Borsuk-Ulam theorem for the loopspace of a sphere

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Abstract
We study some Borsuk-Ulam type results for the loopspace of an euclidean sphere without loops equal to their inverses.

1 Introduction.
Let $s_0 \in \mathbb{S}^n$ and
$$\Omega \mathbb{S}^n := \{ \alpha : I \to \mathbb{S}^n : \alpha(0) = \alpha(1) = s_0 \}.$$  

The group $\mathbb{Z}_2$ acts on $\Omega \mathbb{S}^n$ by
$$\alpha \mapsto \alpha^*, \quad \alpha^*(t) := \alpha(1 - t).$$

The set of fixed points of this action is denoted by
$$\Omega_{tf} := \{ \alpha \in \Omega \mathbb{S}^n : \alpha = \alpha^* \},$$
where tf is an abbreviation for "to and fro".

For any mapping $f: \Omega \mathbb{S}^n \setminus \Omega_{tf} \to \mathbb{R}^k$ denote by $A_f$ the set
$$A_f := \{ \alpha \in \Omega \mathbb{S}^n \setminus \Omega_{tf} : f(\alpha) = f(\alpha^*) \}.$$  

Main result is the following theorem:

**Theorem 1** (A) If $k < n$ then for every $f: \Omega \mathbb{S}^n \setminus \Omega_{tf} \to \mathbb{R}^k$  \hspace{1em} $\dim A_f = \infty$.

(B) If $k = n$ then for every $f: \Omega \mathbb{S}^n \setminus \Omega_{tf} \to \mathbb{R}^k$  \hspace{1em} $A_f \neq \emptyset$.

\footnote{2000 Mathematics Subject Classification: 55M20}
Proof is based on the classical Borsuk-Ulam theorem \cite{1} and on the Jaworowski-Nakaoka theorem \cite{4, 5}. The next proposition needs the following

**Lemma 1** The inclusion $\Omega X \setminus \Omega_{tf} \hookrightarrow \Omega X$ is a homotopy equivalence for every space $X$ with a nontrivial path from its base point.

Denote by $w_1$ the first Stiefel-Whitney class of the double covering $p : E \to B$ with

$$E := \Omega \mathbb{S}^n \setminus \Omega_{tf} \text{ and } B := E/\mathbb{Z}_2.$$ 

It appears that $(w_1)^{n-1} \neq 0$ and

**Proposition 1** If $n \geq 3$ then $(w_1)^n = 0$.

**Corollary 1** There is no an equivariant mapping $\mathbb{S}^n \to \Omega \mathbb{S}^n \setminus \Omega_{tf}, \ (n \geq 3)$.

**Corollary 2** The proof of $A_f \neq \emptyset$ based on the commutative diagram

$$\begin{array}{ccc}
E & \xrightarrow{\varphi} & \mathbb{S}^{k-1} \\
p & \downarrow & \downarrow \gamma \\
B & \xrightarrow{\bar{\varphi}} & \mathbb{R}P^{k-1}
\end{array}$$

with $\varphi(\alpha) = ||f(\alpha) - f(\alpha^*)||^{-1}(f(\alpha) - f(\alpha^*))$ and on the contradiction:

$$0 = \bar{\varphi}^* ((w_1(\gamma)^k) = (w_1)^k \neq 0$$

works for $k < n$ only, giving a weaker result than Theorem 1.

## 2 Proof of Lemma 1.

Without loss of generality consider $X := \mathbb{S}^n$. We start with the description of the way, how to push a loop off the set $\Omega_{tf}$. Fix two arcs $\mu, \nu$ of a great circle of $\mathbb{S}^n$ going in opposite directions from $s_0$, as their parameter increases from 0 to 1. Assume that $\mu, \nu$ are of the same length. For this proof it suffices that $\mu \neq \nu$. We made stonger assumptions on $\mu, \nu$, which will be essential for further considerations. Put $\delta := \frac{1}{4}$. For every $\alpha \in \Omega \mathbb{S}^n$ define $\alpha_{\mu} \in \Omega \mathbb{S}^n \setminus \Omega_{tf}$ as the concatenation of paths:

$$\begin{align*}
\mu & \text{ on } [0, \delta], \ \mu^* \text{ on } [\frac{\delta}{2}, \delta], \ \alpha \text{ on } [\delta, 1-\delta], \ \nu \text{ on } [1-\delta, 1-\frac{\delta}{2}], \ \nu^* \text{ on } [1-\frac{\delta}{2}, 1].
\end{align*}$$
The same schema defines \( \alpha_{s\mu} \) for \( s \in [0, 1] \) with
\[
\delta := \frac{s}{4}, \quad \mu(t) := \mu(st), \quad \nu(t) := \nu(st).
\]

