Dirac’s “magnetic monopole” in pyrochlore ice $U(1)$ spin liquids: Spectrum and classification

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We study the $U(1)$ quantum spin liquid on the pyrochlore spin ice systems. For the non-Kramers doublets such as Pr$_2$TM$_2$O$_7$ and Tb$_2$TM$_2$O$_7$, we point out that the inelastic neutron scattering result not only detects the low-energy gauge photon, but also contains the continuum of the “magnetic monopole” excitations. Unlike the spinons, these “magnetic monopoles” are purely of quantum origin and have no classical analogue. We further point out that the “magnetic monopole” experiences a background dual “$\pi$” flux due to the spin-1/2 nature of the local moment when the “monopole” hops on the dual diamond lattice. We then predict that the “monopole” continuum has an enhanced spectral periodicity with a folded Brillouin zone. This prediction can be examined among the existing data on other non-Kramers doublet spin liquid candidate materials like Pr$_2$TM$_2$O$_7$ and Tb$_2$TM$_2$O$_7$ (with TM = “transition metal”). The application to the Kramers doublet systems and numerical simulation is further discussed. Finally, we present a general classification of distinct symmetry enriched $U(1)$ quantum spin liquids based on the translation symmetry fractionalization patterns of “monopoles” and “spinons”.

I. INTRODUCTION

There has been a tremendous activity in the field of pyrochlore ice material [1–4]. Besides the early efforts in classical spin ice and dipolar spin ice where quantum effects are negligible [5–7], a recent motivation of this exciting area is to search for the three-dimensional $U(1)$ quantum spin liquid (QSL) [8] in the pyrochlore quantum spin ice systems where quantum effects are significant [9–11]. The existence of this exotic quantum phase of matter has been firmly established by the theoretical studies of the relevant and even realistic spin models on the pyrochlore lattice [12–15]. The experimental confirmation of this interesting phase of matter, however, is still open. Even if this phase may have already existed in some candidate materials since the original proposal in Tb$_2$Ti$_2$O$_7$ [16] and Yb$_2$Ti$_2$O$_7$ [17], the firm identification of this exotic phase requires the strong mutual feedback between the experimental progress and the theoretical development that provides and clarifies unique and clear physical observables for the experiments.

The pyrochlore spin ice $U(1)$ QSL is described by the emergent compact $U(1)$ lattice gauge theory with deconfined and fractionalized excitation [18–20]. There are three elementary excitations, namely, spinon, “magnetic monopole”, and gauge photon in this $U(1)$ QSL. Here the nomenclature for the excitations follows from the original work by Hermele, Fisher and Balents [21] (see Table I). To confirm the realization of the $U(1)$ QSL, one would need at least observe one such emergent and exotic excitation. Inelastic neutron scattering, that is a spectroscopic measurement, is likely to provide much richer information than any other experimental probes for the pyrochlore spin ice systems [22]. It is thus of great importance to understand how the neutron moments are coupled to the microscopic degrees of freedom and how the inelastic neutron scattering (INS) results are related to the emergent and exotic properties of the pyrochlore ice $U(1)$ QSL. It is this purpose that motivates our work in this paper.

We mainly deal with the non-Kramers doublets in most parts of this paper. The non-Kramers doublets on the pyrochlore system have been discussed by several previous works. In particular, the generic spin model was introduced and studied in Refs. [4, 6] and [51] and more recently, the random strain effect was discussed for Pr$_2$Tm$_2$O$_7$ in Refs. [32] and [43]. In Ref. [13], we have pointed out the magnetic transition out of $U(1)$ QSL should be a confinement transition by a simple symmetry analysis. For a non-Kramers doublet [46] that is described by a pseudospin-1/2 operator $S$, the time reversal symmetry, $T$, acts rather peculiarly such that

$$T : \quad S^{x,y} \rightarrow S^{x,y}, \quad S^z \rightarrow -S^z.$$ (1)

This property means the neutron moments would merely pick up the $S^z$ component and naturally measure the $S^z$ correlation. By examining the connection with the emergent variables such as gauge fields and matter fields, we point out that, the $S^z$ correlation should detect the gauge photons as well as the “magnetic monopoles”. The “mag-

| Excitations (notation 1) | Excitations (notation 2) |
|-------------------------|-------------------------|
| Spinon                  | Magnetic monopole       |
| “Magnetic monopole”     | Electric monopole       |
| Gauge photon            | Gauge photon            |

