UNBOUNDED VARIATION AND SOLUTIONS OF IMPULSIVE CONTROL SYSTEMS

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Abstract. We consider a control system with dynamics which are affine in the (unbounded) derivative of the control $u$. We introduce a notion of generalized solution $x$ on $[0,T]$ for controls $u$ of bounded total variation on $[0,t]$ for every $t < T$, but of possibly infinite variation on $[0,T]$. This solution has a simple representation formula based on the so-called graph completion approach, originally developed for BV controls. We prove the well-posedness of this generalized solution by showing that $x$ is a limit solution, that is the pointwise limit of regular trajectories of the system. In particular, we single out the subset of limit solutions which is in one-to-one correspondence with the set of generalized solutions. The controls that we consider provide the natural setting for treating some questions on the controllability of the system and some optimal control problems with endpoint constraints and lack of coercivity.

Introduction

We consider a control system of the form

$$\dot{x}(t) = g_0(x(t),u(t),v(t)) + \sum_{i=1}^{m} g_i(x(t),u(t))\dot{u}_i(t), \quad t \in [0,T],$$

$$x(0) = x_0, \quad u(0) = u_0,$$

where $x \in \mathbb{R}^n$ and the measurable control pair $(u,v)$ ranges over a compact set $U \times V \subset \mathbb{R}^m \times \mathbb{R}^q$. Due to the presence of the derivatives $\dot{u}_i$, (1) is a so-called impulsive control system, where a solution $x$ can be provided by the usual Carathéodory solution only if $u$ is an absolutely continuous control. For less regular $u$, several concepts of solutions have been introduced in the literature, either for commutative systems, where the Lie brackets $[(e_i,g_i),(e_j,g_j)] = 0$ for all $i,j = 1,\ldots,m$ (see e.g. [BR1], [D], [Sa], [AR]).

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or assuming that \( u \) and \( x \) are functions of bounded variation, when the Lie Algebra is non trivial (see e.g. \([BR, MR]\)). These solutions are described by different authors in fairly equivalent ways, and we will refer to them as graph completion solutions, since they are obtained completing the graph of \( u \) (see e.g., \([R], [Wa], [GS, KDP], [SV, WS, AKP, K, PS, MS, BR, MiRu], \) for numerical approximations \([CF]\), for extensions to stochastic control \([MS1, DM]\)). In the less studied noncommutative case with controls \( u \) of unbounded variation, let us mention the notion of looping controls \([BR2]\), the definition of limit solution \([AR]\), and the theory of rough paths (for continuous \( u \)) \([LQ]\). Differently from the cases of commutative systems and of bounded variation controls \( u \), in the general case no (simple) explicit representation formula of the solution is known.

In this paper we focus on the noncommutative case for controls \( u : [0, T] \to U \) with total variation bounded on \([0, t]\) for every \( t < T \) but possibly infinite on \([0, T]\), in short \( u \in BV_{loc}(T) \). We extend the graph completion approach to such controls and for any \( u \in BV_{loc}(T) \) and measurable \( v \), we introduce a notion of solution \( x \) to (1)–(2) on \([0, T]\), which we call BV\(_{loc}\) graph completion solution (see Definitions 1.6, 1.7). In particular, we first define an AC\(_{loc}\) solution \( x \) on \([0, T]\), obtained by extending \((x, u)\) to be absolutely continuous on \([0, t]\) for \( t < T \) to \([0, T]\), by choosing \((x, u)(T) = \lim_j (x, u)(\tau_j)\) for some sequence \( \tau_j \nearrow T \). Hence we prove that the concept of BV\(_{loc}\) graph completion solution \( x \) is:

i) \textit{well defined}, since for any \( u \in BV_{loc}(T) \) and measurable \( v \) a corresponding a BV\(_{loc}\) graph completion solution does exist (Theorem 2.1);

ii) \textit{consistent} with that of AC\(_{loc}\) solution, in the sense that if the pair \((x, u)\) is absolutely continuous on \([0, t]\) for \( t < T \) and \( x \) is a BV\(_{loc}\) graph completion solution, then \( x \) is an AC\(_{loc}\) solution (Theorem 2.2);

iii) \textit{well posed}, since \( x \) is the pointwise limit of Carathéodory solutions \( x_k \) to (1), (2) corresponding to inputs \((u_k, v)\), with the controls \( u_k \) absolutely continuous on \([0, T]\) and pointwisely converging to \( u \). In this sense it is a simple limit solution, as recently defined in \([AR]\) (see Definition 3.1). Actually, in Theorem 4.1 we prove something more, in that we characterize the specific subclass of simple limit solutions, that we call BV\(_{loc}\)S limit solutions, corresponding to BV\(_{loc}\) graph completion solutions.

With respect to more general concepts, the BV\(_{loc}\) graph completion solution has a nice representation formula, suitable to derive necessary and sufficient optimality conditions for several optimization problems, both in terms of Pontrjagin Maximum Principle and of Hamilton-Jacobi-Bellman equations (some results in the last direction have been already obtained in \([MS2]\)). Moreover, controls \( u \in BV_{loc}(T) \) are relevant in controllability issues, like approaching a target set, and in optimization problems with endpoint constraints and certain running costs lacking coercivity, as in the following example (see also Example 3.1).
Example 0.1. Let $C \subset \mathbb{R}^n \times U$ be a closed subset, the target, and let $d(\cdot)$ denote the Euclidean distance from $C$. Let us minimize
\begin{equation}
\int_0^T \left[ \ell_0(x(t), u(t), v(t)) + \ell_1(x(t), u(t)) |\dot{u}| \right] dt,
\end{equation}
over trajectory-control pairs $(x, u, v)$ of (1), (2) such that
\begin{equation}
d((x(t), u(t))) > 0 \forall t < T, \liminf_{t \to T^-} d((x(t), u(t))) = 0,
\end{equation}
assuming that $\ell_0 \geq 0$ and $\ell_1$ verifies $\ell_1(x, u) \geq c(d(x, u))$,
for some strictly increasing, continuous function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with $c(0) \geq 0$. In this case, only controls $u \in BV_{loc}(T)$ may have finite cost. The above hypothesis on $\ell_1$ generalizes the so-called weak coercivity condition $\ell_1 \geq C_1 > 0$, assumed in many applications in order to rule out controls with unbounded variation. Notice that, as the variation of $u$ is unbounded, we expect chattering phenomena as $t$ tends to $T$ (see e.g. [CGPT] and the references therein), which in impulsive control systems will affect both $u$ and $x$. It is thus natural to replace the usual endpoint condition $(x(T), u(T)) \in C$ with (4) (see Remark 0.1).

The paper is organized as follows. We end this section with some notation and the precise assumptions that are needed in the paper. In Section 1 we define $AC_{loc}$ solutions and introduce the notion of $BV_{loc}$ graph completion solution. Existence of such solutions and their consistency with regular, $AC_{loc}$ solutions are established in Section 2. In Section 3 we define $BV_{loc}S$ limit solutions and in Section 4 we obtain our main result: the equivalence between $BV_{loc}$ graph completion solutions and $BV_{loc}S$ limit solutions. Section 5 is devoted to the proofs of some technical results.

0.1. Notation. Let $E \subset \mathbb{R}^N$. For any $f : [a, b] \to E$, $Var_{[a,b]}(f)$ denotes the (total) variation of $f$ on $[a, b]$. When $E$ is bounded, we call diameter of $E$ the value $\text{diam}(E) := \sup\{|u_1 - u_2| : u_1, u_2 \in E\}$. For $T > 0$, let $AC([0, T], E)$, $BV([0, T], E)$ denote the set of absolutely continuous and BV functions $f : [0, T] \to E$, respectively, and let us set
\begin{align*}
AC_{loc}([0, T], E) &:= \{ f \in AC([0, t], E) \forall t < T, \lim_{t \to T^-} Var_{[0, t]}[f] \leq +\infty\}, \\
BV_{loc}([0, T], E) &:= \{ f \in BV([0, t], E) \forall t < T, \lim_{t \to T^-} Var_{[0, t]}[f] \leq +\infty\}.
\end{align*}
The set $L^1([0, T], E)$ is the usual quotient with respect to the Lebesgue measure. When no confusion on the codomain may arise, in what follows in place of the above sets we will simply write $AC(T)$, $BV(T)$, $AC_{loc}(T)$, $BV_{loc}(T)$, and $L^1(T)$, respectively.
We set $I \mathbb{R}_{+} := [0, +\infty[$ and call \textit{modulus} (of continuity) any increasing, continuous function $\omega : I \mathbb{R}_{+} \rightarrow I \mathbb{R}_{+}$ such that $\omega(0) = 0$ and $\omega(r) > 0$ for every $r > 0$.

0.2. Assumptions. Throughout the paper we assume the following hypotheses:

(i) the sets $U \subset \mathbb{R}^{m}$ and $V \subset \mathbb{R}^{l}$ are compact;
(ii) the control vector field $g_{0} : \mathbb{R}^{n} \times U \times V \rightarrow \mathbb{R}^{n}$ is continuous and $(x, u) \mapsto g_{0}(x, u, v)$ is locally Lipschitz on $\mathbb{R}^{n} \times U$, uniformly in $v \in V$;
(iii) for each $i = 1, \ldots, m$ the control vector field $g_{i} : \mathbb{R}^{n} \times U \rightarrow \mathbb{R}^{n}$ is locally Lipschitz continuous;
(iv) there exists $M > 0$ such that
$$\left| \left( g_{0}(x, u, v), g_{1}(x, u), \ldots, g_{m}(x, u) \right) \right| \leq M(1 + |(x, u)|),$$
for every $(x, u, v) \in \mathbb{R}^{n} \times U \times V$.

In the main results we will use the following condition.

\textbf{Definition 0.1 (Whitney property).} A compact subset $U \subset \mathbb{R}^{m}$ has the Whitney property if there is some $C \geq 1$ such that for every pair $u_{1}, u_{2} \in U$,
$$\exists \tilde{u} : [0, 1] \rightarrow U \text{ verifying}$$

(5) $\tilde{u}(0) = u_{1}, \quad \tilde{u}(1) = u_{2}, \quad \text{Var}[\tilde{u}] \leq C|u_{1} - u_{2}|$.

For instance, compact, star-shaped sets verify the Whitney property.

1. BV$_{\text{loc}}$ graph completion solutions

For any control $(u, v) \in AC_{\text{loc}}(T) \times L^{1}(T)$ with $u(0) = \overline{u}_{0}$, let 
$$x = x[\overline{x}_{0}, \overline{u}_{0}, u, v]$$
denote the unique Carathéodory solution to (1)–(2), defined on $[0, T[$.

1.1. AC$_{\text{loc}}$ controls and solutions. Let us introduce the set of controls $u \in AC_{\text{loc}}(T)$ extended to $[0, T[$:

(6) $\overline{AC}_{\text{loc}}(T) := \left\lbrace u \in AC_{\text{loc}}(T) : u(T) := \lim_{j} u(\tau_{j}), \text{ for some } \tau_{j} \nearrow T \right\rbrace$

and the corresponding extended solutions:

\textbf{Definition 1.1 (AC$_{\text{loc}}$ solution).} Let $(u, v) \in \overline{AC}_{\text{loc}}(T) \times L^{1}(T)$ with $u(0) = \overline{u}_{0}$, and set 
$$x := x[\overline{x}_{0}, \overline{u}_{0}, u, v].$$
When $x$ is bounded on $[0, T[$, we introduce an extension of $x$ to $[0, T]$, such that

(7) $(x(T), u(T)) \in (x, u)_{\text{set}}(T) := \left\lbrace \lim_{j} (x, u)(\tau_{j}), \text{ for some } \tau_{j} \nearrow T \right\rbrace$.

We call $x$ a (single-valued) AC$_{\text{loc}}$ solution on $[0, T]$ and $(x, u, v)$ an AC$_{\text{loc}}$ trajectory-control pair.

Clearly, the extension of $(x, u)$ to $[0, T]$ is not unique, in general.
Remark 1.1. In order to motivate the above extension, let us consider AC_{loc} trajectory-control pairs \((x, u, v)\) defined on \([0, T]\) as above, verifying the final constraint
\[ (8) \quad (x, u)(T) \in \mathcal{C}, \]
where \(\mathcal{C} \subset \mathbb{R}^n \times U\) is a closed set, which we call the \textit{target}. Condition (8) turns out to be verified when \((x, u)_{\text{set}}(T) \cap \mathcal{C} \neq \emptyset\) and this is equivalent to have
\[ \liminf_{t \to T^-} d((x(t), u(t)), \mathcal{C}) = 0. \]
Incidentally, the stronger condition \((x, u)_{\text{set}}(T) \subseteq \mathcal{C}\) is instead equivalent to
\[ (9) \quad \lim_{t \to T^-} d((x(t), u(t)), \mathcal{C}) = 0 \]
and this limit holds true if and only if for every increasing sequence \((\tau_j)\) converging to \(T\) there exists a subsequence such that \(\lim_{j \to \infty} (x(\tau_j'), u(\tau_j')) = (\bar{x}, \bar{u}) \in \partial \mathcal{C}\). Definition [1.1] can be easily adapted to applications where \(\mathcal{C}\) has to be interpreted as in \([7]\).

1.2. \textbf{Space-time controls and solutions.} For \(L > 0\) and \(0 < S \leq +\infty\), let \(\mathcal{U}_L(S)\) denote the subset of \(L\)-Lipschitz maps
\[ (\varphi_0, \varphi) : [0, S] \to \mathbb{R}^n \times U, \]
such that \(\varphi_0(0) = 0\), and \(\varphi_0'(s) \geq 0, \varphi_0'(s) + |\varphi'(s)| \leq L\) for almost every \(s \in [0, S]\); the apex \('\) denotes differentiation with respect to the \textit{pseudo-time} \(s\). Let \(\mathcal{M}(S)\) denote the set of measurable functions \(\psi : [0, S] \to V\).

Definition 1.2 (Space-time control and solution). \textit{We will call space-time controls the elements \((\varphi_0, \varphi, \psi, S)\), where \(0 < S \leq +\infty\) and \((\varphi_0, \varphi, \psi)\) belongs to the set \(\bigcup_{L \geq 0} \mathcal{U}_L(S) \times \mathcal{M}(S)\). Given \((\overline{\pi}_0, \overline{\pi}_0) \in \mathbb{R}^n \times U\) and a space-time control \((\varphi_0, \varphi, \psi, S)\) such that \(\varphi(0) = \overline{\pi}_0\), the space-time control system is defined by
\[ (10) \quad \begin{cases} \xi'(s) = g_0(\xi(s), \varphi(s), \psi(s))\varphi_0'(s) + \sum_{i=1}^m g_i(\xi(s), \varphi(s))\varphi_i'(s) & s \in [0, S], \\ \xi(0) = \overline{\pi}_0. \end{cases} \]
We will write \(\xi[\overline{\pi}_0, \overline{\pi}_0, \varphi_0, \varphi, \psi]\) to denote the solution of \((10)\).

