Phase Space on a Surface with Boundary via Symplectic Reduction

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Abstract

We describe the symplectic reduction construction for the physical phase space in gauge theory and apply it for the BF theory. Symplectic reduction theorem allows us to rewrite the same phase space as a quotient by the gauge group action, what matches with the covariant phase space formalism. We extend the symplectic reduction method to describe the phase space of the initial data on a slice with boundary. We show that the invariant phase space has description in terms of generalized de Rham cohomology, what makes the topological properties of BF theory manifest. The symplectic reduction can be done in multiple steps using different decompositions of the gauge group with interesting finite-dimensional intermediate symplectic spaces.
1 Introduction

Recently there is a lot of interest in phase space construction for the slices with boundary. The applications involve phase space of matter fields for the “island” approach to the information puzzle [1–3], asymptotic symmetries and soft theorems [4–6] and entanglement entropy computations [7,8]. The common feature of such phase spaces is existence of the degrees of freedom associated to the boundary, which require a proper inclusion into physical phase space. There are multiple different ways of doing so, yet the resulting phase spaces are remarkably similar.

The most common approach is the covariant phase space formalism [9]. The phase space is identified with the space of classical solutions for the equations of motion. Such solutions can be labeled by the boundary data on Cauchy slice, given a uniqueness of the Cauchy problem. The boundary data is constrained by certain components of the equations of motion. The symplectic form on space of solutions, written in terms of the boundary data is degenerate along the gauge transformation directions, so an extra projection on gauge orbits is required.
The covariant phase space approach can be generalized to the slices with boundary. In that case we need to properly analyze the Cauchy problem on space-time with corners, to figure out the relevant initial data and constraints, what turns to be a hard problem. Alternatively, we can use some physical arguments such as edge modes or asymptotic symmetries to modify the phase space. Unfortunately, such modifications rely on fine details of the problem at hand such as existence of a good gauge, reasonable guess on boundary conditions etc.

In this paper we propose a different approach to construction of the physical phase space: symplectic reduction. Physical phase space is identified with the symplectic reduction of the bare phase space, the space of the boundary values of fields, with respect to the gauge symmetry action. The symplectic reduction theorem allows us to rewrite the symplectic reduction as a quotient space by the gauge group action. The quotient representation for the phase space matches with the covariant phase space construction.

The advantage of using the symplectic reduction approach is due to the existence of the reduction in stages procedure. We can split the full gauge group of the theory into several subgroups and perform a consecutive reductions with respect to them. In particular, we can use the subgroup of gauge transformations trivial on the boundary to construct the phase space with asymptotic symmetry action. Unfortunately, most of the symplectic reduction results require finite-dimensional phase spaces. To counter this issue we used the BF theory as the prime example in our analysis. The BF theory is a topological theory with finite-dimensional invariant phase space. We can arrange the reduction in stages in a such a way that the interesting features such as edge modes and asymptotic symmetry can be realized on a finite-dimensional phase space.

In absence of boundary the invariant phase space for BF theory has nice algebraic topology description in terms of de Rham cohomology, what makes topological properties of the theory manifest. We show that this feature of the phase space can be extended to the case with nontrivial boundary. The ordinary de Rham cohomology become modified to the mapping cone, relative or compactly supported de Rham cohomology depending on a particular choice reduction.

2 Phase space in QFT

The modern approach to the QFT uses the path integral as a guiding principle, so in this section we want to describe the field theory phase space using the path integral data.
\subsection{Path integral}

Let us consider QFT on a manifold \( M \). We can define a space of fields \( F(M) \) on \( M \) and describe the partition function \( Z(M) \) for the theory with action \( S : F(M) \to \mathbb{R} \) as the path integral over \( F(M) \)

\[
Z(M) = \int_{F(M)} DA \, e^{iS(A)}, \tag{2.1}
\]

For \( M \) with nontrivial boundary \( \Sigma = \partial M \) we need to additionally specify boundary conditions for the fields on \( \Sigma \) and the partition function \( Z(M) \) acquires dependence on the boundary values of fields. Furthermore the dependence is very special: partition function \( Z(M) \) becomes an element of the Hilbert space \( Z(\Sigma) \) associated with the boundary \( \Sigma \)

\[
Z(M) \in Z(\Sigma). \tag{2.2}
\]

The (2.2) is well known in topological quantum field theory (TQFT) as one of Atiyah axioms \cite{10}. There are many more examples where (2.2) holds, so it is reasonable to conjecture it being the universal property of the path integral. The natural question is:

\textit{How do we construct \( Z(\Sigma) \) from the path integral data?}

The short answer is that the \( Z(\Sigma) \) is constructed by quantization of the boundary phase space \( (\mathcal{M}_\Sigma, \omega_\Sigma) \). Such approach works well for Chern-Simons theory \cite{11}, where the boundary Hilbert space \( Z(\Sigma) \) is a (geometric) quantization of the moduli space of flat connections \( \mathcal{M}_\Sigma \).

In a path integral description the \( \mathcal{M}_\Sigma \) is the space of the boundary value of fields, i.e the pull back \( i_\Sigma^* F(M) \) of the configuration space \( F(M) \) using a natural embedding \( i_\Sigma : \Sigma \hookrightarrow M \). The symplectic structure \( \Omega_\Sigma \) on \( \mathcal{M}_\Sigma \) is encoded in a path integral (2.1) as follows: An action in (2.1)

\[
S(\mathcal{A}) = \int_M L(\mathcal{A}) \tag{2.3}
\]

is an integral of a Lagrangian density \( L(\mathcal{A}) \)\footnote{We use calligraphic letters \( \mathcal{A}, \mathcal{B} \) for the fields on \( M \), capital letters \( A, B \) for fields on \( \Sigma \), the boundary of \( M \).} over \( M \). The variation of \( L(\mathcal{A}) \)

\[
\delta L(\mathcal{A}) = L(\mathcal{A} + \delta \mathcal{A}) - L(\mathcal{A}) = \mathcal{E}(\mathcal{A}) \delta \mathcal{A} + d\Theta(\mathcal{A}, \delta \mathcal{A}) \tag{2.4}
\]
can be rearranged into (Euler-Lagrange) equations $\mathcal{E}(A)$ and a boundary term $\Theta(A, \delta A)$. We can pull back the boundary term to $\Sigma$

$$\theta(A, \delta A) = i_\Sigma^* \Theta(A, \delta A), \quad A = i_\Sigma^* A = A|_\Sigma$$

and identify it with the symplectic potential while the symplectic form $\omega_\Sigma$ on $\mathcal{M}_\Sigma$

$$\omega_\Sigma = \int_\Sigma \delta \theta(A, \delta A).$$

2.2 Gauge theories

The path integral for the gauge theory is defined as an integration over the gauge invariant configurations. Two configurations related by a gauge group are considered identical

$$\Phi_g : A \mapsto A^g = g^{-1}Ag + g^{-1}dg,$$

with $g$ being an element of $G^M$, the gauge group\textsuperscript{2}, a space of maps from $M$ to a Lie group $G$ i.e.

$$g \in G^M = \text{Maps}(M, G).$$

For $M$ without boundary the $Z(M)$ is a number, so the gauge invariance of the path integral (2.1) implies that the action is invariant under the gauge transformation. The invariance of the action implies that the Lagrangian $L(A)$ may change by a total derivative

$$L^g(A) = L(A^g) = L(A) + dJ(A, g).$$

The corresponding symplectic potential density on $\Sigma$ changes by the $\delta$-exact term

$$\theta^g(A, \delta A) = \theta(A, \delta A) + \delta J(A, g)$$

The symplectic form $\omega_\Sigma$ defined in (2.6) stays invariant, so the boundary gauge group $G^\Sigma$ action is a symplectic symmetry of the boundary phase space $(\mathcal{M}_\Sigma, \omega_\Sigma)$.

\textsuperscript{2}In our discussion we use the math literature notation $G^\Sigma$ for the gauge group acting on $\Sigma$, what is very convenient to distinguish the gauge groups associated to $\Sigma$ and boundary $\partial \Sigma$. 

5
2.3 Invariant phase space

The algebra of classical observables is the algebra of functions on phase space $C^\infty(\mathcal{M}_\Sigma)$. The gauge invariant classical observables are $G^\Sigma$-invariant functions. We can describe the gauge invariant observables on $\mathcal{M}_\Sigma$ in terms of arbitrary observables on a small symplectic space

$$\mathcal{M}^{\text{inv}}_\Sigma = \mathcal{M}_\Sigma / G^\Sigma,$$  \hspace{1cm} (2.11)

known as the symplectic reduction of $\mathcal{M}_\Sigma$ with respect to the action of $G^\Sigma$. The symplectic reduction theorem allows us to describe the $\mathcal{M}^{\text{inv}}_\Sigma$ as a quotient space

$$\mathcal{M}^{\text{inv}}_\Sigma = \mu_G^{-1}(0) / G^\Sigma.$$  \hspace{1cm} (2.12)

The moment map $\mu_G : \mathcal{M}_\Sigma \to g^{\Sigma^*}$ takes values in dual Lie algebra $g^{\Sigma^*}$ of $G^\Sigma$. The symplectic reduction realization of the phase space in field theory is well known in case of Chern-Simons theory [12].

3 Phase space construction in presence of boundary

Our phase space construction from section 2 provides a phase space for the manifold $\Sigma$, which is the boundary of the space-time manifold $M$. Being boundary implies that $\Sigma$ does not have the boundary itself i.e. $\partial \Sigma = 0$. Our goal is to generalize the invariant phase space construction to include $\Sigma$ with boundaries. Indeed, for the construction we only need $\Sigma$ to be part of the boundary $\Sigma \subset \partial M$ of codimension 1. Since $\Sigma$ has codimension 1, it is very natural to call it a hyperfurface, while we further will refer to it as a surface for simplicity. The generalization of the construction from section 2 requires to deal with certain problems that we outline below.

3.1 Infinite-dimensional phase spaces

Most of the classical results in symplectic geometry, we review in section 4, especially the symplectic reduction theorem, require finite-dimensional symplectic manifolds. Unfortunately, the phase spaces we typically use in QFT are infinite-dimensional, so there could be potential issues with the symplectic reduction. In present paper we do not try to construct a theory of infinite-dimensional symplectic reduction, but rather we hope that infinite-dimensional reduction works for one of the simples QFT: the BF theory. Furthermore, the BF theory is known to be topological theory with finite-dimensional invariant phase
space for the compact $\Sigma$. We use the topological theory features as a consistency check for infinite-dimensional symplectic reduction results with details described in sections 5 and 8.5.

### 3.2 Boundary terms in symplectic form

Our definition (2.4) of symplectic potential allows for an arbitrary shift of the form

$$\Theta \rightarrow \Theta + dK,$$

so the symplectic form (2.6) is defined up to a possible boundary term

$$\omega_{\Sigma} = \int_{\Sigma} \delta \theta(A, \delta A) + \int_{\partial \Sigma} \delta K(A, \delta A).$$

The boundary terms in symplectic form play the key role in our discussion. There are various ways to fix the boundary terms. One popular approach $[13]$ is to require that the presymplectic potential is invariant under field-dependent gauge transformations. We propose to use the non-degeneracy of $\omega_{\Sigma}$ to fix the boundary terms. The details of our approach are presented in section 7.

