On the two-photon contributions to $e^+e^- \rightarrow \eta\gamma$ and $e^+e^- \rightarrow \eta'\gamma$

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Abstract

Motivated by recent BABAR measurements of the $\gamma^* \rightarrow \gamma\eta$ and $\gamma^* \rightarrow \gamma\eta'$ transition form factors, we estimate two-photon exchange contributions to the corresponding cross sections. By using a phenomenological model, based on the vector meson dominance, it is argued that the expected contributions are small enough not to effect the BABAR results. As a by product we predict $Br(\eta' \rightarrow \mu^+\mu^-) = (1.4 \pm 0.2) \times 10^{-7}$. Our results might be useful also in high precision calculations of radiative corrections to the Dalitz decays of pseudoscalar mesons.
INTRODUCTION

Recently, the BABAR collaboration reported measurements of the $e^+e^- \rightarrow \eta\gamma$ and $e^+e^- \rightarrow \eta'\gamma$ cross sections and the corresponding $\gamma^* \rightarrow \gamma P$ transition form factors at a center-of-mass energy $\sqrt{s} = 10.58$ GeV. In the asymptotic limit of large photon virtualities these form factors are determined by perturbative QCD in terms of the decay constants of pseudoscalar mesons. However, the comparison of the BABAR result with the QCD predictions is hampered by the fact that the determination of the $\eta$ and $\eta'$ decay constants requires taking into account the mixing between these two states and different phenomenological models, cited in [1], give different results. Amusingly, different models are needed to reconcile asymptotic QCD predictions for the $\eta$ and $\eta'$ transition form factors with the BABAR result and neither model can explain the measured ratio of these form factors: $1.10 \pm 0.17$ versus theoretical predictions in the range from 1.6 to 2.3. This discrepancy, first of all, calls for a careful examination of possible sources of unaccounted background.

Two-photon contribution to the $e^+e^- \rightarrow P\gamma$ reaction, described by the diagrams shown in Fig.1 is one such potential background source. As was mentioned in [1], this background is expected to be very small. But no detailed calculations exist in the literature, to our knowledge, to support this conclusion. In this paper, we will try to fill up this gap.

Naively one can think that this contribution shares the $(m_e/M_P)$ suppression factor, inherent to the $P \rightarrow e^+e^-$ decay, because the graphs in Fig.1 are obtained from $e^+e^- \rightarrow \gamma^*\gamma^* \rightarrow P$ by the insertion of a bremsstrahlung photon on the electron line. But this is not the case as was realized long ago [2, 3, 4] in the context of the $P \rightarrow e^+e^-\gamma$ decay.
As we see from Fig.1, off-shell \( P \rightarrow \gamma^*\gamma^* \) amplitude is needed to calculate the two-photon contribution to the \( e^+e^- \rightarrow P\gamma \) reaction. Therefore we first discuss this amplitude.

**VECTOR MESON DOMINANCE MODEL**

We define \( \eta(Q) \rightarrow \gamma^*(k_1, \mu)\gamma^*(k_2, \nu) \) amplitude \( A_{\mu\nu} \) through the \( F_{\eta\gamma^*\gamma^*} \) form factor as follows

\[
A_{\mu\nu} = -ie^2f_\pi F_{\eta\gamma^*\gamma^*}(k_1^2, k_2^2) \epsilon_{\mu\nu\sigma\tau}k_{1\sigma}k_{2\tau}.
\]  

(1)

The pion decay constant \( f_\pi \approx 93 \text{ MeV} \) is introduced to make the form factor dimensionless. For on-shell photons and in the chiral limit, \( Q^2 = 0 \), the \( F_{\eta\gamma^*\gamma^*} \) form factor is fixed by chiral anomaly \([5, 6, 7, 8, 9]\)

\[
F_{\eta\gamma^*\gamma^*}(0, 0) = \frac{\alpha}{\sqrt{3} \pi},
\]

while for large photon virtualities QCD predicts \([10, 11, 12, 13]\) \( \sim 1/k^2 \) asymptotic behaviour.

However, in the interim region the fundamental theories tell little about the form factor behaviour and one has to rely on phenomenological models like the Vector Meson Dominance (VMD) model \([14, 15]\). Historically this model emerged in attempts to understand photon interactions with hadronic matter and in its extreme form asserts that hadronic interactions of the photon proceed exclusively through known vector mesons. At low energy region of the lightest vector mesons (\( \rho, \omega \) and \( \phi \)) VMD was extremely successful in describing a wide range of experimental data \([16]\). This success naturally rises a question whether there is a theoretical justification of the VMD from modern perspective of the Standard Model.

One of the cornerstones of the Standard Model is gauge principle. The naive \( \gamma - \rho \) direct coupling is not compatible with gauge invariance as it leads to the photon acquiring an imaginary mass. Nevertheless it was shown by Kroll, Lee, and Zumino \([17]\) that the complete vector meson dominance is consistent with gauge invariance provided these mesons are coupled only to conserved currents. In phenomenological VMD based applications an additional terms in the interaction Lagrangian implied by the Kroll-Lee-Zumino analysis (for example a photon mass term) are usually neglected as they are of higher order in \( \alpha \).

Another guiding principle of the Standard Model is symmetry. At low energies we cannot solve QCD effectively but its symmetries still guide us in constructing effective theories of colorless hadrons, which are the only relevant QCD degrees of freedom in this non-perturbative
region. Pseudoscalar mesons play a special role in this game as they are Goldstone bosons associated with the spontaneous breaking of QCD chiral symmetry (in fact would-be Goldstone bosons because the chiral symmetry is explicitly broken by nonzero quark masses). Therefore their low energy interactions, encoded into a chiral effective Lagrangian, are uniquely determined from symmetry considerations in terms of a few phenomenological parameters like the pion decay constant \( f_\pi \). The resulting Chiral Perturbation Theory (CHPT) is commonly considered nowadays as the effective field theory of the Standard Model at low energies (for review see, for example, \[18, 19, 20\]).

Although the general method how to add vector mesons (and any other particles) to chiral Lagrangians was formulated long ago \[21, 22\], usually some dynamical principle is needed to reduce number of the free parameters and enhance predictability of the theory. Hidden Local Symmetry (HLS) approach \[23, 24\] is considered to be the most convenient scheme to deal with vector mesons. Hidden local symmetries were initially discovered in the supergravity theories and after realizing that they are a common feature of any nonlinear sigma model Bando et al. suggested to consider vector mesons as dynamical gauge bosons of hidden local symmetry of nonlinear chiral Lagrangian \[23, 24\]. There is nothing special about this “hidden symmetry”. Simply it is a language that makes power counting in derivative expansion more convenient when vector mesons are light \[25\].

When electromagnetism is introduced in the HLS Lagrangian \[26\] one finds that generally there is a direct coupling of photons to charged pseudoscalars. Only for particular choice of parameters,

\[
\left( \frac{f_\pi g_{\rho\pi\pi}}{M_\rho} \right)^2 = \frac{1}{2},
\]

one recovers the complete vector meson dominance. Equation \(2\) is the celebrated Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation \[27\]. Therefore, from this perspective, VMD is not fundamental derivative of the Standard Model but rather just a lucky dynamical accident of the three flavour QCD \[28\].

Nevertheless the KSRF relation \(2\) and hence VMD is experimentally well satisfied. Therefore in our estimates we will use VMD form factors. We do not assume \( SU(3) \) relations between various coupling constants except relative phases, but determine their magnitudes from phenomenology. The relevant coupling constants are defined as follows.

