ON Baire CATEGORY PROPERTIES OF FUNCTION SPACES $C'_k(X,Y)$

TARAS BANAKH AND LEIJIE WANG

ABSTRACT. We prove that for a stratifiable scattered space $X$ of finite scattered height, the function space $C_k(X)$ endowed with the compact-open topology is Baire if and only if $X$ has the Moving Off Property of Gruenhage and Ma. As a byproduct of the proof we establish many interesting Baire category properties of the function spaces $C'_k(X,Y) = \{f \in C_k(X,Y) : f(X') \subset \{\ast_Y\}\}$, where $X$ is a topological space, $X'$ is the set of non-isolated points of $X$, and $Y$ is a topological space with a distinguished point $\ast_Y$.

Contents

1. Introduction and Main Results 1
2. The discrete moving off property 4
3. A convenient base for the function space $C'_k(X,Y)$ 6
4. A function space characterization of DMOP 8
5. A function space characterization of WDMOP 12
6. Introducing properties of $X$, responsible for the metrizability of $C'_k(X,Y)$ 15
7. The metrizability of the function spaces $C'_k(X,Y)$ 17
8. The (almost) complete-metrizability of the function spaces $C'_k(X,Y)$ 20
9. Countable networks in function spaces $C'_k(X,Y)$ 22
10. Recognizing $\infty$-meager function spaces $C'_k(X,Y)$ 25
11. A dichotomy for analytic function spaces $C'_k(X,Y)$ 27
12. The interplay between the function spaces $C'_k(X,Y)$ and $C'_k(X/X', Y)$ 28
13. Characterizing stratifiable scattered spaces of finite scattered height 31
14. On function spaces with values in rectifiable pointed spaces 32
15. Function spaces $C_k(X,Y)$ for scattered $X$ and rectifiable $Y$ 33
References 35

1. Introduction and Main Results

This paper was motivated by the problem of characterization of scattered topological spaces $X$ whose function space $C_k(X)$ is Baire. Here $C_k(X)$ is the space of real-valued continuous functions on $X$, endowed with the compact-open topology.

A topological space $X$ is Baire if for any sequence $(U_n)_{n \in \omega}$ of open dense sets in $X$, the intersection $\bigcap_{n \in \omega} U_n$ is dense in $X$. In [17] Gruenhage and Ma made a conjecture that for a Tychonoff space $X$, the function space $C_k(X)$ is Baire if and only if $X$ has the Moving Off Property (abbreviated MOP), which is defined as follows.

A family $\mathcal{F}$ of subsets of a topological space $X$ is called

- discrete if each point $x \in X$ has a neighborhood $O_x \subset X$ that meets at most one set of the family $\mathcal{F}$.

2000 Mathematics Subject Classification. Primary 54C35; Secondary 54E52.

Key words and phrases. Function space, compact-open topology, Moving Off Property, meager, Baire.
A topological space $X$ is defined to have the Moving Off Property (abbreviated MOP) if each moving off family $F$ of compact sets in $X$ contains an infinite strongly discrete subfamily $D \subset F$.

In Theorem 2.1 of [17] Gruenhage and Ma observed that a Tychonoff space $X$ has MOP if its function space $C_k(X)$ is Baire, and made the following conjecture (see also [18, Question 4.7]).

**Conjecture 1.1.** A Tychonoff space $X$ has MOP if and only if its function space $C_k(X)$ is Baire.

In [17] this conjecture was confirmed for all $q$-spaces, i.e., spaces whose any point $x \in X$ admits a sequence $(U_n)_{n \in \omega}$ of neighborhoods such that each sequence $(x_n)_{n \in \omega} \in \prod_{n \in \omega} U_n$ has an accumulation point in $X$. The class of $q$-spaces includes all locally compact and all first-countable spaces. In [15] Conjecture 1.1 has been confirmed for subspaces of linearly ordered spaces. More extensive information on MOP can be found in the Ph.D. dissertation of Hughes [21]. Some set-theoretical questions related to MOP were studied by Tall in [31].

In this paper we confirm Conjecture 1.1 for stratifiable scattered spaces of finite scattered height.

Let us recall that a topological space $X$ is scattered if each non-empty subspace of $X$ contains an isolated point. A point $x$ of a topological space $X$ is called isolated if its singleton $\{x\}$ is clopen in $X$. A subset of a topological space is clopen if it is both closed and open. For a topological space $X$ by $\hat{X}$ we denote the set of isolated points in $X$.

The complexity of a scattered space $X$ can be measured by an ordinal number $h(X)$, called the scattered height of $X$. It is defined as follows.

For a subspace $A \subset X$ of $X$ denote by $A' := A \setminus \hat{A}$ the set of non-isolated points of $A$. Let $X^{[0]} = X$ and for any ordinal $\alpha > 0$ define the $\alpha$-th derived set $X^{[\alpha]}$ of $X$ by the recursive formula

$$X^{[\alpha]} = \bigcap_{\beta < \alpha} (X^{[\beta]})'.$$

The smallest ordinal $\alpha$ with $X^{[\alpha]} = X^{[\alpha+1]}$ is called the scattered height of $X$ and is denoted by $h(X)$. Observe that a topological space $X$ is scattered if and only if $X^{[h(X)]} = \emptyset$.

A regular topological space $X$ is called stratifiable if to each point $x \in X$ it is possible to assign a decreasing sequence $(U_n(x))_{n \in \omega}$ of neighborhoods such that each closed set $F$ is equal to the intersection $\bigcap_{n \in \omega} \overline{U_n[F]}$ where $U_n[F] = \bigcup_{x' \in F} U_n(x)$. The class of stratifiable spaces includes all metrizable spaces and has many nice properties, see [16, §5]. In Theorem 1.2.2 we shall prove that a scattered space $X$ of finite scattered height is stratifiable if and only if for every $n < h(X)$ the set $X^{[n]}$ is a retract of $X$ and $X^{[n]}$ is a $G_{\delta}$-set in $X$.

One of the principal results of this paper is the following theorem.

**Theorem 1.2.** For a stratifiable space $X$, the function space $C_k(X)$ is Baire if and only if $X$ has MOP and the function space $C_k(X')$ is Baire.

Applying this theorem by induction, we get the following corollary.

**Corollary 1.3.** For a stratifiable space $X$, the space $C_k(X)$ is Baire if and only if $X$ has MOP and for some $n \in \omega$ the function space $C_k(X^{[n]})$ is Baire.

In its turn, Corollary 1.3 implies

**Corollary 1.4.** For a stratifiable scattered space $X$ of finite scattered height, the space $C_k(X)$ is Baire if and only if $X$ has MOP.
Theorem 1.2 and Corollary 1.3 will be proved in Section 15 after extensive preliminary work made in Sections 2–14. Namely, in Section 2 we introduce a discrete version of MOP, called DMOP and prove a game characterization of DMOP, resembling the game characterization of MOP, found by Gruenhage and Ma in [17] (our game $G_{KF}(X)$ resembles also the games $\Gamma_1(X)$ and $\Gamma_2(X)$, studied by McCoy and Ntansu [25, §8]). In Section 4 we apply the game characterization of DMOP to prove that a topological space $X$ has DMOP if and only if for any Polish space $Y$ with a distinguished point $*Y$ the function space
\[
C'_k(X, Y) = \{ f \in C_k(X, Y) : f(X') \subset \{ *Y \} \}
\]
is Baire. In Section 5 we study a “winning” modification of DMOP, called WDMOP and prove that a topological space $X$ has WDMOP if and only if for any pointed Polish space $(Y, *_Y)$ the function space $C'_k(X, Y)$ is Choquet. In Sections 7 and 8 we characterize topological spaces $X, Y$ for which the function space $C'_k(X, Y)$ is metrizable or complete-metrizable. In Section 9 we characterize topological spaces $X$ for which the function space $C'_k(X, Y)$ has a countable network for any second-countable pointed space $Y$. As an application of the mentioned characterizations of Baire category properties of function spaces $C'_k(X, Y)$, in Section 11 we prove the following dichotomy: if for a topological space $X$ and a pointed Polish space $Y$ the function space $C'_k(X, Y)$ is analytic, then it is either Polish or meager (more precisely, $\infty$-meager). Also we prove that for a regular pointed space $Y$ the function space $C'_k(X, Y)$ is analytic if and only if it is cosmic and the function space $C'_p(X, Y)$ is analytic.

The equivalence of meager and $\infty$-meager properties in the function spaces $C'_k(X, Y)$ is established in Section 10. In Section 12 we observe that for any topological space $X$ and pointed topological space $Y$, the function space $C'_k(X, Y)$ admits a continuous bijective map onto the function space $C'_k(X/X', Y)$ defined on the quotient space $X/X'$ having a unique non-isolated point, and give conditions under which this bijective map $C'_k(X, Y) \to C_k(X/X', Y)$ is (or is not) a homeomorphism. In Section 13 we prove that a scattered space $X$ of finite scattered height is stratifiable if and only if for every $k < h[X]$ the set $X^{[k]}$ is a $G_\delta$-retract in $X$. In Section 14 we study function spaces with values in rectifiable spaces $Y$ and prove that if $X'$ is a retract of $X$, then the function space $C_k(X, Y)$ is homeomorphic to $C_k(X', Y) \times C'_k(X, Y)$. In the final section 15 we return back to studying the function spaces $C_k(X, Y)$ over stratifiable scattered spaces with values in rectifiable spaces and using the game characterization of DMOP, prove a more general version of Theorem 1.2 announced in the introduction. It should be mentioned that the results on Baire category properties of the function spaces $C'_k(X, Y)$ obtained in this paper are essentially used in our forthcoming paper [32], devoted to studying the function spaces $C_{1F}(X, Y)$ endowed with the Fell hypograph topology.

The characterization theorems proved in Sections 4, 10 can be unified in the following

**Theorem 1.5.** Let $Y$ be a pointed metrizable space containing more than one point and $X$ be a topological space containing an isolated point.

1. If $X$ does not have DMOP, then $C'_k(X, Y)$ is meager and $\infty$-meager.
2. If $X$ has DMOP and $Y$ is Choquet, then the space $C'_k(X, Y)$ is Baire.
3. $C'_k(X, Y)$ is Choquet if and only if $Y$ is Choquet and $X$ has WDMOP.
4. $C'_k(X, Y)$ is complete-metrizable if and only if $Y$ is complete-metrizable and $X$ is a $\kappa$-space.
5. $C'_k(X, Y)$ is almost complete-metrizable if and only if $Y$ is Choquet and $X$ is a $\kappa$-space.
6. $C'_k(X, Y)$ is Polish if and only if $Y$ is Polish and $X$ is a $\kappa$-space with countable set $\hat{X}$ of isolated points.
7. $C'_k(X, Y)$ is almost Polish if and only if $Y$ is almost Polish and $X$ is a $\kappa$-space with countable set $\hat{X}$.
8. $C'_k(X, Y)$ is metrizable if and only if $X$ is a hemi-$\kappa_\omega$-space.
9. $C'_k(X, Y)$ is metrizable and separable if and only if $Y$ is separable and $X$ is a hemi-$\kappa_\omega$-space with countable set $\hat{X}$.
(10) \( C'_k(X,Y) \) has a countable network if and only if \( Y \) is separable and \( X \) has a countable \( \kappa \)-network.

The statements (1)–(10) of this theorem are proved in Theorems 10.2, 4.4, 5.3, 8.1, 8.4, 8.2, 8.5, 7.4, 7.9, and 9.4, respectively. All undefined notions appearing in Theorem 1.5 can be found in the corresponding sections.

2. The discrete moving off property

In this section we discuss the discrete modification of MOP, called DMOP.

**Definition 2.1.** A topological space \( X \) has the discrete moving off property (briefly, DMOP) if any moving off family \( F \) of finite subsets of \( \hat{X} \) contains an infinite subfamily \( D \subset F \), which is discrete in \( X \).

We recall that \( \hat{X} \) denotes the set of all isolated points in \( X \), i.e. points \( x \in X \) whose singleton \( \{x\} \) is clopen in \( X \).

It is clear that each space with MOP has DMOP.

**Definition 2.2.** A topological space \( X \) is defined to have the property of discrete diagonalization if for any sequence \( F_0,F_1,F_2,\ldots \) of infinite discrete families \( F_n \) of finite subsets of \( \hat{X} \) there exists a sequence \( (F_n)_{n \in \omega} \in \prod_{n \in \omega} F_n \) such that the indexed family \( \{F_n\}_{n \in \omega} \) is discrete in \( X \).

**Lemma 2.3.** Each topological space \( X \) with DMOP has the property of discrete diagonalization.

**Proof.** Assume that \( X \) has DMOP and let \( (F_n)_{n \in \omega} \) be a sequence of infinite discrete families of finite subsets of \( \hat{X} \). Fix a countable subfamily \( \{D_n\}_{n \in \omega} \subset F_0 \) consisting of pairwise disjoint non-empty finite sets \( D_n \) in \( \hat{X} \). For every \( n \in \mathbb{N} \) let

\[ \mathcal{E}_n = \{D_n \cup F_1 \cup \cdots \cup F_n : (F_i)_{i=1}^n \in \prod_{i=1}^n F_i \}. \]

It is easy to see that \( \mathcal{E} := \bigcup_{n=1}^{\infty} \mathcal{E}_n \) is a moving off collection of non-empty finite subsets of \( \hat{X} \). By DMOP, there exists an infinite discrete subfamily \( \{E_k\}_{k \in \omega} \subset \mathcal{E} \) consisting of pairwise disjoint sets \( E_k, k \in \omega \). For every \( k \in \omega \) find a number \( n(k) \in \mathbb{N} \) with \( E_k \in \mathcal{E}_{n(k)} \) and observe that the correspondence \( k \mapsto n(k) \) is injective (as the sets \( E_k \) are pairwise disjoint). Replacing \( (E_k)_{k \in \omega} \) by a suitable subsequence, we can assume that the sequence \( n(k) \) is increasing and hence \( n(k) \geq k \) for all \( k \in \omega \). Then for any \( k \in \omega \) there exists a set \( F_k \in F_k \) such that \( F_k \subset E_k \in \mathcal{E}_{n(k)} \). The discreteness of the family \( \{E_k\}_{k \in \omega} \) implies that discreteness of the family \( \{F_k\}_{k \in \omega} \). \( \square \)

Now we are going to present a game characterization of DMOP with help of the game \( G_{KF}(X) \), played by two players, \( K \) and \( F \) on a topological space \( X \). The player \( K \) starts the game. At the \( n \)-th inning the player \( K \) chooses a compact subset \( K_n \subset X \) and player \( F \) responds by choosing a finite subset \( F_n \) of \( X \) such that \( F_n \cap K_n = \emptyset \). At the end of the game, the player \( K \) is declared the winner if the family \( \{F_n\}_{n \in \mathbb{N}} \) is discrete in \( X \); otherwise the player \( F \) wins the game.

For a topological space \( X \) by \( \mathcal{F}(X) \) and \( \mathcal{K}(X) \) we denote the families of finite and compact subsets of \( X \), respectively.

For set \( A \) by \( A^{\leq \omega} \) we denote the family \( \bigcup_{n \in \omega} A^n \) of all finite sequences \( (a_0,\ldots,a_{n-1}) \) of elements of \( A \). The set \( A^{< \omega} \) is a tree with respect to the partial order \( \leq \) defined by \( (a_0,\ldots,a_n) \leq (b_0,\ldots,b_m) \) iff \( n \leq m \) and \( a_i = b_i \) for \( i \leq n \). For a sequence \( s = (a_0,\ldots,a_{n-1}) \) and a number \( n \leq m \) let \( s|n = (a_0,\ldots,a_{n-1}) \) be the initial segment of \( s \) of length \( n \).

The following theorem is just a suitable modification of the game characterization of MOP, proved by Gruenhage and Ma in [17 Theorem 2.3].
Theorem 2.4. A topological space $X$ has DMOP if and only if the player $F$ has no winning strategy in the game $G_{K_{F}}(X)$.

Proof. If $X$ does not have DMOP, then there exists a moving off family $F$ of finite sets in $\hat{X}$ containing no infinite discrete subfamily. Then the player $F$ can win the game $G_{K_{F}}(X)$ by always choosing distinct members of $F$.

Now assume that $X$ has DMOP and let $S_{F}: K(X)^{<\omega} \rightarrow F(\hat{X})$ be any strategy of the player $F$ in the game $G_{K_{F}}(X)$. The strategy $S_{F}$ is a function assigning to each finite sequence $(K_{1}, \ldots, K_{n}) \in K(X)^{<\omega}$ a finite set $S_{F}(K_{1}, \ldots, K_{n}) \subset \hat{X}$ that is disjoint with the compact set $K_{n}$. Since the player $K$ starts the game, we can assume that $S_{F}$ assigns to the unique sequence $K(X)^{0}$ of length zero the empty subset of $\hat{X}$.

We shall inductively construct a countable subtree $T \subset K(X)^{<\omega}$ such that for any sequence $(K_{1}, \ldots, K_{n}) \in T$ the family $\{S_{F}(K_{1}, \ldots, K_{n}, K) : (K_{1}, \ldots, K_{n}, K) \in T\}$ is infinite and discrete. The tree $T$ will be constructed as the union $T = \bigcup_{n \in \omega} T_{n}$ of trees $T_{n}$ of height $n$. To start the inductive construction, we put $T_{0} := K(X)^{0} = \{\emptyset\}$. Assume that for some $n \in \omega$ the subtree $T_{n} \subset \bigcup_{k \leq n} K(X)^{k}$ has been defined. For any node $(K_{1}, \ldots, K_{n}) \in T_{n}$, consider the family $\{S_{F}(K_{1}, \ldots, K_{n}, K) : K \in K(X)\}$ and observe that it is moving off. Since $X$ has DMOP, this family contains an infinite discrete subfamily $\{S_{F}(K_{1}, \ldots, K, K) : K \in T_{n+1}(K_{1}, \ldots, K_{n})\}$ for some countable infinite subset $T_{n+1}(K_{1}, \ldots, K_{n}) \subset K(X)$. Finally, put

$$T_{n+1} = T_{n} \cup \bigcup_{(K_{1}, \ldots, K_{n}) \in T_{n}} \{(K_{1}, \ldots, K_{n}, K) : K \in T_{n+1}(K_{1}, \ldots, K_{n})\}.$$ 

It is clear that the countable tree $T = \bigcup_{n \in \omega} T_{n}$ has the required property: for every sequence $(K_{1}, \ldots, K_{n}) \in T$ the indexed family $\{S_{F}(K_{1}, \ldots, K_{n}, K) : (K_{1}, \ldots, K_{n}, K) \in T\}$ is infinite and discrete.

By Lemma 2.5, the space $X$ has the discrete diagonalization property. So to each sequence $(K_{1}, \ldots, K_{n}) \in T$ we can assign a compact set $K_{n+1} = \kappa(K_{1}, \ldots, K_{n})$ such that $(K_{1}, \ldots, K_{n+1}) \in T$ and the indexed family

$$D = \{S_{F}(K_{1}, \ldots, K_{n}, \kappa(K_{1}, \ldots, K_{n})) : (K_{1}, \ldots, K_{n}) \in T\}$$

is discrete. Now consider the sequence $(K_{n})_{n \in \omega} \in K(X)^{\omega}$ defined recursively as $K_{1} = \kappa(\emptyset)$ and $K_{n+1} = \kappa(K_{1}, \ldots, K_{n})$ for $n \in \mathbb{N}$. Also put $F_{n} = S_{F}(K_{1}, \ldots, K_{n})$ for every $n \in \mathbb{N}$. Observe that the indexed family $\{F_{n}\}_{n \in \mathbb{N}}$ is discrete, being a subfamily of the discrete family $D$. Now we see that the sequence

$$K_{1}, F_{1}, K_{2}, F_{2}, K_{3}, F_{3}, \ldots$$

is a sequence of moves of the players $K, F$ in which the player $F$ plays according to the strategy $S_{F}$. Since the family $\{F_{n}\}_{n \in \omega}$ is discrete, the player $K$ wins, which means that the strategy $S_{F}$ of the player $F$ is not winning.

Definition 2.5. A topological space $X$ is defined to have the winning discrete moving off property (abbreviated WDMOP) if the player $K$ has a winning strategy $S_{K}$ in the game $G_{K_{F}}(X)$.

This winning strategy is a function $S_{K} : F(\hat{X})^{<\omega} \rightarrow K(X)$ assigning to any finite sequence $(F_{0}, \ldots, F_{n-1})$ of finite subsets of $\hat{X}$ a compact subset $S_{K}(F_{0}, \ldots, F_{n-1})$ of $X$ such that a family $(F_{n})_{n \in \omega}$ of finite subsets of $\hat{X}$ is discrete in $X$ if for every $n \in \omega$ the set $F_{n}$ is disjoint with the compact set $S_{K}(F_{0}, \ldots, F_{n-1})$.

It is clear that WDMOP implies DMOP.

Lemma 2.6. If a topological space $X$ has WDMOP, then the player $K$ has a strategy $S_{K} : F(\hat{X})^{<\omega} \rightarrow K(X)$ in the game $G_{K_{F}}(X)$ such that for any sequence $(F_{n})_{n \in \omega}$ of finite subsets of $\hat{X}$ the indexed family $\{F_{n} \setminus S_{K}(F_{0}, \ldots, F_{n-1})\}_{n \in \omega}$ is discrete in $X$. 


