HAMiltonian loops on the symplectic blow up along a submanifold

Andrés Pedroza

Abstract. We prove that the fundamental group of the group of Hamiltonian diffeomorphisms of the symplectic manifold that is obtained by blowing up a submanifold contains an element of infinite order. We prove this using Weinstein’s morphism and by constructing explicitly such loop of Hamiltonian diffeomorphisms.

1. Introduction

In [12], it was shown that one-point blow up \((\tilde{M}, \tilde{\omega}_\rho)\) of a closed 4-dimensional symplectic manifold \((M, \omega)\) has a loop \(\{\tilde{\psi}_t\}\) of Hamiltonian diffeomorphisms such that the class \([\tilde{\psi}_1]\) has infinite order in \(\pi_1(\text{Ham}(\tilde{M}, \tilde{\omega}_\rho))\). Actually the argument presented in [12] works for any dimension as long as it is greater than two. The restriction to the 4-dimensional case had a specific purpose; it was related to the study of the abelian group \(\pi_1(\text{Ham}(\tilde{M}, \tilde{\omega}_\rho))\). Namely, for any positive integer \(k\) that there exists closed symplectic 4-manifold such that the rank of \(\pi_1(\text{Ham})\) is at least \(k\). This phenomena about the rank of \(\pi_1(\text{Ham}(\tilde{M}, \tilde{\omega}_\rho))\) being positive was first discovered by D. McDuff in [9]; the difference between the results of [12] and those of [9] is that the element of infinite order in \(\pi_1(\text{Ham})\) is written explicitly in [12]. The goal of this article consists on extending this result. Instead of blowing up a zero-dimensional submanifold, we blow up a symplectic submanifold \(N \subset (M, \omega)\) of codimension \(2k > 2\). In this way we also obtain a loop of Hamiltonian diffeomorphisms on \((M, \omega)\), that induces an element of infinite order in \(\pi_1(\text{Ham}(\tilde{M}, \tilde{\omega}_{N, \rho}))\) where \((\tilde{M}, \tilde{\omega}_{N, \rho})\) stands for the symplectic blow up along the submanifold \(N\).

Theorem 1.1. Let \((M, \omega)\) be a rational closed symplectic manifold and \(N \subset (M, \omega)\) a closed symplectic submanifold of codimension \(2k > 2\). Let \((\tilde{M}, \tilde{\omega}_{N, \rho})\) be the symplectic blow up of \((M, \omega)\) along \(N\). Then for some small values of \(\rho\) the rank of \(\pi_1(\text{Ham}(\tilde{M}, \tilde{\omega}_{N, \rho}))\) is positive.

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In \((\widetilde{M},\tilde{\omega}_{N,\rho})\), the symplectic blow up of \((M,\omega)\) along \(N\), the parameter \(\rho\) in the description of \((\widetilde{M},\tilde{\omega}_{N,\rho})\) is related to the \(\omega^n\)-volume of a tubular neighborhood of \(N \subset (M,\omega)\). Therefore, in the case when \(N\) is a single point, it corresponds to the weight of the one-point blow up.

Although, Theorem 1.1 is a statement about the topology of the group \(\text{Ham}(\widetilde{M},\tilde{\omega}_{N,\rho})\), that is as much as we know about its topology. Nonetheless, the study of the homotopy type of \(\text{Ham}(M,\omega)\) has succeeded in some cases, all of which are 4-dimensional. For instance the homotopy type of \((\mathbb{C}P^2,\omega_{FS})\) and \((\mathbb{C}P^1 \times \mathbb{C}P^1,\omega_{FS} \oplus \omega_{FS})\) was computed by M. Gromov \([5]\); in \([1]\), M. Abreu and D. McDuff settled the problem for the one-point blow up of \((\mathbb{C}P^2,\omega_{FS})\). For other examples of 4-dimensional symplectic manifolds see the works of S. Anjos and S. Eden \([2]\), S. Anjos and M. Pinsonnault \([3]\), J. Evans \([4]\) and F. Lalonde and M. Pinsonnault \([6]\). As for computations of the fundamental group of \(\text{Ham}(M,\omega)\), there are the works of J. Li and T.-J. Li \([7]\) and J. Li, T.-J. Li and W. Wu \([8]\).

The techniques that are used to prove Theorem 1.1 are soft techniques. Furthermore, we give an explicit description of the loop \(\{\tilde{\psi}_{N,\rho}\}\) of Hamiltonian diffeomorphisms on \((\widetilde{M},\tilde{\omega}_{N,\rho})\) that induces the element of infinite order in its fundamental group. In fact, the loop \(\{\tilde{\psi}_{N,\rho}\}\) is induced from a Hamiltonian loop \(\{\psi_t\}\) on \((M,\omega)\) that is supported in a tubular neighborhood of \(N\). Finally, in order determine that an element on the fundamental group of \(\text{Ham}(M,\omega)\) has infinite order, we relay on Weinstein’s morphism,

\[ A : \pi_1(\text{Ham}(M,\omega)) \to \mathbb{R}/\mathcal{P}(M,\omega). \]

In this direction we have the following relation between Hamiltonian diffeomorphism on \((M,\omega)\) and on \((\widetilde{M},\tilde{\omega}_{N,\rho})\) for a particular class of Hamiltonian loops. In the formula of the above result, \(\Phi(\mathcal{U}_\rho(\nu_0))\) stands for a particular tubular neighborhood of the symplectic submanifold \(N \subset (M,\omega)\).

**Theorem 1.2.** Let \((M,\omega)\) be a closed symplectic manifold and \(N \subset (M,\omega)\) a symplectic submanifold of codimension \(2k > 2\). If \(\{\psi_t\}\) is a loop of Hamiltonian diffeomorphisms on \((M,\omega)\) that is \(N\)-liftable to a loop \(\{\tilde{\psi}_{N,\rho}\}\) on \((\widetilde{M}_N,\tilde{\omega}_{N,\rho})\), then

\[ A(\tilde{\psi}_{N,\rho}) = A([\psi_t]) + \frac{1}{\text{Vol}(\tilde{M}_N,\tilde{\omega}_{N,\rho})} \int_0^1 \int_{\Phi(\mathcal{U}_\rho(\nu_0))} H_t \omega^n dt \]

in \(\mathbb{R}/\mathcal{P}(\tilde{M}_N,\tilde{\omega}_{N,\rho})\), where \(H_t\) is the normalized Hamiltonian function of the loop \(\{\psi_t\}\).

Loosely speaking, the reason why the results of \([12]\) extend to the case of a submanifold of positive dimension is because many of the arguments that are involved in the definition of symplectic blow up are in a sense \(U(k)\)-equivariant.
Furthermore, the diffeomorphisms \( \{ \psi_t \} \) and \( \{ \tilde{\psi}_{N,t} \} \) that appear in the above result are induced by matrices that lie in the center of \( U(k) \). This condition turns to be crucial, since in many instances the commutativity of the matrices is fundamental.

The paper is organized as follows. In Section 2 we review the definition of the symplectic one-point blow up as well as some facts of [11] and [12]. The definition of the symplectic blow up along a submanifold is reviewed in Section 3. In Section 4 we give conditions that guarantee that a Hamiltonian diffeomorphism can be lifted to the symplectic blow up. Finally, Sections 5 and 6 deal with normalization of Hamiltonian functions and the proofs of the main results respectively.

2. The case of a one-point blow up

2.1. Symplectic structure on the one-point blow up. In this section we review how the symplectic structure on the blow up of the origin in \((\mathbb{C}^k, \omega_0)\) is defined. Following [11, Sec. 2], we also review the conditions that a Hamiltonian diffeomorphism on \((\mathbb{C}^k, \omega_0)\) needs to satisfy in order to induced a Hamiltonian diffeomorphism on the one-point blow up. Later, we will impose similar conditions on a Hamiltonian diffeomorphism in order to induced a Hamiltonian on the symplectic manifold that is obtain by blowing up a compact symplectic submanifold.

As a manifold, the one-point blow up of \( \mathbb{C}^k \) at the origin is the total space of the tautological line bundle,

\[
\tilde{\mathbb{C}}^k := \{(z, \ell) \in \mathbb{C}^k \times \mathbb{C}P^{k-1} | z \in \ell\}.
\]

The projection maps are denoted by \( \text{pr}: \tilde{\mathbb{C}}^k \to \mathbb{C}P^{k-1} \) and \( \pi: \tilde{\mathbb{C}}^k \to \mathbb{C}^k \). Here, \( \pi \) and \( E := \pi^{-1}(0) \) are called the blow up map is the exceptional divisor respectively. Let \( \omega_{FS} \) be the Fubini-Study symplectic form on \( \mathbb{C}P^{k-1} \) normalized so that \( \langle \omega_{FS}, \mathbb{C}P^1 \rangle = \pi \). For each \( \rho > 0 \) define on \( \tilde{\mathbb{C}}^k \) the symplectic form

\[
\omega_{\rho} := \pi^* \omega_0 + \rho^2 \text{pr}^* \omega_{FS}.
\]

Notice that the \( \omega_{\rho} \)-area of a line in the exceptional divisor is \( \pi \rho^2 \). Denote by \( B_r \subset \mathbb{C}^k \) the open ball of radius \( r \) centered at the origin and its preimage under \( \pi \) by

\[
L_r := \pi^{-1}(B_r).
\]

According to [10, Prop. 7.1.13] for each \( \varepsilon \in (0, 1) \) there exists a symplectic form \( \tilde{\omega}_{\rho,\varepsilon} \) on \( \tilde{\mathbb{C}}^k \) such that

- \( \tilde{\omega}_{\rho,\varepsilon} = \omega_{\rho} \) on \( L_\varepsilon \) and
- \( \tilde{\omega}_{\rho,\varepsilon} = \pi^*(\omega_0) \) on \( \tilde{\mathbb{C}}^k \setminus \overline{L}_{\rho+\varepsilon} \).
Later in this note it will be important to know specific details on how the symplectic form \( \tilde{\omega}_{\rho, \epsilon} \) is defined. To that end, one considers a diffeomorphism \( F_{\rho, \epsilon}: \mathbb{C}^k \setminus \{0\} \to \mathbb{C}^k \setminus B_\rho \) that consists of stretching the punctured space, defined as

\[
F_{\rho, \epsilon}(z) := f_{\rho, \epsilon}(|z|) \frac{z}{|z|}
\]

where \( f_{\rho, \epsilon}: \mathbb{R} \to \mathbb{R} \) is a smooth function such that \( f_{\rho, \epsilon}(x) = \sqrt{\lambda^2 + x^2} \) on \((0, \epsilon)\) and is equal to the identity on \([\rho + \epsilon, \infty)\). Then the above symplectic form \( \tilde{\omega}_{\rho, \epsilon} \) is defined as \( \tilde{\omega}_{\rho, \epsilon} := (\pi^* \circ F_{\rho, \epsilon}^*)(\omega_0) \) on \( \tilde{\mathbb{C}}^k \setminus E \).

