Criteria for gravitational instability and quasi-isolated gravitational collapse in turbulent medium

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ABSTRACT
We study the evolution of structures in turbulent, self-gravitating media, and present an analytical criterion $M_{\text{crit}} \approx \epsilon_c^{2/3} \eta \sigma^2 G^{-1} l^{5/3}$ (where $M_{\text{crit}}$ is the critical mass, $l$ is the scale, $\epsilon_c$ is the turbulence energy dissipation rate of the ambient medium, $G$ is the gravitational constant, $\sigma$ is the velocity dispersion, $l$ is the scale and $\eta \approx 0.2$ is an efficiency parameter) for an object to undergo quasi-isolated gravitational collapse. The criterion also defines the critical scale ($l_{\text{crit}} \approx \epsilon_c^{1/2} \eta^{1/2} l^{1/3} G^{-3/4} \rho^{-3/4}$) for turbulent gravitational instability to develop. The analytical formalism explains the size dependence of the masses of the progenitors of star clusters ($M_{\text{cluster}} \sim R_{\text{cluster}}^{1.67}$) in our Galaxy.

Key words: turbulence – gravitation – ISM: kinematics and dynamics – instabilities – methods: analytical

1 INTRODUCTION
Astrophysical fluid systems are characterised by large sizes and long evolution times. The Reynolds number, $Re = UL/v = L^2 T / \nu$ (where $U$ is velocity, $L$ is scale, $T$ is time and $\nu$ is the viscosity) is typically large. Gravity drives the formation of structures. One thus needs to understand the interplay between the two. We consider an object embedded in a turbulent ambient flow, and are concerned with this question: Under what conditions can an object be considered “detached” from the ambient medium, such that its evolution is “quasi-isolated”? What is the appropriate condition for gravitational instability to develop?

One possible criterion is the virial parameter (Bertoldi & McKee 1992), which quantifies the relative importance of gravity and turbulence in a given structure. However, one limitation of the virial parameter in its basic form is that it neglects the dynamical interaction between the structure and the ambient environment (Ballesteros-Paredes 2006). Turbulence is a process where energy has been transferred from larger to smaller scales. Since this energy transfer has not been explicitly considered, the virial parameter in its basic form can not be used to study the interaction between the object and the ambient environment.

We derive an analytical criterion for an object to be considered as quasi-isolated in a turbulent flow. The criterion is derived by explicitly considering the interplay between turbulence and gravity at the boundary of the object. We apply the criterion to the evolution of structures in the turbulent molecular interstellar medium, and find that the observed properties of clumps hosting proto-star clusters can be explained by our formalism. The criterion can be used to study the development of gravitational instability in a turbulent medium (Chandrasekhar 1951; Parker 1952).

2 OVERALL PICTURE
In our picture, the medium is composed of two phases: in the dense phase, gravity determines the level of turbulent motion, and in the diffuse phase, the level of turbulent motion is almost universal. Our “object” is composed of gas in the dense phase, surrounded by the ambient medium that belongs to the diffuse phase. The gas in the dense phase is “quasi-isolated” in the sense that the ambient medium is not able to influence its evolution significantly. We assume that turbulence in the dense phase is viralised, and turbulence in the ambient medium is characterised by a constant energy dissipation rate $\epsilon_{\text{amb}}$. This illustrated in Fig. 1.

In the case of the Jeans instability (Jeans 1902), one is mainly concerned with the interplay between gravity and thermal support in terms of pressure – the instability occurs when the internal gas pressure is not sufficient to prevent gravitational collapse. In the case where the ambient medium is turbulent, one needs to consider a different picture.

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In our proposed picture, turbulence can provide internal support against gravitational collapse. However, this type of support must be distinguished from e.g. support from thermal motion. Unless cooling is extremely efficient, the support from thermal motion does not need to be sustained by an external energy source, and for thermal support to be effective, one only need it to satisfy the pressure equilibrium $p_{\text{thermal}} \approx p_{\text{energy}}$. When an object collapses, $p_{\text{thermal}} < p_{\text{energy}}$. Support from turbulence has a different nature, in that turbulent motion does not sustain by itself. Turbulent motion is dissipative, and without energy injection, turbulence would decay within a few crossing times. Thus, it is necessary to sustain the turbulent motion for it to be effective in supporting against collapse.

