We construct non-BPS dyon solutions of $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory. These solutions are the worldvolume solitons which describe non-BPS Type IIB non-planar string junctions connecting $N$ parallel D3-branes. The solutions are smooth deformations of the 1/4 BPS states which describe planar string junctions.
1 Introduction

Solitons in gauge theories have a geometric interpretation as worldvolume solutions describing the intersection of various brane configurations. For example, the $SU(2)$ BPS monopole describes a D-string stretched between two D3-branes and this can be seen explicitly by graphing the eigenvalues of the Higgs field over $\mathbb{R}^3$. The worldvolume theory of $N$ parallel D3-branes is $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills and the $6N$ vacuum expectation values (vevs) of the Higgs scalars give the positions of the D3-branes in the six-dimensional space which is transverse to the worldvolume of the D3-branes. The $SU(N)$ dyon solutions (an overall $U(1)$ decouples as the centre of mass) describe bound states of D-strings and fundamental strings stretched between the D3-branes.

The string spectrum also contains states, known as string junctions, in which three or more strings meet at a point. If the string junction is planar, then it can preserve 1/4 of the underlying supersymmetry and it has been demonstrated that these states correspond to 1/4 BPS dyons in $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills, where $N \geq 3$. Recently classical dyon solutions have been constructed which correspond to these planar string junctions. Furthermore, it is expected that if $N \geq 4$ then there exist stable non-BPS states, which correspond to string junctions connecting four or more D3-branes which are non-planar. In this paper we present a class of static non-BPS dyon solutions which describe such non-planar string junctions. They are smooth deformations of the 1/4 BPS states which describe planar string junctions.

2 Non-BPS Dyons

The bosonic part of the (3+1)-dimensional $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills Lagrangian is

$$\mathcal{L} = \text{tr}\{-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \sum_{I=1}^{6} D_{\mu} \Phi^I D^{\mu} \Phi^I + \frac{1}{2} \sum_{I,J=1}^{6} [\Phi^I, \Phi^J]^2\}$$

(2.1)

where $\Phi^I$, $I = 1, ..., 6$ are the six Higgs scalars and $D_{\mu} \Phi^I = \partial_{\mu} \Phi^I - i[A_{\mu}, \Phi^I]$. The equations of motion which follow from (2.1) are

$$D_i D_i \Phi^I = D_0 D_0 \Phi^I + \sum_{I=1}^{6} [\Phi^I, [\Phi^I, \Phi^J]]$$

(2.2)

$$D_i F_{ij} - D_0 E_i = \sum_{I=1}^{6} i[\Phi^I, D_j \Phi^I]$$

(2.3)

$$D_i E_i = \sum_{I=1}^{6} i[\Phi^I, D_0 \Phi^I]$$

(2.4)

where Latin subscripts run over the space indices 1,2,3 and we recognise the last equation as Gauss’ law for the electric field $E_i = F_{0i}$.

From (2.2) it is clear that a consistent reduction can be made by setting any of the Higgs scalars to zero. If we work with $M \leq 6$ active scalars, ie. $\Phi^{M+1} = \ldots = \Phi^6 = 0$, then in the gauge $A_0 = -\Phi^1$ the static version of the above equations reduces to

$$D_i D_i \Phi^I = \sum_{I=2}^{M} [\Phi^I, [\Phi^I, \Phi^J]], \quad D_i F_{ij} = \sum_{I=2}^{M} i[D_j \Phi^I, \Phi^I].$$

(2.5)
In this gauge $E_i = D_i \Phi^1$ and $D_0 \Phi^I = -i[\Phi^I, \Phi^1]$ so Gauss’ law is equivalent to the first equation in (2.3) with $J = 1$.

From the above it is now clear that the case $M = 2$ is very special, since in this case the equations reduce to

$$D_i D_i \Phi^2 = 0, \quad D_i F_{ij} = i[D_j \Phi^2, \Phi^2], \quad D_i D_i \Phi^1 = [\Phi^2, [\Phi^2, \Phi^1]].$$  \hspace{1cm} (2.6)

