Tangent Algebras

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Abstract

We study the Zariski tangent cone \(T_X \xrightarrow{\pi} X\) to an affine variety \(X\) and the closure \(\overline{T}_X\) of \(\pi^{-1}(\text{Reg}(X))\) in \(T_X\). We focus on the comparison between \(T_X\) and \(\overline{T}_X\), giving sufficient conditions on \(X\) in order that \(T_X = \overline{T}_X\). One aspect of the results is to understand when this equality takes place in the presence of the reducedness of the Zariski tangent cone. Our other interest is to consider conditions on \(X\) in order that \(\overline{T}_X\) be normal or/and Cohen–Macaulay, and to prove that they are met by several classes of affine varieties including complete intersection, Cohen–Macaulay codimension two and Gorenstein codimension three singularities. In addition, when \(X\) is the affine cone over a smooth arithmetically normal Calabi–Yau projective variety, we establish when \(\overline{T}_X\) is also (the affine cone over) an arithmetically normal Calabi–Yau like (projective) variety.

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1 Introduction

Let $A$ be an affine algebra over a perfect field $k$, and let $\Omega_{A/k}$ be the $A$-module of Kähler $k$-differentials. Classically, the properties of this module are closely related to the local singularities of $A$, embodying in particular the well-known Jacobian criterion for the smoothness of $A$. Its sheaf version for an algebraic variety over a field is fundamental in intersection theory and its cohomology is a major vehicle for the study of the global geometry of the variety.

In this paper we focus on two basic algebras associated to $\Omega_{A/k}$, where $A$ is essentially of finite type over a perfect field $k$:

1. The Rees algebra $R_{A/k}$ of $\Omega_{A/k}$ and its close predecessor $S_{A/k}$, the symmetric algebra of $\Omega_{A/k}$. If $A$ is regular these algebras coincide, but otherwise they may be quite apart and their respective properties have different impact on the nature of $A$. In [21] the ring $S_{A/k}$ was called the Zariski tangent algebra of $A$ because the closed fibers of the map $\text{Spec}(S_{A/k}) \longrightarrow \text{Spec}(A)$ are the Zariski tangent spaces to closed points of $\text{Spec}(A)$, when $A$ is an affine algebra over an algebraically closed field. Alternatively, $\text{Spec}(S_{A/k})$ is the first jet scheme of $\text{Spec}(A)$. As to $R_{A/k}$ it plays the role of the coordinate ring of a correspondence in biprojective space (see [18]). A variation on $R_{A/k}$ is the Rees algebra of the top wedge module of $\Omega_{A/k}$ which, as was argued in [20], gives a full-fledged algebraic version of Zak’s inequality for the dimension of the Gauss image of a projectively embedded variety.

2. If $A$ is reduced, $\Omega_{A/k}$ is generically free and there is a natural surjection $S_{A/k} \twoheadrightarrow R_{A/k}$. A great deal of the present work has to do with this map, whose kernel measures the failure of describing $R_{A/k}$ solely in terms of linear equations, or to use a recent terminology for modules, the failure of $\Omega_{A/k}$ being of linear type. We also study how the singularities of $A$ are reflected in the normality, Cohen–Macaulayness and Gorensteiness of $R_{A/k}$.

Let us now describe the main results in some detail.

The paper is divided into two sections. In the first section we focus on the Zariski tangent algebra $S_{A/k}$. Recall that $\Omega_{A/k} \simeq D/D^2$, where $D$ is the kernel of the multiplication map $A \otimes_k A \longrightarrow A$. Quite generally, let $S$ be a Noetherian ring and let $D \subset S$ be an ideal such that the symmetric algebra $S_A(D/D^2)$ is torsionfree over $A = S/D$. If one assumes, moreover, that $D$ is generically a complete intersection and $A$ is reduced then $S_A(D/D^2)$ is reduced. The converse to this statement, namely, that $S_A(D/D^2)$ is torsionfree if it is reduced, is known to be false in general. The corresponding question for the associated graded ring $\text{gr}_D(S)$ was treated in [15] and shown to be affirmative provided $A$ has finite projective dimension over $S$. Although the diagonal ideal $D$ does not have finite projective dimension for singular $A$, our goal in the first part of the section is to consider this converse in the framework of $S_{A/k} = S_A(D/D^2)$, trading off the homological restriction on $A$ for a condition on its defining equations (Theorem 2.1 and Corollary 2.2).
In the second part of this section we deal with the same question in the case of algebroid curves. Here, without any assumptions on the defining equations, we prove that such curves are non-singular provided $S_{A/k}$ is reduced (Theorems 2.6 and 2.8). This can be regarded as an analogue of Berger’s conjecture (see [4]) in which the reducedness of $S_{A/k}$ replaces the torsionfreeness of $\Omega_{A/k}$.

Looking for other natural regularity assumptions on $\Omega_{A/k}$ which imply, in the spirit of Berger’s conjecture, the regularity of $A$, we also prove a version in which $\Omega_{A/k}$ modulo its torsion is a Cohen–Macaulay $A$-module (Theorem 2.9). Here $A$ is no longer one-dimensional, instead the standing assumption is that it has embedding codimension 2 and is sufficiently well-behaved locally in codimension 2.

In the last part of the section we study the relationship between the reflexivity and the normality of $S_{A/k}$. Quite generally, the former is known to imply the latter. We will prove that the converse holds under suitable conditions on the defining equations of $A$ of order 2 (Proposition 2.10 and Theorem 2.11).

The second section is mainly devoted to the behavior of the Rees algebra $R_{A/k}$.

Our first fundamental result deals with the case where $A$ is locally everywhere a complete intersection over a field of characteristic zero and $R_{A/k}$ is Cohen–Macaulay (Theorem 3.1). Notice that, in this case, the module of differentials $\Omega_{A/k}$ has projective dimension one. To better situate the reader, we recall the parallel case of the symmetric algebra. Quite generally, if $A$ is a local Cohen–Macaulay ring and $E$ is a finite $A$-module of projective dimension, then $E$ satisfies condition $(F_0)$ if and only if the symmetric algebra $S_A(E)$ is a complete intersection and $E$ satisfies condition $(F_1)$ if and only if $S_A(E)$ is $A$-torsionfree (see [11, Proposition 4], [13, 1.1], [23, 3.4]; cf. Section 3 for the notion of condition $(F_1)$). As the torsionfreeness of $S_A(E)$ gives an isomorphism $S_A(E) \cong R_A(E)$, it follows that condition $(F_1)$ implies the Rees algebra $R_A(E)$ to be Cohen–Macaulay. However, the converse does not hold in general, even if one assumes the preliminary condition $(F_0)$. Nevertheless we are able to prove this converse in the case where $E$ is the module of differentials of a complete intersection over a field of characteristic zero (Theorem 3.1). Our result shows that Cohen–Macaulayness is a rather restrictive property for $R_{A/k}$. It implies for instance that if $X \subset \mathbb{P}^{2d+1}_k$ is a $d$-dimensional smooth non-degenerate complete intersection over a field $k$ of characteristic zero with homogeneous coordinate ring $A$, then $R_{A/k}$ is never a Cohen–Macaulay ring. It would be interesting to have a geometric explanation of this phenomenon.

The second kind of structural results concerns the normality of $R_{A/k}$, in which the condition $(F_2)$ will play a predominant role. Throughout $A$ is a normal domain. To enlarge the picture we consider yet another algebra associated to an $A$-module $E$, namely, $B_A(E) := \bigoplus_{i \geq 0} (E^i)**$ which could be dubbed the reflexive closure of the Rees algebra $R_A(E)$ – note it is a Krull domain with the same divisor class group as $A$, but may fail to be Noetherian. This algebra has been studied earlier in [11] (see also [25, Chapter 7]). Here we focus more closely on the case where $E = \Omega_{A/k}$. First is a basic criterion for the
equality $R_A(E) = B_A(E)$ assuming that $R_A(E)$ is normal or just satisfies Serre’s condition $(S_2)$ (Proposition 3.3). The criterion is given in terms of a new invariant that controls the growth of the local analytic spreads of $E$. We then move on to an encore of modules of projective dimension one over Cohen–Macaulay normal domains. For such a module $E$ it is known that the condition $(F_2)$ holds if and only if $S_A(E) \simeq B_A(E)$ (see [11, Proposition 4]). In particular, if $E$ satisfies $(F_2)$ then $R_A(E)$ is normal. Surprisingly, the converse holds as well provided the non-free locus of $E$ be contained in the singular locus of $A$ (Theorem 3.7).

An application to $\Omega_A/k$ when $A$ is a normal complete intersection now ensues (Corollary 3.8).

In fact it turns out that the inequalities $\text{edim } A_p \leq 2 \dim A_p - 2$ locally on the singular locus of $A$ become equivalent to the powers of $\Omega_A/k$ being integrally closed in the range $1 \leq i \leq \text{ecodim } A$.

A substantial part of the section is a push to extend the previous results to situations other than complete intersections. We have succeeded in the cases where $A$ is of low dimension or embedding codimension or is sufficiently structured. Thus, we find satisfactory answers to to the codimension 2 perfect and codimension 3 Gorenstein cases (Propositions 3.10 and 3.11). Naturally the last part requires several developments and the results become somewhat too technical to be described in this introduction. We refer the reader to the appropriate places in the last part of this work.

The section ends with some consequences of the theory so far regarding its relation to Calabi–Yau varieties. We argue that if $A$ is the homogeneous coordinate ring of a Calabi–Yau variety often $S_A/k$ or $R_A/k$ are the homogeneous coordinate rings of “Calabi–Yau like” varieties.

