Representations of fundamental groups of 3-manifolds into PGL(3, C): exact computations in low complexity

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Abstract In this paper we are interested in computing representations of the fundamental group of SL 3-manifold into PGL(3, C) (in particular in PGL(2, C), PGL(3, R) and PU(2, 1)). The representations are obtained by gluing decorated tetrahedra of flags as in Falbel (J Differ Geom 79:69–110, 2008), Bergeron et al. (Tetrahedra of flags, volume and homology of SL(3), 2011). We list complete computations (giving 0-dimensional or 1-dimensional solution sets (for unipotent boundary holonomy) for the first complete hyperbolic non-compact manifolds with finite volume which are obtained gluing less than three tetrahedra with a description of the computer methods used to find them. The methods we use work for non-unipotent boundary holonomy as shown in some examples.

Keywords PGL(3, C) · Flag structures · CR structures · 3-manifolds · Representations · 0-dimensional variety solving

Mathematics Subject Classification 57N10 · 57Mxx · 57M50 · 14Q99 · 68W30 · 32V05

1 Introduction

In this paper we are interested in obtaining representations of the fundamental group of a 3-manifold into PGL(3, C). They will be defined via certain topological triangulations with additional geometric data carried by the 0-skeleton of the triangulation.

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The most important example being hyperbolic geometry we will mainly consider 3-manifolds which carry a complete hyperbolic structure. That is, complete riemannian manifolds with constant sectional curvature equal to $-1$. 

In order to simplify the description we first treat open manifolds which are the interior of manifolds with boundary. If the hyperbolic manifold is complete and of finite volume one shows that its ideal boundary is a union of tori. In that case we consider ideal triangulations such that the 0-skeleton is in the ideal boundary. That is, identifying each boundary to a point, we consider a triangulation of the quotient space such that the 0-skeleton coincides with the set of boundary points.

In hyperbolic space the ideal tetrahedra are described by a configuration of four points in the boundary of hyperbolic space, that is, $\mathbb{CP}^1$. The cross-ratio is sufficient to parameterize ideal tetrahedra up to the isometry group. Ideal triangulations were used by Thurston to obtain examples of hyperbolic manifolds (see [23]). A complete hyperbolic structure is obtained on the manifold once a system of equations on the cross-ratio coordinates of those ideal tetrahedra are solved. A very successful computational tool, SnapPea, was developed by Jeff Weeks (see its very readable description in [24]) which solves numerically the system of equations.

Other geometric structures in 3-manifolds are associated to subgroups of $\text{PGL}(3, \mathbb{C})$, namely, flag structures which are associated to $\text{PGL}(3, \mathbb{R})$ and spherical Cauchy-Riemann structures (CR structures) which are associated to the group $\text{PU}(2, 1)$. These geometric structures are not well understood, in particular one does not know which 3-manifolds can carry one of them.

The first step in trying to find a structure is to obtain a representation of the fundamental group in one of these groups (see [9] for the case $\text{PU}(2, 1)$). We will use the raw data of triangulation of 3-manifolds as obtained in SnapPea. Then we proceed as in [2]; to a triangulation we associate a decoration given by a choice of flags at each vertex of the tetrahedra. We set equations imposing that the gluing of the tetrahedra is compatible with the decoration of flags. The solution of the equations would give immediately a representation of the fundamental group of the 3-manifold into $\text{PGL}(3, \mathbb{C})$ (called the holonomy representation). Imposing that the holonomy restricted to the boundary tori be unipotent is a natural condition which is analogous to the condition of completeness in the hyperbolic case. But more general conditions are very important too. In particular, a condition, which is appropriate to the CR case is that the eigenvalues be of absolute value one. Indeed, parabolic boundary conditions in the CR case imply that condition and it would be interesting to determine if there exist CR structures corresponding to boundary holonomies given by parabolic conditions. Our methods to solve these equations are the same and we give in Sect. 7.5 solutions to some non-unipotent systems. More general boundary conditions could be treated by our methods.

A generalization of the gluing equations for higher dimensions was described in [14], and a simplified set of equations describing the particular case of representations into $\text{SL}(n, \mathbb{C})$ is also described in [15]. These equations are based on decorations of tetrahedra by affine flags (see also the $a$-coordinates for affine flags in [2]) but, as there were initially designed for finding representations into $\text{SL}(n, \mathbb{C})$, they miss solutions related to $\text{PGL}(n, \mathbb{C})$ representations obtained via projective flags. However, the introduction of a cocycle and its computation gives access to all boundary unipotent representation in $\text{PGL}(n, \mathbb{C})$. This interesting method has the advantage to introduce quadratic equations as systems to be solved but is not adapted to non unipotent decorations and thus to describe other representations. On the other hand, it is not clear whether the computations are actually simpler in the affine flag case as shown by limitations in the computations in [16] (even for $\text{PGL}(3, \mathbb{C})$). Indeed, the use of Gröbner bases discards immediately the initial structure of the equations (symetries, sparsity, etc.) and the small degrees of the initial equations does not prevent the appearance of very large
Table 1 Description of the solutions

| Name   | 1-D 0-dimensional prime components | Number of Solutions | Ext. Degrees | PGL(3, \(\mathbb{C}\)) | PSL(3, \(\mathbb{R}\)) | PGL(2, \(\mathbb{C}\)) | PSL(2, \(\mathbb{R}\)) | PU(2, 1) | Volumes |
|--------|-----------------------------------|---------------------|--------------|--------------------------|------------------------|--------------------------|------------------------|-----------|----------|
| m003   | 2, 2, 2, 8, 8, 20                 | 2                   | 2            | 20                       | 0                      | 2                        | 0                      | 2         | 0.648847 | 2.029883 |
| m004   | 2, 2, 2, 2, 8                    | 0                   | 0            | 0                        | 0                      | 2                        | 0                      | 6         | 2.029883 |
| m006   | 6, 6, 12, 28, 43                 | 2                   | 0            | 2                        | 0                      | 3                        | 1                      | 15        | 0.707031 | 0.719829 | 0.971648 | 1.284485 | 2.568971 |
| m007   | 3, 6, 8, 8, 8, 33                | 0                   | 0            | 0                        | 0                      | 3                        | 1                      | 15        | 0.707031 | 0.822744 | 1.336688 | 2.568971 |
| m009   | 2, 4, 4, 6, 8, 28                | 0                   | 0            | 2                        | 2                      | 0                        | 0                      | 8         | 0.507471 | 0.791583 | 1.417971 | 2.666745 |
| m010   | 2, 6, 6, 12, 38                  | 2                   | 0            | 2                        | 2                      | 0                        | 0                      | 4         | 0.251617 | 0.791583 | 0.809805 | 0.982389 | 2.666745 |
| m011   | 3, 4, 16, 64, 87                 | 1                   | 5            | 7                        | 3                      | 21                       | 0                      | 0         | 0.226838 | 0.251809 | 0.328272 | 0.397457 | 0.452707 | 0.524707 | 0.588598 | 0.643302 | 0.685598 | 0.700395 | 0.724553 | 0.770297 | 0.879768 | 1.099133 | 1.184650 | 1.417971 | 2.781834 |
| m015   | 3, 4, 6, 6, 23                   | 0                   | 3            | 3                        | 1                      | 11                       | 0                      | 0         | 0.794323 | 1.583167 | 2.828122 |
| m016   | 3, 3, 10, 66, 66                 | 1                   | 4            | 6                        | 4                      | 24                       | 0                      | 0         | 0.296355 | 0.403707 | 0.710033 | 0.753403 | 0.773505 | 0.796590 | 0.886451 | 1.135560 | 1.422985 | 1.505989 | 2.828122 |
| m017   | 3, 4, 6, 6, 64                   | 3                   | 1            | 3                        | 1                      | 21                       | 0                      | 0         | 0.527032 | 0.794323 | 0.801984 | 0.828705 | 1.252969 | 1.588647 | 2.828122 |
| m019   | 4, 4, 22, 84, 114                | 1                   | 6            | 8                        | 4                      | 24                       | 0                      | 0         | 0.027351 | 0.062112 | 0.323395 | 0.332856 | 0.347159 | 0.411244 | 0.467624 | 0.524801 | 0.544151 | 0.599455 | 0.638404 | 0.738805 | 0.758111 | 0.798098 | 0.851139 | 0.916588 | 1.101800 | 1.130263 | 1.190919 | 1.263709 | 1.340255 | 2.111776 | 2.944106 |
| Wh. link | 2, 2, 4, 10, 32                  | 0                   | 2            | 0                        | 14                     | 1                        | 0                      | 0         | 1.132196 | 1.683102 | 3.663862 |

degrees during the computations. Even if some classes of examples are known to have a good behaviour (see [18] for 0-dimensional systems), up to our knowledge, there does not exist any criteria on the initial equations that might help to decide if such algorithm will be well behaved or not.

In this paper we deal with methods to solve the equations and obtain a list of solutions for manifolds with low complexity. In particular we obtain all solutions for ideal triangulations of complete cusped hyperbolic manifolds with less than four tetrahedra.
In the first sections, in order to make the paper self-contained, we review results in [2] (see also [9] for the \( \text{PU}(2, 1) \) case). That contains the parametrization of decorated tetrahedra (configurations of four flags in \( \mathbb{C}P^2 \)), the description of decorated ideal triangulations, the compatibility equations which will lead us to a system of equations and the computation of the holonomy representation. The special case we deal mostly in this paper has unipotent boundary holonomy.

In Sect. 6 we describe the methods used to solve the system of equations.

In the remaining sections we describe several important examples which illustrate the methods and the results. The complete census up to three tetrahedra is shown in Table 1 (more details can be obtained in our webpage ../SGT).

A very important observation which came out from the examples is that solutions of the gluing equations with unipotent boundary holonomy might not be 0-dimensional even in low complexity. Computations for representations in \( \text{PGL}(2, \mathbb{C}) \) show that, for cusped hyperbolic manifolds obtained with four tetrahedra, only two have a one parameter family component (there are no examples with less than four tetrahedra), but we do not have an efficient method to decide when the solution will have a positive dimensional component.

We verified that the one dimensional components (in all examples) have at most a finite number of \( \text{PSL}(2, \mathbb{C}) \) solutions. Moreover, they are all in \( \text{PSL}(2, \mathbb{R}) \) so the volumes presented in the table concern indeed all \( \text{PSL}(2, \mathbb{C}) \) representations arising from the given triangulation. On the other hand, \( \text{PU}(2, 1) \) solutions might appear in families (of real dimension one or two) inside 1-dimensional components (see the worked example m003 in [3] where there is a whole complex 1-dimensional component consisting of CR solutions). In any case, all these families arise from degenerate representations. Indeed, either they give rise to reducible representations or they have trivial boundary holonomy.

