Hidden continuous quantum phase transition without gap closing in non-Hermitian transverse Ising model

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Abstract
Continuous phase transition in quantum matters is a significant issue in condensed matter physics. In general, the continuous quantum phase transitions in many-body systems occur with gap closing. On the other hand, non-Hermitian systems could display quite different properties as their Hermitian counterparts. In this paper, we show that a hidden, continuous quantum phase transition occurs without gap closing in non-Hermitian transverse Ising model. By using a projected Jordan–Wigner transformation, the one-dimensional (1D) non-Hermitian transverse Ising model with ferromagnetic order is mapped on to 1D non-Hermitian Kitaev model with topological superconducting order and becomes exactly solvable. A hidden, continuous quantum phase transition is really normal–abnormal transition for fermionic correlation in the 1D non-Hermitian Kitaev model. In addition, similar hidden, continuous quantum phase transition is discovered in two-dimensional non-Hermitian transverse Ising model and thus becomes a universal feature in certain non-Hermitian many-body systems.

1. Introduction

Landau’s spontaneous symmetry breaking theory is a fundamental theory that plays an important role in condensed matter physics. According to Landau’s spontaneous symmetry breaking theory, quantum phases are classified by different symmetries and different quantum phases are divided by (continuous) quantum phase transitions. Thus, continuous phase transition in quantum matters becomes a significant issue in condensed matter physics. Generally, continuous quantum phase transitions are relevant to the ground state properties of many-body systems and are accompanied by the gap closure for elementary excitations.

Recently, non-Hermitian (NH) systems attract great research interest from different fields in recent years. In certain non-Hermitian systems, when the non-Hermitian effects are considered, there exist many exotic physics properties, especially its topology properties [1–29]. As long as non-Hermitian parity-time spontaneous breaking [30], non-Hermitian skin effects [31–33], anomalous edge-bulk correspondence [34–39], and one way propagation [40–42]. In reference [43], a continuous quantum phase transition without gap closing in Kitaev’s toric-code model was found due to nonorthogonality of eigenstates. This result indicates that the usual relationship between the correlation length and the energy gap could be changed in non-Hermitian systems.

In this paper, we ask the following questions: ‘does there exist continuous quantum phase transitions without gap closing in other NH systems?’ and ‘Do there exist new universal features of these types of quantum phase transition in NH systems?’. Motivated by the above questions, we investigate the non-Hermitian transverse Ising model, a hidden, continuous quantum phase transition without gap closing is studied. By using a projected Jordan–Wigner transformation, the one-dimensional (1D) non-Hermitian transverse...
Ising model with ferromagnetic order is mapped to 1D non-Hermitian Kitaev model with topological superconducting order. According to exactly solvable properties of 1D non-Hermitian Kitaev model, the universal features of the hidden, continuous quantum phase transitions is explored that is really normal–abnormal transition.

The remainder of the paper is organized as follows. In section 2, we reviewed the theory of an open quantum system of a multi-spin system under post-selection sub-system and derived an effective 1D non-Hermitian transverse Ising model. In this section, we discussed the (global) similarity of non-Hermitian transverse Ising model. In section 3, we studied the quantum phase transition of spontaneous PT symmetry breaking. In section 4, we studied the quantum phase transition with gap closing in the PT symmetric phase. This is an order–disorder transition for the non-Hermitian system. In section 5, we studied the hidden quantum phase transition without gap closing in the PT symmetric phase with long range order by mapping the original 1D non-Hermitian transverse Ising model to an 1D non-Hermitian Kitaev model with the help of projected Jordan–Wigner transformation. In this section, the universal feature for a hidden quantum phase transition is discovered. In section 6, for the 1D non-Hermitian transverse Ising model, a complex, global phase diagram with four phases was obtained. In section 7, similar hidden, continuous quantum phase transition was studied in two-dimensional (2D) non-Hermitian transverse Ising model. Finally, the conclusions are given in section 8.

2. One-dimensional non-Hermitian transverse Ising model

In open quantum system, the quantum dynamics is effectively described by a non-Hermitian Hamiltonian by projecting out quantum jumping processes. We consider an open quantum system of a multi-spin system under post-selection sub-system. The total Hamiltonian of the whole system is

\[ \hat{H} = \hat{H}_S \otimes 1 + 1 \otimes \hat{H}_B + \hat{H}_{BS}, \]

(1)

where \( \hat{H}_S = \sum_j (-J \sigma_j^x \sigma_{j+1}^x + h^x \sigma_j^y) \) is the Hamiltonian of the multi-spin system, \( \hat{H}_B \) is the Hamiltonian of the environment and \( \hat{H}_{BS} = \sum_j \sigma_j^y \otimes \hat{B}_j + \text{h.c.} \) represents the coupling between them. \( \hat{B}_j \) is the operator of the environment. \( N \) is the number of lattice sites.

Within the Markovian approximation, the non-unitary dynamics of an open quantum system is generated by a Liouvillian superoperator \( \mathcal{L} \) acting on the reduced density matrix \( \rho_S(t) \) of the system [44, 45],

\[ \mathcal{L} \rho_S(t) = -i [\hat{H}_S, \rho_S(t)] + \sum_{j=1}^N \left( \hat{L}_j \rho_S(t) \hat{L}_j^\dagger - \frac{1}{2} \{ \hat{L}_j^\dagger \hat{L}_j, \rho_S(t) \} \right), \]

(2)

that is described by Gorini–Kossakowski–Sudarshan–Lindblad equation

\[ \frac{d\rho_S(t)}{dt} = \mathcal{L} \rho_S(t). \]

(3)

For the model in equation (1), the Lindblad operator \( \hat{L}_j = 2\sqrt{\hbar} \sigma_j^z = \sqrt{\hbar} \left( \sigma_j^z - i \sigma_j^x \right) \) describes loss from the coupling with the environment. Due to weak measurement, the amplitudes of quantum states are changed and the amplitude changing is described by the term of \( -\frac{i}{2} \sum_{j=1}^N \hat{L}_j^\dagger \hat{L}_j \). The term \( \sum_{j=1}^N (\hat{L}_j \rho_S(t) \hat{L}_j^\dagger) \) describes the quantum jumping processes that would cause the system decoherence.

