All phase-space linear bosonic channels are approximately Gaussian dilatable

Ludovico Lami\textsuperscript{1,6}, Krishna Kumar Sabapathy\textsuperscript{2,3} and Andreas Winter\textsuperscript{1,2,4}

1 School of Mathematical Sciences and Centre for the Mathematics and Theoretical Physics of Quantum Non-Equilibrium Systems, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom
2 Quantum Information and Communication, École polytechnique de Bruxelles, CP 165, Université libre de Bruxelles, B-1050 Bruxelles, Belgium
3 Xanadu, 372 Richmond St W, Toronto ON, M5V 2L7, Canada
4 Física Teòrica: Informació i Fenòmens Quàntics, Departament de Física, Universitat Autònoma de Barcelona, ES-08193 Bellaterra (Barcelona), Spain
5 ICREA—Institució Catalana de Recerca i Estudis Avançats, Pg. Lluis Companys 23, ES-08010 Barcelona, Spain
6 Author to whom any correspondence should be addressed.

E-mail: ludovico.lami@gmail.com, krishnakumar.sabapathy@gmail.com and andreas.winter@uab.cat

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Abstract

We compare two sets of multimode quantum channels acting on a finite collection of harmonic oscillators: (a) the set of linear bosonic channels, whose action is described as a linear transformation at the phase space level; and (b) Gaussian dilatable channels, that admit a Stinespring dilation involving a Gaussian unitary. Our main result is that the set (a) coincides with the closure of (b) with respect to the strong operator topology. We also present an example of a channel in (a) which is not in (b), implying that taking the closure is in general necessary. This provides a complete resolution to the conjecture posed in Sabapathy and Winter (2017 Phys. Rev. A \textbf{95} 062309). Our proof technique is constructive, and yields an explicit procedure to approximate a given linear bosonic channel by means of Gaussian dilations. It turns out that all linear bosonic channels can be approximated by a Gaussian dilation using an ancilla with the same number of modes as the system. We also provide an alternative dilation where the unitary is fixed in the approximating procedure. Our results apply to a wide range of physically relevant channels, including all Gaussian channels such as amplifiers, attenuators, phase conjugators, and also non-Gaussian channels such as additive noise channels and photon-added Gaussian channels. The method also provides a clear demarcation of the role of Gaussian and non-Gaussian resources in the context of linear bosonic channels. Finally, we also obtain independent proofs of classical results such as the quantum Bochner theorem, and develop some tools to deal with convergence of sequences of quantum channels on continuous variable systems that may be of independent interest.

1. Introduction

In this work we investigate fundamental properties of quantum transformations on modes of electromagnetic radiation field, an example of a continuous variable system, i.e. systems whose degrees of freedom are continuous in nature [1]. In particular, we study optical implementations of what are known as linear bosonic channels, as introduced by Holevo and Werner [2]. The term ‘linear’ has many connotations in quantum optics, but in this article linear channels are those for which the input signal undergoes a linear transformation when described at the level of phase space characteristic functions, i.e. phase-space linear (see equation (15) below). Linearity at the level of density operators is always assumed.

Although the class of linear bosonic channels is special in many respects, it turns out to encompass many examples of physically relevant channels. For instance, all bosonic Gaussian channels [3] are linear, as well as some non-Gaussian operations such as general additive classical noise channels [4], photon-added Gaussian
channels [5], to list a few. Linear channels are instrumental in obtaining benchmarks for teleportation and storage of squeezed states [6], and have been investigated from an information-theoretic point of view with respect to reversibility [7] and extremality [8]. Capacities of the general additive noise channels have been studied in both the classical [9] and quantum settings [10].

An important observation is that the set of linear channels contains all so-called 'Gaussian dilatable channels', defined as those that can be obtained through quadratic interactions between the system and an ancillary environment prepared in an arbitrary state [5]. We will explore this connection in detail in this article. Note that Gaussian dilatable channels can be implemented in a relatively easy way when compared to general non-Gaussian transformations, while still retaining some of the interesting features of the latter. This is particularly important as it is known that non-Gaussian resources, notwithstanding the complexity of their harnessing, are indispensable for many quantum information processing and quantum computation protocols [11].

To motivate the need to go beyond the Gaussian formalism in quantum optics, consider e.g. that in spite of the rich structure Gaussian entanglement exhibits [12–16], it turns out that it can not be distilled using Gaussian operations alone [17–19]. More generally, a similar no-go result holds for generic state conversion tasks in arbitrary Gaussian resource theories [20]. On the other hand, non-Gaussian operations are provably necessary to realize universal quantum computation [21] and many other quantum information processing tasks [22–30]. In view of these limitations, it will not come as a surprise that a substantial effort has been put into developing a consistent resource theory of non-Gaussianity. Many non-Gaussianity measures have been proposed and studied in the past decade [31–36], that can be applied e.g. to bound the conversion rates between arbitrary states by means of Gaussian operations [35]. Recently, a resource theory of non-Gaussianity for channels has also been put forth [37].

Another reason to study Gaussian dilatable channels besides the fact that they constitute one of the few analytically treatable classes of non-Gaussian operations, is that they provide a systematic way of investigating classes of operations with certain physically meaningful properties, e.g. those that can be implemented by means of passive optics and arbitrary states [4] or passive optics with passive ancillary states [38], the latter being motivated from a thermodynamic context, and also for obtaining the operator-sum representations of the corresponding channels [5, 39].

In view of their operational and theoretical importance, in this paper we study the set of linear bosonic channels in detail. Our main result establishes that every linear bosonic channel can be approximated by a sequence of Gaussian dilatable channels. Mathematically speaking, we prove that the closure of the set of Gaussian dilatable channels in the strong operator topology coincides with the set of linear bosonic channels (theorem 8). Taking the closure is necessary, as we show that there are linear bosonic channels that have no Gaussian dilations, even when one allows the ancillary state to have infinite energy. An example is provided by the 'binary displacement channel', which displaces the input state by $\pm s$ or $-s$ (with $s$ fixed) with equal probabilities (corollary 1). Our results solve the question posed in [5], Conjecture 1 (see also [7], Remark 5).

Remarkably, our solution is entirely explicit, and in fact it gives also a feasible experimental procedure to approximate the action of any desired linear bosonic channel by means of a single (possibly non-Gaussian) state and Gaussian unitary dynamics. If this unitary is allowed to vary with the degree of approximation, the ancillary system can be chosen to have the same number of modes as the system on which the channel acts (corollary 9). Incidentally, this also entails that although $2n$ modes may be required for an exact dilation of an $n$-mode Gaussian channel [40], only $n$ ancillary modes suffice if we choose to approximate instead (remark 9). In order to accommodate possible experimental feasibility of our approximation procedure, we also consider the case when the Gaussian unitary is necessarily fixed, and one can only vary the ancillary state. In this setting we are able to construct strong operator approximations of any given $n$-mode linear bosonic channel that require an ancillary system with $n + k$ modes, where $k$ is a number that depends only on the matrix that implements the (linear) phase space transformation induced by the channel (proposition 12).

Our exposition is meant to be entirely self-contained. Along the way, we give independent and simplified proofs of many classical results such as the quantum Bochner theorem (lemma 2) and the complete positivity condition for linear bosonic channels (lemma 3), which may be of independent interest. Our aim is to guide the reader through some of the subtleties of convergence in infinite dimension, and to provide a small handbook of handy convergence results that are of broad applicability in quantum optics. Most notably, we dig out of previous literature a very handy lemma to establish convergence of sequences of density operators in various topologies (the 'SWOT convergence' lemma 4), and we use it to give an analogous criterion for the convergence of sequences of bosonic channels (the 'SWOTTED convergence' lemma 5). We demonstrate how this tool can be used to verify convergence in the strong operator topology—equivalently, uniformly on energy-bounded states—almost effortlessly by applying it to the family of Gaussian additive noise channels that model the transformations induced by the Braunstein–Kimble [41] continuous variable teleportation protocol (remark 4).
The rest of the paper is structured as follows. In section 2 we introduce the basic formalism (section 2.1), define linear bosonic channels (section 2.2), and discuss operator topologies (section 2.3). The following section 3 is devoted to the presentation of our main results. In section 3.1 we start by showing that linear bosonic channels form a strong operator closed set, and in section 3.2 we construct Gaussian dilatable approximations for all such channels. In section 3.3 we discuss an example of a linear bosonic channel that is not exactly Gaussian dilatable, while in section 3.4 we provide a more experimentally feasible procedure to implement the above approximations. Finally, in section 4 we summarise our contributions and highlight some open problems.

2. Bosonic states and channels

2.1. Phase-space formalism

Let us consider a system of \(n\) electromagnetic modes described as quantum harmonic oscillators. The associated Hilbert space is the space of square integrable functions in \(2n\) real variables, denoted by \(H_n := L^2(\mathbb{R}^{2n})\). The quadrature operators \(x_j, p_k (j, k = 1, \ldots, n)\) can be conveniently grouped together to form the vector \(r = (x_1, p_1, x_2, p_2, \ldots, x_n, p_n)^T\). The canonical commutation relations then take the form

\[
[r_j, r_k] = i\Omega_{jk},
\]

where

\[
\Omega := \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}^{\otimes n}
\]

is the standard symplectic form. In what follows we denote by \(\mathcal{H}_n\) the space of trace-class operators acting on \(H_n\).

A particularly important role is played by the family of unitary Weyl–Heisenberg displacement operators, defined as

\[
D(\xi) := e^{i\xi \cdot r},
\]

for all \(\xi \in \mathbb{R}^{2n}\). Applying the Baker–Campbell–Hausdorff formula and making use of equation (1) we see that

\[
D(\xi_1)D(\xi_2) = e^{-\frac{i}{2} \xi_2 \cdot \Omega \cdot \xi_1}D(\xi_1 + \xi_2),
\]

referred to as the Weyl form of the canonical commutation relations. In fact, the formulation in equation (4) is preferable to that in equation (1) in many respects, not least because it involves bounded (unitary) operators instead of unbounded ones.

Among the important properties of these operators, we recall that the associated coherent states \(|\lambda\rangle := D(\lambda)|0\rangle\) satisfy the completeness relation

\[
\langle \alpha | \beta \rangle = \int \frac{d^{2n}\lambda}{(2\pi)^n} \langle \alpha | \lambda \rangle \langle \lambda | \beta \rangle \quad \forall \ |\alpha\rangle, |\beta\rangle \in H_n,
\]

which we can also symbolically write as

\[
\int \frac{d^{2n}\lambda}{(2\pi)^n} |\lambda\rangle \langle \lambda| = I,
\]

where \(I\) is the identity operator. At this level, we regard equation (6) as a purely formal representation of equation (5), so that we do not need to worry about the convergence of the above integral in the operator sense. Mathematically, one says that the integral in equation (6) is understood to converge in the weak operator topology (WOT). For a brief introduction to operator topologies, we refer the reader to section 2.3.

Quantum states are described by density operators, i.e. positive semidefinite trace-class operators of unit trace. The set of all density operators on a Hilbert space \(\mathcal{H}\) will be denoted by \(\mathcal{D}(\mathcal{H})\). To every trace-class operator \(T \in \mathcal{H}_n\) we can associate a characteristic function \(\chi_T : \mathbb{R}^{2n} \rightarrow \mathbb{C}\), defined as

\[
\chi_T(\xi) := \operatorname{Tr}[TD(\xi)].
\]

Characteristic functions are important because they encode all the information about their parent operator, which can be reconstructed as \([42, \text{corollary 5.3.5}]\)

\[
T = \int \frac{d^n\xi}{(2\pi)^n} \chi_T(\xi) D(-\xi),
\]

A positive operator \(A\) acting on a Hilbert space is said to be trace-class if it has finite trace, i.e. if \(\sum_n \langle m|A|m\rangle < \infty\) for some (and thus all) orthonormal bases \(\{|m\rangle\}_m\).
where again the integral converges in the WOT. As originally proved in [43] (see also [42], section 5.3), the above correspondence $T \mapsto \chi_T$, which we have defined only for trace-class $T$, can in fact be extended to an isometry between the Hilbert space of Hilbert–Schmidt operators on $\mathcal{H}_n$ and that of square-integrable functions on $\mathbb{R}^{2n}$, denoted by $L^2(\mathbb{R}^{2n})$. The fact that the mapping is an isometry can be expressed through the noncommutative Parseval’s identity [42, equation (5.3.22)]

$$\text{Tr}[T^* T] = \int \frac{d^2\xi}{(2\pi)^n} \chi_T(\xi) \chi_T(\xi).$$

We can ask what are the conditions a given function $f : \mathbb{R}^{2n} \to \mathbb{C}$ must satisfy to ensure that it is the characteristic function of a quantum state. To answer this question we introduce some terminology.

