THE LINEAR SAMPLING METHOD FOR KIRCHHOFF-LOVE INFINITE PLATES

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Abstract. This paper addresses the problem of identifying impenetrable obstacles in a Kirchhoff-Love infinite plate from multistatic near-field data. The Linear Sampling Method is introduced in this context. We firstly prove a uniqueness result for such an inverse problem. We secondly provide the classical theoretical foundation of the Linear Sampling Method. We lastly show the feasibility of the method with the help of numerical experiments.

1. Introduction. In this contribution we consider the inverse problem of finding an impenetrable obstacle in an infinite elastic plate from multistatic scattering data in the frequency domain. Assuming that the thickness of the plate is small with respect to the wavelength, we consider that the behavior of the elastic plate is governed by the classical Kirchhoff-Love model in the purely bending case. The impenetrable obstacle \( D \subset \mathbb{R}^2 \) is supposed to be a bounded open domain of class \( C^3 \) which is either characterized by a Dirichlet or a Neumann boundary condition. More precisely, by using the notations of [2], in particular \( \Omega = \mathbb{R}^2 \setminus \overline{D} \), the scattered field \( v^s \) satisfies in the unbounded domain \( \Omega \) the problem

\[
\left\{ \begin{array}{ll}
\Delta^2 v^s - k^4 v^s = 0 & \text{in } \Omega \\
B_1(v^s + u^i) = B_2(v^s + u^i) = 0 & \text{on } \partial \Omega \\
\lim_{r \to +\infty} \int_{\partial B_r} \left| \frac{\partial v^s}{\partial n} - i k v^s \right|^2 ds = 0.
\end{array} \right.
\]

Here \( k > 0 \) is the wave number, \( u^i \) is an incident field which satisfies \( \Delta^2 u^i - k^4 u^i = 0 \) in a domain including \( \overline{D} \), \( B_r \) is the open ball centered at 0 and of radius \( r \), \( n \) is the outward normal to \( B_r \) and \( s \) is the measure on \( \partial B_r \). The reader will refer to [2] for a short justification of how the system (1) is derived. Roughly speaking, the first line describes the motion of the plate in the frequency domain, the second one characterizes the boundary conditions on the boundary of the obstacle while the third one is the radiation condition, which specifies that only outgoing scattering waves are admissible. It is shown in [2] that the classical Sommerfeld condition for

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the Helmholtz equation is also valid for the bilaplacian case. In order to specify the surface differential operators \( B_1 \) and \( B_2 \), we need to introduce some notations. A generic point \( x \in \mathbb{R}^2 \) has Cartesian coordinates \((x_1, x_2)\). The outward unit normal to \( \Omega \) is denoted \( n \) (note that \( n \) is oriented inside \( D \)). The unit tangent vector is denoted \( t \) and is such that the angle formed by the vectors \((n, t)\) is \( \pi/2 \). The curvilinear abscissa associated with vector \( t \) is denoted \( s \) and coincides with the measure on \( \partial D \). With the classical definitions

\[
\frac{\partial}{\partial n} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial s} = -n_2 \frac{\partial}{\partial x_1} + n_1 \frac{\partial}{\partial x_2},
\]

either \((B_1, B_2) = (I, \partial n)\) (\( I \) is the identity), which corresponds to the Dirichlet boundary condition, or \((B_1, B_2) = (M, N)\), which corresponds to the Neumann boundary condition, where the operators \( M \) and \( N \) are defined as follows:

\[
\begin{align*}
M_0u &= \nu \Delta u + (1 - \nu)M_0u, \\
N_0u &= -\frac{\partial}{\partial n} \Delta u - (1 - \nu) \frac{\partial}{\partial s} N_0u.
\end{align*}
\]

Here, \( \nu \in [0, 1/2) \) is the Poisson’s ratio and \( M_0 \) and \( N_0 \) are given by

\[
\begin{align*}
M_0u &= \frac{\partial^2 u}{\partial x_1^2} n_1^2 + 2 \frac{\partial^2 u}{\partial x_1 \partial x_2} n_1 n_2 + \frac{\partial^2 u}{\partial x_2^2} n_2^2, \\
N_0u &= \frac{\partial^2 u}{\partial x_1 \partial x_2} (n_1^2 - n_2^2) - \left( \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} \right) n_1 n_2.
\end{align*}
\]

We mention that the Dirichlet boundary condition amounts to specify the out of plane displacement and the angle of rotation of the plate, while the Neumann boundary condition amounts to specify the bending moment and the shear force. In other words, the Dirichlet boundary condition corresponds to the clamped plate while the Neumann boundary condition corresponds to the free plate. Let us now consider the bounded domain \( \Omega_R = \Omega \cap B_R \), where \( R > 0 \) is such that \( B_R \) contains the obstacle \( D \). It is proved in [2] that the problem (1) is equivalent to the following problem set in the bounded domain \( \Omega_R \): find \( u^s \in H^2(\Omega_R) \) such that

\[
\begin{align*}
\Delta^2 u^s - k^4 u^s &= 0 & \text{in } \Omega_R, \\
B_1(u^s + u^t) &= B_2(u^s + u^t) = 0 & \text{on } \partial \Omega, \\
(Nu^s) &= T \left( \frac{u^s}{\partial \Omega_R} \right) & \text{on } \partial B_R,
\end{align*}
\]

where \( T : H^{3/2}(\partial B_R) \times H^{1/2}(\partial B_R) \to H^{-3/2}(\partial B_R) \times H^{-1/2}(\partial B_R) \) is a Dirichlet-to-Neumann operator defined as follows. Assume that \( (f, g) \in H^{3/2}(\partial B_R) \times H^{1/2}(\partial B_R) \) have the Fourier decomposition

\[
f = \sum_{m \in \mathbb{Z}} f_m \xi_m \quad \text{and} \quad g = \sum_{m \in \mathbb{Z}} g_m \xi_m \quad \text{with} \quad \xi_m(\theta) = e^{im\theta}/\sqrt{2\pi},
\]

where

\[
f_m = \frac{1}{R} (f, \xi_m)_{L^2(\partial B_R)} \quad \text{and} \quad g_m = \frac{1}{R} (g, \xi_m)_{L^2(\partial B_R)}.
\]

We have

\[
T \left( \begin{array}{c} f \\ g \end{array} \right) = \sum_{m \in \mathbb{Z}} \xi_m(\theta) T_m \left( \begin{array}{c} f_m \\ g_m \end{array} \right), \quad T_m = \begin{pmatrix} T_m^{11} & T_m^{12} \\ T_m^{21} & T_m^{22} \end{pmatrix},
\]
The problem (1) has a unique solution in Theorem 1.1. of a Fredholm analysis of the equivalent problem (2). The posedness of the forward problem (1) is the main purpose of [2]. The proof consists of a Fredholm analysis of the equivalent problem (2).

**Theorem 1.1.** The problem (1) has a unique solution in $H^2_{loc}(\Omega)$

- for any $k$ in the clamped case,
- for $k \notin \mathcal{K}_0$ in the free case, where the set $\mathcal{K}_0$ is formed by some numbers $k_n > 0$, $n \in \mathbb{N}$, such that $k_n \to +\infty$.

Note that we ignore if the restriction $k \notin \mathcal{K}_0$ is purely technical or not. However, it is proved in [2] that for some particular obstacles $D$, for example a disc, we have $\mathcal{K}_0 = \emptyset$.

The inverse problem we consider is the following. We assume that $D$ is unknown but a priori contained in $B_R$. Let us denote by $G(\cdot, y)$ the fundamental solution associated with the operator $\Delta^2 - k^4$, that is the unique solution in $H^2_{loc}(\mathbb{R}^2)$ of the system

$$
\Delta^2 G(\cdot, y) - k^4 G(\cdot, y) = \delta_y \quad \text{in } \mathbb{R}^2
$$

$$
\lim_{r \to +\infty} \int_{\partial B_r} \left| \frac{\partial G(\cdot, y)}{\partial n} - i k G(\cdot, y) \right|^2 \, ds = 0.
$$

(5)

It is well-known that $G$ is given by

$$
G(x, y) = \frac{i}{8k^2} \left( H^1_0(k|x - y|) - H^1_0(i k|x - y|) \right).
$$

(6)

