Thermodynamical and spectral phase transition for local diffeomorphisms in the circle

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Abstract

It is known that all uniformly expanding dynamics have no phase transition with respect to Hölder continuous potentials. In this paper we show that given a local diffeomorphism $f$ on the circle, that is neither a uniformly expanding dynamics nor invertible, the topological pressure function $\mathbb{R} \ni t \mapsto P_{\text{top}}(f, -t \log |Df|)$ is not analytical. In other words, $f$ has a thermodynamic phase transition with respect to geometric potential. Assuming that $f$ is transitive and that $Df$ is Hölder continuous, we show that there exists $t_0 \in (0, 1]$ such that the transfer operator $L_t e^{-t \log |Df|}$, acting on the space of Hölder continuous functions, has the spectral gap property for all $t < t_0$ and has not the spectral gap property for all $t \geq t_0$. Similar results are also obtained when the transfer operator acts on the space of bounded variations functions and smooth functions. In particular, we show that in the transitive case $f$ has a unique thermodynamic phase transition and it occurs in $t_0$. In addition, if the loss of expansion of the dynamics occurs because of an indifferent fixed point or the dynamics admits an absolutely continuous invariant probability with positive Lyapunov exponent then $t_0 = 1$.

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(Some figures may appear in colour only in the online journal)
1. Introduction

This paper is concerned with the phase transition problem of smooth dynamical systems. From a statistical mechanics point of view, phase transition means a dramatic change on a given medium after the action of certain external conditions. In the pioneering works of Sinai, Ruelle and Bowen [Rue68, S72, Bow75] the thermodynamical formalism was brought from statistical mechanics to dynamical systems. From a dynamical systems point of view, phase transitions are often associated to the lack of differentiability or analyticity of the topological pressure, as a function of the potential.

More precisely, given \( f : \Lambda \to \Lambda \) a continuous dynamical system on a compact metric space \( \Lambda \) and \( \phi : \Lambda \to \mathbb{R} \) a continuous potential, the variational principle for the pressure asserts that

\[
P_{\text{top}}(f, \phi) = \sup \left\{ h_\mu(f) + \int \phi \, d\mu : \mu \text{ is an } f \text{- invariant probability} \right\}
\]

where \( P_{\text{top}}(f, \phi) \) denotes the topological pressure of \( f \) with respect to \( \phi \) and \( h_\mu(f) \) denotes the Kolmogorov–Sinai’s metric entropy. An equilibrium state \( \mu_\phi \) for \( f \) with respect to \( \phi \) is a probability measure that attains the supremum. In the special case that \( \phi \equiv 0 \), the topological pressure \( P_{\text{top}}(f, \phi) \) coincides with the topological entropy \( h_{\text{top}}(f) \), which is one of the most important topological invariants in dynamical systems (see e.g. [W82]). Thereby, we say that \( f \) has a phase transition with respect to \( \phi \) if the topological pressure function

\[
\mathbb{R} \ni t \mapsto P_{\text{top}}(f, t\phi)
\]

is not analytic. It is known from classical Ergodic theory that if \( f \) has finite topological entropy then the pressure function \( \mathbb{R} \ni t \mapsto P_{\text{top}}(f, t\phi) \) is convex and Lipschitz continuous. Then, it follows from Rademacher’s theorem that \( \mathbb{R} \ni t \mapsto P_{\text{top}}(f, t\phi) \) is differentiable except in at most a countable number of points. By [W92], if \( f \) is expansive and has finite entropy then the lack of differentiability of the topological pressure function is related to non uniqueness of the equilibrium states associated to potential \( \phi \) (for generalizations of this result see e.g. [BCMV20, IT20]).

The notion of uniform hyperbolicity, that was introduced in the seventies by Smale [Sm67], drew the attention of many researchers and influenced the way of studying differentiable dynamical systems. In the context of non-invertible maps, hyperbolic dynamics correspond to hyperbolic endomorphism (see [P76]) and uniformly expanding maps. It follows from [Rue68, S72, Bow75] that if \( f \) is a transitive hyperbolic or expanding dynamics and \( \phi \) is Hölder continuous then \( f \) does not have phase transition with respect to \( \phi \). On the other hand, several non-uniformly hyperbolic examples are known to present phase transitions with respect to regular potentials. For example:

- Manneville–Pomeau maps and geometric potential [Lo93],
- large class of maps of the interval with indifferent fixed point and geometric potential [PS92],
- certain quadratic maps and geometric potential [CRL13],
- certain non-degenerate smooth interval maps and geometric potential [RL21],
- porcupine horseshoes and geometric potential [DGR14],
- geodesic flow on Riemannian non-compact manifolds with variable pinched negative sectional curvature and suitable Hölder continuous potential [IRV18],
- geodesic flow on certain M-puncture sphere and geometric potential [V17].
However, a characterization of the set of dynamics which have phase transitions seems still out of reach.

It is important to emphasize that when the dynamics is restricted, for example a shift or suspension flows defined over countable alphabets, it is necessary that in the phase transition problem the potentials have low regularity (see [S01, S06, CL15, L15, IJ13]).

In view of this discussion, we propose the following problem:

**Problem A.** Given \( f \) a local diffeomorphism on a compact manifold, with positive topological entropy, and \( \phi \) a Hölder continuous potential, can we characterize when \( f \) has a thermodynamical phase transition with respect to \( \phi \)?

The entropy requirement above is to assume some chaoticity. Indeed, if \( f \) is the irrational rotation on the circle then \( f \) is not hyperbolic or expanding, \( h_{top}(f) = 0 \) however \( \mathbb{R} \ni t \mapsto P_{top}(f, t\phi) = t \int \phi \, dm \), where \( m \) denotes the Lebesgue measure in the circle. In particular \( f \) has not phase transition for any continuous potential.

When \( f \) is a mixing expanding or hyperbolic dynamics and \( \phi \) is a suitable potential the thermodynamical properties can be recovered through the Ruelle–Perron–Frobenius operator or transfer operator \( L_{\phi} \) acting on functions \( g : M \to \mathbb{C} \) defined as the following:

\[
L_{f,\phi}(g)(x) := \sum_{f(y)=x} e^{\phi(y)} g(y).
\]

Indeed, using the fact that \( L_{\phi} \) has the spectral gap property (see precise definition in section 2) acting on a suitable Banach space\(^1\) it is possible to show that \( f \) has not phase transitions with respect to suitable potentials (see by e.g. [PU10]).

It is important to note that the spectral gap property is also useful to the study of finer statistical properties of thermodynamical quantities as equilibrium states, mixing properties, large deviation and limit theorems, stability of the topological pressure and equilibrium states or differentiability results for thermodynamical quantities (see e.g. [Ba00, GL06, BCV16, BC19]).

Furthermore, in [PS92] it shown that for a large class of maps \( f \) which are piecewise monotone on interval with an indifferent fixed point and the geometric potential \( \phi := -\log |Df| \) the associated transfer operator has not the spectral gap property when acting on the space of bounded variation functions.

In view of this discussion, we propose a conjecture:

**Conjecture A.** Let \( f : M \to M \) be a \( C^2 \)–local diffeomorphism on a Riemannian manifold \( M \). If \( f \) is not a uniformly expanding dynamic then there exists a Hölder continuous potential \( \phi \) such that \( L_{f,\phi} \) does not have the spectral gap property acting on a Hölder continuous function space.

In [GKLM18], under the hypothesis that for every potential \( \phi \) the transfer operator \( L_{f,\phi} \) has the spectral gap property acting on some suitable Banach space, it is studied the geometry of the set of potentials and consequences for the understanding of the thermodynamical quantities of \( f \). Continuing this work in [LR20], it is studied the curvature of the equilibrium states space for a complete shift.

On the other hand, it follows from [KI20] that for any map of the circle which is expanding outside an arbitrarily flat neutral point, the set of Hölder potentials such that \( L_{f,\phi} \) exhibits the

\(^1\) If \( M \) is one dimensional manifold then usually the suitable Banach space is for example the bounded variations space and in the general case the Hölder continuous spaces, smooth function spaces, distributions spaces, among others.
spectral gap property on the Hölder functions is dense in the uniform topology. A natural question is what happens for one parameter families of potentials, in particular families induced by the geometric potential, which allow us to build absolutely continuous invariant probabilities with respect to Lebesgue measure (a.c.i.p.) and physical measures:

**Question A.** If \( f : S^1 \to S^1 \) is a \( C^1 \)–local diffeomorphism in the circle, with positive topological entropy and \( Df \) is Hölder continuous, then the set

\[
\{ t \in [0, 1] \mid L_{f, -t \log |Df|} \text{ has the spectral gap property on a Hölder continuous function space} \}
\]

is a dense subset of \([0, 1]\)?

Remember that for a large class of dynamics of the interval with indifferent fixed point, in [PS92], is proved that except for \( t \geq 1 \) we have that \( L_{f, -t \log |Df|} \) has the spectral gap property on the space of functions with bounded variations.

In [CS09], spectral gap property is investigate for countable shifts, and the following problem is proposed:

*When does a potential \( \phi \) satisfy the SGP? How common is this phenomenon? What are the most important obstructions?*

It is in fact proved that SGP holds densely and a description is given to when it holds.

As a first step in studying these issues for smooth maps, in this paper we obtained answers to the conjecture and question for dynamics in the circle.

This paper is organized as follows. In section 2 we provide some definitions and the statement of the main results on thermodynamical and spectral phase transition. In section 3 are summarized the key ideas of the proofs of each main result. In section 4 we recall the necessary framework on topological dynamics, thermodynamical formalism and transfer operator. In section 5 the main results are proved. Finally, in the section 6 some explicitly examples are given, discussions on the main results are realized and additional questions are proposed. In particular, in section 6.2 is calculated the phase transition parameter, in section 6.3 some comments about the non-transitive case are made and section 6.4 it is discussed briefly on the higher dimensional context.

2. **Statement of the main results**

This section is devoted to the statement of the main results.

Throughout the paper we shall denote the circle \( \{ z \in \mathbb{C} : |z| = 1 \} \) by \( S^1 \).

Our first result ensures thermodynamical phase transition. In fact, the phase transition occurs with respect to geometric potential.

**Theorem A.** Let \( f : S^1 \to S^1 \) be a non invertible \( C^1 \) local diffeomorphism. If \( f \) is not an expanding dynamics then \( f \) has phase transition with respect to \( -\log |Df| \).

In fact; if \( f \) has any negative Lyapunov exponent, we show that the pressure function is not even differentiable at one point.

**Remark 2.1.** Since \( f : S^1 \to S^1 \) is a local diffeomorphism, it follows from classical Ergodic theory and [MP77] that \( h_{\text{top}}(f) = \log \deg(f) \), where \( \deg(f) \) is the topological degree of \( f \). In particular, \( f \) is not invertible if and only if \( h_{\text{top}}(f) > 0 \). Note also that if \( f \) has zero topological
entropy it is easy to see that given a continuous potential \( \phi \) the topological pressure function \( \mathbb{R} \ni t \mapsto P_{\text{top}}(f, t\phi) \) is linear in each of the intervals \((-\infty, 0]\) and \([0, \infty)\).

The previous theorem was expected and its proof is not difficult, however it is the starting point for the finer understanding obtained in the next results.

Given \( E \) a complex Banach space and \( T : E \to E \) a bounded linear operator, we say that \( T \) has the \textit{(strong) spectral gap property} if there exists a decomposition of its spectrum \( \text{sp}(T) \subset \mathbb{C} \) as follows: \( \text{sp}(T) = \{ \lambda_1 \} \cup \Sigma_1 \) where \( \lambda_1 > 0 \) is a leading eigenvalue for \( T \) with one-dimensional associated eigenspace and there exists \( 0 < \lambda_0 < \lambda_1 \) such that \( \Sigma_1 \subset \{ z \in \mathbb{C} : |z| < \lambda_0 \} \).

Given \( r \geq 1 \) an integer and \( \alpha \in (0, 1] \) we denote by \( C^r(\mathbb{S}^1, \mathbb{C}) \) and \( C^\alpha(\mathbb{S}^1, \mathbb{C}) \) the Banach spaces of \( C^r \) functions and \( \alpha \)-Hölder continuous complex functions whose domain is \( \mathbb{S}^1 \), respectively. Furthermore, we denote by \( BV[\mathbb{S}^1] \) the Banach space of bounded variations complex functions on \( \mathbb{S}^1 \).

