Forces inside hadrons: pressure, surface tension, mechanical radius, and all that

Maxim V. Polyakov\textsuperscript{1,2} and Peter Schweitzer\textsuperscript{3}

\textsuperscript{1}Petersburg Nuclear Physics Institute, Gatchina, 188300, St. Petersburg, Russia
\textsuperscript{2}Institut für Theoretische Physik II, Ruhr-Universität Bochum, D-44780 Bochum, Germany
\textsuperscript{3}Department of Physics, University of Connecticut, Storrs, CT 06269, USA

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The physics related to the form factors of the energy momentum tensor spans a wide spectrum of problems, and includes gravitational physics, hard exclusive reactions, hadronic decays of heavy quarkonia, and the physics of exotic hadrons described as hadroquarkonia. It also provides access to the “last global unknown property:” the $D$-term. We review the physics associated with the form factors of the energy-momentum tensor and the $D$-term, their interpretations in terms of mechanical properties, their applications, and the current experimental status.

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I. INTRODUCTION

Such fundamental properties of a particle as mass and spin can be viewed as the response of a particle to external gravitational field (change of space-time metric). As the gravity couples to matter via the energy-momentum tensor (EMT), the mass and the spin of a particle can be obtained from the matrix elements of EMT \cite{1, 2}. However, the EMT matrix elements contain one more fundamental information: the $D$-term \cite{3}.\(^1\) The corresponding quantity is related to the variation of the spatial components of the space-time metric, i.e. to spatial deformations. Therefore, the $D$-term is expressed in terms of the stress-tensor which characterizes the distribution of forces inside the system (see, e.g. \cite{5}). This allows one to define “mechanical properties” of hadrons \cite{6}.

The $D$-term is a characteristic which is as fundamental as the mass and the spin of a particle. For free particles (and also for Goldstone bosons) its value is fixed by general principles — in the same way as, e.g. the gyromagnetic ratio of the free electron is fixed. In strongly interacting systems the $D$-term is sensitive to correlations in the system. For example, the baryon $D$-term behaves as $\sim N_c^2$ whereas all other global observables (mass, magnetic moments, axial charge, etc.) behave at most as $\sim N_c$ in the limit of a large number of colors $N_c$.

The $D$-term is a contribution to unpolarized generalized parton distributions (GPDs) in the region $-\xi \leq x \leq \xi$ \cite{3} which can be accessed through studies of hard exclusive reactions \cite{7–21}, see \cite{22–28, 30} for reviews. The $D$-term determines the asymptotics of GPDs in the limit of renormalization scale $\mu \to \infty$ \cite{24}, appears in Radon transforms \cite{31}, and emerges as a subtraction constant in fixed-$t$ dispersion relations for DVCS amplitudes\(^2\) \cite{32–35}.

The physics of EMT form factors is important not only for the description of hadrons in strong gravitational fields and hard exclusive processes. It is also relevant for the QCD description of hadronic decays of heavy quarkonia \cite{36–38}, played an important role for the physics of light Higgs bosons (at a time when a light Higgs was not excluded) \cite{39, 40}, and for the quantification of the picture of pentaquarks and tetraquarks with hidden charm as hadroquarkonia \cite{41}.

The purpose of this article is to provide a concise review of the physics associated with the $D$-term and EMT form factors, their interpretation in terms of mechanical properties, their applications, and the current experimental status.

II. THE EMT OF QCD

The EMT of QCD can be obtained by varying the action $S_{\text{grav}}$ of QCD coupled to a weak classical torsionless gravitational background field with respect to the metric $g_{\mu\nu}(x)$ of this curved background field according to

$$\hat{T}_{\mu\nu}(x) = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{grav}}}{\delta g^{\mu\nu}(x)}$$  \hspace{1cm} (1)

where $g$ denotes the determinant of the metric. This procedure (see e.g. App. E of \cite{26} for a pedagogical description) yields a symmetric Belinfante-Rosenfeld EMT. The quark and gluon contributions to the total EMT operator are given by

$$T^\mu_{\nu q} = \frac{1}{4} \bar{\psi}_{q} \left(-i\gamma^\nu \gamma^\mu + iD^\nu \gamma^\mu + iD^\mu \gamma^\nu \right) \psi_q - g^{\mu\nu} \bar{\psi}_{q} \left(-\frac{i}{2} \not{D} + \frac{i}{2} \not{m}_q \right) \psi_q$$  \hspace{1cm} (2a)

$$T^\mu_{\nu g} = F^{a,\mu\nu} F_{a,\nu\mu} + \frac{1}{4} g^{\mu\nu} F_{a,\kappa\eta} F^{a,\kappa\eta}$$  \hspace{1cm} (2b)

\(^1\) The name “$D$-term” was coined in Ref. \cite{4} and can be traced back to the notation chosen in Ref. \cite{3}. The similarity to the “$D$-term” used in the terminology of supersymmetric theories is mostly accidental \cite{4}.

\(^2\) It seems for the first time this was shown in Ref. \cite{6} for dispersion relations at small Bjorken $x$ (see Eq. (4) of that paper).
Here $\vec{D}_\mu = \vec{\partial}_\mu + ig t^a A^a_\mu$ and $\vec{D}_\mu = \vec{\partial}_\mu - ig t^a A^a_\mu$ with arrows indicating which fields are differentiated, $F_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$, and the SU(3) color group generators satisfy the algebra $[t^a, t^b] = i f^{abc} t^c$ and are normalized as $\text{tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$. The total EMT is conserved

$$\partial^\mu \tilde{T}_{\mu\nu} = 0, \quad \tilde{T}_{\mu\nu} = \sum_q \tilde{T}^q_{\mu\nu} + \tilde{T}^g_{\mu\nu}. \quad (3)$$

On “classical level” and for massless quarks QCD is invariant under conformal (“scale”) transformations: the associated current $j^\mu = x_i \tilde{T}^\mu_{\nu\nu}$ satisfies $\partial_\mu j^\mu = \tilde{T}^\mu_{\mu\mu}$ and is conserved for light quarks in the chiral limit due to $\tilde{T}^\mu_{\mu\nu} = \sum_q m_q \bar{\psi}_q \psi_q$. Quantum corrections break the conformal symmetry due to the trace anomaly [42-45]

$$\tilde{T}^\mu_{\mu\nu} = \frac{\beta(g)}{2g} F^{a A}_{\mu A} F^{a A}_{\mu A} + (1 + \gamma_m) \sum_q m_q \bar{\psi}_q \psi_q,$$  

where $\beta(g)$ is the $\beta$-function of QCD and $\gamma_m$ is the anomalous dimension of the mass operator. The trace anomaly has the same form in QED.

The vacuum matrix elements of the EMT are related to the quark and gluon vacuum condensates with important applications in the QCD sum rule approach [46]. The EMT vacuum expectation values were also explored, e.g., to prove low energy theorems for correlators containing the gluon operator $F^2$ [47]. The emergence of vacuum condensates reflects the rich and non-trivial structure of the QCD vacuum. In literature it is sometimes said that the vacuum is trivial in the light-front quantization approach, but this is strictly speaking not the case, see Ref. [48] for a review.

The main focus of this review are hadronic properties as encoded in the matrix elements of $\tilde{T}^\mu_{\nu\nu}$ in one-particle states. The matrix elements of the total EMT in one-particle states define the EMT form factors which are Lorentz scalars. Due to the EMT conservation (3) the form factors of the total EMT are renormalization scale invariant.

Since the individual quark and gluon operators, $\tilde{T}^q_{\mu\nu}$ and $\tilde{T}^g_{\mu\nu}$, are each separately gauge-invariant, we can also define quark and gluon EMT form factors which are also Lorentz scalars. Since the separate quark and gluon EMT operators are not conserved additional form factors appear in the decompositions of their matrix elements, and all individual quark and gluon form factors acquire scale- and scheme-dependence.

### III. DEFINITION OF EMT FORM FACTORS

We use the covariant normalization $\langle p' | p \rangle = 2p^0 (2\pi)^3 \delta^{(3)} (p' - p)$ of one-particle states, and introduce the kinematic variables $P = \frac{1}{2} (p' + p)$, $\Delta = p' - p$, $t = \Delta^2$. The EMT form factors of a spin-$\frac{1}{2}$ hadron in QCD are defined as

$$\langle p' ,s'| \tilde{T}^a_{\mu\nu}(x) | p,s \rangle = \bar{u}' \left[ A^a(t) \frac{\gamma(\mu P_v)}{2} + B^a(t) \frac{i P_{(\mu} \bar{\sigma}_{\nu)} \eta^{\rho} \Delta^\rho + D^a(t) \frac{\Delta_{\mu} \Delta_{\nu} - g_{\mu\nu} \Delta^2}{4m} + m \bar{c}^a(t) g_{\mu\nu} \right] u e^{i(p' - p) \cdot x}). \quad (5)$$

where the normalization of spinors is $\bar{u}(p,s) u(p,s) = 2m$, and we introduced the notation $a_1(\mu b_1) = a_\mu b_1 + a_\nu b_\nu$. Exploring the Gordon identity $2m \bar{\psi} \gamma^\nu \gamma^\kappa u = \bar{u}' (2P^\rho + i \sigma^{\alpha\kappa} \Delta_\alpha) u$ an alternative decomposition is obtained

$$\langle p' ,s'| \tilde{T}^a_{\mu\nu}(x) | p,s \rangle = \bar{u}' \left[ A^a(t) \frac{P_{\mu} P_{\nu}}{m} + J^a(t) \frac{i P_{(\mu} \bar{\sigma}_{\nu)} \eta^{\rho} \Delta^\rho + D^a(t) \frac{\Delta_{\mu} \Delta_{\nu} - g_{\mu\nu} \Delta^2}{4m} + m \bar{c}^a(t) g_{\mu\nu} \right] u e^{i(p' - p) \cdot x}). \quad (6)$$

The two representations are equivalent, and the form factors are related as $A^a(t) + B^a(t) = 2 J^a(t)$. Form factors of the non-symmetric canonical EMT were discussed in [49]. For a discussion of classical aspects of the EMT we refer to [50, 51]. Let us remark that if the symmetries of QCD, parity and time reversal, are relaxed, then more form factors are possible. For instance Dirac neutrinos have 5 EMT form factors [1, 52], and Majorana neutrinos even 6 [53].

Spin-0 hadrons like pion, $^4\text{He}$, etc have 3 EMT form factors in QCD which can be defined as follows

$$\langle p' | \tilde{T}^a_{\mu\nu}(x) | p \rangle = \left[ 2P_{\mu} P_{\nu} A^a(t) + \frac{1}{2} (\Delta_{\mu} \Delta_{\nu} - g_{\mu\nu} \Delta^2) D^a(t) + m^2 \bar{c}^a(t) g_{\mu\nu} \right] e^{i(p' - p) \cdot x}. \quad (7)$$

The individual quark and gluon form factors $A^a(t)$, $B^a(t)$ or $J^a(t)$, $D^a(t)$, $\bar{c}^a(t)$ depend on the renormalization scale which we do not indicate for brevity. Due to EMT conservation, Eq. (3), the constraint $\sum_a c^a(t) = 0$ holds, and the total form factors $A(t)$, $B(t)$, $D(t)$ are renormalization scale invariant where we defined $A(t) = \sum_a A^a(t)$ with $a = q, u, d, \ldots$ and analog for other form factors.

The total EMT form factors were introduced by Kobzarev and Okun [1] and by Pagels [2] already in the 1960’s for both spin-0 and spin-$\frac{1}{2}$ hadrons in somewhat different notation. (We refer to Ref. [54] for an overview of the different notations used in literature.) Hadrons with higher spins have additional EMT form factors because, for instance in the spin-1 case the polarization vectors $\epsilon^{\mu \nu} e^\nu$ can be used to generate further symmetric Lorentz structures.
Assuming adequate normalization this form factor satisfies at zero momentum transfer the constraint
\[ A(0) = 1. \] (8)

This is a consequence of translational invariance and reflects the fact that in the limit \( p, p' \to 0 \) only the 00-component remains in Eqs. (5–7), and \( H = \int d^4x \, \hat{T}_{00}(x) \) is the Hamiltonian of the system with \( H \{ p \} = m \{ p \} \) in the rest frame of the particle [2]. The physical meaning of this constraint is that 100% of the hadron momentum is carried by its constituents in the infinite momentum frame, or in lightcone quantization [55].

If we define \( A^Q(t) = \sum_q A^q(t) \) (and analog for other form factors below) the quark and gluon form factors at zero-momentum transfer take in the limit of asymptotically large renormalization scale \( \mu \) the following values [56, 57]
\[
\lim_{\mu \to \infty} A^Q(0) = \frac{N_f}{N_f + 4C_F}, \quad \lim_{\mu \to \infty} A^g(0) = \frac{4C_F}{N_f + 4C_F}.
\] (9)

Here \( C_F = (N_c^2 - 1)/(2N_c) \) where \( N_c = 3 \) is the number of colors in QCD, \( N_f \) is the number of active quark flavors. The results in Eq. (9) refer to the leading-order in the QCD coupling \( \alpha_s \), and show how quarks and gluons share the momentum of the hadron at asymptotically large scales. The values of \( A^Q(0) \) and \( A^g(0) \) can be obtained from parameterizations of the parton distribution functions \( f_1^q(x) \) extracted from deep-inelastic scattering experiments,
\[
A^Q(0) = \sum_q \int_0^1 dx \, x(f_1^q + f_3^q)(x), \quad A^g(0) = \int_0^1 dx \, x f_1^g(x).
\] (10)

The quantity \( f_1^q(x) \) describes the probability to find (at a given normalization scale not indicated here for brevity) a quark or gluon carrying the fraction \( x \) of the hadron’s momentum in infinite-momentum frame/lightfront quantization in the interval \( [x, x + dx] \). Parameterizations of parton distribution functions are available for nucleon [58–61], nuclei [62–64], pion [65, 66], and kaon [67]. For instance, in the nucleon quarks carry about 54% and gluons about 46% of the nucleon momentum at a scale of \( \mu^2 = 4 \text{GeV}^2 \) according to the leading order parameterizations of Ref. [59]. The partonic interpretation is valid in leading order of \( \alpha_s \), and has restrictions for nuclei [68].

