SPLICE DIAGRAM SINGULARITIES AND THE UNIVERSAL ABELIAN COVER OF GRAPH ORBIFOLDS

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Abstract. Given a rational homology sphere $M$, whose splice diagram $\Gamma(M)$ satisfy the semigroup condition, Neumann and Wahl were able to define a complete intersection surface singularity called splice diagram singularity from $\Gamma(M)$. They were also able to show that under an additional hypothesis on $M$ called the congruence condition, the link of the splice diagram singularity is the universal abelian cover of $M$. In this article we generalize the congruence condition to the class of orbifolds called graph orbifolds. We show that under a small additional hypothesis, this orbifold congruence condition implies that the link or the splice diagram equations is the universal abelian cover. We also show that any two node splice diagram satisfying the semigroup condition, is the splice diagram of a graph orbifold satisfying the orbifold congruence condition.

1. Introduction

The topology of an isolated complex surface singularity is determined by its link, which all turn to be among the class of 3-manifolds called graph manifolds, that is the manifolds which only have Seifert fibered pieces in their JSJ-decomposition. There are several graph invariants of graph manifolds which is used to study them. The first is the plumbing diagram of a plumbed 4-manifold $X$ such that our graph manifold $M$ is the boundary of $X$, this does give a complete invariant if one assumes the plumbing diagram is in a normal form (see [Neu81]), of which there are several different. Now the plumbing diagrams can be quite large and does not always display the properties of the manifold clearly, so we are interested in an other invariant called splice diagram. Splice diagrams where original introduced in [ENS5] and [Sie80] for integer homology sphere graph manifolds. It was then later generalized by Neumann and Wahl to rational homology spheres in [NW02]. They used it extensively in [NW05a] and [NW05b], and especially their use in [NW05a] is of interest to us.

In [NW05a] they use the splice diagram of a singularity link $M$ satisfying what they call the semigroup condition, to construct a set of equations called splice diagram equations defining an isolated complete intersection surface singularity $X$. They then showed that if $M$ satisfy an additional hypothesis called the congruence condition, the link of $X$ is the universal abelian cover of $M$. In [Ped10a] I showed that the splice diagram of any graph manifold $M$ always determines the universal abelian cover of $M$. So combining these to result one gets a nice description of the universal abelian cover of a graph manifold $M$, as the link of complete intersection, provided that there is some graph manifold $M'$ with the same splice diagram satisfying the congruence condition. This already implies that the congruence condition
might not be needed, moreover the following splice diagram

\[ \Gamma = \]

have no manifolds with it as its splice diagram satisfying the congruence condition, even though it satisfy the semigroup condition and it is the splice diagram of 4 different manifolds. Nonetheless one can construct a plumbing diagram of the abelian cover use the algorithm derived from my proof of Theorem 6.3 in [Ped10a] which is explained in more detail in [Ped10b] where this example is explicitly constructed, and one can find a dual resolution graph for the resolution of the splice diagram equation of \( \Gamma \) by hand, and it shows that also in this case is the link of the splice diagram singularity the universal abelian cover. This again indicates that the congruence condition is not needed. Even more interesting is the next example. The following splice diagram

\[ \Gamma' = \]

satisfy the semigroup condition, and the universal abelian cover is the link of the splice diagram singularity. But there are no manifolds with \( \Gamma' \) as its splice diagram. So what is the link the universal abelian cover of? To prove Theorem 6.6 of [Ped10a] I had to generalize the notion of splice diagram to a class of 3-dimensional orbifolds which I called graph orbifolds, and \( \Gamma' \) is the splice diagram of several graph orbifold.

This leads to the purpose of this article, to generalize the congruence condition to graph orbifolds, which is done in Section \[ 6 \]. Show that, under a small extra hypothesis, the link of at splice diagram equation of \( \Gamma(M) \) is the universal abelian cover of \( M \) if \( M \) satisfy the orbifold congruence condition in Section \[ 7 \]. In section \[ 8 \] we show that this is indeed an extension of the results of [NW05a], by given any two nodes splice diagram \( \Gamma \) satisfying the semigroup condition constructing a graph orbifold \( M \) satisfying the orbifold congruence condition, with \( \Gamma \) as its splice diagram. Section \[ 2 \] introduces graph orbifolds, Section \[ 3 \] introduces splice diagram, and Section \[ 4 \] the splice diagram equations. Section \[ 5 \] introduces the discriminant group which is needed in the definition of the congruence condition.

2. **Graph Orbifolds**

To generalize the conditions for the splice diagram equation to define the universal abelian cover, we have to extend the notion of splice diagrams to graph orbifolds, so in this section we define graph orbifolds.

**Definition 2.1.** A **graph orbifold** is a 3-dimensional orbifold \( M \), in which there exist a finite collection \( \{ T_i \}_{i \in 1, \ldots, n} \) of smoothly embedded tori, such that \( M - \bigcup_{i=1}^{n} T_i \) is a collection of \( S^1 \) orbifold fibrations over orbifold surfaces.

We will only consider graph orbifolds which has compact closure which is also a graph orbifold, and the boundary components will always be smoothly embedded tori. Notice that if \( M \) is smooth then \( M \) is a graph manifold, since any \( S^1 \) orbifold fibration over an orbifold surface with smooth total space is Seifert fibered.

Next we want to describe how a graph orbifold look locally, so we will look at a \( S^1 \) orbifold fibration over orbifold surface \( \pi: M \to \Sigma \). If \( x \in \Sigma \) is not an orbifold point then there is a disk neighborhood \( D \) of \( x \) such that \( M|_{\pi^{-1}(x)} \) is a trivial fibered solid torus. So the interesting situation is when \( x \in \Sigma \) is an orbifold point. This means that a neighborhood \( U \) of \( x \) is homeomorphic to \( \mathbb{R}^2/(\mathbb{Z}/\alpha \mathbb{Z}) \) for a \( \alpha > 1 \), where the group acts as rotation. Then \( M|_{\pi^{-1}(U)} \) is homeomorphic to
Proposition 2.2. \( H^1_\partial(M) = H_1(M) \) if \( M \) is smooth.

Proof. Since \( \pi_1^\text{orb}(T_{\alpha,\beta}) \) can be presented as \( \langle q, h, t \mid q^\alpha h^\beta = 1, q^\alpha t^h = 1 \rangle \), it is not hard to show that is indeed a presentation of \( \mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z} \).

This can then be used to show the following relating orbifold homology of \( M \) and \( M_K \) defined above which is Proposition 5.2 of \[Ped10\].
Proposition 2.3. Let $K \in M$ be an orbifold curve of degree $q$, then $|H^1_{orb}(M)| = q|H^1_{orb}(M_k)|$.

It then follows that $|H^1_{orb}(M)| = Q|H^1_{orb}(\mathcal{M})| = \mathcal{Q}[H_1(M)]$, where $Q = \prod_{k} q_K$ with the product is taking over all orbifold curves $K$ of orbifold degree $q_K$. It should be mentioned that our $H^1_{orb}(M)$ is part of a homology theory for orbifolds in general see [ALR07].

3. Splice Diagrams

A splice diagram is a tree with no vertices of valence 2, which is decorated by signs on vertices who have valence greater than 2, we call such vertices nodes, and non negative integer weights on edges adjacent to nodes. We will call vertices of valence 1 for leaves, and will in general not distinguish between a leaf and the edge connecting the leaf to a node.

