How to quantify a dynamical resource?

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We show that the generalization of the relative entropy of a resource from states to channels is not unique, and there are at least six such generalizations. We then show that two of these generalizations are asymptotically continuous, satisfy a version of the asymptotic equipartition property, and their regularizations appear in the power exponent of channel versions of the quantum Stein’s Lemma. To obtain our results, we use a new type of “smoothing” that can be applied to functions of channels (with no state analog). We call it “liberal smoothing” as it allows for more spread in the optimization. Along the way, we show that the relative entropy of a resource can be expressed as a $D_{\text{max}}$ distance to the set of quantum channels, and prove a variety of properties of all six generalizations of the relative entropy of a resource.

Introduction– In recent years it has been recognized that many properties of physical systems, such as quantum entanglement, asymmetry, coherence, athermality, contextuality, and many others, can be viewed as resources circumventing certain constraints imposed on physical systems (see \cite{1} and references therein). Each resource can be classified as being classical or quantum, static (e.g. entangled state) or dynamic (e.g. quantum channel), noisy or noiseless, leading to numerous interesting quantum information processing tasks \cite{2} (e.g. quantum teleportation \cite{3}). While there are many ways to quantify the resourcefulness of such properties, all quantifiers of a resource must satisfy certain conditions such as monotonicity under the set of free operations. Typically, there are numerous measures that satisfy these conditions, but what can single out a given measure is an operational interpretation, giving it meaning beyond its sheer ability to quantify somewhat vaguely the resource.

The relative entropy of a resource, which was originally defined in \cite{4} for entanglement theory, is an example of a measure that has such an operational interpretation in many quantum resource theories (QRTs). First, it was shown in \cite{5,6} to be a unique measure in reversible QRTs, and then was shown to be the unique asymptotic rate of interconversion among static resources under resource non-generating operations \cite{6}. Moreover, it was shown very recently \cite{8,9} that resource erasure as a universal operational task leads to the (regularized) relative entropy of a resource as the optimal rate (this idea was first laid out in \cite{10}). In addition, this measure satisfies the asymptotic equipartition property (AEP) \cite{11}, appears as an optimal rate in the generalized quantum Stein’s Lemma \cite{11}, and is asymptotically continuous \cite{12,13}, a property linked to it also being a non-lockable measure \cite{14}. Due to all of these properties, the relative entropy of a resource plays a major role in many QRTs \cite{1}.

In this paper we study six generalizations of the quantum relative entropy of a resource from static resources (i.e. states) to dynamic ones (i.e. channels). Four of these measures were introduced very recently in \cite{12,15}. We show that for two of them, the relative entropy of the dynamical resource is asymptotically continuous, satisfies a version of the AEP, and a version of their regularization appear as optimal rates in a version of the quantum Stein’s Lemma for channels. In addition, we show that all these measures are indeed generalizations to dynamical resources in the sense that they reduce to the relative entropy of a static resource for replacement (i.e. constant) channels.

Resource theories of quantum processes– A quantum resource theory (QRT), consists of a function $\mathcal{F}$ taking any pair of physical systems $A$ and $B$ to a subset of completely positive and trace preserving (CPTP) maps $\mathcal{F}(A \to B) \subset \text{CPTP}(A \to B)$, where $\text{CPTP}(A \to B)$ is the set of all CPTP maps (i.e. quantum channels) from $B(A)$ (bounded operators on Hilbert space of system $A$) to $B(B)$. The mapping $\mathcal{F}$ is a quantum resource theory if the following two conditions hold:

1. For any physical system $A$ the set $\mathcal{F}(A \to A)$ contains the identity map $\text{id}_A$.
2. For any three systems $A, B, C$, if $\mathcal{M} \in \mathcal{F}(A \to B)$ and $\mathcal{N} \in \mathcal{F}(B \to C)$ then $\mathcal{N} \circ \mathcal{M} \in \mathcal{F}(A \to C)$.

Denoting by $\text{id}$ the trivial Hilbert space we identify $\mathcal{F}(1 \to A)$ with the set of free density matrices in $B(A)$. That is, a density matrix $\rho \in \mathcal{F}(1 \to A)$ can be viewed as the CPTP map $\rho(z) = z \rho$ for all $z \in C$. For simplicity, we will write $\mathcal{F}(1 \to A) \equiv \mathcal{F}(A)$. Typically, QRTs are physical in the sense that they arise from some physical constraints, and therefore admit a tensor product structure. That is, the set of free operations $\mathcal{F}$ satisfies the following additional conditions:

3. The free operations are “completely free”: For any three physical systems $A$, $B$, and $C$, if $\mathcal{M} \in \mathcal{F}(A \to B)$ then $\text{id}_C \otimes \mathcal{M} \in \mathcal{F}(CA \to CB)$. 

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4. Discarding a system (i.e. the trace) is a free operation: For any system $A$, the set $\mathcal{F}(A \to 1)$ is not empty.

The above additional conditions are very natural and satisfied by almost all QRTs studied in literature. They implies the following properties [1]:

- If $\mathcal{M}_1$ and $\mathcal{M}_2$ are free channels then also $\mathcal{M}_1 \otimes \mathcal{M}_2$ is free.
- Appending free states is a free operation: For any given free state $\sigma \in \mathcal{F}(B)$, the CPTP map $\mathcal{M}_\sigma(\rho) := \rho \otimes \sigma$ is a free map, i.e., it belongs to $\mathcal{F}(A \to AB)$.
- The replacement map $\mathcal{M}_\sigma(\rho) := \sigma$, for any density matrix $\rho \in \mathcal{B}(A)$ and a fixed free state $\sigma \in \mathcal{F}(B)$, is a free channel; i.e. $\mathcal{M}_\sigma \in \mathcal{F}(A \to B)$.

It is also physical to assume that $\mathcal{F}(A \to B)$ is a closed set, since otherwise there exists a sequence of free channels whose limit is a resource channel. Finally, we will assume that for any integer $n$, free channel $N \in \mathcal{F}(A_1 \cdots A_n \to B_1 \cdots B_n)$, and two permutation channels $\mathcal{P}_A^\pi$ and $\mathcal{P}_B^\pi$ corresponding to a permutation $\pi$ on $n$ elements, we have

$$\mathcal{P}_B^\pi \circ N_{A_1 \cdots A_n \to B_1 \cdots B_n} \circ \mathcal{P}_A^\pi \in \mathcal{F}(A_1 \cdots A_n \to B_1 \cdots B_n).$$

Note that almost all QRTs discussed in literature satisfy this last condition including entanglement theory, coherence, athermality, etc. In the rest of this paper we will assume that $\mathcal{F}$ satisfies all the above conditions.

The most general physical operation that can be performed on a dynamical resource $\mathcal{N} \in \mathcal{CPTP}(A \to B)$ can be characterized with a superchannel [17, 18], $\Theta$, defined for all $\mathcal{N} \in \mathcal{CPTP}(A \to B)$ as a transformation of the form

$$\Theta[\mathcal{N}_{A \to B}] = \mathcal{E}_{B \to B'}^{\text{post}} \circ \mathcal{N}_{A \to B} \circ \mathcal{E}_{A' \to AE}^{\text{pre}},$$

where $\mathcal{E}_{B \to B'}^{\text{post}} \in \mathcal{CPTP}(BE \to B')$ and $\mathcal{E}_{A' \to AE}^{\text{pre}} \in \mathcal{CPTP}(A' \to AE)$ are quantum channels. We say that the superchannel $\Theta$ is free if in addition $\mathcal{E}_{B \to B'}^{\text{post}} \in \mathcal{F}(BE \to B')$ and $\mathcal{E}_{A' \to AE}^{\text{pre}} \in \mathcal{F}(A' \to AE)$ (i.e. $\mathcal{E}_{B \to B'}^{\text{post}}$ and $\mathcal{E}_{A' \to AE}^{\text{pre}}$ are free). Therefore, any measure of a resource $E : \mathcal{CPTP} \to \mathbb{R}$ must satisfy

$$E(\Theta[\mathcal{N}_{A \to B}]) \leq E(\mathcal{N}_{A \to B}),$$

for all $\mathcal{N} \in \mathcal{CPTP}(A \to B)$ and all free superchannels $\Theta$. In addition, we require that $E(\mathcal{N}) = 0$ if $\mathcal{N} \in \mathcal{F}(A \to B)$. This condition implies that $E$ is non-negative. To see it, take $\mathcal{E}_{B \to B'}^{\text{post}}$ in [14] to be the replacement map whose output is some free state in $\mathcal{F}(B')$, and observe that for this case $0 = E(\Theta[\mathcal{N}]) \leq E(\mathcal{N})$ for all $\mathcal{N} \in \mathcal{CPTP}(A \to B)$.

The relative entropy of a resource– We will consider here two generalization of the relative entropy of a resource from the state domain to the channel domain, and leave four further generalizations to the supplemental material (SM). The first relative entropy of a dynamical resource $\mathcal{N} \in \mathcal{CPTP}(A \to B)$ is defined as

$$D_3(\mathcal{N}) := \inf_{\mathcal{M} \in \mathcal{F}(A \to B)} D(\mathcal{N} || \mathcal{M}),$$

with the channel divergence [18–20]

$$D(\mathcal{N} || \mathcal{M}) := \max_{\varphi \in \mathcal{D}(\mathcal{RA})} D(\mathcal{N}_{A \to B}(\varphi_{RA}) || \mathcal{M}_{A \to B}(\varphi_{RA})),$$

and $D(\rho || \sigma) = \text{Tr}[\rho \log \rho - \rho \log \sigma]$ is the relative entropy. The optimization is over all states $\varphi_{RA}$, where w.l.o.g. we can take $R \cong A$ and $\varphi_{RA}$ is pure [19, 20]. If the optimization over $\mathcal{D}(\mathcal{RA})$ is replaced with optimization over the set of all density matrices $\mathcal{F}(\mathcal{RA})$, then one gets the second generalization [15]

$$E_3(\mathcal{N}) := \min_{\mathcal{M} \in \mathcal{F}(A \to B)} \sup_{\rho \in \mathcal{F}(\mathcal{RA})} D(\mathcal{N}_{A \to B}(\rho_{RA}) || \mathcal{M}_{A \to B}(\rho_{RA})).$$

where the supremum is over all free states $\rho \in \mathcal{F}(\mathcal{RA})$ and all dimensions $|R|$, and the minimum is over all free channels in $\mathcal{F}(A \to B)$. Both $D_3$ and $E_3$, as well as other generalizations, were introduced very recently in [15, 16], and in the SM we list all of them along with a few new ones and discuss some of their properties. For clarity, we leave the technical details of all proofs to the SM.

**Theorem 1.** The above relative entropies have the following properties:

1. **[Monotonicity]** $D_3$ and $E_3$ behave monotonically under free superchannels. Specifically, let $\mathcal{E}_{\text{post}} \in \mathcal{CPTP}(BE \to B')$ and $\mathcal{E}_{\text{pre}} \in \mathcal{CPTP}(A' \to AE)$ be completely resource RNG channels, and let $\Theta$ has the form [14]. Then, for all $\mathcal{N} \in \mathcal{CPTP}(A \to B)$

$$D_3(\Theta[\mathcal{N}]) \leq D_3(\mathcal{N}) ; \quad E_3(\Theta[\mathcal{N}]) \leq E_3(\mathcal{N}).$$

2. **[Reduction]** Let $\mathcal{N} \in \mathcal{CPTP}(A \to B)$ be a constant channel $\mathcal{N}(X_A) = \text{Tr}[X_A \omega_B]$ for all $X_A \in \mathcal{B}(A)$ and a fixed density matrix $\omega_B \in \mathcal{D}(B)$. Then,

$$D_3(\mathcal{N}) = E_3(\mathcal{N}) = D_3(\omega_B) := \min_{\sigma \in \mathcal{F}(B')} D(\omega_B || \sigma_B).$$

3. **[Faithfulness]** $D_3(\mathcal{N}_{A \to B}) = 0$ if and only if $\mathcal{N} \in \mathcal{F}(A \to B)$. If $E_3(\mathcal{N}) = 0$ for some $\mathcal{N} \in \mathcal{CPTP}(A \to B)$ then $\mathcal{N}$ must be completely resource non-generating (RNG). Moreover, if for $|R| = |A|$ the set $\mathcal{F}(\mathcal{RA})$ contains a pure state with full Schmidt rank, then

$$E_3(\mathcal{N}_{A \to B}) = 0 \iff \mathcal{N} \in \mathcal{F}(A \to B).$$
In contrast to the monotonicity property above, the function $D_{\bar{3}}$ behaves monotonically under any RNG superchannel. This follows directly from the following:

$$D_{\bar{3}}(\Theta|N) = \min_{\Omega \in \mathcal{F}(A' \to B')} D(\Theta|N_{A' \to B}||\Omega_{A' \to B'}) \leq \min_{\mathcal{M} \in \mathcal{F}(A \to B)} D(\Theta|N_{A' \to B}||\Theta|M_{A' \to B}) \leq \min_{\mathcal{M} \in \mathcal{F}(A \to B)} D(N_{A' \to B}|M_{A' \to B}) = D_{\bar{3}}(N) ,$$

where the first inequality follows from the fact that $\Theta$ is RNG, and the second from the data processing inequality of the channel divergence \cite{13}. Note also that from their definitions we always have

$$E_{\bar{3}}(N) \leq D_{\bar{3}}(N) \quad \forall N \in \text{CPTP}(A \to B). \quad (9)$$

One may wonder if exchanging the min-max order in \cite{3} and \cite{5} would yield other relative entropy based measures that are in general different than $D_{\bar{3}}$ and $E_{\bar{3}}$. However, in the following theorem we show that this is not the case.

**Theorem 2.** Let $d: \mathcal{D}(A) \times \mathcal{D}(A) \to \mathbb{R}$ be any function satisfying non-negativity, contractivity (monotonicity) under CPTP maps, and joint concavity under orthogonally flagged mixtures: This means that for any two families $\{\rho_x\}$ and $\{\sigma_x\}$ of states, and any probability distribution $\{p_x\}$,

$$d \left( \sum_x p_x \rho_x \otimes |x\rangle\langle x|, \sum_x p_x \sigma_x \otimes |x\rangle\langle x| \right) \geq \sum_x p_x d(\rho_x, \sigma_x) , \quad (10)$$

where $|x\rangle$ are orthonormal basis states of an auxiliary system. Moreover, suppose $d$ is convex in the second argument, and suppose $\mathcal{F}(A \to B)$ is convex. Then,

$$\inf_{\mathcal{M} \in \mathcal{F}(A \to B)} \sup_{\rho \in \mathcal{F}(A)} d(\mathcal{M}_{A \to B}(\rho_{RA}), \mathcal{M}_{A \to B}(\rho_{RA})) = \sup_{\rho \in \mathcal{F}(RA)} \inf_{\mathcal{M} \in \mathcal{F}(A \to B)} d(\mathcal{M}_{A \to B}(\rho_{RA}), \mathcal{M}_{A \to B}(\rho_{RA})) .$$

Note that the relative entropy $D$ (as well as the trace distance and all the Renyi diversities) satisfies \cite{10} with equality, and therefore $E_{\bar{3}}$ and $D_{\bar{3}}$ will not change by swapping the min-max order.

**Asymptotic continuity:** Since we only consider here QRTs that admits the tensor product structure, the replacement channels $\mathcal{M}_\sigma(X) = \text{Tr}[X]\sigma$ are free (i.e. in $\mathcal{F}(A \to B)$) for any free $\sigma \in \mathcal{F}(B)$. In the SM we show that this implies that $E_{\bar{3}}$ is bounded as long as the set of free states contains a full rank state. For example, if $\mathcal{F}(B)$ contains the maximally mixed (uniform) state $I_B/|B|$ (were $|B|$ is the dimension of system $B$), then

$$E_{\bar{3}}(N) \leq D_{\bar{3}}(N) \leq \log (|B|^2|A|) . \quad (11)$$

The fact that $E_{\bar{3}}$ and $D_{\bar{3}}$ are bounded enable us to prove that they are also asymptotically continuous.

**Definition 3.** A function $E: \text{CPTP} \to \mathbb{R}_+$ is said to be asymptotically continuous if for any $\mathcal{M}, N \in \text{CPTP}(A \to B)$,

$$|E(\mathcal{M}) - E(N)| \leq \log (|AB|)|\mathcal{M} - N|_\infty , \quad (12)$$

where $f: \mathbb{R} \to \mathbb{R}$ is some function independent on the dimensions and satisfies $\lim_{\epsilon \to 0^+} f(\epsilon) = 0$.

**Theorem 4.** Suppose that for any system $A$, $\mathcal{F}(A)$ contains a full rank state. Then, $D_{\bar{3}}$ is asymptotically continuous. Moreover, if in addition, for any system $A$ the extreme points of $\mathcal{F}(A)$ are pure states (e.g. entanglement theory, coherence, etc), then $E_{\bar{3}}$ is also asymptotically continuous.

**Remark.** The proof of the theorem above is based on a key observation that the diamond norm can be expressed in terms of the $D_{\max}$ distance of $N - M$ to the set of all quantum channels $Q(A \to B)$ (see SM for more details). For $E_{\bar{3}}$ the condition that the extreme points of the set of free states are pure states, ensures that the supremum in \cite{5} can be replaced with a maximum since in this case $|R|$ can be shown to be bounded by $|A|$. If the extreme points of the set of free states are not pure states but $|R|$ is polynomially bounded in $|AB|$, then also in this case $E_{\bar{3}}$ is asymptotically continuous. This happens for example in the QRT of thermodynamics.