TABLE I. Two different but equivalent notations for the excitations in the pyrochlore ice $U(1)$ QSL. The notation 1 was introduced in Ref. [46] and is adopted in this paper. The notation 2 can be found in Ref. [50] and the magnetic monopole in this notation has a classical analogue that is a defect tetrahedron with either “3-in 1-out” or “1-in 3-out” spin configurations [19].
netic monopole" is the topological defect of the emergent vector gauge potential in the compact U(1) quantum electrodynamics and has no classical analogue. Even though the spinon and the “magnetic monopole" can be interchanged by the electromagnetic duality of the lattice gauge theory, the “magnetic monopole" might be more close in spirit to the Dirac’s magnetic monopole\(^1\)\(^2\)\(^3\) from the original definition and theory of the pyrochlore U(1) QSL\(^5\). The existence of the “magnetic monopole" is one of the key properties of the compact U(1) lattice gauge theory\(^5\) and the pyrochlore ice U(1) QSL\(^5\), and it is of great importance to demonstrate that the “magnetic monopole" is a real physical entity rather than any artificial or fictitious excitation.

So far, there were only limited studies of “monopole” physics in the U(1) QSL of the pyrochlore ice context\(^4\)\(^6\)\(^7\)\(^8\). We here realize that the “magnetic monopole" could manifest itself as the “monopole" continuum in the INS result on the non-Kramers doublet pyrochlore spin ice systems. Our renewed understanding of the INS measurement for non-Kramers doublets is further extended to the Kramers doublets and the quantum Monte carlo simulation, and henceforth provides a new insight for the experimental observation and the numerical simulation. Moreover, the “magnetic monopole" experiences a background \(\pi\) flux as the “magnetic monopole" hops around the perimeter on the elementary plaquette of the dual diamond lattice. We then point out that the background \(\pi\) flux immediately modulates the spectral structure of the “monopole" continuum by enhancing the spectral periodicity. This is an unique experimental signature for the “monopole" continuum in the INS measurement. More generally, this is an example of translation symmetry fractionalization in topologically ordered phases\(^10\)\(^11\)\(^12\). Combining with the prior work on the translation symmetry fractionalization of the spinon\(^13\)\(^14\)\(^15\) we establish a general classification for the pyrochlore ice U(1) QSLs based on the translation symmetry and list their relevant spectral properties.

The following part of the paper is organized as follows. In Sec. \[\text{II}\] we introduce the microscopic model for the non-Kramers doublets, and explain the application of several effective models. In Sec. \[\text{III}\] we point out the presence of the “monopole” dynamics in the spin correlation function from the INS measurements. In Sec. \[\text{IV}\] we establish the spectral structure of the “monopole" continuum. In Sec. \[\text{V}\] we carry out the “monopole" mean field theory and explicitly compute the “monopole" dynamics. Finally in Sec. \[\text{VI}\] we give a broad discussion about the spectral properties of non-Kramers doublet and Kramers doublet spin ice materials and present a classification of the U(1) QSLs based on the translation symmetry fractionalization patterns of the “magnetic monopoles" and the spinons.

II. MODEL FOR NON-KRAMERS DOUBLETS AND THE LOW-ENERGY FIELD THEORY

Due to the peculiar property of the non-Kramers doublets under the time reversal symmetry, the generic spin model, that describes the interaction between these doublets on the pyrochlore lattice, is actually simpler than the usual Kramers doublets and is given by\(^16\)\(^17\)

\[
H = \sum_{\langle ij \rangle} J_{zz} S_i^z S_j^z - J_{\pm} (S_i^+ S_j^- + h.c.) + J_{\pm\pm} (\gamma_{ij} S_i^x S_j^x + h.c.) + \text{dipolar interaction, (2)}
\]

where \(S_i^\pm \equiv S_i^x \pm i S_i^y\) and \(\gamma_{ij}\) is the bond-dependent phase variable that arises from the spin-orbit-entangled nature of the non-Kramers doublet. The dipolar interaction includes the further neighbor interactions between the \(S^z\) components since only \(S^z\) is time reversely odd and contributes to the dipole moment. It has been shown in Ref. \[6\] that, in the perturbative Ising limit with \(|J_{zz}| \ll J_{zz}\) and \(|J_{\pm\pm}| \ll J_{zz}\), the system realizes the U(1) QSL. Moreover, it was demonstrated that the U(1) QSL is more robust on the frustrated side\(^18\) with \(J_{\pm} < 0\) and along the axis of \(J_{\pm\pm}\).

Throughout the paper, we deliver our theory through the non-Kramers doublet system. Only in the Sec. \[\text{VI}\] we extend our theory to the Kramers doublet system.

