CLASSICAL SOLVABILITY OF THE RELATIVISTIC VLASOV-MAXWELL SYSTEM WITH BOUNDED SPATIAL DENSITY

REINEL SOSPEDRA-ALFONSO AND REINHARD ILLNER

Abstract. In [3], Glassey and Strauss showed that if the growth in the momentum of the particles is controlled, then the relativistic Vlasov-Maxwell system has a classical solution globally in time. Later they proved that such control is achieved if the kinetic energy density of the particles remains bounded for all time [4]. Here, we show that the latter assumption can be weakened to the boundedness of the spatial density.

1. Introduction

The relativistic Vlasov-Maxwell (RVM) system describes the time evolution of a collisionless plasma whose particles interact through the self-induced electromagnetic field. The plasma is assumed to be at high temperatures, thus the particles may travel at speeds comparable to the speed of light. For a single species, the model equations are

\begin{align*}
\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_p f &= 0 \\
\partial_t E - \nabla \times B &= -j \\
\partial_t B + \nabla \times E &= 0 \\
\nabla \cdot E &= \rho, \quad \nabla \cdot B = 0.
\end{align*}

We have set to unity the mass and charge of the particles as well as the speed of light. The density function \( f = f(t,x,p) \) depends on time \( t \in (0, \infty) \), position \( x \in \mathbb{R}^3 \) and momentum \( p \in \mathbb{R}^3 \). \( E = E(t,x) \) and \( B = B(t,x) \) denote the electric and magnetic fields, respectively, and \( v \) stands for the relativistic velocity, i.e.,

\[ v = \frac{p}{\sqrt{1 + |p|^2}}. \]

The system (1.1)-(1.4) is coupled via the charge (or spatial) density \( \rho = \rho(t,x) \), and the current density \( j = j(t,x) \), given by

\[ j = \int_{\mathbb{R}^3} v f dp, \quad \rho = \int_{\mathbb{R}^3} f dp. \]

The Cauchy problem to the RVM system is (1.1)-(1.4) with initial data

\[ f|_{t=0} = f_0, \quad E|_{t=0} = E_0, \quad B|_{t=0} = B_0 \]

satisfying the constraints (1.4).
The global in time classical solvability for this system remains an open problem. On the other hand, the existence of local solutions was first proved in [9] for \( f_0 \) having compact support. The key result on global existence is due to Glassey and Strauss and can be found in [3]. They showed that local solutions may be extended globally in time if the momentum support of \( f \) is controlled for all \( t \). Hence the implication that, for smooth initial data, a singularity could occur only if some particles travel at speeds arbitrarily close to the speed of light. Two different versions of this result are given in [7, 1]. An extensive review on the RV M system is provided in the monograph [5].

In [4], Glassey and Strauss weakened the assumption made in [3]. They showed that if the kinetic energy density of the particles (defined in (2.5) below) is assumed to be bounded for all times, then the problem can be reduced to the situation studied in [3], i.e., to the boundedness of the momentum support of \( f \). Recently, Pallard improved this result for compactly supported initial data. In [8], he showed that if

\[
\sup_{0 \leq t \leq T} \left\| \left( 1 + |p|^2 \right)^{\frac{\theta}{2}} f(t, \cdot, p) \right\|_{L_1^2} < \infty,
\]

with \( 6 \leq q \leq \infty \) and \( \theta > 4/q \) (strict), then the \( p \)-support of \( f \) is uniformly bounded for all \( 0 \leq t \leq T \). The assumption made in [4] corresponds to the case \( (\theta = 1, q = \infty) \). Here, we weaken the assumption in [8] a little bit more: we prove that if the spatial density of the particles remains bounded for all times, i.e., if (1.7) holds for \( (\theta = 0, q = \infty) \), then the problem can be reduced to that in [3] and therefore, the existence of classical solutions globally in time follows.

