ON FOURIER RE-EXPANSIONS

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ABSTRACT. We study an extension to Fourier transforms of the old problem on absolute convergence of the re-expansion in the sine (cosine) Fourier series of an absolutely convergent cosine (sine) Fourier series. The results are obtained by revealing certain relations between the Fourier transforms and their Hilbert transforms.

1. Introduction

In 50-s (see, e.g., [5] or in more detail [6, Chapters II and VI]), the following problem in Fourier Analysis attracted much attention:

Let \( \{a_k\}_{k=0}^{\infty} \) be the sequence of the Fourier coefficients of the absolutely convergent sine (cosine) Fourier series of a function \( f : \mathbb{T} = [-\pi, \pi) \to \mathbb{C} \), that is \( \sum |a_k| < \infty \). Under which conditions on \( \{a_k\} \) the re-expansion of \( f(t) \) (\( f(t) - f(0) \), respectively) in the cosine (sine) Fourier series will also be absolutely convergent?

The obtained condition is quite simple and is the same in both cases:

\[
\sum_{k=1}^{\infty} |a_k| \ln(k+1) < \infty.
\]

In this paper we study a similar problem for Fourier transforms defined on \( \mathbb{R}_+ = [0, \infty) \). Let

\[
F_c(x) = \int_0^\infty f(t) \cos xt \, dt
\]

be the cosine Fourier transform of \( f \) and

\[
F_s(x) = \int_0^\infty f(x) \sin xt \, dt
\]

be the sine Fourier transform of \( f \), each understood in certain sense.

Let

\[
\int_0^\infty |F_c(x)| \, dx < \infty,
\]

and hence

\[
f(t) = \frac{1}{\pi} \int_0^\infty F_c(x) \cos tx \, dx,
\]

or, alternatively,

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and hence

\[
(5) \quad f(t) = \frac{1}{\pi} \int_0^\infty F_s(x) \sin tx \, dx.
\]

1) Under which (additional) conditions on \( F_c \) we get (4),
or, in the alternative case,

2) under which (additional) conditions on \( F_s \) we get (2)?
The answer is given by the following theorem. To formulate it we turn to the Hilbert transform of an integrable function \( g \)

\[
(6) \quad Hg(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(t)}{x-t} \, dt,
\]
where the integral is understood in the improper (principal value) sense, as \( \lim_{\delta \to 0^+} \int_{|t-x|>\delta} \). It is not necessarily integrable, and when it is, we say that \( g \) is in the (real) Hardy space \( H^1(\mathbb{R}) \).

If \( g \in H^1(\mathbb{R}) \), then

\[
(7) \quad \int_{\mathbb{R}} g(t) \, dt = 0.
\]

It was apparently first mentioned in [8].

**Theorem 1.** In order than the re-expansion \( F_s \) of \( f \) with the integrable cosine Fourier transform \( F_c \) be integrable, it is necessary and sufficient that its Hilbert transform \( H F_c(x) \) be integrable.

Similarly, in order than the re-expansion \( F_c \) of \( f \) with the integrable sine Fourier transform \( F_s \) be integrable, it is necessary and sufficient that its Hilbert transform \( H F_s(x) \) be integrable.

2. Proof of Theorem [1]

Let (2) holds true. Then we can rewrite

\[
(8) \quad F_s(x) = \int_0^\infty \left[ \frac{1}{\pi} \int_0^\infty F_c(u) \cos tu \, du \right] \sin xt \, dt.
\]

The right-hand side can be understood in the \((C, 1)\) sense as

\[
\frac{1}{\pi} \lim_{N \to \infty} \int_0^N \left(1 - \frac{t}{N}\right) \int_0^\infty F_c(u) \cos tu \, du \sin xt \, dt.
\]

In virtue of (2) we can change the order of integration:
\[
\frac{1}{\pi} \lim_{N \to \infty} \int_0^\infty F_c(u) \int_0^N \left(1 - \frac{t}{N}\right) \cos tu \, \sin xt \, dt \, du = \frac{1}{\pi} \lim_{N \to \infty} \int_0^\infty F_c(u) \left(1 - \frac{t}{N}\right) \sin(u + x)t - \sin(u - x)t \] dt \, du.
\]

