Nonlinear Supersymmetry in Quantum Mechanics: 
Algebraic Properties and Differential Representation

A. A. ANDRIANOV\textsuperscript{a,b,*} and A. V. SOKOLOV\textsuperscript{b,†}
\textsuperscript{a} INFN, Sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy
\textsuperscript{b} V.A.Fock Institute of Physics, Sankt-Petersburg State University, 198504 Sankt-Petersburg, Russia

Abstract

We study the Nonlinear (Polynomial, $N$-fold,...) Supersymmetry algebra in one-dimensional QM. Its structure is determined by the type of conjugation operation (Hermitian conjugation or transposition) and described with the help of the Super-Hamiltonian projection on the zero-mode subspace of a supercharge. We show that the SUSY algebra with transposition symmetry is always polynomial in the Super-Hamiltonian if supercharges represent differential operators of finite order. The appearance of the extended SUSY with several (complex or real) supercharges is analyzed in details and it is established that no more than two independent supercharges may generate a Nonlinear superalgebra which can be appropriately specified as $\mathcal{N} = 2$ SUSY. In this case we find a non-trivial hidden symmetry operator and rephrase it as a non-linear function of the Super-Hamiltonian on the physical state space. The full $\mathcal{N} = 2$ Non-linear SUSY algebra includes ”central charges” both polynomial and non-polynomial (due to a symmetry operator) in the Super-Hamiltonian.

\textit{PACS}: 03.65.Ge; 03.65.Fd, 11.30.Pb

\textit{Key words}: Supersymmetric Quantum Mechanics, Nonlinear Supersymmetry, Quasi-solvability, Hidden symmetry.

1. Introduction

Supersymmetric Quantum Mechanics \cite{1,2} has been well proven as providing efficient non-perturbative methods to explore new isospectral quantum systems \cite{3-9} (see reviews \cite{10-14} and references therein) and to design nuclear potentials with required properties \cite{15,16} as

\textsuperscript{*}andrianov@bo.infn.it
\textsuperscript{†}sokolov@mph.phys.spbu.ru
well as to find the SUSY induced hidden dynamical symmetries [17]-[19] and, more specifically, to search for new exactly or quasi-exactly [20, 21] solvable problems in QM [11]-[13], [17, 19], [22]-[27].

When being written in the fermion number representation the one-dimensional SUSY QM assembles a pair of isospectral Hamiltonians $h^+$ and $h^-$ into the matrix Schrödinger operator, a Super-Hamiltonian,

$$
H = \begin{pmatrix}
    h^+ & 0 \\
    0 & h^-
\end{pmatrix} = \begin{pmatrix}
    -\partial^2 + V_1(x) & 0 \\
    0 & -\partial^2 + V_2(x)
\end{pmatrix} \equiv -\partial^2 \mathbf{1} + \mathbf{V}(x),
$$  

(1)

where $\partial \equiv d/dx$. The isospectral connection between components of the Super-Hamiltonian is provided by intertwining relations with the help of Crum-Darboux (see [30] and references therein) differential operators $q^{\pm}$,

$$
h^+ q^+ = q^+ h^-, \quad q^- h^+ = h^- q^-, \quad q^- q^+ = q^+ q^-, \quad q^- q^- = q^+ q^+ = 0.
$$  

(2)

which, in the framework of SUSY QM, are components of the supercharges,

$$
Q = \begin{pmatrix}
    0 & q^+ \\
    0 & 0
\end{pmatrix}, \quad \bar{Q} = \begin{pmatrix}
    0 & 0 \\
    q^- & 0
\end{pmatrix}, \quad Q^2 = \bar{Q}^2 = 0.
$$  

(3)

The isospectral shift (2) entails the conservation of supercharges or the supersymmetry of the Super-Hamiltonian,

$$
[H, Q] = [H, \bar{Q}] = 0,
$$  

(4)

which represents the basis of the SUSY algebra. However its algebraic closure is given, in general, by a non-linear SUSY relation,

$$
\{Q, \bar{Q}\} = \mathcal{P}(H),
$$  

(5)

where $\mathcal{P}(H)$ is a function of the Super-Hamiltonian. In this extended form the Polynomial (or Higher-derivative) SUSY algebra was systematically introduced in [28, 29]: its supercharges were realized by certain $N$-th order differential operators,

$$
q^\pm_N = \sum_{k=0}^N w^\pm_k(x) \partial^k, \quad w^\pm_N = \text{const} \equiv (\mp 1)^N.
$$  

(6)

The coefficient functions $w^\pm_k(x)$ are, in general, complex and sufficiently smooth. If they are real then from the hermiticity of the Hamiltonians and from Eqs. (2) it follows that, $q^- \equiv (q^\dagger)^\dagger = (q^+)^\dagger$ with the notations: $\dagger$ for the Hermitian conjugation and $\dagger$ for transposition. But in the complex case the four types of SUSY algebra can be introduced, based on four intertwining

---

1 In a different context certain higher derivative SUSY charges can be also associated with higher-order Darboux-Crum transformations [30].
operators and, respectively, four supercharges,

\[
q^- = (q^+)^\dagger, \quad \bar{Q} = Q^\dagger, \\
q_c^- = (q^+)^\dagger, \quad Q_c = Q^\dagger, \\
q_c^+ = (q^-)^\dagger = (q^+)^*, \quad Q_c = Q^*,
\]

(7)

where * obviously stands for the complex conjugation of coefficient functions. These algebras are generated by the pairs:

\[
\mathcal{A}_1 = (\bar{Q}, Q), \quad \mathcal{A}_2 = (\bar{Q}_c, Q), \\
\mathcal{A}_3 = (\bar{Q}, Q_c), \quad \mathcal{A}_4 = (\bar{Q}_c, Q_c).
\]

(8)

These sets are eventually united in the complex, nonlinear \( \mathcal{N} = 2 \) supersymmetry (see Sec. 5).

Recently the Polynomial (or Higher-derivative) SUSY algebra has attracted much interest \[31\]-\[50\] being a natural algebraic realization of the ladder \[4\] or dressing chain \[51\] algorithms. It was rediscovered under the name of Nonlinear SUSY \[26\] \[43\] \[49\] and of \( \mathcal{N} \)-fold SUSY \[23\] \[44\] \[45\] \[46\] \[47\] \[48\]. Perhaps the label of Nonlinear SUSY (which we adopt further on) serves better to reflect its essence as one could easily produce non-polynomial examples by means of limiting procedure applied to a Polynomial SUSY algebra when supercharges would become pseudo-differential operators or by considering SUSY algebras with complex-valued supercharges (Sec. 3 and 5).

Meantime it was claimed in \[46\] \[48\] that there exists a \( \mathcal{N} \)-fold SUSY which generalizes the Polynomial SUSY in a non-trivial way. To be precise the theorem formulated and proven in \[46\] states the following.

**Aoyama-Sato-Tanaka (AST) theorem**

*Let \( \phi_n^\pm(x) (n = 1, \cdots N) \) be two sets of \( N \) linearly independent functions, zero-modes of the supercharge components \( \mathcal{A} \),

\[
q_N^\pm \phi_n^\pm = 0, \quad q_N^- = (q_N^+)^\dagger.
\]

(9)

Then the following propositions hold\(^2\)

1) The Hamiltonians \( h^\pm \) have finite matrix representations when acting on the set of functions \( \phi_n^\pm(x) \),

\[
h^\pm \phi_n^\pm = \sum_m \bar{S}_m^\pm \phi_m^\pm.
\]

(10)

\(^2\)The first proposition is a necessary condition for the Hamiltonian system to be quasi-exactly solvable \[20\] \[21\] and it was investigated recently \[33\] \[51\] \[55\] within the notion of “conditional symmetry”.


2) With the help of the $N \times N$ matrices $\bar{S}^\pm$, the SUSY algebra closure takes the general form,

$$\{Q, Q^\dagger\} = \begin{pmatrix} \det M^+_N(h^+) + \pi^+_N q^-_N & 0 \\ 0 & \det M^-_N(h^-) + \pi^-_2 q^+_N \end{pmatrix},$$

(11)

where

$$Q = \begin{pmatrix} 0 & q^+_N \\ 0 & 0 \end{pmatrix}, \quad M^\pm_N(E) \equiv E I - \bar{S}^\pm,$$

(12)

and $\pi^\pm_{1,2}$ are differential operators of lower orders $N_{1,2} \leq N-1$ generating at least one more SUSY algebra (intertwining relations) for the same Super-Hamiltonian $H$, Eq. (11), $[H, P_{1,2}] = [H, \bar{P}_{1,2}] = 0$ with supercharges,

$$P_{1,2} = \begin{pmatrix} 0 & \pi^+_N \\ 0 & 0 \end{pmatrix}; \quad \bar{P}_{1,2} = (P_{1,2})^\dagger; \quad \pi^-_{1,2} = (\pi^+_N)^\dagger.$$

(13)

3) In the case of non-vanishing $\pi^\pm_{1,2}$ the SUSY algebra (11) coexists at least with one Polynomial SUSY of lower order $N_{1,2} \leq N$. 
4) If for given $h^\pm$ the $N$-fold supercharges are uniquely determined then $\pi^\pm_{1,2}$ must be zero and the superalgebra closure leads to the Polynomial SUSY of order $N$,

$$\{Q, Q^\dagger\} = \det M^+_N(H) = \det M^-_N(H) \equiv P_N(H).$$

(14)

Certainly the AST theorem is helpful to make links to quasi-exact solvability of particular quantum Hamiltonians [25]. However it does not explain the origin of generators of “small” supersymmetries $P_{1,2}$. It does not elucidate also the relationship between them and between the matrices $\bar{S}^+$ and $\bar{S}^-$. As well it does not give a hint on what is the maximal order of a coexisting Polynomial SUSY and how many supercharges may commute with a given Super-Hamiltonian. From (11) it is evident that the operators $\pi^+_N q^-_N$ and $\pi^-_2 q^+_N$ are related to conserved symmetries of the Hamiltonians $h^\pm$. However neither genuine hermitian symmetry generators nor the very meaning of such symmetries have not been obtained in the framework of the AST theorem.

The main aim of the present work is to clarify the above mentioned, missing points and furthermore to prove the following.

1. For a given pair of isospectral systems intertwined by differential operators of order $N$ there is always a choice of certain intertwining operators with real coefficients (not necessarily unique) which lead to supercharges of a Polynomial SUSY of the same order $N$. Thereby the $N$-fold SUSY of [15, 16] for a given quantum system always coexists with a Polynomial SUSY of the same order (and possibly few other polynomial SUSY of different orders).

---

3The “Mother Hamiltonian” in the terminology of [35, 36].
4It is just an algebra of $A_1$ type with $Q = Q^\dagger$. 
2. The complex extension of Nonlinear SUSY may bring a SUSY algebra different from a polynomial one just containing non-trivial symmetry operators. These differential operators of odd order can be replaced by non-polynomial functions of a Super-Hamiltonian being defined in the Hilbert space spanned on eigenfunctions of the Super-Hamiltonian. Thus for the same Super-Hamiltonian one can simultaneously introduce the nonlinear SUSY in both a polynomial and a non-polynomial form. In particular it covers the propositions of the AST theorem.