### 3.3 Non-degenerate pairing

By definition the symplectic form is a non-degenerate two form. In our analysis we want make the non-degeneracy manifest. In case of abelian BF theory the phase space is a linear symplectic space of the form $V \oplus W$, with vector spaces $V, W$ being differential forms of a certain degree. In section 4.1 we show how to define the symplectic form on $V \oplus W$ from a non-degenerate pairing $V \times W \rightarrow \mathbb{R}$. The pairing for BF theory is the integration of the differential forms over $\Sigma$ and requires a proper modification in presence of boundary, what we discuss in section 7. The gauge symmetries of BF theory can also be identified with the differential forms, while the moment map requires the dual to the gauge symmetry algebra. The dual algebra can be constructed using the same differential form pairing.

### 3.4 Central extension of the symmetry algebra

The Poisson bracket realization of phase space symmetries allows for the central extension, what we briefly review in section 4.2. The phase space for the surface with boundary requires edge mode inclusion, what leads to additional symmetries and centrally extended symmetry algebra. This feature was observed in various situations like Chern-Simons theory $[12]$,
general relativity [14] and other theories. In section 8.1 we show that the BF theory also has centrally-extended algebra of symmetries. In case of centrally extended symmetry the symplectic reduction theorem requires a generalization, which we briefly review in sections 4.5, 4.6 and implement for BF theory in section 8.2.

3.5 Duality in BF theory

The abelian BF theory of $p$-form in $d+1$-dimensional space, at least on the classical level, is the same as BF theory of $(d-p)$-form on the same space. This property follows from the integration by parts

$$S_{BF}^{(p)} = \int_M dA \wedge B = (-1)^{1+p(d-p)}S_{BF}^{(d-p)} + \int_{\partial M} A \wedge B. \quad (3.3)$$

The symplectic form $\omega_\Sigma$, defined from the action above, is independent on the boundary terms, so both $p$-form and $(d-p)$-form theories have the same phase space. Indeed, an explicit constriction in section 5 shows that phase space is manifestly symmetric under the $p \to d-p$ exchange. In a presence of boundary, when we add edge modes and modify symmetries, it is far from obvious that the invariant phase space remains invariant under the duality. In a literature [15] the proposed phase space for BF theory does not have this symmetry, while we show in section 8.5 that there is a phase space with such symmetry.

4 Symplectic geometry review

The symplectic reduction requires certain terminology from symplectic geometry, which we briefly review in this section.

4.1 Symplectic manifolds

A pair $(\mathcal{M}, \omega)$ defines a symplectic manifold, if $\mathcal{M}$ is a smooth manifold, endowed with a non-degenerate closed two-form $\omega$.

There is a particular type of the symplectic manifolds, useful for our analysis, constructed from a vector space and its dual. Given a finite-dimensional vector space $V$, the vector space $\mathcal{M} = V \oplus V^*$ admits a canonical symplectic structure

$$\omega_0((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_2(v_1) - \alpha_1(v_2) = \langle \alpha_1, v_2 \rangle - \langle \alpha_1, v_2 \rangle, \quad (4.1)$$
where we used standard notation for the canonical pairing

$$\langle \cdot, \cdot \rangle : V \oplus V^* \to \mathbb{R}. \quad (4.2)$$

The pair $$(\mathcal{M}, \omega_0)$$ defines a symplectic manifold with linear structure often referred as *linear symplectic space*.

In our analysis we need an infinite-dimensional generalization of the construction above. Given a pair of infinite-dimensional vector spaces $V$ and $W$ and a non-degenerate pairing between them

$$\langle \cdot, \cdot \rangle : V \oplus W \to \mathbb{R}. \quad (4.3)$$

we can define the symplectic form on $\mathcal{M} = V \oplus W$ to be

$$\omega((v_1, w_1), (v_2, w_2)) = \langle w_1, v_2 \rangle - \langle w_1, v_2 \rangle. \quad (4.4)$$

We can write $\omega$ as a 2-form on $\mathcal{M}^3$

$$\omega = \langle \delta v, \delta w \rangle. \quad (4.5)$$

### 4.2 Symmetries

A smooth map $\Phi : \mathcal{M} \to \mathcal{M}$ is called a *symplectomorphism* (or a *canonical transformation*) if it is a diffeomorphism and it preserves the symplectic form i.e.

$$\Phi^* \omega = \omega. \quad (4.6)$$

An infinitesimal version of symplectomorphism, is a vector field $\xi$ such that

$$0 = \mathcal{L}_\zeta \omega = \tau_\zeta (\delta + \delta \arctan) \omega = \delta (\mathcal{L}_\zeta \omega). \quad (4.7)$$

Such vector fields form a Lie algebra $\text{sym}(\mathcal{M})$ with bracket being the vector field bracket. Locally, we can write any closed form as an exact form

$$\tau_\zeta \omega = \delta H_\zeta. \quad (4.8)$$

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$^3$Here we are using $\delta$ for differential on the field space to reserve the usual $d$ for the differential on a space-time.
We can define the *Hamiltonian vector field* as the vector field $\xi$ such that there is a global function $H_\xi$. Hamiltonian vector fields form a Lie algebra $ham(\mathcal{M})$, which is subalgebra of all vector fields on $\mathcal{M}$. Similarly, we can turn smooth functions $C^\infty(\mathcal{M})$ on $\mathcal{M}$ into Lie algebra with *Poisson bracket* defined as

$$\{H_\eta, H_\xi\} = \iota_\xi \iota_\eta \omega. \tag{4.9}$$

The Lie algebra $C^\infty(\mathcal{M})$ in general is a central extension of $ham(\mathcal{M})$ i.e.

$$\{H_\eta, H_\xi\} = H_{[\eta, \xi]} + c(\eta, \xi). \tag{4.10}$$

The simple example of central extension uses $\mathcal{M} = \mathbb{R}^2$ with coordinates $p, q$ and canonical symplectic form $\omega = \delta p \wedge \delta q$. The pair of commuting vector fields $\partial_p$ and $\partial_q$ are Hamiltonian vector fields

$$H_{\partial_p} = q, \quad H_{\partial_q} = -p, \tag{4.11}$$

while the Poisson bracket between them is nonzero

$$\{H_{\partial_p}, H_{\partial_q}\} = 1 \neq 0 = H_{[\partial_p, \partial_q]}. \tag{4.12}$$

### 4.3 Lie group action

The action of a Lie group $G$ on a symplectic manifold $(\mathcal{M}, \omega)$ is a collection of symplectomorphisms

$$\Phi : G \to \text{Diff}(\mathcal{M}) : g \mapsto \Phi_g, \quad \Phi_g : \mathcal{M} \to \mathcal{M}, \quad \Phi_h \circ \Phi_g = \Phi_{gh}, \quad \Phi^* \omega = \omega. \tag{4.13}$$

An infinitesimal version of the Lie group action is a Lie-algebra morphism

$$V : \mathfrak{g} \to \text{sym}(\mathcal{M}) : \epsilon \mapsto V(\epsilon), \quad V([\epsilon, \eta]) = [V(\epsilon), V(\eta)], \tag{4.14}$$

that we will call *weakly hamiltonian action* if the image of $V$ is in $ham(\mathcal{M}, \omega) \subset \text{sym}(\mathcal{M}, \omega)$. The *moment map* $\mu$ for weakly hamiltonian action $G$ on $(\mathcal{M}, \omega)$ is a smooth map $\mu : \mathcal{M} \to \mathfrak{g}^*$ such that

$$\langle \xi, \mu \rangle = H_{V(\xi)}, \quad \forall \xi \in \mathfrak{g}, \tag{4.15}$$

where $\langle \cdot, \cdot \rangle$ is the canonical paring between $\mathfrak{g}$ and $\mathfrak{g}^*$. The dual Lie algebra $\mathfrak{g}^*$, is equipped with the canonical coadjoint action $Ad_{g^{-1}}^*$ of $G$, so we can define the *equivariant moment*
The action of a Lie group $G$ on $(\mathcal{M}, \omega)$ is called *Hamiltonian action* if there exists an equivariant moment map for this action. An existence of equivariant moment map is equivalent to the absence of central extension in Poisson algebra (4.10).

### 4.4 Symplectic reduction

The Hamiltonian action of a Lie group $G$ on a symplectic manifold $(\mathcal{M}, \omega)$ allows us to define new symplectic manifold $(\mathcal{M}^{\text{red}}, \omega^{\text{red}})$, the *symplectic reduction* of $\mathcal{M}$ by $G$ and denoted using $//\$ notation i.e

$$\mathcal{M}^{\text{red}} = \mathcal{M} // G.$$  \hfill (4.17)

The Marsden-Weinstein theorem [16] provides us with explicit construction of $(\mathcal{M}^{\text{red}}, \omega^{\text{red}})$ in the form of quotient space

$$\mathcal{M}^{\text{red}} = \mu^{-1}(0)/G.$$  \hfill (4.18)

The symplectic form on $\mathcal{M}^{\text{red}}$

$$\omega^{\text{red}} = s^* i^*_{\mu} \omega$$  \hfill (4.19)

is expressed in terms of canonical embedding $i_\mu : \mu^{-1}(0) \hookrightarrow \mathcal{M}$ and a section $s : \mathcal{M}^{\text{red}} \to \mu^{-1}(0)$ of a principal $G$-bundle $\pi : \mu^{-1}(0) \to \mathcal{M}^{\text{red}}$. Furthermore the $\omega^{\text{red}}$ is independent of choice of a section.

### 4.5 Non-equivariant symplectic reduction

We can relax an assumption of the moment map being equivariant and define *non-equivariant symplectic reduction*. We can define the measure of non-equivariance by

$$c_g = \mu(\Phi_g(x)) - Ad_{g^{-1}}^* \mu(x)$$  \hfill (4.20)

and use it to define the *affine coadjoint action*

$$G^a : \xi \mapsto g^a \cdot \xi = Ad_{g^{-1}}^* \xi + c_g,$$  \hfill (4.21)
The moment map $\mu$ becomes equivariant with respect to the affine action, so we can modify the quotient space description

$$\mathcal{M}^{red} = \mu^{-1}(0)/G_0^a,$$

(4.22)

where

$$G_0^a = Stab_{G^a}(0)$$

(4.23)

is the stabilizer subgroup of $0 \in g^*$ under the affine action (4.21).