We will now explain how to assign a splice diagram as an invariant to any graph orbifold. Let $M$ be a rational homology sphere (QHS) graph orbifold, we the construct the splice diagram $\Gamma(M)$, by first taking a node for each piece of the JSJ-decomposition of $M$, connect two nodes if the corresponding $S^1$-fibred pieces of the decomposition are glued to create $M$. We will in general not distinguish between a node in $\Gamma(M)$ and the corresponding $S^1$-fibred piece. This will result in a tree since the pieces of the JSJ-decomposition of $M$ corresponds to the pieces of the JSJ-decomposition of the underlying manifold $\mathcal{M}$, and since $M$ is a QHS it follows from the comments after Proposition 2.3 that $\mathcal{M}$ is a QHS, and hence its JSJ-decomposition gives a tree like structure. We add a leaf to a node for each singular fiber of the node.

Next we want to add the decorations. First the signs at a node $v$ is going to be the sign of the linking number of two non singular fibers at $v$, for precise definition of this see section 2 of [Ped10a], there we do only define it for manifolds, but the definition carries over to orbifolds, even though one has to be careful since linking number is not going to be symmetric any more in the case of orbifolds. One can calculate the linking number in the underlying manifold, and it follows from Lemma 6.3 that signs will be the same.

Last thing to define is the edge weights. Let $v$ be a node and $e$ an edge adjacent to $v$, then we will do the following construction to define the edge weight $d_{ve}$ adjacent to $v$ on $e$. Let $T \subset M$ be the torus corresponding to the edge $e$, cut $M$ along $T$ and let $M'$ be the piece not containing $v$. Let $M_{ve} = M' \bigcup_{e}(S^1 \times D^2)$ by gluing a meridian of the solid torus to the image of a fiber of $v$ in $\partial M'$. Then $d_{ve} = |H^1_{orb}(M_{ve})|$. 

The above definition defines a splice diagram where all weight at leaves are greater than 1, we call such splice diagram reduced, we will not always assume that our splice diagram are reduced, i.e. we will sometimes allow weights at leaf to be 1. A leaf of weight 1 will correspond to a non singular fiber, so if one has a non reduced splice diagram of a graph orbifold $M$, one gets the reduced splice diagram of $M$ by removing all leaves of weight 1.

A general edge between two nodes in a splice diagram looks like

\[
\begin{array}{c}
\vdots \\
 n_{0k_0} \\
 \vdots \\
 v_0
\end{array}
\begin{array}{c}
 r_0 \\
 v_1 \\
 \vdots \\
 n_{1k_1}
\end{array}
\]

\[
\begin{array}{c}
 r_1
\end{array}
\begin{array}{c}
 v_{01}
\end{array}
\]

Having such an edge we associate a number called the edge determinant, which we define as $r_0 r_1 = \varepsilon_0 \varepsilon_1 (\prod_{i=1}^{k_0} n_{0i}) (\prod_{j=1}^{k_1} n_{1j})$, where $\varepsilon_i$ is the sign on the $i$’th node. This
number is important since it helps determining which graph manifolds arises as
singularity links, by the following theorem from [Ped10a]

**Theorem 3.1.** Let $M$ be a QHS graph manifold with splice diagram $\Gamma(M)$. Then
$M$ is the link of an isolated complex surface singularity if and only if, there are no
negative signs on nodes and all edge determinants of $\Gamma(M)$ are positive.

Not all combinatorial splice diagram arises as the splice diagram of a graph
orbifold. We will next introduce an important condition which the splice diagram
of any graph orbifold satisfy. Let $\Gamma$ be a splice diagram, and let $v$ and $w$ be two
vertices in $\Gamma$. Then one defines the linking number $l_{vw}$ of $v$ and $w$ to be the product
of all edge weights adjacent to but not on the shortest path from $v$ to $w$. Similarly
$l'_{vw}$ is defined the the same way except we exclude the weights adjacent to $v$ and $w$.
Let $e$ be an edge at $v$ then $\Gamma_{ve}$ is the connected subgraph of $\Gamma$ one get by removing
$v$, which includes the edge $e$. We then define ideal generator at $v$ in direction of $e$ $d_{ve}$, to be the positive generator of the following ideal in $\mathbb{Z}$

$$\langle l'_{vw} \mid w \text{ is a leaf of } \Gamma_{ve} \rangle$$

**Definition 3.2.** A splice diagram $\Gamma$ satisfy the ideal condition if every edge weight
$d_{ve}$ is divisible by the corresponding ideal generator $d_{ve}$

In section 12.1 in [NW05a] Neumann and Wahl proves that every splice diagram
coming from a singularity link satisfy the ideal condition, and the proof also works
in the general setting of any graph orbifold. So satisfying the ideal condition is a
necessary condition for a splice diagram to come from graph orbifold, in the one
node case the ideal condition is void and we show in section 8 the in the two node
case the ideal condition is also sufficient, unfortunately this is not the case if the
splice diagram have more than two nodes, and the lack of understanding which
splice diagram that in general arises as invariant of graph orbifolds is one of the
reasons we have not been able to extend the main result to more than two nodes
splice diagrams.

4. **Splice Diagram Equations**

We are in general going to be be interested in splice diagram satisfying the
following stronger condition than the ideal condition.

**Definition 4.1.** A splice diagram $\Gamma$ is said to satisfy the semigroup condition if for every node $v$ and edge $e$ at $v$. The edge weight lies in the following semigroup
of $\mathbb{N}$:

$$d_{ve} \in \mathbb{N} \langle l'_{vw} \mid w \text{ a leaf in } \Gamma_{ve} \rangle$$

The semigroup is only interesting if one has no negative signs at nodes, so we will
assume that splice diagrams satisfying the semigroup satisfy this. The semigroup
condition is strictly stronger than the ideal condition for splice diagram with more
than one node.

If $M$ satisfy the semigroup condition, then given a node $v$ and adjacent edge $e$, one can write the corresponding edge weight as

$$d_{ve} = \sum_{w \text{ is a leaf of } \Gamma_{ve}} \alpha_{vw} l'_{vw},$$

where the $\alpha_{vw}$’s are non negative integers. We call the collection of $\alpha_{vw}$ semigroup coefficients of $d_{ve}$. It not hard to see that (1) is equivalent to

$$d_v = \sum_{w \text{ is a leaf of } \Gamma_{ve}} \alpha_{vw} l_{vw},$$

(2)
where $d_v$ is the product of all edge weights adjacent to $v$.

From now on we will assume that $\Gamma$ satisfy the semigroup condition, we then associate to each leaf $w$ of $\Gamma$ a variable $z_w$, and have the following definitions.

**Definition 4.2.** Let $v$ be a node of $\Gamma$ and assume $\Gamma$ has leaves $w_1, \ldots, w_n$, then a $v$-weighting, or $v$-filtration, of $\mathbb{C}[z_{w_1}, \ldots, z_{w_n}]$ is to associate the weight $l_{v,w_i}$ to $z_{w_i}$.

**Definition 4.3.** Let $v$ be a node of $\Gamma$ an $e$ an adjacent edge, then an admissible monomial associated to $v$ and $e$, is a monomial on the form $M_{ve} = \prod w z_{\alpha_w} w$, where the product is taken over all leaves in $\Gamma_{ve}$, and the $\alpha_w$’s is a choice of semigroup coefficients of $d_{ve}$.

It is clear that each admissible monomial is $v$-weighted homogeneous, of total $v$-weight $d_v$.

