ON THE UNBOUNDEDNESS OF THE RATIO OF SPECIES AND RESOURCES FOR THE DIFFUSIVE LOGISTIC EQUATION

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ABSTRACT. Concerning a class of diffusive logistic equations, Ni [1, Abstract] proposed an optimization problem to consider the supremum of the ratio of the $L^1$ norms of species and resources by varying the diffusion rates and the profiles of resources, and moreover, he gave a conjecture that the supremum is 3 in the one-dimensional case. In [1], Bai, He and Li proved the validity of this conjecture. The present paper shows that the supremum is infinity in a case when the habitat is a multi-dimensional ball. Our proof is based on the sub-super solution method. A key idea of the proof is to construct an $L^1$ unbounded sequence of sub-solutions.

1. Introduction. This paper is concerned with the following stationary problem for a diffusive logistic equation

$$
\begin{cases}
    d \Delta u + u(m(x) - u) = 0 & \text{in } \Omega, \\
    \partial_\nu u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(1)

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial \Omega$; $d$ is a positive constant; $m(x)$ is a measurable function belonging to

$$
L^\infty_+(\Omega) := \{ f \in L^\infty(\Omega) \mid f(x) \geq 0 \text{ a.e. } x \in \Omega, \| f \|_{L^\infty} > 0 \}.
$$

From the viewpoint of an ecological model, (1) is expected to realize the stationary distribution of species in the habitat $\Omega$. In this sense, the unknown function $u(x)$ represents the distribution of species and $m(x)$ can be interpreted as the distribution of resources (feed). The boundary condition assumes that there is no flux of species on the boundary $\partial \Omega$ of the habitat. In the field of reaction-diffusion equations, the existence, uniqueness and stability of positive solutions are obtained by Cantrell and Cosner [2]. Besides [2], series of works by them Cantrell and Cosner [3, 4, 5],

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Taira [27, 28] gave a great contribution to the research field for a class of diffusive logistic equations with spatial heterogeneous terms.

**Proposition 1** ([2]). For each $d > 0$ and each $m \in L^\infty_+(\Omega)$, (1) has a unique positive solution $u_{d,m}(x)$ in the class of $W^{2,p}(\Omega)$ for any $p \geq 1$. Furthermore, $u_{d,m}(x)$ is globally asymptotically stable (GAS) in the sense that it attracts all positive solutions of the corresponding parabolic problem as $t \to \infty$.

In the sense of Proposition 1, one can say that, in order to know or design the final state of the distribution of species, it is important to derive mathematical effects of the diffusion rate $d$ and the distribution $m(x)$ of resources on the profile of $u_{d,m}(x)$. As a trigger to know such effects, the following mathematical procedure for (1) was introduced by Lou [19] (see also Ni [26] and references therein): Dividing the first equation of (1) by $u(x)$ and integrating the resulting expression gives

$$d \int_\Omega \frac{\Delta u}{u} = \|u\|_{L^1(\Omega)} - \|m\|_{L^1(\Omega)}.$$  

By the boundary condition, integration by parts in the left-hand side leads to

$$\|u\|_{L^1(\Omega)} - \|m\|_{L^1(\Omega)} = d \int_\Omega \left( \frac{|\nabla u|}{u} \right)^2 \geq 0.$$  

Then one can see that

$$\frac{\|u_{d,m}\|_{L^1(\Omega)}}{\|m\|_{L^1(\Omega)}} \geq 1 \quad \text{for any } (d, m) \in (0, \infty) \times L^\infty_+(\Omega),$$  

where the equality holds only when $m(x)$ and $u_{d,m}(x)$ identically equal to a positive constant $m_0$. In the ecological sense, we can regard $\|u_{d,m}\|_{L^1(\Omega)}$ and $\|m\|_{L^1(\Omega)}$ as the total population of species and the total amount of resources, respectively. Then (2) means that the heterogeneity of resource can benefit species. Motivated by this fact, some optimization problems concerning (1) have been studied in the field of elliptic equations. We refer to [17, 18, 19] and [7, 23, 24, 25] for the dependence of $u_{d,m}$ upon $d > 0$ (for fixed $m$) and $m$ (for fixed $d$), respectively. See [9, 10, 11, 12, 13, 14, 21, 22] for applications of information on $u_{d,m}$ to the dynamics of solutions to a class of diffusive Lotka-Volterra systems. We also refer to book chapters [16], [20] and [26] to know trends of studies for (1) and related problems.