For sake of self-containment, the well-posedness of problem (5) and the expression (6) are proved in Lemma 2.2 hereafter. For some point $y \in \partial B_R$, we denote $u^s(\cdot, y)$ the scattered field which is associated with the incident field $u^i = G(\cdot, y)$ via (1). We also denote $\tilde{u}^s(\cdot, y)$ the scattered field which is associated with the incident field $u^i = \partial_{n_y} G(\cdot, y)$ via (1), where $n_y$ is the outward normal to $B_R$ at point $y \in \partial B_R$. The function $G(\cdot, y)$ can be seen as a point source located at $y$, while the function $\partial_{n_y} G(\cdot, y)$ can be seen as a dipole located at $y$. For all points $y \in \partial B_R$, we measure the scattered fields $u^s(\cdot, y)$ and $\tilde{u}^s(\cdot, y)$ as well as their normal derivatives at all points $x \in \partial B_R$. All these measurements constitute the so-called multistatic data. The goal of the inverse problem is to retrieve the obstacle $D$ from those multistatic data. It arises in the framework of Non Destructive Testing, which is for example quite common in the aircraft industry. The bilaplacian model is interesting when the structure to inspect is thin and the frequency is low, which enables us to replace the 3D elastic model by such 2D approximate model. For example, Structural Health Monitoring would be a nice application. If we think of the SHM of the fuselage or the wings of an aircraft, even if some small hole in the skin would be visible to the
naked eye when the aircraft is on the ground, it cannot be seen when the aircraft flies. Because structures may be complicated in real life, a possible extension of our present work would be to consider obstacles in a plate which has non homogeneous properties in a bounded region. Besides, in a view to consider defects such as corrosion or delamination, which could be modeled by a modified thickness of the plate or modified constants of elasticity, it would be useful to extend the present study to penetrable obstacles instead of impenetrable ones. All these extensions are non trivial, both concerning the uniqueness results and the effective reconstruction methods.

In order to address the inverse problem, we adapt the classical Linear Sampling Method introduced by Colton and Kirsch in [8] to the case of plates. Since this pioneering paper, the Linear Sampling Method has been applied in a large number of situations (see for example [9]), in particular in elasticity (see for example [1, 10, 12, 3, 4]). But the case of Kirchhoff-Love plates is new as far as we can judge. The Linear Sampling Method consists in testing, for all point of a sampling grid, if some test function depending on that point belongs to the range of an integral operator, the kernel of which only depends on the multistatic data. The LSM is both simple and efficient. In addition, the main feature of such method is that it works even if the nature of the obstacle is as priori unknown. To the best of our knowledge, the number of articles concerning inverse scattering problems in some infinite Kirchhoff-Love plate is very small. We highlight the very recent contribution [21] on that subject in the particular case of a bilaplacian operator with zero and first order perturbations. Concerning the static case in a bounded domain, we also mention recent contributions on uniqueness and stability issues for unknown boundaries in [17, 18]. In order to build some synthetic data we need to solve the forward problem (1). To do so, we use a Finite Element Method based on the weak formulation associated with the problem (2) and a discretization of the Dirichlet-to-Neumann operator (3). The practical use of such a D-t-N operator to numerically compute a scattering solution in an infinite plate is new, as far as we know. Note that alternative approaches are the use of Perfectly Matched Layers, as in [13], or Boundary Integral Methods, as in [20].

The paper is organized as follows. The second section is devoted to the treatment of the inverse problem: we first derive an integral representation formula considered here as a preliminary tool, then prove the identifiability of the obstacle from the prescribed data, and lastly provide the justification of the Linear Sampling Method for the Dirichlet case and give some indications for the Neumann case. The third section presents some numerical results: we firstly describe the Finite Element Method we use to produce the artificial data with an example, then show the identification results obtained with the Linear Sampling Method, and lastly complete this numerical section by a short conclusion.

2. The inverse problem.

2.1. Integral representation. The integral representation formula is a basic tool in the proof of the identifiability of the obstacle and also in the justification of the Linear Sampling Method to retrieve it. Let us first recall some classical plate-oriented Green formula (see [14]).
**Lemma 2.1.** In a bounded domain $\mathcal{O}$ of class $C^2$, we denote $H^2(\mathcal{O}, \Delta^2) = \{ u \in H^2(\mathcal{O}), \Delta^2 u \in L^2(\mathcal{O}) \}$. The linear mapping

$$ T : H^2(\mathcal{O}, \Delta^2) \rightarrow H^{-3/2}(\partial \mathcal{O}) \times H^{-1/2}(\partial \mathcal{O}) $$

$$ u \mapsto \left( \begin{array}{c} Nu \\ Mu \end{array} \right) $$

is continuous. Moreover, for all $v \in H^2(\mathcal{O})$, we have the Green formula

$$ \int_\mathcal{O} \Delta^2 u v \, dx = a(u,v) - \int_{\partial \mathcal{O}} \left( \frac{\partial v}{\partial n} Mu + v Nu \right) \, ds, $$

where we have introduced the bilinear form

$$ a(u,v) = \int_\mathcal{O} \left\{ \nu \Delta u \Delta v + (1-\nu) \sum_{i,j=1}^2 \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \right\} \, dx, $$

and $n$ is the outward normal to $\mathcal{O}$. Here the first part of the integral on $\partial \mathcal{O}$ has the meaning of duality pairing between $H^{1/2}(\partial \mathcal{O})$ and $H^{-1/2}(\partial \mathcal{O})$ while the second part has the meaning of duality pairing between $H^{3/2}(\partial \mathcal{O})$ and $H^{-3/2}(\partial \mathcal{O})$.

Given some functions $(\phi, \psi) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$, let us consider the following interior and exterior problems: find $u^- \in H^2(D)$ and $u^+ \in H^2_{\text{loc}}(\Omega)$ such that

$$ \Delta^2 u^- - k^4 u^- = 0 \quad \text{in } D $$

$$ (u^-, \partial_n u^-) = (\phi, \psi) \quad \text{on } \partial D \quad (8) $$

and

$$ \Delta^2 u^+ - k^4 u^+ = 0 \quad \text{in } \Omega $$

$$ (u^+, \partial_n u^+) = (\phi, \psi) \quad \text{on } \partial \Omega $$

$$ \lim_{r \to +\infty} \int_{\partial B_r} \left| \frac{\partial u^+}{\partial n} - ik u^+ \right|^2 \, ds = 0, \quad (9) $$

where the unit normal vector $n$ is oriented outside $D$ in both problems (8) and (9). From Theorem 1.1, the exterior problem is well-posed. Concerning the interior problem, it is straightforward that well-posedness holds if and only if $k \notin \mathcal{K}_D$, where $\mathcal{K}_D$ is the set formed by the fourth roots of the Dirichlet eigenvalues of operator $\Delta^2$ in domain $D$. Before proving our integral representation formula, let us derive the fundamental solution for operator $\Delta^2 - k^4$.

**Lemma 2.2.** Problem (5) has a unique solution which is given by (6).

**Proof.** Let $u = G(\cdot,y)$ be a solution to problem (5). We use the factorization

$$ \Delta^2 u - k^4 u = (\Delta - k^2)(\Delta + k^2)u = (\Delta + k^2)(\Delta - k^2)u. $$

By setting $U = \Delta u + k^2 u$ and $V = \Delta u - k^2 u$, we hence obtain that

$$ \Delta U - k^2 U = \delta_y, \quad \Delta V + k^2 V = \delta_y, $$

In addition, by using a decomposition in the form of a series as in [2], it can be seen that if $u$ satisfies the Sommerfeld radiation condition, the functions $U$ and $V$ both satisfy the radiation condition as well. This implies from the case of Helmholtz equation that

$$ U(x) = -\frac{i}{4} H^1_0(ik|x-y|), \quad V(x) = -\frac{i}{4} H^1_0(k|x-y|). $$
Remark 1. Since \( u_{368} \) Laurent Bourgeois and Arnaud Recoquillay

Proposition 1. Let us consider a function \( H \)

where \((11)\)

have \( u \) solves problem (8) and \( u \)

solves problem (8) and \( u = u|\Omega \) solves problem (9). Then \( G(\cdot, y) \in H^2_{\text{loc}}(\mathbb{R}^2) \). Then \( G(\cdot, y) \not\in H^3_{\text{loc}}(\mathbb{R}^2) \).

**Proof.** Let us consider \( x \in \Omega \) and \( r > 0 \) such that \( B(x, 2r) \in \Omega \). Next we define \( g_x \)

and \( \tilde{g}_x \) such that

\[
G_0(x, y) = \frac{1}{8\pi} |x - y|^2 \log |x - y|.
\]

It is clear that the function \( G(\cdot, y) - G_0(\cdot, y) \) is infinitely smooth and it is easy to check by using polar coordinates that \( G_0(\cdot, y) \) is in \( H^2_{\text{loc}}(\mathbb{R}^2) \). Then \( G(\cdot, y) \in H^2_{\text{loc}}(\mathbb{R}^2) \), which completes the proof. \( \square \)

**Remark 1.** We also check from the comparison with \( G_0(\cdot, y) \) that \( G(\cdot, y) \not\in H^3_{\text{loc}}(\mathbb{R}^2) \).