The first two equations are the usual ones for a Yang-Mills theory with a single Higgs field and hence are solved by any static monopole solution of the Bogomolny equation

$$D_i \Phi^2 = -B_i$$  \hspace{1cm} (2.7)

where $B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$ is the magnetic field. The final equation in (2.6) is then a linear equation for $\Phi^1$ in the background of the monopole. Given the Higgs vevs for $\Phi^1$ and $\Phi^2$ (which give the positions of the $N$ D3-branes in a two-dimensional space transverse to the D3-brane worldvolume) then for each solution of the Bogomolny equation (2.7) there is a unique solution to the linear equation for $\Phi^1$, from which the electric charge of the dyons can be calculated. These are the 1/4 BPS states which have been constructed recently [8, 9, 10, 11], following their predicted existence from string theory as the worldvolume solitons describing planar string junctions [2]. Due to the fact that the solutions (for fixed Higgs vevs) are parametrized by Bogomolny monopoles then expressions for the mass and electric charge of these 1/4 BPS states can be obtained as functions on the monopole moduli space [12, 13].

To describe non-planar string junctions requires at least three active scalars, in order to assign positions to the D3-branes which are not co-planar in the transverse space. With three active scalars, ie. $M = 3$, equations (2.5) become

$$D_i D_i \Phi^2 = [\Phi^3, [\Phi^3, \Phi^2]]$$  \hspace{1cm} (2.8)

$$D_i D_i \Phi^3 = [\Phi^2, [\Phi^2, \Phi^3]]$$  \hspace{1cm} (2.9)

$$D_i F_{ij} = i[D_j \Phi^2, \Phi^2] + i[D_j \Phi^3, \Phi^3]$$  \hspace{1cm} (2.10)

$$D_i D_i \Phi^1 = [\Phi^2, [\Phi^2, \Phi^1]] + [\Phi^3, [\Phi^3, \Phi^1]].$$  \hspace{1cm} (2.11)

The final equation is again a linear equation for $\Phi^1$ (which determines the electric field) in the background of the other fields. However, the remaining equations are more complicated than the usual equations for a Yang-Mills theory with a single Higgs scalar and are not solved by first order Bogomolny equations (except for special cases such as $\Phi^3 = 0$, when the string junction is again planar and we recover the 1/4 BPS states). This is to be expected since it is known from string theory that non-planar string junctions break all supersymmetries and hence it is no surprise that we need to look for non-BPS configurations which are solutions of the full second order equations rather than first order Bogomolny equations. In the next section we shall present some solutions of equations (2.8)–(2.11), which describe non-planar string junctions.
3 The Harmonic Map Ansatz

In this section we shall construct spherically symmetric dyon solutions of equations (2.8)–(2.11) using the harmonic map approach introduced in [10].

Coordinates \( r, z, \bar{z} \) are used on \( \mathbb{R}^3 \), where \( r \) is the radial coordinate and \( z \) is the Riemann sphere angular coordinate given by \( z = e^{i\varphi} \tan(\theta/2) \), where \( \theta, \varphi \) are the standard angular coordinates on the two-sphere. In terms of these coordinates equations (2.5) become

\[
\sum_{I=2}^{M} [D_r \Phi^I, \Phi^J] = \frac{i(1 + |z|^2)^2}{2r^2} (D_z F_{r\bar{z}} + D_{\bar{z}} F_{rz}) \quad (3.1)
\]

\[
-i \sum_{I=2}^{M} [D_{\bar{z}} \Phi^I, \Phi^J] + D_r F_{rz} = \frac{1}{2r^2} D_z ((1 + |z|^2) F_{z\bar{z}}) \quad (3.2)
\]

\[
\frac{1}{r^2} D_r (r^2 D_r \Phi^J) + \frac{(1 + |z|^2)^2}{2r^2} (D_z D_{\bar{z}} \Phi^I + D_{\bar{z}} D_z \Phi^J) = \sum_{I=2}^{M} [\Phi^I, [\Phi^J, \Phi^I]] \quad (3.3)
\]

The harmonic map ansatz to obtain \( SU(N) \) dyons is a simple generalization of the one employed in [10] and is given by

\[
\Phi^I = \sum_{j=0}^{N-2} \beta^I_j (P_j - \frac{1}{N}), \quad A_z = i \sum_{j=0}^{N-2} \gamma^I_j [P_j, \partial_z P_j], \quad A_r = 0. \quad (3.4)
\]

Here \( \beta^I_j(r), \gamma^I_j(r) \) are real functions depending only on the radial coordinate \( r \), and \( P_j(z, \bar{z}) \) are \( N \times N \) hermitian projectors, that is, \( P_j = P_j^\dagger = P_j^2 \), which are independent of the radius \( r \). The set of \( N - 1 \) projectors are taken to be orthogonal, so that \( P_i P_j = 0 \) for \( i \neq j \). Note that we are working in a real gauge, so that \( A_{\bar{z}} = A^z_r \). In (3.4), and for the remainder of the paper, we drop the summation convention.