When an turbulence-dominated object is collapsing, it is often not the case that the turbulent pressure is much lower than the pressure required to support against gravity. For a system where the turbulence is virialised (such that $\sigma_v^2 \approx G m/r$, $m$ is the mass and $r$ is the size of the object, $\sigma_v$ is the velocity dispersion), the ram-pressure of the internal turbulent motion ($p_{\text{turb}} \approx \rho \sigma_v^2$, where $\rho$ is the density) is always comparable to the pressure from self-gravity ($p_{\text{gravity}} \equiv G m^2/2 r \approx 2G \rho^2 r^2$). However, turbulence keeps dissipating kinematic energy from the system. When one is continuously injecting energy to the system, such that the energy injection rate is comparable to or larger than the energy dissipation rate of the viralised turbulence in the object, the object will be “supported” against collapse. When the energy injection rate is not able to compensate for the internal energy dissipation, the object should collapse, such that the gravitational energy released during this collapse would be able to compensate for the additional energy dissipated by the viralised turbulence. In contrast to the case of Jeans instability where pressure plays a crucial role, to decided if a turbulent object will collapse, one need to come up with an energy-based criterion, which we derive in the next section.

Turbulence in astrophysical systems can be either supersonic or subsonic. For subsonic turbulence, turbulent motion would lead to energy cascade, but the gas compression from the turbulent motion is in general insignificant. In this case, one would expect gravitational instability to develop gradually, perhaps limited by a typical scale. Fragments developed from the instability can still have different masses provided that they have different ages. Yet, one still expect to observe a limit below which the instability can not grow.

When the turbulence is supersonic, such as the case of the Milky Way interstellar medium, turbulence itself creates density fluctuations, and the subsequent interplay between turbulence and gravity determines the subsequent growth of the perturbations created by turbulence. In the supersonic case, one does not expect to observe the same limiting scale for gravitational instability to grow. The instability can grow over a variety of scales provided that the initial density fluctuations are large enough. However, the critical condition for the density fluctuations to grow should still be determined by the properties of the ambient turbulent flow.

The central part of our formalism is the “condition for quasi-isolated gravitational collapse”, as it is the theoretical boundary that separates the dense, collapsing phase from an ambient diffuse phase of a turbulent medium. In Sec. 3 we derive the condition for the quasi-isolated gravitational collapse. When applied to a medium with subsonic turbulence, this condition allows us to determine the critical condition for a perturbation to grow (stability criterion). When applied to a medium where the turbulence is supersonic or the density enhancements are pre-existing, the condition allows us to determine if the structures are sufficiently condensed, such that they would ignore the energy cascade from the ambient medium and would evolve on their own.

### 3 THE FORMALISM

We consider the evolution of a dense object in a turbulent ambient medium. All the quantities have been listed in in Table 1. In this section we will present our formulation, and an example is given in Sec. 3.1. The object has mass $m$ and size $l$. The ambient medium has an almost-uniform mean density $\rho_\text{medium}$.

The energy dissipation rate $\dot{\epsilon}$ of the turbulent medium

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1 When the external energy injection rate is much larger than the energy dissipation rate of the viralised turbulence, the object would be disrupted by turbulence cascade.

2 Here, the energy dissipation rate is obtained by averaging over a few crossing times.
can be expressed as
\[ \epsilon_{\text{cascade}} \approx \eta \times \frac{U_1^3}{l} \approx \text{Constant}, \]  
where \( U_1 \) is the velocity at scale \( l \), \( \eta \) is the efficiency of energy dissipation from the turbulence cascade. \( \epsilon_{\text{cascade}} \) is roughly a constant, and is independent on the scale (and thus holds for any given scale, see e.g. the energy dissipation law, and Kritsuk et al. (2007) for the supersonic case).

Then we consider the effect of gravity on such an object, and temporarily neglects the effect of external turbulence. Because of gravity, the object would be dominated by a turbulent motion that is virialised, such that the internal velocity dispersion is \( U_{\text{vir}} = \sqrt{G m/l} \). The energy dissipation rate of such a turbulent, self-gravitating system is
\[ \epsilon_{\text{vir}} \approx \eta \times \frac{U_{\text{vir}}^3}{l} = G^{3/2} \rho^{5/2} l^{3/2} \times \eta. \]  

