ON PRO-\textit{p}-IWAHORI INVARIANTS OF $R$-REPRESENTATIONS OF REDUCTIVE $p$-ADIC GROUPS

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Abstract. Let $F$ be a locally compact field with residue characteristic $p$, and $G$ a connected reductive $F$-group. Let $U$ be a pro-$p$ Iwahori subgroup of $G = G(F)$. Fix a commutative ring $R$. If $\pi$ is a smooth $R[G]$-representation, the space of invariants $\pi^U$ is a right module over the Hecke algebra $H$ of $U$ in $G$.

Let $P$ be a parabolic subgroup of $G$ with a Levi decomposition $P = MN$ adapted to $U$. We complement previous investigation of Ollivier-Vigneras on the relation between taking $U$-invariants and various functor like $\text{Ind}^G_P$ and right and left adjoints. More precisely the authors’ previous work with Herzig introduce representations $I^G_P(M, \sigma, Q)$ where $\sigma$ is a smooth representation of $M$ extending, trivially on $N$, to a larger parabolic subgroup $P(\sigma)$, and $Q$ is a parabolic subgroup between $P$ and $P(\sigma)$. Here we relate $I^G_P(M, \sigma, Q)$ to an analogously defined $H$-module $I^H_P(M, \sigma^U_M, Q)$, where $U_M = U \cap M$ and $\sigma^U_M$ is seen as a module over the Hecke algebra $H_M$ of $U_M$ in $M$. In the reverse direction, if $V$ is a right $H_M$-module, we relate $I^H_P(M, V, Q) \otimes_R \text{c-Ind}^G_Q 1$ to $I^G_P(M, V \otimes_{H_M} \text{c-Ind}^M_Q 1, Q)$. As an application we prove that if $R$ is an algebraically closed field of characteristic $p$, and $\pi$ is an irreducible admissible representation of $G$, then the contragredient of $\pi$ is 0 unless $\pi$ has finite dimension.

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2010 Mathematics Subject Classification. primary 20C08, secondary 11F70.

Key words and phrases. parabolic induction, pro-$p$ Iwahori Hecke algebra.

The first-named author was supported by JSPS KAKENHI Grant Number 26707001.
1. Introduction

1.1. The present paper is a companion to [AHV17] and is similarly inspired by the classification results of [AHV17]; however it can be read independently. We recall the setting. We have a non-archimedean locally compact field $F$ of residue characteristic $p$ and a connected reductive $F$-group $G$. We fix a commutative ring $R$ and study the smooth $R$-representations of $G = G(F)$.

In [AHV17] the irreducible admissible $R$-representations of $G$ are classified in terms of supersingular ones when $R$ is an algebraically closed field of characteristic $p$. That classification is expressed in terms of representations $I_G(P, \sigma, Q)$, which make sense for any $R$. In that notation, $P$ is a parabolic subgroup of $G$ with a Levi decomposition $P = MN$ and $\sigma$ a smooth $R$-representation of the Levi subgroup $M$; there is a maximal parabolic subgroup $P(\sigma)$ of $G$ containing $P$ to which $\sigma$ inflated to $P$ extends to a representation $e_{P(\sigma)}(\sigma)$, and $Q$ is a parabolic subgroup of $G$ with $P \subset Q \subset P(\sigma)$. Then

$$I_G(P, \sigma, Q) = \text{Ind}_{P(\sigma)}^G(e_{P(\sigma)}(\sigma) \otimes \text{St}_Q^{P(\sigma)})$$

where Ind stands for parabolic induction and $\text{St}_Q^{P(\sigma)} = \text{Ind}_Q^{P(\sigma)} R/\sum \text{Ind}_Q^{P(\sigma)} R$, the sum being over parabolic subgroups $Q'$ of $G$ with $Q \subset Q' \subset P(\sigma)$. Alternatively, $I_G(P, \sigma, Q)$ is the quotient of $\text{Ind}_{P(\sigma)}^G(e_{P(\sigma)}(\sigma))$ by $\sum \text{Ind}_Q^{P(\sigma)} e_{Q'}(\sigma)$ with $Q'$ as above, where $e_Q(\sigma)$ is the restriction of $e_{P(\sigma)}(\sigma)$ to $Q$, similarly for $Q'$.

In [AHV17], we mainly studied what happens to $I_G(P, \sigma, Q)$ when we apply to it, for a parabolic subgroup $P_1$ of $G$, the left adjoint of $\text{Ind}_{P_1}^G$, or its right adjoint. Here we tackle a different question. We fix a pro-$p$ parahoric subgroup $\mathcal{U}$ of $G$ in good position with respect to $P$, so that in particular $\mathcal{U}_M = \mathcal{U} \cap M$ is a pro-$p$ parahoric subgroup of $M$. One of our main goals is to identify the $R$-module $I_G(P, \sigma, Q)^{\mathcal{U}}$ of $\mathcal{U}$-invariants, as a right module over the Hecke algebra $\mathcal{H} = \mathcal{H}_G$ of $\mathcal{U}$ in $G$ - the convolution algebra on the double coset space $\mathcal{U}\backslash G/\mathcal{U}$ - in terms on the module $\sigma_{\mathcal{M}}^{\mathcal{M}}$ over the Hecke algebra $\mathcal{H}_M$ of $\mathcal{U}_M$ in $M$. That goal is achieved in section 4 Theorem 4.17.

1.2. The initial work has been done in [OV17] §4 where $(\text{Ind}_P^G \sigma)^{\mathcal{U}}$ is identified. Precisely, writing $M^+$ for the monoid of elements $m \in M$ with $m(\mathcal{U} \cap M)m^{-1} \subset \mathcal{U} \cap N$, the subalgebra $\mathcal{H}_M^+$ of $\mathcal{H}_M$ with support in $M^+$, has a natural algebra embedding $\theta$ into $\mathcal{H}$ and [OV17] Proposition 4.4] identifies $(\text{Ind}_P^G \sigma)^{\mathcal{U}}$ with $\text{Ind}_{\mathcal{H}_M^+}^{\mathcal{H}_M}(\sigma_{\mathcal{M}}^{\mathcal{M}}) = \sigma_{\mathcal{M}}^{\mathcal{M}} \otimes_{\mathcal{H}_M^+} \mathcal{H}$. So in a sense, this paper is a sequel to [OV17] although some of our results here are used in [OV17] §5.

As $I_G(P, \sigma, Q)$ is only a subquotient of $\text{Ind}_P^G \sigma$ and taking $\mathcal{U}$-invariants is only left exact, it is not straightforward to describe $I_G(P, \sigma, Q)^{\mathcal{U}}$ from the previous result. However, that takes care of the parabolic induction step, so in a first approach we may assume $P(\sigma) = G$ so that $I_G(P, \sigma, Q) = e_G(\sigma) \otimes \text{St}_Q^G$. The crucial case is when moreover $\sigma$ is $e$-minimal, that is, not an
extension $e_M(\tau)$ of a smooth $R$-representation $\tau$ of a proper Levi subgroup of $M$. That case is treated first and the general case in section [1] only.

1.3. To explain our results, we need more notation. We choose a maximal $F$-split torus $T$ in $G$, a minimal parabolic subgroup $B = ZU$ with Levi component $Z$ the $G$-centralizer of $T$. We assume that $P = MN$ contains $B$ and $M$ contains $Z$, and that $U$ corresponds to an alcove in the apartment associated to $T$ in the adjoint building of $G$. It turns out that when $\sigma$ is $e$-minimal, the set $\Delta_M$ of simple roots of $T$ in $\text{Lie } N$ is orthogonal to its complement in the set $\Delta$ of simple roots of $T$ in $\text{Lie } U$. We assume until the end of this section [1.3] that $\Delta_M$ and $\Delta_2 = \Delta \setminus \Delta_M$ are orthogonal. If $M_2$ is the Levi subgroup containing $Z$ corresponding to $\Delta_2$, both $M$ and $M_2$ are normal in $G$, $M \cap M_2 = Z$ and $G = M_1 M_2$. Moreover the normal subgroup $M_1'$ of $G$ generated by $N$ is included in $M_2$ and $G = MM_1'$. We say that a right $H_M$-module $V$ is extensible to $H$ if $T_z^M$ acts trivially on $V$ for $z \in z \cap M_1'$ ([1.3]). In this case, we show that there is a natural structure of right $H$-module $e_H(V)$ on $V$ such that $T_g \in H$ corresponding to $UgM$ for $g \in M_1'$ acts as in the trivial character of $G$ ([1.4]). We call $e_H(V)$ the extension of $V$ to $H$ though $H_M$ is not a subalgebra of $H$. That notion is already present in [Abe] in the case where $R$ has characteristic $p$. Here we extend the construction to any $R$ and prove some more properties. In particular we produce an $H$-equivariant embedding $e_H(V)$ into $\text{Ind}_{H_M}^H V$ (Lemma [3.1]). If $Q$ is a parabolic subgroup of $G$ containing $P$, we go further and put on $e_H(V) \otimes (\text{Ind}^G_V R)^{\text{id}}$ and $e_H(V) \otimes (\text{St}^G_V)^{\text{id}}$ structures of $H$-modules (Proposition [3.15] and Corollary [3.17]) - note that $H$ is not a group algebra and there is no obvious notion of tensor product of $H$-modules.

If $\sigma$ is an $R$-representation of $M$ extensible to $G$, then its extension $e_G(\sigma)$ is simply obtained by letting $M_1'$ acting trivially on the space of $\sigma$; moreover it is clear that $\sigma^{\text{id}}$ is extensible to $H$, and one shows easily that $e_G(\sigma^{\text{id}}) = e_H(\sigma^{\text{id}})$ as an $H$-module ([3.5]). Moreover, the natural inclusion of $\sigma$ into $\text{Ind}^G_V \sigma$ induces on taking pro-$p$ Iwahori invariants an embedding $e_H(\sigma^{\text{id}}) \rightarrow (\text{Ind}^G_V \sigma)^{\text{id}}$ which, via the isomorphism of [OV17], yields exactly the above embedding of $\text{H}$-modules of $e_H(\sigma^{\text{id}})$ into $\text{Ind}_{H_M}^H (\sigma^{\text{id}})$. Then we show that the $H$-modules $(e_G(\sigma) \otimes R \text{Ind}^G_V R)^{\text{id}}$ and $e_H(\sigma^{\text{id}}) \otimes R (\text{St}^G_V)^{\text{id}}$ are equal, and similarly $(e_G(\sigma) \otimes R \text{St}^G_V)^{\text{id}}$ and $e_H(\sigma^{\text{id}}) \otimes R (\text{St}^G_V)^{\text{id}}$ are equal (Theorem [4.19]).

1.4. We turn back to the general case where we do not assume that $\Delta_M$ and $\Delta \setminus \Delta_M$ are orthogonal. Nevertheless, given a right $H_M$-module $V$, there exists a largest Levi subgroup $M(V)$ of $G$ containing $Z$ corresponding to $\Delta \cup \Delta_1$ where $\Delta_1$ is a subset of $\Delta \setminus \Delta_M$ orthogonal to $\Delta_M$, such that $V$ extends to a right $H_{M(V)}$-module $e_{M(V)}(V)$ with the notation of section [1.3]. For any parabolic subgroup $Q$ between $P$ and $P(V) = M(V)U$ we put (Definition [4.12])

$$I_H(P, V, Q) = \text{Ind}_{H_M}^H (e_{M(V)}(V) \otimes_R (\text{St}_{\text{Q}\cap M(V)})^{\text{id}}).$$

We refer to Theorem [4.17] for the description of the right $H$-module $I_G(P, \sigma, Q)^{\text{id}}$ for any smooth $R$-representation $\sigma$ of $U$. As a special case, it says that when $\sigma$ is $e$-minimal then $P(\sigma) \supset P(\sigma^{\text{id}})$ and if moreover $P(\sigma) = P(\sigma^{\text{id}})$ then $I_G(P, \sigma, Q)^{\text{id}}$ is isomorphic to $I_H(P, \sigma^{\text{id}}, Q)$.

Remark 1.1. In [Abe] are attached similar $H$-modules to $(P, V, Q)$; here we write them $CI_H(P, V, Q)$ because their definition uses, instead of $\text{Ind}_{H_M}^H$ a different kind of induction,
which we call coinduction. In loc. cit. those modules are use to give, when $R$ is an algebraically closed field of characteristic $p$, a classification of simple $\mathcal{H}$-modules in terms of supersingular modules - that classification is similar to the classification of irreducible admissible $R$-representations of $G$ in [AHHV17]. Using the comparison between induced and coinduced modules established in [Vig15b, 4.3] for any $R$, our corollary [4.21] expresses $CI\mathcal{H}(P,\mathcal{V},Q)$ as a module $I_\mathcal{H}(P_1,\mathcal{V}_1,Q_1)$; consequently we show in §4.5 that the classification of $\text{[Ab}}c$ can also be expressed in terms of modules $I_\mathcal{H}(P,\mathcal{V},Q)$.

1.5. In a reverse direction one can associate to a right $\mathcal{H}$-module $\mathcal{V}$ a smooth $R$-representation $\mathcal{V} \otimes_R R[U\backslash G]$ of $G$ (seeing $\mathcal{H}$ as the endomorphism ring of the $R[G]$-module $R[U\backslash G]$).

If $\mathcal{V}$ is a right $\mathcal{H}_M$-module, we construct, again using [OV17], a natural $R[G]$-map

$$I_\mathcal{H}(P,\mathcal{V},Q) \otimes_R R[U\backslash G] \to \text{Ind}_{P(M)}^G(e_{M(\mathcal{V})}(\mathcal{V}) \otimes_R S_{Q\cap M(\mathcal{V})}(\mathcal{V})),$$

with the notation of [1.4]. We show in §5 that it is an isomorphism under a mild assumption on the $\mathbb{Z}$-torsion in $\mathcal{V}$; in particular it is an isomorphism if $p = 0$ in $R$.

1.6. In the final section §6 we turn back to the case where $R$ is an algebraically closed field of characteristic $p$. We prove that the smooth dual of an irreducible admissible $R$-representation $\mathcal{V}$ of $G$ is 0 unless $\mathcal{V}$ is finite dimensional - that result is new if $F$ has positive characteristic, a case where the proof of Kohlhaase [Koh] for char$(F) = 0$ does not apply. Our proof first reduces to the case where $\mathcal{V}$ is supercuspidal (by [AHHV17]) then uses again the $\mathcal{H}$-module $\mathcal{V}^U$.

2. Notation, useful facts and preliminaries

2.1. The group $G$ and its standard parabolic subgroups $P = MN$. In all that follows, $p$ is a prime number, $F$ is a local field with finite residue field $k$ of characteristic $p$; We denote an algebraic group over $F$ by a bold letter, like $\mathbf{H}$, and use the same ordinary letter for the group of $F$-points, $H = \mathbf{H}(F)$. We fix a connected reductive $F$-group $\mathbf{G}$. We fix a maximal $F$-split subtorus $\mathbf{T}$ and write $Z$ for its $\mathbf{G}$-centralizer; we also fix a minimal parabolic subgroup $\mathbf{B}$ of $\mathbf{G}$ with Levi component $\mathbf{Z}$, so that $\mathbf{B} = \mathbf{ZU}$ where $U$ is the unipotent radical of $\mathbf{B}$. Let $X^*(\mathbf{T})$ be the group of $F$-rational characters of $\mathbf{T}$ and $\Phi$ the subset of roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{G}$. Then $\mathbf{B}$ determines a subset $\Phi^+$ of positive roots - the roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{U}$- and a subset of simple roots $\Delta$. The $\mathbf{G}$-normalizer $\mathbf{N}_\mathbf{G}$ of $\mathbf{T}$ acts on $X^*(\mathbf{T})$ and through that action, $\mathbf{N}_\mathbf{G}/\mathbf{Z}$ identifies with the Weyl group of the root system $\Phi$. Set $\mathcal{N} := \mathbf{N}_\mathbf{G}(F)$ and note that $\mathbf{N}_\mathbf{G}/\mathbf{Z} \simeq \mathcal{N}/\mathbb{Z}$; we write $\mathbb{W}$ for $\mathcal{N}/\mathbb{Z}$.

A standard parabolic subgroup of $\mathbf{G}$ is a parabolic $F$-group containing $\mathbf{B}$. Such a parabolic subgroup $\mathbf{P}$ has a unique Levi subgroup $\mathbf{M}$ containing $\mathbf{Z}$, so that $\mathbf{P} = \mathbf{MN}$ where $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$ - we also call $\mathbf{M}$ standard. By a common abuse of language to describe the preceding situation, we simply say “let $P = MN$ be a standard parabolic subgroup of $G$”; we sometimes write $N_P$ for $\mathbf{N}$ and $M_P$ for $\mathbf{M}$. The parabolic subgroup of $G$ opposite to $P$ will be written $\overline{P}$ and its unipotent radical $\overline{N}$, so that $\overline{P} = MN$, but beware that $\overline{P}$ is not standard! We write $\mathbb{W}_M$ for the Weyl group $(M \cap \mathcal{N})/\mathbb{Z}$.

If $\mathbf{P} = \mathbf{MN}$ is a standard parabolic subgroup of $G$, then $\mathbf{M} \cap \mathbf{B}$ is a minimal parabolic subgroup of $\mathbf{M}$. If $\Phi_M$ denotes the set of roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{M}$, with respect to $\mathbf{M} \cap \mathbf{B}$ we have $\Phi^+_M = \Phi_M \cap \Phi^+$ and $\Delta_M = \Phi_M \cap \Delta$. We also write $\Delta_P$ for $\Delta_M$ as $P$ and $M$ determine each other, $P = MU$. Thus we obtain a bijection $P \mapsto \Delta_P$ from standard parabolic subgroups of $G$ to subsets of $\Delta$, with $B$ corresponds to $\Phi$ and $G$ to $\Delta$. If $I$ is a subset of $\Delta$,
we sometimes denote by \( P_1 = M_1N_1 \) the corresponding standard parabolic subgroup of \( G \). If \( I = \{ \alpha \} \) is a singleton, we write \( P_\alpha = M_\alpha N_\alpha \). We note a few useful properties. If \( P_1 \) is another standard parabolic subgroup of \( G \), then \( P \subset P_1 \) if and only if \( \Delta_P \subset \Delta_{P_1} \); we have \( \Delta_{P \cap P_1} = \Delta_P \cap \Delta_{P_1} \) and the parabolic subgroup corresponding to \( \Delta_P \cup \Delta_{P_1} \) is the subgroup \( \langle P, P_1 \rangle \) of \( G \) generated by \( P \) and \( P_1 \). The standard parabolic subgroup of \( M \) associated to \( \Delta_M \cap \Delta_{M_\alpha} \) is \( M \cap P_1 = (M \cap M_1)(M \cap N_1) \) [Car85 Proposition 2.8.9]. It is convenient to write \( G' \) for the subgroup of \( G \) generated by the unipotent radicals of the parabolic subgroups; it is also the normal subgroup of \( G \) generated by \( U \), and we have \( G = ZG' \). For future references, we give now a useful lemma extracted from [AHHV17]:

**Lemma 2.1.** The group \( Z \cap G' \) is generated by the \( Z \cap M_\alpha \), \( \alpha \) running through \( \Delta \).

**Proof.** Take \( I = \emptyset \) in [AHHV17 II.6 Proposition]. \( \square \)

Let \( v_F \) be the normalized valuation of \( F \). For each \( \alpha \in X^*(T) \), the homomorphism \( x \mapsto v_F(\alpha(x)) : T \to \mathbb{Z} \) extends uniquely to a homomorphism \( Z \to \mathbb{Q} \) that we denote in the same way. This defines a homomorphism \( Z \to X_*(T) \otimes \mathbb{Q} \) such that \( \alpha(v(z)) = v_F(\alpha(z)) \) for \( z \in Z, \alpha \in X_*(T) \).

An interesting situation occurs when \( \Delta = I \cup J \) is the union of two orthogonal subsets \( I \) and \( J \). In that case, \( G' = M_1' M_1', M_1' \) and \( M_1' \) commute with each other, and their intersection is finite and central in \( G \) [AHHV17 II.7 Remark 4].

### 2.2. \( I_G(P, \sigma, Q) \) and minimality.

We recall from [AHHV17] the construction of \( I_G(P, \sigma, Q) \), our main object of study.

Let \( \sigma \) be an \( R \)-representation of \( M \) and \( P(\sigma) \) be the standard parabolic subgroup with

\[
\Delta_{P(\sigma)} = \{ \alpha \in \Delta \setminus \Delta_P \mid Z \cap M_\alpha \text{ acts trivially on } \sigma \} \cup \Delta_P.
\]

This is the largest parabolic subgroup \( P(\sigma) \) containing \( P \) to which \( \sigma \) extends, here \( N \subset P \) acts on \( \sigma \) trivially. Clearly when \( P \subset Q \subset P(\sigma) \), \( \sigma \) extends to \( Q \) and the extension is denoted by \( e_Q(\sigma) \). The restriction of \( e_{P(\sigma)}(\sigma) \) to \( Q \) is \( e_Q(\sigma) \). If there is no risk of ambiguity, we write

\[
e(\sigma) = e_{P(\sigma)}(\sigma).
\]

**Definition 2.2.** An \( R[G] \)-triple is a triple \( (P, \sigma, Q) \) made out of a standard parabolic subgroup \( P = MN \) of \( G \), a smooth \( R \)-representation of \( M \), and a parabolic subgroup \( Q \) of \( G \) with \( P \subset Q \subset P(\sigma) \). To an \( R[G] \)-triple \( (P, \sigma, Q) \) is associated a smooth \( R \)-representation of \( G \):

\[
I_G(P, \sigma, Q) = \text{Ind}_{P(\sigma)}^G(e(\sigma) \otimes \text{St}_{Q(\sigma)}^{P(\sigma)})
\]

where \( \text{St}_{Q(\sigma)}^{P(\sigma)} \) is the quotient of \( \text{Ind}_Q^{P(\sigma)} 1 \), \( 1 \) denoting the trivial \( R \)-representation of \( Q \), by the sum of its subrepresentations \( \text{Ind}_{Q(\sigma)}^{P(\sigma)} 1 \), the sum being over the set of parabolic subgroups \( Q' \) of \( G \) with \( Q \subsetneq Q' \subset P(\sigma) \).

Note that \( I_G(P, \sigma, Q) \) is naturally isomorphic to the quotient of \( \text{Ind}_{Q}^{G}(e_Q(\sigma)) \) by the sum of its subrepresentations \( \text{Ind}_{Q'}^{G}(e_{Q'}(\sigma)) \) for \( Q \subsetneq Q' \subset P(\sigma) \) by Lemma 2.5.

It might happen that \( \sigma \) itself has the form \( e_P(\sigma_1) \) for some standard parabolic subgroup \( P_1 = M_1N_1 \) contained in \( P \) and some \( R \)-representation \( \sigma_1 \) of \( M_1 \). In that case, \( P(\sigma_1) = P(\sigma) \) and \( e(\sigma) = e(\sigma_1) \). We say that \( \sigma \) is **\( e \)-minimal** if \( \sigma = e_P(\sigma_1) \) implies \( P_1 = P, \sigma_1 = \sigma \).
Lemma 2.3 ([AHV17, Lemma 2.9]). Let $P = MN$ be a standard parabolic subgroup of $G$ and let $\sigma$ be an $R$-representation of $M$. There exists a unique standard parabolic subgroup $P_{\text{min,}\sigma} = M_{\text{min,}\sigma}N_{\text{min,}\sigma}$ of $G$ and a unique $e$-minimal representation of $\sigma_{\text{min}}$ of $M_{\text{min,}\sigma}$ with $\sigma = eP(\sigma_{\text{min}})$. Moreover $P(\sigma) = P(\sigma_{\text{min}})$ and $e(\sigma) = e(\sigma_{\text{min}})$.

Lemma 2.4. Let $P = MN$ be a standard parabolic subgroup of $G$ and $\sigma$ an $e$-minimal $R$-representation of $M$. Then $\Delta_P$ and $\Delta_{P(\sigma)} \setminus \Delta_P$ are orthogonal.

That comes from [AHV17, II.7 Corollary 2]. That corollary of loc. cit. also shows that when $R$ is a field and $\sigma$ is supercuspidal, then $\sigma$ is $e$-minimal. Lemma 2.4 shows that $\Delta_{P_{\text{min,}\sigma}}$ and $\Delta_{P(\sigma_{\text{min}})} \setminus \Delta_{P_{\text{min,}\sigma}}$ are orthogonal.

Note that when $\Delta_P$ and $\Delta_\sigma$ are orthogonal of union $\Delta = \Delta_P \cup \Delta_\sigma$, then $G = P(\sigma) = MM'$ and $e(\sigma)$ is the $R$-representation of $G$ simply obtained by extending $\sigma$ trivially on $M'$.

Lemma 2.5 ([AHV17, Lemma 2.11]). Let $(P, \sigma, Q)$ be an $R[G]$-triple. Then $(P_{\text{min,}\sigma}, \sigma_{\text{min}}, Q)$ is an $R[G]$-triple and $I_G(P, l\sigma, Q) = I_G(P_{\text{min,}\sigma}, \sigma_{\text{min}}, Q)$.

2.3. Pro-$p$ Iwahori Hecke algebras. We fix a special parahoric subgroup $\mathcal{K}$ of $G$ fixing a special vertex $x_0$ in the apartment $\mathcal{A}$ associated to $T$ in the Bruhat-Tits building of the adjoint group of $G$. We let $B$ be the Iwahori subgroup fixing the alcove $\mathcal{C}$ in $\mathcal{A}$ with vertex $x_0$ contained in the Weyl chamber (of vertex $x_0$) associated to $B$. We let $U$ be the pro-$p$ radical of $B$ (the pro-$p$ Iwahori subgroup). The pro-$p$ Iwahori Hecke ring $\mathcal{H} = \mathcal{H}(G, U)$ is the convolution ring of compactly supported functions $G \to \mathbb{Z}$ constant on the double classes of $G$ modulo $U$. We denote by $T(g)$ the characteristic function of $ugU$ for $g \in G$, seen as an element of $\mathcal{H}$. Let $R$ be a commutative ring. The pro-$p$ Iwahori Hecke $R$-algebra $\mathcal{H}_{M, R}$ is $R \otimes_{\mathbb{Z}} \mathcal{H}_M$. We will follow the custom to still denote by $h$ the natural image $1 \otimes h$ of $h \in \mathcal{H}$ in $\mathcal{H}_R$.

