Rodrigues Formula for Hi-Jack Symmetric Polynomials Associated with the Quantum Calogero Model

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The Hi-Jack symmetric polynomials, which are associated with the simultaneous eigenstates for the first and second conserved operators of the quantum Calogero model, are studied. Using the algebraic properties of the Dunkl operators for the model, we derive the Rodrigues formula for the Hi-Jack symmetric polynomials. Some properties of the Hi-Jack polynomials and the relationships with the Jack symmetric polynomials and with the basis given by the QISM approach are presented. The Hi-Jack symmetric polynomials are strong candidates for the orthogonal basis of the quantum Calogero model.

KEYWORDS: quantum Calogero model, Hi-Jack symmetric polynomials, Rodrigues formula, quantum inverse scattering method (QISM), Dunkl operator

§1. Introduction

Exact solutions for the Schrödinger equations have provided important significance in physics and mathematical physics. Most of us have studied the Laguerre polynomials and the spherical harmonics in the theory of the hydrogen atom, and the Hermite polynomials and their Rodrigues formula in the theory of the quantum harmonic oscillator. The former is also a good example that shows the role of conserved operators in quantum mechanics. The hydrogen atom has three, independent and mutually commuting conserved operators, namely, the Hamiltonian, the total angular momentum and its z-axis component. The simultaneous eigenstates for the three conserved operators give the orthogonal basis of the hydrogen atom. A classical system with a set of independent and mutually Poisson commuting (involutive) conserved quantities whose number of elements is the same as the degrees of freedom of the system can be integrated by quadrature. This is guaranteed by the Liouville theorem. Such a system is called the completely integrable system. Quantum systems with enough number of such conserved operators are analogously called quantum integrable systems. The hydrogen atom is a simple example of the quantum integrable system.

Among the various quantum integrable systems, one-dimensional quantum many-body systems with inverse-square long-range interactions are now attracting much interests of theoretical physicists. Of the various integrable inverse-square-interaction models, the quantum Calogero model has the longest history. Its Hamiltonian is expressed as

$$\hat{H}_C = \frac{1}{2} \sum_{j=1}^{N} (p_j^2 + \omega^2 x_j^2) + \frac{1}{2} \sum_{j,k=1, j \neq k}^{N} \frac{a^2 - a}{(x_j - x_k)^2}, \quad (1.1)$$

where the constants N, a and ω are the particle number, the coupling parameter and the strength of the external harmonic well respectively. The momentum operator $p_j$ is given by a differential operator, $p_j = -i \frac{\partial}{\partial x_j}$. This model is known to be a quantum integrable system in the sense that it has sufficient number of independent and mutually commuting conserved operators. On the other hand, the Sutherland model, which is a one-dimensional quantum integrable system with inverse-square interactions,

$$\hat{H}_S = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + \frac{1}{2} \sum_{j,k=1, j \neq k}^{N} \frac{a^2 - a}{\sin^2(x_j - x_k)}, \quad (1.2)$$

has been thoroughly investigated and its orthogonal basis is known to consist of the Jack symmetric polynomials. The quantum inverse scattering method and the Dunkl operator (exchange operator) formalism showed that these two models share the same algebraic structure. The fact strongly suggests some similarities in the structures of their Hilbert spaces. In order to clarify this problem, we shall apply a naive approach that we use in the study of the hydrogen atom to the quantum Calogero model and study the deformed multivariable extension of the Hermite polynomials, namely, the Hi-Jack (hidden-Jack) symmetric polynomials.

The Jack symmetric polynomials are uniquely determined by three properties. First, they are the eigenfunctions of the differential operator that is derived from the Hamiltonian of the Sutherland model. Second, they possess triangular expansions in monomial symmetric poly-
nomials with respect to the dominance ordering. And last, they are properly normalized. For detail, see [2.34].

Quite recently, Lapointe and Vinet discovered the Rodrigues formula for the Jack symmetric polynomials using the Dunkl operator for the Sutherland model [14].

In our previous letter [15], we extended their results to the quantum Calogero model and gave the Rodrigues formula for the Hi-Jack symmetric polynomials. Since the Dunkl operators for the Sutherland model and the Calogero model share the same algebraic relations, all the results in our letter was obtained by translating the corresponding results in ref. [16]. However, some parts of their proofs rely on the explicit expressions of the Dunkl operators for the Sutherland model and cannot be translated into the Dunkl operators for the Calogero model. We noticed that the proof of the Rodrigues formula only needs the commutator algebra among the Dunkl operators. One of the aims of this paper is to present a proof of the Rodrigues formula for the Hi-Jack symmetric polynomials. Another aim is to investigate the Hi-Jack symmetric polynomials and to compare them with the basis of the model that was given by QISM [17] and the Dunkl operators [18].

The algebraic construction of the eigenstates for the Hamiltonian (the first conserved operator) of the quantum Calogero model has already been given. Thus the Hi-Jack symmetric polynomials must be linear combinations of them. We shall specify the linear combinations that relate the Hi-Jack polynomials and the Jack polynomials. We have presented the Rodrigues formula for the Hi-Jack “polynomials,” but we have not explicitly shown that they are really polynomials. We shall present a clear answer to these questions.

The outline is as follows. In §2 we summarize and reformulate the results of QISM and Dunkl operators’ approach to the Calogero and Sutherland models. The Jack symmetric polynomials are also introduced. In §3 we present the Rodrigues formula for the Hi-Jack symmetric polynomials and introduce some propositions that guarantee the results. Some properties of the Hi-Jack polynomials are also presented. In §4 we prove the propositions. And in the final section, we give a brief summary and discuss future problems.

§2. Models and Formulations

In our derivation of the Rodrigues formula for the Hi-Jack symmetric polynomials, we do many computations involving the Dunkl operators for the Calogero model. We also compare the Hi-Jack polynomials with the Jack polynomials and with the basis of the Calogero model given by the QISM approach. Thus we need a summary of the Calogero model, the Sutherland model, the QISM approach and the Dunkl operator formalism.

First, we reformulate the QISM and the Dunkl operators for the Calogero model. The Lax matrices for the $N$-body system are given by $N \times N$ operator-valued matrices. To express them, we have to introduce two operator-valued matrices:

\[ \hat{L}_{jk} = ip_j \delta_{jk} - a(1 - \delta_{jk}) \frac{1}{x_j - x_k}, \]
\[ \hat{Q}_{jk} = x_j \delta_{jk}, \]

where $j, k = 1, 2, \cdots, N$. The above $\hat{L}$-matrix is for the (rational) Calogero-Moser model whose Hamiltonian is obtained by taking $\omega = 0$ of the Calogero model or the rational limit of the Sutherland model. The Lax matrices for the Calogero model are

\[ \hat{L}^- = \hat{L} + \omega \hat{Q}, \]
\[ \hat{L}^+ = - \frac{1}{2\omega}(L - \omega \hat{Q}). \]

In eqs. (2.1) - (2.4) above, we introduced unusual normalizations and accent marks for convenience of later discussions. Then the Hamiltonian (1.1) is expressed by the above Lax matrices as

\[ H_C = \omega \left( \sum_{j<k} \hat{L}^+ \hat{L}^- + \frac{1}{2} N \omega (Na + (1 - a)) \right) = \omega \left( \sum_{j<k} \hat{L}^+ \hat{L}^- + E_g \right), \]

where $T_\Sigma$ denotes a sum over all the matrix elements, $T_\Sigma A = \sum_{j<k} A_{jk}$, and $E_g$ is the ground state energy. Note that the first term of the r.h.s of eq. (2.3) is a nonnegative hermitian operator. Thus the ground state is the solution of the following equations,

\[ \sum_{k=1}^N \hat{L}_{jk} \hat{g}_k = 0, \text{ for } j = 1, 2, \cdots, N \]
\[ \Rightarrow \hat{H}_C \hat{g}_k = E_g \hat{g}_k. \]

The ground state wave function is the real Laughlin wave function:

\[ \hat{g}_k = \prod_{1 \leq j < k \leq N} |x_j - x_k|^a \exp(-\frac{1}{2\omega} \sum_{j=1}^N x_j^2). \]

A short note might be in order. The phase of the difference product of the above real Laughlin wave function, which determines the statistics of the particles, or in other words, the symmetry of all the eigenstates, can be arbitrary. We can assign any phase factor to all the exchanges of particles. However, we must introduce a phase factor to the definition of the Dunkl operators. To avoid unnecessary complexity, we fix the phase unity.

The eigenstate of the Calogero model is factorized into an inhomogeneous symmetric polynomial and the ground state wave function. For convenience of investigations on the inhomogeneous symmetric polynomials, we redefine the Lax matrices (2.2) by the following similarity transformation:

\[ L^- = \hat{g}_k^{-1} \hat{L}^- \hat{g}_k, \]
\[ L^+ = \hat{g}_k^{-1} \hat{L}^+ \hat{g}_k. \]

Any operator with a hat, $\hat{O}$, is related to an operator $O$ by the similarity transformation using the ground state wave function $\hat{g}_k$:

\[ O = \hat{g}_k^{-1} \hat{O} \hat{g}_k. \]
\[ \hat{O} = \hat{\phi}_g \hat{O} \hat{\phi}_g^{-1}. \] (2.7b)

A set of mutually commuting conserved operators of the Calogero model \( \{I_n|n = 1, 2, \cdots, N\} \) is given by
\[ I_n = T_\Sigma (L^+ L^-)^n. \] (2.8)

The Hamiltonian \( H_C \) is equal to \( \omega I_1 + E_g \). We regard the first conserved operator \( I_1 \) as the Hamiltonian of the Calogero model. The Heisenberg equations for the \( L^- \) and \( L^+ \) matrices are expressed in the forms of the Lax equation. Moreover, we have more general relations for a class of operators,
\[ V^m_p = T_\Sigma (L^-)^m (L^+)^p |W, \] (2.9)
where the subscript \( W \) means the Weyl ordered product. The class of operators naturally includes the Hamiltonian, \( H_C = \omega V^1_1 \). The generalized Lax equations are
\[ [V^m_p, L^-] = [L^-, Z^m_p] - p[(L^-)^m (L^+)^p-1]|W, \] (2.10a)
\[ [V^m_p, L^+] = [L^+, Z^m_p] + m[(L^-)^m-1 (L^+)^p]|W, \] (2.10b)
where the symbol \( Z^m_p \) is an \( N \times N \) operator-valued matrix that satisfies the sum-to-zero condition:
\[ \sum_{j=1}^N (Z^m_p)_{jk} = \sum_{j=1}^N (Z^m_p)_{kj} = 0, \quad \text{for } k = 1, 2, \cdots, N. \] (2.11)

The generalized Lax equations exhibit that the operators (2.9) satisfy the commutation relations of \( \hat{W} \)-algebra.

The operators with \( m = 0 \) are important in the construction of the eigenstates of the Hamiltonian, because they satisfy
\[ [I_1, V^0_p] = pV^0_p, \quad \text{for } p = 1, 2, \cdots. \] (2.12)

Thus the operators \( V^0_p \) play the same role as the creation operator in the theory of the quantum harmonic oscillator. We call these mutually commuting operators \( V^0_p \) power sum creation operators, whose meaning will be clear later.

