On Yuzvinsky’s lattice sheaf cohomology for hyperplane arrangements

Paul Mücksch

Abstract
We establish the relationship between the cohomology of a certain sheaf on the intersection lattice of a hyperplane arrangement introduced by Yuzvinsky and the cohomology of the coherent sheaf on punctured affine space, respectively projective space associated to the module of logarithmic vector fields along the arrangement. Our main result gives a Künneth formula connecting the cohomology theories, answering a question by Yoshinaga. This, in turn, provides a characterization of the projective dimension of the module of logarithmic vector fields and yields a new proof of Yuzvinsky’s freeness criterion. Furthermore, our approach affords a new formulation of Terao’s freeness conjecture and a more general problem.

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1 Introduction
Let $\mathcal{A}$ be a hyperplane arrangement, i.e. a finite set of codimension one subspaces in a $\mathbb{K}$-vector space $V$ of dimension $\ell \geq 2$ for some field $\mathbb{K}$. The combinatorial structure of $\mathcal{A}$ is encoded in its intersection lattice $L(\mathcal{A})$ which consists of all intersections of subsets of hyperplanes ordered by reverse inclusion. Let $S = \mathbb{K}[x_1, \ldots, x_\ell]$ be the coordinate ring of the vector space $V$. The arrangement $\mathcal{A}$ is called free if the associated graded $S$-module $D(\mathcal{A})$ of logarithmic vector fields along $\mathcal{A}$ or module of $\mathcal{A}$-derivations is a free $S$-module, a notion first introduced and studied by Saito [13] and Terao [17] (see Sect. 2.1). One of the most intricate problems in the study of hyperplane arrangements is to relate properties of $D(\mathcal{A})$ to the combinatorial structure of $\mathcal{A}$ given by its intersection lattice. The ultimate solution is proposed by Terao’s conjecture from

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1 Department of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka 819-0395, Japan

Paul Mücksch
paul.muecksch+uni@gmail.com
the 1980s (see [11, Conj. 4.138]) which asserts that over a fixed field $\mathbb{K}$ the freeness of $\mathcal{A}$ only depends on its intersection lattice $L(\mathcal{A})$. This conjecture still remains open.

Functors defined on the intersection lattice of a hyperplane arrangement and related to the derivation module were already studied by Solomon and Terao [16]. They gave a new proof of Terao’s seminal Factorization Theorem for free arrangements first obtained in [18].

Assume that $\mathcal{A}$ is central and essential, that is $\cap_{H \in \mathcal{A}} H = \{0\}$ and set $L_0 := (L(\mathcal{A}) \setminus \{\{0\}\})^{op}$, i.e. the order relation in $L_0$ is inclusion. In a series of papers [22–24] Yuzvinsky studied the functor $D : L_0 \to \text{Mod}_S$, $(X \subseteq Y) \mapsto (D(X) = D(\mathcal{A}_X) \hookrightarrow D(\mathcal{A}_Y) = D(Y))$ regarded as a sheaf on the finite topological space associated to the poset $L_0$ and its cohomology (see Sects. 3 and 4). An arrangement $\mathcal{A}$ is called locally free if all localization subarrangements $\mathcal{A}_X$ which consist of all hyperplanes from $\mathcal{A}$ containing $X$ are free for all $X \in L_0$. He showed [23, Thm. 1.1] that a locally free hyperplane arrangement $\mathcal{A}$ is free if and only if the lattice sheaf cohomology groups $H^n(L_0, D)$ vanish for all $0 < n < \ell - 1$.

Moreover, in his study of these lattice sheaf cohomology groups, Yuzvinsky showed that free arrangements form a Zariski open subset in the moduli space of arrangements with a fixed intersection lattice, [24, Cor. 3.4]. This is up to date still the strongest general result towards Terao’s conjecture.

A classical theorem by Horrocks [8] asserts that a vector bundle $\mathcal{E}$ on projective space $\mathbb{P}^{\ell - 1} = \text{Proj} \ S$ splits into a direct sum of line bundles if and only if the sheaf cohomology groups $H^n(\mathbb{P}^{\ell - 1}, \mathcal{E}(d))$ vanish for all $0 < n < \ell - 1$ and all $d \in \mathbb{Z}$.

It turns out that the coherent sheaf $\tilde{D}$ on $\mathbb{P}^{\ell - 1}$ associated to the derivation module $D = D(\mathcal{A})$ of a locally free arrangement is a vector bundle, cf. [10, Thm. 2.3]. Applying Horrocks’ criterion to $\tilde{D}$ of a locally free hyperplane arrangement yields a freeness criterion resembling Yuzvinsky’s criterion, cf. [21, Prop. 1.20]. A related similarity with local cohomology was already noticed by Yuzvinsky in [22, Rem. 2.7].

Our aim is to establish the exact relationship between Yuzvinsky’s lattice sheaf cohomology and the sheaf cohomology on projective space and explain the resemblance of Yuzvinsky’s and Horrocks’ criteria for freeness. This clarifies the resemblance with local cohomology already noticed by Yuzvinsky in [22, Rem. 2.7] and answers a question posed by Yoshinaga [21, Prob. 1.49].

Set $\mathcal{X} := \text{Spec} \ S \setminus \{m\}$ where $m = (x_1, \ldots, x_\ell)$ is the homogeneous maximal ideal and let $\mathcal{O}_\mathcal{X} = \mathcal{S}_{|\mathcal{X}}$ be the structure sheaf (the restriction of the structure sheaf of the affine scheme $\text{Spec} \ S$ to the open complement $\mathcal{X}$ of the origin).

Our principal theorem establishes the exact relationship of the cohomology of the sheaf $\mathcal{D}$ on $L_0$ studied by Yuzvinsky with the cohomology of the coherent sheaf $\tilde{D}|_{\mathcal{X}}$ on the punctured spectrum $\mathcal{X}$ associated to the derivation module.

**Theorem 1.1** For all $n \neq \ell - 1$ we have

$$H^n(\mathcal{X}, \tilde{D}|_{\mathcal{X}}) \simeq \bigoplus_{i+j=n} H^i(L_0, D) \otimes_S H^j(\mathcal{X}, \mathcal{O}_\mathcal{X})$$

and for $n = \ell - 1$ we have a short exact sequence

\[ \mathcal{O}_\mathcal{X} \]
0 \rightarrow \bigoplus_{i+j=\ell-1} H^i(L_0, \mathcal{D}) \otimes_S H^j(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \rightarrow H^{\ell-1}(\mathfrak{X}, \tilde{D}|_{\mathfrak{X}}) \\
\rightarrow \text{Tor}^S_1(H^1(L_0, \mathcal{D}), H^{\ell-1}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})) \rightarrow 0.

In particular, \(H^n(\mathfrak{X}, \tilde{D}|_{\mathfrak{X}}) \simeq H^n(L_0, \mathcal{D})\) for \(n < \ell - 1\).

Note that sheaf cohomology on the scheme \(\mathfrak{X}\) and sheaf cohomology on projective space are connected as follows, see e.g. [14, no69: Remarque].

