Group manifolds and homogeneous spaces with HKT geometry: the role of automorphisms.

A.V. Smilga

SUBATECH, Université de Nantes, 4 rue Alfred Kastler, BP 20722, Nantes 44307, France.

Abstract

We present a new simple proof of the fact that certain group manifolds as well as certain homogeneous spaces $G/H$ of dimension $4n$ admit a quaternionic triple of integrable complex structures that are covariantly constant with respect to the same torsionful Bismut connection, i.e. exhibit the HKT geometry. The key observation is that different complex structures are interrelated by automorphisms of the Lie algebra. To construct the quaternion triples, one only needs to construct the proper automorphisms, which is a more simple problem.


1 Motivation

HKT manifolds,\(^1\) discovered first by physicists \(^1\), attracted also a considerable interest of mathematicians \(^2\). Mathematicians got interested in the HKT manifolds because of their rich and nontrivial geometric structure. The physical interest stems from the fact that supersymmetric sigma models defined on HKT target spaces enjoy extended supersymmetries, which makes them relevant for some field theory applications.

I said “field theory”, and there are, indeed, interesting (1+1)-dimensional supersymmetric sigma models that live on the HKT manifolds \(^3\), but there are also more simple mechanical models where the dynamic variables—the coordinates of the manifold and their Grassmann superpartners—depend only on time, and there is no spatial dependence.

Consider, in particular, the sigma model with the following superfield action \(^4\),

\[
S = \frac{i}{2} \int d\theta dt \, g_{MN}(X) \dot{X}^M \mathcal{D}X^N - \frac{1}{12} \int d\theta dt \, C_{KLM} \mathcal{D}X^K \mathcal{D}X^L \mathcal{D}X^M, \tag{1.1}
\]

where \(X^M(t, \theta) = x^M(t) + i\theta \Psi^M(t)\) are \(D = 4n\) \(N = 1\) superfields; \(x^M\) are the coordinates on the manifold (on its one particular chart), \(\Psi^M\) are their Grassmann superpartners, \(g_{MN}\) is the metric, \(C_{KLM}\) is the totally antisymmetric torsion tensor and \(\mathcal{D} = \partial/\partial\theta - i\theta \partial/\partial t\) is the supersymmetric covariant derivative.

The \(N = 1\) supersymmetry of the action (1.1) is manifest.

**Proposition 1.** If the action (1.1) stays invariant under three extra supersymmetries

\[
\delta X^M = \epsilon_p (I_p)_N^M \mathcal{D}X^N, \tag{1.2}
\]

where \(\epsilon_p = 1, 2, 3\) are Grassmann transformation parameters and \(I_p \equiv (I, J, K)\) is a triple of integrable complex structures satisfying the quaternionic algebra

\[
I_p I_q = -\delta_{pq} + \epsilon_{pq} I_s, \tag{1.3}
\]

the target space of this model is an HKT manifold: the complex structures \(I_p\) are covariantly constant with respect to the Bismut connection,

\[
G^M_{NP} = \Gamma^M_{NP} + \frac{1}{2} g^{MK} C_{KNP}, \tag{1.4}
\]

where \(\Gamma^M_{NP}\) are the ordinary symmetric Christoffel symbols and \(C_{KNP}\) is the totally antisymmetric torsion tensor.

This assertion was proved in the component language in \(^1\) and, in the language of \(N = 1\) superfields, in recent \(^5\).

\(^1\)“HKT” means “Hyper-Kähler with Torsion”. This name is somewhat misleading, because these manifolds are not hyper-Kähler and not even Kähler, but it is well established in the literature, no better term has been proposed, and we will use it.
A large class of HKT manifolds are group manifolds. The full list of the HKT group manifolds for the compact groups including only one non-Abelian factor, is given below.

\[ SU(2l + 1), \ SU(2l) \times U(1), \ Sp(l) \times [U(1)]^l, \]
\[ SO(2l + 1) \times [U(1)]^l, \ SO(4l) \times [U(1)]^{2l}, \ SO(4l + 2) \times [U(1)]^{2l-1}, \]
\[ G_2 \times [U(1)]^2, \ F_4 \times [U(1)]^4, \ E_6 \times [U(1)]^2, \ E_7 \times [U(1)]^7, \]
\[ E_8 \times [U(1)]^8. \] (1.5)

The fact that all these manifolds are HKT was proven first in [3]. Another proof was suggested in [7]. In this paper we present still another proof, which includes some salient features of the proofs in [3, 7], but is more simple and explicit. Our explicit constructions are based on the observation that the quaternionic complex structures \((I, J, K)\) can be chosen so that \(J\) and \(K\) are derived from \(I\) by certain automorphisms of the relevant Lie algebra \(g\).

Besides group HKT manifolds, there are also many HKT manifolds representing homogeneous spaces, like \(SU(4)/SU(2)\) [7, 8]. We will discuss them in Sect. 5 of the paper.

2 Basic facts and definitions

**Definition 1.** A complex manifold is a manifold equipped with a tensor field \(I_{MN}\) satisfying the properties

(i) \(I_{MN} = -I_{NM}\);

(ii) \(I_{M}{}^N I_N{}^P = -\delta_P^M;\)

(iii) \(N_{MN}{}^K = \partial_{[M}I_{N]}{}^K - I_{M}{}^P I_N{}^Q \partial_{[P}I_{Q]}{}^K = 0.\) (2.1)

The tensor \(I_{MN}^N\) is called the complex structure.

The first two conditions in this definition imply that the manifold is even-dimentional. The third condition is the requirement for the so-called Nijenhuis tensor to vanish. According to the Newlander-Nirenberg theorem [10] (see also [11]), this is necessary and sufficient for integrability, i.e. for a possibility to introduce the complex coordinates associated with the complex structure \(I\).

Mathematicians often consider real and complex manifolds that are not equipped with a metric. But the group manifolds, which we discuss in this paper, are. Thus, we will always assume that the metric (allowing to lift and to lower the indices) is defined. For a complex manifold (and we will show soon that an even-dimensional group manifold is complex), the metric expressed in complex coordinates \(z^j\) has a Hermitian form:

\[ ds^2 = h_{jk}dz^j d\bar{z}^k, \quad \text{with} \quad \overline{h_{jk}} = h_{kj}. \] (2.2)

The tensor \(I_{MN}^M\) can be represented as

\[ I_{MN}^N = e_{MA}I_{AB}e_B^N, \] (2.3)

where \(e_{MA}\) and \(e_B^N\) are the vielbeins such that \(g_{MN} = e_{MA}e_{NA}\). \(I_{AB}\) is a matrix acting in the tangent space of our group manifold, i.e. on the corresponding Lie algebra. The properties \(I_{AB} = -I_{BA}\) and \(I^2 = -1\) hold. \(I_{AB}\) is the same at all points of the manifold.

Note, however that not all HKT manifolds have group nature. For example, The Delduc-Valent HKT manifold [6] is not associated with any group.
Definition 2. A Kähler manifold is a complex manifold with covariantly constant complex structure, $\nabla^{LC} M I N Q = 0$, where $\nabla^{LC}_M$ is the ordinary Levi-Civita covariant derivative involving the Christoffel symbols.