3. Among the (infinite) variety of supercharges of type \( Q \) (or of type \( \bar{Q} \)) commuting with a given Super-Hamiltonian one can systematically find the optimal set of (no more than) two basic SUSY generators which are differential operators of even and odd order with real coefficients. Respectively all other supercharges of type \( Q \) (or of type \( \bar{Q} \)) represent linear combinations of basic supercharge(s) with coefficient(s) polynomial in the Super-Hamiltonian. For two essentially independent supercharges a dynamical symmetry for the Super-Hamiltonian arises and the related symmetry operator is unique up to a multiplier polynomial in the Super-Hamiltonian.

4. There is a more efficient formulation of the AST theorem which manifestly uses the emerging dynamical symmetry for a Super-Hamiltonian with two supercharges and uniquely specifies the relationship between \( q^\pm, \pi_1^\pm \) and \( \pi_2^\pm \) and between the matrices \( S^+ \) and \( S^- \).

5. For isospectral systems with two independent supercharges the notion of irreducibility for Polynomial SUSY formulated in [29] does not characterize firmly potentials and the same system may be well described by a more reducible and less reducible SUSY algebra.

All theorems and constructions are exemplified by means of an exactly solvable system of second order.

2. Superalgebras with transposition symmetry

The superalgebras with real coefficient functions in the differential representation of supercharges as well as the \( A_{2,3} \) complex superalgebras have the transposition symmetry, \( Q_c = Q^t \). The following theorem is valid for these superalgebras (compare with the AST theorem).

**Theorem on SUSY algebras with T-symmetry**

*Let us again introduce two sets of \( N \) linearly independent functions \( \phi_n^\pm(x) \) \((n = 1, \cdots, N)\) which represent complete sets of zero-modes of the supercharge components \( \mathbf{6} \),

\[
q_N^\pm \phi_n^\pm = 0, \quad q_N^- = (q_N^+)^t. \tag{15}
\]
Then:
1) the Hamiltonians $h^\pm$ have finite matrix representations when acting on the set of functions $\phi^\pm_n(x)$,

$$h^\pm \phi^\pm_n = \sum_m S^\pm_{nm} \phi^\mp_m,$$

(16)

2) the SUSY algebra closure with $\bar{Q}_c = Q^t$ takes the polynomial form,

$$\{Q, Q^t\} = \det [EI - S^+]_{E=H} = \det [EI - S^-]_{E=H} = \mathcal{P}_N(H),$$

(17)

irrespectively on whether the supercharge of order $N$ is unique or there exist several supercharges for a given Super-Hamiltonian $H$.

We stress that the matrix $S^-$ is the same as in the AST theorem, $S^- = \bar{S}^-$ whereas the matrix $S^+$ is different$^5$ from $\bar{S}^+$ due to Eq. (15).

The proof of the first statement of the theorem is analogous to that one of the AST theorem. Namely, one has to act by the operator intertwining relations (2) on the zero-mode functions $\phi^\pm_n(x)$,

$$q^\pm_N h^\mp \phi^\pm_n = h^\pm q^\mp_N \phi^\pm_n = 0,$$

(18)

and conclude therefrom that $h^\pm \phi^\pm_n$ is also a zero-mode solution, i.e. can be expressed as a linear combination (16) of a complete set of $\phi^\pm_n(x)$.

The proof of the second part of this theorem is based on the properties assembled into the lemma.

Lemma

Let $\lambda^\pm_1, ..., \lambda^\pm_N$ be two sets of eigenvalues of matrices $S^\pm$ as being introduced in the above formulated theorem. Then there exist two sets of first-order differential operators $r^\pm_1, ..., r^\pm_N$ such that:

1) they have the canonical form,

$$r^\pm_l = \mp \partial + \chi^\pm_l(x), \quad l = 1, ..., N,$$

(19)

where the functions $\chi^\pm_l(x)$ may be complex and/or singular at some points;

2) the factorizations hold,

$$q^\pm_N = r^+_1 \cdots r^+_N, \quad q^-_N = r^-_1 \cdots r^-_N;$$

(20)

3) the chain relations take place,

$$(r^+_l)^t \cdot r^+_l + \lambda^+_l = r^+_l \cdot (r^+_l)^t + \lambda^+_l \equiv h^+_l, \quad l = 1, ..., N - 1$$

$$r^+_N \cdot r^+_N + \lambda^+_N = h^+_N,$$

$$r^+_1 \cdot (r^+_1)^t + \lambda^+_1 = h^+_1 \equiv h^+_0, \quad l = 1, ..., N - 1$$

From the definitions (9) and (15) it follows that the spectra of two matrices $S^+$ and $S^-$ are mutually complex conjugated.
4) the intermediate Hamiltonian operators have the Schrödinger form,

\[ h_{l}^{\pm} = -\partial^2 + v_1^{\pm}(x); \quad v_1^{\pm}(x) = (\chi_l^{\pm}(x))^2 \mp (\chi_l^{\pm}(x))' + \lambda_l^{\pm} = (\chi_{l+1}^{\pm}(x))^2 \mp (\chi_{l+1}^{\pm}(x))' + \lambda_{l+1}^{\pm}, \] (22)

but, in general, with complex and/or singular potentials;

5) the intertwining relations are valid,

\[ h_{l-1}^{\pm} \cdot r_{l}^{\mp} = r_{l}^{\mp} \cdot h_{l}^{\pm}, \quad (r_{l}^{\mp})' \cdot h_{l-1}^{\pm} = h_{l}^{\pm} \cdot (r_{l}^{\mp})'. \] (23)

The proof of this lemma is based on the quasi-diagonalization of matrices \( S^{\pm} \), i.e., on their reduction to the Jordan canonical form \( \tilde{S}^{\pm} \) which is block-diagonal and contains the Jordan cells with eigenvalues on the main diagonal and unities on the upper subdiagonal. This diagonalization can be realized by nondegenerate linear transformations \( \Omega^{\pm} \) of the zero-mode sets which induce the similarity transformations of matrices \( S^{\pm} \),

\[
\tilde{\phi}_{l}^{\pm} = \sum_{m=1}^{N} \Omega_{lm}^{\pm} \phi_{m}^{\pm}, \quad h^{\pm} \tilde{\phi}_{l}^{\pm} = \sum_{m=1}^{N} \tilde{S}_{lm}^{\pm} \tilde{\phi}_{m}^{\pm}, \\
\tilde{S}^{\pm} = \Omega^{\pm} S^{\pm} (\Omega^{\pm})^{-1}. \] (24)

Certainly it is sufficient to elaborate the factorization of the operator \( q_{N}^{-} \). The last line in the matrix \( \tilde{S}^{\pm} \) contains the Hamiltonian eigenvalue \( \lambda_{N}^{+} \),

\[ h^{+} \tilde{\phi}_{N}^{-} = \lambda_{N}^{+} \tilde{\phi}_{N}^{-}. \] (25)

Next we define,

\[ r_{N}^{-} \tilde{\phi}_{N}^{-} = 0, \quad \chi_{N}^{-} \equiv -\left( \frac{\tilde{\phi}_{N}^{-}}{\phi_{N}^{-}} \right)'. \] (26)

From Eq. (25) it follows that

\[ h^{+} \equiv h^{+}_{N} = (r_{N}^{-})' \cdot r_{N}^{-} + \lambda_{N}^{+}. \] (27)

Furthermore, the intermediate Schrödinger-like Hamiltonian can be introduced,

\[ h^{+}_{N-1} = r_{N}^{-} \cdot (r_{N}^{-})' + \lambda_{N}^{+} = -\partial^2 + (\chi_{N}^{-})^2 + (\chi_{N}^{-})' + \lambda_{N}^{+}, \] (28)

which is obviously involved in the intertwining relations with \( h_{N}^{+} \),

\[ h_{N-1}^{+} \cdot r_{N}^{-} = r_{N}^{-} \cdot h_{N}^{+}, \quad (r_{N}^{-})' \cdot h_{N-1}^{+} = h_{N}^{+} \cdot (r_{N}^{-})'. \] (29)

The combination of Eq. (15) and Eq. (26) yields the factorization,

\[ q_{N}^{-} = q_{N-1}^{-} \cdot r_{N}^{-}. \] (30)
Indeed if (15) and (26) are valid then in (6)

\[
    w_0^-(x) = \sum_{k=1}^{N} w_k^-(x) \xi_k(x), \quad \xi_k(x) \equiv (\hat{\partial} - \chi^-_N)^{k-1}\chi^-_N,
\]

where we have introduced the notation \(\hat{\partial}\) for the derivative of coefficient functions to make a clear distinction from the differential operator \(\partial\). Therefore

\[
    q^-_N = \sum_{k=1}^{N} w_k^-(x) \left( \partial^k + (\hat{\partial} - \chi^-_N)^{k-1}\chi^-_N \right).
\]

Respectively the factorization is realized in each component,

\[
    \left( \partial^k + (\hat{\partial} - \chi^-_N)^{k-1}\chi^-_N \right) = \left[ \partial^{k-1} - \sum_{m=0}^{k-2} \partial^{k-m-2}\xi_{m+1} \right] (\partial + \chi^-_N).
\]

Now it is straightforward to show that:

1) the intertwining relation holds,

\[
    h^- \cdot q^-_{N-1} = q^-_{N-1} \cdot h^+_{N-1},
\]

2) the functions,

\[
    \psi^-_n = r^-_N \hat{\psi}^-_n, \quad 1 \leq n \leq N - 1,
\]

form the complete, linearly independent set of solutions of the equation \(q^-_{N-1}\psi = 0\);

3) the matrix \(\hat{S}^+_{N-1}\) which is uniquely determined from relations,

\[
    h^+_{N-1}\psi^-_n = \sum_{m=1}^{N-1} (\hat{S}^-_{N-1})_{nm} \psi^-_m,
\]

in fact, is derived from \(\hat{S}^+\) after deleting of the last column and the last line: thereby the matrix \(\hat{S}^+_{N-1}\) still has the Jordan canonical form and its spectrum consists of \(\lambda_1^+, \ldots, \lambda_{N-1}^-\).

The first two statements are direct consequences of the basic relations (2) and (15) respectively, whereas the third one can be obtained when acting by the operator \(r^-_N\) on the definition (23) of the matrix \(\hat{S}^+\).

Thus we have reduced the factorization problem of order \(N\) to the latter one of order \(N - 1\) having proved the statements of the lemma on this step. Evidently one can proceed recursively further on and prove completely the lemma by induction.