Proof. Since the space $X$ has WDMOP, the player $K$ has a winning strategy in the game $G_{KF}(X)$. This winning strategy is a function $W : \mathcal{F}(X)^{<\omega} \to \mathcal{K}(X)$ such that a sequence $(F_n)_{n \in \omega}$ of finite subsets of $X$ is discrete in $X$ if $F_n \cap W(F_0,\ldots,F_{n-1}) = \emptyset$ for every $n \in \omega$. Define a strategy $S_K : \mathcal{F}(X)^{<\omega} \to \mathcal{K}(X)$ of the player $K$ in the game $G_{KF}(X)$ assigning to each finite sequence $(F_0,\ldots,F_{n-1}) \in \mathcal{F}(X)^{<\omega}$ the last element $K_n$ of the sequence $(K_0,\ldots,K_n)$ of compact subsets of $X$, defined by the recursive formula: $K_i = W(F_0 \setminus K_0,\ldots,F_{i-1} \setminus K_{i-1})$ for $i \leq n$.

We claim that the strategy $S_K$ has the required property. Indeed, given any sequence $(F_n)_{n \in \omega}$ of finite subsets of $X$, define the sequence $(K_n)_{n \in \omega}$ of compact subsets of $X$ by the recursive formula $K_n = W(\bigcap_{i=0}^{n} K_i \setminus F_i,\ldots,\bigcap_{i=0}^{n} K_i \setminus F_{n-1})$ for $n \in \omega$. The definition of the strategy $S_K$ ensures that $K_n = S_F(F_0,\ldots,F_{n-1})$ for every $n \in \omega$. For every $n \in \omega$ consider the finite subset $E_n := F_n \setminus K_n$ of $F_n \subset X$ and observe that $E_n$ is disjoint with the compact set $K_n = W(E_0,\ldots,E_{n-1})$. Since the strategy $W$ is winning, the indexed family $(E_n)_{n \in \omega} = (F_n \setminus K_n)_{n \in \omega} = (F_n \setminus \bigcap_{i=0}^{n} K_i \setminus F_{n-1})_{n \in \omega}$ is discrete in $X$. \hfill \Box

3. A CONVENIENT BASE FOR THE FUNCTION SPACE $C^*_k(X,Y)$

For topological spaces $X,Y$, let $C^*_k(X,Y)$ be the space of continuous functions from $X$ to $Y$, endowed with the compact-open topology. This topology is generated by the subbase consisting of the sets

$$[K;U] := \{ f \in C^*_k(X,Y) : f(K) \subset U \}$$

where $K$ is a compact subset of $X$ and $U$ is an open subset of $Y$.

By a pointed topological space we understand a topological space $Y$ with a distinguished point $*_Y \in Y$.

A pointed topological space $Y$ is defined to be

- $*-first$-$countable$ if $Y$ is first-countable at its distinguished point $*_Y$;
- $*-admissible$ if the distinguished point $*_Y$ of $Y$ has a neighborhood $U_*$ which is not dense in $Y$.

It is easy to see that a pointed topological space $Y$ is $*-admissible$ if $Y$ is Hausdorff and contains more than one point.

For a topological space $X$ and a pointed topological space $Y$ we shall study the Baire category properties of the subspace

$$C^*_k(X,Y) := \{ f \in C^*_k(X,Y) : f(X') \subset \{*_Y\} \} \subset C^*_k(X,Y),$$

where $X' := X \setminus \hat{X}$ is the set of non-isolated points of $X$.

First we describe a convenient base of the topology of the function space $C^*_k(X,Y)$. For two sets $K \subset X$ and $U \subset Y$ we keep the notation

$$[K;U] := \{ f \in C^*_k(X,Y) : f(K) \subset U \}.$$ 

Fix a base $B_Y \ni Y$ of the topology of the space $Y$ and consider the family $Q$ of all quadruples $(K,U,F,u)$, where

- $K$ is a compact subset of $X$;
- $U \in B_Y$ is a neighborhood of the distinguished point $*_Y$ of $Y$;
- $F \subset X \setminus K$ is a finite set of isolated point in $X$;
- $u : \hat{X} \to B_Y$ is a function assigning to each point $x \in \hat{X}$ a non-empty basic open set $u(x) \in B_Y$ such that $u(x) = Y$ if $x \notin F$. 
For any quadruple \((K, U, F, u) \in \mathcal{Q}\), consider the open set

\[ [K; U|F; u] := [K; U] \cap \bigcap_{x \in F} \{x; u(x)\} \]

in the function space \(C'_k(X, Y)\), and observe that this open set is not empty.

The following lemma shows that the family \([\{K; U|F; u\} : (K, U, F, u) \in \mathcal{Q}\}\) is a base of the topology of the function space \(C'_k(X, Y)\).

**Lemma 3.1.** For any quadruple \((K, U, F, u) \in \mathcal{Q}\), function \(f \in C'_k(X, Y)\) and neighborhood \(O_f \subset C'_k(X, Y)\) of \(f\) there exists a quadruple \((\bar{K}, \bar{U}, \bar{F}, \bar{u}) \in \mathcal{Q}\) such that

\[ F \subset \bar{F}, \quad K \cup F \subset \bar{K} \cup \bar{F}, \quad \bar{U} \subset U \quad \text{and} \quad f \in [\bar{K}; \bar{U}|\bar{F}; \bar{u}] \subset O_f. \]

**Proof.** By definition of the compact-open topology on \(C'_k(X, Y)\), there are compact sets \(K_1, \ldots, K_n \subset X\) and open sets \(U_1, \ldots, U_n \subset X\) such that \(f \in \bigcap_{i=1}^n [K_i; U_i] \subset O_f\).

Choose any basic open set \(\bar{U} \in B_Y\) such that

\[ *_Y \in \bar{U} \subset U \cap \bigcap_{i=1}^n \{U_i : 1 \leq i \leq n, *_Y \in U_i\}. \]

Consider the compact set \(C = K \cup F \cup \bigcup_{i=1}^n K_i\) in \(X\) and observe that the open set \(V = (f|C)^{-1}(\bar{U})\) contains the set \(X' \cap C\). The set \(C \setminus V \subset \bar{X}\) is finite, being a closed discrete subspace of the compact space \(C\). Then the union \(\bar{F} := (C \setminus V) \cup F \subset \bar{X}\) is clopen in \(X\) and its complement \(\bar{K} := C \setminus \bar{F}\) is a compact subset of \(X\).

For every \(x \in \bar{X} \setminus \bar{F}\) put \(\bar{u}(x) = Y\) and for every \(x \in \bar{F}\) choose a basic neighborhood \(\bar{u}(x) \in B_Y\) of the point \(f(x) \in Y\) such that \(\bar{u}(x) \subset \bigcap\{U_i : 1 \leq i \leq n, f(x) \in U_i\}\). It is easy to see that \(f \in [\bar{K}; \bar{U}|\bar{F}; \bar{u}] \subset \bigcap_{i=1}^n [K_i; U_i] \subset O_f\).

As a first application of the above base, we find a conditions of the topological spaces \(X, Y\) ensuring that the function space \(C'_k(X, Y)\) has countable cellularity.

We recall that a topological space has **countable cellularity** if it contains no uncountable family of pairwise disjoint open sets.

**Proposition 3.2.** If the space \(Y\) is second-countable and the set \(\bar{X}\) of isolated points of \(X\) is of type \(F_\sigma\) in \(X\), then the function space \(C'_k(X, Y)\) has countable cellularity.

**Proof.** To derive a contradiction, assume that \(C'_k(X)\) contains an uncountable family \(\{W_i\}_{i \in \omega_1}\) of non-empty pairwise disjoint open sets. Fix a countable base \(B_Y\) of the topology of the space \(Y\). By Lemma 3.1, we can assume that each set \(W_i\) is of basic form \(W_i = [K_i; U_i|F_i; u_i]\) for some quadruple \((K_i, U_i, F_i, u_i) \in \mathcal{Q}\). By the \(\Delta\)-Lemma [22 9.18], there exists an uncountable subset \(\Omega \subset \omega_1\) and a finite set \(F\) such that \(f_i \cap F_j = F\) for any distinct elements \(i, j \in \Omega\). By Pigeonhole Principle, for some function \(u : F \to B_Y\) the set \(\{i \in \Omega : u_i|F = u\}\) is uncountable. Replacing \(\Omega\) by this uncountable set, we can assume that \(u_i|F = u\) for all \(i \in \Omega\).

By our assumption, the set \(\bar{X}\) is an \(F_\sigma\)-set in \(X\), so \(\bar{X} = \bigcup_{n \in \omega} D_n\) for some increasing sequence \((D_n)_{n \in \omega}\) of closed (discrete) sets \(D_n \subset X\). By the Pigeonhole Principle, for some \(n \in \omega\) the set \(\Omega_n := \{i \in \Omega : F_i \subset D_n\}\) is uncountable. Since the set \(D_n\) is closed and discrete in \(X\), for every \(i \in \Omega_n\) the compact set \(E_i := K_i \cap D_n\) is finite. By the \(\Delta\)-Lemma [22 9.18], there exists an uncountable subset \(\Lambda \subset \Omega_n\) and a finite set \(E\) such that \(E_i \cap E_j = E\) for any distinct elements \(i, j \in \Lambda\). Now take any element \(i \in \Lambda\) and observe that

\[ F \cap E \subset F_i \cap E_i \subset F_i \cap K_i = \emptyset. \]

Since each of the families \((F_j \setminus F)_{j \in \Lambda}\) and \((E_j \setminus E)_{j \in \Lambda}\) is disjoint, and the sets \(F_i, E_i\) are finite, the set

\[ \lambda := \{j \in \Lambda : (F_j \setminus F) \cap E_i \neq \emptyset\} \cup \{j \in \Lambda : F_i \cap (E_j \setminus E) \neq \emptyset\} \]
is finite. Take any element \( j \in \Lambda \setminus (\lambda \cup \{i\}) \) and observe that
\[
F_i \cap K_j = F_i \cap D_n \cap K_j = F_i \cap E_j = (F_i \cap (E_j \setminus E)) \cup (F_i \cap E) \subset \emptyset \cup (F_i \cap E_i) = \emptyset.
\]
By analogy we can check that \( F_j \cap K_i = \emptyset \). This allows us to choose a (necessarily continuous) continuous function \( f : X \to Y \) such that
\[
\begin{align*}
&\bullet \ f(x) \in u(x) \text{ for all } x \in F; \\
&\bullet \ f(x) \in u_i(x) \text{ for all } x \in F_i \setminus F; \\
&\bullet \ f(x) \in u_j(x) \text{ for all } x \in F_j \setminus F; \\
&\bullet \ f(x) = *y \text{ for all } x \in X \setminus (F_i \cup F_j).
\end{align*}
\]
It is clear that
\[
f \in [K_i; U_i | F_i, u_i] \cap [K_j; U_j; F_j; u_j] = W_i \cap W_j,
\]
which is a desired contradiction. \( \square \)

4. A FUNCTION SPACE CHARACTERIZATION OF DMOP

In this section we shall characterize DMOP in terms of Baire category properties of function spaces \( C_b^\circ(X, Y) \).

We shall use the classical Oxtoby’s characterizations of Baire and meager spaces in terms of the Choquet games \( G_{NE}(X) \) and \( G_{EN}(X) \), which are played by two players \( E \) and \( N \) (abbreviated from Empty and Non-Empty) on a topological space \( X \).

The game \( G_{NE}(X) \) is started by the player \( N \) whose chooses a non-empty open set \( U_1 \) of \( X \). Then player \( E \) responds selecting a non-empty open set \( V_1 \). In the \( n \)th inning the player \( N \) selects a non-empty open set \( U_n \subset V_{n-1} \) and the player \( E \) responds selecting a non-empty open set \( V_n \subset U_n \). At the end of the game the player \( E \) is declared the winner if the intersection \( \bigcap_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} V_n \) is empty; otherwise the player \( N \) wins the game \( G_{NE}(X) \).

The game \( G_{EN}(X) \) is started by the player \( E \) whose chooses any non-empty open set \( U_1 \) of \( X \). Then player \( N \) responds selecting a non-empty open set \( V_1 \). In the \( n \)th inning the player \( E \) selects a non-empty open set \( U_n \subset V_{n-1} \) and the player \( N \) responds selecting a non-empty open set \( V_n \subset U_n \). At the end of the game the player \( E \) is declared the winner if the intersection \( \bigcap_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} V_n \) is empty; otherwise the player \( N \) wins the game \( G_{EN}(X) \).

The following classical characterization can be found in [29].

**Theorem 4.1** (Oxtoby). A topological space \( X \) is
\[
\begin{align*}
&\bullet \ \text{meager if and only if the player } E \ \text{has a winning strategy in the game } G_{NE}(X); \\
&\bullet \ \text{Baire if and only if the player } E \ \text{has no winning strategy in the game } G_{EN}(X).
\end{align*}
\]

A topological space \( X \) is defined to be **Choquet** if the player \( N \) has a winning strategy in the Choquet game \( G_{EN}(X) \).

Oxtoby’s Theorem [1.1] implies that
\[
\text{Choquet } \Rightarrow \text{ Baire } \Rightarrow \text{ non-meager}.
\]

By [23, 8.17] (see also [4, 7.3]), a metrizable topological space is Choquet if and only if it almost complete-metrizable; moreover, any open continuous image of a Choquet space is Choquet and the Tychonoff product of any family of Choquet spaces is Choquet [33].

A topological space \( X \) is defined to be
\[
\begin{align*}
&\bullet \ \text{complete-metrizable if } X \text{ is homeomorphic to a complete metric space}; \\
&\bullet \ \text{Polish if } X \text{ is separable and complete-metrizable}; \\
&\bullet \ \text{almost Polish if } X \text{ contains a dense Polish subspace}; \\
&\bullet \ \text{almost complete-metrizable if } X \text{ contains a dense complete-metrizable subspace}.
\end{align*}
\]
For every topological space we have the implications

\[
\text{Polish} \quad \xrightarrow{\text{complete-metrizable}} \quad \text{non-meager} \quad \xrightarrow{\text{almost Polish}} \quad \text{almost complete-metrizable} \quad \xrightarrow{\text{Choquet}} \quad \text{Baire}
\]

We recall that a pointed topological space \( Y \) is \( \ast \)-admissible if its distinguished point \( \ast_Y \) has a neighborhood \( U_* \) which is not dense in \( Y \).

**Lemma 4.2.** Let \( X \) be a topological space and \( Y \) be a \( \ast \)-admissible pointed topological space. If the function space \( C_k'(X,Y) \) is non-meager, then the space \( X \) has DMOP.

**Proof.** Assume that the function space \( C_k'(X,Y) \) is non-meager. By Theorem [4.4,1] the player \( E \) has no winning strategy in the Choquet game \( G_{\text{NE}}(C_k'(X,Y)) \). By Theorem [2,3] the DMOP for the space \( X \) will follow as soon as we show that the player \( F \) has no winning strategy in the game \( G_{KF}(X) \). Let \( S_F : \mathcal{K}(X)^{<\omega} \to \mathcal{F}(\hat{X}) \) be any strategy of the player \( F \) in the game \( G_{KF}(X) \). The strategy \( S_F \) assigns to each finite sequence of compact sets \( (K_0,\ldots,K_n) \in \mathcal{K}(X)^{<\omega} \) a finite set \( S_F(K_0,\ldots,K_n) \subset \hat{X} \setminus K_n \).

Define a new strategy \( \tilde{S}_F \) of the player \( F \) in the game \( G_{KF} \) assigning to each sequence \( (K_0,\ldots,K_n) \in \mathcal{K}(X)^{<\omega} \) the sequence \( S_F(K_0,\ldots,K_n) \) where \( \tilde{K}_i = \bigcup_{j<i} S_F(K_0,\ldots,K_j) \) for \( 1 < i \leq n \).

Now we use the strategy \( \tilde{S}_F \), to describe a strategy \( S_E \) of the player \( E \) in the Choquet game \( G_{\text{NE}}(C_k'(X,Y)) \).

Let \( \tau \) denote the family of all non-empty open sets in \( C_k'(X,Y) \). By our assumption, the pointed space \( Y \) is \( \ast \)-admissible. So, there exists a non-empty open set \( W \subset Y \) whose closure does not contain the distinguished point \( \ast_Y \) of \( Y \).

The definition of the compact-open topology ensures that for every \( U \in \tau \) there exists a compact set \( \kappa(U) \subset X \) such that for any finite set \( F \subset \hat{X} \setminus \kappa(U) \) the open set \( U \cap [F;W] \) is not empty.

For any sequence \( (U_0,\ldots,U_n) \in \tau^{<\omega} \) let \( S_E(U_0,\ldots,U_n) = U_n \cap [F_n;W] \) where

\[
F_n = \tilde{S}_F(\kappa(U_0),\ldots,\kappa(U_n)) \subset \hat{X} \setminus \kappa(U_n)
\]

is the answer of the player \( E \) to the moves \( (\kappa(U_0),\ldots,\kappa(U_n)) \) of the player \( K \) in the game \( G_{KF}(X) \), according to the strategy \( \tilde{S}_F \). Since \( F_n \cap \kappa(U_n) = \emptyset \), the open set \( U_n \cap [F_n;W] \) is non-empty and hence it is a legal move of the player \( E \) in the Choquet game \( G_{\text{NE}}(C_k'(X,Y)) \). Since the player \( E \) has no winning strategy in the game \( G_{\text{NE}}(C_k'(X,Y)) \), the strategy \( S_E \) is not winning. So there exists an infinite sequence \( (U_n)_{n\in\omega} \in \tau^{<\omega} \) such that \( U_{n+1} \subset S_E(U_0,\ldots,U_n) \) for all \( n \in \mathbb{N} \) and the intersection \( \bigcap_{n=1}^{\infty} U_n \) is non-empty and hence contains some function \( f \in C_k'(X,Y) \).

Let \( F_0 = \emptyset \) and for every \( n \in \mathbb{N} \) let \( K_n = \kappa(U_n) \cup \bigcup_{i<n} F_i \) and

\[
F_n = \tilde{S}_F(\kappa(U_1),\ldots,\kappa(U_n)) = S_F(K_1,\ldots,K_n) \subset \hat{X} \setminus K_n \subset \hat{X} \setminus \bigcup_{i<n} F_i.
\]

It follows that the family \( (F_n)_{n\in\mathbb{N}} \) is disjoint and \( f \in \bigcap_{n\in\mathbb{N}} [F_n;W] \). Since \( f(\bigcup_{n\in\omega} F_n) \subset W \) and \( \ast_Y \notin \overline{W} \), the continuity of \( f \) guarantees that the closure of the set \( \bigcup_{n=1}^{\infty} F_n \) does not intersect the set \( X' \subset f^{-1}(\ast_Y) \) and hence the disjoint family \( \{F_n\}_{n\in\mathbb{N}} \) is discrete in \( X \).

Observe that

\[
K_1,F_1,K_2,F_2,\ldots
\]

is the sequence of the moves of the players \( K \) and \( F \) in the game \( G_{KF}(X) \), where the player \( F \) plays according to the strategy \( S_F \) and eventually loses as the family \( \{F_n\}_{n\in\mathbb{N}} \) is discrete. So, the strategy \( S_F \) is not winning.

**Lemma 4.3.** Let \( Y \) be a \( \ast \)-first-countable pointed Choquet space. If a topological space \( X \) has DMOP, then the function space \( C_k'(X,Y) \) is Baire.
The definition of $S_Q$ Assume that for some satisfying the inductive conditions (a)–(e) have been defined. Fix a sequence $Q$ are satisfied:

for some quadruple $(\kappa, \kappa, \kappa, \kappa)$ for every $n$ of non-empty open sets in $X$ a non-empty open set $S_N(U_0, \ldots, U_n) \subset U_n$ such that for any infinite sequence $(U_n)_{n \in \omega} \in \tau_Y$ the intersection $\bigcap_{n \in \omega} S_N(U_0, \ldots, U_n)$ is not empty if $U_n \subset S_N(U_0, \ldots, U_{n-1})$ for all $n \in \mathbb{N}$. Since the player $E$ starts the game $\mathcal{G}_E(Y)$, it is convenient to assume that $S_N(\emptyset) = Y$ for the empty sequence $\emptyset \in \tau_Y$. We can also assume that $S_N(U_0, \ldots, U_{n-1}) = Y$ if $U_i = Y$ for all $i < n$.

To derive a contradiction, assume that the function space $C_\ell(X,Y)$ is not Baire. Then $C_\ell(X,Y)$ contains a non-empty open meager subset $W$, which is contained in the countable union $\bigcup_{n \in \omega} W_n$ of an increasing sequence $(W_n)_{n \in \omega}$ of closed nowhere dense sets in $C_\ell(X,Y)$.

