FINITE MIXTURE MODELS: A BRIDGE WITH STOCHASTIC GEOMETRY AND CHOQUET THEORY

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Abstract. In the context of a finite mixture model whose components and weights are unknown, if the number of components is a function of the amount of data collected, we use techniques from stochastic convex geometry to find the growth rate of its expected value. We also show that by placing a Dirichlet process prior on the densities supported on the unit simplex, we are able to retrieve the Dirac measure at the Choquet measure supported on the components. In turn, this gives us the mixture weights. Finally, we propose a novel algorithm that identifies the model capturing the complexity of the data using only the strictly necessary number of mixture components.

1. Introduction

Finite mixture models go back at least to [34, 35] and have served as a workhorse in stochastic modeling [13, 26, 31]. Applications include clustering [29], hierarchical or latent space models [27], and semiparametric models [30] where a mixture of simple distributions is used to model data that is putatively generated from a complex distribution. In finite mixture models, the mixing distribution is over a finite number of components; there are also many examples of infinite mixture models in the Bayesian nonparametrics literature [3, 43].

In general, a finite mixture distribution of \( m \) components for a random vector \( Y \) is given by

\[
Y \sim \sum_{k=1}^{m} p_k f(y; \theta_k), \quad \sum_{k=1}^{m} p_k = 1, \quad p_k \geq 0,
\]

where the elements of the probability vector \( p = (p_1, \ldots, p_m)^\top \) are mixture weights and \( \theta_k \) denotes the parameter values for the \( k \)-th component.

Inference on the number of mixture components for finite mixture models can be difficult. In the Bayesian setting one can place a prior on the number of mixture components [32]. In [19], the authors study the consistency of the posterior distribution of the number of clusters; they also propose a merge-truncate-merge procedure to consistently estimate the number of clusters from Dirichlet process mixture models. In [15], the authors introduce the use of non-local priors for choosing the number of components in finite mixture models.

Another approach to inference on the number of components is to test whether the number of components is a given \( k \) or \( k' > k \). Such literature is quite rich: classical results are summarized in [41], while more modern works include [21] and [7, 25]. In the former, an estimator for the number of components is provided based on transformations of the observed

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data. The latter two propose an EM test for testing whether the number of true components in the mixture is some $k_0 > 0$, or is larger than $k_0$.

A further recent work of interest is [33], where the authors use a data dependent prior and achieve optimal estimation of mixing measures, as well as posterior consistency for the number of clusters. They also consider a Dirichlet Process mixture to estimate a finite mixture model and show that the number of clusters can be used for consistent estimation on the number of components.

Rather than developing new tools for working with or applying finite mixture models, the main goal of this paper is to establish connections between finite mixture models, stochastic convex geometry, and Choquet theory.

1.1. Choquet theory. Choquet theory, named after French mathematician Gustave Choquet, is an area of functional and convex analyses concerned with measures which have support on the extreme points of a convex set [36]. Its fundamental tenet is that we can represent every element in a convex set $C$ via a weighted average of the extrema of the set. Here weighted average is to be understood as a generalization of the usual notion of convex combination to an integral taken over the set $E$ of extreme points of $C$. The formal, central result to Choquet theory is the following.

**Theorem 1.** (Choquet, cf. [36]) Let $C$ be a metrizable compact convex subset of a locally convex space $V$. Pick any $c \in C$. Then, there exists a probability measure $\nu$ on $C$ which represents $c$ and is supported on $E$, that is,

$$f(c) = \int_E f(e) \nu(de),$$

for any affine function $f$ on $C$.

Choquet also characterized those compact convex sets $C$ with the property that for every $c \in C$ there is a unique probability measure $\nu_c$ supported on $E$ that represents $c$. The necessary and sufficient condition is based on the concept of Choquet simplex.

**Definition 2.** A nonempty convex set $C$ (not necessarily compact) of a locally convex space $V$ is a Choquet simplex if it has the following property. Under the embedding of $V$ as the hyperplane $V \times \{1\}$ in the space $V \times \mathbb{R}$, the projecting cone

$$\tilde{C} := \{\alpha c \in V \times \mathbb{R} : c \in C \subset V \times \{1\}, \alpha \geq 0\}$$

of $C$ transforms the space $V \times \mathbb{R}$ into a partially ordered space $P$ such that the space of differences $\tilde{C} - \tilde{C}$ generated by $P$ is a vector lattice in the order induced by $C$. That is, each pair $c_1, c_2 \in \tilde{C} - \tilde{C}$ has at least upper bound $c_1 \vee c_2 \in \tilde{C} - \tilde{C}$.

In the case when $V$ is finite-dimensional, a Choquet simplex is an ordinary simplex with number of vertices equal to $\dim(V) + 1$, where $\dim(V)$ denotes the dimension of space $V$. The characterization of $C$, then, is the following.

**Theorem 3.** (Choquet, cf. [36]) Let $C$ be a metrizable closed convex subset of a locally convex space $V$. Then, $C$ is a Choquet simplex if and only if for every $c$ in $C$, there exists a
unique measure $\nu_c$ which represents $c$ and is supported on $E$, that is,

$$f(c) = \int_E f(e)\nu_c(de),$$

for any affine function $f$ on $C$.

We call $\nu_c$ the Choquet measure for $c$. These results entail that studying the extrema of a convex set gives us important results concerning the (elements of the) whole set. Choquet theory in the context of finite mixture models has been inspected by [22]. There, the author develops an approach that uses Choquet’s theorems for inference with the goal of estimating probability measures constrained to lie in a convex set, for example mixture models. The key observation in [22] is that inference over a convex set of measures can be made via unconstrained inference over the set of extreme measures. The main difference between this work and the approach developed in [22] is that we consider a convex hull of points in a unit simplex rather than the convex hull of probability measures. Furthermore, our goal is different: we use a result from Choquet theory to retrieve the weights in the finite mixture model at hand. Notice also that de Finetti’s theorem [8, 9, 10] can be given a geometric interpretation – inspected in Appendix A – that is heuristically similar to that of Choquet theory.

1.2. Stochastic convex geometry. This paper also establishes a bridge between finite mixture models and stochastic convex geometry that allows to view finite mixture models as well-studied geometric objects. This insight allows to closely relate the number of components in a finite mixture model to the number of extrema of a convex body. Thereby, it facilitates studying the asymptotic growth rate and the asymptotic distribution of the number of components. The geometry of finite mixture models has primarily been studied in two contexts: differential geometry [2, 24] and convex geometry [26, 28]. The approach in this paper is based on (stochastic) convex geometry. The first to observe that a mixture model can be seen as an element of the unit simplex in some Euclidean space $\mathbb{R}^J$ was [26]. The focus was on identifiability of the weights of the mixture, a Carathéodory representation theorem for multinomial mixtures, and the asymptotic mixture geometry. In [28], the author bridges the differential and convex geometric approaches to identify restrictions for which the mixture can be written as more tractable geometric quantitites that can simplify inference problems. This paper is similar in spirit to Lindsay’s work, but uses more modern techniques from [5] and [38].