For a generic complex manifold, the complex structure is not covariantly constant in this sense, but it is covariantly constant with respect to an infinite set of torsionful connections. In particular,

Proposition 2. The complex structure of a generic complex manifold is covariantly constant with respect to the Bismut connection involving the structure (1.4). The totally antisymmetric torsion tensor is expressed as

$$C_{MNP} = I^Q M^S R^P (\nabla^Q I S R + \nabla^S I R Q + \nabla^R I Q S).$$

Definition 3. A hyper-Kähler manifold is a manifold admitting a triple of covariantly constant complex structures, $\nabla^{LC}_M I^p = 1, 2, 3$ $N Q = 0$, that satisfy the quaternion algebra (1.3).

Definition 4. An HKT manifold is a manifold admitting three integrable quaternionic complex structures that are covariantly constant with respect to the same universal torsionful Bismut connection.

One can easily prove that the dimension of both hyper-Kähler and HKT manifolds is a multiple integer of 4.

For the group theory considerations, we use the “physical” convention where the generators $t_A$ are Hermitian. We use the Cartan-Weyl basis where the set of all $t_A$ is subdivided into the orthonormal basis $t_{a=1,...,r}$ in the Cartan subalgebra (CSA) $H$, the positive root vectors $E_{\alpha_j}$, and the negative root vectors $E_{-\alpha_j}$ that are Hermitianally conjugate to $E_{\alpha_j}$. Then for any $h \in H$, the commutation relations

$$[h, E_{\pm \alpha_j}] = \pm \alpha_j(h) E_{\pm \alpha_j}$$

hold. Here $\alpha_j(h)$ are linear forms on $H$ called the roots. We define also the coroots $\alpha_j^\vee$ as the elements of $H$ satisfying $\alpha_j(\alpha_j^\vee) = 2\delta_{jk}$. The coroots also satisfy the following noteworthy property:

$$\omega \in G = e^{2i \pi \alpha_j^\vee} = 1, \quad \text{but} \quad e^{i \phi \alpha_j^\vee} \neq 1 \quad \text{if} \quad \phi < 2\pi.$$ 

For any positive root $\alpha$ the commutator $[E_{\alpha}, E_{-\alpha}]$ is proportional to $\alpha^\vee$. We choose the Chevalley normalization of the root vectors where

$$[E_{\alpha}, E_{-\alpha}] = \alpha^\vee.$$ 

For any two positive or negative roots $\alpha, \beta$ with $\alpha + \beta \neq 0$, the commutator $[E_{\alpha}, E_{\beta}]$ is proportional to $E_{\alpha+\beta}$ if the root $\alpha + \beta$ exists. Otherwise, this commutator vanishes. In the Chevalley normalization, the following nice relation holds [9]:

$$[E_{\alpha}, E_{\beta}] = \pm(q + 1) E_{\alpha+\beta},$$

where $q$ is the greatest positive integer such that $\alpha - q\beta$ is a root.

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3The Bismut connection was introduced in [12]. The beautiful formula (2.4) was written in [8, 13].
3 Samelson theorem

We choose a matrix representation of the group and parametrize the group elements as
\[ \omega \in G = \exp\{i t_M x^M\}, \tag{3.1} \]
where \( t_M \) are the generators satisfying \( \text{Tr}\{t_M t_N\} = C g_{MN} \) and \( x^M \) are the coordinates of the group manifold. We endow the manifold with the Killing metric
\[ g_{MN} = \frac{1}{C} \text{Tr}\{\partial_M \omega \partial_N \omega^{-1}\}. \tag{3.2} \]
Then \( ds^2 = g_{MN} dx^M dx^N \) is invariant under multiplication of \( \omega \) by any group element on the left or on the right. In the vicinity of unity, \( x^M \ll 1 \), the metric (3.2) acquires the form
\[ g_{MN} = \delta_{MN} - \frac{1}{12} f_{MPQ} f_{NPQ} x^P x^Q + o(x^2), \tag{3.3} \]
where \( f_{MPQ} \) are the structure constants. The Samelson theorem says:

**Proposition 3.** [13] Any even-dimensional group manifold is complex.

**Proof.** To prove the theorem, we should define an almost complex structure \( I_{MN} \) and show that the Nijenhuis tensor for this structure vanishes. We take care in this definition that the components of the tensor \( I_{MN} \) in the different points of the manifold are related to each other by the coordinate transformations generated by, say, a right group multiplication \( \omega \to \omega V \). For the close points, this gives
\[ I_{MN}(x) = I_{MN}(0) + \frac{1}{2} I_M^Q(0) f_{NQP} x^P - \frac{1}{2} I_N^Q(0) f_{MPQ} x^P + o(x). \tag{3.4} \]
The relation (3.4) can be alternatively written as \( I_{MN} = e_{MA} e_{NB} I_{AB} \), where
\[ e_{MA} = \delta_{MA} + \frac{1}{2} f_{MAP} x^P - \frac{1}{6} f_{AMR} f_{ANQ} x^R x^Q + o(x^2) \tag{3.5} \]
are the vielbeins satisfying \( e_{MA} e_{NA} = g_{MN} \) and \( I_{AB} \equiv I_{MN}(0) \) is the tangent space projection of the complex structure, the same at all the points.

The complex structure (3.4) is covariantly constant with respect to the Bismut connection with the torsion tensor
\[ C_{MNP} = f_{MNP}. \tag{3.6} \]
Indeed, it is straightforward to see that at the origin \( x = 0 \),
\[ \nabla^B_P I_{MN} = \partial_P I_{MN} - \frac{1}{2} f_{QPM} I_{QN} - \frac{1}{2} f_{QPN} I_{M}^Q = 0, \tag{3.7} \]
where we neglected the contribution of the ordinary Christoffel symbols \( \Gamma^Q_{PM} \), which are of order \( O(x) \). Note that the torsion tensor (3.6) is invariant under group rotations, like the metric is, and does not depend on \( x \).

\(^4C = 1/2 \) in the defining representation of \( SU(N) \).
Let us now substitute the complex structure (2.3) with the vielbeins (3.5) in the integrability condition (2.1) and consider there only leading terms $\propto O(1)$. After some simple transformations, we arrive at the identity
\begin{equation}
 f_{ABC} - I_{AD}I_{BE} f_{DEC} - I_{BD}I_{CE} f_{DEA} - I_{CD}I_{AE} f_{DEB} = 0.
\end{equation}
(3.8)
This was derived by considering the condition (2.1) at the vicinity of one particular point of the manifold with $\omega = 1$. But, bearing in mind the fact that $I_{AB}$ is the same everywhere and the isometry $G_L \times G_R$ of the metric (3.2), so that any point of the group manifold can be brought to unity (where $g_{MN} = \delta_{MN}$ and $e_{AM} = \delta_{AM}$) by a group rotation, the identity (3.8) implies the fulfillment of the condition (2.1) at all points.

Thus, we have to find an antisymmetric matrix $I$ that squares to $-1$ and satisfies the condition (3.8). The geometric problem is reduced to a pure group theory one!

