Quantum Field Theory with Extra Dimensions

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Abstract: We explain that a bulk with arbitrary dimensions can be added to the space over which a quantum field theory is defined. This gives a TQFT such that its correlation functions in a slice are the same as those of the original quantum field theory. This generalizes the stochastic quantization scheme, where the bulk is one dimensional.
1 Introduction

In recent works [1][2], the basic ideas of stochastic quantization [3][4] have been elaborated in a systematic approach, called bulk quantization. These papers give a central role to the introduction of a symmetry of a topological character. The correlation functions for equal values of the bulk time define the correlations of the physical theory. Perturbatively, bulk quantization and the usual quantization method are equivalent because the observables satisfy the same Schwinger-Dyson equations in both approaches; basic concepts such as the definition of the S-matrix (in the LSZ sense) and the Cutkowskii rules can be also directly addressed in 5 dimensions [1]. We believe that the difficulty of giving a consistent stochastic interpretation to all details of the formalism, especially in the case of gauge theories, justifies to directly postulate that bulk quantization is a particular type of a topological field theory. Moreover, there is an interesting geometrical interpretation for many of the ingredients that are needed in bulk quantization. The idea of a topological field theory is actually relevant, since one wishes to define observable that are independent of most of the details of the bulk, such as the metrics components $g_{tt}$ and $g_{\mu t}$. Quite interestingly, the interpretation of anomalies in gauge theories is that, the limit of taking the limit of an equal bulk time can be ambiguous.

As shown in [1], the additional dimension $t$ does not take part in the Poincaré group of symmetries for the case of an additional non compact dimension. The homogeneity of the Lagrangian requires that $t$ has the dimension of the square of the ordinary coordinates. We investigated how this method can be applied to gravity and supersymmetric theories in [5][6]. Here, we will show that bulk quantization can be generalized to the case where the bulk has an arbitrary dimension $n \geq 1$, according to the following picture:

![Diagram showing bulk quantization](image)
The interest of this generalization, when \( n \) is larger than one, is yet to be discovered, but we find that the aesthetics of the whole construction makes it worth being presented.

\section{The fields}

We consider for simplicity the case of a commuting scalar field \( \phi(x) \) in 4 dimensions, with a Lagrangian \( L_0(\phi) \) and action \( I_0[\phi] = \int d^4x L_0(\phi) \). We want to find another formulation of this theory with an action in a space with dimensions \( 4+n \), where \( n \) denotes generically the dimension of the bulk, with observables that are defined in a slice of dimension 4. The number of bosonic fields that are needed depends on \( n \), and increases as \( 2^n \), according to:

\[
\begin{align*}
\text{level } n = 0, & \quad \Phi_0 = \phi_1(x) = \phi(x) \\
\text{level } n = 1, & \quad \Phi_1 = \phi_1(x, t^1), \phi_2(x, t^1) \\
& \quad \vdots \\
\text{level } n, & \quad \Phi_n = \phi_1(x, t^1, \ldots, t^n), \phi_2(x, t^1, \ldots, t^n), \ldots \phi_{2^n}(x, t^1, \ldots, t^n) \\
& \quad \vdots 
\end{align*}
\]

Indeed, if we apply from level \( n \) to \( n + 1 \) the process explained in \cite{1}, the number of field degrees of freedom that are needed doubles. Thus, we have a multiplet with \( 2^n \) components when the bulk has dimension \( n \). For \( n = 1 \), \( \phi_2(x, t^1) \) can be heuristically interpreted as the Gaussian noise of a stochastic process. A more fundamental interpretation is that \( \phi_2(x, t^1) \) is the canonical moment with respect to the bulk time of the field \( \phi_1 = \phi \), and so on.

