Waveguide-QED-based measurement of a reservoir spectral density

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The spectral density (SD) function has a central role in the study of open quantum systems (OQSs). We discover a method allowing for a “static” measurement of the SD – i.e., it requires neither the OQS to be initially excited nor its time evolution tracked in time – which is not limited to the weak-coupling regime. This is achieved through one-dimensional photon scattering for a zero-temperature reservoir coupled to the OQS via the rotating wave approximation. We find that the SD profile is a universal simple function of the photon’s reflectance and transmittance. As such, it can be straightforwardly inferred from photon’s reflection and transmission spectra.

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Mostly because of progress in quantum technologies that is easing access to a variety of single small systems [1], the study of open quantum systems (OQSs) [2-3] has become topical over the last few years. At present, fundamental questions, such as understanding and tackling quantum non-Markovianity [5-6], as well as more applicative concerns, such as developing strategies to counteract the detrimental effect of decoherence in quantum information processing (QIP) tasks [7], are prompting research in this field.

A key concept in the study of OQSs is the spectral density (SD) function [2-5,8]. At any given frequency, the SD essentially measures the interaction strength between the OQS and all the reservoir’s modes at that frequency. Currently, the concept of SD is growing in importance also due to the increasing interest in non-Markovian (NM) OQS dynamics (such as exciton transport in protein complexes [9] where dedicated numerical methods are used to compute the SD [10]). Indeed, structured (namely non-flat) SDs in general entail that the celebrated Kossakowski–Lindblad Markovian master equation (KLME) is not effective [2-5]. This is a linear first-order differential equation having the system’s state as the only unknown and depending on a set of rates (e.g. the familiar spontaneous emission rate of an atom). The open dynamics governed by the KLME is the prototype of a quantum Markovian, namely “memoryless”, dynamics. Despite the easiness to handle it, the KLME arises from a number of approximations. As such, it can be quite ineffective in a number of relevant, known scenarios featuring non-negligible NM effects [2]. In such cases, only the knowledge of the full reservoir spectral density (SD) [2-5,8] guarantees a reliable description of the OQS dynamics. This relies on the crucial property that if the SD is known then the OQS dynamics is fully determined. As a major consequence, the knowledge of the SD is key to designing strategies to hamper decoherence in QIP.

Measuring the SD thereby is a task of utmost importance. A possible method [11] is to measure the relaxation rate of a probe qubit - i.e., a two-level system embodying the OQS – as a function of its Bohr frequency (when this is tunable). This approach relies on the Fermi golden rule, hence on the assumption that the QOS-reservoir interaction is weak (weak-coupling regime). Other schemes [12], which address purely dephasing noise, exploit external pulsing to modify the OQS dynamical evolution. Measurements on the OQS are then used to infer the underlying SD. Such methods are dynamical in nature in that the diagnostic process underpinning them is essentially the OQS time evolution. This typically brings about, in particular, the need for initialising and measuring the OQS in suitable states.

Can one devise a SD measurement strategy with no need for triggering a dynamical evolution of the probe OQS (“static” measurement) and which is effective beyond the weak-coupling regime? In this Letter, we discover that this is achievable for an important class of quantum reservoirs. This encompasses dissipative, zero-temperature baths coupled to a two-level OQS via the rotating wave approximation (RWA). Prominent environmental models extensively investigated in the literature are included, e.g. lossy cavities and photonic band-gap mediums coupled to a quantum emitter [2,8]. The method for such static SD measurement is spectroscopic in nature: it exploits light scattering from the OQS, the outcomes of which are used to extract informations on the SD associated with the OQS dressed with its own reservoir. As a distinctive trait of the scheme is that it employs light that is constrained to travel in a one-dimensional (1D) waveguide. We find that reflection and transmission spectra, which can be recorded through standard intensity measurements, are enough for fully reconstructing the SD in a surprisingly straightforward fashion. This conclusion relies on an equation that directly maps the SD into a simple combination of reflectance and transmittance of the probing photon. The OQS needs not be initially excited neither its dynamics tracked in time, which embodies the static nature of the SD measurement method.

