FEEDBACK STABILIZATION WITH ONE SIMULTANEOUS
CONTROL FOR SYSTEMS OF PARABOLIC EQUATIONS

CĂTĂLIN-GEORGE LEFTER* AND ELENA-ALEXANDRA MELNIG

Faculty of Mathematics, University “Al. I. Cuza” Iași, Romania
Octav Mayer Institute of Mathematics, Romanian Academy, Iași Branch

Dedicated to Professor Jiongmin Yong on the occasion of his 60th anniversary

Abstract. In this work controlled systems of semilinear parabolic equations
are considered. Only one control is acting in both equations and it is distributed
in a subdomain. Local feedback stabilization is studied. The approach is
based on approximate controllability for the linearized system and the use of
an appropriate norm obtained from a Lyapunov equation. Applications to
reaction–diffusion systems are discussed.

1. Introduction. In this paper we study the local feedback stabilization of systems
of parabolic equations in a domain \( \Omega \subset \mathbb{R}^n \), with only one internally distributed
control, supported in a bounded subdomain \( \omega \subset \Omega \). Because of the reduced
number of controls, these systems may not be small time local controllable or,
more precisely, the linearized system is not controllable. In fact, controllability for
the linearized system is usually an argument, via appropriate algebraic Riccati or
Lyapunov equations, to construct stabilizing feedbacks.

Controllability of linear parabolic equations with internally distributed controls
supported in a subdomain was established by O.Yu.Imanuvilov using appropriate
global Carleman estimates for the adjoint equation (see [9] for an introduction to the
field). Local controllability of nonlinear equations or systems may be deduced from
controllability of the linearized system. In such a situation the nonlinear system is
small time local controllable.

Controllability of systems by a reduced number of controls is a challenging prob-
lem and positive results may be obtained under appropriate conditions on the cou-
pling terms, as it is the case for example in the phase field models studied in [1].
When a good coupling is not verified for the linearized system, an issue to exploit
the nonlinearity is the return method of J.-M.Coron, which is linearization along
special solutions, constructed in such a way to fulfill coupling requirements for the
linearized system (see [6, 7, 8]).

In this paper the strategy for stabilization is, in some sense, similar to the one
in [11] and is based not on the controllability of the linearized system but on its
approximate controllability. In fact, exact controllability for the linearized system

2010 Mathematics Subject Classification. 35K40, 35K57, 93D15, 93B52, 93B18.
Key words and phrases. Reaction-diffusion equations, unique continuation, feedback stabilization,
Lyapunov equation.

The second author was supported by a grant of the Ministry of Research and Innovation,
CNCs - UEFISCDI, project number PN-III-P4-ID-PCE-2016-0011.
* Corresponding author: Cătălin-George Lefter (catalin.lefter@uaic.ro).
or, equivalently, observability for the adjoint system, seems not to rely on standard Carleman estimates.

The first step in our approach is to linearize the system around the stationary state. Stabilization for the linear system uses a spectral decomposition of the space, with respect to the elliptic part, in a direct sum of two invariant closed subspaces. One of these subspaces is unstable, but finite dimensional, and the other is an exponentially stable infinite dimensional subspace. The system splits into two independent, controlled systems, one of which finite dimensional.

One key point in our analysis is to prove approximate controllability for the linearized system. This implies exact controllability for the finite dimensional system and, consequently, this has the property of complete stabilization. We may thus construct a feedback law stabilizing the finite dimensional part and then prove that this is stabilizing the full linearized system.

The fact that the feedback law constructed in the linear case is also stabilizing the nonlinear system is proved by using the solution of an appropriate Lyapunov equation.

The systems we want to approach, which are reaction-diffusion type systems, may not be small time local controllable if there is only one control acting in only one equation and we propose a strategy in which we still have one control but acting simultaneously in both equations. We obtain stabilization results if also the diffusion coefficients are different.

For other results concerning stabilization of parabolic equations and systems, with feedback supported in a subdomain, we refer to [3, 2].

2. Preliminaries and main result. Let $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, be a bounded connected domain with the boundary $\partial \Omega$ of class $C^2$ and let $\omega \subset \subset \Omega$ be a nonempty open subset of $\Omega$. We consider the following controlled reaction-diffusion type system:

$$
\begin{align*}
&y_t - d_1 \Delta y = f(y, z) + f_1 + \psi_\omega u, &\text{in } (0, T) \times \Omega, \\
&z_t - d_2 \Delta z = g(y, z) + g_1 + \psi_\omega u, &\text{in } (0, T) \times \Omega, \\
&y(t, x) = 0, \quad z(t, x) = 0, &\text{on } (0, T) \times \partial \Omega, \\
&y(0, x) = y_0(x), \quad z(0, x) = z_0(x), &\text{in } \Omega.
\end{align*}
$$

(2.1)

where $d_1, d_2 \in \mathbb{R}_+$ are the diffusion coefficients, $f, g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are $C^\infty$ coupling nonlinearities, $f_1, g_1 \in L^\infty(\Omega)$, $\psi_\omega \in C^\infty(\Omega)$, supp $\psi_\omega = \omega$, $\psi_\omega > 0$ in $\omega$. $u(t, \cdot)$ is the control which belongs to $L^2(\omega)$ and by $\psi_\omega u$ we denote the extension by 0 of $u$ to $\Omega$ multiplied by $\psi_\omega$.

