GRAPH CLUSTERING VIA GENERALIZED COLORINGS

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ABSTRACT. We propose a new approach for defining and searching clusters in graphs that represent real technological or transaction networks. In contrast to the standard way of finding dense parts of a graph, we concentrate on the structure of edges between the clusters, as it is motivated by some earlier observations, e.g. in the structure of networks in ecology and economics and by applications of discrete tomography. Mathematically special colorings and chromatic numbers of graphs are studied.

1. Introduction and Results

One of the main tasks in network theory is clustering vertices, see Newman [8]. Graph clustering is a well-studied problem and has important applications in graph mining or model construction. The usual methods try to achieve many edges inside clusters and only a few between distinct clusters [9]. This approach generally works well for so-called social graphs, which usually contain more triangles than a random graph with similar edge density or degree properties. In contrast technological or transaction graphs contain fewer triangles and often display tree-like structures. To measure the algorithms’ efficacy the parameter known as Newman modularity is commonly used [8].

However, this standard approach is not always justified. Certain bipartite graphs, e.g. those that describe pollinator networks or trade networks, suggest the presence of different structures, like the notion of embeddedness, see Uzzi [10]. That is, the vertices of each color class can be ordered, and the smaller ranked vertex neighborhood contains the neighborhood of any higher ranked one. In the context of image processing, Juntila and Kaski [6] call a binary matrix \( A \) (that is, a matrix whose entries are either zero or one) fully nested if its rows and columns can be reordered such that the

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ones are in an echelon form. Let \( G_A \) be the bipartite graph whose adjacency matrix is \( A \). Then \( A \) being fully nested is equivalent to \( G_A \) satisfying embeddedness.

Let \( X \) (the columns) and \( Y \) (the rows) be the bipartition of \( G_A \). The matrix \( A \) and the graph \( G_A \) are each said to be \( k \)-nested with respect to \( X \) if \( X \) can be partitioned as \( X_1, \ldots, X_k \) such that all subgraphs spanned by \((X_i, Y)\) are fully nested for \( i = 1, \ldots, k \). The quantity of interest for any \( G_A \) is smallest \( k \) for which \( G_A \) is \( k \)-nested.

We present a new kind of clustering of general (that is, not necessary bipartite) transaction graphs via a certain class of proper colorings. The clusters are the color classes, since we do not want edges inside a cluster, and we restrict the structure of the edges between the pairs of classes. The above examples suggest that in some cases there should be a fully nested or, equivalently, embeddedness relation among any two color classes. We generalize this notion to an arbitrary host graph \( G \) and a forbidden bipartite subgraph \( H \) as follows.

**Definition 1.** Fix a bipartite graph \( H \). A proper coloring of a graph \( G \) is an \( H \)-avoiding coloring if the union of any two color classes spans an induced \( H \)-free graph. Let \( \chi_H(G) \) be the minimum number of colors in an \( H \)-avoiding coloring of \( G \).

Note that the function \( \chi_H(G) \) is not necessarily monotone either in \( H \) or in \( G \). However, we have a useful property:

**Observation 2.** For any graphs \( H \) and \( G \), \( \chi(G) \leq \chi_H(G) \). If \( G \) is \( H \)-free, then \( \chi(G) = \chi_H(G) \).

1.1. **Complexity issues.** We show that that the computation of \( \chi_H(G) \) is NP-hard for some graphs, and polynomially computable for others. The most interesting case, when \( H = 2K_2 \), gives back embeddedness as described above. For these generalized chromatic numbers we derive some theoretical extremal results as well as results on complexity.

In the following we use \( K_n \), \( P_n \) and \( C_n \) for the complete graph, path and cycle on \( n \) vertices, respectively. For graphs \( H_1 \) and \( H_2 \) on disjoint vertex sets, \( H_1 \oplus H_2 \) denotes their disjoint union. Theorem \( 3 \) gives a characterization of the complexity issues in computing \( \chi_H(G) \) depending on the graph \( H \).

**Theorem 3.** The computation of \( \chi_H(G) \) is polynomial-time solvable if \( H \) is \( K_1 \oplus K_1 \), \( K_2 \), or \( K_2 \oplus K_1 \) and is NP-hard for all other graphs.

It is valuable to spell out special cases since the proofs of these are needed in proving Theorem \( 3 \).

**Lemma 4.** It is NP-complete to decide if \( \chi_{P_3}(G) \leq 5 \), while it is polynomial time decidable if \( \chi_{P_3}(G) \leq 3 \).

**Lemma 5.** It is NP-complete to decide if \( \chi_{P_4}(G) \leq 3 \).

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\(^1\)Heuristics for finding \( H \)-avoiding colorings and case studies will be presented in a future paper.
Lemma 6. There is a unique $H$-avoiding coloring of $G$ using exactly $\chi_H(G)$ colors if $H = K_2 \oplus K_1$. One can find this coloring in polynomial time.