Each photon-vector meson vertex gives a \(-ie g_{V\gamma} M_V^2\) factor in the matrix element. The
coupling constants $g_{V\gamma}$ are assumed to be positive and can be determined from the electronic widths $\Gamma(V \to e^+e^-)$:

\[
\Gamma(V \to e^+e^-) = \frac{4\pi\alpha^2}{3} M_V g_{V\gamma}^2 \left(1 + 2 \frac{m_e^2}{M_V^2}\right) \sqrt{1 - \frac{4m_e^2}{M_V^2}} \approx \frac{4\pi\alpha^2}{3} g_{V\gamma} M_V.
\]

Using the PDG data \cite{29}, we get

\[ g_{\rho\gamma} = 0.2014 \pm 0.0016, \quad g_{\omega\gamma} = 0.0586 \pm 0.0010, \quad g_{\phi\gamma} = 0.0747 \pm 0.0012. \quad (3) \]

The $\eta \to V_{\mu}(p)V_{\nu}(q)$ transition gives a factor

\[-i\varepsilon_V \frac{g_{\eta VV}}{f_\pi} \epsilon^{\mu\nu\sigma\tau} p_\sigma q_\tau.\]

This defines dimensionless positive constants $g_{\eta VV}$. Here $\varepsilon_{\rho,\omega} = 1$ and $\varepsilon_\phi = -1$, which together with the definition of the $g_{V\gamma}$ constants corresponds to relative phases expected from the $SU(3)$ symmetry with standard mixing angles. Neglecting $g_{\eta\phi\omega}$, which vanishes for the ideal $\phi - \omega$ mixing and nonet symmetry and hence is expected to be small, other coupling constants can be determined from the $\Gamma(V \to \eta\gamma)$ decay widths assuming VMD:

\[
\Gamma(V \to \eta\gamma) = \frac{\alpha}{24} g_{\eta VV}^2 g_{V\gamma}^2 \left(\frac{M_V}{f_\pi}\right)^2 \left(1 - \frac{M_\eta^2}{M_V^2}\right)^3 M_V.
\]

The results are

\[ g_{\eta\rho\rho} = 0.723 \pm 0.067, \quad g_{\eta\omega\omega} = 0.735 \pm 0.054, \quad g_{\eta\phi\phi} = 0.858 \pm 0.019. \quad (4) \]

Now VMD completely determines the $F_{\eta\gamma\gamma}$ form factor in terms of the above given coupling constants:

\[
F_{\eta\gamma\gamma}(k_1^2, k_2^2) = \frac{g_{\eta\rho\rho}^2 M_\rho^4}{(k_1^2 - M_\rho^2)(k_2^2 - M_\rho^2)} + \frac{g_{\eta\omega\omega}^2 M_\omega^4}{(k_1^2 - M_\omega^2)(k_2^2 - M_\omega^2)} - \frac{g_{\eta\phi\phi}^2 M_\phi^4}{(k_1^2 - M_\phi^2)(k_2^2 - M_\phi^2)}. \quad (5)
\]

As an immediate check, one can calculate two photon decay width $\Gamma(\eta \to 2\gamma)$ using this form factor. It is convenient to express the result as the following sum rule

\[
\Gamma(\eta \to 2\gamma) = \frac{9M_\eta^3}{2\alpha} \sum_{V=\rho,\omega,\phi} \varepsilon_V \frac{M_V}{M_\eta^2 - M_V^2} \left[ \frac{\Gamma(V \to \eta\gamma)}{M_\eta^2 - M_V^2} \right]^2 \left[ \frac{\Gamma(V \to e^+e^-)}{M_V^2 - M_\eta^2} \right]. \quad (6)
\]

This relation, which remains valid even for non-zero $g_{\eta\phi\omega}$, is well satisfied experimentally: it gives $\Gamma(\eta \to 2\gamma) = (0.582 \pm 0.085)$ keV, while the experimental width is \cite{29} $\Gamma(\eta \to 2\gamma) = (0.510 \pm 0.026)$ keV.
For $\eta'$ couplings, we can use $\Gamma(\phi \to \eta'\gamma)$, $\Gamma(\eta' \to \omega\gamma)$ and $\Gamma(\eta' \to \rho\gamma)$ as inputs, along with

$$\Gamma(\eta' \to V\gamma) = \frac{\alpha}{8} g_{\eta'VV}^2 g_{V\gamma}^2 \left( \frac{M_{\eta'}}{f_\pi} \right)^2 \left( 1 - \frac{M_V^2}{M_{\eta'}^2} \right)^3 M_{\eta'},$$

to obtain

$$g_{\eta'\rho\rho} = 0.624 \pm 0.044, \quad g_{\eta'\omega\omega} = 0.724 \pm 0.067, \quad g_{\eta'\phi\phi} = 0.886 \pm 0.059. \quad (7)$$

In terms of these coupling constants, the $F_{\eta'\gamma\rightarrow\gamma}$ form factor has the form

$$F_{\eta'\gamma\rightarrow\gamma}(k_1^2, k_2^2) = \frac{g_{\eta'\rho\rho} g_{\eta'\rho\rho}^2 M_{\rho}^4}{(k_1^2 - M_{\rho}^2)(k_2^2 - M_{\rho}^2)} + \frac{g_{\eta'\omega\omega} g_{\eta'\omega\omega}^2 M_{\omega}^4}{(k_1^2 - M_{\omega}^2)(k_2^2 - M_{\omega}^2)} + \frac{g_{\eta'\phi\phi} g_{\eta'\phi\phi}^2 M_{\phi}^4}{(k_1^2 - M_{\phi}^2)(k_2^2 - M_{\phi}^2)}. \quad (8)$$

Relative phases are again the ones that follow from the nonet-ansatz SU(3) symmetric interaction Lagrangian

$$\mathcal{L} = g e^{\mu\nu\tau} S_p[(\partial_\mu W_\nu)(\partial_\sigma W_\tau)P],$$

where

$$W = \frac{1}{\sqrt{2}} (\rho^3 \lambda_3 + \omega(8) \lambda_8) + \frac{1}{\sqrt{3}} \omega(1), \quad P = \frac{1}{\sqrt{2}} \eta(8) \lambda_8 + \frac{1}{\sqrt{3}} \eta(1),$$

$\lambda_3, \lambda_8$ being the standard Gell-Mann matrices, and the $\eta - \eta'$ mixing is given by

$$\eta(8) = \cos \theta_P \eta + \sin \theta_P \eta', \quad \eta(1) = \cos \theta_P \eta' - \sin \theta_P \eta, \quad \theta_P \approx -20^\circ,$$

while the $\phi - \omega$ mixing is assumed to be ideal:

$$\omega(8) = \sqrt{\frac{2}{3}} \phi + \sqrt{\frac{1}{3}} \omega, \quad \omega(1) = \sqrt{\frac{2}{3}} \omega - \sqrt{\frac{1}{3}} \phi.$$
$\eta \to \mu^+\mu^-$ DECAY

There is an intensive literature devoted to the rare decays of pseudoscalar mesons into a lepton pair (for a review and references see, for example, [30, 31, 32, 33, 34]). Detailed pedagogical calculation in the framework of VMD is given in [35]. In somewhat different manner, this calculation is reproduced in [36]. We follow these references in spirit but differ in technical details.

The leading contribution to the $\eta(Q) \to \mu^+(Q - p, s_+) + \mu^-(p, s_-)$ decay comes from the diagram shown in Fig.2

![Diagram](image)

FIG. 2: The leading contribution to the $\eta \to \mu^+\mu^-$ decay.

The corresponding Feynman amplitude is

$$A = \frac{e^4}{f^2} \bar{u}(p, s_-) M v(Q - p, s_+),$$

where

$$M = \int \frac{dk}{(2\pi)^4} \frac{\gamma_\mu (\hat{p} - \hat{k} + m_\mu) \gamma_\nu e^{\mu_\sigma \tau k_\sigma} Q_\tau}{[(p - k)^2 - m_\mu^2] k^2 (Q - k)^2} F_{\eta^{++}, \eta^+}(k^2, (Q - k)^2).$$

The lepton pair from this decay has total angular momentum $J = 0$ and hence is either in singlet $^1S_0$ or in triplet $^3P_0$ state. But the triplet state has CP-parity $(-1)^{s+1} = 1$ which does not matches the negative CP-parity of the $\eta$ meson. Therefore, assuming CP invariance, $v(Q - p, s_+)\bar{u}(p, s_-)$ in (10) can be replaced by the projection operator to the singlet state for the outgoing $\mu^+\mu^-$ system

$$P(Q - p, p) = \frac{1}{\sqrt{2}} [v(Q - p, +)\bar{u}(p, -) + v(Q - p, -)\bar{u}(p, +)].$$

This projection operator was calculated in [31] with the result

$$P(p_+, p_-) = \frac{1}{2\sqrt{2t}} \left[ -2m_\mu (\hat{p}_+ + \hat{p}_-) \gamma_5 + \frac{1}{2} \epsilon_{\mu \nu \sigma \tau} (p_-^\tau p_+^\nu - p_+^\nu p_-^\tau) \sigma^{\mu \nu} + t \gamma_5 \right].$$
where \( t = (p_+ + p_-)^2 \) and \( \sigma^{\mu\nu} = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \). Therefore

\[
\mathcal{A} = \frac{e^4}{f_\pi} \int \frac{dk}{(2\pi)^4} \frac{\epsilon^{\mu\nu\sigma} k_\sigma Q_\tau L_{\mu\nu}}{[(p - k)^2 - m_\mu^2] k^2 (Q - k)^2} F_{\eta\gamma^*\gamma^*}(k^2, (Q - k)^2),
\]

with

\[
L_{\mu\nu} = \text{Sp} \left[ \mathcal{P} \gamma_\mu (\not p - \not k + m_\mu) \gamma_\nu \right] = -2i\sqrt{2} \frac{m_\mu}{M_\eta} \epsilon_{\mu\nu\sigma} k^\sigma Q^\tau.
\]

But then

\[
\epsilon^{\mu\nu\sigma} k_\sigma Q_\tau L_{\mu\nu} = \frac{8i m_\mu}{\sqrt{2} M_\eta} \left( M_\eta^2 k^2 - (k \cdot Q)^2 \right).
\]

In the above expressions REDUCE [37] was used to perform Dirac algebra and calculate traces.