**Definition 4.4.** Let $\Gamma$ with leaves $w_1, \ldots, w_n$ be a splice diagram satisfying the semigroup condition, then a set of splice diagram equations for $\Gamma$, is the following set of equations in the variables $z_{w_1}, \ldots, z_{w_n}$.

$$\sum_{e} a_{v,ie} M_{ve} + H_{vi}, \quad v \text{ a node of valence } \delta_v, \ e \text{ an adjacent edge, } i = 1, \ldots, \delta_v - 2$$

where

- $M_{ve}$ is an admissible monomial
- for every $v$, all maximal minors of the $(\delta_v - 2) \times \delta_v)$-matrix $(a_{v,ie})$ has full rank
- $H_{vi}$ is a convergent power series in the $z_{w_i}$’s all of whose monomials has $v$-weight higher than $d_v$.

This defines $n-2$ equations in $n$ variables, and the corresponding subscheme $X(\Gamma) \subset \mathbb{C}^n$ is called a splice diagram surface singularity.

We have the first important result concerning splice diagram equations, which is Theorem 2.6 of [NW05a].

**Theorem 4.5.** Let $\Gamma$ be a splice diagram satisfy the semigroup condition, and let $X = X(\Gamma)$ be an associated splice diagram surface singularity. Then $X$ is a two-dimensional complete intersection, with an isolated singularity at the origin.

5. **Plumbing and The Discriminant Group**

From now $M$ will always be a graph orbifold with splice diagram $\Gamma(M)$ and $\overline{M}$ is the underlying manifold, hence $\Gamma(\overline{M})$ is equal to $\Gamma(M) \whit$ any edge-weight $d_{ve}$, replaced by $d_{ve}/o_1 o_2 \cdots o_n$, where $o_1, \ldots, o_n$ are the orbifold degrees of all orbifold curves in $M_{ve}$ by Proposition 2.3.

Now this of course do not always produce an reduced splice diagram of $\overline{M}$, since there could be leaves with weight 1.

**Example 5.1.** Assume $M$ is a graph manifold with following splice diagram

$$\Gamma(M) = \begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{figure.png}}
\end{array}$$

and assume that at the leaves named $w_1, w_2$ and $w_3$, we have orbifold curves of orbifold degrees $3, 2$ and $5$ respectively. Then $\overline{M}$ is going to have the following...
splice diagram

If we want the reduced splice diagram of $M$, we just remove the leaf $w_3$, and the the central vertex becomes a valence two vertex, and therefore has to be suppressed, and we get

Let $\Delta(M)$ be a plumbing diagram of $\overline{M}$, then one can get a plumbing diagram $\Delta(M)$ for $M$, by adding a arrow weighted with the orbifold degree at the corresponding vertex for each orbifold curve of $M$. By blowing up if necessary, we can assume that each vertex has at most one arrow attached. If $v$ is a vertex of $\Delta(M)$, then the orbifold degree $o_v$ of $v$ is the orbifold degree of the arrow attached to $v$, if no arrow is attached to $v$ then $o_v = 1$. When ever we use that notation $\Delta(M)$ and $\Delta(M)$, we will assume that they are connected in this way.

Example 5.2. The graph orbifold $M$ from Example 5.1 has the following plumbing diagram

Let $E_v \subset X$ be the curve where $X$ is an analytic surface with $\partial X = M$ corresponding to the vertex $v \in \Delta(M)$ and $E_v$ the corresponding surface in $X$ with $\partial X = M$. Then let

\begin{align*}
E : &= \bigoplus_{v \in \text{vert}(\Delta(M))} \mathbb{Z} \cdot E_v \\
\overline{E} : &= \bigoplus_{v \in \text{vert}(\Delta(M))} \mathbb{Z} \cdot \overline{E}_v
\end{align*}

Now $E$ and $\overline{E}$ are the same as $\mathbb{Z}$-modules, but they have different intersection pairings defined by $\Delta(\overline{M})$ and $\Delta(M)$. Let $A(\overline{M})$ be the intersection matrix of $\overline{E}$ in the basis given by $E_v$. Then the intersection matrix for $E$ in the basis $E_v$ is given by the matrix $A(M)$ which is gotten from $A(\overline{M})$ by multiplying each column, which corresponds to a vertex $v$, with the orbifold degree $o_v$ of $v$ as defined above.

One can construct $\Gamma(M)$ from $\Delta(M)$ by suppressing all vertices of valence 2 in $\Delta(M)$ to get the tree structure, and using the following propositions from Ped10a to get the decorations.

Proposition 5.3. Let $v$ be a node in $\Gamma(M)$, and $e$ be a edge on that node. We get the weight $d_{ve}$ on that edge by $d_{ve} = |\det(-A(\Delta(M)_{ve}))|$, where $\Delta(M)_{ve}$ is is the connected component of $\Delta(M) - e$ which does not contain $v$. 
Proposition 5.4. Let \( v \) be a node in \( \Gamma(M) \). Then the sign \( \varepsilon \) at \( v \) is \( \varepsilon = -\text{sign}(a_{vv}) \), where \( a_{vv} \) is the entry of \( A(M)^{-1} \) corresponding to the node \( v \).

Let \( \{ e_v \} \subset \hom(\mathbb{E}, \mathbb{Q}) \subset \mathbb{E} \otimes \mathbb{Q} \) and \( \{ \tilde{e}_v \} \subset \mathbb{E}^* = \hom(\mathbb{E}^*, \mathbb{Z}) \subset \mathbb{E}^* \otimes \mathbb{Q} \) be the dual bases.

Define the discriminant group as the finite abelian group
\[ D(\Delta(M)) := \mathbb{E}^*/\mathbb{E}, \]
the order of \( D(\Delta(M)) \) is \( \det(M) := |\det(-A(M))| \). The intersection pairing of \( \Delta(M) \) induces pairings of \( \mathbb{E} \otimes \mathbb{Q} \) into \( \mathbb{Q} \) and \( D(M) \) into \( \mathbb{Q}/\mathbb{Z} \). We then get the following facts about discriminant groups from Section 5 of [NW05a].

Proposition 5.5. Consider a collection \( e_w \), where \( w \) runs over all leaves of \( \Gamma(M) \). Then \( D(M) \) is generated by the images of these \( e_w \).

Proposition 5.6. Let \( e_1, \ldots, e_n \) be the elements of the dual basis of \( \mathbb{E}^* \) corresponding to the leaves of \( \Gamma(M) \). Then the homomorphism \( \mathbb{E}^* \to \mathbb{Q}^n \) defined by
\[ e \mapsto (e \cdot e_1, \ldots, e \cdot e_n) \]
induces an injection
\[ D(M) \hookrightarrow (\mathbb{Q}/\mathbb{Z})^n. \]
In fact, each non-trivial element of \( D(M) \) gives an element of \( (\mathbb{Q}/\mathbb{Z})^n \) with at least two non-zero entries.

We will embed \( (\mathbb{Q}/\mathbb{Z})^n \) into \( \mathbb{C}^n \) via the following map
\[ (\ldots, r, \ldots) \mapsto (\ldots, \exp(2\pi ir), \ldots) =: [\ldots, r, \ldots]. \]

Proposition 5.7. Let \( w_1, \ldots, w_n \) be the leaves of \( \Gamma(M) \), then the discriminant group \( D(M) \) is naturally represented by a diagonal action on \( \mathbb{C}^n \), where the entries are \( n \)-tuples of \( |\det(M)| \)’th roots of unity. Each leaf \( w_j \) corresponds to an element
\[ [e_{w_j} \cdot e_{w_1}, \ldots, e_{w_j} \cdot e_{w_n}] := \{ \exp(2\pi i e_{w_j} \cdot e_{w_1}), \ldots, \exp(2\pi i e_{w_j} \cdot e_{w_n}) \}, \]
and any \( n-1 \) of these generate \( D(M) \). The representation contains no pseudoreflections, i.e. non-identity elements fixing a hyperplane.

