1. Introduction

In this paper we prove a general result establishing \textit{a priori} $L^p$ estimates for solutions of Riemann-Hilbert Problems (RHP’s) in terms of auxiliary information involving an associated “conjugate” problem (see Conjugation Lemma 1.39 below). We then use the result to obtain uniform estimates for a RHP (see Theorem 1.48) that plays a crucial role in analyzing the long-time behavior of solutions of the perturbed nonlinear Schrödinger equation on the line. Theorem 1.48 is proved by combining Conjugation Lemma 1.39 with the steepest-descent method for RHP’s introduced by the authors in [DZ1]. We do not apply the steepest-descent method directly to Theorem 1.48. Rather, as explained in the text, we proceed by rephrasing Theorem 1.48 as an equivalent inhomogeneous RHP in which the underlying objects $M_\pm$ (see Theorem 1.52) have appropriate analyticity properties and can be deformed around the stationary phase point much as in the manner of the classical method of stationary-phase/steepest-descent.

We begin by introducing a variety of definitions and results that arise in the theory of Riemann-Hilbert Problems (RHP’s).

Let $\Sigma$ be an oriented contour in $\mathbb{C}$ and consider the associated Cauchy operator

$$Ch(z) = C_\Sigma h(z) \equiv \int_\Sigma \frac{h(s)}{s-z} \frac{ds}{2\pi i}, \quad z \in \mathbb{C} \setminus \Sigma.$$ 

If we move along the contour in the direction of the orientation, we say, by convention, that the (+)-side (resp. (−)-side) lies to the left (resp. right). The following properties and estimates which will be used without further comment throughout the text are true for a very general class of contours (see, for example, [DS], [Dur]). This class certainly includes contours that are finite unions of smooth curves in $\mathbb{C}$ such that $\mathbb{C} \setminus \Sigma$ has a finite number of components as in Figures 1.35, 1.38 and 3.35 below.

1.1 Let $h \in L^p(\Sigma, |dz|), 1 \leq p < \infty$. Then

$$C^\pm h(z) \equiv \lim_{z' \to z \atop z' \in (\pm)\text{-side of } \Sigma} (Ch)(z')$$

exists as a non-tangential limit for a.e. $z \in \Sigma$.

1.2 Let $h \in L^p(\Sigma, |dz|), 1 < p < \infty$. Then

$$\|C^\pm h\|_{L^p(\Sigma)} \leq c_p \|h\|_{L^p(\Sigma)}$$

for some constant $c_p$.

1.3 $C^\pm = \pm 1/2 - H/2$, where $H$ is the Hilbert transform

$$Hf(z) = \text{P.V.} \int_{\Sigma} \frac{f(s)}{z-s} \frac{ds}{i\pi}.$$ 

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By (1.2), for \( h \in L^p(\Sigma, |dz|) \), \( 1 < p < \infty \),

\[
\|Hh\|_{L^p(\Sigma)} \leq c_p \|h\|_{L^p(\Sigma)}
\]

for some constant \( c_p \).

(1.4) \( C^+ - C^- = 1 \), \( 1 \leq p < \infty \).

Let \( v \) be a \( k \times k \) jump matrix on \( \Sigma \) i.e. \( v \) is a measurable map from \( \Sigma \to GL(k, \mathbb{C}) \) with \( v, v^{-1} \in L^\infty(\Sigma \to GL(k, \mathbb{C})) \). Define the associated singular integral operator

\[
(1.5) \quad C_v h = C^- (h(v - I))
\]

acting on \( L^p(\Sigma) \)-matrix-valued functions. Clearly \( C_v \) is bounded from \( L^p \to L^p \) for all \( 1 < p < \infty \).

Given \( \Sigma \) and \( v \), the operator \( C_v \) is intimately connected with the solution of associated RHP’s on \( \Sigma \). For \( 1 < p < \infty \), we say that a pair of \( L^p(\Sigma) \)-functions \( f^\pm \in \partial C(\mathcal{L}^p) \) if there exists a (unique) function \( h \in L^p(\Sigma) \) such that

\[
(1.6) \quad f^\pm(z) = (C^\pm h)(z), \quad z \in \Sigma.
\]

In turn we call \( f(z) = Ch(z), z \in \mathbb{C} \setminus \Sigma \), the extension of \( f^\pm = C^\pm h \in \partial C(\mathcal{L}^p) \) off \( \Sigma \). Observe that if \( F(z) = (Cf)(z) \) for \( f \in L^p(\Sigma) \) and \( G(z) = (Cg)(z) \) for \( g \in L^q(\Sigma) \), \( 1 < p, q < \infty \), then a simple computation shows that if \( 1/r = 1/p + 1/q \leq 1 \),

\[
FG(z) = (Ch)(z),
\]

where

\[
(1.7) \quad h = -\frac{1}{2}((Hf)g + fHg) \in L^r(\Sigma).
\]

If \( F = Cf(z) \) for \( f \in L^p(\Sigma) \), \( 1 < p < \infty \), and \( G \) is analytic and bounded in \( \mathbb{C} \setminus \Sigma \), then

\[
(1.8) \quad FG(z) = Ch(z),
\]

where

\[
(1.9) \quad h = (C^+ f)G_+ - (C^- f)G_- \in L^p(\Sigma)
\]

and \( G_\pm \) denote the non-tangential limits (see [Dur]) of \( G \) on \( \Sigma \).

**Definition 1.10.** Inhomogeneous Riemann–Hilbert Problem IRHP\(_{L^p}\) (see [CG], [DZ5], [DZ7]). Fix \( 1 < p < \infty \). Given \( \Sigma, v \) and a function \( F \in L^p(\Sigma) \), we say that \( M^\pm \in \partial C(\mathcal{L}^p) \) solves an IRHP\(_{L^p}\) if

\[
(1.11) \quad M^+(z) = M^-(z)v(z) + F(z), \quad z \in \Sigma.
\]

If \( M(z) \) is the extension of \( M^\pm \) off \( \Sigma \), we see that

- \( M(z) \) is analytic in \( \mathbb{C} \setminus \Sigma \),
- \( M^+(z) = M^-(z)v(z) + F(z), z \in \Sigma \), where \( M^\pm(z) = \lim_{z' \to z} M(z') \),
- \( M(z) \to 0 \) as \( z \to \infty \) in any non-tangential direction.

**Remark.**

In [DZ4] and [DZ5], IRHP\(_{L^p}\) is referred to as an inhomogeneous RHP of type 2, IRHP\(_{2,L^p}\). There is also the notion of an inhomogeneous RHP of type 1, IRHP\(_{1,L^p}\). As we only need type 2 in this paper, we simply drop the “2”.

The following relations are basic.
**Proposition 1.12.**

Let $1 < p < \infty$. Then $1 - C_v$ is a bijection in $L^p(\Sigma)$ if and only if IRHP$_{L^p}$ has a unique solution for all $F \in L^p(\Sigma)$. Moreover if $(1 - C_v)^{-1}$ exists, then for $h \in L^p(\Sigma)$

\[(1 - C_v)^{-1}h = (M_+ + h)v^{-1} = (M_- + h)\]

where $M_\pm$ solves IRHP$_{L^p}$ with $F = h(v - I)$. Conversely, if $M_\pm$ solves IRHP$_{L^p}$ with $F \in L^p(\Sigma)$, then

\[(1.14) \quad M_+ = ((1 - C_v)^{-1}(C^*F))v + F \quad \text{and} \quad M_- = (1 - C_v)^{-1}C^*F.\]

Suppose $v = (v^-)^{-1}v^+$ is a factorization of $v$ with $v^\pm, (v^\pm)^{-1} \in L^\infty(\Sigma, GL(k, C))$. Set $w^+ \equiv v^+ - I$, $w^- \equiv I - v^-$ and let $w = (w^-, w^+)$. Define the associated singular integral operator (cf. (1.5))

\[(1.15) \quad C_wh = C^+(hw^-) + C^-(hw^+)\]

acting on $L^p(\Sigma)$-matrix-valued function $h$. Note that for the trivial factorization $v = (I)^{-1}v$, $w = (0, v - I)$, we have $C_w = C_v$.

The following result describes the relation between $1 - C_w$ and $1 - C_{w'}$ for two different factorizations $v = (v^-)^{-1}v^+ = (v^+)^{-1}v^+$ of $v$.

**Proposition 1.16 (see [Z1],[DZ4], [DZ5]).**

Suppose $1 < p < \infty$. The operator $1 - C_w$ is bijective in $L^p(\Sigma)$ for all factorization $v = (v^-)^{-1}v^+ = (I - w^-)^{-1}(I + w^+)$, if and only if $1 - C_{w'}$ is bijective for at least one factorization $v = (v'^-)^{-1}v'^+ = (I - w'^-)^{-1}(I + w'^+)$.

Moreover, for $f \in L^p(\Sigma)$

\[(1 - C_w)^{-1}f = ((1 - C_{w'})^{-1}f)b\]

where $b = v'^+(v'^-)^{-1} = v^-(v^-)^{-1}$.

**Duality.** A non–degenerate bilinear pairing for vector functions in $L^p(\Sigma)$ and $L^q(\Sigma)$, $1 < p, q < \infty$, $1/p + 1/q = 1$, is given by

\[(1.17) \quad (f, g) = \int_{\Sigma} \text{tr}(f(z)g(z)\bar{z})dz.\]

Using (1.3), we see that with respect to this pairing, the dual operators $(C^\pm)'$ are given by

\[(C^\pm)' = -C^{\mp}\]

and

\[(1.19) \quad C_{v'}'h = (C^+h)(I - v^T) = R_{I-v^T}C^+h,\]

where $R_A$ denotes multiplication on the right by a matrix $A$. Now $1 - C_v$ is bijective in $L^p(\Sigma)$ if and only if $1 - C_v'$ is bijective in $L^q(\Sigma)$, $1/p + 1/q = 1$, $1 < p, q < \infty$, and

\[(1.20) \quad \| (1 - C_v')^{-1} \|_{L^q \mapsto L^q} = \| (1 - C_v)^{-1} \|_{L^p \mapsto L^p}.\]

On the other hand, it is well-known that if $K$ and $L$ are bounded linear operators in a Banach space, then (see e.g. [D] and the references therein) $1 - KL$ is invertible if and only if $1 - K$ is invertible and $(1 - KL)^{-1} = 1 + K(1 - LK)^{-1}L$. Setting $K = C^+$ and $L = R_{I-v^T}$, we find that $1 - C^+ R_{I-v^T}$ is a bijection in $L^q(\Sigma)$ and

\[(1.21) \quad (1 - C^+ R_{I-v^T})^{-1} = 1 + C^+(1 - C_v')^{-1}R_{I-v^T}\]
Observe that \((C^+R_{I-vT})h = C^+h(I - v^T)\), which is precisely an operator of the form \(Cw\) in (1.15) above corresponding to the jump matrix \((v^T)^{-1} = (v^-)^{-1}v^+\) with \(w = (w^-, w^+) = (I - v^T, 0)\).

We need the following simple facts. If \(|\cdot|\) denotes any sub-multiplicative norm on an algebra, \(|AB| \leq |A||B|\), then \(|Id| \geq 1\) and if \(A\) is invertible,

\[
|A||A^{-1}| \geq 1
\]

and hence

\[
\langle v \rangle_\infty \equiv \max(\|v\|_\infty, \|v^{-1}\|_\infty) \geq 1.
\]

If \(1 - C_v\) is invertible in \(L^p(\Sigma)\), then a simple calculation using the identity \((1 - C_v)^{-1}(1 - C_v) = 1\), together with (1.23), shows that

\[
0 < c \leq \|(1 - C_v)^{-1}\|_{L^p(\Sigma)\rightarrow L^p(\Sigma)}\langle v \rangle_\infty
\]

for some constant \(c = c_p\) independent of \(v\).

Throughout the text constants \(c\) will be used generically. Statements such as \(\|f\| \leq 2c(1 + e^c) \leq c\), for example, should not cause any confusion.

**Normalized RHP.** Let \(\Sigma\) be an oriented contour with associated jump matrix \(v\). Suppose \(v - I \in L^p(\Sigma)\) for some \(1 < p < \infty\). We say that \(\phi_{\pm}\) solves the normalized RHP \((\Sigma, v)_{L^p}\) if \(\phi_{\pm} - I \in \partial C(L^p)\) solves the IRHP\(_{L^p}\)

\[
\phi_{+} - I = (\phi_{-} - I)v + (v - I) \quad \text{in} \quad \Sigma
\]

or equivalently

\[
\phi_{+} = \phi_{-}v \quad \text{on} \quad \Sigma.
\]

If \(\phi(z) - I\) is the extension of \(\phi_{\pm} - I\) off \(\Sigma\), we see that

- \(\phi(z)\) analytic in \(\mathbb{C} \setminus \Sigma\),
- \(\phi_{+}(z) = \phi_{-}(z)v(z), z \in \Sigma\), where \(\phi_{\pm}(z) = \lim_{z' \to z} \text{side of} \ \Sigma \phi(z')\),
- \(\phi(z) \to I\) as \(z \to \infty\) in any non-tangential direction.