In this case \((\tilde{\mathbb{C}}^k, \tilde{\omega}_{\rho, \epsilon})\) is defined as the symplectic blow up of the origin in \((\mathbb{C}^k, \omega_0)\) of weight \( \rho \). It is important to note that the blow up map \( \pi: (\tilde{\mathbb{C}}^k, \tilde{\omega}_{\rho, \epsilon}) \to (\mathbb{C}^k, \omega_0) \) is a symplectic diffeomorphism on the complement of \( T_{\rho + \epsilon} \subset \tilde{\mathbb{C}}^k \).

The complete details of symplectic one-point blow up, including the exact definitions of \( \tilde{\omega}_{\rho, \epsilon} \) and \( f_{\rho, \epsilon} \) appear in [10, Ch. 7].

### 2.2. Induced Hamiltonian on the one-point blow up.

Next we review some results of [11] that deal with the problem of lifting a Hamiltonian diffeomorphism on \((\mathbb{C}^k, \omega_0)\) to a Hamiltonian on \((\tilde{\mathbb{C}}^k, \tilde{\omega}_{\rho, \epsilon})\).

Let \( \phi \) and \( \tilde{\phi} \) be Hamiltonian diffeomorphisms on \((\mathbb{C}^k, \omega_0)\) and \((\tilde{\mathbb{C}}^k, \tilde{\omega}_{\rho, \epsilon})\) respectively. We say that the diffeomorphism \( \tilde{\phi} \) lifts the diffeomorphism \( \phi \) if the following diagram commutes

\[
\begin{array}{ccc}
(\tilde{\mathbb{C}}^k, \tilde{\omega}_{\rho, \epsilon}) & \xrightarrow{\tilde{\phi}} & (\tilde{\mathbb{C}}^k, \tilde{\omega}_{\rho, \epsilon}) \\
\pi & & \pi \\
(\mathbb{C}^k, \omega_0) & \xrightarrow{\phi} & (\mathbb{C}^k, \omega_0).
\end{array}
\]

This means that \( \pi \circ \tilde{\phi} = \phi \circ \pi \).

Consider \( \{\psi_t\}_{0 \leq t \leq 1} \) a path of Hamiltonian diffeomorphisms on \((\mathbb{C}^k, \omega_0)\) with Hamiltonian function \( H_t \) such that \( \psi_0 = 1_{\mathbb{C}^k} \). The goal is to define a path of Hamiltonian diffeomorphisms \( \{\tilde{\psi}_t\} \) on \((\tilde{\mathbb{C}}^k, \tilde{\omega}_{\rho, \epsilon})\) that lifts the path \( \{\psi_t\} \); that is \( \pi \circ \tilde{\psi}_t = \psi_t \circ \pi \) for each \( t \in [0, 1] \). At this point it is important to note that the Hamiltonian path induced by the function \( H_t \circ \pi: (\tilde{\mathbb{C}}^k, \tilde{\omega}_{\rho, \epsilon}) \to \mathbb{R} \) fails to lift the path \( \{\psi_t\} \). Away from the exceptional divisor true that the path induced by \( H_t \circ \pi \) is a lift of the original path; the problem occurs near the exceptional divisor. Note for instance that \( H_t \circ \pi \) is independent of the parameter \( \rho \) and such parameter is relevant in the symplectic structure on the one-point blow up. As we will see, the Hamiltonian function that induces the desired Hamiltonian path will depend on the parameter \( \rho \).
The required condition for a path of Hamiltonian diffeomorphisms to lift to the one-point blow up is that for each \( t \in [0, 1] \) there exists a matrix \( U_t \in U(k) \) such that \( \psi_t = U_t \) on \( B_{\rho+\epsilon} \subset \mathbb{C}^k \).

**Lemma 2.3.** Let \( \psi_t : (B_r, \omega_0) \to (B_r, \omega_0) \) be a Hamiltonian path given by unitary matrices starting at the identity matrix with Hamiltonian function \( H_t \). Then

\[
H_t(z) = H_t(\lambda z)
\]

for \( z \in B_r \) and \( \lambda \in S^1 \).

**Proof.** Denote by \( X_t \) the time-dependent vector field of the path \( \{\psi_t\} \). For \( \lambda \in S^1 \), let \( \phi_\lambda : B_r \to B_r \) be matrix multiplication by \( \lambda I \). Since \( \phi_\lambda \) is in the center of \( U(k) \) it follows that

\[
X_t \circ \phi_\lambda = \frac{d}{ds} \bigg|_{s=t} \psi_s \circ \psi_t^{-1} \circ \phi_\lambda = \frac{d}{ds} \bigg|_{s=t} \phi_\lambda \circ \psi_s \circ \psi_t^{-1} = (\phi_\lambda)_* X_t.
\]

Therefore

\[
d(H_t \circ \phi_\lambda) = \omega_0(X_t, (\phi_\lambda)_*(\cdot)) = (\phi_\lambda)^* \omega_0(X_t, \cdot) = dH_t.
\]

Since both functions \( H_t \) and \( H_t \circ \phi_\lambda \) agree at the origin, it follows that \( H_t(z) = H_t(\lambda z) \). \( \square \)

Let \( \{\widetilde{\psi}_t\} \) be a path of Hamiltonian diffeomorphisms on \((\mathbb{C}^k, \omega_0)\) such that for each \( t \) the restriction of \( \psi_t \) to \( B_{\rho+\epsilon} \subset \mathbb{C}^k \) is given by a unitary matrix. If \( H_t : (\mathbb{C}^k, \omega_0) \to \mathbb{R} \) is the Hamiltonian function of the path, define the function \( \widetilde{H}_t : (\widetilde{\mathbb{C}}^k, \widetilde{\omega}_{\rho, \epsilon}) \to \mathbb{R} \) by

\[
(2) \quad \widetilde{H}_t(z, \ell) := \begin{cases} 
H_t \circ F_{\rho, \epsilon} \circ \pi(z, \ell) & \text{if } (z, \ell) \in \widetilde{\mathbb{C}}^k \setminus E \\
H_t \left( \frac{\rho}{|w|} w \right) & \text{if } z = 0 \text{ and } |w| = \ell \in \mathbb{C}P^{k-1}.
\end{cases}
\]

It follows from Lemma 2.3 and the definition of \( F_{\rho, \epsilon} \) that \( \widetilde{H}_t \) is well defined. Note that \( \widetilde{H}_t \) depends on the diffeomorphism \( F_{\rho, \epsilon} \) and hence on the parameter \( \rho \). The path of Hamiltonian diffeomorphisms on \((\widetilde{\mathbb{C}}^k, \widetilde{\omega}_{\rho, \epsilon})\) induced by \( \widetilde{H}_t \) is the path that lifts the initial path on \((\mathbb{C}^k, \omega_0)\). The proof of this fact as well as the proof of the next proposition appears in [11, Sec. 3].

**Proposition 2.4.** The time-dependent vector field \( \widetilde{X}_t \) and path of diffeomorphisms \( \{\widetilde{\psi}_t\} \) induced by the function \( \widetilde{H}_t : (\widetilde{\mathbb{C}}^k, \widetilde{\omega}_{\rho, \epsilon}) \to \mathbb{R} \) defined in (2), are such that

1. \( \pi_*(\widetilde{X}_t) = X_t \) for each \( t \in [0, 1] \) and
2. \( \widetilde{\psi}_t \) is a lift of \( \psi_t \) for each \( t \in [0, 1] \).

In particular, \( \{\widetilde{\psi}_t\} \) is a Hamiltonian path of \((\widetilde{\mathbb{C}}^k, \widetilde{\omega}_{\rho, \epsilon})\) that lifts the Hamiltonian path \( \{\psi_t\} \).
Thus if $\psi$ is a Hamiltonian diffeomorphism on $(\mathbb{C}^k, \omega_0)$ such that it can be joined to the identity by a path of Hamiltonian diffeomorphisms such that each one of them is equal to a unitary matrix on the ball $B_{\rho+\epsilon}$, then $\psi$ admits a Hamiltonian lift to the one-point blow up $(\tilde{\mathbb{C}}^k, \tilde{\omega}_{\rho,\epsilon})$.

This definition can be extended to the one-point blow up of weight $\rho$ of an arbitrary symplectic manifold $(M, \omega)$. For the purpose of this note it is not relevant to review such definition. For further details see [11].

3. Review of the blow up along a submanifold

Following [10, Ch. 7], we review the construction of the symplectic blow up along a compact symplectic submanifold. The complete details of this construction are relevant for the Hamiltonian diffeomorphism that we will defined on the blown up manifold.