In turn, the proof of the above formulated Theorem uses the factorization and intertwining relations provided by the Lemma,

\[
    q^+_{N} \cdot q^-_{N} = (r^+_N)^t \cdot \cdots \cdot (r^+_1)^t \cdot r^-_N = (r^-_N)^t \cdot \cdots \cdot (r^-_2)^t \cdot (h^+_N - \lambda_1^+) \cdot r^+_2 \cdots r^+_N \nonumber
\]

\[
    = (r^-_N)^t \cdot \cdots \cdot (r^-_2)^t \cdot r^+_2 \cdots r^+_N (h^+ - \lambda_1^+) = \cdots = (h^+ - \lambda_1^+) \cdots (h^+ - \lambda_N^-) \nonumber
\]

\[
    = \det [EI - S^+]_{E=h^+};
\]

\[
    q^-_{N} \cdot q^+_{N} = r^-_1 \cdots r^-_{N} \cdot (r^-_N)^t \cdot \cdots \cdot (r^-_1)^t = r^-_1 \cdots r^-_{N-1} (h^+_N - \lambda_N^-) \cdot (r^-_{N-1})^t \cdot \cdots \cdot (r^-_1)^t \nonumber
\]

\[
    = \cdots = (h^- - \lambda_1^-) \cdots (h^- - \lambda_N^-) = \det [EI - S^+]_{E=h^-}.
\]
The same relation can be derived for the factorization of the operator \( q^+_N \) into a product of \( r^+_l \) using the Jordan form for the matrix \( \tilde{S}^- \). It leads to the equivalent representation of the polynomial algebra,

\[
q^+_N \cdot q^-_N = (h^+ - \lambda_1^-) \cdots (h^+ - \lambda_N^-) = \det (E I - S^-)_{E = h^\pm}.
\]

(38)

As the relations (37) and (38) are operator ones they hold for any values of spectral parameter \( h^\pm \psi = \epsilon \psi \). Therefore the eigenvalues of matrices \( S^+ \) and \( S^- \) and their corresponding degeneracies coincide.

In general, the eigenvalues of matrices \( S^\pm \) are complex. For real components of supercharges \( q^\pm_N \), the complex eigenvalues obviously appear in complex conjugated pairs providing the real polynomial \( P_N(x) \) but for complex supercharge components \( q^\pm_N \) the resulting polynomial contains complex coefficients. In next sections we examine the non-uniqueness of a complex supercharge describing a given hermitian Super-Hamiltonian.

### 3. Several supercharges and Extended SUSY

The non-uniqueness of supercharges was mentioned for the first time\(^6\) in [56]. It was observed that for a hermitian Super-Hamiltonian \( H \) the conserved supercharges \( Q, \bar{Q} \) with complex intertwining components \( q^\pm_N \) always generate two SUSY algebras: one for their “real” parts \( K, \bar{K} \) and another one for their “imaginary” parts \( P, \bar{P} \) where the corresponding labels are referred to the real and imaginary parts of coefficients in the differential intertwining operators \( q^\pm_N = k^\pm_N + ip^\pm_{N_1} \),

\[
Q = K + iP; \quad [H, Q] = [H, K] = [H, P] = [H, \bar{Q}] = [H, \bar{K}] = [H, \bar{P}] = 0.
\]

(39)

Evidently the conjugated operators can be defined uniquely, \( \bar{K} = K^\dagger = K^\dagger \), \( \bar{P} = P^\dagger = P^\dagger \), independently on what a choice is taken from (8) for the operators \( Q, \bar{Q} \). One can always employ the normalization (6) of the senior derivative in \( q^\pm_N \) on a real constant. Then the second supercharge \( P \) appears to be a differential operator of lower order \( N_1 < N \).

The appearance of the second supercharge conventionally implies the extension of SUSY algebra. To close the algebra one has to include all anticommutators between supercharges, \( i.e. \) the full algebra based on two supercharges \( K \) and \( P \) with real intertwining operators. Two supercharges generate two Polynomial SUSY,

\[
\{ K, K^\dagger \} = \bar{P}_N(H), \quad \{ P, P^\dagger \} = \bar{P}_{N_1}(H).
\]

(40)

\(^6\)See the related Section in the E-archive version of [56] as it was eliminated from the final journal paper under the severe pressure of a referee.
This SUSY algebra has to be embedded into a $\mathcal{N} = 2$ SUSY\textsuperscript{7}. The closure of the extended, $\mathcal{N} = 2$ SUSY algebra is given by

$$
\{P, K^\dagger\} \equiv \mathcal{R} = \begin{pmatrix}
p_N^+ k_N & 0 \\
0 & k_N p_N^+
\end{pmatrix},
$$

$$
\{K, P^\dagger\} \equiv \mathcal{\bar{R}} = \begin{pmatrix}
k_N p_N^+ & 0 \\
0 & p_N^+ k_N
\end{pmatrix}.
$$

(41)

Evidently the components of operators $\mathcal{R}, \mathcal{\bar{R}} = \mathcal{R}^\dagger = \mathcal{R}^t$ are differential operators of $N + N_1$ order commuting with the Hamiltonians $h^\pm$, hence they form symmetry operators $\mathcal{R}, \mathcal{\bar{R}}$ for the Super-Hamiltonian. However, in general, they are not polynomials of the Hamiltonians $h^\pm$ and these symmetries impose certain constraints on potentials\textsuperscript{8}.

All four operators $\hat{P}_N(H), \hat{P}_{N_1}(H), \mathcal{R}, \mathcal{\bar{R}}$ commute each to other. Moreover the hermitian matrix describing this $\mathcal{N} = 2$ SUSY,

$$
\mathcal{Z}(H) = \begin{pmatrix}
\hat{P}_N(H) & \mathcal{R} \\
\mathcal{\bar{R}} & \hat{P}_{N_1}(H)
\end{pmatrix}, \quad \det[\mathcal{Z}(H)] = \hat{P}_N \hat{P}_{N_1} - \mathcal{R} \mathcal{\bar{R}} = 0,
$$

(42)

is degenerate. Therefore it seems that the two supercharges are not independent and by their redefinition (unitary rotation) one might reduce the extended SUSY to an ordinary $\mathcal{N} = 1$ one. However such rotations cannot be global and must use non-polynomial, pseudo-differential operators as “parameters”. Indeed, the operator components of the “central charge” matrix $\mathcal{Z}(H)$ have different order in derivatives. Thus, globally the extended nonlinear SUSY deals with two sets of supercharges but when they act on a given eigenfunction of the Super-Hamiltonian $H$ one could, in principle, perform the energy-dependent rotation and eliminate a pair of supercharges. Nevertheless this reduction can be possible only after the constraints on potentials have been resolved.

Let us find the formal relation between the symmetry operators $\mathcal{R}, \mathcal{\bar{R}}$ and the Super-Hamiltonian. These operators can be decomposed into a hermitian and an antihermitian parts,

$$
B \equiv \frac{1}{2}(\mathcal{R} + \mathcal{\bar{R}}) \equiv \begin{pmatrix}
b^+ & 0 \\
0 & b^-
\end{pmatrix}, \quad i\mathcal{E} \equiv \frac{1}{2}(\mathcal{R} - \mathcal{\bar{R}}) \equiv i \begin{pmatrix}
e^+ & 0 \\
0 & e^-
\end{pmatrix}.
$$

(43)

\textsuperscript{7}There is a misinterpretation concerning the classification of extended SUSY in QM. The conventional $\mathcal{N} = 1$ SUSY deals with non-hermitian \textit{nilpotent} supercharges $Q, \bar{Q}$ whereas the $\mathcal{N} = 2$ SUSY should employ two pairs of nilpotent supercharges $Q_j, \bar{Q}_j$ satisfying the extended SUSY algebra with certain central charges. We are grateful to A. Smilga for the discussion of this point. However there are papers (see discussion in [14]) where the SUSY QM algebra is defined as $\mathcal{N} = 2$ SUSY in terms of hermitian supercharges $Q_1 = Q + \bar{Q}, Q_2 = i(Q - \bar{Q})$. We would like to stress that an elementary SUSY charge is \textit{nilpotent} carrying fermion quantum numbers. Moreover a nontrivial dynamics cannot be obtained with one real SUSY charge (as it was recently mentioned in [14]).

\textsuperscript{8}Such type of symmetries in one-dimensional QM and their possible relation to the Lax method in the soliton theory was discussed in [20, 53].
The operator $B$ is a differential operator of even order and therefore it may be a polynomial of the Super-Hamiltonian $H$. But if the operator $E$ does not vanish identically it is a differential operator of odd order and cannot be realized by a polynomial of $H$.

The first operator plays essential role in the one-parameter non-uniqueness of the SUSY algebra. Indeed, one can always redefine the higher-order supercharge as follows,

$$K^{(c)} = K + \zeta P, \quad \big\{ K^{(c)}, K^{(c)\dagger} \big\} = \hat{P}^{(c)}_N (H),$$

keeping the same order $N$ of Polynomial SUSY for arbitrary real parameter $\zeta$. From (44) one gets,

$$2\zeta B (H) = \hat{P}^{(c)}_N (H) - \hat{P}_N (H) - \zeta^2 \hat{P}_N^1 (H),$$

thereby the hermitian operator $B$ is a polynomial of the Super-Hamiltonian of the order $N_b \leq N - 1$. Let’s use it to unravel the Super-Hamiltonian content of the operator $E$,

$$E^2 (H) = \hat{P}_N (H) \hat{P}_N^1 (H) - B^2 (H),$$

which follows directly from (42) and (43). As the (nontrivial) operator $E (H)$ is a differential operator of odd order $N_e$ it may have only a realization non-polynomial in $H$ being a square root of (46) in an operator sense. This operator is certainly non-trivial if the sum of orders $N + N_1$ of the operators $\hat{k}^\pm_N$ and $p^{\pm}_{N_1}$ is odd and therefore $N_e = N + N_1$. For an even sum $N + N_1$ we cannot in general make any definite conclusion concerning the non-triviality of $E (H)$. However it will be shown in Sec. 7 that if the symmetry operator is non-zero then for any choice of the operators $\hat{k}^\pm_N$ and $p^{\pm}_{N_1}$ an optimal set of independent supercharges (possibly of lower orders) can be obtained which is characterized by an odd sum of their orders.

The existence of a nontrivial symmetry operator $E$ commuting with the Super-Hamiltonian results in common eigenstates which however are not necessarily physical wave functions. In general they may be combinations of two solutions of the Shrödinger equation with a given energy, the physical and unphysical ones. But if the symmetry operator $E$ is hermitian in respect to the scalar product of the Hilbert space spanned on the eigenfunctions of the Super-Hamiltonian $H$ then both operators have a common set of physical wave functions. This fact imposes quite rigid conditions on potentials.

