Isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$ and perturbation of Hopf tori

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Abstract

We produce a new general family of flat tori in $\mathbb{R}^4$, the first one since Bianchi’s classical works in the 19th century. To construct these flat tori, obtained via small perturbation of certain Hopf tori in $S^3$, we first present a global description of all isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$ with flat normal bundle.

1 Introduction

Isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$ appear as a critical situation in one of the central problems of the differential geometry of submanifolds, namely, the problem of classifying isometric immersions between space forms. Let $Q^n(c), \bar{Q}^{n+p}(c)$ be two space forms of dimensions $n, n + p$ respectively, and with the same constant curvature $c$. If $c \geq 0$ and $p < n$ there exist very few isometric immersions $f : Q^n \rightarrow \bar{Q}^{n+p}$, which can be thought of as trivial. However, if $c \geq 0$ and $p \geq n$, or if $c < 0$, the situation changes completely and there exists a wide variety of isometric immersions from $Q^n$ into $\bar{Q}^{n+p}$. This leaves the case $c = 0$ and $n = p = 2$, that is, isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$, as the limit case between both situations.

There is another fact that motivates the study of isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$. Recall that a complete flat surface must be homeomorphic to a plane, a cylinder, a torus, a Möbius strip or a Klein bottle. Among these five topological cases, only the plane and the cylinder can be realized as complete flat surfaces in $\mathbb{R}^3$. On the other hand, all of them occur when one considers flat surfaces in $\mathbb{R}^4$. This indicates that
$\mathbb{R}^4$ is the most natural ambient space in where to consider a flat surface. One further motivation in connection with this point is given by Tompkins theorem, which states that there are no compact flat $n$-manifolds isometrically immersed in $\mathbb{R}^{2n-1}$. Again, the existence of flat tori in $\mathbb{R}^4$ makes isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$ a critical situation.

Motivated by all of this, the present paper studies isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$ with flat normal bundle, i.e. $R^\perp \equiv 0$. This geometric setting contains a large number of examples, and has been widely studied (see [Bia, Bor, CaDa, DaTo, DaTo2, Eno, HKP, Kit1, Kit2, Spi, Ten, Wei1, Wei2, Wei3]). We note that the flat normal bundle condition is the most natural hypothesis when dealing with isometric immersions between space forms with large codimension, as it is our case.

In this work we give a global description of all simply connected flat surfaces with flat normal bundle in $\mathbb{R}^4$, and we use it as main application to study flat tori in $\mathbb{R}^4$ with $R^\perp \equiv 0$ and regular Gauss map.

Up to now, and except for some isolated examples with $R^\perp \neq 0$, only two families of flat tori in $\mathbb{R}^4$ have been found, and both were already known in the 19th century. One of these families is made up by the product tori $\gamma_1 \times \gamma_2 \subset \mathbb{R}^2 \times \mathbb{R}^2 \equiv \mathbb{R}^4$, where $\gamma_1, \gamma_2$ are regular closed curves in $\mathbb{R}^2$. The other family consists of flat tori in the 3-sphere $S^3$. In [Bia] Bianchi showed that the multiplication in the quaternionic model of $S^3$ of two closed curves in $S^3$ satisfying certain conditions gives rise to a flat torus in $S^3$ for which these curves are asymptotic curves. Much later, Kitagawa proved in [Kit1] that the asymptotic curves of flat tori in $S^3$ are periodic, and thus the classical Bianchi method describes all flat tori in $S^3$. In addition, Weiner [Wei1] classified all flat tori in $S^3$ by means of their Gauss maps, thus solving an open problem posed by S.T. Yau [Yau].

In this way, no new flat torus in $\mathbb{R}^4$ with $R^\perp \equiv 0$ has been discovered since the 19th century, even though this family has been deeply studied. Besides, Weiner showed in [Wei1] that there exist no flat Klein bottles in $\mathbb{R}^4$ with $R^\perp \equiv 0$ and regular Gauss map. These facts created the impression, explicitly stated as a conjecture in [Bor], that there do not exist flat tori in $\mathbb{R}^4$ with $R^\perp \equiv 0$ apart from the previous two families.

The main result of the present work makes use of the above mentioned global description of flat surfaces with $R^\perp \equiv 0$ in $\mathbb{R}^4$ to construct infinitely many new flat tori in $\mathbb{R}^4$ with flat normal bundle and regular Gauss map, that do not lie in any of these two families. This produces the first general family of flat tori in $\mathbb{R}^4$ that comes out since the 19th century.

We have organized this paper as follows. In Section 2 we study the Gauss map of flat surfaces $f : \Sigma \to \mathbb{R}^4$ with $R^\perp \equiv 0$. For that purpose, we introduce the notion of flat map, which is essentially a map $F : \Sigma \to S^3$ such that at its regular points it is a flat surface in $S^3$. Through this concept it is shown that if the Gauss map $G$ of $f$ is regular, the surface $\Sigma$ inherits via $G$ a canonical Lorentz surface structure. Moreover, there exists an orthonormal basis $\{N, \vec{N}\}$ of $\mathcal{X}^\perp(f)$ such that both $N, \vec{N}$ are flat maps whose asymptotic directions at their regular points coincide with the null directions of the Lorentz surface $\Sigma$. This fact is used to endow $\Sigma$ with a global coordinate immersion that generalizes the well known existence of Tschebysheff coordinates on any complete flat surface in $S^3$. As applications, we show that global Tschebysheff coordinates exist on
any simply connected analytic flat surface in $S^3$, and that every analytic flat surface in $S^3$ is orientable, regardless of its completeness. In contrast, we produce $C^\infty$ flat Möbius strips in $S^3$ via the Hopf fibration.

In Section 3 we obtain a global representation for all simply connected flat surfaces in $\mathbb{R}^4$ with flat normal bundle and regular Gauss map, in terms of flat maps. This representation depends on the integration of a hyperbolic linear differential system, of which we present some partial integrations, by using both analytic and geometric methods. It is important to observe that, for the local situation, there already exists a representation formula due to do Carmo and Dajczer [CaDa]. In there, the authors describe (except for compositions) all isometric immersions of $U \subset \mathbb{R}^2$ into $\mathbb{R}^4$, with $U$ sufficiently small.

Finally, in Section 4 we construct a large family of new flat tori in $\mathbb{R}^4$, all of them with flat normal bundle and regular Gauss map. They are not contained in any affine 3-sphere, and cannot be expressed as the product of two curves. To do so, we first apply the representation formula of Section 3 to obtain a procedure of unfolding a Hopf torus in $S^3$, so that one gets a flat surface in $\mathbb{R}^4$ (possibly with singular points) with flat normal bundle that does not lie in any affine 3-sphere of $\mathbb{R}^4$. Then we show that for a certain family of Hopf tori this procedure generates flat tori in $\mathbb{R}^4$. At last, we collapse the flat torus we have just constructed into the Hopf torus we started with, and we get rid of singular points. This finishes the construction. We also show in this Section how this ideas can be used to construct new complete flat cylinders in $\mathbb{R}^4$.

While this paper was being prepared, the authors knew about an interesting paper by Weiner [Wei3], in where the author reports on his advances in the problem of describing all isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$. As a matter of fact, he mainly focuses on the $R^\perp \equiv 0$ case, which is the situation studied in the present work, but our approach has important differences with respect to Weiner’s one. Indeed, our description recovers the proper immersion $f : \Sigma \to \mathbb{R}^4$, and not just its differential $df$, and this is crucial in the construction of new flat tori in $\mathbb{R}^4$ that we achieve here.

2 Study of the Gauss map

All along this paper, and unless otherwise stated, $\Sigma$ will denote a two-dimensional simply connected smooth manifold endowed with a Riemannian flat metric $\langle , \rangle$. We shall consider isometric immersions $f : \Sigma \to \mathbb{R}^4$ with flat normal bundle, that is to say, isometric immersions whose normal curvature tensor vanishes identically, $R^\perp \equiv 0$. It is then well known that there exists a unit normal vector field $N \in \mathfrak{X}^\perp(f)$ globally defined on $\Sigma$ that is parallel in the normal bundle, i.e. $dN(X) \in \mathfrak{X}(f)$ for all $X \in \mathfrak{X}(f)$ (see for instance [Ten, Eno]). Any such unit normal vector field of $\mathfrak{X}^\perp(f)$ will be called a special section on $\Sigma$.

It is immediate that special sections come in pairs. In fact, if $N \in \mathfrak{X}^\perp(f)$ is a special section, its orthonormal complement $\hat{N}$ in $\mathfrak{X}^\perp(f)$ is also a special section. Furthermore, any other special section $N'$ on $\Sigma$ is given by $N' = \cos \theta N + \sin \theta \hat{N}$ for some constant angle $\theta$. 

3
On the other hand we shall denote by \( N^f_1 \) the first normal space of the immersion \( f \) (see [CaDa, DaTo]). As it was explained in [CaDa], the study of isometric immersions from \( \mathbb{R}^2 \) into \( \mathbb{R}^4 \) needs a constancy condition on the dimension of \( N^f_1 \). Since flat surfaces with \( \operatorname{dim}(N^f_1) = 1 \) were completely described in [DaTo2], we shall deal only with isometric immersions such that \( \operatorname{dim}(N^f_1) = 2 \). Rather than necessary, this condition is an easy way to prevent any piece of the surface from lying in a hyperplane of \( \mathbb{R}^4 \).

We plan to study flat surfaces with flat normal bundle in \( \mathbb{R}^4 \) by means of their second fundamental form \( \sigma \). For this, we will first of all show that if \( f : \Sigma \to \mathbb{R}^4 \) is such a surface, we can endow \( \Sigma \) with a natural Lorentz surface structure, induced by \( \sigma \) on \( \Sigma \). Then we shall denote by \( \{N, \hat{N}\} \) a Lorentz surface structure on \( \Sigma \) is the class of all Lorentzian metrics on \( \Sigma \) that are conformal to a specific Lorentzian metric on \( \Sigma \).

If \( f : \Sigma \to \mathbb{R}^4 \) is a flat surface with \( R^\perp \equiv 0 \), and we choose local coordinates \((x, y)\) on \( \Sigma \) so that \( \langle \cdot, \cdot \rangle = dx^2 + dy^2 \), and an orthonormal pair of special sections \( N, \hat{N} \) on \( \Sigma \), then the following structure equations hold.

\[
\begin{align*}
  f_{xx} &= E_1 N + E_2 \hat{N} \\
  f_{yy} &= G_1 N + G_2 \hat{N} \\
  f_{xy} &= F_1 N + F_2 \hat{N}
\end{align*}
\]

Here \( E_i, F_i, G_i \) are smooth functions satisfying the Gauss-Codazzi-Ricci equations, among which we quote the Gauss equation \( E_1 G_1 - F_1^2 = -(E_2 G_2 - F_2^2) \), that is, \( \det(dN) + \det(d\hat{N}) = 0 \).

