Two-loop fermionic electroweak corrections to the Z-boson width and production rate

Ayres Freitas

Pittsburgh Particle-physics, Astro-physics & Cosmology Center (PITP–PACC), Department of Physics & Astronomy, University of Pittsburgh, Pittsburgh, PA 15260, USA

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Improved predictions for the Z-boson decay width and the hadronic Z-peak cross-section within the Standard Model are presented, based on a complete calculation of electroweak two-loop corrections with closed fermion loops. Compared to previous partial results, the predictions for the Z width and hadronic cross-section shift by about 0.6 MeV and 0.004 nb, respectively. Compact parametrization formulae are provided, which approximate the full results to better than 4 ppm.

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The recent discovery of the Higgs boson [1] has been a tremendous success of the Standard Model (SM). Remarkably, the observed mass of the Higgs boson, \( M_H \), agrees very well with the value that has been predicted many years earlier from electroweak (EW) precision observables, which are quantities that have been measured with very high accuracy at lower energies, and that can be calculated comparably precisely within the SM. Besides predicting the Higgs mass, global fits to all available electroweak precision observables are crucial for testing the SM at the quantum level and constraining new physics (see Refs. [2–4] for recent examples).

On the theory side, these precision tests rely on calculations of radiative corrections for the relevant observables. At the current level of precision, two-loop EW and higher-order QCD corrections are numerically important [5,6]. Complete two-loop contributions are known for the prediction of the mass of the W boson, \( M_W \) [5,7–9], and the leptonic effective weak mixing angle, \( \sin^2 \theta_W \) [6,7,10,11], which is related to the ratio of vector and axial-vector couplings of the Z boson to leptons. Furthermore, the leading three- and four-loop corrections to these observables in the limit of large values of the top-quark mass, \( m_t \), have been obtained. These are EW contributions of order \( O(\alpha^2 m_t^2) \) [12] and mixed EW/QCD terms of \( O(\alpha \alpha_2 m_t^2) \) [13], \( O(\alpha^2 \alpha m_t^2) \) [12], and \( O(\alpha \alpha_2 m_t^2) \) [14]. Similarly precise results are available for the effective weak mixing angle of quarks [11,15] and the ratio of the Z-boson partial widths into \( bb \) and all hadronic final states, \( R_b \equiv \Gamma_{Z \rightarrow bb} / \Gamma_{Z \rightarrow had} \) [16], except that the so-called bosonic EW two-loop corrections stemming from diagrams without closed fermion loops are not known for these quantities. However, detailed analyses and experience from the calculation of \( M_W \) and \( \sin^2 \theta_W \) indicate that these are small.

Most of the published results have been implemented into the public code ZFITTER [17] and several private packages [4,18].

However, for the decay width and production cross-section of the Z boson, so far only approximate results for the EW two-loop corrections have been calculated in a large-\( m_t \) expansion, including the next-to-leading order \( O(\alpha^2 m_t^2) \) [19] for leptonic final states and quarks of the first two generations, while for the \( Z \rightarrow bb \) partial width merely the leading \( O(\alpha^2 m_t^2) \) coefficient is known [20]. It was estimated that the missing EW two-loop corrections lead to an uncertainty of several MeV for the prediction of the Z width [21], which is comparable to the experimental error, but has not been properly accounted for in the global SM fits. In this Letter, this gap is filled by presenting the complete fermionic EW two-loop corrections (i.e. from diagrams with one or two closed fermion loops) for the Z-boson width and production rate. With these new results, the theoretical uncertainty from missing higher-order contributions in electroweak fits will be significantly reduced.

The width of the Z boson is defined through the imaginary part of the complex pole \( s_0 = M_Z^2 - i\Gamma_Z T_Z \) of the propagator

\[
[s - M_Z^2 + \Sigma_Z(s)]^{-1},
\]

where \( \Sigma_Z(s) \) is the transverse part of the Z self-energy, resulting in

\[
\Gamma_Z = \frac{1}{M_Z} \text{Re} \Sigma_Z(s_0).
\]

This definition is consistent and gauge-invariant too all orders [22].

Expanding Eq. (2) up to next-to-next-to-leading order in the electroweak coupling and using the optical theorem leads to (see Ref. [23] for details)
\[ T_Z = \sum_f T_f', \quad T_f' = \frac{N_c M_Z}{12\pi} [R_{V}^{f} F_{V}^{f} + R_{A}^{f} F_{A}^{f}], \quad f = s, c, b \]  
\[ F_{V}^{f} = V_{f(0)} [1 - \Re \Sigma^{(1)}_{f} - \Re \Sigma^{(2)}_{f}] + \Re \Sigma^{(2)}_{f}, \quad F_{A}^{f} = \frac{1}{2} M_Z \Gamma_Z V_{f(0)} \Im \Sigma^{(2)}_{f}, \]  
where \( V_f \) is the effective vector \( Zf \bar{f} \) coupling, which includes EW vertex corrections and \( Z-\gamma \) mixing contributions, and \( F_A \) is defined similarly in terms of the axial-vector coupling \( a_f \). The subscripts in brackets indicate the loop order and \( \Sigma^2 \) is the derivative of \( \Sigma^2 \). The radiator functions \( R_{V,A} \) take into account final-state QED and QCD radiation and are known up to \( O(\alpha^4) \) in the limit of massless quarks and \( O(\alpha_s^2) \) for the kinematic mass corrections [24].