**Proposition 1.** Let us consider a function \( u \) defined in \( D \cup \Omega \) such that \( u^- = u|\partial D \) solves problem (8) and \( u^+ = u|\Omega \) solves problem (9). Then for all \( x \in D \cup \Omega \), we have

\[
u(x) = \int_{\partial D} G(x, y) \tau(y) \, ds(y) + \int_{\partial D} \frac{\partial G(x, y)}{\partial n_y} \sigma(y) \, ds(y),
\]

where \((\tau, \sigma) = ([Nu]_1, [Mu])\). Here, \([\cdot]_+ - [\cdot]_-\) denotes the jump across the boundary of \( D \) and \( n_y \) is oriented inside \( D \). The first integral means duality between \( H^{3/2}(\partial D) \) and \( H^{-3/2}(\partial D) \) while the second one means duality between \( H^{1/2}(\partial D) \) and \( H^{-1/2}(\partial D) \).

**Proof.** Let us consider \( x \in \Omega \) and \( r > 0 \) such that \( B(x, 2r) \in \Omega \). Next we define \( g_x \)

and \( \tilde{g}_x \) such that

\[
g_x(y) = \begin{cases} G(x, y) & y \in B(x, r) \\ 0 & y \notin B(x, r) \end{cases}
\]

and

\[
\tilde{g}_x(y) = g_x(y) - G(x, y).
\]

Now we take \( \phi \in C^\infty_0(B(x, 2r)) \). The function \( \tilde{g}_x \) satisfies

\[
\langle (\Delta^2 - k^4) \tilde{g}_x, \phi \rangle = \int_{B(x,2r)} \tilde{g}_x(\Delta^2 - k^4) \phi \, dy,
\]

where we have used the distribution brackets \( \langle \cdot, \cdot \rangle \). It follows that

\[
\langle (\Delta^2 - k^4) \tilde{g}_x, \phi \rangle = -\int_{B(x,2r) \setminus B(x, r)} G(x, y)(\Delta^2 - k^4) \varphi(y) \, dy
\]

\[
= \int_{B(x,2r) \setminus B(x, r)} (\varphi(y)(\Delta^2 y - k^4)G(x, y) - G(x, y)(\Delta^2 - k^4) \varphi(y)) \, dy
\]

\[
= \int_{B(x,2r) \setminus B(x, r)} (\varphi(y)(\Delta^2 y G(x, y) - G(x, y)\Delta^2 \varphi(y)) \, dy.
\]

By using Lemma 2.1, we obtain

\[
\langle (\Delta^2 - k^4) \tilde{g}_x, \phi \rangle = -\int_{\partial B(x, r)} \left( M_y G(x, y) \frac{\partial \varphi}{\partial n_y}(y) + N_y G(x, y) \varphi(y) \right) ds(y)
\]

\[
+ \int_{\partial B(x, r)} \left( \frac{\partial G(x, y)}{\partial n_y} M \varphi(y) + G(x, y) N \varphi(y) \right) ds(y),
\]

\[
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\]
where \( n_y \) is the unit normal oriented inside \( B(x, r) \). As a result,

\[
\langle (\Delta^2 - k^4)g_x, \varphi \rangle = \varphi(x) - \int_{\partial B(x, r)} \left( M_y G(x, y) \frac{\partial \varphi}{\partial n_y}(y) + N_y G(x, y) \varphi(y) \right) ds(y)
+ \int_{\partial B(x, r)} \left( \frac{\partial G(x, y)}{\partial n_y} M \varphi(y) + G(x, y) N \varphi(y) \right) ds(y).
\]

Now let us choose \( \varphi = \theta u \) in the above relationship, for \( \theta \in C_0^\infty(B(x, 2r)) \), with \( \theta = 1 \) in \( \overline{B(x, r)} \). That \( \text{supp}(g_x) \subset \overline{B(x, r)} \) implies that

\[
\langle (\Delta^2 - k^4)g_x, \theta u \rangle = \int_{B(x, 2r)} g_x(\Delta^2 - k^4)(\theta u) dx = \int_{B(x, r)} g_x(\Delta^2 - k^4)u dx = 0,
\]

which implies that

\[
(12) \quad u(x) = \int_{\partial B(x, r)} U(x, y) ds(y),
\]

where we have used the notation

\[
U(x, y) = M_y G(x, y) \frac{\partial u}{\partial n_y}(y) + N_y G(x, y) u(y) - \frac{\partial G(x, y)}{\partial n_y} M u(y) - G(x, y) N u(y).
\]

If we now use Lemma 2.1 in the subdomain \( \Omega_{r, R} = (\Omega \cap B_R) \setminus \overline{B(x, r)} \), we obtain

\[
(13) \quad 0 = \int_{\partial B(x, r)} U(x, y) ds(y) + \int_{\partial D} U(x, y) ds(y) + \int_{\partial B_R} U(x, y) ds(y)
\]

where the normal \( n_y \) involved in \( U(x, y) \) is oriented inside \( B(x, r) \) in the first integral, inside \( D \) in the second integral and outside \( B_R \) in the third one. By using the Dirichlet-to-Neumann operator \( T : H^{3/2}(\partial B_R) \times H^{1/2}(\partial B_R) \to H^{-3/2}(\partial B_R) \times H^{-1/2}(\partial B_R) \), we have

\[
\int_{\partial B_R} U(x, y) ds(y) = \left\langle (u \ - \partial_n u), T_y \left( \begin{array}{c} G(x, y) \\ \partial_n G(x, y) \end{array} \right) \right\rangle 
- \left\langle (G(x, y) \ - \partial_n G(x, y)), T \left( \begin{array}{c} u \\ \partial_n u \end{array} \right) \right\rangle.
\]

From (4) we observe that the operator \( T \) is symmetric, so that

\[
\int_{\partial B_R} U(x, y) ds(y) = 0.
\]

By using (12) and (13) we obtain that

\[
u(x) = - \int_{\partial D} \left( M_y G(x, y) \frac{\partial u^+}{\partial n_y}(y) + N_y G(x, y) u^+(y) \right) ds(y)
+ \int_{\partial D} \left( \frac{\partial G(x, y)}{\partial n_y} M u^+(y) + G(x, y) N u^+(y) \right) ds(y),
\]

where \( n_y \) is oriented inside \( D \). Lastly we use again Lemma 2.1 in domain \( D \), so that

\[
0 = \int_{\partial D} \left( M_y G(x, y) \frac{\partial u^-}{\partial n_y}(y) + N_y G(x, y) u^-(y) \right) ds(y)
- \int_{\partial D} \left( \frac{\partial G(x, y)}{\partial n_y} M u^-(y) + G(x, y) N u^-(y) \right) ds(y),
\]
where \( n_y \) is oriented inside \( D \). From the two equations above, we obtain that for \( x \in \Omega \),
\[
\begin{align*}
    u(x) &= -\int_{\partial D} \left( M_y G(x, y) \left( \frac{\partial u}{\partial n_y}(y) \right) + N_y G(x, y) [u(y)] \right) \, ds(y) \\
    &\quad + \int_{\partial D} \left( \frac{\partial G(x, y)}{\partial n_y} [M u(y)] + G(x, y) [Nu(y)] \right) \, ds(y),
\end{align*}
\]
where \( n_y \) is oriented inside \( D \). It could be similarly proved that the same formula is valid for \( x \in D \). Given the boundary conditions satisfied by \( u^+ \) and \( u^- \) on \( \partial D \), we have that \([u] = 0\) and \([\partial_n u] = 0\) on \( \partial D \), so that the first integral vanishes, which completes the proof. \( \square \)

2.2. Uniqueness. Before we detail the effective reconstruction, we prove uniqueness of the obstacle from the data. In the sequel, we simply denote \( \Gamma = \partial B_R \), which is the support of sources and measurements. More precisely, we have the following result.

**Theorem 2.3.** Let \( D_1 \) and \( D_2 \) denote two obstacles, either of Dirichlet type (that is \((B_1, B_2) = (I, \partial_h)\)) or of Neumann type (that is \((B_1, B_2) = (M, N)\)). Let us consider, for all \( y \in \Gamma \) and \( j = 1, 2 \), the scattered fields \( u_j^s(\cdot, y) \) and \( \tilde{u}_j^s(\cdot, y) \) due to obstacle \( D_j \) and associated via problem \((1)\) with the incident fields \( u^i = G(\cdot, y) \) and \( u^i = \partial_n G(\cdot, y) \), respectively. Assume that for all \( y \in \Gamma \), the fields \( u_1^s(\cdot, y) \) and \( u_2^s(\cdot, y) \) coincide on \( \Gamma \) as well as their normal derivative and that the fields \( \tilde{u}_1^s(\cdot, y) \) and \( \tilde{u}_2^s(\cdot, y) \) coincide on \( \Gamma \) as well as their normal derivative. Then \( D_1 = D_2 \).

