Entanglement Produced by Using Some Biparticle Bose Systems

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Abstract

In this paper, the generating of entanglement by using some biparticle Bose systems acting on vacuum state are investigated. These systems include two-mode squeezed system, thermal system of a free single particle (where the fictitious tilde system is regarded another particle), and the system of two coupling harmonic oscillators. The technique of integration within an ordered product (IWOP) of operators is used.

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Entanglement has been recognized as a central resource in various aspects of quantum information processing. The quantum teleportation [1], quantum cryptography [2], quantum dense coding [3] etc. are all based on the entanglement. So producing entanglement of quantum systems is one of the basic tasks for quantum communication and quantum computation. It has been shown that nonlocal operations can generate entanglement. Recently, many authors have studied the generating of entanglement from simple quantum gates, for example, CNOT gate, double CNOT gate and SWAP gate [4]. The produced entanglement from these quantum gates is the entanglement of qubit systems. However, recently, quantum information protocols have extended from qubit systems to continuous variable systems, such as continuous variable teleportation [5], continuous variable quantum computation [6] and error correction [7], and continuous variable cryptography [8]. Thus, it is interested to produce entanglement from continuous variable quantum systems. In this paper, we will try to study the generating of entanglement from some non-qubit Bose systems by using the technique of integration within an ordered product (IWOP) of operators. Follows the original works of Fan [9] we firstly list the rules of IWOP of operators as follows.

(i) The order of Bose operators within a normally ordered product (denoted by symbol ::) can be permuted.
(ii) If and only if the normally ordered form of operators can be put into and out of the symbol : : unchangeably.
(iii) C numbers can be taken into and out of the symbol : : as one wishes.
(iv) A normally ordered product can be integrated or differentiated with respect to C-number provided the integration is convergent.
(v) The normally ordered form of projector of vacuum state is
\[ |0\rangle \langle 0| = e^{-a_1^\dagger a_1}, \]
where \( a_1^\dagger \) and \( a_1 \) are creation and annihilation operators of Bose system in Fock space.

