CYCLICITY IN DIRICHLET-TYPE SPACES AND EXTREMAL POLYNOMIALS

By

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Abstract. For functions $f$ in Dirichlet-type spaces $D_\alpha$, we study how to determine constructively optimal polynomials $p_n$ that minimize $\|pf - 1\|_\alpha$ among all polynomials $p$ of degree at most $n$. We then obtain sharp estimates for the rate of decay of $\|p_n f - 1\|_\alpha$ as $n$ approaches $\infty$, for certain classes of functions $f$. Finally, inspired by the Brown-Shields conjecture, we prove that certain logarithmic conditions on $f$ imply cyclicity, and we study some computational phenomena pertaining to the zeros of optimal polynomials.

1 Introduction

1.1 Cyclicity in spaces of analytic functions. In this paper, we study certain Hilbert spaces of analytic functions in the open unit disk $\mathbb{D}$, denoted $D_\alpha$ and referred to as Dirichlet-type spaces of order $\alpha$. For $-\infty < \alpha < \infty$, the space $D_\alpha$ consists of all analytic functions $f: \mathbb{D} \to \mathbb{C}$ whose Taylor coefficients in the expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{D},$$

satisfy

$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty} (k + 1)^\alpha |a_k|^2 < \infty.$$

It is easy to see that $D_\alpha \subseteq D_\beta$ if $\alpha \geq \beta$, and $f \in D_\alpha$ if and only if the derivative $f'$ satisfies $f' \in D_{\alpha-2}$.
Three values of $\alpha$ correspond to spaces that have been studied extensively and are often defined in terms of integrability:

- $\alpha = -1$ corresponds to the **Bergman space** $B$, consisting of functions such that
  \[
  \int_{D} |f(z)|^2 dA(z) < \infty, \quad dA(z) = \frac{dxdy}{\pi};
  \]

- $\alpha = 0$ corresponds to the **Hardy space** $H^2$, consisting of functions such that
  \[
  \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta < \infty;
  \]

- $\alpha = 1$ corresponds to the usual **Dirichlet space** $D$ of functions $f$ such that
  \[
  \int_{D} |f'(z)|^2 dA(z) < \infty.
  \]

A description similar to that of the Dirichlet space, in terms of an integral, is possible for the $D_\alpha$ spaces for $\alpha < 2$. Indeed, $f \in D_\alpha$ if and only if

\[
D_\alpha(f) = \int_{D} |f'(z)|^2(1 - |z|^2)^{1-\alpha} dA(z) < \infty.
\]

This expression can be used to define an equivalent norm for $f \in D_\alpha$, which we use in Section 2. We refer the reader to the books [7], [8], and [11] for in-depth treatments of Hardy and Bergman spaces; recent surveys concerning the Dirichlet space $D$ include [1] and [14].

A function $f \in D_\alpha$ is said to be **cyclic** in $D_\alpha$ if the closed subspace generated by polynomial multiples of $f$,

\[
[f] = \text{span}\{z^k f : k = 0, 1, 2, \ldots\},
\]

coincides with $D_\alpha$. Note that cyclicity in $D_\alpha$ implies cyclicity in $D_\beta$ for all $\beta < \alpha$. The **multiplier space** $M(D_\alpha)$ consists of analytic functions $\psi$ such that the induced operator $M_\psi: f \mapsto \psi f$ maps $D_\alpha$ into itself; such a function $\psi$ is called a **multiplier**. Thus cyclic functions are precisely those that are cyclic with respect to the operator $M_z$. Since polynomials are dense in $D_\alpha$, we have $[1] = D_\alpha$. It is well known (see [6]) that an equivalent (and more useful) condition for the cyclicity of $f$ is that there exist a sequence of polynomials $\{p_n\}_{n=1}^\infty$ such that

\[
||p_n f - 1||_\alpha \to 0 \quad \text{as} \quad n \to \infty.
\]

We note that for certain values of $\alpha$, the multiplier spaces of $D_\alpha$ are relatively easy to determine. For $\alpha \leq 0$, we have $M(D_\alpha) = H^\infty$; and for $\alpha > 1$, the multiplier space coincides with $D_\alpha$ itself (see [6, p. 273]).
In general, it is not an easy problem to characterize cyclic functions in a space of analytic functions. However, a complete answer to the cyclicity problem for $H^2$ (the case $\alpha = 0$) is given by a theorem of Beurling (see [7, Chapter 7]): $f$ is cyclic if and only if $f$ is an outer function. In particular, a cyclic function $f \in H^2$ cannot vanish in $\mathbb{D}$. In the Bergman space, the situation is considerably more complicated; see [11, Chapter 7]. An obvious common feature of all $D_\alpha$ is that cyclic functions cannot vanish in $D$, since if $f$ is cyclic, there exist $p_n$ such that $p_n f \to 1$ pointwise. For $\alpha > 1$, being non-vanishing in the closed unit disk or, equivalently, $f$ satisfying $|f(z)| > c > 0$ for all $z \in \mathbb{D}$ is a necessary and sufficient condition (see [6]) for cyclicity. On the other hand, if $\alpha \leq 1$, a function may still be cyclic if its zero set on the boundary, i.e., the unit circle $\mathbb{T}$, is not too large. Here, we define the zero set in an appropriate sense via, for instance, non-tangential limits.

In [6], L. Brown and A. L. Shields studied the phenomenon of cyclicity in the Dirichlet space. In particular, they established the following equivalent condition for cyclicity: $f$ is cyclic in $D_\alpha$ if and only if there exists a sequence of polynomials $\{p_n\}$ such that

\begin{equation}
\sup_n \|p_n f - 1\|_\alpha < \infty
\end{equation}

and, pointwise as $n \to \infty$,

\begin{equation}
{p}_n(z)f(z) \to 1, \quad z \in \mathbb{D}.
\end{equation}

Brown and Shields also obtained a number of partial results towards a characterization of cyclic vectors in the Dirichlet space $D$. Their starting point was a result of Beurling, stating that, for any $f \in D$, the non-tangential limit $f^*(\zeta) = \lim_{z \to \zeta} f(z)$ exists quasi-everywhere, that is, outside a set of logarithmic capacity zero. Brown and Shields proved that if $Z( f^*) = \{\zeta \in \mathbb{T}: f^*(\zeta) = 0\}$ is a set of positive logarithmic capacity, then $f$ cannot be cyclic. On the other hand, they also proved that $(1 - z)^\beta$ is cyclic for all $\beta > 0$ and showed that any polynomial without zeros in $\mathbb{D}$ is cyclic. Hence, they asked if being outer and having $\text{cap}(Z( f^*)) = 0$ is sufficient for $f$ to be cyclic. The positive answer to this question is known as the Brown-Shields conjecture, which remains open; see [10] for recent progress by El-Fallah, Kellay, and Ransford, and for background material. Subsequent to the Brown and Shields paper, Brown and Cohn showed [5] that sets of logarithmic capacity zero do support zeros of cyclic functions, and later Brown proved [4] that if $f \in D$ is invertible, that is $1/f \in D$, then $f$ is cyclic. However, there also exist cyclic functions $f$ for which $1/f \not\in D$, e.g., $f(z) = 1 - z$.

The problem of cyclicity in $D$ has been addressed in many papers. An incomplete list includes [12], where sufficient conditions for cyclicity are given in terms
of Bergman-Smirnov exceptional sets; [9], where these ideas are developed further, and examples of uncountable Bergman-Smirnov exceptional sets are found; and [13], where multipliers and invariant subspaces are discussed, leading, for instance, to a proof that non-vanishing univalent functions in the Dirichlet space are cyclic.

1.2 Plan of the paper. In this paper, we set out to improve understanding of cyclicity by studying certain classes of cyclic functions in detail. Many of the results in this paper are variations of the following questions. Given a cyclic function \( f \in D_\alpha \), can we obtain an explicit sequence of polynomials \( \{p_n\} \) such that \( \|p_n f - 1\|_\alpha \to 0 \) as \( n \to \infty \)? Can we give an estimate on the rate of decay of these norms as \( n \to \infty \)? What can we say about the approximating polynomials?

A natural first guess is to take for \( \{p_n\} \) the Taylor polynomials of \( 1/f \). Since \( 1/f \) is analytic in \( \mathbb{D} \) by the cyclicity assumption, we have \( p_n \to 1/f \) pointwise, and hence (1.3) is satisfied. However, it may be the case that norm boundedness in (1.2) fails. This is certainly true for the Taylor polynomials \( T_n(1/f) \) in the case \( f(z) = 1 - z \); indeed, \( 1/f \notin B \supset H^2 \supset D \), and a computation shows that

\[
\|T_n(1/f)f - 1\|^2_D = \|z^{n+1}\|^2_D = n + 2.
\]

Much of the development that follows is motivated by our goal of finding concrete substitutes for the Taylor polynomials of \( 1/f \).

Definition 1.1. Let \( f \in D_\alpha \). We say that a polynomial \( p_n \) of degree at most \( n \) is an optimal approximant of order \( n \) to \( 1/f \) if \( p_n \) minimizes \( \|pf - 1\|_\alpha \) among all polynomials \( p \) of degree at most \( n \). We call \( \|p_n f - 1\|_\alpha \) the optimal norm of degree \( n \) associated with \( f \).

In other words, \( p_n \) is an optimal polynomial of order \( n \) to \( 1/f \) if

\[
\|p_n f - 1\|_\alpha = \text{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n),
\]

where \( \mathcal{P}_n \) denotes the space of polynomials of degree at most \( n \) and

\[
\text{dist}_X(x, A) = \inf\{\|x - a\|_X : a \in A\}
\]

for any normed space \( X \), \( A \subseteq X \), and \( x \in X \).