Space-time controls and solutions can be seen as an \textit{extension} of regular, that is AC and AC_{loc}, controls and solutions. Indeed, if instead of a control pair \((u, v) \in \overline{AC_{loc}}(T) \times L^1(T)\) we consider any time-reparametrization \(t = \varphi_0(s)\) of its graph \((t, u(t), v(t))\), we obtain a space-time control \((\varphi_0, \varphi, \psi) := (\varphi_0, u \circ \varphi_0, v \circ \varphi_0)^\circ\) and the corresponding space-time solution \(\xi[\overline{\pi}_0, \overline{\pi}_0, \varphi_0, \varphi, \psi]\) is nothing but \(x[\overline{\pi}_0, \overline{\pi}_0, u, v] \circ \varphi_0\). On the other hand, space-time controls \((\varphi_0, \varphi, \psi)\) such that \((\varphi, \psi)\) evolves on the intervals where \(\varphi_0\) is constant, are

\footnote{Since every \(L^1\) equivalence class contains Borel measurable representatives, here and in the sequel we tacitly assume that the maps \(v\) and \(\psi\) are Borel measurable when necessary.}
more general objects than the graphs of a control \((u,v)\) with \(u\) in \(AC(T)\) or in \(AC_{loc}(T)\) (see Proposition 1.1 and Theorem 1.1).
In addition, the space-time system has a parameter-free character. Precisely, if \((\check{\varphi}_0,\varphi,\psi,S)\), \((\check{\varphi}_0,\check{\varphi},\check{\psi},\check{S})\) verify \((\varphi_0,\varphi,\psi) = (\check{\varphi}_0,\check{\varphi},\check{\psi}) \circ \check{s}\) for some reparametrization \(\check{s} : [0,S] \to [0,\check{S}]\), it can be shown that \(\xi = \xi \circ \check{s}\), if \(\xi\) and \(\check{\xi}\) denote the solutions to (10) corresponding to \((\varphi_0,\varphi,\psi)\) and \((\check{\varphi}_0,\check{\varphi},\check{\psi})\), respectively. For these reasons, we consider the following subset of space-time controls.

**Definition 1.3** (Feasible space-time controls). *We call feasible the space-time controls belonging to the subset*

\[
\Gamma(T;\overline{\mu}_0) := \{ (\varphi_0,\varphi,\psi,S) : 0 < S \leq +\infty, \ (\varphi_0,\varphi,\psi) \in U_1(S) \times M(S),
\]

\[
\varphi'_0(s) + |\varphi'(s)| = 1 \ a.e., \ \varphi(0) = \overline{\mu}_0, \ \lim_{s \to S} \varphi_0(s) = T \}.
\]

For any feasible space-time control \((\varphi_0,\varphi,\psi,S)\), the pseudo-time \(s\) coincides with the arc-length parameter of the curve \((\varphi_0,\varphi)\) (with respect to the norm \(\varphi'_0(s) + |\varphi'(s)|\)) and we have the identity

\[
s = \varphi_0(s) + Var_{[0,s]}[\varphi] \ \forall s \in [0,S].
\]

As a consequence, the final pseudo-time is \(S = T + Var_{[0,S]}[\varphi]\) and

\[
S = +\infty \ \text{if and only if} \ \ Var_{[0,S]}[\varphi] = +\infty.
\]

Let us introduce the following notion of feasible space-time trajectory-control pair extended to the closed set \([0,S]\), even in case \(S = +\infty\).

**Definition 1.4** (Feasible space-time trajectory-control pairs). *Let \((\varphi_0,\varphi,\psi,S) \in \Gamma(T;\overline{\mu}_0)\) be a feasible space-time control and set \(\xi := \xi[\overline{\mu}_0,\varphi_0,\varphi,\psi]\). If \(S < +\infty\), we extend \((\xi,\varphi,\psi)\) to \([0,S]\) by continuity. If \(S = +\infty\) and \(\xi\) is bounded, we introduce an extension of \((\xi,\varphi)\) to \([0,+,\infty]\), such that

\[
(\xi,\varphi)(+\infty) \in (\xi,\varphi)_{set}(+\infty) := \{ \lim_j (\xi,\varphi)(s_j) : \text{for some} \ s_j \nearrow +\infty \},
\]

and call \((\xi,\varphi,\psi,S)\) a (single-valued) feasible space-time trajectory-control pair on \([0,S]\).

The next results are easy consequences of the chain rule.

**Proposition 1.1.** (i) *Given \((u,v) \in \overline{AC}_{loc}(T) \times L^1(T)\) with \(u(0) = \overline{\mu}_0\), set \(x := x[\overline{\mu}_0,\overline{u}_0, u,v]\) and

\[
\sigma(t) := \int_0^t (1 + |\dot{u}(\tau)|) d\tau \ \forall t \in [0,T], \ S := \lim_{t \to T} \sigma(t) \ (\leq +\infty)
\]

\[
\varphi_0 := \sigma^{-1}, \ \varphi := u \circ \varphi_0, \ \psi := v \circ \varphi_0, \ \xi := \xi[\overline{\mu}_0,\varphi_0,\varphi,\psi] \ \text{in} \ [0,S].
\]

Then \((\xi,\varphi,\psi,S)\) is a feasible space-time trajectory, \((\xi,\varphi,\psi) \circ \sigma = (x,u,v)\) and, when \(u \in \overline{AC}_{loc}(T) \setminus AC(T)\) (so that \(S = +\infty\) and \(x\) is bounded,
(ξ, ϕ)_{\text{set}}(+\infty) = (x, u)_{\text{set}}(T). In particular, if (x, u)(T) = \lim_j (x, u)(\tau_j) for some \tau_j \nearrow T, we have \lim_j \sigma(\tau_j) = +\infty and we can set

\( (ξ, ϕ)(+\infty) := \lim_j (ξ, ϕ)(\sigma(\tau_j)) = (x, u)(T). \)

(ii) Vice-versa, given \((ϕ_0, ϕ, ψ, S) \in Γ(T; \overline{u}_0)\) with

\( ϕ'_0(s) > 0 \) for a.e. \( s \in [0, S[, \)

let us set \( ξ := ξ[\overline{u}_0, \overline{u}_0, ϕ_0, ϕ, ψ] \) and

\[(u, v) := (ϕ, ψ) ∘ ϕ^{-1}_0, \quad x := x[\overline{u}_0, \overline{u}_0, u, v].\]

Then \((x, u, v)\) is a trajectory-control pair of \([1, 2]\), \((x, u, v) \circ ϕ_0 = (ξ, ϕ, ψ)\), and, when \( S = +\infty \) and \( ξ \) is bounded, \((x, u)_{\text{set}}(T) = (ξ, ϕ)_{\text{set}}(+\infty)\). In particular, if \((ξ, ϕ)(+\infty) = \lim_j (ξ, ϕ)(s_j)\) along some \( s_j \nearrow +\infty \), we have \( \lim_j ϕ_0(s_j) = T \) and we can set

\[ (x, u)(T) := \lim_j (x, u)(ϕ_0(s_j)) = (ξ, ϕ)(+\infty). \]

Owing to Proposition \( [1, 4]\) we can identify any \( AC_{\text{loc}} \) trajectory-control pair with the associated feasible space-time trajectory-control pair:

**Definition 1.5** (Arc-length parametrization). We call arc-length graph-parametrization of an \( AC_{\text{loc}} \) trajectory-control pair \((x, u, v)\) the feasible space-time trajectory-control pair \((ξ, ϕ_0, ϕ, ψ, S)\) defined by \( [14]\).

Proposition \( [1, 4]\) also implies the following equivalence result.

**Theorem 1.1.** The set of \( AC\) [resp., \( AC_{\text{loc}}\ \backslash\ AC\)] trajectory-control pairs of \( [1, 2]\) is in one-to-one correspondence with the subset of feasible space-time trajectory-control pairs \((ξ, ϕ_0, ϕ, ψ, S)\) with \( S < +\infty \) [resp., \( S = +\infty\)] and \( ϕ'_0 > 0 \) a.e..

### 1.3. \( BV_{\text{loc}} \) graph completions.
Let us introduce the basic notions of the graph completion approach, which originally was dealing with inputs \( u \in BV(T) \) and that we now extend to controls \( u \in BV_{\text{loc}}(T) \), where

\[ BV_{\text{loc}}(T) := \{u : u : [0, T] \to U, \ u \in BV_{\text{loc}}(T)\}. \]

We refer to \( [BR]\) for the definition and some basic results on BV graph completions, to \( [MR]\) for BV graph completions with dependence on the ordinary control \( v \) and to \( [AR], [AMR]\) for the concept of clock.

**Definition 1.6** (Graph completion and clock). Let \((u, v) \in BV_{\text{loc}}(T) \times L^1(T)\) and \( u(0) = \overline{u}_0 \in U\). We say that a space-time control \((ϕ_0, ϕ, ψ, S) \in Γ(T; \overline{u}_0)\) with \( S \leq +\infty\), is a \( BV_{\text{loc}} \) graph completion of \((u, v)\) if

i) \( \forall t \in [0, T], \ \exists s \in [0, S[ \text{ such that } (ϕ_0, ϕ, ψ)(s) = (t, u(t), v(t))\);  

ii) when \( S < +\infty\), \((ϕ_0, ϕ)(S) = (T, u(T))\);  

iii) when \( S = +\infty\),

\[ \lim_j ϕ(s_j) = u(T) \text{ for some } s_j \nearrow +\infty. \]

In this case we will write, in short, \((ϕ_0, ϕ)(+\infty) = (T, u(T))\).
We call a clock any increasing function \( \sigma : [0, T] \to [0, S] \) such that
\[
(\varphi_0, \varphi)(\sigma(t)) = (t, u(t)) \quad \text{for every } t \in [0, T], \quad \sigma(0) = 0 \quad \text{and} \quad \sigma(T) = S \quad \text{(1)}
\]

If \((\varphi_0, \varphi, \psi, S)\) is a \( BV_{loc} \) graph completion of a control \((u, v) \in BV_{loc}(T) \times L^1(T)\), then \( \text{Var}_{[0,T]}[u] \leq \text{Var}_{[0,S]}[\varphi] \). Indeed, \((\varphi_0, \varphi)\) is a parametrization of a completion of \((t, u(t))\), where, roughly speaking, a discontinuity of \( u \) at \( t \) is bridged by an arbitrary continuous curve in \( \{t\} \times U \). Therefore, if \( S < +\infty \) the control \( u \) has necessarily bounded variation \( \text{Var}_{[0,T]}[u] \leq \text{Var}_{[0,S]}[\varphi] \), while when \( S = +\infty \), \( \text{Var}_{\mathbb{R}^n}[\varphi] = +\infty \) but the control \( u \) may belong either to \( BV_{loc}(T) \) or to \( BV(T) \).

**Definition 1.7** (Graph completion solution). Let \((\varphi_0, \varphi, \psi, S)\) be a \( BV_{loc} \) graph completion of \((u, v) \in BV_{loc}(T) \times L^1(T)\) with \( u(0) = \varpi_0 \), let \( \sigma \) be a clock and set \( \xi := \xi[\varpi_0, \varpi_0, \varphi_0, \varphi, \psi] \). When \( S = +\infty \), let us suppose that \( \xi \) is bounded.

We define a \( BV_{loc} \) graph completion solution to \((1) \cdot (2)\) associated to \((\varphi_0, \varphi, \psi, S)\) and \( \sigma \), a map
\[
x : [0, T] \to \mathbb{R}^n, \quad x(t) := \xi \circ \sigma(t) \quad \forall t \in [0, T],
\]
and
- i) if \( S < +\infty \), \( x(T) = \xi(S) \);
- ii) if \( S = +\infty \), \( (x(T), u(T)) \in (\xi, \varphi)_{set}(+\infty) \) (see \((13)\)).

Notice that graph completions allow for jumps of the trajectory even at the times \( t \) where \( u \) is continuous (a loop of \( u \) could be considered at these instants, which, owing to the non-triviality of the Lie algebra generated by \( \{e_1, g_1\}, \ldots, \{e_m, g_m\} \) might determine a discontinuity in \( x \)).

Just for regular controls \( u \in AC_{loc}(T) \) let us consider the following, more restrictive notion of graph completion, where essentially the variation added to \( u \) by introducing loops is finite; in other words, the difference between the variation of the completion of \((t, u(t))\) and that of \( u \) is finite. This notion will play an important role in Theorem \(2.1\).

**Definition 1.8** (Graph completion with BV loops). Given a \( BV_{loc} \) graph completion \((\varphi_0, \varphi, \psi, S)\) of a control \((u, v) \in AC_{loc}(T) \times L^1(T)\), we say that it is a graph completion with BV loops if either \( S < +\infty \) or \( S = +\infty \) and
\[
\text{Var}_{\mathbb{R}^n}[\varphi_{|\varphi_0'=0}] = \text{meas}\{s \in \mathbb{R}^+ : \varphi_0'(s) = 0\} < +\infty, \quad \text{(16)}
\]

For instance, the arc-length graph parametrization \((\varphi_0, \varphi, \psi, S)\), \( S \leq +\infty \) of \((u, v)\) with \( u \in AC_{loc}(T) \), is a graph completion with BV loops (actually, with no loops), since \( \varphi_0' > 0 \) a.e.. On the other hand, every graph completion \((\varphi_0, \varphi, \psi, S)\) with \( S = +\infty \) of a control \((u, v) \in AC(T) \times L^1(T)\) has not BV loops.

---

\(^2\)When \( S = +\infty \), the notation \( \sigma(T) = +\infty \) means just that \((\varphi_0, \varphi)(+\infty) = (T, u(T))\) in the sense of \((15)\), but it might be \( \lim_{t \to -T} \sigma(t) < +\infty \).
2. Existence and Consistency

This section is devoted to prove the existence of $\text{BV}_{\text{loc}}$ graph completion solutions (Theorem 2.1), and the consistency of such notion of solution with the extended $\text{AC}_{\text{loc}}$ solutions considered in Subsection 1.1 (Theorem 2.2).

**Theorem 2.1 (Existence).** Let $U$ have the Whitney property. Then for any $(u,v) \in \overline{\text{BV}}_{\text{loc}}(T) \times L^1(T)$, there exists a $\text{BV}_{\text{loc}}$ graph completion $(\varphi_0, \varphi, \psi, +\infty)$, and, for any clock $\sigma$, there is an associated $\text{BV}_{\text{loc}}$ graph completion solution $x$ to (1)–(2) on $[0,T]$.

The following result, whose proof is postponed in Section 5 is the key point for the existence of $\text{BV}_{\text{loc}}$ graph completions with unbounded variation.

**Lemma 2.1.** Let us assume that $U$ has the Whitney property. Then for any $u \in \text{BV}([a,b], U)$ and $\bar{u} \in U$, there exist $\bar{S} > 0$ and a 1-Lipschitz continuous map $(\varphi_0, \varphi) : [0, \bar{S}] \to [a,b] \times U$ such that:

(i) $\varphi_0$ is increasing, $\varphi'_0 + |\varphi'| = 1$ a.e., $(\varphi_0, \varphi)(0) = (a, u(a))$, for any $t \in [a,b]$ there is $s \in [0,\bar{S}]$ such that $(\varphi_0, \varphi)(s) = (t, u(t))$,

$$(\varphi_0, \varphi)(\bar{S}) = (b, u(b)),$$

and

$$\exists \bar{S} \leq S \text{ s.t. } (\varphi_0, \varphi)(S) = (b, \bar{u}).$$

Moreover, setting $V := \text{Var}_{[a,b]}(u)$, one has

$$(b - a) + V + |u(b) - \bar{u}| \leq S \leq \bar{S} \leq (b - a) + 2C(V + |u(b) - \bar{u}|),$$

where $C$ is as in (5);

(ii) $(\varphi_0, \varphi)$ admits a 1-Lipschitz continuous extension to $\mathbb{R}_+$ with $\varphi'_0 + |\varphi'| = 1$ a.e. and 

$$\lim_j (\varphi_0, \varphi)(s_j) = (b, \bar{u}),$$

along some increasing, diverging sequence $(s_j)_j$.

