A PASTICHE ON EMBEDDINGS INTO SIMPLE GROUPS  
(FOLLOWING P. E. SCHUPP)

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Abstract. Let \( \lambda \) be an infinite cardinal number and let \( \mathcal{C} = \{ H_i \mid i \in I \} \) be a family of nontrivial groups. Assume that \( 2 \leq |I| \leq \lambda \), \( |H_i| \leq \lambda \), for \( i \in I \), and at least one member of \( \mathcal{C} \) achieves the cardinality \( \lambda \).

We show that there exists a simple group \( S \) of cardinality \( \lambda \) that contains an isomorphic copy of each member of \( \mathcal{C} \) and, for all \( H_i, H_j \) in \( \mathcal{C} \) with \( |H_j| = \lambda \), is generated by the copies of \( H_i \) and \( H_j \) in \( S \).

This generalizes a result of Paul E. Schupp (moreover, our proof follows the same approach based on small cancelation). In the countable case, we partially recover a much deeper embedding result of Alexander Yu. Ol’shanskii.

1. Background and results

In [Sch76] Schupp used small cancelation theory (construction of Adian-Rabin type) to prove, among other things, the following result.

**Theorem S** (Schupp [Sch76]). Let \( G, H \) and \( K \) be nontrivial groups with \( |G| \leq |H*K| \) and \( |K| \geq 3 \). There exists a simple group \( S \) that contains an isomorphic copy of \( G \) and is generated by isomorphic copies of \( H \) and \( K \).

**Corollary S.** Let \( G \) be a countable group. For all \( p, q \in \{2, 3, \ldots\} \cup \{\infty\} \) with \( q \geq 3 \), there exists a simple group \( S \) that contains an isomorphic copy of \( G \) and is generated by a pair of elements of order \( p \) and \( q \), respectively.

The simple group constructed by Schupp in Theorem S, in addition to being dependent on \( G \), depends on \( H \) and \( K \). Accordingly, the simple group in Corollary S, in addition to being dependent on \( G \), depends on the pair \( (p, q) \).

We will show that the argument used by Schupp can be adapted in such a way that the same simple group can be used even if one considerably varies \( H \) and \( K \) in Theorem S and, consequently, the same simple group can be used independently of the pair \( (p, q) \) in Corollary S.

**Theorem A.** Let \( |I| \geq 2 \) and \( \mathcal{C} = \{ H_i \mid i \in I \} \) be a countable family of countable nontrivial groups, at least one of which has at least 3 elements (the groups may be isomorphic for different values of the index).

There exists a 2-generated simple group \( S \) that contains an isomorphic copy of each member of \( \mathcal{C} \) and, for all \( H_i, H_j \) in \( \mathcal{C} \) with \( |H_j| \geq 3 \), is generated by the copies of \( H_i \) and \( H_j \) in \( S \).

2000 Mathematics Subject Classification. 20F06, 20E32.

Key words and phrases. simple groups, embeddings, small cancelation.

Partially supported by NSF grant DMS-0600975.
Corollary A. Let $G$ be a countable group.

There exists a simple group $S$ that contains an isomorphic copy of $G$ and, for all $p, q \in \{2, 3, \ldots\} \cup \{\infty\}$ with $q \geq 3$, is generated by a pair of elements of order $p$ and $q$, respectively.

Moreover, if $|G| \geq 3$, then, for every $p \in \{2, 3, \ldots\} \cup \{\infty\}$, the simple group $S$ is generated by $G$ and an element of order $p$.

In this note, countable means finite or countably infinite. The countability limitations imposed in Theorem A are natural since every countable group contains only countably many finitely generated subgroups. An extension of Theorem S in which countability assumptions are not used follows.

Theorem B. Let $\lambda$ be an infinite cardinal number and let $\mathcal{C} = \{H_i \mid i \in I\}$ be a family of nontrivial groups. Assume that $2 \leq |I| \leq \lambda$, $|H_i| \leq \lambda$, for $i \in I$, and at least one member of $\mathcal{C}$ achieves the cardinality $\lambda$.

There exists a simple group $S$ of cardinality $\lambda$ that contains an isomorphic copy of each member of $\mathcal{C}$ and, for all $H_i$, $H'_i$, in $\mathcal{C}$ with $|H'_i| = \lambda$, is generated by the copies of $H_i$ and $H'_i$ in $S$.