The form factor \( B(t) \) which exists for hadrons with \( J > 0 \) satisfies at zero-momentum transfer the constraint
\[ B(0) = 0. \] (11)

This constraint is referred to as the vanishing of the “anomalous gravitomagnetic moment” of a spin-$\frac{1}{2}$ fermion and was first proven classically in [1] and discussed in quantum field theories in various contexts [2, 8, 69–73]. There is an interesting analogy to the anomalous magnetic moment of the electron: at one loop order in QED the anomalous magnetic moment of the electron is given by the famous Schwinger term \( \alpha/2\pi \). At this order the fermionic contribution to the anomalous gravitomagnetic moment of the electron is \( \alpha/3\pi \). This is compensated by the bosonic contribution \(-\alpha/3\pi \) [70], because photons also couple to gravity! It was shown that the anomalous gravitomagnetic moment of a composite fermion vanishes order by order in the lightfront Fock expansion. This can be traced back to the Lorentz boost properties of the lightfront Fock representation [70].

In the notation of Eq. (6) the constraint (11) is equivalent to
\[ J(0) = \frac{1}{2}, \] (12)
which reflects the fact that the contributions of quarks and gluons to the spin of the nucleon add up to \( \frac{1}{2} \) [8]. We will follow up on this below. At asymptotically large scales the contributions of quarks and gluons to the nucleon spin is analog to the momentum distribution in Eq. (9), namely [74]
\[
\lim_{\mu \to \infty} J^Q(0) = \frac{1}{2} \frac{N_f}{N_f + 4C_F}, \quad \lim_{\mu \to \infty} J^g(0) = \frac{1}{2} \frac{4C_F}{N_f + 4C_F}.
\] (13)

The constraints in Eqs. (8, 11, 12) were revisited recently in an axiomatic approach in Ref. [73].

The deeper reason for the existence of the constraints at zero-momentum transfer — for the form factors \( A(t) \) of all hadrons and the form factors \( B(t) \) or \( J(t) \) for hadrons with \( J \geq \frac{1}{2} \) — is owing to the fact that these form factors are related to the generators of the Poincaré group which are ultimately related to the mass and spin of the particle.
In contrast to this the value of the form factor \( D(t) \) at zero-momentum transfer is unconstrained. This value is often referred to as the \( D \)-term \( D \equiv D(0) \) [3]. All hadrons independently of their spin 0, \( \frac{1}{2} \), 1, \( \frac{3}{2} \) ... possess a \( D \)-term. But this property is unknown and must be determined from experiment.\(^3\) Below we will see that in contrast to \( A(0) \) and \( B(0) \) the \( D \)-term is not related to “external properties” of a particle like mass and spin, but to the stress tensor and internal forces. The only general information about this property is that at asymptotically large scales the relative contributions of quarks and gluons to the \( D \)-term is the same as in Eqs. (9, 13) and can be expressed as [24]

\[
\lim_{\mu \to \infty} D^Q(0) = D \frac{N_f - 4C_F}{N_f + 4C_F}, \quad \lim_{\mu \to \infty} D^g(0) = D \frac{4C_F}{N_f + 4C_F}.
\]

V. RELATION TO OBSERVABLES

The most natural way to probe EMT form factors, scattering off gravitons in Fig. 1a, is also the least practical one. A practical opportunity to access EMT form factors emerged with the advent of GPDs [7–21], which describe hard-exclusive reactions, such as deeply virtual Compton scattering (DVCS) \( eN \to e'N'\gamma \) sketched in Fig. 1b or hard exclusive meson production \( eN \to e'N'M \). In the case of the nucleon, the second Mellin moments of unpolarized GPDs yield the EMT form factors

\[
\int_{-1}^{1} dx \ x H^a(x, \xi, t) = A^a(t) + \xi^2 D^a(t), \quad \int_{-1}^{1} dx \ x E^a(x, \xi, t) = B^a(t) - \xi^2 D^a(t).
\]

The GPDs can be viewed as “amplitudes” for removing from the nucleon a parton carrying the fraction \( x - \xi \) of the average momentum \( P \) and putting back in the nucleon a parton carrying the fraction \( x + \xi \), while the nucleon receives the momentum transfer \( \Delta \). For \( \xi = 0 \) the momentum transfer is purely transverse and the Fourier transform \( H^a(x, b_\perp) = \int d\Delta_\perp/(2\pi)^2 \exp(-i\Delta_\perp b_\perp) H^a(x, 0, -\Delta_\perp^2) \) describes the probability to find a parton carrying the momentum fraction \( x \) of the hadron and located at the distance \( b_\perp \) from the hadrons (transverse) center-of-mass on the lightcone. This allows one to do femtoscale tomography of the nucleon [75–77].

Adding up the two equations in (15) and extrapolating \( t \to 0 \) provides the key to the nucleon’s spin decomposition: the fraction of the nucleon spin due to the angular momentum of the parton of type \( a = g, u, d \ldots \) is obtained from the Ji sum rule [8]

\[
\lim_{t \to 0} \int_{-1}^{1} dx \ x \left( H^a(x, \xi, t) + E^a(x, \xi, t) \right) = 2J^a(0).
\]

These fractions are currently unknown. An unambiguous decomposition into the “spin contribution” and the “orbital angular momentum contribution” to \( J^a(0) \) is not possible, though different schemes can be introduced. We refer to the review article [78] for a discussion of the controversy of the nucleon spin decomposition.

Note that GPDs are not direct observables and they cannot be fully restored from the DVCS observables, see the detailed discussion in Ref. [79] based on the dual parametrisation of GPDs [80]. In particular, it was shown that the Mellin moments of GPDs (15) are not directly observable. However, the \( D \)-term (in contrast to \( J^a(t) \) or \( A^a(t) \)) can be extracted directly from the DVCS observables through the logarithmic \( Q^2 \) dependence of the subtraction constant in the dispersion relations for the DVCS amplitudes [6, 32–35], see the discussions in Section XIX A.

First experimental results related to nucleon GPDs from studies of hard exclusive reactions came from the HERMES and HERA experiments at DESY and Jefferson Lab [81–101]. Studies of hard-exclusive reactions are an important part of ongoing experimental programs at Jefferson Lab and COMPASS at CERN. A spin-0 hadron like pion has only one GPD, namely \( H^a(x, \xi, t) \), which is per se not directly accessible in experiment. However, hard-exclusive production of two pions provides the possibility to access information on the two-pion distribution amplitude which is an analog of a GPD in the crossed channel, see Fig. 1c. This allows one to extract information on the EMT form factors of the pion (and similarly other hadrons) in the time-like region.

For completeness let us remark that a relation of the fixed poles in the angular momentum plane in virtual Compton scattering [102–104] to the \( D \)-term was discussed in [105]. However, it was shown that the \( J = 0 \) fixed pole universality hypothesis of Ref. [105] is an external assumption and might never be proven theoretically [106].

\(^3\) Also the mass \( m \) must be of course determined from experiment. However, once \( m \) is known and an appropriate normalization introduced, the values of \( A(0) \) and \( J(0) \) are fixed. This is not the case for \( D(0) \) showing that this is an independent property.
Figure 1. (a) A natural but impractical probe of EMT form factors is scattering off gravitons. (b) Hard-exclusive reactions like deeply virtual Compton scattering (DVCS) provide a realistic way to access EMT form factors through GPDs. Here one of the relevant tree-level diagrams is shown. (c) Information on the EMT structure of particles not available as targets, such as e.g. $\pi^0$, can also be accessed from studies of generalized distribution amplitudes (GDAs) which are an “analytic continuation” of GPDs to the crossed channel. The shown reaction $\gamma^*\gamma \rightarrow \pi^0\pi^0$ (and analog for other hadrons) can be studied in $e^+e^-$ collisions.

VI. THE LAST GLOBAL UNKNOWN PROPERTY OF A HADRON

The $D$-term is sometimes referred to as the “last unknown global property.” To explain what this means we recall that the structure of hadrons, the bound states of strong interactions, is most conveniently probed by exploring the other fundamental forces: electromagnetic, weak, and (in principle) gravitational interactions. The particles couple to these interactions via the fundamental currents $J^\mu_{\text{em}}$, $J^\mu_{\text{weak}}$, $T^{\mu\nu}_{\text{grav}}$ which are conserved (in case of weak interactions we deal with partial conservation of the axial current, PCAC). The matrix elements of these currents are described in terms of form factors which contain a wealth of information on the probed particle. The undoubtedly most fundamental information corresponds to the form factors at zero momentum transfer. For the nucleon, these are the “global properties:” electric charge $Q$, magnetic moment $\mu$, axial coupling constant $g_A$, induced pseudo-scalar coupling constant $g_p$, mass $M$, spin $J$, and the $D$-term $D$. These properties, being related to external conserved currents, are scale- and scheme-independent in QCD. All global properties are in principle on equal footing and well-known, see Table I, with one exception: the $D$-term.

\begin{table}[h]
\centering
\begin{tabular}{lll}
| Source         | Expression                        | Values                                                               |
\hline
electromagnetic: & $\partial_\mu J^\mu_{\text{em}} = 0 \quad \langle N'|J^\mu_{\text{em}}|N \rangle \quad \rightarrow \quad Q = 1.602176487(40) \times 10^{-19}\text{C}$ & $\mu = 2.792847356(23)\mu_N$ \\
weak:           & $\langle N'|J^\mu_{\text{weak}}|N \rangle \quad \rightarrow \quad g_A = 1.2694(28)$ & $g_p = 8.06(55)$ \\
gravity:        & $\partial_\mu T^{\mu\nu}_{\text{grav}} = 0 \quad \langle N'|T^{\mu\nu}_{\text{grav}}|N \rangle \quad \rightarrow \quad m = 938.272013(23)\text{MeV}/c^2$ & $J = \frac{1}{2}$ \\
& & $D = \frac{3}{2}$
\end{tabular}
\caption{The global properties of the proton defined in terms of matrix elements of the conserved currents associated with respectively electromagnetic, weak, and gravitational interaction. Notice the weak currents include the partially conserved axial current, and $g_A$ or $g_p$ are strictly speaking defined in terms of transition matrix elements in the neutron $\beta$-decay or muon-capture. The values of the properties are from the particle data book [107] and [108] (for $g_p$) except for the unknown $D$-term.}
\end{table}

In some cases (e.g. free particles, Goldstone bosons) the value of the $D$-term is fixed by general principles (see discussions below). For other particles the $D$-term is not fixed and it reflects the internal dynamics of the system through the distribution of forces. In strongly interacting systems the $D$-term is sensitive to correlations in the system. For example, the baryon $D$-term behaves as $\sim N_c^2$ whereas all other global observables (mass, magnetic moments, axial charge, etc.) behave at most as $\sim N_c$ in the large $N_c$ limit. For a large nucleus the $D$-term shows also anomalously fast increase with the atomic mass number $D \sim A^{7/3}$. 

The total energy density in a hadron is thus given by \[6\] to define the quark and gluon contributions to the EMT-nonconserving term \[\bar{c}\] in Eqs. (17a–17b) with respect to \(\Delta\) as follows: 

\[
\langle p', s' | \bar{T}^{00}_a (0) | p, s \rangle = 2 m E \left[ A^a (t) - \frac{t}{4 m^2} [A^a (t) - 2 J^a (t) + D^a (t)] + c^a (t) \right] \delta_{ss'}
\]

\[
\langle p', s' | \bar{T}^{ik}_a (0) | p, s \rangle = 2 m E \left[ D^a (t) \frac{\Delta^k \delta_{ik} - \delta_{ik} \Delta^2}{4 m^2} - c^a (t) \delta_{ik} \right] \delta_{ss'}
\]

\[
\langle p', s' | \bar{T}^{0k}_a (0) | p, s \rangle = 2 m E \left[ J^a (t) \frac{(-i \Delta \times \sigma_{ss'})^k}{2 m} \right]
\]

Notice that in the Breit frame \(\bar{u}'u \equiv \bar{u}(p', s')u(p, s) = 2E \delta_{ss'}\) and the energy \(E\) is given by \(E = \sqrt{m^2 + \Delta^2}\). Here \(\sigma^j_{ss} = \chi^j_s \sigma^j \chi_s\) with the nucleon Pauli spinors \(\chi_s\) and \(\chi_s\) in the respective rest frames normalized as \(\chi^j_s \chi_s = \delta_{ss'}\). The matrix elements of \(\bar{B}_{a}^{0k}\) vanish if the polarization is along \(\Delta\).

We can now define the static EMT \(T^{\mu\nu}(r, s)\) of a hadron by Fourier transforming the matrix elements of the EMT in Eqs. (17a–17b) with respect to \(\Delta\) as follows [6]: 

\[
T_{a}^{\mu\nu}(r, s) = \int \frac{d^3 \Delta}{(2\pi)^3 2E} e^{-ir\Delta} \langle p'| \bar{T}^{a\mu}_\nu (0) | p \rangle,
\]

where \(s\) denotes the polarization vector of the states \(|p, s\rangle\) and \(|p', s'\rangle\) in the respective rest frames.

Let us first discuss the 00-component of (18), i.e. the energy density. Due to the presence of the EMT-nonconserving term \(\bar{c}\) in the energy density \(T_{00}(r)\) can only be defined for the total system (recall that \(\sum_a \bar{c}^a(t) = 0\), see Sec. III). The total energy density in a hadron is thus given by [6]: 

\[
T_{00}(r) = m \int \frac{d^3 \Delta}{(2\pi)^3} e^{-ir\Delta} \left[ A(t) - \frac{t}{4 m^2} [A(t) - 2 J(t) + D(t)] \right].
\]

The energy density is a function of \(r = |r|\). It is independent of polarization, and normalized as 

\[
\int d^3 r T_{00}(r) = m
\]

due to the constraint (8). The expression (19) is analogous to the electric charge distribution which can be mapped out by means of electron scattering experiments. In an analog way, (hypothetical) scattering off gravitons would allow one to access information on the spatial distribution of the energy inside a hadron. We stress that in this way we can only access the energy density of the total system. The decomposition of the nucleon mass in terms of contributions from quarks and gluons was extensively discussed in Refs. [109–111].