Our second result ensures, assuming that the dynamics is transitive, an effective spectral phase transition.

\textbf{Theorem B.} Let \( E = C^\infty(\mathbb{S}^1, \mathbb{C}) \) or \( C^r(\mathbb{S}^1, \mathbb{C}) \) and let \( f : \mathbb{S}^1 \to \mathbb{S}^1 \) be a transitive non invertible \( C^1 \)-local diffeomorphism with \( Df \in E \). If \( f \) is not an expanding dynamics then there exists \( t_0 \in (0, 1] \) such that:

(i) the transfer operator \( \mathcal{L}_{t,-t\log |Df|} \) has the spectral gap property on \( E \) for all \( t < t_0 \) and has not the spectral gap property for all \( t \geq t_0 \).

(ii) If \( E = BV[\mathbb{S}^1] \) then \( \mathcal{L}_{t,-t\log |Df|} \) has the spectral gap property on \( BV[\mathbb{S}^1] \) for all \( t \in [0, t_0) \) and has not the spectral gap property for all \( t \geq t_0 \).

Note that in the context of theorem B, \( f \) is piecewise monotone with \textit{full branches} and transitive. Hence, it follows directly from [CM86, corollary 4.3] that \( f \) is conjugate to an expanding dynamic.

\textbf{Remark 2.2.} The previous theorem is not direct consequence of [PS92] because they assume that the loss of expansion is only caused by an indifferent fixed point similar to a Maneville–Poumeau map and \( E = BV[\mathbb{S}^1] \). In this sense, the previous theorem can be seen as an extension of [PS92].

As a consequence of the previous theorem we prove an effective thermodynamical phase transition.

\textbf{Corollary A.} Let \( f : \mathbb{S}^1 \to \mathbb{S}^1 \) be a transitive non invertible \( C^1 \)-local diffeomorphism with \( Df \) Hölder continuous. If \( f \) is not an expanding dynamics then the topological pressure function \( \mathbb{R} \ni t \mapsto P_{\text{top}}(f, -t\log |Df|) \) is analytical, strictly decreasing and strictly convex in \((-\infty, t_0)\) and constant equal to zero in \([t_0, +\infty)\) (see figure 1).

We show that the \( t_0 \) in the previous theorem and corollary are the same. Moreover, \( t_0 = \inf\{ t \in \mathbb{R} : P_{\text{top}}(f, -t\log |Df|) = 0 \} \) and it can be calculated using the Hausdorff dimension of the support of invariant probabilities with positive Lyapunov exponents. In particular, if the loss of expansion is only caused by an indifferent fixed point or the dynamics admits an a.c.i.p. with positive Lyapunov exponent then \( t_0 = 1 \). (see details in section 6.2)

Taking into consideration a nonwandering decomposition theorem, we see that the non transitive case can be reduced to a transitive context, however it is not possible to apply directly the theorem B (see details in section 6.3).
3. Strategy of the proofs

In this section we summarize the key ideas behind each proof of the main results.

3.1. Theorem A

We investigate the pressure function associated to probabilities with non-negative and non-positive Lyapunov exponents, which we denote $P_+$ and $P_-$, respectively. The pressure function is always the maximum of these two, with $P_+$ necessarily dominating before of $t = 0$ and then $P_-$ after $t = 1$. Thus the point of transition, which lies between 0 and 1, is exactly where phase transition occurs, since each of these functions must have non-negative and non-positive derivatives, respectively. The characterization of expanding maps via positive Lyapunov exponents [CLR03] implies that an ergodic measure with zero or negative Lyapunov exponent must exist, thus $P_-$ is well defined and transition occurs.

3.2. Theorem B

The proof will be divided into three steps:

- In the first step we will show that in the thermodynamical phase transition parameter $t_0$, obtained in the theorem A, the transfer operator $\mathcal{L}_f^{-t_0 \log |Df|}$ has not the spectral gap property acting on $E$. For that step, following the ideas of the proof for expanding maps (such as in e.g. [OV16]), we use Rokhlin’s formula and prove that the invariant measure obtained via spectral gap property is an equilibrium state with topological pressure related to the transfer operator. Furthermore spectral theory ensures that the leading isolated simple eigenvalue varies analytically with respect to the transfer operator.

- In the second step we will show that, after the parameter $t_0$, the transfer operator $\mathcal{L}_f^{-t \log |Df|}$ has not the spectral gap property acting on $E$. For that step, assuming there is spectral gap for some parameter, we make use of Nagaev’s method for obtaining the central limit theorem to ensure that in the neighborhood of this parameter the geometric pressure function is strictly convex. In particular, the constant behavior of the geometric pressure function after the transition parameter implies that spectral gap does not occur for these parameters.

- In the third step we will show that before the parameter $t_0$ the transfer operator $\mathcal{L}_f^{-t \log |Df|}$ has the spectral gap property acting on $E$. For that step we prove that, in our context, quasi-compactness is sufficient for the transfer operator to have spectral gap. For this operator acting on bounded variation functions, we use the estimates on spectral radius from [BK90,
Ba00] to prove quasi-compactness directly. For Hölder and smooth functions, we use estimates from [BJL96, CL97] to prove the essential spectral radius is bounded above by a translation of the pressure function. Using this estimate, we prove spectral gap for \( t = 0 \), and therefore to \( t \) sufficiently close to zero, by openness of spectral gap. Next we extend this property to all \( t < t_0 \) using the monotonicity of the pressure function and the fact that it is related with the spectral radius, not only when the gap holds but also on the boundary of that region.

3.3. Corollary A

The analyticity of the topological pressure function follows directly from theorem B by analyticity of the leading isolated eigenvalue and its relation with the topological pressure.

4. Preliminary

In this section we provide some definitions and preparatory results needed for the proof of the main results.

4.1. Topological dynamics

Given a continuous map \( f: X \to X \) on a compact metric space \( X \), we start defining some properties from a topological point of view describing the mixing behavior of the system, that is, how much the orbits of points, or even open sets, tend to mix with each other and visit all regions of the space \( X \). The basic expected type of mixing is called transitivity and is defined as:

**Definition 4.1.** The dynamic \( f \) is transitive if for every open sets \( U \) and \( V \), there exists \( n \in \mathbb{N} \) such that \( f^n(U) \cap V \neq \emptyset \) or, equivalently, \( f \) admits a dense orbit.

Other concepts of mixing behavior arise naturally, such as strong transitivity, mixing, all the way up to topological exactness, also called locally eventually onto:

**Definition 4.2.** The dynamic \( f \) is topologically exact if for every open set \( U \) there exists \( n \in \mathbb{N} \) such that \( f^n(U) = X \).

Evidently, topologically exactness implies transitivity, the converse need not be true. A consequence of topological exactness is that every set of pre-orbits \( \bigcup_{n \geq 0} f^{-n}\{x\} \) is uniformly dense on \( X \) for every \( x \in X \).

Regarding the distances of orbits, two concepts are essential. A map is said to be expanding if the distance of two sufficiently close points are expanded by \( f \):

**Definition 4.3.** The dynamic \( f \) is called expanding if \( f \) is open map and there are constants \( \sigma > 1, r > 0 \) and \( n \geq 0 \) such that

\[
d(f^n(x), f^n(y)) \geq \sigma d(x, y) \text{ for all } d(x, y) < r.
\]

Note that if \( X \) is a compact Riemannian manifold and \( f \) is a local diffeomorphism then: \( f \) is an expanding dynamics if, only if, \( \inf_{x \in X} |Df(x)| > 1 \).

A dynamic is said to be expansive if the orbits of any two points become distinguishable after enough time:
**Definition 4.4.** We say that $f$ is (positively) **expansive** if there is a constant $\varepsilon_0 > 0$, called the **expansivity constant**, such that for every pair of points $x \neq y$ in $X$ there is $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) \geq \varepsilon_0$.

Every expanding dynamics is expansive. In fact, every expanding dynamics defined on a connected domain will be expansive and topologically exact.

An example of expanding dynamics are the $d$-adic transformations or **Bernoulli maps**: for a given $d \geq 2$ define $f_d : S^1 \rightarrow S^1$ as

$$f_d(x) = \begin{cases} x(1 + 2^{a}x^a), & \text{if } x \in [0, 1/2] \\ 2x - 1, & \text{if } x \in (1/2, 1] \end{cases}$$

which are topologically exact, expansive but not expanding maps. Since $f_2(0) = 0$ is an **indifferent fixed point**, that is $Df_2(0) = 1$.

### 4.2. Ergodic theory

Now we state some classical definitions, notations and results from Ergodic theory (for more details see e.g. [OV16]).

One fundamental result is the called **Birkhoff’s Ergodic theorem** which relates time and space averages of a given potential $\psi : X \rightarrow \mathbb{R}$.

**Theorem 4.5 (Birkhoff).** Let $f : X \rightarrow X$ be a measurable transformation and $\mu$ be an $f$-invariant probability. Given any integrable function $\phi : X \rightarrow \mathbb{R}$, the limit:

$$\bar{\phi}(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x))$$

exists in $\mu$-a.e. $x \in X$. Furthermore, the function $\bar{\phi}$ defined this way is integrable and satisfies

$$\int \bar{\phi}(x) \, d\mu(x) = \int \phi(x) \, d\mu(x).$$

Additionally, if $\mu$ is $f$-ergodic, then $\bar{\phi} \equiv \int \phi \, d\mu$ for $\mu$–a.e.

We denote the $f$-invariant probabilities space by $\mathcal{M}_1(f)$ and the $f$-invariant and ergodic probabilities space by $\mathcal{M}_e(f)$.

An important example of time average are the **Lyapunov exponents** which translate the asymptotical rates of expansion and contraction of a map. On a broader context, these are defined via the Oseledets multiplicative ergodic theorem. For our context, smooth local diffeomorphisms on the circle, the Lyapunov exponents are defined simply as:

$$\lambda(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(x)| = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{n} \log |Df(f^j(x))|,$$
whether the limit exists. That is, the Lyapunov exponents \( \lambda \) coincide with the time average for the continuous potential \( \log |Df| \) in each point \( x \) where the limit exists. On the other hand, given an ergodic measure \( \mu \), by Birkhoff’s Ergodic theorem, we have
\[
\lambda(x) = \int \log |Df(x)| \, d\mu \quad \text{for } \mu - \text{a.e.}
\]
and we define the Lyapunov exponent for this measure \( \chi(\mu) := \int \log |Df| \, d\mu \). Expanding dynamics are characterized by having strict positive Lyapunov exponents and that will play a significant role later on (see [CLR03]).

Now, we present two estimates for the metric entropy that will be crucial later on the proof of main results. First the relation between entropy and Lyapunov exponents, that in our context of smooth local diffeomorphisms on the circle, guarantees us:

**Theorem 4.6 (Margulis–Ruelle inequality, [Rue78]).** Let \( f: S^1 \to S^1 \) be a \( C^1 \)-local diffeomorphism that preserves an \( f \)-invariant and ergodic probability \( \mu \). Then
\[
h(\mu(f)) \leq \max \{ 0, \chi(\mu) \}.
\]

Next, we study the relation between entropy and the Jacobian. Let \( f \) be a local homeomorphism and a given probability \( \nu \) (not necessarily invariant), define the Jacobian of \( f \) with respect to \( \nu \) as the measurable function \( J_{\nu}(f) \), which is essentially unique, satisfying:
\[
\nu(f(A)) = \int_A J_{\nu}(f) \, d\nu
\]
for any measurable invertibility domain \( A \). Up to restricting \( f \) to a full measure subset, \( f \) always admits a Jacobian with respect to \( \mu \), an invariant measure.

When we have a partition that generates the Borelian \( \sigma \)-algebra, we have a more direct way of obtaining the entropy of \( f \) with respect to the measure \( \mu \) using that partition. Given two partitions \( P \) and \( Q \) of \( X \), we define a new partition \( P \cup Q \) as the following:
\[
P \cup Q := \{ A \cap B; A \in P \text{and } B \in Q \}.
\]

**Definition 4.7.** Given \( f: X \to X \) a measurable map, we say that a finite partition \( P \) of \( X \) is a generating partition if \( \bigcup_{i=0}^{m f} f^{-i}(P) \) generates the \( \sigma \)-algebra of \( X \), with \( m = -\infty \) or \( m = 0 \) for invertible and non invertible systems, respectively.

