How to measure the spreading width for decay of superdeformed nuclei

D. M. Cardamone, C. A. Stafford, B. R. Barrett

Physics Department, P. O. Box 210081, University of Arizona, Tucson, AZ 85721
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A new expression for the branching ratio for the decay via the E1 process in the normal-deformed band of superdeformed nuclei is given within a simple two-level model. Using this expression, the spreading or tunneling width \( \Gamma^\downarrow \) for superdeformed decay can be expressed entirely in terms of experimentally known quantities. We show how to determine the tunneling matrix element \( V \) from the measured value of \( \Gamma^\downarrow \) and a statistical model of the energy levels. The accuracy of the two-level approximation is verified by considering the effects of the other normal-deformed states.

Since the first discovery of superdeformation in \(^{152}\text{Dy}\) \(^1\), one of the principal challenges has been to develop a consistent theory regarding the decay-out mechanism of the superdeformed (SD) rotational band into the normal-deformed (ND) band. Although much experimental progress has been made since this first discovery, e.g., Lauritsen et al. \(^2\) and references therein, no consistent theory has been achieved for this decay-out process, and, in fact, considerable confusion still exists regarding the application of the current theoretical interpretations. The purpose of this Letter is to report on new theoretical developments, which permit a direct determination of the spreading (or tunneling) width \( \Gamma^\downarrow \) for decay out of the SD band in nuclei in terms of experimental quantities and, thereby, to obtain the magnitude of the tunneling matrix element.

In an earlier publication \(^2\), two of us presented a simple two-level model to explain the decay out of the SD band in nuclei. Employing a retarded Green’s function approach, we obtained an exact solution for the branching ratio, \( F_N \), for decay via the E1 process in the ND band, in terms of the decay widths \( \Gamma_S \) and \( \Gamma_N \) in the SD and ND potential wells, respectively; \( V \), the tunneling matrix element connecting the state with which it mixes most strongly; and \( \Delta = \epsilon_N - \epsilon_S \), the difference between the unperturbed energies of these two states (see Fig. 1 for a graphical representation of these quantities). Our result yielded

\[
F_N = \frac{(1 + \Gamma_N/\Gamma_S)V^2}{\Delta^2 + \Gamma^2(1 + 4V^2/\Gamma_N\Gamma_S)}.
\]  

where \( \Delta = \Gamma_S + \Gamma_N \)/2 and \( V \) is taken to be positive definite without loss of generality.

We have recently observed that Eq. (1) can also be rewritten in the form

\[
F_N = \frac{\Gamma_N \Gamma^\downarrow/(\Gamma_N + \Gamma^\downarrow)}{\Gamma_S + \Gamma_N \Gamma^\downarrow/(\Gamma_N + \Gamma^\downarrow)},
\]  

where

\[
\Gamma^\downarrow = \frac{2\Gamma V^2}{\Delta^2 + \Gamma^2}
\]  

is the correct expression from Fermi’s golden rule for the spreading (or tunneling) width \( \Gamma^\downarrow \). Eq. (2) clearly shows that decay into the ND band is a two-step process, and that \( \Gamma^\downarrow \) is a real, physical rate, not a mere theoretical construct. Note that

\[
\Gamma^\downarrow \neq 2\pi (V^2)/D_N,
\]  

as was employed by Vigezzi et al. \(^4\). In Eq. (4), \( \langle V^2 \rangle \) is the mean square of the coupling matrix elements connecting the SD and ND states and \( D_N \) is the average spacing of the ND states. In fact, Eq. (4) gives the average spreading width over a flat distribution in \( \Delta \), which can deviate drastically from the exact value given by Eq. (5).

Importantly, \( \Gamma^\downarrow \), as given by Eq. (5), is a measurable quantity, which can be determined from Eq. (2):

\[
\Gamma^\downarrow = \frac{F_N \Gamma_N \Gamma_S}{\Gamma_N - F_N (\Gamma_N + \Gamma_S)}.
\]  

\( \Gamma^\downarrow \) is not a free parameter as was stated in Ref. \(^2\).

