AN APPLICATION OF AN AVERY TYPE FIXED POINT THEOREM TO A SECOND ORDER ANTIpériodIC BOUNDARY VALUE PROBLEM

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ABSTRACT. In this article, we show the existence of an antisymmetric solution to the second order boundary value problem $x'' + f(x(t)) = 0$, $t \in (0, n)$ satisfying antiperiodic boundary conditions $x(0) + x(n) = 0$, $x'(0) + x'(n) = 0$ using an Avery et. al. fixed point theorem which itself is an extension of the traditional Leggett-Williams fixed point theorem. The antisymmetric solution satisfies $x(t) = -x(n-t)$ for $t \in [0, n]$ and is nonnegative, nonincreasing, and concave for $t \in [0, n/2]$. To conclude, we present an example.

1. Introduction. The study of the existence of solutions to boundary value problems has long been an interesting and well-researched area within differential equations. In particular, we can see that antiperiodic boundary conditions have been an important part of the literature, [8, 12, 13, 14]. Recently, Avery et. al. have published several articles which extend the original Leggett-Williams fixed point theorem, [3, 9, 4, 5, 6]. The extension does not require the functional boundaries of the arguments to be invariant. Also quite interestingly, Avery et. al. provide a topological proof for some of their results instead of using index theory arguments. There has been a significant amount of work published utilizing Avery fixed point theorems to prove the existence of solutions, typically positive solutions, to differential, difference and dynamic equations with varying types of boundary conditions. For a small sample see, [1, 2, 10, 7, 11, 16, 17]. In this paper, we will apply the Avery fixed point theorem, [4], to a second order boundary value problem with antiperiodic boundary conditions to prove the existence of an antisymmetric solution in the sense that $x(t) = -x(n-t)$ for $t \in [0, n]$. Of note, in many related papers, the authors utilize a concavity like property of the Green’s function. Here the approach is similar, but the property is somewhat different due to the antisymmetric nature of our solution.

For Section 2, we provide much of the background information required for the problem and define a few important sets. In Section 3, we present the fixed point theorem. Quickly followed by Section 4 where the BVP, Green’s function, and operator are defined. Here we will state and prove a Lemma involving the crucial concavity like property. We also note the importance of the midpoint, $n/2$, of the interval $[0, n]$ Finally, in Section 5, we apply the fixed point theorem to the BVP and conclude in Section 6 with an example.

2. Definitions.

Definition 2.1. Let $\mathcal{B}$ be a real Banach space. A nonempty closed convex set $\mathcal{P} \subset \mathcal{B}$ is called a cone provided:

(i) $x \in \mathcal{P}$, $\lambda \geq 0$ implies $\lambda x \in \mathcal{P}$;

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Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. A map $\alpha$ is said to be a nonnegative continuous concave functional on a cone $P$ of a real Banach space $B$ if $\alpha : P \to [0, \infty)$ is continuous and
\[
\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y)
\]
for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map $\beta$ is a nonnegative continuous convex functional on a cone $P$ of a real Banach space $B$ if $\beta : P \to [0, \infty)$ is continuous and
\[
\beta(tx + (1 - t)y) \leq t\beta(x) + (1 - t)\beta(y)
\]
for all $x, y \in P$ and $t \in [0, 1]$.

Next, we define sets that are integral to the fixed point theorem. Let $\psi$ and $\delta$ be nonnegative continuous functionals on $P$. Then, we define the sets:
\[
P(\psi, b) = \{x \in P : \psi(x) \leq b\}
\]
\[
P(\psi, \delta, a, b) = \{x \in P : a \leq \psi(x) \text{ and } \delta(x) \leq b\}
\]

3. The Fixed Point Theorem. The following fixed point theorem is attributed to Anderson, Avery, and Henderson [4] and is an extension of the original Leggett-Williams fixed point theorem [15].

