A conjecture on independent sets and graph covers

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Abstract

In this article, I present a simple conjecture on the number of independent sets on graph covers. The conjecture implies that the partition function of a binary pairwise attractive model is greater than that of the Bethe approximation.

Key words: graph cover, independent set, Bethe approximation,

1. Terminologies

Throughout this article, \( G = (V, E) \) is a finite graph with vertices \( V \) and undirected edges \( E \). For each undirected edge of \( G \), we make a pair of oppositely directed edges, which form a set of directed edges \( \vec{E} \). Thus, \( |\vec{E}| = 2|E| \).

An \( M \)-cover of a graph \( G \) is its \( M \)-fold covering space. All \( M \)-covers are explicitly constructed using permutation voltage assignment as follows [1]. A permutation voltage assignment of \( G \) is a map \( \alpha : \vec{E} \to \mathfrak{S}_M \) s.t. \( \alpha(u \to v) = \alpha(v \to u)^{-1} \) \( \forall uv \in E \),

\[
\alpha : \vec{E} \to \mathfrak{S}_M \quad \text{s.t.} \quad \alpha(u \to v) = \alpha(v \to u)^{-1} \quad \forall uv \in E,
\]

where \( \mathfrak{S}_M \) is the permutation group of \( \{1, \ldots, M\} \). Then an \( M \)-cover \( \tilde{G} = (\tilde{V}, \tilde{E}) \) of \( G \) is given by \( \tilde{V} := V \times \{1, \ldots, M\} \) and

\[
(v, k)(u, l) \in \tilde{E} \Leftrightarrow uv \in E \text{ and } l = \alpha(v \to u)(k).
\]

If an \( M \)-cover is \( M \) copies of \( G \) then it is called trivial \( M \)-cover and denoted by \( G \oplus M \). This is obtained by identity permutations. The natural projection, \( \pi \), from a cover \( \tilde{G} \) to \( G \) is obtained by forgetting the “layer number”. That is, \( \pi : \tilde{V} \to V \) is given by \( \pi(u, i) = u \) and \( \pi : \tilde{E} \to E \) is given by \( \pi((v, k)(u, l)) = vu \).

An independent set \( I \) of a graph \( G \) is a subset of \( V \) such that none of the elements in \( I \) are adjacent in \( G \). Formally, \( I \) is an independent set iff \( u, v \in I \Rightarrow uv \notin E \). The multivariate independent set polynomial of \( G \) is defined by

\[
p(G) := \sum_{I: \text{independent set}} \prod_{v \in I} x_v,
\]

with indeterminates \( x_v \quad (v \in V) \).

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1Interpret graphs as topological spaces.
2. The conjecture

We extend the definition of the projection map \( \pi \) over the multivariate polynomial ring. First, let us define a map \( \Pi \) for each indeterminate by \( \Pi(x_v) = x_{\pi(v)} \), where \( v \in \tilde{V} \). Then this is uniquely extended to the polynomial ring as a ring homomorphism; for example \( \Pi(x_v + x_{v'}) = \Pi(x_v) + \Pi(x_{v'}) \) and \( \Pi(x_v x_{v'}) = \Pi(x_v) \Pi(x_{v'}) \).

**Conjecture 1.** For any bipartite graph \( G \) and its \( M \)-cover \( \tilde{G} \), we conjecture the following relation:

\[
\Pi(p(\tilde{G})) \preceq p(G)^M,
\]

where \( \Pi \) is defined as above and the symbol \( \preceq \) means the inequalities for all coefficients of monomials.

We can interpret the conjecture more explicitly as follows. For a subset \( U \) of \( V \), define \( \mathcal{I}(\tilde{G}, U) \) := \( \{ I \subset \tilde{V} | I \text{ is independent set, } \pi(I) = U \} \), where \( \pi(I) \) is the image of \( I \). Since \( p(G)^M = \Pi(p(G^{\oplus M})) \), the conjecture is equivalent to the following statement:

\[
|\mathcal{I}(\tilde{G}, U)| \leq |\mathcal{I}(G^{\oplus M}, U)| \text{ for all } U \subset V.
\]

**Example 1.** Let \( G \) be a cycle graph of length four and let \( \tilde{G} \) be its 3-cover that is isomorphic to the cycle of length twelve. Then,

\[
p(G) = 1 + x_1 + x_2 + x_3 + x_4 + x_1 x_3 + x_2 x_4,
\]

\[
p(\tilde{G}) = 1 + \sum_{v=1}^{4} \sum_{m=1}^{3} x(v,m) + \ldots,
\]

\[
\Pi(p(\tilde{G})) = 1 + 3(x_1 + x_2 + x_3 + x_4) + \ldots.
\]

It takes time and effort to check the conjecture, however, it is true in this case.

