WEIGHT HARDY-LITTLEWOOD INEQUALITIES FOR DIFFERENT POWERS.

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Abstract.
In this short article we obtain the non-asymptotic upper and low estimations for linear and bilinear weight Riesz’s functional through the Lebesgue spaces.

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1. Introduction. Statement of problem.

The linear integral operator $I_{\alpha,\beta,\lambda}f(x)$, or, more precisely, the family of operators of a view

$$u(x) = I_{\alpha,\beta,\lambda}f(x) = I[f](x) = |x|^{-\beta} \int_{R^d} \frac{f(y) |y|^{-\alpha}}{|x-y|^{\lambda}} dy$$

is called Weight Riesz’s integral operator, or simply Weight Riesz’s potential, or weight fractional integral.

Here $|x|$, $x \in R^d$ denotes usually Euclidean norm of the vector $x$, $d = 1, 2, 3, \ldots$; $\alpha, \beta, \lambda = \text{const}, \alpha, \beta \geq 0$, $\lambda > 0$; $\alpha + \beta + \lambda < d$.

The bilinear Weight Riesz’s functional $B_{\alpha,\beta,\lambda}(f, g) = B(f, g)$ may be defined as follows:

$$B(f, g) = \int_{R^d} \int_{R^d} \frac{f(y) |x|^{-\beta} g(x) \ dx \ dy}{|y|^{\alpha} |x-y|^{\lambda}}.$$

It is evident that

$$B_{\alpha,\beta,\lambda}(f, g) = (I_{\alpha,\beta,\lambda}f, g),$$

where $(f, g)$ denotes ordinary the inner product of a (measurable) functions $f$ and $g$:

$$(f, g) = \int_{R^d} f(x) \ g(x) \ dx.$$
which is defined, e.g. when $f \in L_m$, $g \in L_l$, $m, l > 1, 1/m + 1/l = 1$, $L_m = L_m(R^d)$ is the classical Lebesgue space of all the measurable functions $f : R^d \to R$ with finite norm

$$|f|_m \overset{\text{def}}{=} \left( \int_{R^d} |f(x)|^m \, dx \right)^{1/m}; \quad f \in L_m \Leftrightarrow |f|_m < \infty.$$  

Obviously,

$$\sup_{|g|_l = 1} |B_{\alpha,\beta,\lambda}(f, g)| = |I_{\alpha,\beta,\lambda}f|_m.$$  

The operators $I_{\alpha,\beta,\lambda}$ and correspondingly the functionals $B_{\alpha,\beta,\lambda}$ are used in the theory of Fourier transform, theory of Partial Differential Equations, probability theory (study of potential functions for Markovian processes and spectral densities for stationary random fields), in the functional analysis, in particular, in the theory of interpolation of operators etc., see for instance [1], [2], [25], [26], [11], [13], [24], [18], [19], [20], [21], [6], [7] etc.

We denote also $L(a,b) = \bigcap_{p \in (a,b)} L_p$.

We will investigate the estimations of a view:

$$|I_{\alpha,\beta,\lambda}f|_q \leq K_{\alpha,\beta,\lambda}(p) |f|_p, \quad (1.a)$$

$$|B_{\alpha,\beta,\lambda}(f, g)| \leq K_{\alpha,\beta,\lambda}(p) |f|_p |g|_{q/(q-1)}, \quad (1.b)$$

with asymptotically exact values of coefficient $K$.

Note that the case $\alpha = \beta = 0$, i.e. the case of classical Riesz potential, is considered in many publications [1], [5], [11], [12], [13], [14], [10], [26], [27], [31] etc. General case $\alpha^2 + \beta^2 > 0$ is partially considered, e.g. in [4], [5], [25], [11], [17], [8], [23], [22].

In the recent publication [16] is considered the weight Riesz potential for the so-called radial function, i.e. for the functions $f(x)$ which dependent only on the Euclidean norm of a vector $x : f(x) = H(|x|)$. It is obtained in [16] the upper bound for the weight Riesz potential without the constants $K_{\alpha,\beta,\lambda}(p)$ estimations.

We intend to improve the results of these works and to obtain the low bounds for the functionals $|B_{\alpha,\beta,\lambda}(f, g)|$ and $|I_{\alpha,\beta,\lambda}(f)|$.

In order to formulate the main result, we must introduce some notations. We define first of all the following function $q = q(p)$ as follows:

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{\alpha + \beta + \gamma}{d}. \quad (2)$$

We will denote the set of all such a values $(p, q)$ as $G(\alpha, \beta, \lambda)$ or for simplicity $G = G(\alpha, \beta, \lambda)$.

FURTHER WE WILL SUPPOSE THAT $(p, q) \in G(\alpha, \beta, \lambda) = G$.

It is known [11], [25], [17] that the inequalities (1a) and (1b) are possible only in the case when $(p, q) \in G(\alpha, \beta, \lambda)$.