2 The symmetric algebra of the module of differentials

We look at the symmetric algebra $S_A/k := S_A(\Omega_A/k)$ of the module of Kähler differentials $\Omega_A/k$ as an ancestor of the corresponding Rees algebra, which will be the topic of the next section.

2.1 Reduced Zariski tangent algebras

The main result of this part has a curious geometric consequence. Let $A$ be a local domain with an isolated singularity that is essentially of finite type over a perfect field. If the Zariski tangent algebra of $A$ is reduced, but not a domain, then the defining ideal of $A$ has to contain “many” quadrics. In the language of jet schemes, the same conclusion holds if the first jet scheme of Spec$(A)$ is reduced, but not irreducible.

**Theorem 2.1** Let $A$ be a reduced ring essentially of finite type over a perfect field $k$. Assume that for every non-minimal $p \in \text{Spec}(A)$, $A_p \simeq R/I$ where $(R,n)$ is a regular local ring.
essentially of finite type over $k$ and $I \subset n^2$ is an $R$-ideal satisfying

$$\mu(I + n^3/n^3) \leq \dim R - 1.$$ 

Then $S_{A/k}$ is reduced (if and only if it is $A$-torsionfree.

Proof. The “if” statement was explained above.

To show the converse, let $T \subset S_{A/k}$ denote the $A$-torsion submodule of $S_{A/k}$ and suppose that $T \neq 0$. Then there exists an associated prime of $S_{A/k}$ contracting to a non-minimal prime of $A$. We may further take such a non-minimal prime to be minimal among all non-minimal primes of $A$ that are contracted from some minimal prime of $S_{A/k}$. By localizing at this non-minimal prime, we do not change either the hypotheses or the conclusion of the statement, so we can reduce the argument to the situation in which $(A,m,K)$ is local and $m$ is the contraction of minimal prime of $S_{A/k}$. Moreover, every minimal prime of $S_{A/k}$ not containing $m$ must contract to a minimal prime of $A$, hence contains the torsion $T$. From this follows, since $S_{A/k}$ is reduced, the crucial relation

$$T \cap mS_{A/k} = 0. \quad (1)$$

Now $S_{A/k}/mS_{A/k}$ is a standard graded polynomial ring over $K$ in $n = \mu(\Omega_{A/k})$ variables. The equality (1) implies that $T$ is mapped isomorphically onto its image in this polynomial ring. Let $h(t) = \dim_K T_t$ and $r = \min\{t \geq 0 \mid T_t \neq 0\}$. It follows that $h(t)$ is at least the number of monomials of degree $t - r$, in other words,

$$h(t) \geq \left( \frac{t - r + n - 1}{n - 1} \right).$$

Now consider a presentation $A \simeq R/I$ as given by assumption, where $(R,n)$ is a regular local ring essentially of finite type over $k$ and $I \subset n^2$. Set $m = \mu(I + n^3/n^3)$ and $(A',m') = (A/m^2, m/m^2)$. The usual $A$-free presentation of $\Omega_{A/k}$ by means of a Jacobian matrix induces a presentation

$$A'^m \rightarrow A'^n \rightarrow \Omega_{A/k} \otimes_A A' \rightarrow 0,$$

yielding, for every $t \geq 1$, an exact sequence

$$A'^m \otimes_{A'} S_{t-1}(A'^n) \rightarrow S_t(A'^n) \rightarrow S_t(\Omega_{A/k}) \otimes_A A' \rightarrow 0.$$

The equality (1) also says that, for every $t \geq 0$, the graded piece $T_t$ of $T$ is a $K$-vector space of dimension $h(t)$ and a direct summand of $S_t(\Omega_{A/k})$ as an $A$-module. Hence $S_t(\Omega_{A/k}) \otimes_A A'$ too admits $K^{\oplus h(t)}$ as a direct summand. Therefore $m'^{\oplus h(t)}$ is a direct summand of the image of $A'^m \otimes_{A'} S_{t-1}(A'^n)$ in $S_t(A'^n)$, which implies that

$$m \left( \frac{t - 1 + n - 1}{n - 1} \right) \geq \mu(m') h(t) \geq \mu(m') \left( \frac{t - r + n - 1}{n - 1} \right),$$

which completes the proof.
for every $t$. Observing that $\dim R = \mu(m) = \mu(m')$, this inequality contradicts the assumption $m \leq \dim R - 1$. 

**Corollary 2.2** Let $(R, n)$ be a regular local ring essentially of finite type over a perfect field $k$, and let $I \subset R$ be an ideal such that $A = R/I$ is reduced. Assume that one of the following conditions holds:

(a) $\mu(I_p) \leq \dim R_p - 1$ for every non-minimal $p \in V(I)$;
(b) $I \subset n^3$ and $A$ is an isolated singularity.

Then $S_{A/k}$ is reduced (if and only if it is $A$-torsionfree).

**Remark 2.3** The full force of Theorem 2.1 takes form in the case of a homogeneous ideal $I \subset k[X] = k[X_1, \ldots, X_n]$ such that $A = k[X]/I$ is reduced and regular on the punctured spectrum. If $S_{A/k}$ is reduced, but not torsionfree, then $\text{Proj } (k[X]/I) \subset \mathbb{P}_k^n$ lies on the intersection of $n$ independent quadrics because in this situation $\dim_k[I]_2 \geq n$ by Theorem 2.1.

There is a converse when $\dim_k[I]_2 \leq 2 \text{height } I$, namely, if $S_{A/k}$ is torsionfree then $n \geq 2 \text{height } I + 1$ (cf. the beginning of Section 3), hence $\text{Proj } (k[X]/I) \subset \mathbb{P}_k^n$ cannot lie on the intersection of $n$ independent quadrics.

The next two examples illustrate the above results. The first shows that Theorem 2.1 is sharp, i.e., that the assumption on the numbers of generators cannot be relaxed.

**Example 2.4** Consider the homogeneous coordinate ring $A$ of the Veronese surface in $\mathbb{P}_k^5$. Here $A = k[X]/I_2(X)$ where $X$ is a symmetric $3 \times 3$ matrix of indeterminates over $k$ (assumed to be of characteristic $\neq 2$). Notice that $A$ is a Cohen-Macaulay domain with an isolated singularity, $I_2(X)$ is generated by quadrics, and $\mu(I_2(X)) = 6 = \dim k[X]$. It can be seen with the aid of Macaulay that $S_{A/k}$ is reduced, but not $A$-torsionfree. In fact, $S_{A/k}$ has exactly two minimal primes, $T$ and $(X)S_{A/k}$, the first of which defines $\mathbb{R}_{A/k}$.

Another computation with Macaulay shows that $S_{A/k}$ is Cohen–Macaulay. This example incidentally answers a question posed in [10, Remark after (2)] concerning the existence of a Cohen–Macaulay generic complete intersection ideal $D \subset S$ for which the symmetric algebra $\mathcal{S}(D/D^2)$ is Cohen–Macaulay, but the symmetric algebra $\mathcal{S}(D)$ is not. Mark Johnson obtained examples of the latter behavior even in a polynomial ring.

**Example 2.5** Let $A = k[X]/I_2(X)$ where $(X)$ is the $r$-catalecticant matrix

$$
\begin{pmatrix}
X_1 & X_2 & X_3 & X_4 \\
X_{r+1} & X_{r+2} & X_{r+3} & X_{r+4}
\end{pmatrix}
$$

with $1 \leq r \leq 4$. Note that for $r = 4$ we obtain the $2 \times 4$ generic matrix and for $r = 1$, the usual $2 \times 4$ generic Hankel matrix. Also $A$ is a Cohen–Macaulay (domain) in all cases, being a specialization of the generic situation. Clearly $A$ is an isolated singularity. For values $1 \leq r \leq 4$
2 the Zariski algebra $S_{A/k}$ is neither Cohen–Macaulay nor torsionfree. Geometrically, we see the reason for $S_{A/k} \neq R_{A/k}$ since in this range the tangential variety $\text{Proj}(k \otimes_A R_{A/k}) \subset \mathbb{P}_{k}^{r+3}$ to $\text{Proj}(k[X]/I_2(X)) \subset \mathbb{P}_{k}^{r+3}$ is a proper subvariety of the ambient space. For values $3 \leq r \leq 4$ the Zariski algebra $S_{A/k}$ is Cohen–Macaulay (computer calculation), hence torsionfree because $r + 4 \geq 2 \cdot \text{height } I_2(X) + 1 = 7$ (see [23, 3.3]).

2.2 Analogues of Berger’s conjecture

When $\dim A = 1$ one can make Theorem 2.1 more precise. Here it is natural to work in the more general setting of complete $k$-algebras. Instead of the module of Kähler differentials we use the universally finite module of differentials, defined in a n analogous way as $\mathbb{D}/\mathbb{D}^2$ where $\mathbb{D}$ is the kernel of the multiplication map $A \otimes_k A \rightarrow A$. Our result is reminiscent of Berger’s well-known conjecture asserting that the module of differentials of a reduced curve singularity over a perfect field cannot be torsionfree (see [4]). It is clear at least that the entire symmetric algebra $S_{A/k}$ cannot be $A$-torsionfree, as otherwise the module of differentials would satisfy condition $(F_1)$, which translates into the inequality $\text{edim } A \leq 2 \dim A - 1 = 1$ (see the discussion at the beginning of Section 3).

Theorem 2.6 Let $k$ be an algebraically closed field and assume that $A = k[[x_1, \ldots, x_n]]$ is a one–dimensional domain. Then $S_{A/k}$ is reduced (if and) only if $A$ is regular.