2. Preliminaries

Consider a \( C^1 \) solution of the system (1.1-1.5). We abbreviate the Lorentz force by \( K = E + v \times B \). It clearly satisfies an estimate \( |K| \leq |E| + |B| =: |\bar{K}| \) since \( |v| \leq 1 \). Further, let \( \omega = (x - y) / |x - y| \). Following [3] and [4], we represent the components of the electric field by

\[
4\pi E^i(t, x) = (E^i)_0(t, x) + E^i_T(t, x) + E^i_S(t, x),
\]

where \( (E^i)_0 \) depends on the initial data and

\[
E^i_T(t, x) = \int_{|x-y| \leq t} \int_{\mathbb{R}^3} a^i(\omega, v) f(t - |x-y|, y, p) dp \frac{dy}{|x-y|^2},
\]

\[
E^i_S(t, x) = \int_{|x-y| \leq t} \int_{\mathbb{R}^3} \nabla_p b^i(\omega, v) \cdot (K f)(t - |x-y|, y, p) dp \frac{dy}{|x-y|},
\]

with kernels \( a^i(\omega, v) \) and \( b^i(\omega, v) \) satisfying the estimate

\[
|a^i(\omega, v)| + \left| \nabla_p b^i(\omega, v) \right| \leq c \sqrt{1 + |p|^2}.
\]
If we denote the kinetic energy density\footnote{Actually, this is the mechanical energy density. The kinetic energy differs from the mechanical energy by the energy of the particles at rest, which is a constant. Nevertheless, we follow the previous convention.} of the particles by

\begin{equation}
(2.5) \quad h(t, x) = \int_{\mathbb{R}^3} \sqrt{1 + |p|^2} f(t, x, p) dp,
\end{equation}

and use the substitution \( y = x - \omega (t - s) \) in \((2.2)\) and \((2.3)\), we see that

\begin{equation}
(2.6) \quad |E_T(t, x)| \leq c \int_0^t \int_{|\omega| = 1} h(s, x - \omega (t - s)) d\omega ds
\end{equation}

and

\begin{equation}
(2.7) \quad |E_S(t, x)| \leq c \int_0^t (t - s) \int_{|\omega| = 1} (h \bar{K})(s, x - \omega (t - s)) d\omega ds.
\end{equation}

Similarly, we can represent the magnetic field as in \((2.1-2.3)\), with kernels \( a^i(\omega, v) \) and \( b^i(\omega, v) \) that can be bounded as in \((2.4)\). The estimates \((2.6)\) and \((2.7)\) remain valid for \( B_S \) and \( B_T \) as well. Now, if we gather all these estimates and use the representation \((2.1)\) for \( E \) and the corresponding representation for \( B \), we observe that

\begin{equation}
(2.8) \quad \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^3} h(t, x) \leq c_T
\end{equation}

implies

\begin{equation}
(2.9) \quad \sup_{x \in \mathbb{R}^3} (|E(t, x)| + |B(t, x)|) \leq c_T \left\{ 1 + \int_0^t \sup_{x \in \mathbb{R}^3} (|E(s, x)| + |B(s, x)|) ds \right\}.
\end{equation}

The Gronwall lemma applies and the field remains bounded for all \( t \leq T \).

On the other hand, the characteristics associated to the equation \((1.1)\) are solutions of

\begin{align}
(2.10) \quad \dot{X}(t, x_0, p_0) &= V(P(t, x_0, p_0)) \\
(2.11) \quad \dot{P}(t, x_0, p_0) &= K(t, X(t, x_0, p_0), P(t, x_0, p_0)),
\end{align}

with \((X(0, x_0, p_0), P(0, x_0, p_0)) \equiv (x_0, p_0) \in \text{supp}f_0\). Thus,

\begin{equation}
(2.12) \quad |P(t, \cdot)| \leq |p_0| + \int_0^t (|E(s, X(s, \cdot))| + |B(s, X(s, \cdot))|) ds.
\end{equation}

Hence, the boundedness of the field implies the boundedness of the momentum support of \( f \), as long as \( f_0 \) has compact support in \( p \). Now, if we apply the above reasoning to the approximate sequence of solutions \( \{(f^n, E^n, B^n)\} \) introduced in \cite{3} (and which we will present in the last section), we have that the \( p \)-supports of \( \{f^n\} \) are uniformly bounded in \( n \). But this guarantees the global existence and uniqueness of classical solutions to the system \((1.1,1.6)\), as stated in \cite{3} Theorem 1]. Therefore, the assumption \((2.8)\) (as well as for its approximations \( h^n \), uniformly in \( n \)) implies the existence and uniqueness of classical solutions to the RVM system for all time. This is the result proved in \cite{1}. In this article we show that the condition \((2.8)\) can be relaxed to the boundedness of the spatial density. In detail, we prove
Theorem 1. Let \( f_0 \in C^1_0(\mathbb{R}^6) \) and \((E_0, B_0) \in C^2_0(\mathbb{R}^3)\) satisfy the constraints (1.4). For \( T > 0 \), assume that
\[
\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} f(t, x, p) dp \leq c_T,
\]
as well as for its approximations \( f^n \) defined in (4.1), uniformly in \( n \). Then, there exists a unique \( C^1 \) solution for all time of the corresponding Cauchy problem to the RVM system.

