Motion of Spin 1/2 Massless Particle in a Curved Spacetime.

II. Field Lagrangian Approach

A.T. Muminov

Ulugh-Beg Astronomy Institute, Astronomicheskaya 33, Tashkent 100052, Uzbekistan
amuminov2002@yahoo.com

Abstract

Earlier we obtained quasi-classical equations of motion of spin 1/2 massless particle in a curved spacetime on base of simple Lagrangian model [10]. Now we suggest an approach to derive the equations in framework of field theory. Noether theorem formulated in terms of Cartan’ formalism of orthonormal frames gives equations for current of spin of the field and tensor of stress-energy. It is shown that under eikonal approximation the above mentioned equations can be reduced to equations for worldline of the particle and equation of spin of the particle along the worldline. This way conformity between corpuscular considerations of spin 1/2 massless particle and approach in framework of spinor field theory in curved spacetime is demonstrated.

Keywords: Spin 1/2 massless particle; Noether theorem; Papapetrou equation.

1 Introduction

Complete description of motion of a non scalar wave in gravitational field is given by its covariant field equations. Under quasi-classical consideration when wave length is sufficiently less than typical scales of observations, propagation of the wave is substituted by motion of a particle. Besides, often it is assumed that such a particle moves along geodesic line. Frenkel was first who pointed to fact that spin changes trajectory of motion of particles in external field[1]. Motion of extended spinning particle in curved spacetime was studied by Papapetrou[2] and Dixon[3]. Similar problem was studied by Turakulov[4] by means of classic Hamiltonian formalism in approximation of spinning rigid body in tangent space. In the mentioned works it was shown that equation of motion differs from geodesic equation due to term which is contraction the curvature with spin and velocity. A number of attempts to describe motion of quantum particles with spin on base of Lagrangian models were made for last eight decades[5]. However, a satisfactory description was not obtained[6]. Nevertheless, studies both Maxwell and Dirac equations point to the fact that equations of motion might include contraction spin with the curvature[7, 8].

An approach to derive the equations of motion for spin 1 massive and massless quantum particles by means of classical Lagrange formalism are shown in our paper[6]. Recently, we extend area of the studies to spin 1/2 fields[9, 10] by constructing classical Lagrangian models describing motion of spin 1/2 massive and massless particles. It was shown there that quasi-classical equations
of motion of such particles have form analogous to Papapetrou equations obtained us earlier for spin 1 particles.

Since complete description of motion of a non scalar wave in curved spacetime is given by its covariant field equations it is evident that a confirmation of the results obtained in framework of classical Lagrange formalism should be given by consideration in framework of field theory. Such a confirmation for spin 1 massless field was achieved in our work[11], where a derivation of Papapetrou equations for photon on base of field variational principles was completed. Preliminary analysis showed that straightforward extent of approach[11] developed for photons to spin 1/2 fields meets serious mathematical difficulties. In our opinion, this is caused by significant difference between nature of fields with noninteger spin and spin 1 fields. Goal of present work is to modify a field approach[11] to derive quasi-classical equations of motion of particle with spin 1/2. Currently we confine our considerations by spin 1/2 massless field.

Spin 1/2 field is described by elements of spinor fibre bundle which consist of local spinor spaces. Determination of the spaces demands specifying an orthonormal frame in considered domain of spacetime. We introduce an orthonormal frames especial way which allows to obtain quasi-classical equations of motion in simplest form. For this end we specify field of comoving orthonormal frames such a way that 0-vectors \( \vec{n}_- \) are tangent to 0-curves of the way propagation, and complimentary to them 0-vectors \( \vec{n}_+ \) \( (\vec{n}_- , \vec{n}_+ = 1) \) specify oscillation of the wave. In turn, vectors \( \vec{n}_\alpha, \alpha = 1,2 \) describes polarization of the wave. Procedure of definition the frame is analogous to the one described in our work[11]. Local spinor spaces are constructed as follows. Dirac matrices \( \{ \gamma^a \} \) referred to the frame \( \{ \nu^a \} \) are introduced. The matrices generate local Clifford algebras referred to the frame. Besides, local Clifford algebra introduced this way specifies two local spinor spaces referred to the frame. These spaces are two spaces of spinor representations of the group \( SO(1,3) \) which are isomorphic to each other under Dirac conjugation[12,13]. Union of the local spinor spaces constitute spinor fibre bundle in considered domain of spacetime.