Proof. Assume that $S_{A/k}$ is reduced. To argue by way of contradiction assume that $n = \text{edim } A \geq 2$. Letting $k[[t]]$ denote the integral closure of $A$, we may arrange so that in the $t$–adic valuation one has

$$v(x_1) < v(x_2) < v(x_3) \leq \cdots \leq v(x_n)$$

and $v(x_2)/v(x_1)$ is not an integer. Write $A = k[[X_1, \ldots, X_n]]/(f_1, \ldots, f_m)$, with $f_i \in (X_1, \ldots, X_n)^2$, $T_j = dX_j$ and $t_j = dx_j$. Notice that

$$S_{A/k} \simeq A[t_1, \ldots, t_n] : = A[T_1, \ldots, T_n]/(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} T_j \mid 1 \leq i \leq m).$$

Here $\partial f_i/\partial x_j$ denotes the image of $\partial f_i/\partial X_j$ in $A$. Clearly these elements are contained in the maximal ideal $m$ of $A$.

Next, let $J$ be an $A$–ideal isomorphic to $\Omega_{A/k}$ modulo its torsion and let $T$ be the kernel of the natural map from $S_{A/k}$ onto the Rees algebra $R_A(J)$. By the reducedness of $S_{A/k}$, one has $T \cap mS_{A/k} = 0$ as in the previous section (see [14]). In particular $\text{Supp}(mS_{A/k}) \cap V(mS_{A/k}) \subset V(T + mS_{A/k})$ as subsets of $\text{Spec}(S_{A/k})$. On the other hand,

$$\dim S_{A/k}/(T + mS_{A/k}) = \dim R_A(J)/mR_A(J) = 1,$$
and therefore \((mS_{A/k}, t_3, \ldots, t_n) \notin \text{Supp}(mS_{A/k})\).

Thus \((mS_{A/k}) (mS_{A/k}, t_3, \ldots, t_n) = 0\). From the presentation
\[ S_{A/k}/(t_3, \ldots, t_n) = A[T_1, T_2]/L \]
with
\[ L = \left( \frac{\partial f_i}{\partial x_1} T_1 + \frac{\partial f_i}{\partial x_2} T_2 \mid 1 \leq i \leq m \right), \]
we see that
\[ mA[T_1, T_2] \cdot m = LA[T_1, T_2] \cdot mA[T_1, T_2]. \]
In particular, there exists a polynomial \(g(T_1, T_2) \in A[T_1, T_2] \setminus mA[T_1, T_2]\) with
\[ g(T_1, T_2) \cdot m \subset \left( \frac{\partial f_i}{\partial x_1} T_1 + \frac{\partial f_i}{\partial x_2} T_2 \mid 1 \leq i \leq m \right). \]
Comparing coefficients, we conclude that
\[ m = \left( \frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2} \mid 1 \leq i \leq m \right). \]

Since \(v(x_1)\) is the smallest positive element in the value semigroup of \(A\), there exists an \(i\) such that \(v(x_1) = v(\partial f_i/\partial x_1)\), or \(v(x_1) = v(\partial f_i/\partial x_2)\). Write \(f = f_i\). Assuming the first, we have \(f = aX_1^2 + g\), with \(a \in k\) and \(g \in (X_1^2) + (X_1, \ldots, X_n)(X_2, \ldots, X_n)\). Obviously \(v(g(x_1, \ldots, x_n)) > 2v(x_1)\) and hence \(v(\partial g/\partial x_1) > v(x_1)\). Therefore \(v(\partial f/\partial x_1) = v(x_1)\) implies \(a \neq 0\), while \(f(x_1, \ldots, x_n) = 0\) implies \(a = 0\). This is a contradiction.

Next assume \(v(\partial f/\partial x_2) = v(x_1)\) and write
\[ f = \sum_{i=2}^{\infty} a_i X_1^i + bX_1X_2 + h, \]
where \(a_i \in k\), \(b \in k\) and \(h \in (X_1^2, X_2)X_2 + (X_1, \ldots, X_n)(X_3, \ldots, X_n)\). Here \(v(h(x_1, \ldots, x_n)) > v(x_1) + v(x_2)\) and hence \(v(\partial h/\partial x_2) > v(x_1)\). Therefore \(v(\partial f/\partial x_2) = v(x_1)\) implies \(b \neq 0\). But then \(f(x_1, \ldots, x_n) = 0\) implies
\[ v(x_1) + v(x_2) = v(bx_1x_2) = v\left( \sum_{i=2}^{\infty} a_i x_1^i \right) = lv(x_1) \]
for some integer \(l\). But this contradicts our assumption that \(v(x_2)/v(x_1)\) is not an integer. \(\Box\)

**Remark 2.7** The above result means that the first jet scheme of an algebroid curve singularity over an algebraically closed field cannot be reduced.
The proof of Theorem 2.6 also gives the following stronger result:

**Theorem 2.8** Let $k$ be a perfect field, let $A = k[[x_1, \ldots, x_n]]$ be a one–dimensional ring and assume that $S_{A/k}$ is reduced. Then $A/p$ is regular for every minimal prime ideal $p$ of $A$. If in addition $\text{char}(k) = 0$ and $A$ is quasi–homogeneous (i.e., the completion of a positively graded $k$–algebra), then $A$ is regular.

**Proof.** We use the notation of the previous proof. Of course $A$ is reduced and we may assume that $k$ is algebraically closed. Suppose there exists a minimal prime $p$ of $A$ such that $A/p$ is not regular. Then choose a minimal generating set $x_1, \ldots, x_n$ of $m$ such that

$$v(x_1 + p) < v(x_2 + p) < v(x_3 + p) \leq \cdots \leq v(x_n + p) \leq \infty$$

and the ratio of the first two integers is not an integer (here $v$ is the $t$–adic valuation on the integral closure $k[[t]]$ of $A/p$). The rest of the proof proceeds as before replacing $S_{A/k}/(t_3, \ldots, t_n)$ by $S_{A/k}/(pS, t_3, \ldots, t_n)$.

Now assume that $\text{char}(k) = 0$ and $A$ is quasi–homogeneous. Let $\tau$ denote the kernel of the Euler map $\Omega_{A/k} \rightarrow m$. Since $\tau$ is the torsion of $\Omega_{A/k}$, by the reducedness of $S_{A/k}$ we have $\tau \cap m\Omega_{A/k} = 0$ as in the proof of Theorem 2.1 hence $\tau$ is a direct summand of $\Omega_{A/k}$. It follows that $\tau = 0$ since $\mu(\Omega_{A/k}) = \text{edim } R = \mu(m)$. But then $A$ is regular because in the quasi–homogeneous case the Berger conjecture is true (see [17, 4.4]).

We shall now consider another version of Berger’s conjecture. Throughout $\tau_A(E)$ will denote the $A$-torsion of an $A$-module $E$. If $(A, m)$ is a Noetherian local ring, we write $\text{ecodim } A = \mu(m) - \dim A$. Let $A \simeq R/I$ be a Cohen–Macaulay normal ring, where $R$ is a regular local ring essentially of finite type over a perfect field $k$ and $I$ is an ideal of height 2 (i.e., $\text{ecodim } A \leq 2$). Here the first Koszul homology $H_1(I)$ of a generating set of $I$ is Cohen–Macaulay (see [2, 2.1(a)]), in particular the natural complexes

$$0 \rightarrow H_1(I) \rightarrow A^m \rightarrow I/I^2 \rightarrow 0 \quad (2)$$

and

$$0 \rightarrow I/I^2 \rightarrow A^n \rightarrow \Omega_{A/k} \rightarrow 0 \quad (3)$$

are exact.

**Theorem 2.9** Let $(A, m)$ be a normal local Cohen–Macaulay ring essentially of finite type over a perfect field $k$ satisfying the following conditions:

(a) $\text{ecodim } A = 2$;

(b) $A$ is an almost complete intersection locally in codimension 2.
If $\Omega_{A/k}/\tau_A(\Omega_{A/k})$ is a Cohen–Macaulay module then $A$ is regular.

Proof. Set $\Omega = \Omega_{A/k}$ and let $\omega = \omega_A$ denote the canonical module of $A$. Write $A = R/I$, where $R$ is a regular local ring of essentially of finite type and of transcendence degree $n$ over $k$ and $I$ is a perfect ideal of height 2. It suffices to prove that $I$ is a complete intersection. For in this case $\Omega$ is torsionfree of finite projective dimension – thus $\Omega = \Omega/\tau_A(\Omega)$ is a maximal Cohen–Macaulay module of finite projective dimension, hence necessarily free.

We first consider the case $\dim A = 2$. Supposing $I$ is not a complete intersection we have that this ideal is minimally generated by 3 elements. Consider a minimal presentation

$$0 \rightarrow R^2 \xrightarrow{\varphi} R^3 \rightarrow I \rightarrow 0.$$  \hfill (4)

From (2) and (3) we obtain an exact sequence

$$0 \rightarrow \omega \simeq H^1(I) \rightarrow A^3 \xrightarrow{\Theta} A^n \rightarrow \Omega \rightarrow 0,$$  \hfill (5)

with $\Theta$ the transposed Jacobian matrix of the 3 generators of $I$. Moreover, since $\Omega/\tau_A(\Omega)$ is Cohen–Macaulay and $\tau_A(\Omega)$ has grade at least 2, $\Ext^1_A(\Omega, \omega) = 0$. Thus applying $\sim = \Hom_A(-, \omega)$ to (5) one can see that there is an exact sequence

$$\omega^n \xrightarrow{\Theta^\vee} \omega^3 \rightarrow \omega^\vee \simeq A \rightarrow A/I_1(\varphi)A \rightarrow 0.$$  

Hence we obtain the exact sequence

$$\omega^n \xrightarrow{\Theta^\vee} \omega^3 \rightarrow I_1(\varphi)/I \rightarrow 0.$$  \hfill (6)

As $I$ is a perfect ideal of height 2, but not a complete intersection, one has $I \subset I_1(\varphi)^2$. In particular, $I_1(\Theta) \subset I_1(\varphi)$. Thus tensoring (3) with $R/I_1(\varphi)$ gives

$$(\omega/I_1(\varphi)\omega)^3 \simeq I_1(\varphi)/I \otimes_R R/I_1(\varphi) \simeq I_1(\varphi)/I_1(\varphi)^2.$$  

On the other hand, dualizing (4) into $R$ yields $\omega \otimes_R R/I_1(\varphi) \simeq (R/I_1(\varphi))^2$. It follows that

$$I_1(\varphi)/I_1(\varphi)^2 \simeq (\omega/I_1(\varphi)\omega)^3 \simeq (R/I_1(\varphi))^6.$$  

Since $R$ is regular, [24, 1.1] implies that $I_1(\varphi)$ is generated by a regular sequence of length 6. But this is impossible in $R$, a ring of dimension four.

