The effect of Kolmogorov (1962) scaling on the universality of turbulence energy spectra

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December 24, 2018

Abstract

It has long been established that turbulence energy spectra scale on the Kolmogorov (1941) variables over a wide range of Reynolds numbers and in vastly different physical systems, depending only on the dissipation rate, the kinematic viscosity and the wavenumber. On the other hand, the analogous study of structure functions in real space is strongly influenced by the Kolmogorov (1962) refined theory, which introduced a dependence on a large length scale $L_{ext}$, characteristic of the system size. If such a dependence exists it is surprising that it does not show up in the study of wavenumber spectra, where the different physical systems suggest that $L_{ext}$ can vary by up to five orders of magnitude. Here we use an order of magnitude calculation to suggest that scaling according to Kolmogorov (1962) would destroy the observed asymptotic universality of energy spectra at large wavenumbers.
1 Introduction

There is a curious dichotomy at the heart of turbulence theory between real-space and wavenumber-space formulations, one being merely the Fourier transform of the other. It has its origins in the work of Kolmogorov, who in 1941 obtained expressions for the second- and third-order structure functions \([1, 2]\): referred to as K41. In particular, he used dimensional analysis in combination with Richardson’s picture of a cascade of eddies, to obtain an expression for the second-order structure function of the form:

\[
S_2(r) = C \varepsilon^{2/3} r^{2/3}.
\]  

(1)

This applied to a range of length scales \(r\) intermediate between the large scales associated with the size of the physical system and the small scales determined by the dissipative action of viscosity. The prefactor \(C\) is taken to be constant and \(\varepsilon\) is the mean energy dissipation rate.

In 1945, Onsager \([3]\) applied similar arguments in wavenumber space to obtain the corresponding energy spectrum in the form:

\[
E(k) = \alpha \varepsilon^{2/3} k^{-5/3},
\]  

(2)

where \(\alpha\) is constant and \(k\) is the wavenumber. This is the legendary Kolmogorov \(-5/3\) law, while \(\alpha\) is the Kolmogorov constant. This result can also be obtained by Fourier transformation of (1).

The subsequent history of the subject is of interest. The work of Kolmogorov, originally only known in Russia, was brought to the wider fluid dynamics community by Batchelor, beginning with a paper discussing the real-space treatment of the subject in 1947 \([4]\). However, after this Batchelor worked in wavenumber (\(k\)) space, particularly in his classic monograph \([5]\), and this set a pattern for later researchers. The situation may be summarised by Figure 2.4 of Reference \([6]\), which is reproduced here for the reader’s convenience as Fig. 1. What is actually shown here is the one-dimensional projection of the spectrum, \(\phi_1(k)\), which has been made dimensionless using the Kolmogorov variables and plotted against wavenumber divided by the Kolmogorov dissipation wavenumber (\(k_d\)). The various flows are characterised by their value of the Taylor-Reynolds number \(R_{\lambda}\), with values shown over a range from 14 to 2000. The existence of universal asymptotic behaviour at high wavenumbers is clearly established by this set of graphs, as is the \(-5/3\) law. In particular, the results of Grant et al. in 1962, showing the \(-5/3\) law over many decades of log wavenumber, clearly established the correctness of the Kolmogorov theory. Paradoxically, in the same year Kolmogorov decided that his theory was wrong \([7]\).

2 Kolmogorov’s refined (sic) theory of 1962

Kolmogorov’s original theory \([1, 2]\) was based on the so-called Richardson picture, in which the turbulence energy transfer proceeded by a step-wise cascade from large eddies to small. Bearing in mind that this is a random process, it could be argued that the average effect would be that detailed information about the large scales would be lost. Accordingly the behaviour of the small scales would be universal. However, Kolmogorov (following a rather obscure criticism of K41 by Landau) introduced a length scale \(L_{\text{ext}}\) said to be characteristic of the size of the system and modified the result \([1]\) for the structure function to

\[
S_2(r) = C \varepsilon^{2/3} r^{2/3} (L_{\text{ext}} / r)^{-\mu}.
\]  

(3)
Accordingly the title of the later work [7], which refers to ‘A refinement of previous hypotheses . . .’, is something of a misnomer. In fact the basic idea of independence from initial conditions has been abandoned. Evidently the presence of $L_{ext}$ in the result destroys any claim to universality.

As equation (2) may be derived from (1) by Fourier transformation, so may we derive the K62 version of the energy spectrum the same way by transforming equation (3) to obtain

$$E(k) = \alpha \varepsilon^{2/3} k^{-5/3} (L_{ext} k)^{-\mu}. \quad (4)$$

It is intuitively obvious that this form, if correct, could pose problems for the spectral scaling shown in Figure 1. Also, we should not forget that K41 is an asymptotic result which applies also to the dissipation region, where the spectrum (irrespective of its detailed form) must scale as a function $f(k/k_d)$. The main objective of this paper is to make a quantitative test of this, but first we will briefly review the development of the subject subsequent to 1962. In particular, we shall discuss intermittency corrections and anomalous exponents.

