TURBULENT ENERGY TRANSPORT IN NONRADIATIVE ACCRETION FLOWS

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Received 2003 June 9; accepted 2003 September 20

ABSTRACT

Just as correlations between fluctuating radial and azimuthal velocities produce a coherent stress contributing to the angular momentum transport in turbulent accretion disks, correlations in the velocity and temperature fluctuations produce a coherent energy flux. This nonadvective thermal energy flux is always of secondary importance in thin radiative disks, but cannot be neglected in nonradiative flows, in which it completes the mean field description of turbulence. It is nevertheless generally ignored in accretion flow theory, with the exception of models explicitly driven by thermal convection, for which it is modeled phenomenologically. This flux embodies both turbulent thermal convection and wave transport, and its presence is essential for a proper formulation of energy conservation, whether or not thermal convection is present or not. The sign of the thermal flux is likely to be outward in real systems, but the restrictive assumptions used in numerical simulations may lead to inward thermal transport, in which case qualitatively new effects may be exhibited. We find, for example, that a static solution would require inward, not outward, thermal transport. Even if it were present, thermal convection would be unlikely to stifle accretion but would simply add to the outward rotational energy flux that must already be present.

Subject headings: accretion, accretion disks — black hole physics — instabilities — MHD — turbulence

1. INTRODUCTION

The fate of gas in the vicinity of a black hole remains a topic of intense astrophysical interest. The classical motivation for black hole accretion is, of course, the possibility of prodigious energy output (Lynden-Bell 1969), but in more recent years attention has also been drawn to a class of flows that radiate inefficiently (Abramowicz et al. 1988; Narayan & Yi 1995; Narayan, Igumenshchev, & Abramowicz 2000). Since the unambiguous Chandra finding that the Galactic Center source Sgr A* is severely underluminous (Baganoff et al. 2001), these models have now acquired a compelling observational motivation.

Analytic models of nonradiating accretion flows (hereafter NRAFs) are generally based on the classical $\alpha$ prescription, whereby all transport is subsumed into a single enhanced viscosity parameter. NRAFs are very likely to be turbulent, of course, because the combination of differential rotation and a magnetic field is prone to the magnetorotational instability (MRI, Balbus & Hawley 1991). The simplifying assumption that has almost universally been made is that the sole dynamical effect of this turbulence is to act as the enlarged $\alpha$ viscosity parameter. In earlier NRAF models, this assumption applied both to energy as well as to angular momentum transport (Narayan & Yi 1994; Blandford & Begelman 1999).

More recently, a class of accretion flow has appeared that specifically invokes thermal convection as the dominant mode of energy transport, while at the same time arguing that the angular momentum stress tensor effectively vanishes (Narayan et al. 2000).

The purpose of this paper is to elucidate, from first principles, the nature of turbulent transport couplings in an NRAF. The formalism that allows a connection to be made between phenomenological transport modeling and the fundamental fluid equations is weak-turbulence theory. This is a less restrictive limitation than it may sound. It is equivalent to retaining the quadratic correlations in the turbulent fluctuations and is a mathematically self-consistent approach. Phenomenological models of turbulent accretion flows certainly do not attempt higher order accuracy, and often settle for far less.

Applications of this type of formalism to nonradiative flows were carried out by Quataert & Narayan (1999) and to black hole accretion more generally by Kato, Fukue, & Mineshige (1998). However, in its emphasis on modes of transport rather than difficult questions of closure, our approach more closely follows that of Balbus & Papaloizou (1999). These authors used weak turbulence theory to elucidate the foundations of $\alpha$ modeling in thin radiative disks and found that whereas MHD processes naturally lend themselves to such a description, self-gravity generally does not. A related finding of this study was that the only correlation tensor of dynamical importance that emerges in an MHD turbulent Keplerian disk is the Maxwell-Reynolds stress. This fact, in turn, can be traced to the assumption that the ratio of the isothermal sound speed $c_s$ to the Keplerian rotation velocity $v_K$ satisfies $c_s/v_K \ll 1$. When this assumption no longer holds, other transport coefficients become dynamically significant (as happens in self-gravitating disks), in which case the simple viscous model itself breaks down.

The regime $c_s/v_K \sim 1$ coincides with the NRAF regime. We are led, therefore, to consider flows in which the energy extracted from the differential rotation is, in addition to being locally dissipated, nonlocally transported by other correlations associated with a turbulent or wavelike energy flux. This form of transport has the potential to be important in any NRAF, whether or not thermal convection is associated with the accretion flow. Its role is quite distinct from that of the dynamical stress tensor, and the two should not be confused.

