Phase Transitions for quantum Ising model with competing XY-interactions on a Cayley tree

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Abstract

The main aim of the present paper is to establish the existence of a phase transition for the quantum Ising model with competing XY interactions within the quantum Markov chain (QMC) scheme. In this scheme, we employ the $C^*$-algebraic approach to the phase transition problem. Note that these kinds of models do not have one-dimensional analogues, i.e. the considered model persists only on trees. It turns out that if the Ising part interactions vanish then the model with only competing XY-interactions on the Cayley tree of order two does not have a phase transition. By phase transition we mean the existence of two distinct QMC which are not quasi-equivalent and their supports do not overlap. Moreover, it is also shown that the QMC associated with the model have clustering property which implies that the von Neumann algebras corresponding to the states are factors.

1 Introduction

The study of magnetic systems with competing interactions in ordering is a fascinating problem of condensed matter physics. One of the most canonical examples of such systems are frustrated Ising models which demonstrate a plethora of critical properties\cite{20, 28, 30}. The frustrations can be either geometrical or brought about the next-nearest neighbor NNN interactions. Competing interactions frustrations can result new phases, change the Ising universality class, or even destroy the order at all. Another interesting aspect of the criticality in the frustrated Ising
models is an appearance of quantum critical points at spacial frustration points of model’s high
degeneracy, and related quantum phase transitions [40]. The Ising models with frustrations
can be thought as perturbation of the classical Ising model. If the perturbation terms do
not commute with the Ising pieces, it outcomes quantum effects. In particular case, if the
perturbation is the XY interaction, then the model become more interesting (see [12, 21, 30, 36]
for a systematic study (physical approach) of the Ising model with quantum frustration on
2D lattices). However, a rigorous (mathematical) investigation of the quantum Ising model
with competing XY interactions does not exist yet in the literature. We notice that XY-
interactions are truly quantum, (i.e. contain pieces not commuting with each other). In the
present paper, we propose to investigate the phase transition problem for the mentioned model
on the Cayley tree or Bethe lattice [35] within quantum Markov chains (QMC) scheme. Here,
the QMC scheme is based on the $C^*-$algebraic approach. We notice that the Ising model with
Ising type competing interactions (with commuting interactions) has been recently studied in
[31, 32, 33, 34] by means of QMC. As we mentioned, in the current paper, the commuting
interactions are non-commutative, and this makes big difference between those papers.

On the other hand, our investigation will allow to construct quantum analogous of Markov
fields (see [23, 27, 37, 42, 41]) which is one of the basic problems in quantum probability.
We notice that quantum Markov fields naturally appear in quantum statistical mechanics and
quantum fields theories [22, 24].

We point out, even in classical setting, for models over integer lattices, there do not exist
analytical solutions (for example, critical temperature) on such lattices. Therefore, it was pro-
posed [15] to consider spin models on regular trees for which one can exactly calculate various
physical quantities. One of the simplest tree is a Cayley tree [35]. In [31, 32] we have established
that Gibbs measures of the Ising model with competing (Ising) interactions on a Cayley trees,
can be considered as QMC. Note that if the perturbation vanishes then the model reduces to
the classical Ising one which was also examined in [13] by means of $C^*$-algebra approach.

In the present paper, we are going to study the Ising model with XY-competing interactions
on a Cayley trees of order two. We point out that this model has non-commutative interactions,
i.e. XY ones, therefore, the investigation of this model is tricky. We notice that, in general,
QMC do not have KMS property (see [19]), therefore, general theory of KMS-states is not
applicable for such kind of chains. One of the main questions of this paper is to know whether
the considered model exhibits two different QMC associated with the mentioned model on the
Cayley trees. Our main result is the following one.

\textbf{Theorem 1.1.} For the Ising model with competing XY- interactions (18), (20), $J_0 > 0$, $J \in \mathbb{R} \setminus \{-J_0, J_0\}$, $\beta > 0$ on the Cayley tree of order two, the following statements hold:

\begin{itemize}
  \item if $\Delta(\theta) \leq 0$, then there is a unique QMC;
  \item if $\Delta(\theta) > 0$, then there occurs a phase transition,
\end{itemize}

where

$$
\Delta(\theta) = \frac{\theta^{2J_0} - \theta^{J_0}(\theta^J + \theta^{-J})}{\theta^{2J_0} - \theta^{J_0}(\theta^J + \theta^{-J}) + 1}, \quad \theta = e^{2\beta}.
$$

\textsuperscript{1}The quantum analogues of Markov chains were first constructed in [1], where the notion of quantum Markov
chain (QMC) on infinite tensor product algebras was introduced. Later on, in [26], finitely correlated states were
introduced and studied, which are related to each other. However, satisfactory constructions of such kind of fields
were not established, since most of the fields were considered over the integer lattices [3, 4].
To establish result, we will prove Theorem [5.3] from which we conclude that there are three coexisting phases in the region $\Delta(\theta) > 0$, and one of it, i.e. $\varphi_{\alpha}$, survives in the region $\Delta(\theta) < 0$. This leads us that the state $\varphi_{\alpha}$ describes the disordered phase of the model, which shows a similar behavior with the classical Ising model [16, 38]. In comparison with the Ising model, we stress that in the present model, we have a similar kind of phases (translation invariant ones) when $\Delta(\theta) > 0$. From Figure 2 (see below), one concludes that the phase transition occurs except for a ”triangular region”. This shows how the competing interactions effect to the existence of other phases. Notice that if $J = 0$, then we obtain the classical Ising model for which the existence of a disordered phase coexisting with two ordered phases is well-known [15, 38].

On the other hand, we emphasize that both problems, i.e. a construction and phase transitions are non-trivial and, to a large extent, open. In fact, even if several definitions of quantum Markov fields on trees (and more generally on graphs) have been proposed, a really satisfactory, general theory is still missing and physically interesting examples of such fields in dimension $d \geq 2$ are very few.

In order to get the existence of the phase transition (see [31]), one needs to check several conditions, and one of them based on a notion of the quasi-equivalence of quantum Markov chains which essentially uses $C^*$-algebraic approach and techniques. This situation totally differs from the classical (resp. quantum) cases, where it is sufficient to prove the existence of at least two different solutions (resp. KMS states) of associated renormalized equations (see [38]). Therefore, even for classical models, to check the existence of the phase transition (in the sense of our paper) is not a trivial problem. Here we mention that the quasi-equivalence of product states (which correspond to the classical model without interactions) was a tricky job and investigated in [29, 39]. In this paper, we are considering more complicated states (which are QMC associated with the model) than product ones, and for these kind of states we are going to obtain their non-quasi equivalence. We will first show that these states have clustering property, and hence they are factor states. We point out that even this fact presents its own interests since these states associates with non-commutative Hamiltonians having non-trivial interactions.

Let us outline the organization of the paper. After preliminary information (see Section 2), in Section 3 we provide a general construction of backward quantum Markov chains on Cayley tree. Moreover, in this section we give the definition of the phase transition. Using the provided construction, in Section 4 we consider the Ising model with competing $XY$-interactions on the Cayley tree of order two. Section 5 is devoted to the existence of the three translation-invariant QMC $\varphi_{\alpha}$, $\varphi_{1}$ and $\varphi_{1}$ corresponding to the model. Section 6 is devoted to the proof of Theorem [11]. In this section we will prove that states $\varphi_{1}$ and $\varphi_{2}$ do not have overlapping supports. Before, to establish their non-quasi equivalence, we first prove that these states have the clustering property. Section 7 we study a particular case $J_0 = 0$, which means that we only have $XY$ interactions. In the considered setting, it turns out that the phase transition does not occur.

2 Preliminaries

Let $\Gamma^k_+ = (L, E)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root $x^0$ (i.e. each vertex of $\Gamma^k_+$ has exactly $k + 1$ edges, except for the root $x^0$, which has $k$ edges). Here $L$ is the set of vertices and $E$ is the set of edges. The vertices $x$ and $y$ are called nearest neighbors and they are denoted by $l = \langle x, y \rangle$ if there exists an edge connecting them. A collection of the
Figure 1: The first levels of $\Gamma_+^2$

pairs $<x,x_1>, \ldots, <x_{d-1},y>$ is called a path from the point $x$ to the point $y$. The distance $d(x,y), x,y \in V$, on the Cayley tree, is the length of the shortest path from $x$ to $y$.

Recall a coordinate structure in $\Gamma_+^k$: every vertex $x$ (except for $x^0$) of $\Gamma_+^k$ has coordinates $(i_1, \ldots, i_n)$, here $i_m \in \{1, \ldots, k\}, 1 \leq m \leq n$ and for the vertex $x^0$ we put $(0)$. Namely, the symbol $(0)$ constitutes level 0, and the sites $(i_1, \ldots, i_n)$ form level $n$ (i.e. $d(x^0, x) = n$) of the lattice (see Fig. 1).

Let us set

$$W_n = \{x \in L : d(x,x^0) = n\}, \quad \Lambda_n = \bigcup_{k=0}^{n} W_k, \quad \Lambda_{[n,m]} = \bigcup_{k=n}^{m} W_k, (n < m)$$

$$E_n = \{<x,y> \in E : x,y \in \Lambda_n\}, \quad \Lambda_n^c = \bigcup_{k=n}^{\infty} W_k$$

For $x \in \Gamma_+^k, x = (i_1, \ldots, i_n)$ denote

$$S(x) = \{(x,i) : 1 \leq i \leq k\}.$$ 

Here $(x,i)$ means that $(i_1, \ldots, i_n, i)$. This set is called a set of direct successors of $x$.

Two vertices $x, y \in V$ is called one level next-nearest-neighbor vertices if there is a vertex $z \in V$ such that $x, y \in S(z)$, and they are denoted by $>x, y<$, and in this case the vertices $x, z, y$ was called ternary and denoted by $<x, z, y>$. Let us define on $\Gamma_+^k$ a binary operation $\circ : \Gamma_+^k \times \Gamma_+^k \to \Gamma_+^k$ as follows: for any two elements $x = (i_1, \ldots, i_n)$ and $y = (j_1, \ldots, j_m)$ put

$$x \circ y = (i_1, \ldots, i_n) \circ (j_1, \ldots, j_m) = (i_1, \ldots, i_n, j_1, \ldots, j_m) \quad (1)$$

and

$$x \circ x^0 = x^0 \circ x = (i_1, \ldots, i_n) \circ (0) = (i_1, \ldots, i_n). \quad (2)$$

By means of the defined operation $\Gamma_+^k$ becomes a noncommutative semigroup with a unit. Using this semigroup structure one defines translations $\tau_g : \Gamma_+^k \to \Gamma_+^k, g \in \Gamma_+^k$ by

$$\tau_g(x) = g \circ x. \quad (3)$$

It is clear that $\tau(0) = id$. 

The algebra of observables \( B_x \) for any single site \( x \in L \) will be taken as the algebra \( M_d \) of the complex \( d \times d \) matrices. The algebra of observables localized in the finite volume \( \Lambda \subset L \) is then given by \( B_\Lambda = \bigotimes_{x \in \Lambda} B_x \). As usual if \( \Lambda^1 \subset \Lambda^2 \subset L \), then \( B_{\Lambda^1} \) is identified as a subalgebra of \( B_{\Lambda^2} \) by tensoring with unit matrices on the sites \( x \in \Lambda^2 \setminus \Lambda^1 \). Note that, in the sequel, by \( B_{\Lambda^+} \) we denote the positive part of \( B_\Lambda \). The full algebra \( B_L \) of the tree is obtained in the usual manner by an inductive limit

\[
B_L = \bigcup_{\Lambda_n} B_{\Lambda_n}.
\]

In what follows, by \( S(B_\Lambda) \) we will denote the set of all states defined on the algebra \( B_\Lambda \).

