SELF-GRAVITATING FLUID SHELLS AND THEIR NONSPHERICAL OSCILLATIONS IN NEWTONIAN THEORY

JIŘÍ BICÁK 1,2 AND BERND G. SCHMIDT 1
Max-Planck Institute for Gravitational Physics, Am Mühlenberg, 14476 Golm, Germany
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ABSTRACT

We summarize the general formalism describing surface flows in three-dimensional space in a form which is suitable for various astrophysical applications. We then apply the formalism to the analysis of nonradial perturbations of self-gravitating spherical fluid shells. Spherically symmetric gravitating shells (or bubbles) have been used in numerous model problems especially in general relativity and cosmology. A radially oscillating shell was recently suggested as a model for a variable cosmic object. Within Newtonian gravity we show that self-gravitating static fluid shells are unstable with respect to linear nonradial perturbations. Only shells (bubbles) with a negative mass (or with a charge the repulsion of which is compensated by a tension) are stable.

Subject headings: gravitation — hydrodynamics — instabilities — stars: oscillations — supernovae: general

1. INTRODUCTION

It is interesting to see how the problem of modeling thin shells whose thickness is being ignored is employed in so many different fields of science, such as general relativity, astrophysics, cosmology, elasticity, or chemical engineering. In general relativity thin spherical shells of dust or perfect fluids 3 have frequently been used to analyze basic issues of gravitational collapse, in both its classical and quantum aspects (see, e.g., Barraès & Israel 1991; Friedmann, Louko, & Winters-Hilt 1997 and references therein); in astrophysics expanding spherical shells model supernovae (e.g., Vishniac 1983; Sato & Yamada 1991); the chief motivation for studying shells in cosmology has been not only the observations of bubble-like structures in the distribution of galaxies (e.g., Peebles 1993; Turok 1997), but also the physics of the early universe in which a region of false vacuum is separated by a domain wall (modeled usually as a spherical shell) from a region of true vacuum (e.g., Blau, Guendelman, & Guth 1987; Berezin, Kuzmin, & Tkachev 1987; Kolotoch & Eardley 1997a, 1997b); in elasticity the theory of rods and thin shells goes back to the last century (cf. Love 1944); and in chemical engineering the mathematical description of the dynamics of an interface is important in such problems as the calming of water waves by oil or in distillation and liquid extraction (Scriven 1960). A theoretical physicist of the new age would of course add membranes (or rather D-branes) moving in higher dimensional spacetimes in superstring theories.

We, as relativists, worked on various problems connected with thin shells (e.g., Bicák & Ledvinka 1993; Hájíček & Bicák 1997). Recently, one of us investigated nonradial oscillations of static self-gravitating spherical fluid shells in general relativity and their Newtonian limit (Schmidt 1999). However, we could not find a reference to this problem solved within Newtonian theory. Radial oscillations of spherical shells in which gravity is balanced by the surface pressure were analyzed in both the Newtonian and relativistic cases by several authors, even for shells surrounding a compact object (Brady, Louko, & Poisson 1991 and references therein). Most recently, in this journal such radially oscillating shells have been suggested as a model for variable cosmic objects (Núñez 1997). Nonradial oscillations are more difficult and even in Newtonian theory require some differential geometry because, for example, the correct form of the equation of continuity for a surface flow depends on the second fundamental form of an embedded surface in R 3.

In this work we investigate the nonradial oscillations of self-gravitating (or charged) spherical shells in Newtonian theory in detail. In § 2 we review the formalism needed to describe surface flows. Here we essentially follow the exposition given by Aris (1989) in chapter 10 of his book which, in turn, “is in the nature a somewhat extended gloss” on a paper by Scriven (1960) (both works thus emanating from a chemical engineering department). Our discussion is, of course, much shorter; however, it generalizes both mentioned works in two respects. We define a “coordinate system fixed in the surface moving in space” (Aris 1989), and we show how the equation of motion and the continuity equation get modified if other (general) coordinates are used within the surface since in concrete problems the “fixed” (Gaussian) coordinates are not practical at all. Second, although we use index notation we make occasional contact with the formulation of mathematical elasticity theory by Marsden & Hughes (1983) which is based on the index-free formulation of modern differential geometry. In fact, the text of Marsden & Hughes also touches on shell theory, but it does not give the equations of motion, and whenever it uses coordinates these are again only the “Gaussian-type” coordinates. 4

In § 3 we first discuss the general dynamics of a self-gravitating shell which is topologically spherical but may largely deviate from a sphere. We believe that § 2 and the

1 Albert Einstein Institute.
2 Department of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovická 2, 180 00 Prague 8, Czech Republic.
3 By “surface perfect fluids” we mean the surface distributions of matter with isotropic stress distribution tangent to the surface.
4 In Marsden & Hughes (1983), the comprehensive work by Naghdi (1972) is quoted as the standard reference for shells. As much as this work may be preferable for dealing with many aspects of elasticity problems, for our purpose, we found Aris (1989) more useful.
beginning of § 3 may serve as a basic formalism for analyzing, for example, expanding or collapsing nonspherical self-gravitating shells in Newtonian theory in astrophysically realistic situations. In the second part of § 3 we derive the conditions for a spherical shell to be in equilibrium. Section 4 is devoted to the derivation of the equations of motion and the continuity equation for linear perturbations of the static solution obtained in § 3. In § 5 the stability of the static solution is analyzed. We prove that although, with an appropriate equation of state, the shell is stable with respect to radial oscillations, it is unstable if it is perturbed nonradially. It can thus hardly serve as a model for variable cosmic objects as suggested recently (Núñez 1997).

At the end we notice the fictitious, but amusing, case of shells with negative gravitational and inertial mass. Since the time of the interesting work of Bondi (1957) it has been well known that, in principle, a negative mass can exist in the sense that it is not forbidden by classical physics. Some of its amusing properties were recently described by Price (1993). We show that spherical static shells with negative mass are, in fact, stable with respect to nonradial oscillations! Although there is no evidence that a negative mass exists in the real universe, in numerical relativity spacetimes containing negative mass solutions of Einstein’s equations serve as test beds.

Finally, by formally considering the gravitational constant to be negative, we show that charged shells, in which the repulsive effects of the charges are compensated by tension, are stable. We also give intuitive physical arguments for the results of the stability analysis in all three cases.

2. EQUATIONS OF MOTION FOR SURFACE FLUIDS

The flow in a surface is more complicated than an infinitely extended three-dimensional flow because the two-dimensional surface (shell) can move in the three-dimensional space which surrounds it.

Let (t, x) be inertial coordinates in Newtonian spacetime. The metric $g_{a\beta}$ is the time-independent metric on the flat Euclidian three-space $R^3$ (depending on the application one may use Cartesian, polar, or other coordinates).

Functions $x^i = \tilde{x}^i(t, a^\alpha), \alpha = 1, 2$ describe the world lines of the particles of the fluid; $a^\alpha$ are thus comoving (Lagrangian) coordinates. We assume that for fixed times $t$ the points $\tilde{x}(t, a^\alpha)$ form a two-surface $\Sigma_t$ in Euclidian space. We may think of the shell as a three-surface in four-dimensional spacetime which is formed by the flow lines or as a sequence of two-surfaces $\Sigma_t$ in $R^3$.