**Asymptotic Equipartition Property (AEP)**- The logarithmic robustness of a dynamical resource $N \in \text{CPTP}(A \to B)$ is defined as \cite{16}

$$LR_{\bar{3}}(N_{A \to B}) := \min_{\mathcal{M} \in \mathcal{F}(A \to B)} D_{\max}(N_{A \to B}||M_{A \to B}) = \log_2 \min \{ t : t \mathcal{M} \geq N ; \mathcal{M} \in \mathcal{F}(A \to B) \} , \quad (13)$$

where the ordering $t \mathcal{M} \geq N$ means that $t \mathcal{M} - N$ is completely positive (CP). We also define here

$$LR_{\bar{3}}(\hat{N}_{A \to B}) := \min_{\mathcal{M} \in \mathcal{F}(A \to B)} D_{\max}(\hat{N}_{A \to B}(\varphi_{RA}), \mathcal{M}_{A \to B}(\varphi_{RA})) . \quad (14)$$

Like $D_{\bar{3}}$ and $E_{\bar{3}}$, the functions $LR_{\bar{3}}$ and $LR_{\bar{3}}$ are resource monotones (see SM). Note that by Theorem 2 the order sup-min can be exchanged, and furthermore,

$$LR_{\bar{3}} \leq LR_{\bar{3}} , \quad (15)$$

with equality if $\mathcal{F}(RA)$ contains a pure state of full Schmidt rank. For example, in entanglement theory, system $A$ is replaced with $AB$ and $R$ with $R_AR_B$ so that $\mathcal{F}(\hat{R}_AR_BAB) = \phi_{\hat{R}_AR_B}^+ \otimes \phi_{R_B}^+$, where $\phi^+$ stands for the maximally entangled state between the respective spaces. Hence, $\phi_{\hat{R}_AR_B}^+(R_AR_BAB)$ has full Schmidt rank between $R_AR_B$ and $AB$ (even though it is a product state between Alice ($R_AA$) and Bob ($R_BB$)). Therefore, in entanglement theory $LR_{\bar{3}} = LR_{\bar{3}}$. 


The smoothed version of the logarithmic robustness can be defined as \[ LR_{\hat{\delta}}(N) := \min_{N' \in B_\epsilon(N)} LR_{\hat{\delta}}(N') , \] (16)
where
\[ B_\epsilon(N) := \left\{ N' \in \text{CPTP}(A \to B) : \|N' - N\|_\infty \leq \epsilon \right\}. \] (17)

The above diamond-smoothed log-robustness is a straightforward generalization from states to channels, and has an operational interpretation in the setting of resource erasure \[ [14] \], generalizing the single-shot part of \[ [3] \]. However, our goal here is to define a method for smoothing that is the least restrictive possible. This will be necessary for a proof of an AEP for the logarithmic robustness of channels.

For this reason, we consider another (more “liberal”) way to define smoothing for channels for which there is no analog in the state domain. For any state \( \varphi \in D(RA) \) and a channel \( N \in \text{CPTP}(A \to B) \) define \( B_\epsilon^c(N) \) to be the set of all CP maps (not necessarily trace preserving) \( N' \in \text{CP}(A \to B) \) satisfying \[ \|N'_{A \to B}(\varphi_{RA}) - N_{A \to B}(\varphi_{RA})\|_1 \leq \epsilon . \] (18)

Clearly, \( B_\epsilon(N) \subseteq \bigcap_{\varphi \in D(RA)} B_\epsilon^c(N) \). We define the smoothing of \( LR_{\hat{\delta}} \) as \[ LR_{\hat{\delta}}^c(N) := \max_{\varphi \in D(RA)} \min_{N' \in B_\epsilon^c(N)} LR_{\hat{\delta}}(N') . \] (19)

Similarly, we denote by \( LR_{\hat{\delta}}^{\epsilon,n} \) the above smoothing of \( LR_{\hat{\delta}}^c \). Note that the above types of smoothing respect the condition that for \( \epsilon = 0 \) the smoothed quantities reduce to the non-smoothed ones. Furthermore, from its definition it follows that (see SM for more details)
\[ LR_{\hat{\delta}}^c(N) \leq LR_{\hat{\delta}}(N) , \] (20)
justifying the name “liberal smoothing”.

In the SM we show that \( LR_{\hat{\delta}}^c(N) \) is a resource monotone, and the regularized versions
\[ LR_{\hat{\delta}}^{\infty,c}(N) := \lim_{n \to \infty} LR_{\hat{\delta}}^{\epsilon,n}(N^{\otimes n}) / n ; D_{\hat{\delta}}^{\infty,c}(N) := \lim_{n \to \infty} D_{\hat{\delta}}^{\epsilon}(N^{\otimes n}) / n , \]
satisfy \( D_{\hat{\delta}}^{\infty,c}(N) \leq LR_{\hat{\delta}}^{\infty,c}(N) \). We believe that in general this inequality can be strict. However, as we show now, if we revise also the type of regularization, then it is possible to get an equality.

The type of regularization that we consider here is as follows. For each \( n \in \mathbb{N} \), and a channel \( N \in \text{CPTP}(A \to B) \), we define the quantities
\[ D_\delta^{(n)}(N) := \frac{1}{n} \max_{\varphi \in D(RA)} \min_{M \in \hat{\delta}(A^n \to B^n)} D\left( N_{A \to B}^{\otimes n}(\varphi_{RA}) |\!| M_{A^n \to B^n}(\varphi_{RA}) \right) , \] (21)
and \( E_\delta^{(n)} \) is defined exactly as above with \( \hat{\delta}(RA) \) replacing \( D(RA) \).

In the SM we show that the limit \( n \to \infty \) of \( D_\delta^{(n)} \) and \( E_\delta^{(n)} \) exists. We therefore define the “regularized” version of \( D_{\hat{\delta}}^\infty \) and \( E_{\hat{\delta}}^\infty \) to be
\[ D_{\hat{\delta}}^{\infty}(N) = \lim_{n \to \infty} D_\delta^{(n)}(N) ; E_{\hat{\delta}}^{\infty}(N) = \lim_{n \to \infty} E_\delta^{(n)}(N) . \]

We can use this regularization method also for the liberal smoothed logarithmic robustness quantities \( LR_{\hat{\delta}}^c \) and \( LR_{\hat{\delta}}^{\infty,c} \). We define
\[ LR_{\hat{\delta}}^{\epsilon,n}(N) := \frac{1}{n} \max_{\varphi \in D(RA)} \min_{N' \in B_\epsilon^c(N^{\otimes n})} LR_{\hat{\delta}}(N') , \]
\[ LR_{\hat{\delta}}^{\infty}(N) := \lim_{\epsilon \to 0} \lim_{n \to \infty} LR_{\hat{\delta}}^{\epsilon,n}(N) . \] (22)

The quantities \( LR_{\hat{\delta}}^{\epsilon,n} \) and \( LR_{\hat{\delta}}^{\infty} \) are defined analogously with \( \hat{\delta}(RA) \) replacing \( D(RA) \).

**Theorem 5.** For all \( N \in \text{CPTP}(A \to B) \)
\[ D_{\hat{\delta}}^{\infty}(N) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} LR_{\hat{\delta}}^{\epsilon,n}(N^{\otimes n}) = LR_{\hat{\delta}}^{\infty}(N) . \]

Moreover, if for any system \( A \) the extreme points of \( \hat{\delta}(A) \) are pure states then
\[ E_{\hat{\delta}}^{\infty}(N) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} LR_{\hat{\delta}}^{\epsilon,n}(N^{\otimes n}) = LR_{\hat{\delta}}^{\infty}(N) . \]

**Quantum Channel Stein’s Lemma–** (See related work \[ [18] [22] \). Consider the task of discriminating between \( n \) copies of a fixed channel \( N \in \text{CPTP}(A \to B) \) and one of the free channels in \( \hat{\delta}(A^n \to B^n) \). There are two types of errors in such a task:

1. The observer guesses that the channel belongs to \( \hat{\delta}(A^n \to B^n) \) while the channel is \( N_{A \to B}^{\otimes n} \). This occurs with probability \( \alpha_{\hat{\delta}}(N, P_n, \varphi_{RA}) := \text{Tr} \left[ N_{A \to B}^{\otimes n}(\varphi_{RA}) (I - P_n) \right] . \)

Here we consider the “parallel” case, in which the observer only provides \( n \) copies of a free state \( \varphi \in \hat{\delta}(RA) \), and \( 0 \leq P_n \leq I_{A^n \otimes B^n} \).

2. The observer guesses that the channel is \( N_{A \to B}^{\otimes n} \) while the channel is some \( M_n \in \hat{\delta}(A^n \to B^n) \). This occurs with probability
\[ \beta_{\hat{\delta}}(P_n, M_n, \varphi_{RA}) := \text{Tr} \left[ M_n(\varphi_{RA}^{\otimes n}) P_n \right] , \]
and the worst case for a given \( \varphi \in \hat{\delta}(RA) \) is
\[ \beta_{\hat{\delta}}(P_n, \varphi_{RA}) := \max_{M_n \in \hat{\delta}(A^n \to B^n)} \text{Tr} \left[ M_n(\varphi_{RA}^{\otimes n}) P_n \right] . \]
We further define
\[ \beta^{(n)}_{\delta,e}(N, \varphi_{RA}) := \min_{P_n} \beta^{(n)}_{\delta,e}(P_n, \varphi_{RA}), \]
where the minimum is over all \( P_n \) satisfying \( \alpha^{(n)}(N, P_n, \varphi_{RA}) \leq \epsilon \) and \( 0 \leq P_n \leq I_{R^nB^n} \).

**Theorem 6.** Let \( \mathcal{R} \) be a convex resource theory satisfying all the conditions discussed in the introduction, and suppose further that the set of free states contains a full rank state. Then, for all \( \epsilon \in (0,1) \),
\[
\tilde{E}^{(\infty)}_{\delta}(N) = \max_{\varphi \in \mathcal{R}(RA)} \lim_{n \to \infty} -\frac{\log \beta^{(n)}_{\delta,e}(N, \varphi_{RA})}{n},
\]
where
\[
\tilde{E}^{(\infty)}_{\delta}(N) := \max_{\varphi \in \mathcal{R}(RA)} \lim_{n \to \infty} \min_{M \in \mathcal{R}((A^n \to B^n))} D(N^{\otimes n}(\varphi_{RA}^{\otimes n}) || M(\varphi_{RA}^{\otimes n})).
\]

Note that the only difference between \( \tilde{E}^{(\infty)}_{\delta}(N) \) and \( E^{(\infty)}_{\delta}(N) \) is the order between the limit and the maximum. Therefore, we must have \( \tilde{E}^{(\infty)}_{\delta}(N) \leq E^{(\infty)}_{\delta}(N) \), and it is left open to determine if this inequality can be strict. If the latter holds that would mean that \( \tilde{E}^{(\infty)}_{\delta}(N) \) is yet another (distinct) generalization of the relative entropy of a resource.

**Conclusions**– We have seen that \( D_{\delta} \) and \( E_{\delta} \) are asymptotically continuous, satisfy the AEP, and are related to a channel-version of the quantum Stein’s Lemma. To establish these results, we had to adopt two unconventional strategies, liberal smoothing and product-state channel regularization. In this way, lots of the properties in the state domain carry over to the channel domain. In the SM we also introduce additional four generalizations of the relative entropy of a resource. This variety of generalizations indicates that in the channel domain things are much more complicated. We believe that the results and techniques presented here will provide an initial step towards the development of QRT with dynamical resources.

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**Supplemental Material**

**How to quantify a dynamical resource?**

**A ZOO OF RELATIVE ENTROPIES FOR A DYNAMICAL RESOURCE**

We introduce here six functions that generalize the relative entropy measure of static resources (i.e., states) to channels. We start with $\mathcal{D}_\omega$ and $\mathcal{E}_\omega$, and prove Theorem 1. For any $\rho \in \mathcal{D}(A)$, denote the relative entropy of resourceness by

$$D_\omega(\rho) := \min_{\sigma \in \mathcal{D}(A)} D(\rho \| \sigma).$$

(26)

We first show that both $D_\omega(\mathcal{N})$ and $\mathcal{E}_\omega(\mathcal{N})$ reduces to this function when $\mathcal{N}_{A \rightarrow B}$ is the replacement channel that always output a fixed state $\omega_B$.

Indeed, in one direction we have

$$E_\omega(\mathcal{N}) = \min_{\mathcal{M} \in \mathcal{D}(A \rightarrow B)} \sup_{\rho, \omega_B} D(\rho \otimes \omega_B) \| \mathcal{M}_{A \rightarrow B}(\rho_{RA}) \leq \min_{\mathcal{M} \in \mathcal{D}(A \rightarrow B)} \sup_{\rho, \omega_B} D(\rho \otimes \omega_B) \| \rho_{RA} \otimes \gamma_B) \quad (27)$$

where the inequality follows from the restriction of the minimization over $\mathcal{F}(A \rightarrow B)$ to minimization over replacement channels in $\mathcal{F}(A \rightarrow B)$.

For the other direction,

$$E_\omega(\mathcal{N}) = \min_{\mathcal{M} \in \mathcal{D}(A \rightarrow B)} \sup_{\rho, \omega_B} D(\rho \otimes \omega_B) \| \mathcal{M}_{A \rightarrow B}(\rho_{RA}) \geq \min_{\mathcal{M} \in \mathcal{D}(A \rightarrow B)} \max_{\rho} D(\omega_B \| \mathcal{M}_{A \rightarrow B}(\rho_{A})) \quad (28)$$

where the inequality follows from the monotonicity of the divergence under partial trace. This proves that $E_\omega(\mathcal{N}) = D_\omega(\omega_B)$. The proof that $D_\omega(\mathcal{N}) \leq D_\omega(\omega_B)$ follows the exact same lines as above, and the proof that $D_\omega(\mathcal{N}) \geq D_\omega(\omega_B)$ follows from the fact that $D_\omega(\mathcal{N}) \geq E_\omega(\mathcal{N})$. Hence, we also have $D_\omega(\mathcal{N}) = D_\omega(\omega_B)$.

The function $E_\omega$ satisfies (29) for any $\Theta$ of the form (11) with $\mathcal{E}_{\text{post}} \in \text{CPTP}(BE \rightarrow B')$ and $\mathcal{E}_{\text{pre}} \in \text{CPTP}(A' \rightarrow AE)$ both being completely RNG. To see it, note that

$$E_\omega(\Theta[\mathcal{N}]) = \min_{\mathcal{M} \in \mathcal{D}(A \rightarrow B)} \sup_{\rho, \omega_B} D(\Theta[N_{A \rightarrow B}](\rho_{RA}) \| \Omega_{A' \rightarrow B'}(\rho_{RA})) \quad (29)$$

The first inequality follows from the assumption that $\Theta$ is RNG so that $\Theta[\mathcal{M}] \in \mathcal{F}(A' \rightarrow B')$, the second inequality from data processing of $D$, and the third inequality from the assumption that $\mathcal{E}_{\text{pre}}$ is completely RNG.

The faithfulness of $D_\omega$ follows directly from the definition. To prove the faithfulness of $E_\omega$ note that if $E_\omega(\mathcal{N}) = 0$ for some $\mathcal{N} \in \text{CPTP}(A \rightarrow B)$ then from the Klein’s inequality (applied to the relative entropy) for all $\rho \in \mathcal{F}(\mathcal{R}A)$ there exists $\mathcal{M} \in \mathcal{F}(A \rightarrow B)$ such that

$$N_{A \rightarrow B}(\rho_{RA}) = M_{A \rightarrow B}(\rho_{RA}) \in \mathcal{F}(\mathcal{R}B). \quad (30)$$
Lemma 8. In the case that \( \mathcal{N} \) contains a pure state with full Schmidt rank then the equation above (with \( \rho_{RA} \) being that pure state) implies that \( \mathcal{N} = \mathcal{M} \); i.e. \( \mathcal{N} \in \mathfrak{F}(A \rightarrow B) \).

Other relative entropies of a dynamical resource

In addition to \( D_{\mathfrak{F}} \) and \( E_{\mathfrak{F}} \), there are other functionals that extend the relative entropy of a resource from states to channels. Here we discuss four additional generalizations.

Two state-based measures

There are two resource monotones that involve no optimization over channels in \( \mathfrak{F}(A \rightarrow B) \), but only optimization over states. They were introduced very recently in [15, 16]. Let \( \mathcal{N} \in \text{CPTP}(A \rightarrow B) \) and define

\[
R_{\mathfrak{F}}(\mathcal{N}_{A \rightarrow B}) := \sup_{\sigma \in \mathcal{D}(RA)} \left( D_{\mathfrak{F}}(\mathcal{N}_{A \rightarrow B}(\sigma_{RA})) - D_{\mathfrak{F}}(\sigma_{RA}) \right),
\]

(31)

\[
\tilde{R}_{\mathfrak{F}}(\mathcal{N}_{A \rightarrow B}) := \sup_{\sigma \in \mathcal{F}(RA)} D_{\mathfrak{F}}(\mathcal{N}_{A \rightarrow B}(\sigma_{RA})).
\]

(32)

Note that \( \tilde{R}_{\mathfrak{F}} \) can be obtained from the expression above for \( R_{\mathfrak{F}} \), by restricting the supremum over \( \sigma \in \mathcal{D}(RA) \) to \( \sigma \in \mathfrak{F}(RA) \). Hence, we always have \( R_{\mathfrak{F}}(\mathcal{N}) \leq \tilde{R}_{\mathfrak{F}}(\mathcal{N}) \). We now show that both \( R_{\mathfrak{F}} \) and \( \tilde{R}_{\mathfrak{F}} \) behave monotonically under completely RNG superchannels.