Notice that, given \( f \in D_\alpha \setminus \{0\} \), the existence and uniqueness of an optimal approximant of order \( n \) to \( 1/f \) follows immediately from the fact that \( f \cdot \mathcal{P}_n \) is a finite-dimensional subspace of the Hilbert space \( D_\alpha \). Thus, \( f \) is cyclic if and only if the optimal approximants \( p_n \) of order \( n \) to \( 1/f \) satisfy \( \|p_n f - 1\|_\alpha \to 0 \) as
n → ∞. Furthermore, since \(\|p_n f - 1\|_\alpha \leq \|f - 1\|_\alpha\), it follows from (1.2) and (1.3) that \(f\) is cyclic if and only if the sequence of optimal approximants \(\{p_n\}_{n=1}^{\infty}\) converges pointwise to \(1/f\).

In Section 2, we describe a constructive approach for computing the coefficients of the optimal approximant of order \(n\) to \(1/f\) for a general function \(f\). In particular, Theorem 2.1 below states that the coefficients of the optimal approximants can be computed as ratios of determinants of matrices whose entries can be explicitly computed via the moments of the derivative of \(f\). If \(f\) itself is a polynomial, these matrices are banded (see Proposition 2.2). As a simple but fundamental example, we compute optimal approximants to the function \(1/f\) when \(f(z) = 1 - z\).

We are also interested in the rate of convergence of optimal norms. Since optimal norms decay exponentially for any function \(f\) such that \(1/f\) is analytic in the closed unit disk, functions that have zeros on the unit circle are of particular interest.

In Section 3, we examine the question of whether all functions with no zeros in the open unit disk but with zeros on the boundary, admitting an analytic continuation to the closed disk, have optimal norm achieving a similar rate of decay. In Theorem 3.7, we prove that when \(\alpha \leq 1\), this is indeed the case and obtain sharp estimates for the rate of decay associated with these functions. This can be viewed as the main result of the paper.

In Section 4, we deal with a generalization to all \(D_\alpha\) of a subproblem of the Brown-Shields conjecture: must a function \(f\) satisfying \(f \in D_\alpha\) and \(\log f \in D_\alpha\) be cyclic in \(D_\alpha\)? The answer is yes in case of \(\alpha = 0\) or \(\alpha > 1\). In Theorem 4.4, we show that this remains true in the case \(\alpha = 1\), and in Theorem 4.5 that it continues to hold for \(\alpha < 1, \alpha \neq 0\) under an additional technical condition. We do not know if this condition is necessary; however, it is satisfied by a large class of examples, namely, all of the functions constructed in Brown-Cohn [5].

We conclude, in Section 5, by presenting some open questions and basic computations connected to the zero sets \(\mathcal{Z}(p_n)\) of the optimal approximants \(p_n\) of \(1/f\) for cyclic functions \(f\).

## 2 Construction of optimal approximants

The optimal approximants \(p_n\) of order \(n\) to \(1/f\) are determined by the fact that \(p_n f\) is the orthogonal projection of 1 onto the space \(f \cdot \mathcal{P}_n\). Hence, in principle, if \(f \in D_\alpha \setminus \{0\}\), the optimal approximants can be computed using the Gram-Schmidt process. More precisely, once a basis for \(f \cdot \mathcal{P}_n\) is chosen, one can construct an
orthonormal basis for $f \cdot P_n$ and then compute the coefficients of $p_n$ with respect to this orthonormal basis.

In this section, we present a simple method, which yields the optimal approximants $p_n$ without using the Gram-Schmidt process, for $\alpha < 2$. To this end, we employ the integral norm of $D_\alpha$, namely,

$$\|f\|_\alpha^2 = |f(0)|^2 + D_\alpha(f),$$

where $D_\alpha(f)$ is as defined in (1.1). Notice that the explicit approximants we obtain in this section depend on this choice of norm.

Recall that we seek an explicit solution to the following problem.

**Problem 1.** Let $n \in \mathbb{N}$. Given $f \in D_\alpha \setminus \{0\}$ such that $1 \notin f \cdot P_n$, minimize $\|pf - 1\|_\alpha$ over $p \in P_n$.

2.1 **Construction of optimal approximants via determinants.** As mentioned in Subsection 1.2, there exists a unique optimal approximant $p_n \in P_n$ of order $n$ to $1/f$ that solves Problem 1, that is,

$$\|p_nf - 1\|_\alpha = \text{dist}_{D_\alpha}(1, f \cdot P_n).$$

Observe that for each polynomial $p(z) = \sum_{k=0}^n c_k z^k \in \mathcal{P}_n$,

$$\|pf - 1\|_\alpha^2 = |p(0)f(0) - 1|^2 + \int_{\mathbb{D}} |(pf)'|^2 d\mu_\alpha$$

$$= |p(0)f(0) - 1|^2 + \int_{\mathbb{D}} \left| \sum_{k=0}^n c_k (z^k f)' \right|^2 d\mu_\alpha,$$

where $d\mu_\alpha(z) = (1 - |z|^2)^{1-\alpha} dA(z)$. It follows that if the optimal approximant of order $n$ to $1/f$ vanishes at the origin, then $\|pf - 1\|_\alpha^2$ is minimal if and only if $c_0 = c_1 = \ldots = c_n = 0$. Consequently, we may assume without loss of generality that the optimal approximant $p_n$ of order $n$ to $1/f$ does not vanish at the origin. By replacing $f$ with $p_n(0)f$, we may also assume that $p_n(0) = 1$ because the optimal approximant of order $n$ to $1/(p_n(0)f)$ is $[p_n(0)]^{-1}p_n$. Hence, under this latter assumption, $p_n(z) = 1 + \sum_{k=1}^n c_k z^k$ is the optimal approximant of order $n$ to $1/f$ if and only if $(c_1^*, \ldots, c_n^*) \in \mathbb{C}^n$ is the unique solution to the following problem

**Problem 2.** Let $n \in \mathbb{N}$. Given $f \in D_\alpha \setminus \{0\}$ such that $1 \notin f \cdot \mathcal{P}_n$, minimize

$$\int_{\mathbb{D}} \left| f' + \sum_{k=1}^n c_k (z^k f)' \right|^2 d\mu_\alpha$$

over $(c_1, \ldots, c_n) \in \mathbb{C}^n$. 
Let us call this minimum $I_n(f)$. It is evident that $(c_1^*, \ldots, c_n^*) \in \mathbb{C}^n$ is the unique solution to Problem 2 if and only if
\[
g = \sum_{k=1}^{n} c_k^*(z^k f)' \text{ satisfies } \|f' + g\|_{L^2(\mu_a)} = \text{dist}_{L^2(\mu_a)}(f', Y),
\]
where $Y = \text{span}\{(z^k f) : 1 \leq k \leq n\}$. Equivalently, $f' + g$ is orthogonal to $Y$ with respect to the $L^2(\mu_a)$ inner product; that is, for each $j$, $1 \leq j \leq n$,
\[
\langle -f', (z^j f)' \rangle_{L^2(\mu_a)} = \langle g, (z^j f)' \rangle_{L^2(\mu_a)}.
\]
Hence, $(c_1^*, \ldots, c_n^*) \in \mathbb{C}^n$ is the unique solution to Problem 2 if and only if it is the solution of the non-homogeneous system of linear equations
\[
(2.1) \quad \sum_{k=1}^{n} c_k \langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_a)} = \langle -f', (z^j f)' \rangle_{L^2(\mu_a)}, \quad 1 \leq k \leq n,
\]
with $(c_1, \ldots, c_n) \in \mathbb{C}^n$.

**Theorem 2.1.** Let $n \in \mathbb{N}$ and $f \in D_a \setminus \{0\}$. Suppose $1 \notin f \cdot \mathcal{P}_n$ and let $M$ denote the $n \times n$ matrix with entries $\langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_a)}$. Then the unique $p_n \in \mathcal{P}_n$ satisfying
\[
\|p_n f - 1\|_a = \text{dist}_{D_a}(1, f \cdot \mathcal{P}_n)
\]
is given by
\[
(2.2) \quad p_n(z) = p_n(0) \left(1 + \sum_{k=1}^{n} \frac{\det M^{(k)}}{\det M} z^k \right),
\]
where $M^{(k)}$ denotes the $n \times n$ matrix obtained from $M$ by replacing the $k$th column of $M$ by the column with entries $\langle -f', (z^j f)' \rangle_{L^2(\mu_a)}$, $1 \leq j \leq n$, and
\[
(2.3) \quad p_n(0) = \frac{\overline{f(0)}}{|f(0)|^2 + I_n(f)}.
\]

**Proof.** As mentioned before, if $p_n$ is the optimal approximant of order $n$ to $1/f$ and $p_n(0) \neq 0$, then the optimal approximant of order $n$ to $1/f_n$ is $[p_n(0)]^{-1} p_n$, where $f_n = p_n(0)f$. If $[p_n(0)]^{-1} p_n(z) = 1 + \sum_{k=1}^{n} c_k^* z^k$, then $(c_1^*, \ldots, c_n^*) \in \mathbb{C}^n$ is the unique solution of the system (2.1) because
\[
\langle (z^k f_n)', (z^j f_n)' \rangle_{L^2(\mu_a)} = |p_n(0)|^2 \langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_a)}
\]
for $0 \leq k \leq n$ and $1 \leq j \leq n$. It follows now that the $n \times n$ matrix $M$ with entries $\langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_a)}$ has non-zero determinant, and thus
\[
c_k^* = \frac{\det M^{(k)}}{\det M}, \quad 1 \leq k \leq n,
\]
by Cramer’s rule, where $M^{(k)}$ denotes the $n \times n$ matrix obtained from $M$ by replacing the $k$th column of $M$ by the column with entries $\langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)}$, $1 \leq j \leq n$. Hence $p_n$ is given by (2.2).