We now need the next simple formula

\[
\int_0^N (1 - \frac{t}{N}) \sin At \, dt = \frac{1}{A} - \frac{\sin NA}{NA^2}.
\]

Applying it yields

\[
F_s(x) = \frac{1}{\pi} \lim_{N \to \infty} \int_0^\infty F_c(u) \left[ \frac{1}{u + x} - \frac{\sin(u + x)N}{N(u + x)^2} \right] du - \frac{1}{\pi} \lim_{N \to \infty} \int_0^\infty F_c(u) \left[ \frac{1}{u - x} - \frac{\sin(u - x)N}{N(u - x)^2} \right] du = I_1 + I_2.
\]

Let us begin with \(I_2\). Substituting \(u - x = t\), we obtain

\[
I_2 = -\frac{1}{\pi} \lim_{N \to \infty} \int_0^\infty F_c(x + t) \left[ \frac{1}{t} - \frac{\sin Nt}{Nt^2} \right] dt.
\]

For \(I_1\), we first substitute \(u = -v\). Thus

\[
I_1 = \frac{1}{\pi} \lim_{N \to \infty} \int_{-\infty}^0 F_c(-v) \left[ \frac{1}{-v + x} - \frac{\sin(-v + x)N}{N(-v + x)^2} \right] dv
\]

\[
= -\frac{1}{\pi} \lim_{N \to \infty} \int_{-\infty}^0 F_c(-v) \left[ \frac{1}{v - x} - \frac{\sin(v - x)N}{N(v - x)^2} \right] dv
\]

\[
= -\frac{1}{\pi} \lim_{N \to \infty} \int_{-\infty}^0 F_c(v) \left[ \frac{1}{v - x} - \frac{\sin(v - x)N}{N(v - x)^2} \right] dv.
\]

The last equality follows from the evenness of \(F_c\). Substituting \(v - x = t\), we obtain

\[
I_1 = -\frac{1}{\pi} \lim_{N \to \infty} \int_{-\infty}^- F_c(x + t) \left[ \frac{1}{t} - \frac{\sin Nt}{Nt^2} \right] dt.
\]

Therefore,

\[
F_s(x) = -\frac{1}{\pi} \lim_{N \to \infty} \int_{-\infty}^\infty F_c(x + t) \left[ \frac{1}{t} - \frac{\sin Nt}{Nt^2} \right] dt.
\]

We are now in a position to apply the following result (see [17, Vol.II, Ch.XVI, Th. 1.22]; even more general result can be found in [15, Th.107]).

**Theorem A.** If \(\frac{|f(t)|}{1 + |t|}\) is integrable on \(\mathbb{R}\), then the \((C, 1)\) means

\[
-\frac{1}{\pi} \int_{-\infty}^\infty f(x + t) \left[ \frac{1}{t} - \frac{\sin Nt}{Nt^2} \right] dt
\]
converge to the Hilbert transform $\mathcal{H}f(x)$ almost everywhere as $N \to \infty$.

It follows from Theorem A that for almost all $x$

(11) $F_s(x) = \mathcal{H}F_c(x)$.

We remark that any integrable function satisfies the assumption of Theorem A.

Now, let (4) holds true. Then we can rewrite

(12) $F_c(x) = \int_0^\infty \left[ \frac{1}{\pi} \int_0^\infty F_s(u) \sin tu \, du \right] \cos xt \, dt$.

The right-hand side can be understood in the $(C,1)$ sense as

$$\frac{1}{\pi} \lim_{N \to \infty} \int_0^N (1 - \frac{t}{N}) \int_0^\infty F_s(u) \sin tu \, du \cos xt \, dt.$$ 

In virtue of (4) we can change the order of integration:

$$\frac{1}{\pi} \lim_{N \to \infty} \int_0^\infty F_s(u) \int_0^N (1 - \frac{t}{N}) \sin tu \, dt \cos xt \, du = \frac{1}{\pi} \lim_{N \to \infty} \int_0^\infty F_s(u) \frac{1}{2} \int_0^N (1 - \frac{t}{N}) [\sin(u + x)t + \sin(u - x)t] \, dt \, du.$$ 

