GENERALIZED SOLUTION TO A SYSTEM OF CONSERVATION LAW WHICH IS NOT STRICTLY HYPERBOLIC

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ABSTRACT. In this paper we study a non strictly system of conservation law when viscosity is present and viscosity is zero, which is studied in [10]. We show the existence and uniqueness of the solution in the space of generalized functions of Colombeau for the viscous problem and construct a solution to the inviscid system in the sense of association. Also we construct a solution using shadow wave approach [5] and Volpert product which was partly determined as vanishing viscosity limit in [10].

1. Introduction

Conservation laws come in applications are sometime not strictly hyperbolic. Classical theory of Lax[17] and Glimm [6] does not apply in this case. In general such systems do not admit distributional solutions. As product of distributions arises, one can not search solutions in the space of distributions. The ideal space where one should search solutions is the Colombeau algebra of generalized functions. For detail, see Colombeau [2, 3, 4] and Oberguggenberger [11].

Our interest is to study the inviscid partial differential equation

\begin{align*}
    u_t + \left(\frac{u^2}{2}\right)_x &= 0, \quad v_t + (uv)_x = 0 \\
    w_t + \left(\frac{v^2}{2} + uw\right)_x &= 0, \quad z_t + (vw + uz)_x = 0.
\end{align*}

(1.1)

and viscous regularization of (1.1) with coefficient of viscosity a generalized constant $\gamma$,

\begin{align*}
    u_t + \left(\frac{u^2}{2}\right)_x &= \frac{\gamma}{2} u_{xx}, \quad v_t + (uv)_x = \frac{\gamma}{2} v_{xx} \\
    w_t + \left(\frac{v^2}{2} + uw\right)_x &= \frac{\gamma}{2} w_{xx}, \quad z_t + (vw + uz)_x = \frac{\gamma}{2} z_{xx},
\end{align*}

(1.2)

with initial conditions

\begin{align*}
    (u(x,0), v(x,0), w(x,0), z(x,0)) = (u_0, v_0, w_0, z_0)
\end{align*}

(1.3)

are generalized functions of Colombeau.

The system which we have considered here is $n = 4$ case of the following system.

\begin{align*}
    (u_j)_t + \sum_{i=1}^{j} \left(\frac{u_iu_{j-i+1}}{2}\right)_x = \frac{c}{2}(u_j)_{xx}, \quad j = 1, 2, \ldots n.
\end{align*}

(1.4)

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where $\epsilon > 0$ is a small parameter. A method to write down an explicit formula for the solution of (1.4), with initial condition of the form
\[
u_j(x, 0) = u_{j0}(x), \quad j = 1, 2, \ldots, n
\]  
(1.5)
is given in [8]. It is well-known that the corresponding inviscid system
\[
(u_j)_t + \sum_{i=1}^{j} \left( \frac{u_i u_{i+1}}{2} \right)_x = 0, \quad j = 1, 2, \ldots, n.
\]  
(1.6)
does not have smooth global solution, even if the initial data (1.5) is smooth; one has to seek solution in a weak sense and weak solutions are not unique. Additional conditions are required to pick the unique physical solution. Vanishing viscosity method is one of the ways to select the physical weak solution of (1.4). That is, the solution of the inviscid system is constructed as the limit $\epsilon$ goes to zero of solutions $u_j^\epsilon(x, t)$ of (1.4), with suitable initial conditions. This was successfully carried out for the cases $n = 1$ and $n = 2$ for general initial data. For $n = 3$, only partial results are available. In fact it was observed in [9] that the order of singularity of vanishing viscosity limit of solutions of (1.4) increases as $n$ increases.

More precisely, when $n = 1$, with $u = u_1$, (1.4) become the celebrated Burgers equation,
\[
u_t + \left( \frac{u^2}{2} \right)_x = \frac{\epsilon}{2} \nu_{xx},
\]
which was explicitly solved for the initial value problem by Hopf [16] and Cole [4]. Hopf [16] showed that the vanishing viscosity limit of its solution with a given bounded measurable initial data is a bounded measurable and locally a BV function which is the weak entropy solution to the inviscid Burgers equation
\[
u_t + \left( \frac{u^2}{2} \right)_x = 0.
\]
Viscous and inviscid Burgers equation in Colombeau setting has studied in [12]. For $n = 2$, with $u = u_1, v = u_2$ the system (1.4) becomes
\[
u_t + \left( \frac{u^2}{2} \right)_x = \frac{\epsilon}{2} \nu_{xx}, \quad v_t + (uv)_x = \frac{\epsilon}{2} v_{xx},
\]
in $\{(x, t) : -\infty < x < \infty, t > 0\}$ and is well studied [7, 9]. The case $n = 2$ is a one dimensional modelling for the large scale structure formation of universe, see [18].

In [9], the existence of the solution for the case $n = 3$ in Colombeau class is shown for the bounded measurable initial data and solution to the inviscid case is considered in the sense of association. However this method can not be carried out for the case $n = 4$ as the solution become more and more singular as $n$ increases.

In [10], the vanishing viscosity limit for the Riemann type initial data is studied for the case $n = 4$. Calculation of vanishing viscosity limit for $n > 2$ is an open question for general type initial data. Here our aim is to study viscous system for the case $n = 4$, i.e., (1.4) and inviscid case (1.1) with initial datas are generalized functions of Colombeau.

The present paper is organised in the following way. In section 2, we recall some definition and results of algebra of generalized functions of Colombeau. In section 3, we discuss existence and uniqueness of the equation (1.2) when initial data belongs to the generalized space of Colombeau and viscosity parameter is a positive generalized constant. In section 4, we construct a macroscopic solution to
the problem (1.1) when initial data are bounded measurable functions. In section 5, we construct a shadow wave solution and a solution using Volpert product for Riemann type initial data when the first component develops rarefaction. It is remarkable that the solution constructed using Volpert product agrees with the vanishing viscosity limit, obtained in the special case when the first component has rarefaction. We conclude in section 6 with some remarks.

2. The algebra of generalized functions of Colombeau

In this section we introduce generalized functions of Colombeau in the domain $\Omega_T = \{(x,t) : -\infty < x < \infty, 0 < t < T\}$, for $T > 0$, containing the space of bounded distributions, see [1, 2, 3, 11]. Here we shall use a version which is sufficient for our purpose. This algebra we denote it by $G_{s,g}(\Omega_T)$. Let $E_{s,g}(\Omega_T) = \{(u^{\epsilon})_{0 \leq \epsilon < 1} : u^{\epsilon} \in C^\infty(\bar{\Omega}_T)\}$.