The space component of the vector tangent to the curves in four-space, i.e., the velocity of a particle of the fluid in $R^3$, is

$$U^i = \left(\frac{\partial \tilde{x}^i}{\partial t}\right)_a,$$

and its acceleration is

$$A^i = \left(\frac{\partial^2 \tilde{x}^i}{\partial t^2}\right)_a.$$

We are free to use arbitrary coordinates $a^\alpha$ on $\Sigma_t$ (Objects intrinsic to $\Sigma_t$ will have Greek indices; in $R^3$ we use Latin indices.) In particular, we will use coordinates $y^\alpha$, obeying the condition that the velocities of the points with $y^\alpha = \text{const}$ are orthogonal to $\Sigma_t$ for any $t$. In these coordinates we describe the shell by

$$x^i = f^i(t, y^\alpha).$$

Then the vectors in $R^3$ given by

$$t^i_a = \left(\frac{\partial f^i}{\partial y^\alpha}\right)_a,$$

are tangent to $\Sigma_t$ and the velocities of the points $y^\alpha = \text{const}$ are parallel to $t^i_a$:

$$g_{ij} t^i_a \left(\frac{\partial f^j}{\partial y^\alpha}\right)_a = 0,$$

where $g_{ij}$ is the metric in $R^3$. These coordinates can be constructed by drawing the orthogonal curves to the family of two-surfaces $\Sigma_t$ in $R^3$. Choosing some coordinates $y^\alpha$ on one surface and taking $y^\alpha$ constant along the orthogonal congruence defines the coordinates $y^\alpha$. These coordinates are unique up to a transformation $y^\alpha(y^\beta)$, independent of $t$. (In general relativity such a coordinate system is called “with vanishing shift”; see, e.g., Wald 1984.)

In general, it may be more convenient to use other coordinates in $\Sigma_t$, say $z^\alpha$, and to describe the moving $\Sigma_t$ in $R^3$ by

$$x^i = \zeta^i(t, z^\alpha), \quad t^i_a = \left(\frac{\partial \zeta^i}{\partial z^\alpha}\right)_a,$$

although $(\partial \zeta^i/\partial t)_a$ is not perpendicular to $t^i_a$; imagine, for example, a motion of the spherical surface into a highly oblate ellipsoidal surface which is described by

$$r = R(t, \theta, \varphi)$$

in the standard spherical coordinates in $R^3$, with $z^\alpha = (\theta, \varphi)$.

We assume $y^\alpha = \tilde{y}^\alpha(t, a^\alpha)$ and, inversely, $a^\alpha = \tilde{a}^\alpha(t, y^\alpha)$; the same for $z^\alpha$. The two-dimensional metric, $h_{\alpha\beta}(t, y^\alpha)$, determines the line element in $\Sigma_t$,

$$dl^2 = h_{\alpha\beta} dy^\alpha dy^\beta,$$

which can be considered as induced by (pull-back of) the metric $g_{ij}$ of $\Sigma_t$,

$$h_{\alpha\beta} = \frac{\partial f^i}{\partial y^\alpha} \frac{\partial f^j}{\partial y^\beta} g_{ij},$$

similar to $z^\alpha$. The surface velocity of the fluid is defined by

$$V^\alpha = \left(\frac{\partial \tilde{y}^\alpha}{\partial t}\right)_a,$$

and the acceleration by

$$A^\alpha = \left(\frac{\partial V^\alpha}{\partial t}\right)_a + V^\alpha \Box V^\beta.$$

where the vertical bar denotes the covariant derivative with respect to $h_{\alpha\beta}$. Such quantities defined analogously in general surface coordinates $z^\alpha$ do not have a natural geometrical (physical) meaning as in the equations (10) and (11).

In order to see this, imagine a particle of the fluid with fixed $a^\alpha$ moving in $R^3$ according to $x^i = \tilde{x}^i(t, a^\alpha)$. Its velocity $U^i$ and acceleration $A^i$ are given by equations (1) and (2). Using the coordinates $y^\alpha$, these may be written as

$$U^i = \frac{\partial f^i}{\partial t} + t^i_a V^\alpha,$$

$$A^i = \frac{\partial \tilde{y}^i}{\partial t} + \left(\frac{\partial f^j}{\partial y^\alpha}\right)_a t^a_j V^\alpha,$$

and
\[ A^i = \frac{\partial U^i}{\partial t} + U^i \cdot V^a, \]

where \( t^i, V^a \) are given by equations (4) and (10) and the covariant derivative is defined by \( U^i_{,a} = U^i_{,a} + \Gamma^i_{jk} V^j H^k_0 \). Let \( n^i \) be the unit normal to \( \Sigma \). Regarding equations (12) and (5), we find

\[ t^i U_i = V_a, \quad n_i U^i = n_i \frac{\partial f^i}{\partial t}, \]

i.e., equation (12) represents the decomposition of the velocity in \( R^3 \) into its normal and tangential parts with respect to \( \Sigma \). (In geometrical language, \( V_a \) is the pull-back of \( U_i \) to \( \Sigma \).) It is easy to see that one can also write

\[ A^i = (n_i A^i)n^i + t^i A^a, \]

where \( A^i \) is given by equation (11) so that it represents the particle’s acceleration along \( \Sigma \). Now, using general coordinates \( z^a \) for which equation (5) is not satisfied, we can still write

\[ \tilde{V}^a = \left( \frac{\partial z^a}{\partial t} \right)_a, \]

for \( \tilde{A}^a \) analogously with equation (11), and we obtain \( U^i \) in the form (see eq. [6])

\[ U^i = \frac{\partial t^i}{\partial t} + \tau^i \tilde{V}^a. \]

However, we now get

\[ \tau^i U_i = \tau^i_0 g_{ij} \left( \frac{\partial z^j}{\partial \tau} \right)_a + \tilde{V}^a, \]

so that \( \tilde{V}^a \) is not a total projection of \( U^i \) on \( \Sigma \) (and neither is it its pull-back) because \( (\partial z^j/\partial \tau)_a \) also has a nonvanishing projection onto \( \Sigma \). When using such coordinates we just have to remember that the surface velocity of the fluid is given by the whole right-hand side of equation (18).

Before considering the dynamics let us point out that \( h_{ab} \) and \( k_{ab} \) are geometric quantities depending only on the position of the surface \( x^i = (t^i, z^a) \) at a given time but independent of the particles’ flow. In fact, the set of surfaces intrinsically defines the normal velocity field \( U^i(t, z^a) \) at each point of the surface at a given time. The explicit expression for this field can be given in terms of our Gaussian coordinates by \( (\partial f^i/\partial t)_a = U_i n^i \) After introducing particles, their \( U^i \) can be decomposed as \( U^i = n^i U_0 + V^i \), and \( V^a \) can be determined intrinsically by \( V_a = (V_{\tan}), t_a \).