Let us note that a $P_3$-avoiding coloring of $G$ has a nice combinatorial meaning, it represents the edges of $G$ as the union of independent matchings. The computation of $\chi_{P_3}(G)$ can be reduced to the normal chromatic number. Let $P_3(G)$ be a graph which made from $G$ by adding an edge to every induced $P_3$, i.e. making a triangle out of these $P_3$.

Observation 7. $\chi(P_3(G)) = \chi_{P_3}(G)$.

Note that if a bipartite graph $G_A$ is $k$-nested then it has a similar reduction as in Observation 7.

For a bipartite graph $G_A$ with bipartition $(X, Y)$, define the conflict graph $co(X)$ on $X$ such that $(x, x')$ is an edge in $co(G)$ for $x, x' \in X$ if there are $y, y' \in Y$ such that $\{x, x', y, y'\}$ spans a $2K_2$ in $G_A$.

Observation 8. The bipartite graph $G_A$ is exactly $k$-nested for $X$ if $\chi(co(X)) = k$.

For applications the computation of $\chi_{2K_2}(G)$ seems to be the most important case.

Theorem 9. It is polynomial time decidable if $\chi_{2K_2}(G) \leq 3$.

For a fix graph $H$ there is a linear upper bound on the value of $\chi_H(G)$. In this paper, $\log n$ is the logarithm of $n$ in base 2 and $\log n$ is the natural logarithm of $n$.

Proposition 10. Let $H$ be a bipartite graph and let $k_1$ be the smallest positive integer such that each bipartition of $H$ has a part with size at least $k_1$. Let $k_2$ be the smallest positive integer such that each bipartition of $H$ has both parts of size at least $k_2$. Let $G$ be an $n$-vertex graph with chromatic number $\chi$ and independence number $\alpha$. If $k_2 \geq 3$, then

$$\chi_H(G) \leq \min \left\{ \frac{n}{k_1 - 1} + \frac{k_1 - 2}{k_1 - 1} \chi, \frac{n}{k_2 - 1} \left( 1 - \frac{1}{\chi} \right) + \frac{k_2 - 2}{k_2 - 1} (\chi - 1) + 1 \right\}.$$  

If $k_2 = 2$, then

$$\chi_H(G) \leq \min \left\{ \frac{n}{k_1 - 1} + \frac{k_1 - 2}{k_1 - 1} \chi, n - \alpha + 1 \right\}.$$  

1.2. Random graphs. In the case where $G$ is a random graph drawn from $G(n, p)$, the Erdős-Rényi random graph on $n$ vertices with edge probability $p$, we establish tight bounds for $\chi_H(G)$. The distribution of $\alpha(G)$, where $G \sim G(n, p)$ for $p$ fixed, was determined by Bollobás and Erdős [3]. The distribution of $\chi(G)$ was first proven by in a classic result by Bollobás [2] and the error terms have been further refined by various authors (see [11]).

To be precise, whp means with high probability, i.e. a probability arbitrarily close to one, provided that the number of vertices (or other natural parameter) is large enough.
Theorem 11. Let $H$ be a bipartite graph with $k_1$ and $k_2$ defined as in Proposition 10. Fix $p \in (0, 1)$, let $d = 1/(1 - p)$, and let $G \sim G(n, p)$. If $k_1 \geq 3$ and $k_2 \geq 2$, then there is a $C = C(H, p)$ such that whp
\[ \frac{n}{k_1 - 1} - C \log n \leq \chi_H(G) \leq \frac{n}{k_1 - 1} + O\left(\frac{n}{\log_d n}\right). \]
If $k_1 = k_2 = 2$, then there exists a $C = C(H, p)$ such that whp
\[ n - C \log n \leq \chi_H(G) \leq n - 2 \log_d n + O\left(\log_d \log n\right). \]
In particular, if $H = 2K_2$, then whp
\[ n - 8 \log_1/Q n + \Omega\left(\log_1/Q \log n\right) \leq \chi_H(G) \leq n - 2 \log_d n + O\left(\log_d \log n\right), \]
where $Q = 1 - 2p^2(1 - p)^2$.

Finally, we mention a useful observation on the $H$-avoiding chromatic number, see its consequences in Section 5.

Observation 12. If $G$ is a graph such that every $H$-free induced subgraph has at most $\ell$ edges, then $\chi_H(G)$ satisfies
\[ \ell\left(\frac{\chi_H(G)}{2}\right) \geq e(G). \]

In the rest of the paper we provide the proofs of the main results. In Section 2 we prove Theorem 3. Section 3 contains the proof of Theorem 9 while Section 4 contains the proofs of Proposition 10 and of Theorem 11. In Section 5 we show some of the consequences of Observation 12. Finally, in Section 6 we list some unsolved questions which naturally came up during the research.

2. Proof of Theorem 3

We start the proof with the cases in which the graph $H$ is equal to either $K_1 \oplus K_1$, $K_2$ or $K_2 \oplus K_1$. The graph $G$ has $K_2$-avoiding coloring if and only if $G$ is the empty graph.