Successive operations of the power sum creation operators generate all the eigenstates of the Hamiltonian, which are labeled by the Young tableaux. The Young tableau \( \lambda \) is a non-increasing sequence of \( N \) nonnegative integers:
\[ \lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0\}. \] (2.13)

Then the polynomial part of the excited state \( \phi_\lambda \) is given by
\[ \phi_\lambda = (V^0_0)^{\lambda_1} (V^0_{N-1})^{\lambda_{N-1}} \cdots (V^0_1)^{\lambda_1} \lambda_{N-1} \cdot 1 \]
\[ = \prod_{p=1}^N (V^0_p)^{\lambda_k - \lambda_{k+1}} \cdot 1, \] (2.14)
where \( \lambda_{N-1} = 0 \) and the eigenvalue for the first conserved operator is
\[ I_1 \phi_\lambda = \sum_{k=1}^N \lambda_k \phi_\lambda = E_1(\lambda) \phi_\lambda. \] (2.15)

It is not trivial that \( \phi_\lambda \) is indeed an inhomogeneous symmetric polynomial. We shall prove it in \( \mathbb{H} \). Note that the eigenstate of the original Hamiltonian \( H_C (\mathbb{H}) \) and its eigenvalue are \( \phi_\lambda = \hat{\phi}_g \phi_\lambda \) and \( \omega E_1(\lambda) + E_g \). These eigenstates give the complete set of the eigenstates. However, they are not orthogonal because of the remaining large degeneracy.

Using the Dunkl operator formalism, we can do an analogous investigation on the Calogero model. The Dunkl operators for the model are
\[ \alpha_i = i(p_i + ia \sum_{k=1}^N \frac{1}{x_i - x_k} (K_{ik} - 1)), \] (2.16a)
\[ \alpha^i = -\frac{i}{2\omega} (p_i + ia \sum_{k=1}^N \frac{1}{x_i - x_k} (K_{ik} - 1) + 2i\omega x_l), \] (2.16b)
\[ d_i = \alpha^i \alpha_i, \] (2.16c)
where \( K_{ik} \) is the coordinate exchange operator. The operator \( K_{ik} \) has the properties
\[ K_{ik} = K_{ki}, \quad (K_{ik})^2 = 1, \quad K^i_{ik} = K_{ik}, \quad K_{ik} \cdot 1 = 1, \]
\[ K_{ik} A_l = A_k K_{ik}, \quad K_{ik} A_j = A_j K_{ik}, \quad \text{for } j \neq l, k. \] (2.17)

Here \( A_j \) is either a partial differential operator \( \frac{\partial}{\partial x_j} \) (or equivalently, a momentum operator \( p_j \)), a particle coordinate \( x_j \) or coordinate exchange operators \( K_{jk} \), \( k = 1, 2, \cdots, N, k \neq j \). Note that the action on the ground state of the above Dunkl operators has already been removed by the similarity transformation (2.7). The Dunkl operators satisfy the relations,
\[ [\alpha_i, \alpha_m] = 0, \quad [\alpha^i, \alpha^m] = 0, \] (2.18a)
\[ [\alpha_i, \alpha^m] = \delta_{lm} (1 + a \sum_{k=1}^N K_{ik}) - a(1 - \delta_{lm}) K_{lm}, \] (2.18b)
\[ [d_i, d_m] = a(d_m - d_l) K_{lm}, \] (2.18c)
\[ \alpha_i \cdot 1 = 0. \] (2.18d)

As we have mentioned, the phase factor of the difference product part of the ground state wave function can be arbitrary. This phase factor affects the definition of the Dunkl operators and coordinate exchange operators with hat, i.e., \( \hat{\alpha}_i, \hat{\alpha}^i, \hat{d}_i \) and \( \hat{K}_{ik} \). We have to introduce a phase factor in the defining relations of the coordinate exchange operators (2.17) and the commutation relations of the Dunkl operators (2.13). This modification is naturally introduced by the inverse of the similarity transformation of the Dunkl operators (2.7b). Using the above relations, we can check that the eigenstates of the Hamiltonian \( I_1 \) are given by (2.18).
where \( m_j(x_1, x_2, \cdots, x_N) \) is a monomial symmetric polynomial term in the In terms of the Dunkl operators, we can express the commuting conserved operators \( I_n \) (2.18) and the operators \( V_p^m \) (2.19) as

\[
I_n = \sum_{l=1}^{N} (d_l)^n |_{\text{Sym}}, \quad n = 1, 2, \cdots, N, \tag{2.21}
\]

\[
V_p^m = \sum_{l=1}^{N} \left[ (\alpha_l^m)^*(\alpha_l^m)^p \right] |_{\text{Sym}}, \tag{2.22}
\]

where the symbol \( |_{\text{Sym}} \) means that the action of the operator is restricted to symmetric functions. Then the power sum creation operators are expressed by

\[
V_n^0 = \sum_{l=1}^{N} (\alpha_l^m)^n |_{\text{Sym}} = p_n(\alpha_1^p, \alpha_2^p, \cdots, \alpha_N^p) |_{\text{Sym}}, \tag{2.23}
\]

where \( p_n(x_1, x_2, \cdots, x_N) \) is the power sum symmetric polynomial of degree \( n \). This shows that two kinds of eigenstates (2.14) and (2.19) are related by the transformation between the power sum symmetric polynomials and the monomial symmetric polynomials.

Next, we consider the QISM and the Dunkl operator formalism for the Sutherland model. The Lax matrix for the Sutherland model is

\[
\hat{L}_{jk} = p_j \delta_{jk} + ia(1 - \delta_{jk}) \cot(x_j - x_k). \tag{2.24}
\]

The above Lax matrix gives the Hamiltonian of the Sutherland model by

\[
\hat{H}_S = \frac{1}{2} T_{\Sigma} \hat{L}^2 + \frac{1}{6} a^2 N(N-1)(N+1) + \epsilon_g \tag{2.25}
\]

where \( \epsilon_g \) is the ground state energy. As is similar to eq. (2.4), the ground state of the Sutherland model satisfies the following equations,

\[
\sum_{k=1}^{N} \hat{L}_{jk} \hat{\psi}_g = 0, \quad \text{for } j = 1, 2, \cdots, N
\]

\[
\Rightarrow \hat{H}_S \hat{\psi}_g = \epsilon_g \hat{\psi}_g, \tag{2.26}
\]

because the first term of the r.h.s. of eq. (2.24) is a nonnegative operator. The ground state is given by the trigonometric Jastraw wave function:

\[
\hat{\psi}_g = \prod_{1 \leq j < k \leq N} |\sin(x_j - x_k)|^a. \tag{2.27}
\]

The phase factor of the above trigonometric Jastraw wave function can be arbitrary. By the change of the variables,

\[
\exp 2ix_j = z_j, \quad j = 1, 2, \cdots, N, \tag{2.28}
\]

the Hamiltonian of the Sutherland model (2.2) is transformed to

\[
\hat{H}_S = -2 \sum_{j=1}^{N} (z_j p_{zj})^2 + (a^2 - a) \sum_{j,k=1}^{N} \frac{z_j z_k}{(z_j - z_k)^2}, \tag{2.29}
\]

where \( p_{zj} = -i \frac{\partial}{\partial z_j} \). The ground state wave function (2.26) is transformed to

\[
\hat{\psi}_g = \prod_{1 \leq j < k \leq N} |z_j - z_k|^a \prod_{j=1}^{N} z_j^{-\frac{1}{2}a(N-1)}. \tag{2.31}
\]

Here we do not mind the difference of the scalar factor of the ground state wave function. The similarity transformation of the above Hamiltonian yields

\[
H_S - \epsilon_g
\]

\[
= \hat{\psi}_g^{-1}(\hat{H}_S - \epsilon_g) \hat{\psi}_g
\]

\[
= -2 \sum_{j=1}^{N} (z_j p_{zj})^2 + ia \sum_{j,k=1}^{N} z_j + z_k (z_j p_{zj} - z_k p_{zk}). \tag{2.33}
\]

The above projected Hamiltonian can be derived from the Lax matrices (2.3). We define the “ground state” for the \( \hat{L} \)-matrix (2.1a) by the solution of the equations,

\[
\sum_{k=1}^{N} \hat{L}_{jk} \Delta^a = 0, \quad \text{for } j = 1, 2, \cdots, N, \tag{2.34}
\]

and their solution is

\[
\Delta^a = \prod_{1 \leq j < k \leq N} |x_j - x_k|^a. \tag{2.35}
\]

The phase of the above Jastraw wave function also can be arbitrary. The effect to the Dunkl operators for the Sutherland model, which is made explicit by the similarity transformation, is also the same as that of the Calogero model. By the similarity transformation using the above Jastraw function, we define \( L \) and \( Q \) as

\[
L = \Delta^{-a} \hat{L} \Delta^a, \tag{2.36}
\]

\[
Q = \Delta^{-a} \hat{Q} \Delta^a = \hat{Q}. \tag{2.37}
\]

Then we get the projected Hamiltonian (2.30), whose variables are not \( \{z_j\} \) but \( \{x_j\} \) by

\[
H_S - \epsilon_g = 2 T_{\Sigma} (QL)^2 \equiv 2 \hat{L}_2. \tag{2.38}
\]

From now on, we take \( \hat{L}_2 \) as the Hamiltonian of the Sutherland model. The Jack symmetric polynomials \( J_\lambda(x; 1/a) \) are uniquely defined by

\[
\mathcal{I}_2 J_\lambda(x; 1/a)
\]

\[
= \sum_{k=1}^{N} (\lambda_k^a + a(N + 1 - 2k)\lambda_k) J_\lambda(x; 1/a), \tag{2.39}
\]

(triangularity)
\[ J_\lambda(x; 1/a) = \sum_{\mu \leq \lambda} v_{\lambda\mu}(a) m_\mu(x), \quad (2.34b) \]

(normalization)

\[ v_{\lambda\lambda}(a) = 1, \quad (2.34c) \]

where \( x = (x_1, x_2, \cdots, x_N) \) and \( \lambda \) and \( \mu \) are the Young tableaux (2.13). The symbol \( \preceq \) is the dominance ordering among the Young tableaux (2.13).

Note that the dominance ordering is not a total ordering but a partial ordering. A total ordering among Young tableaux is given by the lexicographic ordering:

\[ \mu \preceq \lambda \iff \sum_{k=1}^{N} \mu_k = \sum_{k=1}^{N} \lambda_k \text{ and } \sum_{k=1}^{l} \mu_k \leq \sum_{k=1}^{l} \lambda_k \text{ for all } l. \quad (2.35) \]

Commuting conserved operators of the Sutherland model are given by

\[ \mathcal{I}_n = T \Sigma(QL)^n. \quad (2.37) \]

We have similar relations to the generalized Lax equations for the Calogero model (2.10).

\[ [U_q^{l}, L] = [L, Y_q^{l}] - q(lQ)^{q-1}|_W, \quad (2.38a) \]

\[ [U_q^{l}, Q] = [Q, Y_q^{l}] + m[(L)^{(q)^{q}}]|_W, \quad (2.38b) \]

where the operator \( U_q^{l} \) is defined by

\[ U_q^{l} = T \Sigma[(L)^{(q)}]|_W. \quad (2.39) \]

The operator-valued matrix \( Y_q^{l} \) also satisfies the sum-to-zero condition,

\[ \sum_{j=1}^{N} (Y_q^{l})_{jk} = \sum_{j=1}^{N} (Y_q^{l})_{kj} = 0, \quad \text{for } k = 1, 2, \cdots, N. \quad (2.40) \]

Since eqs. (2.10) and (2.38) has the same form, we notice the correspondence between the Calogero model and the Sutherland model:

\[ L^+ = L, L^- \leftrightarrow Q, I_k \leftrightarrow \mathcal{I}_k. \quad (2.41) \]

This means that the two QISM’s for the Calogero and Sutherland models give two different representations of the same commutator algebra.

