In the high density, low temperature limit, Quantum Chromodynamics exhibits a transition to phases characterized by color superconductivity and energy gaps in the fermion spectra. We review some fundamental results obtained in this area and in particular we describe the low energy effective lagrangian describing the motion of the quasi-particles in the high density medium (High Density Effective Theory).

1 Lecture I: Color superconductivity

1.1 Introduction

At high baryonic densities and small temperatures the color interaction favors the formation of quark-quark condensates in the color attractive antisymmetric channel:

\[ \Delta = \langle \psi^T_i \sigma^a \psi_j \rangle \epsilon^{abc} \epsilon^{ijk} \Omega_{\gamma k} \neq 0 \]  

(\( \alpha, \beta, \gamma = 1, 2, 3 \) color indices; \( i, j, k = 1, 2, 3 \) flavor indices). The condensates depend on the matrix \( \Omega \) and act as order parameters of new phases where the \( SU(3) \) color symmetry is spontaneously broken. The densities are so high that these phenomena might probably occur only in the core of neutron stars. Since this mechanism is similar to electron superconductivity it is referred to as color superconductivity (CSC). This is one of the most fascinating advances in Quantum Chromodynamics (QCD) in recent years (for reviews see [5]). The aim of these lectures is to describe an approach to this aspect of QCD that is based on the method of the effective lagrangians and to stress possible astrophysical consequences.

Different phenomena take place depending on the value of the order parameter \( \Omega \). One could have: \( \Omega_{\gamma k} = \delta_{\gamma 3} \delta_{k 3} \), which corresponds to

\[ \epsilon^{abc} \epsilon^{ij} < \psi^T_i \sigma^a \psi_j > = \Delta \delta^{33} \]  

where the sum over the flavor indices run from 1 to 2 and \( \psi \) represents a left handed 2-component Weyl spinor (the right handed field satisfies a similar relation); moreover a sum over spinor indices is understood and \( \sigma = i \sigma_2 \). This
case correspond to decoupling of the strange quark ($m_s \to \infty$; $m_u = m_d = 0$) and is called 2SC model. From dynamical analyses one knows that, for $\mu$ sufficiently large, the condensate (2) is non vanishing. Therefore it breaks the original symmetry group $SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_B$ down to $SU(2)_c \otimes SU(2)_L \otimes SU(2)_R \otimes \mathbb{Z}_2$. The chiral group remains unbroken, while the original color symmetry group is broken to $SU(2)_c$, with generators $T^A$ corresponding to the generators $T^1, T^2, T^3$ of $SU(3)_c$; an unbroken $U(1)_B$ also remains. As a consequence, three gluons remain massless whereas the remaining five acquire a mass. One can construct an effective theory to describe the emergence of the unbroken subgroup $SU(2)_c$ and the low energy excitations, much in the same way as one builds up chiral effective lagrangians with effective fields at zero density. For the two flavor case this development can be found in [10].

For the three flavor case ($m_u = m_d = m_s = 0$) the following case has been widely discussed [3]:

$$\langle \psi_{i\alpha}^L \psi_{j\beta}^L \rangle = -\langle \psi_{i\alpha}^R \psi_{j\beta}^R \rangle = \Delta \sum_{K=1}^{3} \epsilon_{\alpha\beta K} \epsilon_{ijK}.$$  \hspace{1cm} (3)

The condensate (3) breaks the original symmetry group $SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_B$ down to $SU(3)_c \otimes L + R \otimes \mathbb{Z}_2$. Both the chiral group, $U(1)_B$ and the color symmetry are broken but a diagonal $SU(3)$ subgroup remains unbroken in a way that locks together color and flavor (Color-Flavor-Locking=CFL model). There are 17 broken generators; since there is a broken gauge group, 8 of these generators correspond to 8 longitudinal degrees of freedom of the gluons, because the gauge bosons acquire a mass; there are 9 Nambu-Godstone Bosons (NGB’s) organized in an octet associated to the breaking of the flavor group and in a singlet associated to the breaking of the baryonic number. The effective theory describing the NGB for the CFL model was studied in [11].

Another interesting possibility arises when there is a difference $\delta \mu$ between the chemical potentials of the two gapped quarks (for references see [12]; see also [13] and for earlier works [14]). As shown in [12] when the two fermions have different chemical potentials $\mu_1 \neq \mu_2$, for $\delta \mu$ of the order of the gap the vacuum state is characterized by a non vanishing expectation value of a quark bilinear which breaks translational and rotational invariance. The appearance of this condensate is a consequence of the fact that for $\mu_1 \neq \mu_2$, and in a given range of $\delta \mu = |\mu_1 - \mu_2|$, the formation of a Cooper pair with a total momentum $\vec{p}_1 + \vec{p}_2 = 2\vec{q} \neq \vec{0}$ is energetically favored in comparison with the normal BCS state. The possible form of these condensates is discussed in [12] and [13] (see also [14]); it is worthwhile to note that, for simplicity, these authors assume only two flavors. In [14] the ansatz of a plane wave behavior for the condensate is made: $\propto e^{2i\vec{q} \cdot \vec{r}}$; in [1] it is shown that the configuration
that is energetically favored is a face centered cubic crystal. For simplicity I will consider here only the plane wave:

\[-<0|\epsilon_{ij}\epsilon_{\alpha\beta}\psi^{i\alpha}(\vec{x})C\psi^{j\beta}(\vec{x})|0>=2\Gamma_L^i\epsilon^{2i\vec{q} \cdot \vec{x}}; (4)\]

besides the scalar condensate there is also a vector condensate that can be however neglected since it is numerically small. The 2SC and/or the LOFF phases might exist for intermediate values of the chemical potentials, while for very high \(\mu\) the CFL phase should set in.