**Remark 1.2** Let \(M\) be a finitely generated graded \(S\)-module. Denote by \(\tilde{M}|_{\mathfrak{X}}\) the coherent sheaf associated to \(M\) on \(\text{Spec} \ S\) restricted to the open subset \(\mathfrak{X} = \text{Spec} \ S \setminus \{m\}\) and by \(\tilde{M}\) the coherent sheaf on \(\mathbb{P}^{\ell-1} = \text{Proj} \ S\) associated to \(M\). Then \(H^n(\mathfrak{X}, \tilde{M}|_{\mathfrak{X}}) \simeq \bigoplus_{d \in \mathbb{Z}} H^n(\mathbb{P}^{\ell-1}, \tilde{M}(d))\) for \(n \geq 0\).

As a direct consequence of Remark 1.2 and Theorem 1.1 we obtain the following result which establishes the relationship between the lattice sheaf cohomology studied by Yuzvinsky and the sheaf cohomology on projective space. This completely resolves a problem stated by Yoshinaga [21, Prob. 1.49] and readily yields another proof of Yuzvinsky’s freeness criterion using Horrocks’ theorem.

**Theorem 1.3** For \(n < \ell - 1\) we have

\[H^n(L_0, \mathcal{D}) \simeq \bigoplus_{d \in \mathbb{Z}} H^n(\mathbb{P}^{\ell-1}, \tilde{D}(d)).\]

The connection to local cohomology and projective dimension is as follows.

**Remark 1.4** Recall that local cohomology is related to the cohomology on punctured affine space as follows, cf. [6, Prop. 2.2]. For \(i > 0\) we have:

\[H^{i+1}_{m}(D) \simeq H^i(\mathfrak{X}, \tilde{D}|_{\mathfrak{X}}).\]

Furthermore, by [6, Thm. 3.8] the depth, respectively the projective dimension \(\text{pd}(D)\) (by the Auslander–Buchsbaum formula) of the module \(D\) is tied to local cohomology by

\[\text{pd}(D) \leq p \quad \text{if and only if} \quad H^i_{m}(D) = 0 \quad \text{for} \quad i < \ell - p.\]

The module \(D\) is reflexive (cf. [13, p. 268]) and as such, it is the dual of another finitely generated module. So \(D\) has projective dimension at most \(\ell - 2\). Consequently, Theorem 1.1 together with the preceding remark directly yields the following characterization of the projective dimension of \(D\).

**Theorem 1.5** The following two conditions are equivalent:

(i) \(\text{pd}(D) \leq p\);

(ii) \(H^n(L_0, \mathcal{D}) = 0\) for \(0 < n < \ell - 1 - p\).
Finally, as a consequence to Theorem 1.5, we obtain the following stronger form of Yuzvinsky’s freeness criterion [23, Thm. 1.1] showing the assumption of $\mathcal{A}$ being locally free to be superfluous.

**Corollary 1.6** The arrangement $\mathcal{A}$ is free if and only if

$$H^n(L_0, \mathcal{D}) = 0 \quad \text{for} \quad 0 < n < \ell - 1.$$  

This paper is organized as follows. In Sect. 2 we review some basic notions from the theory of hyperplane arrangements. Furthermore, we recall some results from homological algebra and sheaf theory. In Sect. 3 we review sheaves on posets and their cohomology. Section 4 gives further details about some special sheaves on the intersection lattice of an arrangement. In Sect. 5 we prove Theorem 1.1 and finally, in Sect. 6 we comment on possible generalizations of our approach and related problems.

## 2 Recollection and preliminaries

In this note $V \cong \mathbb{K}^\ell$ always denotes an $\ell$-dimensional $\mathbb{K}$-vector space over some field $\mathbb{K}$ where $\ell \geq 2$.

Let $S = \mathbb{K}[x_1, \ldots, x_\ell]$ be the coordinate ring of $V$. The ring $S$ is graded: $S = \bigoplus_{p \in \mathbb{Z}} S_p$ where $S_p$ is the $\mathbb{K}$-space of homogeneous polynomials of degree $p$ (along with 0) and $S_p = \{0\}$ for $p < 0$.

If $f \in S$ then we write $S_f = S[\frac{1}{f}]$ for the localization of $S$ by $f$ and similarly for an $S$-module $M$ we write $M_f = M \otimes_S S_f$.

### 2.1 Hyperplane arrangements

As a general reference for hyperplane arrangements we refer to the book by Orlik and Terao [11].

Let $\mathcal{A} = (\mathcal{A}, V)$ be a hyperplane arrangement in $V$, that is a finite set of codimension one subspaces of $V$. The intersection lattice of $\mathcal{A}$ is

$$L(\mathcal{A}) = \{ \cap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \}$$

with the partial order

$$X \leq Y : \iff X \supseteq Y \quad (X, Y \in L(\mathcal{A})).$$

In this note we always assume $\mathcal{A}$ to be essential, that is for the maximal element $T(\mathcal{A}) := \cap_{H \in \mathcal{A}} H$ in $L(\mathcal{A})$ we have $T(\mathcal{A}) = \{0\}$.

For $X \in L(\mathcal{A})$ the localization $\mathcal{A}_X$ of $\mathcal{A}$ at $X$ is

$$\mathcal{A}_X := \{ H \in \mathcal{A} \mid H \supseteq X \}.$$
If \( X, Y \in L(\mathcal{A}) \) then \( X \wedge Y := \text{sup}\{Z \in L(\mathcal{A}) \mid Z \leq X \text{ and } Z \leq Y\} \). Note that we have \( A_{X \wedge Y} = A_X \cap A_Y \).

For all hyperplanes \( H \in \mathcal{A} \) we fix \( \alpha_H \in V^* \) with \( H = \ker(\alpha_H) \). The defining polynomial \( Q(\mathcal{A}) \) of \( \mathcal{A} \) is

\[
Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H.
\]

A \( \mathbb{K} \)-linear map \( \theta : S \to S \) which satisfies \( \theta(fg) = \theta(f)g + f\theta(g) \) is called a \( \mathbb{K} \)-derivation. Let \( \text{Der}_{\mathbb{K}}(S) \) be the \( S \)-module of \( \mathbb{K} \)-derivations of \( S \). It is a free \( S \)-module with basis \( \partial/\partial x_1, \ldots, \partial/\partial x_\ell \).

**Definition 2.1** The module of \( \mathcal{A} \)-derivations is the \( S \)-submodule of \( \text{Der}_{\mathbb{K}}(S) \) defined by

\[
D(\mathcal{A}) := \{ \theta \in \text{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A} \}.
\]

In particular, if \( B \subseteq \mathcal{A} \), then \( D(\mathcal{A}) \subseteq D(B) \).

We say that \( \mathcal{A} \) is free if the module of \( \mathcal{A} \)-derivations is a free \( S \)-module.

**Definition 2.2** For \( X \in L(\mathcal{A}) \) we define

\[
Q(X) := \prod_{H \in \mathcal{A} \setminus A_X} \alpha_H = \frac{Q(\mathcal{A})}{Q(A_X)}.
\]

The following observation provides a crucial ingredient in the proof of Theorem 1.1.

