Lower $N$-weighted Ricci curvature bound with $\varepsilon$-range and displacement convexity of entropies

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Abstract
In the present article, we provide a characterization of a lower $N$-weighted Ricci curvature bound for $N \in ]-\infty, 1] \cup [n, +\infty]$ with $\varepsilon$-range introduced by Lu-Minguzzi-Ohta [15] in terms of a convexity of entropies over Wasserstein space. We further derive various interpolation inequalities and functional inequalities.

Keywords: $N$-weighted Ricci curvature, Optimal transport theory.
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1 Introduction
In this paper, we present a characterization of a lower $N$-weighted Ricci curvature bound for $N \in ]-\infty, 1] \cup [n, +\infty]$ with $\varepsilon$-range introduced by Lu-Minguzzi-Ohta [15] by a convexity of entropies on the Wasserstein space via mass transport theory.

1.1 Background
We first recall the formulation of the weighted Ricci curvature, and some works on the comparison geometry. Let $(M, d, m)$ denote an $n$-dimensional weighted Riemannian manifold, namely, $M = (M, g)$ is an $n$-dimensional complete Riemannian manifold, $d$ is the Riemannian distance on $M$, and $m := e^{-f} \text{vol}_g$ for $f \in C^\infty(M)$. For $N \in ]-\infty, +\infty]$, the associated $N$-weighted Ricci curvature $\text{Ric}_f^N$ is defined as follows ([II, III]):

$$\text{Ric}_f^N := \text{Ric}_g + \nabla^2 f - \frac{df \otimes df}{N - n}.$$

Here when $N = +\infty$, we interpret the last term of the right hand side as the limit 0, and when $N = n$, we only consider a constant function $f$, and set $\text{Ric}_f^n := \text{Ric}_g$.  

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It is well-known that lower weighted Ricci curvature bounds imply various comparison geometric results. In the classical case of $N \in \mathbb{N}$, under a curvature condition
\[ \text{Ric}_f^N \geq K g \] (1.1)
for $K \in \mathbb{R}$, such investigations have been done by [13], [25], [31], and so on.

In recent years, the validity of the $N$-weighted Ricci curvature with $N \in [-\infty, n]$ has begun to be pointed out (see e.g., [5], [6], [7], [8], [9], [12], [15], [16], [17], [19], [21], [22], [23], [24], [26], [27], [32], [33]). Wylie-Yeroshkin [33] have proposed a curvature condition
\[ \text{Ric}_1^N \geq (n-1)\kappa e^{-\frac{4f}{n-1}} g \] (1.2)
for $\kappa \in \mathbb{R}$ in view of the study of projectively equivalent affine connection, and established an optimal Laplacian comparison theorem, Bonnet-Myers theorem, Bishop-Gromov volume comparison theorem. Remark that before the work of them, Wylie [32] has obtained a splitting theorem of Cheeger-Gromoll type for $\kappa = 0$. For $N \in [-\infty, 1]$, the first named author and Li [7] have extended the condition (1.2) to
\[ \text{Ric}_f^N \geq (n-N)\kappa e^{-\frac{4f}{n-N}} g, \] (1.3)
and generalized the comparison theorems in [33].

Very recently, Lu-Minguzzi-Ohta [15] have suggested a new approach that enables us to investigate the conditions (1.1) with $K = (N - 1)\kappa$, (1.2) and (1.3) in a unified way. For $N \in [-\infty, 1] \cup [n, +\infty]$, they have introduced the notion of the $\varepsilon$-range:
\[ \varepsilon = 0 \text{ for } N = 1, \quad \varepsilon \in [-\varepsilon_0, \sqrt{\varepsilon_0}] \text{ for } N \neq 1, n, \quad \varepsilon \in \mathbb{R} \text{ for } N = n, \] (4.4)
where
\[ \varepsilon_0 := \frac{N - 1}{N - n}. \]
When $N = +\infty$, we interpret $\varepsilon_0$ as the limit 1; in particular $\varepsilon \in [-1, 1]$. Within this $\varepsilon$-range, they have considered a curvature condition
\[ \text{Ric}_f^N \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)^f}{n-1}} g \] (1.5)
for $\kappa \in \mathbb{R}$, which covers the previous curvature conditions by running $\varepsilon$ over $\varepsilon$-range. Here $c = c_{N, \varepsilon} \in [0, 1]$ is the associated positive constant defined by
\[ c := \frac{1}{n-1} \left( 1 - \varepsilon \frac{2N - n}{N - 1} \right) \] (1.6)
if $N \neq 1$, and $c := (n-1)^{-1}$ if $N = 1$. Here we interpret $c$ as the limit $(n-1)^{-1}(1-\varepsilon^2)$ in the case of $N = +\infty$. When $N \in [n, +\infty]$ and $\varepsilon = 1$ with $c = (N - 1)^{-1}$, the curvature condition (1.5) covers (1.1) with $K = (N - 1)\kappa$. Also, when $N = 1$ and $\varepsilon = 0$ with $c = (n-1)^{-1}$, it does (1.2), and when $N \in [-\infty, 1]$ and $\varepsilon = \varepsilon_0$ with $c = (n - N)^{-1}$, it does (1.3). Under the condition (1.5), they have developed comparison geometry in the framework of weighted Finsler manifolds and weighted Finsler space-times.
1.2 Main results

Let us introduce our main results. Lower $N$-weighted Ricci curvature bounds are well-known to be characterized by convexities of entropies on the Wasserstein space. In the classical case of $N \in [n, +\infty[$, the characterization of \((1.1)\) is due to Sturm \cite{28, 29}, and Lott-Villani \cite{14}. Based on such a result, they have independently introduced the so-called curvature-dimension condition \(CD(K, N)\) for metric measure spaces that is equivalent to \((1.1)\) in the smooth setting. The second named author \cite{27} gave a characterization of \(\rho\) where

\[ \psi \mid_{\mathbb{R}^n} \text{Lipschitz} \]

with respect to \(N\), defined by \(\psi(r) = r\) for \(r \in \mathbb{R}\), and \(\psi(0) = 0\). For \(t \in [0, 1]\), the function \(\psi(r)\) is called the \(\text{Rényi entropy}\).

We now aim to provide a characterization of the curvature condition \((1.5)\). Let \(U \in DC_{N, \varepsilon}\), a functional \(U_m\) on \(\mathcal{P}_2(M)\) is defined by

\[ U_m(\mu) := \int_M U(\rho)d\mu, \quad (1.7) \]

where \(\rho\) is the density of the absolutely continuous part in the Lebesgue decomposition of \(\mu\) with respect to \(m\). For a function \(H \in DC_{N, \varepsilon}\) defined by \(H(r) := c^{-1}(c+1)r(1-r^{c+1})\), the functional \(H_m\) on \(\mathcal{P}_2(M)\) defined as \((1.7)\) is called the \(\text{Rényi entropy}\).

Following \cite{27}, we introduce a twisted coefficient in our setting. We define two lower semi continuous functions \(d_{N, \varepsilon, f, t}, d_{N, \varepsilon, f} : M \times M \to \mathbb{R}\) by

\[ d_{N, \varepsilon, f, t}(x, y) := \inf_{\gamma} \int_0^1 dt(x, y) \exp \left\{ \frac{2(1-\varepsilon)T(t, y)}{\varepsilon - 1} \right\} d\xi, \quad d_{N, \varepsilon, f} := d_{N, \varepsilon, f, 1} \]

for \(t \in [0, 1]\), where the infimum is taken over all unit speed minimal geodesics \(\gamma : [0, d(x, y)] \to M\) from \(x\) to \(y\). The function \(d_{N, \varepsilon, f}\) is called the \(\text{re-parametrized distance}\) (cf. \cite{33}). Note that for \(t \in [0, 1]\), the function \(d_{N, \varepsilon, f, t}\) is not always symmetric. For \(\kappa \in \mathbb{R}\), let \(s_{\kappa}(s)\) stand for a unique solution of the Jacobi equation \(\psi''(s) + \kappa \psi(s) = 0\) with \(\psi(0) = 0, \psi'(0) = 1\), and \(C_{\kappa}\) the diameter of the space form of constant curvature \(\kappa\). More precisely, they can be written as

\[ s_{\kappa}(s) = \begin{cases} \sin \frac{\sqrt{\kappa}s}{\kappa} & \text{if } \kappa > 0, \\ s & \text{if } \kappa = 0, \\ \sinh \frac{\sqrt{|\kappa|}s}{\sqrt{|\kappa|}} & \text{if } \kappa < 0, \end{cases} \]

\[ C_{\kappa} = \begin{cases} \frac{\pi}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ \infty & \text{if } \kappa \leq 0. \end{cases} \]

For \(t \in ]0, 1]\), we define the \(\text{twisted coefficient}\) \(\beta_{\kappa, N, \varepsilon, f, t} : M \times M \to \mathbb{R} \cup \{+\infty\}\) by

\[ \beta_{\kappa, N, \varepsilon, f, t}(x, y) := \left( \frac{s_{\kappa}(d_{N, \varepsilon, f, t}(x, y))}{t s_{\kappa}(d_{N, \varepsilon, f}(x, y))} \right)^{-1} \]

if \(d_{N, \varepsilon, f}(x, y) \in ]0, C_{\kappa}\); \(\beta_{\kappa, N, \varepsilon, f, t}(x, y) = 1\) if \(x = y\); otherwise, \(\beta_{\kappa, N, \varepsilon, f, t}(x, y) := +\infty\).
Remark 1.1 The definition of the twisted coefficient for \( x = y \) is reasonable since we see \( \beta_{\kappa,N,\varepsilon,f,t}(x,y) \to 1 \) as \( d(x,y) \to 0 \) (see Appendix for the proof).