To determine the value of $p_n(0)$, notice that if the minimal value $I_n(f)$ in Problem 2 is attained (uniquely) at $d_1, \ldots, d_n$, then the optimal approximant $p_n$ satisfies $d_k = c_k/c_0$ and $\|p_n f - 1\|_2^2 = |c_0 f(0) - 1|^2 + |c_0|^2 I_n(f)$. In particular, $c_0$ is the unique number that minimizes the expression $|cf(0) - 1|^2 + |c|^2 I_n(f)$ over all complex numbers $c$. The global minimum, easily found by completing the square or setting the partial derivatives equal to 0, gives $p_n(0)$ as in (2.3).

When $f$ is a polynomial, the computation of the determinants appearing in (2.2) can be simplified in view of the following proposition.

**Proposition 2.2.** If $f$ is a polynomial of degree $t$, the matrix $M$ in Theorem 2.1 is banded with bandwidth at most $2t + 1$.

**Proof.** The orthogonality of $z^l$ and $z^m$ for $l \neq m$ (under the $L^2(\mu_\alpha)$ inner product) implies that the $(j, k)$-entry of $M$ equals 0 if the degree of $(z^k f)'$ is strictly less than $j - 1$ or if the degree of $(z^j f)'$ is strictly less than $k - 1$; that is, $k + t - 1 < j - 1$ or $j + t - 1 < k - 1$. Therefore, the only entries of $M$ that do not necessarily vanish are those whose indices $j$ and $k$ satisfy $-t \leq j - k \leq t$. Thus, $M$ is banded and has bandwidth at most $2t + 1$. □

2.2 An explicit example of optimal approximants. Let us calculate explicitly optimal approximants to $1/f$, where $f(z) = 1 - z$. This example is already interesting because $f$ is cyclic in $D_\alpha$ for $\alpha \leq 1$, although it is not invertible for any $\alpha \geq -1$.

We begin with some general computations. Let $\beta = 1 - \alpha$. Then

$$\|z^m\|_{L^2(\mu_\alpha)}^2 = \int_0^1 u^{1-a}(1-u)^m \, du = \frac{1}{m+1+\beta} \prod_{\ell=1}^m \frac{\ell}{\ell+\beta}$$

for any non-negative integer $m$. Therefore, if $f(z) = \sum_{i=0}^t a_i z^i$, we have

$$\langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)} = \sum_{i=0}^t \sum_{\ell=0}^t a_i \bar{a}_{\ell}(i+k)(\ell+j) \langle z^{i+k-1}, z^{\ell+j-1} \rangle_{L^2(\mu_\alpha)}$$

(2.4)

$$= \sum_{i=0}^t a_i \bar{a}_{i+k-j}(i+k)^2 \|z^{i+k-1}\|_{L^2(\mu_\alpha)}^2$$

$$= \sum_{i=0}^t a_i \bar{a}_{i+k-j}(i+k) \prod_{\ell=1}^{i+k} \frac{\ell}{\ell+\beta}$$

\[\text{under the usual convention that } a_i = 0 \text{ for any integer } i < 0 \text{ or } i > t.\]
since $z^l$ and $z^m$ are orthogonal for $l \neq m$ under the $L^2(\mu_\alpha)$ inner product.

We simplify notation by setting, for $k \in \mathbb{N},$

$$\Lambda_\beta(k) = k \prod_{\ell=1}^{k} \frac{\ell}{\ell + \beta}.$$  

Since $a_0 = 1$ and $a_1 = -1$, it follows from (2.4) that

$$\langle (z^k f)', (z^{k-1} f)' \rangle_{L^2(\mu_\alpha)} = -\Lambda_\beta(k),$$

$$\langle (z^k f)', (z^k f)' \rangle_{L^2(\mu_\alpha)} = \Lambda_\beta(k) + \Lambda_\beta(k + 1),$$

$$\langle (z^k f)', (z^{k+1} f)' \rangle_{L^2(\mu_\alpha)} = -\Lambda_\beta(k + 1),$$

and

$$\langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)} = \begin{cases} 
\Lambda_\beta(1) & \text{if } j = 1 \\
0 & \text{if } j \geq 2.
\end{cases}$$

Thus, in view of (2.1), the coefficients of $p_n$ satisfy the system of equations

$$c_1 \left[ \Lambda_\beta(1) + \Lambda_\beta(2) \right] - c_2 \left[ \Lambda_\beta(2) \right] = \Lambda_\beta(1)$$

$$-c_{j-1} \left[ \Lambda_\beta(j) \right] + c_j \left[ \Lambda_\beta(j) + \Lambda_\beta(j + 1) \right] - c_{j+1} \left[ \Lambda_\beta(j + 1) \right] = 0$$

$$-c_{n-1} \left[ \Lambda_\beta(n) \right] + c_n \left[ \Lambda_\beta(n) + \Lambda_\beta(n + 1) \right] = 0.$$  

Interpreting $c_{n+1} = 0$, we have, equivalently, for all $2 \leq j \leq n + 1,$

$$\Lambda_\beta(j)(c_j - c_{j-1}) = \Lambda_\beta(1)(c_1 - 1).$$

For fixed $k$, $2 \leq k \leq n + 1$, we obtain, by a repeated use of the previous identity,

$$(2.5) \quad c_k = \left[ \Lambda_\beta(1) \sum_{j=1}^{k} \frac{1}{\Lambda_\beta(j)} \right] (c_1 - 1) + 1.$$  

In particular, $c_1 - 1 = -\Lambda_\beta(n + 1)c_n$, so

$$c_1 - 1 = -\frac{1}{\Lambda_\beta(1) \sum_{j=1}^{n+1} \frac{1}{\Lambda_\beta(j)}}.$$  

Finally, we obtain the explicit solution, which can be expressed as follows. For $1 \leq k \leq n,$

$$(2.6) \quad c_k = \left[ \sum_{j=k+1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left( 1 + \frac{\beta}{\ell} \right) \right] \left[ \sum_{j=1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left( 1 + \frac{\beta}{\ell} \right) \right]^{-1}.$$  

Alternatively, in the case of the Dirichlet space, we can compute the coefficients $c_k$, $1 \leq k \leq n$, using determinants as follows. For $n \in \mathbb{N}$, let $M_n = M$
and $M_n^{(k)} = M^{(k)}$ be the $n \times n$ matrices corresponding to $f$ as in Theorem 2.1. By Proposition 2.2, the matrix $M_n$ is tridiagonal, and so it suffices to compute the coefficients above and below each entry of its main diagonal. The coefficients in the $j$th column of $M_n$ are given by

$$\langle (z^{j+\ell}f)', (z^j f)' \rangle_{L^2} = a_0 \bar{a}_\ell (j + \ell) + a_1 \bar{a}_{1+\ell} (j + \ell + 1)$$

where $\ell = -1, 0, 1$. Since $a_0 = 1$ and $a_1 = -1$, we obtain

$$\langle (z^{j-1}f)', (z^j f)' \rangle_{L^2} = -j,$$

$$\langle (z^j f)', (z^j f)' \rangle_{L^2} = 2j + 1,$$

$$\langle (z^{j+1}f)', (z^j f)' \rangle_{L^2} = -(j + 1),$$

and $\langle -f', (z^j f)' \rangle_{L^2} = \bar{a}_{1-j}$.

Consequently,

$$M_1 = 3, \quad M_1^{(1)} = 1,$$

$$M_2 = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 7 \end{pmatrix}, \quad M_2^{(1)} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 5 & -3 \\ 0 & -3 & 7 \end{pmatrix}, \quad M_2^{(2)} = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 3 & -2 & 0 & 0 \\ -2 & 5 & -3 & 0 \\ 0 & -3 & 7 & -4 \\ 0 & 0 & -4 & 9 \end{pmatrix}, \quad M_3^{(1)} = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & -3 & 7 & -4 \\ 0 & 0 & -4 & 9 \end{pmatrix}, \quad \ldots$$

$$M_4 = \begin{pmatrix} 3 & -2 & 0 & 0 & 0 \\ -2 & 5 & -3 & 0 & 0 \\ 0 & -3 & 7 & -4 & 0 \\ 0 & 0 & -4 & 9 & 12 \end{pmatrix}, \quad M_4^{(1)} = \begin{pmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 \\ 0 & -3 & 7 & -4 & 0 \\ 0 & 0 & -4 & 9 & 12 \end{pmatrix}, \quad \ldots$$

Thus, the optimal approximants to $f$ of orders 1, 2, 3, and 4 are

$$p_1(z) = p_1(0) \left( 1 + \frac{1}{3}z \right),$$

$$p_2(z) = p_2(0) \left( 1 + \frac{5}{11}z + \frac{2}{11}z^2 \right),$$

$$p_3(z) = p_3(0) \left( 1 + \frac{13}{25}z + \frac{7}{25}z^2 + \frac{3}{25}z^3 \right), \quad \text{and}$$

$$p_4(z) = p_4(0) \left( 1 + \frac{77}{137}z + \frac{47}{137}z^2 + \frac{27}{137}z^3 + \frac{12}{137}z^4 \right).$$
What we have shown is that, for any integer \( n \), the optimal approximant for the Dirichlet space is an example of a generalized Riesz mean polynomial: more specifically, defining \( H_0 = 0 \) and \( H_n = \sum_{j=1}^{n} \frac{1}{j} \) for positive integer \( n \), we have

\[
p_n(z) = p_n(0) \left( \sum_{k=0}^{n} \left( 1 - \frac{H_k}{H_{n+1}} \right) z^k \right).
\]

Moreover, for the Hardy space, the optimal approximant is the modified Cesàro mean polynomial

\[
p_n(z) = p_n(0) \left( \sum_{k=0}^{n} \left( 1 - \frac{k + H_k}{n + 1 + H_{n+1}} \right) z^k \right),
\]

and for the Bergman space, the optimal approximants are

\[
p_n(z) = p_n(0) \left(1 + \sum_{k=1}^{n} \left( 1 - \frac{k(k + 7) + 4H_k}{(n + 1)(n + 8) + 4H_{n+1}} \right) z^k \right).
\]

We return to these polynomials in Section 3.