Our proof relies on two key observations. The first one is that we could avoid the uniform estimates on the field used to bound the \( p \)-support of \( f \) in (2.12), (as, for instance, the one that derives from (2.9)), if instead we use estimates on the time-integral of the field along characteristics. This was already noticed in [8] and here we use modifications to some of the estimates provided there. Estimates of this type were successfully used in [2] as well.

As for the second observation, define
\[
W(p) = \sqrt{1 + |p|^2}.
\]
Clearly, \( \dot{W}(p) = v \cdot \dot{p} \). Thus, since \( v \cdot (v \times B) \equiv 0 \), along the characteristics solving (2.10-2.11) we have that
\[
(2.13) \quad \dot{W}(P(t, x_0, p_0)) = V(P(t, x_0, p_0)) \cdot E(t, X(t, x_0, p_0)).
\]
Since \( |v| \leq 1 \), we obtain
\[
(2.14) \quad W(t) \leq W(0) + \int_0^t |E(s, X(s))| ds,
\]
where we have abbreviated \( W(t) = W(P(t, \cdot)) \) and \( E(s, X(s)) = E(s, X(s, \cdot)) \), respectively.

Our strategy is to reduce (2.14) to a Gronwall-type inequality by using estimates on the time-integral of the field along characteristics mentioned above. In turn, this will provide the uniform bound on the \( p \)-support of \( f \) that guarantee the existence of classical solutions to the RVM system for all time. We remark that (2.13) arises naturally in the covariant formulation of the electrodynamics of the relativistic particle [6, Chapter 12]. The quantity \( W(p) \) is the kinetic energy of a single particle, and the set of equations (2.11) and (2.13) describes the general motion of a charged particle in an external electromagnetic field. For the present case that field is induced by the remaining charges of the system and it is computed by means of the Maxwell equations (1.2-1.4).

3. Bounding \( W(p) \)

Let \((f, E, B)\) be a \( C^1 \) solution of the system (1.1-1.5). Define
\[
\begin{align*}
\bar{p}(t) &= \sup \{ |p| : 0 \leq s \leq t, x \in \mathbb{R}^3 : f(s, x, p) \neq 0 \} \\
&= \sup \{ |P(s, x_0, p_0)| : 0 \leq s \leq t, (x_0, p_0) \in \text{supp} f_0 \},
\end{align*}
\]
and set
\[
(3.1) \quad \bar{v}(t) = \frac{\bar{p}(t)}{\sqrt{1 + \bar{p}^2(t)}}, \quad \bar{W}(t) = \sqrt{1 + \bar{p}^2(t)}.
\]
Throughout this section, we assume that for each $0 \leq t \leq T$ and $(x_0, p_0) \in \text{supp} f_0$
\begin{equation}
(3.2) \quad \sup_{0 \leq s \leq t} |\dot{X}(s, x_0, p_0)| \leq \bar{v}(t) < 1,
\end{equation}
and that
\begin{equation}
(3.3) \quad \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^3} \int f(t, x, p) dp \leq c_T.
\end{equation}
The assumption (3.3) readily implies
\begin{equation}
(3.4) \quad h(t, x) \leq \int_{|p| \leq \bar{v}(t)} \sqrt{1 + |p|^2} f(t, x, p) dp \leq c_T W(t).
\end{equation}

Less straightforward is the following implication:

**Lemma 1.** If (3.3) holds, then
\begin{equation}
(3.5) \quad \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \left( |E(t, x)|^2 + |B(t, x)|^2 \right) dx \leq c_T.
\end{equation}

**Proof.** The structure of the Vlasov equation implies that $f$ is constant along characteristics. Therefore, all its $L^q$-norms are preserved in time. In particular, we have that $\|f_0\|_{L^q_{x,p}} = \|f(t)\|_{L^q_{x,p}}, \; 0 \leq t \leq T$. Since $|\nu| \leq 1$, then $|j| \leq \rho$ and therefore
\begin{equation}
\|j(t)\|_{L^2_x}^2 \leq \|\rho(t)\|_{L^\infty} \|\rho(t)\|_{L^1_x} \leq c_T.
\end{equation}

