Fully discrete finite element approximation for a family of degenerate parabolic mixed equations

Ramiro Acevedo∗ Christian Gómez† Bibiana López-Rodríguez‡

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Abstract

The aim of this work is to show an abstract framework to analyze the numerical approximation for a family of linear degenerate parabolic mixed equations by using a finite element method in space and a Backward-Euler scheme in time. We consider sufficient conditions to prove that the fully-discrete problem has a unique solution and prove quasi-optimal error estimates for the approximation. Furthermore, we show that mixed finite element formulations arising from dynamics fluids (time-dependent Stokes problem) and from electromagnetic applications (eddy current models), can be analyzed as applications of the developed theory. Finally, we include numerical tests to illustrate the performance of the method and confirm the theoretical results.

Keywords: Degenerate parabolic equations, mixed problems, finite element method, fully-discrete approximation, error estimates.

1 Introduction

The classical (non-degenerate) mixed parabolic equations, mainly inspired on the Stokes Problem, have been widely studied. For instance, in [9] the authors have introduced some abstract framework for that kind of problems. However, mixed formulations arising from electromagnetic problems (see, for instance [1, 2, 18]), can not fit in that aforementioned theory, because in these cases the first term inside of the time-derivative is not an inner product, which implies that the resulting problem is degenerate.

A degenerate parabolic mixed problem consists in finding $u \in L^2(0,T;X)$ and $\lambda \in H^1(0,T;M)$ such that:

$$\frac{d}{dt} [ \langle Ru(t),v \rangle_Y + b(v,\lambda(t)) ] + \langle Au(t),v \rangle_X = \langle f(t),v \rangle_X \quad \forall v \in X \text{ in } D'(0,T), \quad b(u(t),\mu) = \langle g(t),\mu \rangle_M \quad \forall \mu \in M,$$

(1.1)

where $X$, $Y$ and $M$ are real Hilbert spaces with the imbedding $X \subseteq Y$ is continuous and dense, $R : Y \to Y'$, $A : X \to X'$ are linear and bounded operators, $b : X \times M \to \mathbb{R}$ is a bounded bilinear form, $f \in L^2(0,T;X')$ and $g \in L^2(0,T;M')$. This kind of problems appears in several applications, for instance, to the approximation of the heat equation by means of Raviart-Thomas (RT) elements (see [26]), to the fluid dynamic equations (see [9] and [10]), and to electromagnetic applications (see [1, 2, 18, 28]). Sufficient conditions to the well-posedness of Problem (1.1) with an

∗Universidad del Cauca, Popayán, Colombia, email: rmacevedo@unicauca.edu.co.
†Universidad del Cauca, Popayán, Colombia, email: christiancamilo@unicauca.edu.co.
‡Universidad Nacional sede Medellín, Colombia, email: blopezr@unal.edu.co.
appropriate initial condition, are given in the recent paper [3]. In that work, the authors combine the linear degenerate parabolic equation theory with the classical Babuška-Brezzi Theory to prove the existence and uniqueness of solution, by assuming some reasonable conditions inspired by the application problems.

The purpose of this paper is to provide the analysis for a fully-discrete approximation of the abstract Problem (1.1). This approximation is obtained by using the finite element method in space with a backward Euler in time. In order to develop the analysis, it is necessary to assume the conditions considered in [3] to ensure the well-posedness of the continuous problem (1.1). Furthermore, to obtain the existence and uniqueness for the fully discrete solutions, the bilinear form induced for the operator $A$ have to satisfy a discrete Gårding-type inequality in the discrete kernel of the bilinear form $b$ and this bilinear form $b$ must satisfy the discrete inf-sup condition. The discrete inf-sup condition of $b$ plays an important role to adapt the techniques from the error analysis for finite element approximations for classical parabolic problems. In fact, this discrete inf-sup condition allows to define some projection operator to the orthogonal of the discrete kernel of $b$, which it is necessary to obtain the suitable split of the error to prove quasi-optimal error estimates for the approximation of the main variable of the problem. Moreover, by using again the discrete inf-sup condition, we can obtain the intermediate term to get the corresponding split for the approximation error of the Lagrange Multiplier to show the theoretical convergence of the method.

About the applications for the theory of the fully-discrete approximation of Problem (1.1), we present some problems that arise from dynamic fluids and electromagnetic models. Firstly, we give the convergence analysis of the approximation of the time-dependent Stokes problem. Thus, it can be inferred that the fully-discrete analysis for the non-degenerate mixed problems is a particular case of the theory studied in this work. Many of the real degenerate parabolic problems mainly come from electromagnetic applications, because the existence of two kind of materials (conductors and insulators) lead to the problem has a degenerate character. In fact we show two applications for an electromagnetic problem called the eddy current model. The formulations studied here are based in a time primitive of the electric field and they were studied respectively in [1] (for the case of internal density current source) and in [18] (for the case of density sources with current excitations). The use of the time-primitive of the electric field as main unknown is called modified magnetic vector potential in the electrical engineering literature (see, for instance [13]). Additionally, we perform some numerical results that corroborate the convergence order given for the theory for the model studied in [1] (see Section 5.2.1), since their authors did not present numerical simulation in that work.

The outline of the paper is as follows: In Section 2 we recall the main results given in [3] about the well-posedness of the abstract problem (1.1) and the corresponding analysis for its fully-discrete approximation scheme is presented in Section 3. The results concerning for error estimates of the fully-discrete approximation of the problem are shown in Section 4. The applications of the theory to the time-dependent Stokes problem and to the eddy current model are studied in Section 5, where we use the developed abstract theory to deduce the well-posedness of the discrete problems and the theoretical convergence for their approximations. Furthermore, we show some numerical results for the first of the eddy current models that confirm the expected convergence of the method according to the theory.

2 An abstract degenerate mixed parabolic problem

Let $X$ and $Y$ be two real Hilbert spaces such that $X$ is contained in $Y$ with a continuous and dense imbedding. Furthermore, let $M$ be real reflexive Banach space. Then, we consider the continuous operators $R : Y \to Y'$, $A : X \to X'$ and $b : X \times M \to \mathbb{R}$ be a continuous bilinear form. Let $V$ be the
kernel of the bilinear form $b$, i.e.,

$$V := \{ v \in X : b(v, \mu) = 0 \ \forall \mu \in M \} ,$$

and denote by $W$ its closure with respect to the $Y$-norm, i.e.,

$$W := \overline{V}^{\| \cdot \|_Y}.$$

We consider now the following abstract problem

Given $u_0 \in Y$, $f \in L^2(0,T;X')$ and $g \in L^2(0,T;M')$, the continuous problem is

**Problem 2.1.** Find $u \in L^2(0,T;X)$ and $\lambda \in L^2(0,T;M)$ satisfying the following equations:

$$\frac{d}{dt} [(Ru(t), v)_Y + b(v, \lambda(t))] + (Au(t), v)_X = (f(t), v)_X \quad \forall v \in X \quad \text{in } D'(0,T),$$

$$b(u(t), \mu) = (g(t), \mu)_M \quad \forall \mu \in M,$$

$$(Ru(0), v)_Y = (Ru_0, v)_Y \quad \forall v \in Y.$$

In this order, the hypotheses that guarantee is a well-posed problem are given by

H1. The bilinear form $b$ satisfies a continuous inf-sup condition, i.e., there exists $\beta > 0$ such that

$$\sup_{v \in X} \frac{b(v, \mu)}{\|v\|_X} \geq \beta \|\mu\|_M \quad \forall \mu \in M.$$

H2. $R$ is self-adjoint and monotone on $V$, i.e.,

$$(Rv, w)_Y = (Rw, v)_Y, \quad (Rv, v)_Y \geq 0 \quad \forall v, w \in V.$$

H3. The operator $A$ is self-adjoint on $V$, i.e.,

$$(Av, w)_X = (Aw, v)_X \quad \forall v, w \in V$$

H4. There exist $\gamma > 0$ and $\alpha > 0$ such that

$$(Av, v)_X + \gamma (Rv, v)_Y \geq \alpha \|v\|_X^2 \quad \forall v \in V.$$

H5. The initial data $u_0$ belongs to $W$.

H6. The data function $g$ belongs to $H^1(0,T;M')$.

**Theorem 2.1.** Let us assume that assumptions H1–H6 hold true. Then the Problem 2.1 has a unique solution $(u, \lambda) \in L^2(0,T;X) \times H^1(0,T;M))$ and there exists a constant $C > 0$ such that

$$\|u\|_{L^2(0,T;X)} + \|\lambda\|_{L^2(0,T;M)} \leq C \left\{ \|f\|_{L^2(0,T;X')} + \|g\|_{H^1(0,T;M')} + \|u_0\|_Y \right\}.$$

Moreover, $\lambda(0) = 0$. 

**Proof.** (see [3, Theorem 2.1])

In the following section we present the fully discrete analysis for the Problem 2.1.
3 Fully-discrete approximation for degenerate mixed parabolic problem

Let \( \{X_h\}_{h>0} \) and \( \{M_h\}_{h>0} \) be sequences of finite-dimensional subspaces of \( X \) and \( M \), respectively, and let \( \{t_n := n\Delta t : n = 0,\ldots,N\} \) be a uniform partition of \([0,T]\) with a step size \( \Delta t := T/N \). For any finite sequence \( \{\theta^n : n = 0,\ldots,N\} \) we denote
\[
\bar{\partial} \theta^n := \frac{\theta^n - \theta^{n-1}}{\Delta t}, \quad n = 1,\ldots,N.
\]

The fully-discrete approximation of the Problem 2.1 reads as follows:

**Problem 3.1.** Find \( u^n_h \in X_h, \lambda^n_h \in M_h, n = 1,\ldots,N \), such that
\[
\langle R\bar{\partial}u^n_h, v \rangle_Y + b(v, \bar{\partial}\lambda^n_h) + \langle Au^n_h, v \rangle_X = \langle f(t_n), v \rangle_X \quad \forall v \in X_h,
\]
\[
b(u^n_h, \mu) = \langle g(t_n), \mu \rangle_M \quad \forall \mu \in M_h,
\]
\[
u_0^h = u_{0,h},
\]
\[
\lambda_0^h = 0.
\]

This scheme is obtained by using a backward Euler discrete approximation for the time-derivatives. Furthermore, the third equation the Problem 3.1 includes a suitable approximation \( u_{0,h} \) of the initial data \( u_0 \) to obtain the convergence of the scheme (see Subsection 4 below).