Let $(\overline{y}, \overline{z}) \in (L^\infty(\Omega))^2$ be a stationary state of the system, that is

$$
\begin{align*}
-d_1 \Delta \overline{y}(x) &= f(\overline{y}, \overline{z}) + f_1(x), &\text{in } \Omega, \\
-d_2 \Delta \overline{z}(x) &= g(\overline{y}, \overline{z}) + g_1(x), &\text{in } \Omega, \\
\overline{y} &= 0, \quad \overline{z} = 0, &\text{on } \partial \Omega
\end{align*}
$$

(2.2)

We want to find a finite dimensional feedback law $u = K(y, z)$, such that system (2.1) becomes locally exponentially stable around $(\overline{y}, \overline{z}) = (\overline{y}, \overline{z})^\top$ in an appropriate space.

Remark here that by elliptic regularity $\overline{y}, \overline{z} \in W^{2,p}(\Omega)$ for $p < \infty$ and thus $\overline{y}, \overline{z} \in C^{1,\theta}(\Omega)$ for all $\theta \in (0, 1)$.

We make here the following convention: if a function $y$ belongs to a space, let us say $H^1(\Omega)$, the norm will be denoted, for simplicity, omitting to write the domain: $\|y\|_{H^1}$. In the same spirit, if a vectorial function $Y = (y, z)^\top \in L^\infty(\Omega) \times L^\infty(\Omega)$,
the norm will be denoted by $\|Y\|_{L^\infty}$. If a function belongs to the intersection of some Banach spaces $y \in X_1 \cap X_2$, then the norm $\|y\|_{X_1 \cap X_2} = \max\{\|y\|_{X_1}, \|y\|_{X_2}\}$.

The first step is to linearize the system around $(y, z)$ and to construct a feedback that stabilizes the linearized system around zero.

Next step is to show that the same feedback stabilizes locally the nonlinear system. Stabilization for the nonlinear systems occurs in $H^1$ norm. For this step we use an equivalent norm given by the solution to an adapted Lyapunov equation. Stabilization in $L^\infty$ norm is obtained by using the regularizing effect of parabolic systems.

The controlled linearized system is

$$
\begin{cases}
\xi_t - d_1 \Delta \xi = a(x)\xi + b(x)\eta + \psi_\omega u, \text{ in } (0, T) \times \Omega, \\
\eta_t - d_2 \Delta \eta = c(x)\xi + d(x)\eta + \psi_\omega u, \text{ in } (0, T) \times \Omega, \\
\xi = 0, \quad \eta = 0, \quad \text{on } (0, T) \times \partial \Omega, \\
\xi(0, x) = \xi_0(x), \quad \eta(0, x) = \eta_0(x), \quad \text{in } \Omega,
\end{cases}
$$

(2.3)

where

$$a(x) := \frac{\partial f}{\partial y}(y, z), \quad b(x) := \frac{\partial f}{\partial z}(y, z),
$$

$$c(x) := \frac{\partial g}{\partial y}(y, z), \quad d(x) := \frac{\partial g}{\partial z}(y, z).$$

Let $H$ be the Hilbert space $L^2(\Omega) \times L^2(\Omega)$ and consider the operators

$$A : D(A) \subset H \rightarrow H, \quad D(A) = (H^1_0(\Omega) \cap H^2(\Omega))^2, \quad A = \begin{pmatrix} d_1 \Delta & 0 \\ 0 & d_2 \Delta \end{pmatrix},
$$

$$A_0 : D(A_0) = H \rightarrow H, \quad A_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
$$

$$B : L^2(\omega) \rightarrow H, \quad Bu = \begin{pmatrix} \psi_\omega u \\ \psi_\omega u \end{pmatrix} \quad \text{and} \quad \mathcal{A} := A + A_0.
$$

In operatorial form, with $Y = (y, z)^\top$, $F(Y) = (f(y, z), g(y, z))^\top$ and $F_1 = (f_1, g_1)^\top$, the controlled system (2.1) is written:

$$
\begin{aligned}
Y' &= AY + F(Y) + F_1 + Bu \\
Y(0) &= Y_0.
\end{aligned}
$$

(2.4)

The linearized system (2.3), with $W = (\xi, \eta)^\top$ is

$$
\begin{aligned}
W' &= (A + A_0)W + Bu \\
W(0) &= W_0.
\end{aligned}
$$

(2.5)

Denote by

$$\gamma(x) = a(x) + b(x) - c(x) - d(x),
$$

$$\alpha(x) := (c - a) - \frac{d_1 \gamma(x)}{d_2 - d_1} = (b - d) - \frac{d_2 \gamma(x)}{d_2 - d_1}
$$

and

$$L_T v := \Delta v + \frac{\gamma(x)}{d_1 - d_2} v, \quad D(L_T) = H^1_0(\Omega) \cap H^2(\Omega).$$

The stabilization result that will be proved in §3 for the linearized system is the following:

**Theorem 2.1.** Suppose that the diffusion coefficients are distinct $d_1 \neq d_2$ and one of the following assumptions is true:
• α is not identically constant in ω, or
• α is a constant in Ω and 0 ∉ σ(L_T).