**Lemma 2.3** For all \( X, Y \in L(\mathcal{A}) \) we have:

\[
D(\mathcal{A} Y)_{Q(X)} = D(\mathcal{A} X \wedge Y)_{Q(X)}.
\]

**Proof** For each \( H \in \mathcal{A} \) we define the \( S \)-module homomorphism \( M_H : S^\ell \to S/\alpha_H S \) by

\[
M_H(f_1, \ldots, f_\ell) := \sum_{i=1}^\ell f_i \frac{\partial \alpha_H}{\partial x_i} + \alpha_H S.
\]

For \( X \in L(\mathcal{A}) \) we set

\[
M_X := \sum_{H \in A_X} M_H : S^\ell \to \bigoplus_{H \in A_X} S/\alpha_H S.
\]

From the definition of \( D(\mathcal{A} Y) \) we have the following short exact sequence

\[
0 \longrightarrow D(\mathcal{A} Y) \longrightarrow S^\ell \xrightarrow{M_Y} \bigoplus_{H \in A_Y} S/\alpha_H S \longrightarrow 0.
\]
If we localize at $Q(X)$, for each $H \in \mathcal{A}\backslash\mathcal{A}_X$ we have

\[ S/\alpha_H S \otimes_S S_{Q(X)} = 0, \]
\[ M_H \otimes_S \text{id}_{S_{Q(X)}} \equiv 0. \]

Recall, that $\mathcal{A}_{X \land Y} = \mathcal{A}_X \cap \mathcal{A}_Y$, thus

\[ \bigoplus_{H \in \mathcal{A}_Y} S/\alpha_H S \otimes_S S_{Q(X)} = \bigoplus_{H \in \mathcal{A}_{X \land Y}} S/\alpha_H S \otimes_S S_{Q(X)}. \]

Further, recall that there is a natural inclusion $i: D(A_Y) \hookrightarrow D(A_{X \land Y})$.

Since localization at $Q(X)$ is an exact functor, we obtain the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \rightarrow & D(A_Y)_{Q(X)} & \rightarrow & S^\ell_{Q(X)} & \rightarrow & \bigoplus_{H \in \mathcal{A}_Y} S/\alpha_H S \otimes_S S_{Q(X)} & \rightarrow & 0 \\
\downarrow & & \downarrow i \otimes \text{id} & & \downarrow M_Y \otimes \text{id} & & \downarrow M_{X \land Y} \otimes \text{id} & & \downarrow \bigoplus_{H \in \mathcal{A}_{X \land Y}} S/\alpha_H S \otimes_S S_{Q(X)} & \rightarrow & 0 \\
0 & \rightarrow & D(A_{X \land Y})_{Q(X)} & \rightarrow & S^\ell_{Q(X)} & \rightarrow & \bigoplus_{H \in \mathcal{A}_{X \land Y}} S/\alpha_H S \otimes_S S_{Q(X)} & \rightarrow & 0
\end{array}
\]

Hence $i \otimes \text{id}$ yields the equality (e.g. by the five-lemma and extending the diagram by additional zeros to the left).

We record the following special case of Lemma 2.3.

**Corollary 2.4** Let $X \in L(A)$. Then we have

\[ D(A_X)_{Q(X)} = D(A)_{Q(X)}. \]

**Proof** Let $Y = T(A)$ in Lemma 2.3 and note that then $X \land Y = X$. \qed

**Remark 2.5** The preceding lemma and corollary can also be seen as a consequence of the local property of the functor or sheaf $\mathcal{D}$ [16, Prop. 6.6]: for a generic point $p \in X$ we have $D(A_X)_p \simeq D(A)_p$.

### 2.2 Homological algebra

For the basics we refer to [12]. Let

\[ C^\bullet = \cdots \rightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \rightarrow \cdots \]

be a cochain complex of abelian groups ($S$-modules). Then we write

\[ H^n(C^\bullet) = Z^n / B^n, \]

for the $n$-th cohomology group (module), where $Z^n = \ker(d^n)$ is the group ($S$-module) of $n$-cocycles and $B^n = \text{im}(d^{n-1})$ is the group ($S$-module) of $n$-coboundaries. Note
On Yuzvinsky's lattice sheaf cohomology for hyperplane arrangements}

that with the zero-coboundary maps \( B^n \xrightarrow{d^n|B^n=0} B^{n+1} \) and the inclusion maps \( i^n : B^n \hookrightarrow C^n \) the complex \( B^* \) is a subcomplex of \( C^* \) called the coboundary-subcomplex.

Let \( A^* \) and \( C^* \) be two cochain complexes of \( S \)-modules with coboundary maps \( d_A \) and \( d_C \) respectively. By \( A^* \otimes_S C^* \) we denote their tensor product which is defined as the total complex of the associated bicomplex, i.e.

\[
(A^* \otimes_S C^*)^n := \bigoplus_{i+j=n} A^i \otimes_S C^j
\]

with coboundary maps

\[
d^n(a \otimes_S c) = d^i_A(a) \otimes_S c + (-1)^i a \otimes_S d^j_C(c)
\]

for \( a \in A^i, c \in C^j \) and \( i + j = n \).

To later guarantee the exactness of a tensor product of two special resolutions of sheaves, we require the following special case of the acyclic assembly lemma, cf. [19, Lem. 2.7.3].

**Lemma 2.6** Let \( C^{\cdot,\cdot} \) be a bounded first quadrant bicomplex in an abelian category. If \( C^{\cdot,\cdot} \) has exact rows or columns then \( \text{Tot}(C^{\cdot,\cdot})^n \) which is given by

\[
\text{Tot}(C^{\cdot,\cdot})^n = \bigoplus_{i+j=n} C^{i,j}
\]

is also exact.

One crucial ingredient for the proof of our main Theorem 1.1 is the following Künneth formula for the cohomology of the tensor product of two complexes.

**Theorem 2.7** ([12, Thm. 10.81]) Let \( A^* \) and \( C^* \) be two cochain complexes of \( S \)-modules. Suppose that all terms of \( C^* \) and all terms of its coboundary-subcomplex are flat \( S \)-modules.

