On Algebraic Supergroups, Coadjoint Orbits and Their Deformations

R. Fioresi$^1$, M.A. Lledó$^2$

$^1$ Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato, 5, 40126 Bologna, Italy. E-mail: fioresi@dm.UniBo.it

$^2$ INFN, Sezione di Torino, Italy and Dipartimento di Fisica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy. E-mail: lledo@athena.polito.it

Received: 10 February 2003 / Accepted: 3 September 2003

Published online: 16 December 2003 – © Springer-Verlag 2003

Abstract: In this paper we study algebraic supergroups and their coadjoint orbits as affine algebraic supervarieties. We find an algebraic deformation quantization of them that can be related to the fuzzy spaces of non-commutative geometry.

1. Introduction

The use of “odd variables” in physics is very old and quite natural. Since then the concept of supermanifold in mathematics has evolved to a very precise formulation, starting with the superanalysis of Berezin [1], and together with other works by Kostant [2], Leites [3], Manin [4], Bernstein, Deligne and Morgan [5] to mention some of the most representative. Supermanifolds are seen roughly as ordinary manifolds together with a sheaf of superalgebras. The odd coordinates appear in the stalks of the sheaf. This approach allows considerable freedom. Following Manin [4], one can define different kinds of superspaces, supermanifolds or algebraic supervarieties by choosing a base space with the appropriate topology. The sheaves considered are superalgebra valued, so that one can generalize the concepts of complex and algebraic geometry to this new setting.

In this paper we will deal strictly with algebraic supervarieties, but the concept of supermanifold has been treated extensively in the literature mentioned above.

It is interesting to note that there is an alternative definition of algebraic variety in terms of its functor of points. Essentially, an algebraic variety can be defined as a certain functor from some category of commutative algebras to the category of sets. It is then very natural to substitute the category of algebras by an appropriate category of commutative superalgebras and to call this a supervariety. The same can be done for supergroups, super Lie algebras, coadjoint orbits of supergroups and other “super” objects. The elegance of this approach cannot hide the many non-trivial steps involved in the generalization. As an example, it should be enough to remember the profound
differences between the classifications of semisimple Lie algebras and semisimple Lie superalgebras [1, 6, 7].

The purpose of this paper is to study the coadjoint orbits of certain supergroups, to establish their structure as algebraic supermanifolds and to obtain a quantum deformation of the superalgebras that represent them.

Recently there has been a growing interest in the physics literature on noncommutative spaces (also called “fuzzy spaces”). The idea that spacetime may have non-commuting coordinates which then cannot be determined simultaneously has been proposed at different times for diverse motives [10–13]. In particular, these spaces have been considered as possible compactification manifolds of string theory [13].

In general, we can say that a “fuzzy” space is an algebra of operators on a Hilbert space obtained by some quantization procedure. This means that it is possible to define a classical limit for such an algebra, which will be a commutative algebra with a Poisson structure, that is, a phase space. As it is well known, the coadjoint orbits of Lie groups are symplectic manifolds and hamiltonian spaces with the Kirillov Poisson structure of the corresponding Lie group. There is extensive literature on the quantization of coadjoint orbits, most of the works based on the Kirillov-Kostant orbit principle which associates to every orbit (under some conditions) a unitary representation of the group. Let us consider as an example the sphere $S^2$, a regular coadjoint orbit of the group SU(2). In physical terms, the quantization of $S^2$ is the quantization of the spin; it is perhaps the most classical example of quantization, other than flat space $\mathbb{R}^2$. In the approach of geometric quantization (see Ref. [8] for a review) the sphere must have half integer radius $j$ and to this orbit it corresponds the unitary representation of SU(2) of spin $j$ (which is finite dimensional). Let us take the classical algebra of observables to be the polynomials on the algebraic variety $S^2$. Given that the Hilbert space of the quantum system is finite dimensional, the algebra of observables is also finite dimensional. It is in fact of dimension $(2j+1)^2$ and isomorphic to the algebra of $(2j+1)\times(2j+1)$-matrices [9, 10]. After a rescaling of the coordinates, we can take the limit $j \to \infty$ maintaining the radius of the sphere constant. In this way the algebra becomes infinite dimensional and all polynomials are quantized. This procedure is most appropriately described by Madore [10], being the algebras with finite $j$ approximations to the non commutative or fuzzy sphere.

The deformation quantization approach [14] is inspired in the correspondence principle: to each classical observable there must correspond a quantum observable (operator on a Hilbert space). This quantization map will induce a non-commutative, associative star product on the algebra of classical observables which will be expressed as a power series in $\hbar$, having as a zero order term the ordinary, commutative product and as first order term the Poisson bracket. One can ask the reverse question, and explore the possible deformations of the commutative product independently of any quantization map or Hilbert space. It is somehow a semiclassical approach. Very often, the star product is only a formal star product, in the sense that the series in $\hbar$ does not converge. It however has the advantage that many calculations and questions may be posed in a simpler manner (see [22] for the string theory application of the star product).

The deformation quantization of the sphere and other coadjoint orbits has been treated in the literature [15–19]. The immediate question is if deformation theory can give some information on representation theory and the Kirillov-Kostant orbit principle [20, 21].

In Ref. [18] a family of algebraic star products is defined on regular orbits of semisimple groups. The star product algebra is isomorphic to the quotient of the enveloping algebra by some ideal. It is in fact possible to select this ideal in such a way that it is in the kernel of some unitary representation. The image of this algebra by the representation