The orthogonality of the projectors \( P_j \) means that the Higgs fields \( \Phi^I \) are mutually commuting, ie. \( [\Phi^I, \Phi^J] = 0 \), so they are simultaneously diagonalizable and this allows the eigenvalues to be interpreted as the positions of the strings in the transverse space.

The explicit form of the projectors is given as follows. Let \( f \) be the holomorphic vector

\[
f = (f_0, ..., f_j, ..., f_{N-1})^t, \quad \text{where} \quad f_j = z^j \sqrt{\binom{N-1}{j}} \quad (3.5)
\]

and \( \binom{N-1}{j} \) denote the binomial coefficients. Define the operator \( \Delta \), acting on a vector \( h \in \Phi^N \) as

\[
\Delta h = \partial_z h - \frac{h (h^\dagger \partial_z h)}{|h|^2} \quad (3.6)
\]

then \( P_j \) is defined as

\[
P_j = \frac{(\Delta^j f)(\Delta^j f)^\dagger}{|\Delta^j f|^2}. \quad (3.7)
\]
The particular form of these projectors corresponds to the requirement that the associated dyons are spherically symmetric (see [10] for more details).

As stated above, these projectors form an orthogonal set, and have a number of other special properties, such as the fact that each of them solves the harmonic map equation

\[ [P, \partial_x \partial_{\bar{x}} P] = 0 \]  

of the two-dimensional $\mathbb{C}P^{N-1}$ sigma model (see [17] for more details). Using these properties it can be shown (the analysis follows almost immediately from that given in [10]) that substitution of the ansatz (3.4) into the equations (3.1), (3.2), (3.3) results in a set of coupled ordinary differential equations for the profile functions $\beta_j^I(r), \gamma_j(r)$. In fact it is convenient to make a change of variables to the following linear combinations

\[ \beta_j^I = \sum_{k=j}^{N-2} b_k^I, \quad c_j = 1 - \gamma_j - \gamma_{j+1}, \quad \text{for } j = 0, \ldots, N-2 \]  

where we have defined $\gamma_{N-1} = 0$.

The magnetic charges, $n_k$, for $k = 1, \ldots, N-1$, can be read off from the large $r$ behaviour of the magnetic field

\[ B_i \sim \frac{\hat{x}_i}{2r^2} G \]  

where $G$ is in the gauge orbit of

\[ G_0 = \text{diag}(n_1, n_2 - n_1, \ldots, n_{N-1} - n_{N-2}, -n_{N-1}). \]  

In the case of maximal symmetry breaking, which we shall consider here, they are given by

\[ n_k = k(N-k), \quad k = 1, \ldots, N-1. \]  

Similarly, the large $r$ asymptotics of the electric field

\[ E_i \sim \frac{\hat{x}_i}{2r^2} H \]  

allow the electric charges (which classically are real-valued) to be found from the eigenvalues of $H$. From our ansatz

\[ H = \sum_{j=0}^{N-2} 2(P_j - \frac{1}{N})(r^2 \beta_j^I)'|_{r=\infty} \]  

so the electric charges are related to the $1/r$ coefficients of $b_j^I$ in a large $r$ expansion.

We shall now restrict to the simplest case in which a non-planar string junction can exist, that is $M = 3$ active scalars, as described earlier. The lowest rank gauge group we can consider is $SU(4)$, corresponding to the fact that we require at least four D3-branes otherwise they will always be co-planar. In this case the equations for the profile functions are (for $I = 1, 2, 3$)

\[ (r^2 b_0^I)' = 6c_0^2 b_0^I - 4c_1^2 b_1^I \]  

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\[ (r^2b_{1I})' = 8c_1^2b_1' - 3c_0^2b_0' - 3c_2^2b_2' \]
\[ (r^2b_{2I})' = 6c_2^2b_2' - 4c_1^2b_1' \]
\[ r^2c_0'' = c_0(3c_0^2 - 2c_1^2 - 1 + r^2(b_0^3)^2 + r^2(b_0^3)^2) \]
\[ r^2c_1'' = c_1(4c_1^2 - \frac{3}{2}c_0^2 - \frac{3}{2}c_2^2 - 1 + r^2(b_1^3)^2 + r^2(b_1^3)^2) \]
\[ r^2c_2'' = c_2(3c_2^2 - 2c_1^2 - 1 + r^2(b_2^3)^2 + r^2(b_2^3)^2). \]

