Diffeomorphism invariant subspaces in Witten’s 2+1 quantum gravity on $\mathbb{R} \times T^2$

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Abstract

We address the role of large diffeomorphisms in Witten’s 2+1 gravity on the manifold $\mathbb{R} \times T^2$. In a “spacelike sector” quantum theory that treats the large diffeomorphisms as a symmetry, rather than as gauge, the Hilbert space is shown to contain no nontrivial finite dimensional subspaces that are invariant under the large diffeomorphisms. Infinite dimensional closed invariant subspaces are explicitly constructed, and the representation of the large diffeomorphisms is thus shown to be reducible. Comparison is made to Witten’s theory on $\mathbb{R} \times \Sigma$, where $\Sigma$ is a higher genus surface.

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I. INTRODUCTION

In Witten’s 2+1 gravity \[1,2\] on spacetimes of the form \( \mathcal{M} = \mathbb{R} \times \Sigma \), where \( \Sigma \) is a closed orientable two-manifold of genus \( g \), the simplest nontrivial case of \( g = 1 \) is known to differ in several qualitative ways from the generic case of \( g > 1 \) \[3–8\]. One facet of this is the way that large diffeomorphisms (i.e., diffeomorphisms disconnected from the identity) appear in the theory. For \( g > 1 \), the geometrodynamically relevant connected component of the classical solution space is the cotangent bundle over the Teichmüller space \( T^g \) of \( \Sigma \). The quotient group \( G \) of all diffeomorphisms modulo diffeomorphisms connected to the identity is the modular group, and the quotient space \( T^g/G \) is the Riemann moduli space, which is a smooth manifold everywhere except at isolated singularities. Upon quantization, one option is to take the Hilbert space to be \( L^2(T^g) \) with respect to a natural volume element on \( T^g \) \[4,7,11,14\], and to let the large diffeomorphisms act on this space as symmetries. Another option is to treat the large diffeomorphisms as gauge, in which case they should be factored out from the quantum theory; this can be achieved by taking the Hilbert space to be \( L^2(T^g/G) \).

For \( g = 1 \) the classical solution space is not a manifold, nor is the subset that corresponds to conventional geometrodynamics \[4,11,14\]. The attention has therefore often been fixed to the so-called “spacelike sector” of the theory, where the classical solution space consists of two copies of the “square root” geometrodynamical theory \[12,13\] glued together by a lower-dimensional non-geometrodynamical part \[7\]. The configuration space of this sector can be regarded as \( \mathcal{N}_S := (\mathbb{R}^2 \setminus \{(0,0)\}) / \mathbb{Z}_2 \), where the \( \mathbb{Z}_2 \) action on \( \mathbb{R}^2 \setminus \{(0,0)\} \) is generated by the map \((x^1, x^2) \mapsto (-x^1, -x^2)\), and the phase space is the cotangent bundle over \( \mathcal{N}_S \). \( \mathcal{N}_S \) is equipped with the volume element \( d\mu := dx^1 dx^2 \) \[11,14\]. The modular group is now \( SL(2,\mathbb{Z}) \), and its action on \( \mathcal{N}_S \) is induced from the action on \( \mathbb{R}^2 \) given by

\[
\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \mapsto M \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \quad M \in SL(2,\mathbb{Z}) ,
\]

where \( M \) on the right hand side acts on the column vector by usual matrix multiplication. (Clearly, this \( SL(2,\mathbb{Z}) \) action on \( \mathcal{N}_S \) reduces to an action of the factor group \( PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\mathbb{1}, -\mathbb{1}\} \), where \( \mathbb{1} \) stands for the two by two unit matrix.) If now the large diffeomorphisms are understood as a symmetry, the Hilbert space can be taken to be \( \mathcal{H}_S := L^2(\mathcal{N}_S; d\mu) \) \[1-3,7,11,14\], and the action of \( SL(2,\mathbb{Z}) \) on \( \mathcal{N}_S \) clearly induces a unitary representation \( \mathcal{T}_{SL(2,\mathbb{Z})}^S \) of \( SL(2,\mathbb{Z}) \) on \( \mathcal{H}_S \). Treating the large diffeomorphisms as gauge is more problematic, however. One attempt might be to follow the logic of the higher genus surfaces and regard the quotient space \( \mathcal{N}_S/SL(2,\mathbb{Z}) \) as a configuration space on which the quantum theory is to be built. However, the action of \( SL(2,\mathbb{Z}) \) on \( \mathcal{N}_S \) is not properly discontinuous. In fact, each half-line with rational \( x^2/x^1 \) is fixed by an infinite Abelian subgroup, whereas the half-lines with irrational \( x^2/x^1 \) are fixed only by \( \pm \mathbb{1} \) \[15\]. This implies that the isomorphism class of stabilizer subgroups is nowhere locally constant. The quotient space \( \mathcal{N}_S/SL(2,\mathbb{Z}) \) is thus nowhere locally a manifold, and it is not obvious whether one could use such a space as a configuration space for the quantum theory. An alternative attempt might be to employ the Hilbert space \( \mathcal{H}_S \), but to restrict observables to those that commute with the given unitary action of \( SL(2,\mathbb{Z}) \). This prevents observables from having non-zero matrix elements between states belonging to inequivalent subrepresentations. Therefore, no state
that can be decomposed into the sum of states belonging to inequivalent subrepresentations can be a pure state. Here, “state” is understood to mean “state for the algebra of observables;” see e.g. [10]. However, from the results in Sections [1] and [11] it follows that every state can be written in such a form (as will be discussed more explicitly in the following paragraph). The attempt to reduce the gauge redundancy in this fashion thus leads to the somewhat paradoxical result that there are no pure states.

