On generalized probabilities: correlation polytopes for automaton logic and generalized urn models, extensions of quantum mechanics and parameter cheats

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Abstract

Three extensions and reinterpretations of nonclassical probabilities are reviewed. (i) We propose to generalize the probability axiom of quantum mechanics to self-adjoint positive operators of trace one. Furthermore, we discuss the Cartesian and polar decomposition of arbitrary normal operators and the possibility to operationalize the corresponding observables. Thereby we review and emphasize the use of observables which maximally represent the context. (ii) In the second part, we discuss Pitowsky polytopes for automaton logic as well as for generalized urn models and evaluate methods to find the resulting Boole-Bell type (in)equalities. (iii) Finally, so-called “parameter cheats” are introduced, whereby parameters are transformed bijectively and nonlinearly in such a way that classical systems mimic quantum correlations and vice versa. It is even possible to introduce parameter cheats which violate the Boole-Bell type inequalities stronger than quantum ones, thereby trespassing the Tsirelson limit. The price to be paid is nonuniformity.
1 Hilbert space extensions

Since Planck’s introduction of the quantum 100 years ago [1], quantum mechanics has developed into a fantastically successful theory which appears to be stronger than ever. Despite its obvious relevance and gratifying predictive power, the question of how to proceed to theories beyond the quantum still remains not totally unjustified and is asked by eminent and prominent researchers in the area [2, 3].

Indeed, from an informal, historic perspective, it is quite likely that there will be a successor theory of quantum mechanics of some sorts which will extend this theory in many, hitherto unknown, aspects. Such a theory might even be more mindboggling than quantum theory; or it may be based on simple concepts such as information [4]; or it may be a reinterpretation of the standard quantum formalism in another set theoretic context [5, 6].

A very radical and original approach to nonclassicality has been investigated by John Harding who argues that “absolutely none of the structure of a Hilbert space is necessary to produce an orthomodular poset. [...] Rather] it is a consequence of arithmetical properties of relations” [7]. In what follows, a much more humble extension is pursued which respects the established framework of quantum mechanics.

Von Neumann’s Hilbert space formalism [8] of quantum theory is extended by considering more general forms of operators as proper realizations of physical observables. From the point of view of vector space theory, these extensions reflect well-known properties of the algebraic structures arising in quantum mechanics [9, 10]. It may nevertheless be worthwhile to review them for a proper understanding of the underlying physics.

Let us from now on consider finite dimensional Hilbert spaces. The quantum probability $P(\psi, A)$ of a proposition $A$ given a state $\psi$ is usually introduced as the trace of the product of the state operators $\rho_\psi$ and the projection operator $E_A$; i.e., $P(\psi, A) = \text{Tr}(\rho_\psi E_A)$. (This “axiom” of quantum probability has been derived for Hilbert spaces of dimension larger than two from reasonable basic assumptions by Gleason [11, 12].) The state operator $\rho_\psi$ must be (i) self-adjoint; i.e., $\rho_\psi = \rho_\psi^\dagger$, (ii) positive; i.e., $\langle x | \rho_\psi | x \rangle \geq 0$ for all $x$, and (iii) of trace one; i.e., $\text{Tr}(\rho_\psi) = 1$. Since one criterion for a pure state is its idempotence; i.e., $\rho_\varphi \rho_\varphi = \rho_\varphi$, one way to interpret $E_A$ is a measurement apparatus in a pure state $\rho_\varphi$. But while pure states can be interpreted as a system being in a given property, not every state is pure...
and thus corresponds to a projection.

We propose here to generalize quantum probabilities to properties corresponding also to nonpure states, such that the general form of quantum probabilities can be written as

$$P(\psi, \varphi) = \text{Tr}(\rho_\psi \rho_\varphi),$$

(1)

where again we require that $\rho_\varphi$ is self-adjoint, positive and of trace one. One immediate advantage is the equal treatment of the object and measurement apparatus. They appear interchangeably, stressing the conventionality of the measurement process \[13\].

