Nonlinear normal modes in electrodynamic systems: A nonperturbative approach

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The concept of nonlinear normal modes has intensely been developed in the theory of mechanical oscillating systems. At the same time, the normal-mode dynamics in nonlinear electrodynamic systems has received little attention in the literature. We consider electromagnetic nonlinear normal modes in cylindrical cavity resonators filled with a nonlinear nondispersive medium. The key feature of the analysis is that exact analytical solutions of the nonlinear field equations are employed to study the mode properties in detail. Based on such a nonperturbative approach, the considered nonlinear modes are shown to possess energy orthogonality.

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I. INTRODUCTION

The concept of normal modes is a milestone in the theory of linear oscillating systems and has had a significant impact on all fields of physics [1–4]. As is known, a linear normal mode (LNM) is a free sinusoidal oscillation in a conservative dynamical system with constant parameters. In bounded distributed dynamical systems, an infinite but countable set of LNMs exists. In such systems, each LNM is characterized by its frequency and shape, and satisfies homogeneous partial differential equations (PDEs) of motion with given boundary conditions. A family of LNMs possesses the following important properties, which allows one to solve a whole set of problems related to the calculation of free and forced motions in a linear system.

1. Invariance. Each LNM can be excited independently of other LNMs by the specific choice of the initial conditions.

2. Completeness. An arbitrary oscillatory process in the system can be expressed as a superposition of LNMs.

3. Energy orthogonality. The total energy present in the system due to a free oscillatory process is the sum of the LNM energies.

Since nature is nonlinear, LNMs can only be regarded as very useful mathematical models describing actual oscillations of nonlinear systems in the weak-amplitude limit. However, the following question naturally arises: Do strongly nonlinear systems admit such specific motions that their properties allow one to consider them as nonlinear normal modes (NNMs), i.e., nonlinear generalizations of the LNMs of the underlying linear systems? An affirmative answer to this question with respect to the lumped systems has been done in the seminal works of Lyapunov [5] and Rosenberg [6–8]. Rosenberg defined an NN as a vibration in unison of the mechanical system, i.e., a synchronous oscillation during which all the displacements of the material points of the system reach their extreme values and pass through zero simultaneously. A definition of the NNM in an autonomous distributed system in terms of the dynamics on a two-dimensional invariant manifold in phase space has been proposed by Shaw and Pierre [9]. Using the invariant manifold approach, they have also developed the technique of asymptotic series expansions for constructing NNMs for a rather wide class of nonlinear 1 + 1D autonomous systems. In the past decades, the concept of NNMs in mechanical systems has been studied extensively by a large number of workers (see, e.g., works [10–16] and references therein). At the same time, NNMs in electrodynamic systems remain poorly studied.

In this work, we present a nonperturbative approach to the concept of NNMs in an exactly integrable, nonlinear 2 + 1D electrodynamic system. It should be emphasized that all the forthcoming results are exact, i.e., no asymptotic expansions will be used. Our approach is based on the theory developed in Refs. [17, 18]. The nonlinear PDEs considered in Refs. [17, 18] and herein depend explicitly on the independent variable (radial coordinate) and can formally be regarded as 1 + 1D nonautonomous systems. Therefore, the approach proposed in Ref. [3], which is restricted to the autonomous systems, cannot be applied directly. In the present study, we define an NNM as follows: The nonlinear normal mode (NNM) of a bounded distributed conservative system is an oscillatory motion in which all of the field quantities oscillate periodically in time with the same constant period in the whole volume of the system, except for the countable set of spatial points (nodes of the oscillation, at which some of the field quantities can vanish). Each of the NNM fields must exactly satisfy the nonlinear PDEs of motion and the boundary conditions, and reduces to the corresponding LNM field in the weak-field limit. Note that a similar generalized definition of the NNM as an unnecessarily synchronous periodic motion in a mechanical system was proposed in [16].

In what follows, we give examples of the electromagnetic NNMs in cavity resonators filled with a nonlinear nondispersive medium and discuss their properties in detail. It will be shown that the considered NNMs, as their linear counterparts, exactly satisfy the first above-
mentioned property of invariance and, what is quite remarkable, the third (energy orthogonality) property.