Recall that the flat surface \( f \) has \( \operatorname{dim}(N^f_1) = 2 \). We claim that for any \( p \in \Sigma \) there exists a special section on \( \Sigma \) which is an immersion over a neighbourhood of \( p \). Indeed, if this is not true, it follows from (1) and the fact that every special section \( \xi \) is written as \( \xi = \cos \theta N + \sin \theta \hat{N} \) for some \( \theta \in \mathbb{R} \) that equations

\[
E_1 G_1 - F_1^2 = 0, \quad E_2 G_2 - F_2^2 = 0, \quad E_1 G_2 + E_2 G_1 - 2F_1 F_2 = 0
\]

must hold at \( p \).

This implies the existence of \( \lambda_i, \mu_i \in \mathbb{R}, i = 1, 2 \), so that \( \lambda_i(E_i, F_i) + \mu_i(F_i, G_i) = 0 \) and \( \lambda_i, \mu_i \) do not both vanish simultaneously. Besides, we have from \( \operatorname{dim}(N^f_1)(p) = 2 \) and (2) that \( E_i \) and \( G_i \) do not vanish at \( p \). In this way, by writing \( \nu_i = \mu_i/\lambda_i \in \mathbb{R} \), it is obtained

\[
E_i = \nu_i^2 G_i, \quad F_i = -\nu_i G_i, \quad i = 1, 2.
\]

Finally, this last expression together with the third equation in (2) give \( \nu_1 = \nu_2 \), a contradiction with \( \operatorname{dim}(N^f_1)(p) = 2 \). Summing up, we have ensured the existence of a special section that is an immersion in a neighbourhood of \( p \) for every \( p \in \Sigma \).

Now assume that \( N \) is regular at \( p \), and that \( \{\xi, \hat{\xi}\} \) is another pair of special sections on \( \Sigma \). Then there exists a smooth function \( \lambda \) on a neighbourhood of \( p \) verifying \( \langle d\xi, d\hat{\xi} \rangle = \lambda \langle dN, d\hat{N} \rangle \), that is, the Lorentzian pseudometric \( \langle d\xi, d\hat{\xi} \rangle \) belongs to the Lorentz surface structure induced by \( \langle dN, d\hat{N} \rangle \). To see this, we first note the existence of vectors \( v_1, v_2 \in \mathbb{R}^4 \)
\[ T_p \Sigma \text{ that verify} \]
\[
\begin{cases}
\langle dN_p(v_1), dN_p(v_1) \rangle = 1 \\
\langle dN_p(v_1), dN_p(v_2) \rangle = 0 \\
d\widehat{N}_p(v_i) = k_i dN_p(v_i)
\end{cases}
\]

where \( k_1, k_2 \in \mathbb{R} \). Since \( \xi = \cos \theta N + \sin \theta \widehat{N} \) for some \( \theta \), we get that \( \langle d\xi_p(v_1), d\xi_p(v_2) \rangle = 0 = \langle dN_p(v_1), d\widehat{N}_p(v_2) \rangle \), and that \( \langle d\xi, d\xi \rangle = \lambda \langle dN, d\widehat{N} \rangle \) holds at \( p \) if and only if the identity
\[
\sin \theta \cos \theta (k_1^2 k_2 - k_2) = \sin \theta \cos \theta (k_2^2 k_1 - k_1)
\]
holds. But this is true since the Gauss equation gives \( \det(dN) + \det(d\widehat{N}) = 0 \), i.e. \( 1 + k_1 k_2 = 0 \).

**Remark 1** Since at the regular points of \( N \), its unit normal in the 3-sphere is \( \widehat{N} \), the relation \( 1 + k_1 k_2 \) ensures that \( N \) is a flat surface in \( S^3 \). In other words, every special section of \( f \) defines, at its regular points, a flat surface in \( S^3 \). For any such surface, its second fundamental form \( \langle dN, d\widehat{N} \rangle \) is a Lorentzian metric, and it is classically known that the null directions of this metric point at the asymptotic directions of the surface.

To sum up, we have just proved the following result.

**Lemma 2** Let \( f : \Sigma \rightarrow \mathbb{R}^4 \) be a flat surface with flat normal bundle in \( \mathbb{R}^4 \). There exists a natural Lorentz surface structure on \( \Sigma \) such that at every point \( p \) its null directions are the asymptotic directions of any of the special sections of \( f \) that are regular at \( p \).

Motivated by the above situation, next we introduce a generalization of the flat surfaces in \( S^3 \), which we call flat maps, and for which we allow the presence of singular points. These flat maps will be our basic tool to construct flat surfaces in \( \mathbb{R}^4 \) with \( R^4 \equiv 0 \).

**Definition 3** Let \( \Sigma \) be a smooth simply connected surface. A map \( F : \Sigma \rightarrow S^3 \) on \( \Sigma \) is called a flat map if there exist \( \widehat{F} : \Sigma \rightarrow S^3 \), \( \omega : \Sigma \rightarrow \mathbb{R} \), and a canonical immersion \( \Sigma \rightarrow u,v\text{-plane} \) so that the following relations hold.

\[
\begin{align*}
\langle dF, dF \rangle &= du^2 + 2 \cos \omega \, dudv + dv^2, & \langle F, \widehat{F} \rangle &= 0, \\
\langle dF, d\widehat{F} \rangle &= 2 \sin \omega \, dudv, & \langle dF, \widehat{F} \rangle &= \langle F, d\widehat{F} \rangle = 0, \\
\langle d\widehat{F}, d\widehat{F} \rangle &= du^2 - 2 \cos \omega \, dudv + dv^2, & \omega_{uv} \equiv 0.
\end{align*}
\]

The immersion from \( \Sigma \) into the \( u,v \)-plane must be thought of as a coordinate immersion, but not as proper coordinates. This coordinate immersion is inspired in the well known existence of a global Tschebysheff net on every complete flat surface in \( S^3 \) (see [Spi]). We shall refer to \( \widehat{F} \) (resp. \( \omega \)) as the polar map (resp. the angle) of \( F \). It is
immediate that $\hat{F}$ is also a flat map with polar map $-F$ and angle $\hat{\omega} = \pi + \omega$. Moreover, if $F$ is an immersion, the angle $\omega(u, v)$ can be chosen so that $0 < \omega < \pi$, and $F$ is a flat surface in $S^3$ whose unit normal is $\hat{F}$.

At the end of the present Section we will show how to adapt the known procedures for constructing complete flat surfaces in $S^3$ to the context of flat maps.

Next we turn our attention to the Gauss map, and its relation to the Lorentz surface structure that we have just defined on $\Sigma$. Let $G_{2,4}$ denote the Grassmannian of oriented 2-planes in $\mathbb{R}^4$. The Gauss map of the immersion $f : \Sigma \to \mathbb{R}^4$ is defined as the map $G : \Sigma \to G_{2,4}$ assigning to each $p \in \Sigma$ the oriented tangent plane of $\Sigma$ at $p$. Thus we have a Riemannian pseudometric on $\Sigma$ induced by $G$, and given by

$$\langle \cdot, \cdot \rangle^* := \langle dG, dG \rangle_{G_{2,4}},$$

where $\langle \cdot, \cdot \rangle_{G_{2,4}}$ is the Riemannian metric of $G_{2,4}$. In this way $G : (\Sigma, \langle \cdot, \cdot \rangle^*) \to G_{2,4}$ becomes an isometric immersion at its regular points.

Let us describe the pseudometric $\langle \cdot, \cdot \rangle^*$. For this we first identify in the usual way $G_{2,4}$ with the complex quadric $Q_2$ of $\mathbb{CP}^3$ given in coordinates by $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ (see [HoOs]). Then $Q_2$ is endowed with a Riemannian metric, the so-called Fubini-Study metric, given by

$$ds^2 = 2 \sum_{j<k} |z_j dz_k - z_k dz_j|^2 \frac{\mathcal{T}(z, z) \mathcal{T}(dz, dz) - |\mathcal{T}(z, dz)|^2}{\mathcal{T}(z, z)^2},$$

where we are denoting by $\mathcal{T}$ the usual Hermitian product in $\mathbb{C}^4$, that is,

$$\mathcal{T}(z, w) = \sum_{k=1}^4 z_k \overline{w}_k.$$

If we choose local coordinates $(x, y)$ for the flat surface $f : \Sigma \to \mathbb{R}^4$ such that $\langle \cdot, \cdot \rangle = dx^2 + dy^2$, then

$$\langle \cdot, \cdot \rangle^* = \mathcal{T}(d(f_x + if_y), d(f_x + if_y)) = \langle d(f_x), d(f_x) \rangle + \langle d(f_y), d(f_y) \rangle.$$

From here and the structure equations (1) for $f$ we find that

$$\langle \cdot, \cdot \rangle^* = \langle dN, dN \rangle + \langle d\hat{N}, d\hat{N} \rangle$$

where $N, \hat{N}$ is an arbitrary orthonormal pair of special sections on $\Sigma$.

From now on, we shall assume that the Gauss map of the surface is regular, and thus that $\langle \cdot, \cdot \rangle^*$ is a Riemannian metric on $\Sigma$. This is not restrictive at all, since the following result holds, even in the case in which $R \perp \neq 0$ (see [Wei3]).

**Lemma 4** The Gauss map of $f$ is regular if and only if $\dim(N_1^f) \equiv 2$ on $\Sigma$. 
In this way, as it was shown above, the regularity of \( G \) implies that for every \( p \in \Sigma \) there exists a special section \( N \) on \( \Sigma \) that is an immersion over a neighbourhood of \( p \). Thus there is an open set \( U \subset \Sigma \) containing \( p \) so that \( N : U \to \mathbb{S}^3 \) is a flat surface, and there exist local coordinates \( s, t \) over \( U \) for which (see [Spi] for instance)

\[
\begin{align*}
\langle dN, dN \rangle &= ds^2 + 2 \cos \rho \, ds \, dt + dt^2 \\
\langle dN, d\hat{N} \rangle &= 2 \sin \rho \, ds \, dt \\
\langle d\hat{N}, d\hat{N} \rangle &= ds^2 - 2 \cos \rho \, ds \, dt + dt^2
\end{align*}
\]

hold on \( U \). Here \( \rho : U \to \mathbb{R} \) satisfies \( 0 < \rho < \pi \) and \( \rho_{st} = 0 \), and \( \hat{N} \) is the orthogonal complement of the special section \( N \). With this, it follows directly from (4) that

\[
\langle \cdot, \cdot \rangle^* = 2(ds^2 + dt^2)
\]

on \( U \). Particularly, \( G : \Sigma \to G_{2,4} \) is a flat immersion. In addition, since every orthonormal pair \( (\xi, \hat{\xi}) \) of special sections on \( \Sigma \) is given by \( (\xi, \hat{\xi}) = R_\theta(N, \hat{N}) \), \( R_\theta \) being a rotation of angle \( \theta \), then every such pair satisfies

\[
\langle \xi_s, \hat{\xi}_s \rangle = \langle \xi_t, \hat{\xi}_t \rangle = 0
\]

on \( U \). In other words, \( (\partial_s, \partial_t) \) point at the null directions of the Lorentz surface \( \Sigma \). It becomes clear then that properties [5] and [6] characterize the parameters \( s, t \) on \( U \). In this way, if \( s', t' \) constitute a local coordinate system satisfying both conditions on an open set \( U' \) in \( \Sigma \) such that \( U \cap U' \neq \emptyset \), then \( (\partial_{s'}, \partial_{t'}) = (\partial_{s'}, \partial_{t'}) \) over \( U \cap U' \). From here we are led to the main conclusion of this Section, that is stated in terms of the notion of flat map, which we introduced in Definition 3.

