LEBESGUE INTEGRATION AND OTHER ALGEBRAIC THEORIES

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Abstract. The $\sigma$-algebra definition of measurability does not generalize well to function spaces, and the notion of a Lebesgue integral with values in an infinite-dimensional space $V$ is somewhat problematic unless $V$ is a separable Banach space. In practice, however, the functional-analysis pathologies are usually irrelevant. We propose a non-topological notion of a “space with Lebesgue integration” and define a category of linear spaces and linear maps which embeds faithfully into another category of nonlinear spaces and bounded maps. We do this by viewing the category of standard Borel spaces and measurable kernels as a type of generalized Lawvere-Linton theory. More generally, for any site of definition $\mathcal{C}$ of a Grothendieck topos $\mathcal{E}$, we define a notion of a $\mathcal{C}$-ary Lawvere theory $\tau : \mathcal{C} \to \mathcal{T}$ whose category of models is a stack over $\mathcal{E}$. If $\tau$ is a commutative theory in a certain sense, then we obtain a “locally monoidal closed” structure on the category of models. This generalizes the classical situation, where $\mathcal{C} = \text{FinSet}$ and $\mathcal{E} = \text{Set}$.

1. Introduction

1.1. Spaces with Lebesgue integration. Our motivation is to define a category of linear spaces which have a well-behaved notion of Lebesgue integral. One would imagine that if $V$ is a reasonable linear space and $c : \mathbb{R} \to V$ is a bounded measurable function, then for any finite measure $\mu$, there is a well-defined integral

$$\int c(x) \, d\mu(x) \in V.$$ 

Unless $V$ is a separable Banach space, there does not seem to be a reasonable abstract condition on $c$ which ensures that the integral exists. For example, it is not enough, in general, for $c$ to be weakly measurable.

In many concrete situations, the correct definition of the Lebesgue integral does not have much connection with the topological structure of $V$. For example, suppose we take $V = \mathcal{L}^\infty(\mathbb{R}, \mathbb{R})$ to be the space of bounded Borel functions $f : \mathbb{R} \to \mathbb{R}$. If $c : \mathbb{R} \to V$ is such that $c(-)(-): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is bounded and measurable, then we may define

$$\left(\int c(x) \, d\mu(x)\right)(y) = \int c(x)(y) \, d\mu(x),$$

which is bounded and measurable by Fubini’s theorem and the triangle inequality. This definition is pointwise, and does not involve a topological structure on $V$.

The idea of a locally convex space is surprisingly pathological, and has probably contributed a general distrust of functional analysis among analysts. Furthermore, the definitions are divorced from mathematical practice. In fact, it is only by

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coincidence that bounded linear maps between normed linear spaces are continuous maps as well. For a locally convex space, a map which is bounded with respect to the von Neumann bornology is not necessarily continuous. In applications, the notion of boundedness is the more fundamental one.

Even in the setting of Banach spaces, there are often too many bounded linear maps. For example, let $M(\mathbb{R})$ be the space of Radon measures on $\mathbb{R}$. This is a Banach space with respect to the total variation norm. Let

$$\delta : \mathbb{R} \to M(\mathbb{R})$$

be the Dirac function, sending $x \in \mathbb{R}$ to the Dirac mass at $x$. We will see that $\delta$ is not weakly measurable with respect to the total variation norm. This is connected to the failure of $L^\infty(\mathbb{R}, \mathbb{R})$ to be the dual of $M(\mathbb{R})$. In symbols, there is a mismatch

$$\text{Ban}(M(\mathbb{R}), \mathbb{R}) \neq L^\infty(\mathbb{R}, \mathbb{R})$$

between the space of bounded linear functionals on $M(\mathbb{R})$ and the space of bounded linear maps $\mathbb{R} \to \mathbb{R}$. This can be contrasted with the coincidence

$$\text{Ban}(M_c(\mathbb{R}), \mathbb{R}) = \ell^\infty(\mathbb{R}, \mathbb{R}),$$

where $M_c(\mathbb{R})$ is the space of countably-supported finite measures on $\mathbb{R}$ and $\ell^\infty(\mathbb{R}, \mathbb{R})$ is the space of not-necessarily-measurable bounded maps from $\mathbb{R}$ to $\mathbb{R}$. These non-measurable functions are a source for some of the bad linear functionals on $M(\mathbb{R})$.

This situation may be rectified by considering instead the locally convex space $M_w(\mathbb{R})$ which coincides as a vector space with $M(\mathbb{R})$ and has the topological dual

$$\text{Lcvx}(M_w(\mathbb{R}), \mathbb{R}) = L^\infty(\mathbb{R}, \mathbb{R}).$$

However, it is readily apparent that the bounded sets in $M_w(\mathbb{R})$ are the same as the bounded sets in $M(\mathbb{R})$, so we are now in the pathological situation where the bounded functionals and the continuous functionals do not coincide.

The goal of this paper is to avoid all of these topological and measure-theoretic complications by taking the collection of bounded measurable curves $c : \mathbb{R} \to V$ as part of the intrinsic data of a linear space $V$. More generally, we will have a related notion of a nonlinear space equipped with curves which we imagine to be bounded and measurable. A bounded measurable map between spaces $X$ and $Y$ will then be a map $f : X \to Y$ which preserves boundedness and measurability of curves, but not necessarily any other structure. Our categories $\text{Lin}$ and $\text{Space}$ of linear and nonlinear spaces are designed with the express purpose of having (by definition) an isomorphism

$$\text{Lin}(M(\mathbb{R}), V) \cong \text{Space}(\mathbb{R}, V)$$

for any linear space $V$. This allows us to conceptualize the space of Radon measures as the “free linear space” generated by the points of $\mathbb{R}$. Moreover, we will be able to replace $\mathbb{R}$ by any nonlinear space $X$ and obtain a free linear space $X$ such that

$$\text{Lin}(M(X), V) \cong \text{Space}(X, V).$$

We will not use any topological or analytic miracles besides for the existence and good properties of Lebesgue measure on the real line. That is, we will take the theory of “Lebesgue integration” as a wholesale replacement for the theory of vector spaces.
1.2. **Finitary theories.** Our definitions of linear and nonlinear spaces will be algebraic, so that results on the category of spaces will follow from general results in universal algebra. Classically, the object of study in universal algebra is a *variety of algebras*, which we will call \( \tau \). A model of \( \tau \) is a set \( M \) equipped with \( n \)-ary operations

\[
\alpha : M^n \rightarrow M
\]
satisfying axioms of the form

\[
(\forall m_1, \ldots, m_n \in M) [t_1(m_1, \ldots, m_n) = t_2(m_1, \ldots, m_n)],
\]
where \( t_1 \) and \( t_2 \) are terms built up out of the \( n \)-ary operations by substitution. The canonical example is the theory of groups, which has a 0-ary operation \( e : 1 \rightarrow G \), a unary operation \((-)^{-1} : G \rightarrow G \) and binary operation \( \times : G \times G \rightarrow G \). A variety of algebras is distinguished from a more general type of theory in that the \( n \)-ary operations are globally defined and the axioms are all universally quantified. The theory of fields is a canonical non-example of a variety of algebras. Varieties of algebras were defined and studied by Birkhoff [Bir35], who characterized them in terms of properties of their categories of models.

1.3. **Functorial semantics.** Lawvere, in his thesis [Law63], observed that the theorems of \( \tau \) are completely determined by its category \( \mathcal{T} \) of finitely-generated free models. More significantly, he showed an equivalence

\[
\text{Mod}^\tau \cong \text{Mod}^\mathcal{T} := [\mathcal{T}^{\text{op}}, \text{Set}]_\times
\]

between the category \( \text{Mod}^\tau \) of \( \tau \)-algebras and \( \tau \)-homomorphism, and the category \( \text{Mod}^\mathcal{T} = [\mathcal{T}^{\text{op}}, \text{Set}]_\times \) of finitely multiplicative presheaves on \( \mathcal{T} \) and natural transformations. Furthermore, suppose given a finitely additive identity-on-objects functor

\[
\tau : \aleph_0 \rightarrow \mathcal{T}
\]
from the category \( \aleph_0 \) of finite sets to some other category. Then \( \tau \) is the free model functor for some variety of algebras, whose category of models is thus \( [\tau(\aleph_0)^{\text{op}}, \text{Set}]_\times \). Thus a variety of algebras is precisely the same thing as a finitely-additive identity-on-objects functor \( \tau : \aleph_0 \rightarrow \mathcal{T} \).

1.4. **C-ary theories.** The role of the category of finite sets in Lawvere’s functorial semantics is closely connected to the equivalence

\[
\text{Set} \cong [\aleph_0^{\text{op}}, \text{Set}]_\times.
\]
This equivalence yields, for any Lawvere theory \( \tau \), a forgetful functor

\[
U : \text{Mod}^\mathcal{T} \rightarrow \text{Set}
\]
which is obtained by composition with \( \tau \). Indeed, the functor \( \tau \) is additive, so composition with \( \tau^{\text{op}} \) preserves multiplicative functors.

Linton [Lin69] defined a notion of Lawvere theory with models in a category \( \mathcal{E} \) of *spaces* instead of in the category of sets. He assumes that \( \mathcal{E} \) has a dense subcategory

\[
j : \mathcal{C} \rightarrow \mathcal{E}
\]
so that there is a full and faithful embedding

\[
\iota : \mathcal{E} \hookrightarrow \text{Psh}^\mathcal{C},
\]
where \( \text{Psh}^\mathcal{C} \) is the category of presheaves on \( \mathcal{C} \).
If \( X \) is a set, then we may define an \((n,m)\)-ary operation on \( X \) to be a function in \( \text{Set}(X^n, X^m) \), or equivalently an \( m \)-tuple of \( n \)-ary operations. Similarly, if \( F \in \text{PshC} \) is a presheaf and \( I \) is an object of \( C \), then an \((I,J)\)-ary operation on \( F \) is a function in \( \text{Set}(F^I, F^J) \). Since a space is an example of a presheaf on \( C \), it already has some operations built into its definition. In fact, for each \( I \) we have a map

\[ F_{IJ} : C(J, I) \to \text{Set}(F^I, F^J), \]

which assigns to every function \( \varphi : J \to I \) a substitution operation \( c \mapsto c \circ \varphi \).

A \( C \)-ary theory is, to first approximation, an identity-on-objects functor \( \tau : C \to T \). If \( M \) is a presheaf on \( T \), then it restricts via \( \tau \) to a presheaf \(|M|\) on \( C \). The fact that \(|M|\) extends to a presheaf on \( T \) means, for example, that there are maps

\[ M_{IJ} : T(J, I) \to \text{Set}(|M|^I, |M|^J), \]

interpreting each element of \( T(J, I) \) as an \((I,J)\)-ary operation on the presheaf \(|M|\).

A model of the theory is a presheaf on \( T \) such that \(|M|\) is a space. Thus we obtain a full subcategory

\[ \text{Mod}^T \hookrightarrow \text{Psh}^T, \]

and by definition we have a forgetful functor

\[ U : \text{Mod}^T \to \mathcal{E} \]

sending a model to its underlying space. In order for the objects of \( T \) to be free models, Linton assumes in addition that the Yoneda embedding \( T \to \text{Psh}^T \) factors through \( \text{Mod}^T \).

For Lawvere theories, we have \( \mathcal{E} = \text{Set} \), and \( J \) is the dense inclusion \( j_0 : \aleph_0 \to \text{Set} \).

We may replace \( \aleph_0 \) with a regular cardinal \( \lambda \); the inclusion \( j : \lambda \to \text{Set} \) is still dense. In this case Linton’s theories are functors preserving \( \lambda \)-ary sums, and their models are presheaves preserving \( \lambda \)-ary products.

### 1.5. Sketches

The theory of sketches was developed by Ehresmann [Ehr68], Kennison [Ken68] and Gabriel-Ulmer [GU71]. A sketch generalizes the notion of a variety of algebras. Recall that a model of a \( \lambda \)-ary theory \( \tau : \lambda \to T \) is a presheaf on \( T \) which preserves \( \lambda \)-ary products in \( T^{op} \). It is natural to consider instead any category \( S \) together with a set \( \Phi \) of cocones or cocylinders in \( S \). The pair \((S, \Phi)\) is called a sketch. The category \( \text{Mod}^S \) of models consists of presheaves which send the cocones in \( \Phi \) to limits. The inclusion

\[ \iota : \text{Mod}^S \hookrightarrow \text{Psh}^S \]

admits a reflection, a fact which is of fundamental importance in the theory of sketches.

### 1.6. Topologies

The most familiar types of sketch is a category \( C \) endowed with a Grothendieck topology [AGV72, SGA 4.II]. The sketch \( C \) is then called a site, and a model of the site is a sheaf. The category \( \text{Sh}^C \) of sheaves is \( \text{Set} \)-like and behaves like a category of generalized spaces. For example, it is locally cartesian closed, which is closely connected to the fact that the reflection preserves finite limits.

The Yoneda embedding

\[ y : C \to \text{Sh}^C \]
is a natural setting for Linton’s $C$-ary theories. In this context a $C$-ary theory is a identity-on-objects functor $\tau : C \to T$, and a model of the theory is a presheaf $M$ whose restriction $|M|$ along $\tau$ is a sheaf. This idea is made explicit in [BMW12], where it is also assumed that the left adjoint to $U : \text{Mod}^T \to \text{Sh}^C$ is computed by Kan extension.

Requiring that $|M|$ is a sheaf is the same as requiring that $M$ is a model of a certain colimit sketch on $T$. This means that we can deduce many facts about the category of models of $C$-ary theories from general facts about sketches.

1.7. Indexed categories. To see why the category of sheaves on a site is locally cartesian closed, it is helpful to study the localizations of the site itself. If $C$ is any category, there is a functor

$$\mathcal{C}_- : C \to \text{CAT}$$

sending $I \in C$ to the slice category $C/I$ of objects over $I$. If $C$ has pullbacks, then $\mathcal{C}_\varphi : C_I \to C_J$ has a right adjoint, which we denote by $\mathcal{C}^\varphi : C^I \to C^J$ (here $C^I$ and $C_I$ are different names for the same category). Thus the assignment $I \mapsto C^I$ is the object function of a contravariant pseudofunctor

$$\mathcal{C}^- : C^{\text{op}} \to \text{CAT}.$$ 

If we think of $C$ as a category of spaces, then an object of $C^I$ should be thought of as an $I$-indexed family of spaces $\{C^i\}_{i \in I}$, and the functor $\mathcal{C}^\varphi$ sends $\{C^j\}_{j \in J}$ to $\{C^{\varphi(i)}\}_{i \in I}$.

More generally, according to the philosophy of Bénabou [Bén85], we can think of a pseudofunctor

$$\mathcal{F}^- : C^{\text{op}} \to \text{CAT}$$

as sending each $I \in C$ to the category $\mathcal{F}^I$ of $I$-indexed families $\{F^i\}_{i \in I}$, where each $F^i$ is imagined to be an object of $\mathcal{F}^I$. For example, the category $\mathcal{F}^I$ may consist of vector bundles over a topological space $I$, which we think of as a collection of vector spaces indexed by the points of $I$.

1.8. Fibered categories. For any pseudofunctor $\mathcal{F} : C^{\text{op}} \to \text{CAT}$, we may collect all of the categories $\{\mathcal{F}^I\}_{I \in C}$ into a single category $\int \mathcal{F}$, together with a functor

$$p : \int \mathcal{F} \to C$$

Following [GR71, SGA 1.VI], we will say that say call an object of $\text{CAT}_{/C}$ a $C$-category. A $C$-category is fibered if it isomorphic to the Grothendieck construction of some pseudofunctor $\mathcal{F}$, and split if $\mathcal{F}$ can be chosen to be a strict functor. By abuse of notation, we can identify the fibered $C$-category $p$ with the pseudofunctor $\mathcal{F}$. The basic example is the $C$-category of arrows

$$\text{cod} : [\_ , C] \to C,$$

sending each arrow in $C$ to its codomain. If $C$ has pullbacks, then this is the fibered $C$-category corresponding to the (contravariant) pseudofunctor $\mathcal{C}^- : I \to C/I$ described above.

Dually, we say that a $C$-category is opfibered or opsplit if the corresponding $C^{\text{op}}$-category is fibered or split.
1.9. Localization. Let $\mathcal{E} = \text{Sh}^C$ be a Grothendieck topos. Since $\mathcal{E}$ has pullbacks, we have a codomain fibration $\mathcal{E}$, each fiber of which is a Grothendieck topos. In fact, there is an equivalence of $\mathcal{E}$-indexed categories $[\text{AGV72}, \text{SGA 4.III.5}]$

$$\mathcal{E}^P \cong \text{Sh}^{C/P},$$

for the induced topology on $C/P$.

