Propagation of infinitely narrow $\delta$-solitons

V. G. Danilov
Moscow Technical University of Communication and Informatics, Russia
danilov@amath.msk.ru

V. M. Shelkovich
St.-Petersburg State Architecture and Civil Engineering University, Russia
shelkv@svm.abu.spb.ru

Abstract
We construct a definition of the weak solution to KdV type equations with small dispersion admitting the zero dispersion limit for soliton-like solutions. Using this definition, we obtain a system of equations (the limit problem as the dispersion tends to zero) that describes the soliton dynamics.

1 Introduction and basic results
1. It is well known that the Korteweg-de Vries (KdV) equation
\[ L_{KdV}[u] = u_t + (u^2)_x + \varepsilon^2 u_{xxx} = 0 \] (1.1)
has the one-soliton solution
\[ u(x, t, \varepsilon) = \frac{3v}{2} \text{ch}^{-2}\left(\frac{\sqrt{v}}{2}(x - vt) / \varepsilon\right), \quad x \in \mathbb{R}, \] (1.2)
where $v$ is the soliton velocity. The pointwise limit as $\varepsilon \to +0$ of solution (1.2) to the KdV equation is the discontinuous function $\frac{3v}{2} \chi(x - vt)$, where $\chi(\xi) = 1$, if $\xi = 0$ and $\chi(\xi) = 0$ if $\xi \neq 0$. The weak asymptotics (1.2) as $\varepsilon \to +0$, up to $O_{D'}(\varepsilon^2)$, becomes the infinitely narrow $\delta$-soliton
\[ u_{\varepsilon}(x, t) = A\varepsilon \delta(x - vt), \quad \varepsilon \to +0, \quad A = \frac{3v}{2} \int \text{ch}^{-2}(\xi) \, d\xi = 6\sqrt{v}, \] (1.3)
and $\delta(x)$ is the Dirac delta function. Here and in what follows $\int$ denotes an improper integral from $-\infty$ to $+\infty$. By $O_{D'}(\varepsilon^\alpha)$ we denote a distribution from $D'(\mathbb{R})$ such that for any test function $\varphi(x) \in D$
\[ \langle O_{D'}(\varepsilon^\alpha), \varphi(x) \rangle = O(\varepsilon^\alpha), \] and $O(\varepsilon^\alpha)$ is understood in the ordinary sense.

We stress once more that here all generalized functions (distributions) are treated as functionals on the space $D(\mathbb{R})$ and these functionals depend on the other variables as on parameters.

It follows from (1.2), (1.3) that we have $u(x, t, \varepsilon) = O_{D'}(\varepsilon)$ as $\varepsilon \to +0$ and $\varepsilon^2 u_{xxx} = O_{D'}(\varepsilon^3)$. Therefore, the limit expression (1.3) was interpreted by V. P. Maslov and V. A. Tsupin [1]-[2], V. P. Maslov and G. A. Omelyanov [3]-[4] as an asymptotic up to $O_{D'}(\varepsilon^2)$ general solution of the Hopf equation
\[ L_H[u] = u_t + (u^2)_x = 0, \] (1.4)
which is the limit problem for the KdV equation. In the same works the corresponding generalized Hugoniot conditions, of the type of those for the shock wave front, were obtained.
It is easy to see that in the framework of the above-mentioned approach, the exact solution (1.3) to the KdV equation (1.1) satisfies the Hopf equation (1.4) in the following sense:

$$L_H[u] = O_D(\varepsilon^2),$$

because the following equality holds:

$$L_{KdV}[u] - L_H[u] = O_D(\varepsilon^2).$$  (1.6)

If, instead of expression (1.2) which is the exact solution of the KdV equation and approximates the weak asymptotics (1.3), we consider the function

$$\tilde{u}(x, t, \varepsilon) = A\omega\left(\frac{x - vt}{\varepsilon}\right),$$  (1.7)

where \(\omega(z) \in C^\infty(\mathbb{R})\) has a compact support or rapidly decreases as \(|z| \to \infty\), \(\int \omega(z) \, dz = 1\), then:
1) expression (1.7) also has asymptotics (1.3) in the sense of \(D'\) as \(\varepsilon \to +0\) (see Section 3 for details);
2) substituting (1.7) into the Hopf equation (1.4), provided that a certain correlation between the constants \(v\) and \(A\) is true (the generalized Hugoniot condition), we have \(L_H[\tilde{u}] = O_D(\varepsilon^2)\);
3) and \(u(x, t, \varepsilon) = \tilde{u}(x, t, \varepsilon) = O_D(\varepsilon^2)\).

Therefore, an asymptotics up to \(O_D(\varepsilon^2)\), i.e., an infinitely narrow \(\delta\)-soliton-type solution of the KdV equation (or the Hopf equation which is the limit problem of the KdV equation), can be sought starting not from the exact solution (1.2) of the KdV equation (which is a regularization of the Hopf equation) but from an ansatz of the form (1.7) substituted directly into the Hopf equation.

Generalizing (1.3), one can seek the solution in the form

$$u^*(x, t, \varepsilon) = u_0(x, t) + g(t)\omega\left(\frac{x - \phi(t)}{\varepsilon}\right),$$

by substituting this singular ansatz into the Hopf equation.

However, as we shall see in Section 3.1, such an ansatz results in the solution with constant amplitude \(g = \text{const}\) of the soliton for \(u_0 \neq \text{const}\), which contradicts the well-known results about the soliton behaviour. Therefore, generalizing formulae (1.2), (1.7), (1.3), we can attempt to construct an asymptotic solution to the KdV equation (or the Hopf equation) of the form

$$u^*_\varepsilon(x, t, \varepsilon) = u_0(x, t) + g(t)\varepsilon\delta(x - \phi(t)) + c(x, t)\varepsilon\theta(x - \phi(t)), \quad \varepsilon \to +0,$$  (1.8)

where \(\theta\) is the Heaviside function. A solution in such a form will be called an infinitely narrow \(\delta\)-soliton.

In order to consider the function \(u^*_\varepsilon(x, t)\) as a solution to a nonlinear equation we need to define the rules of substitution of this function into the nonlinear term, that is, to include the functions \(\varepsilon\delta(x - \phi(t)), \varepsilon\theta(x - \phi(t))\), where \(\varepsilon \to +0\), into an algebra with differentiation.

2. The rules of substitution of singular ansatzs into nonlinear equations are finally reduced to the definition of multiplication of distributions from certain classes. Various approaches to this problem are discussed, for example, in [3]. A well-known approach to the problem of multiplication of distributions related to the name of J. Colombeau [8], [9] is based on the construction of an algebra of new generalized functions. The spaces \(C^\infty\) and \(D'\) are embedded into \(G\), and moreover, the first embedding is an algebraic isomorphism. Some of the new generalized functions are associated with distributions but, in the general case, they are not directly related to the Schwartz distributions.

Note that H. A. Biagioni and M. Oberguggenberger ([8]) studied the solution to the KdV equation in the form of an infinitely narrow soliton (1.3) in terms of the theory of Colombeau new generalized functions.

Another approach to this problem is associated with the names of J. B. Livchak [9], Li Bang-He [10], V. K. Ivanov [11]. The main idea of this approach is that products of distributions from a certain class are defined as asymptotic decompositions, in the weak sense, of products of approximations of these distributions as \(\varepsilon \to +0\), where \(\varepsilon\) is an approximation parameter, and the coefficients are distributions.

Development of ideas of this approach in [12], [13] and especially in [14] has led to the construction of the weak asymptotics method which enables one to obtain substantial analytical results for discontinuous solutions to nonlinear equations and, in particular, to study the dynamics of propagation and interaction of singularities of various types. In the framework of this approach the construction of a singular solution
to a nonlinear equation is reduced to the choice of singular generators of the ansatz, that is, distributions (generalized functions), to defining the rules of substitution the solution with the chosen structure into the nonlinear equation, to setting up a system of equations for unknown smooth functions and to investigation of the system obtained.

An essential technical progress achieved in these papers, as compared with the results of [3] – [5], is the obtaining of complete asymptotic decompositions of products of approximations of distributions and, on this basis, of associative and commutative differential algebras of asymptotic distributions including subspaces of distributions. The elements of these algebras are weak asymptotics whose coefficients are distributions. Thus, in [5] there was constructed an associative and commutative algebra $E$ of asymptotic distributions (weak asymptotics) generated by the space of linear combinations of homogeneous or associated homogeneous distributions. For example, the singular ansatz (1.3) is an asymptotic distribution belonging to this algebra.

An asymptotic generalized infinitely narrow $\delta$-soliton-type solution to the Hopf equation (1.2) with constant amplitude, and the generalized Hugoniot condition were constructed in the framework of the technique of substitution of a singular ansatz into (12), (13).

The application of the weak asymptotics method in [14] has made it possible to obtain some substantial results about the structure of singular solutions to quasilinear strictly hyperbolic systems and, with certain restrictions, to find the solution of the problem of separating self-similar singularities formulated in [5], [6].

3. Thus, in order to substitute the singular ansatz (1.8) into the KdV equation (1.1) or the Hopf equation (1.2), we first substitute there a smooth ansatz of the form

$$u^*(x, t, \varepsilon) = u_0(x, t) + g(t)\varepsilon\delta(x - \phi(t), \varepsilon) + e(x, t)\varepsilon\theta(x - \phi(t), \varepsilon), \quad \varepsilon > 0, \quad (1.9)$$

where $u_0(x, t)$, $g(t)$, $e(x, t)$, $\phi(t)$ are the desired smooth functions, and

$$\varepsilon\delta(x, \varepsilon) = \omega(\frac{x}{\varepsilon}), \quad \varepsilon\theta(x, \varepsilon) = \varepsilon\omega(\frac{x}{\varepsilon})$$

are smooth approximations of the asymptotic distributions $\varepsilon\delta(x)$ and $\varepsilon\theta(x)$, respectively.

Here, the function $\omega(z) \in C^\infty(\mathbb{R})$ either has a compact support or decreases sufficiently rapidly as $|z| \to \infty$, for example, $|\omega(z)| \leq C(1 + |z|)^{-1}$ and $\int \omega(z) \, dz = 1$; $\omega_0(z) \in C^\infty(\mathbb{R})$, $\lim_{z \to +\infty} \omega_0(z) = 1$, $\lim_{z \to -\infty} \omega_0(z) = 0$.

