Brauer groups and étale homotopy type

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Abstract
Extending a result of Schröer on a Grothendieck question in the context of complex analytic spaces, we prove that the surjectivity of the Brauer map \( \delta : \text{Br}(X) \to H^2_{\text{ét}}(X, \mathbb{G}_m, X)^{\text{tors}} \) for schemes depends on their étale homotopy type. We use properties of algebraic \( K(\pi, 1) \) spaces to apply this to some classes of proper and smooth algebraic schemes. In particular, we recover a result of Hoobler and Berkovich for abelian varieties.

KEYWORDS
Brauer groups, Brauer map, étale homotopy groups, \( K(\pi, 1) \) spaces

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1 | INTRODUCTION

In [14], Grothendieck established a general formalism for the theory of Azumaya algebras, which allows to construct the Brauer group \( \text{Br}(X) \) of a scheme \( X \) (or more generally of a locally ringed topos), and hence generalizing the previous construction of Azumaya for local rings and that of Auslander–Goldman for arbitrary commutative rings. He defined \( \text{Br}(X) \) as the set of classes of Azumaya algebras on \( X \) modulo Morita equivalence, or equivalently the set of equivalence classes of principal \( \text{PGL}_n \)-bundles. In a part of his works, he constructed via nonabelian cohomology an injective homomorphism of groups \( \delta : \text{Br}(X) \to \text{Br}'(X) \) called the Brauer map (see Theorem 2.1), where \( \text{Br}'(X) := H^2_{\text{ét}}(X, \mathbb{G}_m, X)^{\text{tors}} \) is the torsion part of the cohomological Brauer group \( H^2_{\text{ét}}(X, \mathbb{G}_m, X) \), and asked in which case this map is a bijection, in other words, for a given scheme \( X \), does any cohomological Brauer class \( \beta \in \text{Br}'(X) \) come from an Azumaya algebra? When \( X \) is a complex analytic space endowed with the sheaf of holomorphic functions, one can define, as a special case, the analytic Brauer group \( \text{Br}_a(X) \) of \( X \), and hence we get in terms of complex cohomology a well-defined injective Brauer map \( \delta : \text{Br}_a(X) \to \text{Br}'_a(X) := H^2_{\text{ét}}(X, \mathbb{G}_m, X)^{\text{tors}} \) (cf. [21, 30]).

A positive answer to this question for any class of schemes will be of a big interest when it comes to the computation of \( \text{Br}(X) \), this is due to the fact that the cohomological Brauer group \( \text{Br}'(X) \) appears in many fundamental exact sequences with various other cohomology groups (e.g., Kummer sequence, Artin–Schreier sequence, exponential exact sequence). The question is also partially related to the problem of determining weather an algebraic stack is a quotient stack (see [9]).

The Brauer map is known to be surjective for the following classes of schemes:

1. Regular schemes of dimension \( \leq 2 \): Grothendieck [14].
2. Abelian varieties: Berkovich [5], and more generally abelian schemes: Hoobler [19].
3. Affine schemes, and separated unions of two affine schemes: Gabber [12]. Simplified proofs were given by Hoobler [20] with more additional results.
4. Schemes with ample invertible sheaf (this contains in particular quasi-projective varieties): Proved by Gabber (unpublished). An alternative proof was given by de Jong [8] (see also [7, section 4.2]).
(5) Separated geometrically normal algebraic surfaces: Schröer [29].

For complex spaces, we have the following treated cases:

(1) Complex torus: Elencwajg and Narasimhan [10].
(2) Analytic $K3$ surfaces, Ricci-flat compact Kähler surfaces: Huybrechts and Schröer [21].
(3) Hopf manifolds, complex Lie groups, and elliptic surfaces: Schröer [30]. These are particular cases of a general statement (see Theorem 3.1) for complex analytic spaces proved by the author in *loc. cit.* via homotopy theory.

The equality $\text{Br}(X) = \text{Br}'(X)$ does not hold in general. Indeed, an example of a nonseparated normal surface for which $\text{Br}(X) \neq \text{Br}'(X)$ was constructed in [9] by arguments from quotient stacks theory.

For a given cohomological Brauer class $\beta \in \text{Br}'(X)$, it is difficult to find explicitly an Azumaya algebra on $X$ whose image under $\delta$ is $\beta$. However, many tools have been introduced to ensure the existence of the required algebra. In [22], Lieblich proved that for a nice scheme for which étale cohomology can be computed in Čech terms, the class $\beta$ lies in $\text{Br}(X)$ if and only if there exists a finite locally free $\beta$-twisted étale sheaf on $X$ of positive rank (see also [8]). Using this fact, he recovered Grothendieck and Gabber’s results by simplifications of Hoobler arguments in [20]. Another important tool is the geometric interpretation of the cohomology groups $H^1(\mathbb{G}_m, X)$ and $H^2(\mathbb{G}_m, X)$ via PGL$_n$-torsors and $\mathbb{G}_m$-gerbes. More precisely, one can associate to any Azumaya algebra $A$ a $\mathbb{G}_m$-gerbe $G_A$ and a PGL$_n$-torsor $P_A$ such that the class $[G_A] \in H^2(\mathbb{G}_m, X, \mathbb{G}_m)$ is equal to the image of the class $[P_A] \in H^1(\mathbb{G}_m, X, \text{PGL}_n(\mathcal{O}_X))$ under the boundary map $\delta_1 : H^1(\mathbb{G}_m, X, \text{PGL}_n(\mathcal{O}_X)) \to H^2(\mathbb{G}_m, X, \mathbb{G}_m)$ (see [26, Chapter 12]). In the light of this interpretation, authors in [9] showed that the class $\beta$ lies in $\text{Br}(X)$ if only if the $\mathbb{G}_m$-gerbe $X_\beta$ associated to $\beta$ is a quotient stack. A useful technical tool used by Berkovich, Hoobler, and Gabber, which we shall adopt in this paper states that if there exists a finite étale (or more generally a finite locally free) cover $\pi : Y \to X$ trivializing each class $\beta$ in $H^2(\mathbb{G}_m, Y)$, then $\text{Br}(X) = \text{Br}'(X)$. When $X$ is a complex analytic space, Schröer [30] proved that such a cover can be obtained, and hence one has $\text{Br}_{an}(X) = \text{Br}'_{an}(X)$, if the topological fundamental group $\pi_1(X)$ is a good group, and the subgroup of $\pi_1(X)$-invariants inside the Pontryagin dual $\text{Hom}(\pi_1(X), \mathbb{Q}/\mathbb{Z})$ is trivial.

The aim of this paper is to extend Schröer’s result to the algebraic setting. In this context, for a pointed connected noetherian scheme $(X, \bar{x})$, we are going to work with the Grothendieck étale fundamental group $\pi_1^{\text{ét}}(X, \bar{x})$ introduced in [15], and the higher étale homotopy groups $\pi_1^{\text{ét}}(X, \bar{x})$ $(n \geq 0)$ as defined by Artin and Mazur in [4]. For our purpose, since $\pi_1^{\text{ét}}(X, \bar{x})$ is always profinite, we just have to deal with the higher groups $\pi_n^{\text{ét}}(X, \bar{x})$ $(n \geq 2)$. We adapt an algebraic version of Schröer’s argument to prove—by means of Galois–Grothendieck theory—the following main result (Theorem 4.2):

**Theorem 1.1.** Let $X$ be a connected noetherian scheme with a geometric base point $\bar{x} \to X$. Let $\text{char}(X)$ be the set of residue characteristics of $X$. Suppose that $\pi_2^{\text{ét}}(X, \bar{x}) = 0$, then $\text{Br}(X) = \text{Br}'(X)$ up to a char $(X)$-torsion.

As in the topological context, the calculation of the higher étale homotopy groups $\pi_n^{\text{ét}}(X, \bar{x})$ is in general much more difficult. However, if $X$ is in particular a geometrically unibranch scheme with $\pi_2^{\text{ét}}(X, \bar{x}) = 0$ for all $n \geq 2$, then this is equivalent to say that $X$ is an algebraic $K(\pi, 1)$ space (Definition 4.4). This class of spaces was largely studied by Achinger [1, 2] in addition to other variants (logarithmic [1], rigid analytic, and mixed characteristic [2]).

When it comes to schemes over subfields of $\mathbb{C}$, or algebraically closed fields in the proper case, the étale fundamental group behaves nicely with base change (see [11]). By introducing the notion of Schröer space (Definition 3.4) and by using descent arguments, this behavior can be employed together with properties of algebraic $K(\pi, 1)$ spaces to prove the following result for proper and smooth schemes (see Propositions 5.3 and 5.7).

**Theorem 1.2.** Let $X$ be a geometrically connected scheme of finite type over a field $k$. Suppose that $k$ can be embedded as a subfield of $\mathbb{C}$ such that $X_C$ becomes a Schröer space. Then for a given geometric base point $\bar{x} \to X_C$, we have $\pi_2^{\text{ét}}(X, \bar{x}) = 0$ in the following cases:

(i) $X$ proper and geometrically unibranch with $k$ algebraically closed.
(ii) $X$ smooth, with $k$ finitely generated over $\mathbb{Q}$.

Under some assumptions, our results can be extended to a scheme $X$ defined over an algebraically closed field $k$ of characteristic zero (Proposition 5.4), or more generally over a noetherian scheme $S$ (Proposition 5.9).
By a theorem of Artin [3, Exp XI, Proposition 3.3], any smooth scheme over an algebraically closed field \( k \) of characteristic 0 can be covered by Artin neighborhoods. This was generalized by Achinger [2], by proving that any smooth scheme over a field of positive characteristic admits a cover by \( K(\pi, 1) \) open subschemes. This gives us the possibility to make legitimate assumptions on one piece of such a cover. Therefore, by using some purity theorems, we get in Section 6 our third main result:

**Theorem 1.3.** Let \( X \) be a smooth variety over an algebraically closed field \( k \) of characteristic \( p \geq 0 \). Suppose that there exists an open subscheme \( Y \subseteq X \) such that for every \( z \in Z := X - Y \), the local ring \( \mathcal{O}_{X, z} \) has dimension \( \geq 2 \), and \( \pi^\mathrm{ét}_2(Y, \bar{y}) = 0 \) for some geometric point \( \bar{y} \to Y \). Then, \( Br(X) = Br'(X) \).

