The Casimir-Aharonov-Bohm effect?

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Abstract

The combined effect of the magnetic field background in the form of a singular vortex and the Dirichlet boundary condition at the location of the vortex on the vacuum of quantized scalar field is studied. We find the induced vacuum energy density and current to be periodic functions of the vortex flux and holomorphic functions of the space dimension.

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The emergence of calculable and detectable vacuum energy in quantum field theory under the influence of boundary conditions was predicted first by Casimir \[1\]. Let \(X\) be the base space manifold and \(Y\) be a submanifold of dimension less than that of \(X\). Then the matter field in \(X\) is quantized under a certain boundary condition imposed at \(Y\) and the vacuum polarization effects are studied. Usually \(Y\) is chosen to be noncompact disconnected (e.g. two parallel infinite plates, as generically in Ref. \[1\]) or closed compact (e.g. box or sphere), see Refs. \[2\] and \[3\]. However, as it seems to us, one possibility has been overlooked, namely that \(Y\) is noncompact connected. Therefore we propose to consider such a possibility, moreover to choose \(Y\) as a manifold that confines a singular magnetic vortex in itself: this will allow us to find one more manifestation of the Aharonov-Bohm effect \[4\] in vacuum polarization \[5\].

To be more precise, we start from the operator of the secondly quantized complex scalar field in an external static background

\[
\Psi(x, t) = \sum_{\lambda} \frac{1}{\sqrt{2E_{\lambda}}} \left[ e^{-iE_{\lambda}t} \langle x|\lambda \rangle a_{\lambda} + e^{iE_{\lambda}t} \langle x| -\lambda \rangle b_{\lambda}^+ \right],
\]

where \(a_{\lambda}^+\) and \(a_{\lambda}\) (\(b_{\lambda}^+\) and \(b_{\lambda}\)) are the scalar particle (antiparticle) creation and annihilation operators satisfying the commutation relation

\[
[a_{\lambda}, a_{\lambda'}^+] = [b_{\lambda}, b_{\lambda'}^+] = \langle \lambda|\lambda' \rangle,
\]

\(\lambda\) is the set of parameters (quantum numbers) specifying the state, \(E_{\lambda} = E_{-\lambda} > 0\) is the energy of the state; symbol \(\sum_{\lambda}\) implies the summation over the discrete and the integration (with a certain measure) over the continuous values of \(\lambda\). The wave function \(\langle x|\lambda \rangle\) satisfies the stationary Klein-Gordon equation

\[
\left( -\nabla^2 + m^2 \right) \langle x|\lambda \rangle = E_{\lambda}^2 \langle x|\lambda \rangle,
\]

(\(\nabla\) is the covariant derivative in an external static background) and the Dirichlet boundary condition

\[
\langle x|\lambda \rangle \big|_{x \in Y} = 0.
\]
We take Euclidean d-dimensional space as a base space \( X \) and define \( Y \) as follows

\[
Y : \ x^1 = x^2 = 0, \quad (5)
\]

i.e. the point in the case of \( d = 2 \), the line in the case of \( d = 3 \) and the \((d - 2)\)-dimensional hypersurface in the case of \( d > 3 \). We take classical static magnetic field as an external background, thus the covariant derivative is defined as

\[
\nabla = \partial - iV(x), \quad (6)
\]

where \( V(x) \) is the vector potential of the magnetic field; note that in the \( d \)-dimensional space the magnetic field strength is an antisymmetric tensor of the rank \( d - 2 \)

\[
B^{\nu_1 \cdots \nu_{d-2}}(x) = \left[ \partial_{\mu_1} V_{\mu_2}(x) \right] \epsilon^{\mu_1 \mu_2 \nu_1 \cdots \nu_{d-2}}, \quad (7)
\]

\( \epsilon^{\mu_1 \cdots \mu_d} \) is the totally antisymmetric tensor, \( \epsilon^{12\cdots d} = 1 \). The magnetic field configuration is chosen to be that of a singular vortex placed at \( Y \):

\[
V_1(x) = -\Phi \frac{x^2}{(x^1)^2 + (x^2)^2}, \quad V_2(x) = \Phi \frac{x^1}{(x^1)^2 + (x^2)^2}, \quad V_\nu(x) = 0, \quad \nu = 3, d, \quad (8)
\]

\[
B^{3\cdots d}(x) = 2\pi \Phi \delta(x^1) \delta(x^2), \quad (9)
\]

\( \Phi \) is the total flux (in the units of \( 2\pi \)) of the vortex. In this Letter we shall find how the vacuum of quantized scalar field is polarized under the boundary condition \( (4) \) and \( (5) \) in the magnetic field background \( (6) \) and \( (8) \).

