Twisted Quasi-elliptic cohomology and twisted equivariant elliptic cohomology

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Abstract. In this paper we construct a twisted version of quasi-elliptic cohomology \cite{Hua18}. This theory can be constructed as a $K$-theory of a loop space. After establishing basic properties of the theory, including restriction, change-of-group and induction maps, we construct the Chern character map. And we compute the twisted quasi-elliptic cohomology theories of representation 4-spheres acted by the finite subgroups of $SU(2)$, which, as conjectured in \cite{SS24}, can produce good observables on M-brane charge in a Tate-elliptic enhancement of D-brane charge in twisted equivariant $K$-theory.

1. Introduction

In this paper we construct a twisted version of elliptic cohomology. Quasi-elliptic cohomology, introduced by the author in \cite{Hua18}, is a variant of Tate $K$-theory, which is the generalized elliptic cohomology associated to the Tate curve. The Tate curve $Tate(q)$ is a generalized elliptic curve over $\text{Spec}\mathbb{Z}((q))$, which is classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity \cite[Section 2.6]{AHS01}. The relation between quasi-elliptic cohomology and Tate $K$-theory can be expressed by

\begin{equation}
QEll^*(X//G) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{Z}((q)) \cong K_{Tate}(X//G).
\end{equation}

Quasi-elliptic cohomology is not an elliptic cohomology but it contains all the information of equivariant Tate $K$-theory. That’s how it got its name. In addition, it can be constructed as the orbifold $K$-theory of a loop groupoid $\Lambda(X//G)$, which partially proved a conjecture by Witten \cite{Lan88} emphasizing the relation between elliptic cohomology and circle-equivariant $K$-theory of a free loop space.

One classical example of twisted cohomology theories is twisted $K$-theory \cite{DK70,AS05}, which admits a geometric construction. The relation between twisted $K$-theory and physics has been observed for decades. It is conjectured in \cite{MMS01} that it can classify D-branes, Ramond-Ramond field strengths and spinors in type II string theory under some conditions. Since elliptic cohomology is a higher version of $K$-theory, it is natural to expect a relation between a twisted version of elliptic cohomology and physics.

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Let $G$ be a finite group acting on a space $X$. The twisting that we use to construct twisted quasi-elliptic cohomology $Q\text{Ell}^{\alpha}(\_)$ corresponds to an element $\alpha \in H^3(BG; U(1))$. Motivated by the relation between elliptic cohomology and loop groupoid, we construct $Q\text{Ell}^{\alpha}(\_)$ in Section 5 as the orbifold K-theory of a twisted version of the loop groupoid $\Lambda(X//G)$, which embodies both the loop rotation and the central extension classified by the 2-cocycle obtained from the loop transgression of the twist $\alpha$. We prove that $Q\text{Ell}^{\alpha}(\_)$ has many parallels with twisted equivariant K-theory, including the existence of K"unneth maps, induction maps, change-of-group isomorphisms and Chern characters. It has the relation with the twisted equivariant Tate K-theory $K^{\alpha++}(\_//G)$ defined in [Dov19] as below.

\[ Q\text{Ell}^{\alpha++}_G(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) \cong K^{\alpha++}_{\text{Tate}}(X//G) \]

The author gives a loop space construction of quasi-elliptic cohomology via bibundles in [Hua18] Section 2. We recall the model $\text{Loop}(X//G)$ in Section 2.2 which is constructed from the category of bibundles from $S^1//s$ to $X//G$ enriched by the rotation of loops. In Section 5.3 we construct a groupoid $\text{Loop}^{\text{twist}}(X//G)$ of twisted equivariant loops. A subgroupoid of it consisting of constant loops in $\text{Loop}^{\text{twist}}(X//G)$ provides a loop space model for twisted quasi-elliptic cohomology, thus, for twisted equivariant Tate K-theory as well.

Sati and Schreiber conjecture in [SS24] that the twisted quasi-elliptic cohomology of representation 4-spheres of finite subgroups of $SU(2)$ can produce good observables on M-brane charges in a Tate-elliptic enhancement of D-brane charge in twisted equivariant K-theory. In Section 6.2 and 6.3 we show the study of this conjecture is feasible mathematically by computing the twisted quasi-elliptic cohomology theories of representation 4-spheres acted by any finite subgroup of $SU(2)$. The theories are computed by applying the properties of quasi-elliptic cohomology theories and equivariant K-theories, especially the conclusions in Appendix A which are corollaries of the decomposition formula in [AGU17] Theorem 3.6 and Corollary 3.7]. The interpretation of the results from the perspective of mathematical physics is not included in this paper.

In Section 2 we give a sketch of quasi-elliptic cohomology, including its definition, basic properties, and the loop space construction. In Section 3 we review Devoto’s equivariant elliptic cohomology. In Section 4 we recall the definition of twisted equivariant elliptic cohomology. In Section 5 we construct twisted quasi-elliptic cohomology. In Section 5.3 we define a model of twisted loop space, with which we can construct twisted quasi-elliptic cohomology. In Section 5.4.2 based on the Chern character of quasi-elliptic cohomology, we construct a Chern character map of twisted quasi-elliptic cohomology. In Section 6 we compute more examples of quasi-elliptic cohomology and twisted quasi-elliptic cohomology. Especially in Section 6.2 and 6.3 we compute the twisted quasi-elliptic cohomology of representation 4-spheres acted by all the finite subgroups of $SU(2)$.

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2. Quasi-elliptic cohomology

In this section we review quasi-elliptic cohomology, the main reference of which is [Hua18]. It is a variant of Tate K-theory [AHS01] [Gan14]. Many constructions on it can be made neater than Tate K-theory and most elliptic cohomology theories. As shown in Section 2.2, it can be constructed as orbifold K-theory of a loop space.

2.1. Definition. In this section we recall the definition of quasi-elliptic cohomology in term of equivariant K-theory and state the conclusions that we need in this paper. For more details on quasi-elliptic cohomology, please refer to [Hua18].

Let $G$ be a compact Lie group and $X$ a $G$-space. Let $G^{\text{tors}} \subseteq G$ denote the set of torsion elements of $G$. For any $g \in G^{\text{tors}}$, the fixed point space $X^g$ is a $C_G(g)$-space where $C_G(g) = \{ h \in G \mid hg = gh \}$. This group action can be extended to that by the group

$$\Lambda_G(g) := C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle,$$

which is given explicitly by

$$[h, t] \cdot x := h \cdot x,$$

for any $[h, t] \in \Lambda_G(g)$ and $x \in X^g$.

To give a complete description of the loop groupoid $\Lambda(X//G)$, we need the following definitions.

**Definition 2.1.**

1. Let $g, g'$ be two elements in $G$. Define $C_G(g, g')$ to be the set $\{ h \in G \mid g'h = hg \}$.
2. Let $\Lambda_G(g, g')$ denote the quotient of $C_G(g, g') \times \mathbb{R}/\langle t \rangle$ under the equivalence

$$(\alpha, t) \sim (g'\alpha, t - 1) = (ag, t - 1),$$

where $l$ is the order of $g$ in $G$.

**Definition 2.2.** Define $\Lambda(X//G)$ to be the groupoid with

- **objects**: the space $\bigcoprod_{g \in G^{\text{tors}}} X^g$
- **morphisms**: the space $\bigcoprod_{g, g' \in G^{\text{tors}}} \Lambda_G(g, g') \times X^g$. 
For an object \( x \in X^g \), the morphism \(([\alpha, t], x) \in \Lambda_G(g, g') \times X^g \) is an arrow from \( x \) to \( \alpha \cdot x \in X^{g'} \). The composition of the morphisms is defined by
\[
([\alpha_1, t_1], [\alpha_2, t_2]) \circ ([\alpha_3, t_3], x) = ([\alpha_1 \alpha_2 t_1 + t_2, x_1 + x_3] \cdot x).
\]

Let \( T \) denote the circle group \( \mathbb{R}/\mathbb{Z} \). We have a homomorphism of orbifolds
\[
\pi : \Lambda(X//G) \to B T
\]
sending all the objects to the single object in \( B T \), and a morphism \(([\alpha, t], x) \) to \( e^{2\pi it} \) in \( T \).

**Definition 2.3.** The quasi-elliptic cohomology \( QEll^*_G(X) \) is defined to be \( K^*_\text{orb}(\Lambda(X//G)) \).

The groupoid \( \Lambda(X//G) \) is equivalent to the disjoint union of action groupoids
\[
\bigoplus_{g \in G^\text{tors}_{\text{conj}}} X^g//\Lambda_G(g)
\]
where \( G^\text{tors}_{\text{conj}} \) is the set of a family of representatives of the \( G \)-conjugacy classes in \( G^\text{tors} \). Thus, we can unravel Definition 2.3 and express it via equivariant K-theory.

**Definition 2.4.**
\[
QEll^*_G(X) := \prod_{g \in G^\text{tors}_{\text{conj}}} K^*_\Lambda_G(g)(X^g) = \left( \prod_{g \in G^\text{tors}} K^*_\Lambda_G(g)(X^g) \right)^G,
\]
where \( G^\text{tors}_{\text{conj}} \) is a set of representatives of \( G \)-conjugacy classes in \( G^\text{tors} \).

Consider the composition
\[
\mathbb{Z}[q^\pm] = K^0_{T}(pt) \xrightarrow{\pi^*} K^0_{\Lambda_G(g)}(pt) \to K^0_{\Lambda_G(g)}(X)
\]
where \( \pi : \Lambda_G(g) \to T \) is the projection \([a, t] \mapsto e^{2\pi it}\) and the second map is defined via the collapsing map \( X \to pt \). Via it, \( QEll^*_G(X) \) is naturally a \( \mathbb{Z}[q^\pm] \)-algebra.

**Proposition 2.5.** The relation between quasi-elliptic cohomology and equivariant Tate K-theory \( K^*_Tate(-//G) \) is
\[
QEll^*_G(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) \cong K^*_Tate(X//G).
\]

This is the main reason why the theory is called quasi-elliptic cohomology.

In addition, we give an example computing quasi-elliptic cohomology, which is [Hua18] Example 3.3. The conclusions in Example 2.6 is applied in Section 6.

**Example 2.6 (\( G = \mathbb{Z}/N \)).** Let \( G = \mathbb{Z}/N \) for \( N \geq 1 \), and let \( \sigma \in G \). Given an integer \( k \in \mathbb{Z} \) which projects to \( \sigma \in \mathbb{Z}/N \), let \( x_k \) denote the representation of \( \Lambda_G(\sigma) \) defined by
\[
\Lambda_G(\sigma) = (\mathbb{Z} \times \mathbb{R})/(\mathbb{Z}(N, 0) + \mathbb{Z}(k, 1)) \xrightarrow{[a, t] \mapsto [kt - a]/N} \mathbb{R}/\mathbb{Z} = T \xrightarrow{q} U(1).
\]

For any finite abelian group \( G = \mathbb{Z}/N_1 \times \mathbb{Z}/N_2 \times \cdots \times \mathbb{Z}/N_m \), let \( \sigma = (k_1, k_2, \cdots, k_m) \in G \). We have
\[
\Lambda_G(\sigma) \cong \Lambda_{\mathbb{Z}/N_1}(k_1) \times T \cdots \times T \Lambda_{\mathbb{Z}/N_m}(k_m).
\]
Then

\[ R_{\Lambda}(\sigma) \cong RA_{\mathbb{Z}/N_1}(k_1) \otimes \mathbb{Z}[q^{\pm}] \cdots \otimes \mathbb{Z}[q^{\pm}] RA_{\mathbb{Z}/N_m}(k_m) \]

\[ \cong \mathbb{Z}[q^{\pm}, x_{k_1}, x_{k_2}, \cdots x_{k_m}]/(x_{k_1}^{N_1} - q^{k_1}, x_{k_2}^{N_2} - q^{k_2}, \cdots x_{k_m}^{N_m} - q^{k_m}) \]

where all the \( x_{k_j} \)'s are defined as \( x_k \) in (2.4).

2.2. Loop space. In [Hua18] Section 2] The author provides a loop space construction for quasi-elliptic cohomology. We review that model in this section.