1.3. Setup of our work. We consider a finite mixture of multinomials. We start with the basic multinomial model where our observations $X$ take on $J$ possible values $\{1, \ldots, J\}$ and $X \sim \text{Mult}(\pi)$, with $\pi \equiv (\pi_1, \ldots, \pi_J)^T$ where $\pi_j = \mathbb{P}(X = j)$, with $\pi_j \geq 0$ for all $j$ and $\sum_{j=1}^J \pi_j = 1$. A mixture of $L$ multinomials can be specified as follows

$$X_i \sim \text{Mult}(\pi_i), \quad \pi_i = \sum_{\ell=1}^L \phi_{i,\ell} f_\ell,$$
where probability vector \( \phi_i \equiv (\phi_{i1}, \ldots, \phi_{iL})^\top \) assigns the probability of the \( i \)-th observation coming from the \( \ell \)-th mixture component with multinomial parameters
\[
  f_\ell = (f_{\ell1}, \ldots, f_{\ellL})^\top.
\]
We have that \( \sum_{j=1}^J f_{\ell j} = 1 \) with \( f_{\ell j} \geq 0 \), and \( \sum_{\ell=1}^L \phi_{i,\ell} = 1 \) with \( \phi_{i,\ell} \geq 0 \). An important point throughout the paper is that \( \pi_i \) belongs to the convex hull of probability vectors \( \{f_1, \ldots, f_L\} \). The convex hull of \( \{f_1, \ldots, f_L\} \) is a function of the identifiable elements of \( \{f_1, \ldots, f_L\} \), that is, those elements that cannot be written as a convex combination of the other \( f_\ell \)'s. Hence, understanding the identifiable elements of this set provides information about the key model parameters.

The finite mixture model we stated is an example of an admixture model; the most popular admixture model is the latent Dirichlet allocation (LDA) model [6, 37]. A classic application of an admixture model is a generative process for documents. Consider a document as a collection of words; LDA posits that each document is a mixture of a small number of topics, and that these latter can be modeled by a multinomial distribution on the presence of a word in the topic. The hierarchical Dirichlet process [40], and generalizations thereof, may be considered as the natural nonparametric counterpart of the LDA model.

The \( \pi_i \)'s and the \( f_\ell \)'s are all elements of \( \Delta^{J-1} \), the unit simplex on \( \mathbb{R}^J \). Again, each of the \( \pi_i \) belong to the convex hull of \( \{f_\ell\}_{\ell=1}^L \), or \( \pi_i \in \text{Conv}(f_1, \ldots, f_L) \). Hence, an element of a convex hull in the Euclidean unit simplex represents (the distribution of) a finite mixture model.

Notice that the number of extrema of \( \text{Conv}(f_1, \ldots, f_L) \), which we denote as \( M \), will probably be less than \( L \) because some of the components \( f_\ell \) are likely to be a convex combination of the others. A key concept in this paper is what we call the richest cheap model (RCM) representing \( \pi_i \), that is, the finite mixture model representing \( \pi_i \) whose mixture components are \( \{f_k\}_{k \in \mathcal{I}} \) such that \( f_k \not\in \text{Conv}(f_{\mathcal{I}\setminus\{k\}}) \), for all \( k \in \mathcal{I}, \mathcal{I} \subset \{1, \ldots, L\} \), and \( \#\mathcal{I} = M \), where \( \# \) denotes the cardinality operator. These conditions tell us that the \( M \) components of the richest cheap model are a subset of \( \{f_1, \ldots, f_L\} \) and cannot be written as a convex combination of one another. By assuming – without loss of generality – that the identifiable elements in \( \{f_1, \ldots, f_L\} \) are the first \( M \) ones, we can write the richest cheap model as
\[
  \pi_i = \sum_{\ell=1}^M \varphi_{i,\ell} f_\ell,
\]
where we denote by \( \varphi_i \equiv (\varphi_{i1}, \ldots, \varphi_{iM})^\top \) the probability vector that assigns the probability of the \( i \)-th observation coming from the \( \ell \)-th identifiable mixture component with multinomial parameters \( f_\ell = (f_{\ell1}, \ldots, f_{\ellL})^\top, \ell \in \{1, \ldots, M\} \). Of course the \( \varphi_{i,\ell} \)'s are such that, for all \( i, \sum_{\ell=1}^M \varphi_{i,\ell} = 1 \), and \( \varphi_{i,\ell} \geq 0 \), for all \( \ell \). As we can see, the RCM captures the underlying complexity associted with the data at hand, using only the strictly necessary number of components.

1.4. Main results and structure of the paper. We provide three main results. The first two – Theorems 8 and 9 – are very general, and state the following. Suppose we do not know what the components and the weights in our admixture model are, and we also do not know the number of components. Then, if we assume that the number of identifiable components
M is a function $M(n)$ of the amount $n$ of data we gather, we are able to tell the speed at which its expected value grows. The other main result – Theorem 15 – is more practical in nature. It states that we can place a Dirichlet process on the densities supported on $\Delta^{J-1}$, which eventually retrieves the weights. We also show how looking for the richest cheap model can be seen as an optimization problem, and we propose an algorithm to solve it.

The paper is organized as follows. In section 2, we let the number of identifiable mixture components $M$ depend on the sample size $n$, that is, we let $M = M(n)$. We study the behavior of $M(n)$ as the number of observations increases. In Theorem 4 we show that if $M(n)$ is given by the cardinality of the extremal set of the convex hull of $n$ elements sampled iid from the uniform over the simplex $\Delta^{J-1}$, then the asymptotic growth rate of $\mathbb{E}[M(n)]$ is $(\log n)^{J-1}$. In Theorem 5 we state, retaining the same assumption on $M(n)$, a central limit theorem (CLT) for the distribution of the number of identifiable components of the admixture model. In Theorem 6 we prove that, as number of extrema of the convex hull grows to infinity, the convex hull tends to an apeirogon, a polytope with infinitely many sides. In Theorem 8 we state that the $(\log n)^{J-1}$ asymptotic growth rate of the expectation of the number of identifiable components holds also when the $n$ elements are drawn from a generic distribution, under a very mild assumption. We relax this latter in Theorem 9.

In section 3 we consider inference when the number of identifiable admixture components is equal to $J$, but the admixture components and weights are unknown. In Theorem 15, we use Theorem 3 to show that a Dirichlet process always retrieves our admixture weights. We also give the rate of convergence of the Dirichlet process posterior to the (Dirac at the) weights.

In section 4, we use the idea of mixture models based on the extremal set to formulate a novel algorithm that outputs an admixture model composed of only extremal elements. We state the objective function the algorithm optimizes, and provide a two-stage procedure. We apply this latter to the Associated Press data from the First Text Retrieval Conference (TREC-1), a large collection of terms used in 2246 documents.

Section 5 is a discussion. In appendix A we give an approximation of the joint distribution of the components of our finite mixture model and we provide the number of extrema of the convex hull within a unit simplex having the least amount of vertices. We also clarify the meaning of “almost surely” in equations (6) and (7). We prove our results in appendix B.