The same situation can also be observed in the Dunkl operators’ approach. We introduce the Dunkl operators for the Sutherland model whose action on the ground state is removed in a similar way to deal with the Dunkl operators for the Calogero model:

\[ \nabla_l = i(p_l + ia \sum_{k=1}^{N} \frac{1}{x_l - x_k}(K_{lk} - 1)), \quad (2.42a) \]

\[ x_l, \quad (2.42b) \]

These Dunkl operators satisfy the following relations,

\[ [\nabla_l, \nabla_m] = 0, \quad [x_l, x_m] = 0, \quad (2.43a) \]

\[ [\nabla_l, x_m] = \delta_{lm}(1 + a \sum_{k=1}^{N} K_{ik}) - a(1 - \delta_{lm})K_{lm}, \quad (2.43b) \]

\[ [D_l, D_m] = a(D_m - D_l)K_{lm}, \quad (2.43c) \]

\[ \nabla_l \cdot 1 = 0, \quad (2.43d) \]

which are completely the same as those of Dunkl operators for the Calogero model (2.18). Commuting conserved operators (2.37) are written by the Dunkl operator as

\[ \mathcal{I}_n = \sum_{l=1}^{N} (D_l)^n |_{\text{Sym}}, \quad n = 1, 2, \cdots, N. \quad (2.44) \]

Thus we notice the correspondence between the two sets of Dunkl operators:

\[ \alpha_l = \nabla_l, \alpha_l^\dagger \leftrightarrow x_l, d_l \leftrightarrow D_l. \quad (2.45) \]

Moreover, in the limit \( \omega \to \infty \), the Lax matrices and the Dunkl operators for the Calogero model reduce to those of the Sutherland model. Thus our theory for the Hi-Jack symmetric polynomials described by the Dunkl operators for the Calogero model contains the results for the Jack symmetric polynomials written by the Dunkl operators for the Sutherland model.

We have summarized the QISM and the Dunkl operators’ approach to the Calogero and Sutherland models. They give two different representations of the same commutator algebra. The QISM and the Dunkl operators for the Calogero model include those for the Sutherland model. Thus our theory for the Calogero model as a special case.

\[ 3. \quad \text{Hi-Jack Symmetric Polynomials} \]

Following the definition of the Jack symmetric polynomials (2.33), we define the Hi-Jack symmetric polynomials \( j_\lambda(x; \omega, 1/a) \) by

(eigenfunction)

\[ I_1 j_\lambda(x; \omega, 1/a) = \sum_{k=1}^{N} \lambda_k j_\lambda(x; \omega, 1/a) \]

\[ \quad = E_1(\lambda) j_\lambda(x; \omega, 1/a), \quad (3.1a) \]

\[ I_2 j_\lambda(x; \omega, 1/a) \]

\[ \quad = \sum_{k=1}^{N} (\lambda_k^2 + a(N + 1 - 2k)\lambda_k) j_\lambda(x; \omega, 1/a) \]

\[ \quad = E_2(\lambda) j_\lambda(x; \omega, 1/a), \quad (3.1b) \]
(triangularity)
\[ j_\lambda(x; \omega, 1/a) = \sum_{\mu \leq \lambda, \mu \neq \emptyset}^{} w_{\lambda\mu}(a, 1/2\omega) m_\mu(x), \quad (3.1c) \]

(normalization)
\[ w_{\lambda\lambda}(a, 1/2\omega) = 1, \quad (3.1d) \]

where \(|\lambda|\) is the weight of the Young tableau, \(|\lambda| = \sum_{k=1}^{N} n_k\). In order to write down the Rodrigues formula for the Hi-Jack polynomials, it is convenient to introduce the following operators (cf. eq. (2.16)):
\[ \alpha_j^\dagger = \prod_{j \in J} \alpha_j^\dagger, \quad (3.2a) \]
\[ d_{m,j} = (d_{j,1} + ma)(d_{j,2} + (m + 1)a) \]
\[ \cdots (d_{j,k} + (m + k - 1)a), \quad (3.2b) \]

where \( J \) is a subset of a set \( \{1, 2, \cdots, N\} \) whose number of elements \(|J|\) is equal to \( k \). From eq. (2.18c), we can verify an identity,
\[ (d_i + ma)(d_i + (m + 1)a) \bigg|_\text{Sym} \{^{(i,j)}_{\lambda, \mu} \} = (d_j + ma)(d_i + (m + 1)a) \bigg|_\text{Sym} \{^{(i,j)}_{\lambda, \mu} \}, \quad (3.3) \]

where \( m \) is some integer. The symbol \( \bigg|_\text{Sym} \{^{(i,j)}_{\lambda, \mu} \} \) where \( J \) is some set of integers means that the action of operators is restricted to the space that is symmetric with respect to exchange of any indices in the set \( J \). This identity guarantees that the operator \( d_{m,j} \) does not depend on the ordering of the elements of a set \( J \). The generators of the Hi-Jack polynomials are expressed as
\[ b_k^+ = \sum_{J \subseteq \{1, 2, \cdots, N\}}^{\lambda \subseteq \{1, 2, \cdots, N\}} \prod_{j \in J} \alpha_j^\dagger d_{1,j}, \quad \text{for } k = 1, 2, \cdots, N - 1, \quad (3.4a) \]
\[ b_N^+ = \alpha_1^\dagger \alpha_2^\dagger \cdots \alpha_N^\dagger. \quad (3.4b) \]

Using the generators (3.4), we can write down the Rodrigues formula for the Hi-Jack polynomials \( j_\lambda(x; \omega, 1/a) \) as
\[ j_\lambda(x; \omega, 1/a) = C^{-1}_\lambda (b_N^+)^{\lambda_N} (b_{N-1}^+)^{\lambda_{N-1} - \lambda_N} \]
\[ \cdots (b_1^+)^{\lambda_1 - \lambda_2} \cdot 1, \quad (3.5) \]

with the constant \( C_\lambda \) given by
\[ C_\lambda = \prod_{k=1}^{N-1} C_k(\lambda_1, \lambda_2, \cdots, \lambda_{k+1}; a), \quad (3.6) \]

where
\[ C_k(\lambda_1, \lambda_2, \cdots, \lambda_{k+1}; a) = (a)_{\lambda_k - \lambda_{k+1}}(2a + \lambda_{k-1} - \lambda_k)_{\lambda_k - \lambda_{k+1}} \]
\[ \cdots (ka + \lambda_1 - \lambda_k)_{\lambda_k - \lambda_{k+1}}. \quad (3.7) \]

In the above expression, the symbol \((\beta)_n\) is the Pochhammer symbol, that is, \((\beta)_n = \beta(\beta+1)\cdots(\beta+n-1)\). What we want to prove is summarized as the following proposition.

**Proposition 3.1** The symmetric polynomials generated by the Rodrigues formula (3.5) satisfy the definition of the Hi-Jack symmetric polynomials (5.4).

The first two out of four requirements (3.1) are derived from the following propositions.

**Proposition 3.2**
\[ [I_1, b_k^+]_{\text{Sym}} = k b_k^+_{\text{Sym}}. \quad (3.8) \]

**Proposition 3.3** The null operators \( n_{i+1,J} \), which are defined by
\[ n_{k+1,J} = d_{0,J}, \quad J \subseteq \{1, 2, \cdots, N\}, \quad |J| = k + 1, \quad (3.9) \]
satisfy
\[ n_{k+1,J}(b_k^+)_{\lambda_k}^\lambda (b_{k-1}^+)_{\lambda_2 - \lambda_k} \cdots (b_1^+)_{\lambda_1 - \lambda_2} \cdot 1 = 0. \quad (3.10) \]

**Proposition 3.4**
\[ [I_2, b_k^+]_{\text{Sym}} = \left\{ b_k^+ (2I_1 + k + ak(N - k)) \right\} + \sum_{J \subseteq \{1, 2, \cdots, N\}}^{\lambda \subseteq \{1, 2, \cdots, N\}} g_{k+1,J} n_{k+1,J} \bigg|_{\text{Sym}}, \quad (3.11) \]

where \( g_{k+1,J} \) is an unspecified nonsingular operator that satisfies \( g_{N+1,J} = 0 \).

The first requirement directly follows from Proposition 3.2. For a while, we forget about the normalization constant. From the l.h.s. of eq. (3.1a), we get
\[ I_1 (b_N^+)^{\lambda_N} (b_{N-1}^+)^{\lambda_{N-1} - \lambda_N} \cdots (b_1^+)_{\lambda_1 - \lambda_2} \cdot 1 \]
\[ = \left( [I_1, (b_N^+)^{\lambda_N} (b_{N-1}^+)^{\lambda_{N-1} - \lambda_N} \cdots (b_1^+)_{\lambda_1 - \lambda_2}] \right) \cdot 1. \quad (3.12) \]

Because of eq. (2.18d), the second term of the above equation vanishes:
\[ I_1 \cdot 1 = \sum_{k=1}^{N} \alpha_k^\dagger \alpha_k \cdot 1 = 0. \quad (3.13) \]

Then using Proposition 3.2, we get the expected result:
\[ I_1 (b_N^+)^{\lambda_N} (b_{N-1}^+)^{\lambda_{N-1} - \lambda_N} \cdots (b_1^+)_{\lambda_1 - \lambda_2} \cdot 1 \]
\[ = \left( \sum_{k=1}^{N-1} k(\lambda_k - \lambda_{k+1}) + N\lambda_N \right) \]
\[ (b_N^+)^{\lambda_N} (b_{N-1}^+)^{\lambda_{N-1} - \lambda_N} \cdots (b_1^+)_{\lambda_1 - \lambda_2} \cdot 1 \]
\[ = \sum_{k=1}^{N} \lambda_k (b_N^+)^{\lambda_N} (b_{N-1}^+)^{\lambda_{N-1} - \lambda_N} \cdots (b_1^+)_{\lambda_1 - \lambda_2} \cdot 1. \quad (3.14) \]

The second requirement (3.1b) is shown by induction. It is easy to show it for \( \lambda = 0 \),
\[ I_2 \cdot j_0(x; \omega, 1/a) = E_2(0)j_0(x; \omega, 1/a) = 0, \quad (3.15) \]
by using eq. (2.18d) because \( j_0(x; \omega, 1/a) \) is equal to 1 as a polynomial. By inductive assumption, eq. (3.1b) holds up to \( \lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0\} \). Then for \( \lambda = \{\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_k + 1, 0, \ldots, 0\} \), we have

\[
I_2 b_k^+(b_k^+)^{\lambda_k - \lambda_k} \cdots (b_k^+)^{\lambda_1 - \lambda_2} \cdot 1
= \left( E_2(\{\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0\})
+ 2 \sum_{j=1}^k \lambda_j + k + ak(N - k) \right)
\times (b_k^+)^{\lambda_k - \lambda_k} \cdots (b_k^+)^{\lambda_1 - \lambda_2} \cdot 1
= E_2(\{\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_k + 1, 0, \ldots, 0\})
\times (b_k^+)^{\lambda_k - \lambda_k} \cdots (b_k^+)^{\lambda_1 - \lambda_2} \cdot 1,
\]  

(3.16)

which completes the proof.

