Pairing and Quantum Double of Finite Hopf 
C*-Algebras *

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Abstract This paper defines a pairing of two finite Hopf C*-algebras $A$ and $B$, and 
investigates the interactions between them. If the pairing is non-degenerate, then the 
quantum double construction is given. This construction yields a new finite Hopf C*- 
algebra $D(A, B)$. The canonical embedding maps of $A$ and $B$ into the double are both iso 
metric.

Key words Hopf C*-algebra, paring, quantum double, GNS representation

MR (2000) Subject Classification 46K70;16W30;81R05

Abbreviated title: Pairing of Hopf C*-algebras

*Supported by National Natural Science Foundation of China (No.10301004)and Excellent Young Scholars Research Fund of Beijing Institute of Technology(000Y07-25)
1 Introduction

The Yang-Baxter equation firstly came up in a paper ([1]) as a factorization condition of the scattering \( S \) matrix in the many-body problems in one dimension and in Baxter’s work on exactly solvable models in statistical mechanics. The equation also plays an important role in the quantum inverses scattering method created by Feddeev, Sklyamin and Takhtadjian for the construction of quantum integrable systems. Since braided Hopf algebras ([2]) can provide solutions for the Yang-Baxter equation, attempts to find its solutions in a systematic way have led to the construction of braided Hopf algebras, and moreover, led to the theory of quantum groups. Based on ([3]), Woronowicz ([4]) exhibited C*-algebra structures of quantum groups in the framework of C*-algebra. From then on, the research on Hopf algebra has always been going with that on C*-algebra. This leads to the concept of Hopf C*-algebra ([5]). Indeed, by the Gelfand-Naimark Theorem, an abelian C*-algebra can be understood as the space of all complex continuous functions vanishing at infinity on a locally compact space, and for this reason a C*-algebra can be considered as a noncommutative locally compact quantum space. Henceforth, a Hopf C*-algebra can be regarded as a noncommutative locally compact quantum group.

In this paper, we are interested in finite Hopf C*-algebras. Firstly, we propose the definition of Hopf *-algebra.

Definition 1.1 ([6, 7]) Let \( A \) be a *-algebra with a unit \( 1 \). Suppose that \( \Delta : A \to A \otimes A \) is a *-homomorphism such that
\[
(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta,
\]
and \( \varepsilon : A \to \mathbb{C} \) is a *-homomorphism such that
\[
(\varepsilon \otimes \iota)\Delta = (\iota \otimes \varepsilon)\Delta = \iota,
\]
where \( \iota \) denotes the identity map. Finally, assume that \( S : A \to A \) is a linear, anticommutative map so that for all \( a \in A \),
\[
S(S(a)^*)^* = a,
\]
\[
m(S \otimes \iota)\Delta(a) = m(\iota \otimes S)\Delta(a) = \varepsilon(a)1,
\]
where \( m : A \otimes A \to A \) is the multiplication defined by \( m(a \otimes b) = ab \). Then \( A \) is called a Hopf *-algebra, and \( \Delta, \varepsilon, S \) are called comultiplication, counit and antipode respectively.

Besides the assumption that \( A \) is a finite dimensional Hopf *-algebra, if \( A \) is also a C*-algebra, \( A \) is called a finite Hopf C*-algebra, which satisfies the convolution property, i.e., \( S^2 = \iota \) naturally. And by ([8]), there exists an invariant functional \( \varphi \) on \( A \) so that \( \forall a \in A, \varphi = \varphi \circ S \) and
\[
(\varphi \otimes \iota)\Delta(a) = (\iota \otimes \varphi)\Delta(a) = \varphi(a)1.
\]
After posing the notion of pairing of finite Hopf C*-algebras and exhibiting the actions of dually paired Hopf C*-algebras on each other, this paper gives the quantum double construction ([9]) out of the underlying two Hopf C*-algebras and shows that the consequence of this construction is again a finite Hopf C*-algebra with an invariant integral.
Much of our work is inspired by the work of ([10, 11]). All algebras in this paper will be algebras over the complex field \( \mathbb{C} \). Please refer to ([12]) for general results on Hopf algebra. In many of our calculations, we use the standard Sweedler notation ([13]). For instance, formula like \( m(S \otimes \iota)\Delta(a) = \varepsilon(a)1 \) can be written as
\[
\sum_{(a)} S(a^{(1)})a^{(2)} = \varepsilon(a)1.
\]

2 Pairing of Finite Hopf C*-Algebras

In this section, we consider a bilinear form between finite Hopf C*-algebras.