On the other hand, to obtain the well-posedness of the Problem 3.1, we first notice that by rewriting equations, the solution \( (u^n_h, \lambda^n_h) \in X_h \times M_h \) at each time step have to satisfy the following classical mixed problem:
\[
\mathcal{A}(u^n_h, v) + b(v, \lambda^n_h) = F_n(v) \quad \forall v \in X_h,
\]
\[
b(u^n_h, \mu) = \langle g(t_n), \mu \rangle_M \quad \forall \mu \in M_h,
\]
where
\[
\mathcal{A}(w, v) := \langle Rw, v \rangle_Y + \Delta t \langle Aw, v \rangle_X
\]
\[
F_n(v) := \Delta t \langle f(t_n), v \rangle_X + \langle Ru^{n-1}_h, v \rangle_Y + b(v, \lambda^{n-1}_h).
\]

Hence, the existence and uniqueness of solution of the Problem 3.1 is obtained by assuming the following conditions and using the classical Babuska-Brezzi Theory:

H7. There exist \( \xi_h > 0 \) and \( \alpha_h > 0 \) such that
\[
\langle Av, v \rangle_X + \xi_h \langle Rv, v \rangle_Y \geq \alpha_h \| v \|_X^2 \quad \forall v \in V_h,
\]
where \( V_h \) denotes the discrete kernel of \( b \) in \( X_h \), i.e.,
\[
V_h := \{ v \in X_h : b(v, \mu) = 0 \quad \forall \mu \in M_h \}.
\]

H8. The bilinear form \( b : X_h \times M_h \to \mathbb{R} \) is bounded and it satisfies the discrete \textit{inf-sup} condition, i.e., there exists \( \beta_h > 0 \) such that
\[
\sup_{v \in X_h} \frac{b(v, q)}{\| v \|_X} \geq \beta_h \| q \|_M \quad \forall q \in M_h.
\]
4 Error estimates for the fully-discrete approximation

4.1 Error estimates for main variable $u$

We need to introduce the operators $\tilde{B}_h : X \rightarrow M'_h$ and $B_h : X_h \rightarrow M'_h$ defined as follows

$$\langle \tilde{B}_h v, \mu \rangle_{M'} := b(v, \mu) \quad \forall v \in X, \quad \forall \mu \in M_h,$$

$$\langle B_h v, \mu \rangle_{M'} := b(v, \mu) \quad \forall v \in X_h, \quad \forall \mu \in M_h.$$ 

Now, we notice that if we consider the operator $\Pi_h : X \rightarrow X_h$ characterized by

$$\Pi_h w \in X_h : \quad (\Pi_h w, z)_X = (w, z)_X \quad \forall z \in X_h,$$

then there exists $C > 0$ independent on $h$ satisfying

$$\|w - \Pi_h w\|_X \leq C \inf_{z \in X_h} \|w - z\|_X. \quad (4.1)$$

From the second and third equation of the Problem 3.1, we obtain

$$b(u(t_n) - u^n_h), \mu) = 0 \quad \forall \mu \in M_h,$$

and therefore

$$b(u(t_n) - \Pi_h u(t_n), \mu) = b(u^n_h - \Pi_h u(t_n), \mu) \quad \forall \mu \in M_h.$$ 

On the other hand, since $B_h$ satisfies the discrete inf-sup condition, for all $w \in X$ there exists a unique $\tilde{P}_h w \in V_h^1 \subset X_h$ such that

$$B_h \tilde{P}_h w = \tilde{B}_h (w - \Pi_h w),$$

or equivalently,

$$b(\tilde{P}_h w, \mu) = b(w - \Pi_h w, \mu) \quad \forall \mu \in M_h.$$ 

Furthermore, for all $w \in X$, we have

$$\|\tilde{P}_h w\|_X \leq \frac{1}{\beta_h} \left| B_h \tilde{P}_h w \right|_{M'} \leq \frac{1}{\beta_h} \sup_{\mu \in M_h} \frac{b(\tilde{P}_h w, \mu)}{\|\mu\|_M} = \frac{1}{\beta_h} \sup_{\mu \in M_h} \frac{b(w - \Pi_h w, \mu)}{\|\mu\|_M} \leq \frac{\|b\|}{\beta_h} \|w - \Pi_h w\|_X.$$

Thus, thanks to the triangle inequality, it follows that

$$\|w - \tilde{P}_h w - \Pi_h w\|_X \leq \left(1 + \frac{\|b\|}{\beta_h} \right) \|w - \Pi_h w\|_X.$$ 

Hence, if we define

$$\tilde{P}_h w := \tilde{P}_h w + \Pi_h w \quad \forall w \in X,$$

then

$$\|w - \tilde{P}_h w\|_X \leq \left(1 + \frac{\|b\|}{\beta_h} \right) \|w - \Pi_h w\|_X,$$

and using (4.1), we deduce

$$\|w - \tilde{P}_h w\|_X \leq C \left(1 + \frac{\|b\|}{\beta_h} \right) \inf_{z \in X_h} \|w - z\|_X \quad \forall w \in X. \quad (4.2)$$

Therefore, we can consider the following split of the error

$$e^n_h := u(t_n) - \tilde{P}_h u^n = \rho^n_h + \epsilon^n_h, \quad n = 1, \ldots, N,$$ 

(4.3)
where
\[ \rho_h(t) := u(t) - \mathcal{P}_h u(t), \quad \rho_h^n := \rho_h(t_n), \quad \sigma_h^n := \mathcal{P}_h u(t_n) - u_h^n. \tag{4.4} \]

Next, in order to obtain the convergence of the method, we first prove the estimate for the last term in (4.3) as in the following Lemma.

**Lemma 4.1.** For \( n = 1, \ldots, N \), let \( \rho_h^n \) and \( \sigma_h^n \) as in (4.3) and
\[ \tau^n := \frac{u(t_n) - u(t_{n-1})}{\Delta t} - \partial_t u(t_n). \]

Assume that \( \lambda \in C^1(0, T; M) \) and \( \{ \xi_h \}_{h > 0} \), \( \{ \alpha_h \}_{h > 0} \) (see H7) are bounded uniformly in \( h \), then provided \( \Delta t \) is small enough, there exists a constant \( C > 0 \) independent of \( h \) and \( \Delta t \), such that
\[ \langle R\sigma_h^n, \sigma_h^n \rangle_Y + \Delta t \sum_{k=1}^n \| \sigma_h^n \|_X^2 \]
\[ \leq C \left( \langle R\sigma_h^n, \sigma_h^n \rangle_Y + \Delta t \sum_{k=1}^n \left[ \| \tau^k \|_Y^2 + \| \partial \rho_h^k \|_Y^2 + \| \rho_h^k \|_X^2 + \left( \sup_{v \in V_h} b(v, \partial \lambda(t_k)) \right)^2 \right] \right). \tag{4.5} \]

**Proof.** Let \( n \in \{ 1, \ldots, N \} \) and \( k \in \{ 1, \ldots, n \} \). Using (2.1) and (3.1), it is straightforward to show that
\[ \langle R\sigma_h^n, v \rangle_Y + \langle A\sigma_h^k, v \rangle_X = \langle R\tau^k, v \rangle_Y - \langle R\sigma_h^k, v \rangle_Y - \langle A\rho_h^k, v \rangle_X - b(v, \partial \lambda(t_k)) \quad \forall v \in V_h. \]

Then, by taking \( v := \sigma_h^k \in V_h \) in this last identity, we have
\[ \langle R\sigma_h^k, \sigma_h^k \rangle_Y + \langle A\sigma_h^k, \sigma_h^k \rangle_X = \langle R\tau^k, \sigma_h^k \rangle_Y - \langle R\sigma_h^k, \sigma_h^k \rangle_Y - \langle A\rho_h^k, \sigma_h^k \rangle_X - b(\sigma_h^k, \partial \lambda(t_k)) \tag{4.6} \]

Let \( \xi := \sup_{h > 0} \xi_h > 0 \) and \( \alpha := \inf \alpha_h > 0 \). Since \( R \) is monotone (see H2), it is easily seen that
\[ \langle R\sigma_h^k, \sigma_h^k \rangle_Y \geq \frac{1}{2\Delta t} \left[ \langle R\sigma_h^k, \sigma_h^k \rangle_Y - \langle R\sigma_h^{k-1}, \sigma_h^{k-1} \rangle_Y \right], \]

thus, from (4.6) we deduce
\[ \frac{1}{2\Delta t} \left[ \langle R\sigma_h^k, \sigma_h^k \rangle_Y - \langle R\sigma_h^{k-1}, \sigma_h^{k-1} \rangle_Y \right] + \alpha \| \sigma_h^k \|_X^2 - \xi \langle R\sigma_h^k, \sigma_h^k \rangle_Y \]
\[ \leq \langle R\tau^k, \sigma_h^k \rangle_Y - \langle R\sigma_h^k, \sigma_h^k \rangle_Y - \langle A\rho_h^k, \sigma_h^k \rangle_X - b(\sigma_h^k, \partial \lambda(t_k)). \]

Now, since the operator \( R \) is monotone, it satisfies the following Cauchy-Schwarz type inequality
\[ |\langle Rv, w \rangle_Y|^2 \leq \langle Rv, v \rangle_Y \langle Rw, w \rangle_Y, \]

then, we have
\[ \langle R\sigma_h^k, \sigma_h^k \rangle_Y - \langle R\sigma_h^{k-1}, \sigma_h^{k-1} \rangle_Y + \alpha \Delta t \| \sigma_h^k \|_X^2 \]
\[ \leq (1 + 2\xi) \Delta t \langle R\sigma_h^k, \sigma_h^k \rangle_Y + 2\Delta t \langle R\tau^k, \sigma_h^k \rangle_Y + 2\Delta t \langle R\sigma_h^k, \sigma_h^k \rangle_Y \]
\[ + \frac{2}{\alpha} \Delta t \| A \|_2^2 \| \rho_h^k \|_X^2 + \frac{2}{\alpha} \Delta t \left( \sup_{v \in V_h} \frac{b(v, \partial \lambda(t_k))}{\| v \|_X} \right)^2, \]
and using the continuity of $R$, it follows that
\[
\langle R\sigma_h^k, \sigma_h^k \rangle_Y - \langle R\sigma_h^{k-1}, \sigma_h^{k-1} \rangle_Y + \alpha \Delta t \|\sigma_h^k\|_X^2 \\
\leq (1 + 2\xi)\Delta t \langle R\sigma_h^k, \sigma_h^k \rangle_Y + C \Delta t \left(\|t^{-1}\|_Y^2 + \|\bar{\rho}_h^k\|_Y^2 + \|\rho_h^k\|_X^2 + \left(\sup_{v \in V_h} \frac{b(v, \partial_t \lambda(t_k))}{\|v\|_X}\right)^2\right).
\]

Then, by summing over $k$, we obtain
\[
\langle R\sigma_h^n, \sigma_h^n \rangle_Y - \langle R\sigma_h^0, \sigma_h^0 \rangle_Y + \alpha \Delta t \sum_{k=1}^{n} \|\sigma_h^k\|_X^2 \leq (1 + 2\xi) \Delta t \sum_{k=1}^{n} \langle R\sigma_h^k, \sigma_h^k \rangle_Y + C \Delta t \sum_{k=1}^{n} \Theta_k^2,
\]
where
\[
\Theta_k^2 := \|t^{-1}\|_Y^2 + \|\bar{\rho}_h^k\|_Y^2 + \|\rho_h^k\|_X^2 + \left(\sup_{v \in V_h} \frac{b(v, \partial_t \lambda(t_k))}{\|v\|_X}\right)^2.
\]

Hence, if $1 - (1 + 2\xi) \Delta t \geq \frac{1}{2}$ then
\[
\langle R\sigma_h^n, \sigma_h^n \rangle_Y + 2\alpha \Delta t \sum_{k=1}^{n} \|\sigma_h^k\|_X^2 \leq 2\langle R\sigma_h^0, \sigma_h^0 \rangle_Y + 2(1 + 2\xi) \Delta t \sum_{k=1}^{n-1} \langle R\sigma_h^k, \sigma_h^k \rangle_Y + C \Delta t \sum_{k=1}^{n} \Theta_k^2,
\]
and, in particular
\[
\langle R\sigma_h^n, \sigma_h^n \rangle_Y \leq 2\langle R\sigma_h^0, \sigma_h^0 \rangle_Y + 2(1 + 2\xi) \Delta t \sum_{k=1}^{n-1} \langle R\sigma_h^k, \sigma_h^k \rangle_Y + C \Delta t \sum_{k=1}^{n} \Theta_k^2.
\]

