Solvable groups definable in o-minimal structures

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January 24, 2000

Abstract

Let \( \mathcal{N} \) be an o-minimal structure. In this paper we develop group extension theory over \( \mathcal{N} \) and use it to describe \( \mathcal{N} \)-definable solvable groups. We prove an o-minimal analogue of the Lie-Kolchin-Mal’cev theorem and we describe \( \mathcal{N} \)-definable \( G \)-modules and \( \mathcal{N} \)-definable rings.
1 Introduction

We will work inside an o-minimal structure $\mathcal{N} = (\mathbb{N}, <, \ldots)$ and therefore definable will mean $\mathcal{N}$-definable. We will assume the readers familiarity with basic o-minimality (see [vdd]). We will start by recalling some basic notions and results on definable groups that will be used through the paper.

Pillay in [p] adapts Hrushovski’s proof of Weil’s Theorem that an algebraic group can be recovered from birational data to show that a definable group $G$ can be equipped with a unique definable manifold structure making the group into a topological group, and definable homomorphisms between definable groups are topological homomorphisms. In fact, as remarked in [pps1], if $\mathcal{N}$ is an o-minimal expansion of a real closed field $G$ equipped with the above unique definable manifold structure is a $C^p$ group for all $p \in \mathbb{N}$; and definable homomorphisms between definable groups are $C^p$ homomorphisms for all $p \in \mathbb{N}$. Moreover, again by [pps1], the definable manifold structure on a definable subgroup is the sub-manifold structure.

By [p] definable groups satisfies the descending chain condition (DCC) on definable subgroups. This is used to show that the definably connected component of the identity $G^0$ of a definable group $G$ is the smallest definable subgroup of $G$ of finite index. Also infinite such groups have infinite definable abelian subgroups; a definable subgroup $H$ of $G$ is closed and the following are equivalent (i) $H$ has finite index in $G$, (ii) $\dim H = \dim G$, (iii) $H$ contains an open neighbourhood of the identity element of $G$ and (iv) $H$ is open in $G$. Finally, by [p] an infinite abelian definable group $G$ has unbounded exponent and the subgroup $Tor(G)$ of torsion points of $G$ is countable. In particular, if $\mathcal{N}$ is $\aleph_0$-saturated then $G$ has an element of infinite order.

One dimensional definable manifolds are classified in [p] and the following is deduced. Suppose that $G$ is one-dimensional definably connected definable group. Then by [p] $G$ is abelian, and $G$ is torsion-free or for each prime $p$ the set of $p$-torsion points of $G$ has $p$ elements. In the former case $G$ is an ordered abelian divisible definably simple group.

Note that if $I$ is a one-dimensional definably connected ordered definable group, then the structure $\mathcal{I}$ induced by $\mathcal{N}$ on $I$ is o-minimal. In particular, we have the following results from [ms]. Suppose that $(I, 0, 1, +, <)$ is a one-dimensional definably connected torsion-free definable group, where $1$ is a fixed positive element. Let $\Lambda(\mathcal{I})$ be the division ring of all $\mathcal{I}$-definable
endomorphisms of \((I, 0, +)\). Then exactly one of the following holds: (1) \(\mathcal{I}\) is \textit{linearly bounded} with respect to + (i.e., for every \(\mathcal{I}\)-definable function \(f : I \rightarrow I\) there is \(r \in \Lambda(\mathcal{I})\) such that \(\lim_{x \rightarrow +\infty}[f(x) - rx] \in I\), or (2) there is a \(\mathcal{I}\)-definable binary operation \(\cdot\) such that \((I, 0, 1, \cdot, <)\) is a real closed field. Also, up to \(\mathcal{I}\)-definable isomorphism there is at most one \(\mathcal{I}\)-definable group \((I, 0, \ast)\) such that \(\mathcal{I}\) is linearly bounded with respect to \(\ast\) and at most one \(\mathcal{I}\)-definable (real closed) field \((I, 0, 1, \oplus, \otimes)\).

Moreover, the following are equivalent: (i) \(\mathcal{I}\) is linearly bounded with respect to +, (ii) for every \(\mathcal{I}\)-definable function \(f : A \times I \rightarrow I\), where \(A \subseteq I^n\), there are \(r_1, \ldots, r_l \in \Lambda(\mathcal{I})\) such that for every \(a \in A\) there is \(i \in \{1, \ldots, l\}\) with \(\lim_{x \rightarrow +\infty}[f(a, x) - r_i x] \in I\) and (iii) there is no infinite definable subset of \(\Lambda(\mathcal{I})\).

Let \((I, 0, 1, +, <)\) be as above and let \(\Lambda := \Lambda(\mathcal{I})\). Then \(\mathcal{I}\) is called \textit{semibounded} if every \(\mathcal{I}\)-definable set is already definable in the reduct \((I, 0, 1, +, <, (B_k)_{k \in K}, (\lambda)_{\lambda \in \Lambda})\), of \(\mathcal{I}\) where \((B_k)_{k \in K}\) is the collection of all bounded \(\mathcal{I}\)-definable sets. According to [3], the following are equivalent: (i) \(\mathcal{I}\) is semi-bounded, (ii) there is no \(\mathcal{I}\)-definable function between a bounded and an unbounded subinterval of \(I\), (iii) there is no \(\mathcal{I}\)-definable (real closed) field with domain an unbounded subinterval of \(I\), (iv) for every \(\mathcal{I}\)-definable function \(f : I \rightarrow I\) there are \(r \in \Lambda\), \(x_0 \in I\) and \(c \in I\) such that for all \(x > x_0\), \(f(x) = rx + c\) and (v) \(\mathcal{I}\) satisfies the “structure theorem”.

Let \((I, 0, 1, +, <)\) be a real closed field definable in \(\mathcal{N}\). Let \(\mathcal{K}(\mathcal{I})\) be the ordered field of all \(\mathcal{I}\)-definable endomorphisms of the multiplicative group \((I^0, 1, \cdot)\). Note that \(\mathcal{K}(\mathcal{I}) \rightarrow I\), \(\alpha \rightarrow \alpha'(1)\) is an embedding of ordered fields. The elements of \(\mathcal{K}(\mathcal{I})\) are called \textit{power functions} and for \(\alpha \in \mathcal{K}(\mathcal{I})\) with \(\alpha'(1) = r\) we write \(\alpha(x) = x^r\). By [3] exactly one of the following holds: (1) \(\mathcal{I}\) is \textit{power bounded} (i.e., for every \(\mathcal{I}\)-definable function \(f : I \rightarrow I\) there is \(r \in \mathcal{K}(\mathcal{I})\) such that ultimately \(|f(x)| < x^r\)) or (2) \(\mathcal{I}\) is \textit{exponential} (i.e., there is an \(\mathcal{I}\)-definable ordered group isomorphism \(e : (I, 0, +, <) \rightarrow (I^0, 1, \cdot, <)\)). Moreover, the following are equivalent: (i) \(\mathcal{I}\) is power bounded, (ii) for every \(\mathcal{I}\)-definable function \(f : A \times I \rightarrow I\), where \(A \subseteq I^n\), there are \(r_1, \ldots, r_l \in \mathcal{K}(\mathcal{I})\) such that for every \(a \in A\) the function \(x \mapsto f(a, x)\) is ultimately nonzero then, there is \(i \in \{1, \ldots, l\}\) with \(\lim_{x \rightarrow +\infty}[f(a, x)/x^{r_i}] \in I\) and (iii) there is no infinite definable subset of \(\mathcal{K}(\mathcal{I})\).

If \(\mathcal{I}\) is power bounded, then we know that \((I, 0, +, <)\) and \((I^0, 1, \cdot, <)\) are the only (up to \(\mathcal{I}\)-definable isomorphism) \(\mathcal{I}\)-definable one-dimensional torsion-free ordered groups. The \textit{Miller-Starchenko conjecture} says that in an
o-minimal expansion $\mathcal{I}$ of an ordered field every $\mathcal{I}$-definable one-dimensional torsion-free ordered group is $\mathcal{I}$-definable isomorphic to either $(I, 0, +, <)$ or $(I^0, 1, \cdot, <)$. (In the general case we only know (see [ms]) that up to $\mathcal{I}$-definable isomorphisms there are at most two $\mathcal{I}$-definably connected, $\mathcal{I}$-definable one-dimensional torsion-free ordered groups). Suppose that the Miller-Starchenko conjecture does not hold for $\mathcal{I}$, then we call the unique $\mathcal{I}$-definable group $(I, 0, \oplus, <)$ which is not $\mathcal{I}$-definably isomorphic to $(I, 0, +, <)$ or $(I^0, 1, \cdot, <)$ the \textit{Miller-Starchenko group of $\mathcal{I}$}. Note the following: if $G$ is an $\mathcal{I}$-definable one-dimensional torsion-free ordered group, then we can assume that $G = (I, 0, \oplus, <)$, and $\alpha : G \to (I, 0, +)$ is an abstract $C^1$ isomorphism iff $\forall s \in G, \alpha'(s)\frac{\partial}{\partial x}(0, s) = \alpha'(0)$ where for all $t, s \in G$, $\oplus(t, s) := t \oplus s$ i.e., $\alpha$ is Pfaffian over $(I, 0, 1, \cdot, \oplus, <)$ (note that, by associativity of $\oplus$, for all $s \in G$, $\frac{\partial}{\partial x}(0, s) \neq 0$).

The notion of definably compact groups was introduced in [ps]. Let $G$ be a definable group. We say that $G$ is \textit{definably compact} if for every definable continuous embedding $\sigma : (a, b) \subseteq \mathbb{N} \to G$, where $-\infty \leq a < b \leq +\infty$, there are $c, d \in G$ such that $\lim_{x \to a^+} \sigma(x) = c$ and $\lim_{x \to b^-} \sigma(x) = d$, where the limits are taken with respect to the topology on $G$. In [ps] the following result is established. Let $G$ be a definable group which is not definably compact. Then $G$ has a one-dimensional definably connected torsion-free (ordered) definable subgroup.

The trichotomy theorem [psl] and the theory of non orthogonality from [pps1] are used to prove the following (see theorem 2.4 and theorem 3.9).

\textbf{Fact 1.1} Let $U$ be a definable group and let $A$ be a definable normal subgroup of $U$. Then we have a definable extension $1 \to A \to U \to G \to 1$ with a definable section $s : G \to U$.

If we take in fact 1.1 $A$ to be the definable radical of $U$ i.e., the maximal definable solvable normal subgroup of $U$ we get that $G$ is either finite or definably semisimple i.e., it has no infinite proper abelian definable normal subgroup. Definable definably semisimple groups are classified in [pps1] (see also [pps2] and [pps3]). Below, $G$ is the structure $(G, \cdot)$ where $\cdot$ is the group operation of $G$.

\textbf{Fact 1.2} [pps1] and [pps3]. Let $G$ be a definably semisimple $G$-definably connected definable group. Then $G = G_1 \times \cdots \times G_l$ and for each $i \in \{1, \ldots, l\}$
there is an o-minimal expansion $I_i$ of a real closed field definable in $\mathcal{N}$ such that there is no definable bijection between a distinct pair among the $I_i$’s, $G_i$ is $I_i$-definably isomorphic to a $I_i$-semialgebraic subgroup of $GL(n_i, I_i)$ which is a direct product of $I_i$-semialgebraically simple, $I_i$-semialgebraic subgroups of $GL(n_i, I_i)$.

Fact 1.1 allows us to develop group extension theory with abelian and non abelian kernel over $\mathcal{N}$. We use this theory to prove the following result for definable solvable groups (see theorem 5.12).

Fact 1.3 Let $U$ be a definable solvable group. Then $U$ has a definable normal subgroup $V$ such that $U/V$ is a definably compact definable solvable group and $V = K \times W_1 \times \cdots \times W_s \times V'_1 \times V_1 \times \cdots \times V'_k \times V_k$ where $K$ is the definably connected definably compact normal subgroup of $U$ of maximal dimension and for each $j \in \{1, \ldots, s\}$ (resp., $i \in \{1, \ldots, k\}$) there is a semi-bounded o-minimal expansion $J_j$ of a group (resp., an o-minimal expansion $I_j$ of a real closed field) definable in $\mathcal{N}$ such that there is no definable bijection between a distinct pair among the $J_j$’s and $I_i$’s, $W_j$ is a direct product of copies of the additive group of $J_j$, $V'_j$ is a direct product of copies of the linearly bounded one-dimensional torsion-free $I_i$-definable group and $V_i$ is an $I_i$-definable group such that $Z(V_i)$ has an $I_i$-definable subgroup $Z_i$ such that $Z(V_i)/Z_i$ is a direct product of copies of the linearly bounded one-dimensional torsion-free $I_i$-definable group and there are $I_i$-definable subgroups $1 = Z_i^0 < Z_i^1 < \cdots < Z_i^{m_i} = Z_i$ such that for each $l \in \{1, \ldots, m_i\}$, $Z_i^l/Z_i^{l-1}$ is the additive group of $I_i$, and $V_i/Z(V_i)$ $I_i$-definably embeds into $GL(n_i, I_i)$.

We also prove the following result about definably compact definable groups (see corollary 4.3).

Fact 1.4 Let $U$ be a definably compact, definably connected definable group. Then $U$ is either abelian or $U/Z(U)$ is a definable semi-simple group. In particular, if $U$ is solvable then it is abelian.

Fact 1.3 gives a partial solution to the Peterzil-Steinhorn splitting problem for solvable definable group with no definably compact parts (see [18]). We say that a definable abelian group $U$ has no definable compact parts if there are definable subgroups $1 = U_0 < U_1 < \cdots < U_n = U$ such that for each $j \in \{1, \ldots, n\}$, $U_j/U_{j-1}$ is a one-dimensional definably connected torsion-free
definable group. We say that a definable solvable group \( U \) has no definable compact parts if \( U \) has definable subgroups \( 1 = U_0 \leq U_1 \leq \cdots \leq U_n = U \) such that for each \( i \in \{1, \ldots, n\} \), \( U_i/U_{i-1} \) is a definable abelian group with no definable compact parts. Peterzil and Steinhorn ask in [28] if a definable abelian group \( U \) of dimension two and with no definably compact parts is a direct product of one-dimensional definably connected torsion-free definable groups. Fact 1.3 above reduces this problem to the case where \( U \) is a group definable in a definable o-minimal expansion \( \mathcal{I} \) of a real closed field \( (I, 0, 1, +, \cdot, <) \) and we have an \( \mathcal{I} \)-definable extension \( 1 \to A \to U \to G \to 1 \) where \( A = (I, 0, +, <) \) and \( G = (I, 0, *, <) \) is a one-dimensional torsion-free \( \mathcal{I} \)-definable group. We prove (see lemma 5.9) that in this case, there is an \( \mathcal{I} \)-definable 2-cocycle \( c \in Z^2_\mathcal{I}(G, A) \) for \( U \) such that \( U \) is \( \mathcal{I} \)-definably isomorphic to \( A \times G \) if there is an \( \mathcal{I} \)-definable function \( \alpha : G \to A \) such that \( \forall s \in G, \ \alpha'(s)\frac{\partial \alpha}{\partial s}(0, s) = \alpha'(0) + \frac{\partial \alpha}{\partial s}(0, s) \).

Let \( \mathcal{I} \) be an o-minimal expansion of a real closed field \( (I, 0, 1, +, \cdot, <) \) and suppose that we have an abelian \( \mathcal{I} \)-definable extension \( 1 \to A \to U \to G \to 1 \) where \( A = (I, 0, +, <) \) and \( G = (I, 0, *, <) \) is a one-dimensional torsion-free \( \mathcal{I} \)-definable group. We shall say that \( U \) is a Peterzil-Steinhorn \( \mathcal{I} \)-definable group if \( U \) is not \( \mathcal{I} \)-definably isomorphic to \( A \times G \). A corollary of our main result is the following fact (see corollary 5.14).

