Effects of electric field and anisotropy on the mass-radius relationship of a particular class of compact stars

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Abstract

For a static, spherically symmetric, anisotropic and charged distribution of matter, we present a new class of exact solutions to the Einstein-Maxwell system. By assuming specific forms of the electric field and anisotropy, we transform the master equation of the Einstein-Maxwell system to Bessel and modified Bessel differential equations. The subsequent solutions turn out to be generalizations of the isotropic and uncharged stellar model of Finch and Skea (1989), the isotropic and charged stellar model of Hansraj and Maharaj (2006) and the anisotropic and charged stellar model of Maharaj et al. (2017). We analyze the physical viability of the solutions and utilize one particular class of solutions to examine the effects of charge and anisotropy on the mass-radius relationship of compact stars.

Keywords

Einstein-Maxwell system · Exact solution · Compact star · Mass-radius relationship

1 Introduction

A pulsar is a compact object where GR finds its applications. With technological advancements, accuracy in observational data from pulsars has improved immensely over the years since its first discovery. Hence, pulsars continue attracting huge research interests to theoretical astrophysicists. Studies of highly dense compact stars like pulsars, in the presence of an electromagnetic field, is possible by solving the relevant Einstein-Maxwell field equations. In a series of papers, two of the current authors have published new class of exact solutions to the Einstein-Maxwell system by employing different techniques and exploring their physical applicability in the context of superdense stars (Komathiraj and Sharma 2018; Komathiraj et al. 2019; Komathiraj and Sharma 2020, 2021). Motivation for the consideration of electromagnetic field with or without anisotropic stress has been dealt with in details in these papers. In an Einstein-Maxwell system, the gravitational attraction is counterbalanced by the Coulomb repulsion which prevents the system from collapsing to a point singularity. A wide range of solutions to the Einstein-Maxwell system was compiled by Ivanov (2002). On the other hand, Bowers and Liang (1974) for the first time examined the effects of anisotropy on compact stars which eventually prompted many investigators to develop and analyze anisotropic stellar models. Stability analysis and determination of bounds on the mass to radius ratio $M/R$ had been some of the key objectives in such investigations (Dev and Gleiser 2002, 2003; Mak and Harko 2002, 2003; Chaisi and Maharaj 2005, 2006; Maharaj and Chaisi 2006).

In this paper, in continuation with our previous works, we propose a new set of exact solutions to the Einstein-Maxwell system and utilize them to analyze the effects of charge and anisotropy on the mass-radius relationship of relativistic compact stars.

The paper has been organized as follows: In Sect. 2, we write down the Einstein-Maxwell field equations for a static, spherically symmetric, charged and anisotropic matter distribution. We obtain an equivalent system of the equations by adopting the Durgapal and Bannerji (1983) transformations. In Sect. 3, we choose a physically reasonable
form of the gravitational potential $g_{rr}$, the electric field intensity and the measure of anisotropy and obtain a master equation which becomes integrable for particular choices of one of the model parameters $\alpha$. In Sect. 4, we consider the case $\alpha = 1$ and solve the Einstein-Maxwell system in terms of elementary functions. We show that a plethora of physically reasonable stellar solutions can be regained by suitable parametrization of the new class of solutions. In Sect. 6, we present solutions for $\alpha > 1$ case. In Sect. 7, we analyze physical viability of the solutions and discuss the effects of charge and anisotropy on the mass-radius relationship of such class of stars. We conclude by summarizing our results in Sect. 8.

2 Einstein-Maxwell system

The exterior space-time of a static spherically symmetric relativistic charged fluid distribution is uniquely described by the Reissner-Nordström metric

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2d\Omega^2,$$

where $M$ and $Q$ are the total mass and charge of the configuration. The interior space-time of the charged sphere, however, is not unique. Let that the interior space-time be described by the line element (in Schwarzschild coordinates $(x^a) = (t, r, \theta, \phi)$)

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2d\Omega^2,$$

where $\nu(r)$ and $\lambda(r)$ are yet to be specified.

The matter composition of the charged sphere is assumed to be an anisotropic imperfect fluid having energy momentum tensor

$$T^i_j = \text{diag}(\rho - \frac{1}{2}E^2, p_r - \frac{1}{2}E^2, p_t + \frac{1}{2}E^2, p_t + \frac{1}{2}E^2),$$

where $\rho$ is the energy density, $p_r$ is the radial pressure, $p_t$ is the tangential pressure and $E$ electric field intensity that is measured relative to the comoving fluid 4-velocity $u^i = e^{-\nu}u_0^i$.

For the above distribution, the Einstein-Maxwell field equations are obtained as

$$\frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} = \rho + \frac{1}{2}E^2,$$

where $\sigma$ is the charge density and a prime denotes derivative with respect to the radial coordinate $r$. Here we use the unit system where the constant $8\pi G = 1$ and the speed of light $c = 1$. The system of equations determines the behaviour of the gravitational field for an anisotropic charged perfect fluid source.