The crucial role is played by the zeta function of the operator \((-\nabla^2 + m^2)\)

\[
\zeta(s) = \int d^d x \zeta_x(s), \quad (10)
\]

where the zeta function density can be presented in the form

\[
\zeta_x(s) = \lim_{x' \to x} \langle x | (-\nabla^2 + m^2)^{-s} | x' \rangle, \quad (11)
\]

In the free field case, i.e. in the absence of any boundary condition and any background field, one has

\[
\zeta_x^{(0)}(s) = \lim_{x' \to x} \langle x | (-\partial^2 + m^2)^{-s} | x' \rangle = (2\pi)^{-d} \int d^d p (p^2 + m^2)^{-s}. \quad (12)
\]
The integral in Eq. (12) is convergent at $s > \frac{d}{2}$ when it can be easily evaluated:

$$\zeta^{(0)}_x(s) = \frac{m^{d-2s}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)},$$

(13)

where $\Gamma(z)$ is the Euler gamma function. We have calculated the zeta function density under the conditions specified in Eqs. (4)-(6) and (8) (details will be published elsewhere):

$$\zeta_x(s) = \frac{4 \sin(F\pi)}{(4\pi)^{\frac{d}{2}+1}} \frac{m^{d-2s}}{\Gamma(s)} \int_0^\infty du e^{-u} \left[ K_F(u) + K_{1-F}(u) \right] \gamma \left( s - \frac{d}{2}, \frac{m^2 r^2}{2u} \right),$$

(14)

where

$$\gamma(z, w) = \int_0^w du u^{z-1} e^{-u}$$

is the incomplete gamma function and $K_\omega(z)$ is the McDonald function of the order $\omega$.

$w, r = \sqrt{(x^1)^2 + (x^2)^2}$ and

$$F = \Phi - \left[\Phi\right], \quad 0 \leq F < 1,$$

(15)

$\left[\Phi\right]$ is the integer part of the quantity $\Phi$ (i.e. the integer which is less than or equal to $\Phi$; note that $\zeta_x(s)$ (14) is a periodic function of the vortex flux $\Phi$, since it depends only on $F$ (being symmetric under $F \to 1 - F$).

Although $\zeta_x(s)$, as well as $\zeta^{(0)}_x(s)$, has been calculated at $s > \frac{d}{2}$, both results (13) and (14) can be extended analytically to the whole complex $s$-plane; for $\zeta^{(0)}_x(s)$ the analytic continuation is given obviously by the gamma function, while for $\zeta_x(s)$ the analytic continuation is obtained with the help of the incomplete gamma function (15), i.e. using repeatedly the recurrence relation

$$\gamma(z, w) = \frac{1}{z} \left[ \gamma(z+1, w) + w^z e^{-w} \right].$$

(17)

Moreover, both products $\zeta_x(s)\Gamma(s)$ and $\zeta^{(0)}_x(s)\Gamma(s)$ have the same singularity structure comprising of simple poles on the real axis at $s = \frac{d}{2} + 1 - N (N = 1, 2...).$

Therefore, the difference

$$\zeta_x^{\text{ren}}(s) = \zeta_x(s) - \zeta^{(0)}_x(s)$$

(18)
and even the product $\zeta^{\text{ren}}(s)\Gamma(s)$ appear to be holomorphic on the whole complex s-plane.

The renormalized zeta function density (18) can be presented in the form

$$
\zeta^{\text{ren}}_x(s) = -\frac{16 \sin(F\pi)}{(4\pi)^{\frac{d}{2}+1}\Gamma(s)} \left(\frac{r}{m}\right)^{\frac{s-d}{2}} \times
\int_1^\infty \frac{dv}{\sqrt{v^2-1}} \cosh \left[(2F-1)\text{Arcosh}v\right] v^{s-\frac{d}{2}-1} K_{s-\frac{d}{2}}(2mrv). \quad (19)
$$