This partially extends a result of Grothendieck [14, II, Corollary 2.2] for regular noetherian schemes of dimension less than 2 to varieties of greater dimensions.

Our main example of application is the case of an abelian variety \( A \). In Section 7, we apply our results along with a general version of the Riemann existence theorem for smooth algebraic groups (Lemma 7.1), to prove that \( Br(A) = Br'(A) \). This is an alternative proof to the ones proposed by Hoobler [19] and Berkovich [5].

Notations. Throughout this paper we consider the following notations:

1. A variety over a field \( k \) is a separated, geometrically integral scheme of finite type over \( k \). In particular a variety is quasi-compact and quasi-separated.
2. \( \mu_n, X \): The étale sheaf of \( n \)-th roots of unity on \( X \).
3. \( \mathbb{G}_m, X \): The étale sheaf of multiplicative groups on \( X \).
4. For an abelian group \( A \), \( n \) an integer, the notations \( nA \) and \( A_n \) will stand for the kernel and the cokernel of the endomorphism \( a \to na \) of \( A \).
5. For a field \( k \), \( \bar{k} \) is the separable closure of \( k \). If \( X \) is a scheme over \( k \), and \( k \subseteq K \) a field extension, \( X_K = X \times_k K \) denotes the base change of \( X \) to \( K \).

2 | PRELIMINARIES

Following Grothendieck [14] and Milne [24, Chapter IV], we recall some elementary facts needed in the sequel about Brauer groups of schemes.

Let \( X \) be a scheme. An Azumaya algebra \( A \) on \( X \) is an \( \mathcal{O}_X \)-algebra, which is a locally free \( \mathcal{O}_X \)-module and satisfies one of the following equivalent conditions:

1. For every point \( x \in X \), \( A_x \) is an Azumaya algebra over \( \mathcal{O}_{X, x} \).
2. For every point \( x \in X \), the fiber \( A_x \otimes k(x) \) is a central simple algebra over the residue field \( k(x) \).
3. The natural morphism \( A \otimes_{\mathcal{O}_X} A^{\text{op}} \to \text{End}_{\mathcal{O}_X}(A) \) is an isomorphism.
4. There is a covering \((U_i \to X)\) in the étale topology on \( X \) such that for each \( i \) there exists an \( n_i \) such that \( A \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong M_{n_i}(\mathcal{O}_{U_i}) \).

If \( X \) is connected, then integers \( n_i \) in (iv) are constant. Their value \( r \) is the rank of the Azumaya algebra \( A \).

Two Azumaya algebras \( A_1 \) and \( A_2 \) are called Morita equivalent if there exist locally free \( \mathcal{O}_X \)-modules \( M_1 \) and \( M_2 \) of finite rank, and an isomorphism

\[
A_1 \otimes_{\mathcal{O}_X} \text{End}_{\mathcal{O}_X}(M_1) \cong A_2 \otimes_{\mathcal{O}_X} \text{End}_{\mathcal{O}_X}(M_2).
\]

The set of classes of Azumaya algebras on \( X \) is a group called the Brauer group of \( X \) and denoted by \( Br(X) \). The group law is given by the tensor product, the inverse of a class \([A]\) is the class of its opposite algebra \([A^{\text{op}}]\), and the unit element has the form \( \text{End}_{\mathcal{O}_X}(E) \), where \( E \) is a locally free \( \mathcal{O}_X \)-module.

Fix an integer \( n > 1 \), and consider the following exact sequence of étale sheaves on \( X \):

\[
1 \to \mathbb{G}_{m, X} \to \text{GL}_n(\mathcal{O}_X) \to \text{PGL}_n(\mathcal{O}_X) \to 1.
\]
Nonabelian cohomology yields an exact sequence of Čech cohomology (cf. [24, Chapter IV, p. 143])

\[ \ldots \rightarrow H^1_{\text{ét}}(X, G_{m,X}) \rightarrow H^1_{\text{ét}}(X, GL_n(O_X)) \rightarrow H^1_{\text{ét}}(X, PGL_n(O_X)) \overset{\delta_n}{\rightarrow} H^2_{\text{ét}}(X, G_{m,X}). \]  

The following is a fundamental result in the theory of Brauer groups of schemes.

**Theorem 2.1.** Let \( X \) be a scheme, then we have the following statements:

1. The set of classes of Azumaya algebras of rank \( n^2 \) is isomorphic to the cohomology set \( H^1_{\text{ét}}(X, PGL_n(O_X)) \).
2. The maps \( \delta_n \) induce a group homomorphism \( \delta' : \text{Br}(X) \rightarrow H^2_{\text{ét}}(X, G_{m,X}) \).
3. This homomorphism \( \delta' : \text{Br}(X) \rightarrow H^2_{\text{ét}}(X, G_{m,X}) \) is injective.
4. The image of \( \delta_n \) is killed by \( n \) in \( H^2_{\text{ét}}(X, G_{m,X}) \).

**Proof.** This is the original Grothendieck statement [14, I, Proposition 1.4]. Milne gave in [24, Chapter IV, Theorem 2.5 and Proposition 2.7] a proof for Čech cohomology and another general proof for étale cohomology by means of gerbes theory.

The group \( H^2_{\text{ét}}(X, G_{m,X}) \) is called the cohomological Brauer group, or Brauer–Grothendieck group. As observed by Grothendieck, the map \( \delta' : \text{Br}(X) \rightarrow H^2_{\text{ét}}(X, G_{m,X}) \) is not bijective in general. Indeed, for quasi-compact schemes, \( \text{Br}(X) \) is always torsion [14, I, Section 2], while there exists a normal surface \( S \) such that \( H^2_{\text{ét}}(S, G_{m,S}) \) is not torsion [14, II, 1.11.b].

Let \( X \) be a quasi-compact scheme. Denote by \( \text{Br}'(X) := H^2_{\text{ét}}(X, G_{m,X})_{\text{tors}} \) the torsion part of the cohomology group \( H^2_{\text{ét}}(X, G_{m,X}) \), and consider the map

\[ \delta : \text{Br}(X) \rightarrow \text{Br}'(X), \]

which is called the Brauer map. Grothendieck asked the following question:

**Question:** Is \( \delta : \text{Br}(X) \rightarrow \text{Br}'(X) \) surjective for quasi-compact schemes?

To answer this question, the majority of our results in this paper will be based on the following fundamental lemma. Recall that a finite morphism of schemes \( f : Y \rightarrow X \) is locally free if the direct image sheaf \( f_* \mathcal{O}_Y \) is locally free. If \( X \) is locally noetherian, then \( f \) is finite locally free if and only if it is finite and flat.

**Lemma 2.2.** Let \( X \) be a locally noetherian scheme, and \( \alpha \in H^2_{\text{ét}}(X, G_{m,X}) \). If there exists a finite locally free morphism \( f : Y \rightarrow X \) such that \( f^*(\alpha) = 0 \) in \( H^2_{\text{ét}}(Y, G_{m,Y}) \), then \( \alpha \in \text{Br}(X) \).

**Proof.** This is a well-known lemma of Gabber [12, p. 165, Lemma 4], he proved it by using arguments from theory of gerbes. An elementary proof was given by Hoobler [20, Proposition 3] under the assumption that any finite set of points of \( X \) is contained in an affine open subscheme. A more general version of the statement was established by Lieblich [22, 3.1.3.5] by means of twisted sheaves.

**Remark 2.3.** If \( \beta \in \text{Br}'(X) \), then by Theorem 2.1(iv) there is an integer \( n \) with \( n \beta = 0 \), and a class \( \alpha \in H^2_{\text{ét}}(X, \mu_{n,X}) \) mapping to \( \beta \). Therefore, the result of Lemma 2.2 follows when the morphism \( f : Y \rightarrow X \) sufficiently trivializes classes of \( H^2_{\text{ét}}(X, \mu_{n,X}) \). This comes with very nice consequences, since the étale sheaf \( \mu_{n,X} \) belongs to the category of locally constant constructible torsion étale sheaves, which will play a crucial role in this paper.

### 3 CASE OF SCHEMES OVER \( \mathbb{C} \)

In this section, we give some elementary results on the analytic Brauer group, which are closely related to the Brauer group of schemes over complex numbers.

Let \( (X, \mathcal{O}_X) \) be a complex analytic space, where \( \mathcal{O}_X \) is the sheaf of holomorphic functions. An Azumaya algebra on \( X \) is an associative (noncommutative) \( \mathcal{O}_X \)-algebra \( A \), which is locally (in the analytic topology) isomorphic to a matrix algebra \( M_n(O_X) \) for some \( n > 0 \). Working with cohomology of sheaves, all facts in the previous section could be applied
to define the analytic Brauer group $\Br_{an}(X)$ of $X$, and hence we get a well-defined injective Brauer map $\delta : \Br_{an}(X) \to \Br'(X) := H^2(X, \mathbb{G}_m, X)_{\text{tors}}$ (see [21] for more details). Equivalently, one can define $\Br_{an}(X)$ as the set of equivalence classes of principal $\text{PGL}_n$-bundles via the boundary maps $\delta_n : H^1(X, \text{PGL}_n(\mathcal{O}_X)) \to H^2(X, \mathbb{G}_m, X)$ (cf. [30]).