**Fact 1.5** Let \( \mathcal{I} = (I, 0, 1, +, \cdot, <, \ldots) \) be an o-minimal expansion of a real closed field with no Peterzil-Steinhorn \( \mathcal{I} \)-definable groups. Then every \( \mathcal{I} \)-definable solvable group \( U \) with no \( \mathcal{I} \)-definable compact parts is \( \mathcal{I} \)-definably isomorphic to a group definable of the form \( U' \times G_1 \cdots G_k \cdot G_{k+1} \cdots G_l \) where \( U' \) is a direct product of copies of linearly bounded one-dimensional torsion-free \( \mathcal{I} \)-definable groups, for \( i = 1, \ldots, k \), \( G_i = (I^0, 0, +) \) and for \( i = k+1, \ldots, l \), \( G_i = (I^0, 1, \cdot) \). In particular, \( G := G_1 \cdots G_k \cdot G_{k+1} \cdots G_l \) \( \mathcal{I} \)-definably embeds into some \( GL(n, I) \) and \( U \) is \( \mathcal{I} \)-definably isomorphic to a group definable in one of the following reducts \( (I, 0, 1, +, \cdot, \oplus) \), \( (I, 0, 1, +, \cdot, \oplus, e^t) \) or \( (I, 0, 1, +, \cdot, \oplus, t^{b_1}, \ldots, t^{b_n}) \) of \( \mathcal{I} \) where \( (I, 0, \oplus) \) is the Miller-Starchenko group of \( \mathcal{I} \), \( e^t \) is the \( \mathcal{I} \)-definable exponential map (if it exists), and the \( t^{b_i} \)'s are \( \mathcal{I} \)-definable power functions. Moreover, if \( U \) is nilpotent then \( U \) is \( \mathcal{I} \)-definably isomorphic to a group definable in the reduct \( (I, 0, 1, +, \cdot, \oplus) \) of \( \mathcal{I} \).

In section 8 we use our main result to classify definable \( G \)-modules (see theorem 6.1), this is then used to prove the o-minimal version of the Lie-Kolchin-Mal’cev theorem (see theorem 6.3).
Another application of fact 1.3 is the following result (see theorem 7.1): Let $U$ be a definable group and let $\{T(x) : x \in X\}$ be a definable family of non empty definable subsets of $U$. Then there is a definable function $t : X \rightarrow U$ such that for all $x, y \in X$ we have $t(x) \in T(x)$ and if $T(x) = T(y)$ then $t(x) = t(y)$. This result shows that the many of the theorems from [pst2] can be obtained without the assumption that $N$ has definable Skolem functions. We include here direct proofs (avoiding the use of $\forall$-definability theory) of some of these results, namely fact 1.4 above, corollary 6.2 and corollary 7.2.

In section 8 we apply the main theorem to describe definable rings (see theorem 8.1 and theorem 8.2).

### 2 Definable quotients

**Definition 2.1** Let $S$ be a definable set and let $T := \{T(x) : x \in X\}$ be a definable family of non empty definable subsets of $S$. We say that $T$ has **definable choice** if there is a definable function $t : X \rightarrow S$ such that for all $x \in X$, $t(x) \in T(x)$. If in addition, $t$ is such that for all $x, y \in X$, if $T(x) = T(y)$ then $t(x) = t(y)$, then we say that $T$ has **strong definable choice**. The function $t$ is called a (strong) **definable choice for the family** $T$.

We say that the definable set $S$ has **(strong) definable choice** if every definable family $T$ of non empty definable subsets of $S$ has a (strong) definable choice.

The following fact is easy to prove.

**Fact 2.2** The following hold: (i) if $f : R \rightarrow S$ is a definable map such that for all $s \in S$, $f^{-1}(s)$ is finite and $S$ has (strong) definable choice then $R$ has (strong) definable choice; (ii) if $g : S \rightarrow R$ is a surjective definable map and $S$ has (strong) definable choice then $R$ has (strong) definable choice; (iii) if $S := S_1 \times \cdots \times S_k$ is definable and each $S_i$ is definable and has (strong) definable choice then $S$ has (strong) definable choice.

For the prove of the next lemma we need to recall some definitions from [pps1]: an open interval $I \subseteq N$ is **transitive** if for all $x, y \in I$ there are definably homeomorphic subintervals $I_x, I_y$ of $I$ containing $x$ and $y$ respectively; an open rectangular box $I_1 \times \cdots \times I_n$ is transitive if all the intervals $I_k$ are transitive.
Lemma 2.3 A definable group $U$ has a definable neighbourhood $O$ of 1 (the identity) with strong definable choice.

Proof. Since it is sufficient to prove the lemma for an $\omega_1$-saturated elementary extension of $\mathcal{N}$, we will assume that $\mathcal{N}$ is $\omega_1$-saturated.

By lemma 1.28[pps1], there is a definable chart $(O', \phi)$ on $U$ at 1 such that $\phi(O')$ is a transitive rectangular box, say $I_1 \times \cdots \times I_n$. Let $\phi(1) := (a_1, \ldots, a_n)$. Then by the trichotomy theorem [pst1], the definable structure $\mathcal{J}_i$ induced by $\mathcal{N}$ on some open subinterval $J_i$ of $I_i$ containing $a_i$ is either an $\omega$-minimal expansion of a real closed field or an $\omega$-minimal expansion of an ordered partial group. Without loss of generality we may assume that $(J_i, a_i, +, -)$ is a definable ordered partial group with zero $a_i$ and $J_i = (e, e)$. By fact reffact definable choice its enough to show that $J'_i = (-e, e)$ has strong definable choice. This is follows from the fact that there are definable functions $l, r : J'_i \rightarrow J'_i$ and $m : J'_i \times J'_i \rightarrow J'_i$ such that for all $x, y \in J'_i$, we have $l(x) < x$, $x < r(x)$ and if $x < y$ then $x < m(x, y) < y$: take $l(x) := x + |\frac{e-x}{2}|$; $r(x) := x - |\frac{e-x}{2}|$ and $l(x) := x + |\frac{e-x}{2}|$.

Recall that, if we have a definable set $S$ and a definable equivalence relation $E$ on $S$ then, we say that $S/E$ is definable if there is a definable map $\pi : S \rightarrow T$ such that $\forall x, y \in S, xEy \iff \pi(x) = \pi(y)$. Note that this is the case, if the definable family $\{x/E : x \in S\}$ has a strong definable choice. If $S$ is a definable group, $E$ a definable normal subgroup and the set $S/E$ is definable then, $S/E$ becomes in a natural way a definable group.

Theorem 2.4 Let $U$ be a definable group and let $V$ be a definable normal subgroup of $U$. Then $U/V$ is definable.

Proof. Suppose that $U \subseteq N^m$ and for each $l \in \{1, \ldots, m\}$ let $\pi_l : N^m \rightarrow N^l$ be the projection onto the first $l$ coordinates and let $\pi^l : N^m \rightarrow N$ be the projection onto the $l$th coordinate.

The existence of a strong definable choice $l := (l_1, \ldots, l_m)$ for the family $\{xV : x \in U\}$ follows from the claim below. In fact the claim implies the existence of $l$ on a large definable subset $U_m$ of $U$ (i.e., $dim(U \setminus U_m) < dimU$), but by lemma 2.4 [3], there are $u_1, \ldots, u_n \in U$ such that $U = u_1U_m \cup \cdots \cup u_nU_m$ and so we can extend $l$ from $U_m$ to $U$.

Claim: For each $k \in \{1, \ldots, m\}$ there is a definable subset $U_k$ of $U$ such that (i) $dim(U \setminus U_k) < dimU$ and (ii) if $x \in U_k$ and $y \in U$ is such that $xV =
yV then \( y \in U_k \). Moreover, there are definable functions \( l_1, \ldots, l_k : U_k \rightarrow N \) such that for each \( x \in U_k \) there is \( z \in xV \) such that \( \pi_k(z) = (l_1(x), \ldots, l_k(x)) \) and for all \( y \in U \) if \( xV = yV \) then \( (l_1(x), \ldots, l_k(x)) = (l_1(y), \ldots, l_k(y)) \).

**Proof of Claim:** We will do this by induction on \( k \). Suppose that \( k = 1 \). For the induction let us introduce the following notation: \( U_0 := U \) and for each \( x \in U_0 \), let \( V_0(x) := xV \).

We have a definable function \( \alpha_1 : U_0 \rightarrow N \cup \{+\infty\} \) given by, for each \( x \in U_0 \), \( \alpha_1(x) = \sup \pi^1(V_0(x)) \). Note that, if \( V_0(x) = V_0(y) \) then \( \alpha_1(x) = \alpha_1(y) \). Now if \( x \in U_0 \) is such that \( \alpha_1(x) \in \pi^1(V_0(x)) \) then we can take \( l_1(x) := \alpha_1(x) \). Let \( U_0' = U_0 \setminus M_1 \) where \( M_1 := \{ x \in U_0 : \alpha_1(x) \in \pi^1(V_0(x)) \} \) and suppose that \( U_0' \) is non empty. By o-minimality, the set \( I_1 \) of end points of \( \alpha_1(U_0) \) in \( \alpha_1(U_0) \) is finite. Suppose that \( I_1 \) is non empty and let \( a \in I_1 \). Consider the definable sub family \( \{ V_0(x) : \alpha_1(x) = a \} \) of \( \{ V_0(x) : x \in U_0 \} \). Let \( x_0 \in U_0 \) such that \( \alpha_1(x_0) = a \) and define for all \( x \in U_0 \) such that \( V_0(x) = V_0(x_0) \), \( l_1(x) := a_0 \) where \( a_0 \) is some fixed element of \( \pi^1(V_0(x_0)) \). For each \( x \in U_0 \) such that \( \alpha_1(x) = a \) let \( \gamma_1(x) := \inf \{ z : a_0 \leq z < a, \ (z,a) \subseteq \pi^1(V_0(x)) \} \). If \( V_0(x) = V_0(y) \) then \( \gamma_1(x) = \gamma_1(y) \). For \( x \in U_0 \) with \( \alpha_1(x) = a \) let \( K_1(x) := \{ z \in O : \alpha_1(zx) \in (\gamma_1(x), a) \} \) where \( O \) is the definable neighbourhood of \( 1 \) in \( U \) with strong definable choice (see lemma 2.3). This is a definable family of definable non-empty sets such that if \( V_0(x) = V_0(y) \) then \( K_1(x) = K_1(y) \).

On \( \{ x \in U_0 : \alpha_1(x) = a \} \) define \( l_1(x) := \alpha_1(k_1(x)x) \) where \( k_1(x) \) is a strong definable choice for \( K_1(x) \).

If \( X_1 \cup M_1 \) is large in \( U_0 \) then the claim is proved for \( k = 1 \). Otherwise, we have \( \text{dim}(U_0 \setminus (X_1 \cup M_1)) = \text{dim}U_0 \). Now let \( J_1 := \alpha_1(U_0) \setminus I_1 \). Suppose that \( J_1 \) is non empty. Then \( J_1 \) is a finite union of open intervals. Let \( Y_1 \) be the definable set of all \( x \in U_0 \) such that \( \alpha_1(x) \in J_1 \) and there is (equivalently, for all) \( y \in U_0 \) such that \( V_0(y) = V_0(x) \) and \( \alpha_1 \) is continuous at \( y \). O-minimality implies that \( Y_1 \) is large in \( U_0 \setminus (X_1 \cup M_1) \) and so, \( Y_1 \cup X_1 \cup M_1 \) is large in \( U_0 \).

Let \( A_1 \) be the definable subset of \( Y_1 \) of all \( x \in Y_1 \) such that there is a definable open neighbourhood \( B \) of \( x \) in \( U \), such that \( \alpha_1(B) \subseteq \{ z \in J_1 : \alpha_1(x) \leq z \} \). If \( V_0(x) = V_0(y) \) and \( x \in A_1 \) then \( y \in A_1 \). Clearly, by o-minimality, \( \alpha_1(A_1) \) is finite and as before we can construct \( l_1 \) on \( A_1 \).

Let \( B_1 := Y_1 \setminus A_1 \) and suppose that \( B_1 \) is non empty. Then we have a definable family \( \{ T_1(x) : x \in B_1 \} \) of definable subsets of \( O \), the definable neighbourhood of \( 1 \) in \( U \) with strong definable choice (see lemma 2.3) given by \( T_1(x) := \{ z \in O : \alpha_1(zx) \in S_1(x) \} \) where \( S_1(x) := \pi^1(V_0(x)) \cap \{ z \in J_1 : z < \).
normal subgroup $B$, under $\gamma$:

$G$ submodules. A special definable ($G, \gamma$) prove based on DCC.

Let $\alpha_1(x)$}.

By construction, for all $x \in B_1$, $S_1(x)$ is infinite and if $V_0(x) = V_0(y)$, then $y \in B_1$, $S_1(x) = S_1(y)$ and $T_1(x) = T_1(y)$. We now show that $T_1(x)$ is infinite for all $x \in B_1$: let $z' < \alpha_1(x)$ such that $(z', \alpha_1(x)) \subseteq S_1(x)$, then by continuity of $\alpha_1$ (and the fact that $x \in B_1$) there is a definable open neighbourhood $B$ of $x$ such that $\alpha_1(B) \cap (z', \alpha_1(x))$ is infinite. But then, since $\alpha_1(Ox \cap B) \cap (z', \alpha_1(x))$ is infinite (because, otherwise we would have $x \in A_1$), $T_1(x)$ is infinite as well. Since $O$ has strong definable choice, we have a strong definable choice $l_1'$ for the definable family $\{T_1(x) : x \in B_1\}$ and from this we get $l_1$ for the definable family $\{V_0(x) : x \in B_1\}$ by setting $l_1(x) := \alpha_1(l_1'(x)x)$. Note that if $V_0(x) = V_0(y)$ then $V_0(l_1'(x)x) = V_0(l_1'(y)y)$.

Let $U_1 := X_1 \cup Y_1 \cup M_1$ then $U_1$ is large in $U_0$ and the claim is proved for $k = 1$.

Suppose that the claim is true for $k$. We will show that it is true for $k + 1$. For this consider the definable family $\{V_k(x) : x \in U_k\}$ of non empty definable subsets of $U$, where $V_k(x) := \{u \in xV : \pi_k(u) = (l_1(x), \ldots, l_k(x))\}$ (note that we have $xV = yV$ iff $V_k(x) = V_k(y)$), and substitute in the proof for the case $k = 1$, 0 by $k$ and 1 by $k + 1$. □

3 Definable extensions

3.1 Definable $G$-modules

Definition 3.1 Let $G$ be a definable group. A definable $G$-module is a pair $(A, \gamma)$ where $A$ is a definable abelian group and $\gamma : G \to Aut_N(A)$ is a homomorphism form $G$ into the group of all definable automorphisms of $A$, such that the map $\gamma : G \times A \to A, \gamma(x, a) := \gamma(x)(a)$ is definable.

We say that $A$ is trivial if $\forall x \in G \forall a \in B, \gamma(x)(a) = a$, $A$ is faithful if $\gamma : G \to Aut_N(A)$ is injective. A definable $G$-submodule of $A$ is a definable normal subgroup $B$ of $A$ such that $\forall x \in G, \gamma(x)(B) \subseteq B$ (i.e., $B$ is invariant under $\gamma$). We then have natural induced definable $G$-modules $(B, \gamma|_B)$ and $(A/B, \gamma|_{A/B})$. We say that $A$ is irreducible if it has no proper definable $G$-submodules. A special definable $G$-submodule of $A$ is $A^G := \{a \in A : \forall x \in G, \gamma(x)(a) = a\}$.