Let \( \mathcal{P}_2^\infty(M) \) denote the set of all Borel probability measures in \( \mathcal{P}_2(M) \) that are absolutely continuous with respect to \( m \). We now introduce the following convexity properties:

**Definition 1.2** Let \( \kappa \in \mathbb{R}, N \in ]-\infty, 1[ \cup [n, +\infty], \) and \( \varepsilon \in \mathbb{R} \) in the range (1.4). We say that \((M,d,m)\) satisfies the twisted curvature-dimension condition \( \text{TwCD}(\kappa,N,\varepsilon) \) if for every pair \( \mu_0, \mu_1 \in \mathcal{P}_2^\infty(M) \),

\[
U_m(\mu_t) \leq (1-t) \int_{M^2} U \left( \frac{\rho_0(x)}{\beta_{\kappa,N,\varepsilon,f,1-t}(y,x)} \frac{\beta_{\kappa,N,\varepsilon,f,1-t}(y,x)}{\rho_0(x)} \right) \pi(dx,dy) + t \int_{M^2} U \left( \frac{\rho_1(y)}{\beta_{\kappa,N,\varepsilon,f,1-t}(x,y)} \frac{\beta_{\kappa,N,\varepsilon,f,1-t}(x,y)}{\rho_1(y)} \right) \pi(dx,dy) \tag{1.9}
\]

for all \( U \in DC_{N,\varepsilon} \) and \( t \in ]0,1[ \), where \( \rho_i \) is the density of \( \mu_i \) with respect to \( m \) for each \( i = 0, 1 \), and \( \pi \) is a unique optimal coupling of \((\mu_0,\mu_1)\), and \((\mu_t)_{t \in [0,1]}\) is a unique minimal geodesic in the \( L^2 \)-Wasserstein space \((\mathcal{P}_2(M),W_2)\) from \( \mu_0 \) to \( \mu_1 \), which lies in \( \mathcal{P}_2^\infty(M) \).

Remark 1.3 In Definition 1.2 we only consider \( U \in DC_{N,\varepsilon} \) such that (1.9) makes sense for all \( \mu_0, \mu_1 \in \mathcal{P}_2^\infty(M) \). Notice that for \( H \in DC_{N,\varepsilon} \) defined as \( H(r) := c^{-1}(c+1)r(1-\frac{r^2}{\varepsilon^2}) \), (1.9) makes sense for all \( \mu_0, \mu_1 \). For general \( U \in DC_{N,\varepsilon} \), such a property is guaranteed by a condition

\[
\int_M \frac{1}{(1+d(x,x_0)^2)^{1/\varepsilon}} m(dx) < +\infty
\]

for some \( x_0 \in M \) concerning the reference measure \( m \) (cf. [30] Theorems 17.8, 17.28).

**Definition 1.4** Let \( \kappa \in \mathbb{R}, N \in ]-\infty, 1[ \cup [n, +\infty], \) and \( \varepsilon \in \mathbb{R} \) in the range (1.4). We say that \((M,d,m)\) satisfies the relaxed twisted curvature-dimension condition \( \text{TwCD}_{\text{rel}}(\kappa,N,\varepsilon) \) if the inequality (1.2) holds for \( H \in DC_{N,\varepsilon} \) defined as \( H(r) := c^{-1}(c+1)r(1-\frac{r^2}{\varepsilon^2}) \).

Remark 1.5 In the case of \( N \in [n, +\infty[ \) and \( \varepsilon = 1 \), the condition \( \text{TwCD}(\kappa,N,1) \) coincides with the curvature-dimension condition \( \text{CD}((N-1)\kappa,N) \) in the sense of Lott-Villani [14]. Similarly, \( \text{TwCD}_{\text{rel}}(\kappa,N,1) \) coincides with \( \text{CD}((N-1)\kappa,N) \) in the sense of Sturm [28, 29]. In the case of \( N = 1 \) with \( \varepsilon = 0 \), the conditions \( \text{TwCD}(\kappa,1,0) \) and \( \text{TwCD}_{\text{rel}}(\kappa,1,0) \) coincide with the \( \kappa \)-twisted curvature bound and the relaxed one in [27], respectively.

We now state our main theorem.

**Theorem 1.6** Let \( \kappa \in \mathbb{R}, N \in ]-\infty, 1[ \cup [n, +\infty], \) and \( \varepsilon \in \mathbb{R} \) in the range (1.4). We additionally assume that if \( N \neq 1,n, \) then \( \varepsilon \neq 0. \) Then the following are equivalent:

1. \( \text{Ric}^N_f \geq c^{-1} \kappa e^{\frac{4(1-\varepsilon)t}{\kappa(1-\varepsilon)}} g; \)
2. \((M,d,m)\) satisfies \( \text{TwCD}(\kappa,N,\varepsilon); \)

for some \( c \in \mathbb{R} \) and \( t \in [0,1] \).
(3) \((M,d,m)\) satisfies TwCD\(_{\text{rel}}(\kappa,N,\varepsilon)\).

**Remark 1.7** The restriction \(\varepsilon \neq 0\) is due to a technical issue, which is not a natural requirement. The authors do not know whether this can be removed.

In the case of \(N \in [n, +\infty[\) and \(\varepsilon = 1\), Theorem 1.6 is nothing but the well-known characterization of the curvature condition (1.1) with \(K = (N - 1)\kappa\) by CD\((N - 1)\kappa,N)\) (see [14, Theorem 4.22], and also [29, Theorem 1.7]). When \(N \in [n, +\infty[\), Theorem 1.6 for \(\varepsilon \neq 1\) is new and not treated in the literature.

The second named author [27] has shown Theorem 1.6 when \(N = 1\) (see [27, Theorem 1.4]). Theorem 1.6 for \(N \in ]-\infty, 1[\) is a new result; in particular, by letting \(\varepsilon = \varepsilon_0\), one can obtain the following characterization of the condition (1.3):

**Corollary 1.8** Under the same setting as in Theorem 1.6, if \(N \in ]-\infty, 1[\), then the following statements are equivalent:

1. \(\text{Ric}_f^N \geq (n - N)\kappa e^{-\frac{4}{n-\kappa}}g;\)
2. \((M,d,m)\) satisfies TwCD\((\kappa,N,\varepsilon_0)\); 
3. \((M,d,m)\) satisfies TwCD\(_{\text{rel}}(\kappa,N,\varepsilon_0)\).

We also notice that as a corollary of the proof of Theorem 1.6, we obtain the following (see Proposition 4.1 below):

**Corollary 1.9** Under the same setting as in Theorem 1.6, the implication from (1) to (2) always holds (without the restriction \(\varepsilon \neq 0\)).

From this viewpoint, under the curvature condition (1.5), we derive several interpolation inequalities such as \(p\)-mean inequality, Prékopa-Leindler inequality, Borell-Brascamp-Lieb inequality, and Brunn-Minkowski inequality (see Subsection 4.3), and also study functional inequalities (see Section 5).

## 2 Preliminaries

This section is devoted to basics on optimal transport theory and comparison geometry.

### 2.1 Optimal transport theory

We recall some basic facts on the optimal transport theory. Referring to [3], [20], [30], we use the same notation and terminology as in the preliminaries of [27] (see [27, Subsection 2.2]). On a metric space \((Z,d_Z)\), a curve \(\gamma : [0,l] \to Z\) is said to be a minimal geodesic if there is \(a \geq 0\) such that \(d_Z(\gamma(t_0),\gamma(t_1)) = a|t_0 - t_1|\) for all \(t_0, t_1 \in [0,l]\). Moreover, if \(a = 1\), then \(\gamma\) is said to be a unit speed minimal geodesic.

Let \(\mathcal{P}(M)\) be the set of all Borel probability measures on \(M\). For \(\mu, \nu \in \mathcal{P}(M)\), a Borel probability measure \(\pi\) on \(M \times M\) is said to be a coupling of \((\mu, \nu)\) if \(\pi(X \times M) = \mu(X)\) and \(\pi(M \times X) = \nu(X)\) for all Borel subsets \(X \subset M\). Let \(\Pi(\mu, \nu)\) stand for the set of all
couplings of \((\mu, \nu)\). Recall that \(\mathcal{P}_2(M)\) denotes the set of all Borel probability measures on \(M\) with finite second moment, namely, \(\mu \in \mathcal{P}_2(M)\) if
\[
\int_M d(x, x_0)^2 \mu(dx) < +\infty
\]
for some \(x_0 \in M\). The \(L^2\)-Wasserstein distance function \(W_2\) is defined as
\[
W_2(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left( \int_{M^2} d(x, y)^2 \pi(dx dy) \right)^{1/2}.
\]  
(2.1)

The pair \((\mathcal{P}_2(M), W_2)\) is known to be a complete separable metric space (see e.g., [30, Theorem 6.18]), and called the \(L^2\)-Wasserstein space. A coupling \(\pi \in \Pi(\mu, \nu)\) is said to be optimal if it attains the infimum of (2.1). Recall the following fundamental result on the optimal coupling in smooth setting due to Brenier [2], McCann [18], and Figalli-Gigli [4] (see [2], [4, Theorem 1], [18, Theorem 3]):

**Theorem 2.1** For \(\mu \in \mathcal{P}^{ac}_2(M)\) and \(\nu \in \mathcal{P}_2(M)\), there is a locally semi-convex function \(\phi\) on an open subset \(\Omega\) of \(M\) with \(\mu(\Omega) = 1\) such that a map \(F_t\) defined by
\[
F_t(z) := \exp_z(t \nabla \phi(z))
\]
provides a unique optimal coupling \(\pi\) of \((\mu, \nu)\) via the pushforward measure \(\pi := (F_0 \times F_1)_* \mu\) of \(\mu\) by \(F_0 \times F_1\), and also determines a unique minimal geodesic \((\mu_t)_{t \in [0, 1]}\) in \((\mathcal{P}_2(M), W_2)\) from \(\mu\) to \(\nu\) via \(\mu_t := (F_t)_* \mu_0\).

The function \(\phi\) provided in Theorem 2.1 is called the **Kantorovich potential**, which is twice differentiable \(\mu\)-almost everywhere as a consequence of the Alexandrov-Bangert theorem. The Kantorovich potential \(\phi\) has the following properties (see [4, Theorem 1.1], [3, Proposition 4.1, Corollary 5.2]): If \(\phi\) is twice differentiable at \(x\), then \(F_t(x)\) does not belong to the cut locus \(\text{Cut}(x)\) of \(x\), and the differential \((dF_t)_x\) is well-defined for every \(t \in [0, 1]\). Also, \(\phi\) satisfies the following (see [30, Theorem 8.7]): The curve \((\mu_t)_{t \in [0, 1]}\) lies in \(\mathcal{P}^{ac}_2(M)\). We finally recall the Monge-Ampère equation (see [30, Theorem 11.1]):

**Theorem 2.2** Let \(\mu, \nu \in \mathcal{P}^{ac}_2(M)\), and let \(\phi\) be the Kantorovich potential obtained in Theorem 2.1. Then for \(\mu\)-almost every \(x\), we have:

1. \(\phi\) is twice differentiable at \(x\);
2. the determinant \(\det(dF_t)_x\) is positive for every \(t \in [0, 1]\);
3. \(\rho_0(x) = \rho_1(F_1(x)) e^{-f(F_1(x)) + f(x)} \det(dF_1)_x\), where \(\rho_0\) and \(\rho_1\) are the densities of \(\mu\) and of \(\nu\) with respect to \(\mathfrak{m}\), respectively.
2.2 Comparison geometric results

We next review one of comparison geometric results, which will be used in the proof of the main theorem. Let $N \in (-\infty, 1] \cup [n, +\infty)$, and $\varepsilon \in \mathbb{R}$ in the range (1.4).

