Analytic Resolving Families for Equations with Distributed Riemann–Liouville Derivatives

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Abstract: Some new necessary and sufficient conditions for the existence of analytic resolving families of operators to the linear equation with a distributed Riemann–Liouville derivative in a Banach space are established. We study the unique solvability of a natural initial value problem with distributed fractional derivatives in the initial conditions to corresponding inhomogeneous equations. These abstract results are applied to a class of initial boundary value problems for equations with distributed derivatives in time and polynomials with respect to a self-adjoint elliptic differential operator in spatial variables.

Keywords: Riemann–Liouville derivative; distributed order equation; analytic resolving family of operators; generator of resolving family; perturbation theorem

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1. Introduction

The main goal of this work is the study of the unique solvability issues for a special initial value problem to a class of equations with a distributed Riemann–Liouville derivative. The concept of distributed derivative is firstly encountered, apparently, in the works of A.M. Nakhushev [1,2]. Equations with distributed fractional derivatives appear in various fields of investigations applied to the mathematical modelling of some real processes, when an order of a fractional derivative in a model continuously depends on the process parameters: in the kinetic theory [ 3], in the theory of viscoelasticity [ 4] and so on [5–7]. Numerical methods of solving such equations were developed in the last decades; see [8,9] and the references therein. The qualitative properties of equations with distributed fractional derivatives are investigated in the works of A.M. Nakhushev [1,2], A.V. Pskhu [10,11], S. Umarov and R. Gorenflo [12], T.M. Atanacković, Lj. Oparnica and S. Pilipović [13], A.N. Kochubei [14] and others.

Consider the distributed order equation

\[ \int_b^c \omega(\alpha) D_\alpha^\alpha z(t) d\alpha = Az(t) + g(t), \quad t \in (0, T], \]

with the Riemann–Liouville derivative \( D_\alpha^\alpha \) and with a closed linear operator \( A \) in a Banach space \( Z \), where \(-\infty < b \leq 0 \leq m - 1 < c \leq m \in \mathbb{N}, \omega : (b, c) \rightarrow \mathbb{C}, \omega \neq 0, T > 0,\)
\[ g : [0, T] \rightarrow \mathcal{Z}. \] The Cauchy problem for such an equation with the Gerasimov–Caputo distributed derivative was studied in the case of a bounded operator \( A \) in \([15]\). A special initial value problem

\[
\int_{m-1-k}^{c} \omega(t)D_t^{\alpha-m+k}z(0)dt = z_k, \quad k = 0, 1, \ldots, m - 1, \tag{2}
\]

for Equation (1) with the Riemann–Liouville distributed derivative was researched in \([16]\) with a bounded operator \( A \). Similar results for initial value problems to equations with a degenerate linear operator at the distributed derivative are also obtained in \([15]\).

Necessary and sufficient conditions on a closed operator \( A \) for the existence of an analytic in a sector resolving operators family are obtained for homogeneous Equation (1) with the Gerasimov–Caputo distributed derivative in \([17]\) with \( c \in (0, 1] \) and in \([18]\) with \( c > 1 \). In \([19]\) analogous result was obtained for Equation (1) with a discretely distributed Gerasimov–Caputo derivative; Reference \([20]\) is devoted to the existence issues for strongly continuous resolving operators family of the homogeneous Equation (1) with the Gerasimov–Caputo derivative. The obtained results on resolving operators families allowed, in \([17–20]\), the research of the unique solvability of inhomogeneous Equation (1) and to investigate some properties of the equation, such as the continuity in the operator norm at zero of a resolving family, conditions for the boundedness of a generating operator \( A \), a perturbation theorem for a class of generators \( A \) and others.

All the mentioned results were obtained for \( b = 0 \); here, we will consider case \( b \leq 0 \), but this will not bring any significant changes to our reasoning.

In the second section, the statement of initial value problem (2) for Equation (1) with the Riemann–Liouville derivative is obtained and properties of functions, which arise when applying the Laplace transform to the distributed fractional derivative, are investigated. In the third section, the theorem on analytic in a sector inverse Laplace transforms is generalized to the case of functions with a power singularity at zero. A theorem on conditions for the operator \( A \), which are necessary and sufficient for the existence of analytic in a sector resolving family of operators of homogeneous Equation (1) is proved in the fourth section. This result was applied to studying problem (1), (2) in the fifth section. The last section contains an application of obtained abstract results to a study of a class of initial boundary value problems for equations with a distributed fractional derivative in time and polynomials with respect to a self-adjoint elliptic differential operator in spatial variables.

### 2. Equation with Distributed Riemann–Liouville Derivative

Let \( \mathcal{Z} \) be a Banach space. Denote at \( \beta > 0 \), \( t > 0 \), \( h : \mathbb{R}_+ \rightarrow \mathcal{Z} \), for example, \( h \in C(\mathbb{R}_+; \mathcal{Z}) \), the fractional Riemann–Liouville integral is defined by

\[
\Lambda_{\beta}^{\gamma}h(t) := \frac{1}{\Gamma(\beta)} \int_{0}^{t} (t-s)^{\beta-1}h(s)ds,
\]

where \( \Gamma(\cdot) \) is the Euler gamma function. Let \( m - 1 < \alpha \leq m \in \mathbb{N} \), \( D^{\alpha}_{t}h(t) \) be the usual derivative of the \( m \)-th order of \( h \), \( D^{\alpha}_{t}h(t) := D^{m}_{t}(\Lambda_{\beta}^{\gamma}h(t)) \) be the Riemann–Liouville fractional derivative.

The Laplace transform of a function \( h : \mathbb{R}_+ \rightarrow \mathcal{Z} \) will be denoted by \( \mathring{h} \) or \( \text{Lap}[h] \), if an expression \( h \) is too long. By \( \mathcal{Z} \) denote the set of functions \( h : \mathbb{R}_+ \rightarrow \mathcal{Z} \), such that the Laplace transform \( \mathring{h} \) is defined. The Laplace transform of the Riemann–Liouville fractional derivative of an order \( \alpha > 0 \) satisfies the equality (see \([21]\)):

\[
\mathring{D^{\alpha}_{t}h}(\lambda) = \lambda^\alpha \mathring{h}(\lambda) - \sum_{k=0}^{m-1} D^{\alpha-1-k}_{t}h(0)\lambda^k. \tag{3}
\]
Here and further $D^\beta_t h(0) := \lim_{t \to 0^+} D^\beta_t h(t)$.

Denote by $\mathcal{L}(\mathcal{Z})$ the Banach space of all linear continuous operators from $\mathcal{Z}$ to $\mathcal{Z}$; $\mathcal{C}(\mathcal{Z})$ stands for the set of all linear closed operators, densely defined in $\mathcal{Z}$, acting to the space $\mathcal{Z}$. We supply the domain $D_A$ of an operator $A \in \mathcal{C}(\mathcal{Z})$ by the norm of its graph. Thus, we have the Banach space $D_A$.

Consider a distributed order equation:

$$
\int_{b}^{c} \omega(\alpha) D_t^\alpha z(t) d\alpha = Az(t), \quad t > 0,
$$

where $-\infty < b \leq 0 \leq m - 1 < c \leq m \in \mathbb{N}$, $\omega : (b,c) \to \mathbb{C}$, $\omega \neq 0$, $A \in \mathcal{C}(\mathcal{Z})$. Note that, due to equality (3),

$$
\int_{b}^{c} \omega(\alpha) D_t^\alpha z(\lambda) d\alpha = \int_{b}^{c} \omega(\alpha) \lambda^\alpha \tilde{z}(\lambda) d\alpha - \sum_{l=0}^{m-1} \int_{l-1}^{l} \omega(\alpha) \sum_{k=0}^{l-1} D_t^\alpha z(0) \lambda^k d\alpha - \int_{m-1}^{c} \omega(\alpha) \sum_{k=0}^{m-1} D_t^{\alpha-k} z(0) \lambda^k d\alpha = \int_{b}^{c} \omega(\alpha) \lambda^\alpha \tilde{z}(\lambda) d\alpha - \sum_{k=0}^{m-1} \int_{k}^{m-1-k} \omega(\alpha) D_t^{\alpha-k} z(0) \lambda^m-1-k d\alpha.
$$

Therefore, the natural initial value conditions for Equation (4) are:

$$
\int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(0) d\alpha = z_k, \quad k = 0, 1, \ldots, m - 1,
$$

where

$$
\int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(0) d\alpha := \lim_{t \to 0^+} \int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(t) d\alpha, \quad k = 0, 1, \ldots, m - 1.
$$

By a solution of problem (4), (6) we mean a function $z \in C(\mathbb{R}_+; D_A)$, such that there exists $\int_{b}^{c} \omega(\alpha) D_t^\alpha z(t) d\alpha \in C(\mathbb{R}_+; \mathcal{Z})$, $\lim_{t \to 0^+} \int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(t) d\alpha, \ k = 0, 1, \ldots, m - 1$, and equalities (4) and (6) are fulfilled.

Denote $S_{\theta, a} := \{ \mu \in \mathbb{C} : |\arg(\mu - a)| < \theta, \mu \neq a \}$ at $\theta \in (\pi/2, \pi)$, $a \in \mathbb{R}$,

$$
W(\lambda) := \int_{b}^{c} \omega(\alpha) \lambda^\alpha d\alpha, \quad W_k(\lambda) := \int_{k}^{c} \omega(\alpha) \lambda^\alpha d\alpha, \quad k = 0, 1, \ldots, m - 1.
$$

The following properties of these functions are available.

**Lemma 1.** [17,20]. Let $-\infty < b \leq 0 \leq m - 1 < c \leq m \in \mathbb{N}$, $\omega \in L_1(b,c)$. Then $W, W_k, k = 0, 1, \ldots, m - 1$, are analytic on the set $S_{\pi, \theta}$.