In the case of the figure eight knot the non-hyperbolic representations in \( \text{PGL}(3, \mathbb{C}) \) were obtained in [9]. They happen to be discrete representations in \( \text{PU}(2, 1) \) and two of them give rise to spherical structures on the complement of the figure eight knot (see [8,11]). In the case of the Whitehead link complement, Schwartz ([21], see also [22]) was able to analyse one representation (obtained in a completely independent way as a subgroup of \( \text{PU}(2, 1) \) generated by reflections) and showed that the complement of the link has a spherical CR structure. Other representations of the fundamental group of the complement of the Whitehead link obtained similarly using reflection groups were analysed in [19]. In their case the representations form a one parameter family parameterized by the boundary holonomy. We obtain here independently the unipotent representation which is the holonomy of a CR structure on the complement of the Whitehead link.

It turns out that for the examples of CR structures obtained until now, the boundary holonomy of the \( \text{PU}(2, 1) \) representations are abelian of rank one. This fact made us try to chase representations into \( \text{PU}(2, 1) \) with rank one boundary holonomy in the hope that they will correspond to CR structures. We enumerate all these solutions. For the ideal triangulation of the Whitehead link complement with four tetrahedra there is only one unipotent representation with rank one boundary holonomy (it is the same one obtained independently in [19]).

Above all, the computations encourage us to study CR structures. There is a great proportion of \( \text{PU}(2, 1) \) representations obtained among all \( \text{PGL}(3, \mathbb{C}) \) representations and some of them might correspond to holonomies of geometric structures. On the other hand it is a challenging problem to decide if a 3-manifold has a geometric structure. It is not clear when the representations obtained here are discrete (for general discrete subgroups of \( \text{PGL}(3, \mathbb{C}) \) see [6]). On the other hand, a criterium for rigidity of representations is given in [3] and a discussion of generic discreteness of the boundary holonomy is given in [17]. In this paper we don’t address these questions.
Another interesting observation from Table 1 is the fact that the hyperbolic volume is the maximum among the volumes of all unipotent \( \text{PGL}(3, \mathbb{C}) \) representations. This was shown recently using methods of bounded cohomology in [5]. On the other hand, \( \text{PU}(2, 1) \) representations have null volume by cohomological reasons (see [12]).

A final remark is the fact that no numerical method is available to solve these equations. It is a remarkable fact that, due to Mostow’s rigidity, one knows that there is at most one complete hyperbolic structure on a manifold. This structure corresponds to the unique solution with positive imaginary parts of the variables associated to a triangulation. The numerical scheme introduced in SnapPea should converge to this solution. Then, the LLL-algorithm makes the method very efficient in finding exact solutions corresponding to the hyperbolic structures. On the other hand, there is no knowledge of the number or location of solutions to the equations for representations in \( \text{PGL}(3, \mathbb{C}) \). Therefore, there is no numerical scheme to obtain solutions and we are, for the moment, condemned to solve the equations exactly.

We thank M. Thistlethwaite for introducing us to SnapPea and making available his list of cusped 3-manifolds with particularly simple generators for the boundary fundamental group. We used his list for our computations. We also thank N. Bergeron, M. Deraux, A. Guilloux, M. Thistlethwaite and S. Tillmann for all the stimulating discussions.

2 Three geometric structures on three manifolds

Geometric structures on three manifolds have been extensively studied. The usual setting is an action of a Lie group \( G \) on a homogeneous 3-manifold \( X \). An \( (X, G) \) structure on a 3-manifold \( M \) being a family of charts \( \phi_i : U_i \to X \) from open subsets forming a cover of \( M \) with transition functions \( g_{ij} \) (given by \( \phi_j = g_{ji} \circ \phi_i \)) in the Lie group \( G \).

Hyperbolic structures, that is, \((\mathbb{H}^3_{\mathbb{R}}, \text{PSL}(2, \mathbb{C}))\) structures (where \( \mathbb{H}^3_{\mathbb{R}} \) is hyperbolic 3-space) were shown by Thurston to be very important and his theory made possible, as a far reaching consequence of his ideas, to solve Poincaré’s conjecture. The boundary of the 3-dimensional hyperbolic space can be identified to \( \mathbb{CP}^1 \). By embedding \( \mathbb{CP}^1 \) into \( \mathbb{CP}^2 \) as a conic we observe that a point in \( \mathbb{CP}^1 \) defines a point and a line containing it in \( \mathbb{CP}^2 \). Namely the point obtained by the embedding and the tangent line to the embedding passing through that point. This justifies considering configuration of flags associated to triangulations (recall that a pair consisting of a point and a projective line containing the point is called a flag). We will associate to each vertex of a tetrahedron a flag in the following section.

CR geometry is modelled on the sphere \( S^3 \) equipped with a natural \( \text{PU}(2, 1) \) action. More precisely, consider the group \( \text{U}(2, 1) \) preserving the Hermitian form \( \langle z, w \rangle = w^* J z \) defined on \( \mathbb{C}^3 \) by the matrix

\[
J = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

and the following cones in \( \mathbb{C}^3 \):

\[
V_0 = \{ z \in \mathbb{C}^3 \setminus \{0\} : \langle z, z \rangle = 0 \},
\]

\[
V_- = \{ z \in \mathbb{C}^3 : \langle z, z \rangle < 0 \}.
\]

Let \( \pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2 \) be the canonical projection. Then \( \mathbb{H}^2_{\mathbb{C}} = \pi(V_-) \) is the complex hyperbolic space and its boundary is

\[
\partial \mathbb{H}^2_{\mathbb{C}} = S^3 = \pi(V_0) = \{ [x, y, z] \in \mathbb{CP}^2 : |x\bar{z} + |y|^2 + z\bar{x} = 0 \}.
\]
The group of biholomorphic transformations of $\mathbb{H}_3^2$ is then $\text{PU}(2, 1)$, the projectivization of $\text{U}(2, 1)$. It acts on $S^3$ by CR transformations. An element $x \in S^3$ gives rise to a flag in $\mathbb{CP}^2$ where the line corresponds to the unique complex line tangent to $S^3$ at $x$.

A third real 3-dimensional geometry is the geometry of real flags in $\mathbb{R}^3$. That is the geometry of the space of all couples $[p, l]$ where $p \in \mathbb{RP}^2$ and $l$ is a projective line containing $p$. The space of flags is identified to the quotient

$$\text{SL}(3, \mathbb{R})/B$$

where $B$ is the Borel group of all upper triangular matrices.

In the next section we describe the space of flags in $\mathbb{CP}^2$ which will be the common framework to describe the three geometries based on $\text{PSL}(2, \mathbb{C})$, $\text{PU}(2, 1)$ and $\text{PGL}(3, \mathbb{R})$.

3 Flag tetrahedra

In this section we recall the parametrization of configurations of four flags in the projective space $\mathbb{CP}^2$ (more details can be seen in [2]). Let $V = \mathbb{C}^3$. A flag in $V$ is usually seen as a line and a plane, the line belonging to the plane. Using the dual vector space $V^*$ and the projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, define the spaces of flags $\mathcal{Fl}$ by the following:

$$\mathcal{Fl} = \{([x], [f]) \in \mathbb{P}(V) \times \mathbb{P}(V^*) \mid f(x) = 0\}.$$

The natural action of $\text{SL}(3, \mathbb{C})$ on $\mathcal{Fl}$ makes us identify the space of flags with the homogeneous space $\text{SL}(3, \mathbb{C})/B$, where $B$ is the Borel subgroup of upper-triangular matrices in $\text{SL}(3, \mathbb{C})$. A generic configuration of flags $([x_i], [f_i])$, $1 \leq i \leq n + 1$ is given by $n + 1$ points $[x_i]$ in general position and $n + 1$ lines $f_i$ in $\mathbb{P}(V)$ such that $f_j(x_i) \neq 0$ if $i \neq j$. A configuration of ordered points in $\mathbb{P}(V)$ is said to be in general position when they are all distinct and no three points are contained in the same line.

Up to the action of $\text{PGL}(3, \mathbb{C})$, a generic configuration of three flags $([x_i], [f_i])_{1 \leq i \leq 3}$ has only one invariant given by the triple ratio

$$X = \frac{f_1(x_2)f_2(x_3)f_3(x_1)}{f_1(x_3)f_2(x_1)f_3(x_2)} \in \mathbb{C}^\times$$

3.1 Coordinates for a tetrahedron of flags

We recall the parametrization used in [2]. We refer to Fig. 1 which displays the coordinates.

Let $([x_i], [f_i])_{1 \leq i \leq 4}$ be a generic tetrahedron. We define a set of 12 coordinates on the edges of the tetrahedron (one for each oriented edge). To define the coordinate $z_{ij}$ associated to the edge $i j$, we first define $k$ and $l$ such that the permutation $(1, 2, 3, 4) \mapsto (i, j, k, l)$ is even. The pencil of (projective) lines through the point $x_i$ is a projective line $\mathbb{P}_1(\mathbb{C})$. We have four points in this projective line: the line $\ker(f_i)$ and the three lines through $x_i$ and one of the $x_l$ for $l \neq i$. We define $z_{ij}$ as the cross-ratio of four flags by

$$z_{ij} := [\ker(f_i), (x_i, x_j), (x_i, x_k), (x_i, x_l)].$$

Note that we follow the convention that the cross-ratio of four points $x_1, x_2, x_3, x_4$ on a line is the value at $x_4$ of a projective coordinate taking value $\infty$ at $x_1$, 0 at $x_2$, and 1 at $x_3$. So we employ the formula

$$[x_1, x_2, x_3, x_4] := \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}$$
At each face \((ijk)\) (oriented as the boundary of the tetrahedron \((1234)\)), the 3-ratio is the opposite of the product of all cross-ratios “leaving” this face:

\[
zh_{ijk} := \frac{f_i(x_j) f_j(x_k) f_k(x_i)}{f_i(x_k) f_j(x_i) f_k(x_j)} = -zh_{il}zh_{jl}zh_{kl}.
\]

This follows from a direct computation (see [2]). Observe that if the same face \((ikj)\) (with opposite orientation) is common to a second tetrahedron \(T'\) then

\[
zh_{ijk}(T) = \frac{1}{zh_{ijk}(T')}.
\]

Of course there are relations between the whole set of coordinates, namely, the three cross-ratio leaving a vertex are algebraically related:

\[
zh_{ik} = \frac{1}{1 - zh_{ij}},
zh_{il} = 1 - \frac{1}{zh_{ij}}.
\] (3.1)

Observe that the relations follow a cyclic order around each vertex which is defined by the orientation of the tetrahedron. We also have

\[
zh_{ij} zh_{ik} zh_{il} = -1.
\] (3.2)

The next proposition shows that a tetrahedron is uniquely determined, up to the action of 
\(\text{PGL}(3, \mathbb{C})\), by four numbers. One can pick a variable at each vertex. In fact, the space of flags is a complex manifold \((\text{PGL}(3, \mathbb{C})/B)\) and, therefore, configurations of flags have a natural complex structure.