That the gauge interpretation of large diffeomorphisms leads to the absence of pure states is, in fact, not at all surprising. This can be seen from a more familiar example: Consider the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^2; dx^1 dx^2) \) and the unitary action of the translation group in \( x^1 \)-direction. We want to regard the translations (for simplicity we shall refer to the \( x^1 \)-translations simply as translations) as gauge, and hence require all observables to commute with them. Their integral kernels, \( O(x^1, x^2; y^1, y^2) \), thus only depend on \( x^1 - y^1, x^2, \) and \( y^2 \). On the other hand, each translation invariant subspace, \( \mathcal{H}_\Delta \subset \mathcal{H} \), is uniquely given by those functions whose Fourier transform in \( x^1 \) vanishes almost everywhere outside the measurable set \( \Delta \subset \mathbb{R} \) in the transformed \( x^1 \) coordinate [17]. Given two disjoint measurable sets, \( \Delta_1 \) and \( \Delta_2 \), it is easy to see that all matrix elements of observables between states in \( \mathcal{H}_{\Delta_1} \) and \( \mathcal{H}_{\Delta_2} \) vanish. This is a direct consequence of the fact that translation invariant operators cannot increase the support of the Fourier transform, and can also be checked by direct calculation using the property of integral kernels given above. On the other hand, given a measurable set \( \Delta \) of non-zero measure, it is always possible to find a measurable subset \( \Delta_1 \subset \Delta \) of non-zero but strictly smaller measure. (A short proof of this fact will be given in Appendix [A]) Denoting by \( \Delta_2 \) the complement of \( \Delta_1 \) in \( \Delta \), we have a decomposition \( \Delta = \Delta_1 \cup \Delta_2 \) of \( \Delta \) into disjoint subsets of non-zero measures. If we take for \( \Delta \) the support in the Fourier transformed \( x^1 \) coordinate of an arbitrary element in \( \mathcal{H} \), we immediately infer that any vector is the sum of two vectors which lie in different and strictly smaller invariant subspaces. Hence there are no pure states. A similar argument applies to our gravitational case, using the diffeomorphism invariant subspaces of Theorem [11]2 in Section [11].

In the above simple example it is clear how the redundant translations are properly eliminated: instead of \( \mathcal{H} \) one considers the Hilbert space \( L^2(\mathbb{R}; dx^2) \) of square integrable functions over the classical reduced configuration space, which may be identified with the \( x^2 \) axis. However, in our gravitational case the analogous option is not at our direct disposal, due to the complicated structure of \( \mathcal{N}_S/SL(2, \mathbb{Z}) \). In this paper we shall therefore concentrate on the theory in which the large diffeomorphisms are treated as symmetries. The Hilbert space is thus \( \mathcal{H}_S \), the large diffeomorphisms act on \( \mathcal{H}_S \) by \( T^S_{SL(2, \mathbb{Z})} \), and the algebra of observables is taken to be the full algebra \( B(\mathcal{H}_S) \) of bounded operators on \( \mathcal{H}_S \). Note that this means allowing, in principle, observables that do not commute with the large diffeomorphisms. At the fundamental level this theory has no superselection rules, and rays in \( \mathcal{H}_S \) are in bijective correspondence to pure states.

The purpose of this paper is to make two observations about the unitary representation \( T^S_{SL(2, \mathbb{Z})} \) of \( SL(2, \mathbb{Z}) \) on \( \mathcal{H}_S \). On the one hand, we point out that \( T^S_{SL(2, \mathbb{Z})} \) is reducible, and we exhibit a class of infinite dimensional closed invariant subspaces. On the other hand, we
demonstrate that \( \mathcal{H}_S \) contains no nontrivial finite dimensional invariant subspaces.

Our starting point is the decomposition \([18,19]\) of the standard unitary representation \( T_{SL(2,R)} \) of \( SL(2,R) \) on \( L^2(R^2) \) into a direct integral of irreducible unitary representations, all of whom belong to the principal series \([20–23]\). This decomposition yields an obvious construction of closed infinite dimensional subspaces of \( L^2(R^2) \) that are invariant under \( T_{SL(2,R)} \). Projecting \( L^2(R^2) \) to the subspace \( \mathcal{H}_S \) then produces closed infinite dimensional subspaces of \( \mathcal{H}_S \) that are invariant under \( T_{SL(2,R)} \), and hence also under \( T_{SL(2,Z)}^S \). This is the first claim above.

