Cybersecurity in Distributed and Fully-Decentralized Optimization: Distortions, Noise Injection, and ADMM*

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Abstract—As problems in machine learning, smartgrid dispatch, and IoT coordination problems have grown, distributed and fully-decentralized optimization models have gained attention for providing computational scalability to optimization tools. However, in applications where consumer devices are trusted to serve as distributed computing nodes, compromised devices can expose the optimization algorithm to cybersecurity threats which have not been examined in previous literature. This paper examines potential attack vectors for generalized distributed optimization problems, with a focus on the Alternating Direction Method of Multipliers (ADMM), a popular tool for convex optimization. Methods for detecting and mitigating attacks in ADMM problems are described, and simulations demonstrate the efficacy of the proposed models. The weaknesses of fully-decentralized optimization schemes, in which nodes communicate directly with neighbors, is demonstrated, and a number of potential architectures for providing security to these networks is discussed.

I. MOTIVATION

With increased computing power, convex optimization tools have gathered attention as a fast method for solving constrained optimization problems. However, some problems are still too large to be solved centrally—e.g. scheduling the energy consumption of millions of electric vehicles.

In these applications, decentralized optimization techniques break the problem into a set of subproblems which can be rapidly solved on distributed computing resources, with an aggregator or fusion node bringing the distributed problems into consensus on a global solution. This distribution can yield significant computational benefits, greatly speeding computation times.

A further development on this model, fully-decentralized optimization techniques remove the aggregator, and instead let nodes reach consensus with neighbors, in a process which gradually brings the entire system into consensus (see Figure 1). This approach significantly reduces communication requirements relative to aggregator-coordinated systems.

The potential for reducing computation time and communication burden makes these distributed and fully-decentralized models particularly compelling in smartgrid applications, where systems may need to scale to millions of devices. However, by moving the computation from enterprise servers to consumer devices, significant new security weaknesses are exposed. This article outlines architectures and algorithms for addressing these weaknesses.

II. BACKGROUND LITERATURE

The vulnerability of physical infrastructure to cyberattacks was dramatically highlighted in 2000 when the SCADA system controlling the Maroochy Water Plant in Australia was remotely hijacked, resulting in the release of over one million gallons of untreated sewage [1]. A decade later, the US government developed the Stuxnet worm to attack programmable logic controllers which operated uranium enrichment centrifuges managed by the Iranian government [2]; the Iranian government recently launched similar attacks against targets in Saudi Arabia [3]. A descendant of the Stuxnet worm was used more dramatically by the Russian government to repeatedly cause widespread blackouts in the Ukrainian electricity during 2016 and 2017 [4], [5]. Recent incursions into the networks of American electricity operators by Russian hackers may be laying the foundation for future incursions against U.S. infrastructure [6], [7].

Against this backdrop of cyberattacks used as national security threats, President Obama issued an Executive Order [8] and Presidential Policy Directive [9], [10] directing the executive branch to improve cybersecurity of critical infrastructure. In the face of proven vulnerabilities in consumer-level smart grid devices [11] it seems critical that distributed algorithms for smartgrid controls be designed to be able to identify, localize, and mitigate the effects of these attacks.

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Fig. 1: Different structures for solving mathematical optimization problems, from left: Centralized optimization, where all details of objective and constraints are held by a central entity. Decentralized optimization (also called aggregator-coordinated optimization) where local nodes hold local objective and constraint information, and an aggregator brings nodes into consensus on shared constraints. Fully-decentralized optimization, where no centralized entity exists but neighbors communicate directly with each other to achieve consensus.
Fig. 2: Publication frequency for the topics of ‘Distributed Optimization’, ‘Cybersecurity’, and the intersection of the two fields, 2010-2017. Legend captions include the total number of publications over this period; the number of publications at the intersection is three orders of magnitude less than each field.

Prompted by these concerns, cybersecurity research has increased in recent years, as shown in Figure 2. However, this research is far outpaced by the development of new distributed optimization algorithms, and research on the intersection of the two fields - the security of distributed optimization algorithms - is still nascent. Instead, the most cutting-edge research on distributed optimization techniques under adversarial attacks is found not in power systems or optimal control, but in the field of artificial intelligence.

The deployment of distributed machine learning algorithms to very large computation networks or federated machine learning platforms (in which consumer feedback e.g. in smartphone apps directly trains an estimator) has led researchers to work on algorithms which can be resilient to software crashes, network dropouts, and adversarial attacks on local nodes.

Most of these machine learning problems are solved with consensus algorithms, in which local nodes solve private problems and reach consensus on a global estimator through iteration with a fusion node (aggregator-based model) or with their neighbors (fully-decentralized model). Although widely used, it has been shown that arbitrary attacks by a single corrupted node can lead consensus algorithms to fail [12], [13]- making it critical to understand potential attack vectors, detection algorithms, and mitigation strategies.

As a result, research to combat these approaches has considered a number of methods for developing techniques to allow the consensus algorithm to converge to a best estimate under attack [12]:

- **Filtering techniques** remove outliers from the set of proposed updates [14], [12], [15], in extreme cases using just the median estimator [16], [17] in order to provide tolerance to an attack of half of the nodes. These approaches can be computationally intensive, and may require that the local update functions are all drawn from the same distribution- feasible in statistical estimation, but unlikely in device scheduling.

- **Nonlinear weighting schemes** take advantage of all estimators, but dynamically scale down the impact of suspicious updates. While these are guaranteed to move towards optimum [13], [18], [19], they have a larger optimality gap than other techniques [20].

- **Round-robin techniques** seek to detect and remove attacked updates by iteratively computing the consensus estimate with different combinations of dropped-out nodes, and using the most stable estimate [21], [22], [15]. This is algorithmically simple but computationally intensive, and is most feasible when the number of attackers is well-bounded.

Additionally, [20] demonstrates the improved convergence of these algorithms when compromised nodes can be switched off.

While many of these algorithms were developed for aggregator-coordinated machine learning problems, many of them can be adapted to fully-decentralized structures, as in the estimation problems studied in [23], [19] and the optimization problem studied in [21].

Some of these techniques are beginning to also be deployed in smartgrid optimization problems: [24] demonstrates the hurdle which compromised nodes create for scheduling, much like [25] does for generalized consensus problems.

However, most research on cybersecurity in power system applications has not considered scheduling and optimization problems, but rather state estimation in transmission networks. Liu et al demonstrate in [26] that a single compromised node could introduce an arbitrarily large estimation error, similar to the generalized results in [25]. Subsequent research demonstrated the limitations of attack detection and mitigation strategies under a number of centralized state estimation models [27], [28], [29].

These approaches have also been extended to distributed optimization techniques in [22], where a round-robin ADMM method is used to identify compromised nodes; this approach is demonstrated experimentally in [30]. Similar research has also developed fully-decentralized algorithms for state estimation, with [31] utilizing compressed sensing techniques to recover an estimate of system state when noise injection is bounded, though [32] demonstrates that these approaches still fail when a node is able to inject arbitrary noise.