In particular, for intertwining operators with sufficiently smooth coefficient functions having constant asymptotics at large coordinates the symmetry operator $E$ has the similar properties and is evidently hermitian. In this case one has non-singular potentials with constant asymptotics at large $x$ and therefore a continuum energy spectrum of $H$ with wave functions satisfying the scattering conditions. Thus the incoming and outgoing states, $\psi_{in} (x)$ and $\psi_{out} (x)$, at large $x$ are conventionally represented by combinations of plane waves which are solutions of the Schrödinger equation for a free particle,

$$\psi (x)|_{x \to -\infty} \longrightarrow \exp (ik_{in} x) + R (k_{in}) \exp (-ik_{in} x),$$
$$\psi (x)|_{x \to +\infty} \longrightarrow (1 + T (k_{out})) \exp (ik_{out} x),$$

(47)
where the reflection, \( R(k_{in}) \), and transmission, \( T(k_{out}) \), coefficients are introduced. Since the symmetry is described by a differential operator of odd order which at large \( x \) tends to an antisymmetric operator with constant coefficients the eigenstates of this operator at large coordinates approach to individual plane waves \( \sim \exp(\pm ikx) \) with opposite eigenvalues \( \sim \pm kf(k^2) \) and cannot be their combinations. Hence the eigenstate of the Super-Hamiltonian with a given value of the operator \( E \) may characterize only the transmission and cannot have any reflection, \( R(k_{in}) = 0 \). We conclude that the corresponding potentials \( V_{1,2} \) in (1) inevitably belong to the class of transparent or reflectionless ones [57].

As the symmetry operator \( E \) is hermitian its eigenvalues are real but, by construction, its coefficients are purely imaginary. Since the wave functions of bound states of the system \( H \) can be always chosen real we conclude that they must be zero-modes of the operator \( E(H) \),

\[
E(H)\psi_i = E(E_i)\psi_i = 0, \quad \tilde{P}_N(E_i)\tilde{P}_{N_i}(E_i) - B^2(E_i) = 0, \quad (48)
\]

which represents the algebraic equation on bound state energies of a system possessing two supersymmetries. Among solutions of (48) one reveals also a zero-energy state at the bottom of continuum spectrum. On the other hand one could find also the solutions which are not associated to any bound state. The very appearance of such unphysical solutions is accounted for by the trivial possibility to replicate supercharges by their multiplication on the polynomials of the Super-Hamiltonian and it is discussed in Sec. 6.

4. Example: \( N = 2, N_1 = 1 \)

Let us examine the algebraic structure of the simplest non-linear SUSY with two supercharges,

\[
k^\pm \equiv \partial^2 \mp 2f(x)\partial + \tilde{b}(x) \mp f'(x); \quad p^\pm \equiv \mp \partial + \chi(x), \quad (49)
\]

induced by the complex supercharge of second order in derivatives [56]. The supersymmetries (39) generated by \( K, \bar{K} \) and \( P, \bar{P} \) prescribe that

\[
V_{1,2} = \chi^2 \mp \chi' = \mp 2f' + f^2 + \frac{f''}{2f} - \left( \frac{f'}{2f} \right)^2 - \frac{d}{4f^2} - a, \\
\tilde{b} = f^2 - \frac{f''}{2f} + \left( \frac{f'}{2f} \right)^2 + \frac{d}{4f^2}, \quad (50)
\]

where \( \chi, f \) are real functions and \( a, d \) are real constants. The related superalgebra closure for \( K, \bar{K} \) and \( P, \bar{P} \) takes the form,

\[
\{K, \bar{K}\} = (H + a)^2 + d, \quad \{P, \bar{P}\} = H, \quad (51)
\]

the latter one clarifies the role of constants \( a, d \).
The compatibility of two supersymmetries is achieved on solutions of the following equations

\[ \chi = 2f + \chi_0, \quad f^2 + \frac{f''}{2f} - \left( \frac{f'}{2f} \right)^2 - \frac{d}{4f^2} - a = \chi^2 = (2f + \chi_0)^2, \quad (52) \]

where \( \chi_0 \) is an arbitrary real constant. The latter one represents a nonlinear second-order differential equation which solutions are parameterized by two integration constants. Therefore as it was advertised the existence of two SUSY constrains substantially the class of potentials for which they may hold.

Let us use the freedom to redefine the higher-order supercharge \( (44) \) for eliminating the constant \( \chi_0 \) in \( (52) \). After this simplification the equation \( (52) \) is integrated into the following, first-order one,

\[ \chi = 2f; \quad (f')^2 = 4f^4 + 4af^2 + 4G_0f - d \equiv \Phi_4(f), \quad (53) \]

where \( G_0 \) is a real constant.

The solutions of this equation are elliptic functions which can be easily found in the implicit form,

\[ \int_{f_0}^{f(x)} \frac{df}{\sqrt{\Phi_4(f)}} = \pm(x - x_0), \quad (54) \]

where the lower limit of integration \( f_0 \) and \( x_0 \) are real constants.

It can be shown that they may be nonsingular in three situations.

a) The polynomial \( \Phi_4(f) \) has four different real roots \( f_1 \leq f_2 \leq f_3 \leq f_4 \) and \( f_0 \) is chosen between two roots \( f_2 \) and \( f_3 \). The corresponding potentials are periodic. This case will not be examined here.

b) \( \Phi_4(f) \) has three different real roots and the double root \( \beta/2 \) is either the maximal one or a minimal one,

\[ \Phi_4(f) = 4(f - \frac{\beta}{2})^2 \left( (f + \frac{\beta}{2})^2 - (\beta^2 - \epsilon) \right), \quad 0 < \epsilon < \beta^2. \quad (55) \]

Then there exists a relation between constants \( a, d, G_0 \) in terms of coefficients \( \beta, \epsilon \),

\[ a = \epsilon - \frac{3\beta^2}{2} < 0, \quad G_0 = \beta(\beta^2 - \epsilon), \quad d = \beta^2 \left( \frac{3\beta^2}{4} - \epsilon \right). \quad (56) \]

Besides the constant \( f_0 \) is taken between the double root and a nearest simple root.

c) \( \Phi_4(f) \) has two different real double roots which corresponds in \( (55), (56) \) to \( G_0 = 0, \ \beta^2 = \epsilon > 0, \ \ a = -\epsilon/2, \ \ d = -\epsilon^2/4. \) The constant \( f_0 \) is taken between the roots.

The corresponding potentials \( V_{1,2} \) are well known \[57\] and in the cases b) and c) are reflectionless, with one bound state at the energy \( (\beta^2 - \epsilon) \) and with the continuum spectrum starting from \( \beta^2 \). Respectively the scattering wave function is proportional to \( \exp(ikx) \) with \( k = \sqrt{E - \beta^2} \).

In particular, in the case b) the potentials coincide in their form and differ only by shift in the coordinate,

\[ V_{1,2} = \beta^2 - \frac{2\epsilon}{\text{ch}^2\left( \sqrt{\epsilon}(x - x_0^{(1,2)}) \right)}, \quad x^{(1,2)} = x_0 \pm \frac{1}{4\sqrt{\epsilon}} \ln \frac{\beta - \sqrt{\epsilon}}{\beta + \sqrt{\epsilon}}, \quad (57) \]
and in the case c) one of the potentials can be chosen constant,
\[ V_1 = \beta^2, \quad V_2 = \beta^2 \left( 1 - \frac{2}{\text{ch}^2(\beta(x - x_0))} \right), \]  
(58)
For these potentials one can illustrate all the relations of extended SUSY algebra.

The initial algebra is given by the relations (51). The first, polynomial symmetry operator turns out to be constant, \( B(H) = G_0 \) when taking into account (49) and (53). The second symmetry operator reads,
\[ \mathcal{E}(H) = i \left[ \mathbf{1} \partial^3 - \left( a\mathbf{1} + \frac{3}{2} V(x) \right) \partial - \frac{3}{4} V'(x) \right], \]  
(59)
in terms of the potential (1). From the identity (46) or directly from Eq. (59) one derives with the help of Eqs. (53) and (56) that,
\[ \mathcal{E}^2(H) = H \left[ (H + a)^2 + d \right] - G_0^2 = (H - E_b)^2(H - \beta^2), \]  
(60)
where \( E_b = \beta^2 - \epsilon \) is the energy of a bound state. Thus (some of) the zero modes of \( \mathcal{E}(H) \) characterize either bound states or zero-energy states in the continuum. However there exist also the non-normalizable, unphysical zero-modes corresponding to \( E = E_b, \beta^2 \). We remark that in the case c) only the Hamiltonian \( h^- \) has a bound state. Hence the physical zero modes of \( \mathcal{E}(H) \) may be either degenerate (case b), broken SUSY) or (one of them) non-degenerate (case c), unbroken SUSY).

The square root in (60) can be carried out,
\[ \mathcal{E}(H) = (H - E_b)\sqrt{H - \beta^2}. \]  
(61)
We notice that the symmetry operator (59), (61) is irreducible, i.e. the binomial \( (H - E_b) \) cannot be “stripped off” (the exact meaning of this operation see in Sec. 6). Indeed the elimination of this binomial would lead to an essentially nonlocal operator. The sign of square root in (61) is fixed from the conventional asymptotics of scattering wave functions \( \sim \exp(ikx) \) and the asymptotics \( V_{1,2} \rightarrow \beta^2 \) by comparison of this relation with Eq. (59).

When taking Eq. (61) into account one finds the non-polynomial relations of the extended SUSY algebra,
\[ \{ K, P^\dagger \} = \{ K^\dagger, P \}^\dagger = G_0 - i(H - E_b)\sqrt{H - \beta^2}. \]  
(62)
The hermitian matrix \( Z(H) \), Eq. (42) is built of the elements (51) and (62) and evidently cannot be diagonalized by a unitary rotation with elements polynomial in \( H \). Thus the algebra must be considered to be extended in the class of differential operators of finite order.

It remains to clarify the very non-uniqueness of the higher-order supercharge, namely, its role in the classification of the Polynomial SUSY. For arbitrary \( \zeta \) in (44) one obtains
\[ \{ K^{(\zeta)}, K^{(\zeta)^\dagger} \} = H^2 + (2a + \zeta^2)H + a^2 + d + 2\zeta G_0 = (H + a\zeta)^2 + d\zeta, \]
\[ a\zeta = a + \frac{1}{2}\zeta^2, \quad d\zeta = d + 2\zeta G_0 - a\zeta^2 - \frac{1}{4}\zeta^4 \equiv -\Phi_4(-\frac{\zeta}{2}), \]  
(63)
where $\Phi_4(f)$ is defined in Eq. (53).

One can see that the sign of $d_\zeta$, in general, depends on the choice of $\zeta$. For instance, let us consider the case b) when

$$d_\zeta = -\frac{1}{4} (\zeta + \beta)^2 \left[ (\zeta - \beta)^2 - 4(\beta^2 - \epsilon) \right].$$

(64)

Evidently if $\zeta$ lies in between the real roots of the last factor in (64) then $d_\zeta$ is positive and otherwise it is negative. But two real roots always exist because $\beta^2 > \epsilon$. Thereby the sign of $d_\zeta$ can be freely negative or positive without any change in the Hamiltonians. Hence in the case when the Polynomial SUSY is an extended one, with two sets of supercharges, the irreducibility or reducibility of a Polynomial SUSY algebra does not signify any invariant characteristic of potentials.