**Lemma 2.** Let $-\infty < b \leq 0 \leq m - 1 < c \leq m \in \mathbb{N}$, $\omega \in L_1(b,c)$, $\omega$ be continuous from the left at the point $c$, $\lim_{\alpha \to c^-} \omega(\alpha) \neq 0$. Then,

$$
\exists \epsilon_0 > 0 \forall \epsilon \in (0, \epsilon_0) \exists C > 0 \exists \epsilon > 0 \forall \lambda \in S_{\pi,0} \setminus \{ \lambda \in \mathbb{C} : |\lambda| < \epsilon \} \left| W(\lambda) \right| \geq C|\lambda|^\epsilon - \epsilon. \quad (7)
$$
Proof. For $c_1 \in (b, c)$, which is close enough to $c$, take arbitrary $\varepsilon_1 \in (0, c - c_1)$; then for large enough $|\lambda|$, the mean value theorem implies:

$$\left| \int_{c_1}^{c} \omega(\alpha) \lambda^a d\alpha \right| = |\omega'(\xi)| \int_{c_1}^{c} |\lambda|^\alpha d\alpha = |\omega'(\xi)| \frac{|\lambda|^\varepsilon_1 - |\lambda|^{\varepsilon_1}}{\ln |\lambda|} \geq C_1 |\lambda|^\varepsilon_1,$$

with some $\xi \in (c_1, c)$ and $C_1 = C_1(\varepsilon_1)$. Therefore, for every $\varepsilon \in (\varepsilon_1, c - c_1)$ there exists $C = C(\varepsilon) > 0$ such that for large enough $|\lambda|

$$\left| \int_{b}^{c} \omega(\alpha) \lambda^a d\alpha \right| \geq C_1 |\lambda|^{\varepsilon - \varepsilon_1} - |\lambda|^\varepsilon_1 \left| \int_{b}^{c_1} \omega(\alpha) d\alpha \right| \geq C |\lambda|^{\varepsilon}.$$

☐

Lemma 3. Let $-\infty < b \leq 0 \leq m - 1 < c \leq m \in \mathbb{N}$, $\omega \in L_1(b, c)$. Then, for all $k = 0, 1, \ldots, m - 1$

$$\exists C > 0 \quad \forall \lambda \in S_{\pi, b} \setminus \{ \lambda \in \mathbb{C} : |\lambda| < 1 \} \quad |W(\lambda) - W_k(\lambda)| \leq C|\lambda|^k;$$

$$\forall \lambda \in S_{\pi, b} \setminus \{ \lambda \in \mathbb{C} : |\lambda| < 1 \} \quad |W(\lambda)| \leq \|\omega\|_{L_1(b, c)}|\lambda|^\varepsilon.$$

Proof. We have at $|\lambda| \geq 1$ the evident inequalities $|W(\lambda)| \leq \|\omega\|_{L_1(b, c)}|\lambda|^\varepsilon$ and

$$|W(\lambda) - W_k(\lambda)| \leq \left| \int_{b}^{c} \omega(\alpha) \lambda^a d\alpha \right| \leq \left| \int_{b}^{c} |\omega(\alpha)| d\alpha \right| |\lambda|^k = C |\lambda|^k$$

for all $k = 0, 1, \ldots, m - 1$. ☐

3. Analytic in a Sector Function with a Power Singularity at Zero

Let us introduce the notation $\Sigma_\psi := \{ t \in \mathbb{C} : \arg t < \psi, t \neq 0 \}$ for $\psi \in (0, \pi/2]$ and prove an important for further considerations assertion.

Theorem 1. Let $\theta_0 \in (\pi/2, \pi], a \in \mathbb{R}$, $\beta \in [0, 1]$, $\mathcal{X}$ be a Banach space, a map $H : (a, \infty) \to \mathcal{X}$ be set. The next assertions are equivalent.

(i) There exists an analytic function $F : \Sigma_{\theta_0 - \pi/2} \to \mathcal{X}$, for every $\theta \in (\pi/2, \theta_0)$ there exists such $C(\theta) > 0$, that for all $t \in \Sigma_{\theta - \pi/2}$ the inequality $\|F(t)\|_{\mathcal{X}} \leq C(\theta)|t|^{-\beta}e^{\operatorname{Re} t}$ is satisfied;

$\hat{F}(\lambda) = H(\lambda)$ at $\lambda > a$.

(ii) The map $H$ is analytically continued on $S_{\theta_0, a}$; for every $\theta \in (\pi/2, \theta_0)$ there exists $K(\theta) > 0$, such that for all $\lambda \in S_{\theta, a}$

$$\|H(\lambda)\|_{\mathcal{X}} \leq \frac{K(\theta)}{|\lambda - a|^{1-\beta}}.$$

Proof. For $\beta = 0$ this statement was proved in ([22], Theorem 0.1, p. 5), ([23], Theorem 2.6.1, p. 84) directly, using properties of analytic functions and estimates for the Laplace transform and contour integrals. We will carry out arguments similar to the proof of Theorem 2.6.1 in [23], but in the case $\beta \in (0, 1)$.

Let assertion (i) hold, $\pi/2 < \theta < \theta_0 \leq \pi$, $\gamma_+^\delta = (0, \delta] \cup \{ \delta + re^{\pm i(\theta - \pi/2)} : r \in (0, \infty) \}$. By the Cauchy theorem for all $\lambda > a$

$$\hat{F}(\lambda) = \int_{\gamma_+^\delta} F(t)e^{-\lambda t} dt = \int_{\gamma_+^\delta} F(\tau)e^{-\lambda \tau} d\tau =$$
Mathematics defines an analytic function in the sector is absolutely convergent, uniformly over compact subsets of continuation theorem they define an analytic function since \(\lambda\) is real and \(\delta > 0\), then

\[
\tilde{F}(\lambda) = e^{\pm i(\theta - \pi/2)} \int_0^\infty F(e^{\pm i(\theta - \pi/2)}) e^{-\lambda \cos(\theta - \pi/2)} \, dr
\]

so, \(\|F(t)\|_{\tilde{H}} \leq C(\theta - \epsilon) t^{-\delta} e^{\text{Re}(\text{arg } t) \sin \epsilon} \) for all \(t \in \Sigma_{\theta - \pi/2 - \epsilon/2}\).
By the Fubini theorem and the Cauchy residue theorem we have for $\lambda > a$

$$\tilde{F}(\lambda) = \frac{1}{2\pi i} \int \frac{H(\mu)d\mu}{\lambda - \mu} = H(\lambda) - \lim_{R \to \infty} \left( \int_{\Gamma_R^0} \frac{H(\mu)d\mu}{\lambda - \mu} - \int_{\Gamma_R^+} \frac{H(\mu)d\mu}{\lambda - \mu} - \int_{\Gamma_R^-} \frac{H(\mu)d\mu}{\lambda - \mu} \right),$$

where $\Gamma_R^0 := \{a + Re^{i\varphi} : \varphi \in (-\theta, \theta)\}$, $\Gamma_R^\pm := \{a + re^{\pm i\theta} : r \in [R, \infty)\}$. Then,

$$\left\| \int_{\Gamma_R^0} \frac{H(\mu)d\mu}{\lambda - \mu} \right\|_{X} \leq \frac{\theta}{-\theta} \frac{R^\beta K(\theta)d\varphi}{|a + Re^{\pm i\theta} - \lambda|} \to 0,$$

$$\left\| \int_{\Gamma_R^\pm} \frac{H(\mu)d\mu}{\lambda - \mu} \right\|_{X} \leq \int_{R}^{\infty} \frac{r^{\beta-1}K(\theta)dr}{|a + re^{\pm i\theta} - \lambda|} \leq C_1 \int_{R}^{\infty} r^{\beta-2}dr \to 0,$$

as $R \to \infty$, since $\beta < 1$. Thus, $\tilde{F} \equiv H$. □

4. k-Resolving Families of Operators

A family of operators $\{S_l(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is called $l$-resolving, $l \in \{0, 1, \ldots, m - 1\}$, for Equation (4), if the next conditions are satisfied:

(i) $S_l(t)$ is strongly continuous at $t > 0$;

(ii) $S_l(t)[D_A] \subset D_A$, $S_l(t)Ax = AS_l(t)x$ for all $x \in D_A$, $t > 0$;

(iii) for every $z_l \in D_A$ $S_l(t)z_l$ is a solution of problem (4), (6) with $z_k = 0, k \in \{0, 1, \ldots, m - 1\} \setminus \{l\}$.

A k-resolving family of operators at $k \in \{0, 1, \ldots, m - 1\}$ is called analytic, if it has the analytic continuation to a sector $\Sigma_{\varphi_0}$ at some $\varphi_0 \in (0, \pi/2)$. An analytic k-resolving family of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ has a type $(\varphi_0, a_0, \beta)$ at some $\varphi_0 \in (0, \pi/2]$, $a_0 \in \mathbb{R}, \beta > 0$, if for all $\psi \in (0, \psi_0)$, $a > a_0$ there exists such $C(\psi, a)$, that for all $t \in \Sigma_{\varphi}$ the inequality $\|S_k(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C(\psi, a) |t|^{-\beta} |\varphi|^a \text{Re}t$ is satisfied.

**Proposition 1.** Let $\theta_0 \in (\pi/2, \pi], a_0 \in \mathbb{R}, \beta \in (0, 1)$, there exist an analytic 0-resolving family $\{S_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ of the type $(\theta_0, a_0, \beta)$ for Equation (4) and k-resolving families $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ for Equation (4), $k = 1, 2, \ldots, m - 1$. Then for $k = 1, 2, \ldots, m - 1$ the k-resolving families $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ are analytic with the type $(\theta_0, a_1, 0)$ at some $a_1 > 0$, $a_1 \geq a_0$. Moreover, for every $k \in \{0, 1, \ldots, m - 1\}$ a k-resolving family is unique and $S_k(t) = \frac{1}{k!} S_0(t), t > 0$.