Proof of Lemma 6. If $H = K_2 \oplus K_1$ then any two color classes in an $H$-avoiding coloring spans either a complete or empty bipartite graph. (In the special case if $H = K_1 \oplus K_1$ then there can be only complete bipartite graphs between any two color classes.) Let us define a binary relation $\rho$ such that for $x, y \in V(G)$ we have $x\rho y$ iff $N(x) = N(y)$. Obviously $\rho$ is an equivalence relation, and the equivalence classes induced by $\rho$ are exactly the color classes of $G$ in the unique $K_2 \oplus K_1$-avoiding coloring of $\chi_{K_2\oplus K_1}(G)$ classes. \(\square\)

Král, Kratochvíl, Tuza and Woeginger [7] studied the hardness of coloring $H$-free graphs. They gave a complete description of the problem in the theorem follows:
Theorem 13 (Král-Kratochvíl-Tuza-Woeginger [7]). The problem $H$-FREE COLORING is polynomial-time solvable if $H$ is an induced subgraph of $P_4$ or of $P_3 \oplus K_1$, and NP-complete for any other $H$.

Combining Theorem 13 and Observation 2, one gets immediately that the computation of $\chi_H(G)$ is NP-complete if $\chi(G)$ is also NP-complete for $H$-free $G$. On the other hand, the polynomial-time computability of $\chi$ for $H$-free graphs does not imply the same for $\chi_H$. Among the polynomial cases of Theorem 13 we have checked already the graphs $K_1 \oplus K_1$, $K_2$ and $K_2 \oplus K_1$. Somehow against intuition, the computation of $\chi_H$ is NP-complete for the remaining $H = P_3$ and $H = P_4$ cases according to Lemmas 4 and 5.

Proof of Lemma 4. We need to show $L_5 = \{ G : \chi_{P_3}(G) \leq 5 \}$ is NP-complete language. We use reduction from the language

$$L_{3,2} = \{ T : T \text{ is a 3-uniform hypergraph, } \chi(T) \leq 2 \},$$

which is a well-known NP-complete problem. Let $T$ be an instance, that is $T \in L_{3,2}$.

We need to assign a graph $G_T$ to $T$ such that $\chi_{P_3}(G_T) \leq 5$ if and only if $\chi(T) \leq 2$. It turns out that the greatest difficulty is to associate the colorings of the graph $G_T$ and the hypergraph $T$. The color of a vertex $t$ of $T$ cannot be encoded in one vertex $x_t$ of $G_T$, since the gadgets constructed in $G_T$ that enforce the good coloring of the edges of $T$ containing $t$ would interfere with each other. The solution is to repeat the actual color of vertex $t$ at least as many times as the number of edges of $T$ that contain $t$. For simplicity we repeat the color of any vertex $t$ a total of $m$ times, where $m$ is the number of edges in $T$, and read out the color of $t$ at most once from each place.

The graph $G_T$ will consist of an $n \times m$ matrix of pentagons, in which the $i$-th row codes the color of the $i$-th vertex in $T$. To assess the coloring of the $j$-th edge of $T$, the $j$-th column of this matrix is read. The usual types of gadgets are used in $G_T$ representing and evaluating the edges of $T$, see Figure 1.

Before examining the coloring of $G_T$, let us examine a $P_3$-avoiding good coloring of just $C_5$, since $C_5$ is the main building block of our construction. The vertices are referenced clockwise. If the first vertex is colored by 1, the second by 2, the third vertex color can be neither 2, because of adjacency, or 1 since it would create a two colored $P_3$. So, without loss of generality, the first three vertices are colored 1, 2, 3 respectively. The fourth vertex needs the fourth color. It cannot be colored by 2 or 3 as before. If it would be colored by 1, the first, fifth and fourth vertices would form a two-colored $P_3$. Finally, the fifth vertex needs to be colored 5, since 1 and 4 are colors of adjacent vertices, while 2 or 3 would create two-colored $P_3$’s.

Assuming that the $n$ vertices of $T$ are $x_1, \ldots, x_n$ and the $m$ edges are $e_1, \ldots, e_m$, the graph $G_T$ is first constructed by taking an $n \times m$ matrix with a $C_5$ in each position $(i, j)$. The $C_5$ in the $(i, j)$ position will be referred to as $C_{i,j}$. Second, connect the third and fourth vertices of $C_{i,j}$ to the first vertex of $C_{i+1,j}$ for $i = 1, \ldots, n - 1$, ...
Figure 1. The graph $G_T$ with the possible coloring, proof of Lemma 4.

Figure 2. The two cases of colorings with gadgets, proof of Lemma 4.
j = 1, . . . , n, and similarly from C_{n,j} to C_{1,j+1} for j = 1, . . . , n − 1. Third, draw edges from the fourth vertex of C_{i,j} to the second vertex of C_{i,j+1} for i = 1, . . . , n and j = 1, . . . , m − 1. See Figure 1.

Without loss of generality, any P_3-avoiding five-coloring should use color 1 at the first vertex of any C_5, should use the colors 2 and 5 in the second and fifth vertices (although in any order) and the colors 3 and 4 in the third and fourth vertices (again their order is arbitrary).