The next lemma follows from theorem 2.4 but we include here a direct prove based on DCC.
Lemma 3.2 Let \((A, \gamma)\) be a definable \(G\)-module. Then \(A/A^G\) is a definable group, \(\text{Ker}\gamma\) is a normal definable subgroup of \(G\), \(\overline{G} := G/\text{Ker}\gamma\) is definable and we have a natural induced faithful definable \(\overline{G}\)-module \((\overline{\gamma}, A)\).

Also, if \(U\) is a definable group and \(A\) is a normal subgroup of \(U\) then \(C_U(A)\) is a normal definable subgroup of \(U\) and \(U/C_U(A)\) is definable. In particular, \(U/Z(U)\) is definable.

Proof. For each \(g \in G\) we have a definable endomorphism \(\alpha(g) : A \rightarrow A\) given by \(\forall a \in A\), \(\alpha(g)(a) := \gamma(g)(a) - a\) and \(A^G = \bigcap_{g \in G} \ker\alpha(g)\). And so by DCC on definable subgroups (see [3]) there are \(g_1, \ldots, g_n \in G\) such that \(A^G = \bigcap_{i=1}^n \ker\alpha(g_i)\). But then, the definable map \(A \rightarrow \alpha(g_1)(A) \times \cdots \times \alpha(g_n)(A)\), \(a \rightarrow (\alpha(g_1)(a), \ldots, \alpha(g_n)(a))\) shows that \(A/A^G\) is definable.

Let \(a \in A\) and consider the definable map \(\beta(a) : G \rightarrow A\), \(g \rightarrow \gamma(g)(a) - a\) then \(\{g \in G : \beta(a)(g) = 0\}\) is a definable subgroup of \(G\) and \(\text{Ker}\gamma = \bigcap_{a \in A} \{g \in G : \beta(a)(g) = 0\}\) and by DCC on definable subgroups there are \(a_1, \ldots, a_n \in A\) such that \(\text{Ker} \gamma = \bigcap_{i=1}^n \{g \in G : \beta(a_i)(g) = 0\}\). The definable map \(G \rightarrow \beta(a_1)(G) \times \cdots \times \beta(a_n)(G), g \rightarrow (\beta(a_1)(g), \ldots, \beta(a_n)(g))\) shows that \(G/\text{Ker}\gamma\) is definable.

If \(U\) is a definable group and \(A\) is a normal subgroup then \(C_U(A) = \bigcap_{a \in A} C_U(a)\) and by DCC on definable subgroups there are \(a_1, \ldots, a_n \in A\) such that \(C_U(A) = \bigcap_{i=1}^n C_U(a_i)\) and so \(C_U(A)\) is definable (and normal) and if for each \(a \in A\) we define \(ad(a) : U \rightarrow U\) by \(\forall u \in U\), \(ad(a)(u) := auu^{-1}u^{-1}\) then the definable map \(U \rightarrow ad(a_1)(U) \times \cdots \times ad(a_n)(U), u \rightarrow (ad(a_1)(u), \ldots, ad(a_n)(u))\) shows that \(U/C_U(A)\) is definable. \(\square\)

3.2 Group cohomology

For the rest of this subsection we assume that \((A, \gamma)\) is a definable \(G\)-module.

Definition 3.3 For each \(n \in \mathbb{N}\) let \(C^n_{\mathcal{N}}(G, A, \gamma)\) denote the abelian group of all definable functions from \(G^n\) into \(A\) with point wise addition. An element of \(C^n_{\mathcal{N}}(G, A, \gamma)\) is called a definable \(n\)-cochain (over \(\mathcal{N}\)).

Definition 3.4 The co-boundary map \(\delta : C^n_{\mathcal{N}}(G, A, \gamma) \rightarrow C^{n+1}_{\mathcal{N}}(G, A, \gamma)\), is defined by

\[
\delta(c)(g_1, \ldots, g_{n+1}) := \gamma(g_1)(c(g_2, \ldots, g_{n+1}))+
\]
\[
\sum_{i=1}^{n} (-1)^i c(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) + (-1)^{n+1} c(g_1, \ldots, g_n).
\]

It is clear that \(\delta(c)\) is also definable.

**Lemma 3.5** \(\delta\delta = 0\).

**Proof.** This is a simple calculation. \(\square\)

**Definition 3.6** We therefore have a complex \(C^*_N(G, A, \gamma)\). \(B^*_N(G, A, \gamma)\) denotes the image of \(\delta : C^1_N(G, A, \gamma) \rightarrow C^2_N(G, A, \gamma)\), \(Z^*_N(G, A, \gamma)\) denotes the kernel of \(\delta : C^1_N(G, A, \gamma) \rightarrow C^2_N(G, A, \gamma)\), and \(H^*_N(G, A, \gamma)\) denotes \(Z^*_N(G, A, \gamma)/B^*_N(G, A, \gamma)\). \(H^*_N(G, A, \gamma)\) is the \(n\)-cohomology group over \(N\), the elements of \(B^*_N(G, A, \gamma)\) are the definable \(n\)-coboundaries and the elements of \(Z^*_N(G, A, \gamma)\) are the definable \(n\)-cocycles.

**Remark 3.7** Let \((A, \gamma)\) be a definable \(G\)-module. Suppose that \(A := A_1 \times A_2\) and that \(A_1\) and \(A_2\) are invariant under the action of \(G\) on \(A\). Then \(H^*_N(G, A, \gamma)\) is isomorphic with \(H^*_N(G, A_1, \gamma|_{A_1}) \times H^*_N(G, A_2, \gamma|_{A_2})\).

### 3.3 Definable extensions

**Definition 3.8** Let \(U\) be a definable group. \((U, i, j)\) is an *definable extension* of \(G\) by \(A\) if we have an exact sequence

\[
1 \rightarrow A \xrightarrow{i} U \xrightarrow{j} G \rightarrow 1
\]

in the category of definable groups with definable homomorphisms. A *definable section* is a definable map \(s : G \rightarrow U\) such that \(\forall g \in G, j(s(g)) = g\).

**Note:** Below we will some times assume that \(A \leq U\), and write \((U, j)\) for \((U, i, j)\).

**Theorem 3.9** Let \(1 \rightarrow A \rightarrow U \xrightarrow{j} G \rightarrow 1\) be a definable extension. Then there is a definable section \(s : G \rightarrow U\).
Proof. Suppose that $U \subseteq N^m$. For each $l \in \{1, \ldots, m\}$ let $\pi_l : N^m \to N^l$ be the projection into the first $l$ coordinates and let $\pi_l : N^m \to N$ be the projection onto the $l$-th coordinate. The proof of the theorem follows from the proof of theorem 2.4 after making the following substitutions: $U_0 := G$, for each $x \in U_0$, $V_0(x) := j^{-1}(x)$ and the definable neighbourhoods in $U$ that appear in the proof of theorem 2.4 are substituted by definable neighbourhoods in $G$. \hfill \Box

Definition 3.10 Two definable extensions $1 \to A \overset{i}{\to} U \overset{j}{\to} G \to 1$ and $1 \to A \overset{i'}{\to} U' \overset{j'}{\to} G \to 1$ are definably equivalent if there is a definable homomorphism $\varphi : U \to U'$ such that

$$
\begin{array}{c}
U \\
\downarrow \varphi \\
1 \to A \quad \varphi \\
\downarrow \\
G \to 1 \\
\downarrow \\
U'
\end{array}
$$

is a commutative diagram.

3.4 Definable $G$-kernels

Notation: Let $A$ be a definable group. $\text{Aut}_N(A)$ denotes the group of all definable automorphisms of $A$, $\text{Inn}(A)$ the group of all inner automorphisms of $A$ and $\text{Out}_N(A) := \text{Aut}_N(A)/\text{Inn}(A)$. Let $\iota : \text{Aut}_N(A) \to \text{Out}_N(A)$ denote the natural homomorphism. If $A \leq U$ and $u \in U$ then we denote by $\langle u \rangle$ the automorphism of $A$ given by $\langle u \rangle(a) := uau^{-1}$ for all $a \in A$.

Definition 3.11 Let $G$ be a definable group. A definable $G$-kernel $(A, \theta)$ is a definable group $A$ with a homomorphism $\theta : G \to \text{Out}_N(A)$ and a homomorphism $\alpha : G \to \text{Aut}_N(A)$ such that $\theta = \iota \circ \alpha$ and the map $\alpha : G \times A \to A, \alpha(g, a) := \alpha(g)(a)$ is definable. Note that $\theta$ induces a definable action $\theta_0 : G \times Z(A) \to Z(A)$ making the center $Z(A)$ of $A$ a definable $G$-module. We say that $\alpha$ as above is a definable representative of the definable $G$-kernel $(A, \theta)$ and we write $\alpha \in \theta$. 

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If $\alpha, \beta \in \theta$ then by theorem \[7.1\] there is a definable function $k : G \to A$ such that $\forall x \in G$, $\beta(x) = k(x) > \alpha(x)$. Note also that, by theorem \[7.1\] there is a definable function $h_\alpha : G \times G \to A$ such that we have $\forall x, y \in G, h_\alpha(x, 1) = h_\alpha(1, y) = 1$, and

\[
\forall x, y \in G, \alpha(x)\alpha(y) = h_\alpha(x, y) > \alpha(xy) \tag{1}
\]

and $\forall x, y \in G, \beta(x)\beta(y) = h_\beta(x, y) > \beta(xy)$ where $h_\beta : G \times G \to A$ is the definable function given by

\[
\forall x, y \in G, h_\beta(x, y) := k(x)\alpha(x)(k(y))h_\alpha(x, y)k(xy)^{-1}.
\]

Note that if $(A, \theta) \in E K_N(G, B)$ and let $(U, \pi)$ is a definable extension of $A$ by $G$ and $s : G \to U$ is a definable section. Then

\[
\forall x, y \in G, h_{\alpha_{U,s}}(x, y) := s(x)s(y)s(xy)^{-1}.
\]

**Definition 3.12** Let $G$ be a definable group and $B$ an abelian definable group. Two definable $G$-kernels $(A_i, \theta_i)$ with $i = 1, 2$ with centre $B$ are **definably equivalent** if there is a definable isomorphism $\sigma : A_1 \to A_2$ such that for all $b \in B$ and for all $x \in G$, $\sigma(b) = b$ and $\sigma\theta_1(x)\sigma^{-1} = \theta_2(x)$. This relation is an equivalence relation and the set of all the classes is denoted by $K_N(G, B)$.

**Remark 3.13** Let $(U, \pi)$ be a definable extension of $G$ by $A$. Then there is a canonical homomorphism $\theta_U : G \to Out_N(A)$ such that $(A, \theta_U)$ is a definable $G$-kernel: take, for each $x \in G$, $\theta_U(x) := \{< u : u \in \pi^{-1}(x)\}$ with definable representative given by $\alpha_{U,s} : G \to Aut_N(A)$, $\alpha_{U,s}(g)(a) := < s(g) > (a)$ where $s : G \to U$ is a definable section.

**Definition 3.14** A definable $G$-kernel $(A, \theta)$ is **definably extendible** if there is a definable extension $(U, \pi)$ of $G$ by $A$ such that $(A, \theta_U)$ is definably equivalent to $(A, \theta)$. We say in this case that $(U, \pi)$ is **compatible** with the $G$-kernel. We denote by $Ext_N(G, A, \theta)$ the set of all equivalence classes of definable extensions of $G$ by $A$ compatible with the $G$-kernel $(A, \theta)$. Let $E K_N(G, B)$ be the subset of $K_N(G, B)$ of all classes $(A, \theta)$ such that $Ext_N(G, A, \theta)$ is nonempty. Note that $E K_N(G, B)$ is a well defined subset of $K_N(G, B)$. 

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3.5 Existence of definable extensions

With the set up we have established, the proof of the following results is now as in the classical case, for details see the relevant lemmas in [em1] and [em2].

**Fact 3.15** There is a canonical map from $K_n(G, B)$ into $H^3_n(G, B, \theta_0)$, sending $(A, \theta)$ into $c_{(A, \theta)}$ and $(A, \theta) \in EK_n(G, B)$ if and only if $c_{(A, \theta)} = 1$.

Let $(A, \theta) \in K_n(G, B)$ and let $\alpha \in \theta$ and let $h_\alpha$ be the corresponding definable function as in equation [1]. For $x, y, z \in G$, using associativity, the product $\alpha(x)\alpha(y)\alpha(z)$ may be calculated in two different ways. The identity of this two results gives for all $x, y, z \in G$, the following identity:

$$< h_\alpha(x, y)h_\alpha(xy, z) > = < \alpha(x)(h_\alpha(y, z))h_\alpha(x, yz) >.$$  

But only the elements of the center $B$ of $A$ determine the identity inner automorphism. Hence there exists a definable 3-cochain $c_\alpha \in C^3_n(G, B, \theta_0)$ such that

$$\forall x, y, z \in G, \alpha(x)(h_\alpha(y, z))h_\alpha(x, yz) = c_\alpha(x, y, z)h_\alpha(x, yh_\alpha(xy, z). \quad (2)$$

Now some calculations show that $c_\alpha \in Z^3_n(G, B, \theta_0)$ and if $\beta \in \theta$ then $h_\beta(x, y) = g_{\alpha, \beta}(x, y)h_\alpha(x, y)$ where $g_{\alpha, \beta} \in C^2_n(G, B, \theta_0))$ and $c_\alpha$ is changed to a cohomologous cocycle $c_\beta$ and by suitably changing the choice of $\alpha \in \theta$, $c_\alpha$ may be changed to any cohomologous cocycle.

Suppose now that $(A, \theta) \in EK_n(G, B)$ and let $(U, \pi)$ be a definable extension of $A$ by $G$ and let $s : G \rightarrow U$ be a definable section. Then a simple calculation shows that

$$\alpha_{U, s}(x)(h_{U, s}(y, z))h_{U, s}(x, yz) = h_{U, s}(x, y)h_{U, s}(xy, z).$$ \quad (3)

and therefore $c_{U, s}(x, y, z) = 1$.

Conversely, suppose that $(A, \theta) \in K_n(G, B)$ is such that $c_{(A, \theta)} = 1$ in $H^3_n(G, B, \theta_0)$. Select $\alpha \in \theta$ such that $c_{\alpha}(x, y, z) = 1$ for all $x, y, z \in G$. The proof of the result below shows that we can find $(U, \pi) \in Ext_n(G, A, \theta)$.

**Fact 3.16** Let $(A, \theta) \in EK_n(G, B)$ and $(U, \pi) \in Ext_n(G, A, \theta)$. Then there is a canonical bijection from $Ext_n(G, A, \theta)$ into $H^3_n(G, B, \theta_0)$ sending $(U, \pi)$ into the identity of $H^3_n(G, B, \theta_0)$.
Let \( s : G \to U \) be a definable section. We construct \((V_s, i_s, j_s) \in \text{Ext}_{\mathcal{N}}(G, A, \theta)\) associated with \( s \) as follows: \( V_s \) as domain \( A \times G \) and multiplication given by

\[
\forall a, b \in A \forall x, y \in G, (a, x)(b, y) = (a[\alpha_{U,s}(x)]h_{\alpha_{U,s}}(x, y), xy).
\]

(4)

From equation (3) and equation (2) \( V_s \) is a definable group, \((1, 1)\) is the identity, and the inverse of \((a, x)\) is \((\alpha_{U,s}(x)^{-1}[h_{\alpha_{U,s}}(x, x^{-1})a]^{-1}, x^{-1})\), \( i_s : A \to V_s \) given by \( \forall a \in A, i(a) := (a, 1) \) and \( j_s : V_s \to G \) given by \( \forall a \in A \forall x \in G, j(a, x) := x \). The map \( t : G \to V_s \) given by \( \forall x \in G, t(x) := (1, x) \) is a definable section, we see that for all \( x \in G \), \( t(x) >\alpha_{U,s}(x) \) and therefore \((V_s, i_s, j_s) \in \text{Ext}_{\mathcal{N}}(G, A, \theta)\). Also, the map \( U \to V_s \), \( u := as(x) \to (a, x) \) is a definable isomorphism.