One property of the extended probability measure is its positive definiteness and boundedness; i.e., $0 \leq P(\psi, \varphi) \leq 1$. The former bound follows from positivity. The latter bound by 1 can be easily proved for finite dimensions by making a unitary basis transformation such that $\rho_\varphi$ (or $\rho_\psi$) is diagonal.

A very simple example of an extended probability is the case of the total ignorance of the state of the measurement apparatus as well as of the measured system. Take $n$ nondegenerate possible outcomes for the apparatus and the state, then $\rho_\psi = \rho_\varphi = 1/n = (1/n) \text{diag}(1, 1, \ldots, 1)$, and the probability to find any combination thereof is $P(\psi, \varphi) = 1/n^2$. The extended probability reduces to the standard form if one assumes total knowledge of the state of the measurement apparatus, since then $\rho_\varphi$ is pure and thus a projection.

Another well known fact is the Cartesian and polar decomposition of an arbitrary operator $A$ into operators $B, C$ and $D, E$ such that

$$A = B + iC = DE,$$

$$B = \frac{A + A^\dagger}{2}, \quad C = \frac{A - A^\dagger}{2i},$$

$$E = \sqrt{A^\dagger A}, \quad D = AE^{-1},$$

where $B, C$ are self-adjoint, $E$ is positive and $D$ is unitary (i.e., an isometry). The last two equations are for invertible operators $A$. These are just the matrix equivalents of the decompositions of complex numbers.

If $A$ is a normal operator; i.e., $AA^\dagger = A^\dagger A$, then $B$ and $C$ commute (i.e., $[B, C] = 0$) and are thus co-measurable. (All unitary and self-adjoint operators are normal.) In this case, also the operators of the polar decomposition $D$ and $E$ are unique and commute; i.e., $[D, E] = 0$, and are thus co-measurable.
We have thus reduced the issue of operationalizability of normal operators to the self-adjoint case; an issue which has been solved positively [14].

Hence, normal operators are operationalizable either by a simultaneous measurement of the summands in the Cartesian decomposition or of the factors in a polar decomposition (cf. also [15, 16]). Indeed, all operators are “measurable” if one assumes EPR’s elements of counterfactual physical reality [10, p. 108f]. In this case, one makes use of the Cartesian decomposition, where $B$ and $C$ not necessarily can be diagonalized simultaneously and thus need not commute. Nevertheless, one may devise a singlet state of two particles with respect to the observables $B$ and $C$, and measure $B$ on one particle and $C$ on the other one.

As an example for the case of a normal operator which is neither self-adjoint nor unitary, consider

$$\text{diag}(2, i) = 1 + \sigma_3 + \frac{i}{2}(1 - \sigma_3)$$

where $\sigma_3 = \text{diag}(1, -1)$ and both summands and factors commute and thus are co-measurable.

Co-measurability is an important issue in the theory of partial algebras [17, 18], where, in accordance with quantum mechanics, operations are only allowed between mutually commuting operators corresponding to co-measurable observables. In particular, let us define the context as the set of all co-measurable properties of a physical system. By a well-known theorem, any context has associated with it a single (though not unique) observable represented by a self-adjoint operator $C$ such that all other observables represented by self-adjoint $A_i$ within a given context are merely functions (in finite dimensions polynomials) $A_i = f_i(C)$ thereof. We shall call $C$ the context operator. Context operators are maximal in the sense that they exhaust their context but they are not unique, since any one-to-one transformation of $C$ such as an isometry yields a context operator as well.

Different operators $A_i$ may belong to different contexts. Actually, the proof of Kochen and Specker [18] (of the nonexistence of consistent global truth values by associating such valuations locally) is based on a finite chain of contexts linked together at one operator per junction which belongs to the two contexts forming that junction. This fact suggests that—rather
than considering single operators which may belong to different contexts—it is more appropriate to consider context operators instead. By definition, they carry the entire context and thus cannot belong to different ones. A graphical representation of context operators has been given by Tkadlec [19], who suggested to consider dual Greechie diagrams which represent context operators as vertices and links between different contexts by edges. A typical application would be the measurement of all the $N$ contexts necessary for a Kochen-Specker contradiction in an entangle $N$ particle singlet state. In such a case, there should exist at least one observable belonging to two different contexts whose outcomes are different (cf also [20] for a similar reasoning).

