PATTERN FORMATION (II): THE TURING INSTABILITY

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Abstract. We consider the classical Turing instability in a reaction-diffusion system as the second part of our study on pattern formation. We prove that nonlinear dynamics of a general perturbation of the Turing instability is determined by the finite number of linear growing modes over a time scale of $\ln \frac{1}{\delta}$, where $\delta$ is the strength of the initial perturbation.

1. Growing modes in a reaction-diffusion system

In this section we summarize the classical linear Turing instability criterion for a reaction-diffusion system. Consider a reaction-diffusion system of 2-species as

\[
\frac{\partial U}{\partial t} = \nabla \cdot (D_1(U,V) \nabla U) + f(U,V),
\]

\[
\frac{\partial V}{\partial t} = \nabla \cdot (D_2(U,V) \nabla V) + g(U,V),
\]

where $U(x,t), V(x,t)$ are concentration for species, $D_1, D_2$ diffusion coefficients, $f, g$ reaction terms.

In this paper we consider a $d$-dimensional box $\mathbb{T}^d = (0, \pi)^d$, $d = 1, 2, 3$, with Neumann boundary conditions for $U$ and $V$, i.e.,

\[
\frac{\partial U}{\partial x_i} = \frac{\partial V}{\partial x_i} = 0 \text{ at } x_i = 0, \pi, \text{ for } 1 \leq i \leq d.
\]

Homogeneous steady state $U = \bar{U}, V = \bar{V}$ forms a steady state provided

\[
0 = f(\bar{U}, \bar{V}) = g(\bar{U}, \bar{V}).
\]

In this article, we study the nonlinear evolution of a perturbation $u(x,t) = U(x,t) - \bar{U}, v(x,t) = V(x,t) - \bar{V}$ around $[\bar{U}, \bar{V}]$, which satisfies the equivalent reaction-diffusion system:

\[
\frac{\partial u}{\partial t} = \nabla \cdot (D_1(u + \bar{U}, v + \bar{V}) \nabla u) + f(u + \bar{U}, v + \bar{V}),
\]

\[
\frac{\partial v}{\partial t} = \nabla \cdot (D_2(u + \bar{U}, v + \bar{V}) \nabla v) + g(u + \bar{U}, v + \bar{V}).
\]
The corresponding linearized system then takes the form
\begin{align}
u_t &= \tilde{D}_1 \nabla^2 u + \tilde{f}_u u + \tilde{f}_v v, \\
v_t &= \tilde{D}_2 \nabla^2 v + \tilde{g}_u u + \tilde{g}_v v, \tag{1.6}
\end{align}
where \( \tilde{D}_1 = D_1 (\bar{U}, \bar{V}) \), \( \tilde{D}_2 = D_2 (\bar{U}, \bar{V}) \), \( \tilde{f}_u = \frac{\partial f}{\partial u} (\bar{U}, \bar{V}) \), \( \tilde{f}_v = \frac{\partial f}{\partial v} (\bar{U}, \bar{V}) \), \( \tilde{g}_u = \frac{\partial g}{\partial u} (\bar{U}, \bar{V}) \), \( \tilde{g}_v = \frac{\partial g}{\partial v} (\bar{U}, \bar{V}) \).

We use \([\cdot, \cdot]\) to denote a column vector, and let
\[ w(x, t) \equiv [u(x, t), v(x, t)], \quad \bar{W} = [\bar{U}, \bar{V}]. \]
Then the original nonlinear system (1.4) and (1.5) can be written in a matrix form:
\begin{align}
\frac{\partial w}{\partial t} &= \nabla \cdot (D \nabla w) + F \\
&= (\bar{D} \nabla^2 w + A w) + \left( \nabla \cdot (D \nabla w) - \bar{D} \nabla^2 w \right) + F - A w \\
&= L(w) + N(w), \tag{1.8}
\end{align}
where
\[ D = \begin{pmatrix} D_1 (w + \bar{W}) & 0 \\ 0 & D_2 (w + \bar{W}) \end{pmatrix}, \quad \bar{D} = \begin{pmatrix} \bar{D}_1 & 0 \\ 0 & \bar{D}_2 \end{pmatrix}, \quad F = \begin{pmatrix} f(w + \bar{W}) \\ g(w + \bar{W}) \end{pmatrix}, \quad A = \begin{pmatrix} \bar{f}_u & \bar{f}_v \\ \bar{g}_u & \bar{g}_v \end{pmatrix}. \]

Let \( q = (q_1, \ldots, q_d) \in \Omega = (\mathbb{N} \cup \{0\})^d \) and let
\[ e_q(x) \equiv \prod_{i=1}^d \cos (q_i x_i), \]
where \( q \in \Omega \). Then \( \{e_q(x)\}_{q \in \Omega} \) forms a basis of the space of functions in \( \mathbb{T}^d \) that satisfy Neumann boundary condition (1.2).