The slicing construction is not appropriate for the algebraic category $\text{Mod}^\mathcal{T}$, because a bundle of algebras is not the same thing as an algebra over a space. It is usually the case, however, that a $P$-indexed family of algebras is a space over $P$ with extra structure. This means that the category of $P$-indexed families of algebras should be the category of models for a Linton theory $\tau_P : C/P \to \mathcal{T}$

For this to work, we require that $\tau_P$ is a natural transformation. This means, in particular, that $\mathcal{T}$ is also a functor

$$\mathcal{T} : C \to \text{Cat},$$

or equivalently a opsplit $\mathcal{E}$-category. For such a $\mathcal{T}$ we obtain a split $\mathcal{E}$-category $\text{Mod}^\mathcal{T}$ corresponding to the functor

$$(\text{Mod}^\mathcal{T})^P := \text{Mod}^{\mathcal{T}_P},$$

which will make sense as long as the transition functors $\mathcal{T}_\phi$ are well-behaved.

1.10. Categories with $C$-sums. We will be interested in $C$-ary theories which are Linton theories from the point of view of ordinary category theory and Lawvere theories from the point of view of $C$-category theory. Our Lawvere theories will be categories with $C$-sums in the sense of [BR70]. Recall that a category $D$ with ordinary sums has, for a set $I$, a sum functor

$$\Sigma_I : D^I \to D$$

sending a family of objects to the corresponding sum. This functor is left adjoint to the diagonal $\Delta_I : D \to D^I$. A fibered $C$-category $\mathcal{F}$ with $C$-sums has dependent sum functors

$$\Sigma_I|_\varphi : \mathcal{F}^I \to \mathcal{F}^J$$

where $\varphi : I \to J$ is an arrow in $C$. The functors $\Sigma_I|_\varphi$ are are left adjoint to the restriction functors $\mathcal{F}^\varphi : \mathcal{F}^J \to \mathcal{F}^I$, so every $C$-sum category is bifibered, although the converse does not hold. A $C$-functor of $C$-sum categories is $C$-additive precisely when it is bicartesian.

1.11. $C$-ary Lawvere theories. We return now to Lawvere’s original definition of a theory as an additive functor, but replace finite additivity with $C$-additivity, where $C$ is a standard site. First, we note that the $C$-category of arrows $\mathcal{C}$ is obtained by freely adding $C$-sums to the unit $C$-category, thus for any $C$-sum category $\mathcal{D}$, we have a fibered equivalence

$$[\mathcal{C}, \mathcal{D}]_+ \cong [1, \mathcal{D}] \cong \mathcal{D},$$

where $[-,-]_+$ denotes the fibered $C$-category of $C$-additive functors and $[-,-]_+$ is the fibered $C$-category of cartesian $C$-functors. In particular, we can view the topos $\mathcal{E}$ as a fibered $C$-product category $\mathcal{E}$, which gives $\mathcal{E} \cong [\mathcal{E}^\text{op}, \mathcal{E}]_\times.$
More generally, a $C$-ary Lawvere theory will be a $C$-additive functor
\[ \tau : C \to T \]
such that each component $\tau_I : C_I \to T_I$ is a $C_I$-ary theory. We will show that for a Lawvere theory $\tau$, we have
\[ \text{Mod}^T \cong [\text{T}^{\text{op}}, E]_x, \]
where $[-,-]$ is the $C$-category of $C$-multiplicative functors between two categories with $C$-products. This gives an internal characterization of $\text{Mod}^T$. It also lets us deduce properties of $\text{Mod}^T$ from properties of $E$ in a straightforward way. In particular, we obtain that $\text{Mod}^T$ is a stack (or 2-sheaf) over $C$.

1.12. Extension to $E$. The comparison lemma for stacks [Gir71, II.3.3.4], says that a stack $F$ over a site $C$ extends in an essentially unique way to a stack $y_* F$ over the topos $\mathcal{E} = \text{Sh}^C$, where $\mathcal{E}$ is endowed with the canonical topology. However, it is certainly not true that $y_* F$ will inherit $\mathcal{E}$-products from $C$-products in $\mathcal{F}$.

For example, an ordinary category with finite products can be thought of as an $\aleph_0$-stack with $\aleph_0$-products, but the canonical extension to a $\text{Set}$-stack will only have $\text{Set}$-products if the underlying category has ordinary small products.

It is true, however, that the category of models for an $\aleph_0$-ary (or $\lambda$-ary) Lawvere theory is $\text{Set}$-complete and cocomplete, and we will prove an analogous result in our setting. Completeness follows in a straightforward way from the intrinsic description $\text{Mod}^T \cong [\text{T}^{\text{op}}, E]_x$, but for cocompleteness we will need the more $\text{Set}$-based description in terms of sketches.

1.13. Commutative theories. Let $\tau : \aleph_0 \to \mathcal{T}$ be a commutative Lawvere theory. It is slightly awkward to say what this means, so we will instead describe a symptom of this condition. Let $F : \text{Set} \to \text{Mod}^T$ be the free model functor. The category $\text{Mod}^T$ admits a tensor product $\otimes$, such that
\[ F(X \times Y) \cong F(X) \otimes F(Y) \]
for any sets $X$ and $Y$. In particular, we find that the functor $\tau : \aleph_0 \to \mathcal{T}$ can be enhanced to a strict monoidal functor with respect to the cartesian product on $\aleph_0$. We can take this to be the definition of a commutative Lawvere theory. In fact, the coherence of the tensor product forces certain operations to commute with each other.

In his work on monoidal completions [Day70, Day74, Day72], Day showed that the tensor product on $\text{Mod}^T$ can be computed in terms of the tensor product on $\mathcal{T}$ in a systematic way and in a much more general context. In particular, Day gives conditions for the tensor product on a sketch to extend to the category of models by means of the convolution product on presheaves. Similar results were obtained in [BE72].

1.14. Monoidal $C$-categories. We will see that Day’s results extend in a straightforward way to our $C$-ary Lawvere theories. For this we will use Shulman’s theory of monoidal $C$-categories in [Shu08]. A $C$-ary Lawvere theory $\tau : C \to \mathcal{T}$ will be commutative if it can be enhanced to a strict monoidal $C$-functor, where $C$ is endowed with its cartesian monoidal structure. We obtain an enhancement of
the free model functor $F : \mathcal{E} = \text{Mod}^\mathcal{C} \to \text{Mod}^\mathcal{F}$ to a strong monoidal $\mathcal{C}$-functor. Moreover, the monoidal $\mathcal{E}$-category $\text{Mod}^\mathcal{F}$ will be an $\mathcal{E}$-cosmos in the sense of [Shu13].

We summarize our results as follows

**Theorem 1.1.** Let $\tau : \mathcal{C} \to \mathcal{T}$ be a $\mathcal{C}$-ary Lawvere theory, and let $\mathcal{E}$ be the $\mathcal{E}$-category of arrows for the topos $\mathcal{E} = \text{Mod}^\mathcal{C}$. Then the category of algebras $\text{Mod}^\mathcal{F}$ is complete and cocomplete as a fibered $\mathcal{E}$-category, and the forgetful functor $U : \text{Mod}^\mathcal{F} \to \mathcal{E}$ has a fibered left adjoint $F \dashv U$. If $\tau$ is a commutative theory, then $\text{Mod}^\mathcal{F}$ has the structure of a closed monoidal $\mathcal{E}$-category, and the left adjoint $F$ is strong monoidal with respect to the cartesian monoidal structure on $\mathcal{E}$.

2. Related work

2.1. Convenient vector spaces. The main inspiration for this paper was the work of Frölicher and Kriegl [FK88] on convenient spaces. They defined a convenient category of topological vector spaces with a closed monoidal structure. These spaces are associated with a cartesian closed category $\mathcal{C}^\infty$ of smooth spaces and a monoidal adjunction $F \dashv U : \text{Con} \rightleftarrows \mathcal{C}^\infty$. A smooth space is a set together with a collection of smooth curves satisfying some axioms. This is close in spirit to the idea of a sheaf, but the setness and the extra axioms prevent $\mathcal{C}^\infty$ from being locally cartesian closed. They also exhibited a monoidal adjunction $\ell^1 \dashv U : \text{Con} \rightleftarrows \ell^\infty$, where $\ell^\infty$ is a locally cartesian closed category of bounded spaces. These spaces are equipped with a collection of bounded sequences satisfying some axioms. The space $\ell^1\mathbb{R}$ consists of countably supported measures on $\mathbb{R}$ with bounded mass. Naively, one might expect to be able to replace $\ell$ with $L$ and obtain a notion of a space with bounded measurable curves. However $\text{Con}$ includes the category of Banach spaces as a full subcategory, and Banach spaces already does not have a good notion of Lebesgue integration in general. Moreover, the whole setup is somewhat baroque and apparently miraculous due to the mixture of algebraic and topological definitions.

2.2. The Giry-Lawvere monad. Lawvere proposed using measure theory as a way to encode algebraic structures on topological spaces, and this was worked out in Giry’s paper [Gir82] in the case of Polish spaces. He defines a monad $M$ on the category $\text{Pol}$ of Polish spaces and continuous maps, sending $X$ to the space $MX$ of probability measures on $X$, whose weak topology makes it a Polish space. Thus one can define a convex Polish space to be an algebra for this monad. However, this entails the rather severe restriction that the underlying space is Polish. It also seems more appropriate to consider a category of measurable spaces and measurable maps.

One might hope to define the Giry-Lawvere monad in the context of measurable spaces and measurable maps. This is not so convenient, however, because the category of measurable spaces and measurable maps is not cartesian closed [Aum61]. For this reason Heunen, Kammar, Staton and Yang [HKSY17] defined a notion of a quasi-Borel space in terms of concrete sheaves. The category of quasi-Borel spaces admits a monad $M$ sending a quasi-Borel space to a space of probability measures. Their definition of a convex space is thus very similar in spirit to our definition of a
linear space, but we have found it more convenient to work with arbitrary sheaves instead of concrete sheaves.

2.3. Enriched and internal theories. The notion of a finitary algebraic theory can be generalized to categories other than Set in various different ways. Borceux and Day [BD80] define a notion of a finitary theory in a certain type of closed category, and the arities of this theory are finite multiples of the unit object. This uses a Set-based notion of finiteness.

There are also intrinsic notions of finiteness generalizing the Set-based notion. Johnstone and Wraith [JW78] define a notion of finitary theory internal to an elementary topos, where the notion of finiteness is connected to the natural numbers object. Kelly in [Kel82a] defines a notion of a finite limit theory in the enriched setting, the limits here are finitely presentable in a certain sense. Similarly, Power [Pow99] defines a notion of a finite product theory, whose models are functors preserving powers indexed by finitely presentable objects.

A very general notion of enriched sketch is given in Kelly’s monograph [Kel82b]. One can associate an enriched sketch to a category in a canonical way by choosing all limits with some predetermined indexing type. For example, this could include finitely-presentable powers, but also $\kappa$-presentable powers for some regular cardinal $\kappa$. In Lack and Rosicky’s paper [LR11] some examples are given of sound limit doctrines in enriched categories, where explicit constructions are available. The notion of Lawvere theory is given relative to such a doctrine. They show that in this setting many of the associated constructions can be made more explicit. However, the notion of a sound limit doctrine is rather inflexible, and it is shown in [ABLR02] that for $\lambda > \aleph_0$, the doctrine of $\lambda$-ary products in Set is not sound in this sense. Lucyshyn-Wright [LW16] defines a general notion of enriched Lawvere theory which allows the category of arities to be more or less arbitrary and works in a (not necessarily cartesian) closed category; the main difference between his setup and ours is that ours is much less general, but also significantly more explicit.

Our approach eschews the use of enriched category theory in favor of the theory of fibered ordinary categories, which we have found to be more straightforward to work with. We also restrict our attention to Grothendieck toposes over Set; this ensures that the models for our theories are reflective subcategories of ordinary presheaf categories. The main conceptual difference between our approach and other approaches to universal algebra is that instead of considering a single theory $\tau : C \to T$, we consider a whole family of theories $\tau_I : C_I \to T_I$. This ensures that the notion of model is stable under localization and gives us a well behaved fibered $\mathcal{E}$-category of models. Of course, this is not a new idea at all, and is closely modeled after [AGV72, SGA 4.IV]. Most of our constructions are well-known and use existing techniques. However, it is difficult to track down all of the necessary results in the literature. As we would like to use the results of this paper in future work, we have found it necessary to write them down.

3. Preliminaries

3.1. Foundations. To avoid distracting headaches surrounding size issues, we work in ZFC+$U$. That is, every set $X$ belongs to some Grothendieck universe $U$. In particular, if $U$ is a universe, we have a universe $U'$ containing $U$. 

A $U$-small set is a set $X$ together with an isomorphism $X \cong \tilde{X}$, where $\tilde{X}$ is an element of $U$. Note that $U$-smallness is a structure of a set and not simply a property, and that the $U$-small sets form a proper class.

3.2. Smallness conditions. A category is a set of objects and arrows satisfying the category axioms. If $U$ is a universe, the category $U\text{Set}$ consists of elements of $U$ and functions between them. If $\mathcal{C}$ is any category, we denote by $\text{Cat}(\mathcal{C})$ the 2-category of internal categories, functors, and natural transformations. When $\mathcal{C} = U\text{Set}$, we write $U\text{Cat} = \text{Cat}(U\text{Set})$.

A $U$-small category is a category $\mathcal{C}$ together with an isomorphism of categories $\mathcal{C} \cong \tilde{\mathcal{C}}$ for some $\tilde{\mathcal{C}}$ in $U\text{Cat}$. The isomorphism determines a canonical hom functor $\mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to U\text{Set}$.

A subuniverse $U \hookrightarrow V$ induces a canonical embedding $U\text{Set} \hookrightarrow V\text{Set}$. Thus we obtain a full 2-category $UV\text{Cat} \hookrightarrow V\text{Cat}$ whose objects are $V$-small categories for which $\mathcal{C}(-, -)$ factors as $\mathcal{C}^{\text{op}} \times \mathcal{C} \to U\text{Set} \to V\text{Set}$.

A category is $UV$-small if it is isomorphic to an object of $UV\text{Cat}$. If $U = V$ then this is the same as being $U$-small.

For the purpose of most discussions, we will fix universes $U \in U' \in U'' \cdots$. We then define categories $\text{Set} = U\text{Set}$, $\text{Cat} = U\text{Cat}$, $\text{CAT} = UU'\text{Cat}$ of small sets, small categories, and locally small categories. Thus the category $\text{Cat}$ of small categories is a locally small category, while the category $\text{CAT}$ of locally small categories is just a $U'U''$-small category.

3.3. Presheaves and fibered categories. For categories $\mathcal{C}$ and $\mathcal{D}$, we write $[\mathcal{C}, \mathcal{D}]$ for the category of functors and natural transformations. When $\mathcal{D} = \text{Set}$, we write $\tilde{\mathcal{C}} = \text{Psh}^\mathcal{C} = [\mathcal{C}^{\text{op}}, \text{Set}]$.

for the category of presheaves on $\mathcal{C}$. Similarly, let $\tilde{\mathcal{C}}$ be a $UW$-small category, and let $U \hookrightarrow V$ be another inclusion of universes. We define $UV\text{Fib}^\mathcal{C} \cong [\mathcal{C}^{\text{op}}, UV\text{Cat}]_{ps}$

to be the 2-category of $UV$-small fibered $\mathcal{C}$-categories, cartesian $\mathcal{C}$-functors and cartesian $\mathcal{C}$-transformations, which is 2-equivalent to the 2-category of pseudofunctors $\mathcal{F} : \mathcal{C}^{\text{op}} \to UV\text{Cat}$, pseudonatural transformations, and modifications. For fixed $U \in U'$ we set $\text{Fib}^\mathcal{C} = UU\text{Fib}^\mathcal{C}$ and $\text{FIB}^\mathcal{C} = UU'\text{Fib}^\mathcal{C}$. The equivalence of fibered $\mathcal{C}$-categories and pseudofunctors is mediated by the Grothendieck construction

$$\int : [\mathcal{C}^{\text{op}}, UV\text{Cat}]_{ps} \to UZ\text{Cat}/\mathcal{C},$$

(where $Z$ is the larger of $V$ and $W$), which sends the pseudofunctor $\mathcal{F}$ to the $\mathcal{C}$-category

$$p : \int \mathcal{F} \to \mathcal{C}.$$ 

Here $\int \mathcal{F}$ is the category of elements of $\mathcal{F}$, and the functor $p$ is the canonical projection. By abuse of notation we will denote a fibered $\mathcal{C}$-category $p : \mathcal{F} \to \mathcal{C}$ by
\[ \int F := F. \]

If \( p(f) = \varphi \), for some \( \varphi : I \to J \), we say that the arrow \( f \) is over \( \varphi \), and we depict this situation by

\[
\begin{array}{ccc}
E & \to & F \\
\downarrow & & \downarrow \\
I & \to & J.
\end{array}
\]

The category of objects over \( I \) is denoted by \( F_I \).