The mass of the gravitating object within a stellar radius $r$ is defined as

$$m(r) = \frac{1}{2} \int_0^r \omega^2 \rho(\omega) d\omega.$$

To solve the system, we invoke the following transformation

$$A^2y^2(x) = e^{2\nu(r)}, \quad Z(x) = e^{-2\lambda(r)}, \quad x = Cr^2,$$

where $A$ and $C$ are arbitrary constants. The above transformation was first used by Durgapal and Bannerji (1983) for the development of a compact relativistic star. Under the transformation, the system (4)–(7) takes the form

$$\frac{1 - Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{E^2}{2C},$$

$$4Z\frac{\dot{y}}{y} + \frac{Z - 1}{x} = \frac{p_r}{C} - \frac{E^2}{2C},$$

$$4Z\dot{x}^2 + 2\dot{Z}x^2 + \left(\dot{Z}x - Z + 1 - \frac{E^2x}{C} - \frac{\Delta x}{C}\right)y = 0,$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}(x\dot{E} + E)^2,$$

where a dot (.) denotes differentiation with respect to the variable $x$. The quantity $\Delta = p_t - p_r$ is defined as the measure of anisotropy. The mass function (8) takes the form

$$m(x) = \frac{1}{4C^3/2} \int_0^x \sqrt{\rho(\omega)} d\omega.$$

As the system (10)–(13) comprises four equations in seven unknowns $Z$, $y$, $\rho$, $p_r$, $\Delta$, $E$ and $\sigma$, one needs to fix three of these variables at this stage to solve the system.
3 Technique used to generate new solutions

We plan to solve the system by specifying physically reasonable forms of one of the gravitational potentials $Z$, radial variation of the electric field $E^2$ and measure of anisotropy $\Delta$ in such a manner that the system becomes integrable and provides a viable model of an anisotropic charged superdense star. In our approach, we choose

$$Z = \frac{1}{1 + ax},$$

(15)

$$\frac{E^2}{C} = \frac{a^2(\alpha - \beta) x}{(1 + ax)^2},$$

(16)

$$\frac{\Delta}{C} = \frac{a^2 \beta x}{(1 + ax)^2} + \frac{a \gamma x \dot{y}}{(1 + ax)^2} \dot{y},$$

(17)

where $a$, $\alpha$, $\beta$, and $\gamma$ are real constants. Note that for $y = 0$, (15)–(17) are similar to the choices studied previously by Maharaj et al. (2017) which is a generalization of the stellar models developed earlier by Hansraj and Maharaj (2006) and Finch and Skea (1989). It is important to note that the solutions of Maharaj et al. (2017) are restricted for specific values of $a$ such that $\frac{a}{2} + 1$ is a half-integer (i.e., $a = -1, 1, 3, \ldots$). We demonstrate in following sections that one can accommodate a much wider range of values than previously used to generate solutions. It should also be stressed that the choice of $Z$ is a generalization of some earlier approaches. For example, setting $a = 1$, one regains the potential form considered earlier by Sharma et al. (2017). The form of $E^2$ in (16) is physically reasonable as it remains regular and continuous throughout the sphere. Similarly, the form (17) chosen for $\Delta$ ensures that anisotropy vanishes at the centre of the star (i.e., $p_r = p_t$ at the centre).

Substituting (15)–(17) into (12), we obtain

$$4(1 + ax)\ddot{y} - a(2 + \gamma)\dot{y} + a^2(1 - \alpha) y = 0,$$

(18)

which, under the transformation

$$1 + ax = X, \quad y(x) = Y(X),$$

(19)

helps us to write the equation in a more convenient form

$$4X \frac{d^2 Y}{dX^2} - (2 + \gamma) \frac{dY}{dX} + (1 - \alpha) Y = 0.$$

(20)

Equation (20) is the master equation of the system which should be integrable so as to find an exact solution for an anisotropic charged sphere. We note that it is possible to generate three categories of solutions to (20) in terms of different values of the parameter $\alpha$. The three possible cases are

$$\alpha = 1, \quad 0 \leq \alpha < 1, \quad \alpha > 1.$$

In the following section, we take up these cases separately.

4 Case I: $\alpha = 1$

In this case, equation (20) is easily integrable and the solution is obtained as

$$y(x) = \frac{4(1 + ax)^{6+\gamma}}{6 + \gamma} + c_2,$$

(21)

where $c_1$ and $c_2$ are constants. The complete set of solutions to the Einstein-Maxwell system along with physical quantities are given below:

$$e^{2\nu} = 1 + ax,$$

(22)

$$e^{2\nu} = A^2 \left( \frac{c_1 4(1 + ax)^{6+\gamma}}{6 + \gamma} + c_2 \right)^2,$$

(23)

$$\rho = a[6 + ax(1 + \beta)]$$

(24)

$$\frac{p_r}{C} = \frac{a}{2} \left( \frac{2}{1 + ax} + \frac{ax(1 - \beta)}{(1 + ax)^2} \right),$$

(25)

$$\frac{p_t}{C} = \frac{a}{2} \left( \frac{2}{1 + ax} + \frac{ax(1 + \beta)}{(1 + ax)^2} \right),$$

(26)

$$\frac{E^2}{C} = \frac{a^2 (1 - \beta) x}{(1 + ax)^2},$$

(27)

$$\frac{\sigma^2}{C} = \frac{a^2 C (1 - \beta)(3 + ax)^2}{(1 + ax)^3},$$

(28)

$$\frac{\Delta}{C} = \frac{a^2 \beta x}{(1 + ax)^2} + \frac{c_1 a^2 \gamma (6 + \gamma) x}{4c_1 (1 + ax)^3 + c_2 (6 + \gamma)(1 + ax)^{\frac{6+\gamma}{2}}}.$$  

(29)

It is interesting to note that for $\beta = 1$, the electric field vanishes and we obtain an uncharged anisotropic model. The form of the exact solution (22)–(29) has a similar form as that of Maharaj et al. (2017) which follows for $\gamma = 0$. In other words, this solution may be regarded as a generalization of the Maharaj et al. (2017) charged stellar model.