Consequently, by using the discrete Gronwall’s inequality, we obtain
\[
\langle R\sigma_h^n, \sigma_h^n \rangle_Y \leq C \left(\langle R\sigma_h^0, \sigma_h^0 \rangle_Y + \Delta t \sum_{k=1}^{n} \Theta_k^2\right),
\]
for $n = 1, \ldots, N$. Thus,
\[
\Delta t \sum_{k=1}^{n-1} \langle R\sigma_h^k, \sigma_h^k \rangle_Y \leq C \Delta t \sum_{k=1}^{n-1} \left(\langle R\sigma_h^0, \sigma_h^0 \rangle_Y + \Delta t \sum_{j=1}^{k} \Theta_j^2\right) \leq C \left(\langle R\sigma_h^0, \sigma_h^0 \rangle_Y + \Delta t \sum_{j=1}^{n-1} \Theta_j^2\right),
\]
and finally, by substituting this inequality into (4.7), it follows (4.5). \(\square\)

**Theorem 4.1.** Under the assumptions of Lemma 4.1, if \(\{\beta_h\}_{h>0}\) is bounded uniformly in $h$ and \(u \in H^1(0,T;X) \cap H^2(0,T;Y)\) then there exists a constant $C > 0$ independent of $h$ and $\Delta t$, such that

\[
\max_{1 \leq n \leq N} \langle R(u(t_n) - u_h^n), u(t_n) - u_h^n \rangle_Y + \Delta t \sum_{n=1}^{N} \|u(t_n) - u_h^n\|_X^2 \leq C \left\{\|u_0 - u_0, h\|_Y^2 + \max_{0 \leq n \leq N} \left(\inf_{z \in X_h} \|u(t_n) - z\|_X\right)^2 \right. \right.
\]
\[
+ \int_0^T \left(\inf_{z \in X_h} \|\partial_t u(t) - z\|_X\right)^2 dt + (\Delta t)^2 \|\partial_t u\|_{L^2(0,T;Y)}^2 + \Delta t \sum_{n=1}^{N} \left(\inf_{\mu \in M_h} \|\partial_t \lambda(t_n) - \mu\|_M\right)^2 \right\}.
\]
Proof. A Taylor expansion shows that
\[
\sum_{k=1}^{N} \| \tau^k \|_Y^2 = \sum_{k=1}^{N} \left\| \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} (t_{k-1} - t) \partial_t u(t) \, dt \right\|_Y^2 \leq \Delta t \int_0^T \| \partial_t u(t) \|_Y^2 \, dt.
\]

It is easy to show that
\[
\sup_{v \in H_h} \frac{b(v, \partial_t \lambda(t_k))}{\| v \|_X} \leq C \inf_{\mu \in M_h} \| \partial_t \lambda(t_k) - \mu \|_M.
\]

On the other hand, from (4.2), (4.4) and recalling that \( \{ \beta_h \}_{h > 0} \) is bounded uniformly in \( h \), we get
\[
\| \rho_h(t) \|_X \leq C \inf_{z \in X_h} \| u(t) - z \|_X.
\]

Furthermore, the regularity assumption about \( u \) implies \( \partial_t \mathcal{P}_h u(t) = \mathcal{P}_h (\partial_t u(t)) \), and consequently
\[
\| \partial_t \rho_h(t) \|_X \leq C \inf_{z \in X_h} \| \partial_t u(t) - z \|_X.
\]

Hence, by recalling (4.4), it is easy to check that
\[
\Delta t \sum_{k=1}^{N} \left\| \partial_t \rho_h^k \right\|_Y^2 = \sum_{k=1}^{N} \left\| \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} \partial_t \rho_h(t) \, dt \right\|_Y^2 \leq \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \| \partial_t \rho_h(t) \|_Y^2 \, dt \leq C \int_0^T \| \partial_t \rho_h(t) \|_X^2 \, dt.
\]

Finally, by writing \( \sigma^0_h = e^0_h - \rho^0_h \) and using the fact that \( R \) is self-adjoint and monotone from from third equation the Problem 2.1, it follows that
\[
\langle R \sigma^0_h, \sigma^0_h \rangle_Y \leq 2 \langle R(u_0 - u_{0,h}), u_0 - u_{0,h} \rangle_Y + 2 \langle R \rho^0_h, \rho^0_h \rangle_Y.
\]

Combining these inequalities and Lemma 4.1, the result follows from the fact that \( u(t_n) - u_h^n = \rho^0_h + \sigma^n_h \) (see 4.3) and the triangle inequality.

4.2 Error estimates for Lagrange multiplier \( \lambda \)

The following theorem is the analogous result to Theorem 4.1 but for the case of the Lagrange multiplier \( \lambda \).

Theorem 4.2. Under the assumptions of Theorem 4.1 if \( f \in H^1(0, T; X') \) then there exist a constant \( C > 0 \) independent of \( h \) and \( \Delta t \), such that
\[
\Delta t \sum_{n=1}^{N} \| \lambda(t_n) - \lambda^n_h \|_M^2 \leq C \left( (\Delta t)^2 \| \partial_t f \|_{L^2(0, T; X')}^2 + (\Delta t)^2 \| \partial_t u \|_{L^2(0, T; X)}^2 \right) + \Delta t \sum_{n=1}^{N} \left( \inf_{\mu \in M_h} \| \lambda(t_n) - \mu \|_M \right)^2 + \Delta t \sum_{n=1}^{N} \| u(t_n) - u_h^n \|_X^2.
\]

Proof. Let \( n \in \{1, \ldots, N\} \). By integrating first equation the Problem 2.1 on \([0, t_n]\), we have
\[
\int_0^{t_n} \frac{d}{dt} \langle (Ru(t), v)_Y + b(v, \lambda(t)) \rangle \, dt + \int_0^{t_n} a(u(t), v) \, dt = \int_0^{t_n} \langle f(t), v \rangle_X \, dt \quad \forall v \in X,
\]
then, using the initial condition for $u$ and recalling that $\lambda(0) = 0$ (see Theorem 2.1), it follows that $\lambda(t_n)$ satisfies
\[ b(v, \lambda(t_n)) = \langle L(t_n), v \rangle_X \quad \forall v \in X, \tag{4.8} \]
where
\[ \langle L(t_n), v \rangle_X := \left( \int_0^{t_n} f(t) \, dt, v \right)_X - \langle R(u(t_n) - u_0), v \rangle_Y - \left( A \int_0^{t_n} u(t) \, dt, v \right)_X. \]

On the other hand, by summing (3.1) from 1 to $n$ and using the Problem 3.1, we deduce that $\lambda^n_h$ verifies
\[ b(v, \lambda^n_h) = \langle L^n_h, v \rangle_X \quad \forall v \in X_h, \tag{4.9} \]
with
\[ \langle L^n_h, v \rangle_X := \left( \Delta t \sum_{k=1}^n f(t_k), v \right)_X - \langle R(u^n_h - u_{0,h}), v \rangle_Y - \left( A \left( \Delta t \sum_{k=1}^n u^n_{0,h} \right), v \right)_X. \]

Now, we need to consider the problem

**Problem 4.1.** Find $\lambda^n_h \in M_h$ such that
\[ b(v, \lambda^n_h) = \langle L(t_n), v \rangle_X \quad \forall v \in X. \]

We can check that $L(t_n) \in V^\perp_h := \{ v \in X : b(v, \mu) = 0 \ \forall \mu \in M_h \}$, i.e.,
\[ \langle L(t_n), v \rangle_X = 0 \quad \forall v \in V_h, \]
thus, the Problem 4.1 is a well-posed problem, since $b$ satisfies the discrete inf-sup condition.

Next, we notice that by using (4.8) and (4.1), the following orthogonality relationship is obtained
\[ b(v, \lambda(t_n) - \lambda^n_h) = 0 \quad \forall v \in X_h, \]
consequently, the discrete inf-sup condition implies
\[ \| \lambda^n_h - \mu \|_M \leq \frac{1}{\beta_h} \sup_{v \in X_h} \frac{b(v, \lambda^n_h - \mu)}{\| v \|_X} = \frac{1}{\beta_h} \sup_{v \in X_h} \frac{b(v, \lambda(t_n) - \mu)}{\| v \|_X} \leq \frac{\| b \|}{\beta_h} \| \lambda(t_n) - \mu \|_M, \]
and therefore
\[ \| \lambda(t_n) - \lambda^n_h \|_M \leq \left( 1 + \frac{\| b \|}{\beta_h} \right) \inf_{\mu \in M_h} \| \lambda(t_n) - \mu \|_M. \]

On the other hand, by using (4.9) and (4.1), it follows that
\[ \| \lambda^n_h - \lambda^n_h \|_M \leq \frac{1}{\beta_h} \sup_{v \in X_h} \frac{b(v, \lambda^n_h - \lambda^n_h)}{\| v \|_X} = \frac{1}{\beta_h} \sup_{v \in X_h} \frac{\langle L(t_n) - L^n_h, v \rangle_X}{\| v \|_X}, \]
hence
\[ \| \lambda(t_n) - \lambda^n_h \|_M \leq \left( 1 + \frac{\| b \|}{\beta_h} \right) \inf_{\mu \in M_h} \| \lambda(t_n) - \mu \|_M + \frac{1}{\beta_h} \sup_{v \in X_h} \frac{\langle L(t_n) - L^n_h, v \rangle_X}{\| v \|_X}, \]
and finally, recalling that $\{ \beta_h \}_{h>0}$ is bounded uniformly in $h$, we can conclude
\[ \Delta t \sum_{n=1}^N \| \lambda(t_n) - \lambda^n_h \|_M^2 \leq C \left( \Delta t \sum_{n=1}^N \left( \inf_{\mu \in M_h} \| \lambda(t_n) - \mu \|_M \right)^2 + \Delta t \sum_{n=1}^N \left( \sup_{v \in X_h} \frac{\langle L(t_n) - L^n_h, v \rangle_X}{\| v \|_X} \right)^2 \right). \]
It remains to estimate the last term in the previous inequality. In fact, using the definitions of \( L(t_n) \) and \( L^n_h \), for all \( v \in X_h \), we deduce

\[
\left( \sup_{v \in X_h} \frac{\langle L(t_n) - L^n_h, v \rangle_X}{\|v\|_X} \right)^2 \leq C \left( \|u_0 - u_{0,h}\|_Y^2 + \int_0^{t_n} f(t) \, dt - \Delta t \sum_{k=1}^n f(t_k) \right)^2_X + \int_0^{t_n} \left( u(t) dt - \Delta t \sum_{k=1}^n u_h^k \right)^2_X + \|e^n_h\|_X^2,
\]

therefore,

\[
\Delta t \sum_{n=1}^N \left( \sup_{v \in X_h} \frac{\langle L(t_n) - L^n_h, v \rangle_X}{\|v\|_X} \right)^2 \leq C \Delta t \left( N \|u_0 - u_{0,h}\|_Y^2 + \sum_{n=1}^N \int_0^{t_n} f(t) \, dt - \Delta t \sum_{k=1}^n f(t_k) \right)^2_X + \sum_{n=1}^N \left( \int_0^{t_n} \left( u(t) dt - \Delta t \sum_{k=1}^n u_h^k \right)^2_X + \|e^n_h\|_X^2 \right). \tag{4.10}
\]

Next, we will show that

\[
\Delta t \sum_{n=1}^N \left( \int_0^{t_n} u(t) \, dt - \Delta t \sum_{k=1}^n u_h^k \right)^2_X \leq 2T^2(\Delta t)^2 \|\partial_t u\|_{L^2(0,T;X)}^2 + 2T^2 \Delta t \sum_{k=1}^N \|e^n_h\|_X^2 \tag{4.11}
\]

and

\[
\Delta t \sum_{n=1}^N \left( \int_0^{t_n} f(t) \, dt - \Delta t \sum_{k=1}^n f(t_k) \right)^2_X \leq T^2(\Delta t)^2 \|\partial_t f\|_{L^2(0,T;X')}^2. \tag{4.12}
\]

