F_{i_2} \cap F_{i_3}).
\]

Because we only consider finite simplicial complex (see Section 1), by iteration we get the formula in the statement.

Finally, in the case \( \Delta \) is the full simplex, there is only one facet, \([n]\), and so there are no double nor triple intersections. Naturally, the entire argument reduces to \( v_\Delta = \sum_{F \in Fs_\Delta} v(F) = v([n]) \). \( \square \)

Another point of view for the previous proof is provide by considering the following subcomplexes of \( \Delta \):

\[
\Delta^{(0)} = \Delta \quad \Delta^{(1)} = \Delta \quad \Delta^{(j)} = \bigcup_{(F_{i_1}, F_{i_2}, \ldots, F_{i_j}) \subseteq Fs_\Delta} F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_j}
\]

and generically

\[
(9) \quad \Delta^{(j)} = \bigcup_{(F_{i_1}, F_{i_2}, \ldots, F_{i_j}) \subseteq Fs_\Delta} F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_j}
\]
With this notation and using the arithmetic trick in equation (6), \( v_\Delta \) is the sum with signs of the worth each of these \( \Delta^{(j)} \):

\[
v_\Delta = v(\Delta^{(0)}) - v(\Delta^{(1)}) + v(\Delta^{(2)}) + \cdots + (-1)^j v(\Delta^{(j)}) + \cdots v(\Delta^{(#F_\Delta)})
\]

where some of the \( \Delta^{(j)} \) can be empty.

**Remark 3.3.** If we want to take into consideration a probabilistic approach, we could provide a probability distribution for the facets of \( \Delta^{(0)} \) as done in Section 4 of [BDJLL01] and generalized in [Mar20a]. Then one could do the same for \( \Delta^{(1)} \) and generically \( \Delta^{(j)} \) and obtain a probabilistic version of (10) and so a probabilistic version of the simplicial efficiency axiom. A coherent choice for such probabilities should be requested.

**Remark 3.4.** Another way of looking the probabilistic efficiency proposed in [BDJLL01] in view of our results is the following. Equation (10) shows how \( v(\Delta) \) (that essentially is the condition requested in [BDJLL01] seems a first approximation of the Combinatorics of the problem. Nevertheless, in the matroidal case, such approximation is not so far from the Simplicial efficiency axiom. We are going to treat this in the Section 3.6.

The following functions encode the coefficients of \( v(T) \) in formula (8). For every \( T \in \Delta \), we denote by \( d_T \) the following number:

\[
d_T \overset{\text{def}}{=} \sum_{k, T = F_1 \cap \cdots \cap F_k} (-1)^{k+1}
\]

where the sum runs over all positive number \( k \) such that \( T \) can be written as a \( k \)-intersection of facets of \( \Delta \). It is useful to consider \( d_T = 0 \), if \( T \) cannot be written as intersection of facets.

**Theorem 3.5.** Let \( \Delta \) be a simplicial complex and let \( \mathcal{S} \) be a cone of cooperative games defined on \( \Delta \) containing the carrier games \( C \) and \( \hat{C} \).

Let \( \phi \) be a group value on \( I \) such that for each \( i \in [n] \) and each \( v \in I \), we can write:

\[
\phi_i(v) = \sum_{T \in \text{Link}_A i} p_T^i (v(T \cup i) - v(T)).
\]

The group value \( \phi \) satisfies the simplicial efficiency axiom if and only if for all non-facet \( T \) in \( \Delta \)

\[
\sum_{i \in T} p_T^i \Delta_{\mid T, i} - \sum_{j, T \in \text{Link}_A \Delta} p_T^j = d_T,
\]

and for all facet \( F \) of \( \Delta \)

\[
\sum_{i \in F} p_F^i = 1.
\]

**Proof.** Apply Theorem 2.1 with \( a_T = d_T \). ⌣

**3.6. The matroid case.** In the case the simplicial complex is a matroid \( \Delta = M \) as treated in [BDJLL02, BDJLL01, MTMZ19, MZ11, FV11], then all the facets (bases) \( \{B_1, \ldots, B_k\} \) have the same cardinality, say \( r \).

Therefore by using (6), we prove that the total number of payoff is completely determined as in equation (10) by the \( \Delta^{(0)} \) and \( \Delta^{(1)} \), that is, by the payoff of the facets and of the intersections of cardinality \( r - 1 \). (We have simplified out notation by denoting \( \Delta^{(1)} \) as \( \Delta' \).)
First let us rewrite (7) and set up some notations. Consider a random order of the bases $B_1, B_2, \ldots, B_k$. The notation $B_0 = \emptyset$ is useful. Let us denote
\[ \tilde{B}_j = B_0 \cup B_1 \cup \cdots \cup B_j. \]
This is the sequential partial union of the facets under the given order. For instance, $\tilde{B}_0 = \emptyset$, $\tilde{B}_1 = B_1$, $\tilde{B}_2 = B_1 \cup B_2$ and $\tilde{B}_k = \Delta$.

Matroids are simplicial complexes with a special property: indeed they are shellable [Bjö80, BW96, BW97, ACS16, AB17, SW17] and, therefore, there exists a shelling order of the facets (bases) $B_1, B_2, \ldots, B_k$ such that $\tilde{B}_{j-1} \cap B_j$ has codimension 1, that is $\tilde{B}_{j-1} \cap B_j$ has dimension $r - 2$ (the intersection is made by cardinality $r - 1$ faces).

**Remark 3.7.** The dimension of a simplicial complex differ by one with respect its rank. For instance, every non-empty graphs have rank 2, because every edge is identified by two vertices, and graphs are one dimensional.

**Theorem 3.8.** Let $(M, v)$ be a cooperative game on a matroid $M$ of rank $r$. Then, there exists an ordering of the facets (the shelling order of the bases) such that the total number of payoff is provided as
\[ v_\Delta = v(\Delta) - v(\Delta') \]
where
\[ \Delta' = \bigcup_{\{F_{i_1}, F_{i_2}\} \subseteq F_s \Delta} F_{i_1} \cap F_{i_2}. \]

In other words,
\[ v_M = \sum_{B \in F_s \Delta} v(B) - \sum_{L} v(L), \]
where the second sum runs over the subcomplex $L$ of $\Delta$ of dimension $r - 2$ that are written as the intersection $\tilde{F}_{j-1} \cap F_j$ for $j = 1, \ldots, k$.

**Proof.** This is proved using shellability together with Theorem 3.2 and equation (10). \qed

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