Unlike the approach shown in[11], this time we put constrains to spinor field after variation the field Lagrangian. For this we consider partial differential equations of covariant conservation of current of spin (CS) and stress-energy tensor (SET) given by Noether theorem formulated in terms of Cartan theory of orthonormal frames. Then, as before, we specify locally plane monochromatic (LP) spinor wave field. Consideration of zeroth order eikonal approximation of Dirac equation under constraints imposed clarifies structure of sought solution by extraction spinor fields of definite helicity. Applying the constraints to conservation equation for current of spin yields equation for ”-” component of the current which could be interpreted as equation for spin of the particle. As it was expected the equation become identical to equation for spin obtained under corpuscular consideration[10].

Analysis of conservation equation for the SET shows that under constraints imposed and some additional requirements which has evident physical meaning the equation is reduced to equation for generalized momentum (conjugated with coordinates \( x^i \)) obtained under corpuscular approach[10]. This way, conformity between corpuscular considerations of spin 1/2 massless particle and approach in framework of spinor field theory in curved spacetime has been shown.

2 Field formalism and Noether invariants

Spin 1/2 field is described by elements of spinor fibre bundle which consist of local spinor spaces. Determination of the spaces requires specifying an orthonormal frame in considered domain of
spacetime. Besides, condition of orthonormality should be understood in extended sense. Namely, if \( \{ \nu_a \} \) is such an orthonormal frame scalar products of the vectors:

\[
< \nu^a, \nu^b > = \eta^{ab},
\]

should constitute an invertible matrix \( (\eta^{ab}) \) with constant elements.

In order to describe spinor fields which are arguments of field Lagrangian we introduce Dirac matrices \( \{ \hat{\gamma}^a \} \) referred to the frame \( \{ \nu^a \} \). The matrices are constant in chosen frame and obey anticommutation rules as follows:

\[
\{ \hat{\gamma}^a, \hat{\gamma}^b \} = 2\eta^{ab}.
\] (1)

Algebraic span of Dirac matrices yields local sample of Clifford algebra. Union of the local Clifford algebras constitute fibre bundle of Clifford algebra in considered domain of spacetime.

Invertible elements \( R \) of Clifford algebra such that

\[
R^{-1} = \tilde{R},
\]

where \( \tilde{R} \) stands for Dirac conjugated matrix, constitute \( Spin(1,3) \) group \[12\]. There is an endomorphism \( R : SO(1,3) \rightarrow Spin(1,3) \) defined by formula:

\[
R_\gamma L^a R_L^{-1} = L_b^a \hat{\gamma}^b, \quad (L^a_b) \in SO(1,3).
\] (2)

Elements of local Clifford algebra are operators on two local spinor spaces referred to the frame. The spaces are local linear spaces of representation of group \( Spin(1,3) \) and \( SO(1,3) \). Elements of the local spaces \( \psi \in S \) and \( \bar{\psi} \in \bar{S} \) are \( 4 \times 1 \) and \( 1 \times 4 \) complex matrices accordingly. This way element \( L \) of Lorentz group \( SO(1,3) \) acts on spaces of representation of the group as follows:

\[
'\psi = R_L \psi, \quad '\bar{\psi} = \bar{\psi} R_L^{-1}, \quad \psi \in S, \quad \bar{\psi} \in \bar{S}.
\] (3)

Union of the local spinor spaces constitute spinor fibre bundle in considered domain of spacetime.

Covariant derivatives of the spinor fields take into account transformations of spinor group:

\[
D_a \psi = \bar{n}_a (\psi) - 1/4 \gamma_{abc} \hat{\gamma}^b \hat{\gamma}^c \psi,
\]

\[
D_a \bar{\psi} = \bar{n}_a (\bar{\psi}) + 1/4 \gamma_{abc} \bar{\psi} \hat{\gamma}^b \hat{\gamma}^c,
\] (4) (5)

where Cartan’ rotation 1-forms \( \omega^a_b = \gamma^a_{bc} \nu^c \) specify rotations of the frame.

