Quantum isometries and loose embeddings

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Abstract

We show that countable metric spaces always have quantum isometry groups, thus extending the class of metric spaces known to possess such universal quantum-group actions.

Motivated by this existence problem we define and study the notion of loose embeddability of a metric space \((X, d_X)\) into another, \((Y, d_Y)\): the existence of an injective continuous map that preserves both equalities and inequalities of distances. We show that 0-dimensional compact metric spaces are “generically” loosely embeddable into the real line, even though not even all countable metric spaces are.

We also prove that compact Riemannian manifolds \((M, d)\) equipped with their geodesic distances do not contain a number of distance patterns that rule out loose embeddability into a finite-dimensional Hilbert space, making \((M, d)\) a good candidate for loose embeddability.

Key words: compact quantum group; Riemannian manifold; Gromov-Hausdorff distance; geodesic; isometry; Baire theorem; Baire space; covering dimension

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Introduction

Let \((X, d)\) be a compact metric space and

\[ \alpha : X \times G \to X \tag{0-1} \]

an isometric action of a compact group \(G\) on \(X\). Every metric space admits a universal \(\alpha\), in the sense that every compact-group isometric action \(X \times H \to X\) arises via a unique compact-group morphism \(H \to G\). Indeed, one simply takes \(G = \text{Iso}(X, d)\) (the group of all self-isometries of \(X\)), equipped with the uniform topology.

The present note is partly motivated by the question of whether such universal isometric actions on compact metric spaces \(X\) exist in the context of compact quantum groups. To make sense of this one dualizes (0-1) to a unital \(C\)-algebra morphism

\[ \rho : C(X) \to C(X) \otimes Q \]

where \(Q\) is a compact quantum group (see §1.2 below for detailed definitions). We then have a concept of \(\rho\) being isometric (§1.2 below), and can similarly pose in fairly guessable manner the question of whether there is a “largest” isometric quantum action (Definition 1.11). If there is, we refer to the quantum group in question as the quantum isometry group of \((X, d)\).

One of the main results of [10] (Theorem 4.8 therein) is that quantum isometry groups always exist for compact metric spaces isometrically embeddable in some finite-dimensional Hilbert space. More is true however: according to (a slightly paraphrased version of) [10, Corollary 4.9], we have
Theorem 0.1 A compact metric space \((X, d)\) has a quantum isometry group provided it embeds into some finite-dimensional Hilbert space by a continuous one-to-one map that preserves equalities and differences of distances. ■

The phrasing above is a bit awkward; what is meant is that we have a map \(f : X \to \mathbb{R}^n\) (for some \(n\)) such that for any four points \(x, z, x', z'\) in \(X\) the distances \(d(x, y)\) and \(d(x', y')\) are equal if and only if their Euclidean counterparts 

\[
|fx - fy| \text{ and } |fx' - fy'|
\]

are equal. This motivates the natural question (now entirely separate of the issue of quantum actions) of which compact metric spaces admit such loose embeddings (a term we introduce in Definition 2.2) into Euclidean spaces. The term is meant to convey the fact that such an embedding demands much less that distance preservation; the defining condition (2-1) is essentially combinatorial in nature, concerned, as it is, only with the pattern of equalities between pairwise distances in \((X, d)\).

In Section 1 we gather some of the necessary background on compact quantum groups and metric geometry.

The main result of the short Section 2 is Theorem 2.1, stating that all countable compact metric spaces have quantum isometry groups. We then transition to the material occupying the bulk of the paper, on loose embeddability (Definition 2.2). Note however that by Example 2.3 that condition is not necessary for the existence of quantum isometry groups (i.e. the converse to Theorem 0.1 is not valid).

Section 3 focuses on loose metric embeddability, containing both positive and no-go results. A first recurring theme throughout the discussion is that Riemannian compact metric spaces, i.e. those arising by equipping a compact Riemannian manifold with its global geodesic distance (Definition 3.1), make for poor counterexamples to loose metric embeddability:

- In Proposition 3.2 we observe that a (compact) Riemannian metric space cannot contain \(n\)-tuples of equidistant points for arbitrarily large \(n\), ruling out the type of pathology provided by Example 2.3.

- Generalizing this, Theorem 3.5 shows that compact Riemannian metric spaces do not contain arbitrarily large sets of pairs \(\{x_i, y_i\}\) of points with \(x_i\) and \(y_j\) both equidistant from \(x_i\) and \(y_i\) for all \(j > i\). This rules out (in the Riemannian case) a subtler class of counterexamples to loose metric embeddability given by Lemma 3.4.

On the other hand, although Example 2.3 shows that loose embeddability is not automatic even for countable metric spaces, Theorem 3.13 proves that “most” 0-dimensional (i.e. totally disconnected) compact metric spaces are loosely embeddable. Formally, Theorem 3.13 (with a fragment of Proposition 1.5 thrown in for clarity) reads:

**Theorem 0.2** The set \(\mathcal{M}_{\leq 0}\) of isometry classes of 0-dimensional compact metric spaces is a dense \(G_\delta\) in the complete Gromov-Hausdorff metric space \(\mathcal{M}\) of isometry classes of all compact metric spaces, and hence \(\mathcal{M}_{\leq 0}\) is a Baire space. Furthermore, the complement in \(\mathcal{M}_{\leq 0}\) of the set of isometry classes of loosely embeddable 0-dimensional compact metric spaces is of first Baire category. ■

Finally, in §3.2 we pose a number of questions related to loose embeddability, and answer a weakened form of Question 3.16 affirmatively in the Riemannian setting: Theorem 3.19 says,
roughly speaking, that if the finite subspaces of a compact Riemannian metric space \((M,d)\) are uniformly loosely embeddable (i.e. loosely embeddable into Hilbert spaces of uniformly bounded dimension) then \((M,d)\) itself is loosely embeddable in a weak sense.

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## 1 Preliminaries

Unless specified otherwise, all algebras and morphisms between them are assumed unital. We will frequently have to take tensor products of \(C^*\)-algebras, in which case the tensor symbol always denotes the minimal (or spatial) tensor product [3, Definition 3.3.4]. On the other hand, between plain, non-topological algebras ‘⊗’ denotes the usual, algebraic tensor product.