We define moderate elements of $E_{s,g}(\Omega_T)$ as

$E_{M,s,g}(\Omega_T) = \{(u^{\epsilon})_{0 \leq \epsilon < 1} \in E_{s,g}(\Omega_T) : \forall (l, m) \in \mathbb{N}_0^2 \exists \ p > 0$ such that $|\frac{\partial^j}{\partial t^j} \frac{\partial^l}{\partial x^l} u^{\epsilon}|_{L^\infty(\Omega_T)} = O(\epsilon^{-p})\}$

and null elements by:

$N_{s,g}(\Omega_T) = \{(u^{\epsilon})_{0 \leq \epsilon < 1} \in E_{s,g}(\Omega_T) : \forall (l, m) \in \mathbb{N}_0^2 \forall \ q > 0$ such that $|\frac{\partial^j}{\partial t^j} \frac{\partial^l}{\partial x^l} u^{\epsilon}|_{L^\infty(\Omega_T)} = O(\epsilon^q)\}$

Note that $E_{M,s,g}(\Omega_T)$ is a differential algebra under component wise addition and multiplication. Also $N_{s,g}(\Omega_T)$ is a differential ideal. So the quotient space $G_{s,g}(\Omega_T) = E_{M,s,g}(\Omega_T) / N_{s,g}(\Omega_T)$ is also a differential algebra, addition and multiplication being defined at coset level.

The space of all bounded distributions on $\bar{\Omega}_T$ can be embedded in this version of algebra and under this embedding the product of two bounded smooth functions is preserved.

We say $u \in G_{s,g}(\Omega_T)$ admits a distribution $v \in D'(\Omega_T)$ in the sense of association or as macroscopic aspect if for all test function $\phi \in D(\Omega_T)$ and a representative $(u^{\epsilon})_{\epsilon > 0}$ (so for all representatives):

$$\lim_{\epsilon \to 0} \int_{\Omega_T} u^{\epsilon} \phi(x,t) dx dt = \langle u, \phi \rangle$$

An element $u \in G_{s,g}(\Omega_T)$ is said to be bounded type if there exist a representative $(u^{\epsilon})_{\epsilon > 0}$ of $u$ which is bounded.

An element $u \in G_{s,g}(\Omega_T)$ is said to be a macroscopic solution or solution in the sense of association to the the differential equation $L(u) = 0$ if $L(u)$ has the macroscopic aspect 0.

A generalized function $u \in G_{s,g}(\Omega_T)$ is said to be a generalized constant if it has a representative which is constant for each $\epsilon > 0$. 
3. Existence and uniqueness

In this section first we write explicit solutions of (1.2) for a representative \( \tilde{\gamma} \) of \( \gamma \). That is,

\[
\begin{align*}
    u_t + \left( \frac{v^2}{2} \right)_x &= \frac{\tilde{\gamma}(\epsilon)}{2} u_{xx}, \\
    v_t + (uv)_x &= \frac{\tilde{\gamma}(\epsilon)}{2} v_{xx}, \\
    w_t + \left( \frac{v^2}{2} + uw \right)_x &= \frac{\tilde{\gamma}(\epsilon)}{2} w_{xx}, \\
    z_t + (vw + uz)_x &= \frac{\tilde{\gamma}(\epsilon)}{2} z_{xx}.
\end{align*}
\]

(3.1)

with initial data,

\[
(u(x, 0), v(x, 0), w(x, 0), z(x, 0)) = (u_0^\epsilon(x), v_0^\epsilon(x), w_0^\epsilon(x), z_0^\epsilon(x))
\]

(3.2)

where \( u_0^\epsilon(x), v_0^\epsilon(x), w_0^\epsilon(x), z_0^\epsilon(x) \) are representative of \( u_0, v_0, w_0, z_0 \) respectively, for each fixed \( \tilde{\gamma}(\epsilon) > 0 \), which can be found in \( [10] \) with \( \tilde{\gamma}(\epsilon) \) replaced by \( \epsilon \). Before the statement of the Theorem, we introduce some notations. Starting from the initial data \( u_0^\epsilon, v_0^\epsilon, w_0^\epsilon, z_0^\epsilon \), we define

\[
\begin{align*}
    U_0^\epsilon(x) &= \int_0^x u_0^\epsilon(y)dy, \\
    V_0^\epsilon(x) &= \int_0^x v_0^\epsilon(y)dy, \\
    W_0^\epsilon(x) &= \int_0^x w_0^\epsilon(y)dy, \\
    Z_0^\epsilon(x) &= \int_0^x z_0^\epsilon(y)dy,
\end{align*}
\]

(3.3)

and the functions \( a, b, c, d \)

\[
\begin{align*}
    a(x, t) &= \frac{1}{\sqrt{2\pi t\tilde{\gamma}(\epsilon)}} \int_{-\infty}^{+\infty} e^{-\frac{1}{\tilde{\gamma}(\epsilon)}(u_0^\epsilon(y) + \frac{\epsilon^2}{2t})^2} dy, \\
    b(x, t) &= -\frac{1}{\tilde{\gamma}(\epsilon)} \sqrt{2\pi t\tilde{\gamma}(\epsilon)} \int_{-\infty}^{+\infty} V_0^\epsilon(y)e^{-\frac{1}{\tilde{\gamma}(\epsilon)}(u_0^\epsilon(y) + \frac{\epsilon^2}{2t})^2} dy, \\
    c(x, t) &= \frac{1}{\sqrt{2\pi t\tilde{\gamma}(\epsilon)}} \int_{-\infty}^{+\infty} \left[ \frac{V_0^\epsilon(y)^2}{2\tilde{\gamma}(\epsilon)^2} - \frac{W_0^\epsilon(y)}{\tilde{\gamma}(\epsilon)} \right] e^{-\frac{1}{\tilde{\gamma}(\epsilon)}(u_0^\epsilon(y) + \frac{\epsilon^2}{2t})^2} dy, \\
    d(x, t) &= \frac{1}{\sqrt{2\pi t\tilde{\gamma}(\epsilon)}} \int_{-\infty}^{+\infty} \left[ -\frac{Z_0^\epsilon(y)}{\tilde{\gamma}(\epsilon)} - \frac{V_0^\epsilon(y)^3}{6\tilde{\gamma}(\epsilon)^3} + \frac{V_0^\epsilon(y)W_0^\epsilon(y)}{\tilde{\gamma}(\epsilon)^2} \right] e^{-\frac{1}{\tilde{\gamma}(\epsilon)}(u_0^\epsilon(y) + \frac{\epsilon^2}{2t})^2} dy.
\end{align*}
\]

(3.4)

**Theorem 3.1.** For the initial data \( u_0^\epsilon, v_0^\epsilon, w_0^\epsilon \) and \( z_0^\epsilon \), there exists a classical solution of (3.1) - (3.2) which is given by

\[
\begin{align*}
    u^\epsilon = -\tilde{\gamma}(\epsilon)(\log(a))_x, \\
    v^\epsilon = -\tilde{\gamma}(\epsilon)\left( \frac{b}{a} \right)_x, \\
    w^\epsilon = -\tilde{\gamma}(\epsilon)\left( \frac{c}{a} - \frac{b^2}{2a^2} \right)_x, \\
    z^\epsilon = -\tilde{\gamma}(\epsilon)\left( \frac{1}{3} \left( \frac{b}{a} \right)^3 - \frac{bc}{a^2} + \frac{d}{a} \right)_x,
\end{align*}
\]

(3.5)

where \( a, b, c \) and \( d \) are given by (3.4).