**Definition 1** [44] Given a skew-symmetric $2n \times 2n$ matrix $A$, a function $f : \mathbb{R}^{2n} \to \mathbb{C}$ is said to be $A$-positive if for all finite collections of vectors $\xi_1, \ldots, \xi_N \in \mathbb{R}^{2n}$ one has

$$f(\xi_1, \ldots, \xi_N) \geq 0,$$

meaning that the matrix on the lhs is positive semidefinite.

**Remark 1.** It is elementary to observe that any $A$-positive function $f : \mathbb{R}^{2n} \to \mathbb{C}$ (for $A$ skew-symmetric) must satisfy $f(-\xi) = f(\xi)$ for all $\xi \in \mathbb{R}^{2n}$, and in particular $f(0)$ must be real (and nonnegative). In fact, it can be seen that an $A$-positive function remains such under inversion of the argument, i.e. the new function $g$ defined by $g(\xi) = f(-\xi)$ is again $A$-positive.

The answer to the above question regarding the physical validity of a characteristic function is then given in terms of the following ‘quantum Bochner theorem’, established in [44, 45] (see also [42, theorem 5.4.1]). In appendix A we provide a direct proof which is independent of the analogous result for classical probability theory and does not seem to have appeared in the literature before.

**Lemma 2 (Quantum Bochner Theorem)** [44, 45]. A complex-valued function $f : \mathbb{R}^{2n} \to \mathbb{C}$ on $\mathbb{R}^{2n}$ is the characteristic function of a density operator if and only if the following three conditions are satisfied:

(i) $f(0) = 1$;

(ii) $f$ is continuous at $0$; and

(iii) $f$ is $\Omega$-positive in the sense of definition 1, where $\Omega$ is given by equation (2).

**Remark 2.** It is known that every function that meets requirements (i)–(iii) of the above lemma 2 is necessarily bounded in modulus by 1 and continuous everywhere (see lemma 14).

We conclude this brief exposition by introducing quantum Gaussian states. These can be equivalently defined as thermal states of Hamiltonians that are quadratic in the canonical operators $r_j$, or as those density operators whose characteristic function is a Gaussian. If

$$\chi_{\mu}(\xi) = e^{-\frac{1}{2} \xi^T V \mu + i\xi^T \Omega \xi}$$

for some $2n \times 2n$ real matrix $V$ and some vector $\mu \in \mathbb{R}^{2n}$, we say that $\rho$ is a Gaussian state with covariance matrix $V$ and mean $\mu$ (compare with [46, equation (4.48)]). Clearly, Gaussian states are uniquely specified by these two quantities. It can be seen that the function on the rhs of equation (11) is the characteristic function of a quantum state if and only if

$$V + i\Omega \succeq 0,$$

which can be regarded as a manifestation of Heisenberg’s uncertainty principle [46], equation (3.77). This well-known fact can also be deduced from lemma 15 in appendix A. The vacuum state $|0\rangle \langle 0|$ is an example of a Gaussian state; its mean is 0 and its covariance matrix is $V = I$, hence

$$\chi_{|0\rangle \langle 0|}(\xi) = e^{-\frac{1}{2} \xi^T \xi}.$$

2.2. Linear bosonic channels

One of the fundamental notions of quantum information theory is that of a quantum channel. A quantum channel describes transformations induced on open physical systems by unitary interactions with an environment that is subsequently discarded [47, chapter 8]. In our context, we can define a bosonic channel acting on $n$ modes as
completely positive and trace-preserving linear map $\Phi$ acting on the set of trace-class operators $T(\mathcal{H}_n)$. It is sometimes convenient to describe the action of the channel at the level of characteristic functions, by writing $\Phi : \chi_\rho(\xi) \mapsto \chi_{\Phi(\rho)}(\xi)$ and giving an explicit expression for this latter function. Equivalently, we can also switch to the Heisenberg picture and specify instead the action of $\Phi^*$ on all displacement operators.

Particularly simple examples of bosonic channels are the so-called linear bosonic channels, that act as
\[
\Phi^*_X f (\xi) := D(X) f (\xi),
\]
where $X$ is a $2n \times 2n$ real matrix, and $f : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ is a complex-valued function. We can rewrite this transformation at the level of characteristic functions as
\[
\chi_\rho(\xi) \mapsto \chi_{\Phi^*(\rho)}(\xi) = \chi_\rho(X f (\xi)).
\]
Because of the simplicity of their phase space action, linear bosonic channels have consistently played a major role in theoretical quantum optics [2,7,8]. It is natural to ask what conditions on $X$ and $f$ ensure that $\Phi^*_{X,f}$ is a legitimate quantum channel. The first solution of this problem was put forward in [48] (see also [49, 50]). We provide a self-contained proof in appendix B.

Lemma 3 [48] A map $\Phi^*_{X,f}$ whose action is given by (15) is completely positive and trace-preserving, and hence a linear bosonic channel, if and only if:

(i) $f(0) = 1$;
(ii) $f$ is continuous at 0; and
(iii) $f$ is $f(X)$-positive according to definition 1, where
\[
J(X) = \Omega - X^T \Omega X.
\]

Observe that every Gaussian channel [46, section 5.3] is a linear bosonic channel, but the converse fails to hold. The simplest example of linear bosonic channel that is not a Gaussian operation is probably the additive noise channel, defined by
\[
\rho \mapsto \int D(-s) \rho D(s) \mu (d^{2n} s),
\]
where $\mu$ is an arbitrary probability measure over $\mathbb{R}^{2n}$, and $f$ is its Fourier transform
\[
f(\xi) = \int e^{i \xi \Omega \mu} d^{2n} \xi.
\]

In a way, linear bosonic channels can thus be thought of as a natural generalisation of Gaussian channels. While this point of view may be mathematically well motivated, it is not operationally satisfactory, because equation (15) does not tell us anything about how to implement a given linear bosonic channel in a physically feasible way. To amend this we can decide to look at Gaussian dilatable channels instead [5]. By definition, a Gaussian dilatable channel acts as
\[
\rho \mapsto \text{Tr}_E [ U_{AE} \rho A \otimes \sigma_E U_{AE}^T ],
\]
where $E$ is an $m$-mode optical system; $U_{AE}$ is a Gaussian unitary on the bipartite system $AE$ (obtained by combining arbitrary displacements on $A$ and $E$ with symplectic unitaries on $AE$ [46, section 5.1.2]); and $\sigma_E$ is an arbitrary state of the system $E$. For a pictorial representation of equation (20), see figure 1. From a practical point of view, remember that the Gaussian unitary $U_{AE}$ can be implemented by means of multimode interferometers (passive optics) and single-mode squeezers [51]. In turn, multimode interferometer can be decomposed into two-mode beam splitters and phase-shifters [52, 53], so that these two operations together with single-mode squeezers suffice to reproduce the action of $U_{AE}$. If we want to specify that an $m$-mode ancillary state suffices to implement a Gaussian dilation, we say that the channel is Gaussian dilatable on $m$ modes. The requirement that $U_{AE}$ is a Gaussian unitary here is crucial; in fact, by Stinespring’s dilation theorem [54] every quantum channel can be represented as in equation (20) for some unitary $U_{AE}$ and some ancillary state $\sigma_E$.

We now derive the explicit action of a Gaussian dilatable channel on characteristic functions. Remember that Gaussian unitaries can be always factorised as $U_{AE} = (D_A(s) \otimes D_E(t)) \hat{U}_{AE}$, where $\hat{U}_{AE}$ is a symplectic unitary whose corresponding symplectic matrix we denote by $S_{AE}$. This means that the transformation $\rho_{AE} \mapsto \hat{U}_{AE} \rho_{AE} \hat{U}_{AE}^T$ translates to $\chi_\rho(\zeta) \mapsto \chi_\rho(S_{AE} \zeta)$ at the phase space level. Decomposing $S_{AE}$ according to the
splitting $A \oplus E$ as

$$
S_{AE} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},
$$

where $X$ is $2n \times 2n$ and $Y$ is $2n \times 2m$, it is not difficult to verify that the channel in equation (20) acts on characteristic functions as [5]

$$
\chi_\rho(\xi) \mapsto e^{i\sum_{j}^{}\chi_\rho (X_\xi) \chi_\sigma (Y_\xi)},
$$

where

$$
X^\dagger \Omega X + Y^\dagger \Omega Y = \Omega.
$$

In particular, every Gaussian dilatable channel is linear bosonic. As we shall see in what follows, the converse is not true (corollary 11). The main result of the present paper is however that every linear bosonic channel can be approximated by Gaussian dilatable channels to any desired degree of accuracy, in a precise sense (theorem 8). See figure 2 for a pictorial representation of all the different classes of channels discussed in this paper.

Note. The two matrices $\Omega$ on the lhs of equation (23) are of sizes $2n$ and $2m$, respectively. Without further specifying it, in what follows we will always assume that all matrices $\Omega$ are of the correct size.

It is worth noting that equation (23) is the only condition to be obeyed in order for the transformation in equation (22) to derive from a Gaussian dilatable channel. In other words, equation (23) implies that one can
find matrices $Z$ and $W$ with the property that the matrix in equation \((21)\) is symplectic. This is a consequence of the completion theorem for symplectic bases [55, theorem 1.15].

As we mentioned before, Gaussian channels are always Gaussian dilatable, and the corresponding state $\sigma_\Phi$ of equation \((20)\) can always be chosen to be Gaussian [40, 56]. Another important subclass of Gaussian dilatable channels is that composed of passive dilatable channels [4, 57], i.e. those for which equation \((20)\) holds with $U_{AE}$ satisfying $[U_{AE}, H_{AE}] = 0$ for the free-field Hamiltonian (number operator) $H_{AE} = \frac{1}{2}r^* r$.

2.3. Operator topologies in a nutshell

As discussed above, the main result we present in this article is that every linear bosonic channel can be approximated to any desired degree of accuracy by Gaussian dilatable channels. To make this statement mathematically rigorous, we need to clarify in what sense this approximation holds. This corresponds to introducing a topology, i.e. a notion of convergence of (generalised) sequences, on the set of quantum channels. Since channels are (super)operators, the relevant concept is that of operator topology.

In the present context, the two most important operator topologies to be given to the space $T(\mathcal{H})$ of trace-class operators on some Hilbert space $\mathcal{H}$ are the norm topology (aka strong topology) and the WOT. A sequence of trace-class operators $T_k$ is said to converge to $T \in T(\mathcal{H})$ in the norm topology (or strongly) if

\[
\|T_k - T\| \to 0, \quad \text{as } k \to \infty,
\]

where $\|T\| := \text{Tr} \sqrt{T^* T}$ is the trace norm. In this case we write also $T_k \overset{\text{strong}}{\longrightarrow} T$. The same sequence is instead said to converge to $T$ in the WOT, and we write $T_k \overset{\text{WOT}}{\longrightarrow} T$, if

\[
\lim_{k \to \infty} \langle \alpha | T_k | \beta \rangle = \langle \alpha | T | \beta \rangle, \quad \forall |\alpha\rangle, |\beta\rangle \in \mathcal{H}.
\]

Remark 3. It is not difficult to come up with sequences of density operators that do not converge strongly but tend to zero in the WOT. An example is given by the sequence $T_k = |k\rangle \langle k|$ of projectors onto the Fock basis states of a single harmonic oscillator. In particular, neither the trace nor the trace norm are continuous with respect to the WOT! For sequences of trace-class operators, it turns out that the above example captures the only way in which the above two notions of convergence can lead to different conclusions.

The following lemma is a well-known result from the theory of operators on Hilbert spaces. The acronym ‘SWOT’ stands for ‘Strong/Weak Operator Topology’.

Lemma 4 (SWOT convergence lemma). Let $(\rho_k)_k \in T(\mathcal{H}_n)$ be a sequence of density operators on the Hilbert space $\mathcal{H}_n$ associated with $n$ harmonic oscillators. Then the following are equivalent:

(i) $(\rho_k)_k$ converges to a density operator in the WOT;

(ii) $(\rho_k)_k$ converges in norm to a trace-class operator;

(iii) the sequence $(\chi_{\rho_k})_k$ of characteristic functions converges pointwise to a function that is continuous at 0.

If any of the above conditions is met, then the two limits in (i) and (ii) are the same, and their characteristic function coincides with the limit in (iii).

The equivalence between conditions (i) and (ii) seems to have been discovered by Davies [58, lemma 4.3], and later generalised by Arazy [59]. Cushing and Hudson [60, theorem 2] proved that (ii) $\iff$ (iii) using Lévy’s continuity theorem [61, theorem 19.1]. The full power of the implication (iii) $\Rightarrow$ (ii) has been exploited for instance in [62]. In appendix C we provide a self-contained proof of lemma 4 for the sake of completeness.