**Proposition 3.1** [2] *The space of generic tetrahedra is biholomorphic to \((\mathbb{C} \setminus \{0, 1\})^4\).*

One can use one cross-ratio coordinate at each vertex, for instance \((zh_{12}, zh_{21}, zh_{34}, zh_{43})\) or \((zh_{13}, zh_{21}, zh_{32}, zh_{43})\).
3.2 Hyperbolic ideal tetrahedra

An ideal hyperbolic tetrahedron is given by 4 points on the boundary of $\mathbb{H}^3_{\mathbb{R}}$, i.e. $\mathbb{P}_1(\mathbb{C})$. Up to the action of $\text{PSL}(2, \mathbb{C})$, these points are in homogeneous coordinates $[0, 1], [1, 0], [1, 1]$ and $[1, z]$ – the complex number $z$ being the cross-ratio of these four points.

We may embed $\mathbb{P}_1(\mathbb{C})$ into $\mathbb{P}_2(\mathbb{C})$ via the Veronese map $h$, given in homogeneous coordinates by

$$h : [x, y] \mapsto [x^2, xy, y^2]$$

and therefore this map and its derivative define a map from $\mathbb{P}_1(\mathbb{C})$ to the variety of flags $\mathcal{F}l$.

Let $T$ be the tetrahedron $h([0, 1]), h([1, 0]), h([1, 1])$ and $h([1, z])$. Its image in the variety of flags given by the above map has coordinates (see [2])

$$z_{12}(T) = z_{21}(T) = z_{34}(T) = z_{43}(T) = z.$$  \hspace{1cm} (3.3)

Conversely, given parameters satisfying the equations above, they define a unique hyperbolic tetrahedron.

3.3 The CR case

Recall that an element $x \in S^3$ gives rise to an element $([x], [f]) \in \mathcal{F}l(\mathbb{C})$ where $[f]$ corresponds to the unique complex line tangent to $S^3$ at $x$.

The following proposition describes the space of generic configurations of four points in $S^3$.

**Proposition 3.2** [2,9] Generic configurations (up to translations by $\text{PU}(2, 1)$) of four points in $S^3$ not contained in an $\mathbb{R}$-circle are parametrized by generic configurations of four flags with coordinates $z_{ij}, 1 \leq i \neq j \leq 4$ satisfying the three complex equations

$$z_{ij}z_{ji} = z_{kl}z_{lk}$$  \hspace{1cm} (3.4)

not all of them being real and such that $z_{ji}z_{ki}z_{li} \neq 1$ for each face with vertices $\{j, k, l\}$.

The conditions (3.4) together with relation (3.2) imply that the triple ratio of each face satisfies $|z_{ijk}| = 1$. For the tetrahedron to be CR we need to verify the condition $z_{ijk} = -z_{ji}z_{ki}z_{li} \neq -1$ for each face with vertices $\{j, k, l\}$ (the triple ratio is never $-1$ for a CR generic triple of flags).

Conditions $z_{ij}z_{ji} = \frac{z_{kl}z_{lk}}{z_{ij}z_{ji}} \in \mathbb{R}$ might describe other configurations of four flags which are not CR. From $z_{ijk}z_{ilj} = \frac{z_{jk}z_{kl}}{z_{ij}z_{ji}}$, we deduce that in this case $z_{ijk}z_{ilj} = 1$ and therefore $z_{ijk} = z_{ilj} = z_{ikl} = z_{jkl} = \pm 1$. If the cross-ratio invariants are all real, either one obtains configurations contained in an $\mathbb{R}$-circle (they coincide with the real hyperbolic ones with real cross ratios, that is, $z_{12}(T) = z_{21}(T) = z_{34}(T) = z_{43}(T) = x \in \mathbb{R} - \{0, 1\}$) or one of the triple ratios is $-1$ and, in that case, they are not CR.

3.4 The $\text{SL}(3, \mathbb{R})$ case

Clearly, a configuration contained in the space of real flags is characterized by having all its invariants $z_{ij}$ real. Observe that the three cases intersect precisely for configurations corresponding to degenerate real hyperbolic configurations.
4 Ideal triangulations by flag tetrahedra

We use the definition of an ideal triangulation of a 3-manifold as it is used usually in computations with SnapPea. It is a union of 3-simplices $K = \bigcup_{v}T_{v}$ with face identifications (which are simplicial maps). We let $K^{(0)}$ be the union of vertices in $K$. The manifold $K - K^{(0)}$ is, topologically, the interior of a compact manifold when the vertices $K^{(0)}$ are deleted. In fact, $K - K^{(0)}$ is an (open) 3-manifold that retracts onto a compact 3-manifold with boundary $M$ (the boundary being the link of $K^{(0)}$).

Given such a compact oriented 3-manifold $M$ with boundary, we call a triangulation $K$ as above an ideal triangulation of $M$. A parabolic decoration of an ideal triangulation is the data of a flag for each vertex (equivalently a map from the 0-skeleton of the complex to $\mathcal{FT}$). A parabolic decoration together with an ordering of the vertices of each 3-simplex equip each tetrahedron with a set of coordinates as defined in Sect. 3.

From now on we fix $K$ a decorated oriented ideal triangulation of a 3-manifold together with an ordering of the vertices of each 3-simplex of $K$. Denote by $T_{\nu}, \nu = 1, \ldots, N$, the tetrahedra of $K$ and $z_{ij}(T_{\nu})$ the corresponding $z$-coordinates (or cross-ratio coordinates). We impose the following compatibility conditions which will imply the existence of a well defined representation of the fundamental group of the manifold $M$ into $\text{PGL}(3, \mathbb{C})$.

4.1 Consistency relations

(cf. We recall the relations explained in [2, 10])

Face equations] Let $T$ and $T'$ be two tetrahedra of $K$ with a common face $(ijk)$ (oriented as a boundary of $T$), then $z_{ijk}(T)z_{ikj}(T') = 1$.

For a fixed edge $e \in K$ let $T_{v_{1}}, \ldots, T_{v_{n_e}}$ be the $n_e$ tetrahedra in $K$ which contain an edge which projects unto $e \in K$ (counted with multiplicity). For each tetrahedron in $K$ as above we consider its edge $ij$ corresponding to $e$.

Edge equations] $z_{ij}(T_{v_{1}}) \cdots z_{ij}(T_{v_{n_e}}) = z_{ji}(T_{v_{1}}) \cdots z_{ji}(T_{v_{n_e}}) = 1$.

The face equations are clearly necessary in order to match a triple of flags from one face to a triple of another face. The edge equations follow by considering the 1-dimensional projective space of complex lines at each vertex. Indeed, take a vertex whose associated flag is $(p_{1}, l_{1})$ on the edge $[(p_{1}, l_{1}), (p_{2}, l_{2})]$. Consider the projective space of all lines through $p_{1}$. The line $l_{1}$ will be identified to $\infty$ and the line $[p_{1}, p_{2}]$ to 0. All the flags (not coinciding to $\infty$ or 0) in the tetrahedra having this edge in common give rise to an ordered sequence of points in this 1-dimensional projective space. Imposing that the the point corresponding to the last vertex of the last tetrahedron coincides with the point corresponding to the first vertex of the first tetrahedron amounts precisely to the first edge condition. Analogously, the second condition follows if we consider the projective space of lines at the other vertex of the edge.

One should be aware that the compatibility conditions do not imply immediately the existence of a geometric structure. One should think of these as sufficient conditions for the existence of a 0-skeleton compatible with the side pairings. Certainly it implies the existence of a representation of the fundamental group but one has yet to construct a compatible extension of the side pairings to 3 simplices. In the hyperbolic case one has the advantage of the existence of convex ideal tetrahedra but in $\text{PU}(2, 1)$ there is no such canonical construction (see the discussion in [9]).
4.2 Volume

We recall the definition of volume of a tetrahedron of flags in [2]. The *Bloch-Wigner dilogarithm* function is

\[ D(x) = \arg (1 - x) \log |x| - \operatorname{Im} \int_0^x \log (1 - t) \frac{dt}{t}, \]

\[ = \arg (1 - x) \log |x| + \operatorname{Im} \ln_2(x). \]

Here \( \ln_2(x) = -\int_0^x \log (1 - t) \frac{dt}{t} \) is the dilogarithm function. The function \( D \) is well-defined and real analytic on \( \mathbb{C} - \{0, 1\} \) and extends to a continuous function on \( \mathbb{C}P^1 \) by defining \( D(0) = D(1) = D(\infty) = 0 \).

Given a tetrahedron of flags with coordinates \((z_1(T), z_2(T), z_3(T), z_4(T))\), we define its volume as

\[ \text{Vol}(T) = \frac{1}{4} (D(z_1) + D(z_2) + D(z_3) + D(z_4)). \]

The volume of a triangulation is the sum of the volumes of its tetrahedra. If the tetrahedron is hyperbolic this definition coincides with the volume of volume in hyperbolic geometry. If the tetrahedron is real its volume is zero. Although CR tetrahedra have generically a non-zero volume, if a triangulation is such that all tetrahedra are CR then the total volume is zero ([12]).

5 Holonomy of a decoration

In this section we recall how to compute the holonomy of a decoration described above (see [2] for more details). A decoration of a triangulation of a manifold \( M \) gives rise to a representation (called holonomy representation)

\[ \rho : \pi_1(M, p_0) \to \text{PGL}(3, \mathbb{C}). \]

The base point is not important if we study representations up to conjugation. In fact, each solution of the consistency equations gives a conjugacy class of representations but a special choice of base point and side pairings is needed in explicit computations. The idea is to follow a path in the fundamental group from face to face keeping track of changes using a coordinate system adapted to the faces.