Finally, we get in the standard way

\[
\Gamma(\eta \to \mu^+ \mu^-) = \frac{2\alpha^4}{\pi} M_\eta \left( \frac{m_\mu}{f_\pi} \right)^2 \sqrt{1 - \frac{4m_\mu^2}{M_\eta^2}} |R|^2,
\]

where \([38, 39]\)

\[
R = \frac{i}{\pi^2} \int \frac{dk}{M_\eta^2} \frac{M_\eta^2 k^2 - (k \cdot Q)^2}{k^2 (Q - k)^2 [(p - k)^2 - m_\mu^2]} F_{\eta\gamma^*\gamma^*}(k^2, (Q - k)^2).
\]

Remembering VMD expression (5) for the \( F_{\eta\gamma^*\gamma^*} \) form factor, we can write

\[
R = g_{\eta\rho\rho} g_\gamma^2 I(M_\rho^2, M_\rho^2) + g_{\eta\rho\omega} g_\omega^2 I(M_\omega^2, M_\omega^2) - g_{\eta\phi\phi} g_\phi^2 I(M_\phi^2, M_\phi^2),
\]

where the master integral has the form

\[
I(M_1^2, M_2^2) = \frac{i}{\pi^2} \int \frac{dk}{M_\eta^2} \frac{M_\eta^2 k^2 - (k \cdot Q)^2}{M_1^2 M_2^2} \frac{M_1^2 M_2^2}{(Q - k)^2 [(p - k)^2 - m_\mu^2] (k^2 - M_1^2) [(Q - k)^2 - M_2^2]}.
\]

The imaginary part of this integral can be calculated by using the Cutkosky rules [40, 41]. When intermediate masses \( M_{1,2} \) are greater than the \( \eta \)-meson mass, only two-photon cut contributes to the discontinuity of \( I(M_1^2, M_2^2) \) (which is twice its imaginary part) and, therefore,

\[
disc I(M_1^2, M_2^2) = \frac{(-2\pi i)^2}{\pi^2} \int \frac{dk}{k^2} \frac{\delta_+(k^2) \delta_+[(Q - k)^2]}{k^2 - 2p \cdot k} \frac{M_1^2 M_2^2}{(k^2 - M_1^2) [(Q - k)^2 - M_2^2]},
\]

where

\[
\delta_+(k^2) = \Theta(k_0) \delta(k^2) = \frac{1}{2k_0} \delta(k_0 - |\vec{k}|).
\]
Because of δ-functions, calculation is straightforward and gives the well known model-independent result

\[ Im I(M_1, M_2) = \frac{1}{2} disc I(M_1^2, M_2^2) = \frac{\pi}{4\beta} \ln \frac{1 - \beta}{1 + \beta}, \quad \beta = \sqrt{1 - \frac{4m_\mu^2}{M_\eta^2}}. \] (16)

Calculation of the real part is much more tricky and as a first step involves the following algebraic identity

\[
\frac{[k^2 Q^2 - (k \cdot Q)^2] M_1^2 M_2^2}{D_1 D_2 D_3 D_4 D_5} = -\frac{\lambda(M_1^2, M_2^2, Q^2)}{4D_3 D_4 D_5} - \frac{(Q^2)^2}{4D_1 D_2 D_3} + \frac{(Q^2 - M_1^2)^2}{4D_2 D_3 D_4} + \frac{(Q^2 - M_2^2)^2}{4D_1 D_2 D_3 D_5} - \frac{M_1^2}{4} \left[ \frac{1}{D_3 D_5} - \frac{1}{D_2 D_3} \right] - \frac{M_2^2}{4} \left[ \frac{1}{D_3 D_4} - \frac{1}{D_1 D_3} \right], \] (17)

where

\[ D_1 = k^2, \quad D_2 = (Q - k)^2, \quad D_3 = k^2 - 2p \cdot k, \quad D_4 = k^2 - M_1^2, \quad D_5 = (Q - k)^2 - M_2^2, \]

and \( \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \) is the triangle function. Using this identity and introducing the dimensionless variables

\[ r = \frac{m_\mu^2}{M_\eta^2}, \quad r_1 = \frac{M_1^2}{M_\eta^2}, \quad r_2 = \frac{M_2^2}{M_\eta^2}, \quad \rho_1 = \frac{m_\mu^2}{M_1^2}, \quad \rho_2 = \frac{m_\mu^2}{M_2^2}, \]

we get

\[ I(M_1^2, M_2^2) = J(r_1, r_2) + J(0, 0) - J(r_1, 0) - J(0, r_2) - \frac{r_1}{4} [f_1(\rho_2) - f_1(0)] - \frac{r_2}{4} [f_2(\rho_1) - f_2(0)], \] (18)

with

\[ f_1(\rho_2) = \frac{i}{\pi^2} \int \frac{dk}{[k^2 - 2p \cdot k][((Q - k)^2 - M_2^2)]}, \quad f_2(\rho_1) = \frac{i}{\pi^2} \int \frac{dk}{[k^2 - 2p \cdot k][k^2 - M_1^2]}; \]

and

\[ J(r_1, r_2) = \frac{1}{4} \lambda(1, r_1, r_2) g(r_1, r_2), \quad g(r_1, r_2) = -\frac{i}{\pi^2} \int \frac{M_\eta^2 dk}{[k^2 - 2p \cdot k][k^2 - M_1^2][((Q - k)^2 - M_2^2)].} \]

Integrals with two denominators \((f_1 \text{ and } f_2)\) are easy to calculate by using

\[ \frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1 - x)B]^2} \]

and the dimensionally regularized integral \((\gamma_E \text{ is the Euler constant})\)

\[ \frac{i}{\pi^2} \int \frac{dk}{[k^2 - A]^2} = -\Gamma(\epsilon/2) / (\pi A)^{\epsilon/2} \approx -\left( \frac{2}{\epsilon - \gamma_E - \ln \pi - \ln A} \right), \quad \epsilon = 4 - d \to 0. \]
The result is\[35, 36, 123\]

\[
f_1(\rho_2) - f_1(0) = \int_0^1 dx \ln \frac{m_1^2 x^2 + M_2^2 (1 - x)}{m_2^2 x^2} = -\frac{1}{2\rho_2} \left[ \ln \rho_2 + \sqrt{1 - 4\rho_2} \ln \frac{1 + \sqrt{1 - 4\rho_2}}{1 - \sqrt{1 - 4\rho_2}} \right]. \quad (19)
\]

and

\[
f_2(\rho_1) - f_2(0) = -\frac{1}{2\rho_1} \left[ \ln \rho_1 + \sqrt{1 - 4\rho_1} \ln \frac{1 + \sqrt{1 - 4\rho_1}}{1 - \sqrt{1 - 4\rho_1}} \right]. \quad (20)
\]

As for the integral with three denominators \(g(r_1, r_2)\), we use Feynman parameterization

\[
\frac{1}{D_3D_4D_5} = 2 \int_0^1 dx \int_0^x dy \frac{1}{[(x-y)D_3 + (1-x)D_4 + yD_5]^{3}}
\]

and the integral

\[
\frac{i}{\pi^2} \int \frac{dk}{[k^2 - A]^3} = \frac{1}{2A}
\]

to get

\[
g(r_1, r_2) = \int_0^1 dx \int_0^x dy \frac{1}{r \left( x^2 + y^2 \right) - \frac{1+\beta^2}{2} xy + y + r_1 x - r_2 y - r_1}
\]

Let us shift the \(y\)-variable as follows:

\[
y \to y + \alpha x, \quad \alpha = -\frac{1 - \beta}{1 + \beta}.
\]

Then

\[
g(r_1, r_2) = \int_0^1 dx \int_{\alpha x}^{(1-\alpha)x} dy \frac{1}{f(y) x + h(y)}
\]

with

\[
f(y) = r_1 + \frac{1 - \beta}{1 + \beta} (r_2 - 1) - \beta y, \quad h(y) = -ry^2 + (1 - r_2)y - r_1.
\]