To prove the second claim, let us denote by \( T_{SL(2,Z)} \) the restriction of \( T_{SL(2,R)} \) to \( SL(2,Z) \). It is known that the principal series irreducible unitary representations of \( SL(2,R) \) restrict to irreducible representations of \( SL(2,Z) \) \([24]\). (Further, two representations of \( SL(2,Z) \) obtained in this fashion are equivalent only if the corresponding representations of \( SL(2,R) \) are \([24,25]\).) This means that the direct integral decomposition of \( T_{SL(2,R)} \) yields, through restriction to \( SL(2,Z) \), a decomposition of \( T_{SL(2,Z)} \) into a direct integral of irreducible infinite dimensional representations. It follows immediately that \( L^2(R^2) \) has no nontrivial finite dimensional subspaces that are invariant under \( T_{SL(2,Z)} \). This implies the claim.

With the decomposition of \( T_{SL(2,Z)} \), one can translate the action of \( SL(2,Z) \) on the configuration space \( \mathcal{N}_S \) into an action of \( SL(2,Z) \) on \( S^1 \), where the \( S^1 \) arises as the configuration space of the constituent irreducible representations of \( SL(2,Z) \) on \( L^2(S^1) \). The problem of building a quantum theory with the configuration space \( \mathcal{N}_S/SL(2,Z) \) is thus translated into the problem of building quantum theories with the configuration space \( S^1/SL(2,Z) \). We show that the difficulty persists: the action of \( SL(2,Z) \) on \( S^1 \) is not properly discontinuous, and \( S^1/SL(2,Z) \) is nowhere locally a manifold.

The rest of the paper is as follows. In Section II we present the decomposition of the standard representation of \( SL(2,R) \) on \( L^2(R^2) \) into a direct integral of irreducible representations, and we use this decomposition to construct a class of closed invariant subspaces. The restriction to the subgroup \( SL(2,Z) \) is addressed in Section III. Section IV contains a brief discussion. Appendix A recalls an elementary property of the Lebesgue measure, and Appendix B contains an analysis of the quotient space \( S^1/SL(2,Z) \).

II. REPRESENTATION OF SL(2,R) ON L^2(R^2)

In this section we first review the decomposition of the standard unitary representation of \( SL(2,R) \) on \( L^2(R^2) \) into irreducible unitary representations \([18]\). We then note that the decomposition presents an obvious way of constructing infinite dimensional closed invariant subspaces of \( L^2(R^2) \).

Let \( (x^1, x^2) \) be a pair of global coordinates on \( R^2 \). The group \( SL(2,R) \) has on \( R^2 \) the natural associative action

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1While this paper was in preparation, an independent argument ruling out nontrivial finite dimensional invariant subspaces was given in the revised version of Ref. \([13]\). We thank Peter Peldán for discussions on this issue.
\[
\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \mapsto M \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \quad M \in \text{SL}(2, \mathbb{R}),
\]

(2.1)

where \( M \) on the right hand side acts on the column vector by usual matrix multiplication. Denoting a point on \( \mathbb{R}^2 \) by \( x \), we write this action as \( x \mapsto Mx \).

Let \( \mathcal{H} := L^2(\mathbb{R}^2) = L^2(\mathbb{R}^2; dx^1 dx^2) \) be the Hilbert space of square integrable functions on \( \mathbb{R}^2 \) with the inner product

\[
(f, g) := \int dx^1 dx^2 \overline{f}g.
\]

(2.2)

We define a representation \( \mathcal{T} \) of \( \text{SL}(2, \mathbb{R}) \) on \( \mathcal{H} \), \( f \mapsto \mathcal{T}(M)f \), by

\[
\mathcal{T}(M)f(x) := f(M^{-1}x).
\]

(2.3)

This representation is clearly unitary, \( (\mathcal{T}(M)f, \mathcal{T}(M)g) = (f, g) \). Our aim is to decompose this representation into its irreducible components. As \( \text{SL}(2, \mathbb{R}) \) is a Type I group, the decomposition is essentially unique [26].

We first rewrite the \( \text{SL}(2, \mathbb{R}) \) action on \( \mathbb{R}^2 \) (2.1) in a more convenient manner. This action clearly leaves the origin invariant. On \( \mathbb{R}^2 \setminus \{(0,0)\} \), introduce the polar coordinates \((r, \theta)\) through

\[
\begin{align*}
x^1 &= r \cos \theta, \\
x^2 &= r \sin \theta,
\end{align*}
\]

(2.4)

where \( r > 0 \), and \( \theta \) is understood periodic with period \( 2\pi \). We parametrize a matrix \( M \in \text{SL}(2, \mathbb{R}) \) as

\[
M = U \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} U^{-1},
\]

(2.5)

where \( U \) is the unitary matrix

\[
U := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},
\]