### 3 Rate of decay of the optimal norms

In this section, we obtain estimates for \( \text{dist}_{D_\alpha}(1, f \cdot P_n) \) as \( n \to \infty \) for \( f \in D_\alpha \). It turns out that the example of \( f(z) = 1 - z \), mentioned in the previous section, is a model for the rate of decay of \( \text{dist}_{D_\alpha}(1, f \cdot P_n) \). Notice that the choice of norm affects the rate of decay only up to a universal multiplicative constant. We first examine the rate of decay for this function, then establish such estimates when \( f \) is a polynomial whose zeros are simple and lie in \( \mathbb{C} \setminus \mathbb{D} \), and then extend our results to arbitrary polynomials. We conclude with estimates on functions that admit an analytic continuation to the closed unit disk having at least one zero on the circle.

To simplify notation, define the auxiliary function \( \phi_\alpha \) on \([0, \infty)\) by

\[
\phi_\alpha(s) = \begin{cases} 
    s^{1-\alpha} & \text{if } \alpha < 1, \\
    \log^+(s) & \text{if } \alpha = 1.
\end{cases}
\]

**Lemma 3.1.** If \( f(z) = \zeta - z \), for \( \zeta \in \mathbb{T} \), then \( \text{dist}^2_{D_\alpha}(1, f \cdot P_n) \) is comparable to \( \phi_\alpha^{-1}(n + 1) \) for all sufficiently large \( n \).

**Proof.** First observe that, for any polynomial \( p \) and \( \zeta \in \mathbb{T} \), the polynomial \( q(z) = \zeta p(\zeta z) \) satisfies \( \|p(z)(\zeta - z) - 1\|_\alpha = \|q(z)(1 - z) - 1\|_\alpha \) since rotation by \( \zeta \) is an isometry in \( D_\alpha \). Therefore, it is enough to consider the case \( \zeta = 1 \), i.e., \( f(z) = 1 - z \).
Now, recall that by (2.6), if $f(z) = 1 - z$, the optimal approximant of order $n$ to $1/f$ is $p_n(z) = p_n(0) + \sum_{k=0}^{n} c_k z^k$, where

$$c_k = \left[ \sum_{j=k+1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left( 1 + \frac{1}{\ell} \right) \right]^{-1} \left[ \sum_{j=1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left( 1 + \frac{1}{\ell} \right) \right] - 1, \quad 0 \leq k \leq n,$$

and $\beta = 1 - \alpha$. We claim that $\|p_n f - 1\|_2^\alpha$ is comparable to $\varphi_\alpha^{-1}(n + 1)$ for all sufficiently large $n$.

First, notice that

$$(3.1) \quad p_n(z) f(z) - 1 = p_n(0) - 1 + p_n(0) \sum_{k=1}^{n} (c_k - c_{k-1}) z^k - c_n z^{n+1}.$$

Let us estimate the norm of the term in the brackets. To simplify notation, define for $1 \leq k \leq n$

$$a_k = c_k - c_{k-1} = - \left[ \frac{1}{k} \prod_{\ell=2}^{k} \left( 1 + \frac{1}{\ell} \right) \right]^{-1} \left[ \sum_{j=1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left( 1 + \frac{1}{\ell} \right) \right]$$

and $a_{n+1} = -c_n$. Then

$$(3.2) \quad \sum_{k=1}^{n} k^\alpha |a_k|^2 = \left[ \sum_{j=1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left( 1 + \frac{1}{\ell} \right) \right]^{-2} \sum_{k=1}^{n} k^{\alpha - 2} \left[ \prod_{\ell=2}^{k} \left( 1 + \frac{1}{\ell} \right) \right]^2.$$

Recalling that $2^{-1} x \leq \log(1 + x) \leq x$ for all $x \in [0, 1]$, we see that

$$\prod_{\ell=2}^{k} \left( 1 + \frac{1}{\ell} \right) = \exp \left[ \sum_{\ell=2}^{k} \log \left( 1 + \frac{1}{\ell} \right) \right]$$

is comparable to $\exp \left[ \beta \sum_{\ell=2}^{k} 1/\ell \right]$, and so comparable to $k^\beta$, when $1 \leq k \leq n + 1$. Thus, the sum in (3.2) and

$$(n + 1)^\alpha |a_{n+1}|^2 = (n + 1)^\alpha \left[ \frac{1}{n + 1} \prod_{\ell=2}^{n+1} \left( 1 + \frac{1}{\ell} \right) \right]^2 \left[ \sum_{j=1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left( 1 + \frac{1}{\ell} \right) \right]$$

are comparable to

$$\left[ \sum_{j=1}^{n+1} \frac{1}{j^\alpha} \right]^{-2} \sum_{k=1}^{n} \frac{1}{k^\alpha} \quad \text{and} \quad \frac{1}{(n + 1)^\alpha} \left[ \sum_{j=1}^{n+1} \frac{1}{j^\alpha} \right]^{-2},$$

respectively. Since $\sum_{j=1}^{n} j^{-\alpha}$ is comparable to $\varphi_\alpha(n + 1)$, the sum $\sum_{k=1}^{n+1} k^\alpha |a_k|^2$ is comparable to $\varphi_\alpha^{-1}(n + 1)$ if $n \geq 2$. This shows that the convergence of $\|p_n f - 1\|_2^\alpha$
in (3.1) can be no faster than $\varphi_\alpha^{-1}(n + 1)$. On the other hand, replacing $p_n(0)$ by 1 in (3.1), we obtain another sequence of polynomials whose rate of convergence is at least as fast as $\varphi_\alpha^{-1}(n + 1)$.

□

Let us now examine the rate of decay of optimal norms for polynomials whose zeros are simple and lie in $C \setminus D$. To begin, we introduce some notation. Denote by $A(\mathbb{T})$ the Wiener algebra of functions $f$, defined on $\mathbb{T}$ whose Fourier coefficients are absolutely summable, equipped with the norm $\|f\|_{A(\mathbb{T})} = \sum_{k=-\infty}^{\infty} |a_k|$. The positive Wiener algebra consists of analytic functions whose Fourier coefficients satisfy $\sum_{k=0}^{\infty} |a_k| < \infty$; in particular, these functions belong to the space $H^\infty$ of bounded analytic functions in $D$, and $\|f\|_{H^\infty} \leq \|f\|_{A(\mathbb{T})}$ for all such $f$, where $\|f\|_{H^\infty} = \sup\{|f(z)| : z \in D\}$.

**Proposition 3.2.** Let $\alpha \leq 1$, $t \in \mathbb{N}$, and $f \in P_t$. If all the zeros of $f$ are simple and lie in $C \setminus D$, then for each $n > t$ there exists $p_n \in P_n$ such that

$$\|p_nf - 1\|_2^2 \leq C\varphi_\alpha^{-1}(n + 1)$$

for some constant $C$ that depends on $f$ and $\alpha$ but not on $n$. The sequence $\{p_n f\}_{n > t}$ is bounded in $A(\mathbb{T})$—norm.

**Proof.** Suppose $f$ has zeros $z_1, \ldots, z_t \in C \setminus D$, all of which are simple. Then there exist constants $d_1, \ldots, d_t$ such that

$$\frac{1}{f(z)} = \sum_{j=1}^{t} \frac{d_j}{z_j - z} = \sum_{k=0}^{\infty} \left( \sum_{j=1}^{t} d_j z_j^{k+1} \right) z^k.$$

Define $b_k = \sum_{j=1}^{t} d_j z_j^{-(k+1)}$ for $k \geq 0$. It follows that the sequence $\{b_k\}_{k=0}^{\infty}$ is bounded in modulus by $\sum_{j=1}^{t} |d_j|$, and the Taylor series representations of $f$ and $1/f$ centered at the origin are of the form

$$f(z) = \sum_{k=0}^{t} a_k z^k \quad \text{and} \quad \frac{1}{f(z)} = \sum_{k=0}^{\infty} b_k z^k$$

for some $a_0, \ldots, a_t \in \mathbb{C}$. Set $a_k = 0$ for $k > t$. Consequently,

$$\sum_{j=0}^{k} b_j a_{k-j} = 0 \quad \text{for} \quad k \in \mathbb{N} \setminus \{0\}.$$

Consider the polynomial $p_n(z) = \sum_{k=0}^{n} c_k z^k$ with coefficients

$$c_0 = a_0^{-1} \quad \text{and} \quad c_k = \left( 1 - \frac{\varphi_\alpha(k)}{\varphi_\alpha(n + 1)} \right) b_k \quad \text{for} \quad 1 \leq k \leq n.$$
For convenience of notation, let \( c_k = 0 \) if \( k > n \). Evidently, \((p_n f)(0) = 1\). Let us prove (3.3). To estimate \( \|p_n f - 1\|_a^2 \), we consider separately the norms of

\[
mp = \sum_{k=t+1}^{n+t} \left( \sum_{i=0}^{k} c_i a_{k-i} \right) z^k \quad \text{and} \quad sp = \sum_{k=1}^{t} \left( \sum_{i=0}^{k} c_i a_{k-i} \right) z^k,
\]

and note that

\[
\|p_n f - 1\|_a^2 = \|mp\|_a^2 + \|sp\|_a^2
\]

and

\[
\sum_{i=0}^{k} c_i a_{k-i} = -\frac{1}{\varphi_a(n+1)} \sum_{i=0}^{k} \varphi_a(i) b_i a_{k-i}
\]

by (3.4). To estimate the norm of \( mp \), we need the following result.

