Non-perturbative renormalization of bilinear operators with Möbius domain-wall fermions in the coordinate space

Masaaki Tomii\textsuperscript{a,b}

with G. Cossu\textsuperscript{b}, S. Hashimoto\textsuperscript{a,b}, J. Noaki\textsuperscript{b}

\textsuperscript{a}: The Graduate University for Advanced Studies (Sokendai)
\textsuperscript{b}: High Energy Accelerator Research Organization (KEK)

LATTICE 2014 @ Columbia University in New York, June 23-28
1. Introduction — NPR by X-space method

Renormalization

\[ \mathcal{O}^{\text{lat}}(1/a) \rightarrow \mathcal{O}^{\overline{\text{MS}}}(\mu) = Z^{\overline{\text{MS}}/\text{lat}}(1/a \rightarrow \mu) \mathcal{O}^{\text{lat}}(1/a) \]

We investigate NPR of quark bilinears by the X-space method
(Martinelli et al ’97, Giménez et al 2004, Cichy et al 2012)

Advantages

✧ able to renormalize by gauge invariant quantities
✧ perturbative matching is available up to 4-loop level
(Chetyrkin-Maier, 2011)

Disadvantages

✧ Suffer from the window problem

Examine the potential of X-space method with the Möbius domain-wall fermions
2. Strategy — sketch of X-space method

X-space method uses correlation functions of quark bilinears

\( \Pi_{\Gamma \Gamma} : \) current-current correlators

\[
\begin{align*}
\Pi_{PP}(x) &= \langle P(x)P(0) \rangle, \\
\Pi_{SS}(x) &= \langle S(x)S(0) \rangle, \\
\Pi_{VV}(x) &= \sum_{\mu=1}^{4} \langle V_{\mu}(x)V_{\mu}(0) \rangle, \\
\Pi_{AA}(x) &= \sum_{\mu=1}^{4} \langle A_{\mu}(x)A_{\mu}(0) \rangle
\end{align*}
\]

Renormalization condition in the X-space scheme

\[
(Z_{\Gamma}^{X/\bullet}(\mu_X = 1/|x|))^{2} \Pi_{\Gamma \Gamma}^{\bullet}(x) = \Pi_{\Gamma \Gamma}^{\text{free}}(x)
\]

\[
\rightarrow Z_{\Gamma}^{X/\bullet}(x) = \sqrt{\frac{\Pi_{\Gamma \Gamma}^{\text{free}}(x)}{\Pi_{\Gamma \Gamma}^{\bullet}(x)}}
\]

Matching through the X-space scheme

\[
Z_{\Gamma}^{\text{MS/lat}}(2 \text{ GeV}) = \frac{Z_{\Gamma}^{X/\text{lat}}(x)}{Z_{\Gamma}^{X/\text{MS}}(x, 2 \text{ GeV})} = \sqrt{\frac{\Pi_{\Gamma \Gamma}^{\text{MS}}(x, 2 \text{ GeV})}{\Pi_{\Gamma \Gamma}^{\text{lat}}(x)}}
\]
2. Strategy — sketch of X-space method

Renormalization constants

\[
Z_{\Gamma}^{\overline{\text{MS}}/\text{lat}}(2 \ \text{GeV}) = \sqrt{\frac{\Pi_{\Gamma\Gamma}^{\overline{\text{MS}}}(x, 2 \ \text{GeV})}{\Pi_{\Gamma\Gamma}^{\text{lat}}(x)}}
\]

to be evaluated at \( m_q = 0 \)

Steps:

① Lattice calculation \( \rightarrow \) \( \Pi_{\Gamma\Gamma}^{\text{lat}}(x) \) in chiral limit

② Perturbation \( \rightarrow \) \( \Pi_{\Gamma\Gamma}^{\overline{\text{MS}}}(x, 2 \ \text{GeV}) \) at \( m_q = 0 \)

Renormalization window

① needs sufficiently large \( x \) in order to avoid discretization effects

② needs sufficiently small \( x \) where perturbation theory is reliable

\[
a \ll x \ll \Lambda_{QCD}^{-1}
\]
2. Strategy — lattice action & ensembles

**Lattice action**
- ♦ 2+1 generalized (Möbius) domain-wall fermions with 3-times stout smearing
- ♦ Symanzik improved gauge action

**Ensembles**

| $\beta$ | $a^{-1}$ [GeV] | Volume | $am_s$ | $am_{ud}$ (Confs/Trajectory) |
|---------|----------------|--------|--------|-------------------------------|
| 4.17    | 2.4            | 32³x64 | 0.040  | 0.0190(20/10000), 0.0120(20/10000), 0.0070(20/10000), 0.0035(20/10000) |
|         |                |        | 0.030  | 0.0190(20/10000), 0.0120(20/10000), 0.0070(20/10000) |
| 4.35    | 3.6            | 48³x96 | 0.025  | 0.0120(20/10000), 0.0080(20/10000), 0.0042(20/10000) |
|         |                |        | 0.018  | 0.0120(20/10000), 0.0080(20/4260), 0.0042(20/10000) |
| 4.47    | 4.5            | 32³x64 | 0.018  | 0.0090(10/10000), 0.0060(10/10000), 0.0040(10/10000) |
|         |                |        | 0.015  | 0.0090(10/10000), 0.0060(10/10000), 0.0030(10/10000) |