For $P = MN$ a standard parabolic subgroup of $G$, the similar objects for $M$ are indexed by $M$, we have $\mathcal{K} = \mathcal{K} \cap M, B_M = B \cap M, U_M = U \cap M$, the pro-$p$ Iwahori Hecke ring $\mathcal{H}_M = \mathcal{H}(M, U_M), T^M(m)$ the characteristic function of $U_MmU_M$ for $m \in M$, seen as an element of $\mathcal{H}_M$. The pro-$p$ Iwahori group $U$ of $G$ satisfies the Iwahori decomposition with respect to $P$:

$$U = U_NU_MU_N^{-1},$$

where $U_N = U \cap N, U_N = U \cap N$. The linear map

$$(2.1) \quad \mathcal{H}_M \to \mathcal{H}, \quad \theta(T^M(m)) = T(m) \quad (m \in M)$$

does not respect the product. But if we introduce the monoid $M^+$ of elements $m \in M$ contracting $U_N$, meaning $mU_Nm^{-1} \subset U_N$, and the submodule $\mathcal{H}_{M^+} \subset \mathcal{H}_M$ of functions with support in $M^+$, we have [Vig15b, Theorem 1.4]:

$\mathcal{H}_{M^+}$ is a subring of $\mathcal{H}_M$ and $\mathcal{H}_M$ is the localization of $\mathcal{H}_{M^+}$ at an element $\tau^M \in \mathcal{H}_{M^+}$ central and invertible in $\mathcal{H}_M$, meaning $\mathcal{H}_M = \cup_{n \in \mathbb{N}} \mathcal{H}_{M^+}(\tau^M)^{-n}$. The map $\mathcal{H}_M \to \mathcal{H}$ is injective and its restriction $\theta|_{\mathcal{H}_{M^+}}$ to $\mathcal{H}_{M^+}$ respects the product.

These properties are also true when $(M^+, \tau^M)$ is replaced by its inverse $(M^-, (\tau^M)^{-1})$ where $M^- = \{m^{-1} \in M \mid m \in M^+\}$.
3. Pro-$p$ Iwahori invariants of $I_G(P,\sigma,Q)$

3.1. Pro-$p$ Iwahori Hecke algebras: structures. We supplement here the notations of [2.1] and [2.3]. The subgroups $Z^0 = Z \cap K = Z \cap B$ and $Z^1 = Z \cap U$ are normal in $N$ and we put $W = N/Z^0$, $W(1) = N/Z^1$, $\Lambda = Z/Z^0$, $\Lambda(1) = Z/Z^1$, $Z_k = Z^0/Z^1$.

We have $N = (N \cap K)Z$ so that we see the finite Weyl group $\mathcal{W} = N/Z$ as the subgroup $(N \cap K)/Z^0$ of $W$; in this way $W$ is the semi-direct product $\Lambda \times \mathcal{W}$. The image $W_{G'} = W'$ of $N \cap G'$ in $W$ is an affine Weyl group generated by the set $S^{aff}$ of affine reflections determined by the walls of the alcove $C$. The group $W'$ is normal in $W$ and $W'$ is the semi-direct product $W' \rtimes \Omega$ where $\Omega$ is the image in $W$ of the normalizer $N_C$ of $C$ in $N$. The length function $\ell$ on the affine Weyl system $(W', S^{aff})$ extends to a length function on $W$ such that $\Omega$ is the set of elements of length 0. We also view $\ell$ as a function of $W(1)$ via the quotient map $W(1) \to W$.

We write

$$(3.1) \quad (\tilde{w}, \tilde{w}, w) \in N \times W(1) \times W$$

corresponding via the quotient maps $N \to W(1) \to W$.

When $w = s$ in $S^{aff}$ or more generally $w$ in $W_{G'}$, we will most of the time choose $\tilde{w}$ in $N \cap G'$ and $\tilde{w}$ in the image of $W_{G'}$ of $N \cap G'$ in $W(1)$.

We are now ready to describe the pro-$p$ Iwahori Hecke ring $\mathcal{H} = \mathcal{H}(G, U)$ [Vig16]. We have $G = UNU$ and for $n, n' \in N$ we have $UnU = Un'U$ if and only if $nZ^1 = n'Z^1$. For $n \in N$ of image $w \in W(1)$ and $g \in UNU$ we denote $T_w = T(n) = T(g)$ in $\mathcal{H}$. The relations among the basis elements $(T_w)_{w \in W(1)}$ of $\mathcal{H}$ are:

1. Braid relations: $T_wT_{w'} = T_{ww'}$ for $w, w' \in W(1)$ with $\ell(ww') = \ell(w) + \ell(w')$.
2. Quadratic relations: $T_{\tilde{s}}^2 = q_{s}T_{\tilde{s}^2} + c_{\tilde{s}}T_{\tilde{s}}$

for $\tilde{s} \in W(1)$ lifting $s \in S^{aff}$, where $q_{s} = q_{G}(s) = |U/\tilde{U} \cap \tilde{U}s^{-1}U|$ depends only on $s$, and $c_{\tilde{s}} = \sum_{t \in Z_k} c_{\tilde{s}}(t)T_t$ for integers $c_{\tilde{s}}(t) \in \mathbb{N}$ summing to $q_{s} - 1$.

We shall need the basis elements $(T_w^*)_{w \in W(1)}$ of $\mathcal{H}$ defined by:

1. $T_w^* = T_w$ for $w \in W(1)$ of length $\ell(w) = 0$.
2. $T_{\tilde{s}}^* = T_{\tilde{s}} - c_{\tilde{s}}$ for $\tilde{s} \in W(1)$ lifting $s \in S^{aff}$.
3. $T_{ww'}^* = T_{w'}T_w^*$ for $w, w' \in W(1)$ with $\ell(ww') = \ell(w) + \ell(w')$.

We need more notation for the definition of the admissible lifts of $S^{aff}$ in $N_G$. Let $s \in S^{aff}$ fixing a face $C_s$ of the alcove $C$ and $K_s$ the parahoric subgroup of $G$ fixing $C_s$. The theory of Bruhat-Tits associates to $C_s$ a certain root $\alpha_s \in \Phi^+$ [Vig16 §4.2]. We consider the group $G'_s$ generated by $U_{\alpha_s} \cup U_{-\alpha_s}$ where $U_{\pm \alpha_s}$ the root subgroup of $\pm \alpha_s$ (if $2\alpha_s \in \Phi$, then $U_{2\alpha_s} \subset U_{\alpha_s}$) and the group $G'_s$ generated by $U_{\alpha_s} \cup U_{-\alpha_s}$ where $U_{\pm \alpha_s} = U_{\pm \alpha_s} \cap K_s$. When $u \in U_{\alpha_s} - \{1\}$, the intersection $N_G \cap U_{-\alpha_s}uU_{-\alpha_s}$ (equal to $N_G \cap U_{-\alpha_s}uU_{-\alpha_s} = \{1\}$) is generated by the group $\tilde{Z}'_k = Z \cap G'_s$ is contained in $Z \cap K_s = Z^0$, its image in $Z_k$ is denoted by $Z'_{k,s}$.

The elements $n_{s}(u)$ for $u \in U_{\alpha_s} - \{1\}$ are the admissible lifts of $s$ in $N_G$; their images in $W(1)$ are the admissible lifts of $s$ in $W(1)$. By [Vig16] Theorem 2.2, Proposition 4.4, when $\tilde{s} \in W(1)$ is an admissible lift of $s$, $c_{\tilde{s}}(t) = 0$ if $t \in Z_k \setminus Z'_{k,s}$, and

$$(3.2) \quad c_{\tilde{s}} \equiv (q_{s} - 1)|Z'_{k,s}|^{-1} \sum_{t \in Z'_{k,s}} T_t \mod p.$$
The admissible lifts of $S$ in $\mathcal{N}_G$ are contained in $\mathcal{N}_G \cap \mathcal{K}$ because $\mathcal{K}_s \subset \mathcal{K}$ when $s \in S$.

**Definition 3.1.** An admissible lift of the finite Weyl group $\mathbb{W}$ in $\mathcal{N}_G$ is a map $w \mapsto \hat{w} : \mathbb{W} \to \mathcal{N}_G \cap \mathcal{K}$ such that $\hat{s}$ is admissible for all $s \in S$ and $\hat{w} = \hat{w}_1 \hat{w}_2$ for $w_1, w_2 \in \mathbb{W}$ such that $w = w_1w_2$ and $\ell(w) = \ell(w_1) + \ell(w_2)$.

Any choice of admissible lifts of $S$ in $\mathcal{N}_G \cap \mathcal{K}$ extends uniquely to an admissible lift of $\mathbb{W}$ ([AHVV17, IV.6], [OV17, Proposition 2.7]).

Let $P = MN$ be a standard parabolic subgroup of $G$. The groups $Z, Z^0 = Z \cap \mathcal{K}_M = Z \cap \mathcal{B}_M, Z^1 = Z \cap \mathcal{U}_M$ are the same for $G$ and $M$, but $\mathcal{N}_M = \mathcal{N} \cap M$ and $M \cap G'$ are subgroups of $\mathcal{N}$ and $G'$. The monoid $M^+$ contains $(\mathcal{N}_M \cap \mathcal{K})$ and is equal to $M^+ = \mathcal{U}_M \mathcal{N}_M^+ \mathcal{U}_M$ where $\mathcal{N}_M^+ = \mathcal{N} \cap M^+$. An element $z \in Z$ belongs to $M^+$ if and only if $v_F(\alpha(z)) \geq 0$ for all $\alpha \in \Phi^+ \setminus \Phi^+_M$ (see [Vig15b, Lemme 2.2]). Put $W_M = N_M/Z^0$ and $W_M(1) = N/Z^1$.

Let $\epsilon = +$ or $\epsilon = -$. We denote by $W_{M,\epsilon}$ the images of $\mathcal{N}_M^+ \subset W_M, W_M(1)$. We see the groups $W_M, W_M(1), W_{M'}$ as subgroups of $W, W(1), W_{M'}$. As $\theta$ (2.3), the linear injective map

$$H_M \xrightarrow{\theta^*} H, \quad \theta^*(T_{w,M}^*) = T_{w}^*, \quad (w \in W_M(1)),$$

respects the product on the subring $H_{M,\epsilon}$. Note that $\theta$ and $\theta^*$ satisfy the obvious transitivity property with respect to a change of parabolic subgroups.

### 3.2. Orthogonal case

Let us examine the case where $\Delta_M$ and $\Delta \setminus \Delta_M$ are orthogonal, writing $M_2 = M_2 \setminus \Delta_M$ as in §1.3.

From $M \cap M_2 = Z$ we get $W_M \cap W_{M_2} = \Lambda, W_M(1) \cap W_{M_2}(1) = \Lambda(1)$, the semisimple building of $G$ is the product of those of $M$ and $M_2$ and $S^\text{aff}$ is the disjoint union of $S^\text{aff}_M$ and $S^\text{aff}_{M_2}$, the group $W_G'$ is the direct product of $W_M'$ and $W_{M_2}'$. For $\hat{s} \in W_M(1)$ lifting $s \in S^\text{aff}$, the elements $T_{s,M}^* \in H_M$ and $T_{s} \in H$ satisfy the same quadratic relations. A word of caution is necessary for the lengths $\ell_M$ of $W_M$ and $\ell_{M_2}$ of $W_{M_2}$ different from the restrictions of the length $\ell$ of $W_M$, for example $\ell_M(\lambda) = 0$ for $\lambda \in \Lambda \cap W_{M_2}$.

**Lemma 3.2.** We have $\Lambda = (W_{M,\epsilon} \cap \Lambda)(W_{M_2} \cap \Lambda)$.

**Proof.** We prove the lemma for $\epsilon = -$. The case $\epsilon = +$ is similar. The map $v : Z \to X_\bullet(T) \otimes \mathbb{Q}$ defined in 2.1 is trivial on $Z^0$ and we also write $v$ for the resulting homomorphism on $\Lambda$. For $\lambda \in \Lambda$ there exists $\lambda_2 \in Z_{M_2} \cap \Lambda$ such that $\lambda_2 \in W_{M_2}$, or equivalently $\alpha(v(\lambda_2)) \leq 0$ for all $\alpha \in \Phi^+ \setminus \Phi^+_M = \Phi^+_{M_2}$. It suffices to have the inequality for $\alpha \in \Delta_{M_2}$. The matrix $(\alpha(\beta'))_{\alpha, \beta \in \Delta_{M_2}}$ is invertible, hence there exist $n_\beta \in \mathbb{Z}$ such that $\sum_{\beta \in \Delta_{M_2}} n_\beta \alpha(\beta') \leq -\alpha(\lambda) \Lambda^\vee$ for all $\alpha \in \Delta_{M_2}$. As $v(W_{M_2} \cap \Lambda)$ contains $\oplus_{\alpha \in \Delta_{M_2}} \mathbb{Z} \alpha^\vee$ where $\alpha^\vee$ is the coroot of $\alpha$ [Vig16, after formula (7.1)], there exists $\lambda_2 \in Z_{M_2} \cap \Lambda$ with $v(\lambda_2) = \sum_{\beta \in \Delta_{M_2}} n_\beta \beta^\vee$. $\square$

The groups $\mathcal{N} \cap M'$ and $\mathcal{N} \cap M_2'$ are normal in $\mathcal{N}$, and $\mathcal{N} = (\mathcal{N} \cap M')\mathcal{N}_G(\mathcal{N} \cap M_2') = Z(\mathcal{N} \cap M')(\mathcal{N} \cap M_2')$.

$$W = W_{M'} \Omega W_{M_2'} = W_M W_{M_2'} = W_{M_+} W_{M_2} = W_{M_+} W_{M_2}$$

The first two equalities are clear, the equality $W_M W_{M_2} = W_{M'} W_{M_2}$ follows from $W_M = W_{M,\epsilon} \Lambda$, $W_M \subset W_{M'}$ and the lemma. The inverse image in $W(1)$ of these groups are

$$W(1) = 1 W_{M'} \Omega(1) 1 W_{M_2'} = W_{M}(1) 1 W_{M_2'} = W_{M_+}(1) 1 W_{M_2'} = W_{M_+}(1) 1 W_{M_2'}.$$
We recall the function \( q_G(n) = q(n) = |U/(U \cap n^{-1}U)| \) on \( N \) \cite[Proposition 3.38]{Vig16} and we extend to \( N \) the functions \( q_M \) on \( N \cap M \) and \( q_{M_2} \) on \( N \cap M_2 \):

\[
q_M(n) = \frac{|U_M/(U_M \cap n^{-1}U_M)|}{|U_M|} = \frac{|U_M/(U_M \cap n^{-1}U_M)|}{|U_M|}
\]

The functions \( q, q_M, q_{M_2} \) descend to functions on \( W(1) \) and on \( W \), also denoted by \( q, q_M, q_{M_2} \).

**Lemma 3.3.** Let \( n \in N \) of image \( w \in W \). We have

1. \( q(n) = q_M(n)q_{M_2}(n) \).
2. \( q_M(n) = q_M(nM) \) if \( n = n_Mn_2 \), \( n_M \in N \cap M, n_2 \in N \cap M_2 \) and similarly when \( M \) and \( M_2 \) are permuted.
3. \( q(w) = 1 \Leftrightarrow q_M(\lambda w_M) = q_{M_2}(\lambda w_{M_2}) = 1 \), if \( w = \lambda w_Mw_{M_2}, (\lambda, w_M, w_{M_2}) \in \Lambda \times \mathbb{W}_M \times \mathbb{W}_M \).
4. On the coset \( (N \cap M_2)N_Cn, q_M \) is constant equal to \( q_M(n_{M'}) \) for any element \( n_{M'} \in M' \cap (N \cap M_2)N_Cn \). A similar result is true when \( M \) and \( M_2 \) are permuted.

**Proof.** The product map

\[
Z^1 \prod_{\alpha \in \Phi_{M,red}} U_\alpha \prod_{\alpha \in \Phi_{M_2,red}} \to U
\]

with \( U_\alpha = U_\alpha \cap U \), is a homeomorphism. We have \( U_M = Z^1Y_{M'}, U_{M'} = (Z^1 \cap M')Y_{M'} \) where \( Y_{M'} = \prod_{\alpha \in \Phi_{M,red}} U_\alpha \) and \( N' \cap M \) normalizes \( Y_{M'} \). Similar results are true when \( M \) and \( M_2 \) are permuted, and \( \mathfrak{U} = U_MU_M = U_MU_{M_2} \).

Writing \( N = Z(N \cap M')(N' \cap M_2') \) (in any order), we see that the product map

\[
Z^1(Y_{M'} \cap n^{-1}Y_{M'2n})(Y_{M_2'} \cap n^{-1}Y_{M_2'_2n}) \to U \cap n^{-1}U_n
\]

is an homeomorphism. The inclusions induce bijections

\[
Y_{M'}/(Y_{M'} \cap n^{-1}Y_{M'2n}) \simeq U_M/(U_M \cap n^{-1}U_Mn) \simeq U_M/(U_M \cap n^{-1}U_Mn),
\]

similarly for \( M_2 \), and also a bijection

\[
U/(U \cap n^{-1}U_n) \simeq Y_{M_2'}/(Y_{M_2'} \cap n^{-1}Y_{M_2'_2n}) \times (Y_{M'}/(Y_{M'} \cap n^{-1}Y_{M'n})).
\]

The assertion (1) in the lemma follows from (3.8), (3.9).

The assertion (2) follows from (3.7); it implies the assertion (3).

A subgroup of \( N \) normalizes \( U_M \) if and only if it normalizes \( Y_{M'} \) by (3.8) if and only if \( q_M = 1 \) on this group. The group \( N' \cap M_2' \) normalizes \( Y_{M'} \) because the elements of \( M_2' \) commute with those of \( M' \) and \( q_{M'} \) is trivial on \( N_C \) by (2). Therefore the group \( (N' \cap M_2')N_{C'} \) normalizes \( U_M \). The coset \( (N' \cap M_2')N_{C}n \) contains an element \( n_{M'} \in M' \). For \( x \in (N' \cap M_2')N_{C}, (xn_{M'})^{-1}Uxn_{M'} = n_{M'}^{-1}U_{M'}n_{M'} \) hence \( q_M(xn_{M'}) = q_M(n_{M'}) \).

3.3. Extension of an \( \mathcal{H}M \)-module to \( \mathcal{H} \). This section is inspired by similar results for the pro-\( p \) Iwahori Hecke algebras over an algebraically closed field field of characteristic \( p \) \cite[Proposition 4.16]{Abe}. We keep the setting of (3.2) and we introduce ideals:

- \( \mathcal{J}_\ell \) (resp. \( \mathcal{J}_r \)) the left (resp. right) ideal of \( \mathcal{H} \) generated by \( T_{w_\ell} - 1_{\mathcal{H}} \) for all \( w \in 1_{W_{M_2'}} \),

- \( \mathcal{J}_{M,\ell} \) (resp. \( \mathcal{J}_{M,r} \)) the left (resp. right) ideal of \( \mathcal{H} \) generated by \( T_{\lambda} - 1_{\mathcal{H}_M} \) for all \( \lambda \) in \( 1_{W_{M_2'}} \cap W_M(1) = 1_{W_{M_2'}} \cap \Lambda(1) \).

The next proposition shows that the ideals \( \mathcal{J}_\ell = \mathcal{J}_r \) are equal and similarly \( \mathcal{J}_{M,\ell} = \mathcal{J}_{M,r} \). After the proposition, we will drop the indices \( \ell \) and \( r \).
Proposition 3.4. The ideals \( J_\ell \) and \( J_r \) are equal to the submodule \( J' \) of \( \mathcal{H} \) generated by \( T_w^* - T_{w,w_2}^* \) for all \( w \in W(1) \) and \( w_2 \in 1W_{M'_2} \).

The ideals \( J_{M,\ell} \) and \( J_{M,r} \) are equal to the submodule \( J'_M \) of \( \mathcal{H}_M \) generated by \( T_w^{M,*} - T_{w,\lambda_2}^{M,*} \) for all \( w \in W_M \) and \( \lambda_2 \in \Lambda(1) \cap 1W_{M'_2} \).

Proof. (1) We prove \( J_\ell = J' \). Let \( w \in W(1) \) and \( w_2 \in 1W_{M'_2} \). We prove by induction on the length of \( w_2 \) that \( T_w^*(T_{w_2}^* - 1) \in J' \). This is obvious when \( \ell(w_2) = 0 \) because \( T_w^*T_{w_2}^* = T_{w_2}^* \).

Assume that \( \ell(w_2) = 1 \) and put \( s = w_2 \). If \( \ell(ws) = \ell(w) + 1 \), as before \( T_w^*(T_s^* - 1) \in J' \) because \( T_w^*T_s^* = T_{ws}^* \). Otherwise \( \ell(ws) = \ell(w) - 1 \) and \( T_w^* = T_{ws}^{-1}T_{ws}^* \) hence

\[
T_w^* (T_s^* - 1) = T_{ws}^{-1}(T_s^* - 1)^2 - T_s^* = w^{-1}(q_\lambda T_{s}^* - T_{s}^*) - T_s^* = q_s T_{ws}^* - T_{ws}^*(c_s + 1).
\]

Recalling from (2.3) that \( c_s + 1 = \sum_{t \in Z_k^e} c_s(t)T_t \) with \( c_s(t) \in \mathbb{N} \) and \( \sum_{t \in Z_k^e} c_s(t) = q_s \),

\[
q_s T_{ws}^* - T_{ws}^*(c_s + 1) = \sum_{t \in Z_k^e} c_s(t)(T_{ws}^* - T_{ws}^*T_t^*) = \sum_{t \in Z_k^e} c_s(t)(T_{ws}^* - T_{ws}^*T_t^*) \in J'.
\]

Assume now that \( \ell(w_2) > 1 \). Then, we factorize \( w_2 = xy \) with \( x, y \in 1W_{M_2} \) of length \( \ell(x), \ell(y) < \ell(w_2) \) and \( \ell(xy) = \ell(x) + \ell(y) \). The element \( T_w^*(T_{w_2}^* - 1) = T_{w_2}^*T_x^*(T_y^* - 1) = T_{w_2}^*(T_x^* - 1) \) lies in \( J' \) by induction.

Conversely, we prove \( T_{w_2}^* - T_w^* \in J_\ell \). We factorize \( w = xy \) with \( x \in 1W_{M_2} \) and \( x \in 1W_M \Omega(1) \). Then, we have \( \ell(w) = \ell(x) + \ell(y) \) and \( \ell(xy) = \ell(x) + \ell(y) \).

\[
T_{w_2}^* - T_w^* = T_x^*(T_{w_2}^* - T_y^*) = T_x^*(T_{w_2}^* - T_y^*) \in J_\ell.
\]

This ends the proof of \( J_\ell = J' \).

By the same argument, the right ideal \( J_r \) of \( \mathcal{H} \) is equal to the submodule of \( \mathcal{H} \) generated by \( T_{w_2}^* - T_w^* \) for all \( w \in W(1) \) and \( w_2 \in 1W_{M'_2} \). But this latter submodule is equal to \( J' \) because \( 1W_{M'_2} \) is normal in \( W(1) \). Therefore we proved \( J' = J_r = J_\ell \).

(2) Proof of the second assertion. We prove \( J_{M,\ell} = J_{M} ' \). The proof is easier than in (1) because for \( w \in W_M(1) \) and \( \lambda_2 \in 1W_{M'_2} \cap \Lambda(1) \), we have \( \ell(w \lambda_2) = \ell(w) + \ell(\lambda_2) \) hence \( T_w^{M,*}(T_{\lambda_2}^{M,*} - 1) = T_w^{M,*} - T_{w\lambda_2}^{M,*} \). We have also \( \ell(\lambda_2 w) = \ell(\lambda_2) + \ell(w) \) hence \( T_{w\lambda_2}^{M,*} = T_{\lambda_2}^{M,*} - T_w^{M,*} \) hence \( J_{M,r} \) is equal to the submodule of \( \mathcal{H}_M \) generated by \( T_{\lambda_2}^{M,*} - T_w^{M,*} \) for all \( w \in W_M(1) \) and \( \lambda_2 \in 1W_{M'_2} \cap \Lambda(1) \). This latter submodule is \( J_{M}' \), as \( 1W_{M'_2} \cap \Lambda(1) = \Lambda(1) \cap 1W_{M'_2} = 1W(1) \) is normal in \( W(1) \). Therefore \( J_{M}' = J_{M,\ell} = J_{M,s} \).

By Proposition 3.1 a basis of \( J \) is \( T_w^* - T_{w_2}^* \) for \( w \) in a system of representatives of \( W(1)/1W_{M'_2} \), and \( w_2 \in 1W_{M'_2} \setminus \{1\} \). Similarly a basis of \( J_M \) is \( T_w^{M,*} - T_{w\lambda_2}^{M,*} \) for \( w \) in a system of representatives of \( W_M(1)/\Lambda(1) \cap 1W_{M'_2} \), and \( \lambda_2 \in \Lambda(1) \cap 1W_{M'_2} \setminus \{1\} \).

Proposition 3.5. The natural ring inclusion of \( \mathcal{H}_M^* \) in \( \mathcal{H}_M \) and the ring inclusion of \( \mathcal{H}_M^* \) in \( \mathcal{H} \) via \( \theta^* \) induce ring isomorphisms

\[
\mathcal{H}_M/\mathcal{J}_M \cong \mathcal{H}_M^*/(\mathcal{J}_M \cap \mathcal{H}_M^*) \cong \mathcal{H}/\mathcal{J}.
\]

Proof. (1) The left map is obviously injective. We prove the surjectivity. Let \( w \in W_M(1) \). Let \( \lambda_2 \in 1W_{M'_2} \cap \Lambda(1) \) such that \( w\lambda_2^{-1} \in W_M^{-1}(1) \) (see (3.4)). We have \( T_{w\lambda_2}^{M,*} \in \mathcal{H}_M^- \) and \( T_w^{M,*} = T_{w\lambda_2}^{M,*} - T_{w\lambda_2}^{M,*} = T_{w\lambda_2}^{M,*} + T_{w\lambda_2}^{M,*} (T_{w\lambda_2}^{M,*} - 1) \). Therefore \( T_w^{M,*} \in \mathcal{H}_M^- + \mathcal{J}_M \). As \( w \) is arbitrary, \( \mathcal{H}_M = \mathcal{H}_M^- + \mathcal{J}_M \).
(2) The right map is surjective: let \(w \in W(1)\) and \(w_2 \in 1W_{M_2}^1\) such that \(ww_2^{-1} \in W_{M_2}^{-}(1)\) (see (3.4)). Then \(T_w^* - T_{ww_2}^{-1} \in J\) with the same arguments than in (1), using Proposition 3.4. Therefore \(H = \theta^*(H_{M^-}) + J\).