But

\[
\int_0^1 dx \int_{\alpha x}^{(1-\alpha)x} dy = \int_0^1 dx \int_0^{(1-\alpha)x} dy - \int_0^1 dx \int_{\alpha x}^{1-\alpha} dy = \int_0^1 dy \int_0^{1-\alpha} dx - \int_{y/(1-\alpha)}^1 dy \int_0^{1-\alpha} dx
\]

and after performing the trivial \(x\)-integration we end up with

\[
g(r_1, r_2) = \int_0^{2/(1+\beta)} \frac{dy}{f(y)} \left\{ \ln [f(y) + h(y)] - \ln \left[ \frac{f(y) y (1 + \beta)}{2} + h(y) \right] \right\} -
\]

\[
\int_0^{(1-\beta)/(1+\beta)} \frac{dy}{f(y)} \left\{ \ln [f(y) + h(y)] - \ln \left[ \frac{f(y) y (1 + \beta)}{1 - \beta} + h(y) \right] \right\}. \quad (21)
\]
We need only the real part and, therefore, integrals in (21) are of the type \((\alpha, a, b, A, B, C\) are some real constants here)

\[
F(\alpha; a, b; A, B, C) = \text{Re} \int_0^\alpha \frac{dy}{ay + b} \ln(Ay^2 + By + C). \tag{22}
\]

A particular case of such type of integral

\[
Z = \int_0^1 \frac{dy}{y - y_0} \left[ \ln (y - y_1)(y - y_2) - \ln (y_0 - y_1)(y_0 - y_2) \right]
\]

was considered in detail in [42] (see also [43]). Using their result

\[
\text{Re} Z = \text{Re} \left[ Li_2 \left( \frac{y_0}{y_0 - y_1} \right) - Li_2 \left( \frac{y_0 - 1}{y_0 - y_1} \right) + Li_2 \left( \frac{y_0}{y_0 - y_2} \right) - Li_2 \left( \frac{y_0 - 1}{y_0 - y_2} \right) \right],
\]

we can readily calculate (22) in terms of dilogarithms:

\[
F(\alpha; a, b; A, B, C) = \frac{1}{a} \text{Re} \left\{ \ln \left( 1 + \alpha \frac{a}{b} \right) \ln \left( A \frac{b^2}{a^2} - B \frac{b}{a} + C \right) + \right. \\
- Li_2 \left( \frac{b}{b + y_1 a} \right) - Li_2 \left( \frac{b + \alpha a}{b + y_1 a} \right) + Li_2 \left( \frac{b}{b + y_2 a} \right) - Li_2 \left( \frac{b + \alpha a}{b + y_2 a} \right) \left\}, \tag{23}
\]

where

\[
y_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \quad y_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.
\]

Returning to (21), we get finally

\[
\text{Re} g(r_1, r_2) = F \left( \frac{2}{1 + \beta}; -\beta, r_1 + \frac{1 - \beta}{1 + \beta} (r_2 - 1); -r, 1 - \beta - r_2, \frac{1 - \beta}{1 + \beta} (r_2 - 1) \right) - \\
F \left( \frac{1 - \beta}{1 + \beta}; -\beta, r_1 + \frac{1 - \beta}{1 + \beta} (r_2 - 1); -r, 1 - \beta - r_2, \frac{1 - \beta}{1 + \beta} (r_2 - 1) \right) - \\
F \left( \frac{2}{1 + \beta}; -\beta, r_1 + \frac{1 - \beta}{1 + \beta} (r_2 - 1); -\frac{(1 + \beta)^2}{4}, \frac{1 + \beta}{2} (1 + r_1 - r_2), -r_1 \right) + \\
F \left( \frac{1 - \beta}{1 + \beta}; -\beta, r_1 + \frac{1 - \beta}{1 + \beta} (r_2 - 1); -\frac{(1 + \beta)^3}{4(1 - \beta)}, \frac{1 + \beta}{1 - \beta} r_1, -r_1 \right). \tag{24}
\]

Some arguments of dilogarithms in (24) are complex. Therefore for numerical evaluation we need an algorithm to evaluate this function for complex argument. We use the algorithm described in [42] (see [44] for another algorithm).

First of all the argument of the dilogarithm is brought in the region \(|x| \leq 1, -1 \leq \text{Re}(x) \leq \frac{1}{2}\) by using the functional identities

\[
\text{Li}_2(x) = -\text{Li}_2(1 - x) + \frac{\pi^2}{6} - \ln(x) \ln(1 - x), \quad \text{Li}_2(x) = -\text{Li}_2 \left( \frac{1}{x} \right) - \frac{\pi^2}{6} - \frac{1}{2} \ln^2(-x).
\]
Then the series expansion
\[ Li_2(x) = \sum_{n=0}^{\infty} B_n \frac{z^{n+1}}{(n+1)!}, \]
truncated at \( n = 20 \), is used with \( B_n \) Bernoulli numbers and \( z = -\ln(1-x) \).

Now we have all ingredients at hand to calculate numerically \( \Gamma(\eta \to \mu^+\mu^-) \). The result is
\[ Br(\eta \to \mu^+\mu^-) = \frac{\Gamma(\eta \to \mu^+\mu^-)}{\Gamma_{\eta}} = (5.2 \pm 1.2) \times 10^{-6}. \] (25)

The quoted uncertainty is dominated by uncertainty in the \( g_{\eta\rho\rho} \) coefficient. The experimental number is \( Br(\eta \to \mu^+\mu^-) = (5.8 \pm 0.8) \times 10^{-6} \) and again we observe a good agreement. However, it should be mentioned that the dominant contribution into (25) is given by the imaginary part of the amplitude which is in fact model independent:
\[ Re(R) \approx -0.015, \quad Im(R) \approx -0.074. \]

To check our formulas and computer code, we have make sure that they reproduce numerical values of \( 2I(M_1^2, M_2^2) \) and of the analogous integral for the \( \pi^0 \to e^+e^- \) decay given in [30].

Analogous considerations apply to the \( \eta' \to \mu^+\mu^- \) decay if we use the form factor (8) with coupling constants (7). But now the imaginary part of the amplitude has an additional contribution from the on-shell \( \rho\gamma \) and \( \omega\gamma \) intermediate states. This contribution is calculated by using the Cutkosky rules with the result
\[ \Delta Im I(M_1^2, M_2^2) = -\frac{\pi}{4\beta} \left[ \left( 1 - \frac{M_1^2}{M_{\eta'}^2} \right)^2 + \left( 1 - \frac{M_2^2}{M_{\eta'}^2} \right)^2 \right] \ln \frac{1-\beta}{1+\beta}. \] (26)

The numerical calculation yields the branching ratio
\[ Br(\eta' \to \mu^+\mu^-) = (1.4 \pm 0.2) \times 10^{-7}. \] (27)

Now the imaginary part of the amplitude is only 1.5-times larger in magnitude than the real part:
\[ Re(R) \approx 0.064, \quad Im(R) \approx -0.093. \]

Unfortunately there is no experimental number for the \( \eta' \to \mu^+\mu^- \) branching ratio to compare with (27).
TWO-PHOTON CONTRIBUTION TO $e^+e^- \rightarrow \eta\gamma$: CONSTANT FORM-FACTOR

The amplitude of the $e^+(p_+, s_+) + e^-(p_-, s_-) \rightarrow \eta(q) + \gamma(k, \epsilon)$ process is given by

$$A = \bar{v}(p_+, s_+) \Gamma^\mu(p_+, p_-, k) u(p_-, s_-) \epsilon_\mu^*,$$

where gauge symmetry and invariance under parity and charge conjugation dictate the following decomposition of $\Gamma^\mu$ in terms of four independent invariant form factors (up to irrelevant additional structures proportional to $k^\mu$) [45]

$$\Gamma^\mu = P(\chi_-, \chi_+) \left[ (k \cdot p_+) p_\mu^- - (k \cdot p_-) p_\mu^+ \right] \gamma_5 - i T(\chi_-, \chi_+) \sigma^{\mu\nu} k_\nu \gamma_5 +$$

$$A_+(\chi_-, \chi_+) \left[ k p_\mu^+ - (k \cdot p_+ \gamma^\mu) \right] \gamma_5 - A_-(\chi_-, \chi_+) \left[ k p_\mu^- - (k \cdot p_- \gamma^\mu) \right] \gamma_5,$$

with

$$P(\chi_-, \chi_+) = P(\chi_+, \chi_-), \quad A_+(\chi_-, \chi_+) = A_-(\chi_+, \chi_-), \quad T(\chi_-, \chi_+) = T(\chi_+, \chi_-).$$