At every level \( n \), there is a hidden BRST symmetry, defined by the graded differential operator \( s_n \):

\[
\begin{align*}
s_n \phi_p &= \psi_p \\
s_n \Psi_p &= 0 \\
s_n \bar{\Psi}_{p+2^n-1} &= \phi_{p+2^n-1} \\
s_n \bar{\Phi}_{p+2^n-1} &= 0 \quad (2)
\end{align*}
\]

where, \( 1 \leq p \leq 2^{n-1} \). The \( \Psi \)'s and \( \bar{\Psi} \)'s are ghosts and antighosts, with the opposite statistics to the \( \phi \)'s. The set of fields \( \phi, \Psi \) and \( \bar{\Psi} \) determine a BRST topological quartet. We can define an anti-BRST operator, which merely interchanges the ghosts and antighosts:

\[
\begin{align*}
\bar{s}_n \phi_p &= \bar{\psi}_{p+2^n-1} \\
\bar{s}_n \Psi_p &= -\phi_{p+2^n-1} \\
\bar{s}_n \bar{\Psi}_{p+2^n-1} &= 0 \\
\bar{s}_n \bar{\Phi}_{p+2^n-1} &= 0 \quad (3)
\end{align*}
\]
The action that describes the formulation of the theory with a bulk of dimension $n$ must be $s_n$-exact, that is, it must be of the form:

$$\int d^4x dt^1 \ldots dt^n s_n (\sum_{p=1}^{2^n-1} \bar{\Psi}_{p+2^n-1} Z_p), \quad (4)$$

that is,

$$\int d^4x dt^1 \ldots dt^n \left( \sum_{p=1}^{2^n-1} \phi_{p+2^n-1} Z_p - \sum_{p=1}^{2^n-1} \bar{\Psi}_{p+2^n-1} s_n Z_p \right) \quad (5)$$

To determine the action, it is sufficient to determine the expression of the functionals $Z_p[\Phi_n]$, for every level $n$. At level $n$, the ghosts have parabolic propagators along $t_n$. Thus, they can be integrated out exactly when one computes correlations functions of the $\Phi$’s. Indeed, the latter cannot contain closed loops of the ghosts. It follows that it is sufficient to determine the following part of the action:

$$I_n = \int d^4x dt^1 \ldots dt^n \sum_{p=1}^{2^n-1} \phi_{p+2^n-1} Z_p. \quad (6)$$

The rest of the action will be determined by the requirement of BRST symmetry. The determination of the factors $Z_p$ is restricted by power counting and by symmetries. The latter include the parity in the bulk, that is, the invariance under $t_p \rightarrow -t_p$, for $1 \leq p \leq n$, and by translation invariance. The covariance of the fields under this parity will be shortly determined.

We define the following self-consistent assignments for the dimensions of the bulk coordinates and of the fields:

$$[t^n]^{-1} = 2^n \quad [\phi_p] = 2p - 1 \quad (7)$$

With this assignments, the dimension of the Lagrangian at level $n$ must be $2 + 2^{n+1}$. This ensures that the theory generated by the action $I_n$ is renormalizable by power counting, as a generalization of $[1]$. It is of relevance to note that:

$$[\phi_1] + [\phi_{2^n-1}] + [t^n]^{-1} = 2 + 2^{n+1} \quad (8)$$

This will imply that, in the formulation at level $n$, $\phi_{2^n-1}$ is the conjugate momentum of $\phi_1$ with respect to $t^n$. This turns out to be one of the key facts for proving by induction that the physical content of the theory at level $n$ is the same physics as for the theory at level $n - 1$, and so on, down to the ordinary formulation with the action $I_0$. 

3
3 The action at level $n$

At level zero, the theory is defined by the standard action $I_0[\phi] = \int d^4x L_0(\phi(x))$. At level $n = 1$, it is defined by:

$$I_1[\phi_1, \phi_2] = \int d^4x dt^1 \left( \phi_2 \partial_1 \phi_1 + \phi_2 \left( \phi_2 + \frac{\delta I_0}{\delta \phi_1} \right) \right)$$  \hspace{1cm} (9)

The exponential of this action must be inserted in the path integral with measure $[d\phi_1](x,t)[d\phi_2](x,t)$. $I_1$ satisfies power counting according to equation (7) (here $[t^1]^{-1} = 2, [\phi_1] = 1, [\phi_2] = 3$) and the bulk-parity symmetry $P_1$ is:

$$t^1 \rightarrow -t^1$$
$$\phi_1 \rightarrow \phi_1$$
$$\phi_2 \rightarrow -\phi_2 - \frac{\delta I_0}{\delta \phi_1}$$  \hspace{1cm} (10)

The action $I_1$ and its symmetry $P_1$ have been discussed in details in [1], where we have also shown that it describes the same physics as the action $I_0$. Notice that the existence of the symmetry $P_1$ is obvious after the elimination of $\phi_2$ by its equation of motion.