This paper focuses on a biological question: “How can we maximize the total population under the limited total resources?”. From such a viewpoint, Ni proposed the following optimization problem: “What is the supremum of

$$\frac{\|u_{d,m}\|_{L^1(\Omega)}}{\|m\|_{L^1(\Omega)}}$$  

for any $d > 0$ and any $m \in L^\infty_+(\Omega)$?”, and moreover, he gave a conjecture that the supremum is 3 in the one-dimensional case when $\Omega = (0, \ell)$ (see [1, Abstract]). Concerning this conjecture, Bai, He and Li [1] proved the validity. The procedure of their proof [1] first shows that $\|u_{d,m}\|_{L^1(0,\ell)} < 3\|m\|_{L^1(0,\ell)}$ for any $(d, m) \in (0, \infty) \times L^\infty_+(0, \ell)$, and next, shows that for

$$d_\varepsilon = \sqrt{\varepsilon}, \quad m_\varepsilon(x) = \begin{cases} 1/\varepsilon & \text{for } x \in [0, \varepsilon], \\ 0 & \text{for } x \in (\varepsilon, \ell] \end{cases}$$  

(4)
with small $\varepsilon > 0$, the solution $u_\varepsilon(x) = u_{d_\varepsilon, m_\varepsilon}(x)$ of (1) with $\Omega = (0, \ell)$ satisfies
\[
\frac{\|u_\varepsilon\|_{L^1(0,\ell)}}{\|m_\varepsilon\|_{L^1(0,\ell)}} = \|u_\varepsilon\|_{L^1(0,\ell)} \sim 3 \quad \text{as} \quad \varepsilon \searrow 0.
\] (5)

It can be verified that $u_\varepsilon(x)$ is monotone decreasing for $x \in (0, \ell)$ and decays to zero over any compact set contained in $(0, \ell]$ as $\varepsilon \to 0$, but $u_\varepsilon(0)$ blows up as $\varepsilon \to 0$. Here it should be noted that their elegant proof using the energy method established (5) without any more detailed profiles of $u_\varepsilon(x)$. In [15], the first author of the present paper derived some detailed information on the profile of $u_\varepsilon(x)$. Among other things, he obtained
\[
\lim_{\varepsilon \to 0} \sqrt{\varepsilon} u_\varepsilon(x) = \frac{3}{2} \quad (0 \leq x \leq \varepsilon).
\] (6)

In the one-dimensional habitat case, (4) tells that a concentration of resources and a suitable small diffusion rate make the total population per the total resources be a maximizing sequence. Furthermore, (6) means that, in the resource interval $[0, \varepsilon]$, the growth rate $O(1/\sqrt{\varepsilon})$ of species is less than that of resource.

This paper considers the supremum of the ratio in (2) in the case when $\Omega$ is a unit ball $B^n_1 := \{ x \in \mathbb{R}^n \mid |x| < 1 \}$. The following theorem is a crucial part of a main result (Theorem 2.2):

**Theorem 1.1.** Let $u_{d,m}(x)$ be a positive solution of (1) with $\Omega = B^n_1$. If $n \geq 2$, then
\[
\sup_{(d,m) \in (d,\infty) \times L^\infty(B^n_1)} \frac{\|u_{d,m}\|_{L^1(B^n_1)}}{\|m\|_{L^1(B^n_1)}} = \infty.
\] (7)

This result is a big contrast to that of the one-dimensional case ([1]) where the above supremum is 3, and moreover, gives a negative answer to an open question in [16, (8.36)]. The proof of Theorem 1.1 is based on the sub-super solution method. We employ a concentration setting of resources near the center as $m_\varepsilon(x) = 1/\varepsilon^n$ for $x \in B^n_\varepsilon := \{ x \in \mathbb{R}^n \mid |x| \leq \varepsilon \}$ and $m_\varepsilon(x) = 0$ otherwise. Then a control of the diffusion rate as $d_\varepsilon = O(1/\varepsilon^{n-2})$ enables us to construct an $L^1$ unbounded sequence of sub-solutions as $\varepsilon \to +0$. This sub-solution also ensures that the growth rate of species in the resource region $B^n_\varepsilon$ is equal to $O(1/\varepsilon^n)$ which is same as that of resources.

This paper consists of three sections. Section 2 is devoted to the proof of the main result. In Section 3, some concluding remarks related to the result will be given.

