Global classical small-data solutions for a three-dimensional Keller–Segel–Navier–Stokes system modeling coral fertilization

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Abstract

We are concerned with the Keller–Segel–Navier–Stokes system

\[
\begin{aligned}
\rho_t + u \cdot \nabla \rho &= \Delta \rho - \nabla \cdot (\rho S(x, \rho, c) \nabla c) - \rho m, \quad (x, t) \in \Omega \times (0, T), \\
m_t + u \cdot \nabla m &= \Delta m - \rho m, \quad (x, t) \in \Omega \times (0, T), \\
c_t + u \cdot \nabla c &= \Delta c - c + m, \quad (x, t) \in \Omega \times (0, T), \\
u_t + (u \cdot \nabla) u &= \Delta u - \nabla P + (\rho + m) \nabla \phi, \quad \nabla \cdot u = 0, \quad (x, t) \in \Omega \times (0, T)
\end{aligned}
\]

subject to the boundary condition \((\nabla \rho - \rho S(x, \rho, c) \nabla c) \cdot \nu = \nabla m \cdot \nu = \nabla c \cdot \nu = 0, u = 0\) in a bounded smooth domain \(\Omega \subset \mathbb{R}^3\). It is shown that the corresponding problem admits a globally classical solution with exponential decay properties under the hypothesis that

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\[ S \in C^2(\Omega \times [0, \infty)^2)^{3\times3} \text{ satisfies } |S(x, \rho, c)| \leq C_S \text{ for some } C_S > 0, \text{ and the initial data satisfy certain smallness conditions.} \]

Keywords: Keller–Segel system; Navier–Stokes; tensor–value sensitivity; decay estimates.

AMS Subject Classification: 35B65; 35B40; 35K55; 92C17; 35Q92.

1 Introduction

Chemotaxis, the biased movement of individuals in response to gradients of certain chemicals, has a significant effect on pattern formation in numerous biological contexts (see [2, 12, 23]). In particular, the chemotaxis plays an important role in the reproduction of some invertebrates such as corals, anemones and sea urchins. Indeed, there is experimental evidence that eggs can release a chemical which attracts sperms during the process of coral fertilization ([5, 6, 21, 24, 25]).

The important effect of chemotaxis on the efficiency of coral fertilization is investigated by Kiselev and Ryzhik ([14, 15]) via the following chemotaxis system (the densities of egg and sperm gametes are assumed to be identical)

\[
n_t + U \cdot \nabla n = \Delta n + \chi \nabla \cdot (n \nabla (\Delta)^{-1} n) - \mu n^q \quad \text{in } \mathbb{R}^N \times (0, T) \tag{1.1}
\]

where \( n \) represents the density of egg (sperm) gametes, \( U \) is a prescribed solenoidal sea fluid velocity, and \( \chi > 0 \) denotes the chemotactic sensitivity constant, \( \epsilon n^q \) \((q \geq 2)\) denotes the fertilization phenomenon. For the Cauchy problem in \( \mathbb{R}^2 \) with initial datum \( n(\cdot, 0) = n_0 \), the global-in-time existence of solutions to (1.1) \((N = 2, 3)\) is proved under the suitable conditions on initial data. In addition, they showed that the total mass

\[
\int_{\mathbb{R}^2} n(x, t) dx \to n_{\infty}(\chi, n_0, U) \quad \text{as } t \to \infty
\]

with \( n_{\infty}(\chi, n_0, U) > 0 \) satisfying \( n_{\infty}(\chi, n_0, U) \to 0 \) as \( \chi \to \infty \) in the case \( q > 2 \) of supercritical reaction ([15]), whereas in the critical case \( q = 2 \), the decay rate of \( \int_{\mathbb{R}^2} n(x, t) dx \) is faster than that of \( 1/\log t \) as \( t \to \infty \), and a weaker effect of chemotaxis is observed within finite time intervals ([14]). Recently, the total mass behavior of solution to (1.1) is investigated in [1, 3, 13]...
when the chemical concentration is governed by a parabolic equation. In particular, the results of [3, 13] indicate that unlike in the Cauchy problem, the dynamical behavior of solution to (1.1) with \( q = 2 \) in the framework of bounded domains is essentially independent of the effect from chemotactic cross-diffusion. More precisely, it is shown in [3, 13] that whenever \( U \) is a bounded and sufficiently regular solenoidal vector field, the component \( n \) of any non-trivial classical bounded solution to

\[
\begin{aligned}
    n_t + U \cdot \nabla n &= \Delta n - \chi \nabla \cdot (n \nabla c) - \mu n^2, & x \in \Omega, t > 0, \\
    c_t + U \cdot \nabla c &= \Delta c - c + n, & x \in \Omega, t > 0
\end{aligned}
\]  

(1.2)
decays to zero in either of the spaces \( L^1(\Omega) \) and \( L^\infty(\Omega) \), which can be controlled by appropriate multiples of \( 1/(t + 1) \) from above and below, respectively.

Experiments indicate that in certain of chemotaxis motion in a liquid environment, the interaction between cells and the surrounding fluid may substantially affect the behavior thereof ([16, 20]). In the style of [7, 28], we hence suppose that this interaction occurs not only through transport but possibly also through a buoyancy-driven feedback of sperm (egg) gametes to the fluid velocity. Accordingly, it leads to a refinement of (1.2) in the framework of chemotaxis–(Navier–)Stokes system

\[
\begin{aligned}
    n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n S(x, n, c) \nabla c) - \mu n^2, \\
    c_t + u \cdot \nabla c &= \Delta c - c + n, \\
    u_t + \kappa(u \cdot \nabla) u &= \Delta u - \nabla P + n \nabla \phi, \\
    \nabla \cdot u &= 0
\end{aligned}
\]  

(1.3)

for the unknown density of sperm (egg) gametes \( n \), the signal concentration \( c \), the fluid velocity \( u \) and the associated pressure \( P \) in the physical domain \( \Omega \subset \mathbb{R}^3 \). Here the evolution of velocity \( u \) is governed by the incompressible (Navier)-Stokes equations, in addition, it is driven by gametes through buoyant forces within a gravitational potential \( \phi \), \( \phi \in W^{2,\infty}(\Omega) \) and the chemotactic sensitivity tensor \( S(x, n, c) = (s_{ij}(x, n, c)) \in C^2(\Omega \times [0, \infty)^2), i, j \in \{1, 2, 3\} \), which reflects that the chemotactic migration may not necessarily be oriented along the gradient of the chemical signal, but may rather involve rotational flux components (see [22, sec. 4.2.1] or [30] for tensor-valued sensitivities in the chemotaxis system).
In view of mathematical analysis, the model (1.3) compounds the known difficulties in the study of the three-dimensional fluid dynamics with the typical intricacies in the study of chemotactic cross-diffusion reinforced by signal production. In fact, three-dimensional Navier–Stokes equations are yet lacking complete existence theory, particularly the global solvability in classes of suitably regular functions is yet left as an open problem except in the cases that the initial data are appropriately small (30). In addition, it is observed that when $S = S(x, \rho, c)$ is a tensor, the corresponding chemotaxis–fluid system loses the natural energy structure, which plays a key role in the analysis of the scalar-valued case (34 32 35 33). Despite these challenges, some comprehensive results on the global-boundedness and large time behavior of solutions are available in the literature (see [4 17 19 26 29 35 37] for example). Indeed, by a continuation argument, authors of [37] established the global classical solutions of (1.3) with $\kappa = 1, \mu = 0$ decaying to $(\bar{n}_0, \bar{\rho}_0, 0)$ exponentially with $\bar{n}_0 = \frac{1}{|\Omega|} \int_{\Omega} n_0(x) dx$ if $\|n_0\|_{L^3(\Omega)}, \|\nabla c_0\|_{L^3(\Omega)}$ and $\|u_0\|_{L^3(\Omega)}$ are small enough. In particular, for the 3D chemotaxis–Stokes variant of (1.3) with $r n - \mu n^2$ instead of $\mu n^2$ and $S = \chi$ in the $n$–equation, the existence of global bounded smooth solutions is proved for appropriately large $\mu > 0$ (26); while the corresponding two-dimensional Navier–Stokes variant thereof possesses a global bounded classical solution for arbitrary $\mu > 0$ (27). In addition, the latter two works also provide some results on the asymptotic decay of solutions when $r = 0$, which, in the light of results of [3 13], indeed seems to decay in time like $\frac{1}{t^{1/4}}$. Furthermore, in the very recent paper [35], Winkler showed that in the delicate three-dimensional setting, the Keller–Segel–Navier–Stokes system considered in [27] possesses at least one globally generalized solution, and that under an explicit condition on the size of $\mu$ this solution approach a spatially homogeneous equilibrium in their first two components.

From a biological point of view, it is more realistic to distinguish between eggs and sperms, and it thereby becomes possible to take into account that only spermatozoids will be affect by chemotactic attraction, whereas the eggs are governed by random diffusion, fluid transport and degradation upon contact with sperms during the coral fertilization process (8 9 15). In addition, the interaction of the gametes and the ambient fluid is not negligible. The gametes are assumed to be transported by the fluid, in turn, the motion of the latter is driven by gametes
through buoyant forces within a gravitational potential \( \phi \).

As an important step toward the comprehensive understanding of the coral fertilization process, we shall consider the large time behavior of the egg–sperm chemotaxis–fluid system. More precisely, this paper is concerned with the following Keller–Segel–Navier–Stokes system in the spatially three-dimensional setting

\[
\begin{align*}
\rho_t + u \cdot \nabla \rho &= \Delta \rho - \nabla \cdot (\rho \mathbf{S}(x, \rho, c) \nabla c) - \rho m, & (x, t) \in \Omega \times (0, T), \\
m_t + u \cdot \nabla m &= \Delta m - \rho m, & (x, t) \in \Omega \times (0, T), \\
c_t + u \cdot \nabla c &= \Delta c - c + m, & (x, t) \in \Omega \times (0, T), \\
u_t + (u \cdot \nabla)u &= \Delta u - \nabla P + (\rho + m) \nabla \phi, & (x, t) \in \Omega \times (0, T), \\
(\nabla \rho - \rho \mathbf{S}(x, \rho, c) \nabla c) \cdot \nu &= \nabla m \cdot \nu = \nabla c \cdot \nu = 0, & (x, t) \in \partial \Omega \times (0, T), \\
\rho(x, 0) &= \rho_0(x), m(x, 0) = m_0(x), c(x, 0) = c_0(x), u(x, 0) = u_0(x), & x \in \Omega,
\end{align*}
\]

where the sperm \( \rho \) chemotactically moves toward the higher concentration of the chemical \( c \) released by the egg \( m \), while the egg \( m \) is merely affected by random diffusion, fluid transport and degradation upon contact with the sperm, \( \mathbf{S} = \mathbf{S}(x, \rho, c) \) satisfies

\[
|\mathbf{S}(x, \rho, c)| \leq C_S \quad \text{for some } C_S > 0,
\]

and

\[
\begin{align*}
\rho_0 \in C^0(\overline{\Omega}), \ & \rho_0 \geq 0 \text{ and } \rho_0 \not\equiv 0, \\
m_0 \in C^0(\overline{\Omega}), \ & m_0 \geq 0 \text{ and } m_0 \not\equiv 0, \\
c_0 \in W^{1,\infty}(\Omega), \ & c_0 \geq 0 \text{ and } c_0 \not\equiv 0, \\
u_0 \in D(A^{\beta}) \text{ for all } \beta \in (\frac{3}{4}, 1),
\end{align*}
\]

where \( A \) denotes the realization of the Stokes operator in \( L^2(\Omega) \).

In the context of these assumptions, our main result can be stated as follows:

**Theorem 1.1.** Suppose that \( 1.5 \) hold and \( \int_\Omega \rho_0 > \int_\Omega m_0 \). Let \( p_0 \in (\frac{3}{2}, 3), \ q_0 \in (3, \frac{3p_0}{3-p_0}) \).