Furthermore, is easy to verify that a proper P_3-avoiding five-coloring must use the same order of colors 2 and 5 within a row, while the order of 2 and 5 can be arbitrary for each row. We will use the i-th row to code the color of the vertex x_i of the hypergraph T. However, when we read this “value,” each C_5 is read only once.

Finally, the gadgets realizing the edges of T are m copies of K_{1,3}. Let \( e_\ell \) be \{x_p, x_q, x_r\} and connect the leaves of the \( \ell \)-th K_{1,3} to the fifth vertex of a yet unused C_5 in the p-th, q-th and r-th rows, respectively. The colors the vertices of \( e_\ell \) receives are the color of the fifth vertices of C_5-s which were connected to the leaves of the representing K_{1,3}.

Let us check if proper five-colorings of the construction and proper two-colorings of T correspond to each other. If, for \( e_\ell \), the vertices in the graph coloring all receive the color, say 5, then the leaves of the representative K_{1,3} can be colored 2 or 3. One of these colors appears two times, and it results in a two-colored P_3 in the graph coloring. If \( e_\ell \) is colored properly, say 5, 5, 2, then the connected vertices in the representative K_{1,3} may get the colors 2, 3, 5. Giving color 4 to the 3-degree vertex of the representative K_{1,3} we get a proper P_3-avoiding five-coloring of G. See Figure 2.

The case \( \chi_{P_3}(P_k) \leq 3 \). If G has a vertex of degree at least three, then \( P_3(G) \) has a clique of size at least four, and by Observation 4 \( \chi_{P_3}(G) \geq 4 \). If all vertices have degree at most two, then the components of G are paths and cycles. The components can be colored independently of each other in that case, so G has a P_3-avoiding 3-coloring if and only if all components have. For all \( k \in \mathbb{N} \), \( \chi_{P_3}(P_k) \leq 3 \), we just repeat the pattern 1, 2, 3, 1, 2, 3 . . . starting from one of the ends. The same can be (and must be) done for C_k by specifying a starting vertex. However, it is successful only if \( k \equiv 0 \mod 3 \). \( \square \)

Proof of Lemma 5. As in the proof of Lemma 4 we use a reduction from the language \( L_{3,2} \), the two-coloring of 3-uniform hypergraphs. Having an instance \( T \in L_{3,2} \) with vertex set \( x_1, . . . , x_n \), \( n \geq 4 \), the reduction to a P_4-avoiding 3-coloring of a graph G_T goes as follows. To each vertex \( x_i \) of T we create a pair of vertices \( x_i, x'_i \) and have the edge \( (x_i, x'_i) \). An additional special vertex z is adjacent to each \( x_i \) and to each \( x'_i \).

For each hyperedge \( e_\ell = \{x_p, x_q, x_r\} \) in T, we define a gadget as follows. Take three disjoint copies of P_3, \( a_i, b_i, c_i \) for \( i = 1, 2, 3 \), and vertices \( w_1, w_2 \), and draw the edges \( (c_1, w_1), (c_2, w_1), (c_2, w_2) \) and \( (c_3, w_2) \). Finally we set the gadget by drawing the edges \( (a_1, x_p), (a_2, x_q) \) and \( (a_3, x_r) \).
We claim that a $G$ has a $P_4$-avoiding 3-coloring if and only if $T$ has a $P_4$-avoiding 2-coloring. We may assume vertex $z$ is colored by 3, so $x_i$ is colored 1 or 2, both in the coloring of $G$ and $T$.

If the vertices of an edge $e_\ell = \{x_p, x_q, x_r\}$ all receive the same color, say 2, then in the gadget associated to $e_\ell$ the vertices $a_1, a_2, a_3$ must receive the color 1. (Indeed, if say $a_1$ would be colored by 3, then take an $x_i \not\in \{x_p, x_q, x_r\}$. Either $x_i$ or $x'_i$ has the color 2, say it is $x_i$. But $a_1, x_p, z, x_i$ is a 2-colored $P_4$.) The vertices $b_1, b_2$ and $b_3$ must get color 3, since if, say $b_1$, is of color 2, then $b_1, a_1, x_p, x'_p$ would induce a 2-colored $P_4$. If $c_2$ has color 1, then both $w_1$ and $w_2$ have color 2, since otherwise $w_1$ or $w_2$, $c_2, b_2, a_2$ would be a 2-colored $P_4$. But in that case, the color of $c_3$ could be only 1, inducing 2-colored $P_4$ on the vertices $c_3, w_2, c_2, w_1$.

If $c_2$ has color 2, and at least one of $c_1$ or $c_3$ has color 1, assume $c_1$, then $w_1$ must be colored 3. But then $w_2, c_1, b_1, a_1$ would be a 2-colored $P_4$. Finally, if all $c_1, c_2, c_3$ has color 2, then $w_1$ and $w_2$ must have different colors in order to avoid the 2-colored $P_4$ on $c_1, w_2, c_2, w_2$. But if, say $w_1$ has color 3, then we see a 2-colored $P_4$ on the vertices $c_1, w_1, c_2, b_2$.