Then for each \( n \) there is a short exact sequence

\[
0 \longrightarrow \bigoplus_{i+j=n} H^i(A^*) \otimes_S H^j(C^*) \longrightarrow H^n(A^* \otimes_S C^*) \longrightarrow \bigoplus_{i+j=n+1} \text{Tor}^S_1(H^i(A^*), H^j(C^*)) \longrightarrow 0.
\]

**Corollary 2.8** Let \( A^* \) and \( C^* \) be two cochain complexes of \( S \)-modules. Suppose that all terms of \( C^* \) and all terms of its coboundary-subcomplex are flat \( S \)-modules. Suppose further that \( H^p(C^*) \) is flat for \( p < k \), \( H^p(A^* \otimes_S C^*) = 0 \) for \( p > k \) and \( \text{Tor}^S_1(H^0(A^*), H^k(C^*)) = 0 \).
Then for all $n \neq k$ we have
\[ H^n(A^\bullet \otimes_S C^\bullet) \cong \bigoplus_{i+j=n} H^i(A^\bullet) \otimes_S H^j(C^\bullet) \]
and for $n = k$ we have a short exact sequence
\[ 0 \to \bigoplus_{i+j=k} H^i(A^\bullet) \otimes_S H^j(C^\bullet) \to H^k(A^\bullet \otimes_S C^\bullet) \to \text{Tor}_1^S(H^1(A^\bullet), H^k(C^\bullet)) \to 0. \]

**Proof** By the flatness of $H^p(C^\bullet)$ for $p < k$ and the vanishing of $\text{Tor}_1^S(H^0(A^\bullet), H^k(C^\bullet))$, for $n < k$ we have
\[ \bigoplus_{i+j=n+1} \text{Tor}_1^S(H^i(A^\bullet), H^j(C^\bullet)) = 0. \]
So by Theorem 2.7 for $n < k$ we have
\[ H^n(A^\bullet \otimes_S C^\bullet) \cong \bigoplus_{i+j=n} H^i(A^\bullet) \otimes_S H^j(C^\bullet) \]
and for $n = k$ we obtain the short exact sequence from Theorem 2.7 by taking the flatness of $H^p(C^\bullet)$ for $p < k$ and again the vanishing of $\text{Tor}_1^S(H^0(A^\bullet), H^k(C^\bullet))$ into account.

Finally, since we assume that $H^p(A^\bullet \otimes_S C^\bullet) = 0$ for $p > k$ by Theorem 2.7 we also have
\[ \bigoplus_{i+j=n} H^i(A^\bullet) \otimes_S H^j(C^\bullet) = 0 \]
for all $n > k$. \qed

The following lemma is helpful to verify the assumptions of Theorem 2.7 respectively Corollary 2.8.

**Lemma 2.9** Let $C^\bullet$ be a bounded cochain complex consisting of flat $S$-modules and assume that $\text{Tor}_1^S(H^i(C^\bullet), M) = 0$ for all $i$, all $j \geq 2$ and every $S$-module $M$. Then the coboundary-subcomplex $B^\bullet$ of $C^\bullet$ also consists of flat $S$-modules.

**Proof** Since $C^\bullet$ is bounded by assumption there is an $m \in \mathbb{Z}$ such that $C^i = 0$ for all $i > m$. In particular $B^i = 0$ for all $i > m$ and is therefore flat. Set $H^i := H^i(C^\bullet)$. For each $i$ we have the following two canonical short exact sequences
\[ 0 \to B^i \to Z^i \to H^i \to 0, \]
\[ 0 \to Z^{i-1} \to C^{i-1} \to B^i \to 0. \]
From these sequences and the associated long exact sequences in $\text{Tor}_j^S(\mathbb{M}, \mathbb{M})$, for every $S$-Module $\mathbb{M}$ we have the following exact sequences for each $j \geq 0$

\[
\text{Tor}_j^S(H^i, \mathbb{M}) \to \text{Tor}_j^S(B^i, \mathbb{M}) \to \text{Tor}_j^S(Z^i, \mathbb{M}), \tag{2.1}
\]

\[
\text{Tor}_{j+1}^S(B^i, \mathbb{M}) \to \text{Tor}_j^S(Z^{i-1}, \mathbb{M}) \to \text{Tor}_j^S(C^{i-1}, \mathbb{M}). \tag{2.2}
\]

Now we do reverse induction on $i$. For $i = m$ we have $Z^m = C^m$, so $\text{Tor}_j^S(Z^m, \mathbb{M}) = 0$ for all $j \geq 1$. By assumption we also have $\text{Tor}_{j+1}^S(H^m, \mathbb{M}) = 0$ for all $j \geq 1$ and by (2.1) we then have $\text{Tor}_j^S(B^m, \mathbb{M}) = 0$ for all $j \geq 1$, that is $B^m$ is flat.

Assume that $B^i$ is flat, that is $\text{Tor}_j^S(B^i, \mathbb{M}) = 0$ for all $j \geq 1$. Then by (2.2), the flatness of $C^{i-1}$ implies the flatness of $Z^{i-1}$. Now, from the first Tor-sequence (2.1) (exchanging $i - 1$ for $i$) and the vanishing of $\text{Tor}_{j+1}^S(H^{i-1}, \mathbb{M})$ for all $j \geq 1$ we similarly see that $B^{i-1}$ is flat which concludes the induction. $\square$}

We note the following property of torsion free $S$-modules.

**Lemma 2.10** Let $\mathbb{M}$ be a torsion free $S$-module and $p \in S \setminus \{0\}$. Then

$$\text{Tor}_1^S(\mathbb{M}, S_p/S) = 0.$$  

**Proof** Let $Q(S)$ be the quotient field of $S$. By [12, Lem. 7.11] for the torsion free $S$-module $\mathbb{M}$ we have $\text{Tor}_1^S(\mathbb{M}, Q(S)/S) = 0$ which is equivalent to the injectivity of the localization map $f : \mathbb{M} = \mathbb{M} \otimes S \to \mathbb{M} \otimes S Q(S)$. Now, the map $f$ apparently factors through the localization map $g : \mathbb{M} \otimes S S \to \mathbb{M} \otimes S S p$ and thus $g$ is also injective which in turn implies $\text{Tor}_1^S(\mathbb{M}, S_p/S) = 0$. $\square$

### 2.3 Sheaves

For basics about sheaves and their cohomology we refer to [7, Ch. II, III] and [12, Ch. 5.4, 6.3].

#### 2.3.1 Sheaf cohomology

Let $\mathcal{F}$ be a sheaf of abelian groups ($S$-modules) on the topological space $\mathcal{X}$. The cohomology groups ($S$-modules) of $\mathcal{F}$ are defined as the images of the right derived functors of the global sections functor $\Gamma(\mathcal{X}, -) : \text{Sh}(\mathcal{X}) \to \text{Ab}(\text{Mod}_S), \mathcal{F} \mapsto \mathcal{F}(\mathcal{X})$, that is

$$H^n(\mathcal{X}, \mathcal{F}) = R^n \Gamma(\mathcal{X}, \mathcal{F}).$$

A sheaf $\mathcal{G}$ is called acyclic if $H^n(\mathcal{X}, \mathcal{G}) = 0$ for all $n > 0$. An acyclic resolution $\mathcal{G}^\bullet$ of $\mathcal{F}$ is an exact sequence of sheaves of abelian groups ($S$-modules) on $\mathcal{X}$

$$\mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \mathcal{G}^2 \to \ldots$$
where \( G^i \) is acyclic for each \( i \geq 0 \). Applying the global sections functor to an acyclic resolution yields a cochain complex of abelian groups (\( S \)-modules) \( A^\bullet = \Gamma(\mathcal{X}, G^\bullet) \) whose cohomology computes the sheaf cohomology of \( \mathcal{F} \):

\[
H^n(\mathcal{X}, \mathcal{F}) \simeq H^n(A^\bullet),
\]

cf. [12, Ch. 6].

