REMARKS ON THE COLLECTIVE QUANTIZATION OF THE SU(2) SKYRME MODEL

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Abstract: We point out the question of ordering momentum operator in the canonical quantization of the SU(2) Skyrme Model. Thus, we suggest a new definition for the momentum operator that may solve the infrared problem that appears when we try to minimize the Quantum Hamiltonian.

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1. Introduction

In the Skyrme Model baryons are treated as a soliton solution in a non-Linear Sigma Model with an additional stabilizer Skyrme term \(^1\). The physical spectrum is obtained performing the collective coordinate quantization. Using the Nucleon and Delta masses as input parameters, we get the principal phenomenological results\(^2\). Although most of the static properties predicted by the Skyrme Model are in a good agreement with the experimental results, certain values like, for example, the pion decay constant, \(F_\pi\), and the axial coupling constant, \(g_A\), present large deviation from their experimental values. However, we can overcome these problems attempting to study with more detail the process of canonical quantization in the rotational mode. In previous works\(^3,4\), some authors have pointed out that the question of the quantization of Skyrmions is a very delicate one. They mentioned that the Skyrmion quantization is a simple example of quantum mechanics on a curved space.

This paper deals with the problem of ordering that appears in the definition of the canonical momentum when we try to use the constraint that is present in the system. Due to its simplicity it is not necessary to employ the Dirac formalism of constraints\(^5,6\). We will observe that when we adopt the correct definition for the momentum operator there is an additional term in the Quantum Hamiltonian, a result that has been also obtained by the authors of ref.3,4 using another calculate procedure.

2. Quantization by Collective Coordinate Expansion

Let us consider the classical static Lagrangian of the Skyrme Model

\[
L = \int d^3r \left[ -\frac{F_\pi^2}{16} \, Tr \left( \partial_i U \partial_i U^+ \right) + \frac{1}{32e^2} Tr \left[ U^+ \partial_i U, U^+ \partial_j U \right]^2 \right], \tag{1}
\]

where \(F_\pi\) is the pion decay constant, \(e\) is a dimensionless parameter and \(U\) is an SU(2) matrix.

Performing the collective semi-classical expansion\(^2\), substituting in (1) \(U(r) by U(r, t) = A(t)U(r)A^+(t)\), where \(A\) is a SU(2) matrix, we obtain:
\[ L = -M + \lambda T r \left[ \partial_0 A \partial_0 A^{-1} \right] . \quad (2) \]

In the last equation, \( M \) is the soliton mass which in the hedgehog representation for \( U, U = \exp (i \tau. \hat{r} F(r)) \), is given by

\[ M = 4\pi \frac{F_\pi}{e} \int_0^\infty x^2 \frac{1}{8} \left[ F'2 + 2 \sin^2 F \right] + \frac{1}{2} \frac{\sin^2 F}{x^2} \left[ \frac{\sin^2 F}{x^2} + 2F'^2 \right] dx , \quad (3) \]

where \( x \) is a dimensionless variable defined by \( x = eF_\pi r \), and \( \lambda \) is called the inertia moment written as

\[ \lambda = \frac{4}{6} \pi (1/e^3 F_\pi) \Lambda , \quad (4) \]

with

\[ \Lambda = \int_0^\infty x^2 \sin^2 F \left[ 1 + 4 \left( \frac{F'^2 + \sin^2 F}{x^2} \right) \right] dx . \quad (5) \]

The SU(2) matrix \( A \) can be written as \( A = a_0 + ia.\tau \), with the constraint

\[ \sum_{i=0}^{i=3} a_i^2 = 0 . \quad (6) \]

The Lagrangian (1) can be written as a function of the \( a' \)'s as:

\[ L = -M + 2\lambda \sum_{i=0}^{3} (\dot{a}_i)^2 . \quad (7) \]

Introducing the conjugate momenta \( \pi_i = \partial L / \partial \dot{a}_i = 4\lambda \dot{a}_i \), we can now rewrite the Hamiltonian as
\[ H = \pi_i \dot{a}_i - L = 4\lambda \dot{a}_i \dot{a}_i - L = M + 2\lambda \dot{a}_i \dot{a}_i = M + \frac{1}{8\lambda} \sum_i \pi_i^2 . \] (8)