If \(1 - C_v\) is invertible in \(L^p\), then, by Proposition 1.12, the normalized solution exists, is unique, and is given by

\[
\phi_{-} = I + (1 - C_v)^{-1}C^- (v - I)
\]

and hence \(\|\phi_{-} - I\|_{L^p} \leq c\|(1 - C_v)^{-1}\|_{L^p\rightarrow L^p}\|v - I\|_{L^p}\). Using the relation (1.26) together with (1.23,1.24), we then obtain the inequalities

\[
\|\phi_{\pm} - I\|_{L^p} \leq c\|(1 - C_v)^{-1}\|_{L^p\rightarrow L^p}\langle v \rangle_\infty\|v - I\|_{L^p}.
\]

As noted above, the operator \(1 - C^+R_{I-vT}\) associated with the jump matrix \((v^T)^{-1}\) is invertible in \(L^q(\Sigma)\). Let \(\psi_{\pm}\) solve the normalized RHP \((\Sigma, v_{L^q})\), \(\psi_{+} = \psi_{-}(v^T)^{-1}, \psi_{\pm} - I \in \partial C(L^q)\). As in (1.27), we have \(\psi_{\pm} = I + (1 - C^+R_{I-vT})^{-1}C^+(I - v^T)\), which leads as above to the bound

\[
\|\psi_{+} - I\|_{L^q} \leq c\|(1 - C^+R_{I-vT})^{-1}\|_{L^q\rightarrow L^q}\|v - I\|_{L^q}
\]

\[
\leq c\|(1 - C_v)^{-1}\|_{L^p\rightarrow L^q(\langle v \rangle_\infty)}\|v - I\|_{L^q}, \quad \text{by (1.21)},
\]

\[
= c\|(1 - C_v)^{-1}\|_{L^p\rightarrow L^q(\langle v \rangle_\infty)}\|v - I\|_{L^q}, \quad \text{by (1.20)},
\]
which leads in turn to the estimate

\[
\|\psi - I\|_{L^q} \leq c \|(1 - C_v)^{-1}\|_{L^p \to L^p} \|v\|_{L^q} \langle v \rangle_{2\infty} \|v - I\|_{L^q}
\]

as before. Using the jump relations for \(\phi_{\pm}\) and \(\psi_{\pm}\), we see that \(\psi_+ \phi_{+}^T = \phi_- v^{-1} \psi_-^T = \phi_- \psi_-^T\). But \(\phi_+ \psi_+^T - I = C \hat{h}\), where \(h \in L^1 + L^p + L^q\), by (1.7). Hence \(h = C^+ h - C^- h = 0\), and so \(\psi_\pm = \phi_\pm^{-1}\). We conclude that

\[
\|\phi_{+}^{-1} - I\|_{L^q} \leq c \|(1 - C_v)^{-1}\|_{L^p \to L^p} \|v\|_{L^q} \langle v \rangle_{2\infty} \|v - I\|_{L^q}
\]

and

\[
\|\phi_{-}^{-1} - I\|_{L^q} \leq c \|(1 - C_v)^{-1}\|_{L^p \to L^p} \|v\|_{L^q} \langle v \rangle_{2\infty} \|v - I\|_{L^q}.
\]

Recall that a contour \(\Gamma \subset \mathbb{C}\) is complete (see e.g. [Z2]) if

\[
\mathbb{C} \setminus \Gamma \text{ is a disjoint union of two, possibly disconnected, open regions } \Omega_+ \text{ and } \Omega_-, \text{ and}
\]

\[
\Gamma \text{ may be viewed as the positively oriented boundary for } \Omega_+ \text{ and also as the negatively oriented boundary of } \Omega_-.
\]

Two examples of such contours are \(\Gamma = \mathbb{R}\) and \(\Gamma = \mathbb{R} \cup i\mathbb{R}\).

![Figure 1.35 Γ = R, Γ = R ⊃∪ iR](image)

Note that a complete contour \(\Gamma\) comes equipped with two natural orientations (we may always relabel \(\Omega_\pm\) as \(\Omega_\mp\)). Unless stated otherwise, we will always choose one of these orientations for the specification of RHP’s on \(\Gamma\). Note also that if \(\Gamma\) is complete, then by Cauchy’s theorem \(C^+ C^- = C^- C^+ = 0\), and hence by the basic relation \(C^+ - C^- = 1\), we see that \(C^+\) and \(-C^-\) are complementary projections in \(L^p(\Gamma)\), \(1 < p < \infty\).

In this section, we will also consider extended contours \(\Gamma_{\text{ext}} = \Gamma \cup \Gamma'\) where

\[
\text{dist}(\Gamma, \Gamma') > 0
\]

\[
\Gamma_{\text{ext}} \text{ is complete.}
\]

Two examples of such extended contours are extensions of \(\mathbb{R}\) and \(\mathbb{R} \cup i\mathbb{R}\) as in Figure 1.38.
Given $\Gamma_{\text{ext}} = \Gamma \cup \Gamma'$, let $R$ and $R^{-1}$ be analytic and bounded matrix functions in $\mathbb{C} \setminus \Gamma_{\text{ext}}$. As noted above following (1.9), $R$ and $R^{-1}$ have non-tangential boundary values almost everywhere on $\Gamma_{\text{ext}}$. We denote the boundary values from $\Omega_+$ by $R_+$, $R^{-1}_+$ and from $\Omega_-$ by $R_-$, $R^{-1}_-$. The following Lemma provides bounds on $\| (1 - C_v)^{-1} \|_{L^p(\Gamma)} \rightarrow L^p(\Gamma)$ in terms of auxiliary information involving an associated “conjugate” problem, and is the main structural result in this paper.

**Conjugation Lemma 1.39.**

Let $2 < p < \infty$ and let $\Gamma_{\text{ext}} = \Gamma \cup \Gamma'$, $R, R^{-1}$ be given as above. Let $v, \breve{v} : \Gamma \rightarrow GL(k, \mathbb{C})$ be jump matrices that are related via via

\begin{equation}
(1.40) \quad v = R^{-1}_- \breve{v} R_+.
\end{equation}

Assume in addition that

\begin{equation}
(1.41) \quad v - I, \breve{v} - I \in L^2(\Gamma).
\end{equation}

Suppose that

\begin{equation}
(1.42) \quad (1 - C_v)^{-1} \text{ exists in } L^2(\Gamma) \text{ and also in } L^p(\Gamma)
\end{equation}

and that

\begin{equation}
(1.43) \quad (1 - C_{\breve{v}})^{-1} \text{ exists in } L^2(\Gamma).
\end{equation}

Then $(1 - C_v)^{-1}$ exists in $L^p(\Gamma)$ and

\begin{equation}
(1.44) \quad \| (1 - C_v)^{-1} \|_{L^p(\Gamma) \rightarrow L^p(\Gamma)} \leq c_v^#, \quad \text{where}
\end{equation}

\begin{equation}
(1.45) \quad c_v^# = c \| R \|_{L^\infty(\mathbb{C} \setminus \Gamma_{\text{ext}})} \| R^{-1} \|_{L^\infty(\mathbb{C} \setminus \Gamma_{\text{ext}})} \| (1 - C_v)^{-1} \|_{L^p(\Gamma) \rightarrow L^p(\Gamma)} \| (1 - C_{\breve{v}})^{-1} \|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \times \| (1 - C_v)^{-1} \|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \| v \|_{L^\infty(\mathbb{C} \setminus \Gamma_{\text{ext}})}^2 \| \breve{v} \|_{L^\infty(\mathbb{C} \setminus \Gamma_{\text{ext}})}^2 (1 + \| v - I \|_{L^2(\Gamma)}^2)^2 (1 + \| v - I \|_{L^2(\Gamma)}^2)^2.
\end{equation}

**Remark 1.46.**
(a) The assumption that $\Gamma$ and $\Gamma_{\text{ext}}$ are complete is not necessary. However, if $\Gamma_{\text{ext}}$ is not complete then the meaning of $R_{\pm}$ is not clear and the precise statement of the theorem becomes more complex. Of course, any contour can be extended trivially to a complete contour.

(b) If $v, \bar{v}$ satisfy all the assumptions in the Conjugation Lemma 1.39, except $\bar{v}$ is replaced by $(\bar{v}^T)^{-1}$ in condition (1.42), then the Conjugation Lemma together with the duality relations given above implies that $(1 - C_{\bar{v}})^{-1}$ exists in $L^q(\Gamma)$, $1/p + 1/q = 1$, and

$$
\|(1 - C_{\bar{v}})^{-1}\|_{L^q(\Gamma) \to L^q(\Gamma)} \leq c^\#_w,
$$

where $c^\#_w$ has similar structure to the constant in (1.45). We leave the details to the reader.

An application of the Conjugation Lemma to PDE’s is given in Theorem 1.48 below, as noted at the beginning of this section.

Let $H^1$ denote the first Sobolev space $\{ f: f, f' \in L^2(\mathbb{R}) \}$ and define $H^1_\ell \equiv H^1 \cap \{ f: \| f \|_{L^\infty(\mathbb{R})} < 1 \}$. For $\Sigma = \mathbb{R}$, oriented from $-\infty \to \infty$, consider the 2 jump matrix

$$
(1.47) \quad v_\theta = \begin{pmatrix} 1 - |r(z)|^2 & r(z)e^{i\theta} \\ -r(z)e^{-i\theta} & 1 \end{pmatrix}, \quad z \in \mathbb{R},
$$

where $r \in H^1_\ell$ and $\theta = xz - tz^2$, $x, t \in \mathbb{R}$.

**Theorem 1.48.**

Suppose $r \in H^1_\ell, \|r\|_{H^1} \leq \lambda, \|r\|_{L^\infty} \leq \rho < 1$. Then for $x, t \in \mathbb{R}$, and for any $2 \leq p < \infty$, $(1 - C_{vy})^{-1}$ exists as a bounded operator in $L^p(\mathbb{R})$, and there are constants $\ell_1 = \ell_1(p), \ell_2 = \ell_2(p) > 0$ (see (3.102) below), and a constant $c = c(p)$, such that

$$
(1.49) \quad \|(1 - C_{vy})^{-1}\|_{L^p \to L^p} \leq \frac{c(1 + \lambda)^\ell_1}{(1 - \rho)^\ell_2}
$$

uniformly for all $x, t \in \mathbb{R}$.

**Remark 1.50.** Using 1.46(b), one easily verifies that estimate 1.49 remains true for $1 < p < 2$.

Theorem 1.48 plays a crucial role, in particular, in analyzing the long-time behavior of solutions of the perturbed NLS equation $i \xi_t + q_{xx} - 2|q|^2 q - \varepsilon |q|^2 q = 0, s > 2, \varepsilon > 0$ (see [DZ3], [DZ4], [DZ5]). But the result is also of independent interest. In the linear case, if $b \in L^\infty(\mathbb{R})$ and $\phi(z)$ is real valued, then by (1.2)

$$
(1.51) \quad \|C^{-}(f \phi e^{i\phi})\|_{L^p} \leq c\|b\|_{L^\infty} \|f\|_{L^p},
$$

where the bound is independent of $\phi$. (Theorem 1.48), in which the bound $c$ is independent of the multiplier $e^{i\phi}$, should be viewed as a non-linear version of such estimates. Indeed if $\rho$ is sufficiently small then we can expand $(1 - C_{vy})^{-1}$ in a Neumann series and a bound of the form (1.49), independent of $e^{i\phi}$, follows from (1.51).

In view of Proposition 1.12, Theorem 1.48 is equivalent to the following result.

**Theorem 1.52.**

Suppose $r \in H^1_\ell, \|r\|_{H^1} \leq \lambda, \|r\|_{L^\infty} \leq \rho < 1$. Let $\Sigma = \mathbb{R}$ and $v = v_\theta$ as in (1.47). Then for $x, t \in \mathbb{R}$ and for any $2 \leq p < \infty$, IRHP$_{\ell, p}$ has a unique solution $M_{\pm}$ for any $F$ and there exist $\ell_1 = \ell_1(p), \ell_2 = \ell_2(p) > 0$, and a constant $c = c(p)$, such that

$$
(1.53) \quad \|M_{\pm}\|_{L^p} \leq \frac{c(1 + \lambda)^\ell_1}{(1 - \rho)^\ell_2} \|F\|_{L^p}
$$

uniformly for all $x, t \in \mathbb{R}$.

This equivalence allows us to prove (1.49) by taking advantage of the analyticity properties of $M_{\pm}$. In particular, we are able to deform the RHP around the stationary phase point $z_0 = x/2t$ of $e^{i\phi}$, in the spirit of
the non-linear steepest—descent method introduced by the authors in [DZ1]. As we will see, this deformation plays a crucial role in the analysis.

