I. INTRODUCTION

The hunt for the first detection of continuous gravitational waves (CW) is under way with many searches published[1–7, 9, 10, 12–23, 26, 27, 29–32, 40] or in progress. Some searches target known sources, such as the Crab Pulsar, but other all-sky searches look for unknown sources over a broad frequency band. These searches require substantial computational resources, so any reduction in computational demands is helpful. In this paper, we examine one aspect of the searches amenable to simplification: calculation of signal time delays received by a gravitational wave detector with respect to the solar system barycenter. A highly non-linear conventional computation is transformed into one that has a pure linear sum in its innermost loop. We discuss application of these results to determination of the maximal useful coherence length of continuous wave searches.

II. MATHEMATICAL MODEL

The emission time \( T \) is a function of detector local time \( t \), source location \( u \) and intrinsic source parameters \( p \) (for a source in motion). Because modern computer architectures are vector-based, it is typically more efficient to compute arrays of values of \( T(t, u, p) \) for sets of times \( \mathcal{T} = \{ t_i \} \) and templates \( \mathcal{S} = \{ (u_j, p_j) \} \).

For a single template \((u_0, p_0)\) the function \( T(t, u_0, p_0) \) has a very non-trivial behaviour due to several nearly periodic influences from the Sun, planets, and the Moon as well as contributions from General Relativity.

Because any analysis method must overlap templates \((u, p)\) closely enough to provide sufficient detection coverage, we can expect to compute arrays \( T(\mathcal{T}, u, p) \) for nearby \((u, p)\).

Therefore, we separate the problem into two parts:
• Computation of $T(\mathcal{T}, u_0, p_0)$ for a fixed template $(u_0, p_0)$.

• Computation of differences $\Delta(\mathcal{T}, u, p; u_0, p_0) = T(\mathcal{T}, u, p) - T(\mathcal{T}, u_0, p_0)$

When sets $\mathcal{T}$ and $\mathcal{S}$ are finite we know that there exists the following decomposition:

$$\Delta(t_i, u_j, p_j) = \sum_{k=1}^{N} f_k(t_i) g_k(u_j, p_j)$$  \hspace{1cm} (1)

where $f_k(t_i)$ and $g_k(u_j, p_j)$ are, in general, arbitrary single-valued functions.

The key to our approach is that it is possible to find an approximately version of Equation 1 with a number of terms $N$ much smaller than the dimensionality of space spanned by $\Delta(t_i, u_j, p_j)$.

Besides providing computational efficiency this analysis identifies analytical functions $f_k$ and $g_k$, paving the way for developing advanced Loosely Coherent [33, 34] semi-analytic statistics.

III. PRACTICAL IMPLEMENTATION

We describe sky mismatch using small rotations along right ascension and small offsets in declination. The latter is not a rotation but a flow from one equatorial pole to another, symmetrical with respect to Earth rotation. A small region near the source and sink is excised from the grids.

1. Pick a set of signal arrival times $\mathcal{T}$.

2. Construct a coarse sky grid $\mathcal{G}_t$ with minimum point separation of $\epsilon$ in spherical distance. Add to these grid of points in a neighborhood $B_{\odot}$ around the Sun’s position at each time $t \in \mathcal{T}$.

3. For every time $t \in \mathcal{T}$ and every point in the coarse sky grid $\mathcal{G}_t$, compute the emission times $t_{e} \in T(t, G_t)$, where $T$ is a function defined in LALBarycenter, returning a vector of emission times corresponding to each arrival time and source location.

4. Introduce a displacement grid $\Delta G$ of small sky rotations.

5. Compute emission times $T(t, G_{t,r})$ for each grid $G_{t,r}$ displaced by rotation $r$ in $\Delta G$.

6. Compute the difference $\Delta(t, G_t, r) \equiv T(t, G_{t,r}) - T(t, G_{t,0})$ in emission times for each rotated and unrotated point at each time.

7. Define a function $\Delta(t, G_t, r) \equiv \sum_k a_k x_k$ with a set of coefficients $\{a_k\}$ for parameters $\{x_k\}$, and use least-squares fitting to compute $\{a_k\}$. Ideally, we would want to find $\{a_k\}$ such that $\max(|\Delta(t, G_t, r) - \Delta(t, G_{t,0})|)$ is minimized, but the computational costs of such a search are too high.