**Proof of Theorem 2.1.** Let $(\bar{t}_i)_i \subset [0,T]$ be a strictly increasing sequence converging to $T$, with $\bar{t}_0 = 0$. For every $i \geq 1$, let us set $I_i := [\bar{t}_{i-1}, \bar{t}_i]$, $|I_i| := \bar{t}_i - \bar{t}_{i-1}$ and $V_i := \text{Var}_{I_i}(u)$. Applying Lemma 2.1 to the restriction $u|_{I_i}$ with $\bar{u} = u(T)$, we can define

$$0 < S_i \leq \tilde{S}_i,$$

such that $\varphi_0_i$ is increasing, $\varphi'_0_i + |\varphi'_i| = 1$ a.e.,

$$(\varphi_0, \varphi_i)(0) = (\bar{t}_{i-1}, u(\bar{t}_{i-1})), \quad (\varphi_0, \varphi_i)(\tilde{S}_i) = (\bar{t}_i, u(\bar{t}_i)), \quad (\varphi_0, \varphi_i)(S_i) = (\bar{t}_i, u(T)),$$

and

$$|I_i| + V_i + |u(\bar{t}_i) - u(T)| \leq S_i \leq \tilde{S}_i \leq |I_i| + 2C(V_i + |u(\bar{t}_i) - u(T)|).$$

Let us set, for $i \geq 0$,

$$\tilde{S}_0 := 0, \quad \tilde{s}_i := \sum_{j=0}^{i+1} \tilde{S}_j, \quad s_{i+1} := \tilde{s}_i + S_{i+1}, \quad \tilde{s}_\infty = \lim_i \tilde{s}_i.$$
and let us introduce the space-time control
\[
(\varphi_0, \varphi, \psi)(s) := \sum_{i=1}^{+\infty} (\varphi_{0,i}, \varphi_i, v \circ \varphi_0)(s - \tilde{s}_{i-1}) \chi_{[\tilde{s}_{i-1}, \tilde{s}_i]}(s) \quad \forall s \in [0, \tilde{s}_\infty],
\]
which can be easily proved to be a $\text{BV}_{\text{loc}}$ graph completion of $(u, v)$. If $\tilde{s}_\infty = +\infty$, the proof of the theorem is concluded. In this case indeed, $(\varphi_0, \varphi, \psi)$ is defined on $\mathbb{R}_+$ and $\lim_i s_i = +\infty$ and
\[
\lim_i (\varphi_0, \varphi)(s_i) = (T, u(T)).
\]
Incidentally, by (17) this is always verified when $\text{Var}_{[0,T]}(u) = +\infty$. If instead $\text{Var}_{[0,T]}(u) < +\infty$ and $\tilde{s}_\infty < +\infty$, we can extend $(\varphi_0, \varphi, \psi)$ to a $\text{BV}_{\text{loc}}$ graph completion defined on $\mathbb{R}_+$ and satisfying (18) by Lemma 2.1 (ii).

**Theorem 2.2 (Consistency).** Let $(u, v) \in \overline{\text{AC}}_{\text{loc}}(T) \times L^1(T)$ with $u(0) = \overline{u}_0$. Let $x$ be a $\text{BV}_{\text{loc}}$ graph completion solution to (11)-(2) on $[0, T]$ belonging to $\overline{\text{AC}}_{\text{loc}}([0, T], \mathbb{R}^n)$. Then
\begin{enumerate}[(i)]  
  \item $x$ coincides with the Carathéodory solution $x_s[\overline{x}_0, \overline{u}_0, u, v]$ on $[0, T]$;  
  \item $x$ is an $\text{AC}_{\text{loc}}$ solution to (11)-(2) on $[0, T]$ if either $(x, u)$ is associated to a graph completion with BV loops or only if $(x, u)(T) \in (x, u)_{\text{set}}(T)$ (see Definition 1.7).
\end{enumerate}

Preliminarily, let us state the following uniform convergence result for space-time trajectory-control pairs on compact sets, proven in Section 5.

**Proposition 2.1.** Let $T > 0$, $(\overline{x}_0, \overline{u}_0) \in \mathbb{R}^n \times U$ and $(\varphi_0, \varphi, \psi, S) \in \mathcal{U}(T; \overline{u}_0)$ for some $S \leq +\infty$. Assume that there exist $\bar{T} \leq T$, $\bar{S} \leq S$ with $\bar{S} < +\infty$ and a sequence $(\varphi_{0,h}, \varphi_h, \psi, S)_h \subset \mathcal{U}(T; \overline{u}_0)$ such that, for every $h$, $\varphi_{0,h}$ is strictly increasing, $\varphi_{0,h}(\bar{S}) = \bar{T}$, and
\[
\sup_{s \in [0, \bar{S}]} |(\varphi_{0,h}, \varphi_h) - (\varphi_0, \varphi)| \to 0 \quad \text{as} \quad h \to +\infty.
\]
Let $\sigma : [0, \bar{T}] \to [0, \bar{S}]$ be any increasing function such that $\varphi_0 \circ \sigma(t) = t$ for every $t \in [0, \bar{T}]$, $\sigma(0) = 0$ and $\sigma(\bar{T}) = \bar{S}$. Then, setting $v := \psi \circ \sigma$, we have
\[
\xi := \xi[\overline{x}_0, \overline{u}_0, \varphi_0, v, \sigma] \equiv \xi[\overline{x}_0, \overline{u}_0, \varphi_0, \varphi, v \circ \varphi_0]
\]
and, if $\xi_h := \xi[\overline{x}_0, \overline{u}_0, \varphi_{0,h}, \varphi_h, v \circ \varphi_{0,h}]$, there exists a subsequence (still denoted by $(\xi_{h})_h$) such that
\[
\sup_{s \in [0, \bar{S}]} |\xi_h(s) - \xi(s)| \to 0 \quad \text{as} \quad h \to +\infty.
\]

**Proof of Theorem 2.2.** Given $(u, v) \in \overline{\text{AC}}_{\text{loc}}(T) \times L^1(T)$ with $u(0) = \overline{u}_0$ and an associated $\text{BV}_{\text{loc}}$ graph completion solution $x$ with $x \in \overline{\text{AC}}_{\text{loc}}(T)$, let $(\varphi_0, \varphi, \psi, S)$ and $\sigma$, be a $\text{BV}_{\text{loc}}$ graph completion of $(u, v)$ and a clock, respectively, such that $x = \xi \circ \sigma$, where $\xi$ is the space-time trajectory of

\[
\frac{\partial x}{\partial s} = \nabla \phi + v(x), 
\]

where $\phi$ is a smooth function.
corresponding to \((\varphi_0, \varphi, \psi, S)\), extended to \([0, S]\) as in Definition 1.4 (see also Definitions 1.6 and 1.7).

The proof of part (i) is a generalization, with some simplifications, of the proof of [AR, Theorem 2.2], dealing with BV inputs and trajectories. For every \(s \in [0, S]\), let us consider the space-time control \((\varphi_0, u \circ \varphi_0, v \circ \varphi_0, S)\) and the associated solution \(\xi\) of (10). Notice that \(\dot{x} := \xi \circ \sigma\) coincides with the usual Carathéodory solution \(x[\varphi_0, \varpi_0, u, v]\) of (11)-(2) on \([0, T]\). Indeed, for every \(t < T\),

\[
\dot{x}(t) = \xi(\sigma(t)) = \varpi_0 + \int_0^{\sigma(t)} [g_0(\xi, u \circ \varphi_0, v \circ \varphi_0) \varphi'_0(s) + \sum_{i=1}^m g_i(\xi, u \circ \varphi_0)(u_i \circ \varphi_0)'(s)] ds,
\]

by the change of variables \(t = \varphi_0(s)\), we get

\[
\dot{x}(t) = \varpi_0 + \int_0^t [g_0(\dot{x}, u, v) + \sum_{i=1}^m g_i(\dot{x}, u) u'_i(t)] dt,
\]

and Gronwall’s Lemma easily implies that \(\dot{x}(t) = x[\varpi_0, \varpi_0, u, v](t)\).

Let us now show that \(x = \dot{x}\) on \([0, T]\). By (19) in Proposition 2.1, it is not restrictive to assume that in \((\varphi_0, \varphi, \psi, S)\), \(\psi = v \circ \varphi_0\). Let \(\mathcal{T} \subset [0, T]\) be the (countable) set of discontinuity points of \(\sigma\). Let us assume that \(\mathcal{T}\) is an infinite set, the proof for \(\mathcal{T}\) finite being similar, and actually simpler. For every \(\tau_j \in \mathcal{T}\), set \(s_{1,j} := \lim_{\tau_j^- \to \tau_j} \sigma(\tau)\) and \(s_{2,j} := \lim_{\tau_j \to \tau_j^+} \sigma(\tau)\). Clearly, \(s_{1,j} < s_{2,j} < S\). Since \(x\) and \(u\) (as \(\xi\) and \(\varphi\)) are continuous functions, by the definition of graph completion solution it follows that

\[
(\xi, \varphi)(s_{1,j}) = (x, u)(\tau_j) = (\xi, \varphi)(s_{2,j}) \quad \text{for every } j.
\]

Let us set \(\varphi_1 := u \circ \varphi_0\) on \([s_{1,1}, s_{2,1}]\) and \(\varphi_1 := \varphi\) otherwise and, for every \(j \geq 1\), let us define \(\varphi_{j+1} := u \circ \varphi_0\) in \([s_{1,j+1}, s_{2,j+1}]\) and \(\varphi_{j+1} := \varphi_j\) otherwise. Let us consider the space-time control \((\varphi_0, \varphi_j, v \circ \varphi_0, S)\) and let \(\xi_j\) denote the associated solution of (10).

For \(j = 1\), \(\xi_1(s) = \xi(s)\) for every \(s \in [0, s_{1,1}]\) by definition, moreover, \(\xi_1(s_{1,2}) = \xi_1(s_{1,1})\) since \((\varphi'_0, \varphi'_1) = (0, 0)\) a.e. on \([s_{1,1}, s_{2,1}]\), so that

\[
\xi(s_{2,1}) = \xi(s_{1,1}) = \xi_1(s_{1,1}) = \xi_1(s_{1,1}) = \xi_1(s_{1,1}).
\]

At this point, \(\xi(s) = \xi_1(s)\) also for \(s > s_{2,1}\), since \(\xi\) and \(\xi_1\) solve on \([s_{2,1}, S]\) the same ODE with the same initial condition. Thus the graph completion solution \(x_1 := \xi_1 \circ \sigma\) coincides with the function \(x\) on \([0, T]\). Given \(j \geq 1\), let us assume that \(x_j = x\) on \([0, T]\). Then by the same arguments it follows that \(x_{j+1} = x_j = x\) on \([0, T]\), and, by induction, this proves that \(x_j = x\) on \([0, T]\) for every \(j\).

For any \(t < T\), let \(\mathcal{T}'\) be the subset of discontinuity points of \(\sigma\) contained on \([0, t]\) and set \(\hat{S} := \sigma(t^+) \subset (S, S)\). By definition, \((\varphi_j)j\) pointwisely converges to \(u \circ \varphi_0\). In order to prove that the sequence \((\varphi_j)j\) converges uniformly in \([0, \hat{S}]\) (to \(u \circ \varphi_0\)), let us define, for every \(j\), \(\bar{\varphi}_j := \varphi_j\) on \([0, \hat{S}]\) and
\[ \tilde{\varphi}_j := \varphi \text{ on } ]S_1, S[. \text{ Then, for every } k \text{ and } j \text{ with } k > j, \]
\[ \sup_{s \in [0,S]} |\tilde{\varphi}_k(s) - \tilde{\varphi}_j(s)| = \sup_{s \in [0,S]} |\tilde{\varphi}_k(s) - \tilde{\varphi}_j(s)| \leq \]
\[ \sum_{i=j}^{k-1} \int_{s_{1,i+1}}^{s_{2,i+1}} |\tilde{\varphi}_s'(s)| \, ds \leq \sum_{i=j}^{k-1} \sum_{\tau_{i+1} \in T'} (s_{2,i+1} - s_{1,i+1}), \]
where the last expression tends to zero as \( j \to +\infty \) since
\[ \sum_{i=1}^{+\infty} (s_{2,i} - s_{1,i}) \leq \tilde{S} < +\infty. \]

Hence, in view of Proposition 2.1, \((\xi_j)_j\) converges uniformly to \( \hat{\xi} \) on \([0, \tilde{S}]\) and we get
\[ x(\tau) = \lim_{j \to +\infty} x_j(\tau) = \lim_{j \to +\infty} \xi_j(\sigma(\tau)) = \hat{\xi}(\sigma(\tau)) = \hat{x}(\tau) \quad \forall \tau \in [0, T]. \]

By the arbitrariness of \( t < T \), this implies (i), namely the equality \( x = \hat{x} \) on \([0, T][\).

If \( S < +\infty \) statement (ii) holds true, since \( x = \hat{x} \) holds on \([0, T]\). When \( S = +\infty \), by definition, (ii) is verified if and only if \((x, u)(T) \in (x, u)_{set}(T)\), being \((x, u)_{set}(T) = (\hat{x}, u)_{set}(T)\) in view of (i). To conclude the proof it remains to show that, if \((\varphi_0, \varphi, \psi, S)\) is a graph completion with BV loops of \((u, v)\) with \( S = +\infty \), then \((x, u)(T) \in (x, u)_{set}(T)\). By (16) it follows that
\[ \text{(21)} \quad \text{Var}_{\mathbb{R}^+}(\varphi|\varphi_0=0) = \sum_{j=1}^{+\infty} (s_{2,j} - s_{1,j}) < +\infty. \]

Let \((s_i)_i\) be an increasing, diverging sequence such that \( \lim_{i}(\xi, \varphi)(s_i) = (x, u)(T)\), existing in view of Definition 1.1. For every \( i \), set \( t_i := \varphi_0(s_i) \). If there is some subsequence of \((s_i)_i\), which we still denote by \((s_i)_i\), such that every \( t_i \) does not belong to \( T \), we have \( t_i \not\in T \) and we get
\[ \text{(22)} \quad \lim_{i} (x, u)(t_i) = \lim_{i} (\xi, \varphi)(s_i) = (x, u)(T). \]

By Definition 1.1 this implies that \((x, u)(T) \in (x, u)_{set}(T)\). Otherwise, possibly disregarding a finite number of terms, we can suppose that \((t)_i \subset T\).

In this case, \( \varphi_0 \) is constant on an interval where \((\xi, \varphi)\) describes a loop. Precisely, if \( t_i \) coincides with the element \( \tau_j \in T \),
\[ \varphi_0(s) = t_i \text{ for all } s \in [s_{1,j}, s_{2,j}], \quad (\xi, \varphi)(s_{2,j}) = (\xi, \varphi)(s_{1,j}) = (x, u)(t_i). \]

By the last equality, if there is some subsequence of \((s_i)_i\) such that every \( s_i \) coincides with either \( s_{1,j} \) or \( s_{2,j} \) for some \( j \), we get (22) and we can conclude as above. If instead, possibly disregarding a finite number of terms, \( s_i \in ]s_{1,j}, s_{2,j}[ \) for every \( i \), recalling that \((\xi, \varphi)\) is bounded, we obtain by standard estimates that \((\xi, \varphi)\) is Lipschitz continuous, so that
\[ |(\xi, \varphi)(s_i) - (x, u)(t_i)| \leq \sup_{s \in [s_{1,j}, s_{2,j}]} |(\xi, \varphi)(s) - (\xi, \varphi)(s_{1,j})| \leq C(s_{2,j} - s_{1,j}), \]

for some $C > 0$. At this point, by \(21\) it easily follows that \(22\) still holds and the proof of (iii) is concluded. □

3. BV\(_{loc}\) simple limit solutions

Let us begin recalling the notion of simple and of BV simple limit solution, given in [AR] for vector fields $g_1, \ldots, g_m$ depending on $x$ only and extended to $(x, u)$-dependent data in [AMR]. We use $\mathcal{L}(T) := L([0, T], U)$ to denote the set of pointwisely defined, Lebesgue integrable inputs.

