EXTENSION DIMENSION AND QUASI-FINITE CW-COMPLEXES

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Abstract. We extend the definition of quasi-finite complexes by considering not necessarily countable complexes. We provide a characterization of quasi-finite complexes in terms of $L$-invertible maps and dimensional properties of compactifications. Several results related to the class of quasi-finite complexes are established, such as completion of metrizable spaces, existence of universal spaces and a version of the factorization theorem. Further, we extend the definition of $UV(L)$-spaces on non-compact case and show that some properties of $UV(n)$-spaces and $UV(n)$-maps remain valid, respectively, for $UV(L)$-spaces and $UV(L)$-maps.

1. Introduction

Extension theory introduced by Dranishnikov [14, 15] unifies the covering dimension and the cohomological dimension. There are two classes of maps which play an important role in extension theory. For a given complex $L$, theses are $L$-invertible and $L$-soft maps. It should be mentioned that universal spaces in dimension $L$ as well as absolute extensors in dimension $L$ are obtained as preimages of Hilbert cube or Hilbert space under maps from the above classes [10]. For a countable complex $L$, existence of $L$-invertible mapping of certain $L$-dimensional compactum onto the Hilbert cube is closely connected with the dimensional properties of compactifications of spaces with extension dimension not grater than $L$ [9]. It turned out that the existence of such $L$-invertible mappings can be characterized in terms of “extensional” properties of a complex. This inspired the concept of quasi-finite countable complexes [20].

In the present paper we extend the definition of quasi-finite complexes by considering not necessarily countable complexes. We also provide a characterization of quasi-finite complexes in terms of $L$-invertible maps and dimensional properties of compactifications. Another interesting observation consists in the fact that many results established for finite or countable complexes remain valid...
for quasi-finite complexes. In particular, quasi-finite complexes possess the $L$-soft map property and every metrizable space of extension dimension $\leq L$ has a completion with the same extensional dimension. We also prove a version of the factorization theorem, and construct universal spaces. Finally, in case $L$ being quasi-finite it is possible to define $UV(L)$-property for non-compact spaces. We show that this property does not depend on the embedding of a space into absolute neighborhood extensor in dimension $L$ and obtain some results about $UV(L)$-maps and $UV(L)$-spaces which were known for $UV(n)$-maps and $UV(n)$-spaces, respectively.

2. **Quasi-finite CW-complexes**

Everywhere in this paper we assume that spaces are Tychonov and maps are continuous. Let $X$ and $Y$ be two spaces, $A \subset X$ and $g: A \to Y$ a map. We write $Y \in ANE(g, A, X)$ if $g$ has a continuous extension $\bar{g}: U \to Y$, where $U$ is a neighborhood of $A$ in $X$ which has the following property: there exists a function $h: X \to [0, 1]$ such that $h^{-1}((0, 1]) = U$ and $h(A) = 1$. If, in the above definition, $U = X$, we write $Y \in AE(g, A, X)$. Let us note that, by [16, Lemma 2.8], $Y \in ANE(g, A, X)$ if and only if $g$ extends to a map $\bar{g}: X \to \text{Cone}(Y)$.

Everywhere below $L$ always denotes a CW-complex.

We say that $L$ is an absolute extensor of $X$, notation $L \in AE(X)$, if $L \in AE(g, A, X)$ for every closed $A \subset X$ and every map $g: A \to L$ with $L \in ANE(g, A, X)$. We say also that the extension dimension of $X$ is not greater than $L$, notation $\text{e-dim}X \leq L$, if $L \in AE(X)$. Using Dydak's version of the Homotopy Extension Theorem [16, Theorem 13.7] one can show that if $L_1$ is homotopy equivalent to $L_2$, then $\text{e-dim}X \leq L_1$ is equivalent to $\text{e-dim}X \leq L_2$ for any space $X$. Moreover, our definition of $\text{e-dim}$ coincides with that one of Chigogidze [8] in case $L$ is countable and with the original definition of Dranishnikov [?] when compact spaces are considered.

A pair of spaces $K \subset P$ is called $L$-connected if whenever $A \subset X$ is a closed subset of a space $X$ with $\text{e-dim}X \leq L$, then every map $g: A \to K$ has an extension $\overline{g}: X \to P$ provided $A$ is normally placed in $X$ with respect to $(g, P)$. The notion of a normally placed set was introduced in [8] under different notation and means that for every continuous function $h$ on $P$ the function $h \circ g$ can be continuously extended over $X$. Obviously, this condition is satisfied for every normal space $X$ and every map $g: A \to K$ with $A \subset X$ closed. We sometimes say that a pair $K \subset P$ is $L$-connected with respect to a given class of spaces $\mathcal{B}$ if the additional requirement $X \in \mathcal{B}$ is imposed in the above definition.

Quasi-finite CW-complexes were introduced in [20] as countable complexes $L$ satisfying the following condition: every finite subcomplex $K$ of $L$ is contained in a finite subcomplex $P \subset L$ such that the pair $K \subset P$ is $L$-connected with respect to Polish spaces. It was also shown in [20] that there exists a countable
quasi-finite complex $M$ extension type $[M]$ of which does not contain a finitely dominated complex (see [10] for more information on extension types). In this note we extend the above definition by considering not necessarily countable complexes. Here is our revised definition: a CW-complex $L$ is quasi-finite if every finite subcomplex $K$ of $L$ is contained in a finite subcomplex $P \subset L$ such that the pair $K \subset P$ is $L$-connected. It is easy to verify that this definition coincides with the definition given in [20] in case $L$ is countable.

We say that a map $f : X \to Y$ is $L$-invertible if for any map $g : Z \to Y$ with $e\dim Z \leq L$ there is a map $h : Z \to X$ such that $g = f \circ h$. If, in addition, $Z$ is required to be from a given class of spaces $\mathcal{B}$, then we say that the map $f$ is $L$-invertible with respect to the class $\mathcal{B}$. Everywhere below $w(X)$ denotes the weight of the space $X$ and $\mathbb{I}^\tau$ denotes Tychonov cube of weight $\tau$.

**Theorem 2.1.** The following conditions are equivalent for any CW-complex $L$ and an infinite cardinal $\tau$:

1. $L$ is quasi-finite.
2. $e\dim \beta X \leq L$ whenever $X$ is a space with $e\dim X \leq L$.
3. There exists an $L$-invertible map $f : Y_\tau \to \mathbb{I}^\tau$ such that $Y_\tau$ is a compact space of weight $\leq \tau$ and $e\dim Y_\tau \leq L$.
4. For every $L$-connected pair $K \subset M$, where $K$ is a compactum of weight $\leq \tau$ and $M$ an arbitrary space, there exists a compactum $P \subset M$ containing $K$ such that $w(P) \leq \tau$ and the pair $K \subset P$ is $L$-connected.

**Proof.** (1) $\Rightarrow$ (2) Suppose $e\dim X \leq L$ and let $f : A \to L$, where $A$ is a closed subset of $\beta X$. It is well known that every CW-complex is an absolute neighborhood extensor for the class of compact spaces, so $L \in ANE(f, A, \beta X)$ and there exists a closed neighborhood $B$ of $A$ in $\beta X$ and a map $g : B \to L$ extending $f$. Because $g(B)$ is compact, it is contained in a finite subcomplex $K$ of $L$. Since $L$ is quasi-finite, there exists a finite subcomplex $P$ of $L$ such that the pair $K \subset P$ is $L$-connected. We can assume that $B$ is a zero-set in $\beta X$. Then $B \cap X$, being a non-empty zero-set in $X$, is normally placed in $X$ with respect to $(g, P)$. Therefore, the map $g : B \cap X \to K$ extends to a map $h : X \to P$ because $e\dim X \leq L$ and the pair $K \subset P$ is $L$-connected. Finally, let $\overline{h} : \beta X \to P$ be the unique extension of $h$. Then $\overline{h}$ extends $f$, so $e\dim \beta X \leq L$.

(2) $\Rightarrow$ (3) We consider the family of all maps $\{h_\alpha : X_\alpha \to \mathbb{I}^\tau\}_{\alpha \in \Lambda}$ such that each $X_\alpha$ is a closed subset of $\mathbb{I}^\tau$ with $e\dim X_\alpha \leq L$. Let $X$ be the disjoint sum of all $X_\alpha$ and the map $h : X \to \mathbb{I}^\tau$ coincides with $h_\alpha$ on every $X_\alpha$. Clearly, $e\dim X \leq L$. Therefore, $e\dim \beta X \leq L$. Consider the extension $\overline{h} : \beta X \to \mathbb{I}^\tau$. Then, by the factorization theorem from [24], there exists a compact space $Y_\tau$ of weight $\leq \tau$ and maps $r : \beta X \to Y_\tau$ and $f : Y_\tau \to \mathbb{I}^\tau$ such that $e\dim Y_\tau \leq L$ and $f \circ r = \overline{h}$.

Let us show that $f$ is $L$-invertible. Take a space $Z$ with $e\dim Z \leq L$ and a map $g : Z \to \mathbb{I}^\tau$. Considering $\beta Z$ and the extension $\overline{g} : \beta Z \to \mathbb{I}^\tau$ of $g$, we can
assume that $Z$ is always compact. We also can assume that the weight of $Z$ is \( \leq \tau \) (otherwise we apply again the factorization theorem from [24] to find a compact space $T$ of weight $\leq \tau$ and maps $g_1: Z \to T$ and $g_2: T \to \mathbb{I}^r$ with $e\dim T \leq L$ and $g_2 \circ g_1 = g$, and then consider the space $T$ and the map $g_2$ instead, respectively, of $(Z$ and $g$)). Therefore, without losing generality, we can assume that $Z$ is a closed subset of $\mathbb{I}^r$. According to the definition of $X$ and the map $h$, there is an index $\alpha \in \Lambda$ such that $Z = X_\alpha$ and $g = h_\alpha$. The restriction $r|Z: Z \to Y_\tau$ is a lifting of $g$, i.e. $f \circ (r|Z) = g$.