As byproducts of the proof of the last two requirements of Proposition 3.3, we notice the following results.

**Proposition 3.5** The expansion coefficients of the Hi-Jack polynomials \( C_{\lambda \omega \lambda'}(a, 1/2\omega) \) are polynomials of \( a \) and \( 1/2\omega \) with integer coefficients. This property is analogous to that stated by the Macdonald-Stanley conjecture for the Jack polynomials.

**Proposition 3.6**

\[
J_\lambda(\alpha^1, \alpha^2, \ldots, \alpha^N; 1/a) \cdot 1 = j_\lambda(x; \omega, 1/a).
\]  

(3.18)

During the discussion in \( \S 3 \), we have noticed that the Hi-Jack polynomials should reduce to the Jack polynomials in the limit \( \omega \to \infty \):

\[
j_\lambda(x; \omega = \infty, 1/a) = J_\lambda(x; 1/a).
\]  

(3.19)

Proposition 3.6 gives another relationship between the Jack polynomials and the Hi-Jack polynomials. The relationship between the eigenstates of the Hamiltonian \( H_C \) given by the QISM (2.14) or the eigenstates (2.19a) and the Hi-Jack polynomials is now clear. Several bases for the ring of homogeneous symmetric polynomials are known \( \S 3 \). The power sums, the monomial symmetric polynomials and the Jack polynomials are examples of such bases. Thus, the transformation between the Hi-Jack polynomials and two kinds of the eigenstates of the Hamiltonian \( H_C \) is the transformation between the bases of homogeneous symmetric polynomials. Defining the transformations by

\[
J_k^{\mu}(\{p_\lambda\}) = J_\lambda,  
J_k^{\mu}(\{m_\lambda\}) = J_\lambda,
\]  

(3.20a)

(3.20b)

we have

\[
f_k^{J}(\{\phi_\lambda\}) = j_\lambda,  
f_k^{J}(\{\varphi_\lambda\}) = j_\lambda.
\]  

(3.21a)

(3.21b)

Note that the transformation (3.20b) is nothing but the expansion of the Jack polynomials in the monomial symmetric polynomials (2.34b).

In the next section, we shall prove Proposition 3.3 and the last two requirements of Proposition 3.1.

**§ 4. Proofs**

### 4.1 Hamiltonian

We shall prove Proposition 3.3. It is easy to prove the case \( k = N \). From the definition of \( I_1 \) and \( b_N^+ \), we have

\[
[I_1, b_N^+] = \sum_{j=1}^N [d_j, \alpha^1 \alpha^2 \cdots \alpha^N] = \sum_{j=1}^N \sum_{i=1}^N \alpha^1 \cdots \alpha^i_{i-1} [d_j, \alpha^i] \alpha^i_{i+1} \cdots \alpha^N.
\]  

(4.1)

Using eq. (2.18c), we get

\[
[I_1, b_N^+] = \sum_{j=1}^N \sum_{i=1}^N \alpha^1 \cdots \alpha^i_{i-1}
\times (N b_N^+ + a \alpha^1 \cdots \alpha^i_{i-1})
\times \sum_{j=1}^N \alpha^i_{j+1} (K_{j+1} - K_j) \alpha^i_{i+1} \cdots \alpha^N
\]  

\[= Nb_N^+,
\]  

(4.2)

which says the validity of eq. (3.20) for the case \( k = N \). Note that we do not have to restrict the action of the operator in the above calculation.

For the case \( k \neq N \), we need more computation. First, we decompose the Hamiltonian \( I_1 \) into two parts:

\[
[I_1, b_N^+] = \sum_{j \in J} \left[ \sum_{i \in J} \delta_{i, j} \delta_{j+1} - \sum_{i, j \in J} \delta_{i, j} \delta_{j+1} \right].
\]  

(4.3)

The first part of the r.h.s. of eq. (4.3) is calculated as

\[
\sum_{i \in J} \delta_{i, j} \delta_{j+1}
\]  

\[
= \sum_{i \in J} \sum_{j=1}^k \left\{ \alpha^1 \cdots \alpha^i_{i-1} [d_i, \alpha^i_{j+1} \alpha^i_{j+2} \cdots \alpha^i_{k+1}]; d_i, \alpha^i_{j+1} \right\}
\times (d_{j+1} + a) \cdots (d_{j+1} + (l - 1)a) [d_i, \alpha^i_{j+1} + l a]
\]  

\[
(d_{j+1} + (l + 1)a) \cdots (d_k + ka)
\]  


\[
\sum_{i \in J} \sum_{l = 1}^{k} \left\{ \alpha^\dagger_{j_1} \cdots \alpha^\dagger_{j_{l-1}} \right. \\
(\delta_{ij} \alpha^\dagger_{j_l} + (1 + a \sum_{j \in J} K_{j,i} + a \sum_{j \notin J} J_{ji}) - (1 - \delta_{ij}) a K_{ji}) \\
\left. \alpha^\dagger_{j_{l+1}} \cdots \alpha^\dagger_{j_k} d_{1,j} \\
+ \alpha^\dagger_{j}(d_{j_1} + a) \cdots (d_{j_{l-1}} + (l - 1)a) \\
((d_{j_l} + la) - (d_i + la)) K_{ij} \\
(d_{j_{l+1}} + (l + 1)a) \cdots (d_{j_k} + ka) \right\}. \tag{4.4}
\]

We move exchange operators to the rightmost and utilize the restriction \( \left| \sum_{i \in J} d_i \right|_{\text{Sym}} \). Using eqs. (2.17) and (3.3), we get

\[
\left[ \sum_{i \in J} d_i \right] \left| \sum_{i \in J} \alpha^\dagger_j d_{1,j} \right|_{\text{Sym}} \\
= k a j d_{1,j} \left| \sum_{i \in J} d_i \right|_{\text{Sym}} + a \sum_{j \notin J} \sum_{l = 1}^{k} \left\{ \alpha^\dagger_{j} \dagger_{j_1} \cdots \alpha^\dagger_{j_{k-1}} \right. \\
+ \alpha^\dagger_{j}(d_{j_1, J_{j_1}} d_i + ka) - d_{1,j_1} (d_i + ka) \left. \right\} \left| \sum_{i \in J} d_i \right|_{\text{Sym}} \\
= k a j d_{1,j} \left| \sum_{i \in J} d_i \right|_{\text{Sym}} + a \sum_{i \notin J} \sum_{l = 1}^{k} \left\{ \alpha^\dagger_{j} \dagger_{j_1} \cdots \alpha^\dagger_{j_{k-1}} \right. \\
+ \alpha^\dagger_{j}(d_{j_1, J_{j_1}} d_i + ka) - d_{1,j_1} (d_i + ka) \left. \right\} \left| \sum_{i \in J} d_i \right|_{\text{Sym}}. \tag{4.5}
\]

The second part of the r.h.s. of eq. (3.3) is calculated as

\[
\left[ \sum_{i \notin J} d_i \right] \left| \sum_{i \notin J} \alpha^\dagger_j d_{1,j} \right|_{\text{Sym}} \\
= \sum_{i \notin J} \sum_{l = 1}^{k} \left\{ \alpha^\dagger_{j_1} \cdots \alpha^\dagger_{j_{l-1}} \right. \\
+ \alpha^\dagger_{j}(d_{j_1, J_{j_1}} d_i + ka) - d_{1,j_1} (d_i + ka) \left. \right\} \left| \sum_{i \notin J} d_i \right|_{\text{Sym}} \\
= \sum_{i \notin J} \sum_{l = 1}^{k} \left\{ -a \alpha^\dagger_{j_1} \cdots \alpha^\dagger_{j_{l-1}} \right. \\
+ \alpha^\dagger_{j}(d_{j_1, J_{j_1}} d_i + ka) - d_{1,j_1} (d_i + ka) \left. \right\} \left| \sum_{i \notin J} d_i \right|_{\text{Sym}}. \tag{4.6}
\]

Substitution of eqs. (4.5) and (4.6) into eq. (3.3) yields

\[
\left[ I_1, b^+_k \right]_{\text{Sym}} \\
= \sum_{J \subseteq \{1, 2, \ldots, N\}} \sum_{i \notin J} \left\{ k a j d_{1,j} \\
+ a \sum_{i \notin J} \sum_{l = 1}^{k} \left\{ (\alpha^\dagger_j - \alpha^\dagger_j \dagger_{j_1} \alpha^\dagger_j) d_{j_1, J_{j_1}} (d_i + ka) \\
+ \alpha^\dagger_j (d_{j_1, J_{j_1}} d_i + (ka) \right\} \right\} \left| \sum_{i \notin J} d_i \right|_{\text{Sym}} \\
= k b^+_k \left| \sum_{i \notin J} d_i \right|_{\text{Sym}}. \tag{4.7}
\]

Thus we have proved Proposition 3.2.

4.2 Null operators

Since the function \((b^+_k)^{\lambda_k} (b^+_k)^{\lambda_{k-1} - \lambda_k} \cdots (b^+_1)^{\lambda_1 - \lambda_2} \cdot 1\) is a symmetric function of \(\{x_1, x_2, \cdots, x_N\}\), it is sufficient to prove the case \(J = \{1, 2, \cdots, k + 1\}\). For brevity, we use the symbol \(n_{k+1} = n_{k+1} \{1, 2, \cdots, k + 1\}\) hereafter. Then the expression to be proved is

\[
n_{k+1} (b^+_k)^{\lambda_k} (b^+_k)^{\lambda_{k-1} - \lambda_k} \cdots (b^+_1)^{\lambda_1 - \lambda_2} \cdot 1 = 0. \tag{4.8}
\]