5 Case II: $0 \leq \alpha < 1$

For $0 \leq \alpha < 1$, equation (20) becomes relatively difficult to solve. However, the master equation can be transformed to a
standard Bessel differential equation if the following transformation is introduced:

\[ Y(X) = X^d U(X), \quad (30) \]

where \( d \) is a constant. A similar kind of transformation may be found in references (Hansraj and Maharaj 2006; Komathiraj and Maharaj 2007; Maharaj and Thirukkanesh 2009; Komathiraj and Sharma 2018) where solutions for charged stars were obtained without considering any anisotropic stress. With the help of (30), the differential equation (20) can be written as

\[
4X^2 \frac{d^2 U}{dX^2} + X[2(4d - 1) - \gamma] \frac{dU}{dX} + [d(4d - 6 - \gamma) + (1 - \alpha)X]U = 0.
\]

(31)

By introducing yet another transform \( W = X^k \), where \( k \) is a constant, we obtain

\[
4k^2W^2 \frac{d^2 U}{dW^2} + kW[4(k - 1) + 2(4d - 1) - \gamma] \frac{dU}{dW} + [d(4d - 6 - \gamma) + (1 - \alpha)W^{1/k}]U = 0.
\]

(32)

A substantial simplification of the above is possible if we choose

\[ k = \frac{1}{2} \quad \text{and} \quad d = \frac{6 + \gamma}{8}. \]

Equation (32) then takes the form

\[
W^2 \frac{d^2 U}{dW^2} + W \frac{dU}{dW} + \left[ (1 - \alpha)W^2 - \left( \frac{6 + \gamma}{4} \right)^2 \right] U = 0.
\]

(33)

By introducing the transformation

\( (1 - \alpha)W^2 = V^2 \),

we obtain

\[
V^2 \frac{d^2 U}{dV^2} + V \frac{dU}{dV} + \left[ V^2 - \left( \frac{6 + \gamma}{4} \right)^2 \right] U = 0,
\]

(34)

which is the Bessel equation of order \( \frac{6 + \gamma}{4} \). In general, the differential equation (34) has linearly independent solutions \( J_{\frac{6+\gamma}{4}}(V) \) and \( J_{-\frac{6+\gamma}{4}}(V) \) which are Bessel functions. The general solution to (34) can, therefore, be written as

\[
U = b_1 J_{\frac{6+\gamma}{4}}(V) + b_2 J_{-\frac{6+\gamma}{4}}(V),
\]

(35)

where \( b_1 \) and \( b_2 \) are constants. It is well known that the Bessel functions of half-integer order can be written in terms of the elementary functions. Consequently, for \( \gamma = -4, 0, 4, \ldots \), the solution (34) can be written as \( J_{\frac{1}{2}} \), \( J_{-\frac{1}{2}} \), \( J_{\frac{1}{2}} \), \( J_{-\frac{1}{2}} \), \( J_{\frac{1}{2}} \), \( J_{-\frac{1}{2}} \), 

... Let us consider the cases \( \gamma = -4, 0, 4 \) as under:

5.1 Model I: \( \gamma = -4 \)

For \( \gamma = -4 \), the solution (35) can be written as

\[
U = b_1 J_{\frac{1}{2}}(V) + b_2 J_{-\frac{1}{2}}(V),
\]

(36)

where

\[
J_{\frac{1}{2}}(V) = \frac{2}{\pi V} \sin V,
\]

\[
J_{-\frac{1}{2}}(V) = -\frac{2}{\pi V} \cos V.
\]

The general solution to (20) is obtained as

\[
y(x) = (1 - \alpha)^{-1/4}(c_1 \sin f + c_2 \cos f),
\]

(37)

where, we set \( f = \sqrt{(1 - \alpha)(1 + \alpha x)} \), \( c_1 = \frac{2}{\pi}b_1 \) and \( c_2 = -\frac{2}{\pi}b_2 \). Subsequently, the solution to the Einstein-Maxwell system takes the form

\[
e^{2\varphi} = 1 + ax,
\]

(38)

\[
e^{2\psi} = A^2(1 - \alpha)^{-1/2}(c_1 \sin f + c_2 \cos f)^2.
\]

(39)

\[
\rho = \frac{a[6 + ax(2 - \alpha + \beta)]}{2(1 + ax)^2},
\]

(40)

\[
p_r = \frac{a}{2} \left[ \frac{-2}{1 + ax} + \frac{a(\alpha - \beta)x}{(1 + ax)^2} \right. \\