Consider a triplet \( C \subset B \subset A \) of unital \( C^*\)-algebras. Recall [2] that a quasi-conditional expectation with respect to the given triplet is a completely positive (CP) linear map \( \mathcal{E} : A \to B \) such that \( \mathcal{E}(ca) = c\mathcal{E}(a) \), for all \( a \in A \), \( c \in C \).

**Definition 2.1** ([3]). A state \( \varphi \) on \( B_L \) is called a backward quantum Markov chain (QMC), associated to \( \{\Lambda_n\} \), if for each \( \Lambda_n \), there exist a quasi-conditional expectation \( \mathcal{E}_{\Lambda_n} \) with respect to the triplet

\[
B_{\Lambda_n-1} \subset B_{\Lambda_n} \subset B_{\Lambda_n+1}
\]

and an initial state \( \rho \in S(B_{\Lambda_0}) \) such that:

\[
\varphi = \lim_{n \to \infty} \rho_0 \circ \mathcal{E}_{\Lambda_0} \circ \mathcal{E}_{\Lambda_1} \circ \cdots \circ \mathcal{E}_{\Lambda_n}
\]

in the weak-* topology.

**Remark 2.2.** We notice that in [6] a more general definition of backward QMC is given on arbitrary quasi-local algebras.

### 3 Construction of Quantum Markov Chains

In this section we are going to provide a construction of a backward quantum Markov chain which contains competing interactions.

Let us rewrite the elements of \( W_n \) in the following lexicographic order (w.r.t. the coordinate system), i.e.

\[
\overrightarrow{W_n} := (x_{W_n}^{(1)}, x_{W_n}^{(2)}, \cdots, x_{W_n}^{(|W_n|)}).
\]

Note that \( |W_n| = k^n \). In this lexicographic order, vertices \( x_{W_n}^{(1)}, x_{W_n}^{(2)}, \cdots, x_{W_n}^{(|W_n|)} \) of \( W_n \) are given as follows

\[
x_{W_n}^{(1)} = (1, 1, \cdots, 1, 1), \quad x_{W_n}^{(2)} = (1, 1, \cdots, 1, 2), \quad \cdots \quad x_{W_n}^{(k)} = (1, 1, \cdots, 1, k),
\]

\[
x_{W_n}^{(k+1)} = (1, 1, \cdots, 2, 1), \quad x_{W_n}^{(2)} = (1, 1, \cdots, 2, 2), \quad \cdots \quad x_{W_n}^{(2k)} = (1, 1, \cdots, 2, k),
\]

\[
\vdots
\]

\[
x_{W_n}^{(|W_n|-k+1)} = (k, k, \cdots, k, 1), \quad x_{W_n}^{(|W_n|-k+2)} = (k, k, \cdots, k, 2), \quad \cdots \quad x_{W_n}^{|W_n|} = (k, k, \cdots, k, k).
\]
Analogously, for a given vertex \( x \), we shall use the following notation for the set of direct successors of \( x \):

\[
S(x) := ((x,1), (x,2), \cdots (x,k)) \quad \tilde{S}(x) := ((x,k), (x,k-1), \cdots (x,1)).
\]

In what follows, by \( \circ \prod \) we denote the lexicographic order, i.e.

\[
\circ \prod_{k=1}^{n} a_k = a_1 a_2 \cdots a_n,
\]

where elements \( \{a_k\} \subset B_L \) are multiplied in the indicated order. This means that we are not allowed to change this order.

Note that each vertex \( x \in L \) has interacting vertices \( \{x, (x,1), \ldots, (x,k)\} \). Assume that, to each edges \( < x, (x,i) \) \( \Rightarrow k = 1, \ldots, k \) an operators \( K_{<x,(x,i)>} \in B_x \otimes B_{(x,i)} \) is assigned, respectively. Moreover, for each competing vertices \( > (x,i), (x,i+1) \) \( \Rightarrow (i = 1, \ldots, k) \) the following operators are assigned:

\[
L_{>(x,i),(x,i+1)} \in B_{(x,i)} \otimes B_{(x,i+1)}, \quad M_{(x,(x,i),(x,i+1))} \in B_x \otimes B_{(x,i)} \otimes B_{(x,i+1)}.
\]

We would like to define a state on \( B_{\Lambda_n} \) with boundary conditions \( \omega_0 \in B_{0,+} \) and \( \{h^x \in B_{x,+} : x \in L\} \).

For each \( n \in \mathbb{N} \) denote

\[
A_{x,(x,1),\ldots,(x,k)} = (\circ \prod_{i=1}^{k} K_{x,(x,i)})(\circ \prod_{i=1}^{k} L_{=(x,i)=(x,i+1)})(\circ \prod_{i=1}^{k} M_{(x,(x,i),(x,i+1)})},
\]

\[
K_{[m,m+1]} := \prod_{x \in W_m} A_{x,(x,1),\ldots,(x,k)}, \quad 1 \leq m \leq n,
\]

\[
h_n^{1/2} := \prod_{x \in W_n} (h^x)^{1/2}, \quad h_n = h_n^{1/2}(h_n^{1/2})^*,
\]

\[
K_n := \omega_0^{1/2}(\circ \prod_{m=0}^{n-1} K_{[m,m+1]})h_n^{1/2},
\]

\[
W_{\Lambda_n} := K_{\omega_0}^{n}K_n
\]

One can see that \( W_{\Lambda_n} \) is positive.

In what follows, by \( \text{Tr}_\Lambda : B_L \to B_\Lambda \) we mean normalized partial trace (i.e. \( \text{Tr}_\Lambda(1_L) = 1_\Lambda \), here \( 1_\Lambda = \bigotimes_{y \in \Lambda} 1 \)), for any \( \Lambda \subseteq \text{fin} \ L \). For the sake of shortness we put \( \text{Tr}_{\Lambda_n} := \text{Tr}_{\Lambda_n} \).

Let us define a positive functional \( \varphi_{w_0,h}^{(n)} \) on \( B_{\Lambda_n} \) by

\[
\varphi_{w_0,h}^{(n)}(a) = \text{Tr}(W_{\Lambda_n+1}(a \otimes 1_{W_{n+1}})),
\]

for every \( a \in B_{\Lambda_n} \). Note that here, \( \text{Tr} \) is a normalized trace on \( B_L \) (i.e. \( \text{Tr}(1_L) = 1 \)).

To get an infinite-volume state \( \varphi \) on \( B_L \) such that \( \varphi|_{B_{\Lambda_n}} = \varphi_{w_0,h}^{(n)} \), we need to impose some constrains to the boundary conditions \( \{w_0, h\} \) so that the functionals \( \{\varphi_{w_0,h}^{(n)}\} \) satisfy the compatibility condition, i.e.

\[
\varphi_{w_0,h}^{(n+1)}|_{B_{\Lambda_n}} = \varphi_{w_0,h}^{(n)}.
\]
Theorem 3.1. Assume that for every $x \in L$ and triple $\{x, (x,i), (x,i+1)\}$ ($i = 1, \ldots, k - 1$) the operators $K_{<x,i>}$, $L_{<x,i>}$, $M_{(x,i),(x,i+1)}$ are given as above. Let the boundary conditions $u_0 \in B_{A_\lambda}$ and $h = \{h_x \in B_{x^+}\}_{x \in L}$ satisfy the following conditions:

$$\text{Tr}(\omega_0 h_0) = 1,$$

$$\text{Tr}[A_{x,(i)} A^*_x A_{x,(i)} A^*_x] = h^x, \text{ for every } x \in L,$$

where as before $A_{x,(i)},(i)\}$ is given by (7). Then the functionals $\{\varphi_{w_0,h}^{(n)}\}$ satisfy the compatibility condition (13). Moreover, there is a unique backward quantum Markov chain $\varphi_{w_0,h}$ on $B_L$ such that $\varphi_{w_0,h} = w - \lim_{n \to \infty} \varphi_{w_0,h}^{(n)}$.

Proof. Let us check that the states $\varphi_{w_0,h}^{(n,b)}$ satisfy the compatibility condition. For $a \in B_{A_\lambda}$, we have:

$$\varphi_{w_0,h}^{(n+1)}(a \otimes 1_{W_{n+1}}) = \text{Tr}(W_{n+2}(a \otimes 1_{A_{[n+1,n+2]}}))$$

$$= \text{Tr}(K^*_{[n+1]} K^*_{[n-1,n]} \cdots K^*_{[0,1]} w_0 K_{[0,1]} K_{[1,2]} \cdots K_{[n+1,n+2]} h^1_{n+2} (a \otimes 1_{A_{[n+1,n+2]}}))$$

$$= \text{Tr}(K^*_{[n+1]} K^*_{[n-1,n]} \cdots K^*_{[0,1]} w_0 K_{[0,1]} K_{[1,2]} \cdots K_{[n+1]} \text{Tr}_{n+1}(K_{[n+1,n+2]} h_{n+2} K^*_{[n+1,n+2]}))$$

$$= \text{Tr}(K^*_{[n+1]} K^*_{[n-1,n]} \cdots K^*_{[0,1]} w_0 K_{[0,1]} K_{[1,2]} \cdots K_{[n+1]} (a \otimes 1_{A_{[n+1,n+2]}}))$$

$$= \text{Tr}(W_{n+1}(a \otimes 1_{A_{[n+1,n+2]}}))$$

$$= \varphi_{w_0,h}^{(n)}(a).$$

To show that $\varphi_{w_0,h}^{(n)}$ is a backward QMC, we define quasi-conditional expectations $E_n$ as follows:

$$E_n(x_{n+1}) = \text{Tr}_n(K_{[n+1,n+2]} h^1_{n+1} x_{n+1} K^*_{[n+1,n+2]}), \quad x_{n+1} \in B_{A_{n+1}}$$

$$E_k(x_{k+1}) = \text{Tr}_k(K_{[k-1+1]} h^1_{k+1} x_{k+1} K^*_{[k-1+1]}), \quad x_{k+1} \in B_{A_{k+1}}, \quad k = 1, 2, \ldots, n - 1$$

Then for any monomial $a_{A_\lambda} \otimes a_{W_2} \otimes \cdots \otimes a_{W_n} \otimes 1_{W_{n+1}}$, where $a_{A_\lambda} \in B_{A_\lambda}, a_{W_k} \in B_{W_k}$ $(k = 2, \ldots, n)$, we have:

$$\varphi_{w_0,h}^{(n)}(a_{A_\lambda} \otimes \cdots \otimes a_{W_n}) = \text{Tr}(W_{n+1}(a_{A_\lambda} \otimes \cdots \otimes a_{W_n} \otimes 1_{W_{n+1}}))$$

$$= \text{Tr}(w_0 K_{[0,1]} \cdots K_{[n+1]} h^1_{n+1} (a_{A_\lambda} \otimes \cdots \otimes a_{W_n} \otimes 1_{W_{n+1}}))$$

$$= \text{Tr}(w_0 K_{[0,1]} \cdots K_{[n-1,n]} \text{Tr}_n(K_{[n+1,n+2]} h^1_{n+1} (a_{A_\lambda} \otimes \cdots \otimes a_{W_n} \otimes 1_{W_{n+1}}))$$

$$= \text{Tr}(w_0 K_{[0,1]} \cdots K_{[n-2,n-1]} \text{Tr}_{n-1}(K_{[n-1,n]} E_n(a_{A_\lambda} \otimes \cdots \otimes a_{W_n} K^*_{[n-1,n]}))$$

$$= \text{Tr}(w_0 E_n \otimes \cdots \otimes E_{n-1} \hat{E}_n(a_{A_\lambda} \otimes \cdots \otimes a_{W_n} K^*_{[n-2,n-1]}))$$

This means that the limit state $\varphi_{w_0,h}$ is a backward QMC. This completes the proof. \qed
We notice that a phase transition phenomena is crucial in higher dimensional quantum models \cite{17,40,19}. In \cite{13}, quantum phase transition for the two-dimensional Ising model using $C^*$-algebra approach. In \cite{25} the VBS-model was considered on the Cayley tree. It was established the existence of the phase transition for the model in term of finitely correlated states which describe ground states of the model. Note that more general structure of finitely correlated states was studied in \cite{26}.

