One-dimensional photonic crystals with a sawtooth refractive index: another exactly solvable potential

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\textbf{Abstract.} Exact analytical results expressed in terms of Bessel functions for the bandgaps, reflectance and transmittance of one-dimensional photonic crystals with a sawtooth refractive index profile on the period are derived, to our knowledge for the first time. This extends the set of exactly solvable models with periodic refractive indices. Asymptotic approximations to the above exact results have been also obtained.

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1. Introduction

Recently there has been a renewal of interest in the properties of one-dimensional (1D) photonic crystals based on graded index slabs. Rauh et al [1] have considered in some detail optical properties of periodic systems consisting of slabs whose permittivities $n^2(z)$ increase linearly with depth $z$. They also considered periodic systems in which alternating layers have quadratically increasing and decreasing permittivities. Their work pointed out the relevance of such 1D photonic crystals for technological applications. They stressed in particular that to a good approximation all the bandgaps have the same width.

Wu et al [2] considered graded crystals with an isosceles triangle profile of the refractive index on the period. They found that the bandgap structure of such crystals is substantially different from the structure of conventional flat-layered crystals: bands can be wider or narrower, depending on the particular form (upward or downward fold) of the triangle. This evidently widens the application areas of 1D photonic crystals in general. However, their results were only obtained numerically. In subsequent publications [3, 4], the same group considered a sawtooth periodic refractive index, and obtained transmissivities greatly exceeding unity, which must raise some doubts about their method.

Very recently, Guasti and Diamant [5] also considered periodic triangular stacks. They found that reflection characteristics are similar to those of a rugate structure if the proper apodization is chosen. The results also illustrate an approximate analytical method for modelling light propagation in inhomogeneous media with discontinuities in the derivatives of the refractive index, developed by the same authors [6].

In this paper we analyse the properties of sawtooth crystals, i.e. 1D photonic crystals constructed of layers with linearly increasing refractive index $n(z)$ in each period, giving a quadratic increase of the permittivity $n^2(z)$. Our approach is substantially analytical and, as a result, possible numerical flaws are avoided.

To be precise, for a slab extending from $z = 0$ to $d$ we take

$$n(z) = n_a + (n_b - n_a) \frac{z}{d} \equiv \frac{n_b - n_a}{d} (z - z_0)$$

(1)

with $z_0 \equiv -n_a d/(n_b - n_a)$. As seen in figure 1 the refractive index is periodic, increasing linearly from $n_a$ to $n_b$ inside each layer, and falls sharply at the cell boundaries. From data
in Adachi [7] one finds that the permittivity of the ternary alloy Al$_x$Ga$_{1-x}$As is approximately linear in the relative content $x > 0.4$ of Al for wavelengths $10000 > \lambda > 6000$ Å, so photonic crystals with linearly graded refractive index layers are within present technological possibilities.

We will express the dispersion equation for the bandgaps of these photonic crystals as well as their reflectance and transmittance in terms of Bessel functions of fractional order. This adds to a group of exactly solvable models of periodic refractive indices which is restricted at the moment to flat-layered periodic structures (binary and ternary photonic crystals) and some specific sinusoidal periodic potentials [8–10]. By means of asymptotic expansions we derive approximate sinusoidal expressions showing that the bandgaps of our sawtooth periodic potential tend to a constant width as the wavelength decreases. For simplicity, as in Rauh’s work [1], we restrict our attention to normal propagation.

2. Exact results

Description of optical wave propagation of linearly polarized light of circular frequency $\omega$ through a periodic structure, with real refractive index $n(z + d) = n(z)$, in case of normal incidence, reduces from Maxwell’s equations to the Helmholtz equation

$$\frac{d^2 E(z)}{dz^2} + k_0^2 n^2(z) E(z) = 0,$$

where the total electric and magnetic fields of propagating light are expressed as

$$E = E(z) \exp(-i\omega t) \hat{y},$$

$$B = \frac{i}{k_0} \frac{dE(z)}{dz} \exp(-i\omega t) \hat{x}.$$  

Applying the $W$ transfer matrix method for periodic potentials [11], the field $E(z)$ and its derivative $dE(z)/dz = E'(z)$ at the edge points of the system $z = 0$ and $Nd$ are related by

$$\begin{bmatrix} E(0) \\ E'(0) \end{bmatrix} = \begin{bmatrix} u'(d) & -v(d) \\ -u'(d) & u(d) \end{bmatrix}^N \begin{bmatrix} E(Nd) \\ E'(Nd) \end{bmatrix}.$$
where \( u(z) \) and \( v(z) \) are the so-called normalized solutions, which satisfy the boundary conditions

\[
\begin{bmatrix}
  u(0) \\
  u'(0)
\end{bmatrix} = \begin{bmatrix}
  1 \\
  0
\end{bmatrix}, \quad
\begin{bmatrix}
  v(0) \\
  v'(0)
\end{bmatrix} = \begin{bmatrix}
  0 \\
  1
\end{bmatrix}.
\]