**Definition 2.1** Let \( A \) and \( B \) be two finite Hopf C*-algebras, and \( \langle \cdot, \cdot \rangle : A \otimes B \to \mathbb{C} \) be a bilinear form. Assume that they satisfy:

\[
\forall a_1, a_2, a \in A, b_1, b_2, b \in B,
\]

\[
\langle \Delta(a), b_1 \otimes b_2 \rangle = \langle a, b_1 b_2 \rangle,
\]

\[
\langle a_1 \otimes a_2, \Delta(b) \rangle = \langle a_1 a_2, b \rangle,
\]

\[
\langle a^*, b \rangle = \langle a, S_B(b)^* \rangle,
\]

\[
\langle a, 1_B \rangle = \varepsilon_A(a),
\]

\[
\langle 1_A, b \rangle = \varepsilon_B(b),
\]

\[
\langle S_A(a), b \rangle = \langle a, S_B(b) \rangle,
\]

where \( \varepsilon_A, S_A \) (resp. \( \varepsilon_B, S_B \)) denote the counit and antipode on \( A \) (resp. \( B \)) respectively. Then \( (A, B, \langle \cdot, \cdot \rangle) \) is called a pairing of finite Hopf C*-algebras.

**Definition 2.2** ([14]) Suppose that \( (A, B, \langle \cdot, \cdot \rangle) \) is a pairing of finite Hopf C*-algebras. If \( B \) (resp. \( A \)) can separate the points of \( A \) (resp. \( B \)) (i.e., if \( a_0 \in A \) (resp. \( b_0 \in B \)) such that \( \forall b \in B \) (resp. \( a \in A \)) \( \langle a_0, b \rangle = 0 \) (resp. \( \langle a, b_0 \rangle = 0 \)), then \( a_0 = 0 \) (resp. \( b_0 = 0 \)) and we call the pairing is non-degenerate.

Similar to the discussions in ([10]), we have the following results. Firstly for any pairing of finite Hopf C*-algebras \( (A, B, \langle \cdot, \cdot \rangle) \), we can define the linear mappings by using the standard Sweedler notation:

\[
\mu^l_{A,B} : A \otimes B \to B, a \otimes b \mapsto \sum_{(b)} b^{(1)} \langle a, b^{(2)} \rangle,
\]

\[
\mu^r_{A,B} : B \otimes A \to B, b \otimes a \mapsto \sum_{(b)} \langle a, b^{(1)} \rangle b^{(2)},
\]

\[
\mu^l_{B,A} : B \otimes A \to A, b \otimes a \mapsto \sum_{(a)} a^{(1)} \langle a^{(2)}, b \rangle,
\]

\[
\mu^r_{B,A} : A \otimes B \to B, a \otimes b \mapsto \sum_{(a)} \langle a^{(1)}, b \rangle a^{(2)}.
\]
Proposition 2.3 Let \((A, B, < \cdot, \cdot >)\) be a pairing of finite Hopf \(C^\ast\)-algebras. Then the maps \(\mu_{A,B}^l\) and \(\mu_{A,B}^r\) are left and right actions of \(A\) on \(B\), i.e., \((B, \mu_{A,B}^l)\) is a left \(A\)-module and \((B, \mu_{A,B}^r)\) is a right \(A\)-module, respectively. Analogously, \(\mu_{B,A}^l\) and \(\mu_{B,A}^r\) are left and right actions of \(B\) on \(A\), respectively.

Proof For all \(a, a' \in A, b, b' \in B\), we can obtain
\[
\mu_{A,B}^l(aa \otimes b) = \sum_{(b)} b_{(1)} < aa', b_{(2)} >
= \sum_{(b)} b_{(1)} < a, b_{(2)} > < a', b_{(3)} >
= \sum_{(b)} b_{(1)} < a, b_{(2)} < a', b_{(3)} >
= \mu_{A,B}^l(a \otimes \mu_{A,B}^l(a' \otimes b)),
\]
which shows that \((B, \mu_{A,B}^l)\) is a left \(A\)-module. In a similar way, we can check other relations and we omit them here.  