Replacing $W$ by a smaller set and applying Lemma 3.1 we can assume that $W$ is of basic form

$$W = [\kappa_0; U_0; F_0; u_0]$$

for some quadruple $(\kappa_0, U_0, F_0, u_0) \in Q$ such that $U_0 \subset O_0$. Here we assume that the base $B_Y$ of the topology of $Y$ coincides with the family $\tau_Y$ of all non-empty open sets in $Y$.

For any $n \in \omega$ and any finite sequence $s = (K_0, \ldots, K_n) \in \mathcal{K}(X)^{n+1}$ we shall define inductively a quadruple $(\kappa_s, U_s, F_s, u_s) \in Q$ such that for the sequence $t = (K_0, \ldots, K_{n-1})$ the following conditions are satisfied:

(a) $F_t \subset F_s$;
(b) $\kappa_t \cup K_n \subset \kappa_s \cup F_s$;
(c) $U_s \subset U_t \cap O_{n+1}$;
(d) for every $x \in F_t$ the set $u_s(x)$ is contained in the set $S_N(u_s|_0(x), \ldots, u_s|_{n}(x))$ where $s|_i := (K_0, \ldots, K_{i-1})$ for $i \leq n$;
(e) $[\kappa_s; U_s; F_s; u_s] \subset [K_n \setminus F_t; U_t \cap O_{n+1}] \cap [\kappa_t; U_t; F_t; u_t] \setminus M_{n+1}$.

Assume that for some $n \in \omega$ and all sequences $s \in \mathcal{K}(X)^n$ the function $f_s$ and quadruple $(\kappa_s, U_s, F_s, u_s) \in Q$ satisfying the inductive conditions (a)–(e) have been defined. Fix a sequence $s = (K_0, \ldots, K_n) \in \mathcal{K}(X)^{n+1}$ and consider the sequence $t = s|n = (K_0, \ldots, K_{n-1})$. For every $x \in X$ consider the non-empty open set

$$w_s(x) = S_N(u_s|_0(x), \ldots, u_s|_{n}(x)) \subset u_s|_{n}(x) = u_t(x).$$

The definition of $Q$ ensures that for any $x \in X \setminus F_t$ we have $u_s|_i(x) = Y$ for all $i \leq n$ and then $w_s(x) = Y$ by the choice of the strategy $S_N$. Then the quadruple $(\kappa_t, U_t, F_t, w_t)$ belongs to the family $Q$.

It is easy to see that the open set $[K_n \setminus F_t; U_t \cap O_{n+1}] \cap [\kappa_t; U_t; F_t; w_t] \subset C_\ell(X,Y)$ is not empty and hence contains some function $f_s \in C_\ell(X,Y)$. Since the closed set $M_{n+1}$ is nowhere dense in $C_\ell(X,Y)$, we can assume that $f_s \notin M_{n+1}$. Using Lemma 3.1 find a quadruple $(\kappa_s, U_s, F_s, u_s) \in Q$ such that

$$f_s \in [\kappa_s; U_s; F_s; u_s] \subset [K_n \setminus F_t; U_t \cap O_{n+1}] \cap [\kappa_t; U_t; F_t; w_t] \setminus M_{n+1}$$

and the conditions (a)–(e) are satisfied.

Now define a strategy $S_F : \mathcal{K}(X)^{<\omega} \to \mathcal{F}(\hat{X})$ of the player $F$ in the game $\mathcal{G}_F$ letting

$$S_F(K_0, \ldots, K_n) = F_s(K_0, \ldots, K_n) \setminus K_n = F_s \setminus K_n$$

for $s = (K_0, \ldots, K_n) \in \mathcal{K}(X)^{<\omega}$. By Theorem 2.1 the strategy $S_F$ is not winning. So there exists an infinite sequence $s = (K_n)_{n \in \omega} \in \mathcal{K}(X)^{<\omega}$ such that the family $\{S_F(s|n)\}_{n \in \omega}$ is discrete in $\mathcal{X}$. Then the set $\bigcup_{n \in \omega} S_F(s|n) \subset \mathcal{X}$ is closed in $\mathcal{X}$.

Consider the countable set $D := \bigcup_{n \in \omega} F_s|n \subset \mathcal{X}$. The inductive condition (e) ensures that $u_s|_n(x) \in S_N(u_s|_0(x), \ldots, u_s|_{n-1}(x))$
for every $n \in \mathbb{N}$. Since the strategy $S_\mathbb{N}$ is winning, the intersection $\bigcap_{n \in \omega} u_{s|n}(x)$ is not empty and hence it contains some point $f_\infty(x) \in Y$.

We claim that the function $f : X \to Y$ defined by

$$f(x) = \begin{cases} f_\infty(x) & \text{if } x \in D; \\ *_Y & \text{otherwise}; \end{cases}$$

is continuous.

It suffices to check that $f$ is continuous at each non-isolated point $x \in X'$. Since $(O_n)_{n \in \omega}$ is a neighborhood base at $*_Y$, for any $k \in \mathbb{N}$ it suffices to find a neighborhood $W_x \subset X$ of $x$ such that $f(W_x) \subset O_k$. Choose a neighborhood $W_x$ of $x$ which is disjoint with the closed set

$$\bigcup_{s \not\in W} S_f(s) = \bigcup_{s \not\in W} S_f(s)^+.$$  

We claim that $f(w) \in O_k$ for all $w \in W$. This is clear if $w \not\in D$. So assume that $w \in D$ and find the smallest number $m \in \omega$ such that $w \in F_{s|m}$. Then $w \not\in F_{s|\omega}$. The choice of the neighborhood $W_x$ ensures that $m \geq k$ and

$$w \in F_{s|m} \setminus S_f(s|m) = F_{s|m} \setminus (F_{s|m} \setminus K_{m-1}) = F_{s|m} \cap K_{m-1}.$$  

Since $w \not\in F_{s|\omega}$, the inductive conditions (e) ensures that

$$f(w) = f_\infty(w) \in u_{s|m}(x) \subset O_m \subset O_k.$$  

So $f$ is continuous and belongs to $C'_k(X,Y)$.

Next, we show that $f \in [\kappa_{s|n}; U_{s|n}|F_{s|n}; u_{s|n}]$ for every $n \in \mathbb{N}$. Given any $x \in X$, we should prove that $f(x) \in U_{s|n}$ if $x \in \kappa_{s|n}$ and $f \in u_{s|n}(x)$ if $x \in F_{s|n}$. In the latter case the inclusion follows from $f(x) = f_\infty(x) \in u_{s|n}(x)$. So, we assume that $x \in \kappa_{s|n}$. If $x \not\in D$, then $f(x) = *_Y \in U_{s|n}$ and we are done. So, we assume that $x \in D$ and hence $x \in F_{s|\omega} \setminus F_{s|m}$ for some $m \in \omega$. It follows from $x \in \kappa_{s|n} \subset X \setminus F_{s|n}$ that $m \geq n$.

The inductive conditions (b), (e) and (c) ensure that $x \in \kappa_{s|n} \setminus F_{s|m} \subset \kappa_{s|m}$ and

$$f(x) = f_\infty(x) \in u_{s|(m+1)}(x) \subset U_{s|m} \subset U_{s|n}.$$  

Therefore,

$$f \in \bigcap_{n \in \omega} [\kappa_{s|n}; U_{s|n}|F_{s|n}; u_{s|n}] \subset \bigcap_{n \in \omega} (W \setminus M_n) = \emptyset$$

and this is a desired contradiction, showing that the function space $C'_k(X,Y)$ is Baire. □

Lemmas 4.2 and 4.3 imply the main result of this section.

**Theorem 4.4.** Assume that $(Y, *_Y)$ is a *-admissible *-first-countable Choquet pointed space. For any topological space $X$ the following conditions are equivalent:

1. $C'_k(X,Y)$ is Baire;
2. $C'_k(X,Y)$ is not meager;
3. $X$ has DMOP.

For a topological space $X$ consider the function space

$$C'_k(X,2) := \{ f \in C_k(X,2) : f(X') \subset \{0\} \} \subset C_k(X,2)$$

where the ordinal $2 = \{0,1\}$ is endowed with the discrete topology.

**Definition 4.5.** A topological space $X$ is defined to be

- $C'_k$-meager if the function space $C'_k(X,2)$ is meager;
- $C'_k$-Baire if the function space $C'_k(X,2)$ is Baire;
\begin{itemize}
  \item $C'_k$-Choquet if the function space $C'_k(X, 2)$ is Choquet.
\end{itemize}

Theorem \ref{thm:compact-open} implies the following function space characterization of DMOP.

**Corollary 4.6.** For a topological space $X$ the following conditions are equivalent:

1. $X$ has DMOP;
2. $X$ is $C'_k$-Baire;
3. $X$ is not $C'_k$-meager.

5. A FUNCTION SPACE CHARACTERIZATION OF WDMOP

In this section we shall prove that a topological space has WDMOP if and only if it is $C'_k$-Choquet.

First, we prove a counterpart of Lemma \ref{lem:compact-open}.

**Lemma 5.1.** Let $X$ be a topological space and $Y$ be a *-admissible pointed topological space. If the function space $C'_k(X, Y)$ is Choquet, then the space $X$ has WDMOP.

**Proof.** Assume that the function space $C'_k(X, Y)$ is Choquet, which means that the player $N$ has a winning strategy $S_N$ in the Choquet game $\mathcal{G}_E(C'_k(X, Y))$. The strategy $S_N$ assigns to each finite sequence of non-empty open sets $(U_0, \ldots, U_n)$ in $C'_k(X, Y)$ a non-empty open set $S_N(U_0, \ldots, U_n) \subseteq U_n$ so that for any infinite sequence $(U_n)_{n \in \omega}$ of non-empty open sets in $C'_k(X, Y)$ the intersection $\bigcap_{n \in \omega} S_N(U_0, \ldots, U_n)$ is not empty if $U_n \subseteq S_N(U_0, \ldots, U_{n-1})$ for every $n \in \mathbb{N}$. Since the player $E$ starts the game $\mathcal{G}_E(C'_k(X, Y))$, it convenient to assume that $S_N(\emptyset) = C'_k(X, Y)$ for the empty sequence of zero length.

By our assumption, the pointed space $Y$ is *-admissible. So, there exists a non-empty open set $W \subseteq Y$ whose closure does not contain the distinguished point $*Y$ of $Y$. By the definition of the compact-open topology, for any non-empty open set $U \subseteq C'_k(X, Y)$ there exists a compact set $\kappa(U) \subseteq X$ such that for any finite set $F \subseteq \overline{X} \setminus \kappa(U)$ the intersection $U \cap \overline{F} \cap W$ is not empty.

Now we define a strategy $S_K$ of the player $K$ in the game $\mathcal{G}_K(X)$. This strategy assigns to any sequence $(F_0, \ldots, F_{n-1})$ of finite sets in $\overline{X}$ the compact set

$$K_n := \kappa(U_n) \cup \bigcup_{i < n} F_i$$

where $(U_0, \ldots, U_n)$ is a decreasing sequence of non-empty open sets in $C'_k(X, Y)$, defined recursively by $U_0 = C'_k(X, Y)$ and

$$U_{i+1} = \begin{cases} S_N(U_0 \cap [F_0; W], \ldots, U_i \cap [F_i; W]), & \text{if } U_i \cap [F_i; W] \neq \emptyset; \\ U_i, & \text{otherwise;} \end{cases} \quad \text{for } i < n.$$

We claim that this strategy $S_K$ of player $K$ is winning. Let $(F_n)_{n \in \omega}$ be any sequence of finite subsets of $\overline{X}$ such that for any $n \in \omega$ the set $F_n$ is disjoint with the compact set $S_K(F_0, \ldots, F_{n-1})$. Let $U_0 = C'_k(X, Y)$ and $(U_n)_{n \in \omega}$ be the sequence of non-empty open sets in $C'_k(X, Y)$ defined by the recursive formula

$$U_{n+1} = S_N(U_0 \cap [F_0; W], \ldots, U_n \cap [F_n; W]) \subseteq U_n$$

for $n \in \omega$. Let us show that for every $n \in \omega$ the intersection $U_n \cap [F_n; W]$ is not empty, which means that the set $U_{n+1}$ is well-defined. The intersection $U_0 \cap [F_0; W]$ is not empty as $U_0 = C'_k(X, Y)$. Assume that for some $n \in \mathbb{N}$ we have proved that the set $U_{n-1} \cap [F_{n-1}; W]$ is not empty. Then the set $U_n = S_N(U_0 \cap [F_0; W], \ldots, U_{n-1} \cap [F_{n-1}; W])$ is not empty and by the definition of the strategy $S_K$, we get $\kappa(U_n) \subseteq S_K(F_0, \ldots, F_{n-1})$. Since

$$F_n \cap \kappa(U_n) \subseteq F_n \cap S_K(F_0, \ldots, F_{n-1}) = \emptyset,$$

the definition of the compact set $\kappa(U_n)$ ensures that the open set $U_n \cap [F_n; W]$ is not empty.
Now observe that for the sequence of non-empty open sets \((U_n \cap [F_n;W])_{n \in \omega}\) we have the inclusion
\[ U_{n+1} \cap [F_{n+1};W] \subset U_{n+1} = S_N(U_0 \cap [F_0;W], \ldots, U_n \cap [F_n;W]) \]
for all \(n \in \omega\). The winning property of the strategy \(S_N\) ensures that the intersection \(\bigcap_{n \in \omega} U_n \cap [F_n;W]\) is not empty and hence contains some function \(f \in C'_k(X,Y)\).

It follows that the family \((F_n)_{n \in \mathbb{N}}\) is disjoint and \(f \in \bigcap_{n \in \mathbb{N}} [F_n;W]\). Since \(f(\bigcup_{n \in \omega} F_n) \subset W\) and \(\ast_Y \not\in W\), the continuity of \(f\) guarantees that the closure of the set \(\bigcup_{n=1}^{\infty} F_n\) does not intersect \(X'\) and hence the disjoint family \(\{F_n\}_{n \in \mathbb{N}}\) is discrete in \(X\).

For every \(n \in \omega\) let \(K_n = S_K(F_0, \ldots, F_{n-1})\) and observe that
\[ K_1, F_1, K_2, F_2, \ldots \]
is the sequence of the moves of the players \(K\) and \(F\) in the game \(G_{KF}(X)\), where the player \(K\) plays according to the strategy \(S_K\) and eventually wins as the family \(\{F_n\}_{n \in \mathbb{N}}\) is discrete. So, the strategy \(S_K\) is winning and the space \(X\) has WDMOP.

Now we prove a “Choquet” version of Lemma 4.3

**Lemma 5.2.** Let \(Y\) be a \(*\)-first-countable pointed Choquet space. If a topological space \(X\) has WDMOP, then the function space \(C'_k(X,Y)\) is Choquet.

**Proof.** By our assumption, the distinguished point \(\ast_Y\) of \(Y\) has a countable neighborhood base \(\{O_n\}_{n \in \omega}\) such that \(O_{n+1} \subset O_n\) for all \(n \in \omega\). Let \(\tau_Y\) be the family of all non-empty open sets in \(Y\). The family \(\tau_Y\) plays the role of the base \(B_Y\) in the definition of the family \(\mathcal{Q}\) of quadruples from Section 3.

Since the space \(Y\) is Choquet, the player \(N\) has a winning strategy \(S_N : \tau_Y^{<\omega} \to \tau_Y\) in the Choquet game \(G_{EN}(Y)\). The strategy \(S_N\) is a function assigning to each finite sequence \((U_0, \ldots, U_n) \in \tau_Y^{<\omega}\) of non-empty open sets in \(Y\) a non-empty open set \(S_N(U_0, \ldots, U_n) \subset U_n\) such that for any infinite sequence \((U_n)_{n \in \omega} \in \tau_Y^{\omega}\) the intersection \(\bigcap_{n \in \omega} S_N(U_0, \ldots, U_n)\) is not empty if \(U_n \subset S_N(U_0, \ldots, U_{n-1})\) for any \(n \in \mathbb{N}\). Since the player \(E\) starts the game \(G_{EN}(Y)\), it is convenient to assume that \(S_N(\emptyset) = Y\) for the empty sequence \(\emptyset \in \tau_Y^0\). We can also assume that \(S_N(U_0, \ldots, U_{n-1}) = Y\) if \(U_i = Y\) for all \(i < n\).

By Lemma 2.6, the player \(K\) has a strategy \(S_K\) in the game \(G_{KF}\) such that for any sequence \((F_n)_{n \in \omega}\) of finite subsets of \(X\) the indexed family \(\{F_0 \setminus S_K(F_0, \ldots, F_{n-1})\}_{n \in \omega}\) is discrete in \(X\).

Let \(\tau\) be the family of all non-empty open sets in \(C'_k(X,Y)\). Let \(\kappa_0 = F_0 = \emptyset\), \(U_0 = Y\) and \(u_0 : X \to \{Y\} \subset \tau_Y\) be the constant function.

For any \(n \in \omega\) and any finite sequence \(s = (W_0, \ldots, W_n) \in \tau^{n+1}\) we shall define inductively a compact set \(K_s \subset X\) and two quadruples \((\bar{\kappa}_s, \bar{U}_s, \bar{F}_s, \bar{u}_s), (\kappa_s, U_s, F_s, u_s)\) in \(\mathcal{Q}\) such that the following conditions are satisfied:

(a) \([\bar{\kappa}_s; \bar{U}_s; \bar{F}_s; \bar{u}_s] \subset W_n\);
(b) \(\bar{U}_s \subset O_{n+1} \cap U_{s[n]}\), \(F_{s[n]} \subset \bar{F}_s\), and \(\kappa_{s[n]} \subset \bar{\kappa}_s \cup \bar{F}_s\);
(c) \(K_s = S_K(\bar{F}_s; 0, \ldots, F_{s[n+1]});\)
(d) \(F_s = \bar{F}_s, \bar{U}_s = U_s\) and \(\kappa_s = \kappa_s \cup K_s \setminus \bar{F}_s;\)
(e) \(u_s(x) = S_N(\bar{u}_s[0](x), \ldots, \bar{u}_s[s[n+1]](x)) \subset u_s([s[n+1]](x)) = \bar{u}_s(x)\) for every \(x \in \bar{X}\);
(f) \([\kappa_s; U_s; F_s; u_s] \subset [\bar{\kappa}_s; \bar{U}_s; \bar{F}_s; \bar{u}_s] \subset W_n\).

Assume that for some \(n \in \omega\) and all sequences \(s \in \tau^n\) a compact set \(K_s\), and quadruples \((\bar{\kappa}_s, \bar{U}_s, \bar{F}_s, \bar{u}_s), (\kappa_s, U_s, F_s, u_s)\) satisfying the inductive conditions (a)–(f) have been defined. Fix a sequence \(s = (W_0, \ldots, W_n) \in \tau^{n+1}\).

Using Lemma 3.1, find a quadruple \((\tilde{\kappa}_s, \tilde{U}_s, \tilde{F}_s, \tilde{u}_s) \in \mathcal{Q}\) satisfying the conditions (a), (b). Define the compact set \(K_s\) by the formula (c). Finally define the quadruple \((\kappa_s, U_s, F_s, u_s)\) by the conditions (d) and (e). The conditions (a), (d), (e) imply the condition (f). This completes the inductive step.
After completing the inductive construction, define a strategy $S_N : \tau^{<\omega} \to \tau$ of the player $N$ in the Choquet game $G_{\mathbb{E}}(C_k'(X, Y))$ letting $S_N(s) = [\kappa_s : U_s | F_s; u_s]$ for any sequence $s = (W_0, \ldots, W_n) \in \tau^{<\omega}$. The inductive condition (f) guarantees that $S_N(W_0, \ldots, W_n) \subset W_n$.

We claim that the strategy $S_N$ is winning. Fix an infinite sequence $s = (W_n)_{n \in \omega} \in \tau^{\omega}$ such that $W_n \subset S_N(W_0, \ldots, W_{n-1}) = S_N(s|n)$ for every $n \in \mathbb{N}$. The condition (f) of the inductive construction ensures that for every $n \in \mathbb{N}$ we have the inclusions

$$[\kappa_s|n; U_s|n| F_s|n; u_s|n] \subset [\kappa_s|n; \tilde{U}_s|n| \tilde{F}_s|n; \tilde{u}_s|n] \subset W_{n-1} \subset S_N(s|(n-1)) = [\kappa_s|(n-1); U_s|(n-1)| F_s|(n-1); u_s|(n-1)],$$

which imply the inclusions

$$u_s|n(x) \subset \tilde{u}_s|n(x) \subset u_s|(n-1)(x) = S_N(u_s|0(x), \ldots, \tilde{u}_s|(n-1)(x))$$

holding for every $x \in \tilde{X}$.

Since the strategy $S_N$ is winning, for every $x \in \tilde{X}$ the intersection

$$\bigcap_{n \in \omega} \tilde{u}_s|n(x) = \bigcap_{n \in \omega} u_s|n(x)$$

is not empty and hence contains some point $f_\infty(x) \in Y$.

The choice of the strategy $S_K$ guarantees that the indexed family

$$\{F_s|n \setminus S_K(F_s|0, \ldots, F_s|(n-1))\}_{n \in \omega}$$

is discrete in $X$.