We denote also

$$p_- := \frac{d}{d - \alpha}, \quad p_+ := \frac{d}{d - \alpha - \lambda};$$

and correspondingly

$$q_- := \frac{d}{\beta + \lambda}, \quad q_+ := \frac{d}{\beta}.$$
where in the case $\beta = 0 \Rightarrow q_+ := +\infty$;

$$\kappa = \kappa(\alpha, \beta, \lambda) := (\alpha + \beta + \lambda)/d.$$  

It is known [25], [17] that if $p \in [1, p_-] \cup [p_+, \infty)$, then $K_{\alpha, \beta, \lambda}(p) = +\infty$. Thus, we confine the values $p$ inside the open interval $p \in (p_-, p_+)$.  

Define the exact value of the constant $K_{\alpha, \beta, \lambda}(p)$, i.e. the value

$$V(p) = V_{\alpha, \beta, \lambda}(p) = \sup_{f \in L(p_-, p_+), f \neq 0} \frac{|I_{\alpha, \beta, \lambda}(f)|}{|f|^p}. \quad (3)$$

We use symbols $C(X, Y), C(p, q; \psi)$, etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like $C_{1}(X, Y)$ and $C_{2}(X, Y)$. The relation $g(\cdot) \approx h(\cdot), p \in (A, B)$, where $g = g(p), h = h(p), g, h : (A, B) \to R_+$, denotes as usually

$$0 < \inf_{p \in (A, B)} h(p)/g(p) \leq \sup_{p \in (A, B)} h(p)/g(p) < \infty.$$  

The symbol $\sim$ will denote usual equivalence in the limit sense.

We will denote as ordinary the indicator function $I(x \in A) = 1, x \in A, I(x \notin A) = 0, x /\in A$; here $A$ is a measurable set.

\[ \square \]

2. Main Result: upper and low bounds for weight Riesz’s potential.

**Theorem 1.** For the values $(p, q) \in G(\alpha, \beta, \lambda)$ and $p \in (p_-, p_+)$ there holds:

$$\frac{C_1(\alpha, \beta, \lambda)}{[(p-p_-) \cdot (p_+ - p)]^\kappa} \leq V_{\alpha, \beta, \lambda}(p) \leq \frac{C_2(\alpha, \beta, \lambda)}{[(p-p_-) \cdot (p_+ - p)]^\kappa}. \quad (4)$$

Recall that for the values $(p, q) \notin G(\alpha, \beta, \lambda)$ or for the values $p \notin (p_-, p_+) V_{\alpha, \beta, \lambda}(p) = \infty$.

**Proof of the upper bound** it follows immediately from [17], pp. 215-219 after simple calculations.

Another method used in [20], [21] based on the theory of maximal operators, see in [1], [3], [4], [22], [23].

**Proof of the low bound.** We will consider two examples of a functions from the set $L(p_-, p_+)$.  

**First example.**

$$f_0(x) = |x|^{-(d-\alpha)} I(|x| > 1).$$

We find by direct calculations using the multidimensional polar coordinates:

$$|x| \geq 1 \Rightarrow u_0(x) := I_{\alpha, \beta, \lambda} f_0(x) \geq C_{\alpha, \beta, \lambda}^{(1)} |x|^{1-\beta-\lambda} \log |x| \cdot I(|x| > 1);$$
\[ |u_0|_q \geq C_{\alpha,\beta,\lambda}^{(2)} \times [q - q_0]^{-1-1/q}, \]

recall that \( q = q(p) \).

**Second example.** We put:

\[ g_0(x) = |x|^{-(d-\alpha-\lambda)} I(|x| < 1), \]

and find:

\[ |g_0|_p \asymp (p_+ - p)^{-1/p} \asymp (p_+ - p)^{-\alpha-\lambda/d}, \]

\[ v_0(x) := I_{\alpha,\beta,\lambda} g_0(x) \geq C^{(3)}(\alpha, \beta, \lambda) \cdot |x|^{-\beta} \cdot |\log |x|| \cdot I(|x| < 1); \]

\[ |v_0|_q \geq C^{(4)}(\alpha, \beta, \lambda) \cdot (q_+ - q)^{-1-1/q}, \quad p \in (p_-, p_+). \]

**Third example. Summing.**

We define:

\[ h(x) := f_0(x) + g_0(x); \quad w(x) := I_{\alpha,\beta,\lambda} |h|(x). \]

It follows for the function \( h = h(x) \), which belongs to the space \( L(p_-, p_+) \), that for the values \( p \) from the considered interval \( p \in (p_-, p_+) \)

\[ \inf_{p \in (p_-, p_+)} \frac{|w|_q \left[ (p - p_-)(p_+ - p) \right]^\kappa}{|h|_p} =: C^{(6)}(\alpha, \beta, \gamma; d) > 0. \]

This completes the proof of theorem 1.

\[ \square \]

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