Field Lagrangian of spin 1/2 massless field includes spinor variables \( \Psi(x^i), \bar{\Psi}(x^i) \) and their derivatives referred to the orthonormal frame \( \{ \nu^b \} \):

\[
\mathcal{L} = \mathcal{T}_{ab} < \nu^b, \nu^a >, \quad \mathcal{T}_{ab} = -\frac{i\hbar}{2} \left\{ \bar{\Psi} \hat{\gamma}_b D_a \Psi - D_a \bar{\Psi} \hat{\gamma}_b \Psi \right\}.
\] (6)

It should be noted that Dirac matrices which have invariant form in given frame also have zero covariant derivative (under taking into account their external index). Action of the field specified by the Lagrangian does not depend on choice of the frame:

\[
\mathcal{A} = \int L \epsilon, \quad \epsilon = \frac{1}{4!} \varepsilon_{abcd} \nu^a \wedge \nu^b \wedge \nu^c \wedge \nu^d.
\]

It is known that variational principles is somewhat universal tool to obtain equations for physical fields. For instance, under infinitesimal change of metric structure of spacetime described
by variations of orthonormal frame \{\nu^a\} we obtain Einstein equations. Particularly such a variation of the action of spin 1/2 field gives stress energy tensor (SET) of the field:

$$T_{ab} = -T_{ab} + T_{c}^{\ c} \eta_{ab},$$  \hspace{1cm} (7)

Besides SET of the spin 1/2 field is not symmetric \[15\], so does \(T_{ab}\) which for simplicity we also call SET.

Another type of variations which do not affect to the geometric structure of spacetime but changes system of observers is considered by Noether theorem. Due to Noether theorem each class of infinitesimal local transformations which left action of the field invariant leads to some differential equation for the field observables.

2.1 Noether theorem and Spin current

Infinitesimal local rotations of field of the orthonormal frames \{\nu^b\}:

$$\delta \nu^a = -\varepsilon^{a}_{\ b} \nu^b, \quad \varepsilon_{ab} + \varepsilon_{ba} = 0, \hspace{1cm} (8)$$

leave action of the field invariant, since the one does not depend on choice of the frames. Thus, owing to Noether theorem, differential equation:

$$D_c [S_{ab}^{\ c}] = T_{[a \ b]}, \hspace{1cm} (9)$$

including covariant divergence of CS\[15\]:

$$S_{abc} = i \hbar \bar{\Psi} \{ \hat{\gamma}^c, \hat{\gamma}^a \hat{\gamma}^b \} \Psi, \hspace{1cm} (10)$$

and commutator of SET of the field should be satisfied.

2.2 Coordinate variation

Infinitesimal vector field \(\vec{\varepsilon}\) drags coordinate hyper-surfaces \(\{x^i\}\) onto \(\{y^i\}\) and elements of orthonormal covector and vector frames \{\vec{n}_a\}, \{\nu^b\} onto dragged frames \{'\vec{n}_a\}', \{'\nu^b\}'. Such a class of local infinitesimal coordinate transformations leaves action invariant and lead to differential equation for covariant divergence of SET as follows:

$$D_b [T^{cb}] = 1/2 R_{ab}^{\ cd} S^{ab}_{\ cd}. \hspace{1cm} (11)$$

Proceeding from (10) and anti-commutation rules for Dirac matrices, we find that CS is double antisymmetric over the indexes:

$$S^{abc} = S^{bca} = S^{cab}. \hspace{1cm} (11)$$

Owing Bianchi identity this reduces RHS of the equation for SET divergence to zero:

$$D_b [T^{cb}] = 0. \hspace{1cm} (12)$$
3 Canonical frame and structure of spinor wave

We confine our consideration by spinor wave fields whose propagation in some 4-tube in a curved spacetime can be described as motion of particle with spin. Namely, the wave propagates along some congruence of 0-curves inside 4-tube representing 0-worldline of the particle. The wave in 4-tube is determined by field of orthonormal frames (associated with the congruence) and components of spinors referred to the frame. Since the wave admits corpuscular description we assume that in first order eikonal approximation interaction of the spinor field with gravitation expresses shape of the 0-curves (to be exact dynamic of the field of orthonormal frames), whereas components of spinors satisfy simple equations similar to analogous equations in flat spacetime. It should be noted that such situation is provided by choice of suitable orthonormal frame. Procedure of choice of such a frame for electromagnetic wave propagating in curved spacetime was described in our earlier work [11]. However it is relevant to explain analogous procedure for the case in question more detail. First of all we consider plane wave of massless spinor field in Minkowski spacetime.