Consider the countable set $D = \bigcup_{n \in \omega} F_s|n \subset \tilde{X}$. We claim that the function $f : X \to Y$ defined by

$$f(x) = \begin{cases} f_\infty(x) & \text{if } x \in D \\ 0 & \text{otherwise} \end{cases}$$

is continuous.

It suffices to check that $f$ is continuous at each non-isolated point $x \in X$. Since $(O_n)_{n \in \omega}$ is a neighborhood base at $\ast Y$, for any $k \in \mathbb{N}$ it suffices to find a neighborhood $W_x \subset X$ of $x$ such that $f(W_x) \subset O_k$. Choose a neighborhood $W_x$ of $x$ which is disjoint with the set

$$\bigcup_{i < k} F_s|(i+1) \cup \bigcup_{i=k}^\infty (F_s|(i+1) \setminus S_K(F_s|0, \ldots, F_s|i)).$$

We claim that $f(z) \in O_k$ for all $w \in W_x$. This is clear if $w \notin D$. So assume that $w \in D$ and find the smallest number $m \in \omega$ such that $w \in F_s|(m+1)$. Then $w \notin F_s|m$. The choice of the neighborhood $W_x$ ensures that $m \geq k$ and

$$w \in S_K(F_s|0, \ldots, F_s|m) \setminus F_s|m = K_s|m \setminus F_s|m \subset \kappa_s|m.$$  

The inductive conditions (b), (d) and the inclusion

$$[\kappa_s|(m+1); U_s|(m+1)| F_s|(m+1); u_s|(m+1)] \subset [\kappa_s|m; U_s|m| F_s|m; u_s|m]$$

ensure that $f(w) = f_\infty(w) \in u_s|(m+1)(w) \subset U_s|m \subset O_m \subset O_k$. So $f$ is continuous and belongs to $C_k'(X, Y)$.

Repeating the argument from the proof of Lemma 4.3, we can show that

$$f \in \bigcap_{n \in \omega} [\kappa_s|n; U_s|n| F_s|n; u_s|n] = \bigcap_{n \in \omega} W_{n-1},$$

so the intersection $\bigcap_{n \in \omega} W_n$ is not empty and the strategy $S_N$ is winning, which means that the function space $C_k'(X, Y)$ is Choquet.
Lemmas 5.1 and 5.2 imply the main result of this section.

**Theorem 5.3.** Assume that $Y$ is a $\ast$-admissible $\ast$-first-countable pointed Choquet space. For any topological space $X$ the following conditions are equivalent:

1. $C_k^\prime(X,Y)$ is Choquet;
2. $X$ has WDMOP;
3. $X$ is $C_k^\prime$-Choquet.

6. **Introducing properties of $X$, responsible for the metrizability of $C_k^\prime(X,Y)$**

In this section we introduce some properties of a topological space $X$, which will be used in subsequent sections for characterizing pairs $X,Y$ for which the function spaces $C_k^\prime(X,Y)$ are metrizable, (almost) complete-metrizable or (almost) Polish.

**Definition 6.1.** A topological space $X$ is called

- $\kappa$-space if for any non-closed subset $D \subset X$ of $X$ there exists a compact set $K \subset X$ such that $K \cap D$ is infinite;
- $\omega$-$\kappa$-space if for any non-closed countable subset $D \subset X$ of $X$ there exists a compact set $K \subset X$ such that $K \cap D$ is infinite;
- $\kappa_\omega$-space if there exists a countable family $\{K_n\}_{n \in \omega}$ of compact subsets of $X$ such that $X \subset \bigcup_{n \in \omega} K_n$ and for any non-closed set $D \subset X$ of $X$ there exists $n \in \omega$ such that $D \cap K_n$ is infinite;
- hemi-$\kappa_\omega$-space if there exists a countable family $\{K_n\}_{n \in \omega}$ of compact subsets of $X$ such that for any compact set $K \subset X$ there exists $n \in \omega$ such that $K \cap X \subset K_n$.

These properties relate as follows

![Diagram](https://via.placeholder.com/150)

Non-trivial implications in this diagram are proved in the following proposition.

**Proposition 6.2.** (1) Each $\kappa_\omega$-space is both a $\kappa$-space and a hemi-$\kappa_\omega$-space.

2. A topological space is a $\kappa_\omega$-space if and only if it is a hemi-$\kappa_\omega$-space and an $\omega$-$\kappa$-space.

3. Each $\kappa_\omega$-space has WDMOP.

4. Each space with WDMOP is an $\omega$-$\kappa$-space.

5. A topological space is a $\kappa_\omega$-space if and only if it is a hemi-$\kappa_\omega$-space with WDMOP.

**Proof.** 1. The definitions imply that each $\kappa_\omega$-space $X$ is a $\kappa$-space. To show that $X$ is a hemi-$\kappa_\omega$-space, take any sequence $(K_n)_{n \in \omega}$ of compact subsets of $X$, witnessing that $X$ is a $\kappa_\omega$-space. The hemi-$\kappa_\omega$-space property of $X$ will follow as soon as we check that for any compact set $K \subset X$ there exists $n \in \omega$ such that $K \cap X \subset \bigcup_{i \leq n} K_i$. Assuming no such $n$ exists, for every $n \in \omega$ we can choose a point $x_n \in K \cap X \setminus \bigcup_{i \leq n} K_i$. Since $X \subset \bigcup_{n \in \omega} K_n$, the set $D = \{x_n\}_{n \in \omega} \subset K$ is infinite and hence has an accumulation point $x' \in K \cap X' \subset K \setminus D$ in the compact space $K$. Therefore, $D$ is not closed in $X$ and the choice of the sequence $(K_n)_{n \in \omega}$ yields a number $n \in \omega$ such that $D \cap K_n$ is infinite. On
the other hand, \(D \cap K_n \subset \{x_0, \ldots, x_{n-1}\}\) by the choice of the sequence \((x_n)_{n \in \omega}\). This contradiction shows that \(X\) is a hemi-\(\kappa_\omega\)-space.

2. Assume that \(X\) is a hemi-\(\kappa_\omega\)-space and an \(\omega\)-\(\kappa\)-space. Let \((K_n)_{n \in \omega}\) be a sequence witnessing that \(X\) is a hemi-\(\kappa_\omega\)-space. We claim that this sequence witness that \(X\) is \(\kappa_\omega\)-space. Since each singleton \(\{x\} \subset X\) is compact, there exists \(n \in \omega\) such that \(\{x\} = \{x\} \cap X \subset K_n\), which implies that \(\hat{X} \subset \bigcup_{n \in \omega} K_n\). Given a non-closed subset \(D \subset X\), it remains to find \(n \in \omega\) such that \(K_n \cap D\) is infinite. To derive a contradiction, assume that \(D \cap K_n\) is finite for every \(n \in \omega\). Then the set \(D = D \cap \hat{X} = \bigcup_{n \in \omega} D \cap K_n\) is countable. Since \(X\) is a \(\omega\)-\(\kappa\)-space, there exists a compact set \(K \subset X\) such that \(K \cap D\) is infinite. The choice of the sequence \((K_n)_{n \in \omega}\) ensures that \(\hat{X} \cap K \subset K_n\) for some \(n \in \omega\). Since \(D \cap K = D \cap \hat{X} \cap K \subset D \cap K_n\), the set \(D \cap K_n\) is infinite, which contradicts our assumption. This contradiction completes the proof.

3. Assume that \(X\) is a \(\kappa_\omega\)-space and let \((K_n)_{n \in \omega}\) be a sequence of compact sets witnessing this fact. It is easy to see that the function \(S_K\) assigning to any finite sequence \((F_0, \ldots, F_n) \in \mathcal{F}(X)^{<\omega}\) the compact set

\[
S_K(F_0, \ldots, F_n) = \bigcup_{i \leq n} (F_i \cup K_i)
\]

is a winning strategy of the player \(K\) in the game \(G_{KF}(X)\).

4. Assume that \(X\) has WDMOP and let \(A \subset \hat{X}\) be a non-closed countable set in \(X\). To derive a contradiction, assume that every compact set \(K \subset X\) the intersection \(A \cap K \) is finite.

By Theorem 5.3, the function space \(C_k^r(X, 2)\) is Choquet. The space \(C_k^r(X, 2)\) is a subspace of the space \(F_k(X, 2) = \{f \in 2^X : f(X') \subset \{0\}\},\) endowed with the compact-open topology. A neighborhood base of this topology at a function \(f \in F_k(X, 2)\) consists of the sets \(O_K(f) = \{g \in F_k(X, 2) : g|K = f|K\}\) where \(K\) runs over compact subsets of \(X\).

Endow the doubleton \(2 = \{0, 1\}\) with a group operation \(\oplus\) in which 0 is a neutral element. This group operation induces a continuous group operation on the space \(F_k(X, 2)\).

Let \(\chi \in F_k'(X, 2)\) be the function defined by \(\chi^{-1}(1) = A\). Since the set \(A\) is not closed in \(X\), the function \(\chi\) is discontinuous. On the other hand, this function belongs to the closure \(\overline{C_k'(X, 2)}\) of the subgroup \(C_k'(X, 2)\) in \(F_k'(X, 2)\) as for any compact set \(K \subset X\) the intersection \(K \cap A \subset \hat{X}\) is finite and hence the restriction \(\chi|K\) is continuous.

By \([\text{1}]\) Theorem 3, each Choquet topological group \(H\) is \(G_\delta\)-dense in its Raikov completion \(\overline{H}\), which means that \(H\) has non-empty intersection with any non-empty \(G_\delta\)-subset of \(\overline{H}\).

This fact implies that the Choquet subgroup \(C_k'(X, 2)\) is \(G_\delta\)-dense in its closure \(\overline{C_k'(X, 2)}\). Then the \(G_\delta\)-set \(G = \{f \in C_k'(X, 2) : f|A = \chi|A\}\) has common point \(f\) with the subgroup \(C_k'(X, 2)\). Now get a desired contradiction:

\[
\emptyset \neq X' \cap A \subset f^{-1}(0) \cap f^{-1}(1) = f^{-1}(0) \cap f^{-1}(1) = \emptyset.
\]

5. If \(X\) is a \(\kappa_\omega\)-space, then it is a hemi-\(\kappa_\omega\)-spaces with WDMOP by the statements (1) and (3), proved above. If \(X\) is a hemi-\(\kappa\)-space with WDMOP, then \(X\) is a \(\kappa_\omega\)-space by the statements (2) and (4), proved above.

For topological spaces with countable set of isolated points we can prove a bit more.

**Theorem 6.3.** Let \(X\) be a topological space with countable set \(\hat{X}\) of isolated points. Then

1. \(X\) is a \(\kappa\)-space if and only if it is a \(\omega\)-\(\kappa\)-space;
2. \(X\) is a \(\kappa_\omega\)-space if and only if \(X\) has WDMOP.

**Proof.** 1. The first equivalent follows directly from the definitions.

2. If \(X\) is a \(\kappa_\omega\)-space, then it has WDMOP by Proposition [\text{6,2}(3)]. Now assume that \(X\) has WDMOP and fix a winning strategy \(S_N : \mathcal{F}(X)^{<\omega} \rightarrow K(X)\) of the player \(K\) in the game \(G_{KF}(X)\). Here \(\mathcal{F}(\hat{X})\) is
the family of finite subsets of $\hat{X}$. Since the set $\hat{X}$ is countable, the family $K = \mathcal{F}(\hat{X}) \cup \{S_K(s) : s \in \mathcal{F}(\hat{X})\}^{\omega}$ is countable, too.

Let $K = \{K_n\}_{n \in \omega}$ be an enumeration of the family $K$. We claim that the sequence $(K_{\leq n})_{n \in \omega}$ of the compact sets $K_{\leq n} = \bigcup_{i \leq n} K_i$ witnesses that $X$ is a $\mathcal{k}_\omega$-space.

Given a non-closed subset $D \subset \hat{X}$, we should show that the intersection $D \cap K_{\leq n}$ is infinite for some $n \in \omega$. By Proposition 6.24, there is a compact set $K \subset X$ such that the intersection $K \cap D$ is infinite. We claim that there exists $n \in \omega$ such that $K \cap \hat{X} \subset K_{\leq n}$. To derive a contradiction, assume that for every $n \in \omega$ the complement $K \cap \hat{X} \setminus K_{\leq n}$ contains some point $x_n \in \hat{X}$. Since each subsequence of the sequence $(x_n)_{n \in \omega}$ has an accumulation point in the compact space $K \subset X$, the player $F$ has a winning strategy by choosing his moves in the set $\{x_n\}_{n \in \omega}$. But this is not possible as player $K$ has a winning strategy in the game $\mathcal{G}_{KF}(X)$. This contradiction shows that $K \cap D \subset K \cap \hat{X} \subset K_{\leq n}$ for some $n \in \omega$. Then the intersection $D \cap K_{\leq n} \supset D \cap K$ is infinite.

Theorem 6.3 implies that for any topological space $X$ with countable set $\hat{X}$ we have the following equivalences and implications:

\begin{align*}
\text{hemi-}$\mathcal{k}_\omega$-space & $\iff$ $\mathcal{k}_\omega$-space $\iff$ WDMOP $\iff$ $\omega$-$\mathcal{k}$-space.
\end{align*}

7. The metrizability of the function spaces $C^*_k(X,Y)$

Lemma 7.1. Let $Y$ be a pointed topological space whose distinguished point $\ast_Y$ has a neighborhood $U_\ast \neq Y$. A topological space $X$ is a hemi-$\mathcal{k}_\omega$-space if the function space $C^*_k(X,Y)$ is first-countable at the constant function $c : X \to \{\ast_Y\}$.

Proof. Being first-countable at $c$, the function space $C^*_k(X,Y)$ has a countable neighborhood base $\{O_n\}_{n \in \omega}$ at $c$. By the definition of the compact-open topology on $C^*_k(X,Y)$, for every $n \in \omega$ we can find a compact subset $K_n \subset X$ and a neighborhood $V_n \subset Y$ of $\ast_Y$ such that $c \in [K_n; V_n] \subset O_n$. We claim that the sequence $(K_n)_{n \in \omega}$ witnesses that $X$ is a hemi-$\mathcal{k}_\omega$-space.

Given a compact set $K \subset X$, we should find $n \in \omega$ such that $K \cap \hat{X} \subset K_n$. By our assumption, the distinguished point $\ast_Y$ has an open neighborhood $U_\ast \neq Y$. Consider the open neighborhood $[K; U_\ast] \subset C^*_k(X,Y)$ of the constant function $c$ and find $n \in \omega$ with $O_n \subset [K; U_\ast]$. Then the inclusion $[K_n; V_n] \subset O_n \subset [K; U_\ast]$ implies $K \cap \hat{X} \subset K_n$. Indeed, assuming that $K \cap \hat{X} \not\subset K_n$, we can find an isolated point $x \in K \cap \hat{X} \setminus K_n$. Fix any point $y \in Y \setminus U_\ast$ and consider the map $\chi_x : X \to \{\ast_Y, y\} \subset Y$ defined by $\chi_x^{-1}(y) = x$. Since $x$ is isolated in $X$, the map $\chi_x$ is continuous. Taking into account that $\chi_x(K_n) \subset \{\ast_Y\}$ and $\chi_x(x) = y \notin U_\ast$, we conclude that $\chi_x \in [K_n; V_n] \setminus [K; U_\ast]$, which contradicts the choice of $n$ (with $[K_n; V_n] \subset O_n \subset [K; U_\ast]$).

For the proof of Corollary 7.5, we shall need the following refined version of Lemma 7.1.

Lemma 7.2. Let $Y$ be a pointed topological space such that each point $y \in Y$ has a neighborhood $O_y$ which is not dense in $Y$. A topological space $X$ is a hemi-$\mathcal{k}_\omega$-space if the function space $C^*_k(X,Y)$ contains a dense subspace $D$, which is first-countable at some point $\delta \in D$.

Proof. Let $\{O_n\}_{n \in \omega}$ be a countable neighborhood base at the point $\delta$ of the space $D$. By Lemma 6.1, for every $n \in \omega$ we can find a quadruple $(K_n, U_n, F_n, u_n) \in \mathcal{Q}$ such that $\delta \in D \cap [K_n; U_n] \subset O_n$. We claim that the sequence $(K_n \cup F_n)_{n \in \omega}$ witnesses that $X$ is a hemi-$\mathcal{k}_\omega$-space.

Given a compact set $K \subset X$, we should find $n \in \omega$ such that $K \cap \hat{X} \subset K_n \cup F_n$. By our assumption, each point $y \in Y$ has a neighborhood $O_y$, which is not dense in $Y$. By Lemma 6.1, there exists a quadruple $(\kappa, V, E, v) \in \mathcal{Q}$ such that $\delta \in [\kappa; V[E; v]], K \subset \kappa \cup E, V \subset O_{\ast_Y}$ and $v(x) \in O_{\delta(x)}$ for all $x \in E$. Since $\{O_n\}_{n \in \omega}$ is a neighborhood base of the space $D$ at $\delta$, there exists $n \in \omega$ such that $O_n \subset [\kappa; V[E; v]]$. 

We claim that \( K \cap \hat{X} \subset K_n \cup F_n \). Assuming that \( K \cap \hat{X} \not\subset K_n \cup F_n \), we can find a point \( x \in \hat{X} \cap K \setminus (K_n \cup F_n) \). Choose a non-empty open set \( W_x \subset Y \) such that \( W_x \cap O_{\delta(x)} = \emptyset \) if \( x \in E \) and \( W_x \cap V = \emptyset \) if \( x \notin E \). By the density of the set \( D \) in \( C_1^*(X,Y) \), the intersection \( D \cap [K_n;U_n,F_n;u_n] \cap \{x\};W_x \) contains some function \( f \). Then \( f \in D \cap [K_n;U_n,F_n;u_n] \subset O_n \subset [\kappa;V,E;v] \). Since \( x \in K \subset \kappa \cup \mathbb{U} \), the inclusion \( f \in [\kappa;V,E;v] \) implies that \( f(x) \in V \) if \( x \in \kappa \) and \( f(x) \in v(x) \in O_{\delta(x)} \) if \( x \in E \). In both cases, we get \( f(x) \notin W_x \), which contradicts the inclusion \( f \in \{x\};W_x \). This contradiction shows that \( K \cap \hat{X} \subset K_n \cup F_n \). Therefore, the sequence of compact sets \( (K_n \cup F_n)_{n \in \omega} \) witnesses that \( X \) is a hemi-\( \kappa \)\(-\)space. \( \square \)

**Lemma 7.3.** For any hemi-\( \kappa \)\(-\)space \( X \) and any metrizable space \( Y \), the function space \( C_1^*(X,Y) \) is metrizable.

**Proof.** Let \( (K_n)_{n \in \omega} \) be a sequence of compact sets witnessing that \( X \) is a hemi-\( \kappa \)\(-\)space. Replacing each compact set \( K_n \) by the union \( \bigcup_{i \leq n} K_i \), we can assume that \( K_n \subset K_{n+1} \) for all \( n \in \omega \).

Let \( d \) be any metric generating the topology of the space \( Y \). For a point \( y \in Y \) and \( \varepsilon > 0 \) denote by \( B(y;\varepsilon) := \{x \in Y : d(y,x) < \varepsilon\} \) the open \( \varepsilon \)\(-\)ball centered at \( y \).

We claim that the metric \( \rho \) on \( C_1^*(X,Y) \) defined by the formula

\[
\rho(f,g) = \max_{n \in \omega} \{\frac{1}{2^n}, \max_{x \in K_n} d(f(x),g(x))\}
\]

for \( f,g \in C_1^*(X,Y) \) generates the compact-open topology of the function space \( C_1^*(X,Y) \).

By [14, 8.2.7], the compact-open topology on \( C_1^*(X,Y) \) coincides with the topology of uniform convergence on compacta, which implies that the metric \( \rho \) is continuous. It remains to check that for any compact set \( K \subset X \) and any open set \( U \subset Y \) the subbasic open set \( [K;U] := \{f \in C_1^*(X,Y) : f(K) \subset U\} \) is open in the metric space \( (C_1^*(X,Y), \rho) \). Fix any function \( f \in [K;U] \). By the choice of the sequence \( (K_n)_{n \in \omega} \), there exists a number \( n \in \omega \) such that \( K \cap \hat{X} \subset K_n \). By the compactness of the set \( f(K) \subset U \), there exists a positive real number \( \varepsilon < \frac{1}{2^n} \) such that \( B(f(K);\varepsilon) \subset U \), where \( B(f(K);\varepsilon) = \bigcup_{y \in f(K)} B(y;\varepsilon) \).

We claim that any function \( g \in C_1^*(X,Y) \) with \( \rho(f,g) < \varepsilon \) belongs to \( [K;U] \). Given any \( z \in K \), we should check that \( g(z) \in U \). If \( z \in X' \), then \( g(z) = \ast_Y = f(z) \in f(K) \subset U \). If \( z \notin X' \), then \( z \in K \cap \hat{X} \subset K_n \) and then \( \varepsilon < \frac{1}{2^n} \) and

\[
\min_{x \in K_n} \{\frac{1}{2^n}, d(f(x),g(x))\} \leq \rho(f,g) < \varepsilon
\]

imply that \( d(g(z),f(z)) \leq \max_{x \in K_n} d(f(x),g(x)) < \varepsilon \) and finally, \( g(z) \in B(f(z);\varepsilon) \subset B(f(K),\varepsilon) \subset U \).