6. Congruence Condition for Graph Orbifolds

The plumbing diagram \( \Delta \) we use will be assumed to be quasi-minimal, this means that all weights on strings of \( \Delta \) have weights less that \(-1\), unless the string consist of a single vertex with weight \(-1\).

To any string
\[ -b_1 - b_2 - \ldots - b_k \]
in \( \Delta \) one associates a continued fraction
\[ n/p := b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \ldots}}. \]
We associate 1/0 to the empty string. We need the following standard facts about this relation ship which proofs are not hard and can be found many places.
Lemma 6.1. Reversing a string with continued fraction \( n/p \) gives one with continued fraction \( n/p' \) with \( pp' \equiv 1 \pmod{n} \). Moreover the following relations hold:

\[
\begin{align*}
    n &= \det \begin{pmatrix} -b_0 & b_2 & \cdots & b_{2k} \\ -b_1 & b_3 & \cdots & b_{2k+1} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{2k-1} & b_{2k+1} & \cdots & b_{2n-1} \end{pmatrix} \\
p &= \det \begin{pmatrix} -b_0 & -b_2 & \cdots & -b_{2k} \\ -b_1 & -b_3 & \cdots & -b_{2k+1} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{2k-1} & -b_{2k+1} & \cdots & -b_{2n-1} \end{pmatrix} \\
p' &= \det \begin{pmatrix} -b_0 & b_2 & \cdots & -b_{2k} \\ -b_1 & -b_3 & \cdots & -b_{2k+1} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{2k-1} & b_{2k+1} & \cdots & -b_{2n-1} \end{pmatrix},
\end{align*}
\]

and the continued fraction in the last case is \( p'/n' \) with \( n' = (pp' - 1)/n \).

For each \( n/p \in [1, \infty) \), there is a unique quasi minimal string, and in this case the continued fraction associated to the reverse direction is \( n/p' \) where \( p' \) is the unique number satisfying \( p' \leq n \) and \( pp' \equiv 1 \pmod{n} \).

As we saw in last section the discriminant group \( D(M) \) acts diagonally on \( \mathbb{C}^{n} \). Viewing the \( z_{v_i} \)'s as linear functions on \( \mathbb{C}^{n} \), \( D(M) \) act naturally on \( \mathbb{C}[z_{w_1}, \ldots, z_{w_n}] \), where \( e \) acts on a monomial as

\[
\prod z_{w_i}^{\alpha_{w_i}} \mapsto \left[ -\sum (e \cdot e_{w_i}) \right] \prod z_{w_i}^{\alpha_{w_i}}.
\]

This is the same as saying that the group transforms each monomial according to the character

\[
e \mapsto \exp \left( -2\pi i \sum (e \cdot e_{w_i}) \alpha_{w_i} \right).
\]

Now we will return the setting of \( \Gamma(M) \) satisfying the semigroup conditions, so we have the notion of admissible monomial.

Definition 6.2. Let \( M \) be a graph orbifold with splice diagram \( \Gamma(M) \) satisfy the semigroup condition. Let \( \Delta(M) \) be a plumbing diagram of \( M \), then \( \Delta(M) \) satisfy the orbifold congruence condition if for each node \( v \) of \( \Gamma(M) \) and adjacent edge \( e \), one can choose admissible monomials \( M_{v_e} \) so that \( D(M) \) transforms these monomials according to the same character.

Notice that if \( M \) is a manifold, then this definition is the same as the definition of congruence condition (Definition 6.3) in [NW05]. Next we write down explicit equation in terms of \( \Gamma(M) \) and \( \Delta(M) \) for the congruence condition.

Lemma 6.3. The matrix \( (e_{v}, e_{v'}) \) where \( v, v' \in \text{vert}(\Delta(M)) \) is the inverse matrix of \( A(M) \), and the matrix \( (\sigma_{v}, \sigma_{v'}) \) is the inverse matrix of \( A(M) \)

Proof. This follows from elementary linear algebra. □

Lemma 6.4. For any \( v, v' \in \text{vert}(\Delta(M)) \), we have that

\[
e_{v} \cdot e_{v'} = \sigma_{v} \cdot \sigma_{v'}^{-1}.
\]

Proof. This follows from Cramer’s rule, i.e. if \( M_{ij} \) is the \( i, j \)th minor of \( A(M) \) and \( \overline{M}_{ij} \) is the \( i, j \)th minor of \( \overline{A(M)} \), then \( M_{ij} = \overline{M}_{ij} \) with each column \( l \) multiplied by the corresponding orbifold degree \( \alpha_{l} \), hence \( \det(M_{ij}) = \left( \prod_{l \neq j} \alpha_{l} \right) \det(\overline{M}_{ij}) \) and

\[
A(M)^{-1}_{ij} = \frac{1}{\det(A(M))} (-1)^{i+j} \det(M_{ij})
\]

\[
= \frac{\prod_{l \neq j} \alpha_{l}}{\prod_{l} \alpha_{l} \det(A(M))} (-1)^{i+j} \det(\overline{M}_{ij}) = \frac{1}{\alpha_{j} A(M)} A(M)^{-1}_{ij}.
\]

Lemma 6.5. If \( v \) and \( v' \) are different vertices in \( \Delta(M) \), corresponding to leaves of \( \Gamma(M) \), then

\[
e_{v} \cdot e_{v'} = -\alpha_{v} e_{v'}/d.
\]
Proof. \(\mathfrak{e}_v \cdot \mathfrak{c}_v' = \frac{I_{vv'}}{d}\) by Lemma 6.4 in \([NW05a]\), but by the definition of \(\mathcal{M}\) it follows that \(I_{vv'} = \mathfrak{e}_v \cdot \mathfrak{c}_v' = \mathfrak{e}_v \cdot \mathfrak{e}_v' = \frac{1}{d} \prod_{v', w' \neq v, w} o_{v'} / o_{w'} - \frac{1}{o_{v'}} \prod_{v', w' \neq v, w} o_{v'} / o_{w'} = -\frac{o_v l_{w'}}{d}.

\[\square\]

**Proposition 6.6.** If \(v\) is a node in \(\Gamma(M)\) and \(w\) is a adjacent leaf and the continued fraction associated to the string in \(\Delta(M)\) \((\Delta(\hat{M}))\) is given by \((n_w/o_w)/p\), where \(n_w\) is the weight of the leaf and \(o_w\) is the orbifold degree of the leaf. Let \(p'\) be the smallest positive integer such that \(p'p = 1 \mod (n_w/o_w)\). Let \(\{n_i\}_{i=1}^{k} = o_w\) be the other weights adjacent to \(v\) and let \(N = \prod_{i=1}^{k} n_i\). Then

\[e_w \cdot e_w' = -\frac{o_w N}{dn_w} - \frac{p'}{n_w}.

