Hodge completed derived de Rham algebra of a perfect ring

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Abstract

Derived de Rham cohomology has been recently used in several contexts, as in works of Beilinson, Bhatt and Morin. Inspired by some results of Morin, we aimed to compute Hodge completed derived de Rham complex in the case of a rings map \( \mathbb{Z} \to k \), factoring through \( \mathbb{F}_p \), with \( k \) a perfect ring (i.e. the Frobenius map is an automorphism).

1 Introduction

The derived de Rham complex has been introduced by Illusie [5, Ch. VIII] as a natural following of the definition of the cotangent complex for a scheme morphism ([4, Ch. II]). It provides a fruitful generalization of the de Rham theory for singular varieties (see for example [2]) as well as new construction of Fontaine\’s period ring ([1]) or numerical invariants for special values of zeta functions of varieties over finite fields ([6],[7]). Inspired by some results of Morin [7], we consider derived de Rham complex relative to \( \text{Spec}(\mathbb{Z}) \).

Our main result is the following

**Theorem 1.1.** Let \( k \) be a perfect ring. There is an equivalence of pro-systems of \( E_\infty \mathbb{Z} \)-algebras

\[
\left( \frac{L\Omega^*_k/\mathbb{Z}}{\mathbb{F}_N^N} \right)_N \simeq (W_N(k))_N,
\]

where on the right-hand-side is the ring of Witt vectors associated to \( k \).

Such result implies in particular that the Hodge completed derived de Rham algebra relative to \( \mathbb{Z} \to k \) is quasi-isomorphic to the ring of Witt vectors \( W(k) \). In particular this result can be seen as a natural geometric definition of the ring of Witt vectors. Bhatt computed the derived de Rham complex (not completed) \( p \)-adically completed in the same case (see [3] Corollary 8.6). In particular he showed that when \( k \) is perfect, the ring \( W(k) \) may be obtained as the largest separated torsion-free quotient of the \( p \)-adically completed derived de Rham complex (ibidem Remark 8.7), while in the Hodge-completed case Witt vectors arises more naturally and without extra parts.

The theorem relies on the base change applied to the (crucial) simple case where \( k = \mathbb{F}_p \). Similar results may be obtained by means of crystalline theory computations, see [3] Ch.VIII Proposition 2.2.8]. Here we give a more direct and elementary proof, which takes into account the multiplicative structure of the \( E_\infty \)-algebras (say also differential graded algebras).

2 Preliminaries

Recall that a differential graded algebra (dga) over some (commutative unitary) ring \( A \) is given by a complex of \( A \)-modules \( C^* \) and a map of cochain complexes

\[
C^* \otimes_A C^* \to C^*,
\]
unital and associative in the obvious sense. Further a dga is graded commutative if it is endowed of a map of cochain complexes
\[ C^* \otimes_A C^* \longrightarrow C^* \otimes_A C^* \]
\[ x \otimes y \longrightarrow (-1)^{\deg x \deg y} y \otimes x. \]
such that compositions with the previous map gives a commutative triangle. For now on we consider differential graded algebra graded commutative, so we may omit to specify it when clear by the context. We note as dga the corresponding category.

Given an A-algebra B, we can consider the associated standard simplicial resolution \( P_* \longrightarrow B \) (\([4, I.1.5]\) or more explicitly \([4, I.1.2.1]\)), by applying the de Rham complex functor \( \Omega^*_{-/A} \) we obtain a simplicial \( A \)-dga. If we take the associated simple cochain complex on the horizontal lines we get the double complex \( \Omega^*_{P/\mathcal{A}} \).

Remark 2.1. Hodge completed derived de Rham complex has a non-completed version \( L\Omega^*_{B/A} \) by taking the total complex associated to \( (\Omega^*_{P/\mathcal{A}}, \partial, d) \). The two constructions provide two very different objects. For example, while the second one is useless in characteristic 0 (see \([3, \text{Corollary 2.5}]\)), the first one has been proved to provide the right cohomology for (singular) varieties in characteristic 0 (see \([2]\) in general, more precisely Corollary 4.27).