We look for a normal mode to the linear reaction-diffusion system (1.6) and (1.7) of the following form:
\begin{align}
w(x, t) &= r_q \exp (\lambda_q t) e_q(x), \tag{1.9}
\end{align}
where \( r_q \) is a vector depending on \( q \). We substitute (1.9) into (1.6)–(1.7) to get
\[ \lambda_q r_q = \begin{pmatrix} \bar{f}_u - \bar{D}_1 q^2 \bar{g}_u \\ \bar{g}_v - \bar{D}_2 q^2 \end{pmatrix} r_q, \]
where \( q^2 = \sum_{i=1}^d q_i^2 \). A nontrivial normal mode can be obtained by setting
\[ \det \begin{pmatrix} \lambda_q - \bar{f}_u + \bar{D}_1 q^2 & -\bar{f}_v \\ -\bar{g}_u & \lambda_q - \bar{g}_v + \bar{D}_2 q^2 \end{pmatrix} = 0. \]
This leads to the following dispersion formula for \( \lambda_q \):

\[
(1.10) \quad \lambda_q^2 + \{-f_u + \bar{D}_1 q^2 - g_v + \bar{D}_2 q^2\} \lambda_q + \{(f_u - \bar{D}_1 q^2)(g_v - \bar{D}_2 q^2) - \bar{f}_v \bar{g}_u\} = 0.
\]

We assume first that without diffusion, the \( \lambda_q \) has negative real part (stable):

\[
(1.11) \quad \text{tr} \, A = f_u + g_v < 0, \quad \det \, A = \bar{f}_u \bar{g}_v - \bar{f}_v \bar{g}_u > 0,
\]

On the other hand, in the presence of diffusion, we assume the following diffusion-driven (linear) instability criterion by requiring there exists a \( q \) such that

\[
(1.12) \quad \left( f_u - \bar{D}_1 q^2 \right) \left( g_v - \bar{D}_2 q^2 \right) - \bar{f}_v \bar{g}_u < 0,
\]

which ensures that (1.10) has at least one positive root \( \lambda_q \).

**Remark 1.** To satisfy (1.11) and (1.12), the discriminant for the quadratic equation for \( q^2 \) in (1.12) must be positive:

\[
(1.13) \quad \left( \bar{f}_u \bar{D}_2 + \bar{g}_v \bar{D}_1 \right) > 2 \sqrt{\bar{D}_1 \bar{D}_2} \det \, A > 0,
\]

which means the range of inhibition \( \sqrt{\bar{D}_2 / |\bar{g}_v|} \) is larger than the range of activation \( \sqrt{\bar{D}_1 / |\bar{f}_u|} \). From (1.11) and (1.13), it follows that

\[
(1.14) \quad \bar{f}_u \bar{g}_v < 0, \quad \text{and} \quad \bar{f}_v \bar{g}_u < 0,
\]

and we have only two cases for \( A \):

\[
A = \begin{pmatrix} + & - \\ + & + \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} + & + \\ - & - \end{pmatrix},
\]

where formal case is called activator-inhibitor (or predator-prey) and the latter positive feedback. It also follows from (1.11) that

\[
D_1 \neq D_2.
\]

For given \( q \in \Omega \), we denote the corresponding eigenvalues by \( \lambda_{\pm}(q) \) and eigenvectors by \( r_{\pm}(q) \). We split into the three cases for the linear analysis:

1. **Generic case** where we have two independent real eigenvectors and we denote

   \[
   \Omega_{\text{generic}} \equiv \{ q \in \Omega \text{ such that } r_+(q) \neq r_-(q) \}.
   \]

   By an elementary computation of the discriminant of (1.11), we have, except for only finitely many \( q \),

   \[
   (\bar{D}_1 - \bar{D}_2) q^4 - \text{tr} \, A \left( \bar{D}_1 + \bar{D}_2 \right) q^2 + 4 \left( \bar{f}_u \bar{D}_2 + \bar{g}_v \bar{D}_1 \right) q^2 + (\text{tr} \, A)^2 - 4 \det \, A > 0,
   \]
since $D_1 - D_2 \neq 0$. Therefore, there are two distinct real roots such that

$$\lambda_-(q) < \lambda_+(q)$$

for large $q$. Since $\bar{f}_v \neq 0$ in (1.14), the corresponding (linearly independent) eigenvectors $r_-(q)$ and $r_+(q)$ are given by

$$(1.15) \quad r_\pm(q) = \left[ 1, \frac{\lambda_\pm(q) - \bar{f}_u + D_1 q^2}{f_v} \right].$$