2. **Construction of an $L^1$-unbounded sequence of solutions.** The proof of Theorem 1.1 is based on the (weak) sub-super solution method for a class of elliptic equations. Since $m(x)$ is allowed to be a discontinuous function, we note a framework of the method. Consider the following Neumann problem for a class of semilinear elliptic equations including (1):
\[
\begin{cases}
d \Delta u + f(x, u) = 0 & \text{in} \quad \Omega, \\
\partial_\nu u = 0 & \text{on} \quad \partial \Omega,
\end{cases}
\] (8)

where $f(x, t)$ is a Carathéodory function for $(x, t) \in \Omega \times \mathbb{R}$, that is, for any fixed $t \in \mathbb{R}$, $x \mapsto f(x, t)$ is a measurable function in $\Omega$ and for any fixed $x \in \Omega$, $t \mapsto f(x, t)$ is a continuous function.
Definition 2.1. (e.g., [8, p.52]) A function $u(x)$ is called a (weak) sub-solution of (8) if $u \in W^{1,p}(\Omega)$ ($p > 1$), $f(x,u(x))$ belongs to $L^{p/(p-1)}(\Omega)$ and

$$d \int_{\Omega} \nabla u \cdot \nabla \varphi \leq \int_{\Omega} f(x,u(x)) \varphi$$

(9)

for any $\varphi \in W^{1,p}(\Omega)$ with $\varphi \geq 0$ a.e. in $\Omega$. If the inequality in (9) is reversed, $u(x)$ is called a (weak) super-solution.

The following proposition is fundamental but useful and will play an important role in the proof of Theorem 1.1.

Proposition 2. (e.g., [8, Theorems 4.9 and 4.12]) Let $\underline{u}(x)$ and $\overline{u}(x)$ be (weak) sub- and super-solutions of (8), respectively, satisfying $\underline{u} \leq \overline{u}$ a.e. in $\Omega$. Suppose that there exists a function $k \in L^{p/(p-1)}(\Omega)$ ($p > 1$) such that $|f(x,t)| \leq k(x)$ for a.e. $x \in \Omega$ and all $t \in [\underline{u}(x),\overline{u}(x)]$.

Then (8) admits a weak solution $u(x)$ satisfying $\underline{u} \leq u \leq \overline{u}$ a.e. in $\Omega$.

Hereafter we consider (1) in the case when $\Omega$ is the multi-dimensional unit ball $B^n_1 = \{ x \in \mathbb{R}^n \ | \ |x| < 1 \}$ with $n \geq 2$. By referring the setting of $m(x)$ in [1] for the one-dimensional case, we set

$$m(x) = m_\varepsilon(x) = \begin{cases} 1/\varepsilon^n & \text{for } x \in B^n_\varepsilon, \\ 0 & \text{for } x \in B^n_1 \setminus B^n_\varepsilon \end{cases}$$

(10)

for any $0 < \varepsilon < 1$. From the viewpoint of the ecological model, the above setting of $m_\varepsilon(x)$ concentrates all resources near the center of the unit ball habitat. This location of resources differs from that in the one-dimensional case where all resources are put near an endpoint of the $(0,\ell)$.

Hence it follows that

$$\|m_\varepsilon\|_{L^1(B^n_1)} = |B^n_1|,$$

where $|B^n_1|$ denotes the volume of $B^n_1$. The following theorem is a main result of this paper which immediately leads to Theorem 1.1.

Theorem 2.2. Suppose that the dimension number $n$ satisfies $n \geq 2$. Then there exist positive constants $c_1$ and $c_2$ depending only on $n$ such that the unique positive solution $u_\varepsilon(x)$ of

$$\begin{cases} \frac{c_1}{\varepsilon^{n-2}} \Delta u + u(m_\varepsilon(x) - u) = 0 & \text{in } B^n_1, \\ \partial_\nu u = 0 & \text{on } \partial B^n_1. \end{cases}$$

(11)

satisfies

$$\frac{\|u_\varepsilon\|_{L^1(B^n_1)}}{\|m_\varepsilon\|_{L^1(B^n_1)}} \geq c_2 \left( 1 - \frac{1}{\varepsilon} + \frac{n}{\varepsilon} \log \varepsilon \right)$$

for any $0 < \varepsilon < 1$.

