FAMILIES OF MONGE-AMPÈRE MEASURES WITH HÖLDER CONTINUOUS 
POTENTIALS

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ABSTRACT. Let $X$ be a compact Kähler manifold of dimension $n$. Let $F$ be a family of 
probability measures on $X$ whose superpotentials are of uniformly bounded $C^{\alpha}$ norms 
for some fixed constant $\alpha \in (0, 1]$. We prove that the corresponding family of solutions of 
the complex Monge-Ampère equations $(dd^c \varphi + \omega)^{n} = \mu$ with $\mu \in F$ is Hölder continuous.

Keywords: Monge-Ampère measure, Monge-Ampère equation, superpotential.

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1. INTRODUCTION

Let $X$ be a compact Kähler manifold of dimension $n$ with a fixed Kähler form $\omega$ so 
normalized that $\int_X \omega^n = 1$. Let $\mu$ be a probability measure on $X$. For every bounded 
$\omega$-psh function $\varphi$ on $X$ and $1 \leq j \leq n$, we put $\omega_j^\varphi := (dd^c \varphi + \omega)^j$ which is well-defined by [1, 10]. Consider the complex Monge-Ampère equation

$$\omega^n_\varphi = \mu,$$

where $\varphi$ is a bounded $\omega$-psh function on $X$ and $\int_X \varphi \omega^n = 0$. The equation (1.1) and its 
variants have been extensively studied and have a wide range of applications. Instead of 
giving details on the development of the research on (1.1), in this short paper, we refer 
the readers to [18, 13, 11, 14, 3, 5, 9, 4, 15, 2] and the references therein for detailed 
information.

In this work, we study the Hölder continuity of solutions of (1.1). Recently, based on [3], Dinh and Nguyën proved in [6] that (1.1) has a unique Hölder continuous solution $\varphi_\mu$ if and only if $\mu$ has a Hölder continuous superpotential $\mathcal{U}_\mu$, see Definition 2.1 below. Precisely, they proved that if $\mathcal{U}_\mu$ is Hölder continuous with Hölder exponent $\alpha \in (0, 1]$, then $\varphi_\mu \in C^\beta(X)$ for any $\beta \in (0, \frac{2\alpha}{n+1})$, where $C^\beta(X)$ denotes the set of Hölder continuous functions with Hölder exponent $\beta$ on $X$. In this case, we call $\varphi_\mu$ the (Monge-Ampère) potential of $\mu$. In view of the last result, we would like to address the question of the stability of the Hölder continuity of the solution of (1.1) with respect to $\mu$ : given a

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family of probability measures with Hölder continuous superpotentials, does \( \varphi_\mu \) depend Hölder continuously on \( \mu \) in that family? Let us be more clear in the next paragraph.

Let \( \alpha \in (0, 1] \). By [6, Pro. 4.1], if \( \varphi \) varies in a bounded subset of \( C^\alpha(X) \), then \( \omega_\mu^\alpha \) has a Hölder continuous superpotential with uniformly bounded Hölder exponent and Hölder constant. Hence, in order to study the above stability problem, it is necessary to consider sets of probability measures having the last property.

Now let \( \mathcal{P}_\alpha \) be a set of probability measures \( \mu \) on \( X \) such that the superpotential \( \mathcal{U}_\mu \) of every \( \mu \in \mathcal{P}_\alpha \) is Hölder continuous with Hölder exponent \( \alpha \) and a Hölder constant independent of \( \mu \). Let \( \beta \in (0, \frac{2\alpha}{n+1}] \). Define \( \Phi : \mathcal{P}_\alpha \to C^\beta(X) \) by sending \( \mu \in \mathcal{P}_\alpha \) to the unique solution \( \varphi_\mu \) of (1.1). Recall that the set of probability measures on \( X \) endowed with the weak topology is a metric space with the distance \( \text{dist} \) defined as follows: for measures \( \mu, \mu' \),

\[
\text{dist}(\mu, \mu') := \sup_{\|v\|_1 \leq 1} |\langle \mu - \mu', v \rangle|,
\]

where \( v \) is a smooth real-valued function on \( X \). The following is our main result.