3.1. The blow up along a compact submanifold. Let $N \subset (M, \omega)$ be a compact symplectic submanifold of codimension $2k > 2$. Then $\pi_N : TN^\omega \rightarrow N$ corresponds to its normal bundle which is a symplectic vector bundle. Since the symplectic linear group contracts to the unitary group, $\pi_N$ is in fact a complex vector bundle. For simplicity, write $\nu$ for $T\mathbb{C}^k$. Thus there exists a Hamiltonian linear action of $U(k)$ on $(\mathbb{R}_{2k}, \omega_0) = (\mathbb{C}^k, \omega_0)$ and a principal $U(k)$-bundle $P \rightarrow N$ such that the associated fibration coincides with the normal bundle, $\nu \simeq P \times U(k) \mathbb{C}^k$. Let $\mu : (\mathbb{C}^k, \omega_0) \rightarrow u(k)^*$ be the corresponding moment map and fix a connection 1-form $A \in \Omega^1(P, u(k))$ on $P \rightarrow N$. On $P \times \mathbb{C}^k$ define the 1-form $\langle \mu, A \rangle_{(p, z)}(u + v) := \langle \mu(z), A_p(u) \rangle$

where $u \in T_p P$ and $v \in T_z \mathbb{C}^k$. Here $\langle \cdot, \cdot \rangle$ stands for the pairing between $u(k)^*$ and $u(k)$. Notice that the 1-form $\langle \mu, A \rangle$ is independent of the vectors that lie in $T\mathbb{C}^k$. For $\xi \in u(k)$, let $X_\xi$ be the vector on $\mathbb{C}^k$ induced by the action of $U(k)$. Then on $P \times \mathbb{C}^k$ define the 2-form

$$\omega_A(u_1 + v_1, u_2 + v_2) := \omega_0(v_1 + X_{A(u_1)}, v_2 + X_{A(u_2)}) = d\langle \mu, A \rangle(u_1 + v_1, u_2 + v_2).$$

So defined, the 2-form $\omega_A$ is closed. Moreover, according to [10, Thm. 6.6.3] $\omega_A$ is $U(k)$-invariant and horizontal. Henceforth, it induces a closed 2-form on $\nu \simeq P \times U(k) \mathbb{C}^k$. In order to avoid cumbersome notation, we will make no distinction between a basic form and the induced form on the orbit space. Note that on each fibre of $\pi_N : \nu \rightarrow N$ the form $\omega_A$ restricts to $\omega_0$. Finally, define the 2-form

$$\omega_{\nu, A} := \omega_A + \pi_N^*(\omega).$$
on \( \nu \). Notice that \( \omega_{\nu,A} \) also restricts to \( \omega_0 \) on each fibre. Furthermore, if \( \nu_0 \) stands for the zero-section of \( \nu \) then \( \omega_{\nu,A} \) restricted to \( \nu_0 \) is equal to \( \pi_N^*(\omega) \).

These observations imply the following lemma. First, for \( \epsilon > 0 \) denote by \( U_\epsilon(\nu_0) := P \times_{U(k)} B_\epsilon \) the \( \epsilon \)-disk subbundle of \( \nu \to N \).

**Lemma 3.5.** Let \( N \subset (M, \omega) \) be a compact symplectic submanifold of codimension \( 2k \). Then there exist \( \epsilon_0 > 0 \) such that the 2-form

\[
\omega_{\nu,A} = \omega_A + \pi_N^*(\omega)
\]

is symplectic on the disk subbundle \( U_\epsilon(\nu_0) \). Moreover \( \omega_{\nu,A} \) restricts to \( \omega_0 \) on each fibre and to \( \pi_N^*(\omega) \) on the zero-section.

It follows by the symplectic neighborhood theorem there exists a symplectic diffeomorphism

\[
\Phi : (U_\epsilon(\nu_0), \omega_{\nu,A}) \to (U(N), \omega)
\]

where \( U(N) \subset (M, \omega) \) is some neighborhood of \( N \).

The outline of the construction of the symplectic blow up of \( (M, \omega) \) along the symplectic submanifold \( N \) is as follows: The neighborhood \( (U(N), \omega) \) of \( N \subset (M, \omega) \) is identified with the open set \( (U_\epsilon(\nu_0), \omega_{\nu,A}) \) via the symplectic diffeomorphism \( \Phi \) and thus the zero-section \( \nu_0 \) in \( \nu \) is identified with the submanifold \( N \) is blown up. Hence we give a description of the blow up of the zero-section in \( \nu \). By the blow up of \( \nu_0 \) we mean the total space of the associated bundle that is obtained by \( \tilde{C}^k \) and \( P \to N \) which is denoted by \( \tilde{\nu} \). Finally, \( \tilde{\nu} \) carries a symplectic form \( \tilde{\omega}_{\nu,A,\rho} \) and such model is glued back to \( (M \setminus N, \omega) \).

The complex-linear action of \( U(k) \) on \( \tilde{C}^k \) has a canonical lift to \( \tilde{C}^k \) defined as

\[
U_\epsilon(z, \ell) \mapsto (U(z), U(\ell)) \quad \text{for } U \in U(k) \text{ and } (z, \ell) \in \tilde{C}^k.
\]

This \( U(k) \)-space together with the \( U(k) \)-principal bundle \( P \to N \) induce the associated fibration \( \tilde{\pi}_N : \tilde{\nu} \to N \) where \( \tilde{\nu} := P \times_{U(k)} \tilde{C}^k \). Since the blow up map \( \pi : \tilde{C}^k \to \mathbb{C}^k \) is \( U(k) \)-equivariant, it induces the map

\[
\tilde{\pi} : P \times_{U(k)} \tilde{C}^k \to P \times_{U(k)} \mathbb{C}^k
\]

defined as \( \tilde{\pi}[p, (z, \ell)] := [p, \pi(z, \ell)] = [p, z] \). Notice that \( \tilde{\pi} : \tilde{\nu} \to \nu \) corresponds to the blow up of the zero-section \( \nu_0 \) in \( \nu \). As in the case of the one-point blow up, \( \tilde{\pi} \) is a diffeomorphism on the complement of \( \tilde{\pi}^{-1}(\nu_0) \subset \tilde{\nu} \).

For any \( \epsilon > 0 \), the ball \( B_\epsilon \subset \mathbb{C}^k \) is \( U(k) \)-invariant and hence \( L_\epsilon \subset \tilde{C}^k \) is also \( U(k) \)-invariant. Write \( U_\epsilon(\tilde{\pi}^{-1}(\nu_0)) \) for the subbundle \( P \times_{U(k)} L_\epsilon \). Notice that \( \tilde{\pi} \) maps \( U_\epsilon(\tilde{\pi}^{-1}(\nu_0)) \) onto \( U_\epsilon(\nu_0) \). Furthermore, \( \tilde{\pi} \) maps \( U_\epsilon(\tilde{\pi}^{-1}(\nu_0)) \setminus \tilde{\pi}^{-1}(\nu_0) \) diffeomorphically onto \( U_\epsilon(\nu_0) \setminus \nu_0 \). Summarizing, we have the following
The preimage of the blown up submanifold along the submanifold $N$ corresponds to the projectivization of the normal bundle $\nu$ to the exceptional divisor, $E$. The parameter $\epsilon_0 > 0$ is as before, the blow up of $M$ along the compact symplectic submanifold $N$ is defined as

$$\tilde{M}_N := M \setminus N \cup_{\Phi \circ \pi} \mathcal{U}_\epsilon(\pi^{-1}(\nu_0))$$

where $\mathcal{U}_\epsilon(\pi^{-1}(\nu_0)) \setminus \pi^{-1}(\nu_0)$ is identified with $\mathcal{U}(N) \setminus N$ via the diffeomorphism $\Phi \circ \pi$.

As in the case of the one-point blow up, there is a projection map $\tilde{\pi}_{(M,N)} : \tilde{M}_N \to M$, also called the blow up map, defined as

$$\tilde{\pi}_{(M,N)}(p) := \begin{cases} p & \text{if } p \in M \setminus N \\ \Phi \circ \pi(p) & \text{if } p = [p', (z, \ell)] \in \mathcal{U}_\epsilon(\pi^{-1}(\nu_0)) \end{cases}$$

Note that $\tilde{\pi}_{(M,N)}$ depends on the disk subbundle $\mathcal{U}_\epsilon(\nu_0)$. We omit such dependency from the notation of the map. The preimage of the blown up submanifold $E_N := \tilde{\pi}_{(M,N)}^{-1}(N)$ is called the exceptional divisor and $\tilde{\pi}_{(M,N)}$ restricted to the exceptional divisor, $E_N \to N$, is a fibration with fiber $\mathbb{CP}^{k-1}$ that corresponds to the projectivization of the normal bundle $\nu \to M$.

It remains to define a symplectic structure on $\tilde{M}_N$. In the definition of $\tilde{M}_N$, the parameter $\epsilon_0 > 0$ of the disk subbundle was irrelevant. Its importance will become apparent in the description of the symplectic structure on it.

### 3.2. The symplectic structure on $\tilde{M}_N$.

Let $\tilde{\mu}_\rho : (\tilde{\mathbb{C}}^k, \tilde{\omega}_{\rho,\epsilon}) \to u(k)^*$ be the moment map of the induced action of $U(k)$ on $(\tilde{\mathbb{C}}^k, \tilde{\omega}_\rho)$ described above. Using the same connection 1-form $A$ on the principal bundle $P \to N$, we have the analog of the form $\omega_A$ defined before, but now on $P \times \tilde{\mathbb{C}}^k$. That is, on $P \times \tilde{\mathbb{C}}^k$ we have the following 2-form

$$\tilde{\omega}_{A,\rho}(u_1 + v_1, u_2 + v_2) := \tilde{\omega}_{\rho,\epsilon}(v_1 + X_A(u_1), v_2 + X_A(u_2)) - d(\tilde{\mu}_\rho, A)(u_1 + v_1, u_2 + v_2)$$

that is closed and descends to the orbit space $\tilde{\nu} := P \times_{U(k)} \tilde{\mathbb{C}}^k$. Moreover it restricts to $\tilde{\omega}_{\rho,\epsilon}$ on each fibre of $\tilde{\pi}_N : \tilde{\nu} \to N$. As before, on $\tilde{\nu}$ we define the 2-form

$$\tilde{\omega}_{\tilde{\nu},A,\rho} := \tilde{\omega}_{A,\rho} + \tilde{\pi}_N^*(\omega).$$
Recall that \( \pi : (\tilde{\mathbb{C}}^k, \tilde{\omega}_{\rho, \epsilon}) \rightarrow (\mathbb{C}^k, \omega_0) \) is a symplectic diffeomorphism on the complement of \( \mathcal{T}_{\rho+\epsilon} \).