The classical loop space is on the level 0: we start with a space \( X \) and define a space of free loops

\[ L_X := C^\infty(S^1, X). \]

It comes with an evident action by the circle group \( \mathbb{T} \) defined by rotating the circle

\[ t \cdot \gamma := (s \mapsto \gamma(s + t)), \ t \in \mathbb{T}, \ \gamma \in L_X. \]

Let \( G \) be a compact Lie group and \( X \) a right \( G \)-space. The free loop space \( L_X \) is equipped with an action by the loop group \( LG \)

\[ \delta \cdot \gamma := (s \mapsto \delta(s) \cdot \gamma(s)), \text{ for any } s \in S^1, \ \delta \in L_X, \ \gamma \in LG. \]

Combining the action by the group of automorphisms \( Aut(S^1) \) on the circle and the action by \( LG \), we get an action by the extended loop group \( \Lambda G \) on \( L_X \), where

\[ \Lambda G := LG \rtimes T \]

is a subgroup of

\[ LG \rtimes Aut(S^1), \ (\gamma, \phi) \cdot (\gamma', \phi') := (s \mapsto \gamma(s)\gamma'(\phi^{-1}(s)), \phi \circ \phi') \]

with \( T \) identified with the group of rotations on \( S^1 \). The group \( \Lambda G \) acts on \( L_X \) by

\[ \delta \cdot (\gamma, \phi) := (t \mapsto \delta(\phi(t)) \cdot \gamma(\phi(t))), \text{ for any } (\gamma, \phi) \in \Lambda G, \ \text{and } \delta \in L_X. \]

Let \( \tilde{\delta} : G \times T \to X \) denote the map \( (g, t) \mapsto \delta(t)g \). The action on \( \delta \) by \( (\gamma, t) \) can be interpreted as precomposing \( \tilde{\delta} \) with a \( G \)-bundle map covering the rotation \( \phi \).

\[ (g, t) \mapsto (\gamma(t)g, \phi(t)) \]

\[ G \times T \xrightarrow{(g, t) \mapsto (\gamma(t)g, \phi(t))} G \times T \xrightarrow{\tilde{\delta}} X \]

More generally, we have the definition of the equivariant loop space \( Loop(X//G) \) below.

**Definition 2.7.** We define the equivariant loop space \( Loop(X//G) \) as the category with objects

\[ T \xleftarrow{\pi} P \xrightarrow{f} X \]

where \( \pi \) is a principal \( G \)-bundle over \( T \) and \( f \) is a \( G \)-map. A morphism

\[ (\alpha, t) : \{ \xrightarrow{\pi} P' \xrightarrow{f'} X \} \to \{ \xrightarrow{\pi} P \xrightarrow{f} X \} \]
consists of a $G$–bundle map $\alpha$ and a rotation $t$ making the diagrams commute.

\[
\begin{array}{c}
P' \xrightarrow{f'} P \xrightarrow{f} X \\
\downarrow \quad \downarrow \\
T \xrightarrow{t} T
\end{array}
\]

In this way, starting from a groupoid $X//G$, we get a loop groupoid $\text{Loop}(X//G)$.

**Remark 2.8.** The category $\text{Loop}(X//G)$ has the same objects as the category of bibundles $\text{Bibun}(\mathbb{T}//*,X//G)$. The morphisms in $\text{Bibun}(\mathbb{T}//*,X//G)$ are those of the form $(\alpha,0)$ in $\text{Loop}(X//G)$.

**Proposition 2.9.** The groupoid $\Lambda(X//G)$ is a subgroupoid of $\text{Loop}(X//G)$ consisting of constant loops.

### 3. Devoto’s equivariant elliptic cohomology over $\mathbb{C}$

For the rest part of the paper, the equivariant group $G$ is always a finite group, unless otherwise specified.

In [Dev96] Devoto provided a $G$-equivariant refinement of the elliptic cohomology defined by Landweber, Ravenel and Stong in [LRS04] for finite groups $G$. In this section, we give a brief overview of his construction. Another reference for this section is [BE22, Section 3].

Let $C(G)$ denote the set of pairs of commuting elements of $G$, and $L \subset \mathbb{C}^2$ the subspace of pairs $(t_1,t_2)$ such that the imaginary part of $t_1/t_2$ is defined and positive. The group $\text{SL}_2(\mathbb{Z})$ acts on $L \times C(G)$ from the right by

\[
((t_1,t_2),(g,h)) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ((at_1 + ct_2, bt_1 + dt_2), (g^d h^{-b}, g^{-c} h^a)).
\]

And the group $G$ acts on $L \times C(G)$ from the right by

\[
((t_1,t_2),(g,h)) \cdot k := ((t_1,t_2), (k^{-1} g k, k^{-1} h k)), \quad \forall k \in G.
\]

Since the two group actions commute, we have a right action of the group $G \times \text{SL}_2(\mathbb{Z})$ on $L \times C(G)$.

Let $O(L)$ denote the ring of holomorphic functions on $L$. Let

\[
O_j(L) := \{ f \in O(L) \mid f(\mu^2 t_1, \mu^2 t_2) = \mu^j f(t_1, t_2), \forall (t_1,t_2) \in L, \forall \mu \in \mathbb{C}^* \},
\]

where $\mu \in \mathbb{C}^*$ acts on the lattices by dilation and rotation. The $\mathbb{C}^*$-action on $L$ given by scaling both $t_1$ and $t_2$ induces a graded ring structure on $O(L)$, i.e.

\[
O(L) = \bigoplus_{j \in \mathbb{Z}} O_j(L),
\]

**Definition 3.1.** The $\text{SL}_2(\mathbb{Z})$-invariant elements of $O_j(L)$ are called the weakly modular forms of weight $-j/2$. We denote this subring of $O_j(L)$ by $MF_{\text{weak}}^j$. 

Let $G$ act on a space $X$ from the right, and denote by $X^{g,h} \subset X$ the subspace of points fixed by both $g$ and $h$. The action of $G$ on $X$ induces homomorphisms

$$X^{g,h} \rightarrow X^{k^{-1}gk^{-1}hk}$$

sending $x \mapsto x \cdot k$ for all $k \in G$.

As indicated in (3.3), there is a $G$-action on $C(G)$. We will use $C[G]$ to denote the orbit space of the $G$-action on $C(G)$, and the symbol $[g,h]$ to denote the orbit of $(g,h)$. The stabilizer of $(g,h)$ is the maximal subgroup

$$C_G^{(2)}(g,h) \subset G$$

that centralizes both $g$ and $h$. The $G$-action induces a $C_G^{(2)}(g,h)$-action on $X^{g,h}$.

**Proposition 3.2.** For any $k \in G$, there is an isomorphism

$$H^*_C(k^{-1}gk^{-1}hk)(X^{k^{-1}gk^{-1}hk}) \rightarrow H^*_C(g,h)(X^{g,h})$$

induced by (3.3).

Then we are ready to give the definition of Devoto’s equivariant elliptic cohomology.

**Definition 3.3.** For any integer $k$, the $k$-th Devoto’s $G$-equivariant elliptic cohomology of $X$ is defined as the abelian group

$${\text{Ell}}_G^k(X) := \bigoplus_{i+j=k} \left( \bigoplus_{[g,h] \in C(G)} H^i(X^{g,h}) \otimes_C O^j(L) \right)^{G \times SL_2(\mathbb{Z})}$$

(3.4)

$$\cong \bigoplus_{i+j=k} \left( \bigoplus_{[g,h] \in C(G)} H^i(X^{g,h}) \otimes_C O^j(L) \right)^{C_G^{(2)}(g,h) \times SL_2(\mathbb{Z})}$$

where the isomorphism follows by choosing a representative pair $(g,h)$ for each conjugacy class $[g,h]$ in $C(G)$.

**Remark 3.4.** Note that $\{g,h\}$ and $\{g^b h^{-b}, g^{-c} h^a\}$ generate the same subgroup of $G$. And we have

$${\text{Ell}}_G^k(X)^{SL_2(\mathbb{Z})} = X^{d^2 h^{-b}, g^{-c} h^a} \text{ and } C_G^{(2)}(g,h) = C_G^{(2)}(g^b h^{-b}, g^{-c} h^a).$$

Thus, $SL_2(\mathbb{Z})$ acts trivially on part of the cohomology $H^i(-)$ in (3.4).

**Remark 3.5.** If $G = \{e\}$ is the trivial group, then

$$\text{Ell}_G^k(X) = \bigoplus_{i+j=k} H^i(X) \otimes_C O^j(L)^{SL_2(\mathbb{Z})} = H^i(X) \otimes_C MF^j_{\text{weak}}$$

where the right hand side is the graded tensor product over $C$ of the cohomology ring of $X$ with the graded ring of weakly modular forms of weight $-j/2$.

**Remark 3.6.** If $X = \text{pt}$, then

$$\text{Ell}_G^k(\text{pt}) = \left( \bigoplus_{[g,h] \in C(G)} O^k(L) \right)^{G \times SL_2(\mathbb{Z})} \cong \left( \bigoplus_{[g,h] \in C(G)} O^k(L) \right)^{SL_2(\mathbb{Z})},$$

which is a direct application of the isomorphism in Definition 3.3. On the right hand side we obtain the direct sum of the ring of weakly modular forms of weight $-k/2$ with itself, indexed over all conjugacy classes of commuting pairs in $G$. 

Remark 3.7. In [Gan07], Ganter discussed the equivariant elliptic cohomology $Ell^k$ by Devoto and provided a loop space model of equivariant Tate K-theory motivated by Devoto’s orbifold loop space. In [Hua18], the author constructed a loop space model motivated by Devoto’s orbifold loop space with the circle rotation added.

4. Twisted equivariant elliptic cohomology over $\mathbb{C}$

In [BE22, Section 3], Berwick-Evans constructed a twisted version of Devoto’s equivariant elliptic cohomology with complex coefficients. We sketch his construction in this section. For convenience, the construction is expressed in term of normalised cocycles.

Definition 4.1. A 3-cocycle on $G$ with values in $U(1)$ is a map

$$\alpha : G \times G \times G \to U(1)$$

satisfying

$$\frac{\alpha(g_1, g_2, g_3)\alpha(g_0, g_1, g_2, g_3)\alpha(g_0, g_1, g_2)}{\alpha(g_0, g_1, g_2, g_3)\alpha(g_0, g_1, g_2, g_3)} = 1$$

for any $g_0, g_1, g_2, g_3 \in G$. Such a cocycle is called normalised if it evaluates to 1 on any triple containing the identity element $e \in G$.

Recall the value of $Ell^k_G$ of the single point space in Remark 3.6. We can use $\alpha$ to twist the $G$-action on

$$\bigoplus_{(g, h) \in \mathcal{C}(G)} \mathcal{O}^k(L)$$

by defining it to be

$$h \cdot f_{g_1, g_2} = \frac{\alpha(g_2, h, g_1)\alpha(h, g_1, g_2)\alpha(g_1, g_2, h)}{\alpha(h, g_1, g_2)\alpha(g_1, h, g_2)\alpha(g_2, g_1, h)}f_{g_1, g_2}, \quad \forall h \in G,$$

where $f_{g_1, g_2}$ is the value of $f \in \mathcal{O}(L)$ at $(g_1, g_2) \in \mathcal{C}(G)$.

Remark 4.2. In [Wil08, Section 1.4.3], Willerton gives the formulas of the transgression

$$\tau_x \theta(g) := \frac{\theta(g, x)}{\theta(x, g)}, \quad \tau_x \alpha(g, h) := \frac{\alpha(g, h, x)\alpha(x, g, h)}{\alpha(g, x, h)}, \quad \forall x \in G, \quad \forall g, h \in \mathcal{C}_G(x)$$

which sends a 2-cocycle $\theta \in Z^2(G)$ to a 1-cocycle $\tau_x \theta \in Z^1(C_G(x))$ and a 3-cocycle $\alpha \in Z^3(G)$ to a 2-cocycle $\tau_x \alpha \in Z^2(C_G(x))$. The relation between the $G$-action and the transgression (4.4) can be interpreted in this way:

$$h \cdot f_{g_1, g_2} = \tau_{g_2} \alpha(g_1, h)\tau_{g_2} \alpha(g_1, h)^{-1}f_{g_1, g_2} = \tau_{g_1}(\tau_{g_2} \alpha(h))f_{g_1, g_2}.$$

This is compatible with the $SL_2(\mathbb{Z})$-action on $\mathcal{C}(G)$.