### 4.6 Reduction by stages

Let $G$ and $K$ be two Lie groups with Hamiltonian action on a symplectic manifold $M, \omega$ with commuting actions. Then the following relation holds

$$\mathcal{M} // (G \times K) = (\mathcal{M} // G) // K = (\mathcal{M} // K) // G.$$  

(4.24)

There is generalization of the reduction by stages to the case of non-commuting actions which we can conjecture to be

$$\mathcal{M} // (G \times K) = (\mathcal{M} // G) // K_0^a = (\mathcal{M} // K) // G_0^a$$

(4.25)

were affine actions

$$G^a : \eta \mapsto g \cdot^a \eta = Ad_{g^{-1}}^* \eta + c^K_g, \quad K^a : \epsilon \mapsto k \cdot^a \epsilon = Ad_{k^{-1}}^* \epsilon + c^G_k,$$

(4.26)

are defined from the non-equivariance

$$c^K_g(x) = \mu_K(\Phi_g(x)) - Ad_{g^{-1}}^* \mu_K(x) \neq 0, \quad c^G_k(x) = \mu_G(\Phi_k(x)) - Ad_{k^{-1}}^* \mu_G(x) \neq 0.$$  

(4.27)

Let us illustrate the conjecture in case of the symplectic reduction of the finite-dimensional linear symplectic space. Let $\mathcal{M}$ be an $2N$-dimensional linear symplectic space such that

$$\mathcal{M} = \mathbb{C}[e_1, f_1, e_2, f_2, ..., e_N, f_N], \quad \omega = \sum_{j=1}^{N} \delta e_j \wedge \delta f_j.$$  

(4.28)

Let us further chose the basis such that symplectic actions of $n$-dimensional group $G$ and
$k$-dimensional group $K$ are of the form

$$G : \delta e_i = \epsilon_i, \quad i = 1, \ldots, n,$$

and

$$K : \delta e_{i+n} = \lambda_{i+n}, \quad i = 1, \ldots, k-m; \quad \delta f_i = -\lambda_{k-m+i}, \quad i = 1, \ldots, m. \quad (4.30)$$

The individual actions of $G$ and $K$ are hamiltonian with equivariant moment maps

$$\mu_G = (f_1, \ldots, f_n), \quad \mu_K = (e_1, \ldots, e_m, f_{n+1}, \ldots, f_{n+k-m}), \quad (4.31)$$

but the actions do not Poisson commute i.e.

$$c^K_g(x) = \mu_K(\Phi_g(x)) - Ad^*_g\mu_K(x) = (\epsilon_1, \ldots, \epsilon_m, 0, \ldots, 0) \neq 0, \quad (4.32)$$

$$c^G_k(x) = \mu_G(\Phi_k(x)) - Ad^*_k\mu_G(x) = (-\lambda_{k-m+1}, \ldots, -\lambda_k, 0, \ldots, 0) \neq 0. \quad (4.33)$$

The stabilizer group of the affine action is

$$(G \times K)_0^a : \quad \delta e_i = \epsilon_i, \quad i = m + 1, \ldots, n, \quad \delta e_{i+n} = \lambda_i, \quad i = 1, \ldots, k-m, \quad (4.34)$$

while the reduced phase space

$$\mathcal{M}/\mathcal{G} = \mu^{-1}_G(0)/\mathcal{G} = \mathbb{C}[e_{n+k-m+1}, f_{n+k-m+1}, \ldots, e_N, f_N]. \quad (4.35)$$

The reduction of $\mathcal{M}$ under the $G$–action is

$$\mathcal{M}/\mathcal{G} = \mu^{-1}_G(0)/\mathcal{G} = \mathbb{C}[e_{n+1}, f_{n+1}, \ldots, e_N, f_N], \quad (4.36)$$

while the $K$-action on reduced space is

$$K : \delta e_{i+n} = \lambda_{i+n}, \quad i = 1, \ldots, k-m, \quad (4.37)$$

which is identical to the $K_0^a$ action on $\mathcal{M}$. The $K$-action on $\mathcal{M}/\mathcal{G}$ is hamiltonian with equivariant moment map

$$\tilde{\mu}_K = (f_{n+1}, \ldots, f_{n+k-m}). \quad (4.38)$$
The reduction in stages becomes

\[(\mathcal{M}/G)/G^a_0/\mu^{-1}_K(0) = \mathbb{C}[e_{n+k-m+1}, f_{n+k-m+1}, \ldots, e_N, f_N], \]

which identical to the reduced phase space (4.35). The reduction in stages

\[(\mathcal{M}/K)/G^a_0 \]

can be performed in similar way with the end-result being identical to the (4.35) as well.

5 BF-theory

The abelian p-form BF theory on \((d+1)\)-dimensional manifold \(M\) is a field theory with field space

\[F(M) = \Omega^p(M, \mathbb{R}) \oplus \Omega^{d-p}(M, \mathbb{R})\]

being the space of differential forms. Instead of usual \(\Omega^p(M, \mathbb{R})\) notation for real-valued \(p\)-forms on \(M\) we will use the simplified notation \(\Omega^p(M)\), since all differential forms in our discussion are real-valued.

The action for the BF-theory

\[S[\mathcal{A}, \mathcal{B}] = \int_M d\mathcal{A} \wedge \mathcal{B} \]

is invariant under the two types of gauge transformations

\[G^M : \mathcal{A} \to \mathcal{A} + d\epsilon, \quad \epsilon \in \Omega^{p-1}(M), \quad K^M : \mathcal{B} \to \mathcal{B} + d\lambda, \quad \lambda \in \Omega^{d-p-1}(M).\]

There are many advantages in using the BF theory as a prime example. It is defined in arbitrary dimension and for arbitrary manifold \(M\). The BF action (5.2) is metric-independent, what makes it into a topological theory. The gauge symmetries make the invariant phase space finite-dimensional.

5.1 Phase space

The symplectic space for the theory (5.2) is the space of boundary values of fields

\[\mathcal{M}_\Sigma = i_\Sigma^* F(M) = \Omega^p(\Sigma) \oplus \Omega^{d-p}(\Sigma)\]
with symplectic form
\[ \omega_{\Sigma} = -\int_{\Sigma} \delta A \wedge \delta B. \quad (5.5) \]

For compact \( \Sigma \), i.e. \( \partial \Sigma = 0 \), the symplectic form (5.5) is the canonical symplectic form (4.5) on a linear symplectic space with a pairing being the Poincare pairing
\[ \langle \cdot, \cdot \rangle : \Omega^p(\Sigma) \times \Omega^{d-p}(\Sigma) \rightarrow \mathbb{R} : (A, B) \mapsto \int_{\Sigma} A \wedge B. \quad (5.6) \]

### 5.2 Invariant phase space

The infinitesimal version of gauge transformations (5.3) for the boundary fields
\[ \delta_{g} A = d \epsilon, \quad \delta_{g} B = d \lambda, \quad \epsilon \in g^{\Sigma} = \Omega^{p-1}(\Sigma), \quad \lambda \in k^{\Sigma} = \Omega^{d-p-1}(\Sigma) \quad (5.7) \]
are generated by the symplectic vector fields, i.e.
\[ i_{V(\epsilon)} \omega_{\Sigma} = -(1)^{d-p} \int_{\Sigma} \epsilon \wedge dA = -(1)^{p+1} \int_{\Sigma} \epsilon \wedge dB, \quad (5.8) \]
\[ i_{V(\lambda)} \omega_{\Sigma} = (-1)^{d-p} \int_{\Sigma} \lambda \wedge \delta B = (-1)^{p+1} \int_{\Sigma} \lambda \wedge dA. \quad (5.9) \]

The generating Hamiltonians on \( M_{\Sigma} \) are
\[ H_{\epsilon} = -(1)^{p+1} \int_{\Sigma} \epsilon \wedge dB, \quad H_{\lambda} = (-1)^{(p+1)(d-p)} \int_{\Sigma} \lambda \wedge dA. \quad (5.10) \]

Using Poincare duality we can express the dual Lie algebras
\[ g^{\Sigma*} = \Omega^{p-1}(\Sigma)^*, \quad k^{\Sigma*} = \Omega^{d-p-1}(\Sigma)^*, \quad (5.11) \]
as differential forms, so the canonical pairing becomes the pairing on differential forms
\[ \langle \cdot, \cdot \rangle : g^{\Sigma} \times g^{\Sigma*} \rightarrow \mathbb{R} : (\epsilon, F) \mapsto \langle \epsilon, F \rangle = \int_{\Sigma} \epsilon \wedge F. \quad (5.12) \]

By definition of the moment map
\[ H_{\epsilon} = \langle \epsilon, \mu_{G} \rangle, \quad H_{\eta} = \langle \eta, \mu_{K} \rangle, \quad (5.13) \]
we can construct

\[
\mu_G : \mathcal{M}_\Sigma \to \mathfrak{g}^{\Sigma^*} : \bigoplus \Omega^p(\Sigma) \oplus \Omega^{d-p}(\Sigma) \to \Omega^{d-p+1}(\Sigma) : (A, B) \mapsto (-1)^{p+1} dB \\
\mu_K : \mathcal{M}_\Sigma \to \mathfrak{k}^{\Sigma^*} : \bigoplus \Omega^p(\Sigma) \oplus \Omega^{d-p}(\Sigma) \to \Omega^{p+1}(\Sigma) : (A, B) \mapsto (-1)^{(p+1)(d-p)} dA. \tag{5.14}
\]

Moment maps in (5.14) are equivariant. Equivalently the Poisson algebra of symmetries is trivial

\[
\{H_\epsilon, H_\eta\} = 0, \tag{5.15}
\]

so we can use the symplectic reduction theorem to describe the invariant phase space as

\[
\mathcal{M}_\Sigma^{inv} = \mathcal{M}_\Sigma / \mu_{G, K} = \mu_G^{-1}(0) \cap \mu_K^{-1}(0) / G \times K. \tag{5.16}
\]

The symplectic reduction can be carried explicitly in the form of de Rham cohomology

\[
\mathcal{M}_\Sigma^{inv} = \frac{Z^p(\Sigma) \oplus Z^{d-p}(\Sigma)}{d\Omega^{p-1}(\Sigma) \oplus d\Omega^{d-p-1}(\Sigma)} = H^p(\Sigma) \oplus H^{d-p}(\Sigma). \tag{5.17}
\]

The pullback of \(\omega_\Sigma\) onto \(\mu_G = \mu_K = 0\),

\[
i_\mu^* \omega_\Sigma = - \int_\Sigma \delta A \wedge \delta B, \tag{5.18}
\]

is well-defined on gauge orbits, i.e. it is the same for \(A\) and \(A + d\epsilon\). The choice of section \(s : \mathcal{M}_\Sigma^{inv} \to \mu_K^{-1}(0) \cap \mu_G^{-1}(0)\) is the same as choice of gauge fixing. The reduced symplectic form is independent on choice of section

\[
\omega_\Sigma^{inv} = s^* i_\mu^* \omega_\Sigma = - \int_\Sigma \delta A \wedge \delta B, \quad A \in H^p(\Sigma), \quad B \in H^{d-p}(\Sigma). \tag{5.19}
\]

The invariant symplectic form is a canonical symplectic form (4.5) on a linear symplectic manifold with pairing being the Poincare paring for de Rham cohomology

\[
\langle \cdot, \cdot \rangle : H^p(\Sigma) \times H^{d-p}(\Sigma) \to \mathbb{R}. \tag{5.20}
\]

5.3 BF theory as a topological theory

The invariant phase space (5.17) is described in terms of well known mathematical objects: de Rham cohomology groups \(H^p(\Sigma)\). The de Rham cohomology are known to be finite-dimensional, what makes \(\mathcal{M}_\Sigma^{inv}\) into finite-dimensional phase space. The diffeomorphism
\( f \colon \Sigma \to \Sigma, \) homotopic to the identity, induces the isomorphism \( f^* : H^p(\Sigma) \to H^p(\Sigma) \) of cohomology groups. The phase space (5.17) depends only on topology of \( \Sigma \), as expected in topological theory.