For $x \in M$, let $U_x M$ be the unit tangent sphere at $x$. For $v \in U_x M$, let $\gamma_v : [0, \infty) \to M$ denote the unit speed geodesic with initial conditions $\gamma_v(0) = x$ and $\dot{\gamma}_v(0) = v$. Define a function $s_{N,\varepsilon,f,v} : [0, +\infty] \to [0, s_{N,\varepsilon,f,v}(+\infty)]$ by

$$s_{N,\varepsilon,f,v}(t) := \int_0^t e^{-\frac{2(1-\varepsilon)f(\gamma_v(\xi))}{n-1}} d\xi.$$ We also set

$$\tau(v) := \sup\{t > 0 \mid d(x, \gamma_v(t)) = t\}, \quad \tau_{N,\varepsilon,f}(v) := s_{N,\varepsilon,f,v}(\tau(v)). \tag{2.3}$$

The authors [8] has shown the following (see [8, Lemma 2.6, Proposition 3.1]):

**Proposition 2.3** For $\kappa > 0$, if $\text{Ric}_f^N \geq c^1 - 4(1-\varepsilon)/(n-1) f$, then for all $x \in M$ and $v \in U_x M$,

$$\tau_{N,\varepsilon,f}(v) \leq C_\kappa.$$ Moreover, for the re-parametrized distance $d_{N,\varepsilon,f}$, we have

$$\sup_{x,y \in M} d_{N,\varepsilon,f}(x,y) \leq C_\kappa.$$

Note that the authors [8] have obtained a similar comparison result in a more general setting such that the density is a vector field, and $\kappa$ is variable.

3 Key inequalities

Hereafter, we always fix $N \in (-\infty, 1] \cup [n, +\infty)$, and $\varepsilon \in \mathbb{R}$ in the range (1.4). Moreover, in the case of $N = n$, the density function $f$ is constant; in particular, the main assertions have been already proved in the works of Sturm [28], [29], and Lott-Villani [14]. Furthermore, in the case of $N = 1$, they have been done by the second named author [27]. Thus, we further suppose $N \neq 1, n$.

The aim of this section is to produce the following key inequality for the proof of our main theorem (cf. [27, Proposition 3.1]):

**Proposition 3.1** Let $\mu, \nu \in \mathcal{P}_2^\infty(M)$, and let $\phi$ be the Kantorovich potential in Theorem 2.1. For a fixed $x \in M$, assume that $\phi$ is twice differentiable at $x$, and $\det(dF_t)_x > 0$ for every $t \in [0,1]$, where $F_t$ is defined as (2.2). For each $t \in [0,1]$, set

$$J_t(x) := e^{-f(F_t(x)) + f(x)} \det(dF_t)_x. \tag{3.1}$$

For $\kappa \in \mathbb{R}$, if $\text{Ric}_f^N \geq c^1 - 4(1-\varepsilon)/(n-1) f$, then for every $t \in [0,1]$,

$$J_t(x) \geq (1-t)\beta_{N,\varepsilon,f,1-t}(F_1(x), x) J_0(x) + t \beta_{N,\varepsilon,f,1-t}(x, F_1(x)) J_1(x).$$

Throughout this section, let $\mu, \nu, \phi, x$ be as in Proposition 3.1.
3.1 Riccati inequalities

Define a curve \( \gamma : [0, 1] \to M \) by \( \gamma(t) := F_t(x) \), and choose an orthonormal basis \( \{e_i\}_{i=1}^n \) at \( x \) with \( e_n = \dot{\gamma}(0)/\|\dot{\gamma}(0)\| \). For each \( i \), we define a Jacobi field \( E_i \) along \( \gamma \) by \( E_i(t) := (dF_t)_x(e_i) \). For each \( t \in [0, 1] \) let \( A(t) = (a_{ij}(t)) \) be an \( n \times n \) matrix determined by

\[
E'_i(t) = \sum_{j=1}^n a_{ij}(t) E_j(t).
\]

Let us consider a function \( h : [0, 1] \to \mathbb{R} \) defined by

\[
h(t) := \log \det(dF_t) - \int_0^t a_{nn}(\xi) \, d\xi,
\]

which enjoys the following Riccati inequality (see e.g., (1.4), (1.9) in [29], and (14.21) in [30]):

**Lemma 3.2** For every \( t \in ]0, 1[ \) we have

\[
h''(t) \leq -\frac{h'(t)^2}{n-1} - \text{Ric}_g(\gamma(t)).
\]

We define a function \( l : [0, 1] \to \mathbb{R} \) by

\[
l(t) := h(t) - f(\gamma(t)) + f(x).
\]

We show the following Riccati inequality, which is compatible with our setting (cf. [27] Lemma 3.3, and also [8] Lemma 2.1 in the literature of comparison geometry).

**Lemma 3.3** For every \( t \in ]0, 1[ \) we have

\[
\left( e^{\frac{2(1-\varepsilon)f_x(t)}{n-1}} l'(t) \right)' \leq -e^{\frac{2(1-\varepsilon)f_x(t)}{n-1}} \left( c l'(t)^2 + \text{Ric}_N^F(\gamma(t)) \right).
\] (3.2)

**Proof.** Set \( f_x := f \circ \gamma \). Lemma 3.2 leads us to

\[
l''(t) = h''(t) - f''_x(t) \leq -\frac{h'(t)^2}{n-1} - \left( \text{Ric}_g(\gamma(t)) + f''_x(t) \right)
\]

\[
= -cl'(t)^2 - \frac{2(1-\varepsilon)l'(t) f'_x(t)}{n-1} - \text{Ric}_N^F(\gamma(t))
\]

\[
- \frac{1}{n-1} \left( \varepsilon \sqrt{\frac{N-n}{N-1}} l'(t) + \sqrt{\frac{N-1}{N-n}} f'_x(t) \right)^2
\]

\[
\leq -cl'(t)^2 - \frac{2(1-\varepsilon)l'(t) f'_x(t)}{n-1} - \text{Ric}_N^F(\gamma(t)).
\]

This implies

\[
e^{-\frac{2(1-\varepsilon)f_x(t)}{n-1}} \left( e^{\frac{2(1-\varepsilon)f_x(t)}{n-1}} l'(t) \right)' = l''(t) + \frac{2(1-\varepsilon)l'(t) f'_x(t)}{n-1} \leq -cl'(t)^2 - \text{Ric}_N^F(\gamma(t)).
\]

We arrive at the desired inequality (3.2). \( \square \)
3.2 Jacobian inequalities

Once we obtain the Riccati inequality (3.2), one can prove Proposition 3.1 by the same argument as in the proof of [27, Proposition 3.1]. Define a function $D : [0, 1] \rightarrow \mathbb{R}$ by

$$D(t) := \exp (c l(t)).$$

In virtue of Lemma 3.3 we have the following (cf. [27, Lemma 3.5]):

**Lemma 3.4** For $\kappa \in \mathbb{R}$, if $\text{Ric}^N_f \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)}{n-1}} g$, then for every $t \in ]0, 1[$ we have

$$D(t) \geq \frac{\mathcal{g}_\kappa(d_{N,\varepsilon,f,1-t}(F_1(x), x))}{\mathcal{g}_\kappa(d_{N,\varepsilon,f}(F_1(x), x))} D(0) + \frac{\mathcal{g}_\kappa(d_{N,\varepsilon,f}(x, F_1(x)))}{\mathcal{g}_\kappa(d_{N,\varepsilon,f}(x, F_1(x)))} D(1).$$

**Proof.** As in the proof of [27, Lemma 3.5], we define

$$s_f : [0, 1] \rightarrow \mathbb{R}$$

by

$$s_f(t) := \int_0^t e^{-\frac{4(1-\varepsilon)}{n-1} f(\gamma(\xi))} n^{-1} d\xi.$$

For $a := s_f(1)$, we further define $\widehat{l}, \widehat{D} : [0, a] \rightarrow \mathbb{R}$ by

$$\widehat{l} := l \circ t_f, \quad \widehat{D} := D \circ t_f,$$

where $t_f : [0, a] \rightarrow [0, 1]$ is the inverse function of $s_f$. For each $s \in ]0, a[$ it holds that

$$e^{-1} \frac{\widehat{D}''(s)}{\widehat{D}(s)} = \widehat{p}''(s) + c \widehat{p}'(s)^2. \quad (3.3)$$

We also define functions $L : [0, 1] \rightarrow \mathbb{R}$ and $\widehat{L} : [0, a] \rightarrow \mathbb{R}$ by

$$L(t) := e^{-\frac{2(1-\varepsilon)f(\gamma(t))}{n-1}} l'(t), \quad \widehat{L} := L \circ t_f.$$

By Lemma 3.3 we see

$$\widehat{p}''(s) = \widehat{L}'(s) = t'_f(s) L'(t_f(s)) \leq -e^{-\frac{4(1-\varepsilon)f(\gamma(t_f(s)))}{n-1}} (c l'(t_f(s))^2 + \text{Ric}^N_f(\gamma(t_f(s))))$$

$$= -c \widehat{p}'(s)^2 - e^{-\frac{4(1-\varepsilon)f(\gamma(t_f(s)))}{n-1}} \text{Ric}^N_f(\gamma(t_f(s))). \quad (3.4)$$

The equality (3.3) together with (3.4) yields

$$e^{-1} \frac{\widehat{D}''(s)}{\widehat{D}(s)} \leq -e^{-\frac{4(1-\varepsilon)f(\gamma(t_f(s)))}{n-1}} \text{Ric}^N_f(\gamma(t_f(s))) \leq -e^{-1} \kappa d(x, y)^2,$$

where $y := F_1(x)$. Hence, $\widehat{D}''(s) + \kappa d(x, y)^2 \widehat{D}(s) \leq 0$ on $]0, a[.$
For all $s$, an elementary comparison argument implies the following (see e.g., [30, Theorem 14.28]):

$$a \, d(x, y) = d_{N, \varepsilon, f}(x, y) < \tau_{N, \varepsilon, f} \left( \frac{\gamma'(0)}{\|\gamma'(0)\|} \right),$$

where $\tau_{N, \varepsilon, f}$ is defined as (2.3). Due to Proposition 2.3, $\kappa \, d(x, y)^2 \in [-\infty, a^{-2} \pi^2]$. Now, an elementary comparison argument implies the following (see e.g., [29], and (14.19) in [30]):

