Quantized Josephson phase battery

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We consider a Josephson junction with two ferromagnetic layers and a spin-flipper (magnetic impurity) sandwiched between two conventional s-wave superconductors. We show that when the magnetic moments of the ferromagnetic layers are misaligned and in presence of spin flip scattering the system acts as a phase battery that can store quantized amounts of superconducting phase difference \( \varphi_0 \) in the ground state of the junction. Moreover, for such \( \varphi_0 \)-Josephson junction an anomalous Josephson current appears at zero phase difference. We study the properties of this quantum spin flip scattering induced anomalous Josephson current, especially its tun-ability via exchange interaction and mis-orientation angle between two Ferromagnet’s.

I. INTRODUCTION

The Josephson Free energy, in general, is minimum when the phase difference across the Josephson junction is either zero for 0-junction or \( \pi \) for a \( \pi \)-junction. In such junctions, the Josephson super-current vanishes when the phase difference between two superconductors is zero as the current-phase relation is sinusoidal \( I(\varphi) = I_c \sin(\varphi) \), where \( \varphi \) is the phase difference across the superconductors\(^1\) and \( |I_c| \) is the maximum current flowing through the junction. However, Josephson energy can sometimes be minimum at a phase difference \( \varphi_0 \) (\( \neq 0 \) or \( \pi \)). The current-phase relation in such \( \varphi_0 \)- Josephson junction satisfies \( I(\varphi) = I_c \sin(\varphi + \varphi_0) \), i.e., there is a phase shift \( \varphi_0 \) in the conventional current-phase relation. This suggests that Josephson current can flow even at zero phase difference \( (\varphi = 0) \) between the two superconducting electrodes\(^2\)\(^3\). This effect is known as anomalous Josephson effect (AJE), and \( I_{an} = I(0) = I_c \sin(\varphi_0) \) is referred to as the anomalous Josephson current.

The physics behind anomalous Josephson effect is naturally linked with breaking of some symmetries of the system\(^4\). One of them is time reversal symmetry and it denotes the condition \( I(\varphi) = -I(\varphi) \), which results in \( I(\varphi = 0) \) being zero. So, when the system preserves time reversal symmetry there is no anomalous current in the system. However, breaking time reversal symmetry is a necessary but not sufficient condition to produce anomalous Josephson current at \( \varphi = 0 \). In junctions with ferromagnetic coupling time reversal symmetry is broken, but there is no anomalous Josephson current\(^6\)\(^7\). This implies some other symmetry is present in the system which prevents the appearance of anomalous Josephson current at \( \varphi = 0 \). This symmetry is called chiral symmetry\(^8\) which ensures that at \( \varphi = 0 \) the tunneling amplitude relating electron tunneling from left superconductor to right superconductor is exactly same as the one relating the tunneling in reverse, i.e., from right superconductor to left superconductor. These leftward and rightward tunneling processes cancel each other, implying no current can flow at \( \varphi = 0 \). Thus to have an anomalous Josephson current at \( \varphi = 0 \), one has to break both symmetries at the same time. Different ways have been suggested earlier to break the symmetries and generate an anomalous Josephson current. These include Josephson junctions with conventional s-wave superconductors in presence of both spin-orbit interaction and Zeeman field\(^9\)\(^10\), ferromagnetic Josephson junctions with non-coplanar magnetization\(^6\), SNS junctions with conventional s-wave superconductor where N region is a magnetic normal metal\(^11\)\(^12\) a quantum dot\(^13\)\(^14\) and a quantum point contact\(^15\)\(^16\). Further, anomalous Josephson effect can also be found in systems with unconventional superconductors\(^21\)\(^22\). Experimentally, \( \varphi_0 \) phase shift has been recently predicted in a Josephson junction based on a nanowire quantum dot\(^23\). More interestingly some Josephson junctions reveal the remarkable feature that the phase shift \( \varphi_0 \) is accompanied by a direction dependent critical current \( I_{c+} \neq I_{c-} \), where \( I_{c+} \) and \( I_{c-} \) are the absolute values of the maximum and minimum Josephson current respectively.

In this work we study the anomalous Josephson effect and the direction dependent critical current in a junction consisting of two Ferromagnet’s with mis-aligned magnetizations and a spin-flipper sandwiched between two s-wave superconductors. This system acts as a quantized phase battery which can supply anomalous current even at zero phase difference. The main advantage of our system is that our system can store quantized amounts of phase \( \varphi_0 \) in the ground state of the junction. The reason we are interested in \( \varphi_0 \) Josephson junction is because of the manifold applications like phase-defined quantum bits\(^24\), superconducting computer memory components\(^25\), also superconducting phase battery\(^26\) and rectifiers\(^27\).

Our manuscript is organized as follows: in section II we present the model Hamiltonian, wave-functions and boundary conditions necessary to calculate the Josephson current and anomalous phase. In section III we discuss our results for the Andreev bound states and anomalous Josephson currents (subsection III A), symmetries broken in our system when anomalous current flows through the junction (subsection III B) and anomalous phase and asymmetry of the critical current (subsection III C). Finally, we conclude with an experimental realization and summary of our work.
II. THEORY

A. Hamiltonian

We consider a system which consists of two Ferromagnet’s ($F_1$ and $F_2$) with a spin-flipper, embedded between two s-wave singlet superconductors. Our set-up is depicted in Fig. 1, it shows a spin-flipper at $x=0$ and two superconductors- one to left $x<-a/2$ and another at right $x>a/2$. There are two Ferromagnet’s in between at $-a/2<x<0$ and $0<x<a/2$. The magnetization vectors of the two Ferromagnet’s make an angle $\theta$ with each other. We take the superconducting gap of the form $\Delta = \Delta_0 e^{i\phi_R} \Theta(-x-a/2) + e^{i\phi_L} \Theta(x-a/2)$, where $\Delta_0$ is temperature dependent and it follows that $\Delta_0 \to \Delta_0 \tanh(1.74\sqrt{(T_c/T-1)})$, where $T_c$ is the superconducting critical temperature\textsuperscript{20}, $\varphi_L$ and $\varphi_R$ being superconducting phases for left and right superconductors respectively.

![Josephson junction with two Ferromagnetic's and a spin-flipper](image)

**FIG. 1:** Josephson junction with two Ferromagnet’s and a spin-flipper ($S$, magnetic moment $m'$) at $x=0$ sandwiched between two s-wave superconductors.