Schröer proved (see the theorem below) that the surjectivity of the Brauer map $\delta : \Br_{an}(X) \to \Br'(X)$ for complex analytic spaces depends only on the homotopy type of their underlying topological space. He used the following notion of good groups introduced by Serre in [31]: Let $G$ be a group endowed with the discrete topology, and $\hat{G} = \lim\leftarrow U\cdot\text{orizo}T\text{tal}A\cdot\text{orizo}T\text{tal}A\cdot G/\mathcal{N}$ its profinite completion, where the limit runs over all normal subgroups $\mathcal{N} \subset G$ of finite index. By construction, the group $\hat{G}$ carries the inverse limit topology. Let $M$ be a finite discrete $G$-module, that is a $G$-module, which is finite as a set. The action of $G$ induces an action of $\hat{G}$ on $M$. We say that $G$ is a good group [31, Chapter I, section 2.6, b.2] if the natural morphism of cohomology groups

$$H^n(\hat{G}, M) \longrightarrow H^n(G, M)$$

induced by the natural morphism $G \longrightarrow \hat{G}$ is an isomorphism for all $n \geq 0$ and all finite $G$-module $M$. The following types of groups are examples of good groups:

1. Free groups, finite groups.
2. Almost free groups, almost polysyclic groups (see [30]).
3. Bianchi groups $\text{PSL}(2, \mathbb{O}_d)$, where $\mathbb{O}_d$ is the ring of integers in an imaginary quadratic number filed $\mathbb{Q}(\sqrt{-d})$ [17].
4. Right-angled Artin groups [23].

**Theorem 3.1** [30, Theorem 4.1]. Let $X$ be a complex analytic space. Suppose that the topological fundamental group $\pi_1(X)$ is good, and that the subgroup of $\pi_1(X)$-invariants inside the Pontryagin dual is trivial, that is, $\text{Hom}(\pi_2(X), \mathbb{Q}/\mathbb{Z})_{\pi_1(X)} = 0$. Then, $\Br_{an}(X) = \Br'(X)$.

Let $X$ be a scheme of finite type over $\mathbb{C}$. There is an associated analytic space $X^{an}$ whose underlying topological space is $X(\mathbb{C})$ the space of $\mathbb{C}$-rational points of $X$. The following is a first elementary result describing the link between Brauer maps for $X$ and $X^{an}$.

**Proposition 3.2.** Let $X$ be a scheme of finite type over $\mathbb{C}$. Suppose that $X^{an}$ is compact. Then, $\Br(X) = \Br'(X)$ if only if $\Br_{an}(X^{an}) = \Br'(X^{an})$.

**Proof.** Consider the following commutative diagram

$$\begin{array}{ccc}
\Br(X) & \longrightarrow & \Br_{an}(X^{an}) \\
\downarrow & & \downarrow \\
\Br'(X) & \longrightarrow & \Br'(X^{an})
\end{array}$$

The upper map is an isomorphism according to [30, Proposition 1.4], and the lower map is an isomorphism by [30, Proposition 1.3]. Hence the assertion.

**Example 3.3.** Let $X$ be an algebraic $K3$ surface over $\mathbb{C}$, which is a complete nonsingular variety of dimension 2 over $\mathbb{C}$ such that $\Omega^2_{X/\mathbb{C}} \cong \mathcal{O}_X$ and $H^1_{an}(X, \mathcal{O}_X) = 0$. Its associated analytic space $Y = X^{an}$ is a complex $K3$ surface, that is, a compact connected complex manifold of dimension 2 such that $\Omega^2_Y \cong \mathcal{O}_Y$ and $H^1(Y, \mathcal{O}_Y) = 0$. According to the Huybrechts–Schröer result for analytic $K3$ surfaces [21, Theorem 1.1], we have $\Br_{an}(Y) = \Br'(Y)$, hence Proposition 3.2 asserts that $\Br(X) = \Br'(X)$.

Recall that a topological space $X$ is called a $K(\pi, 1)$ space if it is weakly homotopy equivalent to the classifying space $B\pi_1(X)$, that is, $\pi_n(X) = 0$ for all $n \geq 2$. An equivalent definition of topological $K(\pi, 1)$ spaces in terms of cohomology is given as follows: Let $x \in X$, there is a fully faithful functor

$$\rho^* : \pi_1(X, x)\text{-Mod} \longrightarrow \text{Sh}(X)$$
from the category of $\pi_1(X, x)$-modules to the category of sheaves on $X$, whose essential image is the subcategory of locally constant sheaves on $X$. It associates to any $\pi_1(X, x)$-module $M$ a locally constant sheaf $\rho^*(M)$, with $(\rho^*(M))_x = M$ and $\Gamma(X, \rho^*(M)) = M_{\pi_1(X, x)}$. Therefore, the formalism of universal $\delta$-functors gives rise to natural morphisms of cohomology groups

$$\rho^q : H^q(\pi_1(X, x), M) \to H^q(X, \rho^*(M)).$$

The space $X$ is a topological $K(\pi, 1)$ space if only if the morphisms $\rho^q$ are isomorphisms for all $q \geq 0$.

In order to apply Theorem 3.1 to the space $X^{an}$, we consider the following class of complex schemes, which we call Schröer spaces.

**Definition 3.4.** A scheme $X$ of finite type over $\mathbb{C}$ is called a Schröer space if the following statements hold:

(i) $X(\mathbb{C})$ is a topological $K(\pi, 1)$ space.

(ii) $\pi_1(X(\mathbb{C}))$ is a good group.

This class contains abelian varieties and smooth connected curves, which are not projective of genus zero (cf. proof of [27, Corollary 5.5]). For our purpose in this paper, which is the surjectivity of the Brauer map, we will see in Section 7 that in the case of smooth algebraic groups the goodness assumption is not needed.

Now we check another important example of these spaces, which is the Artin neighborhood. Following Artin [3, Exp XI, Section 3], a morphism of schemes $f : X \to S$ is called an elementary fibration if there exists a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{j} & \bar{X} & \xleftarrow{i} & Y \\
\downarrow{f} & & \downarrow{\bar{f}} & & \downarrow{g} \\
S & & & & S
\end{array}
$$

such that

(i) $j$ is an open immersion and $X$ is fiberwise dense in $\bar{X}$;

(ii) $\bar{f}$ is a smooth and projective morphism whose geometric fibers are irreducible curves;

(iii) the reduced closed subscheme $Y = \bar{X} \setminus X$ is a finite étale cover of $S$.

Let $k$ be a field. An Artin neighborhood over $k$ is a scheme $X$ over $k$ such that there exists a sequence of $X$-schemes

$$X = X_n, \ldots, X_0 = \text{Spec } k$$

with elementary fibrations $f_i : X_i \to X_{i-1}, i = 1, \ldots, n$.

**Lemma 3.5.** An Artin neighborhood $X$ over $\mathbb{C}$ is a Schröer space.

**Proof.** This is proven by Serre [3, Exp XI, Variant 4.6] as a variant of the proof of Artin comparison theorem [3, Exp. XI, Theorem 4.4]. The result follows from the fact that if $X \to S$ is an elementary fibration, then $X(\mathbb{C}) \to S(\mathbb{C})$ is a locally trivial topological fiber bundle whose fiber $F$ is a topological $K(\pi, 1)$ space and its fundamental group $\pi_1(F)$ is free of finite type. The exact sequence of homotopy groups

$$
\ldots \to \pi_n(F) \to \pi_n(X(\mathbb{C})) \to \pi_n(S(\mathbb{C})) \to \pi_{n-1}(F) \to \ldots
$$

implies that $X(\mathbb{C})$ is a topological $K(\pi, 1)$ space and $\pi_1(X(\mathbb{C}))$ is a succession of extensions of free group of finite type, whence by [31, Chapter I, section 2.6.2.d] it is a good group. □

**Proposition 3.6.** Let $X$ be a scheme of finite type over $\mathbb{C}$. If $X$ is a Schröer space, then $\text{Br}^{an}(X^{an}) = \text{Br}'^{an}(X^{an})$. 
Proof. Since $X(\mathbb{C})$ is a topological $K(\pi, 1)$ space, then in particular $\pi_2(X(\mathbb{C})) = 0$. Hence the assertion follows from Lemma 3.5 and Theorem 3.1.

**Corollary 3.7.** Let $X$ be a proper scheme over $\mathbb{C}$. If $X$ is a Schröer space, then $\text{Br}(X) = \text{Br}'(X)$.

**Proof.** Since $X$ is a proper scheme over $\mathbb{C}$, then $X^{an}$ is compact, hence Proposition 3.2 applies.

**Remark 3.8.** By Gabber theorem [12, p. 163, Theorem 1], for any regular scheme $X$, there is an open $U \subseteq X$ such that $\text{Br}(U) = \text{Br}'(U)$. Corollary 3.7 and Lemma 3.5 provide an alternative construction of $U$ when $X$ is in particular smooth and proper over $\mathbb{C}$. This follows from Artin theorem [3, Exp XI, Proposition 3.3], which asserts that such a scheme admits a cover by Artin neighborhoods.

The purpose of the next sections is to study the case of proper and smooth schemes over subfields of $\mathbb{C}$. This involves the algebraic version of $K(\pi, 1)$ spaces, which is closely related to the notion of the étale homotopy type.

### 4 ÉTALE HOMOTOPY TYPE AND $K(\pi, 1)$ SPACES

We begin this section by a brief summary of Artin and Mazur construction of the étale homotopy type and étale homotopy groups. The standard reference for this is [4].