There exists \( \varepsilon > 0 \) such that for any initial data \( (\rho_0, m_0, c_0, u_0) \) fulfilling \( 1.6 \) as well as

\[
\|\rho_0 - \rho_\infty\|_{L^{p_0}(\Omega)} < \varepsilon, \quad \|m_0\|_{L^{q_0}(\Omega)} < \varepsilon, \quad \|c_0\|_{L^{\infty}(\Omega)} < \varepsilon, \quad \|u_0\|_{L^3(\Omega)} < \varepsilon,
\]

\( 1.4 \) admits a global classical solution \( (\rho, m, c, u, P) \). In particular, for any \( \alpha_1 \in (0, \min\{\lambda_1, \rho_\infty\}) \), \( \alpha_2 \in (0, \min\{\alpha_1, \lambda'_1, 1\}) \), there exist constants \( K_i, \ i = 1, 2, 3, 4 \), such that for all \( t \geq 1 \)

\[
\|m(\cdot, t)\|_{L^{\infty}(\Omega)} \leq K_1 e^{-\alpha_1 t},
\]

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\[\|\rho(\cdot, t) - \rho_\infty\|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha_1 t},\]
\[\|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq K_3 e^{-\alpha_2 t},\]
\[\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha_2 t}.\]

Here \(\lambda'_1\) is the first eigenvalue of \(A\), \(\lambda_1\) is the first nonzero eigenvalue of \(-\Delta\) on \(\Omega\) under the Neumann boundary condition, and \(\rho_\infty = \frac{1}{|\Omega|}(\int_\Omega \rho_0 - \int_\Omega m_0).\)

As for the case \(\int_\Omega \rho_0 < \int_\Omega m_0\), i.e., \(m_\infty = \frac{1}{|\Omega|}(\int_\Omega m_0 - \int_\Omega \rho_0) > 0\), we have

**Theorem 1.2.** Assume that (1.5) and \(\int_\Omega \rho_0 < \int_\Omega m_0\) hold, and let \(p_0 \in (2, 3)\), \(q_0 \in (3, 3 + \frac{3}{2(p_0 - 3)})\).
Then there exists \(\varepsilon > 0\) such that for any initial data \((\rho_0, m_0, c_0, u_0)\) fulfilling (1.6) as well as
\[\|\rho_0\|_{L^{p_0}(\Omega)} \leq \varepsilon, \quad \|m_0 - m_\infty\|_{L^{q_0}(\Omega)} \leq \varepsilon, \quad \|\nabla c_0\|_{L^3(\Omega)} \leq \varepsilon, \quad \|u_0\|_{L^3(\Omega)} \leq \varepsilon,\]
(1.4) admits a global classical solution \((\rho, m, c, u, P)\). Furthermore, for any \(\alpha_1 \in (0, \min\{\lambda_1, m_\infty, 1\})\), \(\alpha_2 \in (0, \min\{\alpha_1, \lambda'_1\})\), there exist constants \(K_i > 0\), \(i = 1, 2, 3, 4\), such that
\[\|m(\cdot, t) - m_\infty\|_{L^\infty(\Omega)} \leq K_1 e^{-\alpha_1 t},\]
\[\|\rho(\cdot, t)\|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha_1 t},\]
\[\|c(\cdot, t) - m_\infty\|_{W^{1,\infty}(\Omega)} \leq K_3 e^{-\alpha_2 t},\]
\[\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha_2 t}.\]

**Remark 1.1.** In our results, we have excluded the case \(\int_\Omega \rho_0 = \int_\Omega m_0\). Indeed, in the light of results of [3, 13], algebraical decay rather than exponential decay of the solutions is expected in this case.

It is noted that a similar result was proved in [18] for the three-dimensional Stoke variant of (1.4). However, as is well-known, the nonlinear convection \((u \cdot \nabla)u\) in the three-dimensional Navier–Stokes equation may enforce the spontaneous emergence of singularities in the sense of blow-up with respect to the norm in \(L^\infty(\Omega)\), we thereby subject the study of classical solutions of (1.4) to small initial data by an essentially one-step contradiction argument, unlike that in the two-dimensional case (9). Moreover, in comparison with the chemotaxis–fluid system considered in [4, 37], due to
\[\|e^{t\Delta} \omega\|_{L^p(\Omega)} \leq C_1 \left(1 + t^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|\omega\|_{L^3(\Omega)}\]
for all $\omega \in L^q(\Omega)$ with $\int_\Omega \omega = 0$, $-\rho m$ in the first equation of (1.4) gives rise to some difficulty in mathematical analysis despite its dissipative feature. Indeed, the core of this argument is to verify that the interval $(0, T)$ on which solutions enjoy some exponential decay properties can be extended to $(0, \infty)$, which accordingly requires an appropriate combination of the mass conservation of $\rho(x, t) - m(x, t)$ with the $L^p - L^q$ estimates for the Neumann heat semigroup.

The plan of this paper is as follows: In Section 2, we give a local existence result and some useful estimates. In Section 3, in the case of $S$ vanishing on the boundary, we give the proof of the main results according to either $\int_\Omega \rho_0 > \int_\Omega m_0$ or $\int_\Omega \rho_0 < \int_\Omega m_0$. In the last section, on the basis of certain a priori estimates, the proof of our main results for the general $S$ satisfying (1.5) is realized via an approximation procedure.

2 Preliminaries

In this section, we provide some preliminary results that will be used in the subsequent sections. We begin by recalling the important $L^p - L^q$ estimates for the Neumann heat semigroup on bounded domains ([31]).

Lemma 2.1. (Lemma 1.3 of [31]) Let $(e^{t\Delta})_{t>0}$ denote the Neumann heat semigroup in the domain $\Omega$ and $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in $\Omega \subset \mathbb{R}^N$ under the Neumann boundary condition. There exists $C_i$, $i = 1, 2, 3, 4$, such that for all $t > 0$,

(i) If $1 \leq q \leq p \leq \infty$, then for all $\omega \in L^q(\Omega)$ with $\int_\Omega \omega = 0$,

$$\|e^{t\Delta} \omega\|_{L^p(\Omega)} \leq C_1 \left( 1 + t^{-\frac{N}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \right) e^{-\lambda_1 t} \|\omega\|_{L^q(\Omega)};$$

(ii) If $1 \leq q \leq p \leq \infty$, then for all $\omega \in L^q(\Omega)$,

$$\|\nabla e^{t\Delta} \omega\|_{L^p(\Omega)} \leq C_2 \left( 1 + t^{-\frac{1}{2} - \frac{N}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \right) e^{-\lambda_1 t} \|\omega\|_{L^q(\Omega)};$$

(iii) If $2 \leq q \leq p < \infty$, then for all $\omega \in W^{1,q}(\Omega)$,

$$\|\nabla e^{t\Delta} \omega\|_{L^p(\Omega)} \leq C_3 \left( 1 + t^{-\frac{N}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \right) e^{-\lambda_1 t} \|\nabla \omega\|_{L^q(\Omega)};$$

(iv) If $1 \leq q \leq p \leq \infty$ or $1 < q < \infty$ and $p = \infty$, then for all $\omega \in (L^q(\Omega))^N$,

$$\|e^{t\Delta} \nabla \cdot \omega\|_{L^p(\Omega)} \leq C_4 \left( 1 + t^{-\frac{1}{2} - \frac{N}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \right) e^{-\lambda_1 t} \|\omega\|_{L^q(\Omega)}.$$
Next we introduce the Stokes operator and recall estimates for the corresponding semigroup. With $L^p_\sigma(\Omega) := \{ \varphi \in L^p(\Omega) | \nabla \cdot \varphi = 0 \}$ and $\mathcal{P}$ representing the Helmholtz projection of $L^p(\Omega)$ onto $L^p_\sigma(\Omega)$, the Stokes operator on $L^p_\sigma(\Omega)$ is defined as $A_p = -\mathcal{P}\Delta$ with domain $D(A_p) := W^{2,p}(\Omega) \cap W^{2,p}_0(\Omega) \cap L^p_\sigma(\Omega)$. Since $A_{p_1}$ and $A_{p_2}$ coincide on the intersection of their domains for $p_1, p_2 \in (1, \infty)$, we will drop the index in the following.

**Lemma 2.2.** (Lemma 2.3 of [11]) The Stokes operator $A$ generates the analytic semigroup $(e^{-tA})_{t \geq 0}$ in $L^p_\sigma(\Omega)$. Its spectrum satisfies $\lambda_1 = \inf \text{Re} \sigma(A) > 0$ and we fix $\mu \in (0, \lambda_1)$. For any such $\mu$, we have

(i) For any $p \in (1, \infty)$ and $\gamma \geq 0$, there is $C_5(p, \gamma) > 0$ such that for all $\phi \in L^p_\sigma(\Omega)$,

$$\|A^\gamma e^{-tA} \phi\|_{L^p(\Omega)} \leq C_5(p, \gamma) t^{-\gamma} e^{-\mu t} \|\phi\|_{L^p(\Omega)};$$

(ii) For any $p, q$ with $1 < p \leq q < \infty$, there is $C_6(p, q) > 0$ such that for all $\phi \in L^p_\sigma(\Omega)$,

$$\|e^{-tA} \phi\|_{L^q(\Omega)} \leq C_6(p, q) t^{-\frac{N}{p} \left( \frac{1}{p} - \frac{1}{q} \right)} e^{-\mu t} \|\phi\|_{L^p(\Omega)};$$

(iii) For any $p, q$ with $1 < p \leq q < \infty$, there is $C_7(p, q) > 0$ such that for all $\phi \in L^p_\sigma(\Omega)$,

$$\|\nabla e^{-tA} \phi\|_{L^q(\Omega)} \leq C_7(p, q) t^{-\frac{1}{2} \left( \frac{N}{p} - \frac{N}{q} \right)} e^{-\mu t} \|\phi\|_{L^p(\Omega)};$$

(iv) If $\gamma \geq 0$ and $1 < p < q < \infty$ satisfy $2\gamma - \frac{N}{q} \geq 1 - \frac{N}{p}$, there is $C_8(\gamma, p, q) > 0$ such that for all $\phi \in D(A^\gamma_q)$,

$$\|\phi\|_{W^{1,p}(\Omega)} \leq C_8(\gamma, p, q) \|A^\gamma \phi\|_{L^q(\Omega)}.$$

**Lemma 2.3.** (Theorem 1 and Theorem 2 of [17]) The Helmholtz projection $\mathcal{P}$ defines a bounded linear operator $\mathcal{P} : L^p(\Omega) \to L^p_\sigma(\Omega)$; in particular, for any $p \in (1, \infty)$, there exists $C_9(p) > 0$ such that $\|\mathcal{P} \varphi\|_{L^p(\Omega)} \leq C_9(p) \|\varphi\|_{L^p(\Omega)}$ for every $\varphi \in L^p(\omega)$.

The following elementary lemma provides some useful information on both the short-time and the large-time behavior of certain integrals, which is used in the proof of the main results.

**Lemma 2.4.** (Lemma 1.2 of [17]) Let $\alpha \in (0, 1), \beta \in (0, 1)$, and $\gamma, \delta$ be positive constants such that $\gamma \neq \delta$. Then there exists $C_{10}(\alpha, \beta, \gamma, \delta) > 0$ such that

$$\int_0^t (1 + s^{-\alpha})(1 + (t - s)^{-\beta}) e^{-\gamma s} e^{-\delta(t-s)} ds \leq C_{10}(\alpha, \beta, \gamma, \delta) (1 + t^{\min\{0, 1-\alpha-\beta\}}) e^{-\min\{\gamma, \delta\} t}.$$
Next we recall the result on the local existence of classical solutions, which can be proved
by a straightforward adaptation of well-known fixed point argument (see [32] for example).

**Lemma 2.5.** Suppose that \( (1.5), (1.6) \) and
\[
S(x, \rho, c) = 0, \quad (x, \rho, c) \in \partial \Omega \times [0, \infty) \times [0, \infty)
\]
hold. Then there exist \( T_{\text{max}} \in (0, \infty) \) and a classical solution \( (\rho, m, c, u, P) \) of \( (1.4) \) on \( (0, T_{\text{max}}) \).

Moreover, \( \rho, m, c \) are nonnegative in \( \Omega \times (0, T_{\text{max}}) \), and if \( T_{\text{max}} < \infty \), then for \( \beta \in (\frac{2}{3}, 1) \),
\[
\|\rho(\cdot, t)\|_{L^\infty(\Omega)} + \|m(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|A^\beta u(\cdot, t)\|_{L^2(\Omega)} \to \infty \quad \text{as} \quad t \to T_{\text{max}}.
\]

This solution is unique, up to addition of constants to \( P \).

The following elementary properties of the solutions in Lemma 2.5 are immediate consequences of the integration of the first and second equations in \( (1.4) \), as well as an application of the maximum principle to the second and third equations.