**Remark.** The above conjecture is claimed for the pair (bipartite graph, independent set). I also conjecture analogous properties for (bipartite graph, matching), (graph with even number of vertices, perfect matching) and (graph, Eulerian set).

3. Implication of the conjecture

The conjecture originates from the theory of the Bethe approximation. The partition function of a binary pairwise model on a graph \( G \) is

\[
Z(G; J, h) := \sum_{s \in \{0,1\}^V} \exp(\sum_{u \neq v} J_{uv} s_u s_v + \sum_{v \in V} h_v s_v),
\]

\( ^2 \) I have checked the conjecture for many examples by computer.

\( ^3 \) A subset of edges is Eulerian if it induces a subgraph that only has vertices of degree two and zero.
where the weights \((J, h)\) are called interactions\(^4\) It is called attractive if \(J_{vu} \geq 0\) for all \(vu \in E\). The Bethe partition function\(^5\) \(Z_B\) is defined by \(2\):

\[
Z_B := \exp \left( - \min_q F_B(q) \right)
\]

\[
= \limsup_{M \to \infty} < Z(\tilde{G}) > ^{1/M},
\]

where \(F_B\) is the Bethe free energy and \(< \cdot >\) is the mean with respect to the \(M!^{|E|}\) covers. (Details are omitted. See \([2]\).)

**Theorem 1.** If Conjecture \(4\) holds, then

\[
Z \geq Z_B
\]

holds for any binary pairwise attractive models\(^6\).

**Proof.** From \(11\), the assertion of the theorem is proved if we show that

\[
Z(G)^M \geq Z(\tilde{G})
\]

for any \(M\)-cover \(\tilde{G}\) of \(G\). In the following, we see that the partition function can be written by the independent set polynomial and thus the above inequality holds under the assumption of Conjecture \(5\).

\[
Z(G) = \sum_{s \in \{0, 1\}^V} \exp \left( \sum_{uv \in E} J_{uv} s_u s_v + \sum_{v \in V} h_v s_v \right)
\]

\[
= \sum_{s \in \{0, 1\}^V} \prod_{uv \in E} (1 + A_{uv} s_u s_v) \prod_{v \in V} \exp(h_v s_v)
\]

\[
= \sum_{S \subseteq E, S \neq \emptyset} A_{vu} \prod_{uv \in S} \left( \prod_{v \in V, s_v \neq 0, 1} s_d v (S) \exp(h_v s_v) \right)
\]

\[
= \prod_{v \in V} \exp(h_v) \sum_{S \subseteq E, U \subseteq V} \prod_{uv \in S} A_{uv} \prod_{v \in U} B_v
\]

\[
= \prod_{v \in V} \exp(h_v) \ p(G' ; A, B),
\]

where \(d_v(S)\) is the number of edges in \(S\) connecting to \(v\), \(A_{uv} = e^{J_{uv}} - 1\), \(B_v = e^{-h_v}\) and \(G'\) is a bipartite graph obtained by adding a new vertex on each edge of \(G\). \(\square\)

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\(^4\)In the following, for a cover \(\tilde{G}\) of \(G\), we think that interactions are naturally induced from \(G\).

\(^5\)This is computed from the absolute minimum of the Bethe free energy; other Bethe approximations of the partition function corresponding to local minima are smaller than \(Z_B\).

\(^6\)In a quite limited situation, the inequality is proved in \([3]\).
Remark. The Bethe approximation can also be applied to the computation of the permanent of non-negative matrices. From the combinatorial viewpoint, this problem is related to the (weighted) perfect matching problem on complete bipartite graphs. Vontobel analyzed this problem and pose a conjecture analogous to Eq. (5) [4]. This conjecture implies the inequality between the permanent and its Bethe approximation, $Z \geq Z_B$, given his formula Eq. (11). The statement $Z \geq Z_B$ is, however, directly proved by Gurvits, generalizing Schrijver’s permanental inequality [5].

References

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