The tangent space of a group manifold is the Lie algebra of the corresponding group. The matrix $I_{AB}$ can be interpreted as a linear operator acting on the generators $t_A$. We define the action of $I$ on the root vectors as
\begin{equation}
 \hat{I}E_\alpha = -iE_\alpha, \quad \hat{I}E_{-\alpha} = iE_{-\alpha}.
\end{equation}
(3.9)
Each root vector can be represented as $E_{\pm\alpha} = t_A \pm it_A$ with Hermitian $t_A$ and $t_A^*$. For example, the algebra $A_2 \equiv su(3)$ involves three positive root vectors:
\begin{align*}
 E_\alpha &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = t_1 + it_2, \\
 E_\beta &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = t_6 + it_7, \\
 E_{\alpha+\beta} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = t_4 + it_5.
\end{align*}
(3.10)

In these terms,
\begin{equation}
 \hat{I}t_A = t_{A^*}, \quad \hat{I}t_{A^*} = -t_A \quad \text{or else} \quad I_{A^*A} = -I_{AA^*} = 1.
\end{equation}
(3.11)

If the group is even-dimensional, its Cartan subalgebra is even-dimentional. Choose there the orthonormal basis $t_a$, order the set \{\text{\{t}_a\} in an arbitrary way and define
\begin{equation}
 \hat{I}t_{a_1} = t_{a_2}, \quad \hat{I}t_{a_2} = -t_{a_1}, \quad \hat{I}t_{a_3} = t_{a_4}, \quad \hat{I}t_{a_4} = -t_{a_3}, \quad \text{etc.}
\end{equation}
(3.12)
It is clear that the matrix $I$ thus defined is antisymmetric and squares to $-1$. Let us prove that it satisfies the condition (3.8).

1. Consider the L.H.S. of (3.8) with an arbitrary $C$ and $A, B$ associated with the same root vector: $A = A, B = A^*$. Bearing in mind that $I_{AA^*}I_{A^*A} = -1$, it is easy to see that the first term in (3.8) cancels the second one, while the third and the fourth term vanish.
2. Let now \( A \) and \( B \) be associated with different root vectors \( E_\alpha \) and \( E_\beta \), with \( t_C \equiv t_c \) belonging to the Cartan subalgebra. The commutator \([h, E_\alpha]\) is proportional to \( E_\alpha \) and hence \( f_{ABC} = 0 \). In this case, all the terms in (3.8) vanish.

3. A somewhat less trivial case is when \( A, B, C \) are associated with three different roots \( \alpha, \beta, \gamma \). Note that for any such triple, one can find a couple \((\alpha, \beta)\) such that the commutator \([E_\alpha, E_\beta]\) has no projection on \( E_\gamma \). Indeed, all the nonzero commutators of the positive root vectors have the form \([E_\alpha, E_\beta]\) = \( C E_{\alpha+\beta} \). Consider then a couple \((\alpha, \alpha+\beta)\).

Let \((\alpha, \beta, \gamma)\) be such a triple. Then \([t_A + it_{A^*}, t_B + it_{B^*}]\) has no projection on \( t_C + it_{C^*} \). It follows that

\[
 f_{ABC^*} - f_{A^*B^*C^*} = f_{A^*BC} + f_{AB^*C} = 0
\]  

(3.13)

(note also that the structures like \( f_{ABC} \) or \( f_{AB^*C^*} \) vanish). Bearing this in mind, it is easy to see that the relation (3.8) holds for all star attributions. For example, for \( \{ABC\} \to \{ABC^*\} \), we deduce

\[
 f_{ABC^*} + f_{B^*A^*C} - f_{CB^*A} - f_{A^*CB} = 0.
\]  

(3.14)

4 Quaternion triples

Our goal is to prove that certain group manifolds of dimension 4\(n\) are HKT, which is tantamount to say that certain groups of dimension 4\(n\) admit quaternion triples of the matrices \(I, J, K\) that satisfy the identity (3.8). Indeed, it follows from the consideration above that the corresponding complex structures are integrable. In addition, the torsion tensors for each such structure are given by (3.6) and coincide.

The basic observation is

**Proposition 4.** Let \( \Omega \) be an automorphism of the algebra \( \mathfrak{g} \). Let \( I \) be an antisymmetric matrix that squares to \(-1\) and satisfies (3.8). Then

\[
 J_{AB} = (\Omega I \Omega^T)_{AB}
\]  

(4.1)

has the same properties.

**Proof.** An automorphism of \( \mathfrak{g} \) is an orthogonal matrix \( \Omega_{AB} \) satisfying the condition

\[
 \Omega_{AD}\Omega_{BE}\Omega_{CF}f_{DEF} = f_{ABC}.
\]  

(4.2)

Then clearly \( J^T = -J \) and \( J^2 = -1 \). The fulfilment of (3.8) for the matrix (4.1) also immediately follows from the invariance of \( f_{ABC} \).

Let \( I \) be a matrix (3.11), (3.12). To prove the assertion, we have to find automorphisms that make \( J \) and \( K \) out of \( I \) in such a way that the triple \( I, J, K \) is quaternionic. Before doing so in a generic case, we consider several examples.
4.1 $SU(2) \times U(1)$

Another name for the manifold $SU(2) \times U(1) \equiv S^3 \times S^1$ is the Hopf manifold [15]. We will prove that this manifold is HKT.

The $su(2)$ algebra has only one positive root vector $E_+ = t_1 + it_2$. The Cartan subalgebra of $su(2) \oplus u(1)$ includes the generator $t_3$ of $su(2)$ and $t_0$ of $u(1)$. The canonical complex structure $\mathcal{J}$ is the matrix

$$\mathcal{J} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (4.3)$$

The first two lines in $\mathcal{J}$ follow from the definition (3.9). The third and the fourth line correspond to the sign convention $\hat{\mathcal{J}}t_3 = t_0, \hat{\mathcal{J}}t_0 = -t_3$ that we adopt.

Consider the automorphism

$$\Omega : \ t_{1,0} \rightarrow t_{1,0}, \ t_2 \rightarrow t_3 \rightarrow -t_2. \quad (4.4)$$

It brings the matrix $\mathcal{J}$ to

$$\mathcal{J} = \Omega \mathcal{J} \Omega^T = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \quad (4.5)$$

The third matrix is

$$\mathcal{K} = \mathcal{J} \mathcal{J} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (4.6)$$

One can be easily convinced that these matrices anticommute and are quaternionic. One can further notice that all of them are self-dual, $I_{AB} = \frac{1}{2} \varepsilon_{ABCD} I_{CD}$ with the convention $\varepsilon_{1230} = 1$. (They would be anti-self-dual if the opposite sign convention in the last two lines of (4.3) were chosen.) A physicist would recognize in these matrices the so-called ‘t Hooft symbols [16].

The automorphism (4.4) can be represented as $t_a \rightarrow U^\dagger t_a U$ with

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \exp \left\{ \frac{i\pi}{4} t_1 \right\} = \exp \left\{ \frac{i\pi}{4} (E_+ + E_-) \right\}. \quad (4.7)$$

If one chooses

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & 1 \end{pmatrix} = \exp \left\{ \frac{i\pi}{4} t_2 \right\} = \exp \left\{ \frac{\pi}{4} (E_+ - E_-) \right\}. \quad (4.8)$$

one arrives to the automorphism $\tilde{\Omega} : t_{2,0} \rightarrow t_{2,0}, \ t_1 \rightarrow -t_3 \rightarrow t_1$. It transforms the complex structure $\mathcal{J}$ to $\tilde{\Omega} \mathcal{J} \tilde{\Omega}^T = \mathcal{K}$.