IV. EXAMPLE

The parameters used in this study are listed in Table I.

We use terms falling into several categories

- Direction-independent terms depending on GPS time and shift in sky position
- Direction differential-independent terms depending on source sky position and GPS time
- Time-independent terms depending on source sky position and shift in position

A. Definitions of Variables

The sky position variables are defined as

$$e_1 = \cos(\delta) \cos(\alpha)$$
$$e_2 = \cos(\delta) \sin(\alpha)$$
$$e_3 = \sin(\delta)$$

with $\alpha \in [-\pi, \pi]$ and $\delta \in [-\frac{\pi}{2} + 0.01, \frac{\pi}{2} - 0.01]$. The adjustment by 0.01 radians prevents flow over the poles, which would lead to ambiguous right ascension. The change in $e_i$ for a shift in right ascension $\Delta \alpha$, and in declination $\Delta \delta$ can be approximated via Taylor expansion:

$$\Delta e_1 = (-\frac{1}{2} \Delta \alpha^2 \cos \alpha \cos \delta - \frac{1}{2} \Delta \delta^2 \cos \alpha \cos \delta$$
$$+ \frac{1}{2} \Delta \alpha \Delta \delta^2 \cos \alpha \sin \delta - \Delta \alpha \cos \delta \sin \alpha$$
$$+ \frac{1}{2} \Delta \alpha \Delta \delta^2 \cos \sin \alpha - \Delta \delta \cos \alpha \sin \delta)$$

$$\Delta e_2 = (\Delta \alpha \cos \alpha \cos \delta - \frac{1}{2} \Delta \alpha \Delta \delta^2 \cos \alpha \cos \delta$$
$$- \frac{1}{2} \Delta \alpha^2 \Delta \delta^2 \cos \sin \alpha - \frac{1}{2} \Delta \delta^2 \cos \sin \alpha$$
$$+ \frac{1}{2} \Delta \alpha^2 \Delta \delta^2 \cos \sin \alpha - \Delta \alpha \Delta \delta \cos \alpha \sin \delta$$

$$\Delta e_3 = (\Delta \delta \cos \sin \alpha + \frac{1}{2} \Delta \alpha \Delta \delta \sin \alpha)$$

The LALBarycenter program provides vectors with information on the state of the Sun and Earth that are useful:

$$\mathbf{S}$$ Vector pointing from Sun to Earth
$$\mathbf{v}$$ Detector velocity vector
$$\Delta t$$ Time since reference point
$$\Omega_{\odot}$$ $2\pi$/sidereal day.

We also define an array of the sin/cos of the reference point’s right ascension and declination:

$$z = \{\sin \alpha, \sin \delta, \cos \alpha, \cos \delta\}$$

and the second-order terms, excepting $\sin^2$ terms because they can be expressed as $1 - \cos^2$:

$$z' = \{\cos^2 \alpha, \cos^2 \delta, \sin \alpha \sin \delta, \sin \alpha \cos \alpha, \sin \alpha \cos \delta, \sin \delta \cos \alpha, \sin \delta \cos \delta, \cos \alpha \cos \delta\}$$
FIG. 1. Example of emission time variation with sky position. The middle panel shows differences between computed emission time and a reference time. The top panel shows differences in emission times between points offset in declination by 0.01 rad, while the bottom panel shows differences in emission times between points offset in right ascension by 0.01 rad. Note that the color scales for difference plots are greatly reduced.

