On \( d = 4 \) Yang-Mills instantons in a spherically symmetric background

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**Abstract**

We present arguments for the existence of self-dual Yang-Mills instantons for several spherically symmetric backgrounds with Euclidean signature. The time-independent Yang-Mills field has finite action and a vanishing energy momentum tensor and does not disturb the geometry. We conjecture the existence of similar solutions for any nonextremal \( SO(3) \)-spherically symmetric background.

**Introduction**—The recent work [1] noticed the existence of a new type of instanton solution extremizing the Yang-Mills (YM) action

\[
S = -\frac{1}{16\pi^2\epsilon^2} \int_M d^4x \sqrt{g} \operatorname{Tr} \{ F_{ab} F^{ab} \},
\]

(with \( \epsilon \) the gauge coupling constant) for an euclideanized Schwarzschild background metric.

This YM instanton solution was found for a SU(2) nonabelian ansatz with no dependence on the Euclidean time

\[
A_a dx^a = \frac{1}{2} \left\{ u(r) \tau_3 d\tau + w(r) \tau_1 d\theta + \left( \cot \theta \tau_3 + w(r) \tau_2 \right) \sin \theta d\varphi \right\},
\]

(2)

(where the \( \tau_i \) are the standard Pauli matrices), described by two functions \( w(r) \) and \( u(r) \) which we shall refer to as magnetic and electric potential, respectively.

The configuration reported in [1] differs from the well-known Charap-Duff solution [2], since it satisfies a different set of boundary conditions and has a different value of the action.

Moreover, a double self-dual Schwarzschild background is not crucial for the existence of this new type of solutions. In this letter we report the existence of YM self-dual instantons with similar properties for other Euclidean backgrounds with the same amount of symmetry as the Schwarzschild metric. These solutions are constructed by directly solving the self-duality equations with a suitable set of boundary conditions.

**The model**—The variation of the action principle (1) with respect to the gauge connection \( A_a \) leads to the Yang-Mills equations

\[
\nabla_a (F^{ab}) - i [A_a, F^{ab}] = 0.
\]

Here we will consider YM fields satisfying the duality condition \( F_{ab} = \pm \frac{1}{2} \sqrt{g} e_{abcd} F^{cd} \), in which case the YM energy-momentum tensor vanishes. Thus these self-dual gauge field configurations will not disturb the geometry, while the YM equations (3) are satisfied automatically.

We consider a general spherically symmetric metric ansatz

\[
ds^2 = \sigma^2(r) N(r) d\tau^2 + \frac{dr^2}{N(r)} + R^2(r) (d\theta^2 + \sin^2 \theta d\varphi^2),
\]

(4)
without fixing the metric gauge. Here $\theta$ and $\varphi$ are the angular coordinates with the usual range, $\tau$ corresponds to the Euclidean time, which has a periodicity $\beta$, while the radial coordinate $r$ varies between some $r_0$ and infinity. As $r \to \infty$, the euclideanized (thermal-)Minkowski background is approached, with $R(r) \to r$, $\sigma(r) \to 1$, $N(r) \to 1 - 2M/r$, with $M$ a positive constant. This type of metric usually corresponds to analytical continuations of Lorentzian black holes and particle-like solutions.

For this metric choice, the self-duality equations read (see also [4] for a discussion of this solution)

$$w' = \pm \frac{wu}{\sigma N}, \quad R^2u' = \pm \sigma(1 - w^2),$$

the expression of the YM action being

$$S = \frac{\beta}{2\pi e^2} \left[u(1 - w^2)\right]_0^\infty,$$

which equals the SU(2) Pontryagin charge

$$P_{YM} = \frac{1}{32\pi^2} \int_\mathcal{M} d^4x \sqrt{g} \epsilon^{abcd} \text{Tr}\{F_{ab}F_{cd}\}. \quad (7)$$

Without any loss of generality, we solve the self-duality equations by taking the upper sign in (5); the anti-instanton solutions are found by reversing the sign of the electric potential.

The expansion of the nonabelian potentials as $r \to \infty$ is

$$w(r) = \frac{e^{-\Phi r}}{r^{2M-1}} + \ldots, \quad u(r) = \Phi - \frac{1}{r} + \ldots, \quad (8)$$

with $\Phi$ an arbitrary positive constant. Thus one may define nonabelian electric and magnetic charges computed $e.g.$ according to

$$\left( \frac{Q_E}{Q_M} \right) = \frac{1}{4\pi} \int dS_\mu \sqrt{g} \text{Tr}\{\left( \frac{F^{\mu\tau}}{\tilde{F}^{\mu\tau}} \right) \tau_3 \}. \quad (9)$$

The instanton solutions discussed in this paper have $Q_M = Q_E = 1$.