### 1.1 Metric geometry

We gather some background material on metric geometry and point-set topology with [4, 15, 9] serving as references. Recall (e.g. [4, Definitions 7.3.1 and 7.3.10])

**Definition 1.1** Let \((Z,d)\) be a metric space and \(X,Y \subseteq Z\) two subsets. The *Hausdorff distance* \(d_{H,Z}(X,Y)\) is the infimum over all \(\varepsilon > 0\) such that \(X\) and \(Y\) are each contained in the other’s \(\varepsilon\)-neighborhood.

For two metric spaces \((X,d_X)\) and \((Y,d_Y)\) the *Gromov-Hausdorff distance* is defined as

\[
d_{GH}(X,Y) = \inf d_{H,Z}(X,Y)
\]

where the infimum is taken over all metric spaces \(Z\) housing \(X\) and \(Y\) as isometrically embedded subspaces.

Gromov-Hausdorff distance is an actual metric on the set of isometry classes of compact metric spaces [4, Theorem 7.3.30], and we write \((\mathcal{M},d_{GH})\) for the resulting metric space. By abuse of notation, we often identify a compact metric space with its corresponding point of \(\mathcal{M}\) (i.e. its isometry class).

We will also need the notion of *dimension* for a topological space. As explained throughout [9, Chapter 1], the various competing definitions do not, in general agree. They do, however, for compact metric spaces (or more generally, separable ones) [9, Theorem 1.7.7]. For that reason, we provide one of the definitions (of what is usually called the *covering dimension*; see [9, Definition 1.6.7] or [15, §50]) and omit all qualifiers preceding the term ‘dimension’.

**Definition 1.2** Let \(X\) be a compact metrizable topological space.

A finite open cover \(\mathcal{U} = \{U_i\}\) of \(X\) has *order* \(n \geq 0\) if there are \(n + 1\) mutually intersecting sets \(U_i\) but no \(n + 2\) sets \(U_i\) intersect. The order is infinite if no such \(n\) exists.

\(X\) is said to have *dimension* \(\dim X = n\) if every finite open cover has a finite refinement of order \(\leq n\), and \(n\) is the smallest integer with this property (\(\dim X = \infty\) if no such integers exist). By convention, \(\dim \emptyset = -1\).

A few remarks worth keeping in mind:
• for manifolds $\dim X$ coincides with the standard concept.

• for compact metric spaces 0-dimensionality means total disconnectedness, i.e. the existence of a basis consisting of clopen sets [9, Theorem 1.4.5].

• for separable metric spaces dimension is (as expected) monotonic with respect to inclusions [9, Theorem 3.1.19].

We indicate dimension constraints for elements of $\mathcal{M}$ by a subscript: $\mathcal{M}_0 \subset \mathcal{M}$ for instance denotes the set (of isometry classes) of 0-dimensional compact metric spaces, $\mathcal{M}_{\leq n}$ that of metric spaces of dimension $\leq n$, etc.

$(\mathcal{M}, d_{GH})$ is known to be a complete (separable) metric space and hence, by the Baire category theorem ([15, Theorem 48.2]), a Baire space in the sense of [15, §48]: countable intersections of dense open subsets are again dense.

Baire’s theorem suggests that sets containing countable intersections of dense open sets should be regarded as “large”. We recall the relevant language:

**Definition 1.3** Let $X$ be a topological space. A subset $Y \subseteq X$ is meager or of first category if it is contained in a countable union of nowhere dense closed subsets of $X$.

A subset that is not meager is of second category, and the complement of a meager set is residual.

**Remark 1.4** In Baire spaces being residual is equivalent to containing a dense $G_\delta$ set, i.e. countable intersection of open subsets.

The Baire theorem applies not only to complete metric spaces but to $G_\delta$ subsets of the latter ([15, §48, Exercise 5]). This makes the following result relevant.

**Proposition 1.5** For every non-negative integer $n$, the subspace $\mathcal{M}_{\leq n} \subset \mathcal{M}$ consisting of (isometry classes of) compact metric spaces of dimension $\leq n$ is dense $G_\delta$, and hence a Baire space.

**Proof** The density claim follows from the fact that the set of finite metric spaces (and hence also $\mathcal{M}_{\leq 0}$) is dense in $\mathcal{M}$.

Fix positive integers $M$, $N$ and let $\mathcal{M}_M^N \subset \mathcal{M}$ be the set of metric spaces $(X, d)$ admitting some finite open cover $\mathcal{U} = \{U_i\}$ such that

- $\mathcal{U}$ has mesh $< \frac{1}{N}$, i.e. the supremum of the diameters $\text{diam } U_i$ is $< \frac{1}{N}$;
- for each $n + 2$-element subset $\mathcal{V} \subset \mathcal{U}$ and each $U \in \mathcal{V}$ we have

$$\inf_{x \in U; y \in \mathcal{V}} d(x, y) > \frac{1}{M}$$

where

$$\mathcal{V} = \bigcap_{U' \in \mathcal{V}; U' \neq U} U'.$$

By the characterization of dimension for compact metric spaces given in [9, Theorem 1.6.12] we have

$$\mathcal{M}_{\leq n} = \bigcap_{N \to \infty} \bigcup_{M \to \infty} \mathcal{M}_M^N.$$
Since the intersection can be indexed by reciprocals \( \varepsilon = \frac{1}{N} \) of positive integers as \( N \to \infty \), the conclusion will follow once we prove

**Claim:** \( \mathcal{M}_M^N \) is open in \( \mathcal{M} \). To verify this, let \( (X, d_X) \in \mathcal{M}_M^N \) and consider a cover \( \mathcal{U} = \{U_i\} \) as in the definition of the latter. Let \( \delta > 0 \) (more on its size later) and suppose the elements

\[
(X, d_X), (Y, d_Y) \in \mathcal{M}
\]

are isometrically embedded in a compact metric space \( (Z, d_Z) \) and \( \delta \)-Hausdorff close therein. The open subsets

\[
U_i^\delta := \{z \in Z \mid d_Z(z, U_i) < \delta\} \subseteq Z
\]

will then cover \( Y \subseteq Z \).