Then, the standard canonical quantization is made where we replace \( \pi_i \) by \(-i\partial/\partial a_i\) in (8) leading to

\[ H = M + \frac{1}{8\lambda} \sum_{i=0}^{3} \left( -\frac{\partial}{\partial a_i^2} \right) . \] (9)

Due to the constraint (6), the operator \( \sum_{i=0}^{3} \left( -\frac{\partial}{\partial a_i^2} \right) \) is known as the Laplacian \( \nabla^2 \) on the three-sphere, with the eigenstates being traceless symmetric polynomials in the \( a_i \). In order to incorporate relation (6) it is more convenient to work with hypersphere coordinates defined by

\[
\begin{align*}
    a_0 &= \cos W \\
    a_1 &= \sin W \cos \theta \\
    a_2 &= \sin W \sin \theta \cos \phi \\
    a_3 &= \sin W \sin \theta \sin \phi .
\end{align*}
\] (10)

Then, the Laplacian written as a function of the hypersphere coordinates is given by

\[
\nabla^2 = \frac{\partial^2}{\partial W^2} + 2\frac{\cos W}{\sin W} \frac{\partial}{\partial W} + \frac{1}{\sin^2 W} \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{\sin^2 W \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 W \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} .
\] (11)

Note that when \( W = \pi/2 \), expression (11) reduces to the classical Laplacian in spherical coordinates. Thus, applying expression (11) to the wave function \((a_0 + ia_1)^l\), we obtain

\[
-\nabla^2(a_0 + ia_1)^l = l(l + 2)(a_0 + ia_1)^l .
\] (12)

If we wish to work with the coordinates \( a_i \) we must be able to obtain an expression for the canonical momentum \( \pi_i \). We must remember that if the commutation relation
\[ [a_i a_i, \pi_j] = a_i [a_i, \pi_j] + [a_i, \pi_j] a_i \] (13)

is valid, then the following relation must hold\(^6\)

\[ [a_i, \pi_j] = \delta_{i,j} - a_i a_j . \] (14)

It is not difficult to see that a possible expression for \(\pi_j\), satisfying eq. (14), is given by\(^5\)

\[ \pi_j = \frac{1}{i} [\delta_{i,j} - a_j a_i] \partial_i . \] (15)

Consequently, \(\pi_j \pi_j\) can be written as

\[ \pi_j \pi_j = -\partial_j \partial_j + 3a_j \partial_j + a_i a_j \partial_i \partial_j . \] (16)

Expression (16) is the three-sphere version of the Laplacian \(\nabla^2\) written as a function of the coordinates \(a_i\). It should be noted that the eigenvalues of the above equation are the same of those obtained using eq (12). At this point we must mention the problem of ordering that appears in the formula (15). As the physical Hamiltonian must be Hermitian, the usual choice for the operator momentum \(\pi_j\), following the prescription of Weyl ordering\(^7\) is given by

\[ \pi_j = \frac{1}{2i} [\delta_{i,j} - a_j a_i] \partial_i + \partial_i (\delta_{i,j} - a_i a_j) . \] (17)

If we substitute \(\pi_j\) in eq.(16), we obtain the following expression

\[ \pi_j \pi_j = -\partial_j \partial_j + 3a_j \partial_j + a_i a_j \partial_i \partial_j + \frac{5}{4} . \] (18)

Comparing expression (18) with (16) we see that an extra term appears in the last equation. So, when we pay attention to the question of ordering in the expression of the canonical momentum \(\pi_j\) in the coordinates \(a_i\), an additional term appears in the three-sphere
Unfortunately, if we want to improve the physical parameters predicted by the Skyrme Model, the signal of this extra term must be negative, as it was also shown by A. Toda\cite{4}.

As it was first point out by Bander and Hayot\cite{8}, if we observe the asymptotic solution of the Euler-Lagrange equation that minimizes the Quantum Hamiltonian (9)

\begin{equation}
-\frac{d^2F}{dx^2} - \frac{2}{x} \frac{dF}{dx} + \frac{2}{x^2} F - k^2 F = 0 , \tag{19}
\end{equation}

where $k^2$ is

\begin{equation}
k^2 = \frac{3l(l+2)e^3F_\pi}{8\pi \left( \int_0^\infty x^2 \sin^2 F \left[ 1 + 4 \left( \frac{F'^{2} + \sin^2 F}{x^2} \right) \right] dx \right)^2} , \tag{20}
\end{equation}

then, we verify that $F$ asymptotically behaves as $\frac{\sin kr}{r}$ or $\frac{\cos kr}{r}$ and the integral in the denominator of eq(20) does not converge. The infrared problem is solved when we require that the sign of the extra term is sufficiently negative in order to modify eq(19), which is now written as

\begin{equation}
-\frac{d^2F}{dx^2} - \frac{2}{x} \frac{dF}{dx} + \frac{2}{x^2} F + k^2 F = 0 . \tag{21}
\end{equation}