Theorem 3.1. Suppose $P$ is a cone in a real Banach space $E$, $\alpha$ is a nonnegative continuous concave functionals on $P$, $\beta$ is a nonnegative continuous convex functionals on $P$, and $T : P \to P$ is a completely continuous operator. If there exists nonnegative numbers $a, b, c,$ and $d$ such that
\[
(A1) \{x \in P : a < \alpha(x) \text{ and } \beta(x) < b\} \neq \emptyset;
\]
\[
(A2) \text{if } x \in P \text{ with } \beta(x) = b \text{ and } \alpha(x) \geq a, \text{ then } \beta(Tx) < b;
\]
\[
(A3) \text{if } x \in P \text{ with } \beta(x) = b \text{ and } \alpha(Tx) < a, \text{ then } \beta(Tx) < b;
\]
\[
(A4) \{x \in P : c < \alpha(x) \text{ and } \beta(x) < d\} \neq \emptyset;
\]
\[
(A5) \text{if } x \in P \text{ with } \alpha(x) = c \text{ and } \beta(x) \leq d, \text{ then } \alpha(Tx) > c;
\]
\[
(A6) \text{if } x \in P \text{ with } \alpha(x) = c \text{ and } \beta(Tx) > d, \text{ then } \alpha(Tx) > c; \text{ and if}
\]
\[
(H1) a < c, b < d, \{x \in P : b < \beta(x) \text{ and } \alpha(x) < c\} \neq \emptyset, \ P(\beta, b) \subset P(\alpha, c), \text{ and } P(\alpha, c)
\]
\[
is bounded, \text{ then } T \text{ has a fixed point } x^* \text{ in } P(\beta, a, b, c);
\]
\[
(H2) c < a, d < b, \{x \in P : a < \alpha(x) \text{ and } \beta(x) < d\} \neq \emptyset, \ P(\alpha, a) \subset P(\beta, d), \text{ and } P(\beta, d)
\]
\[
is bounded, \text{ then } T \text{ has a fixed point } x^* \text{ in } P(\alpha, \beta, a, d);
\]

4. The Antiperiodic Boundary Value Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map and $n > 0$ be fixed in $\mathbb{R}$. We will apply the fixed point theorem to the second order boundary value problem
\[
x'' + f(x(t)) = 0, \ t \in (0, n)
\]
with antiperiodic boundary conditions
\[
x(0) + x(n) = 0, \ x'(0) + x'(n) = 0.
\]

We will show that if $f$ satisfies certain conditions, (1), (2) has an antisymmetric solution in the sense that $x(t) = -x(n - t)$ for $t \in [0, n]$.

Throughout this paper, we will utilize the Banach space $E = C[0, n]$ endowed with the supremum norm.

If $x$ is a fixed point of the operator $T$ defined by
\[
Tx(t) := \int_0^n G(t, s)f(x(s))ds,
\]
where $G(t, s)$, defined on $[0, n] \times [0, n]$ by

$$G(t, s) = \begin{cases} \frac{1}{2} \left( s - t + \frac{n}{2} \right), & 0 \leq s \leq t \leq n, \\ \frac{1}{2} \left( t - s + \frac{n}{2} \right), & 0 \leq t \leq s \leq n, \end{cases}$$

is the Green’s function for the operator $L$ defined by $Lx(t) := -x''$ satisfying antiperiodic boundary conditions (2), then (see [8]) $x$ is a solution of the boundary value problem (1), (2).

**Lemma 4.1.** If $s \in [0, n]$, $G(t, s)$ has the property that $yG(n/2 - w, s) \leq wG(n/2 - y, s)$ for all $w, y \in [0, n/2]$ with $w \geq y$.

**Proof.** If $y = 0$, $0 \leq wG(n/2, s) = w \min\{s/2, (n-s)/2\}$, so we assume $y \neq 0$. We consider 3 cases.

**Case 1:** Let $0 \leq n/2 - w \leq n/2 - y \leq s$. Now,

$$yG(n/2 - w, s) = \frac{1}{2}(-yw + y(n-s)) \leq \frac{1}{2}(-yw + w(n-s)) = wG(n/2 - y, s).$$

**Case 2:** Let $0 \leq n/2 - w \leq s \leq n/2 - y$. Then,

$$\frac{G(n/2 - w, s)}{G(n/2 - y, s)} = \frac{1/2(n - (s + w))}{1/2(s + y)} = \frac{w(n-s) - 1}{y(s+y+1)}.$$