**Lemma 3.6.** Let \( \rho \) and \( \epsilon \in (0, 1) \) such that \( \rho + \epsilon < \epsilon_0 \). Then under the map \( \tilde{\pi} : \tilde{\nu} \rightarrow \nu \) we have that

\[
\tilde{\pi}^* (\omega_{\nu, A}) = \tilde{\omega}_{\nu, A, \rho}
\]

on \( P \times U(k) (\tilde{\mathbb{C}}^k \setminus \tilde{\mathcal{T}}_{\rho+\epsilon}) \).

**Proof.** In order to prove the claim, consider the forms on \( P \times U(k) \mathbb{C}^k \) instead of \( P \times U(k) \tilde{\mathbb{C}}^k \). Similarly for the forms on \( P \times U(k) \tilde{\mathbb{C}}^k \). Hence we must show that

\[
(1 \times \pi)^* (\omega_{\nu, A}) = \tilde{\omega}_{\nu, A, \rho}.
\]

First note that since \( \pi_N \) and \( \tilde{\pi}_N \) are fibrations over \( N \) and \( \tilde{\pi} \) is a bundle map, it follows that \( \tilde{\pi}_N^* (\omega) = \tilde{\pi}^* \circ \pi_N^* (\omega) \) on all \( \tilde{\nu} \).

\[
\begin{array}{ccc}
P \times \tilde{\mathbb{C}}^k & \xrightarrow{1 \times \pi} & \tilde{\nu} \setminus \tilde{\mathcal{T}}_{\rho+\epsilon} \setminus \mathcal{T}_{\rho+\epsilon} \setminus \mathcal{T}_{\rho+\epsilon} \\
1 \times \pi & & \tilde{\pi} \setminus \tilde{\mathcal{T}} \setminus \mathcal{T} \\
\downarrow & & \downarrow \pi_N \setminus N \end{array}
\]

Therefore \( (1 \times \pi)^* (\pi_N^* (\omega)) = \tilde{\pi}_N^* (\omega) \). It remains to show that \( (1 \times \pi)^* (\omega_{\nu, A}) = \tilde{\omega}_{\nu, A, \rho} \). In this step the parameter \( \rho \) will be relevant. Since \( \pi : (\tilde{\mathbb{C}}^k, \tilde{\omega}_{\rho, \epsilon}) \rightarrow (\mathbb{C}^k, \omega_0) \) is a symplectic diffeomorphism on the complement of \( \mathcal{T}_{\rho+\epsilon} \) and is \( U(k) \)-equivariant, the moment maps satisfy \( \tilde{\mu}_\rho = \mu \circ \pi \) on \( P \times (\tilde{\mathbb{C}}^k \setminus \tilde{\mathcal{T}}_{\rho+\epsilon}) \).

This in turn imply that \( (1 \times \pi)^* (\mu, A) = (\tilde{\mu}_\rho, A) \) also on \( P \times (\tilde{\mathbb{C}}^k \setminus \tilde{\mathcal{T}}_{\rho+\epsilon}) \).

Once again, using the fact that the blow up map is a symplectic diffeomorphism on the complement of \( \tilde{\mathcal{T}}_{\rho+\epsilon} \) and the fact that \( \pi_* (\tilde{X}_\xi) = X_\xi \) it follows that

\[
\omega_0 (\pi_* (v_1) + X_{A(u_1)}, \pi_* (v_2) + X_{A(u_2)}) = \omega_0 (\pi_* (v_1) + \pi_* (\tilde{X}_{A(u_1)}), \\
\pi_* (v_2) + \pi_* (\tilde{X}_{A(u_2)})) = \pi^* (\omega_0 (v_1 + \tilde{X}_{A(u_1)}, v_2 + \tilde{X}_{A(u_2)})) = \omega_{\rho, \epsilon} (v_1 + \tilde{X}_{A(u_1)}, v_2 + \tilde{X}_{A(u_2)})
\]

where \( u_j \in TP \) and \( v_j \in T\tilde{\mathbb{C}}^k \).

All these facts show that \( (1 \times \pi)^* (\omega_{\nu, A}) = \tilde{\omega}_{\nu, A, \rho} \) on \( P \times (\tilde{\mathbb{C}}^k \setminus \tilde{\mathcal{T}}_{\rho+\epsilon}) \). Henceforth, \( \tilde{\pi}^* (\omega_{\nu, A}) = \tilde{\omega}_{\nu, A, \rho} \) on \( P \times U(k) (\tilde{\mathbb{C}}^k \setminus \tilde{\mathcal{T}}_{\rho+\epsilon}) \). \( \square \)

For the rest of the article we fix the parameters \( \rho > 0 \) and \( \epsilon \in (0, 1) \) such that \( \rho + \epsilon < \epsilon_0 \). Remember that \( \Phi : (\mathcal{U}_0 (\nu_0), \omega_{\nu, A}) \rightarrow (\mathcal{U}(N), \omega) \) is a symplectic diffeomorphism and by Lemma 3.6 that \( \tilde{\pi}^* (\omega_{\nu, A}) = \tilde{\omega}_{\nu, A, \rho} \) on \( P \times U(k) (\tilde{\mathbb{C}}^k \setminus \tilde{\mathcal{T}}_{\rho+\epsilon}) \).
Therefore on $P \times U(k) (\tilde{C}^k \setminus \overline{T_{\rho+\epsilon}})$ we have that $\tilde{\pi}^* \circ \Phi^*(\omega) = \tilde{\omega}_{\nu,A,\rho}$. Hence it follows that $\tilde{M}_N$ carries a symplectic form $\tilde{\omega}_{N,\rho}$ defined as

\begin{equation}
\tilde{\omega}_{N,\rho} := \begin{cases}
\omega & \text{on } M \setminus \Phi(U_{\nu_0}(\nu_0)) \\
\tilde{\omega}_{\nu,A,\rho} & \text{on } U_{\nu_0}(\overline{\pi^{-1}(\nu_0)})
\end{cases}
\end{equation}

where $\epsilon'$ is such that $\rho + \epsilon < \epsilon' < \epsilon_0$. Note that $\tilde{\pi}_{(M,N)}$ is a symplectic diffeomorphism on $M \setminus \Phi(U_{\nu_0}(\nu_0))$.

4. Induced Hamiltonians on the blown up manifold

In this section we give conditions on a Hamiltonian diffeomorphism on $(M,\omega)$ in order to ensure that it lifts to a Hamiltonian diffeomorphism on the blown up manifold $(\tilde{M}_N, \tilde{\omega}_{N,\rho})$. As stated in the previous section, a diffeomorphism $\bar{\phi}$ on $\tilde{M}_N$ is said to be a lift of the diffeomorphism $\phi$ on $M$ if the following diagram commutes

\begin{equation}
\begin{array}{ccc}
\tilde{M}_N & \xrightarrow{\bar{\phi}} & \tilde{M}_N \\
\pi_{(M,N)} \downarrow & & \downarrow \pi_{(M,N)} \\
M & \xrightarrow{\phi} & M
\end{array}
\end{equation}

By the nature of the blow up construction, these conditions will only take place in a neighborhood of the symplectic submanifold $N \subset M$. To that end, fix a tubular neighborhood $U(N)$ of $N$ and a symplectic diffeomorphism $\Phi : (U_{\nu_0}(\nu_0), \omega_{\nu,A}) \to (U(N), \omega)$ as before.

**Definition 1.** Let $N \subset (M,\omega)$ be a compact symplectic submanifold of codimension $2k$. A symplectic diffeomorphism $\psi : (M,\omega) \to (M,\omega)$ is called $N$-liftable if

a) $\psi(U(N)) \subset U(N)$ and

b) the map $\Phi^{-1} \circ \psi \circ \Phi : (U_{\nu_0}(\nu_0), \omega_{\nu,A}) \to (U_{\nu_0}(\nu_0), \omega_{\nu,A})$ takes the form

\[ [p, z] \mapsto [p, U(z)] \]

where $U$ lies in the center of $U(k)$.

**Remark.**

- The above definition depends on the tubular neighborhood $U(N)$ and the diffeomorphism $\Phi$. In order to avoid cumbersome notation we omit such dependencies from the definition.
- The reason that the matrix $U$ must lie in the center of $U(k)$ is to ensure that $\tilde{\mu}_{\rho}(U(z)) = \tilde{\mu}_{\rho}(z)$, where $\tilde{\mu}_{\rho} : (\tilde{C}^k, \tilde{\omega}_{\rho,\epsilon}) \to u(k)^*$ is the moment map of the induced action used in the definition of the symplectic form on described in Sec. 3.2. It will also be useful when proving that the
induced diffeomorphisms on the blown up manifold by a Hamiltonian is also Hamiltonian.

- If $\phi$ is a gauge transformation of $P \to N$, then the map $[p, z] \mapsto [\phi(p), U(z)]$ is a diffeomorphism of $U(\nu_0)$. However, if $\phi \neq 1_P$, such diffeomorphism on $(U(\nu_0), \omega_{\nu, A})$ will not be symplectic.