If \( \delta = \delta(X, d_X) \) is sufficiently small then we can ensure that every intersection

\[
U_i^\delta \cap \cdots \cap U_t^\delta \subset Z, \quad 1 \leq t \leq n
\]

of at most \( n + 1 \) open sets is contained in the \( \delta' \)-neighborhood of the corresponding intersection

\[
U_i^\delta \cap \cdots \cap U_t^\delta \subset X
\]

for arbitrarily small \( \delta' > 0 \) (that would have to be fixed before \( \delta, Y, Z, \) etc.).

In turn, requiring \( \delta' > 0 \) sufficiently small would ensure that the open cover of \( Y \) by \( V_i := U_i^\delta \cap Y \) satisfies the two requirements in the definition of \( \mathcal{M}_M^N \) and hence witnesses \( Y \)'s membership in that subset of \( \mathcal{M} \).  

In particular:

**Corollary 1.6** The subspace \( \mathcal{M}_0 \subset \mathcal{M} \) is \( G_\delta \) and hence Baire.

### 1.2 Compact quantum groups and actions

For the material in this subsection we refer for instance to \([13, 19, 20, 12]\), recalling only skeletal background here. We will also need the very basics of Hopf algebra theory, for which \([18, 14, 1, 17]\) constitute good references.

**Definition 1.7** A **compact quantum group** is a unital \( C^* \)-algebra \( Q \) equipped with a \( C^* \) morphism \( \Delta : Q \to Q \otimes Q \) which

- is coassociative in the sense that

\[
\begin{array}{ccc}
Q & \xrightarrow{\Delta} & Q \otimes Q \\
\Delta & \downarrow & \Delta \otimes \text{id} \\
Q \otimes Q & \xrightarrow{\text{id} \otimes \Delta} & Q \otimes Q \otimes Q
\end{array}
\]

commutes;

- the spans

\[
\Delta(Q)(\mathbb{C} \otimes Q) \quad \text{and} \quad \Delta(Q)(Q \otimes \mathbb{C})
\]

of the respective products are dense in \( Q \otimes Q \).
In general, we regard unital $C^*$-algebras as objects dual to compact quantum spaces; this terminology will be in use throughout. For that reason, we will write $Q = C(G)$ (and refer to $G$ as the compact quantum group) to emphasize that the compact quantum group $G$ is to be regarded as dual to its algebra $Q$ of continuous functions.

A compact quantum group $C(G)$ has a unique dense $*$-subalgebra

$$\mathcal{O}(G) \subseteq C(G)$$

that becomes a Hopf $*$-algebra when equipped with the comultiplication $\Delta$ inherited from $C(G)$; in particular, this means that

$$\Delta(\mathcal{O}(G)) \subset \mathcal{O}(G) \otimes \mathcal{O}(G) \subset C(G) \otimes C(G);$$

see for instance [12, Theorem 3.1.7]. The antipode $\kappa$ of $\mathcal{O}(G)$ need not extend continuously to $C(G)$.

$C(G)$ has a Haar state $h : C(G) \to \mathbb{C}$, left and right $G$-invariant (as the Haar measure is on classical compact groups) in the sense that

$$h * \varphi = h = \varphi * h, \ \forall \text{ states } \varphi \text{ on } C(G)$$

where

$$\begin{array}{ccc}
C(G) & \xrightarrow{\Delta} & C(G) \otimes C(G) \\
\varphi*\psi & \xrightarrow{\varphi \otimes \psi} & \mathbb{C}
\end{array}$$

defines the convolution product on functionals on $C(G)$.

**Definition 1.8** Let $G$ be a compact quantum group.

$C(G)$ is **reduced** if the Haar state $h : C(G) \to \mathbb{C}$ is faithful.

$C(G)$ is **full** if the map $\mathcal{O}(G) \to C(G)$ is the $C^*$ envelope of the complex $*$-algebra $\mathcal{O}(G)$.

For arbitrary $G$ we write $C(G)_r$ for the reduced version of $G$, i.e. the image of the GNS representation of the Haar state $h : C(G) \to \mathbb{C}$, and $C(G)_u$ for the full (or universal) version of $G$, i.e. the $C^*$ envelope of $\mathcal{O}(G)$ (such an envelope exists for every $G$).

Note that we have quantum group morphisms

$$C(G)_u \to C(G) \to C(G)_r. \quad (1-1)$$

**Definition 1.9** An **action** of a compact quantum group $(Q, \Delta)$ on a compact quantum space $A$ is a $C^*$-morphism

$$\rho : A \to A \otimes Q \quad (1-2)$$

such that

$$\begin{array}{ccc}
A & \xrightarrow{\rho} & A \otimes Q \\
\rho & \xrightarrow{\rho \otimes \text{id}} & A \otimes Q \otimes Q \\
\rho & \xrightarrow{\text{id} \otimes \Delta} & A \otimes Q \otimes Q
\end{array}$$

commutes;
the span $\rho(A)(\mathcal{C} \otimes Q)$ is dense in $A \otimes Q$.

If furthermore the span of
\[\{(\varphi \otimes \text{id})\rho(a) \mid a \in A, \varphi \text{ a state on } Q\} \subset Q\]
is dense then the action is faithful.

Composing
\[A \xrightarrow{\rho} A \otimes C(G) \xrightarrow{\text{id} \otimes \pi} A \otimes C(G)_r\]
with $\pi : C(G) \to C(G)_r$ the canonical surjection from (1-1) produces an action of the reduced version $C(G)_r$, so if needed we can always assume that an acting quantum group is reduced.

Throughout the present paper we in fact only work with classical spaces $A$, i.e. $A = C(X)$ (continuous functions) for some compact Hausdorff $X$. It can be shown [11, Theorem 3.16] that for $G$ acting faithfully on $X$ the antipode $\kappa$ of the Hopf $*$-algebra $O(G)$ extends continuously to $C(G)_r$ (so $G$ is of Kac type, in standard terminology). For that reason, we will henceforth assume all of our compact quantum groups $C(G)$ come equipped with antipodes $\kappa$.

We are interested primarily in compact metric spaces $(X,d)$. In that context, the relevant notion of structure-preserving quantum-group action was introduced in [10, Definition 3.1]:

**Definition 1.10** Let $(X,d)$ be a compact metric space, $A = C(X)$ and $\rho : A \to A \otimes Q$ a compact-quantum-group action on $X$. $\rho$ is isometric if we have
\[\rho(d_x)(y) = \kappa(\rho(d_y))(x) \in Q\]
for all pairs of points $x,y \in X$, where $\kappa : Q \to Q$ is the antipode.