If additionally \( X \) is a metric space, then a partition such that the diameter of the elements of \( \bigcup_{i=0}^{\infty} f^{-i}(P) \) gets arbitrarily small is a generating partition. Note that in our context of local diffeomorphisms on the circle, we will see, that the domains of the full branches form a generating partition. Any such partition gives the following way of calculating the entropy:

**Theorem 4.8 (Rokhlin’s formula).** Let \( f: M \to M \) be a locally invertible transformation, on a separable metric space, and \( \mu \) be an \( f \)-invariant probability. Assume that there is some generating, up to zero measure, partition \( \mathcal{P} \) of \( X \) such that every \( P \in \mathcal{P} \) is an invertibility domain of \( f \). Then
\[
h(\mu(f)) = \int \log J_{\mu}(f) \, d\mu.
\]
This equality will be fundamental, later on the proof of theorem B.
4.3. Transfer operator

In this section we recall some properties of the transfer operators. For more details on the transfer operator see e.g. [S12] or [PU10].

In what follows, given $T : E \to E$ a bounded linear operator we denote its spectral radius by $\rho(T)$.

One of the main tools to study thermodynamical quantities and indeed obtain equilibrium states, as well as its properties, is the Ruelle–Perron–Frobenius operator or transfer operator, which acts on function spaces:

**Definition 4.9.** Let $f : M \to M$ be a local homeomorphism on a compact and connected manifold. Given a complex continuous function $\phi : M \to \mathbb{C}$, define the Ruelle–Perron–Frobenius operator or transfer operator $\mathcal{L}_{f,\phi}$ acting on functions $g : M \to \mathbb{C}$ this way:

$$
\mathcal{L}_{f,\phi}(g)(x) := \sum_{f(y)=x} e^{\phi(y)} g(y).
$$

Classical thermodynamical results for sufficiently chaotic dynamics, derive from good spectral properties from this operator.

If $\phi$ is a real continuous function, via Mazur’s separation theorem, since $\mathcal{L}_{f,\phi}$ is a positive operator then it has $\rho(\mathcal{L}_{f,\phi}|_{C^0})$ as an eigenvalue for its dual operator, that is, there exists a probability $\nu_\phi$ with $(\mathcal{L}_{f,\phi}|_{C^0})^* \nu_\phi = \rho(\mathcal{L}_{f,\phi}|_{C^0}) \nu_\phi$. If additionally $E \subset C^0(M, \mathbb{C})$ is a Banach space continuously immersed in $C^0(M, \mathbb{C})$ and $\mathcal{L}_{f,\phi}|_{E}$ has the spectral gap property, then $\rho(\mathcal{L}_{f,\phi}|_{E}) = \rho(\mathcal{L}_{f,\phi}|_{C^0})$ and $\mathcal{L}_{f,\phi}|_{E}$ admits an eigenfunction $h_\phi \in E$ with respect to $\rho(\mathcal{L}_{f,\phi}|_{E})$, which is the leading eigenvalue. We can assume, up to rescaling, that $\int h_\phi \, d\nu_\phi = 1$. Then the probability $\mu_\phi = h_\phi \cdot \nu_\phi$ is proved to be $f$–invariant and is a candidate for the equilibrium state.

**Remark 4.10.** We recall the notion of analyticity for functions on Banach spaces. Let $E_1, E_2$ be Banach spaces and denote by $\mathcal{L}_i(E_1, E_2)$ the space of symmetric $i$-linear transformations from $E_1$ to $E_2$. For notational simplicity, given $P_i \in \mathcal{L}_i(E_1, E_2)$ and $h \in E_1$ we set $P_i(h) := P_i(h, \ldots, h)$.

We say the function $f : U \subset E_1 \to E_2$, defined on an open subset, is analytic if for all $x \in U$ there exists $r > 0$ and for each $i \geq 1$ there exists $P_i \in \mathcal{L}_i(E_1, E_2)$ (depending on $x$) such that

$$
f(x + h) = f(x) + \sum_{i=1}^{\infty} \frac{P_i(h)}{i!}
$$

for all $h \in B(0, r)$ and the convergence is uniform.

Analytic functions on Banach spaces have completely similar properties to real analytic and complex analytic functions. For instance, if $f : U \subset E \to F$ is analytic then $f$ is $C^\infty$ and for every $x \in U$ one has $P_i = D^i f(x)$. For more details see for example [C85, chapter 12].

Let $T : E \to E$ be a bounded linear operator on a complex Banach space. Suppose that $T$ has the spectral gap property, by [R55] there exists $\delta > 0$ such that if $T : E \to E$ is a bounded linear operator with $||T - \tilde{T}|| < \delta$ then $\tilde{T}$ has the spectral gap property. Moreover, $B(T, \delta) \ni$

---

2 When no confusion is possible, for notational simplicity we omit the dependence of the transfer operator on $f$. 
\( \tilde{T} \mapsto (\rho(\tilde{T}), P_{\rho(\tilde{T})}) \) is analytical, where \( P_{\rho(\tilde{T})} \) is the spectral projection of \( \tilde{T} \) with respect to the leading eigenvalue \( \rho(\tilde{T}) \).

On the other hand, following the same proof of \([BCV16, \text{ proposition 4.1}]\) we have

**Proposition 4.11.** Let \( f : M \to M \) be a local homeomorphism on a compact and connected manifold and \( E \) be a Banach algebra of functions such that \( \mathcal{L}_{f,\phi} \) is a bounded linear operator of \( E \) on \( E \), for all \( \phi \in E \). Then

\[
E \ni \phi \mapsto \mathcal{L}_{f,\phi}
\]

is analytical, where we endow on the counterdomain the topology generated by operator norm. In particular, \( \mathbb{R} \ni t \mapsto \mathcal{L}_{f,\phi} \) is real analytic.

Note that if \( T = \mathcal{L}_{f,\phi} \big|_{E} \) has the spectral gap property we have that \( P_{\rho(T)}(g) = \int g d\nu_{\phi} \cdot h_{\phi} \).

Define \( SG(E) := \{ \phi \in E : \mathcal{L}_{f,\phi} \big|_{E} \text{ has the spectral gap property} \} \), we then concluded:

**Corollary 4.12.** \( SG(E) \subset E \) is an open subset and the following map is analytical:

\[
SG(E) \ni \phi \mapsto (\rho(\mathcal{L}_{f,\phi}), h_{\phi}, \nu_{\phi}) .
\]

One weaker spectral property is called quasi-compactness:

**Definition 4.13.** Given \( E \) a complex Banach space and \( T : E \to E \) a bounded linear operator, we say that \( T \) is quasi-compact if there exists \( 0 < \sigma < \rho(T) \) and a decomposition of \( E = F \oplus H \) as follows: \( F \) and \( H \) are closed and \( T \)-invariant, \( \dim F < \infty \), \( \rho(T|_{F}) > \sigma \) and \( \rho(T|_{H}) \leq \sigma \).

In general, spectral gap implies quasi-compactness. Later we will proof that, in the context of theorem \( B \), quasi-compactness is a sufficient condition for spectral gap.

A definition equivalent to quasi-compactness can be given via the essential spectral radius:

**Definition 4.14.** Given \( E \) a complex Banach space and \( T : E \to E \) an bounded linear operator, define

\[
\rho_{es}(T) := \inf \{ r > 0 : \text{sp}(L) \setminus B(0, r) \text{contains only eigenvalues of finite multiplicity} \}
\]

Thus quasi-compactness is equivalent to having \( \rho_{es}(T) < \rho(T) \), and so estimates on the essential spectral radius and the spectral radius will be of the utmost importance.

### 5. Proof of the main results

This sections is devoted to proving theorems \( A \), \( B \) and corollary \( A \).

#### 5.1. Thermodynamical phase transition

This section is devoted to the proof of theorem \( A \).

Given a unidimensional local diffeomorphism \( f : S^1 \to S^1 \) and an \( f \)-invariant measure \( \mu \), we define the Lyapunov exponent for this measure \( \chi_{\mu}(f) := \int \log |Df| d\mu \). Our goal is to understand the smoothness of the topological pressure function

\[
P(t) := \sup \{ h_{\mu}(f) - t \chi_{\mu}(f) : \mu \text{ is an } f \text{-invariant probability} \} , t \in \mathbb{R},
\]
which is known, by the variational principle (see e.g. [OV16]), that this supremum can be taken only over the space of \(f\)-ergodic probabilities \(M_e(f)\). Thus, denote the following sets of measure with non-negative and non-positive Lyapunov exponents:

\[
M^+_e(f) := \{\mu \in M_e(f), \chi_\mu(f) \geq 0\}
\]

\[
M^-_e(f) := \{\mu \in M_e(f), \chi_\mu(f) \leq 0\},
\]

and the pressure functions restricted to each of these sets

\[
P^+_t := \sup_{\mu \in M^+_e(f)} P_{\mu}(t)
\]

\[
P^-_t := \sup_{\mu \in M^-_e(f)} P_{\mu}(t),
\]

with \(P_{\mu}(t) := h_\mu(f) - t\chi_\mu(f)\). With these notations, we prove the following lemmas

**Lemma 5.1.** Let \(f: S^1 \to S^1\) be a \(C^1\) local diffeomorphism such that \(M^+_e(f)\) is non-empty, then the function \(P^+_t\) is convex, non-increasing and has a zero in \((0, 1]\).

**Proof.** \(P^+_t\) is non-increasing. For each \(\mu \in M^+_e(f)\) fixed, the line \(P_{\mu}\) is non-increasing and thus for any \(s > t\) we have \(P_{\mu}(s) \leq P_{\mu}(t)\). Taking supremum over all \(\mu\), it follows that \(P^+_t(s) \leq P^+_t(t)\) and \(P^+_t\) is non-increasing. \(P^+_t\) is in fact convex. Since \(P^+_t\) is supremum of straight lines \(R \ni t \mapsto P_{\mu}(t)\), where \(\mu \in M^+_e(f)\), then \(P^+_t\) is convex. Finally, note that

\[
P^+_0 = \sup \{h_\mu(f), \mu \in M^+_e(f)\} = h_{\text{top}}(f) > 0
\]

(see remark 2.1) and by Ruelle–Margulis’s inequality (see [Rue78])

\[
P^+_1 = \sup \{h_\mu(f) - \chi_\mu(f), \mu \in M^+_e(f)\} \leq 0.
\]

Thus, as \(P^+_t\) is convex, in particular continuous, it has a zero in \((0, 1]\). 

The figure 2 gives a geometric intuition of the function \(P^+_t\) where each line is the graph of \(t \mapsto P_{\mu}(t)\), for a fixed ergodic probability \(\mu\) in \(M^+_e(f)\):

**Lemma 5.2.** Let \(f: S^1 \to S^1\) be \(C^1\) local diffeomorphism such that \(M^-_e(f) \neq \emptyset\) then \(t \mapsto P^+_t\) is linear and not decreasing.

---

**Figure 2.** Graphic of the function \(P^+_t\)
is not increasing there exists $h > 0$ and thus: $P$ gives a geometric intuition of the function $\chi$.

Since $\chi$ is linear and not decreasing, $\chi_0 := \min_{t \in M(f)} \chi_t(f)$.

Graphic of the function $\chi$ is not identically zero for $t < 0$. Hence $\chi(0) = \chi_{\mu}(f) = 0$ for all $t < 0$.

Case $h = 0$. In this case, $P_+ = P_\mu \equiv 0$. Thus, fixed $t_0 \in (0, 1]$ the lower zero of $P_+$, that is, $t_0 := \min \{ t \in \mathbb{R} : P_+(t) = 0 \}$ we have:

$$P(t) = \begin{cases} P_+(t), & \text{for } t \leq t_0 \\ 0, & \text{for } t > t_0 \end{cases}$$

This means that $P$ cannot be real analytic in $t_0$ because all its lateral derivatives on the right are zero, but $P$ is not identically zero for $t < t_0$.

Case $h > 0$. In this case, $P_+$ is not trivially zero, but an increasing line. Thus, since $P_- \to +\infty$ and $P_+(t)$ is not increasing there exists $t > 0$ such that $P_+(t) \leq P_-(t)$.