In fact, to obtain (4.11), first we notice that

\[
\left( \int_0^{t_n} u(t) \, dt - \Delta t \sum_{k=1}^n u_h^k \right)^2_X = \left( \int_0^{t_n} u(t) \, dt - \Delta t \sum_{k=1}^n u(t_k) \right)^2_X + \Delta t \sum_{k=1}^n \|e^n_h\|_X^2 \leq 2 \int_0^{t_n} \left( u(t) dt - \Delta t \sum_{k=1}^n u(t_k) \right)^2_X + 2T \Delta t \sum_{k=1}^n \|e^n_h\|_X^2,
\]

and

\[
\left( \int_0^{t_n} u(t) \, dt - \Delta t \sum_{k=1}^n u(t_k) \right)^2_X = \left( \sum_{k=1}^n \int_{t_{k-1}}^{t_k} (u(t) - u(t_k)) \, dt \right)^2_X \leq \left( \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|u(t) - u(t_k)\|_X \, dt \right)^2_X \leq n \Delta t \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|u(t) - u(t_k)\|_X^2 \, dt \leq n \Delta t \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|u(t) - u(t_k)\|_X^2 \, dt \leq T \Delta t \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|\partial_t u(s)\|_X^2 \, ds,
\]

which implies (4.11). Next, we can apply similar computations to deduce (4.12). Finally, combining (4.10) and (4.12) and the fact that \( e^n_h = u(t_n) - u^n_h \), we conclude the proof.
5 Applications

5.1 The time-dependent Stokes problem

The time-dependent Stokes is a fundamental model of viscous flow. This problem arises from neglecting the nonlinear terms in Navier-Stokes (see [9]). Stokes flows are important in lubrication theory, in porous media flow, biology applications (see [27]). Now, we proceed to study the numerical approximation of the time-dependent Stokes problem. In this way, we consider \( \Omega \subset \mathbb{R}^d \) be an open, bounded and connected, where \( d \) either 2 or 3 the space dimension. The boundary of \( \Omega \) is denoted by \( \Gamma := \partial \Omega \) and assumed to be Lipschitz continuous. In [3] we have studied the well posedness of the following variational formulation for time-dependent Stokes problem given by

**Problem 5.1.** Find \( u \in L^2(0,T;H^1_0(\Omega)^d) \) and \( P \in H^1(0,T;L^2_0(\Omega)) \) such that

\[
\frac{d}{dt} \left( \int_{\Omega} u(t) \cdot v - \int_{\Omega} P(t) \operatorname{div} v \right) + \nu \int_{\Omega} \nabla u(t) : \nabla v = \int_{\Omega} f(t) \cdot v \quad \forall v \in H^1_0(\Omega)^d,
\]

\[
\int_{\Omega} q \operatorname{div} u = 0 \quad \forall q \in L^2_0(\Omega),
\]

\[
u \int_{\Omega} \nabla u : \nabla v = \int_{\Omega} f(t) \cdot v \quad \forall v \in H^1_0(\Omega)^d,
\]

where \( \nabla u : \nabla v := \sum_{i=1}^d \sum_{j=1}^d z_{ij} w_{ij} \); for any \( z, w \in L^2(\Omega)^d \times d \).

In order to obtain the fully-discrete approximation of Problem 5.1, so we want to use finite element subspaces to define \( X_h \) and \( M_h \), the corresponding families of finite dimensional subspaces of \( X := H^1_0(\Omega)^d \) and \( M := L^2_0(\Omega) \), respectively. To this aim, in what follows we assume that \( \Omega \) is a Lipschitz polygon if \( d = 2 \) or a Lipschitz polyhedra if \( d = 3 \). Likewise, let \( \{ T_h \}_h \) be a regular family of triangles meshes of \( \Omega \) if \( d = 2 \) or of tetrahedral meshes of \( \Omega \) for the case \( d = 3 \).

The spaces \( X_h \) and \( M_h \) should be respectively finite element subspaces of \( H^1_0(\Omega)^d \) and \( L^2_0(\Omega) \) satisfying the discrete inf-sup condition required for the assumption H7 (see Section 3), i.e.,

\[
\sup_{v \in X_h} \frac{-\int_{\Omega} q \operatorname{div} v}{\|v\|_{H^1_0(\Omega)^d}} \geq \beta \|q\|_{L^2(\Omega)} \quad \forall q \in M_h. \quad (5.1)
\]

For the sake of simplicity, we only consider the pair of finite element subspaces so-called the MINI finite element, which was introduced by Arnold, Brezzi and Fortin [29]. In the MINI element the discrete space for the velocity and pressure are respectively defined by

\[
X_h := \left\{ v \in \left[ C(\Omega) \cap H^1_0(\Omega) \right]^d : v|_K \in [P_1 \oplus \mathcal{B}_{d+1}]^d \quad \forall K \in T_h \right\} \quad (5.2)
\]

and

\[
M_h := \left\{ v \in C(\Omega) \cap L^2_0(\Omega) : v|_K \in P_1 \quad \forall K \in T_h \right\}, \quad (5.3)
\]

where \( \mathcal{B}_{d+1} \) are the bubbles functions, which are defined by

\[
\mathcal{B}_{d+1} := \left\{ \gamma \varphi(x_1, \ldots, x_d) : \gamma \in \mathbb{R}, \quad \varphi(x_1, \ldots, x_d) := x_1 \ldots x_d (1 - x_1 - \ldots - x_d) \right\}.
\]

The bubble part of the of the velocity is needed to stabilize the formulation, namely to satisfy the discrete inf-sup condition, which is shown in the following lemma.
Lemma 5.1. Let \( X_h \) and \( M_h \) given by (5.2) and (5.3) respectively. Then, there exists \( \beta > 0 \) independent of \( h \) satisfying the discrete inf-sup condition (5.1).

Proof. See, for instance, [21, Lemma 4.20]. \( \square \)

By using the discrete subspaces given by the MINI elements, we can consider the following fully-discrete approximation of Problem 5.1 given \( u_0, h \in X_h \) and by denoting

\[
\begin{align*}
    u^n_h &:= u_{0,h}^n, & P^n_h &:= 0,
\end{align*}
\]

**Problem 5.2.** Find \( u^n_h \in X_h \) and \( P^n_h \in M_h \) for \( n = 1, \ldots, N \), such that

\[
\begin{align*}
    \int_{\Omega} \left( \frac{u^n_h - u^{n-1}_h}{\Delta t} \right) \cdot v - \int_{\Omega} \left( \frac{P^n_h - P^{n-1}_h}{\Delta t} \right) \text{div} v + \nu \int_{\Omega} \nabla u^n_h : \nabla v &= \int_{\Omega} f(t_n) \cdot v & \forall v \in X_h, \\
    \int_{\Omega} q \text{div} u^n_h &= 0 & \forall q \in M_h.
\end{align*}
\]

Since the MINI element satisfies the discrete inf-sup condition (see Lemma 5.1), the property H7 from Section 3 holds true. Hence, to deduce that the Problem 5.2 has a unique solution, it only remains to prove H6, i.e., the Garding-like inequality

\[
\nu \int_{\Omega} |\nabla v|^2 + \gamma h \int_{\Omega} |v|^2 \geq \alpha h \|v\|_{H^1_0(\Omega)}^2 \quad \forall v \in V_h, \tag{5.4}
\]

where \( V_h \) is the discrete kernel

\[ V_h := \left\{ v \in X_h : \int_{\Omega} q \text{div} v = 0 \quad \forall q \in M_h \right\}. \]

Moreover, (5.4) holds uniformly on \( h \) (more exactly with \( \gamma_h = 0 \) and \( \alpha = \nu \), since the norm in \( H^1_0(\Omega) \) is precisely

\[
\|v\|_{H^1_0(\Omega)} = \left( \int_{\Omega} |
abla v|^2 \right)^{1/2}.
\]

Consequently, Problem 5.2 has a unique solution \((u^n_h)^N_{n=1} \subset X_h\) and \((P^n_h)^N_{n=1} \subset M_h\).

Our next goal is to obtain Céa-like error estimates for the fully-discrete approximation of the Stokes problem. We will first obtain error estimates for the approximation of the velocity. In fact, we have the following result for a direct application of Theorem 4.1.

**Theorem 5.1.** Suppose the assumptions of Theorem 2.1 studied in [23] and let \((u, P)\) and \((u^n_h, P^n_h)^N_{n=1}\) be the solutions of Problem 5.1 and Problem 5.2 respectively. Furthermore, assume that

\[
\begin{align*}
    u &\in H^1(0, T; H^1_0(\Omega))^3 \cap H^2(0, T; L^2(\Omega))^3, & P &\in C^1(0, T; L^2(\Omega))^3,
\end{align*}
\]

Then, for a small enough time step \( \Delta t \), there exists a constant \( C > 0 \), independent of \( h \) and \( \Delta t \), such
\[ \max_{1 \leq n \leq N} \| \mathbf{u}(t_n) - \mathbf{u}_h^n \|_{L^2(\Omega)}^2 + \Delta t \sum_{n=1}^N \| \mathbf{u}(t_n) - \mathbf{u}_h^n \|_{H^1_0(\Omega)}^2 \]

\[ \leq C \left\{ \| \mathbf{u}_0 - \mathbf{u}_{0,h} \|_{L^2(\Omega)}^2 + \max_{1 \leq n \leq N} \inf_{\mathbf{v} \in X_h} \| \mathbf{u}(t_n) - \mathbf{v} \|_{H^1_0(\Omega)}^2 \right\} \]

\[ + \Delta t \sum_{n=1}^N \inf_{\mathbf{v} \in X_h} \| \mathbf{u}(t_n) - \mathbf{v} \|_{H^1_0(\Omega)}^2 + \int_0^T \left( \inf_{\mathbf{v} \in X_h} \| \partial_t \mathbf{u}(t) - \mathbf{v} \|_{H^1_0(\Omega)}^2 \right) \, dt \]

\[ + (\Delta t)^2 \int_0^T \| \partial_t \mathbf{u}(t) \|_{L^2(\Omega)}^2 \, dt + \Delta t \sum_{n=1}^N \left( \inf_{q \in M_h} \| \partial_t P(t_n) - q \|_{L^2(\Omega)} \right)^2 \}. \]

The Cea-like error estimates for the approximation of the pressure is given by the following theorem, which is obtained by a direct application of Theorem 4.2.