For convenience, the previous actions will be denoted by \(\triangleright\) and \(<\) : 
\[
\mu_{A,B}^l(a \otimes b) := a \triangleright b, \quad \mu_{A,B}^r(b \otimes a) := b \triangleright a,
\]
\[
\mu_{B,A}^l(b \otimes a) := b \triangleright a, \quad \mu_{B,A}^r(a \otimes b) := a \triangleright b,
\]
which mean “\(a\) acts from the left or right on \(b\)” and “\(b\) acts from the left or right on \(a\)” respectively, according to the directions of the arrows \(\triangleright\) and \(<\).

Lemma 2.4 Let \((A, B, < \cdot, \cdot >)\) be a pairing of finite Hopf \(C^\ast\)-algebras. Then for all \(a, a' \in A, b, b' \in B\),
\[
< b \triangleright a, b' >= < a, b \triangleright b' >, \quad < a \triangleright b, b' >= < a, b \triangleright b' >,
< a, a' \triangleright b >= < aa', b >, \quad < a, b \triangleright a' >= < a' a, b > .
\]

Proof From the implications of the notations “\(\triangleright\)” and “\(\triangleleft\)”, the proof is obvious. 

From Lemma 2.4, we can get the following proposition at once.

Proposition 2.5 Suppose that \((A, B, < \cdot, \cdot >)\) is a non-degenerate paring of finite Hopf \(C^\ast\)-algebras. Then \((A, \mu_{B,A}^l, \mu_{B,A}^r)\) is a \(B\)-bimodule and \((B, \mu_{A,B}^l, \mu_{A,B}^r)\) is an \(A\)-bimodule.

Proof Let \(a \in A\) and \(b_1, b_2, b_3 \in B\), and check \((b_1 \triangleright a) < b_2\) and \(b_1 \triangleright (a \triangleright b_2)\) paired with \(b_3\). Using Lemma 2.4, the associativity of \(B\) implies
\[
< (b_1 \triangleright a) \triangleright b_2, b_3 >= < a, b_2 b_3 b_1 >= < b_1 \triangleright (a \triangleright b_2), b_3 >,
\]
which proves the proposition for the non-degeneracy of the pairing. 

From Proposition 2.5, we will write \((b_1 \triangleright a) < b_2\) as \(b_1 > a < b_2\) briefly in sequence.

Remark 2.6 (1) The third axiom in Definition 2.1 is also symmetric in \(A\) and \(B\):
\[
< a, b^* > = \sum < a^*, S_B(b^*) > = < S_A^{-1}(a^*), S_B(b^*) > = < S_A(a)^*, b > .
\]
(2) The last three axioms in Definition 2.1 are redundant if the pairing < ·, · > is non-degenerate. From the first three axioms of Definition 2.1 and Lemma 2.4, one can obtain for all \( a, a' \in A, b, b' \in B, < T_2^A(a \otimes a'), b \otimes b' >= < a \otimes a', T_1^B(b \otimes b') > \), which also holds for the inverse mappings of \( T_2^A \) and \( T_1^B \). Put \( a = 1_A, b = 1_B \), by the non-degeneracy of \( < \cdot, \cdot >, < a'_(1), 1_B > a'_(2) = a' \). Applying \( \varepsilon \) to the two sides of this equation yields \( < a', 1_B >= \varepsilon_A(a') \). Similarly, \( < 1_A, b' >= \varepsilon_B(b') \). Using \( < T_2^{-1}(a \otimes a'), b \otimes b' >= < a \otimes a', T_1^B(b \otimes b') > \) and Lemma 2.4, one can get \( < S_A(b \triangleright a'), b \triangleleft a > = < b \triangleright a', S_B(b \triangleleft a) > \), which implies the last axiom.

3 The Quantum Double

In what follows, we will only consider the action of \( B \) on \( A \), where \( A \) and \( B \) are two dually paired finite Hopf C*-algebras. It is easy to see that \( A \otimes B \) can be made into a linear space of finite dimension in a natural way ([15]). Furthermore, we can turn the linear space \( A \otimes B \) into an associative algebra which has an analogous algebra structure to the classical Drinfeld’s quantum double.