Then we have in the sense of $D'(\mathbb{R})$ (see the notation above)

$$\varepsilon\delta(x, \varepsilon) = \varepsilon\delta(x) + O_{D'}(\varepsilon^2), \quad \varepsilon\theta(x, \varepsilon) = \varepsilon\theta(x) + O_{D'}(\varepsilon^2), \quad \varepsilon \to +0.$$ 

For more details, see Section 2.

Now, by analogy to (1.3), (1.4), we can introduce the definition of the asymptotic generalized solution of the form (1.8). Namely, we call asymptotic distribution (1.8) an asymptotic generalized solution of the Hopf equation (1.9) if its approximation (1.9) satisfies the relation

$$L_H[u^*(x, t, \varepsilon)] = O_{D'}(\varepsilon^2) \quad (1.10)$$

and, respectively, an asymptotic generalized solution to the KdV equation (1.1), if

$$L_{K,HV}[u^*(x, t, \varepsilon)] = O_{D'}(\varepsilon^2), \quad (1.11)$$

which, in fact, is the same, since

$$L_{K,HV}[u^*(x, t, \varepsilon)] - L_H[u^*(x, t, \varepsilon)] = O_{D'}(\varepsilon^2).$$

It is easy to see that our definition of the solution can depend on the choice of approximations $\frac{1}{\varepsilon}\omega(\frac{x-\phi(t)}{\varepsilon})$ and $\omega_0(\frac{x-\phi(t)}{\varepsilon})$ to the distributions $\delta(x - \phi(t))$, and $\theta(x - \phi(t))$, respectively. Actually, as we shall see later, the dynamics of solution of the type (1.8) is independent of the approximation of the Heaviside function $\omega_0(\frac{x-\phi(t)}{\varepsilon})$.

Definitions (1.10) and (1.11) also imply that, in fact, in order to construct asymptotic generalized solutions satisfying (1.10) or (1.11), we need not calculate products of generalized functions (and, in general, any other nonlinearities) to a high accuracy. Indeed, the substitution of the exact one-solon solution of the KdV equation into the Hopf or KdV equations provides just the accuracy $(O_{D'}(\varepsilon^2))$
corresponding to \( (1.10) \). On the other hand, it is clear that we cannot deal with a lesser accuracy, since the soliton solution contains terms of order \( O(\varepsilon) \).

In our notation, we can say that, within the framework of algebraic constructions related to generalized functions, nonlinear expressions are usually calculated up to \( O(\varepsilon^\infty) \), which is necessary to define associative and commutative algebras of generalized functions.

A distinction of the method of weak asymptotics (which, undoubtedly, originates from the algebraic constructions mentioned above) is that we actually deal with approximations. In fact, the difference between the method of weak asymptotics and the method of ordinary asymptotic expansions is that the smallness of the remainder is understood in a different way. Usually, the remainder is assumed to be small in some uniform sense with sufficient accuracy. Here we assume exactly the same but in the sense of \( O(\varepsilon) \).

In order to obtain the results known from the KdV equation theory it seems natural to use the function from the formula for the exact one-soliton solution \( (1.2) \) to the KdV equation as an approximation for \( \varepsilon \delta(x - \phi(t), \varepsilon) \).

The system for the functions \( u_0(x, t) \), \( g(t) \), \( e(x, t) \), \( \phi(t) \) follows from Definitions \( (1.10) \) or \( (1.11) \) (this system will be derived in detail in Section \( 1.3 \))

\[
\begin{align*}
\frac{\partial u_0}{\partial x} + (u_0^2)_x & = 0, \\
\phi_t - 2u_0(\phi(t), t) - \frac{2}{3}g(t) & = 0, \\
e(\phi(t), t) - \frac{3\sqrt{6}}{2}g_1(t)/g^{3/2}(t) & = 0, \\
(e_1(x, t) + 2(u_0(x, t)e(x, t))_x)_{|x>\phi(t)} & = 0.
\end{align*}
\] (1.12)

It is easy to verify that under the condition \( g > 0 \) (which is an analog of the admissibility condition in the theory of shock waves) the solution of system \( (1.12) \) exists on any interval \( t \in [0, T] \) such that the smooth solution \( u_0 \) of the Hopf equations exists on this interval.

System \( (1.12) \) can be solved in the following way: first, one finds the smooth solution of the Hopf equation, next, one finds the function \( e(x, t) \) from the last equations (which is uniquely solvable in view of the inequality \( 2u_0(\phi(t), t) < \phi_t \)), then one finds the (positive) function \( g(t) \) from the next to the last equation, and finally, one finds the function \( \phi(t) \).

Note that system \( (1.12) \) contains no obstacles to setting \( e(x, t) = 0 \). If so, \( g(t) = \text{const in the case of an arbitrary (nonconstant) background function} u_0(x, t) \). But this conclusion is contrary to well known properties of soliton solutions of the KdV equation (see, e.g., \([3]\)).

Moreover, under our notation, the weak asymptotics of the asymptotic one-soliton solution to the KdV equation, constructed by V. P. Maslov and G. A. Omelyanov \([3]\), has the form

\[
u^1_{x}(x, t) = u_{01}(x, t) + g_1(t)\varepsilon\delta(x - \phi_1(t)) + e_1(x, t)\varepsilon[1 - \theta(x - \phi_1(t))], \quad \varepsilon \to +0.
\] (1.13)

In other words, in the case \( (1.8) \) the "shock wave" with a small amplitude \( \varepsilon e(x, t)\theta(x - \phi_1(t)) \) propagates in front of the soliton \( \varepsilon\delta(x - \phi_1(t)) \), but in the asymptotic one-soliton solution constructed in \([3]\) the small shock wave \( \varepsilon e_1(x, t)[1 - \theta(x - \phi_1(t))] \) arises behind the soliton.

If we apply Definition \( (1.10) \) or \( (1.11) \) to the asymptotic solution obtained in \([3]\), whose weak asymptotics yields \( (1.13) \), we obtain the following system of equations

\[
\begin{align*}
\frac{\partial u_{01}}{\partial x} + (u_{01}^2)_x & = 0, \\
\phi_{1t} - 2u_{01}(\phi_1(t), t) - \frac{2}{3}g_1(t) & = 0, \\
e_1(\phi(t), t) + \frac{3\sqrt{6}}{2}g_1(t)/g_1^{1/2}(t) & = 0, \\
(e_{11}(x, t) + 2(u_{01}(x, t)e_1(x, t))_x)_{|x<\phi_1(t)} & = 0.
\end{align*}
\] (1.14)

The solution of the last system for \( g_{1t}(t) \neq 0 \) is not uniquely determined by the initial conditions \( e_1(x, 0) \) for \( x \leq \phi_1(0) \), since the velocity along the characteristic \( \dot{x} = 2u_{01}(x(t), t) \) is less (for \( g_{1t}(t) > 0 \)) than that of the soliton \( \phi_{1t} = 2u_{01}(\phi_1(t), t) + \frac{2}{3}g_1(t) \).

Thus, the assumption that the structure of the solution to the KdV equation is specified by \( (1.13) \) due to Definitions \( (1.10), (1.11) \) leads to an ill-posed Cauchy problem (with a nonunique solution) for the functions \( u_0(x, t) \), \( g_1(t) \), \( e_1(x, t) \), \( \phi_1(t) \).
On the other hand, the system of equations obtained in [3] for these functions has the form

\[
\begin{align*}
    u_{01t} + (u_{01}^2)_x &= 0, \\
    \phi_{1t} - 2u_{01}(\phi_1(t), t) - \frac{2}{3}g_1(t) &= 0, \\
    e_1(\phi(t), t) + \frac{3\sqrt{6}}{2}g_{1t}(t)/g_1^{3/2}(t) &= 0, \\
    (e_{1t}(x, t) + 2(u_{01}(x, t)e_1(x, t))_x)\bigg|_{x<\phi_1(t)} &= 0, \\
    g_1(t) + 2u_{01}(\phi_1(t), t) &= \text{const},
\end{align*}
\]

(1.15)

It is evident that this system differs from system (1.14) obtained from (1.10), (1.11), (1.13) by the additional equation \( g_1(t) + 2u_{01}(\phi_1(t), t) = g_1(0) + 2u_{01}(\phi_1(0), 0) \). The presence of this equation implies that system (1.15) splits into the two systems

\[
\begin{align*}
    u_{01t} + (u_{01}^2)_x &= 0, \\
    \phi_{1t} - 2u_{01}(\phi_1(t), t) - \frac{2}{3}g_1(t) &= 0, \\
    g_1(t) + 2u_{01}(\phi_1(t), t) &= \text{const},
\end{align*}
\]

(1.16)

and

\[
\begin{align*}
    (e_{1t}(x, t) + 2(u_{01}(x, t)e_1(x, t))_x)\bigg|_{x<\phi_1(t)} &= 0, \\
    e_1(\phi(t), t) + \frac{3\sqrt{6}}{2}g_{1t}(t)/g_1^{3/2}(t) &= 0,
\end{align*}
\]

(1.17)

and equality (1.18) is the boundary condition for equation (1.17), which turns the Cauchy problem for equation (1.17) into the well-posed one (the Cauchy condition, in view of (1.13), has the form \( e_1(x, 0) = e_1^0(x)[1 - \theta(x - \phi_1(0))] \)).

Moreover, if equation (1.17) is considered formally in the domain \( x > \phi_1(t) \), which corresponds to the solution structure given by formula (1.8), then the "redundant" condition

\[
    e(\phi(t), t) - \frac{3\sqrt{6}}{2}g(t)/g^{3/2}(t) = 0,
\]

(1.18)

analogous to (1.13), overdetermines the problem.

Thus, the weak asymptotics corresponding to the asymptotic solution of the Cauchy problem for the KdV equation constructed in [3] cannot be derived from the solution to the KdV equation with the help of Definitions (1.10) or (1.11), and vice versa.

Why is it so? The essence of the matter lies in the definition of weak (generalized) solution to nonlinear equation. It turns out that the definition of the weak (generalized) solution to nonlinear equation depends on the structure of the kernel of the operator adjoint to the linearized operator of the initial differential equation which arises when constructing the smooth asymptotics. This construction of the definition of weak solutions was previously discussed in [17], [18].

In the present paper we do not come into details of construction of the definition of the weak solution to our problem. We just point out that, in terms of this construction, the KdV equation is analogous to the phase field system discussed in [18].

The difference is that for the KdV equation the kernel of the adjoint operator mentioned above is two-dimensional, which results in the following definition analogous to that of the weak solution in the form of the integral identity from [6].