We now consider the case of arbitrary dimension. By the above, $A$ is regular locally in codimension 2. Therefore (2) and (3) imply that $\Omega$ is torsionfree, hence Cohen–Macaulay by assumption. But then also $I/I^2$ is Cohen–Macaulay by (3), which forces $I$ to be a complete intersection (see [3, 2.4]). 

$\blacksquare$
2.3 Normal Zariski tangent algebras

In this part we give sufficient conditions for the normality of $S_{A/k}$ to imply the reflexivity of it graded components. The conditions are stated in terms of $\mu(I + n^3/n^3)$, where $I$ is the defining ideal of $A$ in $(R, n)$. We will write $\ell(\cdot)$ for the analytic spread of an $R$-ideal (see also the beginning of Section 3).

**Proposition 2.10** Let $(R, n)$ be a regular local ring essentially of finite type over a perfect field $k$ and let $I \subset n^2$ be an ideal such that $A = R/I$ is reduced. Write $g$ for the height of $I$ and $m$ for the maximal ideal of $A$.

(a) If $S_{A/k}$ is equidimensional and $(S_{A/k})(m)$ is regular, then $\mu(I + n^3/n^3) \geq 2g$.

(b) Assume that $\text{char}(k) = 0$ and $R$ is the local ring of the polynomial ring $k[X_1, \ldots, X_n]$ at its homogeneous maximal ideal. Let $I_2$ denote the ideal of $R$ generated by 2-forms in $X_1, \ldots, X_n$ such that $I + n^3 = I_2 + n^3$. If $S_{A/k}$ is equidimensional and $(S_{A/k})(m)$ is regular, then $\ell(I_2) = 2g$.

**Proof.** Knowingly, $S_{R/k}$ is a polynomial ring over $R$. More precisely, $S_{R/k} \simeq R[T]$, where $T = T_1, \ldots, T_n$ may be chosen to be the differentials of a transcendence basis of $R$ over $k$ in part (a) and the differentials of $X_1, \ldots, X_n$ in part (b). Set $K = R/n$, so that the residue field of $R(T)$ is $K(T)$. Write $d := \text{trdeg}_k(R/n) = \text{trdeg}_k(A/m) = \text{trdeg}_k(K)$. There is a presentation

$$(S_{A/k})(m) \simeq R(T)/\mathcal{J}.$$ 

Since $S_{A/k}$ is equidimensional, we have

\[
\text{height } \mathcal{J} = \dim R(T) - \dim S_{A/k} = d + n - (\dim S_{A/k} - n) = d + n - ((d + 2n - 2g) - n)) = 2g,
\]

and since $(S_{A/k})(m)$ is regular, we have

\[
\dim_{K(T)}(\mathcal{J} + n^2/n^2) = 2g.
\]

Now let $f_1, \ldots, f_m \in R$ be so chosen that they generate $I$ modulo $n^3$ for part (a), and for part (b) they be homogeneous quadrics in $k[X_1, \ldots, X_n]$ generating the ideal $I_2$. In either case, one has

\[
\mathcal{J} + n^2/n^2 = \langle df_i \mid 1 \leq i \leq m \rangle + n^2/n^2.
\]

It follows that $m \geq 2g$, thus proving (a).

To prove (b), let $f = f(X)$ denote any of the chosen forms $f_1, \ldots, f_m$ in $k[X] = k[X_1, \ldots, X_n]$. Then

$$X \cdot \varphi \cdot T^t = T \cdot \varphi \cdot X^t,$$

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where \( \varphi \) stands for the Hessian matrix of \( f \). From this one sees that

\[
\sum_{j=1}^{n} \frac{\partial f(X)}{\partial X_j} T_j = \sum_{j=1}^{n} \frac{\partial f(T)}{\partial T_j} X_j,
\]

hence

\[
df = \sum_{j=1}^{n} \frac{\partial f(T)}{\partial T_j} X_j.
\]

We deduce that

\[
\dim_{k(T)}(df_i | 1 \leq i \leq m) + \frac{n^2}{n^2} = \text{rank} \left( \frac{\partial f_i(T)}{\partial T_j} \right)_{i,j} = \text{rank} \left( \frac{\partial f_i(X)}{\partial X_j} \right)_{i,j} = n - \text{rank} \Omega_{R/k[f_1,\ldots,f_m]} = n - \text{trdeg}_{k[f_1,\ldots,f_m]} R
\]

(see also [19, 1.1]). On the other hand, since \( f_1,\ldots,f_m \) are forms of the same degree generating the ideal \( I_2 \), we have \( k[f_1,\ldots,f_m] \simeq R(I_2) \otimes_R k \), hence \( \dim k[f_1,\ldots,f_m] = \ell(I_2) \).

Summing up, we have proved that

\[
2g = \dim_{k(T)}(J + \frac{n^2}{n^2}) = \ell(I_2).
\]

\[\square\]

**Theorem 2.11** Let \( A \) be a normal domain essentially of finite type over a perfect field \( k \). Assume that one of the following conditions holds:

(a) For every non-regular prime \( p \in \text{Spec}(A) \), \( A_p \simeq R/I \) where \((R,n)\) is a regular local ring essentially of finite type over \( k \) and \( I \subset n^2 \) is an \( R \)-ideal satisfying \( \mu(I + n^3/n^3) \leq 2 \text{height} I - 1 \);

(b) \((\text{char}(k) = 0) \) \((R,n) \) is the local ring of \( k[X_1,\ldots,X_n] \) at \( (X_1,\ldots,X_n) \), \( A = R/I \), and \( \ell(I_2) \neq 2 \text{height} I \), where \( I_2 \) denotes the ideal of \( R \) generated by 2-forms in \( X_1,\ldots,X_n \) such that \( I + n^3 = I_2 + n^3 \).

Then \( S_{A/k} \) is normal (if and only if \( S_i(\Omega_{A/k}) \) is a reflexive module for every \( i \geq 1 \).

**Proof.** We prove that the natural inclusion \( S_{A/k} \subset \bigoplus_{i \geq 0} S_i(\Omega_{A/k})^{**} \) is an equality assuming that \( S_{A/k} \) is normal. It suffices to show this after localizing at the codimension one primes of \( S_{A/k} \). Thus, let \( q \in \text{Spec}(S_{A/k}) \) be a prime of height one. We are done once we prove that \( A_{q \cap A} \) is regular. Now, replacing \( A \) by \( A_{q \cap A} \), we may assume that \((A,m)\) is local and \( m \subset q \).

By assumption, \( (S_{A/k})_q \) is regular, hence so is its further localization \( (S_{A/k})_m \). According to Proposition 2.10 this contradicts our assumptions, unless \( A \) is regular. \[\square\]
3 The Rees algebra of the module of differentials

We will draw upon [22] for terminology and basic notions about Rees algebras of modules. Let $E$ be a finite module over a Noetherian ring $A$ and assume that $E$ is generically free (i.e., free locally at every associated prime of $A$). In this setting the Rees algebra $\mathcal{R}_A(E)$ of $E$ is defined as the symmetric algebra of $E$ modulo $A$-torsion, $\mathcal{R}_A(E) = S_A(E)/\tau_A(S_A(E))$. As is known, this definition retrieves the usual notion of the Rees algebra of an ideal containing a regular element. If in addition $E$ is torsionfree, then $E$ can be embedded into a finite free module $F$ and the Rees algebra of $E$ can be identified with the image of $S_A(E)$ in $S_A(F)$. In case $S_A(E) = \mathcal{R}_A(E)$ we say that $E$ is a module of linear type.

We recall the following additional notions. Let $E$ be a finite module over a Noetherian ring $A$. Suppose that $E$ has a rank $r$ (i.e., $E$ is free of rank $r$ locally at every associated prime of $A$). Given an integer $t \geq 0$, we say that $E$ satisfies condition $(F_t)$ if $\mu(E_p) \leq r + \dim A_p - t$ for every $p \in \text{Spec}(A)$ such that $E_p$ is not free. In terms of Fitting ideals this condition is equivalent to the condition that height $\text{Fitt}_i(E) \geq i - r + t + 1$ for $i \geq r$.

By the same token, one can introduce yet another condition based on the analytic spread $\ell(E)$ of a module over a local (or graded) ring $(A, m)$, defined to be the Krull dimension of the residue algebra $\mathcal{R}_A(E)/m\mathcal{R}_A(E)$ (see [22]). We will say that a finite module $E$ over a Noetherian ring $A$ satisfies condition $(L_t)$ to mean that $\ell(E_p) \leq \dim A_p + r - t$ for every $p \in \text{Spec}(A)$ with $\dim A_p \geq t$. Roughly, this condition plays a similar role for the Rees algebra as $(F_t)$ plays for the symmetric algebra. We will only use these conditions in the range $0 \leq t \leq 2$.