(2.6)

and \( \alpha \) and \( \beta \) are complex numbers satisfying \( \alpha \bar{\alpha} - \beta \bar{\beta} = 1 \). This parametrization is one-to-one, and if the matrix \( \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \) is interpreted as an element of \( \text{SU}(1,1) \), (2.5) defines an isomorphism \( \text{SL}(2, \mathbb{R}) \simeq \text{SU}(1,1) \) [20]. The \( \text{SL}(2, \mathbb{R}) \) action (2.1) on \( \mathbb{R}^2 \setminus \{(0,0)\} \) takes then the form \( (r, \theta) \mapsto (Mr, M\theta) \), where

\[\text{Note that functions represent the same element in } L^2 \text{ spaces if they differ at most on a set of measure zero. We shall therefore throughout understand functions to be defined only almost everywhere (a.e.), and pointwise equations for the functions to hold only a.e.}\]
\[ e^{iM\theta} = e^{i\theta} \frac{W(M, \theta)}{|W(M, \theta)|} \quad (2.7a) \]

\[ Mr = r|W(M, \theta)| \quad (2.7b) \]

with

\[ W(M, \theta) := \alpha + \beta e^{2i\theta} \quad (2.8) \]

Next, we perform a radial Mellin transform on \( \mathcal{H} \). For \( f \in \mathcal{H} \), its transform \( \hat{f} \) is defined by

\[ \hat{f}(s, \theta) := \int_0^\infty dr \, r^{is} f(r, \theta) \quad s \in \mathbb{R} \quad (2.9) \]

Here, and from now on, we understand the argument of a function in \( \mathcal{H} \) to be the pair of polar coordinates \((r, \theta)\) \( (2.4) \). The transform \( (2.9) \) defines an isomorphism \( \mathcal{H} \simeq \hat{\mathcal{H}} := L^2(\mathbb{R} \times S^1; (2\pi)^{-1}dsd\theta) \): the inner product \( (2.2) \) can be written as

\[ (f, g) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\pi}^{\pi} d\theta \overline{\hat{f}(s, \theta)} \hat{g}(s, \theta) \quad (2.10) \]

and the inverse transform is

\[ f(r, \theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \, r^{-1-is} \hat{f}(s, \theta) \quad (2.11) \]

These statements follow directly from the observation that in terms of the logarithmic radial coordinate \( t = \ln r \), the inner product \( (2.2) \) reads

\[ (f, g) = \int_{-\infty}^{\infty} dt \int_{-\pi}^{\pi} d\theta \overline{e^{itf(e^t, \theta)}} e^{itg(e^t, \theta)} \quad (2.12) \]

and the transforms \( (2.9) \) and \( (2.11) \) reduce to an ordinary Fourier transform pair,

\[ \hat{f}(s, \theta) = \int_{-\infty}^{\infty} dt \, e^{ist} e^{itf(e^t, \theta)} \quad (2.13a) \]

\[ e^{itg(e^t, \theta)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \, e^{-ist} \hat{f}(s, \theta) \quad (2.13b) \]

By \( (2.7) \) and \( (2.9) \), the Mellin transform maps the representation \( \mathbb{T} \) of SL(2,\( \mathbb{R} \)) into a representation \( \hat{\mathbb{T}} \) on \( \hat{\mathcal{H}} \), given by

\[ \hat{\mathbb{T}}(M) \hat{f}(s, \theta) = |W(M, M^{-1}\theta)|^{1+is} \hat{f}(s, M^{-1}\theta) \quad (2.14) \]

The remarkable property of \( (2.14) \) is that the different values of \( s \) are decoupled. To utilize this, we write \( \hat{\mathcal{H}} \) as the direct integral

\[ \hat{\mathcal{H}} = \int_{-\infty}^{\infty} ds \, \hat{\mathcal{H}}_s \quad (2.15) \]
where $\mathcal{H}_s \simeq L^2(S^1; (2\pi)^{-1} d\theta)$ with the inner product
\[
(f_s, g_s)_s = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \overline{f_s(\theta)} g_s(\theta), \quad f_s, g_s \in \mathcal{H}_s.
\] (2.16)

The representation $\hat{T}$ (2.14) then decomposes into a representation $\hat{T}_s$ on each $\mathcal{H}_s$, given by
\[
\hat{T}_s(M)\hat{f}_s(\theta) = |W(M, M^{-1}\theta)|^{1+is} \hat{f}_s(M^{-1}\theta).
\] (2.17)
It is straightforward to verify that $\hat{T}_s$ is a unitary representation for every $s$. However, it is not irreducible.

To proceed, we write $\mathcal{H}_s = \mathcal{H}_s^+ \oplus \mathcal{H}_s^-$, where $\mathcal{H}_s^\pm$ are the two closed subspaces of $\mathcal{H}_s$ where the functions satisfy respectively $\hat{f}_s(\theta + \pi) = \pm \hat{f}_s(\theta)$. $\mathcal{H}_s^\pm$ are clearly each invariant under $\hat{T}_s$. Therefore, $\hat{T}_s$ decomposes into unitary representations of SL$(2, \mathbb{R})$ on $\mathcal{H}_s^\pm$. We shall show that, with the exception of $\mathcal{H}_0^0$, all these representations are irreducible.