**Proof.** Take for $k \in \{1, 2, \ldots, m - 1\}$ the family $\{S_k(t) := \frac{1}{k!} S_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$. Then condition (i) in the definition of a k-resolving family is satisfied. For $x \in D_A, t > 0$

$$\int_0^t \frac{1}{k!} S_0(s)Axds = \frac{t}{k!} S_0(s)xds = \frac{t}{k!} S_0(s)xds = A\frac{1}{k!} S_0(t)x$$

due to the closedness of the operator $A$. So, condition (ii) holds.

Take $x_k \in D_A, S_0(t)x_k$ is a solution of problem (4), (6) with the initial values $z_0 = x_k, z_1 = z_2 = \cdots = z_{m-1} = 0$, consequently, due to (5) $W(\lambda)S_0x_k - \lambda^{m-1}x_k = A\tilde{S}_0x_k$. Hence,

$$W(\lambda)\frac{1}{k!} S_0x_k - \lambda^{m-1-k}x_k = W(\lambda)\lambda^{-k}\tilde{S}_0x_k - \lambda^{m-1-k}x_k = A\lambda^{-k}\tilde{S}_0x_k = A\frac{1}{k!} S_0x_k.$$

Therefore, if there exists a k-resolving family for $k \in \{1, 2, \ldots, m - 1\}$, then it coincides with $\{\frac{1}{k!} S_0(t) : t > 0\}$ due to the uniqueness of the Laplace transform.
Mathematics 2022, 10, 681

7 of 19

By the conditions of the theorem for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$, for all $\lambda \in S_{\theta,a}$

$$\|S_0(\lambda)\|_{\mathcal{L}(\mathcal{Z})} = \|\lambda^{m-1}(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta,a)}{|\lambda - a|^{1-\beta}}.$$  

Consequently, for all $\theta \in (\pi/2, \theta_0)$, $a > a_1$, $\lambda \in S_{\theta,a}$

$$\|\hat{F}_{\theta}S_0(\lambda)\|_{\mathcal{L}(\mathcal{Z})} = \|\lambda^{m-k}(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta,a,a)}{|\lambda|^{k-\beta}|\lambda - a|} \leq \frac{K_1(\theta,a)}{|\lambda - a|},$$

if we take $a_1 > 0$, $a_1 > a_0$. □

**Remark 1.** Due to Proposition 1 further we will not write about the type of $k$-resolving families, $k = 1, 2, \ldots, m - 1$.

Denote by $\rho(A)$ the resolvent set of an operator $A$. Let an operator $A \in Cl(\mathcal{Z})$ satisfy the following conditions:

1. there exist such $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, that for $\lambda \in S_{\theta_0,a_0}$ we have $W(\lambda) \in \rho(A)$;
2. there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists such $K(\theta,a,\epsilon) > 0$, that for all $\lambda \in S_{\theta_0,a_0}$

$$\|(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta,a,\epsilon)}{|\lambda|^{m-1}|\lambda - a|^{1-\epsilon}},$$

Then we will say that the operator $A$ belongs to the class $\mathcal{A}_{\epsilon}(\theta_0,a_0)$. Here, as before, $m - 1 < c \leq m \in \mathbb{N}$, $c$ is the upper limit of the integration in the definition of $W$.

If condition 2) is valid for $\epsilon = 0$, we will denote such class as $\mathcal{A}_{\epsilon}^{(c)}(\theta_0,a_0)$. Obviously, $\mathcal{A}_{\epsilon}^{(c)}(\theta_0,0) \subset \mathcal{A}_{\epsilon}^{(c)}(\theta_0,0)$.

If $A \in \mathcal{A}_{\epsilon}^{(c)}(\theta_0,a_0)$, the operators,

$$Z_k(t) := \frac{1}{2\pi i} \int_{\Gamma} \lambda^{m-k}(W(\lambda)I - A)^{-1}e^{A\lambda}d\lambda, \quad k = 0, 1, \ldots, m - 1,$$

are defined at $t > 0$. Here $\Gamma := \Gamma_{+} \cup \Gamma_{-} \cup \Gamma_0$, $\Gamma_{\pm} := \{\lambda \in \mathbb{C} : \lambda = a + re^{\pm i\theta}, r \in (\delta, \infty)\}$, $\Gamma_0 := \{\lambda \in \mathbb{C} : \lambda = a + \delta e^{i\theta}, \varphi \in (-\theta, \theta)\}$ for some $\delta > 0$, $a > a_0$, $\theta \in (\pi/2, \theta_0)$.

**Theorem 2.** Let $-\infty < b \leq 0 \leq m - 1 < c \leq m \in \mathbb{N}$, $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, $\omega \in L_1(b,c)$.

(i) If there exists an analytic $0$-resolving family of operators of the type $(\theta_0 - \pi/2, a_0, m + \epsilon - c)$ for every $\epsilon \in (0, \epsilon_0)$, then $A \in \mathcal{A}_{\epsilon}^{(c)}(\theta_0,a_0)$.

(ii) If $A \in \mathcal{A}_{\epsilon}^{(c)}(\theta_0,a_0)$, then there exist an analytic $0$-resolving family of operators $\{S_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ of the type $(\theta_0 - \pi/2, a_0, m + \epsilon - c)$ at every $\epsilon \in (0, \epsilon_0)$, and analytic $k$-resolving families of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$, $k = 1, 2, \ldots, m - 1$, for Equation (4). In this case, for every $k = 0, 1, \ldots, m - 1$ a $k$-resolving family of operators is unique, $S_k(t) \equiv Z_k(t)/Z_0(t)$, $t > 0$, and at any $z_0, z_1, \ldots, z_{m-1} \in D_A$ the function

$$z(t) = \sum_{k=0}^{m-1} Z_k(t)z_k$$

is a unique solution of problem (4), (6) in the space $\hat{\mathcal{Z}}$.

**Proof.** Let $A \in \mathcal{A}_{\epsilon}^{(c)}(\theta_0,a_0)$, $R > \delta$, 

$$\Gamma_R := \bigcup_{k=1}^{4} \Gamma_{k,R}, \quad \Gamma_{1,R} := \Gamma_0, \quad \Gamma_{2,R} := \{\lambda \in \mathbb{C} : \lambda = a + Re^{i\varphi}, \varphi \in (-\theta, \theta)\},$$


\[ \Gamma_{3,R} := \{ \lambda \in \mathbb{C} : \lambda = a + re^{i\theta}, r \in [\delta, R] \}, \quad \Gamma_{4,R} := \{ \lambda \in \mathbb{C} : \lambda = a + re^{-i\theta}, r \in [\delta, R] \}, \]

\( \Gamma_R \) is the positively oriented closed loop,

\[ \Gamma_{5,R} := \{ \lambda \in \mathbb{C} : \lambda = a + re^{i\theta}, r \in (R, \infty) \}, \quad \Gamma_{6,R} := \{ \lambda \in \mathbb{C} : \lambda = a + re^{-i\theta}, r \in [R, \infty) \}, \]

then \( \Gamma = \Gamma_{3,R} \cup \Gamma_{6,R} \cup \Gamma_{R} \setminus \Gamma_{2,R} \).

For \( A \in \mathcal{A}_w^{\phi}(\theta_0, a_0) \) by Theorem 1 with \( X = \mathcal{L}(Z) \) the operator family \( \{Z_0(t) \in \mathcal{L}(Z) : t > 0\} \) is analytic of the type \( (\theta_0 - \pi/2, a_0, m + \epsilon - \kappa) \) at every \( \epsilon \in (0, \epsilon_0) \),

\[ \|Z_0(t)\|_{\mathcal{L}(Z)} \leq C(\theta, a, \epsilon)|t|^{-(m-\epsilon)}e^{aRe\theta} \] for \( \theta \in (\pi/2, \theta_0), a > a_0, \epsilon \in (0, \epsilon_0) \). Then for \( k = 0, 1, \ldots, m-1 \) there exist the Laplace transforms at \( \Re \lambda > a_0 \) \( \hat{Z}_k(\lambda) = \lambda^{m-1-k}(W(\lambda)I - A)^{-1} \),

\[ \text{Lap}[f_{\lambda}^{\tau}Z_k](\lambda) = \lambda^{m-1-k}(W(\lambda)I - A)^{-1}, \quad \alpha < m, \text{ therefore, } Z_k(t) = f_{\lambda}^{\tau}Z_0(t). \]

For \( t \in [0, 1], \lambda \in \Gamma, \alpha \in (b, c), \)

\[ \left\| e^{it\lambda^{a-1-k}(W(\lambda)I - A)^{-1}} \right\|_{\mathcal{L}(Z)} \leq \frac{e^{a+b}K(\theta, a)}{|\lambda|^{m-a+k}\alpha - a|\alpha - 1 - m - \epsilon|} \leq \frac{C_1}{|\lambda|^{1+c-a+\kappa - \epsilon}} \leq \frac{C_1}{|\lambda|^{1+k - \kappa}}. \]

Hence, at \( k = 1, 2, \ldots, m-1 \) the integral \( Z_k(t) \) converges uniformly on \( t \in [0, 1] \) and

\[ \left\| f_{\lambda}^{\tau}Z_k(\lambda) = \frac{1}{2\pi i} \int_{1_{R}} \int_{\Gamma_R} \int_{1_{R}} \int_{\Gamma_R} \frac{\lambda^{a-1-k}(W(\lambda)I - A)^{-1}}{\lambda, t_k}d\lambda d\lambda = \right\| \]

\[ \left\| \int_{1_{R}} \int_{\Gamma_R} \int_{1_{R}} \int_{\Gamma_R} \frac{\lambda^{a-1-k}(W(\lambda)I - A)^{-1}}{\lambda, t_k}d\lambda d\lambda = \right\| \]

\[ \text{since by the Cauchy theorem} \]

\[ \int_{1_{R}} \int_{\Gamma_R} \int_{1_{R}} \int_{\Gamma_R} \frac{\lambda^{a-1-k}(W(\lambda)I - A)^{-1}}{\lambda, t_k}d\lambda d\lambda = 0, \]

for \( t \in [0, 1], \lambda \in \Gamma_{s,R} \)

\[ \left\| \int_{1_{R}} \int_{\Gamma_R} \int_{1_{R}} \int_{\Gamma_R} \frac{\lambda^{a-1-k}(W(\lambda)I - A)^{-1}}{\lambda, t_k}d\lambda d\lambda \right\| \leq \frac{C_2}{R^{1+c}} \quad s = 2, 5, 6. \]