This follows from

\[
n_{i+1} b^+_i \left| \sum_{i \notin J} d_i \right|_{\text{Sym}} \sim n_{k+1} \left| \sum_{i \notin J} d_i \right|_{\text{Sym}}, \text{ for } i \geq k, \tag{4.9}
\]

where the symbol \(\sim\) means that the term on the r.h.s. can be multiplied on the left by some nonsingular operator \(\mathcal{O}, \mathcal{O} \cdot 0 = 0\). We can easily verify

\[
n_{k+1} (b^+_k)^{\lambda_k} (b^+_k)^{\lambda_{k-1} - \lambda_k} \cdots (b^+_1)^{\lambda_1 - \lambda_2} \cdot 1 \\
\sim n_{k+1} (b^+_k)^{\lambda_k} (b^+_k)^{\lambda_{k-1} - \lambda_k} \cdots (b^+_1)^{\lambda_1 - \lambda_2} \cdot 1 \\
\vdots \\
\sim n_1, 1 \\
= 0, \tag{4.10}
\]

using eqs. (2.18d) and (4.4). For convenience of explanation, we introduce a symbol \([m] \) for a set \(\{1, 2, \cdots, m\}\) with some integer \(m\). We also introduce \([J]\) with some set of integers \(J\) that indicates the operand \(a\) is a symmetric function of \(x_j, j \in J\). From the identity (3.3), we have

\[
n_{k+1} \left| [k+1] \right|_{\text{Sym}} \\
n_{k} (d_{k+1} + ka) \left| [k+1] \right|_{\text{Sym}} \\
= \left\{ k a n_k + n_{k-1} d_{k+1} (d_k + (k - 1)a) \\
+ a n_{k-1} (d_k + d_k - d_k) \right\} \left| [k+1] \right|_{\text{Sym}} \\
= \left\{ k a n_k + (a K_{k+1} + a) n_k \\
+ n_{k-1} d_{k+1} (d_k - (k - 1)a) \right\} \left| [k+1] \right|_{\text{Sym}}
which means

\begin{equation}
\frac{\sum_{J,\emptyset \subseteq J \subseteq [k]} a_{J_i} b_{k,i}^{+} b_{k,i}^{-} \left[ \begin{array}{c} \Sym \end{array} \right] [^{[k]}]}{\Sym} \sim n_{k+1}^{[k]} \Sym, \quad M \geq N. \tag{4.14}
\end{equation}

Here, the superscript \([M]\) over Dunkl operators indicates that they are made from the Dunkl operators \([2.16]\) that depend not only on the variables \(x_1, x_2, \ldots, x_N\) but also on \(x_{N+1}, \ldots, x_M\). Note that \(n_{k+1}^{[M]}\) and \(b_{k,i}^{[M]}\) are symmetric under \(S_N\) but not under \(S_M\). We just changed the number of variables of Dunkl operators \([2.16]\) but do not change the definition of operators made from them, such as conserved operators \((2.20)\) and generators \([3.2]\) and \([3.3]\). Namely, the indices and subsets in the summand are included in the set \([N]\).

All the Dunkl operators in the remainder of this \([1.2]\) will always depend on \(x_1, \ldots, x_M\). We shall omit the superscript \([M]\) in the following.

To prove Proposition \(4.1\), we need several lemmas. We define the restricted generator by

\begin{equation}
b_{k,J}^{+} = \sum_{\{j \subseteq J \mid |j| = k\}} a_{j}^{+} d_{j}, \quad k = 0, 1, \ldots, k, \tag{4.15}
\end{equation}

where \(J\) is a set of integers. From the definition of generators \([3.4]\), we can easily verify

**Lemma 4.1**

\begin{equation}
b_{k,J}^{+} = \sum_{l=0}^{k} \sum_{\{j \subseteq J \mid |j| = l\}} a_{j}^{+} b_{k-l,J}^{+} d_{k-l+1,J}, \tag{4.16}
\end{equation}

with \(b_{0,J}^{+} = 1\) and when \(|J| = 0\), \(a_{J}^{+} = 1\) and \(d_{k+1,J} = 1\).

We shall also use the following formulæ:

**Lemma 4.2**

\begin{align}
[d_{i}, a_{j}^{+}] &= -a a_{i}^{+} \sum_{j \in J} a_{j}^{+} K_{ij}, \quad i \notin J, \tag{4.17a} \\
[d_{i}, a_{j}^{+}] &= a_{i}^{+} (1 + a) \sum_{j \in M \setminus \{j\}} K_{ij}, \quad i \in J. \tag{4.17b}
\end{align}

The first two formulæ can be checked from the definition of \(d_{i}\) and \(a_{j}^{+}\) and commutation relations \([2.18]\). The last formulæ \((4.17c)\) is proved by induction on \(k\).

In the following, we often need to identify the terms that do not have \(a_{1}^{+}\) appearing as an explicit factor on the left of the terms. When \(O\) represents a Dunkl operator of our interest, such terms are denoted by \(O\) \(\mid a_{1}^{+} \sim 0\).

**Lemma 4.3** For \(M \geq n \geq k + 1\), we have

\begin{equation}
\left( n_{k+1, \alpha_{2}^{+} \cdots \alpha_{N}^{+}} + a_{1}^{+} \alpha_{k+1, \alpha_{2}^{+} \cdots \alpha_{N}^{+}} d_{1}, d_{2}, \ldots, k+1 \right) \mid a_{1}^{+} \sim 0 \sim d_{1}. \tag{4.18}
\end{equation}

This formula is also proved by induction on \(k\).

As a step toward proving Proposition \(4.1\), we prove the case \(N = k + 1\).

**Proposition 4.2**

\begin{equation}
\left( n_{k+1, \alpha_{2}^{+} \cdots \alpha_{N}^{+}} + a_{1}^{+} \alpha_{k+1, \alpha_{2}^{+} \cdots \alpha_{N}^{+}} d_{1}, d_{2}, \ldots, k+1 \right) \mid a_{1}^{+} \sim 0 \sim d_{1}. \tag{4.19}
\end{equation}

Both sides of the above equation are symmetric under permutations of indices \(1, \ldots, k + 1\). From the first two equations of Lemma \(4.2\), we have

\begin{equation}
\left( n_{k+1, \alpha_{2}^{+} \cdots \alpha_{k+1}^{+}} \right) \mid a_{1}^{+} \sim 0 \sim n_{k+1} \mid a_{1}^{+} \sim 0. \tag{4.20}
\end{equation}

where \(O_{ij}\) is some unspecified operator that can be written by \(d_{i}\) and \(K_{im}\) with \(1 \leq l, m \leq k + 1\) and \(\alpha_{1}^{+} \cdots \alpha_{k+1}^{+} = \prod_{j=1}^{k+1} \alpha_{j}^{+}\). Commutators among the operators made of \(d_{i}\) operators are also operators made of \(d_{i}\) and \(K_{im}\) with \(1 \leq l, m \leq k + 1\). Thus we can say that the generator \(b_{k+[k+1]}^{+}\) and the l.h.s. of eq. \([4.19]\) have factors \(a_{1}^{+} \cdots \alpha_{k+1}^{+}\) on the left. Therefore, in order to prove Proposition \(4.2\), it is sufficient for us to prove the following expression:

\begin{equation}
\left( n_{k+1, \alpha_{k+1}^{+} d_{1}, \alpha_{k+1}^{+} \cdots \alpha_{k+1}^{+}} \right) \mid a_{1}^{+} \sim 0 \sim n_{k+1} \mid a_{1}^{+} \sim 0. \tag{4.21}
\end{equation}

This is proved by induction on \(k\).

We need a few more formulæ to prove Proposition \(4.1\).

**Lemma 4.4** For all sets of positive integers \(J = \{j_1, j_2, \ldots, j_l\}\) such that \(J \cap \{1, 2, \ldots, N\} = \emptyset\), we have

\begin{align}
\left( n_{k+1, \alpha_{j}^{+}} \right) \mid a_{1}^{+} \sim 0 \sim n_{k+1} \mid a_{1}^{+} \sim 0, \tag{4.22a} \\
\left( n_{k+1, \alpha_{j}^{+} d_{k+1-l, j}} \right) \mid a_{1}^{+} \sim 0 \sim n_{k+1} \mid a_{1}^{+} \sim 0. \tag{4.22b}
\end{align}
These formulæ can be straightforwardly verified by using the definitions of operators, the third equation of Lemma 4.2 and the fundamental commutation relations among Dunkl operators (1.18).

Now we are ready to prove Proposition 4.1. We shall prove it by induction on \( k \). First, we shall check the case \( k = 1 \). From the definition of generator (4.4) and (4.15), we have

\[
\left[ n_{k+1}, d_{k+1-l,J} \right]_{\text{Sym}}^{[N]} \sim n_{k+1}^{[N]}_{\text{Sym}}. \quad (4.22c)
\]

Using the Leibniz rule, we decompose the l.h.s. into two parts:

\[
\left[ n_{k+1}, \alpha_{J}^{\dagger} b_{k-l,[k+1]}^{\dagger} d_{k-l+1,J} \right]_{\text{Sym}}^{[N]} + \alpha_{J}^{\dagger} b_{k-l,[k+1]}^{\dagger} \left[ n_{k+1}, d_{k-l+1,J} \right]_{\text{Sym}}^{[N]} \quad (4.30)
\]

Then from the third formula of Lemma 4.4, we notice that the second term is similar to \( n_{k+1} \):

\[
\alpha_{J}^{\dagger} b_{k-l,[k+1]}^{\dagger} \left[ n_{k+1}, d_{k-l+1,J} \right]_{\text{Sym}}^{[N]} \sim n_{k+1}^{[N]}_{\text{Sym}}. \quad (4.31)
\]

Our remaining task is to check the first term. When \( l = |J| = 0 \), the first term is similar to \( n_{k+1} \) because of Proposition 4.2. When \( l \neq 0 \), we have to do some calculation:

\[
\left[ n_{k+1}, \alpha_{J}^{\dagger} b_{k-l,[k+1]}^{\dagger} d_{k-l+1,J} \right]_{\text{Sym}}^{[N]} \sim n_{k+1}^{[N]}_{\text{Sym}}. \quad (4.32)
\]

From the third formula of Lemma 4.4, the first term of the r.h.s. of eq. (4.32) is calculated as

\[
\left[ n_{k+1}, \alpha_{J}^{\dagger} b_{k-l,[k+1]}^{\dagger} d_{k-l+1,J} \right]_{\text{Sym}}^{[N]} \sim n_{k+1}^{[N]}_{\text{Sym}}. \quad (4.34)
\]

Note that the operators \( b_{k-l,[k+1]}^{\dagger} \) and \( d_{k-l+1,J} \) are invariant under the permutations of indices \( 1, \cdots, k+1 \). Using the second formula of Lemma 4.4, we can verify that the first term of the r.h.s. of eq. (4.32) reduces to

\[
\left[ n_{k+1}, \alpha_{J}^{\dagger} b_{k-l,[k+1]}^{\dagger} d_{k-l+1,J} \right]_{\text{Sym}}^{[N]} \sim n_{k+1}^{[N]}_{\text{Sym}}. \quad (4.35)
\]

Then the second term of the r.h.s. of eq. (4.33) is calculated as

\[
\left[ n_{k+1}, \alpha_{J}^{\dagger} b_{k-l,[k+1]}^{\dagger} d_{k-l+1,J} \right]_{\text{Sym}}^{[N]} \sim n_{k+1}^{[N]}_{\text{Sym}}. \quad (4.36)
\]

On the other hand, the second term of the r.h.s. of eq. (4.32) is separated into two parts as

\[
\alpha_{J}^{\dagger} \left[ n_{k+1}, b_{k-l,[k+1]}^{\dagger} d_{k-l+1,J} \right]_{\text{Sym}}^{[N]} \quad (4.37)
\]

Thus we have confirmed that the proposition holds for \( k = 1 \).