The phase transition and the non vanishing condensates result from a mechanism analogous to the formation of an electron Cooper pair in a BCS superconductor. At \(T=0\) the only QCD interactions are those involving fermions near the Fermi surface. Quarks inside the Fermi sphere cannot interact because of the Pauli principle, unless the interactions involve large momentum exchanges. In this way the quarks can escape the Fermi surface, but these processes are disfavored, as large momentum transfers imply small couplings due to the asymptotic freedom property of QCD. Even though interactions of fermions near the Fermi surface involve momenta of the order of \(\mu\), their effects are not necessarily negligible. As a matter of fact, even a small attractive interaction between fermions near the Fermi surface and carrying opposite momenta can create an instability and give rise to coherent effects. This is what really happens and the result is the formation of a diquark condensate, as expressed by (2), (3) or (4). We stress again that the only relevant fermion degrees of freedom are therefore those near the Fermi surface. In an effective theory for the CFL model was discussed, based on the approximation of the neglect of the negative energy states. This results in a rather terse formalism displaying as a characteristic note the existence of a Fermi velocity superselection rule and effective velocity-dependent fermion fields. We will refer to this effective lagrangian as the High Density Effective Theory (HDET). In the 2SC model has been studied by the same formalism, while in this effective theory has been applied to the crystalline color superconducting phase, the so-called LOFF phase.

The aim of these lectures is to review some developments in the description of Color Super Conductivity that are based on the HDET approach. This will be mainly done in the second lecture. Here I wish to discuss some possible astrophysical implications of CSC, in particular in the LOFF phase.

### 1.2 Astrophysical implications of the LOFF phase

Besides its theoretical interest for the study of the phase structure of QCD theory, the crystalline phase may result relevant for astrophysical dense sys-
tems, in particular in the explanation of the glitches in the pulsars.

The pulsars are rapidly rotating stellar objects, characterized by the presence of strong magnetic fields and by an almost continuous conversion of rotational energy into electromagnetic radiation. The rotation periods can vary in the range \(10^{-3}\) sec up to a few seconds; these periods increase slowly and never decrease except for occasional glitches, when the pulsar spins up with a variation in frequency that can be \(\delta \Omega/\Omega \approx 10^{-6}\) or smaller. Glitches are a typical phenomenon of the pulsars, in the sense that probably all the pulsar have glitches.

Pulsar are commonly identified with neutron stars; these compact stars are characterized by a rather complex structure comprising a core, an intermediate region with superfluid neutrons and a metallic crust. The ordinary explanation of the glitches is based on the idea that these sudden jumps of the rotational frequency are due to the movements outwards of rotational vortices in the neutron superfluid and their interaction with the crust. This is one of the main reasons that allow the identification of pulsars with neutron stars, as only neutron stars are supposed to have a metallic crust. Since the conventional models for glitches may be not familiar to an audience of nuclear and/or high energy physicists, I will briefly review them in the sequel.

It is known that a boson liquid at \(T \sim 0\) forms a condensate whose wavefunction

\[
\Xi_0 = \sqrt{n_0(t, \vec{r})} e^{i\Phi(t, \vec{r})}
\]  

(5)

has a macroscopic meaning, due to the large number of particles in it. Also the probability current density

\[
\vec{j}_{\text{cond}} = \frac{i\hbar}{2m} (\Xi_0 \nabla \Xi_0^\ast - \Xi_0^\ast \nabla \Xi_0) = \frac{\hbar}{m} n_0 \nabla \Phi
\]  

(6)

has such macroscopic meaning. Since \(\vec{j}_{\text{cond}} = n_0 \vec{v}_s\), where \(\vec{v}_s\) is the condensate velocity that can be identified with the superfluid velocity, we get

\[
\vec{v}_s = \frac{\hbar}{m} \vec{\nabla} \Phi.
\]  

(7)

The consequence of (7) is

\[
\oint _\gamma \vec{v}_s \cdot d\vec{\ell} = 0,
\]  

(8)

if the domain where the curve \(\gamma\) lies is simply connected. Given the arbitrariness of \(\gamma\), a different way to write this result is

\[
\vec{\nabla} \wedge \vec{v}_s = 0.
\]  

(9)
Let us now suppose that the vessel containing the liquid is put in rotation with angular velocity $\tilde{\Omega}$. A consequence of (8) is that the superfluid cannot rotate; to the same result one arrives by noting that the vessel cannot communicate the rotation to the superfluid component, as there is no friction between the recipient and the liquid. However it can be shown that the absence of rotation in the superfluid does not correspond to a state of minimal energy. As a matter of fact, if $E$ and $\vec{L}$ are energy and angular momentum in an inertial frame, the energy as computed in the rotating frame is $E_{\text{rot}} = E - \vec{L} \cdot \tilde{\Omega}$. If $\Omega$ is sufficiently high, then a lower energy can be achieved with

$$\vec{L} \cdot \tilde{\Omega} > 0$$

instead of $\vec{L} \cdot \tilde{\Omega} = 0$ that corresponds to the absence of rotation [4].

Let us suppose for simplicity that the curve $\gamma$ lies in a plane. For the result (10) to be compatible with (8) one has to suppose that inside the curve $\gamma$ there is one point where (9) is violated. Physically this would correspond to the presence of a point with a normal, not superfluid component. Since now the domain is not simply connected, (8) is substituted by

$$\oint_{\gamma} v_s \cdot d\vec{l} = 2\pi n \kappa,$$

where the integer $n$ is a winding number counting the number of times the curve goes around the singular point, and $\kappa$ is a constant with dimensions of vorticity, i.e. $[L]^{2} \cdot [T]^{-1}$. To the same result one could arrive by noting that, by virtue of (4), as $\Phi$ and $\Phi + 2n\pi$ correspond to the same wavefunction, one may have

$$\oint_{\gamma} \nabla \Phi \cdot d\vec{l} = 2\pi n,$$

which shows that $\kappa = \hbar/m$; $\kappa$ is called quantum of vorticity.