Here $\chi_- = 2k \cdot p_-$ and $\chi_+ = 2k \cdot p_+.$

The invariant form factors $P, A_\pm$ and $T$ can be projected out from $\Gamma^\mu$ by a suitable projector operator $\Lambda^\mu_F$, $F = P, A_\pm, T$ according to the the formula

$$F = Sp \left[ \Lambda^\mu_F (\hat{p}_+ - m_e) \Gamma^\mu (\hat{p}_- + m_e) \right]$$

We are interested in the $m_e \rightarrow 0$ limit and in this limit these projectors have the form (our expressions differ somewhat from ones given in [45])

$$\Lambda^\mu_P = \frac{1}{2\Delta^2 \delta^2} \left[ -i \epsilon^{\mu\nu\sigma\tau} k_\nu p_+ p_\sigma p_- \tau - \left( \frac{\Delta^2}{\delta^2 k \cdot p_+ k \cdot p_-} - 2 \right) \Delta^\mu \gamma_5 \right],$$

$$\Lambda^\mu_{A_\pm} = \frac{-1}{16(\Delta^2)^2} \left\{ 2\Delta^2 \pm p^2 k \cdot (p \mp \delta) k \cdot \delta \left[ k \cdot p \gamma^\mu - \hat{k} p^\mu \right] \right. \pm$$

$$p^2 \left( 2 \frac{\Delta^2}{\delta^2} - k \cdot (p \mp \delta) k \cdot p \right) \left( k \cdot \delta \gamma^\mu - \hat{k} \delta^\mu \right) \right\} \gamma_5,$$

$$\Lambda^\mu_T = \frac{i}{4\Delta^2} \epsilon^{\mu\nu\sigma\tau} k_\nu p_+ p_\sigma p_- \tau,$$

where

$$\Delta = (k \cdot p_+) p_- - (k \cdot p_-) p_+, \quad \delta = p_+ - p_-, \quad p = p_+ + p_-.$$
Using these projector operators and calculating traces by means of REDUCE \(^{37}\), it can be found that \(P\) and \(T\) form factors vanish in the \(m_e \to 0\) limit for diagrams shown in Fig. I. Therefore, in this limit we get in the standard way

\[
\frac{d\sigma^{(2\gamma)}(e^+e^- \to \eta \gamma)}{d\Omega} = \frac{s^2}{4^5 \pi^2} \left(1 - \frac{M_\gamma^2}{s}\right)^3 \left[(1 + \cos \theta)^2 |A_+|^2 + (1 - \cos \theta)^2 |A_-|^2\right],
\]

(32)

where \(\sigma^{(2\gamma)}(e^+e^- \to \eta \gamma)\) denotes purely two-photon contribution into the \(e^+e^- \to \eta \gamma\) cross section, \(s = (p_- + p_+)^2\) and \(\theta\) is the flight angle of the photon with respect to the electron momentum \(\vec{p}_-\) in the center of mass frame.

To calculate (32), we need to find out only the \(A_-(\chi_-, \chi_+)\) form factor, because of the charge conjugation symmetry relations \(^{29}\). The following method \(^{3}\) for the form factor determination seems to be more convenient than the direct use of the suitable projector.

In the massless electron limit

\[
\Gamma^\mu = [A_1 p_-^\mu - A_2 p_+^\mu] \hat{k} \gamma_5 + A_3 \gamma^\mu \gamma_5,
\]

(33)

where \(A_1 = -A_-\), \(A_2 = -A_+\) and \(A_3 = k \cdot p_+ A_2 - k \cdot p_+ A_2\). Using the identity

\[
\gamma_\mu \epsilon^{\mu\nu\sigma\tau} = \frac{1}{2} \left(\gamma^\nu \gamma^\sigma \gamma^\tau - \gamma^\nu \gamma^\sigma \gamma^\tau \right)(i \gamma_5)
\]

it can be shown \(^{3}\) that the first two diagrams from Fig. I contribute to \(A_3\) alone. Therefore to determine \(A_- = -A_1\) it is sufficient to consider only the third diagram with photon emission from the internal line. The contribution of this diagram in the massless electron limit looks like

\[
\Gamma^{(c)}_\mu = \frac{e^5}{F_\pi} \frac{d\gamma_\mu(\hat{p}_+ - \hat{l}) \gamma_\mu(\hat{p}_+ - \hat{l} - \hat{k}) \gamma_\lambda e^{\nu\lambda\sigma\tau} l_{\sigma\tau} F_{\eta\gamma^*\gamma^*}(l^2, (q - l)^2). \quad (34)
\]

As follows from \(^{33}\), the coefficient of the \(k_i p_{-\mu}\) term in \(Sp(\gamma_\mu \hat{p}_- \Gamma_\mu \gamma_5 \hat{p}_+)\) is \(-4 p_- \cdot p_+ A_1 = 4 p_- \cdot p_+ A_-\). Besides, \(^{33}\) is free from ultraviolet divergences even for constant form factor \(F_{\eta\gamma^*\gamma^*}\) (point-like \(\eta\)). Therefore in this section we assume, as in \(^{3}\), that the \(F_{\eta\gamma^*\gamma^*}\) is just a constant. Then, to find out the \(A_-\) form factor, we combine \(D_1 = l^2\), \(D_2 = (q - l)^2\), \(D_3 = (p_+ - l)^2\) and \(D_4 = (p_+ - l - k)^2\) denominators in \(^{33}\) by using

\[
\frac{1}{D_1 D_2 D_3 D_4} = 3! \int_0^1 dx_1 \cdots \int_0^1 dx_4 \delta \left(\sum x_i - 1\right) \frac{1}{[x_1 D_1 + x_2 D_2 + x_3 D_3 + x_4 D_4]^4},
\]

shift the variables in the momentum integral according to

\[
l \to l + x_2 p_- + (1 - x_1)p_+ - (x_2 + x_4)k,
\]

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calculate the trace $Sp(\gamma_\mu \hat{p}_- \Gamma_\mu^{(c)} \gamma_5 \hat{p}_+)$ with the help of REDUCE \cite{37}, replace in resulting momentum integrals

$$l_\mu l_\nu \to \frac{1}{4} g_{\mu\nu} l^2, \quad l_\mu l_\nu l_\lambda \to 0, \quad l_\mu \to 0,$$

and separate the coefficient of the $k_\nu p_{-\mu}$ term. As a result we obtain

$$A_- = \frac{6e^5 F_{\gamma\gamma^*\gamma^*}}{f_\pi} \int_0^1 dx_1 \cdots \int_0^1 dx_4 \delta \left( \sum x_i - 1 \right) \times$$

$$\frac{i}{(2\pi)^4} \int dl \frac{(1 - 4x_2 - 2x_4) l^2 - 2sx_2^2x_3 + 2\chi_+(x_2 + x_4 - 1)(x_2x_3 - x_1x_4)}{[l^2 - x_2x_3\chi_- - x_1x_4\chi_+ + M_\eta^2 x_1x_2]^4}, \quad (35)$$

Our result (35) differs somewhat from the corresponding expression in \cite{3}. For this reason we have cross-checked it by an independent calculation where part of the above given program was performed manually.

Momentum integrals read

$$\frac{i}{\pi^2} \int dl \frac{l^2}{[l^2 - A]^4} = \frac{1}{3A}, \quad \frac{i}{\pi^2} \int dl \frac{1}{[l^2 - A]^4} = -\frac{1}{6A^2}.$$

Therefore

$$A_- = \frac{e^5 F_{\gamma\gamma^*\gamma^*}}{8\pi^2 f_\pi} \int_0^1 dx_1 \cdots \int_0^1 dx_4 \delta \left( \sum x_i - 1 \right) \times$$

$$\left[ \frac{1 - 4x_2 - 2x_4}{A} + \frac{s x_2^2x_3 - \chi_+(x_2 + x_4 - 1)(x_2x_3 - x_1x_4)}{A^2} \right],$$

where

$$A = x_2x_3\chi_- + x_1x_4\chi_+ - M_\eta^2 x_1x_2.$$

Let us introduce dimensionless variables

$$X_+ = 1 + \frac{\chi_+}{M_\eta^2}, \quad X_- = 1 + \frac{\chi_-}{M_\eta^2}, \quad (36)$$

and note that

$$\frac{1}{M_\eta^2} \frac{\partial}{\partial X_+} \frac{1}{A} = \frac{x_1x_4}{A^2}, \quad \frac{1}{M_\eta^2} \frac{\partial}{\partial X_-} \frac{1}{A} = -\frac{x_2x_3}{A^2}.$$

