STABILITY OF THE EINSTEIN-LICHNEROWICZ CONSTRAINTS SYSTEM

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Abstract. We study the Einstein-Lichnerowicz constraints system, obtained through the conformal method when addressing the initial data problem for the Einstein equations in a scalar field theory. We prove that this system is stable with respect to the physics data when posed on the standard 3-sphere.

1. Introduction

The Einstein field equations are a set of 10 equations in the theory of relativity which relate the geometry of a space-time to the distribution of matter and energy. Given a space-time $(\mathcal{M}, \tilde{g})$ – namely a $(3 + 1)$-dimensional Lorentzian manifold – they are written as

$$\text{Ric}(\tilde{g})_{ab} - \frac{1}{2} R(\tilde{g}) \tilde{g}_{ab} = T_{ab},$$

where $\text{Ric}(\tilde{g})$ is the Ricci tensor of $\tilde{g}$, $R(\tilde{g})$ is the scalar curvature of $\tilde{g}$ and $T$ is the stress-energy tensor which depends on the model used to represent the distribution of matter and energy. In the scalar field setting, the stress-energy tensor takes the form:

$$T_{ab} = \nabla_a \psi \nabla_b \psi - \left(\frac{1}{2} |\nabla \psi|^2 + V(\psi)\right) \tilde{g}_{ab},$$

(1.1)

where the scalar-field $\psi$ is a smooth function in $\mathcal{M}$, $V$ - the potential associated to $\psi$ - is a smooth function in $\mathbb{R}$ and $\psi$ and $V$ satisfy:

$$\nabla_i \nabla_i \psi = \frac{dV}{d\psi},$$

(1.2)

where $\nabla_i \nabla_i \psi$ is the laplacian for the Lorentzian metric $\tilde{g}$, so that there holds $\nabla_i T^{ij} = 0$. When looking for solutions of (E), in order to avoid pathological examples from a physical point of view, the theory restricts itself to a class of space-times where the Einstein equations can be seen as an evolution problem from a given initial data. Following standard terminology, an initial data set $(M, h, K, \psi_0, \psi_1)$ consists of $(M, h, K, \psi_0, \psi_1)$ where $(M, h)$ is a 3-dimensional Riemannian manifold, $K$ is a smooth $(2, 0)$-tensor and $\psi_0$ and $\psi_1$ are smooth functions in $M$ satisfying the following constraint equations in $M$:

$$\begin{cases}
R(h) + (\text{tr}_h K)^2 - ||K||^2_h = \psi_1^2 + |\nabla \psi_0|^2 + 2V(\psi_0) \\
\nabla^i K_{ij} - \nabla_i (\text{tr}_h K) = \psi_1 \nabla_i \psi_0.
\end{cases}$$

(1.3)

Then, an initial data set $(M, h, K, \psi_0, \psi_1)$ is said to admit a globally hyperbolic space-time development if there exists a Lorentzian manifold $(\mathcal{M}, \tilde{g})$, a smooth function $\psi$ in $\mathcal{M}$ and an embedding $i : M \to \mathcal{M}$ such that (E) and (1.2) are satisfied,
such that \( i(M) \) is a Cauchy hypersurface of \( \mathcal{M} \) and such that \( i^* \hat{g} = h, i^* \mathcal{K} = K, \)
\( \psi \circ i = \psi_0 \) and \( (N \cdot \psi) \circ i = \psi_1, \) where \( \mathcal{K} \) is the second fundamental form of \( i(M) \)
in \( \mathcal{M} \) and \( N \) is the future directed timelike unit normal to \( i(M) \). Straightforward
application of the Gauss and Codazzi equation shows that a necessary condition for
a globally hyperbolic space-time \( (\mathcal{M}, \hat{g}) \) with a scalar-field \( \psi \) to solve \( (1.2) \) and
\( (1.3) \) is that the system \( (1.3) \) be satisfied on a Cauchy hypersurface of \( \mathcal{M} \). Since
the works of Choquet-Bruhat \( [12] \) and Choquet-Bruhat and Geroch \( [6] \), it is known
that \( (1.3) \) is also a sufficient condition on an initial data set \( (M, h, K, \psi_0, \psi_1) \) to
admit a maximal globally hyperbolic development (MGHD). This object provides
a framework for the main open conjectures in general relativity and for the analysis
of solutions of \( (1.3) \).

The constraint equations \( (1.3) \) are a highly underdetermined system of 4 equations
for 14 unknowns. To overcome this problem, the conformal method, initiated
by Lichnerowicz \( [22] \), freezes some of the unknown variables – considering them
from now on as parameters – and proposes to solve the system for the remaining
set of variables. In this process, one looks for the unknown metric \( h \) in the conformal
class of a reference Riemannian metric \( g \) in \( M \). We consider the case of closed
manifolds in what follows, where closed means compact without boundary. We let
\( (M, g) \) be a closed Riemannian 3-manifold. By the conformal method, in order to
obtain a solution of \( (1.3) \) in \( M \) it is enough to solve the following elliptic system of
equations, where the unknown \( (\varphi, W) \) are a smooth positive function on \( M \) and a
smooth 1-form on \( M \):

\[
\begin{align*}
\Delta_g \varphi + R\varphi \varphi &= B_{\tau,\psi,V} \varphi^5 + \frac{A_{\tau,U}(W)}{\varphi^5}, \\
\Delta_g W &= -\frac{2}{3} \varphi^6 \nabla \tau - \pi \nabla \psi,
\end{align*}
\]

where

\[
R\varphi = \frac{1}{8} \left( R(g) - |\nabla \psi|^2_g \right), \quad B_{\tau,\psi,V} = \frac{1}{8} \left( 2V(\psi) - \frac{2}{3} \tau^2 \right),
\]

\( A_{\tau,U}(W) = \frac{1}{8} \left( |U + \mathcal{L}_g W|^2_g + \pi^2 \right) \),

\( R(g) \) is the scalar curvature of \( (M, g) \) and \( \mathcal{L}_g W \) – the conformal Killing operator –
is a \( (2,0) \)-tensor defined in coordinates by:

\[
\mathcal{L}_g W_{ij} = W_{i,j} + W_{j,i} - \frac{2}{3} (\text{div}_g W) g_{ij}.
\]

Also, in \( (1.5) \), \( \Delta_g = -\text{div}_g(\nabla \cdot) \) is the Laplace-Beltrami operator and we have let:

\[
\Delta_g W = -\text{div}_g(\mathcal{L}_g W)
\]

for any 1-form \( W \). Note that in \( (1.5) \), 6 is the critical exponent for the embedding
of the Sobolev space \( H^1(M) \) into Lebesgue spaces. The initial data in \( (1.4) \) are
\( F = (\tau, \psi, \pi, U) \), the unknowns are \( \varphi \) and \( W \). We let \( \mathcal{F} \) be the initial data set

\[
\mathcal{F} = \left\{ (\tau, \psi, \pi, U)), \, \tau, \psi, \pi \in C^\infty(M), U \in T_{(2,0)}(M) \right\},
\]

where \( T_{(2,0)}(M) \) denotes the set of smooth symmetric traceless and divergence-free
\( (2,0) \)-tensors in \( M \). We endow \( \mathcal{F} \) with the following norm: for \( F = (\tau, \psi, \pi, U) \in \mathcal{F}, \)

\[
\| F \|_\mathcal{F} = \| \tau \|_{C^2(M)} + \| \psi \|_{C^1(M)} + \| \pi \|_{C^0(M)} + \| U \|_{C^0(M)}.
\]
Given an initial data $F = (\tau, \psi, \pi, U)$ and a solution $(\varphi, W)$ of \((C_F)\) one obtains an initial data set $(M, h, K, \psi_0, \psi_1)$ solution of the original constraint equations \((1.3)\) by letting

$$
(h, K, \psi_0, \psi_1) = (\varphi^4 g, \frac{\tau}{3} \varphi^4 g + \varphi^{-2}(U + L_g W), \psi, \varphi^{-6} \pi).
$$

(1.8)

Conversely, starting from a solution of \((1.3)\) one obtains a solution $(\varphi, W)$ of \((C_F)\) for suitably defined parameters $F = (\tau, \psi, \pi, U)$ according to \((1.8)\). Note that then $\tau$ turns out to be the mean curvature of the embedding of $M$ in its globally hyperbolic development.

Different physical setting are roughly distinguished by the sign of the coefficient $B_{\tau, \psi, V}$. In the vacuum case, $B_{\tau, \psi, V}$ is nonpositive while it becomes positive if we allow a cosmological constant $\Lambda$ (corresponding to the case $V(\psi) \equiv \Lambda$) and $\tau$ is not too big with respect to this constant. The $B_{\tau, \psi, V}$ positive case has received increasing attention in recent years with the attempts to use scalar-field theories to model the observed acceleration of the expansion of the universe, see Rendall [25]. Several existence results for \((C_F)\) are known. When $M$ is closed these are of different nature according to the sign of $B_{\tau, \psi, V}$ and the case $B_{\tau, \psi, V} > 0$ turns out to exhibit a rich behavior where solutions are not necessarily unique as shown in Premoselli [23] or Holst - Meier [19]. If $\nabla \tau = 0$ (in which case the system \((C_F)\) is decoupled), existence results are in Isenberg [20] for the $B_{\tau, \psi, V} \leq 0$ case and in Hebey-Pacard-Pollack [17] for the max$_M B_{\tau, \psi, V} > 0$ case. If $\tau$ is not constant, existence results are in Dahl-Gicquaud-Humbert [9] and the references therein if $B_{\tau, \psi, V} \leq 0$ and Premoselli [23] if $B_{\tau, \psi, V} > 0$. As \((1.8)\) shows, solving the system \((C_F)\) provides solutions of \((1.3)\) parameterized by some initial data in $\mathcal{F}$. We are in this work interested in the stability of the system \((C_F)\) with respect to perturbations both of the initial data in $\mathcal{F}$ and of the potentials $V$ in $C^1(\mathbb{R})$. We fully answer the question in the positive case of the round 3-sphere.

**Theorem 1.** Let $(M, g) = (S^3, h)$ be the standard round 3-sphere. Let $F_0 = (\tau, \psi, \pi, U) \in \mathcal{F}$ and $V_0 \in C^1(\mathbb{R})$ be such that $B_{\tau, \psi, V_0} > 0$ in $S^3$, $\triangle_h + R_{\psi}$ is coercive and $\pi \neq 0$. For any sequence $(F_\alpha)$, $F_\alpha = (\tau_\alpha, \psi_\alpha, \pi_\alpha, U_\alpha)$, of initial data in $\mathcal{F}$, any sequence $(V_\alpha)$ of potentials in $C^1(\mathbb{R})$, such that

$$
\|F_\alpha - F_0\|_{\mathcal{F}} + \|V_\alpha - V\|_{C^1(\mathbb{R})} \to 0
$$

as $\alpha \to \infty$, where $\| \cdot \|_{\mathcal{F}}$ is as in \((1.7)\), and any sequence $(\varphi_\alpha, W_\alpha)$ of solutions of the conformal constraint system \((C_F)\) with $F = F_\alpha$ and $V = V_\alpha$, there exists a solution $(\varphi_0, W_0)$ of the limit system \((C_F)\) with $F = F_0$ and $V = V_0$ such that, up to a subsequence, $(\varphi_\alpha, W_\alpha)$ converges to $(\varphi_0, W_0)$ in $C^{1, \theta}(M)$ as $\alpha \to +\infty$, where $0 < \theta < 1$ is arbitrary.

Theorem 1 is of course a compactness result. Sequences of solutions of small perturbations of the constraint system \((C_F)\) with respect to the initial data and the potential converge smoothly to a solution of the unperturbed system. In particular, small perturbations of the initial data do not create sequences of solutions far away from the set of solutions of the original system.

When combined with local well-posedness results for quasilinear hyperbolic systems, Theorem 1 provides a stability result on spacetime developments with respect to the physics data in small time in the sense that the spacetime development corresponding to a perturbation of the initial scalar-field data $(\tau, \psi, \pi, U)$ will be close
in strong sense, for small times, to a spacetime development of the original problem. In particular small numerical errors in \((\tau, \psi, \pi, U)\) do not affect the solutions of (E) for small time.

The regularity of convergence of solutions in Theorem 1 depends on the convergence of the initial data. We stated it under reasonable assumptions on the convergence of the \(F\)'s and \(V\)'s, but due to its elliptic features, the system (F) is regularizing: if the perturbed initial data as well as the potentials \(c\) converge in \(C^{k, \theta}\), \(k \geq 2, 0 < \theta < 1\), then the solutions converge in \(C^{k+1, \theta'}(M)\) for all \(\theta' < \theta\).

We state Theorem 1 in the model case of \(S^3\) in order to slightly simplify the presentation of the proof but our arguments easily extend to the general case of a 3-dimensional locally conformally flat manifold. Also note that the only relevant assumptions in Theorem 1 are \(B_{\tau, \psi, V_0} > 0\) and \(\pi \not\equiv 0\). Assuming that \(B_{\tau, \psi, V_0} > 0\), the coercivity of \(\Delta_g h + R\psi\) is a necessary condition for the existence of solutions of the scalar equation of (F).

The proof of Theorem 1 goes through the proof of an involved stability result concerning a slightly wider class of systems than (F), see (2.1) in section 2 below. Section 4 contains the arguments in the proof of Theorem 1 and uses the pointwise description of sequences of blowing-up solutions of (F) around a concentration point. Such a pointwise description is obtained in section 5 and is the core of the analysis of the paper. It requires a simultaneous investigation of the defects of compactness that may occur in each of the two equations of (F). Finally, sections 6 to 9 gather some technical results used throughout the paper.

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### 2. A PDE result

We shall in fact prove a more general result than Theorem 1. We consider a sequence \((u_\alpha, W_\alpha)\) of solutions on \((S^3, h)\) of

\[
\begin{cases}
\Delta_h u_\alpha + h_\alpha u_\alpha = f_\alpha u_\alpha^5 + \frac{a_\alpha}{u_\alpha^4} \\
\Delta_h^2 W_\alpha = u_\alpha^6 X_\alpha + Y_\alpha
\end{cases}
\]

where

\(a_\alpha = |U_\alpha + \mathcal{L}_h W_\alpha|^2 + b_\alpha\)

and \((h_\alpha, f_\alpha, b_\alpha)\) are smooth functions with \(f_\alpha > 0, b_\alpha > 0\), \(X_\alpha\) and \(Y_\alpha\) are smooth 1-forms and \(U_\alpha\) is a smooth traceless divergence free \((2, 0)\)-tensor field. We assume that

\[(h_\alpha, f_\alpha, X_\alpha, Y_\alpha, U_\alpha, b_\alpha) \rightarrow (h_0, f_0, X_0, Y_0, U_0, b_0)\text{ as } \alpha \rightarrow +\infty\]

where all the convergences take place in \(C^0\), except the convergences of \(X_\alpha\) and \(f_\alpha\) to \(X_0\) and \(f_0\) which take place in \(C^3\) with \(f_0 > 0\). We assume moreover that \(h_0\) is such that \(\Delta_h + h_0\) is coercive and that

\[b_0 \not\equiv 0 .\]
In the following, we may assume that $W_\alpha$ is orthogonal to the set of Killing 1-forms on $S^3$ (see proposition 7.2 for a classification of these Killing 1-forms). Indeed, the only quantity entering into the equations is $\mathcal{L}_h W_\alpha$ so that the system is completely invariant under the addition of a Killing 1-form to $W_\alpha$.

Then we have the following result:

**Theorem 2.** Such a sequence $(u_\alpha, W_\alpha)$ is uniformly bounded in $C^{1,\eta}$ for all $\eta > 0$ and, after passing to a subsequence, $(u_\alpha, W_\alpha) \to (u_0, W_0)$ in $C^{1,\eta}$ where $(u_0, W_0)$ is a solution of the limiting system.

Compactness and stability results have a long time history. A striking result was the recent complete proof of the compactness of the Yamabe equation by Khuri-Marques-Schoen [21] together with its limitation by Brendle [3] and Brendle-Marques [4]. As a remark an equation can be compact but unstable, namely sensitive to changes of the parameters in the equation, and this is the case for the Yamabe equation on the projective space. General references on the stability of elliptic PDEs are by Druet [10] and Heby [16]. The specific case of the Einstein-Lichnerowicz equation is addressed in Druet-Heby [11], Heby-Veronelli [18] and Premoselli [23]. Changing the viewpoint, passing from the elliptic world to the dynamical setting, we mention the striking groundbreaking global stability of the Minkowski space-time obtained by Christodoulou-Klainerman [7] or the amount of work concerning the stability of Kerr space-times (see Dafermos-Rodnianski [8] and the references therein).

Theorem 1 is a direct consequence of Theorem 2. The rest of the paper is devoted to the proof of Theorem 2. By standard elliptic theory, if the convergences in (2.2) take place in more regular spaces, one would get more regularity in the convergence in the conclusion of Theorem 2.

3. Notations

Given $P \in S^3$, we let $\pi_P$ be the stereographic projection of pole $-P$. Then we have that

$$(\pi_P^{-1})^* h = U^4 \xi ,$$

where $\xi$ is the Euclidean metric and

$$U(x) = \sqrt{\frac{2}{1 + |x|^2}} .$$

(3.1)

Given $u \in C^\infty (S^3)$ and $W$ a smooth 1-form on $S^3$, we shall denote by $u[P]$ and $W[P]$ the following function and 1-form on $\mathbb{R}^3$:

$$u[P](x) = u \circ \pi_P^{-1}(x)$$

(3.2)

and

$$(\pi_P)^* W[P] = W .$$

(3.3)

Then we have that

$$\Delta_\xi (u[P]U) = U^5 \left( \Delta_h u + \frac{3}{4} u \right) \circ \pi_P^{-1} ,$$

(3.4)

that

$$(\pi_P)^* \left( U^4 \mathcal{L}_\xi (U^{-4} W[P]) \right) = \mathcal{L}_h W$$

(3.5)
and that
\[ (\pi^*)^* \left( \Delta^* (U^{-4}W[P])_{ij} - 6U^{-1}\mathcal{L}_\xi (U^{-4}W[P])_{ij} \partial^i U \right) = (\Delta_h^* W)_i . \]

(3.6)

4. Proof of Theorem 2

We let \((u_\alpha, W_\alpha)\) be a sequence of solutions on \((S^3, h)\) of (2.1). And we assume that (2.2) and (2.3) hold. We first claim that \(u_\alpha\) stays uniformly positive:

Claim 4.1. There exists \(\varepsilon_0 > 0\) such that \(u_\alpha \geq \varepsilon_0\) on \(S^3\) for all \(\alpha\).