In other words,
\[
\alpha_{s\mu}(t) = \begin{cases} 
\mu(8t) & \text{for } t \in [0, \frac{s}{8}] \\
\mu(2s - 8t) & \text{for } t \in \left[\frac{s}{8}, \frac{2}{4}\right] \\
\alpha\left(\frac{2}{2-s}(t - \frac{s}{4})\right) & \text{for } t \in \left[\frac{s}{4}, 1 - \frac{s}{4}\right] \\
\nu(8[t + \frac{s}{4} - 1]) & \text{for } t \in \left[1 - \frac{s}{4}, 1 - \frac{s}{8}\right] \\
\nu(8(1-t)) & \text{for } t \in \left[1 - \frac{s}{8}, 1\right].
\end{cases}
\]

In particular, \( \alpha_{0\mu} = \alpha \), \( \alpha_{1\mu} = \alpha_{\mu} \). In this way, the mapping
\[
\Omega S^n \ni \alpha \mapsto \alpha_{\mu} \in \Omega S^n \setminus \Omega_{tt}
\]
is a homotopy inverse to the inclusion. □

3 Embeddings of \( \mathbb{S}^{n-1} \) in \( \Omega \mathbb{S}^n \).

Imagine the sphere \( \mathbb{S}^n \) with its equator \( \mathbb{S}^{n-1} \), the south pole \( s_0, (x_{n+1} = -1) \) and another base point \( s_1 \) on the equator.

(3.1) [Embedding \( \alpha \).] If \( x \in \mathbb{S}^{n-1} \) then the loop \( \alpha(x) \) is a (great) circle in the intersection of \( \mathbb{S}^n \) with a plane parallel to the axis \( OX_{n+1} \) and going through points \( s_0, x \). Its orientation is chosen in this way, that \( x \) is passed before \(-x\).

(3.2) [Embedding \( \bar{R} \circ \beta \).] If \( x \in \mathbb{S}^{n-1} \) then the loop \( \beta(x) \) is a circle in the intersection of \( \mathbb{S}^n \) with a plane parallel to the axis \( OX_{n+1} \) and going through points \( s_1, x \). Its orientation is chosen in this way, that the south half-sphere is passed before the north one.

Moving the base point from \( s_0 \) to \( s_1 \) (along the path \( s_\lambda \) on the meridian) we see that there is a homotopy
\[
h_\lambda : \alpha \simeq \beta \text{ in } \{u : \mathbb{I} \to \mathbb{S}^n : u(0) = u(1)\},
\]
\[
h_\lambda : S^{n-1} \to \{u : \mathbb{I} \to S^n : u(0) = u(1) = s_\lambda\} =: \Omega(S^n, s_\lambda)
\]
defined similarly like \(\alpha\) and \(\beta\) above. Of course, there is a rotation \(R\) of \(S^n\) such that
\[
\alpha \simeq \bar{R} \circ \beta \quad \text{in } \Omega S^n
\]
with the homeomorphism \(\bar{R} : \Omega(S^n, s_1) \to \Omega S^n, \bar{R}(u)(t) := R(u(t))\).

**Lemma 2** For every \(n \geq 3\)
\[
\alpha_* : H_{n-1}S^{n-1} \to H_{n-1}(\Omega S^n \setminus \Omega tf)
\]
is an isomorphism.