Clearly we can repeat the argument for any plane parallel to the previous one; we therefore conclude that there is an entire line (vortex line) of singular points. If this line is a straight line, then $\vec{v}_s$ will be perpendicular to the vortex line and also perpendicular to the radius joining the singular point and the point at which we compute $v_s$. At a distance $r$ from the singular point one has

$$v_s = \frac{n\kappa}{r},$$

as can be immediately seen from (11). More generally:

$$\vec{v}_s = \frac{\kappa}{2} \int_{\text{v.l.}} \frac{d\vec{l} \wedge \vec{R}}{R^3},$$

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where $\vec{R}$ is the distance vector from the vortex line (v.l.) to the point at which we compute the superfluid velocity.

We note some properties of the vortex lines. First, the integral (11) is independent of $\gamma$, provided the second curve contains the singular point; this result follows from the Stokes theorem. Second, the vortex line must be closed or it must stop at the boundary; were it open, for example at a point $P$, one might construct a surface $\Sigma$ lying on the contour $\gamma$ but large enough as to have no intersection with the vortex line; therefore by the Stokes theorem one would get that the vorticity constructed by $\vec{v}_s$ would be zero. If the vortex line stops at the boundary and the vessel is rigid then the v.l. is pinned at the boundary, as it will be discussed in more detail below.

Let us compute the critical angular velocity $\Omega$ for the formation of the first vortex line. The formation of a vortex line changes the energy $E_{rot}$ by the amount $\Delta E_{rot} = \Delta E - \Delta (\vec{L} \cdot \vec{\Omega})$. We have $\Delta E = \int \frac{\rho_s v_s^2}{2} dV$. Here the integral is over the entire volume of the vessel that, for simplicity, we assume to be a cylinder of height $b$ and radius $R$; on the other hand $v_s$ is given by the expression valid for a vortex line, i.e. (13). Therefore

$$\Delta E = \frac{\rho_s b}{2} 2\pi \int v_s^2 r dr = b \rho_s \pi \frac{h^2 n^2}{m^2} \ln \frac{R}{a}$$ (15)

where $a$ is a cutoff of the order of the interatomic distances, at which the macroscopic description fails down. The minimal energy $E_{rot}$ is obtained for $\vec{L}$ parallel to $\vec{\Omega}$, with $L = \int \rho_s v_s r dV = b \pi R^2 \frac{nh}{m}$. Vortex lines appear if $\Delta E_{rot} < 0$, i.e. if

$$\frac{nh}{m} \ln \frac{R}{a} - \Omega R^2 < 0,$$ (16)

which corresponds to $\Omega > \Omega_{crit} = \frac{h}{m R^2} \ln \frac{R}{a}$. Incidentally we note that, ceteris paribus, vortex lines with $n = 1$ are more stable than those with $n > 1$, as the positive term in (16) has an extra power of $n$. Therefore from now on we put $n = 1$.

What happens when $\Omega \gg \Omega_{crit}$? Clearly we expect there will be several v.l.’s. During the rotation these vortex lines follow the rotational motion of the vessel, which is clear because they are pinned at the boundary of the superfluid\(^a\). Their motion imitate the motion of the liquid as a whole, as it can

\(^a\)For rotations around an axis, the vortex lines are, by symmetry, straight lines parallel to the rotation axis. For motion inside holes, slits, etc. there can be closed v.l.’s that are called vortex rings.
be seen by the following argument: in the formula for \( \Delta E_{\text{crit}} \) one can forget, for large \( \Omega \), the first term \( \Delta E \) in comparison to the second one. Therefore minimizing \( \Delta E_{\text{crit}} \) correspond to maximize the angular momentum \( L \), which is obtained if the liquid moves as a whole. A consequence of this feature of the motion of the v.l.’s is that also for the superfluid one can use the well known hydrodynamical formula \( \vec{\Omega} = \frac{1}{2} \vec{\nabla} \times \vec{v}_s \), which strictly speaking is valid only for the normal component \( \vec{v}_n \); it can be used here only because it refers to the rotation of the superfluid as a whole.

The number of vortex lines that are present in the superfluid is proportional to \( \Omega \), according to the formula (17):

\[
N = \frac{m \Omega}{\pi \hbar}
\]

which shows that with increasing \( N \) the v.l.’s tend to fill in all the space. To prove (17) we consider a large closed curve \( C \) encircling the area \( A \) and containing in its interior \( NA \) v.l.’s. One has

\[
\oint_C d\vec{\ell} \cdot \vec{v}_s = NA 2\pi \kappa
\]

but also

\[
\oint_C d\vec{\ell} \cdot \vec{v}_s = \int_A d\vec{S} \cdot \vec{\nabla} \times \vec{v}_s = 2A \Omega
\]

from which (17) follows. As an example we can evaluate \( N \) for the pulsar in the Crab nebula. Here \( m = 2m_N \) (the condensate is formed by neutral bosons: pairs of neutrons) and \( \Omega = \Omega_{\text{pulsar}} \) gives \( N \simeq 1.9 \times 10^5 \text{ cm}^{-2} \) with an average distance between vortex lines \( d \sim N^{-1/2} \sim 10^{-2} \text{ cm} \).