A. Effective theories

Our purpose is not to understand the energetics of the relevant microscopic spin model. We assume that the U(1) QSL has been realized in the system and try to understand its manifestation in the physical observables. For the U(1) QSL, we can then start from the ring exchange model that is obtained from the perturbative treatment of the \(J_{\pm}\) and \(J_{\pm\pm}\) interactions in the Ising limit. With the mapping \(S_i^z = E_{rr'} + \frac{1}{2} S_i^x = e^{\pm i A_{rr'}}\) and \(|E_{rr'}, A_{rr'}| = i\), one obtains the U(1) lattice gauge theory on the diamond lattice formed by the tetrahedral centers of the pyrochlore lattice\(^19\)\(^20\). In this lattice gauge theory, the spin excitations that violate the ice rule have been traced out in the perturbative treatment, and thus, the effective model captures the physics below the spinon gap. The lattice gauge theory Hamiltonian is given as\(^20\)

\[
H_{\text{LGT}} = -K \sum_\mathcal{O} \cos(\text{curl} A) + \sum_{(rr')} \frac{U}{2} (E_{rr'} - \eta_r)^2 \tag{3}
\]

where “\(r, r'\)" stand for the diamond lattice sites, \(\eta_r = \pm 1\) for the two sublattices of the diamond lattice, and \(E_{rr'} = -E_{r'r}, A_{rr'} = -A_{r'r} \). Here, \(\text{curl} A\) is defined as

\[
\text{curl} A \equiv \sum_{rr'} A_{rr'}, \tag{4}
\]

and thus corresponds to the magnetic field \(B\) through the hexagon center. The magnetic coupling \(K\) is of the order

\[
|K| \ll J_{zz}, J_{\pm}, J_{\pm\pm}, U.
\]
of the ring exchange coupling in the perturbation theory, and the electric field term is introduced to enforce the spin-1/2 Hilbert space. If one focuses on the low-energy and long-distance physics, one can further coarsen scales and obtain the continuous Maxwell field theory with \[ H_{\text{Maxwell}} \approx \frac{K}{2} B^2 + \frac{U}{2} E^2, \] where \( K \) and \( U \) are coarse-grained magnetic and electric couplings.

### B. Photon in low-energy theory

Based on the mapping from the microscopic spin degrees of freedom to the emergent field variables in the lattice gauge theory, one could establish the connection between the spin correlation functions with the emergent degrees of freedom. For the non-Kramers doublet, the INS measurement would merely pick up the \( S^z \) correlator and thus measure the correlation function of the emergent electric field. It was then shown, within the low-energy Maxwell field theory, that the spin correlation correspond to the electric field correlator, \[ \langle E_{\mu, \mathbf{q}} E_{\nu, \mathbf{q'}} \rangle \sim \left[ \delta_{\mu \nu} - \frac{q^\mu q^\nu}{q^2} \right] \omega \delta(\omega - v |\mathbf{q}|), \] where \( v \) is the speed of the photon mode. Apart from the angular dependence, the spectral weight of the photon mode is suppressed as the energy transfer \( \omega \to 0 \).

### III. THE LOOP CURRENT OF “MAGNETIC MONOPOLES”

The well-known result of the photon modes in the INS measurement was obtained by considering the low-energy field theory that describes the long-distance quantum fluctuation within the spin ice manifold. The actual spin dynamics, that is captured by the \( S^z \) correlation in the INS measurement, operates in a broad energy scale up to the exchange energy and certainly contains more information than just the photon mode from the low-energy Maxwell field theory. What is the other information hidden behind? To address this question, we have to leave the low-energy Maxwell field theory and include the gapped matters into our consideration.

The gapped matters are spinons and “magnetic monopoles”. The spinons are sources and sinks of the emergent \( E \) field and live on the diamond lattice sites or the tetrahedral centers. These spinon are excitations out of the spin ice manifold and are created by the \( S^z \) or \( S^y \) operator. For the non-Kramers’ doublet systems, the neutron scattering does not allow such spin-flipping processes. So we turn to the “magnetic monopoles”. The “magnetic monopole” is the source or the sink of the emergent \( B \) field and is the excitation within the spin ice manifold. Since the “magnetic monopole” is located on the dual diamond lattice site (see Fig. 1), to make the “magnetic monopole” explicit, one needs to do a duality transformation on the lattice gauge Hamiltonian \[ H_{\text{LGT}}. \] This standard procedure yields the following dual theory,

\[ H_{\text{dual}} = \sum_{(\mathbf{R}, \mathbf{R}')}(\Phi_{\mathbf{R}}^\dagger \Phi_{\mathbf{R}}^\prime) \delta(\omega - v|\mathbf{q}|) \]

and is simply the electric field going through the center of the hexagon plaquette on the dual diamond lattice. This dual model describes the coupling between the “magnetic monopoles” and the fluctuating dual U(1) gauge fields, and is the starting point to explore the dynamics of the “magnetic monopoles”. For our purpose to capture the generic spectral structure of the “monopole” dynamics, we here keep only the nearest-neighbor “monopole” hopping.