We prove the injectivity: \(\theta^*(H_{M^-}) \cap J = \theta^*(H_{M^-} \cap J_M)\). Let \(\sum_{w \in W_{M_2}^{-}(1)} c_w T_w^{M,*}\), with \(c_w \in \mathbb{Z}\), be an element of \(H_{M^-}\). Its image by \(\theta^*\) is \(\sum_{w \in W(1)} c_w T_w^*\) where we have set \(c_w = 0\) for \(w \in W(1) \setminus W_{M^-}(1)\). We have \(\sum_{w \in W(1)} c_w T_w^* \in J\) if and only if \(\sum_{w \in W_{M_2}^{-}(1)} c_w T_w^{-1} = 0\) for all \(w \in W(1)\). If \(cw_2 \neq 0\) then \(w_2 \in 1W_{M_2}^1 \cap W_M(1)\), that is, \(w_2 \in 1W_{M_2}^1 \cap \Lambda(1)\). The sum \(\sum_{w \in W_{M_2}^{-}(1)} c_w T_w\) is equal to \(\sum_{w \in W_{M_2}^{-}(1)} c_w \lambda_2\). By Proposition 3.4 \(\sum_{w \in W(1)} c_w T_w^* \in J\) if and only if \(\sum_{w \in W_{M_2}^{-}(1)} c_w T_w^{M,*} \in J_M\).

We construct a ring isomorphism

\[
e^*: H_M / J_M \cong H / J
\]

by using Proposition 3.3. For any \(w \in W(1)\), \(T_w^* + J = e^*(T_{w,M}^{M,*} + J_M)\) where \(w_{M^-} \in W_{M^-}(1) \cap w_{1W_{M_2}^1}\) (see (3.4)), because by Proposition 3.3 \(T_w^* + J = T_{w,M}^{M,*} + J\) and \(T_{w,M}^{M,*} + J = e^*(T_{w,M}^{M,*} + J_M)\) by construction of \(e^*\). We check that \(e^*\) is induced by \(\theta^*\):

**Theorem 3.6.** The linear map \(H_M \xrightarrow{\theta^*} H\) induces a ring isomorphism

\[
e^*: H_M / J_M \cong H / J
\]

**Proof.** Let \(w \in W_{M}(1)\). We have to show that \(T_w^* + J = e^*(T_{w,M}^{M,*} + J_M)\). We saw above that \(T_w^* + J = e^*(T_{w,M}^{M,*} + J_M)\) with \(w = w_{M^-} \lambda_2\) with \(\lambda_2 \in 1W_{M_2}^1 \cap W_M(1)\). As \(\ell_M(\lambda_2) = 0\), \(T_{w,M}^{M,*} = T_{w,M^-}^{M,*} \lambda_2 \in T_{w,M^-}^{M,*} + J_M\). Therefore \(T_{w,M^-}^{M,*} + J_M = T_{w,M}^{M,*} + J_M\), this ends the proof of the theorem. \(\square\)

We wish now to compute \(e^*\) in terms of the \(T_w^*\) instead of the \(T_w^*\).

**Proposition 3.7.** Let \(w \in W(1)\). Then, \(T_w^* + J = e^*(T_{w,M}^{M,*}q_{M_2}(w) + J_M)\), for any \(w_M \in W_M(1) \cap w_{1W_{M_2}^1}\).

**Proof.** The element \(w_M\) is unique modulo right multiplication by an element \(\lambda_2 \in W_{M}(1) \cap 1W_{M_2}^1\) of length \(\ell_M(\lambda_2) = 0\) and \(T_{w,M}^{M,*}q_{M_2}(w) + J_M\) does not depend on the choice of \(w_M\). We choose a decomposition (see (3.4)):

\[
w = \tilde{s}_1 \ldots \tilde{s}_a u \tilde{s}_{a+1} \ldots \tilde{s}_{a+b}, \quad \ell(w) = a + b,
\]

for \(u \in \Omega(1)\), \(\tilde{s}_i \in 1W_{M}\) lifting \(s_i \in S_{M}^{aff}\) for \(1 \leq i \leq a\) and \(\tilde{s}_i \in 1W_{M_2}^1\) lifting \(s_i \in S_{M_2}^{aff}\) for \(a+1 \leq i \leq a + b\), and we choose \(u_M \in W_M(1)\) such that \(u \in u_M \cdot 1W_{M_2}^1\). Then

\[
w_M = \tilde{s}_1 \ldots \tilde{s}_a u_M \in W_M(1) \cap w_{1W_{M_2}^1}
\]

and \(q_{M_2}(w) = q_{M_2}(\tilde{s}_a \ldots \tilde{s}_{a+b})\) (Lemma 3.3). We check first the proposition in three simple cases:

Case 1. Let \(w = \tilde{s} \in 1W_{M}\) lifting \(s \in S_{M}^{aff}\); we have \(T_{\tilde{s}} + J = e^*(T_{\tilde{s}}^M + J_M)\) because \(T_{\tilde{s}} - e^*(T_{\tilde{s}}^M, s) \in J\), \(T_{\tilde{s}} = T_{\tilde{s}}^* + c_{\tilde{s}}\), \(T_{\tilde{s}}^M = T_{\tilde{s}}^M, s + c_{\tilde{s}}\) and \(1 = q_{M_2}(\tilde{s})\).

Case 2. Let \(w = u \in W(1)\) of length \(\ell(u) = 0\) and \(u_M \in W_M(1)\) such that \(u \in u_M \cdot 1W_{M_2}^1\). We have \(\ell_M(u_M) = 0\) and \(q_{M_2}(u) = 1\) (Lemma 3.3). We deduce \(T_u + J = e^*(T_u^M + J_M)\) because \(T_u^* + J = T_{u,M}^* + J = e^*(T_{u,M}^M + J_M)\), and \(T_u = T_u^* T_{u,M} = T_{u,M}^M\).
Case 3. Let \( w = \tilde{s} \in 1WM' \) lifting \( s \in \mathcal{S}_{M}^{\mathfrak{c}} \); we have \( T_{\tilde{s}} + \mathcal{J} = e^{*}(q_{M}(\tilde{s}) + \mathcal{J}) \) because \( T_{\tilde{s}} - 1, c_{\tilde{s}} - (q_{\tilde{s}} - 1) \in \mathcal{J}, T_{\tilde{s}} = T_{\tilde{s}}^{*} + c_{\tilde{s}} \in q_{\tilde{s}} + \mathcal{J} \) and \( q_{\tilde{s}} = q_{M}(\tilde{s}) \).

In general, the braid relations \( T_{w} = T_{s_{1}} \cdots T_{s_{n}} T_{w_{a+1}} \cdots T_{s_{n}+b} \) give a similar product decomposition of \( T_{w} + \mathcal{J} \), and the simple cases 1, 2, 3 imply that \( T_{w} + \mathcal{J} \) is equal to
\[
e^{*}(T_{s_{1}}^{M} + \mathcal{J}) \cdots e^{*}(T_{s_{n}}^{M} + \mathcal{J})e^{*}(T_{w_{a+1}}^{M} + \mathcal{J})e^{*}(q_{M}(\tilde{s}_{a+1}) + \mathcal{J}) \cdots e^{*}(q_{M}(\tilde{s}_{a+b}) + \mathcal{J})
= e^{*}(T_{w}^{M} q_{M}(w) + \mathcal{J}).
\]

The proposition is proved. \( \square \)

Propositions 3.4, 3.5, and Theorem 3.6 are valid over any commutative ring \( \mathbb{R} \) (instead of \( \mathbb{Z} \)).

The two-sided ideal of \( \mathcal{H}_{R} \) generated by \( T_{w}^{*} - 1 \) for all \( w \in 1WM' \) is \( \mathcal{J}_{R} = \mathcal{J} \otimes_{\mathbb{Z}} \mathcal{R} \), the two-sided ideal of \( \mathcal{H}_{M,R} \) generated by \( T_{w}^{*} - 1 \) for all \( \lambda \in 1WM' \cap \Lambda(1) \) is \( \mathcal{J}_{M,R} = \mathcal{J}_{M} \otimes_{\mathbb{Z}} \mathcal{R} \), and we get as in Proposition 3.5 isomorphisms
\[
\mathcal{H}_{M,R}/\mathcal{J}_{M,R} \xrightarrow{\sim} \mathcal{H}_{M,-R}/(\mathcal{J}_{M,R} \cap \mathcal{H}_{M,-R}) \xrightarrow{\sim} \mathcal{H}_{R}/\mathcal{J}_{R},
\]
giving an isomorphism \( \mathcal{H}_{M,R}/\mathcal{J}_{M,R} \rightarrow \mathcal{H}_{R}/\mathcal{J}_{R} \) induced by \( \theta^{*} \). Therefore, we have an isomorphism from the category of right \( \mathcal{H}_{M,R} \)-modules where \( \mathcal{J}_{M} \) acts by 0 onto the category of right \( \mathcal{H}_{R} \)-modules where \( \mathcal{J} \) acts by 0.

**Definition 3.8.** A right \( \mathcal{H}_{M,R} \)-module \( V \) where \( \mathcal{J}_{M} \) acts by 0 is called extensible to \( \mathcal{H} \). The corresponding \( \mathcal{H}_{R} \)-module where \( \mathcal{J} \) acts by 0 is called its extension to \( \mathcal{H} \) and denoted by \( e_{\mathcal{H}}(V) \) or \( e(V) \).

With the element basis \( T_{w}^{*} \), \( V \) is extensible to \( \mathcal{H} \) if and only if
\[
vT_{\lambda_{2}}^{M} = v \text{ for all } v \in V \text{ and } \lambda_{2} \in 1WM' \cap \Lambda(1).
\]
The \( \mathcal{H} \)-module structure on the \( \mathcal{R} \)-module \( e(V) = V \) is determined by
\[
vT_{w_{1}}^{*} = v, \quad vT_{w}^{*} = vT_{w}^{M*}, \quad \text{for all } v \in V, w_{2} \in 1WM', w \in WM(1).
\]
It is also determined by the action of \( T_{w}^{*} \) for \( w \in 1WM' \cup WM(1) \) (or \( w \in 1WM' \cup WM(1) \)). Conversely, a right \( \mathcal{H} \)-module \( W \) over \( \mathcal{R} \) is extended from an \( \mathcal{H}_{M} \)-module if and only if
\[
vT_{w_{1}}^{*} = v, \quad \text{for all } v \in V, w_{2} \in 1WM'.
\]
In terms of the basis elements \( T_{w} \) instead of \( T_{w}^{*} \), this says:

**Corollary 3.9.** A right \( \mathcal{H}_{M} \)-module \( V \) over \( \mathcal{R} \) is extensible to \( \mathcal{H} \) if and only if
\[
vT_{\lambda_{2}}^{M} = v \text{ for all } v \in V \text{ and } \lambda_{2} \in 1WM' \cap \Lambda(1).
\]
Then, the structure of \( \mathcal{H} \)-module on the \( \mathcal{R} \)-module \( e(V) = V \) is determined by
\[
vT_{w_{1}} = vw_{w_{2}}, \quad vT_{w} = vT_{w}^{M} q_{M}(w), \quad \text{for all } v \in V, w_{2} \in 1WM', w \in WM(1).
\]
(\( WM(1) \) or \( WM(1) \) instead of \( WM(1) \) is enough.) A right \( \mathcal{H} \)-module \( W \) over \( \mathcal{R} \) is extended from an \( \mathcal{H}_{M} \)-module if and only if
\[
vT_{w_{1}} = vw_{w_{2}}, \quad \text{for all } v \in V, w_{2} \in 1WM'.
\]
3.4. **σ^{H_M} is extensible to H of extension** \( e(σ^{H_M}) = e(σ)^{H} \). Let \( P = MN \) be a standard parabolic subgroup of \( G \) such that \( Δ_P \) and \( Δ \setminus Δ_P \) are orthogonal, and \( σ \) a smooth \( R \)-representation of \( M \) extensible to \( G \). Let \( P_2 = M_2N_2 \) denote the standard parabolic subgroup of \( G \) with \( Δ_{P_2} = Δ \setminus Δ_P \).

Recall that \( G = M_2M_2' \), that \( M \cap M_2 = Z \cap M_2' \) acts trivially on \( σ \), \( e(σ) \) is the representation of \( G \) equal to \( σ \) on \( M \) and trivial on \( M_2' \). We will describe the \( H \)-module \( e(σ)^{H_M} \) in this section. We first consider \( e(σ) \) as a subrepresentation of \( \text{Ind}_P^G σ \). For \( v ∈ σ \), let \( f_v ∈ (\text{Ind}_P^G σ)^{M_2'} \) be the unique function with value \( v \) on \( M_2' \). Then, the map

\[
v ↦ f_v : σ → \text{Ind}_P^G σ
\]

(3.16)

is the natural \( G \)-equivariant embedding of \( e(σ) \) in \( \text{Ind}_P^G σ \). As \( σ^{H_M} = e(σ)^{H} \) as \( R \)-modules, the image of \( e(σ)^{H} \) in \( (\text{Ind}_P^G σ)^{H_M} \) is made out of the \( f_v \) for \( v ∈ σ^{H_M} \).

We now recall the explicit description of \( (\text{Ind}_P^G σ)^{H_M} \). For each \( d ∈ \mathbb{W}_{M_2} \), we fix a lift \( ˆd ∈ 1W_{M_2} \) and for \( v ∈ σ^{H_M} \) let \( f_{Pd, ˆd, v} ∈ (\text{Ind}_P^G σ)^{H_M} \) for the function with support contained in \( Pd \) and value \( v \) on \( ˆd \). As \( Z \cap M_2' \) acts trivially on \( σ \), the function \( f_{Pd, ˆd, v} \) does not depend on the choice of the lift \( ˆd ∈ 1W_{M_2} \) of \( d \). By [OVI7], Lemma 4.5):

The map \( ⊕_{d ∈ \mathbb{W}_{M_2}} σ^{H_M} → (\text{Ind}_P^G σ)^{H_M} \) given on each \( d \)-component by \( v ↦ f_{Pd, ˆd, v} \), is an \( \mathcal{H}_{M^+} \)-equivariant isomorphism where \( \mathcal{H}_{M^+} \) is seen as a subring of \( \mathcal{H} \) via \( θ \), and induces an \( \mathcal{H}_R \)-module isomorphism

\[
v ⊗ h ↦ f_{Pd, ˆd, v} : σ^{H_M} ⊗ _{\mathcal{H}_{M^+}, θ} \mathcal{H} → (\text{Ind}_P^G σ)^{H_M}.
\]

(3.17)

In particular for \( v ∈ σ^{H_M} \), \( v ⊗ T( ˆd) \) does not depend on the choice of the lift \( ˆd ∈ 1W_{M_2} \) of \( d \) and

\[
f_{Pd, ˆd, v} = f_{P\hat{d}, v}T(\hat{d}).
\]

(3.18)

As \( G \) is the disjoint union of \( Pd \hat{d} \) for \( d ∈ \mathbb{W}_{M_2} \), we have \( f_v = ∑_{d ∈ \mathbb{W}_{M_2}} f_{Pd, ˆd, v} \) and \( f_v \) is the image of \( v ⊗ e_{M_2} \) in (3.17), where

\[
e_{M_2} = ∑_{d ∈ \mathbb{W}_{M_2}} T(\hat{d}).
\]

(3.19)

Recalling (3.16) we get:

**Lemma 3.10.** The map \( v ↦ v ⊗ e_{M_2} : e(σ)^{H} → σ^{H_M} ⊗ _{\mathcal{H}_{M^+}, θ} \mathcal{H} \) is an \( \mathcal{H}_R \)-equivariant embedding.

**Remark 3.11.** The trivial map \( v ↦ v ⊗ 1_\mathcal{H} \) is not an \( \mathcal{H}_R \)-equivariant embedding.

We describe the action of \( T(n) \) on \( e(σ)^{H} \) for \( n ∈ N \). By definition for \( v ∈ e(σ)^{H} \),

\[
vT(n) = ∑_{y ∈ H \setminus Hn^{-1}H(n)} y^{-1}v.
\]

(3.20)

**Proposition 3.12.** We have \( vT(n) = vT^M(n_M)q_{M_2}(n) \) for any \( n_N ∈ N \setminus M \) is such that \( n = n_M(N \cap M_2') \).
Proposition 3.15. The description (3.9) of $U/(U \cap n^{-1}Un)$ gives
\[
vT(n) = \sum_{y_1 \in U_M/(U_M \cap n^{-1}Un)} y_1 \sum_{y_2 \in U_M/(U_M \cap n^{-1}Un)} y_2 n^{-1}v.
\]

As $M_\sigma'$ acts trivially on $e(\sigma)$, we obtain
\[
vT(n) = g_{M_\sigma'}(n) \sum_{y_1 \in U_M/(U_M \cap n^{-1}Un)} y_1 n^{-1}_M y = g_{M_\sigma'}(n) v T^M(n_M).
\]

Proof. The description (3.9) of $U/(U \cap n^{-1}Un)$ gives
\[
vT(n) = \sum_{y_1 \in U_M/(U_M \cap n^{-1}Un)} y_1 \sum_{y_2 \in U_M/(U_M \cap n^{-1}Un)} y_2 n^{-1}v.
\]

As $M_\sigma'$ acts trivially on $e(\sigma)$, we obtain
\[
vT(n) = g_{M_\sigma'}(n) \sum_{y_1 \in U_M/(U_M \cap n^{-1}Un)} y_1 n^{-1}_M y = g_{M_\sigma'}(n) v T^M(n_M).
\]

Theorem 3.13. Let $\sigma$ be a smooth $R$-representation of $M$. If $P(\sigma) = G$, then $\sigma^{U_M}$ is extensible to $\mathcal{H}$ of extension $e(\sigma^{U_M}) = e(\sigma)^\mathcal{H}$. Conversely, if $\sigma^{U_M}$ is extensible to $\mathcal{H}$ and generates $\sigma$, then $P(\sigma) = G$.

Proof. (1) The $\mathcal{H}_M$-module $\sigma^{U_M}$ is extensible to $\mathcal{H}$ if and only if $Z \cap M_2'$ acts trivially on $\sigma^{U_M}$. Indeed, for $v \in \sigma^{U_M}$, $z_2 \in Z \cap M_2'$,
\[
vT^M(z_2) = \sum_{y \in U_M/(U_M \cap z_2^{-1}Un)} y z_2^{-1}v = \sum_{y \in U_M/(U_M \cap z_2^{-1}Un)} y z_2^{-1}v = z_2^{-1}v,
\]
by (3.20), then (3.9), then the fact that $z_2^{-1}$ commutes with the elements of $\mathcal{Y}_M$.

(2) $P(\sigma) = G$ if and only if $Z \cap M_2'$ acts trivially on $\sigma$ (the group $Z \cap M_2'$ is generated by $Z \cap M_2'$, for $\alpha \in \Delta_{M_2}$ by Lemma 2.1). The $R$-submodule $\sigma^{Z \cap M_2'}$ of elements fixed by $Z \cap M_2'$ is stable by $M$, because $M = ZM'$, the elements of $M'$ commute with those of $Z \cap M_2'$ and $Z$ normalizes $Z \cap M_2'$.

(3) Apply (1) and (2) to get the theorem except the equality $e(\sigma^{U_M}) = e(\sigma)^\mathcal{H}$ when $P(\sigma) = G$ which follows from Propositions 3.12 and 3.7.

Let $1_M$ denote the trivial representation of $M$ over $R$ (or 1 when there is no ambiguity on $M$). The right $\mathcal{H}_R$-module $(1_G)^\mathcal{H} = 1_\mathcal{H}$ (or 1 if there is no ambiguity) is the trivial right $\mathcal{H}_R$-module: for $w \in W_M(1)$, $T_w = q_w id$ and $T_w^* = id$ on $1_M$.

Example 3.14. The $\mathcal{H}$-module $(\text{Ind}_G^R 1)^\mathcal{H}$ is the extension of the $\mathcal{H}_{M_2}$-module $(\text{Ind}_{M_2}^{M_2 \cap B} 1)^\mathcal{H}_{M_2}$. Indeed, the representation $\text{Ind}_G^R 1$ of $G$ is trivial on $N_2$, as $G = MM_2$ and $N_2 \subset M'$ (as $\Phi = \Phi_M \cup \Phi_{M_2}$). For $g = mn_2$ with $m \in M, m_2' \in M_2$ and $n_2 \in N_2$, we have $Pgn_2 = Pm'g = Pn_2m_2' = Pm_2' = Pg$. The group $M_2 \cap B = M_2 \cap P$ is the standard minimal parabolic subgroup of $M_2$ and $(\text{Ind}_G^R 1)|_{M_2} = \text{Ind}_{M_2}^{M_2 \cap B} 1$. Apply Theorem 3.13.

3.5. The $\mathcal{H}_R$-module $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G 1)^\mathcal{H}$. Let $P = MN$ be a standard parabolic subgroup of $G$ such that $\Delta_P$ and $\Delta \setminus \Delta_P$ are orthogonal, let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module which is extensible to $\mathcal{H}_R$ of extension $e(\mathcal{V})$ and let $Q$ be a parabolic subgroup of $G$ containing $P$.

We define on the $R$-module $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G 1)^\mathcal{H}$ a structure of right $\mathcal{H}_R$-module:

Proposition 3.15. (1) The diagonal action of $T_w^*$ for $w \in W(1)$ on $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G 1)^\mathcal{H}$ defines a structure of right $\mathcal{H}_R$-module. (2) The action of the $T_w$ is also diagonal and satisfies:
\[
(v \otimes f)T_w = (vT_{uwM'} \otimes ft_{uwM_2'}, vT_{uwM'}^* \otimes ft_{uwM_2'}^*),
\]
where $w = uwM'_wM_2'$ with $u \in W(1), \ell(u) = 0, wM' \in 1W_M, wM_2' \in 1W_{M_2}$.
Proof. If the lemma is true for $P$ it is also true for $Q$, because the $R$-module $e(V) \otimes_R (\text{Ind}_R^G 1)$$^H$ naturally embedded in $e(V) \otimes_R (\text{Ind}_P^G 1)$$^H$ is stable by the action of $H$ defined in the lemma. So, we suppose $Q = P$.

Suppose that $T^*_w$ for $w \in W(1)$ acts on $e(V) \otimes_R (\text{Ind}_P^G 1)$$^H$ as in (1). The braid relations obviously hold. The quadratic relations hold because $T^*_s$ with $s \in 1S^\text{aff}$ acts trivially either on $e(V)$ or on $(\text{Ind}_P^G 1)$$^H$. Indeed, $1S^\text{aff} = 1S^\text{aff}_M \cup 1S^\text{aff}_{M_2}$, $T^*_s$ for $s \in 1S^\text{aff}_M$, acts trivially on $(\text{Ind}_P^G 1)$$^H$ which is extended from a $H_{M_2}$-module (Example 3.14), and $T^*_s$ for $s \in 1S^\text{aff}_{M_2}$, acts trivially on $e(V)$ which is extended from a $H_{M_2}$-module. This proves (1).

We describe now the action of $T^*_w$ instead of $T^*_w$ on the $H$-module $e(V) \otimes_R (\text{Ind}_Q^G 1)$$^H$. Let $w \in W(1)$. We write $w = uw_Mw_{M_2} = uw_{M_2}w_M$ with $u \in W(1), \ell(u) = 0, w_M \in 1W_M, w_{M_2} \in 1W_{M_2}$. We have $\ell(w) = \ell(w_M) + \ell(w_{M_2})$ hence $T^*_w = T^*_uT^*_wT^*_u$.

For $w = u$, we have $T^*_u = T^*_u$ and $(v \otimes f)T^*_u = (v \otimes f)T^*_u = vT^*_w \otimes fT^*_w = vT^*_u \otimes fT^*_w$.

For $w = w_M$, $(v \otimes f)T^*_w = vT^*_w \otimes f$; in particular for $s \in 1S^\text{aff}_M$, $c_s = \sum_{t \in Z(s)_{1M}} c_t(t)T^*_t$, we have $(v \otimes f)T^*_s = (v \otimes f)(T^*_s + c_s) = v(T^*_s + c_s) \otimes f = vT^*_w \otimes f$. Hence $(v \otimes f)T^*_w = vT^*_w \otimes f$.

For $w = w_{M_2}$, we have similarly $(v \otimes f)T^*_w = v \otimes fT^*_w$ and $(v \otimes f)T^*_w = v \otimes fT^*_w$. □

Example 3.16. Let $X$ be a right $H_R$-module. Then $1_H \otimes_R X$ where the $T^*_w$ acts diagonally is a $H_R$-module isomorphic to $X$. But the action of the $T^*_w$ on $1_H \otimes_R X$ is not diagonal.

It is known [LY15] that $(\text{Ind}_Q^G 1)$$^H$ and $(\text{St}_Q^G)$$^H$ are free $R$-modules and that $(\text{St}_Q^G)$$^H$ is the cokernel of the natural $H_R$-map

$$\oplus_{Q \subset Q'}(\text{Ind}_Q^G 1)$$^H \to (\text{Ind}_Q^G 1)$$^H$$

although the invariant functor $(-)^H$ is only left exact.

Corollary 3.17. The diagonal action of $T^*_w$ for $w \in W(1)$ on $e(V) \otimes_R (\text{St}_Q^G)$$^H$ defines a structure of right $H_R$-module satisfying Proposition [LY15] (2).

4. Hecke module $I_H(P, V, Q)$

4.1. Case $V$ extensible to $H$. Let $P = MN$ be a standard parabolic subgroup of $G$ such that $\Delta_P$ and $\Delta \setminus \Delta_P$ are orthogonal, $V$ a right $H_{M,R}$-module extensible to $H_{R}$ of extension $e(V)$, and $Q$ be a parabolic subgroup of $G$ containing $P$. As $Q$ and $M_Q$ determine each other: $Q = M_QU$, we denote also $H_{M,Q} = H_Q$ and $H_{M,Q,R} = H_{Q,R}$ when $Q \neq P, G$. When $Q = G$ we drop $G$ and we denote $e_H(V) = e(V)$ when $Q = G$.

Lemma 4.1. $V$ is extensible to an $H_{Q,R}$-module $e_{H_Q}(V)$.