In particular, for the factorization 
\[
\begin{pmatrix}
1 & -re^{i\theta} \\
0 & 1
\end{pmatrix}^{-1} \begin{pmatrix}
1 & 0 \\
-r\overline{e}^{-i\theta} & 1
\end{pmatrix}
\]

of \(v_\theta\),

\[
w_\theta = (w^-_\theta, w^+_\theta) = \begin{pmatrix}
0 & re^{i\theta} \\
0 & 0 \\
0 & -\overline{r}e^{-i\theta} \\
0 & 0
\end{pmatrix},
\]

and it follows that \((1 - C_{w\theta})^{-1}\) obeys the same bound as \((1 - C_{v\theta})^{-1}\) (cf. (1.49)),

\[
\|(1 - C_{w\theta})^{-1}\|_{L^p} \leq \frac{c(1 + \lambda)^{\ell_1}}{(1 - \rho)^{\ell_2}}
\]

uniformly for all \(x, t \in \mathbb{R}\).

Remark 1.55.

As we will see in Section 3, a simple argument shows that \((1 - C_{v\theta})^{-1}\) exists as a bounded operator in \(L^2(\mathbb{R})\) and satisfies the bound \(\|(1 - C_{v\theta})^{-1}\|_{L^2} \leq c/(1 - \rho)\), uniformly for \(x, t \in \mathbb{R}\). As \(v_\theta(z)\) is continuous in \(z\), it follows by general Fredholm arguments (cf. [CG]) that for any \(x, t\), \((1 - C_{v\theta})^{-1}\) exists as a bounded operator in \(L^p(\mathbb{R})\) for any \(1 < p < \infty\), but the bound may depend on \(x, t\). The point here is that, by the Conjugation Lemma 1.39, for any \(1 < p < \infty\), the bound may be chosen uniformly in \(x, t\), as in (1.49) and Remark 1.50.

The paper is organized as follows. In Section 2 we prove the Conjugation Lemma 1.39). In Section 3 we use Lemma 1.39 to prove Theorem 1.48. As noted above, steepest descent methods play a crucial role (see [DZ1], [DIZ], [DZ2]).

Let \(M_\pm \in \partial C(L^p)\) solve the IRHP, \(M_+ = M_- v_\theta + F\). If \(M_\pm = C^\pm h\), set \(\tilde{M}_\pm = C^\pm \tilde{h}\), where \(\tilde{h}(z) = \overline{h(-z)}\). (Thus \(M(z) = M(-z)\) for the extensions of \(M_\pm, \tilde{M}_\pm\) off \(\mathbb{R}\), respectively.) Then a simple computation shows that \(\tilde{M}_\pm\) solves the IRHP \(\tilde{M}_+ = \tilde{M}_- \tilde{v}_\theta + \tilde{F}\), where

\[
\tilde{v}_\theta(z) = \begin{pmatrix}
1 - |\tilde{r}(z)|^2 \\
\tilde{r}(z)e^{-i\theta(-z)} \\
\tilde{r}(z)e^{i\theta(-z)} \\
1
\end{pmatrix},
\]

and \(\tilde{r}(z) = \overline{r(-z)}, \tilde{F}(z) = \overline{F(-z)}\). As \(\tilde{r} \in H^1_1, \|\tilde{r}\|_{H^1} = \|r\|_{H^1}, \|\tilde{F}\|_{L^\infty} = \|F\|_{L^\infty}, \|\tilde{F}\|_{L^p} = \|F\|_{L^p}\), and as \(-\theta(-z) = xz + tz^2\), it follows from Proposition 1.12 that we only need to prove Theorem 1.48 for \(t \geq 0\). In the text that follows we will assume that \(t \geq 0\) without further comment.

Remark on Notation 1.56.

If \(A = (a_{ij})\) is an \(\ell \times m\) matrix, it is convenient in the remainder of the paper to fix the matrix norm, \(|A| \equiv (\Sigma_{ij}|a_{ij}|^2)^{\frac{1}{2}} = (\text{tr} A^* A)^{\frac{1}{2}}\). We say \(A(z) = (a_{ij}(z))\) is in \(L^p(\Sigma)\) for some contour \(\Sigma \subset \mathbb{C}\) if each of the entries \(a_{ij}(z) \in L^p(\Sigma)\) and we define \(|A|_{L^p(\Sigma)} \equiv \|A\|_{L^p(\Sigma)}\).

2. PROOF OF THE CONJUGATION LEMMA

The proof is in steps.

Step 1. As \((1 - C_{v\theta})^{-1}\) exists in \(L^2\) and \(L^p\), and as \(\tilde{v} - I \in L^2 \cap L^p\), it follows from (1.25)(1.14) that \(\tilde{\phi}_\pm\), the solution of the normalized RHP \((\Gamma, \tilde{v})\), exists in \(I + L^p \cap L^2\) and \(\tilde{\phi}_-\) is given by

\[
(2.1) \quad \tilde{\phi}_- = I + (1 - C_{v\theta})^{-1} C^- (\tilde{v} - I).
\]

Now consider the IRHP \((\Gamma, \tilde{v})\) on \(\Gamma\)

\[
\tilde{M}_+ = \tilde{M}_- \tilde{v} + F, \quad \tilde{M}_\pm \in \partial C(L^p), \quad F \in L^p.
\]

Then
Here we used the fact that 
\[ \tilde{M}_+\tilde{\phi}_+^{-1} = \tilde{M}_-\tilde{\phi}_+^{-1} + F\tilde{\phi}_+^{-1} \]
and by the Plemelj formula,
\[ \tilde{M}_-\tilde{\phi}_-^{-1} = C^-(F\tilde{\phi}_+^{-1}). \]
Thus we obtain by (1.14),
\[ (C^-(F\tilde{\phi}_+^{-1}))\tilde{\phi}_- = \tilde{M}_- = (1 - C_v)^{-1}C^-F, \]
which implies that \((C^-(\cdot\tilde{\phi}_+^{-1}))\tilde{\phi}_-\) is bounded from \(L^p \to L^p\) and
\[ \|(C^-(\cdot\tilde{\phi}_+^{-1}))\tilde{\phi}_-\|_{L^p(\Gamma)} \leq c\|(1 - C_v)^{-1}\|_{L^p(\Gamma)\to L^p(\Gamma)}. \]

**Step 2.** As \((1 - C_v)^{-1}\) exists in \(L^2\), and as \(v - I \in L^2\), it follows as above that \(\phi_\pm\), the solution of the normalized RHP \((\Gamma, v)_{L^2}\) exists and \(\phi_-\) is given by
\[ \phi_- = I + (1 - C_v)^{-1}C^-(v - I). \]
Now let \(M_\pm\) solve the IRHP\(_{L^2}\) on \(\Gamma\)
\[ \begin{align*}
M_+ &= M_-v + G, & \quad M_\pm \in \partial C(L^2) \\
\end{align*} \]
with \(G \in L^p \cap L^2 \subset L^2\). Then arguing as above,
\[ M(z) = (C(G\phi_+^{-1}))(z)\phi(z), \quad z \in \mathbb{C}\setminus\Gamma, \]
where \(M(z), \phi(z)\) are the extensions of \(M_\pm, \phi_\pm\) off \(\Gamma\) respectively. Write
\[ G\phi_+^{-1} = G + G(\phi_+^{-1} - I) \in L^p(\Gamma) + L^p(\Gamma) \]
where \(\frac{1}{q} = \frac{1}{2} + \frac{1}{p}\) i.e. \(1 < q = \frac{2p}{2+p} < 2 < p\). But by standard estimates \(C_{\Gamma \to \Gamma'}\) is bounded from \(L^{p'}(\Gamma) \to L^{p'}(\Gamma')\) for any \(p' > 1\), and hence
\[ \begin{align*}
\|M\|_{L^p(\Gamma')} &\leq c\|\phi\|_{L^\infty(\Gamma')} \left( \|G\|_{L^p(\Gamma)} + \|G\|_{L^p(\Gamma)}\|\phi_+^{-1} - I\|_{L^2(\Gamma)} \right) \\
&\leq c(1 + \|\phi_+ - I\|_{L^2(\Gamma)} + \|\phi_- - I\|_{L^2(\Gamma)})(1 + \|\phi_+^{-1} - I\|_{L^2(\Gamma)})\|G\|_{L^2(\Gamma)} \\
&\leq c\|(1 - C_v)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}\|\phi_+^{-1}\|_{L^2(\Gamma)}^2(1 + \|v - I\|_{L^2(\Gamma)}^2)\|G\|_{L^p(\Gamma)}. \\
\end{align*} \]
Here we used the fact that \(\text{dist}(\Gamma, \Gamma') > 0\) to estimate \(\|\phi\|_{L^\infty(\Gamma')}\) in terms of \(\|\phi_\pm - I\|_{L^2(\Gamma')}\).

**Step 3.** Again consider the IRHP\(_{L^2}\) (2.4). Inserting the relations \(v = R^{-1}\tilde{v}R_+\) and \(\tilde{v} = (\tilde{\phi}_-)^{-1}\tilde{\phi}_+\), we obtain
\[ \begin{align*}
M_+R_+^{-1}\tilde{\phi}_+^{-1} &= M_-R_+^{-1}\tilde{\phi}_+^{-1} + GR_+^{-1}\tilde{\phi}_+^{-1} \quad \text{on } \Gamma. \\
\end{align*} \]
Once again by the Plemelj formula, on \(\Gamma\)
\[ \begin{align*}
(MR_+^{-1}\tilde{\phi}_+^{-1})_- &= C_{\Gamma \to \Gamma'}((MR_-^{-1}\tilde{\phi}_+^{-1})_+ - (MR_-^{-1}\tilde{\phi}_+^{-1})_-) \\
&= C_{\Gamma \to \Gamma'}(MR_+^{-1}\tilde{\phi}_+^{-1} - MR_+^{-1}\tilde{\phi}_+^{-1}) + C_{\Gamma \to \Gamma'}(GR_+^{-1}\tilde{\phi}_+^{-1}). \\
\end{align*} \]
Here we have used (2.6) and the fact that \( M \) and \( \tilde{\phi} \) are analytic across \( \Gamma' \). Thus

\[
M_\gamma = [C_{\gamma'} \to \gamma(MR_+^{-1} \tilde{\phi}^{-1} - MR_-^{-1} \tilde{\phi}^{-1})] \tilde{\phi}_- R_- + (C_{\gamma'} \to \gamma(GR_+^{-1} \tilde{\phi}_+^{-1})) \tilde{\phi}_+ R_+ = I + \Pi.
\]

Now as \( \text{dist}(\Gamma, \Gamma') > 0 \),

\[
\|C_{\gamma'} \to \gamma(MR_+^{-1} \tilde{\phi}^{-1} - MR_-^{-1} \tilde{\phi}^{-1})\|_{L^p(\Gamma)} \leq c\|R_+^{-1}\|_{L^\infty(\gamma')}\|M\|_{L^p(\gamma')}\|\tilde{\phi}^{-1}\|_{L^\infty(\gamma')}.
\]

As in Step 2,

\[
\|\tilde{\phi}\|_{L^\infty(\gamma')} \leq c\|v\|_{L^\infty(\gamma)}\|(1 - C_v)^{-1}\|_{L^2(\gamma')}(1 + \|\tilde{v} - I\|_{L^2(\gamma')}),
\]

and hence, using (2.5), we obtain

\[
\|\Pi\|_{L^p(\Gamma')} \leq \|C_{\gamma'} \to \gamma(MR_+^{-1} \tilde{\phi}^{-1} - MR_-^{-1} \tilde{\phi}^{-1})\|_{L^p(\gamma')}\|R_+^{-1}\|_{L^\infty(\gamma')}\|\tilde{\phi}^{-1}\|_{L^\infty(\gamma')} \leq c\|v\|_{L^\infty(\gamma)}\|(1 - C_v)^{-1}\|_{L^p(\gamma')}\|\tilde{v} - I\|_{L^p(\gamma')}
\]

which yields an \( L^p(\Gamma) \) bound for \( M_\gamma \).

Finally, from (1.13), \((1 - C_v)^{-1}F = M_\gamma + F\), where \( M_\gamma \) solve IRHP \( L^2 \) with \( G = F(v - I) \). For \( F \in L^p(\gamma') \), we then have from (2.11) (2.12), together with (1.23,1.24) and their analogs for \( \tilde{v} \), and also the interpolation estimate \( \|v - I\|_{L^p} \leq c\|v - I\|_{L^\infty} + \|v - I\|_{L^2} \leq c\|v\|_{L^\infty} \), a bound of the form \( \|(1 - C_v)^{-1}F\|_{L^p(\Gamma)} \leq c\|F\|_{L^p(\Gamma)} \) with \( c\|F\|_{L^p(\Gamma)} \) as in (1.45). The result then follows by density.