Let us consider again eqns. (12) and (18). If \( \nu(r) \) is the number of vortices per unit area at a distance \( r \) from the rotation axis, they give, if \( \vec{v} = \vec{v}_s \) is the superfluid velocity,

\[
\oint d\vec{\ell} \cdot \vec{v} = \int_0^r d\vec{S} \cdot \vec{\nabla} \times \vec{v} = 2\pi \kappa \int_0^r 2\pi r' \nu(r')dr'.
\]

We put

\[
k = 2\pi \kappa = \frac{\hbar}{2m_n}
\]

and write (20) as follows:

\[
2\pi r^2 \Omega(r) = k \int_0^r 2\pi r' \nu(r')dr',
\]
which implies
\begin{equation}
  k \nu(r) = 2\Omega(r) + r \frac{\partial \Omega}{\partial r} .
\end{equation}

Since the total number of v.l.’s is conserved, one has
\begin{equation}
  \frac{\partial \nu}{\partial t} + \nabla \cdot (\nu \vec{v}_r) = 0
\end{equation}
where \( \vec{v}_r \) is the radial component of the superfluid velocity. We write \((22)\) as
\begin{equation}
  2\pi r^2 \Omega(r) = k \int_0^r \nu \, dS
\end{equation}
and take the time derivative, using \((23)\) to get
\begin{equation}
  2\pi r^2 \frac{\partial \Omega}{\partial t} = -k \int_0^r dS \nabla \cdot (\nu \vec{v}_r) .
\end{equation}

Using the Gauss theorem one gets
\begin{equation}
  2\pi r^2 \frac{\partial \Omega}{\partial t} = -k 2\pi r \nu v_r ,
\end{equation}
i.e.
\begin{equation}
  \frac{\partial \Omega}{\partial t} = -\frac{k \nu v_r}{r} = -\left(2\Omega(r) + r \frac{\partial \Omega}{\partial r}\right) \frac{v_r}{r} .
\end{equation}

Eq. \((27)\) shows that the only possibility for the superfluid to change its angular velocity \((\Omega \neq 0)\) is by means of a radial motion, i.e. \( v_r \neq 0 \).

Let us now consider a rotating superfluid in contact with rotating normal matter on which an external torque is acting. We denote by \( I_c, \Omega_c \) moment of inertia and angular velocity of the normal components that, in a neutron star, includes the crust and possibly other normal components. The equation of motion of the normal component is
\begin{equation}
  I_c \dot{\Omega}_c(t) = M_{ext} + M_{int} .
\end{equation}
Besides the external torque \( M_{ext} \), basically related to the spin down of the pulsar (or the steady accretion in binary pulsars), we have included internal torque \( M_{int} \):
\begin{equation}
  M_{int} = -\int dI_p \dot{\Omega}(r, t)
\end{equation}
due to the interaction with the superfluid. Eqs.\((24)\)\((25)\) are the equations of motion for the angular velocities \( \Omega \) and \( \Omega_c \) (superfluid and crust). The two velocities are coupled not only through \( M_{int} \), but also by \( v_r \), because we will show below that \( v_r = f(\Omega - \Omega_c) \). We note again that fundamental for this model is the existence of radial motion, for, if \( v_r = 0 \), then \( \Omega = \text{const.} \) and only \( \Omega_c \) changes, due to the external torque alone.
In the neutron star, superfluid neutrons (in Cooper pairs) coexist with nuclei of the crust. Also in the crust there are superfluid neutrons, but they are characterized by a different (and smaller) $\Delta$. For superfluid neutrons in the volume $V$, the total energy can be estimated as follows:

$$E \simeq \frac{V}{(2\pi\hbar)^3} \int_{p_F}^{p_F+\Delta^2/E_F} p^3 dp \frac{d\Omega}{2} = \frac{V}{4\pi^2} \frac{\Delta^2 k_F^3}{E_F}$$

where $p_F = \hbar k_F$; we integrate over half the solid angle as the superfluid neutrons only appear in pairs; we have taken into account that only neutrons in a shell of thickness $\Delta^2/E_F$ participate in the pairing.

$E$ is also approximately given by the difference between the energies of superfluid neutrons outside the vortex line and neutrons inside, because those inside the vortex core have $\Delta \to 0$. Therefore neutrons inside the volume $V$ of the vortex core are repelled from going outside the vortex towards the superfluid phase as it would cost more energy. However, if neutron rich nuclei are present, the repulsion will be less important, as $\Delta_c$, the gap for superfluid neutrons in the nuclei, is much smaller than $\Delta_s$, the gap of superfluid neutrons; therefore there will be a force pulling the vortex toward the nuclei; the pinning energy per nucleus will be

$$\delta E_p = \frac{V}{8} \left[ \left( \frac{\Delta^2 k_F^3}{\pi^2 E_F} \right)_s - \left( \frac{\Delta^2 k_F^3}{\pi^2 E_F} \right)_c \right] \approx \gamma \frac{V}{8} \left( \frac{\Delta^2 k_F^3}{\pi^2 E_F} \right)_s$$

with $\gamma \sim 1$.

Let now $\xi$ be the coherence length, i.e. the spatial extension of the Cooper pair, $\xi = \frac{\hbar v_F}{\pi \Delta}$; it can be proved that it gives an estimate of the radius of the vortex core. The maximum pinning force will be obtained, if $2\xi < b$ (the average distance between the nuclei) when the vortex passes through one layer of the lattice; the average distance between vortex core neutrons and superfluid neutrons is of the order of $\xi$ and therefore the maximum force is

$$F_p \approx \frac{\delta E_p}{\xi}$$

and the maximum force per unit length of vortex line (average distance between two consecutive pinning centers) is

$$f_p \approx \frac{\delta E_p}{b\xi}.$$
the crust velocity $\Omega_c$. Let us suppose that they are initially equal, which is a consequence of the pinning. Due to the spinning down of the star, $\Omega_c$ decreases; as long the vortex cores are pinned to the crust lattice, the neutron superfluid cannot spin down, because the radial motion is forbidden. There is therefore a relative velocity of the superfluid with respect to the pinned vortex core because $\Omega > \Omega_c$: 

$$\delta \vec{v} = (\vec{\Omega} - \vec{\Omega}_c) \wedge \vec{r}.$$  

(33)