**Lemma 2.6.** Suppose that \( (1.5), (1.6) \) and \( (2.1) \) hold. Then for all \( t \in (0, T_{\text{max}}) \), the solution
of \( (1.4) \) from Lemma 2.5 satisfies
\[
\|\rho(\cdot, t)\|_{L^1(\Omega)} \leq \|\rho_0\|_{L^1(\Omega)}, \quad \|m(\cdot, t)\|_{L^1(\Omega)} \leq \|m_0\|_{L^1(\Omega)},
\]
\[
\int_0^t \|\rho(\cdot, s)m(\cdot, s)\|_{L^1(\Omega)} ds \leq \min\{\|\rho_0\|_{L^1(\Omega)}, \|m_0\|_{L^1(\Omega)}\},
\]
\[
\|\rho(\cdot, t)\|_{L^1(\Omega)} - \|m(\cdot, t)\|_{L^1(\Omega)} = \|\rho_0\|_{L^1(\Omega)} - \|m_0\|_{L^1(\Omega)},
\]
\[
\|m(\cdot, t)\|_{L^2(\Omega)}^2 + 2 \int_0^t \|\nabla m(\cdot, s)\|_{L^2(\Omega)}^2 ds \leq \|m_0\|_{L^2(\Omega)}^2,
\]
\[
\|m(\cdot, t)\|_{L^\infty(\Omega)} \leq \|m_0\|_{L^\infty(\Omega)},
\]
\[
\|c(\cdot, t)\|_{L^\infty(\Omega)} \leq \max\{\|m_0\|_{L^\infty(\Omega)}, \|c_0\|_{L^\infty(\Omega)}\}.
\]

### 3 Proof of Theorems for \( S = 0 \) on \( \partial \Omega \)

In this section, we give the proofs of Theorem 1.1 and Theorem 1.2 when \( S = 0 \) on \( \partial \Omega \),
respectively, i.e. the proof of Proposition 3.1 and Proposition 3.2 below, under which the boundary condition for \( \rho \) in \( (1.4) \) actually reduces to the homogeneous Neumann condition
\[
\nabla \rho \cdot \nu = 0.
\]

In the case \( \int_\Omega \rho_0 > \int_\Omega m_0 \), i.e., \( \rho_{\infty} > 0, m_{\infty} = 0 \), we have
Proposition 3.1. Suppose that $|\textbf{L}|$ hold and $\int_\Omega \rho_0 > \int_\Omega m_0$. Let $p_0 \in (\frac{3}{2}, 3)$, $q_0 \in (3, \frac{3p_0}{3-p_0})$. There exists $\varepsilon > 0$ such that for any initial data $(\rho_0, m_0, c_0, u_0)$ fulfilling $(|\textbf{L}|)$ as well as

$$\|\rho_0 - \rho_\infty\|_{L^p(\Omega)} < \varepsilon, \quad \|m_0\|_{L^q(\Omega)} < \varepsilon, \quad \|c_0\|_{L^\infty(\Omega)} < \varepsilon, \quad \|u_0\|_{L^4(\Omega)} < \varepsilon,$$

$(|\textbf{L}|)$ admits a global classical solution $(\rho, m, c, u, P)$. In particular, for any $\alpha_1 \in (0, \min\{\lambda_1, \rho_\infty\})$, $\alpha_2 \in (0, \min\{\alpha_1, \lambda_1', 1\})$, there exist constants $K_i$, $i = 1, 2, 3, 4$, such that for all $t \geq 1$

$$\|m(\cdot, t)\|_{L^\infty(\Omega)} \leq K_1 e^{-\alpha_1 t}, \quad (3.1)$$

$$\|\rho(\cdot, t) - \rho_\infty\|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha_1 t}, \quad (3.2)$$

$$\|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq K_3 e^{-\alpha_2 t}, \quad (3.3)$$

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha_2 t}. \quad (3.4)$$

Proposition 3.1 is the consequence of the following lemmas. In the proof thereof the constants $C_i > 0$, $i = 1, \ldots, 10$, refer to those in Lemma 2.1–2.4, respectively. The following verifiable observations will warrant the choice in these lemmas.

Lemma 3.1. Under the assumptions of Proposition 3.1 and $\sigma = \int_0^\infty (1 + s^{-\frac{2}{\sigma}}) e^{-\alpha_1 s} ds$, there exist $M_1 > 0, M_2 > 0$ and $\varepsilon \in (0, 1)$ such that

$$C_2 + 2C_2C_{10}e^{(1+c_1+c_1|\Omega|^{\frac{1}{p_0} - \frac{1}{q_0}})\sigma} \leq \frac{M_2}{4}, \quad (3.5)$$

$$C_4C_{10}C_2M_2(e^{(1+c_1+c_1|\Omega|^{\frac{1}{p_0} - \frac{1}{q_0}})\sigma} + \rho_\infty|\Omega|^{\frac{1}{q_0}}) \leq \frac{M_1}{8}, \quad (3.6)$$

$$C_6 + 2C_6C_9C_{10}\|\nabla \phi\|_{L^\infty(\Omega)}(M_1 + C_1 + C_1|\Omega|^{\frac{1}{p_0} - \frac{1}{q_0}} + 4e^{(1+c_1+c_1|\Omega|^{\frac{1}{p_0} - \frac{1}{q_0}})\sigma}) < \frac{M_3}{4}, \quad (3.7)$$

$$C_7 + 2C_7C_9C_{10}\|\nabla \phi\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{p_0} - \frac{1}{q_0}}(M_1 + C_1 + C_1|\Omega|^{\frac{1}{p_0} - \frac{1}{q_0}} + 4e^{(1+c_1+c_1|\Omega|^{\frac{1}{p_0} - \frac{1}{q_0}})\sigma}) < \frac{M_4}{4}, \quad (3.8)$$

$$3C_{10}C_4C_2(M_1 + C_1 + C_1|\Omega|^{\frac{1}{p_0} - \frac{1}{q_0}})M_2\varepsilon \leq \frac{M_1}{8}, \quad (3.9)$$

$$3C_{10}C_4(M_1 + C_1 + C_1|\Omega|^{\frac{1}{p_0} - \frac{1}{q_0}})M_3\varepsilon \leq \frac{M_1}{8}, \quad (3.10)$$

$$12C_2C_{10}M_3\varepsilon < 1, \quad (3.11)$$

$$12C_7C_9C_{10}M_3\varepsilon \leq 1, \quad (3.12)$$

$$12C_6C_9C_{10}M_4\varepsilon \leq 1. \quad (3.13)$$
Then $T > 0$ is well-defined by Lemma 2.5 and (1.6). Now we claim that $T = T_{\text{max}} = \infty$ if $\varepsilon$ is sufficiently small. To this end, by the contradiction argument, it only needs to verify that all of the estimates mentioned in (3.14) also hold with even smaller coefficients on the right-side thereof, which mainly rely on $L^p - L^q$ estimates for the Neumann heat semigroup and the fact that the classical solution on $(0, T_{\text{max}})$ can be written as

$$
(\rho - m)(\cdot, t) = e^{t\Delta}(\rho_0 - m_0) - \int_0^t e^{(t-s)\Delta}(\nabla \cdot (\rho \mathcal{S}(x, \rho, c) \nabla c) + u \cdot \nabla (\rho - m))(\cdot, s) ds,
$$

(3.15)

$$
m(\cdot, t) = e^{t\Delta}m_0 - \int_0^t e^{(t-s)\Delta}(\rho m + u \cdot \nabla m)(\cdot, s) ds,
$$

(3.16)

$$
c(\cdot, t) = e^{t(\Delta - 1)}c_0 + \int_0^t e^{(t-s)(\Delta - 1)}(m - u \cdot \nabla c)(\cdot, s) ds,
$$

(3.17)

$$
u(\cdot, t) = e^{-t\Delta}u_0 + \int_0^t e^{-(t-s)\Delta} \mathcal{P}((\rho + m) \nabla \phi - (u \cdot \nabla)u)(\cdot, s) ds
$$

(3.18)

for all $t \in (0, T_{\text{max}})$ according to the variation-of-constants formula.

Although the proof of Lemma 3.2 and Lemma 3.3 below is very similar to that of Lemma 3.11 and Lemma 3.12 in (18), respectively, we give their proofs for the convenience of the interested reader.

**Lemma 3.2.** Under the assumptions of Proposition 3.1, for all $t \in (0, T)$ and $\theta \in [q_0, \infty]$, there exists constant $M_5 > 0$ such that

$$
\| (\rho - m)(\cdot, t) - \rho_\infty \|_{L^q(\Omega)} \leq M_5 \varepsilon (1 + t^{-\frac{3}{2}(\frac{1}{p_0} - \frac{1}{\theta})}) e^{-\alpha_1 t}.
$$

**Proof.** Due to $e^{t\Delta} \rho_\infty = \rho_\infty$ and $\int_\Omega (\rho_0 - m_0 - \rho_\infty) = 0$, the definition of $T$ and Lemma 2.1(i) show that for all $t \in (0, T)$ and $\theta \in [q_0, \infty],

$$
\| (\rho - m)(\cdot, t) - \rho_\infty \|_{L^q(\Omega)}
$$

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Lemma 3.3. Under the assumptions of Proposition 3.1, for any $k > 1$,
\[
\|m(\cdot, t)\|_{L^k(\Omega)} \leq M_6\|m_0\|_{L^k(\Omega)}e^{-\rho_\infty t} \quad \text{for all } t \in (0, T)
\]
(3.19)
with $\sigma = \int_0^\infty (1 + s^{-\frac{2}{2\mu_0}})e^{-\alpha_1 s}ds$ and $M_6 = e^{M_5\sigma \epsilon}$.

Proof. Testing the first equation in (1.1) with $m^k$ ($k > 1$) and integrating by parts, it holds that
\[
\frac{d}{dt} \int_\Omega m^k \leq -k \int_\Omega \rho m^k \quad \text{on } (0, T).
\]
In view of $-\rho \leq |\rho - m - \rho_\infty| - m - \rho_\infty \leq -\rho_\infty + |\rho - m - \rho_\infty|$, Lemma 3.2 yields
\[
\frac{d}{dt} \int_\Omega m^k \leq -k \rho_\infty \int_\Omega m^k + k \int_\Omega m^k|\rho - m - \rho_\infty|
\leq -k \rho_\infty \int_\Omega m^k + k \|\rho - m - \rho_\infty\|_{L^\infty(\Omega)} \int_\Omega m^k
\leq -k \rho_\infty \int_\Omega m^k + k M_5 \epsilon(1 + t^{-\frac{3}{2\mu_0}})e^{-\alpha_1 t} \int_\Omega m^k
\]
and thus
\[
\int_\Omega m^k \leq \int_\Omega m_0^k \exp\{-k \rho_\infty t + k M_5 \epsilon \int_0^t (1 + s^{-\frac{3}{2\mu_0}})e^{-\alpha_1 s}ds\} \leq \|m_0\|_{L^k(\Omega)}^k e^{(M_5\sigma \epsilon - \rho_\infty t)}
\]
from which (3.19) follows immediately.

Lemma 3.4. Under the assumptions of Proposition 3.1, we have
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{M_3}{2} \epsilon \left(1 + t^{-\frac{1}{2} + \frac{3}{2\mu_0}}\right)e^{-\alpha_2 t} \quad \text{for all } t \in (0, T).
\]

Proof. For $\alpha_2 < \lambda'_1$, we fix $\mu \in (\alpha_2, \lambda'_1)$. According to (3.18), Lemma 2.2(ii) and Lemma 2.3, we infer that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)}
\leq C_6 \epsilon^{-\frac{\lambda'_1}{2\lambda_0}} e^{-\mu t} \|u_0\|_{L^\lambda(\Omega)} + \int_0^t e^{-(t-s)A}\|P((\rho + m)\nabla \phi - (u \cdot \nabla)u)(\cdot, s)\|_{L^\infty(\Omega)}ds
\leq C_6 \epsilon^{-\frac{\lambda'_1}{2\lambda_0}} e^{-\mu t} \|u_0\|_{L^\lambda(\Omega)} + C_6 \int_0^t e^{-\mu(t-s)}\|P((\rho + m - \bar{\rho} + m)\nabla \phi)(\cdot, s)\|_{L^\infty(\Omega)}ds
\]
\[ + C_6 \int_0^t e^{-\mu(t-s)} \| \mathcal{P}((u \cdot \nabla)u)(\cdot, s) \|_{L^\infty(\Omega)} ds \]

\[ \leq C_6 t^{-\frac{1}{2} + \frac{3}{2q_0}} e^{-\mu t} \| u_0 \|_{L^1(\Omega)} + C_6 C_9 \| \nabla \phi \|_{L^\infty(\Omega)} \int_0^t e^{-\mu(t-s)} \| (\rho + m - \rho + m)(\cdot, s) \|_{L^\infty(\Omega)} ds \]

\[ + C_6 C_9 \int_0^t (t-s)^{-\frac{1}{2}} e^{-\mu(t-s)} \| (u \cdot \nabla)u(\cdot, s) \|_{\frac{3}{2} + \frac{1}{q_0}} ds \]

\[ =: C_6 t^{-\frac{1}{2} + \frac{3}{2q_0}} e^{-\mu t} \| u_0 \|_{L^1(\Omega)} + J_1 + J_2, \]

where \( \mathcal{P}(\rho + m \nabla \phi) = \rho + m \mathcal{P}(\nabla \phi) = 0 \) is used.

Due to \( \alpha_1 < \rho_\infty \), the application of Lemma 3.2 and Lemma 3.3 shows that

\[ J_1 \leq C_6 C_9 \| \nabla \phi \|_{L^\infty(\Omega)} \int_0^t e^{-\mu(t-s)} \| (\rho + m - \rho + m)(\cdot, s) + 2(m - \bar{m})(\cdot, s) \|_{L^\infty(\Omega)} ds \]

\[ \leq C_6 C_9 \| \nabla \phi \|_{L^\infty(\Omega)} \int_0^t e^{-\mu(t-s)} \| (\rho + m - \rho_\infty)(\cdot, s) \|_{L^\infty(\Omega)} + 2 \| (m - \bar{m})(\cdot, s) \|_{L^\infty(\Omega)} ds \]

\[ \leq C_6 C_9 \| \nabla \phi \|_{L^\infty(\Omega)} M_7^2 \varepsilon \int_0^t e^{-\mu(t-s)} (1 + s^{-\frac{3}{2q_0} + \frac{\varepsilon}{q_0}}) e^{-\alpha_1 s} ds \]

with \( M_7^2 = M_5 + 4 e^{M_5 \varepsilon} \).