We denote by the symbol

$$\bigoplus_{(g, h) \in \mathcal{C}(G)} \mathcal{O}^{k+\alpha}(L)$$
the group of holomorphic functions in \((4.2)\) equipped with the \(G\)-action twisted by \(\alpha\). The \(G\)-invariants of this group are the collections of functions \((f_{g,h})_{(g,h)\in C(G)}\) satisfying the following transformation property.

\[
h \cdot \alpha f_{g_1,g_2} = f_{h^{-1}g_1,h^{-1}g_2,h}.\]

**Definition 4.3** ([BE22]). For a manifold \(X\) with an action of a finite group \(G\), the \(\alpha\)-twisted version of Devoto’s equivariant elliptic cohomology for a \(G\)-space \(X\) is given by

\[
\text{Ell}^{k+\alpha}_{G}(X) := \bigoplus_{i+j=k} \left( \bigoplus_{[g,h] \in C(G)} H^i(X^g,h) \otimes O^{j+\alpha}(L) \right)_{C_G^{(2)}(g,h) \times SL_2(\mathbb{Z})}.\]

**Remark 4.4.** In [BE22, Section 4] Berwick-Evans constructs the induction formula as well as other character formulas in the higher Hopkins-Kuhn-Ravenel character theory for the theory \(\text{Ell}^{*+\alpha}_{G}(-)\). Moreover, he discussed its relation with physics.

5. **Twisted quasi-elliptic cohomology**

5.1. **Definition.** In this section we define twisted quasi-elliptic cohomology \(Q\text{Ell}^{\alpha}(X//G)\) with \(\alpha \in H^3(BG;U(1))\), which is constructed as the orbifold \(K\)-theory of a twisted orbifold \(\Lambda^{\alpha}(X//G)\) and is based on the construction in [FQ93, Section 3].

Let \(G\) be a finite group and \(X\) a \(G\)-space. First, we show that each \(\alpha \in H^3(BG;U(1))\) determines an element \(\theta_g\) in \(H^2(BC_G(g);U(1))\) for each conjugacy class \([g]\) in \(G\). Let \(e\) be the evaluation map

\[
e : B\mathbb{Z} \times \text{Map}(B\mathbb{Z}, BG) \longrightarrow BG
\]

and let \(\pi\) be the projection

\[
\pi : B\mathbb{Z} \times \text{Map}(B\mathbb{Z}, BG) \rightarrow \text{Map}(B\mathbb{Z}, BG).
\]

Define the class

\[
\theta := \pi_* e^* \alpha \in H^2(\text{Map}(B\mathbb{Z}, BG);U(1)) \cong \bigoplus_{[g]} H^2(BC_G(g);U(1)),
\]

where \([g]\) goes over all the conjugacy classes in \(G\). The class \(\theta\) is well-defined with degree two because the formalism of the Pontryagin-Thom construction means that the degree of \(e^* \alpha\) drops by one when we push it forward along \(\pi_*\). Note that we have also used the fact that the mapping space \(\text{Map}(B\mathbb{Z}, BG)\) is homotopy equivalent to

\[
\prod_{[g]} BC_G(g).
\]

In this way, \(\theta\) determines an element

\[
\theta_g \in H^2(BC_G(g);U(1))
\]

for each \([g]\).

We can now define the twisted orbifold. Each 2-cocycle \(\theta_g\) determines a central extension

\[
1 \rightarrow T \rightarrow C^0_{G}(g) \rightarrow C_{G}(g) \rightarrow 1
\]
with group multiplication given by
\[(a, h)(b, k) = (a + b + \theta_g(h, k), hk),\]
for any \((a, h), (b, k)\) in \(C^\alpha_G(g)\). We have a well-defined \(C^\alpha_G(g)\)-action on \(X^g\)
\[(a, h) \cdot x := h \cdot x.\]

For ease of notation, we have written \(C^\alpha_G(g)\) in place of what is actually \(C^{\theta_g}_G(g)\). Similar notational simplifications are also made in the rest part of the paper when there is no confusion.

**Lemma 5.1.** Suppose that \(\theta_g\) has order \(n\), and let \(l\) be the order of \(g\). Then the order of \((0, g)\) divides \(nl\).

**Proof.** Note that \(g^l = e\). We have
\[nl(0, g) = (\theta_g(g, g) + \theta_g(g, g^2) + \ldots + \theta_g(g, g^{nl-1}), g^{nl})\]
\[= ((n-1)(\theta_g(g, g) + \theta_g(g, g^2) + \ldots + \theta_g(g, g^l)) + \theta_g(g, g) + \ldots + \theta_g(g, g^{l-1}), e)\]
\[= (n(\theta_g(g, g) + \theta_g(g, g^2) + \ldots + \theta_g(g, g^{l-1})) + (n-1)\theta_g(g, g^l), e)\]
\[= (0, e).\]

The second equality holds because \(g\) has order \(l\), which means that \(g^{ml+k} = g^k\) for all integers \(m\) and \(k\). The third equality holds since \((nl-1) = (n-1)l + l - 1\), and we get the fourth equality just by rearranging terms. The final equality holds since \(\theta_g\) has order \(n\), and \(\theta_g(g, e) = 0\) since \(\alpha\) is normalised. Therefore, \(nl(0, g)\) is equal to the identity element, and so the order of \((0, g)\) must divide \(nl\). \(\square\)

This result is cited in [Dov19, Remark 6.22].

**Example 5.2 (Twisted Inertia Groupoid \(I^\alpha(X//G)\)).** T. Dove constructed the twisted inertia groupoid in [Dov19]. The twisted inertia groupoid \(I^{\text{tors}}(X//G)\) of the translation groupoid \(X//G\) is the groupoid with

- **objects:** the space \(\coprod_{g \in G} X^g\)
- **morphisms:** the space \(\coprod_{g \in G} C^\alpha_G(g) \times X^g\).

For each \(x \in X^g\), \(C^\alpha_G(g)\) is the automorphism group of it. The twisted inertia groupoid is used in [Dov19, Section 6.4.1] to define the twisted equivariant Tate K-theory.

**Example 5.3 (Twisted orbifold loop space).** In [Gan07, Definition 2.3] Ganter defined orbifold loop space
\[L(X//G) := \coprod_{[g]} L_g X//C_G(g),\]
via which equivariant Tate K-theory can be constructed. In this example we provide the twisted version of it.

The space \(L_g X\) is the space \(\text{Map}_{\mathbb{Z}/l}(\mathbb{R}/l\mathbb{Z}, X)\) where \(l\) is the order of \(g\). There is a well-defined \(C^\alpha_G(g)\)-action on \(L_g X\) by
\[\gamma((a, h))(t) = \gamma(t + a)h\]
for $\gamma \in L_gX$ and $(a, h) \in C_G^\alpha(g)$. It’s straightforward to check that $\gamma((a, h))$ is indeed in $L_gX$. The twisted orbifold loop space is defined as

$$L^\alpha(X//G) := \coprod_{[g]} L_gX/C_G^\alpha(g).$$

Note that on the space of constant loops $X^g$, the action by $C_G^\alpha(g)$ in \cite{Lue22} covers that by $C_G(g)$.

The twisted Inertia groupoid $I^\alpha(X//G)$ is the full subgroupoid of $L^\alpha(X//G)$ consisting of constant loops.

Let $\Lambda_G^\alpha(g)$ denote the quotient

$$\mathbb{R} \times C_G^\alpha(g)/((-1, (0, g))),$$

It fits into the short exact sequence

$$1 \to C_G^\alpha(g) \to \Lambda_G^\alpha(g) \to \mathbb{T} \to 1.$$

We have the well-defined twisted orbifold

$$\Lambda^\alpha(X//G) := \coprod_{g \in G_{\text{reg}}} X^g//\Lambda_G^\alpha(g).$$

In addition, we have the short exact sequence

$$1 \to \mathbb{T} \to \Lambda_G^\alpha(g) \to \Lambda_G(g) \to 1.$$

The surjective map in \cite{Lue22} gives the map between orbifolds

$$\Lambda^\alpha(X//G) \to \Lambda(X//G)$$

which sends a morphism $(x, [r, (a, h)])$ to $(x, [r, h])$. The map \cite{Lue22} gives a $\mathbb{T}$-equivariant graded central extension in the sense of \cite{Lue22}.

**Lemma 5.4.** The central extension $\Lambda_G^\alpha(g)$ is determined by the $2$-cocycle $\hat{\theta}_g \in Z^2(\Lambda_G(g))$ given by

$$\hat{\theta}_g([s_2, t_2], [s_1, t_1]) = \theta_g([s_2], [s_1]).$$

The following definition is \cite{AR03} Definition 7.1.

**Definition 5.5.** Let $\theta$ be an element in $H^2_G(X; \mathbb{Z})$ and $G^\theta$ the group extension which represents it, $1 \to \mathbb{T} \to G^\theta \to G \to 1$. Then $X$ is equipped with a $G^\theta$-action via the projection map $G^\theta \to G$. A $\theta$-twisted $G$-equivariant vector bundle over $X$ is defined to be a $G^\theta$-equivariant vector bundle $V$ over $X$ such that the central circle in $G^\theta$ acts by complex multiplication on the fibers of $V$.

Two $\theta$-twisted $G$-equivariant vector bundles over $X$ are isomorphic if and only if they are isomorphic as $G^\theta$-equivariant vector bundles. With this in mind, we state the following definition, which is \cite{AR03} Definition 7.2.

**Definition 5.6.** The $\theta$-twisted $G$-equivariant $K$-theory of a $G$-space $X$, denoted by $K_G^\theta(X)$, is defined to be the Grothendieck group of isomorphism classes of $\theta$-twisted $G$-equivariant vector bundles over $X$.

Now we are ready to give the definition of twisted quasi-elliptic cohomology.
The twisted quasi-elliptic cohomology $QEll^{α+*}_G(−)$ twisted by $α ∈ H^3(BG; U(1))$ is defined by

$$QEll^{α+*}_G(X) := \prod_{g ∈ G_{\text{conj}}} K^g_{ΛG}(X^g),$$

where each $θ_g$ is the factor in (5.1) corresponding to $g$.

Remark 5.8. Note that $θ$-twisted $G$-equivariant vector bundle is a special case of the twisted vector bundle in Definition 2.5 in [Gom17] with trivial $\mathbb{Z}/2$-grading. And the twisted equivariant K-theory in Definition 5.6 is a special case of Freed-Moore K-theory. Therefore, the Real twisted quasi-elliptic cohomology $QEll^{α+*}_R(−)$, which is constructed in [HY22] as Freed-Moore K-theory of a Real version of the orbifold $Λ(X/G)$, is a generalization of twisted quasi-elliptic cohomology.

Proposition 5.9. We have the relation between $QEll^{α+*}_G(−)$ and the twisted equivariant Tate K-theory $K^{α+*}_{\text{Tate}}(−/G)$ in [Dov19] as below.

$$QEll^{α+*}_G(X) ⊗_{\mathbb{Z}[q^±]} \mathbb{Z}((q)) ≃ K^{α+*}_{\text{Tate}}(X/G)$$

Proof. This follows from the definition of both theories and Proposition 2.3. □

5.2. Examples and Properties. In this section we provide some simple examples of twisted quasi-elliptic cohomology and some properties of it.

Example 5.10. When $G$ is the trivial group and $g$ is the identity element, $QEll^*_G(X) = K^*_X$. In this case, for any 3-cocycle $α$, $QEll^{α+*}_G(X) = K^{α+*}_X$.

Example 5.11. Let $X$ be a CW complex with trivial $G$-action. In this case, for any 3-cocycle $α$, by [AR03] Lemma 7.3,

$$QEll^{α+*}_G(X) = \prod_{g ∈ G_{\text{conj}}} K^g_{ΛG}(X) ≃ \prod_{g ∈ G_{\text{conj}}} K^*_X ⊗ R_{θ_g}ΛG(g) = K^*_X ⊗ \left( \prod_{g ∈ G_{\text{conj}}} R_{θ_g}ΛG(g) \right),$$

where $G_{\text{conj}}$ is the set of a family of representatives of the $G$-conjugacy classes in the finite group $G$ and $R_{θ_g}ΛG(g)$ denotes the ring of the $θ_g$-representations of $ΛG(g)$.

Example 5.12 (Restriction map). Let $X$ be a $G$-space and $Y$ an $H$-space. Let $f : G → H$ denote a group homomorphism and let $h : X → Y$ denote a continuous map which is $G$-equivariant in the sense that

$$h(g · x) = f(g) · h(x).$$

From $f$ we can define the group homomorphisms

$$f_g : ΛG(g) → ΛH(f(g)), \quad [a, t] ↦ [f(a), t]$$

for each $g ∈ G$. 

Let $\alpha \in H^3(BH; U(1))$. Note that we have the commutative diagrams

\[
\begin{array}{ccc}
H^3(BH; U(1)) & \xrightarrow{f^*} & H^3(BG; U(1)) \\
\downarrow \tau & & \downarrow \tau \\
H^2(\text{Map}(BZ, BH); U(1)) & \xrightarrow{f^*} & H^2(\text{Map}(BZ, BG); U(1)) \\
\end{array}
\]

Thus,

\[
f^*(\tau(\alpha)) = \tau(f^*(\alpha)); \quad f^*(\theta_{f(g)}) = (f^*\theta)_g.
\]

where $\tau(\alpha) = \prod_{[h]} \theta_h$ with each $\theta_h \in H^2(BCH(h); U(1))$ and $\tau(f^*(\alpha)) = \prod_{[g]} (f^*\theta)_g$ with each $(f^*\theta)_g \in H^2(BCG(g); U(1))$.