**Definition 1.1** The asymptotic distribution of the form (1.13)

\[
    u^*_\varepsilon(x, t) = u_0(x, t) + g(t)\delta(x - \phi(t)) + c(x, t)\varepsilon\theta(-x + \phi(t))
\]

is a generalized asymptotic (soliton-type) solution to the KdV equation (1.1) for \( t \in [0, T] \) with the initial condition \( u^*_\varepsilon(x) \), if for any constants \( c_1, c_2 \) the following equality holds

\[
\begin{align*}
    (c_1 + c_2u^*(x, t, \varepsilon))L_{KdV}[u^*(x, t, \varepsilon)] &= O_D(\varepsilon^2), \\
    u^*_\varepsilon(x, 0) &= u^*_0(x) + O_D(\varepsilon^2),
\end{align*}
\]

(1.19)

where \( u^*(x, t, \varepsilon) \) is a smooth approximation of the asymptotic distribution \( u^*_\varepsilon(x, t) \), and the first estimate is uniform with respect to \( t \in [0, T] \).
It is clear that (1.19) is equivalent to the following relations

\[ L_{KdV}[u^*(x, t, \varepsilon)] = O_D(\varepsilon^2), \quad u^*(x, t, \varepsilon)L_{KdV}[u^*(x, t, \varepsilon)] = O_D(\varepsilon^2), \]

and the first relation coincides with Definition (1.10).

One can easily see that (1.19) can be rewritten as an integral identity but of an unusual form.

By analogy to what was previously said, the solution depends on the choice of the approximation, and to obtain the results known in the theory of the KdV equation one should choose, as an approximation of the asymptotic distribution \( g(t)\varepsilon\delta(x - \phi(t)) \), the function from the formula for asymptotic solution to the KdV equation 

\[ g(t)\varepsilon\delta(x - \phi(t), \varepsilon) = g(t)\omega\left(\frac{x - \phi(t)}{\varepsilon}\right), \]

where \( \alpha(t) = \sqrt{\frac{g(t)}{6}}, \ \omega(z) = \text{ch}^{-2}(z) \).

It is well known that the function \( \omega(\alpha(t)\tau) \) is a solution of the boundary value problem for the differential equation

\[- \phi_t(t) \frac{d\omega}{dz} + 2(u_0(\phi(t), t) + g(t)\omega)\frac{d\omega}{dz} + \alpha^2(t)\frac{d^2\omega}{dz^2} = 0, \]

(1.20)

where \( \omega(z) \to 0 \) as \( |z| \to \infty \).

It is easy to see that this boundary value problem has the solution given above provided that

\[ \phi_t = 2u_0(\phi(t), t) + \frac{2}{3}g(t). \]

In fact, the equation for the approximation \( \omega \) can be derived by applying an analog of the average procedure, but in the present paper we shall simply postulate the choice of approximation (1.20).

In Section 3 it is shown that for approximation of a soliton of the form (1.20), Definition 1.1 immediately leads to the system of equations (1.15) previously obtained in [3] when constructing the smooth asymptotics of one-soliton solution to the KdV equation.

It should be emphasized that Definition 1.1 for the Hopf equation is not equivalent to that for the KdV equation since the term \( \varepsilon^2u_{xxx} \), for \( c_2 \neq 0 \), makes a contribution of the order of \( O_D(\varepsilon) \) to the left-hand side of the equality.

Thus, it can be said that asymptotic generalized solutions exist for both the Hopf and KdV equations. When \( u_0(x, t) = \text{const} \), the systems describing this solution coincide but, in the general case \( (u_0(x, t) \neq \text{const}) \), they prove to be absolutely distinct in the sense that the limit problem for any one of them is ill-posed for the other.

Moreover, the formal \( \delta \)-soliton generalized asymptotic solution of the Hopf equation is not related (is not an asymptotics) to any exact solution of the Hopf equation. In order to explain this fact, we construct the exact solution of this equation satisfying the Cauchy condition

\[ W(x, t, \varepsilon)|_{t=0} = \varepsilon\delta(x). \]

For \( t > 0 \) this (discontinuous) solution has the form

\[ W(x, t, \varepsilon) = \begin{cases} 
0, & x \leq 0, \\
x/2t, & 0 < x < 2\sqrt{\varepsilon t}, \\
0, & x > 2\sqrt{\varepsilon t}.
\end{cases} \]

The discontinuity points \( \phi(t) = 2\sqrt{\varepsilon t} \) can be calculated from the Rankine–Hugoniot conditions. Let \( \varphi(x) \in D(R_+^{'}) \) be an arbitrary test function. Let us calculate the limit

\[ \lim_{t \to +0} \langle W(x, t, \varepsilon), \varphi(x) \rangle = \lim_{t \to +0} \frac{1}{2t} \int_0^{2\sqrt{\varepsilon t}} x\varphi(x) \, dx. \]

By using the L'Hospital rule, we obtain

\[ \lim_{t \to +0} \langle W(x, t, \varepsilon), \varphi(x) \rangle = \varepsilon\varphi(0). \]
It is clear that for $t = 1$ the support of $W(x, t, \varepsilon)$ is the interval $[0, 2\sqrt{\varepsilon}]$, while for $t = t_0$ the singularity support of the generalized $\delta$-soliton constructed, in view of Definition (1.10), with an arbitrary approximation $\omega(x/\varepsilon)$ of the initial value $\varepsilon\delta$ can be found at the point

$$
\phi(t_0) = t_0 \int \omega^2(z) \, dz,
$$

which lies outside the support of the exact solution $W(x, t, \varepsilon)$ (the method used for deriving this relation will be discussed below).

The results, similar to mentioned above can be obtained for the KdV type equation $u_t + (f(u))_x + \varepsilon^2 u_{xxx} = 0$ with arbitrary smooth nonlinearity, which admits soliton solutions. To verify this assertion, by analogy to (1.20), we consider the boundary value for the ordinary differential equation

$$
-\phi_t d\omega dt + d(u_0(\phi, t) + \omega) + \frac{d^3}{dt^3} \omega = 0,
$$

where $u_0, \phi$ are treated as parameters and the solution $\omega = \omega(\tau, t)$ is chosen as an approximation of the term $\varepsilon g(t)\delta(x - \phi(t))$ in the singular ansatz. Integrating this equation, multiplying the obtained equation by $\omega$, and integrating again, we obtain the energy conservation law

$$
\frac{1}{2} (\omega_\tau)^2 + \Phi(\omega) = E,
$$

where the function $\Phi(\omega, t) = \int_0^\omega f(z + u_0) \, dz - \frac{1}{2}\phi_0 \omega^2$ is considered as a potential and $E$ is an arbitrary constant having the meaning of energy.

Integrating the last equation, we find

$$
\tau - \tau_0 = \pm \int_{\omega_1}^\omega \frac{d\xi}{\sqrt{E - \Phi(\xi, t)}},
$$

(1.21)

where $\omega_1, \omega_2$ are the last two roots of the equation $\Phi(\xi) = E$, the radicand is assumed to be positive when $\omega_1 < \omega(\tau) < \omega_2$, $\tau_0$ is an arbitrary constant.

Using the standart methods, one can show that, for a certain choice of the function $f(u)$ and the constant $E$, equation (1.21) has soliton solutions (see [13, §1]).

For the existence of localized solution it is necessary that one of the roots $\omega_1, \omega_2$ of the equation $\Phi(\xi) = E$ is multiple (that is, the integral in (1.21) is divergent).

Thus, in the case of equations with general nonlinearity, we shall substitute into them the smooth ansatzs

$$
u^*(x, t, \varepsilon) = u_0(x, t) + \omega(\frac{x - \phi(t)}{\varepsilon}, t) + \varepsilon(x, t) e_0(\frac{x - \phi(t)}{\varepsilon}), \quad \varepsilon > 0,
$$

(1.22)

where the function $\omega(\tau, t)$ decreases sufficiently rapidly with respect to $\tau$ and $\omega_0(x/\varepsilon)$ is an approximation of the Heaviside function $\theta(x)$.

Accordingly, in this case the weak asymptotics (1.22), that is, for equations of Hopf or KdV type, the singular ansatz has the form

$$
u^*(x, t) = u_0(x, t) + \Omega_1(t) \varepsilon \delta(x - \phi(t)) + \varepsilon(x, t) \varepsilon \theta(\pm (x - \phi(t))), \quad \varepsilon \to +0,
$$

(1.23)

$$
\Omega_1(t) = \int \omega(\tau, t) \, d\tau.
$$

Note that, generally speaking, we need not use any fixed approximations for the distributions contained in (1.23). In this case, we obtain equations that describe the dynamics and, in general, depend on the approximation of singular generators of the ansatz.

The dependence of the asymptotic discontinuous solution to a nonlinear equation on approximations of singular generators of the solution is substantial for our approach. It is due to the fact that asymptotics of the product of approximations is independent of the choice of these approximations only in specific cases, for example, when the factors are "weakly" singular, as when calculating the asymptotics of the product of the approximation of a distribution by the approximation of a smooth function. Naturally, there is no such dependence if we use this method to find discontinuous solutions of linear equations.
By what has been said, when constructing the set of asymptotic distributions in which singular ansatzs of the form \([1.8], [1.13]\) are embedded all possible approximations will be considered for each distribution. The same idea is basic to the construction of the Colombeau algebra of new generalized functions \([3], [4]\). In the Colombeau theory there are also "many" \(\delta\)-functions associated with "one" Schwartz's \(\delta\)-function.

4. Let us summarize the main results. One of the most important results is Definition 1.1 for generalized asymptotic solutions of KdV type equations. In order to explain how to use this definition, in Section 3 of the present paper we construct an associative algebra of smooth functions \(\mathcal{H}\) in which the smooth ansatzs \((1.10), (1.14)\) or \((1.22)\) are embedded which are approximations of the singular ansatzs \((1.12), (1.13)\) or \((1.23)\). The singular ansatzs on which generalized infinitely narrow soliton-type solutions are constructed belong to the set of asymptotic distributions \(\mathcal{E}\) and are derived as weak asymptotics of elements from \(\mathcal{H}\). In this section we also give a definition of generalized solutions to nonlinear equations, that is, rules of substituting singular ansatzs into these equations.

In Section 3 we derive the above-mentioned systems of equations (1.12) and (1.15) by using different definitions of the generalized solution.

In Section 4 we write out the system of equations determining the dynamics of the infinitely narrow \(\delta\)-soliton to equations of Hopf and KdV types with general nonlinear terms.