2 Pitowsky polytopes for automaton logics and generalized urn models

Let us assume that the chances of sunshine in Vienna $P_s(V)$ as well as in Budapest $P_s(B)$ are 50:50. Would you believe a statement claiming that the joint probability $P_s(A \land B)$ that the sun is shining in Budapest as well as in Vienna is 0.99? No, I guess, you would not, since it appears unreasonable to claim that $P_s(A), P_s(B) \leq P_s(A \land B)$.

In the middle of the 19th century the English mathematician George Boole formulated a theory of "conditions of possible experience" [21, 22, 23]. These conditions are related to relative frequencies of logically connected events and are expressed by certain equations or inequalities. More recently, similar equations for a particular setup which are relevant in the quantum mechanical context have been discussed by Bell, Clauser and Horne and others [24, 25, 26, 27]. Itamar Pitowsky has given a geometrical interpretation of classical Boole-Bell "conditions of possible experience" in terms of correlation polytopes [28, 29, 30, 31, 39]: Take the probabilities $P_1, \ldots, P_n$ of some events $1, 2, \ldots n$ and some (or all) of the joint probabilities $P_1 \land P_2, \ldots, P_{n-1} \land P_n, P_1 \land P_2 \land P_3, \ldots$ and write them in vector form $x = (P_1, \ldots, P_n, P_1 \land P_2, \ldots, P_{n-1} \land P_n, P_1 \land P_2 \land P_3, \ldots)$. Every possible combination of all valuations$^1$ of the $n$ Boolean algebras $2^1$ formed by the

$^1$ In what follows, the terms “two-valued (probability) measure”, “two-valued state”, “valuation”, and “dispersion-free measure (state)” will be used synonymously for a lattice homomorphism $P : L \to 0, 1$ such that $P(L) = 1$. 
atoms \{i, i'\}, \ i = 1, \ldots, n (p' \text{ stands for the complement of } p) \text{ corresponds to one of the } 2^n \text{ vertices (i.e., extreme points) of a classical correlation polytope}

\[
\left\{ \lambda_1 x_1 + \cdots + \lambda_l x_l \mid l = 2^n \geq 1, \lambda_j \geq 0, \sum_{j=1}^{2^n} \lambda_j = 1 \right\}, \tag{3}
\]

where \( x_i \) stands for the truth function corresponding to the \( i \)th valuation. Thus, the vector components of \( x_j \) are either 0 or 1, and the first \( n \) components contain all \( 2^n \) possible distinct combinations thereof.

Every convex polytope in an Euclidean space has a dual description: either as the convex hull of its vertices as in Eq. (3) (V-representation), or as the intersection of a finite number of half-spaces, each one given by a linear inequality (H-representation) This equivalence is known as the Weyl-Minkowski theorem (e.g., [32, p. 29]). The problem to obtain all inequalities from the vertices of a convex polytope is known as the hull problem. One solution strategy is the Double Description Method [33] which we shall use but not review here.

What Boole did not foresee, however, is that certain events in one and the same inequality may be operationally incompatible, and that the event structure may not be a Boolean algebra. This applies to quantum mechanics as well as to Wright’s generalized urn models [34, 35], as well as to automaton partition logics [36, 37, 10].

The importance of correlation polytopes lies in the fact that they fully exploit all consistently conceivable probabilities. Their border faces correspond to Boole-Bell type inequalities. If dispersion-free states exist, every vertex corresponds to a dispersion-free state. In this sense, correlation polytopes define the probabilities of a given formal structure completely.

Classical correlation polytopes corresponding to important quantum cases have already been studied intensively [29, 30] (for a recent investigation, see [38, 39]). Nonclassical correlation polytopes and thus the associated probabilities are less known [10, 11, 12]. In a different context, Ron Wright has investigated states on nonclassical event structures in detail [34]. In what follows, we shall use his analysis to define Pitowsky correlation polytopes of automaton partition logics and generalized urn models.