Applying (refces), we get

$$F_s(x) = \frac{1}{\pi} \lim_{N \to \infty} \int_0^\infty F_s(u) \left[ \frac{1}{u + x} - \frac{\sin(u + x)N}{N(u + x)^2} \right] du$$

$$+ \frac{1}{\pi} \lim_{N \to \infty} \int_0^\infty F_s(u) \left[ \frac{1}{u - x} - \frac{\sin(u - x)N}{N(u - x)^2} \right] du = J_1 + J_2.$$ 

Let us begin with $J_2$. Substituting $u - x = t$, we obtain

$$J_2 = \frac{1}{\pi} \lim_{N \to \infty} \int_{-x}^\infty F_s(x + t) \left[ \frac{1}{t} - \frac{\sin Nt}{Nt^2} \right] dt.$$ 

Treating $J_1$ as $I_1$ above, we get

$$J_1 = -\frac{1}{\pi} \lim_{N \to \infty} \int_{-\infty}^0 F_v(-v) \left[ \frac{1}{v - x} - \frac{\sin(v - x)N}{N(v - x)^2} \right] dv$$

$$= \frac{1}{\pi} \lim_{N \to \infty} \int_{-\infty}^0 F_v(v) \left[ \frac{1}{v - x} - \frac{\sin(v - x)N}{N(v - x)^2} \right] dv.$$ 

The last equality follows from the oddness of $F_s$. Substituting $v - x = t$, we obtain

$$J_1 = \frac{1}{\pi} \lim_{N \to \infty} \int_{-\infty}^{-x} F_s(x + t) \left[ \frac{1}{t} - \frac{\sin Nt}{Nt^2} \right] dt.$$ 

Therefore,
Finally, it follows from Theorem A that for almost all $x$

$$F_c(x) = -\mathcal{H}F_s(x).$$

This completes the proof. □

Let us comment on the obtained results. In fact, the proof of Theorem 1 shows that more general results than stated are obtained. Indeed, formulas (11) and (14) are more informative than the assertion of Theorem 1. To be precise, such formulas are known, see [7, (5.42) and (5.43)]. However, the situation is much more delicate. These formulas are proved in [7] for square integrable functions by applying the Riemann-Lebesgue lemma in an appropriate place (5.44). But in [7, §6.19] more details are given (see also [2]) and it is shown that the possibility to apply the Riemann-Lebesgue lemma in that argument is equivalent to (Carleson’s solution of) Lusin’s conjecture. In our $L^1$ setting this is by no means applicable. And, indeed, our proof is different and rests on less restrictive Theorem A. This is well agrees with what E.M. Dyn’kin wrote in his well-known survey on singular integrals [2]: “In fact, the theory of singular integrals is a technical subject where ideas cannot be separated from the techniques.”

3. Sufficient conditions

Analyzing the proof in [5], one can see that in fact their results are similar to ours, that is, can also be given in terms of the (discrete) Hilbert transform. In that case (11) is simply a sufficient condition for the summability of the discrete Hilbert transform.

An analog of (11) for functions cannot be the only sufficient condition for the integrability of the Hilbert transform. Indeed, a known counter-example of the indicator function of an interval works here as well: of course, it stands up to the multiplication by logarithm.

Thus, we are going to give sufficient conditions for the integrability of the Hilbert transforms in the spirit of those in [5] for sequences supplied by some additional properties.

3.1. General conditions. First of all, examining integrability of the Hilbert transform, one can test the integral over, say, $|t| \leq \frac{3}{2}|x|$. Indeed, for $x > 0$, we have

$$\int_0^\infty \left| \int_{\frac{3}{2}x}^\infty \frac{g(t)}{x-t} \, dt \right| \, dx \leq \int_0^\infty |g(t)| \int_0^{2t/3} \frac{dx}{x-t} \, dt = \ln 3 \int_0^\infty |g(t)| \, dt.$$ 

The rest is estimated in a similar manner.

Further, since

$$\int_{-a}^a \frac{dt}{x-t} = 0$$

for any $a > 0$ when the integral is understood in the principal value sense, we can always consider
\[
\int_{-a}^{a} \frac{g(t) - g(x)}{x - t} \, dt
\]

instead of the Hilbert transform truncated to \([-a, a]\).

When in the definition of the Hilbert transform (6) the function \(g\) is odd, we will denote this transform by \(H_o\), and it is equal to

\[
H_o g(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{tg(t)}{x^2 - t^2} \, dt.
\]  

(16)

When \(g\) is even its Hilbert transform \(H_e\) can be rewritten as

\[
H_e g(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{xg(t)}{x^2 - t^2} \, dt.
\]  

(17)

Of course, both integrals should be understood in the principal value sense (see, e.g., [7, Ch.4, §4.2]).