Moreover, if \( s' : G \to U \) is another definable section and \((V_{s'}, i_{s'}, j_{s'}) \in \text{Ext}_{\mathcal{N}}(G, A, \theta)\) the corresponding definable extension, then there is a definable function \( k_{s, s'} : G \to A \) given by \( \forall x \in G, s'(x) := k_{s, s'}(x)s(x) \) such that

\[
\forall x, y \in G, h_{\alpha_{U,s'}}(x, y) := k_{s, s'}(x)\alpha_{U,s}(x)(k_{s, s'}(y))h_{\alpha_{U,s}}(x, y)k_{s, s'}(xy)^{-1}
\]

(5)

and the map \( V_s \to V_{s'}, (a, x) \to (ak_{s, s'}(x)^{-1}, x) \) is a definable isomorphism.

Also, \( V_s \) (and therefore \( U \)) is definably isomorphic with \( A \times_{\alpha_{U,s}} G \) iff there is a definable function \( g : G \to A \) such that

\[
\forall x, y \in G, h_{\alpha_{U,s}}(x, y) = \alpha_{U,s}(x)(g(y))g(x)g(xy)^{-1},
\]

(6)

since if \( g : G \to A \) satisfying equation (3), then the function \( G \to V_s, x \to (g(x)^{-1}, x) \) is a homomorphism.

Finally, if \((U', \pi') \in \text{Ext}_{\mathcal{N}}(G, A, \theta)\) and \( s' : G \to U' \) is a definable section and \( h_{\alpha_{U,s'}} : G \times G \to A \) is the corresponding definable function then there is \( c \in Z^2_{\mathcal{N}}(G, B, \theta_0) \) such that

\[
\forall x, y \in G, h_{\alpha_{U,s'}}(x, y) = c(x, y)h_{\alpha_{U,s}}(x, y),
\]

(7)

c in \( H^2_{\mathcal{N}}(G, B, \theta_0) \) does not depend on the equivalence class of \((U', \pi')\) or the on the choice of the definable section. Moreover, \( c \) is zero in \( H^2_{\mathcal{N}}(G, B, \theta_0) \) iff \((U, \pi)\) and \((U', \pi')\) are definably equivalent.
Fact 3.17 Let \((A, \gamma)\) be a definable \(G\)-module. Then there is a bijection from \(\text{Ext}_N^2(G, A, \gamma)\) onto \(H^2_N(G, A, \gamma)\) sending the class of \(A \rtimes_\gamma G\) into the identity of \(H^2_N(G, A, \gamma)\).

Let \((U, \pi) \in \text{Ext}_N^2(G, A, \gamma)\) and let \(s : G \to U\) be a definable section. Then there is a canonical definable 2-cocycle \(c \in Z^2_N(G, A, \gamma)\) associated with this definable section given by: \(\forall g, h \in G, c(g, h) := s(g)s(h)s(gh)^{-1}\), and therefore, (in \(A\)) we have

\[
\forall g, h, k \in G, c(h, k)^g - c(gh, k) + c(g, hk) - c(g, h) = 0,
\]

and, there is \((V, i, j) \in \text{Ext}_N^2(G, A, \gamma)\) associated with the definable 2-cocycle \(c\) given by:

\[
\forall a, b \in A, \forall g, h \in G, (a, g)(b, h) := (a + b^g + c(g, h), gh),
\]

from equation (8) \(V\) is a group, with identity \((-c(1, 1), 1)\), \(i : A \to V\) given by \(i(a) := (a - c(1, 1), 1)\) and \(j : V \to G\) by \(j(a, g) := g\). And the map \(U \to V, u := as(g) \to (a, g)\) is a definable isomorphism.

If \(s' : G \to U\) is another definable section, and \(c' \in Z^2_N(G, A, \gamma)\) is the corresponding definable 2-cocycle and \((V', i', j') \in \text{Ext}_N^2(G, A, \gamma)\) is the corresponding definable extension, then there is a definable function \(b : G \to A\) given by, \(\forall g \in G, s'(g) := b(g)s(g)\) such that

\[
\forall g, h \in G, c'(g, h) - c(g, h) = b(h)^g - b(gh) + b(g),
\]

i.e., \(c\) and \(c'\) determine the same element in \(H^2_N(G, A, \gamma)\) and the map \(V \to V', (a, g) \to (a - b(g), g)\) is a definable isomorphism.

Also, \(V\) (and therefore \(U\)) is definably isomorphic with \(A \rtimes_\gamma G\) iff there is a definable function \(a : G \to A\) such that

\[
\forall g, h \in G, c(g, h) = a(h)^g - a(gh) + a(g),
\]

(since if \(a : A \to G\) exists and satisfies equation (11), the function \(G \to V, g \to (-a(g), g)\) is a homomorphism).
Fact 3.18 The set $K_N(G, B)$ can be made into an abelian group in the following way: Let $(A_i, \theta_i) \in K_N(G, B)$ with $i = 1, 2$. Then their product $(A_1, \theta_1) \otimes (A_2, \theta_2) \in K_N(G, B)$ is the element, which is the class of $(A, \theta)$ where $A := A_1 \times A_2/C$ where $C := \{(b, b^{-1}) : b \in B\}$, and $\theta$ is represented by the map $\alpha := (\alpha_1, \alpha_2) : G \times A_1 \times A_2 \rightarrow A_1 \times A_2$, given by $\alpha(g)(a_1, a_2) := (\alpha_1(g)(a_1), \alpha_2(g)(a_2))$, where $\alpha_i$ represents $(A_i, \theta_i)$. The identity of $K_N(G, B)$ is the definable $G$-kernel $(B, \theta_0)$. And the inverse of $(A, \theta) \in K_N(G, B)$ is $(A^*, \theta)$ where $A^*$ is the anti-isomorphic with $A$ with domain $A$.

$EK_N(G, B)$ is a subgroup of $K_N(G, B)$: Let $(A_i, \theta_i) \in EK_N(G, B)$ with definable extensions $(U_i, \pi_i) \in Ext_N(G, A_i, \theta_i)$ where $i = 1, 2$. Then, if $(A, \theta) := (A_1, \theta_1) \otimes (A_2, \theta_2)$ then $(U, \pi) := (U_1, \pi_1) \otimes (U_2, \pi_2)$ is an element of $Ext_N(G, A, \theta)$, where $U := D/E$, $D := \{(u_1, u_2) : u_1 \in U_1, u_2 \in U_2, \pi_1(u_1) = \pi_2(u_2)\}$, $E := \{(b, b^{-1}) : b \in B\}$ and $\pi$ is induced by any of $\pi_i$. If $(A, \theta) \in EK_N(G, B)$ with $(U, \pi) \in Ext_N(G, A, \theta)$ then $(U^*, \pi^*) \in Ext_N(G, A^*, \theta)$ where $U^*$ is the group anti-isomorphic with $U$ with domain $U$ and for all $u \in U^*$, $\pi^*(u) := \pi(u^{-1})$. The group $K_N(G, B)/EK_N(G, B)$ is called the group of similarity classes. Note that $Ext_N(G, B, \theta_0)$ can be made into a group with product $\otimes$ defined above.

The map given in fact 3.17 is a isomorphisms between $H^2_{N}(G, B, \theta_0)$ and $Ext_N(G, B, \theta_0)$. Moreover, the map from $H^2_{N}(G, B, \theta_0)$ into $Ext_N(G, A, \theta)$ for a fixed $(U, \pi)$ in $Ext_N(G, B, \theta)$ of fact 3.13 is the composition of the isomorphism from $H^2_{N}(G, B, \theta_0)$ into $Ext_N(G, B, \theta_0)$ and the map from $Ext_N(G, B, \theta_0)$ into $Ext_N(G, A, \theta)$ which sends $(V, j)$ into $(U, \pi) \otimes (V, j)$. Finally note that the map from fact 3.16 is a homomorphism with kernel $EK_N(G, B)$.

4 Definably compact definable groups

In this section we prove that a definably compact definable group is abelian-by-finite. This will follow after we show that a definably connected definably compact definable $G$-module where $G$ is infinite and definably connected is trivial. Before we proceed, we need the following easy lemma.

Lemma 4.1 Let $U$ be an infinite definable group and let $V$ be a definable subgroup such that $\dim V < \dim U$. Then there is a definable continuous
embedding $\sigma : (a, b) \longrightarrow U$ such that $\lim_{t \to a^+} \sigma(t) = 0$ and $\sigma(a, b) \subseteq U \setminus V$.

**Proof.** Let $(O, \phi)$ be a definable chart of 1 (the identity of $U$). Then $\phi(O)$ is a definable open subset of $N^n$ where $n = \dim U$. Let $e = (e_1, \ldots, e_n) = \phi(1)$, $B = I_1 \times \cdots \times I_n \subseteq \phi(O)$ an open box containing $e$ and for each $i = 1, \ldots, n$ let $\overline{I_i} := \{e_1\} \times \cdots \times \{e_{i-1}\} \times I_i \times \{e_{i+1}\} \times \cdots \times \{e_n\}$ and $J_i := \phi^{-1}(\overline{I_i})$. Let $D := \phi(V \cap O) \cap B$. If there is $i \in \{1, \ldots, n\}$ and an open subinterval $I$ of $\overline{I_i}$ with one endpoint $e$ and such that $I \cap D = \emptyset$ then we are done. So suppose otherwise. Then after substituting each $I_i$ with a smaller interval if necessary, we have that each $J_i \subseteq V$. But this clearly implies that $\dim V = n$. \hfill \square

**Theorem 4.2** Let $(A, \gamma)$ be a definably compact, definably connected definable $G$-module, where $G$ is an infinite definably connected definable group. Then $(A, \gamma)$ is trivial.

**Proof.** Without loss of generality, we can assume that $\mathcal{N}$ is $\aleph_1$-saturated, in particular $|\mathcal{N}| > \aleph_0$. Suppose that $A^G \neq A$, and let $B$ be an infinite minimal definable subgroup of $A/A^G$. Let $C$ be a definable subgroup of $A$ such that $C/A^G = B$ and let $\overline{C}$ be the smallest definable $G$-submodule of $A$ containing $C$. Then $C \in \text{Ext}^1(\mathcal{N}(B), A^G)$ i.e., $C$ is a definable extension of $B$ by $A^G$ and there is a definable section $s : B \longrightarrow C$. Let $c(b, b') := s(b) + s(b') - s(b + b')$ be the corresponding definable 2-cocycle, then we have a definable family $\Gamma : G \times B \longrightarrow \overline{C}$ of definable homomorphisms from $B$ into $\overline{C}$ given by, $\forall g \in G \forall b \in B, \Gamma(g, b) = \gamma(g)(s(b)) - s(b)$ and such that $\forall g \in G \forall b \in B, \Gamma(1, b) = 0 = \Gamma(g, 0)$. To see this, subtract to the equation above for the 2-cocycle the equation obtained from it after applying $\gamma(g)$. Since for each $c \in C$ there are unique $a \in A^G$ and $b \in B$ such that $c = a + s(b)$ and for all $g \in G$, $\gamma(g)(c) = a + \gamma(g)(s(b))$ we must have $\ker C \Gamma \neq G$ where $\ker C \Gamma := \{g \in G : \forall b \in B, \Gamma(g, b) = 0\}$.

Since $B$ has no infinite proper definable subgroups, for each $g \in G$, $\Gamma(g)(B)$ is either 0 or infinite (with the same dimension as $B$) and with no infinite proper definable additive subgroups and so by dimension consideration, there is a minimal $n \geq 1$ such that for each $i \in \{1, \ldots, n\}$ there is $g_i \in G$ such that $\Gamma(g_i)(B) \neq 0$ and $\Gamma(G)(B) \subseteq \Gamma(g_1)(B) + \cdots + \Gamma(g_n)(B)$. Now since $F := \bigcap_{i=1}^n \Gamma(g_i)(B)$ is finite, $D := \Gamma(G)(B)/F$ is definable and we
have a natural induced definable family $\Lambda : G \times B \to D$ of definable homomorphisms from $B$ into $D$. Its easy to see that $\ker G \Lambda \neq G$. Now for each $i \in \{1, \ldots, n\}$ let $D_i := \Gamma(g_i)(B)/F$. Then $D = \bigoplus_{i=1}^{n} D_i$ and we have natural induced definable families $\Lambda_i : G \times B \to D_i$ of definable homomorphisms from $B$ into $D_i$, and there is $i_0 \in \{1, \ldots, n\}$ such that $\ker G \Lambda_{i_0} \neq G$.

Since for each $g \in G \setminus \ker G \Lambda_{i_0}$, $\ker \Lambda_{i_0}(g)$ is finite, and $\text{Tor}(B)$ is countable (see corollary 5.8) therefore $\bigcup \{\ker \Lambda_{i_0}(g) : g \in G \setminus \ker G \Lambda_{i_0}\}$ is finite and so there is a finite additive subgroup $E$ of $B$ such that for all $g \in G \setminus \ker G \Lambda_{i_0}$, $\ker \Lambda_{i_0}(g) \subseteq E$. Let $B' := B/E$. Its easy to see that $\Lambda_{i_0}$ induces a natural definable family $\Phi : G \times B' \to B'$ of definable endomorphisms of $B'$ such that $\ker G \Phi \neq G$ and for each $g \in G \setminus \ker G \Phi$, $\Phi(g)$ is a definable automorphism of $B'$.

Since $\ker G \Phi \neq G$ and $G$ is definably connected, we have $\dim(\ker G \Phi) < \dim G$ and by lemma 1, there is a definable continuous embedding $\sigma : (a, b) \to G$ such that $\lim_{t \to a^+} \sigma(t) = 1$ and $\sigma(a, b) \subseteq G \setminus \ker G \Phi$. Let $x_0 \in B' \setminus \{0\}$. Then for every $t \in (a, b)$ there exists a unique $x \in B'$ such that $\Phi(\sigma(t), x) = x_0$. This gives us a definable function $\tau : (a, b) \to \tau(a, b) \subseteq B'$. Since $B'$ is definably compact, there is an element $c \in B'$ such that $\lim_{t \to a^+} \tau(t) = c$. But then, by continuity of $\Phi$ we have $0 = \Phi(1, c) = x_0$, and so we get a contradiction.

The next corollary was also proved in [pst2] but assuming that $\mathcal{N}$ has definable Skolem functions and using the theory of $\lor$-definable groups.

**Corollary 4.3** Let $U$ be a definably compact, definably connected definable group. Then $U$ is either abelian or $U/Z(U)$ is a definable semi-simple group. In particular, if $U$ is solvable then it is abelian.