The infinitesimal version of a diffeomorphism \( f : \Sigma \to \Sigma \) is vector field \( v \) on \( \Sigma \). The infinitesimal transformation of \( A \) is

\[
\delta_v A = \mathcal{L}_v A = dv_v A + i_v dA
\]

For \( A \in H^p(\Sigma) \) the second term in (5.21) vanishes, since \( A \) is a closed form, while the first term is the shift by a total derivative, which is an equivalence relation in quotient space construction of the \( H^p(\Sigma) \).

### 5.4 Covariant phase space

Let us describe the BF-theory phase space by the means of covariant phase space construction. The solutions to equations of motion

\[
dA = dB = 0
\]

are parametrized by boundary values

\[
A = i^*_\Sigma A = A|_\Sigma, \quad B = i^*_\Sigma B = B|_\Sigma,
\]

subject to the constraint equations

\[
i^*_\Sigma (dA) = d(i^*_\Sigma A) = dA = dB = 0.
\]

Note that the constraint equations are identical to the \( \mu_G = \mu_K = 0 \). Thus the solution space, under the uniqueness of Cauchy problem, is

\[
\text{Sol}(M) = Z^p(\Sigma) \oplus Z^{d-p}(\Sigma).
\]

The pre-symplectic form on solution space

\[
\omega_{\Sigma}^{\text{cov}} = - \int_\Sigma \delta A \wedge \delta B,
\]

\footnote{Good introduction to this method you can find in [14]}
is degenerate due to (boundary) gauge transformations

$$\delta A = d\epsilon, \quad \delta B = d\lambda.$$ (5.27)

The pre-symplectic form $\omega_{\Sigma}^{cov}$ is identical to the pull back of $i^*_\mu \omega_{\Sigma}$ in (5.18). Gauge orbits are naturally parametrized by cohomology groups, so the gauge-invariant solution space

$$\text{Sol}^{inv}(M) = H^p(\Sigma) \oplus H^{d-p}(\Sigma)$$ (5.28)

is identical to the invariant phase space (5.17) we constructed using the symplectic reduction.

### 5.5 Surface with boundary

De Rham cohomology groups $H^p(\Sigma)$ are well defined for $\Sigma$ with nontrivial boundary, so we can try to use the (5.17) in a presence of boundary. Unfortunately, the symplectic form (5.19) becomes degenerate. We can immediately see the degeneracy if we consider $p = 0$ BF theory where symplectic paring is a map

$$H^0(\Sigma) \times H^d(\Sigma) \rightarrow \mathbb{R}.$$ (5.29)

The homology $H_p(\Sigma)$ are canonically dual\(^5\) to the cohomology groups, while much easier to visualize. The $H_0(\Sigma)$ counts the number of connected components of $\Sigma$. Let us, for simplicity, assume that $\Sigma$ is connected then $H_0(\Sigma) = \mathbb{R}$. The top homology $H_d(\Sigma)$ represents $d$-dimensional cells with no boundary. In absence of boundary, $\Sigma$ itself is the generator of $H_d(\Sigma) = \mathbb{R}$, but in presence of boundary $\partial \Sigma \neq \emptyset$ and $H_d(\Sigma) = 0$.

For connected $d$-dimensional $\Sigma$ with boundary

$$H^0(\Sigma) = H_0(\Sigma) = \mathbb{R}, \quad H^d(\Sigma) = H_d(\Sigma) = 0$$ (5.30)

The difference in dimensions between $H^0(\Sigma)$ and $H^1(\Sigma)$ not only exclude the Poincare paring, but any non-degenerate paring (5.29) is excluded as well.

Our simple demonstration leads to the conclusion: Cohomology groups $H^*(\Sigma)$ are well defined in a presence of boundary, but the naive generalization $\mathcal{M}^{inv} = H^p(\Sigma) \oplus H^{d-p}(\Sigma)$ fails, since $H^p(\Sigma)$ and $H^{d-p}(\Sigma)$ in general have different dimensions, so there is no non-degenerate paring.

\(^5\)We are working over $\mathbb{R}$ so $H^p(\Sigma) = \text{Hom}(H_p(\Sigma), \mathbb{R})$. 

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6  Cohomology for manifold with boundary

The invariant phase space for BF theory from section 5 has description in terms of the de Rham cohomology. Since in presence of boundary de Rham cohomology fail to describe the invariant phase space, we can check the algebraic topology for possible generalization. The algebraic topology textbook [17] provides us with several immediate generalizations of de Rham cohomology that reduce to the (5.17) in absence of boundary.

6.1  Cohomology with compact support

Compact support \( p \)-form \( A \in \Omega^p_c(\Sigma) \) is a \( p \)-form on \( \Sigma \) which is zero outside a compact set \( C \subset \Sigma \). The de Rham differential preserves this property, so we can define the corresponding cohomology, denoted as \( H^p_c(\Sigma) \). There is non-degenerate Poincare paring

\[
H^p_c(\Sigma) \times H^{d-p}_c(\Sigma) \to \mathbb{R} : (A, B) \mapsto \int_\Sigma A \wedge B. \tag{6.1}
\]

which provides us with the phase space candidate

\[
\mathcal{M}^\text{comp}_\Sigma = H^p_c(\Sigma) \oplus H^p_c(\Sigma)^* = H^p_c(\Sigma) \oplus H^{d-p}(\Sigma) \tag{6.2}
\]

For connected \( d \)-dimensional \( \Sigma \) with boundary

\[
H^0(\Sigma) = \mathbb{R}, \quad H^d_c(\Sigma) = \mathbb{R}, \tag{6.3}
\]

so the paring (6.1) is non-degenerate.

6.2  Relative cohomology

For \( C \subset \Sigma \) we can define the relative \( p \)-chains as elements of

\[
C_p(\Sigma, C) = C_p(\Sigma)/C_p(C). \tag{6.4}
\]

The boundary operator \( \partial : C_p(\Sigma) \to C_{p-1}(\Sigma) \) naturally descends to quotient \( C_p(\Sigma, C) \), so we can define the corresponding homology \( H_p(\Sigma, C) \), known as the relative homology. The relative \( p \)-cycle \( c \) is a \( p \)-chain, that can be anchored on \( \partial \Sigma \) i.e

\[
c \in Z_p(\Sigma, C) \iff \partial c \in C_{p-1}(\partial \Sigma). \tag{6.5}
\]
The relative cohomology are part of the Lefschetz paring

$$H^p(\Sigma) \times H^{d-p}(\Sigma, \partial \Sigma) \rightarrow \mathbb{R}, \quad (6.6)$$

which we can use to define the symplectic space

$$\mathcal{M}^{rel}_\Sigma = H^p(\Sigma) \oplus H^p(\Sigma)^* = H^p(\Sigma) \oplus H^{d-p}(\Sigma, \partial \Sigma). \quad (6.7)$$

For connected $d$-dimensional $\Sigma$ with boundary

$$H^0(\Sigma) = \mathbb{R}, \quad H^d(\Sigma, \partial \Sigma) = \mathbb{R}, \quad (6.8)$$

so the paring (6.6) is non-degenerate. Author of [15] used relative cohomology to describe the phase space of BF theory.

### 6.3 Mapping cone cohomology

Let $f : S \rightarrow \Sigma$ be a smooth map, then we can define mapping cone differential forms

$$\Omega^p(f) = \Omega^p(\Sigma) \oplus \Omega^{p-1}(S) \quad (6.9)$$

and differential

$$d_f : \Omega^p(f) \rightarrow \Omega^{p+1}(f) : (B, b) \mapsto (dB, f^*B - db), \quad d_f^2 = 0. \quad (6.10)$$

The cohomology of $(\Omega^*(f), d_f)$ are known as the mapping cone de Rham cohomology and denoted as $H^*(f)$. We can use and embedding map $i_{\partial \Sigma} : \partial \Sigma \hookrightarrow \Sigma$ to define the non-degenerate Poincare pairing

$$H^p(\Sigma) \times H^{d-p}(i_{\partial \Sigma}) \rightarrow \mathbb{R} : (A, (B, b)) \mapsto \int_\Sigma A \wedge B + \int_{\partial \Sigma} A \wedge b \quad (6.11)$$

and the symplectic manifold

$$\mathcal{M}^{cone}_\Sigma = H^p(\Sigma) \oplus H^p(\Sigma)^* = H^p(\Sigma) \oplus H^{d-p}(i_{\partial \Sigma}). \quad (6.12)$$

The mapping cone differential forms are very similar to edge modes. Authors of [18] used mapping cone cohomology to analyze the invariant phase space of YM theory.
6.4 Cohomology relations

The three types of cohomology are related. The relative cohomology $H^*(\Sigma, C)$ are the same as compact support cohomology when $C$ is the boundary of $\Sigma$

$$H^*(\Sigma, \partial \Sigma) = H^*_c(\Sigma).$$  \hfill (6.13)

The mapping cone cohomology $H^*(f)$ are identical to the relative cohomology for $f$ being an embedding map $i_C : C \hookrightarrow \Sigma$

$$H^*(i_C) = H^*(\Sigma, C).$$  \hfill (6.14)

Thus we conclude that three types of cohomology are the same

$$H^*_c(\Sigma) = H^*(\Sigma, \partial \Sigma) = H^*(i_{\partial \Sigma}).$$  \hfill (6.15)

6.5 BF theory duality

In case of the surface $\Sigma$ without boundary the invariant phase space (5.17) is invariant under the $p \to d - p$ transformation. Neither of possible generalizations of the invariant phase space (6.2), (6.7) and (6.12) is invariant

$$\mathcal{M}_\Sigma^{(p)c} = H^p_c(\Sigma) \oplus H^{d-p}(\Sigma) \neq \mathcal{M}_\Sigma^{(d-p)c} = H^{d-p}_c(\Sigma) \oplus H^p(\Sigma).$$  \hfill (6.16)

For connected $d$-dimensional $\Sigma$ with boundary

$$H^0(\Sigma) = H^d_c(\Sigma) = \mathbb{R}, \quad H^0_c(\Sigma) = H^d(\Sigma) = 0,$$

$$\mathcal{M}_\Sigma^{(0)c} = H^0_c(\Sigma) \oplus H^d(\Sigma) = \mathbb{R} \oplus \mathbb{R} \neq \mathcal{M}_\Sigma^{(d)c} = H^d_c(\Sigma) \oplus H^0(\Sigma) = 0.$$  \hfill (6.17)

7 Boundary terms and parings

In section 6 we provided a brief review of the de Rham cohomology generalizations for the manifold with boundary. Each of discussed generalizations is connected to the generalization of the Poincare paring. The Poincare paring (5.19) on finite-dimensional de Rham cohomology can be extended to the paring on the infinite-dimensional space of differential forms (5.5). In this section we propose the extension of the generalized Poincare pairings (6.1) and (6.11) to the differential forms. Such extension allows us to describe the phase space for the BF theory in the presence of boundary.
We use the triangulation of the manifolds to turn the infinite-dimensional spaces of differential forms into finite-dimensional. The existence of non-degenerate paring between two spaces require equality of the dimensions of the two spaces, what can be used to conjecture the structure of the paring. Let us note that the BF theory is topological, so the triangulated version of the theory to describe the invariant phase space identical to the continuous version. For simplicity of our argument we will focus on the \( d = 1 \) theory, with only two possible topologies: the circle \( S^1 \) with no boundary and the interval \( I \) with boundary being the pair of points.