Let $X$ be a locally noetherian connected scheme, and let $\text{Cov}(X)$ denote the category of étale coverings of $X$, that is, the collection of families $U \rightarrow X := \{f_i : U_i \rightarrow X\}_{i \in I}$ of étale morphisms $f_i$ with $X = \bigcup_{i \in I} f_i(U_i)$. Let $\text{Hyp}(X)$ be the category of étale hypercoverings of $X$. Any object $U'$ of $\text{Hyp}(X)$ is a simplicial object of $\text{Cov}(X)$ [4, Definition 8.4]. Since every scheme $Y$ étale over $X$ is a disjoint union of connected schemes, we can consider the functor $\pi_0 : \text{Cov}(X) \rightarrow \text{Sets}$, where $\pi_0(Y)$ is the set of connected components of $Y$. It extends to a functor $\pi_0 : \text{Hyp}(X) \rightarrow \text{Sets}$ from the category of simplicial étale hypercoverings of $X$ to the category of simplicial sets, and by taking the quotient with simplicial homotopy, we get a functor $\{\pi_0(–)\} : \text{Ho}(\text{Hyp}(X)) \rightarrow \text{Ho}(\text{Sets})$ of homotopy categories. Since $\text{Ho}(\text{Hyp}(X))$ is cofiltering [4, Corollary 8.13.(i)], then one can define the étale homotopy type $Et$ as an object in pro-$\text{Top}$

$$Et : \text{Ho}(\text{Hyp}(X)) \rightarrow \text{Top}$$

in the following sense: Take an hypercovering $U'$ of $X$, and put $\pi_0(X) := \{\pi_0(U_i')\}$. Then one defines $Et(X) := |\pi_0(X)|$, where $|S|$ is the topological realization of the simplicial set $S$. Such a topological space can be given the structure of a CW-complex, hence $Et(X)$ is an object in pro-$\mathcal{H}$, the pro-category of the homotopy category of CW-complexes.

For any abelian group $A$ we have a canonical isomorphism [4, Corollary 9.3]

$$H^n(\text{Et}(X), A) = H^n_{\text{ét}}(X, A).$$

A given geometric point $\tilde{x}$ of $X$ defines a point $\tilde{x}_{\text{ét}}$ on $\text{Et}(X)$, hence one can define the étale homotopy groups for all $n \geq 0$:

$$\pi^n_{\text{ét}}(X, \tilde{x}) := \pi_n(\text{Et}(X), \tilde{x}_{\text{ét}}).$$

In particular by [4, Corollary 10.7], $\pi^1_{\text{ét}}(X, \tilde{x})$ is the usual étale fundamental group defined by Grothendieck in [15].

**Lemma 4.1.** Let $f : (Y, \tilde{y}) \rightarrow (X, \tilde{x})$ be a finite étale surjective morphism of pointed connected schemes, then

$$\pi^n_{\text{ét}}(Y, \tilde{y}) \simeq \pi^n_{\text{ét}}(X, \tilde{x})$$

for all $n \geq 2$.

**Proof.** For smooth connected quasi-projective varieties over an algebraically closed field $k$, this is [27, Proposition 4.1]. For arbitrary connected schemes, the assertion follows by [28, Lemma 2.1] from the analogous statement for topological covers.

□
The following result is a generalization of Theorem 3.1. Since \( \pi_2^{\text{et}}(X, \bar{x}) \) is always profinite, we use properties of continuous cohomology of profinite groups, and hence we can omit the goodness assumption. Furthermore, Lemma 4.1 will serve to get the desired étale Galois cover, which kills cohomological Brauer classes. In what follows, \( \text{char}(X) \) is the set of residue characteristics of \( X \).

**Theorem 4.2.** Let \( X \) be a connected noetherian scheme with a geometric base point \( \bar{x} \rightarrow X \). Suppose that \( \pi_2^{\text{et}}(X, \bar{x}) = 0 \), then \( \text{Br}(X) = \text{Br}'(X) \) up to a \( \text{char}(X) \)-torsion.

**Proof.** Let \( p : (Et(X)^{\sim}, \bar{x}_{\text{et}}) \rightarrow (Et(X), \bar{x}_{\text{et}}) \) be the universal covering of the étale homotopy type \( Et(X) \). For any locally constant constructible torsion étale sheaf \( \mathcal{F} \) on \( X \), we have a spectral sequence

\[
E_2^{p,q} = H^p(\pi_1^{\text{et}}(X, \bar{x}), H^q(Et(X)^{\sim}, p^* \mathcal{F})) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathcal{F}).
\]

This is in fact a Grothendieck spectral sequence associated to the functor

\[
\Gamma(Et(X)^{\sim}, p^*(-)) : \text{Sh}(Et(X)) \rightarrow \pi_1^{\text{et}}(X, \bar{x})-\text{Mod}
\]

from the category of sheaves on \( Et(X) \) to the category of \( \pi_1^{\text{et}}(X, \bar{x})-\text{modules} \), and the functor

\[
(-)^{\pi_1^{\text{et}}(X, \bar{x})} : \pi_1^{\text{et}}(X, \bar{x})-\text{Mod} \rightarrow \text{Ab}
\]

from the category of \( \pi_1^{\text{et}}(X, \bar{x}) \)-modules to the category of abelian groups. Therefore, we get an exact sequence of low-degree terms

\[
0 \rightarrow H^1(\pi_1^{\text{et}}(X, \bar{x}), H^0(Et(X)^{\sim}, p^* \mathcal{F})) \rightarrow H^1_{\text{ét}}(X, F) \rightarrow H^0(\pi_1^{\text{et}}(X, \bar{x}), H^1(Et(X)^{\sim}, p^* \mathcal{F})) \rightarrow H^2(\pi_1^{\text{et}}(X, \bar{x}), H^0(Et(X)^{\sim}, p^* \mathcal{F})).
\]

We have \( H^0(Et(X)^{\sim}, p^* \mathcal{F}) = F_{\bar{x}} \), and since \( p^* \mathcal{F} \) is locally constant, we get by the universal coefficient theorem \( H^1(Et(X)^{\sim}, p^* \mathcal{F}) = 0 \). By the topological Hurewicz theorem, we get an isomorphism

\[
H_2(Et(X)^{\sim}, \mathbb{Z}) \simeq \pi_2^{\text{et}}(Et(X)^{\sim}, \bar{x}_{\text{et}}) \simeq \pi_2(Et(X), \bar{x}_{\text{et}}) = \pi_2^{\text{et}}(X, \bar{x}).
\]

And again by the universal coefficient theorem, we have

\[
H^2(Et(X)^{\sim}, p^* \mathcal{F}) \simeq \text{Hom}(H_2(Et(X)^{\sim}, \mathbb{Z}), F_{\bar{x}}).
\]

Hence we get a short exact sequence

\[
0 \longrightarrow H^2(\pi_1^{\text{et}}(X, \bar{x}), F_{\bar{x}}) \longrightarrow H^2_{\text{ét}}(X, F) \longrightarrow \text{Hom}(\pi_2^{\text{et}}(X, \bar{x}), F_{\bar{x}}) \stackrel{\pi_1^{\text{et}}(X, \bar{x})}{\longrightarrow}.
\]

By assumption on \( \pi_2^{\text{et}}(X, \bar{x}) \), and by [31, Chapter I, section 2.2, Corollary 1], we have an isomorphism

\[
H^2_{\text{ét}}(X, F) \simeq H^2(\pi_1^{\text{et}}(X, \bar{x}), F_{\bar{x}}) \simeq \lim H^2(\pi_1^{\text{et}}(X, \bar{x})/N, F_{\bar{x}}^N),
\]

where the limit runs over all normal open subgroups \( N \) of \( \pi_1^{\text{et}}(X, \bar{x}) \), and \( F_{\bar{x}}^N \) is the submodule of \( N \)-invariant elements. Next, take \( F = \mu_{n,X} \) for some \( n \) invertible in \( X \) (it is then a locally constant sheaf), and choose a class \( \beta \in H^2_{\text{ét}}(X, \mu_{n,X}) \), it belongs to a group \( H^2(\pi_1^{\text{et}}(X, \bar{x})/N, (\mu_{n,X})^N) \) for some open normal subgroup \( N \). Further, \( N \) is of finite index since it is an open normal subgroup of a profinite group, thus \( G := \pi_1^{\text{et}}(X, \bar{x})/N \) is a finite quotient of \( \pi_1^{\text{et}}(X, \bar{x}) \). Therefore, the fundamental Galois correspondence implies that there exists a pointed étale Galois cover \( f : (Y, \bar{y}) \rightarrow (X, \bar{x}) \) with Galois
group $G$ and $\pi^{\text{et}}_1(Y, \bar{y}) = N$. On the other hand, Lemma 4.1 asserts that $\pi^{\text{et}}_2(Y, \bar{y}) = 0$, hence we get by the same argument an isomorphism

$$H^2_{\text{et}}(Y, \mu_{n,Y}) \simeq H^2(N, (\mu_{n,Y})_{\bar{y}}).$$

Since the map

$$H^2(\pi^{\text{et}}_1(X, \bar{x})/N, (\mu_{n,X})^N) \rightarrow H^2(N, (\mu_{n,X})_{\bar{x}}) = H^2(N, (\mu_{n,Y})_{\bar{y}})$$

is zero, we conclude that the image of $\beta$ under the map

$$f^*: H^2_{\text{et}}(X, \mu_{n,X}) \rightarrow H^2_{\text{et}}(Y, \mu_{n,Y})$$

is zero, hence it follows from Lemma 2.2 that $\beta \in \text{Br}(X)$. □

Remark 4.3.

(a) If $X$ is regular, the $\text{char}(X)$-torsion classes in $\text{Br}'(X)$ are always in $\text{Br}(X)$ since they are trivialized by the absolute Frobenius morphism, which is finite locally free.

(b) The proof of Theorem 4.2 shows in fact that if $(X, \bar{x})$ is a pointed connected noetherian scheme with $\pi^{\text{et}}_1(X, \bar{x}) = 0$, then for every locally constant constructible torsion étale sheaf $F$ on $X$, and every class $\beta \in H^2_{\text{et}}(X, F)$, there exists a finite étale cover $f: Y \rightarrow X$ such that $f^*(\beta) = 0$. The inverse holds when $X$ is geometrically unibranch (see Proposition 4.11).