The constant \( c\# \) in (1.44) depends on \( \Gamma \) and \( \Gamma' \) and the distance \( \text{dist}(\Gamma, \Gamma') \) between them. We are interested in particular in applying the Conjugation Lemma 1.39 for contours of the type that appear in Figure 1.38. For such contours it is easy to show that for \( 1 < q \leq p \leq \infty \),

\[
\|C^\pm\|_{L^q(\gamma') \to L^p(\gamma)}, \quad \|C^\pm\|_{L^q(\gamma') \to L^p(\gamma')} \leq \frac{c_q}{(\text{dist}(\gamma, \gamma'))^{\frac{3}{2} + 1/p}}.
\]

We are particularly interested in the case where \( \text{dist}(\Gamma, \Gamma') \) is bounded, say \( \text{dist}(\Gamma, \Gamma') \leq 1 \). Keeping track of the constants, and using \( \text{dist}(\Gamma, \Gamma') \leq 1 \), we see that the constant \( c \) in (2.5) should be replaced by \( c/\text{dist}(\Gamma, \Gamma') \), and that \( c \) in (2.10) should be replaced by \( c/(\text{dist}(\Gamma, \Gamma'))^{3/2+1/p} \). It follows that in (1.44) we should replace

\[
c\# \to c\#/((\text{dist}(\Gamma, \Gamma'))^{3/2+1/p})
\]

in the case \( \text{dist}(\Gamma, \Gamma') \leq 1 \).
3. Proof of Theorem 1.48

Notational Remark. In this section all jump matrices \( v \) are \( 2 \times 2 \) with determinant 1. Thus if \( v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), then \( v^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \). It follows that we can replace \( \langle v \rangle_\infty \) by \( \| v \|_\infty \) in all the estimates in Section 2, and in particular, in (1.45). We will make this replacement systematically in this section without further comment.

As indicated in Remark 1.55, the proof of Theorem 1.48 in the case \( p = 2 \) follows by a simple argument. Indeed for the operator \( h \) we have for \( v \) and in particular, in (1.45). We will make this replacement systematically in this section without further comment.

We now give a second proof of (3.1) by a more general method that will be useful at various points in the calculations that follow.

Proposition 3.2.

Suppose \( r \in L^\infty(\mathbb{R}) \) and \( \| r \|_\infty \leq \rho < 1 \). Let \( v = \begin{pmatrix} 1 - |r|^2 & r \\ -\bar{r} & 1 \end{pmatrix} \), and let \( C_v \) be the associated Cauchy operator as in (1.5). Then \( (1 - C_v)^{-1} \) exists in \( L^2(\mathbb{R}) \) and

\[
(3.1) \quad \| (1 - C_v)^{-1} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq \frac{c}{1 - \rho}.
\]

Note that the bound depends only on \( \rho \) and not on the \( H^1 \) norm of \( r \).

We now give a second proof of (3.1) by a more general method that will be useful at various points in the calculations that follow.

Proof.

Assume first that \( (1 - C_v)^{-1} \) exists. Then by (1.11), \( (1 - C_v)^{-1}f = (M_+ + f) \) for \( f \in L^2 \), where \( M \) solves the IRHP \( L^2 \), \( M_+ = M_-v + f(v - 1) \), \( M_+ \in \partial C(L^2) \). Now by a simple contour argument

\[
\int_{\mathbb{R}} M_-(z) M_+^*(z) dz = 0,
\]

and hence

\[
\int_{\mathbb{R}} M_-(z)v^*(z)M_+^*(z) dz = \int_{\mathbb{R}} M_-(z)(I - v^*(z))f^*(z) dz.
\]

Taking conjugates and adding we obtain

\[
(3.4) \quad \int_{\mathbb{R}} M_-(z)(v(z) + v^*(z))M_+^*(z) dz = \int_{\mathbb{R}} [M_-(z)(I - v^*(z))f^*(z) + f(z)(I - v(z))M_+^*(z)] dz.
\]
But \( v(z) + v(z)^* = 2 \begin{pmatrix} 1 - |r(z)|^2 & 0 \\ 0 & 1 \end{pmatrix} \), and it then follows directly from (3.4) that \( \|M_-\|_{L^2} \leq \frac{1}{\rho} \|f\|_{L^2} \), which implies in turn the bound (3.3) for \((1 - C_\gamma)^{-1}\). Set \( r_\gamma = \gamma r \), \( 0 \leq \gamma \leq 1 \). We have \( v_\gamma - I = \begin{pmatrix} -\gamma^2 |r|^2 & \gamma r \\ -\gamma \bar{r} & 0 \end{pmatrix} \) and hence \( 1 - C_{v_\gamma} = 1 - C(-v_\gamma - I) \) is invertible for small \( \gamma \). However \( \|r_\gamma\|_{L^\infty} \leq \rho \) for all \( 0 \leq \gamma \leq 1 \), and hence \((1 - C_{v_\gamma})^{-1}\) must satisfy the bound (3.3) whenever it exists. By an elementary continuity argument it then follows that \((1 - C_{v_\gamma})^{-1}\) exists and satisfies (3.3) for all \( 0 \leq \gamma \leq 1 \). \( \square \)

In order to control \((1 - C_{v_\gamma})^{-1}\) in \( L^p \), \( 2 < p < \infty \), uniformly for \( x,t \in \mathbb{R} \), we must control the solution \( M_+ \) of the IRHP for \( v_\theta \). Following the steepest descent method introduced in [DZ1], and applied to the NLS equation in [DIZ], [DZ2], we expect the IRHP to “localize” near the stationary phase point \( z_0 = x/2t \) for \( \theta = xz - tz^2 \), \( \theta'(z_0) = 0 \). Furthermore, the signature table for \( \Re i\theta \)

$$
\begin{array}{c|c|c}
\Re i\theta < 0 & \Re i\theta > 0 & z_0 \\
\hline
\Re i\theta > 0 & \Re i\theta < 0 & \\
\end{array}
$$

Figure 3.5. Signature table for \( \Re i\theta \)

should play a crucial role. The basic idea of the method is to deform the contour \( \Gamma = \mathbb{R} \) so that the exponential factors \( e^{i\theta} \) and \( e^{-i\theta} \) are exponentially decreasing, as dictated by Figure 3.5. In order to make these deformations we must separate the factors \( e^{i\theta} \) and \( e^{-i\theta} \) algebraically, and this is done using the upper/lower and lower/upper factorizations of \( v_\theta \),

$$
(3.6) \quad v_\theta = \begin{pmatrix} 1 & re^{i\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{r}e^{-i\theta} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 - |r|^2 & 1 \end{pmatrix} \begin{pmatrix} 1 - |r|^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{1-|r|^2} e^{i\theta} \end{pmatrix}.
$$

The upper/lower factorization is appropriate for \( z > z_0 \) and the lower/upper factorization is appropriate for \( z < z_0 \). The diagonal terms in the lower/upper factorization can be removed by conjugating \( v_\theta \),

$$
(3.7) \quad \tilde{v}_\theta = \delta_3 v_\theta \delta_+^{-\sigma_3}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \text{third Pauli matrix},
$$

by the solution \( \delta_\pm \) of the scalar, normalized RHP \((\mathbb{R}_-, z_0, 1 - |r|^2)_{L^2} \),

$$
(3.8) \quad \begin{cases} 
\delta_+ = \delta_- (1 - |r|^2), & z \in \mathbb{R}_- + z_0, \\
\delta_+ - 1 \in \partial \mathcal{C}(L^2),
\end{cases}
$$

where the contour \( \mathbb{R}_- + z_0 \) is oriented from \(-\infty\) to \( z_0 \). The properties of \( \delta \) can be read off from the following elementary proposition, whose proof is left to the reader.

**Proposition 3.9.**

Suppose \( r \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \) and \( \|r\|_{L^\infty} \leq \rho < 1 \). Then the solution \( \delta_\pm \) of the scalar normalized RHP (3.8) exists and is unique and is given by the formula

$$
(3.10) \quad \delta_\pm(z) = e^{C_{\pm,z_0}^r C_\pm z_0 \log 1 - |r|^2} = e^{\frac{1}{2\pi} \int_{z_0}^z \frac{\log(1 - |r(s)|^2)}{s - t} ds}, \quad z \in \mathbb{R}.
$$

The extension \( \delta \) of \( \delta_\pm \) off \( \mathbb{R}_- + z_0 \) is given by
\[ \delta(z) = e^{C_{\pm} + i\theta \log(1 - |r|^2)} = e^{\frac{1}{2} \int_{-\infty}^{z_0} \frac{\log(1 - |r(s)|^2)}{1 - |r(s)|^2} \, ds}, \quad z \in \mathbb{C} \setminus (\mathbb{R} + z_0), \]
and satisfies for \( z \in \mathbb{C} \setminus (\mathbb{R} + z_0) \),

\[ \delta(z) \bar{\delta}(\bar{z}) = 1, \]

\[ (1 - \rho)^{\frac{1}{2}} \leq (1 - \rho^2)^{\frac{1}{2}} \leq |\delta(z)|, |\delta(z)| \leq (1 - \rho^2)^{-\frac{1}{2}} \leq (1 - \rho)^{-\frac{1}{2}}, \]
and

\[ |\delta^\pm(z)| \leq 1 \quad \text{for} \quad \pm \text{Im} \, z > 0. \]

For real \( z \),

\[ |\delta_+(z)\delta_-(z)| = 1 \quad \text{and, in particular,} \quad |\delta(z)| = 1 \quad \text{for} \quad z > z_0, \]

\[ |\delta_+(z)| = |\delta_-(z)| = (1 - |r(z)|^2)^{\frac{1}{2}}, \quad z < z_0, \]
and

\[ \Delta \equiv \delta_+ \delta_- = e^{\frac{1}{2} \int_{-\infty}^{z_0} \frac{\log(1 - |r(s)|^2)}{1 - |r(s)|^2} \, ds}, \quad \text{where P.V. denotes the principal value.} \]

Also \( |\Delta| = |\delta_+ \delta_-| = 1 \)

\[ \|\delta_+ - 1\|_{L^2(dz)} \leq \frac{c\|r\|_{L^2}}{1 - \rho}. \]

We obtain the following factorization for \( \tilde{v}_\theta \)

\[ \tilde{v}_\theta = \tilde{v}_\theta^{-1} \tilde{v}_\theta = \begin{pmatrix} 1 & r e^{i\theta} \delta^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{r} e^{-i\theta} \delta^{-2} & 1 \end{pmatrix}, \quad z > z_0, \]

\[ \tilde{v}_\theta = \tilde{v}_\theta^{-1} \tilde{v}_\theta = \begin{pmatrix} -\bar{r} e^{-i\theta} \delta^{-2} & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & r e^{i\theta} \delta^{-2} \\ 0 & 1 \end{pmatrix}, \quad z < z_0. \]

Using Figure 3.5 we observe the crucial fact that the analytic continuations to \( \mathbb{C}_+ \) of the exponentials in the factors on the right in (3.19) and (3.20), are exponentially decreasing, and the same is true for the exponentials on the left, when continued to \( \mathbb{C}_- \).

For later reference, observe that (3.20) can also be written in the form

\[ \tilde{v}_\theta = \begin{pmatrix} 1 & 0 \\ -\bar{r} e^{-i\theta} \delta^{-2} \delta^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & r e^{i\theta} \delta_+ \delta_- \\ 0 & 1 \end{pmatrix}, \quad z < z_0. \]

Now clearly

\[ \tilde{M}_\pm \text{ solves IRHP}_{L^2}((\mathbb{R}, \tilde{v}) \text{ with inhomogeneous term } \tilde{F}) \]

\[ \iff \]

\[ M_\pm = \tilde{M}_\pm \delta_{\pm} \text{ solves IRHP}_{L^p} \text{ with inhomogeneous term } F = \tilde{F} \delta_{\pm} \]

and so to control \( (1 - C_{\theta})^{-1} \), it is sufficient to control \( (1 - C_{\theta})^{-1} \).

The \( L^p \) bound on \( (1 - C_{\theta})^{-1} \), and hence on \( (1 - C_{\theta})^{-1} \), in the general case will be inferred eventually from the following model problem, with the aid of the Conjugation Lemma 1.39.

\( \text{(3.22) Model Problem:} \)

(i) Suppose \( x = 0 \) so that \( z_0 = 0 \) and \( \theta = -t z^2 \)
(ii) \( r(z) = \frac{r(0)}{1 + rz}, \) where \( |r(0)| \leq \rho < 1. \)
Proposition 3.23.
Let $0 < \beta < 1/2$. Then for the model problem (3.22),

\[(3.24)\]  
\[\| (1 - C_{\tilde{v}_\theta})^{-1} \|_{L^p \to L^p} \leq \frac{c}{(1 - \rho)^2 + 5\beta^2}, \quad 2 < p < \infty,\]

where $c$ is independent of $t > 0$ and $\rho < 1$, and depends only on $\beta$ on $p$.

We will prove this Proposition in a series of steps. Let $\delta_{\pm}$ denote the solution of the scalar normalized RHP (3.8) for the model $r$ in (3.22).