**Proposition 5** Let \( f : \Sigma \to \mathbb{R}^4 \) be a simply connected flat surface in \( \mathbb{R}^4 \) with flat normal bundle and regular Gauss map \( G \). There exists an immersion from \( \Sigma \) into the \( u, v \)-plane with respect to which every special section \( N \) on \( \Sigma \) is a flat map whose polar map \( \hat{N} \) is its orthonormal complement in \( X^\perp(f) \).

**Proof.** Since \( G \) is a flat immersion, \( (\Sigma, \langle \cdot, \cdot \rangle^*) \) is a Riemannian flat surface. Hence we can take a \( C^\infty \) global conformal parameter \( z : \Sigma \to \mathbb{C} \) such that \( \langle \cdot, \cdot \rangle^* = e^{2\phi}|dz|^2 \) for some smooth function \( \phi : \Sigma \to \mathbb{R} \). The flatness of \( \langle \cdot, \cdot \rangle^* \) implies that \( \phi_{zz} = 0 \), i.e. \( \phi \) is harmonic. Therefore there exists a holomorphic map \( \phi : \Sigma \to \mathbb{C} \) such that \( \phi = \text{Re}(\phi) \). Thus, if we take the immersion from \( \Sigma \) into the \( u, v \)-plane given by

\[
\zeta = u + iv = \frac{\sqrt{2}}{2} \int e^\phi \, dz
\]

it follows that

\[
\langle \cdot, \cdot \rangle^* = |e^\phi \, d\zeta|^2 = 2|d\zeta|^2 = 2 \left( dv^2 + dv^2 \right).
\]

Note that this last relation remains unchanged if we choose \( e^{i\varphi} \zeta \) instead of \( \zeta \), where \( \varphi \in \mathbb{R} \), and that this change rotates \( (\partial_u, \partial_v) \) by an angle \( \varphi \). Hence we can assume that
at a point \( p \in \Sigma \), the above vectors point at the null directions of the Lorentz surface \( \Sigma \). But in that case, it follows from the existence and uniqueness of the local parameters \((s, t)\) defined above that the coordinate immersion \((u, v)\) satisfies properties (5) and (6) globally on \( \Sigma \).

In other words, we obtain an immersion of \( \Sigma \) into \( \mathbb{R}^2 \) such that

\[
\langle , \rangle^* = 2(du^2 + dv^2)
\]

\[
\langle N_u, \widehat{N}_u \rangle = \langle N_v, \widehat{N}_v \rangle = 0
\]

hold on \( \Sigma \), where here \((u, v)\) are canonical coordinates in \( \mathbb{R}^2 \) and \( N, \widehat{N} \) is an arbitrary orthonormal pair of special sections on \( \Sigma \). This is the canonical immersion appearing in the definition of flat map.

In this manner we have

\[
\langle N_u, N_u \rangle + \langle \widehat{N}_u, \widehat{N}_u \rangle = 2 \quad \text{and} \quad 0 = \langle \frac{(N + \widehat{N})_u}{\sqrt{2}}, \frac{(N - \widehat{N})_u}{\sqrt{2}} \rangle = \frac{1}{2} \left( \langle N_u, N_u \rangle - \langle \widehat{N}_u, \widehat{N}_u \rangle \right).
\]

So, \( \langle N_u, N_u \rangle = \langle \widehat{N}_u, \widehat{N}_u \rangle = 1 \) and in the same way \( \langle N_v, N_v \rangle = \langle \widehat{N}_v, \widehat{N}_v \rangle = 1 \). Besides, from (3) we get \( \langle N_u, N_v \rangle = -\langle \widehat{N}_u, \widehat{N}_v \rangle \).

Summing up all these equations it is elementary to show the existence of a smooth map \( \omega : \Sigma \to \mathbb{R} \) such that all conditions in (4) are satisfied. We remark that equation \( \omega_{uv} \equiv 0 \) holds on the set \( \mathcal{R} \) of regular points of \( N \), since in there \( N \) is a flat surface in \( S^3 \), and it also holds in the exterior of \( \mathcal{R} \), in where \( \omega \) must be constant. Thus, by continuity, we have \( \omega_{uv} \equiv 0 \) in all \( \Sigma \).

Proposition 5 shows as a particular case that one can define a global Tschebysheff coordinate immersion on every simply connected flat surface in \( S^3 \), regardless of its completeness. In other words, we have proved the following

**Proposition 6** Every simply connected flat surface in \( S^3 \) (complete or not) is a flat map with \( 0 < \omega < \pi \).

To end up this Section, we describe two methods for constructing flat maps, inspired in the known constructions of complete flat surfaces in \( S^3 \) given by Bianchi-Spivak and Kitagawa, respectively. These classical constructions can be found in [Spi] [Kit1] [Kit2] [Wei2] among others, and they carry over to the context of flat maps almost unchanged, except for two details. One is that on a flat map the parameters \( u, v \) are not proper coordinates, what produces some technical difficulties. The other one is that on a flat map no regularity assumption is required, and this simplifies the description of the construction method. Hence, we will focus on these two differences, and just give a sketch of the rest of the reasoning.

Let us begin by modifying the most classical construction, due to Bianchi and put afterwards into a modern form by Spivak and Sasaki. Let us identify \( \mathbb{R}^4 \) with the quaternions, so that \( S^2 \) is the set of unit pure quaternions, and \( S^3 \) is that of unit quaternions.
Let \( a_1(u), a_2(v) \) be two regular curves in \( S^3 \) parametrized by arclength, assume that \( a_1(0) = a_2(0) = 1 \), and choose \( \xi_0 \in S^3 \) orthogonal to both \( a'_1(0), a'_2(0) \), and lying in \( T_{a_1(0)}S^3 \). Then define \( \xi_1(u) = a_1(u) \cdot \xi_0 \) and \( \xi_2(v) = \xi_0 \cdot a_2(v) \), the dot denoting the product in \( S^3 \). Besides, consider the pair \( (\Phi, \widetilde{\Phi}) \) given by

\[
\Phi(u, v) = a_1(u) \cdot a_2(v), \\
\widetilde{\Phi}(u, v) = a_1(u) \cdot \xi_0 \cdot a_2(v),
\]

(7)
defined over a rectangle \( R \) of the \( u, v \)-plane.

Finally consider an immersion \( \Psi : \Sigma \to R \), where \( \Sigma \) is a simply connected surface, so that \( \Psi(\Sigma) = R \), and define the pair of maps \( (F, \hat{F}) \) from \( \Sigma \) into \( S^3 \) given by

\[
\left( F, \hat{F} \right) = \left( \Phi \circ \Psi, \widetilde{\Phi} \circ \Psi \right).
\]

(8)

Following in this general context [Kit1, Lemma 4.1] we obtain

**Lemma 7** If \( \langle a'_i, \xi_i \rangle \equiv 0, i = 1, 2 \), then \( F \) is a flat map with polar map \( \hat{F} \).

Note that if we define the equivalence relation \( \sim \) over \( \Sigma \) so that \( p \sim q \) if and only if \( \Psi(p) = \Psi(q) \), then it follows from \( S \) that \( F, \hat{F} \) are well defined over \( \Sigma^* = \Sigma/\sim \). In addition, \( \Psi : \Sigma^* \to R \) is well defined and one-to-one, what shows that \( u, v \) are proper coordinates on \( \Sigma^* \). Also note that \( \Psi(\Sigma^*) = R \).

Next, we shall show that this construction can be reversed for any simply connected analytic flat map, but not in general for smooth flat maps. To do so, we begin by considering a particular situation, in which \( F \) is a flat map with polar map \( \hat{F} \) and angle \( \omega(u, v) \), defined on a simply connected surface \( \mathcal{U} \), and endowed with the canonical immersion \( \Psi : \mathcal{U} \to \mathbb{R}^2 \), so that

1) \( \Psi \) is one-to-one, and thus \( u, v \) are proper coordinates on \( \mathcal{U} \), and
2) \( \Psi(\mathcal{U}) \) is a rectangle \( R \) in the \( u, v \)-plane.

Then we can identify \( \mathcal{U} \) with \( R \), and assume that \( (0, 0) \in R \). We shall also suppose without losing generality that \( F(0, 0) = 1 \) and \( \hat{F}(0, 0) = \xi_0 \in S^3 \). Now define two regular curves in \( S^3 \) as \( a_1(u) = F(u, 0) \) and \( a_2(v) = F(0, v) \). Let us compare, for the parameter \( v \), the curves in \( S^3 \) given by \( \Gamma_1(v) = F(u_0, v) \) and \( \Gamma_2(v) = a_1(u_0) \cdot a_2(v) \).

Consider the frame in \( S^3 \) along \( \Gamma_1 \) given by

\[
\{ T(v) = \Gamma'_1(v), N(v) = \hat{F}_v(u_0, v), B(v) = \hat{F}(u_0, v) \}
\]

and the corresponding frame along \( \Gamma_2 \) defined by

\[
\{ t(v) = a_1(u_0) \cdot a'_2(v), n(v) = a_1(u_0) \cdot \xi_0 \cdot a'_2(v), b(v) = a_1(u_0) \cdot \xi_0 \cdot a_2(v) \}.
\]

Since \( (F, \hat{F}) \) satisfy equations \( S \), we obtain that \( \nabla_{F_u} \hat{F} = F_u \times \hat{F} \), where \( \nabla \) and \( \times \) stand for the Riemannian connection and the cross-product in \( S^3 \), respectively. Thus \( \hat{F} \) is left invariant along \( a_1(u) \), and it can be shown in the same way that it is right invariant along \( a_2(v) \). From there we obtain that both frames coincide at 0. Moreover,
one can check that both frames are the Frenet trihedron of their respective curves. Thus, by deriving we find that the two curves have the same curvature and torsion in $S^3$. Since they have the same initial conditions, we conclude that $\Gamma_1 = \Gamma_2$. In other words, the flat map is recovered by means of equations (7) and (8).

Now assume that $F$ is an analytic flat map, defined on $\Sigma$. Then its polar map $\hat{F}$ as well as the canonical immersion $\Psi : \Sigma \to u,v$-plane are both analytic. Note that there exists $U \subset \Sigma$ such that $\Psi|_U$ is one-to-one, and $\Psi(U)$ is a rectangle. Then, we have just proved above that (8) holds on $U$. Next, extend the analytic curves $a(u), b(v)$ as much as possible, so that $\Phi, \hat{\Phi}$ are defined on a rectangle $R'$, and they cannot be extended any further. Since all maps are analytic, we find that (8) holds not just in $U$, but in the largest $W \subset \Sigma$ in which (8) is well defined. But it follows from the continuity of $F$ and the fact that $\Phi$ does not admit a proper continuous extension away from $R'$ that $\Psi(\Sigma) \subset R'$, and this indicates that $W = \Sigma$. Thus, (8) holds on $\Sigma$ globally. In particular $(F, \hat{F})$ are well defined over $\Sigma^* = \Sigma/\sim$, and $u,v$ are proper coordinates. Note that enlarging $\Sigma^*$ if necessary, the coordinates $u,v$ can be chosen to be defined over all $R'$, so that $\Psi(\Sigma^*) = R'$. This completes the first construction of flat maps.