\left. + \frac{4(1 - \alpha)(c_1 - c_2 \tan f)}{(1 + ax)f(c_1 \tan f + c_2)} \right],
\]

(41)

\[
p_t = \frac{a}{2} \left[ \frac{-2}{1 + ax} + \frac{a(\alpha + \beta)x}{(1 + ax)^2} + \frac{4(1 - \alpha)(c_1 - c_2 \tan f)}{(1 + ax)f(c_1 \tan f + c_2)} \right]

- 2a^2(1 - \alpha)(c_1 - c_2 \tan f) f(c_1 \tan f + c_2)

\]

(42)

\[
E^2 = \frac{a^2(\alpha - \beta)x}{(1 + ax)^2},
\]

(43)

\[
\sigma^2 = \frac{a^2C(\alpha - \beta)(3 + ax)^2}{(1 + ax)^3},
\]

(44)

\[
\Delta = \frac{a^2Bx}{(1 + ax)^2} - \frac{2a^2(1 - \alpha)x(c_1 - c_2 \tan f)}{(1 + ax)^2f(c_1 \tan f + c_2)}.
\]

(45)

This is a new solution. We note that for \( \alpha = \beta \), the model (38)–(45) reduces to a solution for an uncharged anisotropic star. Extensive analyses of anisotropic stellar
models ($\Delta \neq 0$) have been carried out in the past by investigators which includes the works of Dev and Gleiser (2002, 2003), Mak and Harko (2002, 2003), Chaisi and Maharaj (2005, 2006) and Maharaj and Chaisi (2006), amongst others.

### 5.2 Model II: $\gamma = 0$

For $\gamma = 0$, (35) yields

$$U = b_1 J_1(V) + b_2 J_{-1}(V),$$  \hspace{1cm} (46)

where,

$$J_1(V) = \sqrt{\frac{2}{\pi V}} \left[ \sin V - \cos V \right],$$

$$J_{-1}(V) = -\sqrt{\frac{2}{\pi V}} \left[ \cos V + \sin V \right].$$

The general solution in this case takes the form

$$y(x) = (1 - a)^{-3/2}[c_1(\sin f - f \cos f)$$

$$+ c_2(\cos f + f \sin f)],$$  \hspace{1cm} (47)

where $f$, $c_1$ and $c_2$ are as in Sect. 5.1. The complete set of exact solutions to the Einstein-Maxwell system (10)–(13) can be written as

$$e^{2b} = 1 + ax,$$  \hspace{1cm} (48)

$$e^{2\nu} = A^2 (1 - \alpha)^{-3/2}[c_1(\sin f - f \cos f)$$

$$+ c_2(\cos f + f \sin f)]^2,$$  \hspace{1cm} (49)

$$\rho = \frac{a[6 + ax(2 - \alpha + \beta)]}{2(1 + ax)^2},$$  \hspace{1cm} (50)

$$p_r = \frac{a}{2} \left( \frac{-2}{1 + ax} + \frac{a(\alpha - \beta)}{(1 + ax)^2} \right) \left[ 2a(1 - \alpha)(c_1 \tan f + c_2) \right.$$ 

$$\left. + \frac{1}{(1 + ax)[(c_2 - c_1) f + (c_1 + c_2) f \tan f]} \right].$$  \hspace{1cm} (51)

$$p_t = \frac{a}{2} \left( \frac{-2}{1 + ax} + \frac{a(\alpha + \beta)}{(1 + ax)^2} \right) \left[ 2a(1 - \alpha)(c_1 \tan f + c_2) \right.$$ 

$$\left. + \frac{1}{(1 + ax)[(c_2 - c_1) f + (c_1 + c_2) f \tan f]} \right].$$  \hspace{1cm} (52)

$$\frac{E^2}{C} = \frac{a^2(\alpha - \beta)x}{(1 + ax)^2},$$  \hspace{1cm} (53)

$$\frac{\sigma^2}{C} = \frac{a^2C(\alpha - \beta)(3 + ax)^2}{(1 + ax)^3},$$  \hspace{1cm} (54)

$$\frac{\Delta}{C} = \frac{a^2 \beta x}{(1 + ax)^2}.$$  \hspace{1cm} (55)

It should be stressed that the new solution (48)–(55) holds good for isotropic as well as anisotropic; charged as well as uncharged cases unlike the solution presented in Sect. 5.1. Moreover, solution (48)–(55) cannot be regained from Maharaj et al. (2017) charged anisotropic stellar solution except for $a = 1$. This implies that our model is the generalization of the Maharaj et al. (2017) stellar model.

Another interesting feature of the new solutions is that several physically reasonable stellar models can be regained from the general class of solutions (47). To demonstrate this by the following examples:

#### 5.2.1 Maharaj et al. (2017) stellar model

If we set $a = 1$, then (47) can be written as

$$y(x) = (1 - \alpha)^{-3/2}[c_1(\sin \sqrt{(1 - \alpha)(1 + x)}$$

$$- \sqrt{(1 - \alpha)(1 + x)} \cos \sqrt{(1 - \alpha)(1 + x)})$$

$$+ c_2(\cos \sqrt{(1 - \alpha)(1 + x)}$$

$$+ \sqrt{(1 - \alpha)(1 + x)} \sin \sqrt{(1 - \alpha)(1 + x)})].$$  \hspace{1cm} (56)

The solution (56) corresponds to the stellar model of Maharaj et al. (2017) which is the generalization of neutron star model of Finch and Skea (1989).

