On the embeddability of $[3] \ast K^*$

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Abstract

We relate the embeddability of the simplicial complex $[3] \ast K$ into $\mathbb{R}^{n+2}$ to that of $K$ into $\mathbb{R}^n$. In brief, the embeddability of $K$ into $\mathbb{R}^n$, in the metastable range $2n \geq 3(d+1)$, is equivalent to embeddability of $[3] \ast K$ into $\mathbb{R}^{n+2}$. We show moreover than the van Kampen obstruction of $K$ vanishes if and only if the van Kampen obstruction of $[3] \ast K$ vanishes. It follows that for $d = 2$, embeddability of $[3] \ast K$ is equivalent with the vanishing of the van Kampen obstruction for $K$, but not with the embeddability of $K$.

1 Statements

Let $K$ be a simplicial complex of dimension $d$. We say $K$ embeds into $\mathbb{R}^n$ if there exists an injective continuous map $f : K \to \mathbb{R}^n$, here we denote by $K$ also the underlying topological space of $K$. Let $[3] \ast K$ denote the simplicial complex which is the join of $K$ with three vertices. In this note, we relate the embeddability of $K$ into $\mathbb{R}^n$ to that of $[3] \ast K$ into $\mathbb{R}^{n+2}$.

Let $K_2$ denote the deleted product of $K$ and $\bar{K}$ denote the quotient of the deleted product under the involution that exchanges the factors. If $d \neq 2$, the classical van Kampen obstruction, denoted by $\nu(K) \in H^{2d}(\bar{K}, \mathbb{Z})$, is a complete obstruction for existence of a PL embedding of $K$ into $\mathbb{R}^{2d}$. We refer to [8, 12, 2] for definitions and details.

There exist simplicial 2-complexes with vanishing van Kampen obstruction that do not embed into $\mathbb{R}^4$ [2]. In fact, there is no known algorithm for deciding embeddability of a 2-complex into $\mathbb{R}^4$, and this problem is known to be NP-hard [5]. Note that, in contrast, for $d \neq 2$, the embeddability of $K$ into $\mathbb{R}^{2d}$ can be decided in polynomial time by the computation of the van Kampen cohomology class. Refer to [5] for the results on the complexity of embedding problems.

There is also a more geometric criterion for the existence of an embedding in certain dimensions. It is easily seen that if $f : K \to \mathbb{R}^n$ is an embedding, then there exists a $\mathbb{Z}_2$-equivariant map $F : K_2 \to S^{n-1}$, where $S^{n-1}$ is the $(n-1)$-sphere with the antipodal action on it. Conversely, the existence of this $\mathbb{Z}_2$-equivariant map is sufficient for the existence of a PL embedding when $2n \geq 3(d+1)$, the metastable range, see

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Theorem 1. Let $X$ be a $Z_2$-space. Denote by $\pi^m_{Z_2}(X)$ the set of $Z_2$-equivariant homotopy classes of $Z_2$-equivariant maps $f : X \to S^m$. Thus, in the metastable range $2n \geq 3(d+1)$, if $\pi^{n-1}_{Z_2}(K_2^d) \neq \emptyset$ an embedding $K \to \mathbb{R}^n$ exists. See [9, 7].

Since we deal only with $Z_2$ actions we remove the $Z_2$ subscript from $\pi^m_{Z_2}(X)$. Moreover, the involution acting on the spaces we consider is always fixed so we do not write it. In general, the existence of $Z_2$-equivariant maps $X \to S^m$ depends on the action of $Z_2$ on $X$.

Let $K_2^d$ be the simplicial complex which is the deleted join of the simplicial complex $K_2$. The following theorem is the main tool in our arguments. This theorem follows from results of [1] and [9].

**Theorem 1** If $2 \leq m$, and, $d \leq m - 1$ then $\pi^m(K_2^d) \cong \pi^{m+2}([3]*K_2^d)$.

**Corollary 1** Let $K$ be a simplicial complex of dimension $d \geq 3$. If $2n \geq 3(d+1)$, then $K$ embeds into $\mathbb{R}^n$ if and only if $[3]*K$ embeds into $\mathbb{R}^{n+2}$.

Note that the above theorem generalizes the fact that if $K$ is a non-planar graph then $[3]*K$ does not embed into $\mathbb{R}^4$. This latter fact follows from a special case of the Grünbaum theorem [3], and the characterization of non-planar graphs as graphs that contain a subgraph which is topologically $K_5$ or $K_{3,3}$. We note that this special case was the only known case of our results.

We also mention the following corollary which is equivalent to Corollary 1. Analogous propositions can be stated for other results of this paper.