Define $t_1 := \min \{ t \geq 0 : P_+(t) \leq P_-(t) \}$. Since $P_+(0) > P_-(0) = 0$ then $t_1 > 0$. Hence

$$P(t) = \begin{cases} P_+(t), & \text{for } t \leq t_1 \\ P_-(t), & \text{for } t > t_1 \end{cases}$$

**Proof.** By Ruelle–Margulis’s inequality (see [Rue78]), if $\chi_\mu(f) \leq 0$ then $h_\mu(f) = 0$ and $P_\mu(t) = -t \chi_\mu(f)$. Then $P_\mu$ is linear and not decreasing, so $P_- = P_\mu$.

Using again the Ruelle–Margulis’s inequality we have that $h_\mu(f) = \min_{t \in M(f)} \chi_t(f)$. Since $f$ is not an expanding dynamics, it follows from [CLR03, theorem 3] that $\chi_\mu(f) \leq 0$. Thus, we will discuss the possibilities for this ergodic probability $\mu$:

**Proof of theorem A.** Since $h_{\text{top}}(f) > 0$, by Ruelle–Margulis’s inequality (see [Rue78]) $f$ admits some ergodic invariant probability with positive Lyapunov exponent, hence $M^e(f) \neq \emptyset$. Using again the Ruelle–Margulis’s inequality we have that $P_-(0) = 0$ and $h_{\text{top}}(f) = P_+(0)$. On the other hand, by [CLR03, theorem 3] there exists $\mu \in M(f)$ such that $\chi_\mu(f) = \min_{t \in M(f)} \chi_t(f)$. Since $f$ is not an expanding dynamics, it follows from [CLR03, theorem 5] that $\chi_\mu(f) \leq 0$. Thus, we will discuss the possibilities for this ergodic probability $\mu$:
Note that $P$ cannot even be differentiable in $t_1$, in fact suppose by absurd that $P$ is differentiable in $t_1$. Then

$$0 \neq -\chi_\mu(f) = \lim_{t \to t_1} \frac{P(t) - P(t_1)}{t - t_1} = \lim_{t \to t_1} \frac{P(t) - P(t_1)}{t - t_1} \leq 0,$$

where the last inequality follows from $P_+$ being non-increasing. This is absurd.

We concluded in both cases that $f$ has phase transition with respect to $-\log |Df|$.  

The figures 4 and 5 gives a geometric intuition of the pressure function in each case previous.

### 5.2. Effective phase transitions

This section is devoted to the proof of the theorem B. Suppose that $f$ is a transitive local diffeomorphism of the circle with degree at least two. First we remember that $f$ is conjugate to $dx \mod 1$ where $d = \deg(f)$, by [CM86, corollary 4.3].

From this conjugacy, we then find that $f$ is expansive, topologically exact, and it admits generating partition by domains of injectivity. In particular, we can apply Rokhlin’s formula (see 4.8).
Also, since \( f \) is transitive and has positive topological entropy, every ergodic measure has non-negative Lyapunov exponents, by [A20, corollary 1.2]\(^3\), and therefore \( P = P_+ \) and \( P \) is non-increasing, by lemma 5.1. Furthermore, \( t_0 := \inf \{ t \in [0, 1] ; P(t) = 0 \} \) is the parameter of the thermodynamical phase transition in theorem \( A \) and \( P(t) = 0 \) for all \( t \geq t_0 \).

5.2.1. Absence of spectral gap. This section is devoted to the proof of spectral phase transition. In fact, we will show:

**Proposition 5.3.** Let \( E = BV(S^1), C^\alpha(S^1, \mathbb{C}) \) or \( C'(S^1, \mathbb{C}) \) and let \( f : S^1 \rightarrow S^1 \) be a transitive \( C^1 \)-local diffeomorphism with \( Df \in E \). If \( f \) is not an expanding dynamics then \( L_{f,-t_0 \log |Df|} \) has not the spectral gap property acting on \( E \).

Now in order to prove proposition 5.3, we must prove \( P_{\top}(f,-t \log |Df|) = \log \rho(L_{f,-t \log |Df|}|E) \). The case \( E = BV(S^1) \) is a direct consequence of [BK90, theorem 3], since \( f \) is continuous and monotone in each element of the generating partition. For the case \( E = C^\alpha(S^1, \mathbb{C}) \) or \( C'(S^1, \mathbb{C}) \) we followed the ideas from [Ba00, proposition 3.5] and proved the following:

**Lemma 5.4.** Let \( f : S^1 \rightarrow S^1 \) be a transitive \( C^\alpha \)-local diffeomorphism, let \( E = BV(S^1), C^\alpha(S^1, \mathbb{C}) \) or \( C'(S^1, \mathbb{C}) \), and let \( \phi \in E \) be a real continuous function. If \( L_{f, \lambda} \phi = \lambda \phi \), with \( |\lambda| = \rho(L_{f, \phi}|E) \) and \( \varphi \in E \setminus \{0\} \), then \( L_{f, \phi} \varphi = \rho(L_{f, \phi}|E) \varphi \). Furthermore, \( \rho(L_{f, \phi} E) = \rho(L_{f, \phi}|E) \). \( \varphi \) is bounded away from zero and \( \dim \ker(L_{f, \phi}|E - \lambda I) = 1 \).

**Proof.** Take \( \xi \in \mathbb{C} \) such that \( \lambda = \rho(L_{f, \phi}|E) \xi \) and \( |\xi| = 1 \).

By Mazur’s separation theorem, there exists a probability \( \nu \) with

\[
(L_{\phi}(C))^{\ast} \nu = \rho(L_{\phi}(C)) \nu.
\]

Therefore \( \rho(L_{\phi}(C)) \leq \rho(L_{\phi} C) \).

Claim: \( \text{supp} \ \nu = S^1 \), where \( \text{supp} \ \nu \) means the support of the measure \( \nu \).

Indeed, let \( A \subset S^1 \) be an open set. Then

\[
\nu(A) = \int \chi_A d\nu = \int \frac{L_{\phi}^{\ast} \chi_A}{\rho(L_{\phi}(C))} d\nu
\]

for every \( n \in \mathbb{N} \). By topological exactness, take \( N > 0 \) such that \( f^N(A) = S^1 \). Thus given \( x \in S^1 \) we can find \( y_0 \in f^{-N}(x) \cap A \), and

\[
L_{\phi}^{\ast} \chi_A (x) = \sum_{y \in f^N(x)} e^{S_{\phi}(y)} \chi_A (y) \geq e^{S_{\phi}(y_0)} \chi_A (y_0) = e^{S_{\phi}(y_0)} \geq e^{\inf \phi}.
\]

Finally

\[
\nu(A) \geq \frac{e^{\inf \phi}}{\rho(L_{\phi}(C))^{N}} > 0.
\]

Which proves the claim.

\(^3\) See appendix for more details.
Now define
\[ s(x) := \begin{cases} \varphi(x)/|\varphi(x)|, & \text{if } \varphi(x) \neq 0 \\ 1, & \text{if } \varphi(x) = 0. \end{cases} \]

Note that \( s|\varphi| = \varphi \), thus
\[ \mathcal{L}_\varphi (s|\varphi|) = \mathcal{L}_\varphi \varphi = \lambda \varphi = \lambda s|\varphi|. \]

This implies that
\[
\mathcal{L}_\varphi \left( \frac{s}{\lambda(s \circ f)} \cdot |\varphi| \right) (x) = \sum_{f(y) = x} e^{\phi(y)} \frac{s(y)}{\lambda s(x)} |\varphi| (y) = \frac{1}{\lambda s(x)} \mathcal{L}_\varphi (s|\varphi|) (x)
\]
\[
= \frac{1}{\lambda s(x)} \lambda s(x) |\varphi| (x) = \mathcal{L}_\varphi \left( s \frac{1}{\lambda(s \circ f)} |\varphi| \right) = |\varphi|.
\]

Since \( \nu \) is an eigenmeasure:
\[
\int \frac{s}{\rho(\mathcal{L}_\varphi |E|) \xi \cdot s \circ f} |\varphi| \, d\nu \cdot \rho(\mathcal{L}_\varphi |E|) = \int |\varphi| \, d\nu.
\]

Hence
\[
\frac{\rho(\mathcal{L}_\varphi |E|)}{\rho(\mathcal{L}_\varphi |\xi|)} \int \frac{s}{\xi \cdot s \circ f} |\varphi| \, d\nu = \int |\varphi| \, d\nu.
\]

Since \( \frac{s}{\xi \cdot s \circ f} \equiv 1 \) and \( |\varphi| \geq 0 \), we must have \( \rho(\mathcal{L}_\varphi |E|) = \rho(\mathcal{L}_\varphi |\xi|) \) and \( s(x) = \xi \cdot s \circ f(x) \), for \( |\varphi| \, d\nu \text{-a.e. } x \). Then \( s(x)|\varphi|(x) = \xi \cdot s \circ f(x)|\varphi|(x) \) for \( \nu \text{-a.e. } x \), that is, \( \varphi = \xi \cdot s \circ f \cdot |\varphi| \), since \( \text{supp } \nu = S^1 \). Now, applying the transfer operator, we have:
\[
\mathcal{L}_\varphi (|\varphi|) = \mathcal{L}_\varphi \left( \frac{\varphi}{\xi \cdot s \circ f} \right) = \frac{\mathcal{L}_\varphi (\varphi)}{\xi \circ s} = \rho(\mathcal{L}_\varphi |E|) \frac{\varphi}{s} = \rho(\mathcal{L}_\varphi |E|) |\varphi|.
\]

Next we prove that \( \varphi \) is bounded away from zero. Indeed, let \( x \in S^1 \) be such that \( |\varphi|(x) = 0 \). Since
\[
\sum_{f(y) = x} e^{\phi(y)} |\varphi| (y) = \rho(\mathcal{L}_\varphi |E|) |\varphi| (x) = 0,
\]
we must have \( \varphi(y) = 0 \) for each pre-image \( y \) of \( x \), and arguing by induction all pre-orbits \( \{f^n(x) : n \in \mathbb{N} \} \) of \( x \) must be a zero of \( |\varphi| \). Since the set of pre-orbits of \( x \) is dense by topological exactness, we would have \( |\varphi| \equiv 0 \), which is a contradiction. Since \( \varphi \in E \), and \( \varphi \) is bounded away from zero, \( |\varphi| \in E \).

Finally we prove that the eigenspace associated with \( \lambda \) is unidimensional. Let \( \varphi_1, \varphi_2 \) be eigenfunctions of \( \lambda \). Note that there exists \( t \in \mathbb{C} \) and \( x_0 \in S^1 \) such that \( \varphi_1 (x_0) + t \varphi_2 (x_0) = 0 \). Since \( \mathcal{L}_f \varphi (t \varphi_1 + \varphi_2) = \lambda (t \varphi_1 + \varphi_2) \), and by the boundness from zero of non-zero eigenfunctions, we conclude that \( \varphi_1 + t \varphi_2 \equiv 0 \).

\[ \square \]
Corollary 5.5. Let $f : S^1 \to S^1$ be a transitive $C^r$-local diffeomorphism, let $E = BV(S^1), C^0(S^1, C)$ or $C^r(S^1, C)$, and let $\phi \in E$ be a continuous real function. If $L_{f,\phi} : E$ has the spectral gap property then there exists a unique probability $\nu_\phi$ on $S^1$ such that $(L_{f,\phi})^* \nu_\phi = \rho(L_{f,\phi}) \nu_\phi$.

Proof. Note that $L_{\phi}$ is a continuous operator on $C^0(S^1, C)$. Thus, the probability $\nu$ used in the proof of the previous lemma exists and $(L_{\phi})^* \nu = \rho(L_{\phi}) \nu$. The previous lemma assures us that $\rho(L_{\phi}) = \rho(L_{f,\phi})$.