Proof. This is straightforward. By Corollary [LY15] $V$ extensible to $H$ means that $T^{M,s}(z)$ acts trivially on $V$ for all $z \in N_{M_2'} \cap Z$. We have $M_Q = MM'_{2,Q}$ with $M'_{2,Q} \subseteq M_Q \cap M'$ and $N_{M_2',Q} \subseteq N_{M_2'}$; hence $T^{M,s}(z)$ acts trivially on $V$ for all $z \in N_{M_2',Q} \cap Z$ meaning that $V$ is extensible to $H_Q$.

Remark 4.2. We cannot say that $e_{H_Q}(V)$ is extensible to $H$ of extension $e(V)$ when the set of roots $\Delta_Q$ and $\Delta \setminus \Delta_Q$ are not orthogonal (Definition 3.8).

Let $Q'$ be an arbitrary parabolic subgroup of $G$ containing $Q$. We are going to define a $H_R$-embedding $\text{Ind}_{H_{Q'}}^H(e_{H_{Q'}}(V)) \rightarrow \text{Ind}_{H_{Q}}^H(e_{H_{Q}}(V)) = e_{H_{Q}}(V) \otimes_{H_{M_Q}} H$ defining
where each summand is isomorphic to $V^3$. In the extreme case $(Q, Q') = (P, G)$, the $\mathcal{H}_R$-embedding $e(V) \otimes_R (\text{St}_Q^G)^d$. In the following lemma where $f_G$ and $f_{PU} \in (\text{Ind}_P^G 1)^d$ of $PU$ denote the characteristic functions of $G$ and $PU$, $f_G = f_{PU}e_{M_2}$ (see (3.19)).

**Lemma 4.3.** There is a natural $\mathcal{H}_R$-isomorphism

$$v \otimes 1_H \mapsto v \otimes f_{PU} : \text{Ind}_{H_M}^H (V) = V \otimes_{H_{M+}} H \xrightarrow{\kappa_P} e(V) \otimes_R (\text{Ind}_P^G 1)^d,$$

and compatible $\mathcal{H}_R$-embeddings

$$v \mapsto v \otimes f_G : e(V) \mapsto e(V) \otimes_R (\text{Ind}_P^G 1)^d,$$

$$v \mapsto v \otimes e_{M_2} : e(V) \xrightarrow{i(P,G)} \text{Ind}_{H_M}^H (V).$$

**Proof.** We show first that the map

$$(4.1) \quad v \mapsto v \otimes f_{PU} : V \mapsto e(V) \otimes_R (\text{Ind}_P^G 1)^d$$

is $H_{M+}$-equivariant. Let $w \in W_{M+}(1)$. We write $w = uw_{M'}w_{M_2}'$ as in Lemma 3.15 (2), so that $f_{PU}T_w = f_{PU}T_{uw_{M_2}'}$. We have $f_{PU}T_{uw_{M_2}'} = f_{PU}$ because $1W_{M'} \subset W_{M+}(1) \cap W_{M-}(1)$ hence $uw_{M_2}' = uw_{M_2}' \in W_{M+}(1)$ and in $H_M \otimes_{H_{M+}} H$ we have $(1 \otimes 1_H)T_{uw_{M_2}'} = 1T_{uw_{M_2}'} \otimes 1_H$, and $T_{uw_{M_2}'}$ acts trivially in $1_{H_M}$ because $t_M(uw_{M_2}') = 0$. We deduce $(v \otimes f_{PU})T_w = vT_w \otimes f_{PU}T_w = vT_w \otimes f_{PU}.$

By adjunction (4.1) gives an $\mathcal{H}_R$-equivariant linear map

$$(4.2) \quad v \otimes 1_H \mapsto v \otimes f_{PU} : V \otimes_{H_{M+}} H \xrightarrow{\kappa_P} e(V) \otimes_R (\text{Ind}_P^G 1)^d.$$

We prove that $\kappa_P$ is an isomorphism. Recalling $\tilde{d} \in N \cap M_2', \tilde{d} \in W_{M_2}'$ lift $d$, one knows that

$$(4.3) \quad V \otimes_{H_{M+}} H = \oplus_{d \in W_{M_2}} V \otimes_{T_{\tilde{d}}} (\text{Ind}_P^G 1)^d = \oplus_{d \in W_{M_2}} V \otimes f_{PU},$$

where each summand is isomorphic to $V$. The left equality follows from §4.1 and Remark 3.7 in [Vig15b] recalling that $v \in \overline{W}_{M_2}$ is of minimal length in its coset $\overline{W}_{M_2}v = \overline{W}_{M_2}$ as $\Delta_M$ and $\Delta_{M_2}$ are orthogonal; for the second equality see §3.4 [Vig15b]. We have $\kappa_P(v \otimes T_{\tilde{d}}) = (v \otimes f_{PU})T_{\tilde{d}} = v \otimes f_{PU}T_{\tilde{d}}$ (Lemma 3.15). Hence $\kappa_P$ is an isomorphism.

We consider the composite map

$$v \mapsto v \otimes 1 \mapsto v \otimes f_{PU}e_{M_2} : e(V) \mapsto e(V) \otimes R 1_H \mapsto e(V) \otimes_R (\text{Ind}_P^G 1)^d,$$

where the right map is the tensor product $e(V) \otimes_R -$ of the $\mathcal{H}_R$-equivariant embedding $1_H \rightarrow (\text{Ind}_P^G 1)^d$ sending $1_R$ to $f_{PU}e_{M_2}$ (Lemma 3.10); this map is injective because $(\text{Ind}_P^G 1)^d/1$ is a free $R$-module; it is $\mathcal{H}_R$-equivariant for the diagonal action of the $T_w$ on the tensor products (Example 3.16 for the first map). By compatibility with (1), we get the $\mathcal{H}_R$-equivariant embedding $v \mapsto v \otimes e_{M_2} : e(V) \xrightarrow{i(P,G)} \text{Ind}_{H_M}^H (V).$ \hfill $\square$

For a general $(Q, Q')$ the $\mathcal{H}_R$-embedding $\text{Ind}_{H_{Q'+}}^H (e_{H_{Q'}}(V)) \xrightarrow{i(Q,Q')} \text{Ind}_{H_{Q'}}^H (e_{H_{Q'}}(V))$ is given in the next proposition generalizing Lemma 4.3. The element $e_{M_2}$ of $\mathcal{H}_R$ appearing in the
definition of \( \iota(P, G') \) is replaced in the definition of \( \iota(Q, Q') \) by an element \( \theta_Q'(e_Q') \in \mathcal{H}_R \) that we define first.

Until the end of \( \text{[4.4]} \), we fix an admissible lift \( w \mapsto \hat{w} : \mathbb{W} \to \mathcal{N} \cap \mathcal{K} \) (Definition \( \text{[3.1]} \)) and \( \hat{w} \) denotes the image of \( \hat{w} \) in \( W(1) \). We denote \( \mathbb{W}_{M_Q} = \mathbb{W}_Q \) and by \( \mathbb{W}_Q \mathbb{W} \) the set of \( w \in \mathbb{W} \) of minimal length in their coset \( \mathbb{W}_Q w \). The group \( G \) is the disjoint union of \( Q \mathbb{d} U \) for \( d \) running through \( \mathbb{W}_Q \mathbb{W} \) \( [OV17, \text{Lemma 2.18 (2)}] \).

(4.4) \[
Q' \mathcal{U} = \bigsqcup_{d \in \mathbb{W}_Q \mathbb{W}_Q'} Q \mathbb{d} U,
\]

Set

(4.5) \[
e_Q' = \sum_{d \in \mathbb{W}_Q \mathbb{W}_Q'} T_d^{M_Q}.
\]

We write \( e_Q = e_Q' \). We have \( e_P' = \sum_{d \in \mathbb{W}_Q \mathbb{W}_Q'} T_d^{M_Q} \).

Remark 4.4. Note that \( \mathbb{W}_M \mathbb{W} = \mathbb{W}_{M_2} \) and \( e_P = e_{M_2} \), where \( M_2 \) is the standard Levi subgroup of \( G \) with \( \Delta_M = \Delta \setminus \Delta_M \), as \( \Delta_M \) and \( \Delta \setminus \Delta_M \) are orthogonal. More generally, \( \mathbb{W}_Q \mathbb{W}_{M_Q} = \mathbb{W}_{M_2} \mathbb{W}_{M_2} \mathbb{W}_{M_2} \) where \( M_{2Q} = M_2 \cap M_{Q'} \).

Note that \( e_Q' \in \mathcal{H}_{M^+} \cap \mathcal{H}_{M^-} \). We consider the linear map

\[
\theta_Q' : \mathcal{H}_Q \to \mathcal{H}_Q', \quad T_w^{M_Q} \mapsto T_w^{M_Q'} \quad (w \in \mathbb{W}_{M_Q}(1)).
\]

We write \( \theta_Q' = \theta_Q \) so \( \theta_Q(T_w^{M_Q}) = T_w \). When \( Q = P \) this is the map \( \theta \) defined earlier. Similarly we denote by \( \theta_{Q'}^{Q''} \) the linear map sending the \( T_w^{M_{Q''}} \) to \( T_w^{M_{Q''}'} \) and \( \theta_{Q'}^{Q''} = \theta_{Q''} \). We have

(4.6) \[
\theta_Q'(e_Q') = \sum_{d \in \mathbb{W}_Q \mathbb{W}_{Q'}} T_d, \quad \theta_{Q'}(e_{Q'}) = \theta_Q(e_P') \theta_Q'(e_{Q'})
\]

Proposition 4.5. There exists an \( \mathcal{H}_R \)-isomorphism

(4.7) \[
v \otimes 1_H \mapsto v \otimes f_{Q'} : \text{Ind}_{H_Q}(e_{H_Q}(V)) = e_{H_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+}, \iota} \mathcal{H} \xrightarrow{\mathcal{H}} e(\mathcal{V}) \otimes_R (\text{Ind}_{Q'} \mathcal{1})\mathcal{U},
\]

and compatible \( \mathcal{H}_R \)-embeddings

(4.8) \[
v \otimes f_{Q'} \mathcal{U} \mapsto v \otimes f_{Q'} \mathcal{U} : e_{H_{Q'}}(\mathcal{V}) \otimes_R (\text{Ind}_{Q'} \mathcal{1})\mathcal{U} \to e_{H_{Q'}}(\mathcal{V}) \otimes_R (\text{Ind}_{Q'} \mathcal{1})\mathcal{U},
\]

(4.9) \[
v \otimes 1_H \mapsto v \otimes \theta_Q(e_{Q'}) : \text{Ind}_{H_{Q'}}(e_{H_{Q'}}(V)) \xrightarrow{\iota_{Q', Q}'} \text{Ind}_{H_Q}(e_{H_Q}(V)).
\]

Proof. We have the \( \mathcal{H}_{M_Q, R} \)-embedding

(4.10) \[
v \mapsto v \otimes e_{Q'} : e_{H_Q}(V) \to \mathcal{V} \otimes_{\mathcal{H}_{M^+}, \iota} \mathcal{H} = \text{Ind}_{H_M}^H(V)
\]

by Lemma \( \text{[4.3]} \) (2) as \( \Delta_M \) is orthogonal to \( \Delta_{M_Q} \setminus \Delta_M \). Applying the parabolic induction which is exact, we get the \( \mathcal{H} \)-embedding

\[
v \otimes 1_H \mapsto v \otimes e_{Q'} \otimes 1_H : \text{Ind}_{H_Q}(e_{H_Q}(V)) \to \text{Ind}_{H_M(H)}^H(\text{Ind}_{H_M}^H(V)).
\]

Note that \( T_d^{M_Q} \in \mathcal{H}_{M_Q^+} \) for \( d \in \mathbb{W}_{M_Q} \). By transitivity of the parabolic induction, it is equal to the \( \mathcal{H}_R \)-embedding

(4.10) \[
v \otimes 1_H \mapsto v \otimes \theta_Q(e_{Q'}) : \text{Ind}_{H_Q}(e_{H_Q}(V)) \to \text{Ind}_{H_{M}}^H(V).
\]
On the other hand we have the $\mathcal{H}_R$-embedding

\[(4.11) \quad v \otimes f_{QU} \mapsto v \otimes \theta_Q(e_P^Q) : e(V) \otimes_R (\text{Ind}_Q^G 1)^H \rightarrow \text{Ind}_{\mathcal{H}_M}^G (V)\]

given by the restriction to $e(V) \otimes_R (\text{Ind}_Q^G 1)^H$ of the $\mathcal{H}_R$-isomorphism given in Lemma 4.3 (1), from $e(V) \otimes_R (\text{Ind}_Q^G 1)^H$ to $V \otimes_{\mathcal{H}_M} \theta H$ sending $v \otimes f_{PH}$ to $v \otimes 1_H$, noting that $v \otimes f_{QU} = (v \otimes f_{PH}) \theta_Q(e_P^Q)$ by Lemma 4.15. $f_{QU} = f_{PH} \theta_Q(e_P^Q)$ and $\theta_Q(e_P^Q)$ acts trivially on $e(V)$ (this is true for $T_d$ for $d \in 1W_M$). Comparing the embeddings (4.10) and (4.11), we get the $\mathcal{H}_R$-isomorphism (4.7).

We can replace $Q$ by $Q'$ in the $\mathcal{H}_R$-homomorphisms (4.7), (4.10) and (4.11). With (4.10) we see $\text{Ind}_{\mathcal{H}_Q}^G (e_{H Q'}(V))$ and $\text{Ind}_{\mathcal{H}_Q}^G (e_{H Q}(V))$ as $\mathcal{H}_R$-submodules of $\text{Ind}_{\mathcal{H}_M}^G (V)$. As seen in (4.6) we have $\theta_Q'(e_{P}^Q) = \theta_Q(e_{P}^Q) \theta_Q'(e_{P}^Q)$. We deduce the $\mathcal{H}_R$-embedding (4.9).

By (3.13) for $Q$ and (4.3),

$$f_{QU} = \sum_{d \in \omega Q} f_{QU} T_d = f_{QU} \theta_Q'(e_{P}^Q)$$

in $\text{c-Ind}_Q^G 1)^H$. We deduce that the $\mathcal{H}_R$-embedding corresponding to (4.9) via $\kappa_Q$ and $\kappa_Q'$ is the $\mathcal{H}_R$-embedding (4.8).

We recall that $\Delta_P$ and $\Delta \setminus \Delta_P$ are orthogonal and that $V$ is extensible to $\mathcal{H}$ of extension $e(V)$.

**Corollary 4.6.** The cokernel of the $\mathcal{H}_R$-map

$$\oplus_{Q \leq Q' \subset G} \text{Ind}_{\mathcal{H}_Q^G}^G (e_{\mathcal{H}_Q^G H Q'}(V)) \rightarrow \text{Ind}_{\mathcal{H}_M}^G (e_{\mathcal{H}_M}(V))$$

defined by the $i(Q, Q')$, is isomorphic to $e(V) \otimes_R (\text{St}_Q^G 1)^H$ via $\kappa_Q$.

**4.2. Invariants in the tensor product.** We return to the setting where $P = MN$ is a standard parabolic subgroup of $G$, $\sigma$ is a smooth $R$-representation of $M$ with $P(\sigma) = G$ of extension $e(\sigma)$ to $G$, and $Q$ a parabolic subgroup of $G$ containing $P$. We still assume that $\Delta_P$ and $\Delta \setminus \Delta_P$ are orthogonal.

The $\mathcal{H}_R$-modules $e(\sigma)^H$ are equal (Theorem 3.13). We compute $I_G(P, \sigma, Q)^H = (e(\sigma) \otimes_R \text{St}_Q^G 1)^H$.

**Theorem 4.7.** The natural linear maps $e(\sigma)^H \otimes_R (\text{Ind}_Q^G 1)^H \rightarrow (e(\sigma) \otimes_R \text{Ind}_Q^G 1)^H$ and $e(\sigma)^H \otimes_R (\text{St}_Q^G 1)^H \rightarrow (e(\sigma) \otimes_R \text{St}_Q^G 1)^H$ are isomorphisms.

**Proof.** We need some preliminaries. In [GK14], [Ly15], is introduced a finite free $\mathbb{Z}$-module $\mathfrak{M}$ (depending on $\Delta_Q$) and a $\mathcal{B}$-equivariant embedding $\text{St}_Q^G \mathbb{Z} \hookrightarrow C_c^\infty(\mathcal{B}, \mathfrak{M})$ (we indicate the coefficient ring in the Steinberg representation) which induces an isomorphism $\text{St}_Q^G \mathbb{Z}^\mathcal{B} \simeq C_c^\infty(\mathcal{B}, \mathfrak{M})^\mathcal{B}$.

**Lemma 4.8.** (1) $(\text{Ind}_Q^G \mathbb{Z})^\mathcal{B}$ is a direct factor of $\text{Ind}_Q^G \mathbb{Z}$.

(2) $(\text{St}_Q^G \mathbb{Z})^\mathcal{B}$ is a direct factor of $\text{St}_Q^G \mathbb{Z}$.

**Proof.** (1) [AHV17] Example 2.2.

(2) As $\mathfrak{M}$ is a free $\mathbb{Z}$-module, $C_c^\infty(\mathcal{B}, \mathfrak{M})^\mathcal{B}$ is a direct factor of $C_c^\infty(\mathcal{B}, \mathfrak{M})$. Consequently, $\iota((\text{St}_Q^G \mathbb{Z})^\mathcal{B}) = C_c^\infty(\mathcal{B}, \mathfrak{M})^\mathcal{B}$ is a direct factor of $\iota(\text{St}_Q^G \mathbb{Z})$. As $\iota$ is injective, we get (2).
We prove now Theorem 4.7. We may and do assume that $\sigma$ is $e$-minimal (because $P(\sigma) = P(\sigma_{\text{min}}), e(\sigma) = e(\sigma_{\text{min}})$) so that $\Delta_M$ and $\Delta \setminus \Delta_M$ are orthogonal and we use the same notation as in [3,2] in particular $M_2 = M_{\Delta \setminus \Delta_M}$. Let V be the space of $e(\sigma)$ on which $M'_2$ acts trivially. The restriction of $\text{Ind}_{Q}^{G} Z$ to $M_2$ is $\text{Ind}_{Q}^{M_2} Z$, that of $\text{St}_{Q}^{G} Z$ is $\text{St}_{Q}^{M_2} Z$.

As in [AHV17, Example 2.2], $(\text{Ind}_{Q}^{M_2} Z \otimes V)^{\text{U}M_2} \simeq (\text{Ind}_{Q}^{M_2} Z)^{\text{U}M_2} \otimes V$. We have $(\text{Ind}_{Q}^{M_2} Z)^{\text{U}M_2} = (\text{Ind}_{Q}^{M_2} Z)^{\text{U}M_2} = (\text{Ind}_{Q}^{G} Z)^{\text{U}M_2}$. The first equality follows from $M_2 = (Q \cap M_2) Q, U_{M_2} = Z^1 U_{M_2}$ and $Z^1$ normalizes $U_{M_2}$ and is normalized by $Q M_2$. The second equality follows from $\mathcal{U} = \mathcal{U}_{M_2}$ and $\text{Ind}_{Q}^{G} Z$ is trivial on $M'$. Therefore $((\text{Ind}_{Q}^{G} Z) \otimes V)^{\text{U}M_2} \simeq (\text{Ind}_{Q}^{G} Z)^{\text{U}M_2} \otimes V$. Taking now fixed points under $U_{M}$, as $\mathcal{U} = \mathcal{U}_{M_2} U_{M},$

$$((\text{Ind}_{Q}^{G} Z) \otimes V)^{\text{U}M_2} \simeq (\text{Ind}_{Q}^{G} Z)^{\text{U}M_2} \otimes V^{\text{U}M_2}$$

The equality uses that the $Z$-module $\text{Ind}_{Q}^{G} Z$ is free. We get the first part of the theorem as $(\text{Ind}_{Q}^{G} Z)^{\text{U}M_2} \otimes V^{\text{U}M_2} \simeq (\text{Ind}_{Q}^{G} R)^{\text{U}M_2} \otimes V^{\text{U}M_2}.$

Tensoring with $R$ the usual exact sequence defining $\text{St}_{Q}^{G} Z$ gives an isomorphism $\text{St}_{Q}^{G} Z \otimes R \simeq \text{St}_{Q}^{G} R$ and in loc. cit. it is proved that the resulting map $\text{St}_{Q}^{G} R \to C^\infty (\mathcal{B}, \mathcal{M} \otimes R)$ is also injective. Their proof in no way uses the ring structure of $R$, and for any $Z$-module $V$, tensoring with $V$ gives a $\mathcal{B}$-equivariant embedding $\text{St}_{Q}^{G} Z \otimes V \to C^\infty (\mathcal{B}, \mathcal{M} \otimes V)$. The natural map $\text{St}_{Q}^{G} R \otimes V \to \text{St}_{Q}^{G} Z \otimes V$ is also injective by Lemma 4.5 (2). Taking $\mathcal{B}$-fixed points we get inclusions

$$(\text{St}_{Q}^{G} Z)^{\mathcal{B}} \otimes V \to (\text{St}_{Q}^{G} Z \otimes V)^{\mathcal{B}} \to C^\infty (\mathcal{B}, \mathcal{M} \otimes V)^{\mathcal{B}} \simeq \mathcal{M} \otimes V.$$

The composite map is surjective, so the inclusions are isomorphisms. The image of $\iota_V$ consists of functions which are left $Z^0$-invariant, and $\mathcal{B} = Z^0 \mathcal{U}'$ where $\mathcal{U}' = G' \cap \mathcal{U}$. It follows that $\iota$ yields an isomorphism $(\text{St}_{Q}^{G} Z)^{\mathcal{U}'} \simeq C^\infty (Z^0 \mathcal{B}, \mathcal{M} \otimes V)^{\mathcal{U}'}$ again consisting of the constant functions.

So that in particular $(\text{St}_{Q}^{G} Z)^{\mathcal{U}'} = (\text{St}_{Q}^{G} Z)^{\mathcal{B}}$ and reasoning as previously we get isomorphisms

$$(\text{St}_{Q}^{G} Z)^{\mathcal{U}'} \otimes V \simeq (\text{St}_{Q}^{G} Z \otimes V)^{\mathcal{U}'} \simeq \mathcal{M} \otimes V.$$

The equality $(\text{St}_{Q}^{G} Z)^{\mathcal{U}'} = (\text{St}_{Q}^{G} Z)^{\mathcal{B}}$ and the isomorphisms remain true when we replace $\mathcal{U}'$ by any group between $\mathcal{B}$ and $\mathcal{U}'$. We apply these results to $\text{St}_{Q}^{M_2} Z \otimes V$ to get that the natural map $(\text{St}_{Q}^{M_2} Z)^{\text{U}M_2} \otimes V \to (\text{St}_{Q}^{M_2} Z \otimes V)^{\text{U}M_2}$ is an isomorphism and also that $(\text{St}_{Q}^{M_2} Z)^{\text{U}M_2} \otimes V \to (\text{St}_{Q}^{M_2} Z \otimes V)^{\text{U}M_2}$. We have $\mathcal{U} = \mathcal{U}_{M_2} U_{M_2}$ so $(\text{St}_{Q}^{G} Z)^{\mathcal{U}'} = (\text{St}_{Q}^{M_2} Z)^{\text{U}M_2}$ and the natural map $(\text{St}_{Q}^{G} Z)^{\mathcal{U}'} \otimes V \to (\text{St}_{Q}^{G} Z \otimes V)^{\mathcal{U}'}$ is an isomorphism. The $Z$-module $(\text{St}_{Q}^{G} Z)^{\mathcal{U}'}$ is free and the $V^{\mathcal{U}'} = V^{\mathcal{U}}$, so taking fixed points under $U_{M_2}$, we get $(\text{St}_{Q}^{G} Z)^{\mathcal{U}'} \otimes V^{\mathcal{U}} \simeq (\text{St}_{Q}^{G} Z \otimes V)^{\mathcal{U}'}$. We have $\text{St}_{Q}^{G} Z \otimes V = \text{St}_{Q}^{G} R \otimes V$ and $(\text{St}_{Q}^{G} Z)^{\mathcal{U}'} \otimes V^{\mathcal{U}} = (\text{St}_{Q}^{G} R)^{\mathcal{U}'} \otimes V^{\mathcal{U}}$. This ends the proof of the theorem.

**Theorem 4.9.** The $\mathcal{H}_R$-modules $(e(\sigma) \otimes R) \text{Ind}_{Q}^{G} 1)^{\mathcal{U}} = e(\sigma)^{\mathcal{U}} \otimes R (\text{Ind}_{Q}^{G} 1)^{\mathcal{U}}$ are equal. The $\mathcal{H}_R$-modules $(e(\sigma) \otimes R) \text{St}_{Q}^{G} 1)^{\mathcal{U}} = e(\sigma)^{\mathcal{U}} \otimes R (\text{St}_{Q}^{G} 1)^{\mathcal{U}}$ are also equal.

**Proof.** We already know that the $R$-modules are equal (Theorem 4.7). We show that they are equal as $\mathcal{H}$-modules. The $\mathcal{H}_R$-modules $e(\sigma)^{\mathcal{U}} \otimes R (\text{Ind}_{Q}^{G} 1)^{\mathcal{U}} = e_{\mathcal{H}}(\sigma^{\mathcal{U}M})^{\mathcal{U}} \otimes R (\text{Ind}_{Q}^{G} 1)^{\mathcal{U}}$
are equal (Theorem 4.13); they are isomorphic to Ind$^H_{\mathcal{H}Q}(e_{\mathcal{H}Q}(\sigma^{\Delta M}))$ (Proposition 4.15), to (Ind$^G_Q(e_Q(\sigma)))^\mathcal{U}$ [OV17 Proposition 4.4] and to $(e(\sigma) \otimes_R \text{Ind}^G_Q 1)^\mathcal{U}$ [AHV17 Lemma 2.5]). We deduce that the $\mathcal{H}_R$-modules $e(\sigma)^\mathcal{U} \otimes_R (\text{Ind}^G_Q 1)^\mathcal{U} = (e(\sigma) \otimes_R \text{Ind}^G_Q 1)^\mathcal{U}$ are equal. The same is true when $Q$ is replaced by a parabolic subgroup $Q'$ of $G$ containing $Q$. The representation $e(\sigma) \otimes_R \text{St}^G_Q$ is the cokernel of the natural $R[G]$-map

$$\oplus_{Q \subseteq Q'} e(\sigma) \otimes_R \text{Ind}^G_Q 1 \xrightarrow{\alpha_Q} e(\sigma) \otimes_R \text{Ind}^G_Q 1$$

and the $\mathcal{H}_R$-module $e(\sigma)^\mathcal{U} \otimes_R (\text{St}^G_Q)^\mathcal{U}$ is the cokernel of the natural $\mathcal{H}_R$-map

$$\oplus_{Q \subseteq Q'} e(\sigma)^\mathcal{U} \otimes_R (\text{Ind}^G_Q 1)^\mathcal{U} \xrightarrow{\beta_Q} e(\sigma)^\mathcal{U} \otimes_R (\text{Ind}^G_Q 1)^\mathcal{U}$$

obtained by tensoring [3.21] by $e(\sigma)^\mathcal{U}$ over $R$, because the tensor product is right exact. The maps $\beta_Q = \alpha_Q^\mathcal{U}$ are equal and the $R$-modules $(e(\sigma)^\mathcal{U} \otimes_R (\text{St}^G_Q)^\mathcal{U}) = (e(\sigma) \otimes_R \text{St}^G_Q)^\mathcal{U}$ are equal. This implies that the $\mathcal{H}_R$-modules $(e(\sigma)^\mathcal{U} \otimes_R (\text{St}^G_Q)^\mathcal{U}) = (e(\sigma) \otimes_R \text{St}^G_Q)^\mathcal{U}$ are equal.