Finally, we can give

**Definition 1.11** Let $(X,d)$ be a compact metric space and $A = C(X)$. An isometric action (1-2) is universal if every isometric action $\rho' : A \to A \otimes H$ factors as
\[A \xrightarrow{\text{id} \otimes \eta} A \otimes H \xrightarrow{\rho'} A \otimes Q\]
for a unique compact quantum group morphism $\eta : Q \to H$.

If such a universal action exists we say that $(X,d)$ has a quantum isometry group.

### 2 Countable metric spaces

**Theorem 2.1** A countable compact metric space $(X,d)$ has a compact quantum isometry group.

**Proof** Consider an isometric action of a CQG $G$ on $(X,d)$.

Being countable, $X$ must contain isolated points. Each isolated point, in turn, is contained in one of the finite sets
\[X_{\geq r} := \{x \in X \mid d(x,y) \geq r, \forall x \neq y \in X\}\]
for some $r > 0$ (this is the set of points which admit no neighbors at a distance smaller than $r$). [5, Theorem 3.1], for instance, makes it clear that each $X_{\geq r}$, $r \geq 0$ is preserved by the action of $G$. 

\[7\]
Letting $r \to 0$, we see that the action of $G$ leaves invariant the entire set
\[
X_{\text{iso}} = \bigcup_{r \to 0} X_{\geq r}
\]
of isolated points. That set is open because each $X_{\geq r}$ is, so the compact countable metric space $X \setminus X_{\text{iso}}$ is again preserved. Now repeat the procedure with the latter space eliminating its isolated points, and so on. This transfinite recursive procedure, which must terminate after countably many steps, will partition the original metric space $X$ into countably many finite subspaces preserved by $G$. Since these spaces do not depend on $G$ or the action but are rather intrinsic to $X$, every isometric action will preserve them. The conclusion follows from the fact that finite metric spaces have quantum isometry groups.

We will need the following notion.

**Definition 2.2** Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. A loose (or loosely isometric) embedding $X \to Y$ is a one-to-one continuous map $f : X \to Y$ with the property that for every $x, x', z$ and $z'$ in $X$ we have
\[
d_Y(fx, fx') = d_Y(fz, fz') \iff d_X(x, x') = d_X(z, z').
\]

We say that $(X, d)$ is loosely embeddable (or LE for short) if there is a loose embedding into some Euclidean space $\mathbb{R}^n$ with its usual distance function, typically denoted by
\[
|x - y| := d_{\mathbb{R}^n}(x, y) = \sum_{i=1}^{n} (x_i - y_i)^2.
\]

A slight generalization of [10, Corollary 4.9] says that loosely embeddable compact metric spaces have compact quantum automorphism groups. Example 2.3 shows that not every countable compact metric space is loosely embeddable, and hence not all spaces covered by Theorem 2.1 fall within the scope of that result.

**Example 2.3** One can easily construct countable compact metric spaces $(X, d)$ such that for every $n$, $X$ contains regular $n$-simplices, i.e. $(n + 1)$-tuples of equidistant points $x_0, x_1$ up to $x_n$. Such a space cannot admit a loose embedding into any Euclidean space $\mathbb{R}^d$, since the latter cannot house a regular simplex with more than $d + 1$ vertices.

**Remark 2.4** Contrast Example 2.3 to the fact that by [6, Corollary 3] finite metric spaces are always loosely embeddable.

### 3   Loose metric embeddability

It is a natural problem, in view of Example 2.3 and the discussion preceding it, to determine to what extent various classes of metric spaces are loosely embeddable. Of special interest, for instance, are Riemannian manifolds equipped with the geodesic metric. [7, 2] are good sources for the Riemannian geometry we will peruse.

**Definition 3.1** A Riemannian metric space is a Riemannian manifold equipped with the global geodesic metric.
Unless specified otherwise, all of our metric spaces are assumed compact; we thus often drop that adjective for brevity.

The first remark is that the technique used in Example 2.3 for producing non-loosely-embeddable metric spaces will not function in this Riemannian setting.

**Proposition 3.2** Let \( (M, d) \) be a compact Riemannian manifold with its geodesic metric. There is an upper bound on the number of vertices of a regular simplex in \( M \).

**Proof** Now let \( v_1 \) and \( v_2 \) be two other vertices of \( \Delta \), chosen so that the angle \( \varepsilon = \angle v_1 v_0 v_2 \) is sufficiently small (possible for large \( n \)). Since \( M \) is compact there is a global lower bound \( K \) for its sectional curvature. By the Toponogov comparison theorem ([2, §6.4.1, Theorem 73]) the length of \( v_1 v_2 \) is bounded above by the length of the third edge in an isosceles triangle with angle \( \varepsilon \) subtending the two edges of equal length \( \ell = v_0 v_1 = v_0 v_2 \) in the space form [2, §6.3.2] of constant curvature \( K \). This length goes to 0 as \( \varepsilon \) does, contradicting \( v_1 v_2 = \ell \).

Large regular simplices are not the only obstruction to loose embeddability. The somewhat more sophisticated configurations that pose problems involve, roughly speaking, large sets of points each equidistant to large sets of pairs of points. To make sense of this we need some terminology.

**Definition 3.3** Let \( n \) be a positive integer. An \( n \)-flag of median hyperplanes is a collection of points

\[
\{p_i, q_i, \ 0 \leq i \leq n - 1\}
\]

such that

\[
d(z, p_s) = d(z, q_s)
\]

for all \( z = p_i \) or \( q_i \) with \( i > s \).

The term ‘median hyperplane’ is meant to invoke the locus of points in a Euclidean space that are equidistant from two given points, while ‘flag’ means chain ordered by inclusion, as in

\[
\{p_i, q_i\}_{i\geq 0} \supset \{p_i, q_i\}_{i\geq 1} \supset \cdots
\]

The relevance of the concept stems from the following simple remark.

**Lemma 3.4** A compact metric space containing \( n \)-flags of median hyperplanes is not LE.

**Proof** If such a space \((X, d)\) were loosely embeddable in \( \mathbb{R}^d \) say, then each of the sets \((3-2)\) would be contained in a hyperplane of \( \mathbb{R}^d \), namely the median hyperplane of (the images in \( \mathbb{R}^d \) of) \( p_i \) and \( q_i \). These hyperplanes would be orthogonal in the sense that their range projections commute, so any \( >d \) of them would intersect trivially.