In what follows we will often deal with convergence questions also at the level of superoperators acting on $T(\mathcal{H})$ (e.g. quantum channels). The strongest notion of convergence in this context is that of uniform convergence. A sequence $(\Phi_k)_k$ of maps $\Phi_k : T(\mathcal{H}_A) \to T(\mathcal{H}_A)$ is said to converge to $\Phi : T(\mathcal{H}_A) \to T(\mathcal{H}_A)$ uniformly, and we write $\Phi_k \overset{u}{\longrightarrow} \Phi$, if $\|\Phi_k - \Phi\|_D \overset{k \to \infty}{\longrightarrow} 0$, where the diamond norm is defined by [63, 64]

\[
\|\Delta\|_D := \sup \{\|[(\Delta \otimes \rho_0)(\rho_{AB})]\| : \mathcal{H}_B \text{ Hilbert space}, \rho_{AB} \in D(\mathcal{H}_B \otimes \mathcal{H}_A)\}.
\]

The name ‘uniform’ is justified since the action of the map $\Phi$ is approximated by that of the maps $\Phi_k$ independently of the input state and of the correlations it has with any external system.

As detailed in [65, 66], uniform convergence is often too stringent of a requirement to be useful for applications in quantum information theory. A weaker notion of convergence that is still operationally very relevant is the following: $(\Phi_k)_k$ is said to converge to $\Phi$ in the strong operator topology, and we write $\Phi_k \overset{\text{SOT}}{\longrightarrow} \Phi$, if

\[
\lim_{k \to \infty} \|\Phi_k - \Phi\|_S = 0,
\]
Table 1. Summary of the different notions of convergence/operator topologies on the space of trace-class operators and channels (superoperators) acting on a given Hilbert space \( \mathcal{H} \). Here \( \{ T_k \}_k \) and \( \{ \Phi_k \}_k \) are a sequence of operators and superoperators respectively; \( \| \cdot \|_{F} \), \( \| \cdot \|_{WO} \), and \( \| \cdot \|_{SE} \) denote respectively the trace norm of operators, diamond norm and energy-constrained diamond norm of superoperators. The final column provides the how the sequences \( \{ T_k \}_k \) and \( \{ \Phi_k \}_k \) converge respectively to \( T \) and \( \Phi \). The details of the convergence are explained in this subsection.

| Space | Topology/convergence | Notation | Definition |
|-------|-----------------------|----------|------------|
| Operators | Weak |
| | \( k \to \infty \) | \( T_k \to T \) |
| | \( k \to \infty \) | \( \| T_k - T \|_F \to 0 \) |
| | Strong |
| | \( k \to \infty \) | \( T_k \to T \) |
| | \( k \to \infty \) | \( \| T_k - T \|_SOT \to 0 \) |
| Superoperators | Uniform |
| | \( k \to \infty \) | \( \Phi_k \to \Phi \) |
| | \( k \to \infty \) | \( \| \Phi_k - \Phi \|_0 \to 0 \) |
| | Strong |
| | \( k \to \infty \) | \( \Phi_k \to \Phi \) |
| | \( k \to \infty \) | \( \| \Phi_k - \Phi \|_{SE} \to 0 \) |
| on energy-bounded states |
| | \( k \to \infty \) | \( \Phi_k \to \Phi \) |
| | \( k \to \infty \) | \( \| \Phi_k - \Phi \|_{SE} \to 0 \) |

\[
\Phi_k(\rho) \xrightarrow{k \to \infty} \Phi(\rho) \quad \forall \rho \in \mathcal{D}(\mathcal{H}).
\]  

Observe that in the above definition we can replace \( \mathcal{D}(\mathcal{H}) \) with \( \mathcal{T}(\mathcal{H}) \), as every trace-class operator is a linear combination of at most two quantum states. Strong operator convergence implies that the action of the map \( \Phi \) on any known input state can be approximated by that of the maps \( \Phi_k \); for this reason, it is a strictly weaker condition than uniform convergence. The strong operator topology is studied for instance in [67]. On the mathematical level, one of its main strengths is that the set of completely positive and trace-preserving maps is closed with respect to it, i.e. the strong limit of a sequence of quantum channels is another quantum channel. Another important feature is that compositions and tensor products behave well under limits. More precisely, for all pairs of sequences \( \{ \Phi_k \}_k \), \( \{ \Psi_k \}_k \) of trace-norm bounded maps \( \Phi_k \), \( \Psi_k : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}) \) such that \( \Phi_k \xrightarrow{k \to \infty} \Phi \) and \( \Psi_k \xrightarrow{k \to \infty} \Psi \), one has [65, 67, 68]

\[
\Phi_k \circ \Psi_k \xrightarrow{k \to \infty} \Phi \circ \Psi,
\]

\[
\Phi_k \otimes \Psi_k \xrightarrow{k \to \infty} \Phi \otimes \Psi.
\]

Recently, the strong operator topology was also shown to be connected with the notion of uniform convergence on the set of quantum states of bounded energy [65, 66], a result which greatly bolstered its physical interpretation. In our case, the energy can be defined with respect to the free-field Hamiltonian \( H := \frac{1}{2} r^2 \), and uniform convergence on energy-bounded states is given by

\[
\| \Phi_k - \Phi \|_{SE} \to 0,
\]

where the energy-constrained diamond norm is defined via [65, equation (1)]. We remind the reader that for the special case of interest here, i.e. that of a free-field Hamiltonian, a related definition was previously put forward in [69]. The resulting norm is equivalent to the one of [65, 66], but because of some desirable properties of the latter, we have chosen that definition. We briefly summarise the various notions of convergence for operators and superoperators in table 1.

In this paper we will be mostly interested in the convergence of sequences of bosonic channels to other bosonic channels. In this context a similar result to lemma 4 holds, as we now set out to establish.

**Lemma 5 (SWOTTED convergence lemma).** Let \( \{ \Phi_k \}_k \) be a sequence of bosonic channels \( \Phi_k : \mathcal{T}(\mathcal{H}_n) \to \mathcal{T}(\mathcal{H}_n) \).

Then the following are equivalent:

\( i \) \( \Phi_k \) converges to a bosonic channel uniformly on energy-bounded states (i.e. equation (29) is satisfied);

\( ii \) \( \Phi_k \) converges in the strong operator topology;

\( iii \) for all \( |\psi\rangle \in \mathcal{H}_n \) the sequence of characteristic functions \( (\chi_{\Phi_k(\rho)}(\psi))_{k} \) converges pointwise to a function that is continuous at 0.

If any of the above conditions is met, then the limit in (i) is the same as that in (ii) (call it \( \Phi \)), and that in (iii) is \( \chi_{\Phi(\rho)}(\psi) \).

**Note.** The acronym ‘SWOTTED’ stands for ‘Strong/Weak Operator Topology/Toplogy of Energy-constrained Diamond norms’.
Proof. The fact that (i) and (ii) are equivalent is proved in [65, proposition 3(B)]. Observe that our Hamiltonian $H = \frac{1}{2} r^T r$ satisfies the required conditions. Moreover, applying lemma 4 with $\rho_k = \Phi_k(|\psi\rangle\langle\psi|)$ we deduce that (ii) implies (iii). All that is left to prove is that (iii) implies (ii).

Start by observing that if (iii) holds then for all finite-rank operators $A$ the sequence $\chi_{\Phi_k(A)}$ converges pointwise to a function that is continuous at 0. For a given density operator $\rho$ and some fixed $\epsilon > 0$, choose a projector $\Pi$ onto a finite-dimensional space such that $\|\rho - \Pi\rho\Pi\| \leq \epsilon$ (this is possible because $\rho$ is trace-class). Since $\rho_k := \Phi_k(\Pi\rho\Pi)$ satisfies lemma 4(iii), from lemma 4(ii) we deduce that $\Phi_k(\Pi\rho\Pi) \xrightarrow{k \to \infty} \Phi(\Pi\rho\Pi)$, i.e. $\|\Phi_k(\Pi\rho\Pi) - \Phi(\Pi\rho\Pi)\| \leq \epsilon$ for all $k \geq k_0$. Using the fact that quantum channels are trace-norm contractions, we find

$$\|\Phi(\rho) - \Phi_k(\rho)\| \leq \|\Phi(\rho) - \Phi(\Pi\rho\Pi)\| + \|\Phi(\Pi\rho\Pi) - \Phi_k(\Pi\rho\Pi)\| + \|\Phi_k(\Pi\rho\Pi) - \Phi_k(\rho)\|$$

$$\leq 2\|\rho - \Pi\rho\Pi\| + \|\Phi(\Pi\rho\Pi) - \Phi_k(\Pi\rho\Pi)\| + \|\Phi_k(\Pi\rho\Pi) - \Phi_k(\rho)\|$$

$$\leq 3\epsilon$$

for all $k \geq k_0$. Since $\epsilon$ was arbitrary, we deduce that $\Phi_k(\rho) \xrightarrow{k \to \infty} \Phi(\rho)$ for all density operators $\rho$, implying that $\Phi_k \xrightarrow{SOT} \Phi$.

Remark 4. Equipped with lemma 5, it is really elementary to show that the Braunstein–Kimble continuous variable teleportation protocol [41] performed as a resource a two-mode squeezed state of energy $E$ implements a sequence of channels that converge to the identity in the strong operator topology [41, 68] or—equivalently—uniformly on bounded energy states, but not uniformly on all states [41, 70]. Indeed, the transformation in [41, equation (9)] can be rewritten at the level of characteristic functions as $\chi_{\rho}(\alpha) \mapsto \chi_{\rho}(\alpha)e^{-\sigma|\alpha|^2}$, where we temporarily reverted to the complex notation. Here, $\sigma$ is a parameter that will be made to tend to 0 to increase the accuracy of the approximation. Since the function on the rhs clearly converges pointwise to that on the lhs for all fixed input states $\rho$, lemma 5 guarantees that the channel convergence happens with respect to the strong operator topology, or uniformly on energy-bounded states. These statements have been the subject of discussion in some recent papers [68, 70].

Remark 5. Another elementary consequence of lemma 5 concerns the family of Gaussian channels known as quantum limited attenuators. These are channels acting on an arbitrary number of modes and defined by

$$A_{\lambda}: \chi_{\rho}(\xi) \mapsto \chi_{\rho}(\sqrt{\lambda}\xi)e^{-\lambda|\xi|^2},$$

where $0 \leq \lambda \leq 1$. A swift application of lemma 5 shows that $A_{\lambda} \xrightarrow{SOT} \text{id}$, with id being the identity channel, a fact already mentioned in [66, section IV.B]. In [66, proposition 1] it is also shown that the convergence in equation (31) is not uniform.

3. Main results

We have seen that linear bosonic channels constitute a mathematically natural generalisation of the set of Gaussian channels. On the other hand, Gaussian dilatable channels are a physically meaningful class of operations for which there exists an operationally feasible implementation that requires only moderate resources (namely, a single non-Gaussian state and a Gaussian unitary). The main result of the present paper is that these two sets of operations are deeply related with each other, namely the latter is the strong operator closure of the former (theorem 8).

3.1. The set of linear bosonic channels is closed

Implicit in the above statement is the claim that linear bosonic channels form themselves a closed set. Let us start by investigating this question, whose affirmative answer is the content of our first result. Our proof technique rests crucially upon the SWOTTED convergence lemma (lemma 5).

Theorem 6. The set of all linear bosonic channels acting on an $n$-mode system is closed with respect to the strong operator topology.
Proof. Consider a sequence of linear bosonic channels \((\Phi_{X_k})\), and assume that \(\Phi_{X_k} \xrightarrow{\text{SOT}} \Phi\) for some completely positive trace-preserving map \(\Phi : \mathcal{T}(\mathcal{H}_a) \rightarrow \mathcal{T}(\mathcal{H}_a)\). We have to prove that also \(\Phi\) is a linear bosonic channel. To this end, start by considering the sequence of matrices \((X_k)\). If the sequence \((X_k^\dagger X_k)_{j\rightarrow k}\) is unbounded for some fixed \(j = 1, \ldots, 2n\), then for almost all \(\xi\) the sequence \(X_k^\dagger X_k \xi\) is also unbounded (this does not happen only on the zero-measure hyperplane \(\xi = 0\)). Remembering that the characteristic function of the vacuum state is given by equation (13), we deduce that the sequence of complex numbers \(\chi_{\{0\}}(X_k \xi)\) has a vanishing subsequence for almost all \(\xi\). Since all functions \(f_k\) are upper bounded by 1 in modulus, the same is true for the sequence \(\chi_{\{0\}}(X_k^\dagger X_k f_k(\xi)) = \chi_{\Phi_{X_k}(\{0\})}(\{0\})\). This however converges to \(\chi_{\Phi(\{0\})}(\{0\})\) by lemma 5, hence we conclude that \(\chi_{\Phi(\{0\})}(\{0\}) = 0\) for almost all \(\xi\). As \(\chi_{\Phi(\{0\})}(\{0\})\) is necessarily continuous, we conclude that it must vanish everywhere. This is absurd as \(\chi_{\Phi(\{0\})}(\{0\}) = 1\) by the quantum Bochner theorem (lemma 2).