**Step 1 (analytic continuations).** Observe that for $r = \frac{r(0)}{1+i\varepsilon}$ and $z < 0$, $\delta_{\pm} \delta_\rho = \frac{\delta_{\pm}^2}{1-|\rho|^2} = \frac{(z^2 + 1)\delta_{\pm}^2}{z^2 + 1 - |\rho(0)|^2}$, and so $h = \frac{z-i\sqrt{1-|\rho(0)|^2}}{z+i\sqrt{1-|\rho(0)|^2}} \delta_{\pm} \delta_\rho$ has an analytic continuation from $\mathbb{R}_-$ to $C_+$. For $z < 0$, $|h(z)| \leq 1$ by (3.15), and for $z > 0$, $|h(z)| \leq \frac{1}{1 - |\rho(0)|^2} \leq \frac{1}{1 - \rho^2}$, again by (3.15). By a standard Phragmén–Lindelöf argument we conclude that

\[(3.25)\]  
\[| (\delta_{+} \delta_{-})(z) | \leq \left| \frac{z + i\sqrt{1-|\rho(0)|^2}}{z - i\sqrt{1-|\rho(0)|^2}} \right| \frac{1}{(1 - \rho^2)^{1+\frac{\beta}{2}}} \]

for $z \in \mathbb{C}_+$, $0 < \arg z < \pi$. It follows that $(\delta_{+} \delta_{-})^{-1}(z) = (\delta_{+} \delta_{-})(\bar{z})$, $z < 0$, has an analytic continuation from $\mathbb{R}_-$ to $\mathbb{C}_-$ satisfying

\[(3.26)\]  
\[| (\delta_{+} \delta_{-})^{-1}(z) | \leq \left| \frac{z - i\sqrt{1-|\rho(0)|^2}}{z + i\sqrt{1-|\rho(0)|^2}} \right| \frac{1}{(1 - \rho^2)^{1+\frac{\beta}{2}}} \]

for $-\pi < \arg z < 0$. Also the analytic continuations of $\delta(z)^{-2}$, $\delta(z)^2$ to $\mathbb{C}_+$, $\mathbb{C}_-$ respectively satisfy the bounds

\[(3.27)\]  
\[| \delta(z)^{-2} | \leq \frac{1}{(1 - \rho^2)^{\frac{\beta}{2}}}, \quad z \in \mathbb{C}_+, \quad 0 < \arg z < \pi, \]

and

\[(3.28)\]  
\[| \delta(z)^2 | \leq (1 - \rho^2)^{\frac{-\beta}{2}}, \quad z \in \mathbb{C}_-, \quad -\pi < \arg z < 0. \]

It follows that $\tilde{v}_{\theta+}$ and $\tilde{v}_{\theta-}$ in (3.19) and (3.21), have analytic continuations to $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively, where they satisfy the bounds $(0 < \beta < \frac{1}{2})$

\[(3.29)\]  
\[| \tilde{v}_{\theta+}(z) - I | \leq \frac{c}{(1 - \rho)^\beta} \quad \text{for} \quad z \in \mathbb{C}_+, \text{arg} \ z = \beta \pi \text{or } \pi - \beta \pi, \]

\[(3.30)\]  
\[| \tilde{v}_{\theta-}(z) - I | \leq \frac{c}{(1 - \rho)^\beta} \quad \text{for} \quad z \in \mathbb{C}_-, \text{arg} \ z = -\beta \pi \text{or } -\pi + \beta \pi, \]

as indicated in Figure 3.31.
The constants $c$ in (3.30) are independent of $\rho$ and $t > 0$.

**Step 2 (scaling and augmentation).** It is convenient to scale the RHP as follows:

\begin{equation}
\tilde{v}_\theta \rightarrow \tilde{v}_t(z) \equiv \tilde{v}_\theta(z/\sqrt{t}) = e^{-iz^2ad}\sigma \tilde{v}(z/\sqrt{t}).
\end{equation}

If $S_t$ denotes the scaling operator $S_tf(z) = t^{-1/2}f(z/\sqrt{t})$, then $S_t$ is an isometry from $L^p(\mathbb{R})$ onto $L^p(\mathbb{R})$ and

\begin{equation}
\frac{1}{1 - C_{\tilde{v}_t}} = S_t^{-1}\left( \frac{1}{1 - C_{\tilde{v}_\theta}} \right) S_t,
\end{equation}

and hence to prove (3.24) it is enough to replace $\tilde{v}_\theta$ with $\tilde{v}_t$. We denote the associated factors of $\tilde{v}_t$ by $\tilde{v}_t^\pm$ and as the bounds in (3.29)-(3.30) are unaffected by scaling, they remain true for $\tilde{v}_t^\pm$ for all $t > 0$.

Consider the IRHP $L^p$ on $\Gamma = \mathbb{R}$,

\begin{equation}
M_+ = M_- \tilde{v}_t + F, \quad F \in L^p(\mathbb{R}).
\end{equation}

Extend (3.34) trivially to the augmented contour $\Gamma$ in Figure 3.35 with an opening angle $\beta \pi$ as in Figure 3.31

by setting

\begin{equation}
\tilde{v}_t \equiv I, \quad F \equiv 0 \text{ on } \Gamma/\mathbb{R}.
\end{equation}

Clearly the extension $M$ of $M_\pm$ off $\mathbb{R}$ satisfies the augmented IRHP $L^p$.

\begin{equation}
M_+ = M_- \tilde{v}_t + F \text{ on } \Gamma.
\end{equation}

Set

\begin{equation}
\{ \Phi = I \quad \text{in } \Omega_0 \\
\Phi = \tilde{v}_t^\pm \quad \text{in } \Omega_\pm
\end{equation}

and define

\begin{equation}
\dot{\phi} = \Phi_- \tilde{v}_t \Phi_+^{-1} \text{ on } \Gamma.
\end{equation}

Note that

\begin{equation}
\dot{\phi} = I \quad \text{on } \mathbb{R}.
\end{equation}

**Step 3. (Bound for $(1 - C_{\dot{\phi}})^{-1}$ in $L^2(\Gamma)$).** We prove the following Lemma.
Lemma 3.41.

\begin{equation}
(3.42) \quad \|(1 - C_{\tilde{v}})^{-1}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \frac{c}{(1 - \rho)^{1 + 3\beta}}
\end{equation}

where $c$ is independent of $\rho < 1$ and $t > 0$.

Proof.

Using the properties of $\delta$ in Proposition 3.9, one sees from (3.19) and (3.21) that $\tilde{v}_t$ has the form

\begin{equation}
\begin{pmatrix}
1 & |\alpha|^2 \\
-\alpha & 1
\end{pmatrix}
on \mathbb{R}_+ \text{ and } \begin{pmatrix}
1 & \alpha \\
-\bar{\alpha} & 1 - |\alpha|^2
\end{pmatrix}
on \mathbb{R}_-
\end{equation}

for some function $\alpha$ with $\|\alpha\|_{L^\infty(\mathbb{R})} \leq \rho$ for all $t > 0$, and it follows then from (the proof of) Proposition 3.2 that $(1 - C_{\tilde{v}_t})^{-1}$ exists in $L^2(\mathbb{R})$ and

\begin{equation}
(3.43) \quad \|(1 - C_{\tilde{v}_t})^{-1}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq \frac{c}{1 - \rho}
\end{equation}

where $c$ is independent of $t > 0$. Now consider the equation $(1 - C_{\tilde{v}_t})f = g$ in $L^2$ of the augmented contour $\Gamma$. As $\tilde{v}_t = I$ on $\mathbb{R} \setminus \mathbb{R}$, this reduces on $\mathbb{R} \subset \Gamma$ to the equation $(1 - C_{\tilde{v}_t})((\tilde{v}_t - I))(f \restriction \mathbb{R}) = (g \restriction \mathbb{R})$ so that $f \restriction \mathbb{R} = (1 - C_{\tilde{v}_t})^{-1}(\tilde{v}_t - I)(g \restriction \mathbb{R})$. But then for $z \in \Gamma \setminus \mathbb{R}$, $f(z) = (C_{\mathbb{R}}((f \restriction \mathbb{R})(\tilde{v}_t - I)))(z) + g(z)$, and hence $\|f\|_{L^2(\Gamma \setminus \mathbb{R})} \leq \frac{c}{1 - \rho}\|g\|_{L^2(\Gamma)}$ by (3.43), which thereby extends to $\Gamma$,

\begin{equation}
(3.44) \quad \|(1 - C_{\tilde{v}_t})^{-1}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \frac{c}{1 - \rho},
\end{equation}

where again $c$ is independent of $t > 0$.

Finally consider the IRHP$_{L^2}$ on $\Gamma$,

\begin{equation}
(3.45) \quad \tilde{M}_+ = \tilde{M}_-\tilde{v} + \tilde{F}(\tilde{v} - I), \quad \tilde{F} \in L^2(\Gamma).
\end{equation}

By (3.39), $\tilde{M}_+\Phi_+ = \tilde{M}_-\Phi_-\tilde{v} + \tilde{F}(\tilde{v} - I)\Phi_+$, and so by (1.14),

\begin{equation}
\begin{aligned}
\|\tilde{M}_-\|_{L^2(\Gamma)} &\leq \|(1 - C_{\tilde{v}_t})^{-1}C^-(\tilde{F}(\tilde{v} - I)\Phi_+)\|_{L^2(\Gamma)}\|\Phi_-^{-1}\|_{L^\infty(\Gamma)} \\
&\leq c\|(1 - C_{\tilde{v}_t})^{-1}\|_{L^2 \to L^2}\|\tilde{F}\|_{L^2}\|\tilde{v} - I\|_{L^\infty}\|\Phi_+\|_{L^\infty}\|\Phi_-^{-1}\|_{L^\infty} \\
&\leq \frac{c}{(1 - \rho)^{1 + 3\beta}}\|\tilde{F}\|_{L^2}
\end{aligned}
\end{equation}

by (3.38), (3.40), (3.29), (3.30), and (3.43). But then (3.42) follows from (1.13). \qed

Step 4 (Bound for $(1 - C_{\tilde{v}_t})^{-1}$ in $L^p$). As $(1 - C_{\tilde{v}_t})^{-1}$ exists in $L^2(\Gamma)$, we know that the IRHP$_{L^2}$ (3.37) has a solution $M_{\pm} \in \partial C(L^2(\Gamma))$ for $F \in L^p \cap L^2 \subset L^2(\Gamma)$. Inserting (3.39) into (3.37) for such $F \in L^p \cap L^2(\Gamma)$, and using (1.14), we see that

\begin{equation}
(3.46) \quad M_-\Phi_-^{-1} = C^-(F\Phi_+^{-1}) + (1 - C_{\tilde{v}})^{-1}C_{\tilde{v}}(C^- (F\Phi_+^{-1})) \equiv I + \Pi.
\end{equation}

Now

\begin{equation}
(3.47) \quad \|\Pi\|_{L^2(\Gamma)} \leq c\|(1 - C_{\tilde{v}_t})^{-1}\|_{L^2 \to L^2}\|\tilde{v} - I\|_{L^p}\|L^\prime\|_{L^p}\|\Phi_-^{-1}\|_{L^\infty} \\
\leq \frac{c}{(1 - \rho)^{1 + 3\beta}}\|L^\prime\|_{L^p}, \quad 1/p' + 1/p = 1/2,
\end{equation}
where we have used (3.38)–(3.40), (3.29), (3.30), together with (3.42). Note that the exponential factors $e^{\pm iz^2}$ are suitably disposed with respect to the signature table in Figure 3.5 and play an essential role in ensuring that the $L^p$ norm of $\hat{v} - I$ is bounded uniformly in $t$. This fact is at the heart of the utility of the steepest descent method in proving the uniform $L^p$ bounds on $(1 - C_{\nu})^{-1}$, and hence, eventually, the desired $L^p$ bounds on $(1 - C_{\nu})^{-1}$.

Write II = $C_{\nu}(C^{-}(F\Phi_+^{-1})) + C_{\nu}\Pi$. Since $\hat{v} = I$ on $\mathbb{R}$, we obtain by (3.47)

\[(3.48) \quad \|\Pi\|_{L^p(\mathbb{R}\setminus[-1,1])} \leq \frac{c}{(1 - \rho)^{2\beta}} \|F\|_{L^p(\Gamma)} + \frac{c}{(1 - \rho)^{1+6\beta}} \|F\|_{L^p(\Gamma)} \leq \frac{c}{(1 - \rho)^{1+6\beta}} \|F\|_{L^p(\Gamma)}.
\]

But

\[(3.49) \quad \|\Pi\|_{L^p(\mathbb{R})} \leq \frac{c}{(1 - \rho)^{\beta}} \|F\|_{L^p},
\]

and hence from (3.46)

\[(3.50) \quad \|M_-\|_{L^p(\mathbb{R}\setminus[-1,1])} \leq \frac{c}{(1 - \rho)^{1+6\beta}} \|F\|_{L^p(\Gamma)}
\]

as $\|\Phi_-\|_{L^\infty(\mathbb{R})} \leq c$.