Lemma 7. Let \( \Theta : \text{CPTP}(A \rightarrow B) \rightarrow \text{CPTP}(A' \rightarrow B') \) be a superchannel defined by

\[
\Theta[\mathcal{N}_{A \rightarrow B}] := \mathcal{E}_{BE \rightarrow B'}^{\text{post}} \circ \mathcal{N}_{A \rightarrow B} \circ \mathcal{E}_{A' \rightarrow AE}^{\text{pre}},
\]

(33)

with \( \mathcal{E}_{A' \rightarrow AE}^{\text{pre}} \in \text{CPTP}(A' \rightarrow AE) \) and \( \mathcal{E}_{BE \rightarrow B'}^{\text{post}} \in \text{CPTP}(BE \rightarrow B') \) being completely RNG. Then,

\[
R_{\mathfrak{F}}(\Theta[\mathcal{N}]) \leq R_{\mathfrak{F}}(\mathcal{N}) \quad \text{and} \quad \tilde{R}_{\mathfrak{F}}(\Theta[\mathcal{N}]) \leq \tilde{R}_{\mathfrak{F}}(\mathcal{N}).
\]

(34)

Proof. From the definitions we have:

\[
R_{\mathfrak{F}}(\Theta[\mathcal{N}_{A \rightarrow B}]) = \sup_{\sigma \in \mathcal{D}(RA')} \left( D_{\mathfrak{F}} \left( \mathcal{E}_{BE \rightarrow B'}^{\text{post}} \circ \mathcal{N}_{A \rightarrow B} \circ \mathcal{E}_{A' \rightarrow AE}^{\text{pre}}(\sigma_{R'A'}) \right) - D_{\mathfrak{F}}(\sigma_{R'A'}) \right)
\]

\[
\leq \sup_{\sigma \in \mathcal{D}(RA')} \left( D_{\mathfrak{F}} \left( \mathcal{N}_{A \rightarrow B} \circ \mathcal{E}_{A' \rightarrow AE}^{\text{pre}}(\sigma_{R'A'}) \right) - D_{\mathfrak{F}}(\sigma_{R'A'}) \right)
\]

\[
\leq \sup_{\sigma \in \mathcal{D}(RA')} \left( D_{\mathfrak{F}} \left( \mathcal{N}_{A \rightarrow B}(\sigma_{RA'}) \right) - D_{\mathfrak{F}}(\sigma_{R'A'}) \right)
\]

\[
\leq \sup_{\rho \in \mathcal{D}(RA)} \left( D_{\mathfrak{F}}(\mathcal{N}_{A \rightarrow B}(\rho_{RA})) - D_{\mathfrak{F}}(\rho_{RA}) \right)
\]

\[
= R_{\mathfrak{F}}(\mathcal{N}_{A \rightarrow B}).
\]

(35)

In the first inequality we used the fact that \( D_{\mathfrak{F}} \) is monotonic under the RNG map \( \text{id}_R \otimes \mathcal{E}_{BE \rightarrow B'}^{\text{post}} \) (recall that we assume that \( \mathcal{E}_{BE \rightarrow B'}^{\text{post}} \) is completely RNG). Similarly, for the second inequality we used the monotonicity of \( D_{\mathfrak{F}} \) under \( \text{id}_R \otimes \mathcal{E}_{A' \rightarrow AE}^{\text{pre}} \). Finally, we substituted an arbitrary state \( \rho_{RA'} \) instead of \( \mathcal{E}_{A' \rightarrow AE}^{\text{pre}}(\sigma_{R'A'}) \) and set \( R \equiv R'E \). The proof of the monotonicity of \( \tilde{R} \) follows the exact same lines by replacing everywhere the set \( \mathcal{D}(RA) \) with \( \mathfrak{F}(RA) \).

The next lemma shows that \( R_{\mathfrak{F}} \) and \( \tilde{R}_{\mathfrak{F}} \) are indeed generalizations of the relative entropy of a resource.

Lemma 8. In the case that \( \mathcal{N}_{A \rightarrow B} = \omega_B \) is a replacement channel, it holds

\[
R_{\mathfrak{F}}(\mathcal{N}_{A \rightarrow B}) = \tilde{R}_{\mathfrak{F}}(\mathcal{N}_{A \rightarrow B}) = D_{\mathfrak{F}}(\omega_B).
\]

(36)
Proof. We have
\[
R_{\mathfrak{F}}(N_{A\rightarrow B}) = \sup_{\rho \in D(\rho A)} \sup_{\sigma \in \mathfrak{F}(\rho A)} \left( D_{\mathfrak{F}}(\rho R \otimes \omega_B) - D(\rho RA||\sigma RA) \right)
\]
\[
= \sup_{\eta R \in D(\rho A)} \sup_{\sigma \in \mathfrak{F}(\rho A)} \left( D_{\mathfrak{F}}(\rho R \otimes \omega_B) - \inf_{\rho \in D(\rho A); \rho^R = \eta^R} D(\rho RA||\sigma RA) \right)
\]
\[
= \sup_{\eta R \in D(\rho A)} \left( D_{\mathfrak{F}}(\rho R \otimes \omega_B) - \inf_{\sigma \in \mathfrak{F}(\rho A); \rho R = \eta^R} \inf_{\sigma \in \mathfrak{F}(\rho A)} D(\rho RA||\sigma RA) \right).
\]  
(37)

Now, observe that from the data processing inequality
\[
\inf_{\sigma \in \mathfrak{F}(\rho A)} \inf_{\rho \in D(\rho A); \rho^R = \eta^R} D(\rho RA||\sigma RA) \geq \inf_{\sigma \in \mathfrak{F}(\rho A)} D(\eta R||\sigma R) = D_{\mathfrak{F}}(\eta R),
\]  
(38)

where the inequality above is in fact an equality as can be seen by taking \( \rho RA = \eta R \otimes \sigma A \) and \( \sigma RA = \sigma R \otimes \sigma A \). Similarly, by using the subadditivity of \( D_{\mathfrak{F}} \), we get that
\[
D_{\mathfrak{F}}(\eta R \otimes \omega_B) = D_{\mathfrak{F}}(\eta R) + D_{\mathfrak{F}}(\omega_B),
\]  
(39)

so that together with (38) (with the inequality replaced with equality) we conclude
\[
R_{\mathfrak{F}}(N_{A\rightarrow B}) \leq D_{\mathfrak{F}}(\omega_B).
\]  
(40)

To get the other direction, note that restricting \( \eta R \) to \( \mathfrak{F}(\rho A) \) gives
\[
R_{\mathfrak{F}}(N_{A\rightarrow B}) \geq \sup_{\eta R \in \mathfrak{F}(\rho A)} \left( D_{\mathfrak{F}}(\eta R \otimes \omega_B) - \inf_{\sigma \in \mathfrak{F}(\rho A); \rho R = \eta^R} \inf_{\sigma \in \mathfrak{F}(\rho A)} D(\rho RA||\sigma RA) \right)
\]
\[
= \sup_{\eta R \in \mathfrak{F}(\rho A)} D_{\mathfrak{F}}(\eta R \otimes \omega_B)
\]
\[
\geq \sup_{\eta R \in \mathfrak{F}(\rho A)} D_{\mathfrak{F}}(\omega_B)
\]
\[
= D_{\mathfrak{F}}(\omega_B).
\]  
(41)

This completes the proof that \( R_{\mathfrak{F}}(N_{A\rightarrow B}) = D_{\mathfrak{F}}(\omega_B) \). The proof that \( \hat{R}_{\mathfrak{F}}(N_{A\rightarrow B}) = D_{\mathfrak{F}}(\omega_B) \) follows along similar lines.

Two measures that are based on the amortized divergence

There is another way to extend a divergence \( D \) to channels. It was introduced in [23] under the name amortized divergence. It is defined as
\[
D^A(\mathcal{N}||\mathcal{M}) := \sup_{\rho, \sigma \in D(\rho A)} D(\mathcal{N}_{A\rightarrow B}(\rho RA)||\mathcal{M}_{A\rightarrow B}(\sigma RA)) - D(\rho RA||\sigma RA).
\]  
(42)

Like \( D(\mathcal{N}||\mathcal{M}) \), also \( D^A(\mathcal{N}||\mathcal{M}) \) satisfies the generalized data processing inequality [23]. That is, for any superchannel \( \Theta : \text{CPTP}(A \rightarrow B) \rightarrow \text{CPTP}(A' \rightarrow B') \),
\[
D^A(\Theta[\mathcal{N}||\Theta[\mathcal{M}]) \leq D^A(\mathcal{N}||\mathcal{M})).
\]  
(43)

Define two functionals
\[
D^A_{\mathfrak{F}}(\mathcal{N}) := \min_{\mathcal{M} \in \mathfrak{F}(A\rightarrow B)} D^A(\mathcal{N}||\mathcal{M}),
\]  
(44)

\[
E^A_{\mathfrak{F}}(\mathcal{N}) := \min_{\mathcal{M} \in \mathfrak{F}(A\rightarrow B)} \sup_{\rho, \sigma \in D(\rho A)} D(\mathcal{N}_{A\rightarrow B}(\rho RA)||\mathcal{M}_{A\rightarrow B}(\sigma RA)) - D(\rho RA||\sigma RA).
\]  
(45)

Note that for any \( \mathcal{N} \in \text{CPTP}(A \rightarrow B) \), we have by definition
\[
D_{\mathfrak{F}}(\mathcal{N}) \leq D^A_{\mathfrak{F}}(\mathcal{N}) \quad \text{and} \quad E_{\mathfrak{F}}(\mathcal{N}) \leq E^A_{\mathfrak{F}}(\mathcal{N}).
\]  
(46)

Therefore, the faithfulness of these functions follows from that of \( D_{\mathfrak{F}} \) and \( E_{\mathfrak{F}} \). The next lemma shows that they behave monotonically under completely RNG superchannels.
Lemma 9. Let $\Theta : \text{CPTP}(A \rightarrow B) \rightarrow \text{CPTP}(A' \rightarrow B')$ be a superchannel defined by

$$
\Theta[N_{A \rightarrow B}] := \mathcal{E}^{\text{post}}_{BE \rightarrow B'} \circ N_{A \rightarrow B} \circ \mathcal{E}^{\text{pre}}_{A' \rightarrow AE},
$$

with $\mathcal{E}^{\text{pre}} \in \text{CPTP}(A' \rightarrow AE)$ and $\mathcal{E}^{\text{post}} \in \text{CPTP}(BE \rightarrow B')$ being completely RNG. Then,

$$
D_\mathcal{F}^A(\Theta[N]) \leq D_\mathcal{F}^A(\mathcal{N}) \quad \text{and} \quad E_\mathcal{F}^A(\Theta[N]) \leq E_\mathcal{F}^A(\mathcal{N}).
$$

Proof. The monotonicity of $D_\mathcal{F}^A$ follows from the data processing inequality of the amortized divergence. Indeed,

$$
D_\mathcal{F}^A(\Theta[N]) = \min_{\mathcal{F} \in \mathcal{F}(A' \rightarrow B') \cap \mathcal{F}(\mathcal{A} \rightarrow \mathcal{B})} \mathbb{E}_{\mathcal{F}} \left[ D(\Theta[N_{A \rightarrow B}]|\mathcal{F}) \right] \leq \min_{\mathcal{M} \in \mathcal{M}(A \rightarrow B)} D(\Theta[N_{A \rightarrow B}]|\mathcal{M}) = D_\mathcal{F}^A(\mathcal{N}).
$$

The monotonicity of $E_\mathcal{F}^A$ is proved as follows:

$$
E_\mathcal{F}^A(\Theta[N]) = \min_{\mathcal{F} \in \mathcal{F}(A' \rightarrow B') \cap \mathcal{F}(\mathcal{A} \rightarrow \mathcal{B})} \sup_{\rho,\sigma \in \mathcal{F}(\mathcal{A} \rightarrow \mathcal{B})} \mathbb{E}_{\mathcal{F}} \left[ D(\Theta[N_{A \rightarrow B}]|\mathcal{F}) \right] \leq \min_{\mathcal{M} \in \mathcal{M}(A \rightarrow B)} \mathbb{E}_{\mathcal{F}} \left[ D(\Theta[N_{A \rightarrow B}]|\mathcal{M}) \right] = E_\mathcal{F}^A(\mathcal{N}).
$$

The first inequality follows from the assumption that $\Theta$ is RNG, the second and third inequalities follow from data processing inequality of $D$, and the fourth inequality follows from the assumption that $\mathcal{E}^{\text{pre}}$ is completely RNG.

Finally, we show that for a replacement channel $N_{A \rightarrow B}$ that outputs a fixed state $\omega_B$,

$$
D_\mathcal{F}^A(\mathcal{N}) = E_\mathcal{F}^A(\mathcal{N}) = D_\mathcal{F}(\omega_B).
$$

Indeed,

$$
D_\mathcal{F}^A(\mathcal{N}) = \min_{\mathcal{M} \in \mathcal{M}(A \rightarrow B) \cap \mathcal{F}(\mathcal{A} \rightarrow \mathcal{B})} \sup_{\rho,\sigma \in \mathcal{F}(\mathcal{A} \rightarrow \mathcal{B})} \mathbb{E}_{\mathcal{F}} \left[ D(\rho_{R \otimes \omega_B} | \mathcal{M}_{A \rightarrow B} | \mathcal{A} \rightarrow \mathcal{B}) \right] \leq \min_{\mathcal{M} \in \mathcal{M}(A \rightarrow B) \cap \mathcal{F}(\mathcal{A} \rightarrow \mathcal{B})} \mathbb{E}_{\mathcal{F}} \left[ D(\rho_{R \otimes \omega_B} | \mathcal{A} \rightarrow \mathcal{B}) \right] = D_\mathcal{F}(\omega_B),
$$

where the first inequality follows from the restriction of the minimization over $\mathcal{F}(A \rightarrow B)$ to minimization over replacement channels in $\mathcal{F}(A \rightarrow B)$. The second inequality follows from data processing of the relative entropy $D$, and the following equality follows from the additivity of the relative entropy. To prove the other direction, note that $D_\mathcal{F}^A(\mathcal{N}) \geq D_\mathcal{F}(\mathcal{N}) \geq D_\mathcal{F}(\omega_B)$. Hence, $D_\mathcal{F}^A(\mathcal{N}) = D_\mathcal{F}(\omega_B)$.

For the proof the $E_\mathcal{F}^A(\mathcal{N}) = D_\mathcal{F}(\omega_B)$, note that $E_\mathcal{F}^A(\mathcal{N}) \leq D_\mathcal{F}^A(\mathcal{N}) = D_\mathcal{F}(\omega_B)$, and for the other direction, $E_\mathcal{F}^A(\mathcal{N}) \geq E_\mathcal{F}(\mathcal{N}) = D_\mathcal{F}(\omega_B)$. This proves that also $E_\mathcal{F}^A(\mathcal{N}) = D_\mathcal{F}(\omega_B)$. 
Since in the QRT of athermality \(\mathcal{F}(A)\) consists of only one free state, namely the Gibbs state at fixed temperature, some of the relative entropies discussed above take simple forms. Here we discuss a few of them. Let the set of free states consists of a single Gibbs state \(\mathcal{F}(A) = \{\gamma_A\}\) and \(\mathcal{F}(B) = \{\gamma_B\}\). Then,

\[
E_{\mathcal{F}}(\mathcal{N}) = \sup_{|R| \in \mathbb{N}} \inf_{M \in \mathcal{F}(A \rightarrow B)} D(N_{A \rightarrow B}(\gamma_R \otimes \gamma_A) \| M_{A \rightarrow B}(\gamma_R \otimes \gamma_A)) = \inf_{M \in \mathcal{F}(A \rightarrow B)} D(N_{A \rightarrow B}(\gamma_A) \| M_{A \rightarrow B}(\gamma_A)) \leq D(N_{A \rightarrow B}(\gamma_A) \| \gamma_A),
\]

which is the Gibbs free energy of the state \(N_{A \rightarrow B}(\gamma_A)\). Note that this is also the value of \(\tilde{R}_{\mathcal{F}}\) so that in the QRT of athermality we have the collapse

\[
E_{\mathcal{F}}(\mathcal{N}) = \tilde{R}_{\mathcal{F}}(\mathcal{N}) = F(N_{A \rightarrow B}(\gamma_A)),
\]

where \(F\) stands for the free energy.

Finally, we show that \(R_{\mathcal{F}}\) reduces to the thermodynamic capacity in the QRT of athermality.

**Lemma 10.** In the thermodynamic case, in which the set of free states consists of a single Gibbs state \(\mathcal{F}(A) = \{\gamma_A\}\) and \(\mathcal{F}(B) = \{\gamma_B\}\), we have:

\[
R_{\mathcal{F}}(N_{A \rightarrow B}) = \sup_{\sigma \in \mathcal{D}(A)} \left( D(N_{A \rightarrow B}(\sigma_A) \| \gamma_B) - D(\sigma_A \| \gamma_A) \right) = T(N_{A \rightarrow B}),
\]

where \(T(N_{A \rightarrow B})\) is the thermodynamic capacity of the channel as defined in [23] (see also [22], where it is shown that the same quantity is the work cost of implementing \(N_{A \rightarrow B}\) using Gibbs-preserving operations).

**Proof.** In this case,

\[
R_{\mathcal{F}}(N_{A \rightarrow B}) := \sup_{\sigma \in \mathcal{D}(R_A)} \left( D(N_{A \rightarrow B}(\sigma_{RA}) \| \gamma_R \otimes \gamma_B) - D(\sigma_{RA} \| \gamma_R \otimes \gamma_A) \right)
\]

Now, note that

\[
D(N_{A \rightarrow B}(\sigma_{RA}) \| \gamma_R \otimes \gamma_B) - D(\sigma_{RA} \| \gamma_R \otimes \gamma_A) = -H(N_{A \rightarrow B}(\sigma_{RA})) - \text{Tr}[N_{A \rightarrow B}(\sigma_{RA}) \log(\gamma_R \otimes \gamma_B)] + H(\sigma_{RA}) + \text{Tr}[\sigma_{RA} \log(\gamma_R \otimes \gamma_B)]
\]

\[
= D(N_{A \rightarrow B}(\sigma_A) \| \gamma_B) - D(\sigma_A \| \gamma_A) + H(R|A)_{\sigma} - H(R|B)_{N_{A \rightarrow B}(\sigma_{RA})}.
\]

Furthermore, from the data processing inequality we have

\[
H(R|A)_{\sigma_{RA}} \leq H(R|B)_{N_{A \rightarrow B}(\sigma_{RA})},
\]

with equality if \(\sigma_{RA} = \sigma_R \otimes \sigma_A\). This completes the proof.

**MINIMAX THEOREM FOR THE RELATIVE ENTROPY**

Consider a distance parameter \(d : \mathcal{D}(A) \times \mathcal{D}(A) \rightarrow \mathbb{R}_+\) on states that is non-negative and contractive (monotone) under CPTP maps. Let \(\mathcal{S}(RA)\) be a convex set of density matrices. We will take here \(\mathcal{S}(RA) = \mathcal{D}(RA)\) or \(\mathcal{S}(RA) = \mathcal{F}(RA)\). For a channel \(\mathcal{N} \in \text{CPTP}(A \rightarrow B)\), and a QRT \(\mathcal{F}\), define

\[
\mathcal{d}_{\mathcal{F}}(\mathcal{N}) := \sup_{\rho \in \mathcal{S}(RA)} \inf_{M \in \mathcal{F}(A \rightarrow B)} d(N_{A \rightarrow B}(\rho_{RA}), M_{A \rightarrow B}(\rho_{RA})),
\]

\[
\tilde{d}_{\mathcal{F}}(\mathcal{N}) := \inf_{M \in \mathcal{F}(A \rightarrow B)} \sup_{\rho \in \mathcal{S}(RA)} d(N_{A \rightarrow B}(\rho_{RA}), M_{A \rightarrow B}(\rho_{RA})).
\]
By general principles (max-min inequality), $d_{\mathcal{S},S}(\mathcal{N}) \leq d_{\overline{\mathcal{S}},S}(\mathcal{N})$, and we will show equality under mild assumptions on $d$ and the free channels. Concretely, assume that $d$ is jointly concave under orthogonally flagged mixtures: This means that for any two families $\{\rho_x\}$ and $\{\sigma_x\}$ of states, and any probability distribution $\{p_x\}$,

$$d \left( \sum_x p_x \rho_x \otimes |x\rangle \langle x|, \sum_x p_x \sigma_x \otimes |x\rangle \langle x| \right) \geq \sum_x p_x d(\rho_x, \sigma_x),$$

(62)

where $\{|x\rangle\}$ is an orthonormal basis of an auxiliary system. This for example holds with equality for the trace distance, relative entropy, and all the Rényi divergences.