**Theorem 5.2.** Under the assumptions of Theorem 5.1, if \( f \in H^1(0,T;L^2(\Omega)^3) \) then, for a small enough time step \( \Delta t \), there exist a constant \( C > 0 \) independent of \( h \) and \( \Delta t \), such that

\[ \Delta t \sum_{n=1}^N \| P(t_n) - P_h^n \|_{L^2(\Omega)}^2 \]

\[ \leq C \left[ (\Delta t)^2 \| \partial_t f \|_{L^2(0,T;L^2(\Omega)^3)}^2 + (\Delta t)^2 \| \partial_t u \|_{L^2(0,T;H^1_0(\Omega))^3)}]^2 \right] \]

\[ + \Delta t \sum_{n=1}^N \left( \inf_{q \in M_h} \| P(t_n) - q \|_{L^2(\Omega)} \right)^2 + \Delta t \sum_{n=1}^N \| \mathbf{u}(t_n) - \mathbf{u}_h^n \|_{H^1_0(\Omega)}^2 \}. \]

Finally, to obtain the asymptotic error estimate for the velocity approximation, we need to consider the vector-valued functions of the Sobolev space \( H^{1+s}(\Omega)^3 \) for \( 0 < s < 1 \). We recall that the vector Lagrange interpolant \( L_h \mathbf{v} \in X_h \) is obtained by taking the scalar Lagrange interpolant of each component, thus it is well defined for all \( \mathbf{v} \in H^{1+s}(\Omega)^3 \cap H^1_0(\Omega)^3 \). Moreover,

\[ L_h : H^{1+s}(\Omega)^3 \cap H^1_0(\Omega)^3 \to X_{1,h}, \]

where \( X_{1,h} \) is the proper subspace of \( X_h \) given by

\[ X_{1,h} := \left\{ \mathbf{v} \in [C(\Omega) \cap H^1_0(\Omega)]^d : \mathbf{v} |_{K} \in \mathbb{P}_1^d \quad \forall K \in \mathcal{T}_h \right\}. \]

Furthermore, the following estimate holds true (see, for instance, [23])

\[ \| \mathbf{v} - L_h \mathbf{v} \|_{H^1_0(\Omega)^3} \leq Ch^s \| \mathbf{v} \|_{H^{1+s}(\Omega)^3} \quad \forall \mathbf{v} \in H^{1+s}(\Omega)^3 \cap H^1_0(\Omega)^3. \]

On the other hand, to obtain the asymptotic error estimate for the pressure approximation, we notice that the scalar Lagrange interpolant \( L_h : H^{1+s}(\Omega) \cap L^2_0(\Omega) \to M_h \) verifies

\[ \| q - L_h q \|_{L^2(\Omega)} \leq Ch^s \| q \|_{H^{1+s}(\Omega)} \quad \forall q \in H^{1+s}(\Omega) \cap L^2_0(\Omega). \]

Consequently, we have the following result which shows the asymptotic convergence of the fully-discrete approximation for the Stokes problem.
Corollary 5.1. Let $0 < s < 1$ a fixed index. Under the assumptions of Theorem 5.2. Then, if we define

$$u_{0,h} := L_h u_0,$$

for a small enough time step $\Delta t$, there exist a constant $C > 0$ independent of $h$ and $\Delta t$, such that

$$\max_{1 \leq n \leq N} \| u(t_n) - u_h^n \|_{L^2(\Omega)}^2 + \Delta t \sum_{n=1}^N \| u(t_n) - u_h^n \|_{H^1(\Omega)}^2 + \Delta t \sum_{n=1}^N \| P(t_n) - P_h^n \|_{L^2(\Omega^D)}^2 \leq C \left[ h^{2s} + (\Delta t)^2 \right].$$

Proof. Corollary follows immediately by combining Theorem 5.1 and Theorem 5.2.

5.2 Eddy current problem

The eddy current problem arises when the displacement currents can be dropped from Maxwell’s equation (see [22, Chapter 9] and [8]). The aim of the eddy current problem is to determine the eddy currents induced in a three-dimensional conducting domain represented by an open and bounded set $\Omega_C$. In this way, let $\Omega \subset \mathbb{R}^3$ be a bounded domain simply connected with a connected boundary $\partial \Omega$. We suppose that $\Omega$ is divided two regions: a conductor domain $\Omega_C$ and insulator domain $\Omega_D$, where $\Omega_D = \Omega \setminus \Omega_C$.

We refer some papers about the numerical analysis for the time-dependent eddy current model in bounded domains containing conductor and isolator materials [16, 30, 31]. These articles deal with the case where the conducting materials are strictly contained in a three-dimensional domain. In general, these formulations present natural or/and essential boundaries conditions depending on the main variable. For a treatment of time-dependent eddy current model with current or voltage excitations, we refer the reader to [17, 18]. Other parabolic formulations for the time-dependent eddy current problem posed in the conductor domain can be founded in [12, 4].

The abstract theory developed in Section 3 and in Section 4 can be used to study the fully discrete mixed formulations proposed for the eddy current model in [28, 1, 2, 18], but we will only focus in the numerical analysis for the formulations studied in [1] and [18]. The well-posedness of the continuous variational formulations corresponding to [1] and [18] was proved in [3] Theorem 3.2 and Theorem 3.4, respectively.

In both cases, the formulation of the eddy current problem was deduced in terms of a time-primitive of the electric field, i.e., in terms of the unknown $u$ given by

$$u(x, t) := \int_0^t E(x, s) ds.$$ 

In [1], it was proposed for an internal conductor $\partial \Omega_C \cap \partial \Omega = \emptyset$, with the boundary condition

$$E \times n = 0 \quad \text{on} \quad [0, T] \times \partial \Omega.$$ 

Furthermore, to determine the uniqueness of $E$ they must add the following conditions:

$$\text{div}(\varepsilon E) = 0 \quad \text{in} \quad \Omega_D \times [0, T),$$

$$\int_{\Sigma_i} \varepsilon E_{| \Omega_D} \cdot n = 0 \quad \text{in} \quad [0, T), \quad i = 1, \ldots, M_I,$$

where $\Sigma_i$, $i = 1, \ldots, M_I$, are the connected components of $\Sigma := \partial \Omega_C$. 

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On the other hand, in [18] the eddy current model was analyzed with input currents intensities as source data. In this case, the conductor can be not strictly contained in $\Omega$ and $\Gamma_C := \partial\Omega_C \cap \partial\Omega$ can be not empty. They have assumed $\partial\Omega_C \cap \partial\Omega = \Gamma_E \cup \Gamma_J$, where $\Gamma_J$ is the boundary associated with the current entrance (the conductor is connected to a source current) and $\Gamma_E$ is the boundary associated with the current exit. Furthermore, $\Gamma_j = \bigcup_{k=1}^{k=N} \Gamma_j^k$, where $\Gamma_j^k$ are the connected components of $\Gamma_j$ (see Figure 5.1).

![Figure 5.1: Sketch of the domain](image)

The intensities of the input current are imposed by

$$\int_{\Gamma_j} \sigma \mathbf{E} \cdot \mathbf{n} = I_n \quad \text{in } [0, T], \quad n = 1, \ldots, N,$$

where $I_n$ is the current intensity through the surface $\Gamma_j^n$, and the following boundary conditions was proposed

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on } [0, T] \times \Gamma_E,$$

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on } [0, T] \times \Gamma_J,$$

$$\mu \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } [0, T] \times \partial \Omega,$$

where $\mathbf{H}$ is the magnetic field. In this case, $\mathbf{E}$ is uniquely determined provided that

$$\text{div}(\varepsilon \mathbf{E}) = 0 \quad \text{in } [0, T] \times \Omega_D,$$

$$\varepsilon \mathbf{E}|_{\Omega_D} \cdot \mathbf{n} = g \quad \text{on } [0, T] \times \Gamma_I,$$

$$\int_{\Gamma_I} \varepsilon \mathbf{E}|_{\Omega_D} \cdot \mathbf{n} = 0, \quad k = 2, \ldots, M_I, \quad \text{in } [0, T],$$

where $\Gamma_D := \partial \Omega \cap \partial \Omega_D$ and $\Gamma_I^k := \partial \Omega_C^k \cap \partial \Omega_D$, $k = 1, \ldots, M_I$, are the connected components of the interface boundary $\Gamma_I$, which is the boundary between the conductor domain and the insulator domain (see Figure ??, $M_I = 5$). Furthermore, $g$ is an additional data.

To show the application of the abstract theory in the models studied in [1] and [18] is necessary to introduce the following functional spaces:

$$\mathbf{H}(\text{curl}; \Omega) := \{ \mathbf{v} \in L^2(\Omega)^3 : \text{curl} \mathbf{v} \in L^2(\Omega)^3 \},$$

$$\mathbf{H}_0(\text{curl}; \Omega) := \{ \mathbf{v} \in \mathbf{H}(\text{curl}; \Omega) : \mathbf{v} \times \mathbf{n} = 0 \text{ on } \partial \Omega \},$$

$$\mathbf{H}(\text{curl}^0; \Omega) := \{ \mathbf{v} \in \mathbf{H}(\text{curl}; \Omega) : \text{curl} \mathbf{v} = 0 \text{ in } \Omega \}.$$
and 
\[ H_\Lambda(\text{curl}^0; \Omega) := H(\text{curl}^0; \Omega) \cap H_\Lambda(\text{curl}; \Omega). \]

Similarly, for \( H(\text{div}; \Omega), H_0(\text{div}; \Omega), H_\Lambda(\text{div}; \Omega) \) and 
\[ H_\Lambda(\text{div}^0; \Omega) = \{ w \in H_\Lambda(\text{div}; \Omega) : \text{div}(\varepsilon w) = 0 \text{ in } \Omega \}. \]

From now on, we refer to the problem studied in [1] as the input current model and the problem studied in [18] as the internal conductor model. Furthermore, we assume that \( \Omega \) and \( \Omega_c \) are Lipschitz polyhedra and that \( \{ T_h \}_h \) is a regular family of tetrahedral meshes of \( \Omega \) such that each element \( K \in T_h \) is contained either in \( \Omega_c \) or in \( \Omega_d \). As usual, \( h \) stands for the largest diameter of the tetrahedra \( K \) in \( T_h \). Finally, we suppose that the family of triangulations \( \{ T_h(\Sigma) \}_h \) induced by \( \{ T_h \}_h \) on \( \Sigma \) is quasi-uniform.

### 5.2.1 Internal conductor model

Let us define 
\[ M := \{ v \in H^1(\Omega_{\text{in}}) : \mu|_{\partial \Omega} = 0, \mu|_{\Sigma_i} = C_i, i = 1, \ldots, M \} \]
endowed with usual norm in \( H^1(\Omega_{\text{in}}) \). The variational formulation of eddy current model with internal conductor (see [1]) is given by

**Problem 5.3.** Find \( u \in L^2(0, T; H_0(\text{curl}; \Omega)) \) and \( \lambda \in H^1(0, T; M) \) such that

\[
\frac{d}{dt} \left[ \int_{\Omega_c} \sigma u(t) \cdot v + \int_{\Omega_d} \varepsilon v \cdot \nabla \lambda(t) \right] + \int_{\Omega} \frac{1}{\mu} \text{curl} u(t) \cdot \text{curl} v = \int_{\Omega} \mathbf{f}(t) \cdot v \quad \forall v \in H_0(\text{curl}; \Omega),
\]

\[
\int_{\Omega_d} \varepsilon u(t) \cdot \nabla \mu = 0 \quad \forall \mu \in M,
\]

\[ u(t, 0) = 0 \quad \text{in } \Omega_{\text{in}} \quad \text{and} \quad \lambda(0) = 0 \quad \text{in } \Omega_d,
\]

where 
\[ \langle \mathbf{f}(t), v \rangle = \int_{\Omega} \mathbf{f}(t) \cdot v = \int_{\Omega} \text{curl} \mathbf{H}_0 \cdot v - \int_{\Omega} \mathbf{J}(t) \cdot v \quad \forall t \in [0, T] \quad \forall v \in X_h. \]

with \( \mathbf{H}_0 \) the initial magnetic condition and \( \mathbf{J} \in L^2(0, T; L^2(\Omega)) \).

To obtain the fully-discrete approximation for the eddy current formulation given by Problem 5.3 we use finite element subspaces to define \( X_h \) and \( M_h \), the corresponding families of finite dimensional subspaces of \( X := H_0(\text{curl}; \Omega) \) and \( M := M(\Omega_{\text{in}}) \) respectively (see Section 3).