It is easy to see from (1.12) that there exist only finitely many $q$ such that $\lambda_+(q) > 0$. We therefore can denote the largest eigenvalue by $\lambda_{\text{max}} > 0$ and define

$$\Omega_{\text{max}} \equiv \{ q \in \Omega \text{ such that } \lambda_+(q) = \lambda_{\text{max}} \}.$$

We also denote $\nu > 0$ to be the gap between the $\lambda_{\text{max}}$ and the rest. Moreover, there is one $q^2$ (possibly two) having $\lambda_+(q^2) = \lambda_{\text{max}}$ when we regard $\lambda_+(q)$ as a function of $q^2$.

(2) Defective case where we have the repeated real eigenvalues and eigenvectors:

Note that there may be possibly one $q^2$ (so finitely many $q$) such that from (1.11)

$$(1.16) \quad \lambda_+(q) = \lambda_-(q) \equiv \lambda(q) = \{ \bar{f}_u + \bar{g}_v - (\bar{D}_1 + \bar{D}_2)q^2 \}/2 < 0$$

and $r_+(q) = r_-(q) \equiv r(q)$ and we denote

$$\Omega_{\text{defective}} \equiv \{ q \in \Omega \text{ such that } r_+(q) = r_-(q) \}.$$

In this case we find another independent vector

$$r'(q) = [0, \frac{1}{\bar{f}_v}]$$

satisfying $(A - \lambda(q)I) r'(q) = r(q)$.

(3) Complex case where we have complex eigenvalues for $q$ and we denote it by $\Omega_{\text{complex}} \equiv \Omega - (\Omega_{\text{generic}} \cup \Omega_{\text{defective}})$. For $q \in \Omega_{\text{complex}}$, we denote $\lambda_\pm(q) \equiv \operatorname{Re} \lambda(q) \pm i \operatorname{Im} \lambda(q)$ and $r_\pm(q) \equiv \operatorname{Re} r(q) \pm i \operatorname{Im} r(q)$. Then we have $\lambda_-(q) \equiv \operatorname{Re} \lambda(q) - i \operatorname{Im} \lambda(q)$ and $r_-(q) \equiv \operatorname{Re} r(q) - i \operatorname{Im} r(q)$. Notice that $\operatorname{Re} \lambda(q) < 0$ as in (1.16), and $\operatorname{Re} r(q)$ and $\operatorname{Im} r(q)$ are linearly independent vectors.
Given any initial perturbation \( w(x,0) \), we can expand it as

\[
 w(x,0) = \sum_{q \in \Omega} w_q e_q(x) = \sum_{q \in \Omega_{\text{generic}}} \{w_q^\ast r_- (q) + w_q^+ r_+ (q)\} e_q(x)
 + \sum_{q \in \Omega_{\text{defective}}} \{w_q r(q) + w_q' r'(q)\} e_q(x)
 + \sum_{q \in \Omega_{\text{complex}}} \{w_q^\ast \text{Re} r(q) + w_q^\ast \text{Im} r(q)\} e_q(x),
\]

so that

\[
 (1.17) \quad w_q = w_q^\ast r_- (q) + w_q^+ r_+ (q) \quad \text{for} \quad q \in \Omega_{\text{generic}},
 w_q = w_q r(q) + w_q' r'(q) \quad \text{for} \quad q \in \Omega_{\text{defective}},
 w_q = w_q^\ast \text{Re} r(q) + w_q^\ast \text{Im} r(q) \quad \text{for} \quad q \in \Omega_{\text{complex}}.
\]

The unique solution \( w(x,t) = [u(x,t), v(x,t)] \) to (1.6)-(1.7) is given by

\[
 (1.18) \quad w(x,t) = \sum_{q \in \Omega_{\text{generic}}} \{w_q^\ast r_- (q) \exp \left( \lambda_q^- t \right) + w_q^+ r_+ (q) \exp \left( \lambda_q^+ t \right)\} e_q(x)
 + \sum_{q \in \Omega_{\text{defective}}} \{(w_q r(q) + w_q' r'(q)) + w_q' r'(q) t\} \exp \left( \lambda_q t \right) e_q(x)
 + \sum_{q \in \Omega_{\text{complex}}} \{w_q^\ast \text{Re} r(q) \cos \left( \text{Im} \lambda_q t \right) - \text{Im} r(q) \sin \left( \text{Im} \lambda_q t \right) \}
 + w_q^\ast \text{Im} \left( \text{Re} r(q) \sin \left( \text{Im} \lambda_q t \right) + \text{Im} r(q) \cos \left( \text{Im} \lambda_q t \right) \right) \exp \left( \text{Re} \lambda_q t \right) e_q(x)
 \equiv e^{\mathcal{L}t}w(x,0).
\]