Now let \( F(t, y^a) \) be any function defined on \( \Sigma \). Denoting by \( S_t \) any part of \( \Sigma \), then one can derive the following analog of Reynolds’s transport theorem for the material (or convective) derivative of the integral of \( F \) over \( S_t \):

\[ \frac{d}{dt} \int_{S_t} F dS = \int_{S_t} \left[ \left( \frac{\partial F}{\partial t} \right)_a + F \left( \frac{\partial}{\partial t} \ln \sqrt{h} \right)_a \right] dS, \]

\[ = \int_{S_t} \left[ \left( \frac{\partial F}{\partial t} \right)_a + F \frac{\dot{h}}{2h} + (F V^a) \right] dS, \]

where \( h = \det (h_{ab}) \) and the dot means \( \partial/\partial t \). In particular, let \( \sigma(y^a, t) \) be the surface density of the fluid. Then, if the mass of any part of \( \Sigma \) is conserved, the transport theorem of equation (19) implies the continuity equation

\[ \frac{d}{dt} + \sigma V^a \left|_a + \sigma \frac{\dot{h}}{2h} = 0, \]

where \( \sigma/\dot{h} = (\partial \sigma/\partial t)_a + V^a \sigma/\partial y^a \) is the material derivative of \( \sigma \).

Let us define the external curvature tensor \( k_{ab} \) of the surface by

\[ t^i_{a;b} = -k_{ab} n^i, \]

where \( n^i \) is the unit normal to \( \Sigma \) as before and the mean curvature is defined by

\[ H = tr k_{ab} \theta^a \theta^b. \]

A short calculation using standard geometry (see, e.g., exercise 10.41 in Aris 1989) shows that the last term in equation (20) can be written as

\[ \frac{\dot{h}}{2h} = H n^i U_i, \]

where \( U^i \) is the space velocity of a fluid particle given by equation (12). The continuity equation (20) can thus be rewritten in the form

\[ \frac{d}{dt} + \sigma V^a \left|_a + \sigma H n^i U_i = 0, \]

which is the form given by Marsden & Hughes (1983) in theorem 5.15 and written in box 5.2 in the Gaussian coordinate system attached to \( \Sigma \), so that equation (5) is satisfied. In general coordinates \( z^a \) on \( \Sigma \), we still obtain the continuity equation in the form of equation (24) if, instead of \( V^a \), we substitute \( g^{ab} \tau^j_{ab} U_i \) given in equation (18); only the expression \( g^{ab} \tau^j_{ab} U_i \) is the component of \( U^i \) parallel to \( \Sigma \) (geometrically the pull-back of \( U_i \) on \( \Sigma \)).

In order to derive equations of motions for the shell, one starts from balancing the rate of change of momentum of a portion of the shell with the total force acting on it. Let the properties of the fluid be described by the surface stress tensor \( T_{ab}^s \). For example, in the case of a Newtonian surface fluid in which the viscous stress depends linearly on the rate of strain, the stress tensor reads (Aris 1989)

\[ T_{ab}^s = -p g_{ab} + \kappa S_{ab} + \nu E^{ab} \]

where \( p \) is the surface pressure, the surface deformation tensor is \( S_{ab} = \frac{\nu}{\sqrt{2}} g_{ab} \frac{\partial \sigma}{\partial y^a} + \frac{\nu}{\sqrt{2}} (V_{\tan}^a + V_{\tan}^b) \), \( E^{ab} = \delta^a_{\alpha} \delta^b_{\beta} + \delta^a_{\beta} \delta^b_{\alpha} - g^{ab} g_{\alpha\beta} \), and \( \kappa \) and \( \nu \) are the coefficients of dilatational and shear surface viscosity. We wrote down equation (25) just for illustration; in the following we shall consider only ideal surface fluids, i.e., \( \kappa = \nu = 0 \), but at the moment we leave a general \( T_{ab}^s \).

Let us now assume that, besides the internal pressure, there acts a surface force, \( F^s \), per unit area of the fluid. We require the balance of momentum in the direction of an arbitrary smooth covariantly constant vector field \( C^a \) in the form

\[ \frac{d}{dt} \int_{S_t} \sigma V^a C_a dS = \int_{S_t} F^s C_a dS + \int_{\partial S_t} T_{ab}^s v_b C_a dl, \]

5 Our definitions agree with those in Marsden & Hughes (1983), but not with Aris (1989): \( b_{ab} = -k_{ab} H(Aris) = -\frac{1}{2} H \).
where
\[ T^\alpha dl \equiv T^\alpha \nu^\alpha_\beta dl \] (27)
is the surface stress vector acting on a linear element \( dl \) (in \( \partial S_i \) in \( S_i \)) with a unit normal \( \nu^\alpha \). Converting the last integral to a surface integral by Green’s theorem, and using the transport theorem of equation (19) and the continuity equation (20) on the left-hand side, we obtain the intrinsic (surface) equations of motion
\[ \sigma A^\alpha = T^\alpha \big|_\beta + F^\alpha, \] (28)
where \( A^\alpha \) is given by equation (11). (As usual, the balance of angular momentum holds since \( T^\phi = T^\phi_\phi \).)

Finally, consider the motion in \( R^3 \). The external force, \( F^i \), will in general have a component normal to the surface \( \Sigma_i \),
\[ F^i = (n_i F^i)n^i + t^i_\alpha F^\alpha, \] (29)
like the acceleration given by equation (15). Notice that \( t^i_\alpha F^\alpha \), and similarly \( t^i_\alpha T^\alpha \), (see eq. [27]), are just space components of the surface external force and surface stress (in geometrical language they are the push-forward vectors of \( F^\alpha \) and \( T^\alpha \)). Starting from the balance of momentum in \( R^3 \) in the direction of an arbitrary smooth covariantly constant vector field \( K^i \) analogous to equation (26) (now with quantities \( \sigma U^i, F^i \) and \( t^i_\alpha T^\phi \)), we find the complete three-dimensional form of the equations of motion:
\[ \sigma A^i = (t^i_\alpha T^\phi) \big|_\beta + F^i. \] (30)
Expressing \( t^i_\alpha \) by using equation (21) we can write them as
\[ \sigma A^i = (t^i_\alpha T^\phi) \big|_\beta - k^j_\alpha T^\phi n^j + F^i. \] (31)
In the case of a perfect fluid the equations of motion become
\[ \frac{dU^i}{dt} = -t^i_\alpha g^\phi p \big|_\beta + p H n^i + F^i, \] (32)
where \( H \) is the mean curvature of equation (22).

It is instructive to project these equations into the directions tangent and normal to \( \Sigma_i \). The tangential part is given by equations (28); the normal part becomes
\[ \sigma(n_i A^i) = p H + n_j F^j, \] (33)
which demonstrates how the surface pressure influences the motion in the direction normal to the surface if the surface is bent in \( R^3 \) so that it has nonzero external curvature.

The equations of motion (32) and the continuity equation (24) determine the motion of the fluid. In the Lagrangian approach the velocity and acceleration are given by equations (1) and (2); all other quantities are also functions of \( (t, a^\alpha) \), and one seeks solutions \( \tilde{x}(t, a^\alpha), \sigma(t, a^\alpha) \), assuming that some equation of state \( p = p(\sigma) \) and external force \( F^i \) are given. In the Eulerian description one solves for \( f(t, y^a) \) and \( \sigma(t, y^a) \) in terms of which \( U^i, t^i_\alpha, V_\alpha, \) and \( g_{ij} \) are determined by equations (4), (9), (10), and (12). Let us recall that we still need not use “Gaussian-type” coordinates \( y^a \) attached to the surface so that equation (5) is satisfied. We can solve for functions \( \tilde{z}(t, z^a) \) (see eq. [6]); we only have to remember that the surface velocity of the fluid (as it appears, for example, in the continuity eq. [24]) is given by the whole right-hand side of equation (18).