Proof. It follows from Proposition 1 that for each

$$d = d_\varepsilon := \frac{c_1}{\varepsilon^{n-2}}$$

(12)

and $m_\varepsilon(x)$ introduced by (10), there exists a unique positive solution $u_\varepsilon(x)$ of (11). By virtue of Proposition 2, if we can find a super-solution $\overline{u}_\varepsilon(x)$ and a sub-solution $\underline{u}_\varepsilon(x)$ satisfying

$$0 < \underline{u}_\varepsilon \leq \overline{u}_\varepsilon \leq u_\varepsilon \leq \overline{u}_\varepsilon \leq \overline{u}_\varepsilon$$

in $B^n_1$,
then \( \underline{u}_\varepsilon \leq u_\varepsilon \leq \overline{u}_\varepsilon \) in \( B^n_1 \). Since \( \|m_\varepsilon\|_{L^1(B^n_1)} = |B^n_1| \) is independent of \( 0 < \varepsilon < 1 \), then our strategy is to construct a sub-solution \( \underline{u}_\varepsilon(x) \) and a super-solution \( \overline{u}_\varepsilon(x) \) satisfying not only (13) but also

\[
\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^1(B^n_1)} \to \infty.
\]

To do so, we introduce two functions \( \overline{u}_\varepsilon(x) \) and \( \underline{u}_\varepsilon(x) \) defined over \( B^n_1 \) as

\[
\overline{u}_\varepsilon(x) := \frac{1}{\varepsilon^n} \quad \text{for} \ x \in B^n_1
\]

and

\[
\underline{u}_\varepsilon(x) := \begin{cases} \frac{c_2}{\varepsilon^n} e^{-|x|^n/\varepsilon^n} & \text{for} \ x \in B^n_1, \\ \frac{c_2}{\varepsilon^n} & \text{for} \ x \in \overline{B^n_1} \setminus B^n_\varepsilon. \end{cases}
\]

Here \( c_2 \) will be determined later independently of \( 0 < \varepsilon < 1 \). It is easily verified that \( u_\varepsilon(x) \) is in the \( C^2 \) class except for \( |x| = \varepsilon \), but still in the \( C^1 \) class. In what follows, we seek for a range of parameters \( (c_1, c_2) \) so that

(a) \( \overline{u}_\varepsilon(x) \) is a super-solution of (1),

(b) \( u_\varepsilon(x) \) is a sub-solution of that, and

(c) \( 0 < u_\varepsilon \leq \overline{u}_\varepsilon \) in \( B^n_1 \).

Since \( m_\varepsilon(x) \) is defined as (10), then \( \overline{u}_\varepsilon(x) \equiv 1/\varepsilon^n \) satisfies

\[
d_\varepsilon \Delta \overline{u}_\varepsilon + \overline{u}_\varepsilon (m_\varepsilon(x) - u_\varepsilon) = \begin{cases} 0 & \text{for} \ x \in B^n_1, \\ -u_\varepsilon^2 & \text{for} \ x \in \overline{B^n_1} \setminus B^n_\varepsilon. \end{cases}
\]

and \( \partial_\nu \overline{u}_\varepsilon = 0 \) on \( \partial B^n_1 \). Hence \( \overline{u}_\varepsilon(x) \) is a super-solution of (11).

Concerning (b), we have to check the inequality of (9):

\[
d_\varepsilon \int_{B^n_1} \nabla \underline{u}_\varepsilon \cdot \nabla \varphi \leq \int_{B^n_1} \underline{u}_\varepsilon (m_\varepsilon - \underline{u}_\varepsilon) \varphi
\]

for any \( \varphi \in W^{1,p}(B^n_1) \) with \( \varphi \geq 0 \) a.e. in \( B^n_1 \). Thanks to the fact

\[
\underline{u}_\varepsilon \in C^2(B^n_1 \setminus \{|x| = \varepsilon\}) \cap C^1(\overline{B^n_1}),
\]

for the verification of (16), it suffices to show that

\[
d_\varepsilon \Delta \underline{u}_\varepsilon + \underline{u}_\varepsilon (m_\varepsilon - \underline{u}_\varepsilon) \geq 0 \quad \text{for} \ x \in B^n_1 \setminus \{|x| = \varepsilon\}
\]

and

\[
\partial_\nu \underline{u}_\varepsilon \leq 0 \quad \text{on} \ \partial B^n_1.
\]

