Instability of the printing jet during the three-dimensional-printing process

Yuting Zuo¹,² and Hongjun Liu¹,²

Abstract
Euler’s instability criterion is widely used in engineering to design a column. However, this criterion is not suitable for judging the instability of a three-dimensional printing process because the axial motion of the printing jet has to be considered. A variational principle is established, and an equivalent Eulerian load is obtained. The theoretical results show that a higher printing velocity makes the moving jet much more stable, and an experiment is designed to verify our theoretical prediction.

Keywords
Bending energy, electrospinning, electrospraying, printing, variational theory

Introduction
The three-dimensional-printing technology is now playing an important role in materials science, constructional engineering, and food engineering.¹⁻⁵ It can print micro/nano devices, e.g. micro-electromechanical systems⁶ and Fangzhu-like devices for collecting water from air.⁷⁻⁹ An accurate printed object requires an exact sprinting process, and any small instability is not allowed. However, when a column subjects to a buckling load, an instability occurs when the axial pressure reaches the Eulerian load

\[ P_{cr} = \frac{\pi^2 EI}{L^2} \]  

where \( EI \) is the rigidity of the column, \( E \) is the elastic modulus, and \( I \) is section moment of inertia, \( L \) is the length, and \( P_{cr} \) is Eulerian load, see Figure 1. For a cylindrical column, the moment of inertia is

\[ I = \frac{\pi d^4}{64} \]  

and equation (1) becomes

\[ P_{cr} = \frac{\pi^3 Ed^4}{64L^2} \]  

When \( P > P_{cr} \), the column becomes instability, and a low frequency vibration might occur.

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In the three-dimensional-printing process\textsuperscript{1–5} (see Figure 2), an instability often occurs for the slender axially moving jet, and the instability always leads to morphology change of the printed object. The instability also occurs in some spinning process, see, for example, the following works\textsuperscript{10–17} In this paper, the moving jet is approximated as an axially moving cylindrical rod-like viscoelastic fluid, and its Eulerian load is studied.

**Newton’s law and variational principle**

The Newton’s second law is widely used to establish a differential model. Consider the pendulum motion as illustrated in Figure 3, the Newton’s law leads to the following equation

\[
mg\sin\theta = -m\ddot{\theta}l
\]

or

\[
\ddot{\theta} + \frac{g}{l}\sin\theta = 0
\]

where \( L \) is the pendulum’s length.

The governing equation of the pendulum can be also derived from the variational principle, which is

\[
J(\theta) = \int [K - P]dt = \int \left\{ \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta) \right\} dt
\]
Figure 2. Extrusion-based 3D printing.

Figure 3. The pendulum.
where $K$ is the kinetic energy and $P$ is the potential energy. The stationary condition of equation (6) is

$$
\delta J(\theta) = \delta \int_{\theta_1}^{\theta_2} \left\{ \frac{1}{2} m f^2 \dot{\theta}^2 - m g l (1 - \cos \theta) \right\} dt
$$

$$
= \int_{\theta_1}^{\theta_2} \left\{ m f^2 \dot{\theta} \delta \theta - m g l \sin \theta \delta \theta \right\} dt
$$

$$
= \int_{\theta_1}^{\theta_2} \left\{ m f^2 \frac{d}{dt}(\dot{\theta} \delta \theta) - \frac{d(m f^2 \dot{\theta})}{dt} \delta \theta - m g l \sin \theta \delta \theta \right\} dt
$$

$$
= (m f^2 \dot{\theta} \delta \theta) \bigg|_{\theta_1}^{\theta_2} - \int_{\theta_1}^{\theta_2} \left\{ m f^2 \dot{\theta} + m g l \sin \theta \right\} \delta \theta dt = 0
$$

(7)

For any arbitrary $\delta \theta$, from equation (7), we obtain

$$
mf^2 \ddot{\theta} + mg l \sin \theta = 0
$$

(8)

Equation (8) can also be obtained directly from the Euler–Lagrange equation, which reads

$$
\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 0
$$

(9)

where $L$ is the Lagrange function

$$
L = \frac{1}{2} mf^2 \dot{\theta}^2 - m g l (1 - \cos \theta)
$$

(10)

Both Newton’s law and the variational principle can be used to establish a governing equation for simple problems, but the latter is more suitable for a complex problem.

**Instability of the printing jet**

Figure 4 shows the printing process, the printing velocity is $u_1$, and the platform moves at a velocity of $u_2$. According to literature,18–22 the printing jet has the following bending energy similar to an elastic column

$$
E = \frac{1}{2} EI \left( \frac{d w}{dx} \right)^2
$$

(11)

where $w$ is the displacement. The jet moves axially, and its kinetic energy is

$$
K = -\frac{1}{2} \rho A \left( \frac{D w}{Dt} \right)^2
$$

(12)

where $\rho$ is density, $A$ is the sectional area, and $D/Dt$ is the material derivative, and it is defined as

$$
\frac{D w}{Dt} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x}
$$

(13)
where \( u \) is the velocity of the printing jet. For a steady printing process, we have
\[
\frac{Dw}{Dt} = u \frac{\partial w}{\partial x}
\]  
(14)

