INVERSE SCATTERING TRANSFORM AND SOLITON SOLUTIONS OF AN INTEGRABLE NONLOCAL HIROTA EQUATION

YUAN LI AND SHOU-FU TIAN*

School of Mathematics, China University of Mining and Technology
Xuzhou, Jiangsu, 221116, China

(Communicated by Alain Miranville)

Abstract. In this work, we study the inverse scattering transform of a non-local Hirota equation in detail, and obtain the corresponding soliton solutions formula. Starting from the Lax pair of this equation, we obtain the corresponding infinite number of conservation laws and some properties of scattering data. By analyzing the direct scattering problem, we get a critical symmetric relation which is different from the local equations. A novel left-right Riemann-Hilbert problem is proposed to develop the inverse scattering theory. The potentials are recovered and the pure soliton solutions formula is obtained when the reflection coefficients are zero. Based on the zero types of scattering data, nine types of soliton solutions are obtained and three typical types are described in detail. In addition, some dynamic behaviors are given to illustrate the soliton characteristics of the space symmetric nonlocal Hirota equation.

1. Introduction. As one of the most important nonlinear equations, the nonlinear Schrödinger (NLS) equation

\[ iq_t(x,t) = q_{xx}(x,t) \pm 2|q(x,t)|^2 q(x,t), \] (1.1)

which has been studied deeply. Many physical models can be described by the NLS equation, such as plasma physics [11], water wave [7], nonlinear optics [5] and in pure mathematics like motion of curves in differential geometry [19].

Although the NLS equation can accurately describe many physical phenomena, higher order corrections need to be taken into account in some experiments. For these physical reasons, Kodama and Hasegawa proposed the higher order nonlinear
Schrödinger equations [14]

\[ iq_t + \frac{1}{2}q_{xx} + |q|^2q + i\varepsilon(\alpha'q_{xxx} + \beta'|q|^2q_x + \gamma'q|q|^2) = 0, \tag{1.2} \]

with constants \( \varepsilon, \alpha', \beta', \gamma' \in \mathbb{R} \). Besides the NLS equation for \( \varepsilon = 0 \), four cases are known to be integrable. When the ratio of the constants are taken to be \( \alpha' : \beta' : \gamma' = 0 : 1 : 1 \) or \( \alpha' : \beta' : \gamma' = 0 : 1 : 0 \), we can obtain the derivative NLS equation of type I [6] and II [9], respectively. For \( \alpha' : \beta' : \gamma' = 1 : 6 : 3 \), one obtains the Sasa-Satsuma equation [21], and for \( \alpha' : \beta' : \gamma' = 1 : 6 : 0 \), one obtains the Hirota equation [12]. In this work, we focus on the Hirota equation.

In recent years, the nonlocal problems in integrable equations have attracted much attention. The first such equation was the nonlocal NLS equation proposed by Ablowitz and Musslimani:

\[ iq_t(x,t) = q_{xx}(x,t) \pm 2q^2(x,t)q^*(x,-t), \tag{1.3} \]

and they obtained its explicit solution by inverse scattering [1, 2]. The nonlocal NLS equation simultaneously admits both the bright and the dark soliton solutions [20], and the interactions between dark soliton and antidark soliton have been studied by using the Darboux transformation method [16].

A number of other nonlocal integrable equations are considered, including high dimensional equations, discrete equations and others [10, 22]. In order to study the nonlocal integrable equations, some effective methods have been produced, such as inverse scattering transform (IST) [1, 2, 3, 4, 13], Riemann-Hilbert (RH) approach [17], \( \bar{\partial} \)-dressing method [15], Hirota bilinear method [18] and Darboux transformations [22, 23].

Similar to the NLS equation, there are several integrable systems corresponding to nonlocal versions of the Hirota equation. The integrability of these models is established by providing their explicit forms of Lax pairs or zero curvature conditions [8]. According to the different inversion relations, integrable nonlocal Hirota equations roughly include the following categories [8]: the space symmetric nonlocal Hirota equation (S-NHE),

\[ iq_t(x,t) = -\alpha\{q_{xx}(x,t) - 2\kappa q^*(-x,t)q^2(x,t)\} \]
\[ + \beta\{q_{xxx}(x,t) - 6\kappa q(x,t)q^*(-x,t)q_x(x,t)\}, \tag{1.4} \]

where \( \alpha, \beta \in \mathbb{R}, \kappa = \mp 1 \), the time symmetric nonlocal Hirota equation (T-NHE) and the space-time symmetric nonlocal Hirota equation (ST-NHE). The explicit multi-soliton solutions for nonlocal Hirota equations have been obtained by Darboux-Crum transformations and Hirota bilinear method [8]. As we know, exact solutions of the S-NHE (1.4) have not been attained via the inverse scattering transform, which is the work of this paper.

Not all the methods mentioned above for solving the nonlocal equations are effective. The most classical and effective method is IST, which relates the Lax pair to the integrable nonlinear equations. One of the Lax pair equations, termed the scattering problem, is used to study the properties of the eigenfunctions and scattering data. The other linear equation serves to determine the evolution of the scattering data. In addition, as a higher order extension of the nonlocal NLS equation, the study of nonlocal Hirota equation enriches the theory of integrable systems.

This paper is planned as follows. In Section 2, we shall give the Lax pair and Jost solutions of the S-NHE. In Section 3, we derive an infinite number of conservation
laws associated with the S-NHE. By analyzing the direct scattering problem, a critical symmetric relation is obtained in Section 4. The inverse scattering problem is established by using a novel left-right Riemann-Hilbert problem in Section 5. In Section 6, we will recover the potentials \( r(x) \) and \( q(x) \). In Section 7, we show the three typical types of pure soliton solutions for the S-NHE under the reflectionless case. The multi-soliton classification is based on the zero types of scattering data. In addition, the localized structures and dynamic behaviors of one-, two- and three-soliton solutions are shown by some graphic analyses. In Section 8, we will introduce some special initial conditions and take one of them as an example to calculate the conserved quantity. Section 9 is our conclusions.

2. Lax pair and Jost solutions. We begin our discussion by considering the following Lax pair [8]

\[
F_x = LF, \quad \text{(2.1a)}
\]

\[
F_t = MF, \quad \text{(2.1b)}
\]

with

\[
L = -i\lambda \sigma_3 + U = -i\lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{bmatrix},
\]

\[
M = \begin{bmatrix} A(x,t) & B(x,t) \\ C(x,t) & -A(x,t) \end{bmatrix},
\]

\[
A(x,t) = -2i\alpha \lambda^2 - iq(r + \beta(-4i\lambda^3 + qr - qr_x - 2i\lambda qr)),
\]

\[
B(x,t) = i\alpha q_x + 2\alpha \lambda q + i\beta(2q^2r - q_{xx} + 2i\lambda q_x + 4\lambda^2 q),
\]

\[
C(x,t) = -i\alpha r_x + 2\alpha \lambda r + i\beta(2qr^2 - r_{xx} - 2i\lambda r_x + 4\lambda^2 r),
\]

where \( F = F(x,t,\lambda) = (F_1(x,t,\lambda), F_2(x,t,\lambda))^T \) is a two component vector, \( \lambda \) is a spectral parameter. The potential functions \( q(x,t), r(x,t) \) are complex functions and \( \alpha, \beta \) are arbitrary real constants and \( \alpha > 0 \). We assume that when \( |x| \to \infty \), \( q(x,t) \) fast decay to zero. Under the symmetric condition \( r(x,t) = \kappa q^*(-x,t) \) (\( \kappa = \mp 1 \)), we can derive the S-NHE (1.4) through the compatibility condition \( L_t - M_x + [L, M] = 0 \).

Normally when we talk about scattering problem, \( t \) is just a parameter and we’re going to ignore it for the moment. When \( |x| \to \infty \), the asymptotic equation of equation (2.1a) is

\[
F_x(x,\lambda) = -i\lambda \sigma_3 F(x,\lambda).
\]

Obviously the asymptotic solution to equation (2.1a) is

\[
E(x,\lambda) = e^{-i\lambda x \sigma_3}.
\]

Therefore, we can use equation (2.3) and the asymptotic condition to define two asymptotic solutions of the equation (2.1a) as

\[
\Phi(x,\lambda) \to e^{-i\lambda x \sigma_3}, \quad x \to -\infty,
\]

\[
\Psi(x,\lambda) \to e^{-i\lambda x \sigma_3}, \quad x \to +\infty.
\]
and then we have the integral equations
\[
\Phi(x, \lambda) = e^{-i\lambda x} + \int_{-\infty}^{x} e^{-i\lambda(y-x)}U(y)\Phi(y, \lambda)dy, \quad (2.5a)
\]
\[
\Psi(x, \lambda) = e^{-i\lambda x} - \int_{x}^{\infty} e^{-i\lambda(y-x)}U(y)\Psi(y, \lambda)dy. \quad (2.5b)
\]

It’s easy to verify equations (2.5) satisfy the equation (2.1a) and the corresponding asymptotic condition (2.4). We define
\[
\Phi(x, \lambda) = (\phi(x, \lambda), \overline{\phi}(x, \lambda)),
\]
\[
\Psi(x, \lambda) = (\overline{\psi}(x, \lambda), \psi(x, \lambda)),
\]
then we can get the following relationships
\[
\lim_{x \to -\infty} \phi(x, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda x}, \quad \lim_{x \to -\infty} \overline{\phi}(x, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\lambda x}, \quad (2.7a)
\]
\[
\lim_{x \to +\infty} \psi(x, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\lambda x}, \quad \lim_{x \to +\infty} \overline{\psi}(x, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda x}. \quad (2.7b)
\]

Throughout the paper, \( \overline{\phi}(x, \lambda) \) is not the complex conjugate of \( \phi(x, \lambda) \). We shall instead use \( \phi^*(x, \lambda) \) to denote the complex conjugate of \( \phi(x, \lambda) \). The functions \( \phi(x, \lambda), \overline{\phi}(x, \lambda), \psi(x, \lambda), \overline{\psi}(x, \lambda) \) defined by the boundary condition are called the Jost solution of the equation (2.1a). By simple analysis, it is easy to reveal that \( \phi(x, \lambda) \) and \( \psi(x, \lambda) \) are analytic for \( \lambda \in \mathbb{D}^+ \), while \( \overline{\phi}(x, \lambda) \) and \( \overline{\psi}(x, \lambda) \) are analytic for \( \lambda \in \mathbb{D}^- \), where \( \mathbb{C}^+ \) and \( \mathbb{C}^- \) respectively the upper and lower half \( \lambda \)-plane.

