We consider a variant of the BB84 protocol for quantum cryptography, the prototype of tomographically incomplete protocols, where the key is generated by one-way communication rather than the usual two-way communication. Our analysis, backed by numerical evidence, establishes thresholds for eavesdropping attacks on the raw data and on the generated key at quantum bit error rates of 10% and 6.15%, respectively. Both thresholds are lower than the threshold for unconditional security in the standard BB84 protocol.

Keywords: Quantum key distribution, BB84 protocol, eavesdropping, partial tomography

1. Introduction

Quantum cryptography deals with all aspects, both theoretical and experimental, of schemes, or “protocols,” for quantum key distribution. When implementing a quantum key distribution protocol, the two parties — Alice and Bob — exploit the laws of quantum mechanics to gain a string of perfectly secure key bits for subsequent one-time-pad encryption of a private message.

The first such protocol, the celebrated BB84 protocol, was proposed by Bennett and Brassard in 1984. In this brief report, we study raw-data attacks on this protocol in an “Ekert setting,” where eavesdropper Eve is given the privilege of sending entangled qubits to Alice and Bob. This setting is equivalent to the traditional scenario where Alice sends qubits to Bob, and Eve performs a cloning attack on the qubits in transmission.

Alice and Bob concede to the fact that there will always be noise in their channel, but they only accept unbiased noise. Now, owing to the incomplete-tomographic nature of this protocol, Eve is free to send a family of states, with each of them appearing the same to Alice and Bob. For them, this state is only parameterized by one parameter, the amount of noise they see. We ask this question: which state should Eve send to Alice and Bob such that she maximizes her accessible information relative to Alice?
We restrict the discussion to the case where the state Eve sends is symmetric between Alice and Bob. For all such states, a von Neumann type measurement that satisfies the necessary conditions for an optimal POVM — in the sense of maximizing the mutual information — is given. But even in this restricted case we cannot prove that the POVM found is globally optimal. As usual\(^2\) we can verify that we found a local optimum, but any claim on global optimality must rely on a thorough numerical search with negative outcome.

A plausible, self-suggesting answer to that question would be for Eve to send a state to Alice and Bob such that it has maximal entropy. But it turns out that this is not always her best choice. Rather, when attacking the raw data Eve should send the state with the largest degree of separability, or smallest concurrence. From the information that is then accessible to Eve, we deduce the security threshold for the raw-data attacks.

2. Partial tomography

The scenario we shall be discussing is as follows: Alice and Bob are promised a sequence of singlets from Eve, an entangled qubit pair provider of dubious reliability. Due to practical imperfections in their channel, they compromise with an unbiased-noise state\(^a\)

\[
\rho_{AB} = (1 - \epsilon) |\phi_1\rangle \langle \phi_1| + \frac{\epsilon}{4},
\]

Here \(|\phi_1\rangle\) is the singlet state, and \(0 \leq \epsilon \leq 1\) characterizes the amount of noise in the channel; the “quantum bit error rate” that is used to quantify the noise in the standard BB84 protocol equals \(\frac{1}{2} \epsilon\). Alice and Bob independently perform measurements in two complementary bases, the \(x\) and \(z\) basis, each with equal probability. The decision of whether to carry out the measurement in the \(x\) or \(z\) basis is made at random.

To check the reliability of the source they received, Alice and Bob sacrifice a fraction of their qubits to verify that the joint probability table of their measurements is consistent with (1), i.e., they check that their joint probability table looks like Table 1. The quantum version\(^3\) of the de Finetti theorem ensures that the qubit pairs received from the source behave like statistically independent pairs. Therefore, the joint probabilities of Table 1 tell Alice and Bob the statistical properties of those pairs as far as measurements in the \(x\) and \(z\) bases are concerned, but there is no information about the \(y\) bases. Accordingly, the “state tomography” performed by Alice and Bob in this manner is incomplete, or partial.