For all $s_0, s_1 \in [0, a]$ and $\lambda \in [0, 1],

$$\tilde{D}((1 - \lambda)s_0 + \lambda s_1) \geq \frac{s_\kappa((1 - \lambda)|s_0 - s_1|d(x, y))}{s_\kappa(|s_0 - s_1|d(x, y))} \tilde{D}(s_0) + \frac{s_\kappa(\lambda|s_0 - s_1|d(x, y))}{s_\kappa(|s_0 - s_1|d(x, y))} \tilde{D}(s_1).$$

This implies that for every $s \in ]0, a[ \,$ we also see

$$\tilde{D}(s) \geq \frac{s_\kappa((a - s)d(x, y))}{s_\kappa(a \, d(x, y))} \tilde{D}(0) + \frac{s_\kappa(s \, d(x, y))}{s_\kappa(a \, d(x, y))} \tilde{D}(a).$$

It follows that for every $t \in ]0, 1[$

$$D(t) \geq \frac{s_\kappa((a - s_f(t)) \, d(x, y))}{s_\kappa(a \, d(x, y))} D(0) + \frac{s_\kappa(s_f(t) \, d(x, y))}{s_\kappa(a \, d(x, y))} D(1).$$

In view of the uniqueness of the geodesic $\gamma$, for every $t \in [0, 1]$ it holds that

$$(a - s_f(t)) \, d(x, y) = d_{N, \varepsilon, f, 1-t}(y, x), \quad s_f(t) \, d(x, y) = d_{N, \varepsilon, f, t}(x, y).$$

Thus, we complete the proof. \hfill \square

Let us give a proof of Proposition 3.1

**Proof of Proposition 3.1.** For $\kappa \in \mathbb{R}$, we assume $\text{Ric}_{f}^{N} \geq c^{-1} \kappa e^{-\frac{2(1-\varepsilon)}{\pi^2} g}$. Set

$$\overline{D}(t) := \exp \left( \int_{0}^{t} a_{nn}(\xi) d\xi \right).$$

The following is well-known (see e.g., (1.10) in [29], and (14.19) in [30]):

$$\overline{D}(t) \geq (1 - t)\overline{D}(0) + t\overline{D}(1). \quad (3.5)$$

From Lemma 3.4, (3.3), and the Hölder inequality, it follows that

$$J_t(x) \equiv D(t)^{1 - \frac{c}{\pi t}} \overline{D}(t)^{\frac{c}{\pi t}} 
\geq (1 - t)^{\frac{c}{\pi t}} \left( \frac{s_\kappa(d_{N, \varepsilon, f, 1-t}(F_1(x), x))}{s_\kappa(d_{N, \varepsilon, f,C}(F_1(x), x))} \right)^{1 - \frac{c}{\pi t}} J_0(x)^{\frac{c}{\pi t}}$$

$$+ t^{\frac{c}{\pi t}} \left( \frac{s_\kappa(d_{N, \varepsilon, f,C}(F_1(x), x))}{s_\kappa(d_{N, \varepsilon, f,C}(F_1(x), x))} \right)^{1 - \frac{c}{\pi t}} J_1(x)^{\frac{c}{\pi t}}.$$  

This proves Proposition 3.1. \hfill \square
4 Displacement convexity

In this section, we prove Theorem 1.6 with the help of Proposition 3.1.

4.1 Curvature bounds imply displacement convexity

We first show the implication from (1) to (2) in Theorem 1.6, which is also stated as Corollary 1.9 in Subsection 1.2 (cf. [27, Proposition 4.1] for the case of $N = 1$).

**Proposition 4.1** For $\kappa \in \mathbb{R}$, if $\text{Ric}^N_g \geq c^{-1} \kappa \text{e}^{-\frac{4(1-\varepsilon)}{\kappa-1}} g$ holds, then $(M, d, \mathfrak{m})$ satisfies $\text{TwCD}(\kappa, N, \varepsilon)$.

**Proof.** Let $\mu, \nu \in \mathcal{P}^M(M)$, and let $\phi$ be the Kantorovich potential obtained in Theorem 2.1. The map $F_t$ defined as (2.2) provides a unique optimal coupling $\pi$ of $(\mu, \nu)$ via $\pi := (F_0 \times F_t) \# \mu$. It also determines a unique minimal geodesic $\gamma : [0, 1] \rightarrow M$. Moreover, thanks to Theorem 2.2, for a fixed $t \in [0, 1]$, the Monge–Ampère equations

$$\rho_0(x) = \rho_1(F_1(x))J_1(x) = \rho_t(F_t(x))J_t(x)$$

(4.1)

hold for $\mu_0$-almost every $x \in M$, where $\rho_t$ denotes the density of $\mu_t$ with respect to $\mathfrak{m}$. For $U \in DC_{N, \varepsilon}^N$, let $\varphi_U(r) := r^{\frac{4(1-\varepsilon)}{\kappa-1}}U(r^{-\frac{\kappa}{1+\varepsilon}})$. From (4.1) and Proposition 3.1, we deduce

$$U_\mu(\mu_t) = \int_M U \left( \frac{\rho_0(x)}{J_t(x)} \frac{J_t(x)}{\rho_0(x)} \right) \rho_0(\mu_t) \mu_0(\mu_t) \mu_0(dx) = \int \varphi_U \left( \frac{J_t(x)}{\rho_0(x)} \right) \mu_0(dx)$$

$$\leq (1 - t) \int \varphi_U \left( \beta_{\kappa, N, \varepsilon, f, 1-t}(F_1(x), x) \frac{J_t(x)}{\rho_0(x)} \right) \mu_0(dx) + t \int \varphi_U \left( \beta_{\kappa, N, \varepsilon, f, t}(F_1(x), x) \frac{J_t(x)}{\rho_0(x)} \right) \mu_0(dx)$$

$$= (1 - t) \int \varphi_U \left( \frac{\beta_{\kappa, N, \varepsilon, f, 1-t}(F_1(x), x)}{\rho_0(x)} \right) \mu_0(dx) + t \int \varphi_U \left( \frac{\beta_{\kappa, N, \varepsilon, f, t}(F_1(x), x)}{\rho_1(F_1(x))} \right) \mu_0(dx).$$

Here we used the convexity and non-increasing property of $\varphi_U$ in the first inequality (cf. [30, Remark 17.2]). From $\pi = (F_0 \times F_1) \mu_0$, one can conclude the desired inequality. \hfill $\Box$

4.2 Displacement convexity implies curvature bounds

The implication from (2) to (3) is trivial. We now show that from (3) to (1) and complete the proof of Theorem 1.6. For subsets $X, Y \subset M$ and $t \in [0, 1]$, let $Z_t(X, Y)$ be the set of all points $\gamma(t)$, where $\gamma : [0, 1] \rightarrow M$ is a minimal geodesic with $\gamma(0) \in X, \gamma(1) \in Y$. We begin with the following Brunn-Minkowski inequality (cf. [27, Lemma 4.3]):
Lemma 4.2 Let $X, Y \subset M$ be two bounded Borel subsets with $m(X), m(Y) \in [0, +\infty]$. For $\kappa \in \mathbb{R}$, if $(M, d, m)$ satisfies $\text{TwCD}_{rel}(\kappa, N, \varepsilon)$, then for every $t \in ]0, 1[$,
\[
m(Z_t(X, Y)) \geq (1 - t) \left( \inf_{(x,y) \in X \times Y} \beta_{\kappa, N, \varepsilon, f, 1-t}(y, x) \right) m(X) + t \left( \inf_{(x,y) \in X \times Y} \beta_{\kappa, N, \varepsilon, f, t}(x, y) \right) m(Y).
\]

Proof. The proof is similar to that in [27, Lemma 4.3]. We omit it.

Having Lemma 4.2 at hand, let us prove the following (cf. [27, Proposition 4.4]):

Proposition 4.3 We suppose $\varepsilon \neq 0$. For $\kappa \in \mathbb{R}$, if $(M, d, m)$ satisfies $\text{TwCD}_{rel}(\kappa, N, \varepsilon)$, then $\text{Ric}_f^N \geq c^{-1}\kappa e^{-\frac{m(c\kappa \varepsilon/2)}{n-1}} g$.

Proof. We will follow the method of the proof of [20, Theorem 1.2], [21, Theorem 4.10]. Fix $x \in M$ and $v \in U_x M$, and set
\[
\theta_\varepsilon := -\frac{1}{n-1} \frac{1}{\varepsilon} (\varepsilon - \varepsilon_0) g(\nabla f, v).
\]
Here we used the assumption $\varepsilon \neq 0$. For a sufficiently small $t_0 > 0$, let $\gamma : [0, t_0] \to M$ be the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Take $\delta \in ]0, t_0[$ and $\eta \in ]0, \delta]$. We denote by $B_r(o)$ the open geodesic ball of radius $r > 0$ centered at $o \in M$, and put $X := B_{n(1+\theta_\varepsilon)\gamma(-\delta)}(\gamma(-\delta))$ and $Y := B_{N,\varepsilon, f, t}(\gamma(-\delta))$. Lemma 4.2 tells us that
\[
m \left( Z_{\frac{1}{2}}(X, Y) \right) \geq \left( \inf_{(x,y) \in X \times Y} \beta_{\kappa, N, \varepsilon, f, \frac{1}{2}}(x, y) \right) m(X) \frac{1}{c^{1/2}} + \frac{1}{2} \left( \inf_{(x,y) \in X \times Y} \beta_{\kappa, N, \varepsilon, f, \frac{1}{2}}(x, y) \right) m(Y) \frac{1}{c^{1/2}}.
\]
Letting $\eta \to 0$ in the above inequality, we have
\[
\lim_{\eta \to 0} \left( \frac{m \left( Z_{\frac{1}{2}}(X, Y) \right)}{\omega_n \eta^n} \right) \geq \frac{1}{2} \left( e^{-f(\varepsilon)}(1 + \theta_\varepsilon \delta) \beta_{\kappa, N, \varepsilon, f, \frac{1}{2}}(\gamma(\delta), \gamma(-\delta)) \right) \frac{1}{c^{1/2}} + \frac{1}{2} \left( e^{-f(\varepsilon)}(1 - \theta_\varepsilon \delta) \beta_{\kappa, N, \varepsilon, f, \frac{1}{2}}(\gamma(-\delta), \gamma(\delta)) \right) \frac{1}{c^{1/2}},
\]
where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.