At first, we study the generating of entanglement by two-mode squeezed operator acting on vacuum state. The scheme for generating two-mode squeezed operator has been studied in last years and recently by using a pair distant cavities [12]. Because the two-mode squeezed operator is
\[ e^{\lambda (a_1^\dagger a_2^\dagger - a_1 a_2)} \]
the created entangled state from vacuum state is just the two-mode squeezed vacuum state
\[ |\psi\rangle = e^{\lambda (a_1^\dagger a_2^\dagger - a_1 a_2)} |00\rangle, \]
where \( \lambda \) is the squeezing parameter, \( a_1^\dagger \) (\( a_1 \)) and \( a_2^\dagger \) (\( a_2 \)) are the creation and (annihilation) operators of the two particles. The degree of entanglement of \(|\psi\rangle\) has been given by many authors [12], [13], [14]. Where we investigate it with IWOP of operators. It is shown that the normally ordered form of two-mode squeezed operator is [10]
\[ |\psi\rangle = \text{sec} \theta \lambda e^{\lambda (a_1 a_2^\dagger - a_1^\dagger a_2)} : |00\rangle \]
so we have
\[ \rho_{12}^{\text{tms}} = \text{sec} \theta \lambda e^{\lambda (a_1 a_2^\dagger - a_1^\dagger a_2)} |00\rangle \langle 00| e^{\lambda (a_1^\dagger a_2^\dagger - a_1 a_2)} \]
where and in the following we use the superscript “tms” denoting “two-mode squeezed”. The reduce density matrix of \( \rho_{12}^{\text{tms}} \) is
\[ \rho_{1}^{\text{tms}} = tr_2 \left\{ \text{sec} \theta \lambda e^{\lambda (a_1 a_2^\dagger - a_1^\dagger a_2)} |00\rangle \langle 00| e^{\lambda (a_1^\dagger a_2^\dagger - a_1 a_2)} \right\}. \]
By using the completeness of coherent state \(|z_2\rangle\)
\[ \int \frac{d^2 z_2}{\pi} |z_2\rangle \langle z_2| = 1, \]
we have
\[ \rho_{1}^{\text{tms}} = \text{tr}_2 \left\{ \frac{d^2 z_2}{\pi} |z_2\rangle \langle z_2| \text{sec} \theta \lambda e^{\lambda (a_1 a_2^\dagger - a_1^\dagger a_2)} |00\rangle \langle 00| e^{\lambda (a_1^\dagger a_2^\dagger - a_1 a_2)} \right\} \]
\[ = \int \frac{d^2 z_2}{\pi} |z_2\rangle \langle z_2| \text{sec} \theta \lambda e^{\lambda (a_1 a_2^\dagger - a_1^\dagger a_2)} : |00\rangle \langle 00| e^{\lambda (a_1^\dagger a_2^\dagger - a_1 a_2)} |z_2\rangle := \]
\[ = \int \frac{d^2 z_2}{\pi} \text{sec} \theta \lambda e^{\lambda (a_1 a_2^\dagger - a_1^\dagger a_2)} : |0\rangle \langle 0| e^{\lambda (a_1 z_2^\dagger - a_1^\dagger z_2)} |0\rangle \langle z_2| = \]
\[ = \int \frac{d^2 z_2}{\pi} \text{sec} \theta \lambda e^{\lambda (a_1 z_2^\dagger - a_1^\dagger z_2)} : \text{exp} \left( z_2 a_1 + z_2^\dagger a_1^\dagger \right) : \]
\[ \int \frac{d^2 z_2}{\pi} \text{sec} \theta \lambda e^{\lambda (a_1 a_2^\dagger - a_1^\dagger a_2)} : \text{exp} \left( z_2 a_1 + z_2^\dagger a_1^\dagger \right) :. \]
Use of the formula
\[ \int \frac{d^2 z}{\pi} \exp \left( -|z|^2 + \sigma z \right) f(z^*) = f(\sigma), \] (8)
we obtain
\[ \rho_1^{tms} = \text{sech}^2 \lambda : \exp \left( a_1 \dagger a_1 \tanh^2 \lambda \right) e^{-a_1 \dagger a_1} :. \] (9)
Because
\[ \exp \left( a_1 \dagger a_1 \tanh \lambda \right) = \sum_{n=0}^{\infty} \left( a_1 \dagger a_1 \tanh \lambda \right)^n / n!, \] (10)
replacing Eq.(10) into Eq.(9) we have
\[ \rho_1^{tms} = \text{sech}^2 \lambda \left\{ |0\rangle \langle 0| + \tanh^2 \lambda |1\rangle \langle 1| + \ldots + \tanh^{2n} \lambda |n\rangle \langle n| \ldots \right\}. \] (11)
So the entanglement denoted by von Neumann entropy of \( \rho_1^{tms} \) is
\[ E^{tms} = -\frac{1}{\cosh^2 \lambda} \log \left( \frac{\cosh^2 \lambda}{\cosh^2 \lambda} \right) - \frac{\tanh^2 \lambda}{\cosh^2 \lambda} \log \left( \frac{\tanh^2 \lambda}{\cosh^2 \lambda} \right) - \ldots \]
\[ = \frac{1}{\cosh^2 \lambda} \left( \log(\cosh^2 \lambda) - \tanh^2 \lambda \log(\sinh^2 \lambda) \right) \times \left( 1 + 2 \tanh^2 \lambda + \ldots + (n+1) \tanh^{2n} \lambda + \ldots \right). \] (12)
From the relationship
\[ 1 + 2 \tanh^2 \lambda + \ldots + (n+1) \tanh^{2n} \lambda + \ldots = \cosh^4 \lambda, \] (13)
we have
\[ E^{tms} = \cosh^2 \lambda \log(\cosh^2 \lambda) - \sinh^2 \lambda \log(\sinh^2 \lambda) , \] (14)
From Eq.(14) we can plot \( E^{tms} \) with the parameter \( \lambda \) as figure 1

From Fig. 1 we see that the entanglement \( E^{tms} \) increase with the increasing of squeezed parameter \( \lambda \), which is coincided with the result obtained by other methods [1, 2, 3].