On the other hand, if we multiply (1.2) and (1.3) by $E$ and $B$ respectively, add up the resultant equations and integrate over $\mathbb{R}^3$, we find that
\begin{equation}
(3.6) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left( |E(t, x)|^2 + |B(t, x)|^2 \right) dx = -\int_{\mathbb{R}^3} j(t, x) \cdot E(t, x) dx.
\end{equation}
The Cauchy-Schwarz and Young inequalities, together with the above estimate on the current density yield
\begin{equation}
(3.7) \quad \left| \int_{\mathbb{R}^3} j(t, x) \cdot E(t, x) dx \right| \leq c_T + \frac{1}{2} \int_{\mathbb{R}^3} \left( |E(t, x)|^2 + |B(t, x)|^2 \right) dx.
\end{equation}
Hence, we combine (3.6) with (3.7) and invoke the Gronwall lemma to conclude that the uniform estimate (3.5) indeed holds. \qed

We notice that although a $C^1$ solution to (1.1-1.5) preserves the total energy in time [1] (i.e., the quantity
\[ \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \sqrt{1 + |p|^2} f(t, x, p) dx dp + \frac{1}{8\pi} \int_{\mathbb{R}^3} \left( |E(t, x)|^2 + |B(t, x)|^2 \right) dx \]
does not depend on $t$, which in turn implies (3.5), the approximate solutions used in the next section may not. Nevertheless, they do satisfy (3.6), and so we can use the previous lemma instead.

Now, for an arbitrary non-negative function $g$, define the integrals
\[ I_k(g; t) = \int_0^t \int_0^s (s - \sigma)^k \int_{|\omega| = 1} g(\sigma, X(s) - \omega(s - \sigma)) d\omega d\sigma ds \]
\[ = \int_0^t \int_\sigma^t (s - \sigma)^k \int_{|\omega| = 1} g(\sigma, X(s) - \omega(s - \sigma)) d\omega ds d\sigma, \]
Lemma 2. We further define $I_k(g; \sigma, t)$ as

$$I_k(g; \sigma, t) = \int_0^t (s - \sigma)^k \int_{|\omega|=1} g(\sigma, X(s) - \omega(s - \sigma)) d\omega ds$$

such that

$$I_k(g; t) = \int_0^t I_k(g; \sigma, t)d\sigma.$$

If we use the representation of the electric field introduced in the previous section and apply the estimates (2.6) and (2.7) to (2.14), we find that

$$W(t) \leq W(0) + c_T + cI_0(h; t) + cI_1(h \tilde{K}; t).$$

In [8], Pallard derived useful estimates on $I_k(g; \sigma, t)$. However, unlike in [8], we aim for estimates in terms of the function $\tilde{W}(t)$ introduced above. Hence, we shall need some modifications of Pallard’s estimates, which we present in the following lemma.

**Lemma 2.** For $0 \leq \sigma < t$, the integrals $I_k(g; \sigma, t)$, $k = 0, 1$, satisfy

$$I_0(g; \sigma, t) \leq c\|g(\sigma)\|_{L^\infty} t$$

(3.10)

$$I_1(g; \sigma, t) \leq c\frac{\|g(\sigma)\|_{L^2}^2}{t - \sigma} \int_0^t [1 + \ln \tilde{W}(s)] ds.$$  

(3.11)

**Proof.** The estimate (3.10) is straightforward from (3.8) with $k = 0$. As for (3.11), let us first rewrite the integral $I_1$ in spherical coordinates

$$I_1(g; \sigma, t) = \int_0^t (s - \sigma) \int_0^\pi \int_0^{2\pi} g(\sigma, X(s) - \omega(s - \sigma)) \sin \phi d\phi d\theta ds,$$

where $\omega = \omega(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$. As shown in [8, Lemma 2.1], condition (3.2) implies that the transformation $\pi_\sigma$, defined by

$$(s, \theta, \phi) \mapsto X(s) - \omega(s - \sigma)$$

is a $C^1$-diffeomorphism whose Jacobian determinant has the form

$$J_{\pi_\sigma}(s, \theta, \phi) = \left(\frac{X(s)}{X(s)} \cdot \omega - 1\right) (s - \sigma)^2 \sin \phi.$$