7
4.2 $SU(3)$

This is the next in complexity case. We start with constructing the complex structure $I$. The definition (3.9) gives the matrix elements

$$I_{21} = I_{54} = I_{76} = -I_{12} = -I_{45} = -I_{67} = 1.$$  

(4.9)

In the Cartan subalgebra, we choose the basis

$$t_3 = \frac{1}{2} \text{diag}(1,0,-1), \quad t_8 = \frac{1}{2\sqrt{3}} \text{diag}(1,-2,1).$$  

(4.10)

We then define $I_{83} = -I_{38} = 1$.

We see that the $8 \times 8$ matrix $I$ splits into two blocks:

1. The block in the subspace $(4,5,3,8)$ that corresponds to the subalgebra $su(2) \oplus u(1)$ of $su(3)$, with $su(2)$ associated with the highest root $\alpha + \beta$.
2. The block in the subspace $(1,2,6,7)$ acting on the root vectors $E_{\pm \alpha}, E_{\pm \beta}$.

Each block has the form (4.3): $I = \text{diag}(I, I)$.

To find the second complex structure, we calculate $\Omega I \Omega^T$ with the automorphism $\Omega : t_a \to U^\dagger t_a U$, where

$$U = \exp \left\{ \frac{i\pi}{4}(E_{\alpha + \beta} + E_{-\alpha - \beta}) \right\} = \exp \left\{ \frac{i\pi}{2} t_4 \right\} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & \sqrt{2} & 0 \\ i & 0 & 1 \end{pmatrix}.$$  

(4.11)

The generators from the subalgebra $su(2) \oplus u(1)$ transform in the same way as in (4.4): $t_4, 8 \to t_4, 8, t_5 \to t_3 \to -t_5$. To find the transformations of the root vectors $E_{\pm \alpha}, E_{\pm \beta}$, we use the Hadamard formula:

$$e^{RX}e^{-R} = X + [R, X] + \frac{1}{2}[R, [R, X]] + \frac{1}{6}[R, [R, [R, X]]] + \ldots$$  

(4.12)

with $R = -i\pi/4(E_{\alpha + \beta} + E_{-\alpha - \beta})$. The nontrivial commutators are

$$[R, E_{\alpha}] = -\frac{i\pi}{4}E_{-\beta}, \quad [R, E_{\beta}] = \frac{i\pi}{4}E_{-\alpha},$$  

$$[R, E_{-\alpha}] = \frac{i\pi}{4}E_{\beta}, \quad [R, E_{-\beta}] = -\frac{i\pi}{4}E_{\alpha}.$$  

(4.13)

We derive

$$E_{\alpha} \xrightarrow{\Omega} \frac{1}{\sqrt{2}}(E_{\alpha} - iE_{-\beta}), \quad E_{\beta} \xrightarrow{\Omega} \frac{1}{\sqrt{2}}(E_{\beta} + iE_{-\alpha}),$$  

$$E_{-\alpha} \xrightarrow{\Omega} \frac{1}{\sqrt{2}}(E_{-\alpha} + iE_{\beta}), \quad E_{-\beta} \xrightarrow{\Omega} \frac{1}{\sqrt{2}}(E_{-\beta} - iE_{\alpha}).$$  

(4.14)

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5This is not the standard choice adopted in the physical community. Our $t_3$ is one half of the coroot $\alpha^\vee + \beta^\vee$ rather than $\alpha^\vee / 2$, as usual.

6Note that the commutators like $[E_{\alpha + \beta}, E_{\alpha}]$ vanish due to the fact that $\alpha + \beta$ is the highest root. That is why our choice (actually, the choice of Ref. [3]) of the subalgebra $su(2) \oplus u(1) \subset su(3)$ is more clever than the other choices. In fact, for $SU(3)$, one could also choose it in the ordinary way in association with the root $\alpha$ rather than with $\alpha + \beta$. One could thus construct a quaternion triple of the complex structures, but this method is not universal and cannot be easily generalized to an arbitrary group, which is our goal.
As was also the case for $I$, the complex structure $J$ splits into two blocks. The first block describes the action of $J$ in the subspace $(4,5,3,8)$. By construction, it coincides with (4.13). The second block describes the action of $J$ in the subspace $(1,2,6,7)$. Bearing in mind (3.9) and (4.14), we derive
\[
\hat{J}E_\alpha = -E_-\beta, \quad \hat{J}E_\beta = E_-\alpha, \quad \hat{J}E_{-\alpha} = -E_\beta, \quad \hat{J}E_{-\beta} = E_\alpha
\] (4.15)
or
\[
\hat{J}t_1 = -t_6, \quad \hat{J}t_2 = t_7, \quad \hat{J}t_6 = t_1, \quad \hat{J}t_7 = -t_2.
\] (4.16)
This gives the matrix that differs from (4.5) by the sign. In other words, $J = \text{diag}(\beta, -\beta)$. The third complex structure is $K = \text{diag}(K, -K)$.

### 4.3 Higher unitary groups

Consider first $SU(4) \times U(1)$. The positive roots of $SU(4)$ are schematically shown below:
\[
\begin{pmatrix}
* & \alpha & \alpha + \beta & \alpha + \beta + \gamma \\
* & * & \beta & \beta + \gamma \\
* & * & * & \gamma \\
* & * & * & *
\end{pmatrix}
\] (4.17)
The action of the complex structure $I$ on the root vectors is defined in (3.9). We choose the basis of the Cartan subalgebra of $SU(4) \times U(1)$ as follows:
\[
t_{\text{out}} = \frac{1}{2}(\alpha + \beta + \gamma)^\vee = \frac{1}{2}\text{diag}(1, 0, 0, -1), \quad t_{\text{in}} = \frac{1}{2}\beta^\vee = \frac{1}{2}\text{diag}(0, 1, -1, 0),
\]
\[
t_{15} = \frac{1}{2\sqrt{2}}\text{diag}(1, -1, -1, 1), \quad t_0 = \frac{1}{2\sqrt{2}}\text{diag}(1, 1, 1, 1).
\] (4.18)
We then define
\[
\hat{I}t_{\text{out}} = -t_{15}, \quad \hat{I}t_{15} = t_{\text{out}}, \quad \hat{I}t_{\text{in}} = -t_0, \quad \hat{I}t_0 = t_{\text{in}}.
\] (4.19)
The matrix $I$ has a block-diagonal form. We distinguish four $4 \times 4$ blocks: (I) the outer block including the generators $E_{\pm(\alpha + \beta + \gamma)}$, $t_{\text{out}}$, and $t_{15}$; (II) the internal block including the generators $E_{\pm\beta}$, $t_{\text{in}}$ and $t_0$; (III) the block including $E_{\pm\alpha}$ and $E_{\pm(\beta + \gamma)}$ and (IV) the block including $E_{\pm(\alpha + \beta)}$ and $E_{\pm\gamma}$. All these blocks have the form (4.3).