**Definition 3.1** The quantum double \( D(A, B) \) of a non-degenerate pairing of finite Hopf C*-algebras \( (A, B, < \cdot, \cdot >) \) is the algebra \( (A \otimes B, m_D) \) with the multiplication map defined through

\[
m_D((a, b)(a', b')) = \sum_{(b)} (a(b_{(3)}) > a' < (S_B^{-1}b_{(1)}), b_{(2)}b')
\]

\[
= \sum_{(a')(b)} (aa'_{(2)}, b_{(2)}b') < a'_{(1)}, S_B^{-1}b_{(3)} > < a'_{(3)}, b_{(1)} >,
\]

where \((a, b), (a', b')\) are in the linear basis \( B_D := \{(a, b) | a \in A, b \in B\} \) of \( D(A, B) \).

Following, we will write \( m_D((a, b)(a', b')) \) as \((a, b)(a', b')\) directly. It is easy to see \((a, 1_B)(a', b) = (aa', b) \) and \((a, b)(1_A, b') = (a, bb') \). In particular, \((1_A, 1_B)\) is the unit of \( D(A, B) \). Under the canonical embedding maps \( i_A : a \mapsto (a, 1_B) \) and \( i_B : b \mapsto (1_A, b) \), \( A \) and \( B \) become subalgebras of \( D(A, B) \).

**Proposition 3.2** The multiplication \( m_D \) of the quantum double \( D(A, B) \) is non-degenerate.

**Proof** For a fixed element \((a, b) \in D(A, B)\), suppose \((a, b)(a', b') = 0 \) for all \((a', b') \in D(A, B)\). Particularly pick \( a' = 1_A \). Then \((a, b)(a', b') = (a, bb') = 0 \). If \( a \neq 0 \), then \( bb' = 0 \) for all \( b' \in B \), which implies \( b = 0 \) for the non-degeneracy of the product on \( B \). Thus we have \( a = 0 \) or \( b = 0 \), i.e., \((a, b) = 0 \). Similarly one can prove that \((a, b)(a', b') = 0 \) for all \((a, b) \in D(A, B)\) if and only if \((a', b') = 0 \).

In order to avoid using too many brackets, we will use \( Sa \) for \( S(a) \). On the basis \( B_D \), set

\[
* D(a, b) = (a, b)^* := \sum_{(a)(b)} (a'_{(2)}, b_{(2)}^*) < a'_{(3)}, b_{(1)}^* > < a'_{(1)}, S_B^b_{(3)} >,
\]

and extend it anti-linearly to the whole space of \( D(A, B) \). Then \((a, 1_B)^* = (a^*, 1_B)\), \((1_A, b)^* = (1_A, b^*)\). To describe the *-structure of \( D(A, B) \) exactly, we firstly do some preparing work.

**Lemma 3.3** \( \forall (a, b) \in D(A, B), (a, b)^{**} = (a, b) \).
Proof

\((a, b)^*\)

\[= \sum_{\{a, b\}} (a^*_2, b^*_2) < a^*_3, S_B b^*_1 > < a^*_3, b^*_3 >
\]

\[= \sum_{\{a, b\}} (a^*_3, b^*_3) < a^*_4, S_B^2 b^*_2 > < a^*_2, b^*_4 > < a^*_5, S_B b^*_1 > < a^*_3, b^*_5 >
\]

\[= \sum_{\{a, b\}} (a^*_3, b^*_3) < a^*_4, S_B^2 b^*_2 > < a^*_2, S_B b^*_4 > < a^*_5, S_B b^*_1 > < a^*_3, b^*_5 >
\]

\[= \sum_{\{a, b\}} (a^*_2, b^*_2) < a^*_1, b^*_5 > < a^*_2, S_B b^*_4 > < a^*_5, S_B b^*_1 > < a^*_3, b^*_5 >
\]

\[= \sum_{\{a, b\}} (a^*_2, b^*_2) [< a^*_1, b^*_2 > < a^*_2, B b^*_2 ] [< a^*_5, B b^*_1 > < a^*_3, B b^*_5 >
\]

where we use relations \(S_A((S_A^a)^*) = a\) and \(< a^*, b > = < a, S_B^b >\) in the third and forth equations.

Lemma 3.4 \(\forall (a, b), (a', b') \in D(A, B), [(a, b)(a', b')]^* = (a', b')^*(a, b)^*\).

