Topology of the Spaces of Functions
with Prescribed Singularities on Surfaces

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Let $M$ be a smooth connected orientable closed surface and $f_0 \in C^\infty(M)$ a function having only critical points of the $A_\mu$-types, $\mu \in \mathbb{N}$. Let $\mathcal{F} = \mathcal{F}(f_0)$ be the set of functions $f \in C^\infty(M)$ having the same types of local singularities as those of $f_0$. We describe the homotopy type of the space $\mathcal{F}$, endowed with the $C^\infty$-topology, and its decomposition into orbits of the action of the group of “left-right changings of coordinates”.

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Let us give a short historical overview, mostly for the case of a Morse function $f_0$ (see the paper [3] and references therein). A. T. Fomenko posed the question (1997) whether the space $\mathcal{F}$ is arcwise connected; it was answered affirmatively by the author [6] for $M = S^2, \mathbb{R}P^2$, by S. V. Matveev [6] and H. Zieschang in the general case. Open $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$-orbits in $\mathcal{F}$ were counted by V. I. Arnold [7] and E. V. Kulinich (1998). Homotopy type of any $\mathcal{D}^0(M)$-orbit in $\mathcal{F}$ was studied by S. I. Maksymenko [8] (when $f_0$ was allowed to have certain degenerate types singularities) and by the author [3–5]. V. A. Vassiliev [9] proved the parametric $h$-principle and studied cohomology of spaces of smooth $\mathbb{R}^N$-valued functions not having too complicated singularities on any smooth manifold $M$. However the 1-parameter $h$-principle

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fails for the spaces of Morse functions on some $M$ with $\dim M > 5$ [10].

1. MAIN RESULT

For any function $f \in C^\infty(M)$, denote by $C_f$ the set of its critical points, and by $C_f^{\text{triv}}$ the set of critical points of the $A_{2m}$-types, $m \in \mathbb{N}$. Recall that, in a neighbourhood of such a point $x \in C_f$, there exist local coordinates $u, v$ such that $f = \eta(u^{2m+1} + v^2) + f(x)$ for some sign $\eta \in \{+, -\}$. The integer $\eta m$ will be called the level of the point $x$.

Denote by $C_{f}^{\text{min}}$ and $C_{f}^{\text{max}}$ (respectively $C_{f}^{\text{saddle}}$) the set of critical points of $f$ of $A_{2m-1}$-types, $m \in \mathbb{N}$, which are (respectively are not) points of local minima or local maxima. In a neighbourhood of such a point $x$, there exist local coordinates $u, v$ such that $f = \eta(u^{2m} \pm v^2) + f(x)$ where $\eta \in \{+, -\}$. The integer $\eta(m-1)$ will be called the level of the point $x$.

The subset of degenerate critical points (i.e. those of non-zero levels) in $\hat{C}_{\text{extr}}^f := C_{f}^{\text{min}} \cup C_{f}^{\text{max}}$ will be denoted by $\hat{C}_{\text{extr}}^f$.

Suppose that an action of a group $G$ on a topological space $X$, a stratified [11] orbifold $Y$ and a continuous surjection $\kappa : X \to Y$ are given. If every $G$-orbit in $X$ is the full pre-image of a stratum from $Y$, we will say that $\kappa$ classifies $G$-orbits, while $Y$ and $\kappa$ are the classifying space and map.

The group $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$ acts on $M \times F$ by the homeomorphisms $(x, f) \mapsto (h^{-1}(x), h^{-1} \circ f \circ h)$, $(h_1, h) \in \mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$. Define the evaluation functional $\text{Eval} : M \times F \to \mathbb{R}$, $(x, f) \mapsto f(x)$, and

$$s := \max\{0, \chi(M) + 1\} > \chi(M).$$

Theorem. For every function $f_0 \in C^\infty(M)$ whose all critical points are of the $A_{\mu}$-types, $\mu \in \mathbb{N}$, there exist smooth manifolds $\mathcal{B}$ and $\mathcal{E}$ and surjective submersions $k : F \to \mathcal{B}$, $\kappa : M \times F \to \mathcal{E}$, $\pi : \mathcal{E} \to \mathcal{B}$, $\varepsilon : \mathcal{E} \to \mathbb{R}$ such that the diagram

\[
\begin{array}{ccc}
M \times F & \xrightarrow{\text{Eval}} & \mathcal{E} \\
\downarrow \text{Pr} & & \downarrow \pi \\
F & \xrightarrow{k} & \mathcal{B}
\end{array}
\]

commutes, where $\text{Pr} : M \times F \to F$ is the projection and $\dim \mathcal{B} = 2s + 2|C_{f_0}^{\text{triv}}| + |C_{f_0}^{\text{extr}}| + |\hat{C}_{f_0}^{\text{extr}}| + 3|C_{f_0}^{\text{saddle}}| = \dim \mathcal{E} - 2$. Moreover:
(a) the maps $k, \pi$ are homotopy equivalences and classify $D^0(M)$- and $D(\mathbb{R}) \times D^0(M)$-orbits in $\mathcal{F}, M \times \mathcal{F}$ for some stratifications of $\mathcal{B}, \mathcal{E}$ whose all strata are submanifolds; the map $\pi$ is a fibre bundle with fibres diffeomorphic to $M$.