**Definition 3.1** (S and BVS limit solution). Let $(u, v) \in \mathcal{L}(T) \times L^1(T)$ with $u(0) = \pi_0$. A map $x$ is called a simple limit solution, shortly $S$ limit solution, of \(1\) and \(2\), if there exists a sequence of controls $(u_k)_k \subseteq AC(T)$ with $u_k(0) = \pi_0$, pointwisely converging to $u$ and such that

(i) the sequence $(x_k)_k$ of the Carathéodory solutions to \(1\) and \(2\) corresponding to $(u_k, v)$ is equibounded on $[0, T]$;

(ii) for any $t \in [0, T]$, $\lim_k x_k(t) = x(t)$.

We say that an $S$ limit solution $x$ is a BV simple limit solution, shortly a BVS limit solution, of \(1\) and \(2\) if the approximating inputs $u_k$ have equibounded variation.

Let us introduce the new definition of $BV_{loc}S$ limit solution.

**Definition 3.2** ($BV_{loc}S$ limit solution). Let $(u, v) \in \mathcal{L}(T) \times L^1(T)$ with $u(0) = \pi_0$. We say that an $S$ limit solution $x$ is a $BV_{loc}S$ simple limit solution, shortly a $BV_{loc}S$ limit solution, of \(1\) and \(2\):

(i) on $[0, T]$, if there exist a sequence of controls $(u_k)_k$ as in the definition of $S$ limit solution, such that for any $t \in [0, T]$ the approximating inputs $u_k$ have equibounded variation on $[0, t]$;

(ii) on $[0, T]$, if, moreover, $x$ is bounded and there exist a positive, decreasing map $\tilde{\varepsilon}$ with $\lim_{s \to +\infty} \tilde{\varepsilon}(s) = 0$ and two strictly increasing, diverging sequences $(\tilde{s}_j)_j \subseteq \mathbb{R}_+$, $(k_j)_j \subseteq \mathbb{N}$, $k_j \geq j$, such that, for every $k > k_j$:

\[
\exists \tau_k^j < T : \; \tau_k^j + \text{Var}_{[0, \tau_k^j]}(u_k) = \tilde{s}_j, \quad |(x_k, u_k)(\tau_k^j) - (x_k, u_k)(T)| \leq \tilde{\varepsilon}(j).
\]

**Remark 3.1.** By Definition 3.1 it follows that, if $x$ is a BVS limit solution associated to $(u, v)$, then $u \in BV(T)$. Analogously, when $x$ is a $BV_{loc}S$ limit solution corresponding to $(u, v)$, Definition 3.2 implies that $u \in BV_{loc}(T)$.

**Remark 3.2.** The $S$, BVS, and $BV_{loc}S$ limit solution associated to a control $(u, v)$ is not unique, unless the system is commutative. Clearly, any BVS limit solution is a $BV_{loc}S$ limit solution, which is an $S$ limit solution, so that the sets of $S$, $BV_{loc}S$ and BVS limit solutions form a decreasing sequence of sets.

\(^3\)In [AR], [AMR] also more general, not necessarily simple, limit solutions have been defined.
Remark 3.3. Following [AR], in the above definition the approximating regular trajectories \( x_k = x[\overline{x}_0, \overline{u}_0, u_k, v] \) are obtained keeping the ordinary control \( v \) fixed. This is in fact equivalent to consider approximating solution \( x_k = x[\overline{x}_0, \overline{u}_0, u_k, v_k] \), where \( v_k \to v \) in \( L^1 \)-norm (see [MS3]).

Remark 3.4. As we will see in Theorem 4.1, condition (23) guarantees that a \( BV_{loc}S \) limit solution \( x \) is a \( BV_{loc} \) graph completion solution on \([0, T]\), not only on \([0, T]\[. Actually, we will prove that any \( x \) verifying part (i) of Definition 3.2 turns out to be a \( BV_{loc} \) graph completion solution on \([0, T]\[.

Condition (23) is more meaningful once we read it as an hypothesis on the the graphs of the approximating sequence \( (x_k, u_k) \). Precisely, for any trajectory-control pair \( (x_k, u_k, v) \) as in Definition 3.2, let \((\xi, \varphi_0_k, \varphi_k, v \circ \varphi_0_k, S_k)\) be its arc-length graph parametrization (see Definition 1.5). Then (23) is equivalent to:

\[
\text{the existence of a positive, decreasing map } \tilde{\varepsilon} \text{ with } \lim_{s \to +\infty} \tilde{\varepsilon}(s) = 0 \text{ and of two strictly increasing, diverging sequences } (\tilde{s}_j)_j \subset \mathbb{R}_+ \text{ and } (k_j)_j \subset \mathbb{N}, \ k_j \geq j, \text{ such that, for every } k > k_j:}
\]

\[
|((\xi_k, \varphi_k)(\tilde{s}_j) - (\xi_k, \varphi_k)(S_k))| \leq \tilde{\varepsilon}(j).
\]

Clearly, (24) holds true when the sequence \((\xi_k, \varphi_k)_k\) is uniformly convergent on \(\mathbb{R}_+\) (by considering, for every \(k\), the extension \((\xi_k, \varphi_k)(s) := (\xi_k, \varphi_k)(S_k)\) for every \(s \geq S_k\)).

As an immediate consequence of Theorems 2.1 and 4.1 we have the following existence result for \(BV_{loc}S\) limit solutions.

Corollary 3.1. If \( U \) has the Whitney property, then for any \((u, v) \in BV_{loc}(T) \times L^1(T)\) with \( u(0) = \overline{u}_0 \) there exists an associated \( BV_{loc}S \) limit solution \( x \) to (1) - (2) on \([0, T]\) (on \([0, T]\[, when \( x \) is bounded).

As a by-product, we get that every function \( u \in BV_{loc}(T) \) is the pointwise limit on \([0, T]\) of a sequence \((u_k) \subset AC(T)\) with equibounded variation on every interval \([0, t]\) with \( t < T \) and verifying (23).

Let us conclude this section with an example, illustrating the relations between the notions of \( AC_{loc}\) solutions, \( BV_{loc} \) graph completion solutions and of \( BV_{loc}S \) limit solutions considered in Definitions 1.1 1.7 and 3.2 above.

Example 3.1. Let us consider the control system in \( \mathbb{R}^3 \)

\[
\dot{x} = g_1(x) \dot{u}_1 + g_2(x) \dot{u}_2, \quad |u| \leq 1
\]

with \( u \in \mathbb{R}^2 \) and initial conditions

\[
x(0) = (1, 0, 1), \quad u(0) = (1, 0),
\]

where

\[
g_1(x) := \eta(x) \begin{pmatrix} 1 \\ 0 \\ x_3 x_2 \end{pmatrix}, \quad g_2(x) := \eta(x) \begin{pmatrix} 0 \\ 1 \\ -x_3 x_1 \end{pmatrix},
\]
and $\eta$ is a Lipschitz continuous function equal to 1 as $|x| \leq 3$ and equal to 0 as $|x| \geq 4$.

(1) For any control $u \in AC(T)$ verifying $u(0) = (1, 0)$, the corresponding Carathéodory solution to
\begin{equation}
(x_1, x_2, x_3)(t) = \left( u_1(t), u_2(t), e^{-\int_0^t (-u_2 u_1 + u_1^2)(s) \, ds} \right) \quad \forall t \in [0, T].
\end{equation}
In particular, $x_3(T) \geq e^{-\text{Var}_{[0,T]}(u)} > 0$.

Hence, if given a control $u \in BV(T)$ we consider just BVS limit solutions to (25), (26), that is, pointwise limits of Carathéodory solutions corresponding to approximating inputs $u_k$ with **equibounded variation** (see Definition 3.1), we always obtain $x_3(T) > 0$. Similarly, if, we introduce graph completions $(\varphi_0, \varphi) : [0, S] \to [0, T] \times U$ of $u$, with $S$ (and thus $\text{Var}_{[0,S]}(\varphi)$) finite, for any clock $\sigma$ we get a graph completion solution with $x_3(T) > 0$ (see Definitions 1.6, 1.7). Precisely, the space-time system is
\begin{equation}
(\xi_1, \xi_2, \xi_3)' = g_k(\xi) \varphi_1' + g_2(\xi) \varphi_2', \quad \xi(0) = (1, 0, 1), \quad \varphi(0) = (1, 0),
\end{equation}
where $\varphi_0(0) = 0, \varphi_0(S) = T, \varphi_0' \geq 0$ and $\varphi_0' + |\varphi'| = 1$ a.e. on $[0, S]$, so that
\begin{equation}
\xi_3(s) = e^{-\int_0^s (-\varphi_2 \varphi_1' + \varphi_1 \varphi_2')(s) \, ds} \quad \text{for} \quad s \in [0, S],
\end{equation}
and $|\int_0^s (-\varphi_2 \varphi_1' + \varphi_1 \varphi_2')(s) \, ds| \leq S$. Thus the graph completion solution, defined by $x = \xi \circ \sigma$, verifies $x_3(T) = \xi_3(S) \geq e^{-S} > 0$.

Let us now consider inputs $u \in AC_{loc}(T)$. In this case, if we set, for instance,
\begin{equation}
u(t):= \left( \cos \left( \frac{1}{T-t} - \frac{1}{T} \right), \sin \left( \frac{1}{T-t} - \frac{1}{T} \right) \right) \quad \text{for} \quad t \in [0, T], \quad u(T) = (1, 0),
\end{equation}
the corresponding solution to (25), (26) on $[0, T]$ has the third component
\begin{equation}
x_3(t) = e^{-\frac{t}{T-t}}, \quad \text{so that the extension} \quad (x_1, x_2, x_3)(T) \equiv (u_1, u_2, x_3)(T) := (1, 0, 0)
\end{equation}
gives a feasible $AC_{loc}$ trajectory-control pair (see Definition 3.1). In fact, such an extended map $x$ is also a $BV_{loc}$-limit solution (see Definition 3.2). Indeed, for every $k$, let us set
\begin{equation}
t_k := \frac{2k\pi T^2}{1 + 2k\pi T}, \quad u_k(t) := u(t)\chi_{[0,t_k]}(t) + u(t_k)\chi_{[t_k,T]}(t),
\end{equation}
where $u$ is as in (29), so that $u(t_k) = (\cos(2k\pi), \sin(2k\pi)) = (1, 0)$. Then $x$ is the pointwise limit of the Carathéodory solutions $x_k$ of (25), (26) corresponding to the controls $u_k \in AC(T)$, with $\text{Var}_{[0,t]}(u_k) \leq \frac{t}{T(T-t)} \quad \forall t \in [0, T]$ and $(x_k, u_k)(T) = (x_k, u_k)(t_k)$, so that easy calculations yield all the remaining conditions of Definition 3.2 below. In particular (23) is verified if we choose $\tilde{s}_j := t_j + \text{Var}_{[0,t_j]}(u)$, where $t_j = \frac{2j\pi T^2}{1 + 2j\pi T}$, and $k_j := j$, so that if we set
\footnote{The multiplication by the cut-off function $\eta$, while unneeded, is sufficient to guarantee the sublinearity hypothesis on the dynamics.}
\( \tau_k^j := t_j, \) we get \( \tau_k^j + Var_{[0, \tau_k^j]}(u_k) = t_j + Var_{[0, t_j]}(u) = \tilde{s}_j \) and, for every \( k \geq j, \) we have

\[ |(x_k, u_k)(t_j) - (x_k, u_k)(T)| = |(x, u)(t_j) - (x, u)(t_k)| = |x(t_j) - x(t_k)| \leq e^{-\frac{\gamma}{T(t_j - t_k)}}, \]

where the last term, independent of \( k, \) tends to zero as \( j \to \infty. \)

(ii) For \((x, u)\) solution of (25)-(26), let us consider the problem of minimizing the following payoff

\[ J(u) := \int_0^T \left[ |1 - u_1(t)| + |u_2(t)| + |x_3(t)||\dot{u}(t)| \right] dt \]

subject to the constraints

\[(x, u)(T) \in C := (U \times \{0\}) \times U.\]

By (i), no AC trajectory-control pairs \((x, u)\) verifying the constraints exist, hence \( \inf_{u \in AC(T)} J(u) = +\infty. \) In the extended class of \( AC_{loc} \) trajectory-control pairs, as observed in Remark 1.1 the terminal constraint is equivalent to assume that

\[ \liminf_{t \to T^-} d((x(t), u(t)), C) = 0. \]

Hence, for every \( k, \) implementing the control

\[ u_k(t) := (1, 0)\chi_{[0, T-(1/k)]} + \left( \cos \left( \frac{1}{T-t} - k \right), \sin \left( \frac{1}{T-t} - k \right) \right) \chi_{[T-(1/k), T]} \]

we get the solution

\[ x_k(t) = (1, 0, 1)\chi_{[0, T-(1/k)]} + (u_{1k}(t), u_{2k}(t), e^{k - \frac{1}{T-t}}) \chi_{[T-(1/k), T]}, \]

with \((x_k, u_k)\) verifying the constraints and \( 1 \leq J(u_k) \leq 1 + \frac{3}{k}, \) so that \( \lim_k J(u_k) = 1. \) In fact, it is not difficult to prove that 1 is the infimum (not the minimum) cost in the class of \( AC_{loc} \) controls. The minimum does exist, and is equal to 1, over the set of \( BV_{loc} \) graph completions: it suffices to consider the space-time control

(31) \( (\varphi_0, \varphi)(s) := (s, 1, 0)\chi_{[0, T]}(s) + (T, (\cos(s - T), \sin(s - T)))\chi_{[T, +\infty]}(s) \)

and the corresponding trajectory

(32) \( \xi(s) = (1, 0, 1)\chi_{[0, T]}(s) + (\cos(s - T), \sin(s - T), e^{-s+T})\chi_{[T, +\infty]}(s). \)

Notice that, by adding to the system the variable

(33) \( \dot{x}_4 = |1 - u_1(t)| + |u_2(t)| + |x_3(t)||\dot{u}(t)|, \quad x_4(0) = 0 \)

in the space-time setting we can consider the extended payoff

\( \mathcal{J}(\varphi_0, \varphi, S) := \int_0^S \left[ (1 - \varphi_1(s)) + |\varphi_2(s)|\right] \varphi_0'(s) + |\varphi_3(s)||\varphi'(s)| \right] ds, \)

where \( S \leq +\infty \) and \( \lim_{s \to S} \varphi_0(s) = T. \) Hence by (31), (32), we get \( \mathcal{J}(\varphi_0, \varphi, +\infty) = 1. \) Finally, in the class of \( S \) limit solutions, where the
optimization problem is equivalent to minimize $x_4(T)$, the minimum cost is still equal to 1. In particular, for every sequence $(x_k, u_k)_k$ of equibounded, absolutely continuous maps defining an $S$ limit solution verifying the terminal constraint, one has $\lim_k \text{Var}_{[0,T]}(u_k) = +\infty$ and

$$x_{4k}(T) = J(u_k) \geq \int_0^T e^{-\int_0^t |u_k| dr} |\dot{u}_k| dt = 1 - e^{-\text{Var}_{[0,T]}(u_k)} \to 1 \quad \text{as } k \to +\infty.$$  

Actually, in view of Theorem 4.1 below, the minimum value is obtained in the subset of $BV_{loc}$ limit solutions (see Definition 3.2).