3. Infinite number of conservation laws. The Liouville integrability of a finite dimensional dynamical system means that the equations in the system can be expressed as Hamilton equations and there are \( n \) independent conserved quantities. In the study of the integrability of soliton equations, it has been found that many infinite dimensional Lax integrable systems have similar properties.

We set \( \phi(x, \lambda) = (\phi_1(x, \lambda), \phi_2(x, \lambda)) \) and define \( a(\lambda) = \phi_1(x, t)e^{i\lambda x} \), then \( a(\lambda) \) is analytic for \( \lambda \in \mathbb{C}^+ \) and satisfies \( \lim_{|\lambda| \to \infty} a(\lambda) = 1 \). Substituting \( \phi_1(x, \lambda) = e^{-i\lambda x+\varphi(x, \lambda)} \) into (2.1a) we find that the function \( \mu(x, \lambda) = \varphi_x(x, \lambda) \) satisfies the Riccati equation
\[
q \frac{\partial}{\partial x} \left( \frac{\mu}{q} \right) + \mu^2 - qr - 2i\lambda \mu = 0. \quad (3.1)
\]
When \( Im\lambda > 0 \), we have \( \lim_{|\lambda| \to \infty} \varphi(x, \lambda) = 0 \). Substituting the expansion
\[
\mu(x, \lambda) = \sum_{n=0}^{\infty} \frac{\mu_n(x, t)}{(2i\lambda)^{n+1}}, \quad (3.2)
\]
into equation (3.1), then we can get a recursive formula for \( \mu_n(x, t) \)
\[
\mu_0(x, t) = -q(x, t)r(x, t), \quad (3.3)
\]
\[
\mu_1(x, t) = -q(x, t)r_x(x, t), \quad (3.4)
\]
\[
\mu_{n+1}(x, t) = q(x, t)\overline{\partial} \left( \frac{\mu_n}{q} \right) + \sum_{m=0}^{n-1} \mu_m \mu_{n-m-1}, \quad n \geq 1. \quad (3.5)
\]
Combining the asymptotic behaviors of $\phi_1(x, \lambda)$ and $\varphi(x, \lambda)$ together with the definition of $\mu$, we have

$$\ln a(\lambda) = \sum_{n=0}^{\infty} \frac{C_n}{(2i\lambda)^{n+1}},$$

with

$$C_n = \int_{-\infty}^{+\infty} \mu_n(x, t) dx.$$  \hfill (3.7)

According to equations (3.2) and (3.7), we can obtain all conserved quantities of S-NHE (1.4) under the symmetry reduction $r(x, t) = \kappa q^*(-x, t)$ ($\kappa = \mp 1$). Now let’s show the first few conserved quantities:

$$C_0 = -\kappa \int_{-\infty}^{+\infty} q(x, t)q^*(-x, t) dx,$$

$$C_1 = \kappa \int_{-\infty}^{+\infty} q(x, t)q_x^*(-x, t) dx,$$

$$C_2 = -\kappa \int_{-\infty}^{+\infty} \{q(x, t)q_{xx}^*(-x, t) - \kappa q^2(x, t)q^*(-x, t)\} dx,$$

$$C_3 = \kappa \int_{-\infty}^{+\infty} \{q(x, t)q^*(-x, t) + \kappa q(x, t)q_x(x, t)q^2(-x, t) - 4q^2(x, t)q^*(-x, t)q_x^*(-x, t)\} dx,$$

$$C_4 = \kappa \int_{-\infty}^{+\infty} \{q(x, t)q_{xxxx}^*(-x, t) + 5\kappa q^2(x, t)q_x^2(-x, t) + 6\kappa q^2(x, t)q^*(-x, t)q_{xx}^*(-x, t) + \kappa q(x, t)q_{xx}(x, t)q^2(-x, t) - 6q(x, t)q^*(-x, t)q_x(x, t)q_x^*(-x, t) - 2q^3(x, t)q^3(-x, t)\} dx.$$  

In order to obtain the local conservation laws, we consider the equation (2.1b)

$$\phi_{1t} = A\phi_1 + B\phi_2.$$  \hfill (3.8)

Substituting the expression for $\mu$ and $\varphi$ into the above expression and taking the $x$ derivative of the resulting equations we find

$$\partial_t \mu(x, \lambda) = \partial_x \left( A_{\text{nonloc}} + \frac{\mu(x, \lambda)B_{\text{nonloc}}}{q(x, t)} \right),$$

where

$$A_{\text{nonloc}} = -2i\alpha \lambda^2 - i\alpha \kappa q(x, t)q^*(-x, t) + i\beta \{-4i\lambda^3 + \kappa q(x, t)q^*(-x, t) - \kappa q(x, t)q_x^*(-x, t) - 2i\kappa \lambda q(x, t)q^*(-x, t)\},$$

$$B_{\text{nonloc}} = i\alpha q_x(x, t) + 2\alpha \lambda q(x, t) + i\beta \{2\kappa q^2(x, t)q^*(-x, t) - q_{xx}(x, t) + 2\kappa \lambda q_x(x, t) + 4\lambda^2 q(x, t)\}.$$  

Substituting the expansion (3.2) of $\mu$ into (3.9), we can get

$$\partial_t \left( \sum_{n=0}^{\infty} \frac{\mu_n(x, t)}{(2i\lambda)^{n+1}} \right) = \partial_x \left( A_{\text{nonloc}} + \frac{B_{\text{nonloc}}}{q(x, t)} \left( \sum_{n=0}^{\infty} \frac{\mu_n(x, t)}{(2i\lambda)^{n+1}} \right) \right).$$  \hfill (3.10)

By comparing the coefficients before the power of $\lambda$, we have

$$\partial_t (\mu_n) = i\partial_x (\mu_n S_1 + \mu_{n+1} S_2 - i\beta \mu_{n+2}), \quad n = 0, 1, 2, 3, \ldots,$$  \hfill (3.11)
are analytic for $\lambda$. We know that

\[
T = \int_{-\infty}^{\infty} \frac{\partial T}{\partial x} = \int_{-\infty}^{\infty} \frac{\partial X}{\partial x},
\]

Therefore, we can write the conservation law as follows

\[
(3.12)
\]

with $T = \mu_n$ and $X = i\mu_n S_1 + i\mu_{n+1} S_2 + \beta \mu_{n+2}$ ($n = 0, 1, 2, 3, \ldots$). We list the first three local conservation laws

\[
T = \mu_0, \quad X = i\mu_0 S_1 + i\mu_1 S_2 + \beta \mu_2, \quad (3.13)
\]

\[
T = \mu_1, \quad X = i\mu_1 S_1 + i\mu_2 S_2 + \beta \mu_3, \quad (3.14)
\]

\[
T = \mu_2, \quad X = i\mu_2 S_1 + i\mu_3 S_2 + \beta \mu_4, \quad (3.15)
\]

where

\[
\mu_0 = -\kappa q(x, t)q^*(-x, t), \quad \mu_1 = \kappa q(x, t)q_x^*(-x, t),
\]

\[
\mu_2 = -\kappa q(x, t)q_{xx}^*(-x, t) + q^2(x, t)q^2(-x, t), \quad \mu_3 = \kappa q(x, t)q_{xxx}^*(-x, t) + q(x, t)q_x(x, t)q^2(-x, t)
\]

\[
- 4q^2(x, t)q^*(-x, t)q_x^*(-x, t), \quad \mu_4 = -\kappa q(x, t)q_{xxxx}^*(-x, t) + q(x, t)q^2(-x, t)q_{xx}(x, t)
\]

\[
- 6q(x, t)q_x(x, t)q^*(-x, t)q_x^*(-x, t) + 5q^2(x, t)q_x^3(-x, t) + 6q^2(x, t)q^*(-x, t)q_{xx}^*(-x, t) - 2\kappa q^3(x, t)q^3(-x, t).
\]

4. Direct scattering problem. In the following sections, we will focus on the scattering problem (2.1a) under the boundary conditions (2.7). First we define the following functions

\[
M(x, \lambda) = e^{i\lambda x} \phi(x, \lambda), \quad \overline{M}(x, \lambda) = e^{-i\lambda x} \overline{\phi}(x, \lambda), \quad (4.1a)
\]

\[
N(x, \lambda) = e^{i\lambda x} \psi(x, \lambda), \quad \overline{N}(x, \lambda) = e^{-i\lambda x} \overline{\psi}(x, \lambda). \quad (4.1b)
\]