**Novel Contributions**

This work advances prior literature by providing an attack detection algorithm for the alternating direction method of multipliers, a popular optimization algorithm used for
convex optimization and consensus problems. Only requiring that the private objective functions be convex, the attack detection algorithm works for both constrained and unconstrained problems, and both aggregator-coordinated and fully-decentralized architectures. By directly identifying compromised nodes, we bypass many of the objections to the filtering, nonlinear averaging, and round-robin techniques described above. Additionally, the proposed detection algorithm can readily be integrated into computational techniques, such as those described in [20].

In developing this, this paper provides the following novel contributions to existing literature:

- Outline a taxonomy of attack vectors for decentralized optimization algorithms
- Detail attack detection, localization, and mitigation techniques for the alternating direction method of multipliers
- Demonstrate and verify an algorithm for detecting noise-injection attacks in the alternating direction method of multipliers
- Outline the unique security challenges of fully-decentralized optimization
- Describe potential architectures for security in fully-decentralized optimization

III. OUTLINE

The remainder of this article is organized as follows. In Section IV we establish some mathematical background and notation. Section V outlines a taxonomy of methods by which an attacker may seek to compromise a decentralized optimization algorithm, and briefly discuss challenges to detection, localization, and mitigation.

Section VI derives an algorithm for detecting noise-injection attacks in convex optimization problems solved with ADMM, and Section VII provides results for a set of simulations with randomly-generated quadratic programs (QPs). Sections VIII and IX describe limitations and extensions, respectively.

Sections X and XI extend the cyberattack detection algorithm by considering the unique security challenges presented by fully-decentralized optimization algorithms. Section XII outlines architectures which can be used to overcome the weaknesses inherent in fully-decentralized architectures. Finally, Section XIII summarizes the article’s main results.

IV. MATHEMATICAL BACKGROUND AND NOTATION

A. Notation

Without loss of generality, we focus on an aggregator-coordinated system with two nodes which compute the x-update and z-update steps in a decentralized optimization problem. Also without loss of generality, we will consider an attack in which the x-update node has been compromised, and use the following notation:

- \( A, B \) Constraint matrices in ADMM binding constraint
- \( c, d \) Linear cost vectors
- \( f(x), g(z) \) Generalized objective functions
- \( H \) Hessian
- \( i, j \) Iterates
- \( k \) Iterate limit
- \( m, n \) Number of dimensions of \( z \) and \( x \) variables respectively
- \( p \) Number of binding constraints
- \( P, Q \) Quadratic cost functions in sample problems
- \( r, s \) Dimensions in Hessian
- \( u \) Scaled dual variable
- \( w \) Combined variable for centralized solution
- \( x, z \) Optimization variables
- \( y \) Unscaled dual variable
- \( \mathcal{X}, \mathcal{Z} \) The private constraints, knowable only to the compute nodes and not publicly shared
- \( x^k \) The unattacked update, solving: \( x^k := \arg\min_{x \in \mathcal{X}} f(x) + \frac{\rho}{2} \| Ax + Bz^{k-1} - c + u^k \|_2^2 \)
- \( \hat{x}^k \) The attacked signal provided by the z node
- \( x^k \) The variable update received by the z-update node, of unknown validity
- \( \hat{x}^k \) A best response created by the z-update node which has received a signal perceived as being an attack

B. Decentralized Optimization

In general, decentralized optimization problems can be cast as:

\[
x^*, y^* = \arg\min_{x, z} \ W(x, z)
\]

s. t.: \( x, z \in \mathcal{W} \)

where \( x \) and \( z \) further satisfy local problems:

\[
x^* = \arg\min_x \ f(x, z^*)
\]

\( x \in \mathcal{X} \)

\[
z^* = \arg\min_z \ g(z, x^*)
\]

\( z \in \mathcal{Z} \)

In this discussion we focus on iterative methods, where updates \( x^k = \arg\min_{x \in \mathcal{X}} f(x, z^{k-1}), z^k = \arg\min_{z \in \mathcal{Z}} g(z, x^{k-1}) \) solve local updates in an algorithm which converges to a global solution, i.e. \( x^k \to x^*, z^k \to z^* \) as \( k \to \infty \).

In addition to computational benefits, this architecture is also advantageous when the local objective functions \( f(x), g(z) \) or local constraint sets \( x \in \mathcal{X}, z \in \mathcal{Z} \) contain private information which can not be directly shared with other participants or the aggregator. In this scenario, the updates \( x^k, z^k \) do not directly reveal private information, yet still allow the system to converge to the global optimum. This is common in markets and scheduling problems, where participants have private constraints or utility functions which they do not wish to share for economic or security reasons.

Additionally, we assume that some information about the private constraint set is publicly knowable, creating a superset which contains the private constraint set: \( \mathcal{X} \subset \mathcal{X}_{\text{pub}}, \mathcal{Z} \subset \mathcal{Z}_{\text{pub}} \). Note that as all variable updates are in
the private constraint set, they must also be in the public set: $x^k \in \mathcal{X} \subset \mathcal{X}_{\text{pub}}$.

C. The Alternating Direction Method of Multipliers

Although any distributed optimization algorithm may be subject to cyberattack, we specifically consider the Alternating Direction Method of Multipliers (or ADMM) algorithm reviewed in [33], which has gained popularity due to its simple formulation and guarantees of convergence for convex problems. The ADMM algorithm is used to decompose a problem with separable objective and constraints:

$$\min_{x,z} f(x) + g(z)$$  
(1)

subject to $Ax + Bz = c$

(2)

$$x \in \mathcal{X}$$  
(3)

$$z \in \mathcal{Z}$$  
(4)

Note that only equation (2) links $x$ and $z$.

Adding a penalty term for violations of the linking constraint, we can create an augmented Lagrangian defined in the domain $\{x \in \mathcal{X}, z \in \mathcal{Z}\}$:

$$L_\rho(x, z, y) = f(x) + g(z) + y^T (Ax + Bz - c) + (\rho/2)\|Ax + Bz - c\|^2$$  
(5)

The standard ADMM algorithm can then be expressed with respect to this augmented Lagrangean $L_\rho(\cdot)$ as:

$$x^{k+1} := \arg\min_{x \in \mathcal{X}} L_\rho(x, z^k, y^k)$$  
(6)

$$z^{k+1} := \arg\min_{z \in \mathcal{Z}} L_\rho(x^{k+1}, z, y^k)$$  
(7)

$$y^{k+1} := y^k + \rho (Ax^{k+1} + Bz^{k+1} - c)$$  
(8)

or in scaled form:

$$x^{k+1} := \arg\min_{x \in \mathcal{X}} f(x) + \frac{\rho}{2}\|Ax + Bz^k - c + u^k\|^2$$  
(6)

$$z^{k+1} := \arg\min_{z \in \mathcal{Z}} g(z) + \frac{\rho}{2}\|Ax^{k+1} + Bz - c + u^k\|^2$$  
(7)

$$u^{k+1} := u^k + Ax^{k+1} + Bz^{k+1} - c$$  
(8)

The iterations stop when:

- The primal residual $r^k = Ax^k + Bz^k - c$ has a magnitude below a threshold $\epsilon_{\text{pri}}$, i.e. $\|r^k\|_2 \leq \epsilon_{\text{pri}}$
- The dual residual $s^k = \rho A^T B(z^k - z^{k-1})$ has a magnitude below a threshold $\epsilon_{\text{dual}}$, i.e. $\|s^k\|_2 \leq \epsilon_{\text{dual}}$

As described in [33], the ADMM iterations will converge to the optimal value of the objective function, and the primal and dual residuals will converge to zero.