4. Well posedness and characterization

Our main result is the following equivalence between $BV_{loc}$ graph completion solutions and $BV_{loc}$ limit solutions.

**Theorem 4.1.** Let us assume that $U$ has the Whitney property. Let $(u, v) \in BV_{loc}(T) \times L^1(T)$ with $u(0) = u_0$. Then

(i) (Well posedness) a $BV_{loc}$ graph completion solution $x$ to \((1)-(2)\) is a $BV_{loc}$ $S$ limit solution;

(ii) (Characterization) Any $BV_{loc}$ $S$ limit solution $x$ to \((1)-(2)\) is a $BV_{loc}$ graph completion solution.

Theorem 4.1 says that any $BV_{loc}$ graph completion solution is an $S$ limit solution. Vice-versa, given an $S$ limit solution $x$, it is a $BV_{loc}$ graph completion solution if and only if there exists an approximating sequence verifying condition \((23)\). Precisely, $x$ is always a $BV_{loc}$ graph completion solution on $[0,T[$; \((23)\) is needed to guarantee the existence of a $BV_{loc}$ graph completion solution assuming the final value $x(T)$.

In order to prove that a $BV_{loc}$ graph completion solution is a $BV_{loc}$ $S$ limit solution, in Theorem 4.2 below we extend to possibly unbounded maps the crucial approximation result of [AR, Theorem 5.1]. The proof is postponed to Section 5.

**Theorem 4.2.** Let $\sigma : [0,T[ \to \mathbb{R}_+$ be a strictly increasing map such that $\sigma(0) = 0$ and $\sigma(t_2) - \sigma(t_1) > (t_2 - t_1) \quad \forall t_1, t_2 \in [0,T[, \ t_1 < t_2$. Set

$$\lim_{t \to T^-} \sigma(t) =: \bar{S} \quad (\bar{S} \leq +\infty).$$

Let $\varphi_0 : \mathbb{R}_+ \to [0,T[$ be the unique, $(1$-Lipschitz continuous) increasing, surjective map verifying

$$\varphi_0 \circ \sigma(t) = t \quad \forall t \in [0,T[, \quad \text{and, if } \bar{S} < +\infty, \quad \varphi_0(s) := T \quad \forall s \geq \bar{S}.$$  

Then there exists a sequence of absolutely continuous, strictly increasing maps $\sigma_h : [0,T[ \to \mathbb{R}_+$ such that

(i) $\sigma_h(0) = 0$, $\lim_{t \to T} \sigma_h(t) = +\infty$, and

(34) $$\lim_{h} \sigma_h(t) = \sigma(t) \quad \forall t \in [0,T[;$$
(ii) the maps \( \varphi_{0,h} := \sigma_h^{-1} : \mathbb{R}_+ \to [0,T] \) are strictly increasing, 1-Lipschitz continuous, surjective, converge locally uniformly to \( \varphi_0 \) and verify, for every \( t \in [0,T] \),

(ii.1) if \( \bar{S} < +\infty \), setting \( \varepsilon(h) := \sup_{s \in [0,S]} |\varphi_{0,h}(s) - \varphi_0(s)| \):

\[
\sup_{s \in \mathbb{R}_+} |\varphi_{0,h}(s) - \varphi_0(s)| \leq \varepsilon(h) + (T - t_h) \quad (t_h := \varphi_{0,h}(\bar{S}))
\]

and \( t_h < T \) for every \( h \), \( \lim_h t_h = T \), and \( \lim_h \varepsilon(h) = 0 \);

(ii.2) if \( \bar{S} = +\infty \): for every \( S \in ]0, +\infty[ \), setting \( s := \varphi_0(S) \) and \( t_h := \varphi_{0,h}(S) \),

\[
\sup_{s \in \mathbb{R}_+} |\varphi_{0,h}(s) - \varphi_0(s)| \leq \varepsilon_S(h) + (T - \min\{t, t_h\}),
\]

where \( \varepsilon_S(h) := \sup_{s \in [0,S]} |\varphi_{0,h}(s) - \varphi_0(s)| \), \( t, t_h < T \) for every \( h \),

\[
|t_h - t| \leq \varepsilon_S(h) \text{ and } \lim_h \varepsilon_S(h) = 0.
\]

Let us point out that, even in case \( \sigma([0,T]) \) is bounded, we introduce approximating maps \( \sigma_h \) from \([0,T[ \to \mathbb{R}_+\). This is a substantial difference from [AR] Theorem 5.1, where \( \sigma \in L^1([a,b],[0,1]) \) and every approximation \( \sigma_h \) maps \([a,b]\) onto \([0,1]\).

4.1. Proof of Theorem 4.1: Well posedness. Let us begin by showing that a BV\(_{\text{loc}}\) graph completion solution is a BV\(_{\text{loc}}\) limit solution. We limit ourselves to consider just BV\(_{\text{loc}}\) graph completions which are not BV, since this last case was already covered by [AR] Theorem 4.2. Let \( x \) be a BV\(_{\text{loc}}\) graph completion solution to (1)-(2), which, by Definitions 1.4 and 1.7, is associated to a feasible space-time trajectory-control pair \( \varphi_0, \varphi, \psi, +\infty \) \( \in \Gamma(T; \overline{u}_0), \xi := \xi(\overline{u}_0, \varphi_0, \varphi, \psi) \) with \( \xi \) bounded, and to a strictly increasing function \( \sigma : [0,T[ \to \mathbb{R}_+ \), such that:

\[
(\xi, \varphi_0, \varphi, \psi) \circ \sigma(t) = (x(t), t, u(t), v(t)) \quad \forall t \in [0,T],
\]

\[
\lim_j (\xi(s_j), \varphi(s_j)) = (x(T), u(T)) \quad \text{for some } s_j \nearrow +\infty.
\]

Let

\[
\bar{S} := \inf\{s > 0 : \varphi_0(s) = T\}.
\]

We consider separately the two cases \( \bar{S} = +\infty \) and \( \bar{S} < +\infty \), since they require a different construction of the equibounded, approximating sequence \( (x_k, u_k)_k \) of \( (x,u) \). Precisely we will prove the following

**Claim.** There exists a sequence \( (u_k)_k \subset AC(T), u_k(0) = \overline{u}_0 \), and \( x_k := x(x_0, \overline{u}_0, u_k, v) \) verifying these properties:

(i) for every \( t \in [0,T] \),

\[
|(x_k, u_k)(t) - (x, u)(t)| + \|(x_k, u_k) - (x, u)\|_{L^1(T)} \to 0 \quad \text{as } k \to +\infty,
\]

\[
(35)
\]

\[
(36)
\]
(ii) there exists an increasing function $V : [0, T] \to \mathbb{R}_+$ with $V(0) = 0$ and $\lim_{t \to T} V(t) = +\infty$, such that, for every $k$,

$$(38) \quad Var_{[0,t]}(u_k) \leq V(t) \quad \text{for every } t \in [0,T];$$

(iii) in correspondence to the sequence $(s_j)_j$ in (35), there exist a positive, decreasing sequence $\varepsilon$ with $\lim_j \varepsilon(j) = 0$ and a strictly increasing sequence $(k_j)_j$ $(k_j \geq j)$ such that, defining implicitly $\tau^j_k$ by

$$\tau^j_k + Var_{[0,\tau^j_k]}(u_k) = s_j,$$

one has

$$|(x_k, u_k)(\tau^j_k) - (x_k, u_k)(T)| \leq \varepsilon(j),$$

so that $(x, u, v)$ is a BV$_{loc}$ limit solution on $[0, T]$.

In both cases, as a first step, using Theorem 4.2 we define a sequence of strictly increasing, Lipschitzian maps $\varphi_{0h}$ approaching locally uniformly $\varphi_0$ as $h \to \infty$ and consider the trajectory-control pairs $(\xi, \varphi) \circ (\varphi_{0h})^{-1}$. Furthermore, we obtain an equibounded subsequence belonging to $AC(T)$ by truncating and then carefully modifying the (non BV) controls $\varphi \circ \varphi_{0h}^{-1}$, using the property (3). Notice that

$$\lim_{t \to T^-} \sigma(t) = \bar{S}.$$ 

In particular, when $\bar{S} < +\infty$ the pair $(x, u)$ has a jump at the final time $t = T$ from $(x, u)(T^-) = (\xi, \varphi)(\bar{S})$ to $(x, u)(T)$ and $Var_{[\bar{S}, +\infty]}(\varphi) = +\infty$.

**Case 1:** Let $\bar{S} < +\infty$. In view of Theorem 4.2 there exists a sequence of absolutely continuous, strictly increasing functions $\sigma_h$ from $[0, T]$ onto $\mathbb{R}_+$ and pointwisely converging to $\sigma$ such that, for every $h$, the maps $\varphi_{0h} := \sigma_h^{-1} : \mathbb{R}_+ \to [0, T]$ are strictly increasing, 1-Lipschitz continuous, surjective and they verify, for every $h$,

$$(39) \quad \sup_{s \in \mathbb{R}_+} |\varphi_{0h}(s) - \varphi_0(s)| \leq \varepsilon(h) + (T - t_h),$$

where $t_h := \varphi_{0h}(\bar{S})$.

Let us define

$$(x, u, v) := (\xi, \varphi, \psi) \circ \sigma,$$

$$u_h := \varphi \circ \sigma_h, \quad x_h := x(x_0, u_0, u_h, v), \quad \xi_h := \xi(x_0, u_0, \varphi_{0h}, \varphi, v \circ \varphi_{0h}).$$

Clearly, $x_h = \xi_h \circ \sigma_h$ on $[0, T]$. Let $(s_j)_j$ be as in (35) and for every $j, h$, let us set

$$(40) \quad \tau^j_h := \varphi_{0h}(s_j).$$

Since $s_j \nearrow +\infty$, it is not restrictive to assume $s_j > \bar{S}$ for every $j$; hence, for every $h$ the sequence $(\tau^j_h)_j$ is strictly increasing and, for every $j$,

$$t_h \leq \tau^j_h < T \quad \text{and} \quad \lim_{h} \tau^j_h = \lim_{h} t_h = T.$$
In order to construct an equibounded trajectory-control sequence verifying (37) and (38), let us preliminary notice that, for every \( j \), by Proposition 2.1 we have, for any \( h \),

\[
\sup_{s \in [0,s_j]} \left| \xi_h(s) - \xi(s) \right| =: \varepsilon_j^1(h),
\]

with \( \varepsilon_j^1(h) \leq \varepsilon_{j+1}^1(h) \) and \( \lim_h \varepsilon_j^1(h) = 0 \). We define a sequence \((h_j)_j\) as follows. Choose \( h_1 \geq 1 \) verifying \( \varepsilon_1^1(h) \leq 1 \) for every \( h \geq h_1 \) and for any \( j > 1 \) let \( h_j > h_{j-1} \) \((\geq j - 1)\) be such that

\[
(42) \quad \varepsilon(h), \ varepsilon_j^1(h) \leq \frac{1}{j} \quad \text{for every } h \geq h_j.
\]

Using the Whitney property (5), let us set

\[
(43) \quad u_j := u_{h_j} x_{[0,\tau_{h_j}]} + \tilde{u}_j \left( \frac{t - \tau_{h_j}}{\tau_{h_j} - \tau_{h_{j-1}}} \right) x_{[\tau_{h_j},T]},
\]

where \( \tilde{u}_j \in AC(1) \) joins \( u_{h_j}(\tau_{h_j}) = \varphi(s_j) \) to \( u(T) \) and \( \text{Var}_{[0,1]}(\tilde{u}_j) \leq C|\varphi(s_j) - u(T)| \). Since \( u_j(T) = u(T) \) for every \( j \), \( \lim_j u_j(T) = u(T) \) trivially. If \( t \in [0,T] \), there is some \( j \) such that \( t \leq \bar{\tau}_{h_j} \) and we have

\[
(44) \quad \lim_j u_{h_j}(t) = \lim_{h_j} \varphi(\sigma_{h_j}(t)) = \varphi(\sigma(t)) = u(t),
\]

recalling that \( \varphi \) is a \((1\text{-Lipschitz})\) continuous function. Moreover,

\[
|\xi_{h_j}(\sigma_{h_j}(t)) - \xi(\sigma(t))| \leq |\xi_{h_j}(\sigma_{h_j}(t)) - \xi(\sigma(t))| + |\xi(\sigma(t)) - \xi(\sigma(t))|,
\]

where \( \sigma_{h_j}(t) \in [0,s_j] \) and

\[
|\xi_{h_j}(\sigma_{h_j}(t)) - \xi(\sigma_{h_j}(t))| \leq \sup_{s \in [0,s_j]} |\xi_{h_j}(s) - \xi(s)| \leq 1/j.
\]

Since \( \xi \) is continuous, this implies that \( \lim_j x_j(t) = x(t) \) for every \( t \in [0,T] \). Let \( t \in [0,T] \). To prove the existence of a function \( V \) such that (38) holds true, notice that \( \lim_j \sigma_{h_j}(t) = \sigma(t) < +\infty \) (actually, \( \sigma(t) \leq \bar{S} \)). Therefore,

\[
\sigma_{h_j}(t) \leq \sigma(t) + M(t), \quad \text{if } M(t) := \max\{1, \max\{\sigma_{h_j}(t) : j = 1, \ldots, j(t)\}\} < +\infty.
\]

By the above estimate, for any \( j \) such that \( t \leq \bar{\tau}_{h_j} \), we get

\[
\text{Var}_{[0,t]}(u_j) = \text{Var}_{[0,\sigma_{h_j}(t)]}(\varphi) \leq \text{Var}_{[0,\sigma(t) + M(t)]}(\varphi),
\]

for every \( j \).
while if $t > \tau_{h_j}$,

$$Var_{[0,t]}(u_j) \leq Var_{[0,t]}(u_{h_j}) + \int_{\tau_{h_j}}^{t} |\dot{u}_j(t)| \, dt \leq$$

$$Var_{[0,\sigma(t)+M(t)]}(\varphi) + C \text{diam}(U).$$

Therefore $(u_j)_j$ verifies (38) if we choose

$$V(t) := Var_{[0,\sigma(t)+M(t)]}(\varphi) + C \text{diam}(U) \forall t \in [0,T].$$

Let us now prove that the sequence $(x_j)_j$ is equibounded. In view of the boundedness of $\xi$ and $x$ and of the previous estimates, we get

$$\sup_{t \in [0,\tau_{h_j}]} |x_j(t)| \leq \sup_{t \in [0,T]} |x(t)| + 2 \sup_{s \geq 0} |\xi(s)| + (1/j) \leq \sup_{t \in [0,T]} |x(t)| + 2 \sup_{s \geq 0} |\xi(s)| + 1 =: R.$$

If instead $t > \tau_{h_j}$, by standard estimates, we have

$$|x_j(t)| \leq \left\{ |x_j(\tau_{h_j})| + (m+1)M|T-\tau_{h_j} + C|\varphi(s_j) - u(T)| \right\} e^{(m+1)M[T-\tau_{h_j} + C|\varphi(s_j) - u(T)|] \leq$$

$$\{R + (m+1)M[T + C \text{diam}(U)]\} e^{(m+1)M[T + C \text{diam}(U)]} =: R'.$$

Hence, for every $j$,

$$\sup_{t \in [0,T]} |x_j(t)| \leq R'.$$

As a consequence, by the Dominated Convergence Theorem we also have that $\lim_j \| (x_j, u_j) - (x, u) \|_{L^1(T)} \to 0$.