**Theorem 3.2.** Let \( \gamma \) be a generalized constant with a representative \( \tilde{\gamma} \) satisfying : there exist \( N \in \mathbb{N} \) and a \( \eta \) such that \( \tilde{\gamma}(\epsilon) \geq \epsilon^N \) for each \( 0 < \epsilon < \eta \). If the initial data \((u_0, v_0, w_0, z_0) \in (G_{s,\rho}(\mathbb{R}))^4 \), then there exists a solution \((u, v, w, z) \in (G_{s,\rho}(\Omega_T))^4 \) of the equation (1.2). (1.3) whose representative explicitly given by (3.3) - (3.5).

**Proof.** We follow Joseph [9], Biagioni and Oberguggenberger [12]. We show using representation formula (3.3) that \( u^\epsilon, v^\epsilon, w^\epsilon \) and \( z^\epsilon \) satisfy moderate estimates.

So it is enough to show \( \frac{du}{dt}, \frac{b}{a}, \frac{c}{a}, \frac{d}{a} \) satisfy moderate estimates.
From the identities (3.3) - (3.5) it is enough to show the element of the form

\[
\int_{-\infty}^{\infty} k_1^j(x, t) e^{-\frac{1}{2\sigma}(U_0^i(y)+\frac{(x-y)^2}{2t})} dy \in \mathcal{E}_{s,g}(\Omega_T)
\] (3.6)

where \( k_1^j \) is a representative of an element of Colombeau class \( k_1 \in \mathcal{G}_{s,g}(\Omega_T) \) and \( U_0^i \) is a representative of \( U_0 \in \mathcal{G}_{s,g}(\mathbb{R}) \).

Let’s denote

\[
F_1(\epsilon, x, t) = \int_{-\infty}^{\infty} k_1^j(x, t) e^{-\frac{1}{2\sigma}(U_0^i(y)+\frac{(x-y)^2}{2t})} dy
\]

\[
F_2(\epsilon, x, t) = \int_{-\infty}^{\infty} k_1^j(x, t) e^{-\frac{1}{2\sigma}(U_0^i(y)+\frac{(x-y)^2}{2t})} dy
\] (3.7)

Then

\[
\frac{F_1(\epsilon, x, t)}{F_2(\epsilon, x, t)} = \frac{\int_{-\infty}^{\infty} k_1^j(x, t) e^{-\frac{1}{2\sigma}(U_0^i(y)+\frac{(x-y)^2}{2t})} dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma}(U_0^i(y)+\frac{(x-y)^2}{2t})} dy}
\] (3.8)

One can easily show, \( \partial_x^k \frac{F_1(\epsilon, x, t)}{F_2(\epsilon, x, t)} \) is the finite linear combinations of finite products of the elements having the form

\[
\frac{\partial_x^{j_1} F_1(\epsilon, x, t)}{F_2(\epsilon, x, t)} \quad \text{and} \quad \frac{\partial_x^{j_2} F_2(\epsilon, x, t)}{F_2(\epsilon, x, t)},
\] (3.9)

where \( j_1 \) and \( j_2 \) are nonnegative integers. Now using change of variable we get

\[
\partial_x^{j_1} F_1(\epsilon, x, t) = \partial_x^{j_1} \int_{-\infty}^{\infty} k_1^j(x, t) e^{-\frac{1}{2\sigma}(U_0^i(y)+\frac{(x-y)^2}{2t})} dy
\]

\[
= \partial_x^{j_1} \int_{-\infty}^{\infty} k_1^j(x-z, t) e^{-\frac{1}{2\sigma}(U_0^i(x-z)+\frac{(x-z)^2}{2t})} dz
\]

\[
= \int_{-\infty}^{\infty} P(\gamma(\epsilon), \partial_x^{1} k_1^j(x-z, t), ..., \partial_x^{j_1} k_1^j(x-z, t), \partial_x^{1} U_0^i(x-z), ..., \partial_x^{j_1} U_0^i(x-z)) e^{-\frac{1}{2\sigma}(U_0^i(x-z)+\frac{(x-z)^2}{2t})} dz,
\] (3.10)

where \( P \) is a polynomial of \( 2J_1 + 1 \) variables. Since the variables satisfy moderate estimates and assumption on \( \gamma \) implies \( \partial_x^{j_1} F_1(\epsilon, x, t) \) satisfy moderate estimate. Similarly one can show \( \partial_x^{j_1} F_2(\epsilon, x, t) \) also satisfy moderate estimates.

So if we take the class \( u, v, w, z \) in Colombeau space whose representatives are respectively \( u^*, v^*, w^*, z^* \), then \( u, v, w, z \) satisfy (1.2) - (1.3). This completes the proof of the theorem. \( \square \)

Now we show the uniqueness for the Cauchy problem for the equation (1.2). For that we use a modified version of Gronowall inequality from [12].

**Lemma 3.3.** Let \( u \) be a nonnegative, continuous function on \([0, \infty)\) and assume that

\[
u(t) \leq a + b \int_{0}^{t} \frac{u(t_1)}{\sqrt{t-t_1}} dt_1
\]

The proof of this lemma is similar to the proof of Lemma 3.2. \( \square \)
for some constant \(a, b \geq 0\) and every \(t \geq 0\). Then
\[
\begin{align*}
        u(t) & \leq a(1 + 2b\sqrt{t}) \exp(\pi b^2 t)
\end{align*}
\]

**Theorem 3.4.** Let \(\gamma\) be a generalized constant with a representative \(\hat{\gamma}\) satisfying: there exist a \(\eta\) such that \(\hat{\gamma}(\epsilon)\log(\frac{1}{\epsilon}) \geq 1\) for each \(0 < \epsilon < \eta\). Also assume \(u\) is of bounded type. Then for each \(T > 0\), the solutions \(u, v, w, z \in G_{s,g}(\mathbb{R} \times [0, T])\) of (1.2)-(1.3) are unique.

**Proof.** Uniqueness for \(u\) is already known, see Oberguggenberger [12]. Let \(\bar{u}\) is a representative of \(u\) which is bounded.