In order to compute holonomies we first fix a face (with ordered vertices) of one of the tetrahedra and a base point on the face. We then follow paths representing the generators of the fundamental group of the manifold which we decompose into arcs contained in each tetrahedron and which are transverse to their sides. For each arc we write the contribution to the holonomy as is explained in the following.

Given a tetrahedron \((ijkl)\), the arc going from face \((ijk)\) to face \((ijl)\) can be represented by a “left turn” arc around the edge \((ij)\). We will also consider permutations of a face as for example, the permutation \((ijk) \to (jki)\).

To follow the change in the holonomy along these arcs we associate to each face, that is, to a configuration of 3 generic flags \(([x_i], [f_i])_{1 \leq i \leq 3}\) with triple ratio \( X \), a projective coordinate system of \( \mathbb{C}P^2 \); take the one where the point \( x_1 = [1 : 0 : 0]^{\ell}, f_1 = [0 : 0 : 1], x_2 = [0 : 0 : 1]^{\ell}, f_2 = [1 : 0 : 0], \) the point \( x_3 \) has coordinates \([1 : -1 : 1]^{\ell}\) and \( f_3 = [X : X + 1 : 1]^{\ell} \).
The cyclic permutation of the flags \((ijk) \rightarrow (jki)\) with triple ratio \(X\) induces the coordinate change given by the matrix

\[
T(X) = \begin{pmatrix}
X & X+1 & 1 \\
-X & -X & 0 \\
X & 0 & 0
\end{pmatrix}.
\]

It is also useful to consider the transposition \((ijk) \rightarrow (jik)\) which is given by the matrix

\[
I = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

Remark then that

\[
T(\frac{1}{X}) = I \circ T^{-1}(X) \circ I.
\]

If we have a tetrahedron of flags \((ijkl)\) with its \(z\)-coordinates, then the basis related to the triple \((ijl)\) is obtained in the coordinate system related to the triple \((ijk)\) by the coordinate change given by the matrix

\[
(ijk) \rightarrow (ijl) = \begin{pmatrix}
1 & 0 & 0 \\
0 & z_{ji} & 0 \\
0 & 0 & z_{ij}
\end{pmatrix}.
\]

5.1 Representation of the fundamental group

We obtain the change of coordinate matrix of a path in the triangulation by decomposing it in the two elementary steps described in the previous section and multiplying the change of coordinate matrices for each step from left to right. Finally, to obtain the representation of the fundamental group we take the inverse of the coordinate change matrix for each path representative of an element of the fundamental group.

One has to be careful if we want to obtain a representation in \(PU(2, 1)\). In general we need to conjugate the above representation by an element so that the group preserves a chosen hermitian form.

Let \((i_0 j_0 k_0)\) (an ordered triple of points in a face of tetrahedron \(T_0\)) be the base point and \(Z_0 = z_{i_0 j_0 k_0}(T_0)\) be its 3-ratio. We consider the transformation \(A = \begin{pmatrix}
-\frac{1+Z_0}{Z_0} & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}\)

such that \(A \cdot \begin{pmatrix}
1 \\
1+Z_0 \\
1
\end{pmatrix} = \begin{pmatrix}
1 \\
-1 \\
1
\end{pmatrix}\). The hermitian form \(J\) is preserved if we conjugate the group elements by \(A\), or equivalently the group preserves the hermitian form \(J_0 = \overline{A}^{-1} J A^{-1}\) with eigenvalues \(1, 1/|1+Z_0|, -1/|1+Z_0|\). We will show computations in a specific example in a future section.

5.2 Boundary holonomy

If we follow a path in a boundary torus of a triangulated manifold we will obtain, applying the procedure above, a coordinate change. We obtain then a representation of the fundamental
group of the boundary in \( \text{PGL}(3, \mathbb{C}) \). Each torus is associated to a vertex and therefore the representation of its fundamental group fixes a flag. Choosing the flag properly up to conjugation, we can arrange it so that the boundary representation is upper triangular. Given two generators of a torus group represented by paths \( a \) and \( b \) one can compute easily the eigenvalues of the holonomy along them.

We suppose a path is contained in the link of the vertex and is transverse to the edges of the triangulation induced by the tetrahedra. Consider a path segment at the link of vertex \( i \) which is turning left around the edge \( ij \), that is \((ijk) \to (ijl)\). Then, as before, the holonomy change of coordinate matrix is

\[
\begin{pmatrix}
1/z_{ji} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & z_{ij}
\end{pmatrix}.
\]

Consider the path segment at the link of vertex \( i \) which is turning right around the edge \( ij \), that is

\((ilj) \to (ikj)\).

Then the holonomy change of coordinate matrix is

\[
\begin{pmatrix}
z_{kl} & z_{lk} & -1/z_{ij} & -1/z_{ki} \\
0 & z_{ij} & 1 & z_{lj}
\end{pmatrix}.
\]

Both matrices being upper diagonal, to find the eigenvalues we need only to multiply the diagonal terms. The eigenvalues of the boundary holonomy are the diagonal elements obtained as above. We will note \( A^* \) and \( A \) the two eigenvalues (the first and the third with the normalization as above so that the second eigenvalue be one) of the path \( a \) and \( B^* \) and \( B \) the two eigenvalues of the path \( b \).

5.3 Boundary unipotent holonomy equations

The boundary holonomy gives a representation of the fundamental group of the boundary tori into \( \text{PGL}(3, \mathbb{C}) \). If \( t \) is the number of boundary tori, we note

\[
\rho_b : (\mathbb{Z} \oplus \mathbb{Z})^t \to \text{PGL}(3, \mathbb{C}).
\]

For knots we can define canonical generators of the boundary torus called meridian and longitude. But, for simplicity, we will refer to generators of boundary tori for other manifolds as well as meridians and longitudes, denoted \( m \) and \( l \). The abelian group generated by the image of \( \rho_b \) is called the peripheral group. It turns out that an important information on the representations is the rank of the peripheral group. For instance, examples of uniformizable CR manifolds were obtained precisely when the rank of the peripheral group is one (see [8, 19, 21]).

We say the representation is \textit{unipotent} if the boundary holonomy is unipotent. That is, if the boundary holonomy representation can be given by matrices (up to scalar multiplication) of the form
That is, all eigenvalues are equal.

In the following we will find unipotent representations associated to parabolic decorations. That means that for each boundary torus we compute the eigenvalues $A^*, A, B^*, B$ corresponding to generators $a$ and $b$ at each torus and impose the equations

$$A^* = A = B^* = B = 1.$$ 

We remark that each eigenvalue is a rational function of the cross-ratio coordinates.

### 6 Computational steps

Our goal is to obtain the solutions of the system in the cross-ratio coordinates $z_{ij}$ and then compute the holonomy representations. We compute:

1. Solving the constraints on gluing decorated tetrahedra, which reduces to study a constructible set defined implicitly by
   
   (a) The consistency relations which define Edge and Face equations (4.1),
   
   (b) The holonomy equations (see Sect. 5),
   
   (c) The cross-ratio equations (see 3.1),
   
   (d) The restrictions on the coordinate values, precisely $z_{i,j}$ must be different from 0 and 1.

2. Distinguishing hyperbolic, real flags and CR solutions (see 3.4)

3. Computing the resulting volumes (4.2)

4. Computing representations of the fundamental group (see Sect. 5).

Each system depends on the $12n$ variables $Z = \{z_{ij}(T)\}$, where $T$ denotes a tetrahedron in the triangulation (containing $n$ tetrahedra) and $ij$ one of its oriented edges.

- $L_e$ (resp. $L_f$) denotes the set of the $2n$ edge (resp. face) equations;
- $L_h$ denotes the set of the polynomials defining the holonomy equations (their number is $4t$, where $t$ is the number of tori in the boundary and in most cases treated here they correspond to unipotent conditions);
- $L_c$ denotes the polynomials defining the cross-ratio relations, say $z_{ik}(T)(1 - z_{ij}(T)) - 1$ and $z_{il}(T)z_{ij}(T) - z_{ij}(T) + 1$;
- $P_0 = \prod_{z \in Z} (1 - z)$ is the polynomial defining the forbidden values for the $z_{ij}(T)$.

We have to compute an exhaustive and exact description of the constructible set

$$S = \{z \in Z \mid P(z) = 0, P \in L_e \cup L_f \cup L_h \cup L_c, P_0(z) \neq 0\}.$$ 

Some preliminary remarks are essential in order to understand the contents of the present section.

For fixed holonomy values (for example unipotent solutions), a straightforward approach that would consist in first solving the large algebraic system given by the consistency equations, the holonomy equations and the cross-ratio relation would certainly fail because of the number of variables ($12n$) and the degree. Moreover it may happen that this system is not 0-dimensional even if the holonomies are fixed.
We chose to perform exclusively exact computation, using elimination techniques. We will first simplify the system using specific adapted pre-processing, then we will use general methods essentially based on Gröbner basis computations (see [7]).

When the system we obtain is 0-dimensional, we then make use of the Rational Univariate Representation ([20]) to get formal parameterizations of the solutions.

6.1 The pre-processing

Using the cross-ratio relations at each vertex of a tetrahedron one can reduce the number of variables to $4n$. We pick a variable at each vertex and write the equations in terms of these variables. For example, such a substitution reduces the study of the $4_1$ knot complement to the solutions of a 0-dimensional system depending on 8 variables (instead of 24).

After these simplifications, some new simple affine relations $Az_{ij} + B$ may appear, where $A$ divides some power of $P_0$, $A$ being a product of $z$ or $(1 - z)$ for some $z \in \mathbb{Z} - \{z_{ij}\}$. The coordinate $z_{ij}$ may be replaced everywhere by $-B/A$. We then obtain a simplified system.

This simplification pre-processing reduces the study of the $4_1$ knot complement to a simplified system of two equations in two variables. The other coordinates of the solutions can then be recovered from simple relations.

Note that the degree of the equations in the final simplified system depends strongly on the way these simplifications are performed. Blind simplifications may lead to consider systems of equations with very large degrees. The general methods for solving such systems formally have polynomial complexity in the Bézout’s bound. Unappropriate choices in the simplifications will generate huge systems that are impossible to solve with state of the art algorithms.

We have implemented some tricky choice functions that minimize the degree of the final simplified system. There is no interest in describing them in detail here, but this straightforward step must be carefully implemented if one wants to succeed in the determination of the solutions.