Let $s$ be fixed, and consider $\mathcal{H}_s^+$. For $\tilde{f} \in \mathcal{H}_s^+$, we can write $\tilde{f}(\theta) = \tilde{f}(2\theta)$, where $\tilde{f}$ is periodic in its argument with period $2\pi$. (For brevity, we drop the subscript $s$ when there is no danger of confusion.) This gives an isomorphism $\mathcal{H}_s^+ \simeq \mathcal{H}_s^+ := L^2(S^1; (2\pi)^{-1} d\phi)$, where the inner product induced from (2.16) is
\[
(f, g)_s = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \overline{f(\phi)} g(\phi).
\] (2.18)

The resulting representation $\hat{T}_s^+$ of SL$(2, \mathbb{R})$ on $\mathcal{H}_s^+$ is then given by
\[
\hat{T}_s^+(M)\tilde{f}(\phi) = |w(M, M^{-1}\phi)|^{1+is} \tilde{f}(M^{-1}\phi),
\] (2.19)
where
\[
w(M, \phi) := \alpha + \beta e^{i\phi},
\] (2.20)
and the action of SL$(2, \mathbb{R})$ on $\phi$ is determined by (2.7a) and takes the form
\[
e^{iM\phi} = e^{i\phi} \frac{w(M, \phi)}{w(M, \phi)}.
\] (2.21)

This is recognized as the irreducible unitary representation of the continuous class $C^0_q$ constructed in Ref. [20] with the Casimir invariant $q$ taking the value $(1 + s^2)/4$. These representations are known as the principal series of even parity [21, 22].

Let then $s$ again be fixed, and consider $\mathcal{H}_s^-$. For $\tilde{f} \in \mathcal{H}_s^-$, we now can write $\tilde{f}(\theta) = e^{i\theta} \tilde{f}(2\theta)$, where $\tilde{f}$ is periodic in its argument with period $2\pi$. This gives an isomorphism $\mathcal{H}_s^- \simeq \mathcal{H}_s^- := L^2(S^1; (2\pi)^{-1} d\phi)$, where the inner product induced from (2.16) is again given by (2.18). The resulting representation $\hat{T}_s^-$ of SL$(2, \mathbb{R})$ on $\mathcal{H}_s^-$ is then given by

Note that formula (6.11) in Ref. [20] has a typographical error and should read $T_\sigma(a)f(\phi) = \mu(a, a^{-1}\phi)\frac{1}{2} + \sigma f(a^{-1}\phi)$. 

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3Note that formula (6.11) in Ref. [20] has a typographical error and should read $T_\sigma(a)f(\phi) = \mu(a, a^{-1}\phi)\frac{1}{2} + \sigma f(a^{-1}\phi)$. 

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\[ \tilde{T}_s^{-}(M) \hat{f}(\phi) = |w(M, M^{-1}\phi)|^{1+is} \nu(M, M^{-1}\phi) \hat{f}(M^{-1}\phi) , \]  

(2.22)

where

\[ \nu(M, \phi) := \frac{w(M, \phi)}{|w(M, \phi)|} , \]

(2.23)

and the rest of the notation is as with \( \tilde{T}_+ \). For \( s \neq 0 \), \( \tilde{T}_s^- \) is recognized as the irreducible unitary representation of the continuous class \( C^1_0/2 \) constructed in Ref. [20], with the Casimir invariant \( q \) taking the value \((1 + s^2)/4\). These representations are known as the principal series of odd parity [21–23]. The representation \( \tilde{T}_0^- \) decomposes into a direct sum of two irreducible unitary representations, denoted in Ref. [20] by \( D^{1/2}_1 \) and \( D^{1/2}_{-1} \) and known as the limits of the discrete series [21–23].

We thus have a complete decomposition of the representation \( T \) (2.3) of SL(2, R) on \( \mathcal{H} \) into its irreducible components. We collect the statements into a theorem [18].

**Theorem II.1.** The Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^2) \) has a decomposition

\[ \mathcal{H} \simeq \int_{-\infty}^{\infty} ds \left( \hat{\mathcal{H}}_s^+ \oplus \hat{\mathcal{H}}_s^- \right) , \]

(2.24)

where the integral is a direct integral, such that (i) \( \hat{\mathcal{H}}_s^\pm \simeq L^2(S^1) \) for every \( s \); (ii) the unitary representation \( \tilde{T}_s \) of SL(2, R) on \( \mathcal{H} \) decomposes into the unitary representations \( \tilde{T}_s^\pm \) on \( \hat{\mathcal{H}}_s^\pm ; \) (iii) \( \tilde{T}_s^+ \) is an irreducible unitary representation in the principal series with even parity, \( C^0_q \), with \( q = (1 + s^2)/4 \); (iv) \( \tilde{T}_s^- \) with \( s \neq 0 \) is an irreducible unitary representation in the principal series with odd parity, \( C^1_q/2 \), with \( q = (1 + s^2)/4 \). \( \square \)