We combine the above-mentioned results to propose our criterion. At the boundary of an object, the density of the gas that belongs to the object is comparable to the density of the gas of the surrounding. When \( \epsilon_{\text{vir}} < \epsilon_{\text{cascade}} \), energy injection from the ambient medium exceeds much the energy dissipation of the turbulence inside the object, the turbulence is virialised. In this case, self-gravity has an almost negligible effect on the system, and the system is “supported” against collapse by energy cascade from the ambient medium. When \( \epsilon_{\text{vir}} > \epsilon_{\text{cascade}} \), the internal energy dissipation rate of the turbulent motion driven by self-gravity exceeds much the external energy cascade from the ambient medium, and the system would neglect the energy flux from the external cascade and would collapse on its own. Thus we propose a criterion for quasi-isolated gravitational collapse:
\[ \epsilon_{\text{vir}} \geq \epsilon_{\text{cascade}}. \]  

When this is fulfilled, energy contribution from external turbulence cascade would not be able to influence the evolution significantly. The energy dissipation from the internal, virialised turbulence dominates the kinetic energy budget of the system, and the system would undergo gravitational collapse. \(^3\) The critical mass beyond which gravity dominates is (letting \( \epsilon_{\text{grav}} = \epsilon_{\text{cascade}} \)).
\[ m_{\text{crit}} \approx G^{-1} \epsilon_{\text{cascade}}^{2/3} \rho^{-2/3} l^{5/3}, \]  
where \( m_{\text{crit}} \) is the critical mass and \( l \) is the size.

### 3.1 Roles of turbulence and gravitational contraction

In our formalism, turbulent motion prevails throughout the region. However, the function of turbulence is different at different regimes. Outside the object, an almost universal turbulence provides supports against collapse. Inside the object, the turbulence is driven by a combination of cascade from the large scale and gravitational collapse (e.g. turbulence driven by accretion (Klessen & Hennebelle 2010; Elmegreen & Burkert 2010), and the energy from gravitational collapse would dominate the energy budget (such that \( \epsilon_{\text{vir}} - \epsilon_{\text{cascade}} \approx \epsilon_{\text{collapse}} \)).

Thus we propose a two-stages in criterion. At the boundary of an object, the density of the internal, collapse would dominate the energy budget (such that \( \epsilon_{\text{vir}} - \epsilon_{\text{cascade}} \approx \epsilon_{\text{collapse}} \)). As for the evolution of the object, it is likely that how fast the object can collapse is determined by the ability of the turbulent motion to remove the kinetic energy (See an analytical model from Murray & Chang (2015) and discussions therein).

### 4 GRAVITATIONAL INSTABILITY IN A TURBULENT MEDIUM

The development of gravitational instability in a turbulent medium has been studied (Chandrasekhar 1951; Parker 1952). However, these models typically assume a uniform velocity dispersion for the gas. Modern studies of turbulence reveal that it is a multi-scaled process (Frisch 1995), and thus our Eq. 3 is much more accurate than assuming a constant velocity dispersion.

We consider the growth a perturbation on scale \( l \). The energy injection from the turbulence cascade of the ambient medium is \( \epsilon_{\text{cascade}} \), and the total energy dissipation from the viralised turbulence were gravity to dominate is \( \epsilon_{\text{vir}} \). For the instability to grow, one requires \( \epsilon_{\text{vir}} > \epsilon_{\text{cascade}} \), which is identical to our criterion of quasi-isolated gravitational collapse. Thus, Eq. 4 still holds, and at the onset of the gravitational instability, \( \rho_{\text{medium}} \approx \rho_{\text{object}} \approx \rho \). The critical length scale on which the instability can develop is (where we have used Eq. 4 and \( m \approx \rho_{\text{crit}} l_{\text{crit}}^3 \))
\[ l_{\text{crit}} \approx \epsilon_{\text{cascade}}^{1/2} G^{-3/4} \rho^{-3/4}. \]  
and the critical mass
\[ m_{\text{crit}} \approx G^{-1} \epsilon_{\text{cascade}}^{2/3} \rho^{-2/3} l^{5/3} = \rho^{3/4} G^{-3/4} \rho^{-5/4}. \]  
where \( \epsilon_{\text{cascade}} \) is energy dissipation rate of the ambient turbulence, \( \rho \) is the density and \( G \) is the gravitational constant.

We expect the turbulent gravitational instability to develop when the medium is supported by a turbulence that is subsonic. In such a case, Eq. 5 predicts the critical length for the instability to develop. In the supersonic case, because turbulence also creates density fluctuations, one can obtain structures that are much smaller than the scale predicted by Eq. 5.