We define \( X_h \) using Nédélec finite elements, more precisely \( X_h \) is the global Nédélec finite elements subspace, which is defined by

\[ X_h := \{ v \in H_0(\text{curl}; \Omega) : v|_K \in \mathcal{N}_1(K) \; \forall K \in T_h \}, \tag{5.5} \]

where \( \mathcal{N}_1(K) \) is the local representation on \( K \) of the lowest-order Nédélec finite elements subspace 
\[ \mathcal{N}_1(K) := \{ a \times x + b : a, b \in \mathbb{R}^3, x \in K \}. \]

On the other hand, we use standard linear Lagrange finite elements to define \( M_h \), i.e.,

\[ M_h := \{ \mu \in H^1(\Omega_{\text{in}}) : \mu|_K \in \mathbb{P}_1(K) \; \forall K \in T_h, K \subset \Omega_d, \mu|_{\partial \Omega} = 0, \mu|_{\Sigma_i} = C_i, i = 1, \ldots, I \}, \tag{5.6} \]

where \( \mathbb{P}_m \) is the set of polynomials of degree not greater than \( m \).

The corresponding fully-discrete problem of Problem 5.3 is given by
Problem 5.4. Find $u^n_h \in X_h$ and $\lambda^n_h \in M_h$ for $n = 1, \cdots, N$ such that
\[
\left[ \int_{\Omega_C} \frac{u^n_h - u^{n-1}_h}{\Delta t} \cdot v + \int_{\Omega_D} \varepsilon v \cdot \nabla \lambda^n_h - \nabla \lambda^{n-1}_h \right] + \int_{\Omega} \frac{1}{\mu} \text{curl} u^n_h \cdot \text{curl} v = \int_{\Omega} f(t_n) \cdot v \quad \forall v \in X_h,
\]
\[
\int_{\Omega_D} \varepsilon u^n_h \cdot \nabla \mu = 0 \quad \forall \mu \in M_h,
\]
\[
u^n_h = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \nu^0_h = 0 \quad \text{in} \quad \Omega_D.
\]
In this case, we can notice that the discrete kernel of $b$ is defined by
\[
V_h = \{ v \in X_h : b(v, \mu) = 0 \quad \forall \mu \in M_h \} = \left\{ v \in X_h : \int_{\Omega_D} \varepsilon v \cdot \nabla \mu = 0 \quad \forall \mu \in M_h \right\}.
\]

**Existence and uniqueness of fully discrete-solutions.**
In order to deduce the existence and uniqueness of solution for the fully-discrete approximation, we have to show that the hypotheses H7–H8 hold. The proof of discrete \textit{inf-sup} condition H7 is completely analogous to the deduction of its continuous version H1 inside the proof of [3, Theorem 3.2]. For this reason, we only show the proof of H8. To this aim, we first need to recall the following result.

**Lemma 5.2.** Let $X_h(\Omega_C) := \{ v|_{\Omega_C} : v \in X_h \}$. There exists a bounded and linear mapping $\mathcal{E}_h : X_h(\Omega_C) \to V_h$ satisfying:

a) $\mathcal{E}_h$ is bounded uniformly in $h$.

b) $(\mathcal{E}_h v_c)|_{\Omega_C} = v_c$ for all $v_c \in X_h(\Omega_C)$.

**Proof.** See [1, Lemma 5.3].

**Lemma 5.3.** There exist positive constants $\hat{\gamma}$ and $\hat{\alpha}$ such that
\[
\int_{\Omega} \frac{1}{\mu} |\text{curl} v|^2 + \hat{\gamma} \int_{\Omega_C} |\sigma v|^2 \geq \hat{\alpha} ||v||_{H(\text{curl}; \Omega)}^2 \quad \forall v \in V_h,
\]
where $V_h$ is the discrete kernel of $b$ given in (5.7).

**Proof.** Let $v \in V_h$ and define $\tilde{w} := v - \mathcal{E}_h v$. Then $\tilde{w} \in V_h$, $\tilde{w} = 0$ in $\Omega_C$ and $\tilde{w}|_{\Omega_D}$ belongs to the subspace
\[
V_{d,h} = \{ v|_{\Omega_D} : v \in V_h \} \cap H_0(\text{curl}; \Omega_D),
\]
where $X_h(\Omega_D) := \{ v|_{\Omega_D} : v \in X_h \}$. It is well-known that the semi-norm $w \mapsto ||\text{curl} w||_{0, \Omega_D}$ is a norm on $V_{d,h}$ uniformly equivalent to the $H(\text{curl}; \Omega_D)$-norm (see, e.g., Theorem 4.7, [23]). It follows that
\[
||\tilde{w}||_{H(\text{curl}; \Omega)} = ||\tilde{w}|_{\Omega_D}||_{H(\text{curl}; \Omega_D)} \leq C ||(\text{curl} \tilde{w})|_{\Omega_D}||_{L^2(\Omega_D)^3}
\]
\[
\leq C \left\{ ||\text{curl} \mathcal{E}_h v_c||_{L^2(\Omega_D)^3} + ||\text{curl} v||_{L^2(\Omega_D)^3} \right\} \leq C \left\{ ||v_c||_{H(\text{curl}; \Omega_D)} + ||\text{curl} v||_{L^2(\Omega_D)^3} \right\},
\]
Consequently,
\[
||v||_{H(\text{curl}; \Omega)} = ||\mathcal{E}_h v + \tilde{w}||_{H(\text{curl}; \Omega)} \leq 2 \left\{ ||\mathcal{E}_h v||_{H(\text{curl}; \Omega)} + ||\tilde{w}||_{H(\text{curl}; \Omega)} \right\}
\]
\[
\leq C \left\{ ||v||_{H(\text{curl}; \Omega_D)} + ||\text{curl} v||_{L^2(\Omega_D)^3} \right\},
\]
from which the result holds. \qed
Hence, we have deduced the existence and uniqueness of Problem 5.4.

**Error estimates.**

Our next goal is to obtain error estimates for the fully-discrete approximation of the eddy current formulation. Since $\lambda = 0$ and $\lambda_{h}^{n} = 0$, we will only be concerned with error estimates for the main variable $u$. In fact, we have the following result for a direct application of Theorem 4.1.

**Theorem 5.3.** Assume that $u \in H^{1}(0,T; H_{0}(\text{curl}; \Omega)) \cap H^{2}(0,T; L^{2}(\Omega)^{3})$. Then, there exists a constant $C > 0$, independent of $h$ and $\Delta t$, such that

$$
\max_{1 \leq n \leq N} \|u(t_{n}) - u_{h}^{n}\|^{2}_{\sigma, \Omega_{C}} + \Delta t \sum_{k=1}^{N} \|u(t_{k}) - u_{h}^{k}\|^{2}_{H(\text{curl}; \Omega)} \\
\leq C \left\{ \max_{1 \leq n \leq N} \inf_{v \in X_{h}} \|u(t_{n}) - v\|^{2}_{H(\text{curl}; \Omega)} + \Delta t \sum_{n=1}^{N} \inf_{v \in X_{h}} \|u(t_{n}) - v\|^{2}_{H(\text{curl}; \Omega)} \\
+ \int_{0}^{T} \left( \inf_{v \in X_{h}} \|\partial_{t}u(t) - v\|^{2}_{H(\text{curl}; \Omega)} \right) dt + (\Delta t)^{2} \int_{0}^{T} \|\partial_{tt}u(t)\|^{2}_{L^{2}(\Omega)^{3}} dt \right\},
$$

where $\|w\|^{2}_{\sigma, \Omega_{C}} := \int_{\Omega_{C}} \sigma |w|^{2}$.

Finally, to obtain the asymptotic error estimate, we need to consider the Sobolev space

$$
H^{r}(\text{curl}, Q) := \{ v \in H^{r}(Q)^{3} : \text{ curl } v \in H^{r}(Q) \}, \quad r \geq 0
$$

(5.9)

endowed with the norm $\|v\|^{2}_{H^{r}(\text{curl}, Q)} := \|v\|^{2}_{Q} + \|\text{ curl } v\|^{2}_{Q}$, where $Q$ is either $\Omega_{C}$ or $\Omega_{D}$. It is well known that the Nédélec interpolant $I_{h}^{N} v \in X_{h}(Q)$ is well defined for all $v \in H^{r}(\text{curl}, Q)$ with $r > 1/2$, see for instance [6, Lemma 5.1] or [11, Lemma 4.7]. We fix now an index $r > 1/2$ and introduce the space

$$
\mathcal{X} := H^{r}(\text{curl}, \Omega) \cap H_{0}(\text{curl}; \Omega).
$$

Then, the Nédélec interpolation operator $I_{h}^{N} : \mathcal{X} \to X_{h}$ is uniformly bounded and the following interpolation error estimate holds true; see [21, Lemma 5.1] or [6, Proposition 5.6]:

$$
\|v - I_{h}^{N}v\|_{H(\text{curl}; \Omega)} \leq C h^{\min\{r, 1\}} \|v\|_{\mathcal{X}} \quad \forall v \in \mathcal{X}.
$$

(5.10)

Consequently, we have the following result which shows the asymptotic convergence of the fully-discrete approximation.

**Corollary 5.2.** If $u \in H^{1}(0,T; \mathcal{X} \cap H_{0}(\text{curl}; \Omega)) \cap H^{2}(0,T; L^{2}(\Omega)^{3})$, there exists a constant $C > 0$ independent of $h$ and $\Delta t$, such that

$$
\max_{1 \leq n \leq N} \|u(t_{n}) - u_{h}^{n}\|^{2}_{\sigma, \Omega_{C}} + \Delta t \sum_{k=1}^{N} \|u(t_{k}) - u_{h}^{k}\|^{2}_{H(\text{curl}; \Omega)} \\
\leq C \left\{ h^{\ell} \left( \max_{1 \leq n \leq N} \|u(t_{n})\|^{2}_{\mathcal{X}} + \|\partial_{t}u\|^{2}_{L^{2}(0,T; \mathcal{X})} \right) + (\Delta t)^{2} \|\partial_{tt}u\|^{2}_{L^{2}(0,T; L^{2}(\Omega)^{3})} \right\}
$$

with $\ell := \min\{r, 1\}$.

**Proof.** It is a direct consequence of Theorem 5.3 and the interpolation error estimate (5.10). \qed

**Remark 5.4.** By testing (4.6) with $v = \text{ curl } \sigma_{h}^{k}$, considering $\lambda = 0$ and using similar arguments of Section 4, we obtain the following result.
Theorem 5.5. Let us assume the hypothesis of Theorem 4.1. If the Lagrange multiplier \( \lambda \) of the Problem 2.1 vanishes identically and the operator \( A \) is monotone on \( X \), then there exists a constant \( C > 0 \) independent of \( h \) and \( \Delta t \) satisfying

\[
\Delta t \sum_{k=1}^{n} \left( R(\partial_t u(t_k)) - \overline{u_h^k}, (\partial_t u(t_k)) - \overline{u_h^k} \right)_Y \leq C \left[ \left\| \Pi_h u_0 - u_{0,h} \right\|_Y^2 + \max_{1 \leq n \leq N} \left( \inf_{v \in X} \left\| u(t_n) - v \right\|_X^2 \right) + \Delta t \sum_{n=1}^{N} \left( \inf_{v \in X_h} \left\| u(t_n) - v \right\|_X^2 \right) + \int_{0}^{T} \left( \inf_{v \in X} \left\| \partial_t u(t) - v \right\|_X^2 \right) dt + (\Delta t)^2 \left\| \partial_t u \right\|_{L^2(0,T;Y)}^2 \right].
\]

Finally, by applying Theorem 5.5 to the eddy currents problem and using the interpolation error estimate (5.10), we deduce the quasi-optimal convergence for the time derivative approximation.