An outline of this work is as follows. Section 2 presents a simple discussion of correlated fluctuations and mean flow, in particular emphasizing the circumstances in which fluctuations, not just mean flow values, must be retained in the governing equations. Section 3 focuses on turbulent energy transport. The key result is equation (15), which is the fundamental thermodynamic relation for the fluctuations. Section 4 develops simple, one-dimensional solutions to the governing
equations and contrasts them with solutions from the literature. Finally, §5 is a summary of the paper.

2. CORRELATED FLUCTUATIONS AND MEAN FLOW

As stated in §1, our presentation is based on weak turbulence theory, which bridges the divide between phenomenological and first-principle approaches to turbulent flow. Any flow quantity may be decomposed into the sum of a mean value plus a fluctuation with zero mean. We use the notation $\langle X \rangle$ for the mean value of quantity $X$ and $\delta X$ to denote its fluctuating component. For the flow velocity, weak turbulence means that the fluctuations are small compared to the isothermal sound speed $c_s$. For the pressure and density, the fluctuations are to be considered small compared with the mean value. The magnetic field $B$ is assumed to be subthermal, i.e., the Alfvén speed $v_A \ll c_s$. In what follows, we shall not need to distinguish between the mean and fluctuating components of $B$.

Mean velocity components may be very large or very small compared with their rms fluctuations. The rotational velocity will generally greatly exceed its azimuthal velocity fluctuation, but the mean radial drift velocity, a consequence of turbulent flow, has a leading asymptotic form, which is critical to the accretion process itself. Standard treatments in effect retain these correlations by inventing an “anomalous viscosity.” To the extent that the theory is independent of the functional form of the correlations, the fact that at least something is present in the equation where the turbulent stress should appear is sufficient for some purposes. It allows points of principle to be illustrated (e.g., the initial spreading and subsequent concentration of an evolving disk), as well as some robust results to be obtained (e.g., the luminosity-accretion rate relation [Pringle 1981]).

Genuine difficulties with the standard treatment emerge when one attempts to go beyond rotational mechanics. Consider the entropy equation

$$\frac{P}{\gamma - 1} \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \ln P_{\gamma} = Q^+ - Q^-, \quad (4)$$

where $Q^+$ and $Q^-$ are respectively the volume specific entropy gains and losses. The tacit assumption that is almost always made is that the flow quantities appearing in equation (4) may be replaced by their mean values. This is incorrect.

The problem is that the mean radial velocity is a second-order quantity, and the entire entropy equation is therefore also of second order. The quantity $P_{\gamma}$ is not, on average, simply the product of the mean pressure and radial drift velocity, but (in one dimension)

$$\langle P_{\gamma} \rangle = \langle \rho v_R \rangle = \langle \theta \rangle \langle \rho v_R \rangle + \langle \rho \rangle \langle \delta v_R \delta \theta \rangle + \text{higher order terms}, \quad (5)$$

where $\theta \equiv c_s^2$ is the temperature variable. In fact, the full entropy equation is even more complex, because there are correlations between $\delta v_R$ and the fluctuation of the entropy gradient term itself that must be taken into account. There is no a priori reason that all of these correlation coefficients should vanish. Clearly, care must be taken when determining quantities such as the sign of the mean entropy gradient in a turbulent accretion flow. A more systematic approach is needed.

3. TURBULENT TRANSPORT IN NRAFs

3.1. Second-Order Conservative Transport Equations

To investigate the behavior of a model flow with a turbulent thermal energy flux in more detail, we begin with the time-steady form of the angular momentum conservation equation:

$$\nabla \cdot \left( \rho R v_{\phi} \frac{v^2 - R}{4\pi} B \right) = 0, \quad (6)$$

and the energy conservation equation,

$$\nabla \cdot \left[ \rho v \left( \frac{v^2}{2} + \Phi + \frac{\gamma \theta}{\gamma - 1} \right) + \frac{1}{4\pi} B \times (v \times B) \right] = -Q^-. \quad (7)$$

We have neglected the contribution of the particle viscosity and (potentially more seriously) the thermal conductivity. Throughout the remainder of this paper, all equations are
understood to be azimuthally averaged. Unsubscripted bold-face vectors and vector operators should be regarded as poloidal.