Since
\[
d_{N,\varepsilon, f, \frac{1}{2}}(\gamma(\delta), \gamma(-\delta)) = \int_{0}^{\delta} e^{-\frac{2(1-\varepsilon) f(\xi)}{n-1}} d\xi, \quad \text{(4.3)}
\]
\[
d_{N,\varepsilon, f, \frac{1}{2}}(\gamma(-\delta), \gamma(\delta)) = \int_{-\delta}^{0} e^{-\frac{2(1-\varepsilon) f(\xi)}{n-1}} d\xi,
\]
\[
d_{N,\varepsilon, f}(\gamma(\delta), \gamma(-\delta)) = \int_{\delta}^{\delta} e^{-\frac{2(1-\varepsilon) f(\xi)}{n-1}} d\xi,
\]
\[
d_{N,\varepsilon, f}(\gamma(-\delta), \gamma(\delta)) = \int_{-\delta}^{-\delta} e^{-\frac{2(1-\varepsilon) f(\xi)}{n-1}} d\xi,
\]
\[
d_{N,\varepsilon, f}(\gamma(\delta), \gamma(-\delta)) = \int_{\delta}^{\delta} e^{-\frac{2(1-\varepsilon) f(\xi)}{n-1}} d\xi,
\]
\[
d_{N,\varepsilon, f}(\gamma(-\delta), \gamma(\delta)) = \int_{-\delta}^{-\delta} e^{-\frac{2(1-\varepsilon) f(\xi)}{n-1}} d\xi,
\]
\[
d_{N,\varepsilon, f}(\gamma(\delta), \gamma(-\delta)) = \int_{\delta}^{\delta} e^{-\frac{2(1-\varepsilon) f(\xi)}{n-1}} d\xi,
the Taylor series with respect to $\delta$ at 0 are

$$
\beta_{\kappa,N,\varepsilon,f,\theta}(\gamma(\delta), \gamma(-\delta)) = 1 - c^{-1} \frac{1 - \varepsilon}{n - 1} g(\nabla f, v) \delta
$$

$$
+ \left( c^{-1} \kappa e^{-\frac{4(1 - \varepsilon)(f(x))}{n - 1}} + (1 - c) \left( c^{-1} \frac{1 - \varepsilon}{n - 1} \right)^2 g(\nabla f, v)^2 \right) \frac{\delta^2}{2} + O(\delta^3),
$$

and

$$
e^{-f(\gamma(-\delta)) + f(x)} = 1 + g(\nabla f, v) \delta + \left( g(\nabla f, v)^2 - \nabla^2 f(v, v) \right) \frac{\delta^2}{2} + O(\delta^3),
$$

$$
e^{-f(\gamma(\delta)) + f(x)} = 1 - g(\nabla f, v) \delta + \left( g(\nabla f, v)^2 - \nabla^2 f(v, v) \right) \frac{\delta^2}{2} + O(\delta^3),
$$

and

$$(1 + \theta_c \delta)^n = 1 + n \theta_c \delta + \frac{n(n - 1)}{2} \theta_c^2 \delta^2 + O(\delta^3),
$$

$$(1 - \theta_c \delta)^n = 1 - n \theta_c \delta + \frac{n(n - 1)}{2} \theta_c^2 \delta^2 + O(\delta^3).
$$

Substituting these series into (4.2), we have

$$
\lim_{\eta \to 0} \frac{m\left(Z_\frac{1}{2}(X, Y)\right)}{\omega_n \eta^n}
\geq e^{-f(x)} \left\{ 1 + \left( c^{-1} \kappa e^{-\frac{4(1 - \varepsilon)(f(x))}{n - 1}} - \nabla^2 f(v, v) \right) \frac{\delta^2}{2} + O(\delta^3) \right\} + O(\delta^3),
$$

where for $\alpha := (1 - \varepsilon)(n - 1)^{-1}$ we set

$$
F(\theta) := n(n - (n - 1)(c + 1)) \theta^2 + 2n(\alpha - c)g(\nabla f, v)\theta + (\alpha^2 + 2\alpha - c)g(\nabla f, v)^2.
$$

Now, we can calculate

$$
F(\theta) + \frac{c + 1}{N - n} g(\nabla f, v)^2 = \varepsilon \theta + \frac{1}{n - 1} (\varepsilon - \varepsilon_0) g(\nabla f, v) \right\}^2,
$$

and hence

$$
\lim_{\eta \to 0} \frac{m\left(Z_\frac{1}{2}(X, Y)\right)}{\omega_n \eta^n}
\geq e^{-f(x)} \left\{ 1 + \left( c^{-1} \kappa e^{-\frac{4(1 - \varepsilon)(f(x))}{n - 1}} - \nabla^2 f(v, v) + \frac{g(\nabla f, v)^2}{N - n} \right) \frac{\delta^2}{2} \right\} + O(\delta^3)
$$

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by the definition of $\theta$. The detailed calculation can be seen in Appendix.

On the other hand,

$$\lim_{n \to 0} \frac{\omega_n \eta^n}{m(Z^2(X, Y))} \leq e^{-f(x)} \left(1 + \text{Ric}_g(v) \frac{\delta^2}{2}\right) + O(\delta^3). \quad (4.10)$$

By comparing (4.9) and (4.10),

$$\text{Ric}_g(v) \geq c^{-1} \kappa e^{-\frac{4(c+1)f(x)}{n-1}} - \nabla^2 f(v, v) + \frac{g(\nabla f, v)^2}{N-n},$$

which means $\text{Ric}_f^N(v) \geq c^{-1} \kappa e^{-\frac{4(c+1)f(x)}{n-1}}$. This completes the proof. \hfill \Box

We are now in a position to conclude Theorem 1.6.

**Proof of Theorem 1.6.** By Propositions 4.1 and 4.3 we complete the proof. \hfill \Box

### 4.3 Interpolation Inequalities

Under the curvature condition (1.2), the second named author [27] has derived some interpolation inequalities from the proof of the characterization result (see [27, Subsection 4.3]). By the same argument, we can obtain such interpolation inequalities in our setting, and we collect them here. We just present their forms, and the proof is left to the readers.

We start with the $p$-mean inequality (cf. [27, Corollary 4.5]). Let $t \in [0, 1]$ and $a, b \in [0, +\infty[$. For $p \in \mathbb{R} \setminus \{0\}$, the $p$-mean is defined as follows:

$$M^p_t(a, b) := \left((1-t)a^p + tb^p\right)^{\frac{1}{p}}$$

if $ab \neq 0$, and $M^0_t(a, b) := 0$ if $ab = 0$. As the limits, it is defined as

$$M^0_t(a, b) := a^{1-t}b^t, \quad M^\infty_t(a, b) := \max\{a, b\}, \quad M^{-\infty}_t(a, b) := \min\{a, b\}.$$

**Corollary 4.4** For $i = 0, 1$, let $\psi_i : M \to \mathbb{R}$ be non-negative, integrable functions. Let $X, Y \subset M$ be bounded Borel subsets with $\text{supp} [\psi_0] \subset X$, $\text{supp} [\psi_1] \subset Y$. Let $\psi : M \to \mathbb{R}$ be a non-negative function. For $t \in [0, 1]$ and $p \geq -c(c+1)^{-1}$, we assume that for all $(x, y) \in X \times Y$ and $z \in Z_t(\{x\}, \{y\})$, we have

$$\psi(z) \geq M^p_t\left(\frac{\psi_0(x)}{\beta_{k, N, \epsilon, f, 1-t}(y, x)}, \frac{\psi_1(y)}{\beta_{k, N, \epsilon, f, t}(x, y)}\right).$$

For $\kappa \in \mathbb{R}$, if $\text{Ric}_f^N \geq c^{-1} \kappa e^{-\frac{4(c+1)f}{n-1}} g$, then we have

$$\int_M \psi \text{d}m \geq M^{\epsilon p}_{t} \left(\int_M \psi_0 \text{d}m, \int_M \psi_1 \text{d}m\right).$$

Here we set $cp((1 + c)p + c)^{-1} := -\infty$ for $p = -c(c+1)^{-1}$. 

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We next show the Prékopa-Leindler inequality, which is the case of $p = 0$ in Corollary 4.4 (cf. [27, Corollary 4.6]):

**Corollary 4.5** For $i = 0, 1$, let $\psi_i, X, Y, \psi$ be as in Corollary 4.4. For $t \in ]0, 1[,$ we assume that for all $(x, y) \in X \times Y$ and $z \in Z_t(\{x\}, \{y\}),$

$$\psi(z) \geq \left( \frac{\psi_0(x)}{\beta_{\kappa,N,\varepsilon,f,1-t}(y,x)} \right)^{1-t} \left( \frac{\psi_1(y)}{\beta_{\kappa,N,\varepsilon,f,t}(x,y)} \right)^t.$$ 

For $\kappa \in \mathbb{R}$, if $\text{Ric}_f^N \geq c^{-1} \kappa e^{-\frac{4(1-t)}{n-1}} g$, then we have

$$\int_M \psi \, dm \geq \left( \int_M \psi_0 \, dm \right)^{1-t} \left( \int_M \psi_1 \, dm \right)^t.$$

We further possess the Borell-Brascamp-Lieb inequality, which is the case of $p = -c(c+1)^{-1}$ in Corollary 4.4 (cf. [27, Corollary 4.7]):

**Corollary 4.6** For $i = 0, 1$, let $\psi_i, X, Y, \psi$ be as in Corollary 4.4. We suppose $\int_M \psi_0 \, dm = \int_M \psi_1 \, dm = 1$. For $t \in ]0, 1[,$ we assume that for all $(x, y) \in X \times Y$ and $z \in Z_t(\{x\}, \{y\}),$

$$\psi(z)^{\frac{1}{c+1}} \leq (1-t) \left( \frac{\psi_0(x)}{\beta_{\kappa,N,\varepsilon,f,1-t}(y,x)} \right)^{\frac{1}{c+1}} + t \left( \frac{\psi_1(y)}{\beta_{\kappa,N,\varepsilon,f,t}(x,y)} \right)^{\frac{1}{c+1}}.$$ 

For $\kappa \in \mathbb{R}$, if $\text{Ric}_f^N \geq c^{-1} \kappa e^{-\frac{4(1-t)}{n-1}} g$, then we have $\int_M \psi \, dm \geq 1$.