So,

$$\frac{G(n/2 - w, s)}{G(n/2 - y, s)} \leq \frac{w}{y}$$

is equivalent to

$$\frac{n-s}{w} - 1 \leq \frac{s}{y} + 1,$$

or

$$s \geq \frac{y(n-2w)}{w+y}.$$

Well,

$$\frac{y(n-2w)}{w+y} = \frac{2y}{w+y}(n/2 - w) \leq n/2 - w \leq s.$$

Therefore,

$$\frac{G(n/2 - w, s)}{G(n/2 - y, s)} \leq \frac{w}{y}.$$

**Case 3:** Let $0 \leq s \leq n/2 - w \leq n/2 - y$. Then,

$$\frac{G(n/2 - w, s)}{G(n/2 - y, s)} = \frac{1/2(s + w)}{1/2(s + y)} = \frac{w(s+y+1)}{y(s+w+1)} \leq \frac{w}{y}.$$

Notice in Case 2 and Case 3, $y, G(n/2 - y, s) \geq 0$, so the inequalities can be cross multiplied to obtain the desired result. \[\square\]

Let $\tau \in (0, n/2)$ and define the cone $P \subset E = C[0, n]$ by

$$P := \{ x \in E : x \text{ is nonnegative, nonincreasing, and concave on } [0, n/2] \text{ and } x(t) = -x(n-t) \text{ on } [n/2, n] \}.$$  

For $x \in P$, define the concave functional $\alpha$ on $P$ by

$$\alpha(x) := \min_{t \in [0, n/2-\tau]} x(t) = x(n/2 - \tau).$$
and the convex functional $\beta$ on $P$ by
\[
\beta(x) := \max_{t \in [0,n/2]} x(t) = x(0).
\]

**Remark 1.** Notice that if $x \in P$, then for $y, w \in [0,n/2]$ with $y \leq w$,
\[
\frac{x(n/2 - w) - x(n/2)}{(n/2 - w) - n/2} \geq \frac{x(n/2 - y) - x(n/2)}{(n/2 - y) - n/2}
\]
due to the fact that $x$ is nonnegative, nonincreasing, and concave. Now, since $x(n/2) = 0$,
\[
yx(n/2 - w) \leq wx(n/2 - y).
\]

5. **Solutions using (H1).** In the following theorem, we demonstrate how to apply the expansiveness condition (H1) of Theorem 3.1 to prove existence of at least one solution to (1), (2). An application of (H2) is similar.

**Theorem 5.1.** If $\tau \in (0,n/2)$ is fixed, $b$ and $c$ are positive real numbers such that $4b \leq c$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous, odd function such that $f([0,\infty)) \subset [0,\infty)$ and

(a) $f(w) \geq \frac{8cn}{(n^2 - 4\tau^2)}$ for $w \in \left[0,\frac{cn/2}{\tau}\right]$,

(b) $f(w)$ is nondecreasing for $w \in \left[0,\frac{b\tau}{n/2}\right]$ and $f \left(\frac{b\tau}{n/2}\right) \geq f(w)$ for $w \in \left[\frac{b\tau}{n/2}, b\right]$, and

(c) $\int_{n/2-\tau}^{n/2} (n/2 - s) f \left(\frac{bs}{n/2-\tau}\right) ds \leq \frac{8b-f\left(\frac{b\tau}{n/2}\right)(n^2-4\tau^2)}{8}$,

then the antiperiodic boundary value problem (1), (2) has at least one solution $x^* \in P(\beta, \alpha, b, c)$.

**Proof.** Let $a = \frac{b\tau}{n/2}$ and $d = \frac{cn/2}{\tau}$. Then, we have $a < c$ and $b < d$ since $4b \leq c$.

**Claim:** $T : P \to P$ is completely continuous.

Since $x \in P$, $x(t) \geq 0$ for $t \in [0,n/2]$. Also, $(Tx)^\prime(t) = -f(x(t))$, and so, $(Tx)^\prime(t) \leq 0$ for $t \in [0,n/2]$. Thus, $Tx$ is concave on $[0,n/2]$, and $(Tx)^\prime$ is nonincreasing on $[0,n/2]$.

Additionally, since $\frac{a}{n}G(t,s)|_{t=0} = 1/2$ and $f$ is odd, $(Tx)^\prime(0) = 1/2 \int_0^n f(x(s))ds = 0$. So, $(Tx)^\prime(t) \leq 0$ for $t \in [0,n/2]$. Thus, $Tx$ is nonincreasing on $[0,n/2]$. Next, since $Tx(n-t) = -Tx(t)$, for $t \in [0,n]$, $Tx(n/2) = 0$. So $Tx \geq 0$ on $[0,n/2]$.

