COMPLETE SPHERICAL CONVEX BODIES

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Abstract. Similarly to the classic notion in Euclidean space, we call a set on the sphere $S^d$ complete, if adding any extra point increases the diameter. Complete sets are convex bodies of $S^d$. We prove that on $S^d$ complete bodies and bodies of constant width coincide.

1. On spherical geometry

Let $S^d$ be the unit sphere in the $(d + 1)$-dimensional Euclidean space $E^{d+1}$, where $d \geq 2$. By a great circle of $S^d$ we mean the intersection of $S^d$ with any two-dimensional subspace of $E^{d+1}$. The common part of the sphere $S^d$ with any hyper-subspace of $E^{d+1}$ is called a $(d-1)$-dimensional great sphere of $S^d$. By a pair of antipodes of $S^d$ we mean any pair of points of intersection of $S^d$ with a straight line through the origin of $E^{d+1}$.

Clearly, if two different points $a, b \in S^d$ are not antipodes, there is exactly one great circle containing them. By the arc $ab$ connecting $a$ and $b$ we mean the “smaller” part of the great circle containing $a$ and $b$. By the spherical distance $|ab|$, or shortly distance, of these points we understand the length of the arc connecting them. The diameter $\text{diam}(A)$ of a set $A \subset S^d$ is the number $\sup_{a,b \in A} |ab|$. By a spherical ball $B_\rho(r)$ of radius $\rho \in (0, \frac{\pi}{2}]$, or shorter a ball, we mean the set of points of $S^d$ having distance at most $\rho$ from a fixed point, called the center of this ball. Spherical balls of radius $\frac{\pi}{2}$ are called hemispheres. Two hemispheres whose centers are antipodes are called opposite hemispheres.

We say that a subset of $S^d$ is convex if it does not contain any pair of antipodes and if together with every two points $a, b$ it contains the arc $ab$. By a convex body, or shortly body, on $S^d$ we mean any closed convex set with non-empty interior.

Recall a few notions from [6]. If for a hemisphere $H$ containing a convex body $C \subset S^d$ we have $\text{bd}(H) \cap C \neq \emptyset$, then we say that $H$ supports $C$. If hemispheres $G$ and $H$ of $S^d$ are different and not opposite, then $L = G \cap H$ is called a lune of $S^d$. The $(d-1)$-dimensional hemispheres bounding the lune $L$ and contained in $G$ and $H$, respectively, are denoted by $G/H$ and $H/G$. We define the thickness of a lune $L = G \cap H$ as the spherical distance of the centers of $G/H$ and $H/G$. For a hemisphere $H$ supporting a convex body $C \subset S^d$ we define the width $\text{width}_H(C)$ of $C$ determined by $H$ as the minimum thickness of a lune of the form $H \cap H'$, where $H'$ is a hemisphere, containing $C$. If for all hemispheres $H$ supporting $C$ we have $\text{width}_H(C) = w$, we say that $C$ is of constant width $w$. 
2. Spherical complete bodies

Similarly to the traditional notion of a complete set in the Euclidean space $E^d$ (for instance, see [1], [2], [3] and [10]) we say that a set $K \subset S^d$ of diameter $\delta \in (0, \pi)$ is complete provided $\text{diam}(K \cup \{x\}) > \delta$ for every $x \notin K$.

**Theorem 1.** Arbitrary set of a diameter $\delta \in (0, \pi)$ on the sphere $S^d$ is a subset of a complete set of diameter $\delta$ on $S^d$.

We omit the proof since it is similar to the proof by Lebesgue [9] in $E^d$ (it is recalled in Part 64 of [1]). Let us add that earlier Pál [12] proved this for $E^2$ by a different method.

The following fact permits to use the term a complete convex body for a complete set.

**Lemma 1.** Let $K \subset S^d$ be a complete set of diameter $\delta$. Then $K$ coincides with the intersection of all balls of radius $\delta$ centered at points of $K$. Moreover, $K$ is a convex body.

**Proof.** Denote by $I$ the intersection of all balls of radius $\delta$ with centers in $K$.

Since $\text{diam}(K) = \delta$, then $K$ is contained in every ball of radius $\delta$ whose center is at a point of $K$. Consequently, $K \subset I$.

Let us show that $I \subset K$, so let us show that $x \notin K$ implies $x \notin I$. Really, from $x \notin K$ we get $|xy| > \delta$ for a point $y \in K$, which means that $x$ is not in the ball of radius $\delta$ and center $y$, and thus $x \notin I$.