By inductive assumption, the proposition holds up to \( k - 1 \). What we like to show is that the commutator between \( n_{k+1} \) and each term of the decomposition of the generator (4.14) is similar to \( n_{k+1} \), i.e.,

\[
\left[ n_{k+1}, \alpha_{J}^{\dagger} b_{k-l,[k+1]}^{\dagger} d_{k-l+1,J} \right]_{\text{Sym}}^{[N]} \sim n_{k+1}^{[N]}_{\text{Sym}}. \quad (4.29)
\]

Using the Leibniz rule, we decompose the l.h.s. into two parts:

\[
\left[ n_{k+1}, \alpha_{J}^{\dagger} b_{k-l,[k+1]}^{\dagger} d_{k-l+1,J} \right]_{\text{Sym}}^{[N]} \sim n_{k+1}^{[N]}_{\text{Sym}}. \quad (4.32)
\]

Summarizing the results, we have

\[
\left[ n_{2}, b_{1,[2]}^{\dagger} \right]_{\text{Sym}}^{[N]} \sim n_{2}^{[N]}_{\text{Sym}}. \quad (4.28)
\]
\[ -\alpha_j^k b_{k-l+[k+1]} n_{k+1} d_{k-l+1,J} \begin{bmatrix} N \end{bmatrix}_{\text{Sym}} . \] 

(4.37)

In an analogous way to the verification of the first and the second terms in the r.h.s. of eq. (4.33), both the second and the first terms in the r.h.s. of eq. (4.37) are respectively similar to \( n_{k+1} \). Then we obtain

\[ \alpha_j^k [n_{k+1}, b_{k-l+[k+1]}] d_{k-l+1,J} \begin{bmatrix} N \end{bmatrix}_{\text{Sym}} \sim n_{k+1} \begin{bmatrix} N \end{bmatrix}_{\text{Sym}} . \] 

(4.38)

Equations (4.33), (4.36) and (4.38) yield

\[ [n_{k+1}, \alpha_j^k b_{k-l+[k+1]}] d_{k-l+1,J} \begin{bmatrix} N \end{bmatrix}_{\text{Sym}} \sim n_{k+1} \begin{bmatrix} N \end{bmatrix}_{\text{Sym}} . \] 

(4.39)

Thus we have proved Proposition 4.4.

4.3 Generators

Proposition 4.4 is given in a form whose operators depend on \( N \) variables \( x_1, \ldots, x_N \). It is convenient for us to explicitly indicate the number of variables, for we shall change the number of variables during the induction procedure.

Proposition 4.3

\[ \left[ I_1^N(N), b_{k,[N]}^* \right] \begin{bmatrix} N \end{bmatrix}_{\text{Sym}} \]

\[ = \left\{ b_{k,[N]}^* (2 I_1^{[N]}(N) + k + a k(N - k)) + \sum_{J \subseteq [N], |J| = k+1} g_{k+1,J}^N \right\} \begin{bmatrix} N \end{bmatrix}_{\text{Sym}}, N \geq k, (4.40) \]

where \( g_{k+1,J}^N \) is an unspecified nonsingular operator that satisfies \( g_{k+1,J}^N + 1 = 0 \).

Note that we have introduced the notation

\[ I_{k}^N(N) = \sum_{i=1}^{N} (a_j^N)^n. \] 

(4.41)

It is obvious that Proposition 4.4 is tantamount to the above proposition. We shall prove Proposition 4.3 by induction on \( k \) and \( N \). Precisely speaking, we use the induction on \( l \), which relates \( k \) and \( N \) by \( k = l + 1 \) and \( N = l + M \) with arbitrary integer \( M \). The plan requires several lemmas.

Lemma 4.5

\[ \left[ (d_j^M)^2, \alpha_1^M, \ldots, \alpha_N^M \right] \]

\[ = \alpha_1^M \ldots \alpha_N^M \]

\[ \left\{ (1 + a K_{ij} + a \sum_{k=N+1}^{M} K_{jk})^2 \right. \]

\[ + (1 + a K_{ij} + a \sum_{k=N+1}^{N} K_{jk}) d_j^M \]

\[ + d_j^M (1 + a K_{ij} + a \sum_{k=N+1}^{N} K_{jk}) \}, \]

\[ 1 \leq i, j \leq N, i \neq j, \] 

(4.42a)

\[ \left[ (d_j^M)^2, \alpha_1^M, \ldots, \alpha_N^M \right]_{\text{Sym}} \]

\[ = -a \alpha_1^M, \ldots, \alpha_N^M \]

\[ (d_j K_{ij} + K_{ij} d_j + K_{ij}), \]

\[ j = i \text{ or } N + 1 \leq j \leq M. \] 

(4.42b)

Both formulae are readily verified from the first two formulae of Lemma 4.3. Lemma 4.3 leads to the following formula.

Corollary 4.1 For \( 1 \leq i < N \leq M \), we have

\[ \left[ I_1^M(N - 1), \alpha_1^M, \ldots, \alpha_N^M \right] \] 

\[ = \alpha_k^M \left[ I_1^{M-1}(N - 1), \alpha_1^M, \ldots, \alpha_N^M \right] \]

\[ \left| \alpha_i^M \right| \sim 0, \] 

\( O^{M-1} \sim O^M \), \( N \leq k \leq M. \) 

(4.43)

The restriction \( \left| O^{M-1} \sim O^M \right| \) means that we respectively identify the Dunkl operators \( \alpha_1^M, \ldots, \alpha_N^M \) and \( d_i^M \) with \( \alpha_i^M, \ldots, \alpha_N^M \) and \( d_i^M \), where \( i \leq M - 1 \).

The following formulae are valid for arbitrary number of variables.

Lemma 4.6 For \( M \geq N \), we have

\[ \left[ I_1^M(N - 1), \alpha_1^M \right] \] 

\[ = a ((N - 1) (d_j^M)^n - I_1^M(N - 1)) \] 

\[ \left| \alpha_1^M \right| \sim 0, \] 

\[ \left| (d_j^M)^n, d_i^M \right| \] 

\[ = a d_i^M \left[ I_1^M(N - 1) - (N - 1) (d_j^M)^n \right] \] 

\[ \forall N \geq 2. \] 

(4.44b)

The first identity can be checked from eq. (2.18c). The second one is proved by induction on \( N \) from again eq. (2.18c).

Now we shall start the proof of Proposition 4.3. We have to separately prove the case for \( k = N \) because of the difference of the definition of the generator \( \left[ I_1^N(N) \right] \):

Proposition 4.4

\[ \left[ I_1^N(N), b_{N,[N]}^+ \right] = b_{N,[N]}^+ \left[ 2 I_1^N(N) + N \right]. \] 

(4.45)

This formula can be straightforwardly verified from the definition of \( b_{N,[N]}^+ \) (3.4b) and the second formula of Lemma 4.2.

As a ground for inductive assumption, we need a proposition:

Proposition 4.5

\[ \left[ I_1^M(M), b_{1,[M]}^+ \right] \] 

\[ \left| \alpha_1^M \right| \sim 0. \]
This is nothing but Proposition 4.3 for $l = 0$, i.e., $k = 1$ and $N = M \geq 1$. The proof is as follows. Using eq. (2.18c), we can rewrite the second conserved operator $I_2^{[M]}(M)$ as

$$I_2^{[M]}(M) = (d_1^{[M]} + \cdots + d_M^{[M]})^2 - \sum_{1 \leq i < j \leq M} (d_i^{[M]} d_j^{[M]} + d_j^{[M]} d_i^{[M]}) + (d_1^{[M]} + \cdots + d_M^{[M]} + a(M - 1))$$

(4.47)

Then from Proposition 4.2 and Proposition 4.1 for $k = 1$, i.e., eq. (4.28), we obtain the results.

We assume that Proposition 4.3 is valid up to $l$, namely, $k = l + 1$ and $N = l + M$. Assuming the validity of Proposition 4.3 for the case of $k$ and $N$ we shall verify the case of $k + 1$ and $N + 1$:

$$I_2^{[N+1]}(N + 1, b_{k+1,1,[N+1]}^{[N+1]}) = \left\{ \begin{array}{l}
\sum_{J \subseteq [N+1]} g_{k+1,J}^{[N+1]} h_{k+1,J}^{[N+1]} \cdot \left( \sum_{J \subseteq [N+1]} g_{k+1,J}^{[N+1]} h_{k+1,J}^{[N+1]} \right)
\end{array} \right\}^{[N+1]} \text{Sym}
$$

(4.48)

Since both sides of the above expression are invariant under the permutations of indices $1, 2, \cdots, N + 1$, it is sufficient to check the term with the factor $\alpha_1^{[N+1]} \cdot \alpha_{k+2}^{[N+1]} \cdots \alpha_{N+1}^{[N+1]} \sim 0$, which we denote by $\alpha_{1,k+2,\cdots,N+1}^{[N+1]}$. From the definition of the conserved operator $I_2^{[N]}(N)$ (4.41) and Lemma 4.1 we have

$$I_2^{[N+1]}(N + 1) = I_2^{[N+1]}(N) + (d_{k+1,1,[N+1]}^{[N+1]})^2 + \cdots$$

(4.49)

Then the l.h.s. of eq. (4.48) is decomposed as

$$I_2^{[N+1]}(N + 1) = \left\{ \begin{array}{l}
I_2^{[N+1]}(N), \alpha_{N+1}^{[N+1]} b_{k,[N]}^{[N+1]} (d_{k+1,1,[N+1]}^{[N+1]} + (k + 1)a)
\end{array} \right\}^{[N+1]} \text{Sym}
$$

(4.51)

Because of Lemma 4.1, we can easily confirm

$$I_2^{[N+1]}(N), \alpha_{N+1}^{[N+1]} b_{k,[N]}^{[N+1]} = 0$$

(4.52a)

Thus the r.h.s. of eq. (4.51) becomes

$$I_2^{[N+1]}(N), \alpha_{N+1}^{[N+1]} b_{k,[N]}^{[N+1]} = \left\{ \begin{array}{l}
I_2^{[N+1]}(N), \alpha_{N+1}^{[N+1]} b_{k,[N]}^{[N+1]} + (k + 1)a)
\end{array} \right\}^{[N+1]} \text{Sym}
$$

(4.53)

We shall calculate the above equation term by term. Using Corollary 4.4, we can rewrite the commutator of the first term of the r.h.s. of eq. (4.53) as

$$I_2^{[N+1]}(N), \alpha_{N+1}^{[N+1]} b_{k,[N]}^{[N+1]}$$
By using eqs. (4.42a) and (4.44b), the third term is cast

\[
\left( d_{N+1}^{[N+1]} + (k+1)a \right)_{a_{1,k+2,\ldots,N}^{[N+1]} = 0}^{[N+1]} \]

into

\[ \alpha_{N+1}^{[N+1]} \left( I_2^{[N+1]}(N), d_{N+1}^{[N+1]} \right)_{a_{1,k+2,\ldots,N}^{[N+1]} = 0}^{[N+1]} \]

The above commutator allows the use of inductive assumption. Thus we have

\[
\left( d_{N+1}^{[N+1]} + (k+1)a \right)_{a_{1,k+2,\ldots,N}^{[N+1]} = 0}^{[N+1]} = \alpha_{N+1}^{[N+1]} \left[ I_2^{[N+1]}(N), d_{N+1}^{[N+1]} \right]_{\text{Sym}}^{[N+1]} + a^2 \sum_{i,j \in \{1, \ldots, N+1\}, i \neq j} K_{iN+1}K_{j,N+1} \left( d_{i}^{[N+1]} \right)_{a_{i,k+2,\ldots,N}^{[N+1]} = 0}^{[N+1]}
\]