**Self dual instantons in a "bolt" background**— Following [4], we start by considering asymptotically flat background metrics whose fixed point set of the Euclidean time symmetry is of two dimensions (a "bolt") and the range of the radial coordinate is restricted to $r_h \leq r < \infty$, while

$$N(r) = N_1(r - r_h) + O(r - r_h)^2, \quad \sigma(r) = \sigma_h + O(r - r_h), \quad R(r) = R_h + O(r - r_h),$$

where $N_1$, $\sigma_h$, $R_h$ are positive constant, determined by the Einstein equations.

The YM potentials have the following expansion as $r \to r_h$

$$w(r) = w_h + \frac{w_h(w_h^2 - 1)}{R_h^2N_1}(r - r_h) + O(r - r_h)^2, \quad u(r) = \frac{\sigma_h(1 - w_h^2)}{R_h^2}(r - r_h) + O(r - r_h)^2, \quad (10)$$

with $0 \leq w_h \leq 1$. From (10), we find the action of the instanton solutions

$$S = \frac{\beta \Phi Q_M}{2\pi e^2}. \quad (11)$$

The properties of the background metric enters here through the expression of $\beta$. Note that [4] imply the existence of a maximal allowed magnitude of the electric potential at infinity for a given $r_h$

$$\Phi < \int_{r_h}^\infty dr \frac{\sigma(r)}{R^2(r)}. \quad (12)$$

\[1\] The Charap-Duff instanton solution [3] is found for a Schwarzschild background $N = 1 - 2M/r$, $\sigma = 1$, $R(r) = r$ and a different set of boundary conditions. It reads $u(r) = -M/r^2$, $w(r) = \sqrt{N}$, the action being $S = -1/e^2$ in the normalization we use (see also [4] for a discussion of this solution).
Figure 1. The parameters $w_h$, $w'(r_h)$, $u'(r_h)$ and the action $S$ (in units $1/e^2$) of the YM instantons in a Schwarzschild instanton background are plotted as a function of $r_h$.

Figure 2. The value $w_h$ of the magnetic gauge potential at $r = r_h$ is plotted as a function of $r_h$ for a Reissner-Nordström background with different values of the electric charge $Q$.

In the numerical procedure, we set $\Phi = 1$ without any loss of generality, which implies $r_h < 1$ for both Schwarzschild and Reissner-Nordström backgrounds.

Unfortunately, we could not find closed form solutions of the equations (5) for any physical relevant choice of the metric backgrounds, except for the euclideanized Minkowski metric (i.e. $N(r) = \sigma(r) = 1$, $R(r) = r$), in which case we recover the well-known self-dual YM solution $w = r / \sinh r$, $u = \coth r - 1/r$. [5]
Figure 3. The profiles of the gauge functions $w(r)$ and $u(r)$ are plotted for self-dual instantons in the background of euclideanized Bartnik-McKinnon solution ($r_h = 0$) and a "bolt" EYM configuration with $r_h = 1$. The nonself-dual "primary" YM field in the EYM solutions has one node in both cases.

However, these equations can easily be solved numerically.\footnote{The numerical integrations were carried out using both a shooting method as well as applying the numerical program COLSYS [2], with complete agreement to very high accuracy.}

Apart from Schwarzschild instanton, we solved the equations for a Reissner-Nordström instanton background ($N = 1 - 2M/r + Q^2/r^2$, $\sigma = 1$, $R = r$) [7] and the euclideanized black hole solutions of the Einstein-Maxwell-dilaton theory [3, 4], with

$$R(r) = r(1 - \frac{r_+}{r})^{2a/(1+a^2)}, \quad \sigma(r) = 1, \quad N(r) = (1 - \frac{r_+}{r})(1 - \frac{r_+}{r})(1-a^2)/(1+a^2),$$

(13)

where $a$ is the dilaton coupling constant ($a = 0$ corresponding to the Reissner-Nordström case), $r_+$, $r_-$ being positive constant related to the configuration's mass and electric charge (with $r_h = r_+ > r_-$). The periodicity $\beta$ of the Euclidean time coordinate, which enters (11), depends on the parameters of the solution (e.g. $\beta = 4\pi r_h$ for a Schwarzschild geometry).

Another interesting background is provided by the euclideanized "hairy" black hole solutions in Einstein-Yang-Mills (EYM) theory found in [10]. These purely magnetic solutions solve the EYM field equations also for an Euclidean signature [11]. In this case it is convenient to take $R(r) = r$ while no closed form expression for the metric functions $N(r)$ and $\sigma(r)$ exists in the literature. The numerical results we found give evidence that this background support a second YM field. This self-dual field has a vanishing energy-momentum tensor and does not curve the geometry, nor interact with the first, purely magnetic nonself-dual gauge field.