When we do post-selection on the multi-spin system, the quantum jumping terms \( \sum_{j=1}^N (\hat{L}_j \rho_S(t) \hat{L}_j^\dagger) \) are projected out and then an effective non-Hermitian Hamiltonian of multi-spin model is obtained as

\[ \hat{H}_{\text{NTI}} = \hat{H}_S - \frac{i}{2} \sum_{j=1}^N \hat{L}_j^\dagger \hat{L}_j \]

\[ = \sum_{j} \left( -J \sigma_j^z \sigma_{j+1}^z + h^z \sigma_j^x \right) - i\hbar \sum_{j} \left( \sigma_j^x + 1 \right), \]

\[ = \sum_{j} \left( -J \sigma_j^z \sigma_{j+1}^z + h^z \sigma_j^x - i\hbar \sigma_j^y \right), \]

(4)

with \( \hat{H}_{\text{NTI}} \neq \hat{H}_S \). The normalization procedure has been used in the derivation, the constant imaginary shift cancels out in the time evolution [46]. And the plus or minus sign before \( h^z \) indicates the gain or loss process. Here \( J > 0 \) is ferromagnetic Ising coupling constant between two nearest neighbor spins and \( h^z \) is
the strength of a real transverse field along $y$-axis, $h^r$ is the strength of an imaginary transverse field along $z$-axis. In this paper, the coupling parameter $J$ is set to be unit, $J = 1$.

3. Quantum phase transition of spontaneous $\mathcal{PT}$ symmetry breaking

Firstly, we study the quantum phase transition of spontaneous $\mathcal{PT}$ symmetry breaking.

It is obvious that the non-Hermitian Hamiltonian has $\mathcal{PT}$-symmetry, i.e., $[\mathcal{P}, H_{\text{NTI}}] \neq 0$ and $[\mathcal{T}, H_{\text{NTI}}] \neq 0$, but $[\mathcal{PT}, H_{\text{NTI}}] = 0$. Here, the time reversal operator $\mathcal{T}$ is defined as $T i T = -i$ and the parity operator $\mathcal{P}$ is defined by rotating each spin by $\pi$ about the $y$-axis $\mathcal{P} = \Pi_{\alpha=1}^{N}(i\sigma^{\alpha}_{z})$.

A spontaneous $\mathcal{PT}$ symmetry breaking occurs at $|h^r| = |h^i|$. For the case $|h^r| > |h^i|$, the system has $\mathcal{PT}$ symmetry. Now, the energy spectra of $H_{\text{NTI}}$ are same to the usual Hermitian transverse Ising model, $H_{0} = \sum_{j}(-J\sigma^{x}_{j}\sigma^{x}_{j+1} + h\sigma^{y}_{j})$ with $\text{Re} \, h \neq 0$ and $\text{Im} \, h = 0$. So, the energy spectra are real; for the case $|h^r| < |h^i|$, $\mathcal{PT}$ symmetry is broken. Now, the energy spectra of $H_{\text{NTI}}$ are same to a transverse Ising model with imaginary transverse field, $H_{0} = \sum_{j}(-J\sigma^{x}_{j}\sigma^{x}_{j+1} + h\sigma^{y}_{j})$ with $\text{Re} \, h = 0$ and $\text{Im} \, h \neq 0$. So, the energy spectra become complex.

For $H_{\text{NTI}}$, another important property is non-Hermitian similarity. The operation of non-Hermitian similarity transformation $S$ is defined as $S(\beta) = \prod_{j}S_{j}(\beta)$ with $S_{j}(\beta) = e^{\frac{1}{2}\sigma^{y}_{j}}$. Here, $\beta$ denotes non-Hermiticity. In what follows, for clarity, we denote $S$ by $S(\beta)$.

We represent the original non-Hermitian transverse Ising model as

$$H_{\text{NTI}} = \sum_{j}(-J\sigma^{x}_{j}\sigma^{x}_{j+1} + h|\sigma^{y}_{j}|^{2}),$$

where

$$\langle \sigma^{y}_{j} \rangle^{\beta} = S_{j}(\beta)\sigma^{y}_{j}S_{j}^{-1}(\beta)$$

$$= \cosh(\beta)\sigma^{y}_{j} + i \sinh(\beta)\sigma^{z}_{j},$$

and

$$|h| = \sqrt{(h^r)^{2} - (h^i)^{2}},$$

or $h \sinh \beta = h^r$, $h \cosh \beta = h^i$. The non-Hermiticity $\beta$ is obtained as

$$\beta = \frac{1}{2}\ln \left( \frac{h^r + h^i}{h^r - h^i} \right).$$

As a result, under an inverse non-Hermitian similarity transformation $S^{-1}$, $H_{\text{NTI}}$ is transformed as

$$H_{0} = S^{-1}(\beta)H_{\text{NTI}}S(\beta) = \sum_{j}(-J\sigma^{x}_{j}\sigma^{x}_{j+1} + h\sigma^{y}_{j}).$$

Under the inverse non-Hermitian similarity transformation $S^{-1}$, the energy levels $E_{\alpha}(\beta)$ of $H_{\text{NTI}}$ are same to those of the Hermitian model $E_{\alpha}(\beta_{\text{NH}} = 0)$ of $H_{0}$, i.e.,

$$E_{\alpha}(\beta_{\text{NH}}) = E_{\alpha}(\beta_{\text{NH}} = 0).$$

The corresponding eigenstates of $H_{\text{NTI}}$ become

$$|\Psi^{\beta}\rangle = S|\Psi_{0}\rangle,$$

where $|\Psi_{0}\rangle$ denotes eigenstates of $H_{0}$.

4. Order–disorder quantum phase transition with gap closing

Secondly, we study the quantum phase transition with gap closing in the $\mathcal{PT}$ symmetric phase, $|h^r| > |h^i|$. Now, the energy spectra of $H_{\text{NTI}}$ are same to the usual Hermitian transverse Ising model, $H_{0} = \sum_{j}(-J\sigma^{x}_{j}\sigma^{x}_{j+1} + h\sigma^{y}_{j})$ with $\text{Im} \, h = 0$.