**Remark 8** There exist simply connected smooth flat maps that cannot be globally recovered through equations (7) and (8), even if $\Psi$ is one-to-one. To show this, we begin by choosing three curves $a_1, a_2, \tilde{a}_2 : (-1, 1) \to S^3$ parametrized by arclength, so that

1. $a_1$ has torsion $\tau = 1$, and $a_2, \tilde{a}_2$ have $\tau = -1$.
2. $a_1(0) = a_2(0) = \tilde{a}_2(0) = 1$
3. $a_2(t) = \tilde{a}_2(t)$ for all $t \in (-1, 0]$, but $a_2(t) \neq \tilde{a}_2(t)$ for $t \in (0, 1)$.

Then define $\Sigma_0 = (-1, 1) \times (-1, 1) \times \{0\} \subset \mathbb{R}^3$, and $\Sigma_1 = (-1, 1) \times (-1, 1) \times \{1\} \subset \mathbb{R}^3$, and consider the map $\Phi : \Sigma_0 \cup \Sigma_1 \to S^3$ given by

$$
\Phi(u,v,0) = a_1(u) \cdot a_2(v)
\Phi(u,v,1) = a_1(u) \cdot \tilde{a}_2(v).
$$

(9)

It is clear that $\Phi$ is a non-connected flat map. Now define on $\Sigma_0 \cup \Sigma_1$ the equivalence relation that identifies $(u,v,0) \simeq (u,v,1)$ for every $(u,v) \in (-1, 1) \times (-1, 0)$, and consider the sets

$$
R_0 = \left(((-1,1) \times (-1,0) \times \{0\}) \cup \left((-1,-1/2) \times (-1,1) \times \{0\}\right)\right) \subset \Sigma_0,
R_1 = \left(((-1,1) \times (-1,0) \times \{1\}) \cup \left((1/2,1) \times (-1,1) \times \{1\}\right)\right) \subset \Sigma_1.
$$

It then follows that $\Sigma := (R_0 \cup R_1)/\sim$ is a simply connected smooth surface, with proper coordinates $u,v$. In addition $\Phi$ induces a well defined flat map $F$ on $\Sigma$ in the obvious way, for which $u,v$ are Tschebycheff coordinates. However, $F$ cannot be expressed by means of (7) and (8), since it has two non-congruent coordinate $v$-curves, namely, $a_2$ and $\tilde{a}_2$. Note that this process works for smooth flat maps, but not for analytic flat maps, since in this case it is impossible to reach the condition $a_2(t) \neq \tilde{a}_2(t)$. 

10
Another difference between analytic flat maps and smooth flat maps is presented in the next result.

**Corollary 9** Every simply connected analytic flat surface in $S^3$ has globally defined proper Tschebyshev coordinates $(u,v)$, and these can be extended to be defined over a rectangle in the $u,v$-plane. In contrast, there exist smooth simply connected flat surfaces in $S^3$ that cannot be endowed with global proper Tschebyshev coordinates.

**Proof:** It only remains to show the assertion about the smooth case, i.e. we have to construct a smooth flat map $F : \Sigma \to S^3$ endowed with an immersion $\Psi : \Sigma \to u,v$-plane such that $F$ is not well defined over $\Sigma^* = \Sigma/\sim$, where $p \sim q$ if and only if $\Psi(p) = \Psi(q)$.

For that purpose we shall add to the sets $R_0, R_1$ constructed in the above Remark a new set on $\Sigma_0$, 

$$\tilde{R}_0 = R_0 \cup \{(−1,1) \times (1/2,1) \times \{0\}\} \subset \Sigma_0.$$  

Now let $\Sigma := \tilde{R}_0 \cup R_1 / \sim$. Then $\Sigma$ is a simply connected surface, and we can define the map $\Psi : \Sigma \to u,v$-plane that takes each point $p$ of $\Sigma$ into its first two coordinates, when $p$ is regarded as a point of $\Sigma_0 \cup \Sigma_1$. This map is an immersion, which is not one-to-one, and whose image is not simply connected in the $u,v$-plane. Now consider curves $a_1, a_2, \tilde{a}_2 : (−1,1) \to S^3$ as in Remark 8, and define $\Phi(u,v) : \Psi(\Sigma) \to S^3$ as in (9). This map is doubled valued exactly on the set of double points of the immersion $\Psi$. Thus, $\Phi(u,v)$ may be regarded as a single valued map $F : \Sigma \to S^3$, which is obviously a flat map. Moreover, $F$ is not well defined over $\Sigma^*$. This completes the proof.

As a consequence of this Corollary, it is not difficult to modify the proof of [Kit1, Theorem 2.3] to obtain

**Corollary 10** Every analytic flat surface in $S^3$ (complete or not) is orientable.

This result was proved for complete $C^\infty$ flat surfaces in $S^3$ by Kitagawa in [Kit1]. It contrasts with the situations of flat surfaces in $\mathbb{R}^3$ and in $\mathbb{R}\mathbb{P}^3$, where analytic flat Möbius strips are known to exist (see [ChKa, Wun]). As a consequence, we obtain that all analytic flat Möbius strips in $\mathbb{R}\mathbb{P}^3$ come from analytic flat cylinders in $S^3$ that are invariant under the antipodal map.

It is interesting to remark that the analyticity condition in Corollary 10 cannot be weakened to smoothness. Indeed, there are examples of $C^\infty$ embedded flat Möbius strips in $S^3$, as we show next.

Let us define $\text{Ad}(x)y = x \cdot y \cdot \bar{x}$, where $x, y \in S^3$, and recall the Hopf fibration $h : S^3 \to S^2$ given by $h(x) = \text{Ad}(x)i$. Here the bar denotes conjugation in the quaternions.

It was observed by H.B. Lawson that if $c$ is a regular curve in $S^2$, then $h^{-1}(c)$ is a flat surface in $S^3$. It has in general the topology of a cylinder, but if $c$ is closed then $h^{-1}(c)$ is actually a torus. Following [Pin], we shall call in general $h^{-1}(c)$ a Hopf cylinder, and also a Hopf torus in case $c$ is closed.
Figure 1: A $C^\infty$ flat Möbius strip embedded in $S^3$, described as an open set of a Hopf cylinder.

With all of this, Figure 1 shows an example of a $C^\infty$ Möbius strip lying on a Hopf cylinder in $S^3$. For that, we choose a regular smooth curve $c$ in $S^2$ with the shape described in the figure (a curve with this shape obviously cannot be real analytic). Then the shadowed region in the Hopf cylinder $h^{-1}(c)$ clearly has the topology of a Möbius strip, and it is trivially flat. This construction is inspired in the example of a flat Möbius strip in $\mathbb{R}^3$ appearing in [ChKa].

Next, we turn to the second construction procedure of flat maps, which is based on Kitagawa’s work [Kit1]. To explain it we shall use the notation in [Wei2].

Let $US^2$ denote the bundle of unit tangent vectors to $S^2$. Then we can define the double cover $\pi: S^3 \to US^2$ given by

$$\pi(x) = (\text{Ad}(x)i, \text{Ad}(x)\xi_0),$$

where $\xi_0 \in S^3$ is orthogonal to $1$ and $i$. Note that if $a(u)$ is a curve in $S^3$, and $c(u)$ denotes its Hopf projection in $S^2$, it holds that

$$c' = \text{Ad}(a)[\bar{a}a', i]. \quad (10)$$

Let $c_1(u), c_2(v)$ be two regular curves in $S^2$ with $c_i(0) = i$, $c'_i(0) = \xi_0$, and consider $\tilde{c}_i = (c_i, c'_i/||c'_i||)$, with values in $US^2$. Then there exist two regular curves $a_1(u), a_2(v)$ in $S^3$ verifying $\pi(a_i) = \tilde{c}_i$. That is, $c_i$ is the Hopf projection of $a_i$, and so equation (10) is verified, and besides $c'_i$ is collinear with $\text{Ad}(a_i)\xi_0$. These two conditions imply that

$$\langle a'_1, a_1 \cdot \xi_0 \rangle \equiv 0 \equiv \langle (\bar{a}_2)'', \xi_0 \cdot \bar{a}_2 \rangle.$$

At this point, the first procedure to construct flat maps ensures that the map $F$ defined through equations (11) (writing $\bar{a}_2$ instead of $a_2$) and (12) is a flat map, provided that we parametrize first $a_i$ by arclength.
Conversely, suppose that we are given a flat map $F$ constructed by means of (11) and (12), and define $\tilde{c}_1(u), \tilde{c}_2(v)$ two regular curves in $US^2$ as $\pi(a_1) = \tilde{c}_1$, $\pi(a_2) = \tilde{c}_2$. Since $\langle a'_1, a_1 \cdot \xi_0 \rangle \equiv 0 \equiv \langle a'_2, \xi_0 \cdot a_2 \rangle$, the above computations ensure that if $c_1$ (resp. $c_2$) is the Hopf projection of $a_1$ (resp. $a_2$), then $c'_1$ (resp. $c'_2$) is collinear with $\text{Ad}(a_1)\xi_0$ (resp. $\text{Ad}(a_2)\xi_0$). In other words, $\tilde{c}_i = (c_i, c'_i/||c'_i||)$, what show that the above process can be reversed. Summarizing, we have obtained the following generalization of Kitagawa’s construction.

**Theorem 11** Let $c_1(u), c_2(v)$ be two regular curves in $S^2$, with $c_i(0) = i$, $c'_i(0) = \xi_0$, for some $\xi_0 \in S^3$ orthogonal to both $1, i$. Let $\pi : S^3 \to US^2$ be the double cover given by

$$\pi(x) = (\text{Ad}(x)i, \text{Ad}(x)\xi_0),$$

(11)

consider $a_1(u), a_2(v)$ two curves in $S^3$ parametrized by arclength and satisfying $\pi(a_i) = (c_i, c'_i/||c'_i||)$, and define

$$\Phi(u,v) = a_1(u) \cdot a_2(v)$$

$$\widehat{\Phi}(u,v) = a_1(u) \cdot \xi_0 \cdot a_2(v).$$

(12)

on a rectangle $R$ in the $u,v$-plane. If $\Sigma$ is a simply connected surface and $\Psi : \Sigma \to \Psi(\Sigma) = R$ is an immersion, then the pair $(F, \widehat{\Phi})$ given by

$$\left( F, \widehat{\Phi} \right) = \left( \Phi \circ \Psi, \widehat{\Phi} \circ \Psi \right).$$

constitutes a flat map $F$ with polar map $\widehat{\Phi}$. Conversely, every analytic flat map is constructed in this way for some $\xi_0$.