II. BASIC EQUATIONS

Consider electromagnetic fields in a bounded cylindrical cavity of radius \( a \) and height \( L \). We assume that the \( z \) axis of a cylindrical coordinate system \((r, \phi, z)\) is aligned with the cavity axis and limit ourselves to consideration of axisymmetric field oscillations, in which only the \( E_z \) and \( H_\phi \) components are nonzero. We will also assume that the cavity is filled with a nonlinear nondispersive medium in which the longitudinal component of the electric displacement can be represented as

\[
D_z = D_0 + \alpha \varepsilon_0 \varepsilon_1 \exp(\alpha E_z) - 1,
\]

where \( D_0 \), \( \varepsilon_1 \), and \( \alpha \) are certain constants. The possibility of using such a model of nonlinearity for media lacking a center of inversion is discussed in detail in Refs. [17–19]. Note, that the model of a nonlinear distributed system presented herein and its generalizations were also discussed in works [20–23]. In this case, the Maxwell equations read

\[
\begin{align*}
\partial_t H &+ r^{-1} H = \varepsilon(E) \partial_t E, \\
\partial_t E &+ \mu_0 \partial_t H,
\end{align*}
\]

where \( E \equiv E_z(r,t) \), \( H \equiv H_\phi(r,t) \), and

\[
\varepsilon(E) \equiv dD_z/dE = \varepsilon_0 \varepsilon_1 \exp(\alpha E).
\]

The exact solution of the system of Eqs. (1) and (2) can be written in implicit form as [18]

\[
\begin{align*}
\hat{E} &= A^{-1} \mathcal{E} \left( \rho e^{\hat{a} \hat{E}/2}, \tau + \hat{\alpha} \hat{H}/2 \right), \\
\hat{H} &= e^{\hat{a} \hat{E}/2} A^{-1} \mathcal{H} \left( \rho e^{\hat{a} \hat{E}/2}, \tau + \hat{\alpha} \hat{H}/2 \right).
\end{align*}
\]

Hereafter, the tilde field components denote dimensionless quantities normalized to an arbitrary amplitude factor \( A \), i.e., \( \hat{E} = E/A \) and \( \hat{H} = Z_0 H/(A e_z)^{1/2} \), \( \rho = r/a \), \( \tau = t(\varepsilon_0 \varepsilon_1 \mu_0)^{-1/2}/a \), and \( \hat{\alpha} = \alpha A \), where \( Z_0 = (\mu_0/\varepsilon_0)^{1/2} \). The functions \( \mathcal{E} \) and \( \mathcal{H} \) describe the electromagnetic field in a linear medium and satisfy the equations

\[
\partial^2_{\rho} \mathcal{E} + \rho^{-1} \partial_\rho \mathcal{E} = \partial^2_{\tau} \mathcal{E}
\]

and

\[
\partial_\rho \mathcal{E} = \partial_\tau \mathcal{H}.
\]

The energy conversation law in the considered nonlinear medium can easily be derived from the field equations (11) and (12). Multiplying Eq. (11) by \( E \), and Eq. (12) by \( H \), after some algebra we obtain

\[
\partial_t w + \nabla \cdot \mathbf{S} = 0
\]

with the energy density

\[
w = \varepsilon_0 \varepsilon_1 A^2 \left[ \frac{(\hat{\alpha} \hat{E} - 1) \exp(\hat{a} \hat{E}) + 1}{\hat{a}^2} + \frac{\hat{H}^2}{2} \right].
\]

and the Poynting vector

\[
\mathbf{S} = -\mathbf{e}_r EH.
\]

In the weak-field limit \( (\alpha E) \ll 1 \), Eq. (7) reduces to the well-known textbook formula \( w = \varepsilon_0 \varepsilon_1 E^2/2 + \mu_0 H^2/2 \).

Based on formulas (4), the method proposed in Ref. [18] makes it possible to easily generate the NNMs, i.e., exact periodic solutions of the system of Eqs. (1) and (2), starting from the LNM which satisfy the linear wave equation (5).