5. Complex SUSY algebras

If the intertwining operators $q^\pm$ have complex coefficients in (6) then we deal with two supercharges which we adopt to be independent (see Sec. 7 for its exact definition). One can split again the complex supercharge $Q$ into a real, $K$, and an imaginary, $P$ counterparts as in Eq. (39) and normalize them so that the intertwining operator in $K$ has a higher order in derivatives. Two SUSY algebras with transposition symmetry, $A_2$ and $A_3$, are polynomial in virtue of the Theorem of Sec. 2. In terms of real supercharges they have the following structure,

$$\{Q, \bar{Q}_c\} = \tilde{P}_N(H) + \tilde{P}_{N_1}(H) - 2E(H).$$

(65)

$$\{Q_c, \bar{Q}\} = \tilde{P}_N(H) - \tilde{P}_{N_1}(H) - 2B(H),$$

(66)

$$\bar{Q}_c = Q^t = K^\dagger + iP^\dagger; \quad Q_c = Q^* = K - iP; \quad \bar{Q} = K^\dagger - iP^\dagger.$$

(67)

Two more algebras, $A_1$ and $A_4$ can be built with a hermitian closure according to Eq. (8). In particular, the algebra $A_1$ (used in the AST theorem, Sec. 1) is completed by the following closure,

$$\{Q, \bar{Q}\} = \tilde{P}_N(H) + \tilde{P}_{N_1}(H) - 2E(H).$$

(68)

Respectively the algebra $A_4$ is completed by the relation,

$$\{Q_c, \bar{Q}_c\} = \tilde{P}_N(H) + \tilde{P}_{N_1}(H) + 2E(H).$$

(69)

When the symmetry operator $E(H)$ is nontrivial they are essentially non-polynomial (see (46)).

We conclude that for complex intertwining operators the same pair of isospectral Hamiltonians may be induced both by the polynomial SUSY algebra (40) (or (65), (66)) and by the non-polynomial one (59), (59).

All four superalgebras $A_m$ generated by $(Q_1, Q_2) \equiv (Q, Q_c)$ can be assembled into the extended $\mathcal{N} = 2$ SUSY algebra,

$$\{Q_i, \bar{Q}_j\} = \left[ (I + \tau_1)\tilde{P}_N(H) + (I - \tau_1)\tilde{P}_{N_1}(H) - \tau_22B(H) - \tau_32E(H) \right]_{ij}.$$

(70)
with $i, j = 1, 2$. It is equivalent, of course, to the algebra (40) and (41).

Let us illustrate such an algebra using the $N = 2, N_1 = 1$ example of Sec. 4. Thus the intertwining operator $q^+ = k^+ + icp^+$ is composed from the operators (49) where the constant $c$ of mass dimension is introduced from dimensional reasons. Respectively,

$$ \{Q_i, \bar{Q}_j\} = \left[ (I + \tau_1) \left( (H + a)^2 + d \right) + (I - \tau_1) c^2 H \right]_{ij}, \quad (71) $$

i.e. is manifestly non-polynomial in respect to the Super-Hamiltonian $H$.

It remains to clarify the relationship between the hermitian algebra $A_1$ determined in the AST theorem by Eq. (11) and that one given by (68). For this purpose we relate the representation (11) to the algebra with transposition symmetry, Eq. (17). In order to establish the exact correspondence the upper and lower components in the matrix $\{Q, \bar{Q}\}$ have to be treated differently. Namely for the upper component $q^+_N q^-_N$ the suitable decomposition, $q^+_N = (q^-_N)^\dagger + 2ip^+_N$, yields,

$$ q^+_N q^-_N = (q^-_N)^\dagger q^-_N = \det [EI - \tilde{S}^+]_{E=h^+} + 2ip^+_N q^-_N, \quad (72) $$

where from one reproduces the upper component of Eq. (11) after the identification $\pi^+_1 \equiv 2ip^+_N$. We stress that the matrix $\tilde{S}^+$ does not coincide with $S^+$ from Eq. (16) because in (72) $q^-_N = (q^+_N)^\dagger \neq (q^+_N)^t$. But in appropriate bases of mutually complex conjugated functions they may be related by complex conjugation, $\tilde{S}^+ = (S^+)^*$.

Similarly the lower component can be transformed into the form (11) by means of the decomposition, $q^-_N = (q^+_N)^t - 2ip^-_N$, and reads,

$$ q^-_N q^+_N = (q^+_N)^\dagger q^+_N - 2ip^-_N q^+_N = \det [EI - \tilde{S}^-]_{E=h^-} - 2ip^-_N q^+_N, \quad (73) $$

where from one obtains the lower component of Eq. (11) after the identification $\pi^-_2 \equiv -2ip^-_N = \pi^-_1$. The matrix $S^-$ is exactly as $S^-$ in the Theorem on SUSY algebras with T-symmetry, Eq. (16).

The AST decomposition (72), (73) is certainly equivalent to the representation (68) but supplies both the polynomial part and the non-polynomial symmetry operator with imaginary contributions which eventually assemble into $-2e^{\pm}(h^\pm)$. Thereby the hermitian symmetry operator $E(H)$ non-polynomial in $H$ is hidden in Eqs. (72), (73). This is why we give our favor to the representation (68) in the analysis of supersymmetries with several supercharges.

6. “Strip-off” problem

The pair of two supersymmetries analyzed in Sec. 3 and 5 may rigidly determine the class of potentials $V_{1,2}$ in (11) to a specific, transparent type of them contracting the freedom in their choice from a functional one to a parametric one. On the other hand, there exists a trivial
possibility when the intertwining operators $k^\pm_N$ and $p^\pm_{N_1}$ are related by a factor depending on the Hamiltonian,

$$k^\pm_N = F(h^\pm)p^\pm_{N_1} = p^\pm_{N_1}F(h^\mp),$$

where $F(x)$ is assumed to be a polynomial. Obviously in this case the symmetry operator $E(H)$ identically vanishes and the appearance of the second supercharge does not result in any restrictions on potentials.

More generally the orders of polynomial superalgebras and some of the roots of associated polynomials may not be involved in determination of the structure of the potentials. In particular, let the operators $k^\pm_N$ and $p^\pm_{N_1}$ be reducible to some lower-order ones $\tilde{k}^\pm_N$ and $\tilde{p}^\pm_{N_1}$,

$$k^\pm_N = F_k(h^\pm)\tilde{k}^\pm_N = \tilde{k}^\pm_NF_k(h^\mp), \quad p^\pm_{N_1} = F_p(h^\pm)\tilde{p}^\pm_{N_1} = \tilde{p}^\pm_{N_1}F_p(h^\mp),$$

where the numbers in the pairs $N, \tilde{N}$ and $N_1, \tilde{N}_1$ are simultaneously odd or even and $F_k(x), F_p(x)$ are polynomials of order $(N - \tilde{N})/2$ and $(N_1 - \tilde{N}_1)/2$. Then evidently the superalgebra generated by $\tilde{k}^\pm_N$ and $\tilde{p}^\pm_{N_1}$ equally well characterizes the Super-Hamiltonian system with the same potentials.

We have come to the problem of how to discern the nontrivial part of a supercharge and avoid multiple SUSY algebras generated by means of “dressing” (75). It can be systematically performed with the help of the following theorem.

**“Strip-off” Theorem**

Let’s admit the construction of the Theorem on SUSY algebras with T-symmetry. Then the requirement that the matrix $\tilde{S}^-$ (or $\tilde{S}^+$) generated on the subspace of zero-modes of the operator $k^+_N$ (or $k^-_N$) contains $n$ pairs (and no more) of Jordan cells with equal eigenvalues $\lambda_l$ and the sizes $\nu_l$ of a smallest cell in the $l$-th pair is necessary and sufficient to ensure for the intertwining operator $k^+_N$ (or $k^-_N$) to be represented in the factorized form:

$$k^\pm_N = \tilde{k}^\pm_N \prod_{l=1}^n (\lambda_l - h^\mp)^{\nu_l},$$

where $\tilde{k}^\pm_N$ are intertwining operators of order $\tilde{N} = N - 2 \sum_{l=1}^n \nu_l$ which cannot be decomposed further on in the product similar to (75) with $F_k(x) \neq \text{const}$.

We shall perform the proof of the theorem for $\tilde{S}^-$ only as its proof for $\tilde{S}^+$ is similar. It is based on the lemma and two remarks.

1. The matrices $\tilde{S}^\pm$ cannot contain more than two Jordan cells with the same eigenvalue $\lambda$ because otherwise the operator $\lambda - h^\pm$ would have more than two linearly independent

---

9In this theorem the intertwining operators $k^\pm_N, \tilde{k}^\pm_N$ and the parameters $\lambda_l$ may also be complex.
zero-modes (not necessarily normalizable).

**Lemma.**

In order that the intertwining operator $k^+_N$ could be factorized,

$$k^+_N = k^+_{N-2}(\lambda - h^-),$$

(77)

with $k^+_{N-2}$ being another intertwining operator of order $N - 2$, it is necessary and sufficient for the matrix $\tilde{S}^-$ to contain two Jordan cells with the same eigenvalue $\lambda$.

**Proof.** The requirement of this Lemma is sufficient because if $\tilde{S}^-$ contains two Jordan cells with the same eigenvalue $\lambda$ then the kernel of $k^+_N$ includes two linearly independent solutions of the equation $h^- \phi = \lambda \phi$. When repeating the way of proof of the Theorem on SUSY algebras with T-symmetry one can derive that $k^+_N$ is factorized in the form,

$$k^+_N = k^+_{N-2}(\partial - \chi_2(x))(\partial - \chi_1(x)),$$

(78)

where $k^+_{N-2}$ is a differential operator of order $N - 2$ and the functions $\chi_1(x)$ and $\chi_2(x)$ are chosen to provide the equal kernels of operators $\lambda - h^-$ and $(\partial - \chi_2)(\partial - \chi_1)$. As a differential operator of second order with the unit coefficient at $\partial^2$ is uniquely determined by two linearly independent elements of its kernel one concludes that

$$\lambda - h^- = (\partial - \chi_2)(\partial - \chi_1),$$

(79)

and therefore (77) is valid. At last, from the relations,

$$k^+_{N-2}h^- (\lambda - h^-) = k^+_Nh^- = h^+k^+_N = h^+k^+_{N-2}(\lambda - h^-),$$

(80)

one obtains that the operator $k^+_{N-2}$ is intertwining.

The condition (77) is also necessary as in this case the kernel of $k^+_N$ includes two linearly independent solutions of the equation $h^- \phi = \lambda \phi$ which induce two different Jordan cells with the same eigenvalue.