To prove such theorem, we need the following reciprocity relationships.

**Lemma 2.4.** For all \( x, y \in \Omega \) and \( z \in \Gamma \),
\[
u^s(x, y) = u^s(y, x), \quad \tilde{u}^s(x, z) = \frac{\partial u^s(x, z)}{\partial n_z}.
\]

**Proof.** We detail the proof for the Dirichlet case, the Neumann case follows the same lines. From the proof of Proposition 1, for any \( x, z \in \Omega \), we have the integral representation
\[
v^s(x, z) = -\int_{\partial D} \left( M_y G(x, y) \frac{\partial u^s(y, z)}{\partial n_y} + N_y G(x, y) u^s(y, z) \right) \, ds(y)
\]
\[
\quad + \int_{\partial D} \left( \frac{\partial G(x, y)}{\partial n_y} M_y u^s(y, z) + G(x, y) N_y u^s(y, z) \right) \, ds(y),
\]
where \( n_y \) is oriented inside \( D \). From the Green formula in \( D \) we obtain
\[
0 = -\int_{\partial D} \left( M_y G(x, y) \frac{\partial G(y, z)}{\partial n_y} + N_y G(x, y) G(y, z) \right) \, ds(y)
\]
\[
\quad + \int_{\partial D} \left( \frac{\partial G(x, y)}{\partial n_y} M_y G(y, z) + G(x, y) N_y G(y, z) \right) \, ds(y).
\]
By introducing the total field \( u(\cdot, z) = G(\cdot, z) + u^s(\cdot, z) \), which satisfies the Dirichlet condition on \( \partial D \), we obtain by adding the two above relationships that
\[
u^s(x, z) = \int_{\partial D} \left( \frac{\partial G(x, y)}{\partial n_y} M_y u(y, z) + G(x, y) N_y u(y, z) \right) \, ds(y).
\]
We now rewrite the first formula by inverting $x$ and $z$, which yields

$$u^s(x, z) = -\int_{\partial D} \left( M_y G(z, y) \frac{\partial u^s(y, x)}{\partial n_y} + N_y G(z, y) u^s(y, x) \right) ds(y) + \int_{\partial D} \left( \frac{\partial G(z, y)}{\partial n_y} M_y u^s(y, x) + G(z, y) N_y u^s(y, x) \right) ds(y).$$

From the Green formula in $\Omega$ and the radiation condition we obtain

$$0 = -\int_{\partial D} \left( M_y u^s(y, x) \frac{\partial u^s(y, z)}{\partial n_y} + N_y u^s(y, x) u^s(y, z) \right) ds(y) + \int_{\partial D} \left( \frac{\partial u^s(y, x)}{\partial n_y} M_y u^s(y, z) + u^s(y, x) N_y u^s(y, z) \right) ds(y).$$

By subtracting the two above relationships, we obtain by using the fact that $G(y, z) = G(z, y)$,

$$u^s(x, z) = -\int_{\partial D} \left( M_y G(x, y) \frac{\partial u^s(y, z)}{\partial n_y} + N_y G(x, y) u^s(y, z) \right) ds(y)$$

Subtracting (15) and (17) and using the fact that $u(\cdot, x)$ satisfies the Dirichlet condition on $\partial D$ we obtain that $u^s(x, z) - u^s(z, x) = 0$ for all $x, z \in \Omega$.

Let us prove the second relationship. For any $x, z \in \Omega$, we have the integral representation

$$\tilde{u}^s(x, z) = -\int_{\partial D} \left( M_y G(x, y) \frac{\partial \tilde{u}^s(y, z)}{\partial n_y} + N_y G(x, y) \tilde{u}^s(y, z) \right) ds(y) + \int_{\partial D} \left( \frac{\partial G(x, y)}{\partial n_y} M_y \tilde{u}^s(y, z) + G(x, y) N_y \tilde{u}^s(y, z) \right) ds(y),$$

where $n_y$ is oriented inside $D$. By computing the normal derivative of (14) with respect to $z$ at point $z \in \Gamma$, we obtain that for all $x \in \Omega$ and all $z \in \Gamma$,

$$0 = -\int_{\partial D} \left( M_y G(x, y) \frac{\partial^2 G(y, z)}{\partial n_y \partial n_z} + N_y G(x, y) \frac{\partial G(y, z)}{\partial n_z} \right) ds(y) + \int_{\partial D} \left( \frac{\partial G(x, y)}{\partial n_y} M_y \frac{\partial G(y, z)}{\partial n_z} + G(x, y) N_y \frac{\partial G(y, z)}{\partial n_z} \right) ds(y).$$

By adding the above relationships, we get

$$\tilde{u}^s(x, z) = \int_{\partial D} \left( \frac{\partial G(x, y)}{\partial n_y} M_y \tilde{u}(y, z) + G(x, y) N_y \tilde{u}(y, z) \right) ds(y),$$

where we have used the fact that the total field $\tilde{u}(\cdot, z) = \partial G(\cdot, z)/\partial n_z + \tilde{u}^s(\cdot, z)$ satisfies the Dirichlet boundary condition on $\partial D$. Computing the normal derivative of (16) with respect to $z$ at point $z \in \Gamma$, we obtain that for all $x \in \Omega$ and all $z \in \Gamma$,

$$\frac{\partial u^s(z, x)}{\partial n_z} = -\int_{\partial D} \left( M_y \frac{\partial G(z, y)}{\partial n_z} \frac{\partial u^s(y, x)}{\partial n_y} + N_y \frac{\partial G(z, y)}{\partial n_z} u^s(y, x) \right) ds(y) + \int_{\partial D} \left( \frac{\partial^2 G(z, y)}{\partial n_y \partial n_z} M_y u^s(y, x) + \frac{\partial G(z, y)}{\partial n_z} N_y u^s(y, x) \right) ds(y).$$

From the Green formula in $\Omega$ and the radiation condition we obtain

$$0 = -\int_{\partial D} \left( M_y \tilde{u}^s(y, z) \frac{\partial u^s(y, x)}{\partial n_y} + N_y \tilde{u}^s(y, z) u^s(y, x) \right) ds(y)$$
Using the boundary condition for $u$ which implies in particular $\partial u^s(x) \in \Gamma$,

Adding the two above relationships implies that for all $x \in \Omega$ and all $z \in \Gamma$,

\[
\frac{\partial u^s(z,x)}{\partial n_z} = - \int_{\partial D} \left( M_y \tilde{u}(y,z) \frac{\partial u^s(y,x)}{\partial n_y} + N_y \tilde{u}(y,z) u^s(y,x) \right) \, ds(y).
\]

Subtracting (18) and (19) and using the fact that $u(\cdot, x)$ satisfies the Dirichlet condition on $\partial D$ we obtain that $\tilde{u}^s(x,z) - \partial u^s(z,x) / \partial n_z = 0$ for all $x \in \Omega$ and $z \in \Gamma$.

**Proof of Theorem 2.3.** Again, we detail the proof for the Dirichlet case, the Neumann case follows the same lines. Let us denote $\tilde{\Omega}$ the unbounded connected component of $\mathbb{R}^2 \setminus D_1 \cup D_2$. We have that for all $x, y \in \Gamma$,

\[
u^s_1(x,y) = u^s_2(x,y), \quad \frac{\partial u^s_1}{\partial n_x}(x,y) = \frac{\partial u^s_2}{\partial n_x}(x,y)
\]

as well as

\[
\tilde{u}^s_1(x,y) = \tilde{u}^s_2(x,y), \quad \frac{\partial \tilde{u}^s_1}{\partial n_x}(x,y) = \frac{\partial \tilde{u}^s_2}{\partial n_x}(x,y).
\]

From well-posedness of the forward diffraction problem when the obstacle is the ball $B_R$ with Dirichlet boundary condition, we have that for all $x \in \mathbb{R}^2 \setminus \overline{B_R}$, for all $y \in \Gamma$,

\[
u^s_1(x,y) = u^s_2(x,y), \quad \tilde{u}^s_1(x,y) = \tilde{u}^s_2(x,y).
\]

Unique continuation for the operator $\Delta^2 - k^4$ then implies that for all $x \in \tilde{\Omega}$, for all $y \in \Gamma$,

\[
u^s_1(x,y) = u^s_2(x,y), \quad \tilde{u}^s_1(x,y) = \tilde{u}^s_2(x,y).
\]

We now use the reciprocity relationships of Lemma 2.4, which imply that for all $x \in \tilde{\Omega}$, for all $y \in \Gamma$,

\[
u^s_1(y,x) = u^s_2(y,x), \quad \frac{\partial u^s_1}{\partial n_y}(y,x) = \frac{\partial u^s_2}{\partial n_y}(y,x).
\]