2 Asymptotic distributions and generalized solutions to nonlinear equations

1. When solving problems listed in Introduction it is necessary to construct a singular solution with singular generators of the special type \([1.8]\) or \([1.13]\) to the nonlinear equation \(L[u] = 0\). According to our method of weak asymptotics, we substitute into the equation \(L[u] = 0\) smooth ansatzs of the form \((1.9)\) resulting from the replacement of singular generators by their approximations in singular ansatzs of the form \((1.8)\).

It is well known \([21, ch.I, \S 4.6]\) that to each distribution (generalized function) \(f(x) \in \mathcal{D}'\) can be assigned its approximation

\[
f(x, \varepsilon) = f(x) \ast K(x, \varepsilon) = (f(t), K(x - t, \varepsilon)), \quad \varepsilon > 0,
\]

where \(\ast\) is a convolution, the kernel \(K(x, \varepsilon) = \frac{1}{\varepsilon} \omega(\frac{x}{\varepsilon})\) is a \(\delta\)-type function such that \(\omega(z) \in C^\infty(\mathbb{R})\), \(\omega(z)\) has a compact support or decreases sufficiently rapidly as \(|z| \to \infty\), for example, \(|\omega(z)| \leq C(1 + |z|)^{-3}\) and \(\int \omega(z) \, dz = 1\).

For all test functions \(\varphi(x) \in \mathcal{D}\) we have:

\[
\lim_{\varepsilon \to 0} \langle f(x, \varepsilon), \varphi(x) \rangle = \langle f(x), \varphi(x) \rangle.
\]

We cite the approximations of the distributions \(\delta(x), \theta(x)\), which are singular generators of the soliton-type ansatzs \([1.8], (1.13)\).

For the approximation of \(\delta\)-function we have from (2.24):

\[
\delta(x, \varepsilon) = \frac{1}{\varepsilon} \omega\left(\frac{x}{\varepsilon}\right).
\]

For the approximation of the Heaviside function \(\theta(x)\) we have

\[
\theta(x, \varepsilon) = \theta(x) \ast \frac{1}{\varepsilon} \omega\left(\frac{x}{\varepsilon}\right) = \int_0^\infty \omega\left(\frac{x}{\varepsilon} - t\right) \, dt.
\]

Hence we find

\[
\theta(x, \varepsilon) = \omega_0\left(\frac{x}{\varepsilon}\right),
\]

where \(\omega_0(z) = \int_{-\infty}^z \omega(\eta) \, d\eta, \quad \lim_{z \to +\infty} \omega_0(z) = 1, \quad \lim_{z \to -\infty} \omega_0(z) = 0, \quad \omega_0(z) \in C^\infty(\mathbb{R})\).

If the Cauchy kernel \(K(x, \varepsilon) = \frac{1}{\varepsilon} \arctg\left(\frac{x}{\varepsilon}\right)\) is used in (2.24) we obtain harmonic approximations \(f(x, \varepsilon)\) for distributions \(f(x)\). In this case (see \([21, 22]\)):

\[
\delta^{(m-1)}(x, \varepsilon) = \frac{(-1)^m (m - 1)!}{2\pi i} (z^{-m} - \pi^{-m}), \quad \theta(x, \varepsilon) = \left(1 + \frac{2}{\pi} \arctg\left(\frac{x}{\varepsilon}\right)\right).
\]
where $z = x + i\epsilon$, $\overline{z} = x - i\epsilon$, $m = 0, 1, 2, \ldots$.

Denote by $\mathcal{H}^*$ the associative and commutative differential algebra generated by finite sums of finite products of functions $f(x, \varepsilon)$ from the space $\mathcal{H}_0$ of smooth ansatzs $u^*(x, t, \varepsilon)$ of the form (1.3), where approximations of the $\delta$-function and the Heaviside function are defined in (2.25) and (2.26), respectively.

Denote by $\mathcal{E}^*$ the set of weak asymptotics $f^*(x, t)$ derived from elements $f^*(x, t, \varepsilon)$ of the algebra $\mathcal{H}^*$, as $\varepsilon \to +0$, determined up to $O_D(\varepsilon^2)$, and, according to [11], [14], call it the set of \textit{asymptotic (soliton-type) distributions}.

By analogy with [14], one can introduce a structure of the asymptotic algebra on $\mathcal{E}^*$ which is not, however, required in the present paper.

Thus, for example, the one-soliton solution to the KdV equation (1.2) belongs to the algebra $\mathcal{H}^*$ and its weak asymptotics (1.3) and singular ansatzs of the form (1.18) are asymptotic distributions from $\mathcal{E}^*$.

It can be shown (see [14]) that each element $f^*(x, t, \varepsilon)$ from the algebra $\mathcal{H}^*$ has a weak asymptotics of the form

$$f^*(x, t, \varepsilon) = f_0(x, t) + f_1(x, t)\varepsilon\delta\left((x - \phi(t))^{-1}\right) + f_2(x, t)\varepsilon\theta\left((x - \phi(t))^{-1}\right) + O_D(\varepsilon^2), \quad \varepsilon \to +0,$$

where $f_j(x, t)$ are smooth functions, $j = 0, 1, 2$.

Finding weak asymptotics of the elements from the algebra $\mathcal{H}^*$ can be interpreted, in a sense, as multiplications of asymptotic distributions (distributions) (see [10], [11] and [12]–[14]).

Let us give some examples. The following equalities can be readily verified:

$$[\varepsilon\delta(x, \varepsilon)]^n = \Omega_n \varepsilon\delta(x) + O_D(\varepsilon^2),$$

$$\varepsilon\theta(x, \varepsilon) = \varepsilon\theta(x) + O_D(\varepsilon^2),$$

$$[\varepsilon\theta(x, \varepsilon)]^m = O_D(\varepsilon^m),$$

$$[\varepsilon\delta(x, \varepsilon)]^n [\varepsilon\theta(x, \varepsilon)]^{m-1} = O_D(\varepsilon^m).$$

where $n = 1, 2, \ldots$, $m = 2, \ldots$, $\omega(z)$ is a $\delta$-type function from (2.24), $\Omega_n = \int_{-\infty}^{\infty} \omega^n(z) \, dz$.

In particular, it follows herefrom that the asymptotic distributions $\varepsilon\delta(x)$ and $\varepsilon\theta(x)$ constitute an algebra, which is asymptotic mod $O_D(\varepsilon^2)$.

2. Let $F(u)$ be a smooth function of at most exponential growth. We define a function of asymptotic distribution $F(f^*(x, t)) \in \mathcal{E}^*$ as the weak asymptotics of the function $F(f^*(x, t, \varepsilon))$, as $\varepsilon \to +0$, where $f^*(x, t, \varepsilon) \in \mathcal{H}^*$ is the approximation of the asymptotic distribution $f^*_x(x, t) \in \mathcal{E}^*$.

Then, for example, $F(\varepsilon\delta(x))$ is defined as the weak asymptotics of the approximating function $F\left(\frac{\omega(x)}{\varepsilon}\right)$ as $\varepsilon \to +0$. Taking into account the estimate for the function $\omega$ and using the Lagrange theorem

$$F(\omega(z)) - F(0) = F'(\Theta\omega(z))\omega(z), \quad 0 < \Theta < 1,$$

we have

$$|F(\omega(z)) - F(0)| \leq K(1 + |z|)^{-3}.$$

Thus, for the function $F(\omega(z)) - F(0)$ there exists an estimate analogous to that for the function $\omega(z)$ approximating the asymptotic distribution $\varepsilon\delta(x)$. Therefore, applying $F\left(\frac{\omega(x)}{\varepsilon}\right)$ to a test function and performing the change of variables $x = \varepsilon\eta$, we obtain

$$J(\varepsilon) = \left\langle F(0), \varphi(x) \rightangle + \left\langle F\left(\frac{\omega(x)}{\varepsilon}\right) - F(0), \varphi(x) \right\rangle$$

$$= \left\langle F(0), \varphi(x) \right\rangle + \varepsilon \int_{-\infty}^{\infty} \left[ F\left(\omega(\eta)\right) - F(0) \right] \varphi(\varepsilon\eta) \, d\eta$$

$$= \left\langle F(0), \varphi(x) \right\rangle + \varepsilon\varphi(0) \int_{-\infty}^{\infty} \left[ F\left(\omega(\eta)\right) - F(0) \right] \, d\eta + O(\varepsilon^2), \quad \varepsilon \to +0.$$

Thus, in the weak sense, we have

$$F(\varepsilon\delta(x)) = F(0) + \varepsilon\Lambda\delta(x) + O_D(\varepsilon^2), \quad \varepsilon \to +0,$$

where the constant $\Lambda = \int \left[ F(\omega(\eta)) - F(0) \right] \, d\eta$. 

9
3 Infinitely narrow solitons to the Hopf and KdV equations

1. The Hopf equation: \( u_t + (u^2)_x = 0 \). To find an asymptotic \( O_D(\varepsilon^2) \) infinitely narrow \( \delta \)-soliton-type solution to the Hopf equation (1.4) of the form

\[
u^*_\varepsilon(x,t) = u_0(x,t) + g(t)\varepsilon \delta(x - \phi(t)), \quad \varepsilon \to +0, \tag{3.30}\]

where \( u_0(x,t) \in C^\infty(\mathbb{R}^2) \), \( g(t), \phi(t) \in C^\infty(\mathbb{R}) \) are the desired functions, one needs to substitute into the equation its approximation from \( \mathcal{H}^* \) of the form:

\[
u^*_\varepsilon(x,t,\varepsilon) = u_0(x,t) + g(t)\varepsilon \delta(x - \phi(t),\varepsilon), \quad \varepsilon > 0, \tag{3.31}\]

where, according to (2.23), the function \( \varepsilon \delta(x,\varepsilon) = \omega(\frac{x}{\varepsilon}) \) is used as an approximation of the asymptotic distribution \( \varepsilon \delta(x) \).

Using the first formula (2.28) for the squared asymptotic distribution \( \varepsilon \delta(x) \), we obtain

\[
[u^*_\varepsilon(x,t)]^2 = u_0^2 + 2u_0g\varepsilon\delta(x - \phi(t)) + \varepsilon^2\Omega\varepsilon^2\delta(x - \phi(t)) + O_D(\varepsilon^2), \quad \varepsilon \to +0.
\]

Then, substituting \( u^*_\varepsilon(x,t) \) into the Hopf equation we have

\[
u^*_\varepsilon + [(u^*_\varepsilon)^2]_x = u_{0t} + (u_0^2)_x + \varepsilon \delta(x - \phi)\left(g_t + 2g_{0x}\right)
\]

\[
+ \varepsilon \phi(t) - \Omega g(t) - 2u_0(x,t)\right)
\]

\[
\left|\varepsilon\phi(t) - \Omega g(t) - 2u_0(x,t)\right|
\]

where \( \Omega = \int g^2(\eta)\,d\eta \).