Since the functions \(F_c\) and \(F_s\) are even and odd, respectively, their Hilbert transforms can be represented as

\[
H_e F_c(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{xF_c(t)}{x^2 - t^2} \, dt.
\]  

(18)

and

\[
H_o F_s(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{tF_s(t)}{x^2 - t^2} \, dt.
\]  

(19)

These may be useful in some applications.

In fact, the most known condition is the following. If \(g\) is of compact support, a classical Zygmund \(L \log L\) condition (see, e.g., [17]) ensures the integrability of the Hilbert transform. More precisely, the condition is the integrability of \(g \log^+ |g|\), where the \(\log^+ |g|\) notation means \(\log |g|\) when \(|g| > 1\) and 0 otherwise. As E.M. Stein has shown in [13], this condition is necessary on the intervals where the function is positive.

However, this condition looks quite restrictive in our case. We will prove the following result.

**Theorem 2.** Let \(g\) be an integrable function on \(\mathbb{R}\) which satisfies conditions (7),

\[
\int_{|x| \geq 1/2} |g(x)| \log 3|x| \, dx
\]  

(20)

and

\[
\int_{\mathbb{R}} \int_{\frac{1}{2} \min(|x|, 1)}^{\frac{1}{2} \min(|x|, 1)} \left| \frac{g(x + t) - g(x)}{t} \right| \, dt \, dx.
\]  

(21)

Then \(g \in H^1(\mathbb{R})\).

Since each function can be represented as the sum of its even and odd parts, we will prove Theorem 2 separately for odd and even functions. Thus, from now on we can consider \(g\) to be defined on \(\mathbb{R}_+\) and analyze either (16) or (17) rather than the general Hilbert transform.
3.2. Odd functions. Though an odd function always satisfies (7), not every odd integrable function belongs to $H^1(\mathbb{R})$, for a counterexample see, e.g., [10]. Paley-Wiener’s theorem (see [12]; for alternative proof and discussion, see Zygmund’s paper [16]) asserts that if $g \in L^1(\mathbb{R})$ is an odd and monotone decreasing on $\mathbb{R}_+$ function, then $Hg \in L^1$, i.e., $g$ is in $H^1(\mathbb{R})$. Recently, in [11, Thm.6.1], this theorem has been extended to a class of functions more general than monotone ones. However, it is doubtful that these results are really practical in our situation.

Back to Theorem 2, we can consider

$$\frac{2}{\pi} \int_{x/2}^{3x/2} \frac{tg(t)}{x^2 - t^2} dt$$

instead of (16). Indeed, the possibility of restricting to that upper limit has been justified above. Similarly,

$$\int_0^\infty \left| \int_0^{x/2} \frac{tg(t)}{x^2 - t^2} dt \right| dx \leq \int_0^\infty \frac{g(t)}{t} \left( \int_{2t}^\infty \frac{dx}{x^2 - t^2} \right) dt \leq \frac{2}{3} \int_0^\infty \frac{|g(t)|}{t} dt.$$ (22)

Now, like in (15), we have

$$\int_0^1 \left| \int_{-x/2}^{x/2} \frac{tg(t)}{x^2 - t^2} dt \right| dx \leq \int_0^1 \left| \int_{-x/2}^{x/2} \frac{t[g(t) - g(x)]}{x^2 - t^2} dt \right| dx + O(\int_0^\infty |g(t)| dt) \leq \int_0^1 \int_{-x/2}^{x/2} \frac{|g(x + t) - g(x)|}{|t|} dt dx + O(\int_0^\infty |g(t)| dt).$$ (23)

When $x \geq 1$ we first estimate

$$\int_1^\infty \left| \int_{x/2}^{x-1/2} \frac{tg(t)}{x^2 - t^2} dt \right| dx \leq \int_1^\infty \frac{t|g(t)|}{t} \left( \int_{t+1/2}^{2t} \frac{dx}{x^2 - t^2} \right) dt \leq C \int_1^\infty \frac{|g(t)|}{t} \ln 3t dt$$

and

$$\int_1^\infty \left| \int_{x+1/2}^{3x/2} \frac{tg(t)}{x^2 - t^2} dt \right| dx \leq \int_1^\infty \frac{t|g(t)|}{t^{1/2}} \left( \int_{2t/3}^t \frac{dx}{t^2 - x^2} \right) dt \leq C \int_1^\infty \frac{|g(t)|}{t} \ln 3t dt.$$