In the region of $|h^r| > |h^i|$, under non-Hermitian (inverse) similarity transformation, the energy levels $E_{\alpha}(\beta)$ of $H_{\text{NTI}}$ are same to the Hermitian model $E_{\alpha}(\beta = 0)$ of $H_{0} = \sum_{j}(-J\sigma^{x}_{j}\sigma^{x}_{j+1} + h\sigma^{y}_{j})$, i.e.,

$$E_{\alpha}(\beta) = E_{\alpha}(\beta = 0).$$

As a result, the real energy gap is obtained as [47]

$$\Delta m = \sqrt{|J^{2} - h^r|}. \quad (11)$$
Figure 1. The energy difference between the two lowest energy levels (a) and the expectation value of order parameter $A_{bi}$ (b) of 1D non-Hermitian transverse Ising model for $N = 16$. Where the yellow dashed lines are the theoretical prediction of the order–disorder transition which gap closing, the green line is spontaneous $\mathcal{PT}$ symmetry breaking line. Note that longitudinal field $h' = 0.002$.

and the quantum phase transition with the gap closing for $\hat{H}_{\text{NTI}}$ is same to that for $\hat{H}_0$ that is obtained as $|J| = |h|$.

For non-Hermitian systems, an order parameter is defined by calculating the expectation value on ground states in the biorthogonal sets $\{|\text{vac}_R(\beta)\rangle\}$ and $\{|\text{vac}_L(\beta)\rangle\}$ [48], i.e.,

$$\langle \text{vac}_L(\beta) | \sum_j \sigma_x^j N | \text{vac}_R(\beta) \rangle = A_{bi}. \quad (12)$$

Under the condition $S^{-1} \sigma_x S = \sigma_y$, due to the similarity of ground states, $|\text{vac}_R(\beta)\rangle = S|\text{vac}(\beta = 0)\rangle$, $\langle \text{vac}_L(\beta) | = \langle \text{vac}(\beta = 0) | S^{-1}$, we have

$$A_{bi} = A_0, \quad (13)$$

where $A_0$ is the order parameter for the Hermitian model $\hat{H}_0$; i.e.,

$$A_0 = \left\langle \text{vac}(\beta = 0) | \sum_j \sigma_x^j N | \text{vac}(\beta = 0) \right\rangle. \quad (14)$$

In figure 1, from exact diagonal numerical calculation for 1D non-Hermitian transverse Ising model with $N = 16$, in which the yellow dashed lines come from theoretical prediction.

As a result, in the region of $|J| > |h|$, the ground state becomes a long range ferromagnetic order with $A_{bi} = A_0 \neq 0$; in the region of $|J| < |h|$, the ground state is disordered state with $A_{bi} = A_0 = 0$. In addition, near the order–disorder transition for the non-Hermitian system described by $\hat{H}_{\text{NTI}}$, the universal critical phenomenon is the same as that of the Hermitian model $\hat{H}_0$.

5. Hidden, continuous quantum phase transitions without gap closing

Thirdly, we study the hidden, continuous quantum phase transition without gap closing in the $\mathcal{PT}$ symmetric phase with long range order, $|h'| > |h|\}$ and $|J| > |h|$. By mapping the original 1D non-Hermitian transverse Ising model $\hat{H}_{\text{NTI}}$ to an 1D non-Hermitian Kitaev model with the help of projected Jordan–Wigner transformation, a hidden, continuous quantum phase transition is explored.

5.1. Projected Jordan–Wigner transformation

Before we start our study, we will introduce the projected Jordan–Wigner transformation. For 1D non-Hermitian transverse Ising model $\hat{H}_{\text{NTI}} = \sum_j (-J \sigma_x^j \sigma_x^{j+1} + h \sigma_y^j + i h' \sigma_z^j)$, the usual Jordan–Wigner transformation is not applicable [49]. However, we found that in the region with long range ferromagnetic order for the 1D non-Hermitian transverse Ising model $\hat{H}_{\text{NTI}}$, projected Jordan–Wigner transformation can be applied, by which we map the original spin model to a non-Hermitian Kitaev model. Here, ‘projected’ means that the Jordan–Wigner transformation can only available for this non-Hermitian transverse Ising model with long range ferromagnetic order. That means, we should concentrate on the non-Hermitian spin model in region of $|h'| > |h|$ and $|J| > |h|$. See the logical schematic diagram for projected Jordan–Wigner transformation in figure 2.
Let us show the details for projected Jordan–Wigner transformation step by step:

**Step 1**: under a global inverse similarity transformation $S^{-1}(\beta)$, $\hat{H}_{\text{NTI}}$ is transformed into a Hermitian one, i.e.,

$$\hat{H}_{\text{NTI}} \rightarrow \hat{H}_0 = S^{-1}(\beta)\hat{H}_{\text{NTI}}S(\beta) = \sum_j (-J \sigma_j^x \sigma_{j+1}^x + h \sigma_j^y),$$  

where $S(\beta) = \prod_j S_j(\beta)$ is the operator of a global non-Hermitian similarity transformation on spin system and $S_j(\beta)$ is defined as

$$S_j(\beta) = e^{\frac{\beta}{2} \sigma_j^x} = e^{\frac{\beta}{2}} \begin{pmatrix} 1 + e^{-\beta} & 1 - e^{-\beta} \\ 1 - e^{-\beta} & 1 + e^{-\beta} \end{pmatrix}$$

with $\beta = \ln(\frac{h_y + h_z}{h_y - h_z})$.

**Step 2**: we do usual Jordan–Wigner transformation on $\hat{H}_0$ by the string-like annihilation and creation operators

$$c_j = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \frac{\sigma_j^+ + i\sigma_j^y}{2},$$

$$c_j^\dagger = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \frac{\sigma_j^+ - i\sigma_j^y}{2}. $$

Under the usual Jordan–Wigner transformation, the Hamiltonian $\hat{H}_0$ becomes that for a 1D superconducting state

$$\hat{H}_0 = \sum_{j=1}^N \left(c_j c_{j+1}^\dagger - c_j^\dagger c_{j+1} - c_j c_{j+1}^\dagger + c_j^\dagger c_{j+1}^\dagger\right) + h \sum_{j=1}^N \left(2c_j^\dagger c_j - 1\right),$$

$$= \sum_{k>0} \psi_k^\dagger [h_F^{\beta=0}(k)] \psi_k.$$  

The $c_j$ satisfy $c_{N+1} = c_1$ for periodic boundary condition and satisfy $c_{N+1} = -c_1$ for antiperiodic boundary condition. Where

$$h_F^{\beta=0}(k) = (-2J \cos k + 2h)\sigma^x + (2J \sin k)\sigma^y$$

and $\psi_k^\dagger = (c_k^\dagger, c_{-k})$. It is obvious that the energy levels for $\hat{H}_{\text{NTI}}$ can be exactly characterized by the Hermitian superconducting model $\hat{F}_\beta^{\beta=0} = \sum_{k>0} \psi_k^\dagger [h_F^{\beta=0}(k)] \psi_k$.