1Recall that a \((i,j)\)-shuffle consists of a permutation \( (\mu, \nu) := (\mu_1 \ldots \mu_i \nu_1 \ldots \nu_j) \in S_{i+j} \) such that \( \mu_1 < \ldots < \mu_i \) and \( \nu_1 < \ldots < \nu_j \).
Remark 2.2. As we defined it, \( L_\Omega^*_B/A \) is a differential graded algebra over \( A \). However in general it is considered in a broader context as an \( E_\infty \)-algebra (see [2, Remark 4.2] for example). This is also due to the fact that given any simplicial \( A \)-algebra resolution \( P_\bullet \longrightarrow B \) whose terms are free, by replacing it to the standard resolution \( P_\bullet \) in the definition of \( L_\Omega^*_B/A \), the output is naturally quasi-isomorphic, in particular equivalent, to the Hodge completed derived de Rham complex (see [2, Theorem 2.25 and Remark 2.26] for an explicit proof). Through these pages we try to work in the context of classical category theory, as in [9]. We are considering maps of differential graded algebras, which are quasi-isomorphisms as maps of complexes. In the associated derived category, it yields to isomorphisms which satisfies the commutativity of some diagrams induced by the multiplication maps. This means that they induce equivalences of \( E_\infty \)-algebras.

3 The complex \( L_\Omega^*_{F_p/Z} \)

We want to compute the Hodge-completed derived de Rham complex for the map \( \mathbb{Z} \rightarrow F_p \).

**Proposition 3.1.** For any \( N > 0 \), there is an equivalence of \( E_\infty \)-algebras over \( \mathbb{Z} \)

\[
\frac{L\Omega^*_p/F^N}{F^N} \cong \frac{\mathbb{Z}}{p^n\mathbb{Z}}.
\]

**Proof.** First we study \( \text{gr}^n L_\Omega^*_{F_p/Z} \) in order to have some hints of its structure, in particular we are going to prove that its cohomology is concentrated in degree 0. This will provide an equivalence \( L\Omega^*_p/F^N \longrightarrow H^0(L\Omega^*_p/F^N) \) as \( E_\infty \)-algebras. Then we compute \( H^0(L\Omega^*_p/F^N) \) as a filtered ring. Such structure does not appear in the statement of the proposition, but it will be fundamental in order to "reconstruct" the isomorphism \( H^0(L\Omega^*_p/F^N) \cong \mathbb{Z}/p^N\mathbb{Z} \).

**Computing the graded pieces.** First of all, since (3) is a surjective morphism with kernel equal to \( p\mathbb{Z} \), i.e. generated by a regular element, the cotangent complex \( L_{F_p/Z} \cong \frac{\mathbb{Z}}{p^2\mathbb{Z}}[1] \), the trivial complex concentrated in degree –1, (see for example [9] Proposition 2.16).

Note that \( \frac{\mathbb{Z}}{p^2\mathbb{Z}} \cong F_p \) is a free \( F_p \)-module, so that, in the derived category \( D(A) \),

\[
\text{gr}^n \left( L\Omega^*_p/F^N \right) \cong \left( L \wedge^n \left( \frac{\mathbb{Z}}{p^2\mathbb{Z}}[1] \right) \right) [-n]
\]

\[
\cong \left( L\Gamma^n \left( \frac{\mathbb{Z}}{p^2\mathbb{Z}} \right) \right) [n] [-n]
\]

\[
\cong \left( \frac{\mathbb{Z}}{p^2\mathbb{Z}} \otimes_{F_p} \ldots \otimes_{F_p} \frac{\mathbb{Z}}{p^2\mathbb{Z}} \right) S_n \left[ 0 \right],
\]

for \( n < N \) (otherwise it equals 0), where the tensor product is performed \( n \) times. Here (4) follows from (2.1.1.5) in [5], (5) from [4, Ch. 1] Proposition 4.3.2.1., for (6) see, for example, [8] Proposition IV.5. Now we see that by induction

\[
\frac{\mathbb{Z}}{p^2\mathbb{Z}} \otimes_{F_p} \ldots \otimes_{F_p} \frac{\mathbb{Z}}{p^2\mathbb{Z}} \cong \frac{p^n\mathbb{Z}}{p^{n+1}\mathbb{Z}}
\]