The interaction between the normal matter in the core of the v.l. and the rest of normal matter (nuclei in the lattice, electrons, etc.) produces a Magnus force per unit length given by 

$$\vec{f} = \rho \vec{k} \wedge \delta \vec{v},$$  

(34)

where $k$ is the quantum of vorticity and the direction of $\vec{k}$ coincides with the rotation axis. $\vec{f}$ is the force exerted on the vortex line; as it cannot be larger than $f_p$ there is a maximum difference of angular velocity that the system can maintain:

$$\omega_{cr} = (\Omega - \Omega_c)_{max} = \frac{f_p}{\rho kr} = \frac{E_p}{\rho k \xi b}.$$  

(35)

If $\omega < \omega_{cr}$ the vortices remain pinned at the lattice sites instead of flowing with the superfluid as they generally in superfluid (see discussion above). On the contrary, if $\omega > \omega_{cr}$, the hydrodynamical forces arising from the mismatch between the two angular velocities ultimately break the crust and produce the conditions for the glitch.

We stop here this introduction to the standard model for glitches; see [17] for more details. The relevance for CSC is that the LOFF phase provides a lattice structure independently of the crust. Even in quark stars, if one is in a LOFF phase, one has a crystal structure: a lattice characterized by a geometric array where the gap parameter varies periodically. This would avoid the frequently raised objection by which one excludes the existence of strange stars because, if the strange matter exists, quark stars should be rather common, in contrast with the widespread appearance of glitches in pulsars. Therefore, if the color crystalline structure is able to produce glitches, the argument in favor of the existence of strange stars would be reinforced.

In a more conservative vein one can also imagine that the LOFF phase be realized in the inner core of a neutron star; in this case the crystalline color superconductivity could be partly responsible for the glitches of the pulsar. A detailed analysis of this scenario is however premature as one should first complete the study of the LOFF phase by including the third quark and by sorting out the exact form of the color lattice [13, 14].
2 Lecture II: High Density Effective Theory

In this section we derive the general formalism of High Density Effective Theory and we present an example of its use. This effective theory shows some resemblance with the Heavy Quark Effective Theory (see e.g. [4]); it is discussed in detail in the review paper [9].

2.1 General Formalism

The main idea of the effective theory is the observation that the quarks participating in the dynamics have large (\(\sim \mu\)) momenta. Therefore one can introduce velocity dependent fields by extracting the large part \(\mu \vec{v}\) of this momentum. One starts with the Fourier decomposition of the quark field \(\psi(x)\):

\[
\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-i\vec{p} \cdot \vec{x}} \psi(p),
\]

and introduces the quark velocity by

\[
p^\mu = \mu v^\mu + \ell^\mu,
\]

where \(v^\mu = (0, \vec{v})\) with \(|\vec{v}| = 1\). Let us put \(\ell^\mu = (\ell^0, \vec{\ell})\) and \(\hat{\ell} = \vec{v} \ell_\parallel + \ell_\perp\) with \(\ell_\perp = \ell - (\ell \cdot \vec{v}) \vec{v}\). We can always choose the velocity parallel to \(\vec{p}\), so that \(\ell_\perp = 0\) and

\[
\int d^4p = \mu^2 \int d\Omega \int_{-\infty}^{+\infty} d\ell_0 = 4\pi \mu^2 \int \frac{d\vec{v}}{4\pi} \int d\ell_\parallel \int_{-\infty}^{+\infty} d\ell_0.
\]

In this way the Fourier decomposition (36) takes the form

\[
\psi(x) = \sum_{\vec{v}} e^{-i\mu v \cdot x} \int \frac{d^4\ell}{(2\pi)^4} e^{-i\ell \cdot x} \psi_{v}(\ell) = \sum_{\vec{v}} e^{-i\mu v \cdot x} \frac{4\pi \mu^2}{(2\pi)^4} \int d^2\ell e^{-i\ell \cdot x} \psi_{v}(\ell),
\]

where \(\psi_{v}(\ell)\) are velocity-dependent fields. One can write

\[
\psi(x) = \sum_{\vec{v}} e^{-i\mu v \cdot x} \left[ \psi_+ (x) + \psi_- (x) \right],
\]

where \(\psi_\pm\) are velocity dependent fields corresponding to positive and energy solutions of the Dirac equation.

Let us now define \(V^\mu = (1, \vec{v})\), \(\bar{V}^\mu = (1, -\vec{v})\), \(\gamma_\parallel = (\gamma^0, (\vec{v} \cdot \vec{\gamma}) \vec{v})\), \(\gamma_\perp = \gamma^\mu - \gamma_\parallel\). Using simple algebraic relations involving the gamma matrices one
obtains
\[ \mathcal{L}_D = \sum \hat{\psi}_+^\dagger iV \cdot D \psi_+ + \hat{\psi}_-^\dagger (2\mu + i\tilde{V} \cdot D)\psi_- + (\tilde{\psi}_+ iD_\mu \gamma_\mu \tilde{\psi}_- + \text{h.c.}) \] ; (41)

\( D_\mu \) is the covariant derivative: \( D_\mu = \partial_\mu + igA_\mu \). We note that here quark fields are evaluated at the same Fermi velocity; off-diagonal terms are sub-leading due to the Riemann-Lebesgue lemma, as they are cancelled by the rapid oscillations of the exponential factor in the \( \mu \to \infty \) limit. One may call this phenomenon Fermi velocity superselection rule, in analogy with the behaviour of QCD in the \( m_Q \to \infty \) limit, where the corresponding effective theory, the Heavy Quark Effective Theory exhibits a similar phenomenon. By the same analogy we may refer to the present effective theory as High Density Effective Theory (HDET).