Let \(v_1\) and \(v_2\) be solutions in the colombeau class with representatives \(\bar{v}_1\) and \(\bar{v}_2\) respectively. Then we will get,
\[
\{(\bar{v}_1 - \bar{v}_2)_t + (\bar{u}(\bar{v}_1 - \bar{v}_2)) + N\}(\epsilon, x, t) = \hat{\gamma}(\epsilon)(\bar{v}_1 - \bar{v}_2)_{xx}(\epsilon, x, t)
\]
\[
(\bar{v}_1 - \bar{v}_2) = n(\epsilon, x)
\]  
(3.11)

Using Duhamel principle, we have
\[
\begin{align*}
        (\bar{v}_1 - \bar{v}_2)(\epsilon, x, t) &= \int_{-\infty}^{\infty} G(\epsilon, x, t, x_1, 0)n(\epsilon, x_1)dx_1 \\
        &= \int_{0}^{t} \int_{-\infty}^{\infty} G(\epsilon, x, t, x_1, t_1)N(\epsilon, x_1, t_1)dx_1dt_1 \\
        &= \int_{0}^{t} \int_{-\infty}^{\infty} \frac{\partial G}{\partial x_1}(\epsilon, x, t, x_1, t_1)(\bar{u}(\bar{v}_1 - \bar{v}_2))(\epsilon, x_1, t_1)dx_1dt_1
\end{align*}
\]  
(3.12)

Since \(\int_{-\infty}^{\infty} G(\epsilon, x, t, x_1, t_1)dx_1 = 1\) and \(\int_{-\infty}^{\infty} | \frac{\partial G}{\partial x_1} | dx_1 = \frac{1}{\sqrt{\pi(t-t_1)\hat{\gamma}(\epsilon)}}\)

Thus we obtain the estimate,
\[
\begin{align*}
        \sup_x | (\bar{v}_1 - \bar{v}_2)(\epsilon, x, t) | & \leq \sup_{x_1} | n(\epsilon, x_1) | + \int_{0}^{t} \sup_{x_1, t_1} | N(\epsilon, x_1, t_1) | \\
        & + \int_{0}^{t} \sup_{x_1} | (\bar{v}_1 - \bar{v}_2)(\epsilon, x_1, t_1) | \\
        & + \sup_{x_1} | \bar{u}(\epsilon, x, t) |
\end{align*}
\]  
(3.13)

So by lemma (3.3),
\[
\begin{align*}
        \sup_{(x,t) \in \mathbb{R} \times [0,T]} | (\bar{v}_1 - \bar{v}_2)(\epsilon, x, t) | & \leq a(1 + 2b\sqrt{T})\exp(\pi b^2 T)
\end{align*}
\]  
(3.14)

where,
\[
\begin{align*}
        a &= \sup_{x_1} | n(\epsilon, x_1) | + T \sup_{(x_1, t_1) \in \mathbb{R} \times [0,T]} | N(\epsilon, x_1, t_1) |
\end{align*}
\]

and
\[
\begin{align*}
        b &= \frac{1}{2\sqrt{\pi\hat{\gamma}(\epsilon)}} \sup_{(x_1, t_1) \in \mathbb{R} \times [0,T]} | \bar{u}(\epsilon, x_1, t_1) |
\end{align*}
\]

From the assumption of \(\gamma(\epsilon)\) we get the condition:
\[
\begin{align*}
        \sup_{(x,t) \in \Omega_T} | (\bar{v}_1 - \bar{v}_2)(\epsilon, x, t) | = O(\epsilon^m)
\end{align*}
\]
for all non negative integers $m$. Since $(\bar{v}_1 - \bar{v}_2)$ satisfy the above estimate and moderate estimate, so it is a null element by theorem (1.2.3) of [13].

The solution for the component $v$ is unique. Let it has a representative $\bar{v}$. Let $w_1$ and $w_2$ be two solutions for the component having representatives $\bar{w}_1$ and $\bar{w}_2$ respectively.

Then we have,

$$\frac{\partial}{\partial t} \bar{w}_1 + \frac{\partial}{\partial x} \frac{\bar{v}^2}{2} + v \bar{w}_1 + n_1(\epsilon, x, t) = \frac{\gamma(\epsilon)}{2} \frac{\partial^2}{\partial x^2} \bar{w}_1$$

$$\frac{\partial}{\partial t} \bar{w}_2 + \frac{\partial}{\partial x} \frac{\bar{v}^2}{2} + v \bar{w}_2 + n_1(\epsilon, x, t) = \frac{\gamma(\epsilon)}{2} \frac{\partial^2}{\partial x^2} \bar{w}_2,$$

and

$$(\bar{w}_1 - \bar{w}_2) = n_3(\epsilon, x, t)$$

where $n_1(\epsilon, x, t), n_2(\epsilon, x, t)$ and $n_3(\epsilon, x, t)$ are null elements in Colombeau space.

Now subtracting the second equation from first in the equation (3.15), we get

$$\frac{\partial}{\partial t}(\bar{w}_1 - \bar{w}_2) + \frac{\partial}{\partial x}(v(\bar{w}_1 - \bar{w}_2)) + n(\epsilon, x, t) = \frac{\gamma(\epsilon)}{2} \frac{\partial^2}{\partial x^2}(\bar{w}_1 - \bar{w}_2),$$

where $n(\epsilon, x, t) = n_1(\epsilon, x, t) - n_2(\epsilon, x, t)$ is a null element in Colombeau algebra. So the analysis similar to above, and condition on $\bar{\gamma}(\epsilon)$ gives the uniqueness for the component $w$.

Uniqueness for the component $z$ can be similarly handled as for the component $w$. This completes the proof of the theorem.

4. Macroscopic solution of the system

In this section we show the existence of macroscopic solution of the equation (1.1) when the initial data $\{u(x, 0), v(x, 0), w(x, 0), z(x, 0)\} = (u_0(x), v_0(x), w_0(x), z_0(x))$ are bounded measurable functions. For the case $n = 3$, is already considered by Joseph [9]. That method can not be applied for $z$-component. Here we take a slower growth order on $u$ to get the required estimates. Now consider the system

$$u_t + \left(\frac{u^2}{2}\right)_x = \frac{\beta(\epsilon)}{2} u_{xx}, \quad v_t + (uv)_x = \frac{\beta(\epsilon)}{2} v_{xx},$$

$$w_t + \left(\frac{v^2}{2} + uw\right)_x = \frac{\beta(\epsilon)}{2} w_{xx}, \quad z_t + (uz + vw)_x = \epsilon z_{xx}. \quad (4.1)$$

with initial data

$$\{u^{\prime}(x, 0), v^{\prime}(x, 0), w^{\prime}(x, 0), z^{\prime}(x, 0)\} = \{u_0^{\beta(\epsilon)}(x), v_0^{\beta(\epsilon)}(x), w_0^{\beta(\epsilon)}(x), z_0^{\epsilon}(x)\}, \quad (4.2)$$

where $A^{\epsilon}(x) = A * \eta^{\epsilon}(x)$, where $\eta^{\epsilon}$ is the usual Friedrich mollifier. With the notation above, we have following theorem.