3 Anomalous exponents

Although the original K41 theory was restricted to the second- and third-order structure functions, the various arguments and hypotheses were applied to the probability distribution, and so the theory is as applicable to structure functions of all orders. Hence we now introduce the general form of the longitudinal structure functions as:

$$S_n(r) = \langle \delta u^n_L(r) \rangle, \quad (5)$$

where the longitudinal velocity increment is given by

$$\delta u_L(r) = [u(x + r, t) - u(x, t)] \cdot \hat{r}. \quad (6)$$

If for $n \geq 4$ these structure functions are found to exhibit power laws, then dimensional analysis would lead to

$$S_n(r) \sim (\varepsilon r)^{n/3}. \quad (7)$$

However, measurement of the structure functions has repeatedly found a deviation from the above dimensional prediction for the exponents. If instead, the structure functions are taken to scale with exponent $\zeta_n$, thus

$$S_n(r) \sim r^{\zeta_n}, \quad (8)$$

then it has been found [8, 9] that the difference $\Delta_n = |n/3 - \zeta_n|$ is nonzero and increases with order $n$. In particular, Figure 14 of Reference [8] illustrates this behaviour rather nicely.

At first this behaviour was classed as intermittency corrections and the concept of intermittency was associated with the very small scales where the dissipation was mainly concentrated. Later it was recognised that intermittency was present at all scales and nowadays the tendency is to speak of internal rather than small-scale intermittency. In any case, since the mid-1990s, it has become usual to refer to the $\zeta_n$ as anomalous exponents, and they are seen as a subject of fundamental interest to this day (e.g. [10],[11]).

4 Testing the effect of K62 on spectral scaling

In order to assess the effect of changing from K41, as given by (1), to K62, as given by (3), we need to make estimates of $L_{ext}$ and $\mu$. We begin by choosing two disparate investigations in
order to make a comparison. We choose the results of Grant et al. [12], which were taken in a tidal channel at a Taylor-Reynolds number of $R_\lambda = 2000$, and the laboratory results of Comte-Bellot and Corrsin [13] taken in grid turbulence (with two-inch grid) at $R_\lambda = 72$. We plot these results on log scales in Figure 2 as the scaled one-dimensional spectrum $\psi(k') = \phi(k') / (\varepsilon \nu^5)$ against the dimensionless wavenumber $k' = k/k_d$, where $k_d$ is the Kolmogorov dissipation wavenumber. Here we have taken the one-dimensional Kolmogorov constant as $\alpha_1 = 1/2$, as this gives good agreement with both sets of results. This is shown in Figure 3, where we follow the modern practice of plotting the spectrum divided by the Kolmogorov form, such that $K_{41}$ corresponds to a horizontal line at unity.

In the absence of a precise definition of $L_{ext}$, we can determine $L_{ext}' = 2\pi/k_{ext}'$, where $k_{ext}'$ marks the departure of the curve from the $K_{41}$ form, as one goes from high wavenumbers to low. In this case we estimate $L_{ext}' \sim 50$ for the grid turbulence, and $L_{ext}' \sim 2000$ for the tidal channel measurements. In fact, the spectra in the results of Grant et al. [12] do not actually peel off from the $-5/3$ line at low $k$ and so our estimate is actually a lower bound for $L_{ext}$ in this case, and this favours $K_{62}$ in the comparison.

The exponent $\mu$ is seen as a universal feature of the $K_{62}$ theory and generally estimates have been in the range $0.1 - 0.2$. We have taken the value $\mu = 0.1$ as reported by Kaneda et al. [14] from their high-resolution numerical simulation. Again, this use of the lower value favours $K_{62}$ in the comparison.

The result of changing from (2) to (4) can be seen in Figure 4. Plotting a compensated spectrum, based on Kolmogorov variables, for a form like equation (4) evidently leaves a residual slope, given in this case by $\mu = 0.1$. For the values of $L_{ext}$ taken here, the two scaled spectra no longer coincide but indeed differ in a constant ratio of 0.69. It should be noted that the change of ordinate scales in this plot exaggerates the spread of values about the line corresponding to $K_{41}$, and also emphasises the bump at $k' = 0.1$ which is characteristic of spectra at low Reynolds number and which disappears with increasing Reynolds number (e.g. see Reference [15]).