The \(ij\)-components of static EMT define the stress tensor [6]. The total quark + gluon stress tensor can be decomposed in a traceless part associated with shear forces \(s(r)\) and a trace associated with the pressure \(p(r)\),

\[
T^{ij}(r) = \left( \frac{r^i r^j}{r^2} - \frac{1}{3} \delta^{ij} \right) s(r) + \delta^{ij} p(r).
\]

The shear forces are “good observables” and exist also separately for quarks and gluons (although \(T^{a}_{\mu\nu}\) and \(T^{a}_{\mu\nu}\) are not conserved separately, the EMT-nonconserving \(g_{\mu\nu} \bar{c}^a(t)\) in (5) drop out from the traceless part of the stress tensor), i.e. we can define (scale-dependent) partial contributions \(s^a(r)\) for \(a = g, u, d, \ldots\) to the shear forces. This also allows one to define the quark and gluon contributions to the D-term [6],

\[
D^a = -\frac{2}{5} m \int d^3 r T^a_{00} (r) \left( r^i r^j - \frac{1}{3} r^2 \delta^{ij} \right) = -\frac{4}{15} m \int d^3 r r^2 s^a(r).
\]

In contrast to this \(p(r)\) is defined only for the total system, and has no relation to the separate \(D^a\) and \(D^g\) [6].

The pressure \(p(r)\) and shear forces \(s(r)\) can be computed from \(D(t)\) as follows:

\[
s(r) = -\frac{1}{4m} \frac{1}{r} \frac{d}{dr} \frac{d}{dr} \bar{D}(r), \quad p(r) = \frac{1}{6m} \frac{1}{r^2} \frac{d}{dr} \frac{d}{dr} \bar{D}(r), \quad \bar{D}(r) = \int \frac{d^3 p}{(2\pi)^3} e^{-ipr} D(-p^2).
\]
The possibility to access the form factors

\[ J^a(r, t) = \frac{1}{4m} \frac{d}{dr} \frac{d}{dt} \tilde{D}^a(r), \quad p^a(r) = \frac{1}{6m^2} \frac{d^2}{dr^2} \frac{d}{dr} \tilde{D}^a(r) - m \int \frac{d^3p}{(2\pi)^3} e^{-ipr} c^a(-p^2). \]  

Comparing this equation with (23) we see that the shear force distribution for individual components can be obtained from the \( \tilde{D}^a(t) \) form factor, whereas the pressure distribution requires additionally the knowledge of the form factors \( \tilde{c}^a(t) \). The form factors \( \tilde{c}^a(t) \) can be computed in lattice QCD, but we are not aware of a physical processes allowing one to measure them.

The 0\( k \)-components of the EMT are related to the spatial distribution of the nucleon spin. The EMT non-conserving terms drop out in Eq. (17c) since they are polarization independent, such that we can define the individual contributions of quarks and gluons as follows

\[ J_k^a(r, s) = -e^{ijk} r^j T_0^k(r, s). \]  

Inserting the expression (17c) into Eq. (25) yields

\[ J_k^a(r, s) = s^j \int \frac{d^3 \Delta}{(2\pi)^3} e^{-ir\Delta} \left[ \left( J^a(t) + \frac{2}{3} t \frac{dJ^a(t)}{dt} \right) \delta^{ij} + \left( \Delta^i \Delta^j - \frac{1}{3} \Delta^2 \delta^{ij} \right) \frac{dJ^a(t)}{dt} \right] \]  

The first term in the square brackets corresponds to a monopole term [6], and the second term corresponds to a quadrupole term of Ref. [112]. In Ref. [6] only the monopole term was considered as the 3D angular momentum density was implicitly defined as \( J^a(0) = s^j \langle \hat{J}^a(r, s) \rangle \), where \( \langle \ldots \rangle \) denotes averaging over the direction of the vector \( s \). Integrating (26) and summing over quarks and gluons yields

\[ \sum_a \int d^3 r \ J_k^a(r, s) = s^j J(0) \]  

where we used that the individual contributions \( J_k^a(0) \) add up to \( \sum_a J_k^a(0) = J(0) \) with \( J(0) = \frac{1}{2} \) according to Eq. (12).

The possibility to access the form factors \( J_k^a(t) \) from studies of hard exclusive reactions through GPDs [8] will give important insights into the spin decomposition of the nucleon. We refer to Ref. [78] for a review on the nucleon spin decomposition, and to Ref. [112] for a discussion of the angular momentum density in the 3D Breit-frame representation and 2D lightfront representation.

**VIII. EMT DENSITIES IN SPIN-0 HADRONS**

The EMT densities of spin-0 hadrons can be defined in the same way as in Eq. (18) with the obvious difference that \( T_0^0(r) = 0 \) and the form factor \( J(t) \) is absent, which reflects the spin-0 character. The expressions for \( T_0^0(r) \) and \( T_1^0(r) \) look almost exactly as in the fermionic case, in Eqs. (19, 23), — with one subtle difference. In the Breit frame the nucleon spinors \( \tilde{u'} u = 2E \delta_{s's} \) contribute the factor \( 2E \) which cancels out the factor \( 1/(2E) \) in the definition of the static EMT in Eq. (18). In the spin-0 case no such compensation occurs, and the EMT densities satisfy

\[ T_{00}(r) = 2m^2 \int \frac{d^3 \Delta}{2E(2\pi)^3} e^{-ir\Delta} \left[ A(t) - \frac{t}{4m^2} \left[ A(t) + D(t) \right] \right], \quad \text{(spin 0 case)} \]  

\[ T_{ij}(r) = \frac{1}{2} \int \frac{d^3 \Delta}{2E(2\pi)^3} e^{-ir\Delta} \left[ \Delta_i \Delta_j - \delta_{ij} \Delta^2 \right] D(t). \]  

We remark that the pressure and shear forces can be extracted from Eq. (28b) as follows

\[ p(r) = \frac{1}{3} \int \frac{d^3 \Delta}{2E(2\pi)^3} e^{-i\Delta r} P_0(\cos \theta) \left[ t D(t) \right], \]  

\[ s(r) = \frac{3}{4} \int \frac{d^3 \Delta}{2E(2\pi)^3} e^{-i\Delta r} P_2(\cos \theta) \left[ t D(t) \right], \]  

\[ \text{For brevity we suppress the dependence of these quantities on the QCD normalisation scale.} \]
which illustrates their relations to respectively, the trace and the traceless part of the stress tensor. With the replacement \(2E \rightarrow 2m\) the Eqs. (29a, 29b) hold also for the nucleon and are equivalent to (23).

Effectively, the expressions for the polarization-independent 00- and \(ij\)-components of the static EMT in spin \(\frac{1}{2}\) vs spin 0 case differ by rescaling the form factors \(A(t)\) and \(D(t)\) by the factor \(m/E\) with \(E = \sqrt{m^2 + \Delta^2/4}\). Such modifications correspond to relativistic corrections to the static EMT densities (18). (Notice that \(E/m\) is the Lorentz gamma-factor for the boost from rest frame system to the Breit one.) There is no unique field-theoretic way to derive such modifications, see discussions in Refs. [114–117]. For the case of heavy particles \((mR_h \gg 1\) where \(R_h\) is the hadron size\) the relativistic corrections modify the EMT densities at short distances of order \(1/m\) which result in the relative corrections of order \(1/(mR_h)^2\) to the radii of EMT densities.\(^5\) In most physically interesting cases such corrections are very small (see the estimates in Ref. [113]) and we shall systematically neglect them. A special case is \(\pi\)-meson. The pion is the (pseudo)Goldstone boson of spontaneous chiral symmetry breaking, so its mass is parametrically small and we have \(m_\pi R_\pi \ll 1\), i.e. the relativistic corrections are big and the static EMT densities are not properly defined. However, for the pion one can apply the methods of chiral effective field theory, see discussion in Section XVII B.

Let us remark that the stress tensor \(T^{ik}(r)\) has the general form (21) in spin-0 and \(\frac{1}{2}\) case. For higher spin particles additional tensor structures \(S_i S_j\), \((S_i r_j + S_j r_i)(S \cdot r)\) are possible. Also the shear forces and the pressure have additional dependence on \((S \cdot r)^2\). The case of higher spin particles will be considered elsewhere.

### IX. CONSEQUENCES FROM EMT CONSERVATION

The EMT conservation, \(\partial^\mu \hat{T}_{\mu \nu} = 0\), implies for the static EMT \(\nabla^i T_{ij} = 0\). This yields the differential equation

\[
\frac{2}{3} s'(r) + \frac{2}{r} s(r) + p'(r) = 0,
\]

i.e. the shear forces and pressure are not independent functions but connected to each other due to EMT conservation. Another consequence of the EMT conservation is the von Laue condition [118], which shows how the internal forces balance inside a composed particle,

\[
\int_0^\infty dr r^2 p(r) = 0.
\]

This relation implies that the pressure must have at least one node. In all model studies so far it was found that the pressure is positive in the inner region, and negative in the outer region. In our convention the positive sign means repulsion towards outside, the negative sign means attraction directed towards inside.

The von Laue condition (31) is a necessary condition for stability, but not sufficient. We shall see below that also the pressure distribution inside an unstable particle like the \(\Delta\)-resonance satisfies the von Laue condition [119]. In models the von Laue condition is often equivalent to the virial theorem which expresses the fact that one deals with a global mimimum of the action for the given quantum numbers.

Using Eq. (30) we can express the \(D\)-term in two equivalent ways in terms of \(s(r)\) [cf. Eq. (22)] and \(p(r)\) as

\[
D = -\frac{4m}{15} \int d^3r r^2 s(r) = m \int d^3r r^2 p(r).
\]

Using (23) we can derive a number of interesting relations. For example, it is obvious that (cf. Ref. [120], App. B)

\[
\int_0^\infty dr \frac{2s(r)}{r} = p(0).
\]

This relation can be viewed as the generalization of well known Kelvin relation between the pressure \(p_0\) in a liquid spherical drop, its surface tension \((\gamma)\) and the radius \((R_{\text{drop}})\) of the drop \(2\gamma/R_{\text{drop}} = p_0\) [121]. Using this analogy we define the average surface tension of the hadron as \(\gamma = \int_0^\infty dr s(r)\) and the surface tension energy of a hadron as \(\int d^3r s(r)\). For the liquid drop we have \(s(r) = \gamma \delta(r - R_{\text{drop}})\) and thus these definitions correspond to \(\gamma\) and to \(4\pi \gamma R_{\text{drop}}^2\) respectively.

---

5 The \(D\)-term is not modified by the relativistic corrections as it is defined at zero momentum transfer.
Another interesting relation is obtained by integrating the second equation in (23) over a ball of radius $R$. Doing this we get:

$$\int_{|r| \leq R} d^3r \ p(r) = \frac{4\pi R^3}{3} \left( \frac{2}{3} s(R) + p(R) \right). \quad (34)$$

Below we shall see that the combination $\left( \frac{2}{3} s(R) + p(R) \right)$ corresponds to the distribution of the normal component of the force and must be positive to guarantee the stability of the system. This implies that the integral of the pressure over the ball of a finite radius $R$ is also positive and the von Laue condition (31) is realised if $s(r)$ and $p(r)$ drop at the infinity faster than $1/r^3$. This is typically the case, e.g. in the nucleon $s(r)$ and $p(r)$ vanish at large $r$ like $1/r^6$ in the chiral limit [120], and faster for finite $m_\pi$ as we shall see below.

Note that the nontrivial shear forces distribution $s(r)$ is responsible for the structure formation in a hadron. Indeed, from Eqs. (21,30) it follows that $s(r) = 0$ corresponds to isotropic matter with a constant pressure. Anisotropy and non-trivial shape of the pressure distribution (hadron shape) appear due to non-trivial shear forces distribution $s(r)$, the latter is also called pressure anisotropy [122]. Interestingly the pressure anisotropy (shear forces distribution) plays an essential role in astrophysics [122, 123], see the review [124] on the role of pressure anisotropy for self-gravitating systems in astrophysics and cosmology.

X. ENERGY DENSITY AND PRESSURE IN THE CENTER

If the form factors $A(t)$ and $D(t)$ are known, then the energy density and pressure in the nucleon center can be computed directly as:

$$T_{00}(0) = \frac{m}{2\pi^2} \int_{-\infty}^{0} dt \sqrt{-t} \left[ A(t) - \frac{t}{4m^2} D(t) \right], \quad (35a)$$

$$p(0) = \frac{1}{24\pi^2 m} \int_{-\infty}^{0} dt \sqrt{-t} \ t D(t). \quad (35b)$$

These integrals are convergent if $A(t)$ drops faster than $\sim |t|^{-3/2}$ and $D(t)$ drops faster than $\sim |t|^{-5/2}$ at large $t$. However, we have to remember that the integrals above are subject to relativistic corrections, see discussion in Sec. VIII. If the integrals (35a, 35b) are convergent the relative size of the relativistic corrections to $T_{00}(0)$ and $p(0)$ is of the order $1/(mR_h)$ and is small for the nucleon both parametrically (it is $\sim 1/N_c$ in the large $N_c$ limit) and numerically. If the integrals (35a, 35b) are divergent the present formalism is not able to make predictions for the values of $T_{00}(0)$ and $p(0)$. Notice that in Eq. (35a) we consistently neglected the contribution $\frac{t}{4m^2} [A(t) - 2J(t)]$, cf. Eq. (19), which constitutes a relativistic correction of the type discussed above.

XI. THE MEAN SQUARE RADIUS OF THE ENERGY DENSITY

The energy density in a mechanical system satisfies the inequality $T_{00}(r) > 0$. This allows us to introduce the notion of the mean square radius of the energy density as follows

$$\langle r^2 \rangle_E = \frac{\int d^3r \ r^2 T_{00}(r)}{\int d^3r T_{00}(r)}.$$

This definition gives the mean square radius of the energy density in terms of the slope of the form factor $A(t)$ and the $D$-term:

$$\langle r^2 \rangle_E = 6A'(0) - \frac{3D}{2m^2}. \quad (37)$$

Note that the second term in this equation is not a relativistic correction for baryons as $D \sim N_c^2$ and the baryon mass $m \sim N_c$ (we will discuss the large-$N_c$ limit below). However, for a heavy meson like a quarkonium, this term could be neglected as a relativistic correction.