Then the following conclusions hold:

(i) The operator A = A + A_0 has compact resolvent and generates an analytic semigroup in H;
(ii) The linear system (2.3) or (2.5) is approximately controllable in any time T;
(iii) For any δ > 0 there exist C = C(δ) > 0, a finite dimensional subspace U ⊂ L^2(ω) and a linear continuous operator K ∈ L(H, U) such that the operator A + BK generates an analytic semigroup of negative type satisfying

\[ \|e^{t(A+BK)}\|_H \leq Ce^{-\delta t}, \quad t > 0. \]  

(2.6)

The main result of this paper, proved in §4, and concerning the stability around the stationary state of the nonlinear system (2.1) respectively (2.4) is the following.

**Theorem 2.2.** Suppose that the hypotheses of Theorem 2.1 are satisfied. Then there exist ε > 0, δ > 0, C > 0, τ > 0 such that if \( \|y_0 - \overline{y}\|_{L^\infty \cap H^1} + \|z_0 - \overline{z}\|_{L^\infty \cap H^1} \leq \varepsilon \) then, taking in (2.1) or in (2.4) the feedback constructed in Theorem 2.1

\[ u = K(Y - \overline{Y}), \]  

(2.7)

we have local exponential stabilization:

\[ \|Y(t) - \overline{Y}\|_{H^s} \leq Ce^{-\delta t}\|Y_0 - \overline{Y}\|_{H^1 \cap L^\infty}, \quad t > \tau, s \in [0, 2], \]  

(2.8)

\[ \|Y(t) - \overline{Y}\|_{H^1 \cap L^\infty} \leq Ce^{-\delta t}\|Y_0 - \overline{Y}\|_{H^1 \cap L^\infty}, \quad t > 0. \]

3. Feedback stabilization of linearized system. Proof of Theorem 2.1.

Observe that, by elliptic regularity, \( \overline{y}, \overline{z} \in W^{2,p}(\Omega), p > 1 \) and thus they are in \( H^2(\Omega) \cap C^{1,\theta}(\Omega), \forall \theta \in (0, 1) \). It turns out that \( A_0 \in L(H^s(\Omega))^2, s \in [0, 2] \) and \( A_0 \in L((C^{1,\theta}(\Omega))^2), \theta \in (0, 1) \).

The operator \( A = A + A_0 \) with \( D(A) = D(A) \) generates an analytic semigroup, by a standard argument on the lower order perturbations of selfadjoint operators. In fact, \( A_0 \in L(H) \) satisfies an estimate of the type \( \|A_0 y\|_H \leq \varepsilon (|A_0 y|_H + C(\varepsilon))\|y\|_H \) for \( y \in D(A) \) (see [12]).

Moreover, \( A \) has compact resolvent as a consequence of Rellich compact embedding theorem. This fact implies that the spectrum \( \sigma(A) \) is discrete, with no finite accumulation point and is contained in an angular domain \( \sigma(A) \subset V_{\pi_1, \phi} := \{ z \in \mathbb{C}, |arg(z - \overline{z})| \in (\pi - \phi, \pi) \} \) for some \( \overline{z} \in \mathbb{R}, \phi \in (0, \frac{\pi}{2}) \).

Concerning the approximate controllability in time \( T \) for the linear problem (2.3) or (2.5), we know that this is equivalent to the unique continuation property for the backward adjoint problem. For this, consider the dual problem

\[ \begin{cases} -\Xi' = A^* \Xi \\ \Xi(T) = \Xi_0, \end{cases} \]

(3.1)

which, with \( \Xi = (p, q)^T \), corresponds to

\[ \begin{cases} -p_t - d_1 \Delta p = ap + cq, & \text{in } (0, T) \times \Omega \\ -q_t - d_2 \Delta q = bp + dq, & \text{in } (0, T) \times \Omega \\ p = 0, & \text{in } (0, T) \times \partial \Omega \\ q(T) = q_T & \text{in } \Omega. \end{cases} \]

(3.2)
The unique continuation property we have to prove is:

\[
B^* \left( \begin{array}{c} p \\ q \end{array} \right) = 0, \quad t \in (0, T) \implies B^* \left( \begin{array}{c} p \\ q \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \quad t \in (0, T)
\]

which, considering that \( B^* (p, q)^T = [\psi_\omega \cdot (p + q)] |_\omega \), it means that we have to prove

\[ p + q = 0 \text{ in } (0, T) \times \omega \implies p = 0, q = 0, \text{ in } (0, T) \times \Omega. \]