According to the analyticity of $\phi(x, \lambda), \psi(x, \lambda), \overline{\phi}(x, \lambda)$ and $\overline{\psi}(x, \lambda)$, it’s easy to know that $M(x, \lambda)$ and $N(x, \lambda)$ are analytic for $\lambda \in \mathbb{D}^+$, while $\overline{M}(x, \lambda)$ and $\overline{N}(x, \lambda)$ are analytic for $\lambda \in \mathbb{D}^-$. Through some analyses we can find

\[
M(x, \lambda) = \left(1 - \frac{1}{2\lambda} \int_{-\infty}^{\infty} r(z)q(z)dz \right) + O(\lambda^{-2}), \quad (4.2)
\]

\[
N(x, \lambda) = \left(1 - \frac{1}{2\lambda} \int_{-\infty}^{\infty} \frac{q(z)}{q(x)} r(z)q(z)dz \right) + O(\lambda^{-2}), \quad (4.3)
\]

\[
\overline{M}(x, \lambda) = \left(1 + \frac{1}{2\lambda} \int_{-\infty}^{\infty} \frac{q(z)}{q(x)} r(z)q(z)dz \right) + O(\lambda^{-2}), \quad (4.4)
\]

\[
\overline{N}(x, \lambda) = \left(1 + \frac{1}{2\lambda} \int_{-\infty}^{\infty} \frac{r(z)q(z)}{q(x)} dz \right) + O(\lambda^{-2}). \quad (4.5)
\]
Because $\phi(x, \lambda)$ and $\overline{\phi}(x, \lambda)$ have different asymptotic behaviors, they are two linearly independent solutions to the equation (2.1a). Similarly, the solutions $\psi(x, \lambda)$ and $\overline{\psi}(x, \lambda)$ of the scattering problem (2.1a) are linearly dependent. However, the scattering problem (2.1a) is a second order linear ODE, the pairs $\{\phi, \overline{\phi}\}$ and $\{\psi, \overline{\psi}\}$ are linearly dependent and one can express one basis set in terms of the other:

$$\phi(x, \lambda) = a(\lambda)\overline{\psi}(x, \lambda) + b(\lambda)\psi(x, \lambda),$$  

$$\overline{\phi}(x, \lambda) = \overline{a(\lambda)}\psi(x, \lambda) + \overline{b(\lambda)}\overline{\psi}(x, \lambda).$$  

(4.6a) (4.6b)

We define $W(\phi(x, \lambda), \psi(x, \lambda)) = \det(\phi(x, \lambda), \psi(x, \lambda)) = \phi_1\psi_2 - \phi_2\psi_1$ to be the Wronskian determinant of the functions $\phi(x, \lambda)$ and $\psi(x, \lambda)$, and by equation (4.6) we can get

$$a(\lambda) = W(\phi(x, \lambda), \psi(x, \lambda)), \quad \overline{a}(\lambda) = W(\overline{\psi}(x, \lambda), \overline{\phi}(x, \lambda)),\quad \overline{b}(\lambda) = W(\overline{\phi}(x, \lambda), \psi(x, \lambda)).$$  

(4.7a) (4.7b)

It can be seen that the scattering data $a(\lambda)$ and $\overline{a}(\lambda)$ are analytic in the upper half complex plane and the lower half complex plane, respectively, while $b(\lambda)$ and $\overline{b}(\lambda)$ cannot be extend off the real $\lambda$ axis. When $Im \lambda = 0$, the scattering data satisfy the following relation

$$a(\lambda)\overline{a}(\lambda) - b(\lambda)\overline{b}(\lambda) = 1.$$  

(4.8)

Now let’s consider some symmetric relations under the condition of $r(x, t) = \kappa q^*(-x, t)$. If $\phi(x, \lambda) = (\phi_1(x, \lambda), \phi_2(x, \lambda))^T$ is a solution to (2.1a), then $(\phi_2^*(-x, -\lambda^*), -\kappa\phi_1^*(-x, -\lambda^*))^T$ is also a solution to (2.1a). Because the solutions of the scattering problem (2.1a) are uniquely determined by their respective boundary conditions (2.7), so we get the following symmetric relation

$$\psi(x, \lambda) = \begin{pmatrix} 0 & -\kappa \\ 1 & 0 \end{pmatrix} \phi^*(-x, -\lambda^*),$$  

$$\overline{\psi}(x, \lambda) = \begin{pmatrix} 0 & 1 \\ -\kappa & 0 \end{pmatrix} \overline{\phi}^*(-x, -\lambda^*).$$  

(4.9a) (4.9b)

By the definitions (4.1) we can further obtain the following symmetric relations

$$N(x, \lambda) = \begin{pmatrix} 0 & -\kappa \\ 1 & 0 \end{pmatrix} M^*(-x, -\lambda^*),$$  

$$\overline{N}(x, \lambda) = \begin{pmatrix} 0 & 1 \\ -\kappa & 0 \end{pmatrix} \overline{M}^*(-x, -\lambda^*).$$  

(4.10a) (4.10b)

From the symmetry relations (4.9) and the Wronskian expressions (4.7) of the scattering data, and the corresponding Wronskian expressions are independent of $x$, it is easy to derive the following symmetry relations

$$a(\lambda) = a^*(-\lambda^*),$$  

$$\overline{a}(\lambda) = \overline{a}^*(-\lambda^*),$$  

$$b(\lambda) = \kappa b^*(-\lambda^*).$$  

(4.11a) (4.11b) (4.11c)

The relation (4.11a) indicates that if $\lambda_1 = \xi_1 + i\eta_1$ is a zero of $a(\lambda)$ in the upper half $\lambda$ plane then $-\lambda_1^* = -\xi_1 + i\eta_1$ is a zero of $a(\lambda)$ in $\lambda \in \mathbb{C}^+$. Similarly, if $\overline{\lambda}_1$ is a zero of $\overline{a}(\lambda)$ in $\lambda \in \mathbb{C}^-$ so is $-\overline{\lambda}_1$ in the lower half $\lambda$ plane. From the Wronski determinant (4.7) we can see that as $\lambda$ approaches infinity, the limit of both $a(\lambda)$ and $\overline{a}(\lambda)$ is 1, which means that $a(\lambda)$ and $\overline{a}(\lambda)$ have only a finite number of zeros in their respective analytical half-planes. In the subsequent analysis, we only consider
the case of simple zeros and suppose \( a(\lambda) \) has \( J \) simple zeros \( \{ \lambda_l \}_1^J \) in \( \lambda \in \mathbb{C}^+ \), \( \overline{a}(\lambda) \) has \( \overline{J} \) simple zeros \( \{ \overline{\lambda}_l \}_1^{\overline{J}} \) in \( \lambda \in \mathbb{C}^- \).

5. **Inverse scattering problem: A left-right RH approach.** The inverse problem consists of constructing the potential functions \( q(x, t) \) and \( r(x, t) \) from the scattering data. The scattering data includes the reflection coefficients \( \rho(\lambda, t) \) and \( \overline{\rho}(\lambda, t) \) defined on \( \text{Im} \lambda = 0 \), eigenvalues \( \lambda_l \), \( \overline{\lambda}_l \) and norming constants \( C_l(t) \), \( \overline{C}_l(t) \). Our approach is based on solving two Riemann-Hilbert problems, which we call the left and right scattering problems and use the symmetry conditions established in the previous sections to connect the two parts. Now we give the definition of the projection operators.

**Definition 5.1.** For any integrable function \( f(\lambda) \), \( \lambda \in \mathbb{C} \), that rapidly decays to zero as \( |\lambda| \to \infty \) we define the projection operators \( P_\pm \) as

\[
P_\pm f = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(\xi)}{\xi - (\lambda \pm i\epsilon)} d\xi.
\]

One of the most important properties of the projection operators are

\[
P_\pm f_\pm = \pm f_\pm, \quad P_\pm f_{\mp} = 0,
\]

where \( f_+(z) \) and \( f_-(z) \) are analytic functions in the upper and lower complex half plane respectively satisfying \( f_\pm(z) \to 0 \) as \( |z| \to \infty \).

5.1. **Left scattering problem.** From the definitions (2.6) of \( \Phi(x, \lambda) \) and \( \Psi(x, \lambda) \), we can know the equations (4.6) can be written as follows

\[
\Phi(x, \lambda) = \Psi(x, \lambda) S_L(\lambda), \quad (5.3)
\]

where left scattering matrix \( S_L(\lambda) \) is

\[
S_L(\lambda) = \begin{pmatrix} a(\lambda) & \overline{b}(\lambda) \\ b(\lambda) & \overline{a}(\lambda) \end{pmatrix}.
\]

Combining the relations (4.1) and (4.6) we get the following relations

\[
\begin{align*}
\frac{M(x, \lambda)}{a(\lambda)} &= N(x, \lambda) + \rho(\lambda) e^{2i\lambda x} N(x, \lambda), \\
\frac{\overline{M}(x, \lambda)}{\overline{a}(\lambda)} &= \overline{N}(x, \lambda) + \overline{\rho}(\lambda) e^{-2i\lambda x} \overline{N}(x, \lambda),
\end{align*}
\]

where \( \rho(\lambda) \) and \( \overline{\rho}(\lambda) \) are the left reflection coefficients defined by

\[
\rho(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \quad \overline{\rho}(\lambda) = \frac{\overline{b}(\lambda)}{\overline{a}(\lambda)}. \quad (5.6)
\]

Taking into account the corresponding boundary conditions as well as

\[
\lim_{|\lambda| \to \infty} a(\lambda) = 1,
\]

we can find the following asymptotic relation

\[
\lim_{|\lambda| \to \infty} \left[ \frac{M(x, \lambda)}{a(\lambda)} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (5.8)
\]