Then

$$A_- = A_-^{(1)} - A_-^{(2)}, \quad (37)$$

with

$$M_\eta^2 A_-^{(1)} = \frac{e^5 F_{\gamma\gamma^*\gamma^*}}{8\pi^2 f_\pi} \left[ 2 + (X_+ + X_- - 1) \frac{\partial}{\partial X_-} \right] I_2(X_+, X_-) +$$
\[
\left[ 2 + (1 - X_+) \left( \frac{\partial}{\partial X_-} - \frac{\partial}{\partial X_+} \right) \right] [I_2(X_+, X_-) + I_4(X_+, X_-)] ,
\]
and
\[
M_\gamma^2 A^{(2)} = \frac{e^5 F_{\gamma\gamma'}}{8\pi^2 f_\pi} \left[ 1 + (1 - X_+) \left( \frac{\partial}{\partial X_-} - \frac{\partial}{\partial X_+} \right) \right] I_0(X_+, X_-).
\]

Here the master integrals are
\[
I_0(X_+, X_-) = \int_0^1 dx_1 \cdots \int_0^1 dx_4 \delta \left( \sum x_i - 1 \right) \frac{1}{x_1 x_2 + (1 - X_-) x_2 x_3 + (1 - X_+) x_1 x_4},
\]
\[
I_n(X_+, X_-) = \int_0^1 dx_1 \cdots \int_0^1 dx_4 \delta \left( \sum x_i - 1 \right) \frac{x_n}{x_1 x_2 + (1 - X_-) x_2 x_3 + (1 - X_+) x_1 x_4}.
\]

In \(37\) the \(A^{(1)}\) part is the same as in \(3\), while the \(A^{(2)}\) part is absent in \(3\).

The master integrals \(38\) can be considered as integrals over a tetrahedron shaped 3-dimensional domain in the \((x_1, x_2, x_3)\)-space. By means of variable transformation \(39\)
\[
x_1 = xz, \quad x_2 = y(1 - z), \quad x_3 = z(1 - x), \quad x_4 = (1 - z)(1 - y),
\]
with the Jacobian
\[
\left| \frac{\partial(x_1, x_2, x_3)}{\partial(x, y, z)} \right| = z(1 - z),
\]
the integration domain transforms into a unit cube in the \((x, y, z)\)-space and we obtain
\[
I_2 = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{y(1 - z)}{(1 - X_+) x + (1 - X_-) y + (X_- + X_+ - 1) xy} = \frac{1}{2} \int_0^1 dy \ln \left( \frac{1 - X_+ + X_+ y + i\epsilon}{1 - X_+ + (X_- + X_+ - 1) y} \right) - \ln \left( \frac{(1 - X_-) y + i\epsilon}{1 - X_+ + (X_- + X_+ - 1) y} \right)
\]
\[
\frac{1}{2} \int_0^1 dy \ln \left( \frac{1 - X_+ + X_+ y + i\epsilon}{1 - X_+ + (X_- + X_+ - 1) y} \right) - \ln \left( \frac{(1 - X_-) y + i\epsilon}{1 - X_+ + (X_- + X_+ - 1) y} \right) y.
\]

Analogously
\[
I_2 + I_4 = \frac{1}{2} I_0 = \frac{1}{2} \int_0^1 dx \ln \left( \frac{1 - X_+ + X_- x + i\epsilon}{1 - X_+ + (X_- + X_+ - 1) x} \right) - \ln \left( \frac{(1 - X_-) x + i\epsilon}{1 - X_+ + (X_- + X_+ - 1) x} \right).
\]

In the above formulas the \(i\epsilon\) prescription for propagators was explicitly restored.

First of all let us calculate the imaginary part of the amplitude which is in fact model independent as far as it is dominated by the on-shell two-photon intermediate state. In \(40\) and \(41\) the imaginary parts originate from negative arguments of the logarithms. Therefore
\[
Im \left( I_2 + I_4 \right) = -\frac{\pi}{2} \int_{1 - 1/X_-}^1 \frac{dx}{1 - X_+ + (X_- + X_+ - 1) x} = \frac{\pi}{4 X_-}.
\]
\[-\frac{\pi}{2(X_- + X_+ - 1)} \ln \frac{X_- X_+}{(X_- - 1)(X_+ - 1)},\]

and

\[Im I_2 = -\frac{\pi}{2} \int_{1-1/X_+}^{1} \frac{y \, dy}{1 - X_+ + (X_- + X_+ - 1)y} = \]

\[= -\frac{\pi}{2X_+} \left[ \frac{1}{X_+ + X_- - 1} + \frac{X_+(X_+ - 1)}{(X_+ + X_- - 1)^2} \ln \frac{X_- X_+}{(X_- - 1)(X_+ - 1)} \right].\]

Inserting these expressions into (37) we get finally

\[Im A_- = \frac{e^5 F_{\eta\gamma^*\gamma^*}}{8\pi f_\pi M^2_\eta} \frac{X_+ - X_-}{X_- X_+(X_- - 1)} = \]

\[= \frac{e^5 F_{\eta\gamma^*\gamma^*}}{8\pi f_\pi} \frac{M^2_\eta}{s^2} \frac{1}{1 - \cos \theta} \frac{\cos \theta}{(1 + \frac{M^2_\eta}{s})^2 - (1 - \frac{M^2_\eta}{s})^2 \cos^2 \theta},\]

and the corresponding contribution into \(\sigma^{(2\gamma)}(e^+e^- \rightarrow \eta\gamma)\):

\[
\frac{d\sigma^{(2\gamma)}}{d\Omega} = 8\alpha^3 \left( \frac{M^2_\eta}{s} \right)^2 \left( 1 - \frac{M^2_\eta}{s} \right)^3 \frac{\Gamma(\eta \rightarrow \gamma \gamma)}{M_\eta} \frac{\cos^2 \theta}{\left[ (1 + \frac{M^2_\eta}{s})^2 - (1 - \frac{M^2_\eta}{s})^2 \cos^2 \theta \right]^2},
\]

where the constant \(F_{\eta\gamma^*\gamma^*}\) was expressed through the two-photon width \(\Gamma(\eta \rightarrow \gamma \gamma)\). Integrating in the limits \(30^\circ \leq \theta \leq 150^\circ\), and assuming \(s = 112\) GeV\(^2\) we find numerically

\[
\sigma^{(2\gamma)}(e^+e^- \rightarrow \eta\gamma) \approx 3.4 \cdot 10^{-4} \text{ fb}, \quad \sigma^{(2\gamma)}(e^+e^- \rightarrow \eta'\gamma) \approx 4.5 \cdot 10^{-3} \text{ fb}.
\]

These numbers are quite small compared to the reported BABAR cross sections

\[
\sigma(e^+e^- \rightarrow \eta\gamma) = 4.5^{+1.2}_{-1.1} \pm 0.3 \text{ fb}, \quad \sigma(e^+e^- \rightarrow \eta'\gamma) = 5.4 \pm 0.8 \pm 0.3 \text{ fb}.
\]

But the amplitude has also a real part which in fact dominates for the constant \(F_{\eta\gamma^*\gamma^*}\) form factor.

The real part of the integral (31) can be evaluated by using a substitution \(t = 1 - X_- + (X_- + X_+ - 1) x\) and the integral

\[
\int_A^B \frac{\ln (1 - Ct)}{t} \, dt = Li_2(CA) - Li_2(CB),
\]

\(A, B\) and \(C\) being some constants. As a result, we get (note that \(Li_2(1) = \pi^2/6\))

\[
Re (I_2 + I_4) = Re \frac{Li_2 \left( \frac{X_-}{1-X_+} \right) + Li_2 \left( \frac{X_+}{1-X_-} \right) - Li_2 \left( \frac{X_- X_+}{(1-X_+)(1-X_-)} \right) - \frac{\pi^2}{6}}{2(X_- + X_+ - 1)}.
\]
By using the Abel identity \[46, 47\] (still valid for real parts if \(x > 1\) and \(y > 1\))
\[
\ln(1-x) \ln(1-y) = \text{Li}_2 \left(\frac{x}{1-y}\right) + \text{Li}_2 \left(\frac{y}{1-x}\right) - \text{Li}_2(x) - \text{Li}_2(y) - \text{Li}_2 \left(\frac{xy}{(1-x)(1-y)}\right),
\]
the above expression can be rewritten as
\[
\text{Re} (I_2 + I_4) = \frac{\text{Re} G(X_-, X_+)}{2(X_- + X_+ - 1)},
\] (45)
where
\[
G(X_-, X_+) = \ln(1 - X_+) \ln(1 - X_-) + \text{Li}_2(X_+) + \text{Li}_2(X_-) - \frac{\pi^2}{6}.
\] (46)
The real part of the integral (40) can be calculated analogously with the result
\[
\text{Re} I_2 = \frac{1}{2(X_- + X_+ - 1)} \text{Re} \left[ \ln \frac{X_++1}{X_++1} - \ln \frac{X_++1}{X_-+1} + \frac{(X_++1) G(X_-, X_+)}{X_-+X_+ - 1} \right].
\] (47)
Using (45) and (47), we find
\[
\text{Re} A_{(1)} = \frac{e^5 F^{(1)}_{\eta^*\gamma\gamma^*}}{8\pi^2 f_\pi} \left\{ \frac{G(X_-, X_+)}{s} - \frac{1}{4k \cdot p_-} \right\},
\]
in agreement with [3]. But according to our results \(A_{(1)}\) is only a part of the \(A_-\) form factor, for which we obtain
\[
\text{Re} A_- = \frac{e^5 F^{(1)}_{\eta^*\gamma\gamma^*}}{8\pi^2 f_\pi} \frac{1}{M^2_\eta(X_- - 1)} \left[ \ln \frac{X_++1}{X_++1} + \ln \frac{X_++1}{X_-+1} - \ln \frac{X_++1}{X_-+1} - \frac{1}{2} \right].
\] (48)
In contrast to [3], all dilogarithms have canceled out.