Using the well-known equality

\[
a(x)\delta'(x) = a(0)\delta'(x) - a'(0)\delta(x), \tag{3.33}\]

and equating the coefficients of \( \varepsilon^0, \varepsilon\delta \) and \( \varepsilon\delta' \) with zero, we obtain the necessary and sufficient conditions for the right-hand side of equation (3.32) to be of the order of \( O_D(\varepsilon^2) \):

\[
u_{0t} + (u_0^2)_x = 0, \tag{3.34}
\]

\[
\phi(t) - \Omega g(t) - 2u_0(x,t) |_{x=\phi(t)} = 0,
\]

\[
g_t(t) = 0.
\]

In the case of constant background \( u_0 = \text{const} \) we have \( \phi(t) = vt + \phi_0 \), where \( v = \Omega g(0) + 2u_0 \) and \( \phi_0 \) is a constant which has the meaning of the coordinate of the initial position of the soliton.

It is clear that the condition \( g_t = 0 \), when \( u_0 \neq \text{const} \), contradicts the physical intuition and the well-known results about soliton behaviour. However, formally, such a structure can also exist, since system (3.38) is well-defined, but we can try to repair the situation by adding a new term which, after differentiation, has the form \( \varepsilon \delta(x - \phi) \). Therefore, we shall seek an asymptotic solution to the Hopf equation of the form (1.9) by substituting into the equation the smooth ansatz of the type (1.9):

\[
u^*(x,t,\varepsilon) = u_0(x,t) + g(t)\Omega_1\varepsilon \delta(\alpha(t)(x - \phi(t))) + e(x,t)\varepsilon \theta(x - \phi(t)), \quad \varepsilon \to +0, \tag{3.35}\]

where \( \alpha(t) \in C^\infty \), \( \Omega_1 = \int \omega(\eta)\,d\eta \), \( \omega(z) \) is a \( \delta \)-type function which has a compact support or decreases sufficiently rapidly as \(|z| \to \infty \), \( \omega_0(\frac{x}{\varepsilon}) \) is an approximation of the Heaviside function.

The weak asymptotics (3.35) has the form

\[
u^*_\varepsilon(x,t) = u_0(x,t) + g(t)\Omega_1\varepsilon \delta(\alpha(t)(x - \phi(t))) + e(x,t)\varepsilon \theta(x - \phi(t)), \quad \varepsilon \to +0. \tag{3.36}\]

This representation coincide with one that can be obtained when considering the smooth asymptotics of the asymptotic solution to the KdV equation with a small dispersion \( \sim \varepsilon^2 \), in the weak sense as \( \varepsilon \to +0 \).
In this case, according to Section 2,

\[
[u^* (x, t)]^2 = u_0^2 (x, t) + \frac{1}{\alpha (t)} \left( 2u_0 (x, t) g(t) \Omega_1 + g^2 (t) \Omega_2 \right) \varepsilon \delta (x - \phi (t))
\]

\[+ 2u_0 (x, t) e (x, t) \varepsilon \theta (x - \phi (t)) + O_D (\varepsilon^2), \quad \varepsilon \to +0,
\]

(3.37)

where \( \Omega_1 = \int \omega (\eta) d\eta \), \( \Omega_2 = \int \omega^2 (\eta) d\eta \).

Substituting (3.35) into the Hopf equation and using (3.36), (3.37), up to the terms of the order of \( O_D (\varepsilon^2) \), we obtain a relation analogous to (3.32). Then, setting the coefficients of \( \varepsilon^0 \), \( \varepsilon \theta (x - \phi (t)) \), \( \varepsilon \delta (x - \phi (t)) \) and \( \varepsilon \delta (x - \phi (t)) \) equal to zero, we arrive at the following result.

**Theorem 3.1** Suppose that for \( t \in [0, T] \) there exists a smooth solution \( u_0 (x, t) \) to the Hopf equation with the smooth initial condition \( u_0 (x, t) \big|_{t=0} = u_0^0 (x) \). Then, on the closed interval \( [0, T] \), the Hopf equation has a solution in the form of an infinitely narrow \( \delta \)-soliton (3.36) if and only if the unknown smooth functions \( u_0 (x, t) \), \( g(t) \), \( e(x, t) \), \( \alpha (t) \), \( \phi (t) \) satisfy the following system of equations:

\[
\begin{align*}
\frac{u_0 (x)}{\alpha (t)} + \frac{u_0^2 (x)}{\alpha (t)} & = 0, \\
\phi_t - 2u_0 (\phi (t), t) - \frac{\Omega_2}{\Omega_1} g(t) & = 0, \\
e (\phi (t), t) - \frac{\Omega_2^2}{\Omega_1} \left( \frac{g(t)}{\alpha (t)} \right) t & = 0, \\
\left. \left( e_t (x, t) + 2(u_0 (x, t) e (x, t)) \right) \right|_{x>\phi (t)} & = 0,
\end{align*}
\]

(3.38)

where \( \Omega_1 = \int \omega (\eta) d\eta \), \( \Omega_2 = \int \omega^2 (\eta) d\eta \).

**Remark.** System (1.12) studied in the Introduction can be obtained from system (3.38) if we take the solution of equation (1.20) as the approximation \( \omega (\alpha (t) (x - \varphi (t))/\varepsilon) \). In this case we have

\[
\omega = \cosh^{-2} (\tau), \quad \alpha = \sqrt{g(t)/6}
\]

and system (3.38) turns into systems (1.12), where the number of unknown functions is equal to the number of equations.

2. The KdV equation: \( u_t + (u^2)_x + \varepsilon^2 u_{xxx} = 0 \). This equation has the exact one-soliton solution (1.2)

\[
u(x, t, \varepsilon) = g \cosh^{-2} \left( \sqrt{\frac{g}{6}} \left( x - \frac{2}{3} gt \right) / \varepsilon \right),
\]

where \( g = \text{const} \) is the amplitude of the soliton [24]. The weak asymptotics of this solution has the form (1.3)

\[
u(x, t) = g \varepsilon \delta (x - \frac{2}{3} gt), \quad \varepsilon \to +0.
\]

Let us study the dynamics of propagation of an infinitely narrow deformed soliton solution of the KdV equation. To this end, we consider a smooth ansatz of the form

\[
u^* (x, t, \varepsilon) = u_0 (x, t) + g(t) \omega (\alpha (t) \frac{x - \phi (t)}{\varepsilon}) + e(x, t) \varepsilon \omega_0^{-} (\frac{x - \phi (t)}{\varepsilon}), \quad \varepsilon > 0,
\]

(3.39)

where \( \alpha (t) = \sqrt{\frac{g(t)}{6}} \), \( \omega (\eta) = \cosh^{-2} (\eta) \), \( \omega_0^{-} (\frac{\xi}{\varepsilon}) = 1 - \omega_0 (\frac{\xi}{\varepsilon}) \), and, according to (2.26),

\[
\omega_0 (\frac{x}{\varepsilon}) = \int_{-\infty}^{x/\varepsilon} \omega_1 (\eta) d\eta.
\]

Here \( \omega_0 (\frac{\xi}{\varepsilon}) \) and \( \omega_0^{-} (\frac{\xi}{\varepsilon}) \) are approximations of the Heaviside functions \( \theta (x) \) and \( \theta (-x) \), respectively, \( \int \omega_1 (\eta) d\eta = 1 \).

The weak asymptotics of the right-hand side of (3.38) has the form (3.36).
According to Definition 2.1, the infinitely narrow soliton-type solution of the KdV equation is defined
as the solution of the following system of the two equations

$$L_{KdV}[u] = u_t + (u^2)_x + \varepsilon^2 u_{xxx} = O_D(\varepsilon^2),$$
$$u L_{KdV}[u] = (u^2)_x + \frac{4}{3}(u^3)_x + \varepsilon^2 2uu_{xxx} = O_D(\varepsilon^2).$$

(3.40)

Let us write the dispersion term of the second equation in the form $G(u, \varepsilon) = \varepsilon^2 2uu_{xxx} = \varepsilon^2 [(u^2)_{xxx} - 3(u_x)^2]_x$.

Up to the terms whose asymptotics are of the order of $O_D(\varepsilon^2)$ in $D'$ we obtain

$$[u^*(x, t, \varepsilon)]^2 = u_0^2 + g^2 \omega^2 + 2u_0 g \omega + 2u_0 \varepsilon \omega \omega_0,$$
$$[u^*(x, t, \varepsilon)]^3 = u_0^3 + g^2 \omega^3 + 3u_0^2 g \omega^2 + 3u_0^2 g \omega + 3u_0^2 \varepsilon \omega \omega_0.$$  

(3.41)

Then for all $\varphi(x) \in D$:

$$\int \omega \left( \alpha(t) \frac{x - \phi(t)}{\varepsilon} \right) \varphi(x) \, dx = \frac{\varepsilon}{\alpha(t)} \int \omega(\eta) \varphi \left( \phi(t) + \frac{\varepsilon}{\alpha(t)} \eta \right) \, d\eta = \Omega_1 \varepsilon \varphi(\phi(t)) + O(\varepsilon^2),$$

where $\Omega_1 = \int \omega(\eta) \, d\eta = \int \text{ch}^{-2}(\eta) \, d\eta = 2$. Therefore,

$$\omega \left( \alpha(t) \frac{x - \phi(t)}{\varepsilon} \right) = \frac{\Omega_1}{\alpha(t)} \varepsilon \delta(x - \phi(t)) + O_D(\varepsilon^2).$$

In a similar way we find that

$$\omega^2 \left( \alpha(t) \frac{x - \phi(t)}{\varepsilon} \right) = \frac{\Omega_2}{\alpha(t)} \varepsilon \delta(x - \phi(t)) + O_D(\varepsilon^2),$$
$$\omega^3 \left( \alpha(t) \frac{x - \phi(t)}{\varepsilon} \right) = \frac{\Omega_3}{\alpha(t)} \varepsilon \delta(x - \phi(t)) + O_D(\varepsilon^2),$$

where $\Omega_2 = \int \omega^2(\eta) \, d\eta = \int \text{ch}^{-4}(\eta) \, d\eta = \frac{4}{3}$, $\Omega_3 = \int \omega^3(\eta) \, d\eta = \int \text{ch}^{-6}(\eta) \, d\eta = \frac{16}{15}$.