### 7.1 Triangulated phase space

We can triangulate the \( S^1 \) by a graph with \( V \) vertices and \( E \) edges. The spaces of simplexes of dimension 0 and 1 are

\[
C_0(\Sigma) = \mathbb{R}^V, \quad C_1(\Sigma) = \mathbb{R}^E. \tag{7.1}
\]

The forms on the triangulation are functions on simplexes

\[
C^0(\Sigma) = \text{Hom}(C_0(\Sigma), \mathbb{R}) = C_0(\Sigma) = \mathbb{R}^V, \quad C^1(\Sigma) = \text{Hom}(C_1(\Sigma), \mathbb{R}) = C_1(\Sigma) = \mathbb{R}^E. \tag{7.2}
\]

The triangulation of \( S^1 \) is such that \( E = V \), so the spaces of 0-forms and 1-forms are identical

\[
C^0(\Sigma) = \mathbb{R}^V \simeq \mathbb{R}^E = C^1(\Sigma) \tag{7.3}
\]

what we can equivalently reformulate as discrete version of Poincare duality

\[
C^0(\Sigma)^* = C^1(\Sigma), \quad C^0(\Sigma)^* = C^0(\Sigma). \tag{7.4}
\]

In case of an interval \( \Sigma = I = [0, 1] \) with boundary \( \partial I = \{0\} \sqcup \{1\} \) the triangulation has

\[
C_0(\Sigma) = \mathbb{R}^V, \quad C_1(\Sigma) = \mathbb{R}^E, \tag{7.5}
\]

but \( V - E = 1 \), so there is no canonical isomorphism between \( C_0 \) and \( C_1 \). Similar to the cohomology we can try to modify the (7.4) by considering the compact support forms or using the mapping cone construction.
7.2 Compactely supported forms

Relative forms in our discrete model are forms that vanish on the boundary

\[
C^0(\Sigma, \partial \Sigma) = C^0(\Sigma)/C^0(\partial \Sigma) = \mathbb{R}^V/\mathbb{R}^2 = \mathbb{R}^{V-2},
\]
\[
C^1(\Sigma, \partial \Sigma) = C^1(\Sigma)/C^1(\partial \Sigma) = \mathbb{R}^E,
\]

so we can define embeddings

\[
C^0(\Sigma, \partial \Sigma)^* = \mathbb{R}^{V-2} \hookrightarrow \mathbb{R}^{V-1} = \mathbb{R}^E = C^1(\Sigma),
\]
\[
C^1(\Sigma, \partial \Sigma)^* = \mathbb{R}^E \hookrightarrow \mathbb{R}^{E+1} = C^0(\Sigma).
\]

The continuous version can be conjectured being

\[
\Omega^p(\Sigma, \partial \Sigma)^* \sim \Omega^{d-p}(\Sigma),
\]

while the pairing is

\[
\Omega^p(\Sigma, \partial \Sigma) \times \Omega^{d-p}(\Sigma) \to \mathbb{R} : (A; B) \mapsto \int_{\Sigma} A \wedge B.
\]

7.3 Mapping cone forms

Mapping cone forms for the triangulation of an interval \(\Sigma\) with boundary \(\partial \Sigma\)

\[
C^0(i_{\partial \Sigma}) = C^0(\Sigma) \oplus C^{-1}(\partial \Sigma) = \mathbb{R}^V
\]
\[
C^1(i_{\partial \Sigma}) = C^1(\Sigma) \oplus C^0(\partial \Sigma) = \mathbb{R}^E \oplus \mathbb{R}^2 = \mathbb{R}^{E+2},
\]

so can define embeddings

\[
C^1(\Sigma)^* = \mathbb{R}^E \hookrightarrow \mathbb{R}^{E+1} = \mathbb{R}^V = C^0(\Sigma),
\]
\[
C^0(\Sigma)^* = \mathbb{R}^V \hookrightarrow \mathbb{R}^{E+2} = C^1(i_{\partial \Sigma}).
\]

In case or dimensions \(d\) higher then 1 and arbitrary \(p\)

\[
C^p(\Sigma)^* \hookrightarrow C^{d-p}(\Sigma) \oplus C^{d-p-1}(\partial \Sigma) = C^{d-p}(i_{\partial \Sigma}).
\]

The continuous version is conjectured to be

\[
\Omega^p(\Sigma)^* \sim \Omega^{d-p}(\Sigma) \oplus \Omega^{d-p-1}(\partial \Sigma) = \Omega^{d-p}(i_{\partial \Sigma}),
\]
while the pairing is
\[ \Omega^p(\Sigma) \times \Omega^{d-p}(\partial \Sigma) \rightarrow \mathbb{R} : (A; B, b) \mapsto \int_\Sigma A \wedge B + \int_{\partial \Sigma} A \wedge b. \] (7.14)

### 7.4 Phase spaces

Using the two pairings (7.9) and (7.14) we can construct two linear symplectic spaces for the surface \( \Sigma \) with boundary:

- **Compact field phase space**
  \[ \mathcal{M}_\Sigma^{\text{comp}} = \Omega^p(\Sigma, \partial \Sigma) \oplus \Omega^{d-p}(\Sigma), \] (7.15)
  with symplectic form
  \[ \omega^{\text{comp}}_\Sigma = -\int_\Sigma \delta A \wedge \delta B. \] (7.16)

- **Edge mode phase space**
  \[ \mathcal{M}_\Sigma^{\text{edge}} = \Omega^p(\Sigma) \oplus \Omega^{d-p}(\Sigma) \oplus \Omega^{d-p-1}(\partial \Sigma), \] (7.17)
  endowed with symplectic form
  \[ \omega^{\text{edge}}_\Sigma = -\int_\Sigma \delta A \wedge \delta B - \int_{\partial \Sigma} \delta A \wedge \delta b. \] (7.18)

The field \( b \in \Omega^{d-p-1}(\partial \Sigma) \) is commonly referred as the *edge mode*.

The two phase spaces \( \mathcal{M}_\Sigma^{\text{comp}} \) and \( \mathcal{M}_\Sigma^{\text{edge}} \) are related by the symplectic reduction
\[ \mathcal{M}_\Sigma^{\text{comp}} = \mathcal{M}_\Sigma^{\text{edge}} / S^{\partial \Sigma}, \] (7.19)
where \( S^{\partial \Sigma} \) is the *surface symmetry*, describing the redefinition of \( b \)
\[ \delta b = \sigma, \quad \sigma \in \Omega^{d-p-1}(\partial \Sigma). \] (7.20)

The corresponding Hamiltonian is
\[ H_\sigma = (-1)^{p(d-p-1)} \int_{\partial \Sigma} \sigma \wedge A, \] (7.21)
while the moment map is

\[ \mu_S = (-1)^p(d-p-1) i_{\partial \Sigma}^* A = (-1)^p(d-p-1) A|_{\partial \Sigma} \in \mathfrak{g}^{\partial \Sigma} \simeq \Omega^p(\partial \Sigma). \]  

(7.22)

The zero locus of moment map is

\[ \mu_S^{-1}(0) = \Omega^p(\Sigma, \partial \Sigma) \oplus \Omega^{d-p}(\Sigma) \oplus \Omega^{d-p-1}(\partial \Sigma), \]  

(7.23)

while we can use the surface symmetry to set \( b = 0 \), so that

\[ \mathcal{M}^\text{red}_\Sigma = \mathcal{M}^\text{edge}_\Sigma // S^{\partial \Sigma} = \mu_S^{-1}(0) / S^{\partial \Sigma} = \Omega^p(\Sigma, \partial \Sigma) \oplus \Omega^{d-p}(\Sigma) = \mathcal{M}^\text{comp}_\Sigma. \]  

(7.24)

The symplectic form

\[ \omega^\text{red}_\Sigma = \pi^* i_{\mu}^* \omega^\text{edge}_\Sigma = - \int_{\Sigma} \delta A \wedge \delta B = \omega^\text{comp}_\Sigma. \]  

(7.25)

7.5 Gauge symmetries

The algebra of gauge symmetries in BF theory can be identified with differential forms on \( \Sigma \)

\[ \mathfrak{g}^\Sigma = \text{Lie}(G^\Sigma) = \Omega^{p-1}(\Sigma), \quad \mathfrak{t}^\Sigma = \text{Lie}(K^\Sigma) = \Omega^{d-p-1}(\Sigma). \]  

(7.26)

The symplectic reduction theorem requires the moment map, defined from canonical pairing between \( \mathfrak{g} \) and \( \mathfrak{g}^* \). For the surface without boundary the Poincare duality allowed us to identify

\[ \mathfrak{g}^{\Sigma^*} = \Omega^{p-1}(\Sigma)^* = \Omega^{d-p+1}(\Sigma), \quad \mathfrak{t}^{\Sigma^*} = \Omega^{d-p-1}(\Sigma)^* = \Omega^{p+1}(\Sigma), \]  

(7.27)

while in presence of boundary we need to use the modified pairing (7.14) to describe the dual algebras

\[ \mathfrak{g}^{\Sigma^*} = \Omega^{p-1}(\Sigma)^* = \Omega^{d-p+1}(\Sigma) \oplus \Omega^d(\partial \Sigma), \quad \mathfrak{t}^{\Sigma^*} = \Omega^{d-p-1}(\Sigma)^* = \Omega^{p+1}(\Sigma) \oplus \Omega^p(\partial \Sigma). \]  

(7.28)

The moment map for the infinitesimal action of gauge symmetry \( G^\Sigma \) on hypersurface \( \Sigma \) with boundary has two components: one with values in bulk forms \( \Omega^{d-p+1}(\Sigma) \), while the other with values in boundary forms \( \Omega^{d-p}(\partial \Sigma) \).

The bulk form part of the moment map for \( G^\Sigma \) is the moment map for

\[ G^\Sigma_c = \{ g \in G^\Sigma \mid g|_{\partial \Sigma} = 1 \}, \]  

(7.29)
a subgroup of all gauge transformations \( G^\Sigma \), trivial at the boundary \( \partial \Sigma \) of \( \Sigma \). Such subgroup often referred in a literature as a compactly supported gauge transformations. The Lie algebra of \( G^\Sigma_c \) is \((p-1)\)-forms on \( \Sigma \) that vanish on the boundary i.e.

\[
\mathfrak{g}^\Sigma_c = \Omega^{p-1}(\Sigma, \partial \Sigma),
\]

(7.30)

while the dual Lie algebra

\[
\mathfrak{g}^{\Sigma*}_c \simeq \Omega^{d-p+1}(\Sigma)
\]

has natural embedding

\[
i_c : \mathfrak{g}^{\Sigma*}_c \hookrightarrow \mathfrak{g}^{\Sigma*} : f \mapsto (f, 0).
\]

(7.32)

The moment map for \( G^\Sigma_c \)-action is a pullback of the \( G^\Sigma \)-moment map

\[
\mu_{G_c} = i^*_c \mu_G, \quad \mu_{G_c} = i^*_c(f, a) = f.
\]

(7.33)

The subspace \( \mu_{G_c} = 0 \) is the the space of solutions to the Gauss law constraint.