In the analogy with the previous examples, we are trying to construct the second complex structure as $\tilde{J} = \Omega I \Omega^T$ where $\Omega$ is the automorphism $t_a \to U_{a\alpha}^\dagger t_a U_{\alpha}$ with
\[
U_{\text{out}} = \exp \left\{ \frac{i\pi}{4}(E_{\alpha+\beta+\gamma} + E_{-\alpha-\beta-\gamma}) \right\}.
\] (4.20)
One immediately sees that this gives the structure (4.5) in the “outer” sector (I).
The nonzero commutators involving \( E_{\pm(\alpha+\beta+\gamma)} \) are
\[
\begin{align*}
[E_{\pm(\alpha+\beta+\gamma)}, E_{\mp\alpha}] &= \mp E_{\pm(\beta+\gamma)}, \\
[E_{\pm(\alpha+\beta+\gamma)}, E_{\mp\gamma}] &= \pm E_{\pm(\alpha+\beta)}, \\
[E_{\pm(\alpha+\beta+\gamma)}, E_{\mp(\beta+\gamma)}] &= \pm E_{\pm\alpha}, \\
[E_{\pm(\alpha+\beta+\gamma)}, E_{\mp(\beta+\gamma)}] &= \mp E_{\pm\gamma}.
\end{align*}
\] (4.21)
Then the action of \( \hat{\mathcal{J}} \) on the root vectors \( E_{\pm\alpha}, E_{\pm(\beta+\gamma)}, E_{\pm(\alpha+\beta)}, E_{\pm\gamma} \) reads
\[
\begin{align*}
\hat{\mathcal{J}}E_\alpha &= -E_{-\beta-\gamma}, \\
\hat{\mathcal{J}}E_{\beta+\gamma} &= E_{-\alpha}, \\
\hat{\mathcal{J}}E_{-\alpha} &= -E_{\beta+\gamma}, \\
\hat{\mathcal{J}}E_{-\beta-\gamma} &= E_\alpha, \\
\hat{\mathcal{J}}E_{\alpha+\beta} &= -E_{-\gamma}, \\
\hat{\mathcal{J}}E_{-\alpha-\beta} &= E_{-\beta-\gamma}, \\
\hat{\mathcal{J}}E_{\alpha-\beta} &= -E_{\gamma}, \\
\hat{\mathcal{J}}E_{-\alpha-\beta} &= E_{\alpha+\beta}.
\end{align*}
\] (4.22)
We see that \( \hat{\mathcal{J}} \) does not mix the sectors (III) and (IV) neither with the sectors (I), (II), nor with each other, and that the matrix \( \mathcal{J} \) in these sectors has the same form as in (4.16), coinciding up to the sign with (4.5).

However, the internal block (II) of the matrix \( \mathcal{J} \) is not affected by the automorphism (4.20). Indeed, its generators \( E_{\pm\beta}, t_\text{in}, t_0 \) commute with \( E_{\pm(\alpha+\beta+\gamma)} \), they belong to the centralizer of \( E_{\pm(\alpha+\beta+\gamma)} \) in \( SU(4) \). As a result, the internal block in \( \mathcal{J} \) coincides with that in \( I \). Thus, the matrix \( \hat{\mathcal{J}} \) does not anticommute with \( I \) and is not a suitable choice for the second complex structure.

Well, it is easy to understand what we should do next. We should apply \( \mathcal{J} \) the “internal” automorphism \( t^a \rightarrow U^\dagger_\text{in} t_a U_\text{in} \) with
\[
U_\text{in} = \exp \left\{ \frac{i\pi}{4} (E_\beta + E_{-\beta}) \right\}.
\] (4.23)
This automorphism does not act on \( E_{\pm(\alpha+\beta+\gamma)} \) and hence the outer block (I) is left unchanged. It acts on the internal block (II) in the same way as the automorphism (1.7) for \( SU(2) \), transforming \( \mathcal{J} \) to \( \hat{\mathcal{J}} \). But it acts nontrivially on the root vectors \( E_{\pm\alpha}, E_{\pm\gamma}, E_{\pm(\alpha+\beta)}, E_{\pm(\beta+\gamma)} \) with a potential danger that the ”good” structure of the blocks (III) and (IV) would be spoiled. Fortunately, this does not happen.

Indeed, the only nonzero commutators that one has to take into account in order to determine the action of \( \Omega_\text{in} \) on \( E_{\pm\alpha} \) are \( [E_\beta, E_\alpha] = -E_{\alpha+\beta} \) and \( [E_{-\beta}, E_{-\alpha}] = E_{\alpha-\beta} \). We derive
\[
U^\dagger_\text{in} E_\alpha U_\text{in} = \frac{1}{\sqrt{2}} (E_\alpha + iE_{\alpha+\beta}), \quad U^\dagger_\text{in} E_{-\alpha} U_\text{in} = \frac{1}{\sqrt{2}} (E_{-\alpha} - iE_{-\alpha-\beta}),
\] (4.24)
and similarly for \( E_{\pm\gamma}, E_{\pm(\alpha+\beta)}, E_{\pm(\beta+\gamma)} \). In other words, the automorphism \( \Omega_\text{in} \) mixes the positive root vectors from the “outer layer” of \( SU(4) \) with the positive root vectors and the negative root vectors with the negative ones. However, the action (1.22) of \( \hat{\mathcal{J}} \) is the same for the doublets \( (E_\alpha, E_{-\beta-\gamma}) \) and \( (E_{\alpha+\beta}, E_{-\gamma}) \) and is the same for the doublets \( (E_\gamma, E_{-\alpha-\beta}) \) and \( (E_{\beta+\gamma}, E_{-\alpha}) \). Hence the blocks (III) and (IV) in the matrix \( J = \Omega_\text{in} \hat{\mathcal{J}} \Omega^T_\text{in} \) have the same form as in \( \mathcal{J} \), coinciding with \( -\mathcal{J} \). Thus, the matrix
\[
J = (\Omega_\text{in} \Omega_\text{out}) I (\Omega_\text{in} \Omega_\text{out})^T
\] (4.25)
has the form \( \pm I \) in all its four blocks. It anticommutes with \( I \).

The procedure for the higher unitary groups \( SU(2l) \times U(1) \) and \( SU(2l+1) \) is now clear.
• Take e.g. $SU(7)$ with six simple roots $\alpha_j$. Take the highest root $\theta_0 = \sum_{j}^{6} \alpha_j$. Consider the centralizer of $E_{\pm \theta_0}$ in $su(7)$. It is $su(5)$ with the simple roots $\alpha_{j=2,3,4,5}$. Take the highest root there: $\theta_1 = \sum_{j=2}^{5} \alpha_j$. The centralizer of $E_{\pm \theta_1}$ in $su(5)$ is $su(3)$ with the highest root $\theta_2 = \alpha_3 + \alpha_4$. This Russian doll contruction terminates at this point, because the centralizer of $E_{\pm \theta_2}$ in $su(3)$ is $u(1)$, which is Abelian with no further roots. Call the roots $\{\theta_0, \theta_1, \theta_2\}$ the basic roots of $g$ [3]. Endow each pair $(E_{\theta_j}, E_{-\theta_j})$ with the corresponding coroot $\tilde{\theta}_j^{CSA} = \frac{1}{2} \theta_j^{\vee}$. Choose three other orthonormal basic vectors $e_k^{CSA}$ in the Cartan subalgebra in an arbitrary way. Define the action of $\hat{I}$ on the root vectors as in (3.9) and supplement it with

$$\hat{I} e_k^{CSA} = \tilde{e}_k^{CSA}, \quad \hat{I} \tilde{e}_k^{CSA} = - e_k^{CSA}.$$ (4.26)

Then $I$ has a block-diagonal form with the matrix $\mathcal{J}$ in each block.