**Theorem 7.4.** For a pointed topological space \( Y \) containing more than one point, and a topological space \( X \) containing an isolated point, the following conditions are equivalent:

1. the function space \( C_1^*(X,Y) \) is metrizable;
2. the space \( Y \) is metrizable and \( X \) is a hemi-\( \kappa \)\(-\)space.

**Proof.** The implication (2) \( \Rightarrow \) (1) was proved in Lemma 7.3.

To prove that (1) \( \Rightarrow \) (2), assume that the function space \( C_1^*(X,Y) \) is metrizable. By Lemma 7.1 \( X \) is a hemi-\( \kappa \)\(-\)space. By our assumption, the space \( X \) contains an isolated point \( x \in X \). It is easy to see that the function

\[
H : C_1^*(X,Y) \to Y \times C_0^*(X \setminus \{x\}, Y), \ f \mapsto (f(x), f|X \setminus \{x\})
\]

is a homeomorphism. Then the space \( Y \) is metrizable, being homeomorphic to a subspace the metrizable space \( Y \times C_0^*(X \setminus \{x\}, Y) = H(C_1^*(X,Y)) \).

Theorem 7.4 and Lemma 7.2 imply the following corollary.
Corollary 7.5. For a pointed metrizable space $Y$ containing more than one point, and a topological space $X$ containing an isolated point, the following conditions are equivalent:

1. the function space $C_k^\prime(X,Y)$ is metrizable;
2. the function space $C_k^\prime(X,Y)$ contains a dense first-countable subspace;
3. $X$ is a hemi-$\kappa_\omega$-space.

A topological space $X$ is defined to have countable spread if each discrete subspace in $X$ is at most countable.

Lemma 7.6. Let $Y$ be a pointed space such that the singleton $\{*\}$ is not dense in $Y$. A topological space $X$ has countable set $\hat{X}$ of isolated points if the function space $C_k^\prime(X,Y)$ has countable spread.

Proof. By our assumption, the space $Y$ contains a point $y \in Y \setminus \{*\}$. For every $x \in \hat{X}$ let $\delta_x : X \to \{*, y\}$ be the unique function with $\delta_x^{-1}(y) = \{x\}$. Observe that the set $U_x := \{f \in C_k^\prime(X,Y) : f(x) \notin \{*\}\}$ is an open neighborhood of $\delta_x$ in $C_k^\prime(X,Y)$. Consider the subspace $D = \{\delta_x : x \in \hat{X}\}$ and observe that it is discrete, because $U_x \cap D = \{\delta_x\}$ for all $x \in \hat{X}$. Since the space $C_k^\prime(X,Y)$ has countable spread, the discrete subspace $D$ has at most countable cardinality $\omega \geq |D| = |\hat{X}|$.

Lemma 7.7. Let $Y$ be a $*$-admissible pointed space and $X$ be a hemi-$\kappa_\omega$-space. If the function space $C_k^\prime(X,Y)$ has countable cellularity, then $\hat{X}$ is at most countable.

Proof. Assuming that $\hat{X}$ is uncountable and taking into account that $X$ is a hemi-$\kappa_\omega$-space, we can find a compact set $K \subset X$ such that $\hat{X} \cap K$ is uncountable.

Since the pointed space $Y$ is $*$-admissible, the point $*Y$ has a neighborhood $U_*$ which is disjoint with some non-empty open set $V \subset Y$. For every $x \in K \cap \hat{X}$ consider the open set

$$W_x := \{\{x\}; V \cap [K \setminus \{x\}; U_*]\}$$

in $C_k^\prime(X,Y)$ and observe that $(W_x)_{x \in \hat{X} \cap K}$ is an uncountable family of pairwise disjoint open sets in $C_k^\prime(X,Y)$, which means that $C_k^\prime(X,Y)$ has uncountable cellularity.

Lemma 7.8. For any separable pointed space $Y$ and any topological space $X$ with countable set $\hat{X}$ of isolated point, the function space $C_k^\prime(X,Y)$ is separable.

Proof. Fix a countable dense set $D \subset Y$, containing the distinguished point $*Y$ of $Y$. It is easy to see that the countable set

$$\{f \in C_k^\prime(X,Y) : f(X) \subset D, \ |X \setminus f^{-1}(\{*\})| < \omega\}$$

is dense in $C_k^\prime(X,Y)$, which means that the function space $C_k^\prime(X,Y)$ is separable.

Theorem 7.9. For a pointed topological space $Y$ containing more than one point, and a topological space $X$ containing an isolated point, the following conditions are equivalent:

1. the function space $C_k^\prime(X,Y)$ is metrizable and separable;
2. $Y$ is a metrizable separable space, and $X$ is a hemi-$\kappa_\omega$-space with countable set $\hat{X}$ of isolated points.

Proof. To prove that $1) \Rightarrow 2)$, assume that the function space $C_k^\prime(X,Y)$ is separable and metrizable. By Theorem 7.4, the space $Y$ is metrizable and $X$ is a hemi-$\kappa_\omega$-space. By Lemma 7.6, the set $\hat{X}$ is countable. Since the space $X$ has an isolated point $x$, the space $Y$ is separable, being the image of the separable space $C_k^\prime(X,Y)$ under the continuous map $\delta_x : C_k^\prime(X,Y) \to Y$, $\delta_x : f \mapsto f(x)$.

$2) \Rightarrow 1)$ Assume that $Y$ is a separable metrizable space, and $X$ is a hemi-$\kappa_\omega$-space with countable set $\hat{X}$ of isolated points. By Theorem 7.4, the function space $C_k^\prime(X,Y)$ is metrizable and by Lemma 7.8, it is separable.
Theorem 7.9 and Lemmas 7.2, 7.7 imply the following characterization.

**Corollary 7.10.** For a pointed metrizable space \( Y \neq \{ \ast_Y \} \) and a topological space \( X \) containing an isolated point, the following conditions are equivalent:

1. The function space \( C'_k(X,Y) \) is metrizable and separable;
2. \( C'_k(X,Y) \) contains a dense separable first-countable subspace;
3. \( Y \) is separable and \( X \) is a hemi-\( k_\omega \)-space with countable set \( \hat{X} \) of isolated points.

8. The (almost) complete-metrizability of the function spaces \( C'_k(X,Y) \)

In this section we characterize function spaces \( C'_k(X,Y) \) which are (almost) complete-metrizable or (almost) Polish.

**Theorem 8.1.** For a pointed topological space \( Y \) containing more than one point, and a topological space \( X \) containing an isolated point, the following conditions are equivalent:

1. The function space \( C'_k(X,Y) \) is complete-metrizable;
2. The spaces \( Y \) and \( C'_k(X,2) \) are complete-metrizable;
3. The space \( X \) is complete-metrizable and \( Y \) is a \( k_\omega \)-space.

**Proof.** (1) \( \Rightarrow \) (2). Assume that the function space \( C'_k(X,Y) \) is complete-metrizable. Take any isolated point \( x \in \hat{X} \) and observe that the map

\[
H : C'_k(X,Y) \to Y \times C'_k(X \setminus \{ x \}, Y), \quad H : f \mapsto (f(x), f|X \setminus \{ x \}),
\]

is a homeomorphism. Then the space \( Y \) is complete-metrizable, being homeomorphic to a closed subspace of the complete-metrizable space \( C'_k(X,Y) \).

Next, choose any point \( y \in Y \setminus \{ \ast_Y \} \) and consider the bijective map \( e : \{ 0,1 \} \to \{ \ast_Y, y \} \) such that \( e(0) = \ast_Y \) and \( e(1) = y \). The map \( e \) induces a closed topological embedding \( e^* : C'_k(X,2) \to C'_k(X,Y) \), \( e^* : f \mapsto e \circ f \). Then the space \( C'_k(X,2) \) is complete-metrizable, as a closed subspace of the complete-metrizable space \( C'_k(X,Y) \).

(2) \( \Rightarrow \) (3). Assume that the space \( C'_k(X,2) \) is complete-metrizable. We can endow the doubleton \( 2 = \{ 0,1 \} \) with the group operation having 0 as its neutral element and consider \( C'_k(X,2) \) as an abelian metrizable topological group. Being complete-metrizable, this topological group is complete in its uniformity (uniform convergence on compacta).

Since the complete-metrizable space \( C'_k(X,2) \) is first-countable, there exist an increasing sequence \((K_n)_{n \in \omega}\) of compact subsets of \( X \) such that the sets

\[
[K_n; \{ 0 \}] := \{ f \in C'_k(X,Y) : f(K_n) \subset \{ 0 \} \}, \quad n \in \omega,
\]

form a neighborhood base at the constant function \( e : X \to \{ 0 \} \), which is the neutral element of the group \( C'_k(X,2) \).

We claim that the sequence \((K_n)_{n \in \omega}\) witnesses that \( X \) is a \( k_\omega \)-space. First observe that for any compact set \( K \subset X \) we can find \( n \in \omega \) such that \( [K_n; \{ 0 \}] \subset [K; \{ 0 \}] \). The latter inclusion implies that \( K \cap \hat{X} \subset K_n \). So, \( \hat{X} \subset \bigcup_{n \in \omega} K_n \).

Next, we shall prove that a subset \( D \subset \hat{X} \) is closed in \( X \) if \( D \cap K_n \) is finite for every \( n \). For every \( n \in \omega \) consider the continuous function \( f_n \in C'_k(X,2) \) defined by \( f_n^{-1}(1) = D \cap K_n \). We claim that the sequence \((f_n)_{n \in \omega}\) is Cauchy in the uniformity of uniform convergence on compacta. Given a compact set \( K \subset X \), we should find \( n \in \omega \) such that \( f_m[K] = f_n[K] \) for all \( m \geq n \). Consider the open neighborhood \([K; \{ 0 \}] \subset C'_k(X,2)\) of the constant function \( e \) and find \( n \in \omega \) such that \([K_n; \{ 0 \}] \subset [K; \{ 0 \}]\). The latter inclusion implies \( K \cap \hat{X} \subset K_n \). Then for any \( m \geq n \) we get \( D \cap K_n \cap K = D \cap K = D \cap K_n \cap K \), which implies \( f_m[K] = f_n[K] \). By the completeness of the group \( C'_k(X,2) \), the Cauchy sequence \((f_n)_{n \in \omega}\) converges to a continuous function \( f_\infty : X \to 2 \). The only
choice for this limit is the characteristic function of the set \( D \), which implies that the set \( D = f_\infty^{-1}(1) \) is closed in \( X \) by the continuity of \( f_\infty \).

(3) \( \Rightarrow \) (1) Assume that the space \( Y \) is complete-metrizable and \( X \) is a \( \kappa_\omega \)-space. Let \((K_n)_{n \in \omega}\) be an increasing sequence of compact sets, witnessing that \( X \) is a \( \kappa_\omega \)-space. For every \( n \in \omega \), the compactness of \( K_n \) and the complete-metrizability of \( Y \) imply the complete-metrizability of the function space \( C_k(K_n,Y) \). Then the product \( \prod_{n \in \omega} C_k(K_n,Y) \) is complete-metrizable as well. The proof of Lemma 7.3 implies that the map 
\[
\delta : C'_k(X,Y) \to \prod_{n \in \omega} C_k(K_n,Y), \quad \delta : f \mapsto (f|K_n)_{n \in \omega},
\]
is a topological embedding.

We claim that the image \( \delta(C'_k(X,Y)) \) is a closed subset of \( \prod_{k \in \omega} C_k(X_n,Y) \). Take any element \((f_n)_{n \in \omega} \in \delta(C'_k(X,Y)) \subset \prod_{n \in \omega} C_k(K_n,Y) \). It follows from \((f_n)_{n \in \omega} \in \delta(C'_k(X,Y))\) that for every \( n \leq m \) the restriction \( f_m|K_n \) coincides with the function \( f_n \) and moreover \( f_n(K_n \cap X') \subset \{ \ast_Y \} \). So, we can define a function \( f : X \to Y \) by \( f(x') \in \{ \ast_Y \} \) and \( f|K_n = f_n \) for every \( n \in \omega \). We claim that the function \( f \) is continuous. It suffices to prove the continuity of \( f \) at each non-isolated point \( x' \in X' \). Assuming that \( f \) is discontinuous at \( x' \), we can find an open neighborhood \( U_x \subset Y \) of the point \( \ast_Y = f(x') \) whose preimage \( f^{-1}(U_x) \) is not a neighborhood of \( x' \), which means that the set \( D = X \setminus f^{-1}(U_x) \subset X \) contains the point \( x' \) in its closure.

Since the sequence \((K_n)_{n \in \omega}\) witnesses that \( X \) is a \( \kappa_\omega \)-space, for some \( n \in \omega \) the intersection \( D \cap K_n \) is infinite. On the other hand, the set 
\[
D \cap K_n = K_n \setminus f^{-1}(U_x) = K_n \setminus (f|K_n)^{-1}(U_x) = K_n \setminus f_n^{-1}(U_x) \subset K_n \cap X
\]
is closed in \( K_n \) by the continuity of \( f_n \). Being a closed discrete subset of the compact space \( K_n \), the set \( D \cap K_n \) is finite, which contradicts the choice of \( n \). This contradiction shows that the function \( f \) is continuous and hence \((f_n)_{n \in \omega} = (f|K_n)_{n \in \omega} = \delta(f) \in \delta(C'_k(X,Y)) \). So, the set \( \overline{\delta(C'_k(X,Y))} \) is closed in \( \prod_{n \in \omega} C_k(K_n,Y) \) and the space \( C'_k(X,Y) \) is complete-metrizable, being homeomorphic to the closed subspace \( \overline{\delta(C'_k(X,Y))} \) of the complete-metrizable space \( \prod_{n \in \omega} C_k(K_n,Y) \).

Theorems 8.3 and 7.9 imply the following characterization of Polish function spaces \( C'_k(X,Y) \).

**Theorem 8.2.** For a pointed topological space \( Y \) that contains more than one point, and a topological space \( X \) containing an isolated point, the following conditions are equivalent:

1. the function space \( C'_k(X,Y) \) is Polish;
2. the spaces \( Y \) and \( C'_k(X,2) \) are Polish;
3. the space \( Y \) is Polish and \( X \) is a \( \kappa_\omega \)-space with countable set \( X \) of isolated points.

**Theorem 8.3.** For a topological space \( X \) and a pointed Polish space \( Y \) the function space \( C'_k(X,Y) \) is Polish if and only if \( C'_k(X,Y) \) is a Choquet space with countable spread.

**Proof.** The “only if” part is trivial. To prove the “if” part, assume that the function space \( C'_k(X,Y) \) is Choquet and has countable spread. If \( Y = \{ \ast_Y \} \), then the space \( C'_k(X,Y) \) is Polish, being a singleton. So, we assume that \( Y \neq \{ \ast_Y \} \). By Theorem 5.3, the space \( X \) has WDMOP and by Lemma 7.5, the set \( X \) is at most countable. By Theorem 6.3, \( X \) is a \( \kappa_\omega \)-space and by Theorem 8.2, the function space \( C'_k(X,Y) \) is Polish.

Finally, we characterize pairs \( X,Y \) for which the function space \( C'_k(X,Y) \) is almost complete-metrizable or almost Polish.

**Theorem 8.4.** For a pointed metrizable space \( Y \neq \{ \ast_Y \} \) and a topological space \( X \) with \( X \neq \emptyset \), the following conditions are equivalent:

1. the function space \( C'_k(X,Y) \) is almost complete-metrizable;
(2) the space \( Y \) is almost complete-metrizable and \( X \) is a \( \kappa_\omega \)-space.

**Proof.** (1) \( \Rightarrow \) (2) If \( C'_k(X,Y) \) is almost complete-metrizable, then it is Choquet and by Lemma 5.1 \( X \) has WDMOP. By our assumption, the space \( X \) contains an isolated point \( x \). It is easy to see that the map \( \delta_x : C'_k(X,Y) \to Y, \delta_x : f \mapsto f(x) \), is an open surjection, so the space \( Y \) is Choquet. Being metrizable, the Choquet space \( Y \) is almost complete-metrizable, see [23 8.17] or [4 7.3]. Being almost complete-metrizable, the space \( C'_k(X,Y) \) contains a dense first-countable subspace. By Lemma 7.2 \( X \) is a hemi-\( \kappa_\omega \)-space. By Proposition 6.2(5), \( X \) is a \( \kappa_\omega \)-space.

(2) \( \Rightarrow \) (1) Assume that the space \( Y \) is almost complete-metrizable and \( X \) is a \( \kappa_\omega \)-space. Then \( Y \) contains a dense complete-metrizable subspace \( M \subset Y \). Let \( \tilde{Y} \) be any complete-metrizable space containing \( Y \) as a dense subspace. By [23, 3.11], the complete-metrizable space \( M \) is a \( G_\delta \)-set in \( \tilde{Y} \). Since the singleton \( \{ \ast_Y \} \) is a \( G_\delta \)-subset of \( \tilde{Y} \), the union \( M \cup \{ \ast_Y \} \) is a \( G_\delta \)-set in \( \tilde{Y} \). By [23, 3.11], the space \( M \cup \{ \ast_Y \} \) is complete-metrizable. Replacing \( M \) by \( M \cup \{ \ast_Y \} \), we can assume that \( \ast_Y \in M \).

By Theorem 8.1 the function space \( C'_k(X,M) \) is complete-metrizable. Since \( C'_k(X,M) \) is a dense subspace in \( C'_k(X,Y) \), the space \( C'_k(X,Y) \) is almost complete-metrizable. \( \square \)

**Theorem 8.5.** For a pointed metrizable space \( Y \neq \{ \ast_Y \} \) and a topological space \( X \) with \( X \neq \emptyset \), the following conditions are equivalent:

(1) the function space \( C'_k(X,Y) \) is almost Polish;

(2) the space \( Y \) is almost Polish and \( X \) is a \( \kappa_\omega \)-space with countable set of isolated points.

**Proof.** (1) \( \Rightarrow \) (2) If \( C'_k(X,Y) \) is almost Polish, then \( C'_k(X,Y) \) is almost complete-metrizable. Moreover, \( C'_k(X,Y) \) is separable and hence has countable cellularity. By Theorem 8.4 the space \( Y \) is almost complete-metrizable and \( X \) is a \( \kappa_\omega \)-space. By Lemma 7.7 the \( \kappa_\omega \)-space has countable set \( X \) of isolated points.

It remains to prove that the space \( Y \) is almost Polish. We already know that \( Y \) is almost complete-metrizable and hence \( Y \) contains a dense complete-metrizable subset \( M \subset Y \). By [23, 3.11], \( M \) is a \( G_\delta \)-set in \( Y \). By our assumption, the space \( X \) contains an isolated point \( x \). It is easy to see that the map \( \delta_x : C'_k(X,Y) \to Y, \delta_x : f \mapsto f(x) \), is an open surjection. This implies that the preimage \( \delta_x^{-1}(M) \) is a dense \( G_\delta \)-set in \( C'_k(X,Y) \). By our assumption, the space \( C'_k(X,Y) \) is almost Polish and hence contains a dense Polish subspace \( P \). Since the complement \( C'_k(X,Y) \setminus \delta_x^{-1}(M) \) is a meager \( F_\sigma \)-set in \( C'_k(X,Y) \), the set \( P \setminus \delta_x^{-1}(M) \) is a meager \( F_\sigma \)-set in the Polish space \( P \) and its complement \( P \cap \delta_x^{-1}(M) \) is a dense \( G_\delta \)-set in \( P \), by the classical Baire Theorem. Then \( \delta_x(P \cap \delta_x^{-1}(M)) \) is a dense separable set in \( M \), which implies that the complete-metrizable space \( M \) is separable and hence Polish. Consequently, the space \( Y \) is almost Polish.

(2) \( \Rightarrow \) (1) Assume that the space \( Y \) is almost Polish and \( X \) is a \( \kappa_\omega \)-space with countable set of isolated points. Let \( P \) be a dense Polish subspace in \( Y \). Replacing \( P \) by \( P \cup \{ \ast_Y \} \), we can assume that \( \ast_Y \in P \). By Theorem 8.2 the function space \( C'_k(X,P) \) is Polish. Since the subspace \( C'_k(X,P) \) is dense in \( C'_k(X,Y) \), the space \( C'_k(X,Y) \) is almost Polish. \( \square \)

**9. Countable networks in function spaces \( C'_k(X,Y) \)**

A family \( \mathcal{N} \) of subsets of a topological space \( X \) is called

- a network if for any open set \( U \subset X \) and point \( x \in U \) there exists a set \( N \in \mathcal{N} \) such that \( x \in N \subset U \);
- a \( cs^* \)-network if for any open set \( U \subset X \) and sequence \( \{ x_n \}_{n \in \omega} \subset X \) that converges to a point \( x_\infty \in U \) there is a set \( N \in \mathcal{N} \) such that \( x_\infty \in N \subset U \) and \( N \) contains infinitely many points \( x_n, n \in \omega \);
- a \( k \)-network if for any open set \( U \subset X \) and compact subset \( K \subset U \) there exists a finite subfamily \( \mathcal{F} \subset \mathcal{N} \) such that \( K \subset \bigcup \mathcal{F} \subset U \);
• a $\kappa$-network if for any compact set $K \subset X$ and any closed subset $D \subset X \setminus K$ of $X$ there exists a finite subfamily $F \subset \mathcal{N}$ such that $K \cap X \subset \bigcup F \subset X \setminus D$.