III. NONLINEAR NORMAL MODES IN A CYLINDRICAL CAVITY RESONATOR

Analytical solution for the oscillations of the \( E_{000} \) type in a circular cylindrical cavity with perfectly conducting walls and the nonlinear filling medium described by the dynamic permittivity (3) has been found in Ref. [18] and is given by

\[
\begin{align*}
\hat{E} &= J_0(\kappa \rho e^{\hat{a} \hat{E}/2}) \cos(\kappa \theta), \\
\hat{H} &= -e^{\hat{a} \hat{E}/2} J_1(\kappa \rho e^{\hat{a} \hat{E}/2}) \sin(\kappa \theta),
\end{align*}
\]

where \( J_m \) is a Bessel function of the first kind of order \( m \), \( \kappa_n \) is the \( n \)th positive root of the equation \( J_0(\kappa) = 0 \), and \( \theta = \tau + \hat{\alpha} \hat{H}/2 \). The electric and magnetic fields are described by the implicit functions \( E(r,t) \) and \( H(r,t) \) which are solutions of system (9) of two transcendental equations. These implicit functions exactly satisfy Maxwell equations (11) and (12), as well as the boundary conditions

\[
E(a, t) = 0, \quad |E(0, t)| < \infty.
\]

The initial conditions can be obtained by substituting \( \tau = 0 \) into formulas (9) to give

\[
\hat{E} = J_0(\kappa_n \rho e^{\hat{a} \hat{E}/2})
\]

at \( t = 0 \) and

\[
H(r, 0) \equiv 0.
\]

For a sufficiently large index \( n \) such that \( n > n^*(\alpha) \), where \( n^* \) is a certain integer, the functions \( E(r,t) \) and \( H(r,t) \) become ambiguous and solution (9), obtained without allowance for dispersion, becomes inapplicable [18]. For \( n < n^* \), implicit functions \( E \) and \( H \) given by formulas (9) describe continuous periodic oscillations with the time period \( T_n = 2\pi/\omega_n \), where \( \omega_n = \kappa_n (\varepsilon_0 \varepsilon_1 \mu_0)^{-1/2} a^{-1} \), satisfy the NNM definition given above, and correspond to the NNMs of the \( E_{000} \) (TM000) type in the cavity. Specifying the integer index \( n \) and imposing the initial conditions (11) and (12), the motion of the system follows the exact solution (9) of the nonlinear boundary value problem for Eqs. (11) and (12), i.e., no oscillations with indices differing from \( n \) are excited. Hence, the considered NNMs satisfy the invariance property.
Let us now consider some important features of NNMs, which have not been pointed out in our previous work \[18\]. First of all, the electric field in these modes does not oscillate in unison in the whole cavity volume, i.e., the amplitudes of the field at different spatial points can reach their extreme values and pass through zero at different instants of time. The same is true for the magnetic field. It is clearly seen in Fig. 3 in Ref. \[18\] that there are no synchronous oscillations at different spatial points in the \(E_{010}\) mode discussed therein. In this respect, the considered electromagnetic NNMs differ from the well-known NNMs in lumped and 1 + 1D distributed mechanical systems \[8, 9\], and can be called “internally resonant” \[13, 16\]. It will be shown below that the degree of oscillation synchronism at different points of the cavity resonator depends on the shape of the cavity.

We now calculate the total energy \(W\) stored in each NN of the \(E_{0n0}\) type. To this end, one should substitute the implicit functions \(E\) and \(H\) given by formulas (11) into Eq. (7) and integrate \(w\) over the cavity volume:

\[
W = a^2 \int_0^L \int_0^{2\pi} \int_0^1 w(\rho, \tau, \beta) d\rho d\phi dz. \tag{13}
\]

A remarkable result is that the quantity \(W\) is independent of the normalized nonlinearity parameter \(\tilde{a}\) and exactly coincides with the total energy \(W_0^{(n)}\) of the corresponding linear \(E_{0n0}\) mode in the cavity resonator filled with a linear medium that has the permittivity \(\varepsilon = \varepsilon_0 \varepsilon_1 = \text{const}\). The rigorous proof of this fact is given in the Appendix. Note that the quantity \(W_0^{(n)}\) is calculated analytically as

\[
W_0^{(n)} = \pi \varepsilon_0 \varepsilon_1 a^2 L \int_0^1 (E^2 + H^2) d\rho
\]

\[
= \pi \varepsilon_0 \varepsilon_1 a^2 L A^2 \left[ \cos^2(\kappa_n \tau) \int_0^1 J_0^2(\kappa_n \rho) d\rho \right]