On the other hand, by the H"older inequality and definition of \( T \), we have

\[ J_2 \leq C_6 C_9 \int_0^t (t-s)^{-\frac{1}{2}} e^{-\mu(t-s)} \| u(\cdot, s) \|_{L^\infty(\Omega)} \| \nabla u(\cdot, s) \|_{L^1(\Omega)} ds \]

\[ \leq 3 C_6 C_9 M_3 M_4 \varepsilon^2 \int_0^t (t-s)^{-\frac{1}{2}} e^{-\mu(t-s)} (1 + s^{-\frac{3}{2q_0} + \frac{\varepsilon}{q_0}}) e^{-2\alpha_2 s} ds. \]

Now, plugging (3.21), (3.22) into (3.20) and applying Lemma 2.4, we end up with

\[ \| u(\cdot, t) \|_{L^\infty(\Omega)} \]

\[ \leq C_6 t^{-\frac{1}{2} + \frac{3}{2q_0}} e^{-\mu t} \| u_0 \|_{L^3(\Omega)} + C_6 C_9 C_{10} \| \nabla \phi \|_{L^\infty(\Omega)} M_7^2 \varepsilon (1 + t^{\min(0, 1 - \frac{3}{2q_0} + \frac{\varepsilon}{q_0})}) e^{-\alpha_2 t} \]

\[ + 3 C_6 C_9 C_{10} M_3 M_4 \varepsilon^2 (1 + t^{-\frac{1}{2} + \frac{3}{2q_0}}) e^{-\alpha_2 t} \]

\[ \leq C_6 t^{-\frac{1}{2} + \frac{3}{2q_0}} e^{-\mu t} \varepsilon + 2 C_6 C_9 C_{10} \| \nabla \phi \|_{L^\infty(\Omega)} M_7^2 e^{-\alpha_2 t} + 3 C_6 C_9 C_{10} M_3 M_4 \varepsilon^2 (1 + t^{-\frac{1}{2} + \frac{3}{2q_0}}) e^{-\alpha_2 t} \]

\[ \leq \frac{M_3}{2} \varepsilon (1 + t^{-\frac{1}{2} + \frac{3}{2q_0}}) e^{-\alpha_2 t}, \]

where (3.7), (3.13) and the fact that \( \frac{3}{2} \left( \frac{1}{p_0} - \frac{1}{q_0} \right) < 1 \) are used.

The estimate for the gradient is also preserved.

**Lemma 3.5.** Under the assumptions of Proposition 3.1, we have

\[ \| \nabla u(\cdot, t) \|_{L^1(\Omega)} \leq \frac{M_4}{2} \varepsilon (1 + t^{-\frac{1}{2}}) e^{-\alpha_2 t} \] for all \( t \in (0, T) \).
Proof. According to (3.18), we have
\[ \nabla u(\cdot, t) = \nabla e^{-tA}u_0 + \int_0^t \nabla e^{-(t-s)A}(\mathcal{P}((\rho + m) \nabla \phi) - \mathcal{P}((u \cdot \nabla)u))(\cdot, s)ds. \]
Applying Lemma 2.2(iii), Lemma 2.3 and the Hölder inequality, we arrive at
\[
\|\nabla u(\cdot, t)\|_{L^3(\Omega)} \\
\leq C_7 t^{-\frac{1}{4}} e^{-\mu t} \|u_0\|_{L^3(\Omega)} + \int_0^t \|\nabla e^{-(t-s)A}\mathcal{P}((\rho + m) \nabla \phi - (u \cdot \nabla)u)(\cdot, s)\|_{L^3(\Omega)} ds \\
\leq C_7 t^{-\frac{1}{4}} e^{-\mu t} \varepsilon + C_7 \int_0^t (t-s)^{-\frac{1}{2}} e^{-\mu (t-s)} \|\mathcal{P}((\rho + m - \rho + m) \nabla \phi)(\cdot, s)\|_{L^3(\Omega)} ds \\
+ C_7 \int_0^t (t-s)^{-\frac{1}{2}} \frac{1}{\varepsilon} e^{-\mu (t-s)} \|\mathcal{P}((u \cdot \nabla)u)(\cdot, s)\|_{L^3(\Omega)} ds \tag{3.23}
\]
Due to \( C_7 C_9 \|\nabla \phi\|_{L^\infty(\Omega)} \|\Omega\|^{\frac{1}{4}} \frac{1}{\varepsilon^2} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\mu (t-s)} \|(\rho - m - \rho - m)(\cdot, s)\|_{L^3(\Omega)} ds \)
\[
\leq C_7 C_9 \|\nabla \phi\|_{L^\infty(\Omega)} \|\Omega\|^{\frac{1}{4}} \frac{1}{\varepsilon^2} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\mu (t-s)} \|(\rho - \rho - \rho_{\infty})(\cdot, s)\|_{L^3(\Omega)} + 2 \|(m - m)(\cdot, s)\|_{L^3(\Omega)} ds \\
\leq C_7 C_9 \|\nabla \phi\|_{L^\infty(\Omega)} \|\Omega\|^{\frac{1}{4}} \frac{1}{\varepsilon^2} M_\varepsilon^2 \int_0^t e^{-\mu (t-s)} (1 + s^{-\frac{1}{4}} \frac{1}{\varepsilon^2}) (t-s)^{-\frac{1}{2}} e^{-\alpha_1 s} ds. \tag{3.24}
\]
On the other hand, from the Hölder inequality and definition of \( T \), it follows that
\[
J_2 \leq 3 C_7 C_9 C_{10} M_4 \varepsilon (t-s)^{-\frac{1}{4}} \frac{1}{\varepsilon^2} e^{-\mu (t-s)} (1 + s^{-\frac{1}{4}} \frac{1}{\varepsilon^2}) e^{-\alpha_2 s} ds \tag{3.25}
\]
Therefore, inserting (3.25), (3.24) into (3.23) and applying Lemma 2.4, we get
\[
\|\nabla u(\cdot, t)\|_{L^3(\Omega)} \\
\leq C_7 t^{-\frac{1}{4}} e^{-\mu t} \varepsilon + C_7 C_9 C_{10} \|\nabla \phi\|_{L^\infty(\Omega)} \|\Omega\|^{\frac{1}{4}} \frac{1}{\varepsilon^2} M_\varepsilon^2 (1 + t^{-\frac{1}{4}} \frac{1}{\varepsilon^2}) e^{-\alpha_2 t} \\
+ 3 C_7 C_9 C_{10} M_4 \varepsilon^2 (1 + t^{-\frac{1}{4}} \frac{1}{\varepsilon^2}) e^{-\alpha_2 t} \\
\leq C_7 t^{-\frac{1}{4}} e^{-\mu t} \varepsilon + 2 C_7 C_9 C_{10} \|\nabla \phi\|_{L^\infty(\Omega)} \|\Omega\|^{\frac{1}{4}} \frac{1}{\varepsilon^2} M_\varepsilon^2 e^{-\alpha_2 t} + 3 C_7 C_9 C_{10} M_4 \varepsilon^2 (1 + t^{-\frac{1}{4}} \frac{1}{\varepsilon^2}) e^{-\alpha_2 t} \\
\leq \frac{M_4}{2} \varepsilon (1 + t^{-\frac{1}{4}} \frac{1}{\varepsilon^2}) e^{-\alpha_2 t},
\]
Lemma 3.6. Under the assumptions of Proposition 3.1, we have
\[ \| \nabla c(\cdot, t) \|_{L^\infty(\Omega)} \leq \frac{M_2}{2} \varepsilon (1 + t^{-\frac{1}{q}}) e^{-\alpha_1 t} \text{ for all } t \in (0, T). \]

Proof. By (3.17) and Lemma 2.1(ii), we have
\[ \| \nabla c(\cdot, t) \|_{L^\infty(\Omega)} \leq \| e^{\tau (\Delta - 1)} \nabla c_0 \|_{L^\infty(\Omega)} + \int_0^t \| e^{(t-s)(\Delta - 1)} (m - u \cdot \nabla c)(\cdot, s) \|_{L^\infty(\Omega)} ds \]
\[ \leq C_2 (1 + t^{-\frac{1}{q}}) e^{-(\lambda_1 + 1)t} \| c_0 \|_{L^\infty(\Omega)} + \int_0^t \| e^{(t-s)(\Delta - 1)} m(\cdot, s) \|_{L^\infty(\Omega)} ds \]
\[ + \int_0^t \| e^{(t-s)(\Delta - 1)} u \cdot \nabla c(\cdot, s) \|_{L^\infty(\Omega)} ds. \] (3.26)

Now we estimate the last two integrals on the right-side of the above inequality. From Lemma 2.1(ii), Lemma 2.4, Lemma 3.3 with \( k = q_0 \) and the fact that \( q_0 > 3 \), it follows that
\[ \int_0^t \| e^{(t-s)(\Delta - 1)} m \|_{L^\infty(\Omega)} ds \leq C_2 \int_0^t (1 + (t-s)^{-\frac{1}{q} - \frac{3}{2q_0}}) e^{-(\lambda_1 + 1)(t-s)} (m \|_{L^{q_0}(\Omega)} ds \] (3.27)
\[ \leq C_2 M_6 \varepsilon \int_0^t (1 + (t-s)^{-\frac{1}{q} - \frac{3}{2q_0}}) e^{-(\lambda_1 + 1)(t-s)} e^{-\rho \varepsilon s} ds \]
\[ \leq C_2 C_{10} M_6 (1 + t^{\min(0, 1) - \frac{3}{2q_0}}) \varepsilon e^{-\alpha_1 t} \]
\[ \leq 2C_2 C_{10} M_6 \varepsilon e^{-\alpha_1 t}. \]

On the other hand, by Lemma 2.1(ii), Lemma 2.4, Lemma 3.3 and the definition of \( T \), we obtain
\[ \int_0^t \| e^{(t-s)(\Delta - 1)} u \cdot \nabla c \|_{L^\infty(\Omega)} ds \]
\[ \leq C_2 \int_0^t (1 + (t-s)^{-\frac{1}{q} - \frac{3}{2q_0}}) e^{-(\lambda_1 + 1)(t-s)} \| u \cdot \nabla c \|_{L^{q_0}(\Omega)} ds \] (3.28)
\[ \leq C_2 M_3 M_2 \varepsilon^2 \int_0^t (1 + (t-s)^{-\frac{1}{q} - \frac{3}{2q_0}}) e^{-(\lambda_1 + 1)(t-s)} (1 + s^{-\frac{1}{q} + \frac{3}{2q_0}})(1 + s^{-\frac{1}{q} + \frac{3}{2q_0}}) e^{-(\alpha_1 + \alpha_2)s} ds \]
\[ \leq 3C_2 M_3 M_2 \varepsilon^2 \int_0^t e^{-(\lambda_1 + 1)(t-s)} e^{-(\alpha_1 + \alpha_2)s} (1 + (t-s)^{-\frac{1}{q} - \frac{3}{2q_0}})(1 + s^{-1 + \frac{3}{2q_0}}) ds \]
\[ \leq 3C_2 C_{10} M_2 M_3 \varepsilon^2 (1 + t^{-\frac{1}{q}}) e^{-\alpha_1 t}. \]

From (3.26)–(3.28), it follows that
\[ \| \nabla c \|_{L^\infty(\Omega)} \leq (C_2 + 2C_2 C_{10} M_6 + 3C_2 C_{10} M_2 M_3 \varepsilon) (1 + t^{-\frac{1}{q}}) e^{-\alpha_1 t} \]
Lemma 3.7. Under the assumptions of Proposition 3.1, for all \( \theta \in [\theta_0, \infty] \) and \( t \in (0, T) \),
\[
\|(\rho - m)(\cdot, t) - e^t \Delta (\rho_0 - m_0)\|_{L^q(\Omega)} \leq \frac{M_1}{2} \varepsilon (1 + t^{-\frac{1}{2}} (\frac{1}{\theta_0} - \frac{1}{\varepsilon})) e^{-\alpha_1 t},
\]
due to the choice of \( M_2, M_3 \) and \( \varepsilon \) in (3.5) and (3.11), and thereby completes the proof.

Proof. According to (3.15), Lemma 2.1(iv), we have
\[
\|(\rho - m)(\cdot, t) - e^t \Delta (\rho_0 - m_0)\|_{L^q(\Omega)} \leq \frac{M_1}{2} \varepsilon (1 + t^{-\frac{1}{2}} (\frac{1}{\theta_0} - \frac{1}{\varepsilon})) e^{-\alpha_1 t}.
\]

Now we need to estimate \( I_1 \) and \( I_2 \). Firstly, from Lemma 3.2 and Lemma 3.3, we obtain
\[
\|(\rho(\cdot, s))\|_{L^q(\Omega)} \leq \|(\rho - m - \rho_\infty)(\cdot, s)\|_{L^q(\Omega)} + \|m(\cdot, s)\|_{L^q(\Omega)} + \|\rho_\infty\|_{L^q(\Omega)} \tag{3.29}
\]
with \( M_8 = e^{(1+C_1+C_4)[\frac{1}{\theta_0} - \frac{1}{\varepsilon}]} e^{-\alpha_1 s} + M_8 \)
which along with Lemma 3.6 and Lemma 2.1 implies that
\[
I_1 \leq C_4C_S M_8 \int_0^t (1 + (t - s))^{-\frac{1}{2} - \frac{1}{2} (\frac{1}{\theta_0} - \frac{1}{\varepsilon})} e^{-\lambda_1 (t-s)} \|\nabla c\|_{L^\infty(\Omega)} ds \tag{3.30}
\]
where we have used (3.6) and (3.9) and \( \frac{1}{\theta_0} - \frac{1}{\varepsilon} < \frac{1}{2} \).
On the other hand, from Lemma 3.2 and Lemma 3.4 it follows that
\[
I_2 = C_1 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}} \left(\frac{s}{2} + \frac{1}{2} \right)\right) e^{-\alpha_1 (t-s)} \rho - m - \rho_\infty \|u\|_{L^\infty(\Omega)} ds
\]
\[
\leq 3C_4 M_3 M_5 \varepsilon^2 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}} \left(\frac{s}{2} + \frac{1}{2} \right)\right) e^{-\alpha_1 (t-s)} (1 + s^{-\frac{1}{2}} + \frac{s}{2} - \frac{s}{2}) e^{-(\alpha_1 + \alpha_2)^2} ds
\]
\[
\leq 3C_4 C_{10} M_3 M_5 \varepsilon^2 (1 + t^{\min\{0, \left(\frac{1}{2} - \frac{1}{p_0}\right)\}}) e^{-\alpha_1 t}
\]
\[
\leq \frac{M_1}{4} \varepsilon (1 + t^{-\frac{3}{2}}) e^{-\alpha_1 t},
\]
(3.31)
where we have used (3.10) and \(\frac{1}{p_0} - \frac{1}{q_0} < \frac{1}{3}\). Hence combining the above inequalities leads to our conclusion immediately.