Suppose that \( p + q \equiv 0 \text{ in } (0, T) \times \omega \), that is \( q = -p \) there. Since \( p \) satisfies both equations in \((0, T) \times \omega\) we get after adding the two equations multiplied by \( d_2 \), respectively by \( d_1 \), the identity

\[ (d_1 - d_2) p_t + (d_2 a + d_1 b - d_2 c - d_1 d) p + p q \equiv 0, \quad (t, x) \in (0, T) \times \omega. \]

By integration we find that for \((t, x) \in (0, T) \times \omega\)

\[ p(t, x) = \int_0^t (p(t, x) e^{-\alpha(x)(t-T)}) \, dt, \quad (3.3) \]

with \( \alpha \) given in the hypothesis. Denote by

\[ \tilde{p}(t, x) := (p(t, x) e^{-\alpha(x)(t-T)}) \quad (3.4) \]

\[ \tilde{q}(t, x) := (q(t, x) e^{-\alpha(x)(t-T)}). \quad (3.5) \]

Observe that \( \tilde{p}(t, x) = p^T(x), \tilde{q}(t, x) = q^T(x) \) in \((0, T) \times \omega\). and \((\tilde{p}, \tilde{q})\) verifies the system:

\[
\begin{cases}
\tilde{p}_t + d_1 \Delta \tilde{p} + 2(t - T) d_1 \nabla \alpha \nabla \tilde{p} + \theta = 0, \text{ in } (0, T) \times \Omega, \\
\tilde{q}_t + d_2 \Delta \tilde{q} + 2(t - T) d_2 \nabla \alpha \nabla \tilde{q} + \theta = 0, \text{ in } (0, T) \times \Omega.
\end{cases} \quad (3.6)
\]

Suppose first that \( \alpha \) is not identically constant in \( \omega \). Then, \( \text{supp } |\nabla \alpha| \cap \omega \) has nonzero measure and in fact it has nonempty interior because of the \( C^{1,\theta} \) regularity of \( \alpha \). It turns out that for \( x \in \omega \) the equations in \((3.6)\) are second order polynomials in \( t \) which are identically zero and thus all coefficients need to be zero. So, \( |\nabla \alpha|^2 \tilde{p} = |\nabla \alpha|^2 \tilde{q} \equiv 0 \text{ in } \omega \times (0, T) \) and thus \( \tilde{p} = \tilde{q} \equiv 0 \text{ in } (0, T) \times [\text{supp } |\nabla \alpha| \cap \omega] \). By the above observation about the nonempty interior of \( \text{supp } |\nabla \alpha| \cap \omega \), from unique continuation, \( \tilde{p} = \tilde{q} \equiv 0 \text{ in } \Omega \times (0, T) \) and thus this occurs for \( p, q \).

Suppose now that \( \alpha \) is a constant in \( \Omega \) and \( 0 \notin \sigma(L_T) \). System \((3.6)\) becomes

\[
\begin{cases}
\tilde{p}_t + d_1 \Delta \tilde{p} + (\alpha + a) \tilde{p} + c \tilde{q} = 0, \text{ in } (0, T) \times \Omega, \\
\tilde{q}_t + d_2 \Delta \tilde{q} + (\alpha + d) \tilde{q} + b \tilde{p} = 0, \text{ in } (0, T) \times \Omega.
\end{cases} \quad (3.7)
\]

Denote by \( L_1, L_2 \) the operators with domain \( H^1_0(\Omega) \cap H^2(\Omega) \) having the following expressions

\[ L_1 v := d_1 \Delta v + (\alpha + a) v, L_2 v := d_2 \Delta v + (\alpha + d) v. \]

Consider \( p_1 := \frac{\partial}{\partial t} \tilde{p}, \quad q_1 := \frac{\partial}{\partial t} \tilde{q} \) which are solutions to the system

\[
\begin{cases}
(p_1)_t + L_1 p_1 + c q_1 = 0, \text{ in } (0, T) \times \Omega, \\
(q_1)_t + L_2 q_1 + b p_1 = 0, \text{ in } (0, T) \times \Omega, \\
p_1 \equiv 0, \quad q_1 \equiv 0, \text{ in } (0, T) \times \omega.
\end{cases}
\]
By the unique continuation property for systems of parabolic equations, we obtain that \( p_1 = q_1 \equiv 0 \) in \((0, T) \times \Omega\). This implies that \( \tilde{p}, \tilde{q} \) are independent of \( t \) in \((0, T) \times \Omega\) and it means that

\[
\begin{aligned}
p(t, x) &= \tilde{p}(x)e^{\alpha(x)(t-T)} \\
q(t, x) &= \tilde{q}(x)e^{\alpha(x)(t-T)}
\end{aligned}
\]

in \((0, T) \times \Omega\) and, by (3.7), \((\tilde{p}, \tilde{q})\) is a solution to the elliptic system

\[
\begin{aligned}
\Delta \tilde{p} + \alpha + \frac{a}{d_1} \tilde{p} + \frac{c}{d_1} \tilde{q} &= 0, \quad \text{in} \, \Omega, \\
\Delta \tilde{q} + \alpha + \frac{b}{d_2} \tilde{q} + \frac{d}{d_2} \tilde{p} &= 0, \quad \text{in} \, \Omega.
\end{aligned}
\]  