Since \( M(x, \lambda) \) and \( a(\lambda) \) are both analytic functions in the upper half complex \( \lambda \) plane and \( a(\lambda) \) has simple zeros at \( \lambda = \lambda_l \), then \( \frac{M(x, \lambda)}{a(\lambda)} \) is not analytic and has...
simple poles at $\lambda = \lambda_l$. Let $\lambda = \lambda_0$ be a simple zero of $a(\lambda)$, then the Laurent series of the function $M(x, \lambda)/a(\lambda)$ around $\lambda_0$ is expanded as

$$\frac{M(x, \lambda)}{a(\lambda)} = \frac{M(x, \lambda_0)}{a'(\lambda_0) (\lambda - \lambda_0)} + \text{analytic function.} \quad (5.9)$$

We set $\lambda_l$ is a zero of $a(\lambda)$ in the upper half complex $\lambda$ plane, that is $a(\lambda_l) = 0$. By (4.7a), we have

$$M(x, \lambda_l) = b(\lambda_l)N(x, \lambda_l) e^{2i\lambda_l x}. \quad (5.10)$$

Assuming $a(\lambda)$ has $J$ simple zeros, namely the $\frac{M(x, \lambda)}{a(\lambda)}$ has $J$ simple poles. We subtract from both sides of (5.5a) the contributions from all poles and use (5.10) to find

$$\frac{M(x, \lambda)}{a(\lambda)} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{l=1}^{J} \frac{C_l N_l(x) e^{2i\lambda_l x}}{\lambda - \lambda_l} = N(x, \lambda) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{l=1}^{J} \frac{C_l N_l(x) e^{2i\lambda_l x}}{\lambda - \lambda_l} + \rho(\lambda) e^{2i\lambda x} N(x, \lambda), \quad (5.11)$$

where $C_l$ is the left norming constant and is defined as follows

$$C_l = \frac{b(\lambda_l)}{a'(\lambda_l)}. \quad (5.12)$$

The left hand side of (5.11) is an analytic function in the upper half plane and goes to zero as $|\lambda| \to \infty$ hence it forms a “+” function. Similarly, $N(x, \lambda) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ forms a “-” function. Apply $P_-$ on (5.11) to find

$$N(x, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{l=1}^{J} \frac{C_l N_l(x, \lambda_l) e^{2i\lambda_l x}}{\lambda - \lambda_l} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \rho(\xi) e^{2i\xi x} N(x, \xi) \frac{d\xi}{\xi - (\lambda - i0)}. \quad (5.13)$$

Assuming $\overline{a(\lambda)}$ has $\overline{J}$ simple zeros, the following results can be obtained using the same analysis method described above

$$N(x, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{l=1}^{\overline{J}} \frac{\overline{C_l} N_l(x, \overline{\lambda_l}) e^{-2i\overline{\lambda_l} x}}{\lambda - \overline{\lambda_l}} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \overline{\rho(\xi)} e^{-2i\xi x} \overline{N(x, \xi)} \frac{d\xi}{\xi - (\lambda + i0)}, \quad (5.14)$$

where the another left norming constant is given by

$$\overline{C_l} = \frac{\overline{b}(\overline{\lambda_l})}{\overline{a}(\overline{\lambda_l})}. \quad (5.15)$$

According to (2.1b), we derive the time evolution of the left scattering data

$$a(\lambda, t) = a(\lambda, 0), \quad \overline{a}(\lambda, t) = \overline{a}(\lambda, 0), \quad (5.16)$$

$$b(\lambda, t) = b(\lambda, 0) e^{(4i\alpha \lambda^2 - 8\beta \lambda^3) t}, \quad \overline{b}(\lambda, t) = \overline{b}(\lambda, 0) e^{(-4i\alpha \lambda^2 + 8\beta \lambda^3) t}. \quad (5.17)$$

From (5.16) and (5.17), it’s easy to obtain the time evolution of the left norming constants and the left reflection coefficients are respectively given by

$$C_l = C_l(0) e^{(4i\alpha \lambda_l^2 - 8\beta \lambda_l^3) t}, \quad \overline{C_l} = \overline{C_l}(0) e^{(-4i\alpha \lambda_l^2 + 8\beta \lambda_l^3) t}, \quad (5.18)$$

$$\rho(\lambda, t) = \frac{b(\lambda, 0)}{a(\lambda, 0)} e^{(4i\alpha \lambda^2 - 8\beta \lambda^3) t}, \quad \overline{\rho}(\lambda, t) = \frac{\overline{b}(\lambda, 0)}{\overline{a}(\lambda, 0)} e^{(-4i\alpha \lambda^2 + 8\beta \lambda^3) t}. \quad (5.19)$$
5.2. **Right scattering problem.** Now let’s think about $\psi(x, \lambda)$ and $\overline{\psi}(x, \lambda)$ in terms of $\phi(x, \lambda)$ and $\overline{\phi}(x, \lambda)$

\[
\psi(x, \lambda) = \alpha(\lambda)\phi(x, \lambda) + \beta(\lambda)\beta(x, \lambda),
\]

\[
\overline{\psi}(x, \lambda) = \overline{\alpha}(\lambda)\phi(x, \lambda) + \overline{\beta}(\lambda)\overline{\phi}(x, \lambda),
\]

where $\alpha(\lambda)$, $\overline{\alpha}(\lambda)$, $\beta(\lambda)$ and $\overline{\beta}(\lambda)$ are the right scattering data. The above system can be written in the following matrix form

\[
\Psi(x, \lambda) = \Phi(x, \lambda)S_R(\lambda),
\]

where the right scattering matrix $S_R(\lambda)$ is

\[
S_R(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \overline{\beta}(\lambda) & \overline{\alpha}(\lambda) \end{pmatrix}.
\]

Following the same steps as for the left RH above, we can formulate the corresponding RH problem on the right and find the following linear integral equations

\[
\overline{M}(x, \lambda) = \begin{pmatrix} 0 & 1 \\ -\sum_{i=1}^{J} B_i M(x, \lambda_i)e^{-2i\lambda x} & 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \overline{R}(\xi)e^{-2i\xi x}M(x, \xi)\,d\xi \end{pmatrix},
\]

\[
M(x, \lambda) = \begin{pmatrix} 1 & 0 \\ -\sum_{i=1}^{J} B_i M(x, \lambda_i)e^{2i\lambda x} & 1 - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \overline{R}(\xi)e^{2i\xi x}M(x, \xi)\,d\xi \end{pmatrix},
\]

where $R(\lambda)$ and $\overline{R}(\lambda)$ are the right reflection coefficients, $C_r$ and $\overline{C}_r$ are the right norming constants defined by

\[
R(\lambda) = \frac{\beta(\lambda)}{\alpha(\lambda)}, \quad \overline{R}(\lambda) = \frac{\overline{\beta}(\lambda)}{\overline{\alpha}(\lambda)},
\]

\[
B_i = \frac{\beta(\lambda_i)}{\alpha(\lambda_i)}, \quad \overline{B}_i = \frac{\overline{\beta}(\lambda_i)}{\overline{\alpha}(\lambda_i)}.
\]

Similarly, we can get the time evolution of the right scattering data

\[
\alpha(\lambda, t) = \alpha(\lambda, 0), \quad \overline{\alpha}(\lambda, t) = \overline{\alpha}(\lambda, 0),
\]

\[
\beta(\lambda, t) = \beta(\lambda, 0)e^{(-4i\alpha\lambda^2+8\beta\lambda^3)t}, \quad \overline{\beta}(\lambda, t) = \overline{\beta}(\lambda, 0)e^{(4i\alpha\lambda^2-8\beta\lambda^3)t}.
\]

The time evolution of the right norming constants and the right reflection coefficients are respectively given by

\[
B_i(t) = B_i(0)e^{(-4i\alpha\lambda_i^2+8\beta\lambda_i^3)t}, \quad \overline{B}_i(t) = \overline{B}_i(0)e^{(4i\alpha\lambda_i^2-8\beta\lambda_i^3)t},
\]

\[
R(\lambda, t) = \frac{\beta(\lambda, 0)}{\alpha(\lambda, 0)}e^{(-4i\alpha\lambda^2+8\beta\lambda^3)t}, \quad \overline{R}(\lambda, t) = \frac{\overline{\beta}(\lambda, 0)}{\overline{\alpha}(\lambda, 0)}e^{(4i\alpha\lambda^2-8\beta\lambda^3)t}.
\]

Next we consider the relationship between the reflectance coefficients. According to the equations (5.3) and (5.21), we find that the left and right scattering matrices satisfy the relationship $S_R(\lambda) = S_L^{-1}(\lambda)$, namely

\[
\alpha(\lambda) = a(\lambda), \quad \overline{\alpha}(\lambda) = \overline{a}(\lambda),
\]

\[
\beta(\lambda) = -b(\lambda), \quad \overline{\beta}(\lambda) = -\overline{b}(\lambda).
\]
Combining the symmetric relations (4.11) we can get
\[ R(\lambda) = \frac{\beta(\lambda)}{\alpha(\lambda)} = -\frac{\beta(\lambda)}{\alpha(\lambda)} = -\kappa b^*(\lambda^*) = -\kappa b^*(\lambda^*), \] (5.31a)
\[ \overline{R}(\lambda) = \frac{\beta(\lambda)}{\alpha(\lambda)} = -\frac{\beta(\lambda)}{\alpha(\lambda)} = -\kappa \overline{b}^*(\lambda^*) = -\kappa \overline{b}^*(\lambda^*). \] (5.31b)