2. Growth rates for extrema and mixture components

Suppose the number $M$ of identifiable admixture components is a function of the amount $n$ of data $x_1, \ldots, x_n$ we collect that is defined as follows. Let

$$S_1, \ldots, S_n \overset{iid}{\sim} \text{Uniform}(\Delta^{J-1}),$$

and call $K_n := \text{Conv}(s_1, \ldots, s_n)$, where $s_j$ denotes the realization of $S_j$. Then, function $M$ is defined as

$$M : \mathbb{N} \to \mathbb{N}, \quad n \mapsto M(n) := \#\text{ex}(K_n),$$

that is, $M(n)$ is given by the cardinality of the extremal set of $K_n$. Before finding the growth rate of $\mathbb{E}[M(n)]$, we need to introduce the concepts of $i$-face and tower of a polytope.
• As pointed out in [45, Definition 2.1], in higher-dimensional geometry, the faces of a polytope are features of all dimensions. A face of dimension $i$ is called an $i$-face. For example, the polygonal faces of an ordinary polyhedron are 2-faces. For any $g$-dimensional polytope, $-1 \leq i \leq g$, where $-1$ is the dimension of the empty set. Let us give a clarifying example. The faces of a cube comprise the cube itself (3-face), its facets (2-faces), the edges (1-faces), its vertices (0-faces), and the empty set (having dimension $-1$). Given a generic $g$-dimensional polytope $P$, we denote by $\mathcal{F}_i(P)$ one of its $i$-faces, $i \in \{-1, 0, \ldots, g\}$.

• We call $\mathcal{F}_i(P)$ the collection of its $i$-faces, and $F_i(P)$ the number of its $i$-faces, that is, $F_i(P) = \#\mathcal{F}_i(P)$, for all $i$.

• We also call a chain $\mathcal{F}_0(P) \subset \mathcal{F}_1(P) \subset \cdots \subset \mathcal{F}_g(P)$ of $i$-dimensional faces a tower of $P$.

Given these definitions, $F_0(K_n)$ denotes the number of extremal points of $K_n$.

**Theorem 4.** Let $K_n := \text{Conv}(s_1, \ldots, s_n)$, where $S_1, \ldots, S_n$ are sampled as in (1). Then,

$$\lim_{n \to \infty} (\log n)^{-(J-1)} \mathbb{E}[F_0(K_n)] = \frac{1}{(J+1)^{J-1}(J-1)!} T(\Delta^{J-1}) =: c(J),$$

where $T(\Delta^{J-1})$ is the number of towers of $\Delta^{J-1}$.

Notice that $F_0(K_n)$ corresponds to $M(n)$. Theorem 4 tells us that the expected number of identifiable mixture components grows at rate $(\log n)^{J-1}$.

We can also state the limiting distribution of $F_0(K_n)$; specifically, we give the following central limit theorem for $F_0(K_n)$. It immediately implies the same result for $M(n)$. We denote by $\mathbb{V}[F_0(K_n)]$ the variance of the number of extreme points of $K_n$.

**Theorem 5.** Let $K_n := \text{Conv}(s_1, \ldots, s_n)$, where $S_1, \ldots, S_n$ are sampled as in (1). Then,

$$\lim_{n \to \infty} \mathbb{P}\left( \frac{F_0(K_n) - \mathbb{E}[F_0(K_n)]}{\sqrt{\mathbb{V}[F_0(K_n)]}} \leq t \right) = \Phi(t)$$

where $\Phi$ is the cumulative distribution function of the standard normal distribution.

A further interesting result concerns the shape that $K_n$ converges to asymptotically. The next theorem states that, as the number of extreme points goes to infinity, the convex hull of these points converges to an apeirogon, a polytope with infinitely many sides.

**Theorem 6.** Let $K_n := \text{Conv}(s_1, \ldots, s_n)$, where $S_1, \ldots, S_n$ are sampled as in (1). If $F_0(K_n)$ grows to infinity, then $K_n$ tends to an apeirogon.

At this point, an obvious question is what the asymptotic growth function of the expected number of identifiable components based on draws from the uniform distribution on the unit simplex tells us about the asymptotic growth rate based on draws from a generic distribution.

In a more general setting, when we drop the uniform assumption in (1), the number of extrema of the convex hull related to our model may be different from $M(n)$. We denote this quantity by $T$, and it is too going to be a function of the amount $n$ of data we collect,
that is, \( T = T(n) \). In particular, suppose that \( S_1, \ldots, S_n \) are now sampled iid from a generic distribution \( G \) on \( \Delta^{J-1} \). Call then \( \tilde{K}_n := \text{Conv}(s_1, \ldots, s_n) \). Function \( T \) is defined as

\[
T : \mathbb{N} \to \mathbb{N}, \quad n \mapsto T(n) := \#\text{ex}(\tilde{K}_n),
\]

that is, \( T(n) \) is given by the cardinality of the extremal set of \( \tilde{K}_n \).

**Remark 7.** Notice that \( M(n), T(n) \geq J \) (of course, \( J \geq 2 \)). If that is not the case, we can still have a convex hull, but it will be a proper subset of a smaller dimensional Euclidean space, and we are not interested in this eventuality.

In Theorem 8, we require that the expected number of identifiable admixture components (ENIAC) \( \mathbb{E}[T(n)] \) in the general setting is in a fixed linear relation with the ENIAC \( \mathbb{E}[M(n)] \) in the simple uniform model. This can be interpreted as the stochasticity around the number of components entering the general model through the simple uniform one, and then being linearly “passed on”. In Theorem 9, we further weaken this already mild regularity condition.

**Theorem 8.** Suppose that, for all \( n \), we can always find \( \gamma \in \mathbb{R} \setminus \{0\} \) such that \( \mathbb{E}[T(n)] = \gamma \cdot \mathbb{E}[M(n)] \). Then,

\[
\lim_{n \to \infty} (\log n)^{-\gamma} \mathbb{E}[T(n)] = c(J, \gamma) := \gamma \cdot c(J).
\]

(4)

**Theorem 9.** Call \( (\gamma_n) \) a sequence in \( \mathbb{R}^N \) for which 0 is not an accumulation point, and let \( \mathbb{E}[T(n)] = g_n(\mathbb{E}[M(n)]) \), where \( g_n \) is a functional on \( \mathbb{R} \) that depends on \( n \). Then, if there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( g_n(\mathbb{E}[M(n)]) = \gamma_n \mathbb{E}[M(n)] \), the following holds

\[
\lim_{n \to \infty} \frac{1}{\gamma_n(\log n)^{J-1}} \mathbb{E}[T(n)] = c(J).
\]

(5)

Theorem 9 is stating the following: up until the \((N - 1)\)-th data point, the ENIAC in the more general case \( \mathbb{E}[T(n)] \) can take on any possible real value. From the \( N \)-th observation onward, though, it must be in a fixed (possibly highly nonlinear) relationship \( \gamma_n \) with \( \mathbb{E}[M(n)] \). If this happens, we are able to relate their growth rates. The following corollary is a direct consequence of Theorem 9.