Proof We firstly prove the relation \([(1_A, b)(a', b')]^* = (a', b')^*(1_A, b)^*\).
Similarly, \((a, b)^* = [(a, 1_B)(1_A, b)]^* = (1_A, b)^*(a, 1_B)^*\) and then
\[
[(a, b)(a', b')]^* = [(a, 1_B)(1_A, b)(a', b')]^*
= [(1_A, b)(a', b')]^*(a, 1_B)^*
= (a', b')^*(1_A, b)^*(a, 1_B)^*
= (a', b')^*(a, b)^*,
\]
which completes the proof. ■

Using Lemma 3.3 and Lemma 3.4, one can immediately get the following result.

**Proposition 3.5** The involution \(\ast_D\) renders \(D(A, B)\) into a non-degenerate \(*\)-algebra.

Furthermore, one can show that \(D(A, B)\) has a Hopf \(*\)-algebra structure. Indeed, under the following structure maps, \(D(A, B)\) becomes a finite dimensional Hopf algebra naturally ([16]): \(\forall (a, b) \in D(A, B)\),
\[
\Delta_D(a, b) = \sum_{(a)(b)} (a(1), b(1)) \otimes (a(2), b(2)),
\]
\[
\varepsilon_D(a, b) = \varepsilon_A(a)\varepsilon_B(b),
\]
\[
S_D(a, b) = \sum_{(a)(b)} (S_Aa(a^2), S_Bb(b^2)) < a(1), S_Bb(3) > < a(3), b(1) > .
\]

**Theorem 3.6** \(D(A, B)\) is a Hopf \(*\)-algebra.

Proof It suffices to show that \(\Delta_D\) and \(\varepsilon_D\) are \(*\)-homomorphisms and \(\forall (a, b) \in D(A, B)\), \(S_D(S_D(a, b))^*) = (a, b)\).

1. \(\Delta_D\) is a \(*\)-homomorphism.

\[
\Delta_D((a, b)^*) = \Delta_D([(a, 1_B)(1_A, b)]^*)
= \Delta_D([(1_A, b^*)^*(a^*, 1_B)]
= \Delta_D(1_A, b^*)\Delta_D(a^*, 1_B)
= \sum_{(b)} (1_A, b(1)_{(1)}) \otimes (1_A, b(2)_{(2)}) \sum_{(a)} (a^*(1), 1_B) \otimes (a^*(2), 1_B)
= \sum_{(a)(b)} [(a(1), 1_B)(1_A, b(1))]^* \otimes [(a(2), 1_B)(1_A, b(2))]^*
= \sum_{(a)(b)} [(a, b(1))^*] \otimes [(a, b(2))]^*
= (\Delta_D(a, b))^*.
\]

Similarly, \(\varepsilon_D\) is a \(*\)-homomorphism.

2. It is easy to see \(S_D(a, 1_B) = (S_Aa, 1_B)\) and \(S_D(1_A, b) = (1_A, S_Bb)\). Thus
\[
S_D(a, b) = S_D([a, 1_B](1_A, b)] = S_D(1_A, b)S_D(a, 1_B) = (1_A, S_Bb)(S_Aa, 1_B),
\]
and therefore,
\[
(S_D(a, b))^* = (S_Aa, 1_B)^*(1_A, S_Bb)^* = (S_A^*a, S_B^*b).
\]
Using these two relations, we have

\[
S_D(S_D(a, b)^*) \\
= S_D(S_A^* a, S_B^* b) \\
= \sum_{(a,b)} (S_A(S_A^* a)(2), S_B(S_B^* b)(2)) \times \\
< S_A^* a(1), S_B(S_B^* b)(3) >= S_A^* a(3), S_B^* b(1) > \\
= \sum_{(a,b)} (a^*_2, b^*_2) < a^*_3, b^*_1 > < S_A^* a(1), S_B(S_B^* b)(3) > \\
= \sum_{(a,b)} (a^*_2, b^*_2) < a^*_3, b^*_1 > < a^*_1, S_B^* b(3) > \\
= (a, b)^*.
\]

\[\blacksquare\]

Now it is time to consider the C^*-algebra structure of D(A, B).

**Lemma 3.7** Let \( \varphi_A \) and \( \varphi_B \) be invariant integrals on \( A \) and \( B \), respectively. \( \forall (a, b) \in D(A, B) \), set

\[\theta((a, b)) := \varphi_A(a)\varphi_B(b).\]

Then \( \theta \) is a faithful positive linear functional on \( D(A, B) \).