\( n \) times

and the action of the symmetric group \( S_n \) becomes trivial.
We obtain
\[ H^0 \left( \text{gr}^n \left( L\Omega^*_p/\mathbb{Z}/F^N \right) \right) \cong \frac{p^n\mathbb{Z}}{p^{n+1}\mathbb{Z}} \] (7)
if \( n < N \), and 0 else. In particular, since every graded pieces is concentrated in degree 0, the derived de Rham complex is concentrated in degree 0. Now consider the following arrows in filtration, by putting for applying the canonical truncations \( \tau \geq 0, \tau \leq 0 \)
\[
L\Omega^*_p/\mathbb{Z}/F^N \longrightarrow \tau \leq 0 \left( L\Omega^*_p/\mathbb{Z}/F^N \right) \longrightarrow \tau \geq 0 \left( \tau \leq 0 \left( L\Omega^*_p/\mathbb{Z}/F^N \right) \right) = H^0( L\Omega^*_p/\mathbb{Z}/F^N ) .
\]
The two arrows turn out to be quasi-isomorphisms of (commutative) differential graded algebras. In the associated derived category we have an isomorphism \( L\Omega^*_p/\mathbb{Z}/F^N \longrightarrow H^0( L\Omega^*_p/\mathbb{Z}/F^N ) \) and a commutative diagram
\[
L\Omega^*_p/\mathbb{Z}/F^N \otimes_{L\mathbb{Z}} L\Omega^*_p/\mathbb{Z}/F^N \longrightarrow H^0( L\Omega^*_p/\mathbb{Z}/F^N ) \otimes_{\mathbb{Z}} H^0( L\Omega^*_p/\mathbb{Z}/F^N ) \\
L\Omega^*_p/\mathbb{Z}/F^N \longrightarrow H^0( L\Omega^*_p/\mathbb{Z}/F^N )
\]
As \( E_\infty \)-algebras, this means that we have an equivalence \( L\Omega^*_p/\mathbb{Z}/F^N \longrightarrow H^0( L\Omega^*_p/\mathbb{Z}/F^N ) \). It remains to compute \( H^0( L\Omega^*_p/\mathbb{Z}/F^N ) \) as a ring.

**Computation of** \( H^0( L\Omega^*_p/\mathbb{Z}/F^N ) \). \( L\Omega^*_p/\mathbb{Z}/F^N \) is a \( \mathbb{Z} \)-dga, so we have a structure morphism
\[
\mathbb{Z} \longrightarrow L\Omega^*_p/\mathbb{Z}/F^N
\] (8)
sending \( \mathbb{Z} \ni 1 \longrightarrow 1 \in P_0 \subset L\Omega^*_p/\mathbb{Z}/F^N \). Further, calling \( D \) the differential of \( L\Omega^*_p/\mathbb{Z}/F^N \),
\[
D^0(1) = d1 = 0,
\]
so that (8) factors trough \( \text{ker} D^0 \), in particular this induces a morphism of graded rings on cohomology
\[
H^*(\mathbb{Z}) = \mathbb{Z} \longrightarrow H^*( L\Omega^*_p/\mathbb{Z}/F^N )
\]
and in particular we have a morphism of rings at level 0, i.e. for any \( N > 0 \)
\[
H^0(\mathbb{Z}) = \mathbb{Z} \longrightarrow H^0( L\Omega^*_p/\mathbb{Z}/F^N ) .
\] (9)
Now \( H^0( L\Omega^*_p/\mathbb{Z}/F^N ) \) inherits a filtration from the Hodge filtration on the derived de Rham complex (modulation filtration), by putting for \( i < N \)
\[
\text{Fil}^i H^0( L\Omega^*_p/\mathbb{Z}/F^N ) := \frac{F^i Z^0( L\Omega^*_p/\mathbb{Z}/F^N )}{F^i H^0( L\Omega^*_p/\mathbb{Z}/F^N )} = \frac{\text{ker} D^0 \cap F^i \left( L\Omega^*_p/\mathbb{Z}/F^N \right)}{F^{i-1} \left( L\Omega^*_p/\mathbb{Z}/F^N \right) \cap F^i \left( L\Omega^*_p/\mathbb{Z}/F^N \right)}
\]
and \( \text{Fil}^N H^0( L\Omega^*_p/\mathbb{Z}/F^N ) = 0 \).