Whereas for Hilbert logics of Hilbert spaces with dimension higher than or equal to three, no dispersion-free state exists [18], for generalized urn logics and automaton partition logics, two-valued states exist and can be used for
an explicit construction of the respective models. Just as for Boolean algebras every probability can be composed by the convex combination of two-valued states, any probability on orthologics (a bounded, orthocomplemented poset in which orthocomplemented joins exist) (admitting a two-valued state) is a convex combination of two-valued states \[34,\] Theorems 1.6, 1.7. We shall restrict our attention to orthologics \(L\) with a separating set of states; i.e., for every \(s, t \in L\) with \(s \neq t\), there is a two-valued state \(P\) such that \(P(s) \neq P(t)\) (an even weaker criterion would be unitality).

One immediate question is the following one: how do such nonclassical correlation polytopes of orthologics admitting two-valued states relate to classical correlation polytopes? An answer can be given in analogy to the Boolean case: The nonclassical correlation polytope \(\mathcal{C}(L)\) corresponding to some nonboolean lattice \(L\) can be defined as the convex hull of all two-valued states thereon. That is, Eq. (3) also applies for the nonclassical case; with the generalization that the set of vectors \(x_i\) corresponds to the set of all two-valued states thereon. (In the quantum mechanical case, no valuations exist for Hilbert spaces of dimension bigger than two; thus the definition cannot be applied to quantum correlation polytopes.)

Separability (unitality) implies embedability of a the orthologic \(L\) into a Boolean algebra \(B = 2^n\) with \(n\) atoms. We may consider the corresponding correlation polytope \(\mathcal{C}_n\) generated by the subset of its \(2^n\) vertices, which are its extreme points if the truth assignments are identified by vector components. The dimension of the vector space depends on the number of propositions involved. Since not all valuations of the Boolean algebra \(2^n\) need to be valuations of \(L\), \(\mathcal{C}(L)\) is a subset of \(\mathcal{C}_n\).

It should also be noted that the requirement that the automaton system can only be in a single one of the states 1, 2 and 3 imposes additional restrictions and effectively reduces the number of vertices. Let us consider a very simple explicit example. Consider the automaton partition logic

\[L = \{\{1\}\{2, 3\}\}.\{\{2\}\{1, 3\}\}.\{\{3\}\{1, 2\}\}\].\]

\(L = MO_3\) is embedable into \(B = 2^3\) with the set of atoms \(\{1, 2, 3\}\) in a straightforward manner by identifying the automaton states 1, 2, 3 with these atoms; cf. Fig. [4]. The correlation polytope \(\mathcal{C}(L)\) of \(L = MO_3\) consists of 3 vertices \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\). The corresponding (in)equalities can be easily obtained by solving the hull problem: \(P_1, P_2, P_3 \geq 0, P_1 + P_2 + P_3 =\]
1. Since all automaton logics have a set theoretic embedding of the above kind, their corresponding correlation polytopes are subsets of the correlation polytopes of the classical algebras in which they can be embedded.

Another, less trivial example, is a logic which is already mentioned by Kochen and Specker \[18\] (this is a subgraph of their $\Gamma_1$) whose automaton partition logic is depicted in Fig. 2. The correlation polytope of this lattice consists of 14 vertices listed in Table 1, where the 14 rows indicate the vertices corresponding to the 14 dispersion-free states. The columns represent the partitioning of the automaton states. The solution of the hull problem by the $\text{LPoly}$ package due to Maximian Kreuzer and Harald Skarke \[43\] yields the equalities \[44\]

$$
1 = P_1 + P_2 + P_3 = P_4 + P_{10} + P_{13},
1 = P_1 + P_2 - P_4 + P_6 + P_7 = -P_2 + P_4 - P_6 + P_8 - P_{10} + P_{12},
1 = P_1 + P_2 - P_4 + P_6 - P_8 + P_{10} + P_{11},
0 = P_1 + P_2 - P_4 - P_5 = -P_1 - P_2 + P_4 - P_6 + P_8 + P_9 , . \tag{4}
$$