6.2 Solving the reduced system using Gröbner Basis

We now have to solve a system of equations and (simple) inequalities. In the case of systems with fixed holonomy values (for example unipotent solutions), three cases may occur:

- The set of equations defines a 0-dimensional variety (finite set of points);
- The set of equations defines a variety of positive dimension but the full constructible set is 0-dimensional (consider for example the system $\{X(XY - 1) = 0, X(X^2 + Y^2 - 1) = 0\}$ and the constructible set $\{X(XY - 1) = 0, X(X^2 + Y^2 - 1) = 0, X \neq 0\}$).
- The constructible set has positive dimension.

In the case of unipotent solutions for problems with 2 or 3 tetrahedra, the first case happened only for the $m_{004}$ variety ($4_1$ complement), the last case occurs for example for the $m_{003}$ variety ($4_1$-sister) and also for the $m_{006}$. The second case occurs for example for the $m_{007}$ and the $m_{015}$ ($5_2$ complement). The procedure we follow is divided into the following steps:

1. Compute a Gröbner basis of the reduced system of equations.
2. Saturate the ideal by the polynomials defining the inequalities.
3. Compute the dimension of this ideal.

These computations are based on Gröbner Basis computations.

A Gröbner basis of a polynomial ideal $\mathcal{I}$ is a set of generators of $\mathcal{I}$, such that there is a natural and unique way (the normal form) of reducing canonically a polynomial $P$ (mod $\mathcal{I}$).
A Gröbner basis is uniquely defined for a given admissible ordering on the monomials. Given \( I \subset \mathbb{Q}[Y_1, \ldots, Y_k][X_1, \ldots, X_m] \), the use of an elimination ordering such that \( Y_i < X_j \) for all \( i, j \), allows to deduce straightforwardly a Gröbner basis of \( I \cap \mathbb{Q}[Y_1, \ldots, Y_k] \) and therefore to eliminate \( X_1, \ldots, X_m \). (see [7, Ch. 3]).

We obtain the saturation of the ideal \( I \subset \mathbb{Q}[X_1, \ldots, X_m] \) by a polynomial \( P_0 \) by computing a Gröbner basis of \( (I + \langle TP_0 - 1 \rangle) \cap \mathbb{Q}[X_1, \ldots, X_m] \), where \( T \) is a new independent variable. This is the usual way for computing an ideal whose zeroes set is the Zariski closure of a given constructible set (the constructible set itself when it is 0-dimensional).

Even if the basic principles used for computing a Gröbner basis are simple (see [7] for a general description), their effective computation is known to be hard. The first algorithm due to Buchberger has been a lot improved and we use, in practice, the algorithm \( F_4 \) by J.-C. Faugére (see [13]), implemented in recent versions of Maple, and known to be currently the fastest variant.

Once a Gröbner Basis of \( I \) is known, one can compute the associated Hilbert polynomial and then deduce the (Hilbert) dimension and the (Hilbert) degree of \( I \) (see [7] Chapter 9—Sect. 3).

### 6.3 Prime decompositions

In general, the ideal generated by the equations as well as its saturation by the inequalities is not prime. When it is of positive dimension, it has components of mixed dimensions. A key point at this stage is to be able to compute a prime or at least a primary decomposition.

There are several existing strategies for computing such a decomposition and it would be too long to enumerate them in the present article. One can mention two classes: methods based on Gröbner bases with an elimination ordering and factorization (see [7]—chapter 7) and those based on triangular sets (see [1] for example). Note however that the Maple function we use for this operation implements heuristics in order to select a favorable strategy.

### 6.4 Rational univariate representation

When a system is 0-dimensional, the so called Rational Univariate Representation (see [20]) defines a one-to-one correspondence between the solutions and the roots of some univariate polynomial \( f \), preserving the multiplicities. If this systems depends on \( m \) variables \( X_1, \ldots, X_m \), the Rational Univariate Representation consists in a polynomial \( f \in \mathbb{Q}[Y] \) and a set of \( m + 1 \) polynomials such that the solutions are

\[
\{ x_i = \frac{f_i(y)}{f_0(y)}, \quad i = 1, \ldots, m \mid f(y) = 0 \}.
\]

The algorithm is based on the fact that, when \( I \) is 0-dimensional, \( V = \mathbb{Q}[X_1, \ldots, X_m]/I \) is a finite dimensional \( \mathbb{Q} \)-algebra generated by the monomials that are irreducible modulo \( I \). These elements than can be directly obtained from a Gröbner basis.

There exists a linear combination \( Y = \sum_{i=1}^{m} \lambda_i X_i \in V \) which separates the zeroes of \( I \). Then \( f \) is the characteristic polynomial of the map \( V \to V \), \( P \mapsto Y \cdot P \).

The polynomials \( f, f_i, i = 0, \ldots, m \) are then computed by linear algebra.

This full parametrization is computed in two steps: first compute a suitable set of generators for the ideal (the best being a Gröbner basis), then compute a Rational Univariate Representation of the reduced 0-dimensional system.
Factorization of $f$ leads to consider that the solutions are parameterized by a finite number of algebraic numbers $\gamma$, defined by their minimal polynomials $f$ and each coordinate $z$ is a polynomial $f_z(\gamma)$.

Then, we substitute the variables that have been extracted during the pre-processing by their expression in $\mathbb{Q}(\gamma) = \mathbb{Q}[Y]/(f)$ using modular computation. We obtain a rational parametrization of the full system. Each of these parametrization may be described as

$$\text{rur} := \{ z = f_z(\gamma), \ z \in \mathbb{Z} \mid f(\gamma) = 0 \}.$$  

Furthermore, $\gamma \in \mathbb{R}$ if and only if $z(\gamma) = f_z(\gamma) \in \mathbb{R}$ for all $z \in \mathbb{Z}$.

### 6.5 Sorting the solutions

In this section we explain an exact procedure to sort solutions in $\text{SL}(3, \mathbb{R}), \text{PGL}(2, \mathbb{C})$ and $\text{PU}(2, 1)$ for 0-dimensional components. Thanks to the previous computations, the solutions of the initial system are expressed by means of rational parameterizations (here $f$ is a prime polynomial)

$$\text{rur} := \{ z = f_z(Y), \ z \in \mathbb{Z} \mid f(Y) = 0 \}.$$  

1. Real solutions are in correspondence with the real zeroes of $f$. The number of real roots can be determined exactly by Sturm algorithms and their approximate values are computed by certified algorithms (see).
2. The hyperbolic solutions are extracted from the global solutions as follows. Equation 3.3 becomes

$$z_{12} \equiv z_{21} \equiv z_{34} \equiv z_{43} \pmod{f}.$$  

From the Rational Univariate Representation it is just a matter of testing if some formal coordinates are equal in a field extension.
3. For the CR solutions, we consider the $3 \times n$ equations 3.4:

$$z_{ij}z_{ji} = z_{lk}z_{kl}.$$  

Let $\gamma = x + iy$ be one of the root of $f$. We obtain a new 0-dimensional system in $\mathbb{Q}[x, y]$ consisting of the 2 polynomial equations: $\text{Re} f(\gamma) = 0$, $\text{Im} f(\gamma) = 0$ and the $2 \times 3n$ equations:

$$\text{Re} \left[ z_{ij}(\gamma)z_{ji}(\gamma) \right] = \text{Re} \left[ z_{kl}(\gamma)z_{lk}(\gamma) \right], \ \text{Im} \left[ z_{ij}(\gamma)z_{ji}(\gamma) \right] = -\text{Im} \left[ z_{kl}(\gamma)z_{lk}(\gamma) \right].$$  

This last system may be solved by computing a Rational Univariate Representation. There exists an integer $\lambda$ such that the algebraic number $\eta = x + \lambda y$ describes the solutions:

$$\text{rur}_{\text{CR}} := \left\{ x = \frac{g_x(\eta)}{g_1(\eta)}, \ y = \frac{g_y(\eta)}{g_1(\eta)} \mid g(\eta) = 0 \right\}.$$  

Real solutions of the system are in a one-to-one correspondence with the real roots of $g$.

### 6.6 Computing the rank of the boundary holonomy representation

We compute first the meridian and the longitude as upper triangular matrices, say

$$g_M = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad g_L = \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}. $$
We easily obtain that 
\[ g^n_M = \begin{pmatrix} \frac{1}{n} a & n c + \frac{1}{2} n(n - 1) a b \\ 0 & n b \\ 0 & 1 \end{pmatrix} \]. We thus deduce that 
\[ g^n_M = g'^n_L \]
iff \( \frac{a}{a'} = \frac{b}{b'} = \frac{2c - ab}{2c' - a'b'} = \frac{n'}{n} \in \mathbb{Q} \). This can be decided by computing first the reduced row echelon form of 
\[ \begin{pmatrix} a & a' \\ b & b' \end{pmatrix} \]
in the ground field \( \mathbb{Q}[Y]/(f(Y)) \) where \( f \) is the minimal irreducible polynomial parameterizing the solutions. If the reduced row echelon form has rank one, we test if \( a \) and \( a' \) are \( \mathbb{Q} \)-dependant or not.

6.7 Certified numerical approximations

For simplicity, and in order to get a human readable output, we often express the results by means of numerical approximations instead of formal (large) expressions. We assert that all these approximations are certified (in practice, all the digits are correct but the last one). We make use of multi-precision floating point numbers coupled with interval arithmetic or well known results on error control for the evaluation of univariate polynomials that allows to accurately evaluate the Rational Univariate Representations at the roots of a given univariate polynomial. A fully treated example in the case of systems with two variables is described in [4].

7 An example: the figure-eight knot

The figure eight knot is given as a gluing of two tetrahedra as in Fig. 2. We refer to it to set the equations of compatibility.

Let \( z_{ij} \) and \( w_{ij} \) be the coordinates associated to the edge \( ij \) of each of the tetrahedra.