Following Achinger [1, 2], we consider the notion of algebraic $K(\pi, 1)$ spaces, which are defined only for quasi-compact and quasi-separated schemes with finitely many components. In our context, we consider connected noetherian schemes, which belong to this class. Further, we adopt the second definition introduced in [2], which does not require sheaves of order invertible on $X$. Note that algebraic $K(\pi, 1)$ spaces are defined in [28, 2.3] in terms of étale homotopy groups. The two definitions are equivalent in the case of geometrically unibranch schemes (Proposition 4.9).

Let $X$ be a noetherian scheme, and denote by $X_{\text{ét}}$ (resp. $X_{\text{fét}}$) the étale site (resp. the finite étale site) of $X$. The forgetful functor from the category of finite étale covers of $X$ to the category of étale covers induces a natural morphism of sites

$$\rho: X_{\text{ét}} \rightarrow X_{\text{fét}}.$$ 

If $X$ is connected, then for a given geometric point $\bar{x} \rightarrow X$, the site $X_{\text{fét}}$ is equivalent to the classifying site $B\pi^{\text{et}}_1(X, \bar{x})$ whose underlying category is the category of continuous $\pi^{\text{et}}_1(X, \bar{x})$-sets. For every locally constant torsion étale sheaf $F$ on $X$ and $q \geq 0$, we have then a natural morphism

$$\rho^q: H^q(\pi^{\text{et}}_1(X, \bar{x}), F_{\bar{x}}) \approx H^q_{\text{fét}}(X, \rho, F) \rightarrow H^q_{\text{ét}}(X, F).$$

Definition 4.4 [1, 2]. A pointed connected noetherian scheme $(X, \bar{x})$ is an algebraic $K(\pi, 1)$ space if for every locally constant constructible torsion étale sheaf $F$ on $X$, the natural morphisms

$$\rho^q: H^q(\pi^{\text{et}}_1(X, \bar{x}), F_{\bar{x}}) \rightarrow H^q_{\text{ét}}(X, F)$$

are isomorphisms for all $q \geq 0$.

Example 4.5. The following schemes are examples of algebraic $K(\pi, 1)$ spaces:

(1) The spectrum of a field $\text{Spec} k$.
(2) A smooth connected curve $X$ over a field $k$ such that $X_{\bar{k}}$ is not isomorphic to the projective line $\mathbb{P}^1_{\bar{k}}$ [28].
(3) Abelian varieties (see Remark 7.3).
(4) Finite product of geometrically connected and geometrically unibranch $K(\pi, 1)$ varieties over a field $k$ of characteristic zero [28].
(5) Connected affine $F_p$-schemes [2].

**Proposition 4.6** [1, Proposition 3.2]. Let $X$ be a connected noetherian scheme. The following statements are equivalent:

(i) $X$ is an algebraic $K(\pi, 1)$ space.
(ii) For every locally constant constructible torsion étale sheaf $F$ on $X$, and every class $\beta \in H^q_{\text{ét}}(X, F)$ with $q \geq 1$, there exists a finite étale cover $f : Y \to X$ such that $f^*(\beta) = 0$ in $H^q_{\text{ét}}(Y, f^*F)$.

**Corollary 4.7.** For any connected noetherian scheme $X$, which is an algebraic $K(\pi, 1)$ space, one has $\text{Br}(X) = \text{Br}'(X)$ up to a $\text{char}(X)$-torsion.

**Remark 4.8.** It is proven in [2] that connected affine schemes in positive characteristic are $K(\pi, 1)$ spaces, hence we obtain here an alternative proof of Gabber’s result [12, p. 163, Theorem 1] for affine schemes.

Recall that a scheme $X$ is geometrically unibranch if for every $x \in X$ the strict henselization of the local ring $\mathcal{O}_{X, x}$ is irreducible (for the original definition see [13, 6.15.1]). In particular, any normal scheme is geometrically unibranch [13, Proposition 6.15.6].

**Proposition 4.9** [2, Proposition 4.4]. Let $(X, \bar{x})$ be a pointed connected noetherian geometrically unibranch scheme. Then, $X$ is an algebraic $K(\pi, 1)$ space if only if $\pi_1^{\text{ét}}(X, \bar{x}) = 0$ for all $n \geq 2$.

**Lemma 4.10.** Let $(X, \bar{x})$ be a pointed connected noetherian scheme. Then,

(a) for any locally constant constructible torsion étale sheaf $F$ on $X$, we have

$$H^1_{\text{ét}}(X, F) \cong H^1(\pi_1^{\text{ét}}(X, \bar{x}), F_{\bar{x}});$$

(b) if $Y \to X$ is a finite étale cover, then $Y$ is an algebraic $K(\pi, 1)$ space if only if $X$ is.

**Proof.** (a) This follows from the exact sequence (4.1) in the proof of Theorem 4.2

$$0 \to H^1(\pi_1^{\text{ét}}(X, \bar{x}), H^0(\text{Et}(X)^-, p^*F)) \to H^1_{\text{ét}}(X, F) \to H^0(\pi_1^{\text{ét}}(X, \bar{x}), H^1(\text{Et}(X)^-, p^*F))$$

and the fact that $H^0(\text{Et}(X)^-, p^*F) = F_{\bar{x}}$ and $H^1(\text{Et}(X)^-, p^*F) = 0$. For an alternative proof by means of the torsor interpretation of $H^1_{\text{ét}}(X, F)$ see [2, Lemma 4.3].

(b) This is [1, Proposition 3.2.(b)]. Alternatively, since $Y \to X$ is a finite étale surjective morphism. Then $X$ is normal if only if $Y$ is. Therefore, for the normal case, the statement can be deduced from Lemma 4.1 and Proposition 4.9. □

The following proposition combines assertions of Proposition 4.6 and Proposition 4.9 for $q = 2$.

**Proposition 4.11.** For a pointed connected noetherian geometrically unibranch scheme $(X, \bar{x})$, the following statements are equivalent:

(i) $\pi_2^{\text{ét}}(X, \bar{x}) = 0$.
(ii) $H^2_{\text{ét}}(X, F) \simeq H^2(\pi_1^{\text{ét}}(X, \bar{x}), F_{\bar{x}})$ for every locally constant constructible torsion étale sheaf $F$ on $X$.
(iii) For every locally constant constructible torsion étale sheaf $F$ on $X$, and every class $\beta \in H^2_{\text{ét}}(X, F)$, there exists a finite étale cover $f : Y \to X$ such that $f^*(\beta) = 0$ in $H^2_{\text{ét}}(Y, f^*F)$.

**Proof.** (i)$\Rightarrow$(ii)$\Rightarrow$(iii): Follow from the proof of Theorem 4.2.
The finite étale cover \( f : Y \to X \) induces by [28, Lemma 2.1] a connected covering \( Et(f) : Et(Y) \to Et(X) \) through which the universal covering \( p : Et(X)^\sim \to Et(X) \) factors, hence \( H^2(Et(X)^\sim, p^\ast F) = 0 \) for every locally constant constructible torsion étale sheaf \( F \) on \( X \). Thus, \( \text{Hom}(\pi^{\text{ét}}_2(X, \hat{x}), G) = 0 \) for every finite group \( G \). Now \( X \) is geometrically unibranch, which means that \( \pi^{\text{ét}}_2(X, \hat{x}) \) is profinite, thus it must be trivial.  

5 | DESCENT OF \( K(\pi, 1) \) SPACES

5.1 | The smooth case

The aim of this subsection is to prove Proposition 5.3. Let \( X \) be a smooth scheme over a field \( k \), then \( X \) is in particular a regular scheme, and hence geometrically unibranch. Keeping in mind Proposition 4.9, we use descent properties of \( K(\pi, 1) \) spaces to apply Theorem 4.2. We need the following proposition, which is a first descent result concerning Schröer spaces. The notations \( \pi^\text{top}_n \) and \( \pi^\text{ét}_n \) will be used to make the difference between topological and étale homotopy groups.

**Proposition 5.1.** A connected Schröer space \( X \) is an algebraic \( K(\pi, 1) \) space.

**First proof.** Suppose that \( X \) is in addition geometrically unibranch. Let \( x \in X(\mathbb{C}) \), and \( \hat{x} \to X \) the corresponding geometric point. By definition \( X(\mathbb{C}) \) is weakly homotopy equivalent to the classifying space \( B\pi^\text{top}_1(X(\mathbb{C}), x) \). Since \( X \) is geometrically unibranch, by [4, Corollary 12.10] the map \( (X(\mathbb{C}))^\wedge \to Et(X) \) is a \( \wedge \)-isomorphism, where \( X^\wedge \) is the completion of \( X \). Thus, according to [4, Corollary 4.4] \( Et(X) \) is weakly homotopy equivalent to \( (X(\mathbb{C}))^\wedge \). Since \( \pi_1^\text{top}(X(\mathbb{C}), x) \) is a good group, it follows from [4, Corollary 6.6] that \( (B\pi_1^\text{top}(X(\mathbb{C}), x))^\wedge = B(\pi_1^\text{top}(X(\mathbb{C}), x))^\wedge \). On the other hand, by Riemann existence theorem [24, Chapter III, Lemma 3.14], one has \( \pi_1^\text{ét}(X, \hat{x}) = \pi_1^\text{top}(X(\mathbb{C}), x)^\wedge \). Therefore, \( Et(X) \) is weakly homotopy equivalent to \( B\pi_1^\text{ét}(X, \hat{x}) \), which means that \( \pi^{\text{ét}}_n(X, \hat{x}) = 0 \) for all \( n \geq 2 \). Thus, by Proposition 4.9, \( X \) is an algebraic \( K(\pi, 1) \) space.  