(b) the map $k$ (resp. $\pi$) induces a homotopy equivalence between every $D^0(M)$-invariant subset $B \subseteq \mathcal{F}$ (resp. $E \subseteq M \times \mathcal{F}$) and its image, e.g. between every orbit from item (a) and the corresponding stratum.

(c) the group $\text{MCG}(M) = D(M)/D^0(M)$ discretely acts on $\mathcal{B}, \mathcal{E}$ by diffeomorphisms preserving the stratifications from item (a) and the function $\varepsilon$; the maps $p \circ k : \mathcal{F} \to \mathcal{B} := \mathcal{B}/\text{MCG}(M)$ and $P \circ \pi : M \times \mathcal{F} \to \mathcal{E} := \mathcal{E}/\text{MCG}(M)$ classify $D(M)$- and $D(\mathbb{R}) \times D(M)$-orbits in $\mathcal{F}$ and $M \times \mathcal{F}$ for the induced stratifications on $\mathcal{B}'$ and $\mathcal{E}'$, where $p : \mathcal{B} \to \mathcal{B}'$ and $P : \mathcal{E} \to \mathcal{E}'$ are the projections.

Let us explain the term “submersion” in the case of functional spaces. If $Q, R$ are smooth manifolds and $Q := Q \times \mathcal{F}$, denote by $C^\infty(R, Q)$ the preimage of $C^\infty(R, Q) \times C^\infty(R \times M)$ under the inclusion $C(R, Q) \hookrightarrow C(R, Q) \times C(R \times M)$, and by $C^\infty(Q, R)$ the set of maps inducing maps $C^\infty(\mathbb{R}^n, Q) \to C^\infty(\mathbb{R}^n, R)$ for all $n \in \mathbb{N}$. A map $p \in C^\infty(Q, R)$ will be called a submersion if, for any $q \in Q$, there exist a neighborhood $U$ of the point $p(q)$ in $R$ and a map $\sigma \in C^\infty(U, Q)$ such that $p \circ \sigma = \text{id}_U$.

2. CONSTRUCTING THE CLASSIFYING MANIFOLDS AND MAPS

Similarly to [12], by a framed function on an oriented surface $M$ we will mean a pair $(f, \alpha)$ where $f \in C^\infty(M)$ has only the $A_\mu$-types local singularities and $\alpha$ is a closed 1–form on $M \setminus C^\text{extr}_f$ such that (i) the 2-form $df \wedge \alpha$ has no zeros on $M \setminus C_f$ and defines a positive orientation, (ii) in a neighbourhood of every critical point $x \in C_f$ there exist local coordinates $u, v$ such that either $f = \eta(u^{2m+1} + v^2) + f(x)$ and $\alpha = \eta d(v - uv)$, or $f = \eta(u^{2m} - v^2) + f(x)$ and $\alpha = \eta d(uv)$, or $f = \eta(u^{2m} + v^2) + f(x)$ and $\alpha = \eta \omega_x \frac{vdu - vdu}{u^2 + v^2}$ where $\omega_x = \text{const} > 0$, $m \in \mathbb{N}$, $\eta \in \{+, -\}$.

Denote by $\mathcal{F} = \mathcal{F}(f_0)$ the space of framed functions $(f, \alpha)$ such that $f \in \mathcal{F}$. Endow this space with the $C^\infty$-topology [12]. Consider the right actions of $D(\mathbb{R}) \times D(M)$ on $\mathcal{F}$ and $M \times \mathcal{F}$ by the homeomorphisms $(f, \alpha) \mapsto (h_1^{-1} \circ f \circ h, h^* \alpha)$ and $(x, f, \alpha) \mapsto (h^{-1}(x), h_1^{-1} \circ f \circ h, h^* \alpha)$, $(h_1, h) \in D(\mathbb{R}) \times D(M)$.