We obtain the edge and face equations:

\[
(L_e) \begin{cases} 
 z_{23} w_{24} z_{13} w_{31} z_{14} w_{21} = 1 \\
 z_{32} w_{42} z_{31} w_{13} z_{41} w_{12} = 1 \\
 z_{24} w_{23} z_{21} w_{43} z_{34} w_{41} = 1 \\
 z_{42} w_{32} z_{12} w_{34} z_{43} w_{14} = 1,
\end{cases} \quad (L_f) \begin{cases} 
 z_{21} z_{31} z_{41} w_{21} w_{31} w_{41} = 1 \\
 z_{12} z_{32} z_{42} w_{13} w_{23} w_{43} = 1 \\
 z_{13} z_{23} z_{43} w_{12} w_{32} w_{42} = 1 \\
 z_{14} z_{24} z_{34} w_{14} w_{24} w_{34} = 1.
\end{cases}
\]

Fig. 2  Triangulation of m004
7.1 Boundary holonomy representation

The meridian and the longitude are computed from the Snappea triangulation. We follow the algorithm to compute the holonomy along the meridian and the longitude along appropriate paths:

\[(421)^1 \rightarrow (423)^1 \rightarrow (324)^0 \rightarrow (314)^0 \]
\[(421)^1 \rightarrow (431)^1 \rightarrow (214)^0 \rightarrow (234)^0 \rightarrow (243)^1 \rightarrow (241)^1 \rightarrow (134)^0 \rightarrow (132)^0 \rightarrow (312)^1 \rightarrow (342)^0 \rightarrow (412)^0 \rightarrow (134)^1 \rightarrow (132)^1 \rightarrow (312)^0 \rightarrow (314)^0 \]

The corresponding matrices are:

\[g_M = \begin{bmatrix} z_{12}z_{21} & -z_{12}z_{13} + z_{34}z_{24} & -z_{32}w_{42} \\ z_{34}w_{24} & 0 & 1 \\ 0 & 0 & z_{34} \end{bmatrix}, \quad g_L = \begin{bmatrix} z_{13}z_{31}w_{14}w_{23} & * & * \\ w_{31}z_{42}w_{42}z_{24} & 0 & 1 \\ 0 & 0 & \frac{z_{31}w_{13}z_{13}w_{24}}{z_{42}w_{32}z_{24}w_{41}} \end{bmatrix} \]

We therefore deduce the unipotent holonomy equations:

\[A = A^* = B = B^* = 1 \text{ where} \]
\[A = \frac{z_{12}z_{21}}{z_{34}w_{24}}, \quad A^* = \frac{w_{42}}{z_{34}}, \quad B = \frac{z_{13}z_{31}w_{14}w_{23}}{w_{31}z_{42}w_{42}z_{24}}, \quad B^* = \frac{z_{31}w_{13}z_{13}w_{24}}{z_{42}w_{32}z_{24}w_{41}}.\]

7.2 Solutions

Here the pre-processing provides a reduced system of 7 polynomial equations in the variables \{z_{14}, z_{43}\}. All of the 22 other variables belong to \(\mathbb{Q}(z_{14}, z_{43})\). This reduced system is 0-dimensional and one can easily compute Rational Univariate Representations of its zeroes (Fig. 3).

One obtain four sets given by their minimal polynomials.

- \(f_1 = Y^2 - Y + 1\).
  The complete hyperbolic structure on the complement of the figure-eight knot is obtained from two conjugate solutions. In fact, in that case, if \(\omega = \frac{1+\sqrt{3}}{2}\) is one root of \(f_1\), then
  \[z_{12} = z_{21} = z_{34} = z_{43} = w_{12} = w_{21} = w_{34} = w_{43} = \omega^\pm\]
  is a solution of the equations as obtained in [23]. Its volume is 2.029883212 \ldots.

- \(f_2 = Y^2 - Y + 1\).
  \[z_{12} = z_{34} = w_{43} = z_{21} = \overline{z}_{43} = w_{12} = w_{21} = \omega^\pm.\]
This solution corresponds to a discrete representation of the fundamental group of the complement of knot in $\text{PU}(2, 1)$ with faithful boundary holonomy. Moreover, its action on complex hyperbolic space has limit set the full boundary sphere ([9]).

- $f_3 = Y^2 + Y + 2$. Let $y^\pm = -\frac{1}{2} \pm i \frac{1}{2} \sqrt{7}$ be a root of $f_3$. The coordinates are

$$z_{21} = w_{21} = \bar{z}_{43} = \bar{w}_{12} = \frac{5-i\sqrt{7}}{4}, \quad z_{12} = w_{2,1} = \frac{3-i\sqrt{7}}{8}, \quad w_{34} = \bar{w}_{43} = -\frac{1+i\sqrt{7}}{2}.$$

- $f_4 = 2Y^2 + Y + 1$. $1/y^\pm = -\frac{1}{4} \pm i \frac{1}{4} \sqrt{7}$ is a root of $f_4$. We obtain

$$z_{12} = \bar{z}_{34} = \frac{3-i\sqrt{7}}{2}, \quad w_{43} = \bar{w}_{34} = \frac{5+i\sqrt{7}}{8}, \quad z_{21} = w_{21} = \bar{z}_{43} = \bar{w}_{12} = -\frac{1+i\sqrt{7}}{4}.$$

All solutions corresponding to $f_2, f_3, f_4$ were obtained in [9] and $f_3, f_4$ correspond to spherical CR structures with unipotent boundary holonomy of rank one ([8]).

### 7.3 Holonomy representation

In order to obtain the representation of the fundamental group of the complement of the figure eight knot we identify the face (324)$^0$ of the tetrahedron 0 to the face (423)$^1$ of the tetrahedron 1. The generators are explicitly written from the remaining three face identifications following paths which pass through the chosen face (324)$^0$ = (423)$^1$ inside the tetrahedra and transport the projective basis defined by each oriented face.

Here are the paths generating the identifications on the polyhedron:

(1) $(34)^0 \rightarrow (34)^1 \rightarrow (34)^0 \rightarrow (42)^0 \rightarrow (23)^0 \rightarrow (24)^0 = (24)^1 \rightarrow (24)^0$ 
(2) $(21)^0 \rightarrow (42)^0 \rightarrow (42)^0 \rightarrow (23)^0 \rightarrow (24)^0 = (24)^1 \rightarrow (43)^0 \rightarrow (43)^1$ 
(3) $(12)^0 \rightarrow (23)^0 \rightarrow (24)^0 = (24)^1 \rightarrow (43)^0 \rightarrow (32)^1 \rightarrow (32)^1$

In order to obtain paths generating the fundamental group with a base point at face (234)$^0$, the paths have to be conjugated, respectively, by

(1) $(24)^0 \rightarrow (34)^0 \rightarrow (34)^0 \rightarrow (41)^0 \rightarrow (13)^0 \rightarrow (14)^0$ 
(2) $(23)^0 \rightarrow (34)^0 \rightarrow (42)^0 \rightarrow (42)^0 \rightarrow (21)^0 \rightarrow (21)^0$ 
(3) $(23)^0 \rightarrow (23)^0 \rightarrow (31)^0 \rightarrow (31)^0 \rightarrow (12)^0$

The coordinate transformations corresponding to each path are obtained by multiplying the coordinate change matrices. The generators of the group representation are obtained as inverse matrices of the coordinate change matrices.

Applying the computations above to the solution $f_2$ we obtain the generators

$$g_1 = \begin{bmatrix} 1 & 0 & 0 \\ -w^\pm & 1 & 0 \\ -\bar{w}^\pm & -2w^\pm & 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 3w^\pm & 6w^\pm & 1 \\ 3w^\pm & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 1 & 2w^\pm & -w^\pm \\ 0 & 1 & w^\pm \\ 0 & 0 & 1 \end{bmatrix}.$$

The group $\langle g_1, g_2, g_3 \rangle$ preserves the hermitian form

$$\begin{bmatrix} 0 & 0 & -w^\pm \\ 0 & 1 & 0 \\ -\bar{w}^\pm & 0 & 0 \end{bmatrix}.$$
For the solution $f_3$ we obtain

$$g_1 = \begin{bmatrix}
1 & 0 & 0 \\
-\frac{2}{\gamma^\pm} & 1 & 0 \\
-\frac{2}{\gamma^\pm} & \frac{1}{2} \gamma^\pm & 1
\end{bmatrix}, \quad g_2 = \begin{bmatrix}
\frac{8}{\gamma^\pm + 3} & \frac{3}{\gamma^\pm + 2} & -\frac{4}{\gamma^\pm + 2} \\
\frac{3}{\gamma^\pm + 2} & 1 & 0 \\
\frac{2}{\gamma^\pm + 2} & 0 & 0
\end{bmatrix},$$

$$g_3 = \begin{bmatrix}
1 & \frac{2}{\gamma^\pm + 2} & -\frac{1}{\gamma^\pm} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.$$  

The group $\langle g_1, g_2, g_3 \rangle$ preserves the hermitian form

$$\begin{bmatrix}
0 & 0 & \gamma^\pm \\
0 & 1 & 0 \\
\gamma^\pm & 0 & 0
\end{bmatrix}.$$  

Finally, for the solution $f_4$ we obtain

$$g_1 = \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
-\frac{2}{\gamma^\pm} & 1 & 0 \\
-\frac{2}{\gamma^\pm} & \frac{1}{2} \gamma^\pm & 1
\end{bmatrix}, \quad g_2 = \begin{bmatrix}
-2 & -\frac{1}{2} \gamma^\pm - 3 & -1 \\
\frac{8}{\gamma^\pm + 6} & 1 & 0 \\
-1 & 0 & 0
\end{bmatrix},$$

$$g_3 = \begin{bmatrix}
1 & \frac{4}{\gamma^\pm + 3} & -\gamma^\pm \\
0 & 1 & -\frac{2}{\gamma^\pm - 2} \\
0 & 0 & 1
\end{bmatrix}.$$  

The group $\langle g_1, g_2, g_3 \rangle$ preserves the hermitian form

$$\begin{bmatrix}
0 & 0 & -\frac{1}{2} \\
0 & 1 & 0 \\
-\frac{1}{2} & 0 & 0
\end{bmatrix}.$$  

Observe that the three generators satisfy the relations

$$g_3 g_2^{-1} g_1^{-1} g_2 = 1 \quad \text{and} \quad g_3^{-1} g_2 g_1^{-1} g_3 g_1 = 1$$

which give a presentation of the fundamental group of the figure eight knot.