Corollary B. For any group $G$ with $|G| \geq 3$, there exists a simple group $S$ that contains an isomorphic copy of $G$ and, for every $p \in \{2, 3, \ldots\} \cup \{\infty\}$, is generated by $G$ and a single element of order $p$.

In the countable case, the embedding results of Schupp were eventually subsumed by the following result of Ol’shanskiǐ (this result also subsumes our Theorem A, but not Theorem B).

Theorem O (Ol’shanskiǐ [Ol’89]). Let $|I| \geq 2$ and $\mathcal{C} = \{H_i \mid i \in I\}$ be a countable family of countable nontrivial groups.

There exists a 2-generated simple group $S$ that contains an isomorphic copy of each member of $\mathcal{C}$ and, moreover, has the following properties (in what follows, the copy of $H_i$ in $S$ is denoted by $H_i$).

1. If $i, j \in I$, $i \neq j$, $|H_j| \geq 3$, then $S$ is generated by $H_i$ and $H_j$.
2. If $i, j \in I$, $i \neq j$, then $H_i \cap H_j = 1$.
3. Every element of finite order in $S$ is conjugate to an element in $H_i$, for some $i \in I$.
4. Every proper subgroup of $S$ is either infinite cyclic, or infinite dihedral, or it conjugate of a subgroup of $H_i$, for some $i \in I$.
5. If, for some $i \in I$, $x \in H_i$, $x \neq 1$, $y \notin H_i$, then either $S$ is generated by $\{x, y\}$ or both $x$ and $y$ are involutions, or both $x$ and $xy$ are involutions,
6. If $i, j \in I$, $i \neq j$, then $H_i \cap H_j^x = 1$, for every element $x$ in $S$.
7. For every $i \in I$, $H_i$ is malnormal in $S$ (for every $x \in S \setminus H_i$, $H_i \cap H_i^x = 1$).

Thus there is a natural trade off in our approach. We extend Theorem S of Schupp (by adapting his approach using small cancelation theory) to arbitrary families of groups in a way that, in the countable case, partially recovers Theorem O of Ol’shanskiǐ. A modest gain is achieved by the fact that the taken approach allows us to handle families of groups that are not necessarily countable. On the other hand, in the countable case, we recover only a small subset of the conclusions that are obtained by the more powerful (but also more onerous) graded diagram methods introduced by Olshanskiǐ.
2. Proofs and additional comments

Proof of Theorem A. Reindex the family $C$ (if necessary) so that it is indexed by an initial segment $I$ of the set of natural numbers $\mathbb{N} = \{0,1,2\ldots\}$ (including the possibility $I = \mathbb{N}$, if $I$ is infinite). Moreover, in case the cyclic group $C_2$ of order 2 is a member of $C$ set $H_0 = C_2$ and make sure that this is the only copy of $C_2$ in $C$.

For each $i \in I$, embed $H_i$ into a 2-generated simple group $S_i = \langle s_i,t_i \rangle$ (this can be done by Theorem S) and consider the free product $F = A * B * \langle *_{i \in I} S_i \rangle$, where $A = \langle a \mid a^2 \rangle = C_2$, $B = \langle b \mid b^3 \rangle = C_3$.

For each index $i \in I$ define the words
\[
    u_i = (ab)^{(2i+1)n+n} (ab^{-1}) (ab)^{(2i+1)n+n-1} (ab^{-1}) \ldots (ab^{-1}) (ab)^{(2i+1)n+1} s_i,
\]
\[
    v_i = (ab)^{(2i+2)n+n} (ab^{-1}) (ab)^{(2i+2)n+n-1} (ab^{-1}) \ldots (ab^{-1}) (ab)^{(2i+2)n+1} t_i,
\]
where $n$ is a positive integer to be specified at a later stage.