Unlike $\zeta_x(s)$ (14) and $\zeta^{(0)}_x(s)$ (13), $\zeta^{\text{ren}}_x(s)$ is decreasing exponentially at large distances from the vortex

$$
\zeta^{\text{ren}}_x(s) = -\frac{\sin(F\pi)}{(4\pi)^{\frac{d}{2}}\Gamma(s)} e^{-2mr} m^{\frac{d}{2}-s-1} r^{s-\frac{d}{2}-1} \left\{1 + O\left[(m)^{-1}\right]\right\}, \quad mr \gg 1. \quad (20)
$$

At small distances $\zeta^{\text{ren}}_x(s)$ is characterized by the power behaviour

$$
\zeta^{\text{ren}}_x(s) = -\frac{4 \sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}\Gamma(s)} \frac{\Gamma\left(\frac{d}{2} - s + F\right) \Gamma\left(\frac{d}{2} - s + 1 - F\right)}{(d-2s)\Gamma\left(\frac{d+1}{2} - s\right)} r^{2s-d} \left\{1 + O\left[(m)^{2}\right]\right\}, \quad mr \ll 1. \quad (21)
$$

Integrating over the plane which is orthogonal to the vortex, we get

$$
\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \zeta^{\text{ren}}_x(s) = -\frac{m^{d-2s-2}}{2(4\pi)^{\frac{d}{2}-1}} \frac{\Gamma(s - \frac{d}{2} + 1)}{\Gamma(s)} F(1 - F); \quad (22)
$$

the integral in Eq.(22) is evaluated at Re $s > \frac{d}{2} - 1$, however the result can be extended analytically to the whole complex s-plane.

The renormalized vacuum energy density is defined as (see Eqs.(11),(12) and (18))

$$
\varepsilon^{\text{ren}}(x) = \zeta^{\text{ren}}_x\left(-\frac{1}{2}\right), \quad (23)
$$

thus we get

$$
\varepsilon^{\text{ren}}(x) = \frac{16 \sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}} \left(\frac{m}{r}\right)^{\frac{d+1}{2}} \times
\int_1^\infty \frac{dv}{\sqrt{v^2-1}} \cosh \left[(2F-1)\text{Arcosh}v\right] v^{-\frac{d+1}{2}} K_{\frac{d+1}{2}}(2mrv). \quad (24)
$$

The asymptotics of $\varepsilon^{\text{ren}}(x)$ at large and small distances are immediately obtained from Eqs.(20) and (21) at $s = -\frac{1}{2}$. Concerning the dependence on the fractional
part of the vortex flux $F$, the renormalized vacuum energy density is a positive
function, which is convex upwards, symmetric under $F \to 1 - F$, and has maximum
at $F = \frac{1}{2}$. We get

$$\varepsilon_{\text{ren}}(x)|_{F = \frac{1}{2}} = \frac{2m^{2N+1}}{(4\pi)^{N+\frac{3}{2}}} \left\{ \frac{(-1)^N}{\Gamma(N + \frac{3}{2})} \right\} \left\{ \frac{\pi}{2} + \pi m r \left[ K_0(2mr) - K_1(2mr) \right] + \frac{1}{\sqrt{\pi}} \sum_{l=0}^{N} (-1)^l \Gamma \left( l + \frac{1}{2} \right) (mr)^{-l} K_{l+1}(2mr) \right\}, \quad d = 2N, \quad (25)$$

$$\varepsilon_{\text{ren}}(x)|_{F = \frac{1}{2}} = \frac{2m^{2(N+1)}}{(4\pi)^{N+1}} \left\{ \frac{(-1)^N}{\Gamma(N + 2)} \right\} \left\{ - E_1(2mr) + e^{-2mr} \sum_{l=0}^{N} (-1)^l \Gamma \left( l + 1 \right) \sum_{n=0}^{l+1} \frac{\Gamma(l + n + 2)(mr)^{-l-n-1}}{2^{n+1} \Gamma(n+1) \Gamma(l - n + 2)} \right\}, \quad d = 2N + 1, \quad (26)$$

where $L_\omega(z)$ is the modified Struve function of the order $\omega$ and

$$E_1(w) = \int_{w}^{\infty} \frac{du}{u} e^{-u}$$

is the integral exponential function (see e.g. [1]).