#### 5.2.2 Hansraj and Maharaj (2006) stellar model

If we set $a = 1$ and $\beta = 0$ in (47), we obtain

$$y(x) = (1 - \alpha)^{-3/2}[c_1(\sin \sqrt{(1 - \alpha)(1 + x)}$$

$$- \sqrt{(1 - \alpha)(1 + x)} \cos \sqrt{(1 - \alpha)(1 + x)})$$

$$+ c_2(\cos \sqrt{(1 - \alpha)(1 + x)}$$

$$+ \sqrt{(1 - \alpha)(1 + x)} \sin \sqrt{(1 - \alpha)(1 + x)})].$$  \hspace{1cm} (57)

which is the Hansraj and Maharaj (2006) stellar model for a superdense charged star. The form (57) has a similar structure to (56); however, it is important to note that the solution (48)–(55) is different from Maharaj et al. (2017) stellar model due to $\Delta = 0$ (or $\beta = 0$).

#### 5.2.3 Finch and Skea (1989) stellar model

If we set $a = 1$, $\beta = 0$ and $\alpha = 0$, (47) reduces to

$$y(x) = [c_1(\sin \sqrt{(1 + x)} - \sqrt{(1 + x)} \cos \sqrt{(1 + x)})$$

$$+ c_2(\cos \sqrt{(1 + x)} + \sqrt{(1 + x)} \sin \sqrt{(1 + x)})].$$  \hspace{1cm} (58)
5.3 Model III: \( \gamma = 4 \)

For \( \gamma = 4 \), we have
\[
U = b_1 J_\frac{3}{2}(V) + b_2 J_{-\frac{3}{2}}(V),
\]
where
\[
J_\frac{3}{2}(V) = \sqrt{\frac{2}{\pi V}} \left[ \frac{3 \sin V}{V^2} - \frac{3 \cos V}{V} - \sin V \right],
\]
\[
J_{-\frac{3}{2}}(V) = -\sqrt{\frac{2}{\pi V}} \left[ \frac{3 \cos V}{V^2} - \frac{3 \sin V}{V} + \cos V \right].
\]

The general solution to the differential equation (20) in this case is obtained as
\[
y(x) = (1 - \alpha)^{-3/4} [c_1 (3 \sin f - 3 f \cos f - f^2 \sin f) + c_2 (3 \cos f - 3 f \sin f + f^2 \cos f)],
\]
where, \( f, c_1 \) and \( c_2 \) are as given in previous sections. Subsequently, the complete set of solutions (10)–(13) can be written as

\[
\begin{align*}
\psi & = \bar{V} + \frac{1}{2} \left( \frac{2}{1 + x^2} \right) W, \\
\Delta & = \frac{8a^2 x f (1 + ax)^{-3} [(c_1 f^2 + c_2 f) - (c_1 f + c_2 (-7 + ax(\alpha - 1) + \alpha)) \tan f]}{(3c_1 f + c_2(-4 + ax(\alpha - 1) + \alpha)) - (-3c_2 f + c_1(2 + ax(\alpha - 1) + \alpha)) \tan f} + \frac{a^2 \beta x}{(1 + ax)^2},
\end{align*}
\]

This is a new solution to the Einstein-Maxwell system in terms of elementary functions. The charged and anisotropic solution (61)–(68) does not have an isotropic analogue as the measure of anisotropy \( \Delta \) cannot be vanished as in Sect. 5.1.

6 Case III: \( \alpha > 1 \)

We now consider the case \( \alpha > 1 \) and write the differential equation (20) as
\[
4X \frac{d^2 Y}{dX^2} - (2 + \gamma) \frac{dY}{dX} - (\alpha - 1) Y = 0.
\]

The integration process of (69) is similar to that in Sect. 5. Equation (69) can be written as
\[
W^2 \frac{d^2 U}{dW^2} + W \frac{dU}{dW} - \left[ \psi^2 W^2 + \left( \frac{6 + \gamma}{4} \right)^2 \right] U = 0,
\]
where \( \psi^2 = \alpha - 1 \). If we now introduce a new independent variable \( \bar{V} = \tilde{V} \), equation (70) takes the form
\[
\tilde{V}^2 \frac{d^2 U}{d\tilde{V}^2} + \tilde{V} \frac{dU}{d\tilde{V}} - \left[ \tilde{V}^2 + \left( \frac{6 + \gamma}{4} \right)^2 \right] U = 0.
\]
which is the modified Bessel equation of order \( \frac{6+\nu}{4} \). In general, the differential equation (71) has linearly independent solutions \( I_{\frac{6+\nu}{2}}(\nu) \) and \( I_{\frac{6-\nu}{2}}(\nu) \) which are modified Bessel functions. The general solution to (71) can, therefore, be written as

\[
U = b_1 I_{\frac{6+\nu}{2}}(\nu) + b_2 I_{\frac{6-\nu}{2}}(\nu),
\]

where \( b_1 \) and \( b_2 \) are constants. It is well known that the modified Bessel functions of half-integer order can be written in terms of the hyperbolic functions. We consider the cases \( \gamma = -4, \ 0, \ 4 \) as in Sect. 5.