Choose a nontrivial element $h_0$ in $H_0$ and, for each $i > 0$, choose a pair of distinct nontrivial elements $h_i$ and $h_i$ in $H_i$. For each pair of indices $i, j \in I$ with $0 \leq i < j$, define the words

\[
    w_{(a,i,j)} = (h_i h_j)^n (h_i h_j)^{n-1} (h_i h_j) \ldots (h_i h_j) (h_i h_j)^1 a
\]
\[
    w_{(b,i,j)} = (h_i h_j)^n (h_i h_j)^{2n-1} (h_i h_j) \ldots (h_i h_j) (h_i h_j)^{n+1} b.
\]

Let $R$ be the set of words obtained by symmetrization (closure under inversion and conjugation; see Remark 1 for a precise definition) of the set of words

\[
    R' = \{ w_{(a,i,j)}, w_{(b,i,j)} \mid i, j \in I, \ 0 \leq i < j \} \cup \{ u_i, v_i, (h_i a)^n, (h_i b)^n \mid i \in I \}
\]
and let $H = \langle F \mid R \rangle$.

Choose $n$ that is relatively prime to 6 and is sufficiently large to ensure that the set of words $R$ satisfies the small cancelation condition $C'(1/6)$ over the free product $F = A * B * \langle *_{i \in I} S_i \rangle$ (see Remark 1 for a definition of the small cancelation condition over free products). It follows, by a result of Lyndon [Lyn66, Theorem IV] (see [LS01, Section V.9] for an exposition), that all factors in the free product $F$ are embedded in $H = \langle F \mid R \rangle$.

The $u$ relations and the $v$ relations ensure that $H$ is generated by $a$ and $b$. On the other hand, the $w$ relations ensure that $H$ is generated by $H_i$ and $H_j$ for any $i, j \in I$ with $0 \leq i < j$.

Let $M$ be a maximal normal subgroup of $H$ and let $S = H/M$. The group $S$ is simple by the maximality of $M$. We claim that all the factors $S_i$, $i \in I$, are still embedded in $S$. The factor $S_i$, being simple, either intersects $M$ trivially or is contained in $M$. In the former case, the factor $S_i$ is still embedded in $S = H/M$. The latter case implies that $h_i = 1$ in $S$. Because of the relators $(h_i a)^n$ and $(h_i b)^n$, it follows that $a^n = b^n = 1$ in $S$. However, $n$ is chosen to be relatively prime to 6. Thus $a = b = 1$ in $S$, which means that $S$ is trivial, a contradiction.

This completes the proof. $\square$

We note here the crucial role of the embeddings $H_i \hookrightarrow S_i$ in the course of the proof. On one hand, the number of generators needed for each factor in $*_{i \in I} S_i$ is uniformized. This is notationally convenient, but not crucial. More significant is the simplicity of the factors $S_i$, which, helped by the relators $(h_i a)^n$ and $(h_i b)^n$, “protects” the embedded subgroups $H_i$ from “crashing” when $M$ is factored out from $H$. 

Proof of Corollary A. Apply Theorem A to $C = \{H_i \mid i \geq 1\}$, where $H_0 = C_2$, $H_1 = G$, and $H_{2i-4} = H_{2i-3} = C_i$, for $i \geq 3$ ($C_m$ denotes the cyclic group of order $m$).

Proof of Theorem B. Let $J$ be an indexing set of cardinality $\lambda$. For each $i \in I$, embed $H_i$ into a simple group $S_i = \{s_{i,j} \mid j \in J\}$. The cardinality of the simple group $S$ and the generating system of $S_i$ can be chosen to be equal to $\lambda = |J|$ by Theorem S. Consider the free product $F = A * B * (\ast_{i \in I} S_i)$, where $A = \langle a \mid a^2 \rangle = C_2$, $B = \ast_{j \in J} \langle b_j \mid b_j^3 \rangle = \ast_{j \in J} C_3$.

Let $\alpha : I \times J \to J$ be an injective map (such a map exists since $|I| \leq |J|$ and $|J|$ is infinite).

For each pair $(i,j) \in I \times J$, define the word
\[
u_{i,j} = (ab_{\alpha(i,j)})^n (ab_{\alpha(i,j)})^{-1} \cdots (ab_{\alpha(i,j)})^{-1} s_{i,j},
\]
where $n$ is a positive integer to be specified at a later stage.