If $\psi : (M, \omega) \to (M, \omega)$ is $N$-liftable, denote by $\psi_\nu$ the diffeomorphism $\Phi^{-1} \circ \psi \circ \Phi$ defined on $(U_\nu(\nu_0), \omega_{\nu, A})$. Hence $\psi_\nu$ induces the diffeomorphism $\tilde{\psi}_\nu : U_\nu(\tilde{\pi}^{-1}(\nu_0)) \to U_\nu(\tilde{\pi}^{-1}(\nu_0))$ given by

$$\tilde{\psi}_\nu[p, (z, \ell)] := [p, (U(z), U(\ell))].$$

Since $\psi$ is $N$-liftable it maps the complement of $U(N) \subset M$ to itself. Therefore $\tilde{\psi}_\nu$ together with $\psi$ give rise to a well-defined diffeomorphism on the blown up manifold $\tilde{M}_N$ that we denote by $\tilde{\psi}_N$. So defined $\tilde{\psi}_N$ is a lift of $\psi$, that is $\tilde{\pi}_{(M,N)} \circ \tilde{\psi}_N = \psi \circ \tilde{\pi}_{(M,N)}$. It remains to show that it is a symplectic diffeomorphism.

**Proposition 4.7.** If $\psi : (M, \omega) \to (M, \omega)$ is a symplectic diffeomorphism that is $N$-liftable, then $\tilde{\psi}_N$ is a symplectic diffeomorphism of $(\tilde{M}_N, \tilde{\omega}_{N, \rho})$.

**Proof.** By definition of the symplectic form $\tilde{\omega}_{N, \rho}$ in Eq. (8) and of the diffeomorphism $\tilde{\psi}_N$, if follows that $\tilde{\psi}_N$ is a symplectic diffeomorphism on $M \setminus \Phi(U_\gamma(\nu_0))$. It only remains to show that $\tilde{\psi}_N$ is a symplectic diffeomorphism on $U_\nu(\tilde{\pi}^{-1}(\nu_0))$.

Since $\psi$ is $N$-liftable, on $U_\nu(\tilde{\pi}^{-1}(\nu_0))$ the diffeomorphism $\tilde{\psi}_N$ is given by $\tilde{\psi}_\nu$. That is

$$\tilde{\psi}_\nu[p, (z, \ell)] = [p, (U(z), U(\ell))],$$

where $U$ lies in the center of $U(k)$. In order to show that $\tilde{\psi}_\nu^*(\tilde{\omega}_{\nu, A, \rho}) = \tilde{\omega}_{\tilde{\nu}, A, \rho}$, we regard the symplectic form on $P \times L_{\tilde{\nu}_0}$ instead of $P \times U(k) L_{\tilde{\nu}_0}$. In particular, $\tilde{\omega}_{\tilde{\nu}, A, \rho}$ is a basic form. On $P \times L_{\nu_0}$ the diffeomorphism $\tilde{\psi}_\nu$ is given by $1_P \times U$, where $U$ lies in the center of $U(k)$. Thus $(1 \times U)^*\tilde{\omega}_{A, \rho} = \tilde{\omega}_{A, \rho}$ and $\psi_N$ is a symplectic diffeomorphism of $(\tilde{M}_N, \tilde{\omega}_{N, \rho})$. □

Next we prove that the lift of a Hamiltonian diffeomorphism, under some constraints, is again a Hamiltonian diffeomorphism on the blown up manifold. To that end, consider $\psi$ a Hamiltonian diffeomorphism on $(M, \omega)$ such that there exists a path $\{\psi_t\}_{0 \leq t \leq 1}$ that start at the identity, ends at $\psi$ and for each $t$ the diffeomorphism $\psi_t$ is $N$-liftable. Hence, $\{\psi_{t, \nu}\}$ is a Hamiltonian path on $(U(\nu_0), \omega_{\nu, A})$. Moreover for each $t \in [0, 1]$ there exist a matrix $U_t$ in the center of $U(k)$ such that

$$\psi_{t, \nu}[p, z] = [p, U_t(z)].$$
Since each matrix $U_t$ is of the form $\lambda_t \cdot 1_{k \times k}$ for $\lambda_t \in \mathbb{C}$ and $|\lambda_t| = 1$, it follows that the induced Hamiltonian function of the path of unitary matrices is autonomous. Moreover, it is given by $H(z) := c \sum_j (x_j^2 + y_j^2) = c|z|^2$ for some $c \in \mathbb{R}$. Recall from Section 2.2 that such Hamiltonian function $H$ induces the Hamiltonian $H : L_{e_0} \rightarrow \mathbb{R}$ defined in Eq. (2). Further it also satisfies the relation $	ilde{H}(U(z), U(\ell)) = \tilde{H}(z, \ell)$. Henceforth, we define the function $\tilde{H}_{N,t} : (\tilde{M}_N, \tilde{\omega}_{N,\rho}) \rightarrow \mathbb{R}$ by

$$
\tilde{H}_{N,t}(q) := \begin{cases} 
H_t(\hat{\pi}(M,N)(q)) & \text{if } q \in M \setminus \Phi(\mathcal{U}_N(\nu_0)) \\
\tilde{H}(z, \ell) & \text{if } q = [p, (z, \ell)] \in \mathcal{U}_o(\tilde{\pi}^{-1}(\nu_0)).
\end{cases}
$$

Next we show that $\tilde{H}_{N,t}$ is the Hamiltonian function of the lifted path of symplectic diffeomorphisms.

**Proposition 4.8.** Let $\psi_t : (M, \omega) \rightarrow (M, \omega)$ be a path of Hamiltonian diffeomorphisms such that each $\psi_t$ is $N$-liftable and $\psi_0 = 1$. Then $\tilde{\psi}_{N,t} : (\tilde{M}_N, \tilde{\omega}_{N,\rho}) \rightarrow (\tilde{M}_N, \tilde{\omega}_{N,\rho})$ is a Hamiltonian path induced by $\tilde{H}_{N,t}$.

**Proof.** Since the Hamiltonian path $\{\psi_t\}$ is $N$-liftable, by Prop. 4.7 the path $\{\tilde{\psi}_{N,t}\}$ consists of symplectic diffeomorphisms on $(\tilde{M}_N, \tilde{\omega}_{N,\rho})$ and satisfies $\psi_t \circ \hat{\pi}(M,N) = \hat{\pi}(M,N) \circ \tilde{\psi}_{N,t}$ for each $t$. Thus if $\tilde{X}_t$ is the time-dependent vector field induced by $\{\tilde{\psi}_{N,t}\}$, then $(\hat{\pi}(M,N))_* \tilde{X}_t = X_t$. Thus by the definition of the function $\tilde{H}_{N,t}$ on (9) and the fact that $\hat{\pi}(M,N)$ is a symplectic diffeomorphism on $M \setminus \Phi(\mathcal{U}_N(\nu_0))$ it follows that

$$
\tilde{\omega}_N(\tilde{X}_t, \cdot) = d\tilde{H}_{t,N}
$$

holds on $\tilde{M}_N \setminus \mathcal{U}_o(\pi^{-1}(\nu_0))$.

Now on $\mathcal{U}_o(\pi^{-1}(\nu_0)) \simeq P \times U(k) L_{e_0}$ the path $\{\tilde{\psi}_{N,t}\}$ is given by $[p, (z, \ell)] \mapsto [p, (U_t(z), U_t(\ell))]$. Thus the time-dependent vector field is actually independent of time and takes the form $0 + \tilde{X}$ where $\tilde{X}$ is a vector field on $L_{e_0}$. Also, on $\mathcal{U}_o(\pi^{-1}(\nu_0))$ the symplectic form $\tilde{\omega}_N$ is equal to $\tilde{\omega}_{\tilde{v},A,\rho} = \tilde{\omega}_{A,\rho} + \pi^*_N(\omega)$. Therefore,

$$
\tilde{\omega}_{\tilde{v},A,\rho}(\tilde{X}_t, \cdot) = \tilde{\omega}_{\tilde{v},A,\rho}(0 + \tilde{X}, \cdot) \\
= \tilde{\omega}_{A,\rho}(0 + \tilde{X}, \cdot) + \pi^*_N(\omega)(0 + \tilde{X}, \cdot) \\
= \tilde{\omega}_{\tilde{v},\rho}(\tilde{X} + 0, \cdot) + d(\tilde{\mu}_{\rho}, A)(0 + \tilde{X}, \cdot) + \pi^*_N(\omega)(0 + \tilde{X}, \cdot) \\
= \tilde{\omega}_{\tilde{v},\rho}(\tilde{X} + 0, \cdot)
$$

By [11, Prop. 3.7], on $L_{e_0}$ we have that $\tilde{\omega}_{\tilde{v},\rho}(\tilde{X}, \cdot) = d\tilde{H}$ where the function $\tilde{H}$ was defined in Eq. (2). Thus the proposition follows. □
Notice that if \( \{ \psi_t \} \) is a loop of Hamiltonian diffeomorphisms on \((M, \omega)\) that is \( N \)-liftable, then the lift \( \{ \tilde{\psi}_{N,t} \} \) is also loop of Hamiltonian diffeomorphisms. We are interested in understanding the the behavior of the loop \( \{ \tilde{\psi}_{N,t} \} \) in the fundamental group of \( \text{Ham}(\tilde{M}_N, \tilde{\omega}_{N,\rho}) \).

5. Weinstein’s morphism

The period group \( \mathcal{P}(M, \omega) \) of \((M, \omega)\) is defined as the image of the pairing \( \langle \cdot, \cdot \rangle : H_2(M; \mathbb{Z}) \rightarrow \mathbb{R} \). Then Weinstein’s morphism [13],

\[
\mathcal{A} : \pi_1(\text{Ham}(M, \omega)) \rightarrow \mathbb{R}/\mathcal{P}(M, \omega)
\]

is defined via the action functional on a loop \( \psi = \{ \psi_t \} \) as

\[
\mathcal{A}(\psi) = \left[ -\int_D u^*(\omega) + \int_0^1 H_t(\psi_t(x_0))dt \right].
\]

Here \( D \) is the unit closed disk and \( u : D \rightarrow M \) is a smooth function such that \( u(\partial D) \) agrees with the loop \( \{ \psi_t(x_0) \} \), \( x_0 \in M \) a base point and \( H_t \) is the 1-periodic Hamiltonian function induced by the Hamiltonian loop \( \{ \psi_t \} \) subject to the normalized condition

\[
\int_M H_t \omega^n = 0
\]

for every \( t \in [0, 1] \).