Notice for $(t,s) \in [0,n] \times [0,n]$, $G(n-t,n-s) = G(t,s)$. Now,
\[
Tx(n-t) = \int_0^n G(n-t,s)f(x(s))ds
\]
Substitute $s = n - r$. Then,
\[
Tx(n-t) = -\int_0^n G(n-t,n-r)f(x(n-r))dr
= \int_0^n G(t,r)f(-x(r))dr
= -\int_0^n G(t,r)f(x(r))dr
= -Tx(t).
\]

A standard application of the Arzela-Ascoli Theorem may be used to show that $T$ is completely continuous.

**Claim:** (A1) \{ $x \in P$ : $a < \alpha(x)$ and $\beta(x) < b$ \} $\neq \emptyset$.

For $L \in \left(\frac{4b}{n(n-\tau)}, \frac{b\tau}{n^2}\right)$, define
\[
x_L(t) := \int_0^n LG(n/2-t,s)ds = \frac{L}{8}(n-2t)(n+2t), \text{ for } t \in [0,n/2]
\]
and \(x_L(t) = -x_L(n - t)\) for \(t \in [n/2, n]\). Then,
\[
\alpha(x_L) = x_L(n/2 - \tau)
= \frac{L}{8}(2\tau)(2n - 2\tau)
> \frac{4b}{8n(n - \tau)}(2\tau)(2n - 2\tau)
= \frac{b\tau}{n/2} = a,
\]
and
\[
\beta(x_L) = x_L(0) = \frac{L}{8}n^2 < \frac{8b}{8n^2}n^2 = b.
\]
Therefore, \(x_L \in \{x \in P : a < \alpha(x)\) and \(\beta(x) < b\}.

**Claim:** (A2) If \(x \in P\) with \(\beta(x) = b\) and \(\alpha(x) \geq a\), then \(\beta(Tx) < b\).
Let \(x \in P\) with \(\beta(x) = b\) and \(\alpha(x) \geq a\). For \(s \in [n/2 - \tau, n/2],\) \(n/2 - \tau \leq s\). By Remark 1, \((n/2 - \tau)x(s) \leq sx(\tau)\). Thus,
\[
x(s) \leq \frac{x(\tau)s}{n/2 - \tau} < \frac{bs}{n/2 - \tau}.
\]
Also, for \(s \in [0, n/2 - \tau],\) \(b \geq x(s) \geq a = \frac{b\tau}{n/2}\). Finally, using antisymmetry and substitution of \(u = n - s\), we have
\[
\int_{n/2}^{n} G(0, s)f(x(s))ds = \int_{n/2}^{n} \frac{1}{2} (n/2 - s)f(x(s))ds
= \int_{n/2}^{T} \frac{1}{2} (n/2 - s)f(-x(n - s))ds
= - \int_{n/2}^{n} \frac{1}{2} (n/2 - s)f(x(n - s))ds
= \int_{0}^{n/2} \frac{1}{2} (n/2 - u)f(x(u))du
= \int_{0}^{n/2} G(0, u)f(x(u))du.
\]
Hence using properties (b) and (c), we have
\[
\beta(Tx) = \int_{0}^{n} G(0, s)f(x(s))ds
\leq 2 \int_{0}^{n/2} G(0, s)f(x(s))ds
= 2 \int_{0}^{n/2} \frac{1}{2} (n/2 - s)f(x(s))ds
= \int_{0}^{n/2} (n/2 - s)f(x(s))ds + \int_{n/2 - \tau}^{n/2} (n/2 - s)f(x(s))ds
\leq f \left( \frac{b\tau}{n/2} \right) \int_{0}^{n/2 - \tau} (n/2 - s)ds + \int_{n/2 - \tau}^{n/2} (n/2 - s)f \left( \frac{bs}{n/2 - \tau} \right) ds
\leq f \left( \frac{b\tau}{n/2} \right) (n^2 - 4\tau^2) + \frac{8b - f \left( \frac{b\tau}{n/2} \right)}{8} (n^2 - 4\tau^2) = b.
\]