The boundary condition (18) is obviously satisfied. Then our crucial task is to find a parameter range of \( (c_1, c_2) \) satisfying (17). Since \( u_\varepsilon(x) \) is a radial function, we know that the required inequality (17) is equivalent to

\[
\begin{cases} d_\varepsilon \left( v''_\varepsilon + \frac{n-1}{r} v'_\varepsilon \right) + v_\varepsilon (\tilde{m}_\varepsilon(r) - v_\varepsilon) \geq 0 & \text{for} \ 0 < r < 1 \text{ and } r \neq \varepsilon, \\ v'_\varepsilon(0) = 0, \end{cases}
\]

where \( v_\varepsilon(r) := u_\varepsilon(x) \) and \( \tilde{m}_\varepsilon(r) := m_\varepsilon(x) \) for \( r = |x| \in [0, 1] \), that is,

\[
v_\varepsilon(r) = \begin{cases} \frac{c_2}{\varepsilon^n} e^{-r^n/\varepsilon^n} & (0 \leq r \leq \varepsilon), \\ \frac{c_2}{\varepsilon^n} (\varepsilon < r \leq 1), \end{cases} \quad \tilde{m}_\varepsilon(r) = \begin{cases} 1/\varepsilon^n & (0 \leq r \leq \varepsilon), \\ 0 & (\varepsilon < r \leq 1), \end{cases}
\]
and the prime symbol \( ' \) represents the derivative by \( r \). Then straightforward calculations yield

\[
v'_\varepsilon(r) = \begin{cases} 
- \frac{c_2n n^{-1}}{\varepsilon^{2n}} e^{-r^n/\varepsilon^n} & (0 \leq r < \varepsilon), \\
- \frac{c_2n}{e^{r^n+1}} & (\varepsilon < r \leq 1)
\end{cases}
\]

and

\[
v''_\varepsilon(r) = \begin{cases} 
\frac{c_2 n (n-1) r^{n-2}}{\varepsilon^{2n}} - \frac{n r^n}{(n-1)\varepsilon^n} - 1 & (0 \leq r < \varepsilon), \\
\frac{c_2 n (n+1)}{e^{r^n+2}} & (\varepsilon < r \leq 1).
\end{cases}
\]

Here it should be noted that \( \upsilon_\varepsilon \in C^2([0, \varepsilon] \cup (\varepsilon, 1]) \cap C^1([0, 1]) \) and the multi-dimensional situation \( n \geq 2 \) ensures \( \upsilon'_\varepsilon(0) = 0 \). For \( 0 < r < \varepsilon \), one can see

\[
\frac{c_1}{\varepsilon^{n-2}} \left( \upsilon''_\varepsilon + \frac{n-1}{r} \upsilon'_\varepsilon \right) + \upsilon_\varepsilon \left( \frac{1}{\varepsilon^n} - \upsilon_\varepsilon \right) = e^{-r^n/\varepsilon^n} \left( \frac{c_1 c_2 n^2}{\varepsilon^{4n-2}} r^{2n-2} - \frac{2 c_1 c_2 n (n-1)}{\varepsilon^{3n-2}} r^{n-2} + \frac{c_2}{\varepsilon^{2n}} - \frac{c_2^2}{\varepsilon^{2n}} e^{-r^n/\varepsilon^n} \right).
\]

To assure the positive minimum of the right-hand side, we estimate the bracket part as follows

\[
\frac{c_1 c_2 n^2}{\varepsilon^{4n-2}} r^{2n-2} - \frac{2 c_1 c_2 n (n-1)}{\varepsilon^{3n-2}} r^{n-2} + \frac{c_2}{\varepsilon^{2n}} - \frac{c_2^2}{\varepsilon^{2n}} e^{-r^n/\varepsilon^n} > - \frac{2 c_1 c_2 n (n-1)}{\varepsilon^{2n}} + \frac{c_2}{\varepsilon^{2n}} - \frac{c_2}{\varepsilon^{2n}} e^{-r^n/\varepsilon^n} = \frac{c_2}{\varepsilon^{2n}} (1 - 2 c_1 n (n-1) - c_2) \quad \text{for any } 0 < r < \varepsilon.
\]