For the case that $\mathcal{S} = \mathcal{F}$, we will assume (in addition to convexity) that there exists a finite dimensional system $R$ such that $\mathcal{F}(R)$ contains at least two orthonormal pure states. Since $\mathcal{F}$ also admits the tensor product structure, this means that there exists a system $R'$ containing any finite number of orthonormal pure states. Hence, combining it with the convexity property, if $\{\rho^i\} \subset \mathcal{F}(A)$ and $\{p_i\}$ is a probability distribution, then there exists a system $R$ and orthonormal set of pure states $\{|i\rangle\} \subset \mathcal{F}(R)$ such that $\sum_i p_i |i\rangle \langle i| \otimes \rho^i_A \in \mathcal{F}(RA)$.

**Proof of Theorem 2**

**Theorem.** For a distance measure satisfying Eq. (62), and assuming that $\mathcal{F}(A \rightarrow B)$ is convex (and satisfies the property above), and that $d$ is convex in the second argument, it holds $d_{\mathcal{S},S}(\mathcal{N}) = \overline{d}_{\mathcal{S},S}(\mathcal{N})$.

**Proof.** We have automatically “$\leq$”, so we will focus on proving “$\geq$”. Fix $R$ for the moment to be a finite-dimensional system. Since $\mathcal{F}(A \rightarrow B)$ is a convex closed set, any channel $M \in \mathcal{F}(A \rightarrow B)$ can be expressed as a convex combination $M = \sum_j q_j M^j$, where each $M^j$ is an extreme channel of $\mathcal{F}(A \rightarrow B)$. Similarly, since $\mathcal{S}(A)$ is convex, every density matrix $\rho_{RA}$ can be expressed as a convex combination $\rho = \sum_i p_i \rho^i$, where each $\rho^i$ is an extreme state of $\mathcal{S}(A)$. This means that the optimization over all channels and states in $\mathcal{F}(A \rightarrow B)$ and $\mathcal{S}(A)$ can be replaced with optimizations over the probability distributions $\{q_j\}$ and $\{p_i\}$. With this in mind we have

$$\sup_{\rho \in \mathcal{S}(RA)} \inf_{M \in \mathcal{F}(A \rightarrow B)} d(\mathcal{N}_{A \rightarrow B}(\rho_{RA}), M_{A \rightarrow B}(\rho_{RA})) = \sup_{\rho \in \mathcal{S}(RA)} \inf_{M \in \mathcal{F}(A \rightarrow B)} \sum_j q_j d(\mathcal{N}_{A \rightarrow B}(\rho_{RA}), M^j_{A \rightarrow B}(\rho_{RA}))$$

$$\leq \inf_{\{q_j\}} \sup_{\rho \in \mathcal{S}(RA)} \sum_j q_j d(\mathcal{N}_{A \rightarrow B}(\rho_{RA}), M^j_{A \rightarrow B}(\rho_{RA}))$$

$$= \inf_{\{q_j\}} \sup_{\{p_i\}} \sum_j p_i q_j d(\mathcal{N}_{A \rightarrow B}(\rho^i_{RA}), M^j_{A \rightarrow B}(\rho^i_{RA}))$$

$$= \sup_{\{p_i\}} \inf_{\{q_j\}} \sum_j p_i q_j d(\mathcal{N}_{A \rightarrow B}(\rho^i_{RA}), M^j_{A \rightarrow B}(\rho^i_{RA}))$$

$$\leq \inf_{\{p_i\}} \sup_{\{q_j\}} \sum_j q_j d(\mathcal{N}_{A \rightarrow B}(\rho^i_{RA}), M^j_{A \rightarrow B}(\rho^i_{RA}))$$

$$= \sup_{\rho \in \mathcal{S}(RR'A)} \inf_{M \in \mathcal{F}(A \rightarrow B)} d(\mathcal{N}_{A \rightarrow B}(\rho_{RR'A}), M_{A \rightarrow B}(\rho_{RR'A})).$$

The first line is because the optimal ensemble $\{q_j, M^j\}$ of free channels will be a point mass on a single optimal channel; the second is due to the general minimax inequality; the third is by the same principle as the first; the fourth line is due to von Neumann’s minmax theorem, noting that the domains of optimization are both convex, and the objective function is linear in either variable; in the fifth, we use the joint concavity with $\mathcal{F} = \sum_i p_i |i\rangle \langle i| \otimes \rho^i_{RA}$; in the sixth line, we enlarge the maximization to arbitrary states on $\mathcal{S}(RR'A)$; and in the seventh we use once more the convex combination principle from lines 1 and 3.

Now, taking the supremum over auxiliary systems $R$, both the l.h.s. and the r.h.s. yield $d_{\overline{\mathcal{S}},S}(\mathcal{N})$, and all inequalities above turn into equalities. In particular, $d_{\mathcal{S},S}(\mathcal{N})$ equals the term in the second line, which evaluates to

$$d_{\mathcal{S},S}(\mathcal{N}) = \inf_{\{q_j\}} \sup_{\rho \in \mathcal{S}(RA)} \sum_j q_j d(\mathcal{N}_{A \rightarrow B}(\rho_{RA}), M^j_{A \rightarrow B}(\rho_{RA}))$$

$$= \inf_{M \in \mathcal{F}(A \rightarrow B)} \sup_{\rho \in \mathcal{S}(RA)} d(\mathcal{N}_{A \rightarrow B}(\rho_{RA}), M_{A \rightarrow B}(\rho_{RA})) = \overline{d}_{\mathcal{S},S}(\mathcal{N}),$$
because the convexity of $\mathfrak{F}$ and $d$.

Without the convexity of $\mathfrak{F}$ and of $d$ in the second argument, there is still something we can do: simply define

$$d_{\mathfrak{F},S}(N) := \inf_{\{q_i\}_{i \in S(\mathcal{RA})}} \sup_{\rho \in S(\mathcal{RA})} \sum_i q_i d(N_{A \rightarrow B}(\rho_{RA}), M_{A \rightarrow B}^i(\rho_{RA})).$$

then the above proof shows

**Lemma 11.** For a distance measure satisfying Eq. (62), it holds $d_{\mathfrak{F},S}(N) = \tilde{d}_{\mathfrak{F},S}(N)$.

**ASYMPTOTIC CONTINUITY**

In this section we prove that the functions $D_{\mathfrak{F}}$ and $E_{\mathfrak{F}}$ are asymptotically continuous. For this purpose, we first need to check if they are bounded from above. Since $E_{\mathfrak{F}} \leq D_{\mathfrak{F}}$ it is sufficient to bound $D_{\mathfrak{F}}$. Now, recall that we only consider here QRTs that admits the tensor product structure, so that the replacement channels $M_\sigma \in \mathfrak{F}(A \rightarrow B)$ for any $\sigma \in \mathfrak{F}(B)$. Hence,

$$D_{\mathfrak{F}}(N) \leq \min_{\sigma \in \mathfrak{F}(B)} \max_{\varphi \in D(\mathcal{RA})} D(N_{A \rightarrow B}(\varphi_{RA}) || \varphi_R \otimes \sigma_B)$$

$$= \min_{\sigma \in \mathfrak{F}(B)} \max_{\varphi \in D(\mathcal{RA})} \left\{ D(N_{A \rightarrow B}(\varphi_{RA}) || \varphi_R \otimes N_{A \rightarrow B}(\varphi_A)) + D(N_{A \rightarrow B}(\varphi_A) || \sigma_B) \right\}$$

$$\leq \log(|AB|) + \min_{\sigma \in \mathfrak{F}(B)} \max_{\varphi \in D(\mathcal{RA})} D(N_{A \rightarrow B}(\varphi_A) || \sigma_B)$$

$$\leq \log(|AB|) + \min_{\sigma \in \mathfrak{F}(B)} \max_{\varphi \in D(\mathcal{RA})} - \text{Tr}[N_{A \rightarrow B}(\varphi_A) \log \sigma_B]$$

$$\leq \log(|AB|) + \min_{\sigma \in \mathfrak{F}(B)} \min_{\varphi \in D(\mathcal{RA})} \log \| \sigma_B^{-1} \|_\infty,$$

where we assumed w.l.o.g. $R \cong A$, and the second line follows from the following triangle equality property of the relative entropy

$$D(\rho_{AB} || \rho_A \otimes \tau_B) = D(\rho_{AB} || \rho_B) + D(\rho_B || \tau_B).$$

We will therefore assume that $\mathfrak{F}(B)$ contain a full rank state to get that $D_{\mathfrak{F}}(N)$ is bounded. For example, if $\mathfrak{F}(B)$ contains the maximally mixed (uniform) state $I_B/|B|$ then

$$D_{\mathfrak{F}}(N_{A \rightarrow B}) \leq \log(|B|^2|A|).$$

**Proof of Theorem 4**

**Weaker Version**

This version only applies to $D_{\mathfrak{F}}$.

**Theorem.** Let $\mathfrak{F}$ be a convex QRT such that

$$\kappa := \max_{N \in \text{CPTP}(A \rightarrow B)} D_{\mathfrak{F}}(N) \leq c \log |AB|$$

for some constant $c \in \mathbb{R}_+$ independent of dimensions. Then, $D_{\mathfrak{F}}$ is asymptotically continuous. In particular, for two channels $N, M \in \text{CPTP}(A \rightarrow B)$ and with $\epsilon = \frac{1}{2} \| N_{A \rightarrow B} - M_{A \rightarrow B} \|_1$, we have

$$|D_{\mathfrak{F}}(N_{A \rightarrow B}) - D_{\mathfrak{F}}(M_{A \rightarrow B})| \leq (1 + \epsilon) h \left( \frac{\epsilon}{1 + \epsilon} \right) + c \kappa,$$

where $h(x) := -x \log x - (1 - x) \log(1 - x)$. 
Proof. We will be using the notation $\mathcal{N}_{AB}^\varepsilon = \sum_{x,y} |x\rangle \langle y|_A \otimes \mathcal{N}_{A\rightarrow B}(|x\rangle \langle y|_A)$ for the Choi matrix of a quantum channel $\mathcal{N} \in \text{CPTP}(A \rightarrow B)$. The diamond norm has been shown to be an SDP \cite{26}, and in particular can be written as

$$\|\mathcal{N} - \mathcal{M}\|_\diamond = \max_{\varphi} \|\mathcal{N}_{A\rightarrow B}(\varphi_{\mathcal{A}A}) - \mathcal{M}_{A\rightarrow B}(\varphi_{\mathcal{A}A})\|_1 = 2 \min_{\omega_{AB} \geq 0 : \omega_{AB} \geq J_{AB}^{N\rightarrow M}} \|\omega_A\|_\infty. \quad (68)$$

Note that there is always an optimal $\omega_{AB}$ such that $\omega_A = \epsilon I_A$. Therefore, the diamond norm can also be expressed as

$$\frac{1}{2} \|\mathcal{N} - \mathcal{M}\|_\diamond = \min_{\omega_{AB} \geq 0 : \omega_{AB} \geq J_{AB}^{N\rightarrow M}} \|\omega_A\|_\infty \quad (69)$$

$$= \min \left\{ \lambda : \lambda J_{\mathcal{A}\mathcal{B}}^\varepsilon \geq J_{AB}^{N\rightarrow M} : \mathcal{E} \in \text{CPTP}(A \rightarrow B) \right\} \quad (70)$$

$$= \min \left\{ \lambda : \lambda \mathcal{E} \geq \mathcal{N} - \mathcal{M} : \mathcal{E} \in \text{CPTP}(A \rightarrow B) \right\} \quad (71)$$

$$= \min \left\{ \mathcal{E} \in \text{CPTP}(A \rightarrow B) \right\} 2^{D_{\max}(N\rightarrow M)(\mathcal{E})} \equiv 2^{L_{\text{CPTP}}(N\rightarrow M)}. \quad (72)$$

That is, the diamond norm can be viewed as the $2^{D_{\max}}$ distance of $\mathcal{N} - \mathcal{M}$ to the set of all quantum channels $\text{CPTP}(A \rightarrow B)$.

Define the CPTP maps $\Delta_{\pm}$ in terms of the optimal matrix $\omega_{AB}$ as (recall that $\omega_A = \epsilon I_A$)

$$J_{AB}^{\Delta_{\pm}} := \epsilon^{-1}\omega_{AB} \quad \text{and} \quad J_{AB}^{\Delta_{\pm}} := J_{AB}^{\Delta_{\pm}} - \epsilon^{-1} J_{AB}^{N\rightarrow M}. \quad (73)$$

Note that

$$J_{AB}^{N\rightarrow M} = \epsilon J_{AB}^{\Delta_{-}}. \quad (74)$$

Dividing both sides by $1 + \epsilon$ gives

$$\Omega_{A\rightarrow B} := \frac{1}{1 + \epsilon} \mathcal{N} + \frac{\epsilon}{1 + \epsilon} \Delta_{-} = \frac{1}{1 + \epsilon} \mathcal{M} + \frac{\epsilon}{1 + \epsilon} \Delta_{+}. \quad (75)$$

Define also

$$\mathcal{E}_{A\rightarrow B} := \frac{1}{1 + \epsilon} \mathcal{E}_{A\rightarrow B}^1 + \frac{\epsilon}{1 + \epsilon} \mathcal{E}_{A\rightarrow B}^2,$$

where $\mathcal{E}_{A\rightarrow B}^1, \mathcal{E}_{A\rightarrow B}^2 \in \mathfrak{F}(A \rightarrow B)$ are free quantum channels. With this at hand, for any channels as above we have

$$D(\Omega_{A\rightarrow B}(\varphi_{RA})) || E_{A\rightarrow B}(\varphi_{RA}) || E_{A\rightarrow B}(\varphi_{RA}) \leq \frac{1}{1 + \epsilon} D(\mathcal{N}_{A\rightarrow B}(\varphi_{RA})) || E_{A\rightarrow B}(\varphi_{RA}) + \frac{\epsilon}{1 + \epsilon} D(\Delta_{A\rightarrow B}(\varphi_{RA})) || E_{A\rightarrow B}(\varphi_{RA})). \quad (76)$$

On the other hand,

$$D(\Omega_{A\rightarrow B}(\varphi_{RA})) || E_{A\rightarrow B}(\varphi_{RA}) = -H(\Omega_{A\rightarrow B}(\varphi_{RA})) - \text{Tr}(\Omega_{A\rightarrow B}(\varphi_{RA})) || E_{A\rightarrow B}(\varphi_{RA})) + \frac{\epsilon}{1 + \epsilon} H(\Delta_{A\rightarrow B}(\varphi_{RA}))$$

$$\geq -h\left(\frac{\epsilon}{1 + \epsilon}\right) - \frac{1}{1 + \epsilon} H(\mathcal{M}_{A\rightarrow B}(\varphi_{RA})) - \frac{\epsilon}{1 + \epsilon} H(\Delta_{A\rightarrow B}(\varphi_{RA}))$$

$$\geq \left(\frac{\epsilon}{1 + \epsilon}\right) - \frac{1}{1 + \epsilon} \text{Tr}(\mathcal{M}_{A\rightarrow B}(\varphi_{RA}) \log \mathcal{E}_{A\rightarrow B}(\varphi_{RA})) - \frac{\epsilon}{1 + \epsilon} \text{Tr}(\Delta_{A\rightarrow B}(\varphi_{RA}) \log \mathcal{E}_{A\rightarrow B}(\varphi_{RA}))$$

$$= h\left(\frac{\epsilon}{1 + \epsilon}\right) + \frac{1}{1 + \epsilon} D(\mathcal{M}_{A\rightarrow B}(\varphi_{RA})) || E_{A\rightarrow B}(\varphi_{RA})) + \frac{\epsilon}{1 + \epsilon} D(\Delta_{A\rightarrow B}(\varphi_{RA})) || E_{A\rightarrow B}(\varphi_{RA})). \quad (77)$$

Combining both (76) and (77) gives

$$\frac{1}{1 + \epsilon} D(\mathcal{M}_{A\rightarrow B}(\varphi_{RA})) || E_{A\rightarrow B}(\varphi_{RA})) - D(\mathcal{N}_{A\rightarrow B}(\varphi_{RA})) || E_{A\rightarrow B}^1(\varphi_{RA})) \right)$$

$$\leq h\left(\frac{\epsilon}{1 + \epsilon}\right) + \frac{\epsilon}{1 + \epsilon} \left( D(\Delta_{A\rightarrow B}(\varphi_{RA})) || E_{A\rightarrow B}(\varphi_{RA}) - D(\Delta_{A\rightarrow B}(\varphi_{RA})) || E_{A\rightarrow B}(\varphi_{RA}) \right). \quad (78)$$
In particular,

\[ D \left( \mathcal{M}_{A \to B}(\varphi_{RA})\|\mathcal{E}_{A \to B}(\varphi_{RA}) \right) - D \left( \mathcal{N}_{A \to B}\|\mathcal{E}_{A \to B}^\lambda \right) \leq (1 + \epsilon)h \left( \frac{\epsilon}{1 + \epsilon} \right) + \epsilon D \left( \Delta_{A \to B}^\lambda \|\mathcal{E}_{A \to B}^\lambda \right). \]  

Finally, choosing \( \varphi_{RA}, \mathcal{E}^1, \) and \( \mathcal{E}^2, \) such that

\[
\begin{align*}
D \left( \mathcal{N}_{A \to B} \|\mathcal{E}_{A \to B}^1 \right) &= D \left( \mathcal{N}_{A \to B} \|\mathcal{E}_{A \to B}^\lambda \right) \\
D \left( \Delta_{A \to B}^\lambda \|\mathcal{E}_{A \to B}^2 \right) &= D \left( \Delta_{A \to B}^\lambda \|\mathcal{E}_{A \to B}^\lambda \right) \\
D \left( \mathcal{M}_{A \to B}(\varphi_{RA})\|\mathcal{E}_{A \to B}(\varphi_{RA}) \right) &= D \left( \mathcal{M}_{A \to B}\|\mathcal{E}_{A \to B}^\lambda \right) \geq D \left( \mathcal{M}_{A \to B} \|\mathcal{E}_{A \to B}^\lambda \right),
\end{align*}
\]

we conclude that

\[ D \left( \mathcal{N}_{A \to B} \right) - D \left( \mathcal{M}_{A \to B} \right) \leq (1 + \epsilon)h \left( \frac{\epsilon}{1 + \epsilon} \right) + \epsilon D \left( \Delta_{A \to B}^\lambda \|\mathcal{E}_{A \to B}^\lambda \right) \leq (1 + \epsilon)h \left( \frac{\epsilon}{1 + \epsilon} \right) + \epsilon k. \]  

This completes the proof.