(5)

The \( N \)th power \( W^N \) of a unimodular \( W \)-matrix as occurs in equation (4) can be expressed in terms of \( W \) and the unit matrix \( \hat{1} \)

\[
W^N = \sin \frac{N\phi}{\sin \phi} \quad W - \sin(N - 1)\phi \quad \left[ \hat{1} \right],
\]

(6)

where the dispersion equation for the Bloch phase \( \phi \) takes the form

\[
\cos \phi = \frac{1}{2} \text{Tr} W = \frac{1}{2} [u(d) + v'(d)].
\]

(7)

The advantage of the transfer matrix method is that once we have solved for \( W \) of a single layer, the result for \( N \) layers is trivial, by use of equation (6).

In dimensionless units, equation (2) with the refractive index defined by equation (1), takes the form

\[
\frac{d^2 E(\tilde{z})}{d\tilde{z}^2} + \frac{1}{4} \tilde{z}^2 E(\tilde{z}) = 0,
\]

(8)

where the dimensionless variable \( \tilde{z} \) is

\[
\tilde{z} = \sqrt{\gamma} (z - z_0), \quad \gamma = \frac{2k_0(n_b - n_d)}{d}.
\]

(9)

It is straightforward to prove (see appendix A) that the normalized solutions of equation (8) are Bessel functions of order \( \nu = \pm 1/4 \), multiplied by \( \tilde{z}^{1/2} \), i.e.

\[
\tilde{u}(\tilde{z}) = \frac{\pi}{2^{1/4}\Gamma(1/4)} \tilde{z}^{1/2} J_{-1/4} \left( \frac{\tilde{z}^2}{4} \right) \sim 1 - \frac{\tilde{z}^4}{48} \ldots ,
\]

\[
\tilde{v}(\tilde{z}) = \frac{\pi}{2^{3/4}\Gamma(3/4)} \tilde{z}^{1/2} J_{1/4} \left( \frac{\tilde{z}^2}{4} \right) \sim -\frac{\tilde{z}^5}{80} \ldots .
\]

(10)

Using standard relations among Bessel functions (see chapter 9 in [12]) one finds

\[
\frac{d\tilde{u}(\tilde{z})}{d\tilde{z}} = -\frac{\pi}{2^{5/4}\Gamma(1/4)} \tilde{z}^{3/2} J_{3/4} \left( \frac{\tilde{z}^2}{4} \right),
\]

\[
\frac{d\tilde{v}(\tilde{z})}{d\tilde{z}} = \frac{\pi}{2^{7/4}\Gamma(3/4)} \tilde{z}^{3/2} J_{-3/4} \left( \frac{\tilde{z}^2}{4} \right).
\]

(11)

If we now change the variable \( \tilde{z} \) in equation (10) back to the original variable \( z \), we obtain another set of the fundamental solutions, \( \tilde{u}(z) \) and \( \tilde{v}(z) \), of equation (2)

\[
\tilde{u}(z) = \frac{\pi}{2^{1/4}\Gamma(1/4)} \left[ \sqrt{\gamma}(z - z_0) \right]^{1/2} J_{-1/4} \left( \frac{\gamma(z - z_0)^2}{4} \right),
\]

\[
\tilde{v}(z) = \frac{\pi}{2^{3/4}\Gamma(3/4)} \left[ \sqrt{\gamma}(z - z_0) \right]^{1/2} J_{1/4} \left( \frac{\gamma(z - z_0)^2}{4} \right).
\]

(12)
To construct the matrix $W$ of equation (4), one has to express $u(z)$ and $v(z)$ in terms of $\tilde{u}(z)$ and $\tilde{v}(z)$. Taking into account equation (5), one obtains

\begin{align*}
    u(z) &= \frac{\tilde{v}'(0)}{\tilde{w}(0)} \tilde{u}(z) - \frac{\tilde{u}'(0)}{\tilde{w}(0)} \tilde{v}(z), \\
    v(z) &= -\frac{\tilde{v}(0)}{\tilde{w}(0)} \tilde{u}(z) + \frac{\tilde{u}(0)}{\tilde{w}(0)} \tilde{v}(z),
\end{align*}

where the Wronskian $\tilde{w}(z)$ of the fundamental solutions $\tilde{u}(z)$ and $\tilde{v}(z)$ of equation (2) at the point $z = 0$ is

\begin{equation}
    \tilde{w}(0) = \tilde{u}(0) \tilde{v}'(0) - \tilde{v}(0) \tilde{u}'(0) = \sqrt{\gamma}.
\end{equation}