7.4 Boundary holonomy

For all solutions in $\text{PU}(2, 1)$ we always obtain $g_L = g_1^{-1}$. The meridian is respectively

$$g_M = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\pm \sqrt{3} & 0 & 1
\end{bmatrix}, \quad g_M = g_L^{-3}, \quad g_M = g_L^3.$$  

7.5 Some non-unipotent configurations of the figure-eight knot

In this section we give examples of non-unipotent representations in order to show how our method can be applied without modification to a general holonomy condition. We will only deal with a few examples for the figure eight knot and reserve a complete study of that knot for a later paper.
We keep the notations of Sect. 7:

\[ A = \frac{z_{12}z_{21}}{z_{34}w_{24}}, \quad A^* = \frac{w_{42}}{z_{34}}, \quad B = \frac{z_{13}z_{31}w_{14}w_{23}}{w_{31}z_{42}w_{42}z_{24}}, \quad B^* = \frac{z_{31}w_{13}z_{13}w_{24}}{z_{42}w_{32}z_{24}w_{41}}. \]

We will first impose that \( A \) and \( A^* \) are primitive cubic roots of unity, and we find a finite set of solutions. In the second case we will impose that \( A \) and \( A^* \) are primitive sixth roots of unity. We find here a finite number of isolated solutions as well a one 1-dimensional component.

We start with the case when \( A \) and \( A^* \) are primitive cubic roots of unity. Here the pre-processing provides a reduced system of 5 polynomial equations in the variables \( \{z_{14}, z_{24}, z_{43}, w_{14}\} \). All of the 20 other variables belong to \( \mathbb{Q}(z_{14}, z_{24}, z_{43}, w_{14}) \). This reduced system is 1-dimensional. After saturation the system becomes 0-dimensional. It has 6 components.

Let \( \zeta \) be a primitive twelfth root of unity.

The first component corresponds to representations in \( \text{PGL}(2, \mathbb{Q}(\zeta^2)) \) and we get \( A = 1/A^* = \zeta^4, \quad B = B^* = 1 \). The volume is 0, we have \( g_M^3 = g_L = 1 \).

The second and the third component give 8 solutions in \( \text{PGL}(3, \mathbb{Q}(\zeta)) \). We get \( A = 1/A^* = \zeta^4, \quad B = B^* = 1 \). Their volumes are equal to 0 and we have \( g_M^3 = g_L = 1 \).

The fourth and the fifth components give 8 solutions in \( \text{PGL}(3, \mathbb{Q}(\alpha)) \) where \( \alpha = \pm\sqrt{5} \pm i\sqrt{3} \). We get \( A = A^* = \zeta^4, \quad B = B^* = 1 \). In this case we have \( g_M^3 = g_L \) or \( g_M^3 = g_L^{-1} \).

The last component gives 4 solutions in \( \text{PGL}(3, \mathbb{C}) \) and 4 solutions in \( \text{PU}(2, 1) \). We get \( A = A^* = \zeta^4 \) and \( B = B^* \) is a root of \( X^4 - 175 X^3 - 327 X^2 - 175 X + 1 \). This last polynomial has 2 conjugated roots with moduli 1 and two real roots. Therefore, the boundary holonomy group has rank 2. These solutions belong to \( \text{PU}(2, 1) \) when \(|B| = 1 \) and the four others give a volume \( 973235 \cdots \), when \( B \) is real.

We now look at solutions with \( A \) and \( A^* \) primitive sixth roots of unity. We find here seven 0-dimensional components and one 1-dimensional component.

There are 4 hyperbolic solutions corresponding to \( A = 1/A^* = \zeta^2 \) and \( B = 1/B^* = 7 \pm 4\sqrt{3} \). Their volume is 1.2212874588 \cdots \). The meridian satisfies \( g_M^6 = 1 \).

There are two components in \( \text{PGL}(3, \mathbb{Q}(\zeta)) \), satisfying \( A = 1/A^* = \zeta^2, \quad B = B^* = -1 \). They all satisfy \( g_M^3 = g_L, \quad g_L^2 = 1 \). None of these belong to \( \text{PU}(2, 1) \).

There are three components in \( \text{PGL}(3, \mathbb{Q}(\beta)) \) where \( \beta \) is a root of \( X^4 + X^3 - X^2 - X + 1 \). We have \( A = 1/A^* = \zeta^2, \quad B = B^* = -1 \). They all satisfy \( g_M^3 = g_L, \quad g_L^2 = 1 \). None of these belong to \( \text{PU}(2, 1) \).

There is a component giving 4 solutions in \( \text{PGL}(3, \mathbb{C}) \) and 4 solutions in \( \text{PU}(2, 1) \). We get \( A = A^* = \zeta^4 \) and \( B = B^* \) is a root of \( X^4 - 27X^3 + 25X^2 - 27X + 1 \). This last polynomial has 2 conjugated roots with moduli 1 and two real roots. Therefore, the boundary holonomy group has rank 2. These solutions belong to \( \text{PU}(2, 1) \) when \(|B| = 1 \) and the four others give a volume 1.730258 \cdots \), when \( B \) is real.

The 1-dimensional component corresponds to reducible representations. We have: \( A = 1/A^* = \zeta^2 \) and \( B = 1/B^* = -1 \). The volume of these configurations is 0. For all these configurations, the face variables are all equal to \(-1 \). We obtain \( z_{ij}z_{ji} = \zeta^{\pm 2}, \quad w_{ij}w_{ji} = \zeta^{\pm 2} \) for all \( i < j \). We always obtain \( g_M^3 = g_L \) and \( g_L^2 = 1 \).

### 8 Description of the unipotent solutions for the first hyperbolic manifolds

As explained in the last section we solve the system consisting of compatibility equations and the unipotent holonomy conditions plus the inequalities which prevent the \( z \)-coordinates to be 0 or 1. The solutions will, in turn, give rise to representations in \( \text{PSL}(3, \mathbb{C}) \).
We have computed an exhaustive list of solutions for the eleven first 3-manifolds and for the Whitehead link complement. Their main properties are summarized in the Table 1. For these examples the dimension of solutions is at most 1 and there are always isolated solutions besides the hyperbolic one. Five of them are 0-dimensional: m004, m007, m009, m015 and the Whitehead link complement. All others are 1-dimensional: m003, m006, m010, m011, m016, m017, m019. We further analyse the solutions to decide if the corresponding representation is in PSL(3, \mathbb{R}), PSL(2, \mathbb{C}), PSL(2, \mathbb{R}) or PU(2, 1) and compute their volume. Observe also that, by Mostow rigidity, only one solution corresponds to the complete hyperbolic structure. This solution is identified as the only one having all positive imaginary parts of the z-coordinates.

In Table 1, we indicate the name of the variety, the number 1-D of 1-dimensional prime components, the degrees of the prime 0-dimensional components. The isolated solutions studied are the ones not contained in the 1-dimensional components. We indicate the total number of solutions such that the corresponding representations are in PSL(3, \mathbb{C}), PSL(3, \mathbb{R}), PSL(2, \mathbb{C}), PSL(2, \mathbb{R}) and PU(2, 1). Observe that PSL(3, \mathbb{R}) \cap PSL(2, \mathbb{C}) = PSL(2, \mathbb{R}) and PSL(3, \mathbb{R}) \cap PU(2, 1) = PSL(2, \mathbb{R}) so PSL(2, \mathbb{R}) solutions are being also redundantly counted in PSL(3, \mathbb{R}), PSL(2, \mathbb{C}) and PU(2, 1). We also indicate the different positive volumes we obtain, writing in boldface the volumes of representations in PSL(2, \mathbb{C}). Keep in mind though that real solutions as well as solutions giving rise to PU(2, 1) representations have zero volume.

For the 0-dimensional case, we briefly describe the solutions and in some cases the group representations. In particular, we enumerate in the examples all cases when the boundary holonomy has rank one. More details can be obtained in the website, .../SGT including matrix representations of the fundamental groups. In the following description, for the sake of simplicity, we will say that a solution is CR (short for Cauchy-Riemann), when the corresponding representation is conjugated to a representation in PU(2, 1). For each variety we singled out a PU(2, 1) representation with boundary holonomy of rank one. We think that they are natural candidates for being holonomies of a spherical CR structure on the variety. Again, the difficulty in obtaining a CR structure lies in the definition of an appropriate 2-skeleton (see [9] for a discussion).

The 1-dimensional components in the examples are not completely described. They are all degenerate, that is, they are either reducible representations or they have trivial unipotent boundary holonomy and their image is a cyclic group (we thank M. Deraux, A. Guilloux and C. Zickert for discussions about these components which helped us to understand their meaning). The reducible families arise from our definition of tetrahedra. In fact we don’t impose that the lines of the quadruple of flags be in general position and the solutions with all the lines of all flags passing through a fixed point gives the family of reducible representations. We don’t enumerate the PSL(2, \mathbb{C}), PU(2, 1) or PSL(3, \mathbb{C}) solutions in these components. But an important observation is that in all examples up to three tetrahedra the 1-dimensional components contain at most a finite number of PSL(2, \mathbb{C}) solutions which turn out to be real so the volumes presented in the table contain indeed all volumes of PSL(2, \mathbb{C}) representations arising from the given triangulation. On the other hand, the 0-dimensional solutions are never reducible nor have trivial boundary holonomy.

8.1 The variety m007

There are three simplices with parameters u_{ij}, v_{ij} and w_{ij}, 1 \leq i, j \leq 4.

- f_1 = Y^3 - 2Y^2 - 1.
The solutions giving rise to PSL(2, C) representations (containing the two conjugate complete hyperbolic ones) are

\[ u_{12} = u_{21} = u_{34} = u_{43} = \gamma, \]
\[ v_{12} = v_{21} = v_{34} = v_{43} = 2\gamma - \gamma^2 = -1/\gamma, \]
\[ w_{12} = w_{21} = w_{34} = w_{43} = \gamma. \]

where \( \gamma \) is a root of \( f_1 = Y^3 - 2Y^2 - 1. \) The volume of the hyperbolic solution is \( \text{vol} = 2.568970609 \cdots. \)

- \( f_2 = Y^6 - Y^5 + Y^4 - 2Y^3 + Y^2 + 1. \)
  2 solutions are CR. The four others define representations with volume \( 0.70703052208 \cdots. \)
- \( f_3 = Y^8 - 2Y^7 + 2Y^6 - 6Y^5 + 7Y^4 - 2Y^3 + 3Y^2 - Y + 1. \)
  4 solutions are CR. The volume of the other representations is \( 0.8274406556 \cdots. \)
- \( f_4 = 3Y^8 - 6Y^7 + 7Y^6 - 4Y^5 + 4Y^4 + 4Y^3 + 3Y^2 + 1. \)
  4 solutions are CR. Their volume is \( 0.82274406556 \cdots. \)
- \( f_5 = 16Y^8 - 20Y^7 + 23Y^6 - 27Y^5 + 10Y^4 - 5Y^3 + 9Y^2 + 2Y + 4. \)
  4 solutions are CR. The four others have volume \( 1.3366875264 \cdots. \)

In conclusion, there are 33 solutions, 15 being CR (Fig. 4).