The proof above can be adjusted in order to prove the asymptotic continuity of \( E_\lambda. \) However, it will be very useful to prove a slightly stronger version of the asymptotic continuity that incorporate both \( D_\lambda \) and \( E_\lambda \) as special cases. We will use this version in the subsequent sections.

**Stronger version**

Let \( \mathcal{S}(RA) \) be a set of density matrices in \( \mathcal{D}(RA). \) For any \( \mathcal{N} \in \text{CPTP}(A \to B) \) denote

\[ E_{\lambda, \mathcal{S}}(\mathcal{N}) := \min_{\mathcal{M} \in \mathcal{S}(A \to B)} \max_{\rho_{RA} \in \mathcal{S}(RA)} D \left( \mathcal{N}_{A \to B}(\rho_{RA})\|\mathcal{M}_{A \to B}(\rho_{RA}) \right). \]  

We will assume here that the extreme points of \( \mathcal{S}(RA) \) are pure states, so that w.l.o.g. \( |R| = |A| \) and there is no need to take supremum over \( |R|. \)

**Lemma 12.** Let \( \mathcal{G} \) be a convex resource theory admitting the tensor product structure. Suppose also that for any system \( B, \mathcal{G}(B) \) contains a full rank state. For a fixed dimension \( |R|, \) let \( \mathcal{S}(RA) \) be a set of density matrices in \( \mathcal{D}(RA), \) whose extreme points are pure states. Further, let \( \mathcal{N} \in \text{CPTP}(A \to B), \) and let \( \{\mathcal{M}_{\varphi}\}_{\varphi \in \mathcal{S}(RA)} \) be a set of CP maps (not necessarily channels) with the property

\[ \|\mathcal{N}_{A \to B}(\varphi_{RA}) - \mathcal{M}_{A \to B}^\varphi(\varphi_{RA})\|_1 \leq \epsilon \quad \forall \varphi \in \mathcal{S}(RA). \]  

Then,

\[ E_{\lambda, \mathcal{S}}(\mathcal{N}_{A \to B}) - \max_{\varphi \in \mathcal{S}(RA)} \min_{\mathcal{M} \in \mathcal{S}(A \to B)} D \left( \mathcal{M}_{A \to B}^\varphi(\varphi_{RA})\|\mathcal{E}_{A \to B}(\varphi_{RA}) \right) \leq f(\epsilon) \log |AB| + \text{Tr} \left( [\gamma_R - \text{Tr}_B [\mathcal{M}_{A \to B}^\varphi(\gamma_{RA})]] \log \gamma_R^{-1} \right), \]  

where \( f(\epsilon) \) is independent on the dimensions and satisfies \( \lim_{\epsilon \to 0} f(\epsilon) = 0. \) and \( \gamma_{RA} \in \mathcal{S}(RA) \) is a pure state defined below in (80).

**Remark.** For the case that for all \( \varphi \in \mathcal{S}(RA), \mathcal{M}_{\varphi} = \mathcal{M} \in \text{CPTP}(A \to B) \) is CPTP and \( \mathcal{S}(RA) = \mathcal{D}(RA) \) with \( |R| = |A|, \) Eq. (83) reduces to \( \|\mathcal{M} - \mathcal{N}\|_1 \leq \epsilon, \) and since \( \mathcal{M} \) is trace preserving, Eq. (84) reduces to

\[ D_{\lambda}(\mathcal{N}_{A \to B}) - D_{\lambda}(\mathcal{M}_{A \to B}) \leq f(\epsilon) \log |AB| \]  

That is, we reproduce that \( D_{\lambda}(\mathcal{N}_{A \to B}) \) is asymptotically continuous.

**Remark.** For the case that for all \( \varphi \in \mathcal{S}(RA), \mathcal{M}_{\varphi} = \mathcal{M} \in \text{CPTP}(A \to B) \) is CPTP and \( \mathcal{S}(RA) = \mathcal{G}(RA), \) the lemma above gives

\[ E_{\lambda}(\mathcal{N}_{A \to B}) - E_{\lambda}(\mathcal{M}_{A \to B}) \leq f(\epsilon) \log |AB| \]  

That is, \( E_{\lambda} \) is also asymptotically continuous.
Moreover, since the trace norm is contractive under partial trace, from \( (83) \) it follows that
\[
\| \gamma_R - \text{Tr}_B [\mathcal{M}_{A \rightarrow B}(\gamma_{RA})] \|_1 \leq \epsilon .
\] (87)
Therefore, we have the bound
\[
\text{Tr} \left[ (\gamma_R - \text{Tr}_B [\mathcal{M}_{A \rightarrow B}(\gamma_{RA})]) \log \gamma_R^{-1} \right] \leq \epsilon \log \| \gamma_R^{-1} \|_{\infty} .
\] (88)

**Proof.** Denote by \( J_{RB}^N := \mathcal{N}_{A \rightarrow B}(\phi_{RA}) \) the Choi matrix of a quantum channel \( \mathcal{N} \in \text{CPTP}(A \rightarrow B) \), and by
\[
\kappa := \min_{\omega \in \mathfrak{S}(B)} \log \| \omega_B^{-1} \|_{\infty} .
\] (89)
Furthermore, for any \( \varphi \in \mathfrak{S}(RA) \) denote by
\[
\tau_{\varphi}^{\pm} = \left( \mathcal{M}_{A \rightarrow B}^{\varphi} (\varphi_{RA}) - \mathcal{N}_{A \rightarrow B}(\varphi_{RA}) \right) \pm
\] (90)
and observe that \( \text{Tr} [\tau_{\varphi}^{\pm} + \tau_{\varphi}^{-\varphi}] \leq \epsilon \). By definition, \( \mathcal{M}_{A \rightarrow B}^{\varphi} (\varphi_{RA}) - \mathcal{N}_{A \rightarrow B}(\varphi_{RA}) = \tau_{\varphi}^{+\varphi} - \tau_{\varphi}^{-\varphi} \) so that
\[
\omega_{RB} := \frac{1}{1 + \epsilon} \mathcal{M}_{A \rightarrow B}^{\varphi} (\varphi_{RA}) + \frac{\epsilon}{1 + \epsilon} \left( \frac{1}{\epsilon} \tau_{RB}^{-\varphi} \right) = \frac{1}{1 + \epsilon} \mathcal{N}_{A \rightarrow B}(\varphi_{RA}) + \frac{\epsilon}{1 + \epsilon} \left( \frac{1}{\epsilon} \tau_{RB}^{-\varphi} \right) .
\] (91)
Also, define
\[
\mathcal{E}_{A \rightarrow B} := \frac{1}{1 + \epsilon} \mathcal{E}_{A \rightarrow B}^1 + \frac{\epsilon}{1 + \epsilon} \mathcal{E}_{A \rightarrow B}^2 ,
\]
where \( \mathcal{E}_{A \rightarrow B}^1, \mathcal{E}_{A \rightarrow B}^2 \in \mathfrak{S}(A \rightarrow B) \) are free quantum channels. With these definitions, for any channels as above we have from the joint convexity of the relative entropy
\[
D \left( \omega_{RB} \| \mathcal{E}_{A \rightarrow B}(\varphi_{RA}) \right) \leq \frac{1}{1 + \epsilon} D \left( \mathcal{M}_{A \rightarrow B}^{\varphi}(\varphi_{RA}) \| \mathcal{E}_{A \rightarrow B}^1(\varphi_{RA}) \right) + \frac{\epsilon}{1 + \epsilon} D \left( \frac{1}{\epsilon} \tau_{RB}^{-\varphi} \| \mathcal{E}_{A \rightarrow B}^2(\varphi_{RA}) \right)
\]
\[
= \frac{1}{1 + \epsilon} D \left( \mathcal{M}_{A \rightarrow B}^{\varphi}(\varphi_{RA}) \| \mathcal{E}_{A \rightarrow B}^1(\varphi_{RA}) \right) - \frac{1}{1 + \epsilon} \text{Tr} \left[ \frac{1}{\epsilon} \tau_{RB}^{-\varphi} \log \mathcal{E}_{A \rightarrow B}^1(\varphi_{RA}) \right] - \frac{\epsilon}{1 + \epsilon} H \left( \frac{1}{\epsilon} \tau_{RB}^{-\varphi} \right) .
\] (92)
On the other hand,
\[
D \left( \omega_{RB} \| \mathcal{E}_{A \rightarrow B}(\varphi_{RA}) \right) = -H \left( \omega_{RB} \right) - \text{Tr} \left( \omega_{RB} \| \mathcal{E}_{A \rightarrow B}(\varphi_{RA}) \right)
\]
\[
\geq -h \left( \frac{\epsilon}{1 + \epsilon} \right) - \frac{1}{1 + \epsilon} H \left( \mathcal{N}_{A \rightarrow B}(\varphi_{RA}) \right) - \frac{\epsilon}{1 + \epsilon} H \left( \frac{1}{\epsilon} \tau_{RB}^{+\varphi} \right)
\]
\[
- \frac{1}{1 + \epsilon} \text{Tr} \left( \mathcal{N}_{A \rightarrow B}(\varphi_{RA}) \log \mathcal{E}_{A \rightarrow B}(\varphi_{RA}) \right) - \frac{1}{1 + \epsilon} \text{Tr} \left( \tau_{RB}^{+\varphi} \log \mathcal{E}_{A \rightarrow B}(\varphi_{RA}) \right)
\]
\[
= \frac{1}{1 + \epsilon} D \left( \mathcal{N}_{A \rightarrow B}(\varphi_{RA}) \| \mathcal{E}_{A \rightarrow B}(\varphi_{RA}) \right) - \frac{1}{1 + \epsilon} \text{Tr} \left( \tau_{RB}^{+\varphi} \log \mathcal{E}_{A \rightarrow B}(\varphi_{RA}) \right) - \frac{\epsilon}{1 + \epsilon} H \left( \frac{1}{\epsilon} \tau_{RB}^{+\varphi} \right) - h \left( \frac{\epsilon}{1 + \epsilon} \right) .
\] (93)
Combining both \((92)\) and \((93)\) gives
\[
D \left( \mathcal{N}_{A \rightarrow B}(\varphi_{RA}) \| \mathcal{E}_{A \rightarrow B}(\varphi_{RA}) \right)
\]
\[
\leq D \left( \mathcal{M}_{A \rightarrow B}^{\varphi}(\varphi_{RA}) \| \mathcal{E}_{A \rightarrow B}^1(\varphi_{RA}) \right) + \text{Tr} \left[ \tau_{RB}^{+\varphi} \log \mathcal{E}_{A \rightarrow B}(\varphi_{RA}) \right] - \text{Tr} \left( \tau_{RB}^{-\varphi} \log \mathcal{E}_{A \rightarrow B}(\varphi_{RA}) \right)
\]
\[
+ \left( 1 + \epsilon \right) h \left( \frac{\epsilon}{1 + \epsilon} \right) + \epsilon \left( H \left( \frac{1}{\epsilon} \tau_{RB}^{+\varphi} \right) - H \left( \frac{1}{\epsilon} \tau_{RB}^{-\varphi} \right) \right) .
\] (94)
(95)
(96)
We now make a few observations. First, note that the last term in the equation above is bounded by
\[
\epsilon \left( H \left( \frac{1}{\epsilon} \tau_{RB}^{+\varphi} \right) - H \left( \frac{1}{\epsilon} \tau_{RB}^{-\varphi} \right) \right) \leq \epsilon \log |RB| .
\] (97)
Second, denote by $t \equiv 2^{D_{\text{max}}(\mathcal{E}^1\|\mathcal{E}^2)}$ the smallest number satisfying $t \mathcal{E}^2 \succeq \mathcal{E}^1$, and observe that

$$\begin{align*}
\text{Tr} \left[ \tau_{RB}^+ \log \mathcal{E}_{A\to B}(\varphi_{RA}) \right] &= \text{Tr} \left[ \tau_{RB}^+ \log \left( \frac{1}{1+\epsilon} \mathcal{E}_{A\to B}^1(\varphi_{RA}) + \frac{\epsilon}{1+\epsilon} \mathcal{E}_{A\to B}^2(\varphi_{RA}) \right) \right] \\
&\leq \text{Tr} \left[ \tau_{RB}^+ \log \left( \frac{t+\epsilon}{1+\epsilon} \mathcal{E}_{A\to B}^2(\varphi_{RA}) \right) \right] \\
&= \text{Tr} \left[ \tau_{RB}^+ \log \left( \frac{t+\epsilon}{1+\epsilon} \mathcal{E}_{A\to B}^2(\varphi_{RA}) \right) + \text{Tr} \left[ \tau_{RB}^+ \log \mathcal{E}_{A\to B}^2(\varphi_{RA}) \right] \right] \\
&\leq \epsilon \log t + \epsilon \log(1 + \epsilon) + \text{Tr} \left[ \tau_{RB}^+ \log \mathcal{E}_{A\to B}^2(\varphi_{RA}) \right],
\end{align*}$$

where the first inequality follows from the operator monotonicity of the log function. Therefore,

$$D \left( \mathcal{N}_{A\to B}(\varphi_{RA}) \| \mathcal{E}_{A\to B}(\varphi_{RA}) \right) \leq D \left( \mathcal{M}_{A\to B}^\varphi(\varphi_{RA}) \| \mathcal{E}_{A\to B}^1(\varphi_{RA}) \right) + (1 + \epsilon) \log \left( \frac{\epsilon}{1+\epsilon} \right) + \epsilon \log(1 + \epsilon) + \epsilon \log |AB| + \epsilon D_{\text{max}}(\mathcal{E}^1\|\mathcal{E}^2) + \text{Tr} \left[ (\tau_{RB}^+ - \tau_{RB}^-) \log \mathcal{E}_{A\to B}^2(\varphi_{RA}) \right].$$

Now, take $\mathcal{E}_{A\to B}^2(X) := \text{Tr}[X]_\omega_B$ to be a constant channel with the full rank state $\omega_B \in \mathcal{B}(B)$ optimizing (89). Then,

$$D_{\text{max}}(\mathcal{E}^1\|\mathcal{E}^2) = \log \min \left\{ t \geq 0 : t I_A \otimes \omega_B \succeq J_{AB}^{\mathcal{E}^1} \right\}.$$

Hence, after minimizing both sides of (100) over $\mathcal{E}^1 \in \mathcal{B}(A \to B)$ we get

$$\min_{\mathcal{E} \in \mathcal{B}(A \to B)} D \left( \mathcal{N}_{A\to B}(\varphi_{RA}) \| \mathcal{E}_{A\to B}(\varphi_{RA}) \right) \leq \min_{\mathcal{E} \in \mathcal{B}(A \to B)} D \left( \mathcal{M}_{A\to B}^\varphi(\varphi_{RA}) \| \mathcal{E}_{A\to B}^1(\varphi_{RA}) \right) + (1 + \epsilon) \log \left( \frac{\epsilon}{1+\epsilon} \right) + \epsilon \log(1 + \epsilon) + \epsilon \log |A|^2 |B| + \epsilon \kappa + \text{Tr} \left[ (\tau_{RB}^+ - \tau_{RB}^-) \log (\varphi_R \otimes \omega_B) \right],$$

Furthermore, let $\gamma_{RA} \in \mathcal{S}(RA)$ be such that

$$E_{\mathcal{B},\mathcal{S}}(\mathcal{N}_{A\to B}) = \min_{\mathcal{E} \in \mathcal{B}(A \to B)} D \left( \mathcal{N}_{A\to B}(\varphi_{RA}) \| \mathcal{E}_{A\to B}(\varphi_{RA}) \right).$$

W.l.o.g. we can assume that $\gamma_{RA}$ is pure since the extreme points of $\mathcal{S}(RA)$ are pure states. With this choice we have

$$E_{\mathcal{B},\mathcal{S}}(\mathcal{N}_{A\to B}) \leq \max_{\varphi \in \mathcal{S}(RA)} \min_{\mathcal{E} \in \mathcal{B}(A \to B)} D \left( \mathcal{M}_{A\to B}^\varphi(\varphi_{RA}) \| \mathcal{E}_{A\to B}(\varphi_{RA}) \right) + (1 + \epsilon) \log \left( \frac{\epsilon}{1+\epsilon} \right) + \epsilon \log(1 + \epsilon) + \epsilon \log |A|^2 |B| + \epsilon \kappa + \text{Tr} \left[ (\tau_{RB}^+ - \tau_{RB}^-) \log (\gamma_R \otimes \omega_B) \right].$$