The operational meaning of $P_i = P_{a_i}$ is “the probability to find the automaton in state $a_i$.” Eqs. \[4\] are equivalent to all probabilistic conditions on the contexts (subalgebras) $1 = P_1 + P_2 + P_3 = P_4 + P_5 + P_6 + P_7 = P_7 + P_8 + P_9 = P_9 + P_{10} + P_{11} = P_4 + P_{10} + P_{13}.$

Let us now turn to the joint probability case. Notice that formally it is possible to form a statement such as $a_1 \land a_{13}$ (which would be true for measure number 1 and false otherwise), but this is not operational on a single
Figure 2: Greechie diagram of automaton partition logic with a nonfull set of dispersion-free measures.

Table 1: Truth table of a logic with 14 dispersion-free states. The rows, interpreted as vectors, are just the vertices of the corresponding correlation polytope in 13 dimensions.

| #  | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_1 \land a_2$ | ⋯  |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|----------|----------|---------------|----|
| 1  | 1     | 0     | 0     | 1     | 0      | 0      | 1      | 0      | 0      | 0        | 1        | 0        |          |               |    |
| 2  | 1     | 0     | 0     | 1     | 0      | 1      | 0      | 1      | 0      | 0        | 0        | 0        |          |               |    |
| 3  | 1     | 0     | 0     | 1     | 0      | 0      | 1      | 0      | 1      | 0        | 0        | 0        |          |               |    |
| 4  | 0     | 1     | 0     | 0     | 1      | 0      | 0      | 1      | 0      | 0        | 1        | 1        |          |               |    |
| 5  | 0     | 1     | 0     | 0     | 1      | 0      | 0      | 1      | 0      | 1        | 0        | 1        |          |               |    |
| 6  | 0     | 1     | 0     | 1     | 0      | 1      | 0      | 0      | 1      | 0        | 0        | 1        |          |               |    |
| 7  | 0     | 1     | 0     | 1     | 0      | 0      | 1      | 0      | 0      | 1        | 0        | 0        |          |               |    |
| 8  | 0     | 1     | 0     | 1     | 0      | 1      | 0      | 0      | 1      | 0        | 0        | 0        |          |               |    |
| 9  | 0     | 1     | 0     | 0     | 1      | 0      | 1      | 0      | 1      | 0        | 1        | 0        |          |               |    |
| 10 | 0     | 0     | 1     | 0     | 0      | 0      | 1      | 0      | 0      | 1        | 0        | 1        |          |               |    |
| 11 | 0     | 0     | 1     | 0     | 0      | 1      | 0      | 0      | 1      | 0        | 1        | 0        |          |               |    |
| 12 | 0     | 0     | 1     | 0     | 0      | 1      | 0      | 0      | 1      | 0        | 1        | 1        |          |               |    |
| 13 | 0     | 0     | 1     | 0     | 0      | 0      | 1      | 0      | 0      | 1        | 0        | 1        |          |               |    |
| 14 | 0     | 0     | 1     | 0     | 0      | 1      | 0      | 0      | 1      | 0        | 1        | 0        |          |               |    |
automaton subject to the Moore measurement conditions \[15\], since no experiment can decide such a proposition on a single automaton. Nevertheless, if one considers a “singlet state” of two automata which are in an unknown yet identical initial state, then an expression such as \( a_1 \land a_{13} \) makes operational sense if property \( a_1 \) is measured on the first automaton and property \( a_{13} \) on the second automaton. Indeed, all joint probabilities \( a_i \land a_j \land \ldots a_n \) make sense in an \( n \)-automaton singlet context.

3 Parameter cheats

In this section, certain bijective (one-to-one) parameter transformations will be performed which artificially give classical systems a quantum flavor; and conversely, seemingly make quantum systems behave classically, at least with respect to joint probabilities. Since such transformations have other, undesirable features, we shall call them “parameter cheats.”