Thus, we obtain a map for each $g \in G$

\[
h^*_g : K_{\Lambda H(f(g))^+}(Yf(g)) \rightarrow K_{\Lambda G(g)}^{f^*(\theta_{f(g)})^+}(X^g)
\]

and the restriction map

\[
h^* = \prod_{[h]} h^*_g : QEll_G^{\alpha^+}(Y) \rightarrow QEll_G^{f^*(\alpha)^+}(X).
\]

In [HY22 Section 3.4], the basic properties of twisted Real quasi-elliptic cohomology are discussed and proved in detail. Since twisted quasi-elliptic cohomology is equivalent to twisted Real quasi-elliptic cohomology of an orbifold with trivial $\mathbb{Z}/2$-grading, we can obtain the corresponding basic properties of twisted quasi-elliptic cohomology straightforward. We sketch them below.

**Proposition 5.13.** Let $G$ be a finite group and $H$ a subgroup of $G$. Let $X$ be a $G$-space and $\alpha \in H^3(BG; U(1))$. Then the change-of-group map

\[
\rho^G_H : QEll_G^{\alpha^+}(X \times_H G) \rightarrow QEll_H^{\alpha^+}(X \times_H G) \rightarrow QEll_H^{\alpha^+}(X)
\]

is an isomorphism, where the first map is induced by the inclusion $i : H \hookrightarrow G$ of groups and the second map is induced by the inclusion

\[
X \hookrightarrow X \times_H G, \quad x \mapsto [x, e].
\]

**Example 5.14.** The induction for $QEll^{\alpha^+}$ can be defined as the composition

\[
\tau^G_H : QEll_H^{\alpha^+}(X) \rightarrow QEll_G^{\alpha^+}(X \times_H G) \rightarrow QEll_G^{\alpha^+}(X),
\]

where the second map is the induction map for $K$-theory induced by the finite covering $\Lambda((X \times_H G)/G) \rightarrow \Lambda(X/G)$ given on objects by $([x, g], \sigma) \mapsto (xg, \sigma)$ and morphisms by $([g', t], ([x, g], \sigma)) \mapsto ([g', t], (xg, \sigma))$.

**5.3. Twisted Loop space.** In this section we give a loop space construction of twisted quasi-elliptic cohomology other than the twisted orbifold loop space in Example 5.3. This is a twisted version of the loop space in Definition 2.7.
**DEFINITION 5.15.** For a $G$-space $X$, we define a category of twisted equivariant loop space $\text{Loop}^{\text{twist}}(X//G)$. The objects of it are diagrams

$$
\begin{array}{c}
\mathbb{T} \leftarrow P \xrightarrow{\pi} Q \xrightarrow{f} X \\
\end{array}
$$

consisting of a principal $G$-bundle $P$ over $\mathbb{T}$, a principal $T$-bundle $Q$ over $P$, a $G$-equivariant map $f'$ and $f = \pi \circ f'$.

A morphism from one object $\begin{array}{c}
\mathbb{T} \leftarrow P_1 \xrightarrow{\pi_1} Q_1 \xrightarrow{f_1} X \\
\end{array}$ to another object $\begin{array}{c}
\mathbb{T} \leftarrow P_2 \xrightarrow{\pi_2} Q_2 \xrightarrow{f_2} X \\
\end{array}$ is of the form $(\alpha, \beta, t)$

$$
\begin{array}{c}
\begin{array}{c}
Q_1 \xrightarrow{\alpha} Q_2 \\
\pi_1 \\
P_1 \xrightarrow{\beta} P_2 \\
\rho_1 \\
\mathbb{T} \xrightarrow{t} \mathbb{T}
\end{array}
\end{array}
$$

where $t$ is a rotation on the circle $\mathbb{T}$, $\beta$ is a bundle isomorphism covering $t$ and $\alpha$ is a bundle isomorphism covering $\beta$. In addition, $f_1 = f_2 \circ \alpha$ and $f_1' = f_2' \circ \beta$.

The constant twisted equivariant loops are those objects in $\text{Loop}^{\text{twist}}(X//G)$

$$
\begin{array}{c}
\mathbb{T} \leftarrow P \xrightarrow{\pi} Q \xrightarrow{f} X \\
\end{array}
$$

with both $f$ and $f'$ constant maps. If $P$ is the principal $G$-bundle $P_g$ classified by $g \in G$, then the image of $f$ consists of a single point $x \in X^g$. Each element $[t, h] \in \Lambda_G(g)$ gives a bundle isomorphism from $P$ to itself covering the rotation $t$ of $\mathbb{T}$. In addition, each element $[a, [t, h]] \in \Lambda^g_G(g)$ gives a bundle isomorphism from $Q$ to itself covering $[t, h]$, where $\theta_g$ is some element in $H^2(BC_G(g); U(1))$. Thus, we have the conclusion below.

**PROPOSITION 5.16.** The groupoid $\Lambda^\alpha(X//G)$ in (5.3) is a subgroupoid of $\text{Loop}^{\text{twist}}(X//G)$ with the constant loops $\prod_{g \in G^\text{tor}^s} X^g$ as objects.

**5.4. The Chern Character map for twisted quasi-elliptic cohomology.** In this subsection we construct the Chern character maps for quasi-elliptic cohomology and twisted quasi-elliptic cohomology.
5.4.1. The Chern Character map for $QEll_l$. We first define a Chern character of $QEll^*_G(X)$. The construction is given below.
Consider the diagram

$$
\begin{array}{cccccc}
1 & \rightarrow & C_G(g) & \rightarrow & \mathbb{T} \times C_G(g) & \rightarrow & \mathbb{T} & \rightarrow & 1 \\
\downarrow & & \downarrow c_\theta & & \downarrow & & \downarrow \\
1 & \rightarrow & C_G(g) & \rightarrow & \Lambda_G(g) & \rightarrow & \mathbb{T} & \rightarrow & 1
\end{array}
$$

where the middle vertical map sends $(t, g)$ to $[lt, g]$ and the right vertical map sends $e^{2\pi it}$ to $e^{2\pi il}$ with $l$ the order of $g$. Let

$$c_g^* : K^*_\Lambda(G)(X^g) \otimes \mathbb{C} \rightarrow K^*_G(X^g) \otimes \mathbb{C}$$

denote the corresponding restriction map.

The Chern character map is constructed as the composition

$$QEll^*_G(X) \otimes \mathbb{C} = \prod_{[g] \in G_{conj}} K^*_\Lambda(G)(X^g) \otimes \mathbb{C}$$

$$\xrightarrow{c^*} \prod_{[g] \in G_{conj}} K^*_G(X^g) \otimes \mathbb{C} \xrightarrow{\cong} \prod_{[g] \in G_{conj}} K^*_C(G)(X^g) \otimes \mathbb{Z}[q^\pm] \otimes \mathbb{C}$$

$$\xrightarrow{AS} \prod_{[g] \in G_{conj}} (\prod_{[h] \in G_{conj}} K^*(X^{g,h}) \otimes \mathbb{C}) c_G^*(g,h) \otimes \mathbb{Z}[q^\pm]$$

$$\xrightarrow{ch} \prod_{[g,h] \in C[G]} (H^*(X^{g,h}) \otimes \mathbb{C}) c_G^*(g,h) \otimes \mathbb{Z}[q^\pm]$$

The first map $c^*$ is the product $\prod_{[g] \in G_{conj}} c_g^*$ of restriction maps. The property of equivariant K-theory implies that the second map is an isomorphism. The third map is the product of Atiyah-Segal maps in [AS89, Theorem 2]

$$K^*_G(X) \otimes \mathbb{C} \xrightarrow{\cong} \prod_{[g] \in G_{conj}} (K^*(X^g) \otimes \mathbb{C}) c_G^*(g).$$

The fourth map is the product of Chern character maps of K-theory.

Note that, other than the first map $c^*$, all the other maps in the composition are isomorphisms.

5.4.2. The Chern Character map for twisted $QEll$. In this subsection we construct the twisted Chern character map of the twisted quasi-elliptic cohomology. Based on the map $c^*$ in the construction of Chern character map in Section 5.4.1 we construct a map $p^*$. Let

$$p_\sigma : \mathbb{T} \times C^\theta_G(\sigma) \rightarrow \Lambda^\theta_G(\sigma)$$

denote the map sending $(t, (a, g))$ to $[Nt, (a, g)]$, where $\theta_\sigma$ is the 2–cocycle defined in Section 5 and $N$ is the order of $(0, \sigma)$ in $C^\theta_G(\sigma)$. Let

$$p^*_\sigma : K^\theta_{\Lambda(\sigma)}(X^\sigma) \otimes \mathbb{C} \rightarrow K^\theta_{T \times C(\sigma)}(X^\sigma) \otimes \mathbb{C}$$
denote the restriction map. Define

\[ p^* := \prod_{[\sigma] \in G_{\text{conj}}} p^*_\sigma. \]

The twisted Chern character map is constructed as the composite

\[ QEll_G^{\oplus+}(X) \otimes \mathbb{C} = \prod_{[\sigma]} K_G^{\oplus+}(X^\sigma) \otimes \mathbb{C} \]

\[ \xrightarrow{\cong} \prod_{[\sigma] \in G_{\text{conj}}} K^G_{\text{conj}}(X^\sigma) \otimes \mathbb{C} \]

\[ \xrightarrow{\cong} \prod_{[\sigma] \in G_{\text{conj}}} K^G_{\text{conj}}(X^\sigma) \otimes \mathbb{C} \otimes \mathbb{Z}[q^{\pm}] \]

\[ \xrightarrow{\text{AS}} \prod_{[\sigma] \in G_{\text{conj}}} \left( \prod_{[\tau] \in G_{\text{conj}}} (K^*(X^{\sigma,\tau}) \otimes L^0_G^{\tau}) \mathbb{C}^{(2)}_G(\sigma,\tau) \otimes \mathbb{Z}[q^{\pm}] \right) \]

\[ \xrightarrow{\text{ch}} \prod_{[\sigma,\tau] \in \mathbf{C}(G)} (H^*(X^{\sigma,\tau}) \otimes L^0_G^{\tau}) \mathbb{C}^{(2)}_G(\sigma,\tau) \otimes \mathbb{Z}[q^{\pm}] \]

where \( h \in C_G^{(2)}(\sigma,\tau) \) acts on \( L^0_G^{\tau} \cong \mathbb{C} \) as multiplication by \( \theta_\sigma(h,\tau)\theta_\tau(\tau,\tau)^{-1} \).

In the composition above, the first map \( p^* \) is the product of the restriction maps \( p^*_\sigma \) induced by the group homomorphisms \( p_\sigma \). If \( M \) is the order of \( (0,\sigma) \), then the kernel of \( p^*_\sigma \)

\[ \ker(p^*_\sigma) = \{ (e^{\pi i - \frac{m}{M}}, (\theta_\sigma(\sigma,\sigma) + ... + \theta_\sigma(\sigma,\sigma^{-1}), \sigma^{-1})) \in \mathbb{T} \times C_G^{(2)}(\sigma) \mid m \in \mathbb{Z} \}, \]

which acts trivially on \( X^\sigma \). The image of \( p^*_\sigma \) is generated by the \( \mathbb{T} \times C_G^{(2)}(\sigma) \)-vector bundles with trivial \( \ker(p^*_\sigma) \)-action on fibers. The second map is the isomorphism in [AR03] Lemma 7.3 since the circle group \( \mathbb{T} \) acts on each \( X^\sigma \) trivially. The third map is the twisted Atiyah-Segal map for twisted equivariant K-theory, which is proved in [AR03] Theorem 7.4. The fourth map is the product of the Chern character maps of twisted K-theory.

Below we provide the explicit description of the twisted Chern character map. Let \( \bigoplus_{\sigma} E_\sigma \) be an element in twisted quasi-elliptic cohomology. Recall that \( q \) is the character of the defining representation of \( \mathbb{T} \). The pullback

\[ \bigoplus_{\sigma} p^*_\sigma E_\sigma \]

splits as a direct sum

\[ (5.13) \quad \bigoplus_{[\sigma]} \bigoplus_{n \in \mathbb{Z}} (p^* E_\sigma)_n \otimes q^n, \]

where we have written \( p \) for \( p_\sigma \), to simplify notation.