**Theorem 4.1.** Let $u_0, v_0, w_0$ and $z_0$ are bounded measurable functions. Then there exists $\beta(\epsilon)$ such that $(u^\epsilon, v^\epsilon, w^\epsilon, z^\epsilon)$ of (1.1) satisfy moderate estimates and $(u, v, w, z) \in G_{s, \beta}(\Omega_T)$ corresponding to the representative $(u^\epsilon, v^\epsilon, w^\epsilon, z^\epsilon)$ is a macroscopic solution to the system (1.1) with initial data (1.3).

To prove the above theorem we need the following lemma whose prove can be found [14, Chap. I, Theorem 2.5].
Lemma 4.2. Let \( u \) satisfy
\[
L(u) = u_t - \sum_{ij} a_{ij}(x,t)u_{x_i x_j} + a_i(x,t)u_{x_i} + a(x,t)u = f(x,t),
\]
where \( u \) is continuous at all point \( (x,t) \in \mathbb{R}^n \times [0,T] \), has continuous derivative \( u_t, u_{x_i}, \) and \( u_{x_i x_j} \) satisfies the equation for \( 0 < t \leq T \), is bounded, the moduli of the coefficients \( a_{ij}, a_i \) do not exceed \( c \) and \( a(x,t) \geq -a_0 \), where \( c \) and \( a_0 \) are nonnegative constants, then the following estimate holds.

\[
\sup_{x \in \mathbb{R}^n, 0 \leq t \leq T} |u(x,t)| \leq (\sup_{x \in \mathbb{R}^n} |u(x,0)| + T \sup_{x \in \mathbb{R}^n, 0 \leq t \leq T} |f(x,t)|) \exp(a_0 T) \tag{4.4}
\]

Proof of the theorem:

Proof. Applying the lemma to the component \( z \) and observing that
\[
(u^*)_x = \frac{O(1)}{\sqrt{\beta(\epsilon)}} \quad (\partial_x)^j v = O(\beta(\epsilon)^k(j)), \quad (\partial_x)^j w = O(\beta(\epsilon)^l(j))
\]
for some non negative integers \( k(j) \) and \( l(j) \).

\[
\sup_{x \in \mathbb{R}, 0 \leq t \leq T} |z(x,t)| \leq (\sup_{x \in \mathbb{R}} |z(x,0)| + T \sup_{x \in \mathbb{R}, 0 \leq t \leq T} |(vw)_x|) \exp\frac{O(1)}{\sqrt{\beta(\epsilon)}} \tag{4.5}
\]
for some negative \( m \). Now chose \( \beta(\epsilon) = (\frac{O(1)}{\log(\frac{1}{\epsilon})})^2 \), then \( \frac{a_0 T}{\beta(\epsilon)} = \frac{1}{\sqrt{\epsilon}} \).

So it is clear that \( z \) satisfies
\[
\sup_{x \in \mathbb{R}, 0 \leq t \leq T} |z(x,t)| = O\left(\frac{1}{\sqrt{\epsilon}}\right) \tag{4.6}
\]

Differentiating the fourth equation of (4.1), with with respect to \( x \), we get
\[
(\epsilon z)_t + 2u_x z_x + u(z)_x + (vw)_x + u_{xx} z = \epsilon z_{xxx}.
\]

Again applying the lemma to the above equation, we get
\[
\sup_{x \in \mathbb{R}, 0 \leq t \leq T} |z_x(x,t)| = O\left(\frac{1}{\epsilon}\right)
\]

Proceeding inductively and using the equation, we get following estimates on all combination of operators \( \partial_t \) and \( \partial_x \).
\[
\sup_{x \in \mathbb{R}, 0 \leq t \leq T} |\partial^m_t \partial_x^n z(x,t)| = O\left(\frac{1}{\epsilon^r}\right),
\]
for some nonnegative integer \( r \) depending only on \( l \) and \( m \). So \( (z^*) \) satisfies moderate estimates. Now we show the colombeau class \( u, v, w, z \) corresponding to the representative \( (u^*, v^*, w^*, z^*) \) is a macroscopic solution to the fourth equation of (1.1).

Multiplying with test function \( \phi \in (-\infty, \infty) \times (0, \infty) \) in the equation for component \( z \) and integrating over \( (-\infty, \infty) \times (0, T) \), we have
\[ \int_{0}^{\infty} \int_{-\infty}^{\infty} \left( z_{t} + (vw + uz)_{x} \right) \phi(x,t) dx dt = \epsilon \int_{0}^{\infty} \int_{-\infty}^{\infty} z_{xx} \phi(x,t) dx dt \quad (4.7) \]

Using integration by parts twice in right hand side of the equation (4.7), we have,

\[ \int_{0}^{\infty} \int_{-\infty}^{\infty} \left( z_{t} + (vw + uz)_{x} \right) \phi(x,t) dx dt = \epsilon \int_{0}^{\infty} \int_{-\infty}^{\infty} z \phi(x,t)_{xx} dx dt. \quad (4.8) \]

By equation (4.6), right hand side of equation (4.7) implies

\[ \lim_{\epsilon \to 0} \int_{0}^{\infty} \int_{-\infty}^{\infty} \left( z_{t} + (vw + uz)_{x} \right) \phi(x,t) dx dt = 0 \]

By the estimate (4.1), the above limit tends to zero as \( \epsilon \) tends to zero.

Since \( \epsilon \) tends to zero imply \( \beta(\epsilon) \) tends to zero, we get from (9) that \((u, v, w)\) is a solution to the first three equations of (1.1) in the sense of association.

Hence \((u, v, w, z) \in G_{s,g}(\Omega_{T})^{4}\) corresponding to the representative \((u', v', w', z')\)

is a macroscopic solution to the system (1.1). \(\square\)

5. Explicit Solution for Riemann Type Data

To understand the macroscopic aspect of the solution obtained in section 3 is an open question for general initial data. However for Riemann type data and when \( u \) develops shock or contact discontinuity it is completely solved in [10]. But for the case when \( u \) develops rarefaction it is partly solved. We describe the results obtained for this case:

For \( u_{l} < u_{r}, v_{l} = v_{r} = \bar{v}, w_{l} = w_{r} = \bar{w}, z_{l} = z_{r} = \bar{z} \),

\[ \lim_{\epsilon \to 0} (w', z') = \begin{cases} 
  (\bar{w}, \bar{z}), & \text{if } x \leq u_{l}t \\
  \left( \frac{\bar{v}^{2}}{2} t \delta_{x=u_{l}t}, \bar{v} \bar{w} t \delta_{x=u_{l}t} - \frac{\bar{v}^{3}}{6} t^{2} \delta'_{x=u_{l}t} \right), & \text{if } x = u_{l}t \\
  (0, 0), & \text{if } u_{l}t < x < u_{r}t \\
  \left( -\frac{\bar{v}^{2}}{2} t \delta_{x=u_{r}t}, -\bar{v} \bar{w} t \delta_{x=u_{r}t} + \frac{\bar{v}^{3}}{6} t^{2} \delta'_{x=u_{r}t} \right), & \text{if } x = u_{r}t \\
  (\bar{w}, \bar{z}), & \text{if } x \geq u_{r}t
\end{cases} \quad (5.1) \]