Consider a singlet state for which the sum of all angular momenta and spins is zero. In the quantum mechanical case, let us assume two particles of spin 1/2 in an EPR-Bohm configuration. Then the probability \( P^=\left(\theta\right) \) to find the angular momentum or spin of both particles measured along two axis which are an angle \( \theta \) apart in the same direction is given by \[14\]

\[
P_{qm}^=\left(\theta\right) = \sin^2\left(\theta/2\right) \tag{5}
\]

\[
P_{cl}^=\left(\theta\right) = \frac{\theta}{\pi} \tag{6}
\]

\[
P_s^=\left(\theta\right) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{n>1} \sin \left[(2k + 1) \left(\frac{2\Delta}{\pi} - 1\right)\right] \quad \text{as } n \to \infty
\]

\[
H\left(2\theta/\pi - 1\right) = \frac{1}{2}(1 + \text{sgn}(2\theta/\pi - 1)) \tag{7}
\]

for \( 0 \leq \theta \leq \pi \). \( P_{qm}^=\left(\theta\right) \), \( P_{cl}^=\left(\theta\right) \), \( P_s^=\left(\theta\right) \) stand for the joint classical, quantum and stronger-than-quantum probabilities, respectively. Figure 3 represents different joint probability functions of the parameter \( \theta \).

3.1 Quantum cheat for classical system

Then, in order to be able to fake a quantum form of the classical expression, we introduce a “cheat parameter” \( \delta \), which is obtained from the angular
Figure 3: Different joint probability functions of the parameter $\theta$. The solid, dashed and dot-dashed lines indicate classical, quantum and stronger-than-quantum behavior ($n = 11$), respectively.
Figure 4: a) Evaluation of the deformed parameter scale $\delta$ versus $\theta$. b) Evaluation of the linear reference parameter $\delta$.

Parameter $\theta$ by a nonlinear transformation $T : \theta \mapsto \delta$ from the Ansatz

$$P_{cl}(\theta(\delta)) = P_{cl}(\delta) = \frac{\theta(\delta)}{\pi} = \sin^2 \left( \frac{\delta}{2} \right).$$

The right hand side of Eq. (8) yields

$$\theta = \pi \sin^2 \left( \frac{\delta}{2} \right)$$

$$\delta = 2 \arcsin \sqrt{\frac{\theta}{\pi}}$$

where $0 \leq \delta \leq \pi$. Figure 4 represents a numerical evaluation of the deformed parameter scale $\delta$ in terms $\theta$.

3.2 Classical cheat for quantum system

In order to be able to fake a classical form of the quantum expression, we introduce a “cheat parameter” $\phi$, which is obtained from the angular parameter
Figure 5: Evaluation of the deformed parameter scale $\phi$ versus $\theta$.

$\theta$ by a nonlinear transformation $T : \theta \mapsto \phi$ from the Ansatz

$$P_{qm}(\theta(\phi)) = P_{qm}(\phi) = \frac{\phi}{\pi} = \sin^2 \left( \frac{\theta(\phi)}{2} \right). \quad (11)$$

The right hand side of Eq. (11) yields

$$\theta = 2 \arcsin \sqrt{\frac{\phi}{\pi}} \quad (12)$$

$$\phi = \pi \sin^2 \left( \frac{\theta}{2} \right) \quad (13)$$

where $0 \leq \phi \leq \pi$. Figure 5 represents a numerical evaluation of the deformed parameter scale $\phi$ in terms $\theta$.