Our goal in this paper is to establish the existence of phase transition for the given family of operators. Heuristically, the phase transition means the existence of two distinct $B$ backward states which describe ground states of the model. Note that more general structure of finitely correlated states was studied in \cite{26}.

In this section, we define the model and formulate the main results of the paper. In what follows we consider a semi-infinite Cayley tree $\Gamma^2 = (L, E)$ of order two. Our starting $C^*$-algebra is the same $\mathcal{B}_L$ but with $\mathcal{B}_x = M_2(\mathbb{C})$ for all $x \in L$. By $\sigma^u_x$, $\sigma^u_y$, $\sigma^u_z$ we denote the Pauli spin operators for a site $u \in L$, i.e.

$$
\mathbf{1}^{(u)} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad 
\sigma^u_x = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad 
\sigma^u_y = \begin{pmatrix}
i & 0 \\
0 & -i
\end{pmatrix}, \quad 
\sigma^u_z = \begin{pmatrix}1 & 0 \\
0 & -1\end{pmatrix}.
$$

For every vertice $(u, (u, 1), (u, 2))$ we put

$$
K_{<u,(u,1)>} = \exp\{J_0 \beta \mathcal{H}_{<u,(u,1)>}\}, \quad i = 1, 2, \quad J_0 > 0, \quad \beta > 0, \quad (18)
$$

$$
L_{<(u,1),(u,2)<} = \exp\{J \beta \mathcal{H}_{<(u,1),(u,2)<}\}, \quad J \in \mathbb{R}, \quad (19)
$$

where

$$
\mathcal{H}_{<u,(u,1)>} = \frac{1}{2} \left( \mathbf{1}^{(u)} \otimes \mathbf{1}^{(u,1)} + \sigma_x^{(u)} \otimes \sigma_x^{(u,1)} \right), \quad (20)
$$

$$
\mathcal{H}_{<(u,1),(u,2)<} = \frac{1}{2} \left( \sigma_x^{(u,1)} \otimes \sigma_x^{(u,2)} + \sigma_y^{(u,1)} \otimes \sigma_y^{(u,2)} \right). \quad (21)
$$

Recall that a representation $\pi_1$ of a $C^*$-algebra $\mathfrak{A}$ is normal w.r.t. another representation $\pi_2$, if there is a normal $*-\$epimorphism $\rho : \pi_2(\mathfrak{A})' \rightarrow \pi_1(\mathfrak{A})'$ such that $\rho \circ \pi_2 = \pi_1$. Two representations $\pi_1$ and $\pi_2$ are called quasi-equivalent if $\pi_1$ is normal w.r.t. $\pi_2$, and conversely, $\pi_2$ is normal w.r.t. $\pi_1$. This means that there is an isomorphism $\alpha : \pi_1(\mathfrak{A})' \rightarrow \pi_2(\mathfrak{A})'$ such that $\alpha \circ \pi_1 = \pi_2$. Two states $\varphi$ and $\psi$ of $\mathfrak{A}$ are said be quasi-equivalent if the GNS representations $\pi_\varphi$ and $\pi_\psi$ are quasi-equivalent \cite{18}.
Furthermore, for the sake of simplicity, we assume that $M_{(u,(u,i),(u,i+1))} = I$ ($i = 1, 2, \ldots, k$) for all $u \in L$.

The defined model is called the Ising model with competing $XY$-interactions per vertices $(u, (u,1),(u,2))$.

For each $m \in \mathbb{N}$, from (20), (21) it follows that

\begin{equation}
H_{<u,v>}^m = H_{<u,v>} = \frac{1}{2} (I(u) \otimes I(v) + \sigma_z^u \otimes \sigma_z^v),
\end{equation}

\begin{equation}
H_{>(u,1),(u,2)}^{2m} = H_{>(u,1),(u,2)}^2 = \frac{1}{2} (I(u,1) \otimes I(u,2) - \sigma_z^{(u,1)} \otimes \sigma_z^{(u,2)})
\end{equation}

\begin{equation}
H_{>(u,1),(u,2)}^{2m-1} = H_{>(u,1),(u,2)}
\end{equation}

Therefore, one finds

\begin{align*}
K_{<u,(u,i)>} &= K_0 I(u) \otimes I(u,i) + K_3 \sigma_z^u \otimes \sigma_z^{(u,i)} \\
L_{>(u,1),(u,2)} &= I(u,1) \otimes I(u,2) + \sinh(J \beta) H_{>(u,1),(u,2)} + (\cosh(J \beta) - 1) H_{>(u,1),(u,2)}^2
\end{align*}

where

\begin{align*}
K_0 &= \frac{\exp(J \beta) + 1}{2}, & K_3 &= \frac{J \exp \beta - 1}{2},
\end{align*}

Hence, from (7) for each $x \in L$ we obtain

\begin{align*}
A_{(u,(u,1),(u,2))} &= \gamma_1 I(u) \otimes I(u,1) \otimes I(u,2) + \gamma_2 I^u \otimes \sigma_y^{(u,1)} \otimes \sigma_y^{(u,2)} \\
&+ \gamma_2 I(u) \otimes \sigma_y^{(u,1)} \otimes \sigma_y^{(u,2)} + \gamma_3 I(u) \otimes \sigma_z^{(u,1)} \otimes \sigma_z^{(u,2)} \\
&+ \delta_1 \sigma_z^u \otimes I(u,1) \otimes \sigma_z^{(u,2)} + \delta_1 \sigma_z^u \otimes \sigma_z^{(u,1)} \otimes I(u,2)
\end{align*}

(25)

where

\begin{align*}
\gamma_1 &= \frac{1}{4} \exp(2J \beta) + 1 + 2 \exp(J \beta) \cosh(J \beta), & \gamma_2 &= \frac{1}{2} \exp(J \beta) \sinh(J \beta). \\
\gamma_3 &= \frac{1}{4} \exp(2J \beta) + 1 - 2 \exp(J \beta) \cosh(J \beta), & \delta_1 &= \frac{1}{4} (\exp(2J \beta) - 1).
\end{align*}

Recall that a function $\{h^u\}$ is called translation-invariant if one has $h^u = h^{\tau^u}$, for all $u, v \in \Gamma^2_\Lambda$. Clearly, this is equivalent to $h^u = h^v$ for all $u, v \in L$.

In what follows, we restrict ourselves to the description of translation-invariant solutions of (14), (15). Consequently, we assume that: $h^u = h$ for all $u \in L$, here

\begin{equation}
h = \begin{pmatrix}
    h_{11} & h_{12} \\
    h_{21} & h_{22}
\end{pmatrix}.
\end{equation}

Then, equation (15) reduces to

\begin{align*}
h &= \text{Tr}[A_{(u,(u,1),(u,2))} (I(u) \otimes h \otimes h) A_{(u,(u,1),(u,2))}^*] \\
&= [C_1 \text{Tr}(h)^2 + C_2 \text{Tr}(\sigma_z h)^2] I(u) + C_3 \text{Tr}(h) \text{Tr}(\sigma_z h) \sigma_z^u.
\end{align*}

(26)
where

\[
\begin{align*}
C_1 &= \frac{1}{4}(\exp(4J\beta) + 1) + \frac{1}{2} \exp(2J\beta) \cosh(2J\beta); \\
C_2 &= \frac{1}{4}(\exp(4J\beta) + 1) - \frac{1}{2} \exp(2J\beta) \cosh(2J\beta); \\
C_3 &= \frac{1}{4}(\exp(4J\beta) - 1).
\end{align*}
\]

Now taking into account

\[\text{Tr}(h) = \frac{h_{11} + h_{22}}{2}, \quad \text{Tr}(\sigma_z h) = \frac{h_{11} - h_{22}}{2}\]

the equation (26) reduces to the following one

\[
\begin{align*}
\text{Tr}(h) &= C_1 \text{Tr}(h)^2 + C_2 \text{Tr}(\sigma h)^2, \\
\text{Tr}(\sigma h) &= C_3 \text{Tr}(h) \text{Tr}(\sigma h), \\
h_{21} &= 0, \quad h_{12} = 0.
\end{align*}
\]

This equation implies that a solution \(h\) is diagonal, and through the equation (14), \(\omega_0\) could be chosen diagonal as well. In the next sections we are going to examine (27).

5 Existence of QMC associated with the model.

In this section we are going to solve (27), which yields the existence of QMC associated with the model. We consider two distinct cases.

5.1 Case \(h_{11} = h_{22}\) and associate QMC

We assume that \(h_{11} = h_{22}\), then (27) is reduced to

\[h_{11} = h_{22} = \frac{1}{C_1}.
\]

Then putting \(\alpha = \frac{1}{C_1}\) we get

\[h_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}
\]

**Proposition 5.1.** The pair \((\omega_0, \{h^u = h_\alpha | u \in L\})\) with \(\omega_0 = \frac{1}{\alpha} \mathbf{1}\), \(h^u = h_\alpha, \forall u \in L\), is a solution of (14), (15). Moreover the associated backward QMC can be written on the local algebra \(B_{L,\text{loc}}\) by

\[
\varphi_\alpha(a) = \alpha^{2^n-1} \text{Tr} \left( \prod_{i=0}^{n-1} K_{[i,i+1]} a \prod_{i=0}^{n-1} K^*_{[n-i-1,n-i]} \right), \quad \forall a \in \mathcal{B}_\Lambda.
\]

5.2 Case \(h_{11} \neq h_{22}\) and associate QMC

Now we suppose that \(h_{11} \neq h_{22}\), and put \(\theta = \exp(2\beta)\). Then the equation (27) reduces to

\[
\begin{align*}
\frac{h_{11} + h_{22}}{2} &= \frac{1}{C_4}, \\
\left(\frac{h_{11} - h_{22}}{2}\right)^2 &= \frac{C_4 - C_3}{C_2 C_3}.
\end{align*}
\]
Denote
\[ \Delta(\theta) = \frac{C_3 - C_1}{C_2} = \frac{\theta^{2J_0} - \theta^{-J_0}(\theta^J + \theta^{-J}) - 3}{\theta^{2J_0} - \theta^{-J_0}(\theta^J + \theta^{-J}) + 1} \] (31)

**Proposition 5.2.** If \( \Delta(\theta) > 0 \), then the equation  (27) has two solutions given by

\[ h = \xi_0 \mathbb{I} + \xi_3 \sigma, \] (32)

\[ h' = \xi_0 \mathbb{I} - \xi_3 \sigma, \] (33)

where

\[ \xi_0 = \frac{1}{C_3} = \frac{2}{\theta^{2J_0} - 1}, \quad \xi_3 = \frac{\sqrt{\Delta(\theta)}}{C_3} = \frac{2\sqrt{\Delta(\theta)}}{\theta^{2J_0} - 1} \] (34)

**Proof.** Assume that \( \Delta(\theta) > 0 \). Then one can conclude that (30) is equivalent to the following system

\[
\begin{aligned}
& h_{1,1} + h_{2,2} = 2\xi_0, \\
& h_{1,1} - h_{2,2} = \pm 2\xi_3
\end{aligned}
\]

It is easy to see that \( h_{1,1} = \xi_0 - \xi_3, h_{2,2} = \xi_0 + \xi_3 \). Hence, we get (32),(33). \( \square \)

From (14) we find that \( \omega_0 = \frac{1}{\xi_0} \mathbb{I} \in \mathcal{B}^+ \). Therefore, the pairs  \( \omega_0, \{h^{(u)} = h, u \in L\} \) and  \( \omega_0, \{h^{(u)} = h', u \in L\} \) define two solutions of (14),(15). Hence, they define two backward QMC \( \varphi_1 \) and \( \varphi_2 \), respectively. Namely, for every \( a \in \mathcal{B}_n \) one has

\[ \varphi_1(a) = \text{Tr}(\omega_0 K_{[0,1]} \cdots K_{[n-1,n]} h^{1/2}_n a h^{1/2}_n K^*_n \cdots K^*_0) \] (35)

\[ \varphi_2(a) = \text{Tr}(\omega_0 K_{[0,1]} \cdots K_{[n-1,n]} h^{1/2}_n a h^{1/2}_n K^*_n \cdots K^*_0). \] (36)

Hence, we summarize this section in the following result.