The twisted Atiyah-Segal map sends an element of the form \((5.13)\) to

\[ (5.14) \quad \bigoplus_{[\sigma,\tau]} \bigoplus_{n \in \mathbb{Z}} \xi(\tau)(p^* E_\sigma)_n |_{X^{\tau,\tau}} \otimes \theta_\sigma(-,\tau)\theta_\tau(\tau,-)^{-1} \otimes q^n \]

In \((5.14)\) above, \( \xi(\tau) \) runs over the eigenvalues of \( \tau \) and \( n \) denotes the component where \( \mathbb{T} \) acts by the eigenvalue \( e^{i\theta n} \).
Finally, the Chern character map sends the element (5.14) to

(5.15) \[
\bigoplus_{[\sigma, \tau]} \bigoplus_n \xi(\tau) \text{ch}((p^* E_\sigma)_n|_{X^\sigma, \tau}) \otimes \theta_\sigma(\tau, -)^{-1} \otimes q^n.
\]

Remark 5.17. The composite of the final two maps of the twisted Chern character map is the same as the map in [FHT07, Theorem 3.9], tensored with \(Z[q^\pm]\). The twisted cohomology

\[\theta_\sigma H(X^{\sigma, \tau}; \mathcal{L}(\tau))\]

which is the target of the map in [FHT07, Theorem 3.9] is the same as

\[H(X^{\sigma, \tau}) \otimes L^{\theta_\sigma}.\]

Remark 5.18. In [HY22, Section 3.6], Young and the author construct the elliptic Pontryagin character for the Real twisted quasi-elliptic cohomology, which is the Real version of the twisted Chern character in this section.

6. Some Computations of Twisted Quasi-elliptic cohomology

Elliptic cohomology theory is usually very difficult to compute. The computation of twisted equivariant elliptic cohomology theory is even much harder. However, twisted quasi-elliptic cohomology theory, constructed as orbifold K-theory of a loop space, is computable. As indicated in [SS24], twisted quasi-elliptic cohomology of 4-spheres has significant physical meaning. This motivates the author to compute the twisted quasi-elliptic cohomology of 4-sphere with the desired group action in Section 6.2 and 6.3.

6.1. Quasi-elliptic cohomology of \(S^1\). We start the computation with two basic examples.

Example 6.1 (Rotation on \(S^1\)). Let \(Z/N\) denote the cyclic group with \(N\) elements. Consider the rotation action of it on the circle \(S^1\). Since the rotation action is free. The fix point space \((S^1)^m\) by \(m \in Z/N\) is

\[(S^1)^m = \begin{cases} S^1, & \text{if } m = 0 \\ \emptyset, & \text{otherwise.} \end{cases}\]

In addition,

\[\Lambda_{Z/N}(0) \cong Z/N \times \mathbb{T}.\]

Thus, by the properties of equivariant K-theories,

\[QEll_{Z/N}(S^1) \cong K_{\Lambda_{Z/N}(0)}(S^1) \cong K_{Z/N}(S^1) \otimes K_{\mathbb{T}}(pt)\]

\[\cong K(S^1/(Z/N)) \otimes Z[q^\pm] \cong K(S^1) \otimes Z[q^\pm] = Z \otimes Z[q^\pm]\]

\[= Z[q^\pm].\]

Example 6.2 (Reflection on \(S^1\)). Let \(Z/2 = \{1, \tau\}\) act on \(S^1\) by reflection. The fixed point spaces are

\[(S^1)^1 = S^1; \quad (S^1)^\tau = S^0.\]

And

\[\Lambda_{Z/2}(1) \cong Z/2 \times \mathbb{T}; \quad \Lambda_{Z/2}(\tau) \cong (Z/2 \times \mathbb{R})/\langle(\tau, -1)\rangle.\]
The two factors in $QEll_{Z/2}(S^1)$ are computed below. By Example 2.6

$$K_{Z/2(\tau)}((S^1)^\tau) \cong K_{Z/2(\tau)}(pt) \oplus K_{Z/2(\tau)}(pt)$$

$$\cong \mathbb{Z}[q^\pm, x]/\langle x^2 - q \rangle \oplus \mathbb{Z}[q^\pm, x]/\langle x^2 - q \rangle.$$

In addition

$$K_{Z/2(1)}((S^1)^1) \cong K_{Z/2}(S^1) \otimes K_{\tau}(pt) \cong \mathbb{Z} \otimes \mathbb{Z}[q^\pm] \cong \mathbb{Z}[q^\pm].$$

We can view $S^1$ as two copies of $D^1$ glued via the boundary $S^0$. A complex vector bundles over $S^1$ with the reflection is two copies of complex vector bundles over $D^1$ glued via the fibres on $S^0$ by identity. Thus, the $\mathbb{Z}/2$-equivariant vector bundle over $S^1$ is equivalent to the trivial bundle over $S^1$. So we have $K_{Z/2}(S^1) = \mathbb{Z}$ in \((6.1)\).

In conclusion,

$$QEll_{G}(S^1) = K_{Z/2(\tau)}((S^1)^\tau) \oplus K_{Z/2(1)}((S^1)^1)$$

$$\cong (\mathbb{Z}[q^\pm, x]/\langle x^2 - q \rangle \oplus z[2^\pm, x]/\langle x^2 - q \rangle) \times \mathbb{Z}[q^\pm].$$

**6.2. Quasi-elliptic cohomology of 4-sphere acted by a finite subgroup of $SU(2)$**. In this subsection, we compute all the quasi-elliptic cohomology theories $QEll_{G}(S^1)$

where $G$ goes over all the finite subgroups of $SU(2) \cong Spin(3)$.

First, we explain how the group $G$ acts on $S^4$. We have the standard orthogonal $SO(5)$-action on $\mathbb{R}^5$ and also on the subspace $S^4 \subset \mathbb{R}^5$. The covering map

$$Spin(5) \longrightarrow SO(5)$$

makes $S^4$ a well-defined $Spin(5)$-space. The $G$-action on $S^4$ is induced by the composition

\[(6.2) \quad i_G : G \hookrightarrow Spin(3) \xrightarrow{p_1} Spin(3) \times Spin(3) = Spin(4) \hookrightarrow Spin(5)\]

where $p_1$ is the projection to the first factor of the product group.

We give the explicit formula of the $G$-action below. The quaternion $\mathbb{H}$ is isomorphic to $SU(2) \cong Spin(3)$ via the correspondence

$$a + bi + cj + dk \mapsto \begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix}.$$

In view of this, $Spin(4)$ can be described as the group

$$\{ \begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix} \mid q, r \in \mathbb{H}, |q| = |r| = 1 \},$$

and $Spin(5)$ can be identified with the quaternionic unitary group. Thus, as indicated in [Por95] pp.263, the inclusion from $Spin(4) \hookrightarrow Spin(5)$ is given by the formula

\[(6.3)\]

$$\begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix} \mapsto \begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix}.$$

In addition, as shown in [Por95] pp.151, the rotation of $\mathbb{R}^4$ represented by

$$\begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix} \in Spin(4)$$

is described by

$$\begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix} \mapsto \begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix}.$$
is given by the map

\[(6.4) \quad \begin{bmatrix} y & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{bmatrix} \mapsto \begin{bmatrix} q & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^q \end{bmatrix}^{-1} = \begin{bmatrix} q & 0 & 0 \\ 0 & r & 0 \\ 0 & r^{-q} & 1 \end{bmatrix}. \]

where $\mathbb{H}^4$ is identified with the linear space

\[
\{ \begin{bmatrix} y & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{bmatrix} \mid y \in \mathbb{H} \}.
\]

Then, the group $Spin(4) \subset Spin(5)$ acts on $S^4 \subset \mathbb{R}^5$ via the composition

\[(6.5) \quad Spin(4) \to SO(4) \xrightarrow{A \mapsto} \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \to SO(5)
\]

and the standard orthogonal action.

Before we compute the examples, we recall the classification of the finite subgroups of $Spin(3) \cong SU(2)$. There are many references for the classification, [Dic14 Chapter XIII], [Ste08], [nLa23] etc. The finite subgroups of $SU(2)$ are classified as:

- the cyclic group of order $n$
  \[G_n := \{ e^{\frac{2\pi i}{n}}k \mid k \in \mathbb{Z} \};\]
- the dicyclic group of order $4n$
  \[2D_{2n} := \langle G_{2n}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rangle;\]
- the binary tetrahedral group $E_6$;
- the binary octahedral group $E_7$;
- the binary icosahedral group $E_8$;

where $n$ is any positive integer.

In the computation below, we use the symbol

\[A_\theta\]

to denote the matrix

\[\begin{bmatrix} e^{\theta i} & 0 \\ 0 & e^{-\theta i} \end{bmatrix}.\]

**Example 6.3.** In this example, we compute $QEll_{G_n}(S^4)$, where

\[G_n := \{ e^{\frac{2\pi i}{n}}k \mid k \in \mathbb{Z} \}.\]

An element in $G_n$ is of the form $A_\theta$ with $\theta = \frac{2\pi m}{n}$ for some integer $m$. Since $G_n$ is abelian, each conjugacy class in it has exactly one element. Below we compute the factor in $QEll_{G_n}(S^4)$ corresponding to each conjugacy class.

If $n \nmid m$, $(S^4)^{A_\theta} \cong S^0$. We have

\[K_{\Lambda_{\alpha}(A_\theta)}((S^4)^{A_\theta}) \cong K_{\Lambda_{\alpha}(m)}(S^0) \cong RA_{\alpha}(m) \oplus RA_{\alpha}(m) \cong \mathbb{Z}[q^\pm, x]/(x^n - q^m) \oplus \mathbb{Z}[q^\pm, x]/(x^n - q^m).\]

The last step is by the computation in Example 2.6.
If $n \mid m$, $(S^4)^{A_n} \cong S^4$.

$$K_{\Lambda_{G_n}(A_n)}((S^4)^{A_n}) \cong K_{Z/n \times T}(S^4) \cong K_{Z/n}(S^4) \otimes \mathbb{Z}[q^{\pm}]$$

$$\cong K_{Z/n}(S^0) \otimes \mathbb{Z}[q^{\pm}] \cong (R(Z/n) \oplus R(Z/n)) \otimes \mathbb{Z}[q^{\pm}]$$

$$\cong \mathbb{Z}[q^{\pm}, x]/\langle x^n - 1 \rangle \oplus \mathbb{Z}[q^{\pm}, x]/\langle x^n - q^m \rangle.$$

where the isomorphism $(\ast)$ is obtained from the equivariant Bott periodicity $[\text{Ati68}]$ Theorem 4.3] and that the action of $\mathbb{Z}/n$ on the north pole and south pole is trivial.

In conclusion,

$$QEll_{G_n}(S^4) = \prod_{m=0}^{n} K_{\Lambda_{G_n}(A_{\frac{2m}{n}})}((S^4)^{A_{\frac{2m}{n}}})$$

$$\cong \prod_{m=0}^{n} \mathbb{Z}[q^{\pm}, x]/\langle x^n - q^m \rangle \oplus \mathbb{Z}[q^{\pm}, x]/\langle x^n - q^m \rangle.$$

**Example 6.4.** In this example we compute $QEll_{2D_{2n}}(S^4)$, where

$$2D_{2n} = \langle G_{2n}, \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right] \rangle$$

with $n$ a positive integer, and $G_{2n}$ is the cyclic group generated by $A_{\frac{2n}{n}}$.

We will use $\tau$ to denote the matrix

$$\left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right].$$

In $2D_{2n}$ there are $n + 3$ conjugacy classes. They are:

1. $\{I\}$,
2. $\{-I\}$,
3. $\{A_{\frac{n}{n}}, A_{\frac{n}{n}}^{-1}\}, \{A_{\frac{n}{n}}, A_{\frac{n}{n}}^{-2}\}, \cdots, \{A_{\frac{n}{n}}, A_{\frac{n}{n}}^{-(n-1)}\}$,
4. $\{\tau, \tau A_{\frac{n}{n}}, \tau A_{\frac{n}{n}}^{-1}, \cdots, \tau A_{\frac{n}{n}}^{2n-2}\}$,
5. $\{\tau A_{\frac{n}{n}}, \tau A_{\frac{n}{n}}^{3}, \cdots, \tau A_{\frac{n}{n}}^{2n-1}\}$,

where the first two form the centre of the group.

Next we compute the factor in $QEll_{2D_{2n}}(S^4)$ corresponding to each conjugacy class below.