---

6 A constant pressure is also obtained if $s(r) \propto \frac{1}{r^3}$ which could hold in a finite region $0 < r < \infty$. It would be interesting to explore e.g. soliton models with piece-wise defined potentials where in a certain region(s) of $r$ this situation could be realized.
Due to the trace anomaly (4) we can express (in the chiral limit of QCD) the mean square radius of the gluon operator $F^2 \equiv F^{\alpha \mu \nu} F^{\alpha \mu \nu}$ in terms of EMT form factors

$$\langle r^2 \rangle_F^2 = 6A'(0) - \frac{9D}{2m^2} = \langle r^2 \rangle_E - \frac{3D}{m^2}. \quad (38)$$

Below in Sections XII and XIII we shall see that the $D$-term is always negative, which implies that $\langle r_{F}^2 \rangle > \langle r_{E}^2 \rangle$.

### XII. THE MECHANICAL MEAN SQUARE RADIUS

The normal component of the total force exhibited by the system on an infinitesimal piece of area $dS^j$ at the distance $r$ has the form $F^i(r) = T^{ij}(r) dS^j = \left[ \frac{2}{3} s(r) + p(r) \right] dS^i$ where $dS^j = dS r^j / r$. In Ref. [119] we argued that for the mechanical stability of the system the corresponding force must be directed outwards. Therefore the local criterion for the mechanical stability can be formulated as the inequality

$$\frac{2}{3} s(r) + p(r) > 0. \quad (39)$$

This inequality implies that the $D$-term for any stable system must be negative [119],

$$D < 0. \quad (40)$$

The positive combination $\left[ \frac{2}{3} s(r) + p(r) \right]$ has the meaning of the normal force distribution in the system. This allows us to introduce the notion of the mechanical radius for hadrons:

$$\langle r^2 \rangle_{\text{mech}} = \frac{\int d^3 r \ r^2 \left[ \frac{2}{3} s(r) + p(r) \right]}{\int d^3 r \ \left[ \frac{2}{3} s(r) + p(r) \right]} = \frac{6D}{\int_{-\infty}^{0} dt \ D(t)}, \quad (41)$$

where in the numerator we used Eqs. (32) as $\int d^3 r \ r^2 \left[ \frac{2}{3} s(r) + p(r) \right] = -3D/2m$, while in the denominator we used the von Laue condition (31) and the relation of the surface tension energy of the system $\int d^3 r \ s(r)$ to $D(t)$ given by

$$\int d^3 r \ s(r) = -\frac{3}{8m} \int_{-\infty}^{0} dt \ D(t), \quad (42)$$

We can conjecture that the surface tension energy must be smaller than the total energy of the system, i.e. $\int d^3 r \ s(r) \leq m$. This would imply at least two things. First, the integral $\int_{-\infty}^{0} dt \ D(t)$ must converge which implies that $D(t)$ must drop at large $|t|$ faster than $-1/t$. Second, the mechanical radius is bound from below by $\langle r^2 \rangle_{\text{mech}} \geq -9D/(4m^2)$. We note however that our conjecture is still lacking a rigorous proof.

We see that $D$-term also determines the mechanical radius of the hadrons. Notice the unusual definition: unlike e.g. the mean square charge radius, the mechanical mean square radius is not related to the slope of a form factor. Also note that one obtains a non-zero value of the mechanical radius if $D(t)$ drops faster than $1/t$ at large (negative) values of $t$. For the multipole Ansatz $D(t) = D/(1 - t/\Lambda^2)^n$ the resulting mechanical mean square radius is $\langle r^2 \rangle_{\text{mech}} = 6(n - 1)/\Lambda^2$.

In Ref. [125], where an extraction of EMT form factors of $\pi^0$ in the time-like region was reported, also a definition of a “mechanical radius” was proposed, however, in terms of the slope of the form factor $D(t)$. This differs from our definition (41), and is not an appropriate measure of the true mechanical radius of a hadron [126].

### XIII. NORMAL AND TANGENTIAL FORCES IN HADRONS

Let us consider the slice of a hadron defined in spherical coordinates by the equation $\theta = \pi/2$. Other choices of the slice are equivalent to this for spherically symmetric hadrons, i.e. hadrons with spins 0 and 1/2. At any point on the slice the force (pressure) acting on the infinitesimal area element $dS = dS_r e_r + dS_\theta e_\theta + dS_\phi e_\phi$ has the following spherical components:

$$\frac{dF_r}{dS_r} = \frac{2}{3} s(r) + p(r), \quad \frac{dF_\theta}{dS_\theta} = \frac{dF_\phi}{dS_\phi} = -\frac{1}{3} s(r) + p(r). \quad (43)$$

The normal ($dF_r$) and tangential ($dF_\theta$, $dF_\phi$) forces are the eigenvalues of the stress tensor $T_{ij}$ with $e_r$, $e_\theta$, $e_\phi$ being the corresponding eigenvectors, which define the local principal axes. The positive (negative) eigenvalues correspond to
“stretching” (“squeezing”) along the corresponding principal axes. For spherically symmetric hadrons (spin-0, spin-1/2) the tangential forces \( dF_\phi, dF_\theta \) are equal to each other, for higher spin hadron, generically, they are different and spin dependent. In a stable spherically symmetric system the normal component of the force \( dF_r / dS_r = \frac{2}{3} s(r) + p(r) \) must correspond to “stretching” forces (to be positive, see Sec. XII) otherwise the system would collapse in “squeezing” direction. The tangential forces change sign with the distance \( r \) because possible “squeezing” is averaged to zero for spherically symmetric systems. It would be interesting to consider the local stability conditions for higher spin hadrons.

The positivity of the normal component of the force \( dF_n \) allowed us to define the notion of the mechanical radius in Sec. XII. The tangential components of the force \( dF_\phi \) and \( dF_\theta \) also allow us to define additional stability conditions with a nice, physically intuitive interpretation as the mechanical stability of lower dimensional subsystems of the whole 3D system. The \( \theta \)-component of the force acts perpendicular to the \( \theta = \pi/2 \) hadron’s slice. Let us consider the \( \theta \) component of the force acting on the infinitesimally narrow ring of the width \( dr \). The corresponding force is:

\[
dF = \left[ -\frac{1}{3} s(r) + p(r) \right] 2\pi r dr.
\]

As the pressure is the isotropic part of the stress tensor, the above expression can be written for the 2D subsystem (slice of the hadron \( \theta = \pi/2 \)) as follows:

\[
(2D \text{ pressure}) \times d(2D \text{ volume}).
\]

Then the 2D von Laue stability condition for the 2D subsystem has the form:

\[
2\pi \int_0^\infty dr \ r \left[ -\frac{1}{3} s(r) + p(r) \right] = 0.
\]

This is a new tangential stability condition. If we further consider the 1D subsystem (e.g. a ray \( \phi = 0 \)) of the 2D slice \( \theta = \pi/2 \) we can compute the 2D force acting on the line element\(^7\) \( dr \) of 1D ray \( \phi = 0, \theta = \pi/2 \) as:

\[
dF^{(2D)} = \left[ -\frac{4}{3} s(r) + p(r) \right] dr,
\]

again it has the structure \((1D \text{ pressure}) \times d(1D \text{ volume})\). The corresponding 1D von Laue stability condition reads:

\[
\int_0^\infty dr \left[ -\frac{4}{3} s(r) + p(r) \right] = 0.
\]

Using Eq. (23) we can relate the normal and tangential pressures to the \( D \)-form factor:

\[
\frac{2}{3} s(r) + p(r) = \frac{1}{2m} \frac{1}{r} \frac{d}{dr} \tilde{D}(r), \quad -\frac{1}{3} s(r) + p(r) = \frac{1}{4m} \frac{1}{r} \frac{d}{dr} \frac{d}{dr} \tilde{D}(r), \quad -\frac{4}{3} s(r) + p(r) = \frac{1}{2m} \frac{d^2}{dr^2} \tilde{D}(r).
\]

From these expressions we see immediately that the tangential stability conditions (46) and (48) are satisfied automatically. Interestingly we can write these expressions and Eq. (23) as

\[
p^{(nD)}(r) = \frac{1}{2nm} \partial^2_{nD} \tilde{D}(r),
\]

where \( \partial^2_{nD} \) is the \( n \)-dimensional \((nD)\) Laplace operator and the pressures in the \( n \)-dimensional subsystems are:

\[
p^{(3D)}(r) = p(r), \quad p^{(2D)}(r) = -\frac{1}{3} s(r) + p(r), \quad p^{(1D)}(r) = -\frac{4}{3} s(r) + p(r).
\]

The von Laue stability condition in \( n \)-dimensions has the form:

\[
\int d^n r \ p^{(nD)}(r) = 0,
\]

which is automatically satisfied due to Eq. (50).

\(^7\) A line (1D) element plays the role of a “volume element” for a 1D system and is the “surface element” from the point of view of the 2D system.
Generically for the $n$-dimensional spherically symmetric subsystem the conserved stress tensor has the form:

\[ T_{ij}^{(nD)}(r) = \left( \frac{r_ir_j}{r^2} - \frac{1}{n} \delta_{ij} \right) s^{(nD)}(r) + \delta_{ij} p^{(nD)}(r). \]  

(53)

The naive restriction of $T_{ij}^{(nD)}(r)$ to the $(n-1)D$ subspace leads to a non-conserved stress tensor. To construct a conserved stress tensor one has to take into account the forces acting on the $(n-1)D$ subsystem from outside. The corresponding conserved stress tensor for the $(n-1)D$ subsystem, i.e. a stress tensor satisfying $\nabla^\alpha T_{\alpha\beta}^{(n-1)D}(r) = 0$, has the form:

\[ T_{\alpha\beta}^{(n-1)D}(r) = T_{\alpha\beta}^{(nD)}(r) + \left( \delta_{\alpha\beta} - \frac{r_\alpha r_\beta}{r^2} \right) \frac{s^{(nD)}(r)}{n-2}, \]

where $\alpha, \beta = 1, \ldots, n-1$ and $r \in (n-1)D$ subspace. The last term in this equation takes into account the forces experienced by $(n-1)D$ subsystem from outside. From this one obtains the following relations between pressures and shear forces of subsystems with different dimensions as:

\[ p^{(n-1)D}(r) = -\frac{1}{n} s^{(nD)}(r) + p^{(nD)}(r), \quad s^{(n-1)D}(r) = \frac{n-1}{n-2} s^{(nD)}(r). \]

(55)

Interesting is that we can consider a hadron as a 3D subsystem of some $n$-dimensional mechanical system with pressure and shear force distributions:\n
\[ p^{(nD)}(r) = p(r) + \frac{2(n-3)}{3n} s(r), \quad s^{(nD)}(r) = \frac{2}{n-1} s(r), \]

where $p(r)$ and $s(r)$ are pressure and shear forces distribution in a 3D hadron. Such a picture can have interesting connections to AdS/QCD.

From the positivity of the normal forces and the first equation in (49) we obtain an important constraint for the 3D Fourier transform of the $D$-form factor, $\tilde{D}(r)$ defined in Eq. (23): it must be monotonically increasing function of $r$. For hadrons we expect that $\tilde{D}(r) \to 0$ for $r \to \infty$, thus $\tilde{D}(r)$ must be negative! So we obtain that not only $D$-term must be negative, but also the whole form-factor in the coordinate space is negative.

The tangential stability condition (46) implies that the tangential forces $(dF_\delta, dF_\theta)$ must at least once change their signs. This implies that the tangential forces inside a hadron must switch their nature from “strecthing” to “squeezing”. Such pattern is illustrated on Fig. 6 of Section XVII C where a model prediction for the distribution of the tangential forces inside the nucleon is shown. One clearly sees that the tangential forces are “stretching” inside the sphere of radius $\sim 0.5$ fm and are “squeezing” outside it.

We obtained here the von Laue (31) and tangential (46,48) stability conditions from physical considerations about forces. Technically they can be traced back to the EMT conservation $\nabla^i T^{(n)}(r) = 0$. Using the conservation of EMT we can derive a set of interesting relations which are presented in Appendix A. In particular, we can obtain the non-linear integral relation among EMT densities (A3) which can be rewritten equivalently as:

\[ \int d^3r \left[ \text{tr}(T(r))^2 - \frac{1}{2} \text{tr}(T(r))^2 \right] = 0, \]

(57)

where $T(r)$ is the matrix corresponding to $T_{ij}(r)$. Such kind of relations can be very useful for analysis of dispersion relations for EMT form factors.

XIV. D-TERM IN FREE THEORIES: DISTINGUISHING BOSONS AND FERMIONS

Before discussing the phenomenologically interesting cases it is instructive to inspect free field theories. Remarkably, the particle property $D$-term can “distinguish” between non-interacting pointlike (“elementary”) bosons and fermions in the following sense. A free spinless boson has a non-zero intrinsic $D$-term $D = -1$. In sharp contrast to this, the $D$-term of a free spin-$\frac{1}{2}$ fermion is zero, see [113, 127] and references therein.

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8 The general relations are $p^{(nD)}(r) = p^{(kD)}(r) + \frac{(k-1)(n-k)}{k_n} s^{(kD)}(r)$ and $s^{(nD)}(r) = \frac{k-1}{n-1} s^{(kD)}(r)$.
This unexpected finding deserves a comment. One can give a pointlike boson a “finite, extended, internal structure” (by “smearing out” its energy density $T_{\text{bo}}(r) = m \delta^{(3)}(r)$ with e.g. a narrow Gaussian, and analog for other densities) such that the property $D = -1$ is preserved. This yields automatically the characteristic shapes for $p(r)$ and $s(r)$ found in dynamical model calculations [113]. Interesting field theories of extended (solitonic $Q$-ball type) solutions with $D = -1$ can be constructed where such “smearing out” is implemented dynamically [113]. In general interactions modify the free theory value $D = -1$, see next section. But the point is: a spin-zero boson has an intrinsic D-term.