With the aid of newly introduced dimensionless parameters,

\begin{align*}
    z_a &= -\sqrt{\gamma} z_0 = \left( \frac{2k_0 d n_a^2}{n_b - n_a} \right)^{1/2}, \\
    z_b &= \sqrt{\gamma} (d - z_0) = \left( \frac{2k_0 d n_b^2}{n_b - n_a} \right)^{1/2},
\end{align*}

the values of $\tilde{u}(0)$, $\tilde{v}(0)$, $\tilde{u}'(0)$, $\tilde{v}'(0)$ take the form

\begin{align*}
    \tilde{u}(0) &= \frac{\pi}{2^{1/4} \Gamma(1/4)} z_a^{1/2} J_{-1/4} \left( \frac{z_a^2}{4} \right), \\
    \tilde{v}(0) &= \frac{\pi}{2^{3/4} \Gamma(3/4)} z_a^{1/2} J_{1/4} \left( \frac{z_a^2}{4} \right), \\
    \tilde{u}'(0) &= \frac{\text{d} \tilde{u}(z)}{\text{d} z} \bigg|_{z=0} = -\sqrt{\gamma} \frac{\pi}{2^{5/4} \Gamma(1/4)} z_a^{3/2} J_{3/4} \left( \frac{z_a^2}{4} \right), \\
    \tilde{v}'(0) &= \frac{\text{d} \tilde{v}(z)}{\text{d} z} \bigg|_{z=0} = \sqrt{\gamma} \frac{\pi}{2^{7/4} \Gamma(3/4)} z_a^{3/2} J_{-3/4} \left( \frac{z_a^2}{4} \right),
\end{align*}

while at $z = d$, $\tilde{u}(d)$, $\tilde{v}(d)$, $\tilde{u}'(d)$, $\tilde{v}'(d)$ are given by

\begin{align*}
    \tilde{u}(d) &= \frac{\pi}{2^{1/4} \Gamma(1/4)} z_b^{1/2} J_{-1/4} \left( \frac{z_b^2}{4} \right), \\
    \tilde{v}(d) &= \frac{\pi}{2^{3/4} \Gamma(3/4)} z_b^{1/2} J_{1/4} \left( \frac{z_b^2}{4} \right), \\
    \tilde{u}'(d) &= \frac{\text{d} \tilde{u}(z)}{\text{d} z} \bigg|_{z=d} = -\sqrt{\gamma} \frac{\pi}{2^{5/4} \Gamma(1/4)} z_b^{3/2} J_{3/4} \left( \frac{z_b^2}{4} \right), \\
    \tilde{v}'(d) &= \frac{\text{d} \tilde{v}(z)}{\text{d} z} \bigg|_{z=d} = \sqrt{\gamma} \frac{\pi}{2^{7/4} \Gamma(3/4)} z_b^{3/2} J_{-3/4} \left( \frac{z_b^2}{4} \right).
\end{align*}
The elements of the $W$-matrix are then

$$W_{11} = v'(d) = \frac{\pi}{2^{5/2}} \sqrt{\frac{n_b}{n_a}} z_a z_b [J_{1/4}(z_a^2/4) J_{3/4}(z_b^2/4) + J_{-1/4}(z_a^2/4) J_{-3/4}(z_b^2/4)],$$

$$W_{21} = -u'(d) = \frac{\pi}{2^{5/2}} \sqrt{n_a n_b z_a z_b [J_{-3/4}(z_a^2/4) J_{3/4}(z_b^2/4) - J_{3/4}(z_a^2/4) J_{-3/4}(z_b^2/4)]},$$

$$k_0 W_{12} = -k_0 v(d) = \frac{\pi}{2^{5/2}} \sqrt{\frac{1}{n_a n_b}} z_a z_b [J_{1/4}(z_a^2/4) J_{-1/4}(z_b^2/4) - J_{-1/4}(z_a^2/4) J_{1/4}(z_b^2/4)],$$

$$W_{22} = u(d) = \frac{\pi}{2^{5/2}} \sqrt{\frac{n_a}{n_b}} z_a z_b [J_{-3/4}(z_a^2/4) J_{1/4}(z_b^2/4) + J_{3/4}(z_a^2/4) J_{1/4}(z_b^2/4)],$$

while the dispersion equation (7) takes the form

$$\cos \phi = \frac{\pi}{2^{5/2}} z_a z_b \left[ \frac{n_a}{n_b} J_{-3/4}(z_a^2/4) J_{1/4}(z_b^2/4) + \frac{n_a}{n_b} J_{3/4}(z_a^2/4) J_{1/4}(z_b^2/4) \right]$$

$$+ \sqrt{\frac{n_b}{n_a}} J_{1/4}(z_a^2/4) J_{3/4}(z_b^2/4) + \sqrt{\frac{n_b}{n_a}} J_{-1/4}(z_a^2/4) J_{-3/4}(z_b^2/4) \right].$$