We have used Eq. (5) to determine \( \Gamma^\downarrow \) from the values of \( F_N \), \( \Gamma_S \), and \( \Gamma_N \) given in Table I of Ref. \(^2\) and obtain values differing radically from those listed in Ref. \(^2\), by three to five orders of magnitude (see our Table I). This is a direct contradiction of the statement in Ref. \(^2\) that “For cases encountered in experiments, their results (i.e.,
the results of our Ref. 3 agree with those obtained with the model of Vigezzi et al., i.e., the model that was employed in Ref. 2.

It should be noted that we obtain a negative value for $\Gamma^d$ in $^{152}$Dy with $I = 26$. This is, of course, physically impossible: $\Gamma^d$ is a positive definite quantity. Eq. (5) therefore requires

$$F_N < \frac{\Gamma_N}{\Gamma_N + \Gamma_S}.$$  

(6)

Experimental results in which the inequality (6) is violated may indicate that the experimentally measured value of $F_N$ is too large or that the SD state is mixing with only one ND state. At the present time, however, uncertainties in the known values of $\Gamma_N$ and $\Gamma_S$, which are of the order of 100% for $\Gamma_N$ and of the order of 10% for $\Gamma_S$, or more, mean that $\Gamma^d$ cannot be meaningfully determined in cases such as $^{152}$Dy(I=26).

In order to determine the tunneling matrix element $V$ from $\Gamma^d$ via Eq. (4), we must know $\Delta$, which generally is not experimentally known. We therefore compute the expected value for $\Delta$ based on the assumption that the states to which the SD state decays in the ND well are distributed according to a Gaussian orthogonal ensemble (GOE). In the GOE, the probable spacing between levels is given by the distribution

$$P(s) = \frac{\pi}{2} e^{-\pi s^2/4},$$

(7)

where $s$ is the spacing in units of $D_N$.

In the absence of tunneling ($V = 0$), the energy spectra of the ND and SD wells are uncorrelated. Given a spacing $sD_N$ between the nearest ND levels above and below the decaying SD level, then, $\Delta$ is given by the rectangular probability distribution

$$P_s(\Delta) = \frac{1}{sD_N} \Theta \left( \frac{s}{2} - \frac{\Delta}{D_N} \right).$$

(8)

Here $\Theta$ is the Heaviside step function, which simply ensures that $\Delta$ is the nearest neighbor. The probability distribution for $\Delta$ is then

$$\mathcal{P}(\Delta) = \int_0^{\infty} P_s(\Delta) P(s) ds = \frac{\pi}{2D_N} \text{erfc} \left( \sqrt{\frac{\Delta}{D_N}} \right).$$

(9)

Figure 2 shows $\mathcal{P}(\Delta)D_N$ plotted as a function of $\Delta/D_N$. From Eq. (9), it is easy to compute the average detuning $\langle |\Delta| \rangle = D_N/4$.

Our ultimate goal is to find the probability distribution $\mathcal{P}(V)$ for given values of $\Gamma^d$, $\Gamma$, and $D_N$, which in general is given by $\mathcal{P}(V) = 2\mathcal{P}(\Delta)|\Delta/V|$. From Eq. (4) we can obtain $|\Delta|$ as a function of $V$,

$$|\Delta| = \sqrt{\frac{2\Gamma}{\Gamma^d} \left( V^2 - \frac{\Gamma^d}{2} \right)}. $$

(10)

$V$ obviously has a lower bound of $V_{\text{min}} = \sqrt{\frac{1}{2} \Gamma^d \Gamma}$ due to the requirement that $|\Delta|$ be real. Computing $\mathcal{P}(V)$ for the allowed region, we find

$$\mathcal{P}(V \geq V_{\text{min}}) = \frac{2\Gamma V \pi}{\Gamma^d |\Delta| D_N} \text{erfc} \left( \sqrt{\pi \frac{|\Delta|}{D_N}} \right),$$

(11)

where $|\Delta|$ is given by Eq. (10). The average value of $V$ is

$$\langle V \rangle = \sqrt{\frac{\Gamma^d}{2\Gamma} \left[ \frac{D_N}{4} + \mathcal{O} \left( \frac{\Gamma^2}{D_N} \right) \right]}. $$

(12)

Earlier attempts to consider a statistical theory of SD decay-out [4, 6] have been handicapped by their focus on average values of $\Gamma^d$ and $F_N$. As was shown in Ref. 2, the statistical fluctuations of these quantities are much greater than their averages, making such computations almost meaningless. This is not the case with the distribution of $V$ given by Eq. (11), however, for which the fluctuations are of order $\langle V \rangle$. Furthermore, $\Gamma^d$ and $F_N$ are experimentally fixed, whereas the uncertainty in $V$ is due to lack of knowledge of the unperturbed energies.
of the ND and SD states. Eq. \( 11 \) therefore represents a central result, since it is the maximal information we can have about \( V \) without prior knowledge of the shape of the potential.