For any \( u(\cdot,t) \in [L^2(\mathbb{T})]^2 \), we denote \( \|u(\cdot,t)\| \equiv \|u(\cdot,t)\|_{L^2} \). Our main result of this section is

**Lemma 1.** Assume that (1.11) and the instability criterion (1.12) are valid. Suppose

\[
 w(x,t) = [u(x,t), v(x,t)] \equiv e^{\mathcal{L}t}w(x,0)
\]

as in (1.18) is a solution to the linearized reaction-diffusion system (1.7)-(1.7) with initial condition \( w(x,0) \). Then there exists a constant \( C_1 \geq 1 \) depending on \( U, V, D_1, D_2, A \) such that

\[
 \|w(\cdot,t)\| \leq C_1 \exp (\lambda_{\text{max}} t) \|w(\cdot,0)\|,
\]

for all \( t \geq 0 \).
Proof. We first notice that from the quadratic formula for \((1.10)\), for \(q\) large,
\[
|\det[r_-(q), r_+(q)]| = \frac{\lambda_q^+ - \lambda_q^-}{|f_v|} \geq c \frac{|\bar{D}_1 - \bar{D}_2|}{|f_v|} q^2.
\]
Thus solving \((1.17)\) yields, due to \(\bar{D}_1 \neq \bar{D}_2\),
\[
|w_q^\pm| \leq \frac{1}{\det[r_-(q), r_+(q)]} |r_\pm(q)| \times |w_q| \\
\leq C |w_q|.
\]
Since \(\lambda_q < 0\), for \(q \in \Omega_{\text{defective}}\), we have
\[
t \exp (\lambda_q t) \leq C.
\]
Moreover, recall \(\text{Re} \lambda(q) < 0\) for \(q \in \Omega_{\text{complex}}\). Thus we deduce the Lemma on the linear growth rate by the formula \((1.18)\). \(\Box\)

2. Main Result

Let \(\theta\) be a small fixed constant, and \(\lambda_{\text{max}}\) be the dominant eigenvalue which is the maximal growth rate. We also denote the gap between the largest growth rate \(\lambda_{\text{max}}\) and the rest by \(\nu > 0\). Then for \(\delta > 0\) arbitrary small, we define the escape time \(T^\delta\) by
\[
(2.1) \quad \theta = \delta \exp (\lambda_{\text{max}} T^\delta),
\]
or equivalently
\[
T^\delta = \frac{1}{\lambda_{\text{max}}} \ln \frac{\theta}{\delta}.
\]

Our main theorem is

**Theorem 1.** Assume \((1.11)\) and that there exists \(q^2 = \sum_{i=1}^d q_i^2\) satisfying instability criterion \((1.12)\). Let
\[
\begin{align*}
w_0(x) &= \sum_{q \in \Omega} \{w^r_q r_-(q) + w^r_q r_+(q)\} e_q(x) \\
&\quad + \sum_{q \in \Omega_{\text{defective}}} \{w_q r(q) + w_q' r'(q)\} e_q(x) \\
&\quad + \sum_{q \in \Omega_{\text{complex}}} \{w_q^\text{Re} \text{Re} r(q) + w_q^\text{Im} \text{Im} r(q)\} e_q(x).
\end{align*}
\]
\( \in H^2 \) such that \( ||w_0|| = 1 \). Assume \( D_1, D_2, f, g \in C^2 \) near \( W \), so that there exists \( \eta > 0 \)

\[
C_\eta \equiv \max_{||w||_{\infty} \leq \eta} \left\{ \sum_{i=1}^{2} ||D_i(W+w)||_{C^2} + ||f(W+w)||_{C^2} + ||g(W+w)||_{C^2} \right\} < \infty.
\]

Then there exist constants \( \delta_0 > 0, C > 0, \) and \( \theta > 0 \), depending on \( U, V, D_1, D_2, f, g \), such that for all \( 0 < \delta \leq \delta_0 \), if the initial perturbation of the steady state \([U, V]\) in (1.3) is

\[
w_\delta(x, 0) = \delta w_0,
\]

then its nonlinear evolution \( w_\delta(t, x) \) satisfies

\[
||w_\delta(t, x) - \delta e^{\lambda_{\max}t} \sum_{q \in \Omega_{\max}} w_q^+ r_+(q)e_q(x)||
\leq C\left\{ e^{-\nu t} + \delta||w_0||^2_{H^2} + \delta e^{\lambda_{\max}t}\right\} \delta e^{\lambda_{\max}t}
\]

for \( 0 \leq t \leq T^\delta \), and \( \nu > 0 \) is the gap between \( \lambda_{\max} \) and the rest of \( \text{Re} \lambda_q \) in (1.10).