The aim is to relate Weinstein’s morphism on \( \pi_1(\text{Ham}(M, \omega)) \) and \( \pi_1(\text{Ham}(\tilde{M}_N, \tilde{\omega}_{N,\rho})) \) on a particular class of loops. To that end, recall that we must consider a Hamiltonian diffeomorphisms on \((M, \omega)\) that are \( N \)-liftable. As seen above, such Hamiltonians are induced by Hamiltonians functions on \((\mathbb{C}^n, \omega)\) that on a neighborhood of the origin take the form \( c \sum_j (x_j^2 + y_j^2) = c|z|^2 \) for some \( c \in \mathbb{R} \).

**Lemma 5.9.** Let \( H : M \rightarrow \mathbb{R} \) be a smooth function that is induced by a Hamiltonian diffeomorphism that is \( N \)-liftable, \( \rho \) as before and \( \tilde{H}_N : \tilde{M}_N \rightarrow \mathbb{R} \) the induced function defined in (9). Then

\[
\int_{\tilde{M}_N} \tilde{H}_N \tilde{\omega}_N^n = \int_M H \omega^n - \int_{\Phi(U_\rho(\nu_0))} H \omega^n.
\]

**Proof.** First note that \( \tilde{M}_N \) is the union of the disjoint sets \( M \setminus \Phi(U_\rho(\nu_0)) \) and \( U_{\epsilon'}(\bar{\pi}^{-1}(\nu_0)) \). Recall that \( \epsilon' \) is such that \( \rho + \epsilon < \epsilon' < \epsilon_0 \). Then from the definition
of $\tilde{H}_N$ in (9) and of $\tilde{H}$ in (2) it follows that
\[
\int_{M_N} \tilde{H}_N \tilde{\omega}^n_{N,\rho} = \int_{M'_N} \frac{\tilde{H}_N}{\Phi(U_{\epsilon}(v_0))} \tilde{\omega}^n_{N,\rho} + \int_{U_{\epsilon}(\tilde{\pi}^{-1}(v_0))} \tilde{H}_N \tilde{\omega}^n_{N,\rho}
\]
\[
= \int_{M'_N} \frac{H \circ \tilde{\pi}(M,\omega)}{\Phi(U_{\epsilon}(v_0))} \tilde{\omega}^n_{N,\rho} + \int_{U_{\epsilon}(\tilde{\pi}^{-1}(v_0))} \tilde{H} \tilde{\omega}^n_{N,\rho}
\]
\[
= \int_{M'_N} \frac{H \omega^n}{\Phi(U_{\epsilon}(v_0))} + \int_{U_{\epsilon}(\tilde{\pi}^{-1}(v_0))} \tilde{H} \tilde{\omega}^n_{N,\rho}
\]
\[
= \int_{M'_N} \frac{H \omega^n}{\Phi(U_{\epsilon}(v_0))} + \int_{(P \times U(k)L_{\epsilon'})\backslash v_0} \tilde{H} \tilde{\omega}^n_{N,\rho,\rho}
\]

Recall that the diffeomorphism $F_{\rho,\epsilon} : \mathbb{C}^k \setminus \{0\} \to \mathbb{C}^k \setminus \overline{B}_\rho$ defined in (1) is $U(k)$-equivariant. Hence it induces a diffeomorphism $F_{\rho,\epsilon} : P \times U(k) (\mathbb{C}^k \setminus \{0\}) \to P \times U(k) (\mathbb{C}^k \setminus \overline{B}_\rho)$ and
\[
F_{\rho,\epsilon}^* (\omega_{\nu,A}) = F_{\rho,\epsilon}^* (\omega_{A}) + F_{\rho,\epsilon}^* (\pi_N^*(\omega))
\]
\[
= \tilde{\omega}_{A,\rho} + \overline{\tilde{\pi}_N^*(\omega)}
\]
\[
= \tilde{\omega}_{\nu,A,\rho}.
\]

It then follows that
\[
\int_{(P \times U(k)L_{\epsilon'})\backslash v_0} \tilde{H} \tilde{\omega}^n_{\nu,A,\rho} = \int_{U_{\epsilon}(\tilde{\pi}^{-1}(v_0))} \tilde{H} \tilde{\omega}^n_{\nu,A,\rho}
\]
\[
= \int_{P \times U(k) (B_{\epsilon'} \setminus B_{\rho})} H \omega^n_{\nu,A,\rho}.
\]

Hence
\[
\int_{M_N} \tilde{H}_N \tilde{\omega}^n_{N,\rho} = \int_{M'_N} \frac{H \omega^n}{\Phi(U_{\epsilon}(v_0))} + \int_{P \times U(k) (B_{\epsilon'} \setminus B_{\rho})} H \omega^n_{\nu,A,\rho}
\]
\[
= \int_{M} H \omega^n - \int_{\Phi(U_{\rho}(v_0))} H \omega^n.
\]

6. Proof of Theorems 1.1 and 1.2

Now we prove the main results of this article that appeared at the Introduction.

**Theorem 1.2.** Let $(M,\omega)$ be a closed symplectic manifold and $N \subset (M,\omega)$ a symplectic submanifold of codimension $2k > 2$. If $\{\psi_t\}$ is a loop of Hamiltonian
diffeomorphisms on $(M, \omega)$ that is $N$-liftable to a loop $\{\tilde{\psi}_{N,t}\}$ on $(\tilde{M}_N, \tilde{\omega}_{N,\rho})$, then

$$\mathcal{A}([\tilde{\psi}_{N,t}]) = \left[ \mathcal{A}([\psi_t]) + \frac{1}{\text{Vol}(\tilde{M}_N, (\tilde{\omega}_{N,\rho})^n)} \int_0^1 \int_{\phi(U_{\rho}(\nu_0))} H_t \omega^n dt \right]$$

in $\mathbb{R}/\mathcal{P}(\tilde{M}_N, \tilde{\omega}_{N,\rho})$, where $H_t$ if the normalized Hamiltonian function of the loop $\{\tilde{\psi}_t\}$.

Proof. Let $\{\psi_t\}$ be a loop of Hamiltonian diffeomorphisms on $(M, \omega)$ that is $N$-liftable with normalized Hamiltonian function $\tilde{H}_t$. If $\{\tilde{\psi}_{N,t}\}$ is the induced loop on $(\tilde{M}_N, \tilde{\omega}_{N,\rho})$, denote by $\tilde{H}_{N,t}$ its Hamiltonian function defined by Eq. (9).

It follows from Lemma 5.9 that for each $t$,

$$\int_{\tilde{M}_N} \tilde{H}_{N,t} \tilde{\omega}_{N,\rho} = -\int_{\phi(U_{\rho}(\nu_0))} H_t \omega^n. \tag{10}$$

Denote such number by $c(\rho, t) \in \mathbb{R}$. Hence the normalized Hamiltonian function of the loop $\{\tilde{\psi}_{N,t}\}$ is

$$\tilde{H}_{N,t} := \tilde{H}_{N,t} - \frac{1}{\text{Vol}(\tilde{M}_N, \tilde{\omega}_{N,\rho})} c(\rho, t).$$

Thus, taking a disk $D$ and a point $x_0$ in $\tilde{M}_N \setminus U_{\varepsilon_0}(\tilde{\pi}^{-1}(\nu_0))$,

$$\mathcal{A}([\tilde{\psi}_{N,t}]) = \left[ -\int_D u^*(\tilde{\omega}_{N,\rho}) + \int_0^1 \tilde{H}_{N,t}(\tilde{\psi}_{N,t}(x_0)) dt \right]$$

$$= \left[ -\int_D u^*(\tilde{\omega}) + \int_0^1 \tilde{H}_{N,t}(\tilde{\psi}_{N,t}(x_0)) dt \right.$$

$$- \frac{1}{\text{Vol}(\tilde{M}_N, \tilde{\omega}_{N,\rho})} \int_0^1 c(\rho, t) dt \right].$$

Hence from Eq. (10) we get

$$\mathcal{A}([\tilde{\psi}_{N,t}]) = \left[ \mathcal{A}([\psi_t]) + \frac{1}{\text{Vol}(\tilde{M}_N, \tilde{\omega}_{N,\rho})} \int_0^1 \int_{\phi(U_{\rho}(\nu_0))} H_t \omega^n dt \right].$$

□

In order to give meaning to the expression that was proven above, now we relate the periods groups of $(\tilde{M}_N, \tilde{\omega}_{N,\rho})$ and $(M, \omega)$. Recall that the blow up map $\tilde{\pi}_{(M,N)} : \tilde{M}_N \to M$ is a diffeomorphism on the complement of $\tilde{\pi}_{(M,N)}^{-1}(N)$.
Moreover, $\bar{\pi}(M,N)$ induces a symplectic diffeomorphism on the complement of a neighborhood of $\bar{\pi}^{-1}(N)$. Therefore, the maps

$$\langle \tilde{\omega}_{N,\rho}, \cdot \rangle : H_2(\tilde{M}_N \setminus \bar{\pi}^{-1}(N); \mathbb{Z}) \to \mathbb{R} \quad \text{and} \quad \langle \omega, \cdot \rangle : H_2(M_N; \mathbb{Z}) \to \mathbb{R}$$

have the same image. It remains to determine the value of $\langle \tilde{\omega}_{N,\rho}, \cdot \rangle$ on the kernel of $\bar{\pi}(M,N)* : H_2(\tilde{M}_N; \mathbb{Z}) \to H_2(M; \mathbb{Z})$. From the commutative diagram

\[
\begin{array}{cccc}
H_1(\nu \setminus \nu_0; \mathbb{Z}) & \longrightarrow & H_2(\tilde{M}_N; \mathbb{Z}) & \longrightarrow & H_2(M \setminus N; \mathbb{Z}) \oplus H_2(\nu; \mathbb{Z}) \\
1 & \downarrow & \bar{\pi}_{(M,N)*} & \downarrow & 1 \oplus \bar{\pi}_* \\
H_1(\nu \setminus \nu_0; \mathbb{Z}) & \longleftarrow & H_2(M; \mathbb{Z}) & \longleftarrow & H_2(M \setminus N; \mathbb{Z}) \oplus H_2(\nu; \mathbb{Z})
\end{array}
\]

it suffices to look at the map $\bar{\pi}_* : H_2(\tilde{\nu}; \mathbb{Z}) \to H_2(\nu; \mathbb{Z})$.