We next expand all quantities into mean plus fluctuating components. The largest terms in the energy flux are second order in the fluctuating amplitudes. (Recall that we are assuming that the Alfvén speed is of the same order as a kinetic velocity fluctuation.) The mass flux is

$$\langle \rho \mathbf{v} \rangle = \langle \rho \rangle \langle \mathbf{v} \rangle + \langle \delta \rho \delta \mathbf{v} \rangle,$$

and satisfies

$$\nabla \cdot \langle \rho \mathbf{v} \rangle = 0. \quad (9)$$

If we now define the stress vector $\mathbf{W}$ to be

$$\mathbf{W} = \left\langle \delta \mathbf{v} \cdot \delta \mathbf{v} - \frac{B \cdot B}{4 \pi \rho} \right\rangle,$$  

then the angular momentum equation becomes

$$\nabla \cdot \left( \langle \rho \mathbf{v} \rangle R^2 \Omega + \langle \rho \rangle \mathbf{R} \mathbf{W} \right) = 0. \quad (11)$$

The energy equation, through second order in weak turbulence theory, reads

$$\nabla \cdot \left[ \langle \rho \mathbf{v} \rangle \left( \frac{R^2 \Omega^2}{2} + \Phi + \frac{\gamma \langle \theta \rangle}{\gamma - 1} \right) \right. 
+ \left. \langle \rho \rangle \mathbf{R} \Omega \mathbf{W} + \frac{\gamma}{\gamma - 1} \langle \rho \rangle \langle \delta \rho \delta \mathbf{v} \rangle \right] = -Q^- \quad (12)$$

The first group of terms multiplying the mass flux $\langle \rho \mathbf{v} \rangle$ corresponds to the Bernoulli constant in polytropic spherical flow. In general, of course, it need not be constant. The key component of the energy flux that distinguishes transport in a nonradiative flow from a thin Keplerian disk involves only one new correlation product: the thermal energy flux proportional to $\langle \delta \rho \delta \mathbf{v} \rangle$. This term is generally ignored in $\alpha$ models of nonradiative accretion flows. Instead, these models retain a form of the energy flux appropriate for a radiative thin disk in which the dominant nonadvective energy flux comes from the rotational transport term $\mathbf{W}$. In general, the ratio of the thermal energy flux to rotational stress will be, as noted, of order $c_s/R \Omega$. While this justifies neglect of the thermal energy flux in thin-disk models, it also shows that this correlation must be retained in any flow with comparable rotation and sound speed. Indeed, retention of the energy flux terms is essential to formulating a thermodynamically self-consistent model for the energetics of turbulent fluctuations.

3.2. Fluctuation Thermodynamics

The result of combining mass conservation, equation (9), with radial hydrostatic equilibrium, equation (2), in the energy conservation equation is

$$\frac{\langle \rho \mathbf{v} \rangle}{2R^2} \frac{d(R^4 \Omega^2)}{dR} + \frac{\langle \theta \rangle}{\gamma - 1} \langle \rho \mathbf{v} \rangle \cdot \nabla S$$

$$+ \nabla \cdot \left( \langle \rho \rangle \mathbf{R} \Omega \mathbf{W} + \frac{\gamma}{\gamma - 1} \langle \rho \rangle \langle \delta \rho \delta \mathbf{v} \rangle \right) = -Q^- \quad (13)$$

where

$$S \equiv \ln \left( \langle P \rangle \langle \rho \rangle^{-\gamma} \right). \quad (14)$$

If we now combine this equation with angular momentum conservation, equation (11), our result simplifies to

$$\nabla \cdot \left( \frac{\gamma}{\gamma - 1} \langle \rho \rangle \langle \delta \rho \delta \mathbf{v} \rangle \right) + \frac{\langle \theta \rangle}{\gamma - 1} \langle \rho \mathbf{v} \rangle \cdot \nabla S$$

$$= - \left( Q^- + \langle \rho \rangle W_{\Omega} \frac{d \Omega}{d \ln R} \right). \quad (15)$$

Equation (15), which is the internal energy equations for the fluctuations, is a key theoretical result of this paper. It is readily interpreted. The right-hand side is the rate at which fluctuations exchange energy with their “surroundings.” The first term $(Q^-)$ represents bulk radiative losses, and the final term is the rate at which energy is supplied by the free energy reservoir of differential rotation. In a classical thin Keplerian disk, these terms together comprise the dominant energy balance, and the entire left side of the equation may be ignored.

In a nonradiative flow, however, none of the free energy of differential rotation escapes, but it is instead routed through two possible thermodynamic paths. The first possibility, embodied by the divergence term on the left-hand side of equation (15), is direct mechanical transport of the thermal energy content of a fluctuation, including simple advection plus $P \ dV$ work. The form of the flux is

$$\frac{\gamma}{\gamma - 1} \langle \rho \rangle \langle \delta \rho \delta \mathbf{v} \rangle = \langle \rho \rangle C_P \langle \delta T \delta \mathbf{v} \rangle,$$

where $C_P$ is the mass specific heat capacity at constant pressure (cf. Schwarzschild 1958). In the case of an adiabatic sound wave, this expression is equivalent to $\langle \delta P \delta \mathbf{v} \rangle$, precisely the formal rate of work done by the wave. The balance set by equating this term with the final term on the right-hand side of equation (15) corresponds to wave action conservation for adiabatic disturbances (Lighthill 1978).