**Theorem 5.3.** The following statements hold:

(i) if \( \Delta(\theta) \leq 0 \), then there is a unique translation invariant QMC \( \varphi_\alpha \);

(ii) if \( \Delta(\theta) > 0 \), then there are at least three translation invariant QMC \( \varphi_\alpha, \varphi_1 \) and \( \varphi_2 \).

From this theorem we conclude that there are three coexisting phases in the region \( \Delta(\theta) > 0 \), and one of it, i.e. \( \varphi_\alpha \), survives in the region \( \Delta(\theta) < 0 \). This leads us that the state \( \varphi_\alpha \) describes the disordered phase of the model, which shows a similar behavior with the classical Ising model [15, 38]. In comparison with the Ising model, we stress that in the present model, we have a similar kind of phases (translation invariant ones) when \( \Delta(\theta) > 0 \). From Figure 2 (see below), one concludes that the phase transition occurs except for a “triangular region”. This shows how the competing interactions effect to the existence of other phases. Notice that if \( J = 0 \), then we obtain the classical Ising model for which the existence of a disordered phase coexisting with two ordered phases is well-known [15, 38].

Next auxiliary fact gives an equivalent condition for \( \Delta(\theta) > 0 \).

**Lemma 5.4.** \( \Delta(\theta) > 0 \) iff one of the following statements hold:
(i) $J > J_0$ or $J < -J_0$;
(ii) $-J_0 < J < J_0$ and
\[ J_0 > \frac{1}{2\beta} \ln \left( \sinh(2J\beta) + \sqrt{\sinh^2(2J\beta) + 3} \right) \]

**Proof.** We know that $\theta = \exp(2\beta)$, $\beta > 0$ and $J_0 > 0$. Then one finds
\[
\Delta = \frac{\theta^{2J_0} - \theta^J}{\theta^{2J_0} - \theta^J} \quad \left( \frac{\theta^J + \theta^{-J} - 3}{\theta^J + \theta^{-J} + 1} \right) = \frac{R(J) - 4}{R(J)}
\]
where
\[ R(J) = \theta^{2J_0} - \theta^J \quad \left( \frac{\theta^J + \theta^{-J} + 1}{\theta^J + \theta^{-J} + 1} \right) = (\theta^{J_0-J} - 1)(\theta^{J_0+J} - 1) \]

Thanks to $\theta > 1$, we have
\[ R(J) \left\{ \begin{array}{ll} > 0, & \text{if } (J_0 - J)(J_0 + J) > 0 \\
< 0, & \text{if } (J_0 - J)(J_0 + J) < 0 \end{array} \right. \]

**Case** $(J_0 - J)(J_0 + J) < 0$. In this setting, one can see that $\Delta(\theta) > 0$.

**Case** $(J_0 - J)(J_0 + J) > 0$ Note that $\Delta(\theta) > 0$ if and only if $R(J) > 4$. For convenience, we denote $\theta^{J_0} = a$ and $\theta^J = b$. And consider the following equation
\[ a^2 - (b + b^{-1})a - 3 = 0. \]

Then
\[ a_1 = \frac{b + b^{-1} + \sqrt{D}}{2} > 0 \]
\[ a_2 = \frac{b + b^{-1} - \sqrt{D}}{2} < 0 \]

where $D = (b + b^{-1})^2 + 12$. Due to $a > 0$, we may conclude that
\[ a^2 - (b + b^{-1})a - 3 \left\{ \begin{array}{ll} > 0, & \text{if } a > a_1 \\
< 0, & \text{if } 0 < a < a_1 \end{array} \right. \]

This means that $\Delta(\theta) > 0$ if and only if
\[ J_0 > \frac{1}{2\beta} \ln \left( \sinh(2J\beta) + \sqrt{\sinh^2(2J\beta) + 3} \right) \]

which completes the proof.

From this lemma we infer that the phase transition exists in the shaded region shown in the Figure 2 (see $(J, J_0)$ plane).
6 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To realize it, we first show not overlapping supports of the states $\varphi_1$ and $\varphi_2$. Then we show that these states satisfy the clustering property, which yields that they are factor states, and this fact allows us to prove their non-quasi-equivalence.

6.1 Not overlapping supports of $\varphi_1$ and $\varphi_2$

As usual we put

$$e_{11} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \quad e_{22} = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right).$$

Now for each $n \in \mathbb{N}$, we denote

$$P_n := \left( \bigotimes_{x \in \Lambda_n} e_{11}^{(x)} \right) \otimes 1, \quad Q_n := \left( \bigotimes_{x \in \Lambda_n} e_{22}^{(x)} \right) \otimes 1.$$

Clearly, $P_n$ and $Q_n$ are orthogonal projections in $\mathcal{B}_{\Lambda_n}$.

**Lemma 6.1.** For every $n \in \mathbb{N}$, one has

(i) $\varphi_1(P_n) = \varphi_2(Q_n) = \frac{1}{2\xi_0} (\xi_0 + \xi_3)^{2^n} \left( \frac{C_1 + C_2 + C_3}{4} \right)^{2^n - 1},$

(ii) $\varphi_1(Q_n) = \varphi_2(P_n) = \frac{1}{2\xi_0} (\xi_0 - \xi_3)^{2^n} \left( \frac{C_1 + C_2 + C_3}{4} \right)^{2^n - 1}.$

**Proof.** (i). From [35] we find

$$\varphi_1(P_n) = \text{Tr}(\omega_0 K_{[0,1]} \cdots K_{[n-1,n]} h_n^{1/2} P_n h_n^{1/2} K_{[n-1,n]}^* \cdots K_{[0,1]}^*),$$

$$\varphi_2(Q_n) = \text{Tr}(\omega_0 K_{[0,1]} \cdots K_{[n-1,n]} h_n P_n K_{[n-1,n]}^* \cdots K_{[0,1]}^*).$$
Thanks to $he_{11} = (\xi_0 + \xi_3)e_{11}$ and (25) one gets

\[
\text{Tr}_{n-1}(K_{[n-1,n]}h_nP_nK_{[n-1,n]}^*) = P_{n-2} \otimes \prod_{u \in W_{n-1}} \text{Tr}_{[u]}(A_{(u,(u,1),(u,2))}e_{11}^{(u)} \otimes h e_{11}^{(u,1)} \otimes h e_{11}^{(u,2)} A_{(u,(u,1),(u,2))})
\]

\[
= (\xi_0 + \xi_3)^2|W_{n-1}|\left(\frac{C_1 + C_2 + C_3}{4}\right)^{|W_{n-1}|} P_{n-1}.
\]

Hence,

\[
\varphi_1(P_n) = (\xi_0 + \xi_3)^{|W_n|}\left(\frac{C_1 + C_2 + C_3}{4}\right)^{|W_{n-1}|+\ldots+|W_0|} \text{Tr} [\omega_0 K_{[0,1]} \cdots K_{[n-2,n-1]} P_{n-1} K_{[n-2,n-1]}^* \cdots K_{[0,1]}^*]
\]

\[
= (\xi_0 + \xi_3)^{|W_n|}\left(\frac{C_1 + C_2 + C_3}{4}\right)^{|\Lambda_{n-1}|} \text{Tr} [\omega_0 P_0]
\]

\[
= \frac{1}{2\xi_0}(\xi_0 + \xi_3)^{2n}\left(\frac{C_1 + C_2 + C_3}{4}\right)^{2n-1}
\]

Analogously, using $h'e_{22} = (\xi_0 + \xi_3)e_{22}$ we obtain

\[
\text{Tr}_{n-1}(K_{[n-1,n]}h'_nQ_nK_{[n-1,n]}^*) = Q_{n-2} \otimes \prod_{u \in W_{n-1}} \text{Tr}_{[u]}(A_{(u,(u,1),(u,2))}e_{2,2}^{(u)} \otimes h' e_{2,2}^{(u,1)} \otimes h' e_{22}^{(u,2)} A_{(u,(u,1),(u,2))})
\]

\[
= (\xi_0 + \xi_3)^2|W_{n-1}|\left(\frac{C_1 + C_2 + C_3}{4}\right)^{|W_{n-1}|} Q_{n-1}.
\]

which yields

\[
\varphi_2(Q_n) = \frac{1}{2\xi_0}(\xi_0 + \xi_3)^{2n}\left(\frac{C_1 + C_2 + C_3}{4}\right)^{2n-1}.
\]

(ii) Now from $he_{22} = (\xi_0 - \xi_3)e_{22}$ and $h'e_{11} = (\xi_0 - \xi_3)e_{11}$, we obtain

\[
\text{Tr}_{n-1}(K_{[n-1,n]}h_nQ_nK_{[n-1,n]}^*) = (\xi_0 - \xi_3)^2|W_{n-1}|\left(\frac{C_1 + C_2 + C_3}{4}\right)^{|W_{n-1}|} Q_{n-1}.
\]

\[
\text{Tr}_{n-1}(K_{[n-1,n]}h'_nP_nK_{[n-1,n]}^*) = (\xi_0 - \xi_3)^2|W_{n-1}|\left(\frac{C_1 + C_2 + C_3}{4}\right)^{|W_{n-1}|} P_{n-1}.
\]

The same argument as above implies (ii). This completes the proof.

\[\square\]

**Theorem 6.2.** For fixed $n \in \mathbb{N}$, one has

\[
\varphi_1(P_n) \to 1, \quad \varphi_2(Q_n) \to 1 \quad \text{as} \quad \beta \to +\infty.
\]
Proof. We know that $\theta = \exp(2\beta) \to +\infty$ as $\beta \to +\infty$. Hence, one finds

$$\frac{1}{2\xi_0} = \frac{\theta^{2J_0} - 1}{4} \sim \frac{\theta^{2J_0}}{4}, \quad \text{as } \theta \to +\infty$$

$$(\xi_0 + \xi_3)^2 = \left(\frac{2}{\theta^{2J_0} - 1}\right)^n \left(1 + \frac{\theta^{2J_0} - \theta^{J_0}(\theta^J + \theta^{-J}) + 1}{\theta^{2J_0} - \theta^{J_0}(\theta^J + \theta^{-J}) - 3}\right)^{2^n} \sim \left(\frac{4}{\theta^{2J_0}}\right)^{2^n}, \quad \text{as } \theta \to +\infty$$

$$\left(\frac{C_1 + C_2 + C_3}{4}\right)^{2^n - 1} = \left(\frac{\theta^{2J_0}}{4}\right)^{2^n - 1}. $$

Hence, we obtain

$$\varphi_1(P_n) = \varphi_2(Q_n) \sim \frac{\theta^{2J_0}}{4} \left(\frac{4}{\theta^{2J_0}}\right)^{2^n} \left(\frac{\theta^{2J_0}}{4}\right)^{2^n - 1} = 1.$$ 

So,

$$\lim_{\theta \to \infty} \varphi_1(P_n) = \lim_{\theta \to \infty} \varphi_2(Q_n) = 1.$$ 

This completes the proof.