We notice that for \( 0 \leq t \leq T^\delta \), \( \delta e^{\lambda_{\max}t} \leq \theta \), is sufficiently small. The initial profile \( w_0 \) is any \( H^2 \) function. In particular, as long as \( w_{q_0}^+ \neq 0 \) for at least one \( q_0 \in \Omega_{\max} \) (generic for a general \( H^2 \) perturbation), the part of its fastest growing modes satisfies

\[
||\delta e^{\lambda_{\max}t} \sum_{q \in \Omega_{\max}} w_q^+ r_+(q)e_q|| \geq \delta e^{\lambda_{\max}t}||w_{q_0}^+||r_+(q_0)||
\]

which has the dominant leading order of \( \delta e^{\lambda_{\max}t} \). Our estimate (2.3) implies that the dynamics of a general perturbation can be characterized by such linear dynamics over a long time period of \( \varepsilon T^\delta \leq t \leq T^\delta \), for any fixed constant \( \varepsilon > 0 \). In particular, choose a fixed \( q_0 \in \Omega_{\max} \) and let

\[
w_0(x) = \frac{r_+(q_0)}{|r_+(q_0)|} e_{q_0}(x)
\]

then if \( t = T^\delta \),

\[
||w_\delta(t, \cdot) - \delta e^{\lambda_{\max}T^\delta} \frac{r_+(q_0)}{|r_+(q_0)|} e_{q_0}(\cdot)|| \leq C\left\{ \delta^{\nu/\lambda_{\max}} + \theta^2 \right\},
\]

hence

\[
||w_\delta(t, \cdot)|| \geq \theta - C\left\{ \delta^{\nu/\lambda_{\max}} + \theta^2 \right\} \geq \theta/2 > 0,
\]

which implies nonlinear instability as \( \delta \to 0 \). The instability occurs before the possible blow-up time.
Reaction-diffusion systems are often employed to study chemical and biological pattern formation and have received much attention from scientists \cite{3}, \cite{4}, \cite{14}, \cite{13}, \cite{16}, since the pioneering work of Turing \cite{17} in 1951. This symmetry breaking instability is called diffusion-driven instability, since the presence of diffusion and the difference of diffusion coefficients are essential for the instability mechanism and nonuniform pattern formation. After some experimental results such as in \cite{2}, \cite{12}, \cite{15}, more extensive and serious works began towards this Turing-like pattern formation across many fields of study. Our result can be interpreted as a mathematical description of early pattern formation. Each initial perturbation can be drastically different from another, which gives rise to the richness of the pattern; on the other hand, the finite number maximal growing modes determine the common characteristics of the pattern, over the time scale of $\ln \frac{1}{\delta}$. In comparison with an earlier different result along this direction \cite{18}: First of all, the reaction-diffusion system considered here is not scaled. Secondly, our initial perturbation is more general, need not be close to the space of finite number of maximal growing modes. Thirdly, a precise estimate of the time scale ($\ln \frac{1}{\delta}$) for pattern formation is given here, without an a-priori assumption for the smallness of the perturbation later in time as in \cite{18}. Lastly, based on Guo-Strauss’ bootstrap argument, our proof is much simpler and direct.

3. Bootstrap Lemma

We state existence of local-in-time solutions for \eqref{1.4}-\eqref{1.5}.

**Lemma 2.** (Local existence) For $s \geq 1$ ($d = 1$) and $s \geq 2$ ($d = 2, 3$), there exist a $T > 0$ and a constant $C$ depending on $\bar{U}, \bar{V}, D_1, D_2, f, g$ such that $\|w(t)\|_{H^s}$ is continuous in $[0, T)$, and

$$\|w(t)\|_{H^s} \leq C \|w(0)\|_{H^s}.$$  

We now derive the following energy estimates for $d$-dimensional reaction-diffusion system with $d = 1, 2, 3$.