Proof. We let \(G_\alpha\) be the Green function of \(\Delta_g + h_\alpha\) on \(S^3\) which is uniformly positive thanks to (2.2) and the fact that \(\Delta_h + h_0\) is coercive (see [26]). Then we can use Green’s representation formula to write that
\[ u_\alpha(x) \geq C \int_{S^3} (f_\alpha u_\alpha^5 + a_\alpha u_\alpha^{-7}) \, dv_h. \]

Since
\[ f_\alpha u_\alpha^5 + a_\alpha u_\alpha^{-7} \geq f_\alpha u_\alpha^5 + b_\alpha u_\alpha^{-7} \geq \frac{12b_\alpha}{5} \left( \frac{7b_\alpha}{5f_\alpha} \right)^{-\frac{72}{7}}, \]
the claim follows from the assumptions \(f_0 > 0\) and \(b_0 \neq 0\).

Claim 4.2. If there exists \(C > 0\) such that \(u_\alpha \leq C\) on \(S^3\) for all \(\alpha\), then, for all \(0 < \eta < 1\), there exists \(D > 0\) such that
\[ \|W_\alpha\|_{C^{1,\eta}} + \|u_\alpha\|_{C^{1,\eta}} \leq D. \]

In particular, the conclusion of Theorem 2 holds in this case.

Proof. This follows from standard elliptic theory as developed in section 7, see proposition 7.1.

From now on, we assume that
\[ \sup_{S^3} u_\alpha \to +\infty \text{ as } \alpha \to +\infty. \]

(4.1)

Claim 4.3. There exist \(N_\alpha \in \mathbb{N}^*\) and \((x_{1,\alpha}, x_{2,\alpha}, \ldots, x_{N_\alpha,\alpha})\) points in \(S^3\) such that:

i) \(\nabla u_\alpha(x_{i,\alpha}) = 0\) for \(i = 1, \ldots, N_\alpha\).

ii) \(d_h(x_{i,\alpha}, x_{j,\alpha})^\frac{1}{2} u_\alpha(x_{i,\alpha}) \geq 1\) for all \(i, j \in \{1, \ldots, N_\alpha\}, i \neq j\).

iii) There exists \(C > 0\) such that
\[ \left( \min_{i=1,\ldots,N_\alpha} d_h(x_{i,\alpha}, x) \right)^3 (u_\alpha(x)^6 + |\mathcal{L}_h W_\alpha(x)|) \leq C \]
for all \(\alpha\) and all \(x \in S^3\).

Proof. We start with lemma 1.1 of [11] which gives us \(N_\alpha \in \mathbb{N}^*\) and points \((x_{1,\alpha}, x_{2,\alpha}, \ldots, x_{N_\alpha,\alpha})\) in \(S^3\) such that:

i) \(\nabla u_\alpha(x_{i,\alpha}) = 0\) for \(i = 1, \ldots, N_\alpha\).

ii) \(d_h(x_{i,\alpha}, x_{j,\alpha})^\frac{1}{2} u_\alpha(x_{i,\alpha}) \geq 1\) for all \(i, j \in \{1, \ldots, N_\alpha\}, i \neq j\).

iv) For any \(x \in S^3\) with \(\nabla u_\alpha(x) = 0,
\[ \left( \min_{i=1,\ldots,N_\alpha} d_h(x_{i,\alpha}, x) \right)^\frac{1}{2} u_\alpha(x) \leq 1. \]
The aim is to replace iv) by iii). Assume by contradiction that iii) is false and let \( x_\alpha \in S^3 \) be such that
\[
\Phi_\alpha (x_\alpha) = \sup_{S^3} \Phi_\alpha (x) \to +\infty \text{ as } \alpha \to +\infty ,
\]
where
\[
\Phi_\alpha (x) = \left( \min_{i=1,\ldots,N_\alpha} \, d_h (x_{i,\alpha}, x) \right)^3 \left( u_\alpha (x)^6 + |\mathcal{L}_h W_\alpha |_h (x) \right) .
\]
We let \( \mu_\alpha > 0 \) be such that
\[
u_\alpha (x_\alpha)^6 + |\mathcal{L}_h W_\alpha |_h (x_\alpha) = \mu_\alpha^{-3} .
\]
Since \( S^3 \) is compact, it is then clear that
\[
\mu_\alpha \to 0 \text{ as } \alpha \to +\infty .
\]
We also have thanks to (4.2) that
\[
d_h (x_\alpha, S_\alpha) \to +\infty \text{ as } \alpha \to +\infty ,
\]
where
\[
S_\alpha = \{ x_{1,\alpha}, \ldots, x_{N_\alpha,\alpha} \} .
\]
We set
\[
\tilde{u}_\alpha = \frac{\Phi_\alpha}{\mu_\alpha} [x_\alpha] (\mu_\alpha x) U (\mu_\alpha x)
\]
and
\[
\tilde{W}_\alpha = \mu^2 \left( \mu_\alpha x \right)^{-4} W_\alpha [x_\alpha] (\mu_\alpha x) .
\]
Then, equation (2.1) leads with (3.4), (3.5) and (3.6) to
\[
\begin{cases}
\Delta_\xi \tilde{u}_\alpha + \mu_\alpha^2 \tilde{h}_\alpha \tilde{u}_\alpha = \tilde{f}_\alpha \tilde{u}_\alpha^{-5} + \tilde{\lambda}_\alpha \tilde{u}_\alpha^{-2}, \\
\left( \Delta_\xi \tilde{W}_\alpha \right)_i = -6 \mu_\alpha^2 \frac{x^j}{1 + \mu_\alpha^2 |x|^2} \left( \mathcal{L}_\xi \tilde{W}_\alpha \right)_{ij} + \mu_\alpha \tilde{u}_\alpha \tilde{X}_\alpha (\tilde{X}_\alpha, \tilde{Y}_\alpha) + \mu_\alpha \tilde{X}_\alpha (\tilde{Y}_\alpha)
\end{cases}
\]
where
\[
\tilde{h}_\alpha (x) = \left( h_\alpha [x_\alpha] (\mu_\alpha x) - \frac{3}{4} \right) U (\mu_\alpha x)^4 ,
\]
\[
\tilde{f}_\alpha (x) = f_\alpha [x_\alpha] (\mu_\alpha x) ,
\]
\[
\tilde{a}_\alpha (x) = \left| \mu^3 \tilde{U}_\alpha + U (\mu_\alpha x)^6 \mathcal{L}_\xi \tilde{W}_\alpha \right|_{\xi}^2 + \mu_\alpha^6 U (\mu_\alpha x)^{12} b_\alpha [x_\alpha] (\mu_\alpha x) ,
\]
\[
\tilde{U}_\alpha (x) = U (\mu_\alpha x)^2 (\pi_{x\alpha})_* U_\alpha (\mu_\alpha x) ,
\]
\[
\tilde{X}_\alpha (x) = U (\mu_\alpha x)^{-6} (\pi_{x\alpha})_* X_\alpha (\mu_\alpha x) ,
\]
\[
\tilde{Y}_\alpha (x) = (\pi_{x\alpha})_* Y_\alpha (\mu_\alpha x) .
\]
We know thanks to (4.2), (4.3), (4.5), (4.6) and (4.7) that
\[
\sup_{B_h (R)} \left( \frac{1}{8} \tilde{u}_\alpha^6 + \left| \mathcal{L}_\xi \tilde{W}_\alpha \right|_{\xi} \right) \leq 1 + o (1)
\]
for all $R > 0$ and that

$$
\frac{1}{8} \tilde{u}_\alpha(0)^6 + \left| \mathcal{L}_\xi \hat{W}_\alpha \right|_\xi (0) = 1. 
\tag{4.11}
$$

Given $y \in \mathbb{R}^3$ and $R > 0$, let us use the Green representation formula to write that

$$
\tilde{u}_\alpha (y) \geq \frac{1}{4\pi} \int_{B_y(2R)} \left( \frac{1}{|x - y|} - \frac{1}{2R} \right) \Delta_\xi \tilde{u}_\alpha \, dx
$$

since $\tilde{u}_\alpha \geq 0$. Using equation (4.8) and the fact that $\hat{f}_\alpha > 0$, we get that

$$
\tilde{u}_\alpha (y) \geq \frac{1}{4\pi} \int_{B_y(2R)} \left( \frac{1}{|x - y|} - \frac{1}{2R} \right) \tilde{u}_\alpha (x) \, dx
\tag{4.10}
$$

$$- \frac{\mu_\alpha^2}{4\pi} \int_{B_y(R)} \left( \frac{1}{|x - y|} - \frac{1}{2R} \right) \tilde{h}_\alpha (x) \tilde{u}_\alpha (x) \, dx.
$$

We deduce that

$$
\int_{B_y(R)} |x - y|^{-1} \tilde{u}_\alpha (x) \leq 8\pi \left( \sup_{B_y(2R)} \tilde{u}_\alpha \right)^8 \left( 1 + 2R^2 \mu_\alpha^2 \sup_{B_y(2R)} |\tilde{h}_\alpha| \right).
$$

Using (4.10), we get that

$$
\limsup_{\alpha \to +\infty} \int_{B_y(R)} |x - y|^{-1} \tilde{u}_\alpha (x) \, dx \leq C_1
$$

for any $R > 0$ and any $y \in \mathbb{R}^3$ where $C_1 > 0$ is some constant independent of $R$ and $y$. Thanks to (2.2) and (4.9), this leads to the existence of some $C_2 > 0$ independent of $R > 0$ and $y \in \mathbb{R}^3$ such that

$$
\limsup_{\alpha \to +\infty} \int_{B_y(R) \setminus B_y(\frac{R}{2})} |x - y|^{-1} \left| \mathcal{L}_\xi \hat{W}_\alpha \right|_\xi (x) \, dx \leq C_2
$$

for all $R > 0$ and all $y \in \mathbb{R}^3$. We deduce easily that: for any $y \in \mathbb{R}^3$ and for any $R > 0$, there exists $\frac{R}{2} \leq r_\alpha \leq R$ such that

$$
\int_{\partial B_y(r_\alpha)} \left| \mathcal{L}_\xi \hat{W}_\alpha \right|_\xi^2 (x) \, dx \leq \frac{2C_2 r_\alpha}{R}.
\tag{4.12}
$$

We use now the Green representation formula, see Proposition 9.1, to write that

$$
\left| \mathcal{L}_\xi \hat{W}_\alpha \right|_\xi (y) \leq C_3 \int_{B_y(r_\alpha)} |x - y|^{-2} \left| \Delta_\xi \hat{W}_\alpha \right| (x) \, dx + C_3 r_\alpha^{-2} \int_{\partial B_y(r_\alpha)} \left| \mathcal{L}_\xi \hat{W}_\alpha \right| \, d\sigma
\leq C_3 \int_{B_y(r_\alpha)} |x - y|^{-2} \left| \Delta_\xi \hat{W}_\alpha \right| (x) \, dx + \frac{4C_3 \sqrt{\pi C_2}}{R}
$$

thanks to (4.12) and to the fact that $2r_\alpha \geq R$. Using (2.2), (4.8), (4.9) and (4.10), we get that

$$
\int_{B_y(r_\alpha)} |x - y|^{-2} \left| \Delta_\xi \hat{W}_\alpha \right| (x) \, dx \to 0
$$
as $\alpha \to +\infty$. Thus we obtain that

$$
\limsup_{\alpha \to +\infty} \left| \mathcal{L}_\xi \hat{W}_\alpha \right|_\xi (y) \leq \frac{4C_3 \sqrt{\pi C_2}}{R}.
\tag{4.13}
$$

Since this holds for all $R > 0$, we have proved that

$$
\mathcal{L}_\xi \hat{W}_\alpha \to 0 \text{ in } L_{\text{loc}}^\infty (\mathbb{R}^3) \text{ as } \alpha \to +\infty.
$$
Then we have thanks to (2.2) and (4.9) that
\[ \mu^2 \hat{h}_\alpha \to 0, \hat{j}_\alpha \to f_0(x_0) \] and \( \tilde{a}_\alpha \to 0 \) in \( L^\infty_{\text{loc}}(\mathbb{R}^3) \) as \( \alpha \to +\infty \)
where \( x_\alpha \to x_0 \) as \( \alpha \to +\infty \) (up to a subsequence). By (4.11) and (4.13), there holds \( \tilde{u}_\alpha(0) = \sqrt{2} + o(1) \). Thanks to (4.10) and (4.13), the Harnack inequality of Proposition 6.1 shows that \( (\tilde{u}_\alpha) \) is uniformly positive in any compact set of \( \mathbb{R}^3 \) so that we can pass to the limit in equation (4.8) to get that, after passing to a subsequence,
\[ \tilde{u}_\alpha \to \tilde{u} \in C^{1,\eta}_{\text{loc}}(\mathbb{R}^3) \] as \( \alpha \to +\infty \),
where
\[ \Delta \xi \tilde{u} = f_0(x_0) \tilde{u}^5 \] in \( \mathbb{R}^3 \)
and
\[ 0 < \tilde{u} \leq \tilde{u}(0) = \sqrt{2} \]
thanks to (4.10), (4.11) and (4.13). Then, by the classification result of Caffarelli-Gidas-Spruck [5], we have that \( \tilde{u}(x) = U(\sqrt{4f_0(x_0)} \cdot x) \) so that it possesses a strict maximum in 0. Thus \( \tilde{u}_\alpha \) necessarily possesses a critical point in the neighbourhood of 0 for \( \alpha \) large, which means that \( u_\alpha \) possesses a critical point \( y_\alpha \in S^3 \) with \( d_h(y_\alpha, x_\alpha) = o(\mu_\alpha) \). This critical point \( y_\alpha \) clearly violates point (iv). This proves that (iii) must hold and this end the proof of the claim.  

For the following claims (4.4 and 4.5), we consider \( x_\alpha \in S^3 \) such that \( \nabla u_\alpha(x_\alpha) = 0 \) and \( \rho_\alpha > 0 \) such that there exists \( C > 0 \) such that
\[ d_h(x_\alpha, x)^3 (u_\alpha(x)^6 + |L_h W_\alpha|_h(x)) \leq C \]
and such that
\[ \frac{1}{C} \leq \rho_\alpha^\frac{1}{2} u_\alpha(x_\alpha) \geq \frac{1}{C} \quad \text{as in claim 4.3.} \]

Claim 4.4. If
\[ \rho_\alpha^\frac{3}{2} \sup_{B_{2\rho}(x_\alpha)} (u_\alpha^6 + |L_h W_\alpha|_h) \leq C \]
for some \( C_1 > 0 \), then there exists \( C_2 > 0 \) such that
\[ \rho_\alpha^\frac{1}{2} u_\alpha \geq C_2 \text{ in } B_{2\rho}(4\rho_\alpha). \]

Proof - If \( \rho_\alpha \not\to 0 \) as \( \alpha \to +\infty \), it is a simple consequence of Claim 4.1. If \( \rho_\alpha \to 0 \) as \( \alpha \to +\infty \), we want to apply Harnack’s inequality to \( \rho_\alpha^\frac{1}{2} u_\alpha(x_\alpha) \). It is uniformly bounded in the \( \rho_\alpha^{-1} \)-dilated image by the stereographic projection of the ball \( B_{2\rho}(8\rho_\alpha) \) and \( \rho_\alpha^\frac{1}{2} u_\alpha(x_\alpha)(0) \geq \sqrt{2} \) thanks to the assumption of the claim. In this same ball, \( \rho_\alpha^\frac{3}{2} [L_h W_\alpha(x_\alpha)] \) is also uniformly bounded thanks to the assumption of the claim. Thus we can use Harnack’s inequality, see proposition 6.1, together with (2.2) to conclude.  

\[ \Box \]
Claim 4.5. If
\[
\rho^3 \sup_{B_{\alpha}(8\rho)} (u^6_{\alpha} + |L^h W_{\alpha}|^2) \to +\infty \text{ as } \alpha \to +\infty ,
\]
then \(\rho \to 0\) as \(\alpha \to +\infty\) and
\[
\sup_{B_0(\rho_0)} \frac{u_{\alpha} [x_{\alpha}] U}{B_0} - 1 \to 0 \text{ as } \alpha \to +\infty ,
\]
where
\[
B_\alpha(x) = \sqrt{2} \mu_\alpha^\frac{1}{2} \left( \mu_\alpha^2 + \frac{4 f_\alpha(x)}{3} |x|^2 \right)^{-\frac{1}{2}}
\]
with
\[
u_\alpha(x) = \mu_\alpha^{-\frac{1}{2}} \to +\infty \text{ as } \alpha \to +\infty
\]
and \(\rho_\alpha / \mu_\alpha \to +\infty\) as \(\alpha \to +\infty\).

Proof - We postpone the proof of this claim to Section 5. This is the core of the analysis of this paper.