**Corollary 10.** Suppose the assumptions of Theorem 9 hold. Then, if there exists a sequence \( \varpi_n \in \mathbb{R}^N \) such that \( \gamma_n = \mathcal{O}(\varpi_n) \), then the growth rate of \( \mathbb{E}[T(n)] \) is \( \varpi_n(\log n)^{J-1} \).

**Remark 11.** It is immediate to see that there is a universal upper bound for the Euclidean distance between two points in a unit simplex: for all \( x, y \in \Delta^{J-1} \), \( d(x, y) \equiv \|x - y\| \leq 2 \). This gives us an interesting result: the Hausdorff distance between \( K_n \) and \( \tilde{K}_n \) has a universal upper bound as well. Indeed,

\[
d_H(K_n, \tilde{K}_n) = \max \left\{ \sup_{x \in K_n} \inf_{y \in K_n} d(x, y), \sup_{y \in \tilde{K}_n} \inf_{x \in K_n} d(x, y) \right\} \leq 2.
\]

Notice also that in Theorem 8 if – instead of requiring \( \mathbb{E}[T(n)] = \gamma \cdot \mathbb{E}[M(n)] \) – we are willing to make the slightly stronger assumption that \( T(n) = \rho \cdot M(n), \rho \in \mathbb{R} \) possibly
different from $\gamma$, then we retrieve Theorem 5. This because, since $F_0(K_n) \equiv M(n)$, we have that
\[
\frac{T(n) - \mathbb{E}[T(n)]}{\sqrt{\mathbb{V}[T(n)]}} = \frac{\rho F_0(K_n) - \mathbb{E}[\rho F_0(K_n)]}{\sqrt{\mathbb{V}[\rho F_0(K_n)]}} = \frac{F_0(K_n) - \mathbb{E}[F_0(K_n)]}{\sqrt{\mathbb{V}[F_0(K_n)]}},
\]
and so Theorem 5 follows. In a similar fashion, if in Theorem 9 we require that, for all $n \geq N$, $T(n) = \rho_n \cdot M(n)$, $(\rho_n) \in \mathbb{R}^N$ possibly different from $(\gamma_n)$, then we retrieve Theorem 5.

3. The Choquet measure and a prior on the extremal points

In this section we build a bridge between finite mixture models and Choquet theory. We show how, thanks to a uniqueness result by Gustave Choquet, a Dirichlet process can be used to retrieve the mixture weights in a finite admixture model. We also give the rate of convergence of the Dirichlet process posterior to the Dirac measure on the weights.

By Theorem 3, we have that for every element $p$ in a simplex $C$, there exists a unique measure – that we call the Choquet measure associated with $p$, and denote by $\nu_p$ – supported on the extrema $E = ex(C)$ such that $p = \sum_{e \in E} e \cdot \nu_p(e)$,\footnote{We write $\nu_p(e)$ in place of $\nu_p(\{e\})$ for notational convenience. We stick to this abuse of notation for the rest of the paper.} In our analysis, $p$ corresponds to $\pi_i$, the elements $e$ in $E = ex(C)$ correspond to the identifiable $f_i$’s, and the $\nu_p(e)$’s correspond to the weights of the identifiable $f_i$’s, that is, $\nu_{\pi_i}(f_\ell) = \phi_i^\ell$, for every identifiable $f_\ell$.

In Theorem 15, we show that if we only assume that the number $M$ of components is known and equal to $J$, a Dirichlet process retrieves $\nu_{\pi_i}$. The $\phi_i^\ell$’s represent the weights of the richest cheap finite mixture model representing $\pi_i$, so $\pi_i = \sum_{\ell=1}^M f_\ell \nu_{\pi_i}(f_\ell)$. By retrieving, we mean that given a Dirichlet process prior $DP(\alpha P_0)$ specified on the distributions supported on $\Delta^{J-1}$ having parameter $\alpha > 0$ and $P_0$ as base measure, its posterior converges weakly almost surely to the Dirac at $\nu_{\pi_i}$.

Remark 12. Recall that $\pi_i = \sum_{\ell=1}^M f_\ell \phi_i^\ell = \sum_{\ell=1}^L f_\ell \phi_i^\ell$, where we labeled the unidentifiable components as $f_{M+1}, \ldots, f_L$, $M \leq L$. This is without loss of generality.

3.1. Choquet theory and extrema of convex bodies. Let us denote by
\[
\mathcal{K}_M := \text{Conv}(f_1, \ldots, f_M) = \text{Conv}(f_1, \ldots, f_L)
\]
the convex hull generated by the $M$ identifiable components of our finite mixture model, and assume $M = J$, so that $\mathcal{K}_M$ is a (Choquet) simplex. An example of a (Choquet) simplex within the unit 2-simplex in $\mathbb{R}^3$ is given in Figure 1. Our first goal is to learn about distributions supported on the extremum of $\mathcal{K}_M$, $E_M := ex(\mathcal{K}_M)$.

Since $\Delta^{J-1}$ is locally convex, and $\mathcal{K}_M \subset \Delta^{J-1}$ is a metrizable compact convex set, then thanks to Theorems 1 and 3, we know that for every $\pi_i \in \mathcal{K}_M$, there exists a unique probability measure $\nu_{\pi_i}$ supported on $E_M$ such that $\pi_i = \sum_{f_\ell \in E_M} f_\ell \cdot \nu_{\pi_i}(f_\ell)$.

Proposition 13. If $\mathcal{K}_M$ is a (Choquet) simplex, then every element in $\pi_i \in \mathcal{K}_M$ can be represented by a unique measure $\nu_{\pi_i}$ (the Choquet measure representing $\pi_i$) supported on $E_M$.
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3.2. Choquet theory for mixture weights. The Dirichlet process (DP) is of fundamental
importance in the Bayesian nonparametrics literature [14, 16, 17]. It is a default prior on
spaces of probability measures, and a building block for priors on other structures. Because
it “selects” almost surely discrete distributions, it is a sensible choice for our case, since
the Choquet measure is supported on the finite set $E_M$. The DP possesses the conjugacy
property, as shown in the following theorem from [14].

Theorem 14. Suppose that $P \sim DP(\alpha P_0)$, for some $\alpha > 0$ and base measure $P_0$, and we
collect iid observations $X_1, \ldots, X_n \mid P \sim P$. Then, (a version of) the posterior for $P$ is given
by $DP(\alpha P_0 + \sum_{j=1}^{n} \delta_{X_j}) \equiv DP(\alpha P_0 + n P_n)$.

Here $P_n = 1/n \sum_{j=1}^{n} \delta_{X_j}$ denotes the empirical distribution of the observations. An exten-
sive treatment of DP’s is given in [16, Chapter 4]. The following is the main result of the
section.