6. Recovery of the potentials \( r(x) \) and \( q(x) \). According to the equation (5.13), we can obtain the large \( \lambda \) asymptotic behavior of \( \mathcal{N}_2(x, \lambda) \)
\[ \mathcal{N}_2(x, \lambda) \sim \frac{1}{\lambda} \sum_{i=1}^{J} C_i N_2(x, \lambda_i) e^{2i\lambda_i x} - \frac{1}{2\pi i \lambda} \int_{-\infty}^{\infty} \rho(\xi) e^{2i\xi x} N_2(x, \xi) d\xi, \] (6.1)
from (4.5) we have
\[ \mathcal{N}_2(x, \lambda) \sim \frac{r(x)}{2i \lambda}. \] (6.2)
Comparing (6.1) with (6.2) we have the following result
\[ r(x) = -2i \left( \sum_{i=1}^{J} C_i N_2(x, \lambda_i) e^{2i\lambda_i x} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \rho(\xi) e^{2i\xi x} N_2(x, \xi) d\xi \right). \] (6.3)
From the symmetric relation (4.10b) we can get
\[ \mathcal{M}_1(x, \lambda) = -\kappa \mathcal{N}_2(-x, -\lambda^*), \] (6.4)
by (4.4) we have
\[ \mathcal{M}_1(x, \lambda) \sim \frac{q(x)}{2i \lambda}, \] (6.5)
that is
\[ \mathcal{N}_2(-x, -\lambda^*) \sim -\kappa \frac{q(x)}{2i \lambda}. \] (6.6)
From the equation (6.1) we know
\[ \mathcal{N}_2(-x, -\lambda^*) \sim -\frac{1}{\lambda} \sum_{i=1}^{J} C_i^* N_2^*(-x, \lambda_i) e^{2i\lambda_i^* x} - \frac{1}{2\pi i \lambda} \int_{-\infty}^{\infty} \rho^*(\xi) e^{2i\xi x} N_2^*(-x, \xi) d\xi. \] (6.7)
Comparing (6.6) with (6.7) we find
\[ q(x) = 2i\kappa \sum_{i=1}^{J} C_i^* N_2^*(-x, \lambda_i) e^{2i\lambda_i^* x} + \frac{\kappa}{\pi} \int_{-\infty}^{\infty} \rho^*(\xi) e^{2i\xi x} N_2^*(-x, \xi) d\xi. \] (6.8)
According to (6.3) and (6.8), it can be seen that the symmetry relation \( r(x) = \kappa q^*(-x) \) still holds.

7. Soliton solutions. In order to obtain the pure soliton solutions, in this section we consider the case where the reflection coefficients are 0, that is \( \rho(\lambda) = \overline{\rho}(\lambda) = 0 \). On the one hand, observing (4.10a) we can get
\[ N_1(x, \lambda_l) = -\kappa M_2^*(-x, -\lambda_l^*), \] (7.1)
where \( \lambda = \lambda_l \) is a zero of \( a(\lambda) \). By (5.10) we have
\[ M_2(x, \lambda_l) = b_l e^{2i\lambda_l x} N_2(x, \lambda_l), \] (7.2)
where \( b_l = b(\lambda_l) \). According to (7.1) and (7.1) we have
\[ N_1(x, \lambda_l) = -\kappa b_l^* e^{-2i\lambda_l x} N_2^*(-x, -\lambda_l^*). \] (7.3)
On the other hand, by (4.10a) we can know
\[ N_2(x, \lambda_i) = M_1^*(-x, -\lambda_i^*) , \]  
(7.4)
by (5.10) we have
\[ M_1(x, \lambda_i) = b_i e^{2i\lambda_i x} N_1(x, \lambda_i) . \]  
(7.5)
Equations (7.4) and (7.5) give
\[ N_2(x, \lambda_i) = b_i^* e^{-2i\lambda_i x} N_1^*(-x, -\lambda_i^*) . \]  
(7.6)
Substituting the equation (7.6) into (7.3), we can get
\[ - \kappa |b_i|^2 = 1 . \]  
(7.7)
This result implies that these types of soliton solutions can only be obtained when \( \kappa = -1 \), and \( b_i \) is arbitrary complex number of length 1, so let us set
\[ b_i = e^{i\theta i} . \]  
(7.8)
Similarly, we can set
\[ \bar{b}_i = e^{i\bar{\theta} i} , \]  
(7.9)
where \( \bar{b}_i = \bar{b}(\lambda_i) \).

In order to get norming constants \( C_l \) and \( \bar{C}_l \), we need to calculate the values of \( a_i \) and \( \bar{a}_i \). We assume that \( J = \bar{J} \), that is \( a(\lambda) \) and \( \bar{a}(\lambda) \) have simple zeros \( k_l, \bar{k}_l \), \( (l = 1, 2 \ldots J) \) respectively, then we can get the following formulas
\[ a' (\lambda_n) = \lim_{\lambda \to \lambda_n} \frac{\prod_{j=1}^{J} (\lambda - \lambda_j)}{\prod_{j=1}^{J} (\lambda - \bar{\lambda}_j)} \sum_{l=1}^{J} \frac{\lambda_l - \bar{\lambda}_l}{(\lambda - \lambda_l)(\lambda - \bar{\lambda}_l)} , \]  
(7.10a)
\[ \bar{a}' (\bar{\lambda}_n) = \lim_{\lambda \to \bar{\lambda}_n} \frac{\prod_{j=1}^{J} (\lambda - \lambda_j)}{\prod_{j=1}^{J} (\lambda - \bar{\lambda}_j)} \sum_{l=1}^{J} \frac{\bar{\lambda}_l - \lambda_l}{(\lambda - \lambda_l)(\lambda - \bar{\lambda}_l)} . \]  
(7.10b)
So we can get the time evolution formulas of norming constants \( C_l \) and \( \bar{C}_l \)
\[ C_l = \frac{\bar{b}(\lambda_l)}{d (\lambda_l)} e^{(4i\alpha \lambda_l^2 - 8i\nu \lambda_l^3)t} , \]  
(7.11a)
\[ \bar{C}_l = \frac{\bar{b} (\bar{\lambda}_l)}{\bar{a} (\lambda_l)} e^{(-4i\alpha \bar{\lambda}_l^2 + 8i\nu \bar{\lambda}_l^3)t} . \]  
(7.11b)
Finally, we can obtain the soliton solutions formula for the S-NHE (1.4) as
\[ q(x, t) = -2i \sum_{l=1}^{J} C_l^* N_2^*(-x, \lambda_l) e^{2i\lambda_l x} , \]  
(7.12)
where the norming constants can be obtained by (7.11), and \( N_2^*(-x, \lambda_l) \) can be obtained by solving the following linear equations
\[ N_2(-x, \lambda_l) = 1 + \sum_{l=1}^{J} \frac{C_l e^{2\bar{\lambda}_l x}}{\lambda_l - \bar{\lambda}_l} \sum_{l=1}^{J} C_j N_2(-x, \lambda_j) e^{-2i\lambda_j x} \]  
(7.13)
\[ = 1 + \sum_{l=1}^{J} \frac{C_l C_j e^{2\bar{\lambda}_l x - 2i\lambda_j x}}{(\lambda_l - \lambda_j)(\lambda_l - \bar{\lambda}_l)} N_2(-x, \lambda_j) . \]

Next, we classify the scattering data \( a(\lambda) \) and \( \bar{a}(\lambda) \) according to their zero point types. First, we divide the simple zeros of \( a(\lambda) \) in \( \mathbb{C}^+ \) into three categories:
(i) $J = N_1$: $a(\lambda)$ has $N_1$ simple zeros $\lambda_l \ (1 \leq l \leq N_1)$ in $\mathbb{C}^+$, and $\lambda_l$ are pure imaginary, that is $\lambda_l = i\eta_l \ (\eta_l > 0)$.

(ii) $J = 2N_2$: $a(\lambda)$ has $2N_2$ simple zeros $\lambda_l \ (1 \leq l \leq 2N_2)$ in $\mathbb{C}^+$, where $\lambda_l = \xi_l + i\eta_l \ (\xi_l \neq 0, \eta_l > 0)$ and $\lambda_{N_2+j} = -\lambda_j^* \ (1 \leq j \leq N_2)$.

(iii) $J = N_1 + 2N_2$: $a(\lambda)$ has $N_1 + 2N_2$ simple zeros $\lambda_l \ (1 \leq l \leq N_1 + 2N_2)$ in $\mathbb{C}^+$, where $\lambda_l = i\eta_l \ (1 \leq l \leq N_1)$ are pure imaginary and $\lambda_l = \xi_l + i\eta_l \ (N_1 + 1 \leq l \leq N_1 + 2N_2)$ are not pure imaginary, and we have $\lambda_{N_1+N_2+j} = -\lambda_j^* \ (1 \leq j \leq N_2)$.

Similarly, the simple zeros of $\pi(\lambda)$ in $\mathbb{C}^-$ can be divided into the above three categories. We have assumed that $a(\lambda)$ has the same number of zeros as $\pi(\lambda)$, that is, $J = J$. Combining $a(\lambda)$ and $\pi(\lambda)$ zeros classification, we can finally get nine types of the solutions. We will list only three of these special cases, namely, $a(\lambda)$ has the same number of pure imaginary zeros as $\pi(\lambda)$. Corresponding to these three cases, three types of soliton solutions can be obtained for the S-NHE (1.4).