We then conclude that \((X_k^\dagger X_k)_{j\rightarrow k}\) is bounded for all \(j\), and hence there exists \(M > 0\) such that \(\|X_k\|_{\text{inf}} = \max_j\|X_k^\dagger X_k\| \leq M\). This means that the sequence \((X_k^\dagger X_k)_{j\rightarrow k}\) is itself bounded, because \(\|\|_{\text{inf}}\) is a norm on the set of matrices. Since we are in a finite-dimensional space, there will exist a converging subsequence \(X_{k_m} \rightarrow X\).

We now show that \(X\) is the limit of \((X_k)\). In fact, if this were not the case there would exist another limit point \(X' \neq X\), i.e. \(X_{k_n} \rightarrow X'\) for some other subsequence \((X_{k_n})\). We can use the fact that \(\Phi_{X_k}(\rho) \rightarrow \Phi(\rho)\) for all density operators \(\rho\) to deduce a contradiction. Choose as \(\rho\) a Gaussian state with covariance matrix \(1\) and mean \(s\), i.e. \(\rho = |s\rangle\langle s|\) where \(|s\rangle\langle s|\) is a coherent state. Using the explicit expression in equation (11) for the characteristic function together with lemma 5, we obtain

\[
\lim_{k} e^{-\frac{1}{2} \text{tr} X_k^\dagger X_k \xi + i\omega' \Omega X_k \xi} f_k(\xi) = \lim_{k} \chi_{\Phi_{X_k}(|s\rangle\langle s|)}(\xi) = \chi_{\Phi(|s\rangle\langle s|)}(\xi).
\]

Since on the two subsequences \((k_m)\) and \((k_n)\) the exponential factor on the lhs of the above equation converges to a nonzero limit, also the sequences \((f_{k_m})_m\) and \((f_{k_n})_n\) have to converge. Let us define

\[
g(\xi) : = \lim_{m} f_{k_m}(\xi), \quad g'(\xi) : = \lim_{n} f_{k_n}(\xi).
\]

Taking the limit in equation (32) over these two subsequences yields

\[
e^{-\frac{1}{2} \text{tr} X^\dagger X \xi + i\omega' \Omega X \xi} g(\xi) = \lim_{m} e^{-\frac{1}{2} \text{tr} X_{k_m}^\dagger X_{k_m} \xi + i\omega' \Omega X_{k_m} \xi} f_{k_m}(\xi)
= \chi_{\Phi(|s\rangle\langle s|)}(\xi)
= \lim_{m} e^{-\frac{1}{2} \text{tr} X_{k_m}^\dagger X_{k_m} \xi + i\omega' \Omega X_{k_m} \xi} f_{k_m}(\xi)
= e^{-\frac{1}{2} \text{tr} X^\dagger X \xi + i\omega' \Omega X \xi} g'(\xi).
\]

Since \(\chi_{\Phi(|s\rangle\langle s|)}(\xi) = 1\) and \(\chi_{\Phi(|s\rangle\langle s|)}(\xi)\) is continuous at \(0\) by the quantum Bochner theorem (lemma 2), in an appropriate neighbourhood \(U\) of \(0\) in \(\mathbb{R}^{2n}\) it must hold that \(\chi_{\Phi(|s\rangle\langle s|)}(\xi) = 0\). From the above chain of equalities we deduce also that \(g(\xi) = 0\) for all \(\xi \in U\). Hence

\[
e^{-\frac{1}{2} \text{tr} X^\dagger X \xi + i\omega' \Omega X \xi} = \frac{g'(\xi)}{g(\xi)} \quad \forall \xi \in U.
\]

The problem with equation (34) is that the rhs does not depend explicitly on \(s\). The only way in which this can happen on the lhs as well is if \(X = X'\), which is what we wanted to prove.

We have established that \(X_k \rightarrow X\). Then, the identity in equation (32) can only hold if the sequence \((f_k)\) converges pointwise to some function \(f\) i.e. \(f_k(\xi) \rightarrow f(\xi)\) for all fixed \(\xi\). For an arbitrary state \(\rho\) we then obtain

\[
\chi_{\Phi(\rho)}(\xi) = \lim_{k} \chi_{\Phi_{X_k}(\rho)}(\xi) = \lim_{k} \chi_{\rho}(X_k \xi) f_k(\xi) = \chi_{\rho}(X \xi) f(\xi),
\]

where the last equality follows from the continuity of \(\chi_{\rho}\). This shows that also \(\Phi\) is a linear bosonic channel. \(\square\)

3.2. Approximate Gaussian dilatation for any linear bosonic channel

Before we state our main result, we look at the simplified case where the matrix \(J(X)\) of equation (16) is invertible. When this happens, it turns out that the problem of writing linear bosonic channels as (limits of) Gaussian dilatatable channels simplifies considerably.

Lemma 7 [2] Any linear bosonic channel \(\Phi_{X_k}\) that acts on an \(n\) -mode system and satisfies \(\det J(X) = 0\), where \(J(X)\) is defined by equation (16), is Gaussian dilatable using \(n\) auxiliary modes.
Proof. It is a well-known fact from elementary linear algebra that all invertible skew-symmetric matrices are equivalent up to congruences [71, Corollary 2.5.14(b)]. In particular, if \( \det J(X) \neq 0 \) there will exist a \( 2n \times 2n \) (invertible) matrix \( Y \) such that

\[
\Omega - X^T \Omega X = J(X) = Y^T \Omega Y.
\]  

(35)

Remember that the complete positivity conditions for \( \Phi_{X,f} \) as expressed by lemma 3 imply that \( f \) is \( J(X) \)-positive. It is not difficult to verify that the function \( g : \mathbb{R}^{2n} \rightarrow \mathbb{C} \) given by

\[
g(\xi) = f(Y^{-1}\xi)
\]  

(36)

will then be \( \Omega \)-positive. Since \( g(0) = f(0) = 1 \) and moreover \( g \) is clearly continuous at 0 as the same is true for \( f \), all conditions of the quantum Bochner theorem (lemma 2) are met, and hence \( g(\xi) = \chi_x(\xi) \) for some \( n \)-mode quantum state \( \sigma \). Substituting \( \xi \mapsto Y^*\xi \) we can then rewrite equation (36) as \( f(\xi) = \chi_{Xf}(Y^*\xi) \). Inserting this into equation (15) we obtain equation (22) for \( s = 0 \). Since \( X \) and \( Y \) satisfy equation (23), the channel \( \Phi_{X,f} \) is Gaussian dilatable. This is summarised as algorithm 1.

\[ \square \]

Remark 6. The above argument shows that any pair \( X, f \) that satisfies conditions (i)--(iii) of lemma 3 and such that \( \det J(X) \neq 0 \) induces a map \( \Phi_{X,f} \) that admits the representation in equation (20) for some Gaussian unitary \( U_{AE} \) and some quantum state \( \sigma \). In particular, \( \Phi_{X,f} \) must be a bosonic channel (completely positive and trace-preserving). This proves that conditions (i)--(iii) of lemma 3 are sufficient in order for \( \Phi_{X,f} \) to be a bosonic channel, at least when \( \det J(X) \neq 0 \). This observation can be used to give an independent proof of lemma 3, see remark 7 and appendix B.

Algorithm 1. Obtaining an exact Gaussian dilation for a linear bosonic channel with invertible \( J(X) \).

| Input: X, f of a linear bosonic channel \( \Phi_{X,f} \). |
| 1 Compute the matrix \( Y \) whose existence is guaranteed by equation (35). |
| 2 Obtain \( g \) using equation (36). |
| 3 Determine \( \sigma \) (the ancilla state) from \( g \) using equation (8); this is possible by lemma 2. |
| 4 Complete \( \begin{pmatrix} X & Y \end{pmatrix} \) to a symplectic matrix \( S_{AE} \). |
| 5 Obtain the Gaussian unitary \( U_{AE} \) for the dilation from \( S_{AE} \). |

Output: Gaussian dilation of \( \Phi_{X,f} \).

Lemma 7 solves our problem as long as \( J(X) \) is an invertible matrix. However, a quick inspection reveals that this is not always the case, the most notable counterexample being the identity channel, for which we have \( X = 1 \) and hence \( J(X) = 0 \). We are now ready to state and prove our main result.

Theorem 8. The set of linear bosonic channels coincides with the closure of the set of Gaussian dilatable channels with respect to the strong operator topology.

Proof. Since we know from theorem 5 that the set of linear bosonic channels is strong operator closed, we only have to show that the closure of the set of Gaussian dilatable channels contains it. To this end, pick a linear bosonic channel \( \Phi_{X,f} \). For all sufficiently small \( \epsilon > 0 \), let us consider the channels \( \Phi_{X,f} \circ A_{(1-\epsilon^2)} \), where \( A_{\epsilon} \) is a quantum limited attenuator as defined by equation (30), and \( \circ \) denotes composition. It can be readily verified by concatenating the transformations in equation (30) and (15) that

\[
\Phi_{X,f} \circ A_{(1-\epsilon^2)} = \Phi_{X,f},
\]  

(37)

is another linear bosonic channel of parameters

\[
X_{\epsilon} := (1 - \epsilon)X, \quad f_{\epsilon}(\xi) := f(\xi) e^{-\frac{1}{2} (1-(1-\epsilon^2))\xi^T X_{\epsilon} \xi}.
\]  

(38)

(39)

Even if the matrix \( J(X) \) of equation (16) fails to be invertible, we claim that \( J(X_{\epsilon}) \) is invertible for all sufficiently small \( \epsilon > 0 \). This can be easily deduced by observing that \( p(\epsilon) := \det J(X_{\epsilon}) \) is a polynomial in \( \epsilon \) which satisfies

\[
p(1) = \det J(0) = \det \Omega = 1, \quad \text{hence it is not identically zero and thus has only isolated zeros.}
\]

Now, since \( J(X_{\epsilon}) \) is invertible, lemma 7 implies that \( \Phi_{X,f} \) is Gaussian dilatable. Moreover, since \( A_{(1-\epsilon^2)} \xrightarrow{\text{SOT}} \text{id} \) by equation (31), and channel composition behaves well under strong topology convergence, as formalised in equation (27), we have that

\[ \square \]
This shows that any linear bosonic channel is the strong operator limit of a sequence of Gaussian dilatable channels, concluding the proof.

**Remark 7.** Continuing along the lines of remark 6, we argue that the above construct in fact proves also that conditions (i)-(iii) of lemma 3 are sufficient in order for $\Phi_{X,f}$ to be a bosonic channel. This is because we constructed a sequence of strong operator approximations to $\Phi_{X,f}$ that are guaranteed to be bosonic channels (because they are Gaussian dilatable), and the set of all channels is closed with respect to the strong operator topology. For further details see appendix B.

Our proof of theorem 8 is constructive, in that we give an explicit sequence of Gaussian dilatable channels that approximates a given linear bosonic channel. Among other things, this shows that $n$-mode ancillary states suffice to implement Gaussian dilatable approximations. We formalise this observation below and summarise the steps for obtaining the approximate dilation in algorithm 2.

**Corollary 9.** Every linear bosonic channel on $n$ modes can be approximated to any desired degree of accuracy in the strong operator topology by channels that are Gaussian dilatable on $n$ modes only.

**Algorithm 2.** Obtaining an approximate Gaussian dilation for any linear bosonic channel using $n$ auxiliary modes.

**Input:** $X, f$ of a linear bosonic channel $\Phi_{X,f}$.

1. Choose $\epsilon > 0$ such that $f((1 - \epsilon)X)$ is non-singular.
2. Compose the original channel with an attenuator: $\Phi_{X,f} = \Phi_{X,f} \circ A_{(1-\epsilon)}$.
3. Obtain a Gaussian dilation for $\Phi_{X,f}$ using algorithm 1.
4. Limit $\epsilon \rightarrow 0$ provides the required approximation to $\Phi_{X,f}$ via Gaussian dilatable channels $\Phi_{X,f}$.

**Output:** Approximate Gaussian dilation of $\Phi_{X,f}$.

**Remark 8.** Corollary 9 also applies to channels that are already Gaussian dilatable, but whose corresponding ancillary state $\sigma_f$ (of equation (20)) is specified on a large number of modes ($> n$) irrespective of it being Gaussian or not. The resulting resource compression could prove useful from a practical point of view.

**Remark 9.** A special case that is of great interest is that of Gaussian channels. In this context, corollary 9 should be compared with the results of [40]. There the authors give an upper bound on the number of modes that are needed in order to construct a Gaussian dilation of any given Gaussian channel (with the ancillary state in equation (20) being also Gaussian). For instance, for a Gaussian additive noise channel, i.e. an additive noise channel as in equation (18) whose function $f$ is Gaussian, a $2n$-mode ancilla is required [40, equation (48)]. However, our findings show that it is possible to construct approximate Gaussian dilations of any $n$-mode Gaussian channel by means of an $n$-mode ancillary system only. Moreover, the proof of theorem 8 shows that in this case the corresponding states $\sigma$ can be chosen to be Gaussian.