**Step 3**: we then consider the projection condition in ferromagnetic ordered phase. The two degenerate ground states can be phenomenologically described by the product states of $|0\rangle_i$ or $|1\rangle_i$, i.e., $|\text{vac}_+\rangle = \prod_i |0\rangle_i$
and $|\text{vac}_-\rangle = \prod_i^N |0\rangle_i$. Due to the similarity, the basis of two degenerate ground states become

\begin{align}
|\text{vac}_+^R(\beta)\rangle &= \mathcal{S}(\beta)|\text{vac}_+\rangle = e^{\beta N/2} \prod_i^N |1\rangle_i, \\
|\text{vac}_-^R(\beta)\rangle &= \mathcal{S}(\beta)|\text{vac}_-\rangle = e^{-\beta N/2} \prod_i^N |0\rangle_i.
\end{align}

(21)  
(22)

Now, for the ground state with uniform distribution of spin direction, a moving Fermion (that is really a kink) will reverse the spin direction. Each reversed spin will contribute an additional factor $e^{\pm\beta}$ on the weight of fermion’s operators. A fermion at site $i$ is accompanied by an ‘amplitude’ string with length $j$, i.e., $e^{\pm\beta}, e^{\pm\beta}, \ldots, e^{\pm\beta}$. By considering the contribution from the ‘amplitude’ string, there exists an additional weight $e^{\pm\beta}$ on fermion’s operators from the similarity transformation $\mathcal{S}(\beta)$,

$$
\tilde{c}_j = \mathcal{S}(\beta)c_j\mathcal{S}^{-1}(\beta) \Rightarrow e^{\beta_j}c_j, \\
\tilde{c}'_j = \mathcal{S}^{-1}(\beta)c'_j\mathcal{S}(\beta) \Rightarrow e^{\beta'}c'_j.
$$

(23)  
(24)

The arrow ‘$\Rightarrow$’ denotes that this equation is under the ferromagnetic ordered condition (or the protected condition).

According to the result from this projected Jordan–Wigner transformation, one can see that the fermions have an additional ‘imaginary’ wave vector $k_0$, i.e.,

$$
k_0 = i\beta = \frac{i}{2} \ln \left( \frac{h^y + h^z}{h^y - h^z} \right).
$$

(25)

**Step 4:** finally, under the order condition, the resulting 1D non-Hermitian transverse Ising model corresponding to the 1D non-Hermitian Kitaev model $\hat{H}_{\text{NTI}}$ is obtained as

$$
\hat{H}_F^{\beta=0} \rightarrow \hat{H}_{\text{NTI}} = \hat{H}_F^{\beta=0}\mathcal{S}^{-1}(\beta) = \sum_{k>0} \tilde{\psi}_k^\dagger \left[ h_F^{\beta=0}(k) \right] \tilde{\psi}_k,
$$

(26)

where

$$
h_F^{\beta}(k) = h_F^{\beta=0}(k) = (-2J \cos k + 2h)\sigma^z + (2J \sin k)\sigma^y
$$

(27)

and $\tilde{\psi}_k = (\tilde{c}^+_k, \tilde{c}^-_k) = (c^+_{k+i\beta}, c^-_{k-i\beta})$.

### 5.2. Hidden quantum phase transition as normal–abnormal transition for fermionic correlation

After the projected Jordan–Wigner transformation, the non-Hermitian spin model is mapped into a 1D non-Hermitian Kitaev model, we can study the correlation between two fermions for the Kitaev model.

Firstly we discuss the correlation between two fermions far away in the 1D Hermitian Kitaev model $\hat{H}_F^{\beta=0}$.

For the 1D Hermitian Kitaev model $\hat{H}_F^{\beta=0}$, the fermion correlation is defined by $\langle \text{vac}|c_i^\dagger c_j|\text{vac}\rangle$. The finite energy gap indicates a finite correlation length $\xi$, i.e.,

$$
\langle \text{vac}|c_i^\dagger c_j|\text{vac}\rangle \sim \left( \frac{\hbar}{\xi} \right)^{|i-j|} = \exp(- |i-j|/\xi).
$$

(28)

Here, the correlation length is obtained as $1/\xi = \ln(\xi)$. We call the exponential decay of $\langle \text{vac}|c_i^\dagger c_j|\text{vac}\rangle$ to be normal correlation between two fermions far away, i.e., $\langle \text{vac}|c_i^\dagger c_j|\text{vac}\rangle \rightarrow 0$ in the limit of $|i-j| \rightarrow \infty$.

Next, we consider the case of 1D non-Hermitian Kitaev model, $\hat{H}_F^{\beta}=\sum_{k>0}\tilde{\psi}_k^\dagger [h_F^{\beta}(k)]\tilde{\psi}_k$ where $h_F^{\beta}(k) = (-2J \cos k + 2h)\sigma^z + (2J \sin k)\sigma^y$ and $\tilde{\psi}_k = (\tilde{c}^+_k, \tilde{c}^-_k) = (c^+_{k+i\beta}, c^-_{k-i\beta})$.