We are now in position to conclude the proof of Theorem 2. We let
\[
16 d_\alpha = \min_{i \neq j} d_h (x_{i,\alpha}, x_{j,\alpha}) \quad (4.17)
\]
and we assume that
\[
d_\alpha \to 0 \text{ as } \alpha \to +\infty , \quad (4.18)
\]
where the \(x_{i,\alpha}\)'s, \(i = 1, \ldots, N_\alpha\), are those of claim 4.3 and we assume, up to reordering, that
\[
d_h (x_{1,\alpha}, x_{2,\alpha}) = 16 d_\alpha.
\]
Note that, by Claim 4.5 above and thanks to Claim 4.1, we know that \(N_\alpha \geq 2\) for \(\alpha\) large. We let
\[
\tilde{u}_\alpha(x) = d_\alpha^\frac{1}{2} u_{x_{1,\alpha}}(d_\alpha x) U(d_\alpha x).
\]
Equation (2.1) becomes
\[
\Delta \xi \tilde{u}_\alpha + d_\alpha^2 \tilde{h}_\alpha \tilde{u}_\alpha = \tilde{f}_\alpha \tilde{u}_\alpha^5 + \tilde{a}_\alpha \tilde{u}_\alpha^{-7}, \quad (4.19)
\]
where
\[
\tilde{a}_\alpha = (64 + o(1)) \left| \frac{L^\xi \tilde{W}_\alpha}{\xi} \right|^2 + o(1)
\]
in any compact set with
\[
\tilde{W}_\alpha(x) = d_\alpha^2 U(d_\alpha x)^{-4} W_{x_{1,\alpha}}(d_\alpha x),
\]
\[
\tilde{f}_\alpha(x) = f_\alpha [x_{\alpha}](d_\alpha x),
\]
\[
\tilde{h}_\alpha(x) = h_\alpha [x_{\alpha}](d_\alpha x).
\]
We shall also let
\[
\tilde{x}_{i,\alpha} = d_\alpha^{-1} \pi_{x_{i,\alpha}}(x_{i,\alpha})
\]
for \(i = 1, \ldots, N_\alpha\). Let us fix \(R > 0\) large. We reorder the concentration points such that
\[
|x_{i,\alpha}| \leq R \text{ if and only if } 1 \leq i \leq N_R
\]
for some $N_R \geq 2$ fixed (up to a subsequence and for $\alpha$ large). For any $i \in \{1, \ldots, N_R\}$, we have the following alternative, thanks to Claims 4.4 and 4.5:

$$\text{either there exists } C_i > 0 \text{ such that } C_i^{-1} \leq \tilde{u}_\alpha \leq C_i \text{ in } B_{\tilde{x},\alpha} \left(\frac{1}{2}\right)$$

or

$$\sup_{B_{\tilde{x},\alpha} \left(\frac{1}{2}\right)} \left| \frac{\tilde{u}_\alpha}{\tilde{B}_{i,\alpha}} - 1 \right| \to 0 \text{ as } \alpha \to +\infty,$$

where

$$\tilde{B}_{i,\alpha}(x) = \sqrt{2\mu_{i,\alpha}^\frac{1}{4}} \left( \mu_{i,\alpha}^2 + \frac{4f_\alpha(x_{i,\alpha})}{3} |x - \tilde{x}_{i,\alpha}|^2 \right)^{-\frac{1}{2}}$$

with $\sqrt{2\mu_{i,\alpha}^\frac{1}{4}} = \tilde{u}_\alpha(\tilde{x}_{i,\alpha})$. These two alternatives are exclusive one to each other and the second one appears if and only if $\tilde{u}_\alpha(\tilde{x}_{i,\alpha}) \to +\infty$ as $\alpha \to +\infty$.

Using the Green’s representation formula and (4.19), we can write that

$$\tilde{u}_\alpha(x) \geq \frac{1}{4\pi} \int_{B_{\alpha}(10R)} \left( \frac{1}{|x - y|} - \frac{1}{10R} \right) \tilde{f}_\alpha(y) \tilde{u}_\alpha(y) dy - \frac{d_\alpha^2}{4\pi} \int_{B_{\alpha}(10R)} \left( \frac{1}{|x - y|} - \frac{1}{10R} \right) \tilde{h}_\alpha(y) \tilde{u}_\alpha(y) dy$$

since $\tilde{f}_\alpha > 0$ and $\tilde{u}_\alpha > 0$. By the dominated convergence theorem, since

$$\tilde{u}_\alpha \leq D_1 d_x(x, [\tilde{x}_{i,\alpha}])^{-\frac{1}{2}}$$

for some $D_1 > 0$, we have that there exists $D_2 > 0$ which does not depend on $\alpha$ and $x$ such that

$$\frac{d_\alpha^2}{4\pi} \int_{B_{\alpha}(10R)} \left( \frac{1}{|x - y|} - \frac{1}{10R} \right) \tilde{h}_\alpha(y) \tilde{u}_\alpha(y) dy \leq D_2 d_\alpha^2$$

for all $x \in B_0(R)$. Thus the above becomes

$$\tilde{u}_\alpha(x) \geq \frac{1}{4\pi} \int_{B_{\alpha}(10R)} \left( \frac{1}{|x - y|} - \frac{1}{10R} \right) \tilde{f}_\alpha(y) \tilde{u}_\alpha(y) dy - D_2 d_\alpha^2$$

for all $x \in B_0(R)$.

If the first alternative in (4.20) holds for some $i \in \{1, \ldots, N_R\}$, then we can write that

$$\frac{1}{4\pi} \int_{B_{\alpha}(10R)} \left( \frac{1}{|x - y|} - \frac{1}{10R} \right) \tilde{f}_\alpha(y) \tilde{u}_\alpha(y) dy \geq \frac{1}{4\pi} C_i^{\frac{1}{2}} \int_{B_{\tilde{x},\alpha} \left(\frac{1}{2}\right)} \left( \frac{1}{|x - y|} - \frac{1}{10R} \right) \tilde{f}_\alpha(y) dy$$

$$\geq D_3$$

for some $D_3 > 0$ independent of $\alpha$ and $x$ for all $x \in B_0(R)$ since $\tilde{f}_\alpha$ is uniformly positive. This implies thanks to (4.18) and (4.21) that

$$\tilde{u}_\alpha \geq \frac{1}{2} D_3$$

for $\alpha$ large for all $x \in B_0(R)$. This proves that the second alternative in (4.20) can not happen for any $j \in \{1, \ldots, N_R\}$.
So far we have proved that either the first alternative in (1.20) holds for all \( i \in \{1, \ldots, N_R\} \), in which case \( \tilde{u}_\alpha \) is uniformly bounded in \( B_0 \left( \frac{R}{2} \right) \), or that the second alternative in (1.20) holds for all \( i \in \{1, \ldots, N_R\} \).

We claim now that we are in the first situation, namely that the second alternative in (1.20) can not hold for all \( i \in \{1, \ldots, N_R\} \). Indeed, assume it holds for some \( i \in \{1, \ldots, N_R\} \), then it holds for all of them. Then, we can use the estimate of the second alternative in (1.20) to write that

\[
\frac{1}{4\pi} \int_{B_{i,\alpha}(\frac{1}{4})} \left( \frac{1}{|x-y|} - \frac{1}{10R} \right) \tilde{f}_\alpha(y)\tilde{u}_\alpha(y)^5 \, dy \geq (1 + o(1)) \tilde{B}_{i,\alpha}(x) - \frac{D_4}{R} \mu_{i,\alpha}^\frac{4}{5}
\]

for \( i = 1, 2 \) where \( D_4 > 0 \) is some constant independent of \( R, \alpha \) and \( x \). This proves thanks to (4.21) that

\[
\tilde{u}_\alpha(x) \geq (1 + o(1)) \left( \tilde{B}_{i,\alpha}(x) + \tilde{B}_{2,\alpha}(x) \right) - \frac{D_4}{R} \left( \sqrt{\mu_{1,\alpha}} + \sqrt{\mu_{2,\alpha}} \right) - D_2 d_\alpha^2.
\]

Applying this to some \( x \) such that \(|x| \leq \frac{1}{4}\), we get thanks to the second alternative of (1.20) for \( i = 1 \) that

\[
1 + o(1) \geq (1 + o(1)) \left( 1 + \frac{\tilde{B}_{2,\alpha}(x)}{\tilde{B}_{1,\alpha}(x)} \right) - \frac{D_4}{R} \left( \frac{\sqrt{\mu_{1,\alpha}} + \sqrt{\mu_{2,\alpha}}}{\tilde{B}_{1,\alpha}(x)} \right) - D_2 d_\alpha^2 \frac{\tilde{B}_{1,\alpha}(x)}{\tilde{B}_{1,\alpha}(x)}.
\]

Thus we get that

\[
\frac{\tilde{B}_{2,\alpha}(x)}{\tilde{B}_{1,\alpha}(x)} \leq o(1) + \frac{D_4}{R} \left( \frac{\sqrt{\mu_{1,\alpha}} + \sqrt{\mu_{2,\alpha}}}{\tilde{B}_{1,\alpha}(x)} \right) - D_2 d_\alpha^2 \frac{\tilde{B}_{1,\alpha}(x)}{\tilde{B}_{1,\alpha}(x)}.
\]

If \( x \neq 0 \), we have that

\[
\frac{\tilde{B}_{2,\alpha}(x)}{\tilde{B}_{1,\alpha}(x)} = (1 + o(1)) \frac{|x|\sqrt{\mu_{2,\alpha}}}{|x - \tilde{x}_2|\sqrt{\mu_{1,\alpha}}},
\]

where \( \tilde{x}_2 = \lim_{\alpha \to +\infty} \tilde{x}_{2,\alpha} \) is such that \(|\tilde{x}_2| = 16\). We also have that

\[
\left( \frac{\sqrt{\mu_{1,\alpha}} + \sqrt{\mu_{2,\alpha}}}{\tilde{B}_{1,\alpha}(x)} \right) \leq D_5 |x| \left( 1 + \frac{\sqrt{\mu_{2,\alpha}}}{\sqrt{\mu_{1,\alpha}}} \right)
\]

for some \( D_5 > 0 \) independent of \( x \) and \( \alpha \). At last, we know that

\[
\frac{d_\alpha^2}{\tilde{B}_{1,\alpha}(x)} \to 0 \quad \text{as} \quad \alpha \to +\infty
\]

thanks to claim 4.1 and the second alternative in (1.20) which tells us that \( d_\alpha = O(\mu_{1,\alpha}) \). Thus we arrive to

\[
\frac{|x|\sqrt{\mu_{2,\alpha}}}{|x - \tilde{x}_2|\sqrt{\mu_{1,\alpha}}} \leq o(1) + \frac{D_4 D_5}{R} |x| \left( 1 + \frac{\sqrt{\mu_{2,\alpha}}}{\sqrt{\mu_{1,\alpha}}} \right) (1 + o(1)).
\]

Up to choose \( R \) large enough, this leads, letting \( x \to 0 \), to

\[
\limsup_{\alpha \to +\infty} \frac{\sqrt{\mu_{2,\alpha}}}{\sqrt{\mu_{1,\alpha}}} \leq \frac{16D_4 D_5}{R - 16D_4 D_5}.
\]

Using the same arguments exchanging the role of \( \tilde{x}_{1,\alpha} \) and \( \tilde{x}_{2,\alpha} \), we obtain also that

\[
\limsup_{\alpha \to +\infty} \frac{\sqrt{\mu_{1,\alpha}}}{\sqrt{\mu_{2,\alpha}}} \leq \frac{16D_4 D_5}{R - 16D_4 D_5}.
\]
This clearly leads to a contradiction as soon as $R$ is chosen large enough so that the right-hand side is less than 1.

Thus we have proved that, up to choose $R$ large enough, only the first alternative in (4.20) can happen for all $i \in \{1, \ldots, N_R\}$ and Claim 4.3 and Proposition 6.1 show that

$$C_R^{-1} \leq \tilde{u}_\alpha \leq C_R$$

for all compact set $K$ of $\mathbb{R}^3$.

Moreover, we know that $\mathcal{L}_\xi \tilde{W}_\alpha$ is uniformly bounded in any compact set of $\mathbb{R}^3$. Then we can prove as during claim 4.3 that, after passing to a subsequence,

$$\mathcal{L}_\xi \tilde{W}_\alpha \to 0 \text{ in } C^0_{\text{loc}}(\mathbb{R}^3) \text{ as } \alpha \to +\infty$$

and that

$$\tilde{u}_\alpha \to \tilde{u} \text{ in } C^1_{\text{loc}}(\mathbb{R}^3) \text{ as } \alpha \to +\infty$$

with

$$\Delta \xi \tilde{u} = f_0(x_1) \tilde{u}^5 \text{ in } \mathbb{R}^3.$$ 

Here, $x_1 = \lim_{\alpha \to +\infty} x_{1,\alpha}$. Thus $\tilde{u}$ has only one critical point in $\mathbb{R}^3$, thanks to the classification result of [5], but it should have at least two since 0 and $\tilde{x}_{2,\alpha}$ were critical points of $\tilde{u}_\alpha$. This is absurd and we have contradicted the fact that $d_\alpha \to 0$ as $\alpha \to +\infty$.

Thus $d_\alpha \geq \delta_0$ and there is only a finite number of concentration points and for each of them, it is clear that the sequence $(u_\alpha)$ remains bounded in $B_0(\delta_0)$. Otherwise, we would be in the situation of Claim 4.3 and $d_\alpha$ would have to go to 0. It is then clear thanks to this and (iii) of Claim 4.3 that the sequence $(u_\alpha)$ is uniformly bounded on $S^3$. And this ends the proof of the theorem thanks to Claim 4.2.

5. Local blow up analysis - Proof of claim 4.4

In this section, we perform the local blow up analysis needed to prove claim 4.4. We assume that we have a sequence $(u_\alpha, W_\alpha)$ of solutions of (2.1) with the convergences (2.2) and (2.3). We also assume that we have sequences $(x_\alpha)$ of critical points of $u_\alpha$ and $(\rho_\alpha)$ of positive real numbers with $0 < \rho_\alpha \leq \frac{\pi}{16}$ such that

$$d_h(x_\alpha, x)^3 (u_\alpha(x)^6 + |\mathcal{L}_h W_\alpha(x)|_h) \leq C_1 \text{ for all } x \in B_{x_\alpha}(8\rho_\alpha) \tag{5.1}$$

for some $C_1 > 0$ independent of $\alpha$. We assume moreover that

$$\rho_\alpha^3 \sup_{B_{x_\alpha}(8\rho_\alpha)} (u_\alpha^6 + |\mathcal{L}_h W_\alpha|_h) \to +\infty \text{ as } \alpha \to +\infty. \tag{5.2}$$

In this section, the constants $C_i$’s will always denote constants independent of $\alpha$. The constants $D_i$’s also but they will have nothing to do one with the other when changing of claims.

Claim 5.1. We have that, up to a subsequence,

$$\mu_\gamma u_\alpha \cdot [x_\alpha] (\mu_\alpha x) \to \left(1 + \frac{4f(x_0)}{3} |x|^2\right)^{-\frac{1}{2}} \text{ in } C^1_{\text{loc}}(\mathbb{R}^3) \text{ as } \alpha \to +\infty$$

and that

$$\mu_\alpha^3 \mathcal{L}_\xi W_\alpha \cdot [x_\alpha] (\mu_\alpha x) \to 0 \text{ in } C^0_{\text{loc}}(\mathbb{R}^3) \text{ as } \alpha \to +\infty,$$
where
\[ u_\alpha(x_\alpha) = \mu_\alpha^{-\frac{1}{2}} \to +\infty \text{ as } \alpha \to +\infty \]
and \( \lim_{\alpha \to +\infty} x_\alpha = x_0 \).

**Proof.** It is really similar to that of claim 4.3. We let \( y_\alpha \in S^3 \) be such that
\[ u_\alpha(y_\alpha)^6 + |L_hW_\alpha|_h(y_\alpha) = \sup_{B_{x_\alpha}(\rho_\alpha)} (u_\alpha^6 + |L_hW_\alpha|_h) . \]  
(5.3)
And we set
\[ u_\alpha(y_\alpha)^6 + |L_hW_\alpha|_h(y_\alpha) = \tilde{\mu}_\alpha^{-3} . \]  
(5.4)
By (5.2), we know that
\[ \rho_\alpha \rightarrow +\infty \text{ as } \alpha \to +\infty . \]  
(5.5)
We also have thanks to (5.1) that
\[ \| \tilde{u}_\alpha \|_\infty \leq C_1 \]  
(5.6)
We set
\[ \tilde{u}_\alpha = \tilde{\mu}_\alpha \frac{u_\alpha}{\mu_\alpha} \]  
(5.7)
and
\[ \tilde{W}_\alpha = \tilde{\mu}_\alpha^2 U(\tilde{\mu}_\alpha x)^{-4} W_\alpha(x_\alpha) \]  
(5.8)
for \( x \in \mathbb{R}^3 \). We set
\[ \tilde{y}_\alpha = \tilde{\mu}_\alpha^{-1} \pi x_\alpha(y_\alpha) \]  
(5.9)
and we know thanks to (5.6) that
\[ |\tilde{y}_\alpha| = O(1) \]  
(5.10)
so that, after passing to a subsequence,
\[ \tilde{y}_\alpha \to \tilde{y}_0 \text{ as } \alpha \to +\infty . \]  
(5.11)
Thanks to (5.5) (5.3) and (5.4), we know that
\[ U(\tilde{\mu}_\alpha x)^{-6} \tilde{u}_\alpha(x)^6 + |\nabla \tilde{W}_\alpha|_\xi(x) \leq 1 = U(\tilde{\mu}_\alpha \tilde{y}_\alpha)^{-6} \tilde{u}_\alpha(\tilde{y}_\alpha)^6 + |\nabla \tilde{W}_\alpha|_\xi(\tilde{y}_\alpha) \]  
(5.12)
for all \( x \in \mathbb{R}^3 \) such that \( d_h(x_\alpha, \pi x_\alpha^{-1}(\mu_\alpha x)) \leq 8 \rho_\alpha \). We also have since \( x_\alpha \) is a critical point of \( u_\alpha \) that
\[ \nabla \tilde{u}_\alpha(0) = 0 . \]  
(5.13)
Then, equation (2.1) leads with (3.4), (3.5) and (3.6) to
\[
\begin{cases}
\Delta \xi \tilde{u}_\alpha + \tilde{\mu}_\alpha^2 \nabla \tilde{u}_\alpha = \tilde{f}_\alpha \tilde{u}_\alpha + \frac{\tilde{a}_\alpha}{\tilde{u}_\alpha^2} \\
\left( \Delta \xi \tilde{W}_\alpha \right)_i = -6 \tilde{\mu}_\alpha^3 \frac{x^i}{1 + \tilde{\mu}_\alpha^2 |x|^2} \left( \nabla \tilde{W}_\alpha \right)_i + \tilde{\mu}_\alpha \tilde{u}_\alpha^6 \left( \tilde{X}_\alpha \right)_i + \tilde{\mu}_\alpha^4 \left( \tilde{Y}_\alpha \right)_i
\end{cases}
\]  
(5.14)
where
\[ \tilde{h}_\alpha(x) = \left( h_\alpha [\mu_\alpha x] (\tilde{\mu}_\alpha x) - \frac{3}{4} \right) U (\tilde{\mu}_\alpha x)^4 , \]
\[ \bar{f}_\alpha(x) = f_\alpha [\mu_\alpha x] (\tilde{\mu}_\alpha x) , \]
\[ \bar{a}_\alpha(x) = \left| \dot{\mu}_\alpha^3 \tilde{U}_\alpha + U (\tilde{\mu}_\alpha x)^6 L_\xi \tilde{W}_\alpha \right|^2 + \dot{\mu}_\alpha^6 U (\tilde{\mu}_\alpha x)^{12} b_\alpha [\mu_\alpha x] (\tilde{\mu}_\alpha x) , \]
\[ (5.15) \]
\[ \ddot{U}_\alpha(x) = U (\tilde{\mu}_\alpha x)^2 (\pi_{x_\alpha})^* U_\alpha (\tilde{\mu}_\alpha x) , \]
\[ \ddot{X}_\alpha(x) = U (\tilde{\mu}_\alpha x)^{-6} (\pi_{x_\alpha})^* X_\alpha (\tilde{\mu}_\alpha x) , \]
\[ \ddot{Y}_\alpha(x) = (\pi_{x_\alpha})^* Y_\alpha (\tilde{\mu}_\alpha x) . \]