Secondly, we study the generating of entanglement of free single particle in thermal environment. In 1975, Takahashi and Umezawa [4] pointed out that the statistical average of physical quantity \( Q \) at the temperature \( T \) can be expressed as
\[ \langle Q \rangle = \langle 0(\beta) | Q | 0(\beta) \rangle, \] (15)
where the thermal quantum vacuum state \( |0(\beta)\rangle \) is defined by
\[ |0(\beta)\rangle = Z(\beta)^{-\frac{1}{2}} \sum_n e^{-\beta E_n / 2} |n, \bar{n}\rangle. \] (16)
Here, we have
\[ Z(\beta) = Tr e^{-\beta H}, \quad \beta = 1/k_BT, \tag{17} \]
for the relevant Hamiltonian \( H \) and \( |n\rangle \) denotes the \( n \)-th eigenstate of the Hamiltonian with the eigenvalue \( E_n \), and the \( |\bar{n}\rangle \) is the corresponding eigenstate in the fictitious dynamical system identical to the original system. It is recently pointed out that the fictitious (tilde) system \( \tilde{H} \) in fact denotes the system of environment \([16]\) (in which it is regarded as one particle). The thermal vacuum state \( |0(\beta)\rangle \) is usually an entangled state and the entanglement is that between the system of interest and its environment. In the following we calculate the entanglement of thermal state of a free boson. The Hamiltonian of free boson is \( H = \omega a^\dagger a \) and its thermal vacuum state is
\[ |0(\beta)\rangle = (1 - e^{-\beta\omega})^\frac{1}{2} \exp(-\frac{\beta\omega}{2} a^\dagger a^\dagger) |0\rangle, \tag{18} \]
then
\[ \rho_{12}^{tv} = (1 - e^{-\beta\omega}) \exp(-\frac{\beta\omega}{2} a^\dagger a^\dagger) |0\rangle \langle 0| \exp(-\frac{\beta\omega}{2} a a), \tag{19} \]
where and in the following the superscript “tv” denotes thermal vacuum state. The reduced density matrix of the system (indicated by subscript 1) is
\[ \rho_1^{tv} = Tr_2 \{ (1 - e^{-\beta\omega}) \exp(-\frac{\beta\omega}{2} a^\dagger a^\dagger) |0\rangle \langle 0| \exp(-\frac{\beta\omega}{2} a a) \}, \tag{20} \]
By using the completeness of coherent state \( |\tilde{z}\rangle \) of the fictitious system
\[ \int \frac{d^2\tilde{z}}{\pi} |\tilde{z}\rangle \langle \tilde{z}| = 1, \tag{21} \]
we have
\[ \rho_1^{tv} = Tr_2 \{ \int \frac{d^2\tilde{z}}{\pi} |\tilde{z}\rangle \langle \tilde{z}| (1 - e^{-\beta\omega}) \exp(-\frac{\beta\omega}{2} a^\dagger a^\dagger) |0\rangle \langle 0| \exp(-\frac{\beta\omega}{2} a a) \} \]
\[ = \int \frac{d^2\tilde{z}}{\pi} (\tilde{z}| (1 - e^{-\beta\omega}) \exp(-\frac{\beta\omega}{2} a^\dagger a^\dagger) |0\rangle \langle 0| \exp(-\frac{\beta\omega}{2} a a) |\tilde{z}\rangle \]
\[ = (1 - e^{-\beta\omega}) \exp(-\beta\omega a^\dagger a) |0\rangle \langle 0| \]
\[ = (1 - e^{-\beta\omega}) \sum_{n=0}^{\infty} e^{-n\beta\omega} |n\rangle \langle n|. \tag{22} \]
So we can calculate its entanglement as
\[ E^{tv} = -\log (1 - e^{-\beta\omega}) - \frac{e^{-\beta\omega}}{1 - e^{-\beta\omega}} \log e^{-\beta\omega}. \tag{23} \]
The entanglement \( E^{tv} \) with temperature \( T \) plotted in figure 2.