Recall that the closed 2-form $\tilde{\omega}_{\tilde{\nu},A,\rho}$ on the total space of the fibration $\tilde{\pi}_N : \tilde{\nu} \to N$ restricts to $\tilde{\omega}_{\tilde{\nu},A,\rho}$ on each fibre.

The relation between the homology of $\tilde{M}_N$ and $M$ is as follows

$$H_2(\tilde{M}_N, \mathbb{Z}) \simeq H_2(M, \mathbb{Z}) \oplus \ker\{\bar{\pi}_{(M,N)*} : H_2(\tilde{\nu}, \mathbb{Z}) \to H_2(N, \mathbb{Z})\}.$$  

Moreover, if $\alpha \in H_2(\tilde{M}_N; \mathbb{Z})$ takes the form $\bar{\pi}_{(M,N)*}(\alpha) + 0$ in the above decomposition, then

$$\langle \tilde{\omega}_{N,\rho}, \alpha \rangle = \langle \omega, \bar{\pi}_{(M,N)*}(\alpha) \rangle.$$  

Since $\bar{\pi}_{(M,N)*} : H_2(\tilde{M}_N; \mathbb{Z}) \to H_2(M; \mathbb{Z})$ is surjective, we have that

$$\mathcal{P}(M, \omega) \subset \mathcal{P}(\tilde{M}_N, \tilde{\omega}_{N,\rho}).$$

Finally, $\ker\{\bar{\pi}_{(M,N)*} : H_2(\tilde{\nu}; \mathbb{Z}) \to H_2(N; \mathbb{Z})\}$ is generated by a single class that its restriction to the fiber is the generator of $H_2(\mathbb{C}P^{k-1}; \mathbb{Z})$. Hence we have the following result.

**Lemma 6.10.** Let $(M, \omega)$ and $(\tilde{M}_N, \tilde{\omega}_{N,\rho})$ as above. Assume that $N$ is compact, then

$$\mathcal{P}(\tilde{M}_N, \tilde{\omega}_{N,\rho}) = \mathcal{P}(M, \omega) + \mathbb{Z}(\pi \rho^2).$$

Recall from Lemma 3.5 that on a neighborhood $P \times_{U(k)} B_{\alpha_0}$ of the zero-section of $\nu = P \times_{U(k)} \mathbb{C}^k$ the symplectic form is given by

$$\omega_\nu,A(u_1 + v_1, u_2 + v_2) = \omega_0(v_1 + X_A(u_1), v_2 + X_A(u_2)) - d\langle \mu, A \rangle(u_1 + v_1, u_2 + v_2) + \pi_N^*(\omega)(u_1 + v_1, u_2 + v_2)$$  

(11)
for \( u_j \in TP \) and \( v_j \in TC^k \). Hence on the normal bundle \( \nu \) we have that

\[
\omega^n_{\nu,A} = \sum_{j=0}^{n} \omega^j_0 \pi_N^*(\omega)^{n-j} + d\alpha
\]  

(12)

for some \((2n - 1)\)-form \( \alpha \). (Keep in mind that in this equation the 2-form \( \omega_0 \) must be evaluated as in Eq. (11)).

**Lemma 6.11.** Let \( H : \nu \to \mathbb{R} \) be a smooth function. Then,

\[
\int_\nu H \omega^n_{\nu,A} = \int_N (\pi_N)_*(H \omega^k_0) \omega^{n-k}.
\]

**Proof.** First we integrate along the fiber with respect to \( \pi_N : \nu \to N \) and then use the commutativity of the diagram

\[
\begin{array}{c}
\nu \\
\downarrow \pi_N \\
N \end{array} \quad \begin{array}{c}
\downarrow \pi_N \\
\{pt\}
\end{array}
\]

Therefore by the projection formula we get that

\[
(\pi_N)_*(H \omega^n_{\nu,A}) = \sum_{j=0}^{n} (\pi_N)_*(H \omega^j_0 \pi_N^*(\omega)^{n-j})
\]

\[
= \sum_{j=0}^{n} (\pi_N)_*(H \omega^j_0) \omega^{n-j}.
\]

Note that in this expression, the symplectic form \( \omega \) is restricted to the submanifold \( N \).

Next we integrate on the submanifold \( N \),

\[
\int_\nu H \omega^n_{\nu,A} = \sum_{j=0}^{n} \int_N (\pi_N)_*(H \omega^j_0) \omega^{n-j}
\]

\[
= \sum_{j=0}^{n} \int_N (\pi_N)_*(H \omega^j_0) \omega^{n-j}
\]

\[
= \int_N (\pi_N)_*(H \omega^k_0) \omega^{n-k}.
\]

The last equality follows from the fact that the form \( \omega_0 \) comes from the fiber \( \mathbb{C}^k \) of \( \pi_N \). \qed

Notice that from this result the evaluation of \( \omega_0 \) as in Eq. (11) becomes irrelevant.
In particular, restrict the above integral to $U_\rho(\nu_0)$ and take $H \equiv 1$. Then we are able to compute the volume of $U_\rho(\nu_0)$, 

$$\text{Vol}(U_\rho(\nu_0), \omega^n_{\nu,A}) = \int_{U_\rho(\nu_0)} \omega^n_{\nu,A}$$

$$= \int_N (\pi N|U_\rho(\nu_0))_* (\omega^k_0) \omega^{n-k}$$

$$= \text{Vol}(B_\rho, \omega^k_0) \cdot \text{Vol}(N, \omega^{n-k})$$

$$= \frac{\pi^k \rho^{2k}}{k!} \cdot \text{Vol}(N, \omega^{n-k}).$$

6.1. Description of the loop. In order to prove Thm. 1.1, we need to defined a loop of Hamiltonian diffeomorphisms on $(M, \omega)$ supported on $P \times U(k) B_{\epsilon_0}$, a neighborhood of $N$. The definition of such loop follows the same line of thought as the Hamiltonian loop constructed in [12, Sec. 2].

Let $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(0) = 0$ and $\alpha(1) \neq 0$. Then consider the path of diagonal matrices $A_t := \exp \left(2\pi i \alpha(t)\right) \cdot 1_{k \times k}$. Hence $A_t$ lies in the center of $U(k)$ and it induces on $(\mathbb{C}^k, \omega^k_0)$ a path $\{\psi^\alpha_{\nu_0}\}_{0 \leq t \leq 1}$ of Hamiltonian diffeomorphisms starting at the identity and with Hamiltonian function

$$H^\alpha_t(z_1, \ldots, z_k) := \pi \alpha'(t) \sum_{j=1}^k |z_j|^2.$$ 

As before we have the positive parameters $\rho, \epsilon_0$ and $\epsilon \in (0, 1)$ such that $\rho + \epsilon < \epsilon_0$, that are used in the definition of $(\tilde{M}_N, \tilde{\omega}_{N,\rho})$. Now fix a $U(k)$-invariant smooth function $g : \mathbb{C}^k \to \mathbb{R}$ supported in $B_{\epsilon_0}$ such that $g \equiv 1$ on $B_{\rho}$. Consider the Hamiltonian function

$$H^\alpha_{t,g} := g \cdot H^\alpha_t$$

and let $\{\psi^\alpha_{t,g}\}_{0 \leq t \leq 1}$ be the induced Hamiltonian path. Since $\omega_0$ and $H^\alpha_{t,g}$ are $U(k)$-equivariant, it follows that each $\psi^\alpha_{t,g}$ is also $U(k)$-equivariant. Furthermore, on $B_{\rho}$ we have that $\psi^\alpha_{t,g} \equiv \psi^\alpha_t$ and $H^\alpha_{t,g} \equiv H^\alpha_t$. Note that as in [12, Lemma 2.4], 

$$\int_0^1 \int_{B_{\rho}} H^\alpha_{t,g} \omega^k_0 \, dt = (\alpha(1) - \alpha(0)) \pi \int_{B_{\rho}} \sum_{j=1}^k x_j^2 + y_j^2 \frac{1}{k!} \, dx_1 \cdots dx_k$$

$$= \alpha(1) \cdot \frac{\pi^k \rho^{2k+2}}{k!}.$$

Since each $\psi^\alpha_{t,g}$ is $U(k)$-equivariant, it induces a path of diffeomorphisms $1 \times \psi^\alpha_{t,g}$ on $\nu = P \times U(k) \mathbb{C}^k$. Furthermore, each diffeomorphism is supported on $U_\epsilon(\nu_0) = P \times U(k) B_{\epsilon_0}$. 


Lemma 6.12. The path $1 \times \psi_t^{\alpha, g}$ restricted to $U_{c_0}(v_0)$ is a Hamiltonian path with Hamiltonian function $1 \times H_t^{\alpha, g}$.

Proof. Recall that the 2-form $\omega_{\nu, A}$ is a basic form on $P \times B_{c_0}$, in particular is $U(k)$-invariant. Therefore, $1 \times \psi_t^{\alpha, g}$ preserves the form $\omega_{\nu, A}$ and is a symplectic diffeomorphism on $U_{c_0}(v_0)$.