Given \( y \in \mathbb{R}^3 \) and \( R > 0 \), let us use the Green representation formula to write that
\[ \bar{u}_\alpha(y) \geq \frac{1}{4\pi} \int_{B_y(2R)} \left( \frac{1}{|x-y|} - \frac{1}{2R} \right) \Delta_\xi \bar{u}_\alpha \, dx \]
since \( \bar{u}_\alpha \geq 0 \). Using equation \((5.14)\) and the fact that \( \bar{f}_\alpha \geq 0 \), we get that
\[ \bar{u}_\alpha(y) \geq \frac{1}{8\pi} \int_{B_y(R)} \frac{1}{|x-y|} \frac{\tilde{a}_\alpha(x)}{\bar{u}_\alpha(x)^2} \, dx - \frac{\tilde{\mu}_\alpha^2}{4\pi} \int_{B_y(R)} \left( \frac{1}{|x-y|} - \frac{1}{2R} \right) \tilde{h}_\alpha(x) \bar{u}_\alpha(x) \, dx . \]

We deduce that
\[ \int_{B_y(R)} |x-y|^{-1} \tilde{a}_\alpha(x) \leq 8\pi \left( \sup_{B_y(R)} \tilde{u}_\alpha \right)^8 \left( 1 + 2R^2 \tilde{\mu}_\alpha^2 \sup_{B_y(R)} |\tilde{h}_\alpha| \right) . \]

Using \((5.12)\), we get that
\[ \limsup_{\alpha \to +\infty} \int_{B_y(R)} |x-y|^{-1} \tilde{a}_\alpha(x) \, dx \leq D_1 \]
for any \( R > 0 \) and any \( y \in \mathbb{R}^3 \) where \( D_1 > 0 \) is some constant independent of \( R \) and \( y \). Thanks to \((5.2)\) and \((5.13)\), this leads to the existence of some \( D_2 > 0 \) independent of \( R > 0 \) and \( y \) such that
\[ \limsup_{\alpha \to +\infty} \int_{B_y(R) \setminus B_y(\frac{3}{2})} |x-y|^{-1} \left| L_\xi \tilde{W}_\alpha \right|^2 (x) \, dx \leq D_2 \]
for all \( R > 0 \) and all \( y \in \mathbb{R}^3 \). We deduce easily that: for any \( y \in \mathbb{R}^3 \) and for any \( R > 0 \), there exists \( \frac{R}{2} \leq r_\alpha \leq R \) such that
\[ \int_{\partial B_y(r_\alpha)} \left| L_\xi \tilde{W}_\alpha \right|^2 (x) \, dx \leq 2D_2 . \]
\[ (5.16) \]
We use now the Green representation formula, see Proposition 9.1, to write that
\[
\left|L_\xi \tilde{W}_\alpha\right|(y) \leq D_3 \int_{B_y(r_\alpha)} |x-y|^{-2} \left|\Delta_\xi \tilde{W}_\alpha(x)\right| \, dx + D_3 r_\alpha^{-2} \int_{\partial B_y(r_\alpha)} \left|L_\xi \tilde{W}_\alpha\right| \, d\sigma
\]
\[
\leq D_3 \int_{B_y(r_\alpha)} |x-y|^{-2} \left|\Delta_\xi \tilde{W}_\alpha(x)\right| \, dx + \frac{2D_3 \sqrt{8\pi D_2}}{R}.
\]

thanks to (5.16) and to the fact that \(2r_\alpha \geq R\). Using (2.2), (5.12), (5.14), (5.15) and the fact that \(\tilde{\mu}_\alpha \to 0\) as \(\alpha \to +\infty\), we get that
\[
\int_{B_y(r_\alpha)} |x-y|^{-2} \left|\Delta_\xi \tilde{W}_\alpha(x)\right| \, dx \to 0
\]
as \(\alpha \to +\infty\). Thus we obtain that
\[
\limsup_{\alpha \to +\infty} \left|L_\xi \tilde{W}_\alpha\right|(y) \leq \frac{2D_3 \sqrt{8\pi D_2}}{R}.
\]
Since this holds for all \(R > 0\), we have proved that
\[
L_\xi \tilde{W}_\alpha \to 0 \text{ in } L^\infty_{\text{loc}}(\mathbb{R}^3) \quad \text{as } \alpha \to +\infty.
\] (5.17)

Then we have thanks to (2.2) and (5.15) and since \(\tilde{\mu}_\alpha \to 0\) as \(\alpha \to +\infty\) that
\[
\mu_2^2 \tilde{h}_\alpha \to 0, \quad \tilde{f}_\alpha \to f(x_0) \quad \text{and} \quad \tilde{a}_\alpha \to 0 \text{ in } L^\infty_{\text{loc}}(\mathbb{R}^3) \quad \text{as } \alpha \to +\infty,
\]
where \(x_\alpha \to x_0\) as \(\alpha \to +\infty\) (up to a subsequence). By (5.12) and (5.17), there holds \(\tilde{u}_\alpha(0) = \sqrt{2} + o(1)\). Thanks to (5.12) and (5.17), the Harnack inequality of Proposition 6.1 shows that \((\tilde{u}_\alpha)\) is uniformly positive in any compact set of \(\mathbb{R}^3\) so that we can pass to the limit in equation (5.14) to get that, after passing to a subsequence,
\[
\tilde{u}_\alpha \to \tilde{u} \text{ in } C^{1,\eta}_{\text{loc}}(\mathbb{R}^3) \quad \text{as } \alpha \to +\infty,
\] (5.18)

where
\[
\Delta_\xi \tilde{u} = f_0(x_0) \tilde{u}^5 \text{ in } \mathbb{R}^3
\]
and
\[
0 < \tilde{u} \leq \tilde{u} (\tilde{y}_0) = \sqrt{2}
\]
thanks to (5.12) and (5.17). Then, by the classification result of Caffarelli-Gidas-Spruck [5], we have that
\[
\tilde{u} = \left(1 + \frac{4f(x_0) |x-y_0|^2}{3} \right)^{-\frac{1}{2}}. \tag{5.19}
\]
Since \(\nabla \tilde{u}(0) = 0\) thanks to (5.13), we deduce that \(\tilde{y}_0 = 0\) and thus that \(\frac{\mu}{\tilde{\mu}_\alpha} \to 1\) as \(\alpha \to +\infty\). The claim follows. \(\diamond\)

We set in the following
\[
B_\alpha(x) = \sqrt{2\mu_\alpha} \left(\mu_\alpha^2 + \frac{4f_\alpha(x_\alpha)}{3} |x|^2\right)^{-\frac{1}{2}}. \tag{5.19}
\]
satisfying $\Delta_\xi B_\alpha = f_\alpha (x_\alpha) B_\alpha^5$ in $\mathbb{R}^3$. We also set for $x \in B_0(1)$:

$$v_\alpha(x) = u_\alpha (x_\alpha) U(x) ,$$

$$\hat{W}_\alpha(x) = U(x)^{-4} W_\alpha (x_\alpha) ,$$

$$\hat{h}_\alpha(x) = \left( h_\alpha (x_\alpha) - \frac{3}{4} \right) U(x)^4 ,$$

$$\hat{f}_\alpha(x) = f_\alpha (x_\alpha) ,$$

$$\hat{a}_\alpha(x) = \left| U^2 (\pi_{x_\alpha})_* U_\alpha + U^6 \mathcal{L}_\xi \hat{W}_\alpha \right|^2 (x) + U(x)^{12} h_\alpha (x_\alpha) ,$$

$$\hat{X}_\alpha(x) = U(x)^{-6} (\pi_{x_\alpha})_* X_\alpha(x) ,$$

$$\hat{Y}_\alpha(x) = (\pi_{x_\alpha})_* Y_\alpha(x) .$$

so that equation (2.1) becomes thanks to (3.4) and (3.6) the following:

$$\begin{cases}
\Delta_\xi v_\alpha + \hat{h}_\alpha v_\alpha = \hat{f}_\alpha v_\alpha^5 + \hat{a}_\alpha v_\alpha^{-7} \\
\left( \Delta_\xi \hat{W}_\alpha \right)_i = -\frac{6x^j}{1 + |x|^2} \left( \mathcal{L}_\xi \hat{W}_\alpha \right)_{ij} + v_\alpha^6 \left( \hat{X}_\alpha \right)_i + \left( \hat{Y}_\alpha \right)_i
\end{cases} \quad (5.21)$$

Note that claim 5.1 tells us that

$$\mu^\frac{1}{2} v_\alpha (\mu_\alpha x) \to U \left( \sqrt{\frac{4f(x_0)}{3}} \right) \text{ in } C^1_{\text{loc}} (\mathbb{R}^3) \text{ as } \alpha \to +\infty \quad (5.22)$$

and that

$$\mu^3 \left| \mathcal{L}_\xi \hat{W}_\alpha \right|_{\xi} (\mu_\alpha x) \to 0 \text{ in } L^\infty_{\text{loc}} (\mathbb{R}^3) \text{ as } \alpha \to +\infty . \quad (5.23)$$

We set also

$$\hat{\rho}_\alpha = \tan (4\rho_\alpha) \quad (5.24)$$

so that (5.1) becomes

$$|x|^3 \left( v_\alpha(x)^6 + \left| \mathcal{L}_\xi \hat{W}_\alpha \right|_{\xi} (x) \right) \leq C_2 \quad (5.25)$$

in $B_0(8\hat{\rho}_\alpha)$ for some $C_2 > 0$ independent of $\alpha$ and $x$.

Fix $\varepsilon > 0$. We define now $r_\alpha > 0$ by

$$r_\alpha = \sup \{ 0 \leq r \leq \hat{\rho}_\alpha \text{ s.t. } v_\alpha < (1 + \varepsilon) B_\alpha \text{ in } B_0(r) \} \quad (5.26)$$

where $\hat{\rho}_\alpha$ is as in (5.24). Thanks to (5.22), it is clear that

$$\frac{r_\alpha}{\mu_\alpha} \to +\infty \text{ as } \alpha \to +\infty . \quad (5.27)$$

Claim 5.2. There exists $C_3 > 0$ such that

$$v_\alpha \leq C_3 B_\alpha$$

in $B_0(7r_\alpha)$. 
We know moreover that
\[ \text{where} \]
\[ D \]
Then \( \tilde{v}_\alpha \) satisfies in \( B_0(16) \) the equation
\[ \Delta \tilde{v}_\alpha + \frac{1}{4} \partial^2 \tilde{v}_\alpha \left( \frac{1}{2} r_\alpha x \right) \tilde{v}_\alpha = f_\alpha \left( \frac{1}{2} r_\alpha x \right) \tilde{v}_\alpha + \left( \frac{r_\alpha}{2} \right) 6 \partial_\alpha \left( \frac{r_\alpha x}{\tilde{v}_\alpha} \right). \]
Thanks to \( 2.2 \) and \( 5.25 \), we know that there exists \( D_1 > 0 \) such that
\[ |x|^2 \tilde{v}_\alpha(x) + |x|^6 a_\alpha \left( \frac{1}{2} r_\alpha x \right) \leq D_1 \]
in \( B_0(16) \). In particular, this implies that
\[ \tilde{v}_\alpha \leq D_1 \text{ in } B_0(16) \setminus B_0(1). \]
We know moreover that
\[ \tilde{v}_\alpha \leq D_2 \mu_\alpha r_\alpha^{-\frac{1}{2}} \text{ in } B_0(2) \setminus B_0(1) \]
for some \( D_2 > 0 \) thanks to the definition of \( r_\alpha \). Applying the Harnack inequality of Proposition 6.1 on \( B_y(1) \) for any \( y \in B_0(3) \setminus B_0(2) \), we get that
\[ \tilde{v}_\alpha \leq D_3 \mu_\alpha^{-\frac{1}{2}} r_\alpha^{-\frac{1}{2}} \text{ in } B_0(3) \setminus B_0(2) \]
for some \( D_3 > 0 \) independent of \( \alpha \). We can then repeat the argument on the annuli \( B_0(4) \setminus B_0(3), \ldots, \) to finally get the existence of some \( D_4 > 0 \) independent of \( \alpha \) such that
\[ \tilde{v}_\alpha \leq D_4 \mu_\alpha^{-\frac{1}{2}} r_\alpha^{-\frac{1}{2}} \text{ in } B_0(14) \setminus B_0(2). \]
This clearly leads to
\[ v_\alpha \leq D_5 B_\alpha \text{ in } B_0(7r_\alpha) \setminus B_0(r_\alpha) \]
for some \( D_5 > 0 \) independent of \( \alpha \). Since, by the definition of \( r_\alpha \), we also have that
\[ v_\alpha \leq (1 + \varepsilon) B_\alpha \text{ in } B_0(r_\alpha), \]
this ends the proof of the claim.
\[ \Box \]
Note that, as a consequence of Claim 4.1 and Claim 5.2 we know that
\[ r_\alpha = O \left( \sqrt{\mu_\alpha} \right). \]
(5.28)

Claim 5.3. There exists \( \eta_\alpha \to 0 \) such that
\[ v_\alpha(y_\alpha) \geq (1 - \eta_\alpha) B_\alpha(y_\alpha) \]
for all sequences \( (y_\alpha) \) of points in \( B_0(8r_\alpha) \).

Proof. We let \( G_\alpha \) be the Green function of \( \Delta + h_\alpha \) on \( S^3 \). Let \( y_\alpha \in B_0(8r_\alpha) \).
Since \( \Delta h_\alpha + h_\alpha u_\alpha \geq 0 \) thanks to \( 2.1 \) and the fact that \( f_\alpha > 0 \) and \( b_\alpha \geq 0 \), we can write with the Green representation formula that
\[ v_\alpha(y_\alpha) \geq \mu_\alpha^\frac{1}{2} U(y_\alpha) \int_{B_0(8r_\alpha)} f_\alpha(\mu_\alpha x) \left( \mu_\alpha^\frac{1}{2} v_\alpha(\mu_\alpha x) \right)^5 H_\alpha(y_\alpha, \mu_\alpha x) \, dx, \]
where
\[ H_\alpha(y_\alpha, y) = U(y) G_\alpha \left( \pi_{x_\alpha}(y_\alpha), \pi_{x_\alpha}(y) \right). \]
We know thanks to (2.2) and to (5.28) that
\[ |y - \mu_\alpha x| H_\alpha(y, \mu_\alpha x) - \frac{\sqrt{2}}{8\pi} \to 0 \]
uniformly for \( x \in B_0 \left( \frac{8r_0}{\mu_\alpha} \right) \). Thus we have that
\[
\frac{v_\alpha(y_\alpha)}{B_\alpha(y_\alpha)} \geq (1 + o(1)) \frac{1}{8\pi} U(y_\alpha) \left( \mu_\alpha^2 + \frac{4f_\alpha(x_\alpha)}{3} |y_\alpha|^2 \right)^{\frac{1}{2}}
\times \int_{B_0 \left( \frac{8r_0}{\mu_\alpha} \right)} f_\alpha(x_\alpha) \left( \mu_\alpha^2 v_\alpha(\mu_\alpha x) \right)^{\frac{1}{2}} |y_\alpha - \mu_\alpha x|^{-1} \, dx.
\]
Fatou’s lemma together with simple computations lead then to the desired result thanks to Claim 5.1.

\[ \diamond \]