**Lemma 3.3.** Under the assumptions of Proposition 3.2, if \( k > t \), there exists a constant \( C = C(\alpha, f) \) such that

\[
\left| \sum_{i=0}^{k} \varphi_a(i) b_i a_{k-i} \right| \leq \frac{C}{(k+1)^\alpha}.
\]

We finish the proof of Proposition 3.2 before proving Lemma 3.3.

By (3.6) and Lemma 3.3,

\[
\|mp\|_a^2 = \sum_{k=t+1}^{n+t} \left| \sum_{i=0}^{k} c_i a_{k-i} \right|^2 (k+1)^\alpha = \frac{1}{\varphi_a^2(n+1)} \sum_{k=t+1}^{n+t} \left| \sum_{i=0}^{k} \varphi_a(i) b_i a_{k-i} \right|^2 (k+1)^\alpha
\]

\[
\leq \frac{C_1}{\varphi_a^2(n+1)} \sum_{k=t+1}^{n+t} \frac{1}{(k+1)^\alpha}
\]

for some constant \( C_1 = C_1(\alpha, f) \). It follows now from the estimates

\[
\sum_{k=t+1}^{n+t} \frac{1}{(k+1)^\alpha} \leq \begin{cases} 
(n(n+t+1)^{-\alpha} & \text{if } \alpha \leq 0, \\
(1-\alpha)^{-1}[(n+t+1)^{1-\alpha} - (t+1)^{1-\alpha}] & \text{if } 0 < \alpha < 1, \\
\log(n+t+1) - \log(t+1) & \text{if } \alpha = 1,
\end{cases}
\]

and the elementary inequalities

\[
(n+t+1)^{-\alpha} \leq 2^{-\alpha}(n+1)^{-\alpha} \quad \text{for } \alpha \leq 0,
\]

\[
(n+t+1)^{1-\alpha} \leq 2^{1-\alpha}(n+1)^{1-\alpha} \quad \text{for } \alpha > 0, \quad \text{and}
\]

\[
\log \frac{n+t+1}{t+1} \leq \log(n+1)
\]
that there exists a constant $C_2 = C_2(\alpha, f)$ such that

\begin{equation}
\sum_{k=t+1}^{n+t} \frac{1}{(k+1)^\alpha} \leq C_2 \phi_\alpha(n+1),
\end{equation}

and so

\begin{equation}
\|mp\|_\alpha^2 \leq \frac{C_1 C_2}{\phi_\alpha(n+1)}.
\end{equation}

Next, we study the norm of $\mathbf{s}p$. Recalling (3.6), we see that

\begin{equation}
\|sp\|_\alpha^2 = \frac{1}{\phi_\alpha^2(n+1)} \sum_{k=1}^{t} \left| \sum_{i=0}^{k} \phi_\alpha(i) b_i a_{k-i} \right|^2 \leq \frac{C_3}{\phi_\alpha^2(n+1)}.
\end{equation}

Hence, (3.3) follows from (3.5), (3.8), and (3.9).

Finally, we show that the sequence $\{p_{nf}\}_{n>t}$ is bounded in $A(\mathbb{T})$. By the triangle inequality and since $\phi$ is increasing,

\begin{equation}
\left| \sum_{i=0}^{k} c_i a_{k-i} \right| \leq \|b\|_{\ell^\infty} \|a\|_{\ell^\infty(t+1)\phi_\alpha(t)}
\end{equation}

if $1 \leq k \leq t$, where $a = \{a_k\}_{k=0}^\infty$ and $b = \{b_k\}_{k=0}^\infty$. Notice that, for $1 \leq k \leq t$, (3.6) and (3.10) imply

\begin{equation}
\left| \sum_{i=0}^{k} c_i a_{k-i} \right| \leq \|b\|_{\ell^\infty} \|a\|_{\ell^\infty(t+1)}
\end{equation}

since $\phi_\alpha(t) \leq \phi_\alpha(n+1)$. On the other hand, for $t < k \leq n+t$, (3.6) and Lemma 3.3 imply

\begin{equation}
\left| \sum_{i=0}^{k} c_i a_{k-i} \right| \leq \frac{C}{(k+1)^\alpha \phi_\alpha^{-1}(n+1)}
\end{equation}

for some constant $C = C(\alpha, f)$. Therefore, by (3.11), (3.12), and (3.7),

\[ \|p_{nf}\|_{A(\mathbb{T})} \leq \sum_{k=1}^{n+t} \left| \sum_{i=0}^{k} c_i a_{k-i} \right| \]

\[ \leq \sum_{k=1}^{t} \|b\|_{\ell^\infty} \|a\|_{\ell^\infty(t+1)} + \frac{C}{\phi_\alpha(n+1)} \sum_{k=t+1}^{n+t} \frac{1}{(k+1)^\alpha} \]

\[ \leq \|b\|_{\ell^\infty} \|a\|_{\ell^\infty(t+1)} t; + C \cdot C_2 \]

and so $\{p_{nf}\}_{n>t}$ is bounded in $A(\mathbb{T})$. □
Proof of Lemma 3.3. For \( k - t \leq s \leq k \),

\[
\phi'(s) \leq \begin{cases} 
(1 - \alpha)k^{-\alpha} & \text{if } \alpha < 0, \\
(1 - \alpha)(k - t)^{-\alpha} & \text{if } 0 \leq \alpha < 1, \\
(k - t)^{-1} & \text{if } \alpha = 1.
\end{cases}
\]

Thus, the Mean Value Theorem, (3.13), and the inequality

\[
(k - t)^{-\alpha} \leq (t + 2)^{\alpha}(k + 1)^{-\alpha}
\]

imply the existence of a constant \( C = C(\alpha, t) \) such that

\[
\varphi_a(k) - \varphi_a(i) \leq C(k - i)(k + 1)^{-\alpha} \quad \text{for } k \geq i.
\]

Recalling (3.4) and that \( a_i = 0 \) for \( i > t \), we obtain

\[
\left| \sum_{i=0}^{k} \varphi_a(i) b_i a_{k-i} \right| \leq \left| \sum_{i=0}^{k} [\varphi_a(k) - \varphi_a(i)] b_i a_{k-i} \right|
\]

\[
\leq \sum_{i=k-t}^{k} [\varphi_a(k) - \varphi_a(i)] \cdot |b_i a_{k-i}|
\]

\[
\leq \|a\|_{\ell^\infty}\|b\|_{\ell^\infty} C(k + 1)^{-\alpha} \sum_{i=k-t}^{k} (k - i),
\]

where \( a = \{a_i\}_{i=0}^\infty \) and \( b = \{b_i\}_{i=0}^\infty \). This establishes the lemma with constant \( \|a\|_{\ell^\infty}\|b\|_{\ell^\infty} C t(t + 1)/2 \). \( \square \)

It seems natural to ask whether the proof of Proposition 3.2 can be extended to polynomials \( f \) whose zeros are not necessarily simple. However, even in the simple case of \( f(z) = (1 - z)^2 \), the coefficients of the Taylor series representation centered at the origin of \( 1/f \) are not bounded. Consequently, our proof of Proposition 3.2 cannot be extended directly, as it uses the boundedness of these coefficients. Nevertheless, for an arbitrary polynomial \( f \), we can obtain an estimate for \( \text{dist}_{D_a}(1, f \cdot P_m) \). Moreover, using Lemma 3.1, we are able to show that this rate of decay is sharp.

**Theorem 3.4.** Let \( \alpha \leq 1 \). If \( f \) is a polynomial all of whose zeros lie in \( \mathbb{C} \setminus \mathbb{D} \), there exists a constant \( C = C(\alpha, f) \) such that

\[
\text{dist}_{D_a}^2(1, f \cdot P_m) \leq C \varphi_a^{-1}(m + 1)
\]

for all sufficiently large \( m \). Moreover, this estimate is sharp in the sense that if \( f \) has at least one zero on \( \mathbb{T} \), there exists a constant \( \tilde{C} = \tilde{C}(\alpha, f) \) such that

\[
\tilde{C} \varphi_a^{-1}(m + 1) \leq \text{dist}_{D_a}^2(1, f \cdot P_m).
\]
Proof. Suppose \( f \) has factorization \( f(z) = K \prod_{k=1}^{s}(z-z_k)^{\gamma_k} \) with \( K \in \mathbb{C} \setminus \{0\}, \)
\( r_1, \ldots, r_s \in \mathbb{N}, \) and \( z_1, \ldots, z_s \in \mathbb{C} \setminus \mathbb{D} \) distinct. Define
\[
g(z) = \prod_{k=1}^{s}(z-z_k) \quad \text{and} \quad h(z) = K^{-1} \prod_{k=1}^{s}(z-z_k)^{\gamma-r_k},
\]
where \( \gamma = \max\{r_1, \ldots, r_s\}, \) and let \( d \) equal the degree of \( h. \) Then \( fh = g^{\gamma}, \)
\[
(3.15) \quad \text{dist}_{D_a}(1, f \cdot \mathcal{P}_{n+d}) \leq \text{dist}_{D_a}(1, fh \cdot \mathcal{P}_n) \quad \text{for } n \in \mathbb{N},
\]
and the zeros of \( g \) are simple and lie in \( \mathbb{C} \setminus \mathbb{D}. \)

By Proposition 3.2, for \( n > s, \) we can choose \( q_n \in \mathcal{P}_n \) such that \( (q_ng)(0) = 1, \)
\[
(3.16) \quad \|q_ng - 1\|_a^2 \leq C_1 \varphi_a^{-1}(n+1)
\]
for some \( C_1 = C_1(\alpha, g), \) and the sequence \( \{q_ng\}_{n>s} \) is bounded in \( A(\mathbb{T}). \)