Corollary 5.3. Let \( u \) be the solution of the Problem 5.3. If \( u \in H^1(0,T;X \cap H_0(\text{curl};\Omega)) \cap H^2(0,T;L^2(\Omega)^3) \), there exists a constant \( C > 0 \) independent of \( h \) and \( \Delta t \), such that

\[
\Delta t \sum_{k=1}^{N} \left\| \partial_t u(t_k) - \overline{u_h^k} \right\|_{L^2(\Omega)}^3 \leq C \left\{ h^{2\ell} \left( \max_{1 \leq n \leq N} \left\| u(t_n) \right\|_X^2 + \left\| \partial_t u \right\|_{L^2(0,T;X)}^2 \right) + (\Delta t)^2 \left\| \partial_t u \right\|_{L^2(0,T;L^2(\Omega)^3)}^2 \right\}
\]

with \( \ell := \min\{r,1\} \).

Remark 5.6. At each time step \( t = t_k \), we can approximate the eddy currents \( \sigma E(x,t_k) \) and the magnetic field \( H(x,t_k) \) by means of \( \sigma E_h^k = \sigma \overline{u_h^k} \) and \( \mu H_h^k = \text{curl} u_h^k - \mu H_0 \), respectively. Then the Corollary 5.3 and the Theorem 4.1 yields the following error estimates

\[
\Delta t \sum_{k=1}^{N} \left\| \sigma E(t_k) - \sigma E_h^k \right\|_{0,\Omega_c}^2 \leq C \left[ h^{2\ell} + (\Delta t)^2 \right],
\]

\[
\Delta t \sum_{k=1}^{N} \left\| \mu H(t_k) - \mu H_h^k \right\|_{0,\Omega}^2 \leq C \left[ h^{2\ell} + (\Delta t)^2 \right].
\]

Numerical results.

Now, we will present some numerical results obtained with a MATLAB code which implements the numerical method described above. First, we solve a test problem with a known analytical solution. Next, we describe a problem with cylindrical symmetry and compare the results with those obtained with an axisymmetric code.

Test 1: A test with known analytical solution

Let us consider \( \Omega := [0,3]^3 \), \( \Omega_c := [1,2]^3 \) and \( T = 10 \). The right hand is chosen so that

\[
u(x_1, x_2, x_3, t) = \sin(\pi t) \begin{bmatrix} x_1^2 x_2 x_3 (x_1 - 3)^2 (2x_2 - 3)(x_2 - 3)(x_3 - 3) \\
-x_1 x_2^2 x_3 (x_1 - 3)(2x_1 - 3)(x_2 - 3)^2 (x_3 - 3) \\
0 \end{bmatrix}
\]

is solution of Problem 5.3. Furthermore, we have assumed without loss of generality that \( \mu = \sigma = 1 \). The numerical method has been applied with several successively refined meshes and time-steps. The
computed solutions have been compared with the analytical one, by calculating the relative percentage error in time-discrete norms from Remark 5.6. More exactly, we have computed the relative percentage error for the physical variables of interest, namely

$$\frac{100}{\Delta t} \sum_{k=1}^{N} \| H(t_k) - H_k \|^2_{0,\Omega}$$

$$\frac{100}{\Delta t} \sum_{k=1}^{N} \| H(t_k) \|^2_{0,\Omega}$$

which are time-discrete forms of the errors in $L^2(0, T; L^2(\Omega))$ and $L^2(0, T; L^2(\Omega_C))$ norms, respectively. To show the linear convergence with respect to the mesh-size and the time step, we have computed the relative percentage error for the physical variables to $h, \Delta t_n, n = 2, \cdots, 7$. Figure 5.2 shows log-log plots the magnetic field and electric field in the conductor domain in the discrete norms considered above versus the number of degrees of freedom (d.o.f).

Figure 5.2: Percentage discretization error curves for $H$ (left) and $E$ (right) versus number of d.o.f. (log-log scale).

**Test 2: A comparison with axisymmetric problem.**

We consider the geometry sketched in Figure 5.3, which corresponds to a typical EMF (electromagnetic forming) setting. Thus $\Omega_C$ is the cylinder of radius $R_P = 1\, \text{m}$ and its $z$-coordinate varies between $(1.2, 1.4)$. We assume that $J$ is supported in $\Omega_s$ where $\Omega_s \subset \Omega$, $\Omega_s$ is a toroidal core of rectangular cross section $S$, with inner radius equal to $R_s = 0.5\, \text{m}$, outer radius $R_S = 1\, \text{m}$ and height $A_s = 0.5\, \text{m}$.  

Figure 5.3: Sketch of the domain $\Omega$ (left) and its meridian section (right).
The source current density is supported in $\Omega_s$ and given by

$$J(t, x) = \frac{I(t)}{\text{meas}(S)} \begin{pmatrix} -\frac{x_2}{\sqrt{x_1^2 + x_2^2}} \\ \frac{\tilde{x_2}}{\sqrt{x_1^2 + x_2^2}} \\ 0 \end{pmatrix} \text{ in } \Omega_s,$$

where the current intensity $I(t)$ is shown in Figure 5.4. Note that, since the source current density field has only azimuthal non-zero component, the solution will be axisymmetric. In particular, we can solve the problem in the meridional section depicted in Figure 5.3 (right). In this case, there is no analytical solution, so we will assess the behavior of the method by comparing the computed results with those obtained with an axisymmetric code on the very fine mesh shown in Figure 5.5 (right) which will be taken as ‘exact’ solution.

Figure 5.4: Source current intensity (A) vs. time (s).

The axisymmetric problem has been solved by using a scalar formulation written in terms of the azimuthal component of a magnetic vector potential $A_\theta$. The corresponding weak formulation, although with boundary conditions different from those of our case, has been analyzed in [19], [20] with moving domains. In particular, the method was proved to converge with optimal order error estimates in terms of $h$ and $\Delta t$ under appropriate assumptions.

Figure 5.5 (right) shows the mesh used in the axisymmetric code. Concerning the 3D mesh, we have exploited the symmetry of the problem and solved it in 1/8 of the whole domain to reduce the number of degrees of freedom. The used mesh is shown in Figure 5.5 (left).
We have solved the problem with several successively refined meshes and a time-step conveniently reduced to analyze the convergence with respect to both, the mesh-size and the time-step simultaneously. Figure 5.6 shows a log-log plot of the relative error for the electric field, versus the number of degrees of freedom (d.o.f.). The curve shows that the obtained results converge to the ‘exact’ ones as $h$ and $\Delta t$ go to zero.

![Log-log plot](image)

Figure 5.6: $100 \times \max_{1 \leq k \leq M} \frac{\|E(t_k) - E_k\|_{L^2(\Omega_C)^3}}{\|E(t_k)\|_{L^2(\Omega_C)^3}}$ versus number of d.o.f. (log-log scale).

### 5.2.2 Input current model

As another application of our theoretical framework, we present the fully-discrete analysis for the input current model proposed in [18]. To this aim, we introduce the following Hilbert spaces

\[ X := \{ w \in H(\text{curl}; \Omega) : \quad w \times n = 0 \text{ on } \Gamma_C, \quad \text{curl } w \cdot n = 0 \text{ on } \partial \Omega \} \]

and

\[ M := \{ \varphi \in H^1(\Omega_D) : \quad \varphi|_{\Gamma_{I_1}} = 0; \quad \varphi|_{\Gamma_{k}} = C_I, \quad k = 2, \ldots, M_I \} \]

with their usual norms in $H(\text{curl}; \Omega)$ and $H^1(\Omega_D)$, respectively.

Let $g \in L^2(0,T;L^2(\Gamma))$, $I_n \in H^2(0,T)$, $n = 1, \cdots, N$ and $H_0 \in H(\text{curl}; \Omega)$ the initial magnetic field. We denote

\[ \langle f(t), v \rangle = \int_{\Omega} f(t) \cdot v := \sum_{n=1}^{N} L_n(v)(I_n(t) - I_n(0)) + \int_{\Omega} \text{curl } H_0 \cdot v \quad \forall v \in X, \]

for any $t \in [0,T]$. The variational formulation for the input current model (see [18]) is:

**Problem 5.5.** Find $u \in L^2(0,T;X)$ and $\lambda \in H^1(0,T;M)$ such that

\[
\frac{d}{dt} \left[ \int_{\Omega_C} \sigma u(t) \cdot v + \int_{\Omega_D} \varepsilon v \cdot \nabla \lambda(t) \right] + \int_{\Omega} \frac{1}{\mu} \text{curl } u(t) \cdot \text{curl } v = \langle f(t), v \rangle \quad \forall v \in X,
\]

\[
\int_{\Omega_D} \varepsilon u(t) \cdot \nabla \mu = \int_{\Gamma_{d}} \left( \int_{0}^{t} g(s) ds \right) \mu \quad \forall \mu \in M,
\]

$u(\cdot, 0) = 0 \quad \text{in } \Omega_C$ and $\lambda(0) = 0 \quad \text{in } \Omega_D$. 

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where we have introduced the time primitive
\[ \lambda(x,t) = \int_0^t \xi(x,s)ds \quad x \in \Omega, \quad t \in [0,T]. \]
of the original lagrange multiplier \( \xi \) of the model in [13]. As before, to get the fully-discrete approximation for Problem 5.5, it is necessary to employ \( X_h \) and \( M_h \), finite-dimensional subspaces of \( X \) and \( M \), respectively. Thus, we define the following spaces
\[ X_h := \{ w \in N_h(\Omega) : w \times n = 0 \text{ on } \Gamma_c \text{ and } \text{curl } w \cdot n = 0 \text{ on } \partial \Omega \}, \]
\[ M_h := \{ \mu \in L(\Omega) : \mu|_{\Gamma_1} = 0, \mu|_{\Gamma_k} = C_k, \quad k = 2, \cdots, M \}, \]
where \( N_h(\Omega) \) and \( L(\Omega) \) are Nédélec (see (5.3)) and Lagrange finite element spaces, respectively.