Proof. By using \([NW05a]\) and the formula for \(\mathfrak{e}_w \cdot \mathfrak{e}_w\) given by proposition 6.6 in \([NW05a]\) we get

\[e_w \cdot e_w' = \frac{\mathfrak{e}_w}{o_w} = -\frac{1}{o_w} \left(\frac{n_w/o_w}{(n_w/o_w)^2} \prod_{i=1}^{k} n_i/o_i + \frac{p'}{n_w/o_w} \right) = -\frac{o_w n_w \prod_{i=1}^{k} n_i}{n_w^2 d} - \frac{p'}{n_w}.

\[\square\]

**Corollary 6.7.** The class of \(e_w'\), where \(w'\) is a leaf, transforms the monomial \(\prod z_{v,w}^n\) by multiplication by the root of unity

\[\left(\sum_{w' \neq w} \alpha_{w'} o_{w'}/\det(\Gamma) - \alpha_{w'} e_{w'} \cdot e_{w'}\right).

We are now able to give formulas for checking the congruence condition.

**Proposition 6.8.** Let \(\Delta\) be an orbifold plumbing diagram which splice diagram \(\Gamma\) satisfy the semigroup condition. Then the orbifold congruence condition is equivalent to the following: for every node \(v\) and adjacent edge \(e\), there is an admissible monomial \(M_{ve} = \prod z_{w,v}^n\) where \(w\) is a leaf in \(\Gamma_{ve}\), so that for every leaf \(w'\) of \(\Gamma_{ve},

\[\left(\sum_{w' \neq w} \alpha_{w'} o_{w'}/\det(\Gamma) - \alpha_{w'} e_{w'} \cdot e_{w'}\right) = \left(\frac{o_{w'} l_{w'}}{\det(\Gamma)}\right).

Proof. The proof follows exactly as the prof given for the manifold case in Proposition 6.8 in \([NW05a]\). \[\square\]

We will now look closer to how the congruence condition looks in the two node case. Let \(\Gamma\) be the splice diagram

\[
\begin{array}{c}
\text{\(w_{01}\)} & \text{\(n_{01}\)} & \text{\(r_0\)} & \text{\(v_0\)} \\
\vdots & & & \\
\text{\(w_{11}\)} & \text{\(n_{11}\)} & \text{\(r_1\)} & \text{\(v_1\)} \\
\end{array}
\]

and let the orbifold degree corresponding to \(n_{ij}\) be \(o_{ij}\). Let \(\Delta\) be a plumbing diagram where we have made sure by blowing up that all the vertices \(v\) with \(o_v \neq 1\) have valence one. Then \(\Delta\) is going to look like
Let \( p'_{ij} \) be the unique positive integer satisfying \( p_j p'_{ij} \equiv 1 \) mod \((n_{ij}/o_{ij})\) and \( p'_{ij} < (n_{ij}/o_{ij}) \). Let \( N_i = \prod_j n_{ij} \). Then the equations for the congruence condition becomes

\[
\Delta = \frac{o_{ij} \cdot N_0 N_1}{n_{ij} d} = \frac{k}{\sum_{i=1}^{k} \alpha_{wi} o_{ij} N_r l}{n_{ij} n_{ij} d} - \alpha_{ij} e_{wij} \cdot e_{wij}
\]

\[
= \frac{k}{\sum_{i=1}^{k} \alpha_{wi} o_{ij} N_r l}{n_{ij} n_{ij} d} - \alpha_{ij} \left( \frac{\mathcal{O}_{ij} N_r l}{n_{ij} n_{ij}} + \frac{p'_{ij}}{n_{ij}} \right)
\]

\[
= \frac{o_{ij} r_0 r_1}{n_{ij} d} + \frac{p'_{ij}}{n_{ij}}
\]

Where we use that \( r_{1-t} = \sum_{i=1}^{k} \alpha_{wi} N_{ri} \) be the choice of admissible monomials. This equality is of course equivalent to

\[
\left[ 0 \right] = \left[ \frac{\mathcal{O}_{ij} (r_0 r_1 - N_0 N_1)}{n_{ij} d} + \frac{p'_{ij}}{n_{ij}} \right] = \left[ \frac{\mathcal{O}_{ij} n}{n_{ij}} + \frac{p'_{ij}}{n_{ij}} \right]
\]

where we use that \( nd = r_0 r_1 - N_0 N_1 \) by the edge determinant equation (Corollary 3.3 in [Ped10a]). Since \( n_{ij}/o_{ij} \) is an integer, the equation becomes equivalent to the following

\[
\frac{\mathcal{O}_{ij} p'_{ij}}{o_{ij}} \equiv -n \mod \left( n_{ij}/o_{ij} \right).
\]

Using the definition of \( p'_{ij} \) gives us the following set of equations one need to check for the congruence to be satisfied

\[
\frac{\mathcal{O}_{ij} p'_{ij}}{o_{ij}} \equiv -n p_{ij} \mod \left( n_{ij}/o_{ij} \right).
\]

7. **Splice diagram equations determining the universal abelian cover**

Let \( M \) be a graph orbifold with splice diagram \( \Gamma = \Gamma(M) \) satisfying the semigroup condition. Assume that \( \{o_w\} \) is a choice of semigroup coefficients such that \( \Delta(M) \) satisfying the orbifold congruence condition.

**Definition 7.1.** The set of semigroup coefficients \( \{o_w\} \) is said to be \( M \) reducible if the orbifold degree \( o_w \) divides \( o_w \) for each leaf \( w \in \Gamma \).

Let \( \overline{M} \) be the underlying manifold of \( M \), then the splice diagram \( \overline{\Gamma} = \Gamma(\overline{M}) \) is equal to \( \Gamma \) except that given an edge \( e \in \Gamma \) at a node \( v \), then the edge weight \( \overline{d}_{ve} \) in \( \overline{\Gamma} \) is given by \( \overline{d}_{ve} = \frac{d_{ve}}{\prod_w \alpha_w o_w} \) where \( d_{ve} \) is the edge weight in \( \Gamma \). If \( \{o_w\} \) is a set on \( M \) reducible semigroup coefficients then \( \{\overline{o_w}\} \), where \( \overline{o_w} = o_w/o_w \), is a set of semigroup coefficients for \( \overline{M} \) since

\[
\overline{d}_{ve} = \frac{d_{ve}}{\prod_w \alpha_w o_w} = \sum_{w \in \Gamma_v} \alpha_w \overline{o_w} \in \overline{\Gamma} = \sum_{w \in \Gamma_v} \alpha_w/o_w \prod_{w' \in \Gamma_v, w' \neq w} \alpha_w' \overline{o_w'} = \sum_{w \in \Gamma_v} \overline{o_w} \overline{\Gamma}.
\]
Definition 7.3. Let \( \{\alpha_w\} \) satisfy the orbifold congruence condition for \( \Delta(M) \) then \( \{\tau_w\} \) satisfy the congruence condition for \( \Delta(M) \).

Proof. Let \( v \in \Gamma \) be a node and \( w' \in \Gamma_{ve} \) be a leaf, then

\[
\left[ \sum_{w \neq w'} \tau_w \frac{1}{\text{det}(M)} - \tau_{w'} \right] = \left[ \sum_{w \neq w'} \alpha_w / \text{det}(M) \text{det}(M)^{-1} \alpha_w \right] = \left[ \frac{1}{\text{det}(M)} \text{det}(M)^{-1} \alpha_w \right] = \left[ \frac{\ell_{w'}}{\text{det}(M)} \right] = \left[ \frac{\ell_{w'}}{\text{det}(M)} \right].
\]

Where we use that \( \{\alpha_w\} \) satisfy the orbifold congruence condition to get from the third line to the fourth.