Equation (19) is the exact dispersion relation for our sawtooth 1D photonic crystal. The amplitude reflection $r$ and transmission $t$ coefficients are

$$r = \frac{W_{11} = \frac{n_a}{n_a} W_{22} + i \left[ k_0 n_{ex} W_{12} + \frac{1}{k_0 n_{in}} W_{21} \right]}{W_{11} = \frac{n_a}{n_a} W_{22} + i \left[ k_0 n_{ex} W_{12} - \frac{1}{k_0 n_{in}} W_{21} \right]},$$

$$t = \frac{2}{W_{11} = \frac{n_a}{n_a} W_{22} + i \left[ k_0 n_{ex} W_{12} - \frac{1}{k_0 n_{in}} W_{21} \right]},$$

where the elements of the $W$-matrix are given in exact form by equation (18) and the elements of the $W_N$-matrix are obtained by use of equation (6). The indices of refraction in the incident and exit media are $n_{in}$ and $n_{ex}$ as seen in figure 1.

The above results can easily be extended to include an isosceles-triangle refractive index profile, as explained in appendix B.

### 3. Asymptotic estimates

The motivation for exploring those is to have simple analytic approximations for the Bloch phase $\cos \phi$, as well as for the reflection and transmission coefficients. Using Hankel’s asymptotic formula (section 9.2 of [12]) one obtains the elements of the $W$-matrix in the...
form

\[ W_{11} = \sqrt{\frac{n_b}{n_a}} \left[ P_{1/4}(z_a^2/4) P_{3/4}(z_b^2/4) + Q_{1/4}(z_a^2/4) Q_{3/4}(z_b^2/4) \right] \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \\
+ \sqrt{\frac{n_b}{n_a}} \left[ P_{1/4}(z_a^2/4) Q_{3/4}(z_b^2/4) - Q_{1/4}(z_a^2/4) P_{3/4}(z_b^2/4) \right] \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right), \]

\[ k_0 W_{12} = -\frac{1}{\sqrt{n_a n_b}} \left[ P_{1/4}(z_a^2/4) P_{1/4}(z_b^2/4) + Q_{1/4}(z_a^2/4) Q_{1/4}(z_b^2/4) \right] \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \\
+ \frac{1}{\sqrt{n_a n_b}} \left[ P_{1/4}(z_a^2/4) P_{3/4}(z_b^2/4) - P_{1/4}(z_a^2/4) Q_{3/4}(z_b^2/4) \right] \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right), \]

\[ W_{22} = \sqrt{\frac{n_a}{n_b}} \left[ P_{1/4}(z_a^2/4) P_{3/4}(z_b^2/4) + Q_{1/4}(z_a^2/4) Q_{3/4}(z_b^2/4) \right] \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \\
+ \sqrt{\frac{n_a}{n_b}} \left[ P_{1/4}(z_a^2/4) Q_{3/4}(z_b^2/4) - P_{1/4}(z_a^2/4) Q_{3/4}(z_b^2/4) \right] \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right), \]

where

\[ P_v(y) = 1 - \frac{(4v^2 - 1)(4v^2 - 9)}{2!(8y)^2} - \frac{(4v^2 - 1)(4v^2 - 9)(4v^2 - 25)(4v^2 - 49)}{4!(8y)^4} \ldots, \]

\[ Q_v(y) = \frac{4v^2 - 1}{8y} - \frac{(4v^2 - 1)(4v^2 - 9)(4v^2 - 25)}{3!(8y)^3} \ldots \]

and the dispersion equation in the form

\[ \cos \phi = \frac{1}{2} \sqrt{\frac{n_b}{n_a}} \left[ P_{1/4}(z_a^2/4) P_{3/4}(z_b^2/4) + Q_{1/4}(z_a^2/4) Q_{3/4}(z_b^2/4) \right] \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \\
+ \frac{1}{2} \sqrt{\frac{n_b}{n_a}} \left[ P_{1/4}(z_a^2/4) P_{3/4}(z_b^2/4) + Q_{1/4}(z_a^2/4) Q_{3/4}(z_b^2/4) \right] \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \\
+ \frac{1}{2} \sqrt{\frac{n_b}{n_a}} \left[ P_{1/4}(z_a^2/4) Q_{3/4}(z_b^2/4) - Q_{1/4}(z_a^2/4) P_{3/4}(z_b^2/4) \right] \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \\
+ \frac{1}{2} \sqrt{\frac{n_b}{n_a}} \left[ P_{1/4}(z_a^2/4) Q_{3/4}(z_b^2/4) - Q_{1/4}(z_a^2/4) Q_{3/4}(z_b^2/4) \right] \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right). \]