Theorem 15. Let $K_M$ be a (Choquet) simplex. If $e_1, \ldots, e_k$ are an iid sample of elements
of $E_M$ from $\nu_{\pi_i}$, then

$$DP \left( \alpha P_0 + \sum_{j=1}^{k} \delta_{e_j} \right) \xrightarrow{w} \delta_{\nu_{\pi_i}} \quad \text{a.s.}$$

where $\alpha$ is a positive real, $P_0$ is a base measure supported on $\Delta^{J-1}$, $\xrightarrow{w}$ denotes the weak
convergence, and $\delta_{\nu_{\pi_i}}$ is the Dirac measure at $\nu_{\pi_i}$. In addition, the rate of convergence relative
to the total variation metric is given by $k^{-1/2}$.

The idea is that we recover the Choquet measure $\nu_{\pi_i}$. Because the DP is a measure over
measures, the formal statement is convergence to $\delta_{\nu_{\pi_i}}$. 

Figure 1. A triangular-shaped convex hull within the unit 2-simplex in $\mathbb{R}^3$.
It is a simplex because it is the convex hull of its vertices.
Remark 16. Let supp\((P)\) denote the support of a generic measure \(P\). Notice that in Theorem 15 \(\text{supp}(P_0) = \Delta^{j-1}\), while \(\text{supp}(\nu_\pi) = E_M \subset \Delta^{j-1}\). This is not a problem since by [16, Theorem 4.15] the weak support of a Dirichlet process on measures on \(\Delta^{j-1}\) is given by \(\{ P : \text{supp}(P) \subset \text{supp}(P_0) \}\).

It is also worth to mention that the Choquet measure can be retrieved using a different approach than Theorem 15. Suppose again that \(\mathcal{K}_M\) is a Choquet simplex, and \(E_M\) is the set of its extremal points. Consider a sample \(e_1, \ldots, e_k \sim \nu_\pi\text{id}\); because every \(e_j\) corresponds to an identifiable element of the mixture model, we can write \(e_j = f_{\ell_j}\), where label \(\ell_j\) belongs to \(\{1, \ldots, M\}\), for all \(j \in \{1, \ldots, k\}\). Now, call \(\zeta\) the distribution of the labels; we immediately notice that there is a one-to-one correspondence between \(\zeta\) and \(\nu_\pi\) since \(\zeta(\ell) = \nu_\pi(f_{\ell})\), for all \(\ell \in \{1, \ldots, M\}\). Call then \(F_\zeta\) the cdf of \(\zeta\), let \(\ell_1, \ldots, \ell_k \sim \zeta\text{id}\), and for all \(x \in \mathbb{R}\), denote by

\[
F_k(x) := \frac{1}{k} \sum_{j=1}^k \mathbb{I}_{[\ell_j, \infty)}(x)
\]

the empirical cdf of \(\ell_1, \ldots, \ell_k\), where \(\mathbb{I}_A(x)\) stands for the indicator function of \(x\) belonging to a generic set \(A\). Then, by Glivenko-Cantelli’s Theorem [42], we have that \(F_k\) converges to \(F_\zeta\) uniformly almost surely, that is,

\[
\sup_{x \in \mathbb{R}} |F_k(x) - F_\zeta(x)| \xrightarrow{a.s.} 0.
\]  

(7)

In addition, [44, Chapter 1, Remarks 1 and 2] shows that the rate of convergence is \(ke^{-k}\). This rate is faster than that of Theorem 15, but \((f_1, \ldots, f_M)\) is not an exchangeable sequence since in this alternative procedure the labels “matter”.\(^2\)

As usual, there is no free lunch.

4. A PROCEDURE TO FIND THE RICHEST CHEAP MODEL

In our admixture model, there are two sets of parameters:

(1) the mixing weights for each individual, that can be arranged in an \(n \times L\) matrix \(\Phi\) whose components \(\Phi_{i,j}\) represent the probability that the \(i\)-th sample is drawn from the \(j\)-th component. Each row of \(\Phi\) is the mixture vector of the \(i\)-th observation \(\phi_i = (\phi_{i,1}, \ldots, \phi_{i,L})\);

(2) the probability vectors parameterizing each mixture component, which we can write as an \(L \times J\) matrix \(F\) whose \(j\)-th row is \(f_j\).

The relation between admixture modeling and sparse factor analysis (SFA) has been explored in detail in [12]. There, conditions are provided when SFA and LDA have very similar results, and the implications for population genetics are discussed. The key insight in [12] is that given an \(n \times J\) observation matrix \(X\) (whose \(i\)-th row is \(x_i\)) from a binomial admixture model, learning an admixture model amounts to the following minimization procedure

\[
\min_{F,\Phi} \| \mathbb{E}[X] - \Phi F \|^2.
\]  

(8)

The SFA framework can be summarized as minimizing (8) with the constraint that many of the elements of \(\Phi\) will be zero, or that every observation is a sparse combination of

\(^2\)For a definition of exchangeable sequence, see appendix A.1.
each component. The spirit behind the algorithm proposed in this section is to think of sparsity as the extremal set: we want to find a set of components that are extremal yet still accurately solves the above minimization. We first state the likelihood for the admixture model, assuming a maximum of $L$ components,

$$
\mathcal{L}(X_1, \ldots, X_n; \{\phi_1, \ldots, \phi_n\}, \{f_1, \ldots, f_L\}) = \prod_{i=1}^{n} \text{Mult} \left( \pi_i = \sum_{\ell=1}^{L} \phi_{i,\ell} f_\ell \right).
$$

The maximum likelihood estimator (MLE) for the above model is

$$
\{\hat{\phi}_1, \ldots, \hat{\phi}_n\}, \{\hat{f}_1, \ldots, \hat{f}_L\} \equiv \text{arg max} \ \mathcal{L}(X_1, \ldots, X_n; \{\phi_1, \ldots, \phi_n\}, \{f_1, \ldots, f_L\}). \quad (9)
$$

A notion of sparsity related to the SFA framework is to maximize the likelihood subject to the constraint that components are identifiable, that is, no component can be represented as a convex combination of other components. We consider a procedure that maximizes the following objective function

$$
\text{argmax} \ \prod_{i=1}^{n} \text{Mult} \left( \pi_i = \sum_{k \in I} \phi_{i,k} f_k \right) \quad (10)
$$

subject to $f_k \notin \text{Conv}(f_{I \setminus \{k\}})$, $\forall k \in I$,

where $I$ is a subset of the set $\{1, \ldots, L\}$ and is the collection of the indices of the extremal set. Constraint $f_k \notin \text{Conv}(f_{I \setminus \{k\}})$, for all $k \in I$, ensures that no mixture component is contained in the convex combination of the others. Notice that the cardinality of $I$ represents the number of components $M$ of the richest cheap model, as we described in section 1.3.

The maximization specified by equation (10) is non-convex and finding the global optima is difficult; we propose a two-step procedure to solve it.