3.1 Plane wave in Minkowski spacetime

Flat space time admits cartesian coordinates \( \{x^i\} \) and by suitable transformation of the coordinates we can put that direction of \( x^3 \) coordinate axis is coincide with direction of propagation of the wave, whilst \( x^0 = ct \) is time coordinate. This way we obtain an orthonormal vector frame \( \{\vec{n}_i\} = \{\partial / \partial x^i\} \). In turn, dual to the frame covector frame \( \nu^a \) specifies spinor fiber bundle in the space by map \( T^* \to CL(1,3) \) which specified by its images on elements of frame of cotangent space: \( \nu^a \to \hat{\gamma}^a \). The images are Dirac matrices. In chiral representation they take matrix form as follows [12]:

\[
\hat{\gamma}^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\gamma}^\mu = \begin{pmatrix} 0 & \tilde{\sigma}^\mu \\ -\tilde{\sigma}^\mu & 0 \end{pmatrix}, \quad \mu = 1, 2, 3; \quad \hat{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};
\]

(13)

where \( \tilde{\sigma}^\mu \) are Pauli matrices. Since we consider spinor plane wave \( \Psi \) propagating in \( \vec{n}_3 \) direction, massless Dirac equation for spinor becomes:

\[
0 = \hat{\gamma}^a \vec{n}_a \circ \Psi = (\hat{\gamma}^0 \vec{n}_0 + \hat{\gamma}^3 \vec{n}_3) \circ \Psi.
\]

The equation gets simplified if we formulate the one in terms of null vectors:

\[
\nu^+ = \nu^0 - \nu^3, \quad \nu^- = 1/2(\nu^0 + \nu^3);
\]

\[
\vec{n}_+ = 1/2(\vec{n}_0 - \vec{n}_3), \quad \vec{n}_- = \vec{n}_0 + \vec{n}_3.
\]

Actually, frames \( \{\vec{n}_\pm, \vec{n}_\alpha\} \) and \( \{\nu^\pm, \nu^\alpha\}, \alpha = 1, 2 \) are dual to each other and orthonormal in sense:

\[
< \nu^\pm, \nu^\pm > = < \nu^\pm, \nu^\alpha > = < \nu^1, \nu^2 > = 0, \quad < \nu^+, \nu^- > = - < \nu^\alpha, \nu^\alpha > = 1.
\]

(14)

So, the map between Grassman and Clifford algebras may be equivalently specified by images of generalized Dirac matrixes referred to frame \( \{\nu^\pm, \nu^\alpha\} \):

\[
\nu^+ \to \hat{\gamma}^+ = \hat{\gamma}^0 - \hat{\gamma}^3, \quad \nu^- \to \hat{\gamma}^- = 1/2(\hat{\gamma}^0 + \hat{\gamma}^3);
\]
whilst $\hat{\gamma}^\alpha$ remain unchanged. Now Dirac equation for plane spinor wave is written as:

$$(\hat{\gamma}^+ \vec{n}_+ + \hat{\gamma}^- \vec{n}_-) \circ \Psi = 0.$$ 

Since dependence of the spinor plane wave on $x^3, t$ variables is given by their combination $ct - x^3$, i.e.

$$\Psi = \exp(i\varphi)\Psi_c, \quad \varphi = \omega(t - x^3), \quad \Psi_c = \text{const}$$ \hspace{1cm} (15)

only derivative along vector $\vec{n}_+$ contributes to the equation for plane spinor wave which due to (15) is reduced to simple matrix equation:

$$\hat{\gamma}^+ \vec{n}_+ \circ \Psi = i\omega \hat{\gamma}^+ \Psi = 0.$$ 

This equation gives us solution of massless Dirac equation in form of plane spinor wave:

$$\Psi_c = \begin{pmatrix} \chi \\ \phi \end{pmatrix}, \quad (1 - \hat{\sigma}^3)\phi = (1 + \hat{\sigma}^3)\chi = 0;$$

$$\phi = \text{const} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi = \text{const} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. $$

Besides solutions $\begin{pmatrix} 0 \\ \phi \end{pmatrix}$ and $\begin{pmatrix} \chi \\ 0 \end{pmatrix}$ describes waves with helicity $\pm 1$ accordingly.