**Lemma 3.** Suppose that $[u(x,t), v(x,t)]$ is a solution to the full system \eqref{1.4}-\eqref{1.5}. Then for $\|w(t)\|_{H^2} \leq \eta$,

$$\frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ |\partial u|^2 + |\partial v|^2 \right\} dx$$

$$+ \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ \frac{D_1}{2} |\nabla \partial u|^2 + \tilde{D}_2 |\nabla \partial v|^2 \right\} dx + \frac{|\bar{g}_v|}{2} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} |\partial v|^2$$

$$\leq C_0 C_1 \|w\|_{H^2} \|\nabla^3 w\|^2 + C_2 \|u\|^2.$$
where $C_0$ is the universal constant while $C_1 = C_0 C_\eta (1 + \eta)$ and

$$C_2 = \frac{\left( \frac{(f_0 + \eta \bar{u})}{2|\eta|} + \bar{f}_u \right)^3}{D_1^2}.$$ 

**Proof.** We first notice that the reaction-diffusion system (1.4)-(1.5) preserves the evenness of the solution $w(x, t)$, i.e., if $w(x, t)$ is a solution, then $w(-x, t)$ is also a solution. We can regard the Neumann problem as a special case with evenness of the periodic problem by standard way of even extension $w(x, t)$ with respect to one of the $x_i$. For this reason we may assume periodicity at the boundary of the extended periodic box $2T^3 \equiv (-\pi, \pi)^d$. Since now there is no contributions from the boundaries, we can take second order $\partial$-derivative of (1.8) to get

$$\frac{1}{2} \frac{d}{dt} \int_{2T^d} |\partial w|^2 = \int_{2T^d} \partial w^T \partial L (w) + \int_{2T^d} \partial w^T \partial N (w).$$

We first treat the last nonlinear term:

$$- \int_{2T^d} \{\nabla \partial w\}^T \left[ \partial \{D (w + \bar{W}) \nabla w\} + \bar{D} \nabla \partial w + \{\nabla \partial w\}^T \partial (F - Aw) \right] \leq C \left\| D (w + \bar{W}) - \bar{D}\right\|_{\infty} \|\nabla \partial w\|^2 + C \left\| (\nabla D) (w + \bar{W}) \right\|_{\infty} \|\nabla w\|_{\infty} \|\partial w\| \|\nabla \partial w\|

+ C \left\| (\partial D) (w + \bar{W}) \right\|_{\infty} \|\nabla w\|^2_{L^4} \|\nabla w\|_{\infty} \|\nabla \partial w\|

+ C \left\| (\partial F) (w + \bar{W}) \right\|_{\infty} \|\nabla w\|_{\infty} \|\nabla w\| \|\nabla \partial w\| + C \left\| \nabla F (w + \bar{W}) - A \right\|_{\infty} \|\partial w\| \|\nabla \partial w\|.$$

We apply the following the Sobolev imbedding to control $\|w\|_{\infty}$

(3.2) \[ \|g\|_{L^\infty (2T^d)} \leq C_0 \|g\|_{H^2 (2T^d)}, \]

for $d \leq 3$. Moreover, from the periodic boundary conditions,

$$\int_{2T^d} \nabla u = \int_{2T^d} \nabla v = 0,$$

we also use the Poincare inequality

(3.3) \[ \|g\| \leq \|g\|_{L^4 (2T^d)} \leq C_0 \|\nabla g\| \quad \text{if } d \leq 3. \]

to further get

$$\|\nabla w\|_{\infty} \leq C_0 \|\nabla w\|_{H^2} \leq C_0 \sum_{|\partial| = 2} \|\partial \nabla w\|.$$

where $C_0$ is a universal constant. From (2.2) and the assumption $\|w\|_{H^2} \leq \eta$, the last nonlinear term in (3.1) is bounded by

$$C_0 C_\eta (1 + \eta) \|w\|_{H^2} \|\nabla \partial w\|^2.$$
We now estimate the second quadratic term in (3.1)

\[- \int_{2T} \{ \bar{D}_1 |\nabla \partial u|^2 + \bar{D}_2 |\nabla \partial v|^2 \} + \bar{g}_v \int_{2T} |\partial v|^2 \]

\[+ (\bar{f}_v + \bar{g}_u) \int_{2T} \partial u \partial v + \bar{f}_u \int_{2T} |\partial u|^2. \]

The last two terms are bounded by

\[(\bar{f}_v + \bar{g}_u) \int_{2T} \partial u \partial v + \bar{f}_u \int_{2T} |\partial u|^2 \leq \frac{|\bar{g}_v|}{2} \int_{2T} |\partial v|^2 + \left\{ \frac{(\bar{f}_v + \bar{g}_u)^2}{2 |\bar{g}_v|} + \bar{f}_u \right\} \int_{2T} |\partial u|^2. \]

Thus we can bound the linear term in (3.1) by (\(\bar{g}_v < 0\))

\[- \int_{2T} \{ \bar{D}_1 |\nabla \partial u|^2 + \bar{D}_2 |\nabla \partial v|^2 \} - \frac{|\bar{g}_v|}{2} \int_{2T} |\partial v|^2 \]

\[+ \left\{ \frac{(\bar{f}_v + \bar{g}_u)^2}{2 |\bar{g}_v|} + \bar{f}_u \right\} \int_{2T} |\partial u|^2. \]