**Algorithm 1** 2-step EM algorithm

**Step 0** Set the number of components $L$ to be arbitrarily large

**Step 1** Compute the MLE specified in (9) so to obtain the estimated parameters

$$
\{\hat{\phi}_1, \ldots, \hat{\phi}_n\}, \{\hat{f}_1, \ldots, \hat{f}_L\}
$$

**Step 2** Compute $\text{Conv}(\{\hat{f}_1, \ldots, \hat{f}_L\})$ and consider the cardinality $M$ of its extremal set, $M := \#\text{ex}(\text{Conv}(\{\hat{f}_1, \ldots, \hat{f}_L\})$

if $M < L$ then rerun the MLE in (9) with $M$ components

else stop and keep the current parameters

end if

The parameters obtained after Step 2 of Algorithm 1 are estimates of the parameters of the richest cheap model. Notice that computing the convex hull is evocative of the Choquet procedure described in section 3. If $M = J$, at the end of this algorithm, we have an estimate $\hat{\nu}_{\pi_i}$ of the Choquet measure for $\pi_i$, since $\hat{\nu}_{\pi_i}(f_\ell) = \hat{\phi}_{i,\ell}$, for all $\ell \in \{1, \ldots, M\}$, where we denote the estimates of the weights of the richest cheap model as $\hat{\phi}_{i,\ell}$, for all $\ell$. 

We applied our two-step procedure to a well studied dataset [4] which is a document-term matrix consisting of term frequencies of 10473 terms in 2246 documents collected from Associated Press documents [20]. We used the latent Dirichlet allocation (LDA) function in the R package topicmodels [18] to compute the MLE. We used the convex hull function in the R package geometry to compute the convex hull. Computing the convex hull over the full topic frequency vectors – elements belonging to simplex $\Delta_{10472}$ – is prohibitive and also does not make sense when the number of topics are less than 10472. We used principal components analysis (PCA) to project the frequency vectors of the topics onto a lower dimensional space and then computed the convex hull of the projections. We used a simple scree plot to notice that 3 – 5 dimensions are sufficient to capture 30% of the variation when we carry out our analysis specifying $L = 200$ initial topics. If we choose $L < 200$ initial topics, we have that 3 – 5 dimension explain more than 30% of the variation. We only need to compute the number of extrema of the convex hull and not the extremal elements themselves in our procedure, so it suffices to compute the convex hull in the low dimensional space.

We ran our algorithm on the document-term matrix specifying the initial number of topics as 50, 100, 150, and 200. Given the results in the PCA step, we projected down to 5 dimensions. The number of extremal points – i.e. the number of topics – we obtained were 12, 11, 9, 8 with initialization 50, 100, 150, and 200, respectively. The number of topics we obtained is similar to the number obtained in previous studies, the majority of which use 9 to 12 topics [4, 23]. The reason why the number of extremal points seems to decrease as $L$ increases will be studied in future work.

5. Conclusion

In this paper there are two key ideas. The first one is that we can use techniques from stochastic convex geometry on the growth rate of the expected number of extrema of random polytopes to provide insights into the asymptotic growth rate of the expected number of components in a finite admixture model. We prove that the ENIAC grows at rate $(\log n)^{J-1}$ where $J$ is the dimension of the Euclidean space we work with. We also provide a central limit theorem for the distributions of the number of extrema. The other key concept is that we can retrieve admixture weights using techniques from Choquet theory. In particular, we show that if the convex hull $K_M$ generated by the identifiable elements of the finite mixture model is a (Choquet) simplex, a Dirichlet process recovers the Choquet measure for $\pi_i$, for any $\pi_i \in K_M$. We also give an algorithm to find the richest cheap admixture model. An interesting open question is whether there are other instances in Bayesian inference where coupling results from stochastic (convex) geometry with results from Choquet theory allows to develop novel analyses, insights, models, or algorithms.

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Appendix A. Further results

A.1. Distribution of our sequence of random points. In this appendix, we assume that the number $L$ of admixture components is known, but the parameters $(f_1, \ldots, f_L)$ of the multinomial components are not. We assume they are identically distributed random vectors, but we do not require independence. After realizing that collection $(f_1, \ldots, f_L)$ can be seen as a finite exchangeable sequence, we inspect how to approximate its joint distribution applying de Finetti’s theorem and a result by Diaconis and Freedman [11].

As pointed out in [1], we can state de Finetti’s result from a functional analytic viewpoint as follows. Let $S \equiv \Delta^{J-1} \subset \mathbb{R}^J$, and recall that a sequence of random variables $X_i$’s is exchangeable if $(X_i)_{i \geq 1} \overset{d}{=} (X_{\text{perm}(i)})_{i \geq 1}$, for any finite permutation $\text{perm}$, where $d$ denotes equality in distribution. We can assume that the elements $f_1, \ldots, f_L$ form a finite exchangeable sequence because the order in which they appear provides no additional information about the finite mixture model.

Let $\Delta(S) \equiv \Delta(S, \mathcal{B}(S))$ be the set of probability measures on $(S, \mathcal{B}(S))$, where $\mathcal{B}(S)$ is the Borel sigma-algebra for $S$. Let then $\Delta(\Delta(S))$ be the set of probability measures on $\Delta(S)$. When we define an infinite exchangeable sequence of $S$-valued random variables, we are actually defining an exchangeable measure, say $\Theta$, on $\Delta(S^\infty)$, where $\Theta$ is the distribution of the sequence.

Consider the set $\mathfrak{M} := \{\mu^\infty := \mu \times \mu \times \cdots \text{s.t. } \mu \in \Delta(S)\} \subset \Delta(S^\infty)$, that is the set of extrema of the convex set of exchangeable elements of $\Delta(S^\infty)$. Then, we have

$$\Theta(A) = \int_{\Delta(S)} \mu^\infty(A) \Lambda(d\mu), \quad \forall A \subset S^\infty.$$ 

Hence, there is a bijection between $\Lambda \in \Delta(\Delta(S))$ and $\Theta \in \Delta(S)$. Notice that a consequence of Proposition 13 is that for all $x \in \Delta^{J-1}$, there exists a unique Choquet measure $\tilde{\nu}_x$, whose support are the extrema of $\Delta^{J-1}$, denoted as $\text{ex}(\Delta^{J-1})$, that allows to represent $x$. Then, there always exists a probability measure $\hat{\nu}_x$ supported on the whole simplex $\Delta^{J-1}$, whose restriction to $\text{ex}(\Delta^{J-1})$ is given by $\tilde{\nu}_x$. To this extent, $\hat{\nu}_x \times \hat{\nu}_x \times \cdots =: \hat{\nu}_x^\infty$ belongs to $\mathfrak{M}$.

As we pointed out before, we can assume $(f_1, \ldots, f_L)$ to be a finite exchangeable sequence. Suppose, without loss of generality, that it is part of a much longer sequence of $m$ components $(f_1, \ldots, f_L, \ldots, f_m)$. Then, we can use [11, Theorem 13] to compute an approximation of $\Theta_L$, the distribution of our finite sequence. Let us denote by $\Theta_m$ the distribution of $(f_1, \ldots, f_L, \ldots, f_m)$; it is an exchangeable probability on $S^m$. Then, $\Theta_L$, $L \leq m$, is the projection of $\Theta_m$ onto $S^L$. Define the value $\beta(m, L)$ as

$$\beta(m, L) := 1 - \frac{m^{-L}m!}{(m-L)!}.$$
and notice that $\beta(m, L) \leq \frac{1}{2} \frac{L(L-1)}{m}$.