3.4. **Restriction and extension.** If \( E \in F_J \) and \( \varphi : I \to J \) are given, then by fiberedness there is an essentially unique cartesian arrow \( \mathcal{F}^\varphi E \to E \) over \( \varphi \), which we call a *restriction* of \( E \) along \( \varphi \). If we choose a restriction \( \mathcal{F}^\varphi E \to E \) for each \( E \in F_J \), we obtain a functor \( \mathcal{F}^\varphi : F_J \to F_I \) which we call a *restriction functor* for \( \varphi \). A *cleavage* is a choice of a restriction functor \( \mathcal{F}^\varphi \) for each \( \varphi \), which determines a pseudofunctor \( \mathcal{F}(-) : C^{op} \to \text{CAT} \).

If this choice is strictly functorial, we say that it is a *splitting*. Dually, let \( p : F \to C \) be opfibered, which means that \( p^{op} : F^{op} \to C^{op} \) is fibered. We write \( F_I \) for the fiber of \( p \) over \( I \). A left extension of \( E \in F_I \) along \( \varphi : I \to J \) is an opcartesian arrow \( E \to \mathcal{F}_\varphi E \). Similarly, we can define left extension functors and opcleavages. We will always assume that a fibered or opfibered category is equipped with a specific choice of cleavage or opcleavage; this does not affect in any way the concept of cartesian functor or transformation.

A \( C \)-category is *bifibered* if it is simultaneously fibered and opfibered. If \( F \) is bifibered and \( \varphi : I \to J \) is an arrow of \( C \), then for any choice of \( \mathcal{F}^\varphi \) and \( \mathcal{F}_\varphi \) we have an adjunction

\[
\mathcal{F}_\varphi \dashv \mathcal{F}^\varphi : \mathcal{F}_J \rightleftarrows \mathcal{F}_I,
\]

where the unit and counit are determined by the universal properties of cartesian and opcartesian arrows. Conversely, if \( F \) is fibered and each restriction \( \mathcal{F}^\varphi \) has a left adjoint, then \( F \) is opfibered as well.

3.5. **The dual fibration.** If \( \mathcal{K} \) is a 2-category, we can define a 2-category \( \mathcal{K}^{op} \) by reversing the 1-cells, but we can also define a 2-category \( \mathcal{K}^{co} \) by reversing the 2-cells. A functor \( F : C \to D \) is the same thing as a functor \( F^{op} : C^{op} \to D^{op} \), but a natural transformation \( \alpha : F \Rightarrow G \) is a natural transformation \( \alpha^{op} : G^{op} \Rightarrow F^{op} \). One checks that \( (-)^{op} \) is a 2-functor

\[
(-)^{op} : \text{UVCat} \to \text{UVCat}^{co}.
\]

In particular, for any category \( C \), we have a 2-functor

\[
[\mathcal{C}^{op}, (-)^{op}] : [\mathcal{C}^{op}, \text{UVCat}]_{ps} \to [\mathcal{C}^{op}, \text{UVCat}^{co}]_{ps} \cong [\mathcal{C}^{op}, \text{UVCat}^{ps}]_{co},
\]

and by the transporting the Grothendieck construction we have a dual category 2-functor

\[
(-)^{op}_{\mathcal{C}} : \text{UFib}^{\mathcal{C}} \to (\text{UFib}^{\mathcal{C}})^{co}.
\]

If \( \mathcal{F} \) is a fibered \( C \)-category, then \( \mathcal{F}^{op} \) is opfibered if and only if each restriction \( \mathcal{F}^{op} \) has a *right adjoint*, which we denote by \( \mathcal{F}_{\varphi^*} \). We call the functor \( \mathcal{F}_{\varphi^*} \) a *right extension functor*. If both \( \mathcal{F} \) and \( \mathcal{F}^{op} \) are bifibered, we say that \( \mathcal{F} \) is trifibered.
When the fibered $C$-category $\mathcal{F}$ is clear from context, we will use the standard notation
\[
\varphi! = \mathcal{F}_\varphi, \quad \varphi* = \mathcal{F}_\varphi, \quad \varphi_\ast = \mathcal{F}_{\varphi_\ast}.
\]

3.6. **Change of base.** Suppose given a pullback square of functors and categories
\[
\begin{array}{ccc}
F & \xrightarrow{p} & G \\
\downarrow p & & \downarrow q \\
C & \xrightarrow{F} & D
\end{array}
\]
If $G$ underlies a fibered $D$-category, then $F$ underlies a fibered $D$-category. Thus we have a restriction 2-fuctor
\[
F^* : \text{Fib}^D \to \text{Fib}^C,
\]
and we can think of $\text{Fib}$ as a fibered $\text{Cat}$-category. The restriction $F^*$ has left and right 2-adjoints $F_!$ and $F_\ast$ (constructed in [Gir71, I.2.4], where the left 2-adjoint $F_!$, for example, is characterized by equivalences
\[
\text{Fib}^D(F_!, \mathcal{F}, G) \cong \text{Fib}^C(F, F^*G).
\]
Note that these are not isomorphisms, and $\text{Fib}$ does not have left and right extensions in the sense we have earlier described.

3.7. **Mates and $C$-sums.** If $\eta_i$ and $\epsilon_i$ are the unit and counit of adjunctions $l_i \dashv r_i$, then the mate [KS74] of a square
\[
\begin{array}{ccc}
l_1 & \xrightarrow{f} & r_1 \\
g & \uparrow \eta \downarrow \lambda & \\
l_2 & \xrightarrow{g} & r_2
\end{array}
\]
is the pasting composite
\[
\begin{array}{ccc}
l_1 & \xrightarrow{f} & r_1 \\
g & \uparrow \eta \downarrow \mu & \\
l_2 & \xrightarrow{g} & r_2
\end{array}
\]
Analogously, if we start with a square $\mu$, we can define its comate $\lambda$. The mate and comate operations are mutually inverse.

Let $\mathcal{F}$ be a fibered category, and suppose given a pullback diagram
\[
\begin{array}{ccc}
X \times_I Y & \xrightarrow{\pi_X} & X \\
\downarrow \pi_Y & & \downarrow k \\
Y & \xrightarrow{g} & I
\end{array}
\]
in $C$. For any choice of restriction functors, there is a canonical isomorphism
\[
\begin{array}{ccc}
\mathcal{F}X \times_I Y & \xrightarrow{\pi_X} & \mathcal{F}X \\
\pi_Y & \cong & k^* \\
\mathcal{F}Y & \xrightarrow{g^*} & \mathcal{F}I.
\end{array}
\]
If $F$ is bifibered, then the mate of this diagram across the adjunctions $\pi_! \dashv \pi^*$ and $k \dashv k^*$ determines a comparison cell

$$
\begin{array}{ccc}
\mathcal{F} \times \mathcal{Y} & \xleftarrow{\pi_!} & \mathcal{F} \\
\pi_Y & \downarrow \uparrow & \downarrow k \\
\mathcal{F} & \xleftarrow{g^*} & \mathcal{F}.
\end{array}
$$

We say that $\mathcal{F}$ has $C$-sums if it is bifibered and the comparison cells for pullback diagrams in $C$ are all isomorphisms. Dually, if $\mathcal{F}^{op}$ has $C$-sums then we say that $\mathcal{F}$ has $C$-products.

If $\mathcal{F}$ is a trifibration, then it has $C$-sums if and only if it has $C$-products. A cartesian $C$-functor is $C$-additive if it is bicartesian. Dually a cartesian $C$-functor $F : \mathcal{F} \to \mathcal{G}$ is $C$-multiplicative if $F^{op} : \mathcal{F}^{op} \to \mathcal{G}^{op}$ is $C$-additive.

3.8. **Kan extensions.** If $\mathcal{C}$ is a small category, then $\text{Psh}^C = [\mathcal{C}^{op}, \text{Set}]$ is locally small, and $\text{Psh}$ is thus a functor

$$
\text{Psh} : \text{Cat}^{op} \to \text{CAT}.
$$

We identify $\text{Psh}$ with a split $\text{Cat}$-category of the same name. The fibered category $\text{Psh}$ has left extensions and right extensions. These are usually called *Kan extensions* [Kan58], but it will be more convenient to preserve the notation $\text{Psh}_F \dashv \text{Psh}^F \dashv \text{Psh}_{F^*}$ in this case as well.

3.9. **Discrete and representable $C$-categories.** Any presheaf $F : \mathcal{C}^{op} \to \text{Set}$ can be viewed as a presheaf of categories $dF$ (or equivalently a split $C$-category) by means of the discrete category functor $d : \text{Set} \to \text{Cat}$. In particular, this is true of representable presheaves, and the fibered Yoneda lemma gives an equivalence of categories (in fact of $C$-categories)

$$
\mathcal{F}^I \cong \text{FIB}^C(dyI, \mathcal{F})
$$

for every $I \in \mathcal{C}$. Thus the objects in the fiber $\mathcal{F}^I$ can be thought of as maps $F : I \to \mathcal{F}$. Note that the left-hand side of the equation is manifestly a split $C$-category. Every fibered $C$-category is thus equivalent, but not necessarily isomorphic to, a split $C$-category. If $\mathcal{C}$ is small, we have a canonical extension of $\mathcal{F}$ along the Yoneda embedding $\mathcal{C} \to \text{Psh}^\mathcal{C}$, where we now define the split $\mathcal{C}$-category $y_*\mathcal{F}$ by

$$
y_*\mathcal{F}^P = \text{FIB}^C(dp, \mathcal{F}).
$$

As suggested by the notation, the 2-functor $y_* : \text{FIB}^\mathcal{C} \to \text{FIB}^{\mathcal{C}}$ is the right 2-adjoint of the restriction $y^* : \text{FIB}^{\mathcal{C}} \to \text{FIB}^\mathcal{C}$.

3.10. **Cartesian closedness.** The 2-category of fibered $C$-categories has finite products, which are computed as in $\text{CAT}^{/\mathcal{C}}$. Given a small fibered $C$-category $\mathcal{F}$, the endofunctor $\mathcal{F} \times -$ on $\text{FIB}^\mathcal{C}$ has a right 2-adjoint $[\mathcal{F}, -]$, which means that for a small fibered $C$-categories $\mathcal{G}$ and a fibered $C$-category $\mathcal{H}$, we have an equivalence

$$
\text{FIB}^\mathcal{C}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) \cong \text{FIB}^\mathcal{C}(\mathcal{G}, [\mathcal{F}, \mathcal{H}])
$$

for every $\mathcal{G}$. Thus the objects in the fiber $\mathcal{F} \times \mathcal{G}$ can be thought of as maps $F \times G : I \to \mathcal{F} \times \mathcal{G}$. Note that the left-hand side of the equation is manifestly a split $C$-category.
Here the split $\mathcal{C}$-category $[\mathcal{F}, \mathcal{H}]$ is defined for $I \in \mathcal{C}$ by

$$[\mathcal{F}, \mathcal{H}]^I = [\mathcal{F} \times I, \mathcal{H}] \cong [\mathcal{F}, \mathcal{H}^I].$$

Here by $\mathcal{H}^I$ we mean, of course, the fibered category $[I, \mathcal{H}]$.

4. Fibered sketches

4.1. Weighted limits. For easy reference we briefly recall some notation for weighted limits; we refer to [Kel82b] for more details. A weight of shape $\mathcal{J}$ is a diagram $W : \mathcal{J} \to \text{Set}$, where $\mathcal{J}$ is small. For $F : \mathcal{J} \to \text{Set}$ a diagram of sets, there is a set of natural transformations $\alpha : W \Rightarrow F$, which we may denote by

$$\{W, F\}_\mathcal{J} = \{\mathcal{J}, \text{Set}\}(W, F).$$

If $F : \mathcal{J} \to \mathcal{D}$ takes values in some other category, then the weighted limit $\{W, F\}_\mathcal{J}$, if it exists, is characterized by the existence of a natural isomorphism $	heta : \mathcal{D}(D, \{W, F\}_\mathcal{J}) \cong \{W, \mathcal{D}(D, F)\}_\mathcal{J}$, where the diagram $\mathcal{D}(D, F) : \mathcal{J} \to \text{Set}$ is defined for $D \in \mathcal{D}$ by

$$\mathcal{D}(D, F)_J = \mathcal{D}(D, F_J).$$

The elements of the set $\{W, \mathcal{D}(D, F)\}_\mathcal{J}$ are cylinders in $\mathcal{D}$. If $\mu : W \Rightarrow \mathcal{D}(D, F)$ is a cylinder corresponding to an isomorphism $\theta(\mu) : D \cong \{W, F\}_\mathcal{J}$, we say that $\mu$ is limiting. Here $\mu$ is shorthand for the data $(\mu, W, F, D)$.

If $G : \mathcal{D} \to \mathcal{E}$ is any functor, then for any cylinder $\mu : W \Rightarrow \mathcal{D}(D, F)$ in $\mathcal{D}$, we have a cylinder $G\mu : W \Rightarrow \mathcal{E}(GD, GF)$ in $\mathcal{E}$. If $G\mu$ is limiting, we say that $G$ is $\mu$-continuous. Here we do not require that $\mu$ be limiting. Note, however, that a cylinder is limiting if and only if all of the representable functors $y^D : \mathcal{D} \to \text{Set}$ are $\mu$-continuous.

Dually, a colimiting cocylinder is a limiting cylinder in $\mathcal{D}^{\text{op}}$, and we write $W * \mathcal{J} F$ for the colimit of $F : \mathcal{J} \to \mathcal{D}$ weighted by $W : \mathcal{J}^{\text{op}} \to \text{Set}$.

4.2. Sketches. A sketch $\mathcal{S}$ consists of a category $\mathcal{S}$ and a set $\Phi$ of distinguished cocylinders on $\mathcal{S}$. A colimit sketch is a sketch whose distinguished cylinders are colimit cylinders.

Let $\mathcal{E}$ be some category, and let $\mathcal{S}$ be a sketch. A functor $F : \mathcal{S}^{\text{op}} \to \mathcal{E}$ is a model of $\mathcal{S}$ in $\mathcal{E}$ if it is $\mu$-continuous for every cocylinder $\mu$ in $\Phi$. We say that $F$ is a model of $\mathcal{S}$ in $\mathcal{E}$. The category of models of $\mathcal{S}$ in $\mathcal{E}$ is the full subcategory

$$\text{Mod}(\mathcal{S}; \mathcal{E}) \hookrightarrow [\mathcal{S}^{\text{op}}, \mathcal{E}]$$

generated by the $\Phi$-continuous functors. When $\mathcal{E} = \text{Set}$ we write

$$\text{Mod}^\mathcal{S} := \text{Mod}(\mathcal{S}; \text{Set}),$$

and write $\text{Psh}^\mathcal{S} = \text{Mod}^\mathcal{S}$ when $\Phi$ is empty. By a standard theorem, the inclusion

$$i^\mathcal{S} : \text{Mod}^\mathcal{S} \hookrightarrow \text{Psh}^\mathcal{S}$$

admits a left adjoint.

Let $\mathcal{S}$ and $\mathcal{T}$ be two sketches. We say that a functor $F : \text{Psh}^\mathcal{S} \to \text{Psh}^\mathcal{T}$ preserves models if $FM$ is a model $\mathcal{T}$ whenever $M$ is a model of $\mathcal{S}$. Similarly, we say that $F$ reflects models if $M$ is a model of $\mathcal{S}$ whenever $FM$ is a model of $\mathcal{T}$.
Given a functor $G : S \to T$ between sketches, we say that $G$ preserves (reflects) models if $\text{Psh}^G : \text{Psh}^S \to \text{Psh}^T$ preserves (reflects) models. We define a sketchy functor between sketches to be a functor which preserves models. Thus if $G$ is sketchy we have a commutative diagram

$$
\begin{array}{ccc}
\text{Mod}^T & \xrightarrow{\text{Mod}^C} & \text{Mod}^S \\
\downarrow & & \downarrow \\
\text{Psh}^T & \xrightarrow{\text{Psh}^G} & \text{Psh}^S
\end{array}
$$

In particular, we have a 2-functor

$$\text{Mod} : \text{Sketch}^{\text{coop}} \to \text{CAT},$$

where $\text{Sketch}$ is the 2-category of small sketches, sketchy functors, and natural transformations. We say a category or functor is sketchable if it lies in (strict) image of $\text{Mod}$. A category is locally presentable if it is equivalent to the category of models of a small sketch.