Remark 6.3. We note that from $P_n \leq 1 - Q_n$ one gets

$$\lim_{\theta \to \infty} \varphi_1(Q_n) = \lim_{\theta \to \infty} \varphi_2(P_n) = 0.$$ 

This implies that the states $\varphi_1$ and $\varphi_2$ have non overlapping supports.

6.2 Clustering Property for $\varphi_1$ and $\varphi_2$

In this subsection, we are going to prove that the states $\varphi_1$, $\varphi_2$ satisfy the clustering property.

Recall that a state $\varphi$ on $B_L$ satisfies the clustering property if for every $a,f \in B_L$ one has

$$\lim_{|g| \to \infty} \varphi(a \tau_g(f)) = \varphi(a) \varphi(f). \quad (37)$$

Thanks to Theorem (5.3) there are two solutions of (14), and (15), these two solutions can be written as follows: $(\omega_0, \{h^u = h, \; u \in L\})$ and $(\omega_0, \{h^u = h', \; u \in L\})$, where

$$\omega_0 = \frac{1}{\xi_0} \mathbf{1}, \quad h = \xi_0 \mathbf{1} - \xi_3 \sigma_z,$$

$$\omega_0 = \frac{1}{\xi_0} \mathbf{1}, \quad h' = \xi_0 \mathbf{1} + \xi_3 \sigma_z.$$

By $\varphi_1$, $\varphi_2$ we denote the corresponding backward quantum Markov chains. To prove the clustering property we need to study the following matrix:

$$A := \begin{pmatrix} c_1 \xi_0 & -c_2 \xi_3 \\ -\frac{c_2}{\xi_3} & \frac{c_1}{\xi_0} \end{pmatrix}$$

One can easily prove the following fact.
Proposition 6.4. The above given matrix $A$ is a diagonalizable matrix, and can be written as follows:

$$A := P \begin{pmatrix} 1 & 0 \\ 0 & (c_1 - \frac{\delta_2}{2})\xi_0 \end{pmatrix} P^{-1}$$

where

$$P := \begin{pmatrix} \frac{c_2 \xi_3}{c_1 \xi_0 - 1} & 2c_2 \xi_3 \\ 1 & 1 \end{pmatrix}, \quad \det(P) = \frac{c_2 \xi_3 (3 - 2c_1 \xi_0)}{c_1 \xi_0 - 1}.$$ 

Lemma 6.5. Let $a \in B_{A_{N_0}}$, for some $N_0 \in \mathbb{N}$, and $f_n = \bigotimes_{x \in W_n} f^{(x)} = f^{x_{W_n}} \otimes \mathds{1}_{W_n \setminus \{x_{W_n}\}} \in \mathcal{B}_{W_n},$

where $f^{x_{W_n}} = f$, then for each backward quantum Markov chains $\varphi_1, \varphi_2$ we have

$$\lim_{n \to \infty} \varphi_k(a \otimes \mathds{1} \cdots \otimes \mathds{1} \otimes f_n) = \varphi_k(a) \varphi_k(f), \quad k = 1, 2.$$ 

Proof. By symmetry of calculations, it is enough to prove the result for $(\omega_0, \{h^{(u)} = h, \ u \in L\}).$

From (12) and (16) we have:

$$\varphi^{(n)}(a_{N_0} \otimes \mathds{1} \cdots \otimes \mathds{1} \otimes f) = \text{Tr}(\omega_0 \mathcal{E}_0 \cdots \mathcal{E}_{N_0} (a \otimes \mathcal{E}_{N_0+1} (\mathds{1} \otimes \cdots \mathcal{E}_{n-1} (\mathds{1} \otimes \hat{\mathcal{E}}_n (\mathds{1} \otimes f)) \cdots))$$

First, let us calculate $\hat{\mathcal{E}}_n (\mathds{1} \otimes f)$. From (16) it follows that

$$\hat{\mathcal{E}}_n (f \otimes \mathds{1}) = \bigotimes_{x \in W_n} \text{Tr}_x (A_{x \vee S(x)} f^{(x)} \otimes h_{S(x)} A_{x \vee S(x)}^*)$$

$$= \text{Tr}_{x_{W_n}} (A_{x_{W_n} \vee S(x_{W_n})} f^{x_{W_n}} \otimes h_{S(x_{W_n})} A_{x_{W_n} \vee S(x_{W_n})}^*) \bigotimes_{x \in W_n \setminus \{x_{W_n}\}} \text{Tr}_x (A_{x \vee S(x)} h_{S(x)} A_{x \vee S(x)}^*)$$

$$= \left( \alpha_1 f^{x_{W_n}} + \alpha_2 (f^{x_{W_n}} \sigma_z^{x_{W_n}} + \sigma_z^{x_{W_n}} f^{x_{W_n}}) \right) \otimes \bigotimes_{x \in W_n \setminus \{x_{W_n}\}} h_x$$

$$= g^{x_{W_n}} \otimes \bigotimes_{x \in W_n \setminus \{x_{W_n}\}} h_x,$$

where

$$g^{x_{W_n}} = \alpha_1 f^{x_{W_n}} + \alpha_2 (f^{x_{W_n}} \sigma_z^{x_{W_n}} + \sigma_z^{x_{W_n}} f^{x_{W_n}}) + \alpha_3 \sigma_z^{x_{W_n}} f^{x_{W_n}} \sigma_z^{x_{W_n}},$$

and

$$\left\{ \begin{array}{l} \alpha_1 = (C_1 - 2\delta_2^2)\xi_0^2 + (C_2 - 2\delta_1^2)\xi_3^2 \\ \alpha_2 = -\frac{\delta_3^2}{2} \xi_0 \xi_3 \\ \alpha_3 = 2\delta_1^2(\xi_0^2 + \xi_3^2) \end{array} \right.$$
Hence, one has
\[
\mathcal{E}_{n-1}(1 \otimes \hat{\mathcal{E}}_n(f \otimes 1)) = \operatorname{Tr}_{x(1)} \left( A_{x(1)} \otimes g(x_{W_n-1}^{(1)}) \otimes hA_{x(1)}^{*} \otimes S(x_{W_n-1}^{(1)}) \right) \otimes h(x) \\
\otimes x \in W_{n-1} \setminus \{x_{W_n-1}^{(1)}\} \\
= \left( \alpha_1 g(x_{W_n-1}^{(1)}) + \alpha_2 \sigma_z(x_{W_n-1}^{(1)}) \right) \otimes h(x) \\
\otimes x \in W_{n-1} \setminus \{x_{W_n-1}^{(1)}\} \\
+ \alpha_2 \sigma_z(x_{W_n-1}^{(1)}) \otimes h(x) \\
\otimes x \in W_{n-1} \setminus \{x_{W_n-1}^{(1)}\}
\]

So, one finds
\[
\mathcal{E}_{n-1}(1 \otimes \hat{\mathcal{E}}_n(f \otimes 1)) = v_1 1(x_{W_n-1}^{(1)}) \otimes h(x) + v_1' 1(x_{W_n-1}^{(1)}) \otimes h(x) \\
\otimes x \in W_{n-1} \setminus \{x_{W_n-1}^{(1)}\}
\]

where
\[
\begin{align*}
\alpha_{1,X} &= C_1 \operatorname{Tr}(X) \xi_0 - C_2 \operatorname{Tr}(\sigma_z X) \xi_3 \\
\alpha_{2,X} &= \frac{C_3}{2} (\operatorname{Tr}(\sigma_z X) \xi_0 - \operatorname{Tr}(X) \xi_3) \\
v_1 &= \alpha_1 g \\
v_1' &= \alpha_2 g
\end{align*}
\]

Then by iteration we obtain
\[
\mathcal{E}_{n-k}(1 \otimes \ldots \mathcal{E}_{n-1}(1 \otimes \hat{\mathcal{E}}_n(f \otimes 1))) = v_k 1(x_{W_n-1}^{(1)}) \otimes h(x) \\
\otimes x \in W_{n-1} \setminus \{x_{W_n-1}^{(1)}\} \\
+ v_k' 1(x_{W_n-1}^{(1)}) \otimes h(x) \\
\otimes x \in W_{n-1} \setminus \{x_{W_n-1}^{(1)}\}
\]

where
\[
\begin{align*}
v_k &= v_{k-1} C_1 \xi_0 - C_2 \xi_3 v_{k-1}' \\
v_k' &= -\frac{C_3}{2} \xi_3 v_{k-1} + \frac{C_3}{2} \xi_0 v_{k-1}'
\end{align*}
\]

Now let calculate the explicit form of the sequence \(v_k\), we can see :
\[
\begin{pmatrix}
v_k \\
v_k'
\end{pmatrix} = A \begin{pmatrix}
v_{k-1} \\
v_{k-1}'
\end{pmatrix} \\
\vdots \\
= A^{k-1} \begin{pmatrix}
v_1 \\
v_1'
\end{pmatrix}
\]
Then by (38) we get,
\[
\begin{pmatrix}
  v_k \\
v_k'
\end{pmatrix} = P \begin{pmatrix}
  1 & 0 \\
  c_1 - \frac{c_3}{2} & 1
\end{pmatrix} P^{-1} \begin{pmatrix}
  v_1 \\
v_1'
\end{pmatrix}
\]
\[
= \begin{pmatrix}
  \frac{1}{3 - 2c_1 \xi_0} & \frac{2c_2 \xi_0}{3 - 2c_1 \xi_0} \\
  \frac{c_1 \xi_0 - 1}{2c_2(3 - 2c_1 \xi_0)} & \frac{3 - 2c_1 \xi_0}{2c_2(3 - 2c_1 \xi_0)}
\end{pmatrix} \begin{pmatrix}
  v_1 \\
v_1'
\end{pmatrix}
\]
\[
= \begin{pmatrix}
  \eta_1 + \tilde{\eta}_1 \xi_0^{k-1} (c_1 - \frac{c_3}{2}) & \eta_2 - \eta_2 \xi_0^{k-1} (c_1 - \frac{c_3}{2}) \\
  \tilde{\eta}_2 - \tilde{\eta}_2 \xi_0^{k-1} (c_1 - \frac{c_3}{2}) & \eta_1 + \eta_1 \xi_0^{k-1} (c_1 - \frac{c_3}{2})
\end{pmatrix} \begin{pmatrix}
  v_1 \\
v_1'
\end{pmatrix}
\]
where
\[
\eta_1 = \frac{1}{3 - 2c_1 \xi_0}, \quad \tilde{\eta}_1 = \frac{2(1 - c_1 \xi_0)}{3 - 2c_1 \xi_0}
\]
\[
\eta_2 = -\frac{2c_2 \xi_0}{3 - 2c_1 \xi_0}, \quad \tilde{\eta}_2 = \frac{-c_1 \xi_0 - 1}{c_2 \xi_0(3 - 2c_1 \xi_0)}
\]
Hence,
\[
\begin{cases}
v_k = \left( \eta_1 + \tilde{\eta}_1 \xi_0^{k-1} (c_1 - \frac{c_3}{2}) \right) v_1 + \left( \eta_2 - \eta_2 \xi_0^{k-1} (c_1 - \frac{c_3}{2}) \right) v_1' \\
v_k' = \left( \tilde{\eta}_2 - \tilde{\eta}_2 \xi_0^{k-1} (c_1 - \frac{c_3}{2}) \right) v_1 + \left( \eta_1 + \eta_1 \xi_0^{k-1} (c_1 - \frac{c_3}{2}) \right) v_1'
\end{cases}
\]
So, one finds
\[
\varphi_{\nu_0}^{(n)}(a \otimes I \cdots \otimes I \otimes f) = v_{n-N_0-1} \text{Tr} \left( \omega_0 \mathcal{E}_0 \circ \cdots \circ \mathcal{E}_{N_0} \left( a \otimes I_x^{(x_{W_{N_0+1}})} \otimes h_x \right) \right)
\]
\[
+ v_{n-N_0} \text{Tr} \left( \omega_0 \mathcal{E}_0 \circ \cdots \circ \mathcal{E}_{N_0} \left( a \otimes \sigma_z^{(x_{W_{N_0+1}})} \otimes h_x \right) \right)
\]
\[
= \left( \eta_1 + \tilde{\eta}_1 \xi_0^{n-N_0} (c_1 - \frac{c_3}{2}) \right) v_1 \text{Tr} \left( \omega_0 \mathcal{E}_0 \circ \cdots \circ \mathcal{E}_{N_0} \left( a \otimes I_x^{(x_{W_{N_0+1}})} \otimes h_x \right) \right)
\]
\[
+ \left( \eta_2 - \eta_2 \xi_0^{n-N_0} (c_1 - \frac{c_3}{2}) \right) v_1' \text{Tr} \left( \omega_0 \mathcal{E}_0 \circ \cdots \circ \mathcal{E}_{N_0} \left( a \otimes I_x^{(x_{W_{N_0+1}})} \otimes h_x \right) \right)
\]
\[
+ \left( \tilde{\eta}_2 - \tilde{\eta}_2 \xi_0^{n-N_0} (c_1 - \frac{c_3}{2}) \right) v_1 \text{Tr} \left( \omega_0 \mathcal{E}_0 \circ \cdots \circ \mathcal{E}_{N_0} \left( a \otimes \sigma_z^{(x_{W_{N_0+1}})} \otimes h_x \right) \right)
\]
\[
+ \left( \tilde{\eta}_2 - \tilde{\eta}_2 \xi_0^{n-N_0} (c_1 - \frac{c_3}{2}) \right) v_1' \text{Tr} \left( \omega_0 \mathcal{E}_0 \circ \cdots \circ \mathcal{E}_{N_0} \left( a \otimes \sigma_z^{(x_{W_{N_0+1}})} \otimes h_x \right) \right)
\]
One can see that $\xi_{0-N_0} \to 0$, as $n \to \infty$, which implies