Having at hand \(\text{Re} A_-\), we can find the corresponding contribution in the \(\sigma^{(2\gamma)}(e^+e^- \rightarrow \eta\gamma)\) cross section:
\[
\frac{d\sigma^{(2\gamma)}}{d\Omega} = -\frac{\alpha^3}{4\pi^2 M^2_\eta} \left(1 - \frac{M^2_\eta}{s}\right) \frac{\Gamma(\eta \rightarrow \gamma\gamma)}{M_\eta} \left[ |F(\cos \theta)|^2 + |F(-\cos \theta)|^2 \right],
\] (49)
where
\[
F(\cos \theta) = \ln \frac{1 + \cos \theta}{1 - \cos \theta} + \ln \frac{X_++1}{X_-} - \ln \frac{X_++1}{X_-} - \frac{1}{2}
\]
and
\[
X_\pm = 1 + \frac{s}{2M^2_\eta} \left(1 - \frac{M^2_\eta}{s}\right) (1 \pm \cos \theta).
\] (50)
Numerically, for \(30^\circ \leq \theta \leq 150^\circ\) and \(s = 112\text{ GeV}^2\), we find
\[
\sigma^{(2\gamma)}(e^+e^- \rightarrow \eta\gamma) \approx 0.5\text{ fb}, \quad \sigma^{(2\gamma)}(e^+e^- \rightarrow \eta\gamma) \approx 0.8\text{ fb}.
\] (51)
These cross sections are quite substantial and they could cause problems with the BABAR analysis if the constant \(F^{(1)}_{\eta^*\gamma\gamma^*}\) form factor would not be unphysical. In reality, however, the form factor drops quickly at large photon virtualities. Therefore in the next section we consider more realistic VMD form factor.
TWO-PHOTON CONTRIBUTION TO $e^+e^- \rightarrow \eta\gamma$: VECTOR MESON DOMINANCE MODEL

The VMD $F_{\eta\gamma\gamma}$ form factor is given by (39). Its insertion into (34) produces momentum integrals with six denominators. Their treatment becomes less formidable if we use the identity

$$\frac{M^4}{D_1D_2D_3D_4D_5D_6} = \frac{1}{D_1D_2D_3D_4} + \frac{1}{D_3D_4D_5D_6} - \frac{1}{D_1D_3D_4D_6} - \frac{1}{D_2D_3D_4D_5},$$

with

$$D_1 = l^2, \ D_2 = (q-l)^2, \ D_3 = (p_+ - l)^2, \ D_4 = (p_+ - l - k)^2, \ D_5 = l^2 - M^2, \ D_6 = (q-l)^2 - M^2.$$

Then the procedure described in the previous section gives

$$A_- = \frac{e^5}{8\pi^2f_\pi} \left[ g_{\eta\rho\rho} g_{\rho\gamma}^2 \tilde{I}(M_1^2, M_2^2) + g_{\eta\omega\omega} g_{\omega\gamma}^2 \tilde{I}(M_1^2, M_2^2) - g_{\eta\phi\phi} g_{\phi\gamma}^2 \tilde{I}(M_1^2, M_2^2) \right], \quad (52)$$

with

$$\tilde{I}(M_1^2, M_2^2) = \tilde{J}(M_1^2, M_2^2) + \tilde{J}(0, 0) - \tilde{J}(M_1^2, 0) - \tilde{J}(0, M_2^2). \quad (53)$$

Here

$$M_1^2 \tilde{J}(M_1^2, M_2^2) = 2 + (X_+ + X_- - 1) \frac{\partial}{\partial X_-} \left[ \tilde{I}_2(X_+, X_-) + \tilde{I}_4(X_+, X_-) \right] - \left[ 1 + (1 - X_-) \left( \frac{\partial}{\partial X_-} - \frac{\partial}{\partial X_+} \right) \right] \tilde{I}_0(X_+, X_-), \quad (54)$$

and the new master integrals are

$$\tilde{I}_0(X_+, X_-) = \int_0^1 dx_1 \cdots \int_0^1 dx_4 \frac{\delta (\sum x_i - 1)}{x_1x_2 + (1 - X_-)x_2x_3 + (1 - X_+)x_1x_4 - x_1r_1 - x_2r_2},$$

$$\tilde{I}_n(X_+, X_-) = \int_0^1 dx_1 \cdots \int_0^1 dx_4 \frac{\delta (\sum x_i - 1) x_n}{x_1x_2 + (1 - X_-)x_2x_3 + (1 - X_+)x_1x_4 - x_1r_1 - x_2r_2}. \quad (55)$$

The presence of the additional $x_1r_1 + x_2r_2$ term in the denominators compared to (48) makes the evaluation of these integrals, of course, more complicated but not substantially different from the evaluation of (58). Therefore we present only final results.

$$\text{Re} \ (\tilde{I}_2 + \tilde{I}_4) = \frac{1}{(X_- + X_+ - 1)} \text{Re} \int_0^1 dz \ (1 - z) G(X'_-, X'_+).$$
\[ Re \tilde{I}_2 = \frac{1}{(X_- + X_+ - 1)} Re \int_0^1 dz (1-z) \left[ \ln \frac{X'_- - 1}{X'_-} - \ln \frac{X'_+ - 1}{X'_+} + \frac{(X'_- - 1) G(X'_-, X'_+)}{X'_+ + X'_- - 1} \right], \]
\[ \text{Re} \tilde{I}_0 = \frac{1}{(X_- + X_+ - 1)} Re \int_0^1 dz G(X'_-, X'_+). \]  

(56)

Here the function \( G(x, y) \) is defined by (46) and

\[ X'_- = \frac{z [r_1 - (1-z) X_-]}{r_1 z + r_2 (1-z) - z(1-z)}, \quad X'_+ = \frac{(1-z) [r_2 - z X_+]}{r_1 z + r_2 (1-z) - z(1-z)}. \]  

(57)

We need only the real parts because the imaginary part of the amplitude is dominated by the two-photon cut and, therefore, is the same as for the constant \( F_{\eta' \gamma^* \gamma^*} \) form factor.

Using (56), we get after some algebra

\[ M^2_\eta Re \tilde{J}(M^2_1, M^2_2) = \frac{M^2}{s} \text{Re} \int_0^1 dz (1-2z) G(X'_-, X'_+) + \]
\[ \text{Re} \int_0^1 dz \frac{z(1-z)}{r_2 + z(X_- - 1)} \left[ \frac{r_1 + X_+ - 1}{r_1 + (X_+ - 1)(1-z)} L(X'_-, X'_+) - 1 \right], \]  

(58)

where

\[ L(x, y) = \ln \frac{y - 1}{x - 1} + \frac{\ln (x - 1)}{x} - \frac{\ln (y - 1)}{y}. \]

Note that

\[ M^2_\eta \left[ r_2 + z(X_- - 1) \right] = \frac{s}{2} \left[ \frac{2M^2_\eta}{s} + z \left( 1 - \frac{M^2_\eta}{s} \right) (1 - \cos \theta) \right], \]

and therefore

\[ \text{Re} \tilde{J}(M^2_1, M^2_2) = \frac{2}{s} \text{Re} j(r_1, r_2), \]

where the dimensionless function \( j(r_1, r_2) \) is given by

\[ j(r_1, r_2) = \frac{1}{2} \int_0^1 dz (1-2z) G(X'_-, X'_+) + \]
\[ \int_0^1 dz \frac{z(1-z)}{2M^2_\eta/s + z \left( 1 - \frac{M^2_\eta}{s} \right) (1 - \cos \theta)} \left[ \frac{r_1 + X_+ - 1}{r_1 + (X_+ - 1)(1-z)} L(X'_-, X'_+) - 1 \right]. \]  