### 5 Functional Inequalities

In this last section, we discuss functional inequalities under the curvature condition (1.5). For $\kappa \in \mathbb{R}$, let $c_\kappa := s_\kappa$. Following [27, Section 5], for $x, y \in M$ we define

$$b_{\kappa,N,\varepsilon,f}(x, y) := \frac{e^{\frac{4(1-\varepsilon)f(x)}{n-1}} d(x, y)}{s_\kappa(d_{N,\varepsilon,f}(x, y))} c^{-1},$$

$$b_{\kappa,N,\varepsilon,f}(x, y) := \frac{1}{c+1} \left( e^{\frac{4(1-\varepsilon)f(x)}{n-1}} d(x, y)c_\kappa(d_{N,\varepsilon,f}(x, y)) - 1 \right)$$

if $d_{N,\varepsilon,f}(x, y) \in]0, C_\kappa[$; $b_{\kappa,N,\varepsilon,f}(x, y) := 1$ and $b_{\kappa,N,\varepsilon,f}(x, y) := 0$ if $x = y$; otherwise, $b_{\kappa,N,\varepsilon,f}(x, y) := +\infty$ and $b_{\kappa,N,\varepsilon,f}(x, y) := +\infty$ (cf. Remark 1.1).

One can verify the following fact (cf. [27, Lemma 5.2]). The proof is left to the readers.

**Lemma 5.1** Let $\kappa \in \mathbb{R}$. Let $x, y \in M$ satisfy $d_{N,\varepsilon,f}(x, y) \in ]0, C_\kappa[$. If $y \notin \text{Cut}(x)$, then as $t \to 0$, we have

$$\beta_{\kappa,N,\varepsilon,f,t}(x, y) \to b_{\kappa,N,\varepsilon,f}(x, y), \quad \frac{1 - \beta_{\kappa,N,\varepsilon,f,1-t}(y,x)}{t} \to b_{\kappa,N,\varepsilon,f}(x, y).$$
For a non-negative Lipschitz function $\rho$ on $M$ with $\int_M \rho \, dm = 1$, set $\mu := \rho \, m$. The generalized Fisher information $I_m(\mu)$ of $\mu$ is defined as

$$I_m(\mu) := \int_M \frac{\| \nabla \rho \|^2}{\rho} \, dm.$$ 

In general, $I_m(\mu) \in [0, +\infty]$. We present the following (cf. [27, Proposition 5.4]):

**Proposition 5.2** Suppose that $m(M) < +\infty$ and $m$ has finite second moment. For $i = 0, 1$, let $\rho_i : M \to \mathbb{R}$ be non-negative Lipschitz functions with $\int_M \rho_i \, dm = 1$. We assume that $\mu := \rho_0 \, m$ and $\nu := \rho_1 \, m$ belong to $\mathcal{P}_2^\infty(M)$. For $\kappa \in \mathbb{R}$, if $\text{Ric}_f^N \geq c^{-1} \kappa e^{-\frac{\left(1-f\right)}{n-1}} \g$, then

$$H_m(\mu) \leq \sqrt{I_m(\mu)} W_2(\mu, \nu) + \frac{c + 1}{c} \int_{M^2} \rho_0(x)^{-\frac{1}{c+1}} b_{\kappa,N,\varepsilon,f}(x, y) \pi(dx, dy)$$

$$- \frac{c + 1}{c} \int_{M^2} \rho_1(y)^{-\frac{1}{c+1}} \left( b_{\kappa,N,\varepsilon,f}(x, y) \frac{1}{c+1} - 1 \right) \pi(dx, dy)$$

$$- \frac{c + 1}{c} \int_{M^2} \left( \rho_1(y)^{-\frac{1}{c+1}} - 1 \right) \pi(dx, dy),$$

where $\pi$ is a unique optimal coupling of $(\mu, \nu)$ with respect to the square of distance. When $I_m(\mu) = +\infty$ and $\mu = \nu$, we use the convention $I_m(\mu) W_2(\mu, \mu) = 0$.

**Proof.** If $I_m(\mu) = +\infty$ and $\mu \neq \nu$, the inequality trivially holds. We first assume $I_m(\mu) < +\infty$. By virtue of Corollary 1.9, $(M, d, m)$ satisfies TwCD$_{\text{rel}}(\kappa, N, \varepsilon)$, and hence

$$H_m(\mu(t)) \leq \frac{c + 1}{c} \int_{M^2} \rho_0(x)^{-\frac{1}{c+1}} b_{\kappa,N,\varepsilon,f,1-t}(y, x) \frac{1}{c+1} \pi(dx, dy)$$

$$- \frac{c + 1}{c} \int_{M^2} \rho_1(y)^{-\frac{1}{c+1}} b_{\kappa,N,\varepsilon,f,t}(x, y) \frac{1}{c+1} \pi(dx, dy),$$

here $(\mu(t))_{t \in [0, 1]}$ is a unique minimal geodesic in $(\mathcal{P}_2(M), W_2)$ from $\mu$ to $\nu$. Therefore,

$$\frac{H_m(\mu_t) - H_m(\mu)}{t} \leq \frac{c + 1}{c} \int_{M^2} \rho_0(x)^{-\frac{1}{c+1}} \frac{1 - \beta_{\kappa,N,\varepsilon,f,1-t}(y, x)}{t} \frac{1}{c+1} \pi(dx, dy)$$

$$+ \frac{c + 1}{c} \int_{M^2} \rho_0(x)^{-\frac{1}{c+1}} \left( \beta_{\kappa,N,\varepsilon,f,1-t}(y, x) \frac{1}{c+1} - 1 \right) \pi(dx, dy)$$

$$- \frac{c + 1}{c} \int_{M^2} \rho_1(y)^{-\frac{1}{c+1}} \left( \beta_{\kappa,N,\varepsilon,f,t}(x, y) \frac{1}{c+1} - 1 \right) \pi(dx, dy)$$

$$- \frac{c + 1}{c} \int_{M^2} \left( \rho_1(y)^{-\frac{1}{c+1}} - 1 \right) \pi(dx, dy) - H_m(\mu).$$

Let $F_1$ be the map defined as (2.2). We can deduce $d_{N,\varepsilon,f}(x, F_1(x)) \in [0, C_\kappa]$ for $\mu$-almost every $x \in M$ from Theorem 2.2 and Proposition 2.3. Lemma 5.1 and $\pi = (F_0 \times F_1)_{\sharp} \mu$.
\[
\lim_{t \to 0} \frac{H_m(\mu_t) - H_m(\mu)}{t} \leq \frac{c + 1}{c} \int_{M^2} \rho_0(x)^{-\frac{c}{c+1}} b_{\kappa,N,\varepsilon,f}(x,y) \pi(dx,dy) \\
- \frac{c + 1}{c} \int_{M^2} \rho_1(y)^{-\frac{c}{c+1}} \left( b_{\kappa,N,\varepsilon,f}(x,y) \pi(dx) - 1 \right) \pi(dx,dy) \\
- \frac{c + 1}{c} \int_{M^2} \left( \rho_1(y)^{-\frac{c}{c+1}} - 1 \right) \pi(dx,dy) - H_m(\mu).
\]

Now we assume that \( \rho_0 \) is bounded below away from 0. Then one can apply [30, Theorem 20.1] so that
\[
\lim_{t \to 0} \frac{H_m(\mu_t) - H_m(\mu)}{t} \geq -\sqrt{I_m(\mu)} W_2(\mu, \nu)
\]
under \( I_m(\mu) < +\infty \) (cf. [30, Remark 20.2]). Note that the condition \( m(M) < +\infty \) assures the integrability conditions \( H(\rho_0), \rho_0 H'(\rho_0) \in L^1(M; m) \) in [30, Theorem 20.1], because
\[
\int_M \rho_0(x)^{\frac{1}{c+1}} m(dx) \\
\leq \left( \int_M (1 + d(x, x_0)^2) \rho_0(x) m(dx) \right)^{\frac{1}{c+1}} \left( \int_M \frac{m(dx)}{(1 + d(x, x_0)^2)^{1/c}} \right)^{\frac{c}{c+1}} < +\infty.
\]
Then we have the conclusion. Next we prove the assertion without assuming the existence of positive lower bound for \( \rho_0 \). Now we set
\[
\rho_0^i := \frac{i \rho_0 + 1}{i + m(M)} \quad \text{and} \quad \mu^i := \rho_0^i m \in \mathcal{P}_2^c(M).
\]
Then we have the conclusion by replacing \( \mu \) (resp. \( \rho_0 \)) with \( \mu^i \) (resp. \( \rho_0^i \)). Since \( m \) has finite second moment, we see \( W_2(\mu^i, \mu) \to 0 \) as \( i \to +\infty \). So the conclusion can be obtained under \( I_m(\mu) < +\infty \), because of the lower semi continuity of \( \nu \mapsto H_m(\nu) \) in \( \mathcal{P}(M, W_2) \). If \( \mu = \nu \), then \( \mu_t = \mu = \nu \), hence (5.1) also trivially holds even if \( I_m(\mu) = +\infty \). \( \square \)

We will show three functional inequalities under the curvature condition (1.5). In what follows, we always assume \( m \in \mathcal{P}_2^c(M) \). To state our results, we introduce the following condition, which seems to be quite strong: We say that \( \mu \in \mathcal{P}_2^c(M) \) is \( m \)-constant if \( d_{N,\varepsilon,f}(x, F_1(x)) = e^{\frac{2(1-\varepsilon)f(x)}{n-1}} d(x, F_1(x)) \) on \( M \), where \( F_1 \) is the map defined as (2.2) for \( \nu = m \). We obtain the following (cf. [30, Theorems 20.10, 21.7]):

**Corollary 5.3** We assume \( m \in \mathcal{P}_2^c(M) \). Let \( \rho : M \to \mathbb{R} \) be a non-negative Lipschitz function with \( \int_M \rho dm = 1 \). Assume that \( \mu := \rho m \) belongs to \( \mathcal{P}_2^c(M) \). We further assume that \( (1-\varepsilon)f \leq (n-1)\delta \) for \( \delta \in \mathbb{R} \), and \( \mu \) is \( m \)-constant. For \( \kappa > 0 \), if \( \text{Ric}^N \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)}{n-1}f} g \), then we have

1. **The HWI inequality**
\[
H_m(\mu) \leq \sqrt{I_m(\mu)} W_2(\mu, m) - \frac{\kappa e^{-4\delta}}{6c} \left( 1 + 2(\sup \rho)^{-\frac{c}{c+1}} \right) W_2(\mu, m)^2;
\]
the Logarithmic Sobolev inequality

\[ H_m(\mu) \leq \frac{3c \left( 1 + 2(\sup \rho)^{-\frac{1}{c+1}} \right)^{-1}}{2\kappa e^{-4\delta}} I_m(\mu). \]