In this ADMM formulation, we refer to the “aggregator” (or “fusion node”) as the agent responsible for updating $u$. The aggregator:

1) receives updates from the $x$-update and $z$-update steps,
2) computes the $u$-update
3) broadcasts the updated value of $u$ to the nodes responsible for computing the $x$- and $z$-updates.

1) Consensus problems: ADMM has become a popular tool for solving consensus problems in both machine learning and power systems engineering, as both fields deal with problems where computation may be spread across thousands (or millions) of nodes. In a consensus problem, local variables $x_i$ and objective functions $f_i(x_i), i \in \{1, \ldots, N\}$ are united by the constraint that at optimality, the local variables mirror the global variable $z$:

$$\min_{x, z} \sum_{i=1}^{N} f_i(x_i)$$  
\text{s.t. } x_i = z, \quad i = 1, \ldots, N$$

Collected, the ADMM form of this is:

$$x_i^{k+1} := \arg\min_{x_i \in \mathcal{X}} f_i(x_i) + y_i^k x_i + (\rho/2)\|x_i - z^k\|^2$$  
(3)

$$z^{k+1} := \frac{1}{N} \sum_{i=1}^{N} x_i^{k+1} + (1/\rho) y_i^k$$  
(4)

$$y_i^{k+1} := y_i^k + \rho (x_i^{k+1} - z^{k+1})$$  
(5)

Note that this structure generalizes minibatch stochastic gradient descent, in which the local objectives are the gradients of the local estimators.

As this consensus algorithm is a specific example of ADMM, results for the general ADMM problem will also apply to the ADMM consensus algorithm. For clarity, in this paper we will continue to refer to $x$- and $z$-update nodes, even though a consensus problem may have thousands or millions of $x$-nodes and a single $z$-update (aggregator) node. Despite this difference in scale, the results we derive for the general 2-node system will still be applicable to the consensus problem.

We will also note that consensus problems are of particular interest in many power system and machine learning problems, as well as in fully-decentralized optimization problems, which include consensus networks.

V. Attack Vectors in Decentralized Optimization

When local problems contain private information, the central coordinator can only check $x_i, z_i \in \mathcal{W}$ and cannot directly verify that the updates $x^{k+1}_i, z^{k+1}_i$ in fact solve the private optimization problems. In this scenario, a malicious node may submit a distorted update $\tilde{x}^k_i$ in order to mislead the central coordinator with the goal of creating a sub-optimal solution, an infeasible solution, or prevent convergence of the iterative algorithm.

In this section, we briefly outline potential attack vectors and methods for addressing them. Of these attacks, the most generalizable but difficult to identify is zero-mean noise injection, which we consider in detail below. These attack vectors can be combined, but detection and mitigation efforts will usually address these separately.
Sub-optimal solution: The compromised node solves a modified objective function \( \tilde{f}(x) \) resulting in \( f(x^k) < \tilde{f}(x^k) \) and consequently \( W(\tilde{x}^k, z^k) > W(x^k, z^k) \) as \( k \to \infty \). As this does not change the problem structure (e.g. convexity, private constraints), it is difficult to discern from an equivalent problem with slightly modified characteristics, e.g. different consumer preferences. This makes attack prevention a matter of appropriately structuring game-theoretic incentives to avoid malicious distortion. As such, we do not consider this in greater detail here.

Infeasible Private Constraints: An attacked node may replace the constraints \( x \in \mathcal{X} \) with a modified constraint set \( \tilde{x} \in \tilde{\mathcal{X}} \) in order to result in an update which is not feasible \( \tilde{x} \not\in \mathcal{X} \), as shown in Figure 3. This leads the system to converge to a point which does not reflect actual conditions, e.g. a schedule which is not operationally feasible or a machine learning estimator distorted by false data. Because these constraints may arise from stochastic processes (user preferences, input data) the aggregator cannot directly discern an attack from an unattacked update with different underlying data. As a result, defenses from this attack are limited to cases where the aggregator can bound the support of the problem, i.e. a publicly knowable constraint set \( \mathcal{X}_{\text{pub}} \). In this scenario, attacks can be detected when updates lie out of the support region, and the effect of the attack can be mitigated by projecting the update back onto the support region.

Infeasible Linking Constraint: Knowing that a solution to the master problem must satisfy \( Ax + Bz = c \), an attacker may present an update which it knows to be unreachable with these constraints, i.e. \( \tilde{x} \in \{ \tilde{Z} | A\tilde{x} + Bz = c \} \). In this case, convergence stops as the primal residual \( r^k = ||Ax^k + Bz^k - c||_2^2 \) remains nonzero. This attack relies on the attacker knowing some bounds on the set reachable by its neighbor \( Z \subseteq \mathcal{Z}_{\text{pub}} \) in order to shape this attack. Similar to above, detection relies on the same knowledge of the public constraint set, and mitigation relies on projecting the attacked signal back onto the feasible space. If the aggregator has knowledge of the public constraints \( \mathcal{Z}_{\text{pub}} \), then detection and mitigation can easily be implemented at the aggregator level for each signal.

Non-Convergence: An attacker may add a noise term which varies with each iteration, i.e. \( \tilde{x} = x^* + \delta(i) \). When the noise term is larger than the rate of convergence of the problem, this results in a persistent residual which remains above the convergence tolerance, as shown in Figure 4. We examine in detail the case where the noise is unbiased (zero-mean). When the noise is biased, the problem can be considered a combination of a zero-mean noise injection attack and one of the attacks described above, and addressed accordingly.

VI. DEVELOPING A DETECTION ALGORITHM FOR NOISE-INJECTION ATTACKS

Because the ADMM algorithm relies on private actors computing \( x- \) and \( z- \) updates, these actors can distort the problem by providing inaccurate updates, with the goal of...

![Diagram](image-url)

Fig. 3: Graphical example of how an attacker may distort the constraint set from \( \mathcal{X} \) to \( \tilde{\mathcal{X}} \) to create an optimum outside of the truly feasible set, and the limited ability to mitigate these impacts by projecting onto a publicly knowable constraint set \( \mathcal{X}_{\text{pub}} \).

![Graph](image-url)

Fig. 4: Plot of local residuals in a 3-node system under noise-injection attack. Injection of noise prevents problem convergence at nodes across the network.
creating a suboptimal or infeasible solution or preventing convergence. We study the case of distorting the $x$-update through injection of zero-mean noise, which is intended to prevent convergence of the algorithm. The new update becomes:

$$x^{i+1} := \text{argmin}_{x \in X} f(x) + \frac{\rho}{2} \|Ax + Bz^i - c + u^i\|^2 + \delta(i)$$

Where $\delta(i)$ is a noise term which changes on each iteration. The $z$-actor or the system aggregator is challenged to identify the attack and take preventative actions before convergence is prevented.