Finally let us show the uniqueness. Let $\eta$ be probability on $S^1$ such that $(L_{\phi})^* \eta = \rho(L_{\phi}) \eta$. The spectral gap property implies that given $\phi \in E$ then for some scalar $z$:

$$
\lim_{n \to \infty} \frac{L^n_{\phi} \phi}{\rho(L_{\phi})} = z \cdot \lim_{n \to \infty} \frac{L^n_{\phi} \eta}{\rho(L_{\phi})} \Rightarrow \eta(\phi) = z = \nu(\phi) \Rightarrow \eta = \nu.
$$

Remark 5.6. Given $L_{f,\phi} : E$ with the spectral gap property, by previous lemma there exists an unique $h_\phi \in E$ such that $\int h_\phi d\nu_\phi = 1$. Moreover, denote the $f^*$ invariant probability $h_\phi d\nu_\phi$ by $\mu_\phi$. Note that $\supp \nu_\phi = S^1$ and $h_\phi > 0$, by previous lemma, thus $\supp \mu_\phi = S^1$.

For the next lemma we follow the proof for expanding maps whose main part is Rohklin’s formula (see eg [OV16]). For convenience of the reader we will present the highlights of the proof.

Lemma 5.7. Let $f : S^1 \to S^1$ be a transitive $C^r$-local diffeomorphism, let $E = C^0(S^1, C)$ or $C^r(S^1, C)$ and let $\phi \in E$ be a real function. If $L_{f,\phi} : E$ has the spectral gap property, then $P_{\nu_\phi}(f, \phi) = \log \rho(L_{f,\phi} \phi)$ and the probability $\mu_\phi$ is an equilibrium state for $f$ with respect to $\phi$.

Proof. From lemma 5.4 the transfer operator $L_{\phi}$ has bounded away from 0 eigenfunction $h_\phi \in E$ for the eigenvalue $\lambda_\phi = \rho(L_{\phi} \phi)$:

$$
L_{\phi}(h_\phi) = \lambda_\phi h_\phi,
$$

and the dual operator $L^*_{\phi} : L_{\phi}$ has an eigenmeasure $\nu_\phi$ for the eigenvalue $\lambda_\phi$, in particular

$$
\int L_{\phi}(g) d\nu_\phi = \lambda_\phi \int g d\nu_\phi \quad \forall g \in E.
$$

Furthermore $\mu_\phi = h_\phi \nu_\phi$ defines an invariant probability. Note that

$$
L^*_{\phi} + \log h_\phi - \log (h_\phi \phi) \mu_\phi = \lambda_\phi \mu_\phi,
$$

by [PU10, corollary 4.2.7] it follows that $f$ admits Jacobians relatively to $\mu_\phi$ given by

$$
J_{\mu_\phi} f = \lambda_\phi \frac{e^{-\phi} h_\phi \phi f}{h_\phi}.
$$

See that $f$ admits generating partition, then Rohklin’s formula (see theorem 4.8) holds and we get:

$$
h_{\mu_\phi} (f) = \int \log J_{\mu_\phi} f d\mu_\phi = \log \lambda_\phi - \int \phi d\mu_\phi + \int (\log h_\phi \phi f - \log h_\phi) d\mu_\phi.
$$
Since $\mu_\phi$ is invariant and $\log h_\phi$ is bounded ($h_\phi$ is bounded away from 0), the last term is zero and thus

$$h_{\mu_\phi}(f) + \int \phi d\mu_\phi = \log \lambda_\phi.$$

Now we prove that $\mu_\phi$ is an equilibrium state. Let $\eta$ be an $f$–invariant probability satisfying

$$h_\eta(f) + \int \phi d\eta \geq \log \lambda_\phi.$$

Let $g_\eta := \frac{1}{\eta}$ be, and let $g_{\mu_\phi} := \frac{1}{\mu_\phi}$ be. Note that

$$\sum_{y \in f^{-1}(x)} g_{\mu_\phi}(y) = \frac{1}{\lambda_\phi h_\phi(x)} \sum_{y \in f^{-1}(x)} e^{\phi(y)} h(y) = \frac{L_{f,\phi} h_\phi(x)}{\lambda_\phi h_\phi(x)} = 1$$

for all $x \in S^1$. Also, since $\eta$ is invariant by $f$

$$\sum_{y \in f^{-1}(x)} g_\eta(y) = 1 \text{ for } \eta - \text{a.e.} x.$$

By Rohklin’s formula,

$$0 \leq h_\eta(f) + \int \phi d\eta - \log \lambda_\phi = \int (-\log g_\eta + \phi - \log \lambda_\phi) d\eta.$$

Since $\eta$ is invariant and by definition of $g_{\mu_\phi}$, the integral above equals

$$\int (-\log g_\eta + \log g_{\mu_\phi} + \log h_\phi \circ f - \log h_\phi) d\eta = \int \log \frac{g_{\mu_\phi}}{g_\eta} d\eta.$$

Then, by definition of $g_\eta$

$$\int \log \frac{g_{\mu_\phi}}{g_\eta} d\eta = \int \left( \sum_{y \in f^{-1}(x)} g_\eta(y) \log \frac{g_{\mu_\phi}}{g_\eta}(y) \right) d\eta(x).$$

Since $t \mapsto \log t$ is concave, we have:

$$\sum_{y \in f^{-1}(x)} g_\eta(y) \log \frac{g_{\mu_\phi}}{g_\eta}(y) \leq \log \sum_{y \in f^{-1}(x)} g_\eta(y) \frac{g_{\mu_\phi}}{g_\eta}(y) = \log \sum_{y \in f^{-1}(x)} g_{\mu_\phi}(y) = 0$$

for $\eta$-a.e. $x$. Finally we get

$$h_\eta(f) + \int \phi d\eta - \log \lambda_\phi = \int \frac{g_{\mu_\phi}}{g_\eta} d\eta = 0.$$

Thereby $\mu_\phi$ is an equilibrium state for $f$ with respect to $\phi$ and from the variational principle

$$P(f, \phi) = \log \rho (L_\phi).$$
Proof of the proposition 5.3. If $\mathcal{L}_{f, -t\log|Df|}^\phi$ had the spectral gap property on $(0, 1]$, then by lemma 5.7 and corollary 4.12, the topological pressure function

$$ (-\varepsilon + t_0, t_0 + \varepsilon) \ni t \mapsto P_{\text{top}}(f, -t\log|Df|) = \log \left( \rho \left( \mathcal{L}_{f, -t\log|Df|}^\phi \right) \right) $$

would be analytical, contradicting theorem A. $\square$

5.2.2. Lack of spectral gap after transition

Proposition 5.8. Let $E = BV[S^1], C^0(S^1, \mathbb{C})$ or $C^r(S^1, \mathbb{C})$ and let $f : S^1 \to S^1$ be a transitive $C^1$-local diffeomorphism not invertible, with $Df \in E$. Suppose also that $f$ is not an expanding dynamics. If $\mathcal{L}_{f, -t\log|Df|}$ has the spectral gap property on $E$, for some $s \in \mathbb{R}$, then the topological pressure function $t \mapsto P(t)$ is strictly convex in a neighborhood of $s$. Moreover, $\mathcal{L}_{f, -t\log|Df|}^\phi$ has not spectral gap property for all $t \geq t_0$.

Proof. Suppose that $\mathcal{L}_{f, -s\log|Df|}^\phi$ has the spectral gap property on $E$, for some $s \in \mathbb{R}$. Then again, by corollary 4.12, spectral gap holds for any small perturbation of this operator and thus, since the operator varies analytically with the potential, for any small perturbation of the potential. Let $\lambda_\phi$ be denoting $\rho(\mathcal{L}_{f, \phi}|_{E})$ and fix $\psi \in E$. Then the function $T(t) := \frac{\lambda_{-\log|Df|}^\phi + it\psi}{\lambda_{-\log|Df|}}$ is well defined for $t \in E$ in a small neighborhood of zero and is analytical. By Nagaev’s method we have $\sigma^2_{f, -s\log|Df|}^\phi(\psi) = -D^2_T(t)|_{t=0}$ where $\sigma^2 := \sigma^2_{f, -s\log|Df|}^\phi(\psi)$ is the variance of the Central Limit Theorem with respect to dynamics $f$, probability $\mu_{-s\log|Df|}$ and observable $\psi$. Moreover $\sigma = 0$ if, only if, there exists $c \in \mathbb{R}$ and $u \in E$ such that

$$ \psi(x) = c + u \circ f(x) - u(x), \text{ for } \mu_{-s\log|Df|} = \text{ a.e. } x,$$

(see e.g. [S12, lecture 4] for more details). Note that

$$ \frac{d\lambda_{-\log|Df|}^\phi + it\psi}{dt} = \frac{d\lambda_\phi}{dt} |_{\phi = -s\log|Df| + it\psi} \Rightarrow$$

$$ \frac{d^2T(t)}{dt^2}|_{t=0} = -\frac{d^2\lambda_{-\log|Df|}^\phi | \phi = -s\log|Df| + it\psi}{d^2\psi} = -\frac{d^2\lambda_{-\log|Df|}^\phi | \phi = -s\log|Df|}{\lambda_{-\log|Df|}} = \frac{d^2\lambda_{-\log|Df|}}{d^2t}|_{t=0} \frac{\lambda_{-\log|Df|}}{\lambda_{-\log|Df|}}.$$

In particular, we will take $\psi = -\log|Df| + \int \log|Df|d\mu_{-s\log|Df|}.$

Define $G(t) := P_{\text{top}}(f, -s\log|Df| + t\psi).$ Note that

$$ G(t) = P(t + s) + t \int \log|Df|d\mu_{-s\log|Df|}, $$

for all $t \in \mathbb{R}$. Furthermore, for $t \in \mathbb{R}$ close enough to zero we have that $\mathcal{L}_{f, -s\log|Df| + t\psi}$ has the spectral gap property and thus $G(t) = \log \lambda_{-s\log|Df| + t\psi}$, by lemma 5.7. Hence:

$$ \frac{d^2\lambda_{-s\log|Df| + t\psi}}{d^2t} = e^{G(t)} \left( (G'(t))^2 + G''(t) \right) \Rightarrow$$

$$ \sigma^2 = \sigma^2_{f, -s\log|Df|}^\phi(\psi) = \frac{d^2\lambda_{-s\log|Df|}}{d^2t}|_{t=0} \frac{\lambda_{-s\log|Df|}}{\lambda_{-s\log|Df|}} = (G'(0))^2 + G''(0).$$
It follows from [W92] that $G'(0) = \int \psi d\mu = 0$, we conclude then that $\sigma^2 = G''(0)$.

Suppose by contradiction that $G''(0) < 0$. Hence, by Nagaev's method, there exists $c \in \mathbb{R}$ and $u \in E$ such that

$$\log |Df(x)| = c + u \circ f(x) - u(x),$$

for $\mu_{-t \log |Df|} = \text{a.e.}$, $x$, where $\mu_{-t \log |Df|}$ is the equilibrium state obtained via the spectral gap property, which we know has full support. Therefore $\log |Df| \equiv c + u \circ f - u$, and thus the Birkhoff time average $\frac{1}{n} S_n \log |Df|$ converges uniformly to $c$. Consequently, $c$ is the unique Lyapunov exponent of $f$.

Since $\text{supp } \nu$ we have:

$$\text{since } \mu_{-t \log |Df|} \text{ has full support, then } \theta (\mu_{-t \log |Df|}, E) = 0 \text{ for all } t \geq 0.$$

Thus $\phi$ is the only peripheral eigenvalue.

By quasi-compactness of $\nu$ we have that the function $\nu$ coincides a.e. Now applying the transfer operator $\phi$ we have:

$$\phi \in \text{sp}(\mu) \supseteq \mu,$$

and the claim holds.

5.2.3. Spectral gap before of transition. First we prove, for transitive local diffeomorphism in the circle, that if the transfer operator is quasi-compact then it has the spectral gap property.

**Proposition 5.9.** Let $E = BV(S^1)$, $C^*(S^1, C)$ or $C(S^1, C)$, let $f : S^1 \rightarrow S^1$ be a transitive $C^*$-local diffeomorphism and let $\phi \in E$ be a real function. If $\theta_{f,\phi}(E)$ is quasi-compact, then it has the spectral gap property.

**Proof.** By quasi-compactness of $\theta_{f,\phi}$ then it admits a peripheral eigenvalue $\xi \rho(\theta_{f,\phi}|E)$, with $|\xi| = 1$. It follows from lemma 5.4 that $\lambda = \rho(\theta_{f,\phi})$ is a simple eigenvalue. It remains to show that it is the only peripheral eigenvalue.