### 2.3.2 Čech cohomology

Let \( \mathcal{F} \) be a sheaf of abelian groups on a topological space \( \mathcal{X} \) and let \( \mathcal{U} = \{ U_i \mid i \in I \} \) be an open cover of \( \mathcal{X} \). Fix a linear order on the index set \( I \).

The Čech complex \( C^\bullet(\mathcal{U}, \mathcal{F}) \) is defined as follows. Set

\[
U_{i_0, \ldots, i_n} := U_{i_0} \cap \ldots \cap U_{i_n}.
\]

The terms are

\[
C^n(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < i_1 < \ldots < i_n} \mathcal{F}(U_{i_0, \ldots, i_n})
\]

and the coboundary maps are given by

\[
d^n(\alpha)_{i_0 < \ldots < i_{n+1}} = \sum_{k=0}^{n+1} (-1)^k \rho|_{U_{i_0, \ldots, i_k, \ldots, i_n+1}} (\alpha_{i_0 < \ldots < i_k < \ldots < i_{n+1}}).
\]

Then the Čech cohomology groups (modules) are

\[
\check{H}^n(\mathcal{U}, \mathcal{F}) = H^n(C^\bullet(\mathcal{U}, \mathcal{F})).
\]

The sheaf version of the Čech complex is defined as follows. If \( j : U_{i_0, \ldots, i_n} \hookrightarrow \mathcal{X} \) is the inclusion, define

\[
C^n(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < i_1 < \ldots < i_n} j_!(\mathcal{F}|_{U_{i_0, \ldots, i_n}}).
\]

The coboundary maps \( d^n \) are defined by the same formula as above. By [7, Lem. III.4.2] we have a resolution \( \mathcal{F} \to C^\bullet(\mathcal{U}, \mathcal{F}) \) of \( \mathcal{F} \). It is not acyclic in general.

**Definition 2.11** Suppose we have \( H^k(U_{i_0, \ldots, i_n}, \mathcal{F}|_{U_{i_0, \ldots, i_n}}) = 0 \) for all finite intersections \( U_{i_0, \ldots, i_n} \) of open subsets in \( \mathcal{U} \) and all \( k > 0 \). Then \( \mathcal{U} \) is called a Leray cover for \( \mathcal{F} \).

Provided the right setting, the Čech complex computes sheaf cohomology by the following classical result due to Leray, see e.g. [12, Thm. 10.79].

**Theorem 2.12** If \( \mathcal{U} \) is a Leray cover then \( \check{H}^n(\mathcal{U}, \mathcal{F}) \simeq H^n(\mathcal{X}, \mathcal{F}) \) for all \( n \geq 0 \).
3 Sheaves on posets

Let \((P, \leq)\) be a finite poset. The point set of the finite topological space associated to \(P\) (also denoted by \(P\)) is given by the ground set of \(P\). A topology on \(P\) consists of the open subsets which are increasing subsets, i.e. \(U \subseteq P\) is open if for all \(x \in U\) and \(y \in P\) with \(x \leq y\) we have \(y \in U\). Finite topological spaces of this kind were first studied by Alexandroff [3] and the topology on \(P\) just described is called the Alexandroff topology.

**Definition 3.1** For \(x \in P\) an open subset of the form \(U_x := \{y \in P \mid x \leq y\}\) is called **principal**. The open cover \(U_P := \{U_x \mid x \in P\}\) is called the **principal open cover** of \(P\).

Recall that a poset \(P\) can be regarded as a small category with objects the elements of \(P\) and exactly one morphism \(x \to y\) for each \(x \leq y\). Every covariant functor \(F : P \to \text{Ab (Mod}_S\)) gives rise to a sheaf \(\mathcal{F}\) of abelian groups (\(S\)-modules) on the associated finite topological space as follows. The principal open subsets \(U_x = \{y \in P \mid x \leq y\}\) form a basis for the topology on \(P\). We define the sections on these principal open sets as

\[ \mathcal{F}(U_x) := F(x) \]

with restriction maps

\[ \rho_{U_x}^{U_y} := F(x \leq y) : \mathcal{F}(U_x) \to \mathcal{F}(U_y). \]

We leave it to the reader to verify that this extends uniquely to a sheaf \(\mathcal{F}\) on the Alexandroff space \(P\), cf. [4]. For the stalks \(\mathcal{F}_x\) we have

\[ \mathcal{F}_x = \lim_{\longrightarrow \substack{x \in U \subseteq P \text{ open}}} \mathcal{F}(U) = \mathcal{F}(U_x) = F(x). \]

Recall that a join-semilattice is a poset \(L\) where every pair of elements \(x, y \in L\) has a least upper bound denoted by \(x \vee y = \inf\{z \in L \mid z \geq x \text{ and } z \geq y\}\). We note the following.

**Remark 3.2** Let \(L\) be a join-semilattice. Then for two principal open subsets \(U_x, U_y \subseteq L\) we have \(U_x \cap U_y = U_{x \vee y}\).

We now discuss the cohomology of sheaves on posets. We recall the following fundamental lemma.

**Lemma 3.3** ([22, Lem. 1.1]) Let \(P\) be a finite poset with a unique minimal element and \(\mathcal{F}\) a sheaf of abelian groups (\(S\)-modules) on \(P\). Then \(\mathcal{F}\) is acyclic.

As a direct consequence of the previous lemma and Remark 3.2 we obtain the following.
Corollary 3.4 Let \( L \) be a finite join-semilattice. The principal open cover \( U := \{ U_x \mid x \in L \} \) is a Leray cover for any sheaf on \( L \).

Lemma 3.5 Let \( \mathcal{F} \) be a sheaf on a finite join-semilattice \( L \), \( U = U_x \subseteq L \) a principal open subset and \( i : U \rightarrow L \) the inclusion. Then

\[
H^n(U, \mathcal{F}|_U) \cong H^n(L, i_*(\mathcal{F}|_U))
\]

for each \( n \geq 0 \), in particular, \( i_*(\mathcal{F}|_U) \) is acyclic.

Proof Note that for a principal open subset \( W = U_y \subseteq L \) by Remark 3.2 \( W \cap U = U_x \lor y \) is also a principal open subset. Thus, by [7, Prop. III.8.1] and Lemma 3.3 for the higher direct image sheaves (that is the right derived functors of the direct image functor) we have

\[
R^k i_*(\mathcal{F})(W) = H^k(W \cap U, \mathcal{F}|_{W \cap U}) = 0
\]

for \( k > 0 \) and for each \( W \) in the principal open cover of \( L \). Hence \( R^k i_*(\mathcal{F}) \equiv 0 \) for all \( k > 0 \) and the isomorphism of the cohomology groups follows with [7, Ex. III.8.1]. \( \square \)

Now let \( X \) be a topological space and assume that we have a finite open cover \( \mathcal{U} \) of \( X \) which is closed under taking intersections, that is \( U \cap U' \in \mathcal{U} \) for all \( U, U' \in \mathcal{U} \). Suppose further that \( \mathcal{F} \) is a sheaf of abelian groups (\( S \)-modules) on \( X \). In this case, we can associate a finite poset \( (P_{\mathcal{U}}, \leq) \) to \( \mathcal{U} \) with elements the open sets contained in \( \mathcal{U} \) and order relation given by reverse inclusion. In this setting \( \mathcal{F} \) induces a functor \( P_{\mathcal{U}} \rightarrow \text{Ab}(\text{Mod}_S) \) and hence also a sheaf \( \mathcal{F}_P \) of abelian groups (\( S \)-modules) on \( P_{\mathcal{U}} \).