Note that the mass and width defined through the complex pole of (1) differ from the experimentally reported values, \( M_Z \) and \( \Gamma_Z \), since the latter have been obtained using a Breit–Wigner formula with a running width. The two are related via
\[ \bar{M}_Z = M_Z/\sqrt{1 + \Gamma_Z^2/M_Z^2}, \quad \bar{\Gamma}_Z = \Gamma_Z/\sqrt{1 + \Gamma_Z^2/M_Z^2}. \]  

amounting to \( \bar{M}_Z \approx M_Z - 34 \text{ MeV} \) and \( \bar{\Gamma}_Z \approx \Gamma_Z - 0.9 \text{ MeV} \). Now let us turn to the \( Z \)-boson cross-section. After subtracting contributions from photon exchange and box diagrams, the amplitude for \( e^+e^- \to f \bar{f} \) can be written as an expansion about the complex pole,
\[ A_Z(s) = \frac{R}{s - s_0} + S(s - s_0) \delta s + \cdots. \]  

Instead of the total cross-section, it is customary to use the hadronic peak cross-section defined as
\[ \sigma_{\text{had}}^0 = \frac{1}{64\pi^2 M^2_Z} \sum_{f=u,d,c,b} \int d\Omega |A_Z(M^2_Z)|^2. \]  

Starting from (6), an explicit calculation yields
\[ \sigma_{\text{had}}^0 = \sum_{f=u,d,c,b} \frac{12\pi}{16\pi^2 M^2_Z} \frac{T_{e f}}{T_{e e}} (1 + \delta \chi), \]  
where \( \delta \chi \) occurs first at two-loop level [25] and is given by
\[ \delta \chi_{(2)} = -\left(\Re \Sigma^{(1)}_{f}\right)^2 - 2 \bar{\Gamma}_Z M_Z \Im \Sigma^{(2)}_{f}. \]  

The calculation of the \( O(\alpha^2) \) corrections to \( T_Z \) and \( \sigma_{\text{had}}^0 \) was carried out as follows: Diagrams for the form factors \( v_{f(n)} \) and \( a_{f(n)} \) were generated with FeynArts 3.3 [26]. For the renormalization of the on-shell scheme has been used, as described in Ref. [5]. Two-loop self-energy integrals and vertex integrals with sub-loop self-energy bubbles have been evaluated with the method illustrated in Section 3.2 of Ref. [11], while for vertex diagrams with sub-loop triangles the technique of Ref. [27] has been employed. As a cross-check, the results for \( \sin^2 \theta_W^2 \) [6], \( \sin^2 \theta_W^2 \) [15], and \( R_b \) [16] have been reproduced using the effective couplings \( v_{f(n)} \) and \( a_{f(n)} \), and very good agreement within theory uncertainties has been obtained.

For the presentation of numerical results, the one- and two-loop EW corrections have been combined with virtual loop corrections of order \( O(\alpha \alpha_s) \) [7], which have been re-computed for this work, and further higher-order corrections proportional to \( \alpha_s \equiv m_t^2 \), of order \( O(\alpha_s^2) \) [13], \( O(\alpha_s^3) \) [12], and \( O(\alpha_s^3) \) [14]. Final-state QED and QCD radiation has been included via the radiator functions \( R_{V,A} \) as described after Eq. (4). However, the factorization between EW corrections \( F_{V,A} \) and final-state radiation \( (\alpha_{\text{ew}}) \alpha_{\text{QCD}} \) in (3) is not exact, but additional non-factorizable contributions appear first at \( O(\alpha_s \alpha_t) \) [28,29]. All fermion masses except for \( m_t \) have been neglected in the EW two-loop corrections, but a finite \( b \) quark mass has been retained in the \( O(\alpha_s) \) and \( O(\alpha_t) \) contributions, and non-zero bottom, charm and tau masses are included in the radiators \( R_{V,A} \).

The final combined result is evaluated as a perturbative expansion in \( \alpha \) and \( \alpha_s \), rather than the Fermi constant \( G_F \). Using the input parameters in Table 1, except \( G_F \), the size of various loop contributions is shown in Table 2.