By repeating the same arguments as above we obtain that for all $x, y \in \tilde{\Omega}$,

\[
u^s_1(x,y) = u^s_2(x,y).
\]

Assume that $D_1 \not\subset D_2$. Since $\mathbb{R}^2 \setminus \overline{D_2}$ is connected, there exists some non empty open set $\Gamma_* \subset (\partial D_1 \cap \partial \tilde{\Omega}) \setminus \overline{D_2}$. We now consider some point $x_* \in \Gamma_*$ and the sequence

\[
x_m = x_* + \frac{n_1(x_*)}{m}, \quad m \in \mathbb{N} \setminus \{0\},
\]

where $n_1(x_*)$ denotes the unit normal to $\Gamma_*$ at point $x_*$. For sufficiently large $m$, $x_m \in \tilde{\Omega}$, so that for all $x \in \tilde{\Omega}$ and all $m$,

\[
u^s_1(x,x_m) = u^s_2(x,x_m),
\]

which implies in particular

\[
u^s_1(\cdot,x_m)\big|_{\Gamma_*} = u^s_2(\cdot,x_m)\big|_{\Gamma_*}, \quad \frac{\partial u^s_1}{\partial n_1}(\cdot,x_m)\big|_{\Gamma_*} = \frac{\partial u^s_2}{\partial n_1}(\cdot,x_m)\big|_{\Gamma_*}.
\]

Using the boundary condition for $u^s_1$, we obtain

\[
G(\cdot,x_m)\big|_{\Gamma_*} = -u^s_2(\cdot,x_m)\big|_{\Gamma_*}, \quad \frac{\partial G}{\partial n_1}(\cdot,x_m)\big|_{\Gamma_*} = -\frac{\partial u^s_2}{\partial n_1}(\cdot,x_m)\big|_{\Gamma_*}.
\]
Passing to the limit \( m \to +\infty \), we get
\[
G(\cdot, x_s)|_{\Gamma_s} = -u_2^5(\cdot, x_s)|_{\Gamma_s}, \quad \frac{\partial G(\cdot, x_s)}{\partial n_1}|_{\Gamma_s} = -\frac{\partial u_2^5}{\partial n_1}(\cdot, x_s)|_{\Gamma_s}.
\]

The function \( u_2^5 \) is infinitely smooth in a vicinity of \( x_s \), while \( \Gamma_s \) is a subset of the boundary of the \( C^3 \) domain \( \Omega_1 := \mathbb{R}^2 \setminus \overline{D} \). This implies that
\[
(u_2^5(\cdot, x_s)|_{\Gamma_s}, \partial_n u_2^5(\cdot, x_s)|_{\Gamma_s}) \in H^{5/2}(\Gamma_s) \times H^{3/2}(\Gamma_s),
\]
which is also the regularity of \((G(\cdot, x_s)|_{\Gamma_s}, \partial_n G(\cdot, x_s)|_{\Gamma_s})\). Hence the function \( G(\cdot, x_s) \), which solves the equation \( \Delta^2 G(\cdot, x_s) - k^4 G(\cdot, x_s) = 0 \) in \( \tilde{\Omega} \), is \( H^3 \) in a vicinity of \( x_s \) in \( \tilde{\Omega} \), from standard regularity results for elliptic problems [15]. But this is a contradiction in view of Remark 1. We conclude that \( D_1 \subset D_2 \) and we prove the same way that \( D_2 \subset D_1 \). Eventually, \( D_1 = D_2 \). \( \square \)

2.3. Justification of the Linear Sampling Method. We detail the classical theory of the Linear Sampling Method in the Dirichlet case, that is \((B_1, B_2) = (I, \partial_n)\). The Neumann case follows the same lines and will be presented without justification in the next section. Let us start by introducing the mapping \( S_D : H^{-3/2}(\partial D) \times H^{-1/2}(\partial D) \to H^{3/2}(\partial D) \times H^{1/2}(\partial D) \) such that, for all \( x \in \partial D \),

\[
\begin{align*}
(20) \quad S_D \left( \begin{array}{c}
\tau \\
\sigma
\end{array} \right)(x) &= \left( \int_{\partial D} \left( G(x, y) \tau(y) + \frac{\partial G(x, y)}{\partial n_y} \sigma(y) \right) ds(y) \right) \\
&\qquad \left( \int_{\partial D} \left( G(x, y) \tau(y) + \frac{\partial G(x, y)}{\partial n_y} \sigma(y) \right) ds(y) \right).
\end{align*}
\]

We have the following property.

**Proposition 2.** The mapping \( S_D \) is an isomorphism if \( k \notin \mathcal{K}_D \).

**Proof.** The proof is based on a comparison between the mapping \( S_D \) and the analogue mapping \( S_{D,0} \) when the fundamental solution \( G \) of operator \( \Delta^2 - k^4 \) is replaced by the fundamental solution \( G_0 \) of operator \( \Delta^2 \) (see [14]) given by (10). The kernel \( G - G_0 \) is infinitely smooth, so that since the mapping \( S_{D,0} : H^{-3/2}(\partial D) \times H^{-1/2}(\partial D) \to H^{3/2}(\partial D) \times H^{1/2}(\partial D) \) is continuous (see for example [14]), this is also the case for \( S_D \). From [11, Lemma 3.1], the operator \( S_{D,0} \) is Fredholm of index 0. Since \( S_D - S_{D,0} \) is compact, this implies that the operator \( S_D \) is Fredholm of index 0 as well. Let us prove that \( S_D \) is surjective. We will then conclude that \( S_D \) is an isomorphism. Assume that \((\phi, \psi) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)\). We have to prove that there exits \((\tau, \sigma) \in H^{-3/2}(\partial D) \times H^{-1/2}(\partial D)\) such that \( S_D(\tau, \sigma) = (\phi, \psi) \). In this view, assuming that \( k \notin \mathcal{K}_D \), let us consider the well-defined solutions \( u^- \) in \( D \) and \( u^+ \) in \( \Omega \) to the interior problem (8) and to the exterior problem (9) associated with the same Dirichlet data \((\phi, \psi)\). From proposition 1, the solution \( u = (u^-, u^+) \) has expression (11). The trace and normal derivative of such solution, namely \((u, \partial_n u)\), are continuous across \( \partial D \) and coincide with \((\phi, \psi)\), so that \((\phi, \psi) = S_D(\tau, \sigma)\), where we have set \((\tau, \sigma) = ([Nu], [Mu]) \in H^{-3/2}(\partial D) \times H^{-1/2}(\partial D)\). The proof is complete. \( \square \)
We introduce the following operators: the obstacle-to-data operator \( F_D : H^{-3/2}(\partial D) \times H^{-1/2}(\partial D) \to L^2(\Gamma) \times L^2(\Gamma) \) such that, for all \( x \in \Gamma \),
\[
(21) \quad F_D \left( \frac{\tau}{\sigma} \right)(x) = \left( \int_{\partial D} \left( G(x, y)\tau(y) + \frac{\partial G(x, y)}{\partial n_y} \sigma(y) \right) ds(y) \right)
\]
the data-to-obstacle operator \( H_D : L^2(\Gamma) \times L^2(\Gamma) \to H^{3/2}(\partial D) \times H^{1/2}(\partial D) \) such that, for all \( x \in \partial D \),
\[
(22) \quad H_D \left( \frac{h}{t} \right)(x) = \left( \int_{\Gamma} \left( G(x, y)h(y) + \frac{\partial G(x, y)}{\partial n_y} t(y) \right) ds(y) \right)
\]
and the near-field operator \( N_D : L^2(\Gamma) \times L^2(\Gamma) \to L^2(\Gamma) \times L^2(\Gamma) \) such that, for all \( x \in \Gamma \),
\[
(23) \quad N_D \left( \frac{h}{t} \right)(x) = \left( \int_{\Gamma} \left( u^*(x, y)h(y) + \tilde{u}^*(x, y)t(y) \right) ds(y) \right)
\]
where \( u^*(\cdot, y) \) and \( \tilde{u}^*(\cdot, y) \) are the solutions to problem (1) associated with the point source \( u^i = G(\cdot, y) \) and the dipole \( \partial_n G(\cdot, y) \), respectively. Lastly, let us define the solution operator \( B_D : H^{3/2}(\partial D) \times H^{1/2}(\partial D) \to L^2(\Gamma) \times L^2(\Gamma) \) such that for \( (\phi, \psi) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D) \), \( B_D(\phi, \psi) \) is formed by the trace and normal derivative on \( \Gamma \) of the unique solution \( v \in H^2_{loc}(\Omega) \) to the problem
\[
(24) \quad \left\{ \begin{array}{ll}
\Delta^2 v - k^4 v = 0 & \text{in } \Omega \\
v = \phi, \quad \frac{\partial v}{\partial n} = \psi & \text{on } \partial \Omega \\
\lim_{r \to +\infty} \int_{\partial B_r} \left| \frac{\partial v}{\partial n} - ikv \right|^2 ds = 0.
\end{array} \right.
\]

**Proposition 3.** Let us assume that \( k \notin K_D \). The operators \( B_D, F_D, H_D \) and \( N_D \) satisfy \( F_D = \overline{F_D} \), \( F_D = B_DS_D \) and \( N_D = -B_DH_D \), where \( A^* \) denotes the adjoint of operator \( A \). In addition, these four operators are compact, injective with dense range.