It is conjectured there that the distributions

\[(w(x,t), z(x,t))\]

\[= \begin{cases} 
  (w_{l}, z_{l}), & \text{if } x < u_{l}t \\
  \left( \frac{v_{l}^{2}}{2} t \delta_{x=u_{l}t}, v_{l} w_{l} t \delta_{x=u_{l}t} - \frac{v_{l}^{3}}{6} t^{2} \delta'_{x=u_{l}t} \right), & \text{if } x = u_{l}t \\
  (0, 0), & \text{if } u_{l}t < x < u_{r}t \\
  \left( -\frac{v_{l}^{2}}{2} t \delta_{x=u_{r}t}, -v_{r} w_{r} t \delta_{x=u_{r}t} + \frac{v_{r}^{3}}{6} t^{2} \delta'_{x=u_{r}t} \right), & \text{if } x = u_{r}t \\
  (w_{r}, z_{r}), & \text{if } x > u_{r}t.
\end{cases} \quad (5.2) \]

is the macroscopic aspect when \( u \) develops rarefaction.

In this section we construct shadow wave solution[5] and solution using Volpert product [15] for Riemann type data when \( u \) develops rarefaction\((u_{l} < u_{r})\).
First we recall some definition from [5]. We keep our discussions in a general level.

**Definition 5.1.** Let $u_\epsilon$ and $u_0$ are given by

$$u^\epsilon(x,t) = \begin{cases} u_1, & \text{if, } x < (c(t) - \epsilon) \\ u_{1\epsilon}, & \text{if, } (c(t) - \epsilon) < x < c(t) \\ u_{2\epsilon}, & \text{if, } (c(t) - \epsilon) < x < (c(t) + \epsilon) \\ u_1, & \text{if, } x > (c(t) + \epsilon), \end{cases}$$

(5.3)

$$u_0(x) = \begin{cases} u_1, & \text{if, } x < 0 \\ u_2, & \text{if, } x > 0, \end{cases}$$

(5.4)

where $u_1, u_2, u_{1\epsilon}$ and $u_{2\epsilon}$ are constants and are in $\mathbb{R}^n$, $(x,t) \in R \times (0, \infty)$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth. The line $x = c(t)$ has its initial point at origin. Let the distributional limit of $u^\epsilon(x,t)$ exists and is $u$. If $(u^\epsilon)_t + f(u^\epsilon)_x$ tends to 0, in the sense of distribution. Then we say $u$ is a Shadow wave solution to the conservation law

$$u_t + f(u)_x = 0$$

with initial data

$$u(x,0) = u_0(x).$$

Also there is an entropy concept for this, see [5], page[500].

**Definition 5.2.** Let $\eta(u)$ be a convex entropy with the entropy flux $q(u)$. Then $u^\epsilon$ is said to be entropy admissible if

$$\liminf_{\epsilon \to 0} \int_0^T \int_{\mathbb{R}} \eta(u^\epsilon) \partial_t \phi dx dt + \int_{\mathbb{R}} \eta(u^\epsilon(x,0)) dx \geq 0$$

(5.5)

for all non-negative test functions $\phi \in C_0^\infty(\mathbb{R} \in (0,T)).$

The above definition is equivalent to:

$$\limsup_{\epsilon \to 0} -c(\eta(u_2) - \eta(u_1)) + c(\eta(u_{1\epsilon}) + \eta(u_{1\epsilon})) + q(u_2) - q(u_1) \leq 0$$

$$\lim_{\epsilon \to 0} -c(\eta(u_{1\epsilon}) + \eta(u_{1\epsilon})) + c(q(u_{1\epsilon}) + q(u_{1\epsilon})) = 0$$

(5.6)

Now we construct shadow wave solution in the following theorem.

**Theorem 5.3.** If $u_l < u_r$, then a shadow wave solution to the equation (1.2) with initial data

$$(u(x,0), v(x,0), w(x,0), z(x,0)) = \begin{cases} (u_l, v_l, w_1, z_1), & \text{if, } x < 0 \\ (u_r, v_r, w_r, z_r), & \text{if, } x > 0 \end{cases}$$

is given by

$$(w, z) = \begin{cases} (w_l, z_l), & \text{if, } x < u_l t \\ -\frac{v^2}{2} t \delta_x = u_l t, v_l u_l t \delta_x = u_l t, & \text{if, } x = u_l t \\ 0,0 & \text{if, } u_l t < x < u_r t \\ -\frac{v^2}{2} t \delta_x = u_r t, -v_r w_r t \delta_x = u_r t, & \text{if, } x = u_r t \\ (w_r, z_r), & \text{if, } x > u_r t \end{cases}$$

(5.7)

The Solution is entropy admissible.
Proof. By vanishing viscosity limit, see [10] the limit \((u, v)\) for the rarefaction case of \(u\) is given by

\[
(u, v) = \begin{cases} 
(u_l, v_l), & \text{if} \quad x < u_l t \\
\left(\frac{u_l}{c}, 0\right), & \text{if} \quad u_l t < x < u_r t \\
(u_r, v_r), & \text{if} \quad x > u_r t
\end{cases}
\]  
(5.8)

So we guess the following ansatz for \((u, v, w, z)\) for the possible shadow wave approximation.

\[
(u_e, v_e, w_e, z_e)(x, t) = \begin{cases} 
(u_l, v_l, w_l, z_l), & \text{if} \quad x < (u_l - \epsilon) t \\
(u_l, \frac{v_l}{c}, \frac{w_l}{c}, \frac{z_l}{c}), & \text{if} \quad (u_l - \epsilon) t < x < u_l t \\
\left(\frac{v}{c}, 0, 0, 0\right), & \text{if} \quad u_l t < x < u_r t \\
(u_r, \frac{v_r}{c}, \frac{w_r}{c}, \frac{z_r}{c}), & \text{if} \quad u_r t < x < (u_r + \epsilon) t \\
(u_r, v_r, w_r, z_r), & \text{if} \quad x > (u_r + \epsilon) t
\end{cases}
\]  
(5.9)