On the other hand, Riemannian manifolds can still not be discounted as LE on the basis of Lemma 3.4.

**Theorem 3.5** A compact Riemannian manifold \((X, d)\) equipped with the geodesic metric cannot contain \( n \)-flags of median hyperplanes for arbitrarily large \( n \).

This will require some amount of preparation.

First, we will have some make some size estimates (for angles, distances, etc.). This raises the usual issue of starting with quantities that are within \( \varepsilon > 0 \) of each other and then obtaining new estimates in terms of \( \varepsilon \) such as, say \( C\varepsilon \) for some constant \( C \). In order to avoid such irrelevancies we make the following
Convention 3.6 \( \varepsilon \) will typically denote a small positive real, and whenever a new small quantity depending on \( \varepsilon \) is introduced, we denote it by decorating \( \varepsilon \) with the usual symbols used to indicate differentiation. So for instance \( \varepsilon', \varepsilon'', \varepsilon^{(5)} \), etc. all denote small positive reals depending on \( \varepsilon \) in some unspecified fashion.

The same notational convention applies to other symbols meant to denote small positive reals.

In the discussion below we will modify the Riemannian tensor \( g \) on a geodesic ball of a Riemannian manifold \((M,g)\) so as to “flatten” said ball. The relevant concept is

Definition 3.7 Let \( B \subset M \) be a geodesic ball in a Riemannian manifold \( M \) with tensor \( g \), and suppose we have fixed a coordinate system for \( B \). We say that \( g \) is \( \varepsilon \)-Euclidean to order \( k \) along \( B \) if the derivatives of orders \( \leq k \) of \( g \) within \( \varepsilon \) of their usual Euclidean counterparts, uniformly on \( B \), in the respective coordinate system.

We typically omit \( k \) from the discussion, simply assuming it is large enough (\( k \geq 2 \) will do for most of our purposes); for that reason, we abbreviate the phrase as \( \varepsilon \)-Euclidean.

The specific \( \varepsilon > 0 \) will also depend on the chosen coordinates, but we ignore this issue too, as the discussion below will only require \( \varepsilon \) sufficiently small, and the various coordinate choices will not affect this.

As a consequence of the smooth dependence of ODE solutions on the initial data (e.g. [8, Theorem B.3]), “sufficiently Euclidean” Riemannian metrics in the sense of Definition 3.7 have “sufficiently straight” geodesics. More formally (keeping in mind Convention 3.6):

Proposition 3.8 Let \((M,g)\) be a Riemannian manifold, \( \varepsilon \)-Euclidean with respect to some coordinate system. Then, for every geodesic \( \gamma \) in \( M \), parallel transport of vectors along \( \gamma \) does not alter angles by more than \( \varepsilon' \)

Notation 3.9 Let \( M \) be a Riemannian manifold with metric tensor \( g \) and geodesic distance \( d \). We write

\[
\text{inj}(M) := \text{injectivity radius of } M
\]

([7, p.271] or [2, p.142, Definition 23]): the largest number such that all pairs of points less than \( \text{inj}(M) \) apart are joined by a unique geodesic segment.

For points \( p, q \) in a Riemannian manifold \( M \) with

\[
\ell := d(p, q) < \text{inj}(M)
\]

we write

\[
\gamma^q_p : [0, \ell] \to M
\]

for the geodesic arc from \( p \) to \( q \), parametrized by arclength. We will also abuse notation and denote the image of \( \gamma^q_p \) by the same symbol.

Definition 3.10 Let \( p, q \) be points in a Riemannian manifold \( M \), less than \( \text{inj}(M) \) apart. The angle

\[
\angle(v_p, v_q)
\]

between two tangent vectors \( v_p \in T_p M \) and \( v_q \in T_q M \) is defined by

- parallel-transporting ([7, Chapter 2, Proposition 2.6 and Definition 2.5] or [2, p.264, Proposition 61]) the unit velocity vector \( v_q \) to a vector \( v \in T_p M \) along \( \gamma^q_p \);
set \[ \angle(v_p,v_q) := \text{angle between } v_p \text{ and } v, \]
computed in \( T_p(M) \) as usual, via the Riemannian tensor.

For points \( p, q, p', q' \) in \( M \), each two less than \( \text{inj}(M) \) apart, the angle \( \angle(\gamma_{p'}^{q'}, \gamma_p^{q'}) \) is the angle (defined as above) between the unit velocity vectors \( (\gamma_{p'}^{q'})'(0) \) and \( (\gamma_p^{q'})'(0) \).

\[ \diamondsuit \]

Remark 3.11 Although Definition 3.10 appears to bias one of the pairs \( p,q \) and \( p',q' \) over the other, the notion is in fact symmetric: because parallel transport is an isometry between tangent spaces, whether we parallel-transport \( (\gamma_{p'}^{q'})'(0) \) to \( T_p M \) or \( (\gamma_p^{q'})'(0) \) to \( T_p'M \) does not affect the value of the angle.

\[ \diamondsuit \]

For an \( n \)-dimensional Riemannian metric space \( (M,d=d_M) \) with a basepoint \( z \in M \) we will consider small geodesic balls \( B_r = B_r(z) := \{ q \in M \mid d(z,q) \leq r \} \) centered at \( z \), parametrized with normal coordinates \([2, §4.4.1] x^i, 1 \leq i \leq n \) (so \( z \) is identified with the origin \( (0, \cdots, 0) \)). Recall that this means the geodesics emanating from \( z \) are identified with straight line segments.

Having fixed such a coordinate system, we can speak about segments in \( B \), angles between those segments, etc.; it will be clear from context when these are actual segments in the ambient \( \mathbb{R}^n \) housing \( B \) rather than, say, geodesic segments in \( M \).

Typically, the radius \( r \) decorating \( B_r \) will be small. We will occasionally have to normalize the Riemannian metric in \( B_r \), scaling distances from the origin \( z = 0 \in B \) by \( \frac{1}{r} \) so that the new ball \( nB_r \) (‘n’ for ‘normalized’) has radius 1.

This normalization procedure has the effect of “flattening” the Riemannian metric, in the sense that the Riemannian structure can be made arbitrarily \( \varepsilon \)-Euclidean (Definition 3.7) as \( r \to 0 \).