What is therefore left is to bound the last term in the RHS of (103). For pure $\gamma_{RA}$ (with $|R| = |A|$) set $|\gamma_{RA}| = \gamma_{RB}^{1/2} \otimes I_R |\varphi_{RA}^+\rangle$, so that

$$\begin{align*}
\text{Tr} \left[ (\tau_{RB}^+ - \tau_{RB}^-) \log (\gamma_R \otimes \omega_B) \right] &= \text{Tr} \left[ (\mathcal{M}_{A\to B}^\varphi(\varphi_{RA}) - \mathcal{N}_{A\to B}(\gamma_{RA})) \log (\gamma_R \otimes \omega_B) \right] \\
&= \text{Tr} \left[ (\mathcal{M}_{A\to B}^\varphi(\gamma_A) - \mathcal{N}_{A\to B}(\gamma_A)) \log (\omega_B) \right] + \text{Tr} [\eta_R \log (\gamma_R)],
\end{align*}$$

where

$$\eta_R := \text{Tr}_B \left[ \mathcal{M}_{A\to B}^\varphi(\gamma_{RA}) - \mathcal{N}_{A\to B}(\gamma_{RA}) \right] = \text{Tr}_B \left[ \mathcal{M}_{A\to B}^\varphi(\gamma_{RA}) - \gamma_R \right] = \gamma_{RB}^{1/2} \left( J_{RB}^{\mathcal{M}^\varphi} - I_R \right) \gamma_{RB}^{1/2},$$

where $J_{RB}^{\mathcal{M}^\varphi}$ is the marginal of the Choi matrix of $\mathcal{M}^\varphi$. Further, using the fact that for any Hermitian operator $X$ we have $X \leq |X| = X_+ + X_-$ and $\text{Tr}[X] = ||X||_1$,

$$\begin{align*}
\text{Tr} \left[ (\mathcal{M}_{A\to B}^\varphi(\gamma_A) - \mathcal{N}_{A\to B}(\gamma_A)) \log (\omega_B) \right] &= \text{Tr} \left[ (\mathcal{N}_{A\to B}(\gamma_A) - \mathcal{M}_{A\to B}^\gamma(\gamma_A)) \log (\omega_B^{-1}) \right] \\
&\leq \text{Tr} \left[ |\mathcal{N}_{A\to B}(\gamma_A) - \mathcal{M}_{A\to B}^\gamma(\gamma_A)| \log (\omega_B^{-1}) \right] \\
&\leq ||\mathcal{N}_{A\to B}(\gamma_A) - \mathcal{M}_{A\to B}^\gamma(\gamma_A)||_1 \log ||\omega_B^{-1}||_\infty \\
&\leq \epsilon \kappa.
\end{align*}$$
Combining everything we get
\[ E_{\mathcal{S}, \mathcal{S}}(\mathcal{N}_{A\to B}) \leq \max_{\varphi \in \mathcal{S}(\mathcal{R}A)} \min_{\mathcal{E} \in \mathcal{S}(A\to B)} D\left(\mathcal{M}_{\mathcal{A}\to B}(\varphi_{RA}) \| \mathcal{E}_{A\to B}(\varphi_{RA})\right) + \text{Tr}\left[\gamma_R^{1/2} \left(I_R - J_R^{\mathcal{M}_R}\right) \gamma_R^{1/2} \log \gamma_R^{-1}\right] + g(\epsilon), \]
(110)

where
\[ g(\epsilon) := \epsilon \left(\log |A|^2|B| + 2\kappa\right) + (1 + \epsilon)h\left(\frac{\epsilon}{1+\epsilon}\right) + \epsilon \log (1 + \epsilon). \]
(111)

This completes the proof.}

\[ \text{THE ASYMPTOTIC EQUIPARTITION PROPERTY (AEP)} \]

As defined in the main text the logarithmic robustness of a dynamical resource \( \mathcal{N} \in \text{CPTP}(A \to B) \) is defined as
\[ LR_{\mathcal{S}}(\mathcal{N}_{A\to B}) := \min_{\mathcal{E} \in \mathcal{S}(A\to B)} D_{\text{max}}(\mathcal{N} \| \mathcal{E}) = \log_2 \min\left\{ t : t\mathcal{E}_{A\to B} \geq \mathcal{N}_{A\to B} ; \mathcal{E} \in \mathcal{S}(A \to B) \right\}, \]
(112)

where the notation \( t\mathcal{E}_{A\to B} \geq \mathcal{N}_{A\to B} \) means that \( t\mathcal{E}_{A\to B} - \mathcal{N}_{A\to B} \) is a CP map. We also define
\[ LR_{\mathcal{S}}^r(\mathcal{N}_{A\to B}) := \min_{\mathcal{E} \in \mathcal{S}(A\to B)} \sup_{\varphi \in \mathcal{S}(\mathcal{R}A)} D_{\text{max}}(\mathcal{N}_{A\to B}(\varphi_{RA}) \| \mathcal{E}_{A\to B}(\varphi_{RA})). \]
(113)

We will assume here that the extreme point of \( \mathcal{S}(\mathcal{R}A) \) are pure states so that the optimization above over \( \mathcal{S}(\mathcal{R}A) \) can be taken to be over pure states with \(|R| = |A|\).

\[ \text{Standard Smoothing} \]

The smoothed version of the logarithmic robustness can be defined as
\[ LR_{\mathcal{S}}^r(\mathcal{N}) := \min_{\mathcal{N}' \in B_\epsilon(\mathcal{N})} LR_{\mathcal{S}}(\mathcal{N}'), \]
(114)

with the diamond-norm ball
\[ B_\epsilon(\mathcal{N}) := \left\{ \mathcal{N}' \in \text{CPTP}(A \to B) : \|\mathcal{N}' - \mathcal{N}\|_\diamond \leq \epsilon \right\}. \]
(115)

The above smoothing of \( LR_{\mathcal{S}} \) is a straightforward generalization from states to channels. While we will adopt a different type of smoothing later on, we start by showing that the regularization of \( LR_{\mathcal{S}}^r \) provides an upper bound on the regularization of \( D_{\mathcal{S}} \).

\[ \text{Lemma 13. Let } \mathcal{S} \text{ be a convex QRT, and define} \]
\[ D_\mathcal{S}^\infty(\mathcal{N}) := \lim_{n \to \infty} \frac{1}{n} D_{\mathcal{S}}(\mathcal{N}^\otimes n) ; \quad LR_{\mathcal{S}}^\infty(\mathcal{N}) := \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} LR_{\mathcal{S}}^r(\mathcal{N}^\otimes n). \]
(116)

Then,
\[ D_\mathcal{S}^\infty(\mathcal{N}) \leq LR_{\mathcal{S}}^\infty(\mathcal{N}). \]
(117)

\[ \text{Proof. Let } \mathcal{N}_n^\epsilon \in \text{CPTP}(A^n \to B^n) \text{ and } \mathcal{E}_n \in \mathcal{S}(A^n \to B^n) \text{ be optimal channels such that } \|\mathcal{N}_n^\epsilon - \mathcal{N}^\otimes n\|_\diamond \leq \epsilon \text{ and } LR_{\mathcal{S}}^r(\mathcal{N}^\otimes n) = D_{\text{max}}(\mathcal{N}_n^\epsilon \| \mathcal{E}_n). \]

Using the fact that \( D_{\text{max}} \) is always greater that the relative entropy \( D \), we conclude that
\[ \frac{1}{n} D_{\mathcal{S}}(\mathcal{N}_n^\epsilon) \leq \frac{1}{n} D(\mathcal{N}_n^\epsilon \| \mathcal{E}_n) \leq \frac{1}{n} D_{\text{max}}(\mathcal{N}_n^\epsilon \| \mathcal{E}_n) = \frac{1}{n} LR_{\mathcal{S}}^r(\mathcal{N}^\otimes n). \]
(118)
Now, since $D_3$ is asymptotically continuous there exists a function $f : \mathbb{R} \to \mathbb{R}$ with the property $\lim_{\epsilon \to 0} f(\epsilon) = 0$ such that

$$\frac{1}{n} D_3(N^\otimes n) \leq \frac{1}{n} D_3(N_n^\sigma) + f(\epsilon).$$

Therefore, taking the limit $n \to \infty$ followed by $\epsilon \to 0$ on both sides gives

$$\liminf_{n \to \infty} \frac{1}{n} D_3(N^\otimes n) \leq \liminf_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} LR_3^F(N^\otimes n).$$

This completes the proof. $lacksquare$

### Liberal Smoothing

Let

$$B_\epsilon^F(N) := \left\{ N' \in \text{CP}(A \to B) : \|N'_{A\to B}(\varphi_{RA}) - N_{A\to B}(\varphi_{RA})\|_1 \leq \epsilon \right\},$$

and consider the following types of smoothing:

$$LR_3^F(N) := \max_{\varphi \in \mathcal{D}(RA)} \min_{N' \in B_\epsilon^F(N)} \min_{E \in \bar{\mathcal{E}}(A \to B)} D_{\text{max}}(N'_{A\to B}\|E_{A\to B}),$$

$$LR_3^F(N) := \sup_{\varphi \in \mathcal{D}(RA)} \min_{N' \in B_\epsilon^F(N)} \min_{E \in \bar{\mathcal{E}}(A \to B)} D_{\text{max}}(N'_{A\to B}(\varphi_{RA})\|E_{A\to B}(\varphi_{RA})).$$

Note that both of the above smoothings respect the condition that for $\epsilon = 0$,

$$LR_3^{F=0}(N) = LR_3^F(N),$$

$$LR_3^{F=0}(N) = LR_3^F(N_{A\to B}).$$

For each $\varphi \in \mathcal{D}(RA)$, it holds $B_\epsilon(N) \subset B_\epsilon^F(N)$, hence we have

$$\overline{LR}_3^F(N) \geq \min_{N' \in B_\epsilon^F(N)} LR_3^F(N').$$

Furthermore, since the above equation holds for all $\varphi \in \mathcal{D}(RA)$ we must have

$$\overline{LR}_3^F(N) \geq LR_3^F(N).$$

The above equation holds also even if we define $B_\epsilon^F$ with respect to CPCP maps. That is, define

$$B_\epsilon^{\sigma}(N) := \left\{ N' \in \text{CPTP}(A \to B) : \|N'_{A\to B}(\varphi_{RA}) - N_{A\to B}(\varphi_{RA})\|_1 \leq \epsilon \right\}$$

and

$$LR_3^{\sigma}(N) := \max_{\varphi \in \mathcal{D}(RA)} \min_{N' \in B_\epsilon^{\sigma}(N)} \min_{E \in \bar{\mathcal{E}}(A \to B)} D_{\text{max}}(N'_{A\to B}\|E_{A\to B}).$$

Then, we also have

$$\overline{LR}_3^F(N) \geq LR_3^{\sigma}(N).$$

We now show that if the inequality above is strict, then also the inequality in is strict, and consequently the AEP cannot hold with standard smoothing.

**Lemma 14.** Let $\mathfrak{F}$ be a convex QRT, and define $LR_3^{\infty}(N) := \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} LR_3^F(N^\otimes n)$. Then,

$$D_3^{\infty}(N) \leq LR_3^{\infty}(N).$$
Using the fact that \( D \) Combining this with Lemma 12 we have
\[
\min_{\mathcal{N}' \in B^\epsilon (\mathcal{N}^n)} \min_{\mathcal{E} \in \mathcal{F}(A^n \to B^n)} D_{\max}(\mathcal{N}'||\mathcal{E}) = D_{\max}(\mathcal{M}^n_{\varphi}||\mathcal{E}^\varphi_{n}) .
\]
Using the fact that \( D_{\max} \) is always greater than the relative entropy \( D \), we conclude that
\[
\frac{1}{n} \min_{\mathcal{E} \in \mathcal{F}(A^n \to B^n)} D (\mathcal{M}^n_{\varphi}(\varphi_{R^nA^n})|\mathcal{E}(\varphi_{R^nA^n})) \leq \frac{1}{n} D (\mathcal{M}^n_{\varphi}|\mathcal{E}^\varphi_{n}) \leq \frac{1}{n} D_{\max}(\mathcal{M}^n_{\varphi}|\mathcal{E}^\varphi_{n}) = \frac{1}{n} \min_{\mathcal{E} \in \mathcal{F}(A^n \to B^n)} \min_{\mathcal{N}' \in B^\epsilon (\mathcal{N}^n)} D_{\max}(\mathcal{N}'||\mathcal{E}) .
\]
Combining this with Lemma 12 we have
\[
\frac{1}{n} D_{\delta}(\mathcal{N}^n) \leq \frac{1}{n} \max_{\varphi \in \mathcal{D}(R^nA^n)} \min_{\mathcal{E} \in \mathcal{F}(A^n \to B^n)} D (\mathcal{M}^n_{\varphi}(\varphi_{R^nA^n})|\mathcal{E}(\varphi_{R^nA^n})) + f(\epsilon) \leq \frac{1}{n} LR^\varphi_{\delta}(\mathcal{N}^n) + f(\epsilon) .
\]
Therefore, taking the limit \( n \to \infty \) followed by \( \epsilon \to 0 \) on both sides gives
\[
\lim_{n \to \infty} \frac{1}{n} D_{\delta}(\mathcal{N}^n) \leq \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} LR^\varphi_{\delta}(\mathcal{N}^n) .
\]
This completes the proof.

The above lemma demonstrates that if the standard smoothing leads to different quantities than the liberal smoothing then AEP cannot hold when the quantities are defined with respect to the standard smoothing. This is the reason why we adopt this new type of smoothing.

The liberal smoothing is strongly connected to the underlying QRT. In particular, the functions \( LR^\varphi_{\delta}(\mathcal{N}) \) and \( LR^\varphi_{\delta}(\mathcal{N}) \) remain resource monotones (see the lemma below). Note also that by definition
\[
LR^\varphi_{\delta}(\mathcal{N}) \leq LR^\varphi_{\delta}(\mathcal{N}) .
\]

**Lemma 15.** Let \( \Theta : \mathcal{CPTP}(A \to B) \to \mathcal{CPTP}(A' \to B') \) be a superchannel defined by
\[
\Theta[\mathcal{N}_{A-B}] := \mathcal{E}_{BE \to B'}^\text{post} \circ \mathcal{N}_{A-B} \circ \mathcal{E}_{A' \to AE}^\text{pre},
\]
with \( \mathcal{E}_{pre} \in \mathcal{CPTP}(A' \to AE) \) and \( \mathcal{E}_{post} \in \mathcal{CPTP}(BE \to B') \) being completely RNG. Then,
\[
LR^\varphi_{\delta}(\Theta[\mathcal{N}_{A-B}]) \leq LR^\varphi_{\delta}(\mathcal{N}_{A-B}) ; \ LR^\varphi_{\delta}(\Theta[\mathcal{N}_{A-B}]) \leq LR^\varphi_{\delta}(\mathcal{N}_{A-B}) .
\]

**Proof.** For any channel \( \mathcal{N} \in \mathcal{CPTP}(A \to B) \), we have
\[
LR^\varphi_{\delta}(\Theta[\mathcal{N}_{A-B}]) = \sup_{\varphi \in \mathcal{D}(R^nA'n)} \min_{\mathcal{M} \in \mathcal{CP}(A \to B)} \min_{\mathcal{E} \in \mathcal{F}(A' \to B')} D_{\max}(\mathcal{N}_{A' \to B'}(\varphi_{R'\mathcal{A}'})|\mathcal{M}_{\mathcal{A}'B'}(\varphi_{R'\mathcal{A}'})|\mathcal{E}) .
\]

The second line follows by restricting \( \mathcal{N} \) to have the form \( \Theta[\mathcal{M}] \). The third line by restricting \( \Phi \) to have the form \( \Theta[\Omega] \). The fourth line from data processing inequality of \( D_{\max} \). The fifth line by substituting \( \mathcal{E}_{pre} = \mathcal{E}_{A' \to AE}^\text{pre}(\varphi_{R'\mathcal{A}'}) \) and then optimizing over all \( \sigma \in \mathcal{F}(R'AE) \). The sixth line from the contractivity of the trace norm, and finally, the seventh and eighth by definition. The monotonicity of \( LR^\varphi_{\delta} \) follows similar lines.
Product-State Regularization

One can define the regularized version of $D_\delta$ and $LR_\delta^\psi$ as in \[116\]. Note, however, that unlike the analogous quantity in the state domain, for channels the limit $n \to \infty$ of $\frac{1}{n} D_\delta(N^{\otimes n})$ may not exist in general, so we had to take in \[116\] the lim inf instead. Moreover, it could even be that for some $\mathcal{N}$

$$D_\delta^\infty(\mathcal{N}) > D_\delta(\mathcal{N}) \quad \text{and even} \quad D_\delta^\infty(\mathcal{N}^{\otimes 2}) > 2D_\delta^\infty(\mathcal{N})!$$

(137)

Therefore, this type of regularization does not seem to be very promising, and we will adopt a different type of regularization that avoid these complications.

The type of regularization that we consider here is as follows. For each $n \in \mathbb{N}$, and a channel $\mathcal{N} \in \text{CPTP}(A \to B)$, we define the quantities

$$D_\delta^{(n)}(\mathcal{N}) := \frac{1}{n} \max_{\varphi \in \mathcal{D}(RA)} \min_{\varepsilon \in \mathcal{E}(A^{n} \to B^n)} D \left( (\mathcal{N}^{\otimes n}_{A \to B}(\varphi^{\otimes n}_{RA}))\|\varepsilon_{A^n \to B^n}(\varphi^{\otimes n}_{RA}) \right),$$

$$E_\delta^{(n)}(\mathcal{N}) := \frac{1}{n} \max_{\varphi \in \mathcal{D}(RA)} \min_{\varepsilon \in \mathcal{E}(A^{n} \to B^n)} D \left( (\mathcal{N}^{\otimes n}_{A \to B}(\varphi^{\otimes n}_{RA}))\|\varepsilon_{A^n \to B^n}(\varphi^{\otimes n}_{RA}) \right).$$

(138)

To motivate these definition, we first discuss some of their properties.

First, note that if $\mathcal{N} \in \text{CPTP}(A \to B)$ is the constant channel $\mathcal{N}_{A \to B}(X_A) = \text{Tr}[X_A] \omega_B$ then

$$D_\delta^{(n)}(\mathcal{N}) = E_\delta^{(n)}(\mathcal{N}) = \frac{1}{n} D_\delta(\omega^{\otimes n}_B) := \frac{1}{n} \min_{\sigma \in \mathcal{E}(B^n)} D(\omega^{\otimes n}_B \| \sigma_{B^n}) .$$

(139)

since both $D_\delta(\mathcal{N})$ and $E_\delta(\mathcal{N})$ reduces to $D_\delta(\omega_B)$ for replacement channels. Therefore, this type of regularization, reduces to the standard one when $\mathcal{N}$ is a replacement channel. Next, we prove the following lemma.

**Lemma 16.** For any $\mathcal{N} \in \text{CPTP}(A \to B)$ we have

$$D_\delta^{(n+m)}(\mathcal{N}_{A \to B}) \leq \frac{n}{n+m} D_\delta^{(n)}(\mathcal{N}_{A \to B}) + \frac{m}{n+m} D_\delta^{(m)}(\mathcal{N}_{A \to B}).$$

(140)

The same relation also holds for $E_\delta^{(n)}$.

**Proof.** We have

$$(n+m)E_\delta^{(n+m)}(\mathcal{N}_{A \to B}) = \sup_{\varphi \in \mathcal{D}(RA)} \min_{\varepsilon \in \mathcal{E}(A^{n+m} \to B^{n+m})} D \left( (\mathcal{N}^{\otimes (n+m)}_{A \to B}(\varphi^{\otimes (n+m)}_{RA}))\|\varepsilon(\varphi^{\otimes (n+m)}_{RA}) \right)$$

$$\leq \sup_{\varphi \in \mathcal{D}(RA)} \min_{\varepsilon \in \mathcal{E}(A^{n+m} \to B^{n+m})} D \left( (\mathcal{N}_{A \to B}(\varphi_{RA}))^{\otimes (n+m)}\|\varepsilon(\varphi_{RA})^{\otimes (n+m)} \| \varepsilon_1(\varphi^{\otimes n}_{RA}), \varepsilon_2(\varphi^{\otimes m}_{RA}) \right)$$

$$(141)$$

The same lines of reasoning holds for $D_\delta$ as well.