• Calculate $J^{(0)} = \Omega_0 I \Omega_0^{T}$ where $\Omega_0$ is the automorphism $t_a \to U_0^{\dagger} t_a U_0$ with

$$U_0 = \exp \left\{ \frac{i \pi}{4} (E_{\theta_0} + E_{-\theta_0}) \right\}.$$ (4.27)

This automorphism transforms $\mathcal{J}$ to $\mathcal{J}$ or to $-\mathcal{J}$ for the blocks of the “outer layer”. One of these blocks includes $E_{\pm \theta_0}, t_0^{CSA}$ and $e_0^{CSA}$. For $SU(7)$, the outer layer also includes five blocks associated with the doublets $(E_{\alpha_1}, E_{-\sum_{j=2}^{5} \alpha_j})$, ..., $(E_{\sum_{j=1}^{5} \alpha_j}, E_{-\alpha_6})$ and their complex conjugates.

Six other $4 \times 4$ blocks in $J^{(0)}$ still have the form $\mathcal{J}$.

• We transform $J^{(0)}$ further going to $J^{(1)} = \Omega_1 J^{(0)} \Omega_1^{T}$ with

$$U_1 = \exp \left\{ \frac{i \pi}{4} (E_{\theta_1} + E_{-\theta_1}) \right\}.$$ (4.28)

After that, the block $(E_{\pm \theta_1}, t_1^{CSA}, e_1^{CSA})$ and three other blocks in the “middle layer” of $su(7)$ [and the outer layer of $su(5) \subset su(7)$] are converted from $\mathcal{J}$ to $\pm \mathcal{J}$. The blocks in the outer layer do not change their form.

• Two remaining unconverted blocks in the internal $su(3)$ — the block involving the generators $(E_{\pm \theta_2}, t_2^{CSA}, e_2^{CSA})$ and the block $(E_{\pm \alpha_3}, E_{\pm \alpha_4})$ — are converted by the automorphism involving

$$U_2 = \exp \left\{ \frac{i \pi}{4} (E_{\theta_2} + E_{-\theta_2}) \right\}.$$ (4.29)

We obtain thus the matrix

$$J = (\Omega_2 \Omega_1 \Omega_0) I (\Omega_2 \Omega_1 \Omega_0)^T,$$ (4.30)

which anticommutes with $I$.

• The third member of the quaternion triple we were looking for is $K = I J$ or else

$$K = (\tilde{\Omega}_2 \tilde{\Omega}_1 \tilde{\Omega}_0) I (\tilde{\Omega}_2 \tilde{\Omega}_1 \tilde{\Omega}_0)^T$$ (4.31)

where $\tilde{\Omega}_k$ is the automorphism $t_a \to \tilde{U}_k^{\dagger} t_a \tilde{U}_k$ with

$$\tilde{U}_k = \exp \left\{ \frac{\pi}{4} (E_{\theta_k} - E_{-\theta_k}) \right\}.$$ (4.32)
simple roots are presented by the matrices

\[ T \]

We recall the salient features of this algebra and the corresponding group.

The rank of \( \text{spin} \) \((\text{number of } U(2) \text{ Lie group. Let the group } G \text{ it is not so difficult to generalize the construction outlined in the previous section to an arbitrary algebra. The group } G \text{ will illustrate it in the nontrivial example of the algebra } B_3 \equiv \text{spin}(7). \) Before doing that, let us recall the salient features of this algebra and the corresponding group.

The spinor representation of this group is 8-dimensional, and its generators may be represented by the matrices \( T_{jk} = i \gamma_j \gamma_k /2 \), where \( T_{jk} \) is the generator of the spinor rotations in the \((jk)\) plane. \( \gamma_j \) are the Euclidean Dirac matrices satisfying the Clifford algebra \( \gamma_j \gamma_k + \gamma_k \gamma_j = 2 \delta_{jk} \). The commutation relations are

\[
[T_{jk}, T_{mn}] = -i (\delta_{jm} T_{kn} - \delta_{jn} T_{km} + \delta_{kn} T_{jm} - \delta_{km} T_{jn}).
\]

The rank of \( \text{spin}(7) \) is 3, the generators \( T_{12}, T_{34} \) and \( T_{56} \) constitute the basis of the CSA. The simple roots are

\[
\alpha = (1, -1, 0), \quad \beta = (0, 1, -1), \quad \gamma = (0, 0, 1).
\]

Two of them are long and the third is short. The corresponding simple coroots are

\[
\alpha^\vee = T_{12} - T_{34}; \quad \beta^\vee = T_{34} - T_{56}; \quad \gamma^\vee = 2 T_{56}.
\]

Two of them are short and the third is long. Besides the simple roots, the algebra includes two other short and four long roots:

\[
\beta + \gamma = (0, 1, 0); \quad \alpha + \beta + \gamma = (1, 0, 0); \quad \alpha + \beta = (1, 0, -1);
\]

\[
\beta + 2 \gamma = (0, 1, 1); \quad \alpha + \beta + 2 \gamma = (1, 0, 1); \quad \theta = \alpha + 2 \beta + 2 \gamma = (1, 1, 0).
\]

The corresponding coroots are

(\[
(\beta + \gamma)^\vee = 2 T_{34}; \quad (\alpha + \beta + \gamma)^\vee = 2 T_{12}; \quad (\alpha + \beta)^\vee = T_{12} - T_{56};
\]

\[
(\beta + 2 \gamma)^\vee = T_{34} + T_{56}; \quad (\alpha + \beta + 2 \gamma)^\vee = T_{12} + T_{56}; \quad (\alpha + 2 \beta + 2 \gamma)^\vee = T_{12} + T_{34}.
\]

All the coroots satisfy (2.6). We choose the Chevalley normalization for the root vectors. Then, for example, \( E_\gamma \) may be chosen as \( E_\gamma = T_{57} - iT_{67} \) so that \( [E_\gamma, E_{-\gamma}] = \gamma^\vee \).

We proceed with our construction.

- Take the highest root \( \theta \) in \( g \). Consider the centralizer \( g^{(1)} \) of \( E_{+\theta} \) in \( g \). An essential difference with the situation for the unitary groups is that this centralizer includes generically several non-Abelian factors. The simplest algebra when it happens is \( \text{spin}(7) \). Its extended Dynkin diagram including the simple roots and the lowest root is drawn in Fig. 10.