The Hamiltonian in Bogoliubov-de Gennes formalism for our system is-

\[
\begin{pmatrix}
H & i\Delta \sigma_y \\
-i\Delta^* \sigma_y & -H^* I
\end{pmatrix}
\psi(x) = E \psi(x),
\]

with $H = p^2/2m^* + V[\delta(x+a/2) + \delta(x-a/2)] - J_0 \delta(x) \vec{S} \cdot \vec{\hat{S}} - \vec{h} \cdot \sigma \Theta(x+a/2) + \Theta(a/2-x)] - E_F$, with $p^2/2m^*$ being the kinetic energy of electron with mass $m^*$, $V$ denotes the strength of $\delta$ potentials at the interfaces between Ferromagnet’s and Superconductor, $J_0$ denotes strength of exchange coupling between electron/hole with spin $\vec{s}$ and spin-flipper\textsuperscript{31} with spin $\vec{S}$. $\psi(x)$ defines a four-component spinor, while $E_F$ is the Fermi energy, $\sigma$’s are Pauli spin matrices and $\hat{I}$ is $2 \times 2$ identity matrix. The magnetization vector ($\vec{h}$) of left ferromagnetic layer ($F_1$) is at an angle $\theta$ with $z$ axis in the $y-z$ plane, while that of right ferromagnetic layer ($F_2$) is fixed along the $z$ axis. Thus, $\vec{h} \cdot \sigma = h \sin \theta \sigma_y + h \cos \theta \sigma_z$\textsuperscript{32}.

B. Wave-functions

If a spin up electron is incident at the $x=-a/2$ interface from left superconductor. Then wave function in the left superconductor ($S_L$) is given by\textsuperscript{31}

\[
\psi_{S_L}(x) = \begin{pmatrix}
u \\ 0 \\ 0 \\ \nu \end{pmatrix} e^{ik_x x} \phi_{m'}^{S} + v_{ee}^\dagger \begin{pmatrix} u \\ 0 \\ 0 \\ v \end{pmatrix} e^{-ik_x x} \phi_{m'}^{S} + r_{ee}^\dagger \begin{pmatrix} 0 \\ u \\ -v \\ 0 \end{pmatrix} e^{-ik_x x} \phi_{m'+1}^{S} + r_{eh}^\dagger \begin{pmatrix} 0 \\ v \\ 0 \\ u \end{pmatrix} e^{ik_x x} \phi_{m'+1}^{S},
\]

(2)
where \( r_{ee}^{\uparrow\uparrow}, r_{ee}^{\uparrow\downarrow}, t_{eh}^{\uparrow\uparrow}, r_{eh}^{\uparrow\downarrow} \) are the amplitudes for normal reflection without flip, normal reflection with spin flip, Andreev reflection with spin flip and Andreev reflection without flip respectively. The corresponding wave function in the right superconductor (\( S_R \)) is given by:

\[
\psi_{SR}(x) = t_{ee}^{\uparrow\uparrow} \begin{pmatrix} u e^{i\varphi} \\ v \\ \psi \end{pmatrix} e^{ik_{x\uparrow}x} \phi_{m'}^S + t_{ee}^{\uparrow\downarrow} \begin{pmatrix} 0 \\ -ve^{i\varphi} \\ 0 \end{pmatrix} e^{-ik_{x\downarrow}x} \phi_{m'+1}^{S'} + t_{eh}^{\uparrow\uparrow} \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} e^{-ik_{x\uparrow}x} \phi_{m'}^S + t_{eh}^{\uparrow\downarrow} \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} e^{-ik_{x\downarrow}x} \phi_{m'+1}^S,
\]

where \( t_{ee}^{\uparrow\uparrow}, t_{ee}^{\uparrow\downarrow}, t_{eh}^{\uparrow\uparrow}, t_{eh}^{\uparrow\downarrow} \) are the transmission amplitudes, corresponding to the reflection process described above. \( \varphi = \varphi_R - \varphi_L \) represents the phase difference between right and left superconductors. \( \phi_{m'}^S \) is the spinor of the spin-flipper, with its \( S^z \) operator acting as \( S^z \phi_{m'}^S = m' \phi_{m'}^S \), with \( m' \) being the spin magnetic moment of the spin flipper.

\[
u = \sqrt{\frac{1}{2} \left( 1 + \sqrt{\frac{E^2 - \Delta^2}{E}} \right)} \quad \text{and} \quad v = \sqrt{\frac{1}{2} \left( 1 - \sqrt{\frac{E^2 - \Delta^2}{E}} \right)}
\]

are the BCS coherence factors. \( k_\pm = \sqrt{\frac{2m^*}{\hbar^2}(E_F \pm \sqrt{E^2 - \Delta^2})} \) is the wave-vector for electron-like quasi-particle (\( k_+ \)) and hole-like quasi-particle (\( k_- \)) in the left and right superconducting wave-functions, \( \psi_{SL} \) and \( \psi_{SR} \) respectively.

The wave-function in the left Ferromagnet (\( F_1 \)) is given by-

\[
\psi_{F_1}(x) = (ee^{iq_1^+}x + fe^{-iq_1^+}x) \begin{pmatrix} \cos \frac{\theta}{2} \\ 0 \\ 0 \end{pmatrix} \phi_{m'}^S + (e'e^{iq_1^+}x + f'e^{-iq_1^+}x) \begin{pmatrix} 0 \\ \cos \frac{\theta}{2} \\ 0 \end{pmatrix} \phi_{m'+1}^S
\]

\[
+ (e'0e^{-iq_1^+}x + f'0e^{iq_1^+}x) \begin{pmatrix} 0 \\ 0 \\ -i \sin \frac{\theta}{2} \end{pmatrix} \phi_{m'}^{S'} + (e0e^{-iq_1^+}x + f0e^{iq_1^+}x) \begin{pmatrix} 0 \\ 0 \\ i \sin \frac{\theta}{2} \end{pmatrix} \phi_{m'+1}^{S'}.
\]

Similarly the wave-function in the right Ferromagnet (\( F_2 \)) is given by-

\[
\psi_{F_2}(x) = (a_0e^{iq_1^+}x + b_0e^{-iq_1^+}x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \phi_{m'}^S + (a'e^{iq_1^+}x + b'e^{-iq_1^+}x) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \phi_{m'+1}^S
\]

\[
+ (ae^{-iq_1^+}x + be^{iq_1^+}x) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \phi_{m'}^{S'} + (a'e^{-iq_1^+}x + b'e^{iq_1^+}x) \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \phi_{m'+1}^{S'},
\]

and \( q_\sigma^\pm = \sqrt{\frac{2m^*}{\hbar^2}(E_F + \rho_\sigma h \pm E)} \) is the wave-vector for electron (\( q_\sigma^+ \)) and hole (\( q_\sigma^- \)) in the Ferromagnet, with \( \rho_\sigma = +1(-1) \) when- \( \sigma = \uparrow(\downarrow) \). In our work we have used the Andreev approximation: \( k_+ = k_- = \sqrt{\frac{2m^*E_F}{\hbar^2}} = k_F \) and \( q_{\uparrow\downarrow} = k_F \sqrt{1 \pm \frac{1}{E_F}} \), where \( k_F \) is the Fermi wave-vector, with \( E_F >> \Delta, E \).