**Second proof (Achinger).** In general, one can deduce the statement from the following commutative diagram

\[
\begin{array}{ccc}
H^n(\pi^{\text{ét}}_2(X, \hat{x}), F_x) & \longrightarrow & H^n(\pi_1^\text{top}(X(\mathbb{C}), x), (p^\ast F)_x) \\
\downarrow & & \downarrow \\
H^q(X, F) & \longrightarrow & H^q(X(\mathbb{C}), p^\ast F)
\end{array}
\]

where \( F \) is a locally constant constructible torsion étale sheaf on \( X \). Indeed, the map on the right is an isomorphism because \( X(\mathbb{C}) \) is a topological \( K(\pi, 1) \) space. The lower map is an isomorphism by Artin comparison theorem [3, Exp. XVI, Theorem 4.1], and since \( \pi_1^\text{ét}(X, \hat{x}) = \pi_1^\text{top}(X(\mathbb{C}), x)^\wedge \), and \( F_x = (p^\ast F)_x \), it follows from the definition of good groups that the upper map is an isomorphism. Hence the assertion.  

The next result generalizes the statement of Corollary 3.7. It follows immediately from Proposition 5.1 and Corollary 4.7.

**Corollary 5.2.** Let \( X \) be a connected scheme of finite type over \( \mathbb{C} \), if \( X \) is a Schröer space, then \( \text{Br}(X) = \text{Br}'(X) \).

**Proposition 5.3.** Let \( X \) be a smooth geometrically connected scheme of finite type over a field \( k \), which is finitely generated over \( \mathbb{Q} \). Suppose that there is an embedding of \( k \) in \( \mathbb{C} \) such that \( X_\mathbb{C} \) becomes a Schröer space. Then, for a geometric point \( \hat{x} \to X_\mathbb{C} \), we have \( \pi^{\text{ét}}_2(X, \hat{x}) = 0 \).

**Proof.** Let \( p : X_\mathbb{C} \to X_\bar{k} \) be the natural map. For every locally constant constructible torsion étale sheaf \( F \) on \( X_\bar{k} \) and \( q \geq 0 \), we have the following commutative diagram:

\[
\begin{array}{ccc}
H^q(\pi^{\text{ét}}_2(X_\mathbb{C}, \hat{x}), F_x) & \longrightarrow & H^q(\pi^{\text{ét}}_1(X_\mathbb{C}, \hat{x}), F_x) \\
\downarrow & & \downarrow \\
H^q(X_\mathbb{C}, F) & \longrightarrow & H^q(X_\bar{k}, p^\ast F)
\end{array}
\]
Since $k$ is finitely generated over $\mathbb{Q}$, then according to [11, Proposition 6.1], we have $\pi_1^{\text{ét}}(X_k, \bar{x}) \simeq \pi_1^{\text{ét}}(X, \bar{x})$, hence the map
\[ H^q(\pi_1^{\text{ét}}(X_k, \bar{x}), F_{\bar{x}}) \longrightarrow H^q(\pi_1^{\text{ét}}(X, \bar{x}), F_{\bar{x}}) \]
is an isomorphism. The map $H^q_{\text{ét}}(X_k, F) \to H^q_{\text{ét}}(X, p^*F)$ is an isomorphism by the smooth base change theorem [3, Exp. XVI, Corollary 1.6]. By Proposition 5.1, $X_k$ is an algebraic $K(\pi, 1)$ space, which means that the map $H^q(\pi_1^{\text{ét}}(X_k, \bar{x}), F_{\bar{x}}) \to H^q_{\text{ét}}(X, p^*F)$ is an isomorphism. It follows that
\[ H^q(\pi_1^{\text{ét}}(X_k, \bar{x}), F_{\bar{x}}) \longrightarrow H^q_{\text{ét}}(X_k, F) \]
is an isomorphism. Thus, $X_k$ is an algebraic $K(\pi, 1)$ space. We finish the proof by following an idea of Mochizuki [25]; consider the Grothendieck homotopy exact sequence (cf. [15, Exp. IX, Theorem 6.1])
\[ 1 \longrightarrow \pi_1^{\text{ét}}(X, \bar{x}) \longrightarrow \pi_1^{\text{ét}}(X, \bar{x}) \longrightarrow \pi_1^{\text{ét}}(\text{Spec } k, \bar{x}) \longrightarrow 1. \]
The injectivity of the map $\pi_1^{\text{ét}}(X_k, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x})$ means (after possible base change to a finite étale cover of Spec $k$) that any finite étale cover of $X_k$ can be realized as the restriction to $X_k$ of a finite étale cover of $X$ (cf. [25, section 4, p. 151]).

Now suppose that $F$ is a locally constant constructible torsion étale sheaf on $X$. Take a class $\beta \in H^2_{\text{ét}}(X, F)$, and consider the Leray spectral sequence associated to the morphism $X \to \text{Spec } k$
\[ E_2^{p,q} = H^p_{\text{ét}}(k, H^q_{\text{ét}}(X_k, \rho^*F)) \Rightarrow H^{p+q}_{\text{ét}}(X, F), \]
where $\rho : X_k \to X$ is the natural morphism. As in [25, section 4, p. 152], the cohomology group $E_2 = H^2_{\text{ét}}(X, F)$ has a filtration $F_n(E_2)_{n \geq 0}$ with three highest subquotients, which are in fact submodules of $E^{1,1}, E^{1,0},$ and $E^{2,0}$, respectively. Therefore, we may assume that the class $\beta$ belongs to one of these three submodules.

If $\beta \in H^0_{\text{ét}}(k, H^2_{\text{ét}}(X_k, \rho^*F))$, then since $X_k$ is a $K(\pi, 1)$ space, there exists a finite étale cover $g : Y \to X_k$ trivializing $\beta$ in $H^2_{\text{ét}}(Y, g^*\rho^*F)$. As mentioned above, there exists a finite étale cover $f : Y' \to X$ such that $Y = Y' \times_X X_k$.

Hence we may assume that $f^*(\beta)$ is zero in $H^2_{\text{ét}}(Y', f^*F)$.

If $\beta \in H^1_{\text{ét}}(k, H^0_{\text{ét}}(X_k, \rho^*F))$, it follows from Proposition 4.6 and Lemma 4.10 (or particularly form the fact that Spec $k$ is an algebraic $K(\pi, 1)$ space) that there exists a finite étale cover $g : Y \to X$ trivializing $\beta$. Therefore, if we consider the finite étale cover $f : Y' = X \times_k Y \to X$, then we may assume that $f^*(\beta) = 0$ in $H^2_{\text{ét}}(Y', f^*F)$.

If $\beta \in H^2_{\text{ét}}(k, H^0_{\text{ét}}(X_k, \rho^*F))$, by the same argument, we may assume the existence of a finite étale cover killing $\beta$.

The up-shot is that for every class $\beta \in H^0_{\text{ét}}(X, F)$, there exists a finite étale cover $f : Y \to X$ such that $f^*(\beta) = 0$ in $H^2_{\text{ét}}(Y, f^*F)$, which means by Proposition 4.11 that the map $H^2(\pi_1^{\text{ét}}(X, \bar{x}), F_{\bar{x}}) \to H^2_{\text{ét}}(X, F)$ is an isomorphism, and hence $\pi_2^{\text{ét}}(X, \bar{x}) = 0$. \qed

Now if $X$ is a smooth proper geometrically connected scheme over an algebraically closed field $k$ of characteristic 0, then there is a subfield $F \subset k$ finitely generated over $\mathbb{Q}$ and $X$ is defined over $F$, that is, there exists a smooth proper geometrically connected scheme $Y$ of finite type over $F$ such that $Y_k = X$. We can prove the following.

**Proposition 5.4.** Let $X$ and $Y$ be as above. If there exists an embedding $i : F \to \mathbb{C}$ such that $Y_\mathbb{C}$ becomes a Shröder space, then $\text{Br}(X) = \text{Br}'(X)$.

**Proof.** Let $\bar{y} \to Y_\mathbb{C}$ be a geometric point of $Y_\mathbb{C}$. By Proposition 5.1 and Proposition 4.9, $\pi_n^{\text{ét}}(Y_\mathbb{C}, \bar{y}) = 0$ for all $n \geq 2$. On the other hand, since $Y$ is proper, it follows from [4, Corollary 12.12] and [4, Corollary 4.4] that $\text{Et}(Y_F)$ is weakly homotopy equivalent to $\text{Et}(Y_\mathbb{C})$ and $\text{Et}(Y_\mathbb{C}) = \text{Et}(X)$ is weakly homotopy equivalent to $\text{Et}(Y_F)$, hence $\pi_n^{\text{ét}}(X, \bar{x}) = 0$ for all $n \geq 2$, where $\bar{x}$ is a geometric point above $\bar{y}$. Now Theorem 4.2 applies. \qed

### 5.2 The proper case

Proper schemes over algebraically closed fields have nice properties such that the stability of the étale fundamental group and étale cohomology groups after base changing to another algebraically closed field. The cohomological
Brauer group behaves in the same way in this case. In what follows, separable extensions of fields are in the sense of [32, Tag 0301].

**Proposition 5.5.** Let \( k \) be a separably closed field of characteristic \( p \), and \( K/k \) a separable extension of fields. Let \( X \) be a proper connected scheme over the field \( k \). Then, \( Br'(X) = Br'(X_K) \) up to a \( p \)-torsion.

To prove the proposition, we need the following lemma, which is due to Ben Lim from MathOverflow.

**Lemma 5.6.** Let \( K/k \) be a separable extension of fields with \( k \) separably closed. Let \( X \) be a quasi-compact and quasi-separated scheme over \( k \). Then, the natural morphism \( H^2_{\text{ét}}(X, \mathbb{G}_m, X) \to H^2_{\text{ét}}(X_K, \mathbb{G}_m, X_K) \) is injective.