We also denote the fermionic correlation by $\tilde{C}_{ij} = \langle \text{vac}|\tilde{c}_i^\dagger \tilde{c}_j|\text{vac}\rangle$, which $|\text{vac}\rangle = |\text{vac}(\beta=0)\rangle$. Due to the existence of additional ‘imaginary’ wave vector $k_0 = i\beta$, we have

$$
\tilde{C}_{ij} = \langle \text{vac}|\tilde{c}_i^\dagger \tilde{c}_j|\text{vac}\rangle = \langle \text{vac}|c_i^\dagger c_j e^{-i\beta} e^{i\beta} |\text{vac}\rangle
$$

$$
\sim \left( \frac{\hbar}{\xi} \right)^{|i-j|} e^{-|i-j|\beta} = e^{-(\beta+1/\xi)|i-j|},
$$

(29)
and
\[ \tilde{C}_{ij} = \langle \text{vac} | \tilde{c}^\dagger_{i} \tilde{c}^\dagger_{j} | \text{vac} \rangle = \langle \text{vac} | \tilde{c}^\dagger_{i} e^{\beta |j|} \text{vac} \rangle \]
\[ \sim (\hbar / J)^{|j|-|j|} e^{\beta (|j| - |j|)} = e^{-\beta (|j| - |j|)}. \]  

(30)

The non-Hermitian effect is embodied into the creation and annihilation operators of excitations. From the fermionic correlations, there are two phases: for the case \( \pm \beta + 1 / \xi > 0 \) or \(|h^r / J| > h^r / J| > 1\), we have a normal correlation between two fermions far away, i.e.,
\[ \langle \text{vac} | \tilde{c}^\dagger_{i} \tilde{c}^\dagger_{j} | \text{vac} \rangle \sim e^{-\beta (|j| - |j|)} \rightarrow 0, \]
\[ \langle \text{vac} | \tilde{c}^\dagger_{i} \tilde{c}^\dagger_{j} | \text{vac} \rangle \sim e^{-\beta (|j| - |j|)} \rightarrow 0, \]

(31)

(32)

where \(|l - j| \rightarrow \infty\); for the case of \( \pm \beta + 1 / \xi < 0 \) or \(|h^r| > |h^r| > J\), we have an abnormal correlation between two fermions far away, i.e.,
\[ \tilde{C}_{ij} \rightarrow \infty, \]
\[ \tilde{C}_{ij} \rightarrow 0 \]

(33)

(34)

or
\[ \tilde{C}_{ij} \rightarrow 0, \]
\[ \tilde{C}_{ij} \rightarrow \infty, \]

(35)

(36)

where \(|l - j| \rightarrow \infty\). As a result, a quantum phase transition may occur at
\[ \pm \beta + 1 / \xi = 0 \]

(37)

since \(1 / \xi = \ln(\frac{J}{k})\), combined with equations (6) and (25), we find that
\[ |h^r| + |h^r|^2 = J. \]

(38)

According to the energy dispersion of quasi-particles \( \pm 2 \sqrt{(-J \cos k + h^r)^2 + (J \sin k)^2} \), there indeed does not exist energy closing at \(|h^r| + |h^r| = J\). As a result, the continuous quantum phase transition at \(|h^r| + |h^r| = J\) is an unusual phase transition without gap closing.

### 5.3. Topological superconducting order for 1D non-Hermitian Kitaev model

We study the topological property of the 1D non-Hermitian Kitaev model that is described by
\[ \tilde{H}_F = \sum_{k=0}^{\beta} v_{\beta} |h_F(k)| \tilde{\psi}_k. \]

To describe this topological property of of \(h_F^c(k)\), we define \(Z_2\) topological invariant,
\[ w = \text{sgn}(\eta_{k=0} \cdot \eta_{k=\pi}), \]

(39)

where \(\eta_{k=0} = \langle \text{vac} | \tilde{c}^\dagger_{l=0} \tilde{c}_{l=0} | \text{vac} \rangle = \langle \text{vac} | \tilde{c}^\dagger_{l=0} \tilde{c}_{l=0} | \text{vac} \rangle\) and \(\eta_{k=\pi} = \langle \text{vac} | \tilde{c}^\dagger_{l=\pi} \tilde{c}_{l=\pi} | \text{vac} \rangle = \langle \text{vac} | \tilde{c}^\dagger_{l=\pi} \tilde{c}_{l=\pi} | \text{vac} \rangle\). So we have \(\eta_{k=0} = \text{sgn}(-J + h), \) and \(\eta_{k=\pi} = \text{sgn}(J + h). \) So, there are two phases: topological superconducting order with \(w = -1\) in the region of \(|h| < |J|\) and trivial superconducting order with \(w = 1\) in the region of \(|h| > |J|\).

Thus, in the region of \(|J| > |h|\), the 1D non-Hermitian Kitaev model has topological superconducting order. The ground state on a closed chain is two-fold degenerate, one is a ground state for fermions under periodic boundary condition, \(|\text{vac}_0^R\rangle = |0\rangle\); the other is a ground state for fermions under anti-periodic boundary condition, \(|\text{vac}_0^S\rangle = |\pi\rangle\).

### 5.4. Effective Hamiltonian for the two degenerate ground states

From above discussion, the ferromagnetic order \(|h| < |J|\) in non-Hermitian transverse Ising model \(H_{\text{NTI}}\) corresponds to the topological phase in non-Hermitian 1D Kitaev model \(H_F^c\). The ground state on a closed chain has two-fold degeneracy: \(|0\rangle\) and \(|\pi\rangle\). To observe the hidden, continuous quantum phase transition without gap closing, we derive the effective Hamiltonian for the two degenerate ground states in topological superconducting order for 1D non-Hermitian Kitaev model.
To accurately characterize the physics of the degenerate ground states, we introduce an two-level effective Hamiltonian [50]

$$\hat{H}_{\text{GS}} = \begin{pmatrix} h_{++} & h_{+-} \\ h_{-+} & h_{--} \end{pmatrix},$$

(40)

where

$$h_{ij} = \langle \text{vac}_I | \hat{H}_{\text{NH}} | \text{vac}_J \rangle,$$

(41)

$$I, J = +, -.$$ (42)

are the basis of the degenerate ground states.

It is known that when a virtual fermion (or a virtual kink) moves around the closed chain, the periodic boundary condition is switched to the anti-periodic boundary condition. Consequently, after the quantum tunneling process one ground state changes to the other, i.e., $|0\rangle \to |\pi\rangle$ or $|\pi\rangle \to |0\rangle$. See the illustration of the quantum tunneling processes from virtual fermion (or a virtual kink) around the closed chain in figure 3. Thus, for the 1D non-Hermitian Kitaev model, the quantum tunneling process between two degenerate ground states comes from virtual fermion moving around the closed chain. To show the physical consequences of the hidden quantum phase transition, we obtain the effective Hamiltonian for the two degenerate ground states by calculating the contribution from quantum tunneling processes step by step.