(3) $\Rightarrow$ (4) Suppose that $K$ is a compact subset of the space $M$ with $w(K) \leq \tau$ and $K \subset M$ being $L$-connected. We embed $K$ in $\mathbb{I}^r$ and consider an $L$-invertible mapping $f: Y_\tau \to \mathbb{I}^r$ such that $Y_\tau$ is compact and $e\dim Y_\tau \leq L$. Let $\bar{K} = f^{-1}(K)$ and $\bar{h} = f|\bar{K}$. Obviously, $\bar{K}$ is normally placed in $Y_\tau$ with respect to $(h, M)$. Consequently, $\bar{h}$ extends to a map $\bar{h}: Y_\tau \to M$ and let $P = \bar{h}(Y_\tau)$. Obviously, $w(P) \leq \tau$, so that it remains only to show that $K \subset P$ is $L$-connected. For this end, let $g: A \to K$, where $A \subset X$ is a closed normally placed subset of $X$ with respect to $(g, P)$ and $e\dim X \leq L$. This implies that $A$ is normally placed in $X$ with respect to $(g, \mathbb{I}^r)$. Since $\mathbb{I}^r$ is an absolute extensor, there exists an extension $g_1: X \to \mathbb{I}^r$ of $g$. Next, we lift $g_1$ to a map $g_2: X \to Y_\tau$ such that $f \circ g_2 = g_1$ (recall that $f$ is $L$-invertible) and let $\bar{g} = \bar{h} \circ g_2$. Clearly, $\bar{g}$ is a map from $X$ into $P$ extending $g$. Therefore, $K \subset P$ is $L$-connected.

(4) $\Rightarrow$ (1) Take a finite subcomplex $K$ of $L$. Let us first show that the pair $K \subset L$ is $L$-connected. Suppose $Z$ is a space with $e\dim Z \leq L$, $A \subset Z$ closed and $g: A \to K$ a map such that $A$ is normally placed in $Z$ with respect to $(g, L)$. Since $K$ is $C$-embedded in $L$, $A$ is normally placed in $Z$ with respect to $(g, K)$. The last condition together with the fact that $K$ is an absolute neighborhood extensor for all separable metric spaces implies that $K \in \text{ANE}(g, A, Z)$. Indeed, we embed $K$ in $\mathbb{R}^\omega$ and fix a retraction $r: U \to K$, where $U$ is a neighborhood of $K$ in $\mathbb{R}^\omega$. Since $A$ is normally placed in $Z$ with respect to $(g, K)$, we can find a map $h: Z \to \mathbb{R}^\omega$ extending $g$. Then $h^{-1}(U)$ is a co-zero neighborhood of $A$ in $Z$ which contains the zero-set $h^{-1}(K)$ and $r \circ h: h^{-1}(U) \to K$ extends $g$. Hence, $K \in \text{ANE}(g, A, Z)$ which yields $L \in \text{ANE}(g, A, Z)$. Since $e\dim Z \leq L$, $g$ can be extended to a map $\bar{g}: Z \to L$. Thus, $K \subset L$ is an $L$-connected pair. Therefore there exists a compact set $H \subset L$ containing $K$ such that the pair $K \subset H$ is $L$-connected. Finally, we take a finite subcomplex $P$ of $L$ which contains $H$ and observe that the pair $K \subset P$ is also $L$-connected. Hence, $L$ is quasi-finite.

**Corollary 2.2.** None of the Eilenberg-MacLane complexes $K(G, n)$, $n \geq 2$ and $G$ an Abelian group, is quasi-finite.

**Proof.** This follows from Theorem 2.1(2) and the following statement (see [22, Theorem 1.4]): there exists a separable metric space $X$ with $\dim_G X \leq 2$ and
e-dim$\beta X > L$ for every Abelian group $G$ and every non-contractible CW-complex $L$. Here $\dim_G X$ denotes the cohomological dimension of $X$ with respect to the group $G$.

Let us also observe that for every quasi-finite complex $L$ there exists a compact metrizable space which is universal for the class of all separable metric spaces of $e$-$\dim \leq L$, in particular every space from this class has a compactification of $e$-$\dim \leq L$. Indeed, let $Y_\omega$ be the space from Theorem 2.1(3). Then, for every $X$ from the above class we take an embedding $i: X \to \mathbb{I}^\omega$ and lift $i$ to a map $j: X \to Y_\omega$. The required compactification of $X$ is the closure of $j(X)$ in $Y_\omega$.

Next corollary provides a characterization of quasi-finite countable complexes in terms of compactifications.

**Corollary 2.3.** For a countable complex $L$ the following conditions are equivalent:

(a) $L$ is quasi-finite.

(b) For every separable metrizable space $X$ with $e$-$\dim X \leq L$ and its metrizable compactification $c(X)$ there exists a metrizable compactification $c^*(X)$ such that $e$-$\dim c^*(X) \leq L$ and $c^*(X) \geq c(X)$ (i.e., there is a map from $c^*(X)$ onto $c(X)$ which is the identity on $X$).

**Proof.** (a) $\Rightarrow$ (b) Let $L$ be quasi-finite and $X$ a separable metric space with $e$-$\dim X \leq L$. We take a metric compactification $c(X)$ of $X$ and a map $f: \beta X \to c(X)$ such that $f(x) = x$ for every $x \in X$. Since, by Theorem 2.1, $e$-$\dim \beta X \leq L$, $f$ can be factored through a metrizable compactum $Z$ with $e$-$\dim Z \leq L$. Clearly, $Z$ is a compactification of $X$ which is $\geq c(X)$.

(b) $\Rightarrow$ (a) According to [17, Corollary 3.4], there exists a metrizable compactum $Y$ with $e$-$\dim Y \leq L$ and a surjective map $f: Y \to \mathbb{I}^\omega$ such that for any map $g: X \to \mathbb{I}^\omega$, $X$ being separable metrizable with $e$-$\dim X \leq L$, there exists an embedding $i: X \to Y$ lifting $g$, i.e. $f \circ i = g$. Hence, $f$ is $L$-invertible with respect to separable metric spaces. By Theorem 2.1(3), it suffices to show that $f$ is $L$-invertible. Consider $g: Z \to \mathbb{I}^\omega$ where $e$-$\dim Z \leq L$. According to [8, Proposition 4.9], there exist a Polish space $P$ with $e$-$\dim P \leq L$ and maps $h: Z \to P$ and $q: P \to \mathbb{I}^\omega$ with $g = q \circ h$. We lift $q$ to a map $\overline{q}: P \to Y$ such that $f \circ \overline{q} = q$. Then $\overline{q} \circ h$ is the required lifting of $g$. □

Here is another property of quasi-finite complexes:

**Proposition 2.4.** Every quasi-finite complex $L$ has the following connected-pairs property:

$\text{(CP)}$ For any metrizable compactum $K$ with $e$-$\dim K \leq L$ there exists a metrizable compactum $P$ containing $K$ such that $e$-$\dim P \leq L$ and the pair $K \subset P$ is $L$-connected.
Proof. Suppose $K$ is a metrizable compactum with $e\dim K \leq L$. We embed $K$ into the Hilbert cube $\mathbb{I}^n$ and take an $L$-invertible map $f: Y \to \mathbb{I}^n$ such that $Y$ is a metrizable compactum with $e\dim Y \leq L$ (see Theorem 2.1(3)). Consider the adjunction space $Y \cup_f K$, i.e. the disjoint union of $Y - f^{-1}(K)$ and $K$ with the topology consisting of the usual open subsets of $Y - f^{-1}(K)$ together with sets of the form $f^{-1}(U - K) \cup (U \cap K)$ for open subsets $U$ of $\mathbb{I}^n$. There are two associated maps $p_K: Y \to Y \cup_f K$ and $f_K: Y \cup_f K \to \mathbb{I}^n$ such that $f = f_K \circ p_K$. Since $f$ is $L$-invertible, so is $f_K$. Moreover, $Y - f^{-1}(K)$, being open in $Y$, is the union of countably many compact sets each with $e\dim \leq L$. Hence, by the countable sum theorem, $e\dim Y \cup_f K \leq L$.

We need only to show that the pair $K \subset Y \cup_f K$ is $L$-connected. Let $g: A \to K$ be a map from a closed subset $A \subset Z$ such that $e\dim Z \leq L$ and $A$ is normally placed in $Z$ with respect to $(g, Y \cup_f K)$. Then, considering $g$ as a map from $A$ into $K \subset \mathbb{I}^n$, we obviously have that $A$ is normally placed in $Z$ with respect to $(g, \mathbb{I}^n)$. Since $\mathbb{I}^n$ is an absolute extensor, there exists a map $\overline{g}: Z \to \mathbb{I}^n$ extending $g$. Finally, since $f_K$ is $L$-invertible, we lift $\overline{g}$ to a map $h: Z \to Y \cup_f K$ with $f_K \circ h = \overline{g}$. Clearly, $h$ extends $g$. \hfill $\square$

**Proposition 2.5.** For every $n \geq 2$ there is no $K(\mathbb{Z}, n)$-connected pair $K \subset P$ of compact sets such that $K$ is homeomorphic to the $n$-dimensional sphere $S^n$ and $\dim_\mathbb{Z} P \leq n$.

Proof. We use the arguments from the proof of [17, Theorem 3.5]. Suppose for some $n \geq 2$ there is a $K(\mathbb{Z}, n)$-connected compact pair $S^n \subset P$ with $\dim_\mathbb{Z} P \leq n$. We choose a complex $L$ of type $K(\mathbb{Z}, n)$ and having finite skeleta. It was shown in [18] that there exist metrizable compacta $X_k$, $k \geq 1$, such that:

- $\dim_\mathbb{Z} X_k \leq n$ for each $k$;
- each $X_k$ contains a copy of $S^n$;
- the inclusion $i: S^n \hookrightarrow L$ cannot be extended over $X_k$ so that the image of the extension is contained in the $k$-skeleton $L^{(k)}$ of $L$.

We take an extension $h: P \to L$ of the inclusion $i: S^n \hookrightarrow L$, and $m$ such that $h(P) \subset L^{(m)}$. This means that the inclusion $j: S^n \hookrightarrow P$ cannot be extended to a map from $X_m$ into $P$ which contradicts the fact that $S^n \subset P$ is $L$-connected. \hfill $\square$

The problem [27] whether, for any fixed $n \geq 2$ there is a universal space in the class of all metrizable compacta $X$ with $\dim_\mathbb{Z} \leq n$ is still unsolved. Zarichnyi [28] observed that each of the above classes does not have an universal element which is an absolute extensor for the same class. Proposition 2.5 yields a little bit stronger observation.