We will mainly focus on the case where $E = \Omega_{A/k}$. Let $A$ be a reduced algebra essentially of finite type over a perfect field $k$ – hence $\Omega_{A/k}$ is generically free. We denote the Rees algebra of $\Omega_{A/k}$ by $\mathcal{R}_{A/k}$. If in addition $(A, m)$ is equidimensional then $\Omega_{A/k}$ has a rank and this rank equals $\dim A + \text{trdeg}_k(A/m)$. In this case some of the above Fitting conditions can be expressed in terms of local embedding dimensions. Namely, $\Omega_{A/k}$ satisfies $(F_t)$ if and only if $\text{edim} A_p \leq 2 \dim A_p - t$ for every non-regular prime $p \in \text{Spec}(A)$ (see [21, the proof of 2.3]). The result of the next subsection will be stated in these terms.

3.1 Cohen–Macaulayness of $\mathcal{R}_{A/k}$

Let $A$ be a local Cohen–Macaulay ring and let $E$ be a finite $A$-module of projective dimension one. If $E$ satisfies condition $(F_1)$ then the symmetric algebra $S_A(E)$ is a Cohen–Macaulay torsionfree $A$-algebra (see [11 Proposition 4], [13 1.1], [24 3.4]) – in particular, the Rees algebra $\mathcal{R}_A(E)$ is Cohen–Macaulay. The question as to whether, conversely, $\mathcal{R}_A(E)$ being Cohen–Macaulay implies the condition $(F_1)$ fails in general even if $E$ satisfies $(F_0)$ (see [22 4.7]). The theorem below will show that this converse holds in the case of the module of differentials of a complete intersection over a field of characteristic zero. This shows that Cohen–Macaulayness is a rather restrictive property for $\mathcal{R}_{A/k}$. 

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Theorem 3.1 Let $k$ be a field of characteristic zero and let $A$ be a $k$-algebra essentially of finite type which is locally a complete intersection. Assume the following:

(i) $\text{edim} \ A_p \leq 2 \dim A_p$ for every prime $p \in \text{Spec}(A)$;

(ii) $\mathbb{R}_{A/k}$ is Cohen–Macaulay.

Then $\text{edim} \ A_p \leq 2 \dim A_p - 1$ for every non-minimal prime $p \in \text{Spec}(A)$.

Proof. Arguing by way of contradiction, let $p \in \text{Spec}(A)$ be minimal such that $\text{edim} \ A_p = 2 \dim A_p \geq 2$. By localizing at this prime, we may assume that $(A, m)$ is local and that $\text{edim} \ A = 2 \dim A$.

Next we reduce the argument to the case where the residue field $A/m$ is algebraic over $k$. To do this, let $r$ be the transcendence degree of $A/m$ over $k$ and suppose $r \geq 1$. Write $A = k[x_1, \ldots, x_m]$ and pick $r$ general $k$-linear combinations $y_1, \ldots, y_r$ of $x_1, \ldots, x_m$ such that, in particular, their residues yield a transcendence basis of $A/m$ over $k$. Note that $K := k(y_1, \ldots, y_r)$ is a subfield of $A$. Furthermore, $\Omega_{A/K} = \Omega_{A/k}/(A y_1 + \cdots + A y_r)$. Hence, by the general choice of $y_1, \ldots, y_r$, the Rees algebra

$$\mathcal{R}(\Omega_{A/K}) \simeq \mathcal{R}(\Omega_{A/k})/(dy_1, \ldots, dy_r)$$

is again Cohen–Macaulay (see [12, 2.2(f)]). Thus, replacing $k$ by $K$, we may assume henceforth that $A/m$ is algebraic over $k$.

Set $d = \dim A$ and induct on $d$. If $d = 1$, the Cohen–Macaulayness of $\mathbb{R}_{A/k}$ implies that $\Omega_{A/k}$ modulo torsion is free (see [22, 4.3]), which would imply that $A$ is regular (see [16]), thus contradicting the equality $\text{edim} \ A = 2$.

Thus we may suppose that $d \geq 2$. Write $A = R/I$, where $(R, n)$ is a regular local ring essentially of finite type over $k$ with $n := \dim R = \text{edim} A = 2d$. Recall that $n = \text{trdeg}_k R$, hence $\Omega_{R/k} = RdX_1 + \cdots + RdX_n$. By assumption, $I$ is generated by an $R$-regular sequence $f_1, \ldots, f_d$. Denoting the images of $\frac{\partial f_j}{\partial X_i}$ in $A$ by $\frac{\partial f_j}{\partial x_i}$ we consider the $n \times d$ matrices

$$\Theta = \left(\frac{\partial f_j}{\partial X_i}\right), \quad \theta = \left(\frac{\partial f_j}{\partial x_i}\right).$$

Now $\theta$ presents the $A$–module $\Omega_{A/k}$, which has projective dimension one, rank $d$, and satisfies condition $(F_1)$ locally in codimension $d - 1 \geq 1$ by the inductive hypothesis. Since $\mathbb{R}_{A/k}$ is Cohen–Macaulay, we may apply [22, 4.6(a) and 4.7] with $s = 2d - 1$ to show that after a linear change of variables,

$$I_1(\theta) = \left(\frac{\partial f_1}{\partial x_n}, \ldots, \frac{\partial f_d}{\partial x_n}\right).$$

Thus

$$I_1(\Theta) \subset \left(\frac{\partial f_1}{\partial X_n}, \ldots, \frac{\partial f_d}{\partial X_n}, I\right).$$
Set \( J = I_1(\Theta) \) and \( J_0 = (\frac{\partial f_1}{\partial X_n}, \ldots, \frac{\partial f_d}{\partial X_n}) \), and notice that \( J + I = J_0 + I \) by the above. Since \( R/\mathfrak{n} \) is algebraic over \( k \) and \( \text{char}(k) = 0 \), we have \( I \subset \overline{\mathfrak{n}J} \), where \( \overline{\cdot} \) denotes integral closure (see [4, Exercise 5.1] in the case where \( R \) is a power series ring over \( k \)). Therefore

\[
J + I = J_0 + I \subset J_0 + \mathfrak{n}(J + I).
\]

The valuative criterion of integrality now yields \( J + I \subset J_0 \). Therefore \( \text{height}(J + I) = \text{height} J_0 \leq d \). Hence \( \dim R/(J + I) \geq 2d - d = d \geq 2 \). But this is impossible because \( R/(J + I) \cong A/I_1(\theta) \) has dimension zero by assumption (i).

\[ \Box \]

**Remark 3.2** The theorem implies the following geometric result: let \( X \subset \mathbb{P}^{2d+1}_k \) denote a \( d \)-dimensional smooth complete intersection over a field \( k \) of characteristic zero, with homogeneous coordinate ring \( A \). If \( R_A/k \) is Cohen–Macaulay then \( X \) is degenerate.

### 3.2 Normality of \( R_A/k \)

Let \( A \) be a normal domain and let \( E \) be a finitely generated \( A \)-module of rank \( r \). Throughout this and the subsequent sections, we set \( E^* := \text{Hom}_A(E, A) \) – the \( A \)-dual module of \( E \). We write \( E^i \) for the \( i \)th graded piece \( R_A(E)_i \) of \( R_A(E) \) and call it the \( i \)th graded power of \( E \). Notice that \( E^i = S_i(E)/\tau_A(S_i(E)) \). Let \( \overline{R_A(E)} \) denote the integral closure of \( R_A(E) \) in its field of fractions. Since \( E \) has rank \( r \), \( E^1 \) can be embedded into \( A^r \) and any such embedding induces an embedding of \( R_A(E) \) into a polynomial ring \( A[t] = A[t_1, \ldots, t_r] \). Since \( A \) is normal, this further induces an embedding of \( \overline{R_A(E)} \) into \( A[t] \) as a graded \( A \)-subalgebra. We denote by \( E^i \subset A[t]_i \) the \( i \)th graded piece of \( \overline{R_A(E)} \) and call it the \( i \)th normalized power of \( E \). One can see that there are inclusions

\[
E^i \subset E^i \subset (E^i)^{**} \subset A[t]_i.
\]

The algebra \( B_A(E) := \bigoplus_{i \geq 0} (E^i)^{**} \) is a Krull domain, not necessarily Noetherian. An important feature of this ring is that it has the same divisor class group as \( A \). We say that \( E \) is integrally closed if \( E^1 = E^1 \). Finally, \( E \) is said to be normal if \( E^i = E^i \) for every \( i \geq 1 \), or equivalently, if \( R_A(E) \) is normal.

**Proposition 3.3** Let \( A \) be a universally catenary normal domain and let \( E \) be a finitely generated \( A \)-module. The following conditions are equivalent:

(i) \( R_A(E) \) is normal and \( E \) satisfies condition \( (L_2) \);

(ii) \( R_A(E) \) satisfies condition \( (S_2) \) of Serre and \( E \) satisfies condition \( (L_2) \);

(iii) \( R_A(E) = B_A(E) \).

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Definition 3.4

Let \( (A, \mathfrak{m}) \) be a Noetherian local ring with infinite residue field and let \( E \subset A^r \) be a submodule of rank \( r \). One says that \( E \) is \( \mathfrak{m} \)-full if there exists an element \( x \in \mathfrak{m} \) such that

\[ \mathfrak{m}E := A^r x = E. \]

It can be shown that an integrally closed module \( E \subset A^r \) is \( \mathfrak{m} \)-full provided the residue field of the local ring \( A \) is infinite. Most properties of this notion, including Theorem 3.6
below, arise from the following result (see [26, 8.101] for details).