### 3.2 Spinor fields and Locally plane wave in curved spacetime

Spin 1/2 field is described by elements of spinor fibre bundle which consist of local spinor spaces. Determination of the spaces demands specifying an orthonormal frame in considered domain of spacetime. Since the wave field propagates along some congruence of 0-curves, it is convenient to specify field of comoving orthonormal (in sense (14)) frames such a way that 0-vectors $\vec{n}_-$ are tangent to the 0-curves, and complimentary to them vectors $\vec{n}_+$ specify oscillation of the wave:

$$\Psi_{+, \alpha} = i\omega \Psi, $$ \hspace{1cm} (16)

where frequency of the oscillations $\omega$ characterizes value of energy $E = \hbar \omega$ in the frame [11]. In turn, vectors $\vec{n}_\alpha, \alpha = 1, 2$ describes polarization of the wave. Vector frame $\{\vec{n}_a\}, a = \pm, 1, 2$ determines dual to it covector frame $\{\nu^a\}$.

It should be noted that pair of vectors $\vec{n}_\pm$ is defined with accuracy up to arbitrary Lorentzian rotations $SO(1, 1)$. Rotating vectors $\vec{n}_\pm$ such a way that value of energy $E$ in the frame becomes a constant, we define canonical frame so that vector $\vec{n}_-$ tangent to the 0-curves becomes an analog of canonical velocity. The equations of motion in quasi-classical approximation take their simplest form in such a canonical frame.

Since interaction the wave with gravitation is expressed by shape of the 0-curves (i.e. by rotations of the canonical frame along $\vec{n}_-$) we expect that components of spinor have no changes in spatial directions inside the 4-tube:

$$\Psi_{+, \alpha} = 0.$$ \hspace{1cm} (17)
3.3 Eikonal approximation

Essence of the approximation is assumption that the derivative $\vec{n}_+ \Psi = i\omega \Psi$ is a dominating derivative. Accordingly (17) this means:

$$|\gamma_{abc}| \ll \omega, \quad |\Psi_\pm| \ll \omega|\Psi|.$$  \hspace{1cm} (18)

Substitution this requirements to Dirac equation gives:

$$0 = \gamma^a D_a \Psi = i\omega \gamma^+ \Psi + o(\omega \Psi) = 0.$$

Thus in zeroth order of eikonal approximation we obtain:

$$\gamma^+ \Psi = 0, \quad \bar{\Psi} \gamma^+ = 0.$$  \hspace{1cm} (19)

This condition clarifies structure of LP spinor wave.

3.4 Structure of the spinor field

Condition (19) allows to recognize structure of components of locally plane spinor wave in the frame. Actually, let us consider matrix

$$\gamma^5 = -i/2 \gamma^1 \gamma^2 [\gamma^+, \gamma^-] = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$  \hspace{1cm} (20)

If we put spinor $\Psi$, satisfying (19) on the right side of matrix $\gamma^5$, due to (20) we obtain:

$$\gamma^5 \Psi = -i\gamma^1 \gamma^2 \Psi = -\hat{H} \Psi, \quad \hat{H} = \left( \begin{array}{cc} \sigma^3 & 0 \\ 0 & \sigma^3 \end{array} \right), \quad \hat{H}^2 = \hat{1};$$  \hspace{1cm} (21)

where $\hat{H}$ is helicity operator. This shows that locally plane spinor waves may be eigenfunctions of helicity operator according to eigenvalues $\pm 1$. Owing to definition, only non zero component of CS $S^{12-}$ (with accuracy up to cyclic permutation of the indexes) can be expressed as averaged value of operator of helicity $\hat{H}$ (multiplied to $\hbar/2$) in state described by spinor $\Psi$:

$$S^{12-} = i\hbar/2 \bar{\Psi} \gamma^- \gamma^1 \gamma^2 \Psi = i\hbar/4 \Psi^\dagger \gamma^1 \gamma^- \gamma^1 \gamma^2 \Psi = i\hbar/2 \Psi^\dagger \gamma^1 \gamma^2 \Psi = \hbar/2 \Psi^\dagger \hat{H} \Psi.$$

As usual we introduce projectors:

$$\hat{P}_\pm = \hat{1} \mp \hat{H} \rightarrow \hat{1} \mp \gamma^5 \Rightarrow \hat{P}_\pm^2 = \hat{1} \pm 2\hat{H} + \hat{H}^2 = 2\hat{P}_\pm, \quad \hat{P}_+ \hat{P}_- = \hat{P}_- \hat{P}_+ = \hat{1} - \hat{H}^2 \equiv 0,$$

$$\hat{H} \hat{P}_\pm = \hat{H} \mp \hat{1} = \pm \hat{P}_\pm, \quad 1/2(\hat{P}_+ + \hat{P}_-) = \hat{1}.$$