Applying formula 3.2 from [5], with \(a_\epsilon = \epsilon, b_\epsilon = 0, c = w_l\) near the discontinuity line \(x = u_l t\) and \(a_\epsilon = 0, b_\epsilon = \epsilon, c = u_r\) near the discontinuity line \(x = u_r t\), we get

\[
w_t \approx (u_l w_l + w_l)\delta_{x=u_l t} - u_l w_1 t \delta'_{x=u_l t} \\
+ (-u_r w_r + w_2)\delta_{x=u_r t} - u_r w_2 t \delta'_{x=u_r t}
\]  
(5.10)

\[
\partial_x \left(\frac{v^2}{2} + u^r w^r\right) \approx \left(-\frac{v_1^2}{2} - u_l w_1\right)\delta_{x=u_l t} + \left(-\frac{v_2^2}{2} + u_l w_1\right) t \delta'_{x=u_l t} \\
+ \left(-\frac{v_2^2}{2} - u_r w_r\right)\delta_{x=u_r t} + \left(-\frac{v_2^2}{2} + u_r w_2\right) t \delta'_{x=u_r t}
\]  
(5.11)

The relation \(w_t + \partial_x \left(\frac{v^2}{2} + u^r w^r\right) \approx 0\) implies

\[
v_1 = v_2 = 0, \quad w_1 = \frac{v_1^2}{2}, \quad w_2 = -\frac{v_2^2}{2}.
\]

Now we calculate the distributional limit of \(w^r\). Let \(\phi\) be a real valued test function supported in \((-\infty, \infty) \times (0, \infty)\).

\[
\int_0^\infty \int_{-\infty}^{u_l t} w^r(x, t) \phi(x, t) dx dt \\
= \int_0^\infty \int_{-\infty}^{(u_l - \epsilon) t} w^r(x, t) \phi(x, t) dx dt + \int_0^\infty \int_{(u_l - \epsilon) t}^{u_l t} w^r(x, t) \phi(x, t) dx dt \\
+ \int_0^\infty \int_{(u_l - \epsilon) t}^{(u_l + \epsilon) t} w^r(x, t) \phi(x, t) dx dt + \int_0^\infty \int_{(u_l + \epsilon) t}^{\infty} w^r(x, t) \phi(x, t) dx dt \\
= \int_0^\infty \int_{-\infty}^{(u_l - \epsilon) t} w_1 \phi(x, t) dx dt + \int_0^\infty \int_{(u_l - \epsilon) t}^{u_l t} \frac{v_1^2}{2} \phi(x, t) dx dt \\
- \int_0^\infty \int_{(u_l - \epsilon) t}^{(u_l + \epsilon) t} \frac{v_1^2}{2} \phi(x, t) dx dt + \int_0^\infty \int_{(u_l + \epsilon) t}^{\infty} w_2 \phi(x, t) dx dt
\]  
(5.12)
As ε tends to 0, we have

\[
\lim_{\varepsilon \to 0} \int_0^\infty \int_{-\infty}^\infty w^\varepsilon(x,t)\phi(x,t)dxdt = \int_0^\infty \int_{-\infty}^\infty w_1\phi(x,t)dxdt + \int_0^\infty \frac{\varepsilon t^2}{2}\phi(u_1t,t)dt - \int_0^\infty \frac{\varepsilon t^2}{2}\phi(u_1t,t)dt + \int_0^\infty w_\varepsilon(x,t)dxdt
\]

(5.13)

Proceeding as above we get,

\[
z_1 \approx (u_1z_1 + z_1)\delta_{x=u_1t} - u_1z_1t\delta'_{x=u_1t}
\]

(5.14)

\[
\partial_t^\varepsilon (v^\varepsilon w^\varepsilon + u^\varepsilon z^\varepsilon) \approx (-v_1w_1 - u_1z_1)\delta_{x=u_1t} + \epsilon\left(-\frac{v_1w_1}{\varepsilon} + \frac{u_1z_1}{\varepsilon}\right)t\delta'_{x=u_1t}
\]

(5.15)

\[
\partial_t^\varepsilon (v^\varepsilon w^\varepsilon + u^\varepsilon z^\varepsilon) \approx (-v_1w_1 - u_1z_1)\delta_{x=u_1t} + \epsilon\left(-\frac{v_2w_2}{\varepsilon} + \frac{u_2z_2}{\varepsilon}\right)t\delta'_{x=u_1t}
\]

From the calculation for \(w\), we had \(v_1 = v_2 = 0\), So

\[
\partial_t^\varepsilon (v^\varepsilon w^\varepsilon + u^\varepsilon z^\varepsilon) \approx (-v_1w_1 - u_1z_1)\delta_{x=u_1t} + u_1z_1t\delta'_{x=u_1t}
\]

\[
+ (v_1w_1 + u_1z_1)\delta_{x=u_1t} + u_1z_1t\delta'_{x=u_1t}
\]

The relation \(z_t + \partial_x (v^\varepsilon w^\varepsilon + u^\varepsilon z^\varepsilon) \approx 0\) implies

\[
z_1 = v_1w_1, \quad z_2 = -v_1w_1.
\]

Following the calculation as in \(w^\varepsilon\), we get

\[
\lim_{\varepsilon \to 0} \int_0^\infty \int_{-\infty}^\infty z^\varepsilon(x,t)\phi(x,t)dxdt = \int_0^\infty \int_{-\infty}^\infty z(x,t)\phi(x,t)dxdt.
\]

(5.16)

To show the solution is entropy admissible, we first determine entropy and entropy flux pair for the system \(1.2\).

Note that if we take the transformation \((u,v,w,z) \rightarrow (2u,v,4w,24z)\), the system \(1.2\) transforms to prolonged system \(n = 4\) with \(f(u) = u^2\), see [19]. Convex entropy for such a system is given by, see [19],

\[
\eta(u) = \tilde{\eta}(u) + c_1v + c_2w + c_3z,
\]

where \(\eta(u)\) is convex.

Since the transformation \((u,v,w,z) \rightarrow (2u,v,4w,24z)\) is linear, so convex entropy and flux of the \(1.2\) is :

\[
\eta(u) = \tilde{\eta}(u) + c_1v + c_2w + c_3z
\]

\[
q = \int w'f(u)udu + c_1wu + c_2\left(\frac{u^2}{2} + uw\right) + c_3(vw + uz)
\]

Since \(u_1^2 + u^2 + c_2^2\), \(v_1^2 + v^2\), \(w_1^2 + w^2\) and \(z_1^2 + z^2\) tends to 0 in the sense of distribution.

So the entropy condition \(5.8\) reduces to the usual entropy condition for the first component \(u\). This completes the proof of the theorem. □
Now we construct a solution to the problem \((1.1)\) using Volpert product for the component \(w\) and \(z\).