A. Attack Detection Overview

We play the part of the aggregator or $z$-actor, and wish to detect an attack of $x$, using the values of the $x^i$, $z^i$, $u^i$, $i = 1 \ldots k$ iterates. We rely on a priori knowledge that $f(x)$ must be convex, we trust the computations of $z^i$, and we can verify the $u^i$ values directly by using the other iterates.

We will detect attacks by assessing the convexity of $f(x)$ implied by the $x^i$ iterates, without being able to directly assess $f(x)$.

To do this, we will use the $z$- and $u$-update values to evaluate the local gradient of $f(x)$, then use these local gradients to construct a finite-differences approximation of the Hessian of $f(x)$. Testing whether this Hessian is symmetric positive semi-definite will then allow us to test the convexity of the $x$-updates; this is visualized in Figure [3]. Given our a priori knowledge that $f(x)$ must be convex, any updates which result in a nonconvex Hessian must represent an attack.

The algorithm progresses in three steps:

- Assess gradient
- Construct Hessian
- Evaluate eigenvalues of Hessian

1) Assess Gradient: We rely on the proof of ADMM convergence provided by [33], and while we focus on the $x$-update the same approach can be used for security checks conducted by the $x$-update node. The gradient of $f(x)$ evaluated at $x^i$ is found as:

$$0 \in \partial L_{\rho}(x^i, z^{i-1}, u^{i-1})$$
$$0 \in \partial f(x^i) + AT + \rho AT(Ax^i + Bz^{i-1} - c)$$
$$\partial f(x^i) = -AT(y^i - \rho B(z^i - z^{i-1}))$$
$$\partial f(x^i) = -AT(\rho u^i - \rho B(z^i - z^{i-1}))$$

where the last equality simply uses the scaled dual variable.

We will interchangeably use the notation $f(x^i)$ and $f(x)|_{x^i}$ to both indicate the definite evaluation of $f(x)$ at the point $x^i$. The iterates $i = 1, \ldots, k$ thus give us a sequence of gradient evaluations at points $x^i, i = 1, \ldots, k$. From this sequence of gradients, we wish to construct the Hessian.

2) Construct Hessian: We wish to construct the Hessian matrix

$$H = \begin{bmatrix}
\partial^2 f(x) & \partial f(x) & \cdots & \partial f(x) \\
\partial x_1 f(x) & \partial^2 f(x) & \cdots & \partial f(x) \\
\partial x_2 f(x) & \partial^2 f(x) & \cdots & \partial f(x) \\
\vdots & \vdots & \ddots & \vdots \\
\partial x_n f(x) & \partial^2 f(x) & \cdots & \partial^2 f(x) \\
\end{bmatrix}$$

To construct the Hessian, we utilize information gained from the gradient evaluations at each of the iterates to compute a numeric approximation of the local Hessian, using a Taylor series expansion of the Hessian around that point.

Example: 2-Dimension Case: For clarity, we begin with a two-dimensional case, i.e. $x^i = [x_1^i, x_2^i]^T$ and visualized in Figure [6]. We will consider two points $x^1$ and $x^2$, and will note the dimensions of these points using the subscripts 1 and 2.

In this case, the change in gradient from point $x^i$ to $x^j$ can be approximated using the following difference equations:

![Fig. 5: Conceptualization of the attack detection process for a $n = 2$ toy problem. Although the validator cannot directly assess the private objective function $f(x_1, x_2)$ (shown in red-blue gradient), they are provided with a sequence of iterates $x^k$, and can to use a subset of these for the detection algorithm, shown here as $(x_1, x_2)$ points with $f(x) = 0$ due to the unknown objective value. Using information from the sequence of iterates, the validator can assess the gradient of $f(x)$ at each of these points. Finally, from these gradients, the validator can estimate the Hessian, and use this to evaluate the convexity of the (unknown) objective function. This can be conceptualized as a local quadratic approximation of the objective function at the reference point, shown in blue-green.](image)
The problem is to find an approximation of the Hessian. Hessian cannot be directly evaluated, the change in gradient evaluations between \( x^i \) and \( x^j \) can be used to derive an approximation of the Hessian.

\[
\frac{\partial^2 f(x)}{\partial x^i \partial x^j} = \frac{\partial^2 f(x)}{\partial x^i} + \frac{\partial^2 f(x)}{\partial x^j} (x^j - x^i) + \frac{\partial^2 f(x)}{\partial x^i \partial x^j} (x^j - x^i)^2
\]

This is visualized in Figure 6 where \( \Delta x_1 = (x^j - x^i) \) and \( \Delta x_2 = (x^2 - x^1) \). From the previous step, we can directly evaluate the gradient terms

\[
\frac{\partial g}{\partial x_r} f(x)|^s
\]

and wish to solve for the second-order terms which will become the entries in the Hessian matrix. However, we have \( n^2 \) unknown second-order terms but only \( n \) equations, making this system underdefined. In order to have a fully-defined system, we need another set of difference equations, found by considering the difference with respect to another point \( k \). Using the notation \( (x^j)^k = x^j_k - x^j \) and \( \frac{\partial}{\partial x^j} f(x)|^k = \frac{\partial}{\partial x^j} f(x) - \frac{\partial}{\partial x^j} f(x)^j \) we can now rearrange the problem into the matrix equation

\[
\frac{\partial^2 f(x)}{\partial x^i \partial x^j} = \frac{\partial^2 f(x)}{\partial x^i} + \frac{\partial^2 f(x)}{\partial x^j} (x^j - x^i) + \frac{\partial^2 f(x)}{\partial x^i \partial x^j} (x^j - x^i)^2
\]

Expanding to \( n \) dimensions: Generalized to \( r,s \in 1 \) dimensions (where \( l \leq n \)), this becomes:

\[
\frac{\partial^2 f(x)}{\partial x^i \partial x^j} = \frac{\partial^2 f(x)}{\partial x^i} + \frac{\partial}{\partial x^j} f(x)^r \sum_{s=1}^{n} \frac{\partial}{\partial x^s} f(x)^s (x^s - x^s)
\]

As there are \( n \) dimensions in \( x \), there are a total of \( n^2 \) unknown terms in the Hessian. As each new point \( x^j \) has \( n \) dimensions, it adds a new set of \( n \) difference equations. In order to fully define the set of equations for computing the Hessian, we need to collect a total of \( n + 1 \) linearly independent points (reference point plus differences with \( n \) points) to solve for the \( n^2 \) terms in \( H \). Since computing the gradient utilizes values from two iterates, this means that for a problem with \( n \) dimensions, the detection algorithm can be conducted on iterations \( n + 2 \) and beyond.