### 5 OBSERVATIONAL TEST

One example to consider is the evolution of star cluster-forming clumps in molecular clouds. Clumps (the definition can be found in Williams et al. (2000)) are condensations of dense gas. They are thought to be the progenitors of star clusters. Here, star cluster-forming clumps are our objects of interest, and gas in the molecular clouds serves as the ambient medium.

The Milky Way molecular clouds are turbulent. Observationally, molecular clouds follow the Larson’s relation...
Star cluster-forming clumps (sometimes simply called “clumps”) are dense gas condensations in molecular clouds. Pfalzner et al. (2015) pointed out the resemblance between the mass-size relation of the clumps $M_{\text{clump}}/M_\odot = 2500 R/\text{pc}$ (Larson 1981) and the mass-size relation of embedded clusters (Lada & Lada 2003), where $M_{\text{cluster}}/M_\odot = 359 R/\text{pc}$ (Larson 1981). One attractive possibility proposed by these authors is that these clumps would collapse and form individual star clusters. If this is the case, the two above-mentioned mass-size relations should share a common origin, which we would now explore.

We interpret the clumps as objects that undergo quasi-isolated gravitational collapse in the molecular ambient medium. This will simultaneously explain why these clumps would collapse to form individual star clusters, as well as the observed mass-size relation. For clumps to be gravitationally isolated, our formalism requires the sizes and the masses to obey Eq. 4, which is

$$M \sim R^{1.3} \sim R^{1.67}.$$  

(7)

The derived scaling index matches exactly with that of the observed mass-size relation of the clumps (where the scaling index is $1.67 \pm 0.01$) as well as that of the embedded star clusters (where the scaling index is $1.71 \pm 0.07$). One can also derive the normalisation. We first estimate the expected turbulence energy dissipation rate. Using Eq. 1 and put $\sigma_v \approx 1 \text{ km/s, } l \approx 1 \text{ pc}$ (Larson 1981), we derive an energy dissipate rate of $3.3 \times 10^{-4} \text{ cm}^2 \text{ s}^{-3} = 3.3 \times 10^{-4} \text{ erg s}^{-1} \text{ g}^{-1} (1\text{erg} = 1\text{ g cm}^2 \text s^2)$. This gives us a mass-size relation $M_{\text{clump}}/M_\odot = 7000 M_{\odot} (R/\text{pc})^{1.67}$, which agrees with the measured relation of embedded star clusters (where the normalisation is $357 M_\odot$, Lada & Lada (2003)), and does not contradict the observed mass-size relation of the clumps (where the normalisation is $2500 M_\odot$, Pfalzner et al. (2015); Urquhart et al. (2013)). There are some differences. As Pfalzner et al. (2015) have emphasised, the differences might arise simply because of the different ways to define the radii.

Out result thus offers an explanation to the observed mass-size relation, and shed light on the nature of these objects – the so-called cluster-forming clumps are objects that undergo quasi-isolated gravitational collapse, and their sizes are the boundaries beyond which gravity from the central object ceases to dominate. Based on this interpretation, we expect the clump evolution to be dynamically detached from the ambient medium, and thus they should collapse individually and form star clusters. This also explains the resemblance between the mass-size relation of the clumps and that of the embedded star clusters.

One clarification should be made: the clumps are self-gravitating, and are dynamically detached from their ambient environment, and this does not imply that the interactions with the ambient medium are completely negligible. We still expect the clumps to interchange matter with the ambient medium. In short, the object is not completely isolated, but is quasi-isolated, that the energy exchange between

$\epsilon_{\text{scale}}$ energy dissipation rate of external turbulence of order $n U_1^2 / l$ $L^{-3} T^{-4}$

$\epsilon_{\text{vir}}$ energy dissipation rate of the virialised turbulence $L^{-3} T^{-4}$

$\eta = 0.2$ efficiency of turbulent energy dissipation 1

$m_{\text{crit}}$ critical mass for turbulent gravitational instability $M$

$l_{\text{crit}}$ critical scale for turbulent gravitational instability $L$

$R$ radius of the star cluster-forming clumps $L$

Table 1. List of definitions of mathematical symbols. In the right column, $L$ stands for scale, $T$ stands for time, and $M$ stands for mass.