The theorem states that there exists $\tilde{\Lambda} \in \Delta(\Delta(S))$ such that the probability $\Theta_{\mu L}$ defined on $S^L$ as

$$\Theta_{\mu L}(A) = \int_{\Delta(S)} \mu^L(A) \tilde{\Lambda}(d\mu), \quad \forall A \subset S^L$$

is such that $d_{TV}(\Theta_L, \Theta_{\mu L}) \leq 2\beta(m, L)$, for all $L \leq m$. We denoted by $\mu^L$ the distribution of $L$ independent picks from $\mu$, that is, $\mu^L((s_1, \ldots, s_L)) = \prod_{j=1}^L \mu(s_j)$, and by $d_{TV}$ the total variation distance

$$d_{TV}(\Theta_L, \Theta_{\mu L}) := \sup_{A \subset S^L} |\Theta_L(A) - \Theta_{\mu L}(A)|.$$  

Notice that $\tilde{\Lambda}$ depends on $m$ and $\Theta_m$, but not on $L$, and its analytical form is given in [11, Proof of Theorem 13].

### A.2. Number of extrema of the convex hull having the least amount of vertices.

The following is an interesting result dealing with the number of extrema of a convex hull in $\Delta^{J-1}$ – but not in any smaller-dimensional Euclidean simplex – having the least amount of vertices.

**Proposition 17.** Call $\mathcal{H} \subset \Delta^{J-1}$, $J \in \mathbb{N}$, a polytope such that

$$\tilde{\varepsilon} := \#ex(\mathcal{H}) = \min_{n \in \mathbb{N}} n$$

subject to $\#q \in \{2, \ldots, J\}$ : $\mathcal{H} \subset \Delta^{J-q}$

Then, $\tilde{\varepsilon} = J$.

### A.3. Clarification of equations (6) and (7).

In this appendix, we elucidate the meaning of “almost surely” in equations (6) and (7). Let us start with the former. Consider measurable space $(E_M, \mathcal{B}(E_M) = 2^{E_M})$, call $\mathcal{M}_{E_M} \equiv \Delta(E_M, \mathcal{B}(E_M))$, and equip this latter with sigma-algebra $\mathcal{B}(\mathcal{M}_{E_M})$. A random measure $\mathcal{P}$ is a function on a generic probability space $(\Omega_1, \mathcal{F}_1, P_1)$,

$$\mathcal{P} : (\Omega_1, \mathcal{F}_1, P_1) \rightarrow (\mathcal{M}_{E_M}, \mathcal{B}(\mathcal{M}_{E_M}))$$

so $\mathcal{P}(\omega) \in \mathcal{M}_{E_M}$, for all $\omega \in \Omega_1$. Let now $\mathcal{P}_k \sim DP\left(\alpha P_0 + \sum_{j=1}^k \delta_{\epsilon_j}\right)$ and $\mathcal{P}_\delta \sim \delta_{\nu_{\epsilon_1}}$. Then, equation (6) means that for all $\varepsilon > 0$ and all continuous and bounded functionals $g$ on $E_M$,\(^\text{3}\)

$$P_1\left(\left\{\omega \in \Omega : \left| \sum_{e \in E_M} g(e)\mathcal{P}_k(\omega)(e) - \sum_{e \in E_M} g(e)\mathcal{P}_\delta(\omega)(e) \right| > \varepsilon \right\}\right)$$

$$= P_1\left(\left\{\omega \in \Omega : \left| \sum_{e \in E_M} g(e)\mathcal{P}_k(\omega)(e) - \sum_{e \in E_M} g(e)\nu_{\epsilon_1}(e) \right| > \varepsilon \right\}\right) \xrightarrow{k \to \infty} 0.$$  

Let us then turn our attention to equation (7). In the alternative procedure explained in Remark 16, we have that the labels are treated as random variables, so $\ell$ is regarded as a

\(^3\)The space of continuous and bounded functionals $g$ on $E_M$ is usually denoted by $C_b(E_M)$.
function on a generic probability space \((\Omega_2, \mathcal{F}_2, P_2)\),
\[
\ell : (\Omega_2, \mathcal{F}_2, P_2) \to (\{1, \ldots, M\}, 2^{\{1, \ldots, M\}}),
\]
hence \(\ell(\omega) \in \{1, \ldots, M\}\), for all \(\omega \in \Omega_2\). Let now \(\ell_1, \ldots, \ell_k \sim \zeta\) iid; then, equation (7) coupled with [44, Chapter 1, Remarks 1 and 2] means that for all \(\varepsilon > 0\),
\[
P_2 \left( \left\{ \omega \in \Omega : \sup_{x \in \mathbb{R}} \left| \frac{1}{k} \sum_{j=1}^{k} \mathbb{E} \left[ \ell(\omega)_j \right] (x) - F_\zeta(x) \right| > \varepsilon \right\} \right) \leq \frac{8k + 4}{\exp \left( \frac{\varepsilon}{\delta} k \right)} = \mathcal{O} \left( \frac{k}{\varepsilon^k} \right) \xrightarrow{k \to \infty} 0.
\]

**APPENDIX B. PROOFS**

**Proof of Theorem 4.** In [39, Theorem 6] and [5, Theorem 5], the authors show that, given a convex polytope \(P\) in \(\mathbb{R}^d\), if we call \(P_n\) the convex hull of \(n\) points sampled iid from a uniform on \(P\), then
\[
\mathbb{E} [F_0(P_n)] = \frac{1}{(d + 1)d-1!} T(P)(\log n)^{d-1} + O \left( (\log n)^{(d-2)} \log \log n \right).
\]
Then, since \(\Delta^{J-1}\) is a convex polytope in \(\mathbb{R}^J\), and given the way we defined \(K_n\), equation (2) follows immediately. \(\square\)

**Proof of Theorem 5.** Let us denote by \(\mathbb{K}^2_+\) the set of compact convex sets in \(\mathbb{R}^d\), \(d \geq 2\), having nonempty interior, boundary of differentiability class \(C^2\), and positive Gaussian curvature. Pick any \(K \in \mathbb{K}^2_+\), and sample \(n\) points iid from the uniform on \(K\). Call their convex hull \(P_n\). Then, in [38, Theorem 6], the author shows that there are numbers \(d_n\) bounded between two positive constants depending on \(K\), and a constant \(c(K)\), such that
\[
\left| \mathbb{P} \left( \frac{F_i(P_n) - \mathbb{E}[F_i(P_n)]}{\sqrt{d_n n^{1-2/\pi}}} \leq t \right) - \Phi(t) \right| \leq c(K) n^{-1/(2+3\pi)} (\log n)^{2+3\pi/2}, \tag{12}
\]
The denominator \(\sqrt{d_n n^{1-2/\pi}}\) is of the same asymptotic order as the standard deviation of \(F_i(P_n)\), so the inequality in (12) implies a central limit theorem for \(F_i(P_n)\).