(3.8)

By computation we have that

\[
\frac{\alpha}{d_1} + \frac{a}{d_1} + \frac{b}{d_2} = \frac{\alpha}{d_2} + \frac{d}{d_2} + \frac{c}{d_1}
\]

and by adding the equations in (3.8) we get:

\[
\Delta (\tilde{p} + \tilde{q}) + \left( \frac{\alpha}{d_1} + \frac{a}{d_1} + \frac{b}{d_2} \right)(\tilde{p} + \tilde{q}) = 0 \text{ in } \Omega.
\]  

(3.9)

Using the unique continuation hypothesis \( p + q = 0 \) in \((0, T) \times \omega\), we deduce that \( \tilde{p} + \tilde{q} = 0 \) in \( \omega \) and we conclude, by the unique continuation property for elliptic equations, that

\[
\tilde{p} + \tilde{q} = 0 \text{ in } \Omega.
\]  

(3.10)

This means, again by (3.8), that we have in the whole \( \Omega \)

\[
\Delta \tilde{p} + \frac{\gamma(x)}{d_1 - d_2} \tilde{p} = 0.
\]  

(3.11)

Since \( 0 \notin \sigma(L_T) \) we deduce that \( \tilde{p} = 0 \) and thus \( \tilde{q} = 0 \) and \( p = q \equiv 0 \) in \((0, T) \times \Omega\). Unique continuation is proved and consequently (ii).

In what follows the strategy for stabilization is very similar to what was done in [10, 11] but we present it for the sake of completeness. Since \( \mathcal{A} \) generates an analytic semigroup in \( H \) and has compact resolvent, \( \sigma(\mathcal{A}) \) lies in a cone \( V_{\lambda, \phi} := \{ z \in \mathbb{C}, |\text{arg}(z - \lambda)| \in (\pi - \phi, \pi) \} \), for some \( \lambda \in \mathbb{R}, \phi \in (0, \frac{\pi}{2}) \).

Consider some \( \delta > 0 \) and choose \( \delta_2 > \delta \) such that \( \sigma(\mathcal{A}) \cap \{ \text{Re} \lambda = -\delta_2 \} = \emptyset \),

\[
\begin{aligned}
\sigma_1 &= \sigma(\mathcal{A}) \cap \{ \lambda \in \mathbb{C}, \text{Re} \lambda > -\delta_2 \} \\
\sigma_2 &= \sigma(\mathcal{A}) \cap \{ \lambda \in \mathbb{C}, \text{Re} \lambda < -\delta_2 \}.
\end{aligned}
\]  

(3.12)

Observe that \( \sigma_1 \) has a finite number of elements and, moreover, it contains also all the eigenvalues \( \lambda \) with \( \text{Re} \lambda > 0 \), corresponding to the unstable states. Observe also that, for some \( \phi' \in (0, \frac{\pi}{2}) \), \( \sigma_2 \subseteq V_{-\delta_2, \phi'} \).

Corresponding to this decomposition of the spectrum we can split the complexified space \( H^c \) into a direct sum \( H^c = H_1 \oplus H_2 \), where the subspaces \( H_1, H_2 \) are invariant under the complexified operator denoted also by \( \mathcal{A} \) and \( \sigma(\mathcal{A}|_{H_1}) = \sigma_1, \sigma(\mathcal{A}|_{H_2}) = \sigma_2 \).

Let \( P \) be the projector onto the space \( H_1 \) corresponding to this decomposition, \( Q = I - P, \mathcal{A}_1 = P \mathcal{A}, \mathcal{A}_2 = Q \mathcal{A} \).