Let $x_1, x_2, \ldots \in M$ be pairwise distinct points. Denote by $D^0_1(M)$ the identity component
of the group $\mathcal{D}_r(M) := \{ h \in \mathcal{D}(M) \mid h(x_i) = x_i, 1 \leq i \leq r \}$, $r \in \mathbb{Z}_+$, whence $\mathcal{D}_0(M) = \mathcal{D}(M)$.

Define the classifying manifolds $\mathcal{B}$ and $\mathcal{E}$ as $\mathcal{B} := \mathcal{B}_s$, $\mathcal{E} := \mathcal{E}_s$, where $\mathcal{B}_r$ and $\mathcal{E}_r$ are the universal moduli spaces

$$\mathcal{B}_r := \mathbb{F}/\mathcal{D}_r^0(M), \quad \mathcal{E}_r := (M \times \mathbb{F})/\mathcal{D}_r^0(M)$$

of framed functions (resp. framed functions with one marked point) in $\mathcal{F}$, $r \in \mathbb{Z}_+$. One shows similarly to [3, 4] that $\mathcal{B}_r$ and $\mathcal{E}_r$ are orbifolds of dimensions $\dim \mathcal{B}_r = 2r + 2|C^\text{triv}| + |C^\text{extr}| + |\hat{C}^\text{extr}| + 3|C^\text{saddle}| = \dim \mathcal{E}_r - 2$. For every group $\mathcal{G} \in \{ \mathcal{D}_r^0(M), \mathcal{D}(\mathbb{R}) \times \mathcal{D}_r^0(M) \}$, we endow $\mathcal{B}_r$ and $\mathcal{E}_r$ with the stratifications whose every stratum is the full preimage of a point under the projection $\mathcal{B}_r \to \mathcal{F}/\mathcal{G}$ and $\mathcal{E}_r \to (M \times \mathcal{F})/\mathcal{G}$.

Due to the $\mathcal{D}(M)$-equivariance of the projection $M \times \mathbb{F} \to \mathbb{F}$ and the $\mathcal{D}(M)$-invariance of the evaluation functional $M \times \mathbb{F} \to \mathbb{R}$, $(x, f, \alpha) \mapsto f(x)$, they induce some maps $\pi_r : \mathcal{E}_r \to \mathcal{B}_r$ and $\varepsilon_r : \mathcal{E}_r \to \mathbb{R}$. Put $\pi = \pi_s$, $\varepsilon = \varepsilon_s$.

Similarly to [12, Theorem 2.5] and [3, Statement 5.3], one proves the following lemmata which readily imply the theorem.

**Lemma 1.** The projection $\text{Forg} : \mathbb{F} \to \mathcal{F}$, $(f, \alpha) \mapsto f$, is a homotopy equivalence and has a homotopy inverse map $i : \mathcal{F} \to \mathbb{F}$ and corresponding homotopies that respect the projections $q : \mathcal{F} \to \mathcal{F}/\mathcal{D}_r^0(M)$ and $q \circ \text{Forg} : \mathbb{F} \to \mathcal{F}/\mathcal{D}_r^0(M)$.

**Lemma 2.** If $r \geq s$ then $\mathcal{B}_r$ is a smooth manifold, while the projection $\text{Ev}_r : \mathbb{F} \to \mathcal{F}_r$ is a homotopy equivalence and has a homotopy inverse map $i_r : \mathcal{F}_r \to \mathbb{F}$ and corresponding homotopies that respect $\text{Ev}_r$ (whence $\text{Ev}_r \circ i_r = \text{id}_{\mathcal{F}_r}$).

Put $k_r = \text{Ev}_r \circ i : \mathcal{F} \to \mathcal{F}_r$. One defines similarly $\kappa_r$. Define the classifying maps $k = k_s$, $\kappa = \kappa_s$.

### 3. REDUCING TO THE CASE OF MORSE FUNCTIONS

If $f_0$ is a Morse function and $s = 0$, then the space $\mathcal{B}$ from §2 coincides with the smooth stratified manifold $\widetilde{\mathcal{M}}$ (the universal moduli space of framed Morse functions) studied in [3–5]. It happens that every $\mathcal{B}_r$ and $\mathcal{E}_r$ can be described in terms of Morse functions.

Recall that a function $f \in C^\infty(M)$ is said to be Morse if all its critical points are nondegenerate (i.e. have the $A_1$-type, cf. §1). Denote by $\text{Morse}(f_0)$ the space of Morse functions on
having exactly $|C_{f_0}^{\text{min}}|$ and $|C_{f_0}^{\text{max}}|$ points of local minima and maxima and $|C_{f_0}^{\text{saddle}}|$ saddle points.