Photon scattering from quantum emitters (even a single one) in 1D waveguides, which we harness as the scheme diagnostic tool, is currently a hot field of research, often dubbed waveguide Quantum ElectroDynamics (QED) [13-17]. Technologic advancements make such processes by now experimentally observable, or next to be so, in a broad variety of different setups, such as open transmission lines coupled to superconducting qubits [19,21] or nanowires (alternatively, photonic-crystal waveguides) coupled to quantum dots [22] (for a more comprehensive review of possible implementations see e.g. Refs. [17,18]). The 1D confinement of light gives rise to unique interference effects such as the perfect re-
flection of a single photon from a quantum emitter \[13\]. There is growing evidence that the rich physics of waveguide QED can be harnessed for a number of promising applications in photonics, e.g., light switches \[15\] and single-photon transistors \[16\], as well as QIP, such as quantum gates \[23\] [24]. The scheme to be presented here further witnesses the potential of waveguide QED.

We consider a two-level OQS called $S$ (e.g., an artificial atom) in dissipative contact with a quantum reservoir $R$ with the joint Hamiltonian modelled as $(\hbar = 1$ throughout \[2\])

$$\hat{H}_{SR} = \omega_0 \hat{\sigma}_+ \hat{\sigma}_- + \sum_i \omega_i \hat{b}_i^\dagger \hat{b}_i + \sum_i \mu_i (\hat{b}_i \hat{\sigma}_+ + \hat{b}_i^\dagger \hat{\sigma}_-) .$$

(1)

Here, $\omega_0$ is the energy separation between the $S$'s excited and ground states $|e\rangle$ and $|g\rangle$, respectively, while $\hat{\sigma}_- = \hat{\sigma}_+^\dagger = |g\rangle \langle e|$ are the usual ladder spin operators. $R$ instead comprises a very large numbers of independent modes, $\omega_i$ of which is a harmonic oscillator with associated frequency $\omega_i$ and bosonic annihilation and creation operators $\hat{b}_i$ and $\hat{b}_i^\dagger$, respectively. In the last sum of Eq. (1), the $i$th term accounts for the interaction (under RWA) between $S$ and the $i$th mode of $R$ with corresponding coupling strength $\mu_i$. A sketch of $S$ and $R$ is shown in Fig. 1 (top part). Under the usual assumption that $R$ features a continuum of modes instead of a discrete set, $\omega_i$ becomes the continuous frequency $\omega$. Accordingly, $\mu_i \rightarrow \mu(\omega)$ and $\sum_i \rightarrow \int d\omega \rho(\omega)$, where $\rho(\omega)$ is the reservoir's density of states.

Consider now the emission process where $S$ is initially excited while $R$ is in the vacuum state $|\text{vac}\rangle_R$ (i.e., at zero temperature) and call $\varepsilon(t)$ the probability amplitude to find $S$ still in the excited state $|e\rangle$ at time $t$, i.e., $\varepsilon(t) = S(e|\text{vac}|\Psi(t))_{SR}$ with $|\Psi(t)\rangle_{SR}$ the joint state of $S$–$R$. An equivalent representation of $\varepsilon(t)$ is its Laplace transform $\tilde{\varepsilon}(z) = \int dt \varepsilon(t)e^{zt}$, where $z$ is a complex variable. It can be shown \[3\] [8] that the general solution for $\tilde{\varepsilon}(z)$ reads

$$\tilde{\varepsilon}(z) = \frac{1}{z - \omega_0 - \int d\omega \frac{\rho(\omega)}{\omega^2}} ,$$

(2)

where $J(\omega)$ is the SD defined as $J(\omega) = \rho(\omega) \langle \mu(\omega) \rangle^2$. Eq. (2) shows that $\tilde{\varepsilon}(z)$, hence $\varepsilon(t)$, is “shaped” by $J(\omega)$: as anticipated, the SD function fully determines the $S$ open dynamics.