**Corollary 2.6.** None of the complexes $K(\mathbb{Z}, n)$, $n \geq 2$, have the $(CP)$-property.

Recall that a map $f: X \to Y$ between metrizable spaces is called uniformly 0-dimensional [21] if there exists a metric on $X$ generating its topology such that
for every $\epsilon > 0$ every point of $f(X)$ has a neighborhood $U$ in $Y$ with $f^{-1}(U)$
being the union of disjoint open subsets of $X$ each of diameter $< \epsilon$. It is well
known that every metric space admits uniformly 0-dimensional map into $l_2$.

**Proposition 2.7.** Let $L$ be a quasi-finite CW-complex. Then for every $\tau \geq \omega$
there exists a perfect $L$-invertible surjection $f_{(L,\tau)} : Y_{(L,\tau)} \to l_2(\tau)$ such that:

(a) $Y_{(L,\tau)}$ is a completely metrizable space of weight $\tau$ with $e\text{-dim} Y_{(L,\tau)} \leq L.$

(b) Every (completely) metrizable space of weight $\leq \tau$ and extension dimension
$\leq L$ can be embedded as a (closed) subspace of $Y_{(L,\tau)}$.

**Proof.** By Theorem 2.1(3), there exists an $L$-invertible map $f : Y \to \mathbb{I}^\omega$, where
$Y$ is a metrizable compactum with $e\text{-dim} Y \leq L$. We embed $l_2$ in $\mathbb{I}^\omega$ and let
$Y_{(L,\omega)} = f^{-1}(l_2)$ and $f_{(L,\omega)} = f|Y_{(L,\omega)}$. Then $e\text{-dim} Y_{(L,\omega)} \leq L$ and since $f$ is
$L$-invertible, so is $f_{(L,\omega)}$.

If $\tau > \omega$, we take a metric $d_1$ on $l_2(\tau)$ and a uniformly 0-dimensional map
g : $l_2(\tau) \to l_2$ with respect to $d_1$. Denote by $Y_{(L,\tau)}$ the fibered product of $l_2(\tau)$
and $Y_{(L,\omega)}$ with respect to the maps $g$ and $f_{(L,\omega)}$. We also consider the projections
$f_{(L,\tau)} : Y_{(L,\tau)} \to l_2(\tau)$ and $h : Y_{(L,\tau)} \to Y_{(L,\omega)}$. Since $f_{(L,\omega)}$ is a perfect and
$L$-invertible surjection, so is $f_{(L,\tau)}$. If $d_2$ is any metric on $Y_{(L,\omega)}$, then $h$ is uni-
formly 0-dimensional with respect to the metric $d = \sqrt{d_1^2 + d_2^2}$ on $Y_{(L,\tau)}$ (see [4]).
Thus $Y_{(L,\tau)}$ admits a uniformly 0-dimensional map into the space $Y_{(L,\omega)}$ having
extension dimension $\leq L$. Hence, by [23, Theorem 1.2], $e\text{-dim} Y_{(L,\tau)} \leq L$. Ob-
serve that $Y_{(L,\tau)}$ is completely metrizable as a perfect preimage of the completely
metrizable space $l_2(\tau)$.

To prove the second item, suppose $M$ is a metrizable space of weight $\leq \tau$ and
e-$\text{dim} M \leq L$. We consider $M$ as a subset of $l_2(\tau)$ and use the $L$-invertibility of
$f_{(L,\tau)}$ to lift the identity map on $M$. Obviously this lifting is an embedding of
$M$ into $Y_{(L,\tau)}$. Moreover, if $M$ is completely metrizable, then we can embed it
in $l_2(\tau)$ as a closed subspace. This implies that the corresponding embedding
of $M$ in $Y_{(L,\tau)}$ is also closed. \hfill $\square$

A completion theorem for $L$-dimensional metric spaces, where $L$ is any count-
able CW-complex, was established in [26]. It follows from Proposition 2.7 that
this is also true for quasi-finite (not necessarily countable) complexes $L$.

**Corollary 2.8.** Let $L$ be a quasi-finite complex. Then every metrizable space
$X$ with $e\text{-dim} X \leq L$ has a completion with extension dimension $\leq L$.

**Corollary 2.9.** Let $L$ be a quasi-finite complex and $X$ a metrizable space. Then
e-$\text{dim} X \leq L$ if and only if $X$ admits a uniformly 0-dimensional map into a
separable metrizable space of extension dimension $\leq L$.

**Proof.** In one direction (sufficiency) this follows from the mentioned above re-
sult of Levin [23, Theorem 1.2]. Suppose $X$ is a metrizable space of weight $\tau$
with e-dim$X \leq L$. By Proposition 2.7, $X$ can be embedded in the space $Y_{(L,\tau)}$. It follows from the construction of $Y_{(L,\tau)}$ that the map $h: Y_{(L,\tau)} \to Y_{(L,\omega)}$ is uniformly 0-dimensional. Then the restriction $h|X$ is also uniformly 0-dimensional which completes the proof.

A general factorization theorem for $L$-dimensional compact spaces, where $L$ is an arbitrary complex, was proved in [24]. We provide here a factorization theorem for $L$-dimensional metrizable spaces with $L$ being quasi-finite (see [23, Theorem 1.5] for similar result with $L$ countable).

**Proposition 2.10.** Let $L$ be a quasi-finite complex and let $f: X \to Y$ be a map with $Y$ metrizable. If e-dim$X \leq L$, then $f$ factors through a metrizable space $Z$ such that e-dim$Z \leq L$ and $w(Z) \leq w(Y)$.

**Proof.** Let us first show how to reduce this proposition to the case $Y$ is separable. This reduction is well known (see, for example, [4]), but we present it here for the reader’s convenience. Suppose the result holds when the range space is separable and metrizable. We take a uniformly 0-dimensional map $g: Y \to l_2$ and apply the “separable factorization theorem” to the map $g \circ f: X \to l_2$ to obtain a separable metrizable space $M$ and maps $q: X \to M$ and $h: M \to l_2$ with e-dim$M \leq L$ and $h \circ q = g \circ f$. Let $p_M: Z \to M$ and $p_Y: Z \to Y$ be the pullbacks of $g$ and $h$ respectively. Clearly, $Z$ is a metrizable space of weight $w(Z) \leq w(Y)$. Since $g$ is uniformly 0-dimensional, so is $p_M$. Then, by [23, Theorem 1.2], e-dim$Z \leq L$.

Now we prove the “separable case”. Let $\hat{Y}$ be a metrizable compactification of $Y$ and $\hat{f}: \beta X \to \hat{Y}$ be the Čech-Stone extension of $f$. Since $L$ is quasi-finite, e-dim$\beta X \leq L$. Therefore we can apply the factorization theorem of Levin-Rubin-Schapiro [24] to obtain a metrizable compactum $\hat{Z}$ and maps $\hat{f}_1: \beta X \to \hat{Z}$ and $\hat{f}_2: \hat{Z} \to \hat{Y}$ such that $\hat{f}_2 \circ \hat{f}_1 = \hat{f}$ and e-dim$\hat{Z} \leq L$. Then the space $Z = \hat{f}_1(X)$ and the maps $f_1 = \hat{f}_1|X$ and $f_2 = \hat{f}_2|Z$ form the required factorization. □

We say that a map $f: X \to Y$ is $L$-soft, where $L$ is a $CW$-complex, if for any space $Z$ with e-dim$Z \leq L$, any closed set $A \subset Z$ and any two maps $h: Z \to Y$ and $g: A \to X$, where $A$ is normally placed in $Z$ with respect to $(g, X)$ and $f \circ g = h|A$, there exists a map $\overline{g}: Z \to X$ extending $g$ such that $f \circ \overline{g} = h$. If, in the above definition, we additionally require $Z$ to be from a given class of spaces $\mathcal{A}$, then we say that $f$ is $L$-soft with respect to the class $\mathcal{A}$. It was established in [11] that for every countable complex $L$ and every metric space $Y$ there exists an $L$-soft map $f: X \to Y$ such that $X$ is a metric space of extension dimension $\leq L$ and $w(X) = w(Y)$. We are going to show that quasi-finite complexes also have this property.
Proposition 2.11. Let $L$ be a quasi-finite CW-complex. Then for every $\tau \geq \omega$ there exists an $L$-soft map $p_{(L,\tau)} : X_{(L,\tau)} \to l_2(\tau)$ such that:

(a) $X_{(L,\tau)}$ is a completely metrizable space of weight $\tau$ with $\text{e-dim} X_{(L,\tau)} \leq L$.
(b) $X_{(L,\tau)}$ is an absolute extensor for all metrizable spaces of $\text{e-dim} \leq L$.
(c) $p_{(L,\tau)}$ is a strongly $(L,\tau)$-universal map, i.e. for any open cover $U$ of $X_{(L,\tau)}$, any (complete) metrizable space $Z$ of weight $\leq \tau$ with $\text{e-dim} Z \leq L$ and any map $g : Z \to X_{(L,\tau)}$ there exists a (closed) embedding $h : Z \to X_{(L,\tau)}$ which is $U$-close to $g$ and $p_{(L,\tau)} \circ g = p_{(L,\tau)} \circ h$.

Proof. Using Proposition 2.11 and following Zarichnyi’s idea from [28] (see also [8]) that invertibility generates softness, we can show the existence of a complete separable metrizable space $X$ with $\text{e-dim} X \leq L$ and an $L$-soft map $f : X \to l_2$. Then, as in [11], we construct the space $X_{(L,\tau)}$ and the map $p_{(L,\tau)} : X_{(L,\tau)} \to l_2(\tau)$ possessing the desired properties. □

3. SOME MORE PROPERTIES OF QUASI-FINITE COMPLEXES

In this section, all spaces and all CW-complexes, unless stated otherwise, are, respectively, metrizable and quasi-finite. We are going to show that some properties of finitely dominated complexes remain valid for quasi-finite complexes.