Equations (21)–(23) are the expressions we sought. By truncating the terms \( P_v \) and \( Q_v \), various asymptotic approximations are obtained. In particular, after some trigonometric simplifications,
the matrix elements become in the first approximation \((P = 1, Q = 0)\)

\[
W_{11} = \sqrt{\frac{n_b}{n_a}} \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right),
\]

\[
W_{21} = -\sqrt{\frac{n_a}{n_b}} \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right),
\]

\[
k_0 W_{12} = \frac{1}{\sqrt{n_a n_b}} \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right),
\]

\[
W_{22} = \sqrt{\frac{n_a}{n_b}} \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right),
\]

in the second \((P = 1, Q = (v^2 - 0.25)/2z)\)

\[
W_{11} = \sqrt{\frac{n_b}{n_a}} \left[ \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) + \left( \frac{3}{8z_a^2} + \frac{5}{8z_b^2} \right) \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \right],
\]

\[
W_{21} = -\sqrt{\frac{n_a}{n_b}} \left[ \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) + \left( \frac{5}{8z_a^2} - \frac{5}{8z_b^2} \right) \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \right],
\]

\[
k_0 W_{12} = \frac{1}{\sqrt{n_a n_b}} \left[ \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) - \left( \frac{3}{8z_a^2} - \frac{3}{8z_b^2} \right) \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \right],
\]

\[
W_{22} = \sqrt{\frac{n_a}{n_b}} \left[ \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) - \left( \frac{5}{8z_a^2} + \frac{3}{8z_b^2} \right) \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \right],
\]

while in the third

\[
W_{11} = \sqrt{\frac{n_b}{n_a}} \left[ \left(1 - \frac{105}{128z_a^4} + \frac{135}{128z_b^4} - \frac{15}{64z_a^2z_b^2} \right) \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right)
+ \left( \frac{3}{8z_a^2} + \frac{5}{8z_b^2} \right) \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \right],
\]

\[
W_{21} = -\sqrt{\frac{n_a}{n_b}} \left[ \left(1 + \frac{135}{128z_a^4} + \frac{135}{128z_b^4} + \frac{25}{64z_a^2z_b^2} \right) \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right)
+ \left( \frac{5}{8z_a^2} - \frac{5}{8z_b^2} \right) \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \right],
\]

\[
k_0 W_{12} = \frac{1}{\sqrt{n_a n_b}} \left[ \left(1 - \frac{105}{128z_a^4} - \frac{105}{128z_b^4} + \frac{9}{64z_a^2z_b^2} \right) \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right)
- \left( \frac{3}{8z_a^2} - \frac{3}{8z_b^2} \right) \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \right],
\]

\[
W_{22} = \sqrt{\frac{n_a}{n_b}} \left[ \left(1 + \frac{135}{128z_a^4} - \frac{105}{128z_b^4} - \frac{15}{64z_a^2z_b^2} \right) \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right)
- \left( \frac{5}{8z_a^2} + \frac{3}{8z_b^2} \right) \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right) \right].
\]
As one can see, the dispersion equation \( \cos \phi = \frac{1}{2}(W_{11} + W_{22}) \) takes a particularly simple form in the first approximation

\[
\cos \phi_1 = \frac{1}{2} \left( \sqrt{\frac{n_a}{n_b}} + \sqrt{\frac{n_b}{n_a}} \right) \cos \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right).
\]

(27)

Due to its simplicity equation (27) may be particularly useful in photonic crystal design. Let us estimate its range of validity. Using expressions for \( W_{11} \) and \( W_{22} \) in equations (24) and (25), we find that \( \cos \phi \) in the second approximation can be written as

\[
\cos \phi_2 = \cos \phi_1 + |\epsilon| \sin \left( \frac{z_a^2}{4} - \frac{z_b^2}{4} \right),
\]

(28)

where

\[
\epsilon = \frac{1}{2} \sqrt{\frac{n_b}{n_a}} \left( \frac{3}{8} \frac{z_a^2}{z_a^2} + \frac{5}{8} \frac{z_b^2}{z_b^2} \right) - \frac{1}{2} \sqrt{\frac{n_a}{n_b}} \left( \frac{5}{8} \frac{z_a^2}{z_a^2} + \frac{3}{8} \frac{z_b^2}{z_b^2} \right)
\]

\[
\leq \frac{1}{2} \left( \frac{5}{8} \sqrt{\frac{n_b}{n_a}} - \frac{3}{8} \sqrt{\frac{n_a}{n_b}} \right) \left( \frac{1}{z_a^2} + \frac{1}{z_b^2} \right) \frac{n_b - n_a}{2k_0 d}
\]

\[
\leq \frac{143 \sqrt{5}}{250} \frac{1}{4} \frac{n_b - n_a}{k_0 d} < \frac{1}{3} \frac{n_b - n_a}{k_0 d}.
\]

(29)

At the last line we took into account that for typical dielectrics the refractive indices are in the range \( n_{a,b} = 1 \ldots 5 \). As a result, it is safe to use equation (27) for design purposes in the range of wavenumbers \( k_0 \) such as

\[
k_0 d > \frac{|n_b - n_a|}{3}.
\]

(30)

Figures 2–4 illustrate all of the above approximations. In the second and third approximations, \( \cos \phi \) can also be written as

\[
\cos \phi_{2,3} = A_{2,3} \cos(\frac{z_a^2}{4} - \frac{z_b^2}{4} + \delta_{2,3}),
\]

(31)

which allows for a shift of both the band centres and their widths. For our choice of \( n_b/n_a = 2 \), the amplitude in the first approximation \( A_1 = 1.06 \) accounts for the narrow bandgaps observed in the drawings.