C. Boundary conditions

The boundary conditions at \( x = -a/2 \) is- \( \psi_{SL}(x) = \psi_{F_1}(x) \) (continuity of wave-functions) and, \( \frac{d\psi_{F_1}}{dx} - \frac{d\psi_{SL}}{dx} = 2m^*V_{\psi_{F_1}} \) (discontinuity in first derivative), and at \( x = 0 \) similarly is- \( \psi_{F_1}(x) = \psi_{F_2}(x) \) and \( \frac{d\psi_{F_2}}{dx} - \frac{d\psi_{F_1}}{dx} = -2m^*H_aS\hat{S}\psi_{F_1} \) where \( \hat{S}S = s^X \hat{S} + \frac{1}{2}(s^-S^+ + s^+S^-) \), \( s^X = s^x \pm is^y \) and \( S^X = S^X \pm is^y \), with \( s^k = \frac{h}{2}\sigma_k, k = x, y, z \) are the exchange operator due to spin flipper in the Hamiltonian, the spin raising and lowering operators for electron/hole and spin flipper respectively with-

\[
\hat{S}S \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \phi_{m'}^S = \begin{pmatrix} m'/2 \\ 0 \\ 0 \end{pmatrix} \phi_{m'}^S + \begin{pmatrix} f_2/2 \\ 0 \\ 0 \end{pmatrix} \phi_{m'+1}^S.
\]
\[
\begin{align*}
\mathbf{s}.\mathbf{S} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \phi_{m'+1} &= -\frac{(m'+1)}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \phi_{m'+1} + \frac{f_2}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \phi_{m'}. 
\end{align*}
\]

Here, \( f_2 = \sqrt{(S - m')(S + m' + 1)} \) is the spin-flip probability\[33\] for spin flipper.

On the other hand when there is no spin-flip scattering, i.e., if \( S = m' \), then spin flip probability of spin flipper: \( f_2 = \sqrt{(S - m')(S + m' + 1)} = 0 \).

Thus, for no-flip process: \( \mathbf{s}.\mathbf{S} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \phi_{m'} = \frac{m'}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \phi_{m'} \).

The Hamiltonian therefore for no-flip process then is- \( H_0 = p^2/2m^* + V[\delta(x + a/2) + \delta(x - a/2)] - J_0 \delta(x)\mathbf{s}.\mathbf{S} - \hbar \sigma \left[ \Theta(x + a/2) + \Theta(a/2 - x) \right] - E_F \), while flip Hamiltonian for spin flip process is- \( H_0 = p^2/2m^* + V[\delta(x + a/2) + \delta(x - a/2)] - J_0 \delta(x)\mathbf{s}.\mathbf{S} - \hbar \sigma \left[ \Theta(x + a/2) + \Theta(a/2 - x) \right] - E_F \).

Finally, at \( x = a/2 \), the boundary conditions are- \( \psi_{F_2}(x) = \psi_{S_R}(x) \), \( \frac{d\psi_{S_R}}{dx} - \frac{d\psi_{F_2}}{dx} = \frac{2m^*V}{\hbar k_F} \psi_{F_2} \). We use the dimensionless parameters \( J = \frac{m^*J}{\hbar k_F} \) as a measure of strength of exchange interaction and \( Z = \frac{m^*V}{\hbar k_F} \) as a measure of interface transparency\[33\].

### D. Josephson current

To calculate bound state contribution to Josephson current we follow the procedure established in Ref. [30]. We neglect the contribution from incoming quasi-particle, i.e., first term \( (u \ 0 \ 0 \ v)^T e^{i k_x x} \phi_{m'} \) of Eq. (2) and insert the wave-functions into boundary conditions defined in section II. C., we get a homogeneous system of linear equations for the scattering amplitudes,

\( R \mathbf{x} = 0 \),

where \( \mathbf{x} \) is a \( 8 \times 1 \) column vector, given by \( \mathbf{x} = \left( r_{e c}^{\dagger} r_{e c}^{\dagger} r_{e h}^{\dagger} r_{e h}^{\dagger} r_{e c}^{\dagger} r_{e c}^{\dagger} r_{e h}^{\dagger} r_{e h}^{\dagger} \right)^T \), \( R \) is a \( 8 \times 8 \) matrix obtained by eliminating the scattering amplitudes for the two Ferromagnet’s via the scattering amplitudes of the left and right superconductor. For a nontrivial solution of this system, the determinant of \( R = 0 \), Thus, we get the Andreev bound state energy spectrum \( E_i \), \( i = \{1, ..., 8\}[33] \). We find that the Andreev energy bound states \( E_i(i = 1, ..., 8) \) can be written as \( \xi_i^T = \pm \epsilon_l(l = 1, ..., 4) \). From Andreev bound state energies we get Free energy of our system, given as\[33\]

\( F = -\frac{1}{\beta} \ln \left[ \prod_l (1 + e^{-\beta E_l}) \right] = -\frac{2}{\beta} \sum_{l=1}^{4} \ln \left[ 2 \cosh \left( \frac{\beta \xi_l}{2} \right) \right] \)

We consider only the short junction limit, such that the total Josephson current can be determined by considering the bound state contribution only. The Josephson current at finite temperature is defined as the derivative of the Free energy \( F \) of our system with respect to the phase difference \( \varphi \) between left and right superconductor\[33\],

\( I = \frac{2e}{\hbar} \frac{\partial F}{\partial \varphi} = -\frac{2e}{\hbar} \sum_{l=1}^{4} \tanh \left( \frac{\beta \xi_l}{2} \right) \frac{\partial \xi_l}{\partial \varphi}, \)

wherein \( e \) is the charge of electron. Eq. [11] is the main working formula of our paper. Using Eq. [11] we can calculate the anomalous Josephson current, which is given as- \( I_{an} = I(\varphi = 0) \). In case interfaces are completely transparent, i.e., \( Z = 0 \), we have-