**Corollary 2** Let $K$ be a simplicial complex of dimension $d + 1$, such that $2n \geq 3(d+1)$ and $d \geq 3$. If $K$ embeds into $\mathbb{R}^{n+2}$ the triple intersection of links of any three vertices is a simplicial $d$-complex that embeds into $\mathbb{R}^n$.

**Theorem 2** Let $K$ be a simplicial complex of dimension $d \geq 1$. $\nu(K) \neq 0$ if and only if $\nu([3]*K) \neq 0$.

**Corollary 3** If $d = 2$ and $\nu(K) = 0$ then $\nu([3]*K) = 0$ and hence $[3]*K$ embeds into $\mathbb{R}^6$ regardless of the embeddability of $K$ into $\mathbb{R}^4$.

**Corollary 4** Let $K$ be a simplicial complex of dimension $d \geq 1$. $\nu(K) = 0$ if and only if $[3]*K$ embeds into $\mathbb{R}^{2d+2}$.

Corollary 4 might be of interest since it gives a totally geometric characterization of the vanishing of the van kampen obstruction.

## 2 Proofs

**Proof of Theorem 1** Let $L$ be an arbitrary polyhedral complex of dimension $s$. By Theorem 2.5 [1], if $s \leq 2m - 2$ there is an isomorphism $\pi^m(L) \cong \pi^{m+1}(\Sigma L)$ where $\Sigma L$ denotes the suspension of $L$. Setting $L = K_2^d$ we get $\pi^m(K_2^d) \cong \pi^{m+1}(\Sigma K_2^d)$ if $2d \leq 2(m-1)$.
By [9] Lemma 4.2.1, there is an equivariant surjection $p: (cK)^2 \to \Sigma K^2$, where the maximum dimension of a non-trivial preimage is $d$. Here, $cK$ denotes the cone over $K$.

Then by [9] 2.5, if $d \leq m - 1$ then $\pi^{m+1}((cK)^2) \cong \pi^{m+1}((cK)^2_\lambda)$. By [4] Exercise 4, Section 5.5, there is an isomorphism $\pi^{m+1}((cK)^2_\lambda) \cong \pi^{m+1}(K^{2*})$. Therefore, if $d \leq m - 1$, $\pi^{m}(K^{2*}) \cong \pi^{m+1}(K^{2*})$.

We now have to prove that $\pi^{m+1}(K^{2*}) \cong \pi^{m+3}([3] \ast K^{2*})$ whenever $d \leq m - 1$. But it is well-known that for simplicial complexes $K_1$ and $K_2$ and for the actions of $\mathbb{Z}_2$ that exchange the factors, $(K_1 \ast K_2)_\lambda^{2*} \cong S_{K1}K^{2*} \ast K^{2*}$. For a proof see [3]. However, $[3]^{2*} \cong S_{1*}^{13}$ and therefore $([3] \ast K)_\lambda^{2*} \cong S_{1*}^{1} \ast K^{2*} \cong S_{K}^{2*}$. The latter space, $\Sigma^2 K^{2*}$, is the result of twice suspension of $K^{2*}$. By two applications of the equivariant suspension theorem, [1] Theorem 2.5, it follows that if $d \leq m - 1$ then $\pi^{m+1}(K^{2*}) \cong \pi^{m+3}([3] \ast K^{2*})$.

**Proof of Corollary**[1] It is easily seen that if $K$ embeds in $\mathbb{R}^n$ then $[k] \ast K$ embeds in $\mathbb{R}^{n+2}$, for any positive integer $k$. Hence we focus on the other direction.

The conditions of Theorem [1] for $m = n - 1$ become $n \geq 3$ and $d \leq n - 2$. If $d \geq 3$ and $2n \geq 3(d + 1)$ then the conditions of the theorem are satisfied. Now if $[3] \ast K$ embeds in $\mathbb{R}^{n+2}$ then $\pi^{n+1}([3] \ast K)_\lambda \neq 0$, it follows that $\pi^{n-1}(K^{2*}) \neq 0$. Thus in the metastable range we can deduce the existence of the embedding $K \to \mathbb{R}^n$.

**Proof of Theorem**[2] If $d = 1$, $\nu(K) \neq 0$ implies $K$ non-planar. Therefore it has a subgraph which is topologically $K_5$ or $K_{3,3}$. Hence $[3] \ast K$ contains as a subspace $[3] \ast K_5$ or $[3] \ast K_{3,3}$. It’s enough to show that the obstruction class for these subspaces is non-zero. Let $L$ be one of these two graphs. By [3], $([3] \ast L)_\lambda^{2*} \cong S_1^3$. If $\nu([3] \ast L) = 0$ then, by the standard theory of the van Kampen obstruction (codimension 3), there exist a map $[3] \ast L \to \mathbb{R}^5$ in which the image of disjoint simplices do not intersect. It follows that then there is an equivariant mapping $F: ([3] \ast L)^2_\lambda \to S^3$. Then by proof of Theorem 1 there will be an equivariant map $([3] \ast L)^2_\lambda \to S^3$. But this is impossible, for instance using the Borsuk–Ulam Theorem. It follows that $\nu([3] \ast K) \neq 0$.