\]

\[
+ \sin^2(\kappa_n \tau) \int_0^1 J_1^2(\kappa_n \rho) d\rho \right]

\]

\[
= \frac{\pi}{2} \varepsilon_0 \varepsilon_1 a^2 L A^2 J_1^2(\kappa_n). \tag{14}
\]

It is worth also nothing that the fundamental frequencies \(\omega_n\) of the NNMs are independent of the field amplitude and the total energy, and coincide with the eigenfrequencies of the LNMs of the \(E_{0n0}\) type in the underlying linear system. In the next section, we will show that the above-mentioned notable features of the NNMs hold for another electromagnetic system described by Eqs. (1) and (2).

IV. NONLINEAR NORMAL MODES IN A COAXIAL RESONATOR

Assume that a coaxial cylindrical inner conductor of radius \(b\) \(0 < b < a\) is inserted inside the cavity considered in the previous section. The NNMs of the \(E_{0n0}\) type in the nonlinear coaxial resonator can readily be constructed using formulas (11) from the corresponding LNMs in the linear resonator with a constant permittivity \(\varepsilon = \varepsilon_0 \varepsilon_1\) \((\alpha = 0)\). The electric fields of the NNMs of the \(E_{0n0}\) type must satisfy Eq. (13) and the following boundary conditions on the perfectly conducting walls of the coaxial volume:

\[
\mathcal{E}(b, t) = \mathcal{E}(a, t) = 0. \tag{15}
\]

The LNMs fields are given by

\[
\mathcal{E} = A \left[ J_0(\mu_n \rho) Y_0(\mu_n) - J_0(\mu_n) Y_0(\mu_n \rho) \right] \cos(\mu_n \tau),

\]

\[
\mathcal{H} = -A \left[ J_1(\mu_n \rho) Y_0(\mu_n) - J_0(\mu_n) Y_1(\mu_n \rho) \right] \sin(\mu_n \tau). \tag{16}
\]

where \(Y_m\) is a Bessel function of the second kind of order \(m\) and \(\mu_n\) is the \(n\)th positive root of the equation

\[
J_0(\beta \mu) Y_0(\mu) - J_0(\mu) Y_0(\beta \mu) = 0, \tag{17}
\]

where \(\beta = b/a\).

Substituting functions (16) into formulas (11), we obtain an exact solution to Eqs. (1) and (2) in implicit form. It can be verified that the boundary conditions (15) remain valid for the implicit functions \(E(r, t)\) and \(H(r, t)\) given by Eqs. (1) and (16). These implicit functions are periodic in time with the period \(T_n = 2\pi/\omega_n\), where \(\omega_n = \mu_n (\varepsilon_0 \varepsilon_1 \mu_0)^{-1/2} a^{-1}\), since the transcendental equations (11) are invariant with respect to the time shifts \(\tau \rightarrow \tau + nT_n\) with integer \(n\). Therefore, Eqs. (1) and (16) describe the fields of NNMs in the coaxial resonator with perfectly conducting walls and a nonlinear nondispersive filling medium.

The fields of NNMs satisfy the initial conditions

\[
\dot{\mathcal{E}} = J_0(\mu_n \rho \varepsilon_1^{\tilde{a} \tilde{E}/2}) Y_0(\mu_n) - J_0(\mu_n) Y_0(\mu_n \rho \varepsilon_1^{\tilde{a} \tilde{E}/2}) \tag{18}
\]

at \(t = 0\) and \(\dot{\mathcal{H}}(r, 0) \equiv 0\), and possess the invariance property.