For the other direction, assume that $e_\ell = \{x_p, x_q, x_r\}$ received two colors in the hypergraph coloring. Without loss of generality, we may assume two vertices are colored 1, and one with 2. The vertex colored by 2 is either on one of the side, $x_p, x_r$ or the in the middle, $x_q$. Let us say $x_p$ has color 2 and $x_q, x_r$ received color 1. Then the coloring extends to the gadget of $e_\ell$ by coloring $a_1, c_2, c_3$ by 1, $c_1, a_2, a_3, w_2$ by 2, and $b_1, b_2, b_3, w_1$ by 3. If $x_q$ has color 2 and $x_p, x_r$ have color 1, then the extension is $c_1, a_2, c_3$ is of color 1, $a_1, c_2, a_3$ is of color 2, and $b_1, b_2, b_3, w_1, w_2$ is of color 3. Notice
that in both cases all \( b \) types vertices received color 3, which “insulates” the gadgets from each other, so the defined 3-coloring is a \( P_4 \)-avoiding one.

\[ \square \]

3. Proof of Theorem 9

**Proof of the case** \( H = 2K_2 \). To see if \( \chi_{2K_2}(G) \leq 3 \) for a given graph \( G \), first we check if \( G \) contains \( 4K_2 \) as an induced subgraph. This requires no more than \( O(n^4) \) time. If \( G \) does contain a \( 4K_2 \), then \( \chi_{2K_2}(G) \geq 4 \), since between two color classes there can be only one of those four independent edges. Assume \( G \) does not contain \( 4K_4 \), and recall a result of Farber, Hujter and Tuza [3]:

**Theorem 14** (Farber-Hujter-Tuza [3]). *If the graph \( G \) does not contain \( (t + 1)K_2 \) as an induced subgraph, then the number of maximal independent sets in \( G \) is at most \( \binom{n}{t} \).*

The following ideas are well-known and perhaps motivated Theorem 14. The set \( \mathcal{M} \) of all maximal independent sets can be found by, for example, a DFS tree algorithm, and can be listed in no more than \( O(n^2|\mathcal{M}|) \) time. The decision problem of whether \( \chi(G) \leq k \) can be solved by checking if there is \( k \)-set from \( \mathcal{M} \) covering the vertex set of \( G \). This still can be done in \( O(\binom{|\mathcal{M}|}{k}) \) time.

Applying Theorem 14 to \( G \), \( |\mathcal{M}| \leq \binom{n/3}{3} < n^3/162 \), so for a possible 3-coloring we have to check a configuration of size no larger than \( O(n^3) \). A configuration consists of three maximal independent sets \( X_1, X_2 \) and \( X_3 \). First, \( \cup_i X_i \) should contain all vertices of \( G \). If this holds, it readily gives a 3-coloring, however it is not necessarily \( 2K_2 \)-avoiding. Indeed we are looking for \( Y_i \subset X_i \) for \( i = 1, 2, 3 \) such that \( \cup_i Y_i \) contains all vertices of \( G \), \( Y_i \cap Y_j = \emptyset \) if \( i \neq j \), and the partition \( \{Y_1, Y_2, Y_3\} \) is \( 2K_2 \)-avoiding.

We can assume that \( \cap_i X_i = \emptyset \), if not, these vertices are isolated, and can be assigned to any \( Y_i \) in the end. Then we start with the sets \( Y_1 := X_1 \setminus (X_2 \cup X_3) \), \( Y_2 := X_2 \setminus (X_1 \cup X_3) \) and \( Y_3 := X_3 \setminus (X_1 \cup X_2) \) and try to put the leftover vertices into those. The triple \( \{Y_1, Y_2, Y_3\} \) should be \( 2K_2 \)-avoiding, otherwise we discard the configuration. Then we have to decide, for example, if a vertex \( x \in X_1 \cap X_2 \) should be put in \( Y_1 \) or \( Y_2 \). If either placement would give a \( 2K_2 \) with the set \( Y_3 \), we discard the configuration; if only one, we place it to the other; if none, we decide about it later.

At the end of this process we have disjoint sets \( Y_1, Y_2, Y_3 \) that are \( 2K_2 \)-avoiding, \( Y_i \subset X_i \), and the vertices of \( R_{1,2} := (X_1 \cap X_2) \setminus (Y_1 \cup Y_2) \) can be placed both \( Y_1 \) or \( Y_2 \) (and same for \( R_{1,3} \) and \( R_{2,3} \)). Let us construct a conflict graph on \( R_{1,2} \) and for other indices do similarly. For \( x, y \in R_{1,2} \) there is an edge \((x, y) \in E(R_{1,2})\) if \( x \) and \( y \) cannot be placed to \( Y_1 \). (That is, they induce a \( 2K_2 \) to \( Y_3 \). It means \( x \) and \( y \) could not be placed in \( Y_2 \) either.) It is easy to see that if all those conflict graphs \( R_{i,j} \), \( i \neq j \) are bipartite, then all vertices can be placed and we are ready. Otherwise the configuration is to be discarded and we have to move to the next one. If none of the configurations can be formed to be a \( 2K_2 \)-avoiding 3-coloring, then \( \chi_{2K_2} > 3. \) \( \square \)
4. Proof of Theorem 11