### 6.1 Model IV: \( \gamma = -4 \)

For \( \gamma = -4 \), the solution (72) takes the form

\[
U = b_1 I_{\frac{1}{2}}(\nu) + b_2 I_{-\frac{1}{2}}(\nu),
\]

where,

\[
I_{\frac{1}{2}}(\nu) = \sqrt{\frac{2}{\pi \nu}} \sinh \nu, \\
I_{-\frac{1}{2}}(\nu) = \sqrt{\frac{2}{\pi \nu}} \cosh \nu.
\]

The general solution to (20) is obtained as

\[
y(x) = (\alpha - 1)^{-1/4}(c_1 \sinh g + c_2 \cosh g),
\]

where, we set \( g = \sqrt{(\alpha - 1)(1 + ax)} \), \( c_1 = \sqrt{\frac{2}{\pi \nu}} b_1 \) and \( c_2 = \sqrt{\frac{2}{\pi \nu}} b_2 \). Subsequently, the complete solutions to the Einstein-Maxwell system (10)–(13) can be written as

\[
e^{2\nu} = 1 + ax,
\]

\[
e^{2\nu} = A^2 (\alpha - 1)^{-1/2}(c_1 \sinh g + c_2 \cosh g)^2,
\]

\[
\rho \frac{\dot{r}}{C} = \frac{a[6 + ax(2 - \alpha + \beta)]}{2(1 + ax)^2},
\]

\[
p_r \frac{\dot{t}}{C} = \frac{a}{2} \left[ \frac{-2}{1 + ax} + \frac{a(\alpha - \beta) x}{(1 + ax)^2} \right] + \frac{4}{1 + ax} \frac{\dot{y}}{y},
\]

\[
p_t \frac{\dot{t}}{C} = \frac{a}{2} \left[ \frac{-2}{1 + ax} + \frac{a(\alpha + \beta) x}{(1 + ax)^2} \right] + \frac{4}{1 + ax} \frac{\dot{y}}{y},
\]

\[
E^2 \frac{\dot{y}}{C} = \frac{a^2(\alpha - \beta) x}{(1 + ax)^2},
\]

\[
\sigma^2 \frac{\dot{y}}{C} = \frac{a^2 C(\alpha - \beta)(3 + ax)^2}{(1 + ax)^2},
\]

\[
\Delta \frac{\dot{y}}{C} = \frac{a^2 \beta x}{(1 + ax)^2} - \frac{4ax}{(1 + ax)^2} \frac{\dot{y}}{y},
\]

where \( y \) is given in (74). Equations (75)–(82) represent new exact solutions to the Einstein-Maxwell system (10)–(13) in terms of hyperbolic functions for a charged fluid in the presence of anisotropic pressure.

### 6.2 Model V: \( \gamma = 0 \)

For \( \gamma = 0 \), the solution (72) becomes

\[
U = b_1 I_{\frac{1}{2}}(\nu) + b_2 I_{-\frac{1}{2}}(\nu),
\]

where,

\[
J_{\frac{1}{2}}(\nu) = \sqrt{\frac{2}{\pi \nu}} \left[ -\sinh \nu + \cosh \nu \right], \\
I_{-\frac{1}{2}}(\nu) = \sqrt{\frac{2}{\pi \nu}} \left[ -\cosh \nu + \sinh \nu \right].
\]

The general solution of equation (20) is obtained as

\[
y(x) = (\alpha - 1)^{-3/4}[c_1 (g \cosh g - \sinh g)
\]

\[
+ c_2 (g \sinh g - \cosh g)],
\]

and the complete set of solutions to the Einstein-Maxwell system (10)–(13) can be written as

\[
e^{2\nu} = 1 + ax,
\]

\[
e^{2\nu} = A^2 (\alpha - 1)^{-3/2}[c_1 (g \cosh g - \sinh g)
\]

\[
+ c_2 (g \sinh g - \cosh g)]^2,
\]

\[
\rho \frac{\dot{r}}{C} = \frac{a}{2} \left[ \frac{-2}{1 + ax} + \frac{a(\alpha - \beta) x}{(1 + ax)^2} \right] + \frac{4}{1 + ax} \frac{\dot{y}}{y},
\]

\[
p_r \frac{\dot{t}}{C} = \frac{a}{2} \left[ \frac{-2}{1 + ax} + \frac{a(\alpha + \beta) x}{(1 + ax)^2} \right] + \frac{4}{1 + ax} \frac{\dot{y}}{y},
\]

\[
E^2 \frac{\dot{y}}{C} = \frac{a^2(\alpha - \beta) x}{(1 + ax)^2},
\]

\[
\sigma^2 \frac{\dot{y}}{C} = \frac{a^2 C(\alpha - \beta)(3 + ax)^2}{(1 + ax)^2},
\]

\[
\Delta \frac{\dot{y}}{C} = \frac{a^2 \beta x}{(1 + ax)^2} - \frac{4ax}{(1 + ax)^2} \frac{\dot{y}}{y},
\]

where \( y \) is given in (84). Equations (85)–(92) provide new exact solutions to the Einstein-Maxwell system expressed in terms of hyperbolic functions. This, in fact, is a generalization of the Maharaj et al. (2017) model which can be regained by setting \( a = 1 \). For \( a = 1 \) and \( \beta = 0 \), the system becomes isotropic and we regain the charged isotropic stellar model developed by Hansraj and Maharaj (2006).
6.3 Model VI: $\gamma = 4$