**Proof.** Denote by \(JX\) the James reduced product of \(X\). Since the pair
\((JS^{n-1}, S^{n-1})\) is \((2n-3)\)-connected \([7]\) p.52, inclusion induces an isomorphism
\[
H_qS^{n-1} \to H_qJS^{n-1}
\]
for \(q + 1 \leq 2n - 3\). By the James-Puppe Theorem \(JS^{n-1} \simeq \Omega S^n\) and the
homomorphism
\[
\beta_* : H_{n-1}S^{n-1} \to H_{n-1}\Omega(S^n, s_1)
\]
is an isomorphism \([2]4.1, [6]\). \(\Box\)

(3.3) [Embeddings \(\gamma_\omega\).] Fix any \(\omega \in \Omega_{tf}\). For every \(x \in S^{n-1}\) denote by
\(\mu(x)\) the meridian arc from \(s_0\) to \(x\), \(\nu(x) := \mu(-x)\). Then
\[
\gamma_\omega(x) := \omega_\mu(x)
\]
with its definition in 2. Note that embeddings \(\alpha\) and \(\gamma_\omega\) are equivariant.

**4 Proof of Theorem 1.**

(A) Fix a map \(f : \Omega S^n \setminus \Omega_{tf} \to \mathbb{R}^k\) with \(k \leq n - 1\). Then
\[
\{\omega_\mu(x) : \omega \in \Omega_{tf}, x \in S^{n-1}\} \approx \Omega_{tf} \times S^{n-1}.
\]
If \(P(Y, y_0)\) is the space of all paths in \(Y\) from \(y_0\) then
\[
\Omega_{tf} \approx P(S^n, s_0) \supset P(S^1 \setminus \{-s_0\}, s_0) \approx P(\mathbb{R}, 0) \supset \mathbb{R}^{d+1} \supset S^d,
\]
$P(\mathbb{R}, 0)$ is an $\infty$-dimensional vector space. Briefly, $\mathbb{S}^d \subset \Omega_{tf}$ for every $d$. We apply Corollary 1 [5] to the following objects:

$X := \{\omega_{\mu(x)} : \omega \in \mathbb{S}^d, x \in \mathbb{S}^{n-1}\} \approx \mathbb{S}^d \times \mathbb{S}^{n-1}$,

$B = B' := \mathbb{S}^d, \ M := \mathbb{S}^{n-1}, \ X' := \mathbb{S}^d \times \mathbb{R}^k$,

$T : X \to X, \ T(\omega, x) = (\omega, -x)$ i.e. $T(\omega_{\mu(x)}) = (\omega_{\mu(x)})^*$ and to the fibre-preserving map

$X \ni (\omega, x) \mapsto (\omega, f(\omega_{\mu(x)})) \in X'$.

Put $C_F := \{(\omega, x) \in X : F(\omega, x) = F(T(\omega, x))\}$. Then

$C_F = \{\omega_{\mu(x)} \in X : f(\omega_{\mu(x)}) = f((\omega_{\mu(x)})^*)\} \subset A_f$

and by Corollary 1 [5],

$H^{d+n-1-k}(C_F/T; \mathbb{Z}_2) \neq 0, \ H^{d+n-1-k}(C_F'; \mathbb{Z}_2) \neq 0,$

$\dim A_f \geq \dim C_F \geq d + n - 1 - k \geq d$ for every $d$.

(B) Suppose on the contrary that there is an equivariant map

$\varphi : \Omega \mathbb{S}^n \setminus \Omega_{tf} \to \mathbb{S}^{n-1}$.

By Borsuk’s Theorem the mapping $\varphi \circ \gamma_{s_0}$ is essential. But

$\gamma_{s_0}(x) = (s_0)_{\mu(x)} \simeq (s_0)_{\mu(s_1)} = \text{const}$

by homotopy

$((s_0)(1-\lambda)\mu(x))_{\lambda\mu(s_1)},$

a contradiction. □

Remark 1: Minor changes in the above proof show that Theorem 1 holds with $\mathbb{S}^n$ replaced by any $n$-dimensional topological manifold.
5 Proof of Proposition 1.

Consider the Gysin exact sequence of the fibering $\mathbb{S}^0 \to E \to B$ of the form

$$\cdots \to H^{q-2}(E; \mathbb{Z}_2) \to H^{q-2}(B; \mathbb{Z}_2) \xrightarrow{\cup w_1} H^{q-1}(B; \mathbb{Z}_2) \xrightarrow{p^*} H^{q-1}(E; \mathbb{Z}_2) \to \cdots$$