**Case 1** Let’s assume that $a(\lambda)$ and $\pi(\lambda)$ have $J = N_1$ pure imaginary zeros $\lambda_l = i\eta_l \ (\eta_l > 0)$ and $\lambda_l^* = i\eta_l \ (\eta_l < 0)$, $1 \leq l \leq N_1$ in their respective analytic half planes. In this case, the $N_1$-soliton solution formula for the S-NHE (1.4) is

$$q(x, t) = -2i \sum_{l=1}^{N_1} C_l^* N_2^*(-x, i\eta_l) e^{2\eta_l x}. \quad (7.14)$$

Now let’s consider the simplest situation that occurs when $N_1 = 1$ in (7.14). Then the single-soliton solution for the S-NHE (1.4) is

$$q(x, t) = -\frac{2i C_1^* (\eta_1 - \eta_1^*) e^{2\eta_1 x}}{(\eta_1 - \eta_1^*)^2 - C_1^* C_1 e^{2(\eta_1 - \eta_1^*) x}}. \quad (7.15)$$

where the norming constants can be obtained by (7.11)

$$C_1 = i e^{i\theta_1} (\eta_1 - \eta_1^*) e^{(-4i\alpha_0 \eta_1^2 + 8i\beta \eta_1^3) t}, \quad (7.16a)$$

$$C_1^* = i e^{i\theta_1} (\eta_1 - \eta_1^*) e^{(4i\alpha \eta_1^2 - 8i\beta \eta_1^3) t}. \quad (7.16b)$$

And from the formula (7.13), we can obtain

$$N_2^*(-x, i\eta_l) = \frac{(\eta_l - \eta_l^*)^2}{(\eta_l - \eta_l^*)^2 - C_1^* C_1 e^{2(\eta_1 - \eta_1^*) x}}. \quad (7.17)$$

By choosing appropriate parameters, we can get a single-breather solution for the S-NHE (1.4), the plots of this breather behavior are shown in Figure 1. If $a(\lambda)$ and $\pi(\lambda)$ only have pure imaginary zeros, we have the first-order time periodic breathers in Figure 1.

**Case 2** Let’s assume that $a(\lambda)$ and $\pi(\lambda)$ have $J = 2N_2$ simple zeros $\lambda_l = \xi_l + i\eta_l \ (\xi_l \neq 0, \eta_l > 0)$ and $\lambda_l^* = \xi_l + i\eta_l \ (\xi_l \neq 0, \eta_l < 0), 1 \leq l \leq 2N_2$ in their respective analytic half planes. And there’s the relationship $\lambda_{N_2+j} = -\lambda_j^* \ (1 \leq j \leq N_2)$ and $\lambda_{N_1+j} = -\lambda_j^* \ (1 \leq j \leq N_2)$. In this case, the $2N_2$-soliton solution formula for the S-NHE (1.4) is

$$q(x, t) = -2i \sum_{l=1}^{N_2} \{C_l^* N_2^*(-x, \lambda_l) e^{2i\lambda_l x} + C_{l+N_2}^* N_2^*(-x, -\lambda_l^*) e^{-2i\lambda_l x}\}. \quad (7.18)$$

In what follows, we shall illustrate the soliton solutions in this case explicitly. To this end, we set $N_2 = 1$ in (7.18). Then, we obtain two-soliton solution for the
Figure 1. The single-breather solution (7.15) with $\eta_1 = 7$, $\eta_1 = -2$, $\theta_1 = \frac{\pi}{2}$, $\bar{\theta}_1 = \frac{\pi}{5}$, $\alpha = 5$, $\beta = 1$. (a, b, c) The local structure, density and intensity profiles of the single-soliton solution $|q(x, t)|^2$.

S-NHE (1.4)

$$q(x, t) = -2i\{C_1^2 N_2^*(x, \lambda_1)e^{2i\lambda_1^* x} + C_2^2 N_2^*(-x, -\lambda_1^*)e^{-2i\lambda_1 x}\}, \quad (7.19)$$

where

$$C_1 = e^{i\theta_1}(\lambda_1 - \lambda_1^*)(\lambda_1 + \lambda_1^*)/(\lambda_1 + \lambda_1^*),$$

$$C_2 = e^{i\theta_2}(-\lambda_1^* - \lambda_1)(-\lambda_1^* + \lambda_1^*)/(\lambda_1^* - \lambda_1),$$

$$C_1 = e^{i\bar{\theta}_1}(\lambda_1 - \lambda_1^*)(\lambda_1 + \lambda_1^*)/(\lambda_1 + \lambda_1^*),$$

$$C_2 = e^{i\bar{\theta}_2}(-\lambda_1^* - \lambda_1)(-\lambda_1^* + \lambda_1^*)/(\lambda_1^* - \lambda_1).$$

We can obtain the following system of equations by (7.13)

$$\begin{align*}
m_{11} N_2(-x, \lambda_1) + m_{12} N_2(-x, -\lambda_1^*) &= 1, \\
m_{21} N_2(-x, \lambda_1) + m_{22} N_2(-x, -\lambda_1^*) &= 1,
\end{align*} \quad (7.20)$$

where

$$m_{11} = 1 - A_{11}B_{11} - A_{12}B_{12},$$

$$m_{21} = -A_{21}B_{11} - A_{22}B_{12},$$

$$m_{12} = -A_{11}B_{21} - A_{12}B_{22},$$

$$m_{22} = 1 - A_{21}B_{21} - A_{22}B_{22},$$
INVERSE SCATTER TRANSFORM OF NONLOCAL HIROTA EQUATION

\[
A_{11} = \frac{C_1 e^{2i\lambda_1 x}}{\lambda_1 - \lambda_1}, \quad B_{11} = \frac{C_1 e^{-2i\lambda_1 x}}{\lambda_1 - \lambda_1},
\]
\[
A_{12} = \frac{C_2 e^{-2i\lambda_1 x}}{\lambda_1 + \lambda_1}, \quad B_{12} = \frac{C_2 e^{2i\lambda_1 x}}{-\lambda_1 - \lambda_1},
\]
\[
A_{21} = \frac{C_1 e^{2i\lambda_1 x}}{-\lambda_1 - \lambda_1}, \quad B_{21} = \frac{C_2 e^{-2i\lambda_1 x}}{-\lambda_1 + \lambda_1},
\]
\[
A_{22} = \frac{C_2 e^{-2i\lambda_1 x}}{-\lambda_1 + \lambda_1}, \quad B_{22} = \frac{C_2 e^{2i\lambda_1 x}}{-\lambda_1 + \lambda_1}.
\]

Solving the above system of equations, we can get

\[
N_2(-x, \lambda_1) = \frac{m_{22} - m_{12}}{m_{11} m_{22} - m_{12} m_{21}}, \quad (7.21)
\]
\[
N_2(-x, -\lambda_1^*) = \frac{m_{11} - m_{21}}{m_{11} m_{22} - m_{12} m_{21}}. \quad (7.22)
\]

Substituting the above expression into (7.19) gives the final expression for the two-soliton solution. The interaction properties of the two-soliton solution (7.19) are depicted in Figure 2. If \(a(\lambda)\) and \(\pi(\lambda)\) have no pure imaginary zeros, we have the second-order breathers in Figure 2. Different from Figure 1, Figure 2 not only shows the breathing state, but also presents a distinct phenomenon of background waves.

**Figure 2.** The two-soliton solution (7.19) with \(\lambda_1 = 1.1 + 0.8i, \lambda_1 = 2 - i, \theta_1 = \theta_2 = \theta_1 = \bar{\theta}_2 = 2\pi, \alpha = 1, \beta = 1. (a, b, c) The local structure, density and intensity profiles with different time of the two-soliton solution \(|q(x,t)|^2\).**

**Case 3** Supposing that \(a(\lambda)\) and have \(J = N_1 + 2N_2\) simple zeros in their respective analytic half planes, where \(\lambda_l = i\eta_l\) (\(\eta_l > 0\)) and \(\bar{\lambda}_l = i\eta_l\) (\(\eta_l < 0\)) \(1 \leq l \leq N_1\) are pure imaginary. The rest \(\lambda_l = \xi_l + i\eta_l, \lambda_l = \xi_l + i\eta_l, (N_1 + 1 \leq l \leq N_1 + 2N_2)\) are not pure imaginary, and we have \(\lambda_{N_1+N_2+j} = -\lambda_{N_1+j}^* (1 \leq j \leq N_2)\) and \(\lambda_{N_1+N_2+j} = -\lambda_{N_1+j}^* (1 \leq j \leq N_2)\). Then the corresponding \((N_1 + 2N_2)\)-soliton
solution formula is

\[ q(x, t) = -2i \sum_{i=1}^{N_1} C_i^* N_2^*(-x, \lambda_i)e^{2i\lambda_i x} - 2i \sum_{j=1}^{N_2} (C_{j+N_1}^*) N_2^*(-x, \lambda_j+N_1)e^{2i\lambda_j+N_1 x} + C_{j+N_1+N_2}^* N_2^*(-x, -\lambda_j+N_1)e^{-2i\lambda_j+N_1 x}. \] 

(7.23)

We consider the simplest situation that occurs when \( N_1 = 1, N_2 = 1 \) in (7.23). Then we can obtain the following three-soliton solution formula

\[ q(x, t) = -2i \{ C_1^* N_2^*(-x, \lambda_1)e^{2i\lambda_1 x} + C_2^* N_2^*(-x, \lambda_2)e^{2i\lambda_2 x} + C_3^* N_2^*(-x, -\lambda_2)e^{-2i\lambda_2 x} \}, \]