**Claim:** (A3) If \(x \in P\) with \(\beta(x) = b\) and \(\alpha(Tx) < a\), then \(\beta(Tx) < b\).
Let $x \in P$ with $\beta(x) = b$ and $\alpha(Tx) < a$. Then,

$$\beta(Tx) = \int_0^n G(0,s)f(x(s))ds$$

$$\leq \frac{n/2}{\tau} \int_0^n G(n/2 - \tau,s)f(x(s))ds$$

$$= \frac{n/2}{\tau} \alpha(Tx) < \frac{n/2}{\tau} \cdot \frac{a}{\tau} = b.$$ 

Claim: (A4) \{x \in P : c < \alpha(x) \text{ and } \beta(x) < d\} \neq \emptyset. 

For $J \in \left(\frac{4d}{n(\alpha - \tau)}: \frac{8d}{n^2}\right)$, define

$$x_J(t) := \int_0^n JG(n/2 - t,s)ds = \frac{J}{8}(n - 2t)(n + 2t), \text{ for } t \in [0, n/2]$$

and $x_J(t) = -x_J(n - t)$ for $t \in [n/2, n]$. Then,

$$\alpha(x_J) = x_J(n/2 - \tau) = \frac{J}{8}(2\tau)(2n - 2\tau)$$

$$> \frac{4d}{8n(n - \tau)}(2\tau)(2n - 2\tau) = \frac{d\tau}{n/2} = c,$$

and

$$\beta(x_J) = x_J(0) = \frac{J}{8}n^2 < \frac{8d}{8n^2}n^2 = d.$$ 

Therefore, $x_J \in \{x \in P : c < \alpha(x) \text{ and } \beta(x) < d\}$. 

Claim: (A5) If $x \in P$ with $\alpha(x) = c$ and $\beta(x) \leq d$, then $\alpha(Tx) > c$. 

First note that $x(s), G(0,s) \geq 0$ for $s \in [0, n/2]$ and $x(s), G(0,s) \leq 0$ for $s \in [n/2, n]$. Let $x \in P$ with $\alpha(x) = c$ and $\beta(x) \leq d$. Then by (a),

$$\alpha(Tx) = \int_0^n G(n/2 - \tau,s)f(x(s))ds$$

$$\geq \frac{\tau}{n/2} \int_0^n G(0,s)f(x(s))ds$$

$$= \frac{\tau}{n/2} \int_0^{n/2} G(0,s)f(x(s))ds + \frac{\tau}{n/2} \int_{n/2}^n G(0,s)f(x(s))ds$$

$$\geq \frac{\tau}{n/2} \int_0^{n/2} G(0,s)f(x(s))ds$$

$$\geq \frac{16d}{n^2 - 4\tau^2} \cdot \frac{\tau}{n/2} \int_0^{n/2 - \tau} \frac{1}{2}(n/2 - s)ds$$

$$= \frac{16d}{n^2 - 4\tau^2} \cdot \frac{\tau}{n/2} \cdot \frac{1}{2} \cdot \frac{1}{8}(n^2 - 4\tau^2) = \frac{\tau d}{n/2} = c.$$ 

Claim: (A6) If $x \in P$ with $\alpha(x) = c$ and $\beta(Tx) > d$, then $\alpha(Tx) > c$. 

Let $x \in P$ with $\beta(Tx) > d$ and $\alpha(x) = c$. Then,

$$\alpha(Tx) = \int_0^n G(n/2 - \tau,s)f(x(s))ds$$

$$\geq \frac{\tau}{n/2} \int_0^n G(0,s)f(x(s))ds$$

$$= \frac{\tau}{n/2} \beta(Tx) > \frac{\tau d}{n/2} = c.$$
Claim: \( (H1) \) \( a < c, \ b < d, \ \{x \in P : b < \beta(x) \ \text{and} \ \alpha(x) < c\} \neq \emptyset, \ \text{P(\(\beta, b) \subset P(\alpha, c)\), and} \ P(\alpha, c) \ \text{is bounded.} \)

Note that \( a < c \) and \( b < d \) has already been established.