Finally, for every $j$, recalling that $x_j(\tau_{h_j}) = \xi_{h_j}(s_j)$, we have

$$|x_j(T) - x(T)| \leq |x_j(T) - x_j(\tau_{h_j})| + |\xi_{h_j}(s_j) - \xi(s_j)| + |\xi(s_j) - x(T)|$$

where $\lim_j \xi(s_j) = x(T)$ and $|\xi_{h_j}(s_j) - \xi(s_j)| \leq 1/j \to 0$ as $j \to +\infty$. Using (46) together with standard estimates, we get

$$|x_j(T) - x_j(\tau_{h_j})| = \left| \int_{\tau_{h_j}}^{T} [g_0(x_j, u_j, v) + \sum_{i=1}^{m} g_i(x_j, u_j)\dot{u}_j] \, dt \right| \leq$$

$$(m+1)(1+R')M[T - \tau_{h_j} + C|u_{h_j}(\tau_{h_j}) - u(T)|] \leq$$

$$(m+1)(1+R')M[T - t_j + C|\varphi(s_j) - u(T)|],$$

recalling that $h_j \geq j$ so that $t_{h_j} \geq t_j$ and hence $\lim_j |x_j(T) - x_j(\tau_{h_j})| = 0$. Thus $\lim_j x_j(T) = x(T)$ and if we rename the index $j$ in the sequence $(x_j, u_j)_j$ by $k$, we obtain a sequence $(x_k, u_k)$ verifying theses (i) and (ii).

For every $k$, let $(\xi_k, \varphi_k, v_k, v_{\varphi_k}, S_k)$ be the arc-length graph parametrization of $(x_k, u_k, v)$ (see Definition 1.5). In view of Remark 3.4 in order to
prove (iii) we need to estimate $|\hat{\xi}_k(s_j) - \hat{\xi}_k(S_k)|$. By the definition of $(x_k, u_k)$, it follows that

$$(\hat{\xi}_k, \hat{\varphi}_k)(S_k) = (x_k, u_k)(T) = (x_k, u)(T).$$

Moreover, for every $k \geq j$, we have $(\hat{\varphi}_k, \hat{\xi}_k) = (\varphi_{0h_k}, \varphi, \xi_{hk})$ on $[0, s_j]$ (where $\varphi_{0h_k}$, $\xi_{hk}$ are the maps introduced above, with $j$ replaced by $k$) and, by (41), (42),

$$\sup_{s \in [0, s_j]} |\hat{\xi}_k(s) - \xi(s)| \leq \frac{1}{j}.$$

Hence for every $k \geq j$, we get

$$\lim_j |\hat{\varphi}_k(s_j) - \hat{\varphi}_k(S_k)| = \lim_j |\varphi(s_j) - u(T)| = 0$$

independently of $k$, and

$$|\hat{\xi}_k(s_j) - \hat{\xi}_k(S_k)| \leq |\hat{\xi}_k(s_j) - \xi(s_j)| + |\xi(s_j) - x(T)| + |x(T) - x_k(T)| = \tilde{\varepsilon}(j),$$

where $\lim_j \tilde{\varepsilon}(j) = 0$ and $\tilde{\varepsilon}$ does not depend on $k$, since $|\hat{\xi}_k(s_j) - \xi(s_j)| \leq 1/j$, $|\xi(s_j) - x(T)| \to 0$ by hypothesis and $\lim_j |x(T) - x_k(T)| = 0$, being $k \geq j$. The proof of the theorem in Case 1 is thus concluded.

**CASE 2:** Let $\tilde{S} = +\infty$. Let $(\sigma_{jh})$ be the sequence of absolutely continuous, strictly increasing functions from $[0, T]$ onto $\mathbb{R}_+$, pointwisely converging to $\sigma$, whose existence is guaranteed by Theorem 4.2. Let $\varphi_{0h} := \sigma_{jh}^{-1}$ be the sequence of the 1-Lipschitz continuous inverse maps, uniformly converging to $\varphi_0$ on any compact interval. Let $(s_j)$ be as in (35). For every $j$ and $h$, we set

$$\tau^j := \varphi_0(s_j), \quad \tau_h^j := \varphi_{0h}(s_j).$$

Since $\varphi_0(s) < T$ for all $s \geq 0$ and $\lim_{s \to +\infty} \varphi_0(s) = T$, one has $\tau^j < T$ for every $j$ and $\lim_j \tau^j = T$. Passing eventually to a subsequence, it is not restrictive to assume that the sequence $(\tau^j)_j$ is strictly increasing. Clearly, for every $h$ the sequence $(\tau^j_h)_j$ is strictly increasing, $0 < \tau^j_h < T$ and $\lim_j \tau^j_h = T$. In view of Theorem 4.2 for every $j$ and $h$, if we set

$$\varepsilon_j(h) := \sup_{s \in \mathbb{R}_+} |\varphi_{0h}(s) - \varphi_0(s)|,$$

we have that

$$\sup_{s \in \mathbb{R}_+} |\varphi_{0h}(s) - \varphi_0(s)| \leq \varepsilon_j(h) + (T - \min\{\tau^j, \tau^j_h\}),$$

where $\tau^j - \varepsilon_j(h) \leq \tau^j_h < T$, $\lim_h \varepsilon_j(h) = 0$.

Let us set $(x, u, v) := (\xi, \varphi, \psi) \circ \sigma$, $u_h := \varphi \circ \sigma_h$, $x_h := x[\sigma_h, \sigma_0, u_h, v]$ and $\xi_h := [\sigma_h, \sigma_0, \varphi_{0h}, \varphi, v \circ \varphi_{0h}]$, so that $x_h = \xi_h \circ \sigma_h$. Then, by Proposition 2.1 we have

$$\sup_{s \in [0, s_j]} |\xi_h(s) - \xi(s)| =: \varepsilon^1_j(h),$$
where $\lim_k \epsilon_j^1(h) = 0$. Now, similarly to Case 1, let us introduce a sequence $(h_j)_j$ such that

$$
\epsilon_j(h), \ \epsilon_j^1(h) \leq \frac{1}{j} \quad \text{for every } h \geq h_j.
$$

At this point, the sequence of absolutely continuous functions $(x_j, u_j)_j$ defined as in (43) is equibounded and converges pointwisely and in $L^1$-norm to $(x, u)$. Indeed, it is enough to observe that $\tau_j - (1/j) \leq \tau_{h_j}^j < T$, so that $\lim_j \tau_{h_j}^j = T$ and afterwards the proof is the same as in Case 1. \hfill \Box

4.2. **Proof of Theorem 4.1: Characterization.** Let us now prove that a BV$_{loc}$S limit solution $x$ is a BV$_{loc}$ graph completion solution. Let us assume the **Claim** at the beginning of Subsection 4.1 as hypothesis.

For every $k$, set $V_k := \text{Var}_{[0,T]}(u_k)$ ($< +\infty$). Taking eventually a subsequence, we can assume that the sequence $(V_k)_k$ of the variations is increasing. If this sequence is bounded, $x$ is in fact a BV limit solution and it coincides with a BV graph completion solution by [AR theorem 4.2]. Hence let us assume

$$(49) \quad \lim_k V_k = +\infty.$$  

In order to prove that $x$ is a BV$_{loc}$ graph completion solution on $[0, T]$, let us consider the arc-length graph parametrizations of the inputs $u_k$. Precisely, let us define for every $k$, a map $\sigma_k : [0, T] \to [0, T + V_k]$ by setting

$$(50) \quad \sigma_k(t) := t + \text{Var}_{[0,T]}(u_k) \quad (\leq t + V(t))$$

and let $\varphi_{0_k} : \mathbb{R}_+ \to [0, T]$ be the 1-Lipschitz continuous, increasing function such that

$$\varphi_{0_k} := \sigma_k^{-1} \quad \text{on} [0, T + V_k], \quad \text{and} \quad \varphi_{0_k}(s) = T \quad \text{for all} \ s \geq T + V_k.$$  

Set $\varphi_k := u_k \circ \varphi_{0_k}$. Then the sequence of space-time controls $(\varphi_{0_k}, \varphi_k)_k$ is 1-Lipschitz continuous on $\mathbb{R}_+$ and satisfies $\varphi_{0_k}'(s) + |\varphi_k'(s)| = 1$ for a.e. $s \in [0, T + V_k]$ (and $\varphi_{0_k}'(s) + |\varphi_k'(s)| = 0$ for $s > T + V_k$). Therefore by Ascoli-Arzelà’s Theorem, taking if necessary a subsequence which we still denote by $(\varphi_{0_k}, \varphi_k)_k$, it converges uniformly on any compact interval $[0, S]$ and pointwise on $\mathbb{R}_+$ to a Lipschitz continuous function $(\varphi_0, \varphi)$ such that $\varphi_0'(s) + |\varphi'(s)| \leq 1$ for $s \geq 0$.

Let us show that $(\varphi_0, \varphi)$ is a BV$_{loc}$ graph completion of $u$, possibly not feasible (namely, not verifying the equality $\varphi_0'(s) + |\varphi'(s)| = 1$ a.e.). Clearly, $\varphi_0$ is nondecreasing, $\varphi_0(0) = 0$ and $\lim_{s \to +\infty} \varphi_0(s) \leq T$. In fact, let us prove that

$$\lim_{s \to +\infty} \varphi_0(s) = T.$$  

For any $\varepsilon > 0$ we show that there exists some $S_\varepsilon > 0$ such that $\varphi_0(s) > T - \varepsilon$ for all $s \geq S_\varepsilon$. Let $T - \varepsilon < t_\varepsilon < T$ and define, for every $k$, $S_{\varepsilon,k} := \sigma_k(t_\varepsilon)$. Notice
Therefore, by the uniform convergence of \( S_{\varepsilon,k} = \varepsilon + Var_{[0,t_\varepsilon]}(u_k) \leq \varepsilon + V(t_\varepsilon) =: S_\varepsilon \), so that \( t_\varepsilon = \varphi_0(S_{\varepsilon,k}) \leq \varphi_0(S_\varepsilon) < T \). Therefore, for any \( s \geq S_\varepsilon \), we obtain that
\[
\varphi_0(s) \geq \varphi_0(S_\varepsilon) = \lim_k \varphi_0(S_{\varepsilon,k}) \geq t_\varepsilon > T - \varepsilon
\]
and the limit above is proved. For every \( t \in [0,T] \), by (50) there exist a subsequence \( (\sigma_k(t))_{k'} \) and \( \sigma(t) \in [0,t + V(t)] \) such that \( \lim_{k'} \sigma_k(t) = \sigma(t) \). Therefore, by the uniform convergence of \((\varphi_0, \varphi_k)\) on \([0, t + V(t)]\), recalling (37), it follows that

\[
(\varphi_0, \varphi) \circ \sigma(t) = \lim_{k'} (\varphi_0, \varphi_k) \circ \sigma_k(t) = (t, u(t)).
\]

Hence \((\varphi_0, \varphi)\) is a (possibly not feasible) \( BV_{loc} \) graph completion of \( u \) on \([0, T]\).

Let \( \xi_k := \xi[\overline{0}, \overline{t}, \varphi_k, \varphi_k, v \circ \varphi_0] \) and \( \xi := \xi[\overline{0}, \overline{t}, \varphi_0, \varphi, v \circ \varphi_0] \) be the corresponding solutions of (10). Clearly, \( \xi_k = x_k \circ \varphi_0_k \). We set
\[
\tilde{x}(t) := \xi \circ \sigma(t) \quad \forall t \in [0,T],
\]
so that \( \tilde{x} \) is a \( BV_{loc} \) graph completion solution (on \([0, T]\)). Actually, \( \tilde{x}(t) = x(t) \) for any \( t \in [0,T] \), since
\[
x(t) = \lim_{k'} x_{k'}(t) = \lim_{k'} \xi_{k'} \circ \sigma_{k'}(t) = \xi \circ \sigma(t) = \tilde{x}(t),
\]
where we used the uniform convergence of \( \xi_k \) to \( \xi \) on \([0, t + V(t)]\), guaranteed by Proposition 2.1, together with the pointwise convergence of \( \sigma_{k'}(t) \) to \( \sigma(t) \).

In order to conclude the proof that \( x \) is a \( BV_{loc} \) graph completion solution, let us show that \( \lim_j (\xi, \varphi)(\tilde{s}_j) = (x, u)(T) \), where \((\tilde{s}_j)_j \) is the same as in (iii) of the Claim. In view of Remark 3.4, hypothesis (iii) implies that
\[
|\xi_k, \varphi_k)(\tilde{s}_j) - (\xi_k, \varphi_k)(S_k)| \leq \tilde{\varepsilon}(j)
\]
with \( S_k := \sigma_k(T) = T + V_k \), for every \( k > k_j (\geq j) \), for some positive, decreasing sequence \( \tilde{\varepsilon} \) with \( \lim_j \tilde{\varepsilon}(j) = 0 \). Notice that, for every \( j \),
\[
\sup_{[0, \tilde{s}_j)} |(\xi, \varphi)(s) - (\xi_k, \varphi_k)(s)| \leq \varepsilon_j(k)
\]
for some positive, decreasing sequence \( \varepsilon_j \) with \( \lim_k \varepsilon_j(k) = 0 \), because of the uniform convergence of \((\xi_k, \varphi_k)\) to \((\xi, \varphi)\) on compact intervals. Hence we can define a sequence \( (\tilde{k}_j)_j \subset \mathbb{N} \) with \( \tilde{k}_j \geq k_j \) and such that \( \varepsilon_j(k) \leq 1/j \) for all \( k > \tilde{k}_j \). Taking into account that \((\xi_k, \varphi_k)(S_k) = (x_k, u_k)(T) \), for every \( k > \tilde{k}_j \), we get
\[
|(\xi, \varphi)(\tilde{s}_j) - (x(T), u(T))| \leq |(\xi, \varphi)(\tilde{s}_j) - (\xi_k, \varphi_k)(\tilde{s}_j)| +
\]
\[
(51) \quad |(\xi_k, \varphi_k)(\tilde{s}_j) - (x_k(T), u_k(T))| + |(x_k(T), u_k(T)) - (x(T), u(T))| \leq 1/j + \tilde{\varepsilon}(j) + |(x_k(T), u_k(T)) - (x(T), u(T))|,
\]
where \( \lim_j |(x_k(T), u_k(T)) - (x(T), u(T))| = 0 \), being \( k > k_j \geq j \). Therefore
\[ \lim_j (\xi, \varphi)(\tilde{s}_j) = (x(T), u(T)). \]

At this point we can recover a feasible space-time control in \( \Gamma(T; \pi_0) \) by introducing the change of variable
\[ \eta(s) := \int_0^s [\varphi'(r) + |\varphi'(r)|] \, dr \quad \forall s \geq 0, \quad \hat{V} := \lim_{s \to +\infty} \eta(s) - T \leq +\infty, \]
considering, e.g. \( s(\cdot) : [0, T + \hat{V}] \to \mathbb{R}_+ \), the strictly increasing right-inverse of \( \eta \) and defining
\[ (\varphi_0, \varphi, \psi, \bar{S}) := (\varphi_0 \circ s, \varphi \circ s, \psi \circ s, T + \hat{V}). \]