Alice and Bob, being paranoid, assume that the noise in (1) is an artifact of Eve’s eavesdropping on their communication. So, if the probability table they obtain is

\(^a\)In practice, Alice and Bob would not expect to receive such a state with perfectly unbiased noise. However, they can include a controlled effective source of noise in their apparatus, by post-processing the measured data, and so bring any state they receive to the unbiased-noise state.
Table 1. Joint probability table between Alice and Bob who communicate with each other to check that the statistical properties of their measurement results are consistent with this table.

|   | Bob | Alice |   |   |
|---|-----|-------|---|---|
|   | z+  | z−    | x+| x−|
| z+| ε/16 | 2−ε/16 | 1/16 | 1/16 |
| z−| 2−ε/16 | ε/16 | 1/16 | 1/16 |
| x+| 1/16 | 1/16 | ε/16 | 2−ε/16 |
| x−| 1/16 | 1/16 | 2−ε/16 | ε/16 |

skewed, or if they find that the noise level ε is too large, Alice and Bob abandon their communication. Otherwise they will use the correlations in their data to establish a cryptographic key for one-time-pad encryption.

By means of the partial tomography, Alice and Bob establish the values of eight independent parameters of their joint two-qubit state. Explicitly, if we write the two-qubit state between Alice and Bob as

$$\rho_{AB} = \frac{1}{4} \sum_{j,k=0}^{3} c_{jk} \sigma_j \otimes \sigma_k,$$

(2)

where $\sigma_0 = 1$ and $\sigma_j$ are the three Pauli matrices, the tomographic constraint of consistency with (1) imposes eight constraints on the coefficients $c_{jk} = \langle \sigma_j \otimes \sigma_k \rangle$,

$$c_{01} = c_{03} = c_{10} = c_{30} = c_{13} = c_{31} = 0 \text{ and } c_{11} = c_{33} = -(1 - \epsilon).$$

(3)

With $c_{00} = 1$ for normalization, there remain seven parameters that are inaccessible to Alice and Bob who, therefore, can ascertain the state they receive only partially. Hence we speak of partial tomography. The remaining seven parameters are independent and are constrained only by the positivity of $\rho_{AB}$.

This is in contrast to, say, the “six-state protocol” or the “minimal qubit protocol” where Alice and Bob perform full tomography. There they can check all 15 parameters of their joint two-qubit state, and tomography uniquely characterizes the state. In addition to (3), Alice and Bob then have the luxury of insisting on

$$c_{02} = c_{20} = c_{12} = c_{21} = c_{23} = c_{32} = 0 \text{ and } c_{22} = -(1 - \epsilon)$$

(4)
as well.

In the partially tomographic BB84 protocol, these parameters are hidden from Alice and Bob. A whole family of distinct states appears equivalent to them. They
would be wise to assume that Eve uses her freedom to manipulate these hidden parameters to her full advantage.

As the scheme in the BB84 protocol goes, Alice and Bob would reveal publicly the bases of their independent measurements. After this announcement, qubit pairs measured in mismatched bases are discarded, while the qubit pairs in matched bases give them a string of sifted data with stronger correlations. In the absence of noise, these sifted data would have perfect correlations, and the resulting key is guaranteed to be secure. However, in the presence of noise, a shorter but still perfectly secure key can still be distilled by means of classical error correcting codes and privacy amplification.

But Alice and Bob can just as well devise a protocol that uses the raw data itself; that is, they exploit the correlations available directly from Table 1 to distill the same amount of secure key bits. Alice and Bob then need not announce their measurement bases. In the subsequent analysis, it is about this raw data that eavesdropper Eve wishes to learn. This is equivalent to Eve attempting to eavesdrop on the sifted data if she does not have the means to store her ancilla qubits until after Alice and Bob will have announced their choice of bases.

According to the Csiszár–Körner theorem\(^6\) of classical information theory, this amount of distillable secure key is measured by the difference in the Shannon’s mutual information between Alice and Bob and between Alice and Eve. The mutual information between Alice and Bob is

\[
I_{AB}(\epsilon) = \frac{1}{2} \Phi (1 - \epsilon) ,
\]

where

\[
\Phi (x) = \frac{1}{2} \left[ (1 - x) \log (1 - x) + (1 + x) \log (1 + x) \right].
\]