The situation is fundamentally distinct for fermions: here interactions do not modify the $D$-term, they generate it. A non-zero fermionic $D$-term is purely of dynamical origin [127]. Recalling that all known matter is fermionic, this indicates the importance to study the physics of the $D$-term.

XV. D-TERM IN WEAKLY INTERACTING THEORIES

In this section we review the $D$-term in theories with an interaction so weak that a perturbative treatment is justified, starting with the $\Phi^4$ theory, see [128] for a review, defined by $\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi)(\partial^\mu \Phi) - V(\Phi)$ with $V(\Phi) = \frac{1}{2} m^2 \Phi^2 + \frac{\lambda}{3!} \Phi^3$. The EMT of this theory was studied in Ref. [129], see also [130–135]. For scalar fields the canonical EMT obtained from the Noether theorem is symmetric, $\tilde{T}^{\mu\nu}(x) = (\partial^\mu \Phi)(\partial^\nu \Phi) - g^{\mu\nu} \mathcal{L}$, and yields $D = -1$ in free Klein-Gordon theory [2, 113]. This symmetric EMT follows from Eq. (1) with a “minimal coupling” of the theory to gravity $S_{\text{grav}, \text{min}} = \int d^4 x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu}(\partial_\mu \Phi)(\partial_\nu \Phi) - V(\Phi) \right)$. However, already on classical level this EMT is not conformally invariant, not even for $m \to 0$. This can be remedied by working with a non-minimal coupling term of the scalar field to the curvature

$$S_{\text{grav}} = \int d^4 x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu}(\partial_\mu \Phi)(\partial_\nu \Phi) - V(\Phi) - \frac{1}{2} h R \Phi^2 \right), \quad h = \frac{n}{4} \left( \frac{n - 2}{n - 1} \right),$$

(58)

where $R$ is the Riemann scalar, $n$ denotes the number of space-time dimensions. The effect of this non-minimal coupling is to add an “improvement term”: $T^{\mu\nu} \to T^{\mu\nu} + \Theta^{\mu\nu}_{\text{improve}}$ with $\Theta^{\mu\nu}_{\text{improve}} = - h(\partial^\mu \partial^\nu - g^{\mu\nu} \Box) \phi(x)^2$ [129]. It is often said one can add to the EMT operator “any quantity whose divergence is zero and which does not contribute to Ward identities” [135]. In fact, adding this $\Theta^{\mu\nu}_{\text{improve}}$ preserves $\partial_\mu T^{\mu\nu} = 0$, has the important virtue of making $T^{\mu\nu}$ a finite renormalized operator at 1, 2, 3 loop level [135], and does not affect the 00- and 0k-components such that mass and spin of the particle are not altered. However, it modifies the stress tensor components, and the $D$-term is changed from its free field theory value $D_{\text{free}} = -1$ to $D_{\text{interact}} = -1 + 4h = -\frac{4}{3}$ in $n = 4$ space-time dimensions.

This is a remarkable result [113]. As the $D$-term is an observable it must be uniquely defined. Thus, total derivatives cannot be arbitrarily added to the EMT. Rather it is essential to establish a unique definition of the EMT and verify (in QCD) whether the matrix elements of this EMT operator are probed in experiment.

Notice that the gravitational background field was used here only to derive the EMT operator: after the variation in Eq. (1) the metric $g^{\mu\nu}$ was set to flat space. However, the renormalizability of the $\Phi^4$ theory can be studied also in weakly curved gravitational background fields: the same improvement term is needed there to make $T^{\mu\nu}$ finite [136]. Since no quantum theory of gravity is known, it is of course also not known whether the term $\Theta^{\mu\nu}_{\text{improve}}$ would ensure renormalizability if quantum gravity effects were included. At this point one might be tempted to think that gravity is far too weak to be of relevance in particle physics. However, the lesson we learn is that even infinitesimally small interactions in $\Phi^4$ theory can impact the $D$-term. So why not infinitesimally small gravitational interactions? We note that the $D$-term emerges to be strongly sensitive to interactions. One must consistently include all forces, perhaps even gravity, to determine the true improvement term and the true value of the $D$-term [113].

At this point it is instructive to mention also studies in the $\Phi^3$ theory defined by $\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi)(\partial^\mu \Phi) - V(\Phi)$ with $V(\Phi) = \frac{1}{2} m^2 \Phi^2 + \frac{\lambda}{3} \Phi^3$. This is not a realistic theory with a potential not even bound from below. However, it is a popular toy model in many situations including studies of GPDs [9] where GPDs $\propto \mathcal{O}(\lambda^2)$ can be computed in one-loop order with no $D$-term [137]. Historically a lot of intuition about GPDs, and their analytic properties was gained in [9] on the basis of diagrams from the $\Phi^4$ theory, see [137] for further applications of the $\Phi^4$ theory. Such diagrams give rise to GPDs $\propto \mathcal{O}(\lambda^2)$ where $\lambda = \lambda(\mu)$ has a renormalization scale dependence. Hence a GPD being proportional to $\mathcal{O}(\lambda^2)$ cannot contribute to the renormalization scale independent $D$-term. Only the tree level diagram can do this, see previous section. But such diagrams are not interesting from the diagramatic point of view, and bear no insights on analytic properties of GPDs. Consequently, they were not considered and the $D$-term was overlooked in [9]. This point was clarified in [3]. Untill today the modelling of GPDs is most conveniently done in the so-called “double distribution Ansatz” developed on the basis of the perturbative calculations in $\phi^3$ theory in [9] supplemented by the $D$-term [3]. This story underlines the deep relation of the $D$-term to non-perturbative physics.

The weakest interaction in nature is gravity. Although a theory of quantum gravity is not yet known, the leading quantum corrections can be computed from the known low energy structure of the theory [138]. These calculations are
challenging [139–142]. The loop corrections to the Reissner-Nordström and Kerr-Newman metrics [140–142] show how (QED, gravity) interactions generate quantum long-range contributions to the stress tensor and other EMT densities. A consistent description of the D-term requires the inclusion of all contributions: also the short-distance contributions which cancel exactly the long-distance ones in the von Laue condition. The results of these works therefore do not allow us to gain insights on how much quantum gravity corrections contribute to the D-terms of elementary (charged) fermions. However, these results give us a feeling of gravitational infrared corrections at large distances.

XVI. ILLUSTRATION OF FORCES IN THE Q-BALL TOY MODEL

Before discussing hadrons it is instructive to review another strongly interacting theory, namely Q-balls [143–145]. Q-balls are non-topological solitons in theories with global symmetries, e.g. U(1) [146–148]. They might have played a role in the early universe and are dark matter candidates [149–151]. Q-balls exhibit a variety of families of solutions including stable, metastable, unstable solitons and radial excitations. In Ref. [143–145] the EMT of Q-balls was studied in the complex scalar theory \( \mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - V \) with \( V = A \Phi^* \Phi - B (\Phi^* \Phi)^2 + C (\Phi^* \Phi)^3 \) where \( A, B, C \) are positive constants. This theory is not renormalizable and understood as an effective theory [147]. The soliton solutions of the type \( \Phi(t, r) = e^{i\omega t} \phi(r) \) where \( \omega_{\text{min}} < \omega < \omega_{\text{max}} \) with \( \omega_{\text{min}}^2 = 2A(1 - B^2/4AC) \) and \( \omega_{\text{max}}^2 = 2A \). For \( \omega \to \omega_{\text{min}} \) one deals with absolutely stable solitons [147]. For \( \omega \to \omega_{\text{max}} \) one deals with Q-clouds, extremely unstable solutions which delocalize, and dissolve into an infinitely dilute system of free quanta [152].

In Fig. 2 we show results from [143, 144] obtained with the parameters \( [A, B, C] = [1.1 \text{GeV}^2, 2.0, 1.0 \text{GeV}^{-2}] \) for the ground state \( N = 0 \) and first excited state \( N = 1 \) with the charge \( Q = 342 \). The absolutely stable ground state has \( m = 286 \text{ GeV} \) and \( D = -2.5 \times 10^3 \). The mechanical radius, mean square radius of the energy density, and charge radius are given by \( \langle r^2 \rangle_{\text{ch}}, \langle r^2 \rangle_{\text{mech}}, \langle r^2 \rangle_{\text{E}} \rangle = [3.8, 4.0, 4.4] \text{ GeV}^{-1} \). The \( N = 1 \) state has \( m = 461 \text{ GeV} \) and \( D = -4.8 \times 10^5 \) with radii \( \langle r^2 \rangle_{\text{ch}}, \langle r^2 \rangle_{\text{mech}}, \langle r^2 \rangle_{\text{E}} \rangle = [4.9, 3.6, 5.1] \text{ GeV}^{-1} \).

The charge distribution of the ground state in Fig. 2a is uniform in the interior and drops to zero over a relatively narrow “edge-region.” \( T_{00}(r) \) exhibits a similar behavior except for the characteristic peak at the “edge” of the system due to the surface tension, see Fig. 2b. The shear forces indicate most clearly the position of the “edge,” Fig. 2c. The pressure is positive and constant in the interior, crosses the zero in the edge region and stays negative for large \( r \), see Fig. 2d. The normal forces are positive, Fig. 2e, and comply with the local stability requirement (39). The tangential forces are positive in the inner region, change sign in the edge region, and remain negative at large \( r \), see Fig. 2f.

For the \( N = 1 \) excited state the situation is much different: a portion of the charge is carried by an outer shell which is separated from the interior, see Fig. 2a. The shell structure is visible also in \( T_{00}(r) \) and \( s(r) \), see Figs. 2b, c. In general, the \( N^{1\text{th}} \) excited state exhibits \( N \) shells, and \( p(r) \) has \((2N + 1)\) zeros: always with the pattern of being positive in the center and negative in the asymptotic large-\( r \) region. The excited solution is much heavier, but not significantly larger: as a consequence the energy density and the magnitude of the forces in its interior are much larger compared to the ground state. Remarkably, the mechanical radius of the excited solution is smaller than that of the ground state solution. This is due to the fact that the normal forces diminish with \( r \) much earlier for \( N = 1 \) as compared to \( N = 0 \).

This shows that the mechanical radius of a system gives a much different view as compared to the charge radius or
energy density radius. In general, the mass and charge of the \( N^{th} \) excited state grow like \( N^3 \) and the radii grow like \( N \). But the \( D \)-term exhibits the strongest growth with \( N^8 \) [144].

In the strict \( Q \)-ball limit \( \varepsilon_{\text{max}}^2 = \omega_{\text{max}}^2 - \omega^2 \rightarrow 0 \) [147] the properties scale as \( \gamma \propto \varepsilon_{\text{max}}^0 \), radii \( \varepsilon_{\text{max}}^{-1} \), \( m \) and \( Q \propto \varepsilon_{\text{max}}^{-3} \), while \( D \propto \varepsilon_{\text{max}}^{-3} \) [143]. In the opposite \( Q \)-cloud limit \( \varepsilon_{\text{min}}^2 = \omega^2 - \omega_{\text{min}}^2 \rightarrow 0 \) [152] the properties behave as \( \gamma \propto \varepsilon_{\text{min}}^{-1} \), radii \( \varepsilon_{\text{min}}^{-1} \), \( m \) and \( Q \propto \varepsilon_{\text{min}}^{-1} \), while the strongest behavior is exhibited by \( D \propto \varepsilon_{\text{min}}^{-2} \) [145]. The \( D \)-term emerges as the property which is most strongly sensitive to the dynamics, and always negative.

If even such extremely unstable systems as \( Q \)-clouds have negative \( D \)-terms, the question emerges whether a physical system exists with a positive \( D \)-term. No such system with a positive \( D \)-term is known so far.

The \( Q \)-ball system was very useful to educate our intuition, and to prepare the discussion of hadronic EMT properties.

**XVII. D-TERM AND STRONG FORCES INSIDE VARIOUS HADRONS**

In this Section we review calculations of \( D \)-terms of hadrons including nuclei, Goldstone bosons, nucleon, vector mesons, deuteron and the \( \Delta \)-resonance.

**A. Nuclei in liquid drop model**

A liquid drop is the simplest mechanical system which can provide us with physical intuition. It can also serve as a model for large atomic nucleus. The pressure and shear forces distributions in the liquid drop are the following [6]:

\[
p(r) = p_0 \theta(r - R) - \frac{p_0 R}{3} \delta(r - R), \quad s(r) = \gamma \delta(r - R), \tag{59}
\]

where \( R \) is the radius of the drop, \( p_0 \) is the pressure inside the drop, and \( \gamma \) is the surface tension coefficient related to \( p_0 \) and \( R \) by the Kelvin relation: \( p_0 = 2 \gamma / R \) [121]. Obviously the von Laue and tangential stability conditions, Eqs. (31, 46), are satisfied automatically. One also sees immediately from Eq. (59) that the distribution of the normal pressure:

\[
\frac{dF_r}{dS_r} = \frac{2}{3} s(r) + p(r) = p_0 \theta(r - R), \tag{60}
\]

is positive (as required by the local stability of the system) and has step-like form.

This form is what one expects intuitively for a liquid drop. Using the definition of the mechanical radius (41) we obtain the intuitively clear result

\[
\langle r^2 \rangle_{\text{mech}} = \frac{3}{5} R^2. \tag{61}
\]

The \( D \)-term for a liquid drop is

\[
D = -\frac{4}{5} \left( \frac{4 \pi}{3} \right) m \gamma R^4. \tag{62}
\]

A more realistic description of a large nucleus is obtained by considering finite-skin effects which make the \( \Theta \)- and \( \delta \)-functions in \( p(r) \) and \( s(r) \) smooth, see Fig. 3. The finite-skin effects make the \( D \)-term more negative [6].