**Remark 2.** Within the Lemma let us put the lower lines of two Jordan cells with the eigenvalue $\lambda$ into the $i$-th and $j$-th line ($i < j$) of $\tilde{S}^-$ respectively and introduce the functions,

$$\tilde{\psi}^+_l(x) = \begin{cases} (\lambda - h^-)\tilde{\phi}^+_l, & 1 \leq l \leq i - 1, \\ (\lambda - h^-)\tilde{\phi}^+_{l+1}, & i \leq l \leq j - 2, \\ (\lambda - h^-)\tilde{\phi}^+_{l+2}, & j - 1 \leq l \leq N - 2, \end{cases}$$

(81)

where the functions $\tilde{\phi}^+_m$ form the basis of the kernel of the operator $k^+_N$ which supply the matrix $\tilde{S}^-$ with the Jordan form. Evidently the functions $\tilde{\psi}^+_l(x)$ are linearly independent because in
the opposite case a nontrivial linear combination of \( \tilde{\phi}^+_i \cdot \cdots \cdot \tilde{\phi}^+_i + 1 \), \( \cdot \cdots \cdot \tilde{\phi}^+_j - 1 \), \( \tilde{\phi}^+_j + 1 \), \( \cdot \cdots \cdot \tilde{\phi}^+_N \) must be a combination of zero-modes \( \tilde{\phi}^+_i \) and \( \tilde{\phi}^+_j \). Thus they form the complete set of solutions of the equation \( k_{N-2}^+ \tilde{\psi} = 0 \) which comes from (77). The matrix \( \tilde{S}^-_{N-2} \) which describes the Hamiltonian action on the zero-mode subspace \( \{ \tilde{\psi}_k^+(x) \} \) of the intertwining operator \( k_{N-2}^+ \), can be produced from the matrix \( \tilde{S}^- \) by deleting both the \( i \)-th and \( j \)-th columns and lines. Thereby it has a Jordan form.

Now from the Lemma and the Remarks 1 and 2 one derives all statements of the “Strip-off” Theorem.

This theorem naturally supplements the Theorem on SUSY algebras with T-symmetry as it entails the essential identity of the Jordan forms \( \tilde{S}^- \) and \( \tilde{S}^+ \) (up to transposition of certain Jordan cells).

Let us illustrate the Theorem by the Example: the matrix \( \tilde{S}^- \) for the intertwining operator \( k_3^+ \) with Jordan cells of different size having the same eigenvalue. It is generated by the operators,

\[
p^\pm = \mp \partial + \chi, \quad h^\pm = p^\mp p^\mp + \lambda, \quad k_3^+ = -p^+ p^- p^+ = p^+ (\lambda - h^-). \tag{82}
\]

Respectively:

\[
\begin{align*}
\tilde{\phi}_2^- & : \quad \begin{array}{c} p^+ \tilde{\phi}_2^- = 0 \end{array} \quad \rightarrow \quad h^- \tilde{\phi}_2^- = \lambda \tilde{\phi}_2^-; \\
\tilde{\phi}_3^- & : \quad \begin{array}{c} p^+ \tilde{\phi}_3^- \neq 0, \quad p^- p^+ \tilde{\phi}_3^- = 0 \end{array} \quad \rightarrow \quad h^- \tilde{\phi}_3^- = \lambda \tilde{\phi}_3^-; \\
\tilde{\phi}_1^- & : \quad \begin{array}{c} p^- p^+ \tilde{\phi}_1^- = \tilde{\phi}_2^- \end{array} \quad \rightarrow \quad h^- \phi_1^- = \lambda \phi_1^- + \tilde{\phi}_2^-.
\end{align*} \tag{83}
\]

Thus,

\[
\tilde{S}^- = \begin{pmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda
\end{pmatrix} \tag{84}
\]

As a consequence of the “Strip-off” Theorem one finds that the supercharge components cannot be factorized in the form (75) if the polynomial \( \tilde{P}_N(x) \) in the SUSY algebra closure (10) does not reveal the degenerate zeroes. Indeed the SUSY algebra closure contains the square of polynomial \( F(x) \), for instance,

\[
k_N^- k_N^+ = F_k(h^-) \tilde{k}_N^- k_N^+ F(h^-) = F^2_k(h^-) \tilde{P}_N(h^-), \tag{85}
\]

where \( \tilde{P}_N(x) \) is a polynomial of lower order, \( \tilde{N} \leq N - 2 \). Therefore each zero of the polynomial \( F_k(x) \) will produce a double zero in the SUSY algebra provided by (85).

Thus the absence of double zeroes is sufficient to deal with the SUSY charges non-factorizable in the sense of Eq. (75). However it is not necessary because the degenerate zeroes may well arise in the ladder (dressing chain) construction giving new pairs of isospectral potentials (see, for instance, [29] for the Polynomial SUSY of second order).

Further on we consider only the stripped-off supercharges. In this case the existence of two intertwining operators results in more equations on their coefficient functions and thereby on the
potentials in $h^{\pm}$. When they are compatible they rigidly dictate the form of potentials leaving only the parametric freedom for their choice.

Still the stripped-off supercharges do not necessarily represent an optimal set of them and provide an optimal structure of the symmetry operator $\mathcal{E}(H)$.

Let us illustrate it with the sample intertwining operators $t^{\pm} = p_{N_1}^{\pm} k_N^\mp p_{N_1}^\pm$ made of two stripped-off supercharges with components $k_N^{\pm}$ and $p_{N_1}^{\pm}$. If the differential operators $k_N^{\pm}$ and $p_{N_1}^{\pm}$ have even and odd order respectively and the operators $b^{\pm}$ are non-zero the operators $t^{\pm}$ cannot be stripped off till $k_N^{\pm}$ or $p_{N_1}^{\pm}$ because the components of symmetry operator $e^{\pm} = \pm i_2 (k_N^{\pm} p_{N_1}^{-} - p_{N_1}^{\pm} k_N^{\mp})$ are surely non-trivial and therefore the operators $k_N^{\pm} p_{N_1}^{-}$ and $p_{N_1}^{\pm} k_N^{\mp}$ are not polynomials of the Hamiltonian. One can see it manifestly for the supercharges (49) of the Example in the case b) when $G_0 \neq 0$.

Meantime the system composed of two supercharges with components $t^{\pm}$ and $p_{N_1}^{\pm}$ has the symmetry realized by the operator $\mathcal{E}_t(H)$ with components,

$$e_{t}^{\pm} = \pm i 2 (t_{N_1}^{\pm} - p_{N_1}^{\pm} t^{\mp}) = \mp i_2 (k_N^{\pm} p_{N_1}^{-} - p_{N_1}^{\pm} k_N^{\mp}) p_{N_1}^{\pm} p_{N_1}^{-} = -e^{\pm} \tilde{P}_{N_1}(h^{\pm}), \quad (86)$$

where to obtain it the commutation of the Super-Hamiltonian with operators $\mathcal{R}$, $\tilde{R}$, Eq. (21) has been used. Thus the symmetry operator $\mathcal{E}_t(H)$ contains a polynomial factor $\tilde{P}_{N_1}(H)$ which zeroes are not in general related to any bound state energies. The lesson is that the symmetry operator must be stripped-off in addition to the intertwining operators in order to avoid unphysical zeroes of the function $\mathcal{E}(E)$. In order to perform the minimization of the symmetry operator one can employ the "Strip-off" Theorem and analyze the Jordan form of the relevant Hamiltonian projection $\tilde{S}_e^{\pm}$ on the zero-mode subspace of the operator $e^{\pm}$. Then the elimination of pairs of equal eigenvalues from different Jordan cells would essentially reduce the spectrum of the projected Hamiltonian towards a set of its bound state energies.

On the other hand the intertwining operators $t^{\pm}$ represent a linear combination of $k_N^{\pm}$ and $p_{N_1}^{\pm}$ with coefficients depending on the Hamiltonian,

$$t^{\pm} = 2b^{\pm}(h^{\pm}) p_{N_1}^{\pm} - \tilde{P}_{N_1}(h^{\pm}) k_N^{\pm}. \quad (87)$$

If the polynomials $b^{\pm}$ and $\tilde{P}_{N_1}$ do not have common roots one may find the intertwining operators $t^{\pm}$ which cannot be stripped off till a combination of $k_N^{\pm}$ and $p_{N_1}^{\pm}$ of lower order in derivatives (see again the case b) in Sec. 4).

However one can easily build the equivalent supercharges $\tilde{t}^{\pm} = t^{\pm} - 2b^{\pm}(h^{\pm}) p_{N_1}^{\pm}$ which can be stripped off till $k_N^{\pm}$. Thus in order to construct the optimal basis of supercharges one should not only factorize out the polynomials of the Super-Hamiltonian but also examine their various linear combinations with coefficients polynomial in the Super-Hamiltonian.
7. Optimization of supercharges

As it follows from the previous discussion the existence of several supercharges is controlled by a non-trivial symmetry operator. If there are several SUSY generators the necessity arises: a) to introduce the notion of (in)dependence of intertwining operators; b) to find out how many independent supercharges can coexist; c) to define an optimal basis of intertwining operators.

Let us extrapolate the relation \((87)\) and define the intertwining operators \(q_i^\pm, i = 1, \ldots n\) to be dependent if and only if the polynomials \(\alpha_i^\pm(y)\) exist such that not all of them are vanishing and
\[
\sum_{i=1}^{n} \alpha_i^\pm(h^\pm)q_i^\pm = 0.
\] (88)

If the relation \((88)\) results in \(\alpha_i^\pm(y) = 0\) for all \(i\) the corresponding SUSY generators are independent. Evidently the (in)dependence of \(q_i^+\) entails the (in)dependence of \(q_i^-\) and vice versa.

The following theorem plays a key role in resolution of how many independent supercharges can commute with a given Super-Hamiltonian.

**Theorem on (in)dependence of supercharges**

Consider two non-trivial intertwining operators \(q_i^\pm, i = 1, 2\) with transposition symmetry \(q_i^+ = (q_i^-)^t\) which in general may have complex coefficients and let us normalize them in accordance to \((6)\). Then the stripped-off intertwining operators \(\tilde{q}_i^\pm\) coincide if and only if the symmetry operator made of \(q_i^\pm\) vanishes, \(q_1^+q_2^- - q_2^+q_1^- = 0\) (or equivalently \(q_1^-q_2^+ - q_2^-q_1^+ = 0\)).

The proof of this theorem is based on the Lemma.

**Lemma**

If \(q_1^+q_2^- - q_2^+q_1^- = 0\) (or equivalently \(q_1^-q_2^+ - q_2^-q_1^+ = 0\)) then the sets of Jordan cells (and their sizes) in the matrices \(\tilde{S}_1^\pm\) and \(\tilde{S}_2^\pm\) for the operators \(\tilde{q}_1^\pm\) and \(\tilde{q}_2^\pm\) are identical.