This lemma implies that the limits of $D_\delta^{(n)}$ and $E_\delta^{(n)}$, as $n \to \infty$, exist. We therefore define the regularized version of $D_\delta$ and $E_\delta$ to be

$$D_\delta^{(\infty)}(\mathcal{N}) = \lim_{n \to \infty} E_\delta^{(n)}(\mathcal{N}) \quad \text{and} \quad E_\delta^{(\infty)}(\mathcal{N}) = \lim_{n \to \infty} E_\delta^{(n)}(\mathcal{N}).$$

(142)

From the lemmas above, the regularized quantities above satisfy

$$D_\delta^{(\infty)}(\mathcal{N}) \leq D_\delta^{(n)}(\mathcal{N}) \quad \text{and} \quad E_\delta^{(\infty)}(\mathcal{N}) \leq E_\delta^{(n)}(\mathcal{N}) \quad \forall \ n \in \mathbb{N}, \ \forall \ \mathcal{N} \in \text{CPTP}(A \to B)$$

(143)

and they are also resource monotones. Furthermore, note that the product-state regularization, $D_\delta^{(\infty)}(\mathcal{N})$, is no greater than the standard regularization $D_\delta^\infty(\mathcal{N})$ as defined in \[116\].
We can use this regularization method also for the smoothed logarithmic robustness quantities $LR^\epsilon_\delta$ and $LLR^\epsilon_\delta$. Define
\begin{equation}
LR^\epsilon_\delta(n)(\mathcal{N}) := \frac{1}{n} \sup_{\varphi \in \mathcal{D}(RA)} \min_{\mathcal{N}' \in B_{\epsilon}^{\otimes n}(\mathcal{N}^{\otimes n})} \min_{\mathcal{E} \in \mathcal{F}(A^n \rightarrow B^n)} D_{\max} \left( \mathcal{N}'_{A^n \rightarrow B^n} \| \mathcal{E}_{A^n \rightarrow B^n} \right),
\end{equation}
\begin{equation}
LLR^\epsilon_\delta(n)(\mathcal{N}) := \frac{1}{n} \sup_{\varphi \in \mathcal{D}(RA)} \min_{\mathcal{N}' \in B_{\epsilon}^{\otimes n}(\mathcal{N}^{\otimes n})} \min_{\mathcal{E} \in \mathcal{F}(A^n \rightarrow B^n)} D_{\max} \left( \mathcal{N}'_{A^n \rightarrow B^n} (\varphi^{\otimes n}) \| \mathcal{E}_{A^n \rightarrow B^n} (\varphi^{\otimes n}) \right),
\end{equation}
\begin{equation}
LR^{(\infty)}_\delta(n)(\mathcal{N}) := \lim_{\epsilon \to 0} \lim_{n \to \infty} LR^\epsilon_\delta(n)(\mathcal{N}),
\end{equation}
\begin{equation}
LLR^{(\infty)}_\delta(n)(\mathcal{N}) := \lim_{\epsilon \to 0} \lim_{n \to \infty} LLR^\epsilon_\delta(n)(\mathcal{N}).
\end{equation}

**Proof of Theorem**

**Theorem.** For all $\mathcal{N} \in \text{CPTP}(A \rightarrow B)$,
\begin{equation}
D^{(\infty)}_\delta(\mathcal{N}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} LR^\epsilon_\delta(n)(\mathcal{N}) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} LLR^\epsilon_\delta(n)(\mathcal{N}) = LR^{(\infty)}_\delta(\mathcal{N}),
\end{equation}
\begin{equation}
E^{(\infty)}_\delta(\mathcal{N}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} LLR^\epsilon_\delta(n)(\mathcal{N}) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} LR^\epsilon_\delta(n)(\mathcal{N}) = LLR^{(\infty)}_\delta(\mathcal{N}).
\end{equation}

**Proof.** We prove the theorem in two steps. First we prove the inequality
\begin{equation}
E^{(\infty)}_\delta(\mathcal{N}) \leq LLR^{(\infty)}_\delta(\mathcal{N}) \quad \forall \mathcal{N} \in \text{CPTP}(A \rightarrow B).
\end{equation}
Let $\epsilon > 0$ and $\varphi \in \mathcal{D}(RA)$. Let $\mathcal{M}^\varphi_n \in \text{CP}(A^n \rightarrow B^n)$ be the optimal CP map such that
\begin{equation}
||\mathcal{M}^\varphi_n(\varphi^{\otimes n}) - \mathcal{N}^{\otimes n}(\varphi^{\otimes n})||_1 \leq \epsilon
\end{equation}
and
\begin{equation}
\min_{\mathcal{N}' \in B_{\epsilon}^{\otimes n}(\mathcal{N}^{\otimes n})} \min_{\mathcal{E} \in \mathcal{F}(A^n \rightarrow B^n)} D_{\max} \left( \mathcal{N}'_{A^n \rightarrow B^n} (\varphi^{\otimes n}) \| \mathcal{E}(\varphi^{\otimes n}) \right) = \min_{\mathcal{E} \in \mathcal{F}(A^n \rightarrow B^n)} D_{\max} \left( \mathcal{M}^\varphi_n(\varphi^{\otimes n}) \| \mathcal{E}(\varphi^{\otimes n}) \right).
\end{equation}
Since $D_{\max}(\rho||\sigma) \geq D(\rho||\sigma)$ for all $\rho$ and $\sigma$ it follows from the above equation that
\begin{equation}
\frac{1}{n} \min_{\mathcal{E} \in \mathcal{F}(A^n \rightarrow B^n)} D(\mathcal{M}^\varphi_n(\varphi^{\otimes n})||\mathcal{E}(\varphi^{\otimes n})) \leq \frac{1}{n} \min_{\mathcal{N}' \in B_{\epsilon}^{\otimes n}(\mathcal{N}^{\otimes n})} \min_{\mathcal{E} \in \mathcal{F}(A^n \rightarrow B^n)} D_{\max} \left( \mathcal{N}'_{A^n \rightarrow B^n} (\varphi^{\otimes n}) \| \mathcal{E}(\varphi^{\otimes n}) \right).
\end{equation}
Therefore, taking the maximum over $\varphi \in \mathcal{D}(RA)$ on both sides gives
\begin{equation}
\frac{1}{n} \max_{\varphi \in \mathcal{D}(RA)} \min_{\mathcal{E} \in \mathcal{F}(A^n \rightarrow B^n)} D(\mathcal{M}^\varphi_n(\varphi^{\otimes n})||\mathcal{E}(\varphi^{\otimes n})) \leq \frac{1}{n} LR^\epsilon_\delta(n)(\mathcal{N}^{\otimes n}).
\end{equation}
Combining this with the asymptotic continuity (see Lemma 12 with $\mathcal{S}$ being the set whose extreme points are the states of the form $\varphi^{\otimes n}$ with $\varphi \in \mathcal{D}(A \rightarrow B)$) gives
\begin{equation}
\frac{1}{n} E^{(n)}_\delta(\mathcal{N}) \leq \frac{1}{n} LR^\epsilon_\delta(n)(\mathcal{N}) + f(\epsilon) \log |AB| + \frac{1}{n} \text{Tr} \left( \gamma_{R}^{\otimes n} - \text{Tr}_{B} \left[ \mathcal{M}^\varphi_n(\gamma_{RA}^{\otimes n}) \right] \log (\gamma_{1}^{\otimes n}) \right),
\end{equation}
where $\gamma_{RA}$ is defined such that
\begin{equation}
E^{(n)}_\delta(\mathcal{N}) = \max_{\varphi \in \mathcal{D}(RA)} \min_{\mathcal{E} \in \mathcal{F}(A^n \rightarrow B^n)} D \left( \mathcal{N}^{\otimes n}(\varphi^{\otimes n}) \| \mathcal{E}(\varphi^{\otimes n}) \right) = \min_{\mathcal{E} \in \mathcal{F}(A^n \rightarrow B^n)} D \left( \mathcal{N}^{\otimes n}(\gamma_{RA}^{\otimes n}) \| \mathcal{E}(\gamma_{RA}^{\otimes n}) \right).
\end{equation}
All that is left to show is that the last term in (154) goes to zero. Note that $\gamma_{RA}$ can depend on $n$. Therefore, we will use the notation $\omega_n \equiv \gamma_{R} \in \mathcal{D}(R)$ to emphasize this dependence.
Let \( \{k \} \) be a subsequence such that
\[
\lim_{k \to \infty} \frac{1}{k} \text{Tr} \left[ R^{\circ,k}_\alpha (N^{\otimes k}) \right] = \liminf_{n \to \infty} \frac{1}{n} \text{Tr} \left[ R^{\circ,n}_\alpha (N^{\otimes n}) \right].
\] (156)

To simplify the notations, we used the notation \( k \) instead of something like \( n_k \). Now recall that
\[
\| \mathcal{M}_n^\gamma (\gamma^{\otimes n}_{RA}) - N^{\otimes n} (\gamma^{\otimes n}_{RA}) \|_1 \leq \epsilon,
\]
and in particular, from the contractivity of the trace norm,
\[
\| \text{Tr}_{B^n} \left[ \mathcal{M}_n^\gamma (\gamma^{\otimes n}_{RA}) \right] - \gamma^{\otimes n}_R \|_1 \leq \epsilon \quad \forall \ n \in \mathbb{N}.
\] (157)

Therefore, if \( \| (\omega^{-1}_k)^{\otimes k} \|_k \) is bounded, then
\[
\frac{1}{k} \text{Tr} \left[ (\omega_k^{\otimes k} - \text{Tr}_{B} [\mathcal{M}_k^\gamma \gamma^{\otimes k}_{RA}]) \log (\omega^{-1}_k)^{\otimes k} \right] \leq \epsilon \log \| \omega^{-1}_k \|_\infty
\] (158)
is bounded and goes to zero as \( \epsilon \to 0 \). We therefore assume now that \( \| \omega^{-1}_k \|_k \) is not bounded. Then, there exists a subsequence \( \{ j \} \subset \{ k \} \) such that \( \lambda_{\min}(\omega_j) \to 0 \) as \( j \to \infty \). Next, we continue to check if there exists a subsequence of \( \{ \omega_j \} \) for which the second smallest eigenvalue of \( \omega_j \) also goes to zero. If there isn’t then we stop. Otherwise, we continue in this way until we find a subsequence of \( n \), let’s call it again for simplicity \( \{ k \} \), such that the first \( m \) largest eigenvalues of \( \omega_k \) are bounded from below, and the remaining \( |R| - m \) eigenvalues are all going to zero in the limit \( k \to \infty \).

We now bound the term
\[
\frac{1}{k} \text{Tr} \left[ (\omega_k^{\otimes k} - \text{Tr}_{B} [\mathcal{M}_k^\gamma \gamma^{\otimes k}_{RA}]) \log (\omega^{-1}_k)^{\otimes k} \right],
\] (159)
which can be expressed equivalently as
\[
\frac{1}{k} \text{Tr} \left[ (\omega_k^{1/2} \otimes k (I_R^{k_1} - J_{R^{k_1}}^{M_2}) (\omega_k^{1/2})^{\otimes k} \log (\omega^{-1}_k)^{\otimes k} \right],
\] (160)
where \( J_{R^{k_1}}^{M_2} \) is the marginal of the Choi matrix of \( \mathcal{M}_k^\gamma \). Next, observe that
\[
\log (\omega^{-1}_k)^{\otimes k} = (\log \omega^{-1}_k \otimes I_R \otimes \cdots \otimes I_R) + \cdots + (I_R \otimes \cdots \otimes I_R \otimes \log \omega^{-1}_k)
\] (161)
It is therefore enough to bound each of the terms
\[
\text{Tr} \left[ (\omega_k^{1/2})^{\otimes k} (I_R^{k_1} - J_{R^{k_1}}^{M_2}) (\omega_k^{1/2})^{\otimes k} \log \omega^{-1}_k \otimes I_R \otimes \cdots \otimes I_R \right] = \text{Tr} \left[ \xi_{R^{k_2}} \omega^{-1}_k \log \omega^{-1}_k \right],
\] (162)
where
\[
\xi_{R^{k_2}} \equiv \text{Tr}_{\neq 1} \left[ \left( I_R \otimes (\omega_k^{1/2})^{(k-1)} \right) (I_R^{k_1} - J_{R^{k_1}}^{M_2}) \left( I_R \otimes (\omega_k^{1/2})^{(k-1)} \right) \right],
\] (163)
with \( \text{Tr}_{\neq 1} \) denoting a trace over all the \( k \) \( R \)-systems except for the first one. Note that from (157) we have \( \text{Tr}[\omega_k^{1/2} \xi_{R}\omega_k^{1/2}] \leq \epsilon \). Now, decompose \( \omega_k = \alpha_k + \beta_k \), where \( \alpha_k = \omega_k P_k \), and \( P_k \) is the projection to the eigenspace of the \( m \) largest eigenvalues of \( \omega_k \), and \( \beta_k = \omega_k (I_R - P_k) \). Since \( \alpha_k \beta_k = \beta_k \alpha_k = 0 \) we have
\[
\text{Tr} \left[ \omega_k^{1/2} \xi_{R} \omega_k^{1/2} \log \omega^{-1}_k \right] = \text{Tr} \left[ \alpha_k^{1/2} \xi_{R} \alpha_k^{1/2} \log \alpha_k^{-1} \right] + \text{Tr} \left[ \beta_k^{1/2} \xi_{R} \beta_k^{1/2} \log \beta_k^{-1} \right],
\] (164)
where the inverses of \( \alpha_k \) and \( \beta_k \) understood as the generalized inverses. Now, observe that
\[
\text{Tr} \left[ \alpha_k^{1/2} \xi_{R} \alpha_k^{1/2} \log \alpha_k^{-1} \right] = \text{Tr} \left[ \omega_k^{1/2} \xi_{R} \omega_k^{1/2} P_k \log \alpha_k^{-1} \right] \leq \text{Tr}[\omega_k^{1/2} \xi_{R} \omega_k^{1/2})_+] \| \alpha_k^{-1} \|_\infty \leq \epsilon \log \| \alpha_k^{-1} \|_\infty,
\] (165)
where \( \| \alpha_k^{-1} \|_\infty \) is bounded. For the other term, note that by definition, since \( \mathcal{M}_k \) is a CP map, its Choi matrix is positive semidefinite so that \( \xi_{R} \leq I_R \). Hence,
\[
\text{Tr} \left[ \beta_k^{1/2} \xi_{R} \beta_k^{1/2} \log \beta_k^{-1} \right] = \text{Tr} \left[ \xi_{R} \beta_k^{1/2} (\log \beta_k^{-1}) \beta_k^{1/2} \right] \leq \text{Tr} \left[ \beta_k^{1/2} (\log \beta_k^{-1}) \beta_k^{1/2} \right] \to 0
\] (166)
as \( k \to \infty \) (since \( \beta_k \to 0 \) as \( k \to \infty \)). To summarize, there exists some constant \( c > 0 \) such that for sufficiently large \( k \)

\[
\text{Tr} \left[ \omega_k^{1/2} \eta R \omega_k^{1/2} \log \omega_k^{-1} \right] \leq c.
\]  

(167)

Since this bound holds for each of the \( k \) terms, we conclude that

\[
\frac{1}{k} \text{Tr} \left[ \left( \omega_k^{1/2} \right)^{\otimes k} \left( I_{R^k} - J^{M_k^1}_R \right) \left( \omega_k^{1/2} \right)^{\otimes k} \log \left( \omega_k^{-1} \right)^{\otimes k} \right] \leq c
\]

(168)

Therefore, by taking on both sides of \( \[164] \) the limit \( n \to \infty \) followed by \( \epsilon \to 0 \) gives

\[
E_3^{(\infty)}(N) \leq \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} LR_3^n(N) \equiv LR_3^{(\infty)}(N).
\]

(169)

We next prove the inequality

\[
E_3^{(\infty)}(N) \geq \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} LR_3^n(N) \quad \forall N \in \text{CPTP}(A \to B).
\]

(170)

This inequality follows by a reasoning very similar to that given in \( \[11] \) for the state domain. Let \( \epsilon > 0 \) and define

\[
r_m := E_3^{(m)}(N) + \epsilon = \max_{\varphi \in \mathcal{F}(RA)} \min_{E \in \mathcal{F}(A \to B^m)} D \left( (N_{A \to B}(\varphi_{RA}))^{\otimes m} \| E_{A \to B^m}(\varphi_{RA}^{\otimes m}) \right) + \epsilon,
\]

(171)

We will also denote by \( \mathcal{E}^\varphi \in \mathcal{F}(A^m \to B^m) \) the optimal channel in \( \mathcal{F}(A^m \to B^m) \) that satisfies

\[
\min_{E \in \mathcal{F}(A^m \to B^m)} D \left( (N_{A \to B}(\varphi_{RA}))^{\otimes m} \| E(\varphi_{RA}^{\otimes m}) \right) = D \left( (N_{A \to B}(\varphi_{RA}))^{\otimes m} \| \mathcal{E}^\varphi(\varphi_{RA}^{\otimes m}) \right).
\]

(172)

For every \( n \in \mathbb{N} \) and \( \varphi_{RA} \in \mathcal{F}(RA) \), we have

\[
(N(\varphi_{RA}))^{\otimes mn} \leq 2^{n r_m} \left( \mathcal{E}^\varphi(\varphi_{RA}^{\otimes m}) \right)^{\otimes n} + \left( N(\varphi_{RA}) \right)^{\otimes mn} - 2^{n r_m} \left( \mathcal{E}^\varphi(\varphi_{RA}^{\otimes m}) \right)^{\otimes n}.
\]

(173)

Denote by

\[
d_m := \text{Tr} \left( (N(\varphi_{RA}))^{\otimes mn} - 2^{n r_m} \left( \mathcal{E}^\varphi(\varphi_{RA}^{\otimes m}) \right)^{\otimes n} \right)_+.
\]

(174)

From \( \[27] \) we have

\[
d_m \leq 2^{-n(r_m t - f(t))},
\]

(175)

where \( t \in [0, 1] \) and

\[
f(t) = \log \text{Tr} \left[ (N^{\otimes m}(\varphi_{RA}^{\otimes m}))^{1+t} \left( \mathcal{E}^\varphi(\varphi_{RA}^{\otimes m}) \right)^{-t} \right].
\]

(176)

Note that \( f(0) = 0 \) and

\[
f'(0) = D(N^{\otimes m}(\varphi_{RA}^{\otimes m}) \| \mathcal{E}^\varphi(\varphi_{RA}^{\otimes m})) \leq E_3^{(m)}(N) = r_m - \epsilon.
\]

(177)

Hence, for small enough \( \epsilon > 0 \) we get that \( r_m t - f(t) > 0 \) which together with \( \[176] \) proves that \( \lim_{n \to \infty} d_m = 0 \) for all \( m \in \mathbb{N} \) and all \( \varphi \in \mathcal{F}(RA) \). Now, recall the following lemma.