(4.54)

Using eq. (4.42b), we get the following expression from the fourth term:

\[
\left( d_{N+1}^{[N+1]} + (k+1)a \right)_{a_{1,k+2,\ldots,N}^{[N+1]} = 0}^{[N+1]} = -a \sum_{i \in \{1, k+2, \ldots, N\}} \alpha_i^{[N+1]} \left( d_{i}^{[N+1]} \right)_{a_{i,k+2,\ldots,N}^{[N+1]} = 0}^{[N+1]}
\]

(4.55)

The second term is straightforwardly calculated from eq. (4.44a):

\[
\alpha_{N+1}^{[N+1]} \alpha_2^{[N+1]} \cdots \alpha_k^{[N+1]} d_{1,\ldots,k}^{[N+1]}
\]

(4.56)

By using eqs. (4.42a) and (4.44b), the third term is cast into

\[
\left( d_{N+1}^{[N+1]} \right)^2 \alpha_{N+1}^{[N+1]} \left[ d_{1,\ldots,k}^{[N+1]} \right]_{a_{1,k+2,\ldots,N}^{[N+1]} = 0}^{[N+1]}
\]

(4.57)

Assembling eqs. (4.55) – (4.58) and doing some calculation with the help of Lemma 4.6 and the definition of null operators (3.9), we obtain

\[
\left( I_2^{[N+1]}(N+1), b_{k+1,\ldots,N}^{[N+1]} \right)_{a_{1,k+2,\ldots,N}^{[N+1]} = 0}^{[N+1]}
\]

(4.59)

which proves Proposition 4.3.

4.4 Normalization and triangularity

We shall prove the last two properties of the Hi-Jack polynomials (3.1c) and (3.1d) and Propositions 3.2 and 3.6. Here again we implicitly assume that the Dunkl operators depend only on \( N \) variables \( x_1, \ldots, x_N \). The restriction symbol without explicit indication of indices also means the restriction to symmetric functions of indices \( 1, 2, \ldots, N \).
First we shall prove the following proposition.

**Proposition 4.6** Operation of symmetric polynomials of Dunkl operators \( \{a_1^\dagger, a_2^\dagger, \ldots, a_N^\dagger\} \) on any symmetric polynomials of \( \{x_1, x_2, \ldots, x_N\} \) yields symmetric polynomials of \( \{x_1, x_2, \ldots, x_N\} \).

Let us denote an arbitrary polynomial of \( x_1, x_2, \ldots, x_N \) by \( P(x_1, x_2, \ldots, x_N) \). Acting the Dunkl operator \( a_j^\dagger \) on \( P(x_1, x_2, \ldots, x_N) \), we have

\[
a_j^\dagger P(x_1, x_2, \ldots, x_N) = \left( x_j - \frac{1}{2\omega} \frac{\partial}{\partial x_j} \right) P(x_1, x_2, \ldots, x_N) + \frac{a}{2\omega} \sum_{i=1 \atop i \neq j}^N \frac{1}{x_i - x_j} \left( P(x_1, x_2, \ldots, x_N) - P(x_i \leftrightarrow x_j) \right).
\]

It is obvious that the first term of the r.h.s. of eq. (4.60) is a polynomial of \( x_1, x_2, \ldots, x_N \). Since the difference of polynomials \( P(x_1, x_2, \ldots, x_N) - P(x_i \leftrightarrow x_j) \) has a zero at \( x_i = x_j \), the second term is also a polynomial of \( x_1, x_2, \ldots, x_N \). Proposition 4.6 follows from this property.

As a basis of symmetric polynomials, we employ the monomial symmetric polynomials \( \{m_\lambda\} \) defined by

\[
m_\lambda(x_1, x_2, \ldots, x_N) = \sum_{\sigma: \text{distinct permutation}} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \ldots x_{\sigma(N)}^{\lambda_N},
\]

where \( \lambda \) is a Young tableau \((2.13)\). By definition, monomial symmetric polynomials are symmetric polynomials with respect to any exchange of indices \( 1, 2, \ldots, N \).

From eqs. (4.60) and (4.61), we have a following result as a special case of Proposition 4.4.

**Corollary 4.2**

\[
m_\lambda(a_1^\dagger, a_2^\dagger, \ldots, a_N^\dagger) \cdot 1 = m_\lambda(x_1, x_2, \ldots, x_N) + \sum_{|\mu| < |\lambda|} u_{\lambda\mu}(a, \frac{1}{2\omega}) m_\mu(x_1, x_2, \ldots, x_N),
\]

where an unspecified coefficient \( u_{\lambda\mu}(a, 1/2\omega) \) is an integer coefficient of polynomial of \( a \) and \( 1/2\omega \).

Next we shall consider the action of \( d_i \) operator on the monomial symmetric polynomials of \( a_1^\dagger, a_2^\dagger, \ldots, a_N^\dagger \). We shall consider the case where the length of the Young tableau \( \lambda \) is \( l \leq N \), i.e., \( \lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0\} \). From the monomial symmetric polynomial \( m_\lambda(a_1^\dagger, a_2^\dagger, \ldots, a_N^\dagger) \), we single out a monomial \( (a_{\sigma(1)}^\dagger)^{\lambda_1} (a_{\sigma(2)}^\dagger)^{\lambda_2} \ldots (a_{\sigma(l)}^\dagger)^{\lambda_l} \). Because of eq. (2.18d), we have

\[
d_i (a_{\sigma(1)}^\dagger)^{\lambda_1} (a_{\sigma(2)}^\dagger)^{\lambda_2} \ldots (a_{\sigma(l)}^\dagger)^{\lambda_l} = d_i (a_{\sigma(1)}^\dagger)^{\lambda_1} (a_{\sigma(2)}^\dagger)^{\lambda_2} \ldots (a_{\sigma(l)}^\dagger)^{\lambda_l}, \quad l \geq s, e < s \]

From eq. (2.18c), we can easily verify

\[
[d_i, a_{\sigma(l)}^\dagger] = \lambda_i \left( d_i + \frac{N}{k=1} K_{ik} - a(1 - \delta_{ij})K_{ij} \right). \quad (4.64)
\]

Using the above formula, we get the following expressions:

\[
\begin{align*}
&[d_i, (a_{\sigma(1)}^\dagger)^{\lambda_1} (a_{\sigma(2)}^\dagger)^{\lambda_2} \ldots (a_{\sigma(l)}^\dagger)^{\lambda_l}]_{\text{Sym}} \\
&= \left\{ (\lambda_h + (N - l) a)(a_{\sigma(1)}^\dagger)^{\lambda_1} (a_{\sigma(2)}^\dagger)^{\lambda_2} \ldots (a_{\sigma(l)}^\dagger)^{\lambda_l} \\
&+ a \sum_{k=1}^{\lambda_h - l} \sum_{j=l+1}^{N} (a_{\sigma(1)}^\dagger)^{\lambda_1} \ldots (a_{\sigma(j-1)}^\dagger)^{\lambda_{j-1}} \ldots (a_{\sigma(l)}^\dagger)^{\lambda_l} \\
&+ a \sum_{j=h+1}^{l} (a_{\sigma(1)}^\dagger)^{\lambda_1} \ldots (a_{\sigma(j)}^\dagger)^{\lambda_j} \ldots (a_{\sigma(l)}^\dagger)^{\lambda_l} \\
&\lambda_h - \lambda_l \sum_{k=1}^{\lambda_h - l} (a_{\sigma(h)}^\dagger)^{\lambda_h - k + 1} (a_{\sigma(j)}^\dagger)^{\lambda_{j + k - 1}} \\
&- a \sum_{j=1}^{h-1} (a_{\sigma(1)}^\dagger)^{\lambda_1} \ldots (a_{\sigma(j)}^\dagger)^{\lambda_j} \ldots (a_{\sigma(l)}^\dagger)^{\lambda_l} \\
&\lambda_h - \lambda_h \sum_{k=1}^{\lambda_h - l} (a_{\sigma(h)}^\dagger)^{\lambda_h - k + 1} (a_{\sigma(j)}^\dagger)^{\lambda_{j + k - 1}} \right\}_{\text{Sym}},
\end{align*}
\]

where \( i = \sigma(h), 1 \leq h \leq l \).

\[
\begin{align*}
&[d_i, (a_{\sigma(1)}^\dagger)^{\lambda_1} (a_{\sigma(2)}^\dagger)^{\lambda_2} \ldots (a_{\sigma(l)}^\dagger)^{\lambda_l}]_{\text{Sym}} \\
&= \left\{ -a \sum_{k=1}^{l} (a_{\sigma(1)}^\dagger)^{\lambda_1} \ldots (a_{\sigma(k)}^\dagger)^{\lambda_k} (a_{\sigma(h)}^\dagger)^{\lambda_h} \\
&- a \sum_{k=1}^{l} \sum_{m=1}^{\lambda_h - 1} (a_{\sigma(1)}^\dagger)^{\lambda_1} \ldots (a_{\sigma(k)}^\dagger)^{\lambda_k} (a_{\sigma(h)}^\dagger)^{\lambda_h - m} \right\}_{\text{Sym}},
\end{align*}
\]

where \( i = \sigma(h), l + 1 \leq h \leq N \).

Note that the summation \( \sum_{k=s}^{e} \), \( e < s \) means zero, which occurs in the third and fourth term of eq. (4.65a) when \( \lambda_h = \lambda_l \) or \( h = 1, l \). From the above calculation, we get the following result.

**Proposition 4.7** No higher order monomial in the dominance ordering is generated by the action of \( a_j \) operator on a monomial \( (a_{\sigma(1)}^\dagger)^{\lambda_1} (a_{\sigma(2)}^\dagger)^{\lambda_2} \ldots (a_{\sigma(N)}^\dagger)^{\lambda_N} \cdot 1 \), where \( \lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0\} \) and \( \sigma \in S_N \), and the coefficients of the monomials are integer coefficient polynomials of \( a \). The weight of the Young tableaux of the monomials are the same as that of the original monomial.

This property causes the triangularity of the Hi-Jack polynomials (3.1c). From the definition of the generator of the Hi-Jack polynomial and the above proposition, we notice the weak form of the third requirement of Proposition 3.1.

**Proposition 4.8**

\[
C_{\lambda, j} (x; \omega, 1/a)
\]
Combining Corollary 4.2 and Proposition 4.8, we can express the terms that come from the third and the fourth terms to calculate the coefficient of \((\sigma_1)^{\lambda_1} \cdots (\sigma_l)^{\lambda_l} \cdot 1\),
\begin{equation}
\sum_{\mu \leq \lambda} v_{\lambda\mu}(a)m_{\mu}(\alpha_1^\dagger, \ldots, \alpha_N^\dagger) \cdot 1 \tag{4.66}
\end{equation}
where \(v_{\lambda\mu}(a)\) is an unspecified integer coefficient polynomial of \(a\).