3. THE SELF-GRAVITATING SHELL

The gravitational field determined by a surface distribution of matter has a potential \( \Phi \) which is continuous at the surface (see, e.g., Kellog 1967). Assume that the shell \( \Sigma_i \) is topologically a sphere. The derivatives of \( \Phi \) have limits on both sides of the shell, and the (covariant component of) gravitational force at a point of the shell is given by the mean
\[ F_i = -\frac{1}{2} \sigma(\Phi_i + \phi_i), \] (34)
where \( + \Phi, - \phi \) are the potentials on the two sides of \( \Sigma \) and commas denote partial derivatives (see, e.g., Purcell 1965 for a derivation of eq. [34] in the analogous electric case).

Hence, if the shell of perfect fluid moves under its own gravitational field, the continuity equation has the form of equation (24) and the equations of motion (32) become
\[ \frac{dU^i}{dt} = -t^i_\alpha g^\phi p \big|_\beta + p H n^i - \frac{1}{2} \sigma g^{ij}(\Phi_j + \phi_j). \] (35)

In order that these equations indeed determine the motion of the shell, we need to know \( \Phi \) in terms of the shell’s variables.

Let us assume that, in general, the position of the shell is given in spherical coordinates by \( r = R(t, \theta, \phi) \), as in equation (7). Then we want to solve the Poisson equation
\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 4\pi G \rho(t, \theta, \phi), \] (36)
where the matter density is nonvanishing only on \( \Sigma_i \). Introducing the surface matter density \( \sigma(t, \theta, \phi) \), we find
\[ \rho(t, \theta, \phi) = \sigma(t, \theta, \phi) \sqrt{h(t, \theta, \phi)} \frac{R(t, \theta, \phi)}{R^2(t, \theta, \phi)} \sin \theta \delta[r - R(t, \theta, \phi)], \] (37)
where \( \delta \) is the Dirac delta and \( h = \det(\delta_{ij}) \) with \( \delta_{ij} \) being the metric on \( \Sigma_i \) induced by the spatial metric \( g_{ij} \) as in equation (9). Equation (37) follows from the general equations for the three-dimensional distribution \( \delta_r \) with support on a two-surface \( \Sigma_i \), given by \( F(r, \theta, \phi) = 0 \), which for any nice function \( f(r, \theta, \phi) \) requires
\[ \int_{R^3} f(r, \theta, \phi) \delta_r r^2 \sin \theta dr d\theta d\phi = \int_{\Sigma} f |_{\Sigma} dS, \] (38)
where \( f |_{\Sigma} \) is the restriction of \( f \) to \( \Sigma \). We shall return to the solution of equation (36) in the next section. Now we consider the simplest case—that of spherical symmetry.

We thus assume \( \sigma, p \) independent of \( \theta, \phi \); \( U^i = (R, 0, 0) \), and the mean curvature of a sphere of radius \( r = R(t) \) is
\[ H = \frac{2}{R}, \] (39)
the normal \( n^i = (1, 0, 0) \), and the potential vanishes inside and reads
\[ + \Phi = -\frac{GM}{r}, \] (40)
outside the shell. The equations of motion (35) thus reduce to
\[ \sigma R = \frac{2p}{R} - \frac{1}{2} \sigma \frac{GM}{R^2}, \] (41)
and the continuity equation (24) is
\[ \frac{dR}{dt} + 2\sigma R = 0. \] (42)
The continuity equation can be immediately integrated to yield
\[ \sigma R^2 = \sigma_0 R_0^2, \quad (43) \]
where the right-hand side denotes values at a fixed time (say \( t = 0 \)). It is evident that equation (43) means the conservation of mass; it is also instructive to see how the outward-directed radial force due to the surface pressure, i.e., the term \( 2p/R \) in equation (41), can be derived from elementary considerations of the force acting on a surface element \( dS = R^2 \sin \theta \, d\theta \, d\phi \) from its surrounding.

Substituting \( M = 4\pi \sigma R^2 \), an equation of state \( p = \rho(\sigma) \), and eliminating \( \sigma \), by equation (43), from the equation of motion (41), we can solve equation (41) for the radius of the shell \( R(t) \). We shall discuss radial oscillations in the next section. Now we just notice that equation (41) admits a unique static solution for a given \( M \) and \( R = R_0 \). In this static case the pressure is
\[
\bar{p} = \frac{1}{4} \frac{G \bar{M}}{R} = \frac{\pi G \sigma^2}{16\pi} \frac{GM^2}{R^3}.
\]

The positive surface pressure is needed to balance the inward-directed gravitational force.

It is amusing to observe that exactly the same static situation arises if \( M < 0 \), \( \sigma < 0 \). How is it possible that a negative gravitational mass \( \bar{M} \) which repels all masses and thus pushes the elements of the shell outward is compensated by a positive pressure which, as we saw above, exerts, apparently, an outward-directed radial force on each element? The resolution of this paradox comes from the fact that we assumed both gravitational and inertial mass of the shell to be negative. A nongravitational force (as the pressure) acting in a given direction on a negative inertial mass gives it an acceleration in the opposite direction (just by \( F = ma \)!) so the positive pressure, in fact, accelerates the elements of the shell in the inward direction and is just compensated by the repulsive gravitational action of the negative gravitational mass. We shall see how these effects can influence the stability of the static shell in the following sections.

Another alternative is to change the sign of \( G \). Gravity then becomes like electricity in which the charges of the same sign repel each other. Equation (44) says that we need a negative \( \bar{p} \), a tension as in soap bubbles, to balance such a shell.

4. THE LINEARIZED EQUATIONS OF MOTION

In general, the solution of the coupled system of nonlinear equations (24), (35), and (36) describing the motion of a highly deformed shell is complicated. Our goal here is to investigate linearized perturbations (oscillations) of a static spherical shell satisfying the conditions in equation (44). Since the background solution is spherically symmetric, we can, without loss of generality, assume that the perturbations are axisymmetric, i.e., independent of the azimuthal coordinate \( \phi \). The position \( \Sigma_r \) of the shell at time \( t \) is thus given by
\[ r = R(t, \theta), \quad (45) \]
with \( t \) fixed. If we let \( \theta \) and \( t \) change, equation (45) describes a three-surface in (Newtonian) spacetime. Now the fluid can move in the shell; denote its surface velocity by
\[ \tilde{V} = W(t, \theta) \quad (46) \]
The vector
\[ \tau^\theta = \frac{\partial x^\phi}{\partial \theta} = (R, \theta, 1, 0) \quad (47) \]
is tangent to \( \Sigma_r \). The fluid velocity in space, given by \( U^i = (\partial x^i/\partial t) + \tau^i \tilde{V}^\theta \) (cf. eqs. [6] and [17]), reads
\[ U^i(t, \theta) = (\tilde{R} + R_\theta W, W, 0, 0) \]
where \( \tilde{R} = \partial R/\partial t \). The surface metric \( g_{\phi\phi} \) induced on \( \Sigma_r \) is
\[ g_{\phi\phi} = R_\theta^2 + \tilde{R}^2 \]
and equation (18) is indeed satisfied.