It is clear that for any family $\mathcal{N}$ we have the implications

$$k\text{-network} \Rightarrow cs^*\text{-network} \Rightarrow \text{network}.$$ 

In case of countable networks, we have the following equivalence, which was proved for Hausdorff spaces in [20].

**Lemma 9.1.** Each countable $cs^*$-network is a $k$-network.

**Proof.** Let $\mathcal{N}$ be a countable $cs^*$-network for a topological space $X$. To prove that it is a $k$-network, fix an open set $W \subset X$ and a compact subset $K \subset W$. Consider the countable subfamily $\mathcal{N}_W := \{N \in \mathcal{N} : N \subset W\}$ and write it as $\mathcal{N}_W = \{N_k\}_{k \in \omega}$. We claim that $K \subset \bigcup_{i \leq n} N_i$ for some $n \in \omega$.

In the opposite case we can find a sequence of points $x_n \in K \setminus \bigcup_{i \leq n} N_i$. Observe that the family $\{N \cap K : N \in \mathcal{N}\}$ is a network for the compact space $K$, which implies that $K$ is hereditarily Lindelöf and hence sequentially compact, by a result of Alas and Wilson [11]. This allows us to find an increasing number sequence $(k_n)_{n \in \omega}$ such that the subsequence $(x_{k_n})_{n \in \omega}$ of the sequence $(x_k)_{k \in \omega}$ converges to some point $x_\infty \in K$. Since $\mathcal{N}$ is a $cs^*$-network, there exists a set $N \in \mathcal{N}$ such that $N \subset W$ and the set $\Omega = \{n \in \omega : x_{k_n} \in N\}$ is infinite. Since $N \in \mathcal{N}_W$, there exists a number $m \in \omega$ such that $N = N_m$ and then $x_i \notin N_m = N$ for all $i \geq m$. In particular, $\Omega \cap \{n \in \omega : k_n < m\}$ is finite, which contradicts the choice of $N$. This contradiction shows that $K \subset \bigcup_{i \leq n} N_i \subset W$ for some $n$, which means that the family $\mathcal{N}$ is a $k$-network. \hfill $\square$

Now we prove some results on networks in the function spaces $C_k^r(X,Y)$.

**Lemma 9.2.** Let $Y$ be a pointed topological space whose distinguished point $*Y$ has a neighborhood $U_* \neq Y$. A topological space $X$ has a countable $\kappa$-network if the function space $C_k^r(X,Y)$ has a countable network.

**Proof.** Let $\mathcal{N}$ be a countable network of the function space $C_k^r(X,Y)$. For every $N \in \mathcal{N}$ let $N^* = \{x \in X : N \subset \{x\}; U_*\}$, where $\{x\}; U_* = \{f \in C_k^r(X,Y) : f(x) \in U_*\}$. We claim that the countable family $\mathcal{N}^* = \{N^* : N \in \mathcal{N}\}$ is a $\kappa$-network for the space $X$.

Given a compact set $K \subset X$ and a closed subset $D \subset X \setminus K$ of $X$, it suffices to find a set $\mathcal{N} \subset \mathcal{N}^*$ such that $K \subset \mathcal{N}^* \subset X \setminus D$. Fix any point $y \in Y \setminus U_*$. Taking into account that the closed set $D \subset X$ consists of isolated points of $X$, we conclude that $D$ is clopen in $X$. Then its characteristic function $\chi : X \rightarrow \{*Y, y\}$ defined by $\chi(y) = D$ is continuous and the set $[K, U_*]$ is a neighborhood of $\chi$ in $C_k^r(X,Y)$. Since $\mathcal{N}$ is a network of the topology of $C_k^r(X,Y)$, there exists $N \in \mathcal{N}$ such that $\chi \in N \subset [K, U_*]$. Observe that for each point $x \in K$ and any function $f \in N \subset [K, U^*]$ we get $f(x) \in U_*$ and hence $f \in \{x\}; U_*$, which implies that $N \subset \{x\}; U_*$ and hence $K \subset N^*$. On the other hand, for every $x \in D$ the function $\chi \not\in N$ does not belong to $\{x\}; U_*$, which implies $N \not\subset \{x\}; U_*$ and hence $x \notin N^*$. This means that $N^* \subset X \setminus D$. \hfill $\square$

**Lemma 9.3.** For any topological space $X$ with a countable $\kappa$-network and any pointed topological space $Y$ with a countable base, the function space $C_k^r(X,Y)$ has a countable network $\mathcal{N}$ (which is a countable $k$-network for $C_k^r(X,Y)$ if $X$ is a $\omega$-$\kappa$-space).

**Proof.** Let $N_X$ be a countable $\kappa$-network for the space $X$ and $B_Y$ be a countable base for the space $Y$. We lose no generality assuming that the family $N_X$ is closed under finite unions. By the definition of a $\kappa$-network, for any point $x \in \bar{X}$ there exists a set $N_x \in N_X$ such that $\{x\} \subset N_x \subset \bar{X} \setminus (X \setminus \{x\}) = \{x\}$, which implies that $N_x = \{x\}$ and hence the family $N_X$ contains all singletons $\{x\} \subset \bar{X}$. Now the countability of $N_X$ implies the countability of the set $\bar{X}$.

Consider the family $Q$ of quadruples $(N,U,F,u)$ where $N \in N_X$, $U \in B_Y$ is a neighborhood of $*Y$, $F \subset \bar{X} \setminus N$ is a finite subset, and $u : \bar{X} \rightarrow B_Y$ is a function such that $u(x) = Y$ for all $x \in \bar{X} \setminus F$. 

Since the sets $\mathcal{N}_X$, $\hat{X}$, $\mathcal{B}_Y$ are countable, so is the family $\mathcal{Q}'$. For any quadruple $(N,U,F,u) \in \mathcal{Q}'$ consider the set

$$[N;U|F;u] := \{f \in C^*_k(X,Y) : f(N) \subset U\} \cap \bigcap_{x \in F} \{f \in C^*_k(X,Y) : f(x) \in u(x)\}. $$

We claim that the countable family $\mathcal{N} = \{[N;U|F;u] : (N,U,F,u) \in \mathcal{Q}'\}$ is a network for the space $C^*_k(X,Y)$.

Take any function $f \in C^*_k(X,Y)$ and a neighborhood $O_f \subset C^*_k(X,Y)$ of $f$. By Lemma 3.1 there exists a quadruple $(K,U,F,u) \in \mathcal{Q}$ such that $f \in [K;U|F;u] \subset O_f$. It follows that $K \subset f^{-1}(U)$ and the set $D = X \setminus f^{-1}(U) \subset X \setminus K$ is closed in $X$. By the definition of the $\kappa$-network $\mathcal{N}_X$, there exists a finite subfamily $\mathcal{F} \subset \mathcal{N}_X$ such that $K \cap \hat{X} \subset \bigcup \mathcal{F} \subset X \setminus (D \cup F) = f^{-1}(U) \setminus \mathcal{F}$. Since the family $\mathcal{N}_X$ is closed under finite unions, the set $N = \bigcup \mathcal{F}$ belongs to $N$. Now it is easy to see that

$$f \in [N;U|F;u] \subset [K;U|F;u] \subset O_f.$$ 

Next, assuming that $X$ is an $\omega$-$\kappa$-space, we shall prove that the family $\mathcal{N}$ is a $cs^*$-network for $C^*_k(X,Y)$. Given any open set $W \subset C^*_k(X,Y)$ and a sequence of functions $\{f_n\}_{n \in \omega} \subset W$ that converge to a function $f_\infty \in W$, we should find a quadruple $(N,U,F,u) \in \mathcal{Q}'$ such that $f_\infty \in [N;U|F;u] \subset W$ and $[N;U|F;u]$ contains infinitely many functions $f_n$. By Lemma 3.1 there exists a quadruple $(K,U,F,u) \in \mathcal{Q}$ such that $f_\infty \in [K;U|F;u] \subset W$. Since $[K;U|F;u]$ is an open neighborhood of $f_\infty$, we can find a number $m \in \omega$ such that $\{f_n\}_{n \geq m} \subset [K;U|F;u]$. We claim that the set $D = \bigcup_{n \in \omega} f^{-1}_n(Y \setminus U) \subset \hat{X}$ is closed in $X$. Taking into account $X$ is an $\omega$-$\kappa$-space with countable set $\hat{X}$ of isolated points, we conclude that $X$ is a $\kappa$-space. Assuming that the set $D$ is not closed in $X$, we can find a compact set $C \subset X$ such that $C \cap \hat{X}$ is infinite. Observe that the set $C \setminus f^{-1}_\infty(U)$ is finite (being discrete and closed in the compact space $C$). So, we can replace $C$ by the compact set $C \cap f^{-1}_\infty(U)$ and assume that $f_\infty(C) \subset U$. Since the sequence $(f_n)_{n \in \omega}$ converges to $f_\infty \in [C;U]$, there exists a number $l \in \omega$ such that $f_n \in [C;U]$ for all $n \geq l$. Then $C \cap D = \bigcup_{n \leq l} C \setminus f^{-1}_n(U)$ is finite, being a finite union of finite sets $C \setminus f^{-1}_n(U)$. This contradiction finishes the proof of the closedness of the set $D$. Since $\mathcal{N}_X$ is a $\kappa$-network for $X$, there exists a set $N \in \mathcal{N}_X$ such that $K \cap \hat{X} \subset N \subset X \setminus (D \cup F)$. It is easy to see that

$$\{f_n\}_{n \geq m} \subset [N;U|F;u] \subset W.$$ 

Therefore, $\mathcal{N}$ is a countable $cs^*$-network. By Lemma 3.1 $\mathcal{N}$ is a countable $k$-network for the function space $C^*_k(X,Y)$. \hfill $\square$

Lemmas 9.2 and 9.3 imply the following characterization.

**Theorem 9.4.** For a pointed metrizable space $Y \neq \{\ast_Y\}$ and a topological space $X \neq X'$ the following conditions are equivalent:

1. the function space $C^*_k(X,Y)$ has a countable network;
2. the space $Y$ is separable and the space $X$ has a countable $k$-network.

If $X$ is a $\omega$-$\kappa$-space, then the conditions (1),(2) are equivalent to

3. the function space $C^*_k(X,Y)$ has a countable $k$-network.

**Problem 9.5.** Is the condition (3) in Theorem 9.4 equivalent to the conditions (1) and (2) for any topological space $X$?

**Theorem 9.6.** For a topological space $X$ with DMOP the following conditions are equivalent:

1. $X$ has a countable $\kappa$-network;
2. $X$ is a hemi-$\kappa$-space with countable set $\hat{X}$ of isolated points.
Proof. (2) ⇒ (1) First assume that \( X \) is a hemi-\( \kappa \)-\( \omega \)-space with countable set \( \hat{X} \) of isolated points, and find a sequence \((K_n)_{n\in\omega}\) of compact sets witnessing that \( X \) is hemi-\( \kappa \)-\( \omega \)-compact. Replacing each compact set \( K_n \) by the union \( \bigcup_{i\leq n} K_i \), we can assume that \( K_n \subset K_{n+1} \) for all \( n \in \omega \). We claim that the countable family
\[
\mathcal{N} = \{ K_n \setminus F : n \in \omega, F \in \mathcal{F}(\hat{X}) \}
\]
is a \( \kappa \)-network for \( X \). Indeed, for any compact set \( K \subset X \) and any closed subset \( D \subset \hat{X} \setminus K \) of \( X \), we can find \( n \in \omega \) with \( K \cap \hat{X} \subset K_n \) and observe that the set \( F = K_n \cap D \) is finite (being closed and discrete in the compact space \( K_n \)). Then the set \( N = K_n \setminus F \in \mathcal{N} \) has the required property: \( K \cap \hat{X} \subset N \subset X \setminus D \).

(1) ⇒ (2) Assume that the space \( X \) has a countable \( \kappa \)-network. By Corollary 4.16 and Theorem 9.4.1, the function space \( C_k^*(X,2) \) is Baire and has a countable network. By [8], each Baire topological group with countable network is metrizable and separable. Applying this result to the space \( C_k^*(X,2) \) (carrying a structure of a topological group), we conclude that \( C_k^*(X,2) \) is metrizable and separable. By Theorem 7.9, \( X \) is a hemi-\( \kappa \)-\( \omega \)-space with countable set of isolated points. \( \square \)

10. Recognizing \( \infty \)-meager function spaces \( C_k^*(X,Y) \)

Lemma 4.2 implies that for any \( \ast \)-admissible pointed space \( Y \) and any topological space \( X \) that does not satisfy DMOP the function space \( C_k^*(X,Y) \) is meager. In this section we prove that the meagerness of \( C_k^*(X,Y) \) in this result can be improved to a stronger property, called the \( \infty \)-meagerness.

Definition 10.1. A subset \( A \) of a topological space \( X \) is called

- \( \infty \)-dense in \( X \) if for any compact Hausdorff space \( K \) the subset \( C_k(K,A) = \{ f \in C_k(K,X) : f(K) \subset A \} \) is dense in \( C_k(K,X) \);
- \( \infty \)-codense in \( X \) if the complement \( X \setminus A \) is \( \infty \)-dense in \( X \);
- \( \infty \)-meager in \( X \) if \( A \) is contained in a countable union of closed \( \infty \)-codense subsets of \( X \).

A topological space \( X \) is \( \infty \)-meager if it is an \( \infty \)-meager subset of \( X \).

It is easy to see that each closed \( \infty \)-codense set is nowhere dense, so each \( \infty \)-meager set is meager. For future applications, \( \infty \)-meager spaces are important as the \( \infty \)-meagerness implies the \( \sigma Z \)-space property, which is a key ingredient in many characterization results of Infinite-Dimensional Topology, see [5], [7], [10], [12], [27], [28], [30].

Theorem 10.2. If a topological space \( X \) does not have DMOP, then for any \( \ast \)-admissible pointed topological space \((Y,\ast Y)\), the function space \( C_k^*(X,Y) \) is \( \infty \)-meager.

Proof. Assuming that \( X \) does not have DMOP and applying Lemma 4.2, we conclude that the function space \( C_k^*(X,2) \) is meager and hence can be written as the countable union \( C_k^*(X,2) = \bigcup_{n\in\omega} M_n \) of closed nowhere dense subsets \( M_n \) in \( C_k^*(X,2) \).

Since the pointed space \( Y \) is \( \ast \)-admissible, the point \( \ast Y \) has an open neighborhood \( U_\ast \subset Y \) which is not dense in \( Y \). Let \( \chi : Y \to \{0,1\} \) be a (unique) function such that \( \chi^{-1}(0) = U_\ast \). Observe that for every function \( f \in C_k^*(X,Y) \) the composition \( \chi \circ f : X \to 2 \) is a continuous function that belongs to the space \( C_k^*(X,2) \). So, for every \( n \in \omega \) we can consider the set
\[
Z_n := \{ f \in C_k^*(X,Y) : \chi \circ f \in M_n \} \subset C_k^*(X,Y)
\]
and its closure \( \bar{Z}_n \) in \( C_k^*(X,Y) \). It is clear that \( C_k^*(X,Y) = \bigcup_{n\in\omega} \bar{Z}_n \).

It remains to prove that each set \( \bar{Z}_n \) is \( \infty \)-codense in \( C_k^*(X,Y) \). Given any compact Hausdorff space \( K \), continuous map \( \mu : K \to C_k^*(X,Y) \), and neighborhood \( O_\mu \subset C_k(K,C_k^*(X,Y)) \) of \( \mu \), we need to find a continuous map \( \mu' \in O_\mu \) such that \( \mu'(K) \cap \bar{Z}_n = \emptyset \).

On the function space \( C_k^*(X,Y) \) consider the base \( \mathcal{B} \) of the topology, consisting of the sets \([K_1; U_1] \cap \cdots \cap [K_m; U_m]\) where \( K_1, \ldots, K_m \) are non-empty compact sets in \( X \) and \( U_1, \ldots, U_m \) are non-empty open sets in \( Y \).
We lose no generality assuming that the neighborhood $O_\mu$ is of basic form

$$O_\mu = \bigcap_{i=1}^m [K_i; B_i]$$

where $K_1, \ldots, K_m$ are compact sets in $K$ and $B_1, \ldots, B_m \in B$. For any $i \leq m$ find a finite family $\mathcal{K}_i$ of non-empty compact sets in $X$ and a function $u_i : \mathcal{K}_i \to \tau_Y$ to the topology $\tau_Y$ of $Y$ such that $B_i = \bigcap_{k \in \mathcal{K}_i} [k; u_i(\kappa)]$. Consider the compact set $C := \bigcup_{i=1}^m \bigcup \mathcal{K}_i$ and the open neighborhood

$$V_* = U_* \cap \left( \bigcap_{i=1}^m \{ u_i(\kappa) : \kappa \in \mathcal{K}_i, \ast_Y \in u_i(\kappa) \} \right)$$

of $\ast_Y$.

For a point $z \in K$ it will be convenient to denote the function $\mu(z) \in C'_k(X,Y)$ by $\mu_z$. Since $\mu_z(C \cap X') \subset \mu_z(X') \subset \{ \ast_Y \} \subset V_*$, there exists a finite set $F_z \subset C \cap X$ such that $\mu_z(C \setminus F_z) \subset V_*$. Then $[C \setminus F_z; V_*]$ is an open neighborhood of the function $\mu_z$ in $C'_k(X,Y)$ and by the continuity of the map $\mu : K \to C'_k(X,Y)$, there exists an open neighborhood $O_z \subset K$ of $z$ such that $\mu(O_z) \subset [C \setminus F_z; V_*]$. By the compactness of $K$, the open cover $\{ O_z : z \in K \}$ of $K$ has a finite subcover $\{ O_z : z \in E \}$. Then for the finite set $F = \bigcup_{z \in E} F_z \subset C \cap X$ we have the inclusion $\mu(K) \subset [C \setminus F; V_*]$.

Observe that the map

$$H : C'_k(X,2) \to 2^F \times C'_k(X \setminus F,2), \quad H : f \mapsto (f|F, f|X \setminus F),$$

is a homeomorphism. For every function $v : F \to 2$ consider the open embedding $e_v : C'_k(X \setminus F,2) \to C'_k(X,2)$ assigning to each function $f \in C'_k(X \setminus F,2)$ the function $e_v f : X \to 2$ such that $e_v f|X \setminus F = f$ and $e_v f|F = v$. The nowhere density of the closed set $M_n$ in $C'_k(X,2)$ implies that the finite union

$$M = \bigcup_{v \in 2^F} e_v^{-1}(M_n)$$

is closed and nowhere dense in $C'_k(X \setminus F,2)$. So, we can choose a function

$$h \in [C \setminus F; \{ 0 \}] \cap C'_k(X \setminus F,2) \setminus M$$

and using the definition of the compact-open topology on $C'_k(X \setminus F,2)$, find a compact set $A \subset X \setminus F$ such that the neighborhood

$$O_h := \{ f \in C'_k(X \setminus F,2) : f|A = h|A \}$$

does not intersect the set $M$. Replacing $A$ by $A \cup (C \setminus F)$, we can assume that $C \setminus F \subset A$. It follows that the set $\bigcup_{v \in 2^F} e_v(O_h)$ is disjoint with the set $M_n$. The continuity of the function $h \in C'_k(X \setminus F,2)$ implies that the set $A_* := A \cap h^{-1}(0)$ is compact and has finite complement $A \setminus A_*$. Moreover, since $h \in [C \setminus F; \{ 0 \}]$, the set $A_*$ contains $C \setminus F$, which implies that $C \cap (A \setminus A_*) = \emptyset$.

Since the set $U_*$ is not dense in $Y$, there exists a point $y \in Y \setminus \bar{U}_*$. Now consider the map $\mu' : K \to C'_k(X,Y)$ assigning to each $z \in K$ the function $\mu'_z : X \to Y$ defined by the formula

$$\mu'_z(x) = \begin{cases} \mu_z(x) & \text{if } x \in F \\ y & \text{if } x \in A \setminus A_* \\ \ast_Y & \text{otherwise}. \end{cases}$$

It is easy to see that the map $\mu' : K \to C'_k(X,Y)$ is continuous and $\mu' \in O_\mu$. It remains to prove that $\mu'(K) \cap Z_n = \emptyset$. Observe that the set $\mu'(K)$ is contained in the open subset $W := [A_*; V_*] \cap [A \setminus A_*; Y \setminus U_*]$ of $C'_k(X,Y)$. So, it suffices to prove that $W \cap Z_n = \emptyset$. Take any function $w \in W$ and consider the function $v = \chi \circ w|F$. The inclusion $w \in W$ yields the equality $\chi \circ w|A = h|A$ implying the inclusion $\chi \circ w|X \setminus F \subset O_h \subset C'_k(X \setminus F,2) \setminus M$ and $\chi \circ w = e_v(\chi \circ w|X \setminus F) \in e_v(O_h) \subset C'_k(X,2) \setminus M_n$ and hence $w \notin Z_n$. 