In the discussion below, for a Riemannian manifold \( M \) with geodesic metric \( d = d_M \), we write \[ \eta(p,q) = \eta_M(p,q) := d(x,y)^2 \] for the squared-distance function (the notation matches that in [16] for instance, where this function features prominently).

Lemma 3.12 Let \( M \) be a Riemannian manifold and \( B = B_r(z) \) a sufficiently small geodesic ball equipped with normal coordinates around \( z \in M \). Let also \( p \in B \) be a point and consider the function \[ \psi : x \mapsto \eta(x,p). \] with \( \eta \) as in (3-3). Denoting by \( v \in T_zM \) the unit vector tangent to the geodesic \( z \to p \), the gradient \( \nabla \psi \) at \( z \) equals \( -2d(z,p)v \).
Proof This is immediate after choosing a normal coordinate system around \( p \), whereupon \( \psi \) becomes
\[
\psi : (x^1, \ldots, x^n) \to \sum_{i=1}^{n} (x^i)^2.
\]
\[\blacksquare\]

Proof of Theorem 3.5 Suppose we do have arbitrarily large flags of median hyperplanes in our compact Riemannian space \((M,d)\). Since \( M \) is compact, we can assume that some large flag \((3-1)\) is contained entirely within some small geodesic ball \( B_r \) centered at a point \( z := p_n \) constituting the flag.

We can assume \( r \) is small enough that the normalized ball \( \frac{nB}{r} \) is \( \epsilon \)-Euclidean in the sense of Definition 3.7. Furthermore, because the size of the flag can also be chosen arbitrarily large, we can also assume that
\[
\angle (\gamma_{p_1}^q, \gamma_{q_0}^p) < \epsilon', \quad \forall 0 \leq i \neq j < n
\]
Henceforth, it will be enough to work with \( p_i \) and \( q_i \) for \( i = 0, 1 \). By the flag condition, both \( p_1 \) and \( q_1 \) are equidistant from \( p_0 \) and \( q_0 \). Additionally, we have
\[
\angle (\gamma_{p_1}^q, \gamma_{q_0}^p) < \epsilon'
\]
Furthermore, because the metric is \( \epsilon \)-Euclidean, the unit-length velocity vectors \( v_x \) along
\[
\gamma := \gamma_{p_1}^q
\]
stay within an angle of \( \epsilon'' \) of the initial velocity vector \((\gamma_{p_1}^q)'(0)\) (by Proposition 3.8), so \((3-4)\) implies that
\[
\angle (v_x, (\gamma_{q_0}^p)'(0)) < \epsilon''(3), \quad \forall x \in \gamma_{p_1}^q.
\]
For each \( x \in \gamma \), we saw in Lemma 3.12 that the gradient of the function
\[
\psi : x \mapsto d(x, p_0)^2 - d(x, q_0)^2
\]
is
\[
2d(x, q_0)(\gamma_{x}^p)'(0) - 2d(x, p_0)(\gamma_{x}^q)'(0).
\]
This is the parallel transport of \( 2p_0q_0 \) to \( x \) in the usual, Euclidean metric, so by our assumption that the original metric is \( \epsilon \)-Euclidean the angle between \((3-7)\) and \((\gamma_{p_0}^q)'(0)\) is \(< \epsilon''(4)\). To summarize, we have

- a small angle between each gradient \( \nabla_x \psi \) of \( \psi \) along \( \gamma \), given by \((3-7)\), and \((\gamma_{p_0}^q)'(0)\);
- a small angle between the latter and the unit tangent vectors \( v_x \) at \( x \in \gamma \), by \((3-5)\).

In particular, at each \( x \) along \( \gamma \) the gradient \( \nabla_x \psi \) and the velocity along \( \gamma \) have positive inner product. This means that the function \( \psi \) in \((3-6)\) increases strictly along the geodesic \( \gamma \), contradicting the fact that it must take the value 0 at both endpoints \( p_1 \) and \( q_1 \). \[\blacksquare\]
3.1 Positive results

Clearly, loose embeddability of a compact metric space in $\mathbb{R}^n$ entails covering dimension $\leq n$ (e.g. [9, §1.6] and [9, Theorem 3.1.19], which applies to compact metric spaces). On the other hand, we will prove that if the dimension is zero then loose embeddability holds “generically” with respect to the Gromov-Hausdorff distance.

**Theorem 3.13** The isometry classes of 0-dimensional compact metric spaces loosely embeddable in $\mathbb{R}$ is a residual set in $(\mathcal{M}_0, d_{GH})$.

We need some preparation.

**Definition 3.14** A distance function $d$ on a metric space $X$ is injective if its restriction to the off-diagonal set $$X \times X \setminus \Delta = \{(x, y) \in X \times X \mid x \neq y\}$$
is one-to-one.

First, note the following sufficient criterion for loose embeddability in $\mathbb{R}$.

**Proposition 3.15** A 0-dimensional compact metric space $(X, d) \in \mathcal{M}_0$ with injective $d$ is loosely embeddable in $\mathbb{R}$.

**Proof** The definition of loose embeddability simply requires a homeomorphism of $X$ onto a subset $Y \subset \mathbb{R}$ so that the usual real line distance $d_{\mathbb{R}}$ on $Y$ is injective. This will be a familiar clopen cover recursive “branching” procedure familiar in working with 0-dimensional compact spaces:

In first instance, cover $X$ with disjoint clopen sets $U_i$ of diameter $\leq 1$ and match them to mutually disjoint compact intervals $I_i \subset \mathbb{R}$ of length $\leq 1$. We arrange furthermore that the intervals $I_i$ are chosen generically, in the sense that if $x \in I_i$, $y \in I_j$, $i \neq j$

$$x' \in I_{i'}, y' \in I_{j'}, i' \neq j'$$

then

$$d_{\mathbb{R}}(x, y) = d_{\mathbb{R}}(x', y') \Rightarrow i = i' \text{ and } j = j'.$$

Next, cover each $U_i$ with finitely many disjoint clopen subsets $U_{ij}$ of diameter $\frac{1}{2}$ and choose corresponding disjoint compact sub-intervals $I_{ij} \subset I_i$ of length $\leq \frac{1}{2}$, again ensuring that for each $i$ the family consisting of all $I_{ij}$ is generic in the above sense. Now continue the procedure, partitioning each $U_{ij}$ into clopen subsets $U_{ijk}$, etc.