The variational principle\(^{22-28}\) for the bending jet is
\[
J(w) = \int_0^L \left( E + K - \frac{1}{2} P_{cr} w^2 \right) dx
\]  
(15)

or
\[
J(w) = \int_0^L \left\{ \frac{1}{2} EI \left( \frac{dw}{dx} \right)^2 + \frac{1}{2} \rho A u^2 \left( \frac{dw}{dx} \right)^2 - \frac{1}{2} P_{cr} w^2 \right\} dx
\]  
(16)

The stationary condition of equation (16) is
\[
\frac{\partial L}{\partial w} - \frac{d}{dx} \left( \frac{\partial L}{\partial w_x} \right) = 0
\]  
(17)

where the Lagrange function is given as follows
\[
L = \frac{1}{2} EI \left( \frac{dw}{dx} \right)^2 + \frac{1}{2} \rho A u^2 \left( \frac{dw}{dx} \right)^2 - \frac{1}{2} P_{cr} w^2
\]  
(18)

It is obvious that
\[
\frac{\partial L}{\partial w} = - P_{cr} w, \quad \frac{\partial L}{\partial w_x} = EI \frac{dw}{dx} + \rho A u^2 \frac{dw}{dx}
\]  
(19)
According to equation (17), the governing equation is

$$\frac{d}{dx} \left[ (EI + \rho Au^2) \frac{dw}{dx} \right] + P_{crw} = 0$$

(20)

We assume that the velocity at the nozzle is $u_1$ and the receptor’s moving velocity is $u_2$, and the jet velocity changes gradually from $u_1$ to $u_2$, so the jet velocity can be written as

$$u(x) = u_1 + \frac{u_2 - u_1}{L} x$$

(21)

In view of equation (21), equation (20) becomes

$$\left( EI + \rho A \left( u_1 + \frac{u_2 - u_1}{L} x \right)^2 \right) \frac{d^2 w}{dx^2} + 2\rho \frac{u_2 - u_1}{L} \left( u_1 + \frac{u_2 - u_1}{L} x \right) \frac{dw}{dx} + P_{crw} = 0$$

(22)

with the following boundary conditions

$$w(0) = 0, \quad \frac{dw}{dx} (L) = u_2$$

(23)

In practical applications, we always assume

$$u_1 = u_2 = u$$

(24)

If $u_1 \gg u_2$, the printed jet will be piled up, and if $u_1 \ll u_2$, the jet will be broken. The bending equation becomes

$$(EI + \rho Au^2) \frac{d^2 w}{dx^2} + P_{crw} = 0$$

(25)

We introduce an equivalent rigidity

$$(EI)_{eq} = EI + \rho Au^2$$

(26)

and equation (25) becomes

$$(EI)_{eq} \frac{d^2 w}{dx^2} + P_{crw} = 0$$

(27)

This is exactly same with the Euler equation, so the critical Eulerian load reads

$$P_{cr} = \frac{\pi^2 (EI)_{eq}}{L^2} = \frac{\pi^2 (EI + \rho Au^2)}{L^2} = \frac{\pi^2 (Ed^4 + 16\rho \pi d^2 u^2)}{64L^2}$$

(28)

Equation (28) predicts that a larger nozzle diameter, or a higher printing velocity, or a shorter printing distance between the nozzle and the receptor leads to a more stable printing process.

**Experimental verification**

SiC/graphene composites can be printed by the 3D-printing technology. In our experiment, SiC paste was prepared with 54.8 wt.% SiC particles, 4 wt.% TMAH, 0.8 wt.% PEG1500, 3.2 wt.% glycerol, 6.4 wt.% carrageenan, 30.6 wt.% water, and 0.2 wt.% graphene.
Figure 5. Effect of the printing distance on the printing instability with $u = 6$ mm/s and $d = 0.86$ mm: (a) $L = 1$ mm and (b) $L = 5$ mm.

Figure 6. Velocity-induced stability, $L = 1$ mm, $d = 0.86$ mm: (a) $u = 2$ mm/s and (b) $u = 4$ mm/s.

Figure 7. Effect of the nozzle diameter on morphology of the printed object, $u = 4$ mm/s, $L = mm$: (a) $d = 0.5$ mm and (b) $d = 0.86$ mm.

Figure 5 shows the effect of the printing distance between the nozzle and the receptor on the printed objects. A shorter distance always leads to a stable printing process, see Figure 5(a), while a longer distance results in instability (see Figure 5(b)).

The printing velocity also affects the printing instability, and a higher velocity leads to a more stable printing process (see Figure 6).
As shown in equation (28), the nozzle diameter also affects the printing instability. Figure 7 shows the experimental results for the different nozzle diameters, which agree with our theoretical prediction.

**Discussion and conclusion**

For the first time ever, this paper suggests an instability criterion for axially moving jet of a 3D-printing process. A longer printing distance always results in a more unstable printing process, see Figure 5, while a higher velocity always makes the moving jet much more stable, see Figure 6, and we call this phenomenon as the motion stability. Due to extremely low elastic modulus of the most printing materials, it is necessary to increase the printing velocity to make the printing process stable. However, due to the solvent evaporation is incomplete for a fast printing process, the printed object is easy to be deformed. We will discuss the problem in a forthcoming paper by establishing a fractal vibration model.\(^3\),\(^30\),\(^31\) The instability of the printing jet will also affect the mechanical and electrical properties of graphene/sic composites.\(^2\)

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