We say that a space $X$ is an absolute (neighborhood) extensor in dimension $L$ (notation $X \in A(N)E(L)$) if for every space $Z$ of extension dimension $\leq L$ and every map $g : A \to X$, where $A$ is a closed subset of $Z$, there exists an extension of $g$ over $Z$ (resp., over a neighborhood of $A$ in $Z$).

Everywhere below $\text{cov}(X)$ denotes the family of all open covers of $X$. Two maps $f_0, f_1 : X \to Y$ are $L$-homotopic [10] if for any map $h : Z \to X \times [0,1]$, where $Z$ is a space with $\text{e-dim} Z \leq L$, the composition $(f_0 \oplus f_1) \circ h \circ (h^{-1}(X \times \{0,1\})) : h^{-1}(X \times \{0,1\}) \to Y$ admits an extension $H : Z \to Y$. If $U \in \text{cov}(X)$ and the extension $H$ in the above definition can be chosen such that the collection $\{H(h^{-1}\{(x) \times [0,1]\}) : x \in X\}$ refines $U$, then $f_0$ and $f_1$ are called $(U,L)$-homotopic.

The following three propositions were given in [10] for finitely dominated countable complexes $L$ and Polish $A(N)E(L)$-spaces $X$. Because of Proposition 2.7, one can show they also hold for quasi-finite complexes $L$ and arbitrary (not necessarily Polish) $A(N)E(L)$-spaces.

Proposition 3.1. Let $X$ be an $A(N)E(L)$-space and $U \in \text{cov}(X)$. Then there exists a cover $V \in \text{cov}(X)$ such that any two $V$-close maps of any space into $X$ are $(U,L)$-homotopic.

Proposition 3.2. Let $X \in A(N)E(L)$ and $U \in \text{cov}(X)$. Then there exists a cover $V \in \text{cov}(X)$ refining $U$, such that the following condition holds:

(H) For any space $Z$ with $\text{e-dim} Z \leq L$, any closed $A \subset Z$, and any two $V$-close maps $f, g : A \to X$ such that $f$ has an extension $F : Z \to X$,
it follows that $g$ also can be extended to a map $G: Z \to X$ which is $(U, L)$-homotopic to $F$.

**Proposition 3.3.** Let $X \in ANE(L)$, $Z$ be a space with $e\dim Z \leq L$ and $A \subset Z$ closed. If $f, g: A \to X$ are $L$-homotopic and $f$ admits an extension $F: Z \to X$, then $g$ also admits an extension $G: Z \to X$, and we may assume that $F$ and $G$ are $L$-homotopic.

A pair of closed subsets $X_0 \subset X_1$ of a space $X$ is called $UV(L)$-connected in $X$ if every neighborhood $U$ of $X_1$ in $X$ contains a neighborhood $V$ of $X_0$ such that $V \subset U$ is $L$-connected with respect to metrizable spaces, i.e. any map $g: A \to V$, where $A$ is a closed subset of a space $Z$ with $e\dim \leq L$, admits an extension $\overline{g}: Z \to U$. When $X_0 \subset X_1$ is $UV(L)$-connected in $X$, we say that $X_0$ is $UV(L)$ in $X$. If in the above definition all pairs under consideration are $L$-connected with respect to a given class $A$, we obtain the notion of $UV(L)$-sets with respect to $A$. If instead of $L$-connectedness of the pair $V \subset U$ we require the inclusion $V \subset U$ to be $L$-homotopic to a constant map in $U$ then the pair $X_0 \subset X_1$ (resp. the set $X_0$) is called $UV(L)$-homotopic in $X$. Obviously, every $UV(L)$-connected pair is $UV(L)$-homotopic. Next corollary, which follows from Proposition 3.3, shows that these two properties are equivalent in case $X \in ANE(L)$.

**Corollary 3.4.** Let $X$ be an $ANE(L)$-space. A pair $X_0 \subset X_1$ of closed subsets of $X$ is $UV(L)$-connected in $X$ if and only if it is $UV(L)$-homotopic in $X$.

**Lemma 3.5.** Let $X_0 \subset X_1 \subset X \subset E$, where both $X$ and $E$ are $ANE(L)$-spaces and $X \subset E$ is closed. Then the pair $X_0 \subset X_1$ is $UV(L)$-connected in $X$ if and only if it is $UV(L)$-connected in $E$.

**Proof.** By Proposition 2.7, there exists a perfect $L$-invertible surjection $f: \bar{E} \to E$ with $e\dim \bar{E} \leq L$, and let $\bar{X} = f^{-1}(X)$. Since $X \in ANE(L)$, we can extend $f|\bar{X}$ to a map $g: W \to X$ with $W$ being a neighborhood of $\bar{X}$ in $\bar{E}$. Since $f$ is closed, we may assume that $W = f^{-1}(G)$ for some neighborhood $G$ of $X$ in $E$. The claim below follows from our constructions.

**Claim.** For every open $O \subset X$ the set $O^* = G - f(g^{-1}(X - O))$ is open in $G$ and has the following two properties: $O^* \cap X = O$ and $g(f^{-1}(O^*)) = O$.

Suppose $X_0 \subset X_1$ is $UV(L)$-connected in $X$. We are going to show that this pair is $UV(L)$-connected in $E$. To this end, let $U \subset G$ be a neighborhood of $X_1$ in $E$. Then there is a neighborhood $O$ of $X_0$ in $X$ such that $O \subset U \cap X$ is $L$-connected. Since $U$ is an $ANE(L)$ (as an open subset of $E$), we can apply Proposition 3.2 for the space $U$ and the one-element cover $U = \{U\}$ to find an open cover $V = \{V_\alpha : \alpha \in \Lambda\}$ of $U$ satisfying the condition (H). For every $\alpha$ let $G_\alpha = V_\alpha \cap (V_\alpha \cap X)^* \cap O^*$ and $V = \bigcup\{G_\alpha : \alpha \in \Lambda\}$. Obviously, $V \subset U$ is open and contains $X_0$. The pair $V \subset U$ is $L$-connected. Indeed, let $h: A \to V$
be a map, where $A \subset Z$ is closed and $\text{e-dim}Z \leq L$. Since $f$ is $L$-invertible, $h$ admits a lifting $h_1: A \to f^{-1}(V)$, i.e. $h = f \circ h_1$. According to the Claim, $g\left(f^{-1}(G_\alpha)\right) \subset V_\alpha \cap X$, $\alpha \in \Lambda$, and $V \cap X \subset O$. This implies that $h$ and the map $h_2 = g \circ h_1: A \to V \cap X$ are $V$-close. Since the pair $O \subset U \cap X$ is $L$-connected, $h_2$ can be extended to a map from $Z$ into $U \cap X$. This yields, according to Proposition 3.2, that $h$ also can be extended to a map from $Z$ into $U$.

Now, suppose the pair $X_0 \subset X_1$ is $UV(L)$-connected in $E$. To show this pair is $UV(L)$-connected in $X$, let $U$ be a neighborhood of $X_1$ in $X$. Then $U^* \subset G$ is open in $E$, and we can find a neighborhood $V$ of $X_0$ in $E$ such that $V \subset U^*$ is $L$-connected. The pair $V \cap X \subset U$ is $L$-connected. Indeed, any map $h: A \to V \cap X$, where $A \subset Z$ is closed and $\text{e-dim}Z \leq L$, admits an extension $h_1: Z \to U^*$. Then the map $\overline{h} = g \circ h_2: Z \to U$, where $h_2: Z \to f^{-1}(U^*)$ is a lifting of $h_1$, extends $h$.

**Theorem 3.6.** Suppose $X$ is an $ANE(L)$-space and the pair $X_0 \subset X_1$ is $UV(L)$-connected in $X$. Then it is $UV(L)$-connected in any $ANE(L)$-space in which $X_1$ is embeddable as a closed subspace.

**Proof.** Let $i: X_1 \to Y$ be a closed embedding, where $Y \in ANE(L)$, and $M$ be the space obtained from the disjoint union $X \uplus Y$ by identifying all pairs of points $x \in X_1 \subset X$ and $i(x) \in Y$. The space $M$ is metrizable and if $p: X \uplus Y \to M$ is the quotient map, then $p(X)$, $p(Y)$ and $p(X_1)$ are closed sets in $M$ homeomorphic, respectively, to $X$, $Y$ and $X_1$. Moreover, $p(X_1)$ is the common part of $p(X)$ and $p(Y)$. We embed $M$ in a normed space $E$ as a closed subspace. Every normed space is an absolute extensor for the class of metrizable spaces, so $E \in ANE(L)$. Since the pair $p(X_0) \subset p(X_1)$ is $UV(L)$-connected in $p(X)$, by Lemma 3.5 it is also $UV(L)$-connected in $E$. This implies, again by Lemma 3.5, that $p(X_0) \subset p(X_1)$ is $UV(L)$-connected in $p(Y)$. □

**Corollary 3.7.** If a space $X$ is $UV(L)$ in a given $ANE(L)$-space, then $X$ is $UV(L)$ in any $ANE(L)$-space in which $X$ is embeddable as a closed subset.

In the existing literature, the $UV^\infty$-property, and more general, the $UV(L)$-property, is defined for compact spaces, see [10] and [6]. We extend this definition to arbitrary (metrizable) spaces: $X$ is a $UV(L)$-space if it is $UV(L)$ in some $ANE(L)$-space containing $X$ as a closed subspace. According to Corollary 3.7, the $UV(L)$-property does not depend on the embeddings in $ANE(L)$-spaces (for compact spaces and finite complexes $L$ this was done in [6]). It follows from Corollary 3.4 that $X$ is a $UV(L)$-space if and only if $X$ is $UV(L)$-homotopic in every space $Y \in ANE(L)$ containing $X$ as a closed subset.