Let us turn to the results of some calculations by formulas (11) and (16). Fig. 1(a) shows the snapshots of the normalized electric and magnetic fields (\(\mathcal{E}\) and \(\mathcal{H}\)) of the NNM of the \(E_{010}\) type in the coaxial resonator with \(\beta = 0.001\) \((n = 1, \mu_1 = 2.65...)\) as functions of \(\rho\) at fixed instants of time \(\tau\). This case corresponds to a thin inner conductor (coaxial wire) inside the cylindrical cavity. For comparison, similar plots for the NN of the \(E_{010}\) type in the cavity with \(\beta = 0.999\) \((\mu_1 = 3141.59...)\) are presented in Fig. 1(b). In the limit \(\beta \rightarrow 1\), a coaxial geometry tends to a thin flat layer. Note that Fig. 1 corresponds to the case of strong nonlinearity where \(\tilde{a} = 1\).

The presented plots show that the electric fields at different spatial points do not oscillate in unison. However, it is seen in Fig. 1 that the degree of synchronism of the field in the NNM of the \(E_{010}\) type inside the cavity turns out to be dependent on the value of \(\beta\). The oscillations are closer to synchronous ones for \(\beta = 0.001\).

The deviations of \(\dot{\mathcal{E}}\) and \(\dot{\mathcal{H}}\) from their values corresponding to the \(E_{010}\) mode in a cavity with \(\varepsilon = \varepsilon_0 \varepsilon_1 = \text{const}\) \((\alpha = 0)\) for \(\beta = 0.999\) are more significant than...
onator. Let us state the following initial and boundary conditions for the linear wave equation (5):

\[ E(0, 0) = \Phi(0), \quad \partial_\tau E(0, 0) = \Psi(0), \quad 0 \leq \rho \leq 1, \quad (19) \]

\[ E(1, \tau) = 0, \quad |E(0, \tau)| < \infty, \quad 0 < \tau < \infty, \quad (20) \]

where \( \Phi(0) \) and \( \Psi(0) \) are given functions. The boundary value problem defined by Eqs. (15), (19), and (20) describes free electromagnetic oscillations with the given initial field distribution in a cylindrical cavity specified by the relations \( \rho = r/a \leq 1 \) and \( 0 \leq z \leq L \), which is filled with a linear medium (\( \alpha = 0 \)). The solution to the linear boundary value problem described by Eqs. (15), (19), and (20) can be found in a standard way by the method of separation of variables (24). As a result, the functions \( E \) and \( H \) are written as

\[ E(\rho, \tau) = \sum_{n=1}^{\infty} J_0(\kappa_n \rho) \left[ B_n \cos(\kappa_n \tau) + C_n \sin(\kappa_n \tau) \right], \]

\[ H(\rho, \tau) = -\sum_{n=1}^{\infty} J_1(\kappa_n \rho) \left[ B_n \sin(\kappa_n \tau) - C_n \cos(\kappa_n \tau) \right], \quad (21) \]

where

\[ B_n = \frac{2}{J_1(\kappa_n)} \int_0^1 \rho \Phi(\rho) J_0(\kappa_n \rho) d\rho, \]

\[ C_n = \frac{2}{\kappa_n J_1(\kappa_n)} \int_0^1 \rho \Psi(\rho) J_0(\kappa_n \rho) d\rho. \quad (22) \]

Substituting series (21) into formulas (4), we obtain an exact solution to Eqs. (14) and (2) in implicit form. Such an implicit solution describes free electromagnetic oscillations which correspond to the presence of an infinite set of NNMs in the nonlinear resonator. The implicit functions \( E(\rho, \tau) \) and \( H(\rho, \tau) \) determined by formulas (4) and (21) satisfy the boundary conditions (20), but correspond to somewhat different initial conditions compared with Eq. (19).

For example, let us specify the functions \( \Phi \) and \( \Psi \) in the simple form

\[ \Phi(\rho) = A(1 - \rho^2), \quad (23) \]

\[ \Psi(\rho) = 0. \quad (24) \]

This leads to

\[ B_n = 8A \kappa_n^{-3} [J_1(\kappa_n)]^{-1}, \quad C_n = 0. \quad (25) \]

At the initial time \( \tau = 0 \), the electric field distribution \( \tilde{E}(\rho, 0) \) in the nonlinear resonator is defined by the transcendental equation

\[ \tilde{E} = A^{-1} \sum_{n=1}^{\infty} B_n J_0(\kappa_n \rho) e^\alpha \tilde{E}/2, \quad (26) \]

while the magnetic field \( \tilde{H} \equiv 0 \) as in the “seeding” linear problem [see Eq. (23)].