Associate to each leaf \( w \in \Gamma \) a variable \( z_w \), for each node \( v \in \Gamma \) let \( \{\alpha_{wv}\} \) be a reducible choice of semigroup coefficients satisfying the orbifold congruence condition, and let \( \tau_{wv} = \frac{1}{\alpha_{wv}} \). Let \( \tau = \prod_{w \in \Gamma} z_w \) and \( \tau = \prod_{w \in \Gamma} z_w \). Let \( b \) be the number of leaves of \( \Gamma \) and let \( V \) be the subvariety of \( \mathbb{C}^b \) defined by the equations

\[
\Sigma_{e \in \mathcal{V}} Z_{wv} = 0, \quad v \text{ a node, } i = 1, \ldots, \delta_v - 2,
\]

where for all \( v \) the maximal minors of the \((\delta_v - 2) \times \delta_v\)-matrix \( \{a_{wv}\} \) have maximal rank. Likewise let \( \mathcal{V} \) be the subvariety of \( \mathbb{C}^i \) defined by the equations

\[
\Sigma_{e \in \mathcal{V}} Z_{wv} = 0, \quad v \text{ a node, } i = 1, \ldots, \delta_v - 2,
\]

with the same choice of \( a_{wv} \) as for \( V \). Define the map \( F: \mathbb{C}^i \to \mathbb{C}^i \) by

(11) \[
F(z_{w1}, \ldots, z_{wi}) = (z_{w1}'', \ldots, z_{wi}'').
\]

Then \( F(V) = \mathcal{V} \), and \( F \) is a branched abelian cover of \( \mathbb{C}^i \) with \( \text{deg}(F) = \prod_w \alpha_w \), branched over \( B = \bigcup_v \{z_v = 0 \mid \alpha_w > 1\} \). Now \( F|V: V \to \mathcal{V} \) is a branched abelian cover, branched over \( \mathcal{V} \cap B \).

Let \( X \) be a singularity which has resolution \( \Delta(M) \) and hence has link \( M \). Since the equations for \( \mathcal{V} \) satisfy the congruence conditions for \( \Delta(M) \), \( \mathcal{V} \) defines the universal abelian cover of \( X \) branched over the origin by the work of Neumann and Wahl [NW05a]. Let \( \pi: \mathcal{V} \to X \) denote the covering map, then \( \text{deg}(\pi) = |H_1(M)| = \prod_w \alpha_w \).

Now \( M \) embeds into \( X \), so choose a small enough embedding \( i \) and let \( L(V) = \pi^{-1}(i(M)) \), then \( L(V) \) is homeomorphic to the link of \( \mathcal{V} \). Let \( L(V) = \mathcal{V}^{-1}(L(V)) \), then by choosing small enough embedding \( L(V) \) is homeomorphic to the link of \( V \). Then the restrictions of the maps \( F|_{L(V)}: L(V) \to L(V) \) and \( i^{-1} \circ \pi|_{L(V)}: L(V) \to \mathcal{V} \) are abelian covers, the first branched over \( L(V) \cap B \).

Let \( f: M \to M \) be the homeomorphism which identifies \( M \) and \( M \) as topological spaces.

Definition 7.3. Let \( \pi: L(V) \to M \) be defined as \( \pi = f \circ i^{-1} \circ \pi|_{L(V)} \circ F|_{L(V)} \).

A priori \( \pi \) is just a continues map, we next turn to prove that \( \pi \) is an abelian orbifold cover.

Let \( K_w \subset M \) be the singular fiber corresponding to the leaf \( w \in \Gamma \). Let \( S \) be the singular set of \( M \), then \( S = \bigcup_{w: \alpha_w > 1} K_w \). We want to determine \( \pi^{-1}(S) \). First
Now we only need to see that the diagrams commute for each of these components.

Let $\pi: (\mathcal{M} - f^{-1}(S)) \to (M - S)$ be an abelian cover, since $f$ is the identity away from $S$. $\pi|_{L(M)}: (L(M) - B) \to (\mathcal{M} - f^{-1}(S))$ is an abelian cover, since it is the restriction of an abelian cover to a union of fibers. And $F|_{L(V)}: (L(V) - F^{-1}(B)) \to (L(M) - B)$ is an abelian cover, since it is the restriction of a branched abelian cover to the complement of the branched locus. Hence

$$\pi|_{L(V)}: (L(V) - \pi^{-1}(S)) \to (M - S)$$

is an abelian cover of degree $\deg(F) \cdot \deg(\pi) = (\prod_{w:o_w > 1} \frac{\deg(M)}{\deg(\pi)}) |H^{1,0}(M)|$.

So next we turn to what happens in the neighborhood of an orbifold curve.

**Proposition 7.4.** Let $K_w \subset M$ be an orbifold curve of degree $o_w$ and let $N_{K_w}$ be a solid torus neighborhood of $K_w$, then $\pi|_{\pi^{-1}(N_{K_w})}: \pi^{-1}(N_{K_w}) \to N_{K_w}$ is an abelian orbifold cover of degree $|H^{1,0}(M)|$.

**Proof.** We both need to show that the exist open set $U \subset \mathbb{R}^3$ and $D \subset \mathbb{R}^2$ and a branched abelian cover $\tilde{\pi}: U \to D \times S^1$ and a homomorphism $\psi: \mathbb{Z}/o\mathbb{Z} \to \mathbb{Z}/\alpha\mathbb{Z}$ such that $\tilde{\pi}$ is equivariant with respect to $\psi$ and the following diagram commutes

$$\begin{array}{ccc}
U & \xrightarrow{\tilde{\pi}} & D \times S^1 \\
\downarrow & & \downarrow \\
U/(\mathbb{Z}/a\mathbb{Z}) & \xrightarrow{\pi} & (D \times S^1)/(\mathbb{Z}/\alpha\mathbb{Z}) \\
\downarrow & & \downarrow \\
\pi^{-1}(N_{K_w}) & \xrightarrow{\pi} & N_{K_w}.
\end{array}$$

The vertical maps are the ones given from the orbifold structures of $\pi^{-1}(N_{K_w})$ and $N_{K_w}$, i.e. the upper maps are quotient maps and lower maps homeomorphisms. We can choose $D$ to be a disk, and by choosing it small enough (i.e. choosing $N_{K_w}$ small enough) we get that $\pi^{-1}(N_{K_w})$ is a disjoint union of solid torus neighborhoods $V_K$ of an singular fiber $K$, hence we can choose $U$ to be a disjoint unions of $D \times S^1$.

We now only need to see that the diagrams commute for each of these component.

Let $t_n = e^{2\pi i/n}$, then $\mathbb{Z}/a\mathbb{Z}$ acts on $(D \times S^1)$ by $(x,s) \to (t^n \cdot x, t^{o_w} \cdot s)$ where $\gcd(o_\alpha,v_\alpha) = 1$. Likewise $\mathbb{Z}/a\mathbb{Z}$ acts on $(D \times S^1)$ by $(x,s) \to (t^n \cdot x, t_s)$.