Recall that a normal space $X$ is a $C$-space [1] if for any sequence $\{\omega_n\}$ of open covers of $X$ there exists a sequence $\{\gamma_n\}$ of open disjoint families such that each $\gamma_n$ refines $\omega_n$ and $\cup \gamma_n$ covers $X$. Every finite-dimensional paracompactum, as well as every countable-dimensional metrizable space has property $C$ [19].
We say that a complex \( L \) (not necessarily quasi-finite) possesses the soft map property if for every space \( X \) there exists a space \( Y \) with \( \text{e-dim} Y \leq L \) and an \( L \)-soft map from \( Y \) onto \( X \). Every countable complex has the soft map property (see [11]), as well as every quasi-finite complex (by Proposition 2.11).

A pair of spaces \( \tilde{V} \subset \tilde{U} \) is called an \( L \)-extension of the pair \( V \subset U \) [7] if \( \tilde{U} \in AE(L) \) and there exists a map \( q: \tilde{U} \to U \) such that the restriction \( q|\tilde{V} \) is an \( L \)-soft map onto \( V \). The following property of \( L \)-extension pairs was established in [7].

**Lemma 3.8.** Let \( L \) be a complex (not necessarily quasi-finite) with the soft map property and \( \tilde{V} \subset \tilde{U} \) an \( L \)-extension of the pair \( V \subset U \). Let also \( A \subset B \) be a pair of closed subsets of a space \( X \) with \( \text{e-dim} X \leq L \). Suppose we have maps \( f: B \to U \) and \( g: A \to \tilde{U} \) such that \( q \circ g = f|A \) and \( f(B \setminus A) \subset V \). Then there exists a map \( h: X \to \tilde{U} \) such that \( q \circ (h|B) = f \).

**Lemma 3.9.** Let \( L \) be a complex (not necessarily quasi-finite). Every \( L \)-connected pair \( V \subset U \) of spaces admits an \( L \)-extension provided \( L \) has the soft map property.

**Proof.** We take a normed space \( E \) containing \( V \) as a closed subspace and an \( L \)-soft surjection \( g: \tilde{U} \to E \) such that \( \tilde{U} \) is a space of \( \text{e-dim} \leq L \). Since \( V \subset U \) is \( L \)-connected, there exists a map \( q: \tilde{U} \to U \) extending the map \( g|\tilde{V} \), where \( \tilde{V} = g^{-1}(V) \). Moreover, \( \tilde{U} \in AE(L) \) because \( E \) is an absolute extensor for the class of metrizable spaces and \( g \) is \( L \)-soft. Therefore, \( \tilde{V} \subset \tilde{U} \) is an \( L \)-extension of \( V \subset U \).

If \( A \) is a subset of a space \( X \) we denote the star of \( A \) with respect to a cover \( \omega \in \text{cov}(X) \) by \( \text{St}(A, \omega) \). We say that \( \nu \in \text{cov}(X) \) is a strong star-refinement of \( \omega \in \text{cov}(X) \) if for each \( V \in \nu \) there exists \( W \in \omega \) such that \( \text{St}(V, \nu) \subset W \).

**Auxiliary Construction.** Suppose we are given the spaces \( X, Z \) and the map \( g: A \to X \), where \( A \subset Z \) is closed. Let \( \alpha_n = \{ U_n(x) : x \in X \}, \beta_n = \{ V_n(x) : x \in X \}, n \geq 0 \), be two sequences of open covers of \( X \) and \( \mu_n^*, n \geq 1 \), be a sequence of disjoint open families in \( A \) such that:

1. \( \alpha_n \) is a strong star refinement of \( \beta_{n-1} \) for any \( n \geq 1 \).
2. Each \( \mu_n^*, n \geq 1 \), refines \( g^{-1}(\beta_n) \) and \( \cup \{ \mu_n^* : n \geq 1 \} \) is a locally finite cover of \( A \).

We are going first to construct open and disjoint families \( \mu_n, n \geq 1 \), in \( Z \) satisfying the following condition:

3. \( \mu = \cup \{ \mu_n : n \geq 1 \} \) is locally finite in \( Z \) and the restriction of each \( \mu_n \) on \( A \) is \( \mu_n^* \).

To this end, we choose an upper semi-continuous (br., u.s.c.) set-valued map \( r: Z \to A \) such that each \( r(z) \) is a finite set and \( r(z) = \{ z \} \) for \( z \in A \) (see [25] for the existence of such \( r \)). Recall that \( r \) is upper semi-continuous means that
$r^*(T) = \{ z \in Z : r(z) \subset T \}$ is open in $Z$ whenever $T$ is open in $A$. Obviously, $r^*(T) \cap A = T$ and $r^*(T_1) \cap r^*(T_2) \neq \emptyset$ if and only if $T_1 \cap T_2 \neq \emptyset$ for any open subsets $T$, $T_1$ and $T_2$ of $A$. Therefore all families $\mu_n = \{ r^*(T) : T \in \mu_n^* \}$, $n \geq 1$, are open and disjoint in $Z$. Since $\mu^*$ is locally finite in $A$ and $r$ is finite-valued, the family $\mu = \cup \{ \mu_n : n \geq 1 \}$ is locally finite in $Z$.

The second part of our construction is to find points $x_W \in X$ such that

(4) $St(g(W \cap A), \alpha_n) \subset V_{n-1}(x_W)$ for every $W \in \mu_n$ and $n \geq 1$

This can be done as follows. Since $\alpha_n$ is a strong star refinement of $\beta_n-1$ and $\mu_n$ refines $g^{-1}(\beta_n)$, for every $n \geq 1$ and $W \in \mu_n$ there exist $S \in \beta_n$ and a point $x_W \in X$ such that $St(g(W \cap A), \alpha_n) \subset St(S, \alpha_n) \subset V_{n-1}(x_W)$. The auxiliary construction is completed.

**Lemma 3.10.** Let $L$ be a complex (not necessarily quasi-finite) with the soft map property and $f : M \to X$ be a surjection with the following property:

(UV) for every $x \in X$ and its neighborhood $U(x)$ in $X$ there exists a smaller neighborhood $V(x)$ of $x$ such that the pair $V(x) = f^{-1}(V(x)) \subset U(x) = f^{-1}(U(x))$ is $L$-connected with respect to the class of metrizable spaces.

Suppose $p : Y \to Z$ is a surjective map with $\text{e-dim} Y \leq L$. Then, for any $\omega \in \text{cov}(X)$ and any map $g : A \to X$, where $A$ is a closed subset of $Z$ such that either $A$ or $g(A)$ is a $C$-space, there is a neighborhood $G$ of $A$ in $Z$ and a map $h : p^{-1}(G) \to M$ with $(f \circ h)|p^{-1}(A)$ being $\omega$-close to $g \circ p$.

**Proof.** For every $x \in X$ and $n = 0, 1, 2, \ldots$ we choose a point $P(x) \in f^{-1}(x)$ and neighborhoods $U_n(x)$ and $V_n(x)$ of $x$ in $X$ such that the cover $\alpha_0 = \{ U_0(x) : x \in X \}$ refines $\omega$, each pair $V_n(x) \subset U_n(x)$ is $L$-connected with respect to all metrizable spaces and the covers $\alpha_n = \{ U_n(x) : x \in X \}$, $\beta_n = \{ V_n(x) : x \in X \}$ satisfy condition (1) from the auxiliary construction. Since either $A$ or $g(A)$ is a $C$-space, there exists a sequence of disjoint open families $\{ \mu_n^* : n \geq 1 \}$ in $A$ satisfying condition (2) above. Therefore, according to the auxiliary construction, we can extend each $\mu_n^*$ to a disjoint open family $\mu_n$ in $Z$ such that $\mu = \cup \{ \mu_n : n \geq 1 \}$ is locally finite in $Z$ and let $G$ be the union of all elements of $\mu$.

We introduce the following notations: $B = p^{-1}(A)$, $\overline{g} = g \circ (p|B)$, $\Omega = p^{-1}(G)$, and $\nu_n = p^{-1}(\mu_n)$. Obviously, each $\nu_n$ is a disjoint open family in $Y$ and $\nu = \cup \{ \nu_n : n \geq 1 \}$ is a locally finite cover of $\Omega$. Let us also consider the open covers $\tilde{\omega} = f^{-1}(\omega)$, $\tilde{\alpha}_n = \{ \tilde{U}_n(x) : x \in X \}$ and $\tilde{\beta}_n = \{ \tilde{V}_n(x) : x \in X \}$ of $M$ corresponding, respectively, to $\omega$, $\alpha_n$ and $\beta_n$. According to Lemma 3.9, every pair $\tilde{V}_n(x) \subset \tilde{U}_n(x)$ has an $L$-extension $\tilde{V}_n(x) \subset \tilde{U}_n(x)$ with a corresponding map $q_{n,x} : \tilde{U}_n(x) \to \tilde{U}_n(x)$ such that $(q_{n,x})|\tilde{V}_n(x)$ is an $L$-soft surjection onto $\tilde{V}_n(x)$.
Consider the nerve $\mathcal{R}$ of $\nu$ and a barycentric map $\theta: \Omega \to |\mathcal{R}|$. Any simplex $\sigma = \langle W_0, W_1, \ldots, W_k \rangle$ from $\mathcal{R}$, where $W_i \in \nu_{n(i)}$, can be ordered such that $n(0) < n(1) < \ldots < n(k)$. This is possible because $\cap\{W_i : i = 0, 1, \ldots, k\} \neq \emptyset$, so the numbers $n(i)$ are different. It is easily seen that, for fixed $k \geq 1$ and $W \in \nu_k$, condition (4) from the auxiliary construction implies the following one

\[(5) \ St(\mathcal{F}(W \cap B), \alpha_k) \subset V_{k-1}(x_W), \text{ and therefore } St(f^{-1}(\mathcal{F}(W \cap B)), \alpha_k) \subset \tilde{V}_{k-1}(x_W).\]