It remains to estimate $\|M_-\|_{L^p(-1,1)}$. Write $\hat{v}_t$ in the form $\hat{v}_t = \delta t^3_{t-} \begin{pmatrix} 1 & r_t \\ 0 & 1 \end{pmatrix} \delta t^3_{t+}$, where $r_t(z) = e^{-iz^2} r(z/\sqrt{t})$, $\delta_t(z) = \delta(z/\sqrt{t})$. Inserting this factorization into (3.34), we obtain on $\mathbb{R}$

\[(3.51) \quad M_+ \delta t^3_{t+} \begin{pmatrix} 1 \\ \bar{r}_t \end{pmatrix} = M_- \delta t^3_{t-} \begin{pmatrix} 1 & r_t \\ 0 & 1 \end{pmatrix} + F \delta t^3_{t+} \begin{pmatrix} 1 \\ \bar{r}_t \end{pmatrix}.
\]

From the explicit form of $r$ in (3.22) we see that $M_+ \delta t^3_{t+} \begin{pmatrix} 1 \\ \bar{r}_t \end{pmatrix}$ has an analytic continuation to $\mathbb{C}_+$, and $M_- \delta t^3_{t-} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has an analytic continuation to $\mathbb{C}_-$. Denote these continuations by $N$ in $\mathbb{C}\setminus\mathbb{R}$. By Cauchy’s formula for $|z| < 2$, $z \notin \mathbb{R}$, we obtain from (3.51)

\[(3.52) \quad N(z) = \oint_{|s|=2} \frac{N(s)}{s - z} \frac{ds}{2\pi i} + \int_{-2}^{2} F(s) \delta t^3_{t+}(s) \begin{pmatrix} 1 \\ r_t(s) \end{pmatrix} \frac{ds}{s - z} \frac{2\pi i}{2\pi i}.
\]

Thus

\[(3.53) \quad \left\|M_- \delta t^3_{t-} \begin{pmatrix} t \\ 0 \end{pmatrix}\right\|_{L^p(-1,1)} \leq \left\|\oint_{|s|=2} \frac{N(s)}{s - \diamond} \frac{ds}{2\pi i}\right\|_{L^p(-1,1)} + c \|\delta t^3_{t+}\|_{L^\infty(\mathbb{R})} \|F\|_{L^p(\mathbb{R})}.
\]

Now from (3.46), (3.47), (3.49), we have $M_- = I' + II'$, where

\[(3.54) \quad \|I'\|_{L^p(\mathbb{R})} \leq \frac{c}{(1 - \rho)^{2\beta}} \|F\|_{L^p(\mathbb{R})}, \quad \|II'\|_{L^p(\mathbb{R})} \leq \frac{c}{(1 - \rho)^{1+6\beta}} \|F\|_{L^p(\mathbb{R})},
\]
and using (3.34) we also obtain $M_+ = I^0 + II^0$, where

\begin{equation}
||I^0||_{L^p(R)} \leq \frac{c}{(1 - \rho)^{1/2}} \|F\|_{L^p(R)}, \quad ||II^0||_{L^2(R)} \leq \frac{c}{(1 - \rho)^{1/2}} \|F\|_{L^p(R)}.
\end{equation}

But then using the Cauchy formula $M(z) = (C(M_+ - M_-))(z)$, we can write $N$ as a sum of two parts, $I^N + II^N$, and

\begin{align*}
||I^N||_{L^p(|z|=2)} &\leq \frac{c}{(1 - \rho)^{3/2}} \|F\|_{L^p(R)}, \\
||II^N||_{L^2(|z|=2)} &\leq \frac{c}{(1 - \rho)^{5/2}} \|F\|_{L^p(R)}.
\end{align*}

The extra factor 1/2 comes from (3.13), whereas $\tilde{r}$ and $r$ are uniformly bounded in $C_+$ and $C_-$ respectively, for all $t > 0$. Inserting these bounds in (3.53) we obtain

\begin{align*}
||M^-||_{L^p(-1,1)} &\leq \frac{c}{(1 - \rho)^{1/2}} \left( \frac{1}{(1 - \rho)^{1/2}} + \frac{1}{(1 - \rho)^{3/2}} \right) \|F\|_{L^p} \\
&\leq \frac{c}{(1 - \rho)^{2+5\beta}} \|F\|_{L^p}.
\end{align*}

Together with (3.50), this implies

\begin{equation}
||M^-||_{L^p(R)} \leq \frac{c}{(1 - \rho)^{2+5\beta}} \|F\|_{L^p}
\end{equation}

as $1 + 6\beta < 2 + 5\beta$ for $\beta < \frac{1}{2} < 1$. As before, the same bound is true for all $F \in L^p(R)$, by density. Finally, by (1.13) and (3.33), this completes the proof of Proposition 3.23.

We now consider the general case where $r \in H^1_1$, $\|r\|_{H^1} \leq \lambda$, $\|r\|_{L^\infty} \leq \rho < 1$. We also continue to assume that $x = 0$, and hence $z_0 = 0$ and $\theta = -t\bar{z}^2$.

Given $r$ in $H^1_1$ as above, define

\begin{equation}
r^#(z) = \frac{r(0)}{\bar{z}}.
\end{equation}

Then $r^#$ corresponds to a model problem of the form (3.22) (ii) with $|r^#(0)| = |r(0)| \leq \rho < 1$, for which Proposition 3.23 applies. Let $\delta, \delta^#$ be the solution of the scalar normalized RHP’s (3.8) associated with $r, r^#$ respectively. Set

\begin{equation}
\delta_1 = \delta(\delta^#)^{-1}.
\end{equation}

Then $\delta_1$ solves the scalar normalized RHP with jump

\begin{equation}
\delta_{1+} = \delta_1 - \frac{1 - |r|^2}{1 - |r^#|^2} \text{ on } \mathbb{R}_.
\end{equation}

By (3.19), (3.20), (3.21), the jump matrix $\tilde{v}_\theta^#$ associated with $r^#$ has the form
\[ (3.60) \quad \tilde{v}_\theta^\# = \begin{pmatrix} 1 & r^\# e^{i\theta} \delta^\# e^{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -r^\# e^{-i\theta} \delta^{-2} & 1 \end{pmatrix}, \quad z > 0, \]

\[ (3.61) \quad \tilde{v}_\theta^\# = \begin{pmatrix} 1 & 0 \\ -r^\# e^{-i\theta} \delta^{-2} & 1 \end{pmatrix} \begin{pmatrix} 1 & r^\# e^{i\theta} \delta^\# e^{2} \\ 0 & 1 \end{pmatrix}, \quad z < 0. \]

Set \( v_1 = \delta_1^\sigma \tilde{v}_\theta^\# \delta_1^{-\sigma} \) and consider the IRHP \( L_\sigma \)

\[ (3.62) \quad M_+ = M_- v_1 + F(v_1 - I), \quad F \in L^p(\mathbb{R}). \]

Using (1.14), we obtain

\[ M_- = [(1 - C_{\tilde{v}_\theta^\#})^{-1} C^- (F(\delta_1^\sigma \tilde{v}_\theta^\# - \delta_1^{-\sigma}))] \delta_1^{-\sigma}. \]

Then by (3.24) and Proposition 3.9,

\[ \| M_- \|_{L^p} \leq c \| (1 - C_{\tilde{v}_\theta^\#})^{-1} \|_{L^p \to L^p} \| F \|_{L^p} (\| \delta_1^\sigma \|_{L^\infty} + \| \delta_1^{-\sigma} \|_{L^\infty}) \| \delta_1^{-\sigma} \|_{L^\infty} \leq \frac{c}{(1 - \rho)^{3+5\beta}} \| F \|_{L^p}, \]

and hence by (1.13)

\[ (3.63) \quad \| (1 - C_{v_1})^{-1} \|_{L^p \to L^p} \leq \frac{c}{(1 - \rho)^{3+5\beta}}, \quad 2 < p < \infty. \]

Furthermore \( v_1 \) has the form

\[ \begin{pmatrix} 1 - |r|^2 \\ -r^\# e^{-i\theta} \delta^{-2} \end{pmatrix}, \quad \text{for } z > 0 \]

\[ \begin{pmatrix} 1 - |r|^2 \\ -r^\# e^{-i\theta} \delta^{-2} \end{pmatrix}, \quad \text{for } z < 0. \]

Hence by (the proof of) Proposition 3.2,

\[ (3.64) \quad \| (1 - C_{v_1})^{-1} \|_{L^2 \to L^2} \leq \frac{c}{1 - \rho}. \]

Note that for \( z > 0 \), \( v_1 \) can be written in the form

\[ (3.65) \quad v_1 = \begin{pmatrix} g & g(r^\# - r) e^{i\theta} \delta^2 \\ 0 & g^{-1} \end{pmatrix} \tilde{v}_\theta \begin{pmatrix} g & 0 \\ -g(r^\# - r) e^{-i\theta} \delta^{-2} & g^{-1} \end{pmatrix} \]

and for \( z < 0 \)

\[ (3.66) \quad v_1 = \begin{pmatrix} g & 0 \\ -g^{-1}(r^\# - r) e^{-i\theta} \delta^2 \delta^{-2} & g^{-1} \end{pmatrix} \tilde{v}_\theta \begin{pmatrix} g & g^{-1}(r^\# - r) e^{i\theta} \delta^2 \delta^{-2} \\ 0 & g^{-1} \end{pmatrix} \]

where

\[ (3.67) \quad g(z) = \begin{pmatrix} 1 - |r|^2 \\ 1 - |z|^2 \end{pmatrix} \frac{1}{2} \quad \text{for } z < 0, \]

\[ = 1 \quad \text{for } z > 0. \]
\[ \bar{v}_0 = \delta^{\sigma_3} v_0 \delta^{-\sigma_3} \] as in (3.7).

Extend \( v_1 \) to the complete, oriented contour \( \Gamma = \mathbb{R} \cup i\mathbb{R} \) on the RHS of Figure 1.35, by setting

\[
(3.68) \quad v^c(z) \equiv v_1(z), \quad z > 0 \\
\equiv v_1^{-1}(z), \quad z < 0 \\
\equiv I, \quad z \in i\mathbb{R}.
\]

The following fact is simple to prove (cf. [DZ4], [DZ5]). Suppose \( \Sigma \) is an oriented contour in \( \mathbb{C} \) with associated jump matrix \( v \), and suppose we reverse the orientation on some subset \( \Sigma' \subset \Sigma \). Denote the new contour by \( \hat{\Sigma} \) and set \( \hat{v} = v \) on \( \Sigma \setminus \Sigma' \), \( \hat{v} = v^{-1} \) on \( \Sigma' \). Then the operators \( C_\hat{v} \) and \( C_v \) on \( L^p(\hat{\Sigma}) \) and \( L^p(\Sigma) \) respectively, \( 1 < p < \infty \), are the same i.e. \( C_\hat{v} f = C_v f \) for all \( f \in L^p(\hat{\Sigma}) \equiv L^p(\Sigma) \). Together with the fact that \( v^c \equiv I \) on \( \Gamma \setminus \mathbb{R} \), this implies by (3.63), (3.64) that

\[
(3.69) \quad \| (1 - C_v) \|_{L^p(\Gamma)} \leq \frac{c}{(1 - \rho)^{\beta + \delta}}, \quad 2 < p < \infty,
\]

and

\[
(3.70) \quad \| (1 - C_v) \|_{L^2(\Gamma)} \leq \frac{c}{1 - \rho}
\]

for some constants \( c \).