For each infinite word $ijk\cdots$ we obtain a point

$$\{p\} = U_i \cap U_{ij} \cap U_{ijk} \cap \cdots \subset X$$
mapped by our embedding to the corresponding unique point

$$I_i \cap I_{ij} \cap I_{ijk} \cap \cdots .$$

The generic condition imposed on our intervals at each step then ensures that indeed the restriction of $d_{\mathbb{R}}$ to the image of $X$ is injective in the sense of **Definition 3.14**. ■
Proof of Theorem 3.13 In view of Proposition 3.15, it will suffice to prove the stronger claim that the collection of \((X,d) \in \mathcal{M}_0\) with injective \(d\) is residual.

Let \(M\) and \(N\) be two positive integers and define \(\mathcal{M}_{N,M} \subset \mathcal{M}_0\) be the collection of 0-dimensional metric spaces \((X,d)\) admitting a partition into clopen subsets \(U_i\) such that

- each \(U_i\) has diameter \(< \frac{1}{N}\);
- whenever the two-element sets \(\{i,j\}\) and \(\{i',j'\}\) are distinct we have
  \[|d(x,y) - d(x',y')| > \frac{1}{M}\]

for all \(x \in U_i, y \in U_j\) and similarly for primed symbols.

\(\mathcal{M}_{N,M}\) is easily seen to be open in the Gromov-Hausdorff distance and the subset of \(\mathcal{M}_0\) consisting of injective-distance metric spaces is \(\bigcap_N \bigcup_M \mathcal{M}_{N,M}\). To conclude, we have to prove

**Claim:** \(\bigcup_M \mathcal{M}_{N,M}\) is dense in \(\mathcal{M}_0\). This, however, is immediate: simply approximate an arbitrary compact metric space in the Gromov-Hausdorff topology by finite metric spaces, which can be chosen to have injective distance functions by effecting small perturbations on said distance functions if needed.

3.2 Questions

Proposition 3.2 and theorem 3.5 appear to suggest that compact Riemannian metric spaces are particularly amenable to loose metric embeddability. I do not know whether they are always LE, but that problem decomposes naturally: first,

**Question 3.16** Let \((X,d)\) be a compact metric space and \(N \in \mathbb{Z}_{>0}\) a positive integer such that every finite subspace of \((X,d)\) is loosely embeddable into \(\mathbb{R}^N\). Does it follow that \((X,d)\) itself is LE?

In other words, does uniform loose embeddability for the finite subspaces of \((X,d)\) entail the LE property for \(X\) as a whole?

Secondly, to circle back to the Riemannian context:

**Question 3.17** Do compact Riemannian metric spaces satisfy the hypothesis of Question 3.16?

We conclude with a partial answer to Question 3.16. First, we need

**Definition 3.18** A metric space \((X,d_X)\) is weakly loosely embeddable (or weakly LE) in the metric space \((Y,d_Y)\) if there is an injective map \(f : X \to Y\) satisfying only the backwards implication of the LE condition (2-1):

\[d_Y(fx,fx') = d_Y(fz,fz') \leq d_X(x,x') = d_X(z,z').\]

**Theorem 3.19** Under the hypotheses of Question 3.16, a compact Riemannian metric space is weakly LE in \(\mathbb{R}^N\).

**Proof** Let \((M,d)\) be a compact Riemannian manifold with its geodesic metric and denote by \((\mathcal{F},\subseteq)\) the poset of finite subsets \(F \subset M\) (ordered by inclusion). For each \(F \in \mathcal{F}\) we fix a map \(\psi_F : F \to B := \text{origin-centered unit ball in } \mathbb{R}^N\)

such that
ψ_F is a loose embedding of (F,d), rescaled if needed so as to ensure it lands in the ball B;

the diameter of ψ_F(F) is precisely 1, with ψ_Fp = 0 and ψ_Fq on the unit sphere ∂B for some p,q ∈ F.

This gives us an F-indexed net [15, Chapter 3, p.187] ψ_F of maps F → B, and since

• B is compact;
• every element p ∈ M belongs to sufficiently large F ∈ F, i.e. to the upward-directed set
  \{F ∈ F | p ∈ F\},

we can take the pointwise limit

ψ(p) := lim_F ψ_F(p) ∈ B

to obtain a map ψ : M → B. it remains to prove that ψ

(a) satisfies the weak LE condition (3-8);
(b) is continuous;
(c) is one-to-one.

(a): condition (3-8). We want to prove that

|ψx − ψx'| = |ψz − ψz'| ⇐ d_M(x,x') = d_M(z,z')

holds; this follows by passing to the limit over F ∈ F in the analogous implication for the partially-defined maps ψ_F : F → B.

We can now define a map

φ : (set of distances d_M(p,q)) → R≥0

by

φ(d_M(p,q)) = |ψp − ψq|.

We define the maps φ_F, F ∈ F similarly, substituting ψ_F for ψ in (3-11).

(b): ψ is continuous. We have to argue that

lim_{d→0} φ(d) = 0.

If not, we can find a subnet (F_α) of F and points p_α,q_α ∈ F_α such that

d_M(p_α,q_α) → 0

but

ε := inf_α |ψ_{F_α}p_α − ψ_{F_α}q_α| > 0;

we abbreviate

ψ_α := ψ_{F_α},

and similarly for φ.
If $\ell > 0$ is sufficiently small (smaller than the injectivity radius of $M$, for instance $[\cdot,\cdot]$), then $(M,d_M)$ contains geodesic triangles with edges

$$\ell, \ell, t$$

for every $2\ell > t > 0$. This can easily be seen, for instance, by continuously decreasing the angle between two length-$\ell$ geodesic rays based at a point from $\pi$ to $0$; the distance between the extremities of those geodesic rays will then decrease continuously from $2\ell$ to $0$.

Now fix some $\ell > 0$, sufficiently small. We will have $d_M(p_\alpha,q_\alpha) < 2\ell$ for sufficiently large $\alpha$, and hence, by the preceding remark, we can find geodesic triangles in $M$ with edges $\ell, \ell$ and $d_M(p_\alpha,q_\alpha)$ (assuming also that $\alpha$ is large enough to ensure that $F_\alpha$ contains the tip of that isosceles geodesic triangle).