To acquire information on $S$ “dressed” by its own reservoir $R$, we send a photon on $S$ and study the resulting scattering process (see Fig. 1). As it is constrained to travel through a 1D waveguide, the photon can be either strictly scattered off – reflected or transmitted – or irreversibly absorbed by the $S$–$R$ joint system. Later on, we will see that the possibility of such absorption is key to the scheme working principle. To describe the scattering process, we add further terms to Hamiltonian \[1\] to include the waveguide field $F$ \[13\]. The total Hamiltonian of $S$, $R$ and $F$ reads

$$\hat{H} = \hat{H}_{SR} + \int dk \, \omega |k| \hat{a}^\dagger(k) \hat{a}(k) + V \int dx \delta(x) [\hat{c}(x) \hat{\sigma}_+ + \text{H.c.}] .$$

(3)

Here, the first integral is the free field Hamiltonian, where $\hat{a}(k) [\hat{a}^\dagger(k)]$ is a field bosonic operator annihilating (creating) a photon of wave vector $k$. We have assumed that the waveguide features a linear dispersion law, namely the photon’s energy $\omega$ depends on its wave vector $k$ along the waveguide axis $x$ as $\omega = \nu |k|$ with $\nu$ being the light group velocity. The second integral in (3) instead accounts for the $S$–$F$ coupling occurring at the $S$ position $x = 0$ (see Fig. 1) and features real-space field operators $\hat{c}(x)$ and $\hat{c}^\dagger(x)$, where $\hat{c}(x)$ [$\hat{c}^\dagger(x)$] annihilates (creates) a photon at position $x$. Hence, the last integral in (3) means that a photon lying at $x = 0$ can be absorbed by $S$ with the latter being promoted to state $|e\rangle$ (or the inverse process). The coupling strength associated with such process is measured by parameter $V$.

A photon with wave vector $k > 0$ is sent towards $S$ when the initial state of $S$–$R$ is $|g\rangle_s |\text{vac}\rangle_R$ (subscripts $S$ and $R$ will be omitted henceforth). Note that unlike the emission process corresponding to Eq. (1) here both $S$ and $R$ are initially unexcited. As usual in quantum scattering problems, we now search for a stationary state of the joint system $S$–$R$–$F$ and enforce that the corresponding energy eigenvalue be $\omega = \nu k$, namely the same as the incoming photon energy. As only a single photon is sent and $\hat{H}$ does not feature counter-rotating terms, the state to seek lies in the single-excitation sector of the total Hilbert space. Thus

$$|\Psi\rangle = \sum_{\alpha} \int dx \psi_\alpha(x) \hat{c}^\dagger(x) |\text{vac}\rangle_R + \sum_i \beta_i \hat{b}_i^\dagger |\text{vac}\rangle_R + \alpha |\text{vac}\rangle e^{|\psi_\alpha(x)\rangle} ,$$

(4)

where $|\text{vac}\rangle = |\text{vac}\rangle_R |\text{vac}\rangle_F$ is the state where both $R$ and $F$ are in the respective vacuum states. Here, $\hat{c}_+(x)$ [$\hat{c}^\dagger(x)$] annihilates a right- (left-) moving photon at $x$, the associated creation operator being $\hat{c}_+(x)$ [$\hat{c}^\dagger(x)$], while $\psi_+(x) = [\theta(-x) + \theta(x)]e^{ikx}$ and $\psi_-(x) = r[\theta(-x)]e^{-ikx}$ reflect the usual scattering ansatz \[13\] with $r(t)$ the photon’s reflection (transmission) coefficient. Imposing now, as anticipated, that $\hat{H}|\Psi\rangle = \omega |\Psi\rangle$ yields through standard methods \[13\] [14] a set of coupled equations for $\psi_\alpha(x)$, $[\beta_i]$ and $\alpha$ \[29\]. Once $[\beta_i]$ and $\alpha$ are eliminated by expressing them in terms of $\psi_\alpha(x)$, we end up with a closed equation in $\psi_\alpha(x) = \psi_+(x) + \psi_-(x)$ \[29\]. This reads

