An exactly solvable toy model whose spectrum is topological

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Abstract. In an attempt to regularize a previously known exactly solvable model [Yang and Zhang, Eur. J. Phys. \textbf{40}, 035401 (2019)], we find yet another exactly solvable toy model. The interesting point is that the spectrum of the model is topologically invariant. Precisely, while the Hamiltonian of the model is parameterized by a function $f(x)$ defined on $[0, \infty)$, the spectrum depends only on the end values of $f$, i.e., $f(0)$ and $f(\infty)$. This model can serve as a good exercise in quantum mechanics at the undergraduate level.

PACS numbers: 03.65.-W, 03.65.Ge
1. Introduction

Recently, in studying the quench dynamics of a Bloch state in a one-dimensional tight-binding ring [1, 2, 3], an exactly solvable model was devised and its dynamics was solved. Later, the eigenstates and eigenenergies of the model were solved in closed and neat forms [4].

The model is very simple. It consists of infinitely many levels \( \{|n\rangle, n \in \mathbb{Z} \} \), with the Hamiltonian being
\[
H = \sum_{n=-\infty}^{\infty} n\Delta |n\rangle\langle n| + g \sum_{n_1, n_2 = -\infty}^{\infty} |n_1\rangle\langle n_2|. \tag{1}
\]
The first term is diagonal. It simply means that the levels are equally spaced with the spacing being \( \Delta \). The second term is off-diagonal. Its peculiarity is that it couples two arbitrary levels, regardless of their distance in energy, with a constant strength \( g \). This makes it a rank-1 operator, a fact which was believed to be responsible for the exact solvability of the model.

While the constant coupling simplifies the mathematics, from a physical point of view, it is too restrictive—it is only realizable with a contact potential as in [1, 2, 3]. In a generic case, the coupling between two basis states should decay to zero as their energy difference goes to infinity.‡ We are thus motivated to “regularize” the off-diagonal coupling and consider the following Hamiltonian
\[
H_f = \sum_{n=-\infty}^{\infty} n\Delta |n\rangle\langle n| + \sum_{n_1, n_2 = -\infty}^{\infty} f(n_1 - n_2)|n_1\rangle\langle n_2|, \tag{2}
\]
where \( f(x) \) is an even function defined on the real line satisfying the condition
\[
\lim_{x \to \pm \infty} f(x) = 0.
\]

The coupling term is now no longer a rank-1 perturbation, and the approach in [4] fails. But it is still regular. Actually, it is in the form of a Toeplitz matrix and its action on a wave function is to convolve it with the function \( f \). This observation suggests the Fourier transform, and we easily solve the eigenvalues as
\[
\varepsilon_m = m\Delta + f(0), \quad m \in \mathbb{Z}. \tag{3}
\]
Here it is interesting that while the Hamiltonian [2] is defined with the function \( f \) as a parameter as a whole, its spectrum depends only on the end value \( f(0) \).

The Fourier transform approach works also for the original Hamiltonian [1]. We thus find that for the function \( f \) we can drop the condition that it converges to zero as \( x \to \pm \infty \), and require just that the limit exists. In this case, the spectrum is
\[
\varepsilon_m = m\Delta + f(0) - f(\infty) + \frac{\Delta}{\pi} \arctan \left( \frac{\pi f(\infty)}{\Delta} \right), \quad m \in \mathbb{Z}. \tag{4}
\]
It depends only on the end values \( f(0) \) and \( f(\infty) \). For the original Hamiltonian [1], \( f(0) = f(\infty) = g \), and we recover the results in [4].

‡ Mathematically, this is the Riemann-Lebesgue lemma.
2. Solution of the eigenvalues

First of all, as in [4], we notice that the off-diagonal part is invariant under the translation \(|n⟩ → |n + 1⟩\), which implies that the energy spectrum should be equally spaced with the equal gap being \(Δ\). Formally, let us define the translation or raising operator \(T = \sum_{n=−∞}^{∞} |n + 1⟩⟨n|\). We claim that if \(|ψ⟩\) is an eigenstate of \(H_f\) with eigenvalue \(ε\), then \(T|ψ⟩\) is an eigenstate with eigenvalue \(ε + Δ\). This is because we have the commutation relation \([H_f, T] = ΔT\) and hence

\[
H_f T|ψ⟩ = (TH_f + T)|ψ⟩ = (Tε + ΔT)|ψ⟩ = (ε + Δ)T|ψ⟩.
\]

(5)

To determine the spectrum, we have to consider the Schrödinger equation \(H_f |ψ⟩ = ε |ψ⟩\). Assuming that \(|ψ⟩ = \sum_{n} a_n |n⟩\), we have

\[
ε a_n = nΔa_n + \sum_{m=−∞}^{∞} f(n − m)a_m.
\]

(6)

The second term on the right hand side is in the form of convolution. This suggests the Fourier transform,

\[
A_k = \sum_{n=−∞}^{∞} e^{ikn}a_n, \quad F_k = \sum_{n=−∞}^{∞} e^{ikn}f(n).
\]

(7)