Let \( \Omega_j = \{ z : (j - 1)\frac{\pi}{2} < \arg z < j\frac{\pi}{2} \}, 1 \leq j \leq 4 \), denote the four components of \( \mathbb{C} \setminus \Gamma \) with oriented boundaries

\[
\begin{align*}
\Sigma_1 &= \{ +i\infty \to 0 \rightarrow +\infty \} \\
\Sigma_2 &= \{ +i\infty \to 0 \rightarrow -\infty \} \\
\Sigma_3 &= \{ -i\infty \to 0 \rightarrow -\infty \} \\
\Sigma_4 &= \{ -i\infty \to 0 \rightarrow +\infty \}
\end{align*}
\]

respectively. Note that in the notation of Figure 1.35, \( \Omega_+ = \Omega_1 \cup \Omega_3 \) and \( \Omega_- = \Omega_2 \cup \Omega_4 \). Set

\[
(3.71) \quad v_2(z) = \bar{v}_0(z), \quad z > 0, \\
= \bar{v}_0^{-1}(z), \quad z < 0, \\
= I, \quad z \in i\mathbb{R}.
\]

With this notation, we have

\[
(3.72) \quad v^c = G_- v_2 G_+
\]

where

\[
(3.73) \quad G_+ = G_1 \quad \text{on} \quad \Sigma_1, \\
G_1(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \in i\mathbb{R}_+,
= \begin{pmatrix} \frac{g}{g(r^\sigma - \bar{r})e^{i\theta} \delta^{-2}} & 0 \\ -g(r^\sigma - \bar{r})e^{-i\theta} \delta^{-2} & g^{-1} \end{pmatrix}, \quad z > 0,
\]

and \( \bar{v}_0 = \delta^{\sigma_3} v_0 \delta^{-\sigma_3} \) as in (3.7).
Note that
\[
\parallel (3.78) \quad \text{dist}(\Gamma) \leq \parallel (3.76) \quad H_\text{we construct functions}
\]
\[
G_+ = G_3 \quad \text{on} \quad \Sigma_3,
\]
\[
G_3(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \in i\mathbb{R}_-,
\]
\[
= \begin{pmatrix} \overline{g^{-1}(r^\# - r^\bar{r})e^{-i\theta} \delta_+ \delta_-} & 0 \\ -g^{-1}(r^\# - r) & g \end{pmatrix}, \quad z < 0,
\]

\[
G_- = G_2 \quad \text{on} \quad \Sigma_2,
\]
\[
G_2(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \in i\mathbb{R}_+,
\]
\[
= \begin{pmatrix} g^{-1} & -g^{-1}(r^\# - r) & 0 \\ 0 & g^{-1}(r^\# - r) & g \end{pmatrix}, \quad z < 0,
\]

and
\[
(3.75) \quad G_+ = G_2 \quad \text{on} \quad \Sigma_4,
\]
\[
G_4(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{on} \quad i\mathbb{R}_-
\]
\[
= \begin{pmatrix} g & g(r^\# - r) \delta_+ \delta_- \\ 0 & g^{-1} \end{pmatrix}, \quad z > 0.
\]

Note that \( \parallel G_\pm \parallel_{L^\infty(\Gamma)} \leq \frac{1}{(1-\rho)^{\gamma/2}}. \)

Now extend \( \Gamma \) to the complete, oriented contour \( \Gamma_{\text{ext}} = \Gamma \cup \Gamma' \) on the RHS of Figure 1.38. Choosing \( \text{dist}(\Gamma, \Gamma') < 1 \) sufficiently small, we will show how to construct an invertible matrix function \( H \) with \( H \) and \( H^{-1} \) analytic and bounded in \( \mathbb{C} \setminus \Gamma_{\text{ext}} \), such that \( H_{\pm} = G_\pm \) is as small as desired on \( \Gamma \). Here \( H_{\pm} \) denote the boundary values of \( H(z) \) on \( \Gamma \) from \( \Sigma_2 \) respectively (cf. Figure 1.38). It then follows by perturbation theory that the bounds (3.69), (3.70) remain true for the Cauchy operator with \( \gamma^c = H_-v_2H_+ \) replaced by \( \gamma^H \equiv H_-v_2H_+ \). But then as \( H \) is piecewise analytic, it follows by the Conjugation Lemma 1.39 that similar bounds are true for \( v_2 \), and hence for \( \tilde{v}_\theta \), and hence for \( v_\theta \), as desired.

We will show how to construct \( H = H(z) \) in \( \Omega_2 \). The construction of \( H \) in \( \Omega_1, \Omega_3 \) and \( \Omega_4 \) is similar and left to the reader. As above, \( r \in H_1^1, \parallel r \parallel_{H^1} \leq \lambda, \parallel r \parallel_{L^\infty} \leq \rho < 1. \)

Set \( g(z) = 1 \) on \( i\mathbb{R}_+ \). Note that with this definition (cf. (3.67)) \( g(z) \) is continuous on \( \Sigma_2 \). Note also that the same is true for the function which equals \( -g^{-1}(r^\# - r) \delta_+ \delta_- \) on \( \mathbb{R}_- \) and equals zero on \( i\mathbb{R}_+ \). For some constant \( c \), any \( 0 < \varepsilon < 1 \), and \( \gamma > 0 \) satisfying
\[
(3.77) \quad \sqrt{\gamma} < \frac{\varepsilon(1-\rho)^{5/2}}{\epsilon \lambda},
\]
we construct functions \( H_2, h_2 \) with the following properties:

\[
(3.78) \quad \begin{align*}
(\text{i}) & \quad \parallel H_2^{-\sigma_3} - g^{-\sigma_3} \parallel_{L^\infty(\Sigma_2)} < \varepsilon \\
(\text{ii}) & \quad H_2(z) \to 1 \quad \text{as} \quad z \to \infty \quad \text{in} \quad \Sigma_2 \\
(\text{iii}) & \quad \parallel H_2^{-\sigma_3} - I \parallel_{L^3(\Sigma_2)} \leq \frac{\epsilon}{1-\rho} \\
(\text{iv}) & \quad H_2^{\sigma_3} \text{ is analytic in } \Sigma_{2, \gamma} = \{ z: \text{dist}(z, \Sigma_2) < \frac{\gamma}{2} \} \text{ and } \parallel H_2^{\sigma_3} \parallel_{L^\infty(\Sigma_{2, \gamma})} \leq \frac{\epsilon}{1-\rho} \\
(\text{i}') & \quad \parallel h_2 \parallel_{L^\infty(i\mathbb{R}_+)} + \parallel h_2 - (\gamma g^{-1}(r^\# - r) \delta_+ \delta_-) \parallel_{L^\infty(\mathbb{R}_-)} < \varepsilon \\
(\text{ii}') & \quad h_2(z) \to 0 \quad \text{as} \quad z \to \infty \quad \text{in} \quad \Sigma_2 \\
(\text{iii}') & \quad \parallel h_2 \parallel_{L^2(\Sigma_2)} \leq \frac{\epsilon \lambda}{1-\rho}.
\end{align*}
\]
(iv) \( h_2 \) is analytic in \( \Sigma_{2,\gamma} \) and \( \|h_2\|_{L^\infty(\Sigma_{2,\gamma})} \leq \frac{c}{(1-\rho)^{1/2}} \).

Define \( H_2 \) as follows. For \( z \in \Sigma_2, \)

\[
H_2(z) \equiv -\int_{\Sigma_2} g(s) \left( \frac{1}{s-(z+\gamma e^{i3\pi/4})} - \frac{1}{s-(z-\gamma e^{i3\pi/4})} \right) \frac{ds}{2\pi i}.
\]

Using the fact that \( 1 - \int_{\Sigma_2} \left( \frac{1}{s-(z+\gamma e^{i3\pi/4})} - \frac{1}{s-(z-\gamma e^{i3\pi/4})} \right) \frac{ds}{2\pi i} \), we obtain

\[
|H_2(z) - g(z)| \leq \int_{\Sigma_2} \frac{|g(s) - g(z)|\gamma ds|}{\pi |(s-z)^2 + i\gamma^2|},
\]

and inserting the estimate \( |g(s) - g(z)| \leq |s-z|^{1/2}\|g'\|_{L^2(\mathbb{R}_-)} \) for \( s, z \in \Sigma_2 \), we arrive at the estimate

\[
|H_2(z) - g(z)| \leq c\gamma\|g'\|_{L^2(\mathbb{R}_-)}.
\]

A simple calculation yields \( \|g'\|_{L^2(\mathbb{R}_-)} \leq \frac{\lambda}{(1-\rho)^{3/2}} \) and hence

\[
\|H_2 - g\|_{L^\infty(\Sigma_2)} \leq \frac{c\sqrt{\gamma\lambda}}{(1-\rho)^{3/2}}.
\]

Using the fact that \( |g(z)| \geq \sqrt{1-\rho} \) on \( \Sigma_2 \), a straightforward calculation shows that if we choose \( \gamma > 0 \) such that

\[
2\sqrt{2\gamma\sqrt{\gamma\lambda}} \leq \sqrt{1-\rho} \left( \frac{1-\rho}{\rho} \right)^{3/2} < \epsilon < 1,
\]

then \( |H_2(z)| > \frac{1}{2}\sqrt{1-\rho} \) and

\[
\|H_2 - g\|_{L^\infty(\Sigma_2)} \leq \frac{\epsilon}{\sqrt{2}}\|I\| = \epsilon.
\]

(recall Remark 1.56), which proves (i). The proof of (ii) is elementary and replacing \( g(z) \) by \( 1 \) in (3.80) we obtain by a standard calculation the bound \( \|H_2 - 1\|_{L^2(\Sigma_2)} \leq \epsilon\|g - 1\|_{L^2(\Sigma_2)} \). But then direct estimation shows that \( \|g - 1\|_{L^2(\Sigma_2)} \leq \lambda\sqrt{1-\rho} \) and so using the above fact that \( |H_2(z)| \geq \frac{1}{2}\sqrt{1-\rho} \), we obtain (iii),

\[
\|H_2 - I\|_{L^2(\Sigma_2)} \leq \frac{c\lambda}{1-\rho}.
\]

Finally, it is clear from (3.79) that \( H_2(z) \) extends to an analytic function in \( \Sigma_{2,\gamma} \), and inserting \( z + \mu e^{i3\pi/4}, -\frac{\gamma}{2} < \mu < \frac{\gamma}{2} \), we obtain as before the bound \( \|H_2(z + \mu e^{i3\pi/4}) - g(z)| \leq c\gamma\|g'\|_{L^2(\mathbb{R}_-)} \leq \frac{c\gamma\lambda}{(1-\rho)^{3/2}} \) for \( z \in \Sigma_2 \).

Using (3.83) this then leads to (iv),

\[
\|H_2\|_{L^\infty(\Sigma_{2,\gamma})} \leq \frac{c}{(1-\rho)^{1/2}}
\]

as desired.

We now construct \( h_2 \). Define \( b \) on \( \Sigma_2 \) by \( b(s) = 0 \) for \( s \in i\mathbb{R}_+ \) and \( b(s) = (r - r^\#)g^{-1}\Delta(s) \) for \( s \in \mathbb{R}_- \), where \( \Delta = \delta_+\delta_- \) as in (3.17). For \( z \in \Sigma_2 \), set

\[
h_2(z) = -\int_{\Sigma_2} b(s) \left( \frac{1}{s-(z+\gamma e^{i3\pi/4})} - \frac{1}{s-(z-\gamma e^{i3\pi/4})} \right) \frac{ds}{2\pi i}.
\]
As above, we obtain \( \|h_2 - b\|_{L^\infty(\Sigma_2)} \leq c \sqrt{\gamma} \|b\|_{L^2(\Sigma_\infty)} \). Now for \( z < 0 \), \( b' = (r' - (r')')g^{-1}\Delta + (r - r')(g^{-1}\Delta' \Delta + (r - r')g^{-1}\Delta') = I + II + III \). Clearly \( \|\Pi\|_{L^2(\Sigma_\infty)} \leq \frac{c}{(1 - \rho)^{1/2}} \). As \( g \) and \( g^{-1} \) have the same structure we obtain as above \( \|(g^{-1})'\|_{L^2(\Sigma_\infty)} \leq \frac{c}{(1 - \rho)^{1/2}} \) and so \( \|\Pi\|_{L^2(\Sigma_\infty)} \leq \frac{c}{(1 - \rho)^{1/2}} \). Now from (3.10), (3.17), \( \Delta = e^{-H((log(1 - r^2))\chi_{\mathbb{R}_-})} \), where \( H = -(C^+ + C^-) \) is the Hilbert transform and \( \chi_{\mathbb{R}_-} \) is the characteristic function for \( \mathbb{R}_- \). Hence

\[
\Delta'(z) = -\Delta(z) \frac{d}{dz} H((log(1 - |r|^2))\chi_{\mathbb{R}_-}) = \Delta(z) H \left( \frac{|r|^2}{1 - |r|^2} \chi_{\mathbb{R}_-} \right) - \frac{i\Delta \log(1 - |r(0)|^2)}{\pi}.
\]

Using the \( L^2 \) mapping properties of \( H \) (see (1.1) (ii)), the identity \( |\Delta| = 1 \), and the elementary bound \( |\log(1 - |r(0)|^2)| \leq \frac{|r(0)|^2}{1 - |r(0)|^2} \), we see that \( |\Delta'(z)| \leq I' + II' \) where \( \|I'\|_{L^2} \leq \frac{c}{\sqrt{\gamma}} \) and \( \|II'(z)| \leq \frac{c}{\sqrt{\gamma}}, z < 0. \) Thus \( \|\Pi\|_{L^2(\Sigma_\infty)} \leq \frac{c}{(1 - \rho)^{1/2}} + \frac{c}{\sqrt{\gamma}} \frac{1}{1 - \rho} \frac{1}{(\sqrt{\gamma})^{1/2}} \|\Sigma\|_{L^2(\mathbb{R})} \). But then by Hardy’s inequality, \( \|\text{r} - \text{r}'\|_{L^2(\Sigma_\infty)} \leq 2\|\text{r}' - (r')'\|_{L^2(\Sigma_\infty)} \leq 4\lambda \), and so \( \|\Pi\|_{L^2(\Sigma_\infty)} \leq \frac{c\lambda}{(1 - \rho)^{1/2}} \), and hence \( \|b'\|_{L^2(\Sigma_\infty)} \leq \frac{c\lambda}{(1 - \rho)^{1/2}} \). Choosing \( \frac{c\lambda}{(1 - \rho)^{1/2}} < \varepsilon \), we obtain (i)'. The proof of (ii)' is immediate and as \( \|h_2\|_{L^2(\Sigma_\infty)} \leq c\|b\|_{L^2(\Sigma_\infty)} \leq \frac{c\lambda}{(1 - \rho)^{1/2}} \), we obtain (iii)'. Again it is clear from (3.86) that \( h_2(z) \) extends to an analytic function in \( \Sigma_{2, \gamma} \), and inserting \( z + \mu e^{3\pi i/4}, -\frac{\gamma}{2} < \mu < \frac{\gamma}{2} \), we see that \( |h_2(z + \mu e^{3\pi i/4})| \leq c\|b\|_{L^2(\Sigma_\infty)} \leq \frac{c}{\sqrt{\gamma}} \). This proves (iv)' . Adjusting the constants \( c \) at various points in the above construction, we see that for \( 0 < \varepsilon < 1 \) and \( \gamma \) as in (3.77) we have obtained \( H_2, h_2 \) with the desired properties (3.78).