At the same time,

\[ \int_{m-1}^{c} \omega(\alpha) f_{\lambda}^{\tau}Z_0(\lambda)z_0d\alpha = \frac{1}{2\pi i} \int_{1_{R}} \int_{1_{R}} \int_{1_{R}} \int_{1_{R}} \frac{\omega(\alpha)\lambda^{a-1}d\alpha(W(\lambda)I - A)^{-1}z_0e^{\alpha t}d\lambda =} \]

\[ = \frac{1}{2\pi i} \int_{1_{R}} \frac{W_{m-1}(\alpha) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}z_0e^{\alpha t}d\lambda + \frac{1}{2\pi i} \int_{1_{R}} \left( \frac{e^{\alpha t}z_0}{\lambda} \right) d\lambda + \]

\[ + \frac{1}{2\pi i} \int_{1_{R}} \frac{1}{\lambda}(W(\lambda)I - A)^{-1}Az_0e^{\alpha t}d\lambda \rightarrow z_0, \]

as \( t \rightarrow 0+ \), since

\[ \left\| \frac{W_{m-1}(\alpha) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1} \right\|_{\mathcal{L}(Z)} \leq \frac{C_1}{|\lambda|^{2+c-m-\epsilon}}, \quad 2 + c - m - \epsilon > 1, \]

\[ \left\| \frac{1}{\lambda}(W(\lambda)I - A)^{-1}Az_0 \right\|_{\mathcal{L}(Z)} \leq \frac{C_2}{|\lambda|^{1+c - \epsilon}}, \quad 1 + c - \epsilon > 1. \]
We have at $c > 1$, $\alpha \in (m - 1, c)$, $\lambda \in \Gamma$, $z_0 \in D_A$

$$\text{Lap}[D^{n-m+1}_t Z_0(t)z_0](\lambda) = \lambda^{\alpha}(W(\lambda)I - A)^{-1}z_0 - \int_{m-2}^{c} \mathcal{W}_{m-\lambda} z_0 d\lambda - \frac{1}{2\pi i} \int_{m-1}^{c} \mathcal{W}_{m-\lambda} z_0 d\lambda$$

$$- \frac{1}{2\pi i} \int_{m-1}^{c} \mathcal{W}_{m-\lambda} z_0 d\lambda = \frac{1}{2\pi i} \int_{m-2}^{c} (W_{m-2}(\lambda) - W(\lambda))(W(\lambda)I - A)^{-1}z_0 d\lambda + \frac{1}{2\pi i} \int_{m-2}^{c} \mathcal{W}_{m-\lambda} z_0 d\lambda$$

$$\| (W_{m-2}(\lambda) - W(\lambda)) (W(\lambda)I - A)^{-1}A z_0 \|_{L(Z)} \leq \frac{C_1}{|\lambda|^{m+c-\varepsilon}}.$$

Hence,

$$\lim_{t \to 0^+} \int_{m-2}^{c} \mathcal{W}_{m-\lambda} z_0 d\lambda = 0.$$

At $z_1 \in D_A$, $\alpha \in (m - 1, c)$, $\lambda \in \Gamma$

$$\text{Lap}[D^{n-m+1}_t Z_1](\lambda) = \lambda^{\alpha-1}(W(\lambda)I - A)^{-1}$$

$$\int_{m-2}^{c} \mathcal{W}_{m-\lambda} z_1 d\lambda = \frac{1}{2\pi i} \int_{m-2}^{c} \mathcal{W}_{m-\lambda} z_1 d\lambda = \frac{1}{2\pi i} \int_{m-2}^{c} \mathcal{W}_{m-\lambda} z_1 d\lambda + \frac{1}{2\pi i} \int_{m-2}^{c} \mathcal{W}_{m-\lambda} z_1 d\lambda$$

as $t \to 0^+$, since

$$\| W_{m-2}(\lambda) - W(\lambda) \|_{L(Z)} \leq \frac{C_1}{|\lambda|^{m+c-\varepsilon}}.$$

For $k = 2, 3, \ldots, m - 1$, $\alpha \in (m - 1, c)$, $\lambda \in \Gamma$

$$\text{Lap}[D^{n-m+1}_t Z_k](\lambda) = \lambda^{\alpha-k}(W(\lambda)I - A)^{-1}$$

$$\| \lambda^{\alpha-k} (W(\lambda)I - A)^{-1} A z_k \|_{L(Z)} \leq \frac{C_1}{|\lambda|^{m+c-\varepsilon}}.$$

hence, $D^{n-m+1}_t Z_k(0) = 0$.

Arguing as before, we obtain the equalities:

$$\int_{m-l-2}^{c} \mathcal{W}_{m-\lambda} z_k d\lambda = 0, \quad k \in \{0, 1, \ldots, m - 1\} \setminus \{l - 1\},$$

$$\int_{m-l-2}^{c} \mathcal{W}_{m-\lambda} z_l d\lambda = z_{l-1}.$$

Hence, for $c > l \in \{2, 3, \ldots, m - 1\}$, $k \in \{0, 1, \ldots, l - 1\}$, $\alpha \in (m - l, c)$, $\lambda \in \Gamma$, $z_k \in D_A$

$$\text{Lap}[D^{n-m+l}_t Z_k(t)z_k](\lambda) = \lambda^{\alpha+l-k-1}(W(\lambda)I - A)^{-1}z_k - \lambda^{l-1}D^{n-m+l}_t Z_k(0)z_k -$$
\[-\lambda^{l-2} D_t^{a-m+1} Z_k(0) z_k - \ldots - D_t^{a-m+l-1} Z_k(0) z_k,\]

\[
\int_{m-1}^{c} \omega(\alpha) D_t^{a-m+1} Z_k(t) z_k d\alpha = \frac{1}{2\pi i} \int_{m-1}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d(\lambda) = \frac{1}{2\pi i} \int_{m-1}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda - \frac{1}{2\pi i} \int_{m-2}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda =
\]

\[
- \frac{1}{2\pi i} \int_{m-1}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda - \frac{1}{2\pi i} \int_{m-2}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda =
\]

\[
= \frac{1}{2\pi i} \int_{m-1}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda + \frac{1}{2\pi i} \int_{m-2}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda - \frac{1}{2\pi i} \int_{m-1}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda =
\]

\[
\lambda^{\alpha-l-1} (W_m - \lambda) - W(\lambda)(W(\lambda) I - A)^{-1} z_k d\lambda =
\]

\[
\frac{1}{2\pi i} \int_{m-1}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda \to 0
\]

as \( t \to 0^+ \), since

\[
\left\| \lambda^{\alpha-l-1} (W_m - \lambda) - W(\lambda)(W(\lambda) I - A)^{-1} z_k d\lambda \right\|_2 \leq \frac{C_1}{|\lambda|^{2+\varepsilon - m-\varepsilon}},
\]

\[
\left\| \lambda^{\alpha-l-1} (W(\lambda) I - A)^{-1} z_k d\lambda \right\|_2 \leq \frac{C_2}{|\lambda|^{2+\varepsilon - m-\varepsilon}}.
\]

If in these arguments \( k = l, \alpha \in (m - k, c), \lambda \in \Gamma, z_k \in D_A \), then,

\[
\text{Lap}[D_t^{a-m+k} Z_k(t)](\lambda) = \lambda^{a-1} (W(\lambda) I - A)^{-1} z_k - \lambda^{k-1} D_t^{a-m} Z_k(0) z_k -
\]

\[
- \lambda^{k-2} D_t^{a-m+1} Z_k(0) z_k - \ldots - D_t^{a-m+k-1} Z_k(0) z_k,
\]

\[
\int_{m-k-1}^{c} \omega(\alpha) D_t^{a-m+k} Z_k(t) z_k d\alpha = \frac{1}{2\pi i} \int_{m-k-1}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d(\lambda) = \frac{1}{2\pi i} \int_{m-k-1}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda - \frac{1}{2\pi i} \int_{m-k-2}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda =
\]

\[
- \frac{1}{2\pi i} \int_{m-k-1}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda - \frac{1}{2\pi i} \int_{m-k-2}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda =
\]

\[
= \frac{1}{2\pi i} \int_{m-k-1}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda + \frac{1}{2\pi i} \int_{m-k-2}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda - \frac{1}{2\pi i} \int_{m-k-1}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda =
\]

\[
\frac{1}{2\pi i} \int_{m-k-1}^{c} \omega(\alpha) \lambda^{\alpha-l-1} d\lambda \to 0
\]

as \( t \to 0^+ \). For \( k \in \{l+1, l+2, \ldots, m-1\} \), \( \alpha \in (m - l, c), \lambda \in \Gamma, z_k \in D_A \)

\[
\text{Lap}[D_t^{a-m+l} Z_k(t)](\lambda) = \lambda^{a+l-1} (W(\lambda) I - A)^{-1} z_k - \lambda^{l-1} D_t^{a-m} Z_k(0) z_k -
\]

\[
- \lambda^{l-2} D_t^{a-m+1} Z_k(0) z_k - \ldots - D_t^{a-m+l-1} Z_k(0) z_k,
\]

\[
\int_{m-l-1}^{c} \omega(\alpha) D_t^{a-m+l} Z_k(0) z_k d\alpha = \frac{1}{2\pi i} \int_{m-l-1}^{c} \omega(\alpha) \lambda^{a+l-1} d(\lambda) = \frac{1}{2\pi i} \int_{m-l-1}^{c} \omega(\alpha) \lambda^{a+l-1} d\lambda - \frac{1}{2\pi i} \int_{m-l-2}^{c} \omega(\alpha) \lambda^{a+l-1} d\lambda =
\]