\( I_{an} = \frac{2e \Delta_0}{\hbar} \left( \tanh \left( \frac{\beta A_1}{2} \right) A'_1 + \tanh \left( \frac{\beta A_2}{2} \right) A'_2 + \tanh \left( \frac{\beta A_3}{2} \right) A'_3 + \tanh \left( \frac{\beta A_4}{2} \right) A'_4 \right) \)

where \( A_1, A_2, A_3, A_4, A'_1, A'_2, A'_3 \) and \( A'_4 \) are large expressions that depend on exchange interaction \( (J) \), magnetization of the Ferromagnet’s, spin \( (S) \) and magnetic moment \( (m') \) of spin-flipper, phase \( (k_F a) \) accumulated in
Ferromagnet’s and spin-flip probability of spin-flipper ($f_2$). The explicit forms for $A_k$’s and $A_k'$’s are given in Appendix. In Appendix we show that for no flip ($f_2 = 0$) or absence of spin-flipper ($J = 0$) or $\theta = 0$ (magnetizations of Ferromagnet’s are aligned), anomalous Josephson current vanishes.

For $\varphi_0$ Josephson junction the ground state of the junction is at $\varphi = \varphi_0$ ($\neq 0$ or $\pi$). At $\varphi = \varphi_0$, the free energy of the junction is a minimum. By determining this minimum value of the free energy, one can calculate the anomalous phase $\varphi_0$ numerically.

III. RESULTS

A. Anomalous Josephson current and Andreev bound states

In this subsection, we first show the results for Andreev bound states in our system as well as the Josephson currents. In Fig. 2 we plot the Andreev bound states and Josephson current as a function of phase difference $\varphi$ between two superconductors for both no flip and spin flip case. In Fig. 2(a) we deal with the no flip case, i.e., $f_2 = 0$, and for no flip ($S = m'$) the Josephson current satisfies $I(-\varphi) = -I(\varphi)$ and there is no current flowing through the junction. Thus, the absolute value of maximum Josephson current, $I_{c+}$ is identical with the absolute value of minimum Josephson current, $I_{c-}$. Further, for no flip case Andreev bound states are invariant against the inversion of mis-orientation angle $\theta$ and magnetization of Ferromagnet’s $h$; $\varepsilon_l(\theta, h) = \varepsilon_l(-\theta, -h)$, exchange interaction of spin-flipper $J$; $\varepsilon_l(J) = \varepsilon_l(-J)$, and phase accumulated in Ferromagnet’s $k_F a$; $\varepsilon_l(k_F a) = \varepsilon_l(-k_F a)$. In Fig. 2(b) we deal with the spin-flip case, i.e., $f_2 \neq 0$, in this case $S \neq m'$ ($S = 3/2$, $m' = -1/2$), and phase accumulated in Ferromagnet’s $k_F a$; $\varepsilon_l(k_F a) = \varepsilon_l(-k_F a)$. In Fig. 2(c) for no flip the Josephson current satisfies $I(-\varphi) = -I(\varphi)$ and there is no current flowing through the junction when the phase difference $\varphi$ between two superconductors is zero. Further, for spin-flip case Andreev bound states are invariant with respect to inversion of phase difference $\varphi$, $\varepsilon_l(-\varphi) = \varepsilon_l(\varphi)$. As a result, in Fig. 2(c) for no flip the Josephson current satisfies $I(-\varphi) = -I(\varphi)$ and there is no current flowing through the junction when the phase difference $\varphi$ between two superconductors is zero. Thus, the absolute value of maximum Josephson current, $I_{c+}$ is identical with the absolute value of minimum Josephson current, $I_{c-}$. Further, for no flip case Andreev bound states are invariant against the inversion of mis-orientation angle $\theta$ and magnetization of Ferromagnet’s $h$; $\varepsilon_l(\theta, h) = \varepsilon_l(-\theta, -h)$, exchange interaction of spin-flipper $J$; $\varepsilon_l(J) = \varepsilon_l(-J)$, and phase accumulated in Ferromagnet’s $k_F a$; $\varepsilon_l(k_F a) = \varepsilon_l(-k_F a)$. In Fig. 2(b) we deal with the spin-flip case, i.e., $f_2 \neq 0$, in this case $S \neq m'$ ($S = 3/2$, $m' = -1/2$) for spin-flipper. Thus, there is finite possibility for spin-flipper to flip its own spin while interacting with an electron/hole. However, there is a finite probability of flipping for the spin of either electron or hole due to the misalignment in the magnetization of the Ferromagnet’s. We see that there are four positive and four negative Andreev levels. In systems where time reversal symmetry is not broken Andreev bound states are not invariant with respect to inversion of phase difference $\varphi$, $\varepsilon_l(-\varphi) \neq \varepsilon_l(\varphi)$. As a result, in Fig. 2(c) for spin-flip the Josephson current is the same as the Andreev levels are doubly degenerate and the $\varphi$ inversion symmetry of $\varepsilon_l(-\varphi) = \varepsilon_l(\varphi)$ is broken. As a result, Josephson current obeys $I(-\varphi) = -I(\varphi)$ for spin-flip process in Fig. 2(c), where not only the anomalous current $I(\varphi = 0) \neq 0$ is obtained, but also a difference between the absolute value of maximum and absolute value of minimum Josephson currents, $I_{c+} \neq I_{c-}$ is seen. Further, for spin-flip case ($\theta, h, J$ and $k_F a$) for spin-flipper, the Andreev bound states are all broken. In Ref. [12] Andreev bound states are calculated numerically for a superconductor (S)-semiconductor nanowire (NW)-superconductor (S) junction in presence of both spin-orbit (SO) interaction and Zeeman field. In the absence of spin-orbit (SO) interaction, they see that Andreev bound states are invariant with respect to the inversion of phase difference $\varphi$ between two superconductors and there is no anomalous Josephson current, which is equivalent to what we see for no flip process for our system. In presence of both spin-orbit interaction (SO) and Zeeman field, they observe $E(-\varphi) \neq E(\varphi)$ for Andreev bound states and anomalous Josephson current flows at $\varphi = 0$, which is equivalent to what we see for spin-flip process.