**Remark 3.2.** Consider the spectral sequence associated to the Hodge filtration
\[
E_1^{n,m} = H^{n+m} \left( \text{gr}^n \left( L\Omega^*_p/\mathbb{Z}/F^N \right) \right) \Rightarrow H^{n+m} \left( L\Omega^*_p/\mathbb{Z}/F^N \right) .
\]
We have \( E_1^{n,m} = 0 \) for \( m + n \neq 0 \), so the sequence degenerates and we have
\[
\frac{p^n\mathbb{Z}}{p^{n+1}\mathbb{Z}} \cong H^0 \left( \text{gr}^n_p \left( L\Omega^*_p/\mathbb{Z}/F^N \right) \right) \cong \text{gr}^n_p H^0 \left( L\Omega^*_p/\mathbb{Z}/F^N \right)
\] (10)
for \( n < N \) and 0 otherwise.

\textsuperscript{2}Note that the unity in \( L\Omega^*_B/\mathbb{A}/F^N \) is the element \( (1, 0, \ldots, 0) \in \oplus_{i \leq N-1} \Omega^i_{B/\mathbb{A}} \).
We can also define a filtration on \( \mathbb{Z} \) and \( \mathbb{Z}/p^N \mathbb{Z} \) as follows: \( F^i \mathbb{Z} := p^i \mathbb{Z} \) and \( F^i \mathbb{Z}/p^N \mathbb{Z} := p^i \mathbb{Z}/p^N \mathbb{Z} \).

Now we want to show that the ring morphism (9) induces a filtered ring morphism
\[
\frac{\mathbb{Z}}{p^N \mathbb{Z}} \longrightarrow H^0 \left( \frac{L^N \Omega^*_{F_p/\mathbb{Z}}}{F^N} \right). \tag{11}
\]

It is enough to show that \( H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N) \) is an abelian group of order \( p^n \) for any \( N > 0 \). We prove this by induction. We have \( H^0(L^0 \Omega^*_{F_p/\mathbb{Z}}) = H^0(\text{gr}^0 L^0 \Omega^*_{F_p/\mathbb{Z}}) \cong \mathbb{F}_p \), so we may assume that the statement holds for \( N > 1 \). Consider the following short exact sequence induced in cohomology by the Hodge filtration
\[
0 \longrightarrow H^0 \left( \frac{F^N L^N \Omega^*_{F_p/\mathbb{Z}}}{F^{N+1}} \right) \longrightarrow H^0 \left( \frac{L^N \Omega^*_{F_p/\mathbb{Z}}}{F^{N+1}} \right) \longrightarrow H^0 \left( \frac{L^N \Omega^*_{F_p/\mathbb{Z}}}{F^N} \right) \longrightarrow 0. \tag{12}
\]

By (7) the first group has order \( p \). The third group is of order \( p^N \) by assumption. Thus the one in the middle must have order \( p^{N+1} \), which proves our claim. So we have map (11) for any \( N > 0 \).