Since the neutron picks up the \( S^z \) component for non-Kramers doublets, we want to find what kind of “monopole” operators in the dual theory correspond to the \( S^z \) component. Since this is a gauge theory, only gauge invariant quantity is physical according to Elitzur’s theorem. It has been shown from the Maxwell’s equations in the early studies of critical theories for the “magnetic monopole” condensation transition that the “magnetic monopole” current on a closed hexagon loop of the dual diamond lattice induces the electric field through the center of the loop (see Fig. 1), i.e.

\[ \sum_{\mathbf{R} \in \Omega^*} J_{\mathbf{RR'}} = E \sim S^z, \]

where \( J_{\mathbf{RR'}} \) is the “monopole” current between the nearest neighbors with

\[ J_{\mathbf{RR'}} \equiv \int [\Phi_{\mathbf{R}}^\dagger \Phi_{\mathbf{R}}^\prime e^{-i2\pi \alpha \omega|\mathbf{q}|} - h.c.]. \]
distance and finite energy dynamics of the “magnetic monopoles”. From this relation, we conclude that the \( S^z \) correlation contains the contribution of the “monopole” current correlator.

The above analysis does not provide the information about the spectral weight of the “monopole” continuum in the \( S^z \) correlation. It was pointed out that increasing further neighbor \( S^-S^- \) interaction could drive a quantum phase transition from the U(1) QSL to the Ising order via the “monopole” condensation.\(^\text{13}\) We thus think that the systems with extended \( S^z \) coupling may have more visible “monopole” continuum in the INS result.

\[ T^m \equiv T^m_{\mu} \text{ for } \mu = 1, 2, 3, \text{ and } a_1 = \frac{1}{2}(011), a_2 = \frac{1}{2}(101), a_3 = \frac{1}{2}(110). \]

We use the cubic coordinate system and set the lattice constant to unity throughout the paper. As the “magnetic monopole” hops successively through the parallelogram defined by \( T^m T^{m\nu} (T^m_{\mu})^{-1} (T^m_{\nu})^{-1} \) with \( \mu \neq \nu \), the “monopole” experiences an identical Aharonov-Bohm flux as the background flux trapped in the hexagon plaquette of the dual diamond lattice.\(^\text{13}\) This is because of the lattice geometry of the diamond lattice. Thus, we have the following algebraic relation

\[ T^m T^{m\nu} (T^m_{\mu})^{-1} (T^m_{\nu})^{-1} = e^{i\pi} = -1. \] \(^\text{(12)}\)

This algebraic relation means the lattice translation symmetry is realized projectively for the “magnetic monopoles”. The translation symmetry fractionalization for the “magnetic monopole” is intimately connected to the spectral periodicity of the “monopole continuum”\(^\text{[13-16]}\).

To demonstrate the enhanced spectral periodicity of the “monopole” continuum, we introduce a 2-“monopole” scattering state \( |A\rangle \equiv |q_A; z_A\rangle \), where \( q_A \) is the total crystal momentum of this state and \( z_A \) represents the remaining quantum number that specifies the state.\(^\text{[23]}\) The translation symmetry fractionalization acts on the individual “monopole”, such that

\[ T^m_{\mu} |A\rangle \equiv T^m_{\mu} (1) T^m_{\mu} (2) |A\rangle, \] \(^\text{(13)}\)

where \( T^m_{\mu} \) is the translation operator for the system, and \( |1\rangle \) and \( |2\rangle \) refer to the two “monopoles” of this state. By translating one “monopole” by the basis lattice vector \( a_\mu \), we obtain another three 2-“monopole” scattering states,

\[ |B\rangle = T^m_{\mu} (1) |A\rangle, \] \(^\text{(14)}\)
\[ |C\rangle = T^m_{\mu} (2) |A\rangle, \] \(^\text{(15)}\)
\[ |D\rangle = T^m_{\mu} (1) |A\rangle. \] \(^\text{(16)}\)
It is ready to compare the translation eigenvalues of these four states by making use of Eq. 12 and obtain the following relations for the crystal momentum of these states,
\[
q_B = q_A + 2\pi(100),
\]
\[
q_C = q_A + 2\pi(010),
\]
\[
q_D = q_A + 2\pi(001).
\]
Since these scattering states have the same energy, we thus conclude that the “monopole” spectrum of the two “monopole” excitations have the following enlarged spectral periodicity such that
\[
L_m(q) = L_m(q + 2\pi(100)) = L_m(q + 2\pi(010)) = L_m(q + 2\pi(001)),
\]
where \(q = \mu,\) \(\xi = \pm\) and \(\phi = \pm\) are the momentum and energy transfer and \(\mu = 0, 1, 2, 3\) the nearest-neighbor vectors connecting two sublattices. Here \(e_0 = \frac{1}{2}(111), e_1 = \frac{1}{2}(111), e_2 = \frac{1}{2}(111), e_3 = \frac{1}{2}(111), (\xi_0, \xi_1, \xi_2, \xi_3) = (0, 1, 1, 0)\) and \(\vec{Q} = 2\pi(100)\).