**3.3. Why the closure is necessary**

A very natural question at this point is the following: *is the closure in theorem 8 really necessary?* In other words, are there examples of linear bosonic channels that are not Gaussian dilatable? Here we argue that this is indeed the case by constructing an explicit example. This question has been recently investigated by the authors of [37], who developed a body of techniques to study non-Gaussian operations. Using these techniques, they were able to provide an example of a non-Gaussian channel (the ‘binary phase-shift channel’ that applies a phase space inversion with probability 1/2 on a single mode) that does not admit an exact Gaussian dilation in which the ancilla has finite energy. Observe that the channel they considered is not linear bosonic, so this result does not have immediate implications for the above question. Moreover, it does not seem possible to employ the techniques in [37], which rely crucially on the theory of relative entropy measures of non-Gaussianity [31, 34], to exclude the existence of a Gaussian dilation whose corresponding ancillary state does not have finite first- or second-order moments.

One could argue that an exact Gaussian dilation such as that in equation (20) in which $\sigma$ has unbounded energy can not be physically realised if not approximately. However, in this case the approximation one can aim for is uniform instead of merely in the strong operator topology. Indeed, it is not difficult to realise that if $\Pi$ is a projector onto a finite-dimensional space such that $\sigma' := \frac{\Pi \sigma \Pi}{\text{Tr}[\sigma \Pi]}$ —which has finite energy—satisfies...
\( \| \sigma - \sigma' \|_1 \leq \epsilon \), the Gaussian channel \( \Phi' \) defined by the same formula as in equation (20) with \( \sigma' \) instead of \( \sigma \) (and the same unitary \( U \)) satisfies \( \| \Phi - \Phi' \|_0 \leq \epsilon \), where the diamond norm \( \| . \|_0 \) is defined by equation (25). In view of these considerations, we see that an exact Gaussian dilation for a quantum channel is still operationally meaningful even if the involved ancilla has infinite energy, because it leads to a sequence of uniform approximations via physically implementable Gaussian dilatable channels. This situation should be contrasted with the approximations we constructed in the proof of theorem 8, which are with respect to the strong operator topology and therefore ‘less accurate’ in a precise sense.

In what follows we give an explicit example of a linear bosonic channel that is not exactly Gaussian dilatable, even if the ancillary state used for the dilation is allowed to have infinite energy. The channel we consider is a ‘binary displacement channel’, which is a particular example of an additive noise channel (see equation (17)) and is defined by the action

\[
E(r) := \frac{1}{2} D(-s) \rho D(s) + \frac{1}{2} D(s) \rho D(-s) \quad \forall \rho \in \mathcal{D}(\mathcal{H}_n),
\]

(41)

where \( 0 \neq s \in \mathbb{R}^n \) is a fixed vector. At the phase space level equation (41) translates to

\[
E_r : \chi_r(\xi) \mapsto \chi_r(\xi) \cos(s^T \Omega \xi),
\]

(42)

which proves that \( E_r \) is indeed a linear bosonic channel. We now set out to show that it cannot be Gaussian dilatable.

**Lemma 10.** Any characteristic function \( \chi_r \) of an m-mode quantum state \( \sigma \) satisfies

\[
| \chi_r(\zeta) | < 1 \quad \forall \zeta \in \mathbb{R}^m, \ zeta = 0.
\]

(43)

**Proof.** We already commented on the fact that characteristic functions of quantum states are upper bounded by 1 in modulus (see lemma 14). The problem is to show that there is strict inequality in equation (43) whenever \( \zeta = 0 \). Let \( \sigma = \sum_{\mu=0}^{\infty} p_\mu | \psi_\mu \rangle \langle \psi_\mu | \) be the spectral decomposition of \( \sigma \), where the series converges strongly (i.e. in trace norm). Let us write

\[
| \chi_r(\zeta) | = \text{Tr}[\sigma D(\zeta)] = \sum_{\mu=0}^{\infty} p_\mu | \langle \psi_\mu | D(\zeta) | \psi_\mu \rangle | \leq \sum_{\mu=0}^{\infty} p_\mu | \langle \psi_\mu | D(\zeta) | \psi_\mu \rangle | = 1,
\]

where we used the fact that \( D(\zeta) \) is unitary. From the above chain of inequalities it is clear that \( | \chi_r(\zeta) | = 1 \) if and only if \( D(\zeta) | \psi_\mu \rangle = e^{i \zeta} | \psi_\mu \rangle \) for all \( \mu \) such that \( p_\mu > 0 \), where \( \psi_\mu \in \mathbb{R}^n \). In particular, there exists a vector \( | \psi \rangle \in \mathcal{H}_m \) such that \( D(\zeta) | \psi \rangle = e^{i \zeta} | \psi \rangle \), implying that \( D(\zeta) | \psi \rangle \langle \psi | D(-\zeta) = | \psi \rangle \langle \psi | \). Computing the trace against a displacement operator \( D(\eta) \) of both sides of this identity and using equation (4) we obtain

\[
e^{i \zeta D(\eta)} \chi_r(\zeta \eta) = e^{i \zeta \eta} \text{Tr}[D(\eta) | \psi \rangle \langle \psi |] = e^{i \zeta \eta} \text{Tr}[D(-\zeta) D(\eta) D(\zeta) | \psi \rangle \langle \psi |]
\]

\[
= e^{i \zeta \eta} \text{Tr}[D(\eta) D(\zeta) | \psi \rangle \langle \psi | D(-\zeta)] = e^{i \zeta \eta} \chi_r(\zeta \eta) = \chi_r(\zeta \eta).
\]

from which we deduce that \( \chi_r(\zeta \eta) = 0 \) whenever \( e^{i \zeta \eta} = 1 \), i.e. whenever \( [D(\zeta), D(\eta)] \rangle \leq 0 \). Since \( \zeta = 0 \) and \( \Omega \) is invertible, this happens almost everywhere in \( \eta \). The continuity of \( \chi_r(\zeta \eta) \) as established in lemma 14 entails that \( \chi_r(\zeta \eta) \equiv 0 \) for all \( \zeta \), which is in contradiction with the fact that \( \chi_r(\zeta \eta) = 1 \) as required by the quantum Bochner theorem (lemma 2). \( \square \)

**Corollary 11.** The linear bosonic channel \( E_r \) defined in equation (41) is not Gaussian dilatable for \( s \neq 0 \).

**Proof.** Assume by contradiction that \( E_r \) were Gaussian dilatable. Comparing equation (42) with equation (22), we see that we would be able to find an \( m \)-mode state \( \sigma \) and a \( 2m \times 2n \) matrix \( Y \) such that \( Y^T \Omega Y = 0 \) and

\[
\chi_r(Y \xi) = \cos(s^T \Omega \xi).
\]

(44)

Observe that for all \( \xi \in \mathbb{R}^n \) such that \( s^T \Omega \xi = 0 \) we have that \( \xi = \frac{2 \pi}{s^T \Omega} \xi \) satisfies

\[
\chi_r(Y \xi) = \cos(s^T \Omega \xi) = \cos(2\pi) = 1.
\]

Applying lemma 10, we conclude that this is only possible if \( Y \xi = 0 \) and hence \( Y \xi = 0 \). Since the constraint \( s^T \Omega \xi = 0 \) holds almost everywhere in \( \xi \), and linear transformations are continuous, we conclude that \( Y = 0 \) as a matrix. This is naturally in contradiction with equation (44), as the rhs of the equation is not constant. \( \square \)
Along the same lines, we suspect that the example in equation (41) can be generalised as to encompass all discrete convex combinations of conjugations by different displacement operators. We conjecture that all such linear bosonic channels are not exactly Gaussian dilatable.

3.4. Number of auxiliary modes required

We now discuss the practical feasibility of approximating any linear bosonic channel with Gaussian dilatable maps. The main advantage of the technique we adopted to prove theorem 8 is that it is fully explicit, and that it requires an ancillary system with the same number of modes as the system itself (corollary 9). However, the main disadvantage is that when \( J(X) = 0 \) we had to construct a sequence of Gaussian dilations with varying Gaussian unitaries and varying ancillary states. This may be far from practical from an experimental point of view, as it would be more desirable to fix the unitary while varying only the ancillary state. It turns out that it is possible to do so, as we will now show. The following result is a generalisation of lemma 7, and it is proved in the same spirit.

**Proposition 12.** Let \( \Phi_{X,f} \) be a linear bosonic channel acting on an \( n \)-mode system \( A \). Let \( E \) be an ancillary system composed of \( n + k \) modes, where \( k := \frac{1}{2} \dim \ker J(X) \), and \( J(X) \) is defined by equation (16). Then there is a Gaussian unitary \( U_{AE} \) and a sequence of states \( \sigma_\epsilon(\epsilon) \) such that the corresponding Gaussian dilatable channels defined through equation (20) converge to \( \Phi_{X,f} \) in the strong operator topology for \( \epsilon \to 0^+ \).

**Proof.** We start by looking at the skew-symmetric matrix \( J(X) \) defined by equation (16). We can find an orthogonal matrix \( O \in O(2n) \) such that [71], Corollary 2.5.14(b)

\[
J(X) = O^\top \left( \bigoplus_{j=1}^n \begin{pmatrix} 0 & d_j \\ -d_j & 0 \end{pmatrix} \right) O,
\]

where \( d_j \geq 0 \) are \( n \) real numbers. Defining \( k := \frac{1}{2} \dim \ker J(X) \), we see that exactly \( k \) of these numbers are zero.

We now construct a \( (2n + 2k) \times 2n \) matrix \( Y \) defined as follows:

\[
Y := \left( \bigoplus_{j=1}^n Y_j \right) O,
\]

\[
Y_j = \begin{cases} 
0 & \text{if } d_j = 0, \\
1 & \text{if } d_j > 0,
\end{cases}
\]

We first show that this \( Y \) satisfies equation (23). Before we do that, observe that the standard symplectic form of \( n + k \) modes can be written as

\[
\Omega := \bigoplus_{j=1}^n \Omega_j,
\]

\[
\Omega_j := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

if \( d_j > 0 \),

\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

if \( d_j = 0 \).

With this in mind, it is easy to check that indeed

\[
Y^\top \Omega Y = O^\top \left( \bigoplus_{j=1}^n \Omega_j Y_j \right) O = O^\top \left( \bigoplus_{j=1}^n \begin{pmatrix} 0 & d_j \\ -d_j & 0 \end{pmatrix} \right) O = \Omega - X^\top \Omega X.
\]

Now, consider the Moore–Penrose inverse of \( Y \), call it \( \tilde{Y} \). One has

\[
\tilde{Y} = O^\top \left( \bigoplus_{j=1}^n \tilde{Y}_j \right),
\]

\[
\tilde{Y}_j = \begin{cases} 
\frac{1}{d_j} I_2 & \text{if } d_j > 0, \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } d_j = 0,
\end{cases}
\]

Observe that

\[
\tilde{Y} Y = 1_{2n},
\]
where \(I_{2n}\) denotes the \(2n \times 2n\) identity matrix, while \(P\) is an orthogonal projector acting on \(\mathbb{R}^{2n+2k}\). Define a family of \((2n + 2k) \times (2n + 2k)\) matrices \(W(\epsilon)\) (where \(\epsilon > 0\) will later be taken to 0) via the identities

\[
W(\epsilon) := \bigoplus_{j=1}^{n} W_j(\epsilon),
\]

\[
W_j(\epsilon) := \begin{cases} 
0 & \text{if } d_j > 0, \\
\epsilon & \text{if } d_j = 0.
\end{cases}
\]

Because of the way it is constructed, it is elementary to verify that \(W(\epsilon)\) obeys

\[
W(\epsilon) \geq i(\Omega - P\Omega).
\]

Moreover, note that

\[
Y^\top W(\epsilon) Y = \epsilon Q,
\]

where \(Q\) is an orthogonal projector acting on \(\mathbb{R}^{2n}\) and defined by

\[
Q := O^\top \left( \bigoplus_{j=1}^{n} Q_j' \right) O,
\]

\[
Q_j' := \begin{cases} 
0 & \text{if } d_j > 0, \\
1 & \text{if } d_j = 0.
\end{cases}
\]

We are in position to define the quantum state \(\sigma\), which will pertain to an \((n + k)\)-mode system. Let us set

\[
\chi_{\sigma(\epsilon)}(\eta) := f \left( \tilde{Y} \eta - \epsilon e^{-\frac{1}{2}Q \bar{W}(\epsilon) \eta} \right).
\]