**Theorem 5.4.** Under Volpert product consideration, the solution for the component \(w\) and \(z\) of the equation \((1.1)\), when \(u\) develops rarefaction with initial data

\[
(u(x, 0), v(x, 0), w(x, 0), z(x, 0)) = \begin{cases} (u_l, v_l, w_l, z_l), & \text{if } x < 0 \\ (u_r, v_r, w_r, z_r), & \text{if } x > 0 \end{cases}
\]

is given by,

\[
(w, z) = \begin{cases} (w_l, z_l), & \text{if } x < u_l t \\
\frac{v_l^2}{2} t \delta_{x = u_l t} - v_l w_l t \delta_{x = u_l t} - \left(\frac{v_l^2}{2} + ct^\frac{1}{2}\right) \delta'_{x = u_l t}, & \text{if } x = u_l t \\
(0, 0), & \text{if } u_l t < x < u_r t \\
-\frac{v_r^2}{2} t \delta_{x = u_r t}, & \text{if } u_l t < x < u_r t \\
-v_r w_r t \delta_{x = u_r t} + \left(\frac{v_r^2}{2} + ct^\frac{1}{2}\right) \delta'_{x = u_r t}, & \text{if } x = u_r t \\
(w_r, z_r), & \text{if } x > u_r t, \end{cases}
\]

\[(5.17)\]

for arbitrary real number \(c\). Here \(\delta' = \frac{\partial}{\partial x}\).

**Proof.** Let’s take the following ansatz for \(w\) and \(z\):

\[
w(x, t) = u_l H(u_l t - x) + w_l (1 - H(u_l t - x)) + e_l(t) \delta_{x = u_l t} + e_r(t) \delta_{x = u_r t}
\]

\[
z(x, t) = z_l H(u_l t - x) + z_r (1 - H(u_l t - x)) + g_l(t) \delta_{x = u_l t} + g_r(t) \delta_{x = u_r t}
\]

\[+ h_l(t) \delta'_{x = u_r t} + h_r(t) \delta'_{x = u_r t} \]

Note that,

\[
\frac{\partial}{\partial t}(a(t) \delta_{x = ct}) = a'(t) \delta_{x = ct} - ca(t) \delta'_{x = ct}
\]

\[
\frac{\partial}{\partial t}(a(t) \delta''_{x = ct}) = a'(t) \delta'_{x = ct} - ca(t) \delta''_{x = ct}
\]

we get

\[
\frac{\partial}{\partial t} w = u_l w_l \delta_{x = u_l t} - u_r w_r \delta_{x = u_l t}
\]

\[+ e_l(t) \delta_{x = u_l t} - u_l e_l(t) \delta'_{x = u_l t} + e_r(t) \delta_{x = u_r t} - u_r e_r(t) \delta'_{x = u_r t} \]

\[
\frac{\partial}{\partial x}(uw) = -u_l w_l \delta_{x = u_l t} + u_r w_r \delta_{x = u_l t} + u_l e_l(t) \delta'_{x = u_l t} + u_r e_r(t) \delta'_{x = u_r t}
\]

\[
\frac{\partial}{\partial x} \left(\frac{v^2}{2}\right) = -\frac{v_l^2}{2} \delta_{x = u_l t} + \frac{v_r^2}{2} \delta_{x = u_r t}
\]

Putting all these in third equation of \((1.1)\), we get,

\[e_l(t) = \frac{v_l^2}{2}, \quad e_r(t) = -\frac{v_r^2}{2}\]

Since at time \(t = 0\), there is no concentration, we take \(e_l(0) = 0, e_r(0) = 0\). So we get

\[e_l(t) = \frac{v_l^2}{2}, \quad e_r(t) = -\frac{v_r^2}{2}\]

Now we calculate for the component \(z\):

Using \((5.18)\),
\[
\frac{\partial}{\partial t} z = u_1 z_1 \delta_{x = u_1 t} - u_r z_r \delta_{x = u_r t}
\]
\[+ g_1(t) \delta_{x = u_1 t} - u_1 g_1(t) \delta'_{x = u_1 t} + g'_1(t) \delta_{x = u_r t} - u_r g_r(t) \delta'_{x = u_r t}
\]
\[+ h_1(t) \delta''_{x = u_1 t} - u_1 h_1(t) \delta''_{x = u_1 t} + h'_1(t) \delta_{x = u_r t} - u_r h_r(t) \delta''_{x = u_r t}.
\]
\[
\frac{\partial}{\partial x} (uz) = -u_1 z_1 \delta_{x = u_1 t} + u_r z_r \delta_{x = u_r t}
\]
\[+ u_1 g_1(t) \delta'_{x = u_1 t} - \frac{h_1(t)}{2t} \delta''_{x = u_1 t} + u_r g_r(t) \delta'_{x = u_r t}
\]
\[+ u_1 h_1(t) \delta''_{x = u_1 t} - \frac{h_r(t)}{2t} \delta'_{x = u_r t} + u_r h_r(t) \delta''_{x = u_r t}.
\]
Using Volpert product \[15,\]
\[
\frac{\partial}{\partial x} (vw) = -v_1 w_1 \delta_{x = u_1 t} + v_r w_r \delta_{x = u_r t} + \frac{v_1^3}{4} \delta'_{x = u_1 t} - \frac{v_r^3}{4} \delta'_{x = u_r t}.
\]
Putting all these in fourth equation of (1.1), we get
\[
g_1(t) = v_1 w_1, \quad g'_1(t) = -v_r w_r, \quad h'_1(t) - \frac{h_1(t)}{2t} = \frac{v_1^3}{4}, \quad h''_1(t) - \frac{h_r(t)}{2t} = \frac{v_r^3}{4}.
\]
Since at time \(t = 0\), there is no concentration, we take \(g_1(0) = g_r(0) = h_1(0) = h_r(0) = 0\) So we get
\[
g_1(t) = v_1 w_1 t, \quad g_r(t) = v_r w_r t, \quad h_1(t) = \frac{v_1^3}{6} t^2 + ct, \quad h_r(t) = -\frac{v_r^3}{6} t^2 + ct\]

6. Conclusion

For \(u_1 < u_r, v_1 = v_r, w_1 = w_r, z_1 = z_r\), (5.17) becomes
\[
w(x, t) = \tilde{w} H(u_1 t - x) + \tilde{w}(1 - H(u_r t - x)) + \delta_{x = u_1 t} + \frac{\tilde{\nu}^2}{2} t \delta_{x = u_1 t} - \frac{\tilde{\nu}^2}{2} t \delta_{x = u_r t}
\]
\[z(x, t) = \tilde{z} H(u_1 t - x) + \tilde{z}(1 - H(u_1 t - x)) + \tilde{w} \tilde{w} \delta_{x = u_1 t} - \tilde{w} \tilde{\omega} t \delta_{x = u_r t}
\]
\[- (\frac{\tilde{\nu}^3}{6} t^2 + ct)^2 \delta'_{x = u_1 t} + (\frac{\tilde{\nu}^3}{6} t^2 + ct^2) \delta''_{x = u_1 t}.
\]
Under Volpert product consideration the solution is not unique due to arbitrary \(c\). When \(c = 0\), (5.17) is exactly the vanishing viscosity limit \[5.1\] which is obtained in \[10]. The shadow wave solution for the component \(w\) agrees with the vanishing viscosity limit where as it does not agree for the component \(z\). Macroscopic aspect of the vanishing viscosity approximation is still an open question for the general type initial data.

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