**Remark 4.10.** The proof shows that the representations $e(\sigma) \otimes_R \text{Ind}^G_Q 1$ and $e(\sigma) \otimes \text{St}^G_Q$ of $G$ are generated by their $\mathcal{U}$-fixed vectors if the representation $\sigma$ of $M$ is generated by its $\mathcal{U}M$-fixed vectors. Indeed, the $R$-modules $e(\sigma)^\mathcal{U} = \sigma^{\Delta M}$, $(\text{Ind}^G_Q 1)^\mathcal{U} = (\text{Ind}^G_Q 1)^\mathcal{U}$ are equal. If $\sigma^{\Delta M}$ generates $\sigma$, then $e(\sigma)$ is generated by $e(\sigma)^\mathcal{U}$. The representation $\text{Ind}^G_Q 1|_{M_2}$ is generated by $(\text{Ind}^G_{Q'} 1)^\mathcal{U}$ (this follows from the lemma below), we have $G = MM_2$ and $M_2$ acts trivially on $e(\sigma)$. Therefore the $R[G]$-module generated by $e(\sigma)^\mathcal{U} \otimes_R (\text{Ind}^G_Q 1)^\mathcal{U}$ is $e(\sigma) \otimes_R \text{Ind}^G_Q 1$. As $e(\sigma) \otimes_R \text{St}^G_Q$ is a quotient of $e(\sigma) \otimes_R \text{Ind}^G_Q 1$, the $R[G]$-module generated by $e(\sigma)^\mathcal{U} \otimes_R (\text{St}^G_Q)^\mathcal{U}$ is $e(\sigma) \otimes_R \text{St}^G_Q$.

**Lemma 4.11.** For any standard parabolic subgroup $P$ of $G$, the representation $\text{Ind}^G_P 1|_{G'}$ is generated by its $\mathcal{U}$-fixed vectors.

**Proof.** Because $G = PG'$ it suffices to prove that if $J$ is an open compact subgroup of $\overline{N}$ the characteristic function $1_{\mathcal{J}_J}^P$ of $PJ$ is a finite sum of translates of $1_{\mathcal{P} \mathcal{U} t} = \pi t$ by $G'$. For $t \in T$ we have $P \mathcal{U} t = P t^{-1} \mathcal{U} t$ and we can choose $t \in T \cap J'$ such that $t^{-1} \mathcal{U} t \subseteq J$. \hfill \Box

4.3. **General triples.** Let $P = MN$ be a standard parabolic subgroup of $G$. We now investigate situations where $\Delta_P$ and $\Delta \setminus \Delta_P$ are not necessarily orthogonal. Let $\mathcal{V}$ a right $\mathcal{H}_{M,R}$-module.

**Definition 4.12.** Let $P(\mathcal{V}) = M(\mathcal{V})N(\mathcal{V})$ be the standard parabolic subgroup of $G$ with $\Delta_{P(\mathcal{V})} = \Delta_P \cup \Delta_{\mathcal{V}}$ and

$$\Delta_{\mathcal{V}} = \{ \alpha \in \Delta \text{ orthogonal to } \Delta_M, T^{M,*}(z) \text{ acts trivially on } \mathcal{V} \text{ for all } z \in Z \cap M'_\alpha \}.$$  

If $Q$ is a parabolic subgroup of $G$ between $P$ and $P(\mathcal{V})$, the triple $(P, \mathcal{V}, Q)$ called an $\mathcal{H}_R$-triple, defines a right $\mathcal{H}_R$-module $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ equal to

$$\text{Ind}^\mathcal{H}_{\mathcal{H}(P, \mathcal{V}, Q)}(e(\mathcal{V}) \otimes_R (\text{St}^M_{Q(\mathcal{V})}) 1|_{M(\mathcal{V})}) = (e(\mathcal{V}) \otimes_R (\text{St}^M_{Q(\mathcal{V})}) 1|_{M(\mathcal{V})}) \otimes_{\mathcal{H}(P, \mathcal{V}, Q)} \mathcal{H}$$

where $e(\mathcal{V})$ is the extension of $\mathcal{V}$ to $\mathcal{H}(P, \mathcal{V}, Q)$.

This definition is justified by the fact that $M(\mathcal{V})$ is the maximal standard Levi subgroup of $G$ such that the $\mathcal{H}_{M,R}$-module $\mathcal{V}$ is extensible to $\mathcal{H}_{M(\mathcal{V})}$.
Lemma 4.13. $\Delta_V$ is the maximal subset of $\Delta \setminus \Delta_P$ orthogonal to $\Delta_P$ such that $T^{M,*}_\lambda$ acts trivially on $V$ for all $\lambda \in \Lambda(1) \cap_{1W_{M,\nu}}$.

Proof. For $J \subset \Delta$ let $M_J$ denote the standard Levi subgroup of $G$ with $\Delta_{M_J} = J$. The group $Z \cap M_J'$ is generated by the $Z \cap M'_\alpha$ for all $\alpha \in J$ (Lemma 2.1). When $J$ is orthogonal to $\Delta_M$ and $\lambda \in \Lambda_{M_J}(1)$, $\ell_M(\lambda) = 0$ where $\ell_M$ is the length associated to $S^\text{red}_M$, and the map $\lambda \mapsto T^{M,*}_\lambda = T^{M}_\lambda : \Lambda_{M_J}(1) \to \mathcal{H}_M$ is multiplicative. \hfill $\square$

The following is the natural generalisation of Proposition 4.5 and Corollary 4.6. Let $Q'$ be a parabolic subgroup of $G$ with $Q \subset Q' \subset P(V)$. Applying the results of 4.11 to $M(V)$ and its standard parabolic subgroups $Q \cap M(V) \subset Q' \cap M(V)$, we have an $\mathcal{H}_{M(V),R}$-isomorphism

$$\text{Ind}^{\mathcal{H}_{M(V)}}_{\mathcal{H}_Q}(e_{\mathcal{H}_Q}(V)) \xrightarrow{\kappa_Q} e_Q(V) \otimes_{\mathcal{H}_{M(Q)}} e(V) \otimes_R \text{Ind}^{M(V)}_{\mathcal{H}_{\mathcal{H}_{M(V)}}} 1^\mathcal{H}(V)$$

and an $\mathcal{H}_{M(V),R}$-embedding

$$\text{Ind}^{\mathcal{H}_{M(V)}}_{\mathcal{H}_{Q'}}(e_{\mathcal{H}_{Q'}}(V)) \xrightarrow{\iota(Q \cap M(V),Q' \cap M(V))} \text{Ind}^{\mathcal{H}_{M(V)}}_{\mathcal{H}_Q}(e_{\mathcal{H}_Q}(V))$$

Applying the parabolic induction $\text{Ind}^{\mathcal{H}_{M(V)}}_{\mathcal{H}_Q}$ which is exact and transitive, we obtain an $\mathcal{H}_R$-isomorphism $\kappa_Q = \text{Ind}^{\mathcal{H}_{M(V)}}_{\mathcal{H}_Q}(\kappa_{Q \cap M(V)})$,

$$\text{Ind}^{\mathcal{H}_{M(V)}}_{\mathcal{H}_Q}(e_{\mathcal{H}_Q}(V)) \xrightarrow{\kappa_Q} \text{Ind}^{\mathcal{H}_{M(V)}}_{\mathcal{H}_Q}(e_{\mathcal{H}_Q}(V)) \otimes_R \text{Ind}^{M(V)}_{\mathcal{H}_{\mathcal{H}_{M(V)}}} 1^\mathcal{H}(V)$$

and an $\mathcal{H}_R$-embedding $\iota(Q,Q') = \text{Ind}^{\mathcal{H}_{M(V)}}_{\mathcal{H}_{Q'}}(\iota(Q,Q')^M(V))$

Apply Corollary 4.6 we obtain:

Theorem 4.14. Let $(P,V,Q)$ be an $\mathcal{H}_R$-triple. Then, the cokernel of the $\mathcal{H}_R$-map

$$\bigoplus_{Q \subset Q' \subset P(V)} \text{Ind}^{\mathcal{H}_{Q'}}_{\mathcal{H}_Q}(e_{\mathcal{H}_Q}(V)) \to \text{Ind}^{\mathcal{H}_{M(V)}}_{\mathcal{H}_Q}(e_{\mathcal{H}_Q}(V))$$

defined by the $\iota(Q,Q')$ is isomorphic to $I_{\mathcal{H}}(P,V,Q)$ via the $\mathcal{H}_R$-isomorphism $\kappa_Q$.

Let $\sigma$ be a smooth $R$-representation of $M$ and $Q$ a parabolic subgroup of $G$ with $P \subset Q \subset P(\sigma)$.

Remark 4.15. The $\mathcal{H}_R$-module $I_{\mathcal{H}}(P,\sigma^{M},Q)$ is defined if $\Delta_Q \setminus \Delta_P$ and $\Delta_P$ are orthogonal because $Q \subset P(\sigma)$ \subset $P(\sigma^{M})$. (Theorem 3.3).

We denote here by $P_{\text{min}} = M_{\text{min}}N_{\text{min}}$ the minimal standard parabolic subgroup of $G$ contained in $P$ such that $\sigma = e_P(\sigma|_{M_{\text{min}}})$ (Lemma 2.3.2 we drop the index $\sigma$). The sets of roots $\Delta_{P_{\text{min}}}$ and $\Delta_{P_{\sigma|_{M_{\text{min}}}}} \setminus \Delta_{P_{\text{min}}}$ are orthogonal (Lemma 2.4). The groups $P(\sigma) = P(\sigma|_{M_{\text{min}}})$, the representations $e(\sigma) = e(\sigma|_{M_{\text{min}}})$ of $M(\sigma)$, the representations $I_G(P,\sigma,Q) = $
$I_G(P_{\text{min}}, \sigma|_M, Q) = \text{Ind}_G^P(e(\sigma) \otimes_R S^P_Q(\sigma))$ of $G$, and the $R$-modules $\sigma^M_{\text{min}} = \sigma^M$ are equal. From Theorem 3.13

$$P(\sigma) \subset P(\sigma^M_{\text{min}}), \quad e_{\mathcal{H}_{\text{min}}}(\sigma^M_{\text{min}}) = e(\sigma)^M,$$

and $P(\sigma^M_{\text{min}}) = P(\sigma)$ if $\sigma^M_{\text{min}}$ generates the representation $\sigma|_M$. The $H_R$-module

$$I_H(P_{\text{min}}, \sigma^M_{\text{min}}, Q) = \text{Ind}_{H_M}^H(e(\sigma^M_{\text{min}}) \otimes_R S^P_{Q,M}(\sigma)^M_{\text{min}})$$

is defined because $\Delta_{P_{\text{min}}}$ and $\Delta_{P(\sigma^M_{\text{min}})} \setminus \Delta_{P_{\text{min}}}$ are orthogonal and $P \subset Q \subset P(\sigma) \subset P(\sigma^M_{\text{min}})$.

**Remark 4.16.** If $\sigma^M_{\text{min}}$ generates the representation $\sigma|_M$ (in particular if $R = C$ and $\sigma$ is irreducible), then $P(\sigma) = P(\sigma^M_{\text{min}})$ hence

$$I_H(P_{\text{min}}, \sigma^M_{\text{min}}, Q) = \text{Ind}_{H_M}^H(e_{\mathcal{H}_{\text{min}}}(\sigma^M_{\text{min}}) \otimes_R S^P_{Q,M}(\sigma)^M_{\text{min}}).$$

Applying Theorem 4.13 to $(P_{\text{min}} \cap M(\sigma), \sigma|_M, Q \cap M(\sigma))$, the $H_{M(\sigma),R}$-modules

$$(4.16) \quad e_{\mathcal{H}_{\text{min}}}(\sigma^M_{\text{min}}) \otimes_R S^M_{Q,M}(\sigma)^M_{\text{min}} = (e_{\mathcal{H}_{\text{min}}}(\sigma^M_{\text{min}}) \otimes_R S^M_{Q,M}(\sigma)^M_{\text{min}})$$

are equal. We have the $H_R$-isomorphism [OV17 Proposition 4.4]:

$$I_G(P, \sigma, Q)^H \cong \left( \text{Ind}_P^G(e(\sigma) \otimes_R S^P_Q(\sigma)) \right)^H \cong \text{Ind}_{H_M}^H \left( (e(\sigma) \otimes_R S^M_{Q,M}(\sigma)^M_{\text{min}}) \right).$$

We deduce:

**Theorem 4.17.** Let $(P, \sigma, Q)$ be a $R[G]$-triple. Then, we have the $H_R$-isomorphism

$$I_G(P, \sigma, Q)^H \cong \text{Ind}_{H_M}^H \left( e_{H_{\text{min}}}(\sigma^M_{\text{min}}) \otimes_R S^M_{Q,M}(\sigma)^M_{\text{min}} \right).$$

In particular,

$$I_G(P, \sigma, Q)^H \cong \begin{cases} I_H(P_{\text{min}}, \sigma^M_{\text{min}}, Q) & \text{if } P(\sigma) = P(\sigma^M_{\text{min}}) \\ I_H(P, \sigma^M_{\text{min}}, Q) & \text{if } P = P_{\text{min}}, P(\sigma) = P(\sigma^M_{\text{min}}). \end{cases}$$

4.4. **Comparison of the parabolic induction and coinduction.** Let $P = MN$ be a standard parabolic subgroup of $G$, $\mathcal{V}$ a right $H_R$-module and $Q$ a parabolic subgroup of $G$ with $Q \subset P(\mathcal{V})$. When $R = C$, in [Abe], we associated to $(P, \mathcal{V}, Q)$ an $H_R$-module using the parabolic coinduction

$${\text{Coind}}_{H_M}^H(-) = \text{Hom}_{H_{M^+}(-)}(\mathcal{H}, -) : \text{Mod}_R(H_M) \to \text{Mod}_R(\mathcal{H})$$

instead of the parabolic induction $\text{Ind}_{H_M}^H(-) = - \otimes_{H_{M^+}, \theta} H$. The index $\theta^*$ in the parabolic coinduction means that $H_{M_{\mathcal{Q}}}$ embeds in $\mathcal{H}$ by $\theta^*_Q$. Our terminology is different from the one in [Abe] where the parabolic coinduction is called induction. For a parabolic subgroup $Q'$ of $G$ with $Q \subset Q' \subset P(\mathcal{V})$, there is a natural inclusion of $H_R$-modules [Abe Proposition 4.19]

$$(4.17) \quad \text{Hom}_{H_{M_{\mathcal{Q}'}, \theta^*}}(\mathcal{H}, e_{H_{Q'}}(\mathcal{V})) \longrightarrow \text{Hom}_{H_M, \theta^*}(\mathcal{H}, e_{H_{Q}}(\mathcal{V})).$$

because $\theta^*(H_{M_{\mathcal{Q}'}}) < \theta^*(H_{M_{\mathcal{Q}}})$ as $W_{M_{\mathcal{Q}'}^w}(1) \subset W_{M_{\mathcal{Q}}}(1)$, and $vT_w^{M_{\mathcal{Q}'}*} = vT_w^{M_{\mathcal{Q}}*}$ for $w \in W_{M_{\mathcal{Q}}}(1)$ and $v \in \mathcal{V}$. 
Lemma 4.20. Let $CI_{H}(P, V, Q)$ denote the cokernel of the map
\[ \oplus_{Q \subset P(V)} \text{Hom}_{H_{M_{Q}, Q}}(H, e_{H_{Q}}(V)) \rightarrow \text{Hom}_{H_{M_{Q}, Q}}(H, e_{H_{Q}}(V)) \]
defined by the $H_{R}$-embeddings $i(Q, Q')$.

When $R = C$, we showed that the $H_{C}$-module $CI_{H}(P, V, Q)$ is simple when $V$ is simple and supersingular (Definition 4.25), and that any simple $H_{C}$-module is of this form for a $H_{C}$-triple $(P, V, Q)$ where $V$ is simple and supersingular, $P, Q$ and the isomorphism class of $V$ are unique [Abe]. The aim of this section is to compare the $H_{R}$-modules $I_{H}(P, V, Q)$ with the $H_{R}$-modules $CI_{H}(P, V, Q)$ and to show that the classification is also valid with the $H_{C}$-modules $I_{H}(P, V, Q)$.

It is already known that a parabolically coinduced module is a parabolically induced module and vice versa [Abe, Vig15b]. To make it more precise we need to introduce notations.

We lift the elements $w$ of the finite Weyl group $W$ to $\tilde{w} \in NG \cap K$ as in [AHIV17, IV.6], [OV17] Proposition 2.7: they satisfy the braid relations $\tilde{w}_{1}\tilde{w}_{2} = (w_{1}w_{2})$ when $\ell(w_{1}) + \ell(w_{2}) = \ell(w_{1}w_{2})$ and when $s \in S$, $\tilde{s}$ is admissible, in particular lies in $1W_{G'}$.

Let $w, w_{M}, w_{M}^{M}$ denote respectively the longest elements in $W, W_{M}$ and $wW_{M}$. We have $w = w^{-1} = w^{M}w_{M}, w_{M} = w_{M}^{-1}, \tilde{w} = w^{M}w_{M}$.

\[ w^{M}(\Delta_{M}) = -w(\Delta_{M}) \subset \Delta, \quad w^{M}(\Phi^{+} \setminus \Phi^{+}_{M}) = w(\Phi^{+} \setminus \Phi^{+}_{M}). \]

Let $w, M$ be the standard Levi subgroup of $G$ with $\Delta_{w, M} = w^{M}(\Delta_{M})$ and $w, P$ the standard parabolic subgroup of $G$ with Levi $w, M$. We have
\[ w_{M} = w^{M}M(w^{M})^{-1} = \tilde{w}^{M}M^{-1}, \quad w^{W, M} = w_{M}w = (w^{M})^{-1}. \]

The conjugation $w \mapsto w^{M}w(w^{M})^{-1}$ in $W$ gives a group isomorphism $W_{M} \rightarrow W_{w, M}$ sending $S_{w, M}$ onto $S_{w, M}$, respecting the finite Weyl subgroups $w_{w, M}(w^{M})^{-1} = \tilde{w}_{w, M}M = w_{w, M}w^{-1}$, and exchanging $W_{w, M}$ and $W_{w, M}^{-1} = w_{w, M}w^{-1}$. The conjugation by $w^{M}$ restricts to a group isomorphism $W_{M}(1) \rightarrow W_{w, M}(1)$ sending $W_{M}(1)$ onto $W_{w, M}(1)$. The linear isomorphism
\[ (4.18) \quad H_{M} \xrightarrow{i(w^{M})} H_{w, M} \quad T_{w}^{M} \rightarrow T_{w}^{w, M}(w^{M})^{-1} \text{ for } w \in W_{M}(1), \]
is a ring isomorphism between the pro-$p$-Iwahori Hecke rings of $M$ and $w, M$. It sends the positive part $H_{M}^{+}$ of $H_{M}$ onto the negative part $H_{w, M}^{-}$ of $H_{w, M}$ [Vig15b, Proposition 2.20]. We have $\tilde{w} = w_{M}w^{M}w, \tilde{w}^{M}w_{M} = \tilde{w}^{M}_{M}w_{M}, (w^{M})^{-1} = w_{w, M}^{M}t_{M}$ where $t_{M} = \tilde{w}_{M}^{2}w_{M}^{2} \in \mathbb{Z}_{k}$.

Definition 4.19. The twist $\tilde{w}^{M}, V$ of $V$ by $\tilde{w}^{M}$ is the right $H_{w, M}$-module deduced from the right $H_{M}$-module $V$ by functoriality: as $R$-modules $\tilde{w}^{M}, V = V$ and for $v \in V, w \in W_{M}(1)$ we have $vT_{w}^{w, M} = vT_{w}^{w, M}(w^{M})^{-1} = vT_{w}^{w, M}t_{M}$.

We can define the twist $\tilde{w}^{M}, V$ of $V$ with the $T_{w}^{M, s}$ instead of $T_{w}^{M}$.

Lemma 4.20. For $v \in V, w \in W_{M}(1)$ we have $vT_{w}^{w, M, s} = vT_{w}^{w, M, s}$ in $\tilde{w}^{M}, V$.

Proof. By the ring isomorphism $H_{M} \xrightarrow{i(w^{M})} H_{w, M}$, we have $c_{w}^{w, M}(w^{M})^{-1} = c_{w}^{w, M}w^{M}w_{M}^{-1} = c_{w}^{w, M}$ when $\tilde{s} \in W_{M}(1)$ lifts $s \in S_{w, M}^{\text{aff}}$. So the equality of the lemma is true for $w = \tilde{s}$. Apply the braid relations to get the equality for all $w \in W_{M}(1)$. \qed
We return to the $\mathcal{H}_R$-module $\text{Hom}_R(e^\ast(\mathcal{H}, V))$ parabolically coinduced from $V$. It has a natural direct decomposition indexed by the set $\mathbb{W}_{W,M}$ of elements $d$ in the finite Weyl group $W$ of minimal length in the coset $d\mathbb{W}_M$. Indeed it is known that the linear map

$$f \mapsto (f(T_d))_{d \in \mathbb{W}_{W,M}} : \text{Hom}_{\mathcal{H}_M,e^\ast(\mathcal{H}, V)} \to \bigoplus_{d \in \mathbb{W}_{W,M}} V$$

is an isomorphism. For $v \in V$ and $d \in \mathbb{W}_{W,M}$, there is a unique element

$$f_{d,v} \in \text{Hom}_{\mathcal{H}_M,e^\ast(\mathcal{H}, V)}$$

satisfying $f(T_d) = v$ and $f(T_{d'}) = 0$ for $d' \in \mathbb{W}_{W,M} \setminus \{d\}$.

It is known that the map $v \mapsto f_{\tilde{w}^Mv,T} : \tilde{w}^M \to \text{Hom}_{\mathcal{H}_M,e^\ast(\mathcal{H}, V)}$ is $\mathcal{H}_{(w,M)}$-equivariant: $f_{\tilde{w}^Mv,T_{\tilde{w}^Mw}} = f_{\tilde{w}^Mv,T_w}$ for all $v \in V$, $w \in W_{w,M}$. By adjunction, this $\mathcal{H}_{(w,M)}$-equivariant map gives an $\mathcal{H}_R$-homomorphism from an induced module to a coinduced module:

$$v \otimes 1_H \mapsto f_{\tilde{w}^Mv} : \tilde{w}^M \otimes_{\mathcal{H}_{(w,M)}+\mathcal{H}} \mathcal{H} \to \text{Hom}_{\mathcal{H}_M,e^\ast(\mathcal{H}, V)}$$

This is an isomorphism [Abe, Vig15b].

The naive guess that a variant $\mu_Q$ of $\mu_P$ induces an $\mathcal{H}_R$-isomorphism between the $\mathcal{H}_R$-modules $I_H(w,P,\tilde{w}^M \cdot V, w.Q)$ and $CI_H(P,V,Q)$ turns out to be true. The proof is the aim of the rest of this section.

The $\mathcal{H}_R$-module $I_H(w,P,\tilde{w}^M \cdot V, w.Q)$ is well defined because the parabolic subgroups of $G$ containing $w.P$ and contained in $P(w.M, V)$ are $w.Q$ for $P \subset Q \subset P(V)$, as follows from:

**Lemma 4.21.** $\Delta_{\tilde{w}^M \cdot V} = -w(\Delta_V)$.

**Proof.** Recall that $\Delta_V$ is the set of simple roots $\alpha \in \Delta \setminus \Delta_M$ orthogonal to $\Delta_M$ and $T^{M,\ast}(z)$ acts trivially on $V$ for all $z \in Z \cap M'_{\alpha}$, and the corresponding standard parabolic subgroup $P_{\alpha} = M_{\alpha}N_{\alpha}$. The $Z \cap M'_{\alpha}$ for $\alpha \in \Delta_V$ generate the group $Z \cap M'_{\Delta_V}$. A root $\alpha \in \Delta \setminus \Delta_M$ orthogonal to $\Delta_M$ is fixed by $w_M$ so $w_M(\alpha) = w(\alpha)$ and

$$\tilde{w}^M_M \cdot V = \tilde{w}^M \cdot V \otimes_{\mathcal{H}_{(w,M)}+\mathcal{H}} \mathcal{H} \to \text{Hom}_{\mathcal{H}_M,e^\ast(\mathcal{H}, V)}$$

The proof of Lemma 4.21 is straightforward as $\Delta = -w(\Delta)$, $\Delta_{w,M} = -w(\Delta_M)$. □

Before going further, we check the commutativity of the extension with the twist. As $Q = M_QU$ and $M_Q$ determine each other we denote $w_{M_Q} = w_Q, w_{MQ} = w_Q$ when $Q \neq P, G$.