Let \( d\mu_\alpha(z) = (1 - |z|^2)^{1-\alpha} \, dA(z). \) Recalling that \( \|p\|_a^2 \) is comparable to
\[
|p(0)|^2 + D_a(p) = |p(0)|^2 + \|p'|^2_{L^2(\mu_\alpha)}
\]
for all \( p \in D_a, \) we obtain
\[
\|q_ng^{\gamma} - 1\|_a^2 \leq C_2 \|(q_ng^{\gamma}')\|_{L^2(\mu_\alpha)}^2
\]
\[
= C_2 \|(q_ng)^{\gamma-1}\gamma(q'_ng + qng')\|_{L^2(\mu_\alpha)}^2
\]
\[
\leq C_2 \gamma^2 \|q_ng\|_{H^{\infty}}^2 \|q'_ng + qng\|_{L^2(\mu_\alpha)}^2
\]
\[
\leq C_3 \gamma^2 \|q_ng\|_{H^{\infty}}^2 \|q_ng - 1\|_a^2
\]
\[
\leq C_3 \gamma^2 \|q_ng\|_{A(\mathbb{T})}^2 \|q_ng - 1\|_a^2
\]
as \( (q_ng)(0) = 1, \) for some constants \( C_2 = C_2(\alpha) \) and \( C_3 = C_3(\alpha). \) Therefore, (3.16)
and (3.17) imply the existence of a constant \( C_4 = C_4(\alpha, \gamma, g) \) such that
\[
\text{dist}^2_{D_a}(1, g^{\gamma} \cdot \mathcal{P}_{n^{\gamma}}) \leq C_4 \varphi_a^{-1}(n+1)
\]
because \( q_n^{\gamma} \in \mathcal{P}_{n^{\gamma}} \) and \( \{q_ng\}_{n>s} \) is bounded in \( A(\mathbb{T}). \) Thus, by (3.15),
\[
(3.18) \quad \text{dist}^2_{D_a}(1, f \cdot \mathcal{P}_{n^{\gamma}+d}) \leq C_4 \varphi_a^{-1}(n+1) \quad \text{if } n > s.
\]

Let \( m > d + (s+1)\gamma. \) Then there exist \( a \in \mathbb{Z} \) and \( n \in \mathbb{N} \) such that \( 0 \leq a < \gamma \)
and \( m - d = n\gamma + a. \) In particular, \( n > s, \) and
\[
(3.19) \quad \text{dist}^2_{D_a}(1, f \cdot \mathcal{P}_m) \leq C_4 \varphi_a^{-1}(n+1)
\]
follows from (3.18) since \( m \geq n\gamma + d. \) Finally, the elementary inequalities
\[
(1 + n\gamma + d + a) \leq (\gamma + d)(1 + n) \quad \text{and} \quad (1 + n\gamma + d + a) \leq (1 + n)^{2\gamma+d},
\]

\text{Cyclicity and Extremal Polynomials} 275
which are valid for all \( n \in \mathbb{N} \), imply the existence of a constant \( C_5 = C_5(\alpha, \gamma, d) \) such that

\[
\varphi_\alpha(m + 1) \leq C_5 \varphi_\alpha(n + 1).
\]

Hence, (3.14) holds for \( m > d + (s + 1)\gamma \) by (3.19) and (3.20).

Let us now show that the inequality is sharp. If \( f \) is a polynomial having zeros outside \( \mathbb{D} \) and at least one zero on \( T \), then \( f(z) = h(z)(\zeta - z) \) for some polynomial \( h \) of degree, say, \( d \). Then for every polynomial \( p_m \) of degree at most \( m \),

\[
\|p_m(z)h(z)(\zeta - z) - 1\|_2^2 \geq \text{dist}_{\mathbb{D}_\alpha}^2(1, (\zeta - z) \cdot \mathcal{P}_{m+d}).
\]

By Lemma 3.1, there exists a constant \( C_1 = C_1(\alpha) \) such that

\[
\text{dist}_{\mathbb{D}_\alpha}^2(1, (\zeta - z) \cdot \mathcal{P}_{m+d}) \geq C_1 \varphi_\alpha^{-1}(m + d + 1).
\]

Now, as was done for (3.20), we can choose a constant \( C_2 = C_2(\alpha, d) \) such that

\[
\varphi_\alpha^{-1}(m + d + 1) \geq C_2 \varphi_\alpha^{-1}(m + 1).
\]

Finally, letting \( \tilde{C} = C_1C_2 \) and noting that the polynomial \( p_m \) was arbitrary, we obtain

\[
\text{dist}_{\mathbb{D}_\alpha}^2(1, f \cdot \mathcal{P}_m) \geq \tilde{C} \varphi_\alpha^{-1}(m + 1),
\]

as desired. \( \square \)

In fact, the rates in Theorem 3.4 hold for nonvanishing functions on the open unit disk having analytic continuations to the closed disk. Such functions \( f \) can be factored as \( f(z) = h(z)g(z) \), where \( h \) is a polynomial with a finite number of zeros on the circle and \( g \) is a function analytic in the closed disk with no zeros there, so, the estimates in Theorem 3.4 hold for \( h \). Moreover, we can obtain estimates on \( g \) that allow us to give upper bounds on the product \( h(z)g(z) \). The estimates needed for \( g \) are contained in the following lemma.

**Lemma 3.5.** Let \( g \) be analytic in the closed disk and \( T_n(g) \) be the Taylor polynomial of degree \( n \) of \( g \). Then there exist \( S > 1 \) and a constant \( C = C(\alpha) \) such that

\[
\|g - T_n(g)\|_2^2 = O(S^{-n}) \quad \text{and} \quad \|T_n(g)\|_{M(\mathbb{D}_\alpha)} \leq C.
\]

**Proof.** Suppose \( g(z) = \sum_{k=0}^{\infty} d_k z^k \) converges in the closed unit disk. Then there exist constants \( R > 1 \) and \( C_1 > 0 \) such that \( |d_k| \leq C_1 R^{-k} \). Therefore,

\[
\|g - T_n(g)\|_2^2 = \sum_{k=n+1}^{\infty} (k+1)^\alpha |d_k|^2 \leq C_1 R^{-2n} \sum_{j=1}^{\infty} (j+n+1)^\alpha R^{-2j} \leq C_1 R^{-2n}(n+1)^\alpha C_2,
\]
where \( C_2 = C_2(\alpha, R) = \sum_{j=1}^{\infty} (j + 1)^\alpha R^{-2j} \) if \( \alpha \geq 0 \) and \( C_2 = \sum_{j=1}^{\infty} R^{-2j} \) if \( \alpha < 0 \). In either case, \( C_2 \) is finite and independent of \( n \). Therefore, for all \( \alpha \),

\[
\|g - T_n(g)\|_a^2 \leq C_1 C_2 R^{-2n}(n + 1)^\alpha = O(S^{-n})
\]

for some \( S > 1 \).

The same type of argument can be used to show that the Taylor polynomials \( T_n(g) \) have uniformly bounded multiplier norms. Indeed, suppose that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \in D_\alpha \). Then in a manner similar to that above, using the exponential decay of the coefficients \( d_k \) of \( g \), one can easily show that for each \( k \),

\[
\|d_k z^k \cdot f\|_a \leq R^{-k} C_1 C_3 \|f\|_a,
\]

where

\[
C_3(k, \alpha) = \begin{cases} (k + 1)^{\alpha/2} & \text{if } \alpha \geq 0, \\ 1 & \text{otherwise}. \end{cases}
\]

Therefore,

\[
\|T_n(g) \cdot f\|_a \leq \sum_{k=0}^{n} \|d_k z^k \cdot f\|_a \leq \left( \sum_{k=0}^{n} C_1 C_3 R^{-k} \right) \|f\|_a.
\]

Since the series \( C = \sum_{k=0}^{\infty} C_1 C_3 R^{-k} \) converges, we obtain \( \|T_n(g)\|_{M(D_\alpha)} \leq C \), as desired.

**Remark 3.6.** One can show that the exponential decay on \( \|g - T_n(g)\|_a^2 \) of the previous lemma also holds for \( \|g - T_n(g)\|_{M(D_\alpha)}^2 \). When \( \alpha > 1 \), this is immediate since \( D_\alpha \) is an algebra. If \( \alpha \leq 1 \), the multiplier norm \( \|g - T_n(g)\|_{M(D_\alpha)}^2 \) decays exponentially since it is controlled by the norm in \( D_2 \).

**Theorem 3.7.** Let \( \alpha \leq 1 \). If \( f \) is a function admitting analytic continuation to the closed unit disk whose zeros lie in \( \mathbb{C} \setminus \mathbb{D} \), then there exists a constant \( C = C(\alpha, f) \) such that

\[
\text{dist}_{\mathbb{D}_\alpha}^2(1, f \cdot P_m) \leq C \varphi^{-1}_\alpha(m + 1)
\]

for all sufficiently large \( m \). Moreover, this estimate is sharp in the sense that if such a function \( f \) has at least one zero on \( \mathbb{T} \), there exists a constant \( \check{C} = \check{C}(\alpha, f) \) such that

\[
\check{C} \varphi^{-1}_\alpha(m + 1) \leq \text{dist}_{\mathbb{D}_\alpha}^2(1, f \cdot P_m).
\]

**Proof.** Let us first examine the upper bound. Without loss of generality, assume that \( f \) is not identically 0. It can therefore have only a finite number of zeros
on the unit circle $\mathbb{T}$. Write $f(z) = h(z)g(z)$, where $h$ is the polynomial formed from
the zeros of $f$ that lie on $\mathbb{T}$ and $g$ is analytic in the closed disk with no zeros there.
Then $1/g$ is also analytic in the closed unit disk (and obviously has no zeros there); hence Lemma 3.5 applies to $1/g$. Observe also that $g$ and $g'$ are bounded in the
disk, and therefore $g$ is a multiplier for $D_\alpha$.