As an intersection of balls, $K$ is a convex body. \hfill \Box

**Lemma 2.** If $K \subset S^d$ is a complete body of diameter $\delta$, then for every $p \in \text{bd}(K)$ there exists $p' \in K$ such that $|pp'| = \delta$.

**Proof.** Suppose the contrary, i.e., that $|pq| < \delta$ for a point $p \in \text{bd}(K)$ and for every point $q \in K$. Since $K$ is compact, there is an $\varepsilon > 0$ such that $|pq| \leq \delta - \varepsilon$ for every $q \in K$. Hence, in a positive distance below $\varepsilon$ from $p$ there is a point $s \notin K$ such that $|sq| \leq \delta$ for every $q \in K$. Thus $\text{diam}(K \cup \{s\}) = \delta$, which contradicts the assumption that $K$ is complete. Consequently, the thesis of our lemma holds true. \hfill \Box

For different points $a, b \in S^d$ at a distance $\delta < \pi$ from a point $c \in S^d$ define the piece of circle $P_c(a, b)$ as the set of points $v \in S^d$ such that $cv$ has length $\delta$ and intersects $ab$.

We show the next lemma for $S^d$ despite we apply it later only for $S^2$.

**Lemma 3.** Let $K \subset S^d$ be a complete convex body of diameter $\delta$. Take $P_c(a, b)$ with $|ac|$ and $|bc|$ equal to $\delta$ such that $a, b \in K$ and $c \in S^d$. Then $P_c(a, b) \subset K$.

**Proof.** First let us show the thesis for a ball $B$ of radius $\delta$ in place of $K$. There is unique $S^2 \subset S^d$ with $a, b, c \in S^2$. Consider the disk $D = B \cap S^2$. Take the great circle containing $P_c(a, b)$ and points $a^*, b^*$ of its intersection with the circle bounding $D$. There is unique $c^* \in S^2$ such that $P_c(a, b) \subset P_{c^*}(a^*, b^*)$. Clearly, $P_{c^*}(a^*, b^*) \subset D \subset B$. Hence $P_c(a, b) \subset B$.

By the preceding paragraph and Lemma 1 we obtain the thesis of the present lemma. \hfill \Box
3. Complete and constant width bodies on \( S^d \) coincide

Here is our main result presenting the spherical version of the classic theorem in \( E^d \) proved by Meissner \[11\] for \( d = 2, 3 \) and by Jessen \[5\] for arbitrary \( d \).

**Theorem 2.** A convex body of diameter \( \delta \) on \( S^d \) is complete if and only if it is of constant width \( \delta \).

**Proof.** (\( \Rightarrow \)) Prove that if \( K \subset S^d \) of diameter \( \delta \) is complete, then \( K \) is of constant width \( \delta \).

Suppose the opposite, i.e., that \( \text{width}_I(K) \neq \delta \) for a hemisphere \( I \) supporting \( K \). By Theorem 3 and Proposition 1 of \[6\] \( \text{width}_I(K) \leq \delta \). So \( \Delta(K) < \delta \). By lines 1-2 of p. 562 of \[6\] the thickness of \( K \) is equal to the minimum thickness of a lune containing \( K \). Take such a lune \( L = G \cap H \), where \( G, H \) are different and non-opposite hemispheres. Denote by \( g, h \) the centers of \( G/H \) and \( H/G \), respectively. Of course, \( |gh| < \delta \). By Claim 2 of \[6\] we have \( g, h \in K \). By Lemma 2 there exists a point \( g' \in K \) in the distance \( \delta \) from \( g \). Since the triangle \( ghg' \) is non-degenerate, there is a unique two-dimensional sphere \( S^2 \subset S^d \) containing \( g, h, g' \). Clearly, \( ghg' \) is a subset of \( M = K \cap S^2 \). Hence \( M \) is a convex body on \( S^2 \). Denote by \( F \) this hemisphere of \( S^2 \) such that \( hg' \subset \text{bd}(F) \) and \( g \in F \). There is a unique \( c \in F \) such that \( |ch| = \delta = |cg'| \). By Lemma 3 for \( d = 2 \) we have \( P_c(h, g') \subset M \).