Claim 5.4. There exists \( C_4 > 0 \) such that, for any \( x \in B_0 (7r_\alpha) \),
\[
\int_{B_0 (7r_\alpha)} |x - y|^{-1} \left| \mathcal{L}_\xi \tilde{W}_\alpha \right|^2 (y) B_\alpha(y)^{-7} \, dy \leq C_4 B_\alpha(x).
\]

\[ \text{Proof.} \] We use the Green representation formula for \( \Delta_\xi + \hat{h}_\alpha \) in \( B_0 (8r_\alpha) \) to write that there exists \( D_1 > 0 \), see (20), such that
\[
v_\alpha(x) \geq D_1 \int_{B_0 (7r_\alpha)} \frac{1}{|x - y|} \hat{a}_\alpha(y) v_\alpha(y)^{-7} \, dy
\]
for all \( x \in B_0 (7r_\alpha) \) thanks to the fact that \( v_\alpha \geq 0 \). Here we also used equation (5.21) and the fact that \( \Delta_\xi v_\alpha + \hat{h}_\alpha v_\alpha \geq 0 \). We write now that there exists \( D_2 > 1 \) such that
\[
\hat{a}_\alpha \geq D_2^{-1} \left| \mathcal{L}_\xi \tilde{W}_\alpha \right|^2 - D_2
\]
in \( B_0 (7r_\alpha) \). Using claim 5.2, we get that
\[
C_3 B_\alpha(x) \geq D_1 D_2 C_3 \int_{B_0 (7r_\alpha)} \frac{1}{|x - y|} \left| \mathcal{L}_\xi \tilde{W}_\alpha \right|^2 (y) B_\alpha(y)^{-7} \, dy
\]
\[
- D_2 D_1 C_3 \int_{B_0 (7r_\alpha)} \frac{1}{|x - y|} B_\alpha(y)^{-7} \, dy.
\]
It remains to remark that there exists \( D_3 > 0 \) such that
\[
\int_{B_0 (7r_\alpha)} \frac{1}{|x - y|} B_\alpha(y)^{-7} \, dy \leq D_3 r_\alpha^{9} \mu_\alpha^{-\frac{5}{2}}
\]
and to note thanks to (5.28) that
\[
r_\alpha^{9} \mu_\alpha^{-\frac{5}{2}} \leq D_4 B_\alpha(x)
\]
for some \( D_4 > 0 \) for all \( x \in B_0 (7r_\alpha) \) to conclude.

\[ \diamond \]

Let us define the 1-form \( V_\alpha \) in \( \mathbb{R}^3 \) by
\[
V_\alpha(x)_i = \tilde{X}_\alpha(0)^i \int_{\mathbb{R}^3} B_\alpha(y)^6 \mathcal{H}_{ij}(x, y) \, dy,
\]
where
\[
\mathcal{H}_{ij}(x, y) = \frac{1}{32\pi} \left( \frac{7\delta_{ij}}{|x - y|} + \frac{(x - y)_i (x - y)_j}{|x - y|^3} \right).
\]
We have that
\[ \overline{\Delta}^3 V_\alpha = B_\alpha^6 \overline{X}_\alpha(0) \text{ in } \mathbb{R}^3. \] (5.30)
We refer here to (8.7) in section 8. We also let in the following
\[ \varepsilon_\alpha = \left| \overline{X}_\alpha(0) \right| \] (5.31)
and, if \( \varepsilon_\alpha \neq 0 \),
\[ \zeta = \lim_{\alpha \to +\infty} \frac{\overline{X}_\alpha(0)}{\varepsilon_\alpha} \] (5.32)
which is a vector in \( \mathbb{R}^3 \) of norm 1. Then direct computations give that
\[ |\mathcal{L}_\xi V_\alpha|_{\xi}(x) \leq C_4 \varepsilon_\alpha \frac{\varepsilon_\alpha}{\mu^2 + |x|^2} \] (5.33)
for all \( x \in B_0(8r_\alpha) \) for some \( C_5 > 0 \) independent of \( x \) and \( \alpha \) and that, if \( \varepsilon_\alpha \neq 0 \),
\[ \frac{r^2}{\varepsilon_\alpha} \mathcal{L}_\xi V_\alpha(r_\alpha x) \to P_{ij} \text{ in } C^{0}_{loc}(B_0(4) \setminus \{0\}) \text{ as } \alpha \to +\infty, \] (5.34)
where
\[ P_{ij}(x) = \frac{3\pi}{8} \left( \frac{3}{4f_0(x_0)} \right)^\frac{2}{3} |x|^{-3} \left( \zeta_k x^k \left( \delta_{ij} - \frac{x_i x_j}{|x|^2} \right) - x_i \zeta_j - x_j \zeta_i \right). \] (5.35)

Claim 5.5. There exists \( C_6 > 0 \) such that
\[ |\mathcal{L}_\xi \overline{W}_\alpha|_{\xi}(x) \leq C_6 \frac{\mu^2}{r_\alpha^3 (\mu^2 + |x|^2)} \] for all \( x \in B_0(2r_\alpha) \) and all \( \alpha \).

Proof. Let \( z_\alpha \in B_0(4r_\alpha) \). Thanks to claim 5.4, we know that
\[ \int_{B_0(6r_\alpha) \setminus B_0(5r_\alpha)} |z_\alpha - y|^{-1} \left| \mathcal{L}_\xi \overline{W}_\alpha \right|^2 \frac{1}{B_\alpha(y)^{-7}} dy \leq C_4 B_\alpha(z_\alpha). \]
This leads to the existence of some \( s_\alpha \in (5r_\alpha, 6r_\alpha) \) and of some \( D_1 > 0 \) independent of \( \alpha \) such that
\[ \int_{\partial B_\alpha(s_\alpha)} \left| \mathcal{L}_\xi \overline{W}_\alpha \right|^2 \frac{1}{B_\alpha(y)^{-7}} dy \leq D_1 \mu^2 r_\alpha^{-7} B_\alpha(z_\alpha). \] (5.36)
Thanks to the Green representation formula in \( B_\alpha(s_\alpha) \), see Proposition 9.1, there exists \( D_2 > 0 \) such that
\[ |\mathcal{L}_\xi (\overline{W}_\alpha - V_\alpha)|_{\xi}(z_\alpha) \leq D_2 \int_{B_0(6r_\alpha)} |z_\alpha - y|^{-2} \left| \overline{\Delta} (\overline{W}_\alpha - V_\alpha) \right| \frac{1}{\xi} dy \]
\[ + D_2 \int_{\partial B_\alpha(s_\alpha)} |z_\alpha - y|^{-2} \left| \mathcal{L}_\xi (\overline{W}_\alpha - V_\alpha) \right| \frac{1}{\xi} dy \].
The boundary term can be estimated thanks to (5.33) and (5.36). We obtain that
\[ \int_{\partial B_\alpha(s_\alpha)} |z_\alpha - y|^{-2} \left| \mathcal{L}_\xi (\overline{W}_\alpha - V_\alpha) \right| \frac{1}{\xi} dy \leq D_3 \left( \frac{\varepsilon_\alpha}{r_\alpha^2} + \frac{\mu^2 r_\alpha^{-2}}{B_\alpha(z_\alpha)^{\frac{2}{7}}} \right) \]
for some constant $D_3 > 0$ independent of $\alpha$ so that we can write that

$$\left| L_\xi \left( \hat{W}_\alpha - V_\alpha \right) \right| (z_\alpha) \leq D_2 \int_{B_0(6r_\alpha)} |z_\alpha - y|^{-2} \left| \hat{\Delta} \left( \hat{W}_\alpha - V_\alpha \right) (y) \right| \xi \ dy$$

$$+ D_2 D_3 \left( \frac{\epsilon_\alpha}{r_\alpha} + \mu_\alpha \frac{1}{r_\alpha} \right) B_\alpha (z_\alpha)^{\frac{1}{2}} .$$

Using equations (5.21) and (5.30), we have that

$$\left| \hat{\Delta} \left( \hat{W}_\alpha - V_\alpha \right) (y) \right| \xi \leq \frac{6|y|}{1 + |y|^2} \left| L_\xi \hat{W}_\alpha \right| (y) + \left| \hat{Y}_\alpha \right| (y)$$

$$+ \left| v_\alpha (y)^6 \hat{X}_\alpha (y) - B_\alpha (y)^6 \hat{X}_\alpha (0) \right| \xi .$$

Using (2.2), this leads with the previous inequality to the existence of some $D_4 > 0$ such that

$$\left| L_\xi \left( \hat{W}_\alpha - V_\alpha \right) \right| (z_\alpha) \leq D_4 \left( I_1^1 + I_2^2 + I_3^3 + I_4^4 + \frac{\epsilon_\alpha}{r_\alpha} + \frac{\mu_\alpha}{r_\alpha} B_\alpha (z_\alpha)^{\frac{1}{2}} \right) ,$$

where

$$I_1^1 = \int_{B_0(6r_\alpha)} |z_\alpha - y|^{-2} |y| \left| L_\xi \hat{W}_\alpha (y) \right| \xi \ dy ,$$

$$I_2^2 = \int_{B_0(6r_\alpha)} |z_\alpha - y|^{-2} \left| \hat{Y}_\alpha \right| (y) \ dy ,$$

$$I_3^3 = \epsilon_\alpha \left. \int_{B_0(6r_\alpha)} |z_\alpha - y|^{-2} \left| v_\alpha (y)^6 - B_\alpha (y)^6 \right| \ dy ,$$

$$I_4^4 = \int_{B_0(6r_\alpha)} |z_\alpha - y|^{-2} |y| v_\alpha (y)^6 \ dy .$$

We clearly have that

$$I_2^2 \leq D_5 r_\alpha .$$

Using Claim 5.2, we can write by direct computations that

$$I_4^4 \leq D_0 \frac{\mu_\alpha}{\mu_\alpha^2 + |z_\alpha|^2} .$$

Thanks to Claim 5.1, there exists $R_\alpha \to +\infty$ such that

$$\mu_\alpha \left\| v_\alpha - B_\alpha \right\|_{L^\infty(B_0(R_\alpha \mu_\alpha))} \to 0 \text{ as } \alpha \to +\infty .$$

Then we write using also Claim 5.2 that

$$I_3^3 \leq \epsilon_\alpha \int_{B_0(R_\alpha \mu_\alpha)} |z_\alpha - y|^{-2} \left| v_\alpha (y)^6 - B_\alpha (y)^6 \right| \ dy$$

$$+ O \left( \epsilon_\alpha \int_{B_0(6r_\alpha) \setminus B_0(\alpha \mu_\alpha)} |z_\alpha - y|^{-2} B_\alpha (y)^6 \ dy \right)$$

$$= o \left( \epsilon_\alpha \mu_\alpha^{-\frac{1}{2}} \int_{B_0(\alpha \mu_\alpha)} |z_\alpha - y|^{-2} B_\alpha (y)^5 \ dy \right)$$

$$+ O \left( \epsilon_\alpha \int_{B_0(6r_\alpha) \setminus B_0(\alpha \mu_\alpha)} |z_\alpha - y|^{-2} B_\alpha (y)^6 \ dy \right) .$$
Simple computations lead then to

\[ I_3^\alpha = o \left( \frac{\varepsilon_\alpha}{\mu_\alpha^2 + |z_\alpha|^2} \right). \]  

(5.41)

In order to estimate \( I_1^\alpha \), we use Hölder’s inequalities with exponents 4 and \( \frac{4}{3} \) to write that

\[
I_1^\alpha \leq \left( \int_{B_0(6r_\alpha)} |z_\alpha - y|^{-1} B_\alpha(y)^{-7} \left| \mathcal{L}_\xi \hat{W}_\alpha \right|^2_\xi (y) \, dy \right)^{\frac{4}{3}} \\
\left( \int_{B_0(6r_\alpha)} |z_\alpha - y|^{-\frac{2}{3}} |y|^\frac{2}{3} B_\alpha(y)^{\frac{2}{3}} \left| \mathcal{L}_\xi \hat{W}_\alpha \right|^3_\xi (y) \, dy \right)^{\frac{1}{3}}. 
\]

Using Claim 5.1 and (5.25), we can write that

\[
\left| \mathcal{L}_\xi \hat{W}_\alpha (y) \right|_\xi \leq D_7 (\mu_\alpha^2 + |y|^2)^{-\frac{\alpha}{2}} 
\]

for all \( y \in B_0(6r_\alpha) \). Using this and Claim 5.4 we get that

\[
I_1^\alpha \leq D_8 B_\alpha (z_\alpha)^{\frac{\alpha}{2}} \left( \mu_\alpha \int_{B_0(6r_\alpha)} |y|^\frac{4}{3} |z_\alpha - y|^{-\frac{2}{3}} (\mu_\alpha^2 + |y|^2)^{-\frac{\alpha}{3}} \, dy \right)^{\frac{\alpha}{4}}. 
\]

Simple computations lead then to

\[
I_1^\alpha \leq D_9 \frac{\mu_\alpha}{\mu_\alpha^2 + |z_\alpha|^2} \left( \ln \frac{2 (\mu_\alpha^2 + |z_\alpha|^2)}{\mu_\alpha^2} \right)^{\frac{\alpha}{4}}. \]  

(5.42)

Coming back to (5.37) with (5.38), (5.39), (5.40), (5.41), (5.42) but also with (5.28) and (5.33), we deduce that

\[
\left| \mathcal{L}_\xi \hat{W}_\alpha \right|_\xi (z_\alpha) \leq D_{10} \left( \frac{\mu_\alpha}{\mu_\alpha^2 + |z_\alpha|^2} \left( \ln \frac{2 (\mu_\alpha^2 + |z_\alpha|^2)}{\mu_\alpha^2} \right)^{\frac{\alpha}{4}} + \frac{\varepsilon_\alpha}{\mu_\alpha^2 + |z_\alpha|^2} + \mu_\alpha r_\alpha^{-\frac{2}{3}} B_\alpha (z_\alpha)^{\frac{\alpha}{4}} \right). \]  

(5.43)
Thanks to this estimate on $\hat{W}_\alpha$ in $B_0(4r_\alpha)$, we can sharpen the estimate on $I_\alpha^1$ for $z_\alpha \in B_0(2r_\alpha)$. Indeed, we can write that

$$I_\alpha^1 = O \left( \int_{B_0(4r_\alpha)} |z_\alpha - y|^2 |y|^{-\frac{2}{\mu_\alpha}} \left( \frac{2}{\mu_\alpha + |y|^2} \right)^{\frac{7}{4}} dy \right)$$

$$+ O \left( \mu_\alpha^2 r_\alpha^{-3} \int_{B_0(4r_\alpha)} |z_\alpha - y|^2 |y|^{-\frac{2}{\mu_\alpha}} (\mu_\alpha^2 + |y|^2)^{-\frac{7}{4}} dy \right)$$

$$+ O \left( \varepsilon_\alpha \int_{B_0(4r_\alpha)} |z_\alpha - y|^2 |y|^{-\frac{2}{\mu_\alpha}} (\mu_\alpha^2 + |y|^2)^{-1} dy \right)$$

$$+ O \left( r_\alpha^{-1} \int_{B_0(6r_\alpha) \setminus B_0(4r_\alpha)} \left| L_\xi \hat{W}_\alpha \right|_{\xi} (y) dy \right).$$

Direct computations lead to

$$I_\alpha^1 = o \left( \frac{\mu_\alpha}{\mu_\alpha^2 + |z_\alpha|^2} \right) + O \left( \mu_\alpha^2 r_\alpha^{-3} \right) + O \left( \varepsilon_\alpha \ln \frac{r_\alpha}{\mu_\alpha} \right)$$

$$+ O \left( \mu_\alpha^2 r_\alpha^{-3} \right).$$

In order to estimate the last term, we apply Claim [5.4] for some $|x| = r_\alpha$ to finally obtain

$$I_\alpha^1 = o \left( \frac{\mu_\alpha}{\mu_\alpha^2 + |z_\alpha|^2} \right) + O \left( \varepsilon_\alpha \ln \frac{r_\alpha}{\mu_\alpha} \right) + O \left( \mu_\alpha^2 r_\alpha^{-3} \right). \tag{5.44}$$

Coming back to (5.37) with (5.38), (5.39), (5.40), (5.41), (5.44) but also with (5.28), we deduce that

$$\left| L_\xi \left( \hat{W}_\alpha - V_\alpha \right) \right|_{\xi} (z_\alpha) \leq D_{11} \left( \frac{\varepsilon_\alpha}{r_\alpha^2} + \frac{\mu_\alpha^2 r_\alpha^{-3}}{\mu_\alpha^2 + |z_\alpha|^2} \right) + o \left( \varepsilon_\alpha \right). \tag{5.45}$$

We claim now that

$$\varepsilon_\alpha = O \left( \mu_\alpha^2 r_\alpha^{-3} \right). \tag{5.46}$$

Indeed, we can write thanks to Claim [5.4] applied to some $|x| = 3r_\alpha$ that, for any $\delta > 0$,

$$\int_{B_0(2\delta r_\alpha) \setminus B_0(\delta r_\alpha)} B_\alpha(y)^{-7} \left| L_\xi V_\alpha \right|_{\xi}^2 (y) dy$$

$$\leq D_{12} \mu_\alpha^2 + 2 \int_{B_0(2\delta r_\alpha) \setminus B_0(\delta r_\alpha)} B_\alpha(y)^{-7} \left| L_\xi \left( \hat{W}_\alpha - V_\alpha \right) \right|_{\xi}^2 (y) dy$$
where $D_{12}$ is of course independent of $\alpha$ and $\delta$. This leads with (5.45) to
\[
\int_{B_0(2\delta)} B_\alpha(r_\alpha y)^{-7} |\mathcal{L}_\xi V_\alpha|^2 (r_\alpha y) \, dy \\
\leq D_{13} \left( \mu_\alpha^4 + \frac{\varepsilon_\alpha^2}{r_\alpha^4} \int_{B_0(2\delta)} B_\alpha(r_\alpha y)^{-7} \, dy \\
+ \mu_\alpha^2 r_\alpha^{-9} \int_{B_0(2\delta)} B_\alpha(r_\alpha y)^{-6} \, dy + \int_{B_0(2\delta)} B_\alpha(r_\alpha y)^{-3} \, dy \right) \\
+ \alpha \left( \frac{\varepsilon_\alpha^2}{\mu_\alpha^2} \int_{B_0(2\delta)} B_\alpha(r_\alpha y)^{-3} \, dy \right),
\]
where $D_{13}$ is independent of $\alpha$ and $\delta$. After simple computations, this gives using (5.28),
\[
\int_{B_0(2\delta)} B_\alpha(r_\alpha y)^{-7} |\mathcal{L}_\xi V_\alpha|^2 (r_\alpha y) \, dy \leq D_{14} \left( \mu_\alpha^4 (1 + \delta^9) + \delta^{10} \varepsilon_\alpha^2 r_\alpha^6 \mu_\alpha^{-7} \right)
\]
for some $D_{14}$ independent of $\alpha$ and $\delta$. Using now (5.34) and (5.35), we can write
\[
\int_{B_0(2\delta)} B_\alpha(r_\alpha y)^{-7} |\mathcal{L}_\xi V_\alpha|^2 (r_\alpha y) \, dy \geq D_{15} \varepsilon_\alpha^6 r_\alpha^6 \mu_\alpha^{-7} \delta^6
\]
for some $D_{15}$ independent of $\alpha$ and $\delta$. Up to choose $\delta > 0$ small enough, we thus obtain that
\[
\varepsilon_\alpha^6 r_\alpha^6 \mu_\alpha^{-7} = O \left( \mu_\alpha^4 \right)
\]
which leads to (5.46).