**Proof.** By lemma 3.2, $U/Z(U)$ is definable. Suppose that $U/Z(U)$ is infinite and not semi-simple. Then there is a normal definably connected definable subgroup $X$ of $U$ such that $Z(U) \leq X$ and $X/Z(U)$ is an abelian infinite normal definable subgroup of $U/Z(U)$. Now $X$ is a definable $U$-module by conjugation and by theorem 4.2, $X = X^U \leq Z(U)$ contradiction. \qed

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5 Definable solvable groups

5.1 Preliminary lemmas

The results below are stated for definable $G$-modules, but each one of them has a corresponding analogue for definable $G$-kernels. These are obtained after making the obvious substitutions. Since after these substitutions, the proofs are exactly the same, we omit them. We will be using through this subsection the results of subsection 3.5. We will also often use the following fact:

**Fact 5.1** Let $A := A_1 \times \cdots \times A_k$ and suppose that $(A, \gamma)$ is a definable $G$-module, and let $(U, j) \in \text{Ext}_N(G, A, \gamma)$ with the corresponding canonical definable 2-cocycle $c \in Z^2_N(G, A, \gamma)$. Suppose also that each $A_i$ is invariant under $G$, then $c := (c_1, \ldots, c_k)$ where for each $i \in \{1, \ldots, k\}$, $c_i \in Z^2_N(G, A_i, \gamma|A_i)$. Let $l \in \{1, \ldots, k\}$. If for each $i \in \{1, \ldots, k\} \setminus \{l\}$, $c_i \in B^2_N(G, A_i, \gamma|A_i)$ then clearly, $U$ is definably isomorphic with a definable group of the form $A_1 \times \cdots \times A_{l-1} \times V \times A_{l+1} \times \cdots \times A_k$ where $V$ the definable extension of $G$ by $A_l$ obtained from $c_l$.

**Lemma 5.2** Let $(A, \gamma)$ be a definable $G$-module. Suppose that $G$ is a one-dimensional torsion-free definably connected definable group, and let $c \in Z^n_N(G, A, \gamma)$ (where $n > 0$). If

$\forall g_1, \ldots, g_{n-1} \in G, \lim_{k \to +\infty} c(g_1, \ldots, g_{n-1}, k) \in A$

then $c \in B^n_N(G, A, \gamma)$.

**Proof.** For each $g_1, \ldots, g_{n-1} \in G$ let

$b(g_1, \ldots, g_{n-1}) := \lim_{k \to +\infty} c(g_1, \ldots, g_{n-1}, k) \in A.$

We have,

$$0 = \gamma(g_1)(c(g_2, \ldots, g_{n+1})) + \sum_{i=1}^{n} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) +$$
\((-1)^{n+1}c(g_1, \ldots, g_n).\)

Taking the limit as \(g_{n+1} \longrightarrow +\infty\), we obtain (note that, since \(G\) is an ordered group \(g_n g_{n+1} \longrightarrow +\infty\) as \(g_{n+1} \longrightarrow +\infty\))

\[(-1)^n c(g_1, \ldots, g_n) = \gamma(g_1)(b(g_2, \ldots, g_n)) + \sum_{i=1}^{n-1} (-1)^i b(g_1, \ldots, g_i g_{i+1}, \ldots, g_n) + (-1)^n b(g_1, \ldots, g_{n-1}).\]

Therefore, \(c\) is the coboundary of \((-1)^n b.\)

\[\square\]

**Lemma 5.3** Let \(A\) be an abelian definably compact definable group such that \((A, \gamma)\) is a definable \(G\)-module. Suppose that \(G\) is a one-dimensional torsion-free definably connected definable group, then the action of \(G\) on \(A\) is trivial and \(\text{Ext}_N(G, A, \gamma)\) is trivial.

**Proof.** This follows from lemma 5.2 and the fact that \(A\) is definably compact. \(\square\)

**Remark 5.4** Suppose that we have (definable) extensions \(1 \to A \to U \xrightarrow{\pi} G \to 1\) and \(B \leq G\) is definable then \(C := \pi^{-1}(B) \leq U\) and \(A \leq C\).

Moreover, if we have a (definable) extension \(1 \to B \to G \xrightarrow{j} H \to 1\). Then we have (definable) extensions \(1 \to C \to U \xrightarrow{i} H \to 1\) and \(1 \to A \to C \xrightarrow{\pi|_C} B \to 1\).

**Lemma 5.5** Let \(G\) be a one-dimensional definably connected torsion-free definable group. For each \(i \in \{1, \ldots, l\}\) let \(A_i\) be a definable group such that there are definable subgroups \(1 = A_i^{0} \subset A_i^{1} \subset \cdots \subset A_i^{n_i} = A_i\) such that for each \(j \in \{1, \ldots, n_i\}\), \(A_i^j / A_i^{j-1}\) is definably isomorphic with a one-dimensional definably connected torsion-free definable group with domain \(I_i\). Suppose that for each \(i \in \{1, \ldots, l\}\) there is no definable bijection between \(G\) and \(I_i\). If \(A := A_1 \times \cdots \times A_l\) and \((A, \gamma)\) is a definable \(G\)-module then the action of \(G\) on \(A\) is trivial and \(\text{Ext}_N(G, A, \gamma)\) is trivial.
Proof. It is easy to see that the action of $G$ on $A$ is trivial. We now prove the rest. We have $\Ext^2(G, A, \gamma) = H^2_\mathcal{N}(G, A, \gamma) = H^2_\mathcal{N}(G, A_1, \gamma|_{A_1}) \times \cdots \times H^2_\mathcal{N}(G, A_l, \gamma|_{A_l})$. We now use Lemma 5.2 to conclude: let $c^i = (c^i_1, \ldots, c^i_n) \in Z^2_\mathcal{N}(G, A_i, \gamma|_{A_i})$ we will show that for each $j \in \{1, \ldots, n_i\}$ and each $g \in G$ the definable function $c^i_j(g, -) : G \to A_i$ is such that $\lim_{x \to +\infty} c^i_j(g, x)$ exists.

By the monotonicity theorem $c^i_j(g, -)$ determines a definable bijection between an unbounded interval in $G$ and an interval in $I^n_i$. If this last interval is bounded in $I^n_i$, then we are done. So suppose its unbounded. But since we have definable group structures on $I_i$ and on $G$, this definable bijection can be extended to a definable bijection between $I_i$ and $G$, which is a contradiction. 

Lemma 5.6 Let $\mathcal{I}$ be an o-minimal structure, $A_1 = \cdots = A_l = (I, 0, +)$ and $G = (I, 0, \oplus)$ $\mathcal{I}$-definably connected one-dimensional torsion-free $\mathcal{I}$-definable groups. Let $A := A_1 \times \cdots \times A_l$ and suppose that $(A, \gamma)$ is an $\mathcal{I}$-definable $G$-module. If $\mathcal{I}$ is linearly bounded with respect to $+$ then the action of $G$ on $A$ is trivial and $\Ext^2_\mathcal{N}(G, A, \gamma)$ is trivial.

Proof. The fact that the action is trivial follows from the fact that $\mathcal{I}$ is linearly bounded with respect to $+$. We now need to show that each $H^2_\mathcal{N}(G, A_i, \gamma|_{A_i})$ is trivial. So we may assume without loss of generality that $l = 1$. For this we use lemma 5.2. Let $c \in Z^2_\mathcal{N}(G, A, \gamma)$ be the definable canonical 2-cocycle. Since $\mathcal{I}$ is linearly bounded with respect to $+$, there are $r_1, \ldots, r_l \in \Lambda(\mathcal{I})$ such that for each $x, y \in G$ we have $c(x, y) = r_{xy} + o(x, y)$ where $r_{xy} \in \{r_1, \ldots, r_l\}$ and $o : G \times G \to A$ is a definable function such that for each $x \in G$ the function $o_x : G \to A, y \mapsto o(x, y)$ is bounded (in particular, $\lim_{y \to +\infty} o(x, y) \in A$).

Let $g, h, k \in G$, and suppose $h$ is large enough so that $r_h = r_{g \oplus h} = r$. Then by equation 5 we have $[r_g(h \oplus k) + o(g, h \oplus k)] - [r_g h + o(g, h)] + [o(h, k) - o(g \oplus h, k)] = 0$.

And therefore $\forall g \in G, r_g = 0$, since the above equality implies that $r_g$ is bounded (take $k \to +\infty$). And so, $\forall g \in I, \lim_{h \to +\infty} c(g, h) \in I$. 

Recall the following important result.
Fact 5.7 Let $G$ be a definable group which is not definably compact, and let $\sigma : (a, b) \subseteq N \rightarrow G$ be a definable curve which is not completable in $G$ suppose without loss of generality that $\lim_{x \rightarrow b^-} \sigma(x)$ does not exist in $G$. Let $I := \sigma((a, b))$. Then there is an induce order $<$ on $I$. Let $M$ be an $|N|^+$-saturated extension of $N$; let $I^\infty = \{ x \in I^M : \forall b \in I^N x > b \}$ and for each $\alpha \in G^N$ let $V_\alpha$ be the infinitesimal neighbourhood of $\alpha$ in $M$, i.e., the intersection of all $N$-definable $V \subseteq G^M$ of $\alpha$.

Define an equivalence relation on $G^N$ by $\alpha T I \beta \iff V_\alpha * I^\infty = V_\beta * I^\infty$ where $*$ is the group operation on $G$. Then the $T_I$-equivalence class of the identity element of $G$ is a one-dimensional torsion-free ordered definable subgroup $H_I$ of $G$ and the $T_I$-equivalence classes are exactly the left cosets of $H$.

A corollary of the proof of fact 5.7 is the following remark which shows the limitations of the method of fact 5.7 for finding one-dimensional torsion-free ordered definable group. (We will use the notation of fact 5.7).

Remark 5.8 Suppose that we have a definable extension $1 \rightarrow A \rightarrow U \rightarrow G \rightarrow 1$ where $G$ is a one-dimensional torsion-free ordered definable group. Let $c \in Z^2_M(G, A, \gamma)$ be the corresponding definable 2-cocycle.

We know that we can assume that $U$ is a definable group with domain $A \times G$ and with group operation given by $(a, x)(b, y) = (a + b^q + c(x, y), xy)$. Suppose that $dimU = n$. Then there is a definable open neighbourhood of the identity element of $U$ which is definably homeomorphic to a definable open subset $O \subseteq N^n$. Without loss of generality, we may assume that $O \subseteq U$. For each $t \in G^{>1}$ let $B_t$ be an open rectangular box such that $B_t \cap \{(0, x) : x \in G\} = \{(0, x) : t^{-1} < x < t\}$. Let $t_0 \in G^{>1}$ be such that $B_{t_0} \subseteq O$. For $1 < t < t_0$ let $\overline{B_t}$ be the topological closure of $B_t$ in $O$ and let $bd(B_t)$ be its boundary.

For each $u \in G$ let $S_u := \{(0, x)(0, u)^{-1} : x \in G^{\geq u}\}$. By o-minimality, its easy to see that for all $1 < t < t_0$, $S_u \cap bd(B_t) \neq \emptyset$. Consider the following definable functions $g : (1, t_0) \times G \rightarrow G$, $f : (1, t_0) \times G \rightarrow U$ and $h : (1, t_0) \rightarrow U$ given by

$$g(t, u) := \inf \{ x \in G^{\geq u} : (0, x)(0, u)^{-1} \in bd(B_t) \},$$
$$f(t, u) := (0, g(t, u))(0, u)^{-1} \in bd(B_t) \text{ and }$$
$$h(t) := \lim_{u \rightarrow +\infty} f(t, u).$$
Note that, a simple calculation shows that

\[ f(t, u) = (-c(u, u^{-1})u^{-1} + c(t, u), u^{-1}, g(t, u)u^{-1}) \]

By the proof of fact 5.4 (see claim 3.8.2 in [??]), \(Imh\) is a one-dimensional subset of \(H_1\) where \(I := \{(0, x) : x \in G^{>1}\}\). Therefore, \(U = A \times G\) iff \(Imh \cap A \neq \{1\}\) iff for some \(t \in (1, t_0)\), \(\lim_{u \to +\infty} g(t, u)u^{-1} \neq 1\).

Let \(U\) be a definable abelian group of dimension two and with no definably compact parts. Lemma 5.5 and lemma 5.6 above show that \(U\) is isomorphic to a direct product of two one-dimensional torsion-free definable groups except possibly in the case where \(U\) is a group definable in a definable o-minimal expansion \(I\) of a real closed field \((I, 0, 1, +, <)\) and we have an \(I\)-definable extension \(1 \to A \to U \to G \to 1\) where \(A = (I, 0, +, <)\) and \(G = (I, 0, \oplus, <)\) is a one-dimensional torsion-free \(I\)-definable group.

**Lemma 5.9** Let \(I\) be an expansion of a real closed field. Suppose that we have an \(I\)-definable abelian extension \(1 \to A \to U \to G \to 1\) where \(A = (I, 0, +, <)\) and \(G = (I, 0, \oplus, <)\) is a one-dimensional torsion-free \(I\)-definable group. Then there is a 2-cocycle \(c \in Z^2_I(G, A)\) be the corresponding to this \(I\)-definable extension such that \(U\) is \(I\)-definably isomorphic to \(A \times G\) iff there is an \(I\)-definable function \(\alpha : G \to A\) such that

\[ \forall s \in G, \alpha'(s)\frac{\partial}{\partial x}(0, s) = \alpha'(0) + \frac{\partial}{\partial y}(0, s). \]

**Proof.** Let \(t : G \to U\) be an \(I\)-definable section, then by o-minimality there are \(g_0 > \epsilon > 0\) such that \(t\) is \(C^m\) on \((g_0 \oplus \epsilon, +\infty)\). Let \(s : G \to U\) be the \(I\)-definable section given by: for all \(g \in G\), if \(g \geq \oplus \epsilon\) then \(s(g) := t(g \oplus g_0)t(g_0)^{-1}\) and if \(g \leq \oplus \epsilon\) then \(s(g) := s(\oplus g)^{-1}\). Then \(s(0) = (0, 0)\) and \(s\) is \(C^m\) on \(G\) everywhere except possibly on \(\{\oplus \epsilon\} \times G \cup G \times \{\oplus \epsilon\}\).

By fact 3.14, \(U\) is \(I\)-definably isomorphic with \(A \times G\) if and only if there is an \(I\)-definable function \(\alpha : G \to A\) with \(\alpha(0) = 0\) such that the definable function \(\beta : G \to U, \beta(s) := (\alpha(s), s)\) is a definable homomorphism, equivalently if and only if the \(I\)-definable function \(\alpha : G \to A\) satisfies

\[ \forall t, s \in G, \alpha(t \oplus s) = \alpha(t) + \alpha(s) + c(t, s) \]

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if and only if (to see this use also the fact that $U$ is abelian)

$$\forall t, s \in G, \alpha'(t \oplus s) \frac{\partial c}{\partial x}(t, s) = \alpha'(t) + \frac{\partial c}{\partial x}(t, s).$$

Putting $t = 0$ in the second equation we get

$$\forall s \in G, \alpha'(s) \frac{\partial c}{\partial x}(0, s) = \alpha'(0) + \frac{\partial c}{\partial x}(0, s).$$

We show that this last equation is equivalent to the second equation: putting $t \oplus s$ in the third equation we get $\alpha'(t \oplus s) \frac{\partial c}{\partial x}(0, t \oplus s) = \alpha'(0) + \frac{\partial c}{\partial x}(0, t \oplus s)$; the associativity of $\oplus$ implies that $\frac{\partial c}{\partial x}(0, t \oplus s) = \frac{\partial c}{\partial x}(t, s) \frac{\partial c}{\partial x}(0, t)$; and since $c$ is a 2-cocyle we get $-\frac{\partial c}{\partial x}(t, s) \frac{\partial c}{\partial x}(0, t) + \frac{\partial c}{\partial x}(0, t \oplus s) - \frac{\partial c}{\partial x}(0, t) = 0$. From these equations together with the third equation we get the second equation. \hfill \square

Using lemma 5.9 and results from [sp] and [pss] we get:

**Corollary 5.10** Let $\tilde{\mathbb{R}}$ be an o-minimal expansion of $(\mathbb{R}, 0, +, <)$ the additive group of real numbers. Then there is an o-minimal expansion $\hat{\mathbb{R}}$ of $\tilde{\mathbb{R}}$ such that every $\tilde{\mathbb{R}}$-definable abelian group with no $\tilde{\mathbb{R}}$-definably compact parts is $\hat{\mathbb{R}}$-definably isomorphic to a product of one-dimensional groups $\hat{\mathbb{R}}$-definably isomorphic to $(\mathbb{R}, 0, +)$ and $(\mathbb{R}^+ \times 1, \cdot)$.