Now we are ready to complete the proof of Theorem 1.1 in the case \(S = 0\) on \(\partial \Omega\).

**Proof of Proposition 3.1.** First from Lemma 3.4, 3.7 and Definition 3.14, it follows that \(T = T_{\max}\). It remains to show that \(T_{\max} = \infty\) and convergence result asserted in Proposition 3.1. Supposed that \(T_{\max} < \infty\), we only need to show that for all \(t \leq T_{\max}\),
\[
\|\rho(\cdot, t)\|_{L^\infty(\Omega)} + \|m(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1, \infty}(\Omega)} + \|A^\beta u(\cdot, t)\|_{L^2(\Omega)} < \infty
\]
with \(\beta \in \left(\frac{2}{3}, 1\right)\) according to the extensibility criterion in Lemma 2.5.

Let \(t_0 := \min\{1, \frac{T_{\max}}{3}\}\). Then from Lemma 3.3 there exists \(K_1 > 0\) such that for \(t \in (t_0, T_{\max})\),
\[
\|m(\cdot, t)\|_{L^\infty(\Omega)} \leq K_1 \varepsilon e^{-\rho_\infty t}.
\]
(3.32)
Moreover, from Lemma 3.2 and the fact that
\[
\|\rho(\cdot, t) - \rho_\infty\|_{L^\infty(\Omega)} \leq \|(\rho - m)(\cdot, t) - \rho_\infty\|_{L^\infty(\Omega)} + \|m(\cdot, t)\|_{L^\infty(\Omega)},
\]
it follows that for all \(t \in (t_0, T_{\max})\) and some constant \(K_2 > 0\),
\[
\|\rho(\cdot, t) - \rho_\infty\|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha_1 t}.
\]
(3.33)
Furthermore, Lemma 3.6 implies that there exists \(K_3' > 0\) such that
\[
\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq K_3' e^{-\alpha_1 t} \quad \text{for all} \quad t \in (t_0, T_{\max})
\]
(3.34)
Hence it only remains to show that
\[
\|c(\cdot, t)\|_{L^\infty(\Omega)} + \|A^\beta u(\cdot, t)\|_{L^2(\Omega)} \leq C \quad \text{for all} \quad t \in (t_0, T_{\max}).
\]
for some constant $C > 0$. In fact, we will show that

$$\| A^\alpha u(\cdot, t) \|_{L^2(\Omega)} \leq Ce^{-\alpha_2 t}$$

(3.35)

for $t_0 < t < T_{\text{max}}$ with some constant $C > 0$.

By (3.18), we have

$$\| A^\alpha u(\cdot, t) \|_{L^2(\Omega)} \leq \| A^\alpha e^{-tA} u_0 \|_{L^2(\Omega)} + \int_0^t \| A^\alpha e^{-(t-s)A} P((\rho + m - \rho_\infty) \nabla \phi)(\cdot, s) \|_{L^2(\Omega)} ds$$

$$+ \int_0^t \| A^\alpha e^{-(t-s)A} P((u \cdot \nabla) u)(\cdot, s) \|_{L^2(\Omega)} ds.$$  (3.36)

According to Lemma 2.2,

$$\| A^\alpha e^{-tA} u_0 \|_{L^2(\Omega)} \leq C_5 e^{-\mu t} \| A^\alpha u_0 \|_{L^2(\Omega)} \text{ for all } t \in (0, T_{\text{max}}).$$

From Lemma 2.2, 2.3, and the Hölder inequality, it follows that there exists $l_1 > 0$ such that

$$\int_0^t \| A^\alpha e^{-(t-s)A} P((\rho + m - \rho_\infty) \nabla \phi)(\cdot, s) \|_{L^2(\Omega)} ds$$

$$\leq C_5 C_9 \| \nabla \phi \|_{L^\infty(\Omega)} |\Omega|^{\frac{q_0}{2q_0} - 1} \int_0^t (2m(\cdot, s) \| L^{q_0(\Omega)}(t - s) - \beta e^{-\mu (t-s)} ds$$

$$\leq C_5 C_9 \| \nabla \phi \|_{L^\infty(\Omega)} |\Omega|^{\frac{q_0}{2q_0} - 1} l_1 \int_0^t e^{-\mu (t-s)} (t - s)^{-\beta}(1 + s)^{-\frac{3}{2}(\frac{1}{q_0} - \frac{1}{m})} e^{-\alpha_1 s} ds$$

$$\leq C_5 C_9 l_1 e^{-\alpha_2 t}(1 + t \min\{0, 1 - \frac{3}{2}(\frac{1}{q_0} - \frac{1}{m})\}).$$

On the other hand, let $M(t) := e^{-\alpha_2 t} \| A^\alpha u(\cdot, t) \|_{L^2(\Omega)}$ for $0 < t < T_{\text{max}}$. By Lemma 2.2(iv) and the Gagliardo–Nirenberg type inequality, one can see that

$$\|(u \cdot \nabla) u(\cdot, s)\|_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{q_0}} \| u(\cdot, s) \|_{L^\infty(\Omega)} \| \nabla u(\cdot, s) \|_{L^1(\Omega)}$$

$$\leq l_2 \| A^\alpha u(\cdot, s) \|_{L^2(\Omega)} \| u(\cdot, s) \|_{L^{q_0(\Omega)}} \| \nabla u(\cdot, s) \|_{L^1(\Omega)}$$

for some $l_2 > 0$ with $\theta = \frac{1}{q_0}/(\frac{1}{q_0} - \frac{1}{2} + \frac{2\beta}{3})$, and thereby the application of Lemma 2.2, 2.3, 3.4 and 3.5 gives

$$\int_0^t \| A^\alpha e^{-(t-s)A} P((u \cdot \nabla) u)(\cdot, s) \|_{L^2(\Omega)} ds$$

$$\leq C_5 C_9 l_1 \int_0^t \| A^\alpha u(\cdot, s) \|_{L^2(\Omega)} \| u(\cdot, s) \|_{L^{q_0(\Omega)}} \| \nabla u(\cdot, s) \|_{L^1(\Omega)}$$

$$\leq l_3 \left( \max_{0 \leq s < T_{\text{max}}} M(s) \right) \| e^{-\mu (t-s)} (t - s)^{-\beta}(1 + s)^{-\frac{3}{2}(\frac{1}{q_0} - \frac{1}{m})} e^{-\alpha_1 s} ds$$

$$\leq C_{10} l_3 \left( \max_{0 \leq s < T_{\text{max}}} M(s) \right) e^{-\alpha_2 t}.$$
for some \(l_3 > 0\).

Hence inserting the above inequalities into (3.36), we arrive at

\[
M(t) \leq C_5 \|A^\beta u\|_{L^2(\Omega)} + C_5 C_9 C_{10} \|\nabla \phi\|_{L^\infty(\Omega)} \frac{q_0-2}{q_0} l_1 (1 + t^{\min\{0,1-\frac{2}{2-q_0},\frac{1}{2-q_0},1-\frac{2}{q_0}\}})
+ C_10 l_3 (\max_{0 \leq s < T_{\max}} M(s))^{\vartheta} (1 + t^{\min\{0,1-\frac{2}{2-q_0},\frac{1}{2-q_0},1-\frac{2}{q_0}\}}),
\]

which implies that for some \(l_4 > 0\) depending on \(t_0\), we have

\[
\max_{t_0 \leq t < T_{\max}} M(t) \leq l_4 + l_4 (\max_{0 \leq t < T_{\max}} M(t))^{\vartheta}.
\]

On the other hand, from Lemma 2.5, \(\max_{0 \leq t \leq t_0} M(t) \leq l_5\). Therefore, we get

\[
\max_{0 \leq t < T_{\max}} M(t) \leq l_4 + l_5 + l_4 (\max_{0 \leq t < T_{\max}} M(t))^{\vartheta}.
\]

Due to \(\vartheta < 1\), we infer that \(M(t) \leq l_6\) for all \(t \in (0, T_{\max})\) for some \(l_6 > 0\) independent of \(T_{\max}\) hence arrive at (3.35).

Furthermore, due to \(D(A^\beta) \hookrightarrow L^\infty(\Omega)\) with \(\beta \in (\frac{3}{4}, 1)\) and Lemma 3.4, we get

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha_2 t} \quad \text{for some } K_4 > 0 \text{ and } t \in (0, T_{\max}). \tag{3.37}
\]

Now we turn to show that there exists \(K_3'' > 0\) such that

\[
\|c(\cdot, t)\|_{L^\infty(\Omega)} \leq K_3'' e^{-\alpha_2 t} \quad \text{for all } t \in (0, T_{\max}). \tag{3.38}
\]

From (3.17), it follows that

\[
\|c\|_{L^\infty(\Omega)} \leq \|e^{(\Delta-1)}c_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(\Delta-1)}(m-u \cdot \nabla c)\|_{L^\infty(\Omega)} ds
\leq e^{-t}\|c_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(\Delta-1)}m(\cdot, s)\|_{L^\infty(\Omega)} ds
+ \int_0^t \|e^{(t-s)(\Delta-1)}u \cdot \nabla c(\cdot, s)\|_{L^\infty(\Omega)} ds. \tag{3.39}
\]

An application of (3.19) with \(k = \infty\) yields

\[
\int_0^t \|e^{(t-s)(\Delta-1)}m(\cdot, s)\|_{L^\infty(\Omega)} ds \leq \int_0^t e^{-(t-s)} \|m(\cdot, s)\|_{L^\infty(\Omega)} ds
\leq \|m_0\|_{L^\infty(\Omega)} M_0 \int_0^t e^{-(t-s)} e^{-\rho s} ds
\leq \|m_0\|_{L^\infty(\Omega)} M_0 C_{10} e^{-\alpha_2 t}. \tag{3.40}
\]

On the other hand, from (3.37) and (3.34), we can see that

\[
\int_0^t \|e^{(t-s)(\Delta-1)}u \cdot \nabla c\|_{L^\infty(\Omega)} ds \leq \int_0^t e^{-(t-s)} \|u\|_{L^\infty(\Omega)} \|\nabla c\|_{L^\infty(\Omega)} ds \tag{3.41}
\]
\[
\leq K_3'K_4 \int_0^t e^{-(\alpha_1 + \alpha_2)s} e^{-(t-s)} ds \\
\leq K_3'K_4C_{10} e^{-\alpha_2 t}.
\]

Hence, inserting (3.40), (3.41) into (3.39), we arrive at the conclusion (3.38). Therefore we have \( T_{\text{max}} = \infty \), and the decay estimates in (3.41)–(3.44) follow from (3.32)–(3.35) and (3.38), respectively.

As for the case \( \int_\Omega \rho_0 < \int_\Omega m_0 \), i.e., \( m_\infty > 0, \rho_\infty = 0 \), we also have

**Proposition 3.2.** Assume that (1.5) and \( \int_\Omega \rho_0 < \int_\Omega m_0 \) hold, and let \( p_0 \in (2, 3) \), \( q_0 \in \left( 3, \frac{3p_0}{2(3-p_0)} \right) \). Then there exists \( \varepsilon > 0 \) such that for any initial data \((\rho_0, m_0, c_0, u_0)\) fulfilling (1.6) as well as

\[
\|\rho_0\|_{L^{p_0}(\Omega)} \leq \varepsilon, \quad \|m_0 - m_\infty\|_{L^\infty(\Omega)} \leq \varepsilon, \quad \|\nabla c_0\|_{L^q(\Omega)} \leq \varepsilon, \quad \|u_0\|_{L^q(\Omega)} \leq \varepsilon,
\]

(1.4) admits a global classical solution \((\rho, m, c, u, P)\). Furthermore, for any \( \alpha_1 \in (0, \min\{\lambda_1, m_\infty, 1\}) \), \( \alpha_2 \in (0, \min\{\alpha_1, \lambda_1'\}) \), there exist constants \( K_i > 0, i = 1, 2, 3, 4 \), such that

\[
\|m(\cdot, t) - m_\infty\|_{L^\infty(\Omega)} \leq K_1 e^{-\alpha_1 t}, \quad (3.42)
\]
\[
\|\rho(\cdot, t)\|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha_1 t}, \quad (3.43)
\]
\[
\|c(\cdot, t) - m_\infty\|_{W^{1,\infty}(\Omega)} \leq K_3 e^{-\alpha_2 t}, \quad (3.44)
\]
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha_2 t}. \quad (3.45)
\]

The basic strategy in the proof of Proposition 3.2 parallels that in the proof of Proposition 3.1 to a certain extent. However, due to differences in the properties of \( \rho \) and \( m \), there are significant differences in the details of their proofs. Thus for the convenience of the reader, we will sketch the proof of Proposition 3.2.