We now claim that for all \(\epsilon > 0\): (a) \(\sigma(\epsilon)\) is a legitimate quantum state; and (b) \(\chi_{\sigma(\epsilon)}(\bar{Y} \xi) = f(\xi) e^{-\frac{1}{2}Q \bar{W}(\epsilon) \xi}\) holds for all \(\xi \in \mathbb{R}^{2n}\). To prove (a), it suffices to show that \(\chi_{\sigma(\epsilon)}\) meets the conditions (i)–(iii) of the quantum Bochner theorem (lemma 2). Condition (i) is clear, since \(\chi_{\sigma(\epsilon)}(0) = f(0) = 1\); (ii) is also straightforward, since the continuity at 0 of \(f\) implies that of \(\chi_{\sigma(\epsilon)}\). The problem is to verify (iii), i.e. \(\Omega\)-positivity. For a finite collection of vectors \(\eta_{i1}, \ldots, \eta_{iN} \in \mathbb{R}^{2n+2k}\), let us write

\[
\chi_{\sigma(\epsilon)}(\eta_{i1} - \eta_{i2}) e^{\frac{1}{2} \bar{Y} \eta_{i1} \bar{Y}^\top (\Omega - X^\top \Omega X) \bar{Y} \eta_{i2}} - f(\bar{Y} \eta_{i1} - \bar{Y} \eta_{i2}) e^{\frac{1}{2} \bar{Y} \eta_{i1} \bar{Y}^\top (\Omega - X^\top \Omega X) \bar{Y} \eta_{i2}} = f(\bar{Y} \eta_{i1} - \bar{Y} \eta_{i2}) e^{\frac{1}{2} \bar{Y} \eta_{i1} \bar{Y}^\top (\Omega - X^\top \Omega X) \bar{Y} \eta_{i2}} \\
- \frac{1}{2} \bar{Y} \eta_{i1} \bar{Y}^\top \bar{W}(\epsilon) \eta_{i1} + \frac{1}{2} \bar{Y} \eta_{i2} \bar{Y}^\top \bar{W}(\epsilon) \eta_{i2} + \frac{1}{2} \eta_{i1}^\top \bar{D} \eta_{i1} + \frac{1}{2} \eta_{i2}^\top \bar{D} \eta_{i2}.
\]

Since \(f\) is \(J(\bar{X})\)-positive by lemma 3, we deduce that

\[
( f(\bar{Y} \eta_{i1} - \bar{Y} \eta_{i2}) e^{\frac{1}{2} \bar{Y} \eta_{i1} \bar{Y}^\top (\Omega - \xi \bar{X}^\top \xi X) \bar{Y} \eta_{i2}} - f(\bar{Y} \eta_{i1} - \bar{Y} \eta_{i2}) e^{\frac{1}{2} \bar{Y} \eta_{i1} \bar{Y}^\top (\Omega - X^\top \Omega X) \bar{Y} \eta_{i2}} )_{\mu,\nu} \geq 0.
\]

To see why, it suffices to write out condition in equation (10) with \(A = J(\bar{X})\) and \(\xi = \bar{Y} \eta_{i1}\). Using the fact that the Hadamard product of positive matrices is positive [71], theorem 7.5.3, we conclude that in order to show that \(\chi_{\sigma(\epsilon)}\) is \(\Omega\)-positive it suffices to check that the matrix

\[
M := ( e^{-\frac{1}{4} \bar{Y} \eta_{i1} \bar{Y}^\top \bar{W}(\epsilon) \eta_{i1}} - \frac{1}{2} \bar{Y} \eta_{i1} \bar{Y}^\top (\Omega - \xi \bar{X}^\top \xi X) \bar{Y} \eta_{i1})_{\mu,\nu}
\]

is positive. Observe that

\[
\hat{Y}^\top j(\bar{X}) \hat{Y} = \hat{Y}^\top \hat{Y} \hat{X} \Omega \hat{Y} = (\hat{Y} \hat{Y}^\top \Omega) \hat{Y} = P \Omega \hat{Y}.
\]

This allows us to rewrite

\[
M_{\mu,\nu} = e^{-\frac{1}{4} \bar{Y} \eta_{i1} \bar{Y}^\top \bar{W}(\epsilon) \eta_{i1}} + \frac{1}{2} \eta_{i1} M(\Omega - P\Omega) \eta_{i1}.
\]

Since \(W(\epsilon) \geq i(\Omega - P\Omega)\) by equation (56), it is not difficult to check (lemma 15) that \(M \geq 0\), indeed. This concludes the proof of claim (a). The argument for claim (b) is much easier:

\[
\chi_{\sigma(\epsilon)}(\bar{Y} \xi) = f(\bar{Y} \xi) e^{-\frac{1}{2} \bar{Y} \xi \bar{W}(\epsilon) \xi} = f(\xi) e^{-\frac{1}{2} \bar{Y} \xi},
\]

where in the second step we used equations (52) and (57).
Until now we have shown that if $\Phi_{X,f}$ is a linear bosonic channel then the channels acting as

$$\chi_{\rho}(\xi) \mapsto \chi_{\rho}(Xf(\xi)) e^{-\frac{1}{2}U^TQeU}$$

are Gaussian dilatable on $n + k$ modes for all $\epsilon > 0$. We now complete the argument by proving that these approximate $\Phi_{X,f}$ in the strong operator topology. This is an immediate consequence of lemma 3(iii), as the rhs of equation (61) converges pointwise to $\chi_{\rho}(Xf(\xi))$ as $\epsilon \to 0^+$, which is manifestly continuous at 0. For convenience we summarise in algorithm 3 the steps to obtain the approximate dilation of a linear bosonic where the unitary in the dilation is fixed.

**Algorithm 3.** Obtaining an approximate Gaussian dilation for any linear bosonic channel using a fixed unitary in the dilation.

**Input:** $X, f$ of a linear bosonic channel $\Phi_{X,f}$.
1. Compute the canonical form for $f(X)$ from equation (43); set $k = \frac{1}{2} \dim \ker f(X)$.
2. Obtain $\tilde{Y}$ using equation (50).
3. For any choice of $\epsilon > 0$ define $W(\epsilon)$ as in equations (54)–(55).
4. Obtain the characteristic function of the ancillary $(n + k)$-mode state $\sigma(\epsilon)$ using equation (60).
5. Complete $(\tilde{X}, \tilde{Y})$ to a symplectic matrix $S_{4k}$.
6. Obtain the Gaussian unitary $U_{4k}$ for the dilation from $S_{4k}$.
7. Limit $\epsilon \to 0^+$ provides the required approximation to $\Phi_{X,f}$ via Gaussian dilatable channels.

**Output:** Approximate Gaussian dilation of $\Phi_{X,f}$ with fixed unitary.

4. Discussion and conclusions

In this paper we investigated the set of linear bosonic channels by relating it to the physically meaningful set of Gaussian dilatable channels. Our main result (theorem 8) states that the former set coincides with the strong operator closure of the latter. Operationally, this means that the action of every linear bosonic channel on any fixed state can be very well approximated by that of a sequence of suitable Gaussian dilatable channels (that do not depend on the state). We showed that taking the closure is in general necessary, as there are examples of linear bosonic channels for which no exact Gaussian dilation can be constructed, even if infinite-energy ancillary states are available (corollary 11). Our results solve the open question raised in [5] (see also [7, remark 5]) on the existence of Gaussian dilations for linear bosonic channels. Note that as it is formulated there, [5], Conjecture 1 is false, as corollary 11 shows that there are examples of linear bosonic channels that are not exactly Gaussian dilatable. However, the conjecture is ‘almost’ true (i.e. it is true up to approximations).

Our proof strategy yields an explicit recipe to construct approximate Gaussian dilations of any given linear bosonic channel. We proved that if the associated Gaussian unitary is allowed to vary, an ancillary system with the same number of modes as the input system suffices (corollary 9). We can also require the unitary not to change when the approximation is sharpened, which yields a more experimentally friendly procedure. In this case we are still able to construct approximate Gaussian dilations, albeit with a larger number of ancillary modes (proposition 12). When applied to Gaussian channels, our findings complement those of [40], in which only exact Gaussian dilations are considered (see remark 9).

Some of the technical tools we developed are of broad interest in the field of quantum information with continuous variables. We highlight especially the SWOTTED convergence lemma 5, which gives easily verifiable necessary and sufficient conditions for a sequence of quantum channels to converge in the strong operator topology or—equivalently—uniformly on energy-bounded states. This is based on an analogous result for states (lemma 4) that was known in the literature before. Our techniques also allowed us to give an alternative proof of the quantum Bochner theorem (appendix A). Our argument is entirely elementary and independent of the proof of the analogous result for classical probability theory, which requires—one may argue—more sophisticated measure theory tools. Our main theorem can also be used to prove directly that conditions (i)–(iii) of lemma 3 suffice to ensure that the corresponding linear bosonic map is in fact a quantum channel. That they are also necessary is even easier to prove, as shown in appendix B, thus we also obtain a proof of lemma 3 that is independent of that presented in [48].

We now discuss some open problems. The careful reader may have noticed that in defining Gaussian dilatable channels (see equation (20)) we did not require the ancillary state to have finite energy. It would be interesting to investigate what happens when one has an energy constraint on the ancillary state. Another related aspect still to be explored is the most efficient way to implement linear bosonic channels via approximate Gaussian dilations, especially from an experimental point of view. In this context, we could for instance ask whether the constructions in the proofs of theorem 8 and proposition 12 are in some sense optimal, either from...
the point of view of the energy of the ancillary state, or from that of the number of modes required. Finally, our results could help to solve an interesting open question about the extremality of linear bosonic channels [8, remark 2].

In conclusion, this paper provides a number of novel insights into the structure of linear bosonic channels, further bolstering their status as an important subject of study in continuous variable quantum information.

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Appendix A. A direct proof of quantum Bochner theorem

In this appendix we provide a self-contained proof of the quantum Bochner theorem (lemma 2) that is moreover entirely elementary, in that it requires only widely known results from standard analysis and no notion of measure theory. For comparison, the arguments in [72, proposition 3.4(7)] and [73, section 6.2.3] make use of the classical Bochner theorem (which in turn depends on Helly’s selection principle or the Banach–Alaoglu theorem, see [74, theorem 5.5.3] for a standard proof), while that in [42, theorem 5.4.1] employs the Stone–von Neumann uniqueness theorem. Remarkably, this is one of the few cases in which the quantum version of the statement is actually easier to prove than its classical counterpart.

A.1. On some properties of $\Omega$-positive functions

Before we delve into the proof of lemma 2, it is useful to familiarise with the important notion of $\Omega$-positive function. We start by studying the regularity properties of these functions, of which we made ample use in the main text. We need a preliminary lemma.

Lemma 13. Let

$$A = \begin{pmatrix} a & x & y \\ x^* & a & z \\ y^* & z^* & a \end{pmatrix} \succeq 0 \quad \text{(A1)}$$

be positive semidefinite. Then

$$|y - z|^2 \leq \Re[(a - z)(a + z^* - 2xy^*)] \leq 4a|a - z|. \quad \text{(A2)}$$

Proof. Follows by rearranging the inequality $\det A \geq 0$.

Lemma 14. Let $f : \mathbb{R}^n \to \mathbb{C}$ be an $\Omega$-positive function. Then $|f(\xi)| \leq f(0)$ for all $\xi$, and moreover $f$ is continuous everywhere if and only if it is continuous in 0.

Proof. For $A = \Omega, N = 2, \xi_1 = \xi$, and $\xi_2 = 0$, the condition in equation (10) reads

$$\begin{pmatrix} f(0) & f(\xi) \\ f(\xi)^* & f(0) \end{pmatrix} \succeq 0,$$

from which we deduce that $f(0) \geq |f(\xi)|$. For $N = 3$ and $\xi_1 = 0, \xi_2 = -\xi$ and $\xi_3 = -\xi_2$ we obtain instead

$$\begin{pmatrix} f(0) & f(\xi) & f(\xi) \\ f(\xi)^* & f(0) & f(\xi_2 - \xi_1)e^{i\xi_1\Omega\xi_2} \\ f(\xi_2^* & f(\xi_2 - \xi_1)e^{i\xi_1\Omega\xi_2} & f(0) \end{pmatrix} \succeq 0.$$

Applying lemma 13 to the above matrix yields

$$|f(\xi_2) - f(\xi_2)|^2 \leq 4f(0)|f(0) - f(\xi_2 - \xi_1)e^{i\xi_1\Omega\xi_2}|.$$

This shows that when $\xi_2 \to \xi$ the sequence $f(\xi_2)$ approaches $f(\xi)$ at a rate that does not differ much from that of the convergence $f(\xi - \xi_2) \to 0$. Hence, $f$ is continuous everywhere whenever it is continuous at 0.
Before we move on to the proof of the quantum Bochner theorem, it is useful to make a simple sanity check. The characteristic function of a Gaussian state with zero displacement is given by \( \chi_\mu(x) = e^{-\frac{i}{2} x^t V x} \), where \( V \geq i \Omega \) is the covariance matrix. According to lemma 2, this must be an \( \Omega \)-positive definite function. Is there a way to verify this directly? The answer is affirmative, as we now set out to see.