In the following, we index dimensions by \( 1, 2, \ldots, n \) and points by \( a, b, \ldots, k \) where \( k \) is used as the reference point. The vector \( \frac{\partial}{\partial x} \) is thus composed by stacking the gradient evaluations:

\[
\frac{\partial f(x)}{\partial x} = \left[ \frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right]
\]

the entries of the Hessian matrix can then be found as \( \frac{\partial^2 f(x)}{\partial x^i \partial x^j} = \frac{\partial^2 f(x)}{\partial x^i} + \frac{\partial}{\partial x^j} f(x)^r \sum_{s=1}^{n} \frac{\partial}{\partial x^s} f(x)^s (x^s - x^s) \)

In this form, \( \frac{\partial}{\partial x} \) collects the changes in gradient evaluations induced by each difference equation, the matrix \( D \) arranges the difference in coordinates at the evaluated iterates, and \( \frac{\partial^2 f(x)}{\partial x^i \partial x^j} \) unstacks the entries of the Hessian matrix \( H \).
Fig. 7: The introduction of noise shifts the gradients computed using the above algorithm, shown here as shifting the unattacked (gray) vector to a new (attacked, red) position. It can be readily seen that a sufficient displacement will cause the estimated curvature (Hessian) to become nonconvex, as shown by the inferred surface represented in Subfigure (b). To avoid detection, an attacker would thus need to use a very small injected signal, but this would limit their ability to stop convergence during the attack.

The matrix $D$ is composed of a stack of $n$ block-diagonal matrices, each of which is the Kronecker product of the $n$-dimensional identity matrix and a set of coordinate differences from $x$-iterates. Each block of this is $n \times n^2$, producing $D \in \mathbb{R}^{n^2 \times n^2}$:

$$D = \begin{bmatrix}
I_n \otimes (x^k - x^0)^T \\
I_n \otimes (x^k - x^T)^T \\
\vdots \\
I_n \otimes (x^k - x^{k-1})^T
\end{bmatrix}$$

We solve for the Hessian elements by computing $\overline{H} = D^{-1} \overline{G}$ as before.

3) Assess Convexity: For $f(x)$ to be convex, the Hessian must be positive semi-definite, i.e. $\lambda_{\text{min}}(H) \geq 0$ where $\lambda_{\text{min}}(H)$ is the least eigenvalue of $H$.

After solving for a local approximation of the Hessian as above, the eigenvalues of the Hessian are evaluated. If the Hessian is not found to be positive semi-definite, then we suspect the $x$-actor may be injecting noise into his updates, as shown in Figure [7]

4) Misclassifications: As attack detection can be viewed as a classification problem, it may be subject to two possible errors, depicted in Figure [8]

- False Positives (Type 1 errors): These occur when an attack is detected, even though no attack took place. This can occur when numeric issues create a slight local non-convexity during solution iterations, even though the underlying objective function is, in fact, convex. These numeric issues can result from the $x$-update, $y$-update, gradient calculation, or Hessian calculation, but are aggravated when the algorithm is close to the objective function and the system of equations used to generate the Hessian is poorly conditioned.

- False Negatives (Type 2 errors): In this type of error the algorithm fails to detect an attack, because the magnitude of the attack was not sufficient to create a non-convexity at the tested points. A small number of false positives are unavoidable: some attacks are not able to prevent convergence within a small number of iterations, and the attack detection algorithms may never be able to collect enough points to detect an attack. For problems that converge with a moderate number of iterations under attack, there may not be a combination of iterate points which result in a nonconvex Hessian.

Fig. 8: Depiction of misclassification errors in the noise detection algorithm. In weakly convex regions of a function, numeric conditioning errors can lead to false positives (subfigure a), whereas problems which have a very strong gradient may converge before the noise detection algorithm is able to identify the injection of noise into the algorithm (subfigure b). In both cases, the scheme for selecting points used to assess the gradient can lead to more accurate assessment of the Hessian, reducing misclassification errors.
For other cases, lowering the false negative rate requires testing a variety of point combinations in order to maximize the likelihood of finding a point where the finite-difference Hessian estimate is nonconvex.

VII. SIMULATION AND RESULTS

To verify the effectiveness of this problem under a variety of conditions, we simulate a large number of sample problems (both attacked and unattacked) and evaluate the effectiveness of the attack detection algorithm.

We have chosen to consider quadratic programs (QPs): they are widely used in power systems research (e.g. optimal power flow [31], optimal dispatch [34], [35]), subsume the set of linear programming problems, and yet offer many computational benefits (efficient off-the-shelf solvers, easy analytic solutions, easily visualized). In the future, we would like to extend this to other classes of convex problems (e.g. SOCP, SDP), but simulating hundreds of those problems would be computationally burdensome.

A. Problem: Random Quadratic Programs

We consider unconstrained quadratic programs of the form

\[ \min_{x,z} \quad x^T P x + c^T x + z^T Q z + d^T z \]

s.t. \( Ax + B z = c \)

For simplicity, we examine problems where \( c = 0 \) and \( A \) and \( B \) have a single non-zero value per row, as described below. This can be thought of as consensus problems where some cost information is private.

1) Problem Generation: Problems are generated randomly as follows:

1) The size of the problem is determined by randomly choosing the magnitudes \( m, n, p \). The user defines a maximum size \( \text{maxdim} \) and \( m \) and \( n \) are drawn as integers on the uniform distribution \([1, \text{maxdim}]\). The number of consensus constraints \( p \) is then randomly chosen by drawing an integer on the interval \([1, \min(m, n)]\).

2) The problem will then be characterized by:

- Variables \( x \in \mathbb{R}^n, z \in \mathbb{R}^m \),
- Cost terms \( P \in \mathbb{R}^{n \times n}, Q \in \mathbb{R}^{m \times m}, c \in \mathbb{R}^n, d \in \mathbb{R}^m \),
- Constraint matrices \( A \in \mathbb{R}^{p \times n}, B \in \mathbb{R}^{p \times m} \).

3) Quadratic cost matrices \( P \) and \( Q \) are created by first randomly composing \( L_P \in \mathbb{R}^{n \times n}, L_Q \in \mathbb{R}^{m \times m} \) with entries drawn randomly from the uniform distribution on \([-S, S] \). \( P \) and \( Q \) are then constructed to be symmetric positive semi-definite by construction as \( P = (L_P)^T L_P, Q = (L_Q)^T L_Q \). To confirm that these are positive semi-definite, the eigenvalues are computed and checked to be greater than 0.

4) \( c \) and \( d \) are generated by randomly drawing values from \([-S^2, S^2] \), resulting in entries that are the same magnitude as those in \( P \) and \( Q \).

5) \( A \) is composed as \( A = \lfloor 0 \in \mathbb{R}^{(n-p) \times (n-p)}, I \in \mathbb{R}^{p \times p} \rfloor \) and \( B \) is composed as \( B = [I \in \mathbb{R}^{p \times p}, 0 \in \mathbb{R}^{(m-p) \times (m-p)}] \).

B. Solution Method

We consider the case where consensus is desired between two actors who do not want to share information about their cost information \( P, c \) and \( Q, d \). In this scenario, ADMM is a well-suited tool for achieving optimality without compromising privacy.

For this problem, the ADMM algorithm can be stated as:

\[
\begin{align*}
x^{k+1} &= \arg\min_x \quad x^T P x + c^T x + \frac{\rho}{2} \|Ax + Bz^k - c + u^k\|_2^2 \\
z^{k+1} &= \arg\min_z \quad z^T Q z + d^T z + \frac{\rho}{2} \|Bz + Ax^{k+1} - c + u^k\|_2^2 \\
u^{k+1} &= u^k + Ax^{k+1} + Bz^{k+1} - c
\end{align*}
\]

The \( x \)-update and \( z \)-update steps constitute unconstrained QPs, and can be solved analytically as described in the Appendix. These analytic update steps were used to avoid rounding issues resulting from using an iterative numeric solver.