### 8.2 The variety \( m009 \)

There are three simplices with parameters \( u_{ij}, v_{ij} \) and \( w_{ij}, 1 \leq i, j \leq 4 \) (Fig. 5).

We obtain

- \( f_1 = Y^2 - Y + 2. \)

The PSL(2, C) solutions (containing the two conjugate complete hyperbolic ones) are given by

\[ u_{12} = u_{21} = u_{34} = u_{43} = \gamma, \]
\[ v_{12} = v_{21} = v_{34} = v_{43} = \frac{1}{4}(\gamma + 1) \]
\[ w_{12} = w_{21} = w_{34} = w_{43} = \gamma. \]
where $\gamma = \frac{1}{2}(1 \pm i\sqrt{7})$ is a root of $f_1$. The volume of the hyperbolic solution is $\text{vol} = 2.6667447834 \cdots$

- $f_2 = f_3 = Y^4 + Y^3 - Y^2 - Y + 1$.
  They correspond to 8 solutions with the same volume $0.50747080320 \cdots$.
- $f_4 = Y^4 + 2Y^3 - Y^2 - 2Y - 4 = (Y^2 + Y - 1 - \sqrt{5})(Y^2 + Y - 1 + \sqrt{5})$.
  There are two real solutions corresponding to $\gamma^\pm = -\frac{1}{2} \pm \frac{1}{2}i\sqrt{5 - 4\sqrt{5}}$ and two conjugate CR solutions.

We have

\[
\begin{align*}
  u_{12} &= w_{34} = \frac{\gamma^\pm + 3}{\gamma^\pm + 1},
  u_{21} &= w_{43} = \gamma^\pm, \\
  u_{34} &= w_{12} = \frac{\gamma^\pm - 2}{\gamma^\pm}, \\
  u_{43} &= w_{21} = -1 - \gamma^\pm, \\
  v_{12} &= v_{34} = \frac{1}{\gamma^\pm + 3}, \\
  v_{21} &= v_{43} = \frac{1}{2 - \gamma^\pm}.
\end{align*}
\]

The two real solutions give rise to representations in $\text{PSL}(3, \mathbb{R}) \setminus \text{PSL}(2, \mathbb{R})$. Moreover, for all of these solutions we find that the meridian $g_M$ and the longitude $g_L$ satisfy $g_M g_L^2 = 1$.

We find

\[
g_M = \begin{bmatrix}
  1 & -2 \frac{\gamma^\pm + 2}{\gamma^\pm (\gamma^\pm + 1)} & -2 \frac{\gamma^\pm + 2}{\gamma^\pm (\gamma^\pm + 3)} \\
  0 & 1 & \frac{3 + \gamma^\pm}{\gamma^\pm + 3} \\
  0 & 0 & 1
\end{bmatrix}.
\]

- $f_5 = Y^6 - Y^4 + 2Y^3 + Y^2 - Y + 1$.
  There are two CR solutions and 4 solutions with the same volume $0.79158333031 \cdots$.
- $f_6 = 8Y^8 - 16Y^7 + 22Y^6 - 25Y^5 + 16Y^4 - 6Y^3 + Y^2 + 3Y + 1$.
  There are 4 CR solutions and 4 solutions with the same volume $1.4179708859 \cdots$.

In conclusion we found 28 solutions, 8 being CR. There are 3 different volumes apart from the hyperbolic one.

8.3 The 52 knot complement

There are three simplices with parameters $u_{ij}, v_{ij}$ and $w_{ij}, 1 \leq i, j \leq 4$.

We find 23 solutions in 5 sets of Galois conjugate solutions (Fig. 6).

- $f_1 = Y^3 - Y + 1$.
The solutions correspond to the two conjugate complete hyperbolic solutions and a real one.

\[
\begin{align*}
    u_{12} &= u_{21} = u_{34} = u_{43} = \gamma, \\
    v_{12} &= v_{21} = v_{34} = v_{43} = \gamma, \\
    w_{12} &= w_{21} = w_{34} = w_{43} = \gamma.
\end{align*}
\]

where \(\gamma\) is a root of \(f_1\). The volume is \(2.828122 \cdots\).

- \(f_2 = Y^4 - Y^2 + 1\).
  We obtain 4 solutions (the 4 12th-roots of unity) that are all CR.
- \(f_3 = 4Y^4 - 4Y^3 - Y^2 + Y - 1\).
  We obtain 2 real solutions (giving representations in \(\text{PSL}(3, \mathbb{R}) \setminus \text{PSL}(2, \mathbb{R})\)) that are the roots of
  \[
  4Y^2 - 2Y - (1 - \sqrt{5}),
  \]
  and 2 complex conjugate solutions that are the roots of
  \[
  4Y^2 - 2Y - (1 + \sqrt{5}).
  \]

These two last solutions are CR. We obtain, for these four solutions:

\[
\begin{align*}
    u_{21} &= v_{21} = v_{34} = w_{21} = \frac{1}{1 - \gamma}, \\
    u_{12} &= u_{43} = v_{43} = w_{43} = \frac{2}{2\gamma + 1}, \\
    u_{12} &= u_{12} = -\gamma(\gamma - 1), u_{34} = w_{34} = \frac{1}{2} - \gamma^2.
\end{align*}
\]

Note that in this case the meridian and the longitude are equal:

\[
\begin{bmatrix}
  1 & 2 \\
  0 & 1 \\
  0 & 0
\end{bmatrix}
\frac{1}{\gamma(2\gamma - 1)} \begin{bmatrix}
  1 & \gamma(2\gamma - 1)^2 \\
  \frac{1}{\gamma} & 1
\end{bmatrix}.
\]

The holonomy group, in this case, is generated by

\[
g_1 = g_2^{-1} \cdot g_2 = \begin{bmatrix}
  \frac{1}{2} & 0 \\
  \frac{2}{2\gamma(1 - 2\gamma)} & 1 \\
  \frac{1}{\gamma - 1} & 0
\end{bmatrix}
\frac{2}{2\gamma(1 - 2\gamma)} \begin{bmatrix}
  \frac{2}{2\gamma - 1} & 0 \\
  \frac{1 - 4\gamma^2}{2\gamma} & 1 \\
  \frac{\gamma - 1}{2\gamma} & 0
\end{bmatrix}.
\]

\[
g_3 = \begin{bmatrix}
  -\frac{1}{2\gamma - 1} & 0 \\
  1 & 0 \\
  \frac{2\gamma^2(1 - 2\gamma)}{2\gamma^2(1 - 2\gamma)} & 1
\end{bmatrix},
\quad
\begin{bmatrix}
  \frac{1}{2\gamma(1 - 2\gamma)} & 0 \\
  \frac{2\gamma(2\gamma - 1)}{2\gamma(1 - 2\gamma)} & 1 \\
  \frac{2\gamma}{2\gamma} & 0
\end{bmatrix}.
\]

where \(\gamma\) is one of the four roots of \(f_3\).

- \(f_4 = Y^6 - Y^5 - 3Y^3 + 2Y^2 + Y + 1\).
  Only two of these roots correspond to CR structures. The four others have same volume: \(1.583167 \cdots\).
• $f_5 = 17Y^6 - 5Y^5 + 33Y^4 - 11Y^3 + 26Y^2 - 4Y + 8$.

Only two of these roots correspond to CR structures. The others define representations of volume 0.794323 \ldots.

In conclusion we obtain 23 solutions and 11 of them are CR.

8.4 The whitehead link

Here we have four tetrahedra with parameters $u_{ij}$, $v_{ij}$, $w_{ij}$ and $x_{ij}$ with $1 \leq i, j \leq 4$ glued as in Fig. 7.

There are 6 different groups of solutions.

• $f_1 = Y^2 + 1$. The conjugate complete hyperbolic solutions are given by

\[
\begin{align*}
  u_{12} &= u_{21} = u_{34} = u_{43} = v_{12} = v_{21} = v_{34} = v_{43} = \pm i, \\
  w_{12} &= w_{21} = w_{34} = w_{43} = w_{12} = w_{21} = w_{34} = w_{43} = \pm i.
\end{align*}
\]

• $f_2 = Y^2 + Y + 4$.

We obtain

\[
\begin{align*}
  u_{12} &= u_{21} = u_{34} = u_{43} = \frac{3}{8} - \frac{1}{8} \gamma, \\
  u_{21} &= u_{43} = w_{21} = w_{43} = \gamma, \\
  u_{34} &= v_{12} = v_{21} = v_{34} = v_{43} = \frac{3}{8} - \frac{1}{8} \gamma.
\end{align*}
\]

where $\gamma = -\frac{1}{2} \pm \frac{1}{2} \sqrt{15}$.

The holonomies of the meridian and longitude on the first torus (it corresponds to the vertex 1 of the first tetrahedron) are both equal to

\[
\begin{bmatrix}
  1 & 2 & 1 \\
  0 & 1 & 1 \\
  0 & 0 & 1
\end{bmatrix},
\]

while the holonomies of the meridian and longitude of the second torus are equal to

\[
\begin{bmatrix}
  \frac{1}{4} + \frac{1}{4} i \sqrt{15} & \frac{3}{4} + \frac{3}{4} i \sqrt{15} & \frac{3}{2} + \frac{1}{2} i \sqrt{15} \\
  \frac{3}{4} - \frac{1}{4} i \sqrt{15} & -2 - \frac{1}{2} i \sqrt{15} & -\frac{21}{8} - \frac{3}{8} i \sqrt{15} \\
  \frac{3}{2} & \frac{21}{4} + \frac{1}{2} i \sqrt{15} & \frac{19}{4} + \frac{1}{4} i \sqrt{15}
\end{bmatrix}.
\]

The representation corresponding to this solution was also obtained in [19] where it is shown that it is the holonomy of a spherical CR structure on the complement of the Whitehead link.

Fig. 7 Triangulation of Whitehead link complement

\[\text{Springer}\]
\[ f_3 = f_4 = Y^4 + Y^3 + 2Y^2 - Y + 1. \] All of these 8 solutions are CR.

\[ f_5 = f_6 = Y^{10} - 4Y^8 + 7Y^7 - 7Y^6 - 4Y^5 - 8Y^4 + 4Y^3 + 5Y^2 + 1. \] 4 of these 20 solutions are CR-spherical.

Among these 32 solutions, 14 are CR.

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