Thus far we have treated only a two-level model of the superdeformed decay-out process. It is reasonable to ask what effect the inclusion of more ND states could have. As a first step, we include the next-nearest neighbor state in the ND well.

We can again turn to the assumption of the GOE in order to find typical values for \( \Delta_1 \) and \( \Delta_2 \), the energies of the nearest and next-nearest neighbor states, respectively. \( \mathcal{P}(\Delta_1) \) is simply \( \mathcal{P}(\Delta) \) given by Eq. \( \mathcal{P} \). The calculation of \( \mathcal{P}(\Delta_2) \) is similar to that for \( \mathcal{P}(\Delta_1) \), but we must now concern ourselves with the question of whether \( \Delta_1 \) and \( \Delta_2 \) have the same or opposite signs.

In all cases of physical interest, we find that for a given spacing \( s \), the correction to the two-state result necessitated by the inclusion of a third level is larger when the two ND levels bracket the SD level. This is quite natural, since Eq. \( 11 \) requires that the next-nearest ND level in this configuration lie on average \( 40\% \) closer to the decaying SD state than it would if the nearest two ND levels lay on the same side of the SD level. Consideration of the case in which \( \Delta_1 \) and \( \Delta_2 \) have the same sign, then, only decreases the necessary correction. Since our goal is to set a reasonable upper bound on this quantity, we assume the ND levels lie on opposite sides of the SD level. Decays for which this is not true will in general conform to the two-state approximation with even greater accuracy.

Based on this assumption, we can construct a density function for \( \Delta_2 \) similar to Eq. \( 8 \).

\[
\mathcal{P}(\Delta_2) = \frac{1}{sD_N} \Theta \left( \frac{|\Delta_2|}{D_N} - s \right) \Theta \left( s - \frac{|\Delta_2|}{D_N} \right).
\]

Together with Eq. \( 7 \), this yields a distribution for \( \Delta_2 \):

\[
\mathcal{P}(\Delta_2) = \frac{\pi}{2D_N} \left[ \text{erf}(\sqrt{\pi}|\Delta_2|) - \text{erf}(\frac{\sqrt{\pi}}{2}|\Delta_2|) \right].
\]

This expression for \( \mathcal{P}(\Delta_2) \) is illustrated by Fig. 2. Its average detuning is \( \langle |\Delta_2| \rangle = 3D_N/4 \).

Having computed the average values of \( \Delta_1 \), \( \Delta_2 \), and \( V \), we are now in a position to begin to see the effect of a second ND level. In general, Eq. \( 11 \) would suggest that the contribution to the total branching ratio of a second level is substantially less than that of the nearest neighbor. Since Eq. \( 11 \) was derived in the context of only one ND level, however, we ought to seek a more rigorous theory for the three-state branching ratio. In particular, we should expect that effects such as competition and interference will play a role in the exact result.

The Hamiltonian for the three-state system can be taken to be the sum of two parts, \( H_0 \), which represents the independent SD and ND wells, and \( V \), which mixes the states of the two wells. \( H_0 \) can be written

\[
H_0 = \sum_i \varepsilon_i c_i^\dagger c_i + H_{EM},
\]

where the sum on \( i \) runs over \( S, N_1, \) and \( N_2 \), \( c_i \) is the annihilation operator for state \( i \), and \( H_{EM} \) contains the coupling to the electromagnetic field which gives the states their nonzero widths. Since they occur by the same decay process, we assume the widths of the ND states are equal, i.e. \( \Gamma_{N1} = \Gamma_{N2} = \Gamma_N \).

Taking \( V \) as a perturbation, it is a trivial exercise to use Dyson’s Equation to construct the retarded Green’s function of the system. The result, exact to all orders in \( V \), is given in the \( |S\rangle, |N1\rangle, |N2\rangle \) basis by

\[
G^{-1}(E) = \begin{pmatrix}
E + i\frac{\Gamma_N}{2} & -V_1 & -V_2 \\
-V_1 & E - \Delta_1 + i\frac{\Gamma_N}{2} & 0 \\
-V_2 & 0 & E - \Delta_2 + i\frac{\Gamma_N}{2}
\end{pmatrix},
\]

where \( V_1 \) and \( V_2 \) may be chosen positive without loss of generality. In the following, we assume further that \( V_1 = V_2 \).