### 7.6 Asymptotic symmetries

We can perform the symplectic reduction over \( G^\Sigma \)-action in two steps. The \( G^\Sigma_c \) reduction first

\[
\mathcal{M}^{\text{inv,c}}_{\Sigma} = \mathcal{M}_\Sigma / G^\Sigma_c = \frac{\mathcal{M}_{\Sigma} / (0)}{G^\Sigma_c}
\]

(7.34)

and the second reduction under the action of stabilizer group of zero

\[
G^\Sigma_c / G^\Sigma_c = G^{\partial \Sigma} : \mathfrak{g}^{\Sigma*}/ \mathfrak{g}^{\Sigma*}_c \to \mathfrak{g}^{\Sigma*}/ \mathfrak{g}^{\Sigma*}_c.
\]

(7.35)

According to the reduction in stages the phase space \( \mathcal{M}^{\text{inv,c}}_{\Sigma} \) carries the quotient group action, which can be naturally identified with the asymptotic symmetry group. The commonly used definition of the asymptotic symmetry group (ASG) is

\[
\text{ASG} = \frac{\text{Allowed transformations}}{\text{Trivial transformations}},
\]

(7.36)

where allowed transformations defined as (infinitesimal gauge) transformations \( \epsilon \) with finite values of a certain boundary charge \( Q[\epsilon] \), while the trivial transformations are the ones with zero values of \( Q[\epsilon] \). In our notations we can construct the boundary charge \( Q[\epsilon] \) from the
moment map $\mu_{\partial \Sigma}$ of the $G^{\partial \Sigma}$-action

$$Q[\epsilon] = \langle \epsilon, i_{\partial \Sigma}^* \mu_{\partial \Sigma} \rangle = \int_{\partial \Sigma} i_{\partial \Sigma}^* \epsilon \wedge a,$$

(7.37)

for the gauge transformation parameter $\epsilon \in g^\Sigma$ and embedding

$$i_{\partial}: g^\Sigma / g^\Sigma_e \hookrightarrow g^\Sigma: a \mapsto (0, a).$$

(7.38)

8 Symplectic reduction in presence of boundary

The invariant phase space of the BF theory (5.2) can be defined as the symplectic reduction of the edge-mode extended phase $M^\Sigma_{\text{edge}}$ over the action of gauge symmetries $G^\Sigma \times K^\Sigma$ and surface symmetry $S^{\partial \Sigma}

$$M_{\Sigma}^{\text{inv}} = M_{\Sigma}^{\text{edge}} / (G^\Sigma \times K^\Sigma \times S^{\partial \Sigma}).$$

(8.1)

In this section we will carefully describe the symmetries and describe the symplectic reduction.

8.1 Symmetries

Let us identify the symplectic symmetries of our system. There are two types of gauge symmetries $G^\Sigma$ for field $A$ and $K^\Sigma$ for $B$-field with infinitesimal versions

$$\delta A = d\epsilon, \quad \delta B = d\lambda, \quad \epsilon \in g^\Sigma = \Omega^{p-1}(\Sigma), \quad \lambda \in k^\Sigma = \Omega^{d-p-1}(\Sigma).$$

(8.2)

The corresponding Hamiltonians

$$H_\epsilon = (-1)^{p+1} \int_\Sigma \epsilon \wedge dB - \int_{\partial \Sigma} \epsilon \wedge (B + (-1)^p db),$$

$$H_\lambda = (-1)^{(p+1)(d-p)} \int_\Sigma \lambda \wedge dA + (-1)^p (d-p) \int_{\partial \Sigma} \lambda \wedge A.$$

(8.3)

The surface symmetry $S^{\partial \Sigma}$ is the edge mode redefinition

$$\delta b = \sigma, \quad \sigma \in \Omega^{d-p-1}(\partial \Sigma),$$

(8.4)

with Hamiltonian

$$H_\sigma = (-1)^{p(d-p-1)} \int_{\partial \Sigma} \sigma \wedge A.$$

(8.5)
From (7.28) we can deduce the moment maps

$$
\mu_G = ((-1)^{p+1}dB, -i_{\partial \Sigma}B + (-1)^{p+1}db) \in g^* = \Omega^{d-p+1}(\Sigma) \oplus \Omega^{d-p}(\partial \Sigma),
$$

$$
\mu_K = ((-1)^{(p+1)(d-p)}dA, (-1)^{p(d-p)}i_{\partial \Sigma}A) \in k^* = \Omega^{p+1}(\Sigma) \oplus \Omega^p(\partial \Sigma),
$$

$$
\mu_S = (-1)^{p(d-p-1)}i_{\partial \Sigma}A \in s^* = \Omega^p(\partial \Sigma).
$$

(8.6)

All groups $G^\Sigma, H^\Sigma, S^{\partial \Sigma}$ are abelian so their Lie algebras have trivial brackets. The Poisson algebra of (8.3), (8.5) is centrally extended

$$
\{ H_\epsilon, H_\lambda \} = (-1)^{p(d-p)} \int_{\partial \Sigma} \lambda \wedge d\epsilon,
$$

$$
\{ H_\epsilon, H_\sigma \} = (-1)^{p(d-p-1)} \int_{\partial \Sigma} \epsilon \wedge d\sigma,
$$

$$
\{ H_\lambda, H_\sigma \} = 0,
$$

(8.7)

so the corresponding moment maps are not equivariant.

Let us point out that the central extension in (8.7) is a generic feature of the phase space with edge modes. To describe the phase space for all $A$-field configurations, including the ones not vanishing on the boundary, i.e. $A \in \Omega^p(\Sigma)$ we need to enlarge the ”dual momenta” $B \in \Omega^{d-p}(\Sigma)$ by an edge mode $b \in \Omega^{d-p}(\partial \Sigma)$. We can freely redefine the edge mode $b$ by the surface symmetry action. The moment map for the surface symmetry is the boundary value $i_{\partial \Sigma}A$ of $A$. The boundary value is shifted by the boundary gauge transformations, hence the surface symmetry moment map is not equivariant. The non-equivariance of the moment map is equivalent to the nontrivial central extension for the phase space realization of symmetries.

### 8.2 Reduction in stages

The symplectic space (8.1) is defined as symplectic reduction with respect to the non-equivariant action of gauge and surface symmetries. The non-equivariant reduction theorem in section 4.5 provides us with construction to represent this reduction as a quotient space. However, we are not going to use this method, since one of the goals of this note is to compare our results for BF theory with similar results from the literature. We can decompose the group of all symmetries into parts and perform the reduction in stages. Not only we can mach the results from literature but also we can avoid usage of the non-equivariant reduction.

Let us outline the the two decompositions below, while leaving the details of reductions in following sections.
• **Compact support description:** We can perform a symplectic reduction over $S^\partial \Sigma$ first, so the invariant phase space (8.1) becomes

$$\mathcal{M}^{inv}_\Sigma = \left[ \mathcal{M}^{edge}_\Sigma / S^\partial \Sigma \right] / ((G^\Sigma \times K^\Sigma)_a^0). \quad (8.8)$$

The advantage of this decomposition is that the first reduction is easy to perform as it is a familiar compact support version of the phase space

$$\mathcal{M}^{comp}_\Sigma = \mathcal{M}^{edge}_\Sigma / S^\partial \Sigma = \Omega^p(\Sigma, \partial \Sigma) \oplus \Omega^{d-p}(\Sigma). \quad (8.9)$$

The reduction over $(G^\Sigma \times K^\Sigma)_a^0$ can be further decomposed into the compact support gauge transformations and the boundary support one, what makes it very similar to the asymptotic symmetry approach towards the invariant phase space construction.

• **Edge modes:** We can also observe that both central terms in (8.7) have similar structure so we can cancel them if we mix $K^\Sigma$- and $S^\partial \Sigma$- symmetries in a specific way. Let denote the modified symmetry by $\tilde{K}^\Sigma$, while the reduction in stages gives us another description of the invariant phase space

$$\mathcal{M}^{inv}_\Sigma = \left[ \mathcal{M}^{edge}_\Sigma / (G^\Sigma \times \tilde{K}^\Sigma) \right] / (S^\partial \Sigma)_a^0. \quad (8.10)$$

The phase space in square brackets can be naturally identified with the edge mode construction of phase space.

### 8.3 Invariant phase space with edge modes

Let us define the new symmetry $\tilde{K}^\Sigma$ as a combination of $K^\Sigma$ and $S^\partial \Sigma$ with the infinitesimal action

$$\delta_g(B, b) = (d\lambda, (-1)^{p+1}i^*_\partial \Sigma \lambda + d\xi) = di_{\partial \Sigma}(\lambda, \xi), \quad \lambda \in \Omega^{d-p-1}(\Sigma), \quad \xi \in \Omega^{d-p-2}(\partial \Sigma), \quad (8.11)$$

so it is identical to the de Rham differential for the mapping cone from section 6.3 up to a field redefinition by a multiplicative factor $(-1)^p$. The modified Hamiltonian

$$\tilde{H}_\lambda = (-1)^{(p+1)(d-p)} \int_\Sigma \lambda \wedge dA + (-1)^{(p+1)(d-p-1)} \int_{\partial \Sigma} \xi \wedge dA \quad (8.12)$$
has trivial Poisson bracket with $H_e$

$$\{\hat{H}_\lambda, H_e\} = 0, \quad (8.13)$$

We manage to remove the central extension term in (8.7) by the symmetry modification. The symplectic reduction

$$\mathcal{M}^{\text{edge,inv}}_\Sigma = \mathcal{M}^{\text{edge}}_\Sigma / (G^\Sigma \times \tilde{K}^\Sigma) \quad (8.14)$$

is an equivariant moment map reduction. The moment maps for the group actions

$$\mu_G = ((-1)^{p+1}dB, -i^*_\partial B + (-1)^{p+1}db) \in \Omega^{d-p+1}(\Sigma) \oplus \Omega^{d-p}(\partial \Sigma)$$

$$\mu_{\tilde{K}} = ((-1)^{(p+1)(d-p)}dA, (-1)^{(p+1)(d-p-1)}d(i^*_\partial A)) \in \Omega^{p+1}(\Sigma) \oplus \Omega^{p+1}(\partial \Sigma). \quad (8.15)$$

The zero locus of the moment map

$$\mu_G^{-1}(0) \cap \mu_{\tilde{K}}^{-1}(0) = Z^p(\Sigma) \times Z^{d-p}(i\partial \Sigma), \quad (8.16)$$

with $Z^{d-p}(i\partial \Sigma)$ being the space of closed $(d-p)$-forms in the mapping cone de Rham complex form section 6.3. The invariant phase space phase space after reduction

$$\mathcal{M}^{\text{edge,inv}}_\Sigma = \frac{\mu_G^{-1}(0) \cap \mu_{\tilde{K}}^{-1}(0)}{G^\Sigma \times \tilde{K}^\Sigma} = \frac{Z^p(\Sigma) \oplus Z^{d-p}(i\partial \Sigma)}{d\Omega^{p-1}(\Sigma) \oplus d(i^*_\partial \Omega^{d-p-1}(i\partial \Sigma))} = H^p(\Sigma) \oplus H^{d-p}(i\partial \Sigma) \quad (8.17)$$

is identical to the mapping cone generalization of the invariant phase space (6.2). The symplectic form

$$\omega^{\text{edge,inv}}_\Sigma = s^*i^*_\mu \omega^{\text{edge}}_\Sigma = -\int_\Sigma \delta A \wedge \delta B - \int_{\partial \Sigma} \delta A \wedge \delta b, \quad A \in H^p(\Sigma), \quad (B, b) \in H^{d-p}(i\partial \Sigma) \quad (8.18)$$

is canonical symplectic form for the pairing (6.11) and is similar to the results in a literature [13] for Maxwell theory.