64³x128 on the product run
3. Lattice Part — short-distance correlator

- Discretization effect is very large

![Graph showing the comparison between lattice, interacting and continuum, free models. The graph plots $\alpha^6 \Pi_{PP}$ against $(x/a)^2$ with data points and a fitted line for both lattice and continuum models. The lattice data shows a significant deviation from the continuum model, indicating a large discretization effect.](image-url)
3. Lattice Part — short-distance correlator

Discretization effect is very large

“lattice, interacting” and “lattice, free” are strongly correlated
⇒ discretization effect is similar
We use

\[ \Pi^\text{lat}_\Gamma(x) \rightarrow \Pi'_\Gamma^\text{lat}(x) \]

\[ = \Pi^\text{lat}_\Gamma(x) + \Pi^\text{cont,free}_\Gamma^\text{lat}(x) - \Pi^\text{lat,free}_\Gamma^\text{lat}(x) \]
3. Lattice Part — democratic cut

Ref: Cichy et al, Nucl.Phys.B864(2012)268

- \( \theta \): angle between \( x \) and \((1,1,1,1)\)

- Correlators at large \( \theta \) are more distorted

- Free correlators at small \( \theta \) are closer to those in continuum theory

- We neglect correlators at \( \theta > 30^\circ \)

\[ x^6 \Pi_{PP}^{lat}(x) \]

interacting, lattice
free, continuum

\[
\log(a^6 \Pi_{PP}^{lat}) \quad (x/a)^2
\]

\[
\Pi_{PP} \times x^6 \quad (x/a)^2
\]

0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1

0 5 10 15 20 25 30 35 40

free theory correction & democratic cut
4. Continuum Part — procedure

① Perturbative expansion of correlation functions up to 4-loop order (Chetyrkin-Maier, 2011)

\[ \Pi_{\text{PP,SS}}^{\text{MS}}(x, \mu) = \Pi_{\text{PP,SS}}^{\text{MS}}(x, \tilde{\mu}) = \frac{3}{\pi^4 x^6} \left( 1 + \sum_{n=1}^{\infty} \tilde{C}_n \tilde{a}_s^n \right) \]

\[ \Pi_{\text{VV,AA}}^{\text{MS}}(x) = \Pi_{\text{VV,AA}}^{\text{MS}}(x) = \frac{6}{\pi^4 x^6} \left( 1 + \sum_{n=1}^{\infty} \tilde{C}_n \tilde{a}_s^n \right) \]

\[ \tilde{a}_s = \frac{\alpha_s^{\text{MS}}(\tilde{\mu} = 1/x)}{\pi} = \frac{\alpha_s^{\text{MS}}(\mu = 2e^{-\gamma_E}/x)}{\pi} \]

② Scale evolution

\[ \Pi_{\text{SS,PP}}^{\text{MS}}(x, \tilde{\mu}') = \frac{c(a_s(\tilde{\mu}'))}{c(a_s(\tilde{\mu}))} \Pi_{\text{SS,PP}}^{\text{MS}}(x, \tilde{\mu}) \leftarrow \text{RG equation} \]

\[ c(x) : \text{known up to 4-loop order (Chetyrkin '97, Vermasren et al '97)} \]

\[ \Pi_{\text{VV,AA}}^{\text{MS}}(x) : \text{scale independent} \leftarrow \text{WTI} \]

① + ② → \[ \Pi_{\text{TT}}^{\text{MS}}(x, 2 \text{ GeV}) \]
4. Continuum Part — convergence

Convergence of the perturbative series

Numerical coefficients of $\tilde{\alpha}_s^4$ are relatively large

$$\Pi_{\text{SS}}^{\overline{\text{MS}}}(x, \tilde{\mu}) = \frac{3}{\pi^4 x^6} \left(1 + 0.67\tilde{\alpha}_s - 16.3\tilde{\alpha}_s^2 - 31\tilde{\alpha}_s^3 + 497\tilde{\alpha}_s^4\right)$$

$$\Pi_{\text{VV}}^{\overline{\text{MS}}}(x) = \frac{6}{\pi^4 x^6} \left(1 + \tilde{\alpha}_s - 4\tilde{\alpha}_s^2 - 1.9\tilde{\alpha}_s^3 + 94\tilde{\alpha}_s^4\right)$$

$\alpha_s(\mu) \leftarrow N_f = 3, \ \Lambda_{\text{QCD}} = 340 \text{ MeV}$
4. Continuum Part — improve convergence

- Scale evolution of $a_s \longrightarrow$ a new perturbative series
  - $a_s(\bar{\mu}) = a_s(\bar{\mu}')(1 + \sum_{n=1}^{\infty} \delta_n(l)a_s^n(\bar{\mu}'))$
  - We know $\delta_1, \delta_2, \delta_3, \delta_4$ as functions of $l = 2\ln(\bar{\mu}'/\bar{\mu})$
  - $\longrightarrow \Pi_{\text{MS}}^\Gamma(x, \bar{\mu}) = \frac{3 \text{ or } 6}{\pi^4 x^6} (1 + \sum_{n=1}^{\infty} C_n^\Gamma a_s^n(\bar{\mu}'))$ up to $n = 4$