Now we find the asymptotics of the dispersion term $G(u^*(x, t, \varepsilon), \varepsilon)$. It immediately follows from (3.37) that

$$G(u^*(x, t, \varepsilon), \varepsilon) = \varepsilon^2 [(u^*)^2_{xxx} - 3(u^*_x)^2]_x = -3\varepsilon^2 [(u^*_x)^2]_x + O_D(\varepsilon^2).$$

From (3.39) we have

$$u^*_x(x, t, \varepsilon) = u_0 + g \omega x + \varepsilon \omega \omega_0 + \varepsilon \omega \omega_0 x.$$ 

Squaring this expression and analyzing its components, we see that up to the terms of the order of $O_D(\varepsilon^2)$ we have

$$G(u^*(x, t, \varepsilon), \varepsilon) = -3\varepsilon^2 g^2 [(\omega^*_x)^2]_x + O_D(\varepsilon^2).$$

After calculating the weak asymptotics of this expression, we obtain for all $\varphi(x) \in D$

$$\langle G(u^*(x, t, \varepsilon), \varepsilon), \varphi(\xi) \rangle = -3g^2(t)\varepsilon^2 \int \left[ \omega \left( \alpha(t) \frac{x - \phi(t)}{\varepsilon} \right) \right]^2 \varphi(x) \, dx$$
$$= -3g^2(t)\varepsilon \alpha(t) \int \omega' \eta^2 \varphi(\phi(t) + \frac{\varepsilon}{\alpha(t)} \eta) \, d\eta = -3\Omega_4 g^2(t)\alpha(t)\varepsilon \varphi(\phi(t)) + O(\varepsilon^2),$$

where $\Omega_4 = \int \omega'(\eta)^2 \, d\eta = 4 \int \text{ch}^{-6}(\eta) \text{sh}^2(\eta) \, d\eta = \frac{16}{15}$.

It follows that the weak asymptotics of the dispersion term is

$$G(u^*_x(x, t, \varepsilon), \varepsilon) = -3g^2(t)\alpha(t)\varepsilon \delta(x - \phi(t)) + O_D(\varepsilon^2).$$

(3.42)

Substituting the asymptotics for $\omega$, $\omega^2$, $\omega^3$ obtained above into (3.39)-(3.41) we have

$$u^*_x(x, t, \varepsilon) = u_0 + g \frac{\Omega_1}{\alpha} \varepsilon \delta(x - \phi(t)) + \varepsilon (x, t) \varepsilon \theta(-x + \phi(t)) + O_D(\varepsilon^2),$$
$$[u^*_x(x, t)]^2 = u_0^2 + \left( g^2 \frac{\Omega_2}{\alpha} + 2u_0 g \frac{\Omega_1}{\alpha} \right) \varepsilon \delta(x - \phi(t)) + 2u_0 \varepsilon \theta(-x + \phi(t)) + O_D(\varepsilon^2),$$

where $\alpha$ and $\Omega_i$ are defined as in (3.35).
\[ [u^*_x(x,t)]^3 = u_0^3 + \left\{ g^3 \frac{\Omega_1}{\alpha} + 3u_0g^2 \frac{\Omega_2}{\alpha} + 3u_0^2g \frac{\Omega_1}{\alpha} \right\} \varepsilon \delta(x - \phi(t)) + 3u_0^2 \varepsilon \theta(-x + \phi(t)) + O_\varphi(\varepsilon^2). \]

After the substitution of the coefficients \( \Omega_k, \ k = 1, 2, 3, \) into these expressions we obtain

\[ u^*_x(x,t) = u_0(x,t) + 2\sqrt{6} g^{1/2}(t) \varepsilon \delta(x - \phi(t)) + e(x,t) \varepsilon \theta(-x + \phi(t)) + O_\varphi(\varepsilon^2), \]

\[ [u^*_x(x,t)]^2 = u_0^2(x,t) + \sqrt{6} \left\{ \frac{4}{3} g^{3/2}(t) + 4u_0(\phi(t),t)g^{1/2}(t) \right\} \varepsilon \delta(x - \phi(t)) + 2u_0(x,t)e(x,t) \varepsilon \theta(-x + \phi(t)) + O_\varphi(\varepsilon^2), \]

\[ [u^*_x(x,t)]^3 = u_0^3(x,t) + \sqrt{6} \left\{ \frac{16}{15} g^{5/2}(t) + 4u_0(\phi(t),t)g^{3/2}(t) + 6u_0^2(\phi(t),t)g^{1/2}(t) \right\} \varepsilon \delta(x - \phi(t)) + 3u_0^2(x,t)e(x,t) \varepsilon \theta(-x + \phi(t)) + O_\varphi(\varepsilon^2). \]  

(3.43)

According to Definition 1.3, substituting asymptotics (3.43) into the first equation of system (3.46) and setting the coefficients of \( \varepsilon^0, \varepsilon \delta, \varepsilon \delta' \) and \( \varepsilon \theta \) equal to zero, we obtain the necessary and sufficient conditions for the right-hand side of the equation \( L_{KdV}[u] = 0 \) to be of order \( O_\varphi(\varepsilon^2) \):

\[ \varepsilon^0 : \quad \left( u_0^2 \right)_x + \frac{3}{2} (u_0^3)_x = 0, \]

\[ \varepsilon \delta' : \quad -\phi_t + 2u_0(\phi(t),t) + \frac{2}{3} \theta_g(t) = 0, \]

\[ \varepsilon \delta : \quad \sqrt{6} g^3(t) \theta(t) - (-\phi_t + 2u_0(\phi(t),t)) e_\phi(t) = 0, \]

\[ \varepsilon \theta : \quad \left( e_t(x,t) + 2u_0^2(x,t)e(x,t) \right)_{x\phi(t)} = 0. \]  

(3.44)

The second and third equations of system (3.44) imply the equation for the jump of the amplitude of a small shock wave:

\[ e(\phi(t),t) = -\frac{3}{2} \sqrt{6} \frac{g_\theta(t)}{g^{1/2}(t)}. \]  

(3.45)

Following Definition 1.3, we now substitute asymptotics (3.43) into the second equation of system (3.46) and, setting the coefficients of \( \varepsilon^0, \varepsilon \delta, \varepsilon \delta' \) and \( \varepsilon \theta \) equal to zero, we obtain the necessary and sufficient conditions for the right-hand side of the equation \( uL_{KdV}[u] = 0 \) to be of order \( O_\varphi(\varepsilon^2) \):

\[ \varepsilon^0 : \quad \left( u_0^2 \right)_x + \frac{3}{2} (u_0^3)_x = 0, \]

\[ \varepsilon \delta' : \quad -\phi_t + 2u_0(\phi(t),t) + \frac{2}{3} \theta_g(t) = 0, \]

\[ \varepsilon \delta : \quad \sqrt{6} g^3(t) \theta(t) - e(\phi(t),t) - \frac{3}{2} \sqrt{6} \frac{g_\theta(t)}{g^{1/2}(t)} = 0, \]

\[ \varepsilon \theta : \quad \left( 2u_0^2(x,t)e(x,t) \right)_{t\phi} + 4u_0^2(x,t)e(x,t) = 0. \]  

(3.46)

Since \( u_0(x,t) \) and \( e(x,t) \) are smooth functions, the first and the last equations from systems (3.44) and (3.46) are equivalent. It follows from the third equation of system (3.46) and equation (3.43) that \( g(t) + 2u_0(\phi(t),t) = \text{const} \).

Relations (3.44) – (3.46) imply the following theorem which determines the dynamics of a single deformed soliton to the KdV equation.

**Theorem 3.2** Let us assume that for \( t \in [0, T] \) there exists a smooth solution \( u_0(x,t) \) to the Hopf equation with the smooth initial condition \( u_0(x,t) \big|_{t=0} = u^0_0(x) \).

Then the KdV equation on the closed interval \( [0, T] \), up to \( O_\varphi(\varepsilon^2) \), has an infinitely narrow \( \delta \) -soliton-type solution

\[ u^*_x(x,t) = u_0(x,t) + 2\sqrt{6} g^{1/2}(t) \varepsilon \delta(x - \phi(t)) + e(x,t) \varepsilon \theta(-x + \phi(t)), \]

if and only if the unknown smooth functions \( u_0(x,t), \ g(t), \ e(x,t), \ \phi(t) \) satisfy the system of equations

\[ u_0t + (u_0^2)_x = 0, \]

\[ \phi_t = 2u_0(\phi(t),t) + \frac{2}{3} \theta_g(t), \]

\[ g(t) + 2u_0(\phi(t),t) = g(0) + 2u_0^0(\phi(0)), \]

\[ \left( e_t(x,t) + 2u_0(x,t)e(x,t) \right)_{x\phi(t)} = 0, \]

\[ e(\phi(t),t) = -\frac{3}{2} \sqrt{6} \frac{g_\theta(t)}{g^{1/2}(t)}. \]  

(3.47)
4 Infinitely narrow solitons to the equations $u_t + (f(u))_x = 0$ and $u_t + (f(u))_x + \varepsilon^2 u_{xxx} = 0$

1. The equation $u_t + (f(u))_x = 0$. Consider the infinitely narrow soliton-type solution of this equation, substituting smooth ansatz (1.22) into the equation. This ansatz is a generalization of the smooth one (3.35) and has, up to $O_D(\varepsilon^2)$, a weak asymptotics of the form (1.23). In this case, in (1.22) $\omega_0(\frac{x}{\varepsilon})$ is an approximation of the Heaviside function $\theta(x)$.