Note that the further decomposition of \( \tilde{T}_0^- \) is not relevant in Theorem II.1, as the point \( s = 0 \) is a set of measure zero on the real line and therefore does not contribute to the integral in (2.24). Note also that if we write

\[ \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- , \]

(2.25a)

\[ \hat{\mathcal{H}} = \hat{\mathcal{H}}^+ \oplus \hat{\mathcal{H}}^- , \]

(2.25b)

where \( \mathcal{H}^\pm \) consist of those functions \( f \) in \( \mathcal{H} \) that satisfy \( f(r, \theta + \pi) = \pm f(r, \theta) \), and similarly \( \hat{\mathcal{H}}^\pm \) consist of those functions \( \hat{f} \) in \( \hat{\mathcal{H}} \) that satisfy \( \hat{f}(s, \theta + \pi) = \pm \hat{f}(s, \theta) \), we then have

\[ \mathcal{H}^\pm \simeq \hat{\mathcal{H}}^\pm = \int_{-\infty}^{\infty} ds \hat{\mathcal{H}}_s^\pm \simeq \int_{-\infty}^{\infty} ds \hat{\mathcal{H}}_s^\pm \]  

(2.26)

As SL(2, R) has no nontrivial finite dimensional unitary representations [23], \( \mathcal{H} \) cannot have nontrivial finite dimensional subspaces that are invariant under \( T \). There are, however, infinite dimensional closed subspaces of \( \hat{\mathcal{H}}^\pm \) that are invariant under \( T \). We have the following theorem.

**Theorem II.2.** Let \( E \) be a measurable subset of \( \mathbb{R} \), and let

\[ \hat{\mathcal{H}}_E^\pm = \{ \hat{f} \in \hat{\mathcal{H}}^\pm \mid \hat{f}(s, \theta) = 0 \text{ for a.e. } s \notin E \} \].  

(2.27)
Then $\hat{\mathcal{H}}_E^\pm$ is a closed subspace of $\hat{\mathcal{H}}^\pm$, and it is invariant under $\hat{T}$.

**Proof.** We can assume that $E$ and $R \setminus E$ both have strictly positive measure (since otherwise $\hat{\mathcal{H}}_E^\pm = \{0\}$ or $\hat{\mathcal{H}}_E^\pm = \hat{\mathcal{H}}^\pm$). It is clear that $\hat{\mathcal{H}}_E^\pm$ is an invariant subspace of $\hat{\mathcal{H}}^\pm$. Closedness follows from the observation that $\hat{\mathcal{H}}_E^\pm$ is the orthogonal complement in $\hat{\mathcal{H}}^\pm$ of the subspace $\hat{\mathcal{H}}_{R \setminus E}^\pm$. □

The construction of $\hat{\mathcal{H}}_E^\pm$ closely parallels the construction of closed translationally invariant subspaces of $L^2(R)$. For a measurable set $E \subset R$, the functions in $L^2(R)$ whose Fourier transform vanishes almost everywhere outside $E$ constitute a closed translationally invariant subspace of $L^2(R)$; conversely, every closed translationally invariant subspace of $L^2(R)$ is of this form for some $E$. We shall not address here the question as to whether the spaces $\hat{\mathcal{H}}_E^\pm$ exhaust all closed $T$-invariant subspaces of $\hat{\mathcal{H}}^\pm$.

### III. REPRESENTATION OF SL(2, Z) ON $L^2(R^2)$

We now consider the consequences of the above results when SL(2, R) is restricted to the subgroup SL(2, Z).

It is clear that all the unitary representations of SL(2, R) appearing in Section II restrict to unitary representations of SL(2, Z). The spaces $\hat{\mathcal{H}}_E^\pm$ of Theorem II.2 are therefore invariant also under the representation of SL(2, Z) that is inherited from $T$. In the physical terminology introduced in Section I, this implies that $\hat{\mathcal{H}}_E^\pm$ are closed diffeomorphism invariant subspaces of $\mathcal{H}_S \simeq \hat{\mathcal{H}}^\pm$.

To examine the possibility of finite dimensional diffeomorphism invariant subspaces, let us denote by $\hat{T}'$ and $\hat{T}_s^\pm$ the representations of SL(2, Z) that are obtained by restriction from respectively $\hat{T}$ and $\hat{T}_s^\pm$. Let $\mathcal{F}^\pm$ be a finite dimensional subspace of $\hat{\mathcal{H}}^\pm$, and let \( \{ \hat{f}_k^\pm \in \hat{\mathcal{H}}^\pm \mid k = 1, \ldots, N \} \) be a finite set of vectors spanning $\mathcal{F}^\pm$. Suppose now that $\mathcal{F}^\pm$ is invariant under $\hat{T}'$. It follows that for a.e. $s \in R$, the subspace $\hat{\mathcal{F}}_s^\pm$ of $\hat{\mathcal{H}}_s^\pm \simeq \hat{\mathcal{H}}_s^\pm$ that is spanned by the functions \( \{ \hat{f}_k^\pm(s, \theta) \} \) is invariant under $\hat{T}_s^\pm$. But since $\hat{\mathcal{H}}_s^\pm$ are infinite dimensional and $\hat{T}_s^\pm$ are irreducible (except for $\hat{T}_0^-$) [24], $\mathcal{F}_s^\pm$ must be trivial for a.e. $s$. This implies that every $\hat{f}_k^\pm$ is the zero vector in $\hat{\mathcal{H}}^\pm$, and hence $\mathcal{F}_s^\pm = \{0\}$.