For each $i \in I$, choose a nontrivial element $h_i$ in $H_i$. For each $i' \in I$ such that $|H_i| = \lambda$, choose a nontrivial element $h_{i'}$ in $H_{i'}$ different from $h_i$ and distinct nontrivial elements $h_{i',j}$, $j \in J$, $h_{i',j}$, $j \in J$, in $H_{i'}$ that are also different from $h_{i'}$ and $h_{i'}$. Let $L \subseteq I \times I$ be a set of pairs such that, for each pair of indices $i, i' \in I$ such that $|H_i| < |H_{i'}| = \lambda$, the ordered pair $(i, i')$ is in $L$, and, for each pair of indices $i, i' \in I$ such that $i \neq i'$ and $|H_i| = |H_{i'}| = \lambda$, exactly one of the ordered pairs $(i, i')$ and $(i', i)$ is in $L$ (if more than one member of $C$ has cardinality $\lambda$, there are many possible choices for $L$ and we select one; the set $L$ must be nonempty because $|I| \geq 2$). For every pair $(i, i')$ in $L$, define the words
\[w_{(i,i')}(a) = (h_i h_{i'})^n (h_i h_{i'})^{-1} \cdots (h_i h_{i'})^{-1} a\]
\[w_{(i,i')} = (h_i h_{i'})^{-1} (h_i h_{i'})^{-1} \cdots (h_i h_{i'})^{-1} (h_i h_{i'})^{-1} b_j, j \in J.
\]

For all $i \in I$ and $j \in J$ also define the words
\[(h_i a)^n, (h_i b_j)^n.
\]

The group $H = F/R$ is defined as before ($R$ is obtained by symmetrization of the set of all words defined so far and $n$ is chosen to be relatively prime to 6 and to be sufficiently large to yield the cancelation condition $C'(1/6)$ over the free product $F = A * B * (\ast_{i \in I} S_i)$). The group $S = H/M$, where $M$ is a maximal normal subgroup of $H$ satisfies the required conditions. \hfill \Box

Remark 1. We recall here the definition of the small cancelation property $C'(1/6)$ over free products and specify a value for $n$ in the proof of Theorem A that ensures that this condition is satisfied.

Let $G = \ast_{i \in I} G_i$ be a free product of a nonempty family of nontrivial groups $G_i$, $i \in I$ (here $I$ is an arbitrary nonempty indexing set). The free product $G$ is generated by the set $\Sigma = \cup_{i \in I} G_i \setminus \{1\}$ of nontrivial elements in the (disjoint) union of the factors of $G$. A word $g_1 g_2 \ldots g_k$ over $\Sigma$ is reduced if, for $\ell = 1, \ldots, k-1$, $g_\ell$ and $g_{\ell+1}$ come from a different factor of $G$. Every element $g$ in $G$ can be represented by a unique reduced word over $\Sigma$, called the normal form of $g$. By definition, the length of an element in $G$ is equal to the length of its normal form. A reduced word $g_1 g_2 \ldots g_k$ over $\Sigma$ is weakly cyclically reduced if $k \leq 1$ or $g_k g_1 \neq 1$ (thus $g_k$ and $g_1$ may come from the same factor of $G$, but may not cancel). Let $u = u_1 \ldots u_k$ and $v = v_1 \ldots v_m$ be two reduced words over $\Sigma$. If $k = 0$ or $m = 0$ or $u_k v_1 \neq 1$, we say that the product $uv$ is semi-reduced (thus $u_k$ and $v_1$ may come from the same
factor of $G$, but may not cancel). A set $R$ of words over $\Sigma$ is called symmetrized if it consists of weakly cyclically reduced words, it is closed for inversion, and, for every $r$ in $R$, all weakly cyclically reduced words over $\Sigma$ representing conjugates of $r$ are in $R$. A nonempty reduced word $p$ over $\Sigma$ is a piece in the symmetrized set $R$ if there exist two reduced words $q_1$ and $q_2$ and two distinct words $r_1$ and $r_2$ in $R$ such that $r_1 = pq_1$, $r_2 = pq_2$ in $G$ and the products $pq_1$ and $pq_2$ are semi-reduced. In this case we say that $p$ is a piece in $r_1$ (and in $r_2$). Note that $p$ does not have to be a subword of $r_1$ to be a piece in it. A symmetrized set $R$ of words over $\Sigma$ satisfies the small cancelation property $C'(1/6)$ over the free product $G$ if, every word in $R$ has length greater than $6$ and, for every piece $p$, every reduced word $q$, and every word $r$ in $R$ such that $r = pq$ in $G$ and the product $pq$ is semi-reduced, the inequality $|p| < \frac{1}{6}|r|$ holds.