**Proof of Proposition 10**

Let $G$ have a coloring with part sizes $s_1, s_2, \ldots, s_\chi$ and $s_1$ the largest. First, further partition each color class arbitrarily into subparts of size at most $k_1 - 1$. The number of parts is

$$\sum_{i=1}^{\chi} \left\lceil \frac{s_i}{k_1 - 1} \right\rceil \leq \sum_{i=1}^{\chi} \left( \frac{s_i}{k_1 - 1} + \frac{k_1 - 2}{k_1 - 1} \right) = \frac{n}{k_1 - 1} + \frac{k_1 - 2}{k_1 - 1} \chi,$$

which is an upper bound that holds regardless of the value of $k_2$.

Second, if $k_2 \geq 3$, partition each color class except the largest arbitrarily into subparts of size at most $k_2 - 1$. The number of parts is

$$1 + \sum_{i=2}^{\chi} \left\lceil \frac{s_i}{k_2 - 1} \right\rceil \leq 1 + \sum_{i=2}^{\chi} \left( \frac{s_i}{k_2 - 1} + \frac{k_2 - 2}{k_2 - 1} \right) = 1 + \frac{n - s_1}{k_2 - 1} + \frac{k_2 - 2}{k_2 - 1} (\chi - 1) \leq 1 + \frac{n - n/\chi}{k_2 - 1} + \frac{k_2 - 2}{k_2 - 1} (\chi - 1).$$

Third, if $k_2 = 2$, color $G$ by giving the largest independent set one color and every other vertex an individual color. The number of parts is $n - \alpha + 1$. Trivially, each of these partitions is an $H$-free coloring. All three combined bounds give the result in the proposition.

□

**Proof of Theorem 11.**

To obtain the upper bound, in the case of $k_1 \geq 3$, we use Proposition 10 together with the result from Bollobás [2] that, whp $\chi (G(n, p)) = (1 + o(1)) \frac{n}{2 \log_d n}$. Hence,

$$\chi_H (G) \leq \frac{n}{k_1 - 1} + O \left( \frac{n}{\log_d n} \right).$$

In the case of $k_1 = 2$, the upper bound comes from Proposition 10 together with the result from Bollobás and Erdős [3] that, whp $\alpha (G(n, p)) = 2 \log_d n - 2 \log_d \log n + O(1)$. Hence,

$$\chi_H (G) \leq n - 2 \log_d n + O \left( \log_d \log n \right).$$

Now we proceed to the lower bound. An $(\ell; k)$-complex is a family of $\ell$ disjoint independent sets, each of size $k$, $A_1, \ldots, A_\ell$ such that each pair $(A_i, A_j)$, $1 \leq i < j \leq \ell$ induces a graph that has no induced copy of $H$. The key to the proof is to show that for certain values of $k$ and $\ell = \ell(n)$, the probability that a $(\ell; k)$-complex exists goes to zero.
If no \((\ell; k)\)-complex exists, then whenever there is a coloring with color classes of size \(n_1, \ldots, n_t \geq k\) it is the case that \(\sum_{i=1}^t \left\lfloor \frac{n_i}{k} \right\rfloor < \ell\). Thus,

\[
\frac{1}{k} \sum_{i=1}^t n_i - \frac{k - 1}{k} t \leq \sum_{i=1}^t \left\lfloor \frac{n_i}{k} \right\rfloor < \ell
\]

while the leftover vertices are in color classes of size at most \(k - 1\). So, if there are \(t\) color classes of size at least \(k\), then

\[
\chi_H(G) \geq t + \frac{n - k\ell - (k - 1)t}{k - 1} = \frac{n}{k} - \frac{k}{k - 1} \ell.
\]  

(1)

For the graph \(H\), let \(Q = Q(H, p)\) be the probability that a \(k_1 \times k_1\) random bipartite graph has no induced copy of \(H\). Taking the product over all \(\binom{\ell}{2}\) pairs \((A_i, A_j)\) and multiplying by the probability that each \(G[A_i]\) induces an independent set, we obtain:

\[
\mathbb{P}[\exists \text{ an } (\ell; k_1)\text{-complex}] = \frac{(n)_{k_1\ell}}{\ell! (k_1!)^\ell} Q^{\binom{\ell}{2}}(1 - p)^{\ell \binom{k_1}{2}}
\]

\[
< \left[ \left( \sqrt{\frac{e}{\ell}} \right) n Q^{(\ell - 1)/(2 k_1)} (1 - p)^{(k_1 - 1)/2} \right]^{k_1 \ell},
\]

which is obtained from the inequalities \((n)_{k_1\ell} \leq n^{k_1\ell}, \ell! \geq (\ell/e)^\ell,\) and \(k_1! \geq 1\).