Two last equations shows that arbitrary spinor $\Psi$ can be presented as sum of spinors $\hat{P}_\pm \Psi$ of definite helicity $\pm$. Now let $\Psi$ is spinor representing LP spinor wave, thus (19) and Dirac equation is satisfied. Hence, we obtain:

$$\gamma^a D_a \hat{P}_\pm \Psi = \gamma^a D_a (\hat{1} \mp \gamma^5) \Psi = \gamma^a (\hat{1} \mp \gamma^5) D_a \Psi =$$

$$= (\hat{1} \pm \gamma^5) \gamma^a D_a \Psi = \hat{P}_\pm \gamma^a D_a \Psi \equiv 0,$$
where we used (21) and fact that matrix $\hat{\gamma}^5$ anticommutates with each of Dirac matrices. This way it is shown that each of component $\Psi_\pm = 1/2\bar{P}_\pm\Psi$ (of helicity $\pm 1$) of spinor $\Psi = \Psi_+ + \Psi_-$ representing LP spinor wave also satisfies Dirac equation. Since LP spinor wave admits description in terms of corpuscular theory which considers spin $1/2$ massless particle of definite helicity, hereafter we will consider only LP spinor waves $\Psi = \Psi_\pm$ of definite helicity:

$$\hat{H}\Psi = H\Psi, \quad H = \pm 1.$$  (22)

Thus, accordingly to (21) we can recognize structure of spinor wave in given representation:

$$\Psi_+ = \begin{pmatrix} 0 \\ \phi \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} \chi \\ 0 \end{pmatrix}, \quad \dot{\sigma}^3\phi = \phi,$$

$$\dot{\sigma}^3\chi = -\chi, \quad \phi = F\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi = F\begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$\Psi = F\Psi_c, \quad \Psi_c = \Psi_\pm/F;$$

where $F$ some scalar. Besides, accordingly to our assumption that $\vec{n}_+\Psi = i\omega\Psi$ is dominating derivative we put:

$$F = \exp(i\varphi)f, \quad \varphi_+ = \omega = \text{const}, \quad f_+ = 0, \quad \text{Im}(f) = 0.$$  (24)

Moreover, accordingly (17) we assume:

$$f_{\alpha} = \varphi_{,\alpha} = 0.$$  (25)

Structure of spinor allows to find out view of some observables:

$$\bar{\Psi}\hat{\gamma}^a\Psi = 0, \quad \bar{\Psi}\hat{\gamma}^-\Psi = \Psi^\dagger\Psi = |f|^2, \quad \bar{\Psi}\hat{\gamma}^+\Psi = 0 \Rightarrow \bar{\Psi}\hat{\gamma}^a\Psi = \delta^-|f|^2.$$  (26)

Let’s apply this property to calculation derivative of “-” component of the ”velocity” [14] of spinor field along vector $\vec{n}_-$:

$$\vec{n}_-(\bar{\Psi}\hat{\gamma}^-\Psi) = \vec{n}_a(\bar{\Psi}\hat{\gamma}^a\Psi) - \vec{n}_a(\bar{\Psi}\hat{\gamma}^a\Psi) - \vec{n}_+(\bar{\Psi}\hat{\gamma}^+\Psi) =$$

$$= D_a\bar{\Psi}\hat{\gamma}^a\Psi + \bar{\Psi}\hat{\gamma}^a D_a\Psi - \hat{\gamma}^a_{ac}\bar{\Psi}\hat{\gamma}^c\Psi = -\text{Div}(\vec{n}_-\bar{\Psi}\hat{\gamma}^-\Psi), \quad \text{i.e.}:$$

$$\vec{n}_-|f|^2 + \text{Div}(\vec{n}_-)|f|^2 = 0.$$  (27)

### 3.5 Structure of stress-energy tensor

Structure of spinor wave allows to clarify structure of the SET. Actually, due to (23,24,25) and (28) after simple calculations we find out:

$$T_{ab} = E\eta_{a-}(\bar{\Psi}\hat{\gamma}_b\Psi) + 1/2\gamma_{a\delta}S^\delta_{-b} = Ef\eta_{a-} - 1/2\gamma_{a\delta}S^\delta_{-b},$$

i.e. $T_{a\beta} = \gamma_{a-\beta}S^\beta$, $T_{a-} \equiv 0$, $T_{a+} = Ef\eta_{a-} + 1/2\gamma_{acd}S^d_{-b}$.  (28)