C. Attack Implementation

Attack was simulated by multiplying the optimal \( x \)-update values by a vector with entries randomly selected from \([+10\%, -10\%], \text{i.e. a Bernoulli distribution, at each iteration. It was found that choosing uniformly distributed zero-centered noise was not sufficient to prevent convergence, as small-magnitude noise entries allow the algorithm to converge faster than the attacker is able to delay convergence. A set of 10,000 QPs was simulated without attack, and were found to converge in a median of 63 iterations, with a mode of 16 iterations and a maximum value of 216 iterations. When simulated with the attack described above, convergence was significantly hindered on the same QPs: 86% were not able to converge in 300 iterations, and 87% do not converge in 500 iterations. The 14% of QPs which converge in less than 300 iterations were found to behave similarly to the unattacked QPs, likely due to a set of costs which are strongly convergent even in the presence of attack.

The distribution of the iterations until convergence for both the unattacked and attacked QPs is depicted in Figure 9.

D. Attack Detection Implementation

We implement the algorithm described above. As we initially consider \( p \leq 2 \), we need to complete at least 4 iterations in order to create two sets of difference equations.

To improve conditioning, for iteration \( i \) we consider the difference equations with \( i \) and the reference point, and compare with iteration 2 and iteration \((i/2)\). Using the \( i, i - 1, i - 2 \) iterates was found to result in poor conditioning and a very high (17%) false positive rate, though a slightly lower false negative rate.

To ensure that we can solve for the values of the Hessian entries, we check that the difference vectors are not collinear.
Fig. 9: Convergence results for 10,000 QPs, simulated both with and without noise-injection attack. Without attack, all simulations converge in less than 300 iterations. Under attack, 86% of problems do not converge in 500 iterations (at which time calculation was stopped).

TABLE I: Results for a simulation of attack detection. 500 quadratic programs were generated without attack, and false positives (upper right quadrant) were identified. An additional 500 quadratic programs were generated with simulated attack, and false negatives measured.

(a) Results, total number of simulations

|                     | Attacked | Unattacked |
|---------------------|----------|------------|
| Attack Detected     | 479      | 0          |
| No Attack Detected  | 21       | 500        |
| Total               | 500      | 500        |

(b) Results, simulation error rate

|                     | Attacked | Unattacked |
|---------------------|----------|------------|
| Attack Detected     | 0.958    | 0          |
| No Attack Detected  | 0.042    | 1.0        |
| Total               | 1.0      | 1.0        |

E. Results

We show results for 1000 simulated problems, of which 500 were attacked and 500 were unattacked. The results are presented as a confusion matrix in Table I(a), which shows both accurate classifications and misclassifications. The number of problems of each type is shown in Table I(a), the same results are shown in proportional form in Table I(b).

We consider two types of errors: Type 1 (false positive, upper right quadrant) errors, and Type 2 (false negative, bottom left quadrant) errors. The experimental results were compared when computing the local update with both numeric solvers and with an analytic solution; both approaches were found to give comparable results.

VIII. LIMITATIONS

This algorithm has been tested with constrained and unconstrained QPs of relatively small dimension, and in development was also tested with linear programs. We have not explored other problem types (e.g. second-order cone programs, semi-definite programs) where the Hessian may change over the iterates. We also have not expressly addressed the theoretical implications of using this on strongly-convex problems (e.g. problems which can be bounded below by a QP) compared with weakly-convex problems.

A. High-Dimensional Problems

We have previously assumed that \( n + 1 \) iterates are needed to assess the validity of the \( x \)-update step. Where \( n \) is large, collecting sufficient iterates may be costly in time or computation resources. Further, the problem may converge before \( n + 1 \) iterates are collected (Note, this is not a problem if the goal is to prevent convergence-stopping attacks).

We consider two scenarios: one where the full set of \( x \)-update variables are made public, and one where only the variables associated with the linking constraint are made public.

All \( x \)-variables are public: When all \( x \)-variables are made public, the gradient is constructed as:

\[
A \in \mathbb{R}^{p \times n}
\]

\[
x_{pub} \in \mathbb{R}^n
\]

\[
(z^i - z^{i-1}) \in \mathbb{R}^m
\]

\[
(y^i - \rho B(z^i - z^{i-1})) \in \mathbb{R}^p
\]

\[
-A^T(y^i - \rho B(z^i - z^{i-1})) \in \mathbb{R}^n
\]

The resulting gradient contains all \( x \)-dimensions, but will be rank deficient, having span \( \mathbb{R}^p \) and nullspace \( n - p \). In this case, each new point gives us \( n \) new dimensions, but we also seek to determine the Hessian in \( n \) dimensions and \( n^2 \) unknowns. We thus need to gather \( n + 1 \) points.

Only linking dimensions are public: If only linking dimensions are public, the gradient is constructed as:

\[
A \in \mathbb{R}^{p \times p}
\]

\[
x_{pub} \in \mathbb{R}^p
\]

\[
-A^T(y^i - \rho B(z^i - z^{i-1})) \in \mathbb{R}^p
\]

In this case, the gradient estimate fully spans the linking dimensions, and each new point gives us \( p \) new equations. We seek the Hessian in \( p \) dimensions and \( p^2 \) unknowns, thus needing to gather \( p + 1 \) datapoints.

Either way, we only are able to assess nonconvexity in the linking dimensions, and are unsure of activity in the other dimensions.
IX. Extensions

A. Reducing Error Rates

As described above, the false positive and false negative rates can be reduced by tuning the algorithm:

• Reducing False Positives: As previously described, false positives result from poor numeric conditioning in computing the Hessian from the system of difference equations. Reducing these errors requires testing the condition number of the system of equations, and potentially also testing the condition number of the resulting Hessian estimate. Neither topic has been explored here.

• Reducing False Negatives: False negatives occur when the tested set of points do not indicate a non-convexity; this is most likely to occur when the magnitude of attack is small relative to the curvature of the x-update function. Reducing this error rate requires testing a variety of combinations of points in order to maximize the likelihood of finding a point where magnitude of the attack is large relative to the local curvature of the function.

B. Improving Algorithmic Efficiency

We have not explored opportunities to improve the efficiency of the algorithm, but highlight three opportunities here:

• Exploiting symmetry in Hessian: We have treated the Hessian as having \( n^2 \) unknowns, but as it is symmetric there are actually only \( n(n+1)/2 \) independent terms. Exploiting this structure would mean that only \( [(n+1)/2] + 1 \) points are needed, but also requires pruning the equation set to avoid over-determining the system of equations.

• Exploiting problem-specific structure of Hessian: For quadratic programs, the Hessian matrix is constant throughout the problem, meaning that the difference equations do not need to be constructed with respect to a single point, but rather can be constructed using all combinations of points. In this case, we need to only use \( j \) iterates such that \( \left\lceil \frac{j}{2} \right\rceil > [(n+1)/2] + 1 \), allowing the algorithm to be started faster, and allowing for more robust checking in the case of poor conditioning.