**Proposition 3.5** Let $(A, \mathfrak{m})$ be a Noetherian local ring and $E$ a submodule of $A^r$ such that $mE :_{A^r} x = E$ for some $x \in \mathfrak{m}$. Then

$$E/xE \simeq (E :_{A^r} \mathfrak{m})/E \oplus (E + xA^r)/xA^r \simeq (E :_{A^r} \mathfrak{m})/E \oplus E/(E :_{A^r} \mathfrak{m}).$$

**Theorem 3.6** Let $(A, \mathfrak{m})$ be a Noetherian local ring and $E$ a submodule of $A^r$ such that $mE :_{A^r} x = E$ for some regular element $x \in \mathfrak{m}$. Suppose that $\text{projdim}(E) < \infty$ and $\mathfrak{m} \in \text{Ass}(A^r/E)$. Then $A$ is a regular local ring.

We use this result in the proof of the next theorem. Even for a finite module $E$ of projective dimension one over normal domain $A$, the normality of the Rees algebra $R_A(E)$ does not necessarily imply that $E$ satisfies $(F_2)$ – for an easy example take $E$ to be the homogeneous maximal ideal of a polynomial ring in two variables over a field. Surprisingly though, this implication does hold for modules whose non-free locus is contained in the singular locus of the ring:

**Theorem 3.7** Let $A$ be a Cohen–Macaulay normal domain and let $E$ be a finitely generated $A$-module of rank $r$ such that:

(a) $E$ has a projective resolution $0 \to P_1 \to P_0 \to E \to 0$;
(b) $E_p$ is $A_p$-free for any prime $p \subset A$ such that $A_p$ is regular.

The following are equivalent:

(i) $R_A(E)$ is normal;
(ii) $E^i = E^i$ in the range $1 \leq i \leq \text{rank } P_0 - r$;
(iii) $E$ satisfies condition $(F_2)$, i.e., height $\text{Fitt}_i(E) \geq i - r + 3$ for $i \geq r$;
(iv) $R_A(E) = B_A(E)$.

**Proof.** (i) $\Rightarrow$ (ii): This is obvious.

(ii) $\Rightarrow$ (iii): Let $p \in \text{Spec}(A)$ be a prime minimal with the property that $(F_2)$ fails. Localizing at $p$, we may assume that $(A, \mathfrak{m})$ is a local ring of dimension $d$, $\mu(E) \geq d + r - 1$, and $E$ satisfies $(F_2)$ locally on the punctured spectrum of $A$. Consider a minimal free resolution

$$0 \to A^s \to A^n \to E \to 0.$$

By assumption $n \geq d + r - 1$, hence $s \geq d - 1$. Since $(F_0)$ holds on the punctured spectrum, it follows that the Weyman complex is a minimal free resolution of the symmetric power $S_{d-1}(E)$ (see [28, Theorem 1(b)]). This complex has length $d - 1$ because $s \geq d - 1$. Moreover $E$ satisfies
(F1) on the punctured spectrum, therefore $S_{d-1}(E)$ is torsionfree. Thus $E^{d-1} \simeq S_{d-1}(E)$ has projective dimension $d - 1$. In particular, $m \in \text{Ass}(S_{d-1}(A^r)/E^{d-1})$. Since $s \leq \text{rank } P_0 - r$, we can apply the main assumption to conclude that $E^{d-1}$ is integrally closed. Therefore $A$ is regular by Theorem 3.6. This is a contradiction vis-à-vis (b) since $E$ is not free.

(iii) ⇒ (iv): This is well-known for modules of projective dimension one over Cohen–Macaulay rings (see, e.g., [1, Proposition 4]).

(iv) ⇒ (i): This follows from Proposition 3.3. 

Theorem 3.7 applies naturally to the case of complete intersections via the intervention of the Jacobian criterion, which says that the non-free locus of the module of differentials coincides with the singular locus of the ring.

**Corollary 3.8** Let $A$ be a normal complete intersection domain essentially of finite type over a perfect field $k$. The following are equivalent:

(i) $R_{A/k}$ is normal;

(ii) $\Omega^i_{A/k} = \Omega^i_{A/k}$ in the range $1 \leq i \leq \text{ecodim } A$;

(iii) $\text{edim } A_p \leq 2 \dim A_p - 2$ for every non-regular prime $p \in \text{Spec}(A)$;

(iv) $R_{A/k} = B_{A/k}$.

**Proof.** Note that condition (iii) translates into $\Omega_{A/k}$ satisfying (F2) according to the preliminaries before Section 3.1. The rest follows immediately from Theorem 3.7. 

The following result is reminiscent of Theorem 2.1.

**Proposition 3.9** Let $A = R/I$ be a normal domain, where $R$ is a regular ring essentially of finite type over a perfect field $k$. If $i \geq 1$ is such that $S_i(\Omega_{A/k}) \simeq \Omega^i_{A/k}$ and $\Omega^i_{A/k}$ is integrally closed, then $I \not\subset p^3$ for any $p \in \text{Ass}_R((\Omega^{i}_{A/k})^{**}/\Omega^{i}_{A/k})$.

**Proof.** We replace $R$ by $R_p$ to assume that $(R, \mathfrak{n})$ and $(A, \mathfrak{m})$ are local. We may also suppose that the residue field is infinite. Notice that $\mathfrak{m} \in \text{Ass}_A((\Omega^{i}_{A/k})^{**}/\Omega^{i}_{A/k})$, in particular dim $A \geq 2$. Since $S_i(\Omega_{A/k}) \simeq \Omega^i_{A/k}$, the presentation

$$I/I^2 \rightarrow A^n = \Omega_{R/k} \otimes_R A \rightarrow \Omega_{A/k} \rightarrow 0$$

induces an exact sequence

$$I/I^2 \otimes_A S_{i-1}(A^n) \rightarrow S_i(A^n) \rightarrow \Omega^i_{A/k} \rightarrow 0.$$
Let \( d = \text{trdeg}_k A \). As \( \Omega^i_{A/k} \) is assumed to be integrally closed, it is \( \mathfrak{m} \)-full in any embedding \( \Omega^i_{A/k} \subset A^i \), where \( t = \binom{i+d-1}{d-1} \). Thus there is an element \( x \in \mathfrak{m} \) satisfying the hypothesis of Proposition 3.5 with \( E = \Omega^i_{A/k} \). Clearly \( \mathfrak{m} \in \text{Ass}_A(A^i/\Omega^i_{A/k}) \), hence \( \Omega^i_{A/k} : A^i \mathfrak{m} \neq \Omega^i_{A/k} \). Therefore the proposition implies that \( \Omega^i_{A/k}/x \Omega^i_{A/k} \) has \( A/\mathfrak{m} \) as a direct summand over \( A/(x) \).

On the other hand, tensoring (8) with \( A/(x) \), we see that the syzygies of \( \Omega^i_{A/k}/x \Omega^i_{A/k} \) have coefficients in the \( A/(x) \)-ideal generated by the entries of the Jacobian matrix of the generators of \( I \). As \( \mathfrak{m}/(x) \neq 0 \), we cannot have \( I \subset n^3 \).

\[ \square \]

### 3.3 Algebras of low codimension

We now study the normality of \( R_{A/k} \) in several settings in low dimension or low embedding codimension. A highlight is the closer relationship between the normality of \( R_{A/k} \) and the size of the non-singular locus of \( A \) than is typical of more general Rees algebras.

Let \( A \) be a local Cohen–Macaulay ring essentially of finite type over a perfect field \( k \) and assume \( \text{ecodim} A \leq 2 \). Write \( A = R/I \), where \( R \) is a regular local ring essentially of finite type over \( k \) and \( I \) an ideal of height 2. The presentation

\[ I/I^2 \longrightarrow \Omega_{R/k} \otimes_R A \cong A^n \longrightarrow \Omega_{A/k} \longrightarrow 0 \]

induces a complex of graded modules over the polynomial ring \( B = S_A(A^n) \),

\[ 0 \longrightarrow (\wedge^2 I/I^2)^* \otimes_A B[-2] \longrightarrow I/I^2 \otimes_A B[-1] \longrightarrow B \longrightarrow S_{A/k} \longrightarrow 0. \]  

(9)

We notice that this is the \( \mathbb{Z} \)-complex of the module \( \Omega_{A/k} \) (see [25, Chapter 3] for details). It can be used to derive properties of \( S_{A/k} \) and of \( R_{A/k} \):

**Proposition 3.10** Let \( A \) be a reduced local Cohen–Macaulay ring essentially of finite type over a perfect field \( k \). Assume that \( \text{ecodim} A \leq 2 \) and that \( \text{edim} A_p \leq 2 \dim A_p \) for every prime \( p \in \text{Spec}(A) \).

(a) The sequence (9) is exact and \( S_{A/k} \) is Cohen-Macaulay;

(b) \( \Omega_{A/k} \) is of linear type if and only if \( \text{edim} A_p \leq 2 \dim A_p - 1 \) for every non-minimal prime \( p \in \text{Spec}(A) \);

(c) In case \( A \) is normal, \( \Omega_{A/k} \) is of linear type and normal if and only if \( \text{edim} A_p \leq 2 \dim A_p - 2 \) for every non-regular prime \( p \in \text{Spec}(A) \).

**Proof.** Write \( A = R/I \) as above.

(a) Since \( \Omega_{A/k} \) satisfies \( (F_0) \) it follows that \( A \) is a complete intersection locally in codimension one. Furthermore, (2) implies \( \text{depth} I/I^2 \geq \dim A - 1 \). Finally notice that

\[ (\wedge^2 I/I^2)^* \cong \omega_A^* \cong \text{Hom}_A(\omega_A \otimes_A \omega_A, \omega_A) \cong \text{Hom}_A(S_2(\omega_A), \omega_A). \]
By [27, 1.3], $S_2(\omega_A)$ is a maximal Cohen-Macaulay $A$-module and therefore $(\wedge^2 I/I^2)^{**}$ is Cohen-Macaulay as well. These facts imply that $S_2$ is exact by the acyclicity lemma and that depth $S_{A/k} \geq \dim A + n - 2$. On the other hand, $\dim S_{A/k} = \dim A + n - 2$ since we are assuming that $\Omega_{A/k}$ satisfies condition $(F_0)$ (see [23, 2.2]). Therefore $S_{A/k}$ is Cohen-Macaulay.