The following elementary observations can be also verified easily:

**Lemma 3.8.** Under the assumptions of Proposition 3.2, it is possible to choose \( M_1 > 0, M_2 > 0 \) and \( \varepsilon > 0 \) such that

\[
C_3 + C_2C_{10}(1 + C_1 + C_1|\Omega|^{\frac{1}{p_0}}) \left( \frac{1}{m_0} + \frac{1}{m_0} \right) + M_1 \leq \frac{M_2}{4}, \quad (3.46)
\]
\[
C_6 + 2C_6C_9C_{10}(M_1 + 2 + 2C_1 + 2C_1|\Omega|^{\frac{1}{p_0}}\frac{1}{m_0})\|\nabla \phi\|_{L^\infty(\Omega)} < \frac{M_3}{4} \quad (3.47)
\]
\[
C_7 + 2C_7C_9C_{10}(M_1 + 2 + 2C_1 + 2C_1|\Omega|^{\frac{1}{p_0}}\frac{1}{m_0})\|\nabla \phi\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{p_0}} < \frac{M_4}{4} \quad (3.48)
\]
12C_2 C_{10} M_3 \varepsilon \leq 1, \quad (3.49)
2C_1 + (\min \{1, |\Omega|\})^{-\frac{1}{p_0}} \leq \frac{M_1}{8}, \quad 12C_6 C_9 C_{10} M_4 \varepsilon < 1, \quad (3.50)
24C_4 C_5 C_{10} M_2 \varepsilon < 1, \quad (3.51)
12C_7 C_9 C_{10} M_3 \varepsilon < 1, \quad (3.52)
12C_4 C_{10} C_5 M_1 M_2 \varepsilon < 1, \quad (3.53)
24C_1 C_{10} (1 + C_1 + C_1 |\Omega|^{-\frac{1}{p_0}} - \frac{1}{p_0} + M_1) \varepsilon < 1, \quad (3.54)
18C_4 C_{10} M_3 \varepsilon < 1, \quad (3.55)
12C_{10} C_4 M_3 (1 + C_1 + C_1 |\Omega|^{-\frac{1}{p_0}} - \frac{1}{p_0}) \varepsilon < 1. \quad (3.56)

Define

\[ T := \sup \left\{ \tilde{T} \in (0, T_{\max}) : \begin{align*}
\| (m - \rho)(\cdot, t) - e^{t \Delta} (m_0 - \rho_0) \|_{L^q(\Omega)} &\leq \varepsilon (1 + t^{-\frac{1}{2}}(\frac{1}{p_0} - \frac{1}{q}) \varepsilon^{-\alpha_1 t}; \\
\| \rho(\cdot, t) \|_{L^q(\Omega)} &\leq M_1 \varepsilon (1 + t^{-\frac{1}{2}}(\frac{1}{p_0} - \frac{1}{q}) \varepsilon^{-\alpha_1 t}, \forall \theta \in [0, \infty]; \\
\| \nabla c(\cdot, t) \|_{L^\infty(\Omega)} &\leq M_2 \varepsilon (1 + t^{-\frac{1}{4} + \frac{3}{2p_0}) \varepsilon^{-\alpha_2 t} \text{ for all } t \in [0, \tilde{T}); \\
\| u(\cdot, t) \|_{L^{q_0}(\Omega)} &\leq M_3 \varepsilon \left(1 + t^{-\frac{1}{4} + \frac{3}{2p_0}) \varepsilon^{-\alpha_2 t} \text{ for all } t \in [0, \tilde{T)}; \\
\| \nabla u(\cdot, t) \|_{L^3(\Omega)} &\leq M_4 \varepsilon \left(1 + t^{-\frac{1}{4}} \varepsilon^{-\alpha_2 t} \text{ for all } t \in [0, \tilde{T}).
\end{align*} \right\} \quad (3.57) \]

By Lemma 2.5 and (1.6), \( T > 0 \) is well-defined. As in the previous subsection, we first show \( T = T_{\max} \), and then \( T_{\max} = \infty \). To this end, we will show that all of the estimates mentioned in (3.57) are valid with even smaller coefficients on the right hand side than that in (3.57).

The derivation of these estimates will mainly rely on \( L^p - L^q \) estimates for the Neumann heat semigroup and the corresponding semigroup for Stokes operator, and the fact that the classical solution of (1.1) on \( (0, T) \) can be represented as

\[ (m - \rho)(\cdot, t) = e^{t \Delta} (m_0 - \rho_0) + \int_0^t e^{(t-s) \Delta} (\nabla \cdot (\rho S(x, \rho, c) \nabla c) - u \cdot \nabla (m - \rho))(\cdot, s) ds, \quad (3.58) \]
\[ \rho(\cdot, t) = e^{t \Delta} \rho_0 - \int_0^t e^{(t-s) \Delta} (\nabla \cdot (\rho S(x, \rho, c) \nabla c) + u \cdot \nabla \rho + \rho m)(\cdot, s) ds, \quad (3.59) \]
\[ c(\cdot, t) = e^{t(\Delta - 1)} c_0 + \int_0^t e^{(t-s)(\Delta - 1)} (m - u \cdot \nabla c)(\cdot, s) ds, \quad (3.60) \]
\[ u(\cdot, t) = e^{-t \Delta} u_0 + \int_0^t e^{-(t-s) \Delta} \mathcal{P}((\rho + m) \nabla \phi - (u \cdot \nabla) u)(\cdot, s) ds. \quad (3.61) \]

The proofs of the following two lemmas are same as that of [18], so we omit it here.
Lemma 3.9. (Lemma 3.17 in [18]) Under the assumptions of Proposition 3.2, we have
\[ \|(m - \rho)(\cdot, t) - m_\infty\|_{L^p(\Omega)} \leq M_5 \varepsilon (1 + t^{-\frac{2}{9\alpha} + \frac{1}{70}}) e^{-\alpha_1 t} \]
for all \( t \in (0, T) \) and \( \theta \in [q_0, \infty] \) with \( M_5 = 1 + C_1 + C_1|\Omega|\frac{1}{q_0} - \frac{1}{60} \).

Lemma 3.10. (Lemma 3.18 in [18]) Under the assumptions of Proposition 3.2,
\[ \|m(\cdot, t) - m_\infty\|_{L^p(\Omega)} \leq (M_5 + M_1) \varepsilon (1 + t^{-\frac{2}{9\alpha} + \frac{1}{70}}) e^{-\alpha_1 t} \quad \text{for all} \quad t \in (0, T), \theta \in [q_0, \infty]. \]

Lemma 3.11. Under the assumptions of Proposition 3.2, we have
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{M_3}{3} (1 + t^{-\frac{1}{9\alpha} + \frac{1}{30}}) e^{-\alpha_2 t} \quad \text{for all} \quad t \in (0, T). \]

Proof. For any given \( \alpha_2 < \lambda_1' \), we can fix \( \mu \in (\alpha_2, \lambda_1') \). By (3.61), Lemma 2.2, Lemma 2.3 and \( \mathcal{P}(\nabla \phi) = 0 \), we obtain that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} 
\leq C_6 t^{-\frac{2}{9\alpha} + \frac{1}{70}} e^{-\mu t} \|\nabla u\|_{L^2(\Omega)} + \int_0^t \|e^{-(t-s)A}\mathcal{P}((\rho + m)\nabla \phi - (u \cdot \nabla) u)(\cdot, s)\|_{L^\infty(\Omega)} ds \quad (3.62)
\]
\[
\leq C_6 t^{-\frac{1}{9\alpha} + \frac{1}{30}} e^{-\mu t} \varepsilon + C_6 C_9 \|\nabla \phi\|_{L^\infty(\Omega)} \int_0^t e^{-\mu(t-s)} \|(\rho + m - m_\infty)(\cdot, s)\|_{L^\infty(\Omega)} ds
\]
\[+ C_6 C_9 \int_0^t (t - s)^{-\frac{1}{2}} e^{-\mu(t-s)} \|(u \cdot \nabla) u(\cdot, s)\|_{L^\infty(\Omega)} ds. \]

By Lemma 3.10 and the definition of \( T \), we get
\[ \|(\rho + m - m_\infty)(\cdot, s)\|_{L^\infty(\Omega)} = \|(m - m_\infty)(\cdot, s)\|_{L^\infty(\Omega)} + \|\rho(\cdot, s)\|_{L^\infty(\Omega)} \]
\[\leq (2M_5 + M_1) \varepsilon (1 + s^{-\frac{2}{9\alpha} + \frac{1}{30}}) e^{-\alpha_1 s}. \]

Inserting (3.63) into (3.62), by the definition of \( T \) and noting that \( \frac{2}{9\alpha} - \frac{1}{90} < 1 \), we have
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \]
\[\leq C_6 t^{-\frac{1}{9\alpha} + \frac{1}{30}} e^{-\mu t} \varepsilon + C_6 C_9 (2M_5 + M_1) \|\nabla \phi\|_{L^\infty(\Omega)} \varepsilon \int_0^t (1 + s^{-\frac{2}{9\alpha} + \frac{1}{30}}) e^{-\alpha_1 s} e^{-\mu(t-s)} ds
\]
\[+ C_6 C_9 \int_0^t (t - s)^{-\frac{1}{2}} e^{-\mu(t-s)} \|(u \cdot \nabla) u(\cdot, s)\|_{L^\infty(\Omega)} ds \]
\[\leq C_6 t^{-\frac{1}{9\alpha} + \frac{1}{30}} e^{-\mu t} \varepsilon + C_6 C_9 C_{10} (2M_5 + M_1) \|\nabla \phi\|_{L^\infty(\Omega)} \varepsilon (1 + t^{-\min\{0, 1 - \frac{2}{9\alpha} + \frac{1}{30}\}}) e^{-\alpha_2 t}
\]
\[+ 3C_6 C_9 M_3 M_4 \int_0^t (t - s)^{-\frac{1}{2}} \varepsilon e^{-\mu(t-s)} (1 + s^{-\frac{1}{90}}) e^{-2\alpha_2 s} ds
\]
\[\leq C_6 t^{-\frac{1}{9\alpha} + \frac{1}{30}} \varepsilon e^{-\mu t} + 2C_6 C_9 C_{10} (2M_5 + M_1) \|\nabla \phi\|_{L^\infty(\Omega)} \varepsilon e^{-\alpha_2 t} \]
Lemma 3.12. Under the assumptions of Proposition 3.2, we have
\[ \|\nabla u(\cdot, t)\|_{L^3(\Omega)} \leq \frac{M_4}{2} \varepsilon (1 + t^{-\frac{3}{2}}) e^{-\alpha_2 t} \]
for all \( t \in (0, T) \).

Proof. According to (3.61), and applying Lemma 2.2(iii) and Lemma 2.3, we arrive at
\[ \|\nabla u(\cdot, t)\|_{L^3(\Omega)} \leq C_7 t^{-\frac{1}{2}} e^{-\mu t} \|u_0\|_{L^3(\Omega)} + \int_0^t \|\nabla e^{-(t-s)\lambda} P((\rho + m) \nabla \phi - (u \cdot \nabla) u)(\cdot, s)\|_{L^3(\Omega)} ds \]
\[ \leq C_7 t^{-\frac{1}{2}} e^{-\mu t} \varepsilon + C_7 |\Omega|^{\frac{1}{3}} \varepsilon_0^{\frac{1}{90}} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\mu(t-s)} \|P((\rho + m - m_\infty) \nabla \phi)(\cdot, s)\|_{L^9(\Omega)} ds \]
\[ + C_7 \int_0^t (t-s)^{-\frac{1}{2}} e^{-\mu(t-s)} \|\nabla u(\cdot, s)\|_{L^3(\Omega)} \|u(\cdot, s)\|_{L^{30}(\Omega)} ds, \tag{3.64} \]
where \( P(m_\infty \nabla \phi) = m_\infty P(\nabla \phi) = 0 \) is used.