It is remarkable that the liquid drop model applied to a large nucleus gives \( D_{\text{nucleus}} \propto A^{7/3} \) since the nuclear masses and radii grow like \( m_A \propto A \) and \( R_A \propto A^{1/3} \) with the mass number, while \( \gamma \propto A^{0} \) [6]. Numerical calculations in more sophisticated nuclear models support this prediction. In [153] in the Walecka model the \( D \)-terms of selected isotopes with \( J^\pi = 0^+ \) were studied, see Table IIa. For oxygen \( ^{16}O \) and heavier nuclei it was found \( D \propto A^{2.26} \) [153] in good agreement with [6]. For completeness we remark that in a model based on a non-relativistic nuclear spectral function a different \( A \)-behavior was found [154].

**B. Goldstone bosons**

The \( D \)-terms of Goldstone bosons of spontaneous chiral symmetry breaking are given in the soft pion limit by

\[
D = -1. \tag{63}
\]
This result was obtained in [36–39] and rederived from a soft-pion theorem for pion GPDs in Ref. [155]. For an early study of EMT form factors of pseudoscalar mesons based on current-algebra techniques we refer to [156]. The leading (log enhanced) chiral correction to the $D$-term are given by [157]

$$D = - \left( 1 - \frac{m_\pi^2 \ln(\mu^2/m_\pi^2)}{48\pi^2 f_\pi^2} + \mathcal{O}(m_\pi^2) \right)$$  \hspace{1cm} (64)$$

where $f_\pi \approx 93$ MeV is the pion decay constant. The renormalization scale $\mu$ can be chosen to be of the order of e.g. the $\rho$-meson mass, but the $\mathcal{O}(m_\pi^2)$ corrections in (64) depend on $\mu$ such that the total result for $D$ is scale independent. The formula (64) shows that the leading log chiral correction reduces the absolute value of the $D$-term.

Including the $\mathcal{O}(m_\pi^2)$ corrections in (64) computed in [157] yields the results for the $D$-terms of pions, kaons and $\eta$-mesons shown in Table IIb [113]. The quoted uncertainties reflect the uncertainty of the low energy constants entering to this order [157] and include rough estimates of other chiral corrections (e.g. whether one uses $f_\pi \approx 93$ MeV at the physical value of the pion mass or $f_\pi \approx 88$ MeV in the chiral limit). As expected, the deviations from the chiral limit value (63) increase with mass and range from few percent for pions, to 20 % for kaons, to 30 % for $\eta$-mesons. Electromagnetic and isospin-breaking corrections to pion EMT form factors were studied in [158].

The slopes of the pion EMT form factors were computed in [155] in the large $N_c$ limit using the instanton liquid picture of QCD vacuum with the following result:

$$A'(0) = -D'(0) = \frac{N_c}{48\pi^2 f_\pi^2},$$  \hspace{1cm} (65)$$

where the pion decay constant behaves in the chiral limit as $f_\pi^2 = \mathcal{O}(N_c)$. We see that in the large $N_c$ limit the slopes of $A(t)$ and $D(t)$ form factors are the same. Interesting is that this corresponds to the fourth order effective chiral action for the Goldstone bosons (described by the nonlinear chiral field $U(x) = \exp[\pi^a x^a/f_\pi]$) in external gravitational field of the form:

$$S = -\frac{N_c}{96\pi^2} \int d^4 x \sqrt{-g} \operatorname{tr} \left( \partial_\mu U \partial_\nu U^\dagger \right) \left[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right],$$  \hspace{1cm} (66)$$

in which Goldstone bosons couple not separately to the Ricci tensor $(R_{\mu\nu})$ and the scalar curvature $(R)$ but to their combination which is the Einstein tensor $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$. Probably there is some deep reason for that peculiarity which still remains to be revealed.

While in large $N_c$ limit the slopes of $A(t)$ and $D(t)$ coincide, a drastic difference arises in sub-leading $1/N_c$ order due to pion loop corrections. These corrections lead to a non-analytical dependence of $D'(0)$ on the pion mass, whereas $A'(0)$ remains analytical in $m_\pi$. This non-analyticity leads to a divergence of the $D(t)$ slope in the chiral limit of the form [39]:

$$-D'(0)_{\text{chiral loop}} = \frac{\ln(\mu^2/m_\pi^2)}{24\pi^2 f_\pi^2}.$$  \hspace{1cm} (67)$$

For the numerical estimate of the $D(t)$ slope we can combine the results in Eqs. (65,67):

$$-D'(0) = \frac{N_c}{48\pi^2 f_\pi^2} + \frac{\ln(\mu^2/m_\pi^2)}{24\pi^2 f_\pi^2} = (0.73 + 1.66) \text{ GeV}^{-2} = 2.40 \text{ GeV}^{-2}.$$  \hspace{1cm} (68)$$

| isotope | $^{12}\text{C}$ | $^{16}\text{O}$ | $^{40}\text{Ca}$ | $^{90}\text{Zr}$ | $^{208}\text{Pb}$ |
|---------|----------------|----------------|----------------|----------------|----------------|
| $D$     | -6.2           | -115           | -1220          | -6606          | -39356        |

| Goldstone boson | pion | kaon | $\eta$-meson |
|-----------------|------|------|--------------|
| $D$             | -0.97 ± 0.01 | -0.77 ± 0.15 | -0.69 ± 0.19 |

Table II. (a) $D$-terms of selected nuclear isotopes with spin-parity quantum numbers $J^P = 0^+$ [153]. For $^{16}\text{O}$ and heavier nuclei it is approximately $D \approx -0.246 A^{2/3}$ as predicted in the liquid drop model [6]. (The convention used in Ref. [153] is $d_A = \frac{1}{3}D$.) (b) $D$-terms of (pseudo) Goldstone bosons of chiral symmetry breaking $\pi$, $K$, $\eta$ with $J^P = 0^-$ based on one-loop chiral theory [157] with estimated uncertainties [113]. In the chiral limit the $D$-term of a Goldstone boson is $D = -1$ [36, 37].
Here for the numerical estimate we use $\mu = m_\rho$ and physical pion mass of $m_\pi = 0.140$ GeV. An important conclusion from the consideration of slopes of the EMT form factors in chiral theory is that for the pion $-D'(0)$ should be larger than $A'(0)$ due to the different behavior of these slopes in the chiral limit.

It is also instructive to compare Eq. (68) with the analogous estimate (see section 6.1 of Ref. [155]) for the slope of pion charge form factor $F_{e.m.}(t)$:

$$F_{e.m.}'(0) = \frac{N_c}{24\pi^2 f_\pi^2} + \frac{\ln\left(\frac{\mu^2}{m^2}\right)}{96\pi^2 f_\pi^2} = (1.46 + 0.42) \text{ GeV}^{-2} = 1.88 \text{ GeV}^{-2}. \quad (69)$$

First, the obtained numerical value is in good agreement with the experimental value of $F_{e.m.}'(0) = (1.86 \pm 0.03) \text{ GeV}^{-2}$ [159], which indicates that such an estimate gives sensible results for pion form factors. Second, it is very instructive to compare the expressions (68) and (69) – one sees that the chiral loop corrections to the slope of the pion $D(t)$ form factor are four times larger than the to slope of the pion charge form factor, whereas the large $N_c$ (“core”) contribution is two times smaller – the slope of the pion $D(t)$ form factor is dominated by chiral logs. This demonstrates that the $D$-term is very sensitive to physics of spontaneous breakdown of the chiral symmetry in QCD and study of the $D$-term can provide us with new effective tools for probing the mechanisms of chiral symmetry breaking in QCD.

The low energy effective chiral Lagrangian in curved space-time and the gravitational form factors of the pion were also studied in quark model frameworks [160–162], AdS/QCD models in [163], and covariant and light-front constituent models [164]. The result (65) was rederived in Ref. [160] in the large $N_c$ limit in quark spectral models, where it was noted that the equality of the slopes of $A(t)$ and $D(t)$ was independent of the particular realization of the spectral model. The von Laue condition for the pion was studied in [165]. A study of pion EMT form factors in lattice QCD was reported in Ref. [166].

C. Nucleon

The first model studies of the nucleon $D$-term were performed in the bag model [167], chiral quark soliton model [168], see also [120, 169–172], and Skyrme model [173, 174]. The quark contributions to the $D$-term were also studied in the QCD multi-color limit $N_c \to \infty$ [24], lattice QCD [175–177], dispersion relations [178], and quark models [179–184, 187–189].

The bag and chiral quark soliton model were used in Ref. [127] to illustrate how interactions can generate the $D$-term of a fermion. In the bag model a non-zero $D$-term emerges when interactions are introduced in the shape of the bag boundary condition which is imposed to simulate confinement and bind the otherwise free quarks. In the chiral quark soliton model the $D$-term vanishes when the chiral interactions are “switched off” and the free theory is restored in a limiting procedure. The bag model with massless quarks gives a small value $D = -1.1$ [127, 167]. The chiral models predict a more sizable $D$-term in the range [120, 169–174]

$$-4 \lesssim D \lesssim -2. \quad (70)$$

In the large $N_c$ limit the $D$-term of the nucleon exhibits the flavor hierarchy [24]

$$|D^u(t) + D^d(t)| \sim N_c^2 \gg |D^u(t) - D^d(t)| \sim N_c. \quad (71)$$

This result is supported by numerical calculations in chiral quark soliton model [172] and lattice QCD [175–177].

Fig. 4 shows the EMT densities from the chiral quark soliton model ($\chi$QSM) [120]. In the center $T_{00}(0) = 1.7 \text{ GeV/fm}^3$ which is approximately 13 times the nuclear matter density while $p(0) = 0.23 \text{ GeV/fm}^3$, which corresponds to $3.7 \cdot 10^{29}$ atmospheric pressures. The positive pressure in the center means repulsion, and negative $p(r)$ for $r \gtrsim 0.6 \text{ fm}$ means attraction. Repulsive and attractive forces balance each other exactly according to the von Laue condition (31).

The von Laue condition can be rigorously proven in the $\chi$QSM [120] by exploring a theorem known as “virial theorem:” the soliton mass is a functional of the soliton profile. One may consider a special class of variations of the profile function generated by the dilatational transformations $r \to \lambda r$. This yields an energy functional $m(\lambda)$ which has a minimum at $\lambda = 1$. The von Laue condition can now be expressed as $\int d^3r T_{00}(r) = m'(\lambda)|_{\lambda=1} = 0$ [120]. This shows that this condition is satisfied by any stationary solution: global minimum, local minimum, other extremum, saddle point of the action. This means the von Laue condition is necessary but not sufficient for stability.

The von Laue condition can be proven in exactly the same way also in the Skyrme model [173] and bag model [190]. These models have in common that they describe the nucleon in terms of a static mean field, even though in these models the mean fields are realized in much different ways. The generic mean field picture of the nucleon is justified in QCD in the large-$N_c$ limit [191, 192]. Thus, the connection of the von Laue condition and the virial theorem is of
more general character than the respective models: it holds in the large-$N_c$ limit in QCD. It is not known whether a connection of the von Laue condition and extrema of the action can be established also in QCD with finite $N_c$.

It is interesting to investigate what happens when one increases the value of the current quark masses (as it was routinely done until recently in lattice QCD studies). In this case the hadron masses increase, while their sizes decrease. For the EMT densities it has the following implications: the energy density in the center of the nucleon increases and so does the pressure, see Fig. 5. This implies a more negative $D$-term [171].

Modifications of the $D$-term of the nucleon in nuclear matter were studied in [193, 194]. As the density of the nuclear medium increases, the energy density in the center of the nucleon bound in the medium and the pressure both decrease. The size of the system, however, grows and the $D$-term becomes more negative [193, 194].

Chiral perturbation theory cannot predict the value of the nucleon $D$-term, but it predicts its $m_\pi$-dependence and the small-$t$ behavior of $D(t)$ [195–197]. The slope of $D(t)$ at zero-momentum transfer diverges in the chiral limit as $D'(0) \sim 1/m_\pi$. This behavior is reproduced also in chiral models [120, 173].

In Section XII the mechanical radius of a hadron was defined not in terms of the slope of $D(t)$. Applying the definition of the mechanical radius (41) to the nucleon case, one can see on general grounds that the corresponding

Figure 4. EMT densities of the nucleon from the chiral quark soliton [120]. (a) Energy density $T_{00}(r)$, (b) densities $p(r)$ and $s(r)$ of the stress tensor $T_{ij}(r)$, and (c) $4\pi r^2 p(r)$ where the shaded areas above and below the $x$-axis are exactly equal to each other which demonstrates how the von Laue condition (31) is realized. (d) The integrand of the $D$-term is proportional to $r^4 p(r)$ and yields $D < 0$ upon integration. The negative sign of $D$ emerges as a natural consequence of the “stability pattern” [120].

Figure 5. The pion mass dependence of the nucleon EMT densities from chiral quark soliton [171] from the chiral limit up to $m_\pi$ of the order of magnitude of the kaon mass. (a) Energy density normalized as $4\pi r^2 T_{00}(r)/m$ such that the curves integrate to unity, and (b) $r^2 p(r)$ which integrates to zero. (c) The pressure in the center as function of the energy density in the center.
mechanical radius (in contrast to $D'(0)$ and to the charge radius of the nucleon) is finite in the chiral limit ($m_\pi \to 0$). Therefore, one expects that the nucleon mechanical radius should be smaller than, say, the charge radius. Indeed, the chiral quark soliton model predicts the mechanical radius of the proton to be about 25% smaller than its mean square charge radius: $(r^2)_{\text{mech}} \approx 0.75 (r^2)_{\text{charge}}$.

It is instructive to see details of the strong forces distribution inside the nucleon. The radial (normal) forces in Eq. (43), are always “stretching” (directed outwards the nucleon centre) and monotonically decrease with distance from the centre. The distribution of the tangential forces provides us with further fine details of how the strong forces keep the nucleon together. From the stability condition (46) it is clear that the tangential force must at least once change its direction. Studying these forces one can pose very intriguing questions about nature of strong forces – how many times do the forces change from “stretching” to “squeezing”? What does this number mean? What does distinguish the regions of “stretching” and “squeezing”? What do we learn about the confinement mechanism from this?

Presently we are not able to answer the above posed questions. Here we just report the results on the force distribution in the nucleon from models. In Fig. (6) we plot the vector field of the $\phi$-component of the tangential force (the 2D vector field $4\pi r^2 T_{ij} e^\phi_j$) inside the nucleon obtained from EMT densities from the chiral quark soliton model [120].