Since $\text{Psh}^G$ always has a left adjoint $\text{Psh}_G$ (left Kan extension), and the inclusion of models into presheaves is reflective, we find that the sketchable functor $\text{Mod}^G$ admits a left adjoint $\text{Mod}_G$ as well.

### 4.3. Fibered sketches.

Let $\mathcal{C}$ be a small category. A fibered $\mathcal{C}$-sketch $(\mathcal{T}, \Phi)$ is a small fibered $\mathcal{C}$-category $\mathcal{T}$ together with a collection of sketches $(\mathcal{T}^I, \Phi^I)$ on the fibers of $\mathcal{T}$, such that every restriction is sketchy.

If $(\mathcal{T}, \Phi)$ and $(\mathcal{T'}, \Psi)$ are fibered sketches, we say that an cartesian functor

$$H : \mathcal{T} \to \mathcal{T'}$$

is sketchy if its components $H^I : \mathcal{T}^I \to \mathcal{T'}^I$ are sketchy. We denote by $\text{SkFib}^\mathcal{C}$ the locally full sub-2-category of $\text{Fib}^\mathcal{C}$ generated by fibered sketches and sketchy cartesian functors.

The 2-category $\text{SkOpFib}^\mathcal{C}$ of opfibered $\mathcal{C}$-sketches is defined in a similar way, but we require that the left extensions are sketchy. The 1-cells are opcartesian functors with sketchy components.

For a bifibered $\mathcal{C}$-sketch, we require that $\mathcal{T}$ be bifibered and that the restriction and left extension functors are sketchy. The 2-category $\text{SkBiFib}$ consists of bifibered sketches, sketchy bicartesian functors, and cartesian transformations. For the full sub-2-category $\text{SkFib}_+$ of $\mathcal{C}$-sum sketches, we also require that $\mathcal{T}$ have $\mathcal{C}$-sums.

### 5. Categories of models

#### 5.1. Sketchable $\mathcal{C}$-categories.

Every opfibered $\mathcal{C}$-sketch gives rise to a fibered $\mathcal{C}$-category of models. We will call such a category a sketchable $\mathcal{C}$-category. To be precise, the 2-functor $\text{Mod} : \text{Sketch}^{\text{coop}} \to \text{CAT}$ induces, for any small category $\mathcal{C}$, a 2-functor

$$\text{Mod}_\mathcal{C} := [\mathcal{C}^{\text{op}}, \text{Mod}]_{\text{ps}} : [\mathcal{C}^{\text{op}}, \text{Sketch}^{\text{coop}}]_{\text{ps}} \to [\mathcal{C}^{\text{op}}, \text{CAT}]_{\text{ps}},$$

Identifying pseudofunctors with fibered categories, we have a corresponding 2-functor

$$\text{Mod}_\mathcal{C} : \text{SkOpFib}(\mathcal{C})^{\text{coop}} \to \text{FIB}(\mathcal{C}),$$
which sends an opfibered sketch to its sketchable $\mathcal{C}$-category of models. This depends, though only up to equivalence, on the choice of opcleavage for $\mathcal{F}$; however, our opfibered sketches will be equipped with a canonical splitting. If $\mathcal{F}$ is an opfibered sketch, then we have

$$\text{Mod}_{\mathcal{C}}^{\mathcal{F}, \mathcal{I}} = \text{Mod}_{\mathcal{C}}^{\mathcal{I}},$$

and if $\mathcal{F}_\varphi : \mathcal{I}_I \to \mathcal{I}_J$ is any left extension along $\varphi : I \to J$, then the sketchable functor

$$\text{Mod}_{\mathcal{C}}^{\mathcal{F}, \mathcal{I}_\varphi} := \text{Mod}_{\mathcal{C}}^{\mathcal{F}, \mathcal{I}_J} : \text{Mod}_{\mathcal{C}}^{\mathcal{F}, \mathcal{I}_I} \to \text{Mod}_{\mathcal{C}}^{\mathcal{F}, \mathcal{I}_J}$$

is a restriction along $\varphi$. In particular, every restriction has a left adjoint, so $\text{Mod}_{\mathcal{C}}^{\mathcal{F}}$ is bifibered.

5.2. Models for $\mathcal{C}$-sum sketches. If $\mathcal{F}$ is a bifibered $\mathcal{C}$-sketch, then for each $\varphi : I \to J$ there is an adjunction $\mathcal{F}_\varphi \dashv \mathcal{F}_J : \mathcal{F}_I \cong \mathcal{F}_I$. Since $\mathcal{F}_\varphi$ and $\mathcal{F}_J$ are sketchy, the adjunction lives in the 2-category $\text{Sketch}$. Applying the 2-functor $\text{Mod} : \text{Sketch}^{\text{coop}} \to \text{CAT}$ produces an adjunction

$$\text{Mod}^{\mathcal{F}_\varphi} \dashv \text{Mod}^{\mathcal{F}_J} : \text{Mod}^{\mathcal{F}_I} \cong \text{Mod}^{\mathcal{F}_I}$$

in the 2-category of categories. The left adjoint is, by definition, a restriction functor $\text{Mod}_{\mathcal{C}}^{\mathcal{F}_\varphi}$. Thus every restriction $\text{Mod}_{\mathcal{C}}^{\mathcal{F}_\varphi}$ has a right adjoint, which, moreover, is given by an explicit formula. In particular, we find that $\text{Mod}_{\mathcal{C}}^{\mathcal{F}}$ is trifibered. If, moreover, the Beck-Chevalley condition holds for $\mathcal{F}$, then it evidently holds for the right extensions in $\text{Mod}_{\mathcal{C}}^{\mathcal{F}}$ as well by 2-functoriality. Thus $\text{Mod}_{\mathcal{C}}^{\mathcal{F}}$ has $\mathcal{C}$-sums and $\mathcal{C}$-products as long as $\mathcal{F}$ has $\mathcal{C}$-sums.

5.3. The free model functor. Let $\mathcal{S}$ and $\mathcal{T}$ be bifibered $\mathcal{C}$-sketches. If $H : \mathcal{S} \to \mathcal{T}$ is sketchy and opcartesian, then the components of $\text{Mod}_{\mathcal{C}}^{H}$ are given by

$$\text{Mod}_{\mathcal{C}}^{H, I} = \text{Mod}^{H_I} : \text{Mod}_{\mathcal{C}}^{\mathcal{S}_I} \to \text{Mod}_{\mathcal{C}}^{\mathcal{T}_I}.$$

Thus each of the components of $\text{Mod}_{\mathcal{C}}^{H}$ is an ordinary sketchable functor and admits an ordinary left adjoint. We will show that if $H$ is cartesian, then $\text{Mod}_{\mathcal{C}}^{H}$ has a bifibered left adjoint.

**Proposition 5.1.** Let $\mathcal{S}$ and $\mathcal{T}$ be bifibered sketches over $\mathcal{C}$. If $H : \mathcal{S} \to \mathcal{T}$ is a sketchy bicartesian functor, then the sketchable functor

$$\text{Mod}_{\mathcal{C}}^{H} : \text{Mod}_{\mathcal{C}}^{\mathcal{T}} \to \text{Mod}_{\mathcal{C}}^{\mathcal{S}}$$

has a fibered left adjoint.

**Proof.** For each $I$ in $\mathcal{C}$, we have an adjunction

$$\text{Mod}_{H_I} \dashv \text{Mod}^{H_I} : \text{Mod}_{\mathcal{C}}^{\mathcal{T}_I} \cong \text{Mod}_{\mathcal{S}_I},$$

because $\text{Mod}_{\mathcal{C}}^{H_I}$ is an ordinary sketchable functor. Since $H$ is opcartesian, we have for every $\varphi : I \to J$ in $\mathcal{C}$, two canonical isomorphisms

$$\begin{align*}
\mathcal{F}_I & \xrightarrow{H_I} \mathcal{F}_I \\
\mathcal{F}_J & \xrightarrow{\varphi} \mathcal{F}_J \\
\mathcal{F}_J & \xrightarrow{H_J} \mathcal{F}_J
\end{align*}$$

and

$$\begin{align*}
\text{Mod}_{\mathcal{F}_J} & \xleftarrow{\text{Mod}^{H_J}} \text{Mod}_{\mathcal{F}_I} \\
\text{Mod}_{\mathcal{F}_J} & \xleftarrow{\text{Mod}^{\varphi}} \text{Mod}_{\mathcal{F}_I}
\end{align*}.$$
In order for $\text{Mod}^H$ to have a fibered left adjoint, we must show that mate of $\text{Mod}^\varphi$ under the adjunctions $\text{Mod}_{H_1} \dashv \text{Mod}^{H_1}$ and $\text{Mod}_{H_2} \dashv \text{Mod}^{H_2}$ is an isomorphism.

In fact, the mate of $\alpha_\varphi$ under the adjunctions $\mathcal{I}_\varphi \dashv \mathcal{J}_\varphi$ and $\mathcal{I}_\varphi \dashv \mathcal{J}_\varphi$ is an isomorphism, because $H$ is cartesian. It follows immediately that the mate of $\text{Mod}^\varphi$ under the adjunctions $\text{Mod}^\varphi \dashv \text{Mod}^\varphi$ and $\text{Mod}^\varphi \dashv \text{Mod}^\varphi$ is an isomorphism. By a general property of mates, this implies that the mate of $\text{Mod}^\varphi$ under $\text{Mod}_{H_1} \dashv \text{Mod}^{H_1}$ and $\text{Mod}_{H_2} \dashv \text{Mod}^{H_2}$ is an isomorphism as well. □

6. Sketches under a site

6.1. Standard sites as bifibered sketches. If $(\mathcal{C}, J)$ is a small site, then we can define a sketch $(\mathcal{C}, \Phi)$ whose models are sheaves. We can take $\Phi$ to contain the cocylinders $(\mu, R,i,l_{\mathcal{C}},U)$, where $U \in \mathcal{C}$ is any object and

$$\mu : R \rightarrow \mathcal{C}(-, U),$$

is the inclusion of a covering sieve $R$ into $U$. For sites $(\mathcal{C}, J)$ and $(\mathcal{C}', J')$, a sketchy functor $G : (\mathcal{C}, \Phi) \rightarrow (\mathcal{C}', \Phi')$ is the same thing as a continuous functor of sites. It will be convenient to identify a site with the sketch $(\mathcal{C}, \Phi)$. We say that a sketch $(\mathcal{C}, \Phi)$ is topological if the inclusion $\text{Mod}^\varphi \rightarrow \text{Psh}^\mathcal{C}$ has a left exact reflection, and we write $\mathcal{E} = \text{Sh}^\mathcal{C}$ for the category of models.

Now suppose that $(\mathcal{C}, \Phi)$ is standard. This means that $\mathcal{C}$ has finite limits and every representable functor is a sheaf. Now consider the fibered $\mathcal{C}$-category of arrows, denoted

$$\mathcal{E} = \text{Arr}(\mathcal{C}).$$

Each fiber $\mathcal{E}^I = \mathcal{C}/I$ is equipped with a topology $\Phi^I$ induced by the domain projection $\text{dom} : \mathcal{C}/I \rightarrow \mathcal{C}$. Moreover, if $\varphi : I \rightarrow J$ in an arrow in $\mathcal{C}$, then the restriction functor $\mathcal{E}^\varphi$ is given by pullback and the extension functor $\mathcal{E}_\varphi$ is given by composition with $\varphi$. Both of these define continuous morphisms of sites, which means they are sketchy functors. Thus every standard site gives rise to a bifibered sketch.

We will denote the category of models of the bifibered sketch $\mathcal{E}$ by

$$\mathcal{E}_x = \text{Mod}^{\mathcal{E}}.$$

There is an equivalence of $\mathcal{E}$-fibrated categories

$$\mathcal{E}_x \cong \text{FIB}^y \text{Arr}(\mathcal{E}),$$

where $y : \mathcal{C} \rightarrow \mathcal{E}$ is the Yoneda embedding, and $\text{Arr}(\mathcal{E})$ is the $\mathcal{E}$-category of arrows in $\mathcal{E}$.

6.2. Multi-sorted theories. If $\mathcal{A}$ is a small fibered $\mathcal{C}$-category with $\mathcal{C}$-products, we can enhance the category $\text{FIB}_x^\mathcal{C}(\mathcal{A}, \mathcal{E})$ of $\mathcal{C}$-multiplicative functors into a fibered $\mathcal{E}$-category corresponding to the pseudofunctor

$$[\mathcal{A}, \mathcal{E}]^P_x := \text{FIB}_x^\mathcal{C}(\mathcal{A}, \mathcal{E}^P).$$

Since $\mathcal{E}$ is a topos, every map $\varphi : P \rightarrow Q$ gives rise to a string of fibered adjunctions $\mathcal{E}_\varphi \dashv \mathcal{E}^\varphi \dashv \mathcal{E}_\varphi$. In particular, the functors $\mathcal{E}^\varphi$ and $\mathcal{E}_\varphi$ preserve $\mathcal{E}$-products, so that the fibered adjunctions $\mathcal{E}^\varphi \dashv \mathcal{E}_\varphi$ induce corresponding adjunctions $[\mathcal{A}, \mathcal{E}]^P_x \dashv [\mathcal{A}, \mathcal{E}_\varphi]_x \dashv [\mathcal{A}, \mathcal{E}_\varphi]_x$ in $\text{CAT}$. This implies that the $\mathcal{E}$-category $[\mathcal{A}, \mathcal{E}]_x$ is well-defined and inherits the property of having fibered $\mathcal{E}$-products from $\mathcal{E}$. 
More generally, let $\mathcal{B}$ be any small fibered $\mathcal{C}$-category. Then we have a natural fibered equivalence

$$\mathcal{B}, [\mathcal{A}, \mathcal{E}]_x \cong [\mathcal{A}, [\mathcal{B}, \mathcal{E}]]_x.$$ 

In particular, if $R$ is a covering sieve for $P \in \mathcal{E}$, we have

$$[R, [\mathcal{A}, \mathcal{E}]_x] \cong [\mathcal{A}, [R, \mathcal{E}]]_x \cong [\mathcal{A}, [P, \mathcal{E}]]_x \cong [P, [\mathcal{A}, \mathcal{E}]]_x,$$

because $\mathcal{E}$ is a stack over $\mathcal{E}$. Thus the fibered $\mathcal{E}$-category $[\mathcal{A}, \mathcal{E}]_x$ is also a stack over $\mathcal{E}$.

### 6.3. Lawvere theories

To obtain cocompleteness of categories of models, we will restrict our attention to single-sorted theories of a very particular form. Fix a standard site $(\mathcal{C}, \Psi)$. Now let $\tau : \mathcal{C} \to \mathcal{T}$ be a bicartesian identity-on-objects functor. We may then define sketches $(\mathcal{T}^I, \Phi^I)$ whose distinguished cylinders are precisely those induced by $\tau^I$. A model of $\mathcal{T}^I$ is thus a presheaf on $\mathcal{C}^I$ whose composition with $\tau^I$ is a sheaf on $\mathcal{C}^I$. It easy to see that $\mathcal{T}$ is a bifibered $\mathcal{C}$-sketch. We say that $\tau$ is a $\mathcal{C}$-ary Lawvere theory if $(\mathcal{T}, \Phi)$ is a colimit $\mathcal{C}$-sketch. This means exactly that $\tau^I$ is $\Psi^I$-continuous for each $I$.

Now we show that $\text{Mod}^\mathcal{T}$ admits an intrinsic description as the category of “internal models” for the theory $\tau$.

**Proposition 6.1.** For $J \in \mathcal{C}$, let $\Delta_J : J \to J \times J$ be the diagonal map. If $\tau$ is a Lawvere theory, then the functors $\alpha_J : [\mathcal{T}^{\mathcal{op}}, \mathcal{E}^J]_x \to \text{Mod}^{\mathcal{T}, J}$ defined by

$$\alpha_J(M) = \mathcal{E}^{J \times J}(\Delta_J, M^J -)$$

are the components of a cartesian functor

$$[\mathcal{T}^{\mathcal{op}}, \mathcal{E}]_x \to \text{Mod}^{\mathcal{T}}.$$

Moreover, this cartesian functor is an equivalence in $\text{FIB}^{\mathcal{C}}$.