Next, for \( M \in \left( \frac{8b}{\tau(n - \tau)}, \frac{2c}{\tau(n - \tau)} \right) \), define

\[
x_M(t) := \int_0^t MG(n/2 - t, s) ds = \frac{M}{8}(n - 2t)(n + 2t), \ \text{for} \ t \in [0, n/2]
\]

and \( x_M(t) = -x_M(n - t) \) for \( t \in [n/2, n] \). Then,

\[
\alpha(x_M) = x_M(n/2 - \tau)
\]

\[
= \frac{M}{8}(2\tau)(2n - 2\tau)
\]

\[
= \frac{M}{2}\tau(n - \tau)
\]

\[
\leq \frac{2c}{2\tau(n - \tau)} \cdot \tau(n - \tau) = c,
\]

and

\[
\beta(x_M) = x_M(0) = \frac{M}{8}n^2 \geq \frac{8b}{8\tau(n - \tau)}n^2 > b.
\]

Hence, \( x_M \in \{x \in P : b < \beta(x) \ \text{and} \ \alpha(x) < c\}. \)

Now, let \( x \in P(\alpha, c) \), then

\[
\tau \beta(x) = \tau x(0) \leq \frac{n}{2}x(n/2 - \tau) = \frac{n}{2}\alpha(x) \leq \frac{n}{2}c.
\]

So, \( ||x|| = \beta(x) \leq \frac{\alpha(x)}{\tau} \cdot \frac{n}{2} \leq \frac{n}{2} \cdot \frac{c}{\tau} \). Thus, \( P(\alpha, c) \) is bounded.

Lastly, let \( x \in P(\beta, b) \). Then, \( \alpha(x) \leq \beta(x) \leq b < c \). Therefore, \( P(\beta, b) \subset P(\alpha, c) \).

Since hypotheses (A1)-(A6) and (H1) hold, we have that \( T \) has a fixed point \( x^* \) in \( P(\beta, \alpha, b, c) \). That is, \( x^* \) solves (1), (2), \( x^*(0) = \beta(x^*) \geq b \), and \( x^*(n/2 - \tau) = \alpha(x^*) \leq c \).

6. Example for (H1). Let \( n = 1, \ \tau = \frac{1}{4}, \ b = \frac{1}{4}, \ \text{and} \ c = 1 \). Note that \( 4b \leq c \). Define a continuous, odd function \( f : \mathbb{R} \to \mathbb{R} \) by

\[
f(w) = \begin{cases} 
  w/2 & : w \in [0, 1/8] \\
  1/16 & : w \in [1/8, 1/4] \\
  2045w/35 - 509/36 & : w \in [1/4, \infty),
\end{cases}
\]

and for \( w \in (-\infty, 0) \), \( f(w) = -f(-w) \). Then,

(i) for \( w \in \left[ c, \frac{cn/2}{\tau} \right] = [1, 4] \), \( f(w) \geq \frac{125}{3} = \frac{8cn}{3\tau - 4\tau} \),

(ii) \( f(w) \) is nondecreasing for \( w \in \left[ 0, \frac{b\tau}{n/2} \right] = [0, \frac{1}{8}] \) and \( f \left( \frac{b\tau}{n/2} \right) = f \left( \frac{1}{8} \right) = \frac{1}{16} \geq f(w) \) for \( w \in \left[ \frac{b\tau}{n/2}, \frac{1}{8} \right] = \left[ \frac{1}{8}, \frac{1}{4} \right] \), and

(iii)

\[
\int_{n/2 - \tau}^{n/2} (n/2 - s) f \left( \frac{bs}{n/2 - \tau} \right) ds = \int_{1/4}^{1/2} (1/2 - s) f(s) ds
\]

\[
= \int_{1/4}^{1/2} (1/2 - s) \left( \frac{2045s - 509}{36} \right) ds \approx 0.149884
\]

\[
\leq 0.2441 \cdot \frac{125}{512} = \frac{8b - f \left( \frac{b\tau}{n/2} \right)}{8} (n^2 - 4\tau^2).
\]
Since the hypotheses of Theorem 5.1 are satisfied, the antiperiodic boundary value problem
\[ x'' + f(x) = 0, \quad t \in (0, n), \quad x(0) + x(n) = 0, \quad x'(0) + x'(n) = 0 \]
has at least one antisymmetric solution \( x^* \) with
\[ x(0) \geq \frac{1}{4} \quad \text{and} \quad x \left( \frac{1}{4} \right) \leq 1. \]

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