$$\frac{d^2 \psi}{dx^2} + k^2 \psi(x) = 2k \nu W(\omega) \delta(x) \psi(x) ,$$

(5)

with

$$W(\omega) = V^2 \tilde{\varepsilon}(\omega) ,$$

(6)

where $\tilde{\varepsilon}(\omega)$ is the same function as in Eq. (4) for $z = \omega$. 

FIG. 1: (Color online) Setup for the SD measurement.
The form of Eq. (5) is familiar in many contexts. In classical optics, an analogous equation describes an electromagnetic wave penetrating through a thin dielectric slab, e.g. a mirror [25]. In elementary quantum mechanics, just the same equation is found for a particle of mass $k/\nu$ scattering from a potential barrier $W(x)$. To make the language simpler, in what follows we refer to $W(\omega)$ as the effective potential (this is reminiscent of Ref. [25], where however $R$ was absent).

Note that this is frequency-dependent, such dependance occurring through function $\mathcal{P}(\omega)$ which is associated with $J(\omega)$ [cf. Eq. (2)]. We have thus reduced our problem to the elementary calculation [26] of the reflection and transmission coefficients of a particle scattering from a pointlike potential barrier. These are given by

$$r = 1 - t = -\frac{i W}{\nu} \frac{W}{W_0}.$$  

(7)

It is important to stress that $W(\omega)$ is in general complex. Indeed, through the well-known Sokhotski–Plemelj theorem [3][27] the improper integral appearing in Eq. (2) for $z=\omega$ can be expressed as $\int d\omega' J(\omega')/(\omega-\omega') = \mathcal{P}(\omega) - i\pi J(\omega)$, where $\mathcal{P}(\omega)$ stands for the integral’s principal value. Thus $W(\omega)$ can be decomposed into its real and imaginary parts as $W(\omega) = W_R(\omega) - i W_I(\omega)$ with

$$W_R(\omega) = V^2 \frac{\omega - \omega_0 - \mathcal{P}(\omega)}{[\pi \mathcal{J}(\omega)]^2 + [\omega - \omega_0 - \mathcal{P}(\omega)]^2},$$  

(8)

$$W_I(\omega) = V^2 \frac{\pi \mathcal{J}(\omega)}{[\pi \mathcal{J}(\omega)]^2 + [\omega - \omega_0 - \mathcal{P}(\omega)]^2}.$$  

(9)

Note that $W_I(\omega) \geq 0$. Using the aforementioned optical analogy, it is as if the photon impinges on an effective classical mirror, which besides being refractive is also absorptive. Scattering from complex potentials is a well-known tool [28] arising as an effective description of inelastic scattering channels. Due to such channels, the sum of photon’s reflection and transmission probabilities (reflectance and transmittance, respectively) is lower than one, namely $|r|^2 + |t|^2 < 1$. From Eq. (7), under the replacement $W=W_0-iW_I$, we indeed find

$$1-|r|^2 - |t|^2 = \frac{2 W_I}{W_I^2 + (1+ W_R)^2},$$  

(10)

which shows that $|r|^2 + |t|^2 = 1$ if and only if $W_I = 0$. The complexity of $W$ matches the physical expectation that – due to the $R$ infiniteness – the photon can be reversibly absorbed by the $S-R$ system. A crucial point is that $W_I$ is strictly related to $J(\omega)$ [see Eq. (2)]: the probability to lose the photon is non-zero whenever the SD at the photon’s frequency $\omega$ is non-null, namely when $\omega$ matches a reservoir frequency (if any) that is coupled to $S$. We show next that, upon a perspective reversal, the last fact can be exploited for measuring the SD.