In appendix C we also compare the exact results to approximations provided by binary photonic crystals (replacing the ramp by discrete steps).

4. Bandgap analysis

Figure 2 shows that the dispersion equation for \( \cos \phi \) in the first approximation is already in quite good agreement with the exact results. We now discuss some of the predictions for the bandgaps that follow from it. The bandgap edges are defined by the condition \( \cos \phi = (-1)^q \), where the index \( q = 1, 2, \ldots \) numbers the bandgaps. This condition applied to equation (27) leads to expressions for the right \( k_{r0} \) and left \( k_{l0} \) bandedges in the first approximation:

\[
k_{r0} d = \frac{q \pi + \alpha}{n_{av}}, \quad k_{l0} d = \frac{q \pi - \alpha}{n_{av}},
\]

(32)
Figure 2. Exact results (thicker grey lines) (see equations (18)–(20)) versus the first asymptotic approximation (thinner red lines) (see equation (24)) for the Bloch phase, reflectance and transmittance, for a sawtooth periodic structure with \( n_a = 1.5 \), \( n_b = 3 \) and the number of periods \( N = 6 \), in the case of normal incidence. The refractive indices outside the structure are \( n_{\text{in}} = 1.0 \) for \( z < 0 \) and \( n_{\text{ex}} = 1.0 \) for \( z > N d \).
Figure 3. Exact results (thicker grey lines) (see equations (18)–(20)) versus the second asymptotic approximation (thinner green lines) (see equation (25)) for the Bloch phase, reflectance and transmittance, for a sawtooth periodic structure with parameters as in figure 2.
Figure 4. Exact results (thicker grey lines) (see equations (18)–(20)) versus the third asymptotic approximation (thinner blue lines) (see equation (26)) for the Bloch phase, reflectance and transmittance, for a sawtooth periodic structure with parameters as in figure 2.
Figure 5. Dimensionless difference, \((k_{0,a}^c - k_{0}^c) d\), \(a = 1, 2, 3\), between exact results, \(k_{0}^c d\), and a given asymptotic approximation, \(k_{0,a}^c d\), for bandgap centres of a sawtooth periodic array with refractive index parameters \(n_a = 1.5\), \(n_b = 3.0\) versus the bandgap index \(q\) for the first eight bandgaps. From top to bottom the lines are \((k_{0,1}^c - k_{0}^c) d\) (red), \((k_{0}^c - k_{0}^c) d\) (grey), \((k_{0,3}^c - k_{0}^c) d\) (blue), \((k_{0,2}^c - k_{0}^c) d\) (green).

Figure 6. Dimensionless bandgap widths, \(w d\), for a sawtooth periodic array with refractive index parameters \(n_a = 1.5\), \(n_b = 3.0\) versus the bandgap index \(q\) for the first eight bandgaps. From top to bottom the lines are the second asymptotic approximation (green), the first asymptotic approximation (red), exact result (grey) and the third asymptotic approximation (blue).
where

$$\alpha = \arccos \left( \frac{2}{\sqrt{n_a/n_b + n_b/n_a}} \right), \quad n_{av} = \frac{n_a + n_b}{2}. \quad (33)$$

Therefore, the centres of the bandgaps $k_c^0$ and their widths $w$ are given in the first approximation by

$$k_c^0 d = q \frac{\pi}{n_{av}}, \quad wd = \frac{2\alpha}{n_{av}}. \quad (34)$$

This is qualitatively similar to the asymptotic results obtained by Rauh et al [1]. In figure 5 we compare the accuracy of the three approximations for the bandgap centres, and in figure 6 for their widths. As one can see from equations (24)–(26), each approximation incorporates new corrections of orders $1/\varepsilon^2_{a,b}$ and their squares, respectively. With the exception of the first bandgap (but that is inherent in use of an asymptotic series), all three approximations provide one with quite accurate estimates and the second one balances accuracy and simplicity.

5. Conclusion

We have derived the exact dispersion equation, reflection and transmission coefficients, in terms of Bessel functions, for our one-dimensional photonic crystal with a sawtooth refractive index profile. Using the Hankel expansion, we worked out the first three asymptotic approximations to those coefficients. Even the first approximation, which has a particularly simple analytic form, provides one with very reasonable estimates. Using the results of appendix A, one can easily work out solutions for any monomial behaviour $(k_0 n(z))^2 \sim (z/L)^n$ of the permittivity within a single cell of the periodic system. They always involve Bessel functions $J_{\pm m}(z)$ of order $m = 1/(n + 2)$.