Let us set \( \tilde{\xi} := \xi[\pi_0, \varphi_0, \varphi, \psi] \). Notice that \((\varphi_0, \varphi)\) is constant on any interval \([s_1, s_2]\) where \( \eta \) is constant, so that \((\varphi_0, \varphi, \tilde{\xi}) \circ \eta = (\varphi_0, \varphi, \xi)\). Hence \((\varphi_0, \tilde{\xi})\) turns out to be a feasible BV\(_{loc}\) graph completion of \( u \) on \([0, T]\) with clock \( \bar{\sigma} := \eta \circ \sigma \). Finally, \( x \) is a BV\(_{loc}\) graph completion solution such that \( x = \tilde{\xi} \circ \bar{\sigma} \).

\[
5. \textbf{Technical proofs}
\]

5.1. **Proof of Lemma 2.1** (i) Since \( u \) is a BV function, the set \( \mathcal{T} \subset [a, b] \) of discontinuity points of \( u \) is countable and right and left limits of \( u \) always exist. For every \( \tau_j \in \mathcal{T} \), owing to the Whitney property, we can define the maps \( \tilde{u}_j^- \), \( \tilde{u}_j^+ \), \( \tilde{u}_b : [0, 1] \to U \) verifying
\[
\tilde{u}_j^-(0) = u(\tau_j^-), \quad \tilde{u}_j^-(1) = u(\tau_j); \quad \tilde{u}_j^+(0) = u(\tau_j), \quad \tilde{u}_j^+(1) = u(\tau_j^+);
\]
\[
\tilde{u}_b(0) = u(b), \quad \tilde{u}_b(1) = \tilde{u}_1
\]
and such that
\[
\text{Var}_{[0,1]}(\tilde{u}_j^-) \leq C|u(\tau_j) - u(\tau_j^-)|; \quad \text{Var}_{[0,1]}(\tilde{u}_j^+) \leq C|u(\tau_j^+) - u(\tau_j)|;
\]
\[
\text{Var}_{[0,1]}(\tilde{u}_b) \leq C|u(b) - \tilde{u}_1|.
\]

We introduce the function \( \sigma : [a, b] \to [0, \lambda] \) given by
\[
\sigma(t) = t - a + \text{Var}_{[a,t]}(u) \quad \text{and} \quad \lambda := b - a + \hat{V}.
\]

Notice that \( u \) is continuous, [left-continuous, right-continuous] at \( t \) if and only if \( \sigma \) is continuous, [left-continuous, right-continuous] at \( t \) and let \( \tilde{\varphi}_0 \) be the unique, increasing and continuous function such that \( \tilde{\varphi}_0 \circ \sigma(t) = t \) for
all \( t \in [a, b] \). Similarly to the proof of [AR] Theorem 2.4, let us set

\[
(52) \quad \tilde{u}_j = \begin{cases} 
\sigma - \sigma(j^-) & \text{if } \sigma(j^-) < \sigma(j) \text{ and } \sigma \in [\sigma(j^-), \sigma(j)] \\
\sigma(j) - \sigma(j^+) & \text{if } \sigma(j) < \sigma(j^+) \text{ and } \sigma \in [\sigma(j), \sigma(j^+)] 
\end{cases}
\]

\[
\dot{\sigma}(\tau_j) = \begin{cases} 
u(\tau_j) & \text{if, for some } j, \text{ either } \sigma = \sigma(j^-) = \sigma(j) \text{ or } \sigma = \sigma(j) = \sigma(j^+) \\
\tilde{u}_b(\sigma - \lambda) \chi_{[\lambda, \lambda+1]}(\sigma) + \tilde{u}_b(2 - \sigma + \lambda) \chi_{(\lambda+1, \lambda+2)}(\sigma) & \text{if } \sigma \in [\lambda, \lambda + 2] \\
u \circ \dot{\sigma}_0(\sigma) & \text{if } \sigma \in [0, \lambda] \setminus \sigma(T).
\end{cases}
\]

Setting \( \dot{\sigma}_0(\sigma) = b \) for \( \sigma \in [\lambda, \lambda + 2] \) we have that the function \((\dot{\sigma}_0, \dot{\sigma}) : [0, \lambda + 2] \to [a, b] \times U \) is absolutely continuous, verifies \((\dot{\sigma}_0, \dot{\sigma})(0) = (a, u(a))\),

\( (\dot{\sigma}_0, \dot{\sigma})(\lambda) = (\dot{\sigma}_0, \dot{\sigma})(\lambda + 2) = (b, u(b)) \) and \((\dot{\sigma}_0, \dot{\sigma})(\lambda + 1) = (b, \bar{u}_1) \). Moreover,

\[
\lambda \leq Var_{[0, \lambda]}(\dot{\sigma}_0, \dot{\sigma}),
\]

and

\[
Var_{[0, \lambda + 2]}(\dot{\sigma}_0, \dot{\sigma}) \leq (b - a) + 2C(V + |u(b) - \bar{u}_1|).
\]

Let us now introduce, for \( \sigma \in [0, \lambda + 2] \), the arc-length parametrization

\[
(53) \quad s(\sigma) = \int_0^\sigma (\dot{\sigma}_0(r) + |\dot{\sigma}(r)|) \, dr
\]

and let us set

\[
(54) \quad S := s(\lambda + 1) \quad \text{and} \quad \tilde{S} := s(\lambda + 2),
\]

so that

\[
(b - a) + V + |u(b) - \bar{u}_1| \leq S \leq \tilde{S} \leq (b - a) + 2C(V + |u(b) - \bar{u}_1|).
\]

Let \( \tilde{\sigma} : [0, \tilde{S}] \to [0, \lambda + 2] \) denote the inverse function of \( s(\cdot) \) and define

\[
(55) \quad (\varphi_0, \varphi)(s) := (\dot{\sigma}_0, \dot{\sigma}) \circ \tilde{\sigma}(s) \quad \text{for } s \in [0, \tilde{S}].
\]

We get \( \varphi'_0 + |\varphi'| = 1 \text{ a.e.}, \ (\varphi_0, \varphi)(0) = (a, u(a))\),

\[
(\varphi_0, \varphi)(s(\lambda)) = (\varphi_0, \varphi)(\tilde{S}) = (b, u(b)), \quad (\varphi_0, \varphi)(S) = (b, \bar{u}_1),
\]

and it is easy to see that for any \( t \in [a, b] \) there is \( s \in [0, \tilde{S}] \) (in fact, \( s \in [0, s(\lambda))] \) such that \( (t, u(t)) = (\varphi_0, \varphi)(s) \).

(ii) For \( s > \tilde{S} \), let us consider the periodic extension of the restriction \((\varphi_0, \varphi)\) to the interval \([s(\lambda), s(\lambda + 2)]\), with period \( p = s(\lambda + 2) - s(\lambda) \). Setting, for every \( j \geq 1 \), \( s_j := s(\lambda + 1) + jp \), one clearly has \((\varphi_0, \varphi)(s_j) = (b, \bar{u}_1)\) for all \( j \), so proving (ii).  \( \square \)
5.2. Proof of Proposition 2.1. Let \((\varphi_0, \varphi, \psi), (\varphi_{0h}, \varphi_h, \psi) \in \mathcal{U}(T; \mathfrak{u}_0, S), \xi, \xi_h\) be the given space-time controls and the corresponding solutions, respectively. Since \(\varphi_0'(s) + |\varphi'(s)| = 1\) and \(\varphi_{0h}'(s) + |\varphi_h'(s)| = 1\) a.e. on \([0, S]\), so that in particular they are bounded, by standard estimates it follows that

\[
\sup_{s \in [0, \hat{S}]} |\xi(s)|, \quad \sup_{s \in [0, \hat{S}]} |\xi_h(s)| \leq \hat{M} := (\|\mathfrak{u}_0\| + (m + 1)\hat{M}\hat{S})e^{(m+1)\hat{M}\hat{S}}.
\]

Let us denote by \(\omega\) a modulus of continuity of \(g_0\) and by \(\hat{M}, \hat{L}\) a sup-norm and a Lipschitz constant, respectively, for the vector fields \(g_i, i = 0, \ldots, m\), in the compact set \(\overline{B_0(0, \hat{M})} \times U \times V\).

Let us start by showing that \(\xi = \xi[\mathfrak{u}_0, \mathfrak{u}_0, \varphi_0, \varphi, \psi] \equiv \xi[\mathfrak{u}_0, \mathfrak{u}_0, \varphi_0, \varphi, v \circ \varphi_0] =: \hat{\xi}\). Indeed, there is an at most countable number of disjoint intervals, say \([s_1, s_2]\) for \(j \in J\), where \(\varphi_0\) is constant; moreover, \(\psi\) may differ from \(v \circ \varphi_0\) only on these intervals, for \(\varphi_0^{-1}(\varphi_0(s))\) is single valued outside such set. Hence, for every \(s \in [0, \hat{S}]\), we get

\[
\xi(s) - \hat{\xi}(s) = \int_{[0, s] \setminus \bigcup [s_i, s_{i+1}]} [g_0(\xi(r), \varphi(r), v \circ \varphi_0(r))] dr + \int_{[0, s]} \sum_{i=1}^{m} g_i(\xi(r), \varphi(r)) \varphi_i' dr
\]

and thesis (19) follows easily by Gronwall’s Lemma.

In order to prove (20), for every \(s \in [0, \hat{S}]\) we apply again Gronwall’s Lemma and get

\[
(56) \quad \|\xi_h(s) - \xi(s)\| \leq \left(\int_0^s |g_0(\xi(r), \varphi(r), v \circ \varphi_0(r))(\varphi_{0h}'(r) - \varphi_0'(r)) + 1\right) dr + (m + 1)\hat{L} \int_0^\hat{S} |\varphi_h(r) - \varphi(r)| dr + \int_0^\hat{S} \omega(|v \circ \varphi_{0h}(r) - v \circ \varphi_0(r)|)(\varphi_{0h}'(r) dr)
\]

The uniform convergence of \((\varphi_{0h}, \varphi_h)\) to \((\varphi_0, \varphi)\) on \([0, \hat{S}]\) implies that the maps \((\varphi_{0h}', \varphi_h')\) tend to \((\varphi_0', \varphi')\) in the weak* topology of \(L^\infty([0, \hat{S}], \mathbb{R}^{1+m})\), so that

\[
f_h(s) := \left|\int_0^s [g_0(\xi(r), \varphi(r), v \circ \varphi_0(r))(\varphi_{0h}'(r) - \varphi_0'(r)) + 1\right] dr + \sum_{i=1}^{m} g_i(\xi(r), \varphi(r)) \varphi_i'(r) dr
\]

tends to 0 as \(h \to +\infty\). The uniform convergence to 0 of the \(f_h\)’s now follows from Ascoli-Arzelá Theorem, for the \(f_h\)’s are equibounded and equi-Lipschitzean. The convergence to 0 of the second integral in the r.h.s. of (56) is trivial. It remains to prove the convergence to 0, eventually for a further subsequence, of the last term of (56). Let us set \(\sigma_h := \varphi_{0h}^{-1}\) and observe that

\[
(57) \quad \int_0^\hat{S} |v(t) - v \circ \varphi_0 \circ \sigma_h(t)| dt = \int_0^\hat{S} |v \circ \varphi_{0h}(s) - v \circ \varphi_0(s)| \varphi_{0h}'(s) ds.
\]

Now, it suffices to prove that the expression in (57) tends to 0 as \(h \to +\infty\): in this case, indeed, there exists a subsequence of \((v - v \circ \varphi_0 \circ \sigma_h)\) converging
to 0 a.e. on \([0, \bar{T}]\), and the Dominated Convergence Theorem implies that, for such subsequence,

\[
\int_0^\bar{T} \omega(|v \circ \varphi_{0_h}(s) - v \circ \varphi_0(s)|)\varphi'_{0_h}(s) \, ds = \int_0^\bar{T} \omega(|v(t) - v \circ \varphi_0 \circ \sigma_h(t)|) \, dt \to 0,
\]

so implying \([20]\).

Since \(|\varphi'_0| \leq 1\), when \(v\) is a continuous function \([57]\) holds true, owing to the uniform continuity of \(v\) and to the uniform convergence of \(\varphi_{0_h}\) to \(\varphi_0\) on \([0, \bar{T}]\).

For \(v \in L^1([0, \bar{T}], V)\), \(\forall \varepsilon > 0\) there exists, by density, \(\tilde{v} \in C_c([0, \bar{T}], \mathbb{R})\) such that \(\int_0^\bar{T} |\tilde{v}(t) - v(t)| \, dt \leq \varepsilon\). Hence we get

\[
\int_0^\bar{T} |v \circ \varphi_{0_h}(s) - v \circ \varphi_0(s)|\varphi'_{0_h}(s) \, ds \leq \int_0^\bar{T} |v \circ \varphi_{0_h}(s) - \tilde{v} \circ \varphi_0(s)|\varphi'_{0_h}(s) \, ds +
\]

\[
\int_0^\bar{T} |\tilde{v} \circ \varphi_0(s) - v \circ \varphi_0(s)|\varphi'_{0_h}(s) \, ds + \int_0^\bar{T} |\tilde{v} \circ \varphi_0(s) - v \circ \varphi_0(s)|\varphi'_{0_h}(s) \, ds.
\]

Performing the change of variable \(t = \varphi_{0_h}(s)\), the first integral on the r.h.s. is smaller than \(\varepsilon\), while the second one converges to 0 because \(\tilde{v}\) is continuous. For the third integral on the r.h.s., taking into account that \(|v(t)|, |\tilde{v}(t)| \leq M\) for all \(t \in [0, \bar{T}]\) for some \(\bar{M} > 0\), by the weak* convergence of \(\varphi'_{0_h}\) to \(\varphi'_0\) we derive that

\[
\int_0^\bar{T} |\tilde{v} \circ \varphi_0(s) - v \circ \varphi_0(s)|\varphi'_{0_h}(s) \, ds \to \int_0^\bar{T} |\tilde{v} \circ \varphi_0(s) - v \circ \varphi_0(s)|\varphi'_{0_h}(s) \, ds \quad \text{as} \quad h \to +\infty,
\]

and the last term is smaller than \(\varepsilon\) by the change of variable \(t = \varphi_0(s)\). This concludes the proof of \([57]\) by the arbitrariness of \(\varepsilon > 0\). \(\Box\)

### 5.3. Proof of Theorem 4.2. **Case 1:** \(\lim_{t \to T^-} \sigma(t) = \bar{S} < +\infty\).

Let us extend \(\sigma\) to \([-T, 2T]\) as follows:

\[
\bar{\sigma}(t) = \sigma(t) \chi_{[0,T]}(t) - \sigma(-t) \chi_{[-T,0]}(t) - (\sigma(2T - t) - 2\sigma(T)) \chi_{[T,2T]}(t).
\]

Let \(\rho : \mathbb{R} \to \mathbb{R}_+\), \(\rho \in C^\infty\) be an even map, with compact support contained on \([-T, T]\) and such that \(\int_{\mathbb{R}} \rho(t) \, dt = 1\); for \(h \in \mathbb{N}\) let us set \(\rho_h(t) := 2h \rho(2ht)\) and

\[
\bar{\sigma}_h(t) := \int_{-\infty}^{+\infty} \bar{\sigma}(t - \tau) \rho_h(\tau) \, d\tau.
\]

The fact that \(\rho\) is even together with \([59]\) easily yield, for every \(h \in \mathbb{N}\),

\[
\bar{\sigma}_h(0) = 0, \quad \bar{\sigma}_h(T) = \sigma(T^-) = \bar{S}.
\]