Coming back to (5.45) with (5.46) but also with (5.33), we obtain the claim.

We set now, for $x \in B_0(4)$,
\[
\tilde{v}_\alpha(x) = \mu_\alpha^{-\frac{7}{2}} r_\alpha v_\alpha (r_\alpha x), \\
\tilde{\hat{W}}_\alpha(x) = \tilde{\hat{W}}_\alpha (r_\alpha x), \\
\tilde{\hat{h}}_\alpha(x) = \tilde{\hat{h}}_\alpha (r_\alpha x), \\
\tilde{\hat{f}}_\alpha(x) = \tilde{\hat{f}}_\alpha (r_\alpha x), \\
\tilde{\hat{a}}_\alpha(x) = \tilde{\hat{a}}_\alpha (r_\alpha x), \\
\tilde{\hat{X}}_\alpha(x) = \tilde{\hat{X}}_\alpha (r_\alpha x), \\
\tilde{\hat{Y}}_\alpha(x) = \tilde{\hat{Y}}_\alpha (r_\alpha x), \\
\tilde{\hat{Z}}_\alpha(x) = \tilde{\hat{Z}}_\alpha (r_\alpha x).
\]

The first line of system (5.21) becomes
\[
\Delta_\xi \tilde{v}_\alpha + r_\alpha^2 \tilde{\hat{h}}_\alpha \tilde{v}_\alpha = \left( \frac{\mu_\alpha}{r_\alpha} \right)^2 \tilde{\hat{f}}_\alpha \tilde{v}_\alpha^5 + \frac{r_\alpha^8}{\mu_\alpha^2} \tilde{\hat{a}}_\alpha \tilde{v}_\alpha^7 \quad \text{in } B_0(4).
\] (5.47)
We can now use (2.2), (5.20) and (5.28) together with Claims 5.2 and 5.5 to write that
\[ \frac{r^8}{\mu^3} \delta_x \leq C_7 \left( \frac{\mu^2}{r^2} + |x|^2 \right)^{\frac{3}{2}} \]  
(5.48)
for all \( x \in B_0(2) \).

Thanks to Claims 5.2 and 5.3 we also know that
\[ C_3 \mu_3^{-\frac{3}{2}} r_a B_\alpha (r_a x) \leq \varepsilon r_a x \leq C_3 \mu_3^{-\frac{3}{2}} r_a B_\alpha (r_a x) \leq C_3 \left( \frac{3}{2f_a(x_0)} |x|^{-1} \right) \]  
(5.49)
in \( B_0(4) \setminus \{ 0 \} \). We also have thanks to the definition (5.26) of \( r_a \) that
\[ \varepsilon r_a (x) < (1 + \varepsilon) \mu_3^{-\frac{3}{2}} r_a B_\alpha (r_a x) \in B_0(1) \]  
(5.50)
and that, if \( r_a < \hat{\rho}_\alpha \), there exists \( z_\alpha \in \partial B_0(1) \) such that
\[ \varepsilon r_a (z_\alpha) = (1 + \varepsilon) \mu_3^{-\frac{3}{2}} r_a B_\alpha (r_a z_\alpha) . \]  
(5.51)
By standard elliptic theory, using (5.27), (5.47), (5.48) and (5.49), we obtain that after passing to a subsequence, since \( \varepsilon r_a \) is uniformly bounded from below in every compact subset of \( B_0(2) \setminus \{ 0 \} \) thanks to (5.49),
\[ \varepsilon r_a \to \hat{\varepsilon} \text{ in } C^1_{\text{loc}} (B_0(2) \setminus \{ 0 \}) \text{ as } \alpha \to +\infty . \]  
(5.52)
Moreover, it is easily checked thanks to (5.17) that
\[ \hat{\varepsilon} (x) = \frac{\lambda}{|x|} + \beta (x) , \]  
(5.53)
where \( \beta \in C^1 (B_0(2)) \) is some super-harmonic function and
\[ \lambda = \left( \frac{2f_0(x_0)}{3} \right)^{-\frac{2}{3}} . \]
Moreover, using Claim 5.3 it is also easily checked that
\[ \beta (x) \geq 0 \text{ in } B_0(2) . \]  
(5.54)
Now we clearly have that
\[ \beta > 0 \text{ in } B_0(2) \text{ if } r_a < \hat{\rho}_\alpha \]  
(5.55)
thanks to (5.51) and to the fact that \( \beta \) is super-harmonic.

\textbf{Claim 5.6.} We have that \( \beta (0) = 0 \) so that \( r_a = \hat{\rho}_\alpha \).

\textbf{Proof.} - Let us apply the Pohožaev identity to \( \varepsilon r_a \) in a ball \( B_0 (\delta) \). This reads as
\[ \int_{B_0(\delta)} \left( x k \partial_k \varepsilon r_a + \frac{1}{2} \varepsilon r_a \right) \Delta \xi \varepsilon r_a \, dx = \int_{\partial B_0(\delta)} \left( \frac{1}{2} \delta \left| \nabla \varepsilon r_a \right|^2 - \frac{1}{2} \varepsilon r_a \partial_k \varepsilon r_a - \delta \left( \partial_k \varepsilon r_a \right)^2 \right) \, d\sigma . \]
Using (5.52) and (5.53), we obtain after simple computations that
\[ \lim_{\delta \to 0} \lim_{\alpha \to +\infty} \int_{B_0(\delta)} \left( x k \partial_k \varepsilon r_a + \frac{1}{2} \varepsilon r_a \right) \Delta \xi \varepsilon r_a \, dx = 2\pi \lambda \beta (0) . \]  
(5.56)
In order to estimate the left-hand side, we need some asymptotic of \( x k \partial_k \varepsilon r_a + \frac{1}{2} \varepsilon r_a \) on the ball \( B_0 (\delta) \) for \( \delta > 0 \) small enough. We have thanks to claim 5.1 that
\[ \frac{\mu_\alpha}{r_\alpha} \left( x k \partial_k \varepsilon r_a + \frac{1}{2} \varepsilon r_a \right) \left( \frac{\mu_\alpha}{r_\alpha} \right)^{\frac{1}{2}} \frac{1}{2} \left( \frac{1}{2} + \frac{|x|^2}{\lambda^2} \right)^{\frac{3}{2}} \left( \frac{1}{2} - \frac{|x|^2}{\lambda^2} \right) \]  
(5.57)
in $C^0_{\text{loc}}(\mathbb{R}^3)$ as $\alpha \to +\infty$. We let now $(z_\alpha)$ be a sequence of points in $B_0(\delta)$ such that
\[
\frac{r_\alpha |z_\alpha|}{\mu_\alpha} \to +\infty \text{ as } \alpha \to +\infty
\] (5.58)
and we write with the Green representation formula that
\[
\left(x^k \partial_k \tilde{v}_\alpha + \frac{1}{2} \tilde{v}_\alpha\right)(z_\alpha) = \int_{B_0(1)} H^1(z_\alpha, x) \Delta \xi \tilde{v}_\alpha(x) \, dx + \int_{\partial B_0(1)} H^2(z_\alpha, x) \tilde{v}_\alpha(x) \, d\sigma,
\]
where
\[
H^1(z, x) = \frac{1}{8\pi} \left( \frac{|x|^2 - |z|^2}{|x - z|^3} + \frac{|x|^2 |z|^2 - 1}{|z|^4} \right)
\]
and
\[
H^2(z, x) = \frac{1}{8\pi |z - x|} \left( |z|^4 - 10 |z|^2 + 1 + 4 (1 + |z|^2) (x, z) \right).
\]
Thanks to (5.47) and (5.52), this leads to
\[
\left(x^k \partial_k \tilde{v}_\alpha + \frac{1}{2} \tilde{v}_\alpha\right)(z_\alpha) = \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(1)} H^1(z_\alpha, x) \tilde{f}_\alpha \tilde{v}_\alpha^3 \, dx
\]
\[- r_\alpha^2 \int_{B_0(1)} H^1(z_\alpha, x) \tilde{h}_\alpha \tilde{v}_\alpha \, dx
\]
\[+ \frac{r_\alpha^8}{\mu_\alpha^2} \int_{B_0(1)} H^1(z_\alpha, x) \tilde{a}_\alpha \tilde{v}_\alpha^{-7} \, dx
\]
\[+ \int_{\partial B_0(1)} H^2(z_\alpha, x) \tilde{v}(x) \, d\sigma + o(1), \]
where $z_0 = \lim_{\alpha \to +\infty} z_\alpha$. Direct computations lead thanks to (2.2) and (5.49) to
\[
\int_{B_0(1)} H^1(z_\alpha, x) \tilde{h}_\alpha \tilde{v}_\alpha \, dx = O(1).
\] (5.60)
Using (5.48), we obtain that
\[
\frac{r_\alpha^8}{\mu_\alpha} \int_{B_0(1)} H^1(z_\alpha, x) \tilde{a}_\alpha \tilde{v}_\alpha^{-7} \, dx \leq D_2
\] (5.61)
for some $D_2 > 0$ independent of $z_\alpha \in B_0(\delta)$. Let us take some $R_\alpha \to +\infty$ such that
\[
\frac{r_\alpha |z_\alpha|}{R_\alpha \mu_\alpha} \to +\infty \text{ as } \alpha \to +\infty
\]
and such that
\[
\sup_{B_0(R_\alpha \frac{z_\alpha}{r_\alpha})} \left| \frac{\tilde{v}_\alpha}{\mu_\alpha^2 r_\alpha B_\alpha (r_\alpha x)} - 1 \right| \to 0 \text{ as } \alpha \to +\infty.
\]
Such a sequence $R_\alpha$ clearly exists thanks to claim 5.1 and to (5.58). Then we write thanks to (2.2) and to (5.49) that
\[
\left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(1) \setminus B_0(R_\alpha \frac{z_\alpha}{r_\alpha})} H^1(z_\alpha, x) \tilde{f}_\alpha \tilde{v}_\alpha^3 \, dx = o \left( |z_\alpha|^{-1} \right).
\]
We can also write that
\[
\left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(R, \frac{r_\alpha}{d\alpha})} H^1(z_\alpha, x) \tilde{f}_\alpha \tilde{v}_\alpha^5 \, dx = -\frac{\lambda}{2 |z_\alpha|} + o\left(\frac{1}{|z_\alpha|}\right)
\]
so that
\[
\left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(1)} H^1(z_\alpha, x) \tilde{f}_\alpha \tilde{v}_\alpha^5 \, dx = -\frac{\lambda}{2 |z_\alpha|} + o\left(\frac{1}{|z_\alpha|}\right) .
\] (5.62)

Coming back to (5.59) with (5.60)-(5.62), we obtain that
\[
\left( x^k \partial_k \tilde{v}_\alpha + \frac{1}{2} \tilde{v}_\alpha \right) (z_\alpha) = \frac{1}{2} \mu_\alpha \frac{1}{\left(1 + \frac{\lambda |z_\alpha|^2}{\lambda |z_\alpha|^2}\right)^2} + o(1) + o\left(\frac{1}{|z_\alpha|}\right) .
\] (5.63)

And this holds for any sequence \((z_\alpha)\) in \(B_0(\delta)\) satisfying (5.58). Thus, together with (5.57), we have obtained that for any sequence of points \((z_\alpha)\) in \(B_0(\delta)\),
\[
\left( x^k \partial_k \tilde{v}_\alpha + \frac{1}{2} \tilde{v}_\alpha \right) (z_\alpha) = \frac{1}{2} \mu_\alpha \frac{1}{\left(1 + \frac{\lambda |z_\alpha|^2}{\lambda |z_\alpha|^2}\right)^2} + o(1) + o\left(\frac{1}{|z_\alpha|}\right) .
\] (5.64)

In particular, there exists \(\delta > 0\) such that for \(\alpha\) large,
\[
x^k \partial_k \tilde{v}_\alpha (x) + \frac{1}{2} \tilde{v}_\alpha (x) \leq 0 \text{ in } B_0(\delta) \setminus B_0\left(\frac{\lambda \mu_\alpha}{r_\alpha}\right) .
\] (5.65)

Using now equation (5.47), (5.65) and the fact that \(\bar{a}_\alpha \geq 0\), we can write that
\[
\int_{B_0(\delta)} \left( x^k \partial_k \tilde{v}_\alpha + \frac{1}{2} \tilde{v}_\alpha \right) \Delta \xi \tilde{v}_\alpha \, dx \leq \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(\delta)} \left( x^k \partial_k \tilde{v}_\alpha + \frac{1}{2} \tilde{v}_\alpha \right) \tilde{f}_\alpha \tilde{v}_\alpha^5 \, dx
\]
\[
= \tilde{f}_\alpha(0) \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(\delta)} \left( x^k \partial_k \tilde{v}_\alpha + \frac{1}{2} \tilde{v}_\alpha \right) \tilde{v}_\alpha^5 \, dx
\]
\[
+ \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(\delta)} \left( x^k \partial_k \tilde{v}_\alpha + \frac{1}{2} \tilde{v}_\alpha \right) \left( \tilde{f}_\alpha - \tilde{f}_\alpha(0) \right) \tilde{v}_\alpha^5 \, dx
\]
\[
= o(1) + O\left(\left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(\delta)} \left| x^k \partial_k \tilde{v}_\alpha + \frac{1}{2} \tilde{v}_\alpha \right| |x| \tilde{v}_\alpha^5 \, dx\right) = o(1) .
\]

We also easily get that
\[
\int_{B_0(\delta)} \left( x^k \partial_k \tilde{v}_\alpha + \frac{1}{2} \tilde{v}_\alpha \right) \tilde{h}_\alpha \tilde{v}_\alpha \, dx = O(1) .
\]
At last, using (5.49) and (5.64), we obtain that
\[
\frac{r^8}{\mu^4} \int_{B_0(\lambda \frac{\mu}{r_\alpha})} \left( x^k \partial_k \tilde{v}_\alpha + \frac{1}{2} \tilde{v}_\alpha \right) \tilde{v}_\alpha^{-7} \, dx = O \left( \left( \frac{\mu}{r_\alpha} \right)^5 \right).
\]

Coming back to (5.56) with these last estimates, we obtain that \( \beta(0) \leq 0 \). Thanks to (5.55), this proves that \( r_\alpha = \hat{\rho}_\alpha \). This ends the proof of the claim. ♦

Now the results of this section clearly permit to prove Claim 4.5, combining the definition (5.26) of \( r_\alpha \) and Claims 5.3 and 5.6.

6. A Harnack inequality

We prove in the following a Harnack inequality for solutions of Einstein-Lichnerowicz equation which was used throughout the paper:

**Proposition 6.1.** Let \( a, f, h \) be smooth functions on \( \overline{B_0(2)} \subset \mathbb{R}^3 \) and let \( u \in C^2(B_0(2)) \) be a positive solution in \( B_0(2) \) of
\[
\Delta_\xi u + hu = fu^5 + au^{-7}.
\]

We assume that
- \( u \in L^\infty(B_0(2)) \).
- \( f \geq 0 \) and \( a \geq 0 \), \( a \neq 0 \).

Then there exists \( C > 0 \) depending only on \( \|h\|_{L^\infty(B_0(2))}, \|a\|_{L^\infty(B_0(2))}, \|f\|_{L^\infty(B_0(2))} \) and \( \|u\|_{L^\infty(B_0(2))} \) such that
\[
\sup_{B_0(1)} u \leq C \inf_{B_0(1)} u.
\]

The proof follows the standard Nash-Moser iterative scheme. However, we have to deal with an additional difficulty due to the negative power nonlinearity of \( u \) that makes the proof more involved. In particular, unlike the standard Harnack inequality, proposition 6.1 is no longer an a priori estimate and does not induce a control on the \( L^\infty \)-norm of \( \nabla u \) in \( B_0(1) \).