**Lemma 15.** Let \( V \) and \( A \) be \( 2n \times 2n \) real matrices. Assume \( V \) is symmetric and \( A \) skew-symmetric. The Gaussian function \( f : \mathbb{R}^{2n} \to \mathbb{C} \) defined by \( f(\xi) = e^{-\frac{i}{2} \xi^t V \xi} \) is \( A \)-positive in the sense of definition 1 if and only if \( V \geq iA \).

**Proof.** We first show that if \( f \) is \( A \)-positive then necessarily \( V \geq iA \). We evaluate the \( A \)-positivity condition of equation (10) on an arbitrary complex vector \( x \in \mathbb{C}^N \) such that

\[
\sum_{\mu=1}^{N} x_\mu = 0,
\]

obtaining

\[
\sum_{\mu,\nu} x_\mu^* x_\nu \exp \left( -\frac{i}{4} (t \xi_\mu - \xi_\nu)^t V (\xi_\mu - \xi_\nu) + \frac{it^2}{2} \xi_\mu^t A \xi_\nu \right) \geq 0
\]

for all \( t \in \mathbb{R} \) and finite collections \( \xi_0, \ldots, \xi_N \in \mathbb{R}^{2n} \). Expanding to second order in \( t \) around 0 and using equation (A3) yields

\[
\sum_{\mu,\nu} x_\mu^* x_\nu \left( -\frac{1}{2} (\xi_\mu - \xi_\nu)^t V (\xi_\mu - \xi_\nu) + i t \xi_\mu^t A \xi_\nu \right) \geq 0.
\]

Making use of equation (A3) once again we can further simplify this to

\[
\sum_{\mu,\nu} x_\mu^* x_\nu (\xi_\mu^t V \xi_\nu + i \xi_\mu^t A \xi_\nu) = \left( \sum_{\mu} x_\mu \xi_\mu \right)^t (V + iA) \left( \sum_{\nu} x_\nu \xi_\nu \right) \geq 0.
\]

Since every vector \( z \in \mathbb{C}^{2n} \) can be written as \( z = \sum_{\mu=1}^{N} x_\mu \xi_\mu \) for \( x = \frac{1}{\sqrt{2}} (1, -1, i, -i) \) (which satisfies equation (A3)), \( \xi_1 = -\xi_2 = \Re z \) and \( \xi_3 = -\xi_4 = 2\Im z \), it must hold that \( V + iA \geq 0 \). Conversely, let us prove that if \( V + iA \geq 0 \) then the condition in equation (10) is always met. Start by rewriting

\[
f(\xi_\mu - \xi_\nu) e^{\frac{i}{2} \xi_\mu^t A \xi_\nu} = \exp \left[ \frac{i}{2} \xi_\mu^t V \xi_\nu \right] e^{\frac{i}{2} \xi_\mu^t (V + iA) \xi_\nu} e^{-\frac{i}{2} \xi_\nu^t V \xi_\mu}.
\]

Up to congruences by diagonal matrices, it is enough to show that \( R_{\mu\nu} : \mathbb{R}^n \to \mathbb{C} \) defined by \( R_{\mu\nu}(x) = \exp \left[ \frac{i}{2} \xi_\mu^t (V + iA) \xi_\nu \right] \) identifies a positive semidefinite matrix. Observe that if \( R \) is the Hadamard (i.e., entrywise) exponential of the Gram matrix \( H_{\mu\nu} = \frac{i}{2} \xi_\mu^t (V + iA) \xi_\nu \), then \( R \) is a positive semidefinite form \( V + iA \), hence it is itself positive semidefinite. Since Hadamard exponentials preserve positive semidefiniteness [75, theorem 6.3.6] we conclude that \( R \geq 0 \) as claimed.

**A.2. Proof of quantum Bochner theorem**

We are ready to discuss our proof of lemma 2. Compared to other proofs that have appeared in the previous literature on the subject, ours relies heavily on the following elementary lemma, whose importance seems to have been somewhat overlooked.

**Lemma 16 (Diagonal integration lemma).** Let \( f : \mathbb{R}^p \to \mathbb{C} \) be a bounded integrable function, i.e. let it be measurable and such that \( |f(\xi)| \leq M \) and \( \int d^n \xi |f(\xi)| < \infty \). Then

\[
\int \frac{d^n \xi}{L^{-p}} \int_{-L/2}^{L/2} d\xi_1 d\xi_2 f(\xi_1 - \xi_2),
\]

where it is understood that the integration region on the rhs is \([-L/2, L/2] \) for all components of the vectors \( \xi_1, \xi_2 \).

**Proof.** We limit ourselves to showing the case \( p = 1 \), since the others are analogous and follow by iteration of the same method. Let us write
\[
\lim_{L \to \infty} \frac{1}{L} \int_{-L/2}^{L/2} d\xi_1 \int_{-L/2}^{L/2} d\xi_2 \ f(\xi_1 - \xi_2) \ = \ \lim_{L \to \infty} \frac{1}{L} \int_{-L}^{L} d\xi \int_{-L}^{L} d\eta \ f(\xi) \\
= \ \lim_{L \to \infty} \frac{1}{L} \int_{-L}^{L} d\xi \ (L + |\xi|) f(\xi) \\
= \ \lim_{L \to \infty} \int_L^{\infty} d\xi \left(1 + \frac{|\xi|}{L}\right) f(\xi) \\
\geq \ \int_{x=1}^{\infty} d\xi \ f(\xi).
\]

The justification of the above steps is as follows: 1: we defined \(\zeta := \xi_1 - \xi_2\) and \(\eta := \frac{\zeta + \xi_2}{2}\); 2: we observed that

\[
\left| \int_{-\infty}^{\infty} d\xi \ f(\xi) - \int_L^{\infty} d\xi \left(1 + \frac{|\xi|}{L}\right) f(\xi) \right| \leq \int_{|\xi| > L} d\xi \ f(\xi) + \int_L^{\infty} d\xi \left| \frac{|\xi|}{L} f(\xi) \right| \\
\leq \int_{|\xi| > L} d\xi \ f(\xi) + \int_{L/2 < |\xi| < L} d\xi \left| \frac{|\xi|}{L} f(\xi) \right| + \int_L^{\infty} d\xi \left| \frac{|\xi|}{L} f(\xi) \right| \\
\leq \int_{|\xi| > L} d\xi \ f(\xi) + \int_{L/2}^{L/2} d\xi \left| \frac{|\xi|}{L} f(\xi) \right| + \int_L^{\infty} d\xi \left| \frac{|\xi|}{L} f(\xi) \right| \\
= \int_{|\xi| > L} d\xi \ f(\xi) + \frac{M}{L^{1/3}},
\]

and both terms tend to zero as \(L \to \infty\).

**Proof of lemma 2.** First of all, let us show that conditions (i)–(iii) are necessary for \(f\) to be a characteristic function of a quantum state. This part of the proof is totally standard, see for instance [42, theorem 5.4.1]. If \(\rho\) is a normalised trace-class operator, by putting \(\xi = 0\) in equation (7) we find \(\chi_D(0) = \text{Tr}[\rho D(0)] = \text{Tr} \rho = 1\), which proves the necessity of (i). Now, Stone’s theorem ensures that the correspondence \(\xi \mapsto D(\xi)\) is strongly operator continuous (i.e. \(\lim_{\xi \to 0} \|D(\xi)\psi\| = 0\) for all vectors \(\psi\)), see section 2.3, and in particular continuous in the WOT. This latter fact can be equivalently rephrased as \(\lim_{\xi \to 0} \text{Tr}[\rho D(\xi)] = \text{Tr}[\rho]\) for all trace-class \(\rho\) (see e.g. [76, lemma 7]), which is to say that \(\chi_D\) is continuous at 0 for all quantum states. We move on to showing the necessity of requirement (iii). For \(N \in \mathbb{N}, \xi_0, \ldots, \xi_N \in \mathbb{R}^n\) and \(c \in \mathbb{C}^N\), observe that

\[
0 \leq \text{Tr} \left[ \left(\sum_{\mu,\nu} c_\mu c_\nu^* D(\xi_\nu)\right)^{\dagger} \left(\sum_{\mu,\nu} c_\mu c_\nu^* D(\xi_\nu)\right) \right] \\
= \sum_{\mu,\nu} c_\mu c_\nu^* \text{Tr}[D(-\xi_\nu) \rho D(\xi_\nu)] \\
= \sum_{\mu,\nu} c_\mu c_\nu^* \text{Tr}[\rho D(\xi_\nu) D(-\xi_\nu)] \\
= \sum_{\mu,\nu} c_\mu c_\nu^* c_{\xi_\mu} \text{Tr}[\rho D(\xi_\nu - \xi_\mu)] \\
= \sum_{\mu,\nu} c_\mu c_\nu^* c_{\xi_\mu} \text{Tr}[\rho D(\xi_\nu - \xi_\mu)].
\]

This is equivalent to equation (10) for \(A = \Omega\) and \(f = \chi_D\).

Now we have to show the converse, i.e. that all functions \(f\) satisfying (i)–(iii) are characteristic functions of some quantum state. By lemma 14 we know that \(f\) is continuous and bounded by 1 in modulus. Let us first focus on the case when \(f\) is square-integrable, i.e. such that \(\int d^2\xi \ |f(\xi)|^2 < \infty\). This is more or less the strategy adopted in [74, theorem 5.5.3]. Under this assumption, one can construct

\[
\rho_f := \int \frac{d\Omega(\xi)}{(2\pi)^n} f(\xi) D(-\xi), \quad (A5)
\]

which is well-defined in the sense of weak convergence (see [42], section 5.3, in particular equation (5.3.18) and theorem 5.3.3). Observe that \(\rho_f\) is a Hilbert–Schmidt operator since \(f\) is square-integrable, and that it satisfies
Tr[\rho_f D(\xi)] = f(\xi) \quad (A6)

and

\[ \text{Tr}[\rho_f^2] = \int \frac{d^{2n} \xi}{(2\pi)^n} |f(\xi)|^2, \]

which is just a specialisation of equation (9). We now show that \( \rho_f \) is indeed positive semidefinite, i.e. that \( \langle \psi | \rho_f | \psi \rangle \geq 0 \) for all (normalised) vectors |\psi⟩. Write

\[ \langle \psi | \rho_f | \psi \rangle = \int \frac{d^{2n} \xi}{(2\pi)^n} f(\xi) \langle \psi | D(-\xi) | \psi \rangle \]

where we used equation (\ref{eq:rho_f}). The above steps are justified as follows: 1: is an application of equation (\ref{eq:Tr_rho_f}); 2: follows from lemma 16 because \( \xi \mapsto f(\xi) \langle \psi | D(-\xi) | \psi \rangle \) is integrable, as can be seen by writing

\[ \int \frac{d^{2n} \xi}{(2\pi)^n} |f(\xi)|^2 \langle \psi | D(-\xi) | \psi \rangle \leq \left( \int \frac{d^{2n} \xi}{(2\pi)^n} |f(\xi)|^2 \right)^{1/2} \left( \int \frac{d^{2n} \xi}{(2\pi)^n} \langle \psi | D(-\xi) | \psi \rangle^2 \right)^{1/2} \]

is clearly finite, we are free to exchange the integration order; finally, 6: is a consequence of the fact that

\[ \int_{-L/2}^{L/2} d^{2n} \xi d^{2n} \xi \ f(\xi - \xi) e^{-\frac{i}{2} \xi \cdot \xi} \langle \psi | D(-\xi) | \lambda \rangle \langle \lambda | D(\xi) | \psi \rangle \leq 0 \]

for all \( \lambda \in \mathbb{R}^{2n} \), since this is an integral of a continuous bounded function over a bounded interval, hence it is a limit of Riemann sums, and each sum is positive because it is of the form \( \sum_{\mu} \sum_{\nu} \delta_{\mu \nu} f(\xi_{\mu} - \xi_{\nu}) e^{-\frac{i}{2}(\xi_{\mu} \cdot \xi_{\nu})} \geq 0 \), where the latter inequality holds because \( f \) is assumed to be \( \Omega \)-positive. We have shown that \( \rho_f \) defined in equation (\ref{eq:rho_f}) is positive semidefinite for all square-integrable functions \( f \). In fact, in this case \( \rho_f \) is actually trace-class, because

\[ \text{Tr}[\rho_f] = \text{Tr}[\rho_f D(0)] = f(0) = 1, \]

where we employed equation (\ref{eq:Tr_rho_f}) together with hypothesis (i).