Now we consider the following situation in the category of abelian groups
\[
\frac{\mathbb{Z}}{p^N \mathbb{Z}} \longrightarrow H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N) \longrightarrow \text{gr}^0 H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N),
\]

where the second arrow is the surjection onto
\[
\text{gr}^0_{F \text{fil}} H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N) = \frac{F \text{il}^0 H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N)}{F \text{il}^1} \cong \mathbb{F}_p \tag{13}
\]
as seen in (10). Since \( 1 \notin \text{Im} D^{-1} \) (the cokernel of the map is \( \mathbb{F}_p \), because it is its standard resolution) and neither \( 1 \notin F^1 L^N \Omega^*_{F_p/\mathbb{Z}} \), we are considering non-zero maps of groups. In particular we have that
\[
\mathbb{Z}/pN \mathbb{Z} \ni 1 \mapsto 1 \in \text{gr}^0_{F \text{fil}} H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N) = \mathbb{F}_p
\]
and this implies that \( p \mathbb{Z} \) goes to zero. Hence the inclusion \( p \mathbb{Z}/p^N \mathbb{Z} \hookrightarrow \mathbb{Z}/p^N \mathbb{Z} \) together with the group morphism associated to the map (11) factors as follow
\[
\frac{\mathbb{Z}}{p^N \mathbb{Z}} \longrightarrow H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N) \quad \text{with} \quad \frac{\mathbb{Z}}{p^N \mathbb{Z}} \longrightarrow F \text{il}^1 H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N).
\]

Fix \( N > 0 \). As above, the short exact sequence
\[
0 \longrightarrow F \text{il}^{n-1} \quad \text{Fil}^n \longrightarrow H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N) \quad \text{Fil}^n \longrightarrow H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N) \quad \text{Fil}^{n-1} \longrightarrow 0 \tag{14}
\]
shows that \( H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N) \) is an abelian group of order \( p^n \), hence the map (11) factors for all \( n < N \) (and for \( n = N \) trivially)
\[
\frac{\mathbb{Z}}{p^N \mathbb{Z}} \longrightarrow H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N) \quad \text{with} \quad \frac{\mathbb{Z}}{p^N \mathbb{Z}} \longrightarrow F \text{il}^n H^0(L^N \Omega^*_{F_p/\mathbb{Z}}/F^N).
\]
which means that we have a morphism of $\mathbb{Z}$–modules compatible with the filtrations, that is $\text{(11)}$ is a map of filtered rings. Looking at the morphism induced on the graded pieces, we have for $n < N$

\[
\begin{array}{c}
p^n\mathbb{Z}/p^{n+1}\mathbb{Z} \\
\cong \text{Fil}^n H^0(L\Omega^*_{\mathbb{Z}/\mathbb{Z}}/F^N) \\
\cong \frac{p^n\mathbb{Z}}{p^{n+1}\mathbb{Z}}
\end{array}
\]

which is a non-zero map of cyclic abelian group of the same order. Thus they are isomorphic.

By induction (starting from $N$ to 0) on the following diagram, where the rows are exact sequences, with $n < N$,

\[
\begin{array}{c}
\text{Fil}^n \\
\cong \\
\text{Fil}^{n-1} \\
\text{Fil}^n
\end{array}
\]

\[
\begin{array}{c}
p^n\mathbb{Z}/p^{n+1}\mathbb{Z} \\
\cong \frac{p^n\mathbb{Z}}{p^{n+1}\mathbb{Z}}
\end{array}
\]

we can deduce that for $0 < n \leq N$ there is an isomorphism of abelian groups

\[
\begin{array}{c}
p^n\mathbb{Z}/p^{n+1}\mathbb{Z} \\
\cong \text{Fil}^n H^0(L\Omega^*_{\mathbb{Z}/\mathbb{Z}}/F^N).
\end{array}
\]

Finally, we consider again the short exact sequence $\text{(14)}$ for the case $n = N$, and we get that for any $N > 0$ there is the following isomorphism of rings

\[
\begin{array}{c}
\mathbb{Z}/p^N\mathbb{Z} \\
\cong H^0 \left( L\Omega^*_{\mathbb{Z}/\mathbb{F}}/F^N \right).
\end{array}
\]

\vspace{1em}

\textbf{Remark 3.3.} This result says that $L\widehat{\Omega}^*_{\mathbb{Z}/\mathbb{F}}$ is equivalent to the ring of $p$–adic integers, seen as projective system $\widehat{\mathbb{Z}}_p := (\mathbb{Z}/p^N\mathbb{Z})_N$.