**Proof.** Note first that \( M \) is compact under \( \kappa > 0 \) and \( (1 - \varepsilon)f \leq (n - 1)\delta \) for \( \delta \in \mathbb{R} \) (see [30 Proposition 3.2]). Hence \( \rho \) is bounded. We begin with the HWI inequality. Since \( \pi = (F_0 \times F_1)\# \mu \), and \( \mu \) is \( m \)-constant,

\[
\begin{align*}
\mathbf{b}_{\kappa,N,\varepsilon,f}(x,y) &= \left( \frac{d_{N,\varepsilon,f}(x,y)}{s_\kappa(d_{N,\varepsilon,f}(x,y))} \right)^{c-1}, \\
\mathbf{b}_{\kappa,N,\varepsilon,f}(x,y) &= \frac{1}{c+1} \left( \frac{d_{N,\varepsilon,f}(x,y)}{s_\kappa(d_{N,\varepsilon,f}(x,y))} - 1 \right)
\end{align*}
\]

on the support of \( \pi \). By elementary estimates and \( (1 - \varepsilon)f \leq (n - 1)\delta \), we possess the following (cf. [30 (20.32), (20.34)]):

\[
- \left( \mathbf{b}_{\kappa,N,\varepsilon,f}(x,y) \right)^{\frac{c}{c+1}} \leq -\frac{\kappa}{6(c+1)} d_{N,\varepsilon,f}(x,y)^2 \leq -\frac{\kappa e^{-4\delta}}{6(c+1)} d(x,y)^2,
\]

\[ b_{\kappa,N,\varepsilon,f}(x,y) \leq -\frac{\kappa}{3(c+1)} d_{N,\varepsilon,f}(x,y)^2 \leq -\frac{\kappa e^{-4\delta}}{3(c+1)} d(x,y)^2. \]

Applying Proposition 5.2 to \( \rho_0 = \rho \) and \( \rho_1 = 1 \), we see

\[
H_m(\mu) \leq \sqrt{I_m(\mu) W_2(\mu, m)} - \frac{\kappa e^{-4\delta}}{3c} \int_{M^2} \rho(x)^{-\frac{\varepsilon}{c+1}} d(x,y)^2 \pi(dx dy)
\]

\[ - \frac{\kappa e^{-4\delta}}{6c} \int_{M^2} d(x,y)^2 \pi(dx dy), \]

and hence

\[
H_m(\mu) \leq \sqrt{I_m(\mu) W_2(\mu, m)} - \frac{\kappa e^{-4\delta}}{6c} \left( 1 + 2(\sup \rho)^{-\frac{\varepsilon}{c+1}} \right) \int_{M^2} d(x,y)^2 \pi(dx dy).
\]

By the optimality of \( \pi \), the right hand side of the above inequality is equal to that of the desired one. We next show the Logarithmic Sobolev inequality. Using an elementary inequality, we have

\[
\sqrt{I_m(\mu) W_2(\mu, m)} \leq \frac{3c \left( 1 + 2(\sup \rho)^{-\frac{1}{c+1}} \right)^{-1}}{2\kappa e^{-4\delta}} I_m(\mu) + \frac{\kappa e^{-4\delta}}{6c} \left( 1 + 2(\sup \rho)^{-\frac{\varepsilon}{c+1}} \right) W_2(\mu, m)^2.
\]

From the HWI inequality, one can derive the desired one. This completes the proof. \( \Box \)

Finally, we conclude the following finite dimensional transport energy inequality (cf. [30 Theorem 22.37, Corollary 22.39]):
Corollary 5.4  We assume \( m \in \mathcal{P}_2^{ac}(M) \). Let \( \rho : M \to \mathbb{R} \) be a non-negative Lipschitz function with \( \int_M \rho \, dm = 1 \). Assume that \( \mu := \rho, m \) belongs to \( \mathcal{P}_2^{ac}(M) \), and also assume that \( \mu \) is \( m \)-constant. For \( \kappa > 0 \), if \( \text{Ric}^N \geq c^{-1} \kappa e^{-\frac{4(1-a)}{a-1}g} \), then

\[
H_m(\mu) \geq \frac{1}{2} \cdot \frac{c+1}{c} + \frac{1}{2} \int_M \rho^{\frac{1}{\kappa+1}} \log \rho \, dm \\
- \frac{1}{2} \cdot \frac{c+1}{c} \int_{M^2} \left( b_{\kappa, N, \varepsilon, f}(x, y) + \exp \left( 1 - b_{\kappa, N, \varepsilon, f}(x, y) \frac{1}{\kappa+1} \right) \right) \pi(dx dy),
\]

where \( \pi \) is the unique optimal coupling of \((\mu, m)\).

**Proof.** Under \( m \in \mathcal{P}_2^{ac}(M) \), we see the well-definedness of \( \int_M \rho^{\frac{1}{\kappa+1}} \log \rho \, dm \in ]-\infty, +\infty] \), because \( x^a \log x \) is bounded below for any \( a \in [0, 1] \). We start with

\[
2H_m(\mu) = 2 \cdot \frac{c+1}{c} - 2 \cdot \frac{c+1}{c} \int_{M^2} \rho(x)^{-\frac{1}{\kappa+1}} \pi(dx dy).
\]

(5.2)

Let us recall the following Young inequality:

\[
ab \leq a \log a - 2a + e^{b+1}.
\]

We set \( B(x, y) := b_{\kappa, N, \varepsilon, f}(x, y) \frac{\varepsilon}{\kappa+1} \). From the Young inequality, we derive

\[
\rho(x)^{-\frac{1}{\kappa+1}} \log \rho(x)^{-\frac{1}{\kappa+1}} = \left( \rho(x)^{-\frac{1}{\kappa+1}} e^{-B(x, y)} \right) \left( e^{B(x, y)} \log \rho(x)^{-\frac{1}{\kappa+1}} \right) \\
\leq \left( \rho(x)^{-\frac{1}{\kappa+1}} e^{-B(x, y)} \right) \left( e^{B(x, y)} B(x, y) - 2 e^{B(x, y)} + e^{1-B(x, y)} \right) \\
= \rho(x)^{-\frac{2}{\kappa+1}} B(x, y) - 2 \rho(x)^{-\frac{1}{\kappa+1}} + e^{1-B(x, y)}
\]

on the support of \( \pi \), and hence

\[
-2 \rho(x)^{-\frac{1}{\kappa+1}} \geq \rho(x)^{-\frac{1}{\kappa+1}} \log \rho(x)^{-\frac{1}{\kappa+1}} - \rho(x)^{-\frac{1}{\kappa+1}} B(x, y) - e^{1-B(x, y)}.
\]

(5.3)

By (5.2) and (5.3), we obtain

\[
2H_m(\mu) \geq 2 \cdot \frac{c+1}{c} + \int_{M^2} \rho^{\frac{1}{\kappa+1}} \log \rho \, dm \\
- \frac{c+1}{c} \int_{M^2} \left( \rho(x)^{-\frac{1}{\kappa+1}} b_{\kappa, N, \varepsilon, f}(x, y)^{\frac{1}{\kappa+1}} + \exp \left( 1 - b_{\kappa, N, \varepsilon, f}(x, y)^{\frac{1}{\kappa+1}} \right) \right) \pi(dx dy).
\]

(5.4)

We apply Proposition 5.2 to \( \rho_0 = 1 \) and \( \rho_1 = \rho \). From \( H_m(m) = 0 \) and \( I_m(m) = 0 \),

\[
0 \leq \int_{M^2} \left( b_{\kappa, N, \varepsilon, f}(x, y) - \rho(y)^{-\frac{1}{\kappa+1}} b_{\kappa, N, \varepsilon, f}(x, y)^{\frac{1}{\kappa+1}} \right) \hat{\pi}(dx dy) + 1,
\]

where \( \hat{\pi} \) is a unique optimal coupling of \((m, \mu)\). Since \( \mu \) is \( m \)-constant, \( b_{\kappa, N, \varepsilon, f} \) and \( b_{\kappa, N, \varepsilon, f} \) are symmetric on the support of \( \pi \). It follows that

\[
0 \leq \int_{M^2} \left( b_{\kappa, N, \varepsilon, f}(x, y) - \rho(x)^{-\frac{1}{\kappa+1}} b_{\kappa, N, \varepsilon, f}(x, y)^{\frac{1}{\kappa+1}} \right) \pi(dx dy) + 1,
\]

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which is equivalent to
\[-\int_{M^2} \rho(x)^{-\gamma+1} b_{\kappa,N,\varepsilon,f}(x,y)^{\gamma+1} \pi(dxdy) \geq -1 - \int_{M^2} b_{\kappa,N,\varepsilon,f}(x,y) \pi(dxdy). \quad (5.5)\]

Combining (5.4) and (5.5) leads to the desired inequality. \(\square\)

On Corollaries 5.3 and 5.4, the authors do not know whether the assumption that 
\[(1 - \varepsilon)f \leq (n - 1)\delta \text{ and } \mu \text{ is } m\text{-constant} \]
can be dropped.

Under the curvature condition (1.1), similar functional inequalities are known to be useful to analyze the gradient flow of entropy functionals (see e.g., [30, Chapters 23, 24, 25]). There might be some applications of our inequalities to the analysis of such gradient flow under the curvature condition (1.5).

Concerning the curvature condition (1.1), functional inequalities can be also derived from the so-called Bakry-Émery’s \(\Gamma\)-calculus ([1]). Li-Xia [10] have formulated a Bochner type formula that is associated with \(\text{Ric}^f\). One might be able to develop the \(\Gamma\)-calculus in our framework via their Bochner formula.

6 Appendix

6.1 Twisted coefficients

In this appendix, we give a proof of the assertion stated in Remark 1.1. Namely, we show:

**Proposition 6.1** For the twisted coefficient \(\beta_{\kappa,N,\varepsilon,f,t}(x,y)\) defined as (1.8), it holds that \(\beta_{\kappa,N,\varepsilon,f,t}(x,y) \to 1\) as \(d(x,y) \to 0\).