For $\gamma = 4$, (72) takes the form

$$U = b_1 I_{\frac{1}{2}}(\tilde{V}) + b_2 I_{-\frac{1}{2}}(\tilde{V}),$$

where,

$$I_{\frac{1}{2}}(\tilde{V}) = \frac{2}{\pi \sqrt{V}} \left[ \frac{3 \sin \tilde{V}}{V^{3/2}} - \frac{3 \cos \tilde{V}}{V} + \sin \tilde{V} \right],$$

$$I_{-\frac{1}{2}}(\tilde{V}) = \frac{2}{\pi \sqrt{V}} \left[ \frac{3 \cos \tilde{V}}{V^{3/2}} - \frac{3 \sin \tilde{V}}{V} + \cos \tilde{V} \right].$$

The general solution to (20) is obtained as

$$y(x) = (\alpha - 1)^{-5/4} \left[ c_1 (3 \sinh g - 3g \cosh g + g^2 \sinh g) + c_2 (3 \cosh g - 3g \sinh g + g^2 \cosh g) \right].$$

(94)

Subsequently, the complete solution to the Einstein-Maxwell system (10)–(13) takes the form

$$e^{2\lambda} = 1 + ax,$$

$$e^{2\varphi} = A^2 (\alpha - 1)^{-5/2} \left[ c_1 (3 \sinh g - 3g \cosh g + g^2 \sinh g) + c_2 (3 \cosh g - 3g \sinh g + g^2 \cosh g) \right]^{2},$$

$$\rho = \frac{a[6 + ax(2 - \alpha + \beta)]}{2(1 + ax)^2},$$

$$\frac{p_r}{C} = \frac{a}{2} \left[ \frac{-2}{1 + ax} + \frac{a(\alpha - \beta) x}{(1 + ax)^2} \right] + \frac{4}{(1 + ax)^2} \frac{\dot{y}}{y},$$

$$\frac{p_t}{C} = \frac{a}{2} \left[ \frac{-2}{1 + ax} + \frac{a(\alpha + \beta) x}{(1 + ax)^2} \right] + \frac{4(1 + 2ax)}{(1 + ax)^2} \frac{\dot{y}}{y},$$

$$\frac{E^2}{C} = \frac{a^2 (\alpha - \beta) x}{(1 + ax)^2},$$

$$\frac{\alpha^2}{C} = \frac{a^2 C (\alpha - \beta)(3 + ax)^2}{(1 + ax)^3},$$

$$\frac{\Delta}{C} = \frac{a^2 \beta x}{(1 + ax)^2} + \frac{4ax}{(1 + ax)^2} \frac{\dot{y}}{y},$$

where $y$ is given by (94). This is also a new category of solutions.

In the following section, we explore the physical viability of the new solutions.

7 Physical viability

To examine the physical viability of the new class of solutions, let us consider a particular class of solutions (48)–(55) obtained in the Sect. 5.2 as it already contains some of the previously developed realistic stellar models (Maharaj et al. 2017; Hansraj and Maharaj 2006; Finch and Skea 1989). First, we need to choose the value of $a$ in such a manner that the energy density $\rho$, the radial pressure $p_r$ and the tangential pressure $p_t$ remain positive. The choice of $a$ must also ensure that the gravitational potential $e^{2\lambda}$ remains positive as the other potential $e^{2\varphi}$ is necessarily positive.

In (48) and (49), we note that $(e^{2\lambda})_{r=0} = (e^{2\varphi})_{r=0} = 0$, and $e^{2\lambda(0)} = 1, e^{2\varphi(0)} = A^2(1 - \alpha)^{-5/2} \left[ c_1 (\sin \sqrt{1 - \alpha} - \sqrt{1 - \alpha} \cos \sqrt{1 - \alpha}) + c_2 (\cos \sqrt{1 - \alpha} + \sqrt{1 - \alpha} \sin \sqrt{1 - \alpha}) \right]^2$ is constant. This confirms that the gravitational potentials (48) and (49) are regular at the centre of the star $r = 0$.

Using Equation (50), we obtain the central density $\rho_0 = (\rho)_{r=0}$ which implies that $aC > 0$. Using (51) and (52), we obtain the radial and tangential pressures at $r = 0$ as

$$(p_r)_{r=0} = (p_t)_{r=0} = -aC,$$

$$\frac{E^2}{C} = \frac{a^2 (\alpha - \beta) x}{(1 + ax)^2},$$

$$\frac{\alpha^2}{C} = \frac{a^2 C (\alpha - \beta)(3 + ax)^2}{(1 + ax)^3},$$

$$\frac{\Delta}{C} = \frac{a^2 \beta x}{(1 + ax)^2} + \frac{4ax}{(1 + ax)^2} \frac{\dot{y}}{y},$$

where $y$ is given by (94). This is also a new category of solutions.