1. First we consider the conjugacy class represented by $I$. The centraliser $C_{2D_{2n}}(I) = 2D_{2n}$, and thus, $\Lambda_{2D_{2n}}(I) \cong 2D_{2n} \times \mathbb{T}$. The corresponding equivariant K-theory

$$K_{\Lambda_{2D_{2n}}(I)}((S^4)^I) \cong K_{2D_{2n} \times T}(S^4) \cong K_{2D_{2n}}(S^4) \otimes \mathbb{Z}[q^{\pm}]$$

$$\cong K_{2D_{2n}}(S^0) \otimes \mathbb{Z}[q^{\pm}] \cong (R(2D_{2n}) \oplus R(2D_{2n})) \otimes \mathbb{Z}[q^{\pm}],$$

where the step $(\ast)$ is also obtained from the equivariant Bott periodicity.

2. Then we consider the conjugacy class represented by $-I$. The centraliser $C_{2D_{2n}}(-I) = 2D_{2n}$.
And the fixed point space \((S^4)^{\tau} = S^4\). The group \(2D_{2n}\) fits into the short exact sequence

\[
0 \longrightarrow \mathbb{Z}/2 \longrightarrow 2D_{2n} \xrightarrow{\pi} D_{2n} \longrightarrow 0,
\]

where \(D_{2n}\) is the dihedral group with \(2n\) elements. Then, by Lemma A.2

\[
K_{2D_{2n}}((-1))((S^4)^{\tau}) \cong K_{2D_{2n}}((-1))(S^0) \cong K_{2D_{2n}}((-1))(\text{pt}) \oplus K_{2D_{2n}}((-1))(\text{pt})
\]

\[
\cong K_{2D_{2n}}((\text{pt}) \oplus K_{2D_{2n}}((\text{pt}) \oplus K_{2D_{2n}}((\text{pt}) \oplus K_{2D_{2n}}((\text{pt})
\]

\[
\cong K_{D_{2n} \times \tau}(\text{pt}) \oplus K_{D_{2n} \times \tau}(\text{pt}) \oplus K_{D_{2n} \times \tau}(\text{pt}) \oplus K_{D_{2n} \times \tau}(\text{pt})
\]

\[
\cong (R(D_{2n}) \oplus R(D_{2n})) \oplus Z[q^\pm] \oplus (R(D_{2n}) \oplus R(D_{2n})) \oplus Z[q^\pm].
\]

where \(\rho\) is the sign representation of \(\mathbb{Z}/2\).

(3) Then we consider the conjugacy class represented by the element \(A_m\), \(m = 1, 2, \ldots, n - 1\). The centralizer

\[
C_{2D_{2n}}(A_m) = G_{2n} \cong \mathbb{Z}/2n.
\]

Thus, \(\Lambda_{2D_{2n}}(A_m) \cong \Lambda_{\mathbb{Z}/2n}(2m)\). In addition, the fixed point space

\[
(S^4)^{A_m} \cong S^0.
\]

So we have

\[
K_{\Lambda_{2D_{2n}}(A_m)}((S^4)^{A_m}) \cong K_{\Lambda_{\mathbb{Z}/2n}(2m)}(S^0) \cong RA_{\mathbb{Z}/2n}(2m) \oplus RA_{\mathbb{Z}/2n}(2m)
\]

\[
\cong \mathbb{Z}[x, q^\pm]/\langle x^{2n} - q^{2m} \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^{2n} - q^{2m} \rangle.
\]

(4) Then we consider the conjugacy class represented by \(\tau\). The centralizer \(C_{2D_{2n}}(\tau) = \langle \tau \rangle \cong \mathbb{Z}/4\) and the fixed point space \((S^4)^{\tau}\) is \(S^0\).

Thus,

\[
K_{\Lambda_{2D_{2n}}(\tau)}((S^4)^{\tau}) \cong K_{\Lambda_{\mathbb{Z}/4}(1)}(S^0) \cong RA_{\mathbb{Z}/4}(1) \oplus RA_{\mathbb{Z}/4}(1)
\]

\[
\cong \mathbb{Z}[x, q^\pm]/\langle x^4 - q \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^4 - q \rangle.
\]

(5) Then we deal with the last conjugacy class, which is represented by \(\tau A_m\).

There are \(n\) elements in the conjugacy class, thus, the centralizer of \(\tau A_m\) has 4 elements. Then it’s direct to check that

\[
C_{2D_{2n}}(\tau A_m) = \langle \tau A_m \rangle \cong \mathbb{Z}/4.
\]

In addition, the fixed point space \((S^4)^{\tau A_m}\) is \(S^0\).

Thus,

\[
K_{\Lambda_{2D_{2n}}(\tau A_m)}((S^4)^{\tau A_m}) \cong K_{\Lambda_{\mathbb{Z}/4}(1)}(S^0) \cong RA_{\mathbb{Z}/4}(1) \oplus RA_{\mathbb{Z}/4}(1)
\]

\[
\cong \mathbb{Z}[x, q^\pm]/\langle x^4 - q \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^4 - q \rangle.
\]
Thus, in conclusion,

\[ QEll_{2D_2n}(S^4) = K_{A_{2D_{2n}}(1)}((S^4)^I) \times K_{A_{2D_{2n}}}(-I)((S^4)^{-I}) \]

\[ \times \prod_{m=1}^{n-1} K_{A_{2D_{2n}}(A_{2n}^m)}((S^4)^{A_{2n}^m}) \]

\[ \times K_{A_{2D_{2n}}}((S^4)^{\tau}) \times K_{A_{2D_{2n}}}((S^4)^{\tau A_{2n}^{2m}}) \]

\[ \cong (R(2D_{2n}) \oplus R(2D_{2n})) \otimes \mathbb{Z}[q^{\pm}] \]

\[ \times (R(D_{2n}) \oplus R(D_{2n})) \otimes \mathbb{Z}[q^{\pm}] \oplus (R_{[D_{2n}^+]}(D_{2n}) \oplus R_{[D_{2n}^+]}(D_{2n})) \otimes \mathbb{Z}[q^{\pm}] \]

\[ \times \prod_{m=1}^{n-1} \mathbb{Z}[x, q^{\pm}]/\langle x^{2n} - q^{2m} \rangle \oplus \mathbb{Z}[x, q^{\pm}]/\langle x^{2n} - q^{-2m} \rangle \]

\[ \times \mathbb{Z}[x, q^{\pm}]/\langle x^4 - q \rangle \oplus \mathbb{Z}[x, q^{\pm}]/\langle x^4 - q \rangle \]

\[ \times \mathbb{Z}[x, q^{\pm}]/\langle x^4 - q \rangle \oplus \mathbb{Z}[x, q^{\pm}]/\langle x^4 - q \rangle \]

**Example 6.5.** In this example we compute \( QEll_{E_6}(S^4) \) where \( E_6 \) is the binary tetrahedral group. The quaternion representation of \( E_6 \) is given explicitly at \( \Phi \text{ic} \) and \( \Phi \text{ib} \).

We can compute the conjugacy classes in \( E_6 \) explicitly. A list of representatives are given in Figure 1. This list can be obtained by direct computation. A multiplication table for the binary tetrahedral group is given here \( \Phi \text{ia} \). For the convenience of the readers, we apply the same symbols of the elements as those in \( \Phi \text{ia} \) and \( \Phi \text{ib} \).

By the multiplication table \( \Phi \text{ia} \) and direct computation, we obtain the centralizers of each representative and the corresponding fixed point space, as in Figure 2.

Then the factors in \( QEll_{E_6}(S^4) \) corresponding to each conjugacy class is computed below.

1. For the conjugacy class represented by 1,

\[ K_{A_{E_6}(1)}((S^4)^I) \cong K_{E_6 \times T}(S^4) \cong K_{E_6}(S^4) \otimes \mathbb{Z}[q^{\pm}] \]

\[ \cong K_{E_6}(S^0) \otimes \mathbb{Z}[q^{\pm}] \cong (R(E_6) \oplus R(E_6)) \otimes \mathbb{Z}[q^{\pm}] \]
Centralizers and fixed point spaces

| Representatives $\alpha$ of Conjugacy classes | Centralizers $C_E(\alpha)$ | Fixed point spaces $(S^4)^{\alpha}$ |
|-----------------------------------------------|-----------------------------|---------------------------------|
| 1                                            | $E$                         | $S^4$                           |
| $-1$                                          | $E_6$                       | $S^0$                           |
| $i$                                           | $\{\pm 1, \pm i\} \cong \mathbb{Z}/4$ |                                  |
| $a$                                           | $\{\pm 1, \pm a, \pm a^2\} \cong \mathbb{Z}/6$ | $S^0$                           |
| $-a$                                          | $\{\pm 1, \pm a, \pm a^2\} \cong \mathbb{Z}/6$ | $S^0$                           |
| $a^2$                                         | $\{\pm 1, \pm a, \pm a^2\} \cong \mathbb{Z}/6$ | $S^0$                           |
| $-a^2$                                        | $\{\pm 1, \pm a, \pm a^2\} \cong \mathbb{Z}/6$ | $S^0$                           |

**Figure 2.** Centralizers and fixed point spaces

(2) Then we compute the factor corresponding to the conjugacy class represented by $-1$. We have the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \mathbb{Z}/2 & \to & E_6 & \to & T_6 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{Z}/2 & \to & Spin(3) & \to & SO(3) & \to & 0
\end{array}
\]

where $T_6$ is the tetrahedral group and both the horizontal sequences are exact.

Then, by Lemma A.2, we have

\[
K_{\Lambda E_6}(-1)((S^4)^{-1}) \cong K_{\Lambda E_6}(1)(S^0) \oplus K_{\Lambda E_6}([\Lambda_{E_6(1)}])^{++}(S^0)
\]

\[
\cong K_{T_6 \times \mathbb{T}}(S^0) \oplus K_{T_6 \times \mathbb{T}}([\Lambda_{E_6(1)}])^{++}(S^0)
\]

\[
\cong K_{T_6}(S^0) \otimes \mathbb{Z}[q^\pm] \oplus K_{T_6}([\Lambda_{E_6(1)}])^{++}(S^0) \otimes \mathbb{Z}[q^\pm]
\]

\[
\cong (R(T_6) \oplus R(T_6) \oplus R([\Lambda_{E_6(1)}])(T_6) \oplus R([\Lambda_{E_6(1)}])(T_6)) \otimes \mathbb{Z}[q^\pm].
\]

where $\rho$ is the sign representation of $\mathbb{Z}/2$.

(3) For the conjugacy class represented by $i$, we have

\[
K_{\Lambda E_6(i)}((S^4)^i) \cong K_{\Lambda E_6(1)}(S^0) \cong R(\Lambda_{E_6(1)}(1)) \oplus R(\Lambda_{E_6(1)}(1))
\]

\[
\cong \mathbb{Z}[x, q^\pm]/\langle x^4 - q \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^4 - q \rangle.
\]

(4) For the conjugacy class represented by $a$, we have

\[
K_{\Lambda E_6(a)}((S^4)^a) \cong K_{\Lambda E_6(a)}(S^0) \cong R(\Lambda_{E_6(a)}(1)) \oplus R(\Lambda_{E_6(a)}(1))
\]

\[
\cong \mathbb{Z}[x, q^\pm]/\langle x^6 - q^4 \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^6 - q^4 \rangle.
\]

(5) For the conjugacy class represented by $-a$, we have

\[
K_{\Lambda E_6(-a)}((S^4)^{-a}) \cong K_{\Lambda E_6(4)}(S^0) \cong R(\Lambda_{E_6(4)}(1)) \oplus R(\Lambda_{E_6(4)}(1))
\]

\[
\cong \mathbb{Z}[x, q^\pm]/\langle x^6 - q^4 \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^6 - q^4 \rangle.
\]

(6) For the conjugacy class represented by $a^2$, we have

\[
K_{\Lambda E_6(a^2)}((S^4)^{a^2}) \cong K_{\Lambda E_6(2)}(S^0) \cong R(\Lambda_{E_6(2)}(1)) \oplus R(\Lambda_{E_6(4)}(2))
\]

\[
\cong \mathbb{Z}[x, q^\pm]/\langle x^6 - q^2 \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^6 - q^2 \rangle.
\]
Centralizers

\[
\begin{array}{|c|c|c|}
\hline
\text{Representatives } \beta & \text{Centralizers } C_E(\beta) & \text{Fixed point spaces } (S^4)^\beta \\
\hline
1 & E_7 & S^4 \\
-1 & E_7 & S^0 \\
i = t^2 & \langle t \rangle \cong \mathbb{Z}/8 & S^0 \\
s & \langle s \rangle \cong \mathbb{Z}/6 & S^0 \\
-s = s^4 & \langle s \rangle \cong \mathbb{Z}/6 & S^0 \\
r & \langle r \rangle \cong \mathbb{Z}/4 & S^0 \\
t & \langle t \rangle \cong \mathbb{Z}/8 & S^0 \\
-t = t^5 & \langle t \rangle \cong \mathbb{Z}/8 & S^0 \\
\hline
\end{array}
\]