At level $n = 2$, the action is:

$$I_2[\phi_1, \phi_2, \phi_3, \phi_4] = \int d^4x dt^1 dt^2 \left( \phi_3 \partial_2 \phi_1 + \phi_3 \left( \phi_3 + \frac{\delta I_1}{\delta \phi_1} \right) \right)$$
$$\hspace{6cm} + \phi_4 \left( \phi_2 + \frac{\delta I_0}{\delta \phi_1} + \partial_1 \phi_1 \right)$$  \hspace{1cm} (11)

$I_2$ is invariant under the bulk-parity transformations $P_2$ and $P_1$. $P_2$ is defined as:

$$t_2 \rightarrow -t_2$$
$$\phi_1, \phi_2 \rightarrow \phi_1, \phi_2$$
$$\phi_3 \rightarrow -\phi_3 - \frac{\delta I_1}{\delta \phi_1}$$
$$\phi_4 \rightarrow \phi_4 + \partial_2 \phi_2$$  \hspace{1cm} (12)

The action of the symmetry $P_2$ on $\phi_4$ is such that $\delta I_2 = -\int \partial_2 (I_1)$. $P_1$ is defined as:

$$t_1 \rightarrow -t_1$$

4
\[
\begin{align*}
\phi_1, \phi_3 & \rightarrow \phi_1, \phi_3 \\
\phi_2 & \rightarrow -\phi_2 - \frac{\delta I_0}{\delta \phi_1} \\
\phi_4 & \rightarrow -\phi_4 + \phi_3 \frac{\delta^2 I_0}{\delta \phi_1 \delta \phi_1}
\end{align*}
\]

(13)

\(P_1\) transforms \(\phi_4\) in such a way that the variation of the term \(\phi_4(\delta I_2/\delta \phi_2)\) compensates that of \(\phi_3(\delta I_1/\delta \phi_1)\).

The parity symmetry under \(P_1\) and \(P_2\) implies that \(I_2\) has the form displayed in eq. (11). In this action, power counting implies that \(I_0[\phi]\) is a local functional, which can be identified as an action that is renormalizable in 4 dimensions. Thus, no new parameter of physical relevance can be introduced when one switches from the ordinary formulation to the formulation with a bulk. By a straightforward generalization of (1), one can then prove that the correlations functions, computed from \(I_2\) at a given point of the two-dimensional bulk, satisfy the same Dyson–Schwinger equations as those computed from \(I_1\), at a given point of the one-dimensional bulk. In turn, there is the equivalence of the physics computed either from \(I_1\) from \(I_0\), which gives the desired result that we can use a two-dimensional bulk to compute physical quantities with the same result as in the ordinary formulation.

We can now give the general expression of the action at level \(n\), which satisfies power counting and is invariant under all parity transformations in the bulk, \(t_p \rightarrow -t_p\), for \(1 \leq p \leq n\). It reads:

\[
I_n = \int d^4x dt^1 dt^2 \ldots dt^n \left( \phi_{1+2^{n-1}} \partial_n \phi_1 + \phi_{1+2^{n-1}} \left( \phi_{1+2^{n-1}} + \frac{\delta I_{n-1}}{\delta \phi_1} \right) \right.
\]

\[
+ \sum_{p=2+2^{n-1}}^{2^n} \phi_p \frac{\delta I_{n-1}[\phi_1, \ldots, \phi_{2^{n-1}}]}{\delta \phi_p} \left( \phi_{2^{n-1}-1} \right) \right)
\]

(14)