The acceleration appearing on the left-hand side of equation (35) is (see eq. [13])
\[ \frac{dU^i}{dt} = \frac{\partial U^i}{\partial t} + U^j \tilde{V}^\theta = \frac{\partial U^i}{\partial t} + \left( \frac{\partial U^i}{\partial \theta} + \Gamma^j_{\theta \phi} U^j \right) \tilde{V}^\phi, \quad (50) \]
which in spherical coordinates implies
\[
\frac{dU^r}{dt} = \tilde{R} + 2R_\theta W + R_\theta W + W(R_\theta W_\theta) - RW^2, \quad (51)
\]
\[ \frac{dU^\theta}{dt} = W + W \tilde{R} + WW_\theta + 2W^2 R_\theta - R \quad (52) \]
(Since \( \tilde{V}^\phi, U^\phi \) are coordinate components, their dimension and thus the dimension of \( W \) is \( \text{cm} \, s^{-1} \), whereas \( U^\phi \) has the usual dimension \( \text{cm} \, s^{-1} \) in cgs units.) The normal to the shell is given by
\[ n_i = \frac{1}{\sqrt{1 + (R_\theta)^2}} \left( 1, - R_\theta, 0 \right), \quad (53) \]
\[ n^i = \frac{1}{\sqrt{1 + (R_\theta)^2}} \left( 1, - R^{-2} R_\theta, 0 \right), \quad (54) \]
and the mean curvature turns out to be
\[ H = \frac{1}{R \sqrt{1 + (R_\theta)^2}} \left[ 2 - \frac{1}{R} (R_\theta \cot \theta + R_\theta) \right] + \frac{1}{1 + (R_\theta/R)^2} \left( \frac{R_\theta}{R} \right)^2. \quad (55) \]

We can write down the exact form of the continuity equation (24) in the general case by substituting the expressions above for \( n_i, U^i, \) and \( H \) and by substituting equation (49) for \( V^\alpha \) since the surface coordinates \( \theta \) and \( \phi \) are generalized coordinates as \( \xi^\alpha \) in § 2 rather than \( \phi^\alpha \) [the lines (\( \theta, \phi \)] = const are not perpendicular to the surface \( r = R(t, \theta) \). However, we shall not analyze the general case further; we shall now linearize both the continuity equation and the equations of motion around the static solution satisfying equation (44).

To derive the linearized equations we consider one-parameter families \( R(t, \theta, \epsilon) \), \( W(t, \theta, \epsilon) \) of shell solutions. The coordinates in the shell and in the embedding space are uniquely fixed. Hence, we obtain a description of the linearized equations in a particular (coordinate) gauge. As always we assume that the family is smooth in \( \epsilon \) and that we can interchange \( \epsilon \)-derivatives and spacetime derivatives.

Let \( R(t, \theta, 0) = \tilde{R}, \, W(t, \theta, 0) = 0 \) be a static shell. We denote background quantities with an overbar and the perturbation of a quantity \( Q \) by \( \delta Q \). It is easy to see that in the
linearized case the coordinates become Gaussian in linear order, i.e., as coordinates \(y^i\) used in § 2; indeed, neglecting higher order terms, equation (5) is satisfied. The linearized acceleration of equations (51) and (52) is

\[ \delta A^i = (\delta R^i, W, 0) \]  

(56)

because the background acceleration vanishes. We have now to determine the radial and tangential components of the linearization of all terms in equations (35). For the inner forces due to the pressure gradient we obtain only a tangential component

\[ - \delta(t^i g^{j\beta} \delta p_{,\beta}) = -t^i g^{j\beta} \delta p_{,\beta} = -\frac{1}{R^2} \delta\beta \delta p_{,\beta} \]  

(57)

because the background pressure is independent of \(\theta\). The perturbation of the normal force is

\[ \delta(pH_n) = \delta p\tilde{H}n^i + \tilde{p} \delta Hn^i + \tilde{p}H \delta n^i . \]  

(58)

Regarding equations (53), (54), and (55) we obtain

\[ \delta n^i = (0, -\delta R^i_{,\theta} 0) , \]  

(59)

\[ \delta H = -\frac{1}{R^2} \delta R^i_{,\theta} + \cot \theta \left(\frac{1}{R^2} \delta R_{,\theta}\right) - \frac{2}{R^2} \delta R . \]  

(60)

The linearized equations of motion (35) can thus be written in the form

\[ \delta \dot{A}^i = -\delta \beta \delta R^{-2} \delta p_{,\beta} + \tilde{H}n^i \delta p + \tilde{p} \delta H \delta n^i + \frac{1}{2} \delta \sigma g^{ij}(\hat{\Phi}_j + \hat{\Phi}_i) - \frac{1}{2} \delta \sigma g^{ij}(\hat{\Phi}_j + \hat{\Phi}_i) , \]  

(61)

where \(i, j = r, \theta, \phi\) and \(g^{00} = R^{-2}\) (and we do not need the \(\phi\)-components because of axial symmetry).

Before calculating the perturbations of the gravitational potential, let us make the standard assumption that all quantities can be decomposed into spherical harmonics. Thus, we write

\[ \delta \sigma = \sum_{l=0}^{\infty} \delta \sigma_l(t) Y_l , \delta \beta = \sum_{l=0}^{\infty} \delta \beta_l(t) Y_l , \delta R = \sum_{l=0}^{\infty} \delta X_l(t) Y_l , \delta W = \sum_{l=0}^{\infty} \delta \eta_l(t) Y_{l,\theta} , \]  

(63)

(64)

where \(Y_l = Y_{l,0}(\theta) Y_{l,\theta} = \delta Y_l / \delta \theta\). The form of \(\delta R\) and \(\delta W\) corresponds to the fact that \(\delta R\) describes a shift whereas \(\delta W\) is a \(\theta\)-component of a velocity (cf. eq. [46]). Because \(Y_{l,\theta} + \cot \theta Y_{l,\theta} = -l(l+1)Y_l\), we obtain from equation (61)

\[ \delta H = \sum_{l=1}^{\infty} \delta \eta_l Y_l , \]  

(65)

The perturbations of the potential can be calculated by integrating the Poisson equation (36) with \(\rho\) obtained by perturbing the \(\delta\)-function source of equation (37). We shall proceed somewhat differently, but we checked that both procedures lead to the same result. Decompose the potential inside and outside the shell into spherical harmonics (we are now omitting the argument \(t\) since it is irrelevant here):

\[ \Phi = \sum_{l=0}^{\infty} a_l(r) Y_l , \]  

(66)

\[ \Phi_r = \sum_{l=0}^{\infty} a_l(r) r^{-l-1} Y_l . \]  

(67)

At the shell, \(r = R(\theta, \epsilon)\), the potential is continuous,

\[ -\Phi[R(\theta, \epsilon), \epsilon] = +\Phi[R(\theta, \epsilon), \epsilon] , \]  

(68)

and its gradient satisfies

\[ \{n^i [\Phi[r, \theta, \epsilon], \epsilon] - \Phi[r, \theta, \epsilon, \epsilon] \}_r = R(\theta, \epsilon) = 4\pi G\sigma(\theta, \epsilon) . \]  

(69)

Linearization of these relations with \(\delta R = \sum_{l=0}^{\infty} \epsilon^l Y_l\), \(R(\theta, \epsilon) = R\) and \(\delta \eta_l = 0, \delta \eta_l = 0\) for \(l \geq 1\) implies

\[ \delta \eta_l R^l = 0 \delta R^l - 1 - \delta \eta_l Y_l R^{-2} \epsilon , \]  