Now, for $m \in \mathbb{N}$, let $q_m$ be the optimal approximant of order $m$ to $1/h$, and define $p_m = q_m T_m(1/g)$. By the triangle inequality,

$$
\|p_m f - 1\|_\alpha \leq \|T_m(1/g)g(q_m h - 1)\|_\alpha + \|T_m(1/g)g - 1\|_\alpha.
$$

Since $g$ is a multiplier for $D_\alpha$, the $q_m$ are optimal for $h$, and the $T_m(1/g)$ are uni-
formly bounded in multiplier norm in light of Lemma 3.5, we see that the square of
the first summand on the right-hand side is dominated by a constant times $\varphi_\alpha(m+1)$
independent of $m$. On the other hand, by the second part of Lemma 3.5, the square
of the second summand is $o(\varphi_\alpha(m + 1))$, and thus is negligible by comparison.
Therefore, $\text{dist}_{D_\alpha}^2(1, f \cdot \mathcal{P}_m) \leq C \varphi_\alpha^{-1}(m + 1)$ for some constant $C = C(\alpha, f)$, as desired.

Let us now address the lower bound for $f$. Notice first that if the lower bound
holds for functions of the form $(\zeta - z)g(z)$, where $g$ is analytic and without zeros
in the closed unit disk, then the conclusion holds for $f$. Moreover, as in the proof
of Lemma 3.1, it is enough to consider $\zeta = 1$. Therefore, we write $f(z) = h(z)g(z)$, where $h(z) = 1 - z$ and $g$ is as above. Again, since $g$ is analytic and has no zeros
in the closed disk, both $g$ and $1/g$ are multipliers for $D_\alpha$. Therefore, if $p_m$ is a
polynomial of degree less than or equal to $m$,

$$
\|p_m f - 1\|_\alpha \leq \|g\|_{M(D_\alpha)} \|p_m h - 1/g\|_\alpha \leq \|g\|_{M(D_\alpha)} 1/g \|M(D_\alpha)\|p_m f - 1\|_\alpha.
$$

Now choose $p_m$ to be the optimal approximants to $1/f$ of degree less than or
equal to $m$. Then by the above discussion, we can assume $p_m h - 1/g \rightarrow 0$ in $D_\alpha$;
in particular, the norms $\|p_m h\|_\alpha$ are bounded. We thus obtain

$$
\|p_m f - 1\|_\alpha = \|p_m h(g - T_m(g) + T_m(g)) - 1\|_\alpha \\
\geq \|p_m h T_m(g) - 1\|_\alpha - \|p_m h(g - T_m(g))\|_\alpha
$$

Now, by Lemma 3.1, $\|p_m h T_m(g) - 1\|_\alpha^2$ is greater than or equal to a constant times
$\varphi_\alpha^{-1}(2m + 1)$, which in turn is comparable to $\varphi_\alpha^{-1}(m + 1)$. On the other hand,

$$
\|p_m h(g - T_m(g))\|_\alpha \leq \|p_m h\|_\alpha \|g - T_m(g)\|_{M(D_\alpha)},
$$

so by Remark 3.6 and since the norms of $\|p_m h\|_\alpha$ are bounded, this term decays at
an exponential rate. Therefore, there exist constants $C_1$ and $C_2$ such that

$$
\text{dist}_{D_\alpha}^2(1, f \cdot \mathcal{P}_m) = \|p_m f - 1\|_\alpha^2 \geq C_1 \|p_m h T_m(g) - 1\|_\alpha \geq C_2 \varphi_\alpha^{-1}(m + 1),
$$
as desired. □

The methods used in the proofs of Theorems 3.4 and 3.7 yield an independent proof of the upper bound for the optimal norm in the Dirichlet space (the case \( \alpha = 1 \)), valid for a class of functions with the property that the Fourier coefficients of \( f \) and of \( 1/f \) exhibit simultaneously rapid decay. More specifically, let \( \{a_j\} \) be the sequence of Taylor coefficients of a function \( f \in D \) and \( \{b_k\} \) the sequence Taylor coefficients of \( 1/f \). We say that \( f \) is strongly invertible if it has no zeros in \( \mathbb{D} \) and if for all \( j \) and \( k \), \( |a_j| \leq C(j + 1)^{-3} \), and \( |b_k| \leq C(k + 1)^{-1} \), for some constant \( C \). For example, one can show that if \( f \) is strongly invertible, then \( 1/f \) is in the Dirichlet space. (In fact, much more is true: \( 1/f \in D_2 \); cf. the discussion at the beginning of Section 4.) That is, strongly invertible implies invertible in the Dirichlet space, and such functions are known to be cyclic [6, p. 274] and are therefore of interest.

By defining polynomials analogous to those at the end of Section 2, namely,

\[ P_n(z) = \sum_{k=0}^{n} \left( 1 - \frac{H_k}{H_{n+1}} \right) b_k z^k, \]

we can use the stronger condition on the decay of the coefficients of \( 1/f \) to prove a version of Lemma 3.3 with the coefficients \( H_k \) and then obtain the conclusion of Theorem 3.7 for these strongly invertible functions. In particular, the following result can be shown to hold.

**Proposition 3.8.** Let \( f \) be a strongly invertible function, \( \gamma \in \mathbb{N} \), and \( g = f^{\gamma} \). Then there exist polynomials \( q_n \) of degree at most \( n \) such that

\[ \|q_n g - 1\|^2_D \leq \frac{C}{\log(n+2)}. \]

It is natural to investigate whether these Riesz-type polynomials provide close to optimal approximants for more general functions, in particular functions of the form \( f_\beta(z) = (1 - z)^\beta \), when \( \beta < 1 \). Even in the case of \( H^2 \), we do not know when Cesàro sums for \( 1 = f \) provide close to optimal approximants in \( H^2 \) for an outer function \( f \), nor whether the rate of convergence of the optimal norms can be arbitrarily slow. Another interesting question is whether the rate of decay that we have observed for functions admitting an analytic continuation to the closed disk holds for other functions that vanish on precisely the same set.

**4 Logarithmic conditions**

It is well known that if \( f \) is invertible in the Hardy or Dirichlet space, then it is cyclic in that space. In addition, it is easy to see that if both \( f \) and \( 1/f \) are in \( D_\alpha \)
and $f$ is bounded, then $\log f \in D_{\alpha}$, but the converse does not hold. The condition $\log f \in D_{\alpha}$ can be thought of as intermediate between $f \in D_{\alpha}$ and $1/f \in D_{\alpha}$. Indeed, $\log f \in D_{\alpha}$ is equivalent to $f'/f \in D_{\alpha-2}$. On the other hand, $f \in D_{\alpha}$ if and only if $f' \in D_{\alpha-2}$, while $1/f \in D_{\alpha}$ if and only if $f'/f^2 \in D_{\alpha-2}$. It is therefore natural to study the following question.

**Problem 4.1.** Is every $f \in D_{\alpha}$ such that $q = \log f \in D_{\alpha}$ cyclic in $D_{\alpha}$?

In several cases the statement is true. For example, if $\alpha > 1$ or $\alpha = 0$ and $f \in D_{\alpha}$ is such that $q = \log f \in D_{\alpha}$, then $f$ is cyclic in $D_{\alpha}$. Indeed, for $\alpha > 1$, $\log f \in D_{\alpha}$ implies $1/f \in H^\infty$, which is equivalent to $f$ being cyclic; see [6, p. 274]. For $\alpha = 0$, it is easy to see that if $\log f \in H^1$, then $\log |f(0)| = (1/2\pi) \int_0^{2\pi} \log |f(e^{i\theta})|d\theta$; and therefore $f$ is outer, that is, cyclic in $H^2$. Moreover, the logarithmic condition implies the following interpolation result.

**Lemma 4.2.** Let $\alpha < 2$. Suppose $f \in D_{\alpha}$ is such that $\log f \in D_{\alpha}$. Then, for all $\tau \in (0, 1],

$$D_{\alpha}(f^\tau) \leq \tau^2 (D_{\alpha}(f) + D_{\alpha}(\log f)),$$

and consequently $f^\tau \in D_{\alpha}$.

**Proof.** It suffices to establish the bound on $D_{\alpha}(f^\tau)$. To this end, we write

$$D_{\alpha}(f^\tau) = \int_\mathbb{D} |(f^\tau)'(z)|^2d\mu_{\alpha}(z) = \tau^2 \int_\mathbb{D} \left|\frac{f'(z)}{f(z)}\right|^2 |f(z)|^2 d\mu_{\alpha}(z)
$$

$$= \tau^2 \int_\mathbb{D} \left|\frac{f'(z)}{f(z)}\right|^2 |f(z)|^2 \chi_{\{z \in \mathbb{D} : |f(z)| < 1\}}d\mu_{\alpha}(z)
$$

$$+ \tau^2 \int_\mathbb{D} \left|\frac{f'(z)}{f(z)}\right|^2 |f(z)|^2 \chi_{\{z \in \mathbb{D} : |f(z)| \geq 1\}}d\mu_{\alpha}(z)
$$

$$\leq \tau^2 \int_\mathbb{D} \left|\frac{f'(z)}{f(z)}\right|^2 \chi_{\{z \in \mathbb{D} : |f(z)| < 1\}}d\mu_{\alpha}(z)
$$

$$+ \tau^2 \int_\mathbb{D} |f''(z)|^2 \chi_{\{z \in \mathbb{D} : |f(z)| \geq 1\}}d\mu_{\alpha}(z),$$

and the resulting integrals can be bounded in terms of $D_{\alpha}(f)$ and $D_{\alpha}(\log f)$, as claimed. \qed

This lemma allows us to show that for a function $f$ in the Dirichlet space $D$, which corresponds to the case $\alpha = 1$, the condition $\log f \in D$ implies $f$ is cyclic. The proof relies on the following theorem applied with $\mu$ being Lebesgue measure.