We can get rid of the negative energy solutions by integrating out the \( \psi_- \) fields in the generating functional; in this way we get
\[ \mathcal{L}_D \simeq \mathcal{L}_0 = \sum \hat{\psi}_+^\dagger iV \cdot \partial \psi_+ + \hat{\psi}_-^\dagger i\tilde{V} \cdot \partial \psi_- \] ,
where now \( \psi_\pm \) are both positive energy solutions with \( \psi_\pm = \psi_\pm \hat{v} \). The construction described above is valid for any theory describing massless fermions at high density provided one excludes degrees of freedom very far from the Fermi surface. As discussed in the first lecture however, for small temperature and high density the fermions are likely to be gapped due to the phenomenon of the color superconductivity. We shall examine here the modification in the formalism for the LOFF model, with the condensate
\[ \Delta(\vec{x}) = \Delta \exp\{2i\vec{q} \cdot \vec{x}\} \] .

The effect of the non vanishing vacuum expectation value can be taken into account by adding to the lagrangian the term:
\[ \mathcal{L}_\Delta = -\frac{\Delta}{2} \exp\{2i\vec{q} \cdot \vec{x}\} \epsilon_{\alpha\beta\gamma} \epsilon_{ij} \psi_{\alpha\gamma}^T(x)C\psi_{\beta\gamma}(x) - (L \to R) + \text{h.c.} \] .

In order to introduce velocity dependent positive energy fields \( \psi_{\vec{v}_i;\alpha} \) with flavor \( i \) we decompose both fermion momenta as in (37) and we have:
\[ \mathcal{L}_\Delta = \frac{\Delta}{2} \sum \epsilon_{\alpha\gamma} \epsilon_{ij} \psi_{\alpha\gamma}^T \psi_{\beta\gamma}(x)C\psi_{\beta\gamma}(x) \] ,
where \( \delta(\vec{v}_i, \vec{v}_j, \vec{q}) = 2\vec{q} - \mu_i \vec{v}_i - \mu_j \vec{v}_j \). We define
\[ \mu = \frac{\mu_1 + \mu_2}{2} \quad \delta\mu = -\frac{\mu_1 - \mu_2}{2} \] ,
(46)
and perform the $\mu \to \infty$ limit on a smeared amplitude as follows

$$\lim_{\mu \to \infty} \exp\{i\vec{x} \cdot \vec{a}(\vec{v}_1, \vec{v}_2, \vec{q})\} \equiv \frac{1}{V} \int_{V(x)} d\vec{r} \exp\{i\vec{r} \cdot \vec{a}(\vec{v}_1, \vec{v}_2, \vec{q})\}, \quad (47)$$

where $V(\vec{x})$ is a small volume centered at $\vec{x}$. We evaluate (47) by taking $\vec{q}$ along the $z$-axis, so that we get: $\vec{v}_i \simeq -\vec{v}_j \equiv -\vec{v}$ by the $x$ and $y$ integrations, while the $z$-integration gives $(\vec{n} \equiv \vec{q}/q)$

$$\frac{\pi}{R} e^{i2q\Delta \cdot \vec{n}} \delta_R[h(\vec{v} \cdot \vec{n})] \approx \frac{\pi}{R} \delta_R[h(\vec{v} \cdot \vec{n})]. \quad (48)$$

We have put $R = q|\Delta \ell|$ where $\Delta \ell$ is a smearing distance along the direction of $q$ $(|\Delta \ell| \sim \frac{\pi}{q})$. We have introduced the ”fat delta” $\delta_R(x)$ defined by

$$\delta_R(x) \equiv \frac{\sin(Rx)}{Rx}, \quad (49)$$

which, for large $R$, gives $\delta_R(x) \to \delta(x)$. Moreover in the $\mu \to \infty$ limit $h(x) = 1 - \frac{\delta \mu}{qx}$. Therefore one has

$$\mathcal{L}_\Delta = -\frac{\Delta}{2} \frac{\pi}{R} \delta_R[h(\vec{v} \cdot \vec{n})] \sum_{\vec{v}} \epsilon_{ij} \epsilon_{\alpha \beta \delta} \psi_{i \alpha}(x) C \bar{\psi}_{j \beta}(x) - (L \to R) + h.c.. \quad (50)$$

In an appropriate basis the effective lagrangian is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\Delta = \sum_{\vec{v}} \sum_{A = 0}^5 \chi_A^\dagger \left( \begin{array}{ccc} i \delta_{AB} V \cdot \partial & \Delta_{AB} \delta_{AB} V \cdot \partial & \Delta_{AB} \end{array} \right) \chi_B, \quad (51)$$

where the matrix $\Delta_{AB}$ is as follows: $\Delta_{AB} = 0$ $(A$ or $B = 4$ or $5)$, and, for $A, B = 0, ..., 3$:

$$\Delta_{AB} = \Delta_{eff} \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \end{array} \right)_{AB}, \quad (52)$$

having put

$$\Delta_{eff} = \frac{\Delta}{R} \delta_R[h(\vec{v} \cdot \vec{n})]. \quad (53)$$

From these equations one can derive the propagator for gapped quarks:

$$D_{AB}(\ell) = \frac{1}{V \cdot \ell V \cdot \ell - \Delta_{eff}^2} \left( \begin{array}{cc} V \cdot \ell \delta_{AB} & -\Delta_{AB} \\
-\Delta_{AB} & V \cdot \ell \delta_{AB} \end{array} \right). \quad (54)$$
2.2 NGB and their parameters

Both in the CFL model and in the LOFF model with two flavors there are Nambu-Goldstone Bosons, associated in one case to the breaking of internal symmetries and in the other to the breaking of space-symmetries (phonon). In order to derive an effective low energy lagrangian for the NGB’s one can use the gradient expansion, where the NGB’s are introduced as external fields and acquire a kinetic term, thus becoming dynamical fields, by integrating out the fermion fields. I shall describe in some detail the calculation for the phonon field in the LOFF phase (other examples can be found in [9]).

\[ \mathcal{L}_{\Delta} \] explicitly breaks rotations and translations and induces a lattice structure given by plane waves. The crystal can fluctuate and its local deformations define one phonon field \( \phi \) that is the Nambu-Goldstone boson associated to the breaking of the translational symmetry. It is introduced by the substitution in \( z \rightarrow z + \frac{\phi}{2qf} \), \( (55) \)

with \( \langle \phi \rangle_0 = 0 \). We are interested in an effective description of the field \( \phi \) in the low energy limit, i.e. for wavelengths much longer than the lattice spacing \( \sim 1/q \). In this limit the field \( \phi \) varies almost continuously and we can get rid of the lattice structure and use in the sequel the continuous notation.