(b) From part (a) we have that $S_{A/k}$ is unmixed. In this case, the module $\Omega_{A/k}$ is of linear type if and only if it satisfies $(F_1)$ (see [23, 3.3 and the first remark on page 346]). Alternatively, one could use the exactness of (9).

(c) If $\Omega_{A/k}$ satisfies $(F_2)$ then it is of linear type by part (b), i.e., $S_{A/k} = R_{A/k}$. By (a) $S_{A/k}$ is Cohen–Macaulay. Thus, $S_{A/k} = R_{A/k}$ satisfies condition (ii) of Proposition 3.3 hence the proposition implies that $\Omega_{A/k}$ is normal (and the equality $S_{A/k} = B_{A/k}$ holds).

Conversely, assume that $\Omega_{A/k}$ is of linear type and normal. For any non-regular prime $p \in \text{Spec}(A)$ write $A_p = R/I$, where $(R,n)$ is a regular local ring essentially of finite type over $k$ and $I \subset n^2$ is an $R$-ideal of height $g \leq 2$. By Theorem 2.11, it suffices to show that $\mu(I + n^3/n^2) \leq 2g - 1$. This is clear if $g = 1$ or else $g = 2$ and $\mu(I) \leq 3$. Thus we may assume that $g = 2$ and $\mu(I) \geq 4$. But in this situation the Hilbert–Burch theorem gives $I \subset n^3$. 

We now treat the case where $A$ is a Gorenstein algebra of embedding codimension 3. It has a striking similarity to complete intersections. Write $A = R/I$, where $R$ is a regular local ring essentially of finite type over a perfect field $k$ and $I$ an ideal of height 3. As in (9), starting from a presentation of $\Omega_{A/k}$ we obtain the $\mathcal{Z}$-complex

\[
0 \longrightarrow (\wedge^3 I/I^2)^{**} \otimes_A B[-3] \longrightarrow (\wedge^2 I/I^2)^{**} \otimes_A B[-2] \longrightarrow (I/I^2) \otimes_A B[-1] \longrightarrow B \longrightarrow S_{A/k} \longrightarrow 0. \tag{10}
\]

**Proposition 3.11** Let $A$ be a reduced local Gorenstein ring essentially of finite type over a perfect field $k$. Assume that $	ext{ecodim } A \leq 3$ and that $\Omega_{A/k}$ satisfies $(F_0)$.

(a) The sequence (10) is exact and $S_{A/k}$ is Gorenstein;

(b) $\Omega_{A/k}$ is of linear type if and only if $\text{edim } A_p \leq 2 \dim A_p - 1$ for every non-minimal prime $p \in \text{Spec}(A)$;

(c) In case $A$ is normal, $\Omega_{A/k}$ is of linear type and normal if and only if $\text{edim } A_p \leq 2 \dim A_p - 2$ for every non-regular prime $p \in \text{Spec}(A)$.

**Proof.** Since $\Omega_{A/k}$ satisfies $(F_0)$ it follows that $A$ is a complete intersection locally in codimension two. We note that $(\wedge^3 I/I^2)^{**}$ is the determinant divisor of $I/I^2$, which is $A$ itself, and that the pairing

\[
\wedge^2 I/I^2 \times I/I^2 \rightarrow \wedge^3 I/I^2 \rightarrow A
\]

identifies $(\wedge^2 I/I^2)^{**}$ with $(I/I^2)^*$. Furthermore $I/I^2$ is a Cohen-Macaulay $A$-module (see [23, 3.3(a)]). We conclude that the three left most modules in (10) are maximal Cohen-Macaulay
Now the argument proceeds as in the proof of Proposition 3.10. As for the Gorensteinness of $S_{A/k}$ one uses the fact that (10) is a self-dual complex of $B$-modules.

The previous results motivate the following question:

**Question 3.12** Let $A$ be a local Cohen–Macaulay normal domain essentially of finite type over a perfect field $k$. If $\Omega_{A/k}$ is of linear type and normal does it follow that $\text{edim} A_p \leq 2 \dim A_p - 2$ for every non-regular prime $p \in \text{Spec}(A)$ (or, equivalently, $R_{A/k} = \mathbb{B}_{A/k}$)?

As to the converse, it was remarked earlier in the proof of Proposition 3.10 that, in any codimension, if $S_{A/k}$ satisfies condition $(S_2)$ (e.g., if it is Cohen–Macaulay) then the inequalities $\text{edim} A_p \leq 2 \dim A_p - 2$ for every non-regular prime $p \in \text{Spec}(A)$ imply the normality of $\Omega_{A/k}$ and also the equality $R_{A/k} = \mathbb{B}_{A/k}$.

In the remainder of this section we will prove Theorem 3.16, which is a refined version of Proposition 3.10(c). For this we need several auxiliary results that may be of independent interest. Recall that if $A$ is a Noetherian ring and $U \subset E$ are finite modules having a rank then $U$ is said to be a **reduction** of $E$ if the induced inclusion $R(U) \subset R(E)$ is an integral ring extension (see, e.g., [22]).

**Proposition 3.13** Let $A$ be a Noetherian ring and let $\varphi : A^{g+1} \longrightarrow A^g$ be a homomorphism such that $\text{im}(\varphi)$ has rank $g$ and $\text{Ext}^1_A(\text{im}(\varphi), A) = 0$. Then $I_g(\varphi) \simeq \ker(\varphi)^*$. In particular, if $A$ satisfies $(S_2)$ it follows that $I_g(\varphi)$ is either the unit ideal or an unmixed ideal of height one. If in addition $A$ is Gorenstein locally in codimension one, then $\text{im}(\varphi)$ is a reduction of $\text{im}(\varphi)^*$.

**Proof.** There is an exact sequence

$$A^n \xrightarrow{\psi} A^{g+1} \xrightarrow{\varphi} A^g,$$

so that the entries of the first column of $\psi$ are the signed maximal minors of $\varphi$, hence generate $I_g(\varphi)$, an ideal of positive grade. Set $K = \ker(\varphi)$ and $L = \text{im}(\varphi)$. From the assumption, we obtain a short exact sequence

$$0 \rightarrow L^* \longrightarrow A^{g+1*} \longrightarrow K^* \rightarrow 0,$$

which shows that $K^* \simeq \text{im}(\psi^*)$. Next we project $A^n*$ onto the free module generated by the first basis element, thus getting a commutative diagram

$$\begin{array}{ccc}
A^{g+1*} & \xrightarrow{\psi^*} & A^n* \\
\| & & \downarrow \pi \\
A^{g+1*} & \xrightarrow{\pi \psi^*} & A.
\end{array}$$

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Notice that $\text{im}(\pi\psi^*) = I_g(\varphi)$. The projection $\pi$ induces a surjection $\text{im}(\psi^*) \to \text{im}(\pi\psi^*) = I_g(\varphi)$, which is necessarily an isomorphism since the two modules are torsionfree of rank one.

Moreover, if $A$ satisfies $(S_2)$ then $I_g(\varphi)$ satisfies $(S_2)$, hence it is the unit ideal or an unmixed ideal of height one. The last assertion is a property of the integral closure of modules with divisorial determinantal ideal (see [12, the proof of 2.5]).

\begin{corollary}
Let $R$ be a regular local ring essentially of finite type over a field. Let $A = R/I$ be an almost complete intersection ring which is a complete intersection locally in codimension one and satisfies $(S_2)$. If $A$ is non-obstructed (i.e., $T^2(A/R, A) = \text{Ext}_A^1(I/I^2, A) = 0$), then $I/I^2$ is a reduction of $(I/I^2)^{**}$.
\end{corollary}

\begin{proof}
Write $g = \text{height } I$. Notice that $I/I^2$ has rank $g$ as an $A$-module because $A$ is equidimensional. Since $I$ is generated by $g + 1$ elements, the first Koszul homology $H_1(I)$ of these elements is the canonical module of $A$, hence satisfies $(S_2)$. The exact sequence

$$0 \to H_1(I) \to A^{g+1} \to I/I^2 \to 0$$

shows that $I/I^2$ is a torsionfree $A$-module. Embedding $I/I^2$ into $A^g$ and applying Proposition 3.13 with $\text{im}(\varphi) = I/I^2$, we deduce the result.
\end{proof}

\begin{proposition}
Let $(A, \mathfrak{m})$ be a normal local ring essentially of finite type over a perfect field $k$ satisfying the following conditions:

(a) $\dim A = \text{ecodim } A = 2$;

(b) $A$ is either a complete intersection or else an almost complete intersection defined by an ideal of order $\geq 3$.

Then $\Omega_{A/k}/\tau_A(\Omega_{A/k})$ is not integrally closed.
\end{proposition}

\begin{proof}
Write $A = R/I$, where $(R, \mathfrak{n})$ is a regular local ring essentially of finite type over $k$ and $I$ an ideal of height 2. We may assume that $R$ has infinite residue field. Set $\Omega = \Omega_{A/k}$.

In the complete intersection case, the exact sequence

$$0 \to I/I^2 \to A^n \to \Omega \to 0$$

shows that $\Omega$ is a torsionfree $A$-module of projective dimension one, hence the assertion follows from Theorem 3.6.