Since the theory of (ordered) real closed fields has quantifier elimination in the language of ordered rings, we have:

**Corollary 5.11** Let $\mathbb{R} = (\mathbb{R}, 0, 1, +, \cdot, <)$ be a real closed field. Then every $\mathbb{R}$-definable abelian group with no $\mathbb{R}$-definably compact parts is $\mathbb{R}$-definably isomorphic to a product of one-dimensional groups $\mathbb{R}$-definably isomorphic to $(\mathbb{R}, 0, +)$ and $(\mathbb{R}^+ \times 1, \cdot)$.

### 5.2 The main theorem

We are now read to prove our main theorem.

**Theorem 5.12** Let $U$ be a definable solvable group. Then $U$ has a definable normal subgroup $V$ such that $U/V$ is a definably compact definable solvable group and $V = K \times W_1 \times \cdots \times W_s \times V_1' \times V_1 \times \cdots \times V_k' \times V_k$ where $K$ is
the definably connected definably compact normal subgroup of $U$ of maximal dimension and for each $j \in \{1, \ldots, s\}$ (resp., $i \in \{1, \ldots, k\}$) there is a semi-bounded o-minimal expansion $J_j$ of a group (resp., an o-minimal expansion $I_j$ of a real closed field) definable in $\mathcal{N}$ such that there is no definable bijection between a distinct pair among the $J_j$’s and $I_i$’s, $W_j$ is a direct product of copies of the additive group of $J_j$, $V'_i$ is a direct product of copies of the linearly bounded one-dimensional torsion-free $I_i$-definable group and $V_i$ is an $I_i$-definable group such that $Z(V_i)$ has an $I_i$-definable subgroup $Z_i$ such that $Z(V_i)/Z_i$ is a direct product of copies of the linearly bounded one-dimensional torsion-free $I_i$-definable group and there are $I_i$-definable subgroups $1 = Z_i^0 < Z_i^1 < \cdots < Z_i^{m_i} = Z_i$ such that for each $l \in \{1, \ldots, m_i\}$, $Z_i^l/Z_i^{l-1}$ is the additive group of $I_i$, and $V_i/Z(V_i) I_i$-definably embeds into $GL(n_i, I_i)$.

**Proof.** We prove this by induction on dimension of $U$. The result is clearly true for dimension one. So let $U$ be as above and suppose that the result is true for solvable definable groups of lower dimensions than that of $U$.

Let $K$ be the definably compact, definably connected, definable normal subgroup of $U$ of maximal dimension. This exists: let $K_1$ be a definably compact, definably connected, definable normal subgroup of $U$ and let $U_1 = U/K_1$. Let $K_2$ be a definably compact, definably connected, definable normal subgroup of $U_1$. Now apply remark 5.4 and let $K_3$ be the definable normal subgroup of $U$ which is a definable extension of $K_2$ by $K_1$. $K_3$ is a definably compact, definably connected, definable normal subgroup of $U$ with $dimK_3 \geq dimK_1$. Repeating this process finitely many times we obtain $K$.

Let $U' := U/K$. Then $U'$ is definable and has a definable normal (solvable) subgroup with no definably compact parts, for otherwise the only definable normal (solvable) subgroups of $U'$ would be definably compact and so by remark 5.4, $K$ would not be maximal. Let $Y$ be the maximal definable normal subgroup of $U'$ with no definably compact parts (this exists by an argument similar to that above). Then $U'/Y$ is definable and definably compact, for otherwise by the induction hypothesis $U''/Y$ would have a definable normal subgroup with no definably compact parts and by remark 5.4, $Y$ would not be maximal. Now apply remark 5.4 and let $V$ be the definable normal subgroup of $U$ which is a definable extension of $Y$ by $K$. Note that $U/V = U'/Y$ and so, it is definably compact.
We now proceed with the proof, we will same times use fact 5.1. Its easy to verify that each time we do this, all the hypothesis are satisfied. By repeated application of remark 5.4, lemma 5.3 and fact 5.1 we see that \( V = K \times Y \). By induction hypothesis and by repeated application of remark 5.4, lemma 5.3 and fact 5.1 we see that 
\[
V = K \times Y.
\]
By induction hypothesis and by repeated application of remark 5.4, lemma 5.5 and fact 5.1 we see that 
\[
Y = Y_1 \times \cdots \times Y_r
\]
where for each \( i \in \{1, \ldots, r\} \) there are definable subgroups 
\[
Y_i^0 < Y_i^1 < \cdots < Y_i^{n_i} = Y_i
\]
such that for each \( j \in \{1, \ldots, n_i\} \), 
\[
Y_i^j/Y_i^{j-1}
\]
is definably isomorphic with a one-dimensional definably connected torsion-free definable group with domain \( I_i \) and for \( j \neq i \), there is no definable bijection between \( I_i \) and \( I_j \). For each \( i \in \{1, \ldots, r\} \) let 
\( I_i \) be the definable structure induced by \( N \) on \( I_i \).

If \( I_i \) is a semi-bounded o-minimal expansion of a group then we make \( J_i := I_i \) and \( W_i := Y_i \). And by induction on \( \dim W_i \) and applying (if needed) remark 5.4 and lemma 5.6 several times we are done. So assume that \( I_i \) is an o-minimal expansion of a real closed field. Let \( Y_i' \) be the maximal \( I_i \)-definable normal subgroup of \( Y_i \) which is a direct product of copies of the linearly bounded one-dimensional torsion-free \( I_i \)-definable group. Then \( X_i := Y_i/Y_i' \) is definable and by repeated application of remark 5.4, lemma 5.4 and fact 5.1 we see that \( Y_i = Y_i' \times X_i \). Now put \( V_i' := Y_i' \) and \( V_i := X_i \).

The fact that \( Z(V_i) \) is as described is proved in the same way. The fact that \( V_i/Z(V_i) \) \( I_i \)-definably embeds into some \( GL(n_i, I_i) \) is proved in [opp].

Corollary 5.14 below is an adaption of an argument due to Iwasawa (see the proof of lemma 3.4 [i]). We will need the following result from [s]. Recall that a definable group \( G \) is monogenic if there is \( g \in G \) such that the smallest definable group containing \( g \) (which exists by DCC) is \( G \).

**Fact 5.13 [s]** Let \( A \vdash U \) be definable groups. If \( A \subseteq Z(U) \) and \( U/A \) is monogenic then \( U \) is abelian.

**Corollary 5.14** Let \( \mathcal{L} = (I, 0, 1, +, \cdot, <, \ldots) \) be an o-minimal expansion of a real closed field and suppose that there are no Peterzil-Steinhorn \( \mathcal{L} \)-definable groups. Let \( U \) be an \( \mathcal{L} \)-definable solvable group with no \( \mathcal{L} \)-definable compact parts. Then \( U \) is \( \mathcal{L} \)-definably isomorphic to a group definable of the form \( U' \times G_1 \cdots G_k \cdot G_{k+1} \cdots G_l \) where \( U' \) is a direct product of copies of the linearly bounded one-dimensional torsion-free \( \mathcal{L} \)-definable group, for each \( i \in \{1, \ldots, k\} \), \( G_i = (I, 0, +) \) and for each \( i \in \{k+1, \ldots, l\} \), \( G_i = (I^{>0}, 1, \cdot) \).
particular, $G := G_1 \cdots G_k \cdot G_{k+1} \cdots G_l$ $I$-definably embeds into some $GL(n, I)$ and $U$ is $I$-definably isomorphic to a group definable in one of the following reducts $(I, 0, 1, +, \cdot, \oplus)$, $(I, 0, 1, +, \cdot, \oplus, e')$ or $(I, 0, 1, +, \cdot, t^{b_1}, \ldots, t^{b_r})$ of $I$ where $(I, 0, \oplus)$ is the Miller-Starchenko group of $I$, $e'$ is the $I$-definable exponential map (if it exists), and the $t^{b_i}$’s are $I$-definable power functions. Moreover, if $U$ is nilpotent then $U$ is $I$-definably isomorphic to a group definable in the reduct $(I, 0, 1, +, \cdot)$ of $I$.

**Proof.** By theorem 5.12, we may assume that $U = U' \times G$ where $U'$ is the maximal $I$-definable normal subgroup of $U$ which is a product of copies of the linearly bounded one-dimensional torsion-free $I$-definable group and $G$ is as described there. Furthermore, since there are no Peterzil-Steinhorn $I$-definable groups, every $I$-definable abelian group with no $I$-definably compact parts is a direct product of one-dimensional torsion-free $I$-definable groups and therefore by an argument similar to those used in the proof of theorem 5.12 we can assume that $Z(G)$ is a direct product of copies of additive group of $I$. Moreover, by an argument similar to that used in the proof of theorem 5.12 (substitute “$I$-definably compact $I$-definable group” by “linearly bounded one-dimensional torsion-free $I$-definable group”), there are $I$-definable subgroups $1 = H_0 \leq H_1 \leq \cdots \leq H_{n+1} = G$ such that for each $i \in \{1, \ldots, n\}$, $H_i$ is the smallest definable normal subgroup of $H_{i+1}$ such that $H_{i+1}/H_i$ is abelian, $H_i/H_{i-1}$ is a direct product of copies of additive group of $I$ and $H_{n+1}/H_n$ is a direct product of copies (possibly zero copies) of the linearly bounded one-dimensional torsion-free $I$-definable group.

Let $\overline{G} := G/Z(G)$. Since $\overline{G}$ $I$-definably embeds into some $GL(k, I)$, by the remark above, $\overline{G} = \overline{G}_1 \cdots \overline{G}_k \cdot \overline{G}_{k+1} \cdots \overline{G}_l$ where for each $i \in \{1, \ldots, k\}$, $\overline{G}_i = (I, 0, +)$ and for each $i \in \{k+1, \ldots, l\}$, $\overline{G}_i = (I^{>0}, 1, \cdot)$.

Let $N$ be the $I$-definable extension of $\overline{G}_1 \cdots \overline{G}_k \cdot \overline{G}_{k+1} \cdots \overline{G}_{l-1}$ by $Z(U)$ (and therefore $G/N$ is a one dimensional torsion-free $I$-definably connected $I$-definable group). By induction its enough to show that $G$ contains an $I$-definable subgroup $H$ ($I$-definably isomorphic with $G/N$) such that $G = NH$ and $H \cap N = 1$.

We prove this by induction on $l$. Note that if $l = 0$ or $l = 1$, then $G$ is abelian (in the second case by fact 5.13) and so the claim holds by assumption. Assume that the claim is true all $I$-definable groups with no $I$-definably compact parts and with lower $l$. 

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Suppose that $N$ contains a proper $\mathcal{I}$-definable normal subgroup $N_1$ of $G$. By induction applied to $G/N_1$ there is an $\mathcal{I}$-definable subgroup $G_1$ such that $G = NG_1$, $G_1 \cap N = N_1$ and $G_1/N_1 = G/N$. Again the induction assumption for $G_1$ and $N_1$ gives us an $\mathcal{I}$-definable subgroup $H$ such that $G_1 = N_1H$ and $H \cap N_1 = 1$. This $H$ satisfies the claim.

We can therefore assume that $N$ has no proper $\mathcal{I}$-definable subgroup which is normal in $G$. If $N$ is in the centre of $G$ then by fact [5.13] $G$ is abelian and by assumption the claim is proved. If $N$ is not in the centre of $G$ then, using the decomposition series $1 = K_0 \leq K_1 \leq \cdots \leq K_{m+1} = N$ of $N$ like the one we got above for $G$, we see that $N$ must be a direct product of $k$ copies of the additive group of $\mathcal{I}$. $N$ is therefore an $\mathcal{I}$-definable $G$-module under conjugation and we have a natural $\mathcal{I}$-definable homomorphism $A : G \to GL(k, I)$. $G/N \mathcal{I}$-definably embeds in $GL(k, I)$. We show that that there is $g \in G$ such that $\det(A(g) - I) \neq 0$ and so $[N, g] = N$. Since $N$ is not in the centre of $G$, there is $g \in G$ which does not commute with some element in $N$. Let $N'$ be the eigen-space for the value 1 of the matrix $A(g)$. Since $A(G)$ is abelian, $N'$ is invariant under all the $A(h)$. But this means that the $\mathcal{I}$-definable subgroup $N'$ of $N$ is normal in $G$ and therefore by the assumption we must have either $N' = N$ or $N' = 1$. The first case does not hold since $g$ does not commute with some element of $N$. Therefore $N' = 1$, $\det(A(g) - I) \neq 0$ and $[N, g] = N$.

Now take an arbitrary element $y \in G$ and put $z := gyy^{-1}y^{-1}$. Since $G/N$ is abelian, we have $z \in N$. Take $u \in N$ such that $z = gug^{-1}u^{-1}$ and put $v := u^{-1}y$. It follows that $gv = vg$ and so $G = NC_G(g)$. If $x \in C_G(g) \cap N$, then $gxg^{-1}x^{-1} = 1$ and $\det(A(g) - I) \neq 0$ implies that $x = 1$, i.e., $C_G(g) \cap N = 1$.

We have $G := G_1 \cdots G_k \cdot G_{k+1} \cdots G_l$. An induction on $l$ shows that $G$ $\mathcal{I}$-definably embeds into some $GL(n, I)$ and $G$ is $\mathcal{I}$-definably isomorphic to a group definable in one of the following reducts $(I, 0, 1, +, \cdot)$, $(I, 0, 1, +, \cdot, e^t)$ or $(I, 0, 1, +, \cdot, t^{b_1}, \ldots, t^{b_k})$ of $\mathcal{I}$ where $e^t$ is the $\mathcal{I}$-definable exponential map (if it exists), and the $t^{b_i}$s are $\mathcal{I}$-definable power functions. If $U$ is nilpotent then $G$ is nilpotent and by [pps3], $G$ is $\mathcal{I}$-definably isomorphic to a group definable in the reduct $(I, 0, 1, +, \cdot)$ of $\mathcal{I}$. 

Remark 5.15 [pps3] There are solvable linear groups $U$ and $V$ definable in o-minimal expansions of $(\mathbb{R}, 0, 1, +, \cdot, <)$ by the $exp$ and $t^r$ respectively,
such that $U$ (resp., $V$) is not isomorphic (even abstractly) to a definable in o-minimal expansions of $(\mathbb{R}, 0, 1, +, \cdot, <)$ by some $t^*$ (resp., real semialgebraic group): Let $A = (\mathbb{R}^2, 0, +)$, $G = (\mathbb{R}, 0, +)$ and $H = (\mathbb{R}^{>0}, 1, \cdot)$. Let $U = A \rtimes G$ and $V = A \rtimes H$, where $\alpha(t)(a, b) = (\exp(t)a + t\exp(t)b, \exp(t)b)$ and $\beta(t)(a, b) = (ta, t^r b)$.

We end this subsection with the following result from [ps] which shows that definable abelian groups are not necessarily the direct product of a definable abelian group with no definably compact parts and a definably compact definable abelian group.