Let us emphasize the following remarkable fact: since the first argument of the
incomplete gamma function in Eq.(14) is $s - \frac{d}{2}$ and the analytic continuation in
the variable $s$ is possible, then the analytic continuation in the variable $d$ is also possible.
So, Eq.(24) can be continued analytically in $d$, and the renormalized vacuum energy
density appears to be holomorphic on the complex $d$-plane. Therefore, Eq.(24) can be regarded as a result obtainable in the framework of the dimensional regularization
procedure: the difference between vacuum energy densities in the presence and the
absence of a vortex being calculated at $\text{Re} \ d < -1$ and then continued analytically
to the whole complex $d$-plane.

What else, in addition to the energy density, is induced in the vacuum? Contrary
to the case of the fermionic vacuum (see Refs.[5,7,8]), the charge and the angular
momentum cannot be induced in the bosonic vacuum, so there remains the current

$$j(x) = \lim_{x' \to x} \left. \langle x \big| (-i \nabla) (-\nabla^2 + m^2)^{-\frac{1}{2}} \big| x' \rangle \right.$$  

$$= (28)$$
Under the condition (4)-(6) and (8) only one (angular) component is nonvanishing:
\[
j_\varphi(x) \equiv r^{-1} \left[ x^1 j_2(x) - x^2 j_1(x) \right] = \frac{32 \sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}} m^{\frac{d+3}{2}} r^{-\frac{d+3}{2}} \int_{1}^{\infty} dv \sinh \left[ (2F - 1) \text{Arcosh} \right] v^{-\frac{d+1}{2}} K_{\frac{d+1}{2}}(2mr). (29)
\]
The calculation has been performed at \(\text{Re } d < 3\), and then the result has been continued analytically as a holomorphic function on the whole complex \(d\)-plane.

Unlike the vacuum energy density (24), the vacuum current (29) vanishes at \(F = \frac{1}{2}\), being negative and convex downwards in the interval \(0 < F < \frac{1}{2}\), while positive and convex upwards in the interval \(\frac{1}{2} < F < 1\); the positions of the minimum and the maximum are symmetric with respect to the point \(F = \frac{1}{2}\). The asymptotics at large and small distances from the vortex are the following:

\[
j_\varphi(x) = \frac{2 \sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}} \left( F - \frac{1}{2} \right) e^{-2mr} m^{\frac{d-3}{2}} r^{-\frac{d+3}{2}} \left\{ 1 + O \left[ (mr)^{-1} \right] \right\}, \quad mr \gg 1, \quad (30)
\]
\[
j_\varphi(x) = \frac{4 \sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}} \left( F - \frac{1}{2} \right) \frac{\Gamma \left( \frac{d+1}{2} + F \right) \Gamma \left( \frac{d+1}{2} + 1 - F \right)}{\Gamma \left( \frac{d+1}{2} + 1 \right)} r^{-d} \left\{ 1 + O \left[ (mr)^{2} \right] \right\}, \quad mr \ll 1. \quad (31)
\]

Note, that in the case of odd \(d\) the integral in Eq.(29) can be taken, yielding a quadratic combination of the McDonald functions of the variable \(mr\), whereas in the case of even \(d\) Eq.(29) can be transformed to a linear combination of the integral \(\int_{2mr}^{\infty} \frac{du}{u} K_{2F-1}(u)\) and the McDonald functions of the variable \(2mr\). In particular, we get

\[
j_\varphi(x) = \frac{\sin(F\pi)}{4\pi^{2}r^{2}} \left( F - \frac{1}{2} \right) \left\{ -4 \left[ (F - \frac{1}{2})^{2} + m^{2}r^{2} \right] \int_{2mr}^{\infty} \frac{du}{u} K_{2F-1}(u) + +mr \left[ K_{2F}(2mr) + K_{2(1-F)}(2mr) \right] \right\}, \quad d = 2, \quad (32)
\]
\[
j_\varphi(x) = \frac{\sin(F\pi)}{6\pi^{3}} m \frac{r}{r^{2}} \left\{ F(F - \frac{1}{2}) + m^{2}r^{2} \right\} mr K_{F}^{2}(mr) - [1 - F)] \left( \frac{1}{2} - F \right) + m^{2}r^{2} \right\} mr K_{1-F}^{2}(mr) + +2 \left[ F(1 - F) - m^{2}r^{2} \right] (F - \frac{1}{2}) K_{1-F}(mr) K_{F}(mr), \quad d = 3, \quad (33)
\]
\[
\begin{align*}
    j_\phi(x) &= \frac{\sin(F\pi)}{32\pi^3 r^4} \left( F - \frac{1}{2} \right) \left( 4 \left( F - \frac{1}{2} \right)^2 - \frac{1}{2} + 2m^2 r^2 \right) + m^4 r^4 \times \\
    &\times \int_{2mr}^{\infty} \frac{du}{u} K_{2F-1}(u) + 2m^2 r^2 K_{2F-1}(2mr) - \\
    &- \left( F - \frac{1}{2} \right)^2 - 1 + m^2 r^2 \right) mr \left[ K_{2F}(2mr) + K_{2(1-F)}(2mr) \right], \quad d = 4, \quad (34)
\end{align*}
\]