**Figure 3.** Conjugacy classes, centralizers and fixed point spaces

(7) For the conjugacy class represented by \(-a^2\), we have

\[
K_{\Lambda_E}(a^2)(S^4)^{-a^2} \cong K_{\Lambda_E(\mathbb{Z}/6)}(S^0) \cong R(\Lambda_{\mathbb{Z}/6}(5)) \oplus R(\Lambda_{\mathbb{Z}/6}(5)) \\
\cong \mathbb{Z}[x, q^\pm]/\langle x^6 - q^5 \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^6 - q^5 \rangle.
\]

Thus, in conclusion,

\[
Q\text{Ell}_{E_7}(S^4) = K_{\Lambda_E(1)}((S^4)^1) \times K_{\Lambda_E(-1)}((S^4)^{-1}) \times K_{\Lambda_E(1)}((S^4)^1) \\
\times K_{\Lambda_E(0)}((S^4)^0) \times K_{\Lambda_E(-a)}((S^4)^{-a}) \times K_{\Lambda_E(a)}((S^4)^a) \\
\times K_{\Lambda_E(-a^2)}((S^4)^{-a^2}) \\
\cong (R(E_7) \oplus R(E_7)) \otimes \mathbb{Z}[q^\pm] \\
\times (R(T_0) \oplus R(T_0) \oplus R(T_0)) \otimes \mathbb{Z}[q^\pm] \\
\times \mathbb{Z}[x, q^\pm]/\langle x^4 - q \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^4 - q \rangle \\
\times \mathbb{Z}[x, q^\pm]/\langle x^6 - q \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^6 - q \rangle \\
\times \mathbb{Z}[x, q^\pm]/\langle x^6 - q^4 \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^6 - q^4 \rangle \\
\times \mathbb{Z}[x, q^\pm]/\langle x^6 - q^5 \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^6 - q^5 \rangle.
\]

**Example 6.6.** In this example we compute \(Q\text{Ell}_{E_7}(S^4)\) where \(E_7\) is the binary octahedral group.

A presentation of \(E_7\) is given as

\[
E_7 = \langle s, t \mid i^2 = s^3 = t^4 = rst = -1 \rangle.
\]

We can get immediately that \(r = st\). Equivalently, there is a quaternion presentation of \(E_7\) given by the embedding

\[
E_7 \rightarrow \mathbb{H}
\]

sending \(s\) to \(\frac{1}{2}(1 + i + j + k)\), \(t\) to \(\frac{1}{\sqrt{2}}(1 + i)\), and \(r\) to \(\frac{1}{\sqrt{2}}(i + j)\).

By [McK80] and direct computation, we get Figure 3, which provides a list of the representatives of the conjugacy classes of \(E_7\), the centralizers of each representative, and the corresponding fixed point spaces.

Below we give the factor of \(Q\text{Ell}_{E_7}(S^4)\) corresponding to each conjugacy class.
(1) For the conjugacy class of $1$,
\[
K_{\Lambda_{E_7}}((S^4)^1) \cong K_{E_7 \times T}(S^4) \cong K_{E_7}(S^4) \otimes R^T
\cong K_{E_7}(S^0) \otimes \mathbb{Z}[q^\pm] \cong (RE_7 \oplus RE_7) \otimes \mathbb{Z}[q^\pm].
\]

(2) Then we consider the conjugacy class of $-1$.

There is a commutative diagram with each horizontal sequence exact.

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & E_7 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\pi & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & Spin(3) & \longrightarrow & 0
\end{array}
\]

where $T_7$ is the octahedral group.

Thus, by Lemma $A.2$, we have
\[
K_{\Lambda_{E_7}}(-1)((S^4)^{-1}) \cong K_{\Lambda_{E_7}}(1)(S^0) \oplus K_{\Lambda_{E_7}}(1)\Lambda_{E_7}(1)
\cong K_{T_7 \times T}(S^0) \oplus K_{T_7 \times T}((T_7 \times T)^{1+})*S^0
\cong K_{T_7}(S^0) \otimes \mathbb{Z}[q^\pm] \oplus K_{T_7}(S^0) \otimes \mathbb{Z}[q^\pm]
\cong (R(T_7) \oplus R(T_7) \oplus R((T_7)^{1+})*(T_7)^{1+})*(T_7) \oplus R((T_7)^{1+})*(T_7)) \otimes \mathbb{Z}[q^\pm],
\]

where $\rho$ is the sign representation of $\mathbb{Z}/2$.

(3) For the conjugacy class of $i$,
\[
K_{\Lambda_{E_7}}(i)((S^4)^i) \cong K_{\Lambda_{E_7}}(2)(S^0) \cong RA_{\mathbb{Z}/2}(2) \oplus RA_{\mathbb{Z}/2}(2)
\cong \mathbb{Z}[x, q^\pm]/\langle x^8 - q^2 \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^8 - q^2 \rangle.
\]

(4) For the conjugacy class of $s = \frac{1}{2}(1 + i + j + k)$,
\[
K_{\Lambda_{E_7}}((S^4)^s) \cong K_{\Lambda_{E_7}}(1)(S^0) \cong RA_{\mathbb{Z}/6}(1) \oplus RA_{\mathbb{Z}/6}(1)
\cong \mathbb{Z}[x, q^\pm]/\langle x^6 - q \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^6 - q \rangle.
\]

(5) For the conjugacy class of $-s = -\frac{1}{2}(1 + i + j + k)$,
\[
K_{\Lambda_{E_7}}((S^4)^{-s}) \cong K_{\Lambda_{E_7}}(4)(S^0) \cong RA_{\mathbb{Z}/6}(4) \oplus RA_{\mathbb{Z}/6}(4)
\cong \mathbb{Z}[x, q^\pm]/\langle x^6 - q^4 \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^6 - q^4 \rangle.
\]

(6) For the conjugacy class of $r = \frac{1}{\sqrt{2}}(i + j)$,
\[
K_{\Lambda_{E_7}}((S^4)^r) \cong K_{\Lambda_{E_7}}(4)(S^0) \cong RA_{\mathbb{Z}/4}(1) \oplus RA_{\mathbb{Z}/4}(1)
\cong \mathbb{Z}[x, q^\pm]/\langle x^4 - q \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^4 - q \rangle.
\]

(7) For the conjugacy class of $t = \frac{1}{\sqrt{2}}(1 + i)$,
\[
K_{\Lambda_{E_7}}((S^4)^t) \cong K_{\Lambda_{E_7}}(8)(S^0) \cong RA_{\mathbb{Z}/8}(1) \oplus RA_{\mathbb{Z}/8}(1)
\cong \mathbb{Z}[x, q^\pm]/\langle x^8 - q \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^8 - q \rangle.
\]

(8) For the conjugacy class of $-t = -\frac{1}{\sqrt{2}}(1 + i)$,
\[
K_{\Lambda_{E_7}}((S^4)^{-t}) \cong K_{\Lambda_{E_7}}(5)(S^0) \cong RA_{\mathbb{Z}/8}(5) \oplus RA_{\mathbb{Z}/8}(5)
\cong \mathbb{Z}[x, q^\pm]/\langle x^8 - q^5 \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^8 - q^5 \rangle.
\]
Thus, in conclusion,
\[ QEll_{E_7}(S^4) = K_{A_{E_7}(1)}((S^4)^1) \times K_{A_{E_7}(-1)}((S^4)^{-1}) \times K_{A_{E_7}(t)}((S^4)^t) \times K_{A_{E_7}(r)}((S^4)^r) \]
\[ \cong (RE_7 \oplus RE_7) \otimes Z[q^\pm] \]
\[ \times (R(T_7) \oplus R(T_7) \oplus R([-T_7], T_7) \oplus R([T_7], T_7)) \otimes Z[q^\pm] \]
\[ \times Z[x, q^\pm]/(x^8 - q^2) \oplus Z[x, q^\pm]/(x^8 - q^2) \]
\[ \times Z[x, q^\pm]/(x^6 - q) \oplus Z[x, q^\pm]/(x^6 - q) \]
\[ \times Z[x, q^\pm]/(x^6 - 4q) \oplus Z[x, q^\pm]/(x^6 - 4q) \]
\[ \times Z[x, q^\pm]/(x^4 - q) \oplus Z[x, q^\pm]/(x^4 - q) \]
\[ \times Z[x, q^\pm]/(x^8 - q) \oplus Z[x, q^\pm]/(x^8 - q) \]
\[ \times Z[x, q^\pm]/(x^8 - q^2) \oplus Z[x, q^\pm]/(x^8 - q^2). \]

Example 6.7. In this example we compute \( QEll_{E_8}(S^4) \), where \( E_8 \) is the binary icosahedral group. A presentation of this group is
\[ (r, s, t \mid (st)^2 = s^3 = t^3 = -1). \]
The cardinality of \( E_8 \) is 120. In this example, we use \( \tau \) to denote \( \frac{1 + \sqrt{5}}{2} \) and \( \sigma \) to denote the number \( \frac{1 - \sqrt{5}}{2} \).

By [KAAK07] page 7635, Table 1, and direct computation, we obtain a list of the representatives of the conjugacy classes of \( E_8 \), the centralizers of each representative, and the corresponding fixed point spaces in Figure 4.

Then we compute the factor of \( QEll_{E_8}(S^4) \) corresponding to each conjugacy class of \( E_8 \) one by one.

1. For the conjugacy class of 1, we have
\[ K_{A_{E_8}(1)}((S^4)^1) \cong K_{E_8 \times T}(S^4) \cong K_{E_8}(S^4) \times R \cong K_{E_8}(S^0) \otimes Z[q^\pm] \cong (R + R) \otimes Z[q^\pm]. \]
(2) Then we consider the conjugacy class of $-1$. We have the commutative diagram with both horizontal sequences exact.

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}/2 & \rightarrow & E_8 & \rightarrow & T_8 & \rightarrow & 0 \\
0 & \rightarrow & \mathbb{Z}/2 & \rightarrow & Spin(3) & \rightarrow & SO(3) & \rightarrow & 0,
\end{array}
\]

where $T_8$ is the icosahedral group. Thus, by Lemma A.2 we have

\[
K_{A_{E_8}}(-1)((S^4)^{-1}) \cong K_{A_{E_8}}(S^0) \oplus K_{A_{E_8}}(\rho_{A_{E_8}})^{++}(S^0)
\]

\[
\cong K_{T_8 \times \mathbb{Z}}(S^0) \oplus K_{T_8 \times \mathbb{Z}}^{+++}(S^0)
\]

\[
\cong K_{T_8}(S^0) \otimes \mathbb{Z}[q^\pm] \oplus K_{T_8}^{+++}(S^0) \otimes \mathbb{Z}[q^\pm]
\]

\[
\cong (R(T_8) \oplus R(T_8) \oplus R(T_8) \oplus R(T_8)) \otimes \mathbb{Z}[q^\pm].
\]

where $\rho$ is the sign representation of $\mathbb{Z}/2$.

(3) For the conjugacy class of

\[
y_3 := \frac{1}{2}(\tau + i + \sigma k),
\]

\[
K_{A_{E_8}}(y_3)((S^4)^{y_3}) \cong K_{A_{E_8}/10}(S^0) \cong RA_{E_8/10}(1) \oplus RA_{E_8/10}(1)
\]

\[
\cong \mathbb{Z}[x, q^\pm]/\langle x^{10} - q \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^{10} - q \rangle.
\]

(4) For the conjugacy class of

\[
y_4 := \frac{1}{2}(-\tau + \sigma i - j),
\]

since $y_4 = y_2^2$,

\[
K_{A_{E_8}}(y_4)((S^4)^{y_4}) \cong K_{A_{E_8}/10}(S^0) \cong RA_{E_8/10}(2) \oplus RA_{E_8/10}(2)
\]

\[
\cong \mathbb{Z}[x, q^\pm]/\langle x^{10} - q^2 \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^{10} - q^2 \rangle.
\]

(5) For the conjugacy class of

\[
y_5 := \frac{1}{2}(\sigma + i + \tau j),
\]

\[
K_{A_{E_8}}(y_5)((S^4)^{y_5}) \cong K_{A_{E_8}/10}(S^0) \cong RA_{E_8/10}(1) \oplus RA_{E_8/10}(1)
\]

\[
\cong \mathbb{Z}[x, q^\pm]/\langle x^{10} - q \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^{10} - q \rangle.
\]

(6) For the conjugacy class of

\[
y_6 := \frac{1}{2}(-\sigma + \tau i - k),
\]

since $y_6 = y_3^2$,

\[
K_{A_{E_8}}(y_6)((S^4)^{y_6}) \cong K_{A_{E_8}/10}(S^0) \cong RA_{E_8/10}(2) \oplus RA_{E_8/10}(2)
\]

\[
\cong \mathbb{Z}[x, q^\pm]/\langle x^{10} - q^2 \rangle \oplus \mathbb{Z}[x, q^\pm]/\langle x^{10} - q^2 \rangle.
\]
Thus, for any finite subgroup $G$ of $SU(2)$, the cohomology of a space $X$ acted by a finite subgroup $G$ of $SU(2)$.