Let $\Sigma(\sigma)$, $\sigma \in \mathcal{R}$, be the closed subset $\theta^{-1}(\sigma)$ of $\Omega$ and $\Sigma^k = \theta^{-1}(\mathcal{R}^k)$, where $\mathcal{R}^k$ denotes the $k$-th skeleton of $\mathcal{R}$. For every $k \geq 0$ and $\sigma = \langle W_0, W_1, \ldots, W_k \rangle \in \mathcal{R}^k$ with $W_0 \in \nu_{n(0)}$, we define by induction maps $h_k: \Sigma^k \to M$ and $h_\sigma: \Sigma(\sigma) \to \tilde{U}_{n(0)-1}(x_{W_0})$ such that

\[(6) \ h_k|_{\Sigma^{k-1}} = h_{k-1} \text{ for } k \geq 1 \text{ and } h_k|_{\Sigma(\sigma)} = q_{n(0)-1, x_{W_0}} \circ (h_\sigma|_{\Sigma(\sigma)}) \text{ for } k \geq 0 \text{ and } \]

\[(7) \ f^{-1}(\mathcal{F}(W_0 \cap B)) \bigcup h_k(\Sigma(\sigma)) \subset \tilde{U}_{n(0)-1}(x_{W_0}), \text{ } k \geq 0.\]

We also require that

\[(8) \ h_{\sigma_1}|_{(\Sigma(\sigma_1) \cap \Sigma(\sigma_2))} = h_{\sigma_2}|_{(\Sigma(\sigma_1) \cap \Sigma(\sigma_2))} \text{ for any } \sigma_1 \text{ and } \sigma_2 \text{ from } \mathcal{R}^k \text{ having the same first vertex.}\]

For $k = 0$ we define $h_0: \Sigma^0 \to M$ and $h_{<W>}: \Sigma(< W >) \to \tilde{U}_{n-1}(x_W)$ by $h_0(\Sigma(< W >)) = P(x_W)$ and $h_{<W>}(\Sigma(< W >)) = Q(x_W)$, where $W \in \nu_n$ and $Q(x_W)$ is a point from $\tilde{V}_{n-1}(x_W)$ with $q_{0, x_W}(Q(x_W)) = P(x_W)$. Obviously, $h_0$ restricted on every set $W \cap \Sigma^0$ is constant, so it is continuous. Moreover, every $h_{<W>}$ is also constant satisfying condition (6), and, by (5), $h_0$ satisfies also (7). Note that condition (8) holds for $k = 0$.

Suppose that for some $k \geq 1$ maps $h_{k-1}: \Sigma^{k-1} \to M$ and $h_\sigma: \Sigma(\sigma) \to \tilde{U}_{m-1}(x_W)$ satisfying conditions (6), (7) and (8) have already been defined. Here $\sigma \in \mathcal{R}^{k-1}$ and $W \in \nu_m$ is the first vertex of the simplex $\sigma$.

Now, let $\sigma = \langle W_0, W_1, \ldots, W_k \rangle \in \mathcal{R}^k$ with $W_i \in \nu_{n(i)}$, $i = 0, 1, \ldots, k$. Then $\sigma \cap \mathcal{R}^{k-1}$ consists of the simplexes $\sigma_i = \langle W_0, W_i, W_{i+1}, \ldots, W_k \rangle$, $i = 1, 2, \ldots, k$ and the simplex $\sigma_0 = \langle W_0, W_1, W_2, \ldots, W_k \rangle$.

Claim. $f^{-1}(\mathcal{F}(W_0 \cap B)) \bigcup h_{k-1}(\Sigma(\sigma_0)) \subset \tilde{V}_{n(0)-1}(x_{W_0})$ and $f^{-1}(\mathcal{F}(W_0 \cap B)) \bigcup h_{k-1}(\Sigma(\sigma_i)) \subset \tilde{U}_{n(0)-1}(x_{W_0})$ for every $i = 1, \ldots, k$.

Indeed, by (7) we have $f^{-1}(\mathcal{F}(W_1 \cap B)) \bigcup h_{k-1}(\Sigma(\sigma_0)) \subset \tilde{U}_{n(1)-1}(x_{W_1})$. But $\mathcal{F}(W_1 \cap B) \cap \mathcal{F}(W_0 \cap B) \neq \emptyset$, and hence $f^{-1}(\mathcal{F}(W_1 \cap B)) \bigcup h_{k-1}(\Sigma(\sigma_0))$ is contained in $St(f^{-1}(\mathcal{F}(W_0 \cap B)), \tilde{\alpha}_{n(1)-1})$. Since $n(0) \leq n(1)-1$, $\tilde{\alpha}_{n(1)-1}$ refines $\tilde{\alpha}_{n(0)}$. This fact and the inclusion $St(f^{-1}(\mathcal{F}(W_0 \cap B)), \tilde{\alpha}_{n(0)}) \subset \tilde{V}_{n(0)-1}(x_{W_0})$, which follows from (5), complete the proof of the claim for $i = 0$. Since $W_0$ is a vertex of each $\sigma_i$, $i = 1, 2, \ldots, k$, the other inclusions from the claim follow directly from (7).
Consider the “boundary” \( \partial \Sigma(\sigma) = \bigcup_{i=0}^{k} \Sigma(\sigma_i) \) of \( \Sigma(\sigma) \). According to the claim, \( h_{k-1}(\partial \Sigma(\sigma)) \subset \tilde{U}_{n(0)-1}(x_{W_0}) \) and \( h_{k-1}(\partial \Sigma(\sigma) \setminus \Sigma_0) \subset \tilde{V}_{n(0)-1}(x_{W_0}) \), where \( \Sigma_0 = \bigcup_{i=1}^{k} \Sigma(\sigma_i) \). Since the maps \( h_{\sigma_i} : \Sigma(\sigma_i) \to \tilde{U}_{n(0)-1}(x_{W_0}) \), \( i = 1, \ldots, k \), satisfy condition (8), they determine a map \( h_{\Sigma} : \Sigma_0 \to \tilde{U}_{n(0)-1}(x_{W_0}) \) such that \( h_{\sigma_i}|\Sigma(\sigma_i) = h_{\Sigma}|\Sigma(\sigma_i) \) for each \( i \). Moreover, by (6), \( q_{n(0)-1,x_{W_0}} \circ h_{\Sigma} = h_{k-1}|\Sigma_0 \).

Therefore, we can apply Lemma 3.8 for the pair \( \tilde{V}_{n(0)-1}(x_{W_0}) \subset \tilde{U}_{n(0)-1}(x_{W_0}) \), its \( L \)-extension \( \tilde{V}_{n(0)-1}(x_{W_0}) \subset \tilde{U}_{n(0)-1}(x_{W_0}) \), the sets \( \Sigma_0 \subset \partial \Sigma(\sigma) \subset \Sigma(\sigma) \) and the maps \( h_\Sigma \) and \( h_{k-1}|\partial \Sigma(\sigma) \). In this way we obtain a map \( h_\sigma : \Sigma(\sigma) \to \tilde{U}_{n(0)-1}(x_{W_0}) \) such that \( q_{n(0)-1,x_{W_0}} \circ h_\sigma|\partial \Sigma(\sigma) = h_{k-1}|\partial \Sigma(\sigma) \). Now we define \( h_k : \Sigma^k \to M \) by \( h_k|\Sigma(\sigma) = q_{n(0)-1,x_{W_0}} \circ h_\sigma \). Obviously, \( h_k \) is continuous on every “simplex” \( \Sigma(\sigma) \), \( \sigma \in \mathfrak{H}^k \), and, since the family \( \nu \) is locally finite in \( \Omega \), \( h_k \) is continuous. Moreover, \( h_k \) and \( h_\sigma \) satisfy conditions (6), (7) and (8), and the induction is completed.

Finally, we define \( h : \Omega \to M \) letting \( h|\Sigma^k = h_k \) for each \( k \). Continuity of \( h \) follows from continuity of each \( h_k \) and the fact that \( \nu \) is locally finite. Observe also that \( (f \circ h)|p^{-1}(A) \) is \( \omega \)-close to \( g \circ p \) because of condition (7).

**Proposition 3.11.** Let \( L \) be a complex (not necessarily quasi-finite) with the soft map property and \( f_0 : M \to X \) be a closed map such that each fiber \( f_0^{-1}(x) \), \( x \in X \), is \( UV(L) \)-connected in \( M \). Then for every map \( g_0 : A \to X \), where \( A \) is a closed subset of a space \( Z \) with \( e\dim Z \leq L \) such that either \( A \) or \( \varrho_0(A) \) is a \( C \)-space, there exists a neighborhood \( Q \) of \( A \) in \( Z \) and an u.s.c map \( \Psi : Q \to M \) such that \( \Psi \) is single-valued on \( Q \setminus A \) and \( f_0 \circ \Psi \) is a continuous single-valued map extending \( g_0 \).

**Proof.** Our proof is based on some ideas from [2, proof of Theorem 3.1]. Let \( f_0 \) and \( g_0 \) be as in the proposition. We take sequences \( \{\omega_n\} \subset \text{cov}(X) \) and \( \{\gamma_n\} \subset \text{cov}(A) \), and open intervals \( \{\Delta_n\} \) covering the interval \( J = [0,1] \), with \( 0 \in \Delta_1 \), such that:

- \( \omega_{n+1} \) is a strong star-refinement of \( \omega_n \) and \( \gamma_{n+1} \) is a strong star-refinement of \( \gamma_n \), \( n = 1, 2, 3, \ldots \),
- \( \lim \text{mesh}(\omega_n) = \lim \text{mesh}(\gamma_n) = 0 \)
- \( \Delta_n \cap \Delta_m \neq \emptyset \) if and only if \( n \) and \( m \) are consecutive integers.

Then \( \omega = \{\omega_n \times \Delta_n : n = 1, 2, \ldots\} \) and \( \gamma = \{\gamma_n \times \Delta_n : n = 1, 2, \ldots\} \) are open covers, respectively, of \( X \times J \) and \( A \times J \), satisfying the following conditions:

1. For every point \((x,1) \in X \times I \) and its neighborhood \( U \) in \( X \times I \) there exists another neighborhood \( V \) such that \( St(V, \omega) \subset U \).
2. For every point \((a,1) \in A \times I \) and its neighborhood \( U \) in \( A \times I \) there exists another neighborhood \( V \) such that \( St(V, \gamma) \subset U \).