**Theorem 1.1.** The map \( \Phi \) is Hölder continuous with Hölder exponent \( \alpha' \) for any \( 0 < \alpha' < \beta(\frac{2\alpha}{n+1} - \beta)2^{-n-1} \).

Equivalently, the last theorem says that there is a constant \( C \) (depending on \( \beta, \alpha' \)) such that

\[
\|\varphi_{\mu_1} - \varphi_{\mu_2}\|_{C^\beta} \leq C[\text{dist}(\mu_1, \mu_2)]^{\alpha'},
\]

for every \( \mu_1, \mu_2 \in \mathcal{P}_\alpha \). Consequently, if \( \{\mu_k\}_{k \in \mathbb{N}} \in \mathcal{P}_\alpha \) converges weakly to \( \mu \in \mathcal{P}_\alpha \), then the associated solution \( \varphi_{\mu_k} \) converges to \( \varphi_\mu \) in \( C^\beta(X) \). An interesting feature in the last assertion is that \( \mu_k \) and \( \mu \) can be singular to each other for every \( k \). An imitation of Kołodziej’s arguments in [12] only gives an estimate of type (1.2) but with \( \text{dist}(\mu_1, \mu_2) \) replaced by the mass norm \( \|\mu_1 - \mu_2\| \) of \( \mu_1 - \mu_2 \). A such estimate is not useful when \( \mu_1, \mu_2 \) are singular to each other. For example as in the situation described in Corollary 1.2 below, the supports of measures \( \mu_1, \mu_2 \) in question are disjoint, hence \( \|\mu_1 - \mu_2\| = 2 \) in this case.

We give now an application of our main result. Recall that a real submanifold of \( X \) is said to be Cauchy-Riemann generic if the real tangent space at any point of it isn’t contained in a complex hypersurface of the real tangent space at that point of \( X \). By [17], the restriction of a smooth volume form of an immersed (Cauchy-Riemann) generic submanifold \( Y \) of \( X \) to a compact subset \( K \) of \( Y \) has a Hölder continuous superpotential. It is also clear from the arguments there that if the compact \( K \) depends smoothly on a parameter \( \tau \) then the Hölder exponent and Hölder constant of the superpotential can be chosen to be fixed numbers for every \( \tau \), see Proposition 2.6 below. Precisely, let \( M \) be a compact real manifold and \( Y \) a real Riemannian manifold. Assume that there is a smooth map \( \Psi : Y \times M \to X \) such that \( \Psi_\tau := \Psi|_{Y \times \{\tau\}} : Y \to X \) is an embedding into \( X \) such that \( Y_\tau := \Psi_\tau(Y) \) is a generic submanifold \( Y_\tau \) for every \( \tau \in M \). Then \( \{Y_\tau\}_{\tau \in M} \) is a smooth family of generic submanifolds of \( X \). Note that using local charts of \( X \), we see that such family exists abundantly. With this setting, we get the following nice geometric result.

**Corollary 1.2.** Let \( K \) be a compact subset of \( Y \). For \( \tau \in M \), define \( \mu_\tau \) to be the pushforward measure of the volume form of \( Y \) on \( K \) under \( \Psi_\tau \). Then the family of the Monge-Ampère potential \( \varphi_{\mu_\tau} \) of \( \mu_\tau \) is Hölder continuous in \( \tau \).
Note that as in Theorem 1.1 we can give an explicit Hölder exponent in Corollary 1.2. In the next section, we will give a proof of Theorem 1.1.

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2. PROOF OF THEOREM 1.1

Let \( P_0 \) be the set of \( \omega \)-psh functions \( \varphi \) on \( X \) such that \( \int_X \varphi \omega^n = 0 \). We define the distance \( \text{dist}_{L^1} \) on \( P_0 \) by putting
\[
\text{dist}_{L^1}(\varphi_1, \varphi_2) := \int_X |\varphi_1 - \varphi_2| \omega^n,
\]
for every \( \varphi_1, \varphi_2 \in P_0 \).

**Definition 2.1.** The superpotential of a probability measure \( \mu \) (of mean 0) is the function \( U : P_0 \rightarrow \mathbb{R} \) given by
\[
U(\varphi) := \int_X \varphi d\mu.
\]
We say that \( U \) is Hölder continuous with Hölder exponent \( \alpha \in (0, 1] \) if it is so with respect to the distance \( \text{dist}_{L^1} \). The \( C^\alpha \)-norm of \( U \) is defined as usual.