**Proof.** Let \( p : X_K \to X \) be the canonical map. Since \( K \) is separable over \( k \), we can write \( K = \lim R_i \) as a direct limit of smooth \( k \)-algebras \( R_i \). Now \( X_{R_i} \) are quasi-compact and quasi-separated, thus by [24, Chapter III, Lemma 1.16], we have

\[
H^2_{\text{ét}}(X_K, \mathbb{G}_m, X_K) \cong \lim H^2_{\text{ét}}(X_{R_i}, \mathbb{G}_m, X_{R_i}).
\]

Let \( \alpha \in H^2_{\text{ét}}(X, \mathbb{G}_m, X) \) with \( p^*(\alpha) = 0 \) in \( H^2_{\text{ét}}(X_K, \mathbb{G}_m, X_K) \), then there is a smooth \( k \)-algebra \( R_i \) such that \( p^*(\alpha) = 0 \) in \( H^2_{\text{ét}}(X_{R_i}, \mathbb{G}_m, X_{R_i}) \). Since \( k \) is separably closed, the smoothness of \( R_i \) implies that \( \text{Spec } R_i \) admits a \( k \)-rational point. Therefore, the map \( p_i : X_{R_i} \to X \) admits a section \( g \), which means that the composition of cohomology groups

\[
H^2_{\text{ét}}(X, \mathbb{G}_m, X) \xrightarrow{p_i^*} H^2_{\text{ét}}(X_{R_i}, \mathbb{G}_m, X_{R_i}) \xrightarrow{g^*} H^2_{\text{ét}}(X, \mathbb{G}_m, X)
\]

is the identity. It follows then that \( \alpha = g^* p_i^*(\alpha) = 0 \). \( \Box \)

**Proof of Proposition 5.5.** Fix an integer \( n \) prime to \( p \), and consider the Kummer exact sequence

\[
1 \to \mu_{n,X} \to \mathbb{G}_{m,X} \xrightarrow{x \mapsto x^n} \mathbb{G}_{m,X} \to 1.
\]

(5.1)

The corresponding exact sequence of cohomology yields an exact sequence

\[
0 \to \text{Pic}(X) \to H^2_{\text{ét}}(X, \mu_{n,X}) \to H^2_{\text{ét}}(X, \mathbb{G}_{m,X}) \to H^2_{\text{ét}}(X, \mathbb{G}_{m,X}),
\]

(5.2)

where \( \text{Pic}(X) = H^1_{\text{ét}}(X, \mathbb{G}_{m,X}) \). We have a similar exact sequence for \( X_K \), which gives rise to the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & \text{Pic}(X) \\
\downarrow & & \downarrow \\
\text{Pic}(X_K) & \to & H^2_{\text{ét}}(X_K, \mu_{n,X}) \to H^2_{\text{ét}}(X_K, \mathbb{G}_{m,X}) \\
\downarrow & & \downarrow \\
0 & \to & \text{Pic}(X_K)
\end{array}
\]

(5.3)

The map \( H^2_{\text{ét}}(X, \mu_{n,X}) \to H^2_{\text{ét}}(X_K, \mu_{n,X}) \) is an isomorphism by the proper base change theorem [3, Exp. XII, Corollary 5.4], and by a diagram chasing, the injectivity of the map \( H^2_{\text{ét}}(X, \mathbb{G}_{m,X}) \to H^2_{\text{ét}}(X_K, \mathbb{G}_{m,X}) \) would imply that \( \text{Pic}(X) = \text{Pic}(X_K) \). The exact sequence (5.2) induces a short exact sequence

\[
0 \to \text{Pic}(X) \to H^2_{\text{ét}}(X, \mu_{n,X}) \to \_nBr'(X) \to 0.
\]

(5.4)

The equality \( \_nBr'(X) = \_nBr'(X_K) \) results then form the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & \text{Pic}(X) \\
\downarrow & & \downarrow \\
\text{Pic}(X_K) & \to & \_nBr'(X) \\
\downarrow & & \downarrow \\
0 & \to & \text{Pic}(X_K)
\end{array}
\]

(5.5)

\( \Box \)
Proposition 5.7. Let $X$ be a proper geometrically connected scheme over an algebraically closed field $k$. Suppose that $k$ can be embedded as a subfield of $\mathbb{C}$ such that $X_\mathbb{C}$ becomes a Schröer space. Then, $X$ is an algebraic $K(\pi, 1)$ space.

Proof. Let $p : X_\mathbb{C} \to X$ be the natural map and $\tilde{x} \to X_\mathbb{C}$ a geometric point of $X_\mathbb{C}$. For every locally constant constructible torsion étale sheaf $\mathcal{F}$ on $X$ and $q \geq 0$, we have a commutative diagram

$$
\begin{array}{c}
H^q(\pi^\text{ét}_1(X, \tilde{x}), \mathcal{F}) \ar[r] \ar[d] & H^q(\pi^\text{ét}_1(X_\mathbb{C}, \tilde{x}), \mathcal{F}) \ar[d] \\
H^q_\text{ét}(X, F) \ar[r] & H^q_\text{ét}(X_\mathbb{C}, p^* F)
\end{array}
$$

Due to [11, Proposition 5.3], we have $\pi^\text{ét}_1(X, \tilde{x}) \simeq \pi^\text{ét}_1(X_\mathbb{C}, \tilde{x})$, hence the map

$$
H^q(\pi^\text{ét}_1(X, \tilde{x}), \mathcal{F}) \to H^q(\pi^\text{ét}_1(X_\mathbb{C}, \tilde{x}), \mathcal{F})
$$

is an isomorphism. The proper base change theorem [3, Exp. XII, Corollary 5.4] asserts that $H^q_\text{ét}(X, F) \simeq H^q_\text{ét}(X_\mathbb{C}, p^* F)$. Proposition 5.1 shows that the map $H^q(\pi^\text{ét}_1(X_\mathbb{C}, \tilde{x}), \mathcal{F}) \to H^q_\text{ét}(X_\mathbb{C}, p^* \mathcal{F})$ is an isomorphism. Hence the assertion. □

Remark 5.8. Let $X$ be a scheme as in Proposition 5.7, then by Corollary 4.7 we have $\text{Br}(X) = \text{Br}'(X)$. This can be obtained alternatively by the descent of Brauer maps when in particular the natural morphism $\text{Br}(X) \to \text{Br}(X_\mathbb{C})$ is surjective. Indeed, in the following commutative diagram

$$
\begin{array}{ccc}
\text{Br}(X) & \to & \text{Br}(X_\mathbb{C}) \\
\downarrow & & \downarrow \\
\text{Br}'(X) & \to & \text{Br}'(X_\mathbb{C})
\end{array}
$$

the map on the right is an isomorphism by Corollary 3.7, and Proposition 5.5 asserts that the lower map is also an isomorphism. The injectivity of Brauer maps (Theorem 2.1(iii)) implies that the upper map is bijective. Thus the assertion.

More generally, let us replace the morphism $X \to \text{Spec } k$ by a morphism $f : X \to S$ of arbitrary schemes. We have the following result.

Proposition 5.9. Let $f : X \to S$ be a proper morphism of connected noetherian schemes such that the sequence

$$
1 \to \pi^\text{ét}_1(X, \tilde{x}, \bar{s}) \to \pi^\text{ét}_1(X, \tilde{x}) \to \pi^\text{ét}_1(S, \bar{s}) \to 1
$$

is exact, where $\bar{s} \to S$ is a geometric point of $S$, and $\tilde{x} \to X_\bar{s}$ is a geometric point in the fiber $f^{-1}(\bar{s}) = X_\bar{s}$ above $\bar{s}$. Suppose that $\pi^\text{ét}_2(X, \tilde{x}, \bar{s}) = \pi^\text{ét}_2(S, \bar{s}) = 0$, then $\text{Br}(X) = \text{Br}'(X)$ up to a char($X$)-torsion.

Proof. Consider the Leray spectral sequence associated to the morphism $f : X \to S$

$$
E_2^{p,q} = H^p_\text{ét}(S, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}_\text{ét}(X, F),
$$

where $F$ is a locally constant constructible torsion étale sheaf on $X$, and $p : X_\bar{s} \to X$ is the natural map. Observe first that, after the restriction to some finite étale cover of $S$, we can replace $R^q f_* \mathcal{F}$ by $H^q_\text{ét}(X_\bar{s}, p^* \mathcal{F})$. Indeed, since $f$ is proper, then $R^q f_* \mathcal{F}$ is a locally constant constructible sheaf. Therefore, by [24, Chapter V, Proposition 1.1 and Remark 1.2.(b)] and by the proper specialization [24, Chapter VI, Corollary 2.5], there exists a finite étale cover $g : S' \to S$ such that

$$
g^*(R^q f_* \mathcal{F}) = (R^q f_* \mathcal{F})_{\tilde{x}} = H^q_\text{ét}(X_{\tilde{x}}, p^* \mathcal{F}).
$$

Now let $F = \mu_n, X$ for some $n$ invertible in $X$, and let $\beta \in H^2(X, \mu_n, X)$. The Mochizuki argument used in the last step in the proof of Proposition 5.3 can be applied to get a finite étale cover $f : Y \to X$, which kills the class $\beta$. Now Lemma 2.2 applies. □
Remark 5.10. The sequence (5.9) is always right exact when \( f \) is flat with geometrically connected and reduced fibers (cf. [15, Exp. X, Corollary 1.4]). If moreover \( f \) is smooth and admits a section, and \( Y \subset X \) is a complement of a normal crossing divisor relative to \( S \), then for a geometric point \( \overline{y} \to Y \) with \( g(\overline{y}) = \overline{s} \), where \( g : Y \to S \) is the restriction map, we have an exact sequence (cf. [15, Exp. XIII, Proposition 4.1 and Example 4.4])

\[
1 \longrightarrow \pi^\text{et}_1(Y, \overline{y}) \longrightarrow \pi^\text{et}_1(Y, \overline{y}) \longrightarrow \pi^\text{et}_1(S, \overline{s}) \longrightarrow 1.
\]

6 | LOCAL \( K(\pi, 1) \) CONDITION

Grothendieck proved [14, II, Theorem 2.1] that for a noetherian scheme \( X \), and \( \beta \in \text{Br}'(X) \), there exists an open \( U \subset X \) such that \( X - U \) has codimension \( \geq 2 \), and an Azumaya algebra \( A \) on \( U \) such that \( \delta([A]) = \beta|_U \). He applied this to show that for a regular noetherian scheme of dimension \( \leq 2 \), one has \( \text{Br}(X) = \text{Br}'(X) \) [14, II, Corollary 2.2]. In the next theorem, we prove that the same statement holds for a smooth \( k \)-variety of arbitrary dimension, provided that the subscheme \( U \) has vanishing second étale homotopy group. A key ingredient in the proof is the Zariski–Nagata purity theorem and the following purity theorem for the cohomological Brauer group, which was predicted by Grothendieck in [14, III, Section 6] and proved recently by Česnavičius in the general case.