Define \( H \) as a piecewise analytic matrix function in \( \Sigma_\infty \) as follows: For \( z \in \Sigma_2 \cap \Sigma_{2, \gamma} \subset \Sigma_\infty \) (see cf. Figure 3.18) we set

\[
H(z) = \begin{pmatrix} H_2^{-1}(z) & h_2(z)e^{id} \\ 0 & H_2(z) \end{pmatrix}
\]

and for \( z \in \Sigma_2 \setminus \Sigma_{2, \gamma} \subset \Sigma_+ \) (cf. Figure 3.18) we set

\[
H(z) = I.
\]

Similar constructions taking into account the triangularity of \( G_{\pm} \) (see (3.73), (3.74), (3.76)) yield \( H \) in \( \Omega_1, \Omega_3 \) and \( \Omega_4 \) respectively; the details are left to the reader. We obtain an invertible matrix valued function \( H \) on \( \mathbb{C} \setminus \Gamma_\text{ext} \) with the following properties for some constant \( c > 0 \), and

\[
0 < \frac{c\lambda\sqrt{\gamma}}{(1 - \rho)^{3/2}} < \varepsilon < 1,
\]

\[
H(z) \text{ is analytic and bounded in } \mathbb{C} \setminus \Gamma_\text{ext}, \|H^{z_{\pm}}\|_{\mathbb{C} \setminus \Gamma_\text{ext}} \leq \frac{c}{(1 - \rho)^{1/2}},
\]

\[
\|H_{\pm} - G_{\pm}\|_{L^\infty(\Gamma)} < \varepsilon,
\]

where \( H_{\pm} \) denote the boundary values of \( H(z) \) on the oriented contour \( \Gamma \subset \Gamma_\text{ext} \).

\[
H(z) \rightarrow I \text{ uniformly as } z \rightarrow \infty \text{ in } \mathbb{C} \setminus \Gamma_\text{ext} \text{ and }
\]

\[
\|H_{\pm} - I\|_{L^2(\Gamma)}, \|H_{\pm}^{-1} - I\|_{L^2(\Gamma)} \leq \frac{c\lambda}{1 - \rho}.
\]

Note that in deriving these properties the signature table for \( \text{Re } i\theta \) in Figure 3.5 plays a crucial role.

Now if \( \frac{c\sqrt{\gamma}}{(1 - \rho)^{3/2}} < \varepsilon \), then
where (3.94) holds, (3.95), (3.96) are satisfied. It then follows by (1.44) that analytic and invertible in $\mathbb{R}$ if \( \sqrt{\gamma} \) by (3.71), (3.19), and (3.21). Thus for $2 < p < \infty$, by (3.69),

\[
\| (v^H - v^e)(1 - C_{v^e})^{-1} \|_{L^p(\Gamma) \to L^p(\Gamma)} \leq \frac{c\varepsilon}{\sqrt{1 - \rho}} \cdot \frac{c}{(1 - \rho)^{3+5\beta}} < \frac{1}{2}
\]

if $\varepsilon < c(1 - \rho)^{7/2 + 5\beta}$. Thus if

\[
\sqrt{\gamma} < \frac{c(1 - \rho)^{5/2}}{\lambda} \cdot c(1 - \rho)^{7/2 + 5\beta} = \frac{c(1 - \rho)^{6 + 5\beta}}{\lambda},
\]

we see by the second resolvent identity that $(1 - C_{v^H})^{-1}$ exists in $L^p(\Gamma)$ and

\[
\| (1 - C_{v^H})^{-1} \|_{L^p(\Gamma) \to L^p(\Gamma)} \leq \frac{c}{(1 - \rho)^{3 + 5\beta}}.
\]

Similarly for $\gamma$ satisfying (3.94) (adjust $c$ if necessary) we have from (3.70)

\[
\| (1 - C_{v^H})^{-1} \|_{L^2(\Gamma) \to L^2(\gamma)} \leq \frac{c}{1 - \rho}.
\]

We are now in a position to apply the Conjugation Lemma 1.39. In $\mathbb{C} \backslash \Gamma_{ext}$ (see Figure 1.38), set $R(z) = H(z)$ for $z \in \Omega_+$ and $R(z) = H(z) - \Omega_\infty z$ for $z \in \Omega_-$. Then $v_2 = R_2^{-1} v^H R_+$. Clearly $R(z)$ is analytic and invertible in $\mathbb{C} \backslash \Gamma_{ext}$ and $\| R \|_{L^\infty(\mathbb{C} \backslash \Gamma_{ext})}$, $\| R^{-1} \|_{L^\infty(\mathbb{C} \backslash \Gamma_{ext})} \leq \frac{c}{1 - \rho}$. As in (3.43), we have $\| (1 - C_{v_2})^{-1} \|_{L^2(\Gamma)} \leq \frac{c}{1 - \rho}$, and taking into account the discussion preceding (3.69), (3.70), we obtain $\| (1 - C_{v_2})^{-1} \|_{L^p(\Gamma)} \leq \frac{c}{1 - \rho}$. Also $\| v_2 \|_{L^\infty(\Gamma)} \leq c$, $\| v^H \|_{L^\infty} \leq \frac{c}{1 - \rho}$, $\| v_2 - I \|_{L^2(\Gamma)} \leq \lambda$, $\| v^H - I \|_{L^2(\Gamma)} \leq \frac{c\lambda}{(1 - \rho)^{3/2}}$, and provided (3.94) holds, (3.95), (3.96) are satisfied. It then follows by (1.44) that

\[
\| (1 - C_{v_2})^{-1} \|_{L^p(\Gamma) \to L^p(\Gamma)} \leq c^#,
\]

where

\[
c^# = c_{dist(\Gamma, \Gamma')} \| R \|_{L^\infty(\mathbb{C} \backslash \Gamma_{ext})} \| R^{-1} \|_{L^\infty(\mathbb{C} \backslash \Gamma_{ext})} \| (1 - C_{v^H})^{-1} \|_{L^p(\Gamma)} \| (1 - C_{v^H})^{-1} \|_{L^2(\Gamma)} \times \| (1 - C_{v_2})^{-1} \|_{L^2(\Gamma)} \| v_2 \|_{L^\infty(\Gamma)} \| v^H \|_{L^\infty(\Gamma)} (1 + \| v^H - I \|_{L^2(\Gamma)}^2 (1 + \| v_2 - I \|_{L^2(\Gamma)}^2)^2 (1 + \lambda \| v_2 - I \|_{L^2(\Gamma)}^2)^2
\]

\[
\leq c_{dist(\Gamma, \Gamma')} \frac{1}{(1 - \rho)^{3/2}} \frac{1}{(1 - \rho)^{3/2}} \frac{1}{(1 - \rho)^{3/2+5\beta}} \frac{1}{(1 - \rho)} \frac{1}{(1 - \rho)^2} \frac{1}{(1 - \rho)^2} \left( 1 + \frac{\lambda}{(1 - \rho)^{3/2}} \right)^2
\]

\[
\times (1 + \lambda)^2
\]

\[
\leq c_{dist(\Gamma, \Gamma')} \frac{(1 + \lambda)^4}{(1 - \rho)^{12 + 5\beta}},
\]

where $c_{dist(\Gamma, \Gamma')}$ depends on the distance $\frac{\gamma}{2}$ between $\Gamma$ and $\Gamma'$. By (2.14), $c_{dist(\Gamma, \Gamma')} = \frac{C_{\gamma}}{\gamma^{3/2 + 5\beta}}$, provided $\frac{\gamma}{2} \leq 1$. Choose

\[
\sqrt{\gamma} = \frac{c(1 - \rho)^{6 + 5\beta}}{1 + \lambda}.
\]
where the constant $c$ may be taken as the minimum of 1 and the constant on the RHS of (3.94). Then certainly $\frac{2}{\nu} > 1$, and we obtain the bound
\begin{equation}
(1 - C_{v_2})^{-1} \|_{L^p \to L^p} \leq \frac{c_p(1 + \lambda)^4}{(1 - \rho)^{12 + 3\beta}} \left( \frac{(1 + \lambda)^2}{(1 - \rho)^{12 + 10\beta}} \right) \frac{\delta + \frac{\rho}{\beta}}{1 + \frac{\rho}{\beta}},
\end{equation}
where $\beta' > 0$. But then reversing the discussion preceding (3.69), (3.70), we obtain for $2 < p < \infty$,
\begin{equation}
(1 - C_{v_2})^{-1} \|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq \frac{c_p(1 + \lambda)^{7 + \frac{2}{\beta}}}{(1 - \rho)^{30 + \frac{12}{p} + \beta'}}, \quad \beta' > 0.
\end{equation}
Using the formula
\begin{equation}
(1 - C_{v_2})^{-1} h = [(1 - C_{v_2})^{-1} (C^{-}(h(v_0 - I)\delta_+^{-\sigma_3}))] \delta_+^\sigma + h
\end{equation}
which follows from (1.13), (1.14), we obtain finally that for $2 < p < \infty$
\begin{equation}
(1 - C_{v_2})^{-1} \|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq \frac{c_p(1 + \lambda)^{7 + \frac{2}{\beta}}}{(1 - \rho)^{31 + \frac{12}{p} + \beta'}}, \quad \beta' > 0.
\end{equation}
By (3.3), we have
\begin{equation}
(1 - C_{v_2})^{-1} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq \frac{c}{1 - \rho}.
\end{equation}
This completes the proof of (1.49) in the case that $x = 0$, for all $t \in \mathbb{R}$. For general $x \neq 0$, and $z_0 = \frac{e^{ix}}{2}$, let $T_{z_0} f(z) = f(z + z_0)$. Then $T_{z_0}$ is an isometry from $L^p(\mathbb{R})$ onto $L^p(\mathbb{R})$ and a direct calculation shows that $T_{z_0} \circ C_{v_2} \circ T_{z_0}^{-1} = C_{v_{2,0,0}}$, where
\begin{equation}
v_{z_0,0}(z) = \left( \frac{1 - |r_{z_0}(z)|^2}{r_{z_0}(z)} e^{iz} \frac{r_{z_0}(z)e^{-iz}}{1} \right)
\end{equation}
and $r_{z_0}(z) = r(z + z_0)e^{iz_0}$. As $r_{z_0} \in H^1(\mathbb{R})$ and $\|r_{z_0}\|_{H^1(\mathbb{R})} \leq \|r\|_{L^1(\mathbb{R})} \leq \|r\|_{L^\infty(\mathbb{R})} \leq \rho < 1$, the general case now follows from the case $x = 0$. Thus (3.102), (3.103) are true for all $x, t \in \mathbb{R}$.
We can apply Riesz–Thorin interpolation to (3.102), (3.103). For any $k > 1$, $2 < p < \infty$, we find (denote the constant in the $L^2$ bound by $c_2$)
\begin{equation}
(1 - C_{v_2})^{-1} \|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq \frac{c_p^k c_2^{1-k}(1 + \lambda)^{(7 + \frac{2}{\beta})k}}{(1 - \rho)^{31 + \frac{12}{p} + \beta' k + \frac{2}{\beta} k (1 - k)}}
\end{equation}
where $\xi = \left( 1 - \frac{2}{\nu} \right) \left( 1 - \frac{\nu}{2} \frac{\rho}{\beta} \right)$. For example, for $p = 4$ which is a case of principal interest in [DZ5], given $\nu > 0$, we can choose $\beta'$ sufficiently small and $k$ sufficiently large so that
\begin{equation}
(1 - C_{v_2})^{-1} \|_{L^4(\mathbb{R}) \to L^4(\mathbb{R})} \leq \frac{c_p^4 c_2^{1-k}(1 + \lambda)^{(7 + \frac{2}{\beta})k}}{(1 - \rho)^{31 + \frac{12}{p} + \beta' k + \frac{2}{\beta} k (1 - k)}}
\end{equation}
for some (very large) constant $c_4'$, which should be compared with (3.102) for $p = 4$.

Remark.
In [DZ5], one needs a bound of the form $\| (1 - C_{v_2})^{-1} \|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq c$ for all $x \in \mathbb{R}$ and for all $t \geq t_0$, where the time $t_0$ is large. This is much simpler situation than considered in this paper: to prove this bound one still uses steepest descent methods, but the Conjugation Lemma 1.39 is not needed. We refer the reader to [DZ5] for the details.
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