We now go back to our concrete setup from the proof of Theorem A. The lengths of the words in the set of relators $R'$ are

$$|w_0| = 4n^2i + 3n^2 + 3n - 1, \quad |w_{(a,i,j)}| = n^2 + 3n - 1, \quad |(h_i a)^n| = 2n,$$

$$|v_i| = 4n^2i + 5n^2 + 3n - 1, \quad |w_{(b,i,j)}| = 3n^2 + 3n - 1, \quad |(h_i b)^n| = 2n.$$ 

The reduced word $p = (h_i h_j)^n (h_i h_j)(h_i h_j)^n$ of length $4n + 2$ is a piece in $w_{(a,i,j)}$, since it is a subword of $w_{(b,i,j)}$ (and whence a prefix in a cyclic conjugate of $w_{(b,i,j)}$) and since $u_{(a,i,j)}$ can be written as a semi-reduced product $pq$ (for an appropriately chosen reduced word $q$ with first letter $h_j^{-1} h_j$). It can be easily verified that $w_{(a,i,j)}$ does not have a piece longer than $p$ (although it has other pieces of the same length).

Thus $n$ needs to be selected in such a way that

$$\frac{4n + 2}{n^2 + 3n - 1} < \frac{1}{6}.$$ 

This is true for any $n \geq 22$, but since we require $n$ to be relatively prime to $6$, the smallest good choice is $n = 23$.

We may now fix $n = 23$, consider all other pieces of words, and verify that the $C'(1/6)$ condition is satisfied.

For instance, the word $(ab)^{(2i+1)n+n} (ab^{-1})(ab)^{(2i+1)n+n}$ of length $8ni + 8n + 2$ is a piece of $u_i$ (and this word does not have any longer pieces). Thus we need to verify that

$$\frac{8ni + 8n + 2}{4n^2i + 3n^2 + 3n - 1} < \frac{1}{6},$$

for all $i \geq 0$. Think of the fraction on the left as a function of $i$. Since $8n(3n^2 + 3n - 1) - (8n + 2) \cdot 4n^2 < 0$, this function is decreasing for $i \geq 0$, the maximum is achieved at $i = 0$, and its value is $(8n + 2)/(3n^2 + 3n - 1) = 186/1655 < 1/6$.

We can equally easily verify all other cases. Thus we may take $n = 23$.

Remark 2. Consider again the proof of Theorem A. We used the original work of Schupp not only to model our approach, but also to embed each group $H_i$ into a simple 2-generated group $S_i$ (in order to protect $H_i$ in the quotient $S = H/M$). In turn, in his proof of Theorem S, Schupp uses embeddings of $G$, $H$, and $K$ into countable simple groups. At about the same time Schupp proved his result, Goryushkin also proved that every countable group can be embedded into a 2-generated simple group [Gor74]. Before the results of Schupp and Goryushkin, it was known from the work of P. Hall that every countable group can be embedded into a 3-generated simple group [Hal74, Theorem C2]. However, both Hall and Goryushkin also base their
proofs on the existence of embeddings of countable groups into countable simple groups. Thus to get back on some firm footing one could perhaps go back directly to the classical embedding results of Higman, Neumann and Neumman. Namely they prove [HNN49] that every countable group can be embedded into a countable group in which any two elements that have the same order are conjugate. As a corollary, every countable group can be embedded into a countable simple divisible group (see [LS01, Theorem IV.3.4] for an exposition). Of course, by using such embeddings directly in the course of the proof of Theorem A, we could skip over a layer in the construction at the cost of a mild notational difficulty (one would have to deal with countably many countable generating sets).

Acknowledgments. The author would like to thank Centre Interfacultaire Bernoulli at EPF-Lausanne for the support, the staff members for the hospitality during his stay in May 2007, Goulnara Arzhantseva and Alain Valette for their kind invitation to participate in the program, and the referee for his/her rather useful remarks.

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