By [EG17 Section 5], for any finite subgroup $G$ of $SU(2)$, $H^2(BG; U(1)) = 0$. Thus, for any finite subgroup $G$ of $SU(2)$, the target of the transgression map $H^3(BG; U(1)) \rightarrow \prod_{[g]} H^2(BG; U(1))$.
is zero, where \([g]\) goes over all the conjugacy classes in \(G\). Note that the subgroup \(C_G(g)\) for each \(g \in G\) is still a finite subgroup of \(SU(2)\). Thus, there is only one central extension of \(G\) by the circle group \(\mathbb{T}\), which is the cartesian product \(G \times \mathbb{T}\).

\[
1 \rightarrow \mathbb{T} \rightarrow G \times \mathbb{T} \rightarrow G \rightarrow 1.
\]

To avoid confusion, we use the symbol 
\[
\alpha
\]
to denote the only element \(1\) in each target group 
\[
\prod_{[g]} H^2(BC_G(g); U(1)) \cong \{1\}.
\]

We have 
\[
\begin{align*}
C_\alpha^G(g) &= C_G(g) \times \mathbb{T}; \\
\Lambda_\alpha^G(g) &= \Lambda_G(g) \times \mathbb{T}.
\end{align*}
\]

Then, each factor in the twisted quasi-elliptic cohomology \(QEll_\alpha^+\) 
\[
K^\alpha_{\Lambda_G(g)}(X^g) \cong K^\alpha_{\Lambda_G(g)}(X^g).
\]

Thus, we have the conclusion

**Proposition 6.8.**
\[
(6.9) \quad QEll_\alpha^+(X) \cong QEll^\alpha_G(X).
\]

So we have the corollary below especially for the twisted quasi-elliptic cohomology of \(S^4\) acted by a finite subgroup \(G\) of \(SU(2)\).

**Corollary 6.9.** The twisted quasi-elliptic cohomology of \(S^4\) acted by a finite subgroup \(G\) of \(SU(2)\) in the way as \((6.2)\) is isomorphic to the untwisted theory, i.e.
\[
(6.10) \quad QEll_\alpha^+(S^4) \cong QEll^\alpha_G(S^4).
\]

**Example 6.10.** In addition, we can compute the twisted version of the quasi-elliptic cohomology of \(S^1\) in Section 6.1. Explicitly,

- For \(S^1\) acted by \(\mathbb{Z}/N\) via the rotation,
  
  \[
  QEll_\alpha^+(S^1) \cong \mathbb{Z}[q^\pm],
  \]
  
  for any twist \(\alpha \in H^3(B\mathbb{Z}/N; U(1))\).

- For \(S^1\) acted by \(\mathbb{Z}/2\) via the reflection,
  
  \[
  QEll_{\mathbb{Z}/2}^\alpha(S^1) \cong (\mathbb{Z}[q^\pm, x]/\langle x^2 - q \rangle \oplus \mathbb{Z}[q^\pm, x]/\langle x^2 - q \rangle) \times \mathbb{Z}[q^\pm],
  \]
  
  for any twist \(\alpha \in H^3(B\mathbb{Z}/2; U(1))\).

**Remark 6.11.** One question arising with Proposition 6.8 is how we can get a more interesting example of twisted quasi-elliptic cohomology that can detect deeper physical meaning. The problem does not lie in what the elliptic cohomology theory is, but the property of the equivariant groups directly. One choice of the groups that can provide nontrivial twist as well as good physical meaning is the loop groups \(LG\) with \(G\) a compact Lie group.

However, most loop groups are not compact, which makes the computation of the corresponding equivariant K-theories impossible. Indeed, there are definitions of equivariant K-theories for non-compact Lie groups, as that given in [3SP10], etc.
However, there is usually a compact restriction for such definitions. For instance, in [JSPT10], the isotropy groups are required to be all compact, which does not make the situation any better. In a representation sphere $S^n$, the north pole and south pole are always the fixed points under the group action. This means the isotropy groups of these two points are both the whole group, which is non-compact.

As a result, in the current background of algebraic topology there is not a good definition of loop group equivariant quasi-elliptic cohomology which is well-defined and has the right physical meaning at the same time. In addition, the computation methods in algebraic topology can deal with some ideal cases in mathematical physics. A reference working on this aspect is [Mei11]. To deal with most cases in the real world, we may need to find some ideas in mathematical physics.

Appendix A. Corollaries of Ángel-Gómez-Uribe Decomposition Formula

In this section, we prove some corollaries of [ÁGU17, Theorem 3.6, Corollary 3.7] that are applied in Section 5. The corollaries all apply to compact Lie groups.

**Lemma A.1.** Let $Q$ and $G$ be compact Lie groups. And we have a short exact sequence
\[
1 \longrightarrow \mathbb{Z}/2 \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1
\]
and $l(A)$ is contained in the center of $G$. Let $X$ be a $G$-space with $(\mathbb{Z}/2)$ acting on it trivially. Then, we have the isomorphism
\[
K^*_G(X) \cong K^*_Q(X) \oplus K^*_Q[\tilde{\text{sign}}]^+(X)
\]

**Proof.** As given in [ÁGU17, Section 2.1], there is a well-defined $G$-action on the irreducible $\mathbb{Z}/2$-representations by
\[
(g \cdot \rho)(a) = \rho(g^{-1}ag) = \rho(a),
\]
for any $g \in G$, $a \in \mathbb{Z}/2$ and any irreducible $\mathbb{Z}/2$-representation $\rho$.

Since the irreducible representations $(\rho, V_\rho)$ of $\mathbb{Z}/2$ are all 1-dimensional and fixed by $G$, the group $PU(1)$ of inner automorphism of $U(1)$ consists of exactly one element, i.e. the identity map. As in [ÁGU17 (1), page 6], we use the symbol $\tilde{G}_\rho$ to denote the pullback
\[
\begin{array}{ccc}
\tilde{G}_\rho & \longrightarrow & U(1) \\
\tau_\rho \downarrow & & \downarrow \\
G & \longrightarrow & PU(1)
\end{array}
\]
We have $\tilde{G}_\rho = G \times U(1)$. The map $\tau_\rho$ is the projection map to $G$ and $\tilde{f}$ is the projection map to $U(1)$.

Then we consider the commutative diagram
\[
\begin{array}{ccc}
\mathbb{Z}/2 & \longrightarrow & \tilde{G}_\rho \\
\downarrow & & \downarrow \\
\mathbb{Z}/2 & \longrightarrow & G
\end{array}
\]
where \( \tilde{l} \) is defined to be the unique map so that \( \rho = f \circ \tilde{l} \). Thus, \( \tilde{l} \) is the product of \( l \) and the representation \( \rho \).

Then we consider the commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z}/2 & \to & \mathbb{Z}/2 \\
\downarrow \tilde{l} & & \downarrow \tilde{l} \\
\mathbb{T} & \to & G \\
\downarrow \pi & & \downarrow \pi \\
\mathbb{T} & \to & Q \end{array}
\]

where the vertical sequences are both exact, the horizontal sequences are \( T \)-central extensions and the square is a pullback square. If \( \rho \) is the trivial representation of \( \mathbb{Z}/2 \), \( \tilde{Q} \rho \sim Q \times T \) and, by [AGU17, Proposition 2.2], \( \rho \) extends to an irreducible representation of \( G \). However, if \( \rho \) is the sign representation of \( \mathbb{Z}/2 \), it may not extend to the whole group \( G \). And the central extension

\[
1 \to \mathbb{T} \xrightarrow{i\mathbb{Q}} \tilde{Q}_\rho \xrightarrow{p\mathbb{Q}} Q \to 1
\]

may correspond to a nontrivial element \([\tilde{Q}_\rho]\) in \( H^3(BQ; \mathbb{Z}) \).

By [AGU17, Corollary 3.7],

\[
(A.2) \quad K^*_G(X) \cong \bigoplus_{\rho \in G/Irr(\mathbb{Z}/2)} K^{[\tilde{Q}_\rho]++}(X),
\]

where \( \rho \) runs over representatives of the orbits of the \( G \)-action on the set of isomorphism classes of irreducible \( \mathbb{Z}/2 \)-representations, i.e. \( \{1, \text{sign}\} \), the action of

\( Q_\rho = G_\rho/(\mathbb{Z}/2) \)

on \( X \) is induced from the \( G \)-action on \( X \), and \( G_\rho \) is the isotropy group of \( \rho \) under the \( G \)-action. Note that the two irreducible \( \mathbb{Z}/2 \)-representations are fixed by the \( G \)-action and \( G_\rho = G \) for each \( \rho \). Thus, the isomorphism \((A.2)\) is exactly

\[
K^*_G(X) \cong K^*_Q(X) \oplus K^{[\tilde{Q}_\text{sign}]++}(X)
\]

In each component, the \( Q \)-action on \( X \) is induced from the quotient map \( \pi : G \to Q \).

Let

\[
1 \to \mathbb{Z}/2 \xrightarrow{l} G \xrightarrow{\pi} Q \to 1
\]

be a short exact sequence of compact groups and \( l(A) \) is contained in the center of \( G \). For any torsion element \( \alpha \) in \( G \), we have the short exact sequence

\[
0 \to \mathbb{Z}/2 \xrightarrow{i} \Lambda_G(\alpha) \xrightarrow{[\pi, l]} \Lambda_Q(\pi(\alpha)) \to 0
\]

with

\[
i(\mathbb{Z}/2) = \{[\beta, 0] \in \Lambda_G(\alpha) \mid \beta \in l(\mathbb{Z}/2)\}
\]

contained in the center of \( \Lambda_G(\pi(\alpha)) \). In addition, \( X^\alpha \) is a \( \Lambda_G(\alpha) \)-space with the action by \( i(\mathbb{Z}/2) \) trivial.
Especially, if $\alpha$ is the nontrivial element in $l(\mathbb{Z}/2)$, then $\pi(\alpha) = 1$ and we have

$$\Lambda_Q(\pi(\alpha)) \cong Q \times T; \quad \Lambda_{\hat{Q}}(\pi(\alpha))_\rho \cong \hat{Q}_\rho \times T.$$ 

In this case, the central extension

$$1 \longrightarrow T \longrightarrow \Lambda_{\hat{Q}}(\pi(\alpha))_\rho \longrightarrow \Lambda_Q(\pi(\alpha)) \longrightarrow 1$$

is completely determined by

$$1 \longrightarrow T \longrightarrow \hat{Q}_\rho \longrightarrow Q \longrightarrow 1,$$

thus, by the 3-cocycle $[\hat{Q}_\rho]$.

Then we can get a corollary of Lemma A.1.

**Lemma A.2.** Let

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1$$

be a short exact sequence of compact groups and $l(A)$ is contained in the center of $G$. Let $X$ be a $G$-space with $l(\mathbb{Z}/2)$ acting on it trivially. For any torsion element $\alpha$ in $G$, we have the isomorphism

$$K^*_{\Lambda_G(\alpha)}(X^\alpha) \cong K^*_{\Lambda_Q(\pi(\alpha))}(X^\alpha) \oplus K^*_{\Lambda_{\hat{Q}}(\pi(\alpha))}(X^\alpha).$$

Especially, if $\alpha$ is the nontrivial element in $l(\mathbb{Z}/2)$,

$$K^*_{\Lambda_G(\alpha)}(X^\alpha) \cong K^*_{\hat{Q}}(X^\alpha) \otimes \mathbb{Z}[q^\pm] \oplus K^*_{\hat{Q}}(X^\alpha) \otimes \mathbb{Z}[q^\pm].$$

**Conflicts of Interest Statement**

The author declares that she has no conflicts of interest.

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