Let \(C' = C'(H, p) = \frac{2k_1}{\log(1/Q)}\). For \(n\) sufficiently large, if \(\ell > C' \log n\), then the probability in (2) goes to zero. By (1), it is the case that \textbf{whp}

\[
\chi_H(G) \geq \frac{n}{k_1 - 1} - C' \frac{k_1}{k_1 - 1} \log n.
\]

Thus the general lower bound is satisfied.

In the special case where \(H = 2K_2\), we observe that \(Q(2K_2, p) = 1 - 2p^2(1 - p)^2\).

\[
\mathbb{P}[\exists \text{ an } (\ell; 2)\text{-complex}] = \frac{(n)_{2\ell}}{\ell! 2^\ell} Q^{\binom{\ell}{2}}(1 - p)^\ell
\]

\[
< \left[ \left( \sqrt{\frac{e}{2\ell}} \right) n Q^{(\ell - 1)/4} (1 - p)^{1/2} \right]^{2\ell}.
\]

If \(\ell > \frac{4 \log n}{\log(1/Q)} - \frac{4 \log \log n}{\log(1/Q)} + \log \log \log n\) and \(n\) is sufficiently large, then \textbf{whp} no \((\ell; 2)\)-complex exists. By (1), it is the case that \textbf{whp}

\[
\chi_{2K_2}(G) \geq n - (1 - o(1)) \frac{8 \log n}{\log(1/Q)};
\]

where \(Q = 1 - 2p^2(1 - p)^2\).
Similar results can be obtained as long as $\min\{p, 1 - p\} = \omega\left(\frac{\log n}{n}\right)$ but express our results in the case where $p$ is a fixed constant.

5. Examples for Observation 12

An easy consequence of Observation 12 is as follows:

**Corollary 15.**

$$\chi_{2K_2}(P_n) \geq \sqrt{2 \left\lceil \frac{n - 1}{3} \right\rceil + \frac{1}{4} + \frac{1}{2}}.$$

A more refined argument gives the value of $\chi_{2K_2}(P_n)$ as follows:

**Corollary 16.** If $k$ is the least integer that satisfies

$$\left\lceil \frac{k + 1}{2} \right\rceil (k - 2) \geq \left\lceil \frac{n - 1}{3} \right\rceil,$$

then $\chi_{2K_2}(P_n) = k$.

**Proof.** Let $\ell = \left\lceil \frac{n - 1}{3} \right\rceil$. In particular, this means $n - 1 \leq 3\ell \leq n + 1$.

Choose $k$ to be the least integer so that

$$\left\lceil \frac{k}{2} \right\rceil \geq \ell + 1, \quad \text{if } k \text{ is odd;}$$

$$\left\lceil \frac{k}{2} \right\rceil - \frac{k}{2} + 1 \geq \ell + 1, \quad \text{if } k \text{ is even}.$$  \hspace{1cm} (3)

This value is chosen because the longest Eulerian trail in $K_k$ has $\binom{k}{2}$ edges if $k$ is odd and has $\binom{k}{2} - \frac{k}{2} + 1$ edges if $k$ is even. The latter case occurs when a matching of size $k/2 - 1$ is removed from $K_k$.

Let $a_1, a_2, \ldots, a_\ell, a_{\ell + 1}$ be an Eulerian trail in $K_k$. Enumerate the vertices of $P_n$ as $1, 2, \ldots, n$. Let the coloring of the vertices of $P_n$ be $f : [n] \to \{a_1, a_2, \ldots, a_\ell, a_{\ell + 1}\}$, defined as follows:

$$f(1) = f(3) = a_1,$$

$$f(3i - 4) = f(3i - 2) = f(3i) = a_i, \quad \text{for } i = 2, \ldots, \ell - 1;$$

$$f(3\ell - 4) = f(3\ell - 2) = a_\ell,$$

$$f(3\ell) = a_\ell, \quad \text{if } 3\ell \leq n;$$

$$f(3\ell - 1) = a_{\ell + 1};$$

$$f(3\ell + 1) = a_{\ell + 1}, \quad \text{if } 3\ell + 1 \leq n.$$  

Note that $3\ell + 2 \geq n + 1$ by our choice of $\ell$, so there are no other vertices to color.

For all $j \in \{1, \ldots, \ell - 1\}$ each pair of color classes $a_ja_{j + 1}$ induces a $P_4$ plus some isolated vertices. The pair $a_\ell a_{\ell + 1}$ also induces a path plus isolated vertices and the path is either $P_2$, $P_3$, or $P_4$, depending on the remainder of $n$ modulo 3. Therefore, this coloring is a $2K_2$-free coloring.
To see that equality holds, consider a $2K_2$-free $k$-coloring of $P_n$. Every pair of color classes either induces an empty graph or a graph whose only nontrivial component is $P_2$, $P_3$, or $P_4$. Thus, the edges of $P_n$ can be partitioned according to which unique pair of colors induce a particular subpath. Furthermore, subpaths that share a vertex must share a color. Thus, we can construct an auxiliary graph $\Gamma$ on $\{1, \ldots, k\}$ where $ij$ is an edge if and only if the pair of colors $\{i, j\}$ induces a path on at least two vertices. Because consecutive small paths must share a vertex and hence a color, the edges of $\Gamma$ form a trail on $K_k$. Since each edge of $\Gamma$ corresponds to at most 3 edges of $P_n$, the number of edges in $P_n$ is at most three times the length of a longest trail in $K_k$. That is,

$$n - 1 \leq 3 \cdot \left\{ \begin{array}{ll}
\binom{k}{2}, & \text{if } k \text{ is odd;} \\
\binom{k}{2} - \frac{k}{2} + 1, & \text{if } k \text{ is even.}
\end{array} \right.$$ 