Firstly, we calculate the transition matrix $\Delta^+$ between two degenerate ground states based on the 1D Hermitian Kitaev model $\hat{H}_E=0$. We denote the transition matrix $\Delta^+$ between two degenerate ground states from virtual fermions (or virtual kinks) by

$$\langle \text{vac}|c_{j+N}^{\uparrow} (\langle \cdot \rangle) c_j \text{vac}\rangle,$$

(43)

where $j + N$ and $j$ are the same lattice site and (\langle \cdot \rangle) means the virtual fermion moving along single direction from $j$ to $j + N$. For the quantum tunneling process of a virtual fermion propagating along one direction around the chain, the transition matrix is obtained as

$$\Delta^+ \sim \langle \text{vac}|c_{j+N}^{\uparrow} \langle \cdot \rangle c_j \text{vac}\rangle$$

$$\sim \left(\frac{h}{\xi}\right)^N \exp(-N/\xi).$$

(44)

For the Hermitian model, the conjugate transition matrix $\Delta^-$ is defined by $(\Delta^+)^\dagger \sim \langle \text{vac}|c_{j+N}^{\uparrow} \langle \cdot \rangle c_j + N |\text{vac}\rangle$ where

$$\langle \text{vac}|c_{j+N}^{\uparrow} \langle \cdot \rangle c_j + N |\text{vac}\rangle.$$ (45)

Next, we consider the case of non-Hermitian Kitaev model $\hat{H}_E$. We denote the transition matrix $\Delta^+$ between two degenerate ground states from virtual fermion (or a virtual kink) by

$$\langle \text{vac}|c_{j+N}^{\uparrow} \langle \cdot \rangle c_j \text{vac}\rangle$$

where $j + N$ and $j$ are the same lattice site and (\langle \cdot \rangle) means the virtual fermion moving along single direction from $j$ to $j + N$. We have

$$\Delta^+ \sim \langle \text{vac}|c_{j+N}^{\uparrow} \langle \cdot \rangle c_j \text{vac}\rangle \sim e^{-(\beta+1/\xi)N}$$

(46)

and

$$\Delta^- \sim \langle \text{vac}|c_{j}^{\uparrow} \langle \cdot \rangle c_{j+N} |\text{vac}\rangle \sim e^{-(\beta+1/\xi)N}.$$

(47)

Thirdly, we add a perturbative term slightly breaking the global symmetry, i.e.,

$$\hat{H}_{\text{NH}} \to \hat{H}'_{\text{NH}} = \hat{H}_{\text{NH}} + \delta\hat{H}$$

where $\delta\hat{H}$ slightly breaks the degeneracy of ground states and chooses a particular ground state with lowest energy $|\text{vac}_I^{\uparrow}(\beta)\rangle$. Due to $\delta\hat{H}$, there exists energy difference between two degenerate ground states $\varepsilon$. This process plays a role of $\varepsilon$ on the quantum states.
Finally, for the 1D Kitaev model from non-Hermitian transverse Ising model by projected Jordan–Wigner transformation, the effective Hamiltonian for the degenerate ground states is obtained as

$$\hat{H}_{\text{GS}} = \Delta^+ \tau^+ + \Delta^- \tau^- + \epsilon \tau^z,$$

where $\tau^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\Delta^+ \sim \langle \text{vac|}c_i^+ N \langle \otimes \bar{c}_j \rangle |\text{vac} \rangle$, $\Delta^- \sim \langle \text{vac|}c_i^0 N \langle \otimes \bar{c}_j \rangle |\text{vac} \rangle$.

### 5.5. The observation of the hidden, continuous quantum phase transition

In this section, we show the existence of the hidden, continuous quantum phase transition and provide approaches to observe it.

From above discussion, for the non-Hermitian transverse Ising model, the effective Hamiltonian for the degenerate ground states is reduced into

$$\hat{H}_{\text{GS}} \rightarrow \epsilon \cdot \tau^z.$$

Now, in the thermodynamic limit $N \rightarrow \infty$, the state-similarity $\Upsilon$ of the two degenerate ground states turns to zero, i.e.,

$$\Upsilon = \left| \langle \text{vac}_r \rangle \right| = e^{-2\beta N} \rightarrow 0.$$

On the other hand, one is denoted by $\Delta^+ \rightarrow 0$, $\Delta^- \rightarrow \infty$ or $\Delta^\pm \rightarrow \infty$, $\Delta^\pm \rightarrow 0$. In the region, we have $|e^{N(\pm \beta - \kappa)}| \rightarrow \infty$ or $\pm \beta - \kappa > 0$. The effective Hamiltonian for the degenerate ground states is reduced into

$$\hat{H}_{\text{GS}} \rightarrow e^{\pm 2\beta N} \tau^+ \text{ or } e^{\pm 2\beta N} \tau^-.$$

Now, in the thermodynamic limit $N \rightarrow \infty$, the state-similarity of the two degenerate ground states turns to 1, i.e.,

$$\Upsilon = \left| \langle \text{vac}_r \rangle \right| = |\text{tanh}(2\beta N)| \rightarrow 1.$$

As a result, to observe the existence of the hidden, continuous quantum phase transition, one can detect the sudden change of state-similarity with increasing $\beta$. In figure 4, we numerically calculate the state-similarity of the two degenerate ground states and find the sudden change due to the hidden, continuous quantum phase transition. The numerical results are agree with the theoretical prediction.

In addition, the fidelity of the ground state with lowest energy could illustrates the hidden, continuous quantum phase transition. We calculate the fidelity of the ground state with lowest energy by exactly numerical. The fidelity of ground state is defined by

$$F(\epsilon, \delta) = \left| \langle \text{vac}_r(\epsilon) | \text{vac}_r(\epsilon + \delta) \rangle \right|,$$
Figure 5. (a). The state-similarity $\mathcal{Y}$ of two degenerated ground states for $N = 16$, the red dashed line is $|h| \pm |h'| = J$ of which hidden phase transition happens, the dashed green line is spontaneous $\mathcal{PT}$ symmetry breaking line, the dashed black line is where we calculate the fidelity of ground state for $h_z = 0$. (b) The fidelity of the ground state for $N = 16, h_z = 0$. The divergence of the fidelity in this figure indicates the existence of hidden quantum phase transition, the divergence point is $h_z = 0.4J, h_z = -0.4J$ agrees with theoretical prediction.