Therefore, $\mathcal{H}$ has no nontrivial finite dimensional subspaces that are invariant under $\hat{T}'$. In the physical terminology of Section I, this implies that $\mathcal{H}_S \simeq \hat{\mathcal{H}}^\pm$ has no nontrivial finite dimensional subspaces invariant under the large diffeomorphisms.

### IV. DISCUSSION

In this paper we have addressed the role of large diffeomorphisms in Witten’s 2+1 gravity on the manifold $R \times T^2$. We concentrated on a “spacelike sector” quantum theory that treats the large diffeomorphisms as a symmetry. On the one hand, we showed that the Hilbert space contains no nontrivial finite dimensional subspaces that are invariant under the large diffeomorphisms. On the other hand, we constructed explicitly a class of infinite dimensional closed invariant subspaces. The existence of such subspaces implies, in particular, that the representation of the large diffeomorphisms on the Hilbert space is reducible.
These results shed light on both the similarities and differences between the behavior of Witten’s theory on the manifold $R \times T^2$ and the manifolds $R \times \Sigma$, where $\Sigma$ is a surface of genus $g > 1 \ [2–4,10]$. For $g > 1$, a geometrodynamically relevant quantum theory that treats the large diffeomorphisms as a symmetry is obtained by taking the configuration space to be the Teichmüller space $T^g$. The quotient of $T^g$ under the action of the large diffeomorphisms is the Riemann moduli space: as $T^g$ contains infinitely many copies of the Riemann moduli space, the quantum theory has no nontrivial finite dimensional diffeomorphism invariant subspaces. This is similar to what we have found for the torus. On the other hand, the differences between the torus and the higher genus surfaces manifest themselves when one attempts to treat the large diffeomorphisms as gauge. As the Riemann moduli space is a manifold except at isolated singularities, a higher genus theory that treats the large diffeomorphisms as gauge can be obtained by defining the inner product by an integral over just the Riemann moduli space instead of all of $T^g$. In contrast, the corresponding quotient space for the torus seems too pathological to be employed in a similar fashion [15]. We shall show in Appendix B that this pathology persists even when the torus Hilbert space is decomposed by (2.26) into a direct integral of Hilbert spaces that carry irreducible representations of the large diffeomorphism group. An attempt to reduce the gauge redundancies in the torus theory at the quantum level leads to the absence of any pure states in the theory, as discussed in the Introduction.

For $R \times T^2$, the construction of a connection representation quantum theory where the large diffeomorphisms are treated as gauge remains thus an open problem. In the metric-type representations of Refs. [4–6,8,27], the difficulty does not appear.

Finally, it should be emphasized that we have not attempted to interpret physically the symmetries generated by the large diffeomorphisms. Doing so would require, among other things, a physical interpretation of those observables that do not commute with the large diffeomorphisms. For noncompact two-manifolds, one possibility to approach this might be to introduce boundary conditions that fix additional structure at an asymptotic infinity, and to interpret the large diffeomorphisms in terms of the structure at the infinity [28]. The infinity would then be understood as an ambient physical system. However, for compact two-manifolds no such reference to an outside system is possible. The interpretational issue is therefore not at all obvious.

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**APPENDIX A: A PROPERTY OF THE LEBESGUE MEASURE**

In this appendix we consider the measure space given by the pair $(R, \mu)$, where $R$ is the real line and $\mu$ its Lebesgue measure. We prove that for any given measurable set $\Delta$, where
\[ \mu(\Delta) = d > 0, \text{ there exists a measurable subset } \Delta_1 \subset \Delta, \text{ so that } 0 < \mu(\Delta_1) < \delta, \text{ for any given } \delta < d. \text{ Note that } d \text{ may be } \infty. \]

To prove this, we cover the real line by the intervals \( I_n = [n\delta, (n+1)\delta], n \in \mathbb{Z}. \) We have \( \mu(\Delta) = \sum_n \mu(\Delta \cap I_n) = d. \) Hence there exists an interval, say \( I_k, \) such that \( \mu(I_k \cap \Delta) > 0. \) We then define \( \Delta_1 := \Delta \cap I_k, \) which as intersection of two measurable sets is measurable. It clearly satisfies \( 0 < \mu(\Delta_1) < \mu(I_k) = \delta. \)

**APPENDIX B: SL(2, Z) ACTION ON THE CIRCLE**

We have noted that the principal series irreducible unitary representations \( \hat{T}^+ (s \in \mathbb{R}) \) and \( \hat{T}^- (s \in \mathbb{R} \setminus \{0\}) \) of SL(2, R) on \( L^2(S^1) \) restrict to irreducible unitary representations of SL(2, Z) on \( L^2(S^1) \) [24]. In this appendix we show that the associated action of SL(2, Z) on the “configuration space” \( S^1 \) of \( L^2(S^1) \) is nowhere properly discontinuous, and hence the quotient space \( S^1/SL(2, \mathbb{Z}) \) is nowhere locally a manifold. This reintroduces, at the level of the decomposition (2.26), the difficulties outlined in Section I for constructing a quantum theory in which the large diffeomorphisms are treated as gauge.