Under this above gauge fixing, we have the “monopole” mean-field Hamiltonian,
\[
H_{\text{MFT}} = -\frac{1}{\rho} \sum_{\langle RR' \rangle} e^{-\frac{1}{2}q(\vec{e}_\mu R \cdot \vec{e}_\mu R')} \Phi^\dagger R \Phi R' - \mu \sum_R \Phi^\dagger R \Phi R, \tag{22}
\]
where the “monopole” spectrum is found to be
\[
\Omega^\pm(q) = \pm t \lbrack 4 \pm 2(3 + C_x C_y - C_x C_z + C_y C_z) \rbrack^{\frac{1}{2}} - \mu,
\]
and the “monopole” spectrum by 2 enhanced. The spectrum is invariant if one translates the system in the system, but the spectral periodicity is enhanced. The spectrum is invariant if one translates the spectrum by 2(100), 2(010), or 2(001). This is very different from the conventional case where the spectral periodicity is given by the reciprocal lattice vectors, 2π(111), 2π(111) and 2π(111), for the FCC Bravais lattice. Therefore, the spectral periodicity enhancement with a fold Brillouin zone is a strong indication of the fractionalization in the system.

V. THE “MONOPOLE” MEAN-FIELD THEORY AND THE CONTINUUM

To explicitly compute the “monopole” dynamics and demonstrate the spectral periodicity enhancement, we carry out the mean-field approximation for the “monopole”-gauge coupling. To capture the \(\pi\) background flux, we set the dual gauge potential as a specific choice of “monopole” hopping and chemical potential.

\[
\phi = \frac{\pi}{2}, \tag{23}
\]
\[
E = \Omega^i_1(q_1) + \Omega^i_2(q_2), \tag{24}
\]
where \(\phi = \frac{\pi}{2}\) and \(E\) are the momentum and energy transfer of the neutrons, \(q_1\) and \(q_2\) are the crystal momenta of the two “monopoles”, and the offset \(Q\) arises from the dual gauge link that is present in the “monopole” current. The minimum (maximum) of the energy \(E\) is obtained when \(i_1 = i_2 = -\) and \(j_1 = j_2 = +\) (\(i_1 = i_2 = +\) and \(j_1 = j_2 = -\)). In Fig. 2 we depict the upper and lower excitation edges of the “monopole” continuum for a specific choice of “monopole” hopping and chemical potential. Clearly, the spectral periodicity is enhanced in both plots.
VI. DISCUSSION

A. Non-Kramers doublets

We discuss the application of our results to various pyrochlore ice systems. We begin with the non-Kramers doublets. The continuous excitations have actually been observed from the INS measurements on Pr$_2$Zr$_2$O$_7$, Tb$_2$Ti$_2$O$_7$ and Pr$_2$Hf$_2$O$_7$. In particular, in the INS result for Pr$_2$Hf$_2$O$_7$, besides the very low-energy features that seem to resemble the suppressed spectral intensity of the photon mode, there exists a broad excitation continuum extending to higher energies. This continuum may be attributed to the random strain effect that has already been suggested to Pr$_2$Zr$_2$O$_7$. Nevertheless, the random strain effect was also suggested to create quantum entanglement and induce U(1) QSL phase in non-Kramers doublet system. Therefore, if the underlying systems realize the U(1) QSL, according to our theory, these mysterious continuous excitations may at least contain the contribution from the two-“monopole” continuum that is predicted in this work.

How does one verify the above claim of the “monopole” continuum in the INS measurement? We here propose a scheme to exclude the presence of the spinon continuum in the INS result by conducting a thermal transport measurement. Spinons are higher energy excitations, and their contribution to thermal conductivity should appear at higher temperatures. If one observes that the energy scale of the continuum in the INS measurement is clearly lower than the temperature scale where the spinons contribute to the thermal conductivity, one could then conclude the presence of the spinon excitation in the thermal conductivity results and the absence of the spinon excitation in the continuum of the INS results. The direct measurement would be the confirmation of the enhanced spectral periodicity of the “monopole” continuum in the momentum space. This may be difficult as the low-energy photon excitation is also present in the low-energy INS data. Thus, the higher energy part of the “monopole” continuum may provide more useful information. It is certainly very exciting if all the three excitations, spinon, “magnetic monopole”, and gauge photon are confirmed by a combination of the INS and the thermal transport measurements.