The Dynkin diagram of the non-Abelian part of the centralizer of \( E_{+\theta} \) in any Lie algebra \( g \) is obtained from its extended Dynkin diagram by crossing out the circle \(-\theta\) and also the circles for the simple roots with which \(-\theta\) is connected. For \( su(7) \) such a surgery gives the Dynkin diagram for \( su(5) \). For \( \text{spin}(7) \) we are left with two disconnected circles meaning that the centralizer is \( g^{(1)} = su(2) \oplus su(2) \oplus \text{possible Abelian summands} \), which are absent in this case. Such a centralizer involves two highest roots.
Take the highest root(s) in \( g^{(1)} \) and determine their centralizer \( g^{(2)} \). If \( g^{(2)} \) still includes non-Abelian summands, repeat the procedure. All the highest roots thus found constitute a set of the basic roots. To define the matrix \( I \), we use (3.9) and complement it as in (4.26), where \( t_{k}^{\text{CSA}} \) are the basic coroots (with a factor 1/2) and \( e_{k}^{\text{CSA}} \) are the remaining generators of CSA. The latter can be chosen arbitrary, one has only take care that they are orthogonal to \( t_{k}^{\text{CSA}} \), to each other, and are normalized in the same way as \( t_{k}^{\text{CSA}} \).

Obviously, the number of \( e_{k}^{\text{CSA}} \) should coincide with the number of \( t_{k}^{\text{CSA}} \), and this imposes a constraint on \( G \). The group manifold \( SU(2l + 1) \) is HKT: all the centralizers \( g^{(1)}, g^{(2)}, \ldots \) include only one simple non-Abelian summand with only one highest root, there are altogether \( l \) basic roots, \( l \) basic coroots, and there are exactly \( l \) generators in the CSA, which are left. A similar counting works for \( SU(2l) \times U(1) \). It also works for the group \( Spin(7) \times [U(1)]^{3} \) (the dimension of \( Spin(7) \) is 21, and we need to bring about three extra \( U(1) \) factors to make the net dimension a multiple integer of 4). The set of the basic roots in \( spin(7) \) involves the highest root \( \theta_{spin(7)} = \alpha + 2\beta + 2\gamma \) and the highest roots \( \alpha \) and \( \gamma \) of the centralizer. Three corresponding coroots \( t_{k}^{\text{CSA}} \) are matched by three generators \( e_{k}^{\text{CSA}} \) of the three \( U(1) \) factors.

But not any group manifold of dimension \( 4n \) is HKT. For example, \( Spin(8) \) of dimension 28 is complex due to the Samelson theorem, but not HKT. Indeed, look at the extended Dynkin diagram of \( spin(8) \) in Fig. 1. The centralizer of the highest root is \( su(2) \oplus su(2) \oplus su(2) \). This gives altogether 1 + 3 = 4 basic roots and 4 corresponding coroots. Thereby, the CSA of \( Spin(8) \) is exhausted and we cannot match \( t_{k}^{\text{CSA}} \) with \( e_{k}^{\text{CSA}} \). To get the latter, we have to endow \( Spin(8) \) with four extra \( U(1) \) factors. The manifold \( Spin(8) \times [U(1)]^{4} \) is HKT.

The highest root \( \theta \) can be represented as a sum of two other roots in several different ways. For example, for \( spin(7) \),

\[
\theta = (\alpha + \beta) + (\beta + 2\gamma) = (\alpha + \beta + \gamma) + (\beta + \gamma) = (\alpha + \beta + 2\gamma) + \beta. \quad (4.38)
\]

Let \( \theta = \alpha^* + \beta^* \). In the Chevalley normalization for the root vectors, their commutators

Figure 1: Extended Dynkin diagrams for some algebras. \(-\theta\) is the lowest root. A small circle stands for the short simple root in \( spin(7) \). The numbers in the circles are the Dynkin labels—the factors with which the simple roots enter \( \theta \).
are given by (2.7), (2.8). In particular,

\[
[E_{\pm \theta}, E_{\pm \alpha^*}] = \pm E_{\pm \beta}, \quad [E_{\pm \theta}, E_{\mp \beta^*}] = \mp E_{\pm \alpha^*}.
\]

(4.39)

Proceeding in the same way as for the unitary groups, we apply to \(I\) the automorphism generated by the group element

\[
U^{(0)} = \exp \left\{ \frac{i\pi}{4} (E_{\theta} + E_{-\theta}) \right\}.
\]

(4.40)

Capitalizing on the fact that the commutators (4.39) have exactly the same form as in the unitary case [cf. (4.21)], we deduce that the automorphism (4.40) converts \(J\) to \(\pm J\) in the blocks associated with \(E_{\pm \theta}, \theta^\vee\) and one more element of the CSA, and with the quartets \((E_{\pm \alpha^*}, E_{\pm \beta^*})\).

- The blocks associated with the centralizer \(g^{(1)}\) stay unconverted at this stage. We convert some of them by the automorphism

\[
U^{(1)} = \prod_k \exp \left\{ \frac{i\pi}{4} (E_{\theta_k} + E_{-\theta_k}) \right\}.
\]

(4.41)

where \(\theta_k\) are the highest roots (sometimes, only one highest root) in \(g^{(1)}\). If \(g^{(2)}\) still includes non-Abelian summands, we repeat the procedure. If necessary, repeat it again...

For \(spin(7) \oplus u(1) \oplus u(1) \oplus u(1)\), the automorphism \(U^{(0)}\) converts four outer blocks and the automorphism \(U^{(1)}\) converts two remaining blocks associated with \(E_{\pm \alpha}\) and \(E_{\pm \gamma}\). The complex structure \(J\) thus obtained anticommutes with \(I\).

5 Homogeneous spaces

Not only group manifolds \(G\) display the HKT structure. Some homogeneous spaces \(G/H\) also do [7, 8]. It is rather easy to understand in our approach.

(i) Consider \(SU(4)\). As was explained above, the second complex structure \(J\) is obtained in this case from \(I\) by two consequent automorphisms (4.20) and (4.23). The second automorphism was necessary to convert the internal block associated with the centralizer of \(E_{\pm \theta}\), which for \(su(4)\) is \(su(2) \oplus u(1)\). The summand \(su(2)\) includes \(E_{\pm \beta}\) and \(\beta^\vee\). Suppose, however, that we are interested in the manifold \(SU(4)/SU(2)\). Its tangent space involves only 12 generators of \(SU(4)\), not including \(E_{\pm \beta}\) and \(\beta^\vee\). But then the extra automorphism (4.23) is pointless. The outer automorphism (4.20) converting three blocks in the outer layer of \(I\) is sufficient.

In fact, due to the presence of the \(u(1)\) summand in the centralizer, we have several options at this point.

1. We can quotient \(SU(4)\) over \(SU(2)\) The tangent space of \(SU(4)/SU(2)\) includes only two of the three generators of the CSA of \(SU(4)\). They form together with \(E_{\pm \theta}\) the outer \(4 \times 4\) block.