\[ figure2 \]

It is shown that when \( T \to 0 \), \( E^{tv} \to 0 \) which shows that the free single particle do not entangle with its environment at very low temperature and when
the temperature increase the entanglement emerges. In the room temperature the entanglement emerges with the increasing of temperature. This result is physically reasonable.

Thirdly, we calculate the entanglement of two coupling harmonic oscillators. This system can model some molecules through modeling atoms with harmonic oscillators. The simplest form of two coupling harmonic oscillator system has the Hamiltonian as

\[ H = \frac{1}{2m} (p_1^2 + p_2^2) + \frac{1}{2} m \omega^2 (x_1^2 + x_2^2) - \delta x_1 x_2. \]  

(24)

Here, \( m \) denotes the mass and the \( \omega \) denotes the frequency of particles (suppose they are same for the two particles), \( \delta \) denotes the coupling constant and

\[ x_i = \frac{a_i + a_i^\dagger}{\sqrt{2}}, \quad p_i = \frac{a_i - a_i^\dagger}{j\sqrt{2}} (i = 1, 2, j^2 = -1). \]  

(25)

When this system acting on the vacuum state, the eigenstate of energy is \( U |00\rangle \) where the Hamiltonian

\[ U = \left( \frac{\omega_2}{\omega_1 \omega_2} \right)^{\frac{j}{4}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_1 dx_2 \left| u \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \right\rangle \left\langle u \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \right| \]  

(26)

Here,

\[ \omega_1^2 = \omega^2 - \frac{\delta}{m}, \quad \omega_2^2 = \omega^2 - \frac{\delta}{m}, \]

\[ u = \left( \begin{array}{cc} \sqrt{\frac{\omega_1^2}{\omega^2}} & \sqrt{\frac{\omega_2^2}{\omega^2}} \\ \sqrt{\frac{\omega_2^2}{\omega^2}} & -\sqrt{\frac{\omega_1^2}{\omega^2}} \end{array} \right), \quad \text{det } u = \frac{\omega}{\sqrt{\omega_1 \omega_2}}. \]  

(27)

By using IWOP one can obtain

\[ U = \left( \text{sec } h r_1 \text{ sec } h r_2 \right)^{\frac{j}{4}} \exp \left\{ \frac{1}{4} \left( a_1^\dagger a_1 + a_2^\dagger a_2 \right)^2 \text{tan hr}_1 + \frac{1}{4} \left( a_1^\dagger a_1 - a_2^\dagger a_2 \right)^2 \text{tan hr}_2 \right\} \]

\[ \cdot \exp \left\{ \left( \frac{1}{\sqrt{2}} \text{ sec } h r_1 - 1 \right) a_1^\dagger a_1 + \frac{1}{\sqrt{2}} a_1^\dagger a_1 \text{ sec } h r_1 - \frac{a_1^2}{2} \text{tan hr}_1 \right\} \]

\[ \cdot \exp \left\{ \left( - \left( \frac{1}{\sqrt{2}} \text{ sec } h r_2 + 1 \right) a_2^\dagger a_2 - \frac{1}{\sqrt{2}} a_2^\dagger a_2 \text{ sec } h r_2 - \frac{a_2^2}{2} \text{tan hr}_2 \right) \right\} \]  

(28)

where

\[ \text{tan hr}_i = \frac{\omega - \omega_i}{\omega + \omega_i}, \quad \text{sec } h r_i = \frac{2\sqrt{\omega \omega_i}}{\omega + \omega_i}, \quad i = 1, 2. \]  