\$
\square
\$

\[\text{TARAS BANAKH AND LELIE WANG}\]
11. A dichotomy for analytic function spaces $C'_k(X,Y)$

A topological space $X$ is defined to be analytic if there exists a surjective continuous map $f : P \to X$, defined on some Polish space $P$. By an old result of Christensen [13, Theorem 5.4], each Baire analytic commutative topological group is Polish. We use this fact to prove the following dichotomy, which is the main result of this section.

**Theorem 11.1.** Let $Y$ be a pointed Polish space. If for some topological space $X$ the function space $C'_k(X,Y)$ is analytic, then $C'_k(X,Y)$ is either Polish or ∞-meager.

**Proof.** If $Y = \{*y\}$, then $C'_k(X,Y)$ is a Polish space, being a singleton. So, assume that $Y \neq \{*y\}$ and fix any point $y \in Y \setminus \{*y\}$. Let $e : 2 \to \{*y, y\}$ be the map defined by $e(0) = *y$ and $e(1) = y$. The map $e$ induces a closed embedding $e^* : C'_k(X,2) \to C'_k(X,Y)$, $e^* : f \mapsto e \circ f$. Now we see that the space $C'_k(X,2)$ is analytic, being homeomorphic to a closed subspace of the analytic space $C'_k(X,Y)$. Since $C'_k(X,2)$ carries a natural structure of a commutative topological group, we can apply Christensen’s Theorem 5.4 [13] and conclude that $C'_k(X',2)$ is either Polish or meager.

If $C'_k(X,2)$ is Polish, then by Theorem 8.2, $X$ is a $k$-space with countable set $X$ of isolated points. By Theorem 8.2, the function space $C'_k(X,Y)$ is Polish.

If $C'_k(X,2)$ is meager, then by Corollary 4.6, the space $X$ does not DMOP and by Theorem 10.2 the space $C'_k(X,Y)$ is ∞-meager.

Now we prove that the analyticity of a cosmic function space $C'_k(X,Y)$ is equivalent to the analyticity of the space

$$C'_p(X,Y) = \{f \in C_p(X,Y) : f(X') \subset \{*y\}\} \subset C_p(X,Y).$$

Here by $C_p(X,Y)$ we denote the space $C(X,Y)$, endowed with the topology of pointwise convergence. This topology coincides with the Tychonoff product topology, inherited from $Y^X$.

A topological space $X$ is called cosmic if it is regular and has a countable network. A function $f : X \to Y$ between two topological spaces is called Borel if for any open set $U \subset Y$ the preimage $f^{-1}(U)$ is a Borel subset of $X$.

**Proposition 11.2.** For a pointed regular topological space $Y$ and a topological space $X$, the following conditions are equivalent:

1. the space $C'_k(X,Y)$ is analytic;
2. the space $C'_p(X,Y)$ is analytic.

**Proof.** If $Y = \{*y\}$, then both spaces $C'_k(X,Y)$ and $C'_p(X,Y)$ are singletons and the conditions (1), (2) are satisfied and hence are equivalent. So, we assume that $Y \neq \{*y\}$. The regularity of the space $Y$ implies the regularity of the spaces $C_p(X,Y) \subset Y^X$ and $C_k(X,Y)$, see [14, 3.4.13].

(1) $\Rightarrow$ (2) If the space $C'_k(X,Y)$ is analytic, then it is cosmic, being a continuous image of some Polish space, see [18, 4.9]. The space $C'_p(X,Y)$ is analytic, being a continuous image of the analytic space $C'_k(X,Y)$.

(2) $\Rightarrow$ (1) Now assume that the space $C'_p(X,Y)$ is cosmic and the space $C'_p(X,Y)$ is analytic. Observe that the identity map $C'_k(X,Y) \to C'_p(X,Y)$ is continuous. We claim that the identity map $C'_p(X,Y) \to C'_k(X,Y)$ is Borel. Since the space $C'_k(X,Y)$ is hereditarily Lindelöf (being cosmic), it suffices to check that for any compact set $K \subset X$ and any open set the subbasic open set $[K; U] \subset C_k(X,Y)$ is Borel in $C'_p(X,Y)$. This is trivially true if $[K; U]$ is empty. So we assume that $[K; U] \neq \emptyset$.

The space $C'_k(X,Y)$, being cosmic, has countable spread. Applying Lemma 7.6 we conclude that the set $X$ is at most countable.
If $K \cap X' = \emptyset$, then $K$ is finite and the set $[K; U]$ is open in $C'_p(X, Y)$. If $K \cap X' \neq \emptyset$, then $[K; U] \neq \emptyset$ implies $x \in U$ and

$$[K; U] = \bigcap_{x \in X \cap K} \{x\} \cup U$$

is a $G_\delta$-set in $C'_p(X, Y)$.

Therefore, the identity map $C'_p(X, Y) \to C'_k(X, Y)$ is Borel. By [S], a cosmic space is analytic if and only if it is a Borel image of a Polish space. This characterization implies that the cosmic space $C'_k(X, Y)$ is analytic, being a Borel image of the analytic space $C'_p(X, Y)$. \hfill\Box

12. **The Interplay between the Function Spaces $C'_k(X, Y)$ and $C'_k(X/X', Y)$**

Given a non-discrete topological space $X$, by $X/X'$ we denote the quotient topological space (with the set $X'$, collapsed to the point $\{X'\} \in X/X' := (X \setminus X') \cup \{X'\}$) and observe that $X/X'$ is a Hausdorff space with a unique non-isolated point $\{X'\}$. So, $(X/X')'$ is a singleton.

For every pointed topological space $(X, \ast)$, the quotient map $q : X \to X/X'$ induces a continuous bijective map

$$q^* : C'_k(X/X', Y) \to C'_k(X', Y), \quad q^* : f \mapsto f \circ q.$$

This map is a homeomorphism if the quotient map $q$ is *compact-covering* in the sense that any compact set $K \subset X/X'$ is contained in the image $q(C)$ of some compact set $C \subset X$. In this case the investigation of the function space $C'_k(X, Y)$ can be reduced to studying the function space $C'_k(X/X', Y)$ over the space $X/X'$ with a unique non-isolated point.

However in many natural situations the quotient map $q : X \to X/X'$ is not compact-covering. In particular, it is not compact-covering for the space $X = \omega_1$ of all countable ordinals endowed with the order topology. The reason is that the space $\omega_1$ is pseudocompact but not compact. We recall that a Tychonoff space $X$ is *pseudocompact* if each continuous real-valued function on $X$ is bounded.

**Example 12.1.** If $X$ is a non-compact pseudocompact space $X$ with dense set $\hat{X}$ of isolated points, then

1. the space $X$ does not have DMOP;
2. the space $X/X'$ is compact;
3. the function space $C'_k(X, 2)$ is meager but $C'_k(X/X', 2)$ is discrete;
4. the quotient map $q : X \to X/X'$ is not compact-covering.

**Proof.** 1. To see that $X$ does not have DMOP, observe that the family of singletons $\{\{x\} : x \in \hat{X}\}$ is moving off, but contains no infinite discrete subfamily (otherwise $X$ would admit an unbounded real-valued continuous function).

2. Since the pseudocompactness is preserved by continuous images, the quotient space $Q = X/X'$ is pseudocompact. Since $Q$ has only one non-isolated point, it is compact. Indeed, given any open cover $\mathcal{U}$ of $Q$, we can find an open set $U \in \mathcal{U}$ containing the unique non-isolated point $q$ of the space $X$. We claim that the complement $Q \setminus U$ is finite. Otherwise, we could find a sequence $(x_n)_{n \in \omega}$ of pairwise distinct points in $Q \setminus U$ and define a continuous unbounded function $f : Z \to \mathbb{R}$ by the formula

$$f(x) = \begin{cases} n, & \text{if } x = x_n \text{ for some } n \in \mathbb{N}; \\ 0, & \text{otherwise}. \end{cases}$$

But the existence of such unbounded continuous function $f$ contradicts the pseudocompactness of the space $Q = q(X)$.

3. Since the space $X$ does not have DMOP, the function space $C'_k(X, 2)$ is meager by Theorem [4.4]. Since the space $X/X'$ is compact, the function space $C'_k(X/X', 2)$ is discrete.
4. Assuming that the quotient map \( q : X \to X/X' = Q \) is compact-covering, we could find a compact set \( K \subset X \) such that \( q(K) = Q \). Then \( K \) contains all isolated points of the space. Since the set \( X \) is dense in \( X \), we conclude that \( X \subset K = K \) and the space \( X \) is compact, which contradicts our assumptions. \( \square \)

Now we shall characterize topological spaces \( X \) for which the quotient map \( q : X \to X/X' \) is compact-covering.

A topological space \( X \) is called a \( \mu \)-space if each closed bounded subset of \( X \) is compact. A subset \( B \) of a topological space \( X \) is bounded if for any continuous function \( f : X \to \mathbb{R} \) the image \( f(B) \) is a bounded subset of the real line. It is known [3, 6.9.7] that the class of \( B \)-spaces includes all \( B \)-complete spaces, and consequently, all paracompact spaces [14, 8.5.13] and all submetrizable spaces [3, 6.10.8].

**Definition 12.2.** A topological space \( X \) is called a \( \mu \)-space if for any subset \( A \subset X \) the following conditions are equivalent:

- \( A \) is contained in a compact subset of \( X \);
- for any closed subset \( D \subset X \) of the intersection \( A \cap D \) is finite.

**Theorem 12.3.** Let \( Y \) be a pointed topological space whose distinguished point \( *_Y \) has an open neighborhood \( U_* \neq Y \). For a non-discrete topological space \( X \) the following conditions are equivalent:

1. \( X \) is a \( \mu \)-space;
2. the quotient map \( q : X \to X/X' \) is compact-covering;
3. the map \( q^* : C_k'(X/X', Y) \to C_k'(X, Y) \), \( q^* : f \mapsto f \circ q \), is a homeomorphism.

The conditions (1)–(3) follow from the condition

4. \( X \) is a \( \mu \)-space.

**Proof.** (1) \( \Rightarrow \) (2) Assume that \( X \) is a \( \mu \)-space. Given any compact subset \( K \subset X/X' \), observe that the set \( A = q^{-1}(K) \setminus X' \) has finite intersection \( A \cap D \) with any closed subset \( D \subset X \) of \( X \) (because \( K \) is compact and \( q(X \setminus D) \) is an open neighborhood of the unique non-isolated point of \( X/X' \)). Since \( X \) is a \( \mu \)-space, the set \( A \) is contained in some compact subset \( \tilde{K} \) of \( X \). Replacing \( \tilde{K} \) by a larger compact set, we can assume that \( \tilde{K} \cap X' \neq \emptyset \). Then \( q(\tilde{K}) \) is a compact subset of \( X/X' \), containing \( K \), which means that \( q \) is compact-covering.

(2) \( \Rightarrow \) (1) Assume that the quotient map \( q \) is compact-covering. To prove that \( X \) is a \( \mu \)-space, take any set \( A \subset X \) that has finite intersection \( A \cap D \) with any closed subset \( D \subset X \) of \( X \). The definition of the quotient topology on \( X/X' \) ensures that the subset \( q(A \cup X') \subset X/X' \) is compact. Since \( q \) is compact-covering, there exists a compact set \( K \subset X \) such that \( q(A \cup X') \subset q(K) \) and hence \( A \subset K \).

(2) \( \Rightarrow \) (3) It is easy to see (from the definition of the compact-open topology) that the compact-covering property of the quotient map \( q : X \to X/X' \) implies that the continuous bijective map \( q^* : C_k'(X/X', Y) \to C_k'(X, Y) \) is a homeomorphism.

(3) \( \Rightarrow \) (2) Assuming that \( q^* : C_k'(X/X', Y) \to C_k(X, Y) \) is a homeomorphism, we shall prove that the map \( q : X \to X/X' \) is compact-covering. By our assumption, the distinguished point \( *_Y \) of \( Y \) has an open neighborhood \( U_* \) that does not contain some point \( y \in Y \). Given a compact subset \( K \subset X/X' \), consider the open neighborhood \([K; U_*]\) of the constant function \( X/X' \to \{*_Y\} \subset Y \). Since the map \( q^* \) is open, the image \( q^*([K; U_*]) \) is an open neighborhood of the constant function \( X \to \{*_Y\} \subset Y \). Consequently, there exists a compact set \( \tilde{K} \subset X \) and a neighborhood \( U \subset Y \) of \( *_Y \) such that \([\tilde{K}; U] \subset q^*([K; U_*]) \). Replacing \( \tilde{K} \) by a larger compact set, we can assume that \( \tilde{K} \cap X' \neq \emptyset \).

We claim that \( K \subset q(\tilde{K}) \). Given any point \( x \in K \), we should prove that \( x \in q(\tilde{K}) \). If \( x \) is not isolated in \( X/X' \), then \( x \in q(X') = q(X' \cap \tilde{K}) \). So we assume that \( x \) is isolated in \( X/X' \) and hence \( x = q(\tilde{x}) \) for a unique isolated point \( \tilde{x} \in \tilde{X} \). We claim that \( \tilde{x} \in \tilde{K} \). In the opposite
case, we can consider the function $\chi : X \to \{\ast y, y\}$ defined by $\chi^{-1}(y) = \{\tilde{x}\}$, and observe that $\chi \in \{\tilde{K}; U\} \subseteq q^*(\{K; U\})$, which implies that $y \in U_*$. But this contradicts the choice of the point $y$. This contradiction shows that $\tilde{x} \in \tilde{K}$ and hence $x = q(\tilde{x}) \in q(\tilde{K})$ and the map $q$ is compact-covering.

$(4) \Rightarrow (1)$ Finally, assuming that $X$ is a $\mu$-space, we prove that it is a $\check{\mu}$-space. Fix any subset $A \subset X$ that has finite intersection $A \cap D$ with any closed subset $D \subset X$ of $X$. Assuming that $A$ is not compact in the $\mu$-space $X$, we conclude that the set $A$ is not bounded. So there exists a continuous function $f : X \to \mathbb{R}$ such that the set $f(A)$ is unbounded in the real line. Then we can choose a sequence $\{a_n\}_{n \in \omega} \subset A$ such that $|f(a_n)| > n$ for every $n \in \omega$. The continuity of the function $f$ ensures that the sequence $(a_n)_{n \in \omega}$ has no accumulation points in $X$, which means that the set $D = \{a_n\}_{n \in \omega}$ is closed in $X$ and hence has a finite intersection $D \cap A = D$ with $A$, which is not true. This contradiction shows that $X$ is a $\check{\mu}$-space.

Now we recall the definitions of some familiar properties of topological spaces that correspond to the properties introduced in Definition [6.1]

**Definition 12.4.** A topological space $X$ is called

- a $k$-space if a set $F \subset X$ is closed if for any compact set $K \subset X$ the intersection $F \cap K$ is closed in $K$;
- a $k_\omega$-space if there exists a countable family $\mathcal{K}$ of compact sets in $X$ such that a set $F \subset X$ is closed if for any $K \in \mathcal{K}$ the intersection $F \cap K$ is closed in $K$;
- hemicompact if there exists a countable family $\mathcal{K}$ of compact sets such that each compact subset of $X$ is contained in some set $K \in \mathcal{K}$.

Comparing Definitions [6.1] and [12.4] we can see that a topological space with a unique non-isolated point $X$ is a $\check{k}$-space (resp. a $\check{k}_\omega$-space, a hemi-$\check{k}_\omega$-space) if and only if $X$ is a $k$-space (resp. $k_\omega$-space, a hemicompact space). Using these equivalences, we can prove the following characterization.

**Proposition 12.5.** A $\check{\mu}$-space $X$

1. has DMOP iff $X/X'$ has DMOP iff $X/X'$ has MOP;
2. has WDMOP iff $X/X'$ has WDMOP;
3. is a $k$-space iff $X/X'$ is a $k$-space;
4. is a hemi-$k_\omega$-space iff $X/X'$ is hemicompact;
5. is a $k_\omega$-space iff $X/X'$ is a $k_\omega$-space;
6. has a countable $k$-network iff $X$ has a countable $k$-network.

**Proof.** Notice that a topological space with unique non-isolated point has DMOP if and only if it has MOP, then (1) is follows from Theorems [4.4] The implications (2),(4),(5) and (6) follow directly from Theorems [5.3], [7.4], [8.1] and [9.4] respectively.

(3) Suppose $X$ is a $\check{k}$-space and $D \subset X$ such that for any compact subset $K \subset X'$, the intersection $D \cap K$ is finite. Then for any compact subset $\tilde{K} \subset X$, the intersection $D \cap q(\tilde{K})$ is finite, which implies that $D \cap \tilde{K}$ is finite, so $D$ is closed in $X$ and hence closed in $X/X'$. Consequently, $X/X'$ is $\check{k}$-space. On the other hand, suppose $D \subset X$ such that for any compact subset $\tilde{K} \subset X$, the intersection $D \cap \tilde{K}$ is finite. Since $q$ is compact-covering, for every compact subset $K \subset X/X'$ there exists a compact subset $\tilde{K} \subset X$ such that $K \subset q(\tilde{K})$. So $D \cap K \subset D \cap q(\tilde{K})$ is finite, and $D$ is closed in $X/X'$. It follows that $D$ is closed in $X$ by the definition of quotient map, and hence $X$ is $\check{k}$-space.

Proposition [12.5](6) allows us to give a partial answer to Problem [9.5]

**Corollary 12.6.** For any $\check{\mu}$-space $X$ with a countable $k$-network $X$ and any regular space $Y$ with a countable $k$-network, the function space $C_k(X,Y)$ has a countable $k$-network.
Proof. By Theorem 12.3, the quotient map \( q : X \to X/X' \) is compact-covering and function spaces \( C_k'(X,Y) \) and \( C_k'(X/X',Y) \) are homeomorphic. By Proposition 12.5(6), the space \( X/X' \) has a countable k-network. Being a Hausdorff space with a unique non-isolated point, the space \( X/X' \) is regular. By a result of Michael [26] (see also [18, 11.5]), the function space \( C_k(X/X',Y) \) has a countable k-network. Then its subspace \( C_k'(X,Y) \) has a countable k-network, too. \( \square \)

13. Characterizing stratifiable scattered spaces of finite scattered height

In this section we characterize stratifiable spaces among scattered space of finite scattered height. A subset \( A \) of a topological space \( X \) is called

- a retract of \( X \) if there exists a continuous map \( r : X \to A \) such that \( r(a) = a \) for all \( a \in A \) (this map \( r \) is called a retraction of \( X \) onto \( A \));
- a \( G_\delta \)-retract of \( X \), if \( A \) is a retract in \( X \) and \( A \) is a \( G_\delta \)-subset of \( X \).

**Theorem 13.1.** A non-discrete topological space \( X \) is stratifiable if and only if the set \( X' \) of non-isolated points of \( X \) is a stratifiable space and \( X'/X \) is a \( G_\delta \)-retract of \( X \).

Proof. First, assume that \( X' \) is a stratifiable \( G_\delta \)-retract in \( X \). Fix a retraction \( r : X \to X' \) and write \( X' = \bigcap_{n \in \omega} W_n \) of a decreasing sequence \( (W_n)_{n \in \omega} \) of open sets in \( X \).

By definition, each point \( x \in X' \) of the stratifiable space \( X' \) has a countable family \( (U_n(x))_{n \in \omega} \) of open neighborhoods such that each closed set \( F \subseteq X' \) is equal to the intersection \( \bigcap_{n \in \omega} U_n[F] \).

For every \( x \in X \setminus X' \) put \( W_n(x) = \{ x \} \) for all \( n \in \omega \). Also for every \( x \in X' \) and \( n \in \omega \) let \( W_n(x) = W_n \cap r^{-1}(U_n(x)) \). We claim that the system of neighborhoods \( (W_n(x))_{n \in \omega}, x \in X \) witnesses that the space \( X' \) is stratifiable. Given any closed subset \( F \subseteq X \), we should prove that \( F = \bigcap_{n \in \omega} W_n[F] \) where \( W_n[F] = \bigcup_{x \in F} W_n(x) \). Given any point \( x \not\in F \), we should find \( n \in \omega \) such that \( x \not\in W_n[F] \).

If \( x \not\in X' \), then there exists \( n \in \omega \) such that \( x \not\in W_n \). Then the neighborhood \( \{ x \} \) of \( x \) is disjoint with the set \( W_n[F] \subseteq F \cup W_n \) and we are done.

If \( x \in X' \), then \( x \not\in F \cap X' = \bigcap_{n \in \omega} U_n[F \cap X'] \). So, there exists a number \( n \in \omega \) with \( x \not\in U_n[F \cap X'] \) and then \( x \not\in F \cup r^{-1}(U_n[F \cap X']) \). It remains to observe that

\[
W_n[F] = F \cup W_n[F \cap X'] \subseteq F \cup r^{-1}(U_n[F \cap X']) \subseteq F \cup r^{-1}(U_n[F \cap X'])
\]

and \( W_n[F] \subseteq F \cup r^{-1}(U_n[F \cap X']) \subseteq X \setminus \{ x \} \). This completes the proof of the “if” part.