By the interpolation between \(\|\nabla \partial u\|\) and \(\||u||\), the last term above is bounded by

\[\left\{ \frac{(\bar{f}_v + \bar{g}_u)^2}{2 |\bar{g}_v|} + \bar{f}_u \right\} \{ a \int_{2T} \|\nabla \partial u\|^2 + \frac{1}{4a^2} \int_{2T} ||u||^2 \}\]

for any \(a > 0\). We can choose \(a\) such that

\[\left\{ \frac{(\bar{f}_v + \bar{g}_u)^2}{2 |\bar{g}_v|} + \bar{f}_u \right\} a = \frac{1}{2} \bar{D}_1. \]

Collecting terms, we conclude the proof. \(\square\)

We are now ready to establish the bootstrap lemma, which controls the \(H^2\) growth of \(w(x, t)\) in term of its \(L^2\) growth nonlinearly.

**Lemma 4.** Suppose that \(w(x, t)\) is a solution to the full system (1.4)-(1.7) such that for \(0 \leq t \leq T\)

\[\|w(\cdot, t)\|_{H^2} \leq \min \left\{ \eta, \frac{\bar{D}_1}{2C_0C_1}, \frac{\bar{D}_2}{C_0C_1} \right\}\]

and

\[||w(\cdot, t)|| \leq 2C_1 e^{\lambda_{\text{max}} t} ||w(\cdot, 0)||, \]

then we have for \(0 \leq t \leq T\)

\[||w(t)||_{H^2}^2 \leq C_3 \{ ||w(0)||_{H^2}^2 + e^{2\lambda_{\text{max}} t} ||w(\cdot, 0)||^2 \}\]
where \( C_3 = C_1^2 \max \left\{ \frac{4C_2}{\lambda_{\text{max}}}, 1 \right\} \geq 1. \)

_Proof._ It suffices to only consider the second-order derivatives of \( w(x, t) \).

From the previous lemma and our assumption for \( \|w\|_{H^2} \), we deduce that

\[
\frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \{ |\partial u|^2 + |\partial v|^2 \} \, dx \leq C_2 \| u \|^2.
\]

So that by (3.4) and an integration from 0 to \( t \leq T \), we have

\[
\sum_{|\alpha|=2} \int_{\mathbb{T}^d} \{ |\partial u(t)|^2 + |\partial v(t)|^2 \}
\leq \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \{ |\partial u(0)|^2 + |\partial v(0)|^2 \} + \frac{4C_2C_1^2}{\lambda_{\text{max}}} e^{2\lambda_{\text{max}}t} \| w(\cdot, 0) \|^2.
\]

Thus our lemma follows. \( \square \)

4. **Nonlinear instability and pattern formation**

We now prove our main Theorem 1:

_Proof._ Let \( w^\delta(x, t) \) be the family of solutions to the reaction-diffusion system (1.4)-(1.5) with initial data \( w^\delta(x, 0) = \delta \mathbf{w}_0 \). Define \( T^* \) by

\[
T^* = \sup \left\{ t \mid \| w^\delta(t) - \delta e^{ct} \mathbf{w}_0 \| \leq \frac{C_1}{2} \delta \exp(\lambda_{\text{max}}t) \right\}.
\]

Note that \( T^* \) is well defined. We also define

\[
T^{**} = \sup \left\{ t \mid \| w(t) \|_{H^2} \leq \min \left\{ \eta, \frac{\bar{D}_1}{2C_0C_1}, \frac{\bar{D}_2}{C_0C_1} \right\} \right\}.
\]

We now derive estimates for \( H^2 \) norm of \( w^\delta(x, t) \) for \( 0 \leq t \leq \min\{T^*, T^{**}\} \). First of all, by the definition of \( T^* \), for \( t \leq T^* \) and Lemma 4

\[
\| w^\delta(t) \| \leq \frac{3C_1}{2} \delta \exp(\lambda_{\text{max}}t).
\]

Moreover, using Lemma 4 and applying a bootstrap argument yields

\[
(4.1) \quad \| w^\delta(t) \|_{H^2} \leq \sqrt{C_3} \{ \delta \| w_0 \|_{H^2} + \delta e^{\lambda_{\text{max}}t} \}.
\]