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Appendix A. Solutions of the wave equation for power law potentials

While discussing the Wentzel–Kramers–Brillouin approximation, Schiff [13, p 272] points out that when $k^2(z) = (z/L)^n$, an exact solution of the stationary state wave equation

$$0 = \psi''(z) + k^2(z) \psi(z), \quad (A.1)$$

is given by

$$\psi(z) \equiv A(z) J_m(\xi), \quad A(z) \equiv \sqrt{\xi(z)/k(z)},$$

$$\xi(z) \equiv \int_0^z k(t) dt, \quad m = \frac{1}{n + 2}. \quad (A.2)$$
The task is to show that \( J_m(\xi) \) is a generic Bessel function of order \( m \), a result given originally by Langer [14]. The most used case is a linear turning point with \( n = 1 \Rightarrow m = 1/3 \), where the solution coincides with the Airy function. The general result makes it easy to work out the transfer matrix for a 1D photonic crystal whose refractive index profile has any monomial 'sawtooth shape' in each layer. To begin, we note that for our assumed \( k^2(z) \)

\[
\xi(z) = \frac{2L}{n+2} \left( \frac{z}{L} \right)^{(n+2)/2},
\]

\[
k'(z) = \frac{n}{2L} \frac{z}{\left( \frac{z}{L} \right)^{(n-2)/2}},
\]

\[
A^2(z) = \frac{\xi}{k} = 2mz,
\]

\[
2A(z)A'(z) = 2m = \text{const},
\]

\[
A^3(z)A''(z) = -[A(z)A']^2 = -m^2.
\]

(A.3)

From here we work out \( \psi' \) and \( \psi'' \), and substitute into the wave equation. On functions of \( z \), the prime means \( d/dz \), but on the Bessel function it means \( d/d\xi \). We have

\[
\psi' = A'(z) J_m(\xi) + A(z)k(z)J'_m(\xi),
\]

\[
\psi'' = A''(z)J_m(\xi) + A(z)k^2(z)J''_m(\xi) + [2A'(z)k(z) + A(z)k'(z)]J'_m(\xi).
\]

(A.4)

Substituting into the wave equation and removing the factor \( Ak^2 \), we find that the coefficient of \( J'_m(\xi) \) is

\[
\frac{2A'k + Ak'}{Ak^2} = \frac{1}{\xi}.
\]

(A.5)

see equation (A.8) for details, while the coefficient of \( J_m(\xi) \) is \( 1 + [A''/Ak^2] \), where the variable portion is

\[
\frac{A''}{Ak^2} = \frac{A^3(\xi)A''}{A^4k^2} = -\frac{(A')^2A^2}{A^4k^2} = -\frac{m^2}{\xi^2}.
\]

(A.6)

This verifies that the wave equation has been reduced to Bessel’s equation for the fractional order \( m \),

\[
J''_m(\xi) + \frac{1}{\xi}J'_m(\xi) + \left[ 1 - \frac{m^2}{\xi^2} \right] J_m(\xi) = 0.
\]

(A.7)

The regular solution \( J_m(\xi) \) will vanish at the origin when \( m > 0 \), so the irregular solution \( J_{-m}(\xi) \) is also required if a non-zero initial value is in order. In equation (A.5) we used the following relations:

\[
A^2(z) = \frac{\xi}{k}, \quad 2AA' = 1 - \frac{\xi'k'}{k^2},
\]

\[
\frac{2A'k + Ak'}{Ak^2} = \frac{2AA'}{A^2k} + \frac{k'}{k^2} = \frac{1}{\xi}.
\]

(A.8)
Appendix B. Triangular profile

Our purpose here is to show how the results of the section 2 can be used to describe wave propagation through a periodic structure with a triangular refractive index profile, see figure B.1. Since the refractive index of the period with \(-d \leq z \leq d\) is an even function, the normalized solutions \(u\) and \(v\), relating the field \(E(z)\) and its derivative \(dE(z)/dz\) at the edge points \(z = \pm d\), can be taken to be the even and odd functions, respectively. As a result, we have

\[
\begin{bmatrix}
E(-d) \\
E'(-d)
\end{bmatrix} = W_i \begin{bmatrix}
E(d) \\
E'(d)
\end{bmatrix},
\]

where

\[
W_i = \begin{bmatrix}
u(-d) & v(-d) \\
u'(-d) & v'(-d)
\end{bmatrix} \begin{bmatrix}
v'(d) & -v(d) \\
-u'(d) & u(d)
\end{bmatrix} = \begin{bmatrix}
u(d) & -v(d) \\
-u'(d) & v'(d)
\end{bmatrix} \begin{bmatrix}
v'(d) & -v(d) \\
-u'(d) & u(d)
\end{bmatrix}.
\]

The dispersion equation takes the form

\[
\cos \phi = u(d) v'(d) + v(d) u'(d).
\]

Expressions for the elements \(u(d), u'(d), v(d)\) and \(v'(d)\) are given by equation (18).