**Proof.** Let us prove that \( F_D = \overline{F_D} \). We have, for \((\tau, \sigma) \in H^{-3/2}(\partial D) \times H^{-1/2}(\partial D)\) and \((h, t) \in L^2(\Gamma) \times L^2(\Gamma)\),
\[
\left\langle F_D \left( \frac{\tau}{\sigma} \right), \left( \frac{h}{t} \right) \right\rangle = \int_{\Gamma} \left( \int_{\partial D} \left( G(x, y)\tau(y) + \frac{\partial G(x, y)}{\partial n_y} \sigma(y) \right) ds(y) \right) \overline{h(x)} ds(x)
\]
\[
+ \int_{\Gamma} \frac{\partial_n}{\partial n} \left( \int_{\partial D} \left( G(x, y)\tau(y) + \frac{\partial G(x, y)}{\partial n_y} \sigma(y) \right) ds(y) \right) \overline{t(x)} ds(s)
\]
\[
= \int_{\partial D} \tau(y) \left( \int_{\Gamma} G(x, y)\overline{h(x)} + \partial_n G(x, y) \overline{t(x)} ds(x) \right) ds(y)
\]
\[
+ \int_{\partial D} \sigma(y) \frac{\partial_n}{\partial n} \left( \int_{\Gamma} G(x, y)\overline{h(x)} + \partial_n G(x, y) \overline{t(x)} ds(x) \right) ds(y)
\]
that is \( F_D = \overline{H}_D \). In the computation above, we have used that \( G(x, y) = G(y, x) \) in view of (6). The identities \( F_D = B_D S_D \) and \( N_D = -B_D H_D \) are simple consequences of the very definition of the four operators. That \( B_D \) is compact is a consequence of the interior regularity of operator \( \Delta^2 \). This implies, from \( F_D = \overline{H}_D \), \( F_D = B_D S_D \) and \( N_D = -B_D H_D \), that \( F_D, H_D \) and \( N_D \) are also compact operators. Let us prove that \( B_D \) is injective. If \((\phi, \psi) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)\) is such that \( B_D(\phi, \psi) = 0 \), this means that the solution \( v \) to problem (24) satisfies \((v|_\Gamma, \partial_n v|_\Gamma) = (0, 0)\). From uniqueness of the scattering problem with Dirichlet boundary condition (see again Theorem 1.1), we have that \( v = 0 \in \mathbb{R}^2 \setminus \overline{B_R} \), and then \( v = 0 \in \Omega \) from unique continuation for the operator \( \Delta^2 - k^4 \). We conclude that \( v|_{\partial D} \) and \( \partial_n v|_{\partial D} \) is such that \((\phi, \psi) = (0, 0)\). From \( F_D = B_D S_D \) and the injectivity of \( S_D \), we obtain the injectivity of \( F_D \). Let us prove the injectivity of \( H_D \). Assume that \((h, t) \in L^2(\Gamma) \times L^2(\Gamma)\) is such that \( H_D(h, t) = 0 \). Let us consider the function

\[
v_{h,t}(x) = \int_{\partial D} G(x, y) h(y) ds + \partial_n G(x, y) t(y) ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma.
\]

Such function is a solution to the interior problem (8) with \((\phi, \psi) = (0, 0)\), so that \( v_{h,t} = 0 \) in \( D \) because \( k \notin K_D \). By unique continuation, we also have \( v_{h,t} = 0 \in B_R \). Since we have the jump relationships \( [v_{h,t}] = 0 \) and \( \partial_n [v_{h,t}] = 0 \) across \( \Gamma \), the function \( v_{h,t} \) is also a solution to the exterior problem in \( \mathbb{R}^2 \setminus B_R \) with vanishing Dirichlet boundary condition. Then \( v_{h,t} = 0 \in \mathbb{R}^2 \setminus B_R \). From the jump relationships \( h = [N v_{h,t}] \) and \( t = [M v_{h,t}] \) across \( \Gamma \), we obtain that \((h, t) = (0, 0)\). So \( H_D \) is injective, as well as \( N_D \) since \( N_D = -B_D H_D \). That \( F_D = \overline{H}_D \) implies that \( F_D \) has a dense range, as well as \( B_D = F_D S^{-1}_D \), \( H_D = \overline{H}_D \) and \( N_D = -B_D H_D \).

**Remark 2.** From proposition 3, we derive the classical factorization

\[
N_D = -\overline{H}^*_D S^{-1}_D H_D.
\]

We also need the fundamental range test property.

**Proposition 4.** Assume that \( k \notin K_D \). We have

\[
z \in D \iff \begin{pmatrix} G(\cdot, z)|_\Gamma \\ \partial_n G(\cdot, z)|_\Gamma \end{pmatrix} \in \text{Range}(F_D).
\]

**Proof.** First we observe that since \( S_D \) is an isomorphism, \( \text{Range}(F_D) = \text{Range}(B_D) \). If \( z \in D \), we observe that the function \( G(\cdot, z) \) is the solution to problem (24) with \((\phi, \psi) = (G(\cdot, z)|_{\partial D}, \partial_n G(\cdot, z)|_{\partial D}) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)\).

This implies that

\[
\begin{pmatrix} G(\cdot, z)|_\Gamma \\ \partial_n G(\cdot, z)|_\Gamma \end{pmatrix} \in \text{Range}(B_D).
\]
If on the contrary \( z \notin D \), let us assume that
\[
\left( \frac{G(\cdot, z)\big|_{\Gamma}}{\partial_{n}G(\cdot, z)\big|_{\Gamma}} \right) = B_{D}\left( \frac{\phi}{\psi} \right),
\]
with \((\phi, \psi) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)\). The corresponding solution to problem (24) is denoted \( v \). Since the trace \((v|_{\Gamma}, \partial_{n}|_{\Gamma})\) coincides with \((G(\cdot, z)|_{\Gamma}, \partial_{n}G(\cdot, z)|_{\Gamma})\), we have that \( G(\cdot, z) \) and \( v \) coincide outside the ball \( B_{R} \), and then in \( \Omega \setminus \{z\} \) by a unique continuation argument. The contradiction comes from a regularity comparison at point \( z \): the function \( v \) is locally infinitely smooth while from Remark 1, \( G(\cdot, z) \) is not locally \( H^{3} \).

From propositions 2, 3 and 4, we have the following Theorem. Since the proof mimics the one of [5] and is classical, it is left to the reader.

**Theorem 2.5.** We assume that \( k \notin K_{D}. \)

- If \( z \in D \), then for all \( \varepsilon > 0 \) there exists a solution \( (h_{\varepsilon}(\cdot, z), t_{\varepsilon}(\cdot, z)) \in L^{2}(\Gamma) \times L^{2}(\Gamma) \) of the inequality
  \[
  \left\| N_{D} \left( \begin{array}{c} h_{\varepsilon}(\cdot, z) \\ t_{\varepsilon}(\cdot, z) \end{array} \right) - \left( \frac{G(\cdot, z)\big|_{\Gamma}}{\partial_{n}G(\cdot, z)\big|_{\Gamma}} \right) \right\|_{L^{2}(\Gamma) \times L^{2}(\Gamma)} \leq \varepsilon
  \]
  such that the function \( H_{D}(h_{\varepsilon}(\cdot, z), t_{\varepsilon}(\cdot, z)) \) converges in \( H^{3/2}(\partial D) \times H^{1/2}(\partial D) \) as \( \varepsilon \to 0. \)
  Furthermore, for a given fixed \( \varepsilon > 0 \), the couple of functions \( (h_{\varepsilon}(\cdot, z), t_{\varepsilon}(\cdot, z)) \) satisfies
  \[
  \lim_{z \to \partial D} \|(h_{\varepsilon}(\cdot, z), t_{\varepsilon}(\cdot, z))\|_{L^{2}(\Gamma) \times L^{2}(\Gamma)} = +\infty
  \]
  and
  \[
  \lim_{z \to \partial D} \left\| H_{D} \left( \begin{array}{c} h_{\varepsilon}(\cdot, z) \\ t_{\varepsilon}(\cdot, z) \end{array} \right) \right\|_{H^{3/2}(\partial D) \times H^{1/2}(\partial D)} = +\infty.
  \]
- If \( z \notin D \), then every solution \( (h_{\varepsilon}(\cdot, z), t_{\varepsilon}(\cdot, z)) \) of the inequality (25) satisfies
  \[
  \lim_{\varepsilon \to 0} \|(h_{\varepsilon}(\cdot, z), t_{\varepsilon}(\cdot, z))\|_{L^{2}(\Gamma) \times L^{2}(\Gamma)} = +\infty
  \]
  and
  \[
  \lim_{\varepsilon \to 0} \left\| H_{D} \left( \begin{array}{c} h_{\varepsilon}(\cdot, z) \\ t_{\varepsilon}(\cdot, z) \end{array} \right) \right\|_{H^{3/2}(\partial D) \times H^{1/2}(\partial D)} = +\infty.
  \]