**Proof.** For $M : \mathcal{T}^{\mathcal{op}} \to \mathcal{E}^J$ cartesian and $\mathcal{C}$-multiplicative, the components of $M$ are functors

$$M^I : \mathcal{T}^{\mathcal{op}} \to \mathcal{E}^{I \times J}.$$

First, we claim that $\alpha_J(M) : \mathcal{T}_J^{\mathcal{op}} \to \text{Set}$ is a model of the sketch $\mathcal{T}_J$. This means exactly that $\alpha_J(M)\tau^J$ is a sheaf. To see this, we write $E \in \mathcal{C}_{/J}$ as $E \cong \mathcal{E}_E \mathcal{E}^{E_{1, J}}$ and compute

$$\mathcal{E}^{J \times J}(\Delta_J, M^J \tau^J E) \cong \mathcal{E}^{J \times J}(\Delta_J, M^J \tau^J \mathcal{E}_{E \times J} \mathcal{E}^{E_{1, J}}) \cong \mathcal{E}^J(1_J, \mathcal{E}^{J \times J}(\mathcal{E}_{E \times J} \mathcal{E}^{E_{1, J}})) \cong \mathcal{E}^J(1_J, \mathcal{E}_{E \times J} \mathcal{E}^{(\Delta_J, E)} \mathcal{E}^{E_{1, J}}) \cong \mathcal{E}^J(1_J, \mathcal{E}_{E \times J} \mathcal{E}^{\Delta_J} \mathcal{E}^{E_{1, J}}) \cong \mathcal{E}^J(1_J, \mathcal{E}_{E \times J} \mathcal{E}^{\Delta_J} \mathcal{E}^{E_{1, J}}) \cong \mathcal{E}^J(1_J, \mathcal{E}_{E \times J} \mathcal{E}^{\Delta_J} \mathcal{E}^{E_{1, J}}),$$

which is manifestly a sheaf with respect to $E \in \mathcal{C}_{/J}$. 

Now we need to show that $\alpha_J$ is pseudonatural in $J$. For $\varphi : K \to J$ we claim there is a canonical isomorphism

$$
\begin{array}{ccc}
\mathcal{F}^{\text{op}}, \mathcal{E}^J \times & \xrightarrow{\alpha_J} \text{Mod} \mathcal{F}_J \\
\mathcal{F}^{\text{op}}, \mathcal{E}^K \times & \xrightarrow{\alpha_K} \text{Mod} \mathcal{F}_K
\end{array}
$$

Indeed, for $M : \mathcal{F}^{\text{op}} \to \mathcal{E}^J$ and $T \in \mathcal{F}_K$ we have

$$
(\text{Mod} \mathcal{F}_\varphi \alpha_J M)(T) = (\alpha_J M)(\mathcal{F}_\varphi T) = \mathcal{E}^J \times (\Delta_J, M^J \mathcal{F}_\varphi T) \\
\cong \mathcal{E}^J \times (\Delta_J, \mathcal{E}^J \times J, M^K T) \\
\cong \mathcal{E}^K \times (\mathcal{E}^J \times J, M^K T)
$$

On the other hand, we have

$$
(\alpha_K \mathcal{E}^J \Delta_J M)(T) = \mathcal{E}^K \times (\Delta_K, (\mathcal{E}^J M)^K T) \\
= \mathcal{E}^K \times (\Delta_K, \mathcal{E}^J \times J, M^K T) \\
\cong \mathcal{E}^J \times (\mathcal{E}^J \times J, M^K T)
$$

But we have a canonical isomorphism $\mathcal{E}^J \times J \cong \mathcal{E}^J \times K \Delta_J$, in light of the commutative diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\Delta_K} & K \\
\downarrow \text{trac} & & \downarrow \text{cart} \\
K \times K & \xrightarrow{\varphi \times K} & J \times J
\end{array}
$$

and a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{E} & I \times J \\
\downarrow \text{cart} & & \downarrow \text{trac} \\
X \times J & \xrightarrow{\pi_{X,J}} & I \times J
\end{array}
$$

and a commutative diagram

$$
\begin{array}{ccc}
I \times J & \xrightarrow{\Delta_J} & J \\
\downarrow \text{cart} & & \downarrow \text{cart} \\
I \times J \times J & \xrightarrow{\pi_{J \times J}} & J \times J
\end{array}
$$

Now we construct a quasi-inverse $\beta_J$ to $\alpha_J$. Suppose we have a model $\mathcal{M} : \mathcal{F}^{\text{op}} \to \text{Set}$. We want to find a $\mathcal{C}$-multiplicative $M : \mathcal{F}^{\text{op}} \to \mathcal{E}^J$ such that $\alpha_J M \cong \mathcal{M}$. For every $E : X \to I \times J$ in $\mathcal{C}_{I \times J}$, we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{E} & I \times J \\
\downarrow \text{cart} & & \downarrow \text{trac} \\
X \times J & \xrightarrow{\pi_{X,J}} & I \times J
\end{array}
$$

and a commutative diagram

$$
\begin{array}{ccc}
I \times J & \xrightarrow{\varphi} & J \\
\downarrow \text{cart} & & \downarrow \text{cart} \\
I \times J \times J & \xrightarrow{\pi_{J \times J}} & J \times J
\end{array}
$$
If $M$ is $C$-multiplicative and $\alpha_J M = \overline{M}$ then we find that for $E \in C_{I \times J}$ and $T \in \mathcal{T}$ there is a canonical isomorphism
\[
\mathcal{E}^{I \times J}(E, M^I(T)) \cong \mathcal{E}^{I \times J}(\mathcal{C}_{(\pi_I \circ E)} \times J, \mathcal{C}_{\pi_J} \mathcal{E}^{\pi_I \circ E}(\Delta_J, M^J(T))) \\
\cong \mathcal{E}^{I \times J}(\Delta_J, M^J(\mathcal{T}_{\pi_J \mathcal{T}}(\mathcal{E}^{\pi_I \circ E}(T)))) \\
\cong \overline{M}(\mathcal{T}_{\pi_J \mathcal{T}}(\mathcal{E}^{\pi_I \circ E}(T))).
\] (1)

Thus for a model $\overline{M} : \mathcal{T}^{op} \to \text{Set}$ we define a $C$-functor $M : \mathcal{T}^{op} \to \mathcal{E}$ with components $M^I : \mathcal{T}^{op} \to \mathcal{E}^{I \times J}$ given by
\[
M^I(T)(E) = \overline{M}(\mathcal{T}_{\pi_J \mathcal{T}}(\mathcal{E}^{\pi_I \circ E}(T))).
\]

To see that $M^I(T)$ is in fact a sheaf on $C_{I \times J}$, we recall that every $T$ in $\mathcal{T}$ is of the form $T = \tau^I F$ for some $F \in C_I$. But now consider the pasting of pullback squares
\[
\begin{array}{ccc}
\mathcal{E}^{\pi_I \circ E} & \xrightarrow{\varphi^{F \times J}} & E \\
\downarrow \varphi^{\pi_I \circ E} & & \downarrow \varphi^{F \times J} \text{cart} \\
I \times J & \xrightarrow{\varphi^{F \times J}} & F \\
\end{array}
\]

We have a canonical isomorphism
\[
\mathcal{T}_{\pi_J \mathcal{T}}(\mathcal{E}^{\pi_I \circ E}(\tau^I F)) \cong \tau^J \mathcal{E}_{\pi_J \mathcal{E}} \mathcal{C}_{\pi_I \circ E}(F) \\
\cong \tau^J \mathcal{E}_{\pi_J \mathcal{E}} \mathcal{C}_{\mathcal{E}}^{F \times J} E,
\]

and thus
\[
M^I(\tau^I F)(E) \cong \overline{M}(\tau^J \mathcal{E}_{\pi_J \mathcal{E}} \mathcal{C}_{\pi_I \circ E}(F) \mathcal{C}_{\mathcal{E}}^{F \times J} E).
\]

Since $\overline{M}(\tau^J -)$ is a sheaf on $C_{I \times J}$ by assumption, and all of the restrictions and left extensions of $\mathcal{E}$ are sketchy, it follows that the $\overline{M}$ we have defined is a sheaf on $C_{I \times J}$.

Now we need to show that the $M^I$ we have defined are components of a $C$-multiplicative $C$-functor $M : \mathcal{T}^{op} \to \mathcal{E}^I$. Recall that for $\varphi : I \to K$, the restriction and right extension along $\varphi$ are defined for $P \in \mathcal{E}^{I \times J}$, $Q \in \mathcal{E}^{J \times K}$ by
\[
(\mathcal{E}^{\varphi \times J} P)(-)) = P(\mathcal{E}_{\varphi \times J} -) \\
(\mathcal{E}^{\varphi \times J} Q)(-)) = Q(\mathcal{E}_{\varphi \times J} -).
\]

Then we have
\[
(\mathcal{E}^{\varphi \times J} M^K(T))(E) = \overline{M}(\mathcal{T}_{\varphi \times J \circ \mathcal{T}}(\mathcal{E}^{\pi_K \circ \varphi \times J} \mathcal{E}^{\pi_J \circ \mathcal{T}}(T))) \\
= \overline{M}(\mathcal{T}_{\varphi \circ \mathcal{T}}(\mathcal{E}^{\pi_I \circ \mathcal{T}}(T))
\]
\[
\text{while}
\]
\[
M^I(\mathcal{T}^\varphi(T))(E) = \overline{M}(\mathcal{T}_{\varphi \circ \mathcal{T}}(\mathcal{E}^{\pi_I \circ \mathcal{T}} T)
\]

and thus the isomorphism $\mathcal{T}^{\varphi \circ \pi_I \circ \mathcal{T}} \cong \mathcal{T}^{\pi_I \circ \mathcal{T}} \mathcal{T}^\varphi$ induces an isomorphism
\[
M^I \mathcal{T}^\varphi \cong \mathcal{E}^{\varphi \times J} M^K.
\]
Similarly, let \( E \in \mathcal{E}^{K \times J} \) be given. The pullback diagram

\[
\begin{array}{ccc}
E^* (\varphi \times J) & \xrightarrow{E} & E \\
\downarrow (\varphi \times J)^* E & & \downarrow E \\
I \times J & \xrightarrow{\varphi \times J} & K \times J \\
\pi_I & \downarrow & \pi_K \\
I & \xrightarrow{\varphi} & K
\end{array}
\]

determines a Beck-Chevalley isomorphism

\[
\mathcal{F}^{\pi_K \circ E} \mathcal{F} \varphi \cong \mathcal{F}_{E^* (\varphi \times J)} \mathcal{F}^{\pi_J \circ (\varphi \times J)^* E},
\]

and thus we have

\[
M^K (\mathcal{F}_{\varphi} T)(E) = M (\mathcal{F}_{\pi_J \circ E} \mathcal{F}^{\pi_K \circ E} \mathcal{F}_{\varphi} T) \\
\cong M (\mathcal{F}_{\pi_J \circ E} \mathcal{F}_{E^* (\varphi \times J)} \mathcal{F}^{\pi_J \circ (\varphi \times J)^* E} T) \\
\cong M (\mathcal{F}_{\pi_J \circ (\varphi \times J)^* E} \mathcal{F}^{\pi_J \circ (\varphi \times J)^* E} T), \\
\cong M (\mathcal{F}_{\pi_J \circ (\varphi \times J)^* E} \mathcal{F}^{\pi_J \circ (\varphi \times J)^* E} T) \\
= (\mathcal{E}_{\varphi \times J^* M})^T (T).
\]

Thus \( M \) is both \( C \)-cartesian and \( C \)-multiplicative.

Let

\[
\beta_J : \text{Mod}^{\mathcal{F}, J} \rightarrow [\mathcal{F}^{\text{op}}, \mathcal{E}^{J}]_x
\]

be the functor sending \( M \) to \( M \). We claim that \( \beta_J \) is a quasi-inverse for \( \alpha_J \). Indeed the isomorphism \( \beta_J \alpha_J \cong \text{id} \) is the observation in (1), while the isomorphism \( \alpha_J \beta_J \cong \text{id} \) is given by the formula

\[
\alpha_J \beta_J (M) \cong M (\mathcal{F}_{\pi_J \circ \Delta_J} \mathcal{F}^{\pi_J \circ \Delta_J} T) \\
\cong M (T).
\]

Since each component of \( \alpha_J \) is an equivalence, we conclude that \( \alpha \) is an equivalence as well.

**Corollary 6.2.** If \( \tau : \mathcal{C} \rightarrow \mathcal{F} \) is a Lawvere theory, then \( \text{Mod}^{\mathcal{F}} \) is a stack over \( \mathcal{C} \).

**Proof.** We have already seen that \( [\mathcal{F}^{\text{op}}, \mathcal{E}]_x \) is a stack over \( \mathcal{C} \). \( \square \)

## 7. Cocompleteness of \( \text{Mod}^{\mathcal{F}} \)

### 7.1. Enlarging the sketch

In order to show cocompleteness for the \( \mathcal{E} \)-category \( y_* \text{Mod}^{\mathcal{F}} \), we will use the comparison lemma \([AGV72, SGA 4.III.4]\) for categories of sheaves. Here we will view \( \text{Mod} \) as a 2-functor

\[
\text{Mod} : \text{SKETCH}^{\text{coop}} \rightarrow \text{CAT}'
\]

sending a large sketch to the very large category of models in \( \text{Set} \). If the large sketch is the category of models of a small sketch, the comparison lemma allows us to avoid size issues. We note that the ideas in this section are almost entirely due to Linton \([Lin69]\) and Giraud \([Gir71, II]\).
Lemma 7.1. Let \((\mathcal{C}, \Phi)\) be a small site, and suppose that \(G : \mathcal{C} \rightarrow \mathcal{T}\) is \(\Phi\)-cocontinuous and identity on objects, so that \(G\) induces a colimit sketch on \(\mathcal{T}\). Suppose we have a diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{y_{\mathcal{C}}} & \mathcal{E} \\
\downarrow^{G} & \cong & \downarrow^{G} \\
\mathcal{T} & \xrightarrow{Z} & \mathcal{T} & \xrightarrow{i} \text{Mod}^\mathcal{T},
\end{array}
\]

where \(F\) is the left adjoint to the forgetful functor, the functor \(G\) is identity on objects, and the functors \(Z\) and \(i\) are fully faithful. Then \(\text{Mod}^Z\) is an equivalence of categories.

Proof. Let \(Z^* = \text{Mod}^Z\). Then \(Z^*\) has a right adjoint \(R\), defined by

\[
(RM)(\mathcal{T}) = \text{Mod}^T(i\mathcal{T}, M),
\]

which is essentially the right Kan extension along \(Z\). For this to make sense, we must verify that \(RM\) is a model whenever \(M\) is. Indeed, we have

\[
(RM)(G X) \cong \text{Mod}^T(FX, M) \cong \mathcal{E}(X, UM),
\]

which is a sheaf.

The counit \(Z^* R \rightarrow \text{id}\) is an isomorphism, because \(Z\) is full and faithful. To see that the unit is an isomorphism, we observe that \(G^* : \text{Mod}^\mathcal{T} \rightarrow \text{Mod}^\mathcal{E}\) is conservative because \(G\) is identity on objects, and \(y^* : \text{Mod}^\mathcal{E} \rightarrow \text{Mod}^\mathcal{E}\) is an equivalence by the comparison lemma for categories of sheaves. Thus the unit \(\eta : \text{id} \rightarrow RZ^*\) is an isomorphism if and only if this is true for the pasting diagram

But the left-hand square in the diagram can be inverted by pasting with the counit of the adjunction. It follows that the indicated two-cell is invertible, and therefore the same is true of the counit \(\epsilon\), because \(y^* G^*\) is conservative. \(\square\)

Let \(\tau : \mathcal{C} \rightarrow \mathcal{T}\) be a Lawvere theory, and let \(F : \mathcal{E} \rightarrow \text{Mod}^\mathcal{C}\) be the left adjoint to the forgetful functor. If we define \(\bar{\mathcal{C}} = \mathcal{E}\), we may factor \(F\) as

\[
\begin{array}{ccc}
\bar{\mathcal{C}} & \xrightarrow{\tau} & \mathcal{T} & \xrightarrow{\iota} \text{Mod}^\mathcal{C},
\end{array}
\]

where \(\tau\) is identity on objects and \(\iota\) is fully faithful. An arrow of \(\int \mathcal{T}\) is cartesian or opcartesian iff this is true of the image under \(\int \tau\). In particular, the \(\mathcal{C}\)-category \(\mathcal{T}\) is bifibered, and the functor \(\tau : \mathcal{E} \rightarrow \mathcal{T}\) is bicartesian, because \(F\) a fibered left adjoint and therefore bicartesian.