These two bounds lead to the logarithmic condition (20). The remained integral

$$\int_1^\infty \left| \int_{x+1/2}^{x-1/2} \frac{tg(t)}{x^2 - t^2} dt \right| dx$$

is estimated exactly like that in (23). Applying (15), we obtain
\[
\int_1^\infty \left| \int_{x-1/2}^{x+1/2} \frac{t g(t)}{x^2 - t^2} \, dt \right| \, dx \leq \int_1^\infty \left| \int_{x-1/2}^{x+1/2} \frac{t [g(t) - g(x)]}{x^2 - t^2} \, dt \right| \, dx + O(\int_0^\infty |g(t)| \, dt)
\]
(24)
\[
\leq \int_1^\infty \int_{-1/2}^{1/2} \frac{|g(x + t) - g(x)|}{|t|} \, dt \, dx + O(\int_0^\infty |g(t)| \, dt).
\]
Combining all the obtained estimates, we arrive at the required result.

3.3. Even functions. While an odd function always satisfies (7), in the case of even functions the situation is more delicate: the function must satisfy (7) already on the half-axis. With this in hand, the proof goes along the same lines as that for odd functions. The only problem is that an estimate like (22) does not follow immediately from the formula (17). However, using the above remark on the cancelation property for \(g\) on the half-axis, we can rewrite (17) as

\[
\mathcal{H}_e g(x) = \frac{2}{\pi} \int_0^\infty g(t) \left[ \frac{x}{x^2 - t^2} - \frac{2}{x} \right] \, dt.
\]
(25)

Now,

\[
\int_0^\infty \left| \int_0^{x/2} \frac{t^2 g(t)}{x(x^2 - t^2)} \, dt \right| \, dx \leq \int_0^\infty |g(t)| \frac{t^2}{x(x^2 - t^2)} \, dt \int_0^\infty \frac{dx}{x(x^2 - t^2)} \, dt
\]
(26)
\[
\leq \frac{1}{6} \int_0^\infty |g(t)| \, dt.
\]

This additional term \(\frac{2}{x}\) does not affect the other estimates of the previous subsection.

The proof is complete.

4. Concluding remarks

First of all, the relations (11) and (14) are of interest by their own.

The assertions of Theorem 1 can be reformulated in terms of Hardy spaces: belonging of \(F_c\) (\(F_s\)) to the real Hardy space \(H^1(\mathbb{R})\) ensures the integrability of \(F_s\) (\(F_c\)).

The problem of sharpness is simple in this case: any known counterexample of an integrable function with non-integrable Hilbert transform works perfectly. For example, let \(F_c(x) = \frac{1}{1+x^2}\), the Fourier transform of \(f(t) = e^{-|t|}\). Surely, \(F_c \in L^1(\mathbb{R})\). However, \(f\) cannot be re-expanded in the integrable sine Fourier transform, since \(\mathcal{H}F_c(x) = \frac{x}{1+x^2} \not\in L^1(\mathbb{R})\). That this is true, one can see from the fact that the odd extension of this \(F_c\) from the right half-axis to the whole \(\mathbb{R}\) is not continuous at zero.

More can be said about odd functions. Certain convenient conditions for belonging of such functions to \(H^1(\mathbb{R})\) are known for quite a long time. They are functions (Fourier transform) analogs of important sufficient sequence conditions for the integrability of trigonometric series (see, e.g., [14] and [3]) and can be found, for example, in [9] and in [4]. In fact, many of these subspaces first appeared in [1]. For \(1 < q \leq \infty\), set

\[
\|g\|_{A_q} = \int_0^\infty \left( \frac{1}{u} \int_{u \leq |t| \leq 2u} |g(t)|^q \, dt \right)^{1/q} \, du.
\]
with a standard modification when $q = \infty$. In other words, belonging of $g$ to one of the spaces $A_q$ ensures the integrability of the odd Hilbert transform of $g$.

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