**Lemma** \((11, 28)\). Let \( \rho \) and \( \sigma \) be two density matrices, and \( P \geq 0 \) be some positive semidefinite operator satisfying \( \rho \leq P + \epsilon \sigma \) for some \( \epsilon > 0 \). Then, there exists a density matrix \( \tilde{\rho} \) satisfying

\[
\tilde{\rho} \leq \frac{1}{1 - \epsilon} P \quad \text{and} \quad \| \rho - \tilde{\rho} \|_1 \leq 4\sqrt{\epsilon}.
\]

(178)
From this lemma and \([173]\) it follows that there exists a sequence of density matrices \(\eta_{R^{nm}B^{nm}}\) such that
\[
\| (N(\varphi_{RA}))^{\otimes mn} - \eta_{R^{nm}B^{nm}} \|_1 \leq 4\sqrt{\delta_{mn}} \quad \text{and} \quad \eta_{R^{nm}B^{nm}} \leq \frac{1}{1 - \delta_{mn}} 2^{nr_m} (E_\varphi(\varphi_{RA}))^{\otimes n}. \tag{179}
\]
Now, define the CP map \(M^\varphi_{mn} \in \text{CP}(A^{mn} \rightarrow B^{mn})\) that satisfy
\[
\eta_{R^{nm}B^{nm}} = M^\varphi_{mn} (\varphi_{RA}^{\otimes m}). \tag{180}
\]
Such a CP map always exists as long as the bipartite state \(\varphi_{RA}\) is pure. This also implies that
\[
M^\varphi_{mn} \leq \frac{1}{1 - \delta_{mn}} 2^{nr_m} (E_\varphi)^{\otimes n} \quad \text{and} \quad \| (N_{A \rightarrow B}(\varphi_{RA}))^{\otimes mn} - M^\varphi_{mn} (\varphi_{RA}^{\otimes mn}) \|_1 \leq 4\sqrt{\delta_{mn}}. \tag{181}
\]
Let \(n\) be large enough such that \(4\sqrt{\delta_{nm}} \leq \epsilon\). Hence,
\[
\frac{1}{nm} LR_{\delta}^{\epsilon, mn}(N) = \max_{\varphi \in \mathfrak{F}(RA)} \min_{N' \in B^{\otimes mn}} \min_{\varphi' \in \mathfrak{F}} \min_{N'' \in \mathfrak{F}} D_{\text{max}}(N'(\varphi_{RA}^{\otimes mn}) (\varphi_{RA}^{\otimes mn}) (\varphi_{RA}^{\otimes mn}) ) \tag{182}
\]
\[
\leq \max_{\varphi \in \mathfrak{F}(RA)} \min_{N' \in B^{\otimes mn}} (\varphi_{RA}^{\otimes mn}) (\varphi_{RA}^{\otimes mn}) ) \tag{183}
\]
\[
\leq \max_{\varphi \in \mathfrak{F}(RA)} \min_{N' \in B^{\otimes mn}} (\varphi_{RA}^{\otimes mn}) (\varphi_{RA}^{\otimes mn}) ) \tag{184}
\]
\[
\leq \frac{1}{nm} \log(1 - \delta_{mn}). \tag{185}
\]
Hence,
\[
\frac{1}{nm} LR_{\delta}^{\epsilon, mn}(N) \leq \frac{1}{m} E^{(m)}(N) + \epsilon - \frac{1}{nm} \log(1 - \delta_{mn}). \tag{186}
\]
Now, similar to the arguments given in \([11]\) in the state domain, also here we have for any \(m \in \mathbb{N}\)
\[
\limsup_{n \to \infty} \frac{1}{nm} LR_{\delta}^{\epsilon, mn}(N) = \limsup_{n \to \infty} \frac{1}{n} LR_{\delta}^{\epsilon, n}(N^{\otimes n}). \tag{187}
\]
Hence, taking on both sides of \([187]\) the limit \(n \to \infty\) followed by the limit \(\epsilon \to 0\) gives
\[
\lim\sup_{n \to \infty} \frac{1}{n} LR_{\delta}^{\epsilon, n}(N^{\otimes n}) \leq \frac{1}{m} E^{(m)}(N). \tag{188}
\]
Since the above equation holds for all \(m \in \mathbb{N}\), this completes the proof. The proof of the equality for \(D_{\text{max}}^{(\infty)} = LR_{\delta}^{(\infty)}\) follows the exact same lines with \(\mathcal{D}(RA)\) replacing \(\mathfrak{F}(RA)\) everywhere.

**PROOF OF THEOREM 6.**

Recall the two types of errors:

1. The observer guesses that the channel belongs to \(\mathfrak{F}(A^n \rightarrow B^n)\), while the channel really is \(N_{A \rightarrow B}^{\otimes n}\). This occurs with probability
\[
\alpha^{(n)}(N, P_n, \varphi_{RA}) := \operatorname{Tr} \left[ N_{A \rightarrow B}^{\otimes n} (\varphi_{RA}^{\otimes n}) (I - P_n) \right]. \tag{190}
\]

2. The observer guesses that the channel is \(N_{A \rightarrow B}^{\otimes n}\), while the channel really is some \(M_n \in \mathfrak{F}(A^n \rightarrow B^n)\). This occurs with probability
\[
\beta^{(n)}(P_n, M_n, \varphi_{RA}) := \operatorname{Tr} \left[ M_n (\varphi_{RA}^{\otimes n}) P_n \right], \tag{191}
\]
and the worst case for a given \(\varphi_{RA} \in \mathfrak{F}(RA)\) is
\[
\beta^{(n)}(P_n, \varphi_{RA}) := \max_{M_n \in \mathfrak{F}(A^n \rightarrow B^n)} \operatorname{Tr} \left[ M_n (\varphi_{RA}^{\otimes n}) P_n \right]. \tag{192}
\]
We further define
\[ \beta_{\mathcal{F},\epsilon}^{(n)}(N, \varphi_{RA}) := \min \left\{ \beta_{\mathcal{F}}^{(n)}(P_n, \varphi_{RA}) : \alpha^{(n)}(N, P_n, \varphi_{RA}) \leq \epsilon ; \ 0 \leq P_n \leq I^{R^nB^n} \right\}. \] (193)

**Theorem.** Let \( \mathcal{F} \) be a closed convex resource theory admitting the tensor product structure, with the set of free states containing a full rank state. Then, for all \( \epsilon \in (0, 1) \) and all \( \varphi \in \mathcal{F}(RA) \)

\[ \lim_{n \to \infty} - \frac{\log \beta_{\mathcal{F},\epsilon}^{(n)}(N, \varphi_{RA})}{n} = \min_{n \to \infty} \frac{D(N^{\otimes n}_{A\to B}(\varphi_{RA}^{\otimes n}) || M_{A^n \to B^n}(\varphi_{RA}^{\otimes n}))}{n}. \]

**Proof.** Fix \( \varphi \in \mathcal{F}(RA) \) and define
\[ \mathcal{M}_n(\varphi) := \left\{ M_{A^n \to B^n}(\varphi_{RA}^{\otimes n}) : M \in \mathcal{F}(A^n \to B^n) \right\}. \] (194)

We show that the set \( \mathcal{M}_n(\varphi) \) satisfies the 5 properties of [11]:

1. \( \mathcal{M}_n(\varphi) \subset \mathcal{D}(R^nB^n) \) is closed and convex. This holds trivially since \( \mathcal{F}(A^n \to B^n) \) is closed and convex.

2. \( \mathcal{M}_n(\varphi) \) contains a state \( \sigma_{RB}^{\otimes n} \) with \( \sigma \in \mathcal{D}(RA) \) being full rank. Indeed, note that by taking \( \mathcal{M}_n = \Omega_{A\to B}^{\otimes n} \) with \( \Omega \in \mathcal{F}(A \to B) \) we get that \( (\Omega_{A\to B}(\varphi_{RA}))^{\otimes n} \in \mathcal{M}_n(\varphi) \). Further, taking \( \Omega_{A\to B} \) to be the constant channel outputting the fixed full rank state \( \omega_B \in \mathcal{F}(B) \) we get that \( \Omega_{A\to B}(\varphi_{RA}) = \varphi_R \otimes \omega_B \) is a full rank state.

3. For every \( \gamma \in \mathcal{M}_{n+1}(\varphi) \) then \( \text{Tr}_k(\gamma) \in \mathcal{M}_n(\varphi) \) for any \( k = 1, \ldots, n+1 \). Indeed, suppose \( \gamma \in \mathcal{M}_{n+1}(\varphi) \). Then,
\[ \gamma_{R^{n+1}B^{n+1}} = M_{A^{n+1} \to B^{n+1}}(\varphi_{RA}^{\otimes n} \otimes \varphi_{A^n}). \] (195)

Now, by tracing out the last subsystem \( RB \) we get that
\[ \omega_{R^nB^n} := \text{Tr}_{RB}[\gamma_{R^{n+1}B^{n+1}}] = \text{Tr}_B \circ M_{A^{n+1} \to B^{n+1}}(\varphi_{RA}^{\otimes n} \otimes \varphi_A). \] (196)

Define \( \Omega \in \text{CPT}(A^n \to B^n) \) as
\[ \Omega_{A^n \to B^n}(X_{A^n}) := \text{Tr}_B \circ M_{A^{n+1} \to B^{n+1}}(X_{A^n} \otimes \varphi_A) \quad \forall X_{A^n} \in \mathcal{B}(\mathcal{H}_{A^n}). \] (197)

Now, since \( \mathcal{F} \) is a QRT admitting the tensor product structure, and since \( \varphi_A \) is free, it follows that \( \Omega \in \mathcal{F}(A^n \to B^n) \) (i.e. \( \Omega \) is free). Hence,
\[ \omega_{R^nB^n} = \Omega_{A^n \to B^n}(\varphi_{RA}^{\otimes n}) \in \mathcal{M}_n(\varphi). \] (198)

The same conclusion holds if we traced out from \( \gamma_{R^{n+1}B^{n+1}} \) any of the \( n+1 \) RB systems.

4. If \( \gamma \in \mathcal{M}_n(\varphi) \) and \( \eta \in \mathcal{M}_m(\varphi) \) then \( \gamma \otimes \eta \in \mathcal{M}_{n+m}(\varphi) \). Indeed, write \( \gamma_{R^nB^n} = M_{A^n \to B^n}(\varphi_{RA}^{\otimes n}) \) and \( \eta_{R^mB^m} = \Omega_{A^m \to B^m}(\varphi_{RA}^{\otimes m}) \). Then, denote by \( \Delta_{A^n \to B^{n+m}} := M_{A^n \to B^n} \otimes \Omega_{A^m \to B^m} \in \mathcal{F}(A^{n+m} \to B^{n+m}) \) and note that
\[ \gamma_{R^nB^n} \otimes \eta_{R^mB^m} = \Delta_{A^n \to B^{n+m}}(\varphi_{RA}^{\otimes n+m}) \in \mathcal{M}_{n+m}(\varphi). \] (199)

5. If \( \gamma \in \mathcal{M}_n(\varphi) \) then \( P^n \gamma P^{n-1} \in \mathcal{M}_n(\varphi) \) for every permutation \( \pi \in S_n \). Recall that we assume that \( \mathcal{F} \) has the property that if \( \mathcal{M}_n \in \mathcal{F}(A^n \to B^n) \) then also
\[ \Pi_{B^n \to B^n}^{-1} \circ \mathcal{M}_n \circ \Pi_{A^n \to A^n}^{-1} \in \mathcal{F}(A^n \to B^n), \] (200)

where
\[ \Pi_{A^n \to A^n}(X_{A^n}) = P^n X_{A^n} P^{n-1}, \] (201)

with \( \{P^\pi_{A^n}\} \) a representation of the permutation group in \( \mathcal{H}_{A^n}^{\otimes n} \). Then, for any permutation \( \pi \in S_n \)
\[ P^n R^nB^n \left( M_{A^n \to B^n}(\varphi_{RA}^{\otimes n}) \right) P_{R^nB^n}^{n-1} = \Pi_{R^nB^n}^{\pi} \circ (\text{id}_{R^n} \otimes M_{A^n \to B^n})(\varphi_{RA}^{\otimes n}) \]
\[ = \Pi_{R^nB^n}^{\pi} \circ (\text{id}_{R^n} \otimes M_{A^n \to B^n}) \circ \Pi_{R^nA^n}^{n-1}(\varphi_{RA}^{\otimes n}) \] (202)
\[ \left( \text{id}_{R^n} \otimes \Pi_{R^nA^n}^{\pi} \circ M_{A^n \to B^n} \circ \Pi_{A^n}^{n-1} \right)(\varphi_{RA}^{\otimes n}) \in \mathcal{M}_n(\varphi). \]
Since the set $\mathcal{M}_n(\varphi)$ satisfies all the 5 properties of [11], the main result of [11], which includes both the direct part and strong converse, can be applied to $\mathcal{M}_n(\varphi)$. In particular, it follows that for any $\epsilon \in (0, 1)$

$$
\lim_{n \to \infty} \frac{-\log \beta_{\mathcal{S}, \varphi}^{(n)}(\mathcal{N}, \varphi_{RA})}{n} = \lim_{m \to \infty} \frac{1}{m} \min_{M \in \mathcal{S}(A^m \rightarrow B^m)} D(\mathcal{N}^{\otimes m}_{A^m \rightarrow B^m}(\varphi_{RA}^m) \parallel M(\varphi_{RA}^m)).
$$

(203)

This concludes the proof.

**LOWER BOUND ON THE CHERNOFF BOUND**

Suppose Alice is given with $t_0$ probability the channel $\mathcal{N}_{A \rightarrow B}^{\otimes n}$ and with $t_1$ probability one of the channels in $\mathfrak{F}(A^n \rightarrow B^n)$. Alice’s goal is to determine if she is holding in her lab $\mathcal{N}_{A \rightarrow B}^{\otimes n}$ or one of the channels in $\mathfrak{F}(A^n \rightarrow B^n)$. The probability of error is therefore given by

$$
P_{\text{error}}^{(n)}(\varphi) = \max_{\mathcal{M}_n \in \mathfrak{F}(A^n \rightarrow B^n)} \frac{1}{2} \left( 1 - \max_{\varphi \in \mathfrak{F}(A^n \rightarrow B^n)} \min_{\mathcal{M}_n \in \mathfrak{F}(A^n \rightarrow B^n)} ||t_0 \mathcal{N}_{A \rightarrow B}^{\otimes n}(\varphi_{RA}^n) - t_1 \mathcal{M}_n(\varphi_{RA}^n)||_1 \right). 
$$

(204)

We had to maximize the error over all possible channels in $\mathfrak{F}$ to get the worst case scenario. She will therefore choose $\varphi$ to minimize the above quantity. That is,

$$
P_{\text{error}}^{(n)} = \min_{\varphi \in \mathfrak{F}(A^n \rightarrow B^n)} P_{e}(\varphi) = \frac{1}{2} \left( 1 - \max_{\varphi \in \mathfrak{F}(A^n \rightarrow B^n)} \min_{\mathcal{M}_n \in \mathfrak{F}(A^n \rightarrow B^n)} ||t_0 \mathcal{N}_{A \rightarrow B}^{\otimes n}(\varphi_{RA}^n) - t_1 \mathcal{M}_n(\varphi_{RA}^n)||_1 \right). 
$$

(205)

In [29] it was shown that for any two positive operators $A$ and $B$ and $\alpha \in (0, 1)$ we have

$$
\text{Tr}[A^\alpha B^{1-\alpha}] \geq \frac{1}{2} \text{Tr}[A + B - |A - B|].
$$

(206)

Hence, for any $0 \leq \alpha \leq 1$,

$$
\frac{1}{2} ||t_0 \mathcal{N}_{A \rightarrow B}^{\otimes n}(\varphi_{RA}^n) - t_1 \mathcal{M}_n(\varphi_{RA}^n)||_1 \geq \frac{1}{2} - t_0^\alpha t_1^{1-\alpha} \text{Tr}\left( (\mathcal{N}_{A \rightarrow B}^{\otimes n}(\varphi_{RA}^n))^{\alpha} (\mathcal{M}_n(\varphi_{RA}^n))^{1-\alpha} \right),
$$

(207)

so that

$$
P_{\text{error}}^{(n)} \leq t_0^\alpha t_1^{1-\alpha} \max_{\varphi \in \mathfrak{F}(A^n \rightarrow B^n)} \min_{\mathcal{M}_n \in \mathfrak{F}(A^n \rightarrow B^n)} \text{Tr}\left( (\mathcal{N}_{A \rightarrow B}^{\otimes n}(\varphi_{RA}^n))^{\alpha} (\mathcal{M}_n(\varphi_{RA}^n))^{1-\alpha} \right). 
$$

(208)

We therefore conclude that

$$
\liminf_{n \to \infty} -\frac{1}{n} \log P_{\text{error}}^{(n)} \geq \max_{\alpha \in [0, 1]} (1 - \alpha) G_{\mathfrak{F}, \alpha}^{\infty}(\mathcal{N}),
$$

(209)

where

$$
G_{\mathfrak{F}, \alpha}^{\infty}(\mathcal{N}) := \liminf_{n \to \infty} \frac{1}{n} \min_{\varphi \in \mathfrak{F}(A^n \rightarrow B^n)} \max_{\mathcal{M}_n \in \mathfrak{F}(A^n \rightarrow B^n)} D_\alpha(\mathcal{N}_{A \rightarrow B}^{\otimes n}(\varphi_{RA}^n) \parallel \mathcal{M}_n(\varphi_{RA}^n)),
$$

(210)

where $D_\alpha$ is the Petz quantum Renyi divergence.