Thus, the Cauchy-Schwarz inequality implies that

$$I_1(g; \sigma, t) \leq \left(\int_0^t (s - \sigma) \int_0^\pi \int_0^{2\pi} g^2(\sigma, X(s) - \omega(s - \sigma)) |J_{\pi_\sigma}(s, \theta, \phi)| d\theta d\phi ds\right)^{\frac{1}{2}}$$

$$\leq \frac{\|g(\sigma)\|_{L^2}^2}{t - \sigma} \left(\int_0^t \int_0^\pi \int_0^{2\pi} \frac{\sin \phi d\phi d\theta ds}{1 - X(s) \cdot \omega}\right)^{\frac{1}{2}}.$$

We estimate the integral

$$\int_0^\pi \int_0^{2\pi} \frac{\sin \phi d\phi d\theta}{1 - X(s) \cdot \omega} \leq \int_{-1}^1 \frac{2\pi du}{1 - |X(s)|u} \leq c(1 + |\ln(1 - \bar{v}(s))|) \leq c(1 + \ln \tilde{W}(s)),$$
and therefore,
\[
\mathcal{I}_1(g; \sigma, t) \leq c \|g(\sigma)\|_{L_2^\infty} \left( \int_{\sigma}^{t} \left[ 1 + \ln \bar{W}(s) \right] ds \right)^{\frac{1}{2}}
\]
\[
\leq c \|g(\sigma)\|_{L_2^\infty} \sqrt{t - \sigma} \int_{\sigma}^{t} \left[ 1 + \ln \bar{W}(s) \right] ds.
\]
In the final step we have used that \(0 < t - \sigma \leq \int_{\sigma}^{t} \left[ 1 + \ln \bar{W}(s) \right] ds\). This completes the proof of the lemma. \(\square\)

From (3.14) we have that \(\|h(\sigma)\|_{L_2^\infty} \leq c_T \bar{W}(\sigma)\). Together with Lemma 1 this implies
\[
\|\left( h|\bar{K} \right)(\sigma)\|_{L_2^\infty} \leq \|h(\sigma)\|_{L_2^\infty} \|\bar{K}(\sigma)\|_{L_2^\infty} \leq c_T \bar{W}(\sigma).
\]
Combining these inequalities and Lemma 2, we find that the integrals \(I_0(h; t)\) and \(I_1(h|\bar{K}; t)\) satisfy the estimates
\[
I_0(h; t) \leq c_T t \int_{0}^{t} \bar{W}(s) ds,
\]
\[
I_1(h|\bar{K}; t) \leq c_T \int_{0}^{t} \bar{W}(s) \sqrt{t - \sigma} \left[ 1 + \ln \bar{W}(s) \right] d\sigma ds
\]
\[
= c_T \int_{0}^{t} \int_{0}^{s} \bar{W}(\sigma) \sqrt{t - \sigma} \left[ 1 + \ln \bar{W}(s) \right] d\sigma ds
\]
\[
\leq c_T \sqrt{t} \int_{0}^{t} \bar{W}(s) \left[ 1 + \ln \bar{W}(s) \right] ds,
\]
where in the last inequality we have used that \(\bar{W}(t)\) is non-decreasing in \(t\).

Finally, (3.12) and (3.13) combined with (3.9) yield
\[
\bar{W}(t) \leq \bar{W}(0) + c_T + c_T \int_{0}^{t} \bar{W}(s) \left[ 1 + \ln \bar{W}(s) \right] ds.
\]

4. **Proof of Theorem 1**

We recall the recursive method introduced in [3] to generate a sequence of approximate solutions. A density argument allows to consider the following initial conditions:

Let \(f_0 \in C^2_0(\mathbb{R}^3)\), \((E_0, B_0) \in C^3_0(\mathbb{R}^3)\) and \((\partial_t E, \partial_t B)_0\) be in \(C^2(\mathbb{R}^3)\). Let also \(f_0(t, x, p) = f_0(x, p)\) and \((E_0, B_0)(t, x) = (E_0, B_0)(x)\).