**Remark 1.** Observe that, since \( H_1 \) is finite dimensional and is generated by eigenfunctions or generalized eigenfunctions of \( \mathcal{A} \), by elliptic regularity \( H_1 \subset (C^{1, \theta}(\Omega))^2 \cap (H^c(\Omega))^2, \theta \in (0, 1), s \in [0, 2] \).
We project the linear equation (2.5) on $H_1, H_2$ and denoting by $W_1 := PW, W_2 := (I - P)W$, we have
\[
W_1' = A_1 W_1 + PBu \\
W_2' = A_2 W_2 + (I - P)Bu.
\] (3.13)
The first one is a finite dimensional linear system in the space $H_1$. From the approximate controllability in time $T$ established in (ii) we have that the reachable set
\[
\{W^u(T, \cdot), u \in L^2(0, T, L^2(\omega))\},
\]
with $W^u$ solution to (2.5), is dense in $H$. So, the projection of the reachable set on the finite dimensional space $H_1$ is the entire $H_1$. It means that the first equation is exactly controllable in any time $T$, and so it is completely stabilizable:
\[
\forall \delta_1 > 0, \quad \exists K_1 : H_1 \to L^2(\omega, \mathbb{C}), \quad \exists c = c(\delta_1)
\]
such that we have the following exponential decay of the finite dimensional semigroup in $H_1$
\[
\|e^{t(A_1 + PBK_1)}\|_{L(H_1)} \leq ce^{-\delta_1 t}. \tag{3.14}
\]
We define $\tilde{K} := K_1 \circ P$. Denoting by $W_1^{\tilde{K}}$ the solution of the finite dimensional equation stabilized by $\tilde{K}$, then
\[
\|W_1^{\tilde{K}}(t)\|_{H} \leq Ce^{-\delta_1 t}\|W_0^1\|_{H}. \tag{3.15}
\]
We consider the feedback control $u := \tilde{K}W_1 = \tilde{K}PW$ so the solution of the second equation is, by variation of constants formula,
\[
W_2^{\tilde{K}}(t) = e^{tA_2}W_2^0 + \int_0^t e^{(t-s)A_2}(I - P)BK_1^{\tilde{K}}W_1^{\tilde{K}}(s)ds. \tag{3.16}
\]
Using the fact that $A_2$ generates a stable analytic semigroup on $H_2$ we have
\[
\|e^{tA_2}W_2^0\|_{H} \leq Ce^{-\delta_2 t}\|W_0^2\|_{H}. \tag{3.17}
\]
Now, if we pass to norms in variation of constants formula, using the estimates obtained on $W_1$ and $e^{tA_2}$,
\[
\|W_2^{\tilde{K}}(t)\|_{H} \leq Ce^{-\delta_2 t}\|W_2^0\|_{H} + \int_0^t Ce^{-\delta_2 (t-s)}e^{-\delta_1 s}\|W_1^0\|_{H}. \tag{3.18}
\]
If we choose, in the stabilization of $W_1$, $\delta_1 > \delta_2$ then there exists a constant $C = C(\delta_1, \delta_2)$ such that
\[
\|W_2^{\tilde{K}}(t)\|_{H} \leq Ce^{-\delta_2 t}\|W_2^0\|_{H}. \tag{3.19}
\]
Together with the estimate on $W_1^{\tilde{K}}$ and looking only to the real part of the system, that is taking
\[
K := \text{Re} \, \tilde{K},
\]
we find that $K$ stabilizes the linear system (3.13):
\[
\|e^{t(A+BK)}\|_{L(H)} \leq Ce^{-\delta t}, \quad t > 0, \tag{3.20}
\]
which completes the proof of Theorem 2.1.

\[\square\]

Remark 2. The key argument above is to prove approximate controllability and by projection on $H_1$ to obtain exact controllability. One may use in fact a regularization argument to obtain all this with more regular controls. Thus, we may construct the feedback such that $K \in L(H \cap (C^{1,\theta}(\Omega))^2)$ and $K \in L(H \cap (H^2(\Omega))^2)$. Moreover, since $\psi_0 \in C^\infty(\Omega)$ with supp $\psi_0 = \overline{\Omega}$, $B$ is also a linear continuous operator in $L[(H^s(\omega))^2, (H^s(\Omega))^2]$ and $L[(C^{1,\theta}(\omega))^2, (C^{1,\theta}(\Omega))^2], s \in [0, 2], \theta \in (0, 1)$.
4. Local stabilization of the nonlinear system. Proof of Theorem 2.2.

Stabilization of system (2.4) to the stationary state $\bar{Y}$ with feedback $u = K(Y - \bar{Y})$ is equivalent to proving stability in $0$ for the system satisfied by $Z := Y - \bar{Y}$:

$$
\begin{align*}
Z' &= AZ + F(\bar{Y} + Z) - F(\bar{Y}) + BKZ \\
Z(0) &= Z_0 = Y_0 - \bar{Y}.
\end{align*}
$$

(4.1)

Since our result has as consequence local stabilization in $L^\infty$, it is natural to study a truncated system with $F$ replaced by $F_R = \rho_R F$ with $R > 2\|\bar{Y}\|_{L^\infty}$ and a cutoff function $\rho_R$ satisfying:

$$
\rho_R \in C^\infty(\mathbb{R}^2); \quad \rho_R(w) = 1, \text{ if } |w|_2 \leq R; \quad \rho_R(w) = 0, \text{ if } |w|_2 \geq 2R
$$

where we denoted by $|\cdot|_2$ the Euclidean norm in $\mathbb{R}^2$. The truncated system is

$$
\begin{align*}
\begin{cases}
Z' = AZ + BKZ + R(\bar{Y})(Z) = \bar{A}Z + R(\bar{Y})(Z) \\
Z(0) = Z_0
\end{cases}
\end{align*}
$$

(4.2)

where for $w \in \mathbb{R}^2$, $R_w(Z) = F_R(w + Z) - F_R(w) - DF_R(w)Z$ and $\bar{A} = A + BK$ with $D(\bar{A}) = D(A)$.

Remark 3. Observe that $F_R$ is a $C^2$ function with compact support and so it has bounded derivatives in $\mathbb{R}^2$. Considering a Taylor expansion for $F_R$, the following estimates are both true uniformly for $w$ belonging to bounded subsets of $\mathbb{R}^2$, say $|w|_2 \leq R$:

$$
|R_w(\zeta)|_2 \leq C \min\{|\zeta|_2, |\zeta|^2_2\},
$$

for all $\zeta \in \mathbb{R}^2$, with a constant $C$ depending only on $R$.