By construction, the \(\bar{\sigma}_h\) are continuous, strictly increasing, and, by a property of the convolution product,

\[
\lim_{h} \bar{\sigma}_h(t) = \frac{\sigma(t^+) + \sigma(t^-)}{2} \quad \text{for} \ 0 \leq t < T.
\]
It is easy to show that for any $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, (63) implies
\[
\tilde{\sigma}_h(t_2) - \tilde{\sigma}_h(t_1) = \int_{-\infty}^{t_2} (\tilde{\sigma}(t_2 - \tau) - \tilde{\sigma}(t_1 - \tau)) \rho_h(\tau) d\tau \geq \\
\int_{-\infty}^{t_2} (t_2 - t_1) \rho_h(\tau) d\tau \geq t_2 - t_1.
\]

Let $(\tilde{t}_h)_h$ be a strictly increasing sequence of continuity points of $\sigma$ converging to $T$. By the strict monotonicity of $\sigma$ and (62) it follows that $s_h := \tilde{\sigma}_h(\tilde{t}_h) < \tilde{S}$ and $\lim h s_h = \tilde{S}$. In order to obtain a sequence of strictly increasing maps which are onto on $\mathbb{R}_+$ and converging to $\sigma$, let us set
\[
\sigma_h(t) := \begin{cases} 
\tilde{\sigma}_h(t) & \text{for } t \leq \tilde{t}_h \\
h \sqrt{\frac{T - \tilde{t}_h}{T - t}} & \text{for } \tilde{t}_h \leq t < T.
\end{cases}
\]

Since $\sigma'_h(t) = s_h \sqrt{\frac{T - t}{T - \tilde{t}_h}}$ for $t \in [\tilde{t}_h, T[$, $\sigma'_h \geq \frac{s_h}{2(T - \tilde{t}_h)} \geq 1$ for $h$ large enough and for any $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we get $\sigma_h(t_2) - \sigma_h(t_1) \geq t_2 - t_1$. Moreover, the maps $\sigma_h$ are continuous, onto on $\mathbb{R}_+$, and verify (62) for every $t < T$, since $\sigma_h(t) = \tilde{\sigma}_h(t)$ for all $h$ such that $\tilde{t}_h > t$. The inverse functions
\[
\varphi_{0h}(s) := \sigma_h^{-1}(s) = \begin{cases} 
\tilde{\sigma}_h^{-1}(s) & \text{for } 0 \leq s \leq s_h \\
T - \frac{s_h^2}{2}(T - \tilde{t}_h) & \text{for } s > s_h,
\end{cases}
\]

are $1$-Lipschitz continuous and strictly increasing, so that by Ascoli-Arzelà’s Theorem, taking if necessary a subsequence, they converge uniformly on any compact interval $[0, S]$ and pointwise on $\mathbb{R}_+$ to a $1$-Lipschitz continuous function $\tilde{\varphi}_0$. In fact, $\tilde{\varphi}_0 = \varphi_0$, where $\varphi_0 = \sigma^{-1}$ on $[0, S]$ and $\varphi_0(s) = T$ for all $s \geq \tilde{S}$. Indeed, if $t < T$ is a continuity point of $\sigma$, $\sigma_h(t) = \tilde{\sigma}_h(t) < \tilde{S}$ for $h$ sufficiently large, and
\[
t = \varphi_{0h}(\tilde{\sigma}_h(t)) \leq |\varphi_{0h}(\tilde{\sigma}_h(t)) - \varphi_{0h}(\sigma(t))| + \varphi_{0h}(\sigma(t)) \leq |\tilde{\sigma}_h(t) - \sigma(t)| + \varphi_{0h}(\sigma(t))
\]
which implies that
\[
\varphi_0(\sigma(t)) = t = \lim_h \varphi_{0h}(\tilde{\sigma}_h(t)) = \tilde{\varphi}_0(\sigma(t)).
\]

If $t$ is not a continuity point, then there exist two sequences $t^1_k$ and $t^2_k$ of continuity points of $\sigma$ with
\[
t^1_k < t < t^2_k, \quad t^1_k \to t, \quad t^2_k \to t.
\]

Since the $\varphi_{0h}$ are increasing, then $\tilde{\varphi}_0$ is increasing and
\[
\tilde{\varphi}_0(\sigma(t^1_k)) \leq \varphi_0(\sigma(t)) \leq \tilde{\varphi}_0(\sigma(t^2_k)).
\]

Since $t^1_k, t^2_k$ are continuity points we have $t^i_k = \tilde{\varphi}_0(\sigma(t^i_k)) = \varphi_0(\sigma(t^i_k))$ for $i = 1, 2$ and (64) implies
\[
t^1_k \leq \varphi_0(\sigma(t)) \leq t^2_k.
\]

Passing to the limit, we can conclude that $\tilde{\varphi}_0 = \varphi_0$ on $[0, \tilde{S}]$. 

Moreover, \( \varphi_{0_h}(s) \geq \varphi_{0_h}(\bar{S}) \geq \varphi_{0_h}(s_h) \) for every \( s \geq \bar{S} \) and, setting \( t_h := \varphi_{0_h}(\bar{S}) \), we get \( \varphi_{0_h}(s) \geq t_h \geq \bar{t}_h \) for every \( s \geq \bar{S} \) and
\[
\sup_{s \geq \bar{S}} |\varphi_{0_h}(s) - \varphi_0(s)| = \sup_{s \geq \bar{S}} [T - \varphi_{0_h}(s)] \leq (T - t_h) \leq (T - \bar{t}_h) \to 0 \quad \text{as} \quad h \to +\infty.
\]
Hence \( \varphi_{0_h} \) converges uniformly to \( \varphi_0 \) on \( \mathbb{R}_+ \) and we have
\[
(65) \quad \sup_{s \in \mathbb{R}_+} |\varphi_{0_h}(s) - \varphi_0(s)| \leq \varepsilon(h) + (T - \bar{t}_h)
\]
where \( \varepsilon(h) := \sup_{s \in [0, \bar{S}]} |\varphi_{0_h}(s) - \varphi_0(s)| \).

By \((62)\) the proof is concluded if \( \sigma(t) = \frac{\sigma(t^+) + \sigma(t^-)}{2} \) for every \( t \in [0, T] \).

In the general case, we can adapt the above construction simply by replacing the sequence \( (\bar{t}_h)_h \) on \([0, T]\) by a new sequence of strictly increasing functions, pointwisely converging to the extended map \( \sigma : [0, T] \to [0, \bar{S}] \), \( \sigma(T) = \bar{S} \), and verifying \((61)\) and \((63)\), whose existence easily follows by AR Theorem 5.1.

**Case 2:** \( \lim_{t \to T^-} \sigma(t) = +\infty \). The function \( \sigma \) does not in general belong to \( L^1(T) \), hence the convolution product \((60)\) cannot be defined as in the previous case. Let us choose a strictly increasing sequence \((\bar{t}_i)_i \) (with \( \bar{t}_0 := 0 \)) of continuity points of \( \sigma \), such that \( \lim_i \bar{t}_i = T \). We know that \( \sigma \) is monotone and \( \sigma \in L^1_{\text{loc}}(T) \) and we can perform the convolution of the restriction \( \sigma^i := \sigma|_{I_i} \), where \( I_i := [\bar{t}_{i-1}, \bar{t}_i] \) and \( |I_i| := \bar{t}_i - \bar{t}_{i-1} \) for \( i \geq 1 \).

Let \( \rho^i : \mathbb{R} \to \mathbb{R}_+ \) be an even, \( C^\infty \) function, with compact support contained on \([-|I_i|, |I_i|]\), such that \( \int_{\mathbb{R}} \rho^i(t) \, dt = 1 \) and let us set \( \rho_{h}^i(t) := 2h \rho^i(2ht) \).

Let us extend each function \( \sigma^i \) to \([\bar{t}_{i-1} - |I_i|, \bar{t}_i + |I_i|]\) as follows: for \( 0 < t \leq |I_i| \) and for every \( i \geq 1 \) we set
\[
(66) \quad \tilde{\sigma}^i(\bar{t}_{i-1} - t) := -\sigma^i(\bar{t}_{i-1} + t) + 2\sigma^i(\bar{t}_{i-1}),
\]
\[
\tilde{\sigma}^i(\bar{t}_i + t) := -\sigma^i(\bar{t}_i - t) + 2\sigma^i(\bar{t}_i).
\]

Let us now define for each \( i \) and \( h \geq 1 \)
\[
(67) \quad \tilde{\sigma}^i_h(0) = 0, \quad \tilde{\sigma}^i_h(\bar{t}_{i-1}) = \sigma(\bar{t}_{i-1}), \quad \tilde{\sigma}^i_h(\bar{t}_i) = \sigma(\bar{t}_i).
\]
We set for \( t \in [0, T] \)
\[
(68) \quad \tilde{\sigma}_h(t) := \tilde{\sigma}^i_h(t), \quad \text{if} \quad t \in [\bar{t}_{i-1}, \bar{t}_i]
\]
so that \( \tilde{\sigma}_h(\bar{t}_i) = \sigma(\bar{t}_i) \) for every \( h \) and \( i \). By construction, \( \tilde{\sigma}_h \) is continuous on \([0, T]\), strictly increasing since \( \sigma \) is so, and for \( t \in [0, T] \)
\[
\lim_{h \to +\infty} \tilde{\sigma}_h(t) = \frac{\sigma(t^+) + \sigma(t^-)}{2}
\]
Moreover if $0 \leq t_1 < t_2 < T$ then it is not difficult to prove, that

\begin{equation}
\tilde{\sigma}_h(t_2) - \tilde{\sigma}_h(t_1) \geq t_2 - t_1.
\end{equation}

Indeed if $t_1, t_2 \in I_i$ for some $i$, we can prove that

\begin{equation}
\tilde{\sigma}_h(t_2) - \tilde{\sigma}_h(t_1) = \tilde{\sigma}_i^h(t_2) - \tilde{\sigma}_i^h(t_1) = \int_{-\infty}^{+\infty} (\tilde{\sigma}_i^{\tau}(t_2 - \tau) - \tilde{\sigma}_i^{\tau}(t_1 - \tau)) \rho^h_\tau(\tau) d\tau \geq 
\end{equation}

\[ \int_{-\infty}^{+\infty} (t_2 - t_1) \rho^h_\tau(\tau) d\tau \geq t_2 - t_1. \]

If $t_1 \in I_j$ and $t_2 \in I_i$ and $j \neq i$, the same result can be easily proved, by interpolating a suitable number of $\sigma(I_h) = \tilde{\sigma}_h(I_h)$, since each $\tilde{\sigma}_h$ is continuous and obtained by piecing together the $\tilde{\sigma}_i^h$ restricted to $I_i$. Since $\tilde{\sigma}_h$ is increasing, defined on $[0, T]$ onto $\mathbb{R}_+$ and $\sigma(t)$ holds, the maps $\tilde{\varphi}_{0_h} := \tilde{\sigma}_h^1 : \mathbb{R}_+ \rightarrow [0, T]$ are strictly increasing, surjective and $1$-Lipschitz continuous, so that $\lim_{s \rightarrow \infty} \tilde{\varphi}_{0_h}(s) = T$. Taking if necessary a subsequence, $(\tilde{\varphi}_{0_h})_h$ converges locally uniformly to an increasing $1$–Lipschitz continuous function $\tilde{\varphi}_0$, which can be proven to coincide with $\varphi_0$, arguing similarly to the previous case. Hence for each $t \in [0, T]$, $(\sigma(t^+) < +\infty$ and) we can write

\begin{equation}
\sup_{s \in \mathbb{R}_+} |\tilde{\varphi}_{0_h}(s) - \varphi_0(s)| \leq \sup_{s \in [0, \sigma(t^+)]} |\tilde{\varphi}_{0_h}(s) - \varphi_0(s)| 
\end{equation}

\[ + \sup_{s \geq \sigma(t^+)} |\tilde{\varphi}_{0_h}(s) - \varphi_0(s)| \leq \varepsilon_t(h) + (T - (t_h \land t)), \]

where, setting $\varepsilon_t(h) := \sup_{s \in [0, \sigma(t^+)]} |\tilde{\varphi}_{0_h}(s) - \varphi_0(s)|$ and $t_h := \tilde{\varphi}_{0_h}(\sigma(t^+))$, one has

\[ |t_h - t| \leq \varepsilon_t(h) \text{ and } \lim_{h} \varepsilon_t(h) = 0. \]

Finally, we recover a new sequence, denoted by $(\sigma_h)_h$ with strictly increasing, $1$-Lipschitz continuous inverse functions $\varphi_{0_h}$ verifying $(71)$ and such that $\lim_{h} \sigma_h(t) = \sigma(t)$ at every $t \in [0, T]$. Since $\sigma([0, T]) = \mathbb{R}_+$, differently from the previous case, we cannot apply straightforwardly $[AR$, Theorem 5.1$]$, but we can adapt the arguments of its proof to unbounded maps. Let $\mathcal{T} \subset [0, T]$ be the (countable) set of discontinuity points of $\sigma$. For every $\tau_j \in \mathcal{T}$, set $s_{1,j} := \lim_{\tau \rightarrow \tau_j^-} \sigma(\tau)$ and $s_{2,j} := \lim_{\tau \rightarrow \tau_j^+} \sigma(\tau)$ and define a new sequence $(\varphi_{0_h})_h$ such that $\varphi_{0_h}(s) = \tilde{\varphi}_{0_h}(s)$ for every $s \notin [s_{1,j}, s_{2,j}]$, while $\varphi_{0_h}(s)$ is a suitable strictly increasing, $1$-Lipschitz function obtained, in each interval $[s_{1,j}, s_{2,j}]$, by two concatenated linear interpolations of values of $\tilde{\varphi}_{0_h}$, with range equal to the interval $[\tilde{\varphi}_{0_h}(s_{1,j}), \tilde{\varphi}_{0_h}(s_{2,j})]$ and such that the inverse functions $\sigma_h := \varphi_{0_h}^{-1}$ verify $\lim_{h} \sigma_h(\tau_j) = \sigma(\tau_j)$ for every $j$ (we refer for the precise construction to the proof of $[AR$, Theorem 5.1$]$). At this point, it is not difficult to see that $(\varphi_{0_h})_h$, as $(\tilde{\varphi}_{0_h})_h$, converges locally uniformly to $\varphi_0$ and verifies $(71)$.

In order to show that $\sigma_h$ converges pointwisely to $\sigma$, let us consider the sequence $(\tilde{\sigma}_h)_h$ of continuity points of $\sigma$, converging to $T$, which was used in the definition $(65)$, and set $s_{i} := \sigma(\tilde{\sigma}_i)$. By construction, for all $h$ and $i$, we
have
\[ \varphi_{0,h}(s_i) = \tilde{\varphi}_{0,h}(s_i) = \varphi_0(s_i) = \tilde{t}_i, \]
so that \( \sigma_{h}(\tilde{t}_i) = \tilde{\sigma}_{h}(\tilde{t}_i) = \sigma(\tilde{t}_i) \) and \( \sigma_{h}(0, \tilde{t}_i) = [0, s_i] \). Hence the sequence \( (\sigma_{h})_h \) restricted to \( [0, \tilde{t}_i] \) verifies \( \lim_{h} \sigma_{h}(t) = \sigma(t) \) for \( t \in [0, \tilde{t}_i] \) by the proof of [AR] Theorem 5.1. Since, for every \( t \in [0, T] \) there is some \( i \) such that \( t \in [0, \tilde{t}_i] \), we can conclude that \( \sigma_{h} \) pointwisely converges to \( \sigma \) on the whole interval \( [0, T] \).

\[ \square \]

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