(29)

Thus, the density matrix under the operator \( U \) acting on vacuum state is

\[ \rho_{12}^{\text{cho}} = \text{sec } h r_1 \text{ sec } h r_2 \exp \left\{ \frac{1}{2} \left( a_1^\dagger \text{tan hr}_1 + a_2^\dagger \text{tan hr}_2 \right) \right\} |00\rangle \langle 00| \]

\[ \exp \left\{ \frac{1}{2} \left( a_1^\dagger \text{tan hr}_1 + a_2^\dagger \text{tan hr}_2 \right) \right\} \]  

(30)
where and in the following the superscript “cho” denotes coupling harmonic oscillators. So we have

\[
\rho_{1}^{\text{cho}} = \text{sec} h r_1 : \exp \left\{ \frac{1}{2} \left( a_1^2 + a_1^2 \right) \tanh r_1 \right\} e^{-a_1 a_1^\dagger} : \nonumber \\
= \text{sec} h r_1 \sum_{n,m=0} \frac{(1/2 \tanh r_1)^{m+n}}{m!n!} a_1^{2n} |0\rangle \langle 0| a_1^{2m}. \quad (31)
\]

Suppose the temperature of the two-atom molecule is very low, there are only two level in the atoms then the density matrix becomes

\[
\rho_{1,1}^{\text{cho}} = \text{sec} h r_1 \begin{pmatrix}
1 & 0 & \frac{1}{\sqrt{2}} \tanh r_1 \\
0 & 0 & 0 \\
\frac{1}{\sqrt{2}} \tanh r_1 & 0 & \frac{1}{2} \tanh^2 r_1
\end{pmatrix}. \quad (32)
\]

From this density matrix we can calculate the entropy entanglement of this system, it is

\[
E_{1}^{\text{cho}} = -(\text{sechr} + \frac{1}{2} \text{sechr} \tanh^2 r_1) \log(\text{sechr} + \frac{1}{2} \text{sechr} \tanh^2 r_1). \quad (33)
\]

From Eq.(33) we can plot $E_{1}^{\text{cho}}$ with squeezed parameter $r_1$ as figure 3.

**figure 3**

It is shown that the entanglement take its maximum at $r_1 \approx 2$. A furthermore calculation shows that when the temperature raise and the level of each harmonic oscillators more than two, for example $n$, the density matrix $\rho_{1}^{\text{cho}}$ becomes a matrix of $(2n - 1) \times (2n - 1)$. When $n$ big enough, this problem will become very difficult. So, in order to calculate the entanglement of this system a better method is expected.

In this paper, by using the results of normally ordered forms of operators and the IWOP of operators we calculated the generating of entanglement from biparticle Bose systems acting on vacuum state. These systems include two-mode squeezed system, free single particle in the thermal environment, and two coupling harmonic oscillators system. The generating of entanglement by two mode squeezed operator acting on vacuum state is equal to the entanglement of two-mode squeezed vacuum state. Our result is coincided with that has been obtained by many authors. The entanglement produced from free single particle in thermal environment is one between the particle and the environment, which is obtained rigorously by using our method. The entanglement of coupled two harmonic oscillators is the entanglement between one particle and the other. By using our result one can obtain the approximative degree of entanglement of this system at arbitrarily accurate degree, theoretically. Of course, it is difficult when these particles in a higher temperature. It is shown that IWOP is a useful tool for the investigating of generating of entanglement on biparticle Bose systems.

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Figure 1: $E^{rms}$ with squeezed parameter $\lambda$.
Figure 2: $E^{tv}$ with temperature $T$ where $\omega = 10^{-20}$, $k_B = 1.3806568 \times 10^{-23}$ J K$^{-1}$.
Figure 3: $E^{h_0}$ with the squeezed parameter $r_1$. 