\(I_n\) is invariant under the parity transformation \(P_n\), with:

\[
\begin{align*}
t_n & \rightarrow -t_n \\
\phi_{1+2^{n-1}} & \rightarrow -\phi_{1+2^{n-1}} - \frac{\delta I_{n-1}}{\delta \phi_1} \\
\phi_p & \rightarrow \phi_p \quad \text{for} \quad p < 1 + 2^{n-1} \\
\phi_p & \rightarrow \phi_p + \partial_n \phi_{p-2^{n-1}} \quad \text{for} \quad p > 1 + 2^{n-1}
\end{align*}
\]

(15)

Under the symmetry \(P_n\), the Lagrangian density varies by a pure derivative, \(\mathcal{L}_n \rightarrow \mathcal{L}_n - \partial_n(\mathcal{L}_{n-1})\).
As for the rest of the parity transformations $P_p$ of the fields in the bulk, with $1 \leq p \leq n - 1$, their existence can be proven by induction.

Assume that the full bulk parity symmetry exists at level $n - 1$, that is, field transformations exist that leave invariant $I_{n-1}[\phi_1, \ldots, \phi_{2n-1}]$ for all transformations $t_p \rightarrow -t_p$, $1 \leq p \leq n - 1$. Then, the triangular nature of the Jacobian of the transformation $(\phi_1, \ldots, \phi_{2n-1}) \rightarrow P_p(\phi_1, \ldots, \phi_{2n-1})$ at level $n - 1$ implies that one can extend this transformation law for the new fields that occur at level $n$, $(\phi_1, \ldots, \phi_{2n}) \rightarrow P_p(\phi_1, \ldots, \phi_{2n})$ and that $I_n$, as given in eq. (14), is invariant under $P_p$, in a way that generalizes eq. (13).

Conversely, the parity symmetry and power counting imply that $I_n$ must be of the form (14). This shows that the number of parameters of the theory is the same in bulk quantization, with any given choice of the bulk dimension $n$, as in the standard formulation. These parameters are just those of an action that is renormalizable by power counting in 4 dimensions.

We can now write the action in the following form:

$$\int d^4x dt^1 dt^2 \ldots dt^n s_n \left( \bar{\Psi}_{1+2n-1} \left( \partial_n \phi_1 + \phi_1 + \frac{\delta I_{n-1}[\phi_1, \ldots, \phi_{2n-1}]}{\delta \phi_1} \right) + \sum_{p=2}^{2n-1} \bar{\Psi}_p \frac{\delta I_{n-1}[\phi_1, \ldots, \phi_{2n-1}]}{\delta \phi_p} \right)$$

(16)

A propagation occurs in the new direction $t^n$, while the equation of motions of the formulation at degree $n - 1$ are enforced in a BRST invariant way. Because the action is $s_n$ exact, and $\phi_{1+2n-1}$ is the momentum with respect to $t_n$ of $\phi_1$, the correlation functions that one can compute in the $(4 + n)$-dimensional theory, at an equal bulk component $t_n$, are identical to those computed in the theory defined by $I_{n-1}$. The proof is just as in the case of a one-dimensional bulk, and uses the BRST invariance and the translation and parity symmetries in the bulk. Finally, the correlation functions computed in the $(4 + n)$-dimensional theory, where all argument only involve a single point $T$ in the bulk, are identical to those computed from the basic four dimensional $I_0$, that is,

$$G_{N}^{I_0} \left( x_1, \ldots, x_N \right) = G_{N}^{I_n} \left( (x_1, t^p) \ldots, (x_N, t^p) \right) |_{t^p = \ldots = t^p N = T}$$

(17)

Due to translation invariance, such a correlation function is independent on the choice of $T$. An another interesting expression of the action at level $n$ is:

$$\int d^4x dt^1 dt^2 \ldots dt^n \left( s_n (\bar{\Psi}_{1+2n-1} \partial_n \phi_1) \right. \left. + s_n \bar{s}_n (\bar{\Psi}_{1+2n-1} \Psi_1 + I_{n-1}[\phi_1, \ldots, \phi_{2n-1}] \right)$$

(18)
It shows that the Hamiltonian at level \( n \) is a supersymmetric term, 
\[ H = \frac{1}{2} \{ Q_n, \bar{Q}_n \}, \]
which involves in the very simple way the action at level \( n - 1 \).

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