Combining Corollary 4.2 and Proposition 4.8, we can easily confirm the third requirement of Proposition 4.8 i.e., eq. (3.1c). To verify the fourth requirement (3.1d), we have to show
\begin{equation}
v_{\lambda\lambda}(a) = C_{\lambda}. \tag{4.67}
\end{equation}

Computation of \(v_{\lambda\lambda}(a)\) needs consideration on the cancellation among the monomials in eq. (4.65a).

After summing over the distinct permutation \(\sigma \in S_N\), the terms that come from the third and the fourth terms of (4.65a) cancel out and vanish, for the fourth term with the permutation \(\sigma\) replaced by \(\sigma' = \sigma(jh)\) yields
\begin{equation}
-a(\alpha_{\sigma'(1)}^\dagger)^{\lambda_1} \cdots (\alpha_{\sigma'(j)}^\dagger)^{\lambda_j} \sum_{k=1}^{\lambda_k - \lambda_j} (\alpha_{\sigma'(j)}^\dagger)^{\lambda_k - k + 1}(\alpha_{\sigma'(h)}^\dagger)^{\lambda_j + k - 1} = -a(\alpha_{\sigma(1)}^\dagger)^{\lambda_1} \cdots (\alpha_{\sigma(j)}^\dagger)^{\lambda_j} \sum_{k=1}^{\lambda_k - \lambda_j} (\alpha_{\sigma(j)}^\dagger)^{\lambda_k - k + 1}(\alpha_{\sigma(h)}^\dagger)^{\lambda_j + k - 1}, \tag{4.68}
\end{equation}
which is the summand in the third term of eq. (4.65a) with negative sign. Similar cancellation also occurs between the second term in the bracket of the first term of eq. (4.65a) and the first term of eq. (4.65b) with \(\sigma\) replaced by \(\sigma' = \sigma(kh)\), \(l + 1 \leq k \leq N\), where \(\sigma\) in the r.h.s. is that of eq. (4.65b): \(-a(\alpha_{\sigma'(1)}^\dagger)^{\lambda_1} \cdots (\alpha_{\sigma'(j)}^\dagger)^{\lambda_j} (\alpha_{\sigma'(h)}^\dagger)^{\lambda_h} = -a(\alpha_{\sigma(1)}^\dagger)^{\lambda_1} (\alpha_{\sigma(2)}^\dagger)^{\lambda_2} \cdots (\alpha_{\sigma(j)}^\dagger)^{\lambda_j}. \tag{4.69}\)

Since there are \(N - l\) permutations \(\sigma'\) that yield the monomial \(-a(\alpha_{\sigma(1)}^\dagger)^{\lambda_1} (\alpha_{\sigma(2)}^\dagger)^{\lambda_2} \cdots (\alpha_{\sigma(j)}^\dagger)^{\lambda_j}\) through the commutator (4.65b), we can cancel the second term of the first bracket in the first term of eq. (4.65a). Thus the coefficient of the term \((\alpha_{\sigma(1)}^\dagger)^{\lambda_1} \cdots (\alpha_{\sigma(j)}^\dagger)^{\lambda_j}\) coming from the commutator \([d_i, m_{\lambda}(\alpha_1^\dagger, \alpha_2^\dagger, \ldots, \alpha_N^\dagger)] \big|_{\text{Sym}}\) is \(\lambda_i h\), where \(i = \sigma(h)\).

Since the Hi-Jack polynomial is symmetric with respect to the exchange of indices \(1, \ldots, N\), we have only to calculate the coefficient of \((\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l}\). In the following calculation, we shall omit all the monomials except for \((\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l}\), namely the monomial with identity permutation. Any lower order monomial and the same order monomial with different permutation are omitted in the expression. However, we implicitly sum up over the distinct permutations to use the above cancellation. To know the coefficient of the monomial of interest that is yielded from \(b_i^\dagger m_{\lambda}(\alpha_1^\dagger, \ldots, \alpha_N^\dagger) \cdot 1\), we have only to do the following calculation using eq. 4.67:
\begin{equation}
d_{1, \{1, \ldots, 1\}}(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1
= d_{1, \{1, \ldots, 3\}}(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \bigg|_{\text{Sym}}
+ l a(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1
= d_{1, \{1, \ldots, 2\}}(\lambda_1 + la)(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l}
+ a(N - l)(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l}
+ a \sum_{\lambda_i < \lambda_h}^N (\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1. \tag{4.70}
\end{equation}

Summing over the distinct permutation, we can cancel out the second and third term as has been explained in eqs. (4.68) and (4.69). Next, we operate \(d_2\) on the operand:
\begin{equation}
d_{1, \{1, \ldots, 1\}}(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1
= (\lambda_1 + la)d_{1, \{1, \ldots, 3\}}(\lambda_2 + (l - 1)a)(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l}
+ (N - l)(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l}
- a \sum_{\lambda_i > \lambda_2}^1 (\alpha_1^\dagger)^{\lambda_1} (\alpha_2^\dagger)^{\lambda_2} (\alpha_3^\dagger)^{\lambda_3} \cdots (\alpha_l^\dagger)^{\lambda_l}
+ a \sum_{\lambda_i < \lambda_3}^N (\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1. \tag{4.71}
\end{equation}

Since we have already operated \(d_1\) on the monomial symmetric polynomial \(m_{\lambda}(\alpha_1^\dagger, \ldots, \alpha_N^\dagger)\), summation over the distinct permutation is not anymore invariant under the permutation of the indices \(1, \ldots, N\), but invariant under the permutation of indices \(2, \ldots, N\). To cancel out the second and fourth terms, we only need transpositions (2j), \(3 \leq j \leq N\). The third term, which is a monomial with the same Young tableau and a permutation (12), can not be canceled out because of the break of invariance of summation under the permutation involving the index 1. However, this monomial can not be changed to the monomial with identity permutation by operating \(d_i, i \geq 3\). Thus, we have
\begin{equation}
d_{1, \{1, \ldots, 1\}}(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1
= (\lambda_1 + la)(\lambda_2 + (l - 1)a)
+ d_{1, \{1, \ldots, 3\}}(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1. \tag{4.72}
\end{equation}

Repeating analogous calculations, we get
\begin{equation}
d_{1, \{1, \ldots, 1\}}(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1
= (\lambda_1 + la)(\lambda_2 + (l - 1)a) \cdots (\lambda_l + a)
(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1. \tag{4.73}
\end{equation}

Then the following expansion follows from the above for-
where \( \lambda + 1^l = \{ \lambda_1 + 1, \ldots, \lambda_l + 1 \} \) and \( y_{\lambda+1^l}(a) \) is an unspecified integer coefficient polynomial of \( a \). We remark that 1 is a monomial symmetric polynomial with \( \lambda = 0 \).

Then from the definition of the generator (3.4) and repeated use of eq. (4.74), we finally verify eq. (4.67).

**Lemma 4.7**

\[ v_{\lambda}(a) = C_{\lambda}. \]

Combining Corollary 4.2 Proposition 4.8 and Lemma 4.7, we can confirm the third and fourth requirements of Proposition 3.1, eqs. (3.1c) and (3.1d), and Proposition 3.5.

In the limit \( \omega \to \infty \), eq. (4.66) reduces to

\[ C_{\lambda}j_{\lambda}(x; \omega \to \infty, 1/a) = \left( b_N^+ \right)^{\lambda_N} \left( b_{N-1}^+ \right)^{\lambda_{N-1} - \lambda_N} \cdots \left( b_1^+ \right)^{\lambda_1 - \lambda_2} 1_{\omega \to \infty} = \sum_{\mu \subseteq \lambda} v_{\lambda \mu}(a)m_{\mu}(x_1, \ldots, x_N). \tag{4.75} \]

As has been remarked in [2], the Dunkl operators for the Calogero model reduce to corresponding Dunkl operators [2.43] for the Sutherland model in the limit \( \omega \to \infty \). Then it is obvious that the Hi-Jack polynomials in the limit \( \omega \to \infty \) are the Jack polynomials. Thus we have

\[ C_{\lambda}j_{\lambda}(x; 1/a) = \sum_{\mu \subseteq \lambda} v_{\lambda \mu}(a)m_{\mu}(x_1, \ldots, x_N), \tag{4.76} \]

which means the coefficients \( v_{\lambda \mu}(a) \) in eq. (2.34b) and \( v_{\lambda \mu}(a) \) in (4.66) are essentially same:

\[ v_{\lambda \mu}(a) = C_{\lambda}v_{\lambda \mu}(a). \tag{4.77} \]

This proves Proposition 3.6.

**§5. Summary**

We have studied the Hi-Jack symmetric polynomials, which we proposed in a previous work [8] through their Rodrigues formula that is an extension of the Rodrigues formula for the Jack symmetric polynomials discovered by Lapointe and Viennot [24]. A proof of the formula based on the algebraic relations among the Dunkl operators is given. In the consideration of their normalizations, we have clarified that expansions of the Hi-Jack symmetric polynomials in terms of the monomial symmetric polynomials have triangular forms, as is similar to the Jack symmetric polynomials. We have also confirmed that the Hi-Jack symmetric polynomials exhibit the property corresponding to the weak form of the Macdonald-Stanley conjecture for the Jack symmetric polynomials [16]. The Hi-Jack symmetric polynomials and the eigenstates for the Hamiltonian that was algebraically constructed through QISM are related by the transformation between the Jack symmetric polynomials and the power sum symmetric polynomials.

The orthogonal basis provides a very useful tool for the study of physical quantities in quantum theory. The orthogonality of the Hi-Jack symmetric polynomials is expected to be important in the exact calculation of the thermodynamic quantities such as the Green functions and the correlation functions, as has been done for the Sutherland model using the properties of the Jack polynomials [2.42-43]. However, construction of the simultaneous eigenstates for the first two conserved operators is not sufficient to conclude that the Hi-Jack symmetric polynomials form the orthogonal basis of the quantum Calogero model, because of the remaining degeneracy in the two eigenvalues [2.43]. Orthogonality of the Jack polynomials is proved by showing that all the commuting conserved operators of the Sutherland model \( \{ Z_k | k = 1, 2, \ldots, N \} \) are simultaneously diagonalized by the Jack polynomials [24]. Considering the correspondence between the Calogero model and the Sutherland model, we can expect that all the conserved operators of the Calogero model \( \{ Z_k | k = 1, 2, \ldots, N \} \) are also diagonalized by the Hi-Jack polynomials. However, a proof of the orthogonality still remains open.

The Dunkl operators for the Calogero and Sutherland models are shown to be related to the Yangian symmetry and the Star-Triangule relations (3.1). They are also related to the generators of \( W \)-algebra. Moreover, the Jack polynomials is identified with the singular vectors of the \( W_N \)-algebra. Relationship between the irreducible representation of the Yangian and the eigenstates of the spin-1/2 generalization of the Sutherland model is also claimed. Thus relationships with the representation theory of Yangian and \( W \)-algebra are also interesting.

Some other extensions are also hopeful. For instance, a \( q \)-deformation of the Hi-Jack polynomials associated with the relativistic Calogero model [2.42] ("Hidden-Macdonald" polynomials), "Hi-Jack" polynomials associated with root lattices other than \( A_{N-1} \) and the spin generalization of them sound attractive. We expect some progresses in these directions in the near future.

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