Notice then that \(\Delta^{J-1} \in \mathbb{K}^2_+\) for any \(\mathbb{R}^J\), \(J \geq 2\). Hence, given the way we defined \(K_n\), equation (3) follows immediately. As we can see, the rate of convergence of the distribution of \(F_0(K_n)\) to \(\Phi\) is given by \(n^{-1/(2+3\pi)} (\log n)^{2+3\pi/2} \). \(\square\)

**Proof of Theorem 6.** Call \(E_n\) the extremal set of \(K_n\), that is, \(E_n = \text{ex}(K_n)\). Let \(F_0(K_n) \to \infty\), and call \(\bar{E} \neq \emptyset\) the set that \(E_n\) tends to in the Hausdorff distance
\[
d_H(E_n, \bar{E}) := \max \left\{ \sup_{f \in E_n} \inf_{g \in \bar{E}} d(f, g), \sup_{g \in \bar{E}} \inf_{f \in E_n} d(f, g) \right\}
\]
\[
= \sup_{s \in E_n \cup \bar{E}} \left| \inf_{f \in E_n} d(s, f) - \inf_{g \in \bar{E}} d(s, g) \right|,
\]
\[\text{Possibly different than } (\Omega_1, \mathcal{F}_1, P_1).\]
as the cardinality of $E_n$ approaches infinity. Here $d$ denotes the usual Euclidean distance, and the second equality is an equivalent way of writing the Hausdorff distance. Let $\tilde{K}$ be the convex hull of $\tilde{E}$.

Step 1: We first show that $\tilde{K}$ is well defined. By construction, we know that $\tilde{E} \neq \emptyset$; $\tilde{K}$ is then the convex hull of the points in $\tilde{E}$, which is well defined as we can always construct the convex hull of any given (sub)set of a vector space.

Step 2: Now, we show that $\tilde{K}$ has infinitely many sides. Suppose for the sake of contradiction that $\tilde{K}$ has finitely many sides. Then, it has a finite number of $\ell$-faces, for some $\ell$, which implies a finite number of vertices. But $\tilde{K}$ is the convex hull of the elements in $\tilde{E}$, that are infinite, a contradiction.

Step 3: $\tilde{K}$ is convex: this is immediate from it being the convex hull of $\tilde{E}$.

Step 4: We are left to show that $\tilde{K}$ is the limit of $K_n$. We have seen that $E_n \to \tilde{E}$ in the Hausdorff metric as $n$ goes to infinity; we also know that $K_n = \text{Conv}(E_n)$, for all $n$ (there is a small abuse of notation here: $K_n$ is the convex hull of the elements of $E_n$; since no confusion arises and since we save some notation, we leave it as it is). But then

$$K_n = \text{Conv}(E_n) \xrightarrow{d_H} \text{Conv}(\tilde{E}) = \tilde{K},$$

which concludes our proof.

Proof of Theorem 8. The proof consists of showing that if condition $\mathbb{E}[T(n)] = \gamma \cdot \mathbb{E}[M(n)]$ holds, then the growth rate of the extrema stated in Theorem 4 will hold for a more general procedure. We already know from Theorem 4 that if $S_1, \ldots, S_n \sim \text{Uniform}(\Delta^{J-1})$ iid, then $\lim_{n \to \infty} (\log n)^{-(J-1)} \mathbb{E}[F_0(K_n)] = c(J)$, a value depending on the dimension of the Euclidean space $\mathbb{R}^J$ we work in. Recall that the number of extrema $F_0(K_n)$ of the convex body associated with our mixture model in the uniform case corresponds to $M(n)$. We now relax the assumption in equation (1). Fix any $n \in \mathbb{N}$. Let then $\mathbb{E}[T(n)]$ be the expected number of extrema of $K_n$. Assume that $\mathbb{E}[T(n)] = \gamma \cdot \mathbb{E}[M(n)]$, for some $\gamma \in \mathbb{R}$, $\gamma \neq 0$. Then, by Theorem 4, we have that $\lim_{n \to \infty} (\log n)^{-(J-1)} \mathbb{E}[T(n)] = \gamma \cdot c(J)$. Equation (4) then follows by putting $c(J, \gamma) = \gamma \cdot c(J)$. □

Proof of Theorem 9. By hypothesis, we have that for all $n \geq N$, $\mathbb{E}[T(n)] = \gamma_n \mathbb{E}[M(n)]$. In addition, by Theorem 4 we have that

$$\lim_{n \to \infty} \frac{1}{(\log n)^{J-1}} \mathbb{E}[M(n)] = c(J).$$

Hence we obtain that

$$\lim_{n \to \infty} \frac{\mathbb{E}[T(n)]}{\gamma_n (\log n)^{J-1}} = \lim_{n \to \infty} \frac{\gamma_n \mathbb{E}[M(n)]}{\gamma_n (\log n)^{J-1}} = \lim_{n \to \infty} \frac{\mathbb{E}[M(n)]}{(\log n)^{J-1}} = c(J),$$

concluding the proof. □

Proof of Corollary 10. Suppose that the assumptions of Theorem 9 hold and that $\gamma_n = O(\nu_n)$. This latter means that there exists $M \in \mathbb{R}$ and $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\frac{\gamma_n}{\nu_n} \leq M.$$
Then,
\[ \lim_{n \to \infty} \frac{\mathbb{E}[T(n)]}{\varpi_n (\log n)^{J-1}} = \lim_{n \to \infty} \frac{\gamma_n \mathbb{E}[M(n)]}{\varpi_n (\log n)^{J-1}} = \lim_{n \to \infty} \frac{\gamma_n}{\varpi_n} \lim_{n \to \infty} \frac{\mathbb{E}[M(n)]}{(\log n)^{J-1}} \leq Mc(J), \]
concluding the proof.

Proof of Proposition 13. The proposition is an immediate consequence of Definition 2 and Theorem 3.

Proof of Theorem 15. The need for \( K_M \) to be a (Choquet) simplex comes from Proposition 13. The weak almost sure convergence statement comes from [16, Corollary 4.17], while the rate of convergence comes from [16, Example 8.5]. In this latter, the authors give it relative to the semimetric \( d(P, Q) = |P(A) - Q(A)| \), for a fixed \( A \). But \( \nu_{\pi_k} \) is defined on \( E_M \) that is finite and independent of \( k \), so there is only a bounded number of choices for \( A \). In turn this implies that the rate of convergence holds in terms of the total variation distance as well.

Proof of Proposition 17. Suppose for the sake of contradiction that \( \tilde{e} \neq J \). This means that either \( \tilde{e} > J \), or \( \tilde{e} < J \). If the latter holds, then there exists \( q' \in \{2, \ldots, J\} \) such that \( \mathcal{K} \subset \Delta^{J-q'} \), which contradicts (11). If instead \( \tilde{e} > J \), then we can find \( \mathcal{K}' \subset \mathcal{K} \) such that \( \#ex(\mathcal{K}') < \tilde{e} \), but \( \mathcal{K}' \) is still a proper subset of \( \Delta^{J-1} \), thus again contradicting (11). This concludes the proof.

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