**Remark 12** In the known constructions of flat surfaces in $S^3$ that we have just generalized, one has to assume further conditions to forbid the appearance of singular points. It is not difficult to check that if one assumes those additional conditions in the above descriptions of flat maps, it is obtained a flat surface in $S^3$. It is remarkable that this general global process works for any analytic simply connected flat surface in $S^3$, and not just locally or under a completeness assumption.

In addition, arguing as in Remark 8 we can produce $C^\infty$ simply connected flat surfaces in $S^3$ that cannot be globally obtained through any of these two methods. Indeed, these examples will possess three mutually non-congruent asymptotic curves.

### 3 Construction of flat surfaces

Let $f : \Sigma \to \mathbb{R}^4$ be an isometric immersion of a simply connected flat surface $\Sigma$ into $\mathbb{R}^4$ with flat normal bundle and regular Gauss map $\mathcal{G}$. Let $N : \Sigma \to S^3$ be a special section on $\Sigma$. Then, from Lemma 5 we find that $N$ is a flat map. Besides, if we consider the immersion of $\Sigma$ into the $u,v$-plane constructed in that Lemma, it is obtained that \{${N, \bar{N}, N_u, \bar{N}_u}$\} is an orthonormal frame of $\mathbb{R}^4$. Here $\bar{N}$ stands for the polar map of
$N$, which is also a special section on $\Sigma$. From here the isometric immersion $f$ can be expressed as

$$f = \alpha N + \beta \hat{N} + \alpha_u N_u + \beta_u \hat{N}_u,$$

where $\alpha = \langle f, N \rangle$ and $\beta = \langle f, \hat{N} \rangle$. Noting that

$$\begin{pmatrix} N_v \\ \hat{N}_v \end{pmatrix} = \begin{pmatrix} \cos \omega & \sin \omega \\ \sin \omega & -\cos \omega \end{pmatrix} \begin{pmatrix} N_u \\ \hat{N}_u \end{pmatrix},$$

we obtain that $\alpha, \beta$ satisfy the differential system

$$\begin{pmatrix} \alpha_u \\ \beta_u \end{pmatrix} = \begin{pmatrix} \cos \omega & \sin \omega \\ \sin \omega & -\cos \omega \end{pmatrix} \begin{pmatrix} \alpha_u \\ \beta_u \end{pmatrix}.$$  

This process can be reversed. More concretely, we get the following result, which describes all flat surfaces in $\mathbb{R}^4$ with flat normal bundle and regular Gauss map in terms of flat maps.

**Theorem 13** Let $f : \Sigma \to \mathbb{R}^4$ be an isometric immersion of a simply connected flat surface $\Sigma$ into $\mathbb{R}^4$ with flat normal bundle and regular Gauss map. There exists an immersion of $\Sigma$ into the $u,v$-plane with respect to which every special section $N$ on $\Sigma$ is a flat map, and if $\hat{N}$ is the polar map of any such $N$ the immersion $f$ is expressed as

$$f = \alpha N + \beta \hat{N} + \alpha_u N_u + \beta_u \hat{N}_u.$$  

Besides, if $\omega$ denotes the angle of the flat map $N$, the functions $\alpha, \beta : \Sigma \to \mathbb{R}$ are solution of the differential system

$$\begin{pmatrix} \alpha_u \\ \beta_u \end{pmatrix} = \begin{pmatrix} \cos \omega & \sin \omega \\ \sin \omega & -\cos \omega \end{pmatrix} \begin{pmatrix} \alpha_u \\ \beta_u \end{pmatrix}.$$  

Conversely, let $N : \Sigma \to \mathbb{S}^3$ be a flat map with angle $\omega$ and polar map $\hat{N}$, and let $\alpha, \beta : \Sigma \to \mathbb{R}$ be a solution of the differential system (15). Then the mapping $f : \Sigma \to \mathbb{R}^4$ given by (14) constitutes at its regular points a flat surface in $\mathbb{R}^4$ with flat normal bundle and regular Gauss map, for which $N, \hat{N}$ are orthonormal special sections.

**Remark 14** By considering its universal covering if necessary, every flat surface in $\mathbb{R}^4$ with $R^\perp \equiv 0$ and regular Gauss map is globally described via Theorem 13. Conversely, a flat surface $f : \Sigma \to \mathbb{R}^4$ produced by (14) may actually be well defined on some quotient of $\Sigma$ with non-trivial topology. We shall follow this last strategy in our construction of flat tori and flat cylinders in the next Section.

Let us examine more closely this representation formula. To begin with, it is convenient to characterize the singular points of the flat surface $f$. Actually, the geometric meaning of the system (15) will provide a geometric interpretation of the appearance of singular points in $f$, as we show next.
First of all, note that (15) gives
\[
\begin{align*}
\alpha_{uv} &= \cos \omega (\alpha_{uu} + \omega_u \beta_u) + \sin \omega (\beta_{uu} - \omega_u \alpha_u), \\
\beta_{uv} &= \cos \omega (-\beta_{uu} + \omega_u \alpha_u) + \sin \omega (\alpha_{uu} + \omega_u \beta_u).
\end{align*}
\]
If we denote
\[
A = \alpha + \alpha_{uu} + \omega_u \beta_u, \quad B = \beta + \beta_{uu} - \omega_u \alpha_u,
\]
and
\[
\begin{pmatrix}
\hat{A} \\
\hat{B}
\end{pmatrix} = \begin{pmatrix}
\cos \omega & \sin \omega \\
\sin \omega & -\cos \omega
\end{pmatrix} \begin{pmatrix}
A \\
B
\end{pmatrix},
\]
then we obtain
\[
\begin{pmatrix}
(f_u) \\
(f_v)
\end{pmatrix} = \begin{pmatrix}
A & B \\
\hat{A} & \hat{B}
\end{pmatrix} \begin{pmatrix}
N_u \\
\hat{N}_u
\end{pmatrix}.
\]
That is, if \(x_1, x_2\) are canonical coordinates in the \(\{N_u, \hat{N}_u\}\)-plane at some point \(p \in \Sigma\), then the vectors \(f_u, f_v\) are symmetric with respect to the axis \(\sin(\omega/2)x_1 = \cos(\omega/2)x_2\). Thus the surface \(f : \Sigma \to \mathbb{R}^4\) has a singular point at \(p\) if and only if \(f_u(p)\) lies on the above axis or is perpendicular to it. That is, \(f\) is regular if and only if the relation
\[
(A^2 - B^2) \sin \omega - 2AB \cos \omega \neq 0
\]
holds at every point. Moreover, the metric of \(f\) in the coordinates \((u, v)\) is written as
\[
\langle df, df \rangle = (A^2 + B^2)(du^2 + dv^2) + 2 ((A^2 - B^2) \cos \omega + 2AB \sin \omega) \ dudv.
\]

**Remark 15** If \(\alpha, \beta\) are solutions of (15), then the functions \(A, B\) appearing in (16) are also solutions of (15). This follows from the condition \((f_u)_v = (f_v)_u\) in (17), taking into account that the second derivatives of \(N\) and \(\hat{N}\) can be expressed in terms of \(\{N, \hat{N}, N_u, \hat{N}_u\}\) by differentiation of the system (13).

In the remaining of this Section we shall discuss how to construct solutions of the differential system (15).

In the particular case in which the angle \(\omega\) is constant, the system can be completely integrated, and \((\alpha, \beta)\) are of the form \((\alpha, \beta)(u, v) = (\alpha_1, \beta_1)(u + v) + (\alpha_2, \beta_2)(u - v)\) for arbitrary real functions \(\alpha_i, \beta_i\). Geometrically, the flat surfaces in \(\mathbb{R}^4\) for which \(\omega\) is constant are precisely the products of curves in \(\mathbb{R}^4\), that is, the products \(\gamma_1 \times \gamma_2\) of two regular plane curves in \(\mathbb{R}^4\) lying in orthogonal planes. Those are the flat surfaces whose Gauss image in \(G_{2,4} \equiv S^2 \times S^2\) lies in the product of two great circles.

On the other hand, the system (15) admits constant solutions. If \(\alpha, \beta\) are constant, then \(f(\Sigma)\) lies in a 3-sphere in \(\mathbb{R}^4\) centred at the origin. The situation in which the image of \(f\) lies in an affine 3-sphere not centred at \(0\) is more interesting from the viewpoint of
the system (15), and will be crucial in the construction of new flat tori in $\mathbb{R}^4$ that we will accomplish in the next Section. If $f$ is a flat surface in a 3-sphere of radius $\rho$ and centred at $a \in \mathbb{R}^4$, then there is some special normal section $N$ of $f$ such that $f = a + \rho N$. If $\hat{N}$ denotes the polar map of the flat map $N$, it follows that $\alpha = \langle a, N \rangle + \rho$, and $\beta = \langle a, \hat{N} \rangle$. Thus, we conclude from the linearity of (15) the following.

**Proposition 16** Let $N$ be a flat map with angle $\omega$, and denote its polar map by $\hat{N}$. Then the coordinates $(N_i, \hat{N}_i)$, $1 \leq i \leq 4$, are solutions of the differential system (15).

These solutions, as well as their linear combinations, will be referred to as geometric solutions of the system (15). The above comments ensure that a flat surface in $\mathbb{R}^4$ given through equation (14) lies in a 3-sphere of $\mathbb{R}^4$ if and only if $\alpha, \beta$ are geometric solutions of (15) (or constants).

It is important to observe that in general this geometric solutions cannot be obtained explicitly in analytic terms. For instance, the coordinates of any Hopf torus $h^{-1}(c)$ in $\mathbb{S}^3$ have an explicit formula in terms of the curve $c$ in $\mathbb{S}^2$ that generates it. However, it is not generally possible to derive explicitly the expression of these coordinates with respect to the parameters $(u, v)$.

Apart from these geometric integrations, the system (15) also admits some particular analytic integrations, as we show next.

First, let us fix some matrix notation to treat (15). For that, we define $X = (\alpha, \beta)^T$, and $M$ the matrix such that (15) is $X_v = MX_u$. Also note that, since $\omega_{uv} \equiv 0$, the angle $\omega$ can be written as $\omega(u, v) = \omega_1(u) + \omega_2(v)$. Thus $M(u, v) = L(u)H(v)$, where

$$L(u) = \begin{pmatrix} \cos \omega_1 & \sin \omega_1 \\ \sin \omega_1 & -\cos \omega_1 \end{pmatrix}, \quad H(v) = \begin{pmatrix} \cos \omega_2 & \sin \omega_2 \\ -\sin \omega_2 & \cos \omega_2 \end{pmatrix}.$$

We remark that if $X$ is a solution of (15), then $Z = MX_{uv}$ is another solution of (15).

This fact can be reversed to obtain a more interesting situation. Let us start with a solution $X$ of (15), so that $X$ verifies $X_v = L(u)H(v)X_u$. Then $(LX)_v = (HX)_u$, what ensures the existence of a vector $Y$ such that $Y_u = LX$ and $Y_v = HY$. Thus $(H^{-1}Y)_u = (LY)_v$, and there exists some $Z$ verifying $Z_u = LY$ and $Z_v = H^{-1}Y$. But this finally implies that $Z_v = MZ_u$, that is, $Z$ is again a solution of (15). To sum up, the vector

$$Z(u, v) = \int LXdu + \int H^{-1}Ydv,$$

where $Y = \int LXdu + \int HYdv$ and $X$ is an arbitrary solution of (15), is again a solution of (15).