Thus, we may assume that $A$ is an almost complete intersection and that $I \subset \mathfrak{n}^3$. There is an exact sequence

$$0 \to (I/I^2)^{**} \to A^n \to \Omega^1 = \Omega/\tau_A(\Omega) \to 0. \quad (11)$$

22
By Theorem 2.9, $\Omega^1$ is not Cohen–Macaulay, hence not reflexive. Thus $m \in \text{Ass} (\Omega^1)$.
Suppose that $\Omega^1$ is integrally closed. In particular, it is $m$-full. Thus, according to Proposition 3.5 there exists an $x \in m$ such that $A/m$ is a direct summand of $\Omega^1/x\Omega^1$. Write $(A', m')$ for the local ring $A/(x)$ and let $\{e_1, \ldots, e_n\}$ be the canonical basis of $A'^n$. Tensoring (11) with $A'$ we obtain a presentation

$$0 \rightarrow M \rightarrow A'^n \rightarrow \Omega^1/x\Omega^1 \rightarrow 0,$$

where we may assume that $M = m' e_1 \oplus N$ with $N \subset \bigoplus_{i=2}^n A'e_i$. Hence the image of $M$ under the projection $A'^n \rightarrow A'e_1 \simeq A'$ is $m'$.

On the other hand, as $A$ is a Cohen–Macaulay ring of embedding codimension 2 it is non-obstructed (see [9, 3.2(a)]). Therefore by Corollary 3.14 $I/I^2$ is a reduction of $(I/I^2)^*$. Since $I \subset n^3$ it follows that $I/I^2 \subset n^2 A^n$. Hence, projecting onto $A'e_1 \simeq A'$ we see that $(m')^2$ is a reduction of $m'$. This is impossible because $\dim A' > 0$.

**Theorem 3.16** Let $(A, m)$ be a normal local Cohen–Macaulay ring essentially of finite type over a perfect field $k$ satisfying the following conditions:

(a) $\text{ecodim} A = 2$;

(b) Locally in codimension 2, the ring $A$ is either a complete intersection or else an almost complete intersection defined by an ideal of order $\geq 3$.

Then the following conditions are equivalent:

(i) $\Omega_{A/k}$ is normal;

(ii) $\text{edim} A_p \leq 2 \dim A_p - 2$ for every non-regular prime $p \in \text{Spec}(A)$.

Moreover, under any of the above equivalent conditions $\Omega_{A/k}$ is of linear type and $R_{A/k}$ is Cohen–Macaulay.

**Proof.** (i) $\Rightarrow$ (ii) Proposition 3.15 implies that $\Omega_{A/k}$ satisfies $(F_1)$. Hence by Proposition 3.10(b), $\Omega_{A/k}$ is of linear type and then according to Proposition 3.10(c), it satisfies $(F_2)$.

(ii) $\Rightarrow$ (i) This follows from Proposition 3.10(c).

The remaining assertions follow from the same proposition.

3.4 Relation to Calabi–Yau varieties

In this last part we explain to what extent the present results relate to Calabi–Yau varieties.

Let $X \subset \mathbb{P}^{n-1}_C$ be an arithmetically normal projective variety and let $A$ stand for its homogeneous coordinate ring. We say that $X$ is of Calabi–Yau type if there exists a homogenous isomorphism $\omega_A \simeq A$. The notion of Calabi–Yau variety would also require that
X be smooth and $H^1(X, \mathcal{O}_X) = 0$. If X is of Calabi–Yau type it often turns out that $Y = \text{Proj}(\mathbb{R}_{A/k}) \subset \mathbb{P}^{2n-1}_\mathbb{C}$ has the same property.

We first look into the case of a complete intersection.

**Proposition 3.17** Let $X \subset \mathbb{P}^{n-1}$ be a non-degenerate smooth projective variety that is a complete intersection of hypersurfaces of degrees $d_1 \geq \cdots \geq d_g$, and let $A$ stand for its homogeneous coordinate ring. Assume that $X$ is of Calabi–Yau type and consider the subschemes $Y = \text{Proj}(\mathbb{R}_{A/\mathbb{C}}) \subset Z = \text{Proj}(S_{A/\mathbb{C}}) \subset \mathbb{P}^{2n-1}_\mathbb{C}$.

(a) $Z$ is the complete intersection of $2g$ hypersurfaces of degrees $d_1, \ldots, d_g, d_1, \ldots, d_g$ and $\omega_{S_{A/\mathbb{C}}} \simeq S_{A/\mathbb{C}}$ as graded modules;

(b) If $d_1 = 2$ then $Z$ is neither reduced nor irreducible, and $Y$ is not arithmetically Cohen–Macaulay;

(c) If $d_1 \geq 3$ then $Y = Z$ is reduced and irreducible;

(d) The subscheme $Y$ is arithmetically normal if and only if $d_1 \geq 4$ or else $d_2 \geq 3$, in which case $Y$ is of Calabi–Yau type.

**Proof.** Since $\omega_A \simeq A(-n + \sum_{i=1}^{g} d_i)$ and $X$ is Calabi–Yau, we have $n = \sum_{i=1}^{g} d_i$. Since $X$ is non-degenerate, $d_g \geq 2$. Therefore $n \geq 2g$, hence $\Omega_{A/\mathbb{C}}$ satisfies condition $(F_0)$. Since $\text{projdim}_A(\Omega_{A/\mathbb{C}}) = 1$ this implies that $S_{A/\mathbb{C}}$ is a complete intersection (see [1, Proposition 4]). Moreover, $\omega_{S_{A/\mathbb{C}}} \simeq S_{A/\mathbb{C}}(-2n + 2 \sum_{i=1}^{g} d_i) = S_{A/\mathbb{C}}$ as graded modules. This shows (a).

To prove (b), note that the hypothesis forces the equality $n = 2g$ and thus $\Omega_{A/\mathbb{C}}$ does not satisfy condition $(F_1)$. It follows from [3, 2.2] that $Z$ is not irreducible, whereas Theorem 2.1 implies that $Z$ is not reduced either. Furthermore $Y$ is not arithmetically Cohen–Macaulay according to Theorem 3.1.

As to (c), the assumption implies that $n \geq 2g + 1$, hence $\Omega_{A/\mathbb{C}}$ satisfies condition $(F_1)$. Again since $\text{projdim}_A(\Omega_{A/\mathbb{C}}) = 1$ we have $S_{A/\mathbb{C}} = \mathbb{R}_{A/\mathbb{C}}$ (see [1, Proposition 4], [13, 1.1], [23, 3.4]). Finally, notice that $d_1 \geq 4$ or $d_2 \geq 3$ if and only if $n \geq 2g + 2$. Thus, part (d) follows from Corollary 3.8. \qed

Next is the non-complete intersection case.

**Proposition 3.18** Let $X \subset \mathbb{P}^{n-1}$ be a smooth arithmetically Cohen–Macaulay projective variety that is not a complete intersection. Assume that $\dim X \geq 2, \text{ecodim} X \leq 3$ and $X$ is of Calabi–Yau type. Consider the subschemes $Y = \text{Proj}(\mathbb{R}_{A/\mathbb{C}}) \subset Z = \text{Proj}(S_{A/\mathbb{C}}) \subset \mathbb{P}^{2n-1}_\mathbb{C}$, where $A$ stand for the homogeneous coordinate ring of $X$ in the given embedding.

(a) $Z$ is arithmetically Gorenstein and $\omega_{S_{A/\mathbb{C}}} \simeq S_{A/\mathbb{C}}$ as graded modules;

(b) If $\dim X = 2$ then $Z$ is neither reduced nor irreducible;
(c) If \( \dim X \geq 3 \) then \( Y = Z \) is reduced and irreducible;

(d) \( Z \) is arithmetically normal if and only if \( \dim X \geq 4 \), in which case \( Y = Z \) is of Calabi–Yau type.

**Proof.** Notice that \( A = R/I \), where \( R = \mathbb{C}[X_1, \ldots, X_n] \) and \( I \subset (X_1, \ldots, X_n)^2 \) is a homogeneous Gorenstein ideal of height at most 3, hence exactly of height 3 because \( X \) is not a complete intersection. Since \( X \) is smooth and \( n = \text{height } I + \dim A \geq 3 + 3 = 6 \), the module \( \Omega_{A/C} \) satisfies \((F_0)\). By Proposition 3.11(a), the ring \( S_{A/C} \) is Gorenstein. In fact, as a complex of graded modules over the standard graded polynomial ring \( B = \mathbb{C}[X_1, \ldots, X_n, T_1, \ldots, T_n] \), the exact sequence now reads

\[
0 \longrightarrow (\wedge^3 I/I^2)^* \otimes_A B \longrightarrow (\wedge^2 I/I^2)^* \otimes_A B \longrightarrow (I/I^2) \otimes_A B \longrightarrow B \longrightarrow S_{A/k} \longrightarrow 0,
\]

where \(-^* = \text{Hom}_A(-, A) = \text{Hom}_A(-, \omega_A)\). From this we get \( \omega_{S_{A/C}} \cong S_{A/C} \) as graded modules, proving part (a).

As to (b), if \( \dim X \geq 2 \) then as above we see that \( \Omega_{A/C} \) does not satisfy \((F_1)\). Thus, \( Z \) is not irreducible according to \([3, 2.2]\). On the other hand, by considering the Hilbert function of \( A \) modulo a linear system of parameters one sees that \( \dim_{\mathbb{C}}[I]_2 \leq 5 = n - 1 \). Now an application of Theorem 2.1 yields that \( Z \) is not reduced either, proving (b).

Finally, part (c) follows from Proposition 3.11(b) and part (d) from Proposition 3.11(c). \( \square \)

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