By analogy to (2.29), define the asymptotic distribution $f(u_0(x,t) + \Omega_1(t)\varepsilon\delta(x) + e(x,t)\varepsilon\theta(x))$
as the weak asymptotics of the approximating function as $\varepsilon \to +0$

$$f(u_0(x,t) + \omega(\frac{x}{\varepsilon},t) + e(x,t)\varepsilon\omega_0(\frac{x}{\varepsilon}))$$

It is clear that

$$f(u_0 + \omega(\frac{x}{\varepsilon},t) + e\varepsilon\omega_0(\frac{x}{\varepsilon})) = f(u_0 + \omega(\frac{x}{\varepsilon},t)) + \varepsilon f'(u_0) e\omega_0(\frac{x}{\varepsilon}) + O(\varepsilon^2).$$

In addition, according to the Lagrange theorem, we have

$$f(u_0 + \omega(\frac{x}{\varepsilon},t)) - f(u_0) = f'(u_0 + \Theta\omega(\frac{x}{\varepsilon},t)\omega(\frac{x}{\varepsilon},t),$$

where $0 < \Theta < 1$. Hence, for the function $f(u_0 + \omega(\frac{x}{\varepsilon},t)) - f(u_0)$ we have the same estimation with respect to $\tau$ as for the function $\omega(\tau, t)$,

Applying the function $f(u_0 + \omega(\frac{x}{\varepsilon},t) + e\varepsilon\omega_0(\frac{x}{\varepsilon}))$ to a test function and performing the change of variables $x = \varepsilon\tau$, we obtain, as in deduction of formula (2.29),

$$f(u_0(x,t) + \omega(\frac{x}{\varepsilon},t) + e\varepsilon\omega_0(\frac{x}{\varepsilon}))$$

$$= f(u_0(x,t)) + \Lambda(0,t)\varepsilon\delta(x) + f'(u_0(x,t))e(x,t)\varepsilon\theta(x) + O_D(\varepsilon^2), \quad \varepsilon \to +0,$$

where the function

$$\Lambda(x,t) = \int \left[ f(u_0(x,t) + \omega(\tau,t)) - f(u_0(x,t)) \right] d\tau.$$

Substituting $u_\varepsilon^+(x,t)$, in the form of singular ansatz (1.23), into the initial equation and using (1.48) we obtain

$$u_\varepsilon^+(x,t) = u_0 + (f(u_0))_x + \left\{ - \phi(0)\Omega_1(t) + \Lambda(\phi(t),t) \right\}\varepsilon\delta(x - \phi(t))$$

$$+ \left\{ \left( \Omega_1(t) \right)_x - e(x,t)\phi(0) + e(x,t)f'(u_0(x,t)) \right\}\varepsilon\delta(x - \phi(t))$$

$$+ \left\{ e_1(x,t) + (f'(u_0(x,t))e(x,t))_x \right\}\varepsilon\theta(x - \phi(t)) + O_D(\varepsilon^2), \quad \varepsilon \to +0.$$
Then, for \( t \in [0, T] \), the initial equation has the infinitely narrow \( \delta \)-soliton solution \((1.23)\) if and only if the unknown smooth functions \( u_0(x,t), \epsilon(x,t), \phi(t) \) satisfy the following system of equations:

\[
\begin{align*}
\frac{u_0}{\epsilon} + (f(u_0))_x &= 0, \\
-\phi_1 \Omega_1(t) + \Lambda(\phi(t), t) &= 0, \\
(\Omega_1)_t + \left[ f'(u_0(\phi(t), t)) - \frac{\Lambda(\phi(t), t)}{\Omega_1(t)} \right] \epsilon(\phi(t), t) &= 0, \\
\left[ e_1(x, t) + (f'(u_0(x, t)) \epsilon(x, t)) \right] \bigg|_{x \phi(t)} &= 0,
\end{align*}
\]

where \( \Omega_1(t) = \int \omega(\tau, t) \, d\tau \) and the function \( \Lambda(x, t) \) is defined in \((4.49)\).

It is easy to verify that if the inequality \( f''(u) > 0 \) is satisfied, then for \( \Omega_1(t) > 0 \) system \((4.54)\) has exactly the same properties as system \((1.14)\).

2. The equation \( u_t + (f(u))_x + \epsilon^2 u_{xxx} = 0 \).

As it was said in Introduction, for some convex smooth \( f(u) \) this equation has exact one-soliton solutions. In this case we use our approach to describe the dynamics of propagation of a deformed infinitely narrow soliton. By virtue of Definition \(1.1\), as in the case of the KdV equation, the infinitely narrow soliton-type solution \((1.23)\) is defined as a solution of the two equations

\[
L[u] = u_t + (f(u))_x + \epsilon^2 u_{xxx} = O_D(\epsilon^2),
\]

where \( \tilde{f}(u) = 2uf(u) - 2f'(u) \int_0^u f(\zeta) \, d\zeta \). Note that the ansatz \((1.23)\) has the approximation \((1.24)\), where \( \omega_0 = \omega(\frac{1}{2}) \) is the approximation of the Heaviside function \( \theta(-x) \).

The dispersion term in the second conservation law is just the same as for the KdV equation:

\[
G(u, \epsilon) = \epsilon^2 2u u_{xxx} = \epsilon^2 [(u^2)_{xx} - 3(u_x)^2]_x,
\]

and its weak asymptotics has the form \((3.42)\) and \( \Omega_1(t) = \int \omega_x^*(\tau, t)^2 \, d\tau \).

Substituting the smooth ansatz \((1.22)\) into the initial equation, just as for the KdV equation from Section \(3.2\), we obtain,

\[
[u^*_x(x,t)]^2 = u_0^2 + \left\{ \Omega_2(t) + 2u_0 \Omega_1(t) \right\} \epsilon \delta(x - \phi(t)) + 2u_0 \epsilon \theta(-x + \phi(t)) + O_D(\epsilon^2),
\]

where \( \Omega_1(t) = \int \omega(\tau, t) \, d\tau, \) \( \Omega_2(t) = \int \omega_x^2(\tau, t) \, d\tau \).

From \((1.23)\) and \((1.24)\), using formula \((4.48)\), we find

\[
\tilde{f}(u_0(x,t) + \Omega_1(t) \epsilon \delta(x) + e(x,t) \epsilon \theta(-x))
\]

\[
= \tilde{f}(u_0(x,t)) + \tilde{\Lambda}(0,t) \epsilon \delta(x) + \tilde{f}'(u_0(x,t)) e(x,t) \theta(-x) + O_D(\epsilon), \quad \epsilon \to +0,
\]

where \( \tilde{f}'(u) = 2uf'(u) \) and

\[
\tilde{\Lambda}(x,t) = \int \left[ \tilde{f}(u_0(x,t) + \omega(\tau, t)) - \tilde{f}(u_0(x,t)) \right] d\tau.
\]

Obviously, the system of equations derived from the first equation in \((4.51)\) coincides with system \((4.54)\) from Theorem \(4.1\) which describes the dynamics of the formal soliton solution of the equation \( u_t + (f(u))_x = 0 \).

We find the system of equations which follows from the second conservation law \( aL[u] = 0 \) in \((4.51)\). To this end, substitute the asymptotics \( [u^*_x(x,t)]^2, (4.52) \) and \((3.42)\) into the second equation \((4.51)\) and set the coefficients of \( \epsilon^0, \epsilon^2, \epsilon \delta, \epsilon \theta \) and \( \epsilon \delta \theta \) equal to zero.

Setting the coefficients of \( \epsilon^0 \) and \( \epsilon \theta \) in the obtained system equal to zero, we have the equations

\[
\left. \begin{array}{l}
(u_0^u)_t + (\tilde{f}(u_0))_x = 0, \\
\left[ (2u_0 e)_t + (2u_0 f'(u_0)e) \right] \bigg|_{x<\phi(t)} = 0,
\end{array} \right\}
\]

which, in virtue of smoothness of the functions \( u_0(x,t), e(x,t), f(u) \), coincide with the corresponding equations of system \((4.51)\) of the first conservative law \( L[u] = 0 \).
After setting the coefficient of $\varepsilon \delta$ equal to zero, we find
\[
\left( \Omega_2(t) + 2u_0(\phi(t), t)\Omega_1(t) \right)_t + 2u_0(\phi(t), t)e(\phi(t), t) \left( f'(u_0) - \phi_t(t) \right) = 0.
\]

Using the second and the third equations of system (4.50), we bring the last equation to the form
\[
\frac{d}{dt} \Omega_2(t) + 2\Omega_1(t) \frac{d}{dt} u_0(\phi(t), t) = 0,
\]
which coincides with the third equation from (3.47) for the KdV equation when $\alpha(t) = \sqrt{\frac{8(t)}{6}}$.

Consider the coefficient of $\varepsilon \delta'$ and prove

**Lemma 4.1** The coefficients of $\varepsilon \delta'$ in the conservation laws $L[u] = 0$ and $uL[u] = 0$ (4.54) coincide.

**Proof.** Setting the coefficient of $\varepsilon \delta'$ in the left-hand side of the equation $uL[u] = 0$ equal to zero, we find:
\[
- \phi_t(t) \left( \Omega_2(t) + 2u_0(\phi(t), t)\Omega_1(t) \right) + \tilde{\Lambda}(\phi(t), t) - 3\Omega_\Delta(t) = 0. \tag{4.55}
\]

Let $u(x, t) = u_0(x, t) + \omega(\tau, t)$ be an exact soliton solution to the equation $u_t + (f(u))_x + \varepsilon^2 u_{xxx} = 0$, where $\tau = \frac{x - \phi(t)}{\varepsilon}$. Substitute $u(x, t)$ into this equation and consider the coefficient of $\frac{1}{\varepsilon}$:
\[
- \phi_t \omega_\tau + \left( f(u_0 + \omega) \right)_\tau + \omega_{\tau\tau} = 0. \tag{4.56}
\]

Integrating equation (4.56) with respect to $\tau$ from $-\infty$ to $\tau$ and taking into account that $\omega(\tau, t)$ and its derivatives with respect to $\tau$ tend to zero as $|\tau| \to \infty$, we obtain
\[
- \phi_t \omega + f(u_0 + \omega) + \omega_{\tau\tau} = f(u_0). \tag{4.57}
\]

Here the integration constant $f(u_0)$ has been derived from the boundary conditions.