The remaining part of the surface symmetry $(S^{\partial \Sigma})^0_0$ is defined as the stabilizer of 0 under the the affine action of the $S^{\partial \Sigma}$. The action of $S^{\partial \Sigma}$ on the moment maps

$$\delta_S \mu_G = \delta_S ((-1)^{p+1}dB, -i^*_\partial B + (-1)^{p+1}db) = (0, (-1)^{p+1}d\sigma)$$

$$\delta_S \mu_{\tilde{K}} = \delta_S ((-1)^{(p+1)(d-p)}dA, (-1)^{(p+1)(d-p-1)}d(i^*_\partial A)) = (0, 0). \quad (8.19)$$

The subgroup of $S^{\partial \Sigma}$ that preserves the zero locus of moment map is $Z^{d-p-1}(\partial \Sigma)$, the closed
forms on the boundary. The exact forms on the boundary are already included in (8.11) so we left with the cohomology $H^{d-p-1}(\partial \Sigma)$. The invariant phase space becomes

$$
\mathcal{M}_\Sigma^{inv} = \mathcal{M}_\Sigma^{edge,inv} / / H^{d-p-1}(\partial \Sigma).
$$

(8.20)

The moment map of this action

$$
\mu_{S\partial \Sigma} : \mathcal{M}_\Sigma^{edge,inv} \to H^p(\partial \Sigma) : (A, B, b) \mapsto (-1)^{p(d-p-1)} i_{\partial \Sigma}^* A.
$$

(8.21)

The zero locus of the moment map

$$
\mu_{S\partial \Sigma}^{-1}(0) = \ker [i_{\partial \Sigma}^* : H^p(\Sigma) \to H^p(\partial \Sigma)] \oplus H^{d-p}(i_{\partial \Sigma}),
$$

(8.22)

while the reduced phase space is

$$
\mathcal{M}_\Sigma^{inv} = \frac{\mu_{S\partial \Sigma}^{-1}(0)}{(S\partial \Sigma)^a} = \ker [i_{\partial \Sigma}^* : H^p(\Sigma) \to H^p(\partial \Sigma)] \oplus \frac{H^{d-p}(i_{\partial \Sigma})}{\Im [i : H^{d-p-1}(\partial \Sigma) \to H^{d-p}(i_{\partial \Sigma})]},
$$

(8.23)

where $i : H^{d-p-1}(\partial \Sigma) \to H^{d-p}(i_{\partial \Sigma})$ is an embedding map for short exact sequence associated to the mapping cone cohomology. The mapping cone forms can be organized into a short exact sequence

$$
\begin{array}{cccccc}
0 & \Omega^{d-p-1}(\partial \Sigma) & \Omega^{d-p}(i_{\partial \Sigma}) & \Omega^{d-p}(\Sigma) & 0 \\
b & \pi & \text{Im} \\
(0, b) & B
\end{array}
$$

(8.24)

Given a short exact sequence we can construct long exact sequence of cohomology

$$
\begin{array}{cccccc}
\ldots & H^{d-p-1}(\Sigma) & i_{\partial \Sigma}^* & H^{d-p-1}(\partial \Sigma) & i & H^{d-p}(i_{\partial \Sigma}) & \pi & H^{d-p}(\Sigma) & \ldots
\end{array}
$$

(8.25)

which defines the map $i : H^{d-p-1}(\partial \Sigma) \to H^{d-p}(i_{\partial \Sigma})$.

We can use the long exact sequence (8.25) to simplify our expression (8.23) for the reduced phase space. Since all cohomology are over real numbers the exact sequence splits and we
can further rewrite
\[
\frac{H^{d-p}(i_{\partial \Sigma})}{\text{Im}(i)} = \text{Im}(\pi) = \ker \left[ i_{\partial \Sigma}^* : H^{d-p}(\Sigma) \to H^{d-p}(\partial \Sigma) \right],
\]
so that
\[
\mathcal{M}_{\Sigma}^{\text{inv}} = \ker [i_{\partial \Sigma}^* : H^p(\Sigma) \to H^p(\partial \Sigma)] \oplus \ker [i_{\partial \Sigma}^* : H^{d-p}(\Sigma) \to H^{d-p}(\partial \Sigma)].
\]
Using a pair of long exact sequences (8.25), arranged so that the vertical lines are generalized Poincare duality isomorphisms
\[
\cdots \to H^{p-1}(\Sigma) \xrightarrow{i_{\partial \Sigma}} H^{p-1}(\partial \Sigma) \xrightarrow{i} H^p(i_{\partial \Sigma}) \xrightarrow{\pi} H^p(\Sigma) \xrightarrow{i_{\partial \Sigma}} H^p(\partial \Sigma) \to \cdots
\]
\[
\cdots \to H^{d-p+1}(\partial \Sigma) \xrightarrow{i_{\partial \Sigma}} H^{d-p+1}(\Sigma) \xrightarrow{i} H^{d-p}(\Sigma) \xrightarrow{\pi} H^{d-p}(i_{\partial \Sigma}) \xrightarrow{i_{\partial \Sigma}} H^{d-p-1}(\partial \Sigma) \to \cdots
\]
we can rewrite
\[
\mathcal{M}_{\Sigma}^{\text{inv}} = \ker i_{\partial \Sigma}^* \oplus \text{coker } i = \ker i_{\partial \Sigma}^* \oplus (\ker i^*)^* = \ker i_{\partial \Sigma}^* \oplus (\ker i_{\partial \Sigma}^*)^*,
\]
so it assumes the form of the linear symplectic space
\[
\mathcal{M}_{\Sigma}^{\text{inv}} = V \oplus V^*, \quad V = \ker i_{\partial \Sigma}^*.
\]

8.4 Invariant phase space with compact support

The action of \((G^\Sigma \times K^\Sigma)\) on a moment map (8.6) for the surface symmetry
\[
K^\Sigma \times G^\Sigma : \mu_S = (-1)^{p(d-p-1)} i_{\partial \Sigma}^* A \mapsto (-1)^{p(d-p-1)} i_{\partial \Sigma}^* A + (-1)^{p(d-p-1)} i_{\partial \Sigma}^* d\epsilon,
\]
leads to the stabilizer subgroup of the affine action of the form
\[
(K^\Sigma \times G^\Sigma)_0^a = K^\Sigma \times G^\Sigma_c \times H^{p-1}(\partial \Sigma).
\]
The action of the \((K^\Sigma \times G^\Sigma)_0^a\) is equivariant since the only nontrivial Poisson bracket
\[
\{H_\epsilon, H_\lambda\} = (-1)^{p(d-p)} \int_{\partial \Sigma} \lambda \wedge d\epsilon = 0
\]
vanishes because $i^*_{\partial \Sigma} d\epsilon = 0$ follows from the stabilizer definition.

The compact support phase space

$$\mathcal{M}_{\Sigma}^{\text{edge}} / S^{\partial \Sigma} = \mathcal{M}_{\Sigma}^{\text{comp}} = \Omega^p(\Sigma, \partial \Sigma) \oplus \Omega^{d-p}(\Sigma).$$

is invariant under the compact support gauge transformations $G_c^\Sigma \times K^\Sigma$ with the corresponding moment maps

$$\mu_{G_c} = (-1)^{p+1} dB \in \mathfrak{g}_c^\Sigma = \Omega^{d-p+1}(\Sigma),$$
$$\mu_K = ((-1)^{(p+1)(d-p)} dA, 0) \in \mathfrak{k}^\Sigma = \Omega^{p+1}(\Sigma) \oplus \Omega^p(\partial \Sigma).$$

The reduced phase space

$$\mathcal{M}_{\Sigma}^{\text{comp, inv}} = \frac{\mu_{G_c}^{-1}(0) \cap \mu_K^{-1}(0)}{G_c^\Sigma \times K^\Sigma} = H^p_c(\Sigma) \oplus H^{d-p}(\Sigma)$$

has natural description as a cohomology phase space (6.2). The invariant phase space requires the final reduction

$$\mathcal{M}_{\Sigma}^{\text{inv}} = \mathcal{M}_{\Sigma}^{\text{comp, inv}} / \!/ H^{p-1}(\partial \Sigma).$$

We can describe using the $\mathcal{M}_{\Sigma}^{\text{inv}}$ using the long exact sequence for relative cohomology and identification $H_c(\Sigma) = H(\Sigma, \partial \Sigma)$. The short exact sequence associated to the relative forms

$$\begin{array}{ccccccccc}
0 & \rightarrow & \Omega^p(\Sigma, \partial \Sigma) & \overset{j^*}{\rightarrow} & \Omega^p(\Sigma) & \overset{i^*_{\partial \Sigma}}{\rightarrow} & \Omega^p(\partial \Sigma) & \rightarrow & 0
\end{array}$$

with $i^*_{\partial \Sigma}$ being the restriction map of the differential forms on $\Sigma$ to $\partial \Sigma$. The map $j^* : \Omega^p(\Sigma) \rightarrow \Omega^p(\Sigma, \partial \Sigma)$ is a dual of the quotient map for chains

$$j : C_p(\Sigma, \partial \Sigma) = C_p(\Sigma)/C_p(\partial \Sigma) \rightarrow C_p(\Sigma).$$

The corresponding long exact sequence of cohomology takes the form

$$\begin{array}{ccccccccc}
\ldots & \rightarrow & H^{p-1}(\Sigma) & \overset{i^*_{\partial \Sigma}}{\rightarrow} & H^{p-1}(\partial \Sigma) & \overset{i}{\rightarrow} & H^p(\Sigma, \partial \Sigma) & \overset{j^*}{\rightarrow} & H^p(\Sigma) & \overset{i^*_{\partial \Sigma}}{\rightarrow} & H^p(\partial \Sigma) & \rightarrow & \ldots
\end{array}$$

The connecting homomorphism $i$ defines the action of $H^{p-1}(\partial \Sigma)$ on phase space in (8.36). The moment map $\mu_i$ for this action can be described using the generalized Poincare duality

$$(H^p(\partial \Sigma))^* = H^{d-p}(\partial \Sigma), \quad (H^p(\Sigma, \partial \Sigma))^* = H^{d-p}(\Sigma).$$
so that
\[ i^*: (H^p(\Sigma, \partial \Sigma))^* \to (H^p(\partial \Sigma))^* : H^{d-p}(\Sigma) \to H^{d-p}(\partial \Sigma) \quad (8.41) \]

and
\[ \mu_i : \mathcal{M}^{\text{comp.inv}}_{\Sigma} \to (H^{p-1}(\partial \Sigma))^* : H^p(\Sigma, \partial \Sigma) \oplus H^{d-p}(\Sigma) \to H^{d-p}(\partial \Sigma) : (A, B) \mapsto i^* B. \quad (8.42) \]

Using a pair of long exact sequence (8.39), arranged so that the vertical lines are generalized Poincare duality isomorphisms

\[ 
\begin{array}{cccccccccccc}
\ldots & \quad H_{p-1}(\Sigma) & \xrightarrow{i_{\partial \Sigma}} & H_{p-1}(\partial \Sigma) & \xrightarrow{i} & H^p(\Sigma, \partial \Sigma) & \xrightarrow{j^*} & H^p(\Sigma) & \xrightarrow{i_{\partial \Sigma}^*} & H^p(\partial \Sigma) & \xrightarrow{i} & \ldots \\
& \downarrow & \cong & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\ldots & H_{d-p+1}(\Sigma, \partial \Sigma) & \xrightarrow{i} & H_{d-p}(\partial \Sigma) & \xrightarrow{i_{\partial \Sigma}} & H^p(\Sigma, \partial \Sigma) & \xrightarrow{j} & H^p(\Sigma, \partial \Sigma) & \xrightarrow{i} & \ldots \\
\end{array}
\]

we can argue that \( i^* = i^*_{\partial \Sigma} \), i.e. it is a restriction map for the differential forms.