Thus if \( 1 - 2 c_1 n (n-1) - c_2 \geq 0 \), then the differential inequality (19) holds for \( 0 < r < \varepsilon \). On the other hand, for \( \varepsilon < r < 1 \), we know

\[
\frac{c_1}{\varepsilon^{n-2}} \left( \upsilon''_\varepsilon + \frac{n-1}{r} \upsilon'_\varepsilon \right) - \upsilon'_\varepsilon = \frac{c_2}{e^{r^n+1}} \left( \frac{2 c_1 n}{\varepsilon^{n-2}} - \frac{c_2}{e^{r^n-2}} \right)
\]

and

\[
\frac{2 c_1 n}{\varepsilon^{n-2}} - \frac{c_2}{e^{r^n-2}} \geq \frac{1}{\varepsilon^{n-2}} \left( 2 c_1 n - \frac{c_2}{e} \right).
\]

Thus if \( 2 c_1 n - c_2/e \geq 0 \), then (19) holds for \( \varepsilon < r < 1 \). Therefore, we know that if \( (c_1, c_2) \) satisfies

\[
1 - 2 c_1 n (n-1) - c_2 \geq 0 \quad \text{and} \quad 2 c_1 n - c_2/e \geq 0,
\]

then (17) holds, and thereby, the required (b) is satisfied. Here it is noted that the set

\[
T := \{ (c_1, c_2) \in \mathbb{R}_{\geq 0}^2 \mid (c_1, c_2) \text{ satisfies (21)} \}
\]

forms a triangle whose vertices are

\[
(c_1, c_2) = (0, 0), \left( \frac{1}{2n(\varepsilon + n - 1)}, \frac{\varepsilon}{\varepsilon + n - 1} \right), \left( \frac{1}{2n(n-1)}, 0 \right).
\]

The final condition (c) \( \underline{u}_\varepsilon \leq \overline{u}_\varepsilon \) in \( B^n_1 \) holds true if and only if \( c_2 \leq 1 \). However the condition \( c_2 \leq 1 \) is already necessary for (21).

Consequently, we can deduce that if \( (c_1, c_2) \in T \), then \( \overline{u}_\varepsilon(x) \) and \( u_\varepsilon(x) \) introduced by (14) and (15) satisfies (a)-(c). Therefore, Propositions 1 and 2 imply that the
unique positive solution $u_\varepsilon(x)$ of (11) satisfies $\underline{u}_\varepsilon(x) \leq u_\varepsilon(x) \leq \overline{u}_\varepsilon(x)$ for all $x \in B_1^\varepsilon$.

In view of (20), one can see that

$$\|u_\varepsilon\|_{L^1(B_1^\varepsilon)} = A_n \int_0^1 v_\varepsilon(r)r^{n-1} dr$$

$$= c_2 A_n \left( \int_0^\varepsilon \frac{r^{n-1}}{\varepsilon^n} e^{-r^n/\varepsilon^n} dr + \int_\varepsilon^1 \frac{1}{er} dr \right)$$

$$= c_2 A_n \left( \frac{1}{n} \left( 1 - \frac{1}{e} \right) + \frac{1}{e} |\log \varepsilon| \right),$$

where $A_n$ denotes the surface area of $\partial B_1^n$. Since $\|m_\varepsilon\|_{L^1(B_0(1))} = |B_1^n| = A_n/n$, then we have

$$\frac{\|u_\varepsilon\|_{L^1(B_1^\varepsilon)}}{\|m_\varepsilon\|_{L^1(B_1^\varepsilon)}} = \frac{\|u_\varepsilon\|_{L^1(B_1^\varepsilon)}}{\|m_\varepsilon\|_{L^1(B_1^\varepsilon)}} = c_2 \left( 1 - \frac{1}{e} + \frac{n}{e} |\log \varepsilon| \right).$$

(22)

Thus the proof of Theorem 2.2 is complete.

**Proof of Theorem 1.1.** By setting $\varepsilon \to 0$ in (22), we see that the unique positive solution $u_\varepsilon(x)$ of (11) satisfies

$$\lim_{\varepsilon \to 0} \frac{\|u_\varepsilon\|_{L^1(B_1^\varepsilon)}}{\|m_\varepsilon\|_{L^1(B_1^\varepsilon)}} = \infty,$$

which implies (7). The proof of Theorem 1.1 is complete.

3. **Concluding remarks.** In this section, we give some concluding remarks. For each dimension number $n \geq 1$ and any $(d, m) \in (0, \infty) \times L^\infty_+ (B_1^\varepsilon)$, we define

$$I_n(d, m) := \frac{\|u_{d,m}\|_{L^1(B_1^\varepsilon)}}{\|m\|_{L^1(B_1^\varepsilon)}},$$

where $B_1^n := \{ x \in \mathbb{R}^n \mid |x| < 1 \}$. Concerning the maximizing problem to consider

$$M_n := \sup_{(d,m) \in (0,\infty) \times L^\infty_+ (B_1^\varepsilon)} I_n(d, m),$$

Theorem 1.1 reveals a fact that $M_n = \infty$ if $n \geq 2$, which is a big contrast to the one-dimensional situation $M_1 = 3$ obtained by [1].