**Theorem 4.3 ([13, Theorem 4.3]).** If $f \in D$ is an outer function and $\tau > 0$ is such that $f^\tau \in D$, then $[f] = [f^\tau]$. 

In [13], Richter and Sundberg applied this theorem to show that if \( f \) is univalent and non-vanishing, then \( f^\tau \in D \), and hence is cyclic. In what follows, we do not require univalence.

**Theorem 4.4.** Suppose \( f \in D \) is such that \( \log f \in D \). Then \( f \) is cyclic in the Dirichlet space.

**Proof.** As discussed above, the condition \( \log f \in D \) implies that \( f \) is outer. By Lemma 4.2, \( f^\tau \in D \) for all \( \tau > 0 \), and so \([f] = [f^\tau]\) for each \( \tau \). Since that Lemma also implies \( f^\tau \to 1 \) in \( D \) as \( \tau \to 0 \), we have \([f] = [1]\), and the assertion follows. \( \square \)

For non-negative \( \alpha \), a more general version of Theorem 4.4 was obtained, using a different proof, by Alexandru Aleman in his Habilitationsschrift, Fernuniversität Hagen, 1993. We thank Richard Rochberg for drawing Aleman’s work to our attention.

The following is the main result for the remaining cases \( \alpha < 0 \) and \( 0 < \alpha < 1 \).

**Theorem 4.5.** Let \( f \in H^\infty \) be such that \( q = \log f \in D_\alpha \). Suppose there exists a sequence of polynomials \( \{q_n\} \) that approaches \( q \) in \( D_\alpha \) norm with

\[
2 \sup_{z \in D} \Re(q(z) - q_n(z)) + \log(\|q - q_n\|_2^\alpha) \leq C
\]

for some constant \( C > 0 \). Then \( f \) is cyclic in \( D_\alpha \).

**Remark 4.6.** An immediate consequence of Theorem 4.5 is that if \( q = \log f \) can be approximated in \( D_\alpha \) by polynomials \( \{q_n\} \) with \( \sup_{z \in D} \Re(q(z) - q_n(z)) < C \), then \( f \) is cyclic. Brown and Cohn proved [5, Theorem B] that for any closed set \( E \subset \partial D \) of logarithmic capacity 0, there exists a cyclic function \( f \) in \( D \) such that \( \mathcal{Z}(f^*) = E \). The functions they built satisfy this hypothesis on \( q_n \), and therefore, for any potential cyclic function zero set, these assumptions are always satisfied by at least one cyclic function.

**Proof of Theorem 4.5.** Assume \( \alpha \leq 1 \); otherwise, the statement is immediate. As discussed earlier in this section, \( f \in D_\alpha \). Now, by the triangle inequality, for every sequence of polynomials \( p_n \),

\[
\|p_n f - 1\|_\alpha \leq \|p_n f - e^{-q_n} f\|_\alpha + \|e^{-q_n} f - 1\|_\alpha.
\]

The first summand on the right hand side can be bounded as

\[
\|p_n - e^{-q_n}\|_\alpha \leq \|p_n - e^{-q_n}\|_{M(D_\alpha)} \|f\|_\alpha.
\]
Moreover, for $\alpha \leq 1$, the multiplier norm of a function is controlled by the $H^\infty$ norm of its derivative. Hence, a good choice of approximating polynomials is a sequence $\{p_n\}$ such that $p_n(0) = e^{-q_n(0)}$ and $\|p'_n + q'_n e^{-q_n}\|_{H^\infty} \leq 1/n$. Such a sequence exists by Weierstrass’ Theorem. With such a sequence, the first summand on the right hand side of (4.1) approaches 0 as $n \to \infty$.

Note that the sequence of polynomials $\{p_n\}$ converges pointwise to $1/f$. Therefore, to prove that $f$ is cyclic, it suffices to show that the norms of $p_nf - 1$ stay bounded. So what remains to show is that $\|e^{-q_n f} - 1\|_a^2$ is uniformly bounded as $n \to \infty$. To evaluate this expression for large $n$, we use the norm in terms of the derivative:

$$\|e^{-q_n f} - 1\|_a^2 \approx \|q'_n e^{-q_n f} + e^{-q_n f'}\|_{a-2}^2 + |e^{-q_n(0)} f(0) - 1|^2.$$ 

The last term tends to 0 since $q_n$ approaches $q$ pointwise.

In the first summand on the right hand side, taking out a common factor, we see that

$$\|q'_n e^{-q_n f} + e^{-q_n f'}\|_{a-2}^2 \leq \|e^{q_n f'}\|_{H^\infty} \left\| \frac{f'}{f} - q_n' \right\|_{a-2}^2 \leq C e^{2 \sup \Re(q - q_n)} \|q - q_n\|_a^2$$

for some constant $C$. Given our assumptions on $q_n$, the right hand side is bounded. \hfill \Box

It would be interesting to know whether the required approximation property of the polynomials $q_n$ in Theorem 4.5 is a consequence of the facts that $f \in H^\infty$ and $\log f \in D_\alpha$.

5 Asymptotic zero distributions for approximating polynomials

In this paper, we have primarily been interested in functions $f \in D_\alpha$ that are cyclic and satisfy $f^*(\zeta) = 0$ for at least one $\zeta \in \mathbb{T}$. Prime examples of such a function are

$$f_\beta(z) = (1 - z)^\beta, \quad \beta \in [0, \infty),$$

which we have examined closely in the case $\beta \in \mathbb{N}$.

Numerical experiments, described below, suggest that a study of the zero sets $Z(p_n)$ of approximating polynomials may be interesting from the point of view of cyclicity. It seems that the rate at which zeros approach the circle is related to the extent to which the corresponding polynomials furnish approximants in $D_\alpha$. For
instance, we have compared the zero sets associated with the Taylor polynomials of $1/f_\beta$ with those of Riesz-type polynomials,

$$R_n \left( \frac{1}{f_\beta} \right) (z) = \sum_{k=0}^{n} \left( 1 - \frac{H_k}{H_{n+1}} \right) b_k z^k, \quad n \geq 1. \tag{5.1}$$

Intuitively, since $1/f_\beta$ has a pole at $z = 1$, we should expect the approximating polynomials $p_n$ to be “large” in the intersection of disks of the form $B(1, r)$ with the unit disk. On the other hand, the remainder functions $p_n f - 1$ have to tend to 0 in norm (and hence pointwise). Since $1/f_\beta$ has a pole on $\mathbb{T}$, the Taylor series of $1/f_\beta$ cannot have radius of convergence greater than 1. It therefore follows from Jentzsch’s Theorem [15, Subsection 1.7] that every point on $\mathbb{T}$ is a limit point of the zeros of the sequence $\{T_n(1/f_\beta)\}_{n=1}^\infty$. We refer the reader to [15] for background material concerning sections of polynomials, and for useful computer code.

We start with the simplest case $f_1(z) = 1 - z$. The zeros of the Taylor polynomials $T_n(1/f)$, the Cesàro polynomials $C_n(1/f)$, and the Riesz polynomials $R_n(1/f)$, for $n = 1, \ldots, 50$, are shown in Figure 5.1. All the zeros of these polynomials are located outside the unit disk and inside a certain cardioid-like curve. In the case of the Taylor polynomials, the explicit formula

$$T_n(1/f_1)(z) = \frac{1 - z^{n+1}}{1 - z}$$

holds, and so $\mathcal{Z}(T_n)$ simply consists of the n-th roots of unity minus the point $\zeta = 1$. As noted by R. W. Barnard, J. Cima, and K. Pearce [2, Section 4], one can show that the unit circle is also a subset of the limit set of the zeros of the Cesàro polynomials. However, replacing Taylor polynomials by Cesàro polynomials has the effect of repelling zeros from the unit circle into the exterior of the disk. This effect is even more pronounced for the Riesz polynomials (5.1), where it appears...
that convergence of roots to the unit circle, and the roots of unity in particular, is somewhat slower. Note also the relative absence of zeros close to the pole of $1/f_1$ and the somewhat tangential approach region at $\zeta = 1$.

Next, we turn to a function with two simple zeros on $\mathbb{T}$, namely

$$f = 1 - z + z^2.$$ 

Plots of zeros of successive approximating polynomials are shown in Figure 5.2. While $\mathcal{Z}(T_n)$ is more complicated than in the previous case, the general features of Figure 5.1 persist. We again note a relative absence of zeros close to the two poles of $1/f$ and that the zeros of the Cesàro and Riesz polynomials are again located in the exterior disk and seem to tend to $\mathbb{T}$ more slowly. On the other hand, in this case, the Taylor polynomials do exhibit zeros inside the unit disk. Similar observations are made in the papers by Barnard, Cima, Pearce, and W. Wheeler (see [2, 3]); they prove that for certain subclasses of outer functions, zeros of Cesàro sums are indeed located outside the unit disk. Note that in our setting, the target functions $1/f$ are outer but not in $D_\alpha$, and do not necessarily arise as derivatives of bounded convex functions; cf. the subclass considered in [2, 3]. We observe approach regions with vertices at the symmetrically placed poles, and the angle at these vertices seems to decrease as we move from Taylor polynomials through Cesàro polynomials to the polynomials in (5.1). It seems natural to suspect that locally the picture would be similar for a polynomial with a large number of zeros on the unit circle.

It would be interesting to investigate whether there is a relationship between zeros of approximating polynomials, the region of convergence of the Taylor series of $1/f$, and the cyclicity of $f$ in future work.
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