$$\lim_{n \to \infty} \varphi_n, h_0(a \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes f_n) = (\eta_1 v_1 + \eta_2 v'_1) \operatorname{Tr} \left( \omega_0 \mathcal{E}_0 \circ \cdots \circ \mathcal{E}_N_0 (a \otimes \frac{\xi_{xW_0+1}}{x} \otimes h_x) \right)$$

where

$$\begin{align*}
\eta_1 v_1 + \eta_2 v'_1 &= \frac{\xi_2}{6C_1} [(4C_3 - 2C_1) \operatorname{Tr}(f) - \sqrt{\Delta(\theta)} (4C_2 + C_3) \operatorname{Tr}(\sigma_z f)] \\
\hat{\eta}_1 v_1 + \hat{\eta}_1 v'_1 &= - c_3 \xi_3 (\eta_1 v_1 + \eta_2 v'_1)
\end{align*}$$

On other hand we have

$$\varphi_{w_0, h}(f) = C_3 (\eta_1 v_1 + \eta_2 v'_1),$$

Hence, one gets

$$\lim_{n \to \infty} \varphi_{w_0, h}(a \otimes \cdots \otimes f_n) = \varphi_{w_0, h}(f) \operatorname{Tr} \left( \omega_0 \mathcal{E}_0 \circ \cdots \circ \mathcal{E}_N_0 (a \otimes h_{xW_0+1}) \otimes h_x \right)$$

This completes the prove.

Now we are ready to prove the clustering property.

**Theorem 6.6.** The states $\varphi_1$ and $\varphi_2$ satisfy the clustering property.

**Proof.** Thanks to the density argument, without lost of generality, we may assume $a, f \in \mathcal{B}_{\text{loc}}$. This means that there are $N_0, m_0 \in \mathbb{N}$ such that $a \in \mathcal{B}_{\Lambda N_0}, f \in \mathcal{B}_{\Lambda m_0}$. Moreover, $f$ can be write in the following form

$$f = \bigotimes_{x \in \Lambda_{m_0}} f^{(x)}.$$

By symmetry of calculations, it is enough to prove the result for $(\omega_0, \{h^{(u)} = h, u \in L\}).$
In what follows, we assume that $g \in W_n$. Therefore, we put $g_n := g$. Then one has

$$
\tau_{g_n}(f) = \bigotimes_{x \in \Lambda_{n+0}} f^{(g_n \circ x)}
= f^{(g_n)} \otimes f^{(g_n \circ W_1)} \otimes f^{(g_n \circ W_2)} \otimes \ldots \otimes f^{(g_n \circ W_{m_0})},
$$

where

$$
f^{(g_n \circ W_k)} = \bigotimes_{x \in W_k} f^{(g_n \circ x)} \text{ and } \{g_n, W_k\} = \{(g_n \circ x), \ x \in W_k\}.
$$

We can see $\tau_{g_n}(f)$ as an element of $B_{\Lambda_{n+0}}$, i.e.

$$
\tau_{g_n}(f) = \mathbf{1}_{\Lambda_n} \otimes (f^{(g_n)} \otimes \mathbf{1}_{W_n \setminus \{g_n\}}) \otimes (f^{(g_n \circ W_1)} \otimes \mathbf{1}_{W_{n+1} \setminus \{g_n, W_1\}}) \otimes \ldots \otimes (f^{(g_n \circ W_{m_0})} \otimes \mathbf{1}_{W_{n+m_0} \setminus \{g_n, W_{m_0}\}}).
$$

For the sake of simplicity, let us denote

$$
\tau_{g_n}(f) = \bigotimes_{x \in \Lambda_{n+0}} f^{(x)}.
$$

From (12) and (16) it follows that

$$
\varphi_{w_0, h}(a \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \tau_{g_n}(f)) = \varphi^{(n+m_0)}_{w_0, h}(a \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \tau_{g_n}(f))
= \text{Tr}(\omega_0 \mathcal{E}_0 \circ \ldots \circ \mathcal{E}_{n_0}(a \otimes \mathcal{E}_{n_0+1}(\mathbf{1} \otimes \ldots \otimes \mathcal{E}_{n+m_0-1}(\mathbf{1} \otimes \hat{\mathcal{E}}_{n+m_0}(\tau_{g_n}(f))))\ldots)).
$$

Let us calculate $\hat{\mathcal{E}}_{n+m_0}(\tau_{g_n}(f))$. Indeed, from (16) one gets

$$
\hat{\mathcal{E}}_{n+m_0}(\tau_{g_n}(f) \otimes h_{W_{n+m_0}+1}) = \text{Tr}_{n+m_0}[K_{[n+m_0,n+m_0+1]} \tau_{g_n}(f) \otimes h_{W_{n+m_0}+1} K^*_{[n+m_0,n+m_0+1]}]
= \bigotimes_{x \in \Lambda_{n+m_0-1}} f^{(x)} \bigotimes_{x \in W_{n+m_0}} \text{Tr}_g(A_{g_n \circ x \circ S(x)} f^{(g_n \circ x)} \otimes h^{(S(g_n \circ x))} A^*_{g_n \circ x \circ S(x)})
= \bigotimes_{x \in \Lambda_{n+m_0-1}} f^{(x)} \bigotimes_{x \in W_{n+m_0}} \text{Tr}_g(A_{g_n \circ x \circ S(x)} h^{(x)} A^*_{g_n \circ x \circ S(x)})
\otimes \bigotimes_{x \in W_{n+m_0} \setminus \{g_n, W_{m_0}\}} h^{(x)}
= \bigotimes_{x \in \Lambda_{n+m_0-1}} f^{(x)} \bigotimes_{x \in W_{n+m_0}} T^{(g_n \circ x)}_{m_0} \bigotimes_{x \in W_{n+m_0} \setminus \{g_n, W_{m_0}\}} h^{(x)}
$$

where

$$
T^{(g_n \circ x)}_{m_0} = \text{Tr}_{g_n \circ x}[A_{(g_n \circ x) \circ S(g_n \circ x)} f^{(g_n \circ x)} \otimes h^{(S(g_n \circ x))} A^*_{(g_n \circ x) \circ S(g_n \circ x)}].
$$

Hence,
\begin{align*}
E_{n+m_0-1}(I \otimes \hat{E}_{n+m_0}(\tau_{g_n}(f))) &= \bigotimes_{x \in A_{n+m_0-2}} f_1^{(x)} \text{Tr}_{n+m_0-1}(K_{[n+m_0-1,n+m_0]} \bigotimes_{x \in W_{n+m_0-1}} f_1^{(x)} T_{m_0}^{(g_n \circ x)}) \\
&= \bigotimes_{x \in A_{n+m_0-2}} f_1^{(x)} \text{Tr}_{g_n \circ x}(A_{g_n \circ x} \otimes S(g_n \circ x) f(g_n \circ x)) \\
&= \bigotimes_{x \in A_{n+m_0-2}} f_1^{(x)} \text{Tr}_{m_0-1}(T_{m_0}^{(g_n \circ x)} \bigotimes_{x \in W_{n+m_0-1} \setminus \{g_n,W_{m_0}\}} h^{(x)})
\end{align*}

where

\begin{align*}
T_{m_0-1}^{(g_n \circ x)} &= \text{Tr}_{g_n \circ x}(A_{g_n \circ x} \otimes S(g_n \circ x) f(g_n \circ x) \otimes A_{g_n \circ x}^{*} \otimes S(g_n \circ x) )
\end{align*}

By iteration, we obtain

\begin{align*}
E_n(I \otimes E_{n+1}(I \otimes \ldots \otimes E_{n+m_0-1}(I \otimes \hat{E}_{n+m_0}(\tau_{g_n}(f))) \ldots)) &= \bigotimes_{x \in W_0} T_{0}^{(g_n \circ x)} \bigotimes_{x \in W_n \setminus \{g_n,W_0\}} h^{(x)} \\
&= T_{0}^{(g_n \circ x)} \bigotimes_{x \in W_n \setminus \{g_n,x_0\}} h^{(x)},
\end{align*}

which yields

\begin{align*}
\varphi_{w_0,h}(a_{N_0} \otimes I \otimes I \otimes \ldots \otimes I \otimes \tau_{g_n}(f)) &= \text{Tr}(\omega \circ \ldots \circ E_{n_0} \circ \ldots \circ E_{n_0+1}(I
\otimes E_{n_0+2}(I \otimes \ldots \otimes E_{n-1}(T_{0}^{(g_n \circ x)} \otimes \bigotimes_{x \in W_n \setminus \{g_n,x_0\}} h^{(x)}))
\end{align*}

Then Lemma [6.5] implies

\begin{align*}
\lim_{n \to \infty} \varphi_{w_0,h}(a_{N_0} \otimes I \otimes I \otimes \ldots \otimes I \otimes \tau_{g_n}(f)) &= \varphi_{w_0,h}(a_{N_0}) \text{Tr}(\omega \circ \ldots \circ E_{n-1}(T_{0}^{(g_n \circ x)} \otimes \bigotimes_{x \in W_n \setminus \{g_n,x_0\}} h^{(x)})) \\
&= \varphi_{w_0,h}(a_{N_0}) \varphi_{w_0,h}(\tau_{g_n}(f)).
\end{align*}

This completes the proof.
6.3 Non quasi equivalence of $\varphi_1$ and $\varphi_2$

In this subsection we are going to prove that the states $\varphi_1$ and $\varphi_2$ are not quasi equivalent. To establish the non-quasi equivalence, we are going to use the following result (see [18, Corollary 2.6.11]).