Studying the asymptotic behaviour of $F$ we observe that it behaves as $\exp -\frac{kr}{r}$, and the integral in the denominator of eq(21) converges.

In order to be able to deal with the problems that have been presented by us in the previous lines we suggest a new definition for the canonical operator momentum, which also satisfies the commutation relation (14),

\begin{equation}
\pi_j = \frac{1}{(1+\alpha)^4} \left[ (\delta_{i,j} - a_j a_i) \partial_i + \alpha \partial_i (\delta_{i,j} - a_i a_j) \right] , \tag{22}
\end{equation}
where $\alpha$ is a free parameter. Consequently, $\pi_j\pi_j$ is given by

$$\pi_j\pi_j = -\partial_j\partial_j + 3a_j\partial_j + a_ja_i\partial_j\partial_i - \frac{5\alpha (2\alpha - 3)}{(1 + \alpha)^2}, \quad (23)$$

and the eigenvalues of the extended Quantum Hamiltonian are given by

$$E = M + \frac{1}{8\lambda} \left[l(l + 2) - \frac{5\alpha(2\alpha - 3)}{(1 + \alpha)^2}\right]. \quad (24)$$

In the above equation we observe that when $\alpha > \frac{3}{2}$, the extra term is negative. For the nucleon state, $l=1$, we verify that for $\alpha > \frac{21 + 5\sqrt{21}}{14}$ or $\alpha < \frac{21 - 5\sqrt{21}}{14}$ there is no infrared problem in the quantum Hamiltonian, as we have remarked in (19). Now it is possible to search to a solution $F(x)$ that minimizes the total Quantum Hamiltonian written in (24).

3. Conclusion

We have shown that with the definition of the canonical momentum, $\pi_i$, which rules the Weyl prescription of ordering$^7$, there is an additional term in the usual Skyrmion quantization. It is possible to redefine the expression of the canonical momentum, which also satisfies the commutation relation of a particle in the three-sphere, with the purpose of removing the infrared problem. We hope that, with the use of the quantum variational solution, $F(r)$, one can be able to obtain an improvement of the physical parameters. The behaviour of these solutions$^9$ and the extension of this analysis to the SU(N) Collective Quantization, in particular in the case of the SU(3) Skyrmions$^{10}$ will be objects of forthcoming papers$^{11,12}$.

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References

1. T.H.R. Skyrme, Proc. Roy. Soc. A260 (1961) 127.
2. G.S. Adkins, C.R. Nappi and E. Witten, Nucl. Phys. B228 (1983) 552.
3. K. Fujii, K.I. Sato, N. Toyota and A.P. Kobushkin, Phys. Rev. Lett. 58, 7 (1987)
   and Phys. Rev. D35, 6 (1987) 1896.
4. A. Toda, Prog. Theor. Phys. 84 (1990) 324.
5. H. J. Schnitzer, Nucl. Phys. B261 (1985) 546.
6. H. Verschelde and H. Verbeke, Nucl. Phys. A500 (1989) 573.
   N. Ogawa, K. Fujii and A. Kobushkin, Prog. Theor. Phys. 83 (1990) 894.
7. The same procedure was performed by A. Toda (ref. 4) in another context.
   T. D. Lee, Particle Physics and Introduction to Field Theory p. 476. (Harwood Academic,
   New York, 1981)
8. M. Bander and F. Hayot, Phys. Rev. D30, 8 (1984) 1837.
9. E. Ferreira and J. Ananias Neto, J. Math. Phys. 33 (3) (1992) 1185.
10. H. Yabu and K. Ando Nucl. Phys. B301 (1988) 601.
    C. Callan and I. Klebanov, Nucl. Phys. B262 (1985) 365.
11. Jorge Ananias Neto, work in progress.
12. Jorge Ananias Neto, work in progress.