From (3.63), it follows that
\[ \int_0^t (t-s)^{-\frac{1}{2}} e^{-\mu(t-s)} \|((\rho + m - m_\infty)(\cdot, s)\|_{L^{30}(\Omega)} ds \leq (2M_5 + M_1) \varepsilon \int_0^t (t-s)^{-\frac{1}{2}} e^{-\mu(t-s)} (1 + s^{-\frac{3}{2}}(\frac{c_\lambda}{\rho_0} - \frac{1}{\rho_0})) e^{-\alpha_1 s} ds. \tag{3.65} \]

In addition, an application of the Hölder inequality and definition of \( T \) shows that
\[ \int_0^t (t-s)^{-\frac{1}{2}} e^{-\mu(t-s)} \|u(\cdot, s)\|_{L^{30}(\Omega)} \|\nabla u(\cdot, s)\|_{L^3(\Omega)} ds \leq 3M_4 \varepsilon^2 \int_0^t (t-s)^{-\frac{1}{2}} e^{-\mu(t-s)} (1 + s^{-\frac{3}{2}}(\frac{c_\lambda}{\rho_0} - \frac{1}{\rho_0})) e^{-2\alpha_2 s} ds. \tag{3.66} \]

Therefore, inserting (3.66), (3.65) into (3.64) and applying Lemma 2.4, we get
\[ \|\nabla u(\cdot, t)\|_{L^3(\Omega)} \leq C_7 t^{-\frac{1}{2}} e^{-\mu t} \varepsilon + C_7 C_9 C_{10} \|\nabla \phi\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{3}} \varepsilon_0^{\frac{1}{90}} (2M_5 + M_1) \varepsilon (1 + t^{\min\{0, \frac{1}{2} - \frac{3}{2}(\frac{c_\lambda}{\rho_0} - \frac{1}{\rho_0})\}}) e^{-\alpha_2 t} \]
\[ + 3C_7 C_9 C_{10} M_3 M_4 \varepsilon^2 (1 + t^{-\frac{1}{2}}) e^{-\alpha_2 t} \]
\[ \leq \frac{M_4}{2} \varepsilon (1 + t^{-\frac{1}{2}}) e^{-\alpha_1 t}, \]

where (3.48), (3.52) are used.

**Lemma 3.13.** Under the assumptions of Proposition 3.2, we have
\[ \| \nabla c(\cdot, t) \|_{L^\infty(\Omega)} \leq \frac{M_2}{2} \varepsilon (1 + t^{-\frac{1}{2}}) e^{-\alpha_1 t} \quad \text{for all } t \in (0, T). \]

**Proof.** From (3.66) and the standard regularization properties of the Neumann heat semigroup \((e^{\tau \Delta})_{\tau > 0}\) in [31], one can conclude that
\[
\| \nabla c(\cdot, t) \|_{L^\infty(\Omega)} \leq e^{-t} \| \nabla e^t \Delta c_0 \|_{L^\infty(\Omega)} + \int_0^t \| \nabla \Phi(t-s)(\Delta-1)(m - u \cdot \nabla c)(\cdot, s) \|_{L^\infty(\Omega)} ds
\]
\[
\leq C_3 (1 + t^{-\frac{3}{2}}) e^{-t} \| \nabla c_0 \|_{L^3(\Omega)} + \int_0^t \| \nabla \Phi(t-s)(\Delta-1)(m - m_\infty)(\cdot, s) \|_{L^\infty(\Omega)} ds
\]
\[+ \int_0^t \| \nabla \Phi(t-s)(\Delta-1) u \cdot \nabla c(\cdot, s) \|_{L^\infty(\Omega)} ds. \quad (3.67) \]

In the second inequality, we have used \(\nabla \Phi(t-s)(\Delta-1) m_\infty = 0\).

From Lemma 2.1(ii), Lemma 3.10 and Lemma 2.4, it follows that
\[
\int_0^t \| \nabla \Phi(t-s)(\Delta-1)(m - m_\infty)(\cdot, s) \|_{L^\infty(\Omega)} ds
\]
\[
\leq C_2 \int_0^t (1 + (t - s)^{-\frac{1}{2}} - \frac{1}{2m_0}) e^{-((\lambda_1 + 1)(t-s))(1 + s^{-\frac{1}{2}}(1 - \frac{1}{r_0} - \frac{1}{q_0}))} ds \quad (3.68)
\]
\[
\leq C_2 (M_5 + M_1) \varepsilon \int_0^t (1 + (t - s)^{-\frac{1}{2}} - \frac{1}{2m_0}) e^{-((\lambda_1 + 1)(t-s))(1 + s^{-\frac{1}{2}}(1 - \frac{1}{r_0} - \frac{1}{q_0}))} ds \quad (3.69)
\]
\[
\leq C_2 C_{10} (M_5 + M_1) \varepsilon (1 + t^{-\frac{1}{2}}) e^{-\alpha_1 t}. \]

On the other hand, by Lemma 2.1(ii), Lemma 2.4 and the definition of \(T\), we obtain
\[
\int_0^t \| \nabla \Phi(t-s)(\Delta-1) u \cdot \nabla c(\cdot, s) \|_{L^\infty(\Omega)} ds
\]
\[
\leq C_2 \int_0^t (1 + (t - s)^{-\frac{1}{2}} - \frac{1}{2m_0}) e^{-((\lambda_1 + 1)(t-s))} \| u \cdot \nabla c(\cdot, s) \|_{L^{\infty}(\Omega)} ds \quad (3.69)
\]
\[
\leq C_2 \int_0^t (1 + (t - s)^{-\frac{1}{2}} - \frac{1}{2m_0}) e^{-((\lambda_1 + 1)(t-s))} \| u(\cdot, s) \|_{L^{\infty}(\Omega)} \| \nabla c(\cdot, s) \|_{L^{\infty}(\Omega)} ds
\]
\[
\leq C_2 M_3 M_2 \varepsilon^2 \int_0^t (1 + (t - s)^{-\frac{1}{2}} - \frac{1}{2m_0}) e^{-((\lambda_1 + 1)(t-s))} ds \leq C_2 M_3 M_2 \varepsilon^2 \int_0^t e^{-(\lambda_1 + 1)(t-s)} ds \leq C_2 M_3 M_2 \varepsilon^2 \int_0^t e^{-\alpha_1 t} ds \]
\[
\leq C_2 M_3 M_2 C_{10} \varepsilon^2 (1 + t^{-\frac{1}{2}}) e^{-\alpha_1 t}. \]
Hence combining above inequalities and applying \((3.46)\) and \((3.49)\), we arrive at the conclusion.

**Lemma 3.14.** *Under the assumptions of Proposition 3.2, we have*

\[
\|\rho(\cdot, t)\|_{L^\theta(\Omega)} \leq \frac{M_1}{2} \varepsilon (1 + t^{-\frac{1}{2}} (\frac{1}{\rho_0} - \frac{1}{\theta})) e^{-\alpha_1 t} \quad \text{for all } t \in (0, T), \theta \in [\theta_0, \infty].
\]

**Proof.** From \((3.59)\), we have

\[
\rho(\cdot, t) = e^{\Delta (\Delta - m_\infty)} \rho_0 - \int_0^t e^{(t-s)(\Delta - m_\infty)} (\nabla \cdot (\rho \mathbf{S}(\cdot, \rho, c) \nabla \rho) - u \cdot \nabla \rho)(\cdot, s) ds + \int_0^t e^{(t-s)(\Delta - m_\infty)} \rho(m_\infty - m)(\cdot, s) ds.
\]

By Lemma 2.1, the result in Section 2 of \[31\] and \(\alpha_1 < \min \{\lambda_1, m_\infty\}\), we obtain

\[
\|\rho(\cdot, t)\|_{L^\theta(\Omega)} \leq e^{-m_\infty t} (\|e^{\Delta (\rho_0 - \mathbf{P}_0)}\|_{L^\theta(\Omega)} + \|\mathbf{P}_0\|_{L^\theta(\Omega)}) + \int_0^t \|e^{(t-s)(\Delta - m_\infty)} \nabla \cdot (\rho \mathbf{S}(\cdot, \rho, c) \nabla \rho)(\cdot, s)\|_{L^\theta(\Omega)} ds + \int_0^t \|e^{(t-s)(\Delta - m_\infty)} \rho(m_\infty - m)(\cdot, s)\|_{L^\theta(\Omega)} ds
\]

\[
\leq C_1 (1 + t^{-\frac{3}{2}} (\frac{1}{\rho_0} - \frac{1}{\theta})) e^{-\lambda_1 + m_\infty(t-s)} \|\rho_0 - \mathbf{P}_0\|_{L^{\infty}(\Omega)} + (\min \{1, |\Omega|\})^{-\frac{1}{p_0}} e^{-m_\infty t} \varepsilon
\]

\[
+ C_4 C_S \int_0^t (1 + (t-s))^{-\frac{1}{2} + \frac{3}{2}(\frac{1}{\rho_0} - \frac{1}{\theta})) e^{-\lambda_1 + m_\infty(t-s)} \|\mathbf{P}_0\|_{L^{\infty}(\Omega)} \|\nabla \rho\|_{L^{\infty}(\Omega)} ds + \int_0^t \|e^{(t-s)(\Delta - m_\infty)} \rho(m_\infty - m)(\cdot, s)\|_{L^\theta(\Omega)} ds
\]

\[
\leq 2C_1 + (\min \{1, |\Omega|\})^{-\frac{1}{p_0}} (1 + t^{-\frac{3}{2}} (\frac{1}{\rho_0} - \frac{1}{\theta})) \varepsilon e^{-\alpha_1 t}
\]

\[
+ C_4 C_S \int_0^t (1 + (t-s))^{-\frac{1}{2} + \frac{3}{2}(\frac{1}{\rho_0} - \frac{1}{\theta})) e^{-\lambda_1 + m_\infty(t-s)} \|\mathbf{P}_0\|_{L^{\infty}(\Omega)} \|\nabla \rho\|_{L^{\infty}(\Omega)} ds
\]

\[
+ C_4 \int_0^t (1 + (t-s))^{-\frac{1}{2} + \frac{3}{2}(\frac{1}{\rho_0} - \frac{1}{\theta})) e^{-\lambda_1 + m_\infty(t-s)} \|\rho\|_{L^{\infty}(\Omega)} \|u\|_{L^{\infty}(\Omega)} ds
\]

\[
+ C_1 \int_0^t (1 + (t-s))^{-\frac{3}{2}(\frac{1}{\rho_0} - \frac{1}{\theta})} e^{-m_\infty(t-s)} \|\rho\|_{L^{\infty}(\Omega)} \|m - m_\infty\|_{L^{\infty}(\Omega)} ds.
\]

According to the definition of \(T\), Lemma 3.13 and Lemma 2.4, this shows that

\[
\int_0^t (1 + (t-s))^{-\frac{1}{2} + \frac{3}{2}(\frac{1}{\rho_0} - \frac{1}{\theta})} e^{-\lambda_1(t-s)} e^{-2\alpha_1 s} (1 + s^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{\rho_0} - \frac{1}{\theta})}) ds
\]

\[
\leq 3M_1 M_2 \varepsilon^2 \int_0^t (1 + (t-s))^{-\frac{1}{2} + \frac{3}{2}(\frac{1}{\rho_0} - \frac{1}{\theta})} e^{-\lambda_1(t-s)} e^{-2\alpha_1 s} (1 + s^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{\rho_0} - \frac{1}{\theta})}) ds
\]

\[
\leq 3C_{10} M_1 M_2 \varepsilon^2 (1 + t^{\min \{0, -\frac{3}{2}(\frac{1}{\rho_0} - \frac{1}{\theta})\}}) e^{-\min \{\lambda_1, 2\alpha_1\} t}
\]

Similarly, we can also get

\[
\int_0^t (1 + (t-s))^{-\frac{1}{2} + \frac{3}{2}(\frac{1}{\rho_0} - \frac{1}{\theta})} e^{-\lambda_1 + m_\infty(t-s)} \|\rho\|_{L^{\infty}(\Omega)} \|u\|_{L^{\infty}(\Omega)} ds
\]
\[ \leq 3M_1M_3\varepsilon^2 \int_0^t (1 + (t-s)^{-\frac{1}{2}}(\frac{1}{\rho_0} - \frac{1}{q_0}))(\rho - \rho_0)e^{-\lambda_1(t-s)}e^{-2\alpha_1 s}e^{s^2}(1 + s^{-\frac{1}{2}}(\frac{1}{\rho_0} - \frac{1}{q_0})) ds \]

\[ \leq 3C_{10}M_3M_1\varepsilon^2(1 + t^{\min\{0, -\frac{3}{4}(\frac{1}{\rho_0} - \frac{1}{q_0})\}})e^{-\min\{\lambda_1, 2\alpha_1\}t} \]

and

\[ \int_0^t (1 + (t-s)^{-\frac{1}{2}}(\frac{1}{\rho_0} - \frac{1}{q_0}))e^{-m_\infty(t-s)}||\rho||_{L^\infty(\Omega)}||m - m_\infty||_{L^\infty(\Omega)} ds \]

\[ \leq 3M_1(M_5 + M_1)\varepsilon^2 \int_0^t (1 + (t-s)^{-\frac{1}{2}}(\frac{1}{\rho_0} - \frac{1}{q_0}))e^{-m_\infty(t-s)}e^{-2\alpha_1 s}(1 + s^{-\frac{3}{4} + \frac{3}{2q_0}}) ds \]

\[ \leq 3C_{10}M_1(M_5 + M_1)\varepsilon^2(1 + t^{\min\{0, -\frac{3}{4}(\frac{1}{\rho_0} - \frac{1}{q_0})\}})e^{-\min\{m_\infty, 2\alpha_1\}t}, \]

where the fact that \( q_0 \in (3, \frac{3\rho_0}{2(3-\rho_0)}) \) warrants \(-\frac{3}{\rho_0} + \frac{3}{2q_0} > -1\) is used. Hence the combination of the above inequalities yields \( \|\rho(\cdot, t)\|_{L^\theta(\Omega)} \leq \frac{M_1}{\varepsilon^2}(1 + t^{\min\{0, -\frac{3}{4}(\frac{1}{\rho_0} - \frac{1}{q_0})\}})e^{-\alpha_1 t} \), thanks to (3.55), (3.54) and (3.51).