**Proof** \((a, b)(a, b)^* = (a, b)(1_A, b^*)(a^*, 1_B) = (a, bb^*)(a^*, 1_B)\). In the following, we denote \( bb^* \) by \( c \) briefly.

\[
\theta((a, b)(a, b)^*) \\
= \theta((a, c)(a^*, 1_B)) \\
= \sum_{(a,c)} \theta((aa^*_2, c(2))) < a^*_3, S_Bc(3) > < a^*_3, c(1) > \\
= \sum_{(a,c)} \varphi_A(aa^*_2)\varphi_B(c(2)) < a^*_1, S_Bc(3) >= S_A^* a(3), a(1) > \\
= \sum_{(a,c)} \varphi_A(aa^*_2) < a^*_1, S_Bc(3) > < a^*_3, \varphi_B(c(2)c(1) > \\
= \sum_{(a,c)} \varphi_A(aa^*_2) < a^*_1, S_Bc(2) > < a^*_3, 1_B > \varphi_B(c(1)) \\
= \sum_{(a,c)} \varphi_A(aa^*_2) < a^*_1, S_Bc(2) > \varphi_B(c(1)) \\
= \sum_{(a,c)} \varphi_A(aa^*_2) < a^*_1, S_Bc(2) > \varphi_B(c(1)) \\
= \sum_{(a,c)} \varphi_A(aa^*_2) < a^*_1, S_Bc(2) > \varphi_B(c(1)) \\
= \sum_{(a,c)} \varphi_A(aa^*_2) < a^*_1, 1_B > \varphi_B(c) \\
= \sum_{(a,c)} \varphi_A(aa^*_2) < a^*_1, 1_B > \varphi_B(c) \\
= \varphi_A(aa^*)\varphi_B(c) \\
= \varphi_A(aa^*)\varphi_B(c) \geq 0,
\]

where we use the relation \( \varphi_B \circ S_B = \varphi_B \) for the last third and forth equations.

It is clear that \( \theta((a, b)(a, b)^*) = 0 \) if and only if \( a = 0 \) or \( b = 0 \), which implies \( (a, b) = 0 \). Thus \( \theta \) is a faithful positive linear functional on \( D(A, B) \). \[\blacksquare\]

**Theorem 3.8** \( D(A, B) \) is a finite Hopf C^*-algebra.
Proof Using the result in Lemma 3.7, one can construct the associated GNS representation of $D(A, B)$ ([17]): \( \forall x, y \in D(A, B) \), set

\[
<x, y>_\theta = \theta(y^*x),
\]

where \(<x, y>_\theta\) denotes the inner product of \(x\) and \(y\). Thus \(D(A, B)\) turns into a Hilbert space \(K\). For \(d \in D(A, B)\), define

\[
\pi(d) : K \rightarrow K, \ x \mapsto dx.
\]

Using ([17]), \((\pi, K)\) is a faithful *-representation of \(D(A, B)\), and hence \(D(A, B)\) can embeds into \(B(K)\) isometrically through

\[
\pi : D(A, B) \rightarrow B(K), \ d \mapsto \pi(d).
\]

Again \(D(A, B)\) is finite dimensional, therefore, it is a C*-algebra with C*-norm \(\|(a, b)\| = (\theta((a, b)(a, b)^*))^{1/2}\).

Remark 3.9 A short calculation shows that \(\theta\) coincides with \(\varphi_A \otimes \varphi_B\), which is indeed an integral on \(D(A, B)\). Using the relation \(\theta((a, b)(a, b)^*) = \varphi_A(aa^*)\varphi_B(bb^*)\), one can get \(\|(a, b)\| = \|a\|\|b\|\). In particular, \(\|(a, 1_B)\| = \|a\|\) (resp. \(\|(1_A, b)\| = \|b\|\)), which implies that the canonical embedding \(i_A\) (resp. \(i_B\)) is isometric.

Example 3.10 Let \(H\) be a finite Hopf C*-algebra and \(H'\) be its dual, which is also a finite Hopf C*-algebra by ([8]). They are naturally dually pairing and have invariant integrals, denoted by \(h\) and \(h'\) respectively. Drinfeld's quantum double \(D(H)\) of \(H\), which is defined as the bicrossed product of \(H\) and \(H'\), is a special case of our construction. One ([11]) can prove that it is also a finite Hopf C*-algebra and has an invariant integral \(h \otimes h'\).

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