Multiplying equation (4.57) by $\omega_\tau$ and integrating it again with respect to $\tau$ from $-\infty$ to $\tau$, we have
\[
- \phi_t \omega^2 + 2 \int_{-\infty}^{\tau} f(u_0 + \omega) \omega_\tau d\tau - 2f(u_0) \int_{-\infty}^{\tau} \omega_\tau d\tau + (\omega_\tau)^2 = 0. \tag{4.58}
\]

Let us introduce the function $\tilde{f}_1(u) = \int_0^u f(z) dz$, $\tilde{f}_1(0) = 0$. Since
\[
\frac{d}{d\tau} \tilde{f}_1(u_0 + \omega) = f(u_0 + \omega) \omega_\tau,
\]
and since $\omega(\tau, t) \to 0$ for $\tau \to -\infty$,
\[
\int_{-\infty}^{\tau} f(u_0 + \omega) \omega_\tau d\tau = \int_{-\infty}^{\tau} \frac{d}{d\tau} \tilde{f}_1(u_0 + \omega) d\tau = \tilde{f}_1(u_0 + \omega) - \tilde{f}_1(u_0)
\]

Now equation (4.58) can be rewritten in the form
\[
- \phi_t \omega^2 + 2 \left[ \tilde{f}_1(u_0 + \omega) - \tilde{f}_1(u_0) \right] - 2f(u_0) \omega + (\omega_\tau)^2 = 0.
\]

Integrating this equation with respect to $\tau$ from $-\infty$ to $\infty$, we find
\[
- \phi_t \Omega_2 - 2f(u_0)\Omega_1 + 2\tilde{\Lambda}_1 + \Omega_\Delta = 0, \tag{4.59}
\]
where $\Omega_1(t) = \int \omega(\tau, t) d\tau$, $\Omega_2(t) = \int \omega^2(\tau, t) d\tau$, $\Omega_\Delta(t) = \int (\omega_\tau)^2(\tau, t) d\tau$,
\[
\tilde{\Lambda}_1(x, t) = \int \left[ \tilde{f}_1(u_0(x, t) + \omega(\tau, t)) - \tilde{f}_1(u_0(x, t)) \right] d\tau. \tag{4.60}
\]

Multiplying (4.57) by $\omega$ and integrating the obtained equation with respect to $\tau$ from $-\infty$ to $\infty$, we find
\[
- \phi_t \Omega_2 + \int \left[ f(u_0 + \omega) - f(u_0) \right] \omega d\tau - \Omega_\Delta = 0, \tag{4.61}
\]

\[16\]
where the last term has been derived by integrating the expression \( \int \omega_\tau \omega \, d\tau \) by parts.

Subtracting equation \( (4.59) \) from the doubled equation \( (4.61) \), we have

\[
- \phi_t \Omega_2 + 2f(u_0)\Omega_1 + 2 \int \left[ f(u_0 + \omega) - f(u_0) \right] \omega \, d\tau - 2\Lambda_1 - 3\Omega_\Delta = 0. \tag{4.62}
\]

Since \( \tilde{f}(u) = 2uf(u) - 2\tilde{f}_1(u) \), where \( \tilde{f}_1(u) = \int_0^u f(z) \, dz \), the function \( \tilde{\Lambda}(x, t) \) from \( (4.53) \) can be rewritten in the form

\[
\tilde{\Lambda}(x, t) = 2 \int \left[ (u_0 + \omega)f(u_0 + \omega) - u_0 f(u_0) \right] \, d\tau - 2 \int \left[ \tilde{f}(u_0 + \omega) - \tilde{f}(u_0) \right] \, d\tau.
\]

Adding and subtracting \( f(u_0)\omega \) in the first integral and comparing the obtained expression to formulae \( (4.48) \) and \( (4.60) \) for \( \Lambda(x, t) \) and \( \tilde{\Lambda}_\Delta(x, t) \), respectively, we obtain

\[
\Lambda(x, t) = 2u_0\Lambda(x, t) + 2f(u_0)\Omega_1 + 2 \int \left[ f(u_0 + \omega) - f(u_0) \right] \omega \, d\tau - 2\tilde{\Lambda}_\Delta(x, t). \tag{4.63}
\]

Setting the coefficient of \( \varepsilon \delta^\prime \) in the first conservation law \( \mathcal{L}[u] = 0 \) equal to zero and taking into account Theorem 4.1, we arrive at the equation

\[-\phi_t \Omega_1(t) + \Lambda(\phi(t), t) = 0. \]

Substituting this relation and relation \( (4.63) \) into equation \( (4.62) \), we obtain equation \( (4.55) \), that is, the coefficient of \( \varepsilon \delta^\prime \) in the second conservation law \( u\mathcal{L}[u] = 0 \). Thus, the equations obtained by setting the coefficients of \( \varepsilon \delta^\prime \) in both conservation laws \( (4.51) \) equal to zero are equivalent.

The proof of the lemma is complete.

As in Section 3.2, this implies the theorem which determines the dynamics of propagation of the soliton solution of equation under consideration.

**Theorem 4.2** Suppose that for \( t \in [0, T] \) there exists a smooth solution \( u_0(x, t) \) to the equation \( u_t + (f(u))_x = 0 \) with the smooth initial condition \( u_0(x, t) \big|_{t=0} = u_0^0(x) \). Suppose that the problem \( (1.21) \) has a solution and \( f''(u) > 0 \) for \( u \in [\omega_1, \omega_2] \).

Then the equation \( u_t + (f(u))_x + \varepsilon^2 u_{xx} = 0 \) for \( t \in [0, T] \) has an infinitely narrow delta-soliton-type solution \( (1.23) \) if and only if the unknown smooth functions \( u_0(x, t), \ g(t), \ e(x, t), \ \phi(t) \) satisfy the following system of equations:

\[
\begin{align*}
\frac{\partial u_0}{\partial t} + f(u_0)x_x &= 0, \\
-\phi_t \Omega_1(t) + \Lambda(\phi(t), t) &= 0, \\
\frac{d}{dt} \Omega_2(t) + 2\Omega_1(t) \frac{d}{dt} u_0(\phi(t), t) &= 0, \\
(\Omega_1) + \left[ f'(u_0(\phi(t), t)) - \frac{\Lambda^\prime(\phi(t), t)}{\Omega_1} \right] e(\phi(t), t) &= 0, \\
\left[ e_0(x, t) + (f'(u_0(x, t))e(x, t))_x \right]_{x<\phi(t)} &= 0,
\end{align*}
\]

where \( \Omega_1 = \int \omega(\tau, t) \, d\tau, \ \Omega_2 = \int \omega^2(\tau, t) \, d\tau, \)

\[
\Lambda(\phi(t), t) = \int \left[ f(u_0(\phi(t), t) + g(t)(\omega(\tau, t) - f(u_0(\phi(t), t))) \right] \, d\tau.
\]

It is easy to see that the system given in Theorem 4.2 and system \( (1.13) \) discussed in the Introduction are quite similar.

In conclusion, we note that all constructions carried out in this paper are formal and the weak asymptotics method is still not completely developed.

This work was partially supported by the Russian Foundation for Basic Research (Grant Nos. 97-01-01123 and 99-01-01074).
References

[1] V. P. Maslov and V. A. Tsupin, Necessary conditions for the existence of infinitely narrow solitons in gas dynamics (Russian) Dokl. Akad. Nauk SSSR, 1979, v. 246, 298–300; Soviet Phys. Dokl., 1979, v.24, N 5, 354–356.

[2] V. P. Maslov and V. A. Tsupin, δ-type generalized in the sense of Sobolev solutions of quasilinear equations, Dokl. Akad. Nauk SSSR, 1979, v. 246, N 2, 298–300.

[3] V. P. Maslov and G. A. Omel’yanov, Asymptotic soliton-form solutions of equations with small dispersion, Russian Math. Surveys, 1981, v. 36, N 3, 1981, 73-119; translated from Uspekhi Mat. Nauk., 1981, v. 36, N 3, 63–126.

[4] V. P. Maslov and G. A. Omel’yanov, On Hugoniot-type conditions for infinitely narrow solutions of equations for simple waves, Sibir. Mat. Zh., 1983, v. XXIV, N 5, 172–182.

[5] Yu. V. Egorov, A contribution to the theory of generalized functions, Russian Math. Surveys, 1990, v. 45, N 5, 1981, 1-49; translated from Uspekhi Mat. Nauk., 1990, v.45, N 5, 3–40.

[6] J. F Colombeau, Elementary introduction to new generalized functions. – North Holland, 1985.

[7] M. Oberguggenberger, Multiplication of distributions and applications to partial differential equations. – N.-Y., 1992.

[8] H. A. Biagioni, M. Oberguggenberger, Generalized solution to the Korteweg-De Vries and the regularized long-wave equations, SIAM J. Math. Anal., 1992, v.23, N 4, 923–940.

[9] Yu. B. Livchak, Towards the theory of generalized functions. Trudy Rizhskogo Algebr. Seminar, Riga 1969, 98–164 (in Russian).

[10] Li Bang-He, Non-standard analysis and multiplication of distributions Acta scientia sinica, 1978, v.21, N 5, 561–585.

[11] V. K. Ivanov, Asymptotical approximation to the product of generalized functions. Izv. Vyssh. Uchebn. Zaved. Mat., 1981, N 1, 19–26 (in Russian).

[12] V. M. Shelkovich, An associative algebra of distributions and multipliers, (Russian) Dokl. Akad. Nauk SSSR, 1990, v. 314, 159–164; translated from Soviet Math. Dokl., 1991, v. 42, 409–414.

[13] V. M. Shelkovich, An associative-commutative algebra of distributions that includes multipliers, generalized solutions of nonlinear equations, Mathematical Notices, 1995, v. 57, N 5, 765–783.

[14] V. G. Danilov, V. P. Maslov, V. M. Shelkovich, Algebra of singularities of singular solutions to first-order quasilinear strictly hyperbolic systems, Theor. Math. Phys., 1998, v. 114, N 1, 1–42.

[15] V. P. Maslov, Three algebras corresponding to nonsmooth solutions of systems of quasilinear hyperbolic equations, Uspekhi Mat. Nauk, 1980, v. 35, N 2, 252–253.

[16] V. P. Maslov, Non-standard characteristics in asymptotical problems. – Proceeding of the International Congress of Mathematicians, August 16-24, 1983, Warszawa, v.I, p. 139–185. North-Holland, Amsterdam, N.Y., Oxford, 1984.

[17] V. G. Danilov, A new definition of weak solutions of semilinear equations with a small parameter Uspekhi Mat. Nauk, 51:5 (1997), p. 184 (in Russian). English transl. in Russian Math. Surveys.

[18] V. G. Danilov, G. A. Omelyanov, E. V. Radkevich, Hugoniot-type conditions and weak solutions to the phase-field system, Euro J. Appl. Math., 1999, v.10, 55–77.

[19] I. A. Molotkov, S. A. Vakulenko, Localised nonlinear waves. – Leningrad: Leningrad University, 1988.
[20] G. B. Whitham, Linear and Nonlinear waves. John Wiley and Sons, New York, London, Toronto, 1974.

[21] V. S. Vladimirov, Generalized Functions in Mathematical Physics. Mir Publ., Moscow, 1979.

[22] H. Bremermann, Distributions, Complex Variables, and Fourier Transforms. – Addison-Wesley Publ.Comp, Reading, Massachusetts, 1965.

[23] Joel Smoller, Shock Waves and Reaction-Diffusion Equations. Springer-Verlag, 1983.

[24] G. L. Lamb, Elements of soliton theory. A Wiley – Interscience Publication. John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1980.