Applying $\psi_\alpha$, we have a triangle in $B$ with edges

$$\varphi_\alpha(\ell), \varphi_\alpha(\ell), \varphi_\alpha(d_M(p_\alpha,q_\alpha)).$$

In particular, we have

$$\varphi_\alpha(\ell) \geq \varphi_\alpha(d_M(p_\alpha,q_\alpha)) \geq \frac{\varepsilon}{2} > 0$$

by (3-12). Since $\ell > 0$ was arbitrary (so long as it was small enough), this means that by passing to large enough $\alpha$ we can find arbitrarily large finite subsets $F$ of $M$, of girth $\geq \ell$ (i.e. so that all pairs of points are at least $\ell$ apart), and hence so that (by (3-13))

$$|\psi p - \psi q| \geq \frac{\varepsilon}{2}, \forall p,q \in F.$$ 

Since the cardinality of $F$ (and hence that of $\psi(F)$) can be made arbitrarily large, we are contradicting the compactness of $B$. This completes the proof of (b) above.

(e): $\psi$ is injective. Suppose not. In a sense, this means we are in precisely the opposite situation to that encountered in the proof of part (b): there is a subnet $(F_\alpha)_\alpha$ of $F$ with points $p_\alpha, q_\alpha \in F_\alpha$ such that

$$\ell := \inf_\alpha d_M(p_\alpha,q_\alpha) > 0$$
$$\inf_\alpha |\psi_\alpha p_\alpha - \psi_\alpha q_\alpha| = 0.$$  

(3-14)

By compactness, we can also assume $p_\alpha$ and $q_\alpha$ are convergent and hence in particular that the distances

$$\ell_\alpha := d_M(p_\alpha,q_\alpha)$$

are as well. For sufficiently small $t > 0$ there are triangles in $M$ with edges

$$\ell_\alpha, \ell_\alpha, t$$

for all $\alpha$ (consider two geodesic rays of length $\ell_\alpha$ with common origin, subtending small angles at said origin). But then an application of one of the $\psi_\alpha$ will yield a triangle with edges

$$\varphi_\alpha(\ell_\alpha), \varphi_\alpha(\ell_\alpha), \varphi_\alpha(t)$$

with $\varphi$ and $\varphi_\alpha := \varphi_{F_\alpha}$ as in (3-11) and subsequent discussion, meaning that

$$\varphi_\alpha(t) \leq \frac{\varphi_\alpha(\ell_\alpha)}{2}.$$
Since the right hand side converges to zero by (3-14), we conclude that \( \varphi(t) = 0 \) for all sufficiently small \( t > 0 \). In other words,

\[
\psi \text{ identifies any two points that are sufficiently close. (3-15)}
\]

Now, for each \( \alpha \) we also have, by assumption, points \( x_\alpha, y_\alpha \) in \( F_\alpha \) that achieve distance 1 upon applying \( \psi_\alpha \):

\[
|\psi_\alpha x_\alpha - \psi_\alpha y_\alpha| = 1
\]

By compactness, passage to a subnet if necessary allows us to assume that \( x_\alpha \) and \( y_\alpha \) converge to \( x \) and \( y \) in \( M \) respectively, and limiting over \( \alpha \) produces \( |\psi x - \psi y| = 1 \).

For some distance \( t > 0 \) small enough to qualify for (3-15) we can find a broken geodesic consisting of some finite number \( N \) of length-\( t \) segments

\[
x =: p_0 \rightarrow p_1, \ p_1 \rightarrow p_2, \ldots, \ p_{N-1} \rightarrow p_N := y
\]

connecting \( x \) and \( y \). Applying \( \psi \) we similarly obtain a broken geodesic consisting of \( N \) length-\( \varphi(t) \) segments connecting \( x \) and \( y \), but the latter are distance 1 apart while \( \varphi(t) = 0 \) by (3-15). This gives the contradiction we seek and finishes the proof. ■

References

[1] Eiichi Abe. *Hopf algebras*, volume 74 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge-New York, 1980. Translated from the Japanese by Hisae Kinoshita and Hiroko Tanaka.

[2] Marcel Berger. *A panoramic view of Riemannian geometry*. Springer-Verlag, Berlin, 2003.

[3] Nathaniel P. Brown and Narutaka Ozawa. *\( C^* \)-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[4] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[5] Alexandru Chirvasitu. On quantum symmetries of compact metric spaces. *J. Geom. Phys.*, 94:141–157, 2015.

[6] M. Deza and H. Maehara. Metric transforms and Euclidean embeddings. *Trans. Amer. Math. Soc.*, 317(2):661–671, 1990.

[7] Manfredo Perdigão do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[8] J. J. Duistermaat and J. A. C. Kolk. *Lie groups*. Universitext. Springer-Verlag, Berlin, 2000.

[9] Ryszard Engelking. *Dimension theory*. North-Holland Publishing Co., Amsterdam-Oxford-New York; PWN—Polish Scientific Publishers, Warsaw, 1978. Translated from the Polish and revised by the author, North-Holland Mathematical Library, 19.
[10] Debashish Goswami. Existence and examples of quantum isometry groups for a class of compact metric spaces. *Adv. Math.*, 280:340–359, 2015.

[11] Huichi Huang. Invariant subsets under compact quantum group actions. *J. Noncommut. Geom.*, 10(2):447–469, 2016.

[12] Johan Kustermans and Lars Tuset. A survey of $C^*$-algebraic quantum groups. I. *Irish Math. Soc. Bull.*, (43):8–63, 1999.

[13] Ann Maes and Alfons Van Daele. Notes on compact quantum groups. *Nieuw Arch. Wisk. (4)*, 16(1-2):73–112, 1998.

[14] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.

[15] James R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition of [ MR0464128].

[16] Liviu I. Nicolaescu. Random morse functions and spectral geometry, 2012. arXiv:1209.0639.

[17] David E. Radford. *Hopf algebras*, volume 49 of *Series on Knots and Everything*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[18] Moss E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

[19] S. L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.*, 111(4):613–665, 1987.

[20] S. L. Woronowicz. Compact quantum groups. In *Symétries quantiques (Les Houches, 1995)*, pages 845–884. North-Holland, Amsterdam, 1998.

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