\[
- \frac{1}{2\pi i} \int_{m-l-1}^{c} \omega(\alpha) \lambda^{a+l-1} d\lambda - \frac{1}{2\pi i} \int_{m-l-2}^{c} \omega(\alpha) \lambda^{a+l-1} d\lambda =
\]

\[
= \frac{1}{2\pi i} \int_{m-l-1}^{c} \omega(\alpha) \lambda^{a+l-1} d\lambda + \frac{1}{2\pi i} \int_{m-l-2}^{c} \omega(\alpha) \lambda^{a+l-1} d\lambda - \frac{1}{2\pi i} \int_{m-l-1}^{c} \omega(\alpha) \lambda^{a+l-1} d\lambda =
\]

\[
\frac{1}{2\pi i} \int_{m-l-1}^{c} \omega(\alpha) \lambda^{a+l-1} d\lambda \to 0
\]
Therefore, there exists a unique solution of problem (4), (6) in the space \( \mathbb{C} \). Due to the initial conditions, we get the equality \( z(t) \) at every \( k \). Hence, \( z(t) = \sum_{k=0}^{m-1} Z_k(t)z_k \) satisfies initial conditions (6). Since the operator \( A \) is closed and commutes with the operators \((W(\lambda)I - A)^{-1}\) on \( D_A \), at \( z_k \in D_A, k = 0, 1, \ldots, m - 1 \) the inclusions \( AZ_k(\cdot)z_k = Z_k(\cdot)Az_k \) in \( C(\mathbb{R}_+; \mathbb{Z}) \) are fulfilled also, i.e. \( z(\cdot) \) is a solution of Equation (4). We apply the inverse Laplace transform to both sides of the obtained equality and get equality \( (4) \) at all continuity points of the function \( k \). Using formula (3) for the Laplace transform, we obtain for \( \text{Re} \lambda > a_0 \)

\[
\text{Lap} \left[ \int_{c}^{b} \omega(a)D_t^k S_k(t)z_k dt \right] (\lambda) = \lambda^{m-1-k}W(\lambda)(W(\lambda)I - A)^{-1}z_k - \lambda^{m-k-1}z_k = \lambda^{m-1-k}(W(\lambda)I - A)^{-1}Az_k = \text{Lap}[AZ_k(t)z_k](\lambda).
\]

We apply the inverse Laplace transform to both sides of the obtained equality and get equality (4) at all continuity points of the function \( AZ_k(\cdot)z_k \), that is, for all \( t > 0 \). Hence, \( \{Z_0(t) \in L(\mathbb{Z}) : t > 0\} \) is an analytic 0-resolving family of operators of the type \((\theta - \pi/2, a_0, m - c - \epsilon)\) at every \( \epsilon \in (0, \epsilon_0) \) for Equation (4) and \( \{Z_k(t) \in L(\mathbb{Z}) : t > 0\} \) are analytic k-resolving families of operators for Equation (4), \( k = 1, 2, \ldots, m - 1 \).

Let \( \theta_0 \in (\pi/2, \pi), a_0 \geq 0, \beta \in [0, 1], \) there exists an analytic k-resolving k-families of operators \( \{S_k(t) \in L(\mathbb{Z}) : t > 0\}, k \in \{0, 1, \ldots, m - 1\} \), of the type \((\theta - \pi/2, a_0, m - c - \epsilon)\) at every \( \epsilon \in (0, \epsilon_0) \) at \( k = 0 \) and of the type \((\theta - \pi/2, a_0, 0)\) at \( k = 1, 2, \ldots, m - 1 \) for Equation (4). From Equation (4) due to condition (ii) of the k-resolving family definition we obtain at \( z_k \in D_A \) equalities

\[
\int_{0}^{b} \omega(a)D_t^k S_k(t)z_k dt = AS_k(t)z_k = S_k(t)Az_k,
\]

hence, due to the closedness of the operator \( A \) at \( \lambda > a_0 \) we have \( S_k(t) \equiv Z_k(t) \equiv f^k_0 Z_0(t) \) by virtue of the uniqueness of the inverse Laplace transform.

Therefore, the operator \( W(\lambda)I - A : D_A \rightarrow Z \) is bijective and \( \hat{S}_k(\lambda) = \lambda^{m-1-k}(W(\lambda)I - A)^{-1}, \lambda > a_0 \). For \( k = 0 \) from Theorem 1 it follows that \( A \in \mathcal{B}E^o_{\theta_0,a_0} \); for all \( k \in \{0, 1, \ldots, m - 1\} \) we obtain \( S_k(t) \equiv Z_k(t) \equiv f^k_0 Z_0(t) \) by virtue of the uniqueness of the inverse Laplace transform.

If there exist two solutions \( y_1, y_2 \) of problem (4), (6) from the class \( \hat{Z} \), then their difference \( y = y_1 - y_2 \in \hat{Z} \) is a solution of Equation (4) and satisfies the initial conditions (6) with \( z_k = 0, k = 0, 1, \ldots, m - 1 \). Performing the Laplace transform on both parts of equation (4) and due to the initial conditions, we get the equality \( W(\lambda)\hat{y}(\lambda) = A\hat{y}(\lambda) \). Since \( A \in \mathcal{B}E^o_{\theta_0,a_0} \), at \( \lambda \in \mathcal{S}_{\theta_0,a_0} \) we obtain the identity \( \hat{y}(\lambda) \equiv 0 \). It means that \( y \equiv 0 \). Therefore, there exists a unique solution of problem (4), (6) in the space \( \hat{Z} \).

**Corollary 1.** Let \(-\infty < b \leq 0 \leq m - 1 < c \leq m \in \mathbb{N}, \theta_0 \in (\pi/2, \pi), a_0 \geq 0, \omega \in L_1(b,c)\). If there exists an analytic 0-resolving family of operators of the type \((\theta - \pi/2, a_0, m + c - \epsilon)\) at every \( \epsilon \in (0, \epsilon_0) \) for Equation (4), there exist analytic k-resolving families of operators \( \{S_k(t) \in \mathbb{C}\} \).
\( L(\mathcal{Z}) : t > 0 \), \( k = 1, 2, \ldots, m - 1 \), for Equation (4). In this case, for every \( k = 0, 1, \ldots, m - 1 \) \( S_k(t) \equiv \int_0^t Z_0(t), t > 0 \).

**Remark 2.** If we consider problem (4), (6) on a segment \([0, T]\), then we can continue the function \( y = y_1 - y_2 \) on \([T, \infty)\) by a continuous bounded way. Reasoning in the same way, we get the uniqueness of a solution on a segment not only in the space \( \mathcal{Z} \).

**Remark 3.** In [18] it is shown that there exists a 0-resolving family of Equation (4) with the distributed Gerasimov–Caputo derivative, if and only if an operator \( A \in Cl(\mathcal{Z}) \) satisfy the next conditions:

1. there exist such \( \theta_0 \in (\pi/2, \pi], a_0 \geq 0 \), that for \( \lambda \in S_{\theta_0,a_0} \) we have \( W(\lambda) \in \rho(A) \);
2. for every \( \theta \in (\pi/2, \theta_0) \), \( a > a_0 \) there exists such \( K(\theta, a) > 0 \), that for all \( \lambda \in S_{\theta_0,a_0} \)

\[ \|(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{|\lambda|K(\theta, a)}{|W(\lambda)||\lambda - a|}. \]

The corresponding class of operators is denoted by \( \mathcal{A}_W(\theta_0,a_0) \). It is easy to show that, if \( \omega \in L_1(b,c) \) and is continuous from the left at the point \( c \), \( \lim_{\alpha \rightarrow t^-} \omega(\alpha) \neq 0 \), then due to Lemmas 2 and 3 \( \mathcal{A}_W^B(\theta_0,a_0) \subset \mathcal{A}_W(\theta_0,a_0) \subset \mathcal{A}_W^R(\theta_0,a_0) \).

**Theorem 3.** Let \( -\infty < b \leq 0 \leq m - 1 < c \leq m \in \mathbb{N}, \theta_0 \in (\pi/2, \pi], a_0 \geq 0, \omega \in L_1(b,c), W \) satisfies (7), \( A \in \mathcal{A}_W^R(\theta_0,a_0) \). There exists a limit,

\[ \lim_{t \rightarrow 0^+} \int_{m-1-k}^{c} \omega(\alpha) D^\alpha_t - m + k Z_k(t) d\alpha = I, \tag{8} \]

in the norm of \( \mathcal{L}(\mathcal{Z}) \) at some \( k \in \{0,1,\ldots, m - 1\} \), if and only if \( A \in \mathcal{L}(\mathcal{Z}) \).