In the following section, we explore the physical viability of the new solutions.
The matching conditions determine the constants $A$ as

$$A = \frac{(1 - \alpha)^{3/4}}{\sqrt{1 + ax}} \times \left[ c_1 \sin(fR) - fR \cos(fR) \right] + c_2 \left[ \cos(fR) + fR \sin(fR) \right]^{-1}. \quad (105)$$

Utilizing the above results, we analyze physical viability of the solution (48)–(55) graphically for a given set of choices $\alpha = 0.9$, $c_1 = C = 1$, and $\alpha = 0.5$, $\beta = 0.2$ over the interval $0 \leq r \leq 1$. Using these values in (104) and (105), we determine the remaining constants as $c_2 = 0.799491$ and $A = 0.31395$ which are consistent with the bound (103).

Figures 1–2 show that the gravitational potentials are continuous, regular and well-behaved in the interior of the star. The energy density is a decreasing function of $r$ i.e., $\frac{d\rho}{dr} < 0$ within the star as shown in Fig. 3. Figure 4 shows that the radial pressure and the tangential pressure are continuous and monotonically decreasing towards the surface of the star and the radial pressure vanishes at the boundary $r = 1$. Radial variation of the electric field intensity $E^2$ is shown in Fig. 5 which is similar to our observation in Komathiraj et al. (2019), Komathiraj and Sharma (2020, 2021). The proper charge density $\sigma^2$ is given in Fig. 6 which is positive and monotonically decreasing. In Fig. 7, we show the fall-off behaviour of the anisotropy factor $\Delta$. The anisotropic factor is positive and monotonically increases from the centre until it attains a maximum value at the boundary of the stellar object as expected (Lemaitre 1933). Fulfillment of energy conditions $\rho + p_r + 2p_t > 0$ and $\rho - p_r - 2p_t > 0$ are also shown to be satisfied in Fig. 8. In Fig. 9 we show that respective values of the radial and tangential speed of sound $v_r^2 = \frac{dp_r}{d\rho}$ and $v_t^2 = \frac{dp_t}{d\rho}$ remain less than the speed of light $c = 1$ throughout the interior of the star which implies that the causality condition is satisfied in this model (Delgaty and Lake 1998). The quantity $v_t^2 - v_r^2$ in Fig. 9 is always positive and bounded by unity which shows that the model is stable (Herrera 1992; Abreu et al. 2007). In Fig. 10, we note that both radial adiabatic index $\Gamma_r = \frac{\rho + p_r}{p_r} \frac{dp_r}{d\rho}$ and transverse adiabatic index $\Gamma_t = \frac{\rho + p_t}{p_t} \frac{dp_t}{d\rho}$ are greater than $4/3$ which is the requirement for a stable configuration. The mass function is zero at the centre and increases outward as shown in Fig. 11. In summary, we demonstrate that there exists particular set of values for which the solution (48)–(55) satisfies all the necessary requirements of a realistic star.

7.1 Mass-radius relationship

A particular objective of the current investigation is to analyze the effects of charge and anisotropy on the mass-radius ($M - R$) relationship of compact stars. With observational data providing more and more precisional measurements of masses and radii of pulsars, one hopes to get better understanding of the equation of state (EOS) of such compositions and other factors that can affect the compactness of such stars. In an attempt to examine the impacts of electric field and anisotropy on compactness, we use a particular solution obtained in this paper and adopt numerical techniques to generate the mass-radius relationship.
We assume a reasonable value of the surface density (we choose $\rho(r = R) = 7.5 \times 10^{14}$ gm/cm$^3$) to obtain the mass-radius ($M - R$) relationship for two sets of values (Set-I and Set-II) as shown in Figs. 12 and 13, respectively. We note that the presence of charge leads to an accumulation of more mass within a given radius both in isotropic and anisotropic cases, the compactness being greater in isotropic cases. For an isotropic object, the compactness appears to increase with the increase in the electric field strengths. On the other hand, the presence of anisotropy reduces the compactness of charged compact objects. The effects become more pronounced for comparatively larger values of electric field and anisotropic stress, as shown in Fig. 13. Our study reconfirms Ratanpal et al. (2017) earlier claims that electric field and/or anisotropy can serve as tuning parameters to fine-tune the mass-radius relationship of compact stars.
8 Discussion

In this paper, by introducing a more general form of the electric field and the anisotropic factor than considered earlier, we have managed to generate a much broader class of solutions to the Einstein-Maxwell system. The advantage of the new class of solutions is that the general form of the closed-form solutions can be used to study all possible compositions (isotropic and uncharged, isotropic and charged, anisotropic and uncharged and anisotropic and charged). This facilitates an analysis of the impacts of electric field and anisotropy on the mass-radius relationship of compact stars.

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Declarations

Competing interests The authors declare no competing interests.

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