Lemma 3.6 Suppose \( \mathcal{U} \) is a finite Leray cover for \( \mathcal{F} \) which is also closed under taking intersections. Then

\[
H^n(X, \mathcal{F}) \cong H^n(P_{\mathcal{U}}, \mathcal{F}_P)
\]

for each \( n \geq 0 \).

Proof We consider the principal open cover \( \mathcal{U}' \) of \( P_{\mathcal{U}} \). Recall the definition of the terms in the Čech complexes of \( \mathcal{U} \) respectively \( \mathcal{U}' \). We apparently have

\[
C^n(\mathcal{U}, \mathcal{F}) = C^n(\mathcal{U}', \mathcal{F}_P)
\]

for all \( n \). Note that \( P_{\mathcal{U}} \) is a join-semilattice with \( U \lor W = U \cap W \). The cover \( \mathcal{U} \) is Leray by assumption and the principal cover \( \mathcal{U}' \) of \( P_{\mathcal{U}} \) is Leray by Corollary 3.4. Hence, we obtain the isomorphism of the cohomology groups by Theorem 2.12. \( \square \)

Note that the Alexandroff space of a finite poset \( P \) is a noetherian topological space of dimension \( \text{dim}(P) = d \) the maximal length of a maximal chain \( x_0 < x_1 < \cdots < x_d \) in \( P \). By Grothendieck’s vanishing theorem for the cohomology of sheaves on noetherian spaces [5, Ch. 3.6] we have the following.

Theorem 3.7 For a finite poset \( P \) and for any sheaf \( \mathcal{F} \) of abelian groups (\( S \)-modules) on \( P \) we have \( H^i(P, \mathcal{F}) = 0 \) for all \( i > \text{dim}(P) \).
4 Sheaves on the intersection lattice

Recall that we set $L_0 := (L(A) \setminus T(A))^{op}$ so the order relation of $L_0$ is inclusion. The sheaf on $L_0$ associated to an arrangement considered by Yuzvinsky is defined as follows.

**Definition 4.1** ([22]) Denote by $\mathcal{D}$ the sheaf associated to the functor $L_0 \rightarrow \text{Mod}_S$, $X \mapsto \mathcal{D}(A_X)$ given by $\mathcal{D}(U_X) := D(A_X)$ for a principal open subset $U_X = \{Y \in L_0 \mid X \subseteq Y\} \subseteq L_0$. The restriction maps $\mathcal{D}(U_Y) \rightarrow \mathcal{D}(U_X)$ are given by the inclusions $D(A_Y) \subseteq D(A_X)$ for $Y \subseteq X$.

Note that, since $L(A)$ is a lattice, $L_0$ is a join-semilattice and

$$X \wedge_{L(A)} Y = X \vee_{L_0} Y.$$

From now on until the end of this note let $\mathcal{X} = \text{Spec} \ S \setminus \{m\}$ be the punctured spectrum.

Since we assume the arrangement $A$ to be essential, associated to $A$ is the following affine open cover of $\mathcal{X}$ which is in particular Leray for any coherent sheaf on $\mathcal{X}$ by a classical result due to Serre [15].

**Definition 4.2** Recall that $Q(X) = \prod_{H \in A \setminus A_X} \alpha_H = Q(A \setminus A_X)$. Define the open cover

$$U_A := \{U(X) := \mathcal{X} \setminus V(Q(X)) \mid X \in L_0\}$$

of $\mathcal{X}$ associated to $A$.

Note that $U(X) \cap U(Y) = U(X \vee Y)$ for all $X, Y \in L_0$, i.e. $U_A$ is closed under taking intersections. We obtain a poset $P_{U_A}$ with order relation given by reverse inclusion as discussed in Sect. 3.

We further have $U(X) \leq U(Y)$ if and only if $X \subseteq Y$, thus the map

$$L_0 \rightarrow P_{U_A}$$

$$X \mapsto U(X)$$

is a poset isomorphism. If $\mathcal{F}$ is a coherent sheaf on $\mathcal{X}$, from the general discussion in Sect. 3 and the poset isomorphism above, $\mathcal{F}$ also defines a sheaf on $L_0$.

**Definition 4.3** (i) Let $\mathcal{O}_X$ be the structure sheaf of the punctured affine space. This defines as discussed above a sheaf of $S$-modules on $L_0$ which we denote by $\mathcal{O}_{L_0}$ with

$$\mathcal{O}_{L_0}(U_X) = S_{Q(X)} \ (X \in L_0).$$

(ii) Let $\widetilde{D}|_{\mathcal{X}}$ be the coherent sheaf on $\mathcal{X}$ associated to the derivation module $D = D(A)$. This induces a sheaf of $S$-modules on $L_0$ denoted by $\widetilde{D}_{L_0}$ with

$$\widetilde{D}_{L_0}(U_X) = D_{Q(X)} \ (X \in L_0).$$
The tensor product of two sheaves $\mathcal{F}, \mathcal{G}$ of $S$-modules on a finite poset $P$ is given by

$$(\mathcal{F} \otimes_S \mathcal{G})(U) := \mathcal{F}(U) \otimes_S \mathcal{G}(U)$$

for all $U \subseteq P$ open with restriction maps the tensor product of the restriction maps of $\mathcal{F}$ and $\mathcal{G}$ (it suffices to define this for principal open subsets). A sheaf $\mathcal{F}$ of $S$-modules on $L_0$ is flat if $- \otimes_S \mathcal{F}$ yields an exact functor. This is the case if and only if $\mathcal{F}_X = \mathcal{F}(U_X)$ is a flat $S$-module for all $X \in L_0$.

Remark 4.4 As $\mathcal{O}_{L_0}(U_X) = S_{Q(X)}$ is a localization, it is flat for all $X \in L_0$. In particular, $\mathcal{O}_{L_0}$ is a flat sheaf and for the principal open cover $\mathcal{U}_{L_0}$ of $L_0$ all terms in the complex of sheaves $\mathcal{C}^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})$ are flat as they are finite direct products of the flat sheaves $i^*_X(\mathcal{O}_{L_0}|_{U_X})$ where $i : U_X \to L_0$ is the inclusion of the open subset $U_X$.

The next result gives the cohomology of the structure sheaf.

Proposition 4.5 With the notation as above we have

$$H^n(L_0, \mathcal{O}_{L_0}) \simeq H^n(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \begin{cases} S, & n = 0, \\ S_{x_1 \cdots x_\ell} / S, & n = \ell - 1, \\ 0, & \text{else.} \end{cases}$$

Proof The computation of the cohomology of $\mathcal{O}_{\mathcal{X}} = \widetilde{S}|_{\mathcal{X}}$ is a classical result or exercise, see [6, Thm. 3.8] and [9, p. 9].