**Fact 5.16** [ps] Let $\tilde{\mathbb{R}} := (\mathbb{R}, 0, 1, +, \cdot, <)$. Then for $m, n \in \mathbb{N}$ and $L$ an integral lattice in $\mathbb{R}^n$ there are $\tilde{\mathbb{R}}$-definable abelian groups $T(m, n, L)$ and $T(n, L)$ with dimensions $m + n$ and $n$ respectively, such that we have an $\tilde{\mathbb{R}}$-definable extension $1 \to (\mathbb{R}^m, 0, +) \to T(m, n, L) \to T(n, L) \to 1$. Moreover, if $L$ is "generic" then $(\mathbb{R}^m, 0, +)$ does not have an $\tilde{\mathbb{R}}$-definable complement in $T(m, n, L)$ and $T(n, L)$ does not have $\tilde{\mathbb{R}}$-definable infinite proper subgroups.

The same result holds in $(\mathbb{R}, 0, 1, +, \cdot, <)$.

6 The Lie-Kolchin-Mal’cev theorem

6.1 More on definable $G$-modules

In this subsection we will describe definable $G$-modules, generalising a result from [mmt] describing faithful irreducible definable $G$-modules.

**Notation:** Let $(A, \gamma)$ be a definable $G$-module, for $i = 1, \ldots, m$ let $(A_i, \gamma_i)$ be a definable $G_i$-module and let $(B, \gamma)$ be a definable trivial $G$-submodule. We write $(G, A, \gamma) = (G_1, A_1, \gamma_1) \times \cdots \times (G_m, A_m, \gamma_m)$ if $G = G_1 \times \cdots \times G_m$, $A = B \times A_1 \times \cdots \times A_m$ and for all $g = (g_1, \ldots, g_m) \in G$, for all $a = (b, a_1, \ldots, a_m) \in A$ we have $\gamma(g)(a) = (b, \gamma_1(g_1)(a_1), \ldots, \gamma_m(g_m)(a_m))$. Recall also that $\overline{G}$ denotes $G/\text{Ker}\gamma$ and we have a natural definable $\overline{G}$-module $(A, \overline{\gamma})$. Also, $\overline{A} := A/A^G$ and we have a natural definable $G$-module $(\overline{A}, \overline{\gamma})$.

**Theorem 6.1** Let $(U, \gamma)$ be a definable non trivial $G$-module where, $U$ and $G$ are infinite definably connected definable groups. Then there is a definable
subgroup $V$ of $U$ of the form $K \times W \times V_1 \times \cdots \times V_m$ such that $(K, \gamma)$ is the maximal definably connected definably compact trivial $G$-submodule of $(U, \gamma)$, $(W, \gamma)$ is the maximal trivial $G$-submodule of $(U, \gamma)$ with no definably compact parts and for each $i \in \{1, \ldots, m\}$ there is a definable o-minimal expansion $I_i$ of a real closed field $I_i$ such that if $j \neq i$ then there is no definable bijection between $I_i$ and $I_j$ and (i) $(V_i, \gamma)$ is an $I_i$-definable non trivial $G$-submodule of $(U, \gamma)$ with no definably compact parts and non $I_i$-linearly bounded, (ii) $(\overline{G}, V, \overline{\gamma}) = (G_1, V_1, \gamma_1) \times \cdots \times (G_m, V_m, \gamma_m)$ where for each $i$, $G_i$ is definably isomorphic to an $I_i$-definable subgroup of some $GL(k_i, I_i)$, $(V_i, \gamma_i)$ is a faithful definable $G_i$-module, $\overline{V_i} = (I_i^{>0}, 1_i, \cdot, \cdot)\times (I_i, 0_i, +, i)^{\nu_i}$ and (iii) $(G_i, \overline{V_i}, \gamma_i|_{\overline{V_i}}) = (H_i^1, U_i^1, \alpha_i^1) \times \cdots \times (H_i^{m_i}, U_i^{m_i}, \alpha_i^{m_i})$ where for each $j \in \{1, \ldots, m_i\}$, $(U_i^j, \alpha_i^j)$ is a $I_i$-semialgebraic faithful and irreducible $H_i^j$-module and $H_i^j/Z(H_i^j)$ is a direct product of $I_i$-semialgebraic non abelian $I_i$-semialgebraically simple groups. Moreover, $(U/V, \gamma|_{U/V})$ is a definably compact trivial definable $G$-module.

**Proof.** We will refer to the notation of theorem 5.12. It's clear from theorem 5.12 the existence of $V$ with $K$ and $W$ with the properties mentioned, so to finish the prove of (i) its enough to show that there is a definable non linearly bounded and with no definably compact parts non trivial $G$-submodule. Suppose this is not the case. Then by theorem 4.2 and by fact it follows that $V$ is contained in $U^G$ and so $U := U/U^G$ is a definably compact, definably connected definable group. By theorem 4.2 $(U, \gamma|_{U})$ is a trivial $G$-module and so $\forall g \in G \nu \pi \in U, \gamma|_{U}(g)(\pi^{-1}(\overline{\pi})) \subseteq \pi^{-1}((\overline{\pi})$ (where, $\pi : U \rightarrow U$ is the natural projection) and therefore, if $B$ is an infinite minimal definable subgroup of $U$ we have a definable family $\Gamma : G \times B \rightarrow U^G$ of definable homomorphisms from $B$ into $U^G$ given by, $\forall g \in G \forall b \in B$, $\Gamma(g)(b) := \gamma|_{U}(g)(x) \rightarrow x$ for some $x \in \pi^{-1}(b)$. Now, since $(U, \gamma)$ is a non trivial definable $G$-module, by an argument similar to that in the proof of theorem 4.2 we get a contradiction.

We now prove (ii). Now let $k_i := \dim V_i$. By corollary 2.21 and fact 2.24 in [14] we have, after fixing a basis for the tangent space of each $V_i$ a definable homomorphism $G \rightarrow GL(k_1, I_1) \times \cdots \times GL(k_m, I_m)$ given by $g \rightarrow (d_0(\gamma|_{V_1}(g)), \ldots, d_0(\gamma|_{V_m}(g))$ and with kernel $Ker\gamma$. This shows that $G = G_1 \times \cdots \times G_m$ where each $G_i$ is definably isomorphic with an $I_i$-definable subgroup of $GL(k_i, I_i)$. Since $G$ is definably connected, each $G_i$ is infinite and since for $j \neq i$ there is no definable bijection between $I_i$ and $I_j$, we have $G_i \subseteq Ker\gamma|_{V_j}$, so to prove the first part of (ii), take $\gamma_i := \gamma|_{V_i}$.

Consider $G_i$ as an $I_i$-definable group and consider the $I_i$-definable group.
$V_i \rtimes \gamma_i G_i$ whose center is $V_i^{G_i} \times (\text{Ker} \gamma_i \cap Z(G_i)) = V_i^{G_i} \times \{1\}$. By [opp], we have that $V_i \times G_i / (V_i^{G_i} \times \{1\})$ is $I_i$-definably isomorphic with an $I_i$-definable subgroup of some $GL(l_i, I_i)$ and so by [pps3] $\overline{V_i} = (I_i^{>0}, 1, \cdot)_i \times (I_i, 0, +_i)^{n_i}$.

We will now prove (iii). We clearly have $V_i = U_i^0 \times U_i^1 \times \cdots \times U_i^{m_i}$ where $(U_i^0, \gamma_i|_{V_i})$ is trivial $I_i$-definable $G_i$-submodule of $(\overline{V_i}, \gamma_i|_{\overline{V_i}})$, and for each $j \in \{1, \ldots, m_i\}$, $(U_i^j, \gamma_i|_{\overline{V_i}})$ is a faithful and irreducible $I_i$-definable $G_i$-submodule of $(\overline{V_i}, \gamma_i|_{\overline{V_i}})$ and therefore each such $U_i^j$ is a vector space over the real closed field $I_i$. O-minimality implies that the action of $G_i$ on $U_i^j$ is by vector space automorphisms and so we can easily get $H_i^j$ and $\alpha_i^j$ satisfying the first part of (iii). The rest is proved in proposition 1.3 [mmt].

Peterzil and Starchenko proved in [pst2], using the theory of $\bigvee$-definable groups and assuming that $N$ has definable Skolem functions, that if $U := (U, \cdot)$ is a definable group which is not abelian-by-finite, then a real closed field is interpretable in $U$. Here we get the following.

**Corollary 6.2** Let $U$ be a definable group which is not abelian-by-finite. Then a real closed field is definable in $(N, <, U, \cdot)$.

**Proof.** Suppose that $U$ is definably connected. Let $R(U)$ be the maximal definably connected definable normal solvable subgroup of $U$. If $R(U)$ is abelian then it is a definable $U$-module under conjugation and if it is non-trivial we can apply theorem 6.1, otherwise we have $Z(U) = R(U)$ and $U/Z(U)$ is an infinite definably semi-simple definable group and the result follows from [pps1] and [pps2].

So suppose that $R(U)$ is not abelian. Since it is solvable, it has a definable abelian normal subgroup $X$ such that $Z(R(U)) \leq X$ and $X/Z(R(U))$ is an infinite definable abelian group. $X$ is a non-trivial definable $R(U)$-module and we can apply theorem 6.1. □

### 6.2 The Lie-Kolchin-Mal’cev theorem

Let $G$ be a definable group and $X$ a subset of $G$. By DCC on definable subgroups, the intersection of all definable subgroups of $G$ containing $X$ is a definable subgroup of $G$. This is the smallest definable subgroup of $G$.
Lemma 6.3 Let $G$ be a definable group. Then the following holds: (1) The operator $d$ is a closure operator i.e., for all subsets $X, Y$ of $G$ we have $X \subseteq d(X)$, if $Y \subseteq X$ then $d(Y) \leq d(X)$ and $d(d(X)) = d(X)$. (2) If the elements of $X \subseteq G$ commute with each other, then $d(X)$ is abelian. (3) If a subgroup $A \leq G$ normalises the subset $X \subseteq G$, then $d(A)$ normalises $d(X)$. (4) If $X, Y \leq G$ then $[d(X), d(Y)] \leq d([X,Y])$ in particular, a subgroup $H \leq G$ is solvable (resp., nilpotent) of class $n$ iff $d(H)$ is also solvable (resp., nilpotent) of class $n$.

Proof. (1) is trivial. For (2) and (3) see the proof of lemma 5.35 in [bn]. As for (4), the proof in [bn] for the finite Morley rank analogue (see corollary 5.38 and lemma 5.37 in [bn]) works in our case using the following result (which is a consequence of DCC): if $G$ is a definable group and, $H$ is a definable normal subgroup of $G$, $A$ is a subgroup of $G$ containing $H$ and $Y$ is a subset of $G$ containing $H$ are such that $A/H = C_{G/H}(Y/H)$, then $A$ is definable. \[\square\]

Lemma 6.4 Let $G$ be a definable group. (1) If $G$ is definably connected then, every finite normal subgroup is contained in $Z(G)$ and if $Z(G)$ is finite then $G/Z(G)$ is centerless. (2) If $G$ is infinite and nilpotent then $Z(G)$ is infinite. (3) If $G$ is infinite solvable but not nilpotent then $G$ has an infinite proper maximal normal definable subgroup $H$ such that $G/H$ is abelian.

Proof. (1) is the o-minimal analogue of corollary 1 in [n] and lemma 6.1 in [bn]. The proof is the same. (2) is the o-minimal analogue of lemma 6.2 in [bn] again the proof is the same. (3) is proved by an argument contained in the proof of theorem 2.12 in [pps2]. \[\square\]

We are now ready to prove the o-minimal version of the Lie-Kolchin-Mal’cev theorem. The proof is a modification of that in [n] for the finite Morley rank case.

Theorem 6.5 If $U$ is a definably connected definable solvable group, then $U^{(1)}$ is a $\bigvee$-definable nilpotent normal subgroup and $d(U^{(1)})$ is a definable nilpotent normal subgroup.
Proof. Let $U$ be a minimal counter-example, so both $U^{(1)}$ and $d(U^{(1)})$ are not nilpotent.

Claim (1): We can assume that $Z(U) = Z(U^{(1)}) = 1$.

Proof of Claim (1): The fact that we may assume $Z(U) = 1$ follows from $(U/Z(U))^{(1)} = U^{(1)}Z(U)/Z(U) \simeq U^{(1)}/U^{(1)} \cap Z(U) \supseteq U^{(1)}/Z(U^{(1)})$ (because $U^{(1)} \cap Z(U) \leq Z(U^{(1)})$) and so $Z(U)$ is finite and we can substitute $U$ by $U/Z(U)$ which is centerless by lemma 3.4.

By lemma 3.2, $U/C_U(U^{(1)})$ is definable. We have: $(U/C_U(U^{(1)}))^{(1)} = U^{(1)}C_U(U^{(1)})/C_U(U^{(1)}) \simeq U^{(1)}/U^{(1)} \cap C_U(U^{(1)}) = U^{(1)}/Z(U^{(1)})$. And so, if $C_U(U^{(1)})$ is infinite then $(U/C_U(U^{(1)}))^{(1)}$ is nilpotent and so $U^{(1)}$ is also nilpotent. Therefore, $C_U(U^{(1)})$ is finite and by lemma 3.4 we have $Z(U^{(1)}) \subseteq C_U(U^{(1)}) \subseteq Z(U)$.

Claim (2): $U^{(1)}$ and $d(U^{(1)})$ are torsion-free.

Proof of Claim (2): We have $U^{(1)} \leq d(U^{(1)}) \leq W_1 \times \cdots \times W_s \times V'_1 \times V_1 \times \cdots \times V'_k \times V_k$ and this last group is torsion-free (this can be proved by induction on dimension and using equation (4)).

Claim (3): There is an infinite definable abelian normal subgroup $A$ of $U$ which is an irreducible faithful definable $U/C_U(A)$-module under conjugation.

Proof of Claim (3): Since $U$ is not nilpotent, by lemma 3.4, $U$ has an infinite proper maximal normal definable subgroup $X$ such that $U/X$ is abelian. Therefore, $d(U^{(1)})$ is an infinite definable normal proper subgroup of $U$ and so $U^{(2)} \subseteq d(U^{(1)})^{(1)} \subseteq d(d(U^{(1)}))^{(1)}$ is nilpotent and infinite (for otherwise, $U^{(2)}$ is finite and since by claim (2) $U^{(1)}$ is torsion-free, $U^{(2)} = 1$ and $U^{(1)}$ would be abelian). Now by lemma 3.4, $Z(d(U^{(1)})^{(1)})$ is infinite. Now let $A$ be an infinite definable normal subgroup of $U$ contained in $Z(d(U^{(1)}))$ and minimal for these properties. Note that we have $U^{(2)} \leq C_U(A)$ and $U/C_U(A)$ is infinite because otherwise we would have $A \leq Z(U) = 1$. By minimality of $A$, $A$ is an irreducible faithful definable $U/C_U(A)$-module under conjugation.

By theorem 3.1, $U/C_U(A)$ is abelian (since is solvable) and therefore we have $1 = (U/C(U(A)))^{(1)} = U^{(1)}C_U(A)/C_U(A) \simeq U^{(1)}/C_{U^{(1)}}(A)$ and therefore, $U^{(1)} = C_{U^{(1)}}(A)$ i.e., $A \leq Z(U^{(1)}) = 1$ contradicting claim (3). \qed

We finish this subsection with the following result on definable nilpotent groups. Recall that a group $G$ is the central product of two subgroups $H$
and $K$ if $G = HK$, $H$ and $K$ are normal and $H \cap K \leq Z(G)$. We denote this by $G = H \ast K$. $H$ is \textit{divisible} if for every $n \in \mathbb{N}$ and every $x \in H$ there is $y \in H$ such that $y^n = x$.