Since \( f_0 \) is a closed map all fibers of which are \( UV(L) \)-connected in \( M \), the map \( f = f_0 \times id : M \times J \to X \times J \) has the property \( (UV) \) from Lemma 3.10.
Further, let $g$ denote the map $g_0 \times id : A \times J \to X \times J$ and consider an $L$-soft surjection $p : Y \to Z \times I$, $I = [0, 1]$, such that $Y$ is a space of e-dim$Y \leq L$. We have the following diagram:

$$
\begin{array}{ccc}
Y & \xrightarrow{p \text{ (L-soft)}} & M \times J \\
\downarrow & & \downarrow f = f_0 \times id \\
Z \times I \supset A \times J & g = g_0 \times id & X \times J
\end{array}
$$

Since the product of any metrizable $C$-space and $J$ is also a $C$-space, either $A \times J$ or $g_0(A) \times J$ is a $C$-space. Following the notations from Lemma 3.10, we can apply construction of this lemma by considering the spaces $M \times J$, $X \times J$, $Z \times J$, $A \times J$ and $p^{-1}(Z \times J)$ instead of the spaces $M$, $X$, $Z$, $A$ and $Y$, respectively. Let us also note that in our situation we take $\alpha_n$ and $\beta_n$, $n \geq 0$, to be open covers of $X \times J$ satisfying condition (1) from the auxiliary construction with $\alpha_0$ refining $\omega$. We also require $\mu^*_n$ to be disjoint open families in $A \times J$ satisfying condition (2) such that $\mu^* = \bigcup_{n=1}^\infty \mu^*_n$ is a locally finite open cover of $A \times J$ which, in addition, refines $\gamma$. Then, as in the auxiliary construction, we can extend $\mu^*_n$ to disjoint open families $\mu_n$ in $Z \times J$ by choosing an u.s.c. retraction $r : Z \times I \to A \times I$ such that $r(z,t) \subset A \times \{t\}$ for every $t \in I$. This can be achieved by taking an u.s.c. finite-valued retraction $r_1 : Z \to A$ and letting $r(z,t) = r_1(z) \times \{t\}$. Observe that this special choice of $r$ implies that $r^2(T)$ is open in $Z \times I$ for every open $T \subset A \times I$ and $r^2(T)$ is contained in $Z \times J$ provided $T \subset A \times J$. We also pick the points $x_W \in X \times J$, $W \in \mu$, satisfying condition (4).

According to Lemma 3.10, there exists a map $h : p^{-1}(G) \to M \times J$, where $G = \cup \{ \Lambda : \Lambda \in \mu \}$, such that each $h_k = h|\Sigma^k$ satisfies condition (7) and $(f \circ h)|\{p^{-1}(A \times J)\}$ is $\omega$-close to $g \circ p$. Now, let $H = p^{-1}(G \cup (A \times \{1\}))$ and define the set-valued map $\psi : H \to M \times I$ letting $\psi(y) = h(y)$ if $y \in p^{-1}(G)$ and $\psi(y) = (f_0^{-1}(g_0(p(y))), 1)$ if $y \in p^{-1}(A \times \{1\})$. Let also $\psi_1 = \pi \circ \psi : H \to M$, where $\pi : M \times I \to M$ is the projection.

Claim. The map $\psi_1$ is u.s.c.

Since $\pi$ is continuous, it suffices to prove that $\psi$ is u.s.c. To this end, observe that $p^{-1}(G)$ is open in $H$ and $\psi$ is single-valued and continuous on $p^{-1}(G)$, so that we need to show only that $\psi$ is u.s.c. at the points of $p^{-1}(A \times \{1\})$. Let $\{y_i\} \subset H$ be a sequence converging to a point $y_0 \in p^{-1}(A \times \{1\})$ and $U_0 = V_0 \times (t, 1]$ be a neighborhood of $\psi(y_0) = (f_0^{-1}(g_0(p(y_0))), 1)$ in $M \times I$. We are going to show that $\psi(y_i) \subset U_0$ for almost all $i$ which will complete the proof of the claim. Since $f_0$ is a closed map, $\psi$ is u.s.c. on $p^{-1}(A \times \{1\})$. Therefore we can assume that $\{y_i\} \subset p^{-1}(G)$, hence $\psi(y_i) = h(y_i)$ for all $i$. Thus $p(y_0) = (a, 1) \in A \times \{1\}$ and $p(y_i) \in G$. Since $f_0$ is closed, we can find a neighborhood $V$ of $g_0(p(y_0))$ in $X$ with $f_0^{-1}(V) \subset V_0$. By (9), there exists a
Let \( \varphi \) and \( X \) be established in [6, Corollary 7.5] for finite complexes \( UV \) between Polish spaces \((U, \vartheta)\). Since \( \{p(y_i)\} \) converges to \((a, 1)\), we can assume that \( \{p(y_i)\} \subset r^2(S) \). It suffices to show that \( f(h(y_i)) \in U \) for all \( i \). To this end, fix \( i \) and \( \Lambda_0 \in \mu_k(0) \), \( p(y_i) \), where \( k(0) \) is the minimal \( k \) such that \( p(y_i) \) is contained in some element of \( \mu_k \). Then \( \Lambda_0 = r^2(\Lambda_0^*) \) for some \( \Lambda_0^* \in \mu^* \) and therefore \( p(y_i) \in r^2(\Lambda_0^*) \cap r^2(S) \). Consequently, \( S \) meets \( \Lambda_0^* \) and let \( p(y_i^*) \in \Lambda_0^* \cap S \), where \( y_i^* \in p^{-1}(\Lambda_0^*) \). On the other hand, there exists \( \Gamma \in \gamma \) containing \( \Lambda_0^* \) (recall that \( \mu^* \) refines \( \gamma \)). Therefore, \( p(y_i^*) \in St(S, \gamma) \cap T(a) \times (q, 1] \). Since \( g(p(y_i^*)) = (g_0 \circ id)(p(y_i^*)) \), according to the choice of \( T(a) \times (q, 1] \) we have
\[
(10) \quad g(p(y_i^*)) \in U_1 = V_1 \times (q, 1].
\]
Since \( k(0) \) is the minimal \( k \) such that \( y_i \) is contained in some \( W \in \nu_k \), according to the definition of the maps \( h_k \) and condition (7) from Lemma 3.10, we have \( h(y_i) \in \tilde{U}_{k(0)-1}(x_{W_0}) \), where \( W_0 = p^{-1}(\Lambda_0) \). The last inclusion implies \( f(h(y_i)) \in U_{k(0)-1}(x_{W_0}) \). Also, condition (5) from Lemma 3.10 yields that
\[
(11) \quad g(p(y_i^*)) \in g(p(W_0 \cap p^{-1}(A \times J))) \subset V_{k(0)-1}(x_{W_0}).
\]
Hence, both \( g(p(y_i^*)) \) and \( f(h(y_i)) \) are points from \( U_{k(0)-1}(x_{W_0}) \). But the cover \( \alpha_{k(0)-1} \) refines \( \omega \), and hence \( U_{k(0)-1}(x_{W_0}) \) is contained in an element \( O \) of \( \omega \). Therefore, \( O \) contains \( g(p(y_i^*)) \) and \( f(h(y_i)) \). This means, according to (10), that \( f(h(y_i)) \in St(U_1, \omega) \). Finally, since \( St(U_1, \omega) \subset U \), we obtain \( f(h(y_i)) \in U \) which completes the proof of the claim.

Now we can finish the proof. There exists a decreasing sequence \( \{Q_i\} \) of open subsets of \( Z \) and an increasing sequence of real numbers \( 0 = t_0 < t_1 < \ldots < 1 \) such that \( \bigcap_{i=1}^\infty Q_i = A \), \( \lim t_i = 1 \), \( \overline{Q}_{i+1} \subset Q_i \), and \( Q_i \times [0, t_i] \subset G \) for all \( i \). Let \( \varphi_i: Z \to [t_{i-1}, t_i], i \geq 1 \), be continuous functions such that \( \varphi_i(Z \setminus Q_i) = t_{i-1} \) and \( \varphi_i(z) = t_i \) for \( z \in \overline{Q}_{i+1} \). Then \( \varphi: Z \to [0, 1] \) defined by \( \varphi(z) = \varphi_i(z) \) for \( z \in Q_i \setminus Q_{i+1} \), \( \varphi(Z \setminus Q_1) = 0 \), and \( \varphi(A) = 1 \), is continuous. Consequently, the map \( \theta: Q_1 \to G \cup (A \times \{1\}) \), \( \theta(z) = (z, \varphi(z)) \), is well defined and continuous. Moreover, \( \theta(z) = (z, 1) \) for all \( z \in A \). Since \( p \) is \( L \)-invertible and \( e \)-dim \( Q_1 \) \( \leq L \) (as an open subset of \( Z \)), we can lift \( \theta \) to a map \( \overline{\theta}: Q_1 \to H \). Then \( \Psi = \psi_1 \circ \overline{\theta}: Q \to M \), where \( Q = Q_1 \), is the required map.

Theorem 3.12 below is a generalization of the well known result that if \( G \) is an u.s.c. decomposition of a metrizable space \( X \) such that each element of \( G \) is \( UV^n \) in \( X \), then \( X/G \) is \( LC^n \) [13, Theorem 11]. The result from Theorem 3.12 was also established in [6, Corollary 7.5] for finite complexes \( L \) and proper \( UV(L) \)-maps between Polish spaces \((UV(L)-maps are maps with all fibers being \( UV \)-(spaces). The version of Theorem 3.12 when \( L \) is a point is a generalization
of the well known result of Ancel [3, Theorem C.5.9]. This version was also established in [12, Proposition 3.5].

**Theorem 3.12.** Let \( L \) be quasi-finite and \( f: X \to Y \) be a closed map with all fibers being \( UV(L) \)-connected in \( X \). Then \( Y \) is an \( ANE(L) \) with respect to \( C \)-spaces. If, in addition, \( X \) is \( C^L \) (i.e., every map into \( X \) is \( L \)-homotopic to a constant map in \( X \)), then \( Y \in AE(L) \) with respect to \( C \)-spaces.