One clearly sees that at a distance of $r \approx 0.5$ fm from the nucleon centre the tangential force changes its direction, and turns from “stretching” to “squeezing”. Thus, we see that there are two qualitatively different regions inside the nucleon – they are distinguished by the chirality of the tangential forces. It would be interesting to understand at the microscopic level the physical reasons for the emergence of these two different regions.

In the lattice QCD study [177] a hybrid approach based on domain wall valence quarks with $2 + 1$ flavors of improved staggered sea quarks was used. The range $0.1 \text{GeV}^2 < -t < 1.2 \text{GeV}^2$ was covered for pion masses from

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Figure 6. Visualisation of the $\phi$-component of the tangential force (the 2D vector field $4\pi r^2 T_{ij} e^\phi_j$) distribution in the nucleon from the chiral quark soliton model. The radius of the disc on the figure is 1.5 fm, the colour legend gives the absolute value of the tangential force in GeV/fm.

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$^9$ See also recent lattice measurement of the spatial distribution of forces for the heavy quark $\bar{Q}Q$ pair in Ref. [198]. The formalism provided here paves a way to perform analogous studies on the lattice for hadrons.
760 MeV down to 350 MeV. Depending on the chiral extrapolation method the following values were obtained which do not include disconnected diagrams: \( D^Q = -1.07 \pm 0.25 \) using covariant baryon chiral perturbation theory, and \( D^Q = -1.68 \pm 0.22 \) using heavy baryon chiral perturbation theory at the physical value of the pion mass in \( \overline{MS} \) scheme at \( \mu^2 = 4 \text{GeV}^2 \). The quark contribution to the \( D \)-term from dispersion relations [178] refers to the same \( \mu^2 \) and is in the range \( -1.54 \leq D^Q \leq -2.72 \) in good agreement with the lattice result. Considering that the results from chiral models (70) show the total \( D \)-term, the dispersion relation and lattice result agree well with these models [171, 173].

The nucleon EMT form factors \( A(t) \) and \( B(t) \) were also studied in approaches based on light front wave functions such as AdS/QCD models or spectator models [179–184, 187–189]. Such models are often based on a light-front Fock state expansion. Typically the form factors \( A(t) \) and \( B(t) \) can be evaluated, which are simply related to the helicity non-flip and helicity flip matrix elements of the component \( \hat{T}_{++} \) of the EMT. Being related to the stress tensor \( \hat{T}_{ij} \) the form factor \( D(t) \) naturally “mixes” good and bad light-front components and is described in terms of transitions between different Fock state components in overlap representation. As a quantity intrinsically non-diagonal in a Fock space, it is difficult to study the \( D \)-term in approaches based on light-front wave-functions. This is due to the relation of the \( D \)-term to internal dynamics: a complete description of a hadron requires the inclusion of all Fock components.

D. Size of the forces in the nucleon, and comparison with linear potential confinement forces

Very frequently, e.g. in colour tube models, the confinement forces are related to the linear potential \( V_{\text{conf}}(r) = \sigma r \), where \( \sigma \sim 1 \text{GeV/fm} \) is estimated from the slope of meson Regge trajectories. Recently the spatial distribution of the stress tensor for a heavy quark \( \bar{Q}Q \) pair was directly measured on the lattice: the typical size of the forces \( \sim 1 \text{GeV/fm} \) was confirmed [198]. Such a linear interquark potential corresponds to a constant force between quarks \( F = \sigma \). Our aim is to compare this force with the forces encoded in the stress tensor.

The spherical shell of radius \( r \) in the nucleon experiences the normal force \( F_n = 4\pi r^2 \left[ \frac{2}{3} s(r) + p(r) \right] \) and tangential force \( F_t = 4\pi r^2 \left[ -\frac{1}{3} s(r) + p(r) \right] \). We use the chiral quark-soliton model (\( \chi \)QSM) results of Ref. [120] to compute the corresponding forces. The result is shown on Fig. 7, we see that the maximally achieved strength is five times smaller than the confining forces in a colour tube model.

E. Spin-1 hadrons

Light vector mesons were studied in Ref. [199] using light-front wave-functions obtained from an AdS/QCD model. For the \( \rho \)-meson the mean square radius of the energy density was found to be \( \langle r_{\rho}^2 \rangle = 0.21 \text{fm}^2 \). This is significantly smaller then the mean square charge radius of \( \rho^+ \) determined to be \( \langle r_{\rho^+}^2 \rangle = 0.53 \text{fm}^2 \) in the same approach [200].

The GPDs for the deuteron were introduced in [185] and studied in details in Ref. [186]. The EMT form factors of the deuteron were studied in Ref. [201] using a deuteron wave function from a softwall AdS/QCD model. The \( D \)-term was not computed neither for the vector mesons nor for deuteron. We are not aware of a calculation of the \( D \)-term of a spin-1 hadron. To best of our knowledge, the only spin 1 particle whose \( D \)-term has been studied is the photon, which has a negative \( D \)-term [202, 203].

![Figure 7](image_url)

Figure 7. The normal force \( F_n = 4\pi r^2 \left[ \frac{2}{3} s(r) + p(r) \right] \) (solid) and tangential force \( F_t = 4\pi r^2 \left[ -\frac{1}{3} s(r) + p(r) \right] \) (doted) experienced by a spherical shell of radius \( r \) in the nucleon computed in the \( \chi \)QSM.
F. \( \Delta \)-resonance

The \( D \)-term of the \( \Delta \) was studied in the Skyrme model in Ref. [119]. Despite being an unstable state, the \( \Delta \)-resonance has nevertheless a negative \( D \)-term. In the Skyrme model one finds the intuitive result that the \( \Delta \) appears to have a larger radius than the nucleon, and the internal forces are weaker, see Fig. 8. The \( D \)-term is negative, but its magnitude is about 25\% smaller as a consequence of the reduced forces. Using the formalism of Ref. [119] we can compute also the mechanical radius for \( \Delta \)-resonance, its value is by about 25\% larger than the mechanical radius of the nucleon, that reflects that the normal forces in \( \Delta \)-resonance are more spread out. In [119] not only the \( \Delta \)-resonance was studied, but also higher spin-isospin states which are predicted in the Skyrme model but not observed in nature. The study of the EMT densities of such states provides an explanation why no such states are observed in nature.

Soliton models based on the large-\( N_c \) expansion (chiral quark-soliton model, Skyrme model) describe light baryons with spin and isospin quantum numbers \( S = I = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \) as different rotational states of the same soliton solution. In this “rigid rotator approach” \( 1/N_c \) corrections are considered by expanding the action in terms of the angular velocity of the rotating solitons. The quantum numbers \( S = I = \frac{1}{2} \) and \( \frac{3}{2} \) correspond to the nucleon and \( \Delta \). But rigid rotator states with \( S = I \geq \frac{5}{2} \) are not observed in nature. For a long time this was considered an unsatisfactory artifact of the approach, until it was shown that such states actually do not exist — not even in the rigid rotator framework [119]. Using the Skyrme model it was shown that the \( 1/N_c \) corrections are a small perturbation for the nucleon with \( S = I = \frac{1}{2} \). The corrections are more sizable for \( S = I = \frac{3}{2} \) but the \( \Delta \) still complies with stability criteria. But for the rigid rotator artifact states \( S = I \geq \frac{5}{2} \) the \( 1/N_c \) corrections become so destabilizing, that the basic stability criterion (39) is violated. These are therefore unphysical states (which would have a positive \( D \)-term). The results from the Skyrme model are shown in Fig. 8.

![EMT densities in Skyrme model for the quantum numbers](image)

Figure 8. EMT densities in Skyrme model for the quantum numbers \( S = I = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \). Energy density \( T_{00}(r) \) indicates the states are heavier with increasing spin, and does not reveal anything unusual. More insightful are \( s(r) \) and \( p(r) \) which flip signs and violate the stability criterion \( \frac{2}{3} s(r) + p(r) \geq 0 \) in Eq. (39) for \( S = I \geq \frac{5}{2} \). This calculation explains why \( S = I = \frac{1}{2}, \frac{3}{2} \) are physical states (nucleon, \( \Delta \)) while \( S = I \geq \frac{5}{2} \) are unphysical states and not observed in nature [119].

XVIII. EMT DENSITIES AND APPLICATIONS TO HIDDEN-CHARM PENTA- AND TETRA-QUARKS

The extraction of the \( D \)-term will not only give insights on how the internal strong forces balance inside the nucleon. Knowledge of EMT form factors has also applications to the spectroscopy of some of the recently observed exotic hidden-charm hadrons, pentaquark and tetraquark states.

In the LHCb experiment pentaquark states were observed [204] which decay in \( J/\psi \) and proton. One of these states, the narrow \( P_{11}^+(4450) \) with a width \( \Gamma \approx 40\text{ MeV} \), can be described exploring that quarkonia are small compared to the nucleon size. This justifies a multipole expansion which shows that the baryon-quarkonium interaction is dominated by the emission of two virtual chromoelectric dipole gluons in a color singlet state. The effective interaction [205] can be expressed in terms of the quarkonium chromoelectric polarizability \( \alpha \) and nucleon EMT densities as

\[
V_{\text{eff}}(r) = - \alpha \frac{4\pi}{3} \left( \frac{2}{g_s} \right)^2 \left[ \nu T_{00}(r) - 3 p(r) \right].
\]

Here \( b = \frac{11}{3} N_c - \frac{2}{3} N_f \) is the leading coefficient of the Gell-Mann-Low function, \( g_s \) (\( g \)) is the strong coupling constant at the scale associated with the nucleon (quarkonium) size, and the parameter \( \nu \) was estimated to be \( \nu \approx 1.5 \) [41].

For realistic values of \( \alpha(1S) \), the potential \( V_{\text{eff}}(r) \) is not strong enough to bind nucleon and \( J/\Psi \) with the EMT densities from the chiral quark soliton. But bound states of the mass 4450 MeV exist in the \( \psi(2S) \)-nucleon channel for
Figure 9. The decay width $\Gamma$ vs mass $M$ of tetraquarks obtained in Ref. [210] from varying the parameters which describe the unknown $\phi$-meson EMT form factors within a wide range of values. The crosses on the $M$-axis indicate the kinematic bounds $m_J + m_\phi < M < m_\psi + m_\phi$. For comparison we show the four tetraquarks in the $J/\psi - \phi$ resonance region with their statistical (thin lines) and systematic (shaded areas) uncertainties and spin parity assignments [212]. The state $X(4274)$ emerges as a candidate for the description as a hadrocharmonium. This method can be used to identify other possible hadroquarkonia.

$\alpha(2S) \sim 17 \text{GeV}^{-3}$ [41] which is close to perturbative QCD estimates of this chromoelectric polarizability [206, 207]. The $J^P$ quantum numbers of these $s$-wave bound states are $\frac{1}{2}^-$ and $\frac{3}{2}^-$ which is among the possible spin-parity assignments of the LHCb partial wave analysis [204]. The 2 states are mass degenerate in the heavy quark limit. Their mass splitting was roughly estimated to be of $O(20 \text{MeV})$ [41]. The decay of $P_{c}^{+}(4450)$ is governed by the same $V_{cH}(r)$ but with a much smaller transition polarizability $\alpha(2S \rightarrow 1S) \sim O(1)$ [206–208] which explains the relatively narrow width [41]. The other putative pentaquark state seen by LHCb, $P_{c}^{+}(4380)$, is much broader with $\Gamma \sim 200 \text{MeV}$ and not described by this binding mechanism [41]. These findings are confirmed using Skyrme model predictions for nucleon EMT densities indicating they are largely model-insensitive [119]. The approach predicts also bound states of $\psi(2S)$ with $\Delta$ [119] and hyperons [209] which will allow us to test this theoretical framework.

One can generalise the formalism applied to heavy pentaquarks [209] to the case of $\psi(2S)$-light meson bound states, the tetraquarks. The case of $\psi(2S)$-$\phi$ meson bound states was studied in Ref. [210]. It was assumed there that the chromoelectric polarizability $\alpha(2S) \approx 17 \text{GeV}^{-3}$ is fixed by the pentaquark mass $P_{c}^{+}(4450)$. However, very little is known about the $\phi$-meson EMT densities necessary for the calculation of the effective $\psi(2S)$-$\phi$ potential. In Ref. [210] the corresponding densities were therefore parametrised by flexible Ans"atze whose parameters were varied in a wide range of values for the $\phi$-meson square radii $0.05 \text{fm}^2 < \langle r^2 \rangle_{\text{E,mech}} < 1 \text{fm}^2$ and $D$-term $-15 < D < 0$. Not surprisingly a wide range was obtained for the masses $M$ of the corresponding tetraquarks and their partial $J/\psi - \phi$ decay widths $\Gamma$.

Interestingly, it was found that the tetraquark masses and the partial decay widths are strongly correlated, see Fig. 9. This is remarkable: even though we know nothing about the $\phi$-meson structure, the formalism predicts that $M$ and $\Gamma$ of candidate $\phi$-$\psi(2S)$ tetraquarks are systematically correlated. The Fig. 9 shows also the $J/\psi - \phi$ resonances observed by the LHCb collaboration [212]. The state $X(4274)$ observed in the $J/\psi - \phi$ channel has a width of $\Gamma = 56 \pm 11^{+8}_{-11} \text{MeV}$ [212] and fits exactly in the range predicted by the calculations in Ref. [210].

**XIX. FIRST EXPERIMENTAL RESULTS**

Recently first information on the $D$-terms of the proton and the neutral pion became available from phenomenological analyses of experimental data. In this section we review what is currently known.