Reversing the order of the two matrices whose product gives \(W_i\) in equation (B.2) would give the transfer matrix for the inverted V triangle pattern. That is to say, if figure B.1 is a pattern \(DU DU DU\), the new pattern would be \(UD UD UD\). These are examples of biperiodic superlattices [16], and some interesting physics arises if one of the two components \(U\) or \(D\) is given a slightly different width \(d_U \neq d_D\).

Appendix C. Binary approximations

We consider a binary photonic crystal composed of layers of constant refractive indices \(n_1, n_2\), with corresponding thicknesses \(d_1, d_2\). The elements of a binary \(W\)-matrix are well-known [11]
Figure C.1. Exact results (thicker grey lines) for a sawtooth periodic potential with parameters as in figure 2 versus binary approximation (thinner black lines) with \( n_1 = n_a, n_2 = n_b \).
Figure C.2. Like figure C.1 but with $n_1 = (n_a + 3n_b)/4$, $n_2 = (3n_a + n_b)/4$. 
and given by

\[ W_{11} = \cos(k_0 n_1 d_1) \cos(k_0 n_2 d_2) - \frac{n_2}{n_1} \sin(k_0 n_1 d_1) \sin(k_0 n_2 d_2), \]

\[ \frac{W_{21}}{k_0} = n_1 \sin(k_0 n_1 d_1) \cos(k_0 n_2 d_2) + n_2 \cos(k_0 n_1 d_1) \sin(k_0 n_2 d_2), \]

\[ k_0 W_{12} = -\frac{1}{n_1} \sin(k_0 n_1 d_1) \cos(k_0 n_2 d_2) - \frac{1}{n_2} \cos(k_0 n_1 d_1) \sin(k_0 n_2 d_2), \]

\[ W_{22} = \cos(k_0 n_1 d_1) \cos(k_0 n_2 d_2) - \frac{n_1}{n_2} \sin(k_0 n_1 d_1) \sin(k_0 n_2 d_2). \]  

(C.1)

The simplest binary approximation to the sawtooth profile discussed above is given by \( n_1 = n_a, \) \( n_2 = n_b \) and \( d_1 = d_2 = d/2. \) Figure C.1 compares the exact results for the sawtooth potential with parameters as in figure 2 to that of the simplest binary approximation. As one can see there are some similarities, but the bandgap widths differ strongly; for example, the third bandgap of the sawtooth potential disappears entirely. A better binary approximation, see figure C.2, is obtained if one follows the recipe of Kalotas and Lee [15], taking the layer indices to be \( n_1 = (n_a + 3n_b)/4, \) \( n_2 = (3n_a + n_b)/4 \) and widths \( d_1 = d_2 = d/2. \) (These values give the correct average wave number in each layer.)

References

[1] Rauh H, Yampolskaya G I and Yampolskii S V 2010 New J. Phys. 12 073033
[2] Wu X-Y, Zhang B-J, Yang J-H, Liu X-J, Ba N, Wu Y-H and Wang Q-C 2011 Physica E 43 1694
[3] Wu X-Y, Zhang B-J, Yang J-H, Zhang S-Q, Liu X-J, Wang J, Ba N, Hua Z and Yin X-G 2012 Physica E 45 166
[4] Wu X-Y, Zhang B-J, Liu X-J, Zhang S-Q, Wang J, Ba N, Xiao L and Li H 2012 Physica E 46 133
[5] Fernandez-Guasti M and Diamant R 2013 Optical Interference Coatings (OSA Techn. Dig.) ed M Tilsch and D Ristau (Washington, DC: Optical Society of America) paper TC.3
[6] Diamant R and Fernandez-Guasti M 2013 Opt. Commun. 294 64–72
[7] Adachi S 1985 J. Appl. Phys. 58 R1–29
[8] Casperson L W 1984 Phys. Rev. A 30 2749–51
[9] Wu S M and Shih C C 1985 Phys. Rev. A 32 3736–8
[10] Takayama K 1986 Phys. Rev. A 34 4408–10
[11] Sprung D W L, Wu H and Martorell J 1993 Am. J. Phys. 61 1118–24
[12] Abramowitz M and Stegun I S 1964 Handbook of Mathematical Functions (New York: Dover)
[13] Schiff L I 1968 Quantum Mechanics 3rd edn (New York: McGraw-Hill)
[14] Langer R E 1937 Phys. Rev. 51 669–76
[15] Kalotas T M and Lee A R 1991 Phys. Scr. 44 313–20
[16] Sprung D W L, Vanderspek L W A, Van Dijk W, Martorell J and Pacher C 2008 Phys. Rev. B 77 035333