**Lemma 4.22.** $e_{\mathcal{H}_w \cdot Q}(\tilde{w}^M \cdot V) = \tilde{w}^M \cdot e_{\mathcal{H}_Q}(V)$.

**Proof.** As $R$-modules $V = e_{\mathcal{H}_w \cdot Q}(\tilde{w}^M \cdot V) = \tilde{w}^M \cdot e_{\mathcal{H}_Q}(V)$. A direct computation shows that the Hecke element $T_{\tilde{w}^M}^{w.Q,\ast}$ acts in the $\mathcal{H}_R$-module $e_{\mathcal{H}_w \cdot Q}(\tilde{w}^M \cdot V)$, by the identity if $w \in \tilde{w}^M \cdot w_{MMQ}Q^{-1}$ and by $T_{\tilde{w}^M}^{M,\ast}(w_{MMQ}Q^{-1})$ for $w \in \tilde{w}^M \cdot w_{MMQ}Q^{-1}$ where $M_Q$ denotes the standard Levi subgroup with $\Delta_{M_Q} = \Delta_Q \setminus \Delta_P$. Whereas in the $\mathcal{H}_R$-module $\tilde{w}^M \cdot e_{\mathcal{H}_Q}(V)$, the Hecke element $T_{\tilde{w}^M}^{w.Q,\ast}$ acts by the identity if $w \in 1W_{w,MQ}$ and by $T_{\tilde{w}^M}^{M,\ast}(w_{MMQ}Q^{-1})$ if $w \in W_{w,MQ}(1)$. So the lemma means that

$$1W_{w,MQ} = \tilde{w}^M \cdot w_{MQQ}Q^{-1}, \quad (\tilde{w}^M)^{-1} w_{MQQ}Q = (\tilde{w}^M)^{-1} w_{MQQ}M \if w \in W_{w,MQ}(1)$$

These properties are easily proved using that $w_{GQ}$ is normal in $W(1)$ and that the sets of roots $\Delta_P$ and $\Delta_Q \setminus \Delta_P$ are orthogonal: $w_Q = w_{MQ}w_M$, the elements $w_{MQ}$ and $w_M$ normalise $W_M$ and $W_{MQ}$, the elements of $\mathbb{W}_{MQ}$ commutes with the elements of $\mathbb{W}_M$. □
We return to our guess. The variant \( \mu_Q \) of \( \mu_P \) is obtained by combining the commutativity of the extension with the twist and the isomorphism \([4.19]\) applied to \((Q, e_{H_Q}(V))\) instead of \((P, V)\). The \( H_R \)-isomorphism \( \mu_Q \) is:

\[
(4.20) \quad v \otimes 1_H \mapsto f_{\bar{w}^M, v} : \text{Ind}^H_{H_{w,M}}(e_{H_{w,Q}}(\bar{w}^M, V)) \xrightarrow{\mu_Q} \text{Hom}_{H_{w,Q}}(H, e_{H_Q}(V)).
\]

Our guess is that \( \mu_Q \) induces an \( H_R \)-isomorphism from the cokernel of the \( H_R \)-map

\[
\oplus_{Q \subseteq Q' \subset P(V)} \text{Ind}^H_{H_{w, Q'}}(e_{H_{w, Q'}}(\bar{w}^M, V)) \rightarrow \text{Ind}^H_{H_{w, Q}}(e_{H_{w, Q}}(\bar{w}^M, V))
\]
defined by the \( H_R \)-embeddings \( i(w, Q, w, Q') \), isomorphic to \( I_H(w, P, \bar{w}^M V, w, \overline{Q}) \) via \( \kappa_{w, Q} \) (Theorem 4.14), onto the cokernel \( CI_H(P, V, Q) \) the \( H_R \)-map

\[
\oplus_{Q \subseteq Q' \subset P(V)} \text{Hom}_{H_{w, Q'}}(H, e_{H_Q}(V)) \rightarrow \text{Hom}_{H_{w, Q}}(H, e_{H_Q}(V))
\]
defined by the \( H_R \)-embeddings \( i(Q, Q') \). This is true if \( i(Q, Q') \) corresponds to \( i(w, Q, w, Q') \) via the isomorphisms \( \mu_{Q'} \) and \( \mu_Q \). This is the content of the next proposition.

**Proposition 4.23.** For all \( Q \subseteq Q' \subset P(V) \) we have

\[
i(Q, Q') \circ \mu_{Q'} = \mu_Q \circ i(w, Q, w, Q').
\]

We postpone to section 4.6 the rather long proof of the proposition.

**Corollary 4.24.** The \( H_R \)-isomorphism \( \mu_Q \circ \kappa_{w, Q}^{-1} \) induces an \( H_R \)-isomorphism

\[
I_H(w, P, \bar{w}^M V, w, \overline{Q}) \rightarrow CI_H(P, V, Q).
\]

4.5. **Supersingular \( H_R \)-modules, classification of simple \( H_C \)-modules.** We recall first the notion of supersingularity based on the action of center of \( H \).

The center of \( H \) \([\text{Vig14}]\) Theorem 1.3 contains a subalgebra \( Z_{T^+} \) isomorphic to \( \mathbb{Z}[T^+/T_1] \) where \( T^+ \) is the monoid of dominant elements of \( T \) and \( T_1 \) is the pro-\( p \)-Sylow subgroup of the maximal compact subgroup of \( T \).

Let \( t \in T \) of image \( \mu_t \in W(1) \) and let \((E_o(w))_{w \in W(1)}\) denote the alcove walk basis of \( H \) associated to a closed Weyl chamber \( o \) of \( W \). The element

\[
E_o(C(\mu_t)) = \sum_{\mu} E_o(\mu')
\]
is the sum over the elements in \( \mu' \) in the conjugacy class \( C(\mu_t) \) of \( \mu_t \) in \( W(1) \). It is a central element of \( H \) and does not depend on the choice of \( o \). We write also \( z(t) = E_o(C(\mu_t)) \).

**Definition 4.25.** A non-zero right \( H_R \)-module \( V \) is called supersingular when, for any \( v \in V \) and any non-invertible \( t \in T^+ \), there exists a positive integer \( n \in \mathbb{N} \) such that \( v(z(t))^n = 0 \). If one can choose \( n \) independent on \( (v, t) \), then \( V \) is called uniformly supersingular.

**Remark 4.26.** One can choose \( n \) independent on \( (v, t) \) when \( V \) is finitely generated as a right \( H_R \)-module. If \( R \) is a field and \( V \) is simple we can take \( n = 1 \).

When \( G \) is compact modulo the center, \( T^+ = T \), and any non-zero \( H_R \)-module is supersingular.

The induction functor \( \text{Ind}^H_{H_M} : \text{Mod}(H_{M,R}) \to \text{Mod}(H_R) \) has a left adjoint \( \mathcal{L}^H_{H_M} \) and a right adjoint \( \mathcal{R}^H_{H_M} \) \([\text{Vig15b}]\): for \( V \in \text{Mod}(H_R) \),

\[
(4.21) \quad \mathcal{L}^H_{H_M}(V) = \bar{w}^M \circ (V \otimes_{H(w, M)} \otimes^\theta H_{w, M}) \quad \text{and} \quad \mathcal{R}^H_{H_M}(V) = \text{Hom}_{H_{w, M}}(\theta(H_M, V)).
\]
In the left adjoint, $\mathcal{V}$ is seen as a right $\mathcal{H}_{(w,M)^-}$-module via the ring homomorphism $\theta_{w,M}^*: \mathcal{H}_{(w,M)^-} \rightarrow \mathcal{H}$; in the right adjoint, $\mathcal{V}$ is seen as a right $\mathcal{H}_{M^+}$-module via the ring homomorphism $\theta_M: \mathcal{H}_{M^+} \rightarrow \mathcal{H}$.

**Proposition 4.27.** Assume that $\mathcal{V}$ is a supersingular right $\mathcal{H}_R$-module and that $p$ is nilpotent in $\mathcal{V}$. Then $L_{t}^{H}_{\mathcal{H}_{M}}(\mathcal{V}) = 0$, and if $\mathcal{V}$ is uniformly supersingular $R_{t}^{H}_{\mathcal{H}_{M}}(\mathcal{V}) = 0$.

**Proof.** This is a consequence of three known properties:

1. $\mathcal{H}_{M}$ is the localisation of $\mathcal{H}_{M^+}$ (resp. $\mathcal{H}_{M^+}$) at $T^M_{\mu}$ for any element $\mu \in \Lambda_T(1)$, central in $W_M(1)$ and strictly $N$-positive (resp. $N$-negative), and $T^M_{\mu} = T^{M,*}_{\mu}$. See [Vig15b, Theorem 1.4].

2. When $o$ is anti-dominant, $E_{o}(\mu) = T_{\mu}$ if $\mu \in \Lambda^+(1)$ and $E_{o}(\mu) = T^{*}_{\mu}$ if $\mu \in \Lambda^-(1)$.

3. Let an integer $n > 0$ and $\mu \in \Lambda(1)$ such that the $\mathcal{W}$-orbit of $v(\mu) \in X_*(T) \otimes \mathbb{Q}$ (Definition in [21]) and of $\mu$ have the same number of elements. Then

$$(E_{o}(C(\mu)))^n E_{o}(\mu) - E_{o}(\mu)^{n+1} \in p\mathcal{H}.$$ 

See [Vig15a, Lemma 6.5], where the hypotheses are given in the proof (but not written in the lemma).

Let $\mu \in \Lambda^+_{\mathcal{T}}(1)$ satisfying (1) for $M^+$ and (3), similarly let $w.\mu \in \Lambda^-_{\mathcal{T}}(1)$ satisfying (1) for $(w.M)^-$ and (3). For $(R, \mathcal{V})$ as in the proposition, let $v \in \mathcal{V}$ and $n > 0$ such that $vE_{o}(C(\mu))^n = vE_{o}(C(w.\mu))^n = 0$. Multiplying by $E_{o}(\mu)$ or $E_{o}(w.\mu)$, and applying (3) and (2) for $o$ anti-dominant we get:

$$vE_{o}(\mu)^{n+1} = vT^{n+1}_{\mu} \in p\mathcal{V}, \quad vE_{o}(w.\mu)^{n+1} = v(T^{*}_{w.\mu})^{n+1} \in p\mathcal{V}.$$ 

The proposition follows from: $vT^{n+1}_{\mu}, v(T^{*}_{w.\mu})^{n+1}$ in $p\mathcal{V}$ (as explained in [Abe16, Proposition 5.17] when $p = 0$ in $R$). From $v(T^{*}_{w.\mu})^{n+1}$ in $p\mathcal{V}$, we get $v \otimes (T^{*}_{w.\mu})^{n+1} = v(T^{*}_{w.\mu})^{n+1} = 1_{\mathcal{H}_{w.M}}$ in $p\mathcal{V} \otimes H_{(w,M)^-} \theta_{\mathcal{T}} H_{w.M}$. As $T^{*}_{w.M} = T^{M}_{w}$ is invertible in $H_{w.M}$ we get $v \otimes 1_{\mathcal{H}_{w.M}}$ in $p\mathcal{V} \otimes H_{(w,M)^-} \theta_{\mathcal{T}} H_{w.M}$. As $v$ was arbitrary, $v \otimes H_{(w,M)^-} \theta_{\mathcal{T}} H_{w.M} \subset p\mathcal{V} \otimes H_{(w,M)^-} \theta_{\mathcal{T}} H_{w.M}$. If $p$ is nilpotent in $\mathcal{V}$, then $\mathcal{V} \otimes H_{(w,M)^-} \theta_{\mathcal{T}} H_{w.M} = 0$. Suppose now that there exists $n > 0$ such that $\mathcal{V}(\omega(t))^n = 0$ for any non-invertible $t \in T^*$, then $VT^{n+1}_{\mu} \subset p\mathcal{V}$ where $\mu = \mu_t$; hence $\varphi(h) = \varphi(hT^{n+1}_{\mu}) \in p\mathcal{V}$ for an arbitrary $\varphi \in \text{Hom}_{H_{M^+}, \theta_{\mathcal{T}}}(H_{M}, \mathcal{V})$ and an arbitrary $h \in H_{M}$. We deduce $\text{Hom}_{H_{M^+}, \theta_{\mathcal{T}}}(H_{M}, \mathcal{V}) \subset \text{Hom}_{H_{M^+}, \theta_{\mathcal{T}}}(H_{M}, \mathcal{V})$. If $p$ is nilpotent in $\mathcal{V}$, then $\text{Hom}_{H_{M^+}, \theta_{\mathcal{T}}}(H_{M}, \mathcal{V}) = 0$.

Recalling that $\tilde{w}^{M}.\mathcal{V}$ is obtained by functoriality from $\mathcal{V}$ and the ring isomorphism $i(\tilde{w}^{M})$ defined in [4.13], the equivalence between $\mathcal{V}$ supersingular and $\tilde{w}^{M}.\mathcal{V}$ supersingular follows from:

**Lemma 4.28.**

1. Let $t \in T$. Then $t$ is dominant for $U_{M}$ if and only if $\tilde{w}^{M}(\tilde{w}^{M})^{-1} \in T$ is dominant for $U_{w.M}$.

2. The $R$-algebra isomorphism $H_{M,R} \xrightarrow{i(\tilde{w}^{M})} H_{w,M,R}$, $T^{M}_{w} \mapsto T^{w.M}_{w.M(w^{M})^{-1}}$ for $w \in W_M(1)$ sends $z.M(t)$ to $z^{w.M}(\tilde{w}^{M}(\tilde{w}^{M})^{-1})$ for $t \in T$ dominant for $U_{M}$.

**Proof.** The conjugation by $\tilde{w}^{M}$ stabilizes $T$, sends $U_{M}$ to $U_{w.M}$ and sends the $w.M$-orbit of $t \in T$ to the $w.M$-orbit of $\tilde{w}^{M}(\tilde{w}^{M})^{-1}$, as $\tilde{w}^{M}.\tilde{w}^{M}(\tilde{w}^{M})^{-1} = \tilde{w}^{w.M}$. It is known that $i(\tilde{w}^{M})$ respects the antidominant alcove walk bases [Vig15b, Proposition 2.20]: it sends $E.M(w)$ to $E^{w.M}(\tilde{w}^{M}(\tilde{w}^{M})^{-1})$ for $w \in W_M(1)$. \[\square\]
Corollary 4.29. Let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module. Then $\mathcal{V}$ is supersingular if and only if the right $\mathcal{H}_{w,M,R}$-module $\tilde{w}^M\mathcal{V}$ is supersingular.

Assume $R = C$. The supersingular simple $\mathcal{H}_{M,C}$-modules are classified in \cite{Vig15a}. By Corollaries 4.24 and 4.29, the classification of the simple $\mathcal{H}_C$-modules in \cite{Abc} remains valid with the $\mathcal{H}_C$-modules $I_H(P,V,Q)$ instead of $CI_H(P,V,Q)$:

Corollary 4.30 (Classification of simple $\mathcal{H}_C$-modules). Assume $R = C$. Let $(P, V, Q)$ be a $\mathcal{H}_C$-triple where $V$ is simple and supersingular. Then, the $\mathcal{H}_C$-module $I_H(P, V, Q)$ is simple. A simple $\mathcal{H}_C$-module is isomorphic to $I_H(P, V, Q)$ for a $\mathcal{H}_C$-triple $(P, V, Q)$ where $V$ is simple and supersingular, $P, Q$ and the isomorphism class of $V$ are unique.

4.6. A commutative diagram. We prove in this section Proposition 4.23. For $Q \subset Q' \subset P(V)$ we show by an explicit computation that

$$
\mu^{-1}_Q \circ i(Q, Q') \circ \mu_{Q'} : \text{Ind}_{\mathcal{H}_{w,Q'}}^H(\mathcal{e}_{\mathcal{H}_{w,Q'}}(\tilde{w}^M\mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_{w,Q}}^H(\mathcal{e}_{\mathcal{H}_{w,Q}}(\tilde{w}^M\mathcal{V})).
$$

is equal to $i(w.Q, w.Q')$. The $R$-module $\mathcal{e}_{\mathcal{H}_{w,Q'}}(\tilde{w}^M\mathcal{V}) \otimes 1_H$ generates the $\mathcal{H}_R$-module $\mathcal{e}_{\mathcal{H}_{w,Q'}}(\tilde{w}^M\mathcal{V}) \otimes_{\mathcal{H}_{w,Q,R}\theta+} \mathcal{H}_R = \text{Ind}_{\mathcal{H}_{w,Q'}}^H(\mathcal{e}_{\mathcal{H}_{w,Q'}}(\tilde{w}^M\mathcal{V}))$ and by (4.15)

$$
i(w.Q, w.Q')(v \otimes 1_H) = v \otimes \sum_{d \in \mathbb{W}_{M,Q}^w \mathbb{W}_{M,Q'}} T_d
$$

for $v \in \mathcal{V}$ seen as an element of $\mathcal{e}_{\mathcal{H}_{w,Q'}}(\tilde{w}^M\mathcal{V})$ in the LHS and an element of $\mathcal{e}_{\mathcal{H}_{w,Q}}(\tilde{w}^M\mathcal{V})$ in the RHS.

Lemma 4.31. $(\mu^{-1}_Q \circ i(Q, Q') \circ \mu_{Q'})(v \otimes 1_H) = v \otimes \sum_{d \in \mathbb{W}_{M,Q}^w} q_d T_{w.Q(\tilde{w}Qd)}^{*d-1}$.

Proof. $\mu_{Q'}(v \otimes 1_H)$ is the unique homomorphism $f_{w.M,Q'}) \in \text{Hom}_{\mathcal{H}_{M,Q'}}\mathcal{H}_{w,Q'}(\mathcal{V})$ sending $T_{w.Q}$ to $v$ and vanishing on $T_{d'}$ for $d' \in \mathbb{W}_{M,Q'} \setminus \{w.Q'\}$ by (4.20). By (4.17), $i(Q, Q')$ is the natural embedding of $\text{Hom}_{\mathcal{H}_{M,Q'}}\mathcal{H}_{w,M}(\mathcal{V})$ in $\text{Hom}_{\mathcal{H}_{M,Q'}}\mathcal{H}_{w,Q}(\mathcal{V})$ therefore $i(Q, Q')(f_{w.M,Q'})$ is the unique homomorphism $\text{Hom}_{\mathcal{H}_{M,Q'}}\mathcal{H}_{w,Q}(\mathcal{V})$ sending $T_{w.Q}$ to $v$ and vanishing on $T_{d'}$ for $d' \in \mathbb{W}_{M,Q'} \setminus \{w.Q'\}$. As $\mathbb{W}_{M,Q} = \mathbb{W}_{M,Q'} \mathbb{W}_{M,Q}$, this homomorphism vanishes on $T_{d}$ for $w$ not in $w.M,Q'\mathbb{W}_{M,Q}$. By \cite{Abc} Lemma 2.22, the inverse of $\mu_{Q}$ is the $\mathcal{H}_R$-isomorphism:

$$
\text{Hom}_{\mathcal{H}_{M,Q'}}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V})) \xrightarrow{\mu^{-1}_Q} \text{Ind}_{\mathcal{H}_{w,M}}^H(e_{\mathcal{H}_{w,Q}}(\tilde{w}^M\mathcal{V}))
$$

$$
f \mapsto \sum_{d \in \mathbb{W}_{M}^w} f(T_{d}) \otimes T_{w.Q}^{*d-1},
$$

where $\mathbb{W}_{M}^w$ is the set of $d \in \mathbb{W}$ with minimal length in the coset $d\mathbb{W}_{M}$. We deduce the explicit formula:

$$
(\mu^{-1}_Q \circ i(Q, Q') \circ \mu_{Q'})(v \otimes 1_H) = \sum_{w \in \mathbb{W}_{M,Q}} i(Q, Q')(f_{w.M,Q'}) (T_{\tilde{w}}) \otimes T_{w.M,Q}^{*d-1}.
$$
Some terms are zero: the terms for \( w \in \mathcal{W}_{MQ} \) not in \( \mathcal{W}_{MQ} \mathcal{W}_{MQ}^{\prime} \). We analyse the other terms for \( w \) in \( \mathcal{W}_{MQ} \mathcal{W}_{MQ}^{\prime} \cap \mathcal{W}_{MQ}^{\prime} \mathcal{W}_{MQ} \); this set is \( \mathcal{W}_{MQ}^{\prime} \mathcal{W}_{MQ}^{\prime} \). Let \( w = \mathcal{W}_{MQ}^{\prime} d, d \in \mathcal{W}_{MQ}^{\prime} \), and \( \tilde{w} = \mathcal{W}_{MQ}^{\prime} d \) with \( \tilde{d} \in 1_{W_{G}} \) lifting \( d \). By the braid relations \( T_{\tilde{w}} = T_{\mathcal{W}_{MQ}} T_{\tilde{d}} \). We have \( T_{\tilde{d}} = \theta^{*}(T_{d}) \) by the braid relations because \( d \in \mathcal{W}_{MQ}^{\prime} \), \( S_{MQ} \subset S^{aff} \) and \( \theta^{*}(c_{MQ}) = c_{d} \) for \( s \in S_{MQ} \). As \( \mathcal{W}_{MQ}^{\prime} \subset \mathcal{W}_{MQ}^{\prime} \times \mathcal{W}_{MQ}^{\prime} \), we deduce:

\[
i(Q, Q')(f_{\mathcal{W}_{MQ}, v}(T_{\tilde{w}})) = i(Q, Q')(f_{\mathcal{W}_{MQ}, v}(T_{\mathcal{W}_{MQ}} T_{d}) = i(Q, Q')(f_{\mathcal{W}_{MQ}, v}(T_{\mathcal{W}_{MQ}} T_{d})^{\prime}M_{Q} = vT_{d}^{M_{Q}} = q_{d}w.
\]

Corollary 3.9 gives the last equality. \( \square \)

The formula for \( \left( \mu_{Q}^{-1} \circ i(Q, Q') \circ \mu_{Q} \right)(v \otimes 1_{H}) \) given in Lemma 4.31 is different from the formula (4.22) for \( i(wQ, wQ')(v \otimes 1_{H}) \). It needs some work to prove that they are equal.

A first reassuring remark is that \( \mathcal{W}_{w, Q} \mathcal{W}_{w, Q'} = \{ w^{-1}d \mid d \in \mathcal{W}_{MQ} \} \), so the two summation sets have the same number of elements. But better,

\[
\mathcal{W}_{w, Q} \mathcal{W}_{w, Q'} = \{ w^{Q}(w^{Q}d)^{-1} \mid d \in \mathcal{W}_{MQ} \}
\]

because \( w^{Q} \mathcal{W}_{MQ}^{\prime} w = \mathcal{W}_{MQ}^{\prime} \). To prove the latter equality, we apply the criterion: \( w \in \mathcal{W}_{MQ}^{\prime} \) lies in \( \mathcal{W}_{MQ}^{\prime} \mathcal{W}_{MQ}^{\prime} \) if and only if \( w(\alpha) > 0 \) for all \( \alpha \in \Delta_{Q} \) noticing that \( d \in \mathcal{W}_{MQ}^{\prime} \) implies \( wQ(\alpha) \in \Delta_{Q}, d_{wQ}(\alpha) \in \Phi_{MQ}, w_{MQ}d_{wQ}(\alpha) > 0 \). Let \( x_{d} = w^{Q}(w^{Q}d)^{-1} \). We have \( w^{MQ} \tilde{d}^{-1} = x_{d} \) because the lifts \( \tilde{w} \) of the elements \( w \in \mathcal{W} \) satisfy the braid relations and \( \ell(x_{d}) = \ell_{MQ}d^{-1}wQ' = \ell(wQ') = \ell(wQ) - \ell(d) \). We have \( \sum q_{d}T_{x_{d}^{wQ}w_{MQ}^{\prime}} = \sum q_{wQx_{d}wQ_{MQ}^{\prime}}T_{x_{d}}^{wQ} \).

In the RHS, only \( w^{M}, V, wQ, wQ' \) appear. The same holds true in the formula (4.22). The map \( (P, V, Q, Q') \rightarrow (wP, w^{M}, V, wQ, wQ') \) is a bijection of the set of triples \( (P, V, Q, Q') \) where \( P = MN, Q, Q' \) are standard parabolic subgroups of \( G \), \( V \) a right \( \mathcal{H}_{R} \)-module, \( Q \subset Q' \subset P(V) \) by Lemma 4.21. So we can replace \( (wP, w^{M}, V, wQ, wQ') \) by \( (P, V, Q, Q') \). Our task is reduced to prove in \( e_{HQ}(V) \otimes_{H_{MQ}^{\prime}} H_{R} \):

\[
(4.24) \quad v \otimes \sum_{d \in \mathcal{W}_{MQ} \mathcal{W}_{MQ}^{\prime}} T_{d} = v \otimes \sum_{d \in \mathcal{W}_{MQ} \mathcal{W}_{MQ}^{\prime}} q_{wQd}T_{d}^{wQ}.
\]

A second simplification is possible: we can replace \( Q \subset Q' \) by the standard parabolic subgroups \( Q_{2} \subset Q_{2} \) of \( G \) with \( \Delta_{Q_{2}} = \Delta_{Q} \mathcal{A}_{\Delta_{P}} \) and \( \Delta_{Q_{2}'} = \Delta_{Q_{2}} \mathcal{A}_{\Delta_{P}} \), because \( \Delta_{P} \) and \( \Delta_{P}(V) \mathcal{A}_{\Delta_{P}} \) are orthogonal. Indeed, \( \mathcal{W}_{MQ} = \mathcal{W}_{M} \mathcal{W}_{M}^{Q_{2}} \) and \( \mathcal{W}_{MQ} = \mathcal{W}_{M} \mathcal{W}_{MQ}^{}\mathcal{W}_{MQ}^{Q_{2}} \) are direct products,
the longest elements \( w^*_Q = w_M w^*_Q \), \( w_Q = w_M w_Q^2 \) are direct products and
\[
\mathcal{W}_M \mathcal{W}_Q = \mathcal{W}_M^2 \mathcal{W}_Q^2 \quad \text{and} \quad w_Q w^*_Q = w_Q^2 w^*_Q.
\]
Once this is done, we use the properties of \( e_{H_Q}(\mathcal{V}) \): \( v h \otimes 1_h = v \otimes e_Q(h) \) for \( h \in H_{Q^2}^+ \), and \( T^*_w \) acts trivially on \( e_{H_Q}(\mathcal{V}) \) for \( w \in 1 W_{Q^2} \cup (\Lambda(1) \cap 1 W_{Q^2}) \). Set \( \mathcal{W}_M^* = \{ w \in 1 \mathcal{W}_M^2 | w \text{ is a lift of some element in } \mathcal{W}_M Q^2 \} \) and \( \mathcal{W}_M^* \) similarly. Then \( Z_k \cap 1 \mathcal{W}_M^* \subset (\Lambda(1) \cap 1 W_{Q^2}^2) \cap 1 W_{Q^2}^2 \) and \( \mathcal{W}_M^* \subset 1 W_{Q^2}^* \cap 1 W_{Q^2}^2 \). This implies that \( (4.24) \) where \( Q \subset Q' \) has been replaced by \( Q_2 \subset Q'_2 \) follows from a congruence
\[
(4.25) \quad \sum_{d \in \mathcal{W}_M^2 \mathcal{W}_M^2} T_d \equiv \sum_{d \in \mathcal{W}_M^2 \mathcal{W}_M^2} q_{w_Q^2 d w^*_Q} T^*_d.
\]
in the finite subring \( H(\mathcal{W}_M^2) \) of \( H \) generated by \( \{ T_w \ | \ w \in 1 \mathcal{W}_M^2 \} \) modulo the right ideal \( J_2 \) with generators \( \{ e_Q(T^*_w) - 1 | w \in (Z_k \cap 1 \mathcal{W}_M^2) \cup 1 \mathcal{W}_M Q^2 \} \).