By [6, Le. 3.3], that \( U \) is Hölder continuous with Hölder exponent \( \alpha \in (0, 1] \) is equivalent to having
\[
\int_X |\varphi_1 - \varphi_2| d\mu \leq C \max \{ \|\varphi_1 - \varphi_2\|_{L^1(X)}, \|\varphi_1 - \varphi_2\|_{L^1(X)} \}.
\]
for some constant \( C \) independent of \( \varphi_1, \varphi_2 \). By the arguments in [6], we immediately get the following.

**Lemma 2.2.** Assume that the superpotential \( U \) of a probability measure \( \mu \) on \( X \) is Hölder continuous with Hölder exponent \( \alpha \) and Hölder constant \( C \). Let \( \beta \in \left(0, \frac{2\alpha}{n+1}\right) \). Then the unique solution \( \varphi_\mu \) of (1.1) with \( \int_X \varphi_\mu \omega^n = 0 \) is Hölder continuous with Hölder exponent \( \beta \) and Hölder constant \( \tilde{C} \) depending only on \( \alpha, \beta, C \) and \( X \). In particular, \( \varphi_\mu \) is bounded by \( \tilde{C} \) independent of \( \mu \in \mathcal{P}_\alpha \).

Let \( K \) be a Borel subset of \( X \). The capacity of \( K \) is given by
\[
\text{cap}_\omega(K) := \sup \left\{ \int_K \omega^n : 0 \leq \varphi \leq 1, \varphi \text{ \( \omega \)-psh} \right\}.
\]
The above notion is due to Kołodziej as an analogue to the capacity given by Bedford and Taylor in the local setting. As introduced in [6], a positive measure \( \mu \) is said to be K-moderate if there are positive constants \( A \) and \( \delta_0 \) for which
\[
\int_K \omega^n_{\varphi_1} \leq Ae^{-[\text{cap}_\omega(K)]^{-\delta_0}}.
\]
for every Borel subset \( K \) of \( X \). Recall that if \( \mu \) has a Hölder continuous superpotential with Hölder exponent \( \alpha \) and Hölder constant \( C \), then \( \mu \) is K-moderate by [6, Pro. 2.4]. Moreover, the constants \( A, \delta_0 \) in (2.2) depend only on \( \alpha, C \) and \( X \). The following result is crucial for our later proof.
Lemma 2.3. Let \( \varphi_1, \varphi_2 \) be bounded \( \omega \)-psh functions on \( X \). Let \( s \) be a real number. Assume that the set \( \{ \varphi_1 - s < \varphi_2 \} \) is nonempty and \( \omega_n^\varphi \) is \( K \)-moderate. Let \( \delta \) be a positive number in \((0, 1)\). Then there exists a constant \( A' \) depending only on \( A, \delta_0, \delta \) and \( n \) such that for any \( \epsilon \in (0, 1) \) we have

\[
\text{cap}_\omega \{ \{ \varphi_1 - s - \epsilon < \varphi_2 \} \} \geq A'(1 + \|\varphi_2\|_{L^\infty})^{-\delta} \epsilon^\delta.
\]

Proof. By (2.2), we have

\[
\int_K \omega_n^\varphi \leq A_1 [\text{cap}_\omega(K)]^{-n^2\delta^{-1}},
\]

for some constant \( A_1 \) depending only on \( A, n, \delta_0, \delta \). Define \( h(t) := t^{n^2\delta^{-1}} \) for positive real numbers \( t \). Put \( c_\epsilon := \text{cap}_\omega \{ \{ \varphi_1 - s - \epsilon < \varphi_2 \} \} \). Applying now [12, Le. 2.2] to \( h(t) \) and \( \varphi_1, \varphi_2 \) gives

\[
\int_{c_\epsilon^{-1/n}}^\infty t^{-1}h^{-1/n}(t)dt + h^{-1/n}(c_\epsilon^{-1/n}) \gtrsim (1 + \|\varphi_2\|_{L^\infty})^{-1}\epsilon.
\]

Then the desired inequality follows easily. The proof is finished.