**Proof.** Due to the proof of Theorem 2, if there exists limit in (8), then it equals the identical operator, since it is so on \( D_A \). Let the function

\[ \eta(t) = \left\| \int_{m-1-k}^{c} \omega(\alpha) D^\alpha_t - m + k Z_k(t) d\alpha - I \right\|_{\mathcal{L}(\mathcal{Z})} \]

is continuous on the segment \([0,1]\) and \( \eta(0) = 0 \). Therefore, the function \( \eta \) is bounded on \([0,1]\). Due to the proof of Theorem 2 and Lemma 3 for all \( t > 1 \)

\[ \eta(t) = \left\| \int_{m-1-k}^{c} \omega(\alpha) \frac{1}{2\pi i} \int_{\Gamma} \lambda^{\alpha-1}(W(\lambda)I - A)^{-1} e^{\lambda t} d\lambda d\alpha - I \right\|_{\mathcal{L}(\mathcal{Z})} \]

\[ = \left\| \frac{1}{2\pi i} \int_{\Gamma} \frac{W_{m-1-k}(\lambda)}{\lambda} (W(\lambda)I - A)^{-1} e^{\lambda t} d\lambda - I \right\|_{\mathcal{L}(\mathcal{Z})} \]

\[ \leq C_1 e^{\alpha t} \int_{0}^{\infty} e^{-\theta} \cos \theta d\theta + C_2 e^{\alpha t} \int_{\theta}^{\infty} e^{\delta \cos \varphi} d\varphi + 1 \leq C_3 e^{\alpha t - \varepsilon} + C_4 e^{t(\alpha + \delta)t} + 1 \leq C_5 e^{(\alpha + \delta)t}. \]

Take \( \Re \lambda > a + \delta > a_0 \). Then we obtain, as in the proof of Theorem 2,

\[ \int_{0}^{\infty} e^{-\lambda t} \left( \int_{m-1-k}^{c} \omega(\alpha) D^\alpha_t - m + k Z_k(t) d\alpha - I \right) dt = \]
\[= \int_{m-1-k}^{c} \omega(\alpha) \lambda^{\alpha-1} (W(\lambda) I - A)^{-1} d\alpha - \frac{I}{\lambda} = \frac{W_{m-1-k}(\lambda)}{\lambda} (W(\lambda) I - A)^{-1} - \frac{I}{\lambda} = \]
\[= \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda} (W(\lambda) I - A)^{-1} + \frac{W(\lambda)}{\lambda} (W(\lambda) I - A)^{-1} - \frac{I}{\lambda} \]

(9)

For any \( \chi \in (0, 1) \) take \( \rho > 0 \) such that \( \eta(t) \leq \chi \) for all \( t \in [0, \rho] \). Then due to Lemma 3 and equality (9)
\[
\left\| \frac{W(\lambda)}{\lambda} (W(\lambda) I - A)^{-1} - \frac{I}{\lambda} \right\|_{\mathcal{L}(Z)} \leq \int_{0}^{\rho} e^{-\lambda t} \eta(t) dt + \int_{\rho}^{\infty} e^{-\lambda t} \eta(t) dt + \]
\[
+ \left\| \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda} (W(\lambda) I - A)^{-1} \right\|_{\mathcal{L}(Z)} \leq \frac{\chi}{\lambda} + o \left( \frac{1}{\lambda} \right)
\]
as \( \text{Re}\lambda \to +\infty \). Consequently, for large enough \( \text{Re}\lambda \)
\[
\left\| \frac{W(\lambda)}{\lambda} (W(\lambda) I - A)^{-1} - \frac{I}{\lambda} \right\|_{\mathcal{L}(Z)} < 1;
\]
hence, the operator \( \lambda^{-1} W(\lambda) (W(\lambda) I - A)^{-1} \) is continuously invertible,
\[
[\lambda^{-1} W(\lambda) (W(\lambda) I - A)^{-1}]^{-1} = \lambda W(\lambda)^{-1} (W(\lambda) I - A) \in \mathcal{L}(Z).
\]
Thus, \( A \in \mathcal{L}(Z) \).

Now let \( A \in \mathcal{L}(Z) \), then for \( t > 0 \)
\[
\int_{m-1-k}^{c} \omega(\alpha) D_{t}^{k-m+k} Z_{k}(t) d\alpha = \frac{1}{2\pi i} \int_{\Gamma} \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda} (W(\lambda) I - A)^{-1} e^{\lambda t} d\lambda = \]
\[
= \frac{1}{2\pi i} \int_{\Gamma} \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda} (W(\lambda) I - A)^{-1} e^{\lambda t} d\lambda + I + \frac{1}{2\pi i} \int_{\Gamma} \sum_{n=1}^{\infty} \frac{A_{n}^{n} e^{\lambda t} \lambda}{\lambda W(\lambda) W(\lambda)} d\lambda,
\]
\[
\left\| \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda} (W(\lambda) I - A)^{-1} \right\|_{\mathcal{L}(Z)} \leq \frac{C_{1}}{\lambda^{2+c-m-\varepsilon+k^{2}},\varepsilon}
\]
\[
\left\| \sum_{n=1}^{\infty} \frac{A_{n}^{n} e^{\lambda t} \lambda}{\lambda W(\lambda) W(\lambda)} \right\|_{\mathcal{L}(Z)} \leq \sum_{n=1}^{\infty} C \frac{\| A \|_{\mathcal{L}(Z)}}{\lambda^{c-\varepsilon} t^{n+1}} \leq \frac{C \| A \|_{\mathcal{L}(Z)}}{\lambda^{c-\varepsilon} t^{n+1}} \frac{1}{\lambda^{c-\varepsilon}} \leq \frac{2C \| A \|_{\mathcal{L}(Z)}}{\lambda^{c-\varepsilon+1}} \]
for small enough \( \varepsilon > 0 \) and \( |\lambda| > (2 \| A \|_{\mathcal{L}(Z)})^{1/t} \). Take small \( t > 0 \) and \( R = 1/t > (2 \| A \|_{\mathcal{L}(Z)})^{1/t} \), then
\[
\left\| \int_{\Gamma} \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda} (W(\lambda) I - A)^{-1} e^{\lambda t} d\lambda \right\|_{\mathcal{L}(Z)} = \]
\[
= \left\| \left( \int_{1_{R}} + \int_{1_{R}}^{c} + \int_{1_{R}}^{c} + \int_{1_{R}}^{c} \right) \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda} (W(\lambda) I - A)^{-1} e^{\lambda t} d\lambda \right\|_{\mathcal{L}(Z)} \leq \]
\[
\leq \sum_{s=2,5,6} \left\| \int_{1_{s,R}} \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda} (W(\lambda) I - A)^{-1} e^{\lambda t} d\lambda \right\|_{\mathcal{L}(Z)} \leq \frac{C_{2}}{R^{1+c-m-\varepsilon}} = C_{2} t^{1+c-m-\varepsilon}.\]
Analogously, we obtain:

$$\left\| \sum_{n=1}^{\infty} A^n \frac{e^{\lambda t}}{\lambda W(\lambda)^n} \right\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{C_3}{Re^\varepsilon} = C_4 t^{\varepsilon - \varepsilon},$$

$$\left\| \int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} Z_k(t) d\alpha - I \right\|_{\mathcal{L}(\mathcal{Z})} \leq C_4 t^{\varepsilon - \varepsilon} \to 0$$
as $t \to 0^+$. □

**Remark 4.** Reasoning as in the proof of the previous theorem we can show that if at some $k, l \in \{0, \ldots, m - 1\}$ the family $\{D_t^{\alpha-m+k} \phi(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is continuous at $t = 0$ in the norm of $\mathcal{L}(\mathcal{Z})$, then $A \in \mathcal{L}(\mathcal{Z})$.

**Remark 5.** For $A \in \mathcal{L}(\mathcal{Z})$ the $k$-resolving operators of Equation (4) have the form:

$$Z_k(t) = \frac{1}{2\pi i} \int_{\gamma} \lambda^{m-k-1} \sum_{n=0}^{\infty} \frac{A^n e^{\lambda t}}{\lambda W(\lambda)^n + 1} d\lambda = \sum_{n=0}^{\infty} a_n(t) A^n, \quad a_n(t) = \frac{1}{2\pi i} \int_{\gamma} \frac{\lambda^{m-k-1} e^{\lambda t} d\lambda}{W(\lambda)^{n+1}}, \quad n \in \mathbb{N}_0,$$

where $\gamma = \{ Re^{i\varphi} : \varphi \in (-\pi, \pi) \} \cup \{ xe^{i\pi}, x \in (-\infty, -\infty) \} \cup \{ xe^{-i\pi}, x \in (-\infty, 0) \}$ at $R > 0$ large enough. For equation $D_t^{\alpha} z(t) = Az(t)$ we have $W(\lambda) = \lambda^\alpha$, and we obtain using the Hankel representation for the Euler gamma function, for every $n \in \mathbb{N}_0$,

$$a_n(t) = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} d\lambda / \lambda^{an+a-m+k+1} = \frac{\Gamma(\alpha n + \alpha - m + k + 1)}{\Gamma(\alpha n + \alpha - m + k + 1)}.$$

Thus,

$$Z_k(t) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha n + \alpha - m + k + 1)}{\Gamma(\alpha n + \alpha - m + k + 1)} A^n = \Gamma_{\alpha, \alpha-m+1}(t^\alpha A),$$

where $E_{\beta, \gamma}$ is the Mittag-Leffler function.

**5. Inhomogeneous Equation**

A solution of initial problem,

$$\int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(0) d\alpha = z_k, \quad k = 0, 1, \ldots, m - 1,$$

for the inhomogeneous equation

$$\int_{b}^{c} \omega(\alpha) D_t^{\alpha} z(t) d\alpha = Az(t) + g(t), \quad t \in (0, T),$$

where $-\infty < b \leq 0 \leq m - 1 < c \leq m \in \mathbb{N}$, $\omega : (b, c) \to \mathbb{C}$, $T > 0$, $g \in C([0, T]; \mathcal{Z})$, is a function $z \in C((0, T); \mathcal{A})$, such that there exist $\int_{a}^{b} \omega(\alpha) D_t^{\alpha} z(t) d\alpha \in C((0, T); \mathcal{Z})$,

$$\lim_{t \to 0^+} \int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(t) d\alpha$$

and equalities (10) and (11) are fulfilled.

**Lemma 4.** Let $-\infty < b \leq 0 \leq m - 1 < c \leq m \in \mathbb{N}$, $\omega \in L_1(b, c)$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $A \in A_{\alpha, \alpha}^{\theta_0, a_0}(\theta_0, a_0)$, $g \in C([0, T]; \mathcal{A})$. Then the function:

$$z_g(t) = \int_{0}^{t} Z_{m-1}(t-s) g(s) ds$$
is a unique solution of problem (10), (11) with $z_k = 0, k = 0, 1, \ldots, m - 1$.