**Proof.** Let \( g: A \to Y \) be an arbitrary map, where \( A \) is a closed subspace of a space \( Z \) with \( e\dim Z \leq L \), such that \( A \) is a \( C \)-space. Since \( L \) is quasi-finite, it has the soft mapping property. Therefore we can apply Proposition 3.11 to obtain a neighborhood \( U \) of \( A \) in \( Z \) and an u.s.c. map \( \Psi: U \to X \) such that \( \Psi \) is single-valued outside \( A \) and \( f \circ \Psi \) is a single-valued extension of \( g \). Hence, \( Y \in ANE(L) \) with respect to \( C \)-spaces (actually we proved that \( Y \in ANE(g,A,Z) \) for arbitrary \( g: A \to Y \), where \( A \) is a closed subspace of \( Z \) such that \( e\dim Z \leq L \) and \( A \) is a \( C \)-space).

Suppose now that \( X \) is \( C^L \) and let \( A \subset Z \) and \( g: A \to Y \) be as above. To show that \( Y \in AE(L) \) with respect to \( C \)-spaces, we need to extend \( g \) over \( Z \). Embedding \( Z \) as a closed subset of an \( AE(L) \)-space with \( e\dim L \leq L \), we can assume that \( Z \in AE(L) \). Then, as before, there exists a neighborhood \( U \) of \( A \) in \( Z \) and an u.s.c. map \( \Psi: U \to X \) such that \( \Psi \) is single-valued outside \( A \) and \( f \circ \Psi \) extends \( g \). Take neighborhoods \( V_1 \) and \( V_2 \) of \( A \) in \( Z \) such that \( \overline{V_1} \subset V_2 \subset \overline{V_2} \subset U \). Let \( W = Z \setminus \overline{V_1} \) and \( F = W \cap \overline{V_2} \). Since \( W \cap U \) is open in the \( AE(L) \)-space \( Z \), the cone \( \text{Cone}(W \cap U) \) is an \( AE(L) \). So, the inclusion \( F \subset W \cap U \) can be extended to a map \( \varphi: W \to \text{Cone}(W \cap U) \) because \( F \) is closed in \( W \) and \( e\dim W \leq L \). On the other hand, since \( X \in C^L \), \( \Psi|W(\cap U) \) is \( L \)-homotopic to a constant map in \( X \). Consequently, the map \( \Psi|F \) can be extended to a map \( h: W \to X \). Finally, we define the set-valued map \( \theta: Z \to X \) by \( \theta(z) = h(z) \) if \( z \in Z \setminus \overline{V_2} \) and \( \theta(z) = \Psi(z) \) otherwise. Obviously, \( \theta \) is u.s.c. and single-valued outside \( A \). Moreover, \( f \circ \theta \) is the required extension of \( g \).

We say that a space \( X \) is locally \( ANE(L) \) if every point from \( X \) is \( UV(L) \) in \( X \). Let us mention the following corollary from Theorem 3.12.

**Corollary 3.13.** Let \( Y \) be locally \( ANE(L) \), where \( L \) is quasi-finite. Then \( Y \in ANE(L) \) with respect to \( C \)-spaces. If, in addition, \( Y \in C^L \), then \( Y \in AE(L) \) with respect to \( C \)-spaces.

**Remark.** We can show that if, in Corollary 3.13, the property of \( X \) to be locally \( ANE(L) \) is replaced by the weaker one \( X \) to be \( LC^L \) (every \( x \in X \) is \( UV(L) \)-homotopic in \( X \) [10]), then \( X \) is an \( ANE(L) \) with respect to finite-dimensional spaces (see also [6, Theorem 4.1] for a similar result).

We know that the domain and the range of a \( UV^n \)-map between compacta are simultaneously \( UV^n \) (see, for example [5]). Here is a generalization of this
result for a subclass of quasi-finite complexes. We say that a CW complex $L$ is a $C$-complex if every space of $e$-$\dim \leq L$ is a $C$-space. Each complex $L$ with $L \leq S^n$ for some $n$ (this means that $e$-$\dim Z \leq L$ implies $\dim Z \leq n$ for any space $Z$) is a $C$-complex, in particular every sphere $S^k$ is such a complex. Observe that Lemma 3.10 and Proposition 3.11 remain valid for $C$-complexes $L$ having the soft map property without the requirements either $A$ or $g(A)$ (resp., $g_0(A)$) to be $C$-spaces. This yields that, if in Theorem 3.12 and Corollary 3.13 $L$ is a quasi-finite $C$-complex, then $Y$ is an $A(N)E(L)$.

**Theorem 3.14.** Let $L$ be a quasi-finite $C$-complex and $f : X \to Y$ a closed map with $UV(L)$-fibers. Then $X$ is $UV(L)$ if and only if $Y$ is.

**Proof.** Let $E_X$ be a normed space containing $X$ as a strong $Z$-set. This means that $X \subset E_X$ is closed and for every $\omega \in cov(E_X)$ and every map $g : Q \to E_X$, where $Q$ is an arbitrary space, there is another map $h : Q \to E_X$ which is $\omega$-close to $g$ and $h(Q) \cap X = \emptyset$ (such space $E_X$ can be constructed as follows: embed $X$ as a closed subset of a normed space $F$ and let $E_X$ be the product $F \times l_2(\tau)$, where $w(X) \leq \tau$; then $X \times \{0\}$ is a copy of $X$ which is a strong $Z$-set in $E_X$). Identifying each fiber of $f$ with a point, we obtain space $E_Y$ (equipped with the quotient topology) and let $p : E_X \to E_Y$ be the natural quotient map. Obviously, $p(X) \subset E_Y$ is closed and, since $f$ is a closed map, $p(X)$ is homeomorphic to $Y$. And everywhere below we write $Y$ instead of $p(X)$. Moreover, $p$ is a closed map and $E_Y$ is metrizable. Any fiber of $p$ is either a point or $f^{-1}(y)$ for some $y \in Y$. Hence, $p$ is an $UV(L)$-map. Since $E_X$ is an absolute extensor for metrizable spaces, the fibers of $p$ are $UV(L)$-connected in $E_X$. Consequently, by the modified version of Theorem 3.12 for $C$-complexes, $E_Y \in AE(L)$.

$X \in UV(L) \Rightarrow Y \in UV(L)$. To prove this implication, by Corollary 3.7, it suffices to show that $Y$ is $UV(L)$ in $E_Y$. Let $U$ be a neighborhood of $Y$ in $E_Y$. Since $X$ is $UV(L)$ in $E_X$ (recall that $E_X$ is an absolute extensor) and $p$ is closed, there exists a neighborhood $V$ of $Y$ in $E_Y$ such that the pair $p^{-1}(V) \subset p^{-1}(U)$ is $L$-connected. We choose a neighborhood $V_1$ of $Y$ in $E_Y$ with $V_1 \subset U$ and show that the pair $V_1 \subset U$ is $L$-connected. To this end, take a space $Z$ with $e$-$\dim Z \leq L$ and a map $h : A \to V_1$ with $A \subset Z$ being closed. Since $U$ is an $ANE(L)$, there exists $\omega \in cov(U)$ satisfying condition (H) from Proposition 3.2. Further, let $\beta \in cov(E_Y)$ be the cover $\{G \cap V : G \in \omega\} \cup \{E_Y \setminus V_1\}$. By Lemma 3.10, there exists a map $h_1 : A \to E_X$ such that $p \circ h_1$ is $\beta$-close to $h$. Obviously, $h_1(A) \subset p^{-1}(V)$ and hence there exists an extension $h_2 : Z \to p^{-1}(U)$ of $h_1$. Then $p \circ h_2$ is a map from $Z$ into $U$ such that $(p \circ h_2)\mid A$ is $\omega$-close to $h$. Finally, according to the choice of $\omega$, $h$ admits an extension $\overline{h} : Z \to U$.

$Y \in UV(L) \Rightarrow X \in UV(L)$. As in the previous implication, it suffices to show that $X$ is $UV(L)$ in $E_X$. To this end, let $U$ be a neighborhood of $X$ in $E_X$. We can assume that $U = p^{-1}(U_0)$ for some neighborhood $U_0$ of $Y$ in $E_Y$. 
Choose neighborhoods $V_0$, $G_0$ and $W_0$ of $Y$ such that $V_0 \subset \overline{V_0} \subset G_0 \subset \overline{G_0} \subset W_0 \subset \overline{W_0} \subset U_0$ and the pair $G_0 \subset W_0$ is $L$-connected. Denote by $V$, $G$ and $W$, respectively, the preimages $p^{-1}(V_0)$, $p^{-1}(G_0)$ and $p^{-1}(W_0)$. We claim that the pair $V \subset U$ is $L$-connected. Indeed, consider a map $g_V : A \to V$, where $A$ is a closed subset of a space $Z$ with $\epsilon$-$\dim Z \leq L$. Let $\alpha \in \text{cov}(U)$ satisfy condition (H) from Proposition 3.2 and $\alpha_1 = \{T \cap G : T \in \alpha\} \cup \{E_X \setminus \overline{V}\} \in \text{cov}(E_X)$. Since $X$ is a strong $Z$-set in $E_X$, we can find a map $g_G : A \to E_X$ which is $\alpha_1$-close to $g_V$ and $g_G(A) \cap X = \emptyset$. It is easily seen that $g_G(A) \subset G$ and $g_G$ is $\alpha$-close to $g_V$. The last yields (because of the choice of $\alpha$) that $g_G$ can be extended to a map from $Z$ into $U$ if and only if $g_G$ has such an extension. Hence, our proof is reduced to show that $g_G$ admits an extension from $Z$ into $U$. Obviously, $g_G$ can be considered as a map from $A$ into $G_0$ such that the closure $g_G(A)$ (this is a closure in $E_Y$) does not meet $Y$. Since $G_0 \subset W_0$ is $L$-connected, $g_G$ can be extended to a map $g_W : Z \to W_0$. Finally, consider the cover $\gamma \in \text{cov}(E_Y)$ defined by $\gamma = \{p(T \setminus X) : T \in \alpha\} \cup \{E_Y \setminus \overline{g_G(A)}\} \cup \{E_Y \setminus \overline{W}\}$. According to Lemma 3.10, there exists a map $g_U : Z \to E_X$ such that $p \circ g_U$ is $\gamma$-close to $g_W$. It is easily seen that $g_U(Z) \subset U$ and $g_U|A$ is $\alpha$-close to $g_G$. The last condition implies that $g_G$ admits an extension from $Z$ into $U$ which completes our proof. □

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