Figure 3 shows the dependence of the anomalous Josephson current on the angle $\theta$ between the magnetic moments of the Ferromagnet’s. In Fig. 3(a) we plot anomalous Josephson current as a function of the mis-orientation angle.
(θ) between two ferromagnetic layers for different values of spin-flip probabilities of spin flipper. We see that for no flip, i.e., \( f_2 = 0 \), anomalous current vanishes (no current flows through the junction at \( \varphi = 0 \)). The reason can be understood from Fig. 2(a). Since for no flip process phase inversion symmetry of Andreev levels \((\varepsilon_l(-\varphi) = \varepsilon_l(\varphi))\) is not broken, thus Josephson current satisfies \( I(-\varphi) = -I(\varphi) \) and therefore anomalous current is zero. The magnitude of anomalous current decreases with increasing spin-flip probability. So even though finite flip probability is necessary to see anomalous current in the first place, large flip probability leads to decreasing anomalous current. Further, one can see that the sign of anomalous Josephson current can be tuned via the mis-orientation angle \( \theta \) between two Ferromagnet’s. The anomalous current is periodic as function of mis-orientation angle with period \( 2\pi \). In Fig. 3(b) we plot the absolute value of the anomalous Josephson currents as function of the mis-orientation angle \( \theta \) for some values as in Fig. 3(a). We see that the anomalous current regardless of flip probability is symmetric about \( y \)-axis but asymmetric with respect to \( z \)-axis. From Fig. 3 we also see that when the magnetic moments of the Ferromagnet’s are aligned parallel or anti-parallel \((\theta = 0 \text{ or } \theta = \pi)\), anomalous Josephson current vanishes even when spin flipper flips its spin. Thus, both spin flipper and magnetization orientation of the Ferromagnet’s play an important role in generating anomalous Josephson effect in our system.

Next in Fig. 4 we show the effects of exchange interaction \( J \) of spin-flipper, magnetization of Ferromagnet’s \( h \) and interface transparency \( Z \) on the anomalous Josephson current. In Fig. 4(a) we plot anomalous Josephson current as a function of exchange interaction \( J \) of spin flipper for different values of spin-flip probabilities of spin-flipper. We see that for ferromagnetic coupling \((J > 0)\) there is no sign change of anomalous Josephson current with change in \( J \). However, for anti-ferromagnetic coupling \((J < 0)\) there is a sign change in \( I_{an} \) as \( J \) changes from \( J = 0 \) to \( J = -4 \), implying tun-ability of the sign of anomalous Josephson current via the exchange interaction of spin flipper. We also see that anomalous Josephson current is asymmetric with respect to \( J \) and first increases and then decreases with increasing \( J \) for ferromagnetic coupling. Further, the maximum value of \( I_{an} \) decreases with increase of spin flip probability of spin flipper. In Fig. 4(b) we plot anomalous Josephson current as a function of magnetization \( h \) of the ferromagnetic layers. We see that anomalous Josephson current first increases and attains a maximum value and then decreases with increasing \( h \). In contrast to Fig. 4(a), anomalous Josephson current is symmetric with respect to magnetization \( h \) of the Ferromagnet’s. In Fig. 4(c) we plot \( I_{an} \) as a function of interface barrier strength \( Z \). We see that there is no sign change of \( I_{an} \) with increase of interface barrier strength \( Z \). Further, anomalous Josephson current decreases with increase of \( Z \) and is almost zero in the tunneling regime. It is also evident from Fig. 4(b) and Fig. 4(c) maximum value of \( I_{an} \) first increases and then decreases with increase of spin flip probability of spin flipper.

There are some important differences between our work and other works wherein anomalous Josephson current is seen with ferromagnetic Josephson junction. In our work anomalous Josephson effect arises due to the mutual spin interaction.
flipping of spin polarized Josephson current and spin flipper. On the other hand, in Ref. [3] anomalous Josephson effect arises due to the spin precession of electrons and holes in the ferromagnetic layers. In Ref. [4], anomalous Josephson current is zero in the transparent regime \( Z = 0 \) but in the tunneling limit anomalous Josephson current is finite. But, in our work we have seen opposite behavior, anomalous Josephson current is finite in the transparent regime, but in the tunneling limit anomalous Josephson current vanishes.

### B. Reasons for existence of Anomalous Josephson effect

1. **Explaining quantum spin flip scattering**

The extremely important role played by spin flipper entails a detailed analysis of this process. The Josephson current flowing through either ferromagnetic layer \( (F_1 \text{ or } F_2) \) is spin-polarized in the direction of magnetization of that ferromagnetic layer. Subsequently when this spin polarized Josephson current state, denoted by a macroscopic wave-function \( \sim |\Psi_{s.c.}\rangle e^{i\varphi_P} \approx (u_{00} v)^T e^{i\varphi_P} \) (where \( P = L \text{ or } R \)), interacts with the spin flipper, there is finite probability for mutual spin-flip. One should observe that this is a probability not a certainty, since the interaction of spin polarized Josephson current with spin-flipper is quantum in nature, see Eq. 7. Thus, the combined spin polarized Josephson current and spin-flipper state after interaction is in a superposition of mutual spin-flip as well as no flip state given by the joint entangled wave-function of spin-polarized Josephson current and spin-flipper as:

\[
|s.c\rangle \otimes |\phi_{s.c.}^{m'}\rangle = \sqrt{\frac{f_2}{2}} |\text{Mutual-Flip}\rangle + \sqrt{\frac{m'}{2}} |\text{No flip}\rangle
\]  

(13)

where Josephson current state \( |s.c\rangle \) is spin polarized. Quantum spin flip scattering plays an integral role in observing the anomalous Josephson effect as we discuss later. In absence of spin flip scattering probability \( (f_2 = 0) \) the anomalous current vanishes.