(59)

The corresponding contribution in the \( \sigma^{(2\gamma)} (e^+e^- \rightarrow \eta \gamma) \) cross section looks like

\[ \frac{d\sigma^{(2\gamma)}_{R}(e^+e^- \rightarrow \eta \gamma)}{d\Omega} = \frac{\alpha^5}{16\pi P^2} \left( 1 - \frac{M^2_\eta}{s} \right)^3 \left[ |f(\cos \theta)|^2 + |f(-\cos \theta)|^2 \right], \]  

(60)
where
\[
    f(\cos \theta) = (1 - \cos \theta) \left[ g_{\eta \rho \rho}^2 g_{\rho \gamma}^2 i(M_\rho^2, M_\rho^2) + g_{\eta \omega \omega}^2 g_{\omega \gamma}^2 i(M_\omega^2, M_\omega^2) - g_{\eta \phi \phi}^2 g_{\phi \gamma}^2 i(M_\phi^2, M_\phi^2) \right],
\]
and
\[
i(M_1^2, M_2^2) = j(r_1, r_2) + j(0, 0) - j(r_1, 0) - j(0, r_2).
\]
Now we have all ingredients to find the cross section under interest numerically. The results are (for the same kinematic conditions as before)
\[
    \sigma_R^{(2\gamma)}(e^+ e^- \rightarrow \eta \gamma) \approx 1.6 \cdot 10^{-3} \text{ fb}, \quad \sigma_R^{(2\gamma)}(e^+ e^- \rightarrow \eta' \gamma) \approx 1.2 \cdot 10^{-3} \text{ fb}. \quad (61)
\]
For \(e^+ e^- \rightarrow \eta \gamma\) the real part of the amplitude still dominates although not as drastically as for the constant \(F_{\eta \gamma, \gamma^*}\) form factor. For \(e^+ e^- \rightarrow \eta' \gamma\) the real part is even somewhat smaller than the imaginary part.

**CONCLUSIONS**

In this work we used a phenomenological, VMD inspired \(F_{\eta \gamma, \gamma^*}\) form factor to estimate the expected contributions into the \(e^+ e^- \rightarrow \eta \gamma\) and \(e^+ e^- \rightarrow \eta' \gamma\) amplitudes from the two-photon exchange diagrams of Fig.1. The results are given in (44) and (61). The numbers obtained are too small to be of any importance for the recent BABAR analysis [1], but indicate that interference effects of the order of several percent are expected in future high precision studies of corresponding transition form factors.

Our results might be of some relevance for accurate evaluation of radiative corrections to Dalitz decay \(P \rightarrow e^+ e^- \gamma\) [15], especially in light of some discrepancy with earlier studies [2], but we do not pursue this line of thought further in this article.

**Acknowledgments**

This investigation was initiated by V. Druzhinin. The author thanks him for valuable discussions. J. D. Jackson is acknowledged for sending his unpublished report. The work is supported in part by grants Sci.School-905.2006.2 and RFBR 06-02-16192-a.

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[1] B. Aubert [BABAR Collaboration], arXiv:hep-ex/0605018.
[2] G. B. Tupper, T. R. Grose and M. A. Samuel, Phys. Rev. D 28, 2905 (1983).
[3] G. Tupper, Phys. Rev. D 35, 1726 (1987).
[4] D. S. Beder, Phys. Rev. D 34, 2071 (1986).
[5] J. Wess and B. Zumino, Phys. Lett. B 37, 95 (1971).
[6] E. Witten, Nucl. Phys. B 223, 422 (1983).
[7] O. Kaymakcalan, S. Rajeev and J. Schechter, Phys. Rev. D 30, 594 (1984).
[8] M. J. Savage, M. E. Luke and M. B. Wise, Phys. Lett. B 291, 481 (1992)
   arXiv:hep-ph/9207233.
[9] B. Borasoy and R. Nissler, Eur. Phys. J. A 19, 367 (2004) arXiv:hep-ph/0309011.
[10] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, M. B. Voloshin and V. I. Zakharov, Nucl.
     Phys. B 237, 525 (1984).
[11] J. M. Gerard and T. Lahna, Phys. Lett. B 356, 381 (1995) arXiv:hep-ph/9506255.
[12] M. Hayakawa and T. Kinoshita, Phys. Rev. D 57, 465 (1998) [Erratum-ibid. D 66, 019902
     (2002)] arXiv:hep-ph/9708227.
[13] K. Melnikov and A. Vainshtein, Phys. Rev. D 70, 113006 (2004) arXiv:hep-ph/0312226.
[14] J. J. Sakurai, Currents and mesons, Chicago University Press, Chicago, 1969.
[15] H. B. O'Connell, B. C. Pearce, A. W. Thomas and A. G. Williams, Prog. Part. Nucl. Phys.
     39, 201 (1997) arXiv:hep-ph/9501251.
[16] T. H. Bauer, R. D. Spital, D. R. Yennie and F. M. Pipkin, Rev. Mod. Phys. 50, 261 (1978)
     [Erratum-ibid. 51, 407 (1979)].
[17] N. M. Kroll, T. D. Lee and B. Zumino, Phys. Rev. 157, 1376 (1967).
[18] G. Ecker, Prog. Part. Nucl. Phys. 35, 1 (1995) arXiv:hep-ph/9501357.
[19] J. Bijnens, arXiv:hep-ph/0604043.
[20] S. Scherer, Adv. Nucl. Phys. 27, 277 (2003) arXiv:hep-ph/0210398.
[21] C. G. Callan, S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. 177, 2247 (1969).
[22] S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. 41, 531 (1969).
[23] M. Bando, T. Kugo and K. Yamawaki, Phys. Rept. 164, 217 (1988).
[24] M. Bando, T. Kugo, S. Uehara, K. Yamawaki and T. Yanagida, Phys. Rev. Lett. 54, 1215
     (1985).
[25] H. Georgi, Phys. Rev. Lett. 63, 1917 (1989); Nucl. Phys. B 331, 311 (1990).
[26] J. Schechter, Phys. Rev. D 34, 868 (1986).
[27] K. Kawarabayashi and M. Suzuki, Phys. Rev. Lett. 16, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966).

[28] M. Harada and K. Yamawaki, Phys. Rev. Lett. 87, 152001 (2001) arXiv:hep-ph/0105335.

[29] S. Eidelman et al. [Particle Data Group], Phys. Lett. B 592, 1 (2004).

[30] L. Ametller, A. Bramon and E. Masso, Phys. Rev. D 48, 3388 (1993) arXiv:hep-ph/9302304.

[31] B. R. Martin, E. De Rafael and J. Smith, Phys. Rev. D 2, 179 (1970).

[32] M. Pratap and J. Smith, Phys. Rev. D 5, 2020 (1972).

[33] J. L. Ritchie and S. G. Wojcicki, Rev. Mod. Phys. 65, 1149 (1993).

[34] L. Littenberg and G. Valencia, Ann. Rev. Nucl. Part. Sci. 43, 729 (1993) arXiv:hep-ph/9303225.

[35] C. Quigg and J. D. Jackson, Decays of neutral pseudoscalar mesons into lepton pairs. Lawrence Radiation Laboratory report UCRL-18487 (1968) (unpublished).

[36] G. Valencia, Nucl. Phys. B 517, 339 (1998) arXiv:hep-ph/9711377.

[37] A. C. Hearn, Reduce user’s manual, Rand Publication, 1989.

[38] L. Bergstrom, Z. Phys. C 14, 129 (1982).

[39] L. Bergstrom, E. Masso, L. Ametller and A. Bramon, Phys. Lett. B 126, 117 (1983).

[40] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).

[41] S. Mandelstam, Phys. Rev. 115, 1741 (1959).

[42] G. ’t Hooft and M. J. G. Veltman, Nucl. Phys. B 153, 365 (1979).

[43] S. Weinzierl, arXiv:hep-ph/0604068.

[44] C. Osacár, J. Palacián and M. Palacios, Celest. Mech. Dyn. Astron. 62, 93 (1995).

[45] K. Kampf, M. Knecht and J. Novotny, Eur. Phys. J. C 46, 191 (2006) arXiv:hep-ph/0510021.

[46] A. N. Kirillov, Prog. Theor. Phys. Suppl. 118, 61 (1995) arXiv:hep-th/9408113.

[47] L. Lewin, Polylogarithms and associated functions, North Holland, New York, 1981.