### 2.4. The case of Neumann obstacle.

In the Neumann case, that is \((B_{1}, B_{2}) = (M, N)\), the important point is that the near-field operator \( N_{N} \) is unchanged with respect to \( N_{D} \), provided in this case the scattered fields \( u^{s}(\cdot, y) \) and \( \tilde{u}^{s}(\cdot, y) \) for \( y \in \Gamma \) are obtained for a Neumann obstacle instead of a Dirichlet obstacle. However the operators \( S_{N}, F_{N}, H_{N} \) and \( B_{N} \) which are involved in the theoretical justification of the Linear Sampling Method are modified as follows: \( S_{N} : H^{3/2}(\partial D) \times H^{1/2}(\partial D) \to H^{-3/2}(\partial D) \times H^{-1/2}(\partial D) \) is such that, for all \( x \in \partial D \),

\[
S_{N} \left( \begin{array}{c} \phi \\ \psi \end{array} \right) (x) = \left( \begin{array}{c} N \int_{\partial D} (N_{y}G(x, y)\phi(y) + M_{y}G(x, y)\psi(y)) \, ds(y) \\ M \int_{\partial D} (N_{y}G(x, y)\phi(y) + M_{y}G(x, y)\psi(y)) \, ds(y) \end{array} \right),
\]
the obstacle-to-data operator is \( F_N : H^{3/2}(\partial D) \times H^{1/2}(\partial D) \rightarrow L^2(\Gamma) \times L^2(\Gamma) \) such that, for all \( x \in \Gamma \),

\[
\begin{align*}
F_N \left( \phi, \psi \right) (x) &= \left( \int_{\partial D} (N_y G(x, y) \phi(y) + M_y G(x, y) \psi(y)) \, ds(y), \right. \\
&\quad \left. \int_{\partial D} (N_y G(x, y) \phi(y) + M_y G(x, y) \psi(y)) \, ds(y) \right),
\end{align*}
\]

the data-to-obstacle operator is \( H_N : L^2(\Gamma) \times L^2(\Gamma) \rightarrow H^{-3/2}(\partial D) \times H^{-1/2}(\partial D) \) such that, for all \( x \in \partial D \),

\[
\begin{align*}
H_N \left( h, t \right) (x) &= \left( N \int_\Gamma \left( G(x, y) h(y) + \frac{\partial G(x, y)}{\partial n_y} t(y) \right) \, ds(y), \\
&\quad M \int_\Gamma \left( G(x, y) h(y) + \frac{\partial G(x, y)}{\partial n_y} t(y) \right) \, ds(y) \right) .
\end{align*}
\]

Lastly, the solution operator \( B_N : H^{-3/2}(\partial D) \times H^{-1/2}(\partial D) \rightarrow L^2(\Gamma) \times L^2(\Gamma) \) is such that, for \( (\tau, \sigma) \in H^{-3/2}(\partial D) \times H^{-1/2}(\partial D) \), \( B_N(\tau, \sigma) \) is formed by the trace and normal derivative on \( \Gamma \) of the unique solution \( v \) in \( H^2_{\text{loc}}(\Omega) \) to the problem

\[
\begin{align*}
\Delta^2 v - k^4 v &= 0 \quad \text{in } \Omega \\
M v &= \tau, \quad N v = \sigma \quad \text{on } \partial \Omega \\
\lim_{r \to +\infty} \int_{\partial B_r} \left| \frac{\partial v}{\partial n} - i k v \right|^2 \, ds &= 0.
\end{align*}
\]

It can be checked that the Propositions 2, 3, 4 and Theorem 2.5 are also valid for operators \( S_N, B_N, F_N, H_N \) and \( N_N \) provided \( k \notin \mathcal{K}_N \), where \( \mathcal{K}_N \) is the union of \( \mathcal{K}_0 \) (see Theorem 1.1) and of the fourth roots of Neumann eigenvalues of operator \( \Delta^2 \) in domain \( D \).

3. Numerical experiments.

3.1. Producing artificial data. In all the numerics, the Poisson’s ratio is \( \nu = 0.3 \).

In order to obtain artificial data for the inverse problem, we need to solve the problem (2) numerically in \( \Omega_R \). We hence derive a weak formulation which is equivalent to the strong problem (2). In the Neumann case, for example, such weak formulation is the following: find the scattered field \( u^s \in H^2(\Omega_R) \) such that

\[
b(u^s, v) = \ell(v), \quad \forall v \in H^2(\Omega_R),
\]

where \( b \) is the sesquilinear form defined by

\[
b(u, v) = a(u, v) - k^4 \int_{\Omega_R} u \overline{v} \, dx - t(u, \overline{v}),
\]

\( a \) is given by (7) with \( \mathcal{O} = \Omega_R \), \( t \) is defined by

\[
t(u, v) = \int_{\partial B_R} \left( v \frac{\partial u}{\partial n} \right) T \left( \frac{u}{\partial u}{\partial n} \right) \, ds
\]

and the antilinear form \( \ell \) is defined on \( H^2(\Omega_R) \) by

\[
\ell(v) = - \int_{\partial D} \left( \overline{v} \frac{\partial N}{\partial n} \right) M u^i \, ds.
\]

In order to approximate the solution \( u^s \) to the weak formulation (30), we use a Finite Element Method based on the Morley’s element and by limiting the infinite sum (3) which defines the Dirichlet-to-Neumann operator \( T \) to a finite number \( M \).
of terms. The finite element of Morley was introduced in [19] and analyzed for example in [16, 6]. This non-conforming finite element is probably the most simple element which enables one to solve bilaplacian problems. If $\Omega_R$ is approximated by a polygonal domain and meshed with triangles, the finite element space is formed by functions $u_h$ such that their restriction on a given triangle is a second degree polynomial, so that each triangle has 6 degrees of freedom: the values of $u_h$ at the three vertices of the triangle and the values of the normal derivatives of $u_h$ at the middle of the three sides of the triangle. We use a very refined mesh, which is in particular consistent with the largest value of $k$ ($k = 30$) that is used later on. The integer $M$ has also to be sufficiently large with respect to $k$, we take $M = \lfloor kR \rfloor + 20$.

The computation of all scattered fields that we need to produce the artificial data is implemented using the Matlab software and takes only a few minutes on a regular laptop. In the figure 1, we check the validity of the discretized Dirichlet-to-Neumann operator by computing the scattering solution either in $\Omega_1$ or in $\Omega_2$, that is $\Omega_R$ for $R = 1$ and $R = 2$. We verify that the two solutions coincide in the numerical sense in the intersection $\Omega_1 \cap \Omega_2 = \Omega_1$.

**Figure 1.** Validation of the artificial boundary condition. Left: scattering solution computed in $\Omega_1$. Right: scattering solution computed in $\Omega_2$.