For instance, if we let $X = (0, 0)$, which is a trivial solution of (15), and $v_0$ is a fixed vector of $\mathbb{R}^2$, then

$$Z(u, v) = \left( \int \begin{pmatrix} \cos \omega_1 & \sin \omega_1 \\ \sin \omega_1 & -\cos \omega_1 \end{pmatrix} du + \int \begin{pmatrix} \cos \omega_2 & \sin \omega_2 \\ -\sin \omega_2 & \cos \omega_2 \end{pmatrix} dv \right) v_0.$$
is a new non-constant solution of (15). In particular, choices like
\[
\begin{align*}
\alpha_1 &= \int \cos \omega_1 du + \int \cos \omega_2 dv, \\
\beta_1 &= \int \sin \omega_1 du + \int \sin \omega_2 dv,
\end{align*}
\[
\begin{align*}
\alpha_2 &= \int \sin \omega_1 du - \int \sin \omega_2 dv, \\
\beta_2 &= -\int \cos \omega_1 du + \int \cos \omega_2 dv,
\end{align*}
\]
constitute solutions of (15). These solutions have the particular feature that \((\alpha_i, \beta_i) = (A_i, B_i)\), where \(A_i, B_i\) are defined as in Remark 15 for \(i = 1, 2\).

Next, we shall consider a special case in which all solutions of (15) can be found. As it was shown in the preceding Section, every flat map can be constructed as the product of two curves in \(S^3\) with some special properties, in analogy with the classical Bianchi tori, the most simple situation that is non-trivial is to consider helices in \(S^3\) with torsion 1 and \(-1\), respectively.

Let us consider the curve \(\sigma(s) : \mathbb{R} \to S^3\) given by
\[
\sigma(s) = \frac{1}{\sqrt{1 + r^2}} (r \cos(s/r), r \sin(s/r), \cos(rs), \sin(rs)), \quad r > 1.
\]

Straightforward computations show that \(\sigma(s)\) is parametrized by arclength, its curvature is constant, \(\kappa \equiv (r^2 - 1)/r\), and its torsion \(\tau\) verifies \(\tau^2 = 1\). Thus, it is a helix in \(S^3\). Conversely, every helix in \(S^3\) with \(\tau^2 = 1\) is written in that way, modulo a (possibly orientation-reversing) isometry of \(S^3\). Moreover, \(\sigma(s)\) is periodic if and only if \(r^2 \in \mathbb{Q}\). In the case where \(r \in \mathbb{N}\), \(\sigma(s)\) is a simple closed curve of period \(2\pi r\).

Now consider two of these helices, one with \(\tau = 1\), the other with \(\tau = -1\), and both with the same curvature \(\kappa = (r^2 - 1)/r\), and construct a flat map \(N : \mathbb{R}^2 \to S^3\) following the Bianchi-Spivak method. If we let \(\mu = (r^2 - 1)/2r\), the angle of this flat map is \(\omega(u, v) = 2\mu(u + v)\). Let \(\alpha, \beta\) be solutions of (15) for this angle, and define
\[
\phi(u, v) = \cos \left(\frac{\omega}{2}\right) \alpha + \sin \left(\frac{\omega}{2}\right) \beta.
\]

Then \(\phi_u = \phi_v\), and thus \(\phi(u, v) = 2g(u + v)\) for some smooth real function \(g(t)\). From here, if we define \(\psi(u, v) = 2\mu g(u + v) + h(u - v)\), \(h(t)\) being a smooth real function, the system can be completely integrated, and \(\alpha, \beta\) are given by
\[
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} =
\begin{pmatrix}
\phi(u, v) & \psi(u, v) \\
-\psi(u, v) & \phi(u, v)
\end{pmatrix}
\begin{pmatrix}
\cos (\mu(u + v)) \\
\sin (\mu(u + v))
\end{pmatrix}.
\]

In this way we obtain via (14) the explicit coordinates of infinitely many flat surfaces \(f : \Sigma \to \mathbb{R}^4\), possibly with singular points, with flat normal bundle and regular Gauss map. An analogous integration is obtained if we assume that the angle is \(\omega(u, v) = \mu(u - v)\).

The regularity condition for these surfaces is written as
\[
(2G'(u + v))^2 + (2\mu G(u + v) + H(u - v))^2 \neq 0,
\]
where \(G(t) = (1 + \mu^2)g(t) + g''(t)\) and \(H(t) = (1 + \mu^2)h(t) + h''(t)\). With this, it is easy to arrange \(g, h\) in order to define \(f\) without singular points. Actually, one can construct
to be defined on the whole $u,v$-plane. Furthermore, if we choose $g(t) = 0$ and $\mu$ so that $\mu^2 \in \mathbb{Q}$, then $f : u,v$-plane \to \mathbb{R}^4$ is a cylinder.

Any surface of this kind is characterized by the fact that its Gauss image in $G_{2,4} \equiv \mathbb{S}^2 \times \mathbb{S}^2$ lies in the product of two circles with geodesic curvatures verifying $k_1^2 = k_2^2$. In the case where the domain of the parameters $(u,v)$ is the whole $u,v$-plane, its Gauss map is compact, even though the surface $f$ is not if $\omega \neq \text{const}$. This contrasts with the situation in $\mathbb{S}^3$, where all flat surfaces with globally defined Tschebysheff coordinates and whose Gauss map is a compact surface in $\mathbb{S}^2 \times \mathbb{S}^2$ are actually flat tori [Wei1].

This special shape of the Gauss map implies the following facts:

1. None of these flat surfaces $f : \Sigma \to \mathbb{R}^4$ has a special section without singular points. This situation justifies the way in which we introduced in Section 2 the Lorentz surface structure on $\Sigma$, since $\langle dN, d\hat{N} \rangle$ is not in general a non-degenerate Lorentzian metric for any pair $(N, \hat{N})$ of special sections.

2. These flat surfaces are exactly the ones made up by compositions (see [DaTo]). This follows from the characterization of compositions in terms of their Gauss maps that was observed in [Wei3].

3. In particular, as it was also observed in [Wei3], none of these flat surfaces is complete (unless $\mu = 0$).

To end up this Section, we shall construct explicitly some new flat cylinders in $\mathbb{R}^4$ with $R^\perp \equiv 0$ and regular Gauss map that are not compositions.

To do so, we multiply again two helices in $\mathbb{S}^3$, this time with different curvatures, so that the angle of the flat map that they determine is $\omega(u,v) = 2ru + 2sv$, where $r, s \in \mathbb{R}, r^2 \neq s^2$. It is straightforward to check that

$$\begin{align*}
\alpha &= \exp(su + rv)(\cos(ru + sv) + \sin(ru + sv)) \\
\beta &= \exp(su + rv)(\sin(ru + sv) - \cos(ru + sv))
\end{align*}$$

are solutions of (15) for this angle. Since they satisfy the regularity condition (18), they give rise to a flat surface with $R^\perp \equiv 0$ and regular Gauss map. If $r, s \in \mathbb{N}$, these two helices are closed and the resulting immersion $f : u,v$-plane \to \mathbb{R}^4$ is a cylinder. However, none of this examples is complete. We shall show at the end of the paper how to produce new complete flat cylinders in $\mathbb{R}^4$.

4 New flat tori

In this Section we shall exhibit a general family of new flat tori in $\mathbb{R}^4$. This family, which is the first one that comes out since Bianchi’s works in the 19th century on flat tori in $\mathbb{S}^3$, is made up by flat tori with flat normal bundle, and provides a counterexample to Borisenko’s conjecture (see [Bor]). Specifically, we shall give a constructive proof of the following result.
Theorem 17 There exists a general family of flat tori in $\mathbb{R}^4$ with flat normal bundle that are not product of curves, and do not lie in any affine 3-sphere of $\mathbb{R}^4$.

It is remarkable that the tori of the family that we construct cannot be given in explicit coordinates.

The key step of the proof consists in verifying the following fact, that has interest in its own right.

Theorem 18 Given $n \in \mathbb{N}, n > 1$, there exists a Hopf torus $T$ in $S^3$ with non-constant angle $\omega(u)$ such that the Hopf cylinder in $S^3$ with angle $\tilde{\omega}(u) = \omega(u/n)$ is actually a torus. Moreover, $T$ can be chosen arbitrarily close to any prescribed Clifford torus in $S^3$.

We shall actually show that there are infinitely many Hopf tori in $S^3$ with the above properties for any $n \in \mathbb{N}, n > 1$. Here, by a Clifford torus we mean a product torus $S^1(r) \times S^1(\sqrt{1 - r^2})$ in $S^3$, which can be constructed as the lift via the Hopf fibration $h$ of a circle in $S^2$. The closeness stated in Theorem 18 must be understood in the following sense: the curve $c \in S^2$ that generates the Hopf torus $T$ can be chosen arbitrarily close to a circle in $S^2$.

Remark 19 Let $c(u)$ be a closed regular curve in $S^2$, the parameter $u$ being the arclength parameter of its asymptotic lift in $S^3$ (see [Kit1]). If $k(u)$ denotes the geodesic curvature of $c(u)$, the angle $\omega(u)$ of the Hopf torus $h^{-1}(c)$ is given by

$$\omega(u) = \cot^{-1}(k(u))$$

Thus Theorem 18 asserts the existence of a periodic curve $c(u)$ in $S^2$ such that the only (up to congruence) curve $\tilde{c}(u)$ in $S^2$ with geodesic curvature $k(u/n)$ is periodic. This is strongly related to the following problem posed by S.S. Chern (see [CC]): when is a curve with periodic curvatures in a space form periodic?

Remark 20 It was proved in [Kit1] that the arclength parameter $s$ of a curve $c(s)$ in $S^2$ is expressed in terms of the arclength parameter $u$ of its asymptotic lift in $S^3$ as

$$\frac{ds}{du} = \frac{2}{\sqrt{1 + k(u)^2}},$$

(22)

where $k(u)$ is the geodesic curvature of the curve $c(s)$ at $u = u(s)$.

Proof of Theorem 17: Fix $n \in \mathbb{N}, n > 1$. From Theorem 18 we know the existence of a Hopf torus $N$ in $S^3$ with non-constant angle $\omega(u)$ such that $\tilde{\omega}(u) = \omega(u/n)$ is also the angle of a Hopf torus $\tilde{N}$ in $S^3$. Let $\tilde{\alpha}, \tilde{\beta}$ be geometric solutions corresponding to $\tilde{\omega}$. Then they are of the form $(\tilde{\alpha}, \tilde{\beta}) = (\tilde{a}(u), \tilde{b}(u))$. Now let us define

$$(\alpha, \beta)(u, v) = (\tilde{\alpha}, \tilde{\beta})(nu, nv).$$

(23)
It is easy to check that $\alpha, \beta$ are solutions of (12) for the angle $\omega$ we started with. Moreover, they are not geometric solutions, since they are not of the form $(\overline{\alpha}, \overline{\beta}) = a(u) \sin v + b(u) \cos v$.