Theorem 6.1 [6, Theorem 6.1]. Let \( X \) be a regular noetherian scheme, and \( U \subset X \) an open subscheme such that the complement \( X - U \) has codimension \( \geq 2 \). Then, \( H^2_\text{et}(X, \mathbb{G}_m, X) \simeq H^2_\text{et}(U, \mathbb{G}_m, U) \).

Combined with Grothendieck’s result, this theorem provides the following local statement for Brauer groups.

Proposition 6.2. For any regular noetherian scheme \( X \) with finite Brauer group \( \text{Br}(X) \), there exists an open \( U \subset X \) with codimension of \( X - U \geq 2 \) such that \( \text{Br}(U) = \text{Br}'(U) \).

Now we assert the main theorem.

Theorem 6.3. Let \( X \) be a smooth variety over an algebraically closed field \( k \) of characteristic \( p \geq 0 \). Suppose that there exists an open subscheme \( Y \subset X \) such that \( X - Y \) has codimension \( \geq 2 \), and \( \pi^\text{et}_2(Y, \overline{y}) = 0 \) for some geometric point \( \overline{y} \to Y \). Then \( \text{Br}(X) = \text{Br}'(X) \).

Proof. Notice first that the case of \( p \)-torsion classes follows by Remark 4.3.(a). Now fix an integer \( n \) prime to \( p \), and choose a class \( \beta \in H^2_\text{et}(X, \mu_n, X) \). As in the proof of Proposition 5.5 the Kummer exact sequence (5.1) for both \( X \) and \( Y \) gives rise to the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Pic}(X)_n & \longrightarrow & H^2_\text{et}(X, \mu_n, X) & \longrightarrow & n\text{Br}'(X) & \longrightarrow & 0 \\
\downarrow & & & & & \downarrow & & & \downarrow & \\
0 & \longrightarrow & \text{Pic}(Y)_n & \longrightarrow & H^2_\text{et}(Y, \mu_n, Y) & \longrightarrow & n\text{Br}'(Y) & \longrightarrow & 0.
\end{array}
\]

(6.1)

The map \( n\text{Br}'(X) \to n\text{Br}'(Y) \) is an isomorphism by Theorem 6.1. Since \( X \) is smooth, then we have \( \text{Pic}(X) \simeq \text{CH}^1(X) \simeq \text{Cl}(X) \), where \( \text{CH}^1(X) \) is the Chow group of codimension 1, and \( \text{Cl}(X) \) is the Weil divisor class group of \( X \). As for \( Y \) we have also \( \text{Pic}(Y) \simeq \text{CH}^1(Y) \simeq \text{Cl}(Y) \). Therefore, by [18, II, Proposition 6.5], the map \( \text{Pic}(X) \to \text{Pic}(Y) \) is an isomorphism. It follows from the above commutative diagram that \( H^2_\text{et}(X, \mu_n, X) \simeq H^2_\text{et}(Y, \mu_n, Y) \). On the other hand, by Zariski–Nagata purity theorem, the functor \( S \to S \times_Y Y \) induces an equivalence of categories between finite étale covers of \( X \) and finite étale covers of \( Y \), thus \( \pi^\text{et}_2(Y, \overline{y}) \simeq \pi^\text{et}_2(Y, \overline{y}) \) (cf. [16, Exp X, Theorem 3.10]). This gives an isomorphism

\[
H^2(\pi^\text{et}_2(X, \overline{y}), (\mu_n, X)_\overline{y}) \simeq H^2(\pi^\text{et}_2(Y, \overline{y}), (\mu_n, Y)_\overline{y}).
\]

By assumption \( \pi^\text{et}_2(Y, \overline{y}) = 0 \), which means that the map

\[
H^2(\pi^\text{et}_2(Y, \overline{y}), (\mu_n, Y)_\overline{y}) \longrightarrow H^2_\text{et}(Y, \mu_n, Y)
\]

is an isomorphism.
is an isomorphism. Therefore, from the following commutative diagram,

\[
\begin{array}{ccc}
H^2(\pi_1(X, \bar{y}), (\mu_n, X)) & \longrightarrow & H^2(\pi_1(Y, \bar{y}), (\mu_n, Y)) \\
\downarrow & & \downarrow \\
H^2_\text{ét}(X, \mu_n, X) & \longrightarrow & H^2_\text{ét}(Y, \mu_n, Y)
\end{array}
\]

we get an isomorphism

\[
H^2(\pi_1(X, \bar{y}), (\mu_n, X)) \cong H^2_\text{ét}(X, \mu_n, X).
\]

Due to Proposition 4.11, there exists a finite étale cover \( f : X' \to X \) such that \( f^*(\beta) = 0 \) in \( H^2_\text{ét}(X', \mu_n, X') \). Therefore, in the light of Lemma 2.2, we conclude that \( \text{Br}(X) = \text{Br}'(X) \).

\[\square\]

7 | APPLICATION TO ABELIAN VARIETIES

In [19] (see also [20]), Hoobler showed that \( \text{Br}(A) = \text{Br}'(A) \) for an abelian variety \( A \) over a field \( k \) by proving that \( A \) satisfies the generalized theorem of a cube. An alternative proof was given by Berkovich in [5] by proving the following equality:

\[
H^2_\text{ét}(A, \mu_n, A) = \bigwedge \text{Hom}(\mu_n A, \mu_n A)
\]

when \( A \) is defined over a separably closed field \( k \) and \( n \) is prime to \( \text{char}(k) \). A very strong result established by Gabber states that \( \text{Br}(X) = \text{Br}'(X) \) when \( X \) is a scheme with ample invertible sheaf, and this holds for any quasi-projective variety over an affine scheme. The result is not published and de Jong [8] wrote down a different proof (see also [7, Section 4.2]).

We give in turn another proof based on the étale homotopy type of abelian varieties, and which works for a more general class of smooth algebraic groups. We need the following result of Demarche and Szamuely, which is a general form of the Riemann existence theorem for smooth connected algebraic groups. To simplify notations, we omit geometric base points.

**Lemma 7.1.** Let \( G \) be a connected smooth algebraic group over \( \mathbb{C} \). For all \( n \geq 1 \), there is an isomorphism

\[
\pi_n^\text{ét}(G) \cong \pi_n^\text{top}(G(\mathbb{C}))^\wedge.
\]

**Proof** (Sketch). As in the first proof of Proposition 5.1, \( (G(\mathbb{C}))^\wedge \) is weakly homotopy equivalent to \( Et(G) \). On the other hand, Demarche and Szamuely remarked that the homotopy groups \( \pi_n^\text{top}(G(\mathbb{C})) \) are finitely generated abelian groups. Thus, by a result of Sullivan [33, Theorem 3.1], the natural map \( \pi_n^\text{top}(G(\mathbb{C}))^\wedge \to \pi_n^\text{top}(G(\mathbb{C}))^\wedge \) is an isomorphism. Hence the assertion.

\[\square\]

**Proposition 7.2.** Let \( A \) be an abelian variety over a field \( k \) of characteristic 0. Then, \( \text{Br}(A) = \text{Br}'(A) \).

**Proof.** Note that by a limit argument as in [20, Corollary 4 and A2] we can assume \( k \) algebraically closed. Hence, there is a subfield \( F \subset k \) finitely generated over \( \mathbb{Q} \) and \( A \) is defined over \( F \), that is, there is an abelian variety \( B \) over \( F \) such that \( B_{\overline{F}} = A \). Choose an embedding \( i : \overline{F} \to \mathbb{C} \). Applying the above lemma, we get \( \pi_n^\text{ét}(B_{\mathbb{C}}) \cong \pi_n^\text{top}(B_{\mathbb{C}}(\mathbb{C}))^\wedge \). It follows from [27, Proposition 5.4] that \( Et(B_{\overline{F}}) \) is weakly homotopy equivalent to \( Et(B_{\mathbb{C}}) \) and \( Et(B_{\overline{F}}) = Et(A) \) is weakly homotopy equivalent to \( Et(B_{\overline{F}}) \). Since \( B_{\mathbb{C}}(\mathbb{C}) \) is a complex torus, which is a topological \( K(\pi, 1) \) space, we get \( \pi_n^\text{ét}(A) = \pi_n^\text{top}(B_{\mathbb{C}}(\mathbb{C}))^\wedge \). Hence, Theorem 4.2 implies that \( \text{Br}(A) = \text{Br}'(A) \).

\[\square\]

**Remark 7.3.** The argument sketched in the proof of Proposition 7.2 shows in fact that if \( G \) is a connected smooth algebraic group over an algebraically closed field \( k \) of characteristic 0, such that \( G(\mathbb{C}) \) is a topological \( K(\pi, 1) \) space in the case that \( k = \mathbb{C} \), then \( G \) is an algebraic \( K(\pi, 1) \) space. This generalizes the result of [27, Corollary 5.5.(b)] for an abelian variety \( A \), which requires the goodness of \( \pi_n^\text{top}(A(\mathbb{C})) \).
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