The isomorphism $H^n(L_0, \mathcal{O}_{L_0}) \simeq H^n(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ follows with Lemma 3.6 applied to the open cover $\mathcal{U}_A$. □

5 Proof of Theorem 1.1

Now, all preparations are complete and we put the pieces together to prove Theorem 1.1.

Let $\mathcal{D}, \tilde{\mathcal{D}}_{L_0}$ and $\mathcal{O}_{L_0}$ be the sheaves of $S$-modules on $L_0$ defined in Sect. 4. First, we note the following.

Lemma 5.1 We have $\tilde{\mathcal{D}}_{L_0} = \mathcal{D} \otimes_S \mathcal{O}_{L_0}$.

Proof This follows immediately from Corollary 2.4 and the definition of the tensor product of sheaves on $L_0$. □

Proposition 5.2 Let $\mathcal{O}_{L_0}$ and $\mathcal{D}$ be as before. Then

$$\mathcal{C}^\bullet(\mathcal{U}_{L_0}, \mathcal{D}) \otimes_S \mathcal{C}^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})$$

is an acyclic resolution of $\mathcal{D} \otimes_S \mathcal{O}_{L_0}$ and for all $n \geq 0$ we have

$$H^n(L_0, \mathcal{D} \otimes \mathcal{O}_{L_0}) \simeq H^n(\mathcal{C}^\bullet(\mathcal{U}_{L_0}, \mathcal{D}) \otimes_S \mathcal{C}^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})).$$
Proof Since $\mathcal{U}_{L_0}$ is a finite cover, both of the complexes $C^\bullet(\mathcal{U}_{L_0}, \mathcal{D})$, $C^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})$ are bounded and so is the bicomplex of their tensor product. By Remark 4.4 all terms of $C^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})$ are flat and the exactness of the complex $C^\bullet(\mathcal{U}_{L_0}, \mathcal{D}) \otimes_S C^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})$ follows with Lemma 2.6.

It remains to show that all terms in the tensor product complex are acyclic sheaves.

All terms of $C^\bullet(\mathcal{U}_{L_0}, \mathcal{D}) \otimes_S C^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})$ are finite direct products of sheaves of the form

$$\mathcal{C}_{X,Y} := i_{X*}(\mathcal{D}|_{U_X}) \otimes_S i_{Y*}(\mathcal{O}_{L_0}|_{U_Y})$$

for $X, Y \in L_0$ and inclusion maps $i_X : U_X \rightarrow L_0$, $i_Y : U_Y \rightarrow L_0$. For $Z \in L_0$ we have

$$\mathcal{C}_{X,Y}(U_Z) = i_{X*}(\mathcal{D}|_{U_X})(U_Z) \otimes_S i_{Y*}(\mathcal{O}_{L_0}|_{U_Y})(U_Z)$$

$$= \mathcal{D}(U_{X\cap Z}) \otimes_S \mathcal{O}_{L_0}(U_{Y\cap Z})$$

$$= D(\mathcal{A}_X \cap L(A))Q(\mathcal{Y} \cap L(A))Z$$

$$= D(\mathcal{A}_X \cap L(A))\mathcal{Y} \cap L(A)Z$$

$$= D(\mathcal{A}_X)Q(\mathcal{Y} \cap L(A))Z,$$

where the last two equalities hold thanks to Lemma 2.3. Hence

$$\mathcal{C}_{X,Y} = i_{Y*}(D(\mathcal{A}_X)|_{U_Y})$$

where $D(\mathcal{A}_X)$ is the sheaf associated to the derivation module of the localization $\mathcal{A}_X$ as in Definition 4.3. As a direct image sheaf of an inclusion of a principal open subset it is acyclic by Lemma 3.5 and so are all the terms of $C^\bullet(\mathcal{U}_{L_0}, \mathcal{D}) \otimes_S C^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})$. Consequently, $C^\bullet(\mathcal{U}_{L_0}, \mathcal{D}) \otimes_S C^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})$ is an acyclic resolution of $\mathcal{D} \otimes_S \mathcal{O}_{L_0}$ as desired. Finally, note that by the definition of the tensor product of sheaves on $L_0$ for the global sections we have

$$\Gamma(L_0, C^\bullet(\mathcal{U}_{L_0}, \mathcal{D}) \otimes_S C^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})) = C^\bullet(\mathcal{U}_{L_0}, \mathcal{D}) \otimes_S C^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})$$

and so the cohomology of the tensor product of the two Čech complexes computes the cohomology of $\mathcal{D} \otimes_S \mathcal{O}_{L_0}$.  

It remains to verify the assumptions of the Künneth formula (Theorem 2.7 resp. Corollary 2.8) for the Čech complex of $\mathcal{O}_{L_0}$. This is done by the following proposition.

Proposition 5.3 All terms in $C^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})$, all terms of its coboundary-subcomplex and $H^p(C^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0}))$ ($p < \ell - 1$) are flat $S$-modules.

Proof First note that the complex $C^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})$ is bounded since $\mathcal{U}_{L_0}$ is a finite cover. By Remark 4.4 all $C^i(\mathcal{U}_{L_0}, \mathcal{O}_{L_0})$ are flat and note that since $\mathcal{U}_{L_0}$ is a Leray cover we have $H^p(L_0, \mathcal{O}_{L_0}) \simeq H^p(C^\bullet(\mathcal{U}_{L_0}, \mathcal{O}_{L_0}))$. 

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Recall that $\text{Tor}^j_j(A/B, M) = 0$ for all $j \geq 2$ and every $S$-module $M$ provided $A$ and $B$ are both flat $S$-modules. So by Proposition 4.5 we have

$$\text{Tor}^j_j(H^p(C^\bullet(U_{L_0}, \mathcal{O}_{L_0})), M) = 0$$

for each $j \geq 2$ and every $S$-module $M$. Consequently, the complex $C^\bullet(U_{L_0}, \mathcal{O}_{L_0})$ satisfies the assumptions of Lemma 2.9 and so all terms of the coboundary-subcomplex of $C^\bullet(U_{L_0}, \mathcal{O}_{L_0})$ are flat.

Finally, once more by Proposition 4.5 the modules $H^p(C^\bullet(U_{L_0}, \mathcal{O}_{L_0}))$ are flat $S$-modules for $p < \ell - 1$.