Now, since $N$ is a Hopf torus in $\mathbb{S}^3$, the map $N(u, v)$ is doubly periodic in $u$ and $v$. More exactly, there exists $T > 0$ such that $N(u, v)$ is well defined on the rectangular torus $\mathbb{R}^2/\Lambda$, where $\Lambda = \{(Tp, 2\pi q) \in \mathbb{R}^2 : p, q \in \mathbb{Z}\}$. Thus $\omega(u)$ is $T$-periodic, and $\tilde{\omega}(u)$ is $nT$-periodic. In this way the Hopf torus $\tilde{N}$ is defined on $\mathbb{R}^2/\Gamma$, where $\Gamma = \{(Tknp, 2\pi q) : p, q \in \mathbb{Z}\}$, for some $k \in \mathbb{N}$. Thus $\alpha, \beta$ and $N$ are defined on $\mathbb{R}^2/\Gamma$.

Summing up, the map $f$ given by (14) is well defined on a rectangular torus $\mathbb{T}^2 = \mathbb{R}^2/\Gamma$ with respect to the coordinates $(u, v)$. Note that it is not contained in any affine 3-sphere, since $\alpha, \beta$ are not geometric solutions of (15). In other words, we have obtained a flat torus in $\mathbb{R}^4$, possibly with singular points, that has flat normal bundle and regular Gauss map. Furthermore, it is not a product of curves and it does not lie in any 3-sphere of $\mathbb{R}^4$.

To conclude the proof of the Theorem we need to show that this flat torus can be chosen without singular points. For this purpose, we begin by defining on $\mathbb{T}^2$, for any $\lambda > 0$, the functions $\alpha_\lambda = 1 + \lambda \alpha$ and $\beta_\lambda = \lambda \beta$. It follows that $\alpha_\lambda, \beta_\lambda$ also constitute a solution of (15), and thus they give rise to a new flat surface in $\mathbb{R}^4$, $f_\lambda : \mathbb{T}^2 \to \mathbb{R}^4$, possibly with singular points. Let us also note that $f$ lies in an affine 3-sphere if and only if so does $f_\lambda$. Now observe that $(\alpha_\lambda, \beta_\lambda)$ tends pointwise to $(\alpha_0, \beta_0) := (1, 0)$ as $\lambda$ tends to 0. But since

$$|\alpha_\lambda(u, v) - \alpha_\lambda'(u, v)| = |\lambda - \lambda'| |\alpha(u, v)|,$$

and the same relation holds for $\beta_\lambda$, we infer that $(\alpha_\lambda, \beta_\lambda)$ tends uniformly on $\mathbb{T}^2$ to $(\alpha_0, \beta_0)$. Since $(\alpha_0, \beta_0)$ were chosen to satisfy the regularity condition (18), we obtain the existence of some $\lambda_0 > 0$ such that $f_\lambda$ is an immersion of $\mathbb{T}^2$ into $\mathbb{R}^4$ for any $\lambda < \lambda_0$. This completes the proof.

\qed

Proof of Theorem (18). Let us begin by defining $\mathcal{C}$ as the set of smooth closed regular curves in $\mathbb{S}^2$, parametrized as $\alpha(t) : [0, 1] \to \mathbb{S}^2$, where $t = s/\ell$, $s$ denotes the arclength parameter, and $\ell$ is the length of the curve. On $\mathcal{C}$ we can define a topology through the norm $|| \cdot ||_C$ given by

$$||\alpha(t)||_C = ||\alpha(t)||_\infty + ||\alpha'(t)||_\infty + ||\alpha''(t)||_\infty.$$

The key ingredient of this proof is the construction for each $n \in \mathbb{N}$, $n > 1$, of a continuous map $\mathcal{A}_n$ from $\mathcal{C}$ into $\mathbb{R}$ with the following property (P): if $\omega(u)$ denotes the angle of the Hopf torus $h^{-1}(\alpha(t))$, then the Hopf cylinder in $\mathbb{S}^3$ of angle $\omega(u/n)$ is actually a torus if and only if $\mathcal{A}_n(\alpha(t)) \in \mathbb{Q}$.

Fix $n \in \mathbb{N}$, $n > 1$. To begin the construction of $\mathcal{A}_n$, we let $\xi_0 = j$, and consider the asymptotic lift $\gamma(t)$ of $\alpha(t) \in \mathcal{C}$, given by

$$\pi(\gamma(t)) = (\alpha(t), \alpha'(t)/||\alpha'(t)||),$$
where $\pi$ is as in (11). We infer from this expression that the mapping from $\mathcal{C}$ into $C^\infty([0, 1])$ given by

$$\alpha(t) \in \mathcal{C} \mapsto u_\alpha(t) = \int_0^t \|\gamma'(w)\|dw \in C^\infty([0, 1]) \quad (24)$$

is continuous.

Next, note that for any fixed $\alpha(t) \in \mathcal{C}$, the function $u_\alpha(t)$ is non-negative and strictly increasing. Therefore we can define the function $\hat{t}_\alpha(t) : [0, 1] \to [0, 1]$ given by

$$\hat{t}_\alpha(t) = u_{\alpha}^{-1}\left( \frac{u_{\alpha}(t)}{n} \right), \quad (25)$$

which is smooth and strictly increasing. Hence we may consider the mapping from $\mathcal{C}$ to $C^\infty([0, 1])$ given by

$$\alpha(t) \in \mathcal{C} \mapsto \hat{t}_\alpha(t) \in C^\infty([0, 1]). \quad (26)$$

We claim that this mapping is continuous. In fact, since the mapping (24) is continuous, equation (25) shows that the continuity of (26) would follow from the continuity of a mapping $B$ that we describe next: let $S$ be the subset of $C^\infty([0, 1])$ made up by all strictly increasing smooth functions $f : [0, 1] \to [0, +\infty)$ such that $f(0) = 0$. Then $B$ is the mapping from $S$ into $S$ given by

$$f(t) \in S \mapsto f^{-1}\left( \frac{f(t)}{n} \right) \in S.$$  

It is clear that $B$ is well defined, because of the conditions imposed to $f(t)$. Moreover, standard arguments show that $B$ is continuous, and therefore (26) is also continuous, as we wished.

On the other hand, we can define another continuous mapping from $\mathcal{C}$ into $C^\infty([0, 1])$, namely the one given by

$$\alpha(t) \in \mathcal{C} \mapsto k_\alpha(t) \in C^\infty([0, 1]), \quad (27)$$

where here $k_\alpha(t)$ stands for the geodesic curvature of $\alpha(t)$. Next define, for a fixed element $\alpha(t)$ of $\mathcal{C}$, the function $\tilde{k}_\alpha(t) : [0, 1] \to \mathbb{R}$ given by

$$\tilde{k}_\alpha(t) = k_\alpha(\hat{t}_\alpha(t)).$$

Therefore we can define another mapping from $\mathcal{C}$ into $C^\infty([0, 1])$, this time given by

$$\alpha(t) \in \mathcal{C} \mapsto \tilde{k}_\alpha(t) \in C^\infty([0, 1]) \quad (28)$$

Note that if $\alpha_1(t), \alpha_2(t) \in \mathcal{C}$, the inequality

$$|\tilde{k}_{\alpha_1}(t) - \tilde{k}_{\alpha_2}(t)| \leq |k_{\alpha_1}(\hat{t}_{\alpha_1}(t)) - k_{\alpha_1}(\hat{t}_{\alpha_2}(t))| + |k_{\alpha_1}(\hat{t}_{\alpha_2}(t)) - k_{\alpha_2}(\hat{t}_{\alpha_2}(t))|.$$
holds. As a consequence of this expression and the continuity of the maps (26) and (27), we obtain that (28) is again continuous.

Given \( \alpha(t) \in C \), we plan to construct a curve \( \tilde{\alpha}(t) : [0, 1] \to S^2 \) verifying:
1) \( u_\alpha(t) \) is the arclength parameter of the asymptotic lift in \( S^3 \) of \( \alpha(t) \), and
2) the geodesic curvature of \( \alpha(t) \) is given by \( \tilde{k}_\alpha(t) \).

If \( \tilde{s}_\alpha \) denotes the arclength parameter of a curve \( \tilde{\alpha}(t) \) in \( S^2 \), then equation (22) ensures that \( \tilde{\alpha}(t) \) satisfies condition 1) if and only if \( \tilde{s}_\alpha(t) \) is given by

\[
\frac{d\tilde{s}_\alpha}{dt} = \frac{d\tilde{s}_\alpha}{du_\alpha} \frac{du_\alpha}{dt} = \frac{2||\gamma'(t)||}{\sqrt{1 + \tilde{k}_\alpha(t)^2}}. \tag{29}
\]

Note that the mapping

\[
\alpha(t) \in C \mapsto \left( \frac{d\tilde{s}_\alpha}{dt}, \tilde{k}_\alpha(t) \right) \in C^\infty([0, 1]) \times C^\infty([0, 1]), \tag{30}
\]

where \( \tilde{s}_\alpha(t) \) is defined by (29), is trivially continuous, and that so is also the process assigning to \( (d\tilde{s}_\alpha/dt, \tilde{k}_\alpha(t)) \) the only (up to congruence) curve \( \tilde{\alpha}(t) \) in \( S^2 \) with arclength parameter \( \tilde{s}_\alpha(t) \) and geodesic curvature \( \tilde{k}_\alpha(t) \). This follows from standard results on the regularity of the solutions to ordinary differential systems with respect to initial conditions and parameters.

Therefore, the mapping

\[
\alpha(t) \in C \mapsto \tilde{\alpha}(t) : [0, 1] \to S^2, \tag{31}
\]

obtained via this process is well defined up to congruences in \( S^2 \) and continuous. Moreover, \( \tilde{\alpha}(t) \) satisfies conditions 1) and 2).

Let us also note that if \( \alpha(t) \in C \) and \( k_1, k_2 \) are smooth functions satisfying that \( k_1(u_\alpha(t)) = \tilde{k}_\alpha(t) \), and \( k_2(u_\alpha(t)) = k_\alpha(t) \), then the identity

\[
k_1(u_\alpha(t)) = k_2 \left( \frac{u_\alpha(t)}{n} \right) \tag{32}
\]

holds. This relation ensures that the angle \( \omega(u) \) of the Hopf torus \( h^{-1}(\alpha(t)) \), and the angle \( \tilde{\omega}(u) \) of the Hopf cylinder \( h^{-1}(\tilde{\alpha}(t)) \) are related by \( \tilde{\omega}(u) = \omega(u/n) \).

To sum up, we have shown up to now that the process assigning to each \( \alpha(t) \in C \) the curve \( \tilde{\alpha}(t) : [0, 1] \to S^2 \) such that, with the above notations, \( \tilde{\omega}(u) = \omega(u/n) \) holds, is well defined up to congruences in \( S^2 \), and continuous.

Next extend the parametrization \( \alpha(t) : [0, 1] \to S^2 \) to a periodic function \( \alpha(t) : \mathbb{R} \to S^2 \), so that \( \alpha(t) \) (and hence \( \gamma(t) \)) is 1-periodic.