As \(\text{Mod}^\mathcal{C}\) is a stack over \(\mathcal{C}\), the adjunction \(\varphi \dashv \iota\) in \(\text{STACK}^\mathcal{C}\) induces an adjunction \(y_* F \dashv y_* U\) in \(\text{STACK}^\mathcal{E}\). We can interpret this as an adjunction

\[
\mathcal{F} \dashv \mathcal{U} : y_* \text{Mod}^\mathcal{C} \rightleftarrows \bar{\mathcal{C}}.
\]

Here \(\bar{\mathcal{C}} = \text{Arr}(\mathcal{E})\) is the fibered \(\mathcal{E}\)-category of arrows in \(\mathcal{E}\), which is equivalent to \(y_* \mathcal{E}\). By factoring \(\mathcal{F}\) as

\[
\begin{array}{ccc}
\bar{\mathcal{C}} & \xrightarrow{\tau} & \mathcal{F} & \xrightarrow{\iota} y_* \text{Mod}^\mathcal{C},
\end{array}
\]

we get...

we obtain a Lawvere $\mathcal{E}$-theory. The topology on each site $\mathcal{E}^P$ is the canonical one, and since each functor $(y_* F)^P : (y_* \mathcal{E})^P \to (y_* \text{Mod}^\mathcal{E})^P$ is an ordinary left adjoint we find that all of the colimit cylinders in the sketch for $\mathcal{E}^P$ are preserved by $\tau_P$.

Although we have not shown that $y_* \text{Mod}^\mathcal{E}$ has fibered sums, the functor $F$ does preserve the opcartesian arrows in $\mathcal{T}$, because it is a fibered left adjoint. It follows in particular that $\mathcal{T}$ is a bifibered $\mathcal{E}$-category.

**Proposition 7.2.** There is an equivalence of fibered $\mathcal{E}$-categories

$$\text{Mod}^\mathcal{T} \cong y_* \text{Mod}^\mathcal{E}.$$

**Proof.** In fact, we have an equivalence $\text{Mod}^\mathcal{T} \cong [\mathcal{T}^{\text{op}}, y_* \mathcal{E}]_\times$, which shows that $\text{Mod}^\mathcal{T}$ is a stack over $\mathcal{E}$. By the comparison lemma for stacks, it then suffices to show that $y^* \text{Mod}^\mathcal{T} \cong \text{Mod}^\mathcal{E}$ as fibered $\mathcal{C}$-categories. Now, one can check that

$$y^* \text{Mod}^\mathcal{T} \cong \text{Mod}^{y^* \mathcal{C}} \cong \text{Mod}^\mathcal{E},$$

and we have $Z : \mathcal{T} \to \mathcal{T}$ fitting into a diagram of fibered $\mathcal{C}$-categories and functors

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{y} & \mathcal{E} \\
\downarrow \tau & \cong & \downarrow F \\
\mathcal{T} & \xrightarrow{Z} & \mathcal{T} & \xrightarrow{\tau} & \text{Mod}^\mathcal{E}.
\end{array}$$

We claim that $\text{Mod}^Z : \text{Mod}^\mathcal{T} \to \text{Mod}^\mathcal{E}$ is an equivalence. Equivalently, all of the components $\text{Mod}^{Z_i} : \text{Mod}^{\mathcal{T}_i} \to \text{Mod}^{\mathcal{E}_i}$ are equivalences. But this follows from Lemma 7.1, by applying the 2-functor $\text{ev}_I : \text{FIB}^\mathcal{C} \to \text{CAT}$.

**Proposition 7.3.** The fibered $\mathcal{E}$-category $\text{Mod}^\mathcal{T}$ is bifibered, and its fibers are locally presentable.

**Proof.** Let $\mathcal{F}$ be the opfibered $\mathcal{E}$-category defined by

$$\mathcal{F}_P = \mathcal{C}/P.$$

Note that this is certainly not a fibered $\mathcal{E}$-category. We define an opfibered $\mathcal{E}$-category $\mathcal{F}'$ and opcartesian $\mathcal{E}$-functors $\tau : \mathcal{F} \to \mathcal{F}'$ and $Z : \mathcal{F}' \to \mathcal{T}$ so that we obtain a diagram of opcartesian $\mathcal{E}$-functors

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{y} & y_* \mathcal{E} \\
\downarrow \tau & \cong & \downarrow F \\
\mathcal{F}' & \xrightarrow{Z} & \mathcal{T} & \xrightarrow{\iota} & \text{Mod}^\mathcal{E}
\end{array}$$

as above. By means of this diagram, we deduce that $\text{Mod}^Z : \text{Mod}^\mathcal{T} \to \text{Mod}^{\mathcal{F}'}$ is an equivalence. But $\mathcal{F}'$ is an opfibered sketch with small fibers. Thus each fiber $\text{Mod}^{\mathcal{F}'}_i$ is sketchable, and each restriction functor $\text{Mod}^{\mathcal{F}'}_i$ has a left adjoint, because it is a sketchable functor.

**Corollary 7.4.** The $\mathcal{E}$-category $y_* \text{Mod}^\mathcal{T}$ has $\mathcal{E}$-sums and $\mathcal{E}$-products, and its fibers are locally presentable.
Proof. Indeed, we have an equivalence \( y \cdot \text{Mod}^\mathcal{T} \cong \text{Mod}^\mathcal{T} \). But \( \text{Mod}^\mathcal{T} \) has \( \mathcal{E} \)-products, since it is the category of models for a Lawvere \( \mathcal{E} \)-theory. It is also bifibered, as we have just shown, which means that it has \( \mathcal{E} \)-sums. Moreover, its fibers are locally presentable. All of these properties can be transported to \( y \cdot \text{Mod}^\mathcal{T} \). \( \square \)

8. Monoidal structures

In this section, we show that for a commutative theory, the category of models admits a well-behaved tensor product. The ideas in this section are essentially due to Day [Day70, Day74, Day72] and Day and Street [DS95] (see also [BE72]).

8.1. Ends and coends. Recall that the end of a functor \( H : \mathcal{J}^{op} \times \mathcal{J} \to \mathcal{D} \), if it exists, is the weighted limit

\[
\int_{J \in \mathcal{J}} H^J := \{ J, H \}_{\mathcal{J}^{op} \times \mathcal{J}},
\]

where \( \mathcal{J} : \mathcal{J}^{op} \times \mathcal{J} \to \text{Set} \) is the hom functor. Similarly, the coend is the weighted colimit

\[
\int^{J \in \mathcal{J}} H^J := \mathcal{J}^{*_{\mathcal{J}^{op} \times \mathcal{J}}} H.
\]

In sufficiently complete categories, weighted limits are ends of powers, so that for \( W : \mathcal{J} \to \text{Set} \) and \( H : \mathcal{J} \to \mathcal{D} \), we have

\[
\{ W, H \}_\mathcal{J} = \int_{J \in \mathcal{J}} [W_J, H_J],
\]

when the powers indicated all exist. An adjunction \( \varphi_1 \dashv \varphi^* : \mathcal{J}_1 \leftrightarrows \mathcal{J}_2 \) between two categories induces an adjunction \( [\varphi^*, \mathcal{D}] \dashv [\varphi_1, \mathcal{D}] : [\mathcal{J}_1, \mathcal{D}] \leftrightarrows [\mathcal{J}_2, \mathcal{D}] \) between functor categories, which we may express as a natural isomorphism

\[
[\mathcal{J}_1, \mathcal{D}](\varphi^*, \mathcal{D})F, G) \cong [\mathcal{J}_2, \mathcal{D}](\varphi_1, \mathcal{D})G
\]
or more suggestively by the change of variables formula

\[
\int_{J_1 \in \mathcal{J}_1} \mathcal{D}^{\varphi^* J_1} G_{J_1} \cong \int_{J_2 \in \mathcal{J}_2} \mathcal{D}^{\varphi J_2} G_{\varphi_1 J_2},
\]

which can also be seen as a special case of the mate correspondence. By separation of variables, we deduce a more general change of variables formula for ends, namely that for \( H : \mathcal{J}_2^{op} \times \mathcal{J}_1 \to \mathcal{D} \) we have

\[
\int_{J_1 \in \mathcal{J}_1} H^{\varphi^* J_1} J_1 \cong \int_{J_2 \in \mathcal{J}_2} H^{\varphi J_2} \varphi_1 J_2.
\]

Similarly, for \( H : \mathcal{J}_1^{op} \times \mathcal{J}_2 \to \mathcal{D} \), we have an isomorphism of coends

\[
\int_{J_2 \in \mathcal{J}_2} H^{\varphi_1 J_2} J_2 \cong \int_{J_1 \in \mathcal{J}_1} H^{J_1 \varphi^*} J_1.
\]
8.2. Monoidal adjunctions. In what follows, we will assume that all monoidal categories and functors are symmetric monoidal.

We recall Day’s reflection theorem [Day72]. For a closed monoidal category $V$ and a reflective subcategory $r \dashv \iota : W \leftrightarrows V$, the adjunction can be improved to a monoidal adjunction if and only if $W$ is an exponential ideal in $V$. Assuming that $W$ is replete, this means that $[V,W] \in W$ whenever $W \in W$ and $V \in V$. Under these assumptions, the subcategory $W$ is closed monoidal as well, and $\iota$ is a closed functor.

8.3. Day convolution. We apply the reflection theorem with $V = (\mathbf{Psh}^C, \otimes, y(e))$, where $(C, \otimes, e)$ is a small monoidal category and the Day convolution product [Day70] of presheaves $P, Q \in \mathbf{Psh}^C$ is the presheaf $P \otimes Q$ defined by the coend

\[(P \otimes Q)^C = \int_{C_1, C_2 \in C} P^{C_1} \times Q^{C_2} \times C_{C_1 \otimes C_2}.\]

The internal hom is the presheaf $[P, Q]$ defined by the end

\[[P, Q]^C = \int_{C_1 \in C} [P^{C_1}, Q^{C_1 \otimes C_2}].\]

The Yoneda embedding $y : C \to \mathbf{Psh}^C$ is strong monoidal with respect to the given monoidal structure on $C$ and the Day convolution on $\mathbf{Psh}^C$. Note that when $C$ is cartesian monoidal, the same holds for $\mathbf{Psh}^C$, so this reduces to the fact that the Yoneda embedding preserves finite (indeed any) products.

8.4. Monoidal sketches. Let $(S, \Phi)$ be a small sketch. We say that a monoidal structure $(\otimes, e)$ on $S$ is compatible if the tensor product functors $T \otimes - : S \to S$ are all sketchy. In this case we say that $(S, \Phi, \otimes)$ is a monoidal sketch.

The category $\mathbf{Mod}^S$ is an exponential ideal in $\mathbf{Psh}^S$. Indeed, the category of models is closed under limits. For each $S_1$, the presheaf $S \mapsto N^{S_1 \otimes -}$ is a model, because $S_1 \otimes -$ is sketchy and $N$ is a model. Now the Day internal hom $[M, N]$ is the limit, weighted by $M$, of the diagram $S_1 \mapsto N^{S_1 \otimes -}$, which is valued in $\mathbf{Mod}^S$. Thus $[M, N]$ is a model as well.

By Day’s reflection theorem, the reflective inclusion $r \dashv \iota : \mathbf{Mod}^S \hookrightarrow \mathbf{Psh}^S$ enriches in an essentially unique way to a monoidal adjunction, and the monoidal category $\mathbf{Mod}^S$ thus obtained is closed.

8.5. Functoriality of Day convolution. If $G : S_1 \to S_2$ is any oplax monoidal functor, then $\mathbf{Psh}^G : \mathbf{Psh}^{S_2} \to \mathbf{Psh}^{S_1}$ is lax monoidal with respect to the Day convolution structure. The adjunctions $r_1 \dashv \iota_1 : \mathbf{Mod}^{S_1} \leftrightarrow \mathbf{Psh}^{S_1}$ both live in the 2-category $\mathbf{MonCAT}_{lax}$. Thus, if $G$ is also sketchy, we obtain a lax monoidal structure on $\mathbf{Mod}^G = r_1 \mathbf{Psh}^G \iota_2$ as well. Similarly, a monoidal transformation from $G$ to $G'$ induces a corresponding monoidal transformation from $\mathbf{Mod}^G$ to $\mathbf{Mod}^{G'}$. If we define $\mathbf{MonSketch}$ to be the 2-category of small monoidal sketches, sketchy oplax monoidal functors, and monoidal transformations, then $\mathbf{Mod}$ can be defined as a 2-functor

\[\mathbf{Mod} : \mathbf{MonSketch}^{coop} \to \mathbf{MonCAT}_{lax},\]

which takes values in the 2-category monoidal categories, lax monoidal functors, and monoidal transformations. Note that even if $G$ is a strong monoidal functor, the functor $\mathbf{Psh}^G$ described above is usually only lax.
8.6. **Monoidal C-categories.** The category $\text{UVFib}^C$ has finite products, and we may define a monoidal (fibered) $C$-category to be a pseudomonoid $\mathcal{F}$ in the monoidal 2-category $(\text{UVFib}^C, \times, 1)$. This determines 2-categories $\text{UVMonFib}^C_{\text{lax/oplax/strong}}$ of $UV$-small monoidal $C$-categories, lax/oplax/strong monoidal cartesian functors, and monoidal cartesian transformations.

We can give a more explicit description to make the idea clear. A monoidal $C$-category $\mathcal{F}$ has a cartesian multiplication
\[ \otimes : \mathcal{F} \times \mathcal{F} \to \mathcal{F} \]
and a cartesian unit
\[ e : 1 \to \mathcal{F} \]
along with cartesian associators, unitors and symmetry satisfying the appropriate coherence axioms. We can express this coherence as follows: For $I \in C$, the fiber $\mathcal{F}^I$ is a monoidal category, where the tensor product and unit are the components
\[ \otimes_I : \mathcal{F}^I \times \mathcal{F}^I \to \mathcal{F}^I \]
and
\[ e_I : 1 \to \mathcal{F}^I \]
of the cartesian functors $\otimes$ and $e$. If $\mathcal{F}^\varphi$ is an inverse image functor for $\varphi : I \to J$ in $C$, then $\mathcal{F}^\varphi \times \mathcal{F}^\varphi$ is an inverse image functor as well, and we may define a canonical isomorphism, natural for $A, B \in \mathcal{F}^J$,
\[ \mathcal{F}^\varphi A \otimes I \mathcal{F}^\varphi B \cong \mathcal{F}^\varphi (A \otimes_J B) \]
and similarly
\[ \mathcal{F}^\varphi e_J \cong e_I. \]
This must give a strong monoidal structure on $\mathcal{F}^\varphi$ with respect to the induced monoidal structure on the fibers. Given $\varphi : I \to J$, $\psi : J \to K$ and restrictions $\mathcal{F}^\varphi$, $\mathcal{F}^\psi$, $\mathcal{F}^{\varphi\psi}$, and $\mathcal{F}^{\text{id}}$, the canonical isomorphisms
\[ \mathcal{F}^{\varphi\psi} \cong \mathcal{F}^\psi \mathcal{F}^\varphi \]
and
\[ \mathcal{F}^{\text{id}} \cong \text{id} \]
are monoidal transformations.

Conversely, suppose we are given monoidal structures on the fibers. If every restriction functor in some cleavage has the structure of a strong monoidal functor, and the pseudofunctoriality constraints are all monoidal transformations, then the monoidal structures on the fibers are induced by a monoidal $C$-category structure on $\mathcal{F}$. In fact, the monoidal constraints on the distinguished $\mathcal{F}^\varphi$ correspond to pseudonaturality constraints for the $\otimes_I$ and $e_I$, and the monoidal coherence of the pseudofunctoriality constraints make the associators, unitors, and symmetry into modifications.

Given monoidal $C$-category $\mathcal{F}$ and $\mathcal{G}$, a monoidal $C$-functor $F : \mathcal{F} \to \mathcal{G}$ is a cartesian functor, with monoidal constraints of whatever flavor on the components, such that for restriction functors $\mathcal{F}^\varphi$ and $\mathcal{G}^\varphi$, the isomorphisms $F^I \mathcal{F}^\varphi \cong \mathcal{G}^\varphi F^J$ are monoidal transformations.
8.7. The external product. Shulman [Shu08] has shown that the definition of monoidal $C$-category we have given is equivalent to a somewhat different one. He defines a \textit{monoidal fibration} to be a fibred $C$-category $p : \mathcal{F} \to C$, such that $p$ is strict monoidal and each functor $T \otimes -$ preserves cartesian arrows. The model case is when $\mathcal{F} = \mathcal{C}$, so that $\int \mathcal{C}$ is the $C$ category of arrows $\text{cod} : \text{Arr}(\mathcal{C}) \to C$. For arrows $f_1 : X_1 \to I_1$, we have an \textit{external product} $f_1 \times f_2 : X_1 \times X_2 \to I_1 \times I_2$, and the projection satisfies $\text{cod}(f_1 \times f_2) = \text{cod} f_1 \times \text{cod} f_2$.