Figure 6. The global phase diagram of 1D non-Hermitian transverse Ising model for $J = 1$. There are four different phases: I phase has $\mathcal{PT}$ symmetry and long range ferromagnetic order with normal fermionic correlation; II phase has $\mathcal{PT}$ symmetry and long range ferromagnetic order with abnormal fermionic correlation; III phase has $\mathcal{PT}$ symmetry and short range ferromagnetic order; IV phase has $\mathcal{PT}$ symmetry breaking. There exist three kinds of phase transitions: (a) Spontaneous $\mathcal{PT}$ symmetry breaking at $|h'| = |h|$; (b) Quantum phase transition at $J = |h|$ that corresponds to the order–disorder phase transition with gap closing; (c) Quantum phase transition at $|h'| \pm |h| = J$ that is characterized by normal–abnormal transition for fermionic correlation in 1D non-Hermitian Kitaev model. This corresponds to the hidden, continuous quantum phase transition without gap closing.

where $\epsilon$ is a parameter term of the model, $\delta$ is the increment of the term. See the results in figure 5, one can see that the fidelity becomes divergent at $h_z = \pm 0.5$ for $h_z = 0.5$. The result indicates a quantum phase transition, which is consistent to our theoretical prediction.

6. Global phase diagram

Therefore, as shown in figure 6, for the 1D non-Hermitian transverse Ising model described by $\hat{H}_{\text{NH}}$, we get a complex phase diagram with four phases (see figure 6): I phase has $\mathcal{PT}$ symmetry and long range ferromagnetic order with normal fermionic correlation; II phase has $\mathcal{PT}$ symmetry and long range ferromagnetic order with abnormal fermionic correlation; III phase has $\mathcal{PT}$ symmetry and short range ferromagnetic order; IV phase has $\mathcal{PT}$ symmetry breaking. There exist three kinds of phase transitions:

(a) Spontaneous $\mathcal{PT}$ symmetry breaking at $|h'| = |h|$;
(b) Quantum phase transition at $J = |h|$ that corresponds to the order–disorder phase transition with gap closing;
(c) Quantum phase transition at $|h'| \pm |h| = J$ that is characterized by normal–abnormal transition for fermionic correlation in 1D non-Hermitian Kitaev model. This corresponds to the hidden, continuous quantum phase transition without gap closing.
7. Hidden, continuous quantum phase transitions in two-dimensional non-Hermitian transverse Ising model

At last to generalize our study from a 1D case to two dimensions, we found the hidden, continuous quantum phase transition also exists in two-dimensional non-Hermitian transverse Ising model. In the last section, we study two-dimensional non-Hermitian transverse Ising model \([31–56]\), of which the Hamiltonian is given by

\[
\hat{H}_{\text{NTI}}^{2D} = -J \sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x + \sum_j h' \sigma_j^x + i \sum_j h'' \sigma_j^y,
\]

where \(J > 0\) is ferromagnetic Ising coupling constant between two nearest neighbor spins and \(h'\) is the strength of a real transverse field along \(y\)-axis, \(h''\) is the strength of a imaginary transverse field along \(z\)-axis.

In this paper, the coupling parameter \(J\) is set to be unit, \(J = 1\). This Hamiltonian also shows \(\mathcal{PT}\)-symmetry, i.e., \(|\mathcal{P}, \hat{H}_{\text{NTI}}^{2D}\rangle \neq 0\) and \(|\mathcal{T}, \hat{H}_{\text{NTI}}^{2D}\rangle \neq 0\), but \(|\mathcal{PT}, \hat{H}_{\text{NTI}}^{2D}\rangle = 0\).

We consider a non-Hermitian similarity transformation \(\hat{S}(\beta) = \prod_j S_j(\beta)\) on spin system with \(S_j(\beta)\) as

\[
(s_j^x)^\beta = S_j(\beta) s_j^x S_j^{-1}(\beta) = \cosh(\beta) s_j^x + i \sinh(\beta) s_j^y,
\]

\[
(s_j^y)^\beta = S_j(\beta) s_j^y S_j^{-1}(\beta) = \cosh(\beta) s_j^y - i \sinh(\beta) s_j^x,
\]

\[
(s_j^z)^\beta = S_j(\beta) s_j^z S_j^{-1}(\beta) = s_j^z,
\]

where \(\beta = \ln(\sqrt{h' + i h'}/\sqrt{h''})\) denotes non-Hermiticity. Under the non-Hermitian similarity transformation, the original spin model turns into

\[
\hat{H}_0^{2D} = \hat{S}(\beta) \hat{H}_{\text{NTI}}^{2D} \hat{S}(\beta)^{-1}
\]

\[
= -\sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x + h \sum_j \sigma_j^y,
\]

where \(h = \sqrt{|h'|^2 - |h''|^2}\). For the case of \(|h'| > |h''|\), the energy spectra for excitations are all real; for the case of \(|h'| < |h''|\), the energy spectra for the excitations become complex. At \(|h'| = |h''|\), there exist exceptional points for excitations. In this paper, we focus on the case of \(|h'| > |h''|\).

The order parameter is defined by the expectation value for the ground states in the bioorthogonal sets \(|\text{vac}^R\rangle\) and \(|\text{vac}^L\rangle\), i.e., \(|\text{vac}^R\rangle \frac{1}{N} \sum_j \sigma_j^y |\text{vac}^L\rangle = A^B\). In the region of \(|h'| > |h''|, h < h_c\, A^B \neq 0\), there exists long range ferromagnetic order. In the region of \(|h'| > |h''|, h > h_c\, A^B = 0\), the ground state is a disordered state without degeneracy. At \(h_c \approx 3.04J\) that indicates the order–disorder phase transition occurs.