The second term on the left-hand side of equation (15) is the average rate at which entropy is generated by the mean mass flow. It is the only other thermodynamic channel available to the free energy of differential rotation; if the energy is not carried off by adiabatic processes, then it must be dissipated. In turn, the dissipated energy must, in the absence of radiative losses, drive either an inflow or outflow. Dynamical considerations strongly favor an outflow, at least in the outer flow regions, leading to the scenario envisaged by Blandford & Begelman (1999). To the extent that bulk flow does not occur, a thermal flux must carry the energy away. This term does not appear in the Blandford & Begelman formulation, so this form of transport is precluded in their analysis.

Finally, it should be noted that equation (15) differs somewhat from the internal equation (4) used in Abramowicz et al. (2002) and elsewhere by the same authors. This is because the fluctuation formalism is not used by Abramowicz et al. (2002), so the internal energy input is regarded as a source of particle heating and is therefore dissipative. By contrast, in this work we have granted the fluid the degrees of freedom associated with fluctuations, which allows the free energy of differential rotation to be extracted with or without irreversible dissipation.
3.3. Thermal Convection Versus Thermal Flux

The discussion of the previous section invites comparison with the convection-dominated flows put forth by Narayan et al. (2000). While the dynamical description of these models has been criticized (Balbus & Hawley 2002; see Narayan et al. 2002 for a reply), one should not dismiss the possibility that a thermal energy flux might affect the flow structure under more general circumstances. The point goes beyond the presence or absence of thermal convection and allows us to sharpen the very notion of what it means for a flow to be "convection-dominated."

In an accretion flow whose primary source is the free energy of differential rotation, what does it mean to say that the turbulence is dominated by thermal convection? The unstable linear mode in question is always the same slow-mode branch of the MHD dispersion relation regardless of the mixture of adverse entropy and thermal gradients that may be present. (Recent claims to the contrary in the literature reflect confusion on this point.) If the primary instability is the MRI, which in its nonlinear resolution allows accretion to proceed, and any adverse entropy gradients are present only as a secondary consequence of accretion, in what sense can the flow ever be dominated by convection?

As originally formulated (Narayan et al. 2000; Abramowicz et al. 2002), the role of convection was truly dominant; it was argued that it could cause the stress tensor to vanish and that it could eliminate turbulent dissipation throughout the body of the accretion flow. However, this leads to serious difficulties (Balbus & Hawley 2002). The vanishing of the radial component of $W$ (more precisely a density-weighted average thereof, denoted $T_{\phi\phi}$) leaves the fluctuations without a local source of free energy. Source-free, dissipation-free local turbulence maintained by large dissipative entropy gradients is a thermodynamically dubious proposition. It is therefore not surprising that global, three-dimensional MHD accretion simulations (Hawley, Balbus, & Stone 2001; Hawley & Balbus 2002; Igumenshchev, Narayan, & Abramowicz 2003) all report robust positive values of $T_{\phi\phi}$.

More recently, claims that a simulated accretion flow is dominated by convection have shifted to morphological grounds (Igumenshchev et al. 2003) with only the visual appearance of the flow used as a basis for an identification. This is not the diagnostic of choice. Whether or not thermal convection is of significance in a high temperature NRAF simulation is in fact a well-posed quantitative problem. One must first begin by treating energy and angular momentum transport on the same footing by elevating the energy correlation flux $\langle \delta \theta \delta \nu \rangle$ to the same status as $T_{\phi\phi}$. Second, controlled studies are essential. The visual appearance of turbulence is not a very reliable guide to the presence or absence of thermal convection, especially if the root cause of the turbulence is a magnetorotational process. Simulations must be run with and without a dissipative heat source, and both turbulent correlation coefficients examined in detail for each case. This has in fact already been done for $T_{\phi\phi}$; Stone & Pringle (2001) and Hawley & Balbus (2002) both found very little difference in the behavior of the stress tensor whether or not resistive heating was present in their simulations. The turbulent energy flux has yet to undergo a similar level of scrutiny, however. The question is, how much does this thermal flux correlation change relative to its value in the absence of a dissipative heat source? What matters ultimately is whether the thermal energy flux is an efficient drain for the free energy of differential rotation in any nonradiative flow, convective or otherwise.