(3.15)

The boundary conditions at \( r = 0 \) are that \( b_j'(0) = 0 \) and \( c_j(0) = 1 \) for all \( I, j \), which ensures that the Higgs fields and gauge potential are regular at the origin. We shall restrict to the case of maximal symmetry breaking \( SU(4) \rightarrow U(1)^3 \), (corresponding to the four D3-branes being distinct in transverse space), in which case the boundary conditions on \( c_j(r) \) at infinity are \( c_j(\infty) = 0 \), for all \( j \). The remaining free parameters, \( b_j'(\infty) \), required to specify a unique solution to the set of equations (3.15), determine the vevs of the Higgs fields as follows. Evaluating the ansatz (3.4) along the positive \( x_3 \)-axis, which corresponds to setting \( z = 0 \), and using the change of variables (3.9), results in

\[ \Phi'(r) = \frac{1}{4} \text{diag}(3b_0' + 2b_1' + b_2', 2b_1' + b_2' - b_0', b_2' - b_0' - b_1' - 2b_1' - 3b_2') \]

(3.16)

from which the Higgs vevs can be read off in terms of \( b_j'(\infty) \). Writing the components of (3.16) as

\[ \Phi'(r) = \text{diag}(\Phi_{1I}'(r), \Phi_{2I}'(r), \Phi_{3I}'(r), \Phi_{4I}'(r)) \]

(3.17)

then the positions of the four D3-branes in the three-dimensional transverse space are given by

\[ (x_4^I, x_5^I, x_6^I) = (\Phi_{1I}'(\infty), \Phi_{2I}'(\infty), \Phi_{3I}'(\infty), \Phi_{4I}'(\infty)) \]

for \( I = 1, 2, 3, 4 \).

(3.18)

Applying equation (3.18) for finite values of \( r \) gives the positions of the strings forming the string junction and ending on the D3-branes.

Before considering non-planar string junctions it is perhaps instructive to see how the \( SU(4) \) 1/4 BPS solutions [7, 11], which describe planar junctions, are obtained in this formalism. To construct a planar junction we place the D3-branes in the \((x^4, x^5)\)-plane; as an example we shall take the positions (3.18) to be

\[ (x_4^1, x_5^1, x_6^1) = (14, 3, 0) \]
\[ (x_4^2, x_5^2, x_6^2) = (-18, 1, 0) \]
\[ (x_4^3, x_5^3, x_6^3) = (14, -1, 0) \]
\[ (x_4^4, x_5^4, x_6^4) = (-10, -3, 0). \]

(3.19)

These give the Higgs vevs (via (3.18)) and using the formula (3.16) the boundary conditions are found to be

\[ b_0'(\infty) = 32, b_1'(\infty) = -32, b_2'(\infty) = 24, \]
\[ b_0'(\infty) = b_1'(\infty) = b_2'(\infty) = 0. \]

(3.20)
This is a planar example, and the solution can be obtained explicitly in closed form. Clearly, $b_0^3 = b_1^3 = b_2^3 = 0$, and the functions $c_j, b_j^2$ are those which solve the Bogomolny equation (see [9] for a description of how to obtain these solutions). In this case

$$c_0 = c_1 = c_2 = \frac{2r}{\sinh 2r}, \quad b_0^2 = b_1^2 = b_2^2 = 2 \coth 2r - \frac{1}{r}. \quad (3.21)$$

The linear equations for $b_j^1$ can then be solved by converting to a diagonal form using the methods of [7]. Explicitly, the solution is

$$b_0^1(\infty) = -2, \quad b_1^1(\infty) = 0, \quad b_2^1(\infty) = 2,$$

$$b_0^2(\infty) = 1, \quad b_1^2(\infty) = 0, \quad b_2^2(\infty) = 1,$$

$$b_0^3(\infty) = -1, \quad b_1^3(\infty) = 2, \quad b_2^3(\infty) = -1. \quad (3.24)$$

In fact this example is particularly symmetric, which leads to the reduction $c_2 = c_0, b_1^1 = 0, b_0^1 = b_2^1, b_0^2 = b_2^2, b_0^3 = b_2^3$. The solution satisfying the boundary conditions (3.24) is plotted in Figure 2 (the individual functions can be identified by their asymptotic values). In Figure 3 we graph the positions of the strings corresponding to this non-planar string junction.