2. **Explaining chiral symmetry breaking**

All standard Josephson junctions have a certain symmetry, called chiral symmetry. Due to this symmetry, at \( \varphi = 0 \) one can not distinguish between electron tunneling from left to right superconductor and vice-versa. Thus, the tunneling coefficient which describes electron tunneling from left to right superconductor equals the tunneling coefficient which describes electron tunneling from right to left superconductor when there is no phase difference between two superconductors \( (\varphi = 0) \). Thus for our system, as in Fig. 1, when Ferromagnet’s are aligned \( (\theta = 0) \), \( t_{ee}^{\uparrow \downarrow}(\varphi = 0) \mid_{L \rightarrow R} = t_{ee}^{\downarrow \uparrow}(\varphi = 0) \mid_{L \rightarrow R} \), where \( t_{ee}^{\uparrow \downarrow} \mid_{L \rightarrow R} \) and \( t_{ee}^{\downarrow \uparrow} \mid_{L \rightarrow R} \) are the transmission amplitudes for electron tunneling from left to right superconductor and vice-versa. This implies chiral symmetry is not broken and as a result, \( I(\varphi = 0) \) is strictly zero. But, when Ferromagnet’s are misaligned \( (\theta \neq 0) \), \( t_{ee}^{\uparrow \downarrow}(\varphi = 0) \mid_{L \rightarrow R} \neq t_{ee}^{\downarrow \uparrow}(\varphi = 0) \mid_{L \rightarrow R} \),

FIG. 4: (a) The anomalous Josephson current as a function of exchange interaction \( J \) of spin flipper. Parameters are \( \Delta_0 = 1 \text{meV}, I_0 = e\Delta_0/h, T/T_c = 0.01, h = 0.5E_F, k_Fa = \pi, \varphi = 0, \theta = \pi/2, Z = 0 \). (b) The anomalous Josephson current as a function of magnetization \( (h) \) of the Ferromagnet’s. Parameters are \( \Delta_0 = 1 \text{meV}, I_0 = e\Delta_0/h, T/T_c = 0.01, J = 1, \varphi = 0, k_Fa = \pi, \theta = \pi/2, Z = 0 \). (c) The anomalous Josephson current as a function of the interface barrier strength \( (Z) \). Parameters are \( \Delta_0 = 1 \text{meV}, I_0 = e\Delta_0/h, k_Fa = \pi, T/T_c = 0.01, J = 1, \varphi = 0, \theta = \pi/2, h = 0.8E_F \).
i.e., chiral symmetry is broken. This results in \( I(\varphi = 0) \neq 0 \), i.e., anomalous Josephson current flows across the junction.

3. Explaining time reversal symmetry breaking

The symmetry in BdG Hamiltonian (Eq. 1) is explained here. The Hamiltonian matrix of Eq. 1 is denoted as \( H_{BdG}(\varphi) \), such that \( H_{BdG}\psi(x) = E\psi(x) \). When spin flip probability \( f_2 = 0 \) or Ferromagnet’s are aligned \( \theta = 0 \), \( H_{BdG}(\varphi) \) obeys time reversal symmetry \( (T) \), thus \( TH_{BdG}(\varphi)T^\dagger = H_{BdG}(-\varphi) \), which implies that if \( H_{BdG}(\varphi) \) has an energy eigenvalue \( \varepsilon_i(\varphi) \), then \( H_{BdG}(-\varphi) \) must have the same energy eigenvalue. Then the Andreev bound states satisfy the relation \( \varepsilon_i(\varphi) = \varepsilon_i(-\varphi) \), as a result Josephson current follows \( I(\varphi) = -I(-\varphi) \) and there is no anomalous Josephson effect. In presence of spin flip scattering \( (f_2 \neq 0) \) and also when Ferromagnet’s are misaligned \( (\theta \neq 0) \), time reversal symmetry is broken, as a result, \( \varepsilon_i(\varphi) \neq \varepsilon_i(-\varphi) \), i.e., the Andreev bound states symmetry is also broken and thus Josephson current obeys \( I(-\varphi) \neq -I(\varphi) \), which implies \( I(\varphi = 0) \neq 0 \). Thus, when both \( f_2 \neq 0 \) and \( \theta \neq 0 \), i.e., both time reversal symmetry and chiral symmetry are broken, an anomalous Josephson current flows across the junction. In contrast, when \( f_2 = 0 \) and \( \theta = 0 \), i.e., only chiral symmetry is broken, but time reversal symmetry is not broken then anomalous Josephson current vanishes.

C. Quantized anomalous phase

In the previous subsection we have shown the results of Andreev bound states and Anomalous Josephson current, in this subsection we discuss the results of anomalous phase \( \varphi_0 \) and asymmetry of the critical current \( \Delta I = (I_{c+} - I_{c-})/(I_{c+} + I_{c-}) \). In Fig. 5(a) we plot Free energy as a function of exchange interaction \( J \) and phase difference \( \varphi \). When \( J \neq 0 \), for each particular value of \( J \), the minimum Free energy is at \( \varphi = \varphi_0(\neq 0, \pm \pi) \) and a \( \varphi_0 \)-Josephson junction is realized. Thus, for each particular value of \( J \), we get the anomalous phase \( \varphi_0 \) numerically, where the Free energy of the junction becomes minimum. In Fig. 5(b) we plot Free energy as a function of magnetization \( h \) and phase difference \( \varphi \). When \( h \neq 0 \), for each particular value of \( h \), the minimum Free energy is at \( \varphi = \varphi_0(\neq 0, \pm \pi) \). Thus, again for each value of \( h \), we get an anomalous phase \( \varphi_0 \) numerically, where Free energy of the junction is minimum. This procedure of calculating \( \varphi_0 \) is well known and is also done in Refs. [12, 47].

In Fig. 6 we study anomalous phase \( \varphi_0 \) as a function of exchange interaction \( J \) of spin flipper and magnetization \( h \) of the Ferromagnet’s. In Fig. 6(a), \( \varphi_0 \) is plotted as a function of \( J \) for both anti-ferromagnetic coupling \( (J < 0) \) as well as ferromagnetic coupling \( (J > 0) \). In Fig. 6(a) we see the “quantized” steps in the anomalous phase \( \varphi_0 \) which are
of exactly same magnitude (0.0314 radians) although the width linearly decreases as one goes from anti-ferromagnetic to ferromagnetic coupling. In our work anomalous current is always accompanied by quantized anomalous phase. We never see anomalous current with non quantized anomalous phase. In Fig. 6(b) anomalous phase $\varphi_0$ is shown as a function of the normalized magnetization $h/E_F$ of the Ferromagnet’s. Similar to Fig. 6(a), we also see the quantized steps in anomalous phase $\varphi_0$ which are again exactly of same magnitude (0.0314 radians) although the width initially decreases and then increases as one changes $h/E_F$ from -0.99 to 0 and then from 0 to 0.99. The quantized step magnitude or height remains same for different values of spin, magnetic moment and different spin flip probability of spin flipper. Quantized behavior of $\varphi_0$ is also shown in Fig. 6(c), where we show density plot of $\varphi_0$ as a function of $J$ and $h$. It is also evident from Fig. 6(c) larger values of the exchange interaction and the magnetization correspond to larger magnitudes of the anomalous phase $\varphi_0$. In Ref. [12] anomalous phase $\varphi_0$ changes continuously with change in magnetic field and is not quantized in contrast to our case.