To prove the “only if” part, assume that the space \( X \) is stratifiable. Since the stratifiability is inherited by subspaces, \( X' \) is stratifiable. By definition, each closed subset of a stratifiable space is a \( G_\delta \)-set in \( X \), which implies that the closed subset \( X' \) of \( X \) is a \( G_\delta \)-set in \( X \). It remains to prove that \( X' \) is a retract of \( X \). By definition of stratifiability, each point \( x \) has a countable family neighborhoods \( \{ U_n(x) \}_{n \in \omega} \) such that \( F = \bigcap_{n \in \omega} U_n[F] \) for every closed subset \( F \subseteq X \). We lose no generality assuming that \( U_0(x) = X \) for any \( x \in X \). Replacing each neighborhood \( U_n(x) \) by the intersection \( \bigcap_{i \leq n} U_i(x) \), we can assume that \( U_{n+1}(x) \subseteq U_n(x) \) for all \( n \in \omega \) and \( x \in X \).

Since \( X' = \bigcap_{n \in \omega} U_n[X'] = \bigcap_{n \in \omega} U_n[X'] \), for every \( x \in X \) the set \( N_x = \{ n \in \omega : x \in U_n[X'] \} \) is finite and hence has the largest element \( n_x \). Choose any point \( r[x] \in X' \) with \( x \in U_{n_x}(r[x]) \). We claim that the function \( r : X \to X' \) defined by

\[
r(x) = \begin{cases} r[x] & \text{if } x \in \hat{X}; \\ x & \text{otherwise;}
\end{cases}
\]

is a continuous retraction of \( X \) onto \( X' \).

It suffices to prove the continuity of \( r \) at any point \( x' \in X' \). Take any open neighborhood \( V \subseteq X \) of \( x' \). Then \( X \setminus V = \bigcap_{n \in \omega} U_n[X \setminus V] \) and there exists a number \( n \in \omega \) such that \( x' \notin U_n[X \setminus V] \). We
claim that \( O_{x'} := U_n(x') \setminus \overline{U_n[X \setminus V]} \) is an open neighborhood of \( x' \) such that for any \( x \in O_{x'} \) we get \( r(x) \in V \).

If \( x \in X' \cap O_{x'} \), then \( r(x) = x \in O_{x'} \subset X \setminus U_n[X \setminus V] \subset X \setminus (X \setminus V) = V \).

If \( x \in X \), then the inclusion \( x \in O_{x'} \subset U_n(x') \subset U_n[X] \) implies that \( n_x \geq n \) and \( x \in U_{n_x}(r(x)) \). Assuming that \( r(x) \in X \setminus V \), we conclude that \( x \in U_{n_x}(r(x)) \subset U_{n_x}[X \setminus V] \subset U_n[X \setminus V] \), which contradicts the choice of \( x \in O_{x'} \subset X \setminus U_n[X \setminus V] \).

\[ \Box \]

**Theorem 13.2.** A scattered space \( X \) of finite scattered height is stratifiable if and only if for every \( n < h[X] \) the set \( X^{[n]} \) is a \( G_\delta \)-retract in \( X \).

**Proof.** This characterization will be proved by induction on the scattered height \( h[X] \). If \( h[X] = 0 \), then the space \( X \) is discrete and hence stratifiable.

Assume that for some \( n \in \mathbb{N} \) we have prove that a scattered space \( X \) of scattered height \( h[X] < n \) is stratifiable if and only if the sets \( X^{[k]}, k < h[X], \) are \( G_\delta \)-retracts of \( X \).

Let \( X \) be a scattered space of scattered height \( h[X] = n \). If \( X \) is stratifiable, then \( X' \) is a stratifiable \( G_\delta \)-retract of \( X \), by Theorem 13.1. Since the space \( X' \) has scattered height \( h[X'] \) \(< n \) we can apply the inductive assumption and conclude that for every \( k < h[X'] \) the set \( X^{[k]} \) is a \( G_\delta \)-retract in \( X' \).

Since \( X' \) is a \( G_\delta \)-retract in \( X \), the set \( X^{[k]} \) is a \( G_\delta \)-retract in \( X \).

Now assume that for every \( k < h[X] \) the set \( X^{[k]} \) is a \( G_\delta \)-retract in \( X \). So, there exists a retraction \( r_k : X \to X^{[k]} \) \( G_\delta \)-set in \( X \). Then \( X^{[k]} \) is a \( G_\delta \)-set in \( X' \) and the restriction \( r_k|X' : X' \to X^{[k]} \) is a retraction of \( X' \) onto its closed subset \( X^{[k]} \). So \( X^{[k]} \) is a \( G_\delta \)-retract of \( X' \). By the inductive assumption, the space \( X' \) is stratifiable. Since \( X' \) is a \( G_\delta \)-retract of \( X \), we can apply Theorem 13.1 and conclude that the space \( X \) is stratifiable. \( \Box \)

14. ON FUNCTION SPACES WITH VALUES IN RECTIFIABLE POINTED SPACES

A pointed topological space \( Y \) is called a **rectifiable space** if there exists a homeomorphism \( H : Y \times Y \to Y \times Y \) such that \( H(\{y\} \times Y) = \{y\} \times Y \) and \( H(y, y) = (y, *_Y) \) for every \( y \in Y \). Each topological group \( Y \) is rectifiable because of the homeomorphism \( H : Y \times Y \to Y \times Y, H : (y, z) \mapsto (y, y^{-1}z) \). The unit sphere \( S^7 \) in the space of Cayley octonions is an example of a rectifiable pointed space, which is not homeomorphic to a topological group. By [19], a rectifiable space is regular (and metrizable) if and only if it satisfies the separation axiom \( T_0 \) (and is first-countable). More information on rectifiable spaces can be found in [2], [9], [19].

**Proposition 14.1.** Let \( Y \) be a rectifiable space and \( X \) be a topological space. If the set \( X' \) is a retract of \( X \), then the function space \( C_k(X, Y) \) is homeomorphic to \( C'_k(X, Y) \times C'_k(X, Y) \).

**Proof.** By the rectifiability of the pointed space \( Y \), there exists a homeomorphism \( h : Y \times Y \to Y \times Y \) such that \( h(\{y\} \times Y) = \{y\} \times Y \) and \( h(y, y) = (y, *_Y) \) for every \( y \in Y \). Let \( pr_2 : Y \times Y \to Y, \) \( pr_2 : (x, y) \mapsto y \), be the projection onto the second coordinate. Assume that \( r : X \to X' \) is a retraction of \( X \) onto \( X' \).

We claim that the map
\[
H : C_k(X, Y) \to C_k(X', Y) \times C'_k(X, Y), \quad H : f \mapsto \left( f|X', pr_2 \circ h(f \circ r, f) \right),
\]
is a homeomorphism. Here \( pr_2 \circ h(f \circ r, f) : X \to Y \) is the map assigning to each \( x \in X \) the element \( pr_2 \circ h(f \circ r(x), f(x)) \in Y \). Observe that for any \( x \in X' \) we get
\[
pr_2 \circ h(f \circ r(x), f(x)) = pr_2 \circ h(f(x), f(x)) = pr_2(f(x), *_Y) = *_Y,
\]
which means that \( pr_2 \circ h(f \circ r, f) \in C'_k(X, Y) \).

The map \( H \) has a continuous inverse
\[
H^{-1} : C_k(X', Y) \times C'_k(X, Y) \to C_k(X, Y), \quad H^{-1} : (g, f) \mapsto pr_2 \circ h^{-1}(g \circ r, f),
\]
where the map $\text{pr}_2 \circ h^{-1}(g \circ r, f) : X \to Y$ assigns to each $x \in X$ the element $\text{pr}_2 \circ h^{-1}(g \circ r(x), f(x)) \in Y$.

To show that $H^{-1} \circ H$ is the identity map of $C_k(X,Y)$, take any function $f \in C_k(X,Y)$ and observe that

$$H^{-1} \circ H(f) = H^{-1}(f \mid X', \text{pr}_2 \circ h(f \circ r, f)) = \text{pr}_2 \circ h^{-1}((f \mid X') \circ r, \text{pr}_2 \circ h(f \circ r, f)) =$$

$$= \text{pr}_2 \circ h^{-1}(f \circ r, \text{pr}_2 \circ h(f \circ r, f)) = \text{pr}_2 \circ h^{-1} \circ h(f \circ r, f) = \text{pr}_2(f \circ r, f) = f.$$

Next, we prove that $H \circ H^{-1}$ is the identity map of $C_k(X,Y)$, $\times C_k'(X,Y)$. Given any pair $(\varphi, \psi) \in C_k(X,Y) \times C_k'(X,Y)$, consider the function $f = H^{-1}(\varphi, \psi) = \text{pr}_2 \circ h^{-1}(\varphi \circ r, \psi)$. We claim that $H(f) = (\varphi, \psi)$. Recall that $H(f) = (f \mid X', \text{pr}_2 \circ h(f \circ r, f))$. Then

$$f \mid X' = \text{pr}_2 \circ h^{-1}(\varphi \circ r, \psi)\mid X' = \text{pr}_2 \circ h^{-1}(\varphi, \ast_Y) = \text{pr}_2(\varphi, \varphi) = \varphi.$$

On the other hand,

$$(f \circ r, \text{pr}_2 \circ h(f \circ r, f)) = h(f \circ r, f) = h(\text{pr}_2 \circ h^{-1}(\varphi \circ r, \psi) \circ r, \text{pr}_2 \circ h^{-1}(\varphi \circ r, \psi)) =$$

$$= h(\text{pr}_2 \circ h^{-1}(\varphi \circ r, \ast_Y), \text{pr}_2 \circ h^{-1}(\varphi \circ r, \psi)) = h(\text{pr}_2(\varphi \circ r, \varphi \circ r), \text{pr}_2 \circ h^{-1}(\varphi \circ r, \psi)) =$$

$$= h(\varphi \circ r, \text{pr}_2 \circ h^{-1}(\varphi \circ r, \psi)) = h(\varphi \circ r, \psi) = (\varphi \circ r, \psi)$$

and hence $\psi = \text{pr}_2 \circ h(f \circ r, f)$. \qed

**15. Function spaces $C_k(X,Y)$ for scattered $X$ and rectifiable $Y$**

In this section we shall prove (general versions of) the results announced in the introduction.

**Theorem 15.1.** Let $X$ be a topological space such that $X'$ is a $G_\delta$-retract in $X$ and $Y \neq \{\ast_Y\}$ be a second-countable Choquet rectifiable $T_0$-space. Then the following conditions are equivalent:

1. the function space $C_k(X,Y)$ is Baire.
2. the space $X$ has DMOP and the function space $C_k(X',Y)$ is Baire.

**Proof.** By Proposition 14.1, the function space $C_k(X,Y)$ is homeomorphic to $C_k(X',Y) \times C_k'(X,Y)$. By [19], the second-countable rectifiable $T_0$-space $Y$ is metrizable and separable. Consequently, $Y$ is $\ast$-first-countable. Since $Y \neq \{\ast_Y\}$, the metrizable space $Y$ is $\ast$-admissible.

(1) $\Rightarrow$ (2) Assume that the function space $C_k(X,Y)$ is Baire. Since the Baire space $C_k(X,Y)$ is homeomorphic to $C_k'(X,Y) \times C_k(X',Y)$, the spaces $C_k'(X,Y)$ and $C_k(X',Y)$ are Baire. By Corollary 14.3, the space $X$ has DMOP.

(2) $\Rightarrow$ (1) Assume that the space $X$ has DMOP and the function space $C_k(X',Y)$ is Baire. By Corollary 14.6, the space $C_k'(X,Y)$ is Baire. By our assumption, $X'$ is a $G_\delta$-set in $X$. By Proposition 3.2, the space $C_k'(X,Y)$ has countable cellularity. By Theorem 3.1 of [24], the product of a Baire space and a countably cellular Baire space is Baire. Consequently, the product $C_k(X',Y) \times C_k'(X,Y)$ is Baire and so is its topological copy $C_k(X,Y)$.

Applying Theorem 15.1 inductively, we can prove the following characterization.

**Corollary 15.2.** Let $n \in \mathbb{N}$ and $X$ be a topological space such that for every $k \leq n$ the set $X^{[k]}$ is a $G_\delta$-retract in $X$. Then for any second-countable Choquet rectifiable $T_0$-space $Y \neq \{\ast_Y\}$, the following conditions are equivalent:

1. the function space $C_k(X,Y)$ is Baire;
2. for every $i \leq n$ the space $X^{[i]}$ has DMOP and the space $C_k(X^{[n]},Y)$ is Baire.

Combining Corollary 15.2 with Theorem 14.2, we obtain the following characterization.

**Corollary 15.3.** For any stratifiable scattered space $X$ of finite scattered height and any second-countable Choquet rectifiable space $Y \neq \{\ast_Y\}$, the following conditions are equivalent:
The function space $C_k(X,Y)$ is Baire;
(2) for every $k < h[X]$ the space $X^{[k]}$ has DMOP.

**Corollary 15.4.** For a stratifiable scattered space $X$ of finite scattered height the following conditions are equivalent:

(1) The function space $C_k(X)$ is Baire.
(2) The space $X$ has MOP.
(3) For every $n < h[X]$ the space $X^{[n]}$ has DMOP.

**Proof.** (1) $\Rightarrow$ (2) If the function space $C_k(X)$ is Baire, then the space $X$ has MOP by Theorem 2.1 in [17].

(2) $\Rightarrow$ (3) If $X$ has MOP, then each closed subspace of $X$ has MOP. In particular, for every $k < h[X]$ the closed subspace $X^{[k]}$ has MOP and hence has DMOP (because MOP implies DMOP).

The implication (3) $\Rightarrow$ (1) follows from Corollary 15.3 since the real line is a rectifiable Choquet space.

Analogous result can be proved for the Choquet property.

**Theorem 15.5.** Let $X \neq X'$ be a topological space such that $X'$ is a retract of $X$ and $Y \neq \{\ast_Y\}$ be a second-countable rectifiable $T_0$-space. Then the following conditions are equivalent:

(1) the function space $C_k(X,Y)$ is Choquet;
(2) the spaces $Y$ and $C_k(X',Y)$ are Choquet and the space $X$ has WDMOP.

**Proof.** By Proposition 14.1 the function space $C_k(X,Y)$ is homeomorphic to $C_k(X',Y) \times C_k'(X,Y)$.

(1) $\Rightarrow$ (2) Assume that the function space $C_k(X,Y)$ is Choquet. Since the Choquet space $C_k(X,Y)$ is homeomorphic to $C_k'(X,Y) \times C_k(X',Y)$, the spaces $C_k'(X,Y)$ and $C_k(X',Y)$ are Choquet, being open continuous images of $C_k(X,Y)$. By Theorem 5.3 the space $X$ has WDMOP.

(2) $\Rightarrow$ (1) Assume that $Y$ is Choquet, $X$ has WDMOP and function space $C_k(X',Y)$ is Choquet. By Theorem 5.3 the space $C_k'(X,Y)$ is Choquet. Then the space $C_k(X,Y)$ is Choquet, being homeomorphic to the product $C_k'(X,Y) \times C_k(X',Y)$ of two Choquet spaces.

This theorem has the following corollaries.

**Corollary 15.6.** Let $n \in \mathbb{N}$ and $X \neq X'$ be a topological space such that for every $k \leq n$ the set $X^{[k]}$ is a retract of $X$. Then for any second-countable rectifiable $T_0$-space $Y \neq \{\ast_Y\}$, the following conditions are equivalent:

(1) the function space $C_k(X,Y)$ is Choquet;
(2) the spaces $Y$ and $C_k(X^{[n]},Y)$ are Choquet, and for every $k \leq n$ the space $X^{[k]}$ has WDMOP.

**Corollary 15.7.** For any non-empty stratifiable scattered space of finite scattered height and any second-countable rectifiable $T_0$-space $Y \neq \{\ast_Y\}$, the following conditions are equivalent:

(1) the function space $C_k(X,Y)$ is Choquet;
(2) $Y$ is Choquet and for every $k < h[X]$ the space $X^{[k]}$ has WDMOP.

**Corollary 15.8.** For any stratifiable scattered space $X$ of finite scattered height the following conditions are equivalent:

(1) The function space $C_k(X)$ is Choquet.
(2) For every $n < h[X]$ the space $X^{[n]}$ has WDMOP.
ON BAIRE CATEGORY PROPERTIES OF FUNCTION SPACES $C^*_k(X,Y)$

References

[1] O. Alas, R. Wilson, When is a compact space sequentially compact?, Topology Proc. 29:2 (2005), 327–335.
[2] A. Arhangel’skii, Topological invariants in algebraic environment, in: Recent Progress in General Topology, II, 1–57, North-Holland, Amsterdam, 2002.
[3] A. Arhangel’skii, M. Tkachenko, Topological groups and related structures, Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.
[4] T. Banakh, Quasicontinuous functions with values in Piotrowski spaces, Real Anal. Exchange 43:1 (2018), 77–104.
[5] T. Banakh, R. Cauty, M. Zarichnyi, Open problems in infinite-dimensional topology, in: Open Problems in Topology, II (E. Pearl ed.), Elsevier, (2007) 601–624.
[6] T. Banakh, O. Hryniv, Some Baire category properties of topological groups, https://arxiv.org/abs/1901.01420
[7] T. Banakh, T. Radul, M. Zarichnyi, Absorbing sets in infinite-dimensional manifolds, VNTL Publishers, Lviv, (1996) 240 pp.
[8] T. Banakh, A. Ravsky, Banalytic spaces and characterization of Polish groups, https://arxiv.org/abs/1901.10732
[9] T. Banakh, D. Repovs, Sequential rectifiable spaces with countable $cs^*$-character, Bull. Malaysian Math. Sci. Soc. 40:3 (2017) 975–993.
[10] C. Bessaga, A. Pełczyński, Selected topics in infinite-dimensional topology, PWN, Warsaw, (1975) 353 pp.
[11] C.R. Borges, On stratifiable spaces, Pacific J. Math. 17 (1966) 1–16.
[12] A. Chigogidze, Inverse spectra, North-Holland Publishing Co., Amsterdam, (1996) x+421 pp.
[13] J.P.R. Christensen, Topology and Borel Structure, North-Holland, Amsterdam, (1974) 133 pp.
[14] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[15] M. Granado, G. Gruenhage, Baireness of $C^*_k(X)$ for ordered $X$, Comment. Math. Univ. Carolin. 47:1 (2006) 103–111.
[16] G. Gruenhage, Generalized metric spaces, in: Handbook of Set-Theoretic Topology, Elsevier, (1984) 425–501.
[17] G. Gruenhage, D. Ma, Baireness of $C^*_k(X)$ for locally compact $X$, Topology Appl. 80 (1997) 131–139.
[18] G. Gruenhage, The story of a topological game, Rocky Mountain J. Math. 36:6 (2006) 1885–1914.
[19] A. Gul’ko, Rectifiable spaces, Topology Appl. 68:2 (1996), 107–112.
[20] J.A. Guthrie, A characterization of $K_0$-spaces, General Topology and Appl. 1:2 (1971) 105–110.
[21] G. Hughes, Completeness properties in function spaces with the compact-open topology, Ph. D. Dissertation, Auburn Univ. (2014).
[22] T. Jech, Set Theory, Springer, (2003) xiii+769 pp.
[23] A. S. Kechris, Classical Descriptive Set Theory, Springer-Verlag, New York 1995.
[24] R. Li, L. Žilinskas, More on products of Baire spaces, Topology Appl. 230 (2017) 35–44.
[25] R.A. McCoy, I. Ntantu, Completeness properties of function spaces, Topology Appl. 22 (1986), 191–206.
[26] E. Michael, $K_0$-spaces, J. Math. Mech. 15 (1966) 983–1002.
[27] J. van Mill, Infinite-dimensional topology. Prerequisites and introduction, North-Holland, Amsterdam, (1989), xii+401 pp.
[28] J. van Mill, The infinite-dimensional topology of function spaces, North-Holland Publishing Co., Amsterdam, (2001), xii+630 pp.
[29] J.C. Oxtoby, The Banach-Mazur game and Banach Category Theorem, in: Contributions to the theory of games, Vol. Ill, Annals of Math. Studies, 39 (1957) 159–163.
[30] K. Sakai, Topology of Infinite-Dimensional Manifolds, unpublished book, (2015) 324 pp.
[31] F. Tall, Some observations on the Baireness of $C^*_k(X)$ for a locally compact space $X$, Topology Appl. 213 (2016) 212–219.
[32] L. Wang, T. Banakh, Baire category properties of function spaces with the Fell hypograph topology, preprint.
[33] H.E. White Jr., Topological spaces that are $\alpha$-favorable for a player with perfect information, Proc. Amer. Math. Soc. 50 (1975), 477–482.

Institute of Mathematics, Jan Kochanowski University in Kielce (Ukraine), and Ivan Franko National University in Lviv (Ukraine)
E-mail address: t.o.banakh@gmail.com

Department of Mathematics, Shantou University, Shantou, Guangdong, 515063, PR China
E-mail address: 161jwang@stu.edu.cn