### 4 Equations of motion

In this section we derive equations of motion describing propagation of spin $1/2$ massless wave field in corpuscular terms from differential equations for Noether invariants.
4.1 Equation for spin

Now let’s consider nonzero components $S^{abc}$ of CS. Due to definition and indexes $a, b, c$ should differ each other. Owing to (19), it is seen that no one of them may be equal to ”+”. So only possibility is:

$$S^{abc} \neq 0 \Rightarrow \{a, b, c\} = \{1, 2, -\}. \quad (29)$$

Since the wave is propagating in $\vec{n}_-$ direction it is essential to define spin of the particle describing propagation of the wave as follows:

$$S^{ab} = S^{ab-}. \quad (30)$$

It is seen that only nonzero components of the spin are $S^{12} = -S^{21}$.

Starting from conservation law for current of spin (9) we derive equation for nonzero components of the spin:

$$T^{[\delta\varepsilon]} = -T^{[\delta\varepsilon]} = D_a(S^{\delta\varepsilon})_a = D_a(S^{\delta\varepsilon})_a + \gamma^a \cdot S^{\deltaabc} =$$

$$= D_- (S^{\delta\varepsilon}) + D_+ (S^{\delta\varepsilon})^+ + D_\alpha (S^{\delta\varepsilon})^\alpha + D\text{iv}(\vec{n}_-) S^{\delta\varepsilon} =$$

$$= D_- (S^{\delta\varepsilon}) + D\text{iv}(\vec{n}_-) S^{\delta\varepsilon} + \gamma^a [\delta^a S^{\varepsilon\alpha}],$$

where indexes which stay outside brackets at symbol of covariant derivation $D$ indicate that under calculation the derivative they should not be taken into account. Such a manner we obtain:

$$D_- (S^{\delta\varepsilon}) + D\text{iv}(\vec{n}_-) S^{\delta\varepsilon} = -\gamma^a_- [\delta^a S^{\varepsilon\alpha}] - 1/2 \gamma^a_- S^{\delta\varepsilon}. \quad (31)$$

Let $\delta = 1$, $\varepsilon = 2$. Noting that all nonzero elements of $S^{abc}$ are elements with indexes $\{a, b, c\} = \{1, 2, -\}$ we see that RHS of the above equation is zero:

$$-\gamma^1_- S^{21} + \gamma^2_- S^{12} - \gamma^1_- S^{12} + \gamma^2_- S^{21} = 0.$$

This gives us:

$$D_- (S^{\delta\varepsilon}) + D\text{iv}(\vec{n}_-) S^{\delta\varepsilon} = 0. \quad (32)$$

It is convenient to introduce normed spin:

$$\sigma^{ab} = S^{ab}/(\bar{\Psi}\gamma^- \Psi), \quad (33)$$

which become an analog of spin 1/2 particle. Due to (27) equation for normed spin simplifies:

$$D_- (\sigma^{\delta\varepsilon}) = 0. \quad (34)$$

The equation coincides with analogous equation for spin 1/2 massless particle obtained us earlier under consideration Lagrange model of the particle [10].

4.2 Equation for generalized momentum

Since SET of quantum wave field has components which are analogs of momentum of classic particle equation of continuity for SET is expected to be reduced to equation for momentum of classic particle under some conditions. From the other hand, if we obtain exact solution of Dirac equation and substitute it into the equation for SET continuity we obtain trivial identity $0 = 0$. But essence of our approach is to avoid solving field equation and to extract necessary information
about worldline of propagation of the wave from the equations for observables (SC and SET). We put suitable requirements to the field representing motion of the particle. That allowed to clarify structure of the field. When we substitute such a field into the equation for SET this time we expect to obtain an equation for components of SET which can be interpreted as equation for momentum of the particle. The equation should coincide with obtained us earlier equation for generalized momentum of spin 1/2 massless particle derived in framework of classic Lagrange formalism [10]. Due to the above mentioned arguments let us analyze continuity equation for SET:

\[ 0 = D_b(T_a^b) - D_b(\delta_a^b T_c^c) = D_b(T_a^b) - D_a(T_c^c) = \]

\[ = D_b(T_a^b) - \gamma_{bc}^b T_a^c - D_a(T_c^c) - \gamma_{ab}^c T_c^b = \]

\[ = D_-(T_a)^- + \gamma_{b-}^b T_a^- - D_a(T_\gamma)^- + \Delta = \]

\[ \Delta = D_\beta(T_a)^{-} + \gamma_{b\delta}^b T_a^{-} - D_a(T_\delta)^{-} - \gamma_{ab}^c T_c^b. \] (34)