In all cases, the gauge functions $w(r)$ and $u(r)$ interpolate monotonically between the corresponding values at $r = r_h$ and the asymptotic values at infinity, without presenting any local extrema (this behaviour can be proven analytically from (6)). For small enough values of $r_h$, the solutions look very similar to the flat space self-dual YM configuration. These solutions get deformed increasing the value of $r_h$, while the value of the magnetic potential $w$ at $r = r_h$ steadily decreases. As $r_h$ approaches some maximal value implied by (12), we find $w_h \to 0$ and the solution approaches the limiting abelian configuration

$$w(r) = 0, \quad u(r) = \Phi + \int \frac{\sigma(r)}{R^2(r)} dr.$$  

(14)
For the case of an extremal background with $N(r) \sim O(r - r_h)^2$ as $r \to r_h$ ($\sigma(r)$ and $R(r)$ being nonzero and finite in the same limit), we find the trivial solution $w = 1, u = \Phi$ only.

In Figures 1 and 2 we plotted several relevant parameters of the YM instanton solutions as a function of $r_h$, for the case of Schwarzschild and Reissner-Nordström backgrounds. A typical self-dual YM solution in a euclideanized one-node EYM "hairy" black hole is plotted in Figure 3. The event horizon radius in this case is $r_h = 1$, while $w_0 \simeq 0.9118$, the periodicity of Euclidean time-coordinate $\tau$ being $\beta \simeq 19.644$. These plots retain the generic features of the picture we found in other cases.

**Self dual instantons in a topologically trivial background**– The EYM "hairy" black hole solutions discussed above survive in the limit $r_h \to 0$, when the particle-like Bartnik-McKinnon configurations, indexed by the node number $k$ of the "primary" gauge field [12] are approached. This suggests the existence of self dual instantons for a topologically $R^4$ background, the Killing vector $\partial/\partial \tau$ presenting in this case no fixed points sets (i.e. $g_{\tau\tau} > 0$ for any $r$ and an arbitrary periodicity $\beta$). This type of backgrounds usually corresponds to analytical continuations of Lorentzian globally regular, particle-like solutions. A convenient choice here is $R(r) = r$ and we have $0 \leq r < \infty$, while $N(0) = 1, \sigma(0) = \sigma_0 \neq 0$.

The approximate expression of the YM instanton solutions as $r \to 0$ is

$$w(r) = 1 - br^2 + O(r^4), \quad u(r) = 2b\sigma_0 r + O(r^2), \quad (15)$$

(with $b > 0$), the asymptotic form [3] being valid in this case, too.

We solved the self-duality equations for the case of a a euclideanized Bartnik-McKinnon background with $k = 1, 2, 3$ nodes. The profiles of the gauge functions $u$ and $w$ for the one-node Bartnik-McKinnon background are plotted in Figure 3. The parameter $b$ which enters the expansion at the origin (15) is $b \approx 0.003504$.

We found also solutions with similar properties for a gravitating skyrmion background [13]. Nontrivial solutions of the eqs. [4] are likely to exist for other topologically $R^4$ backgrounds. The action $S$ of these solutions is still given by (11). However, different from the bolt case, $S$ is independent on the background metric, since the parameter $\beta$ is arbitrary for a $R^4$ topology (although the profiles of $u$ and $w$ depend on the details of the geometry).

**Further remarks**– In this letter we have presented numerical arguments for the existence of self-dual instantons for a number of spherically symmetric backgrounds with Euclidean signature. Hopefully, these numerical results will be of help in constructing the solutions analytically.

Based on our results, we conjecture the existence of similar solutions for any spherically symmetric background with no dependence on the Euclidean time function, which approaches at infinity the euclideanized Minkowski background.

The existence of these solutions raise several interesting questions, since in principle the Euclidean action of any spherically symmetric configuration can be adjusted to arbitrary values by including self-dual nonabelian fields in the action principle.

A discussion of possible generalizations of this work should start with axially symmetric generalizations, by including a winding number $n$ in the YM ansatz [14]. Our preliminary numerical results indicate the existence of an axially symmetric YM self-dual solution in a Schwarzschild background, whose action is $n$–times the action of single self-dual (anti)instanton. Such configurations are likely to exist for other background choice as well. Also, it would be interesting to look for self-dual instantons in an axially symmetric background (e.g. Kerr-Newman instanton).

Spherically symmetric YM self-dual solutions exist also for a different asymptotic structure of the background metric. For example, we found solutions with many similar properties for an Euclidean anti-de Sitter background with $N(r) = 1 + r^2/l^2$, $\sigma(r) = 1$ and $R(r) = r$ in [4].

Further details on these solutions including an existence proof for a Schwarzschild background will be presented elsewhere.

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