From Eqs. (8) and (9) immediately follows that the SD can be expressed in terms of the effective potential $W(\omega)$ as

$$J(\omega) = \frac{V^2}{\pi} \frac{W_I(\omega)}{|W(\omega)|^2}.$$  

(11)

On the other hand, by taking the ratio of Eq. (10) to the reflectance $|r|^2$ [cf. Eq. (7)], for any complex potential

$$\frac{W_I}{|W|^2} = \frac{1}{2\nu} \frac{1-|r|^2 - |t|^2}{|r|^2}.$$  

(12)

Combined together, this and Eq. (11) yield

$$J(\omega) = \frac{V^2}{\nu} \frac{1-|r|^2 - |t|^2}{2\pi |r|^2}.$$  

(13)

We thus find that the SD is a universal simple function of the photon reflectance and transmittance. A major immediate consequence of identity (13) is that one can straightforwardly extract the SD profile from reflection and transmission spectra, which are normally easy to record via simple intensity measurements. Note that $V^2/\nu$ works as a sort of magnification knob: the larger $V^2/\nu$ the better the SD profile can be appreciated. This is reasonable since $V^2/\nu$ is the spontaneous emission rate of $S$ into the waveguide [13], i.e., it measures the effective $S-F$ coupling strength: a weakly (strongly) interacting photon is little (highly) sensitive to $S-R$. Note that no approximations on the $S-R$ coupling strength has been made to derive Eq. (13). Hence, it is not limited to the weak-coupling regime. NM effects can thus be significant.

A formally trivial, yet physically noteworthy, consequence of Eq. (13) is that for a flat SD the combination of reflectance and transmittance $f(\omega) = (1-|r|^2 - |t|^2)/|r(\omega)|^2$ is constant in frequency. As a constant SD yields a Markovian dynamics described by the KLME, the flatness of function $f(\omega)$ can be used as a test to assess whether the KLME is effective. To provide a check of Eq. (13) based on real experiments, let us consider microwave photon scattering in a 1D open transmission line from a superconducting qubit (artificial atom), which has been the focus of Refs. [20][21]. In these experiments, it was found that the measured field reflection and transmission coefficients are well described by

$$r(\omega) \approx 1 - t(\omega) = -\frac{\Gamma_{eg}}{2\gamma_{eg}} \frac{1 - i\delta\omega/\gamma_{eg}}{1 + (\delta\omega/\gamma_{eg})^2 + \Omega^2/[\Gamma_{eg} + \Gamma_f \gamma_{eg}]}.$$  

(14)

where $\delta\omega = \omega - \omega_0$, $\Gamma_{eg}$ is the atom’s relaxation rate into the waveguide modes while $\gamma_{eg}$ = $\Gamma_{eg}/2 + \Gamma_{d}$ is the total atom’s decoherence rate. Here, importantly, $\Gamma_f = \Gamma_f/2 + \Gamma_d$ depends on the atom’s coupling to reservoirs external to the waveguide modes: $\Gamma_f$ is the rate of intrinsic losses while $\Gamma_d$ is the pure dephasing rate. $\Omega$ is the Rabi frequency proportional to the input field power. Computing $f(\omega)$ as defined above, we find

$$f(\omega) = \frac{2(\Gamma_f + 2\Gamma_d)}{\Gamma_{eg}} + \frac{4(\Gamma_{eg} + \Gamma_f + 2\Gamma_d) \Omega^2}{\Gamma_{eg} (\Gamma_{eg} + \Gamma_f)^2 (\Gamma_{eg} + \Gamma_f + 2\Gamma_d)^2 + 4\delta\omega^2}.$$  

For a single-photon beam, $\Omega$ is negligible [20][21] and $f(\omega)$ becomes independent of $\omega$ (the second term vanishes). In our framework, this corresponds to a flat SD yielding that the atom’s open dynamics is describable through the KLME. Significantly, this is consistent with coefficients (14) since these can be worked out through a simple semiclassical model [20] based on a KLME for the atom’s density matrix (the input field
being treated as a classical drive). Note that for the present setting, besides the dissipative reservoir associated with rate $\Gamma$, the atom is subject also to purely dephasing noise (corresponding to $g_\phi$). While samples with negligible $g_\phi$ are within reach \cite{23,24}, being thus fully compatible with Hamiltonian (1), it is remarkable that $f(\omega)$ for $\Omega \approx 0$ is flat even for $g_\phi \neq 0$ (suggesting that the scheme might be generalizable to some extent).