The branching ratios of the full three-state system can now be computed from Parseval’s theorem

\[
F_i = \Gamma_i \int_{-\infty}^{\infty} \frac{dE}{2\pi} |\langle S|G(E)|i\rangle|^2,
\]

where \( i = S, N_1, N_2 \). These integrals can be done analytically by Cauchy integration, but the results are alge-
The energy levels were taken to lie at their mean detunings, and constant values of $\Gamma_S$ and $\bar{\Gamma}/D_N = 10^{-4}$ were used. These orders of magnitude are consistent with the combinations of parameters $\Gamma_S$, $\Gamma_N$, $V$, and $D_N$. In all cases of physical interest, however, $\Gamma_S$, $\Gamma_N \ll V$, $D_N$, so we can restrict the relevant parameter space by varying only the combinations of parameters $\Gamma_S/\Gamma_N$ and $V/D_N$ separately. The corrections to the branching ratios required by such a restriction are of order $\Gamma_N/V$.

Furthermore, the order of $\Gamma_S/\Gamma_N$ is determined by the mass region of the nucleus (see Table I). In the results that follow, variation of this parameter over reasonable values does not significantly impact the necessary correction to the two-state result.

Figures 3 and 4 show comparisons of the two- and three-state branching ratios for the $A \approx 190$ and $A \approx 150$ mass regions, respectively. These figures demonstrate that in cases of physical interest, the correction to the two-state system due to the presence of a third level is relatively small. In the $A \approx 190$ region, in particular, we find that $0.9 \lesssim F_{N}^{(2)} / F_{N}^{(3)} \lesssim 1$, where the superscripts on the branching ratios indicate the number of states included in their calculation. In the $A \approx 150$ region, we find that $0.7 < F_{N}^{(2)} / F_{N}^{(3)} < 1$. The increased importance of additional levels in the $A \approx 150$ region arises because the typical tunneling matrix elements at decay out are significantly larger, which follows from Eq. 11 and the relative sizes of $\Gamma_S$ and $\Gamma_N$ in the two mass regions. Constructive interference between the two ND levels, plainly visible in Fig. 4 where $F_{N}^{(3)}$ exhibits a distinct maximum for $V \sim D_N/2$, can also enhance the importance of additional ND levels.

In the 190 mass region, however, quantum interference effects are seen to be negligible, and this approach can be extended to set an approximate upper bound on the total error incurred by ignoring all of the ND states except the nearest neighbor. It is clear from Eq. 1 that, neglecting quantum interference effects, a level’s branching ratio is approximately proportional to $\Delta^{-2}$. Assuming that each level in the infinite ND band lies at its average detuning, we can write the total $n$-level branching ratio as

$$F_{N}^{(n)} \approx \sum_{k=1}^{n-1} \frac{F_{N}^{(2)}(D_N/4)^2}{(2k-1)^2} = \frac{\pi^2 n}{8} F_{N}^{(2)}. \quad (18)$$

We thus have the result

$$F_{N}^{(\infty)} \approx \sum_{k=1}^{\infty} \frac{F_{N}^{(2)}(D_N/4)^2}{(2k-1)^2} = \frac{\pi^2 n}{8} F_{N}^{(2)}. \quad (19)$$

In the 190 mass region, then, the expected correction is no more than about 23% of $F_{N}^{(2)}$.

The three-state results, together with the arguments of Eqs. 15 and 16, demonstrate that the two-state model is sufficient to describe the dominant decay-out process of SD nuclei. Within the two-state model, we have shown that the decay out of an SD level via $E1$ processes in the ND band is a two-step process, whose branching ratio is expressed in terms of three measurable rates, $\Gamma_S$, $\Gamma_N$, and $\Gamma^4$. We have also shown how to determine the tunneling matrix element $V$ [Eqs. 16 and 17] from the measured value of $\Gamma^4$ and a statistical model of the ND band. It is hoped that these results will help clarify the nature of the decay-out process in SD nuclei.

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