At the first order one gets the couplings

\[ \mathcal{L}_{\phi\psi\psi} = -\sum_{\vec{v}} \frac{\pi \Delta}{R} \delta_R [h(\vec{v} \cdot \vec{n})] \frac{i \phi}{f} \epsilon_{ij} \epsilon^{\alpha\beta} \psi_i,\alpha,\vec{v} C \psi_j,\beta,-\vec{v} - (L \rightarrow R) + h.c. \]

\[ (56) \]

\[ \mathcal{L}_{\phi\phi\psi\psi} = \sum_{\vec{v}} \frac{\pi \Delta}{R} \delta_R [h(\vec{v} \cdot \vec{n})] \frac{\phi^2}{2f^2} \epsilon_{ij} \epsilon^{\alpha\beta} \psi_i,\alpha,\vec{v} C \psi_j,\beta,-\vec{v} - (L \rightarrow R) + h.c. \]

\[ (57) \]

In the basis of the \( \chi \) fields one has:

\[ \mathcal{L}_3 + \mathcal{L}_4 = \sum_{\vec{v}} \sum_{A=0}^3 \tilde{\chi}^A_{\dagger} \begin{pmatrix} 0 & g^\dagger \end{pmatrix} \tilde{\chi}^B. \]

\[ (58) \]

Here

\[ g = \left( \frac{\pi \Delta}{R} \delta_R [h(\vec{v} \cdot \vec{n})] \frac{i \phi}{f} - \frac{\pi \Delta}{R} \delta_R [h(\vec{v} \cdot \vec{n})] \frac{\phi^2}{2f^2} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{AB} \]

\[ (59) \]
To compute the effective lagrangian for the phonon field we use the propagator given in Eq. (54) and the interaction vertices in (58). The result of the calculation at the second order in the momentum expansion is

$$\mathcal{L}_{\text{eff}}(p) = i \frac{4 \times 4 \mu^2}{16 \pi^3 f^2} \sum_{\vec{v}} \frac{1}{2} \left( \frac{\pi \Delta R}{R} \delta_R[\bar{h}(\vec{v} \cdot \vec{n}_m)][i \phi] \right)^2 \int d^2 \ell \frac{2 \Delta^2_{\text{eff}} V \cdot p \tilde{V} \cdot p}{[D(\ell)]^3}. $$

One can handle the fat delta according to the Fermi trick in the Golden Rule; in expressions involving the gap parameters one makes in the numerator the substitution \( \delta_R[h(x)] \rightarrow \delta[h(x)] \); one fat delta is substituted by the Dirac delta while the other fat delta gives \( \frac{\pi \delta_R[h(x)]}{R} \rightarrow \frac{\pi \delta_R(0)}{R} \rightarrow 1 \). We finally get the effective lagrangian in the form

$$\mathcal{L}_{\text{eff}}(p) = -\frac{\mu^2 k_R}{2\pi^2 f^2} \sum_{\vec{v}} \delta \left\{ \vec{v} \cdot \vec{n} - \frac{\delta \mu}{q} \right\} V_\mu \tilde{V}_\nu p_\mu \phi p_\nu \phi. $$

Here \( k_R \) is kinematical factor of the order of 1 induced by the approximation of the Riemann-Lebesgue lemma \(^8\). The integration over fermi velocities can be easily performed and one obtains the effective lagrangian in the form

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \left( \dot{\phi}^2 - v_\parallel^2 |\vec{\nabla}_\parallel \phi|^2 - v_\perp^2 |\vec{\nabla}_\perp \phi|^2 \right), $$

if

$$f^2 = \frac{\mu^2 k_R}{2\pi^2}. $$

Here \( \vec{\nabla}_\parallel \phi = \vec{n}(\vec{n} \cdot \vec{\nabla}) \phi \), \( \vec{\nabla}_\perp \phi = \vec{\nabla}_\phi - \vec{\nabla}_\parallel \phi \). Moreover

$$v_\parallel^2 = \frac{1}{2} \sin^2 \theta_q, $$

$$v_\perp^2 = \cos^2 \theta_q $$

and

$$\cos^2 \theta_q = \left( \frac{\delta \mu}{q} \right)^2. $$

Therefore the dispersion law for the phonon is anisotropic.

\(^8\)One can easily control that the Goldstone theorem is satisfied and the phonon is massless.
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References

1. B. Barrois, *Nucl. Phys. B* **129**, 390 (1977); S. Frautschi, in *Proceedings of the workshop on hadronic matter at extreme density, Erice 1978*. See also: D. Bailin and A. Love, *Phys. Rept.* **107**, 325 (1984), and references therein.