Finally, we return to (3) and observe that the condition on $k$ is equivalent to

$$\left\lfloor \frac{k+1}{2} \right\rfloor (k - 2) \geq \ell.$$

The statement then follows. See Table 1 for small values of $\chi_{2K_2}(P_n)$.

**Corollary 17.**

$$\chi_{2K_2}\left(\frac{n}{2} \cdot K_2\right) = \left\lfloor \sqrt{n + \frac{3}{2} + \frac{1}{2}} \right\rfloor.$$

**Proof.** If $k = \chi_{2K_2}\left(\frac{n}{2} \cdot K_2\right)$, then $\binom{k}{2} \geq \frac{n}{2}$ by Observation 12 which establishes a lower bound of $\left\lfloor \sqrt{n + \frac{3}{4} + \frac{1}{2}} \right\rfloor$. This is also sufficient in that the vertices of each edge of $\frac{n}{2} \cdot K_2$ can be independently assigned to distinct endvertices of $K_k$ where $k$ is the value given.

Bounds on some other graphs are given as follows:

**Corollary 18.** Let $n$ be odd and let $T$ be the tree formed when each edge of $K_{1,(n-1)/2}$ is subdivided (by a vertex) exactly once.

$$\chi_{2K_2}(T) = \left\lfloor \sqrt{n - \frac{3}{4} + \frac{1}{2}} \right\rfloor.$$

The proof for Corollary 18 is nearly identical to that of Corollary 17 so it is left to the reader.
Corollary 19. Let $Q_d$ be the $d$-dimensional hypercube. Then
\[
\chi_{2K_2}(Q_2) = 2,
\]
\[
\chi_{2K_2}(Q_3) = 4,
\]
and if $d \geq 4$ then
\[
\chi_{2K_2}(Q_d) \geq \sqrt{\frac{d}{2d-1} \frac{2^d}{2^d} + \frac{1}{2}}
\]
\[
= \sqrt{n \frac{1}{2} \frac{1}{1-(2\log n)} + \frac{1}{2}}.
\]

Proof. The case of $Q_2$ is trivial. It takes some small work to show that for $Q_3$ there is no $Q_3$-avoiding 3-coloring and there is a $Q_3$-avoiding 4-coloring. In the case where $d \geq 4$, we use Observation 12, where $k = \chi_{2K_2}(Q_d)$,
\[
(2d-1) \binom{k}{2} \geq d2^{d-1},
\]
because the only graphs that can occur between pairs of color classes are double-stars (at most $2d-1 \geq 7$ edges) and $K_{2,2}$ (4 edges).

\square

Definition 20. Let $p$ be a prime power and let $G(p)$ be the bipartite graph on $n = 2(p^2 + p + 1)$ vertices defined by the projective plane of order $p + 1$. That is, there are $p^2 + p + 1$ points and $p^2 + p + 1$ lines and a point is adjacent to a line if and only the point is in the line in the projective plane. This graph is $(p+1)$-regular with no $K_{2,2}$.

Corollary 21. If $G(p)$ is the graph in Definition 20 then,
\[
\chi_{2K_2}(G(p)) \geq \sqrt{\frac{2(p^2 + p + 1)(p + 1)}{2p + 1} + \frac{1}{2}}
\]
\[
\geq \sqrt{\frac{n}{2} + \frac{\sqrt{n}}{4} + \frac{1}{2}}.
\]

Proof. With $k = \chi_{2K_2}(G)$, we use the inequality
\[
(2d+1) \binom{k}{2} \geq (p^2 + p + 1)(p + 1).
\]

\square
6. Questions

The proof method of Theorem 9 suggests the following problem:

Question 1. Is it polynomial time decidable if \( \chi_{2K_2}(G) \leq k \) for any fixed \( k \in \mathbb{N} \)?

The structure of \( P_4 \)-free graphs is very nice, which allows easy computation of chromatic number, clique number etc. In fact \( P_4 \)-free bipartite graphs, called difference graphs [5], are well-studied. It is quite surprising that the function \( \chi_{P_4} \) is NP-hard.

Question 2. Is it true that determining whether \( \chi_{P_4}(G) \leq 3 \) is NP-hard?

Question 3. What is \( \chi_{2K_2}(Q_d) \)?

Question 4. What is \( \max\{\chi_{2K_2}(T) : v(T) = n, T \text{ is a tree}\} \)?

Question 5. Is it true that \( \chi_{2K_2}(G) = n - (1 + o(1))2\log_2 n \) whp if \( G \sim G(n, 1/2) \)?

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