To test Eq. (13) in the case of NM noise, let us consider the experiment in Ref. \cite{19}. This differs from the previous one for the fact that the OQS coupled to the 1D line (i.e., system $S$) is a high-finese resonator $C$. Also, the resonator is coupled to a lossy Cooper pair box (CPB) via a Jaynes-Cummings (JC) interaction $\sum \ket{a} \bra{a}$. Here, $R$ is jointly embodied by the CPB and its own reservoir. Note that in such single-photon process, both $C$ and the CPB behave as effective qubits. In Ref. \cite{14}, it was shown that the scattering coefficients in such experiment are reasonably given by

$$r(\omega)=1-t(\omega)=-i\frac{V^2}{\nu}\frac{\omega-\omega_0+i\Gamma_1}{(\omega-\omega_1+i\Gamma_1)(\omega-\omega_0+\frac{\nu^2}{\nu^2})-g^2}, \quad (15)$$

where $\omega_0$ ($\omega_1$) is the frequency of $C$ (CPB), $\Gamma_1$ is the CPB dissipation rate and $g$ is the C-CPB coupling rate. In such case, Eq. (13) yields the Lorentzian SD

$$J_\omega(\omega)=\frac{g^2}{\pi}\frac{\Gamma_1}{(\omega-\omega_0)^2}, \quad (16)$$

which is indeed a signature of a damped JC dynamics \cite{23} (observing a JC coupling is the main focus of Ref. \cite{19}).

It should be noted that Eq. (5), hence Eq. (13), relies on the assumption (which routinely occurs in waveguide QED and beyond) that only a relatively narrow photonic bandwidth is involved \cite{14}. This could be no more valid when the dressing of $S$ by $R$ is very strong. Yet, this does not prevent Eq. (13) from holding for $S$-$R$ couplings as strong as to yield significant NM effects. Indeed, for the SD (16), various non-Markovianity measures \cite{31} are non-zero in the resonant case $\omega_0=\omega_1$ if and only if $4g^2/\Gamma_1^2 >1$. In the setup of Ref. \cite{19} discussed above, $g/(2\pi) \approx 5.8$ MHz while $\Gamma_1/(2\pi) \approx 0.7$ MHz so that one can estimate $4g^2/\Gamma_1^2 \approx 275$, which indicates that NM effects are pronounced.

In conclusion, we have shown a method for measuring the SD of a reservoir in dissipative contact with a small OQS. This is achieved by coupling the OQS to a 1D photonic waveguide and sending through this photons which undergo scattering from the OQS. The SD has been shown to be a simple universal function of photon reflectance and transmittance. As such, it can be easily extracted from reflection and transmission spectra. The result does not rely on the weak-coupling approximation, hence significant NM effects can be present. The SD measurement is “static” since the dynamical evolution of the OQS in contact with $R$ needs not be switched on or monitored. Such dynamics is fully reconstructable via the 1D photon scattering. The scheme diagnostic power has been tested on the basis of two real waveguide-QED experiments, including one exhibiting important NM effects \cite{32}.

A remarkable point is that while – expectably – the scattering coefficients depend on the SD in quite a complicated way [cf. Eqs. (2), (6)–(7)] the SD is instead quite a simple function of them. We envisage that this property, suggesting the perspective reversal at the heart of the method, has the potential to inspire novel approaches to the SD measurement problem in more general situations such as finite-temperature and/or purely dephasing noise.