Given an external product $\boxtimes : \int \mathcal{F} \times \int \mathcal{F} \to \int \mathcal{F}$ making $\mathcal{F}$ into a monoidal fibration, the \textit{internal product} $\otimes_1$ on the fiber $\mathcal{F}^I$ is obtained by taking

$$A \otimes_1 B = \mathcal{F}^{\Delta_1}(A \boxtimes B),$$

where $\Delta : I \to I \times I$ is the diagonal. If $\mathcal{C}$ is cartesian, then this sets up an equivalence between monoidal fibrations and monoidal $C$-categories. A monoidal functor $F : \mathcal{F} \to \mathcal{G}$ is then an ordinary monoidal functor, such that the identity $p_\mathcal{G} F = p_\mathcal{F}$ is a monoidal transformation.

If $\tau : \mathcal{C} \to \mathcal{F}$ is a Lawvere theory, then $\tau$ is commutative if the underlying functor $\int \tau : \int \mathcal{C} \to \int \mathcal{F}$ between the total categories can be improved to a strict monoidal functor. Since $\tau$, $p_\mathcal{G}$ and $p_\mathcal{F}$ are strict, it is clear that the identity $p_\mathcal{G} \tau = p_\mathcal{F}$ is a monoidal transformation. We just need to check that $\tau E \otimes -$ preserves cartesian arrows for every $E \in \text{Ob}(\int \mathcal{C})$. But we can just consider cartesian arrows of the form $\tau f$ for $f \in \int \mathcal{C}$, in which case $\tau E \otimes \tau f = \tau(E \times f)$ is cartesian because $\mathcal{C}$ is a monoidal fibration.

8.8. Distributive $C$-categories. If $\mathcal{V}$ is a closed monoidal category, then the functors $\mathcal{V} \otimes -$ : $\mathcal{V} \to \mathcal{V}$ are left adjoints, and must preserve colimits. Suppose $\mathcal{C}$ is a monoidal category which is not necessarily closed, and $y : \mathcal{C} \to \mathcal{E}$ is an embedding which preserves colimits in some set $\Phi$. We want to extend the monoidal structure on $\mathcal{C}$ to a closed monoidal structure on $\mathcal{E}$. It is clear that this will not be possible unless the functors $\mathcal{C} \otimes \phi : \mathcal{C} \to \mathcal{E}$ preserve colimits in $\Phi$ as well. We have already

If $\mathcal{V}$ is a closed monoidal category, then the functors $\mathcal{V} \otimes \phi : \mathcal{V} \to \mathcal{V}$ are left adjoints, and must preserve colimits. Suppose $\mathcal{C}$ is a monoidal category which is not necessarily closed, and $y : \mathcal{C} \to \mathcal{E}$ is an embedding which preserves colimits in some set $\Phi$. We want to extend the monoidal structure on $\mathcal{C}$ to a closed monoidal structure on $\mathcal{E}$. It is clear that this will not be possible unless the functors $\mathcal{C} \otimes \phi : \mathcal{C} \to \mathcal{E}$ preserve colimits in $\Phi$ as well. This condition is precisely the compatibility condition on a monoidal sketch. For a fibered $C$-category, we also have colimits which are indexed by arrows in $\mathcal{C}$, and for an bifibered sketch we need a comparability condition with respect to these colimits as well.

Let $\mathcal{F}$ be a monoidal $C$-category which is bifibered, and let $\mathcal{F}_\phi$ be a left extension functor for $\phi : I \to J$. Given $A, B \in \mathcal{F}^I$, the tensor product of the opcartesian arrows $A \to \mathcal{F}_\phi A$ and $B \to \mathcal{F}_\phi B$ over $\phi$ is an arrow $A \otimes_1 B \to \mathcal{F}_\phi A \otimes_1 \mathcal{F}_\phi B$ over $\phi$. By the universal property of left extensions, we may define a canonical map

$$\mathcal{F}_\phi (A \otimes_1 B) \to \mathcal{F}_\phi A \otimes_1 \mathcal{F}_\phi B$$

and similarly a canonical map $\mathcal{F}_\phi e_1 \to e_J$. These give every left extension a canonical oplax monoidal structure. Given a restriction $\mathcal{F}^\phi$, which has a canonical strong monoidal structure, we can define a canonical comparison map

$$\mathcal{F}_\phi (\mathcal{F}^\phi T' \otimes_1 T) \to \mathcal{F}_\phi \mathcal{F}^\phi T' \otimes_1 \mathcal{F}_\phi T \to T' \otimes_1 \mathcal{F}^\phi T,$$
and we say that the projection formula holds if this comparison map is an isomorphism.

**Definition 8.1.** Let \( T \) be a monoidal bifibered \( C \)-category. We say that \( T \) is **distributive** if the projection formula

\[
\mathcal{T}_\varphi(T^0 \otimes 1 T) \cong T^0 \otimes J \mathcal{T}_\varphi T
\]

holds for all \( T^0 \in \mathcal{T}^0, T \in \mathcal{T}^I \) and \( \varphi : I \to J \).

**Definition 8.2.** A distributive \( C \)-sketch consists of a \( C \)-sum sketch \((T, \varphi)\) which is also a distributive monoidal \( C \)-category, such that each fiber is a monoidal sketch with respect to the internal monoidal structure.

**8.9. Tensor product of models.** Suppose \( T \) is a distributive \( C \)-sketch. For each fiber we have a monoidal structure \((\otimes I, e_I)\) given by the Day convolution. The functors \((\otimes I, e_I)\) are components of cartesian functors \(\otimes : \text{Mod}^T \times \text{Mod}^T \to \text{Mod}^T\) and \(e : 1 \to \text{Mod}^T\).

To see this, suppose we are given \( \varphi : I \to J \) and left extension and restriction functors \( T_\varphi \) and \( T_{\varphi^*} \). In the adjunction \( T_\varphi \dashv T_{\varphi^*} \), the unit and counit are monoidal transformations, where we consider \( T_\varphi \) and \( T_{\varphi^*} \) with their canonical oplax monoidal structures. It follows that in the induced adjunction \( \text{Mod}^{T_\varphi} \dashv \text{Mod}^{T_{\varphi^*}} \), the functors are lax monoidal and the unit and counit are monoidal transformations. By Kelly’s doctrinal adjunction theorem [Kel74], we can conclude that the left adjoint \( \text{Mod}^{T_\varphi} \) is **strong** monoidal. Thus we have provided a canonical strong monoidal structure for every restriction. Moreover, the pseudofunctoiality constraints of \( \text{Mod}^T \) are induced by the pseudofunctoiality constraints of an opcleavage for \( T \) and thus inherit the monoidal coherence.

In fact, each restriction \( \varphi^* = \text{Mod}^{T_{\varphi^*}} \) is actually a closed functor. For \( \varphi : I \to J \) and \( M, N \in \text{Mod}^{T_\varphi} \), we have

\[
(\varphi^*[M, N])^T_I = [M, N]^T_{\varphi^*T_I}
\]

\[
\int_{T^0 \in \mathcal{T}^I} [M^{T^0}, N^{T^0 \otimes J \varphi^* T_I}]
\]

\[
\cong \int_{T^0 \in \mathcal{T}^I} [M^{T^0}, N^\varphi(\varphi^* T^0 \otimes T_I)]
\]

by the projection formula. But then by change of variables this is canonically isomorphic to

\[
\int_{T^0 \in \mathcal{T}^I} [M^{\varphi^* T^0}, N^{\varphi^*(T^0 \otimes T_I)}] = [\varphi^* M, \varphi^* N]_J^T.
\]

**8.10. Commutative theories.** Recall that if \( C \) has finite limits, then \( C = \text{Arr}(C) \) is a cartesian distributive \( C \)-category. We say that a Lawvere theory \( \tau : C \to \mathcal{T} \) is **commutative** if \( \tau \) enriches to a strict monoidal \( C \)-functor, where the monoidal structure on \( C \) is cartesian. The monoidal \( C \)-category \( \mathcal{T} \) inherits distributivity from \( C \).

**Lemma 8.1.** Let \( \tau : C \to \mathcal{T} \) be a commutative Lawvere theory. Then \( \mathcal{T} \) is a distributive \( C \)-category. For \( f : I \to J \) in \( C^J \) and \( T \in \mathcal{T}^I \), we have a natural isomorphism

\[
\tau^J f \otimes_1 T \cong \mathcal{T}_f \mathcal{T}_f^I T.
\]
In particular, each functor $T \otimes J : \mathcal{F}^J \rightarrow \mathcal{F}^J$ is sketchy, and $\mathcal{F}$ is a distributive $\mathcal{C}$-sketch.

**Proof.** We need to check that the projection formula holds. Each component of $\tau$ is identity on objects, and we need to show that for each $\varphi : I \rightarrow J$ in $\mathcal{C}$, $E' \in \mathcal{C}_{/J}$ and $E \in \mathcal{C}_{/I}$, the canonical map

$$\mathcal{F}_\varphi(\mathcal{F}^\tau J E' \otimes I \tau I E) \rightarrow \tau J E' \otimes J \mathcal{F}_\varphi \tau I E$$

is an isomorphism. But since $\tau$ is strict monoidal and bicartesian, this essentially reduces to the projection formula for $\mathcal{C}$, which is standard.

A consequence of the projection formula is that

$$\tau J f \otimes J T \cong \tau J (\mathcal{F}_f \id_I) \otimes J T$$

$$\cong \mathcal{F}_f (\tau I \id_I) \otimes J T$$

$$\cong \mathcal{F}_f \mathcal{F}^T,$$

again since $\tau$ is bicartesian and strict monoidal.

In particular, the functors $T \otimes J$ are sketchy because $\mathcal{F}_f$ and $\mathcal{F}^T$ are sketchy, and every $T$ is of the form $\tau J f$ for some $f$. □

**8.11. Free models.** Let $\tau : \mathcal{C} \rightarrow \mathcal{F}$ be a commutative Lawvere theory. The free model functor $\text{Mod}_\tau$ is obtained by the composition

$$\text{Sh}_{\mathcal{C}_{/I}} \overset{\iota}{\rightarrow} \text{Psh}_{\mathcal{C}_{/I}} \overset{\tau I}{\rightarrow} \text{Psh}_{\mathcal{F}_{/I}} \overset{r}{\rightarrow} \text{Mod}_{\mathcal{F}_{/I}}.$$

The Day convolution on $\text{Psh}_{\mathcal{C}_{/I}}$ is easily seen to be the cartesian product. Since the cartesian product is preserved by sheafification, the resulting monoidal structure on $\text{Sh}_{\mathcal{C}_{/I}}$ is also the cartesian product. Since the inclusion of sheaves into presheaves is a right adjoint, it preserves the cartesian product, and thus the first functor in this sequence is strong monoidal.

Since $\tau I$ is strong monoidal, the left Kan extension $\text{Psh}_{\mathcal{F}_{/I}}$ is strong monoidal with respect to the Day convolution [DS95]. Finally, the reflection $r$ sending presheaves into models is strong monoidal, because that is how we defined the monoidal structure on models. Thus the free model functor $\text{Mod}_\tau$ is strong monoidal.

The reader can verify that each functor in this sequence is the component of a strong monoidal $\mathcal{C}$-functor.

**8.12. Extension to $\mathcal{E}$.** Let $\tau : \mathcal{C} \rightarrow \mathcal{F}$ be a commutative Lawvere theory. We have shown that $\mathcal{F}$ is a sketchy distributive fibration, and therefore gives rise to a monoidal $\mathcal{C}$-cosmos $\text{Mod}_\mathcal{F}$. We know that $y_\tau \text{Mod}_\mathcal{F}$ is the category of models for a large theory $\mathcal{P} : \mathcal{C} \rightarrow \mathcal{F}$, and $\mathcal{P}$ can be made strict monoidal using the fact that the free model functor is strong monoidal. Formally, it follows that $\text{Mod}_\mathcal{F}$ admits an essentially unique closed monoidal structure extending that of $\text{Mod}_\mathcal{F}$.

To avoid size issues and excessive abstraction, we can simply describe this structure explicitly. The objects of $y_\tau \text{Mod}_\mathcal{F}$ are cartesian functors (pseudonatural transformations) from $\mathcal{P}$ to $\text{Mod}_\mathcal{F}$. Given $M, N \in y_\tau \text{Mod}_\mathcal{F}$, the tensor product is defined pointwise for $f \in \mathcal{P}^I$ by

$$(M \otimes_P N)(f) = M(f) \otimes I N(f),$$
and the internal hom by

\[ [M,N]_P(f) = [M(f), N(f)]_I. \]

These expressions are pseudonatural in \( f \) because \( M \) and \( N \) are pseudonatural in \( f \) and the restrictions in \( \text{Mod}^\mathcal{P} \) are strong monoidal and closed. The unit and counit for the adjunction \( M \otimes_P - \dashv [M,-] \) have as components the unit

\[ \text{ins}_{M(f)} : N(f) \to [M(f), M(f) \otimes I, N(f)]_I \]

and counit

\[ \text{ev}_{M(f)} : M(f) \otimes_I [M(f), N(f)]_I \to N(f) \]

of the adjunction \( M(f) \otimes_I - \dashv [M(f),-]_I \).

We recall the notion of an \( \mathcal{E} \)-cosmos.

**Definition 8.3** ([Shu13]). A \( \mathcal{C} \)-cosmos is a monoidal \( \mathcal{C} \)-category \( V \) such that

1. The \( \mathcal{E} \)-category \( V \) has \( \mathcal{E} \)-sums and \( \mathcal{E} \)-products.
2. The fibers of \( V \) have small limits and colimits.
3. The fibers of \( V \) are closed.
4. The restriction functors closed.

**Theorem 8.2.** Let \( \tau : \mathcal{C} \to \mathcal{T} \) be a commutative Lawvere theory. Then the extension to \( \mathcal{E} \) of the category of models is an \( \mathcal{E} \)-cosmos.

**Proof.** The completeness and cocompleteness properties are true for any Lawvere theory. We have defined a closed monoidal structure on the fibers of \( y_* \text{Mod}^\mathcal{P} \), and it is straightforward to check that this gives \( \mathcal{V} \) the structure of a monoidal \( \mathcal{E} \)-cosmos. \qed

9. **Examples**

9.1. **Finitary algebraic theories.** Let \( \mathcal{C} = \aleph_0 \) be a skeletal category of finite sets and functions. We can define a topology on \( \mathcal{C} \) whose cylinders correspond to all finite sums \( m = m_1 + \cdots + m_k \). We then have \( \mathcal{E} = \text{Set} \). Every ordinary small category \( \mathcal{T} \) corresponds to a small fibered \( \mathcal{C} \)-category \( \mathcal{F} \), where \( \mathcal{F}^n = [n, \mathcal{T}] \). A cartesian functor \( \tau : \mathcal{C} \to \mathcal{E} \) is bicartesian precisely when \( \tau^1 : \mathcal{C} \to \mathcal{T} \) preserves finite sums. Thus we recover the usual notion of a Lawvere theory.

If \( \mathcal{D} \) is an ordinary category with finite products, then a multiplicative functor \( M : \mathcal{F}^{\text{op}} \to \mathcal{D} \) is the same as an functor \( M : \mathcal{T}^{\text{op}} \to \mathcal{D} \) preserving finite products in the ordinary sense. In particular, if \( I \) is any set, then the category \( \text{Mod}^{\mathcal{F},I}_\mathcal{E} \) is equivalent to the category of models of the ordinary Lawvere theory \( \mathcal{T} \) in \( \text{Set}^I \).

9.2. **Lextensive categories.** Let \( \mathcal{C} \) be any category with finite limits and \( \kappa \)-ary disjoint sums which are stable under pullback. Then there is an \( \kappa \)-extensive topology \( \Phi_\kappa \) on \( \mathcal{C} \) generated by the \( \kappa \)-ary sums \( \sum_{i \in \lambda} I_i \), with \( I_i \in \mathcal{C} \) and \( \lambda < \kappa \). If \( \tau : \mathcal{C} \to \mathcal{F} \) is a Lawvere theory, then a model of \( \mathcal{F}^I \) is precisely a functor \( M : (\mathcal{F}^I)^{\text{op}} \to \text{Set} \) preserving \( \kappa \)-ary products.