In the one-dimensional case when $n = 1$, a maximizing sequence $(d_\varepsilon, m_\varepsilon) = (\sqrt{\varepsilon}, \varepsilon^{-1} \chi_{[0,\varepsilon]})$ realizes $I(d_\varepsilon, m_\varepsilon) \nearrow M_1 = 3$ as $\varepsilon \searrow 0$ ([1]), where $\chi_A$ denotes the characteristic function of the set $A$. This result says that, under the concentration setting of resource as $m_\varepsilon = \varepsilon^{-1} \chi_{[0,\varepsilon]}$, a small control of the diffusion rate as $d_\varepsilon = \sqrt{\varepsilon}$ can make $I_1(d_\varepsilon, m_\varepsilon)$ tend to the supremum 3 from below as $\varepsilon \to +0$. In this situation as $\varepsilon \to +0$, the profile of $u_\varepsilon(x)$ obtained by [15] shows that $u_\varepsilon(x)$ with $x \in [0,\varepsilon]$ grows with the order $O(1/\sqrt{\varepsilon})$. This result means that the species in the resource interval $[0,\varepsilon]$ cannot follow the height (the $L^\infty$ norm) $1/\varepsilon$ of resource. Furthermore, the singular limit $d_\varepsilon = \sqrt{\varepsilon} \to 0$ leads to the shrink property that $u_\varepsilon(x) \to 0$ uniformly in any compact set contained in $(0,1] \times \varepsilon$ as $\varepsilon \to +0$.

In the two-dimensional case when $n = 2$, Theorem 2.2 asserts that, under the concentration of resource near the center as $m_\varepsilon = \varepsilon^{-2} \chi_{B_2^\varepsilon}$, a middle control of the diffusion rate as $d_\varepsilon = c_1$ (independent of $\varepsilon$) can make $I_2(c_1, m_\varepsilon)$ tend to infinity as $\varepsilon \to +0$. Furthermore, the profile of the sub-solution $\underline{u}_\varepsilon = c_2 \varepsilon^{-2} \exp(-(|x|/\varepsilon)^2)$ for $x \in B_2^\varepsilon$ ensures that $u_\varepsilon(x) \geq \underline{u}_\varepsilon(x)$ for $x \in B_2^\varepsilon$ grows with the order $O(1/\varepsilon^2)$ as $\varepsilon \to +0$. This fact means that the species in the resource disk $B_2^\varepsilon$ can follow
the height (the $L^\infty$ norm) $1/\varepsilon^2$ of resource. On the other hand, in the no-resource annulus $B^n_\varepsilon \setminus B^2_\varepsilon$, the sub-solution $u_\varepsilon(x) = c_2\varepsilon^{-1}|x|^{-2}$ for $x \in B^n_\varepsilon \setminus B^2_\varepsilon$ implies that $u_\varepsilon(x) \geq u(x)$ for $x \in B^n_\varepsilon \setminus B^2_\varepsilon$. This fact is also a big difference from the one-dimensional case that $u_\varepsilon(x)$ decays to zero in any compact set contained in $(0,1]$ as $\varepsilon \to +0$.

In the higher dimensional case when $n \geq 3$, under the concentration of resource as $m_\varepsilon = \varepsilon^{-n} \chi_{B^n_\varepsilon}$, a large control of the diffusion rate as $d_\varepsilon = c_1/\varepsilon^{n-2}$ can make $I_n(d_\varepsilon, m_\varepsilon)$ tend to infinity as $\varepsilon \to +0$. In this situation as $\varepsilon \to +0$, the profile of the sub-solution $u_\varepsilon(x)$ tells us that $u_\varepsilon(x)$ can follow $m_\varepsilon(x)$ in the resource ball $B^n_\varepsilon$ with the same order $O(1/\varepsilon^n)$ and $u_\varepsilon(x) \geq u(x) = c_2\varepsilon^{-1}|x|^{-n}$ for the no-resource region $B^n_\varepsilon \setminus B^2_\varepsilon$.

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