• Choosing point set: If the Hessian is computed multiple times per iteration to reduce error rates as described above, choosing the iterates wisely can improve numeric conditioning and reduce the need for extra computations. This has not been explored.

X. EXTENSION: FULLY-DECENTRALIZED OPTIMIZATION

As the central information hub, the aggregator in a decentralized optimization problem must be trusted to provide fair updates to all the nodes. When the aggregator and the local nodes have the same incentives – e.g., they are all computing resources owned by the same entity – this is unlikely to present a conflict of interests.

However, when the aggregator has different incentives than the compute nodes, the local nodes may not trust the central aggregator, e.g. if the aggregator can extract additional profits by acting as a market maker. Further, aggregator-based decentralized computation is subject to several other weaknesses: it requires a high communication overhead by coordinating message-passing between all nodes (a significant hurdle for highly distributed systems like energy devices), and introduces a central point of failure in the case of communication/power outage or cyberattack.

To address these issues with aggregator-coordinated decentralized optimization, fully-decentralized algorithms have been developed which take advantage of problem structure to achieve consensus between nodes that directly share constraints, rather than passing information from all nodes to a centralized aggregator [36].

A. Fully-Decentralized ADMM

As an example, the ADMM algorithm from above can be expressed in fully-decentralized form by introducing local copies \( u_z \) and \( u_z \) of the penalty variables:

\[
x^{k+1} := \arg\min_{x \in X} f(x) + \frac{\rho}{2} \|Ax + Bz^k - c + u_z^k\|^2_2
\]

\[
z^{k+1} := \arg\min_{z \in Z} g(z) + \frac{\rho}{2} \|Az^{k+1} + Bz - c + u_z^k\|^2_2
\]

\[
u_z^{k+1} := u_z^k + Az^{k+1} - Bz^{k+1} - c
\]

\[
u_z^{k+1} := u_z^k + Az^{k+1} - Bz^{k+1} - c
\]

Under restrictions described in [36], this can be shown to have the same convergence and optimality guarantees as conventional aggregator-based systems.

In systems where nodes are very sparsely connected, this can produce a significant reduction in communication overhead- e.g. in power systems we may seek an optimum amongst millions of nodes, but each node is only connected to its parent and one or two downstream nodes. Rather than requiring that an aggregator handle millions of connections, nodes pass messages with their neighbors, ultimately bringing the full system into optimum. For examples of this approach applied to power systems, see e.g. [37], [38], [39].

XI. ATTACK VECTORS IN FULLY-DECENTRALIZED OPTIMIZATION

As nodes in a fully-decentralized network are only connected with their neighbors, they must assess whether updates represent the true state of the network, or whether
the neighbor has been compromised and the update is spurious. The following sections highlight the unique challenges of detecting, localizing, and mitigating attacks in a fully-decentralized system, as shown graphically in Figure 11.

A. Attack Vector: Private Infeasibility Attack

In Section V we highlighted how a malicious node may distort its private constraints to shift the equilibrium out of the operationally feasible region.

This attack is not identifiable in a fully-decentralized system, as a node cannot in general discern between the update corrupted by its immediate neighbor

\[
x^{2k} = \arg\min_{x \in \mathcal{X}} L(x, z^{k-1}, w^{k-1}, u^{k-1}_x)
\]

and a best response from a neighbor in reaction to a corrupted signal from the upstream \( h(w) \) node:

\[
x^{2k} = \arg\min_{x \in \mathcal{X}} L(x, z^{k-1}, \tilde{w}^{k-1}, u^{k-1}_x)
\]

Alternatively, on seeing \( x^{2k} \in \{ \mathcal{X} \mid w^{k-1}, z^{k-1} \} \) it is necessary to know \( w^k \) in order to assess whether the problem is truly feasible given the full state of the system, \( x^{2k} \in \{ \mathcal{X} \mid w^{k-1}, z^{k-1} \} \) (this can be extended to more complex system architectures with more upstream/downstream nodes).

However, in a fully-decentralized system, \( w \) is not generally visible to \( x \) and so it is not possible for the \( z \)-update node to assess whether \( x^{2k} \in \{ \mathcal{X}_{\text{pub}} \mid w^k \} \).

This means that detection, localization, and mitigation are all not possible in a conventional fully-decentralized system, as a global information layer is needed to assess the feasibility of the received updates, identify nodes which may be causing infeasibility, and construct a best response.

B. Attack vector: Infeasible Linking Constraint

Similarly, on receiving an update \( x^{2k} \) which would violate the linking constraint \( \{ Ax^{2k} + Bz = c \} \), the \( z \)-update node is not able to discern between an infeasible update created by a neighboring node:

\[
x^{2k} = x^{*k} + \varepsilon^k
\]

and an infeasible signal resulting from a malicious upstream/downstream node:

\[
x^{2k} = \arg\min_{x \in \mathcal{X}} L(x, z^{k-1}, \tilde{w}^{k-1}, u^{k-1}_x)
\]

\[
= \arg\min_{x \in \mathcal{X}} L(x, z^{k-1}, w^{k-1} + \varepsilon, u^{k-1}_x)
\]

However, if information on public constraints \( Z_{\text{pub}}, \mathcal{X}_{\text{pub}} \) is knowable to all participants, these constraints can be integrated into the local optimization problems as additional (publicly known) constraints, converting the update step from

\[
x^{k+1} = \arg\min_{x \in \mathcal{X}} L(\rho(x, z^k, u^k))
\]

to

\[
x^{k+1} = \arg\min_{x \in \mathcal{X} \cap \{ Z_{\text{pub}} \mid A^T x + B^T z = c \}} L(\rho(x, z^k, u^k))
\]

This would mean that any update which violates this constraint must be the result of a malicious node, and the problem can be localized.

**Attack Mitigation:** If a feasible solution exists, it must satisfy the constraints:

\[
x^* \in Z \subset \mathcal{X}_{\text{pub}}
\]

\[
z^* \in Z \subset Z_{\text{pub}}
\]

\[
A^T x^* + B^T z^* = c
\]

Therefore, at each step we can project any iterate onto the feasible set described by the constraints above:

\[
\tilde{x}^k = \arg\min_{z \in \{ Z \mid A^T x + B^T z = c \}} \| x^{2k} - z \|_2
\]

In an unattacked case this projection simply accelerates convergence of the standard ADMM algorithm.

In the case of attack, projection creates a \( \tilde{x}^k \in \{ Z \mid A^T x + B^T z = c \} \) which is the *best response* to the attack \( \tilde{x}^k \). While this best response will in general be suboptimal, it is feasible and will let the primal residual \( r = Ax^{k+1} + Bz^{k+1} - c \) converge to zero even in the presence of attack.
C. Attack Vector: Zero-Mean Noise Injection

Because it is only dependent on local information (its own history and updates received from the neighbors), each node in a fully-decentralized network is able to carry out the detection strategy outlined in Section VI for identifying the presence of a noise-injection attack.