Finally, in Fig. 7 we perform a similar analysis for the asymmetry of the critical current, defined as $\eta \equiv N = (I_{c+} - I_{c-})/(I_{c+} + I_{c-})$. In Fig. 7(a) we plot $\eta$ as a function of the exchange interaction $J$ and see that the maximum values of $\eta$ almost remain same for different spin flip cases. Further, it is also evident from Fig. 7(a), the sign of $\eta$ can be tuned via $J$, and $\eta$ is asymmetric with respect to $J$. Figure 7(b) shows the asymmetry $\eta$ as a function of magnetization $h$ of the Ferromagnet’s. We see that in contrast to Fig. 7(a), the maximum values of $\eta$ are different for different spin flip cases. Further, $\eta$ is symmetric with respect to $h/E_F$. In Fig. 7(c) we show a density plot of asymmetry of the critical current ($\eta$) as a function of the exchange interaction $J$ of spin flipper and magnetization $h$ of the Ferromagnet’s. We see that larger values of the exchange interaction and the magnetization correspond to smaller values of $\eta$. Further, we find maximum values of $\eta \simeq 0.16$ with $\eta$ changing sign from anti-ferromagnetic to ferromagnetic and is not quantized in contrast to our case.
to ferromagnetic coupling. Asymmetry of the critical current is also calculated in Ref. [47] as a function of the spin orbit interaction and the magnetization. But, in contrast to our case, in the presence of larger values of the spin orbit interaction and the magnetization.

IV. EXPERIMENTAL REALIZATION AND CONCLUSIONS

The set-up envisaged in Fig. 1 can be realized in the lab. Superconductor-Ferromagnet-Ferromagnet-Superconductor (S-F-F-S) Josephson junctions have been designed experimentally for quite some time now [48]. Embedding a S-F-F-S junction with a magnetic adatom or spin-flipper at the interface between two ferromagnets shouldn’t be difficult, especially with an s-wave superconductor like Aluminum or Lead it should be perfectly possible. In Ref. [49] local electronic properties of the surface of a superconductor are studied experimentally in the vicinity of a magnetic adatom with a scanning tunneling microscope (STM). Further, in Ref. [50] iron (Fe) chains are doped on the superconducting Pb surface and the subgap spectra is examined using scanning tunneling microscope. Further, in one of our previous works [51] we have shown how presence of magnetic impurity at Metal-Superconductor interface can give rise to almost quantized zero bias YSR conductance peaks, which are non-topological in origin.

To conclude, we have studied anomalous Josephson effect and the direction dependent critical current in S-F₁-spin flipper-F₂-S junction where F₁, F₂ are the two ferromagnetic layers with misaligned magnetization. In absence of spin flip scattering, Andreev bound states are time reversal symmetric, i.e., \( \varepsilon_l(\phi) = \varepsilon_l(-\phi) \). As a result, Josephson current is sinusoidal with \( I(\phi) = -I(-\phi) \) and there is no anomalous Josephson supercurrent at \( \phi = 0 \). But in presence of spin flip scattering, anomalous Josephson effect is seen. Andreev bound states satisfy the relation: \( \varepsilon_l(\phi) \neq \varepsilon_l(-\phi) \) when spin flipper flips it’s spin. As a result, the Josephson current breaks phase inversion symmetry \( I(\phi) \neq -I(-\phi) \), and an anomalous Josephson current flows at \( \phi = 0 \). Further, our system acts as a phase battery which can store quantized amounts of anomalous phase \( \phi_0 \) in the ground state of the junction.

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APPENDIX: EXPLICIT FORM OF ANOMALOUS JOSEPHSON CURRENT

The explicit form of \( A_1, A_2, A_3, A_4, A'_1, A'_2, A'_3, A'_4 \) in Eq. [12] is

\[
A_{1(2)} = \Delta_0 \sqrt{K - \frac{1}{2} \sqrt{L + M} \pm \frac{1}{2} \sqrt{2L - M - \frac{2N}{\sqrt{L + M}}}}
\]

\[
A'_{1(2)} = -\frac{1}{2A_{1(2)}} \left( -K' - \frac{L' + M'}{4\sqrt{L + M}} \pm \frac{2L' - M' + \frac{N(L' + M')}{(L + M)^{3/2}} - \frac{2N'}{\sqrt{L + M}}}{4\sqrt{2L - M - \frac{2N}{\sqrt{L + M}}}} \right)
\]

\[
A_{3(4)} = \Delta_0 \sqrt{K + \frac{1}{2} \sqrt{L + M} \pm \frac{1}{2} \sqrt{2L - M + \frac{2N}{\sqrt{L + M}}}}
\]

\[
A'_{3(4)} = -\frac{1}{2A_{3(4)}} \left( -K' + \frac{L' + M'}{4\sqrt{L + M}} \pm \frac{2L' - M' - \frac{N(L' + M')}{(L + M)^{3/2}} + \frac{2N'}{\sqrt{L + M}}}{4\sqrt{2L - M + \frac{2N}{\sqrt{L + M}}}} \right)
\]
where

\[
L = 4T_1^2 - \frac{2}{3}T_2,
\]
\[
M = \frac{2^{1/3}X_1}{3(X_2 + \sqrt{X_2^2 - 4X_1^3})} - \frac{(X_2 + \sqrt{X_2^2 - 4X_1^3})^{1/3}}{2^{1/3}},
\]
\[
N = 8T_1^3 - 2T_1T_2 + T_3,
\]
\[
K = T_1,
\]
\[
L' = -8T_1U_1 - \frac{2}{3}U_2,
\]
\[
M' = \frac{2^{1/3}X_1}{3(X_2 + \sqrt{X_2^2 - 4X_1^3})^{1/3}} - \frac{2^{1/3}X_1Y}{9(X_2 + \sqrt{X_2^2 - 4X_1^3})^{4/3}},
\]
\[
N' = -192T_1^2U_1 + 16T_2U_1 - 16T_1U_2 + 8U_3,
\]
\[
K' = U_1,
\]