The quotient space description of the invariant phase space (8.36)
\[ \mathcal{M}^{\text{inv}}_{\Sigma} = \mathcal{M}^{\text{comp.inv}}_{\Sigma} / H^{p-1}(\partial \Sigma) = \frac{\mu_i^{-1}(0)}{\text{Im}(i)} = \frac{\ker i^* \oplus H^p(\Sigma, \partial \Sigma)}{\text{Im}(i)} \quad (8.43) \]

In more explicit form the expression above is
\[ \mathcal{M}^{\text{inv}}_{\Sigma} = \ker \left[ i^*_{\partial \Sigma} : H^{d-p}(\Sigma) \to H^{d-p}(\partial \Sigma) \right] \oplus \frac{H^p(\Sigma, \partial \Sigma)}{\text{Im} [i : H^{p-1}(\partial \Sigma) \to H^p(\Sigma, \partial \Sigma)]}. \quad (8.44) \]

Using the long exact sequence (8.39) we can evaluate
\[ \frac{H^p(\Sigma, \partial \Sigma)}{\text{Im} [i : H^{p-1}(\partial \Sigma) \to H^p(\Sigma, \partial \Sigma)]} = \text{Im} [j^* : H^p(\Sigma, \partial \Sigma) \to H^p(\Sigma)] = \ker [i^*_{\partial \Sigma} : H^p(\Sigma) \to H^p(\partial \Sigma)], \]

so that
\[ \mathcal{M}^{\text{inv}}_{\Sigma} = \ker \left[ i^*_{\partial \Sigma} : H^{d-p}(\Sigma) \to H^{d-p}(\partial \Sigma) \right] \oplus \ker [i^*_{\partial \Sigma} : H^p(\Sigma) \to H^p(\partial \Sigma)] \quad (8.45) \]

which is identical to the phase space (8.27), constructed using edge modes in previous section.
8.5 Invariant phase space

We constructed the invariant phase space $\mathcal{M}^{inv}_\Sigma$ for BF theory using two different decompositions of the symmetry group. In both cases we first performed the infinite-dimensional symplectic reduction to get the finite-dimensional spaces $\mathcal{M}^{edge,inv}_\Sigma$ and $\mathcal{M}^{comp,inv}_\Sigma$. On second step we performed the finite-dimensional reductions for both spaces to construct the $\mathcal{M}^{inv}_\Sigma$. We can describe spaces $\mathcal{M}^{edge,inv}_\Sigma$ and $\mathcal{M}^{comp,inv}_\Sigma$ as a “symplectic extension” for the invariant phase space by either the edge modes inclusion or asymptotical symmetries “ungauging”.

The edge mode extension is self-explanatory, while the for the asymptotic symmetries “ungauging” is the following procedure. By construction the invariant phase space

$$\mathcal{M}^{inv}_\Sigma = 
\mathcal{M}^{comp,inv}_\Sigma \big// H^{p-1}(\partial \Sigma),\quad (8.46)$$

is the space of $H^{p-1}(\partial \Sigma)$-invariant observables on $\mathcal{M}^{comp,inv}_\Sigma$. If we turn the $H^{p-1}(\partial \Sigma)$ gauge symmetry into a global symmetry then the invariant phase space becomes $\mathcal{M}^{comp,inv}_\Sigma$. Let us recall that the origin of the $H^{p-1}(\partial \Sigma)$ gauge symmetry was a subgroup of the gauge transformations $G^\Sigma$ on the boundary $\partial \Sigma$. Hence, we can say that the symplectic extension of the $\mathcal{M}^{inv}_\Sigma$ to a bigger phase space $\mathcal{M}^{comp,inv}_\Sigma$ is done by turning the boundary gauge symmetry into a global symmetry.

Let us also recall that the symplectic reduction approach is well developed only in case of the finite-dimensional spaces, while we used it for the infinite-dimensional reduction. In case of BF theory phase space for the surface with no boundary in section 5.2 we observed certain nice properties, so checking this properties for $\mathcal{M}^{inv}_\Sigma$ can be considered as a consistency check.

- **Finite dimensional phase space.** Our expression for the invariant phase space (8.27) is direct sum of the liner map kernels, what makes in a natural subspace of the direct sum of the corresponding domains $H^{d-p}(\Sigma) \oplus H^p(\Sigma)$. The cohomology groups $H^p(\Sigma)$ are know to be finite-dimensional so the $\mathcal{M}^{inv}_\Sigma$ is finite-dimansional as well. Moreover we can use the long exact sequences (8.25) and (8.39) to prove that the symplectic extensions $\mathcal{M}^{edge,inv}_\Sigma$ and $\mathcal{M}^{comp,inv}_\Sigma$ are finite-dimensional as well.

- **Topological theory.** Our expression for the invariant phase space (8.27) is direct sum of the kernels of the linear maps between the cohomology groups. The cohomology groups $H^p(\Sigma)$ are known to be topologically invariant, so the kernel of the linear map between them is also a topologically invariant.

- **Self-dual:** Our expression (8.27) is manifestly invariant under the $p \to d - p$ duality.
Let us observe that the two extensions $\mathcal{M}^{\text{edge,inv}}_{\Sigma}$ and $\mathcal{M}^{\text{comp,inv}}_{\Sigma}$ are related by duality transformation

$$\mathcal{M}^{(p)}_{\Sigma}^{\text{comp,inv}} = \mathcal{M}^{(d-p)}_{\Sigma}^{\text{edge,inv}}$$

(8.47)

Indeed the the $p$-form BF theory compact invariant phase space

$$\mathcal{M}^{(p)}_{\Sigma}^{\text{comp,inv}} = H_p^c(\Sigma) \oplus H_p^c(\Sigma)^* = H_p^c(\Sigma) \oplus H^{d-p}(\Sigma)$$

(8.48)

and the edge mode $d - p$-form BF theory compact invariant phase space

$$\mathcal{M}^{(d-p)}_{\Sigma}^{\text{edge,inv}} = H^{d-p}(\Sigma) \oplus H^{d-p}(\Sigma)^* = H^{d-p}(\Sigma) \oplus H^p(i_{\partial \Sigma})$$

(8.49)

are identical due to the cohomology relation (6.15).

### 9 Conclusion

We used the infinite-dimensional generalization of a symplectic reduction to describe the gauge-invariant phase space for BF theory. In absence of boundary our expression for invariant phase space is identical to the one obtained by covariant phase space formalism. The gauge-invariant phase space is a direct sum of the two cohomology groups, what is compatible with BF theory being topological theory.

In case surface with boundary we proposed a generalization of symplectic reduction construction for the invariant phase space. The invariant phase space preserves the topological features of BF theory as well as the $p \rightarrow d - p$ symmetry invariance. The symplectic reduction for Bf theory can be done in several steps, so that the intermediate phase spaces can be identified with the edge mode and asymptotic symmetry constructions.

Our choice of Bf theory as a prime example allowed us to provide an explicit description of various phase spaces in terms of well known objects from algebraic topology: differential forms and de Rham cohomology. We hope that such explicit description could be useful to check various conjectures and statements about the edge modes and asymptotic symmetries. In particular the symplectic space gluing and TQFT description, which we briefly outline below.
9.1 Symplectic space gluing

Let us consider two surfaces $\Sigma_L$ and $\Sigma_R$ with common boundary

$$C = \partial \Sigma_L = -\partial \Sigma_R,$$

(9.1)

where the minus sign stands for the orientation change. We can glue them together over the common boundary into new surface $\Sigma$

$$\Sigma = \Sigma_L \cup_C \Sigma_R.$$  

(9.2)

There is natural question:

What is the relation between $\mathcal{M}_{\Sigma_L}$, $\mathcal{M}_{\Sigma_R}$ and $\mathcal{M}_{\Sigma}$?

The relation was conjectured by Donnelly and Friedel [13] to be

$$\mathcal{M}_{\Sigma} = (\mathcal{M}_{\Sigma_L} \times \mathcal{M}_{\Sigma_R}) / G_C,$$

(9.3)

with $G_C$ being some kind of diagonal action of the symmetry group of edge modes on $\mathcal{M}_{\Sigma_L}$ and $\mathcal{M}_{\Sigma_R}$.

There are several reasons why our analysis of BF theory can be useful in verifying this conjecture:

• The invariant phase spaces of BF theory are finite dimensional, so we can use finite-dimensional symplectic reduction.

• The phase space has explicit description in terms of de Rham cohomology groups, which are well known algebraic topology objects.

Using all these ideas we are working on gluing conjecture verification for BF theory.

9.2 Extended TQFT

The quantum version of the gluing conjecture (9.3) relates the corresponding Hilbert spaces through entangled product

$$\mathcal{H}(\Sigma_L \cup_C \Sigma_R) = \mathcal{H}(\Sigma_L) \otimes_{G_C} \mathcal{H}(\Sigma_R).$$

(9.4)
In extended TQFT there is a similar gluing axiom [19,20] in the form of the tensor product over algebra $A_C$ of the right $A_C$-module $\mathcal{H}(\Sigma_L)$ left $A_C$-module $\mathcal{H}(\Sigma_R)$

$$\mathcal{H}(\Sigma_L \cup C \Sigma_R) = \mathcal{H}(\Sigma_L) \otimes_{A_C} \mathcal{H}(\Sigma_R) \subset \mathcal{H}(\Sigma_L) \otimes_C \mathcal{H}(\Sigma_R). \quad (9.5)$$

The two gluing operations become the identical if we conjecture that the algebra $A_C$ is the group algebra of $G_C$

$$A_C = \mathbb{C}[G_C] = \left\{ \sum_{i=1}^{|G|} \lambda_i g_i \mid \lambda_i \in \mathbb{C} \right\}. \quad (9.6)$$

The consistency of ETQFT among other conditions require that

$$A_C = \mathcal{H}(C \times [0,1]). \quad (9.7)$$

The form of $A_C$ is fixed from our conjecture (9.6) so given our explicit form $\mathcal{M}^{\text{inv}}_\Sigma$ we can check the consistency (9.7) and analyze remaining extended TQFT axioms.

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