B. Direction-independent terms

The following terms are constant in sky-direction, and can be precomputed for every GPS time, and direction difference.

\[ \sum_{i} a_{1,i} \Delta e_i \]
TABLE I. Parameters used in fits

| Parameter | Value |
|-----------|-------|
| $T$       | Every hour between $t_{\text{min}}$ and $t_{\text{max}}$ |
| $t_{\text{min}}$ | From start to end of O1, spaced every 200,000 seconds |
| $t_{\text{max}}$ | $t_{\text{min}} + 250,000$ sec |
| $\epsilon$ | 0.1040524 rad |
| $\Delta G$ | All combinations of $\Delta \alpha$ and $\Delta \delta$ |
| $G_{t,r}$ | Random subset of $G_t$ with $7.5 \times 10^5$ points |
| $\Delta \alpha$ | $\{-0.01, -0.00667, -0.00333, 0, 0.00333, 0.00667, 0.01\}$ |
| $\Delta \delta$ | $\{-0.01, -0.00667, -0.00333, 0, 0.00333, 0.00667, 0.01\}$ |
| $\epsilon_\odot$ | 0.001 rad |
| $N_\odot$ | 5 |
| $S(t)$ | Sun position at time $t$ |
| $B_\odot$ | A grid of $N_\odot \times N_\odot$ points centered on $S(t)$, evenly spaced in $\alpha$ and $\delta$ with step $\epsilon_\odot$ |

C. Difference-independent terms

The following terms are constant in direction-difference.

$$\sum_i a_{2,i} \sin(\Omega \odot \Delta t) z'_i + b_{2,i} \sin(\Omega \odot \Delta t) z'_i$$  \hspace{1cm}(8)

$$a_{3,1} \Delta t \cos \delta + a_{3,2} \Delta t^2 \cos \delta + a_{3,3} \Delta t \cos^2 \delta$$  \hspace{1cm}(9)

$$\sum_i a_{4,i} \Delta t^2 z'_i$$  \hspace{1cm}(10)

$$\sum_i a_{5,i} S_i e_i$$  \hspace{1cm}(11)

D. Time-independent terms

The following terms vary only in sky-direction.

$$\sum_i a_{6,i} z_i$$  \hspace{1cm}(12)

$$\sum_i a_{7,i} z'_i$$  \hspace{1cm}(13)

Each of the terms in equations 7-13 is multiplied by $\Delta \alpha$, $\Delta \delta$, $\Delta \alpha^2$, $\Delta \delta^2$, $\Delta \alpha \Delta \delta$. In addition, we include direction-time differential terms

$$\sum_i a_{8,i} \Delta t \Delta e_i$$  \hspace{1cm}(14)

without $\Delta \alpha / \Delta \delta$ factors. Each term goes to zero when the rotation angle goes to zero. Note that Sun-Earth and detector velocity vectors are those for the saved points. In each term, any parts greater than order 3 in $\Delta \alpha$ and $\Delta \delta$ are removed. The effects of removing sets of terms are shown in Table II.

An example fit is included in the appendix. This fit has the largest maximum error among the fits, $2.03 \times 10^{-5}$ s. The fit expression is a bilinear product of precomputed fit coefficients and monomials in $\Delta \alpha$, $\Delta \delta$ and $\Delta t$. In a practical implementation the grid of displacements, and thus monomial coefficients, is kept static inside the loop that computes $\Delta t$. The actual computation of $\Delta t$ easily vectorizes and takes few instructions on modern computers. Note that it is not necessary to keep the grid static with respect to all variables. For example, the grid can be static in $\Delta \delta$ and depend on $t$ and $\alpha$ — the monomial grid recompute cost will be amortized away.

V. RESULTS

As a maximum acceptable error, we chose a 30-degree phase difference for a 2 kHz signal, or 42 $\mu$s. The fit terms given above achieved this goal in the fitting set, but we wished to test situations similar to those in which the model would be used. We chose $16 \times 8 = 128$ points on the sky, evenly spaced in right ascension and declination, and

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
Term Group & Equation & Max Fit Error (s) \\
\hline
2nd Order Sinusoids & 13 & 3.4319967642 \\
1st Order Sinusoids & 12 & 0.4335595581 \\
$\Delta t$ & 9 & 0.0235629364 \\
$\Delta t^2$ & 10 & 0.0010017764 \\
Sun Direction & 11 & 0.0002486640 \\
Sidereal Rotation & 8 & 0.0001820357 \\
Direction-difference & 7 & 0.0001655650 \\
\hline
\end{tabular}
\end{table}
to serve as patch centers. For each patch, we shifted the central point by a random value in $[-0.01, 0.01]$ for right ascension, and another for declination. A total of 50 shifted points were generated for each patch.