A Morse function $f \in \text{Morse}(f_0)$ will be called $f_0$-labeled if every its critical point $x \in C_f$ is labeled by an integer and, in the case when this integer does not vanish and $x \in C_{f_0}^{\text{extr}}$, also by a 1-dimensional subspace $\ell_x \subset T_x M$, moreover $|C_{f_0}^{\text{triv}}|$ of non-critical points of $f$ are labeled by non-zero integers in such a way that the level (cf. §1) of every critical point of $f_0$ coincides with the integer label of the corresponding labeled point of $f$, for some bijections $C_{f_0}^{\text{min}} \approx C_{f}^{\text{min}}$, $C_{f_0}^{\text{max}} \approx C_{f}^{\text{max}}$, $C_{f_0}^{\text{saddle}} \approx C_{f}^{\text{saddle}}$ and a bijection between $C_{f_0}^{\text{triv}}$ and the set of labeled non-critical points of $f$.

Denote by $\text{Morse}^*(f_0)$ the space of framed (cf. §2) $f_0$-labeled Morse functions. It is not difficult to construct homeomorphisms

$$B_r \approx \text{Morse}^*(f_0)/\mathcal{D}^0_r(M), \quad \mathcal{E}_r \approx (M \times \text{Morse}^*(f_0))/\mathcal{D}^0_r(M), \quad r \in \mathbb{Z}_+.$$ (2)

4. RELATION WITH MEROMORPHIC FUNCTIONS AND THE CONFIGURATION SPACES

Suppose that $M$ is either a sphere $S^2$ or a torus $T^2$. If $M = S^2$, denote by $\mathbb{A}(f_0)$ the space of rational functions $R$ on the Riemann sphere $\overline{\mathbb{C}}$ such that all poles of the 1-form $\omega = R(z)dz$ are simple and have real residues, being positive at $|C_{f_0}^{\text{min}}|$ poles and negative at $|C_{f_0}^{\text{max}}|$ poles. If $M = T^2$, denote by $\mathbb{A}(f_0)$ the space of pairs $(\lambda, R)$ where $\lambda \in \mathbb{C}$, $\text{Im} \lambda > 0$, and $R$ is a meromorphic function on the torus $T^2_{\lambda} = \mathbb{C}/(\mathbb{Z} + \lambda\mathbb{Z})$, whose poles are all simple, all periods of the meromorphic 1-form $\omega = R(z)dz$ are purely imaginary, and the residues are positive at $|C_{f_0}^{\text{min}}|$ poles and negative at $|C_{f_0}^{\text{max}}|$ poles.

Let $\mathbb{A}_0(f_0)$ be the space of functions $R \in \mathbb{A}(f_0)$ or pairs $(\lambda, R) \in \mathbb{A}(f_0)$ such that $\omega = R(z)dz$ has only simple zeros.

Due to [13, Proposition 3.4], the assignment to a 1-form $\omega$ its poles and residues at them gives a bijection $\varphi : \mathbb{A}(f_0) \xrightarrow{\sim} C(f_0)$, where $C(f_0)$ is the “labeled configuration space” consisting of $|C_{f_0}^{\text{extr}}|$-points subsets of $M$ equipped by $|C_{f_0}^{\text{min}}|$ positive and $|C_{f_0}^{\text{max}}|$ negative real marks with zero total sum. Thus $\mathbb{A}_0(f_0)$ is homeomorphic to the open subset $\varphi(\mathbb{A}_0(f_0)) \subseteq C(f_0)$ consisting of the “labeled configurations” that correspond to 1-forms $\omega$ without multiple zeros.
It is not difficult to derive from (2) with \( r = s \) (cf. (1) and [12, Remark 2.6]) that our manifold \( \mathcal{B} \) is homeomorphic to the space \( \mathcal{A}_0(f_0) \) of functions \( R \in \mathcal{A}_0(f_0) \) or pairs \((\lambda, R) \in \mathcal{A}_0(f_0)\), marked by \( f_0 \)-labels (cf. §3) at zeros and poles of the 1-form \( \omega = R(z)dz \) and at some other \(|C^{\text{triv}}_{f_0}|\) points, as well as by a “vertical” label consisting of (i) a real label and (ii) either a positive real label in the case of \(|C^{\text{saddle}}_{f_0}| = |C^{\text{extr}}_{f_0}| = 0\), or \(|C^{\text{extr}}_{f_0}|\) integral curves of the field \( \ker(\Re \omega) \) separating the poles from other labeled points. Thus, the manifold \( \mathcal{B} \approx \mathcal{A}_0(f_0) \) can be obtained from the “labeled configuration subspace” \( \varphi(\mathcal{A}_0(f_0)) \subseteq C(f_0) \) by assigning the \( f_0 \)-labels and the (topologically inessential) “vertical” label.

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