\[
\begin{align*}
    j_\phi(x) &= \frac{\sin(F\pi) m}{60\pi^4} \left( m \right) \left( F - \frac{1}{2} \right) \left( 1 + F \right) (2 - F) - \left( 2F - \frac{1}{2} \right) m^2 r^2 \right) - \\
    &- m^4 r^4 \right) mr K_F^2(mr) - \left( F - \frac{1}{2} - F \right) \left( 1 + F \right) (1 - F) (2 - F) - \left( \frac{3}{2} - 2F \right) m^2 r^2 \right) - \\
    &- m^4 r^4 \right) mr K_{1-F}^2(mr) + 2 \left( 1 + F \right) F (1 - F) (2 - F) + \\
    &+ \left( 1 - 2F (1 - F) \right) m^2 r^2 + m^4 r^4 \right) F - \left( \frac{1}{2} \right) K_F(mr) K_{1-F}(mr) \right), \quad d = 5; \quad (35)
\end{align*}
\]

the result (33) has been already known for a rather long time [9,10].

Supposing the validity of the Maxwell equation relating the vacuum current to
the vacuum magnetic field with the total flux (in the units of \(2\pi\))

\[
\Phi^{(t)} = \frac{e^2}{2} \int_0^{\infty} dr \, r^2 j_\phi(x)
\]

\((e\) is the coupling constant possessing the dimension \(m^{3-d/2}\)), we get

\[
\Phi^{(t)} = \frac{2e^2 m^{d-3}}{3(4\pi)^{d+1}} \Gamma \left( \frac{d-3}{2} \right) F \left( 1 - F \right) \left( F - \frac{1}{2} \right); \quad (37)
\]

this result is evaluated at \(\text{Re } d < 3\) and can be continued analytically as a meromorphic function on the whole complex \(d\)-plane. Note that in the physically meaningful domain \(d \geq 2\) Eq.(37), as well as Eq.(22) at \(s = -\frac{1}{2}\), has poles at odd \(d\) and is finite at even \(d\).

Completing the analysis of the vacuum polarization effects in the background of
a singular magnetic vortex, we note that the vacuum energy density (24) is even
and the vacuum current (29) is odd under charge conjugation. Note also that in
the case of quantized massless scalar field \((m = 0)\) the expressions for the vacuum

\[\footnote{Note that the results of Ref.\[9\] concerning the vacuum energy density in the case of \(d = 3\) are intrinsically controversial and completely wrong.}\]
energy density and current are simplified considerably, see Eqs.\((21)\) at \(s = -\frac{1}{2}\) and \((31)\).

In conclusion let us compare the above results with the pure Casimir effect, i.e. the effect of the boundary condition \((4)\) and \((5)\) solely (in the absence of any background field) on the bosonic vacuum. Certainly, the vacuum current is vanishing in this case, whereas for the renormalized zeta function density we get the expression

\[
\zeta_{x}^{\text{ren}}(s) = -\frac{1}{(2\pi)^{\frac{d}{2}}} \frac{r^{2s-d}}{2s\Gamma(s)} \int_{0}^{\infty} du \frac{u^{\frac{d}{2}-s-1}}{\exp\left(-u - \frac{m^{2}r^{2}}{2u}\right) I_{0}(u)}, \tag{38}
\]

where \(I_{\omega}(z)\) is the modified Bessel function of the order \(\omega\). The integral in Eq.\((38)\) is divergent at \(\text{Re } s < \frac{d-1}{2}\) and cannot be extended analytically to this domain, thus the vacuum energy density (Eq.\((38)\) at \(s = -\frac{1}{2}\)) is infinite in the physically meaningful case \(d \geq 2\).

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