For the “monopoles continuum”, probably the most positive side in this identification of “monopole continuum” is that weak external magnetic field can be used to manipulate the “monopole” continuum. With weak magnetic fields, the U(1) QSL will not be destroyed, and the “magnetic monopole” remains to be a valid description of the excitation of the system. However, the external magnetic field, that only couples linearly to the $S^z$ components, polarizes $S^z$ slightly and thereby modifies the background dual U(1) gauge flux that is experienced by the “monopole”. As a result, the “monopole” band would probably develop a Hofstadter band, and the spectral structure of the “monopole” continuum is modified. How this “monopole” continuum is modulated depends on the orientation and the amplitude of the external magnetic fields. The detailed behavior of the “monopole” continuum in the weak field will be explored in future works.

B. Kramers doublets and numerical simulation

As for the usual Kramers doublets, all the three components of the local moments are odd under the time reversal symmetry, and the neutron spin would couple to all of them. Therefore, the INS results on the U(1) QSL with the usual Kramers doublets would also detect the spin flipping events out of the spin ice manifold and measure the spinon continuum in addition to the gauge photon and the “monopole” continuum. As we have already pointed out in the previous sections, the visibility of the “monopole” continuum in the INS data depends on how much weight of the “monopole” continuum, and may vary for different materials.

If the neutron energy transfer is located within the “monopole” continuum, the spectral periodicity would experience an enhancement. If the neutron energy transfer is located in the spinon continuum, the spectral periodicity is enhanced (not enhanced) if the spinon experiences a background $\pi$ (0) flux on the diamond lattice.

The U(1) QSL has been explored by quantum Monte Carlo simulation, and the photon mode was identified in the $S^z$ correlation function. It might be of interest to introduce further $S^z$ interactions to possibly enhance and manifest the “monopole” continuum in the $S^z$ correlation.

C. A classification of the U(1) QSLs

Finally, let us remark on the translation symmetry fractionalization patterns for the U(1) QSLs. In this work, we have focused on the “magnetic monopole” exci-
tation and found that the “magnetic monopole” experiences a background dual $U(1)$ flux on the dual diamond lattice. In the previous work\[22\], we studied the spectral periodicity and the translation symmetry fractionalization for the spinon excitation. The combination of the “magnetic monopole” and the spinon symmetry fractionalization patterns results in a classification of the distinct symmetry enriched $U(1)$ QSLs in Table II. Like the classification scheme that was developed for the two-dimensional $Z_2$ QSLs and applied to the $Z_2$ toric code mode\[23\], one could use the result in Table I to further establish the translation symmetry fractionalization for the (fermionic) dyon that is a bound state of the “magnetic monopole” and the spinon symmetry fractionalization for the spinon excitation. The combination not only helps improve the understanding of the crystal symmetry fractionalization in the $U(1)$ QSLs, but also provides unique and detectable experimental signatures for the $U(1)$ QSLs.

\[1\] Hamid R. Molavian, Michel J. P. Gingras, and Benjamin Canals, “Dynamically Induced Friction as a Route to a Quantum Spin Ice State in Tb$_2$Ti$_2$O$_7$ via Virtual Crystal Field Excitations and Quantum Many-Body Effects,” Phys. Rev. Lett. 98, 157204 (2007)

\[2\] M J P Gingras and P A McClarty, “Quantum spin ice: a search for gapless quantum spin liquids in pyrochlore magnets,” Reports on Progress in Physics 77, 056501 (2014)

\[3\] Lucile Savary and Leon Balents, “Quantum spin liquid: a review,” Reports on Progress in Physics 80, 016502 (2016)

\[4\] Shigeki Onoda and Yoichi Tanaka, “Quantum Melting of Spin Ice: Emergent Cooperative Quadrupole and Chirality,” Phys. Rev. Lett. 105, 047201 (2010)

\[5\] Lucile Savary and Leon Balents, “Coulombic Quantum Liquids in Spin-1/2 Pyrochlores,” Phys. Rev. Lett. 108, 037202 (2012)

\[6\] SungBin Lee, Shigeki Onoda, and Leon Balents, “Genuine quantum spin ice,” Phys. Rev. B 86, 104412 (2012)

\[7\] Lucile Savary and Leon Balents, “Spin liquid regimes at nonzero temperature in quantum spin ice,” Phys. Rev. B 87, 205130 (2013)

\[8\] Roger G. Melko, Byron C. den Hertog, and Michel J. P. Gingras, “Long-range order at low temperatures in dipolar spin ice,” Phys. Rev. Lett. 87, 067203 (2001)

\[9\] H. Fukazawa, R. G. Melko, R. Higashina, Y. Maeno, and M. J. P. Gingras, “Magnetic anisotropy of the spin-ice compound Dy$_2$Ti$_2$O$_7$,” Phys. Rev. B 65, 054410 (2002)