**Theorem 6.7.** Let $\varphi_1$, $\varphi_2$ be two factor states on a quasi-local algebra $\mathfrak{A} = \cup_{A} \mathfrak{A}_A$. The states $\varphi_1$, $\varphi_2$ are quasi-equivalent if and only if for any given $\varepsilon > 0$ there exists a finite volume $A \subseteq L$ such that $\|\varphi_1(a) - \varphi_2(a)\| < \varepsilon \|a\|$ for all $a \in \mathcal{B}_A$ with $A \cap A' = \emptyset$.

Now due to Theorem 6.6 the states $\varphi_1$ and $\varphi_2$ have clustering property, and hence they are factor states. Let us define an element of $\mathcal{B}_{\Lambda_n}$ as follows:

$$E_{\Lambda_n} := x_{W_n}^{(1)} \otimes \left( \bigotimes_{y \in \Lambda_n \setminus \{x_{W_n}^{(1)}\}} 1_y \right),$$

where $x_{W_n}^{(1)}$ is defined in (6). Now we are going to calculate $\varphi_1(E_{\Lambda_n})$ and $\varphi_2(E_{\Lambda_n})$, respectively. First consider the state $\varphi_1$, then we know that this state is defined by $\omega_0 = \frac{1}{\xi_0} I$ and $h^x = h = \xi_0 I + \xi_3 \sigma_z$. Define two elements of $\mathcal{B}_{W_n}$ by

$$\hat{\mathfrak{n}} := I_{x_{W_n}^{(1)}} \otimes \bigotimes_{x \in \cup_{n \setminus \{x_{W_n}^{(1)}\}}} h^{(x)}$$

$$\check{\mathfrak{n}} := \sigma_{x_{W_n}^{(1)}} \otimes \bigotimes_{x \in \cup_{n \setminus \{x_{W_n}^{(1)}\}}} h^{(x)}$$

**Lemma 6.8.** Let

$$\hat{\psi}_n := \text{Tr}_{n-1}[\omega_0 K_{[0,1]}^{(1)} \ldots K_{[n-1,n]}^{(1)} \hat{\mathfrak{n}} K_n^{*} \ldots K_{[0,1]}^{*}]$$

$$\check{\psi}_n := \text{Tr}_{n-1}[\omega_0 K_{[0,1]}^{(1)} \ldots K_{[n-1,n]}^{(1)} \check{\mathfrak{n}} K_n^{*} \ldots K_{[0,1]}^{*}]$$

Then there are two pairs of reals $(\hat{\rho}_1, \check{\rho}_2)$ and $(\hat{\rho}_1, \check{\rho}_2)$ depending on $\theta$ such that

$$\begin{cases} \hat{\psi}_n = \hat{\rho}_1 + \check{\rho}_2 (\frac{C_4}{C_3} - 1)^n, \\ \check{\psi}_n = \hat{\rho}_1 + \check{\rho}_2 (\frac{C_4}{C_3} - 1)^n \end{cases}$$

**Proof.** One can see that

$$\begin{pmatrix} \hat{\psi}_n \\ \check{\psi}_n \end{pmatrix} = \begin{pmatrix} \text{Tr}_{n-1}[\omega_0 K_{[0,1]}^{(1)} \ldots K_{[n-2,n-1]}^{(1)} \text{Tr}_{n-1}[K_{[n-1,n]}^{(1)} \hat{\mathfrak{n}} K_n^{*} \ldots K_{[0,1]}^{*}, K_{[n-2,n-1]}^{*} \ldots K_{[0,1]}^{*}] \ldots K_{[n-2,n-1]}^{*} \ldots K_{[0,1]}^{*}] \\ \text{Tr}_{n-1}[\omega_0 K_{[0,1]}^{(1)} \ldots K_{[n-2,n-1]}^{(1)} \text{Tr}_{n-1}[K_{[n-1,n]}^{(1)} \check{\mathfrak{n}} K_n^{*} \ldots K_{[0,1]}^{*}, K_{[n-2,n-1]}^{*} \ldots K_{[0,1]}^{*}] \ldots K_{[n-2,n-1]}^{*} \ldots K_{[0,1]}^{*}] \end{pmatrix}.$$ 

After small calculations, we find

$$\begin{pmatrix} \text{Tr}_{x} \left[ A_{(x,(x,1),(x,2))} (I^{(x)} \otimes I^{(x,1)} \otimes h^{(x,2)}) A_{(x,(x,1),(x,2))} \right] = C_1 \xi_0 I^{(x)} + \frac{1}{2} C_3 \xi_3 \sigma_z^{(x)} \\ \text{Tr}_{x} \left[ A_{(x,(x,1),(x,2))} (I^{(x)} \otimes \sigma^{(x,1)} \otimes h^{(x,2)}(\xi_0, \xi_3)) A_{(x,(x,1),(x,2))} \right] = C_2 \xi_3 I^{(x)} + \frac{1}{2} \sigma_z^{(x)} \end{pmatrix}$$

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Hence, one gets
\[
\begin{align*}
\text{Tr}_{n-1}K_{[n-1,n]}^\dagger \hat{h}_n K_{[n-1,n]}^* &= C_1 \xi_0 \hat{h}_{n-1} + \frac{1}{2} C_3 \xi_3 \hat{h}_{n-1}, \\
\text{Tr}_{n-1}K_{[n-1,n]}^\dagger \hat{h}_n K_{[n-1,n]}^* &= C_2 \xi_3 \hat{h}_{n-1} + \frac{1}{2} \hat{h}_{n-1}.
\end{align*}
\]
Therefore,
\[
\begin{pmatrix}
\hat{\psi}_n \\
\tilde{\psi}_n
\end{pmatrix} = \begin{pmatrix} C_1 \xi_0 \hat{\psi}_{n-1} + \frac{1}{2} C_3 \xi_3 \hat{\psi}_{n-1} \\
C_2 \xi_3 \hat{\psi}_{n-1} + \frac{1}{2} \tilde{\psi}_{n-1}
\end{pmatrix}
\]
\[
\vdots
\]
\[
\begin{pmatrix}
C_1 \xi_0 \\
C_2 \xi_3
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} C_3 \xi_3 \\
\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
\hat{\psi}_n \\
\tilde{\psi}_n
\end{pmatrix}
\]
where
\[
\begin{align*}
\hat{\psi}_0 &= \text{Tr}(\omega_0) = \frac{1}{\xi_0} \\
\tilde{\psi}_0 &= \text{Tr}(\omega_0 \sigma_z) = 0
\end{align*}
\]
The matrix
\[
N := \begin{pmatrix} C_1 \xi_0 & \frac{1}{2} C_3 \xi_3 \\
C_2 \xi_3 & \frac{1}{2}
\end{pmatrix}
\]
can be written in diagonal form by:
\[
N = P \begin{pmatrix} 1 & 0 \\
0 & \frac{C_1}{C_3} - \frac{1}{2}
\end{pmatrix} P^{-1}
\]
where
\[
P = \begin{pmatrix}
\frac{C_1}{2C_2} & -\xi_0 \\
\frac{\xi_0}{C_3} & 1
\end{pmatrix}, \quad \det(P) = \frac{3C_3 - 2C_1}{2C_2}
\]
So,
\[
\begin{pmatrix}
\hat{\psi}_n \\
\tilde{\psi}_n
\end{pmatrix} = P \begin{pmatrix} 1 & 0 \\
0 & \left(\frac{C_1}{C_3} - \frac{1}{2}\right)^n
\end{pmatrix} P^{-1} \begin{pmatrix}
\frac{1}{\xi_0} \\
0
\end{pmatrix}
\]
\[
\begin{pmatrix}
\hat{\rho}_n \\
\tilde{\rho}_n
\end{pmatrix} = \begin{pmatrix}
\hat{\rho}_n + \hat{\rho}_2 \left(\frac{C_1}{C_3} - \frac{1}{2}\right)^n \\
\tilde{\rho}_n + \tilde{\rho}_2 \left(\frac{C_1}{C_3} - \frac{1}{2}\right)^n
\end{pmatrix}
\]
where
\[
\begin{align*}
\hat{\rho}_1 &= \frac{C_2^2}{3C_3 - 2C_1}, \quad \hat{\rho}_2 = \frac{2C_3(C_3 - C_1)}{3C_3 - 2C_1}, \\
\tilde{\rho}_1 &= \frac{2C_2 C_3^2 \xi_3}{3C_3 - 2C_1}, \quad \tilde{\rho}_2 = -\frac{2C_2 C_3^2 \xi_3}{3C_3 - 2C_1}.
\end{align*}
\]
This completes the proof. \(\square\)
Lemma 6.10. Using the same argument like in the proof of Lemma 6.8 we can prove the following auxiliary result.

Proposition 6.9. For each \( n \in \mathbb{N} \) one has

\[
\varphi_1(E_{\Lambda_n}) = \frac{1}{2} \left[ (\xi_0 + \xi_3)(C_1\xi_0 + C_2\xi_3)\hat{\rho}_1 + \frac{C_3}{2} (\xi_0 + \xi_3)^2 \hat{\rho}_1 \right] \\
+ \frac{1}{2} \left[ (\xi_0 + \xi_3)(C_1\xi_0 + C_2\xi_3)\hat{\rho}_2 + \frac{C_3}{2} (\xi_0 + \xi_3)^2 \hat{\rho}_2 \right] \left( \frac{C_1}{C_3} - \frac{1}{2} \right)^n
\]

Proof. From (35) we have

\[
\varphi_1(E_{\Lambda_n}) = \text{Tr}(\omega_0 K_{[0,1]} \cdots K_{[n-1,n]} h_{\Lambda_n}^{1/2} E_{\Lambda_n} h_{\Lambda_n}^{1/2} K^*_{[n-1,n]} \cdots K^*_{[0,1]})
\]

One can calculate that

\[
\text{Tr}_{n-1}(K_{[n-1,n]} h_{\Lambda_n} K^*_{[n-1,n]} E_{\Lambda_n}) = \text{Tr}_{x^{(1)}_{W_{n-1}}} \left( A_{x^{(1)}_{W_{n-1}}^* x^{(1)}_{W_{n}}} (1) (x_{W_{n-1}} \otimes e_{1,1} h^{(4)}_{x_{W_{n}}} \otimes h^{(2)}_{x_{W_{n}}})
\right)
\]

Hence

\[
\varphi_1(E_{\Lambda_n}) = \frac{1}{2} (\xi_0 + \xi_3)(C_1\xi_0 + C_2\xi_3) \text{Tr} \left[ \omega_0 K_{[0,1]} \cdots K_{[n-2,n-1]} h_{n-1} K^*_{[n-2,n-1]} \cdots K^*_{[0,1]} \right]
\]

Now using the values of \( \psi_{n-1} \) and \( \tilde{\psi}_{n-1} \) given by the previous lemma we obtain the result. \( \square \)

Now we consider the state \( \varphi_2 \). Recall that this state is defined by \( \omega_0 = \frac{1}{\xi_0} \mathbb{I} \) and \( h^x = h' = \xi_0 \mathbb{I} - \xi_3 \sigma_z \). Define two elements of \( B_{W_n} \) by

\[
\hat{h}'_n := \mathbb{I}^{x^{(1)}_{W_n}} \otimes \bigotimes_{x \in W_n \setminus \{x^{(1)}_{W_n}\}} h'^{(x)}
\]

\[
\hat{h}''_n := \sigma^{x^{(1)}_{W_n}} \otimes \bigotimes_{x \in W_n \setminus \{x^{(1)}_{W_n}\}} h'^{(x)}
\]

Using the same argument like in the proof of Lemma 6.8 we can prove the following auxiliary fact.