Following the proof of the lemma 5.4, let $\phi \in E$ be an eigenvector, not null, for $\xi \rho(\theta_{f,\phi}|E)$, define

$$s(x) := \begin{cases} \frac{\phi(x)}{\xi \rho(\theta_{f,\phi}|E)} & \text{if } \phi(x) \neq 0 \\ 1 & \text{otherwise,} \end{cases}$$

claim: $\rho(\theta_{f,\phi}|E) = 0 < s^0|\phi|$, for all $n \in \mathbb{Z}$.

Case $E = BV(S^1)$: we know that there exists an eigenmeasure $\nu \in \text{ker } \left( (\theta_{f,\phi}|C^0)^* - \lambda I \right)$ which has full support, since $f$ is topologically exact (see claim in lemma 5.4). Applying [Ba00, proposition 3.5] the claim holds.

Case $E = C^*(S^1, C)$ or $C(S^1, C)$: Following the proof of lemma 5.4 we have $s(x) = \xi \cdot s \circ f(x)$, for $|\phi|\text{a.e.}$. Thus $s^0 = \xi^0 \cdot s |\phi|\text{a.e.}$ which means $s^0|\phi| = \xi^0 \cdot s^0 |\phi|$ and $s^0|\phi|$. Since $\text{supp } \nu = S^1$ we have that the functions coincide a.e. Now applying the transfer operator we have:

$$L_{\phi}(s^0|\phi|) = L_{\phi}(s^0|\phi|) = \xi^0 \cdot s^0 (\phi \circ f^{-1} \cdot s|\phi|)$$

and the claim holds.

By lemma 5.4 $\phi$ is bounded away from zero, thus $s$ and $|\phi| \in E$. Thereby $s^0|\phi| \in E$ and, by previous claim, it follows that $\xi^0 \cdot \rho(\theta_{f,\phi}|E) \in \text{sp}(\theta_{f,\phi})$ for all $n \in \mathbb{Z}$. 

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Since the set \( \{\xi^n : n \in \mathbb{Z}\} \) forms a subgroup of the circle then it is either dense or periodic. However by quasi-compactness of the transfer operator all eigenvalues are isolated and it cannot be a dense subgroup, so there is a \( k > 0 \) such that

\[
\xi^k \rho(\mathcal{L}_\theta|E) = \rho(\mathcal{L}_\theta|E).
\]

We already know \(|\varphi|\) is an eigenvector for the spectral radius, thus we have both

\[
\mathcal{L}_\varphi^k (|\varphi|) = \rho(\mathcal{L}_\varphi)^k |\varphi|
\]

\[
\mathcal{L}_\varphi^k (\varphi) = \rho(\mathcal{L}_\varphi)^k \varphi
\]

and then

\[
\mathcal{L}_\varphi^k (|\varphi| - \varphi) = \rho(\mathcal{L}_\varphi)^k (|\varphi| - \varphi).
\]

We can assume without loss of generality that there exists \( x_0 \) such that \( \varphi(x_0) > 0 \), otherwise it is enough replacing \( \varphi \) by \( \frac{\varphi}{\varphi(0)} \). Thus \( (|\varphi| - \varphi)(x_0) = 0 \). Similarly to the end of the proof of lemma 5.4 we conclude that \(|\varphi| - \varphi\) is null in \( \{y \in f^{-kn}(x_0) : n \in \mathbb{N}\} \). Topological exactness implies that \( \{y \in f^{-kn}(x_0) : n \in \mathbb{N}\} \) is dense. Thus continuity implies that \(|\varphi| - \varphi \equiv 0 \). Hence

\[
\xi^k \rho(\mathcal{L}_\varphi|E) \varphi = \mathcal{L}_\varphi\varphi = \mathcal{L}_\varphi|\varphi| = \rho(\mathcal{L}_\varphi|E) \varphi.
\]

Finally \( \xi = 1 \) and \( \rho(\mathcal{L}_\varphi|E) \) is the unique peripheral eigenvalue. \( \square \)

**Proof of the theorem B.** By proposition 5.8, to prove the Theorem is enough proving that \( \mathcal{L}_{f, t}\log|Df|\|E \) has the spectral gap property for all \( t < t_0 \). The proof will be divided in two cases.

**Case E = BV[\mathbb{S}^1]:**

Let \( R_t := \lim_{n \to \infty} \| \mathcal{L}_{f, -t \log(Df)} \|^1 \) and \( \tilde{R}_t := \lim_{n \to \infty} \| \frac{1}{\mathbb{M}_{n}} \|_{\infty} \) be. Then, by [Ba00, theorem 3.2] and [BK90, theorem 3], \( R_t = \rho(\mathcal{L}_{f, -t \log|Df| |E}) = e^{\log(t)} \) and, by [BK90, theorem 1], \( \tilde{R}_t \geq \rho_{\text{ess}}(\mathcal{L}_{f, -t \log|Df| |E}) \).

**Lemma 5.10.** \( t \geq 0 \Rightarrow \tilde{R}_t \leq 1 \).

**Proof.** Obviously \( \tilde{R}_0 = 1 \). Take \( t > 0 \). Suppose by absurd that \( \tilde{R}_t > 1 \). Then there exists \( \lambda > 1 \) and \( n_0 \in \mathbb{N} \) such that \( \| \frac{1}{\mathbb{M}_{n}} \|_{\infty} \geq \lambda \), for all \( n \geq n_0 \). Thus, for each \( n \geq n_0 \) there exists \( x_n \in \mathbb{S}^1 \) with:

\[
\frac{1}{|Df^n(x_n)|^\frac{1}{N}} \geq \lambda \Rightarrow |Df^n(x_n)| \leq \lambda^{-\frac{1}{N}}.
\]

Define \( \mu_n := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x_n)} \). Since the space of probabilities on \( \mathbb{S}^1 \) is compact, in the weak-* topology, take \( \mu := \lim_{k \to \infty} \mu_{\mu_k} \). Note that \( \mu \) is \( f \)-invariant, furthermore

\[
\int \log |Df| d\mu = \lim_{k \to \infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \log |Df(f^j(x_n))| = \lim_{k \to \infty} \frac{1}{n_k} \log |Df^n(x_n)|
\]

\[
\leq - \frac{1}{N} \log \lambda < 0.
\]
By ergodic decomposition theorem we can write $\mu = \int \mu_x \, d\mu(x)$, where each $\mu_x$ is $f$-ergodic (see e.g. [OV16]). Hence

$$\int \log |Df| \, d\mu = \int \left( \int \log |Df| \, d\mu_x \right) \, d\mu(x) = \int \chi_{\mu_x} (f) \, d\mu(x) \leq - \frac{1}{t} \log \lambda < 0 \Rightarrow \exists \mu_x \text{com} \chi_{\mu_x} (f) < 0.$$

This is absurd because $\mathcal{M}_\infty^c (f) = \emptyset$.

Note that $R_s = e^{\rho(t)} > 1$, for all $t \in [0, t_0)$. Applying the previous lemma, it follows that $\mathcal{L}_{f, t \log |Df|}$ is quasi-compact for all $t \in [0, t_0)$ and by lemma 5.9 it has the spectral gap property for $t \in [0, t_0)$.

Case $E = C^\alpha (S^1, \mathbb{C})$ or $C^r (S^1, \mathbb{C})$:

**Lemma 5.11.** The pressure function $t \mapsto P(t)$ is strictly decreasing in $(\infty, t_0]$.

**Proof.** Since $f$ is expansive, classic Ergodic theory implies the entropy function $\mu \mapsto h_\mu (f)$ is upper semi-continuous (see e.g. [OV16]) and therefore $f$ has at least one ergodic equilibrium state $\mu_\phi$ for a given potential $\phi$. For all $t < t_0$ we have $P(t) > 0$, thus $P(t) = h_t - t \lambda_0$, where $h_t := h_{t \log |Df|}$ and $\lambda_t := \int \log |Df| \, d\mu_{t \log |Df|} > 0$ by Ruelle–Margulis’s inequality. Take $t < s < t_0$, then by supremum definition

$$P(t) = h_t - t \lambda_t \geq h_s - t \lambda_s = h_s - s \lambda_s + (s - t) \lambda_s \geq h_s - s \lambda_s = P(s)$$

and $P$ is strictly decreasing.

**Remark 5.12.** Note that, the previous estimates also shows that if $t < s$ and $\lambda_0 > 0$ then $P(t) > P(s)$.

For the following lemmas we need to estimate the essential spectral radius, which in the given spaces will require different approaches. First for $E = C^\alpha (S^1, \mathbb{C})$, the results [BJL96, theorems 1 and 2] reads, in our context, as the following

**Theorem 5.13.** Let $f : S^1 \to S^1$ be a $C^1$-local diffeomorphism and let $\phi \in C^\alpha (S^1, \mathbb{R})$ be then

$$\rho_{ess} (\mathcal{L}_f, \phi | C^\alpha) = \lim_{n \to \infty} \| \mathcal{L}_f^n, \phi - \alpha \log |Df| 1 \|^{1/n}_\infty \text{ and } \rho (\mathcal{L}_f, \phi | C^\alpha) = \max \left\{ \lim_{n \to \infty} \| \mathcal{L}_f^n, 1 \|^{1/n}_\infty, \rho_{ess} (\mathcal{L}_f, \phi | C^\alpha) \right\},$$

**Remark 5.14.** Note that $\lim_{n \to \infty} \| \mathcal{L}_f^n, 1 \|^{1/n}_\infty = \rho (\mathcal{L}_f, \phi | C^\alpha)$.

For $E = C^r (S^1, \mathbb{C})$, the results [CL97, theorems 1 and 2] summarize to:

**Theorem 5.15.** Let $f : S^1 \to S^1$ be a $C^r$-local diffeomorphism and let $\phi \in C^r (S^1, \mathbb{R})$ be then

$$\rho_{ess} (\mathcal{L}_f, \phi | C^r) \leq \exp \left[ \sup_{\mu \in \mathcal{M}_1 (f)} \left\{ h_\mu (f) + \int \phi d\mu - k \chi_\mu (f) \right\} \right] \text{ and } \rho (\mathcal{L}_f, \phi | C^r) \leq \exp \left[ \sup_{\mu \in \mathcal{M}_1 (f)} \left\{ h_\mu (f) + \int \phi d\mu \right\} \right],$$

for $k = 0, 1, \ldots, r$. 

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Note that in this case
\[ \rho_{\text{ess}} \left( \mathcal{L}_{f,-\log |Df|} | C^s \right) \leq e^{P(k+i)} \quad \text{and} \quad \rho \left( \mathcal{L}_{f,-\log |Df|} | C^s \right) \leq e^{P(i)} \]

Then on continuous Hölder functions,
\[ \rho_{\text{ess}} \left( \mathcal{L}_{f,-\log |Df|} | C^\alpha \right) = \rho \left( \mathcal{L}_{f,-(\tau+\alpha)\log |Df|} | C^\alpha \right) \leq e^{P(\alpha+i)} \]

by theorems 5.13 and 5.15.

In either case we have
**Lemma 5.16.** \( \rho_{\text{ess}} \left( \mathcal{L}_{f,-\log |Df|} | E \right) \leq e^{P(k+i)} \), for some \( k > 0 \).

We start proving spectral gap for non-positive \( t \).

**Lemma 5.17.** \( \mathcal{L}_{t,-\log |Df|} | E \) has spectral gap for all \( t \leq 0 \).

**Proof.** \textit{Claim:} \( \mathcal{L}_{f,0} | E \) has spectral gap.

In fact, since \( P \) is strictly decreasing on \([0,t_0]\)
\[ \rho_{\text{ess}} \left( \mathcal{L}_{f,0} | E \right) \leq e^{P(k)} < e^{\rho(t)} = e^{\deg(f)} = \deg(f) = \rho \left( \mathcal{L}_{f,0} | E \right) . \]

Hence \( \mathcal{L}_{f,0} | E \) is quasi-compact and by lemma 5.9 it has spectral gap.

Let \( \tau_1 := \inf \{ t < 0; \mathcal{L}_{t,-\log |Df|} | E \) has the spectral gap property \} \ be. Suppose by contradiction that \( \tau_1 > -\infty \). Since the spectral gap property is open (see corollary 4.12) then \( \mathcal{L}_{f,-\tau_1 \log |Df|} | E \) has no spectral gap property.