\[10\] S. T. Bramwell, M. J. Harris, B. C. den Hertog, M. J. P. Gingras, J. S. Gardner, D. F. McMorrow, A. R. Wildes, A. L. Cornelius, J. D. M. Champion, R. G. Melko, and T. Fennell, “Spin Correlations in Ho$_2$Ti$_2$O$_7$: A Dipolar Spin Ice System,” Phys. Rev. Lett. 87, 047205 (2001)

\[11\] K. A. Ross, J. P. C. Ruff, C. P. Adams, J. S. Gardner, H. A. Dabkowski, Y. Qiu, J. R. D. Copley, and B. D. Gaulin, “Two-Dimensional Kagome Correlations and Field Induced Order in the Ferromagnetic XY Pyrochlore Yb$_2$Ti$_2$O$_7$,” Phys. Rev. Lett. 103, 227202 (2009)

\[12\] Yi-Ping Huang, Gang Chen, and Michael Hermele, “Quantum Spin Ices and Topological Phases from Dipolar-Octupolar Doublets on the Pyrochlore Lattice,” Phys. Rev. Lett. 112, 167203 (2014)

\[13\] Gang Chen, “Magnetic monopole” condensation of the pyrochlore ice U(1) quantum spin liquid: Application to Pr$_3$Ir$_2$O$_7$ and Yb$_2$Ti$_2$O$_7,” Phys. Rev. B 94, 205107 (2016)

\[14\] Yuan Wan and Oleg Tchernyshyov, “Quantum Strings in Quantum Spin Ice,” Phys. Rev. Lett. 108, 247202 (2012)

\[15\] Yao-Dong Li and Gang Chen, “Symmetry enriched U(1) topological orders for dipole-octupole doublets on a pyrochlore lattice,” Phys. Rev. B 95, 041106 (2017)

\[16\] Han Yan, Owen Benton, Ludovia Jaubert, and Nic Shannon, “Theory of multiple-phase competition in pyrochlore magnets with anisotropic exchange with application to Yb$_2$Ti$_2$O$_7$, Er$_2$Ti$_2$O$_7$, and Er$_2$Sn$_2$O$_7,” Phys. Rev. B 95, 094422 (2017)

\[17\] Lucile Savary, Xiaojun Wang, Hae-Young Kee, Yong Baek Kim, Yue Yu, and Gang Chen, “Quantum spin ice on the breathing pyrochlore lattice,” Phys. Rev. B 94, 075146 (2016)

\[18\] T. Fennell, M. Kenzelmann, B. Roessli, M. K. Haas, and R. J. Cava, “Power-Law Spin Correlations in the Pyrochlore Antiferromagnet Tb$_2$Ti$_2$O$_7,” Phys. Rev. Lett. 109, 017201 (2012)

\[19\] Yukio Yasui, Masaki Kanada, Masafumi Ito, Hiroshi Harashina, Masatoshi Sato, Hajime Okumura, Kazuhsia Kakurai, and Hiroaki Kadowaki, “Static Correlation and Dynamical Properties of Tb$_3$+ -moments in Tb$_2$Ti$_2$O$_7$: Neutron Scattering Study,” Journal of the Physical Society of Japan 71, 599–606 (2002)

\[20\] J. S. Gardner, B. D. Gaulin, A. J. Berlinsky, P. Waldron, S. R. Dunsiger, N. P. Raju, and J. E. Greedan, “Neutron scattering studies of the cooperative paramagnet pyrochlore Tb$_2$Ti$_2$O$_7,” Phys. Rev. B 64, 224416 (2001)

\[21\] Zhihao Hao, Alexandre G. R. Day, and Michel J. P. Gingras, “Bosonic many-body theory of quantum spin ice,” Phys. Rev. B 90, 214430 (2014)

\[22\] Lieh-Jeng Chang, Shigeiki Onoda, Yixi Su, Ying-Jer Kao, Ku-Ding Tsuei, Yukio Yasui, Kazuhsia Kakurai, and Martin Richard Lees, “Higgs transition from a magnetic Coulomb liquid to a ferromagnet in Yb$_2$Ti$_2$O$_7,” Nature...
Communications 3, 992 (2012)

K. Kimura, K. Nakatsuji, J-J. Wen, C. Broholm, M.B. Stone, E. Nishibori, and H. Sawa, “Quantum fluctuations in spin-ice-like Pr$_2$Zr$_2$O$_7$,” Nature Communications 4, 2914 (2013)

Jason S. Gardner, Michel J. P. Gingras, and John E. Greedan, “Magnetic pyrochlore oxides,” Rev. Mod. Phys. 82, 53–107 (2010)