With this, observe that, since \( k_\alpha(t) \) is 1-periodic and \( \tilde{\k}_\alpha(t+n) = 1+\tilde{\k}_\alpha(t) \), the function \( \tilde{k}_\alpha(t) \) is \( n \)-periodic. In this way (29) tells that \( d\tilde{s}_\alpha/dt \) is also \( n \)-periodic with respect to \( t \), from where we obtain that \( \tilde{s}_\alpha(t) = \varrho(t) + c_0 t \), where \( c_0 \in \mathbb{R} \) and \( \varrho \) is \( n \)-periodic. But now, defining \( \mu = c_0 n \), we get that \( \tilde{k}_\alpha(\tilde{s}_\alpha(t) + \mu) = \tilde{k}_\alpha(\tilde{s}_\alpha(t)) \), that is, \( \tilde{k}_\alpha \) is \( \mu \)-periodic with respect to the parameter \( \tilde{s}_\alpha \).
In order to finish the construction of the mapping $A_n$ from $C$ into $\mathbb{R}$ mentioned at the beginning of the proof, let us consider for $\tilde{\alpha}(\tilde{s})$ the only rigid motion $A_\alpha$ in $S^2$ such that $A_\alpha(\tilde{\alpha}(0)) = \tilde{\alpha}(\mu)$, $A_\alpha(\tilde{\alpha}'(0)) = \tilde{\alpha}'(\mu)$, and $A_\alpha(\tilde{\alpha}(0) \times \tilde{\alpha}'(0)) = \tilde{\alpha}(\mu) \times \tilde{\alpha}'(\mu)$. It then holds that $A_\alpha \in SO(3)$, and moreover that $\tilde{\alpha}(\tilde{s} + \mu) = A_\alpha(\tilde{\alpha}(\tilde{s}))$, since both curves have the same geodesic curvature $\tilde{k}_\alpha(\tilde{s})$, and the same initial conditions. Let $\theta_\alpha$ denote the angle of the rotation $A_\alpha$, then it follows from the above comments that $\theta_\alpha/\pi \in \mathbb{Q}$ if and only if $A_\alpha^q = \text{Id}$ for some $q \in \mathbb{N}$, if and only if $\tilde{\alpha}(\tilde{s} + q\mu) = \tilde{\alpha}(\tilde{s})$ for some $q \in \mathbb{N}$. In particular, if $\tilde{k}_\alpha(\tilde{s})$ is not constant, we obtain that $\tilde{\alpha}$ is closed if and only if $\theta_\alpha/\pi$ is rational. Also note that the map
\begin{equation}
\tilde{\alpha}(t) \mapsto \theta_\alpha \in \mathbb{R}
\end{equation}
is continuous, and is invariant under congruences in $S^2$. Thus, we can define the continuous mapping
\begin{equation}
A_n : C \rightarrow \mathbb{R}
\end{equation}
 obtenered as the composition of $\mathbf{31}$ and $\mathbf{33}$. This mapping has been constructed so that it satisfies property (P), as desired.

It is known that there exist two equivalence classes of regular homotopy on the set of smooth regular closed curves in $S^2$. Those correspond to the class of the circle, which will be denoted by $C_0$, and that of the figure eight. One can check that if $\gamma$ is a circle in $S^2$, then $A_n(\gamma) = 0 \in \mathbb{Q}$. Actually, circles are the only elements of $C$ whose image under $A_n$ can be explicitly calculated. Since the mapping $A_n : C \rightarrow \mathbb{R}$ is continuous, the set $J := A_n(C_0) \subseteq \mathbb{R}$ is necessarily an interval $J$, unless $A_n$ is constant on $C_0$, in which case $J = \{0\}$. In any case, there exists an infinite number of curves $c(t) \in C_0$ that are not circles, and such that $A_n(c(t)) \in \mathbb{Q}$. Note that they can be chosen arbitrarily close to any given circle. Finally, the Hopf torus of any of these curves satisfies the conditions asserted in the theorem, and we are done.

\[\square\]

**Remark 21** Since for every $n \in \mathbb{N}$, $n > 1$ there exist infinitely many Hopf tori $h^{-1}(c)$ such that $A_n(c) \in \mathbb{Q}$, and for every such $h^{-1}(c)$ we have created a new flat torus in $\mathbb{R}^4$ via Theorem $\mathbf{17}$, our description actually produces a general family of new flat tori in $\mathbb{R}^4$, and not just isolated examples.

**Remark 22** The construction process described in Theorems $\mathbf{17}$ and $\mathbf{18}$ can be seen as a method of unfolding a Hopf torus in $S^3$ of a particular kind, so that the result is a flat torus in $\mathbb{R}^4$ that does not lie in any affine 3-sphere.

To explain this interpretation, we will assume for simplicity that $n = 2$. So, we start with a Hopf torus $\Sigma_0 = h^{-1}(c)$, with $c$ a closed regular curve in $S^2$, such that $A_2(c) \in \mathbb{Q}$, for the mapping $A_2$ defined in $\mathbf{31}$. Thus $A_2(c) = l/k$, where $l, k \in \mathbb{N}$, and this implies that the curve $\tilde{c}$ constructed from $c$ through $\mathbf{31}$ is closed. Moreover, $c(u)$ winds $k$ times whenever $\tilde{c}(u)$ winds once, where here $u$ is the (common) arclength parameter of their asymptotic lifts in $S^3$. Next, let us consider a trivial $2k$-folded covering of $\Sigma_0$, denoted by$\Sigma^*$, obtained by tracing $c(u)$ $2k$-times, and taking then its Hopf torus in $S^3$. Thus $\Sigma^*$ is
again a torus, and the Hopf torus $\widetilde{T} = h^{-1}(\overline{c})$ can be parametrized as a map from $\Sigma^*$ into $\mathbb{S}^3$ in the obvious way. Now, if we define $\tilde{\alpha}, \tilde{\beta}$ as coordinates of $\widetilde{T}$ and its unit normal in $\mathbb{S}^3$, and we parametrize them at double speed in terms of the asymptotic parameters $u, v$, i.e. we define $(\alpha, \beta)(u, v) = (\tilde{\alpha}, \tilde{\beta})(2u, 2v)$, then $\tilde{\alpha}, \tilde{\beta}, \alpha, \beta$ are all well defined over $\Sigma^*$. Once here we can use formula (14) to obtain a map $f : \Sigma^* \to \mathbb{R}^4$, which will be a flat torus in $\mathbb{R}^4$, possibly with singular points, and not lying in any 3-sphere. But finally, by collapsing this map into $\Sigma_0$ we find that, closely enough to $\Sigma_0$, $f$ does not have singular points.

In other words, we have covered $2k$-times the Hopf torus $\Sigma_0$, and then perturbed this trivial covering in a special way, so that we assign to each one of the $2k$-sheets of the covering a different perturbation. After this deformation, the resulting surface is a regular flat torus in $\mathbb{R}^4$, with flat normal bundle, regular Gauss map, and not lying in any 3-sphere, as we wished.

A simpler version of the process we have just described can be used to generate new complete flat cylinders in $\mathbb{R}^4$ with $R^\perp \equiv 0$ and regular Gauss map, as small perturbations of Hopf cylinders.

To begin with, let $N$ be a Hopf cylinder with angle $\omega(u)$. We shall assume that $N$ is complete, has bounded mean curvature, and its non-trivial family of asymptotic curves have bounded curvature in $\mathbb{S}^3$.

For $n \in \mathbb{N}$, $n > 1$ we consider the Hopf cylinder $\widetilde{N}$ with angle $\widetilde{\omega}(u) = \omega(u/n)$, as well as geometric solutions $\tilde{\alpha}, \tilde{\beta}$ corresponding to $\widetilde{\omega}$. Then (23) gives a solution of (15) for $\omega$, and we obtain via (14) a flat cylinder $f : u, v$-plane $\to \mathbb{R}^4$, with flat normal bundle and regular Gauss map. It may have singular points, but it does not lie in any 3-sphere.

In order to get rid of singular points, we substitute $\alpha, \beta$ by $\alpha_\lambda = 1 + \lambda \alpha$ and $\beta_\lambda = \lambda \beta$, and note that

$$\lim_{\lambda \to 0} \left\{ (A_\lambda^2 - B_\lambda^2) \sin \omega - 2A_\lambda B_\lambda \cos \omega \right\} = \sin \omega$$

(pointwise on the $u, v$-plane. Here $A_\lambda, B_\lambda$ are the functions appearing in (16), but related to $\alpha_\lambda, \beta_\lambda$.

Besides, since $\tilde{\alpha}, \tilde{\beta}$ are geometric solutions, the flat surface in $\mathbb{R}^4$ that they define along with $\widetilde{N}$ lies in a 3-sphere of $\mathbb{R}^4$. Thus, from (14), $\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}_u, \tilde{\beta}_u$ are bounded on the $u, v$-plane. In addition, note that $\langle \widetilde{N}_{uu}, \widetilde{N}_{uv} \rangle = 1 + \widetilde{\omega}_u^2$. As $\tilde{\alpha}, \tilde{\beta}$ can be chosen to be coordinates of $\widetilde{N}$ and its polar map, we obtain $\widetilde{\alpha}_u^2 + \widetilde{\beta}_u^2 \leq 2(1 + \widetilde{\omega}_u^2)$, and this ensures that $\tilde{\alpha}_u, \tilde{\beta}_u$ are bounded, since we required the asymptotic curves of $N$ to have bounded curvature.

Finally, note that

$$\begin{align*}
A_\lambda(u, v) &= 1 + \lambda \left( \tilde{\alpha} + n^2 \tilde{\alpha}_u + n^2 \tilde{\omega}_u \tilde{\beta}_u \right) (nu, nv), \\
B_\lambda(u, v) &= \lambda \left( \tilde{\beta} + n^2 \tilde{\beta}_u - n^2 \tilde{\omega}_u \tilde{\alpha}_u \right) (nu, nv).
\end{align*}$$

All of this shows that $A_\lambda$ and $B_\lambda$ are bounded. Hence, the convergence in (35) is actually uniform over the $u, v$-plane. Since $N$ has bounded mean curvature, there is some $c_0$ such
that $0 < c_0 \leq \sin \omega$. Thus there is some $\lambda_0 > 0$ such that, for all $0 < \lambda < \lambda_0$, $f_\lambda$ preserves all the above mentioned properties of $f$, but has no singular points.

Furthermore, the metric of $f_\lambda$, given by (19) for $A_\lambda, B_\lambda$ and $\omega$, converges uniformly on the $u, v$-plane to the metric of $N$, which satisfies $\langle dN(X), dN(X) \rangle \geq d > 0$ for some $d > 0$, and for every unit tangent vector field $X \in \mathfrak{X}(N)$. This shows that, for $\lambda$ sufficiently small, $f_\lambda : u, v$-plane $\to \mathbb{R}^4$ is complete.

These are, to the best of our knowledge, the first examples of complete flat cylinders in $\mathbb{R}^4$ with flat normal bundle and regular Gauss map that do not lie in any 3-sphere. Nevertheless, it is remarkable that all of them are bounded in $\mathbb{R}^4$.

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