### 3. Constraints on Eve

Eve creates an entangled four-qubit state

\[
|\Psi_{ABE}\rangle = \sum_{j=1}^{4} |\phi_j\rangle |E_j\rangle ,
\]

where the unnormalized kets \(|E_j\rangle\) are the four two-qubit states of her ancilla which record the outcomes of Alice and Bob’s measurements, and

\[
\begin{align*}
|\phi_1\rangle = \left( |z+\rangle |z-\rangle + |z-\rangle |z+\rangle \right) \frac{1}{\sqrt{2}} \\
|\phi_2\rangle = \left( |z+\rangle |z+\rangle - |z-\rangle |z-\rangle \right) \frac{1}{\sqrt{2}}
\end{align*}
\]

and

\[
\begin{align*}
|\phi_3\rangle = \left( |z+\rangle |z-\rangle + |z-\rangle |z+\rangle \right) \frac{1}{\sqrt{2}} \\
|\phi_4\rangle = \left( |z+\rangle |z+\rangle - |z-\rangle |z-\rangle \right) \frac{1}{\sqrt{2}}
\end{align*}
\]

are the four Bell states, which we use as the basis states for the qubit pair received by Alice and Bob. The two-qubit state obtained by performing a partial trace over
Eve’s ancilla,
\[ \rho_{AB} = \text{Tr}_E \left\{ |\Psi_{ABE}\rangle \langle \Psi_{ABE}| \right\}, \]
(9)
is what Eve sends to Alice and Bob. The geometry of Eve’s ancilla states are fully determined by \( \rho_{AB} \), the state Eve chooses, and the free choice of basis we used for Alice and Bob in writing (7). With \( \rho_{AB} \) in the form of (2), Eve is thus constrained by the values of \( c_{jk} \) in (3).

4. Raw-data attacks

Eve’s task is to maximize the mutual information between Alice and herself. For every state \( \rho_{AB} \) that Eve chooses to send, she has a corresponding optimal POVM that maximizes her mutual information. Hence hers is a double optimization problem: first, she has to find the best POVM and, second, she has to choose the most advantageous values for the seven adjustable coefficients that do not appear in (3). The optimal POVM depends, of course, on the parameter choice.

The optimization has to be done with respect to Eve’s four input states. They are the ancilla states conditioned on Alice measuring \( z \pm \) or \( x \pm \). Each of these states has rank two.

In practice, Eve does not have to make use of ancillas for an attack on the raw data. Once she decides which POVM to use on her ancillas, she can trace out her subsystem from the state (7) conditioned on her intended POVM outcomes. The remaining ensemble of states would then give the pre-manufactured states that Eve should send to Alice and Bob.

We restrict our study to the symmetric case of \( c_{02} = c_{20} = c_{12} = c_{21} = c_{23} = c_{32} = 0 \), where \( \rho_{AB} \) is symmetric under the interchange of Alice and Bob, and all expectation values \( c_{jk} = \langle \sigma_j \otimes \sigma_k \rangle \) vanish if either \( \sigma_j = \sigma_2 \) or \( \sigma_k = \sigma_2 \) but not both. Although there exist subspaces outside the symmetric region where the accessed information equals the accessed information in the symmetric region, numerical simulations suggest strongly that Eve has no advantage from nonsymmetric states.

In this symmetric regime, then, Eve’s four ancilla states in the basis specified by (7) are mutually orthogonal,
\[ \langle E_j | E_k \rangle = \delta_{jk} \langle E_j | E_j \rangle, \]
(10)
so that the right-hand side of (7) is the Schmidt decomposition of \( |\Psi_{ABE}\rangle \), and \( \rho_{AB} \) is a weighted sum of projectors on the Bell states (8) with the weights given by
\[
\begin{align*}
\langle E_1 | E_1 \rangle &= \frac{1}{4}(3 - 2\epsilon - c_{22}), \\
\langle E_2 | E_2 \rangle &= \langle E_4 | E_4 \rangle = \frac{1}{4}(1 + c_{22}), \\
\langle E_3 | E_3 \rangle &= \frac{1}{4}(-1 + 2\epsilon - c_{22}).
\end{align*}
\]
(11)
The positivity of $\rho_{AB}$ thus requires
\begin{equation}
-1 \leq c_{22} \leq 2\epsilon - 1,
\end{equation}
which identifies the shaded area in Figure 1.