If the string junction is non-planar, but is close to a planar junction, then an approximate solution can be found as a deformation of a 1/4 BPS state. As an example, consider the small perturbation of the planar junction (3.19) to

$$(x_4^1, x_5^1, x_6^1) = (14, 3, \epsilon)$$

$$(x_4^2, x_5^2, x_6^2) = (-18, 1, -\epsilon)$$

$$(x_4^3, x_5^3, x_6^3) = (14, -1, 0)$$
\[(x_4^4, x_4^5, x_4^6) = (-10, -3, 0). \]  
(3.25)

where \(|\epsilon| \ll 1\) is the constant which describes the small displacement of the first two D3-branes out of the plane. Equations (3.15) can then be solved to first order in \(\epsilon\). The boundary conditions are now modified to

\[b_3^0(\infty) = 2\epsilon, \ b_3^1(\infty) = -\epsilon, \ b_3^2(\infty) = 0.\]  
(3.26)

The equations for \(c_j\) contain only quadratic terms in \(b_3^j\), so the \(c_j\) solutions are not modified to first order in \(\epsilon\). The equations for \(b_1^j, b_2^j\) do not contain \(b_3^j\) so they are also not modified. Thus we are left with solving the linear equations for \(b_3^j\), which are precisely the same as those already solved to obtain \(b_1^j\), the only difference now is the boundary condition (3.26).

The solution in this case is

\[b_3^0(r) = -\frac{6r(2ch^3 + 2ch + 2shch^2 + sh) - sh(10ch^2 + 2 + 9shch)}{12rsh^3}, \]

\[b_3^1(r) = \frac{-2rch(ch^2 + 2) + sh(2ch^2 + 1)}{2rsh^3}. \]  
(3.27)

It can be checked that \(|b_3^j(r)| \leq 2|\epsilon|\) for \(j = 0, 1, 2\) and all \(r\), hence the assumption that \(b_3^j(r)\) is small for all \(r\) if the boundary values are small is a consistent one. This approximate solution is in good agreement with the one obtained numerically for small values of \(\epsilon\).

As we have seen, for junctions which are close to being planar, approximate solutions can be obtained by solving linear equations in the background of a BPS monopole configuration. Thus it should be possible to find an expression, as a function on the monopole moduli space, for the leading order contribution to the mass of deforming a planar junction. Since non-planar junctions are non-BPS states then they are not protected from quantum corrections by supersymmetry, so it would perhaps be worthwhile to investigate further the properties of such states which are perturbations of 1/4 BPS configurations.

### 4 Conclusion

We have described the construction of a class of static non-BPS dyon solutions of \(\mathcal{N} = 4\) \(SU(N)\) supersymmetric Yang-Mills theory, whose existence was predicted by string theory, in which they describe non-planar string junctions connecting \(N\) D3-branes [3].

The solutions we have presented here are spherically symmetric and their electric charge is determined by the Higgs vevs. The same situation arises for 1/4 BPS states describing planar junctions [4, 5, 11] and subsequently more general non-spherical solutions were found [12] containing parameters that allow the electric charge to take a range of values. These more general solutions are required in order to make contact with the quantum theory, since the classical electric charge takes real values but must be restricted to integer values upon quantization. It would therefore be useful to find the generalization of our spherical solutions, corresponding to separating the individual monopoles, though this task may be rather difficult given the non-BPS nature of the solutions.
String theory predicts that non-planar string junctions should be stable configurations, despite the fact that they are non-BPS states [3]. We therefore expect that the Yang-Mills dyons we have presented are stable solutions and it would be interesting to investigate this issue. Although the methods we have used are based upon those developed to find non-Bogomolny monopoles [10], which are expected to be unstable, there is an important difference in the present case in that there are no BPS states with the given charges and Higgs vevs for the non-BPS dyons to decay to.

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Figure 1: A planar string junction connecting four D3-branes (squares).
Figure 2: The functions $c_j, b_j^f$ corresponding to the boundary conditions for the non-planar string junction given in the text. The functions can be identified by their asymptotic values.
Figure 3: A non-planar string junction connecting four D3-branes (squares).