However, in a fully-decentralized network the node would be challenged to identify whether the noise injection attack originates from an immediate neighbor, or from an upstream node.

Specifically, if \( h(w) \) represents the problem solved by the upstream node, the \( z \)-update node cannot differentiate between \( x^k = x^{*k} + \delta(k) \) and

\[
x^k = \arg\min_{x \in X} L_p(x, z^{k-1}, w^{k-1}, u^{k-1}) = \arg\min_{x \in X} L_p(x, z^{k-1}, w^{*k-1} + \delta(k-1), u^{k-1})
\]

Nevertheless, although the malicious node cannot be localized, a zero-mean noise injection attack can be detected.

XII. SUMMARY OF SECURITY CHALLENGES

Table [1(a)] highlights the feasibility of achieving these goals in conventional aggregator-coordinated optimization, and Table [1(b)] shows the same capabilities in a fully-decentralized system. We close by exploring approaches for reaching these goals in a fully-decentralized environment.

TABLE II: Security issues in aggregator-coordinated and fully-decentralized systems

| (a) Detection Feasibility on Aggregator-Coordinated System | Detect | Localize | Mitigate |
|---------------------------------------------------------|--------|---------|---------|
| Private Infeasibility                                  | X      | X       | X       |
| Linking Infeasibility                                  | X      | X       |         |
| Noise Injection                                        | X      |         |         |

(b) Detection Feasibility on Fully-Decentralized System

| Detect | Localize | Mitigate |
|--------|----------|----------|
| Private Infeasibility | X |       |
| Linking Infeasibility  | X |       |
| Noise Injection        | X |       |

A. Architectures for Security

Although aggregator-based systems present some security and trust issues, the aggregator is able to take advantage of global information to check the validity of each node’s updates, enabling detection, localization, and mitigation strategies, which are not available in the fully-decentralized system.

However, if the fully-decentralized system is augmented with a global information layer, the same security checks become possible in a fully-decentralized environment. In this section, we briefly discuss information architectures which could allow a fully-decentralized system to take advantage of the security benefits of an aggregator-coordinated system.

A number of different architectures might provide this security:

1) A centralized database held by a trusted authority, with security checks computed by each node
2) A fully-connected network with securely signed messages, in which each node maintains a database of message history
3) A partially-connected network with bypass connections to allow nodes to bypass suspect neighbors
4) A decentralized peer-to-peer database with message histories, with security checks computed by each node
5) A blockchain used as a decentralized database to store the message histories, with smart contracts used to compute security checks in a decentralized manner.

Option 1 presents trust and monopoly distortion issues described above, in addition to a high communication overhead. Option 2 requires high computational overhead and will fail if there are dropouts in communication with part of the network; Option 3 has gained some attention as a way to approach this while not requiring the same degree of communication redundancy. The decentralized database in Option 4 is appealing, but requires a method for reconciling different versions of the database which might be proposed by different neighbors. Further, both Option 1 and Option 4 do not guarantee that each node conducts the same security checks.

By contrast, blockchains are decentralized databases which rely on securely signed messages and a consensus mechanism that simultaneously guarantees consistency across the network and provides secure timestamping. Smart contracts build on this architecture by guaranteeing the execution of simple computational functions as part of the consensus mechanism, thus guaranteeing consistent execution of security checks.

Figure 13 conceptually shows how a blockchain-based system enables the same type of security checks as are offered by an aggregator system, by providing a cryptographically secured global information layer with guaranteed execution of the security checks outlined above.

Consequently, the features of cyberattack diagnostics algorithms for aggregator-coordinator systems can be accomplished in a fully decentralized setting, if a shared and secure
information layer is provided, such as blockchains.

XIII. CONCLUSION

We have demonstrated the weaknesses of conventional distributed and fully-decentralized architectures to a number of attack vectors, including distortion of the private optimization problem, injection of noise, and distortion of the linking constraints in fully-decentralized networks. As optimization problems become distributed to consumer hardware (e.g. smartphones, smartmeters, autonomous vehicles, IoT devices), optimization algorithms becomes exposed to these threats along an exponentially larger attack surface. We examine methods for detecting and mitigating attacks. In particular, we detail a detection algorithm for noise-injection attacks for a set of stochastically generated ADMM problems. Discussions on limitation and extensions are provided, along with perspectives on challenge and opportunities for fully decentralized systems.

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APPENDIX

The x-update step is an unconstrained quadratic program, and can be solved analytically. For clarity, we let $\gamma = Bz^k + u^k$ and proceed as:

$$x^{k+1} = \arg\min_x x^T P x + c^T x + \frac{\rho}{2} ||Ax + \gamma||_2^2$$

$$x^{k+1} = \arg\min_x x^T P x + c^T x + \frac{\rho}{2} (Ax + \gamma)^T (Ax + \gamma)$$

$$x^{k+1} = \arg\min_x x^T P x + c^T x + \frac{\rho}{2} (x^T A^T Ax + 2\gamma^T Ax + \gamma^T \gamma)$$

$$0 = \frac{\partial}{\partial x} \left( x^T P x + c^T x + \frac{\rho}{2} (x^T A^T Ax + 2\gamma^T Ax + \gamma^T \gamma) \right)$$

$$0 = 2Px + c + \rho A^T Ax + \rho \gamma^T A$$

$$0 = (2P + \rho A^T A)x + c + \rho \gamma^T A$$

$$x = (2P + \rho A^T A)^{-1}(-c - \rho \gamma^T A)$$

Similarly, the z-update step can be solved analytically by letting $\mu = Ax^{k+1} - c + u^k$ and following a similar process to find:

$$z^{k+1} = (2Q + \rho B^T B)^{-1}(-d - \rho \mu^T B)$$

These analytic solutions are used in our implementation to avoid inaccuracies induced from a numeric solution.

A. Comparison with Central Solution

Because the problem is an unconstrained QP and entries with consensus between a subset of variables, a centralized solution can be computed by composing the cost matrices into a single quadratic problem which can be solved analytically. This is shown here for the case where $c = 0$ and $A$ and $B$ are composed as described above, but also can be computed for other $A$, $B$.

We break $P$ and $Q$ into sub-matrices dependent on the number of consensus constraints $p$, where $P_{11}, Q_{00} \in \mathbb{R}^{p \times p}$ and the other dimensions follow accordingly.

$$P = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}$$

$$Q = \begin{bmatrix} Q_{00} & Q_{01} \\ Q_{10} & Q_{11} \end{bmatrix}$$

$$\Pi = \begin{bmatrix} P_{00} & P_{01} & 0 \\ P_{10} & P_{11} + Q_{00} & Q_{10} \\ 0 & Q_{10} & Q_{11} \end{bmatrix}$$

Similarly, the $c$ and $d$ vectors can be combined as

$$\kappa = \begin{bmatrix} c_0 \\ c_1 + d_0 \\ d_1 \end{bmatrix}$$

The problem can then be expressed as an unconstrained minimization problem:

$$\min_w w^T \Pi w + \kappa^T w$$

which is solved by $w^* = -\frac{1}{2}(\Pi^T)^{-1}\kappa$

This central solution was used only for verification.