\vspace{1em}

\section{Main result}

In order to prove Theorem 1.1 we first make the following consideration. Given a perfect $\mathbb{F}_p$–algebra $k$, there is the following diagram of rings

\[
\begin{array}{c}
W \rightarrow k = W \otimes_{\mathbb{Z}_p} \mathbb{F}_p \\
\uparrow \uparrow \\
\mathbb{Z}_p \rightarrow \mathbb{F}_p
\end{array}
\]

where $W = W(k)$. Thus we can apply the base change property (see lemma), giving a result similar to Theorem 1.1 but for a Hodge completed derived de Rham complex over $W(k)$. We first need some lemmas.

\textbf{Lemma 4.1 (Base change).} Suppose that $A \rightarrow B$, $A \rightarrow C$ are Tor–independent (see [4, Chap. II 2.2.2]), the canonical map

\[
L\widehat{\Omega}^*_{B/A} \otimes_A^L C \rightarrow L\widehat{\Omega}^*_{B \otimes_AC/A}
\]

is a quasi-isomorphism.
Proof. Let \( P_* \rightarrow B \) the standard simplicial resolution of the \( A \)-algebra \( B \). Since \( B, C \) are \( \text{Tor} \)-independent the associated chain complex of \( P_* \otimes_A C \) is a free simplicial resolution of the \( C \)-algebra \( B \otimes_A C \) and can be used to compute \( L\bar{\Omega}^{*}_{B \otimes_A C/A} \). On the other hand,

\[
\text{Tot}(\Omega^*_{P_* \otimes_A C/C}) \cong \text{Tot}(\Omega^*_{P_*/A} \otimes_A C) \cong \text{Tot}(\Omega^*_{P_*/A}) \otimes_A C.
\]

Remark 4.2. Being canonical, the base change (quasi-)isomorphism yields a commutative diagram in the associated derived category

\[
\begin{array}{ccc}
\left( L\bar{\Omega}^*_{B/A}/F^N \otimes^L_A C \right) \otimes^L_A \left( L\bar{\Omega}^*_{B/A}/F^N \otimes^L_A C \right) & \rightarrow & L\bar{\Omega}^*_{B\otimes_A C/C}/F^N \otimes^L_C L\bar{\Omega}^*_{B\otimes_A C/C}/F^N \\
L\bar{\Omega}^*_{B/A}/F^N \otimes^L_A C & \rightarrow & L\bar{\Omega}^*_{B\otimes_A C/C}/F^N
\end{array}
\]

The following Lemma 4.3 and Lemma 4.6 are similar (but proofs are a little different), they allow us to change the "ring of coefficients" for the ring \( k \).

Lemma 4.3. Let \( k \) be a perfect ring of characteristic \( p \) and \( W := W(k) \) its Witt vectors ring. Given a ring homomorphism \( W \rightarrow B \), then the canonical map \( L\bar{\Omega}^*_{B/Z} \rightarrow L\bar{\Omega}^*_{B/W} \) is a quasi-isomorphism.

Proof. See [9, Lemma 3.28].

Remark 4.4. Recall that \( W \) is relatively perfect over \( \mathbb{Z}_p \), so that the cotangent complex \( LW/Z \) is acyclic (see [3, Corollary 3.8]).

Remark 4.5. The result holds also for the non-completed case. Bhatt proved it by means of the conjugate filtration, which yields a spectral sequence convergent to the non completed derived de Rham complex (see [3, Proposition 2.3 and Lemma 8.3(5)]).