We divided the span of the first Advanced LIGO data run (∼4 months), O1, into 200,000 second chunks, and took time points from each chunk at 30-minute intervals. We used LALBarycenter to obtain reference values for all points, then applied the fit model to each patch’s points as a deflection from its center.

A plot of the maximum absolute residual for each $\Delta t$ and $\Delta \phi$ is shown in Figure 2. The maximum absolute residual for each reference time is shown in Figure 3. All points fell below the error threshold. We also show a histogram of all errors in Figure 4. The bulk of the errors are well below the threshold, and for a search of this length, any particular point would spend only a small fraction of time in a high-error region.

VI. CONCLUSIONS

The *Loosely Coherent* method of detecting signals analyzes sets of potential templates. For the set based on nearby sky locations it is important to understand the evolution of signal phases for nearby templates. The fit described in this paper explicitly demonstrates that relatively few parameters are needed to describe time arrival differences between nearby templates.

It is well-known that mathematically optimal detection statistic consists of a linear filter followed by a power detector [41]. The linear filter is chosen to match expected signal properties and to reject noise outside of signal bandwidth. As the example shows, the sky position mismatch is equivalent to frequency modulation of the incoming signal. Thus for any search where sky position uncertainty requires multiple templates, the fully coherent search is not the most computationally efficient [33] and it is best to compute the total power of the modulated signal.

For example, if such a search uses one year’s worth of data from a single interferometer, the maximal sensitivity is reached at 6 months coherence length, or even earlier if parameters other than sky position are uncertain. For a search using many interferometers a fully coherent search can be more sensitive, but the gain in sensitivity is smaller than predicted from the increase of coherence time alone.

This development provides an efficient method to compute emission time corrections, provides a basis for extension of the PowerFlux cache to longer coherence times and lays the groundwork for future development of *Loosely Coherent* algorithms.

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Appendix: Example Fit

As an example, we list below the resulting formula from a fit for GPS time 1127833121. The expression is a bilinear product between precomputed fit coefficients and monomials in $\Delta \alpha$, $\Delta \delta$ and $\Delta t$. Only significant terms are shown. This fit has the largest maximum error among the fits, $2.03 \times 10^{-5}$ s.

$$\Delta T = \left[-1.69 \times 10^{-5} \Delta t \Delta e_1 + [8.98 \times 10^{-5} \Delta t \Delta e_2 + [-495 e_2 + 71.2 e_1] \Delta \alpha + [30.8 \cos(\delta) - 71.2 e_3 \sin(\alpha) - 495 e_3 \cos(\alpha)] \Delta \delta + [3.89 \cos(\delta)] \times 10^{-5} \Delta \delta \Delta t + [8.14 e_2 + 83 e_1] \Delta \alpha^2 + [0.0132 e_2 - 0.00634 e_1] \sin(\Omega_{02} \Delta t) \Delta \alpha + [0.0064 e_2 + 0.0132 e_1] \cos(\Omega_{02} \Delta t) \Delta \alpha + [-35.6 e_2 - 2.5 e_3] \delta^2 + [0.00634 e_1 \sin(\alpha) + 0.0132 e_3 \cos(\alpha)] \sin(\Omega_{02} \Delta t) \Delta \delta + [0.0972 e_2 - 0.0157 e_1] \times 10^{-10} \Delta \alpha \Delta t^2 + [-0.0132 e_3 \sin(\alpha) + 0.0064 e_3 \cos(\alpha)] \cos(\Omega_{02} \Delta t) \Delta \delta + [-0.00634 \cos(\delta) + 0.0157 e_3 \sin(\alpha) + 0.0972 e_3 \cos(\alpha)] \times 10^{-10} \Delta \delta \Delta t^2 + [4.37 \Delta \alpha \Delta e_1 + [-331] \Delta \alpha \Delta e_2 + [83.2] \Delta \alpha^2 \Delta e_1 + [-82.2] \Delta \delta^2 \Delta e_1 + [165 e_3 \sin(\alpha) - 27.4 e_3 \cos(\alpha)] \Delta \alpha \Delta \delta + [-0.0972 e_3 \sin(\alpha)] \times 10^{-10} \Delta \alpha \Delta \delta \Delta t^2ight]$$
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