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**Supplemental material**

In this Supplemental Material, we supply some technical details related to some properties discussed the paper’s main text (MT). The derivation follows standard methods that are now routinely used in the literature on waveguide QED.

### A. Schrödinger equation

In the Appendix of Ref. \cite{14}, it is shown that the Schrödinger equation $\dot{H}\Psi=V\Psi$ with $H$ and $\Psi$ given by Eqs. (3) and (4) in the MT, respectively, is equivalent to the set of coupled equations in $\psi_+(x)$, $\psi_-(x)$ and $a$:

$$i\nu \frac{d\psi_+}{dx}(x)+V\delta(x)a = \omega \psi_+(x), \quad (17)$$

$$\omega_0a+V[\psi_+(0)+\psi_-(0)]+\sum_{\mu} \mu_i \beta_i = \omega a, \quad (18)$$

$$\omega_0a+V[\psi_+(x)+\psi_-(x)]+\sum_{\mu} \mu_i \beta_i = \omega a. \quad (19)$$

We point out that in Ref. \cite{14} the authors focus on the implications of these equations on photon transport by restricting to the special case of a reservoir $R$ with a flat SD (Markovian reservoir). In the present work, instead, we tackle the measurement problem of the SD. Thereby, the SD is left fully unspecified throughout.

### B. Proof of Eq. (5)

To derive Eq. (5) in the MT – where $\psi(x)$ is the only unknown function – from Eqs. (17)–(19), we first translate Eq. (17) into a corresponding equation for $\psi_+(x)=\psi_+(x)+\psi_-(x)$ (a similar task was carried out in Ref. \cite{23} but in the absence of the reservoir $R$). Subtracting the equation for $\psi_+(x)$ from that corresponding to $\psi_-(x)$ yields

$$i\nu \frac{d\psi}{dx}(x) = \omega[\psi_-(x)-\psi_+(x)]. \quad (20)$$

Once either side is derived in $x$ and the derivatives of $\psi_+(x)$ on the right-hand side are expressed through Eq. (17), we end up with

$$\frac{d^2\psi}{dx^2}(x)+k^2 \psi(x) = \frac{2kV}{\nu} \alpha \delta(x) \quad (21)$$
where we used $\omega = \nu k$. This equation alongside Eqs. (18) and (19) now form a set of equations having $\psi(x)$, $\alpha$ and $\beta$, as the only unknowns [note that in Eq. (18) the factor multiplying $V$ equals $\psi(0)$].

Solving Eq. (19) for $\beta$, and replacing the result into Eq. (18) gives $\alpha = V\psi(0)/[\omega_0 - \sum_i \mu_i^2/(\omega_0 - \omega_i)]$. In the continuous limit: $\omega_0 \rightarrow \omega', \mu_i \rightarrow \mu(\omega')$ while $\sum_i \int d\omega \rho(\omega')$. Hence, using Eq. (2) in the MT and $J(\omega) = \rho(\omega) [\mu(\omega)]^2$, we obtain $\alpha = V\psi(0)\tilde{\epsilon}(\omega)$. Once this is replaced in Eq. (21), we finally end up with Eq. (5) in the MT.

C. Calculation of $r$ and $t$

The elementary calculation of coefficients (7) in the MT can be solved by imposing that $\psi(x)$ be a continuous function at $x = 0$ and its derivative $d\psi/dx = \psi'$ fulfil $\psi'(0^+) - \psi'(0^-) = 2k/\nu W(0)$ [here $\psi'(0^+)$ is the right-sided (left-sided) limit of $\psi'$ at $x = 0$]. The latter constraint can be straightforwardly obtained from Eq. (5) by integrating either side over an infinitesimal interval $dI$ across $x = 0$ and using that $\int dI \delta(0)\psi(x) = \psi(0)$ [26]. Imposing such boundary conditions to $\psi(x)$ and $\psi'(x)$ (which are functions of $r$ and $t$, see MT) yields coefficients (7).

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