Lemma 2.4. Let \( \mu_1, \mu_2 \in \mathcal{P}_\alpha \) and \( \varphi_1, \varphi_2 \) Hölder continuous solutions of (1.1) for \( \mu_1, \mu_2 \) respectively. Let \( \beta \in (0, \frac{2\alpha}{n+1}) \). Then we have

\[
\|\varphi_1 - \varphi_2\|_{L^1(X)} \leq C \text{dist}(\mu_1, \mu_2)^{\beta 2^{-n}},
\]

for some constant \( C \) independent of \( \mu_1, \mu_2 \).

Proof. By [2, Th. 1.2], we have

\[
\int_X d(\varphi_1 - \varphi_2) \wedge d^c(\varphi_1 - \varphi_2) \wedge \omega_n^{\varphi} \leq C \left( \int_X (\varphi_1 - \varphi_2)(\omega_n^\varphi - \omega_n^{\varphi_2}) \right)^{21^{-n}},
\]

for some constant \( C \) independent of \( \mu_1, \mu_2 \). Now using Poincaré’s inequality (see [3, Th. 1, page 275]) for \( L^2 \)-norm and the fact that \( \varphi_1, \varphi_2 \) are Hölder continuous with Hölder exponent \( \beta \) and a fixed Hölder constant, we get

\[
\|\varphi_1 - \varphi_2\|_{L^1(X)} \lesssim \left( \int_X (\varphi_1 - \varphi_2)(\omega_n^\varphi - \omega_n^{\varphi_2}) \right)^{2^{-n}} \lesssim \text{dist}_\beta(\mu_1, \mu_2)^{2^{-n}},
\]

where

\[
\text{dist}_\beta(\mu_1, \mu_2) := \sup_{\|v\| \leq 1} |\langle \mu_1 - \mu_2, v \rangle|.
\]

Recall from [7, 16] that \( \text{dist}_\beta(\mu_1, \mu_2) \lesssim \text{dist}(\mu_1, \mu_2) \) for \( \beta \in [0, 1] \). This together with (2.6) gives (2.4). The proof is finished.

The following result is the interpolation inequality for Hölder norms of which we include a proof for the readers’ convenience.

Lemma 2.5. Let \( f \) be a Hölder continuous function in \( \mathcal{C}^\beta(X) \) for some positive constant \( \beta \in (0, 1) \). Let \( \epsilon \) be a positive number in \( [0, 1 - \beta] \). Then we have

\[
\|f\|_{\mathcal{C}^\beta} \leq C \|f\|_{\mathcal{C}^\beta_{\epsilon^\beta}} \|f\|_{\mathcal{C}^{\beta + \epsilon}}^\beta,
\]

for some constant \( C \) depending only on \( X \).
Proof. Recall that the $C^\beta$-norm is defined in $X$ by using a fixed cover of $X$ by local charts. Thus without loss of generality, we can assume that $X = \mathbb{C}^n$. For $x, y \in \mathbb{C}^n$, we have

$$
\frac{|f(x) - f(y)|}{|x - y|^3} = |f(x) - f(y)| \frac{\alpha}{\beta + \epsilon} \leq 2\|f\|_{X_0} \|f\|_{C^{\beta + \epsilon}}.
$$

The proof is finished. \qed

End of the proof of Theorem 1.1. Let $\mu_1, \mu_2 \in \mathcal{P}_\alpha (\mu_1 \neq \mu_2)$ and $\varphi_1, \varphi_2$ Hölder continuous solutions of (1.1) for $\mu_1, \mu_2$ respectively. Fix a constant $\beta \in (0, \frac{2\alpha}{n+1})$ and $\delta \in [0, \frac{2\alpha}{n+1} - \beta)$. By Lemma 2.2 and the definition of $\varphi$, there is a positive constant $\tilde{C}$ independent of $\varphi_1, \varphi_2$ such that $\varphi_1, \varphi_2$ are Hölder continuous with Hölder exponent $(\beta + \delta)$ and Hölder constant $\tilde{C}$. Set

$$
N(\varphi_1, \varphi_2) := \max\{\|\varphi_1 - \varphi_2\|_{L^1(X)}, \|\varphi_1 - \varphi_2\|_{L^1(X)}\} \neq 0.
$$