where

\[
X_1 = T_2^2 - 12T_1T_3 - 12T_4,
\]
\[
X_2 = 2T_2^2 - 36T_1T_2T_3 - 432T_2T_4 + 27T_3^2 + 72T_2T_4,
\]
\[
X_1' = 2T_2U_2 + 12T_3U_1 - 12T_1U_3,
\]
\[
Y = Y' + \frac{X_2Y' - 6X_1^2X_1'}{\sqrt{X_2^2 - 4X_1^3}},
\]
\[
Y' = 6T_2^2U_2 - 36T_1T_2U_2 + 36T_3T_2U_1 + 864T_1T_2U_1 + 54T_3U_4 + 72T_2U_4,
\]
\[
T_1 = \left( P_1(S, m', f_2, h, J, kfa) + P_2(S, m', f_2, h, J, kfa) \cos(\theta) + P_3(S, m', f_2, h, J, kfa) \cos(2\theta) + P_4(S, m', f_2, h, J, kfa) \cos(4\theta) \right) \left( Q_1(S, m', f_2, h, J, kfa) + Q_2(S, m', f_2, h, J, kfa) \cos(\theta) + Q_3(S, m', f_2, h, J, kfa) \cos(2\theta) + Q_4(S, m', f_2, h, J, kfa) \cos(3\theta) + Q_5(S, m', f_2, h, J, kfa) \cos(4\theta) \right),
\]
\[
T_2 = \left( P_6(S, m', f_2, h, J, kfa) + P_7(S, m', f_2, h, J, kfa) \cos(\theta) + P_8(S, m', f_2, h, J, kfa) \cos(2\theta) + P_9(S, m', f_2, h, J, kfa) \cos(3\theta) + P_{10}(S, m', f_2, h, J, kfa) \cos(4\theta) \right) \left( Q_1(S, m', f_2, h, J, kfa) + Q_2(S, m', f_2, h, J, kfa) \cos(\theta) + Q_3(S, m', f_2, h, J, kfa) \cos(2\theta) + Q_4(S, m', f_2, h, J, kfa) \cos(3\theta) + Q_5(S, m', f_2, h, J, kfa) \cos(4\theta) \right),
\]
\[
T_3 = \left( P_{11}(S, m', f_2, h, J, kfa) + P_{12}(S, m', f_2, h, J, kfa) \cos(\theta) + P_{13}(S, m', f_2, h, J, kfa) \cos(2\theta) + P_{14}(S, m', f_2, h, J, kfa) \cos(3\theta) + P_{15}(S, m', f_2, h, J, kfa) \cos(4\theta) \right) \left( Q_1(S, m', f_2, h, J, kfa) + Q_2(S, m', f_2, h, J, kfa) \cos(\theta) + Q_3(S, m', f_2, h, J, kfa) \cos(2\theta) + Q_4(S, m', f_2, h, J, kfa) \cos(3\theta) + Q_5(S, m', f_2, h, J, kfa) \cos(4\theta) \right),
\]
\[
T_4 = \left( P_{16}(S, m', f_2, h, J, kfa) + P_{17}(S, m', f_2, h, J, kfa) \cos(\theta) + P_{18}(S, m', f_2, h, J, kfa) \cos(2\theta) + P_{19}(S, m', f_2, h, J, kfa) \cos(3\theta) + P_{20}(S, m', f_2, h, J, kfa) \cos(4\theta) \right) \left( Q_1(S, m', f_2, h, J, kfa) + Q_2(S, m', f_2, h, J, kfa) \cos(\theta) + Q_3(S, m', f_2, h, J, kfa) \cos(2\theta) + Q_4(S, m', f_2, h, J, kfa) \cos(3\theta) + Q_5(S, m', f_2, h, J, kfa) \cos(4\theta) \right),
\]
\[
U_1 = \left( Jf_2 \sin(\theta) \left( P_{21}(S, m', f_2, h, J, kfa) + P_{22}(S, m', f_2, h, J, kfa) \cos(\theta) + P_{23}(S, m', f_2, h, J, kfa) \cos(2\theta) \right) \right) \left( Q_6(S, m', f_2, h, J, kfa) + Q_7(S, m', f_2, h, J, kfa) \cos(\theta) + Q_8(S, m', f_2, h, J, kfa) \cos(2\theta) + Q_9(S, m', f_2, h, J, kfa) \cos(3\theta) + Q_{10}(S, m', f_2, h, J, kfa) \cos(4\theta) \right),
\]
\[ U_2 = \left( J f_2 \sin(\theta) \left( P_{24}(S, m', f_2, h, J, k_F a) \cos(\theta) + P_{25}(S, m', f_2, h, J, k_F a) \cos(2\theta) \right) \right) / \\
\left( Q_1(S, m', f_2, h, J, k_F a) + Q_2(S, m', f_2, h, J, k_F a) \cos(\theta) + Q_3(S, m', f_2, h, J, k_F a) \cos(2\theta) + Q_4(S, m', f_2, h, J, k_F a) \cos(3\theta) + Q_5(S, m', f_2, h, J, k_F a) \cos(4\theta) \right), \]

\[ U_3 = \left( J f_2 \sin(\theta) \left( P_{27}(S, m', f_2, h, J, k_F a) + P_{28}(S, m', f_2, h, J, k_F a) \cos(\theta) + P_{29}(S, m', f_2, h, J, k_F a) \cos(2\theta) \right) \right) / \\
\left( Q_1(S, m', f_2, h, J, k_F a) + Q_2(S, m', f_2, h, J, k_F a) \cos(\theta) + Q_3(S, m', f_2, h, J, k_F a) \cos(2\theta) + Q_4(S, m', f_2, h, J, k_F a) \cos(3\theta) + Q_5(S, m', f_2, h, J, k_F a) \cos(4\theta) \right), \]

Here, \( P_i \) for \( i = 1, 2, \ldots, 29 \) and \( Q_i \) for \( i = 1, 2, \ldots, 10 \) are functions of all parameters like exchange interaction \( (J) \), magnetization of the Ferromagnets \( (h) \), spin \( (S) \) and magnetic moment \( (m') \) of spin flipper, phase \( (k_F a) \) accumulated in ferromagnetic region and spin flip probability of spin flipper \( (f_2) \). Since these are large expressions we do not explicitly write them here. From the above expressions for no flip \( (f_2 = 0) \) or absence of spin flipper \( (J = 0) \) or \( \theta = 0 \) (magnetizations of the Ferromagnets are aligned), \( U_1, U_2, U_3 \) and also \( L', M', N' \) and \( K' \) vanish. Thus, from Eq. 12 \( A_{1(2)} = 0 \) and \( A_{3(4)} = 0 \), implying for no flip case or absence of spin flipper or \( \theta = 0 \) anomalous Josephson current vanishes \( (I_{an} = 0) \).
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