We will use the quadratic growth estimate in the local stabilization arguments. The linear growth estimate will be useful in questions related to bounds in stronger norms and, what is essential, to prove that solutions to the truncated system remain solutions to the nonlinear initial system if the initial data is small enough.

Using the linear growth, one has immediately that

$$
\|R(\bar{Y})(Z)\|_{L^2} \leq C\|Z\|_{L^2}, \quad \|R(\bar{Y})(Z)\|_{L^\infty} \leq C\|Z\|_{L^\infty}
$$

(4.3)

By a simple computation using the $C^{1,\theta}$ regularity of $\bar{Y}$ and the properties of the functions $F_R, R_w$, one also obtains the following estimate:

$$
\|R(\bar{Y})(Z)\|_{H^1} \leq C\|Z\|_{H^1},
$$

(4.4)

with a constant depending only on $R$.

Since $D(\bar{A}) = D(A^*)$ we may use the following result (see e.g. [5] or [10, 11]):

Proposition 1. For $\beta \geq 0$ there exists $P$ an unbounded selfadjoint operator in $H$ such that

$$
(PZ, Z) = \int_0^\infty \|(-\bar{A})^{\beta+\frac{1}{2}}e^{t\bar{A}}Z\|^2_H dt, \quad Z \in D(P)
$$

and $(PZ, Z)^{\frac{1}{2}}_H$ defines an equivalent norm in $D((-\bar{A})^\beta)$. Moreover, $D((-\bar{A})^{2\beta}) \subset D(P)$ with continuous embedding, that is:

$$
\|PZ\|_H \leq C\|(-\bar{A})^{2\beta}Z\|_H.
$$

Moreover, the following Lyapunov algebraic equation is satisfied by $P$

$$
(PZ, \bar{A}Z) = -\frac{1}{2}\|(-\bar{A})^{\beta+\frac{1}{2}}Z\|^2_H,
$$

(4.5)

for $Z \in D(\bar{A}^{\text{max}(2\beta, \beta+\frac{1}{2})})$. 

Consider in the above proposition $\beta = \frac{1}{2}$ and the corresponding operator $P$. Multiply then (4.2) with $PZ$ in $H$ and obtain
\[
\frac{1}{2} \frac{d}{dt} (PZ, Z)_H = (\dot{A}Z, PZ)_H + (R_{\tau}(Z), PZ)_H.
\]

Using the quadratic growth estimate for $R_{\tau}$ and the Lyapunov equation we find:
\[
\frac{d}{dt} (PZ, Z)_H \leq -C\|\dot{A}Z\|_H^2 + C_1\|Z\|_{L^4}^2\|PZ\|_H.
\]

Observe that in space dimension $n \leq 3$, $H^1(\Omega) \subset L^4(\Omega)$ and thus
\[
\frac{d}{dt} (PZ, Z)_H \leq -C\|\dot{A}Z\|_H^2 + C_1\|Z\|_{L^4}^2\|\dot{A}Z\|_H^2.
\]

We denoted by $C, C_1, C_2$ various constants which do not depend on $Z$. It turns out that for some $\delta_1 > 0$ small enough, the neighborhood $\{Z, (PZ, Z)_H < \delta_1\}$ of 0 is invariant for the flow and satisfies.
\[
\frac{d}{dt} (PZ, Z)_H \leq -2\delta(PZ, Z)_H,
\]

for some $\delta > 0$. Exponential decay for the norm $H^1$ follows immediately:
\[
\|Z\|_{H^1} \approx (PZ, Z)_H^\frac{1}{2} \leq C e^{-\delta t}\|Z_0\|_{H^1}.
\]

which is partly the conclusion of the theorem.

Last step of the proof is to show that we may stabilize the truncated system in $L^\infty$-norm and a consequence of this fact is the local $L^\infty$ stabilization of the original nonlinear system.

Rewrite (4.2) as
\[
\begin{aligned}
Z' &= AZ + T(Z) \\
Z(0) &= Z_0,
\end{aligned}
\]

where $T(Z) = (A_0 + BK)Z + R_{\tau}(Z)$, by (4.3),(4.4), satisfies
\[
\|T(Z)\|_X \leq C\|Z\|_X
\]

in either of the norms $X \in \{L^2, L^\infty, H^1\}$.

First, by parabolic maximum principle (see [4]) applied independently to the equations in (4.7), we obtain that
\[
\|Z(t)\|_{L^\infty} \leq \|Z_0\|_{L^\infty} + C t\|T(Z)\|_{L^\infty(0,t);L^\infty(\Omega)}.
\]

It turns out that choosing a $\tau < \frac{1}{2C}$ there exists $\delta_2 > 0$ such that if $\|Z_0\|_{L^\infty} < \delta_2$
\[
\|Z(t)\|_{L^\infty} \leq R, \quad t \in [0,\tau].
\]

The following estimates for solutions to (4.7) is classic for parabolic-like nonhomogeneous equations or systems:
\[
\|Z(t + \tau)\|_{H^2} \leq \frac{1}{\tau^2} \|Z(t)\|_{H^1} + C\|T(Z)\|_{L^2(t,t+\tau; H^1)}.
\]