- Scale evolution of correlation function (for scalar channel)
  - $\Pi_{\text{SS}}^\Gamma(x, \bar{\mu}') = \frac{c(a_s(\bar{\mu}'))}{c(a_s(\bar{\mu}))} \Pi_{\text{SS}}^\Gamma(x, \bar{\mu})$ as a polynomial of $a_s(\bar{\mu}')$
  - Perform the scale evolution: $(\overline{\text{MS}}, \bar{\mu}') \rightarrow (\overline{\text{MS}}, 2 \text{ GeV})$

- We use BLM scale for vector channel (Brodsky-Lepage-Mackenzie ’83)
  - $\bar{\mu}' = \bar{\mu}^* = \bar{\mu} \exp ( -11/6 + 2\zeta(3)) \simeq 1.8\bar{\mu}$
  - $\bar{a}_s^* \equiv a_s(\bar{\mu}^*)$

- Improved perturbative series
  - $\Pi_{\text{SS}}^\Gamma(x, \bar{\mu}^*) = \frac{3}{\pi^4 x^6} (1 + 2.9\bar{a}_s^* + 1.1\bar{a}_s^{*2} - 42\bar{a}_s^{*3} + 24\bar{a}_s^{*4})$
  - $\Pi_{\text{VV}}^\Gamma(x) = \frac{6}{\pi^4 x^6} (1 + \bar{a}_s^* + 0.083\bar{a}_s^{*2} - 6\bar{a}_s^{*3} + 18\bar{a}_s^{*4})$
4. Continuum Part — consistency between lattice & PT

- Perturbative series become much better
- Correlators on lattice & PT are roughly same valued

\[ Z^2 \Pi_{latt}^{\Gamma \Gamma} x^6, \ Z = 0.924 \]

\[ Z^2 \Pi_{MS}^{\Gamma \Gamma} x^6, \ Z = 1.00 \]

- Scalar, Pseudoscalar
- Vector, Axial-Vector

- \[ m_{ud} \rightarrow 0, \ m_s = 320 \text{ MeV}, \ a^{-1} = 3.6 \text{ GeV} \]
5. Result — plot for $\beta = 4.35$, $am_s = 0.025$

$Z_{\Gamma}^{MS/lat}(2 \text{ GeV}) = \sqrt{\frac{\Pi_{\Gamma \Gamma}^{MS}(x, 2 \text{ GeV})}{\Pi_{\Gamma \Gamma}^{lat}(x)}}$ are ideally independent of $x$

- Extract RCs from renormalization window where “$x$-dependence” of $Z_{\Gamma}^{MS}(2 \text{ GeV})$ is roughly absent (i.e. plateau)

$Z_{S}^{MS/lat}(2 \text{ GeV}) = 0.924 \pm 0.021 \pm 0.003$

$Z_{V}^{MS/lat} = 0.990 \pm 0.011 \pm 0.007$

Statistical error

Systematic error
5. Result — preliminary values

- Systematic error $\lesssim 1\%$

- 20 confs $\rightarrow$ 200 confs: statistical error $\rightarrow < 1\%$

| $\beta$ | $a^{-1}$ [GeV] | $a m_s$ | $Z_{MS/lat}^S$ (2 GeV) | $Z_{V/lat}^{MS}$ |
|---------|----------------|---------|------------------------|-----------------|
| 4.17    | 2.4            | 0.040   | 1.144(12)(8)           | 1.033(3)(11)    |
|         |                 | 0.030   | 1.073(28)(13)          | 1.003(9)(5)     |
| 4.35    | 3.6            | 0.025   | 0.924(21)(3)           | 0.990(11)(7)    |
|         |                 | 0.018   | 0.975(15)(13)          | 1.009(4)(3)     |
| 4.47    | 4.5            | 0.018   | on going               | on going        |
|         |                 | 0.015   | on going               | on going        |

Notes:
- The $\beta$ values range from 4.17 to 4.47.
- $a^{-1}$ represents the inverse lattice spacing in GeV.
- $a m_s$ is the bare mass.
- $Z_{MS/lat}^S$ and $Z_{V/lat}^{MS}$ are renormalization factors.
- Systematic and statistical errors are indicated.
- The results are ongoing for some configurations.
We investigate NPR of quark bilinears using X-space method

✧ Discretization effect is reduced by applying the free theory correction and democratic cut
✧ Convergence of perturbative series of correlation functions become better by expanding correlators in a polynomials of coupling $\tilde{a}_s^*$ at an appropriate $\tilde{\mu}^*$

Goal is to obtain RCs within 1% precision

✧ Statistical error is currently larger than systematic error
✧ 20 confs → 200 confs: statistical error would become < 1 %

Furthermore we will try to

✧ extract $\alpha_s$, $\langle \bar{q}q \rangle$, ... from short distance correlators
✧ apply to heavy quark physics