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10 A similar correlation of the mass and width was found in Ref. [211] in a different context.
A. Nucleon

The $D$-term was shown to be of importance for the phenomenological description of hard-exclusive reactions [17–20], see also the reviews [28, 29] and references there in. The $D$-term can be accessed in DVCS with help of fixed-$t$ dispersion relations [32–35], for the LO DVCS Compton form factor $\mathcal{H}(\xi,t) = \int_{-1}^{1} dx \left( \frac{1}{\xi-x-m} - \frac{1}{\xi+x+m} \right) H(x,\xi,t)$ one obtains

$$\text{Re}\mathcal{H}(\xi,t) = \Delta(t) + \frac{1}{\pi} \text{p.v.} \int_{0}^{1} d\xi' \text{Im}\mathcal{H}(\xi',t) \left( \frac{1}{\xi - \xi'} - \frac{1}{\xi + \xi'} \right). \tag{72}$$

The corresponding subtraction constant $\Delta(t)$ in the leading QCD order is related to the $D$-term in the following way:

$$\Delta(t) = 2 \int_{-1}^{1} dz \frac{D(z,t)}{1-z}, \tag{73}$$

with $D(z,t)$ having the following expansion in the Gegenbauer polynomials $C_{n}^{3/2}(z)$:

$$D(z,t) = (1-z^2) \sum_{k=1}^{\infty} \left[ c_{u}^{2} \int_{-1}^{1} d\xi d^{2k-1}_{u}(t) + c_{d}^{2} \int_{-1}^{1} d\xi d^{2k-1}_{d}(t) \right] C_{2k-1}^{3/2}(z), \tag{74}$$

where $e_q$ is the electric charge of the quark with flavour $q$. In the above equation we neglected contributions of strange and heavy quarks. The EMT form factor $D^q(t) = \frac{2}{5} d^q_1(t)$. We remind that the quantities considered here ($d^q_1(t)$, $D(z,t)$, etc.) depend on the QCD normalisation point $\mu^2$. We do not write explicitly this dependence for brevity.

The first experimental access to the subtraction constant $\Delta(t,\mu^2)$ based on the most complete database of DVCS results was obtained in [20] (KM15 fit) in the form:

$$\Delta(t,\mu^2) = -\frac{C}{(1-t/M_C^2)^2}, \tag{75}$$

with parameters $C = 2.768$ and $M_C = 1.204$ GeV at the QCD normalisation point of $\mu^2 = 4$ GeV$^2$. The statistical uncertainty of the parameters are of order 20–30% [213], but the authors of Ref. [20] refrained from publishing the precise value of the statistical error bars due to large systematic uncertainties (see the discussion of this point in relation to the $D$-term in Ref. [214]) \footnote{We are grateful to Kresimir Kumerički for discussion of this point.}

We can relate the LO subtraction constant $\Delta(t,\mu)$ to the EMT form factor $D^q(t,\mu^2) = D^d(t,\mu^2) + D^u(t,\mu^2)$ using the following simplifying assumptions:

- only the first coefficient $d^q_1(t)$ of the Gegenbauer expansion (74) is taken account. In the asymptotic limit of infinitely large renormalization scale $\mu$ all $d^q_i(t)$ for $i > 1$ vanish, except for $d^q_1(t)$ which determines the asymptotic form of GPDs [24] and is related to the EMT form factor $D^q(t) = \frac{2}{5} d^q_1(t)$;

- dominance of the flavour singlet combination of the quark $D$-term $d^u_1 \approx d^d_1 \approx D^Q/2$. This can be justified by in the limit of large number of colours, see Eq. (71).

Under these assumptions we obtain:

$$D^Q(t) = \frac{4}{5} \frac{1}{2(c_u^2 + c_d^2)} \Delta(t) = \frac{18}{25} \Delta(t). \tag{76}$$

The KM15 fit Eq. (75) corresponds to the negative $D$-term of $D^Q = -2.0$ at $\mu^2 = 4$ GeV$^2$ with about 20% statistical uncertainty and unestimated systematic one. The result of the KM15 fit [20] corresponding to Eqs. (75,76) is shown in Fig. 10 in comparison with theoretical predictions and other fits to DVCS data.

Recently an analysis of the JLab data [90, 101]\footnote{These data are included in the experimental database of Ref. [20]} was reported [215] where an experimental information on the quark contribution to the $D$-term was also extracted. Additionally, the pressure distribution in the proton was presented in...
Ref. [215]. Below we compare the theoretical predictions with the data on the form factor, and not with the pressure distribution of [215] as the latter was obtained under model assumptions which are still missing clear justification.

In Ref. [215] the dispersion relations subtraction constant $\Delta(t)$ (see Eq. (72) for the definition) at the normalisation point of $\mu^2 = 1.5 \text{ GeV}^2$ was presented on their Fig. 4 [216]. The main difference of the analysis in [215] with that in [20] is the much smaller systematic uncertainties in the former. This difference calls for a clarification.

The $D_Q^Q(t)$ form factor obtained from the analysis of [215] with help of Eq. (76) is also shown in Fig. 10 where for comparison we include the results for the $D$-term form factor from dispersion relations [178], lattice QCD [177] and models [120, 167, 173].

The dispersion relation study of Ref. [178] used information on pion parton distribution functions which fixes the overall normalization of the form factor: in Fig. 10 the result for $D_Q^Q(t)$ is shown which is normalized as $D^Q = -1.56$. The results from the dispersion relations and lattice QCD show the quark contribution to $D_Q^Q(t)$ and refer to the scale $\mu^2 = 4 \text{ GeV}^2$ [177, 178]. The lattice data were obtained in a hybrid approach using domain wall valence quarks with $2 + 1$ flavors of improved staggered sea quarks not including disconnected diagrams. The “dataset 6” from [177] shown in Fig. 10 was taken on a $28^3 \times 32$ lattice with a lattice spacing $a = 0.124 \text{ fm}$ and a pion mass of $m_\pi = (352.3 \pm 1.4) \text{ MeV}$. The results from bag [167], chiral quark soliton [120] and Skyrme [173] model show the total scale–independent $D(t)$.

Keeping all this in mind, Fig. 10 shows a remarkable agreement. The MIT bag model [167] seems to underestimate the magnitude of the $D$-term form factor, the Skyrme model [173] seems to overestimate it (though, with different parameter fixing than in [173], a better discription may be possible). The results from dispersion relations [178] and chiral quark soliton model [120] compare very well to the experimental results from [215].

![Figure 10](image-url)  
Figure 10. The $D^Q(t)$ form factor obtained from the KM15 fit [20] in comparison to $D^Q(t)$ obtained from Jefferson Lab analysis [215], to calculations from dispersion relations [178], lattice QCD [177], and results from the bag [167], chiral quark soliton [120] and Skyrme [173] model. The JLab data [215] refers to the normalisation point of $\mu^2 = 1.5 \text{ GeV}^2$. KM15 fit, dispersion relations and lattice results show the contribution of quarks to the $D$-term at the QCD scale of $4 \text{ GeV}^2$. The models show the total $D$-term which is renormalization scale independent.
B. Pion

Recently in Ref. [125] the first extraction of the pion EMT form factors from the BELLE data on $\gamma\gamma \to 2\pi^0$ [217] was reported. The results for the quark part of EMT form factors ($A^Q(t)$ and $D^Q(t)$ in our notation, see Eq. (7)) were presented. The results for the form factors at zero momentum transfer are:

$$A^Q(0) \approx 0.70, \quad D^Q(0) \approx -0.75. \quad (77)$$

These results are in agreement with the normalisation condition for the full form factor $A(0) = 1$ and with the soft pion theorem $D = -1$, given that quarks carry only a fraction (about 70% according to the result of Ref. [125]) of the pion mass, and the gluon contribution to the $D$-term is not extracted. Also it is important that the analysis of [125] shows that the $D$-term is definitely negative as it should be for mechanical stability of the pion. The result obtained in [125] for the slopes of the pion EMT form factors are:

$$\frac{1}{A^Q(0)} \frac{d}{dt} A^Q(0) = 1.33 \sim 2.02 \text{ GeV}^{-2}, \quad \frac{1}{D^Q(0)} \frac{d}{dt} D^Q(0) = 8.92 \sim 10.35 \text{ GeV}^{-2}. \quad (78)$$

These results confirm the inequality $-D'(0) > A'(0)$ expected from chiral theory, however the numerical values are in sharp contrast with our estimate (68) based on the instanton picture of QCD vacuum combined with the chiral perturbation theory. It would be very important to understand which dynamical mechanism leads to anomalously large slopes of the pion EMT form factors obtained in analysis of Ref. [125].

XX. CONCLUSIONS

We have reviewed aspects of the physics associated with the $D$-term and other EMT properties. The physics of EMT form factors is important for a variety of problems including the description of hadrons in strong gravitational fields, hard exclusive processes, hadronic decays of heavy quarkonia, and the description of certain exotic hadrons with hidden charm as hadroquarkonia.

The matrix elements of the EMT contain fundamental information on a particle, namely the mass, spin, and $D$-term. While mass and spin are related to the Casimir operators of the Poincaré group, the $D$-term is related to the stress tensor and internal forces inside a composed particle. When interpreted in the Breit frame the Fourier transforms of the EMT form factors give insights on the 3D spatial densities describing the distributions of energy, pressure and shear forces.

In free field theory the $D$-term of a spin-zero boson is negative, but that of a spin $\frac{1}{2}$ fermion is zero. This indicates an interesting distinction of bosons and fermions.

In interacting theories the $D$-term in general is not fixed, except for the Goldstone bosons of chiral symmetry breaking for which the $D$-term is determined by soft-pion theorems to be $D = -1$ in the chiral limit. For other hadrons the $D$-term is not fixed, and reflects the internal dynamics of the system through the distribution of forces, and is sensitive to correlations in the system. For example, the baryon $D$-term behaves as $\sim N_c^2$ whereas all other global observables (mass, magnetic moments, axial charge, etc.) behave at most as $\sim N_c$ in the large $N_c$ limit. For a large nucleus the $D$-term shows also anomalously fast increase with the atomic mass number $D \sim A^{7/3}$.

The form factor $D(t)$ provides the key to introduce mechanical properties. For instance, we have given a definition of the mechanical radius of a hadron, discussed the concepts of normal and tangential forces, and presented (on the basis of model results) a picture of the forces inside the nucleon. Remarkably, the forces change their directions in the inner region of the nucleon vs the outer region. This change of sign is dictated by the conservation of the EMT, but the physics behind it is presently not fully understood.

We have reviewed results from models, lattice QCD, dispersion relations and the experimental information on the $D$-term of the nucleon based on DVCS data, and the neutral pions based on Belle data. The theoretical approaches and the first phenomenological extractions agree that the $D$-term is negative, as predicted on the basis of stability requirements.

Open questions include the issue of the mass decomposition of the nucleon [109, 110] and how it is related to the internal forces [111]. It is a legitimate but unanswered question whether the forces encoded in the EMT can tell us interesting lessons about confinement. Another interesting question is whether one can define further mechanical properties of hadrons such as e.g. speed of sounds inside a hadron. It would be also interesting to investigate the relation (if it exists) to hydrodynamical transport properties such as viscosity in strongly interacting systems in heavy ion collisions. The analogy of the $D$-term to the vacuum cosmological constant observed in [218] is a further topic worthwhile investigating. These topics will be discussed in future studies.
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Appendix A: Relations for EMT densities

Due to EMT conservation we have generically

$$\int_V d^3r \left( \nabla^i T^{ij} \right) X^{klm...} = 0 \iff \int_V d^3r T^{ij}(\nabla^i X^{klm...}) = \oint_{\partial(V)} da^i T^{ij} X^{klm...}$$

(A1)

where $X^{klm...}$ is some differentiable tensor which can be used in integration by parts. $V$ is some finite or infinite volume. In our situation (no spin effects for spin 0 and spin 1/2, spherical symmetry) it makes sense to choose one free index: we define $X^j = r^j f(r)$ where $f(r)$ is an arbitrary function which can be used in integration by parts.

Let us choose now the volume to be infinite and $f(r)$ such that the surface term in Eq. (A1) can be neglected. After performing the integration by parts and inserting the expression (21) for the stress tensor we obtain

$$I[f(r)] = \int d^3r T^{ij}[\nabla^i (x^j f(r))] = \int d^3r \left( \frac{2}{3} r s(r) f'(r) + r p(r) f'(r) + 3 p(r) f(r) \right) = 0.$$  

(A2)

This is a “master integral” for deriving relations for the densities of the stress tensor. Let us explore different choices for the function $f(r)$:

- **choice $f(r) = 1$:** We recover the von Laue condition, $I[1] = 3 \int d^3r p(r) = 0$.

- **choice $f(r) = r^N$:** We obtain $I[r^N] = 4\pi \int_0^\infty dr r^{N+2} [\frac{2}{3} N s(r) + (N + 3) p(r)] = 0$ which is equivalent to the “Mellin-moment-relations” for $p(r)$ and $s(r)$ derived in App. B or Ref. [120]. For $N = 0$ we obtain the previous relation, for $N = -3$ we reproduce the Kelvin relation (33).

- **special case $N = -1$:** this yields $I[1/r] = 4 \times 2\pi \int_0^\infty dr \left[ -\frac{1}{3} s(r) + p(r) \right] = 0$ and we recover the sum rule for tangential forces (46) obtained from physics consideration of forces acting on spherical slice of a hadron.

- **special case $N = -2$:** this case yields $I[1/r^2] = 4 \pi \int_0^\infty dr \left[ -\frac{4}{3} s(r) + p(r) \right] = 0$, which corresponds to the stability condition (48) for the 1D subsystems in a hadron.

- **choice $f(r) = \Theta(R - r)$:** we obtain the relation $\int_{V(R)} d^3r p(r) = V(R) \left[ \frac{2}{3} s(R) + p(R) \right]$ derived in Eq. (34) of this work. Here we defined $V(R) = \frac{4}{3} \pi R^3$. Notice that this “trial function” is not differentiable, but it does not matter. All we need is that it does not contribute to the integral at infinity!

- **choice $f(r) = \delta(r - R)$:** this choice yields the well-known relation $\frac{2}{3} s'(R) + \frac{2}{3} s(R) + p'(R) = 0$ which is equivalent to EMT conservation $\nabla^i T^{ik} = 0$.

- **choice $f(r) = p(r)$:** we obtain the following fascinating non-linear integral sum rule for EMT densities, which was shown above in Eq. (57) in equivalent form,

$$\int d^3r \left( \frac{2}{3} s(r)^2 - \frac{3}{2} p(r)^2 \right) = 0.$$  

(A3)
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