4. ONE-DIMENSIONAL SOLUTIONS

4.1. Formulation of the Problem

To better understand the physical consequences of a thermal energy flux in an NRAF, we apply the formal results of § 3 to some explicit, simple accretion solutions. To do this, we need explicit functional forms for $W$ and $\langle \delta \theta \delta \nu \rangle$. Since our goal is one of illustrating a point of principle, we use the dimensional analysis associated with $\alpha$ formalism for each of these fluxes. That is,

$$\rho \delta \nu \delta \nu = \frac{h R \delta \nu}{4 \pi} = \alpha_{SS} (P),$$

$$\langle \rho \delta \nu \delta \theta \theta \rangle = \alpha_T (P) (c_s),$$

where $\alpha_{SS}$ (the Shakura-Sunyaev $\alpha$ parameter) and $\alpha_T$ are dimensionless constants. Note that $\alpha_{SS} > 0$, but we leave the sign of $\alpha_T$ unconstrained for the moment. The dynamical stress must be positive in order to extract energy from the differential rotation, whereas the thermal flux is a secondary consequence of the turbulence and could, in principle, have either sign, depending on what mode of transport dominates. This is particularly important when one uses this flux as a numerical diagnostic, because the restrictions sometimes used (e.g., two dimensions, adiabatic gas law) can bias the direction of the energy flux.

Next, we take the rather drastic but standard step of height-integrating the equations of motion and assume that what remains is a well-defined, one-dimensional (on average) radial flow (Narayan & Yi 1994; Blandford & Begelman 1999). It is useful to have such a formal solution to compare with others in the literature. Numerical NRAF simulations do show a well-defined thick wedge forming along the midplane, surrounded by a low-density coronal envelope and possessing a relatively autonomous interior (Hawley & Balbus 2002). There is some sense in regarding much of the wedge structure as one-dimensional, although there is also a significant amount of high-latitude backflow that is lost in this approximation. Perhaps the best reason to study these one-dimensional flows is to illustrate a point of principle when they fail to work self-consistently, in contrast, say, to steady viscous disk models.

Our one-dimensional formulation is similar to the “idealized ADAF” analyzed by Blandford & Begelman (1999), with an added thermal energy flux. The equations of motion, which can now be expressed in terms of mean value quantities, are

$$\Sigma R \dot{\Omega}^2 = \frac{d}{dR} \left( \frac{dP}{dR} \right) \frac{dZ}{dR} + \frac{GM\Sigma}{R^2},$$

$$- \frac{m \Omega}{2\pi} + \alpha_{SS} \frac{P}{R^2},$$

$$- \frac{m}{2\pi} \left( \frac{R^2 \dot{\Omega}^2}{2} - \frac{GM}{R} + \frac{\gamma - 1}{\gamma - 1} \frac{P}{\Sigma} \right)$$

$$+ R^2 \Omega \alpha_{SS} \frac{P}{\gamma - 1} \alpha_T \frac{P_{\alpha}^{3/2}}{\Sigma^{3/2}} = C_E.$$  

These are, respectively, the radial equation of motion and the angular momentum and energy conservation equations, where
\( \dot{m} \) is the conserved mass accretion rate, \( C_J \) is the conserved rate of angular momentum transport, \( C_E \) is the conserved energy transport rate, \( \Sigma \) is the height-integrated density (i.e., the column density), and \( \mathcal{P} \) is the height-integrated pressure. We have assumed a central black hole mass \( M \), but, as is customary, have used a gravitational potential depending only on \( R \). An alternative useful form for equation (19) is obtained by eliminating the \( \alpha_{SS} \) term via equation (18):

\[
\frac{\dot{m}}{2\pi} \left( \frac{R^3 \Omega^2}{2} + \frac{GM}{R} - \frac{\gamma}{\gamma - 1} \frac{\mathcal{P}}{\Sigma} \right) + \frac{R^7}{\gamma - 1} \frac{\alpha_T}{\alpha_{SS}} \frac{P_{31/2}}{\Sigma^{3/2}} = C_E - \Omega C_J. \tag{20}
\]

### 4.2. Static Envelope

We begin with a search for solutions with \( \dot{m} = 0 \). Combining equations (17) with (18) gives

\[
\mathcal{P} = \frac{C_J}{\alpha_{SS} R^2}, \quad \langle \mathcal{P} \rangle \propto R^{-3}, \tag{21}
\]

and

\[
\Omega^2 = \frac{GM}{R^3} - \frac{3C_J}{\alpha_{SS} \Sigma R^4}. \tag{22}
\]