**Proof.** Due to Theorem 2 and Proposition 1, $Z_{m-1}(t)$ has an analytic extension to $\Sigma_{\theta_0 - \pi/2}$ and for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists $C(\theta, a)$ such that $\|Z_{m-1}(t)\|_{L(Z)} \leq C(\theta, a)e^{aT}$. Define $g(t) = 0$ at $t \geq T$; then $z_g = Z_{m-1} * g$ is the convolution, $\tilde{z}_g(\lambda) = \tilde{Z}_{m-1}(\lambda)\tilde{g}(\lambda)$. In the proof of Theorem 2 it was shown that $\tilde{Z}_{m-1}(\lambda) = \lambda(W(\lambda)I - A)^{-1}$, $\text{Lap}[D_t^{\alpha + k}Z_{m-1}](\lambda) = \lambda^{\alpha + m+k}(W(\lambda)I - A)^{-1}$, at $\alpha \in (m - 1 - k, c), t \in (0, 1)$

$$
\|D_t^{\alpha + k}Z_{m-1}(t)\|_{L(Z)} \leq C_1 \int_t^0 \frac{ds}{|\lambda|^{m-k-\alpha}}, \quad k = 0, 1, \ldots, m - 2,
$$

hence, $D_t^{\alpha + k}Z_{m-1}(0) = 0$. Therefore, $\tilde{z}_g(\lambda) = (W(\lambda)I - A)^{-1}\tilde{g}(\lambda), \text{Lap}[D_t^{\alpha + k}z_g](\lambda) = \lambda^{\alpha + m}(W(\lambda)I - A)^{-1}\tilde{g}(\lambda)$ at $\alpha < c$, by the mean value theorem:

$$
\|D_t^{\alpha - a}Z_{m-1}(t)\|_{L(Z)} \leq \int_0^t \|D_t^{\alpha - a}Z_{m-1}(t-s)\|_{L(Z)}\|g(s)\|_2 ds = \|D_t^{\alpha - a}Z_{m-1}(t-\xi)\|_{L(Z)} \int_0^t \|g(s)\|_2 ds \rightarrow 0
$$
as $t \rightarrow 0+$, since $\xi \in (0, t), t-\xi \rightarrow 0$.

Further, we have: $\text{Lap}[D_t^{\alpha - a+1}z_g](\lambda) = \lambda^{\alpha - m+1}(W(\lambda)I - A)^{-1}\tilde{g}(\lambda)$ at $\alpha > m - 1$,

$$
\|D_t^{\alpha - a+1}z_g(t)\|_{L(Z)} \leq \|D_t^{\alpha - a+1}Z_{m-1}(t-\xi)\|_{L(Z)} \int_0^t \|g(s)\|_2 ds \rightarrow 0
$$
as $t \rightarrow 0+$.

Reasoning in the same way we obtain $D_t^{\alpha - m+k}z_g(0) = 0$ for $\alpha > m - k, k = 2, 3, \ldots, m - 2$.

Finally,

$$
\text{Lap}[D_t^{\alpha - a}z_g](\lambda) = \lambda^{\alpha - 1}(W(\lambda)I - A)^{-1}\tilde{g}(\lambda)
$$
at $\alpha > 1$,

$$
D_t^{\alpha - a}z_g(t) = \int_0^t D_t^{\alpha - a}Z_{m-1}(t-s)g(s)ds,
$$

$$
\int_0^\varepsilon \omega(\alpha)D_t^{\alpha - a}z_g(t)d\alpha = \frac{1}{2\pi i} \int_\Gamma \omega(\alpha) \lambda^{\alpha - 1}(W(\lambda)I - A)^{-1}\tilde{g}(\lambda)e^{\lambda \varepsilon}d\lambda d\alpha =
$$

$$
= \frac{1}{2\pi i} \int_\Gamma \frac{W_0(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}\tilde{g}(\lambda)e^{\lambda \varepsilon}d\lambda + \frac{1}{2\pi i} \int_\Gamma \frac{W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}\tilde{g}(\lambda)e^{\lambda \varepsilon}d\lambda \rightarrow 0
$$
as $t \rightarrow 0+$, since

$$
\|\tilde{g}(\lambda)\|_2 \leq \max_{\varepsilon \in [0, T]} \|g(t)\|_2 \int_0^T e^{\lambda \varepsilon} dt \leq \frac{C_1}{|\lambda|},
$$

$$
\left\|\frac{W_0(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}\right\|_{L(Z)} \leq \frac{C_2}{|\lambda|^{1+\varepsilon}}, \left\|\frac{W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}\tilde{g}(\lambda)\right\|_{L(Z)} \leq \frac{C_3}{|\lambda|^{2-\varepsilon}}.
$$

Thus, the function $z_g$ satisfies initial conditions (10).

We have:

$$
\text{Lap} \left[ \int_b^c \omega(\alpha)D_t^{\alpha}z_g da \right] (\mu) = W(\mu)(W(\mu)I - A)^{-1}\tilde{g}(\mu) = \tilde{g}(\mu) + A(W(\mu)I - A)^{-1}\tilde{g}(\mu).
$$
Applying the inverse Laplace transform on the both sides of this equality, we get
\[
\int_b^c \omega(a) D_l^2 z_g(t) d\alpha = g(t) + A(Z_{m-1} * g)(t) = g(t) + A z_g(t),
\]
since \( g \in C([0,T]; D_A) \) and due to the closedness of \( A \) the values \( A(Z_{m-1} * g)(t) = Z_{m-1} * A g(t), t \in (0,T) \), are defined.

The proof of the uniqueness of the problem solution can be found in Remark 2. \( \square \)

From Theorem 2 and Lemma 4, we get the following result.

**Theorem 4.** Let \( -\infty < b \leq 0 \leq m - 1 < c \leq m \in \mathbb{N}, \omega \in L_1(b,c), \theta_0 \in (\pi/2, \pi], a_0 \geq 0, A \in \mathcal{A}_c(\theta_0,a_0), g \in C([0,T]; D_A), z_k \in D_A, k = 0,1,\ldots,m-1. \) Then the function
\[
z(t) = \sum_{k=0}^{m-1} z_k(t) z_k + \int_0^t Z_{m-1}(t-s) g(s) ds
\]
is a unique solution of problem (10), (11).

6. Application to a Class of Initial-Boundary Value Problems

Consider polynomials \( P_n(\lambda) = \sum_{j=0}^n c_j \lambda^j \neq 0, Q_n(\lambda) = \sum_{j=0}^n d_j \lambda^j, \) where \( c_j, d_j \in \mathbb{R}, j = 0,1,\ldots,n, d_n \neq 0. \) Let \( \Omega \subset \mathbb{R}^d \) be a bounded region with a smooth boundary \( \partial \Omega, \) an operator pencil \( A_1, A_2,\ldots,A_p \) be regularly elliptic [24], where
\[
(\Lambda u)(s) = \sum_{|q| \leq 2p} a_q(s) \frac{\partial^{q_1} u(s)}{\partial s_1^{q_1}} \cdots \frac{\partial^{q_d} u(s)}{\partial s_d^{q_d}}, a_q \in C^\infty(\Omega),
\]
\[
(B_l u)(s) = \sum_{|q| \leq p_l} b_q(s) \frac{\partial^{q_1} u(s)}{\partial s_1^{q_1}} \cdots \frac{\partial^{q_d} u(s)}{\partial s_d^{q_d}}, b_q \in C^\infty(\partial \Omega), l = 1,2,\ldots,p,
\]
\( q = (q_1,q_2,\ldots,q_d) \in \mathbb{N}_0^d, |q| = q_1 + \cdots + q_d. \) Define an operator \( \Lambda_1 \in CL(L_2(\Omega)) \) with a domain \( D_{\Lambda_1} = H^{2p}_{(B_l)}(\Omega) [24] \) by the equality \( \Lambda_1 u = \Lambda u. \) Let the operator \( \Lambda_1 \) is self-adjoint, then the spectrum \( \sigma'(\Lambda_1) \) of the operator \( \Lambda_1 \) is real and discrete [24]. Suppose, moreover, \( \sigma'(\Lambda_1) \) is bounded from the right and does not contain the origin, \( \{ \lambda_k : k \in \mathbb{N} \} \) is an orthonormal in \( L_2(\Omega) \) system of eigenfunctions of the operator \( \Lambda_1 \) numbered according to the non-increase of the corresponding eigenvalues \( \{ \lambda_k : k \in \mathbb{N} \}, \) taking into account their multiplicities.

Consider the initial-boundary value problem:
\[
\int_1^c \omega(a) D_l^{n-2} u(s,t) d\alpha = u_0(s), \quad \int_0^c \omega(a) D_l^{n-1} u(s,t) d\alpha = u_1(s), \quad s \in \Omega,
\]
\[
B_l \Lambda^k u(s,t) = 0, \quad k = 0,1,\ldots,n-1, \quad l = 1,2,\ldots,p, \quad (s,t) \in \partial \Omega \times (0,T),
\]
\[
\int_0^c \omega(a) D_l^p P_n(\Lambda) u(s,t) d\alpha = Q_n(\Lambda) u(s,t) + f(s,t), \quad (s,t) \in \Omega \times (0,T),
\]
where \( 1 < c < 2, \omega : [0,c] \to \mathbb{R}. \) Denote \( n_0 := \max\{ j \in \{0,1,\ldots,n \} : c_j \neq 0 \}, \)
\[
\mathcal{X} = \{ v \in H^{2n_0}(\Omega) : B_l \Lambda^k v(s) = 0, \quad k = 0,1,\ldots,n_0-1, \quad l = 1,2,\ldots,p, \quad x \in \partial \Omega \},
\]
\[
\mathcal{Y} = L_2(\Omega), \quad L = P_n(\Lambda), \quad M = Q_n(\Lambda),
\]
\[ D_M = \{ v \in H^{2n}(\Omega) : B_l \Lambda^k v(s) = 0, k = 0, 1, \ldots, n - 1, l = 1, 2, \ldots, p, x \in \partial\Omega \}. \quad (17) \]

Then \( L \in \mathcal{L}(\mathcal{X}'; \mathcal{Y}) \), \( M \in \mathcal{C}(\mathcal{X}'; \mathcal{Y}) \) (if \( n_0 = n \), that is, \( c_M \neq 0 \), then \( M \in \mathcal{L}(\mathcal{Y}'; \mathcal{X}) \)). Let \( P_n(\lambda) \neq 0 \) for all \( k \in \mathbb{N} \), then there exists an inverse operator \( L^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{X}) \) and problem (12)–(14) can be represented as problem (10), (11), where \( \mathcal{Z} = \mathcal{X}, A = L^{-1}M \in \mathcal{C}(\mathcal{Z}), D_A = D_M, z_0 = u_0(\cdot), z_1 = u_1(\cdot), g(t) = f(\cdot, t) \).