4 Inside the clump boundaries, $\epsilon_{\text{vir}}$ dominates, and outside the clump boundaries, $\epsilon_{\text{scale}}$ dominates. One thus need to require that the turbulence energy dissipation rate increases with decreasing radii. The dissipation inside the clump can be easily derived assuming some density profiles and with our Eq. 1. When the inside-out density gradient is steeper than $\rho \sim r^{-3}$, the energy dissipation rate increases as one moves inward, and at regions inside the clumps, the external turbulence cascade does not contribute much to the energy budget. This condition is easily fulfilled for the majority of the observed clumps (where, typically, $\rho \sim r^{-2}$, see e.g. Wyrowski et al. (2016)).

5 Our interpretation should be distinguished from the interpretation of Kauffmann et al. (2013), where the mass-size relation originates from the Larson relation, which is an empirical relation derived from molecular cloud observations. We agree that the Larson relation plays an important role, and yet, in our formalism, the mass-size relation is rather a consequence of an almost-uniform turbulence energy dissipation of the Milky Way molecular ISM.
between the object and the ambient medium is not sufficient to influence the gravitational collapse significantly.

For the molecular clouds in the Milky Way, because supersonic turbulence also creates density fluctuations upon which gravity acts, we do not expect them to respect the stability criterion (Eq. 5), but they should still respect the criterion for quasi-isolated gravitational collapse (Eq. 4). Only when the turbulence is subsonic (and is close to be incompressible) do we expect our Eq. 5 to predict the typical mass of the fragments that develop from turbulent gravitational instability.

The importance of magnetic field in molecular clouds has been increasing recognized (Li et al. 2014). Our formalism does not include the magnetic field. However, how magnetic fields evolve in such a turbulent medium is still not clear. Observationally, systematic magnetic field measurements are only available for nearby molecular clouds (Crutcher 1999), and measurements of field strength in massive star-forming regions are still limited to individual sources (Pillai et al. 2015; Li et al. 2015; Zhang et al. 2014). In this sense, a complete picture of the evolution of the field strength in different objects is still missing. One theoretical possibility is that the magnetic fields are maintained by turbulent motion and Galactic shear, and the very process of field amplification by turbulence would lead to turbulent magnetic reconnection, which might enable a relatively fast removal of the magnetic flux (Brandenburg & Subramanian 2005; Lazarian 1993). If this is the case, the effect of magnetic field would be secondary as compared to turbulence. But the issue is still unsettled, and more investigations are needed.

6 CONCLUSIONS

We derive an analytical criterion for an object to undergo quasi-isolated gravitational collapse in a turbulent medium. Different from the previous treatments assuming constant velocity dispersions for the turbulence (Chandrasekhar 1951; Parker 1952), we describe the multi-scaled structure of the turbulent flow. Our main results include a criterion for quasi-isolated gravitational collapse and a condition for turbulent gravitational instability.

The criterion of quasi-isolated gravitational collapse allows one to decide if an object is dynamically detached from the ambient turbulent flow. The critical mass is linked to the size of the object and energy dissipation rate of the ambient medium $e_{\text{cascade}}$ by

$$ m_{\text{crit}} \approx G^{-1/3} e_{\text{cascade}}^{2/3} \eta^{-2/3} \rho^{1/3} $$

where $\eta \approx 0.2$ is an efficiency factor. This result is applicable to both supersonic and subsonic flows.

We also derive a condition for turbulent gravitational instability to develop: the critical scale $l_{\text{crit}}$ is determined by the density of the medium $\rho$ and the energy dissipation:

$$ l_{\text{crit}} \approx e_{\text{cascade}}^{1/2} G^{-1/3} \rho^{-3/4} $$

This formula is applicable to the subsonic case where we expect the instability to develop gradually from a medium of a almost-uniform density.

Our criterion for quasi-isolated gravitational collapse explains the observed mass-size relation of the star clustering $M_{\text{clump}} \sim r_1^{0.7 \pm 0.01}$, and thus supports a scenario that these objects are dynamically detached from their environments, and are undergoing quasi-isolated gravitational collapse.

Note added in proof: A series of efforts have been made by Hennebelle (2012); Lee & Hennebelle (2016a,b), where they consider the interplay between turbulence dissipate and accretion, and derived an analytical mass-size relation $m \sim r^2$. Their picture shares many similarities with ours. However, to derive the mass-size relation, they had to assume Larson’s relation, and in our case, the mass-size relation is a direct consequence of the universal turbulence dissipation in the ambient medium. Future observations of fragmentation in different environments are required to identify the key physical mechanism that lead to the mass-size relation.

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