(70)

\[ - (l + 1)\delta \eta_l R^{-l-2} + 2 \delta \eta_l Y_l R^{-3} \epsilon - l \delta \eta_l R^{-l-1} \epsilon = 4\pi G\delta \epsilon_l , \]  

(71)

where \(Y_l = 1/\sqrt{4\pi}\). We can solve for \(\delta \eta_l, \delta \eta_l\) in terms of \(\epsilon_l\) and \(\delta \epsilon_l\):

\[ \delta \eta_l = \frac{1}{2l + 1} \left[ -(l - 1) \delta \eta_l Y_l R^{-l-2} \epsilon - 4\pi G R^{-l+1} \delta \epsilon_l \right] , \]  

(72)

\[ \delta \eta_l = \frac{1}{2l + 1} \left[ -(l - 1) \delta \eta_l Y_l R^{-l-1} \epsilon - 4\pi G R^{-l+1} \delta \epsilon_l \right] . \]  

(73)

For the static background shell we have \(\delta \eta_l Y_l = -4\pi G R^2 \epsilon_l\). Using this we obtain

\[ \delta \eta_l = \frac{4\pi G}{2l + 1} \left[ -(l - 1) \delta \eta_l Y_l R^{-l-1} \epsilon - 4\pi G R^{-l+1} \delta \epsilon_l \right] . \]  

(74)

\[ \delta \eta_l = \frac{4\pi G}{2l + 1} \left[ -(l - 2) \delta \eta_l Y_l R^{-l-1} \epsilon - 4\pi G R^{-l+1} \delta \epsilon_l \right] . \]  

(75)

The gravitational intensity at the shell is

\[ F_l = -\frac{1}{6} [\Phi[r, \theta, \epsilon], \epsilon]_l = -\Phi[r, \theta, \epsilon, \epsilon] - R(\theta, \epsilon) . \]  

(76)

Inserting \(\hat{\Phi}\), we obtain after linearization, using the background that is spherically symmetric, for the covariant \(\theta\)-component (with fixed angular behavior \(Y_l\)) of the force intensity at the shell

\[ F_{l,0} = -\frac{1}{2} [\delta \eta_l R^l + \delta \eta_l R^{-l-1}] Y_{l,\theta} \]  

(77)

and the radial component is

\[ F_r = -\frac{1}{6} [\delta \eta_l R^{-l-1} - \delta \eta_l R^{-l-2} + 2 \delta \eta_l Y_l R^{-3} \epsilon - 4\pi G R^{-l+1} \delta \epsilon_l] Y_l . \]  

(78)

Inserting \(\delta \eta_l, \delta \eta_l,\tilde{\eta}\—for which in the case of \(l = 0\) we omit the background term, \(-GM/2R^2\), since it drops out as a consequence of equation (44)—we obtain

\[ F_{l,0} = -\frac{1}{2l + 1} \left[ -3 \delta \eta_l - 2 R \delta \epsilon_l \right] Y_{l,\theta} \]  

(79)

\[ F_r = -\frac{1}{2l + 1} \left[ -3 \delta \eta_l - 2 R \delta \epsilon_l \right] \left[ (l - 1) + (l + 1)(l + 2) \right] \]  

\[ \times R^{-l-1} \delta \epsilon_l + 4\pi G \delta \epsilon_l Y_l . \]  

(80)

The linearized form of the continuity equation (24) remains to be considered. Since \(\sigma = const\) and \(\dot{\Phi}^i = \dot{U}^i = 0\), the first term, \(\partial \sigma / \partial t + \dot{V}^i \partial \sigma / \partial x^i\), after linearization just becomes \(\dot{\sigma} = \sigma / \dot{t}\), the second term becomes \(\partial \delta \sigma / \partial t + \delta \sigma / \partial W + \delta \sigma / \partial \dot{W}\) cot \(\theta\), and the third \(\dot{\sigma} / \partial \dot{R}\). Substituting the angular decompositions of equations (63) and (64) and using \(Y_{l,\theta} + \cot \theta Y_{l,\theta} = -l(l+1)Y_l\), we obtain the \(l\)-part of
the continuity equation in the form

$$\delta \dot{\sigma}_i - k(l+1) \delta \eta_i + 2 \delta \dot{R}^{-1} \xi_i = 0 \ .$$

(81)

Integrating, we get

$$\delta \sigma_i - k(l+1) \delta \eta_i + 2 \delta \dot{R}^{-1} \xi_i = 0 \ ,$$

(82)

where we put the integration constants equal to zero since

$$\xi_i = \eta_i = 0 \implies \delta \sigma_i = 0 .$$

Multiplying now the intensity components of equations (79) and (80) by $\dot{\sigma}$, assuming the background conditions of equation (44) are satisfied and substituting the perturbed quantities as given above into the equations of motion (62), we obtain

$$\delta \xi_i = 2 \dot{R}^{-1} \delta p_i + \frac{1}{4} G \delta M \dot{R}^{-3} [l(l+1) - 2] \xi_i$$

$$- \frac{1}{2} \frac{GM \dot{R}^{-2} \delta \sigma_i - \frac{1}{2} \ddot{\sigma} \frac{4\pi G}{2l+1}}{x [2l(l-1) \dot{R}^{-1} \xi_i + \delta \sigma_i]} ,$$

$$\delta \eta_i = - \dot{R}^{-2} \delta p_i - \frac{1}{2} G \delta M \dot{R}^{-4} \xi_i$$

$$+ \frac{1}{2} \ddot{\sigma} \frac{4\pi G}{2l+1} \dot{R}^{-2} [3 \delta \sigma_i + 2 \dot{R} \delta \sigma_i] ,$$

(83)

where the first equation is meaningful for all $l \geq 0$ and the second for $l \geq 1 .

Finally, let us assume that the perturbed pressure and matter densities are connected by a linear relation $\delta p = \alpha \delta \sigma$, so that

$$\delta p_i = \alpha \delta \sigma_i \ .$$

(85)

This, by $p = p(\sigma)$, $\delta p = (d p / d \sigma) \delta \sigma$, corresponds to a general equation of state for barotropic fluids. Substituting for $\delta p$ into equations (83) and (84), writing $\dot{\sigma} = M / 4 \pi \dot{R}^2$, and expressing $\delta \sigma_i$ from the continuity equation (68), we arrive at a system of two equations for just $\xi_i$ and $\eta_i$, as follows:

$$\ddot{\xi}_i = \frac{GM}{R^3} \left[ - \frac{l^2 - 3l - 2}{2l+1} \frac{2l(l+1)}{4} \frac{4 \pi \dot{R}^2}{GM} \right] \xi_i$$

$$+ \frac{GM}{R^3} \left[ - \frac{2l+1}{2l+1} \frac{2l(l+1)}{4} \frac{\dot{R}^2}{GM} \right] \eta_i ,$$

(86)

$$\ddot{\eta}_i = \frac{GM}{R^3} \left[ - \frac{2l+1}{2l+1} \frac{2l(l+1)}{4} \frac{\dot{R}^2}{GM} \right] \xi_i$$

$$+ \frac{GM}{R^3} \left[ \frac{2l(l+1)}{2l+1} \frac{2l(l+1)}{4} \frac{\dot{R}^2}{GM} \right] \eta_i ,$$

(87)

where for $l = 0$ only the first equation is meaningful. Since we already used the continuity equation the last two equations are the only equations to be solved for $\xi_i$ and $\eta_i$ to determine the general axisymmetric linearized perturbations.