\textit{Claim:} \( \rho \left( \mathcal{L}_{f,-\tau_1 \log |Df|} | E \right) = e^{P(\tau_1)} \).

In fact, for all \( t \in (\tau_1,0) \) the transfer operator \( \mathcal{L}_{-\tau_1 \log |Df|} | E \) has the spectral gap property, then \( \rho \left( \mathcal{L}_{-\tau_1 \log |Df|} | E \right) = e^{P(t)} \) and \( \langle \tau_1, 0 \rangle \ni t \mapsto \rho \left( \mathcal{L}_{-\tau_1 \log |Df|} | E \right) \) is decreasing, by lemmas 5.7 and 5.11. Now suppose by contradiction there is a \( t_0 \leq \tau_1 \) such that \( \rho \left( \mathcal{L}_{-t_0 \log |Df|} | E \right) < \rho \left( \mathcal{L}_{-\tau_1 \log |Df|} | E \right) - \epsilon . \) By semi-continuity of spectral components (see e.g. [K95]) we have \( \text{sp} \left( \mathcal{L}_{-t_0 \log |Df|} | E \right) \subset B(0, \rho \left( \mathcal{L}_{-\tau_1 \log |Df|} | E \right) + \epsilon / 2) \). Therefore
\[ \rho \left( \mathcal{L}_{-\tau_1 \log |Df|} | E \right) \geq \sup_{t \in (\tau_1,0)} \rho \left( \mathcal{L}_{-\tau_1 \log |Df|} | E \right) = \sup_{t \in (\tau_1,0)} e^{P(t)} = e^{P(\tau_1)} . \]

On the other hand, by theorem 5.15 \( \rho \left( \mathcal{L}_{-\tau_1 \log |Df|} | C^s \right) \leq e^{P(\tau_1)} \). Furthermore, by theorem 5.13 and, again, theorem 5.15
\[ \rho \left( \mathcal{L}_{-\tau_1 \log |Df|} | C^\alpha \right) = \max \left\{ \rho \left( \mathcal{L}_{-\tau_1 \log |Df|} | C^\alpha \right) , \rho_{\text{ess}} \left( \mathcal{L}_{-\tau_1 \log |Df|} | C^\alpha \right) \right\} \leq \max \left\{ e^{P(\tau_1)} , e^{P(\tau_1 + \alpha)} \right\} . \]

Since \( t \mapsto P(t) \) is decreasing we obtain that \( \rho \left( \mathcal{L}_{-\tau_1 \log |Df|} | C^s \right) \leq e^{P(\tau_1)} \). Thus we conclude \( \rho \left( \mathcal{L}_{-\tau_1 \log |Df|} | E \right) = e^{P(\tau_1)} \).

Therefore,
\[ \rho \left( \mathcal{L}_{-\tau_1 \log |Df|} | E \right) = e^{P(\tau_1)} > e^{P(\tau_1 + k)} \geq \rho_{\text{ess}} \left( \mathcal{L}_{-\tau_1 \log |Df|} | E \right) , \]

and by lemma 5.9 \( \mathcal{L}_{-\tau_1 \log |Df|} | E \) has the spectral gap property, contradicting the minimality of \( \tau_1 \).
Let \( \tau_2 := \sup \{ t > 0; \mathcal{L}_{f,-t\log |Df|}k \text{ has the spectral gap property} \} \), note that \( \tau_2 \) exists and is at most \( t_0 \). Analogously to the second claim in the previous Lemma, it holds \( \rho(\mathcal{L}_{f,-\tau_2 \log |Df|}k) = e^{P(\tau_2)} \). Since the spectral gap property is open, \( \mathcal{L}_{f,-\tau_2 \log |Df|}k \) has not spectral gap and thus \( \rho(\mathcal{L}_{f,-\tau_2 \log |Df|}k) = \rho_{\text{ess}}(\mathcal{L}_{f,-\tau_2 \log |Df|}k) \) by lemma 5.9.

**Lemma 5.18.** \( \tau_2 = t_0 \)

**Proof.** By the essential radius estimates on lemma 5.16, we get:

\[
e^{P(\tau_2)} = \rho(\mathcal{L}_{-\tau_2 \log |Df|}k) = \rho_{\text{ess}}(\mathcal{L}_{-\tau_2 \log |Df|}k) \leq e^{P(\tau_2 + k)}.
\]

Since \( P \) is decreasing we have \( P(\tau_2) = P(\tau_2 + k) \). Therefore \( \tau_2 \geq t_0 \), since \( P(t) \) is strictly decreasing before \( t_0 \). Consequently, \( \tau_2 = t_0 \).

From definition of \( \tau_2 \) we get:

**Corollary 5.19.** \( \mathcal{L}_{f,-t\log |Df|}k \) has spectral gap for all \( t < t_0 \).

This completes the proof of theorem B.

### 5.2.4. Effective thermodynamical phase transition

**Proof of the corollary A.** Just like proposition 5.3, this is a direct consequence of \( \rho(\mathcal{L}_{f,-t\log |Df|}k) = e^{P(t)} \) for \( t < t_0 \) and corollary 4.12. \( \square \)

### 6. Examples, discussions and more questions

#### 6.1. Examples

In this section we present some explicit examples of local diffeomorphisms with different types of phase transition. First, given a \( C^1 \) diffeomorphism \( g : [0,1/2] \to [0,1] \) consider the circle map with identification \( 0 = 1 \) defined as,

\[
f(x) = \begin{cases} g(x), & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 - g(1-x), & \text{if } \frac{1}{2} < x \leq 1. \end{cases}
\]

(6.1)

The map \( f \) defined this way is a \( C^1 \) local diffeomorphism.

**Example 6.1.** Let \( a \in (0,2) \), then \( g_a(x) = ax + (4 - 2a)x^2 \) gives a continuous family of local diffeomorphisms \( f_i(x) \) by equation (6.1). Note that \( f'_a(x) = a + (8 - 4a)x \) and \( f'_a(0) = a \), so that the system undergoes a sort of pitchfork bifurcation:

1. For \( a \in (1,2] \), the fixed point 0 is expanding, and the map \( f_a \) is in fact uniformly expanding. Note that for \( a = 2 \), the map \( f_2 \) is the doubling map \( 2 \text{mod } 1 \) (see figure 7).

   Consequently the geometric pressure for these maps is strictly convex and decreasing and therefore has no phase transition.

2. For \( a = 1 \), the fixed point 0 is indifferent, and the map \( f_1 \) obtained from \( g_1(x) = x + 2x^2 \) is transitive and intermittent, that is, expanding everywhere except for 0. Note that this is a circle version of the Manneville–Pomeau map on the circle.

   Consequently, the Lyapunov exponent associated to the Dirac measure \( \delta_0 \) is zero and therefore the geometric pressure for this map fits into the case of figure 4 of theorem A.
Since this map is of class $C^{1+Lip}$, in particular $\log |Df|$ is Hölder continuous, we have that theorem B and corollary A do apply for this map (see figure 8).

(3) For $a \in (0, 1)$, the fixed point 0 becomes attracting, and two other expanding fixed points emerge. In this case, the map $f_a$ is not transitive, where the small arc containing these three fixed points is invariant.

Consequently the Lyapunov exponent associated to the Dirac measure $\delta_0$ is negative and therefore the pressure for these maps fits into the case of figure 5 of theorem A. However theorem B and corollary A do not apply to this case.
Figure 8. Case $\alpha = 1$ and $g_\alpha(x) = x + 2x^2$ in the equation (6.1).

Figure 9. Intermittent map of class $C^2$.

Example 6.2. As mentioned before, the main models for transitive non-expanding local diffeomorphisms on the circle are the Manneville–Pomeau maps obtained from $g_\alpha(x) = x + 2^\alpha x^{\alpha+1}$ and equation (6.1). This maps are of class $C^{2+\alpha}$ but not of class $C^2$. In order to have both the fixed point 0 and the middle point $1/2$ as roots of the second derivative, and thus proper inflexion points, we need polynomials of higher degree. So a $C^2$ version of the Manneville–Pomeau map is the map $f$ obtained from $g(x) = x + 8x^3 - 8x^4$ in equation (6.1) (see figure 9):

We could again consider two families containing this map. The first is for a given $0 < k \leq 2$ the family of functions $f_k(x) = kx + (16 - 8k)x^3 + (8k - 16)x^4$ that when applied to equation (6.1), gives a family of $C^2$ local diffeomorphisms on the circle. For $k < 1$ the maps...
are non-transitive map with attracting fixed point, \( k = 1 \) the \( C^2 \) Manneville–Pomeau map, and for \( k > 1 \) uniformly expanding maps that converge to the Bernoulli map: \( 2x \mod 1 \). Thus we have again a bifurcation, that is completely analogous to figure 6.

Another interesting family of maps that have all the point \( x = 0 \) as an indifferent fixed point would be the following: consider \( p \in [0, 1] \) and define \( b = b(p) := ((\frac{1}{2})^{3+p} - \frac{4 + p}{4 + 2p})^{-1} \) and \( a = a(p) := \frac{-b(4 + p)}{4 + 2p} \) then \( f(x) = x + ax^{3+p} + bx^{4+p} \) applied to equation (6.1) gives a family of \( C^2 \) maps with indifferent fixed points, that is analogous to the family obtained from \( g_\alpha(x) = x + 2^\alpha x^{\alpha+1} \).

6.2. Calculating the phase transition

Let \( f : S^1 \to S^1 \) be as in corollary A. We obtain the transition parameter

\[
t_0 = \inf \{ t \in \mathbb{R} : P_{\text{top}}(f, -t \log |Df|) = 0 \} = \sup \{ t > 0 : \mathcal{L}_{f, -t \log |Df|} \text{ has the spectral gap property } \}
\]

\[
= \inf \{ t > 0 : \mathcal{L}_{f, -t \log |Df|} \text{ has not the spectral gap property } \},
\]

moreover \( t_0 \leq 1 \).

We are interested in this section to discuss when we can make the transition parameter \( t_0 \) explicit, as well as obtain a characterization of \( t_0 \) in terms of the measures with positive Lyapunov exponent.

Initially we will see cases where \( t_0 = 1 \). The first one is when we have an a.c.i.p. with positive Lyapunov exponent.

**Proposition 6.3.** Let \( f : S^1 \to S^1 \) be as in corollary A. Suppose that \( f \) admits an \( f \)-invariant probability \( \mu \) such that \( \mu \) is absolutely continuous with respect to Lebesgue measure and has positive Lyapunov exponent. Then \( t_0 = 1 \).

**Proof.** Since \( \mu \) is absolutely continuous with respect to Lebesgue measure, by Pesin’s entropy formula (see [P77]), we have that \( \mu \) is an equilibrium state with respect to geometric potential \( -\log |Df| \). By hypothesis, \( \mu \) has positive Lyapunov exponent. Thus, by remark 5.12, we have \( 0 = P(1) < P(t) \) for all \( t < 1 \). It implies that \( t_0 = 1 \). \( \square \)

The second case is associated with the case in which the loss of expansion is only caused by indifferent fixed point. Initially we need to introduce the notion of dynamical dimension. Fix \( f : [0, 1] \to [0, 1] \) a piecewise monotonic map. Given \( \mu \in \mathcal{M}_r(f) \), define \( \text{HD}(\mu) := \inf \{ \text{HD}(Y) : \mu(Y) = 1 \} \), where \( \text{HD}(Y) \) is the Hausdorff dimension of \( Y \). We define \( dD(f) := \sup \{ \text{HD}(\mu) : h_\mu(f) > 0 \} \), the dynamical dimension of \( f \).

**Remark 6.4.** Another way to think of the dynamical dimension of a map, in some contexts, is to consider the supremum of \( \text{HD}(\Lambda) \) for all \( \Lambda \) repellers for \( f \). Take for instance a continuous map \( f : X \to X \) of a compact subspace of the closed complex plane \( X \subset \mathbb{C} \) that can be analytically extended to a neighborhood \( U(f) \) of \( X \). Then [PU10, theorem 10.6.1] guarantees that given any measure with positive Lyapunov exponent we obtain measures supported on repellers that approximate both entropy and Lyapunov exponents of \( \mu \), thus by [HZ12, corollary 4] we obtain an approximation also for the dimension of these measures.