\textbf{Theorem 6.6} Let $B$ be a definable nilpotent group. Then $B = B^0 \ast F$ for some finite subgroup $F$ and $B^0$ is divisible. Moreover, if $B$ is abelian then $B = B^0 \times F$ and if we have an extension $1 \rightarrow A \rightarrow U \rightarrow G \rightarrow 1$ where $A$ is an abelian definable group and $G$ is a finite group then $U$ is definable.

\textbf{Proof.} We will first prove the second part of the theorem. Let $1 \rightarrow A \rightarrow U \rightarrow G \rightarrow 1$ be an extension, where $A$ is an abelian definably connected group and $G$ is a finite group. It’s clear that every abelian definably connected definable group $H$ is divisible: for every $n \in \mathbb{N}$, the kernel of the homomorphism $H \rightarrow nH, h \rightarrow nh$ is a definable subgroup of $H$ with bounded exponent, and therefore by $\mathfrak{S}$ is finite and so $nH = H$. An argument similar to that of lemma 5.2 where we use $\sum_{k \in G}$ instead of $\lim_{y \rightarrow +\infty}$ show that if $A$ is a definable abelian connected group (and therefore divisible) and $G$ is a finite group, then $H^n(G, A)$ is trivial and this proves the second part of the theorem.

Let $B$ be a minimal counterexample to the first part of the theorem. Then by the above, $B$ is not abelian-by-finite, $Z(B)^0$ is infinite and $B/Z(B)^0$ is infinite. And so $B/Z(B)^0 = (B/Z(B)^0)^0 \ast F$. Let $H$ and $K$ be definable normal subgroups of $B$ such that $H/Z(B)^0 = (B/Z(B)^0)^0$ and $K/Z(B)^0 = F$. We have $K \neq B$ and by induction $K = K^0 \ast F_1$. Now we have $B = (K^0H) \ast F_1$ and by exercise 14, page 6 $\mathfrak{P}$, $K^0H$ is divisible and therefore, also definably connected, i.e., $K^0H = B^0$. \hfill \qed

7 \textbf{Existence of strong definable choice}

Here we finally prove that definable groups have strong definable choice.

\textbf{Theorem 7.1} Let $U$ be a definable group and let $\{T(x) : x \in X\}$ be a definable family of non empty definable subsets of $U$. Then there is a definable function $t : X \rightarrow U$ such that for all $x, y \in X$ we have $t(x) \in T(x)$ and if $T(x) = T(y)$ then $t(x) = t(y)$.
Proof. Let \( R(U) \) be the maximal definable solvable normal subgroup of \( U \). Then \( U/R(U) \) is definable and by [pps1] it has the property stated in the theorem. On the other hand, there is a definable section \( s : U/R(U) \to U \) and so \( U \) is definably isomorphic to a definable group with domain \( R(U) \times U/R(U) \) and so, it is sufficient to prove the theorem for definable solvable groups. By theorem 5.12 and an argument similar to the one above, the result is true for definable solvable groups if it is true for definably compact definable abelian groups.

So let \( U \) be a definably compact definable group and let \( \{T(x) : x \in X\} \) be a definable family of non empty definable subsets of \( U \). First note that the (induced) topology for the definable family \( T = \{T(x) : x \in X\} \) is uniformly definable. Let \( T := \{T(x) : x \in X\} \) be the closure of \( T(x) \) in \( U \). Suppose that \( U \subseteq N^m \) and for each \( l \in \{1, \ldots, m\} \) let \( \pi_l : N^m \to N^l \) be the projection onto the first \( l \) coordinates. For each \( x \in X \) let \( Y_m(x) := T(x) \) and for each \( i \in \{1, \ldots, m-1\} \) let \( Y_i(x) := \{a \in \pi_i(U) : \{(a, b) \in \pi_{i+1}(U)\} \cap Y_{i+1}(x) \neq \emptyset\} \) and let \( Y_i := \bigcup_{x \in X} Y_i(x) \). Note that for each \( x \in X \) and each \( a \in Y_{m-1}(x) \) the boundary of \( \{(a, b) \in U \cap T(x) \} \) in \( T(x) \) is finite (with cardinality uniformly bounded) and non empty because \( T(x) \) is closed. We have in this way a definable function \( l_{m-1} : X \times Y_{m-1} \to U \cup \{\infty\} \) such that \( l_{m-1}(x, a) \in T(x) \) iff \( a \in Y_{m-1}(x) \) and \( l_{m-1}(x, a) = \infty \) otherwise. Similarly, for each \( x \in X \) and \( a \in Y_{m-2}(x) \), the definable set \( l_{m-1}(x, \{(a, b) \in Y_{m-1}(x)\}) \) has a finite and non empty boundary in \( T(x) \) and we obtain a definable function \( l_{m-2} : X \times Y_{m-2} \to U \cup \{\infty\} \) such that \( l_{m-2}(x, a) \in T(x) \) iff \( a \in Y_{m-2}(x) \) and \( l_{m-2}(x, a) = \infty \) otherwise. Continuing in this way, we see that for each \( i \in \{1, \ldots, m-1\} \) there is a definable function \( l_i : X \times Y_i \to U \cup \{\infty\} \) such that \( l_i(x, a) \in T(x) \) iff \( a \in Y_i(x) \) and \( l_i(x, a) = \infty \) otherwise. Now, for each \( x \in X \) the definable set \( l_1(x, Y_1(x)) \) has finite and non empty boundary in \( T(x) \) and so we get a definable choice \( l \) for \( T(x) \) which by construction is a strong definable choice.

Now let \( O \) be the definable neighbourhood of 1 in \( U \) which has strong definable choice. And consider the definable family \( S := \{S(x) : x \in X\} \) of non empty definable subsets of \( O \) where \( S(x) := \{z \in O : l(x)z \in l(x)O \cap T(x)\} \). Note that if \( T(x) = T(y) \) then \( S(x) = S(y) \). Let \( s \) be a strong definable choice for \( S \). Then clearly, \( t := s \cdot l \) is a strong definable choice for \( T \). \qed
Corollary 7.2 below was also proved in [pst2] but assuming that $N$ has definable Skolem functions and using the theory of $\lor$-definable groups. By theorem 7.1, the assumption that $N$ has definable Skolem function is unnecessary:

**Corollary 7.2** Let $A$ be a definably compact definable abelian group. Then the following holds. (1) For every definable abelian group $B$, there is no infinite definable family of definable homomorphisms from $A$ into $B$ or vice-versa. (2) There is no infinite definable family of definable subgroups of $A$.

**Proof.** (1) Let $\gamma : S \times A \rightarrow B$ be an infinite definable family of definable automorphisms of $A$. Then by lemma 2.17 of [pst2] there is $\{a_1, \ldots, a_n\} \subseteq A$ such that for $s \in S$, $\gamma(s)$ is determined by its values on this finite set. Therefore, we can identify $S$ with a definable subset of $A \times \cdots \times A$ ($n$ times). Now the rest of the proof is obtained by adapting the proof of (1) in [pst2] and using theorem 7.1.

(2) The argument in the proof of corollary 5.2 [pst2] reduces it to case (1). \(\square\)

## 8 Definable rings

In this section we apply our result on definable abelian groups to describe definable rings. We start by recalling some facts about definable rings.

Let $U$ be a definable ring. Then by [1] and [ppm] $U$ can be equipped with a unique definable manifold structure making the ring into a topological ring, and definable homomorphisms between definable rings are topological homomorphisms. In fact, it follows from the results in [bps], that if $N$ is an o-minimal expansion of a real closed field then, $U$ equipped with the above unique definable manifold structure is a $C^p$ ring for all $p \in \mathbb{N}$ and definable homomorphisms between definable rings are $C^p$ homomorphisms for all $p \in \mathbb{N}$.

It follows from the DCC for definable groups, that $U$ satisfies the descending chain condition (DCC) on definable left (resp., right and bi-) ideals. Let $U^0$ be the definable connected component of zero in the additive group of $U$. Then $U^0$ is the smallest definable ideal of $U$ of finite index. We say that $U$ is definably connected if $U^0 = U$.  

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Finally we mention the following result from \cite{ps} which we generalise below. Let $U$ be an infinite definable associative ring without zero divisors. Then $U$ is a division ring and there is a one-dimensional definable subring $I$ of $U$ which is a real closed field such that $U$ is either $I$, $I(\sqrt{-1})$, or the ring of quaternions over $I$.

We now use our main result (Theorem 5.12), the results from \cite{opp} about rings definable in o-minimal expansions of real closed fields and Wedderburn theory to prove the following.

**Theorem 8.1** Let $U$ be a definable ring. Then there is a definable left ideal $V = K \oplus \bigoplus_{j=1}^m W_j \oplus \bigoplus_{i=1}^n V'_i \oplus \bigoplus_{i=1}^n V_i$ of $U$ such that $K$ is the definably compact, definably connected definable left ideal of $U$ of maximal dimension and for each $j = 1, \ldots, m$ (resp., $i = 1, \ldots, n$) there is a semi-bounded o-minimal expansion $\mathcal{J}_j$ of a group (resp., an o-minimal expansion $\mathcal{I}_i$ of a real closed field) definable in $\mathcal{N}$ such that there is no definable bijection between a distinct pair among the $\mathcal{J}_j$’s and the $\mathcal{I}_i$’s, $W_j$ is a direct product of copies of the additive group of $\mathcal{J}_j$ and has zero multiplication, $V'_i$ is a direct product of copies of the linearly bounded one-dimensional torsion-free $\mathcal{I}_i$-definable group and has zero multiplication, each $V_i$ is an $\mathcal{I}_i$-definable ring such that if $\overline{V}_i := V_i/\text{ann}_{\mathcal{I}_i} V_i$ is non-trivial then $\overline{V}_i$ is a finitely generated $\mathcal{I}_i$-algebra (and therefore $\mathcal{I}_i$-definable) and if it is associative then it is $\mathcal{I}_i$-definably isomorphic to a finitely generated $\mathcal{I}_i$-subalgebra of some $M_{n_i}(\mathcal{I}_i)$ and has a nilpotent finitely generated ideal $Z_i$ such that $\overline{V}_i/Z_i$ is $\mathcal{I}_i$-definably isomorphic to $\bigoplus_{j=1}^{m_i} M_{k_{i,j}}(D_{i,j})$ where for each $j = 1, \ldots, m_i$, $D_{i,j}$ is either $\mathcal{I}_i$, $\mathcal{I}_i(\sqrt{-1})$, or the ring of quaternions over $\mathcal{I}_i$. Moreover, $U/V$ is a definably compact definable ring.

**Proof.** If we consider $U$ as an additive definable group and apply Theorem 5.12 then $U$ has a definable subgroup $V = K \times W_1 \times \cdots \times W_m \times V'_1 \times V_1 \times \cdots \times V'_n \times V_n$ such that $K$ is the definably compact, definably connected definable additive subgroup of $U$ of maximal dimension and for each $j = 1, \ldots, m$ (resp., $i = 1, \ldots, n$) there is a semi-bounded o-minimal expansion $\mathcal{J}_j$ of a group (resp., an o-minimal expansion $\mathcal{I}_i$ of a real closed field) definable in $\mathcal{N}$ such that there is no definable bijection between a distinct pair among the $\mathcal{J}_j$’s and the $\mathcal{I}_i$’s, the additive group $W_j$ is a direct product of copies of the additive group of $\mathcal{J}_j$, the additive group $V'_i$ is a direct product of copies of the linearly bounded one-dimensional torsion-free $\mathcal{I}_i$-definable group and
each $V_i$ is an $\mathcal{I}_i$-definable additive group. Moreover, any definable additive subgroup of $U/V$ is definably compact.

It follows easily from this that $V$, $K$, $W_j$'s, $V'_i$'s and $V_i$'s are all definable left ideals of $U$ and $V = K \oplus \bigoplus_{j=1}^m W_j \oplus \bigoplus_{i=1}^n V'_i \oplus \bigoplus_{i=1}^n V_i$.

We now show that each $W_j$ has zero multiplication (the proof is the same for each $V'_i$): we have a group homomorphism $W_j/\text{ann} W_j W_j^{-} \to \text{End}(W_j)$ of additive groups, where $\text{End}(W_j)$ is the group of all $\mathcal{J}_j$-definable endomorphisms of $W_j$, which is clearly isomorphic with $M_{n_j}(\Lambda(\mathcal{J}_j))$ where $\Lambda(\mathcal{J}_j)$ is the division ring of all $\mathcal{J}_j$-definable endomorphisms of the additive group of $\mathcal{J}_j$. By 
\[\text{Ins},\] $W_j/\text{ann} W_j W_j$ must be finite, and because $W_j$ is $\mathcal{J}_j$-definably connected we have $W_j = \text{ann} W_j$.

By construction of $\mathcal{I}_i$, $V_i$ is a $\mathcal{I}_i$-definable ring. Suppose that $V_i$ is non-trivial. The fact that each $V_i$ is $\mathcal{I}_i$-definably isomorphic with a finitely generated $\mathcal{I}_i$-algebra and that if it is associative then it is $\mathcal{I}_i$-definably isomorphic to a finitely generated $\mathcal{I}_i$-subalgebra of some $M_{n_i}(\mathcal{I}_i)$ follows from (the proof of) lemma 4.3 in [opp] and the rest is just Wedderburn theory (for details see for example the section on Wedderburn theory in [al]).

\[\Box\]

**Theorem 8.2** A definably compact, definably connected definable ring has zero multiplication.

**Proof.** This is a corollary of the proof of theorem 1.2.

\[\Box\]

**Definition 8.3** Recall that a Lie ring is an additive group $L$ with a bilinear product (called bracket) $[x, y]$ such that for all $x, y, z \in L$ (i) $[x, x] = 0$ and (ii) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi identity). $L$ is abelian if for all $x, y \in L$, $[x, y] = 0$.

The following facts are proved exactly as above (using in fact 8.4 the Lie ring analogue of lemma 4.3 in [opp]).

**Fact 8.4** Let $U$ be a definable Lie ring. Then there is a definable left ideal $V = K \oplus \bigoplus_{j=1}^m W_j \oplus \bigoplus_{i=1}^n V'_i \oplus \bigoplus_{i=1}^n V_i$ of $U$ such that $K$ is the definably compact, definably connected definable left ideal of $U$ of maximal dimension and for each $j = 1, \ldots, m$ (resp., $i = 1, \ldots, n$) there is a semi-bounded o-minimal expansion $\mathcal{J}_j$ of a group (resp., an o-minimal expansion $\mathcal{I}_i$ of a
real closed field) interpretable in $N$ such that there is no definable bijection between a distinct pair among the $J_j$’s and the $I_i$’s, $W_j$ is a direct product of copies of the additive group of $J_j$ and is an abelian Lie ring, $V_i'$ is a direct product of copies of the linearly bounded one-dimensional torsion-free $I_i$-definable group and is an abelian Lie ring, each $V_i$ is an $I_i$-definable ring such that $\overline{V_i} := V_i/ann_VV_i$ is $I_i$-definably isomorphic to a finitely generated Lie subalgebra of some $M_n(I_i)$. Moreover, $U/V$ is a definably compact definable Lie ring.

**Fact 8.5** A definably compact, definably connected definable Lie ring is abelian.

**Acknowledgements.** Part of the work presented here is contained in the authors DPhil Thesis which was financially supported by JNICT grant PRAXIS XXI/BD/5915/95. I would like to thank my thesis adviser Professor Alex Wilkie and Kobi Peterzil for their constant support. I would also like to thank the EPSRC for current financial support.

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