We intend to show that \( c \) is not on the great circle \( E \) of \( S^2 \) through \( g \) and \( h \). In order to see this, for a while suppose the opposite, i.e. that \( c \in E \). Then from \( |g'g| = \delta, |g'c| = \delta \) and \( |hc| = \delta \) we conclude that \( \angle ggc = \angle cg'g \). So the spherical triangle \( g'gc \) is isosceles, which together with \( |gg'| = \delta \) gives \( |cg| = \delta \). Since \( |gh| = \Delta(L) = \Delta(K) > 0 \) and \( g \) is a point of \( ch \) different from \( c \), we get a contradiction. Hence, really, \( c \notin E \).

By the preceding paragraph \( P_c(h, g') \) intersects \( \text{bd}(M) \) at a point \( h' \) different from \( h \) and \( g' \). So the non-empty set \( P_c(h, g') \setminus \{ h, h' \} \) is out of \( M \). This contradicts the result of the paragraph before the last. Consequently, \( K \) is a body of constant width \( \delta \).

(\( \Leftarrow \)) Let us prove that if \( K \) is a spherical body of constant width \( \delta \), then \( K \) is a complete body of diameter \( \delta \). In order to prove this, it is sufficient to take any point \( r \notin K \) and show that \( \text{diam}(K \cup \{ r \}) > \delta \).

Take the largest ball \( B_\rho(r) \) disjoint with the interior of \( K \). Since \( K \) is convex, \( B_\rho(r) \) has in common with \( K \) exactly one point \( p \). By Theorem 3 of \[8\] there exists a lune \( L \supset K \) of thickness \( \delta \) with \( p \) as the center of one of the two \( (d - 1) \)-dimensional hemispheres bounding this lune. Denote by \( q \) the center of the other \( (d - 1) \)-dimensional hemisphere. By Claim 2 of \[6\] also \( q \in K \). Since \( p \) and \( q \) are the centers of the two \( (d - 1) \)-dimensional hemispheres bounding \( L \), we have \( |pq| = \delta \). From the fact that \( rp \) and \( pq \) are orthogonal to \( \text{bd}(H) \) at \( p \), we see that \( p \in rq \). Moreover, \( p \) is not an endpoint of \( rq \) and \( |pq| = \delta \). Hence \( |rq| > \delta \). Thus \( \text{diam}(K \cup \{ r \}) > \delta \). Since \( r \notin K \) is arbitrary, \( K \) is complete. \( \square \)

We say that a convex body \( D \subset S^d \) is of constant diameter \( \delta \) provided \( \text{diam}(D) = \delta \) and for every \( p \in \text{bd}(D) \) there is a point \( p' \in \text{bd}(D) \) with \( |pp'| = \delta \) (see \[8\]).

The following fact is analogous to the result in \( E^d \) given by Reidemeister \[13\].
Theorem 3. Bodies of constant diameter on $S^d$ coincide with complete bodies.

Proof. Take a complete body $D \subset S^d$ of diameter $\delta$. Let $g \in \text{bd}(D)$ and $G$ be a hemisphere supporting $D$ at $g$. By Theorem 2 our $D$ is of constant width $\delta$. So width$_G(D) = \delta$ and a hemisphere $H$ exists that the lune $G \cap H \supset D$ has thickness $\delta$. By Claim 2 of [6] centers $h$ of $H/G$ and $g$ of $G/H$ belong to $D$. So $|gh| = \delta$. Thus $D$ is of constant diameter $\delta$.

Consider a body $D \subset S^d$ of constant diameter $\delta$. Let $r \notin D$. Take the largest $B_\rho(r)$ whose interior is disjoint with $D$. Denote by $p$ the common point of $B_\rho(r)$ and $D$. Observe that $D \subset J$ (if not, there is point $v \in D$ out of $J$; clearly $vp$ passes through int$B_\rho(r)$, a contradiction). Since $D$ is of constant diameter $\delta$, there is $p' \in D$ with $|pp'| = \delta$. Observe that $\angle rpp' \geq \frac{\pi}{2}$. If it is $\pi/2$, then $|rp'| > \delta$. If it is over $\pi/2$, the triangle $rpp'$ is obtuse and then by the law of cosines $|rp'| > |pp'|$ and hence $|rp'| > \delta$. By $|rp'| > \delta$ in both cases we see that $D$ is complete. □

By Theorem 2 in Theorem 8 we may exchange “complete” to “constant width”. This form is proved earlier as follows. Any body of constant width $\delta$ on $S^d$ is of constant diameter $\delta$ and the inverse is shown for $\delta \geq \pi/2$, and for $\delta < \pi/2$ if $d = 2$ (see [8]). By [4] the inverse holds for any $\delta$. Our short proof of Theorem 3 is different from these in [8] and [4].

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