In our main application, we take \( \mathcal{C} \) to be the category of standard Borel spaces (see [Kur48, Mac57]). For example, let \( \text{Borel} \) be the category of standard Borel spaces, whose objects are

\[ \text{Borel} = \{0, 1, 2, \ldots, \aleph, \mathbb{R}\}, \]
each considered as a measurable space with the \( \sigma \)-algebra of Borel sets. The maps are the Borel-measurable functions. This category has countable limits and countable sums which are disjoint and stable under pullback, so it is a countable lextensive category.

If we consider \( \text{Borel} \) to be a finitary extensive site, then sheaves on \( \text{Borel} \) include interesting spaces like \( \mathbb{R}^b \), where \( \mathbb{R}^X = L^\infty(X, \mathbb{R}) \) is the set of bounded measurable functions from \( X \) to \( \mathbb{R} \). It is clear that the presheaf \( \mathbb{R}^b \) preserves finite products but not countable products.

9.3. Integral kernels. Let \( Y \) be a standard Borel space, and let \( M(Y) \) be the space of signed Radon measures on \( Y \). Recall that a function \( k : X \to M(Y) \) is a measurable kernel if, for each Borel set \( E \), the map \( x \mapsto k(x, E) \) is Borel.

By standard results on measurable kernels (see, e.g. [Dyn61, Chapter 1, Lemma 1.7]), the composition of kernels is as well-behaved as one might expect. We can thus define a fibered category \( \text{FreeLin} \) of free linear spaces, whose objects are measurable bundles of free linear spaces generated by the standard Borel spaces.

We first define the fibers of \( \text{FreeLin} \). If \( f_i : X_i \to I \) are Borel spaces over \( I \), then a bundle map \( k : X_1 \to MX_2 \) is a measurable kernel \( k : X_1 \to MX_2, \) such that \( \sup_{x_1} ||k(x_1)|| < \infty \) and

\[
k(x_1, x_2) = 1_{X_1 \times I, X_2}(x_1, x_2)k(x_1, x_2).
\]

That is, the kernel \( k \) is supported on the Borel set

\[
X_1 \times I, X_2 = \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}.
\]

We can compose kernels \( k : X_1 \to MX_2 \) and \( k' : X_2 \to MX_3 \) by the rule

\[
(k' \circ k)(x_1, x_3) = \int_{X_2} k(x_1, x_2)k'(x_2, x_3) dx_2,
\]

and the identity for this composition is the Dirac kernel \( \delta : X \to MX \) sending \( x \) to \( \delta_x \).

It is easy to see that if \( k' \) and \( k \) are bundle maps, then the composition \( k' \circ k \) is a bundle map as well. We define \( \text{FreeLin}_I \) to be the category whose objects are Borel spaces over \( I \) and whose arrows are bundle maps.

More generally, if \( f_i : X_i \to I_i \) are bundles of Borel spaces, and \( \varphi : I_1 \to I_2 \) is any Borel map, then say that \( k : X_1 \to MX_2 \) is a bundle map over \( \varphi \) if it is a bundle map with respect to the maps to \( I_2 \). We will notate this situation by

\[
X_1 \xrightarrow{k} MX_2 \\
I_1 \xrightarrow{\varphi} I_2.
\]

Thus we obtain a split opfibration over \( \text{Borel} \). For each \( \varphi : I \to J \) and \( f : X \to I \) we have a canonical opcartesian arrow \( \delta : X \to MX \) over \( \varphi \).
9.4. **Pushforward of measures.** If \( h : X \to Y \) is a Borel map, then family of Dirac masses

\[
M(h)(x, y) = \delta(h(x), y)
\]

is a measurable kernel \( M(h) : X \to M(Y) \). Thus \( M \) defines a functor

\[
M : \text{Borel} \to \text{FreeLin}_1,
\]

Suppose now that

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow{f} & & \downarrow{g} \\
I & \xrightarrow{\varphi} & J
\end{array}
\]

is a commutative diagram of Borel maps. One can check that \( M(h) : X \to M(Y) \) is a bundle map over \( \varphi \). Thus \( M \) extends to a \textbf{Borel}-functor \( \tau : \text{Arr} (\text{Borel}) \to \text{FreeLin} \). The functor \( \tau \) preserves op-cartesian arrows simply because the functor \( M \) preserves identities.

We claim that \( \tau \) preserves cartesian arrows as well. Suppose given a cartesian diagram

\[
\begin{array}{ccc}
X \times_I J & \xrightarrow{p_X} & X \\
\downarrow{p_I} & & \downarrow{g} \\
J & \xrightarrow{\varphi} & I
\end{array}
\]

of Borel spaces, and let

\[
\begin{array}{ccc}
Z & \xrightarrow{k} & MX \\
\downarrow & & \downarrow \\
J & \xrightarrow{\varphi} & I
\end{array}
\]

be any bundle map over \( \varphi \) for some Borel map \( f : Z \to J \). We need to check that \( k \) factors uniquely through the bundle map

\[
\begin{array}{ccc}
X \times_I J & \xrightarrow{Mp_X} & MX \\
\downarrow & & \downarrow \\
J & \xrightarrow{\varphi} & I
\end{array}
\]

Suppose that \( \tilde{k} : Z \to M(X \times_I J) \) is a bundle map. Then, for each \( z \in Z \), the measure \( \tilde{k}(z) \) is supported on the set \( \{(x, j) \in X \times_I J : g(z) = j\} \), and must be of the form

\[
\tilde{k}(z, x) = a(z, x) \otimes \delta(g(z), j),
\]

where \( a(z, x) \) is a kernel determined by

\[
a(z, x) = \int_J \tilde{k}(z, x, j) \, dj = Mp_X \circ \tilde{k}(z, x).
\]

This shows that \( \tilde{k} \) is uniquely determined by the factorization \( k = Mp_X \circ \tilde{k} \). Now, given \( k : Z \to MX \), define the kernel \( \tilde{k} : Z \to M(X \times_I J) \) by

\[
\tilde{k}(z, x, j) = k(z, x) \otimes_I \delta(f(z), j),
\]

where for measures \( \mu \in M(X) \) and \( \nu \in M(J) \), the notation

\[
d\mu(x) \otimes_I d\nu(y) = d(\mu \times \nu)(x,y)|_{X \times_I J}
\]
indicates the restriction to $X \times J$ of the product measure. Then we can compute

$$(M_{p_X} \circ \tilde{k})(z, x) = \int_{X \times J} k(z, x') \otimes \delta(f(z), j') \delta(x', x) \, dx' \, dj'$$

$$= \int_{X \times J} 1_{\varphi(j') = g(x')} 1_{j'' = f(z)} k(z, x') \otimes \delta(f(z), j') \delta(x', x) \, dx' \, dj'$$

$$= \int_{X \times J} k(z, x') \otimes \delta(f(z), j') \delta(x', x) \, dx' \, dj'$$

because $k$ is a bundle map and satisfies the identity

$$1_{\varphi(j') = g(x')} 1_{j'' = f(z)} k(z, x') = 1_{\varphi(f(z)) = g(x')} k(z, x') = k(z, x').$$

But the last integral is just $k(z, x)$, so $k$ factors as $k = M_{p_X} \circ \tilde{k}$, as desired.

We have shown that $\tau$ preserves precartesian arrows. Because $\tau$ is identity on objects, it follows that there are enough precartesian arrows that $\text{FreeLin}$ is a fibred Borel-category. Moreover, the functor $\tau$ is bicartesian.

9.5. Tensor product of measures. We have shown that the identity-on-objects functor $\tau: \text{Arr}(\text{Borel}) \to \text{FreeLin}$ is bicartesian. For each $g: Y \to I$, one checks that the functor sending a space $f: X \to I$ in $\text{Borel}_{/I}$ to the collection of bundle maps $k: X \to MY$ is a sheaf (for the finitary extensive topology) and thus $\tau$ is a Lawvere theory.

The theory $\tau$ is moreover commutative in the sense that the underlying functor $\tau: \text{Arr}(\text{Borel}) \to \int \text{FreeLin}$ is strict monoidal. That is, for spaces $f_i: X_i \to I_i$ and $g_i: Y_i \to J_i$ and bundle maps $k_i: X_i \to MY_i$ over maps $\varphi_i: I_i \to J_i$, we can extend (any given) cartesian tensor product on the arrow category $\text{Arr}(\text{Borel})$ to a tensor product

$$
\begin{array}{ccc}
X_1 \times X_2 & \xrightarrow{k_1 \otimes k_2} & M(Y_1 \times Y_2) \\
\downarrow & & \downarrow \\
I_1 \times I_2 & \xrightarrow{\varphi_1 \times \varphi_2} & J_1 \times J_2
\end{array}
$$

on the total category of $\text{FreeLin}$. The monoidal structure is defined in terms of product measures. For each $(x_1, x_2) \in X_1 \times X_2$, we let

$$k_1 \otimes k_2(x_1, x_2, y_1, y_2) = k_1(x_1, y_1) \otimes k_2(x_2, y_2),$$

which defines a measure on $Y_1 \times Y_2$ in the usual way. The kernel thus defined is a bundle map over $\varphi_1 \times \varphi_2$. If $k_i(x_i, y_i) = \delta(h_i(x_i), y_i)$ for Borel maps $h_i: X_i \to Y_i$ over the $\varphi_i$, then

$$k_1 \otimes k_2(x_1, x_2, y_1, y_2) = \delta(h_1(x_1), y_1) \otimes \delta(h_2(x_2), y_2)$$

$$= \delta((h_1 \times h_2)(x_1, x_2), (y_1, y_2)).$$

This shows that the functor $\tau$ is strict monoidal, so the corresponding Lawvere theory is commutative by our definition.

9.6. Concrete spaces. A sheaf $F$ over the extensive site $\aleph_0$ is determined up to isomorphism by its underlying set, because we have isomorphisms $F(n) \cong F(1)^n$, natural in $n$, and the naturality condition then determines $F(f)$ for any $f$ in $\aleph_0(n, m)$.

For a sheaf over $\text{Borel}$, on the other hand, we have a canonical map

$$F(X) \to F(1)^{|X|},$$
where $|X|$ is the set of points $p : 1 \to X$. However, this map will almost never be an isomorphism. On the other hand, it may happen that it is at least injective for each $X$, and in this case we can view the sheaf $F$ as a set $F(1)$ together with a choice, for each $|X|$, of a set of admissible curves $c : |X| \to F(1)$. A map of spaces $f : F \to G$ is then a map $f_1 : F(1) \to G(1)$ between the underlying sets which that sends admissible curves in $F$ to admissible curves in $G$. We will call such a space a concrete space.

Similarly, given a Lawvere theory $\tau : \text{Arr}(\text{Borel}) \to \text{FreeLin}$, we will say that a model $M$ of $\tau$ is concrete when the underlying sheaf $M\tau$ is concrete.

10. Some linear spaces

In what follows we define

$$\text{Space} = \text{Sh}(\text{Borel})$$

to be the topos of sheaves on the finitary extensive Borel site, and

$$\text{Lin} = \text{Mod}(\text{FreeLin})$$

to be the category of models for the theory of linear spaces defined above.

10.1. Separable Banach spaces. In a separable Banach space $V$, there is a well-behaved notion of measurability. If $X$ is a standard Borel space, a function $f : X \to V$ is said to be measurable precisely when $l \circ f : X \to \mathbb{R}$ is measurable for every bounded linear functional $l : V \to \mathbb{R}$. If in addition $f$ is bounded in norm, the Bochner integral

$$\int f(x) \, d\mu(x)$$

exists for any finite measure $\mu$ and is determined by the requirement that

$$l \left( \int f(x) \, d\mu(x) \right) = \int l(f(x)) \, d\mu(x)$$

for every $l$ in the Banach space dual $V'$ of $V$. Let $\widetilde{V}(X)$ denote the set of measurable bounded functions from $X$ to $V$. If $k : Y \to MX$ is a kernel, then we define an action $\widetilde{V}(k) : \widetilde{V}(X) \to \text{Set}(Y,V)$ by the Bochner integral

$$\widetilde{V}(k)[f](y) = \int f(x) \, k(y,x) \, dx.$$ 

The function $\widetilde{V}(k)[f]$ is a measurable function of $Y$ because for $l \in V'$ we have

$$l(\widetilde{V}(k)[f](y)) = \int l(f(x)) \, k(y,x) \, dx,$$

and this is a composition of measurable kernels because $x \mapsto l(f(x))$ is measurable.

The function $\widetilde{V}(k)f$ is also bounded by the triangle inequality for the Bochner integral. Thus we have a map

$$\widetilde{V}(k) : \widetilde{V}(X) \to \widetilde{V}(Y),$$

and it is easy to check that the assignment $X \mapsto \widetilde{V}(X)$ defines a product-preserving functor from the category $\text{FreeLin}_{\tau}^{\text{op}}$ to $\text{Set}$. 
Now suppose that $T : V \to W$ is an bounded linear map between two separable Banach spaces. Then composition with $T$ clearly preserves the measurable curves, and we define

$$\tilde{T} : \tilde{V} \to \tilde{W}$$

to be composition with $T$. This $\tilde{T}$ is a natural transformation, because Bochner integration commutes with bounded linear maps. Thus we obtain a functor

$$B : \text{SepBan} \to \text{Lin}$$
on the category $\text{SepBan}$ of separable Banach spaces, sending each space $V$ to the linear space $\tilde{V}$ defined above.

**Proposition 10.1.** The functor

$$\tilde{(-)} : \text{SepBan} \to \text{Lin}$$

sending a separable Banach space $V$ to the linear space

$$\tilde{V}(X) = \{ c : X \to V \mid c \text{ is measurable and } \sup_x \|c(x)\| < \infty \}$$

is full and faithful.

**Proof.** Faithfulness is obvious, so it remains to show that the functor is full. Suppose that $\tilde{T} : \tilde{V} \to \tilde{W}$ is a linear space map. Since $\tilde{V}$ and $\tilde{W}$ are concrete, the map $\tilde{T}$ is determined by its action on the underlying sets. Since $\tilde{T}$ commutes with integration, it preserves finite linear combinations, which means it is linear. Moreover, it is bounded, because the image of any bounded sequence of points must be a bounded sequence of points. $\square$

**10.2. Spaces of measures.** The Yoneda embedding gives some trivial examples of linear spaces, namely the spaces $MY$ of Radon measures on $Y$, where $Y$ is a Borel space. We have by definition that

$$\text{Lin}(MY, \mathbb{R}) \cong \text{Borel}(Y, U\mathbb{R}),$$

where $\mathbb{R}$ is the linear space corresponding to the separable Banach space $\mathbb{R}$ and $U\mathbb{R}$ is its underlying space. The right hand side is the set of bounded measurable maps from $Y$ to $\mathbb{R}$.

On the other hand, let $M_{\text{Ban}}Y$ be the Banach space of Radon measures with the total variation norm. The space of arbitrary Banach space maps $I : M_{\text{Ban}}Y \to \mathbb{R}$ contains many unpleasant characters. For example, let $g : Y \to \mathbb{R}$ be an arbitrary bounded function, not necessarily measurable, and let $\mathbb{R}^Y \subset M_{\text{Ban}}Y$ be the linear subspace consisting of finitely-supported measures on $Y$. For any measure $\mu$ in $\mathbb{R}^Y$, the integration

$$(f \mu)(\mu) := \int f(y)d\mu(y)$$

is perfectly well-defined, because $\mu$ is finitely supported. Moreover, it is clear that $f \mu : \mathbb{R}^Y \to \mathbb{R}$ is a bounded linear functional, because $g$ is bounded. By the Hahn-Banach theorem, it extends to a bounded linear functional $\tilde{f} \mu : M_{\text{Ban}}Y \to \mathbb{R}$. Unless $g$ is chosen to be measurable, the function $\tilde{f} \mu$ does not correspond to a morphism in $\text{Lin}(MY, \mathbb{R})$. In fact, composing with $\tilde{f} \mu$ destroys the measurability of curves. We have

$$(\tilde{f} \mu) \circ \delta(y) = g(y)$$
by construction, and \( g : Y \to \mathbb{R} \) is not measurable. Essentially, we need to choose the weak topology on \( MY \) instead of the strong topology, by defining the dual of \( MY \) to be \( \text{Borel}(Y, \mathbb{R}) \). However, the resulting locally convex space will not be bornological, and in particular we have left the setting of “convenient” spaces.

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