5 Discussion

We should point out that the fact that Kaneda et al. [14] measured a slope different from $-5/3$ is not evidence in favour of $K_{62}$. Such measurements, with a finite inertial range, are sensitive to the criteria used to establish the extent of that range. In practice, it is expected that at large wavenumbers spectra will roll off in some form of exponential. It is also worth pointing out that, as a modification to the power law requires the introduction of a length scale in order to preserve the correct dimension, internal scales like the Taylor microscale or the Kolmogorov dissipation length scale, if used for this purpose, will also change the dependence on the dissipation rate.

It seems surprising that there is, and has been, such a large concentration on intermittency, when $K_{41}$ relies on various approximations, in particular the neglect of the viscosity. In 2002, Lundgren offered a rigorous proof that $K_{41}$ was valid in the limit of infinite Reynolds number [16]. This was reinforced by the author’s comparison with the experimental results of Mydlarski and Warhaft [17]. More recently various workers have given some attention to the effects of finite viscosity and finite system size [18]–[22] and of the neglect of the time dependence [23] in $K_{41}$.

Many of the problems in this topic seem to arise from the addiction of the turbulence community to real-space treatments. If the cascade is seen as unsatisfactory then recourse

\footnote{Of course the vast majority of applications in engineering fluid dynamics require real-space treatments, but}
is made to a vortex stretching picture: see Reference [24] for a general discussion. However, although both pictures have their own intuitive appeal, they both suffer from a certain vagueness. Indeed their general ‘hand waving’ characteristics do not provide a satisfactory basis for a mathematical physics approach.

The contrast with the wavenumber picture could not be greater. Here the Fourier modes are the independently excited degrees of freedom of the system. The number of these degrees of freedom increases with the Reynolds number and, in the limit of infinite Reynolds number, there is an infinite number of them and (naturally!) an infinite amount of energy. They are all coupled to each other by the nonlinear term (nonlinear mixing) and, in the language of statistical physics, constitute a many-body problem. The energy-conservation of the nonlinear term can be deduced by inspection, and the existence of the inertial range can be deduced by simple physical reasoning. It corresponds to the case where the energy flux through wavenumber attains a constant, maximum value, which in a steady state is equal to the viscous dissipation rate. All of this can be found in the monograph by Batchelor [5] which was originally published in 1954. It also tells us that the Kolmogorov $-5/3$ spectrum is due to the scale invariance of the energy flux and, as this is a globally averaged quantity, it is unaffected by intermittency, but rather takes it into account. To conclude this point, all fundamental statistical closures of turbulence are formulated in wavenumber space, all are dimensionally compatible with K41, and differences between them boil down to the existence or otherwise of certain integrals.

6 Conclusion

The behaviour of the exponents of the structure functions, as shown in Figure 14 of Reference [8], is often cited as evidence for anomalous exponents, and possibly for K62. However, alternative explanations are the increasing sensitivity of higher-order moments to the approximations made in K41 or to systematic error. In the latter case, given a fixed approximation to the probability distribution, evaluating exponents of progressively higher order should indeed yield progressively larger deviations from the canonical results. This possibility has been tested by McComb et al. [20] for the particular case of the second-order structure function, by making allowance for systematic error. In this way they found that the second-order exponent tended to the K41 form, in the limit of infinite Reynolds numbers.

Taken in conjunction with the results presented in Fig. [4] this suggests a more cautious approach to the whole idea of anomalous exponents. It is our view that the papers published in 1941 represent the true legacy of A. N. Kolmogorov, and that the 1962 ‘refinement’ was unnecessary.

7 Acknowledgements

We would like to thank Bob Antonia, Jorgen Frederiksen, Marcello Meldi and Sam Yoffe for reading a first draft of this work and for making helpful comments and suggestions.

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Figure 1: Measured one-dimensional energy spectra for a wide range of Reynolds numbers and physical situations showing the asymptotic effect of scaling on Kolmogorov (1941) variables. Image reproduced from Figure 2.4 of Reference [6].
Figure 2: Two representative measured one-dimensional energy spectra, one at $R_\lambda = 2000$, taken in a tidal channel [12]; the other at $R_\lambda = 72$, taken in grid generated turbulence [13]; and normalised on Kolmogorov (1941) variables. Note that we plot the scaled one-dimensional spectrum, as given by $\psi(k') = \phi(k')/(\varepsilon \nu^5)^{1/4}$.
Figure 3: The results of Figure 2 plotted in compensated form such that $k^{-5/3}$ spectra appear as a constant. Note that we have taken the one-dimensional spectral constant to be $\alpha_1 = 1/2$, on the basis of Figure 2.
Figure 4: Demonstration of the effect on the compensated spectra of Figure 3 of including the Kolmogorov (1962) corrections to the K41 theory. The departure of the slope from K41 was based on the value of $\mu$ obtained in the numerical simulation of Kaneda et al. [14].