The electric field distributions \( \Phi(\rho)/A \) and \( \tilde{E}(\rho, 0) \) in the linear and nonlinear cases, respectively, are presented in Fig. 2(a). Figure 2(b) shows oscillograms of the field conditions for the linear wave equation (5):

\[ E(\rho, 0) = \Phi(\rho), \quad \partial_\tau E(\rho, 0) = \Psi(\rho), \quad 0 \leq \rho \leq 1, \quad (19) \]

\[ E(1, \tau) = 0, \quad |E(0, \tau)| < \infty, \quad 0 < \tau < \infty, \quad (20) \]

where \( \Phi(0) \) and \( \Psi(0) \) are given functions. The boundary value problem defined by Eqs. (15), (19), and (20) describes free electromagnetic oscillations with the given initial field distribution in a cylindrical cavity specified by the relations \( \rho = r/a \leq 1 \) and \( 0 \leq z \leq L \), which is filled with a linear medium (\( \alpha = 0 \)). The solution to the linear boundary value problem described by Eqs. (15), (19), and (20) can be found in a standard way by the method of separation of variables (24). As a result, the functions \( E \) and \( H \) are written as

\[ E(\rho, \tau) = \sum_{n=1}^{\infty} J_0(\kappa_n \rho) \left[ B_n \cos(\kappa_n \tau) + C_n \sin(\kappa_n \tau) \right], \]

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where

\[ B_n = \frac{2}{J_1(\kappa_n)} \int_0^1 \rho \Phi(\rho) J_0(\kappa_n \rho) d\rho, \]

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Substituting series (21) into formulas (4), we obtain an exact solution to Eqs. (14) and (2) in implicit form. Such an implicit solution describes free electromagnetic oscillations which correspond to the presence of an infinite set of NNMs in the nonlinear resonator. The implicit functions \( E(\rho, \tau) \) and \( H(\rho, \tau) \) determined by formulas (4) and (21) satisfy the boundary conditions (20), but correspond to somewhat different initial conditions compared with Eq. (19).

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The electric field distributions \( \Phi(\rho)/A \) and \( \tilde{E}(\rho, 0) \) in the linear and nonlinear cases, respectively, are presented in Fig. 2(a). Figure 2(b) shows oscillograms of the field

![FIG. 1. (a) Electric and magnetic fields as functions of \( \rho \) (solid and dashed lines, respectively) in the \( n = 1 \) mode of the coaxial resonator with \( \beta = 0.001 \) at times \( T_1 = 2\pi/(7\mu_1), T_2 = 2\pi/(5\mu_1), \) and \( T_3 = \pi/(2\mu_1). \) (b) The same as in Fig. 1(a), but for the cavity with \( \beta = 0.999. \)](image)

V. MORE GENERAL OSCILLATIONS.
ENERGY ORTHOGONALITY OF NONLINEAR NORMAL MODES

In this section, we consider more general electromagnetic oscillations which correspond to the presence of an infinite set of NNMs in a cylindrical (noncoaxial) resonator. Let us state the following initial and boundary
The performed analysis shows that the energy orthogonality property of the NNMs holds for a rather wide class of initial conditions [19]. One should only ensure the convergence of Fourier series [21] and the absence of ambiguity of the implicit solutions.

For the nonlinear system of Eqs. (1) and (2), the principle of superposition does not hold and the NNMs lack the usual orthogonality property. The interaction of the considered NNMs in the forced oscillations can result in complex nonlinear dynamics with a singular-continuous (fractal) Fourier spectrum [11]. Therefore, the observed energy orthogonality of the NNMs seems especially interesting.