(7.24)

where

\[ C_1 = \frac{e^{i\theta_1}(\lambda_1 - \bar{\lambda}_1)(\lambda_1 - \bar{\lambda}_2)(\lambda_1 + \bar{\lambda}_2)}{(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)} e^{(4i\alpha\lambda_1^2 - 8\beta\lambda_1^2)t}, \]

\[ C_2 = \frac{e^{i\theta_2}(\lambda_2 - \bar{\lambda}_1)(\lambda_2 - \bar{\lambda}_2)(\lambda_2 + \bar{\lambda}_2)}{(\lambda_2 - \lambda_1)(\lambda_2 + \lambda_2)} e^{(4i\alpha\lambda_2^2 - 8\beta\lambda_2^2)t}, \]

\[ C_3 = \frac{e^{i\theta_3}(\lambda_2 - \bar{\lambda}_1)(\lambda_2 - \bar{\lambda}_2)(\lambda_2 + \bar{\lambda}_2)}{(\lambda_2 - \lambda_1)(\lambda_2 + \lambda_2)} e^{(4i\alpha\lambda_2^2 + 8\beta\lambda_2^2)t}, \]

\[ \bar{C}_1 = \frac{e^{i\bar{\theta}_1}(\bar{\lambda}_1 - \lambda_1)(\bar{\lambda}_1 - \lambda_2)(\bar{\lambda}_1 + \lambda_2)}{(\bar{\lambda}_1 - \lambda_2)(\bar{\lambda}_1 + \lambda_2)} e^{(-4i\alpha\bar{\lambda}_1^2 + 8\beta\bar{\lambda}_1^2)t}, \]

\[ \bar{C}_2 = \frac{e^{i\bar{\theta}_2}(\bar{\lambda}_2 - \lambda_1)(\bar{\lambda}_2 - \lambda_2)(\bar{\lambda}_2 + \lambda_2)}{(\bar{\lambda}_2 - \lambda_1)(\bar{\lambda}_2 + \lambda_2)} e^{(-4i\alpha\bar{\lambda}_2^2 + 8\beta\bar{\lambda}_2^2)t}, \]

\[ \bar{C}_3 = \frac{e^{i\bar{\theta}_3}(\bar{\lambda}_2 - \lambda_1)(\bar{\lambda}_2 - \lambda_2)(\bar{\lambda}_2 + \lambda_2)}{(\bar{\lambda}_2 - \lambda_1)(\bar{\lambda}_2 + \lambda_2)} e^{(-4i\alpha\bar{\lambda}_2^2 - 8\beta\bar{\lambda}_2^2)t}, \]

We can obtain the following system of equations by (7.13)

\[
\begin{align*}
& m_{11} N_2(-x, \lambda_1) + m_{12} N_2(-x, \lambda_2) + m_{13} N_2(-x, -\lambda_2) = 1, \\
& m_{21} N_2(-x, \lambda_1) + m_{22} N_2(-x, \lambda_2) + m_{23} N_2(-x, -\lambda_2) = 1, \\
& m_{31} N_2(-x, \lambda_1) + m_{32} N_2(-x, \lambda_2) + m_{33} N_2(-x, -\lambda_2) = 1, \\
\end{align*}
\]

(7.25)

with

\[
\begin{align*}
m_{11} &= 1 - A_{11} B_{11} - A_{12} B_{12} - A_{13} B_{13}, \\
m_{21} &= - A_{21} B_{11} - A_{22} B_{12} - A_{23} B_{13}, \\
m_{31} &= - A_{31} B_{11} - A_{32} B_{12} - A_{33} B_{13}, \\
m_{12} &= - A_{11} B_{21} - A_{12} B_{22} - A_{13} B_{23}, \\
m_{22} &= 1 - A_{21} B_{21} - A_{22} B_{22} - A_{23} B_{23}, \\
m_{32} &= - A_{31} B_{21} - A_{32} B_{22} - A_{33} B_{23}, \\
m_{13} &= - A_{11} B_{31} - A_{12} B_{32} - A_{13} B_{33}, \\
m_{23} &= - A_{21} B_{31} - A_{22} B_{32} - A_{23} B_{33}, \\
m_{33} &= 1 - A_{31} B_{31} - A_{32} B_{32} - A_{33} B_{33}, \\
\end{align*}
\]

\[
A_{i,j} = \frac{\bar{C}_j e^{2i\bar{\lambda}_j x}}{\lambda_i - \bar{\lambda}_j}, \quad B_{i,j} = \frac{C_i e^{-2i\lambda_i x}}{\bar{\lambda}_j - \lambda_i}, \quad 1 \leq i, j \leq 3.
\]
Solving the above system of equations, we can get

\[ N_2(-x, i\eta_1) = \frac{\det(M_1)}{\det(M)}, \quad N_2(-x, \lambda_2) = \frac{\det(M_2)}{\det(M)}, \quad N_2(-x, -\lambda_2^*) = \frac{\det(M_3)}{\det(M)}, \quad (7.26) \]

with

\[ M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}, \quad (7.27) \]

\[ M_1 = \begin{pmatrix} 1 & m_{12} & m_{13} \\ 1 & m_{22} & m_{23} \\ 1 & m_{32} & m_{33} \end{pmatrix}, \quad M_2 = \begin{pmatrix} m_{11} & 1 & m_{13} \\ m_{21} & 1 & m_{23} \\ m_{31} & 1 & m_{33} \end{pmatrix}, \quad M_3 = \begin{pmatrix} m_{11} & m_{12} & 1 \\ m_{21} & m_{22} & 1 \\ m_{31} & m_{32} & 1 \end{pmatrix}. \quad (7.28) \]

Substituting the above expression into (7.24), we can get the final expression for the three-soliton solution. The interaction behaviors of this solution are demonstrated in Figure 3. When \( a(\lambda) \) and \( \bar{a}(\lambda) \) have both pure and impure imaginary zeros, we will have first-order time periodic breathers (in Figure 3) similar to Figure 1. Combined with Figures 1, 2 and 3, we guess that as long as \( a(\lambda) \) and \( \bar{a}(\lambda) \) have pure imaginary zeros, the soliton solutions may present first-order breathers.

![Figure 3](image-url)

(3) The three-soliton solution (7.24) with \( \lambda_1 = 1.2i, \lambda_2 = 1.1 + 2i, \lambda_3 = -i, X_2 = 0.8 - i, \theta_j = \bar{\theta}_j = \pi, (1 \leq j \leq 3) \alpha = \beta = 1. \) (a, b, c) The local structure, density and intensity profiles with different time of the three-soliton solution \(|q(x, t)|^2\).

8. Eigenvalues and conserved quantities under some special initial conditions. In the previous sections we considered the pure soliton solutions for cases where the reflection coefficients \( \rho(\lambda) = 0, \bar{p}(\lambda) = 0. \) However, a non-zero initial value of \( q(x, 0) \) will result in a non-zero reflection coefficient, in which case, we can’t use the IST method. In this section we will introduce some special initial conditions, and take one of them as an example to calculate the conserved quantity. We first consider the box initial condition

\[ q(x, 0) = \begin{cases} 0, & x \in (-\infty, 0), \\ h, & x \in (0, L), \\ 0, & x \in (L, \infty). \end{cases} \quad (8.1) \]
where \( h \) and \( L \) are positive real constants. Under the symmetry relation \( r(x,0) = -q^*(-x,0) \), we get the initial data of \( r(x,t) \)

\[
r(x,0) = \begin{cases} 
0, & x \in (-\infty, -L), \\
-h, & x \in (-L, 0), \\
0, & x \in (0, \infty). 
\end{cases} 
\]

(8.2)

Then we consider the following arcuated initial condition

\[
q(x,0) = \begin{cases} 
0, & x \in (-\infty, 0), \\
-x^2 + Lx, & x \in (0, L), \\
0, & x \in (L, \infty), 
\end{cases} 
\]

where \( L \in \mathbb{R} (L > 0) \). Under the symmetry relation \( r(x,0) = -q^*(-x,0) \), we get the initial data of \( r(x,t) \)

\[
r(x,0) = \begin{cases} 
0, & x \in (-\infty, -L), \\
x^2 + Lx, & x \in (-L, 0), \\
0, & x \in (0, \infty). 
\end{cases} 
\]

(8.4)

Finally we consider the following triangular initial condition

\[
q(x,0) = \begin{cases} 
0, & x \in (-\infty, 0), \\
L - |x-L|, & x \in (0, 2L), \\
0, & x \in (2L, \infty), 
\end{cases} 
\]

(8.5)

where \( L \in \mathbb{R} (L > 0) \). Under the symmetry relation \( r(x,0) = -q^*(-x,0) \), we get the initial data of \( r(x,t) \)

\[
r(x,0) = \begin{cases} 
0, & x \in (-\infty, -2L), \\
-L + |x+L|, & x \in (-2L, 0), \\
0, & x \in (0, \infty). 
\end{cases} 
\]

(8.6)

Taking the box initial condition as an example, we introduce the method to solve the conserved quantity of the corresponding equations. According to the time dependent scattering problem (2.1a), we have

\[
\begin{cases} 
F_{1,x} = -i\lambda F_1 + q(x,t)F_2, \\
F_{2,x} = i\lambda F_2 + r(x,t)F_1. 
\end{cases} 
\]

(8.7)

Solving the linear differential equations and considering the boundary condition (2.7), we can obtain

\[
\begin{pmatrix} 
\phi_1(x,\lambda) \\
\phi_2(x,\lambda) 
\end{pmatrix} = \begin{pmatrix} 
\frac{h}{2i\lambda} c_1 e^{i\lambda x} + c_2 e^{-i\lambda x} \\
c_1 e^{i\lambda x} 
\end{pmatrix}, \quad 0 < x < L, 
\]

(8.8)

\[
\begin{pmatrix} 
\hat{\phi}_1(x,\lambda) \\
\hat{\phi}_2(x,\lambda) 
\end{pmatrix} = \begin{pmatrix} 
c_2 e^{i\lambda x} + \frac{h}{2i\lambda} \hat{c}_1 e^{-i\lambda x} 
\end{pmatrix}, \quad -L < x < 0. 
\]

(8.9)