**Proof.** It suffices to prove that
\[
\frac{d_{N,\varepsilon,f,t}(x,y)}{td_{N,\varepsilon,f}(x,y)} \to 1
\]
as \(d(x,y) \to 0\). Fix \(x \in M\), and a sufficiently small \(r > 0\). Take \(y \in B_r(x)\), and a unique minimal geodesic \(\gamma : [0,d(x,y)] \to M\) from \(x\) to \(y\). We set \(a := 2(n - 1)^{-1}(1 - \varepsilon)\). Then it holds that
\[
\left| \frac{d_{N,\varepsilon,f,t}(x,y)}{td_{N,\varepsilon,f}(x,y)} - 1 \right| = \left| \frac{t}{\int_0^{d(x,y)} e^{-af(\gamma(\xi))}d\xi} - 1 \right|
\]
\[
= \left| \frac{1}{\int_0^{d(x,y)} e^{-af(\gamma(\xi))}d\xi} \right| \left( \int_0^{d(x,y)} e^{-af(\gamma(\xi))}d\xi - \int_0^{d(x,y)} e^{-af(\gamma(\xi))}d\xi \right)
\]
\[
\leq \frac{1}{\int_0^{d(x,y)} e^{-af(\gamma(\xi))}d\xi} \left( \int_0^{d(x,y)} e^{-af(\gamma(\xi))}d\xi \right) \left| 1 - e^{-af(\gamma(\xi))} \right| d\xi.
\]
We now recall the following elementary estimate: For all \( b \in \mathbb{R} \),
\[
|1 - e^{-b}| \leq e^{|b|} - 1.
\]
It follows that
\[
|1 - e^{-a(f(\gamma(t\xi)) - f(\gamma(\xi)))}| \leq e^{|a||f(\gamma(t\xi)) - f(\gamma(\xi))|} - 1.
\]
Furthermore, setting
\[
A := (1 - t) \sup_{B_r(x)} \|\nabla f\|,
\]
we obtain
\[
|f(\gamma(t\xi)) - f(\gamma(\xi))| \leq d(\gamma(t\xi), \gamma(\xi)) \sup_{B_r(x)} \|\nabla f\| \leq Ad(x, y).
\]
Therefore, we see
\[
e^{a|f(\gamma(t\xi)) - f(\gamma(\xi))|} - 1 \leq e^{a|Ad(x, y)|} - 1 = \sum_{k=1}^{\infty} \left(\frac{|a| Ad(x, y)|}{k!}\right)^k
\]
\[
= |a| Ad(x, y) \sum_{k=1}^{\infty} \left(\frac{|a| Ad(x, y)|}{k!}\right)^{k-1}
\]
\[
\leq |a| Ad(x, y) \sum_{k=1}^{\infty} \left(\frac{|a| Ar}{(k-1)!}\right) = |a| Ae^{|a| Ar} d(x, y).
\]
Combining the above estimates, we arrive at
\[
\left|\frac{d_{N,\varepsilon,f,t}(x, y)}{td_{N,\varepsilon,f}(x, y)} - 1\right| \leq |a| Ae^{|a| Ar} d(x, y).
\]
This proves the desired claim. \(\Box\)

### 6.2 Taylor series

This appendix is also devoted to a supplemental material for the proof of Proposition 4.3 since the calculation is straightforward but quite complicated. We use the same notation as in the proof.

First, we give an outline of the proof of (4.4). In view of (4.3), one can verify
\[
d_{N,\varepsilon,f,\delta}(\gamma(\delta), \gamma(-\delta)) = e^{-\frac{2(1-\varepsilon)f(\delta)}{n-1}} \delta \left(1 - \frac{(1-\varepsilon)g(\nabla f, v)}{n-1}\delta + A \delta^2 + O(\delta^3)\right),
\]
\[
d_{N,\varepsilon,f,\delta}(\gamma(-\delta), \gamma(\delta)) = e^{-\frac{2(1-\varepsilon)f(\delta)}{n-1}} \delta \left(1 + \frac{(1-\varepsilon)g(\nabla f, v)}{n-1}\delta + A \delta^2 + O(\delta^3)\right),
\]
\[
d_{N,\varepsilon,f}(\gamma(\delta), \gamma(-\delta)) = 2e^{-\frac{2(1-\varepsilon)f(\delta)}{n-1}} \delta \left(1 + A \delta^2 + O(\delta^3)\right),
\]
where
\[
A := \frac{2(1-\varepsilon)^2g(\nabla f, v)^2}{3(n-1)^2} - \frac{(1-\varepsilon)\nabla^2 f(v, v)}{3(n-1)}.
\]
Using $s_\kappa(s) = s - (\kappa/6)s^3 + O(s^3)$, we see
\begin{align*}
  s_\kappa(d_{N,\varepsilon,f}\frac{1}{2}\gamma(\delta), \gamma(-\delta)) \\
  &= e^{-2(1-\varepsilon)f(x)} \left\{ 1 - \frac{(1-\varepsilon)g(\nabla f, v)}{n-1} \delta + \left(A - \frac{\kappa}{6}e^{-2(1-\varepsilon)f(x)}\right) \delta^2 + O(\delta^3) \right\}, \\
  s_\kappa(d_{N,\varepsilon,f}\frac{1}{2}\gamma(-\delta), \gamma(\delta)) \\
  &= e^{-2(1-\varepsilon)f(x)} \left\{ 1 + \frac{(1-\varepsilon)g(\nabla f, v)}{n-1} \delta + \left(A - \frac{\kappa}{6}e^{-2(1-\varepsilon)f(x)}\right) \delta^2 + O(\delta^3) \right\}, \\
  s_\kappa(d_{N,\varepsilon,f}(\gamma(-\delta), \gamma(-\delta))) \\
  &= 2e^{-2(1-\varepsilon)f(x)} \left\{ 1 + \left(A - \frac{2\kappa}{3}e^{-2(1-\varepsilon)f(x)}\right) \delta^2 + O(\delta^3) \right\}.
\end{align*}

By $(1 + s)^{-1} = 1 - s + s^2 - s^3 + O(s^4)$, we obtain
\begin{align*}
  \frac{2s_\kappa(d_{N,\varepsilon,f}\frac{1}{2}\gamma(\delta), \gamma(-\delta))}{s_\kappa(d_{N,\varepsilon,f}\frac{1}{2}\gamma(\delta), \gamma(-\delta))} &= 1 - \frac{(1-\varepsilon)g(\nabla f, v)}{n-1} \delta + \frac{\kappa}{2}e^{-2(1-\varepsilon)f(x)} \delta^2 + O(\delta^3), \\
  \frac{2s_\kappa(d_{N,\varepsilon,f}\frac{1}{2}\gamma(-\delta), \gamma(\delta))}{s_\kappa(d_{N,\varepsilon,f}\gamma(-\delta), \gamma(\delta))} &= 1 + \frac{(1-\varepsilon)g(\nabla f, v)}{n-1} \delta + \frac{\kappa}{2}e^{-2(1-\varepsilon)f(x)} \delta^2 + O(\delta^3).
\end{align*}

From $(1 + s)^a = 1 + as + (a(a - 1)/2)s^2 + O(s^3)$, we conclude (4.4).

We next sketch the proof of (4.7). Combining (4.4), (4.5), (4.6), we have
\begin{align*}
  e^{-f(\gamma(-\delta)+f(x))} (1 + \theta_\varepsilon \delta)^n \beta_{n,\varepsilon,f}\frac{1}{2}\gamma(\delta), \gamma(-\delta)) \\
  &= 1 + \left\{ n\theta_\varepsilon - \left(c^{-1} - \frac{1}{n-1} - 1\right) g(\nabla f, v) \right\} \delta - n\theta_\varepsilon \left(c^{-1} - \frac{1}{n-1} - 1\right) g(\nabla f, v) \delta^2 \\
  &\quad + \left\{ \left(1 - c^{-1} \frac{1 - \varepsilon}{n-1} \right)^2 \delta^2 - c^{-1} \left(\frac{1 - \varepsilon}{n-1}\right)^2 \right\} g(\nabla f, v) \delta^2 \\
  &\quad + \left(e^{-c^{-1} \frac{1}{n-1}} - \nabla^2 f(v, v) + n(n-1)\theta_\varepsilon^2 \right) \delta^2 + O(\delta^3), \\
  e^{-f(\gamma(\delta)+\theta_\varepsilon f(x))} (1 - \theta_\varepsilon \delta)^n \beta_{n,\varepsilon,f}\frac{1}{2}\gamma(-\delta), \gamma(\delta)) \\
  &= 1 - \left\{ n\theta_\varepsilon - \left(c^{-1} - \frac{1}{n-1} - 1\right) g(\nabla f, v) \right\} \delta - n\theta_\varepsilon \left(c^{-1} - \frac{1}{n-1} - 1\right) g(\nabla f, v) \delta^2 \\
  &\quad + \left\{ \left(1 - c^{-1} \frac{1 - \varepsilon}{n-1} \right)^2 \delta^2 - c^{-1} \left(\frac{1 - \varepsilon}{n-1}\right)^2 \right\} g(\nabla f, v) \delta^2 \\
  &\quad + \left(e^{-c^{-1} \frac{1}{n-1}} - \nabla^2 f(v, v) + n(n-1)\theta_\varepsilon^2 \right) \delta^2 + O(\delta^3).
\end{align*}
This implies
\[
\left( e^{-f(\gamma(-\delta))+f(x)} (1 + \theta \epsilon \delta)^n \beta_{\kappa,N,e,f,x} \gamma(\delta), \gamma(-\delta) \right)_{c+1}^{c+1} \\
= 1 + \frac{c}{c+1} \left\{ n \theta \epsilon - (c^{-1} \alpha - 1) g(\nabla f, v) \right\} \delta \\
+ \frac{c}{c+1} \left( c^{-1} \kappa e^{-\frac{f(1-\gamma)/f(x)}{n-1}} - \nabla^2 f(v,v) - \frac{F(\theta)}{c+1} \right) \frac{\delta^2}{2} + O(\delta^3),
\]
where \( F(\theta) \) is defined as (4.8). In particular,
\[
\frac{1}{2} \left( e^{-f(\gamma(-\delta))+f(x)} (1 + \theta \epsilon \delta)^n \beta_{\kappa,N,e,f,x} \gamma(\delta), \gamma(-\delta) \right)_{c+1}^{c+1} \\
= 1 + \frac{c}{c+1} \left\{ n \theta \epsilon - (c^{-1} \alpha - 1) g(\nabla f, v) \right\} \delta \\
+ \frac{c}{c+1} \left( c^{-1} \kappa e^{-\frac{f(1-\gamma)/f(x)}{n-1}} - \nabla^2 f(v,v) - \frac{F(\theta)}{c+1} \right) \frac{\delta^2}{2} + O(\delta^3).
\]
Substituting this equation into (4.2), we arrive at (4.7).

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