Another simplification concerns \( T_d^* \) modulo \( J_2 \) for \( d \in \mathcal{W}_M^2 \). We recall that for any reduced decomposition \( d = s_1 \ldots s_n \) with \( s_i \in S \cap \mathcal{W}_M^2 \), we have \( T^*_d = (T_{s_1} - c_{s_1}) \ldots (T_{s_n} - c_{s_n}) \) where the \( s_i \) are admissible. For \( s \) admissible, by \( (3.2) \)
\[
c_{s_i} \equiv q_s - 1.
\]
Therefore
\[
(4.26) \quad T^*_d = (T_{s_1} - q_{s_1} + 1) \ldots (T_{s_n} - q_{s_n} + 1).
\]
Let \( J' \subset J_2 \) be the ideal of \( H(\mathcal{W}_M^2) \) generated by \( \{ T_t - 1 | t \in Z_k \cap 1 \mathcal{W}_M^2 \} \). Then the ring \( H(\mathcal{W}_M^2)/J' \) and its right ideal \( J_2/J' \) are the specialisation of the generic finite ring \( \mathcal{H}(\mathcal{W}_M^2)^g \) over \( \mathbb{Z}(q_s)_{s \in S} \) where the \( q_s \) for \( s \in S_{M^2} = S \cap \mathcal{W}_M^2 \) are indeterminates, and of its right ideal \( J_2^g \) with the same generators. The similar congruence modulo \( J_2^g \) in \( H(\mathcal{W}_M^2)^g \) (the generic congruence) implies the congruence \( (4.25) \) by specialisation.

We will prove the generic congruence in a more general setting where \( H \) is the generic Hecke ring of a finite Coxeter system \((\mathcal{W}, S)\) and parameters \( (q_s)_{s \in S} \) such that \( q_s = q_{s'} \) when \( s, s' \) are conjugate in \( \mathcal{W} \). The Hecke ring \( \mathcal{H} \) is a \( \mathbb{Z}[\{q_s\}_{s \in S}] \)-free module of basis \( (T_w)_{w \in \mathcal{W}} \) satisfying the braid relations and the quadratic relations \( T_s^2 = q_s + (q_s - 1)T_s \) for \( s \in S \). The other basis \( (T_w^*)_{w \in \mathcal{W}} \) satisfies the braid relations and the quadratic relations \( T_s^2 = q_s - (q_s - 1)T_s \) for \( s \in S \), and is related to the first basis by \( T_s^* = T_s - (q_s - 1) \) for \( s \in S \), and more generally \( T_w T_{w^{-1}} = T_{w^{-1}} T_w = q_w \) for \( w \in \mathcal{W} \) \( [\text{Vig16}, \text{Proposition 4.13}] \).

Let \( J \subset S \) and \( J \) is the right ideal of \( \mathcal{H} \) with generators \( T^*_w - 1 \) for all \( w \) in the group \( W_J \) generated by \( J \).

**Lemma 4.32.** A basis of \( J \) is \( (T^*_w - 1)T^*_w \) for \( w_1 \in W_J \setminus \{ 1 \} \), \( w_2 \in W_J \mathcal{W} \), and adding \( T^*_w \) for \( w_2 \in W_J \mathcal{W} \) gives a basis of \( J \). In particular, \( J \) is a direct factor of \( \mathcal{H} \).

**Proof.** The elements \( (T^*_w - 1)T^*_w \) for \( w \in W_J \) generate \( J \). We write \( w = u_1 u_2 \) with unique elements \( u_1 \in W_J \), \( u_2 \in W_J \mathcal{W} \), and \( T^*_w = T^*_w \). Therefore, \( (T^*_w - 1)T^*_w \). By an induction on the length of \( u_1 \), one proves that \( (T^*_w - 1)T^*_w \) is a linear combination of \( (T^*_w - 1) \).
for \( v_1 \in \mathbb{W}_J \) as in the proof of Proposition 3.4. It is clear that the elements \((T_{w_1}^* - 1)T_{w_2}^*\) and \(T_{w_2}^*\) for \( w_1 \in \mathbb{W}_J \setminus \{1\}, w_2 \in \mathbb{W}_J \mathbb{W} \) form a basis of \( H \).

Let \( w_J \) denote the longest element of \( \mathbb{W}_J \) and \( w = w_S \).

**Lemma 4.33.** In the generic Hecke ring \( H \), the congruence modulo \( \mathcal{J} \)

\[
\sum_{d \in \mathbb{W}_J \mathbb{W}} T_d \equiv \sum_{d \in \mathbb{W}_J \mathbb{W}} q_{w_J dw} T_d^*
\]

holds true.

**Proof.** Step 1. We show:

\[
\mathbb{W}_J \mathbb{W} = w_J \mathbb{W}_J \mathbb{W} w, \quad q_{w_J q_{w_J dw}} T_d^* = T_{w_J} T_{w_J dw} T_{w_J}.
\]

The equality between the groups follows from the characterisation of \( \mathbb{W}_J \mathbb{W} \) in \( \mathbb{W} \): an element \( d \in \mathbb{W} \) has minimal length in \( \mathbb{W}_J \mathbb{W} \) if and only if \( \ell(ud) = \ell(u) + \ell(d) \) for all \( u \in \mathbb{W}_J \). An easy computation shows that \( \ell(uw_J dw) = \ell(u) + \ell(w_J dw) \) for all \( u \in \mathbb{W}_J, d \in \mathbb{W}_J \mathbb{W} \) (both sides are equal to \( \ell(u) + \ell(w_J) - \ell(d) \)). The second equality follows from \( q_{w_J} q_{w_J dw} = q_{dw} \) because \((w_J)^2 = 1 \) and \( \ell(w_J + \ell(w_J dw) = \ell(dw) \) (both sides are \( \ell(w) - \ell(d) \)) and from \( q_{dw} T_d = T_{dw} T_{w_{ud-1}} T_d = T_{dw} T_{w_J} \). We also have \( T_{dw} = T_{w_J} T_{w_J dw} \).

Step 2. The multiplication by \( q_{w_J} \) on the quotient \( H/\mathcal{J} \) is injective (Lemma 4.32) and \( q_{w_J} \equiv T_{w_J} \). By Step 1, \( q_{w_J dw} T_d^* \equiv T_{w_J dw} T_{w_J} \) and

\[
\sum_{d \in \mathbb{W}_J \mathbb{W}} q_{w_J dw} T_d^* \equiv \sum_{d \in \mathbb{W}_J \mathbb{W}} T_d T_{w_J}^*.
\]

The congruence

\[
(4.26) \sum_{d \in \mathbb{W}_J \mathbb{W}} T_d \equiv \sum_{d \in \mathbb{W}_J \mathbb{W}} T_d T_{w_J}^*
\]

for all \( s \in S \) implies the lemma because \( T_{w_J}^* = T_{s_1}^* \cdots T_{s_n}^* \) for any reduced decomposition \( w = s_1 \cdots s_n \) with \( s_i \in S \).

Step 3. When \( J = \emptyset \), the congruence \( (4.26) \) is an equality:

\[
(4.27) \sum_{w \in \mathbb{W}} T_w = \sum_{w \in \mathbb{W}} T_w T_{w_J}^*.
\]

It holds true because \( \sum_{w \in \mathbb{W}} T_w = \sum_{w < w_J} T_w (T_s + 1) \) and \( (T_s + 1) T_{s_J}^* = T_s T_{s_J}^* + T_{s_J}^* = q_s + T_{s_J}^* = T_s + 1 \).

Step 4. Conversely the congruence \( (4.26) \) follows from \( (4.27) \) because

\[
\sum_{w \in \mathbb{W}} T_w = \left( \sum_{u \in \mathbb{W}_J} T_u \right) \sum_{d \in \mathbb{W}_J \mathbb{W}} T_d \equiv \left( \sum_{u \in \mathbb{W}_J} q_u \right) \sum_{d \in \mathbb{W}_J \mathbb{W}} T_d
\]

(recall \( q_u = T_u^*, T_u \equiv T_u \)) and we can simplify by \( \sum_{u \in \mathbb{W}_J} q_u \) in \( H/\mathcal{J} \).

This ends the proof of Proposition 4.23.
5. Universal representation $I_H(P, V, Q) \otimes_H R[U \backslash G]$

The invariant functor $(-)^H$ by the pro-$p$ Iwahori subgroup $U$ of $G$ has a left adjoint

$$- \otimes_H R[U \backslash G] : \text{Mod}_R(H) \to \text{Mod}_R^c(G).$$

The smooth $R$-representation $V \otimes_H R[U \backslash G]$ of $G$ constructed from the right $H_R$-module $V$ is called universal. We write

$$R[U \backslash G] = X.$$

**Question 5.1.** Does $V \neq 0$ imply $V \otimes_H X \neq 0$? or does $v \otimes 1_U = 0$ for $v \in V$ implies $v = 0$? We have no counter-example. If $R$ is a field and the $H_R$-module $V$ is simple, the two questions are equivalent: $V \otimes_H X \neq 0$ if and only if the map $v \mapsto v \otimes 1_U$ is injective. When $R = C$, $V \otimes_H X \neq 0$ for all simple $H_C$-modules $V$ if this is true for $V$ simple supersingular (this is a consequence of Corollary 5.13).

The functor $- \otimes_H X$ satisfies a few good properties: it has a right adjoint and is compatible with the parabolic induction and the left adjoint (of the parabolic induction). Let $P = MN$ be a standard parabolic subgroup and $X_M = R[U_M \backslash M]$. We have functor isomorphisms

$$(- \otimes_H X) \circ \text{Ind}_{H_M}^H \to \text{Ind}_{G}^G (- \otimes_H X_M), \quad (5.1)$$

$$(-)_N \circ (- \otimes_H X) \to (- \otimes_H X_M) \circ \mathcal{L}_{H_M}^H. \quad (5.2)$$

The first one is [OV17 formula 4.15], the second one is obtained by left adjunction from the isomorphism $\text{Ind}_{H_M}^H \circ (-)^{\mu_M} \to (-)^{\mu} \circ \text{Ind}_{G}^G$ [OV17 formula (4.14)]. If $V$ is a right $H_R$-supersingular module and $p$ is nilpotent in $V$, then $\mathcal{L}_{H_M}^H(V) = 0$ if $M \neq G$ (Proposition 4.27).

Applying (5.2) we deduce:

**Proposition 5.2.** If $p$ is nilpotent in $V$ and $V$ supersingular, then $V \otimes_H X$ is left cuspidal.

**Remark 5.3.** For a non-zero smooth $R$-representation $\tau$ of $M$, $\Delta_{\tau}$ is orthogonal to $\Delta_P$ if $\tau$ is left cuspidal. Indeed, we recall from [AHIV17 II.7 Corollary 2] that $\Delta_{\tau}$ is not orthogonal to $\Delta_P$ if and only if it exists a proper standard parabolic subgroup $X$ of $M$ such that $\sigma$ is trivial on the unipotent radical of $X$; moreover $\tau$ is a subrepresentation of $\text{Ind}^M_X(\tau|_X)$, so the image of $\tau$ by the left adjoint of $\text{Ind}^M_X$ is not 0.

From now on, $V$ is a non-zero right $H_{M,R}$-module and

$$\sigma = V \otimes_{H_{M,R}} X_M.$$

In general, when $\sigma \neq 0$, let $P_\perp(\sigma)$ be the standard parabolic subgroup of $G$ with $\Delta_{P_\perp(\sigma)} = \Delta_P \cup \Delta_{\perp, \sigma}$ where $\Delta_{\perp, \sigma}$ is the set of simple roots $\alpha \in \Delta_\sigma$ orthogonal to $\Delta_P$.

**Proposition 5.4.**

1. $P(V) \subset P_\perp(\sigma)$ if $\sigma \neq 0$.
2. $P(V) = P_\perp(\sigma)$ if the map $v \mapsto v \otimes 1_{U_M}$ is injective.
3. $P(V) = P(\sigma)$ if the map $v \mapsto v \otimes 1_{U_M}$ is injective, $p$ nilpotent in $V$ and $V$ supersingular.
4. $P(V) = P(\sigma)$ if $\sigma \neq 0$, $R$ is a field of characteristic $p$ and $V$ simple supersingular.

**Proof.** (1) $P(V) \subset P_\perp(\sigma)$ means that $Z \cap M'_V$ acts trivially on $V \otimes 1_{U_M}$, where $M'_V$ is the standard Levi subgroup such that $\Delta_{M_{\sigma}} = \Delta_{V}$. Let $z \in Z \cap M'_V$ and $v \in V$. As $\Delta_M$ and $\Delta_V$ are orthogonal, we have $T_{M,z}^*(z) = T_{M}(z)$ and $U_M z U_M = U_M z$. We have $v \otimes 1_{U_M} = v T_{M}(z) \otimes 1_{U_M} = v \otimes T_{M}(z) 1_{U_M} = v \otimes 1_{U_M} z = v \otimes z^{-1} 1_{U_M} = z^{-1}(v \otimes 1_{U_M})$. 


(2) If \( v \otimes 1_{U_M} = 0 \) for \( v \in \mathcal{V} \) implies \( v = 0 \), then \( \sigma \neq 0 \) because \( \mathcal{V} \neq 0 \). By (1) \( P(\mathcal{V}) \subseteq P_\perp(\sigma) \).

As in the proof of (1), for \( z \in \mathcal{Z} \cap M'_\perp \), we have \( vT^{M,\ast}(z) \otimes 1_{U_M} = vT^M(z) \otimes 1_{U_M} = v \otimes 1_{U_M} \) and our hypothesis implies \( vT^{M,\ast}(z) = v \) hence \( P(\mathcal{V}) \supseteq P_\perp(\sigma) \).

(3) Proposition 5.2 Remark 5.3 and (2).

(4) Question 5.1 and (3). \( \square \)

Let \( Q \) be a parabolic subgroup of \( G \) with \( P \subseteq Q \subseteq P(\mathcal{V}) \). In this chapter we will compute \( I_H(P, \mathcal{V}, Q) \otimes_R R[\mathfrak{U}/G] \) where \( I_H(P, \mathcal{V}, Q) = \text{Ind}^H_{H_M(\mathcal{V})} (e(\mathcal{V}) \otimes (\text{Ind}^P_Q(\mathcal{V}))^{U_M(\mathcal{V}))} \) (Theorem 5.11). The smooth \( R \)-representation \( I_G(P, \sigma, Q) \) of \( G \) is well defined: it is 0 if \( \sigma = 0 \) and \( \text{Ind}^G_P(e(\sigma) \otimes St_Q(\sigma)) \) if \( \sigma \neq 0 \) because \( (P, \sigma, Q) \) is an \( R[G] \)-triple by Proposition 5.2. We will show that the universal representation \( I_H(P, \mathcal{V}, Q) \otimes_R R[\mathfrak{U}/G] \) is isomorphic to \( I_G(P, \sigma, Q) \), if \( P(\mathcal{V}) = P(\sigma) \) and \( p = 0 \), or if \( \sigma = 0 \) (Corollary 5.12). In particular, when \( R = C \) and \( I_H(P, \mathcal{V}, Q) \otimes_R R[\mathfrak{U}/G] \simeq I_G(P, \sigma, Q) \) when \( \mathcal{V} \) is supersingular.

5.1. \( Q = G \). We consider first the case \( Q = G \). We are in the simple situation where \( \mathcal{V} \) is extensible to \( \mathcal{H} \) and \( P(\mathcal{V}) = P(\sigma) = G \), \( I_H(P, \mathcal{V}, G) = e(\mathcal{V}) \) and \( I_G(P, \sigma, G) = e(\sigma) \). We recall that \( \Delta \setminus \Delta_P \) is orthogonal to \( \Delta_P \) and that \( M_2 \) denotes the standard Levi subgroup of \( G \) with \( \Delta_{M_2} = \Delta \setminus \Delta_P \).

The \( \mathcal{H}_R \)-morphism \( e(\mathcal{V}) \to e(\sigma)^{\mathcal{U}_M} = \sigma^{\mathcal{U}_M} \) sending \( v \) to \( v \otimes 1_{U_M} \) for \( v \in \mathcal{V} \), gives by adjunction an \( R[G] \)-homomorphism

\[
v \otimes 1_U \mapsto v \otimes 1_{U_M} : e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X} \xrightarrow{\Phi^G} e(\sigma),
\]

If \( \Phi^G \) is an isomorphism, then \( e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X} \) is the extension to \( G \) of \( (e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X})|_M \), meaning that \( M'_2 \) acts trivially on \( e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X} \). The converse is true:

**Lemma 5.5.** If \( M'_2 \) acts trivially on \( e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X} \), then \( \Phi^G \) is an isomorphism.

**Proof.** Suppose that \( M'_2 \) acts trivially on \( e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X} \). Then \( e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X} \) is the extension to \( G \) of \( (e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X})|_M \), and by Theorem 3.13 \( (e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X})^{\mathcal{U}_M} \) is the extension of \( (e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X})^{\mathcal{U}_M} \). Therefore

\[
(v \otimes 1_U)T^*_w = (v \otimes 1_U)T^{M,\ast}_w \quad \text{for all } v \in \mathcal{V}, w \in W_M(1).
\]

As \( \mathcal{V} \) is extensible to \( \mathcal{H} \), the natural map \( v \mapsto v \otimes 1_U : \mathcal{V} \xrightarrow{\Psi} (e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X})^{\mathcal{U}_M} \) is \( \mathcal{H}_M \)-equivariant, i.e.:

\[
vT^{M,\ast}_w \otimes 1_U = (v \otimes 1_U)T^{M,\ast}_w \quad \text{for all } v \in \mathcal{V}, w \in W_M(1).
\]

because \((3.11)\) \( vT^{M,\ast}_w \otimes 1_U = vT^*_w \otimes 1_U = v \otimes T^*_w = (v \otimes 1_U)T^*_w \) in \( e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X}. \)

We recall that \( - \otimes_{\mathcal{H}_M,R} \mathcal{X}_M \) is the left adjoint of \( (\cdot)^{\mathcal{U}_M} \). The adjoint \( R[M] \)-homomorphism

\[
\sigma = \mathcal{V} \otimes_{\mathcal{H}_M, R} \mathcal{X}_M \to e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X} \text{ sends } v \otimes 1_{U_M} \text{ to } v \otimes 1_U \text{ for all } v \in \mathcal{V} \text{. The } R[M] \text{-module generated by the } v \otimes 1_U \text{ for all } v \in \mathcal{V} \text{ is equal to } e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X} \text{ because } M'_2 \text{ acts trivially. Hence we obtained an inverse of } \Phi^G. \]

Our next move is to determine if \( M'_2 \) acts trivially on \( e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X} \). It is equivalent to see if \( M'_2 \) acts trivially on \( e(\mathcal{V}) \otimes 1_U \) as this set generates the representation \( e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X} \) of \( G \) and \( M'_2 \) is a normal subgroup of \( G \) as \( M'_2 \) and \( M \) commute and \( G = ZM'M'_2 \). Obviously, \( U \cap M'_2 \) acts trivially on \( e(\mathcal{V}) \otimes 1_U \). The group of double classes \( (U \cap M'_2)/M'_2) \) is generated by the lifts \( s \in \mathcal{N} \cap M'_2 \) of the simple affine roots \( s \) of \( W_{M'_2} \). Therefore, \( M'_2 \) acts trivially on \( e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathcal{X} \) if and only if for any simple affine root \( s \in S_{M'_2}^{\text{aff}} \) of \( W_{M'_2} \), any \( s \in \mathcal{N} \cap M'_2 \) lifting \( s \) acts trivially on \( e(\mathcal{V}) \otimes 1_U \).
Lemma 5.6. Let \( v \in \mathcal{V}, s \in S_{M_2}^{\text{aff}} \) and \( \hat{s} \in \mathcal{N} \cap M_2 \) lifting \( s \). We have
\[
(q_s + 1)(v \otimes 1_U - \hat{s}(v \otimes 1_U)) = 0.
\]

Proof. We compute:
\[
T_s(\hat{s}1_U) = \hat{s}(T_s1_U) = 1_{U\hat{s}U(\hat{s})^{-1}} = \sum_u \hat{s}u(\hat{s})^{-1}1_U = \sum_{u^{\text{op}}} u^{\text{op}}1_U,
\]
\[
T_s(\hat{s}^21_U) = \hat{s}^2(T_s1_U) = 1_{U\hat{s}U(\hat{s})^{-2}} = 1_{U(\hat{s})^{-2}}1_U = \sum_u u\hat{s}1_U.
\]

for \( u \) in the group \( U/(\hat{s}^{-1}U\hat{s} \cap U) \) and \( u^{\text{op}} \) in the group \( \hat{s}U(\hat{s})^{-1}/(\hat{s}U(\hat{s})^{-1} \cap U) \); the reason is that \( \hat{s}^2 \) normalizes \( U_\mathcal{S} \), \( U\hat{s}U(\hat{s})^{-1} \) is the disjoint union of the sets \( U\hat{s}U^{-1}(\hat{s})^{-1} \) and \( U(\hat{s})^{-1}U \) is the disjoint union of the sets \( U(\hat{s})^{-1}U \). We introduce now a natural bijection
\[
(5.3) \quad u \mapsto u^{\text{op}} : U/(\hat{s}^{-1}U\hat{s} \cap U) \to \hat{s}U(\hat{s})^{-1}/(\hat{s}U(\hat{s})^{-1} \cap U)
\]
which is not a group homomorphism. We recall the finite reductive group \( G_{k,s} \) quotient of the parahoric subgroup \( \mathfrak{R}_s \) of \( G \) fixing the face fixed by \( s \) of the alcove \( C \). The Iwahori groups \( Z^0U \) and \( Z^0\hat{s}U(\hat{s})^{-1} \) are contained in \( \mathfrak{R}_s \) and their images in \( G_{k,s} \) are opposite Borel subgroups \( Z_kU_{s,k} \) and \( Z_kU_{s,k}^{\text{op}} \). Via the surjective maps \( u \mapsto \overline{u} : U \to U_{s,k} \) and \( u^{\text{op}} \mapsto \overline{u}^{\text{op}} : \hat{s}U(\hat{s})^{-1} \to U_{s,k}^{\text{op}} \) we identify the groups \( U/(\hat{s}^{-1}U\hat{s} \cap U) \simeq U_{s,k} \) and similarly \( \hat{s}U(\hat{s})^{-1}/(\hat{s}U(\hat{s})^{-1} \cap U) \simeq U_{s,k}^{\text{op}} \). Let \( G_{k,s}' \) be the group generated by \( U_{s,k} \) and \( U_{s,k}^{\text{op}} \), and let \( B_{k,s}' = G_{k,s}' \cap Z_kU_{s,k} = (G_{k,s}' \cap Z_k)U_{s,k} \). We suppose (as we can) that \( \hat{s} \in \mathfrak{R}_s \) and that its image \( \hat{s}_k \) in \( G_{k,s} \). We have \( \hat{s}_kU_{s,k}(\hat{s}_k)^{-1} = U_{s,k}^{\text{op}} \) and the Bruhat decomposition \( G_{k,s}' = B'_{k,s}' \cup U_{s,k}\hat{s}_kB'_{k,s}' \) implies the existence of a canonical bijection \( \overline{u}^{\text{op}} \mapsto \overline{u} : (U_{s,k}^{\text{op}} \cup \{1\}) \to (U_{s,k} \cup \{1\}) \) respecting the cosets \( \overline{B_{k,s}'} = \overline{\hat{s}_k}B'_{k,s}' \). Via the preceding identifications we get the wanted bijection \((5.3)\).

For \( v \in e(V) \) and \( z \in Z^0 \cap M_2' \) we have \( vT_s = v, z1_U = T_z1_U \) and \( v \otimes T_z1_U = vT_z \otimes 1_U \) therefore \( Z^0 \cap M_2' \) acts trivially on \( V \otimes 1_U \). The action of the group \( (Z^0 \cap M_2') U \) on \( V \otimes 1_U \) is also trivial. As the image of \( Z^0 \cap M_2' \) in \( G_{s,k} \) contains \( Z_k \cap G_{s,k}' \),
\[
(5.4) \quad u\hat{s}(v \otimes 1_U) = u^{\text{op}}(v \otimes 1_U)
\]
when \( u \) and \( u^{\text{op}} \) are not units and correspond via the bijection \((5.3)\). So we have
\[
(5.4) \quad v \otimes T_s(\hat{s}1_U) - (v \otimes 1_U) = v \otimes T_s(\hat{s}^21_U) - v \otimes \hat{s}1_U
\]
We can move \( T_s \) on the other side of \( \otimes \) and as \( vT_s = q_s v \) (Corollary \((3.9)\)), we can replace \( T_s \) by \( q_s \). We have \( v \otimes \hat{s}^21_U = v \otimes T_{s^{-2}}1_U \) because \( \hat{s}^2 \in Z^0 \cap M_2' \) normalizes \( U \); as we can move \( T_{s^{-2}} \) on the other side of \( \otimes \) and as \( vT_{s^{-2}} = v \) we can forget \( \hat{s}^2 \). So \((5.4)\) is equivalent to
\[
(5.4) \quad (q_s + 1)(v \otimes 1_U - \hat{s}(v \otimes 1_U)) = 0.
\]
Combining the two lemmas we obtain:

Proposition 5.7. When \( \mathcal{V} \) is extendible to \( \mathcal{H} \) and has no \( q_s + 1 \)-torsion for any \( s \in S_{M_2}^{\text{aff}} \), then \( M_2' \) acts trivially on \( e(V) \otimes_{\mathcal{H}R} X \) and \( \Phi^G \) is an \( R[G] \)-isomorphism.
Example 5.8. Let $G = GL(2, F)$ and $R$ an algebraically closed field where $q_{s_0} + 1 = q_{s_1} + 1 = 0$ and $S_{\text{aff}} = \{s_0, s_1\}$. (Note that $q_{s_0} = q_{s_1}$ is the order of the residue field of $R$.) Then the dimension of $1_H \otimes_{\mathcal{H}_R} X$ is infinite, in particular $1_H \otimes_{\mathcal{H}_R} X \neq 1_G$.