Let $N'_K = f^{-1}(N_{K_w}) \subset M$, and then $\pi^{-1}(N'_K) = \bigcup_{i=1}^{o_w} V_K$, where $V_K$ is a solid torus neighborhood of a singular fiber of degree $\alpha$. Let $\alpha' = \alpha/o_w$ and $\beta' = \beta/o_w$, then the orbifold structure on $N'_K$ is given by $\mathbb{Z}/\alpha'\mathbb{Z}$ acting on $D \times S^1$ by $(x,v_\alpha' \cdot x, t_s)$, where $v_\alpha' \beta' \equiv 1 \mod \alpha'$ and this implies that $v_\alpha \equiv v_\alpha'$ mod $\alpha'$. Since $\pi$ is an abelian orbifold cover there exist an homomorphism $\psi': \mathbb{Z}/\alpha'\mathbb{Z} \to \mathbb{Z}/\alpha'\mathbb{Z}$, and an abelian cover $\tilde{\pi}'$ such that the following diagram commutes

$$\begin{array}{ccc}
D \times S^1 & \xrightarrow{\pi} & D \times S^1 \\
\downarrow & & \downarrow \\
(D \times S^1)/(\mathbb{Z}/a'\mathbb{Z}) & \xrightarrow{\pi} & (D \times S^1)/(\mathbb{Z}/\alpha'\mathbb{Z}) \\
\downarrow & & \downarrow \\
\overline{V_K} & \xrightarrow{\pi} & N'_K.
\end{array}$$
and \( \psi(t_{a'})\overline{\pi}(x, s) = \overline{\pi}(t'_{a'}x, t_{a}s) \).

Now since \( \overline{\pi} \) is an abelian cover, it follows that \( \alpha' = \lambda\alpha' \) for some \( \lambda \) and hence, \( \psi(t_{a'}) = t_{\lambda a}^\alpha \).

Now restricting \( F \) to \( V_K \) one sees that \( F \) is a branched cover and if \( V_K = D \times S^1 \) then \( F(x, s) = (x^{a\alpha}, s) \in V_K = D \times S^1 \). Now the Seifert fibered structure on \( \overline{V}_K \) is defined by the curve of slope \( b'/a' \), and it lifts to the curve of slope \( a\alpha b'/a' \) and hence \( a = a' \) and \( b = a\alpha b' \) and hence \( v_{a'} \equiv v_{a}\mod a' \).

We can now define \( \overline{\pi} \) and \( \psi \). \( \overline{\pi}(x, s) = \overline{\pi}(x^{a\alpha}, s) \) and \( \psi(t_{a}) = t_{a'}^{a\lambda} \). So first we need to check that \( \overline{\pi} \) is equivariant with respect to \( \psi \)

\[
\psi(t_{a})\overline{\pi}(x, s) = t_{a'}^{a\lambda}\overline{\pi}(x^{a\alpha}, s) = \psi(t'_{a'})\overline{\pi}(x^{a\alpha}, t_{a}s) = \overline{\pi}(t'_{a'}x^{a\alpha}, t_{a}s) = \overline{\pi}(t'_{a'}x, t_{a}s)
\]

So what is left is just checking that the diagram commutes. Start by taking \( (x, s) \in D \times S^1 \), then the one composition is taking \( \overline{\pi}(x, s) \) and send it to the class in \( (D \times S^1)/(\mathbb{Z}/a\mathbb{Z}) \) and the other composition is sending \( (x, s) \) to the class in \( (D \times S^1)/(\mathbb{Z}/a\mathbb{Z}) \) then take \( \pi \). If we denote the class in \( (D \times S^1)/(\mathbb{Z}/k\mathbb{Z}) \) where the action is given by the integers \( k, l \) by \( [x, s]_{(k,l)} \), then we need to see that \( \pi(x, s)_{(a,b)} = [\pi(x, s)]_{(a,b)} \). Now \( \pi(x, s)_{(a,b)} = f(\overline{\pi}(F([x, s]_{(a,b)}))) \) by definition. Since the Seifert fibered structure on \ \) is given by pulling back the Seifert fibered structure on \( V_K \) by \( (x^{a\alpha}, s) \), \( F([x, s]_{(a,b)}) = [x^{a\alpha}, s]_{(a',b')} \). By construction we have that \( \overline{\pi}(x, s)_{(a',b')} = \overline{\pi}(x, s)_{(a,b)} \), and by definition \( f([x, s]_{(a',b')}) = [x, s]_{(a,b)} \), hence \( \pi(x, s)_{(a,b)} = \overline{\pi}(x, s)_{(a',b')} \), and the diagram commutes.

Last we need to calculate the degree of \( \pi|_U \). First the degree of \( \pi|_{N_{K_w}}: V_K \to N_{K_w} = o_w\lambda \). The degree of \( \overline{\pi} \) on \( \overline{V}_K \) is \( \overline{\pi}^{-1}(N_{K_w}) = \bigcup_{i=1}^m \overline{V}_K \) is \( |H_1(M)|/\lambda \), hence \( m = |H_1(M)|/\lambda \). \( U = \bigcup_{i=1}^m \prod_{w \neq w}^{o_w} o_w \) and hence the number of component of \( U \) is \( (|H_1(M)|/\lambda) \prod_{w \neq w}^{o_w} o_w \) and \( \deg(\pi|_U) = o_w\lambda(|H_1(M)|/\lambda) \prod_{w \neq w}^{o_w} o_w = |H_1(M)| \prod_{w \neq w}^{o_w} o_w = |H_1^{\text{orb}}(M)| \).

Combining the above results gives us the following theorem

**Theorem 7.5.** Let \( M \) be a rational homology sphere graph orbifold with splice diagram \( \Gamma \) satisfying the semigroup condition. Suppose there exist a graph orbifold \( M' \) also with splice diagram \( \Gamma' \), and a set of reducible semigroup coefficients \( \{\alpha\} \) for \( M' \) satisfying the orbifold congruence condition. Then the link of the complete intersection defined by \( (\Gamma', \{\alpha\}) \) is homeomorphic to the universal abelian cover of \( M \).

**Proof.** The above show that \( \pi: L(V) \to M' \) is an orbifold abelian cover of degree \( |H_1^{\text{orb}}(M')| \), and hence the universal abelian cover of \( M' \), combining this with the second main theorem of [Ped10a] gives the result.

So to prove that the splice diagram always define the universal abelian cover, one just have to show that given \( M \) with \( \Gamma(M) \) satisfying the semigroup condition, there always exist a \( M' \) with a reducible set of admissible monomials satisfying the orbifold congruence condition such that \( \Gamma(M') = \Gamma(M) \). We will in the next section show this is always true in the case of a splice diagram with only two nodes, by constructing such a \( M' \) from any two node splice diagram satisfying the semigroup condition.

**8. Algorithm for Construction an Orbifold with a Given Two Node Splice Diagram**

In this section we will make an algorithm which given any two node splice diagram \( \Gamma \) satisfying the ideal generator condition gives a graph orbifold \( M \) with
\( \Gamma(M) = \Gamma \). We will construct \( M \) by giving a plumbing diagram \( \Delta \) such that \( \Delta = \Delta(M) \). We will not give a complete plumbing diagram, but specify the orbifold degrees, the weight at the nodes, and the continued fraction associated to the strings. From this data one can obtain the complete plumbing diagram if needed. Let the splice diagram look like the following

\[
\Gamma = \begin{array}{c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

Let \( N_j = \prod_i n_{ji} \), and let \( D \) be the edge determinant of the central edge. The plumbing diagram will be given by

\[
\Delta = \begin{array}{c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

So we need to specify \( o_{ji} \), \( p_{ji} \) and \( b_j \), from the information given by \( \Gamma \).