Fix a real number $\tilde{\delta}$ in $(0, 1)$. In order to prove (1.2), it suffices to suppose from now on that $\text{dist}(\mu_1, \mu_2)$ is small. As it will be clear later, we will need that $\text{dist}(\mu_1, \mu_2)$ is less than a positive constant depending on $\tilde{\delta}$ but independent of $\mu_1, \mu_2$. By Lemma 2.4, the quantity $N(\varphi_1, \varphi_2)$ is also small. In what follows, we use the notations $\lesssim$ and $\gtrsim$ to indicate $\leq$ and $\geq$ respectively up to a multiplicative constant independent of $\mu_1, \mu_2$.

Let $\epsilon$ be a positive real number in $(0, 1)$ to be chosen later. Put $E_\epsilon := \{\varphi_1 + \epsilon < \varphi_2\}$. On $E_\epsilon$ we have $\varphi_1 - \varphi_2 \leq -\epsilon < 0$, hence $|\varphi_1 - \varphi_2| \geq \epsilon$. It follows that

$$
\int_{E_\epsilon} d\mu_1 \leq \epsilon^{-1} \int_X |\varphi_1 - \varphi_2| d\mu_1 \lesssim \epsilon^{-1} N(\varphi_1, \varphi_2)
$$

by (2.1). Since $|\varphi_2| \leq \tilde{C}$, for any $\omega$-psh function $\varphi$ on $X$ such that $0 \leq \varphi \leq 1$, we have

$$
E := \{\varphi_1 + (\tilde{C} + 2)\epsilon < \epsilon \varphi + (1 - \epsilon)\varphi_2\} \subset \{\varphi_1 + (\tilde{C} + 2)\epsilon < \epsilon + \varphi + \epsilon \tilde{C}\} = E_\epsilon.
$$

This combined with the comparison principle gives

$$
\int_E \omega^n_{\epsilon \varphi + (1-\epsilon)\varphi_2} \leq \int_E \omega^n_{\varphi_1} \leq \int_{E_\epsilon} \omega^n_{\varphi_1} = \int_{E_\epsilon} d\mu_1.
$$

On the other hand, we also have $E_{2\epsilon(\tilde{C}+1)} \subset E$ and $\omega^n_{\epsilon \varphi + (1-\epsilon)\varphi_2} \geq \epsilon^n \omega^n_{\varphi}$. This yields

$$
\epsilon^n \int_{E_{2\epsilon(\tilde{C}+1)}} \omega^n_{\varphi} \leq \int_E \omega^n_{\epsilon \varphi + (1-\epsilon)\varphi_2} \leq \int_{E_\epsilon} d\mu_1.
$$

Combining the last inequality with (2.7), we obtain

$$
\epsilon^n \int_{E_{2\epsilon(\tilde{C}+1)}} \omega^n_{\varphi} \lesssim \epsilon^{-1} N(\varphi_1, \varphi_2).
$$

Taking the supremum over every $\varphi$ in the last inequality implies

$$
\text{cap}_\omega (E_{2\epsilon(\tilde{C}+1)}) \lesssim \epsilon^{-n-1} N(\varphi_1, \varphi_2).
$$