**Proposition 5.4** For $n \neq \ell - 1$ we have

$$H^n(L_0, \tilde{D}_{L_0}) \simeq \bigoplus_{i+j=n} H^i(L_0, \mathcal{D}) \otimes_S H^j(L_0, \mathcal{O}_{L_0})$$

and for $n = \ell - 1$ we have a short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=\ell-1} H^i(L_0, \mathcal{D}) \otimes_S H^j(L_0, \mathcal{O}_{L_0}) \longrightarrow H^{\ell-1}(L_0, \tilde{D}_{L_0})$$

$$\longrightarrow \text{Tor}^S_1(H^1(L_0, \mathcal{D}), H^{\ell-1}(L_0, \mathcal{O}_{L_0})) \longrightarrow 0.$$

**Proof** By Lemma 5.1, we readily get $H^n(L_0, \tilde{D}_{L_0}) \simeq H^n(L_0, \mathcal{D} \otimes_S \mathcal{O}_{L_0})$. Owing to Proposition 5.2 we have

$$H^n(L_0, \mathcal{D} \otimes_S \mathcal{O}_{L_0}) \simeq H^n(C^\bullet(U_{L_0}, \mathcal{D}) \otimes_S C^\bullet(U_{L_0}, \mathcal{O}_{L_0})).$$

By Proposition 5.3 the coboundary-subcomplex of $C^\bullet(U_{L_0}, \mathcal{O}_{L_0})$ is flat and by Proposition 4.5 $H^p(C^\bullet(U_{L_0}, \mathcal{O}_{L_0}))$ is flat for all $p < \ell - 1$. Further, as the dimension of $L_0$ is $\ell - 1$, by Theorem 3.7 we have

$$H^p(C^\bullet(U_{L_0}, \mathcal{D}) \otimes_S C^\bullet(U_{L_0}, \mathcal{O}_{L_0})) = H^p(L_0, \mathcal{D} \otimes_S \mathcal{O}_{L_0}) = 0$$

for $p > \ell - 1$. Moreover, $H^0(C^\bullet(U_{L_0}, \mathcal{D})) \simeq H^0(L_0, \mathcal{D}) \simeq D(A)$ is a reflexive $S$-module (cf. [13, p. 268]) and as such it is torsion free in particular. Since $H^{\ell-1}(C^\bullet(U_{L_0}, \mathcal{O}_{L_0})) \simeq H^{\ell-1}(L_0, \mathcal{O}_{L_0}) \simeq S_{x_1 \ldots x_\ell}/S$ by Proposition 4.5 we have

$$\text{Tor}^S_1(H^0(C^\bullet(U_{L_0}, \mathcal{D})), H^{\ell-1}(C^\bullet(U_{L_0}, \mathcal{O}_{L_0}))) = 0$$

by Lemma 2.10. Hence, we can applying Corollary 2.8 to the two complexes $C^\bullet(U_{L_0}, \mathcal{D})$ and $C^\bullet(U_{L_0}, \mathcal{O}_{L_0})$ which concludes the proof.

By Lemma 3.6 we have isomorphisms $H^n(L_0, \tilde{D}_{L_0}) \simeq H^n(\tilde{X}, \tilde{D}|_{\tilde{X}})$ and $H^n(L_0, \mathcal{O}_{L_0}) \simeq H^n(\tilde{X}, \mathcal{O}_{\tilde{X}})$ for all $n \geq 0$. Recall that we have $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \simeq S$ by Proposition 4.5. This yields our main theorem.
Corollary 5.5 (Theorem 1.1) For all \( n \neq \ell - 1 \) we have:

\[
H^n(\mathcal{X}, \tilde{D}|\mathcal{X}) \cong \bigoplus_{i+j=n} H^i(L_0, \mathcal{D}) \otimes S H^j(\mathcal{X}, \mathcal{O}_\mathcal{X})
\]

and for \( n = \ell - 1 \) we have a short exact sequence

\[
0 \rightarrow \bigoplus_{i+j=\ell-1} H^i(L_0, \mathcal{D}) \otimes S H^j(\mathcal{X}, \mathcal{O}_\mathcal{X}) \rightarrow H^{\ell-1}(\mathcal{X}, \tilde{D}|\mathcal{X}) \rightarrow \text{Tor}^1_1(H^1(L_0, \mathcal{D}), H^{\ell-1}(\mathcal{X}, \mathcal{O}_\mathcal{X})) \rightarrow 0.
\]

In particular, \( H^n(\mathcal{X}, \tilde{D}|\mathcal{X}) \cong H^n(L_0, \mathcal{D}) \) for \( n < \ell - 1 \).

6 Concluding remarks

In view of Corollary 1.6, we may reformulate Terao’s conjecture as follows:

Conjecture 6.1 Let \( \mathcal{A}, \mathcal{B} \) be two arrangements in \( \mathbb{K}^\ell \) with \( L(\mathcal{A}) \cong L(\mathcal{B}) \) and denote their corresponding sheaves of logarithmic vector fields on \( L_0 \) by \( \mathcal{D}_A \) respectively \( \mathcal{D}_B \). Then \( H^n(L_0, \mathcal{D}_A) = 0 \) for \( 0 < n < \ell - 1 \) if and only if \( H^n(L_0, \mathcal{D}_B) = 0 \) for \( 0 < n < \ell - 1 \).

In his recent work [2], Abe studies the behavior of the projective dimension under addition-deletion operations. He also states the problem [2, Prob. 6.3(3)] whether indeed even the projective dimension of the derivation module is combinatorial. This generalization of Terao’s conjecture can be reformulated with Theorem 1.5 as follows.

Problem 6.2 Let \( \mathcal{A}, \mathcal{B} \) be two arrangements in \( \mathbb{K}^\ell \) with \( L(\mathcal{A}) \cong L(\mathcal{B}) \) and denote their corresponding sheaves of logarithmic vector fields on \( L_0 \) by \( \mathcal{D}_A \) respectively \( \mathcal{D}_B \). Then does \( H^n(L_0, \mathcal{D}_A) = 0 \) if only if \( H^n(L_0, \mathcal{D}_B) = 0 \) (for all \( n \)) hold?

Note that Lemma 2.3 easily generalizes to the case of multi-arrangements, i.e. arrangements equipped with a multiplicity function \( m : \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0} \) which were introduced by Ziegler [25]. Moreover, the other essential properties of \( D(\mathcal{A}) \) used in the proof of Theorem 1.1 hold more generally for the module of multi-derivations \( D(\mathcal{A}, m) \). Hence, all the main results of Sect. 1 extend to multi-arrangements essentially with the same proofs.

In view of the important results about freeness preserved under various addition, deletion and restriction operations, cf. [1, 17, 20, 25], thinking of the long exact sequence in cohomology obtained from a short exact sequence of sheaves, we note the following natural problem.

Problem 6.3 Are there short exact sequences relating the sheaves \( \mathcal{D}, \mathcal{D}', \mathcal{D}'', \mathcal{D}^H, \mathcal{D}'^H, \mathcal{D}''^H, \mathcal{D}^H, \mathcal{D}'^H, \mathcal{D}''^H \) where \( \mathcal{D}', \mathcal{D}'' \) are the sheaves of a deletion, restriction or Ziegler-restriction of the given arrangement, respectively?
Finally, we suspect that Yuzvinsky’s celebrated theorem [24, Thm. 3.3] stating that the subset formed by free arrangements in the moduli space of all arrangements with a given intersection lattice is Zariski open could be generalized with our results: arrangements with projective dimension greater or equal to $p$ constitute a Zariski closed subset in the moduli space of all arrangements with a given intersection lattice, or equivalently, arrangements with projective dimension less or equal to $p$ form a Zariski open subset.

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Declarations

Conflict of interest The author states that there is no conflict of interest.

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