Considering the values of the critical points \( x = -L \) and \( x = 0 \) and the boundary condition (2.7), we obtain the following relation

\[
\hat{c}_1 = 1, \quad \hat{c}_2 = -\frac{h}{2i\lambda} e^{2i\lambda L}, 
\]

(8.10)

\[
c_1 = \frac{h}{2i\lambda} (1 - e^{2i\lambda L}), \quad c_2 = 1 + \left( \frac{h}{2i\lambda} \right)^2 (e^{2i\lambda L} - 1). 
\]

(8.11)
When \( x > L \), we can obtain the following equation by (2.7b) and (4.6a)

\[
\phi(x, t) = \left( \frac{a(\lambda)e^{-i\lambda x}}{b(\lambda)e^{i\lambda x}} \right).
\]  

(8.12)

Combining the values of the equations (8.8) and (8.12) at \( x = L \), we can get

\[
a(\lambda) = 1 - \left( \frac{h}{2i\lambda} \right)^2 (e^{2i\lambda L} - 1)^2,
\]  

(8.13)

\[
b(\lambda) = -he^{i\lambda L} \frac{\sin(\lambda L)}{\lambda},
\]  

(8.14)

then the zeros of \( a(\lambda) \) can be given implicitly by the following equation

\[
e^{2i\lambda L} - 1 \pm \frac{2i\lambda}{h} = 0.
\]  

(8.15)

In addition, the asymptotic behavior of \( a(\lambda) \) for large \( \lambda \) and small \( \lambda \) can be deduced from (8.13)

\[
a(\lambda) \sim 1 - \frac{h^2}{(2i\lambda)^2}, \quad \lambda \to \infty,
\]  

(8.16)

\[
a(\lambda) \sim 1 - h^2 L^2, \quad \lambda \to 0.
\]  

(8.17)

According to the large \( \lambda \) asymptotic behavior of \( a(\lambda) \) and the relation (3.6), we find that the conserved quantity satisfies the following equation

\[
C_{2n} = 0, \quad C_{2n+1} = -\frac{h^{2n+2}}{n+1}, \quad n = 0, 1, 2, \ldots
\]  

(8.18)

Similarly, the same method can be used to obtain the conserved quantity of the equation under the other two initial conditions.

9. Conclusions. The multi-soliton solutions and some properties of the S-NHE (1.4) are obtained by the inverse scattering transform. At the beginning, we treated \( t \) as a parameter and ignore it for the moment. Based on the Lax pair (2.1) of the S-NHE, we have got the Jost solutions and infinite number of conservation laws of this nonlocal equation. We have obtained a critical symmetric relation by analyzing the direct scattering problem. Then we have successively established the left and right scattering problems and obtain the time evolution of the left and right scattering data by considering the time \( t \). Using the corresponding symmetry condition, we have obtained the soliton solutions formula of S-NHE (1.4) in reflectionless case. According to the symmetry of scattering data (4.11), the zeros of \( a(\lambda) \) and \( \bar{a}(\lambda) \) have been divided into three categories, respectively, and the corresponding multi-soliton solutions formulas can be divided into nine categories. Taking three kinds of them as examples, we have written out the specific soliton solutions formulas. In addition, we have used images to describe these solutions by selecting appropriate parameters. Finally, we have given the method of finding the conserved quantity in reflection case.

Taking the parameters of the S-NHE (1.4) as \( \alpha = -1 \) and \( \beta = 0 \), we can obtain the nonlocal NLS equation (1.3) studied by IST method in references [1, 2]. In addition, when \( \alpha = -1 \) and \( \beta = 0 \), the expressions of scattering data, reflection coefficients, norming constants and soliton solutions are also consistent with the results of NLS equation in [1, 2]. On the basis of references [1, 2], we classify soliton solutions according to the zero types of scattering data.
In reference [8], the authors constructed explicit multisoliton solutions for the S-NHE (1.4) by employing Hirota’s direct method as well as Darboux-Crum transformations. They obtained the following two types of one-soliton solutions by employing Hirota’s direct method

\[ q_{st}^{(1)} = \frac{\lambda (\mu - \mu^*)^2 \tau_{\mu,\gamma}}{(\mu - \mu^*)^2 + |\lambda|^2 \tau_{\mu,\gamma} \tau_{\mu,\gamma}^*}, \quad q_{nonst}^{(1)} = \frac{(\mu + \nu) \tau_{\mu,i\gamma}}{1 + \tau_{\mu,i\gamma}^2 - \nu - i\theta}. \]

Moreover, they also obtained two types of one-soliton solutions by Darboux-Crum transformations, see equations (4.27) and (4.29) in reference [8] for more details. By comparing the expressions of the solutions, we found that the two groups of one-soliton solutions in reference [8] are different from the one-soliton solution (7.15) obtained in our work. Similarly, two-soliton solutions and even \( n \)-soliton solutions are also different. Therefore, we get some new results different from reference [8]. In our work, we consider the soliton solutions for the S-NHE (1.4) under the restriction of “when \( |x| \to \infty \), \( q(x,t) \) fast decay to zero \( \)”, and this result may have potential applications in the physical sense of a special framework.

Acknowledgment. The authors would like to thank the editor and the referees for their valuable comments and suggestions.

REFERENCES

[1] M. J. Ablowitz and Z. H. Musslimani, Integrable nonlocal nonlinear Schrödinger equation, Phys. Rev. Lett., 110 (2013), 064105, 5pp.
[2] M. J. Ablowitz and Z. H. Musslimani, Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation, Nonlinearity, 29 (2016), 915–946.
[3] M. J. Ablowitz, B. Feng, X. Luo and Z. H. Musslimani, Inverse scattering transform for the nonlocal reverse space-time nonlinear Schrödinger equation, Theor. Math. Phys., 196 (2018), 1241–1267.
[4] M. J. Ablowitz, X. Luo and Z. H. Musslimani, Inverse scattering transform for the nonlocal nonlinear Schrödinger equation with nonzero boundary conditions, J. Math. Phys., 59 (2018), 011501.
[5] G. P. Agrawal, Nonlinear Fiber Optics, Springer, Berlin, 2000.
[6] D. Anderson and M. Lisak, Nonlinear asymmetric self-phase modulation and self-steepening of pulses in long optical waveguides, Phys. Rev. A, 27 (1983), 1393–1398.
[7] D. J. Benney and A. C. Newell, The propagation of nonlinear wave envelopes, J. Math. Phys., 46 (1967), 133–139.
[8] J. Cen, F. Correa and A. Fring, Integrable nonlocal Hirota equations, J. Math. Phys., 60 (2019), 081508, 18pp.
[9] H. Chen, Y. Lee and C. Liu, Integrability of nonlinear hamiltonian systems by inverse scattering method, Phys. Scr., 20 (1979), 490–492.
[10] A. S. Fokas, Integrable multidimensional versions of the nonlocal nonlinear Schrödinger equation, Nonlinearity, 29 (2016), 319–324.
[11] Martin V. Goldman, Strong turbulence of plasma waves, Rev. Mod. Phys., 56 (1984), 709–735.
[12] Ryogo Hirota, Exact envelope-soliton solutions of a nonlinear wave equation, J. Math. Phys., 14 (1973), 805–809.
[13] J. Ji and Z. Zhu, Soliton solutions of an integrable nonlocal modified Korteweg-de Vries equation through inverse scattering transform, J. Math. Anal. Appl., 453 (2017), 973–984.
[14] Y. Kodama and A. Hasegawa, Nonlinear pulse propagation in a monomode dielectric guide, IEEE J. Quantum Electron., 23 (1987), 510–524.
[15] Z. Q. Li and S. F. Tian, A hierarchy of nonlocal nonlinear evolution equations and \( \hat{\theta} \)-dressing method, Appl. Math. Lett., 120 (2021), 107254, 8pp.
[16] M. Li and T. Xu, Dark and antidark soliton interactions in the nonlocal nonlinear Schrödinger equation with the self-induced parity-time-symmetric potential, Phys. Rev. E, 91 (2015), 033202, 8pp.
[17] W. Ma, Riemann-Hilbert problems and soliton solutions of nonlocal real reverse-spacetime mKdV equations, *J. Math. Anal. Appl.*, **498** (2021), 124980, 13pp.

[18] W. Peng, S. Tian, T. Zhang and Y. Fang, Rational and semi-rational solutions of a nonlocal (2+1)-dimensional nonlinear Schrödinger equation, *Math. Methods Appl. Sci.*, **42** (2019), 6865–6877.

[19] C. Rogers and W. K. Schief, *Bäcklund and Darboux transformations : geometry and modern applications in soliton theory*, Cambridge University Press, Cambridge, UK, 2002.

[20] A. K. Sarma, M. A. Miri, Z. H. Musslimani and D. N. Christodoulides, Continuous and discrete Schrödinger systems with parity-time-symmetric nonlinearities, *Phys. Rev. E*, **89** (2014), 052918, 7pp.

[21] N. Sasa and J. Satsuma, New-type of soliton solutions for a higher-order nonlinear Schrödinger equation, *J. Phys. Soc. Jpn.*, **60** (1991), 409–417.

[22] C. Song, D. Xiao and Z. Zhu, Solitons and dynamics for a general integrable nonlocal coupled nonlinear Schrödinger equation, *Commun. Nonlinear Sci. Num. Simul.*, **45** (2017), 13–28.

[23] Z. Zhou, Darboux transformations and global solutions for a nonlocal derivative nonlinear Schrödinger equation, *Commun. Nonlinear Sci. Num. Simul.*, **62** (2018), 480–488.

Received June 2021; 1st revision September 2021; 2nd revision September 2021; early access October 2021.

*E-mail address: sftian@cumt.edu.cn*

*E-mail address: ly1114981860@163.com*