In the paper [H93] the aim is to find conditions such that \( dD(f_\Lambda) \geq z_\Lambda \geq \text{HD}(A) \), where \( A \) is a compact \( f \)-invariant subset, \( f_\Lambda \) is transitive and \( z_\Lambda := \inf \{ t > 0 : P_{\text{top}}(f_\Lambda, -t \log |Df_\Lambda|) \leq 0 \} \).
In fact, if $|Df| > 1$ except on a finite set of indifferent fixed points then $z_A \geq HD(A)$. In our context $A = S^1$ and $z_A = t_0$, thus applying [H93, theorem 6]:

**Proposition 6.5.** Let $f : S^1 \to S^1$ be as in the corollary A. Suppose that $|Df| > 1$ except on a finite set of fixed points on which $Df$ is equal to 1. Then $t_0 = 1$

In spite of these two cases, in general $t_0$ does not need to be 1 because, as we will see, $t_0$ is related to the support of measures with zero Lyapunov exponent.

**Proposition 6.6.** Let $f : S^1 \to S^1$ be as in the corollary A. Then $t_0 = dD(f)$.

**Proof.** On the one hand, applying [HZ12, corollary 4] in our context we have that: if $\mu$ is an $f$–invariant ergodic Borel probability measure with $\chi_{\mu}(f) > 0$ then $HD(\mu) = \frac{h_\mu(f)}{\chi_{\mu}(f)}$. In particular, $P_\mu(HD(\mu)) = 0$. Thus, take $0 < t < t_0$ and $\mu_1$ such that $P_\mu(t) = P(t) > 0$. Hence, $0 < h_\mu(f) \leq \chi_{\mu}(f)$ and then

$$P_\mu(HD(\mu_i)) = 0 \Rightarrow HD(\mu_i) > t \Rightarrow dD(f) > t, \forall t < t_0 \Rightarrow dD(f) \geq t_0.$$ 

On the other hand, suppose by absurd that $dD(f) > t_0$. Then there exists $\mu \in M_\mu(f)$ with $h_\mu(f) > 0$ and $HD(\mu) > t_0$. Therefore

$$\frac{h_\mu(f)}{\chi_{\mu}(f)} > t_0 \Rightarrow P_\mu(t_0) > 0 \Rightarrow P(t_0) > 0,$$

this is absurd. \qed

The previous result is analogous to the generalized Bowen’s formula obtained in [PRL19] in the context of multimodal maps.

This discussion leads us to the following question:

**Question B.** There exists a dense subset of the space of transitive, non invertible and non uniformly expanding local diffeomorphism on the circle whose thermodynamical phase transition occurs in the parameter $t = I$ ?

### 6.3. Non-transitive case

In theorem B and corollary A it was assumed the transitivity hypothesis, the purpose of this section is to present a classic result on piecewise monotonic maps in order to indicate some perspectives on this case.

Given $f : [0, 1] \to [0, 1]$ a piecewise continuous and monotonic map, the papers [H86, HR89, W88] guarantee us that a decomposition theorem on the nonwandering set $\Omega(f)$ holds:

$$\Omega(f) = \bigcup_{j=1}^{\infty} L_j \cup L_\infty \cup W \cup P$$

where the sets $L_j, L_\infty$ and $P$ are closed and $f$–invariant, and the intersection of two different sets in this union is finite or empty. Moreover,

- $f_{L_j}$ is topologically conjugated to a transitive subshift of finite type.
- $L_{\infty}$ is the disjoint union of finitely many compact, $f$–invariant sets $N_i$, such that $N_i$ is a Cantor-like and $f_{|N_i}$ is minimal. In fact, there are only finitely many ergodic invariant Borel probability measures on each $N_i$ and $h_{top}(f_{|L_\infty}) = 0$. 


• \( W \) consists of nonperiodic points, which are isolated in \( \Omega(f) \).
• \( P \) consists of periodic points contained in nontrivial intervals \( K \subset S^1 \) with the property that \( f^n \) maps \( K \) monotonically into \( K \) for some \( n \).
• Given \( \mu \in \mathcal{M}_e(f) \) there exists \( \eta \in \mathcal{M}_e(f|_{L_j}) \) such that \( h_\mu(f) \leq h_\eta(f) \).

In particular, taking \( f : S^1 \to S^1 \) a local diffeomorphism, define
\[
P_j(t) := P_{\text{top}}\left(f|_{L_j}, -t \log |Df|\right), \quad P_{\infty}(t) := P_{\text{top}}\left(f|_{L_{\infty}}, -t \log |Df|\right)
\]
and
\[
P_p(t) := P_{\text{top}}\left(f|_p, -t \log |Df|\right).
\]
Note that \( P_{\infty} \equiv 0 \) and \( P_p \) is linear in each of the intervals \( (-\infty, 0) \) and \( [0, +\infty) \). In general, we expect that \( f \) can have infinite thermodynamical phase transitions with respect to \( -\log |Df| \).

Suppose now that \( f \) is not invertible and non transitive, \( Df \) is Hölder continuous and that there is a unique \( L_j \). Then
\[
P_{\text{top}}(f, -t \log |Df|) = \max\{P_j(t), P_p(t)\}
\]
and \( h_{\text{top}}(f|_{L_j}) = h_{\text{top}}(f) = \log \deg(f) \). Hence, \( f|_{L_j} \) is topologically conjugated to a full shift. We cannot apply directly theorem \( \text{A} \) on \( f|_{L_j} \) because \( L_j \) will not be the circle or a union of intervals (in fact, typically \( L_j \) will be a Cantor set). Despite this fact, we can follow the proof of theorem \( \text{A} \) and the fundamental point for its conclusion are the estimates about the spectral radius and essential spectral radius of the transfer operator \( \mathcal{L}_{f|_{L_j}, -t \log |Df|}|_{C^0} \). Thereby, if it is possible to overcome such obstacle we would conclude that \( f \) can have thermodynamical phase transitions with respect to \( -\log |Df| \) in at most in three points.

6.4. A little bit about the higher dimensional context

In this section we want to discuss a little about the problem \( \text{A} \) and conjecture \( \text{A} \) presented in the introduction.

We start with an example. Given the expanding dynamics \( f : S^1 \rightarrow S^1 \) and the irrational rotation \( R_\alpha : S^1 \rightarrow S^1 \), take the skew product
\[
F: S^1 \times S^1 \rightarrow S^1 \times S^1
\]
\[(x, y) \mapsto (f(x), R_\alpha(y))\]

Given the Hölder potential \( \phi : S^1 \times S^1 \rightarrow \mathbb{R} \) define the Hölder continuous potential
\[
\tilde{\phi} : S^1 \rightarrow \mathbb{R}
\]
\[x \mapsto \phi(x, y) dy \text{Leb}(y)\]

Note that
\[
P_{\text{top}}(F, \phi) = \sup_{\mu \in \mathcal{M}_e(F)} \left\{ h_\mu(f) + h_{\text{Leb}}(R_\alpha) + \int \phi(x, y) d\text{Leb}(y) d\mu(x) \right\} = P_{\text{top}}\left(f, \tilde{\phi}\right).
\]
Since \( f \) is an expanding dynamics then \( C^0(S^1, \mathbb{R}) \ni g \mapsto P_{\text{top}}(f, g) \) is analytical. We conclude that \( C^0(S^1 \times S^1, \mathbb{R}) \ni \phi \mapsto P_{\text{top}}(F, \phi) \) is analytical, even though \( F \) is not expanding or hyperbolic dynamics.
This example shows in high dimension we cannot expect to obtain thermodynamical phase transition for all local diffeomorphism that are not expanding or hyperbolic. Note that the previous example is not topologically conjugated to an expanding or hyperbolic dynamics. Moreover there is no mixing between the central direction and the expanding direction, that is a rigid condition. Thus, we propose the following questions:

**Question C.** Let \( f : M \to M \) be a local diffeomorphism such that \( f \) is topologically conjugated to an expanding or hyperbolic dynamics, but \( f \) is not an expanding or hyperbolic dynamics. Then \( f \) has thermodynamical phase transition with respect to some Hölder continuous potential?

**Question D.** Generically a local diffeomorphism is expanding, hyperbolic or has thermodynamical phase transition with respect to some Hölder continuous potential?

Remember that the paper [K120] assures us that for any map of the circle which is expanding outside an arbitrarily flat neutral point, the set of Hölder potentials such that \( L_{f,\phi}^p C^\alpha \) has the spectral gap property is dense, in the uniform topology. Thus it leads us to the following question:

**Question E.** Let \( f : M \to M \) be \( C^1 \)-local diffeomorphism on a manifold \( M \), with \( h_{\text{top}}(f) > 0 \). The set of potentials \( \phi_i \), in a suitable Banach space (Hölder continuous or smooth functions), such that \( L_{f,\phi_i} \) has not the spectral gap property acting on a suitable Banach space (Hölder continuous or smooth functions) can be dense or residual in the uniform topology?

### Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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### Appendix. Attracting periodic orbits

In this appendix we first transcribe part of the text from Araújo [A20], concerning attracting periodic orbits, then we apply it to our context.

In what follows \( M \) is a connected compact finite \( d \)-dimensional manifold \( M \), with \( d \geq 2 \). Let \( f : M \to M \) be a \( C^1 \) map such that \( \inf_{x \in M} \|Df(x)\| > 0 \). The subadditive Ergodic theorem ensures that the largest asymptotic growth rate

\[
\chi(x) = \lim_{n \to +\infty} \frac{\ln \|Df^n(x)\|^{1/n}}{n}
\]

exists for all \( x \) on a total probability subset since \( \ln^+ \|Df\| = \max \{0, \ln \|Df\| \} \) is \( \mu \)-integrable for each \( f \)-invariant probability measure \( \mu \).
In what follows we write $A_k^-(x) = \liminf_{n \to +\infty} n^{-1} \sum_{j=0}^{n-1} \ln \| \partial f^j (f^k x) \|$. We recall that $p \in M$ belongs to a periodic orbit (with period $r$) if there exists $r \in \mathbb{Z}$ so that $f^r(p) = p$. This periodic orbit $O(p) = \{ f^0(p), \ldots, f^{r-1}(p) \}$ is attracting (a sink, for short) if there exists a neighborhood $V_p$ of $p$ such that $\| f^r |_{V_p} : V_p \to V_p \|$ is a contraction: there exists $0 < \lambda < 1$ so that $\text{dist}(f^r q, f^r r) < \lambda \text{dist}(q, r), \forall q, r \in V_p$. Equivalently, $\| Df^r(p) \| < \lambda$ for some $\lambda \in (0,1)$.

With this set up, V. Araujo proves that a measure with negative Lyapunov exponents is concentrated on a sink:

**Corollary A.1 ([A20]).** Let $\mu$ be an invariant probability measure with respect to a $C^1$ map $f : M \to M$ such that $\inf_{x \in M} \| Df(x) \| > 0$ and $\chi(x) < 0$, $\mu$-a.e. $x \in M$. Then $\mu$ decomposes as $\hat{\mu} + \sum_{i \geq 1} \mu_i$, where each $\mu_i$ is a Dirac mass equidistributed on a periodic attracting orbit of $f$ (a sink), the sum is over at most countably many such orbits, and $\hat{\mu}$ (which might be the null measure) satisfies $A_k^-(x) \geq 0$, $\mu$-a.e. $x \in M$. In addition, if $\mu$ is $f$-ergodic, then $\mu$ is concentrated on the orbit of a periodic attractor (sink).

Now consider a given $C^1$ -local diffeomorphism on the circle $f : S^1 \to S^1$ and an $f$ -ergodic measure $\mu$ with negative Lyapunov exponent $\chi_\mu(f) < 0$. Given the restriction on the dimension of $M$, in order to apply corollary A.1, it is sufficient to consider the product map $f \times f$ acting on $S^1 \times S^1$ and the ergodic measure $\mu \times \mu$, with $\chi_{\mu \times \mu}(f \times f) = \chi_\mu(f) < 0$. We conclude that $\mu \times \mu$ is concentrated on the orbit of a sink, then so is $\mu$. To sum up, an ergodic measure with negative Lyapunov exponent for a $C^1$ map of the circle is necessarily supported on an attracting orbit.

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