Here we write components \( T_a^- \) of SET which can be interpreted as components of momentum of classic particle separately. To be exact let us define analog of generalized momentum of spinor wave field as:

\[ P_a = T_a^- / |f|^2 = E \eta_{a-} + 1/2 \gamma_{ade} \sigma^{de}. \] (35)

It is to note that such an analog of the momentum has the same structure as generalized momentum derived in our later work [10] under consideration Lagrange model of spin 1/2 massless particle. Thus due to (27)

\[ D_- (T_a)^- + Div (\bar{n}_-) T_a^- = |f|^2 D_- (P_a) \] (34)

After this equation (34) can be rewritten as:

\[ - |f|^2 \{ D_- (P_a) - D_a (P_-) \} = 2 |f|_a P_+ + \]

\[ + [D_\beta (T_a)^{-} + \gamma_{b\delta}^b T_a^{-} - D_a (T_\delta)^{-} - \gamma_{ab}^c T_c^b]. \] (36)

It can be demonstrated that LHS of the equation coincides with LHS of equation for generalized momentum [10]. Besides, under \( a = - \) the LHS turns to equality. If \( a \neq - \), due to [25, 24] \( D_a |f|^2 = 0 \) and all the rest terms of RHS of the above equation can be neglected under some additional requirements. Actually, if we consider locally plane wave propagated along some congruency of 0-worldlines it is essentially to assume that there is some central zone in area of the wave propagating where infinitesimally neighboring worldlines are locally parallel to each other. So that field of vectors \( \bar{n}_- \) has vanishing rotations in directions \( \bar{n}_a \), \( a \neq - \), i.e. rotation coefficients \( \gamma_{a-}^b \) are negligible. If we accept such requirements, accordingly (28), elements \( T_c\beta \) of SET also become negligible and RHS of (36) certainly vanishes.

It should be noted that even the above applied requirements has obvious physical meaning they may be not indisputable from point of view of geometry of spacetime. Nevertheless we daresay that equation

\[ D_- (P_a) - D_a (P_-) = 0, \] (37)

obtained under the requirements represents correctly a behavior of worldline of propagation of the wave. Difficulties appeared under derivation the equation, in our opinion, conditioned by unreality notion of monochromatic wave. For quite correct derivation the equations of motion of quantum particle with non integer spin in curved space time in framework of field theory it is necessary to consider wave packet characterizing by definite spectrum of energies. However, that
is much more difficult procedure requires calculating average momentum, energy and trajectory of the packet. This is why under first stage of consideration the problem we prefer to have deal with monochromatic wave.

Anyhow, more or less correct we obtain equation (37) for analog of generalized momentum (35). Let us rewrite its second term in explicit form:

\[ D_a(P_-) = 1/2 \left( \gamma_{-cd} \sigma^{cd} \right)_a - \gamma_{a-} \cdot P_b = 1/2 \gamma_{-cd,a} \sigma^{cd} - \gamma_{a-} \cdot P_b, \]

where we take into account the fact that coefficients \( \sigma^{cd} \) are constants. After this, finally, equation (37) takes form as follows

\[ D_-(P_a) + \gamma_{a-} \cdot P_b - 1/2 \gamma_{-cd,a} \sigma^{cd} = 0. \] (38)

It is seen that form of the equation coincides with equation for generalized momentum obtained us earlier in work [10] under consideration Lagrangian model for spin 1/2 massless particle. Thus, exactly the same manner, this equation can be reduced to Papapetrou equation for trajectory of the particle derived in our paper [10]:

\[ E \cdot D_-(\vec{n}_-) = 1/2 R_{\delta \epsilon -}^{\ a} \sigma^{\delta \epsilon} \vec{n}_a, \] (39)

where

\[ \Omega_{c.}^{\ d} = d\omega_{c.}^{\ d} + \omega_{e.}^{\ d} \wedge \omega_{c.}^{\ e} = 1/2 R_{c. ab}^{\ d} \nu^{a} \wedge \nu^{b} \]

is 2-form of curvature.

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