The reduced ancilla states that are conditioned on Alice getting one of her four measurement results $z+, z-, x+, x-$ are then given by
\begin{align}
\rho_{z\pm} &= (|E_1\rangle \pm |E_2\rangle)(\langle E_1| \pm \langle E_2|) + (|E_3\rangle \pm |E_4\rangle)(\langle E_3| \pm \langle E_4|), \\
\rho_{x\pm} &= (|E_1\rangle \mp |E_4\rangle)(\langle E_1| \mp \langle E_4|) + (|E_2\rangle \pm |E_3\rangle)(\langle E_2| \pm \langle E_3|),
\end{align}
each of them occurring with probability $\frac{1}{4}$. It is thus Eve’s task to discriminate between these states as best as she can, by a suitable POVM, whereby the figure of merit is the accessed information, equal to the mutual information between Eve and Alice that results from the chosen POVM.
5. POVMs that maximize the mutual information

The mutual information that Eve achieves is maximized, at least locally, by a von Neumann measurement composed of the projectors to the following kets:

\[
\begin{align*}
|P_1\rangle &= |E_1\rangle \frac{1}{\sqrt{3 - 2\epsilon - c_{22}}} \pm |E_2\rangle \sqrt{\frac{3 - 2\epsilon - c_{22}}{1 - c_{22}^2}}, \\
-|P_2\rangle &= -|E_3\rangle \frac{i}{\sqrt{2\epsilon - 1 - c_{22}}} \mp |E_4\rangle i \sqrt{\frac{2\epsilon - 1 - c_{22}}{1 - c_{22}^2}}, \\
\text{and} \quad |P_3\rangle &= |E_1\rangle \frac{1}{\sqrt{3 - 2\epsilon - c_{22}}} \pm |E_2\rangle i \sqrt{\frac{2\epsilon - 1 - c_{22}}{1 - c_{22}^2}}, \\
+|P_4\rangle &= +|E_3\rangle \frac{i}{\sqrt{2\epsilon - 1 - c_{22}}} \mp |E_4\rangle i \sqrt{\frac{3 - 2\epsilon - c_{22}}{1 - c_{22}^2}}. \\
\end{align*}
\]

The mutual information obtained from this POVM is

\[
I_{AE} = \frac{1}{2} \Phi\left(\sqrt{1 - c_{22}^2}\right),
\]

independent of \(\epsilon\).

In fact, this POVM is not the only one that attains this mutual information. Since Eve’s conditioned ancilla states (13) have real coefficients in the basis of the \(|E_j\rangle\) kets, the complex conjugate of (14) gives exactly the same mutual information. And so does any convex combination of these two POVMs, although the members of the so-formed POVM are no longer of rank one. In particular, an equal-weight combination gives an optimal POVM with real coefficients. Consult Ref. 2 for more details about this matter.

Eve makes use of her freedom to select any \(c_{22}\) value within the limits of (12) such that \(I_{AE}\) is largest. This amounts to choosing the smallest permissible value of \(|c_{22}|\). For \(\epsilon \geq \frac{1}{2}\), this is \(c_{22} = 0\), giving the straight line (d) in Fig. 1; for \(\epsilon \leq \frac{1}{2}\) the best choice is \(c_{22} = -(1 - 2\epsilon)\), which traces out line (c) in Fig. 1.

Accordingly, Eve has

\[
I_{AE}(\epsilon) = \begin{cases} 
\frac{1}{2} \Phi\left(2\sqrt{\epsilon(1 - \epsilon)}\right) \text{ for } 0 \leq \epsilon \leq \frac{1}{2}, \\
\frac{1}{2} \text{ for } \frac{1}{2} \leq \epsilon \leq 1,
\end{cases}
\]

after optimizing the value of \(c_{22}\). The comparison with \(I_{AB}(\epsilon)\) of (5) then implies that the BB84 protocol is secure under raw-data attacks when \(\epsilon < \frac{1}{5}\), which corresponds to a quantum bit error rate of 10%.