E. Lhotel, S. R. Giblin, M. R. Lees, G. Balakrishnan, L. J. Chang, and Y. Yasui, “First-order magnetic transition in Yb$_2$Ti$_2$O$_7$,” Phys. Rev. B 89, 224419 (2014)

L. D. C. Jaubert, Owen Benton, Jeffrey G. Rau, J. Oitmaa, Gang Chen, Hae-Young Kee, and Yong Baek Kim, “Frustrating quantum spin ice: a tale of three spin liquids, and hidden order,” Phys. Rev. Lett. 115, 097202 (2015)

Mathieu Taillefumier, Owen Benton, Han Yan, Ludovic Jaubert, and Nic Shannon, “Frustrating quantum spin ice: a tale of three liquid spin, and hidden order,” arXiv:1705.00148 (2017).

Gang Chen, Hae-Young Kee, and Yong Baek Kim, “Unusual Liquid State of Hard-Core Bosons on the Pyrochlore Lattice,” Phys. Rev. Lett. 100, 047208 (2008)

Yasuyuki Kato and Shigeki Onoda, “Numerical Evidence of Quantum Melting of Spin Ice: Quantum-to-Classical Crossover,” Phys. Rev. Lett. 115, 077202 (2015)

Jian-Ping Lv, Gang Chen, Youjin Deng, and Zi Yang Meng, “Coulomb Liquid Phases of Bosonic Cluster Mott Insulators on a Pyrochlore Lattice,” Phys. Rev. Lett. 115, 037202 (2015)

Chong Wang and T. Senthil, “Time-Reversal Symmetric U(1) Quantum Spin Liquids,” Phys. Rev. X 6, 011034 (2016)

Shigeki Onoda and Yoichi Tanaka, “Quantum fluctuations in the effective pseudospin-$\frac{1}{2}$ model for magnetic pyrochlore oxides,” Phys. Rev. B 83, 094411 (2011)

P.A.M. Dirac, “Quantised singularities in the electromagnetic field,” Proc. Roy. Soc. (London) A133 (1931), 10.1098/rspa.1931.0130

Eduardo Fradkin, Field Theories of Condensed Matter Physics, 2nd ed. (Cambridge University Press, 2013).

M. P. Kwasigroch, B. Doucet, and C. Castelnovo, “Semi-classical approach to quantum spin ice,” Phys. Rev. B 95, 134439 (2017)

Andrew M. Essin and Michael Hermele, “Spectroscopic signatures of crystal momentum fractionalization,” Phys. Rev. B 90, 121102 (2014)

Xiao-Gang Wen, “Quantum orders and symmetric spin liquids,” Phys. Rev. B 65, 165113 (2002)

Doron L. Bergman, Gregory A. Fiete, and Leon Balents, “Ordering in a frustrated pyrochlore antiferromagnet prox-
imate to a spin liquid,” [Phys. Rev. B 73, 134402 (2006)]

58 S. Elitzur, “Impossibility of spontaneously breaking local symmetries,” [Phys. Rev. D 12, 3978-3982 (1975)]

59 O. I. Motrunich and T. Senthil, “Origin of artificial electrodynamics in three-dimensional bosonic models,” [Phys. Rev. B 71, 125102 (2005)]

60 Xiao-Gang Wen, “Quantum order: a quantum entanglement of many particles,” [Physics Letters A 300, 175 – 181 (2002)]

61 Romain Sibille, Nicolas Gauthier, Han Yan, Monica Ciomaga Hatnean, Jacques Ollivier, Barry Winn, Geetha Balakrishnan, Michel Kenzelmann, Nic Shannon, and Tom Fennell, “Experimental signatures of emergent quantum electrodynamics in a quantum spin ice,” ArXiv:1706.03604 (2017).

62 Hiroshi Takatsu, Hiroaki Kadowaki, Taku J Sato, Jeffrey W Lynn, Yoshikazu Tabata, Teruo Yamazaki, and Kazuyuki Matsuhira, “Quantum spin fluctuations in the spin-liquid state of Tb$_2$Ti$_2$O$_7$,” Journal of Physics: Condensed Matter 24, 052201 (2012).

63 J. M. Baker and B. Bleaney, “Paramagnetic resonance in some lanthanon ethyl sulphates,” [Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 245, 156–174 (1958)]

64 Yuji Matsuda, Unpublished (2016).

65 Douglas R. Hofstadter, “Energy levels and wave functions of bloch electrons in rational and irrational magnetic fields,” [Phys. Rev. B 14, 2239–2249 (1976)]

66 Andrew M. Essin and Michael Hermele, “Classifying fractionalization: Symmetry classification of gapped $Z_2$ spin liquids in two dimensions,” [Phys. Rev. B 87, 104406 (2013)]