### 3.2. Identification results.

The procedure described in section 3.1 enables us to compute, for all $y \in \Gamma$, the solutions $u^* (\cdot, y)$ and $\tilde{u}^* (\cdot, y)$ to the problem (1) with $u^i = G (\cdot, y)$ and $\partial_n G (\cdot, y)$, respectively. For each sampling point $z \in \Omega_R$ we wish to solve in $L^2 (\Gamma) \times L^2 (\Gamma)$ the near-field equation

$$
N_D \begin{pmatrix} h (\cdot, z) \\ t (\cdot, z) \end{pmatrix} = \begin{pmatrix} G (\cdot, z)|_{\Gamma} \\ \partial_n G (\cdot, z)|_{\Gamma} \end{pmatrix}
$$

for Dirichlet data. That the operator $N_D$ is compact implies that the previous equation is always ill-posed, this is why we solve it in the Tikhonov sense. In the realistic case when the data are perturbed by noise, the regularization parameter in the Tikhonov regularization is chosen as a function of the amplitude of noise by using the Morozov discrepancy principle exactly as in [7]. Of course the operator $N_D$ has to be replaced by $N_N$ for Neumann data. As it is done classically by LSM.
users, for each \( z \in \Omega_R \) we plot

\[
\Psi(z) = \log \left( \frac{1}{\sqrt{\|h_\varepsilon(\cdot, z)\|_{L^2(\Gamma)}^2 + \|t_\varepsilon(\cdot, z)\|_{L^2(\Gamma)}^2}} \right),
\]

where \((h_\varepsilon, t_\varepsilon)\) is the Tikhonov regularized solution. Following Theorem 2.5, the function \( \Psi \) happens to be finite inside the unknown defect \( D \) and \(-\infty\) outside \( D \), which means that imaging the defect \( D \) amounts to plotting the level sets of the function \( \Psi \). In practice, the set of data is finite. In other words, we have to handle discretized versions of operators \( N_D \) and \( N_N \) corresponding to multistatic data \( u^s(x_i, y_j) \) and \( \tilde{u}^s(x_i, y_j) \) where the points \( x_i \) and \( y_j \) for \( i, j = 1, \ldots, I \) are equally distributed on the circle \( \Gamma \) (the locations of points \( x_i \) and points \( y_j \) are the same). Here we have \( I = 500 \) in all the identification experiments. In the figure 2, we have represented the function \( \Psi \) for a Dirichlet obstacle \( D \) formed by the union of two discs and by using exact data, for various wave numbers \( k \), that is \( k = 10 \), \( k = 20 \) and \( k = 30 \). In the figure 3, we show the identification results for an obstacle with the same geometry but with Neumann boundary condition, for \( k = 10 \), \( k = 20 \) and \( k = 30 \). In the figure 4, the identification results are presented for a Dirichlet obstacle formed by three circles with \( k = 30 \) and for a kite-shaped Neumann obstacle.
Figure 3. Function $\Psi$ for a Neumann obstacle, exact data and various wave numbers $k$. Top left: $k = 10$. Top right: $k = 20$. Bottom: $k = 30$.

with $k = 20$. Note that such kite-shaped obstacle is not an obstacle of class $C^3$ as assumed in the introduction. However, the numerical resolution of the forward problem seems to work in that case, as well as the numerical resolution of the inverse one. Data are exact in all the previous cases.

Now, let us analyze the impact of noise on the data $u^s(x_i, y_j)$ and $\tilde{u}^s(x_i, y_j)$ for $i, j = 1, \ldots, I$. This noise is such that for each $j$, the scattered fields $u^s(\cdot, y_j)$ and $\tilde{u}^s(\cdot, y_j)$ as well as their normal derivatives are perturbed at each point $x_i$ by a Gaussian noise which is then rescaled in such a way that the artificial relative errors for the $L^2$ norm of all these fields have a prescribed value $\sigma$. We show in figure 5 the obtained results for the Dirichlet obstacle formed by two circles for $k = 20$ when the amplitude of noise is 5% and 10% (that is $\sigma = 0.05$ and $\sigma = 0.1$). These results have to be compared with the top right part of figure 2 obtained with unperturbed data. The identification is still efficient. However, it has to be noted that the contrast of the function $\Psi$ between points inside and outside the obstacle $D$ is much smaller in the noisy case, which stems from the Tikhonov/Morozov regularization procedure to solve the near-field equation.

We complete this numerical study by some identification attempts with much less data. Indeed, from a practical point of view, it could be considered as a complicated task to produce the field $\tilde{u}^s(\cdot, y)$, which for each $y$ is the diffractive response to a
The Linear Sampling Method for Kirchhoff-Love infinite plates

Figure 4. Left: Function $\Psi$ for a Dirichlet obstacle formed by 3 circles, $k = 30$ and exact data. Right: Function $\Psi$ for a Neumann kite-shaped obstacle, $k = 20$ and exact data.

Figure 5. Function $\Psi$ for a Dirichlet obstacle, $k = 20$ and noisy data. Left: noise of amplitude 5%. Right: noise of amplitude 10%.

dipole, namely $\partial_n G(\cdot, y)$. In addition, not only our sampling method needs the trace of fields $u^s(\cdot, y)$ and $\tilde{u}^s(\cdot, y)$ on $\Gamma$ but also their normal derivatives. Hence it is natural to wonder if one can use a sampling method using only $u^s(x, y)$ for $(x, y) \in \Gamma \times \Gamma$, exactly as if the scattered field $u^s$ solved the Helmholtz equation $\Delta u^s + k^2 u^s = 0$ instead of the true equation $\Delta^2 u^s - k^4 u^s = 0$. In [2], we observed that in $\Omega$, any solution $u$ to the equation $\Delta^2 u - k^4 u = 0$ is given by $u = u_{pr} + u_{ev}$, where $u_{pr}$ satisfies $\Delta u_{pr} + k^2 u_{pr} = 0$ and $u_{ev}$ satisfies $\Delta u_{ev} - k^2 u_{ev} = 0$. If we assume in addition that the function $u$ is radiating, it was proved that outside some ball $B_R$, the function $u_{pr}$ is an infinite linear combination of the functions $H^1_n(kr)e^{in\theta}$, which are oscillating and slowly decaying at infinity, while $u_{ev}$ is an infinite linear combination of the functions $H^1_n(ikr)e^{i\theta}$, which are exponentially decaying at infinity. This is why we call $u_{pr}$ the propagating part of the radiating solution $u$ and $u_{ev}$ the evanescent part. As a conclusion, at a long distance of the obstacle, the scattered field $u$ can be approximated by its propagating part $u_{pr}$, which solves the Helmholtz equation (the evanescent part $u_{ev}$ of $u$ is neglected).
Similarly, the fundamental solution $G(\cdot, y)$ given by (6) can be approximated at long distance of $y$ by its propagating part $i H_0^1(k|\cdot-y|)/8k^2$, which up to a constant coincides with the fundamental solution $\mathcal{G}(\cdot, y)$ to the Helmholtz equation. In the Dirichlet case, it is then tempting, rather than solving the near-field equation (31), to solve the classical near-field equation corresponding to the Helmholtz equation. It consists, for each sampling point $z \in \Omega_R$, to solve in $L^2(\Gamma)$ the near-field equation

$$\mathcal{N} h = \mathcal{G}(\cdot, z)|_{\Gamma},$$

where the operator $\mathcal{N} : L^2(\Gamma) \to L^2(\Gamma)$ is defined, for $h \in L^2(\Gamma)$, by

$$\mathcal{N} h(x) = \int_{\Gamma} u^s(x, y) h(y) \, dy, \quad x \in \Gamma.$$  

We here use the fact that the Dirichlet boundary conditions for problem (1) implies the Dirichlet boundary condition for the Helmholtz equation. Comparing the operator $\mathcal{N}$ and the operator $\mathcal{N}_D$, we see that solving the equation (33) rather than the equation (31) leads us to use one block of data out of four. On the picture 6, we present the identification results for the Dirichlet obstacles formed by two or three circles, in the case of unperturbed data and $k = 30$, when we solve the equation (33) instead of (31). These results have to be compared with the bottom part of picture 2 and the left part of picture 4. We now consider the case when we have

![Image of results](image.png)

**Figure 6.** Function $\Psi$ for a Dirichlet obstacle with less (exact) data, $k = 30$. Left: two circles. Right: three circles.

less data and when those data are noisy. The corresponding results are presented on picture 7 for the Dirichlet obstacle formed by two circles, for $k = 30$, when the amplitude of noise is 5% and 10%. These results have to be directly compared with the ones presented on picture 5.

3.3. **Conclusion.** The numerical experiments of the previous section seem to show that the Linear Sampling Method is effective for Kirchhoff-Love plates, at least with data given on a circle $\Gamma$ surrounding the unknown obstacle. It works both for a Dirichlet obstacle and for a Neumann obstacle, even in the presence of noisy data. As usual, the identification results improve when the wave number increases. Maybe we notice a slight degradation of the quality of the identification if we compare the results with those classically obtained for the Helmholtz equation in two dimensions. If we compare to such case of Helmholtz equation in two dimensions, the LSM...
requires more data in the sense that we need the trace and the normal derivative on \( \Gamma \) of the scattered fields associated with both the point source \( G(\cdot, y) \) and the dipole \( \partial_n G(\cdot, y) \) for all \( y \in \Gamma \). However, we have shown experimentally in the Dirichlet case that using only the trace on \( \Gamma \) of the scattered field associated with the point source \( G(\cdot, y) \) for all \( y \in \Gamma \) and hence proceeding exactly as if we solved the Dirichlet case for the Helmholtz equation, produces almost as good results as when using the complete data. This simplification would deserve a more quantitative justification.

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