**Proof** - First of all, since \( u, a \) and \( f \) are nonnegative there holds
\[
\Delta u + hu \geq 0
\]
in \( B_0(2) \). Hence Theorem 8.18 in Gilbarg-Trudinger [14] applies and shows that for any \( 1 \leq p < 3 \), there exists \( C_1(h, p) \) depending only on \( \|h\|_{L^\infty(B_0(2))} \) and \( p \) such that
\[
\inf_{B_0(1)} u \geq C_1(h, p) \|u\|_{L^p(B_0(2))} \quad (6.1)
\]
We now aim at proving that for any \( p \geq 1 \) there exist \( C = C(a, h, f, u, p) \) such that
\[
\sup_{B_0(1)} u \leq C \|u\|_{L^p(B_0(2))}
\]
which together with (6.1) will conclude the proof of the proposition. We adapt the steps of the proof of Theorem 4.1 in Han-Lin [15]. Let \( k \geq 8 \) be given and
\( \eta \in C_c^\infty(B_0(2)) \) be a smooth positive function with compact support in \( B_0(2) \). Multiplying the equation satisfied by \( u \) by \( \eta^2 u^k \) and integrating yields

\[
\frac{4(k - 1)}{(k + 1)^2} \int_{B_0(2)} \left| \nabla u^{\frac{k+1}{2}} \right|^2 \eta^2 \, dx \leq \int_{B_0(2)} \left( \eta^2 f u^{k+5} + a \eta^2 u^{k-7} - \eta^2 h u^{k+1} \right) \, dx \\
+ \int_{B_0(2)} \left| \nabla \eta \right|^2 u^{k+1} \, dx.
\]

Let \( 0 < r < R \leq 2 \). Assume that \( \eta \) is compactly supported in \( B_0(R) \), is equal to 1 in \( B_0(r) \) and that it satisfies \( |\eta| \leq 1 \) and \( |\nabla \eta| \leq \frac{2}{r} \) in \( B_0(2) \). It is then easily seen that there exists \( C_2 > 0 \), depending only on \( \|h\|_{L^\infty(B_0(2))} \), \( \|a\|_{L^\infty(B_0(2))} \), \( \|f\|_{L^\infty(B_0(2))} \) and \( \|u\|_{L^\infty(B_0(2))} \), such that

\[
\int_{B_0(2)} \left| \eta^2 f u^{k+5} + a \eta^2 u^{k-7} - \eta^2 h u^{k+1} \right| \, dx \leq \frac{C_2}{r} \left( \int_{B_0(r)} u^{k+1} \, dx \right)^{\frac{k+7}{k+1}}
\]

and that

\[
\int_{B_0(2)} \left| \nabla \eta \right|^2 u^{k+1} \, dx \leq \frac{C_2}{r^2} \left( \int_{B_0(r)} u^{k+1} \, dx \right)^{\frac{k+7}{k+1}}.
\]

Independently, Sobolev’s inequality shows that

\[
\int_{B_0(2)} \left( \eta u^{\frac{k+1}{2}} \right)^6 \leq K \left( \int_{B_0(r)} \left| \nabla \left( \eta u^{\frac{k+1}{2}} \right) \right|^2 \, dx \right)^{\frac{1}{3}}
\]

for some \( K > 0 \) which leads to

\[
\left( \int_{B_0(2)} \left( \eta u^{\frac{k+1}{2}} \right)^{3(k+1)} \right)^{\frac{1}{3}} \leq C_3 \left( \int_{B_0(r)} \left| \nabla \eta \right|^2 u^{k+1} \, dx + \int_{B_0(2)} \eta^2 \left| \nabla u^{\frac{k+1}{2}} \right|^2 \, dx \right)
\]

for some positive \( C_3 \). We now let \( \gamma = k + 1 \geq 9 \). Combining the above estimates, we obtain that

\[
\|u\|_{L^{\gamma}(B_0(r))} \leq C_4 \left( \frac{\gamma}{(R-r)^2} \right)^{\frac{1}{3}} \|u\|_{L^{\infty}(B_0(r))}^{\frac{2}{r+3}} \|u\|_{L^{\infty}(B_0(2))}^{\frac{2}{r+3}} \|
\]

for some positive constant \( C_4 \) depending only on \( \|h\|_{L^{\infty}(B_0(2))} \), \( \|a\|_{L^{\infty}(B_0(2))} \), \( \|f\|_{L^{\infty}(B_0(2))} \) and \( \|u\|_{L^{\infty}(B_0(2))} \). We now pick some \( 0 < r < 2 \) and define two sequences \( \gamma_i \) and \( r_i \) by \( \gamma_i = 3^i \gamma \) and \( r_0 = 2, r_{i+1} = r_i - (2 - r)2^{i-1} \). Inequality (6.2) then gives that, for any \( i \geq 0 \):

\[
\|u\|_{L^{\gamma_i}(B_0(r_{i+1}))} \leq C_4 \left( \frac{2 \cdot 6^i \gamma}{(2 - r)^2} \right)^{\frac{1}{3}} \|u\|_{L^{\gamma_i}(B_0(r_i))}^{\frac{1}{r+3}} \|u\|_{L^{\infty}(B_0(r_i))}^{\frac{2}{r+3}}
\]

and we thus obtain that there exists some constant \( C_6 > 0 \) depending on \( \|h\|_{L^{\infty}(B_0(2))} \), \( \|a\|_{L^{\infty}(B_0(2))} \), \( \|f\|_{L^{\infty}(B_0(2))} \) and \( \|u\|_{L^{\infty}(B_0(2))} \) but which does not depend on \( r \) nor on \( \gamma \) such that

\[
\|u\|_{L^{\gamma_i}(B_0(r_i))} \leq \frac{C_6}{(2 - r)^{\frac{1}{3}}} \|u\|_{L^{\gamma}(B_0(2))}^\gamma
\]
for all \( i \geq 0 \), where \( \alpha = \alpha(\gamma) = \prod_{k=0}^{\infty} \left( 1 - \frac{8}{3^k} \right) \). Passing to the limit as \( i \to \infty \) we thus obtain:

\[
\|u\|_{L^\infty(B_0(\gamma))} \leq \frac{C_6}{(2 - r)^\alpha} \|u\|_{L^2(B_0(\gamma))}.
\] (6.3)

To conclude the proof of Proposition 6.1, we need to improve estimate (6.3). Let \( 1 \leq p < \gamma \). We bound \( (2 - r)^{-3} \|u\|_{L^1(B_0(\gamma))} \) using Young’s inequality with exponents \( \frac{2}{\gamma - p\alpha} \) and \( \frac{p}{\gamma} \) and combine with (6.3) to obtain, for any \( \varepsilon > 0 \):

\[
\|u\|_{L^\infty(B_0(\gamma))} \leq C_6 \varepsilon^{-\frac{2}{\gamma - p\alpha}} \|u\|_{L^\infty(B_0(\gamma))} + \frac{C_6}{(2 - r)^p} \varepsilon^{-\frac{p}{\gamma}} \left( \int_{B_0(\gamma)} u^p dx \right)^{\frac{1}{p}}. \tag{6.4}
\]

It is easily seen that \( \alpha(\gamma) \to 1 \) as \( \gamma \to \infty \). Hence, choosing \( \varepsilon = (2C_6)^{-1} \) one can then pick \( \gamma \) large enough (depending on \( a, h, f \) and \( u \)) so as to have, in (6.4):

\[
\|u\|_{L^\infty(B_0(\gamma))} \leq 2 \|u\|_{L^\infty(B_0(\gamma))} + C_7 (2 - r)^{-\beta} \|u\|_{L^p(B_0(\gamma))},
\]

where \( C_7 > 0 \). The conclusion follows using Lemma 4.3 in Han-Lin [15].

7. Standard elliptic theory for the vectorial Laplacian on \( S^3 \)

Equation \( \Delta_\xi X = 0 \) historically appeared in the formulation of mathematical linear elasticity and is sometimes referred to as the Lamé system. We deal in this subsection with several properties of the operator \( \Delta_h \) on \( (S^3, h) \). It is a differential operator between sections of the cotangent bundle \( T^*S^3 \). If \( X \) is a 1-form in \( S^3 \), \( \Delta_h \) writes in coordinates as:

\[
\Delta_h X_i = \nabla^i \nabla_j X_i + \nabla^j \nabla_i X_j - \frac{2}{3} \nabla_i (\text{div} X).
\]

If we write formally \( \Delta_h X(x) = L(x, \nabla)X \) then the principal symbol of the operator \( \Delta_h \) at some point \( x \in S^3 \) and for some \( \xi \in T_xS^3 \) is given by the determinant of the map \( L(x, \xi) \) seen as a linear endomorphism of \( T_xS^3 \). Thus there holds

\[
|L(x, \xi)| = \frac{4}{3} \xi^\xi h \tag{7.1}
\]

which shows that \( \Delta_h \) is uniformly elliptic in \( S^3 \). It also satisfies the so-called strong ellipticity condition (also called Legendre-Hadamard condition) since for any \( x \in S^3 \) and any \( \eta \in T_xS^3 \):

\[
(L(x, \xi) \eta, \eta) = |\xi|^2 h |\eta|^2 h + \frac{1}{3} |\xi, \eta|^2 h \geq |\xi|^2 h |\eta|^2 h. \tag{7.2}
\]

Since \( S^3 \) is closed, integrating by parts, one gets that, for any 1-forms \( X \) and \( Y \),

\[
\int_{S^3} \langle \Delta_h X, Y \rangle_h d\nu_h = \frac{1}{2} \int_{S^3} \langle L_h X, L_h Y \rangle_h d\nu_h. \tag{7.3}
\]

In particular, (7.3) shows that \( \Delta_h \) is self-adjoint on \( H^1(M) \) (we still denote the Sobolev space of 1-forms by \( H^1(M) \) since no ambiguity will occur) and that there holds in \( S^3 \)

\[
\Delta_h X = 0 \iff L_h X = 0 \tag{7.4}
\]
for any 1-form \( X \). Fields of 1-forms in \( S^3 \) satisfying \( \mathcal{L}_h X = 0 \) are called conformal Killing 1-forms and by \( (7.3) \) and standard Fredholm theory the set of those 1-forms is finite dimensional. With \( (7.2) \), \( (7.3) \), and \( (7.4) \) standard results of elliptic theory for elliptic operators acting on vector bundles on closed manifolds apply, see for instance Theorem 27, Appendix H in Besse [2], or Theorem 5.20 in Giaquinta-Martinazi [13]. In particular, for 1-forms which are \( L^2 \)-orthogonal to the subspace of conformal Killing forms, we have the following estimates:

**Proposition 7.1.** For any \( p > 1 \), there exist constants \( C_1 = C_1(h, p) \) and \( C_2 = C_2(h, p) \) depending only on \( h \) and \( p \) such that for any 1-form \( X \) in \( S^3 \):

\[
\|X\|_{W^{2,p}(S^3)} \leq C_1 \|\overline{\Delta}_h X\|_{L^p(S^3)} + C_2 \|X\|_{L^1(S^3)}. \tag{7.5}
\]

If, in addition, \( X \) satisfies

\[
\int_{S^3} \langle X, K \rangle_h \, dv_h = 0 \tag{7.6}
\]

for all conformal Killing 1-form \( K \), then, for any \( p > 1 \), we can choose \( C_2 = 0 \) in \( (7.5) \).

It is in fact possible to fully describe the conformal Killing 1-forms on \( S^3 \) in terms of the conformal Killing forms in \( \mathbb{R}^3 \).

**Proposition 7.2.** Let \( P \in S^3 \). Then any conformal Killing 1-form \( K \) in \( S^3 \) is given by

\[
Z = \pi_P(U^4 L), \tag{7.7}
\]

where \( \pi_P \) is the stereographic projection of pole \(-P\), \( U \) is as in \( (3.1) \) and \( L \) is some conformal Killing 1-form in \( \mathbb{R}^3 \), that is satisfying \( \mathcal{L}_\xi L = 0 \).

**Proof.** Let \( Z \) satisfy \( \mathcal{L}_h Z = 0 \) in \( S^3 \). Using \( (3.5) \), there holds \( \mathcal{L}_\xi(U^4(\pi_P)_*Z) = 0 \). Conversely, let \( I \) be a conformal Killing 1-form in \( \mathbb{R}^3 \). These forms are classified (see for instance Chapter 1 of Schottenloher [27]) and span a 10-dimensional vector space. In particular \( L \) has the following expression:

\[
L(x) = 2(b, x)_\xi x_i - |x|^2 b_i + \lambda x_i + c_i + (\Omega x)_i, \tag{7.8}
\]

where \( \lambda \in \mathbb{R}, b, c \in \mathbb{R}^3 \) and \( \Omega \) is a skew-symmetric matrix. Note that \( (7.8) \) actually describes any conformal Killing 1-form on any open subset of \( \mathbb{R}^3 \). Let \( P \in S^3 \) be some arbitrary point and define \( Z = \pi_P(U^4 L) \). This defines a smooth conformal 1-form in \( S^3 \backslash \{-P\} \). We now show that \( Z \) is in \( L^2(S^3) \) and actually satisfies \( \overline{\Delta}_h Z = 0 \) in \( S^3 \) in a weak sense. Using the regularity theory as in proposition \( 7.1 \) this will show that \( Z \) is a smooth conformal Killing 1-form in \( S^3 \) and \( (7.4) \) will show that \( Z \) is conformal. First, \( Z \in L^2(S^3) \) since using \( (7.8) \) there holds, for any \( y \in S^3 \backslash \{-P\} \):

\[
|Z(y)|_{h(y)} = U^2(\pi_P(y))|L|_{\xi}(\pi_P(y)) \leq C,
\]

where \( C \) is some constant depending only on \( n \) and \( h \). Then, if \( X \) is some smooth 1-form in \( S^3 \), integrating by parts we get for any \( \varepsilon > 0 \):

\[
\int_{S^3 \backslash B_{\varepsilon}(P)} \langle Z, \overline{\Delta}_h X \rangle \, dv_h = \int_{\partial B_{\varepsilon}(P)} \langle \mathcal{L}_h X, Z \otimes \nu \rangle \, d\sigma_h = o(1)
\]

so that letting \( \varepsilon \to 0 \) shows that \( \overline{\Delta}_h Z = 0 \) in a weak sense in \( S^3 \). \( \square \)
8. **Fundamental solution of the Lamé-type system in \( \mathbb{R}^3 \).**

We define, for \( 1 \leq i \leq 3 \), a 1-form \( H_i(y) \) in \( \mathbb{R}^3 \) by:

\[
H_i(y)_j = \frac{1}{32\pi} \left( 7\delta_{ij} + \frac{y_i y_j}{|y|^3} \right) \tag{8.1}
\]

for any \( y \neq 0 \). Note that the matrices \( (H_{ij}(y))_{ij} \) thus defined are symmetric: for any \( y \neq 0 \),

\[
H_{ij}(y) = H_{ji}(y). \tag{8.2}
\]

Let \( X \) be a field of 1-forms in \( \mathbb{R}^3 \). Integrating by parts and using Stoke’s formula it is easily seen that for any \( R > 0 \) and for any \( x \in B_0(R) \) there holds:

\[
X_i(x) = \int_{B_0(R)} H_{ij}(x-y) \nabla_x X(y)_j^i dx + \int_{\partial B_0(R)} \mathcal{L}_x X(y)^{kl} \nu_k(y) H_{il}(x-y) d\sigma
- \int_{\partial B_0(R)} \mathcal{L}_x (H_i(x-\cdot))(y)_k \nu^k(y) X(y)_l^i d\sigma. \tag{8.3}
\]

This means in a distributional sense that

\[
\nabla_x (H_i(x-\cdot)) = \delta_x e_i, \tag{8.4}
\]

where \( e_i \) is the \( i \)-th vector of the canonical basis and there holds, for any 1-form \( Y \):

\[
\langle \delta_x e_i, Y \rangle = Y_i(x). \tag{8.5}
\]

Equivalently, if we write \( H(x,y) = (H_{ij}(x-y))_{1 \leq i,j \leq 3} \), we get that

\[
\nabla_x H(x,\cdot) = \delta_x \text{Id}, \tag{8.6}
\]

where \( \nabla_x \) is now seen as a matrix of differential operators acting on a distribution-valued matrix. Note that the standard results of distribution theory easily extend to distribution-valued matrices, see for instance Schwartz [28].

If now \( X \) is some smooth field of 1-forms in \( \mathbb{R}^3 \) we let, for any \( R > 0 \) and any \( x \in B_0(R) \),

\[
Z_i(x) = \int_{\partial B_0(R)} \mathcal{L}_x X(y)^{kl} \nu_k(y) H_{il}(x-y) d\sigma
- \int_{\partial B_0(R)} \mathcal{L}_x (H_i(x-\cdot))(y)_k \nu^k(y) X(y)_l^i d\sigma. \tag{8.7}
\]

This defines a smooth 1-form in \( B_0(R) \) which satisfies

\[
\nabla_x Z = 0 \quad \text{in} \quad B_0(R) \tag{8.8}
\]

due to (8.3) and (8.4). Similarly, using again (8.3), if \( Y \in L^1(\mathbb{R}^3) \) is a smooth 1-form then the 1-form defined by

\[
W_i(x) = \int_{\mathbb{R}^3} H_{ij}(x-y) Y^j(y) dy = (H \ast Y)_i(x)
\]

satisfies in a weak sense

\[
\nabla_x W_i(x) = Y_i(x). \tag{8.9}
\]
9. Green function with Neumann boundary conditions for Lamé-type systems in $\mathbb{R}^3$.

The system (2.1) we are interested in in this article is invariant up to adding to $W_\alpha$ some conformal Killing 1-form in $S^3$. We exploit this invariance all along the article by noting that the only relevant quantity to investigate is $\mathcal{L}_h W_\alpha$ and not the vector field $W_\alpha$ itself. In particular we use several times a Green identity for $\Delta_\xi$ with Neumann boundary conditions that is proven in what follows. We let

$$K_R = \{X \in H^1(B_0(R)), \mathcal{L}_\xi X = 0\}$$

be the Kernel subspace associated to the Neumann problem for $\Delta_\xi$ in $B_0(R)$. The orthogonal subspace of $K_R$ in $B_0(R)$ is the set of forms $Y \in H^1(B_0(R))$ such that for any $K \in K_R$:

$$\int_{B_0(R)} (Y, K) \xi dx = 0$$

holds.

Elements of $K_R$ are infinitesimal generators of conformal transformations of $B_0(R)$ and are classified, see (7.8). In particular $K_R$ is finite dimensional, $\dim K_R = 10$, and it is spanned by smooth vector fields. Let $(K_j)_{j=1...10}$ be an orthonormal basis of $K_0(R)$ for the $L^2$-scalar product, that is

$$\int_{B_0(R)} (K_i, K_j) \xi dx = \delta_{ip}.$$

The following proposition states the existence of Green 1-forms satisfying Neumann boundary conditions:

**Proposition 9.1.** For any $1 \leq i \leq 3$ and any $R > 0$ there exists a unique $G_{i,R}$ defined in $B_0(R) \times B_0(R) \setminus D$, where $D = \{(x, x), x \in B_0(R)\}$, such that $G_{i,R}(x, \cdot)$ is orthogonal to $K_R$ for any $x \in B_0(R)$ and such that for any smooth 1-form $X$ in $B_0(R)$:

$$(X - \pi_R(X))_i(x) = \int_{B_0(R)} G_{i,R}(x, y) \Delta_\xi X(y) dy$$

$$+ \int_{\partial B_0(R)} \mathcal{L}_\xi X(y)^k \nu_k(y) G_{i,R}(x, y) d\sigma,$$

where $\pi_R(X)$ is the orthogonal projection of $X$ on $K_R$ given by:

$$\pi_R(X) = \sum_{j=1}^{10} \left( \int_{B_0(R)} (K_j, X) dx \right) K_j.$$ 

Moreover $G_{i,R}$ is continuous and continuously differentiable in each variable in $B_0(R) \times B_0(R) \setminus D$. Furthermore, if $K$ denotes any compact set in $B_0(R)$ and if we let

$$\delta = \frac{d(K, \partial B_0(R))}{R} > 0$$

there holds:

$$|x - y| |\nabla G_{i,R}(x, y)| + |G_{i,R}(x, y)| \leq C(\delta)|x - y|^{-1}$$

for any $x \in B_0(K)$ and any $y \in B_0(R)$, whether the derivative in (9.5) is taken with respect to $x$ or $y$, and where $C(\delta)$ is a positive constant that only depends on $\delta$ as in (9.4) (in particular it does not depend on $x$).
Proof. The proof of this proposition goes through a sequence of claims. The techniques used are strongly inspired from Robert \cite{26}.