Recall [29,30] that SL(2, Z) is generated by the two matrices\(^4\)

\[
S := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{(B1a)}
\]

\[
T := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{(B1b)}
\]

whose only independent relations are

\[ S^3 = T^2 = -\mathbb{I} \quad \text{(B2)} \]

Here \( \mathbb{I} \) denotes the two by two identity matrix as before. In terms of the SU(1, 1) parametrization (2.5), \( S \) corresponds to \( \alpha = \frac{1}{2} + i \) and \( \beta = \frac{1}{2}, \) and \( T \) corresponds to \( \alpha = i \) and \( \beta = 0. \)

We are interested in the SL(2, Z) action on \( S^1 \) given by (2.21). It will be useful to identify \( S^1 \) with the one-point compactified real line, \( \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}, \) through

\[ \tan(\phi/2) = t, \quad t \in \hat{\mathbb{R}}, \quad \text{ (B3)} \]

where \( \phi = \pi \) is understood to correspond to \( t = \infty. \)

Let us denote the action of SL(2, Z) on \( \hat{\mathbb{R}} \) arising through (2.21) and (B3) by \( t \mapsto \Omega(M)t, \) \( M \in \text{SL}(2, \mathbb{Z}). \) We have

\[ \Omega(S)t = -1/(1 + t), \quad \text{(B4a)} \]

\[ \Omega(S^{-1})t = -(1 + t)/t, \quad \text{(B4b)} \]

\[ \Omega(T)t = \Omega(T^{-1})t = -1/t, \quad \text{(B4c)} \]

\(^4\)We follow the notation of Ref. [29].
It is clear that $\Omega(S)$ and $\Omega(T)$ both act on $\hat{\mathbb{R}}$ freely, and $\Omega(S)$ and $\Omega(T)$ respectively generate the subgroups $\Gamma_S := \{1_{\hat{\mathbb{R}}}, \Omega(S), \Omega(S^{-1})\} \simeq \mathbb{Z}_3$ and $\Gamma_T := \{1_{\hat{\mathbb{R}}}, \Omega(T)\} \simeq \mathbb{Z}_2$ of $\Gamma := \{\Omega(M) \mid M \in \text{SL}(2, \mathbb{Z})\} \simeq \text{PSL}(2, \mathbb{Z})$. Here $1_{\hat{\mathbb{R}}}$ denotes the identity transformation on $\hat{\mathbb{R}}$. We wish to characterize the quotient space $S^1/\text{SL}(2, \mathbb{Z}) \simeq \hat{\mathbb{R}}/\Gamma$.

Consider first $\hat{\mathbb{R}}/\Gamma_S$. We have

$$\Omega(S)[0, \infty) = [-1, 0) \quad \text{(B5a)}$$
$$\Omega(S)[-1, 0) = [\infty, -1) \quad \text{(B5b)}$$
$$\Omega(S)[\infty, -1) = [0, \infty) \quad \text{(B5c)}$$

Every equivalence class in $\hat{\mathbb{R}}/\Gamma_S$ has therefore a unique representative in $[0, \infty)$. This means that we can identify $\hat{\mathbb{R}}/\Gamma_S \simeq [0, \infty)$, with the circular topology that identifies $[0, \infty) \simeq S^1$.

Next, consider the action of $\Gamma_T$ on $\hat{\mathbb{R}}/\Gamma_S \simeq [0, \infty)$. Composing $\Omega(T)$ with $\Omega(S)$ or $\Omega(S^{-1})$ as appropriate, one finds that this action is

$$t \mapsto \begin{cases} t/(1-t), & 0 \leq t < 1, \\ t-1, & 1 \leq t < \infty. \end{cases} \quad \text{(B6)}$$

For any $t_0 \in [1, \infty)$, iterating (B6) eventually gives the fractional part of $t_0$, $\text{frac}(t_0) \in [0, 1)$. Therefore, $S^1/\text{SL}(2, \mathbb{Z})$ can be identified with the quotient space of $[0, 1)$ under iteration of the map $F : [0, 1) \to [0, 1)$, defined by

$$F(t) = \text{frac}\left(\frac{1}{1-t}\right) \quad \text{(B7)}$$

All the fixed points of $F$ are unstable, and it is clear that $S^1/\text{SL}(2, \mathbb{Z})$ is not a manifold. Note that $F$ is closely related to the Gauss map, $t \mapsto \text{frac}(1/t)$, which is well known in ergodic theory \cite{31,32}, and which has also been encountered in the qualitative analysis of Bianchi type IX cosmology \cite{33,34}. 

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