If $d = 1$, $\nu(K) = 0$ then the graph does not contain any $K_5$ or $K_{3,3}$ as a subspace. Hence it is planar. And $[3] \ast K$ embeds in $\mathbb{R}^4$. It follows that $\nu([3] \ast K) = 0$.

If $d \geq 2$ and $\nu(K) \neq 0$ then $\pi^{2d-1}(K^{2*}) = \emptyset$. Then by Theorem [1] $\pi^{2d+1}([3] \ast K)^2_\lambda = \emptyset$ and hence $[3] \ast K$ is not embeddable $\mathbb{R}^{2d+2}$. Since $d + 1 > 2$ it follows that $\nu([3] \ast K) \neq 0$.

If $d > 2$ and $\nu(K) = 0$ then $K$ embeds into $\mathbb{R}^{2d}$ hence $[3] \ast K$ embeds into $\mathbb{R}^{2d+2}$ and $\nu([3] \ast K) = 0$.

The only remaining case is $d = 2$ and $\nu(K) = 0$. We only sketch a proof here and refer to [6] for complete details. Since $\nu(K) = 0$ there is a PL map $f: K \to \mathbb{R}^4$ such that the algebraic intersection number of the images of any two disjoint 2-simplices is defined and is zero. Fix $\mathbb{R}^4 \subset \mathbb{R}^6$. The set of $\mathbb{R}^5$s that intersect at the fixed $\mathbb{R}^4$ can be parametrized by $S^1$. Take three points $\alpha_1, \alpha_2, \alpha_3 \in S^1$ and map the cone over $K$ from $\nu(K)$ is an obstruction to the existence of an equivariant map $K^2_\lambda \to S^{2d-1}$. More formally, $\nu(K)$ is the image of the non-zero element of $H^{2d}(\mathbb{R}P^\infty, \mathbb{Z})$ under the quotient of any equivariant map $F: K^2_\lambda \to S^\infty$. $\pi^{2d-1}(K^2_\lambda) \neq \emptyset$ implies there is map $F$ into $S^{2d-1}$, thus the images of all higher dimensional homology classes are zero under $F$.
the vertex \(i \in [3]\) into \(\mathbb{R}^5\) with parameter \(\alpha_i\). The map in each \(\mathbb{R}^5\) is the cone of the map \(f\). Let \(F : [3] * K \rightarrow \mathbb{R}^6\) be the resulting map.

The intersections between images of two 3-simplices of \([3] * K\) incident on the same vertex \(i \in [3]\) might be a set of arcs, however, we only care about the intersections between images of disjoint simplices of \([3] * K\). Images of disjoint 3-simplices are in distinct \(\mathbb{R}^5\)'s and they intersect whenever the images of the corresponding 2-simplices intersect in \(\mathbb{R}^4\). We only need to bring these intersections into the interior of 3-simplices without losing any of them, or introducing new ones or changing sign of an intersection. For this purpose, we first change the map \(F\) to make sure that, for any intersection point of images of any two disjoint 2-simplices \(\sigma_1, \sigma_2\) in \(\mathbb{R}^4 \subset \mathbb{R}^6\), in a small ball centered at that point, the images of \(1 * \sigma_1\) and \(2 * \sigma_1\) coincide with the image of \(3 * \sigma_1\), and the same for \(j * \sigma_2\). This is not difficult to do without introducing new intersections between disjoint 3-simplices. Let \(p \in \mathbb{R}^4 \subset \mathbb{R}^6\) be a point of intersection of images of \(\sigma_1\) and \(\sigma_2\). In the resulting map, each of the three pairs of disjoint 3-simplices \(i * \sigma_1, j * \sigma_2\) for \(i \neq j\) intersect at a short arc which ends at \(p\).

After the above modification to \(F\), we can change the map by a small perturbation near the intersection points of (images of) disjoint 2-simplices to make the short arc of intersection points into a single point in the interior of the 3-simplices. Each intersection point of 2-simplices of \(K\) gives rise to a point in which three distinct pairs of disjoint 3-simplices intersect. For any pair of disjoint 3-simplices \(i * \sigma_1, j * \sigma_2\) their algebraic sum of intersection numbers would be the same as the algebraic sum of the intersection numbers of \(\sigma_1\) and \(\sigma_2\).

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