- First chose integer \( \alpha_{ji} \) such that \( r_{1-j} = \sum_i \alpha_{ji} \frac{N_j}{n_{ji}} \); these exist since \( \Gamma \) satisfy the ideal generator condition. If \( \Gamma \) furthermore satisfy the semigroup condition, then the \( \alpha_{ji} \)'s can be chosen to be non negative, and the choice of \( \alpha_{ji} \)'s is a choice of semigroup coefficients for \( r_{1-j} \).
- Let \( \lambda_{ji} \) be the smallest integer, such that \( \lambda_{ji} n_{ji} \geq \varepsilon_j a_{ji} \) if \( D > 0 \), and \( \lambda_{ji} n_{ji} \geq -\varepsilon_j a_{ji} \) if \( D < 0 \).
- Let \( o_{ji} = \gcd(n_{ji}, \alpha_{ji}) \).
- Let \( p_{ji} = \frac{\lambda_{ji} n_{ji} + \varepsilon_j \alpha_{ji}}{\alpha_{ji}} \) if \( D > 0 \), and \( p_{ji} = \frac{\lambda_{ji} n_{ji} - \varepsilon_j \alpha_{ji}}{\alpha_{ji}} \) if \( D < 0 \).
- Let \( b_j = \sum_i \lambda_{ji} \).

Notice that \( \gcd(n_{ji}, o_{ji}, p_{ji}) = 1 \) so these choices gives a well-defined plumbing.

**Proposition 8.1.** Let \( M \) be the graph orbifold given by \( \Delta \) with the above choices, then \( \Gamma(M) = \Gamma \) and \( |H_1^{orb}(M)| = D(e) \).

**Proof.** Since the weight to the leaves in \( \Gamma(M) \) is \( (n_{ji}, o_{ji}) a_{ji} \), it is the same weight as in \( \Gamma \). Next we start by considering the case that \( D > 0 \), then the unnormalized edge determinant equation (Lemma 3.2 in [Ped10a]) implies that \( \text{det}(\Delta) > 0 \), so the only thing to check is that \( \tilde{r}_j \), the weights associated to the central string in \( \Gamma(M) \), is \( \varepsilon_{1-j} r_j \).

\[
\tilde{r}_{1-j} = \left( \prod_i o_{ji} \right) \text{det}(\Delta(M))_{n_{ji}+1} = \left( \prod_i o_{ji} \right) \left( \prod_i n_{ji}/o_{ji} \right) (b_j - \sum_i p_{ji}/n_{ji} o_{ji})
\]

\[
= n_{ji} \left( \sum_i \lambda_{ji} - \sum_i p_{ji} o_{ji}/n_{ji} \right) = \sum_i \left( \lambda_{ji} n_{ji} - \frac{\varepsilon_j \alpha_{ji}}{o_{ji}} N_j n_{ji} \right)
\]

\[
= \sum_i \varepsilon_j \alpha_{ji} \frac{N_j}{n_{ji}} = \varepsilon_j \tilde{r}_{1-j}.
\]

The case with \( D < 0 \) is similar, but now \( \text{det}(\Delta) < 0 \) and \( \tilde{r}_j = -\varepsilon_{1-j} r_j \). The last statement follows from the edge determinant equation (Corollary 3.3 in [Ped10a]), since the fiber intersection number of \( e \) is 1, or by using the above calculation to calculate \( \text{det}(\Delta) \).

Notice that \( M \) do depend on the choice of \( \alpha_{ji} \)'s.
Corollary 8.2. Let $\Gamma$ be a splice diagram satisfying the semigroup condition. Then if $M$ is a graph orbifold given by the above algorithm, then $M$ satisfy the orbifold congruence condition.

Proof. We need to check that the equations $\frac{\alpha_{ji}}{o_{ji}} \equiv -np_{ji} \mod (n_{ji}/o_{ji})$ given by (10) are satisfied. Now $n = 1$ so by definition

$$p_{ji} = \frac{\lambda_{ji}n_{ji} - \alpha_{ji}}{o_{ji}} = \lambda_{ji} \frac{n_{ji}}{o_{ji}} - \frac{\alpha_{ji}}{o_{ji}}.$$

Which implies that $\frac{\alpha_{ji}}{o_{ji}} \equiv p_{ji} \mod (n_{ji}/o_{ji})$, and hence the congruence condition is satisfied. \qed

Notice that the $\alpha_{ji}$’s are a reducible set of semigroup coefficients by the definition of the $o_{ji}$’s. So combining this with Theorem 7.5 we get that given a two node splice diagram $\Gamma$ satisfying the semigroup condition, the link of any splice diagram surface singularity is homeomorphic to the universal abelian cover of any graph orbifold with $\Gamma$ as its splice diagram.

Now this method for proving that the link of the splice diagram equations are the universal abelian covers does not easily generalize to more than two nodes. Already in the 3-node case is the semigroup (or ideal) condition not sufficient for a splice diagram to be realized by a graph orbifold. The following diagram

$$\Gamma =$$

is not the splice diagram of any graph orbifold, even though it satisfy the semigroup condition. The reason is that one has by the edge determinant equation (Corollary 3.3 in [Ped10a]) that the order of $H_{1}^{orb}(M)$ divides all edge determinants, so if $\Gamma$ where the splice diagram for some $M$, then $|H_{1}^{orb}(M)|$ would divide $D(e_{1}) = 26$ and $D(e_{2}) = 20$, and hence divide 2. Now $M$ can not be an integer homology sphere because then all weight adjacent to a node would have to be pairwise coprime according to [EN85], so $H_{1}^{orb}(M) = \mathbb{Z}/2\mathbb{Z}$. Using the topological description of the ideal generator given in section 12.1 in [NW05a], one easily sees that given any edge $e$ in $\Gamma(M)$, the product of the two ideal generators associated to each of the ends of $e$ has to divide the order of $H_{1}^{orb}(M)$, this includes edges to leaves. Now the ideal generator associated to leaf $w$ is 4, and hence do not divide the order of $H_{1}^{orb}(M)$ so we get a contradiction.

The above consideration on ideal generators leads to the following necessary condition for a splice diagram $\Gamma$ to be realized from a graph orbifold: The product of the ideal generators associated to any edge has to divide all the edge determinants.

But even this condition is not sufficient. The following splice diagram satisfy it, but is not realizble by any graph orbifold.

$$\Gamma' =$$
The edge determinant equation and the above mentioned condition implies that $|H_{1\text{orb}}^a(M)| = 30$ if $M$ is a graph orbifold realizing $\Gamma'$. Then using this and the edge determinant equation we can make a splice diagram for $M_{v_1 e_1}$ since we know that $|H_{1\text{orb}}^a(M_{v_1 e_1})| = 90$. We can continue doing this until we get that there exists a graph manifold $M'$ with $|H_1(M')| = 60$ and with the following splice diagram.

\[
\begin{array}{c}
\circ \\
6 & 21 & 10
\end{array}
\]

we now this has to be manifold, since all the singular fibers has come from the process of creating $M_{v_1 e_i}$’s, and hence does not have orbifold curves. This means $M'$ is a Seifert fibered manifold with Seifert invariants $(1, -b), (6, \beta_1), (21, \beta_2), (10, \beta_3)$, a simple calculation shows that such a Seifert fibered manifold can not have first homology group of order 60.

The failure of the last example is not as easy as the first to specify in a nice condition, so do at the moment not have a good idea on a set of necessary conditions for a splice diagram to be realized by graph orbifold. Even without this, it might still be possible to used Theorem 7.5 to prove it for more general graph orbifolds that just the once having two node splice diagram.

Another interesting question is, what are splice diagram singularities coming from diagrams as above.
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