For the non-Hermitian transverse Ising model, in the region of \(|h'| > |h''|, J > h_c\, the ground states have two-fold degeneracy. The effective Hamiltonian for the degenerate ground states is written as

\[
\tilde{H}_{\text{GS}} = \Delta^+ \varepsilon^+ + \Delta^- \varepsilon^- + \varepsilon_i^+\varepsilon_i^-,
\]

where \(\Delta^\pm \sim \Delta_0 e^{\pm \kappa N/4}\) and \(\Delta_0\) is an amplitude from quantum tunneling effect. Here, \(N\) is the total lattice number. Unfortunately, for the 2D non-Hermitian transverse Ising model, the projected Jordan–Wigner transformation fails. In this paper, we just estimate the tunneling amplitude \(\Delta_0\) to be proportional to \((\frac{1}{2})^N\).

As a result, we have \(\Delta^\pm \sim (\frac{1}{2})^N e^{\pm \kappa N/4}\).

In thermodynamic limit \(N \to \infty\), there exists the competition between the exponential decay of \(\Delta \sim e^{-\kappa N} (\kappa = \ln(\frac{1}{2}))\) with the size of the system from quantum tunneling effect and the exponential enhancement of \(e^{\pm \kappa N}\) with the size of the system from non-Hermitian similarity effect. Therefore, in thermodynamic limit \(N \to \infty\), there exist two phases: one is denoted by \(e^{(\kappa N \pm \beta - \kappa N)} \to 0\) or \(|\beta| < |\kappa| = |\ln(\frac{1}{2})|\), the other is denoted by \(e^{(\kappa N \pm \beta - \kappa N)} \to \infty\) or \(|\beta| > |\kappa| = |\ln(\frac{1}{2})|\). At \(|\beta| = |\kappa| = |\ln(\frac{1}{2})|\) or \(|h'| \pm |h''| = |h_c|\), a hidden quantum phase transition without gap closing occurs.

We then study the two dimensional non-Hermitian transverse Ising model by exactly diagonal numerical calculation on 4 \times 4 square lattice. Figures 7 and 8 show the energy gap and order parameter from exact diagonal numerical calculations for 2D non-Hermitian transverse Ising model on 4 \times 4 square lattice, respectively. The energy difference \(\Delta E = E_1 - E_0\) between the lowest two energy levels \(E_1\) and \(E_0\) is numerically obtained from exact diagonalization calculations for the model on \(N = 4 \times 4\) lattice. From figure 7, we can clearly find that the ground states are doubly degenerate in the ordered phase, while the ground state is unique for the disordered phase. This result indicates there is order–disorder phase.
Figure 7. The energy difference of two lowest energy level for 2D non-Hermitian transverse Ising model with $N = 4 \times 4$. Where the green line represents the $PT$ symmetry breaking happens, the yellow dashed line is $\sqrt{|h^y|^2 - |h^z|^2} = |b| = 2$ which the order–disorder phase transition with gap closing happens.

Figure 8. The order parameter $A^h$ of the ground state for 2D non-Hermitian transverse Ising model with $N = 4 \times 4$. Where the green line represents the $PT$ symmetry breaking happens, the blue line is $\sqrt{|h^y|^2 - |h^z|^2} = |b| = 2$ which the order–disorder phase transition with gap closing happens. Note that longitudinal field $h^x = 0.002$.

Figure 9. The state similarity $|\langle \text{vac}^R(\beta) | \text{vac}^L(\beta) \rangle|$ of two degenerated ground state for 2D non-Hermitian transverse Ising model with $N = 4 \times 4$. Where the green line represents the $PT$ symmetry breaking happens, the blue line is $\sqrt{|h^y|^2 - |h^z|^2} = |b| = 2$ which the order–disorder phase transition with gap closing happens, the red line is $|h^y| \pm |h^z| = 2$ that hidden phase transition without gap closing happens. Those are agree with our theoretical prediction by replacing $|h_c|$ as 2 due to the finite size effect.

transition, where the order parameters $\langle \text{vac}^R | \sum_j \sigma_j^z | \text{vac}^L \rangle$ can be calculated by using the biorthogonal basis as shown in figure 8.

We find that the behaviors of state-similarity $|\langle \text{vac}^R(\beta) | \text{vac}^L(\beta) \rangle|$ for the two degenerate ground states persist in 2D non-Hermitian transverse Ising model as demonstrated in figure 9. A hidden quantum phase transition occurs at $|h^y| \pm |h^z| = |h_c|$. We found that at $|\beta| = |\kappa| = |\ln(J_h)|$ (or $|h^y| \pm |h^z| = |h_c|$) there
indeed exists sudden change of state-similarity $\langle \psi^R_{\beta} | \psi^R_{\beta} \rangle$ from 0 to 1, i.e.,

$$\langle \psi^R_{\beta} | \psi^R_{\beta} \rangle = 0, \quad |\beta| < |\kappa|;$$

$$\langle \psi^R_{\beta} | \psi^R_{\beta} \rangle = 1, \quad |\beta| > |\kappa|.$$ 

As a result, the global phase diagram is quite similar to that of the 1D non-Hermitian transverse Ising model, except for the positions of critical values. We note that the critical value $|h_c|$ obtained from our $N = 4 \times 4$ lattice at $h_0 = 0$ is smaller than $|h_c| \approx 3.04$ [57, 58] for the 2D Hermitian transverse Ising model in thermodynamic limit due to the finite-size effect.

Finally, we conclude the hidden, continuous quantum phase transitions without gap closing are universal in non-Hermitian transverse Ising model.

8. Conclusion and discussion

In this paper, we studied the 1D non-Hermitian transverse Ising model, a typical non-Hermitian many-body model with $\mathcal{PT}$ symmetry. For the 1D non-Hermitian transverse Ising model, we show a much more complex phase diagram than that for its Hermitian counterpart. This fact indicates the richness of non-Hermitian many-body physics. In particular, a hidden, continuous quantum phase transition without gap closing is discovered by using projected Jordan–Wigner transformation. The numerical results are all consistent with the theoretical predictions, that indicates the results in this paper are valid and universal for non-Hermitian many-body physics with spontaneous $\mathbb{Z}_2$-symmetry breaking. In the future, the quantum phases and quantum phase transitions in many-body non-Hermitian models will be studied to explore exotic, universal features of the non-Hermitian physics.

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Data availability statement

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

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Usual Jordan-Weigner transformation will introduce the string operators like \( \prod \sigma_i^x \) in non-Hermitian transverse Ising model

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