**Lemma 3.15.** Under the assumptions of Proposition 3.2, we have

\[ \|(m - \rho)(\cdot, t) - e^{t\Delta}(m_0 - \rho_0)\|_{L^\theta(\Omega)} \leq \varepsilon \frac{1}{2}(1 + t^{\frac{3}{4}(\frac{1}{\rho_0} - \frac{1}{q_0})})e^{-\alpha_1 t} \text{ for } \theta \in [q_0, \infty], t \in (0, T). \]

**Proof.** From (3.58) and Lemma 2.1(iv), it follows that

\[ \|(m - \rho)(\cdot, t) - e^{t\Delta}(m_0 - \rho_0)\|_{L^\theta(\Omega)} \]

\[ \leq \int_0^t \| e^{(t-s)\Delta}(\nabla \cdot (\rho \nabla \cdot (\rho \nabla c) - u \cdot \nabla (m - \rho)))(\cdot, s)\|_{L^\theta(\Omega)} ds \]

\[ \leq \int_0^t \| e^{(t-s)\Delta} \nabla \cdot (\rho \nabla \cdot (\rho \nabla c))(\cdot, s)\|_{L^\theta(\Omega)} ds + \int_0^t \| e^{(t-s)\Delta} \nabla \cdot ((m - \rho - m_\infty)u)(\cdot, s)\|_{L^\theta(\Omega)} ds \]

\[ \leq C_4C\int_0^t \left(1 + (t - s)^{-\frac{1}{2}}(\frac{1}{\rho_0} - \frac{1}{q_0})\right)e^{-\lambda_1(t-s)}\|\rho(\cdot, s)\|_{L^\infty(\Omega)}\|\nabla c(\cdot, s)\|_{L^\infty(\Omega)} ds \]

\[ + C_4 \int_0^t \left(1 + (t - s)^{-\frac{1}{2}}(\frac{1}{\rho_0} - \frac{1}{q_0})\right)e^{-\lambda_1(t-s)}\|u(m - \rho - m_\infty)(\cdot, s)\|_{L^\infty(\Omega)} ds \]

\[ =: I_1 + I_2. \]

From the definition of \( T \) and (3.53), we have

\[ I_1 \leq 3C_4C\varepsilon M_1M_2\varepsilon^2 \int_0^t \left(1 + (t - s)^{-\frac{1}{2}}(\frac{1}{\rho_0} - \frac{1}{q_0})\right)e^{-\lambda_1(t-s)}(1 + s^{-\frac{1}{2}}(\frac{1}{\rho_0} - \frac{1}{q_0}))e^{-2\alpha_1 s} ds \]

\[ \leq 3C_4C\varepsilon M_1M_2\varepsilon^2(1 + t^{\min\{0, -\frac{3}{4}(\frac{1}{\rho_0} - \frac{1}{q_0})\}})e^{-\min\{\lambda_1, 2\alpha_1\}t} \]

\[ \leq \frac{\varepsilon}{4}(1 + t^{\frac{3}{4}(\frac{1}{\rho_0} - \frac{1}{q_0})})e^{-\alpha_1 t}. \]
From Lemma 3.16, Lemma 3.11 and (3.56), it follows that
\[ I_2 = C_4 \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{3}{2} (\frac{1}{\alpha} - \frac{1}{2})}) e^{-\lambda_1 (t-s)} \left\| m - \rho - m_\infty \right\|_{L^\infty(\Omega)} \| u \|_{L^6(\Omega)} ds \]
\[ \leq C_4 M_3 M_5 \varepsilon^2 \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{3}{2} (\frac{1}{\alpha} - \frac{1}{2})}) e^{-\lambda_1 (t-s)} (1 + s^{-\frac{3}{2} + \frac{3}{2} \alpha}) e^{-\alpha_2 s} ds \]
\[ \leq 3C_4 M_3 M_5 \varepsilon^2 \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{3}{2} (\frac{1}{\alpha} - \frac{1}{2})}) (1 + s^{-\frac{3}{2} + \frac{3}{2} \alpha}) e^{-\lambda_1 (t-s)} e^{-(\alpha_1 + \alpha_2 s)} ds \]
\[ \leq 3C_10 C_4 M_3 M_5 \varepsilon^2 e^{-\min\{\lambda_1, \alpha_1 + \alpha_2\} t} (1 + t^{\min\{0, \frac{3}{2} (\frac{1}{\alpha} - \frac{1}{2})\}}) \]
\[ \leq \frac{\varepsilon}{4} (1 + t^{-\frac{3}{2} (\frac{1}{\alpha} - \frac{1}{2})}) e^{-\alpha_1 t}. \]

Combining the above inequalities, we arrive at
\[ \left\| (\rho - m) (\cdot, t) - e^{t\Delta} (\rho_0 - m_0) \right\|_{L^6(\Omega)} \leq \frac{\varepsilon}{2} (1 + t^{-\frac{3}{2} (\frac{1}{\alpha} - \frac{1}{2})}) e^{-\alpha_1 t} \]
and thus complete the proof of this lemma.

By the above lemmas, one can see that \( T = T\max \), and the further estimates of solutions are needed to ensure \( T\max = \infty \).

**Lemma 3.16.** Under the assumptions of Proposition 3.2, for all \( \beta \in (\frac{\alpha}{4}, \min\{\frac{5}{4}, \frac{3}{2\alpha}, 1\}) \) there exists \( M_6 > 0 \) such that
\[ \| A^\beta u (\cdot, t) \|_{L^2(\Omega)} \leq \varepsilon M_6 e^{-\alpha_2 t} \quad \text{for } t \in (t_0, T\max) \quad \text{with } t_0 = \min\{\frac{T\max}{6}, 1\}. \]

**Proof.** The proof is similar to that of (3.35), and thus is omitted here.

**Lemma 3.17.** Under the assumptions of Proposition 3.2, there exists \( M_7 > 0 \) such that \( \| c (\cdot, t) - m_\infty \|_{L^\infty(\Omega)} \leq M_7 e^{-\alpha_2 t} \) for all \( (t_0, T\max) \) with \( t_0 = \min\{\frac{T\max}{6}, 1\} \).

**Proof.** We refer the readers to the proof of Lemma 3.24 in [18].

At this position, we can show the proof of Theorem 1.2 in the case \( S = 0 \) on \( \partial \Omega \).

**Proof of Proposition 3.2.** We first show that the solution is global, i.e. \( T\max = \infty \). To this end, according to the extensibility criterion in Lemma 2.5, it suffices to show that there exists \( C > 0 \) such that for all \( t_0 < t < T\max \)
\[ \| \rho (\cdot, t) \|_{L^\infty(\Omega)} + \| m (\cdot, t) \|_{L^\infty(\Omega)} + \| c (\cdot, t) \|_{W^{1, \infty}(\Omega)} + \| A^\beta u (\cdot, t) \|_{L^2(\Omega)} < C. \]

From Lemma 3.14 Lemma 3.14 and Lemma 3.16, there exists \( K_i > 0, i = 1, 2, 3, 4, \) such that
\[ \| m (\cdot, t) - m_\infty \|_{L^\infty(\Omega)} \leq K_1 e^{-\alpha_1 t}, \quad \| \rho (\cdot, t) \|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha_1 t}, \]
\[ \| c (\cdot, t) \|_{W^{1, \infty}(\Omega)} \leq K_3 e^{-\alpha_1 t}, \quad \| A^\beta u (\cdot, t) \|_{L^2(\Omega)} \leq K_4 e^{-\alpha_1 t}, \]
where \( K_1, K_2, K_3, K_4 \) depend on \( \varepsilon \), \( M_6 \), \( M_7 \) and \( T\max \). To establish the boundedness of the solution in \( L^\infty(\Omega) \), we can use the maximum principle, which states that the maximum of a solution is achieved at the boundary of the domain, i.e. \( \partial \Omega \). This implies that \( \rho (\cdot, t) \) and \( m (\cdot, t) \) are bounded, and thus \( \rho (\cdot, t) \) and \( m (\cdot, t) \) are bounded in \( L^\infty(\Omega) \). Consequently, we have
\[ \| \rho (\cdot, t) \|_{L^\infty(\Omega)} + \| m (\cdot, t) \|_{L^\infty(\Omega)} \leq C < \infty, \]
and the solution is globally bounded in \( L^\infty(\Omega) \). Therefore, the solution is global and \( T\max = \infty \).
Using this definition, we regularize (1.4) as follows

\[ \|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq K_3 e^{-\alpha_1 t}, \quad \|A^\beta u(\cdot, t)\|_{L^2(\Omega)} \leq K_4 e^{-\alpha_2 t} \]

for \( t \in (t_0, T_{\text{max}}) \). Furthermore, Lemma 3.17 implies that \( \|c(\cdot, t) - m_\infty\|_{W^{1,\infty}(\Omega)} \leq K'_3 e^{-\alpha_2 t} \) with some \( K'_3 > 0 \) for all \( t \in (t_0, T_{\text{max}}) \). Since \( D(A^\beta) \hookrightarrow L^\infty(\Omega) \) with \( \beta \in (\frac{3}{4}, 1) \), it follows from Lemma 3.16 that \( \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha_2 t} \) for some \( K_4 > 0 \) for all \( t \in (t_0, T_{\text{max}}) \). This completes the proof of Proposition 3.2.

4 Proof of main results for general \( S \)

In this section, we give the proof of our results for the general matrix-valued \( S \) by a rather standard argument, which is accomplished by an approximation procedure (see [4] for example). In order to make the previous results applicable, we introduce a family of smooth functions \( \rho_\eta \in C^\infty_0(\Omega) \) and \( 0 \leq \rho_\eta(x) \leq 1 \) for \( \eta \in (0, 1) \), \( \lim_{\eta \to 0} \rho_\eta(x) = 1 \) and let \( S_\eta(x, \rho, c) = \rho_\eta(x) S(x, \rho, c) \).

Using this definition, we regularize (1.4) as follows

\[
\begin{aligned}
(\rho_\eta)_t + u_\eta \cdot \nabla \rho_\eta &= \Delta \rho_\eta - \nabla \cdot (\rho_\eta S_\eta(x, \rho_\eta, c_\eta) \nabla c_\eta) - \rho_\eta m_\eta, \\
(m_\eta)_t + u_\eta \cdot \nabla m_\eta &= \Delta m_\eta - \rho_\eta m_\eta, \\
(c_\eta)_t + u_\eta \cdot \nabla c_\eta &= \Delta c_\eta - c_\eta + m_\eta, \\
(u_\eta)_t + (u_\eta \cdot \nabla) u_\eta &= \Delta u_\eta - \nabla P_\eta + (\rho_\eta + m_\eta) \nabla \phi, \\
\frac{\partial \rho_\eta}{\partial \nu} &= \frac{\partial m_\eta}{\partial \nu} = \frac{\partial c_\eta}{\partial \nu} = 0, \quad u_\eta = 0 
\end{aligned}
\]

with the initial data

\[
\rho_\eta(x, 0) = \rho_0(x), \quad m_\eta(x, 0) = m_0(x), \quad c(x, 0) = c_0(x), \quad \text{and} \quad u_\eta(x, 0) = u_0(x), \quad x \in \Omega. \quad (4.2)
\]

It is observed that \( S_\eta \) satisfies the additional condition \( S = 0 \) on \( \partial \Omega \). Therefore based on the discussion in Section 3, under the assumptions of Theorem 1.1 and Theorem 1.2, problem (4.1)-(4.2) admits a global classical solution \((\rho_\eta, m_\eta, c_\eta, u_\eta, P_\eta)\) that satisfies

\[
\begin{aligned}
\|m_\eta(\cdot, t) - m_\infty\|_{L^\infty(\Omega)} &\leq K_1 e^{-\alpha_1 t}, &\|\rho_\eta(\cdot, t) - \rho_\infty\|_{L^\infty(\Omega)} &\leq K_2 e^{-\alpha_1 t}, \\
\|c_\eta(\cdot, t) - m_\infty\|_{W^{1,\infty}(\Omega)} &\leq K_3 e^{-\alpha_2 t}, &\|A^\beta u_\eta(\cdot, t)\|_{L^2(\Omega)} &\leq K_4 e^{-\alpha_2 t}
\end{aligned}
\]

for some constants \( K_i, \quad i = 1, 2, 3, 4 \), and all \( t \geq 0 \). Applying a standard procedure such as in Lemma 5.2 and Lemma 5.6 of [3], one can obtain a subsequence of \( \{\eta_j\}_{j \in \mathbb{N}} \) with \( \eta_j \to 0 \) as
\[ j \to \infty \] such that \( \rho_{nj} \to \rho, \ m_{nj} \to m, \ c_{nj} \to c, \ u_{nj} \to u \) in \( C^\infty_{\text{loc}}(\overline{\Omega} \times (0, \infty)) \) as \( j \to \infty \) for some \( \nu \in (0, 1) \). Moreover, by the arguments as in Lemma 5.7, Lemma 5.8 of [4], one can also show that \( (\rho, m, c, u, P) \) is a classical solution of (1.4) with the decay properties asserted in Theorem 1.1 and Theorem 1.2, respectively. The proof of our main results is thus complete.

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