It remains to prove that is Hamiltonian. Note that by definition on $(B_{c_0}, \omega_0)$ we have that $\{\psi_t^{\alpha, g}\}$ is Hamiltonian with Hamiltonian function $H_t^{\alpha, g}$. Let $X_t^{\alpha, g}$ be the associated Hamiltonian vector field, $\omega_0(X_t^{\alpha, g}, \cdot) = dH_t^{\alpha, g}$. Recall from Eq. (11) that

$$\omega_{\nu, A}(u_1 + v_1, u_2 + v_2) = \omega_0(v_1 + X_A(u_1), v_2 + X_A(u_2)) - d\langle \mu, A \rangle(u_1 + v_1, u_2 + v_2) + \pi_N^*(\omega)(u_1 + v_1, u_2 + v_2)$$

for $u_j \in TP$ and $v_j \in TB_{c_0}$. If $X_\xi$ is the vector field on $B_{c_0}$ induced by $\xi \in u(k)$ we have that $\omega_0(X_\xi, X_\xi) = \langle \mu, [\xi_1, \xi_2] \rangle$ for any $\xi_1, \xi_2 \in u(k)$. In particular, the vector field $X_\xi^{\alpha, g}$ is of the form $X_\xi$ for $\xi$ in the center of $u(k)$ and therefore $\omega_0(X_\xi^{\alpha, g}, X_\xi) = 0$ for any $\xi$. Then for $u \in TP$ and $v \in TB_{c_0}$

$$\omega_{\nu, A}(X_t^{\alpha, g}, u + v) = \omega_0(X_t^{\alpha, g}, v + X_A(u))$$

$$= \omega_0(X_t^{\alpha, g}, v)$$

$$= dH_t^{\alpha, g}(v)$$

$$= d(1 \times H_t^{\alpha, g})(u, v)$$

where $\langle \mu, A \rangle$ vanishes since $v_1 = 0$. □

Now we will consider another path of Hamiltonian diffeomorphisms in order to create a loop by concatenating it with the path $\{\psi_t^{\alpha, g}\}_{0 \leq t \leq 1}$. Hence, consider $\beta : \mathbb{R} \to \mathbb{R}$ such that $\beta(0) = 0$ and $\alpha(t) - \beta(t) \equiv 1$ in a neighborhood of 1. As above denote by $\{\psi_t^{\beta, g}\}_{0 \leq t \leq 1}$ the induced path of Hamiltonian diffeomorphisms with Hamiltonian function $H_t^{\beta, g}$ supported on $B_{c_0}$. Then the desired loop of Hamiltonian diffeomorphisms $\{\psi_t^{c_k}\}_{0 \leq t \leq 2}$ is defined as

$$\psi_t^{c_k} := \begin{cases} \psi_t^{\alpha, g} & t \in [0, 1] \\ \psi_2^{\beta, g} & t \in [1, 2] \end{cases}$$

with Hamiltonian function

$$H_t^{c_k} := \begin{cases} H_t^{\alpha, g} & t \in [0, 1] \\ H_2^{\beta, g} & t \in [1, 2]. \end{cases}$$

Furthermore, we have that

$$\int_0^2 \int_{B_{t, v}} H_t^{c_k} \omega_0^k \, dt = (\alpha(1) - \beta(1)) \frac{\pi^{k+1} \rho^{2k+2}}{(k+1)!} = (\pi \rho^2)^{k+1} \frac{(k+1)!}{(k+1)!}.$$
Theorem 1.1. Let $(M, \omega)$ be a rational closed symplectic manifold and $N \subset (M, \omega)$ a closed symplectic submanifold of codimension $2k > 2$. Let $(\widetilde{M}, \widetilde{\omega}_{N,\rho})$ be the symplectic blow up of $(M, \omega)$ along $N$. Then for some small values of $\rho$ the rank of $\pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_{N,\rho}))$ is positive.

Proof. Consider the loop of Hamiltonian diffeomorphisms $\{\psi_t^{C_k}\}_{0 \leq t \leq 2}$ defined above. Since each $\psi_t^{C_k}$ is $U(k)$-equivariant, it induces a loop of diffeomorphisms $\{\psi_t^{\nu}\}$ on $\nu = P \times_{U(k)} \mathbb{C}^k$. Furthermore, each diffeomorphism is supported on $P \times U(k)_B \epsilon_0^2$.

From Lemma 6.12, we have that $\{1 \times \psi_t^{\nu}\}$ is a loop of Hamiltonian diffeomorphisms on $P \times_{U(k)} \mathbb{C}^k$ with Hamiltonian function $H_t^{\nu} = 1 \times H_t^{C_k}$. Since $P \times_{U(k)} \mathbb{C}^k$ is symplectomorphic to a neighborhood $U(N)$, we consider $\{\psi_t^{\nu}\}$ as a loop of Hamiltonian diffeomorphisms on $(M, \omega)$. Denote such loop by $\{\psi_t^M\}$ with Hamiltonian function $H_t^M$. Notice that such loop is contractible in $\text{Ham}(M, \omega)$ and the Hamiltonian function is not normalized. Thus consider the function

\[
\begin{align*}
\langle \pi \rho \rangle_{\nu} := & \int_M H_t^M \omega^n = \int_{U(N)} H_t^M \omega^n. \\
\text{From its definition, the loop } & \{\psi_t^M\}_{0 \leq t \leq 2} \text{ is } N\text{-liftable. Denote by } \{\widetilde{\psi}_{N,t}\}_{0 \leq t \leq 2} \text{ the induced loop of Hamiltonian diffeomorphisms. Hence by Thm 1.2 we have that} \\
\mathcal{A}([\widetilde{\psi}_{N,\ell}]) & = \left[ \frac{1}{\text{Vol}(\widetilde{M}_N, \tilde{\omega}^{n}_{N,\rho})} \int_0^2 \int_{\Phi(\mathcal{U}(\nu_0))} H_t^{M, \text{norm}} \omega^n dt \right]. \\
\text{in } & \mathcal{P}(M, \omega) + \mathbb{Z}(\pi \rho^2). \text{ From the definition of the symplectic blow up we know that } \text{Vol}(\widetilde{M}_N, \tilde{\omega}^{n}_{N,\rho}) = \text{Vol}(M, \omega^n) - \text{Vol}(\mathcal{U}(\nu_0), \omega^{n}_{\nu,A}). \\
\text{Next we compute the integrals. To that end, by the definition of the Hamiltonian function on the neighborhood } & \Phi(\mathcal{U}(\nu_0)) \text{ of } N \text{ we have that} \\
\int_0^2 \int_{\Phi(\mathcal{U}(\nu_0))} H_t^{M, \text{norm}} \omega^n dt & = \int_0^2 \int_{\Phi(\mathcal{U}(\nu_0))} H_t^M - \frac{c_t}{\text{Vol}(M, \omega^n)} \omega^n dt \\
& = \int_0^2 \int_{\mathcal{U}(\nu_0)} 1 \times H_t^{C_k} \omega^n_{\nu,A} dt \\
& \quad - \frac{\text{Vol}(\mathcal{U}(\nu_0), \omega^n_{\nu,A})}{\text{Vol}(M, \omega^n)} \int_0^2 c_t dt. 
\end{align*}
\]
By replacing the normal bundle $\nu$ by the disk bundle $U_\rho(\nu_0)$ on the first sumand, we get by Lemma 6.11 that

$$\int_{U_\rho(\nu_0)} 1 \times H^C_t \omega^n_{\nu,A} = \int_N (\pi_N|_{U_\rho(\nu_0)})_*(1 \times H^C_t \omega^k_0) \omega^{n-k}. $$

Since

$$(\pi_N|_{U_\rho(\nu_0)})_*(1 \times H^C_t \omega^k_0) = \int_{B_\rho} H^C_t \omega^k_0,$$

it follows that

$$\int^2_0 \int_{U_\rho(\nu_0)} 1 \times H^C_t \omega^n_{\nu,A} \, dt = \frac{(\pi \rho^2)^{k+1}}{(k+1)!} \text{Vol}(N, \omega^{n-k})$$

since $\alpha(1) - \beta(1) = 1$.

Notice that $c_t$ can be rewritten as

$$c_t = \int_M H^M_t \omega^n = \int_{U(N)} H^M_t \omega^n = \int_{U(N)} 1 \times H^C_t \omega^n_{\nu,A} = \int_N (\pi_N)_*(1 \times H^C_t \omega^k_0) \omega^{n-k}. $$

Before the fiber was $(B_\rho, \omega_0)$, in this case the fiber is $(C^k, \omega_0)$ and $H^C_t$ has compact support. But on $(C^k, \omega_0)$, we know that the Calabi morphism vanishes. Therefore we have that

$$\int^2_0 c_t \, dt = \int^2_0 (\pi_N)_*(1 \times H^C_t \omega^k_0) \, dt = 0$$

and

$$\mathcal{A}(\tilde{\psi}_{N,t}) = \left[ \frac{1}{\text{Vol}(M, \omega^n)} - \frac{1}{k} \cdot \frac{\text{Vol}(N, \omega^{n-k})}{k!} \cdot (\pi \rho^2)^{k+1} \text{Vol}(N, \omega^{n-k}) \right]$$

in $\mathcal{P}(M, \omega) + \mathbb{Z}(\pi \rho^2)$. By hypothesis, $\text{Vol}(M, \omega^n), \text{Vol}(N, \omega^{n-k})$ and $\mathcal{P}(M, \omega)$ are in $\mathbb{Q}$. Therefore, it follows from the above expression that if $\rho$ is such that $\pi \rho^2$ is a transcendental number then $\mathcal{A}(\tilde{\psi}_{N,t}^m) \neq 0$ in $\mathcal{P}(M, \omega) + \mathbb{Z}(\pi \rho^2)$ for all non zero $m \in \mathbb{Z}$. Henceforth, $\pi_1(\text{Ham}(\tilde{M}, \tilde{\omega}_{N,\rho}))$ has positive rank. $\square$
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Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo No. 340, Colima, Col., Mexico 28045

Email address: andres.pedroza@ucol.mx