2. M. Alford, K. Rajagopal and F. Wilczek, *Phys. Lett.* B **422**, 247 (1998) [arXiv:hep-ph/9711393].

3. M. Alford, K. Rajagopal and F. Wilczek, *Nucl. Phys. B* **537**, 443 (1999) [arXiv:hep-ph/9804403].

4. M. Alford, J. Berges and K. Rajagopal, *Nucl. Phys. B* **558**, 219 (1999) [arXiv:hep-ph/9903502]; T. Schäfer and F. Wilczek, *Phys. Rev. Lett.* **82**, 3956 (1999) [arXiv:hep-ph/9811478]; T. Schäfer and F. Wilczek, *Phys. Rev. D* **60**, 114033 (1999) [arXiv:hep-ph/9906512]; T. Schäfer and F. Wilczek, *Phys. Rev. D* **60**, 074014 (1999) [arXiv:hep-ph/9903503]; T. Schäfer and F. Wilczek, *Phys. Lett.* B **450**, 325 (1999) [arXiv:hep-ph/9810509]; G. W. Carter and D. Diakonov, *Phys. Rev. D* **60**, 16004 (1999) [arXiv:hep-ph/9812445]; E.V. Shuryak and M. Velkovsky, *Phys. Rev. Lett.* **81**, 53 (1998) [arXiv:nucl-th/9711396]; M. Alford, J. Berges K. and Rajagopal, *Nucl. Phys. B* **571**, 269 (2000) [arXiv:hep-ph/9910254]; R. D. Pisarski and D. H. Rischke, *Phys. Rev. Lett.* **83**, 37 (1999) [arXiv:nucl-th/9811104] and *Phys. Rev. D* **61**, 051501 (2000) [arXiv:nucl-th/9907041]; N. O. Agasian, B.O. Kerbikov and V. I. Schevchenko, *Phys. Rep.* **320**, 131 (1999) [arXiv:hep-ph/9902355]; E. Shuster and D.T. Son, *Nucl. Phys. B* **573**, 434 (2000) [arXiv:hep-ph/9905448]; D.K. Hong, V.A. Miransky, I.A. Shovkovy and L. C. R. Wijewardhana, *Phys. Rev. D* **61**, 056001 (2000), erratum, *ibidem*, **62**, 59903 (2000) [arXiv:hep-ph/9906478]; F. Sannino and W. Shafer, *Phys. Lett.* B **527**, 142 (2002) [arXiv:hep-ph/0111095]; F. Sannino and W. Shafer [arXiv:hep-ph/0204353]; J. B. Kogut, M. P. Lombardo and D. K. Sinclair, *Phys. Rev. D* **51**, 1282 (1995) [arXiv:hep-lat/9401039]; *Nucl. Phys. Proc. Suppl.* **42**, 514 (1995) [arXiv:hep-lat/9412057].

5. K. Rajagopal and F. Wilczek, in *Handbook of QCD*, M. Shifman, ed.
6. R. Casalbuoni, R. Gatto and G. Nardulli, Phys. Lett. B 498, 179 (2001) [arXiv:hep-ph/0010324].
7. R. Casalbuoni, R. Gatto, M. Mannarelli and G. Nardulli, Phys. Lett. B 524, 144 (2001) [arXiv:hep-ph/0107024].
8. R. Casalbuoni, R. Gatto, M. Mannarelli and G. Nardulli, Phys. Lett. B 511, 218 (2001) [arXiv:hep-ph/0101326]. G. Nardulli [arXiv:hep-ph/0111178]. R. Casalbuoni, R. Gatto, M. Mannarelli and G. Nardulli [arXiv:hep-ph/0201059].
9. G. Nardulli, to appear in Riv. Nuovo Cimento [arXiv:hep-ph/0202037].
10. R. Casalbuoni, Z. Duan and F. Sannino, Phys. Rev. D 62, 094004 (2000) [arXiv:hep-ph/0004207]; Phys. Rev. D 63, 114026 (2001) [arXiv:hep-ph/0101394].
11. R. Casalbuoni and R. Gatto, Phys. Lett. B 464, 111 (1999) [arXiv:hep-ph/9908227].
12. M. Alford, J. A. Bowers and K. Rajagopal Phys. Rev. D 63, 074016 (2001) [arXiv:hep-ph/0008208]; J. A. Bowers, J. Kundu, K. Rajagopal and E. Shuster, Phys. Rev. D 64, 014024 (2001) [arXiv:hep-ph/0101067]; A. K. Leibovich, K. Rajagopal and E. Shuster Phys. Rev. D 64, 094005 (2001) [arXiv:hep-ph/0104073]; K. Rajagopal Acta Phys. Polon. B 31, 3021 (2000) [arXiv:hep-ph/0009058]; M. Alford, J. A. Bowers and K. Rajagopal J. Phys. G 27, 541 (2001) [arXiv:hep-ph/0009357]; T. Schäfer and E. Shuryak [arXiv:nucl-th/0010049]; K. Rajagopal in QCD@Work, P. Colangelo and G. Nardulli eds. (AIP, 2001) [arXiv:hep-ph/0109135].
13. J.A. Bowers and K. Rajagopal [arXiv:hep-ph/0204079].
14. R. Casalbuoni, R. Gatto and G. Nardulli, [arXiv:hep-ph/0205219].
15. A. I. Larkin and Yu. N. Ovchinikov Zh. Eksp. Teor. Fiz. 47, 1136 (1964) (Sov. Phys. JETP 20, 762 (1965)); P. Fulde and R. A. Ferrell Phys. Rev. 135, A550 (1964).
16. E. M. Lifshitz, L.P. Pitaevskii, Statistical Physics, Part 2 (L.D.Landau and E.M. Lifshitz, Course of Theoretical Physics, Vol. IX), (Butterworth Heinemann, 1996).
17. M. A. Alpar, P.W. Anderson, D.Pines and J. Shaham, Astrophys. J. 276, 325 (1984); see also P.W.Anderson and N.Itoh, Nature 256, 25 (1973).
18. For reviews of the heavy quark effective theory see for example the textbook: A.V. Manohar and M. B. Wise, Heavy Quark Physics (Cambridge Univ. Press, 2000) and R. Casalbuoni et al., Phys. Rept., 281 145 (1997) [arXiv:hep-ph/9605342].