Figure 2 shows \(I_{AE}(\epsilon)\) for the \(c_{22}\) values along curves a, b, and c in Fig. 1, for the relevant range of \(0 \leq \epsilon \leq \frac{1}{2}\). Also shown is \(I_{AB}(\epsilon)\) of (5), which decreases as \(\epsilon\) increases.
Fig. 2. The decreasing function shows the mutual information $I_{AB}$ between Alice and Bob as given in (5). Curve a gives Eve's accessible information (15) when she honestly sends the unbiased-noise state (1). Eve is restricted to this if Alice and Bob were to perform complete tomography on their states. Curve b plots Eve's accessible information when she sends a state with maximum entropy, and curve c applies when she sends the state with the largest degree of separability, or the smallest concurrence, and thus achieves the optimum of (16). The curves a, b, and c intersect Alice and Bob's mutual information at $\epsilon = 1 - \sqrt{1/2} = 0.2929$, $\epsilon = 1 - \sqrt{5/4 - 1/2} = 0.2138$, and $\epsilon = 1/5 = 0.2000$, respectively. — The dashed curve shows $I_{AE}^{(HSV)}(\epsilon)$ of (20).

6. Largest degree of separability, smallest concurrence

As is clearly shown by Figs. 1 and 2, Eve's best choice for $c_{22}$ does not amount to sending the unbiased-noise state (1) to Alice and Bob, for which $c_{22} = -(1 - \epsilon)$, nor the two-qubit state with the largest entropy, for which $c_{22} = -(1 - \epsilon)^2$. Rather, Eve sends the state with the largest degree of separability $S$, a quantity introduced by Lewenstein and Sanpera, and the smallest value of the Hill–Wootters concurrence $C$.

For the two-qubit state $\rho_{AB}$ under consideration, which is diagonal in the Bell-state basis of (8) and thus “self-transposed” in the terminology of Ref. 10, one has

$$S = \min\left\{1, \epsilon + \frac{1}{2}(1 + c_{22})\right\},$$

$$C = \max\left\{0, \frac{1}{2}(1 - c_{22}) - \epsilon\right\},$$

so that $S + C = 1$ and maximizing $S$ is tantamount to minimizing $C$. For other families of states, however, different states may realize the largest $S$ value and the smallest $C$ value. Since the self-transposed states do not offer a clue, we leave it as a moot point which of the two quantities is the crucial one.
7. Maximal entropy

The POVM of Sec. 5 and the thresholds of Fig. 2 apply when it is Eve’s objective to gain maximal knowledge about Alice’s measurement results, for each qubit pair sent to Alice and Bob. In other words, Eve is attacking the raw data that have the correlations of Table 1.

These correlations are turned into a cryptographic key by a suitable error-correcting code for one-way communication from Alice to Bob. For this purpose, the measurement results are identified with the letters of an alphabet — such as $(z+ , z−, x+ , x−)\equiv(A,B,C,D)$ for Alice and $(z+ , z−, x+ , x−)\equiv(B,A,D,C)$ for Bob — and the code words are sequences of A, B, C, and D. Alice chooses at random one of the code words and informs Bob, over a public channel, which of her measurement results make up the code words (“Listen, it’s qubits 101, 17, 53, 2674, . . . ”). Bob’s corresponding measurement results constitute the received word, which he can then decode to the code word Alice sent. The sequence of transmitted code words, or perhaps a single very long code word, are then the key for the one-time-pad encryption.

Clearly, Eve is much more interested in this key than in the raw data from which it is generated. If she has the technical means for storing her ancillas, she will not measure them until after Alice has publicly announced the qubit pairs that contribute to the code words. Only then will Eve measure the respective ancillas jointly, thereby gaining more information per qubit pair, possibly as much as the Holevo–Schumacher–Westmoreland bound\(^{11}\) grants,

$$I_{AE}^{(HSW)} = S(\rho_E) - \frac{1}{4} \sum_{\alpha=\{z\pm, x\pm\}} S(\rho_\alpha),$$

where

$$\rho_E = \frac{1}{4} \sum_{\alpha} \rho_\alpha = \sum_{j=1}^{4} |E_j\rangle\langle E_j|$$

is the over-all ancilla state, and $S(\rho) = -\text{tr}\{\rho \log_2 \rho\}$ is the von Neumann entropy in units of bits. The non-zero eigenvalues of each $\rho_\alpha$ are $1 - \frac{1}{2} \epsilon$ and $\frac{1}{2} \epsilon$, and the eigenvalues of $\rho_E$ are the probabilities of (11), so that $I_{AE}^{(HSW)}$ is readily evaluated.