Choose $\epsilon := C_{\tilde{\delta}} (N(\varphi_1, \varphi_2))^{1/(n+1+\tilde{\delta})} \in (0, 1)$, where $C_{\tilde{\delta}} > 1$ is a constant big enough (depending on $\tilde{\delta}$) which is independent of $\varphi_1, \varphi_2$. Here recall that $N(\varphi_1, \varphi_2)$ was assumed to be small enough at the beginning of the proof.
We claim that \( E_{2s(C+2)} \) is empty. Suppose the contrary. Thus applying Lemma 2.3 to 
\( s := -2(\tilde{C} + 2) \varepsilon \) shows that \( \text{cap}_\omega (E_{2s(C+1)}) \geq A_\delta \varepsilon^\delta \) for some constant \( A_\delta \) independent of \( \varphi_1, \varphi_2 \). This coupled with (2.8) gives 
\[
N(\varphi_1, \varphi_2) \geq A_\delta \varepsilon^{n+1+\delta} = A_\delta C^{n+1+\delta} N(\varphi_1, \varphi_2).
\]
We get a contradiction because \( C_\delta \) can be chosen such that \( A_\delta C^{n+1} > 1 \). Therefore \( E_{2s(C+2)} \) is empty. In other words, we have
\[
\varphi_1 - \varphi_2 \gtrsim (N(\varphi_1, \varphi_2))^{1/(n+1+\delta)}.
\]
By swapping the roles of \( \varphi_1, \varphi_2 \) we also get
\[
\varphi_2 - \varphi_1 \gtrsim (N(\varphi_1, \varphi_2))^{1/(n+1+\delta)}.
\]
This implies that
\[
\|\varphi_1 - \varphi_2\|_{L^\infty(X)} \lesssim (N(\varphi_1, \varphi_2))^{1/(n+1+\delta)} \lesssim \|\varphi_1 - \varphi_2\|_{L^1(X)}^{\alpha/(n+1+\delta)}
\]
which is
\[
\lesssim \text{dist}(\mu_1, \mu_2)^{\alpha \beta / (n+1+\delta)}.
\]
Now applying Lemma 2.5 to \( f = \varphi_1 - \varphi_2 \) and using (2.10) we obtain that for any \( \delta \in [0, \frac{2\alpha}{n+1} - \beta] \),
\[
\|\varphi_1 - \varphi_2\|_{\psi - \beta} \lesssim \|\varphi_1 - \varphi_2\|_{L^\infty(X)}^{\delta/(n+1+\delta)} \lesssim \text{dist}(\mu_1, \mu_2)^{\delta \alpha (n+1+\delta) - 1 - \frac{\beta}{n+1}}.
\]
Letting \( \delta \to (\frac{2\alpha}{n+1} - \beta) \) and \( \tilde{\delta} \to 0 \) gives the desired result. The proof is finished. \( \square \)

**Proposition 2.6.** Let \( M, \{Y_\tau\}_{\tau \in M} \) and \( \Psi \) be as in Introduction. Then the superpotential of \( \mu_\tau \) is Hölder continuous with uniformly bounded Hölder exponent and Hölder constant as \( \tau \) varies in \( M \).

**Proof.** As already mentioned, the desired result can be deduced directly from [17]. We briefly explain it here for the reader’s convenience. We need to prove (2.1) for \( \mu_\tau \) instead of \( \mu \) and the constants \( C, \alpha \) there must be independent of \( \tau \).

Since the problem is local, it is enough to work locally. Each \( Y_\tau \) inherits the metric from \( Y \). Fix \( \tau \in M \) and a point \( a \in Y_\tau \). The crucial point is that the data in [17] Le. 3.1 can be chosen uniformly in \( \tau, a \). To be precise, there exists a local chart \((W, \Phi)\) around \( a \) in \( X \) with \( \Phi : W \to \mathbb{C}^n \) such that the following three properties holds:

(i) \( W \cap K_\tau \) contains a ball \( \mathbb{B}_{Y_\tau}(a, r) \) of radius \( r_a \) centered at \( a \) of \( Y_\tau \), where \( r > 0 \) is a constant independent of \( a, \tau \),

(ii) \( \|\Phi\|_{\psi - \beta} \) and \( \|\Phi^{-1}\|_{\psi - \beta} \) are bounded by a constant independent of \( a, \tau \),

(iii) \( \Phi(W \cap K) \) is the graph over the unit ball \( \mathbb{B} \) of \( \mathbb{R}^n \) of a smooth map \( h : \mathbb{B} \to \mathbb{R}^n \) (\( \mathbb{C}^n \approx \mathbb{R}^n + i\mathbb{R}^n \)) such that \( \|h\|_{\psi - \beta} \) is bounded by a constant independent of \( a, \tau \) and \( D^j h(0) = 0 \) for \( j = 0, 1, 2 \).

By Property (i), the number of local charts \((W, \Phi)\) needed to cover \( K_\tau \) can be chosen to be a fixed number for every \( \tau \). On a such local chart, every constant in [17] Pro. 3.7 can be chosen to be the same for every \( \tau, a \). Now the rest of the proof is done as in [17]. This gives us constants \( C, \alpha \) in (2.1) independent of \( \tau \). The proof is finished. \( \square \)
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