We now estimate the \( L^2 \) norm of \( w^\delta(x, t) \) for \( 0 \leq t \leq \min\{T^*, T^{**}\} \). We apply Duhamel’s principle to obtain

\[
w^\delta(t) = \delta e^{ct} \mathbf{w}_0 - \int_0^t e^{c(t-\tau)} \mathcal{N}(w^\delta(\tau)) \, d\tau,
\]
Using Lemma 1, (3.2), (3.3), and Lemma 4 yields, for \(0 \leq t \leq \min\{T^*, T^{**}\}
\begin{align*}
  &\|w^\delta(t) - \delta e^{Ct}w_0\| \\
  \leq & C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} \|\nabla \cdot (D\nabla w^\delta) - \tilde{D}\nabla^2 w^\delta\| + \|F - Aw^\delta\| d\tau \\
  \leq & C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} \|D\|_{C^1} \|w^\delta(\tau)\|_{\infty} \|w^\delta(\tau)\|_{H^2} d\tau \\
  &+ C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} \|D\|_{C^1} \|\nabla w^\delta(\tau)\|_{L^4} \|\nabla w^\delta(\tau)\|_{L^4} d\tau \\
  &+ C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} \|F\|_{C^2} \|w^\delta(\tau)\|_{\infty} \|w^\delta(\tau)\| d\tau \\
  \leq & C_1 C_0^2 C_\eta \int_0^t e^{\lambda_{\max}(t-\tau)} \|w^\delta(\tau)\|_{H^2}^2 d\tau.
\end{align*}

from assumption (2.2) with \(\|w\|_{H^2} \leq \eta\). We plug (1.1) with \(t = \tau\) to further obtain
\begin{align*}
  (4.2) \quad &\|w^\delta(t) - \delta e^{Ct}w_0\| \\
  \leq & C_1 C_0^2 C_\eta C_3 \int_0^t e^{\lambda_{\max}(t-\tau)} \{\xi^2 \|w_0\|_{H^2}^2 + \delta^2 e^{2\lambda_{\max}T}\} d\tau \\
  \leq & C_1 C_0^2 C_\eta C_3 \{\|w_0\|_{H^2}^2 \theta e^{\lambda_{\max}T} + \frac{1}{\lambda_{\max}} \delta e^{\lambda_{\max}T}\} e^{\lambda_{\max}t}.
\end{align*}

We now choose \(\theta\) in (2.1) to satisfy
\begin{align*}
  (4.3) \quad &C_0^2 C_3 C_\eta \theta < \frac{\lambda_{\max}}{4}, \\
  (4.4) \quad &2\sqrt{C_3} \theta < \min \left\{ \eta, \frac{\tilde{D}_1}{2C_0 C_1}, \frac{\tilde{D}_2}{C_0 C_1} \right\}.
\end{align*}

We now prove by contradiction that for \(\delta\) sufficiently small,
\[ T^\delta \leq \min\{T^*, T^{**}\}, \]
and therefore our theorem follows from (4.2), by further separating \(q \in \Omega_{\max}\) and move \(q \notin \Omega_{\max}\) in (1.18) to the right hand side.

If \(T^{**}\) is the smallest among \(T^\delta, T^*, T^{**}\), we can let \(t = T^{**} < T^\delta\) in (4.1)
\begin{align*}
  \|w^\delta(T^{**})\|_{H^2} < & \sqrt{C_3} \{\xi \|w_0\|_{H^2} + \delta e^{\lambda_{\max}T^\delta}\} \\
  = & \sqrt{C_3} \{\xi \|w_0\|_{H^2} + \theta\} \leq 2\sqrt{C_3} \theta,
\end{align*}
for small $\delta$ such that $\delta \|w_0\|_{H^2} \leq \theta$. By the choice of $\theta$ in (4.4), we have
\[
\|w(T^{**})\|_{H^2} < \min \left\{ \eta, \frac{\bar{D}_1}{2C_0C_1}, \frac{\bar{D}_2}{C_0C_1} \right\}.
\]
This is a contradiction to the definition of $T^{**}$.

On the other hand, if $T^*$ is the smallest among $T^\delta$, $T^*$ and $T^{**}$, we can let $t = T^*$ in (4.2) to get
\[
\|w^\delta(T^*) - \delta e^{\mathcal{L}t}w_0\|
\leq C_1C_0^2C_3C_\eta \left\{ \|w_0\|_{H^2}^2 \delta + \frac{1}{\lambda_{\text{max}}} \delta e^{\lambda_{\text{max}}T^*} \right\} \delta e^{\lambda_{\text{max}}T^*}
\leq C_1C_0^2C_3C_\eta \left\{ \|w_0\|_{H^2}^2 \delta + \frac{\theta}{\lambda_{\text{max}}} \right\} \delta e^{\lambda_{\text{max}}T^*}
< C_1^2 \delta e^{\lambda_{\text{max}}T^*},
\]
for $C_0^2C_3C_\eta \|w_0\|_{H^2}^2 \delta < 1/4$ for $\delta$ small, by our choice of $\theta$ in (4.3). This again contradicts the definition of $T$ and our theorem follows. \qed

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