Lemma 6.10. Let

\[
\hat{\phi}_n := \text{Tr}_{n-1} \left[ \omega_0 K_{[0,1]} \cdots K_{[n-1,n]} h'_{n} K^*_{[n-1,n]} \cdots K^*_{[0,1]} \right]
\]
Then there are two pairs of reals \((\tilde{\pi}_1, \tilde{\pi}_2)\) and \((\hat{\pi}_1, \hat{\pi}_2)\) depending on \(\theta\) such that

\[
\begin{align*}
\hat{\phi}_n &= \hat{\pi}_1 + \hat{\pi}_2 (\frac{C_1}{C_3} - \frac{1}{2})^n, \\
\tilde{\phi}_n &= \tilde{\pi}_1 + \tilde{\pi}_2 (\frac{C_1}{C_3} - \frac{1}{2})^n
\end{align*}
\]

where

\[
\begin{align*}
\hat{\pi}_1 &= \frac{C_2^3}{3C_3 - 2C_1}, \\
\hat{\pi}_2 &= \frac{2C_3(C_3 - C_1)}{3C_3 - 2C_1}, \\
\tilde{\pi}_1 &= -\frac{2C_2^3 \xi_3}{3C_3 - 2C_1}, \\
\tilde{\pi}_2 &= \frac{2C_2^3 \xi_3}{3C_3 - 2C_1}
\end{align*}
\]

**Proposition 6.11.** For each \(n \in \mathbb{N}\) one has

\[
\varphi_2(E_{\Lambda_n}) = \frac{1}{2} \left[ (\xi_0 - \xi_3)(C_1 \xi_0 - C_2 \xi_3) \hat{\pi}_1 + \frac{C_3}{2} (\xi_0 - \xi_3)^2 \hat{\pi}_1 \right] + \frac{1}{2} \left[ (\xi_0 - \xi_3)(C_1 \xi_0 - C_2 \xi_3) \tilde{\pi}_1 + \frac{C_3}{2} (\xi_0 - \xi_3)^2 \tilde{\pi}_1 \right] \left( \frac{C_1}{C_3} - \frac{1}{2} \right)^{n-1}.
\]

**Proof.** From (36) we find

\[
\begin{align*}
\varphi_2(E_{\Lambda_n}) &= \text{Tr}(\omega_0 K_{[0,1]} \cdots K_{[n-1,n]}^* \cdots K_{[n-1,n]}^* E_{\Lambda_n} H_n^{1/2} E_{\Lambda_n} H_n^{1/2} K_{[n-1,n]}^* K_{[n-1,n]}^*) \\
&= \text{Tr}(\omega_0 K_{[0,1]} \cdots K_{[n-1,n]}^* H_n^\prime E_{\Lambda_n} K_{[n-1,n]}^*) \\
&= \text{Tr}(\omega_0 K_{[0,1]} \cdots K_{[n-1,n]}^* H_n^\prime E_{\Lambda_n} K_{[n-1,n]}^*)
\end{align*}
\]

We easily calculate that

\[
\text{Tr}_{n-1}(K_{[n-1,n]}^* H_{n}^\prime E_{\Lambda_n} K_{[n-1,n]}^*) = \frac{1}{2} (\xi_0 - \xi_3)(C_1 \xi_0 - C_2 \xi_3) \hat{\phi}_{n-1} + C_3 \left( \frac{\xi_0 - \xi_3}{2} \right)^2 \hat{H}_{n-1}^\prime.
\]

Hence, from (41) one gets

\[
\varphi_2(E_{\Lambda_n}) = \frac{1}{2} \left[ (\xi_0 - \xi_3)(C_1 \xi_0 - C_2 \xi_3) \hat{\phi}_{n-1} + C_3 (\xi_0 - \xi_3)^2 \hat{\phi}_{n-1} \right].
\]

Using the values of \(\hat{\phi}_{n-1}\) and \(\tilde{\phi}_{n-1}\) given in Lemma 6.10 we obtain the desired assertion. \(\square\)

**Theorem 6.12.** Assume that \(J \in ]- J_0, J_0[\), then the two Backward QMC \(\varphi_1\) and \(\varphi_2\) are not quasi-equivalent.

**Proof.** For any \(\forall n \in \mathbb{N}\) it is clear that \(E_{\Lambda_n} \in B_{\Lambda_n} \setminus B_{\Lambda_{n-1}}\). Therefore, for any finite subset \(\Lambda \in L\), there exists \(n_0 \in \mathbb{N}\) such that \(\Lambda \subset \Lambda_{n_0}\). Then for all \(n > n_0\) one has \(E_{\Lambda_n} \in B_{\Lambda_n} \setminus B_{\Lambda}\). It is clear that

\[
||E_{\Lambda_n}|| = ||e_{1,1}^{(1)}(w_n) \otimes \mathbb{I}_Y|| = ||e_{1,1}|| = \frac{1}{2}.
\]

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From Propositions 6.9 and 6.11 we obtain

\[ |\varphi_1(E_{\Lambda_n}) - \varphi_1(E_{\Lambda_n})| = \frac{1}{2} \left| \left[ (\xi_0 + \xi_3)(C_1\xi_0 + C_2\xi_3)\hat{\rho}_1 + C_3(\xi_0 + \xi_3)^2\hat{\rho}_1 \right] 
- \left[ (\xi_0 - \xi_3)(C_1\xi_0 - C_2\xi_3)\tilde{\pi}_1 + C_3(\xi_0 - \xi_3)^2\hat{\rho}_1 \right] 
+ \left( \left[ (\xi_0 + \xi_3)(C_1\xi_0 + C_2\xi_3)\hat{\rho}_2 + C_3(\xi_0 + \xi_3)^2\hat{\rho}_2 \right] 
- \left[ (\xi_0 - \xi_3)(C_1\xi_0 - C_2\xi_3)\tilde{\pi}_2 + C_3(\xi_0 - \xi_3)^2\hat{\rho}_2 \right] \right) \left( \frac{C_1}{C_3} - \frac{1}{2} \right)^{n-1} \]

\[ \geq I_1 - I_2 \left| \frac{C_1}{C_3} - \frac{1}{2} \right|^{n-1} \]

where

\[ I_1 = \frac{1}{2} \left| \left[ (\xi_0 + \xi_3)(C_1\xi_0 + C_2\xi_3)\hat{\rho}_1 + C_3(\xi_0 + \xi_3)^2\hat{\rho}_1 \right] 
- \left[ (\xi_0 - \xi_3)(C_1\xi_0 - C_2\xi_3)\tilde{\pi}_1 + C_3(\xi_0 - \xi_3)^2\hat{\rho}_1 \right] \right| \]

\[ I_2 = \frac{1}{2} \left| \left[ (\xi_0 + \xi_3)(C_1\xi_0 + C_2\xi_3)\hat{\rho}_2 + C_3(\xi_0 + \xi_3)^2\hat{\rho}_2 \right] 
- \left[ (\xi_0 - \xi_3)(C_1\xi_0 - C_2\xi_3)\tilde{\pi}_2 + C_3(\xi_0 - \xi_3)^2\hat{\rho}_2 \right] \right|. \]

Due to \( \beta > 0, \theta = \exp 2\beta > 1, C_1 > 0, C_3 > 0, \xi_0 > \xi_3 > 0 \), one can find that

\[ I_1 = \frac{C_3\xi_3(2C_2 + C_3)}{3C_3 - 2C_1} > 0. \]

Now we have \( \frac{2C_1 - C_3}{2C_3} = \frac{\theta^{J_0}(\theta^{J_0} + \theta^{-J_0})}{2(\theta^{2J_0} - 1)} \), since \( J \in [-J_0, J_0] \) then

\[ \frac{2C_1 - C_3}{2C_3} = \frac{\theta^{J_0}(\theta^{J_0} + \theta^{-J_0})}{2(\theta^{2J_0} - 1)} \sim \frac{1}{2(\theta^{J_0} - 1)} \leq \frac{1}{2}, \quad \theta \geq \theta_0 \]

Then the following equality is hold:

\[ \left| \frac{C_1}{C_3} - \frac{1}{2} \right| \leq \frac{1}{2} \]

which yields

\[ I_2 \left| \frac{C_1}{C_3} - \frac{1}{2} \right|^{n-1} \to 0 \quad \text{as} \quad n \to +\infty. \]

Then there exists \( n_1 \in \mathbb{N} \) such that \( \forall n \geq n_0 \) one has

\[ I_2 \left| \frac{C_1}{C_3} - \frac{1}{2} \right|^n \leq \frac{\varepsilon_1}{2}. \]

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Hence, for all \( n \geq n_1 \) we obtain
\[
|\phi_1(E_{\Lambda_n}) - \phi_1(E_{\Lambda_{n,1}})| \geq \frac{\varepsilon_1}{2} = \varepsilon_1 \|E_{\Lambda_n}\|.
\]
This, according to Theorem 6.4, means that the states \( \phi_1 \) and \( \phi_2 \) are not quasi-equivalent. The proof is complete.

Now Theorems 5.3, 6.2 and 6.12 imply Theorem 1.1.

7 QMC associated with the XY-interaction model with \( J_0 = 0 \)

In this section, we consider a model which does not contain the classical Ising part, i.e. \( J_0 = 0 \), which means the model has only competing XY-interactions. In this setting, from (7) one gets

\[
A_{(u,(u,1),(u,2))} = L_{(u,1),(u,2)} <
\]

\[
= 1^{(u,1)} \otimes 1^{(u,2)} + \sinh(J\beta)H_{(u,1),(u,2)} <
\]

\[
+ (\cosh(J\beta) - 1)H_{(u,1),(u,2)} <
\]

\[
= R_1 1^{(u)} \otimes 1^{(u,1)} \otimes 1^{(u,2)} + R_2 1^{(u)} \otimes \sigma_y^{(u,1)} \otimes \sigma_y^{(u,2)} + R_3 1^{(u)} \otimes \sigma_z^{(u,1)} \otimes \sigma_z^{(u,2)}
\]

where

\[
\begin{align*}
R_1 &= \frac{1}{4}(\cosh(J\beta) + 1); \\
R_2 &= \frac{\sinh(J\beta)}{2}; \\
R_3 &= \frac{1}{2}(1 - \cosh(J\beta)).
\end{align*}
\]

Therefore, one finds:

\[
h = Tr[A_{(u,(u,1),(u,2))}[1^{(u)} \otimes h \otimes 1]A_{(u,(u,1),(u,2))}^*]
\]

\[
= [(R_1 + 2R_1^2 + R_3^2)Tr(h)^2] 1^{(u)}. \tag{44}
\]

The equation (44) is reduced to the following one

\[
\begin{align*}
\begin{cases}
    h_{11} = h_{22} = \frac{1}{(R_1 + 2R_1^2 + R_3^2)} , \\
    h_{21} = 0 ,
\end{cases}
\end{align*} \tag{45}
\]

Then putting \( \alpha = \frac{1}{(R_1 + 2R_1^2 + R_3^2)} \) we get

\[
h_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \tag{46}
\]

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Proposition 7.1. The pair \((\omega_0, \{h^x = h_\alpha | x \in L\})\) with \(\omega_0 = \frac{1}{n} I\), \(h^x = h_\alpha, \forall x \in L\), is solution of (14), (15). Moreover the associated Backward QMC can be written on the local algebra \(B_{L, loc}\) by:

\[
\varphi_\alpha(a) = \alpha^{2^n - 1} \text{Tr} \left( \prod_{i=0}^{n-1} K_{[i, i+1]} a \prod_{i=0}^{n-1} K_{[n-i, n-i]}^* \right), \quad \forall a \in B_{\Lambda_n}.
\] (47)

In this case, there is no phase transition.

We stress that if one takes nearest neighbor XY interactions on the Cayley tree of order two, still there does not occur a phase transition \([8]\). However, if the order of the tree is three or more then for the mentioned model there exists a phase transition \([9]\).

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