Define \(f^n\) as the solution of
\[
\begin{align*}
\partial_t f^n + v \cdot \nabla_x f^n + K^{n-1} \cdot \nabla_p f^n &= 0 \\
f^n(0, x, p) &= f_0(x, p),
\end{align*}
\]
(which is linear when \(E^{n-1}\) and \(B^{n-1}\) are given). Since \(f_0 \in C^2\), then \(f^n\) is a \(C^2\) function provided that \(E^{n-1}\) and \(B^{n-1}\) are \(C^2\) as well. Moreover, \(f^n\) is constant along the characteristics of (4.1)
\[
\begin{align*}
\dot{X}_n(t, x_0, p_0) &= V(P_n(t, x_0, p_0)) \\
\dot{P}_n(t, x_0, p_0) &= K^{n-1}(t, X_n(t, x_0, p_0), P_n(t, x_0, p_0)).
\end{align*}
\]

Hence, for each \(n\) we have that \(f^n\) has compact support in \(p\) provided that \(E^{n-1}\) and \(B^{n-1}\) are bounded functions. It follows that \(\rho^n = \int f^n dp\) and \(j^n = \int vf^n dp\) are
Theorem 1. Therefore, the sequence \( \{ \\}
\) concludes the proof of the theorem.

Then, owing to the compact \( \rho^n \) with a constant \( c \)

we deduce the logarithmic Gronwall inequality

\[
\| \rho^n \|_{L_{1,p}^\infty} \leq \| f(t) \|_{L_{1,p}^\infty}, \forall 0 \leq t \leq T.
\]

Hence, recalling that \( \bar{W}_n(t) = \sqrt{1 + \rho^n(t)} \), it is not difficult to check that \( (\ref{4.11}) \) becomes

\[
\bar{W}_n(t) \leq \bar{W}_0 + c_T t + c_T \int_0^t \bar{W}_{n-1}(s) \left[ 1 + \ln \bar{W}_n(s) \right] ds,
\]

where \( \bar{W}_0 \equiv \bar{W}_n(0) \). If we set

\[
\tilde{W}_n(t) = \sup_{k \leq n} \bar{W}_k(t)
\]

we deduce the logarithmic Gronwall inequality

\[
\tilde{W}_n(t) \leq \bar{W}_0 + c_T t + c_T \int_0^t \bar{W}_n(s) \left[ 1 + \ln \bar{W}_n(s) \right] ds.
\]

Then, owing to the compact \( p \)-support of \( f_0 \), we obtain the uniform bound

\[
\sup_{0 \leq t \leq T} \sup_{n \in \mathbb{N}} \bar{p}_n(t) \leq \sup_{0 \leq t \leq T} \sup_{n \in \mathbb{N}} \tilde{W}_n(t) \leq c_T
\]

with a constant \( c_T \) not depending on \( n \). But this is the assumption made in \([3\text{ Theorem 1}]\). Therefore, the sequence \( \{ (\!f^n, E^n, B^n) \} \) converges uniformly in \( t \in [0, T] \), \( x \in \mathbb{R}^3 \) and \( p \in \mathbb{R}^3 \) to the unique classical solution of the RVM system. This concludes the proof of the theorem.
References

[1] Bouchut, F., Golse, F. and Pallard, C., Classical solutions and the Glassey-Strauss theorem for the 3D Vlasov-Maxwell system. Arch. Ration. Mech. Anal. 170:1-15, 2003.
[2] Calogero, S., Global Classical Solutions to the 3D Nordström-Vlasov system. Commun. Math. Phys. 266:343-353, 2006.
[3] Glassey, R. and Strauss, W., Singularity formation in a collisionless plasma could occur only at high velocities. Arch. Rational Mech. Anal., 92:59-90, 1986.
[4] Glassey, R. and Strauss, W., High Velocity Particles in a Collisionless Plasma. Math. Meth. in the Appl. Sci., 9:46-52, 1987.
[5] Glassey, R., The Cauchy Problem in Kinetic Theory. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996. ISBN 0-89871-367-6.
[6] Jackson, J. D., Classical Electrodynamics. Wiley, New York, 1962.
[7] Klainerman, S. and Staffilani, G., A new approach to study the Vlasov-Maxwell system. Commun. Pure Appl. Anal., 1:103-125, 2002.
[8] Pallard, C., On the Boundedness of the Momentum Support of Solutions to the Relativistic Vlasov-Maxwell System. Indiana Univ. Math. J., Vol. 54, No.5, 2005.
[9] Wollman, S., Existence and Uniqueness Theory of the Vlasov Equation. Internal Report, Courant Institute of Math. Science, New York, 1982.

Department of Mathematics and Statistics, University of Victoria, PO BOX 3045 STN CSC, Victoria BC V8W 3P4.
E-mail address, R. Sospedra-Alfonso: sospedra@math.uvic.ca
E-mail address, R. Illner: rillner@math.uvic.ca