The radial power law dependence of the pressure suggests we look for similar behavior in \( \Omega \) and \( \Sigma \). This is possible only if \( C_E = 0 \), so that the net energy flux vanishes. Then equation (20) leads directly to

\[
\Omega = \frac{\gamma}{\gamma - 1} \frac{\alpha_T}{\alpha_{SS}} \frac{C_J^{1/2}}{\Sigma^{1/2} R^2}. \tag{23}
\]

Evidently, it is necessary that \( \alpha_T < 0 \), i.e., the heat transport must be inward to ensure there is no energy deposition. Eliminating \( \Omega \) between the last two equations leads to an equation for \( \Sigma \),

\[
\Sigma = \frac{3C_J}{\alpha_{SS} GMR} \left( 1 + \xi \right), \quad \xi \equiv \frac{\alpha_T \gamma^2}{3 \alpha_{SS} (\gamma - 1)^2}, \tag{24}
\]

and \( \Omega \) follows from equation (23),

\[
\Omega^2 = \frac{\xi}{1 + \xi} \frac{GM}{R^3}. \tag{25}
\]

This is a sub-Keplerian profile and is associated with thick-disk structure (Frank, King, & Raine 2002). The temperature is given by

\[
\langle \theta \rangle = \frac{\mathcal{P}}{\Sigma} = \frac{GM}{3R} \left( 1 + \xi \right)^{-1}. \tag{26}
\]

Equations (24), (25), and (26) comprise our one-dimensional static solution. Note that it satisfies the virial equilibrium condition

\[
2T + \Phi + 3\langle \theta \rangle = 0, \tag{27}
\]

where \( T = R^2 \Omega^2 / 2 \) is the rotational kinetic energy, and that it differs from the static convection dominated solutions of Narayan et al. (2000) and Abramowicz et al. (2002) in requiring an inward, rather than an outward, thermal flux. The cause of this difference is that our solution has a positive value of the \( \partial \phi \) stress tensor component, as is required by energy conservation.

### 4.3. Accreting Envelope

Equations (17), (18), and (19) allow for power-law solutions even when \( \dot{m} \) does not vanish, a property shared with NRAFs in their standard formulation (Narayan & Yi 1994; Blandford & Begelman 1999). We require the vanishing of both the energy as well as the angular momentum flux, \( C_E = C_J = 0 \). Since \( W_{R0} \) is no longer directly proportional to \( C_J \), as in the static solution of the previous section, there is no difficulty with simultaneously demanding a positive stress and a vanishing net angular momentum flux. This condition is satisfied in classical Keplerian disks at radii that are large compared with the inner edge.

With

\[
\mathcal{P} \propto R^{-3/2}, \quad \Sigma \propto R^{-1/2}, \quad \Omega \propto R^{-3/2}, \tag{28}
\]

equation (17) becomes

\[
\langle \theta \rangle = \frac{\mathcal{P}}{\Sigma} = \frac{2}{5} \frac{GM}{R} \left( \frac{GM}{R} - R^2 \Omega^2 \right). \tag{29}
\]

and, after some manipulations, the energy equation (19) may be written

\[
\frac{R^2 \Omega^2}{2} \left( \frac{9\gamma}{5} - 1 \right) - \frac{GM}{R} \left( 1 - \frac{3\gamma}{5} \right) + \frac{\gamma \alpha_T}{\alpha_{SS}} \left( \frac{2R^2 \Omega^2}{5} \right)^{1/2} \left( \frac{GM}{R} - R^2 \Omega^2 \right)^{1/2} = 0. \tag{30}
\]

Defining

\[
A = \frac{9\gamma}{5} - 1, \quad B = 1 - \frac{3\gamma}{5},
\]

\[
C = \frac{\alpha_T}{\alpha_{SS}} \left( \frac{2}{5} \right)^{1/2}, \quad x^2 = \frac{R^2 \Omega^2}{GM}, \tag{31}
\]

we find that \( x \) must satisfy

\[
x^4 (C^2 + A^2/4) - x^2 (C^2 + AB) + B^2 = 0. \tag{32}
\]

The choice of sign for this quadratic (in \( x^2 \)) equation must be chosen to be consistent with equation (30), since a spurious root was introduced in the course of squaring a radical.

It is the appearance of \( C \) that distinguishes our solution from those of earlier NRAF studies. If \( C > 0 \), corresponding to outward transport, then it must be large compared with \( A \) to change the \( C = 0 \) solution \( x^2 = 2B/A \) qualitatively. (The novel \( B = 0 \) solution of equation [32] is spurious when \( C > 0 \).) On the other hand, if \( C < 0 \), corresponding to inward transport, a qualitative change does occur: whereas a standard
one-dimensional NRAF has no solution for the important case $\gamma = 5/3$ (i.e., $B = 0$), the presence of an inward energy flux leads to a new sub-Keplerian solution

$$x^2 = (1 + A^2/4C^2)^{-1}, \quad \theta = (2/5)(GM/R)(1 + 4C^2/A^2)^{-1}. \tag{33}$$

An inward thermal flux seems anomalous. Considering also its novel consequences for both static and accreting flows, we are thus led to consider the physical processes leading to either inward or outward transport.