5. THE STABILITY ANALYSIS

Before investigating stability let us cast the last coupled equations (86) and (87) into a still simpler form. Denoting

$$\ddot{\xi}_i = \xi_i , \quad \ddot{\eta}_i = \dot{R} \eta_i , \quad \beta = \alpha \frac{R}{GM} ,$$

(88)

and the dimensionless time coordinate

$$\tau = \left( \frac{GM}{\dot{R}} \right)^{1/2} t ,$$

(89)

we get

$$\frac{d^2 \xi_i}{d \tau^2} \xi_i + A \xi_i + B \eta_i = 0 \ ,$$

(90)

$$\frac{d^2 \eta_i}{d \tau^2} + C \xi_i + D \eta_i = 0 \ ,$$

(91)

where the coefficients are

$$A = \frac{l^2 - 3l - 2}{2l+1} + \frac{2l(l+1)}{4} + 4 \beta ,$$

$$B = \frac{2l(l+1)}{2l+1} \frac{l(l+1)}{2l+1} - 2l \beta ,$$

$$C = \frac{l+1}{2l+1} - 2 \beta ,$$

$$D = \frac{l(l+1)}{2l+1} + l(l+1) \beta \ .$$

(92)

Applying $d^2 / d \tau^2$ to equation (90) and regarding equation (91), we obtain the following fourth-order equation for $\xi_i$,

$$\frac{d^2 \xi_i}{d \tau^2} + (A + D) \frac{d^2 \xi_i}{d \tau^2} + (AD - BC) \xi_i = 0 \ ,$$

(93)

and the same equation for $\eta_i$. Assuming $\xi_i = \Xi_i e^{i \omega \tau}$, the last equation implies

$$\omega_i^2 - (A + D) \omega_i^2 + AD - BC = 0 \ .$$

(94)

Hence, the frequencies of the oscillations are given by

$$\omega^{(1,2)}_i = \pm \left[ \frac{1}{2} (A + D) \pm \left( \frac{1}{2} (A + D)^2 - (AD - BC) \right)^{1/2} \right] ,$$

(95)

where (1, 2) refers to the ± sign inside the bracket; the first sign (outside the bracket) represents a trivial alternative. The system is stable if—a fixed $\beta$—the $\omega_i$-values are real for all $l$.

Let us first look at radial oscillations. With $l = 0$ we have $A = -3 / 2 + 4 \beta$, $B = 0$, $C = -2 \beta$, and $D = 0$, so that equation (95) gives

$$\omega_{i=0} = \pm (2 \beta - \frac{3}{2})^{1/2} .$$

(96)

The second solution of equation (94), $\omega = 0$, has no meaning since for $l = 0$ only the first equation (92) with $\eta = 0$ is valid. Therefore, we conclude that the shell is stable with respect to radial oscillations if

$$\beta > \frac{3}{2} .$$

(97)

Using the equation of state $\delta p = \alpha \delta \sigma$ and equations (44) and (88), we can write this as the condition

$$\alpha = \frac{\delta p}{\delta \sigma} > \frac{3 GM}{8 R} \Leftrightarrow \frac{\delta \sigma}{\delta p} > \frac{3}{2} .$$

This displays the analogy to the standard stability condition, $\Gamma_i = (\rho / \rho) (\delta p / \delta \rho) > 4 / 3$, for radial adiabatic stellar oscillations. For stronger gravity ($3 GM / 8 R$ large) a stiffer equation of state, $\delta p = \alpha \delta \sigma$, is needed to guarantee stability.
Turning next to dipole ($l = 1$) perturbations we get $A = -4/3 + 4\beta = -B$ and $C = 2/3 - 2\beta = -D$ and equation (95) yields

$$\omega_{l=1}^{(1)} = 0, \quad \omega_{l=1}^{(2)} = 2(3\beta - 1)^{1/2}. \quad (98)$$

Regarding equations (90) and (91), we easily find out that the second solution implies a trivial amplitude $\xi_1 = \eta_1 = 0$, whereas the first, $\omega_{l=2}^{(1)} = 0$, implies time-independent amplitudes; by incorporating the angular parts we easily see that, as usual, they just describe a (small) shift of the origin of the coordinates along the axis $\theta = 0, \pi$.

For quadrupole ($l = 2$) perturbations we get

$$\frac{1}{2}(A + D) = -\frac{2}{5} + 5\beta, \quad AD - BC = -\frac{1}{2}\beta, \quad (99)$$

so that the equation for the frequencies, equation (95), implies

$$\omega_{l=2}^{(1)} = \pm \left\{ -\frac{2}{5} + 5\beta + \left[-\frac{2}{5} + 5\beta + \frac{6}{5}\beta\right]^{1/2}\right\}^{1/2},$$

$$\omega_{l=2}^{(2)} = \pm \left\{ -\frac{2}{5} + 5\beta - \left[-\frac{2}{5} + 5\beta + \frac{6}{5}\beta\right]^{1/2}\right\}^{1/2}. \quad (100)$$

(101)

Since for stable radial oscillations we must have $\beta > \frac{5}{6}$, $\omega_{l=2}^{(2)}$ is not real and therefore the self-gravitating fluid shell is unstable with respect to quadrupole perturbations.

Although this result is of course sufficient to prove instability let us look at perturbations with large $l$. We find that equations (92) imply for $l \to \infty$

$$\frac{1}{2}(A + D) \to \frac{1}{2}l^2(\beta - \frac{1}{2}) + \frac{1}{2}(\beta + \frac{3}{2}) + O(1),$$

$$AD - BC \to -\frac{1}{2}\beta^2 + 0(l). \quad (102)$$

Again, $\beta > \frac{1}{2}$ implies $\frac{1}{2}(A + D) > 0$, but $AD - BC$ is negative so that $\omega_{l=2}^{(2)}$ is imaginary—there is an instability.

Let us now turn to the amusing case of shells with negative mass $M$ (and with also $\bar{\sigma} < 0$ as the inertial mass density). Putting $l = 0$ in the original equation (86) for $\xi_l$ and defining $\beta$ again by equation (88), we obtain

$$\xi_0 = \frac{GM}{R^3} \left(-\frac{3}{2} + 4\beta\right) \xi_0 = 0. \quad (103)$$

Introducing dimensionless time

$$\tau = \left(-\frac{GM}{R^3}\right)^{1/2} t, \quad (104)$$

we find

$$\omega_{l=0} = \pm 2(-\beta + \frac{3}{8})^{1/2}. \quad (105)$$

Since $\beta = \alpha R / GM$, we get stability for any $\beta$ such that

$$-\infty < \beta < \frac{3}{8}, \quad \text{i.e.,} \quad -\frac{3G(-M)}{8R} < \alpha < \infty. \quad (106)$$

Introducing $\tau$ from equation (104) into equations (86) and (87), we obtain again the equations (90) and (91), only with negative signs of $A$, $B$, $C$, and $D$ given by equation (92). Thus, the frequencies of oscillations of the shells with $M < 0$ are

$$\omega_{l=1}^{(1,2)} = \pm \left\{ \frac{3}{2}(A + D) \pm \left[\frac{3}{2}(A + D)^2 - (AD - BC)\right]^{1/2}\right\}^{1/2}. \quad (107)$$