To prove its validity we first remark that evidently the symmetry operator \(q_1^+\tilde{q}_2^- - q_2^+\tilde{q}_1^-\) for stripped-off operators \(\tilde{q}_1^\pm\) and \(\tilde{q}_2^\pm\) also vanishes in virtue of intertwining relations and because the factor polynomials for \(q_i^+\) and \(q_i^-\) are equal. Therefore the operator
\[
b_{12}^+ = \tilde{q}_1^+\tilde{q}_2^- - \tilde{q}_2^+\tilde{q}_1^- = \frac{1}{2}[(\tilde{q}_1^+ + \tilde{q}_2^+)(\tilde{q}_1^- + \tilde{q}_2^-) - \tilde{q}_1^+\tilde{q}_1^- - \tilde{q}_2^+\tilde{q}_2^-]
\] (89)
is a polynomial of the Hamiltonian \(h^+\) (compare with \((43)\)). Hence, according to the "Strip-off" Theorem the matrix \(\tilde{S}_{12}^-\) on the basis of zero-modes of the operator \(b_{12}^+\) contains two (and evidently no more) Jordan cells of the same size for each eigenvalue.

Next, from the Theorems of Sec. 2 and 6 one concludes that the spectrum of the matrix \(\tilde{S}_{12}^-\) joins the spectra of \(\tilde{S}_1^+\) and \(\tilde{S}_2^+\) for the operators \(\tilde{q}_1^\mp\) and \(\tilde{q}_2^\mp\). Moreover since the latter ones are stripped off the related matrices \(\tilde{S}_i^\pm\) include one Jordan cell only for each eigenvalue.
At last, taking into account that $\ker(q_i^-) \subset \ker(q_j^+ q_i^-)$ one can derive that for each eigenvalue of $S_i^+$ a related Jordan cell with the same eigenvalue exists in $\tilde{S}_i^-$ and its size is no less than that one of the Jordan cell in $\tilde{S}_i^+$. Thus the number and sizes of Jordan cells in matrices $\tilde{S}_1^+$ and $\tilde{S}_2^+$ are the same.

As a consequence the orders of differential operators $\tilde{q}_1^+$ and $\tilde{q}_2^+$ are the same and they can be combined to form another intertwining operator of a lower order, $\sigma_1^+ = q_1^+ - \tilde{q}_2^+$. If $\sigma_1^+ \neq 0$ one can strip off and normalize this operator $\sigma_1^+ \to \tilde{\sigma}_1^+$ then apply the Lemma to the pair of operators $\tilde{\sigma}_1^+, \tilde{q}_1^+$ which symmetry operator is again trivial. As a result we prove these operators to have the same order in the contradiction with our initial construction unless the operator $\sigma_1^+ = 0$ and thereby $\tilde{q}_1^+ = \tilde{q}_2^+$. The latter completes the proof of the Theorem.

As a corollary of the Theorem one finds that for the stripped-off operators $\tilde{q}_1^+$ and $\tilde{q}_2^+$ of different order the symmetry operator $\tilde{q}_1^+ \tilde{q}_2^- - \tilde{q}_2^+ \tilde{q}_1^- \neq 0$.

Two other consequences of the Theorem concern the structure of symmetry operators and their uniqueness:

a) any symmetric (self-transposed) symmetry operator $B^+ = (B^+)^t$, $B^+ h^+ = h^+ B^+$ is a polynomial of the Hamiltonian, the latter is obtained by substituting a pair of the operators $B^+$ and $1$ (unity) instead of $q_i^+$, $i = 1, 2$ in the formulation of the Theorem;

b) any two antisymmetric symmetry operators $e_i^+ = -(e_i^+)^t$, $e_i^+ h^+ = h^+ e_i^+$, $i = 1, 2$ are dependent, i.e. being stripped off coincide, that follows from the Theorem after substituting them instead of $q_i^+$, $i = 1, 2$.

Now one can answer the question about a maximal number of independent supercharges and their relative oddness. First let us prove that the number of supercharges cannot exceed two. Assume that there exist three independent intertwining operators $q_i^+$, $i = 1, 2, 3$. Then their pairwise symmetry operators $q_i^+ q_j^- - q_j^+ q_i^-$, $i \neq j$ are antisymmetric and non-trivial but dependent in virtue of the consequence b), $q_i^+ q_j^- - q_j^+ q_i^- = \alpha_{ij} (h^+) e^+$. When multiplying these relations on $\alpha_{kl}$ one can assemble the following identity,

$$q_1^+ [\alpha_{13} (h^-) q_2^- - \alpha_{12} (h^-) q_3^-] - [q_2^+ \alpha_{13} (h^-) - q_3^+ \alpha_{12} (h^-)] q_1^- = 0 = q_1^+ q_{23}^- - q_{23}^+ q_1^-;
$$

$$q_2^+ \alpha_{13} (h^-) - q_3^+ \alpha_{12} (h^-) \equiv q_{23}^+,$$  

(90)

where the operator $q_{23}^+$ is non-trivial due to independence of $q_2^+$ and $q_3^+$. Evidently two intertwining operators $q_1^+$ and $q_{23}^+$ satisfy the requirements of the Theorem and therefore are dependent in contradiction with the initial assumption. Thus we have proved that the maximal number of independent supercharges is two.

Next let us consider two normalized independent intertwining operators $q_i^+$, $i = 1, 2$ of orders $N_1$ and $N_2$ such that $N_1 > N_2$ and the sum of their order $N_1 + N_2$ is even. Then evidently the operator

$$q_3^+ = q_1^+ - (-h^+)^{\frac{N_1 - N_2}{2}} q_2^+$$  

(91)
is independent of $q^+_2$ and has the order $N_3$ less than $N_1$. If the sum of orders $N_2 + N_3$ is again even one may normalize $q^+_3$ and apply the above algorithm to derive a lower order independent SUSY generator until the sum of orders became odd. Thus one can always construct the basis of two intertwining operators containing an even and an odd one.

Finally one can search for a set of minimal intertwining operators $k^\pm_N, p^\pm_{N_1}$ just solving the system for two independent intertwining operators,

$$q^\pm_i = \alpha^\pm_i (h^\pm) k^\pm_N + \beta^\pm_i (h^\pm) p^\pm_{N_1}, \quad i = 1, 2;$$  \hspace{1cm} (92)

with coefficients $\alpha^\pm_i (h^\pm), \beta^\pm_i (h^\pm)$ polynomial in the Hamiltonian.

Indeed,

a) among all intertwining operators $q^+$ there exist an unique real operator of lowest order $p^+$ normalized according to (6);

b) among all intertwining operators $q^+$ independent of $p^+_{N_1}$ there exist a real operator of lowest order $k^+_N$ normalized according to (6);

c) with the help of the algorithm (91) one can prove that one of the operators $p^+_{N_1}$ and $k^+_N$ is of even order and another one is of odd order;

d) an arbitrary intertwining operator $q^\pm$ is always decomposed in the form (92) in the unique way which can be proven by a consequent application of the algorithm (91) and taking into account that the one of the terms $\sim p^+_N$ and $\sim k^+_N$ is of even order and another one is of odd order.

Thus the set of $p^+_{N_1}$ and $k^+_N$ form an optimal basis of intertwining operators. As all $q^- = (q^+)^t$ the same results are translated to the set of $p^-_{N_1}$ and $k^-_N$.

8. More about symmetry operators

In the previous Section we have proven that the antisymmetric symmetry operator (in each component) is unique after being stripped off. But the optimization of supercharge basis may not guarantee the minimal form of components of the symmetry operator. The uniqueness of decomposition (92) allows to formulate a necessary condition for the symmetry operator $e^\pm$ made of the minimal operators $k^\pm_N, p^\pm_{N_1}$ to be stripped off further on. Namely if a polynomial $f^\pm(h^\pm)$ can be factorized out of the symmetry operator $e^\pm$ then the same polynomial appears as a multiplier in $\mathcal{P}_N(h^\pm), \mathcal{P}_{N_1}(h^\pm), b^\pm(h^\pm)$. It follows from the relations,

$$\mp ie^\pm k^\pm_N = \frac{1}{2} (k^\pm_N p^\pm_{N_1} - p^\pm_{N_1} k^\pm_N) k^\pm_N = b^\pm(h^\pm) k^\pm_N - \mathcal{P}_N(h^\pm) p^\pm_{N_1},$$

$$\mp ie^\pm p^\pm_{N_1} = \frac{1}{2} (k^\pm_N p^\pm_{N_1} - p^\pm_{N_1} k^\pm_N) p^\pm_{N_1} = \mathcal{P}_{N_1}(h^\pm) k^\pm_N - b^\pm(h^\pm) p^\pm_{N_1}. \hspace{1cm} (93)$$

One can give a more detailed description of each component of the symmetry operator for a particular class of potentials with the help of the Lemma.
Lemma
Assume that:

a) the Hamiltonian $h_a$ commutes with an antisymmetric real operator $R_a$ of order $2n + 1$ which cannot be stripped off;  
b) this Hamiltonian has at least one bound state;  
c) the wave function $\Psi_0$ characterizes a bound state with the lowest energy $E_0$.

Then there exist a non-singular Hamiltonian $h_b$, a non-singular real operator $r_a = \partial - (\Psi_0'/\Psi_0)$ and a non-singular antisymmetric real operator $R_b$ of order $2n - 1$ which cannot be stripped off such that:

$$
\begin{align*}
  h_a &= r_a^t r_a + E_0; &
  h_b &= r_a^t r_a + E_0; &
  r_a h_a &= h_b r_a; \\
  R_a &= r_a^t R_b r_a; &
  R_b h_b &= h_b R_b.
\end{align*}
$$

(94)

(95)

The proof of relations (94) is standard for the SUSY QM \cite{4}. The partial factorization of $R_a = \hat{R}_b r_a$ with a non-singular differential operator $\hat{R}_b$ of order $2n$ is provided by the equations $R_a \Psi_0 = 0$ and $r_a \Psi_0 = 0$. Evidently it is an intertwining operator, $h_a \hat{R}_b = \hat{R}_b h_b$ due to Eqs. (94). As the Hamiltonian $h_b$ does not have the level $E_0$ the latter relation entails $\hat{R}_b \Psi_0 = 0$. Hence the factorization takes place $\hat{R}_b = R_b^l r_a$ with a non-singular differential operator $R_b^l$ of order $2n - 1$. From the intertwining relations it follows that the operator $R_b^l$ is a symmetry operator. At last its antisymmetry under transposition can be easily derived from the similar property of $R_a$. The operator $R_b$ cannot be stripped off if Eqs. (95) hold and the operator $R_a$ has been stripped off already.