2. We can quotient \(SU(4)\) over the full centralizer \(SU(2) \times U(1)\) and multiply the result by \(U(1)\). The outer block includes in this case \(E_{\pm \theta}\), the remaining generator of the CSA of \(SU(4)\), and the generator of \(U(1)\).
3. We can quotient $SU(4)$ over $U(1)$ and multiply the result by $[U(1)]^2$. In this case, the nontrivial internal block in the complex structures is left, and, to derive the complex structure $J$, we need to apply to $I$ both automorphisms (4.20) and (4.23).

4. We can leave $SU(4)$ as it is and multiply it by $U(1)$. This gives the group manifold of the preceding section. All these different manifolds are HKT.

(ii) The simplest example is $SU(3)$. The centralizer of $E_{\pm(\alpha+\beta)}$ in $SU(3)$ is $U(1)$ generated by $t_8 \propto \text{diag}(1,-2,1)$, and we can, besides $SU(3)$, consider the coset $SU(3)/U(1)$, which is also HKT.

(iii) For a higher unitary group like $SU(7)$, we have many possibilities. The centralizer of $E_{\pm\theta}$ in $SU(7)$ is $SU(5) \times U(1)$. This gives us the HKT manifolds $SU(5)$ and $SU(5)/U(1)$, where the complex structures include four $4 \times 4$ blocks: the outer block associated with $\theta$ [it includes besides $E_{\pm\theta}$ the remaining generator of the CSA of $Spin(7)$ and the generator of $U(1)$] and three blocks associated with the pairs $(\alpha^*, \beta^*)$ satisfying $\alpha^* + \beta^* = \theta$.

Alternatively, one can quotient $Spin(7)$ over only one of the $SU(2)$ factors in $G^{(1)}$, for example, over $SU_\alpha(2)$. The relevant automorphism would in this case include the outer automorphism (4.40) and the automorphism

$$U_\gamma = \exp \left\{ \frac{i\pi}{4} (E_\gamma + E_{-\gamma}) \right\}. \quad (5.1)$$

The automorphism (5.1) converts the block associated with the root $\gamma$, while the block associated with the root $\alpha$ is absent, and there is no need to convert it. We obtain the 20-dimensional HKT manifold $Spin(7)/SU(2) \times [U(1)]^2$. The block in the complex structures associated with $\theta$ includes besides $E_{\pm\theta}$ and $\theta^\gamma$ the generator of one of $U(1)$, and the block associated with $\beta$ includes besides $E_{\pm\beta}$ and $\beta^\gamma$ the generator of another $U(1)$.

It is clear now how to construct an arbitrary HKT homogeneous space.

- At level 0, take any simple or semi-simple group $G$. As was shown in the preceding section, the group manifold $G \times [U(1)]^p$ including the appropriate number of the $U(1)$ factors is HKT.

8We hope that the reader understood from the considered examples what “appropriate” means—the dimension of the CSA of $G \times [U(1)]^p$ should be 2 times more than the number of the basic roots in the described Russian doll construction.
Consider the centralizer $G^{(1)}$ of the set of the highest roots in $G$. Even if $G$ were simple, the centralizer of its highest root almost never is\(^9\) it includes several simple factors. Let $\mathcal{S}_1$ be a set of all the products of these factors.

- An HKT homogeneous space of level 1 is the quotient of $G$ over any such product $s_1 \in \mathcal{S}_1$ multiplied by an appropriate number $p$ of $U(1)$ factors. $p$ should be such that the number $p + q$, where $q$ is the number of the CSA generators that are still left in the quotient, is twice as large as the number of the remaining basic roots.

- We may now pick up the highest roots in all the elements of $\mathcal{S}_1$, consider their centralizers, define $\mathcal{S}_2$ as the set of all the products of the simple factors of all these centralizers and repeat the procedure. We thus obtain the HKT homogeneous spaces of level 2.

- If possible, we repeat the procedure again...

The full list of the homogeneous spaces derived from all classical simple Lie groups $G$ is given in Table 2 of Ref. [8].

6 Deformations and supersymmetry

Our proof was constructed assuming the “round” Killing metric (3.2) enjoying rich isometry that corresponds to the left and right group multiplications. But the Killing metric can be deformed so that the deformed metric is still HKT. Take the example of the Hopf manifold $SU(2) \times U(1) \equiv S^3 \times S^1$. Its metric can be written in the form

$$ds^2 = \frac{(dx_M)^2}{r^2},$$

where $M = 1, 2, 3, 4$, $r^2 = (x_M)^2$, and the points with the coordinates $x_M$ and $2x_M$ are identified. This metric is HKT. But it is known [1] that any metric obtained from (6.1) by a conformal transformation is still HKT. The metric “squashed” in such a way would loose its isometries. Note that the triple of the complex structures $(I^{M\bar{N}}, J^{M\bar{N}}, K^{M\bar{N}})$ for all such metrics may be chosen in exactly the same form.

The only known universal way to generate multidimensional HKT metrics is based on the harmonic superspace technique. One starts by choosing certain harmonic prepotentials and then the metric can be found as a solution of a set of complicated harmonic equations. [The problem is thus essentially more intricate than in the Kähler case: any Kähler metric in a given chart can be represented as a double derivative of an arbitrary chosen Kähler potential, $h_{jk} = \partial_j \partial_k K(z^m, \bar{z}^m)$.] We will not describe this method here, referring the reader to the monography [18], to the paper [19], where the problem of reconstructing the HKT metric in a framework of a certain $\mathcal{N} = 4$ supersymmetric sigma model in the harmonic superspace representation was solved, and to [5] with a simple demonstration that the metric thus obtained is HKT, indeed.

The analysis of [19, 5] shows that, while for a 4-dimensional HKT manifolds, the allowed deformations include only one functional parameter, the conformal factor, one has much more freedom in higher dimensions. In dimension 8, one disposes of 6 such parameters. One obtains a

\(^9\)It is only in one case: the centralizer of $E_{\pm(\alpha+\beta)}$ in $SU(3)$ is $U(1)$. 

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large “Obata family” of the HKT manifolds with coinciding Obata connections\footnote{The Obata connection \cite{20} is the torsionless connection with respect to which all three complex structures are covariantly constant. With such a choice, the metric need not be covariantly constant so that not only the directions of a tangent vector, but also its length may be changed under a parallell transport. We address the reader to Ref. \cite{5} for detailed explanations.} and coinciding complex structures.

In recent \cite{21}, the metric that seems to be a particular member of this family, which includes also the “round” $SU(3)$ metric, has been explicitly constructed. This metric does not have the isometry $SU(3) \times SU(3)$, its isometry is only $SU(3) \times SU(2)$. The problem of constructing the Killing metric (3.2) in the superspace approach rests by now unresolved.

At the current state of knowledge, one cannot exclude a logical possibility that such a construction is not possible. This would mean that the supersymmetric technique suggested in \cite{19} is not universal: it allows one to construct some HKT metrics, but not all of them. Personally, I think that this construction is universal, but the rigorous proof of this conjecture has not been given yet. One can mention, however, that such a universality proof exists for a similar supersymmetric construction of of hyper-Kähler metrics: any such metric can be reproduced by choosing an appropriate prepotential \cite{22}. We believe that a more attentive study of the supersymmetric constructions for $SU(3)$ and, eventually, reproducing in this way the Killing $SU(3)$ metric may help to construct a similar proof for all HKT manifolds.

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