However, it is common practice not to use the \( W \) mass, \( M_W \), as an input parameter, but instead to determine \( M_W \) from \( G_F \). Using the results from Refs. [5,8,9] for the computation of \( M_W \), the numbers in Table 3 are obtained. For comparison, also shown are the corresponding results based on the approximation of the EW two-loop corrections for large values of \( m_t \) [19,20]. The difference illustrates the impact of the full fermionic two-loop corrections beyond the large-\( m_t \) approximation, which can be seen to be of moderate size, about 0.6 MeV for \( \bar{\Gamma}_Z \) and about 0.004 nb for \( \sigma_{\text{had}}^0 \).

The new results, including all corrections described above and the currently most precise result for \( M_W \) [9], can be accurately described by the simple parameterization formula
\[ X = X_0 + c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4^2 + c_5 X_5 X_6 + c_6 X_7 + c_7 X_8, \]  
\[ L_H = \log \frac{M_H}{125.7 \text{ GeV}}, \quad \Delta_t = \left( \frac{m_t}{173.2 \text{ GeV}} \right)^2 - 1, \]  
\[ \Delta_{\alpha_s} = \frac{\alpha_s}{0.1184} - 1, \quad \Delta_\mu = \frac{\Delta \mu}{0.059} - 1, \]  
\[ \Delta Z = \frac{M_Z}{91.1876 \text{ GeV}} - 1. \]  

Note that Eq. (9) differs from the expression shown in Ref. [25] since the non-resonant terms in Eq. (5) were not included there. See Ref. [23] for details.

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**Table 1**

| Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|
| \( M_Z \) | 91.1876 GeV | \( m_t^\text{BP} \) | 4.20 GeV |
| \( I_Z \) | 2.4952 GeV | \( m_\tau \) | 1.275 GeV |
| \( M_W \) | 80.385 GeV | \( \Delta \alpha \) | 0.0590 |
| \( m_t \) | 125.7 GeV | \( \alpha_s(M_Z) \) | 0.1184 |

**Table 2**

| Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|
| \( I_Z \) | 60.26 | \( \Delta \alpha \) | 48.85 |
| \( \alpha_s \) | 9.11 | \( \Delta \alpha \) | 3.14 |
| \( \alpha_s \) | 1.20 | \( \Delta \alpha \) | 0.48 |
| \( \alpha_s \) | 5.13 | \( \Delta \alpha \) | 1.03 |
| \( \alpha_s \) | 3.04 | \( \Delta \alpha \) | 9.07 |

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1. These numbers have been kindly supplied by S. Mishima based on the work in Ref. [4].

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The coefficients are given by

$$X = I_2^0 [\text{MeV}]: \quad X_0 = 2494.24, \quad c_1 = -2.0, \quad c_2 = 19.7, \quad c_3 = 58.60, \quad c_4 = -4.0, \quad c_5 = 8.0, \quad c_6 = -55.9, \quad c_7 = 9267;$$

$$X = \sigma_{0_{\text{had}}} [\text{pb}]: \quad X_0 = 41488.4, \quad c_1 = 3.0, \quad c_2 = 69.0, \quad c_3 = -579.4, \quad c_4 = 38.1, \quad c_5 = 7.3, \quad c_6 = 85.4, \quad c_7 = -86.027.$$  \hspace{1cm} (11)

$$X = \sigma_{0_{\text{had}}} [\text{nb}]: \quad X_0 = 006 nb [23].$$

This formula approximates the full results to better than 0.01 MeV and 0.01 pb, respectively, for the input parameters within the ranges $M_H = 125.7 \pm 2.5$ GeV, $m_t = 173.2 \pm 2.0$ GeV, $\alpha_s = 0.1184 \pm 0.0005$, $\Delta \alpha = 0.0590 \pm 0.0005$ and $M_2 = 91.1876 \pm 0.0042$ GeV.

In summary, the complete fermionic two-loop electroweak corrections to the Z-boson decay width and the cross-section for $e^+e^- \to$ hadrons have been computed within the Standard Model. Compared to known previous calculations, the new contributions lead to shifts of 0.6 MeV in $I_2$ and 0.004 nb in $\sigma_{0_{\text{had}}}$, which are smaller than, but of comparable order of magnitude as the current experimental uncertainties (2.3 MeV and 0.037 nb) [30]. Therefore, the new results are important for robust predictions of these quantities, while the remaining theory uncertainty is estimated to be relatively small. It mainly stems from the missing bosonic $O(\alpha^2)$ contributions and $O(\alpha^2 \alpha_s)$, $O(\alpha^3)$ and $O(\alpha^3 \alpha_s)$ corrections beyond the large-$m_t$ approximation, leading to the estimates $\delta I_2 \approx 0.5$ MeV and $\delta \sigma_{0_{\text{had}}} \approx 0.006$ nb [23].

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