From the Lemma one can obtain a certain relationship between the number of bound states of the Hamiltonian and the structure of the symmetry operator. Namely, suppose that the Hamiltonian $h_0$ has $n$ bound states with energies $E_l$, $E_{l+1} > E_l$ and commutes with a antisymmetric real operator $R_0$ of order $2n + 1$ which cannot be stripped off. Then the (normalized) symmetry operator can be factorized,

$$
R_0 = r_0^t \ldots r_{n-1}^t \partial r_{n-1} \ldots r_0; \quad r_l \equiv \partial + \chi_l,
$$

(96)

with non-singular real superpotentials $\chi_l$. Respectively the ladder (dressing chain) relations hold,

$$
\begin{align*}
  h_{l+1} r_l &= r_l h_l; &
  l &= 0, \ldots, n - 1; \\
  h_l &\equiv r_{l-1}^t r_{l-1}^t + E_{l-1} = r_l^t r_l + E_l; &
  l &= 1, \ldots, n - 1; \\
  h_0 &= r_0^t r_0 + E_0; &
  h_n &= r_{n-1}^t r_{n-1}^t + E_{n-1},
\end{align*}
$$

(97)

and the hidden symmetry operators arise for each intermediate Hamiltonian,

$$
\begin{align*}
  R_l &= r_l^t \ldots r_{n-1}^t \partial r_{n-1} \ldots r_l; &
  R_n &= \partial; \\
  R_l h_l &= h_l R_l; &
  l &= 0, \ldots, n.
\end{align*}
$$

(98)
Evidently the Hamiltonian $h_n$ describes a free particle and therefore the Hamiltonian system with a hidden symmetry can be related to the free-particle system.

In the case b) of Sec. 4 each component of the symmetry operator can be also represented in the canonical factorized form (96),

$$e^\pm = i \left[ \partial^3 - (a + \frac{3}{2} V_{1,2}) \partial - \frac{3}{4} V'_{1,2} \right] = -i \left( -\partial - \frac{\Psi'_{1,2}}{\Psi_{1,2}} \right) \partial \left( \partial - \frac{\Psi'_{1,2}}{\Psi_{1,2}} \right),$$

(99)

with the help of the bound-state wave functions, $\Psi_{1,2} = C \sqrt{V_{1,2} - \beta^2}$. In the case c) the potentials exemplify the Lemma as one of them is constant.

One can guess that the above relationship between the Hamiltonian $h_0$ and the symmetry operator $R_0$ is quite general because the algorithm of the Lemma helps to transform the system with $n$ bound states and with a symmetry operator to a system with $n - 1$ bound states and a symmetry operator of order lower in two units. After one removes all bound states with this algorithm the remaining Hamiltonian is still reflectionless and thereby the scattering coefficients are trivial corresponding to a free-particle system.

9. Concluding remarks

We have established that:

a) for supercharges of finite order the Polynomial SUSY can be always realized when the supercharges are related by transposition;

b) in certain cases (for instance, for complex intertwining operators) several supercharges may commute with the Super-Hamiltonian which may yield a non-trivial hidden symmetry of such a system;

c) the maximal number of independent supercharges of type $Q$ (or of type $\bar{Q}$) commuting with a given Super-Hamiltonian is two and in the case of Extended $\mathcal{N} = 2$ superalgebras there exists an optimal set of two real supercharges with components of a minimal order in derivatives, one of which is an odd-order operator and another one is an even-order operator;

d) for Extended $\mathcal{N} = 2$ superalgebras the hidden symmetry operator $\mathcal{E}(H)$ defined in (43) is unique up to a multiplier polynomial in the Super-Hamiltonian and among zero-modes of this operator there are all bound states of the Super-Hamiltonian.

Finally we mention possible extensions of the Theorems and results of this paper. First of all it seems to be straightforward to apply them to the Super-Hamiltonians with complex potentials as the SUSY algebra with transposition symmetry is well defined for such Super-Hamiltonians. The application to matrix Super-Hamiltonians is less trivial but certainly interesting as well as a generalization on multidimensional QM.
Acknowledgments

One of us (A.A.) is grateful to F. Cannata and J.-P. Dedonder for useful discussions. This work was supported by the Grant RFBR 02-01-00499.

References

[1] H. Nicolai, J. Phys. A: Math. Gen. A9 (1976) 1497.
[2] E. Witten, Nucl. Phys. B188 (1981) 513.
[3] F. Cooper and B. Freedman, Ann. Phys. (N.Y.) 146 (1983) 262.
[4] A.A. Andrianov, N.V. Borisov and M.V. Ioffe, JETP Lett. 39 (1984) 93; Phys. Lett. A105 (1984) 19; Theor. Math. Phys. 61 (1985) 1078.
[5] M.M. Nieto, Phys. Lett. B145 (1984) 208.
[6] B. Mielnik, J. Math. Phys. 25 (1984) 3387.
[7] D. Fernández, Lett. Math. Phys. 8 (1984) 337.
[8] A.A. Andrianov, N.V. Borisov, M.V. Ioffe and M.I. Eides, Phys. Lett. 109A (1985) 143; Theor. Math. Phys. 61 (1985) 965.
[9] C.V. Sukumar, J. Phys. A: Math. Gen. 18 (1985) L57; 2917; 2937.
[10] L.E. Gendenshtein and I.V. Krive, Sov. Phys. Usp. 28 (1985) 645.
[11] A. Lahiri, P.K. Roy and B. Bagchi, Int. J. Mod. Phys. A5 (1990) 1383.
[12] F. Cooper, A. Khare and U. Sukhatme, Phys. Rept. 251 (1995) 267.
[13] G. Junker, Supersymmetric Methods in Quantum and Statistical Physics (Springer, Berlin-Heidelberg, 1996).
[14] R. de Lima Rodrigues, hep-th/0205017.
[15] D. Baye, Phys. Rev. Lett. 58 (1987) 2738;
[16] R.D. Amado, F. Cannata and J.-P. Dedonder, Phys. Rev. C41 (1990) 1289; ibid. C43 (1991) 2077; Int. J. Mod. Phys. A5 (1990) 3401.
[17] A.A. Andrianov, M.V. Ioffe and D.N. Nishnianidze, Phys. Lett. A201 (1995) 103; Theor. Math. Phys. 104 (1995) 1129; preprint solv-int/9605007; J. Phys. A: Math. Gen. A32 (1999) 4641.
[18] A. A. Andrianov, F. Cannata, M.V. Ioffe and D.N. Nishnianidze, J. Phys. A: Math. Gen. A30 (1997) 5037.

[19] F. Cannata, M.V. Ioffe and D.N. Nishnianidze, J. Phys. A: Math. Gen. A35 (2002) 1389.

[20] A.V. Turbiner, Commun. Math. Phys. 118 (1988) 467.

[21] M. A. Shifman, Int. J. Mod. Phys. A12 (1989) 2897.

[22] C.M. Bender and G.V. Dunne, J. Math. Phys. 37 (1996) 6.

[23] H. Aoyama, H. Kikuchi, I. Okouchi, M. Sato and S. Wada, Nucl. Phys. B553 (1999) 644.

[24] B. Bagchi, F. Cannata and C. Quesne, Phys. Lett. A269 (2000) 79.

[25] R. Sasaki R. and K. Takasaki, J. Phys. A: Math. Gen. A34 (2001) 9533.

[26] S.M. Klishevich and M.S. Plyushchay, Nucl. Phys. B606 (2001) 583; Nucl. Phys. B616 (2001) 403;

[27] P. Dorey, C. Dunning and R. Tateo, J. Phys. A: Math. Gen. A34 (2001) 5679; L391 .

[28] A.A. Andrianov, M.V. Ioffe and V.P. Spiridonov, Phys. Lett. A174 (1993) 273.

[29] A.A. Andrianov, F. Cannata, J-P. Dedonder and M.V. Ioffe, Int. J. Mod. Phys. A10 (1995) 2683.

[30] V.B. Matveev and M.A. Salle, Darboux Transformations and Solitons (Springer, Berlin-Heidelberg, 1991).

[31] V.G. Bagrov and B.F. Samsonov, Theor. Math. Phys. 104 (1995) 1051.

[32] B.F. Samsonov, Mod. Phys. Lett. A11 (1996) 1563.

[33] A. Gangopadhyaya and U. Sukhatme, Phys. Lett. A224 (1996) 5.

[34] U. Sukhatme, C. Rasinariu and A. Khare, Phys. Lett. A234 (1997) 401.

[35] A. Das and S.A. Pernice, Mod. Phys. Lett. A12 (1997) 581.

[36] D.J. Fernández C., Int. J. Mod. Phys. A12 (1997) 171.

[37] G. Junker and P. Roy, Ann. Phys. 270 (1998) 155.

[38] D.J. Fernández C., V. Hussin and B. Mielnik, Phys. Lett. A244 (1998) 309.

[39] J. O. Rosas-Ortiz, J. Phys. A31 (1998) 10163.

[40] B. Bagchi, A. Ganguly, D. Bhaumik and A. Mitra, Mod. Phys. Lett. A14 (1999) 27.
[41] D. J. Fernández C., J. Negro and L.M. Nieto, Phys. Lett. A275 (2000) 338.

[42] D.J. Fernández C. and V. Hussin, preprint quant-ph/0011004.

[43] M.S. Plyushchay, Int. J. Mod. Phys. A15 (2000) 3679; Phys. Lett. B485 (2000) 187.

[44] H. Aoyama, M. Sato, T. Tanaka and M. Yamamoto, Phys. Lett. B498 (2001) 117.

[45] H. Aoyama, M. Sato and T. Tanaka, Phys. Lett. B503 (2001) 423.

[46] H. Aoyama, M. Sato and T. Tanaka, Nucl. Phys. B619 (2001) 105.

[47] H. Aoyama, N. Nakayama, M. Sato and T. Tanaka, Phys. Lett. B519 (2001) 260; Phys. Lett. B521 (2001) 400.

[48] T. Tanaka, preprint hep-th/0212276.

[49] S.M. Klishevich and M.S. Plyushchay, Nucl. Phys. B628 (2002) 217.

[50] D. J. Fernández C., R. Muñoz and A. Ramos, preprint quant-ph/0212026.

[51] A.P. Veselov and A.B. Shabat, Funct. Anal. Appl. 27 (1993) 81.

[52] V.E. Adler, Funct. Anal. Appl. 27 (1993) 140.

[53] R. Zhdanov, J. of Math. Phys. 37 (1996) 3198.

[54] W. Fushchych, A. Nikitin, J. of Math. Phys. 38 (1997) 5944.

[55] H.-D. Doebner, R. Zhdanov, math-ph/9809021.

[56] A.A. Andrianov, F. Cannata, M.V. Ioffe and D.N. Nishnianidze, preprint quant-ph/9902057; Phys. Lett. A266 (2000) 341.

[57] S. Flügge, Practical Quantum Mechanics (Springer, Berlin-Heidelberg, 1971); B.N. Zakhariev and A.A. Suzko, Direct and Inverse Methods (Springer, Berlin-Heidelberg, 1990).

[58] A.A. Andrianov, F. Cannata, J-P. Dedonder and M.V. Ioffe, Int. J. Mod. Phys. A14 (1999) 2675.