Inserting (4.4) into (4.10), using (4.6) and (4.8), we obtain that
\[
\|Z(t + \tau)\|_{H^2} \leq C(\tau) e^{-\delta t}\|Z_0\|_{H^1}
\]

It is now clear that, since $H^2(\Omega) \subset L^\infty(\Omega)$ with continuous embedding, for some $\delta_3 > 0$, if $\|Z_0\|_{H^1} < \delta_3$ and considering also (4.9) we have
\[
\|Z(t)\|_{L^\infty} \leq R, \quad t > 0.
\]
and exponential stabilization occurs locally, in $H^{2}$ norm for (4.1) and thus we proved the local stabilization around a stationary state for the initial nonlinear system. The proof of Theorem 2.2 is completed.

Example 1. The following reaction-diffusion system, with $d_{1} = d_{2} = 1$, was studied in [7]:

$$
\begin{align*}
&y_{t} - d_{1} \Delta y = g(y, z) + \psi_{\omega} u, \quad t > 0, x \in \Omega \\
&z_{t} - d_{2} \Delta z = y^{m} + Rz, \quad t > 0, x \in \Omega \\
&y = 0, z = 0, \quad t > 0, x \in \partial \Omega.
\end{align*}
$$

(4.12)

$\Omega \subset \mathbb{R}^{n}$ is a nonempty connected bounded domain and $\omega \subset \Omega$, $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given $C^{\infty}$ function vanishing at $(0, 0)$ and $R \in \mathbb{R}$.

The result there is that given $T > 0$ the system with $m = 3$ is null controllable in time $T$, with initial data from a small neighborhood in $L^{\infty}$. The return method is used and linearization is done along a particular nonstationary solution. This kind of linearization is not compatible with our stabilization strategy because spectral decomposition is not working anymore. For $m = 2$ the system is not controllable in any time (see [7]). The result remains true, with unchanged proof, for the case $d_{1} \neq d_{2}$.

The result in this paper applies to the feedback stabilization of system (4.12) but introducing the same control also in the second equation. Local feedback stabilization at $(0, 0)$ is obtained in this way for

$$
\begin{align*}
&y_{t} - d_{1} \Delta y = g(y, z) + \psi_{\omega} u, \quad t > 0, x \in \Omega \\
&z_{t} - d_{2} \Delta z = y^{m} + Rz + \psi_{\omega} u, \quad t > 0, x \in \Omega \\
&y = 0, z = 0, \quad t > 0, x \in \partial \Omega,
\end{align*}
$$

(4.13)

with arbitrary $m \geq 1$, if the diffusion coefficients are different $d_{1} \neq d_{2}$ and hypotheses of Theorem 2.1 for the linearized system are verified. More precisely, when verifying the hypotheses of Theorem 2.1 in this particular case, we see that $\alpha(x)$ is a constant and there is only one value for $R$, say $\overline{R}$, for which $0 \in \sigma(L_{T})$. The stabilization results stated in Theorems 2.1, 2.2 occur if $R \neq \overline{R}$.

REFERENCES

[1] F. Ammar Khodja, A. Benabdallahl, C. Dupaix and I. Kostin, Controllability to the trajectories of phase-field models by one control force, SIAM J. Control Optim., 42 (2003), 1661–1680.

[2] V. Barbu and G. Wang, Feedback stabilization of semilinear heat equations, Abstr. Appl. Anal., 12 (2003), 697–714.

[3] V. Barbu and G. Wang, Internal stabilization of semilinear parabolic systems, J. Math. Anal. Appl., 285 (2003), 387–407.

[4] V. Barbu, Partial Differential Equations and Boundary Value Problems, Dordrecht: Kluwer Academic Publishers, 1998.

[5] A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter, Representation and Control of Infinite Dimensional Systems. Volume I. Boston: Birkhäuser, 1992.

[6] J.-M. Coron, Controllability and nonlinearity, ESAIM, Proc., 22 (2008), 21–39.

[7] J.-M. Coron, S. Guerrero and L. Rosier, Null controllability of a parabolic system with a cubic coupling term, SIAM J. Control Optim., 48 (2010), 5629–5653.

[8] J.-M. Coron and J.-P. Guilleron, Control of three heat equations coupled with two cubic nonlinearities, SIAM J. Control Optim., 55 (2017), 989–1019.

[9] A. V. Fursikov and O. Yu. Imnauvilov, Controllability of Evolution Equations, Seoul: Seoul National Univ., 1996.
[10] C. Lefter, Feedback stabilization of 2D Navier-Stokes equations with Navier slip boundary conditions, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, **70** (2009), 553–562.

[11] C.-G. Lefter, Feedback stabilization of magnetohydrodynamic equations, *SIAM J. Control Optim.*, **49** (2011), 963–983.

[12] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

Received November 2017; revised March 2018.

E-mail address: catalin.lefter@uaic.ro, leftercg@yahoo.com
E-mail address: alex.melnig@yahoo.com