VI. CONCLUSIONS

In this work, to the best of our knowledge, we have presented the first nonperturbative approach to the basic properties of NNMs in a distributed nonlinear system. The approach does not require asymptotic expansions and provides a rigorous theoretical formulation of the NN properties. This formulation is not restricted to consideration of weakly nonlinear systems. We have constructed exact solutions to the electromagnetic fields of NNMs in cylindrical cavity resonators filled with a nonlinear nondispersive medium. It has been shown that the oscillations of the found NNMs are periodic in time, but are not synchronous at different spatial points. We have established that the total energy of any NNM is independent of the nonlinearity parameter and exactly coincides with the energy of the corresponding LNM in the linear resonator. We have also obtained an exact solution which describes a more general oscillatory process corresponding to the presence of a countable set of NNMs in the nonlinear cylindrical resonator. Based on this solution, we have rigorously established the energy orthogonality of the NNM fields. A very intriguing and physically important issue, which naturally arises from the present analysis and still remains open, is whether energy orthogonality of NNMs is a property inherent in the particular model of nonlinearity or can be extended to a wider class of nonlinear distributed systems.

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APPENDIX

Let us show that the total energy of free nonlinear oscillations is independent of the nonlinearity parameter $\alpha$ and coincides with the total energy in the linear case ($\alpha = 0$). Due to the energy conservation in a cavity with perfectly conducting walls and a nondispersive filling medium, it is sufficient to prove this fact for an arbitrary fixed time instant (say, $\tau = 0$). The electric field distribution $\tilde{E}(\rho, 0)$ in the considered nonlinear oscillations is defined by the transcendental equation

$$
\tilde{E} = \alpha^{-1} \mathcal{E}(\rho e^{\tilde{\alpha} E/2}, 0),
$$

(A.1)

while $H(\rho, 0) \equiv 0$. Introducing the notation $R = \rho \exp(\tilde{\alpha} E/2)$, we have

$$
dR = (1 + \tilde{\alpha} \rho \tilde{E}/2)e^{\tilde{\alpha} E/2}d\rho.
$$

(A.2)

The derivative $\tilde{E}''\rho$ can be found from Eq. (A.1) as

$$
\tilde{E}''\rho = \alpha^{-1}[1 - \tilde{\alpha}e^{\tilde{\alpha} E/2}\rho \mathcal{E}''/(2A)]^{-1}e^{\tilde{\alpha} E/2}e_\mathcal{E}''R.
$$

(A.3)

Substituting Eq. (A.3) into Eq. (A.2) yields

$$
d\rho = [1 - \tilde{\alpha} R \mathcal{E}''/(2A)]e^{-\tilde{\alpha} E/2}dR.
$$

(A.4)

Consider for clarity a cylindrical (noncoaxial) resonator. Making the change of variables and using Eq. (A.4), from Eqs. (7) and (13) one obtains the total energy

$$
W = 2\pi \varepsilon_0 \varepsilon_1 a^2 L A^2 e^{-2\alpha - 2} \int_0^1 (\tilde{\alpha} \tilde{E} + e^{-\tilde{\alpha} E} - 1) \times \int_0^1 (\tilde{\alpha} \tilde{E} + e^{-\tilde{\alpha} E} - 1) \times [1 - \tilde{\alpha} R \mathcal{E}''/(2A)]dR.
$$

(A.5)

Integrating the first term in the integrand of Eq. (A.5) by parts and using the boundary condition $\mathcal{E} = 0$ at $R = 1$, we have

$$
-\frac{1}{2} \int_0^1 \mathcal{E}''R^2dR = \frac{1}{2} \int_0^1 \mathcal{E}^2RdR.
$$

(A.7)

Integrating the second and third terms in the integrand of Eq. (A.5) by parts, one can find that the result of integration cancels the last two terms in this integrand. Finally, we get

$$
W = 2\pi \varepsilon_0 \varepsilon_1 a^2 L A^2 \int_0^1 \mathcal{E}^2RdR,
$$

(A.8)

which coincides with the total energy of the linear oscillations. For a coaxial resonator, the proof is similar.

In addition, it should be noted that the total energy of the radially localized field distributions vanishing for $\rho \to \infty$ in an unbounded nonlinear medium, which can be the case for, e.g., cylindrical electromagnetic waves [18], is also independent of the nonlinearity parameter $\alpha$.

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