By Lemma 1 and [3] Example 1.5,

$$H_j E = H_j \Omega S^n = \begin{cases} \mathbb{Z} & \text{for } j \equiv 0 \pmod{(n-1)} \\ 0 & \text{otherwise,} \end{cases}$$

$$H^j(E; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } j \equiv 0 \pmod{(n-1)} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\pi_1 E = \pi_2 S^n = 0$, $\pi_1 B = \mathbb{Z}_2 = H^1(B, \mathbb{Z}_2)$. By Gysin sequence,

$$H^j(B, \mathbb{Z}_2) = \mathbb{Z}_2 \text{ for } j \leq n-2 \text{ and } (w_1)^{n-1} \neq 0.$$  

By Lemma 2 and the Universal Coefficient Theorem for cohomology the homomorphism $\alpha^*$ in the commutative diagram

$$\begin{array}{ccc}
H^{n-1}(E; \mathbb{Z}_2) & \xrightarrow{\alpha^*} & H^{n-1}(\Omega S^{n-1}, \mathbb{Z}_2) \\
p^* \uparrow & & \uparrow \gamma^* \\
H^{n-1}(B, \mathbb{Z}_2) & \xrightarrow{\alpha^*} & H^{n-1}(\mathbb{R}P^{n-1}, \mathbb{Z}_2) 
\end{array}$$

is an isomorphism. Thus $p^* = 0$, because $\gamma^* = 0$. Then $H^{n-1}(B; \mathbb{Z}_2) = \mathbb{Z}_2$ and by Gysin sequence, $H^n(B; \mathbb{Z}_2) = 0$. □

Remark 2 Further application of Gysin sequence yields

$$H^j(B; \mathbb{Z}_2) = 0 \text{ for } n < j < 2n-2,$$

$$H^{2n-2}(B; \mathbb{Z}_2) = \mathbb{Z}_2 \text{ with a generator } u,$$

$$H^j(B; \mathbb{Z}_2) = \mathbb{Z}_2 \text{ for } 2n-2 < j < 3n-3,$$

$$H^{3n-3}(B; \mathbb{Z}_2) \ni u \cup (w_1)^{n-1} \neq 0.$$

It would be interesting to determine $H^j(B; \mathbb{Z}_2)$ for all $j \geq 3n-3.$
6 An example.

For \((f_1, \ldots, f_k) = f : \Omega \mathbb{R}^n \to \mathbb{R}^k\) consider the equation

\[(\star) \quad f(\alpha) = f(\alpha^*), \text{ where} \]

\[f_j(\alpha) = \int_0^1 ||\alpha(t) - \beta_j(t)||^2 dt\]

with fixed paths \(\beta_j : \mathbb{I} \to \mathbb{R}^n, j = 1, \ldots, k\). The equation \((\star)\) is equivalent to the system

\[(\star\star) \quad \int_0^1 \langle \alpha(t), \beta_j(1-t) - \beta_j(t) \rangle dt = 0, \quad j = 1, \ldots, k.\]

To see that \(\dim A_f = \infty\) it suffices to specify

\[\alpha_x(t) := \begin{cases} 
  x(t) & \text{for } t \in [0, \frac{1}{4}] \\
  x\left(\frac{1}{2} - t\right) & \text{for } t \in \left[\frac{1}{4}, \frac{1}{2}\right] \\
  0 & \text{for } t \in \left[\frac{1}{2}, 1\right]
\end{cases}\]

for any \(x : [0, \frac{1}{4}] \to \mathbb{R}^n, x(0) = 0\), and rewrite \((\star\star)\) for \(\alpha = \alpha_x\) in the form of a system of \(k\) homogeneous linear equations

\[\int_0^{\frac{1}{2}} \langle x(t), \beta_j(1-t) - \beta_j(t) + \beta_j\left(\frac{1}{2} + t\right) - \beta_j\left(\frac{1}{2} - t\right) \rangle dt = 0, \quad j = 1, \ldots, k;\]

in the \(\infty\)-dimensional space \(C_0\left([0, \frac{1}{4}]; \mathbb{R}^n\right)\). Of course, \(x \neq 0\) iff \(\alpha_x \neq (\alpha_x)^*\).

In this example \(k\) and \(n\) were any positive integers. The natural question arises: does Theorem 1 hold for \(k > n\)?
References

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\(^2\)Lectures in algebraic topology: elements of homotopy theory.

\(^3\)Lectures in algebraic topology: homotopy theory of cell complexes.