For the left hand side of (6), we get \(εA_k\). For the right hand side, we get

\[
\sum_{n=−∞}^{∞} e^{ikn}Δn a_n + \sum_{n=−∞}^{∞} \sum_{m=−∞}^{∞} e^{ikn}f(n − m)a_m
\]

\[
= −iΔ \frac{∂}{∂k} \sum_{n=−∞}^{∞} e^{ikn}a_n + \sum_{n−m=−∞}^{∞} e^{ik(n−m)}f(n − m) \sum_{m=−∞}^{∞} e^{ikm}a_m
\]

\[
= −iΔ \frac{∂}{∂k} A_k + F_k A_k.
\]

(8)

We thus get the equation of \(A_k\),

\[
ε A_k = −iΔ \frac{∂}{∂k} A_k + F_k A_k.
\]

(9)

Let us first consider the special case of \(f(∞) = 0\). In this case, as long as \(f\) decays sufficiently fast, \(F\) is a regular function of \(k\), and we can solve \(A\) as

\[
A_k = A_0 \exp \left[ \frac{i}{Δ} \int_{0}^{k} dq (ε − F_q) \right].
\]

(10)

The value of \(ε\) is determined by the boundary condition of \(A_0 = A_{2π}\), which is equivalent to

\[
2πmΔ = 2πε − \int_{0}^{2π} F_k dk
\]

(11)

for some integer \(m\). But the integral here is simply \(2πf(0)\). We thus obtain the eigenvalues as in [3]. They are indeed equidistant. But what is interesting is that there is only dependence on the end value \(f(0)\).
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In the more general case of \( f(\infty) \neq 0 \), we can decompose the function into two parts as \( f = [f - f(\infty)] + f(\infty) \). The Fourier transform of \( f \) is then in the form of

\[
F_k = F_k^{(1)} + 2\pi f(\infty) \sum_{m \in \mathbb{Z}} \delta(k - 2\pi m),
\]

(12)

where the first term comes from \( f - f(\infty) \) while the delta functions from the constant part. We still have (9). But now \( A \) is discontinuous at \( k = 0 \). Integrating from \( 0^- \) to \( 0^+ \), we get

\[
0 = -i\Delta(A_0^+ - A_0^-) + 2\pi f(\infty) \int_{0^-}^{0^+} \delta(k) A(k) dk
\]

(13)

On the other hand, integrating from \( 0^+ \) to \( 2\pi^- \), we get

\[
A_0^- = A_{2\pi^-} = A_0^+ \exp \left[ \frac{i}{\Delta} \int_0^{2\pi} dk \left( \varepsilon - F_k^{(1)} \right) \right]
\]

(14)

From (13) and (14), we solve the spectrum in (11).

In hindsight, it is not difficult to recognize the reason why the spectrum depends only on the end values of \( f \). We note that (9) is the Schrödinger equation for a charged particle constrained in a ring which is pierced by a magnetic field. The Hamiltonian is \( \mathcal{K} = -i\Delta \partial / \partial k + F_k \), where \( k \in [0, 2\pi] \) is the angular variable and \( F_k \) is the vector field. Actually, \( H_f \) in (2) is just the representation of \( \mathcal{K} \) in the basis of the plane waves on the ring, i.e., if we identify level \( |n\rangle \) with the basis function \( e^{ink} / \sqrt{2\pi} \).

As is well-known and easily checked, (11) is invariant under the gauge transform

\[
A_k \to A_k e^{i\theta(k)}, \quad F_k \to F_k + \Delta \frac{\partial \theta}{\partial k},
\]

(15)

where \( \theta(k) \) is an arbitrary, smooth, and 2\( \pi \)-periodic function of \( k \). Under this gauge transform, the value of \( f \) at a generic point

\[
f(n) = \frac{1}{2\pi} \int_0^{2\pi} F_k e^{-ink} dk
\]

(16)

of course changes. However, the end values \( f(0) \) and \( f(\infty) \) are invariant. Conversely, any two \( f \)'s with the same end values can be converted into each other with a gauge transform, which leaves the spectrum invariant.

3. Conclusions and discussions

We have tried to regularize a previously known exactly solvable model which is considered to be short of generality. In doing so, we found that it belongs to a class of exactly solvable toy model. It is the Toeplitz property instead of the rank-1 property that has been generalized.

The interesting thing is that while the model is parameterized by a function \( f \), its spectrum depends only on the end values of this function. This topological characteristic
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is best understood by noticing that the model can be realized with a charged particle confined on a ring which is threaded by some magnetic flux. As long as the end values are fixed, different $f$’s just correspond to different choices of the gauge. But the energy spectrum of the system, as a physical observable, apparently should be gauge-independent.

We note that the original model finds use in the quench dynamics of a Bloch state with a contact potential [1][2][3]. Hopefully, the current model is relevant if the quench potential is a more general, extended one. But anyway, the model is of pedagogical value for students of quantum mechanics, as the solution is simple and the result is illuminating.

Acknowledgments

The authors are grateful to K. Jin and Y. Xiang for their helpful comments. This work is supported by the National Science Foundation of China under Grant No. 11704070.

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