Since $S(\rho_\alpha)$ does not depend on $c_{22}$, the $c_{22}$ value for which $I_{AE}^{(HSW)}$ is largest, is the value for which $\rho_E$ has maximal entropy. It is also the $c_{22}$ value for which $\rho_{AB}$ has maximal entropy because $\rho_E$ and $\rho_{AB}$ are unitarily equivalent. As stated above, this happens for $c_{22} = -(1 - \epsilon)^2$. Then

$$I_{AE}^{(HSW)}(\epsilon) = 1 - \Phi(1 - \epsilon) = 1 - 2I_{AB}(\epsilon),$$

so that the corresponding threshold value for $\epsilon$ is determined by $I_{AB}(\epsilon) = \frac{1}{4}$. This gives $\epsilon = 0.1230$, or a quantum bit error rate of 6.15\%, as is illustrated by the dashed curve in Fig. 2.
8. Conclusion

We have considered quantum key distribution from the raw-data correlations of the BB84 scheme by one-way communication, and have established the noise thresholds for eavesdropping attacks on the raw data and on the generated key. Owing to the incomplete state tomography in the BB84 scenario, Eve can choose the two-qubit state she sends to Alice and Bob from a seven-parameter family. Our analysis invokes a plausible symmetry conjecture, which is unproven as yet but backed by numerical evidence, namely that Eve can restrict herself to sending self-transposed states to Alice and Bob, which have only one free parameter.

We find that the noise threshold for the raw-data attack is $\epsilon = \frac{1}{5}$, which Eve achieves by distributing the two-qubit state with the smallest concurrence, or the largest degree of separability, to Alice and Bob. By contrast, the ultimate attack on the generated key is most powerful when the two-qubit state with maximal entropy is sent, and the resulting threshold is at $\epsilon = 0.1230$. The corresponding quantum bit error rates, after the bases matching in the standard BB84 protocol, are 10% and 6.15%, respectively. Comparison with the accepted threshold for unconditional security for BB84, about 12.4%, thus establishes that the key extraction by two-way communication (bases matching etc.) is advantageous for Alice and Bob.

Acknowledgments

J. S. and B.-G. E. wish to thank Hans Briegel for the generous hospitality extended to them at the Institute for Quantum Optics and Quantum Information in Innsbruck, where part of this work was done. This work is supported by A*STAR Temasek Grant No. 012-104-0040 and NUS Grant WBS R144-000-116-101.

References

1. C. H. Bennett and G. Brassard, “Quantum cryptography: Public key distribution and coin tossing,” in IEEE Conference on Computers, Systems, and Signal Processing, Bangalore, India (IEEE, New York, 1984), pp. 175–179.
2. J. Suzuki, S. M. Assad, and B.-G. Englert, “Accessible information about quantum states: An open optimization problem,” in Mathematics of Quantum Computation and Quantum Technology, edited by G. Chen et al. (Chapman & Hall/CRC, Boca Raton, in production).
3. R. Renner, Security of Quantum Key Distribution, Ph.D. thesis (ETH, Zürich 2005), eprint arXiv:quant-ph/0512258.
4. Y. C. Liang, D. Kaszlikowski, B.-G. Englert, L. C. Kwek, and C. H. Oh, Phys. Rev. A 68, 022324 (2003).
5. B.-G. Englert, D. Kaszlikowski, H. K. Ng, W. K. Chua, J. Reháček, and J. Anders, Highly efficient quantum key distribution with minimal state tomography, eprint arXiv:quant-ph/0412075.
6. I. Csiszár and J. Körner, IEEE Trans. Inf. Theory 24, 339 (1978).
7. B.-G. Englert, C. Miniatura, and J. Baudon, J. Phys. II France 4, 2043 (1994).
8. M. Lewenstein and A. Sanpera, Phys. Rev. Lett. 80, 2261 (1998).
9. S. Hill and W. K. Wootters, Phys. Rev. Lett. 78, 5022 (1997).
10. B.-G. Englert and N. Metwally, “Kinematics of qubit pairs,” in Mathematics of Quantum Computation, edited by G. Chen and R. K. Brylinski (CRC Press LLC, Boca Raton, 2002), pp. 25–75.
11. I. Devetak and A. Winter, Proc. Roy. Soc. A 461, 207 (2005).
12. R. Renner, N. Gisin, and B. Kraus, Phys. Rev. A 72, 012332 (2005).