**Problem 5.6.** Find \( u^n_h \in X_h \) and \( \lambda^n_h \in M_h \) for \( n = 1, \cdots, N \) such that
\[ \left[ \int_{\Omega_c} \sigma u^n_h - u^{n-1}_h \Delta t \right] \cdot v + \int_{\Omega_D} \varepsilon \cdot \frac{\nabla \lambda^n_h - \nabla \lambda^{n-1}_h}{\Delta t} \mu + \int_\Omega \frac{1}{\mu} \text{curl } u^n_h \cdot \text{curl } v = \int_\Omega f(t_n) \cdot v \quad \forall v \in X_h, \]
\[ \int_{\Omega_D} \varepsilon u^n_h \cdot \nabla \mu = \int_{\Gamma_d} \left( \int_0^{t_n} g(s)ds \right) \mu \quad \forall \mu \in M_h, \]
\[ u^n_h = 0 \quad \text{in } \Omega \quad \text{and} \quad \lambda^n_h = 0 \quad \text{in } \Omega_D. \]

Next, we deduce the existence and uniqueness of the fully-discrete solution of Problem 5.6. So, we will prove that the hypotheses H7-H8 hold. Consequently, we define the discrete kernel of \( b \) given by
\[ V_h = \{ v \in X_h : b(v,\mu) = 0 \quad \forall \mu \in M_h \}. \]

The proof of discrete inf-sup condition H7 is similar to the deduction of its continuous version (see [3 Theorem 3.2]). Then, we only show the proof of H8. To this end, we need to deduce a discrete version of [14 Proposition 7.4]. To do that, we introduce the following notation:
\[ \mathbb{H} := H^1_0(\{\text{curl},\Omega_D\}) \cap H^1_0(\{\text{div},\Omega_D\}), \]
\[ \tilde{H}^1(\text{curl};\Omega_D) := \{ w \in H^1(\text{curl};\Omega_D) : \text{curl } w \cdot n = 0 \text{ on } \Gamma_D \}, \]
\[ \tilde{H}(\text{div};\Omega_D) := \{ w \in H(\text{div},\Omega_D) : w \cdot n |_{\Gamma_D} \in L^2(\Gamma_D) \}, \]
\[ N(\Omega_D) := \left\{ v|_{\Gamma_h} : v \in N(\Omega) \right\}, \]
\[ N(\Omega_c) := \left\{ v|_{\Gamma_c} : v \in N(\Omega) \right\}, \]
\[ V_h,D := \left\{ w \in \tilde{H}^1(\text{curl};\Omega_D) \cap N(\Omega_D) : b(w,\varphi) = 0 \quad \forall \varphi \in M_h \right\}. \]

**Lemma 5.4.** There exist a constant \( C > 0 \) independent of \( h \) such that
\[ \|v\|_{0,\Omega_D} \leq C \|\text{curl } v\|_{0,\Omega_D} \quad \forall v \in V_{h,D}. \]

**Proof.** The proof is adapted from [7 Lemma 4.7]. The authors have done the case on the which the conductors do not go through the boundary of \( \Omega \). Let \( v \in V_{h,D} \). In virtue of an orthogonal decomposition of \( L^2(\Omega_D) \) (see [14 Proposition 6.4]) we can write \( v = \text{curl } Q + \nabla \chi + k \) with
\[ Q \in H^1_0(\text{curl};\Omega_D) \cap H^1_0(\text{div};\Omega_D) \cap \mathbb{H}^1, \quad \chi \in H^1_1(\Omega_D) \quad \text{and} \quad k \in \mathbb{H}. \]
Lemma 5.5. If we define $u = \text{curl} Q$, it is easy verify the following $\text{curl} u = \text{curl} v$ in $\Omega_D$, $\text{div} u = 0$ in $\Omega_D$, $u \cdot n = 0$ on $\Gamma_D$ and $u \times n = 0$ on $\Gamma_t$. Then,

$$u \in H^1_l(\text{curl}; \Omega_D) \cap H_D^0(\text{div}^0; \Omega_D),$$

and consequently $u \in H^s(\Omega_D)$ for some $s > 1/2$ and there exists $C > 0$ such that

$$\|u\|_{s, \Omega_D} \leq C \|u\|_{H_l(\text{curl}; \Omega_D)}.$$

Moreover, since $u \in \mathbb{H}^1$ and by using [14, Proposition 7.4], we have

$$\|u\|_{s, \Omega_D} \leq C \|\text{curl} u\|_{0, \Omega_D}. \quad (5.13)$$

Furthermore, thanks to that $\text{curl} u = \text{curl} v \in L^\infty(\Omega_D)^3$ then we can define $\Pi_h u \in \mathcal{N}_h(\Omega_D)$. Note that there exists $\phi_h \in M_h$ such that $\nabla \cdot \phi_h = \Pi_h (\nabla \cdot v + k)$, and hence

$$\|v\|_{0, \Omega_D}^2 \leq \|v\|_{0, \Omega_D} \|\Pi_h u\|_{0, \Omega_D}. \quad (5.14)$$

On the other hand, for all $K \in T_h$ con $K \subset \Omega_D$, we obtain

$$\|\Pi h u - u\|_{0,K} \leq C h^s (\|u\|_{s,K} + \|\text{curl} v\|_{s,K}) \leq C (h^s \|u\|_{s,K} + \|\text{curl} v\|_{0,K}),$$

where we have used the local inverse estimate

$$\|\text{curl} v\|_{s,K} \leq C h^{-s} \|\text{curl} v\|_{0,K}.$$

By using (5.13) and triangular inequality, we have

$$\|\Pi h u\|_{0, \Omega_D} \leq C \|\text{curl} v\|_{0, \Omega_D}.$$

Finally, the Lemma follows from (5.14). \hfill \square

Lemma 5.5. If we define

$$X_h(\Omega_c) := \{v \in \mathcal{N}(\Omega_c): v \times n = 0 \quad \text{on} \quad \Gamma_c\}.$$

Then, the lineal mapping $\mathcal{E}_h : X_{C,h}(\Omega_c) \to V_h$ characterized by

$$\mathcal{E}_h(v_c)|_{\Omega_c} = v_c \quad \forall v_c \in X_h(\Omega_c),$$

$$\int_{\Omega_D} (\text{curl} \mathcal{E}_h v_c) \cdot \text{curl} w_D = 0 \quad \forall v_c \in X_h(\Omega_c) \quad \forall w_D \in V_{h,d}.$$

is well defined and bounded.

Proof. Let us denote $\gamma^C_r : \mathcal{N}_h(\Omega_c) \to H^{-1/2}(\text{div}_r; \partial \Omega_c)$ and $\gamma^D_r : \mathcal{N}_h(\Omega_D) \to H^{-1/2}(\text{div}_r; \partial \Omega_D)$ by tangential traces on $H(\text{curl}; \Omega_c)$ and $H(\text{curl}; \Omega_D)$, respectively. It follows that linear operator $\eta : X_h(\Omega_c) \to H^{-1/2}(\text{div}_r; \partial \Omega_D)$ given by

$$\eta(v_c) := \left\{ \begin{array}{ll} \gamma^C_r(v_c)|_{\Gamma_t} & \text{on} \quad \Gamma_t, \\
0, & \text{on} \quad \Gamma_D, \end{array} \right.$$ \hfill \hfill (5.15)

is well defined. Moreover, we have

$$\|\eta(v_c)\|_{H^{-1/2}(\text{div}_r; \partial \Omega_D)} = \|\gamma^C_r(v_c)\|_{H^{-1/2}(\text{div}_r; \partial \Omega_c)} \leq C_1 \|v_c\|_{H(\text{curl}; \Omega_c)} \quad \forall v_c \in H^1_r(\text{curl}; \Omega_c).$$

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After considering the continuous right inverse of tangential operator $\gamma^D$, we can define the continuous linear operator $L_h : X_h(\Omega_c) \to \mathcal{N}(\Omega_d)$ given by

$$L_h(v_c) := (\gamma^D)^{-1}(\eta(v_c)) \quad \forall v_c \in X_h(\Omega_c)$$

there holds

$$L_h(v_c) \times n = 0 \quad \text{on} \quad \Gamma_d \quad \text{and} \quad \frac{d}{n} \quad \text{on} \quad \Gamma_\tau$$

By denoting $H_h(\Omega_d) := \dot{\mathcal{H}}_{\Gamma}^\perp(curl; \Omega_d) \cap \mathbb{H} \cap \mathcal{N}_h(\Omega_d)$, we consider the following mixed problem

**Problem 5.7.** Find $z_D \in H_h(\Omega_d)$ and $\rho \in M_h$ such that

$$
\int_{\Omega_d} \nabla z_D \cdot \nabla w_D + b(w_D, \rho) = -\int_{\Omega_d} \nabla L_h(v_c) \cdot \nabla w_D \quad \forall w_D \in H_h(\Omega_d)
$$

$$b(z_D, \mu) = -b(L_h(v_c), \mu) \quad \forall \mu \in M_h(\Omega_d).$$

Now, we proceed to show that the previous problem is well-posedness. By using the Lemma 5.3, the bilinear form given by

$$\langle v_D, w_D \rangle \mapsto \int_{\Omega_d} \nabla v_D \cdot \nabla w_D,$$

is coercive on $V_{h,d}$. Furthermore, the discrete inf-sup conditions is satisfied. In fact: by noting $\text{grad}(M_h) \subset H_h(\Omega_d)$. Thus, we obtain

$$\sup_{v_D \in H_h(\Omega_d)} \frac{b(v_D, \mu)}{\|v_D\|_{H(\text{curl}, \Omega_d)}} \geq \frac{\|b(\nabla \mu, \mu)\|}{\|\nabla \mu\|_{H(\text{curl}, \Omega_d)}} \quad \forall \mu \in M_h.$$

It follows from the Babuska-Brezzi theory that the Problem 5.7 has a unique solution, which satisfies

$$\|z_D\|_{H(\text{curl}, \Omega_d)} \leq C \|v_c\|_{H(\text{curl}, \Omega_c)} \quad \forall v_c \in X_h(\Omega_c).$$

Hence, we define

$$\mathcal{E}_h v_c := \begin{cases} v_c & \text{in} \quad \Omega_c, \\ z_D + L_h v_c & \text{in} \quad \Omega_d, \end{cases}$$

there holds

$$v_c|_{\Gamma_\tau} \times n = L_h(v_c)|_{\Gamma_\tau} \times n \quad \text{and} \quad z_D|_{\Gamma_\tau} \times n = 0,$$

from which the result follows.

The following result may be proved in much the same way as Lemma 5.15. This is due to Lemma 5.4 and Lemma 5.6.

**Lemma 5.6.** There exist positive constants $\hat{\gamma}$ and $\hat{\alpha}$ such that

$$\int_{\Omega} \frac{1}{\mu} |\nabla v|^2 + \hat{\gamma} \int_{\Omega_c} \sigma|v|^2 \geq \hat{\alpha} \|v\|^2_{H(\text{curl}, \Omega)} \quad \forall v \in V_h. \quad (5.15)$$

**Proof.** Let $v \in V_h$, considering $v_c := v|_{\Omega_c}$ and $\mathcal{E}_h v_c$ given by Lemma 5.5, we can define $\tilde{w} := v - \mathcal{E}_h v_c$, then

$$\tilde{w} = 0 \quad \text{in} \quad \Omega_c, \quad \tilde{w} \in V_h, \quad \tilde{w}|_D \in V_{h,d}.$$

Thus, by using the Lemma 5.4, the continuity of $\mathcal{E}_h$ and proceeding as in Lemma 5.3 it is deduced the result.

\[\square\]
Consequently, we can use the results of Section 3 to conclude that the fully-discrete approximation of the Problem \ref{problem} has a unique solution \((u^h_n, \lambda^h_n) \in X_h \times M_h, \ n = 1, \ldots, N\). Now, our next goal is to obtain error fully-discrete scheme. Before, we recall that \(\lambda = 0\) (see \cite{18, Theorem 2.4}). As in subsection 5.2.1 and by assuming \(u \in H^1(0,T; \mathcal{X}) \cap H^2(0,T; L^2(\Omega)^3)\) (see (5.11)), we can obtain similar results as in Theorem 5.3 and Theorem 5.5. These results allow us to obtain the asymptotic error estimates. In fact, fixing an index \(r > \frac{1}{2}\) and considering \(X_h^r := H^r(\text{curl}, \Omega) \cap X\) (see (5.9)), according to \cite{15, Lemma 2.2}, the Nédélec interpolant operator \(I_h^N : \mathcal{X} \rightarrow X_h^r\) is well defined and we can easily obtain an analogous result to Corollary 5.2. Thus, we easily obtain similar error estimates to those that were given in Remark 5.6 for the approximation of the electric and magnetic field at each time step. Finally, for some numerical results of this subsection that confirm the theoretical result obtained in this work, we refer the reader to \cite{18, Section 4}.

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**Authors contributions**

The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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