**Claim 9.1.** Let $F$ and $G$ be smooth 1-forms, in $B_0(R)$ and in $\partial B_0(R)$ respectively, satisfying:

$$
\int_{B_0(R)} F_l K^l d\xi + \int_{\partial B_0(R)} G_l K^l d\sigma = 0 \quad (9.6)
$$

for any $K \in K_{R}$, where $K_{R}$ is as in \eqref{eq:K_R}. Then there exists a unique smooth 1-form $Z$ orthogonal to $K_{R}$ such that

$$
\begin{cases}
\Delta \xi Z = F & B_0(R) \\
\nu^k \xi Z_{kl} = G_l & \partial B_0(R).
\end{cases} \quad (9.7)
$$

**Proof.** The existence and uniqueness of $Z$ is ensured by the Lax-Milgram theorem applied on the orthogonal complement of $K_{R}$ to the symmetric bilinear form

$$
B(X, Y) = \frac{1}{2} \int_{B_0(R)} \langle L_\xi X, L_\xi Y \rangle dx
$$

and to the linear form:

$$
L(X) = \int_{B_0(R)} F_l X^l d\xi - \int_{\partial B_0(R)} G_l X^l d\sigma.
$$

The coercivity of $B(X, X)$ on the orthogonal complement of $K_{R}$ follows from the definition of $K_{R}$ and is obtained via the direct method. We claim now that $Z$ is smooth in $B_0(R)$. This is a consequence of general elliptic regularity results up to the boundary for elliptic systems satisfying complementing boundary conditions, as stated in Agmon-Douglis-Nirenberg \cite{1}. Due to (7.1) and (7.2) the problem (9.7) is complemented so that Theorem 10.5 in \cite{1} applies and shows that $Z$ is smooth. \hfill \square

For any $1 \leq i \leq 3$ and any $x \in B_0(R)$ we let $U_{i,x}^{R}$ be the unique 1-form in $H^1(B_0(R))$, orthogonal to $K_{R}$, satisfying:

$$
\begin{cases}
\Delta \xi U_{i,x}^{R} = -\sum_{j=1}^{10} (K_j)_i(x) K_j & B_0(R) \\
\nu^k \xi (U_{i,x}^{R})_{kl} = -\nu^k \xi (H_i(x - \cdot))_{kl} & \partial B_0(R).
\end{cases} \quad (9.8)
$$

The existence and smoothness of $U_{i,x}^{R}$ is ensured by Claim \ref{clm:9.1}. Indeed, the compatibility condition \eqref{eq:compatibility} is satisfied by applying \ref{clm:8.3} to any $K \in K_{R}$.

We now let, for $x \neq y$:

$$
G_{i,R}(x, y) = H_i(x - y) + U_{i,x}^{R}(y) - \sum_{j=1}^{10} \left( \int_{B_0(R)} (K_j, H_i(x - \cdot)) dy \right) K_j(y). \quad (9.9)
$$

By construction, $G_{i,R}(x, \cdot)$ is a 1-form defined in $B_0(R) \setminus \{x\}$. It clearly belongs to $L^2(B_0(R))$, is orthogonal to $K_{R}$ and continuously differentiable in $B_0(R) \setminus \{x\}$. Combining \ref{clm:8.3} and \ref{clm:9.8} it is easily seen that \eqref{eq:compatibility_2} holds. The next three claims aim at finishing the proof of the proposition. The first one is a uniqueness result.
Claim 9.2. Assume that for some $1 \leq i \leq 3$ and for some $x \in B_0(R)$ there exists a 1-form $M_i$ in $L^1(B_0(R))$ such that for any $X \in C^2(B_0(R))$ with $\mathcal{L}_\xi X_{kl} = 0$ on $\partial B_0(R)$ there holds:

$$
\int_{B_0(R)} \langle M_i, \Delta_\xi X \rangle \xi d\xi = (X - \pi(X))_i(x).
$$

(9.10)

Then $G_{i,R}(x,\cdot) - M_i \in K_R$, where $K_R$ is as in (9.1).

Proof. Let $F_i = G_{i,R}(x,\cdot) - M_i$. Let $Y$ be a smooth 1-form with compact support in $B_0(R)$. By Claim 9.1 there exists a smooth 1-form $X$ in $B_0(R)$ such that $\Delta_\xi X = Y - \pi_R(Y)$ in $B_0(R)$ and $\mathcal{L}_\xi X_{kl} = 0$ in $\partial B_0(R)$, where $\pi_R$ is as in (9.3). Using (9.2) and (9.10) there holds:

$$
0 = \int_{B_0(R)} \langle F_i, \Delta_\xi X \rangle \xi d\xi = \int_{B_0(R)} \langle F_i, Y - \pi_R(Y) \rangle \xi d\xi = \int_{B_0(R)} \langle F_i - \pi_R(F_i), Y \rangle \xi d\xi
$$

by definition of $\pi_R$. Assume for a while that $F_i$ belongs to $L^p(B_0(R))$ for some $p > 1$. A density argument then shows that $F_i = \pi_R(F_i)$.

It thus remains to prove that $F_i \in L^p(B_0(R))$ for some $p > 1$. We only need to prove this for $M_i$. Let $p \in (1,3)$ and define $q = \frac{p}{p-1}$. Let $Y$ be a smooth 1-form compactly supported in $B_0(R)$. By Claim 9.1, there exists a smooth 1-form $X$ in $B_0(R)$, orthogonal to $K_R$, such that $\Delta_\xi X = Y - \pi_R(Y)$ in $B_0(R)$ and $\mathcal{L}_\xi X_{kl} = 0$ in $\partial B_0(R)$. Then the definition of $\pi_R$ and (9.10) yield:

$$
\int_{B_0(R)} \langle M_i - \pi_R(M_i), Y \rangle \xi d\xi = \int_{B_0(R)} \langle M_i, Y - \pi_R(Y) \rangle \xi d\xi = \int_{B_0(R)} \langle M_i, \Delta_\xi X \rangle \xi d\xi = X_i(x).
$$

Elliptic regularity results for complemented elliptic systems, as those stated in the proof of Claim 9.1, show that there exists a constant $C$ only depending on $q$ such that $\|X\|_{W^{2,q}} \leq C\|Y - \pi_R(Y)\|_{L^q}$ where we omit to say that these norms are taken on $B_0(R)$ for the sake of clarity. Since $q > \frac{3}{2}$ we thus obtain, using the Sobolev embedding of $W^{2,q}$ in $C^0(B_0(R))$:

$$
\left| \int_{B_0(R)} \langle M_i - \pi_R(M_i), Y \rangle \xi d\xi \right| \leq C\|Y - \pi_R(Y)\|_{L^q} \leq C\|Y\|_{L^q}.
$$

A density argument then shows that $M_i - \pi_R(M_i)$, and thus $M_i$, belongs to $L^p(B_0(R))$ for all $p \in (1,3)$.

We now state some rescaling-invariance property of the Green 1-forms $G_{i,R}$:

Claim 9.3. For any $R > 0$ there holds:

$$
G_{i,R}(x,y) = \frac{1}{R} G_{i,1}\left(\frac{x}{R}, \frac{y}{R}\right)
$$

(9.11)

for any $x, y \in B_0(R)$.
Proof. Let \( Y \) be a smooth, compactly supported, 1-form in \( B_0(R) \). We define, in \( B_0(1) \), \( Y_R = Y(R \cdot) \), which is then compactly supported in \( B_0(1) \). Let \( x \in B_0(R) \) and \( 1 \leq i \leq 3 \). Equation (9.12) shows that

\[
(Y_R - \pi_1(Y_R))_i \left( \frac{x}{R} \right) = \int_{B_0(1)} \left< G_{i,1} \left( \frac{x}{R}, y \right), \Delta_{\xi} Y_R(y) \right> \xi \, dy.
\]  

(9.12)

Since for any \( y \in B_0(1) \) there holds \( \Delta_{\xi} Y_R(y) = R^2 \Delta_{\xi} Y(Ry) \) we easily obtain:

\[
\int_{B_0(1)} \left< G_{i,1} \left( \frac{x}{R}, y \right), \Delta_{\xi} Y_R(y) \right> \xi \, dy = \frac{1}{R} \int_{B_0(R)} \left< G_{i,1} \left( \frac{y}{R} \right), \Delta_{\xi} Y(y) \right> \xi \, dy.
\]

Let now \( (L_j)_{1 \leq j \leq 10} \) be an orthonormal basis of \( K_1 \), where \( K_1 \) is as in (9.1). Let \( Z_j = R^{-\frac{2}{3}} L_j \left( \frac{y}{R} \right) \) for any \( 1 \leq j \leq 10 \) and any \( x \in B_0(R) \). Then \( (Z_j)_{1 \leq j \leq 10} \) is an orthonormal basis of \( K_R \) since there holds

\[
\int_{B_0(R)} \left< Z_k, Z_l \right> \xi \, d\xi = \int_{B_0(R)} R^{-3} \langle L_k \left( \frac{y}{R} \right), L_l \left( \frac{y}{R} \right) \rangle \xi \, dy = \int_{B_0(1)} \langle L_k, L_l \rangle \xi \, d\xi = \delta_{kl}.
\]

Hence one has, by definition of \( \pi_R \):

\[
\pi_R(Y)(x) = \sum_{j=1}^{10} \left( \int_{B_0(R)} \left< R^{-\frac{2}{3}} L_j \left( \frac{y}{R} \right), Y(y) \right> \xi \, dy \right) R^{-\frac{2}{3}} L_j \left( \frac{x}{R} \right)
\]

\[
= \sum_{j=1}^{10} \left( \int_{B_0(1)} \left< L_j(y), Y_R(y) \right> \xi \, dy \right) L_j \left( \frac{x}{R} \right) = \pi_1(Y_R) \left( \frac{x}{R} \right).
\]

In the end (9.12) becomes:

\[
(Y - \pi_R(Y))_i (x) = \int_{B_0(R)} \left< \frac{1}{R} G_{i,1} \left( \frac{x}{R}, \frac{y}{R} \right), \Delta_{\xi} Y(y) \right> \xi \, dy.
\]

Finally, \( \frac{1}{R} G_{i,1} \left( \frac{x}{R}, \frac{y}{R} \right) \) is orthogonal to \( K_R \): indeed for \( 1 \leq j \leq 10 \) there holds:

\[
\int_{B_0(R)} \left< \frac{1}{R} G_{i,1} \left( \frac{x}{R}, \frac{y}{R} \right), R^{-\frac{2}{3}} L_j \left( \frac{y}{R} \right) \right> \xi \, dy = R^{\frac{2}{3}} \int_{B_0(1)} \left< G_{i,1} \left( \frac{x}{R}, y \right), L_j(y) \right> \xi \, dy = 0,
\]

where the last equality is true since \( G_{i,1} \left( \frac{x}{R}, \cdot \right) \) is orthogonal to \( K_1 \) by definition. Using Claim (9.2) we then obtain (9.11).

The last ingredient of the proof is a symmetry property of \( G_{i,R} \):

**Claim 9.4.** For any \( x, y \in B_0(R) \) there holds:

\[
G_{i,R}(x, y)_i = G_{j,R}(y, x)_i - \pi_R \left( \tau G_{i, \cdot} (\cdot, x) \right)_i
\]

(9.13)

where we have set:

\[
\tau G_{i, \cdot} (\cdot, x)_j = G_{j,R}(x, y)_i.
\]

(9.14)

Proof. Let \( \Psi \in C^0(B_0(R)) \) be a 1-form orthogonal to \( K_R \). We define a 1-form in \( B_0(R) \) by

\[
H_i(x) = \int_{B_0(R)} G_{j,R}(y, x)_i \Psi(y) \, dy.
\]

(9.15)
By the explicit construction of $G_{j,R}$ in (9.9) it is easily seen that $H$ is continuous in $B_0(R)$. Also, $H$ is orthogonal to the conformal Killing forms since by Fubini’s theorem, for any $K \in K_R$,

$$
\int_{B_0(R)} H_i(y) K^i(y) dy = \int_{B_0(R)} \Psi^j(z) \int_{B_0(R)} G_{j,R}(z,y) K^i(y) dy dz = 0 \quad (9.16)
$$

since by construction $G_{j,R}(z,\cdot)$ is orthogonal to $K_R$ for any $z \in B_0(R)$. By Claim 9.1 and since $\Psi$ is orthogonal to $K_R$ we can let $F$ be the unique $C^1$ 1-form in $B_0(R)$ orthogonal to $K_R$ for any $z \in B_0(R)$. Let also $\Phi$ be a smooth 1-form such that $\mathcal{L}_\xi \Phi^k = 0$ on $\partial B_0(R)$. With Fubini’s theorem, equation (9.2) and using the properties of $\Psi$ and $\Phi$ there holds:

$$
\int_{B_0(R)} H_i(y) \Delta_\xi \Phi^i(y) dy = \int_{B_0(R)} \Psi^j(z) \int_{B_0(R)} G_{j,R}(z,y) \Delta_\xi \Phi^i(y) dy dz
$$

where we integrated by parts to obtain the last inequality since the boundary terms vanish. In particular:

$$
\int_{B_0(R)} (H - F)_j(y) \Delta_\xi \Phi^j(y) dy = 0
$$

for any smooth $\Phi$ with $\mathcal{L}_\xi \Phi^k = 0$ on $\partial B_0(R)$. Note that by a density argument the above inequality remains true for $\Phi \in W^{2,p}(B_0(R))$ for any $p > 1$ and orthogonal to $K_R$. By construction $F$ is orthogonal to $K_R$ and thanks to (9.16) so is $H$. By Claim 9.1 we can thus choose $\Phi$ to be the unique 1-form orthogonal to $K_R$ satisfying $\Delta_\xi \Phi = F - H$ in $B_0(R)$ and $\mathcal{L}_\xi \Phi^k = 0$ in $\partial B_0(R)$ to obtain, with the above inequality, that $F = H$. Independently, using (9.2) gives:

$$
F_i(x) = \int_{B_0(R)} G_{i,R}(x,y) \Psi^j(y) dy
$$

so that

$$
\int_{B_0(R)} (G_{j,R}(y,x)_i - G_{i,R}(x,y)_j) \Psi^j(y) dy = 0 \quad (9.17)
$$
for any continuous, Killing-free $\Psi$. Let now $X$ be any smooth 1-form in $B_0(R)$. Choose $\Psi = X - \pi_R(X)$. There holds

$$
\int_{B_0(R)} G_{j,R}(y,x)\pi_R(X)^j(y)dy
$$

$$
= \sum_{p=1}^{10} \int_{B_0(R)} G_{j,R}(y,x)\left(\int_{B_0(R)} (K_p(z), X(z))\xi dz\right)K_p(y)^jdy
$$

$$
= \sum_{p=1}^{10} \int_{B_0(R)} X_i(z)\left(\int_{B_0(R)} G_{j,R}(y,x)K_p(y)^jdy\right)K_p'(z)dz
$$

$$
= \sum_{p=1}^{10} \int_{B_0(R)} X_i(z)\pi_R \left(\frac{\partial G_i}{\partial x}\right)^j(z)dz,
$$

where $\frac{\partial G_i}{\partial x}$ is as in (9.14). Since $G_{i,R}(x,\cdot)$ has no conformal Killing part equation (9.17) becomes:

$$
\int_{B_0(R)} \left(G_{j,R}(y,x)_{i} - G_{i,R}(y,x)_{j} - \pi_R \left(\frac{\partial G_i}{\partial x}\right)^j(y)\right)X^j(y)dy = 0
$$

for any smooth 1-form $X$ and this concludes the proof of the claim.  

\[
\square
\]

We are now able to end the proof of Proposition 9.1. Let $x \in B_0(1)$ and consider $U_{i,2}^1$ as defined in (9.8). Since $\overline{\Delta}_\xi$ is coercive on the orthogonal of $K_1$ we can use elliptic regularity results – as those stated in the proof of Claim 9.1 – to get that there exist positive constants $C_1, C_2$ that do not depend on $x$ such that

$$
\|U_{i,1}^1\|_{C^1(\partial B_0(R))} \leq C_1 + C_2\|\mathcal{L}_E H_1(x - \cdot), \nu \otimes \cdot\|_{C^1(\partial B_0(R))},
$$

(9.18)

where we have let $(\mathcal{L}_E H_1(x - \cdot), \nu \otimes \cdot)_1 = \nu^k \mathcal{L}_E \left(H_1(x - \cdot)\right)_{kl}$. Let $K$ be some compact set in $B_0(1)$ and assume $x \in K$. It is easily seen by definition of $H_1$ as in (8.1) that

$$
\|\mathcal{L}_E H_1(x - \cdot), \nu \otimes \cdot\|_{C^1(\partial B_0(R))} \leq C_3 d(K, \partial B_0(1))^{-2}
$$

for some positive constant $C_3$ independent of $x$. By the definition of $G_{i,1}(x,\cdot)$ in (9.9) one therefore easily obtains that for $x \in K$ and $y \in B_0(1)$:

$$
|x - y|\|\nabla_x G_{i,1}(x, y)\| + |G_{i,1}(x, y)| \leq C(\delta)|x - y|^{-1},
$$

(9.19)

where $\delta$ is as in (9.4). This gives (9.3) when the derivative is taken with respect to $y$. The same estimate when the derivative is taken with respect to $x$ is obtained differentiating (9.13) and combining with (9.19). Finally, (9.5) for any positive $R$ is obtained combining (9.19) with Claim 9.3.

\[
\square
\]

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