### 3.3 Stronger-than-quantum (STQ) cheat for classical system

In order to be able to fake a STR form of the classical expression, we introduce a “cheat parameter” $\Delta$, which is obtained from the angular parameter $\theta$ by a nonlinear transformation $T : \theta \mapsto \Delta$ from the Ansatz

$$P_{cl}^= (\theta(\Delta)) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{n} \sin \left[ \frac{(2k+1)(2\Delta/\pi - 1)}{2k+1} \right] = \delta(\Delta), \quad (14)$$

where $n \geq 1$. In the limit,

$$\lim_{n \to \infty} \frac{4}{\pi} \sum_{k=0}^{n} \frac{\sin \left[ \frac{(2k+1)(2\Delta/\pi - 1)}{2k+1} \right]}{2k+1} = \text{sgn} \left( \frac{2\Delta}{\pi} - 1 \right).$$

13
The right hand side of Eq. (14) yields

$$\theta = \frac{\pi}{2} + 2 \sum_{k=0}^{n} \frac{\sin [(2k+1)(2\Delta/\pi - 1)]}{2k+1}$$

(15)

3.4 How do the cheats perform?

Cheats perform in a very simple way, which can be best understood if one considers the “proper” physical parameter and compares it to the “cheat” parameter. The cheat parameter effectively deforms the proper parameter range in that measures therein pretend to be in a different parameter range than the one in which the proper parameter is. It is quite clear then, that cheats can mimic almost any behavior as long as the parameter transformation remains one-to-one.

Let us consider the Clauser-Horne (CH) inequality

$$-1 \leq P(A_1B_1) + P(A_1B_2) + P(A_2B_2) - P(A_2B_1) - P(A_1) - P(B_2) \leq 0 \quad (16)$$

and a classical system on which a quantum cheat has been applied. Let the angles be

$$A_1 : \delta_1 = 0,$$
$$B_1 : \delta_2 = \pi/4,$$
$$A_2 : \delta_3 = \pi/2,$$
$$B_2 : \delta_4 = 3\pi/4.$$  

Identify $P(A_1) = P(B_1) = 1/2$ and

$$P(A_1B_1) = P_{cl}^{\pi}((\delta_2 - \delta_1)/2 = \pi/8),$$
$$P(A_2B_2) = P_{cl}^{\pi}((\delta_4 - \delta_3)/2 = \pi/8),$$
$$P(A_1B_2) = P_{cl}^{\pi}((\delta_4 - \delta_1)/2 = 3\pi/8),$$
$$P(A_2B_1) = P_{cl}^{\pi}((\delta_3 - \delta_2)/2 = \pi/8).$$

With a choice of these angles, the right hand side of Eq. (16) is violated.

Of course, we cannot expect from the cheat parameter to inherit the linear behavior of the old parameter; in particular $\delta_3(\theta_3) = \delta_1(\theta_1) + \delta_2(\theta_2)$ does not imply $\theta_3 = \theta_1 + \theta_2$, and $\delta(\theta_1) + \delta(\theta_2) = \delta(\theta_3)$; i.e.,

$$\delta_3(\theta_1 + \theta_2) \neq \delta_1(\theta_1) + \delta_2(\theta_2)$$

(17)
Therefore, one might call a parameter description to be “proper” if it is isotropic and linear with respect to a reference scheme. Of course, this leaves open the question whether or not it makes sense to refer to particular parameter descriptions as absolute ones; yet at least in the typical experimental physical context this seems evident and appropriate for most physical purposes. In such a scheme, the above mentioned cheat parameters are improper.

4 Summary

There exist extensions of quantum mechanics guided by Hilbert space theory which may be considered as generalizations of the standard formalism. All these extensions are operationalizable and may thus contribute to a better understanding of the quantum phenomena.

In the second section we discussed Boole-Bell type inequalities and Pitowsky correlation polytopes as a criterion for probabilities of automaton partition logics and generalized urn models. We find that, although the event structure is nonboolean, the corresponding probabilities can be represented as linear combinations of dispersion-free states and thus by the hull of the vertices defined by them. The corresponding correlation polytope is a subset of the classical correlation polytope of the Boolean algebra in which these logics can be embedded. The issue of Pitowsky correlation polytopes which exceed their classical counterpart remains open.

Finally, parameter transformation have been discussed which translate classical correlations into nonclassical ones and vice versa. While at the first glance the possibility for such a representation seems counterintuitive, a more detailed analysis reveals that the corresponding parameters can be used consistently but have undesirable features such as nonuniformity.

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