The difference between this expression and equation (95), i.e., the sign change at the first term $\frac{3}{2}(A + D)$, is crucial. For dipole perturbations nothing new arises—one solution for $\omega$ leads to vanishing amplitudes, the other represents just a shift of the origin. However, turning to the quadrupole oscillations we now get frequencies

$$\omega_{l=2}^{(1,2)} = \pm \left\{ \frac{3}{2} - 5\beta \pm \left[\frac{3}{2} - 5\beta + \frac{6}{5}\beta\right]^{1/2}\right\}^{1/2}, \quad (108)$$

which are all real provided that

$$\beta < 0, \quad \text{i.e.,} \quad \alpha > 0. \quad (109)$$

Similarly, regarding equations (102) for large $l$ and taking $\beta < 0$ we find that all frequencies are real:

$$\omega_{l=\infty}^{(1,2)} \approx \pm \frac{1}{2} \left[1 - 4\beta\right]^{1/2}, \quad \omega_{l=\infty}^{(3)} \approx \pm \frac{\beta}{1 - 4\beta}. \quad (110)$$

In fact, one can prove that for any integer $l > 1$ the expression in superbrackets under the square root in equation (107) is positive. Indeed, the expression under the square root in the square brackets is a parabola in $\beta$ for any fixed $l$, and one can show that its minimum is always positive.

Next, one finds that

$$-\frac{1}{2}(A + D) = -\frac{1}{2}(l^2 + 4\beta) + \frac{1}{2}(l^2 + l + 6),$$

which is positive for $\beta < 0$. Hence, the expression with the plus sign in the superbrackets is positive. By forming its product with the corresponding expression with the minus sign one obtains

$$AD - BC = \frac{l(l - 1)(l + 1)}{4(2l + 1)} \left[-\beta(2l - 3)(l + 2) + l - 2\right],$$

which is always positive for an integer $l > 1$ (although not, e.g., for $l = 1.2$), and so $\omega_{l=1,2}$ is always real.

Thus, we conclude that self-gravitating spherical shells with both negative inertial and gravitational mass are stable with respect to small oscillations.

Finally, it is of interest to consider an analogous case in electrostatics. We thus neglect all inductive and radiative properties of the electromagnetic field, taking into account only the fact that the same charges repel each other by Coulomb’s law. This should be a reasonable approximation for charged bubbles. Hence, we put $M = Q > 0$, the total charge of the sphere, and consider a negative gravitational constant $-G = \gamma > 0$. In fact, we can take any negative value for $\gamma$, multiply by $\gamma^{-1}$ both original equations (83) and (84), and then identify by

$$\Sigma = \frac{\bar{\sigma}}{\gamma}, \quad \delta \Sigma = \frac{\delta \sigma}{\gamma} \quad (111)$$

the inertial surface mass density and its perturbation, while leaving $\bar{\sigma} = Q/4\pi R^2$ and $\delta \sigma$ as the background charge density and its perturbation. (We thus consider a fluid of charged particles with a fixed value of the specific charge.)

First, from the equilibrium condition in equation (43) we find that in order to have a static equilibrium we need a negative pressure, i.e., a tension,

$$\bar{p} = -\gamma \frac{Q^2}{14R^4}, \quad (112)$$

which balances the repulsive electric force. Next we assume again a linear relationship

$$\delta p_l = \alpha \delta \sigma_l = \alpha \gamma \delta \Sigma_l. \quad (113)$$

The equation of continuity, equation (82), is valid for both inertial mass and charge. However, in the terms on the right-hand side of the equations of motion (83) and (84), giving the electric force, we of course have to substitute $\delta \sigma_l$. 

It is easy to see that the equations (86) and (87) become

\[
\gamma^{-1} \xi_i = -\frac{Q}{R^3} \left[ -\frac{l^2 - 3l - 2}{2l + 1} - \frac{2 - \ell(l + 1)}{4} + \frac{4\alpha R}{\gamma Q} \right] \xi_i \\
- \frac{Q}{R^2} \left[ -\frac{l(l + 1)^2}{2l + 1} - 2\ell(l + 1) \frac{\alpha R}{\gamma Q} \right] \eta_i, 
\]

(114)

\[
\gamma^{-1} \eta_i = -\frac{Q}{R^4} \left[ -\frac{l + 1}{2l + 1} - \frac{2\alpha R}{\gamma Q} \right] \xi_i \\
- \frac{Q}{R^3} \left[ \frac{l(l + 1)}{2l + 1} + \ell(l + 1) \frac{\alpha R}{\gamma Q} \right] \eta_i. 
\]

(115)

Therefore, writing \( \xi_i = \xi_i, \eta_i = R \eta_i \) as in equation (88) and defining now

\[
\beta = -\frac{\alpha R}{\gamma Q} = \frac{\alpha R}{GQ}, 
\]

(116)

\[
\tau = \left( \frac{\gamma Q}{R^3} \right)^{1/2} t, 
\]

(117)

we arrive at

\[
\frac{d^2 \xi_i}{dt^2} - A \xi_i - B \eta_i = 0, 
\]

(118)

\[
\frac{d^2 \eta_i}{dt^2} - C \xi_i - D \eta_i = 0, 
\]

(119)

where \( A, B, C, \) and \( D \) are given again by equation (92). These are exactly the same equations as those we analyzed for the shells with negative mass. Consequently, we can conclude that the charged shells will be stable with respect to radial and nonradial oscillations if the parameter \( \beta \) in equation (116) is negative. Since \( \alpha > 0 \), this requires

\[
\alpha > 0, 
\]

(120)

i.e., regarding equation (113), a decrease of the tension when the mass and the charge density increase. Consider a static charged shell in which the repulsive effects of the charges are compensated by a tension. If the radius of the shell is increased while its total charge is unchanged, the repulsion decreases and a decreased tension can still bring back the shell into the original radius. The results of the stability analysis of all three problems we discussed can, in fact, be made intuitively plausible by considering the following Figure 1.

In Figure 1a a perturbed self-gravitating shell with positive mass is illustrated. Gravity is pointing always inward. But the perturbed element is pushed inward also by the surface pressure exerted by adjacent elements, so an instability arises. In Figure 1b a shell with a negative mass is considered. Gravity now acts outward; the pressure acts also outward, but it causes an acceleration pointing inward because the inertial mass of the element is negative (see the discussion at the end of §3), and thus the shell with negative mass can be in stable equilibrium. If the element is pushed inward as in Figure 1a, both gravity and pressure give it an acceleration pointing outward. With charged shells the situation is similar to that of a shell with a negative mass. It is now a tension that pulls an element inward in Figure 1b and thus acts against the electric repulsion.

In the theory of adiabatic nonradial stellar oscillations the Schwarzschild discriminant, \( A = (d \ln \rho/dr) - \Gamma \frac{1}{3} (d \ln p/dr), \) plays an important role. If the criterion of convective stability, \( A < 0, \) is violated in the whole star, unstable modes exist (e.g., Ledoux & Walraven 1958). This corresponds to our considerations illustrated in Figure 1. In the situation described in Figure 1a the element suffers a “convective instability,” whereas it is stable in the situation depicted in Figure 1b.
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