**Remark 6.** For \( c \in (0, 2) \), \( \omega \in \mathcal{C}([0, c]; \mathbb{R}) \), \( \omega(\alpha) \geq 0 \) for \( \alpha \in [0, c] \), \( \omega(c) > 0 \), the spectrum \( \sigma(\Lambda_1) \) do not contain the origin and zeros of the polynomial \( P_n(\lambda) \), and designations (15)–(17) be valid. Then \( L^{-1}M \in \mathcal{A}_{\mathcal{X}}^R(\theta_0, a_0) \).

**Proof.** Denote \( \mu_k = Q_n(\lambda_k)/P_n(\lambda_k) \) for \( k \in \mathbb{N} \). Since \( (-1)^{n-n_0}a_n/c_n < 0 \), then \( \lim_{k \to \infty} \mu_k = -\infty \) and there exists \( \max \mu_k \). Due to Lemma 1 the function \( W(\lambda) := \int_0^c \omega(\alpha)\lambda^k d\alpha \) is analytic on \( S_{\pi, 0} \). At \( c \in (1, 2) \) take some \( \theta_0 \in (\pi/2, \pi/c) \), then for some large enough \( a_0 > 0 \) and every \( \lambda \in S_{\theta_0, a_0} \) we have \( |W(\lambda)| \geq C(\epsilon)|\lambda|^\epsilon \geq \max_{k \in \mathbb{N}} \mu_k \) and \( W(\lambda) \in S_{c_\theta, a_0, \epsilon} \), since \( \omega(\alpha) \geq 0 \) on \([0, c]\).

For any \( \theta \in (\pi/2, \theta_0) \), \( a > a_0, \lambda \in S_{\theta, a}, v \in \mathcal{X} \)
\[
\| (W(\lambda)I - A)^{-1}v \|_{\mathcal{X}} \leq \frac{\| (W(\lambda) - 1)Q_n(\lambda)^{-1} \|_{\mathcal{X}}}{\| W(\lambda) \|} \leq \frac{C(\epsilon)\| v \|_{\mathcal{X}}^2}{|\lambda|^{2(\epsilon - \eta)}} \]

for every small enough \( \epsilon > 0 \) by Lemma 2. Therefore, \( L^{-1}M \in \mathcal{A}_{\mathcal{X}}^R(\theta_0, a_0) \).

Finally, when \( c \in (0, 1] \), a similar arguments may be used to get the written conclusion. \( \Box \)

**Lemma 5** and **Theorem 4** implies the next unique solvability theorem.

**Theorem 5.** Let \( c \in (1, 2) \), \( \omega \in \mathcal{C}([0, c]; \mathbb{R}) \), \( \omega(\alpha) \geq 0 \) for \( \alpha \in [0, c] \), \( \omega(c) > 0 \), the spectrum \( \sigma(\Lambda_1) \) do not contain the origin and zeros of the polynomial \( P_n(\lambda) \), \( u_0, u_1 \in D_M, \ f \in \mathcal{C}([0, T]; D_M) \). Then there exists a unique solution of problem (12)–(14).

**Remark 6.** For \( c \in (0, 1] \) instead of (12) the initial condition has the form
\[ \int_0^c \omega(\alpha)D_t^{\alpha - 1}u(s, t) d\alpha = u_0(s), \ s \in \Omega. \]

**Example 1.** Take \( P_1(\lambda) \equiv 1, Q_1(\lambda) = \lambda, A = \Delta u, p = 1, B_1 = I \). Then, (12)–(14) is the initial-boundary value problem for the modified equation of the ultraslow diffusion [14]
\[ \int_0^c \omega(\alpha)D_t^{\alpha}u(s, t) d\alpha = \Delta u(s, t), \ (s, t) \in \Omega \times (0, T), \]
\[ u(s, t) = 0, \ (s, t) \in \partial\Omega \times (0, T), \]
\[ \int_0^c \omega(\alpha)D_t^{\alpha - 2}u(s, t) d\alpha = u_0(s), \ \int_0^c \omega(\alpha)D_t^{\alpha - 1}u(s, t) d\alpha = u_1(s), \ s \in \Omega. \]

**7. Conclusions**

Linear differential equations in a Banach space with a distributed Riemann–Liouville derivative and with a closed operator in the right-hand side are studied. It is shown that a
natural initial value problem for this equation is a problem with given values of special form distributed derivatives of a solution at initial time. A theorem on the generation of analytics in a sector resolving families of operators for such equations is proved. It gives necessary and sufficient conditions on the closed operator in the equation for the existence of the resolving family. This result allows us to study the unique solvability of the mentioned initial problem to the corresponding inhomogeneous equation. The abstract results of the work are applied to the research of the unique solvability for initial boundary value problems for a class of partial differential equations with a distributed Riemann–Liouville derivative in time.

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**References**
1. Nakhushev, A.M. On continual differential equations and their difference analogues. *Sov. Math. Dokl.* 1988, 37, 729–732.
2. Nakhushev, A.M. Positiveness of the operators of continual and discrete differentiation and integration, which are quite important in the fractional calculus and in the theory of mixed-type equations. *Differ. Equ.* 1998, 34, 103–112.
3. Sokolov, I.M.; Chechkin, A.V.; Klafter, J. Distributed-order fractional kinetics. *Acta Phys. Pol. B* 2004, 35, 1323–1341.
4. Lorenzo, C.F.; Hartley, T.T. Variable order and distributed order fractional operators. *Nonlinear Dyn.* 2002, 29, 57–98.
5. Caputo, M. Mean fractional order derivatives differential equations and filters. *Ann. Univ. Ferrara* 1995, XLII, 73–84.
6. Bagley, R.L.; Torvik, P.J. On the existence of the order domain and the solution of distributed order equations. Part 1. *Int. J. Appl. Math.* 2000, 2, 865–882.
7. Jiao, Z.; Chen, Y.; Podlubny, I. *Distributed-Order Dynamic System. Stability, Simulations, Applications and Perspectives*; Springer: London, UK, 2012.
8. Diethelm, K.; Ford, N.J. Numerical solution methods for distributed order time fractional diffusion equation. *Fract. Calc. Appl. Anal.* 2001, 4, 531–542.
9. Diethelm, K.; Ford, N.; Freed, A.D.; Luchko, Y. Algorithms for the fractional calculus: A selection of numerical methods. *Comput. Methods Appl. Mech. Eng.* 2003, 194, 743–773.
10. Pskhu, A.V. On the theory of the continual and integro-differentiation operator. *Differ. Equ.* 2004, 40, 128–136.
11. Pskhu, A.V. *Partial Differential Equations of Fractional Order*; Nauka Publ.: Moscow, Russia, 2005. (In Russian)
12. Umarov, S.; Gorenflo, R. Cauchy and nonlocal multi-point problems for distributed order pseudo-differential equations. *Z. Für Anal. Und Ihre Anwendungen* 2005, 24, 449–466.
13. Atanacković, T.M.; Oparnica, L.; Pilipović, S. On a nonlinear distributed order fractional differential equation. *J. Math. Anal. Appl.* 2007, 328, 590–608.
14. Kochubei, A.N. Distributed order calculus and equations of ultraslow diffusion. *J. Math. Appl.* 2008, 340, 252–280.
15. Fedorov, V.E.; Streletskaya, E.M. Initial-value problems for linear distributed-order differential equations in Banach spaces. *Electron. J. Differ. Equ.* 2018, 1–17.
16. Fedorov, V.E.; Abdrahmanova, A.A. A class of initial value problems for distributed order equations with a bounded operator. In *Stability, Control and Differential Games*; Tarasiev, A., Maksimov, V., Filippova, T., Eds.; Springer Nature: Cham, Switzerland, 2020; pp. 251–262.
17. Fedorov, V.E. On generation of an analytic in a sector resolving operators family for a distributed order equation. Zap. POMI 2020, 489, 113–129.
18. Fedorov, V.E. Generators of analytic resolving families for distributed order equations and perturbations. Mathematics 2020, 8, 1306.
19. Fedorov, V.E.; Filin, N.V. Linear equations with discretely distributed fractional derivative in banach spaces. Tr. Instituta Mat. I Mekhaniki UrO RAN 2021, 27, 264–280.
20. Fedorov, V.E.; Filin, N.V. On strongly continuous resolving families of operators for fractional distributed order equations. Fractal Fract. 2021, 5, 20.
21. Bajlekova, E.G. Fractional Evolution Equations in Banach Spaces. Ph.D. Thesis, University Press Facilities, Eindhoven University of Technology, Eindhoven, The Netherlands, 2001.
22. Prüss, J. Evolutionary Integral Equations and Applications; Springer: Basel, Switzerland, 1993.
23. Arendt, W.; Batty, C.J.K.; Hieber, M.; Neubrander F. Vector-valued Laplace Transforms and Cauchy Problems; Springer: Basel, Switzerland, 2011.
24. Triebel H. Interpolation Theory, Functional Spaces, Differential Operators; North Holland Publ.: Amsterdam, The Netherlands, 1978.