4.4. Discussion

It is perhaps surprising that, in these one-dimensional models, inward thermal transport is required to establish either a static profile or a $\gamma = 5/3$ accretion profile. However, given a power-law leading asymptotic order behavior for the thermal and rotational energy fluxes at large $R$, there is little choice. Energy must be strictly conserved. Therefore, either the two fluxes must cancel one other, or their sum must be conserved. Because it forces a unique power-law behavior, the latter possibility overconstrains the problem. Hence the rotational energy flux, which must always be outward, is canceled by an equal and opposite thermal flux.

Second, equation (15), together with static or isentropic flow, implies a very frugal use of free energy: essentially all of it must go into an inward thermal flux. What might be the physical basis for such behavior?

The MRI, acting in the presence of a Schwarzschild-stable entropy gradient, will tend to drive a thermal energy flux down the entropy gradient, which in this case is indeed inwards (Balbus 2000). For the static solution, equations (21) and (24) give

$$S \sim \ln R^{2\gamma-3}$$

for the static solution, which is Schwarzschild-stable for $\gamma > 1.5$. On the other hand, for the $\gamma = 5/3$ accreting solution, $S$ is constant with $R$.

However, there are deeper problems with the mechanism of turbulent mixing, even for the $\gamma > 1.5$ static solution. The inward energy transport is produced by an intrinsically dissipative process, and the entropy of a fluid element would continuously increase. Without radiative losses, a static solution would not persist under these circumstances. This is reflected in equation (15), in which it can be seen that the dissipative term is coupled both to mass flow as well as to a finite entropy gradient.

Might it be possible to continuously extract the free energy of differential rotation and have it go directly into non-dissipative energy transport? Certainly. This is what waves do in the process of conserving their action, and nondissipative wave propagation does not cause systematic mass flow in the background medium. However, in our problem, any waves that are present will be coupled to dissipative turbulence.

This raises another issue. In general, the linear behavior of a rotating magnetized fluid is marked by two incompressible modes. At a fixed wavelength and arbitrarily weak field strengths, one branch is a destabilized slow mode (the MRI) and the other is generally a stable inertial wave (assuming that entropy gradient is dominated by its vertical component), which becomes Alfvénic as the field strengthens. Nonlinear coupling between the unstable and stable modes is almost certain to be present in a turbulent fluid.

Hydrodynamical inertial waves in disks are characterized by frequencies $\omega$ below the epicyclic value (Vishniac & Diamond 1989; Balbus 2003), which in a Keplerian disk is simply $\Omega$. Hence axisymmetric waves propagate inwards, because any initially outward propagating wave at fixed $\omega$ will eventually encounter a turning point at which $\omega$ exceeds $\kappa$, and the wave must reflect. Two-dimensional numerical simulations of the MRI would then be characterized by $\alpha_{SS} > 0$ (outward transport by rotational stress) and, to the extent that inertial forces exceed those of magnetic tension, $\alpha_T < 0$. The MRI dominates the angular momentum transport (axisymmetric inertial waves carry no angular momentum of course) but perhaps not the energy transport. A preliminary numerical investigation of a global, two-dimensional, self-gravitating, adiabatic, magnetized torus does indeed show comparable values of $\alpha_{SS}$ and $|\alpha_T|$ with $\alpha_T < 0$ (S. Fromang 2003, private communication).

In three dimensions, the situation is significantly more complex, because inertial waves are not free to propagate inward without reflection. Rather than $\omega < \kappa$, the condition for propagation is

$$|\omega - m\Omega| < \kappa,$$

where $m$ is the azimuthal wave number. The radius at which equality in the above relation is obtained is known as the Lindblad resonance, and normally there are two such locations, the outer and inner Lindblad resonances, where $\omega - m\Omega$ is respectively positive or negative. The existence of an inner Lindblad resonance is a consequence of non-axisymmetry, and the resulting trapping of any inertial waves undermines the arguments used in the axisymmetric case. One can always appeal to the magnetic field to get past the Lindblad barriers, but there is no guarantee that waves liberated in this way will be heading preferentially inwards.