Lemma 4.6. Given a sequence of morphisms of rings \( Z \rightarrow Z_p \rightarrow F_p \rightarrow B \), there exists a quasi-isomorphism between the Hodge completed derived de Rham (differential graded) algebras

\[
L\bar{\Omega}^*_{B/Z} \rightarrow L\bar{\Omega}^*_{B/Z_p}.
\]

Proof. As a matter of fact, we have a quasi-isomorphism

\[
L_B/Z \rightarrow L_B/Z_p.
\]

If we consider the case \( B = F_p \), such result can be easily proved by direct computation. From the sequence of the statement we may deduce the following diagram of associated exact triangles

\[
\begin{array}{cccc}
L_{F_p/Z} \otimes^L_{F_p} B & \rightarrow & L_B/Z & \rightarrow & L_B/F_p \rightarrow L_{F_p/Z} \otimes^L_{F_p} B[1] \\
\cong & & & & \cong \\
L_{F_p/Z_p} \otimes^L_{F_p} B & \rightarrow & L_B/Z_p & \rightarrow & L_{F_p/Z_p} \otimes^L_{F_p} B[1]
\end{array}
\]

which gives us (15). The quasi-isomorphism on the cotangent complex leads to the claim for the Hodge completed derived de Rham complex (and its quotient pieces).

Hence, by taking the pro-system of complexes associated, we are done.
Now we can compute the derived de Rham complex for a perfect ring $k$.

**Lemma 4.7.** Let $k$ a perfect ring and $W = W(k)$ its ring of Witt vectors. Let $\hat{W} = (W_N(k))_N$ the projective system of truncated pieces of $W$. Then the canonical base-change map $L\hat{\Omega}^*_{k/W} \otimes_{\mathbb{Z}_p} W \rightarrow L\hat{\Omega}^*_{k/W}$ induces an equivalence of $E_\infty$–algebras pro-systems

$$L\hat{\Omega}^*_{k/W} \simeq \hat{W}.$$

**Proof.** The canonical base-change map is induced by the fact that, since $k$ is perfect, $k = \mathbb{F}_p \otimes_{\mathbb{Z}_p} W$. Further $\mathbb{F}_p$ and $W$ are $\mathbb{Z}_p$–algebras Tor-independent. Consider the following exact sequence, coming from the free resolution of $\mathbb{F}_p$, as $\mathbb{Z}_p$–module,

$$0 \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Z}_p \rightarrow \mathbb{F}_p \rightarrow 0.$$

If we tensorize by $\otimes_{\mathbb{Z}_p} W$, we get

$$0 \rightarrow W \stackrel{p}{\rightarrow} W \rightarrow k \rightarrow 0. \quad (*)$$

It is again an exact sequence, so that $0 = H^i(\mathbb{F}_p) = \text{Tor}_N^i(k, \mathbb{F}_p)$, for any $i > 0$, which proves the Tor–independence. Hence we can apply the base change lemma and we get the following equivalences

$$L\hat{\Omega}^*_{k/W} = L\hat{\Omega}^*_{W \otimes_{\mathbb{Z}_p} \mathbb{F}_p/W \otimes_{\mathbb{Z}_p} \mathbb{Z}_p} \quad (16)$$

$$\simeq L\hat{\Omega}^*_{\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W} \quad (17)$$

$$\simeq \left( \frac{\mathbb{Z}_p}{p \mathbb{Z}_p} \right)_N \otimes_{\mathbb{Z}_p} W \quad (18)$$

$$\simeq \left( \frac{W}{p^N W} \right)_N \quad (19)$$

$$\simeq \hat{W}_p, \quad (20)$$

$\hat{W}_p$ being the $p$–adic projective system of the ring of Witt vectors $W$. Since $k$ is perfect, $W/p^N W \cong W_N$, so the projective system $\hat{W}_p$ equals $\hat{W}$.

Now we recollect all these results to get Theorem 1.1.

**Proof.** (Theorem 1.1). By Lemma 4.3, Lemma 4.6 we get $L\hat{\Omega}^*_{k/\mathbb{Z}} \simeq L\hat{\Omega}^*_{k/\mathbb{Z}_p} \simeq L\hat{\Omega}^*_{k/W}$. Then apply Lemma 4.7.

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