Indeed, the Steinberg representation $\text{St}_G = (\text{Ind}_G^G 1_Z)/1_G$ of $G$ is an indecomposable representation of length 2 containing an irreducible infinite dimensional representation $\pi$ with $\pi^\mu = 0$ of quotient the character $(-1)^{\text{val} \circ \text{det}}$. This follows from the proof of Theorem 3 and from Proposition 24 in [Vig89]. The kernel of the quotient map $\text{St}_G \otimes (-1)^{\text{val} \circ \text{det}} \to 1_G$ is infinite dimensional without a non-zero $U$-invariant vector. As the characteristic of $R$ is not $p$, the functor of $U$-invariants is exact hence $(\text{St}_G \otimes (-1)^{\text{val} \circ \text{det}})^\mu = 1_H$. As $- \otimes_{\mathcal{H}_R} R[U \setminus G]$ is the left adjoint of $(-)^\mu$ there is a non-zero homomorphism

$$1_H \otimes_{\mathcal{H}_R} X \to \text{St}_G \otimes (-1)^{\text{val} \circ \text{det}}$$

with image generated by its $U$-invariants. The homomorphism is therefore surjective.

5.2. $\mathcal{V}$ extensible to $\mathcal{H}$. Let $P = MN$ be a standard parabolic subgroup of $G$ with $\Delta_P$ and $\Delta \setminus \Delta_P$ orthogonal. We still suppose that the $\mathcal{H}_{M,R}$-module $\mathcal{V}$ is extensible to $\mathcal{H}$, but now $P \subset Q \subset G$. So we have $\text{Ind}_H(P, \mathcal{V}, Q) = e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G 1)^\mu$ and $\text{Ind}_G(P, \sigma, Q) = e(\sigma) \otimes_R \text{St}_Q$ where $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} X_M$. We compare the images by $- \otimes_{\mathcal{H}_R} X$ of the $\mathcal{H}_R$-modules $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G 1)^\mu$ and $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \sigma)^\mu$ with the smooth $R$-representations $e(\sigma) \otimes_{\mathcal{H}_R} \text{Ind}_Q^G 1$ and $e(\sigma) \otimes_{\mathcal{H}_R} \text{St}_Q$ of $G$.

As $- \otimes_{\mathcal{H}_R} X$ is left adjoint of $(-)^\mu$, the $\mathcal{H}_R$-homomorphism $v \otimes f \mapsto v \otimes 1_{U_M} \otimes f : e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G 1)^\mu \to (e(\sigma) \otimes_{\mathcal{H}_R} \text{Ind}_Q^G 1)^\mu$ gives by adjunction an $R[G]$-homomorphism

$$v \otimes f \otimes 1_H \mapsto v \otimes 1_{U_M} \otimes f : (e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G 1)^\mu) \otimes_{\mathcal{H}_R} X \xrightarrow{\Phi_Q} e(\sigma) \otimes_R \text{Ind}_Q^G 1.$$  

When $Q = G$ we have $\Phi_Q^G = \Phi_Q$. By Remark 3.10, $\Phi_Q^G$ is surjective. Proposition 5.7 applies with $M_Q$ instead of $G$ and gives the $R[M_Q]$-homomorphism

$$v \otimes 1_{U_{M_Q}} \mapsto v \otimes 1_{U_M} : e_{H_Q}(\mathcal{V}) \otimes_{H_Q,R} X_{M_Q} \xrightarrow{\Phi_Q} e_Q(\sigma).$$

Proposition 5.9. The $R[G]$-homomorphism $\Phi_Q^G$ is an isomorphism if $\Phi^Q$ is an isomorphism, in particular if $\mathcal{V}$ has no $q_{s} + 1$-torsion for any $s \in S^\text{aff} \cap M_Q$.

Proof. The proposition follows from another construction of $\Phi_Q^G$ that we now describe. Proposition 4.5 gives the $\mathcal{H}_R$-module isomorphism

$$v \otimes f_{QU} \mapsto v \otimes 1_H : (e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G 1)^\mu) \to \text{Ind}_H^H(e_{H_Q}(\mathcal{V})) = e_{H_Q}(\mathcal{V}) \otimes_{H_{M_Q,R}} H.$$

We have the $R[G]$-isostructure [OV17, Corollary 4.7]

$$v \otimes 1_H \otimes 1_H \mapsto f_{QU, v \otimes 1_{UM_Q}} : \text{Ind}_H^H(e_{H_Q}(\mathcal{V}) \otimes_{H_R} X) \to \text{Ind}_Q^G(e_{H_Q}(\mathcal{V}) \otimes_{H_Q,R} X_{M_Q})$$

and the $R[G]$-isomorphism ***

$$f_{QU, v \otimes 1_{UM}} \mapsto v \otimes 1_{UM} \otimes f_{QU} : \text{Ind}_Q^G(e_Q(\sigma)) \to e(\sigma) \otimes_R \text{Ind}_Q^G 1.$$ 

From $\Phi^Q$ and these three homomorphisms, there exists a unique $R[G]$-homomorphism

$$(e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G 1)^\mu) \otimes_{\mathcal{H}_R} X \to e(\sigma) \otimes_R \text{Ind}_Q^G 1$$

sending $v \otimes f_{QU} \otimes 1_H$ to $v \otimes 1_{U_M} \otimes f_{QU}$. We deduce: this homomorphism is equal to $\Phi_Q^G$, $\mathcal{V} \otimes 1_{QU} \otimes 1_H$ generates $(e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G 1)^\mu) \otimes_{\mathcal{H}_R} X$, if $\Phi^Q$ is an isomorphism then $\Phi_Q^G$ is an
isomorphism. By Proposition 5.7, if $V$ has no $q_s + 1$-torsion for any $s \in S_{M_r \cap M_Q}$, then $\Phi^G_Q$ and $\Phi^G_{Q'}$ are isomorphisms.

We recall that the $H_{M_r}$-module $V$ is extensible to $H$.

**Proposition 5.10.** The $R[G]$-homomorphism $\Phi^G_{Q'}$ induces an $R[G]$-homomorphism

$$(e(V) \otimes_R (St^G_{Q'}))^H \otimes_{H_r} \mathbb{X} \to e(\sigma) \otimes_R St^G_Q,$$

It is an isomorphism if $\Phi^G_{Q'}$ is an $R[G]$-isomorphism for all parabolic subgroups $Q'$ of $G$ containing $Q$, in particular if $V$ has no $q_s + 1$-torsion for any $s \in S_{M_r}^\text{aff}$.

**Proof.** The proof is straightforward, with the arguments already developed for Proposition 4.5 and Theorem 4.9. The representations $e(\sigma) \otimes_R St^G_Q$ and $(e(V) \otimes_R (St^G_{Q'}))^H \otimes_{H_r} \mathbb{X}$ of $G$ are the cokernels of the natural $R[G]$-homomorphisms

$$\oplus_{Q' \leq Q'} e(\sigma) \otimes_R \text{Ind}^G_{Q'} 1 \xrightarrow{id \otimes \alpha} e(\sigma) \otimes_R \text{Ind}^G_Q 1,$$

$$\oplus_{Q' \leq Q'} (e(V) \otimes_R (\text{Ind}^G_{Q'} 1)^H) \otimes_{H_r} \mathbb{X} \xrightarrow{id \otimes \alpha^H \otimes id} (e(V) \otimes_R (\text{Ind}^G_Q 1)^H) \otimes_{H_r} \mathbb{X}.$$ These $R[G]$-homomorphisms make a commutative diagram with the $R[G]$-homomorphisms $\oplus_{Q' \leq Q'} \Phi^G_{Q'}$ and $\Phi^G_Q$ going from the lower line to the upper line. Indeed, let $v \otimes f_{Q'U} \otimes 1_U \in (e(V) \otimes_R (\text{Ind}^G_{Q'} 1)^H) \otimes_{H_r} \mathbb{X}$. One hand, it goes to $v \otimes f_{Q'U} \otimes 1_U \in (e(V) \otimes_R (\text{Ind}^G_{Q'} 1)^H) \otimes_{H_r} \mathbb{X}$ by the horizontal map, and then to $v \otimes 1_{U_M} \otimes f_{Q'U} \otimes (\text{Ind}^G_{Q'} 1^H)$ by the vertical map. On the other hand, it goes to $v \otimes 1_{U_M} \otimes f_{Q'U}$ by the vertical map, and then to $v \otimes 1_{U_M} \otimes f_{Q'U} \otimes (\text{Ind}^G_{Q'} 1^H)$ by the horizontal map. One deduces that $\Phi^G_{Q'}$ induces an $R[G]$-homomorphism $(e(V) \otimes_R (St^G_{Q'})^H) \otimes_{H_r} \mathbb{X} \to e(\sigma) \otimes_R St^G_Q$, which is an isomorphism if $\Phi^G_{Q'}$ is an $R[G]$-isomorphism for all $Q \subset Q'$.

5.3. **General.** We consider now the general case: let $P = MN \subset Q$ be two standard parabolic subgroups of $G$ and $V$ a non-zero right $H_{M_r}$-module with $Q \subset P(V)$. We recall $I_H(P, V, Q) = \text{Ind}^H_{H_r \cap H(M_V)}((e(V) \otimes_R (St^P_{Q'})^H) \otimes_{H_M} X_M)$ and $\sigma = V \otimes_{H_{M_r}} X_M$ (Proposition 5.4). There is a natural $R[G]$-homomorphism

$$I_H(P, V, Q) \otimes_{H_r} \mathbb{X} \xrightarrow{\Phi^G_P} \text{Ind}^G_P(e_M(V)(\sigma) \otimes_R St^P_Q(V))$$

obtained by composition of the $R[G]$-isomorphism [OV17, Corollary 4.7] (proof of Proposition 5.9):

$$I_H(P, V, Q) \otimes_{H_r} \mathbb{X} \to \text{Ind}^G_P((e(V) \otimes_R (St^M_{Q\cap M(V)})^H \otimes_{H_{M_r}} X_M(V)),$$

with the $R[G]$-homomorphism

$$\text{Ind}^G_P((e(V) \otimes_R (St^P_{Q'})^H) \otimes_{H_{M_r}} X_M(V)) \to \text{Ind}^G_P(e_M(V)(\sigma) \otimes_R St^P_{Q'}),$$

image by the parabolic induction $\text{Ind}^G_P$ of the homomorphism

$$(e(V) \otimes_R (St^P_{Q'})^H) \otimes_{H_{M_r}} X_M(V) \to e_M(V)(\sigma) \otimes_R St^P_{Q'},$$

induced by the $R[M(V)]$-homomorphism $\Phi^P_{Q'} = \Phi^M_{Q \cap M(V)}$ of Proposition 5.10 applied to $M(V)$ instead of $G$. 


This homomorphism $\Phi^G_I$ is an isomorphism if $\Phi^P_Q$ is an isomorphism, in particular if $V$ has no $q_s$-torsion for any $s \in S^\text{aff}_{M_2}$ where $\Delta_{M_2} = \Delta_{M(V)} \setminus \Delta_M$ (Proposition 5.10). We get the main theorem of this section:

**Theorem 5.11.** Let $(P = MN, V, Q)$ be an $H_R$-triple and $\sigma = V \otimes_{H_{M,R}} R[\mathcal{U}_{M} \setminus M]$. Then, $(P, \sigma, Q)$ is an $R[G]$-triple. The $R[G]$-homomorphism

$$I_H(P, V, Q) \otimes_{H_R} R[\mathcal{U} \setminus G] \xrightarrow{\Phi^G_I} \text{Ind}^G_P(V)(e_M(V)(\sigma) \otimes_R \text{St}_Q^P(V))$$

is an isomorphism if $\Phi^P_Q$ is an isomorphism. In particular $\Phi^G_I$ is an isomorphism if $V$ has no $q_s + 1$-torsion for any $s \in S^\text{aff}_{M_2}$.

Recalling $I_G(P, \sigma, Q) = \text{Ind}^G_P(e(\sigma) \otimes_R \text{St}_Q^P(\sigma))$ when $\sigma \neq 0$, we deduce:

**Corollary 5.12.** We have:

$$I_H(P, V, Q) \otimes_{H_R} R[\mathcal{U} \setminus G] \simeq I_G(P, \sigma, Q), \text{ if } \sigma \neq 0, P(V) = P(\sigma) \text{ and } V \text{ has no } q_s + 1\text{-torsion for any } s \in S^\text{aff}_{M_2},$$

$$I_H(P, V, Q) \otimes_{H_R} R[\mathcal{U} \setminus G] = I_G(P, \sigma, Q) = 0, \text{ if } \sigma = 0.$$

Recalling $P(V) = P(\sigma)$ if $\sigma \neq 0$, $R$ is a field of characteristic $p$ and $V$ simple supersingular (Proposition 5.4.4)), we deduce:

**Corollary 5.13.** $I_H(P, V, Q) \otimes_{H_R} R[\mathcal{U} \setminus G] \simeq I_G(P, \sigma, Q)$ if $R$ is a field of characteristic $p$ and $V$ simple supersingular.

6. Vanishing of the smooth dual

Let $V$ be an $R[G]$-module. The dual $\text{Hom}_R(V, R)$ of $V$ is an $R[G]$-module for the contragredient action: $gL(gv) = L(v)$ if $g \in G$, $L \in \text{Hom}_R(V, R)$ is a linear form and $v \in V$. When $V \in \text{Mod}^\text{aff}_R(G)$ is a smooth $R$-representation of $G$, the dual of $V$ is not necessarily smooth. A linear form $L$ is smooth if there exists an open subgroup $H \subset G$ such that $L(hv) = L(v)$ for all $h \in H, v \in V$; the space $\text{Hom}_R(V, R)^\infty$ of smooth linear forms is a smooth $R$-representation of $G$, called the smooth dual (or smooth contragredient) of $V$. The smooth dual of $V$ is contained in the dual of $V$.

**Example 6.1.** When $R$ is a field and the dimension of $V$ over $R$ is finite, the dual of $V$ is equal to the smooth dual of $V$ because the kernel of the action of $G$ on $V$ is an open normal subgroup $H \subset G$; the action of $G$ on the dual $\text{Hom}_R(V, R)$ is trivial on $H$.

We assume in this section that $R$ is a field of characteristic $p$. Let $P = MN$ be a parabolic subgroup of $G$ and $V \in \text{Mod}^\infty_R(M)$. Generalizing the proof given in [Vig07] 8.1 when $G = GL(2, F)$ and the dimension of $V$ is 1, we show:

**Proposition 6.2.** If $P \neq G$, the smooth dual of $\text{Ind}^G_P(V)$ is 0.

**Proof.** Let $L$ be a smooth linear form on $\text{Ind}^G_P(V)$ and $K$ an open pro-$p$-subgroup of $G$ which fixes $L$. Let $J$ an arbitrary open subgroup of $K$, $g \in G$ and $f \in (\text{Ind}^G_P(V))^J$ with support $PgJ$. We want to show that $L(f) = 0$. Let $J'$ be any open normal subgroup of $J$ and let $\varphi$ denote the function in $(\text{Ind}^G_P(V))^J$ with support $PgJ'$ and value $\varphi(g) = f(g)$ at $g$. For $j \in J$ we have $L(j\varphi) = L(\varphi)$, and the support of $j\varphi(x) = \varphi(xj)$ is $PgJ'j^{-1}$. The function $f$ is the sum of translates $j\varphi$, where $j$ ranges through the left cosets of the image $X$ of $g^{-1}Pg \cap J$.
in $J/J'$, so that $L(f) = rL(\varphi)$ where $r$ is the order of $X$ in $J/J'$. We can certainly find $J'$ such that $r \neq 1$, and then $r$ is a positive power of $p$. As the characteristic of $C$ is $p$ we have $L(f) = 0$.

The module $R[\mathcal{U}\backslash G]$ is contained in the module $R^{\mathcal{U}\backslash G}$ of functions $f : \mathcal{U}\backslash G \to R$. The actions of $\mathcal{H}$ and of $G$ on $R[\mathcal{U}\backslash G]$ extend to $R^{\mathcal{U}\backslash G}$ by the same formulas. The pairing

$$(f, \varphi) \mapsto \langle f, \varphi \rangle = \sum_{g \in \mathcal{U}\backslash G} f(g)\varphi(g) : R^{\mathcal{U}\backslash G} \times R[\mathcal{U}\backslash G] \to R$$

identifies $R^{\mathcal{U}\backslash G}$ with the dual of $R[\mathcal{U}\backslash G]$. Let $h \in \mathcal{H}$ and $\bar{h} \in \mathcal{H}$, $\bar{h}(g) = h(g^{-1})$ for $g \in G$. We have

$$\langle f, h\varphi \rangle = \langle \bar{h}f, \varphi \rangle.$$ 

**Proposition 6.3.** When $R$ is an algebraically closed field of characteristic $p$, $G$ is not compact modulo the center and $\mathcal{V}$ is a simple supersingular right $\mathcal{H}_R$-module, the smooth dual of $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G]$ is 0.

**Proof.** Let $\mathcal{H}^{\text{aff}}_R$ be the subalgebra of $\mathcal{H}_R$ of basis $(T_w)_{w \in W'(1)}$ where $W'(1)$ is the inverse image of $W'$ in $W(1)$. The dual of $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G]$ is contained in the dual of $\mathcal{V} \otimes_{\mathcal{H}^{\text{aff}}_R} R[\mathcal{U}\backslash G]$; the $\mathcal{H}^{\text{aff}}_R$-module $\mathcal{V}|_{\mathcal{H}^{\text{aff}}_R}$ is a finite sum of supersingular characters [Vig15a]. Let $\chi : \mathcal{H}^{\text{aff}}_R \to R$ be a supersingular character. The dual of $\mathcal{V} \otimes_{\mathcal{H}^{\text{aff}}_R} R[\mathcal{U}\backslash G]$ is contained in the dual of $R[\mathcal{U}\backslash G]$ isomorphic to $R^{\mathcal{U}\backslash G}$. It is the space of $f \in R^{\mathcal{U}\backslash G}$ with $\bar{h}f = \chi(h)f$ for all $h \in \mathcal{H}^{\text{aff}}_R$. The smooth dual of $\chi \otimes_{\mathcal{H}^{\text{aff}}_R} R[\mathcal{U}\backslash G]$ is 0 if the dual of $\chi \otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G]$ has no non-zero element fixed by $\mathcal{U}$. Let us take $f \in R^{\mathcal{U}\backslash G}$ with $\bar{h}f = \chi(h)f$ for all $h \in \mathcal{H}^{\text{aff}}_R$. We shall prove that $f = 0$. We have $T_w = T_{w^{-1}}$ for $w \in W(1)$.

The elements $(T_t)_{t \in Z_k}$ and $(T_s)_{s \in S^{\text{aff}}}$ where $s$ is an admissible lift of $s$ in $W^{\text{aff}}(1)$, generate the algebra $\mathcal{H}^{\text{aff}}_R$ and

$$T_tT_w = T_{tw}, \quad T_sT_w = \begin{cases} T_{\tilde{sw}} & \text{if } \tilde{w} > w, \\ T_{\tilde{sw}} & \text{if } \tilde{w} < w. \end{cases}$$

with $c_s = -|Z'_{s,t}| \sum_{t \in Z_{k,s}} T_t$ because the characteristic of $R$ is $p$ [Vig16] Proposition 4.4].

Expressing $f = \sum_{w \in W(1)} a_wT_w$, $a_w \in R$, as an infinite sum, we have

$$T_tf = \sum_{w \in W(1)} a_{t^{-1}w}T_w, \quad T_sf = \sum_{w \in W(1), \tilde{w} < w} (a_{(\tilde{s})^{-1}w} + a_\tilde{w}c_{\tilde{s}})T_w,$$

where $<$ denote the Bruhat order of $W(1)$ associated to $S^{\text{aff}}$ [Vig16] and [Vig16] Proposition 4.4]. A character $\chi$ of $\mathcal{H}^{\text{aff}}_R$ is associated to a character $\chi_k : Z_k \to R^*$ and a subset $J$ of $S^{\text{aff}}_{\chi_k} = \{ s \in S^{\text{aff}} \mid (\chi_k)|_{Z_k,s} \text{ trivial} \}$ [Vig15a Definition 2.7]. We have

$$\chi(T_t) = \chi_k(t) \quad t \in Z_k, \quad \chi(T_s) = \begin{cases} 0 & s \in S^{\text{aff}} \setminus J, \\ -1 & s \in J. \end{cases} \quad (\chi_k)(c_{\tilde{s}}) = \begin{cases} 0 & s \in S^{\text{aff}} \setminus S^{\text{aff}}_{\chi_k}, \\ -1 & s \in S^{\text{aff}}_{\chi_k}. \end{cases}$$

(6.1)
Therefore $\chi_k(t)f = \tilde{T}t f = T_{t^{-1}}f$ hence $\chi_k(t)a_w = a_{tw}$. We have $\chi(T_{\tilde{s}})f = \tilde{T}_s f = T_{(\tilde{s})^{-1}}f = T_{s} T_{(\tilde{s})^{-2}}f = \chi_k((\tilde{s})^2) T_{s}f$; as $(\tilde{s})^2 \in Z'_{k,s}$ \cite[three lines before Proposition 4.4]{Vig16} and $J \subset S^{\aff}_{\chi_k}$, we obtain

\begin{equation}
T_{\tilde{s}} f = \begin{cases} 0 & s \in S^{\aff} \setminus J, \\ -f & s \in J. \end{cases}
\end{equation}

Introducing $\chi_k(t)a_w = a_{tw}$ in the formula for $T_{\tilde{s}} f$, we get

\begin{align*}
\sum_{w \in W(1), \tilde{s}w < w} a_w c_{\tilde{s}} T_w &= -|Z'_{k,s}|^{-1} \sum_{w \in W(1), \tilde{s}w < w, t \in Z'_{k,s}} a_w T_{tw} \\
&= -|Z'_{k,s}|^{-1} \sum_{w \in W(1), \tilde{s}w < w, t \in Z'_{k,s}} a_{t^{-1}w} T_w \\
&= -|Z'_{k,s}|^{-1} \sum_{t \in Z'_{k,s}} \chi_k(t^{-1}) \sum_{w \in W(1), \tilde{s}w < w} a_w T_w \\
&= \chi_k(c_{\tilde{s}}) \sum_{w \in W(1), \tilde{s}w < w} a_w T_w.
\end{align*}

\begin{equation}
T_{\tilde{s}} f = \sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1}w} + a_w \chi_k(c_{\tilde{s}})) T_w
\end{equation}

\begin{align*}
&= \begin{cases} \sum_{w \in W(1), \tilde{s}w < w} a_{(\tilde{s})^{-1}w} T_w & s \in S^{\aff} \setminus S_{\chi_k}^{\aff}, \\ \sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1}w} - a_w) T_w & s \in S_{\chi_k}^{\aff}. \end{cases}
\end{align*}

From the last equality and (6.2) for $T_{\tilde{s}} f$, we get:

\begin{equation}
a_{\tilde{s}w} = \begin{cases} 0 & s \in J \cup (S^{\aff} \setminus S_{\chi_k}^{\aff}), \tilde{s}w < w, \\ a_w & s \in S_{\chi_k}^{\aff} \setminus J. \end{cases}
\end{equation}

Assume that $a_w \neq 0$. By the first condition, we know that $w > \tilde{s}w$ for $s \in J \cup (S^{\aff} \setminus S_{\chi_k}^{\aff})$. The character $\chi$ is supersingular if for each irreducible component $X$ of $S^{\aff}$, the intersection $X \cap J$ is not empty and different from $X$ \cite[Definition 2.7, Theorem 6.18]{Vig15a}. This implies that the group generated by the $s \in S_{\chi_k}^{\aff} \setminus J$ is finite. If $\chi$ is supersingular, by the second condition we can suppose $w > \tilde{s}w$ for any $s \in S^{\aff}$. But there is no such element if $S^{\aff}$ is not empty. \hfill $\Box$

**Theorem 6.4.** Let $\pi$ be an irreducible admissible $R$-representation of $G$ with a non-zero smooth dual where $R$ is an algebraically closed field of characteristic $p$. Then $\pi$ is finite dimensional.

**Proof.** Let $(P, \sigma, Q)$ be a $R[G]$-triple with $\sigma$ supercuspidal such that $\pi \simeq I_G(P, \sigma, Q)$. The representation $I_G(P, \sigma, Q)$ is a quotient of $\text{Ind}_G^Q e_Q(\sigma)$ hence the smooth dual of $\text{Ind}_G^Q e_Q(\sigma)$ is not zero. From Proposition 6.2, $Q = G$. We have $I_G(P, \sigma, G) = e(\sigma)$. The smooth dual of $\sigma$ contains the smooth linear dual of $e(\sigma)$ hence is not zero. As $\sigma$ is supercuspidal, the $H_M$-module $\sigma^{H_M}$ contains a simple supersingular submodule $V$ \cite[Proposition 7.10, Corollary 7.11]{Vig15a}. The functor $- \otimes_{H_M} R[U_M \setminus M]$ being the right adjoint of $(-)^{H_M}$, the irreducible representation $\sigma$ is a quotient of $V \otimes_{H_M} R[U_M \setminus M]$, hence the smooth dual of
$V \otimes_{\mathcal{H}_{M,R}} R[U_M \backslash M]$ is not zero. By Proposition 6.3, $M = Z$. Hence $\sigma$ is finite dimensional and the same is true for $e(\sigma) = I_G(B, \sigma, G) \simeq \pi$. □

Remark 6.5. When the characteristic of $F$ is 0, Theorem 6.4 was proved by Kohlhaase for a field $R$ of characteristic $p$. He gives two proofs [Koh, Proposition 3.9, Remark 3.10], but none of them extends to $F$ of characteristic $p$. Our proof is valid without restriction on the characteristic of $F$ and does not use the results of Kohlhaase. Our assumption that $R$ is an algebraically closed field of characteristic $p$ comes from the classification theorem in [AHHV17].

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