The case for inward thermal energy transport is even more problematic in real black hole systems. Coulomb conduction, hitherto ignored, is likely to be an important process in the dilute plasmas that characterize nonradiative black hole accretion. The relevant instability criterion for a plasma whose thermal conductivity is dominated by a magnetized Coulomb conductivity is that the temperature, not the entropy, increase outward (Balbus 2001). The combined magnetorotational/magnetothermal instability would result in outward, not inward, thermal transport. A static solution under these conditions is not possible.

Both acoustic waves and turbulence associated with an unstable thermal gradient will tend to move thermal energy outward with high efficiency, since in both cases the correlation between the velocity and temperature fluctuation is strong. In the case of acoustic waves, a net outward flux is expected, since initially inward-propagating waves will tend to refract and head outwards. Outwardly propagating waves, on the other hand, will become increasingly dominated by their radial wavenumber and eventually dissipate or shock. This would seem more likely to result in a wind from the outer regions of the flow, rather than a static or accreting halo.

To conclude, in a height-integrated one-dimensional analytic formalism, inward transport of thermal energy is needed for a static solution or for $\gamma = 5/3$ rotating accretion to exist. Conditions favorable to inward transport include axisymmetric and adiabatic flow, which are restrictive assumptions likely to
Three-dimensional flow and the mixing of magnetothermal with magnetorotational instabilities in the presence of a Coulomb conductivity all tend to produce an outward energy flux. In the presence of a positive rotational stress, outward thermal transport does not stifle accretion. It does, however, lead to accretion flows whose salient properties cannot be captured by one-dimensional height-integrated modeling.

5. SUMMARY

The key point of this paper is that thermal fluctuations must be taken into account in the formulation of governing equations of nonradiative accretion flows and, in particular, that the entropy equation cannot be analyzed directly by assuming all flow quantities take on their mean values. The energy flux

$$
\frac{\gamma}{\gamma - 1} \langle \rho \delta \theta \delta v \rangle
$$

(34)

is an important component in any nonradiative accretion process. Energy conservation, the central assumption of nonradiative flows, is not satisfied unless this term is present; such basic processes as wave action conservation are lost. It is important here to note that the presence of this correlation flux is more fundamental than the mere inclusion of some additional transport process, such as thermal conduction. The issue is one of thermodynamic self-consistency. The energy flux, equation (34), must accompany the advected flux term

$$
\frac{\gamma \langle \theta \delta v \rangle}{\gamma - 1} \langle \rho \delta v \rangle
$$

(35)

for the same reasons and at the same level of formalism that the stress tensor $W$ accompanies the advected angular momentum flux. The results are summarized quantitatively in equation (15), which expresses energy conservation for either wavelike or turbulent fluctuations.

With one exception, no other accretion model familiar to the author includes a nonadvective thermal energy flux. As noted in § 3.1, convection-dominated models (Narayan et al. 2000; Quataert & Gruzinov 2000; Abramowicz et al. 2002) assume that the energetics are dominated by an outward energy flux generated by a convectively unstable accretion process. The central assumption of these models is that convection inhibits angular momentum transport and causes the turbulent stress to vanish. We have argued, on the other hand, that the stress tensor $W_{ij}$ must be positive in order to extract free energy from differential rotation. One consequence of this is that any viable, height-integrated static solution must have an inward flux of thermal energy, rather than an outward flux, as expected in a convectively unstable flow. (The same inward flux requirement holds for the existence of steady, height-integrated. $\gamma = 5/3$ rotating accretion flows.) It therefore seems unlikely that thermal convection, even if it were present, would stifle accretion. Instead, it would simply add to the outward rotational energy transport while negligibly affecting the angular momentum transport (Stone & Pringle 2001; Hawley & Balbus 2002).

Establishing how the thermal flux emerges in a mean flow formalism for weak turbulence is a theoretical result that we believe will foster a deeper understanding of nonradiative accretion, if for no other reason than that it avoids the pitfalls of confusing flow quantities with their mean values. However, the immediate utility of the present work is likely to be as a numerical diagnostic. This is particularly true for those cases in which thermal convection is claimed to be present. In contrast to the Reynolds-Maxwell stress tensor, the energy correlation $\langle \delta v \delta \theta \rangle$ has not been well studied. It is particularly important to know how much of the free energy is ultimately “radiated” by this underexplored form of mechanical luminosity.

It is a pleasure to thank J. Hawley and an anonymous referee for detailed and constructive comments on an earlier draft of this paper. I am also grateful to G. Dubus, S. Fromang, and J.-P. Lasota for stimulating conversations and suggestions and to the Institut d’Astrophysique de Paris, whose hospitality and support allowed much of this work to be completed. This work was supported by NASA grants NAG5-9266 and NAG5-13288, and NAG-10655.

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