We now show that the working assumption that \( f \) is square-integrable is actually a consequence of the properties of \( f \). Since \( |f| \) is bounded by 1 by lemma 14, we can define the sequence of functions

\[ f^r(\xi) = f(\xi) e^{r \xi}, \]

where \( r > 0 \). For these functions, which are square-integrable and furthermore can be shown to be \( \Omega \)-positive as a consequence of lemma 15, one can construct a trace-class, positive semidefinite operator \( \rho_{f^r} \) as in equation (\ref{eq:rho_f}). As all legitimate density matrices, these will satisfy \( \text{Tr}[\rho_{f^r}] \leq 1 \). Then, using equation (\ref{eq:Tr_rho_f}) one has
For all $L > 0$, using Lebesgue’s dominated convergence theorem we can write

$$\int_{L/2}^{L/2} \frac{d^2\xi}{(2\pi)^2} |f(\xi)|^2 = \lim_{\epsilon \to 0} \int_{L/2}^{L/2} \frac{d^2\xi}{(2\pi)^2} |f_{\epsilon}(\xi)|^2 \leq \lim_{\epsilon \to 0} \int \frac{d^2\xi}{(2\pi)^2} |f_{\epsilon}(\xi)|^2 \leq 1.$$ 

Upon taking the limit $L \to \infty$, this shows that $f$ is indeed square-integrable, as claimed.

### Appendix B. Complete positivity of linear bosonic maps

Throughout this appendix we show that an elementary and self-contained proof of lemma 3 can be obtained as a by-product of our main result. Our argument should be compared with the original one by Demoen et al [48] (see also [49, 50]), which requires at the very least quite a few notions of functional analysis.

**Proof of lemma 3.** As observed in remark 7, the proof of theorem 8 combined with that of lemma 7 shows that conditions (i)–(iii) are sufficient for the map $\Phi_{X,f}$ to be a bosonic channel. Indeed, we showed that any such map is in the strong operator closure of the set of Gaussian dilatable channels, and quantum channels form a strong operator closed set.

Now we prove that conditions (i)–(iii) are also necessary. Since $\chi_{\Phi_{X,f}}(\rho) = \chi_{\rho}(X|\xi)f(\xi)$ must be the characteristic function of a quantum state for all input states $\rho$, (i) and (ii) follow in an elementary way from the quantum Bochner theorem (lemma 2). We then move on to (iii). First of all, we employ equation (4) and equation (14) to write

$$e^{-\frac{i}{\hbar} \xi \cdot \Pi_{X,f}} (D(\xi)D(-\xi)) = D(X\xi - X\xi)f(\xi) = D(X\xi)f(-\xi)e^{-\frac{i}{\hbar} \xi \cdot X|\xi}.\chi_{\rho}(\xi),$$

which implies that

$$e^{\frac{i}{\hbar} \xi \cdot X|\xi} f(\xi - \xi) = D(-X\xi)f(\xi) D(-X\xi) \chi_{\rho}(\xi),$$

where on the lhs $I$ stands for the identity operator on $\mathcal{H}_n$. If $\{|\xi\rangle\}$ is an orthonormal basis of $\mathcal{H}_n$ (called from now on the ‘computational basis’), define the truncated maximally entangled states on $\mathcal{H}_n \otimes \mathcal{H}_n$ as

$$|\phi_m\rangle = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} |j\rangle |j\rangle. \quad (B1)$$

Observe that

$$\langle \phi_m | A \otimes B | \phi_m \rangle = \frac{1}{m} \text{Tr}[\Pi_m A B], \quad (B3)$$

where

$$\Pi_m = \sum_{j=0}^{m-1} |j\rangle \langle j|,$$

and the transposition is taken with respect to the computational basis. Using equation (B3) and the fact that $\Pi_m A \Phi_m \xrightarrow{m \to \infty} A$ for all trace-class $A$, it is not difficult to see that

$$\lim_{m \to \infty} m \langle \phi_m | A \otimes B | \phi_m \rangle = \text{Tr}[A B] \quad \forall A \in \mathcal{T}(\mathcal{H}_n), \forall B \in \mathcal{B}(\mathcal{H}_n),$$

where $B(\mathcal{H}_n)$ stands for the set of bounded operators.

Now, take an arbitrary test density matrix $\rho \in \mathcal{D}(\mathcal{H}_n)$, whose specific nature is irrelevant to us. For $\xi_1, \ldots, \xi_N \in \mathbb{R}^{2n}$ and $c \in \mathbb{C}^N$, construct the bounded operator $B = \sum c_{\xi} D(-X\xi) \otimes D(-\xi)$, where $*$ denotes complex conjugation again with respect to the computational basis. Write...
\[ 0 \leq \lim_{m \to \infty} m \operatorname{Tr}[(I \otimes \Phi_{X,f})|\phi_m\rangle\langle \phi_m| B^*(\rho^\top \otimes I)B] \]
\[ \leq \lim_{m \to \infty} m \langle \phi_m| (I \otimes \Phi_{X,f}) (B^*(\rho^\top \otimes I)B) |\phi_m\rangle \]
\[ = \lim_{m \to \infty} m \sum_{\mu,\nu} c_{\mu,\nu}^* \langle \phi_m| D(-X_{\xi_{\mu}}^\top \rho^\top D(-X_{\xi_{\nu}}) \otimes \Phi_{X,f}^* (D(\xi_{\mu})D(-\xi_{\nu})|\phi_m\rangle \]
\[ = \sum_{\mu,\nu} c_{\mu,\nu}^* \lim_{m \to \infty} m \langle \phi_m| D(-X_{\xi_{\mu}}^\top \rho^\top D(-X_{\xi_{\nu}}) \otimes \Phi_{X,f}^* (D(\xi_{\mu})D(-\xi_{\nu})|\phi_m\rangle \]
\[ = \sum_{\mu,\nu} c_{\mu,\nu}^* \operatorname{Tr}[D(X_{\xi_{\mu}}^\top \rho D(-X_{\xi_{\nu}}) \Phi_{X,f}^* (D(\xi_{\mu})D(-\xi_{\nu})] \]
\[ = \sum_{\mu,\nu} c_{\mu,\nu}^* \operatorname{Tr}[\rho D(-X_{\xi_{\mu}}^\top \Phi_{X,f}^* (D(\xi_{\mu})D(-\xi_{\nu})] \]
\[ = 4 \sum_{\mu,\nu} c_{\mu,\nu}^* \operatorname{E}(f(X)\xi_{\mu}^* \langle \xi_{\mu}, \xi_{\nu} \rangle|f_{\mu,\nu} \geq 0, \]
showing that \( f \) is \( J(X) \)-positive.

### Appendix C. Proof of the SWOT convergence lemma

**Proof of lemma 4.** We start by showing that (i) \( \Rightarrow \) (ii). Assume that \( \rho_k \xrightarrow{k \to \infty} \rho \), with \( \rho \) being a density operator. Fix \( \varepsilon > 0 \), and pick a projector \( \Pi \) onto a subspace of finite dimension such that \( \|\rho - \Pi \rho \Pi\| < \varepsilon \). We start by applying the triangle inequality:
\[ \|\rho - \rho_k\| \leq \|\rho - \Pi \rho \Pi\| + \|\Pi \rho \Pi - \Pi \rho_k \Pi\| + \|\Pi \rho_k \Pi - \rho\|, \]
The first term of the sum on the right-hand side is already small, while the second can also be made smaller than \( \varepsilon \) by taking \( k \geq k_0 \) for some large enough integer \( k_0 \) (this is because \( \Pi \rho \Pi - \Pi \rho_k \Pi \) has finite rank). As for the third term, thinking of \( \Pi \rho_k \Pi \) as a post-measurement state we see that it must be close to the initial state \( \rho_k \) whenever the corresponding probability \( \operatorname{Tr}[\rho_k \Pi] \) is close to 1. Since \( \lim \operatorname{Tr}[\rho_k \Pi] = \operatorname{Tr}[\rho \Pi] \) (as follows from convergence in the WOT), we can require that \( \operatorname{Tr}[\rho_k \Pi] > 1 - \varepsilon \) for \( k \geq k_0 \). Then the ‘gentle measurement lemma’ [77, lemma 9] yields
\[ \|\rho_k - \Pi \rho_k \Pi\| \leq 2\sqrt{1 - \operatorname{Tr}[\rho_k \Pi]} < 2\sqrt{\varepsilon}. \]
Putting all together shows that
\[ \|\rho - \rho_k\| < 2\varepsilon + 2\sqrt{\varepsilon}, \]
completing the proof that (i) \( \Rightarrow \) (ii).

The implication (ii) \( \Rightarrow \) (iii) is obvious, since if \( \rho_k \xrightarrow{k \to \infty} \rho \) then for all \( \xi \in \mathbb{R}^{2n} \) one has that
\[ |\chi_{\rho_k}(\xi) - \chi_\rho(\xi)| = \operatorname{Tr}[(\rho_k - \rho)D(\xi)] \leq \|\rho_k - \rho\| \xrightarrow{k \to \infty} 0. \]

The only missing step is thus (iii) \( \Rightarrow \) (i). To show this, assume that \( \chi_{\rho_k} \) converges pointwise to a function \( f \) that is continuous at 0. We start by showing that \( f \) is the characteristic function of a quantum state. First, \( f(0) = \lim_k \chi_{\rho_k}(0) = \lim_k 1 = 1 \), so the normalisation condition is met. Secondly, \( f \) is continuous at 0 by hypothesis. Third, the numbers
\[ f(\xi_{\mu} - \xi_{\nu})e^{i\xi_{\nu}\Omega_{\nu}} = \lim_k \chi_{\rho_k}(\xi_{\mu} - \xi_{\nu})e^{i\xi_{\nu}\Omega_{\nu}} \]
are the entries of an \( N \times N \) positive semidefinite matrix for all \( \xi_0, \ldots, \xi_N \), because the set of positive semidefinite matrices of a fixed size is closed, and the numbers inside the limit on the rhs of the above equation are the entries of a positive semidefinite matrix for all \( k \). By the quantum Bochner theorem, we conclude that \( f \) must be the characteristic function of a quantum state, i.e. \( f = \chi_{\rho} \) for some density operator \( \rho \).

We now show that \( \rho_k \xrightarrow{k \to \infty} \rho \). Pick a finite-rank operator \( A \) and \( \varepsilon > 0 \), and for some radius \( R > 0 \) yet to be determined write
\[ |\text{Tr}(\rho - \rho_k)A| \leq \int_{|\xi| \leq R} d\xi \left| \chi_A(\xi) - \chi_{\rho_k}(\xi) \right| \chi_A(-\xi) \]  

\[ \leq \int_{|\xi| \leq R} d\xi \left| \chi_{\rho_k}(\xi) \right| \chi_A(-\xi) + \int_{|\xi| > R} d\xi \left| \chi_{\rho_k}(\xi) \right| \chi_A(-\xi) \]  

\[ \leq \int_{|\xi| \leq R} d\xi \left| \chi_{\rho_k}(\xi) \right| \chi_A(-\xi) \]  

\[ + \left( \int_{|\xi| > R} d\xi \left| \chi_{\rho_k}(\xi) \right| \right)^{1/2} \left( \int_{|\xi| > R} d\xi \left| \chi_A(\xi) \right|^2 \right)^{1/2} \]  

\[ \leq \int_{|\xi| \leq R} d\xi \left| \chi_{\rho_k}(\xi) \right| \chi_A(-\xi) + 2 \left( \int_{|\xi| > R} d\xi \left| \chi_A(\xi) \right|^2 \right)^{1/2} \]  

where for the last step we observed that  

\[ \left( \int_{|\xi| > R} d\xi \left| \chi_{\rho_k}(\xi) \right| \chi_A(-\xi) \right)^{1/2} \leq \left( \int d\xi \left| \chi_A(\xi) \right|^2 \right)^{1/2} = \left\| \rho - \rho_k \right\|_2 \leq 2. \]

Now, since \( \chi_A \in L^2(\mathbb{R}^{2n}) \) is a square-integrable function, the second addend of equation (C.5) can be made smaller than \( \epsilon/2 \) by taking \( R \) large enough. As for the first addend, we can apply Lebesgue’s dominated convergence theorem and show that it converges to 0 for all fixed \( R \). Indeed, since the integrable functions \( \left| \chi_{\rho_k}(\xi) \right| \chi_A(-\xi) \) tend to 0 pointwise, and moreover are bounded by 2 on the whole ball \( |\xi| \leq R \), we have  

\[ \lim_{k \to \infty} \int_{|\xi| \leq R} d\xi \left| \chi_{\rho_k}(\xi) \right| \chi_A(-\xi) = 0 \]

and therefore  

\[ \int_{|\xi| \leq R} d\xi \left| \chi_{\rho_k}(\xi) \right| \chi_A(-\xi) < \epsilon/2 \]  

for \( n \geq N \). Putting all together, we see that  

\[ |\text{Tr}(\rho - \rho_k)A| < \epsilon \]

when \( k \geq k_0 \), which shows that  

\[ \text{Tr}[\rho_k A] \xrightarrow[k \to \infty]{} \text{Tr}[\rho A] \]  

for all finite-rank operators \( A \). This is the same as saying that \( \rho_k \xrightarrow{\text{WOT}} \rho \), thereby completing the proof. \( \square \)

**ORCID iDs**

Ludovico Lami \( \text{https://orcid.org/0000-0003-3290-3557} \)

Andreas Winter \( \text{https://orcid.org/0000-0001-6344-4870} \)

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