ORTHOGONAL ITERATIONS ON STRUCTURED PENCILS*

R. BEVILACQUA†, G.M. DEL CORSO‡, AND L. GEMIGNANI§

Abstract. We present a class of fast subspace tracking algorithms based on orthogonal iterations for structured matrices/pencils that can be represented as small rank perturbations of unitary matrices. The algorithms rely upon an updated data sparse factorization—named LFR factorization—using orthogonal Hessenberg matrices. These new subspace trackers reach a complexity of only \(O(nk^2)\) operations per time update, where \(n\) and \(k\) are the size of the matrix and of the small rank perturbation, respectively.

Key words. Subspace tracking, Orthogonal iteration, Eigenvalues, Eigenvectors, Unitary matrices, Low rank correction

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1. Introduction. Subspace tracking is an important tool in modern adaptive systems. The goal is the recursive estimation of the \(s\) largest or smallest eigenvalues and the associated eigenvectors of a possibly time-varying matrix/pencil. In this paper we are concerned with the design of fast subspace trackers for a certain class of structured matrices (pencils) that can be represented as small rank perturbations of unitary matrices. The paramount example is (block) companion matrices/pencils. Applications arise in several different frameworks. Subspace tracking is often required in signal processing, especially in multidimensional harmonic retrieval and system identification algorithms because the characteristics of the signal can be retrieved from the roots of some associated matrix polynomial [22, 33]. The computation of the roots of a matrix polynomial also plays an important role in the stability analysis of time-varying dynamical systems, which amounts to establishing whether a linear system affected by some time-varying parameters is asymptotically stable for all the admissible values of the parameters [20]. More generally, subspace tracking is relevant for model approximation and model reduction of dynamic systems [16, 19].

Linearization and discretization are basic techniques used for the search of a reduced or approximated model. As result, we usually determine a matrix polynomial which capture the dominant features of the original model. A motivating application of this approach can be pursued for solving nonlinear eigenvalue problems (NEP) of the form \(T(z)v = 0, \ v \neq 0\), where \(T : \Omega \to \mathbb{C}^{k \times k}\) is a holomorphic matrix-valued function and \(\Omega \subseteq \mathbb{C}\) is a connected and open set. The pair \((\lambda, v), \ v \neq 0\), is an eigenpair of \(T\) if it satisfies \(T(\lambda)v = 0\), i.e., \(\det(T(\lambda)) = 0\) and \(v \in \text{Ker}(T(\lambda))\). Nonlinear eigenvalue problems arise in many applications [8, 23]. The most studied case is the polynomial eigenvalue problem (PEP) that can be tackled by finding suitable linearizations [29] to convert PEP into an equivalent generalized eigenproblem. Linearization methods using block companion forms [24] allow the design of fast and stable methods [5, 9, 12] which exploit the unitary plus low rank structure [5]. Various methods have been also proposed in the literature to solve a NEP directly, for example Newton’s method [28] and contour integrals [13, 21, 36] techniques. For a general matrix-valued function \(T(z)\) the associated eigenvalue problem might have infinitely many eigenvalues, and, hence, the usual scenario is to focus on computing a few eigenvalues located in a

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†Dipartimento di Informatica, Università di Pisa, Pisa, Italy, roberto.bevilacqua@unipi.it
‡Dipartimento di Informatica, Università di Pisa, Pisa, Italy, gianna.delcorso@unipi.it
§Dipartimento di Informatica, Università di Pisa, Pisa, Italy, luca.gemignani@unipi.it
certain subset $\Delta \subset \Omega$. A possible approach consists first of approximating $T(z)$ with a matrix polynomial $P_\ell(z)$ inside $\Delta$, and then of computing the eigenvalues of $P_\ell(z)$ to provide numerical approximations of the eigenvalues of $T(z)$ in $\Delta$. Since in general we are interested only in a few eigenvalues of $T(z)$, it is convenient to approximate only the eigenvalues of interest rather than approximate the all spectrum of $P_\ell(z)$.

Subspace trackers based on orthogonal iterations can be numerically accurate and backward stable. The method of orthogonal iteration goes back to Bauer (see [32] and the reference given therein). If $s$ is the dimension of the subspace we want to approximate, a fast $O(ns)$ tracker for scalar companion matrices based on orthogonal iteration first appeared in [35]. The algorithm is basically a game of orthonormal Givens plane rotations moved from one side to the other side of orthogonal factors. More recently these algorithms have been termed core-chasing ones [4]. In this paper we extend the game to more general matrices $A \in \mathbb{C}^{n \times n}$ which are unitary plus some low rank-$k$ correction term. The development follows by exploiting the properties of a suitable LFR factorization [11, 12] of some bordered extension $\tilde{A}$ of $A$, that is, $\tilde{A} = LFR$, where $L$ ($R$) is a unitary $k$-lower ($k$-upper) Hessenberg matrix and $F = U + EZ$ is a unitary plus rank-$k$ matrix where $U$ is a block diagonal unitary matrix of the form $\begin{bmatrix} I_k & \tilde{U} \end{bmatrix}$ and $E = [I_k, 0]^T$. The unitary matrix $\tilde{U}$ can be expressed as product of $\ell < n - k$ unitary Hessenberg matrices. It is shown that the shape of $U$ determines the shape of $\tilde{A}$. In particular $\tilde{A}$ and, a fortiori, $A$ is upper triangular if and only if $\tilde{U}$ is upper triangular and hence diagonal. Based on this property it makes possible to design an implementation of the inverse orthogonal iteration scheme applied to $A$ using only $O(n \max\{s, \ell\}k)$ ops per iteration. The resulting algorithm is maximally fast w.r.t. the size of the correction term and it is backward stable. Moreover, it can be easily generalized to deal with both the orthogonal iteration and the inverse orthogonal iteration method for structured pencils $(A, B)$ where $A$ and/or $B$ are perturbed unitary matrices.

The paper is organized as follows. In Section 2 we recall the theoretical background concerning the orthogonal iteration methods and the properties of modified unitary matrices. Section 3 presents the derivation of our fast adaptations of the orthogonal iteration methods for modified unitary matrices. In Section 4 we show the results of numerical experiments that lend support to the theoretical findings. Finally, Section 5 summarizes conclusions and future work.

2. Preliminaries.

In this section we recall some preliminary results concerning the formulation of both direct and inverse orthogonal iteration schemes for matrix pencils as well as the structural properties and data-sparse representations of modified unitary matrices.

2.1. The Method of Orthogonal Iteration for Matrix Pencils.

The method of orthogonal iteration (sometimes called subspace iteration or simultaneous iteration) can be easily generalized for matrix pencils. Let $A - \lambda B$, $A, B \in \mathbb{C}^{n \times n}$, be a regular matrix pencil with $A$ or $B$ invertible. The orthogonal iteration method can be applied for approximating the largest or smallest magnitude eigenvalues of the matrix pencil by working on the matrices $B^{-1}A$ or $A^{-1}B$.

If $B$ is nonsingular then a generalization of the orthogonal iteration method to compute the $s$-largest (in magnitude) generalized eigenvalues and corresponding eigen-
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vectors of the matrix pencil $A - \lambda B$ proceeds as follows:

\begin{equation}
\begin{aligned}
AQ_i &= BZ_{i+1} \\
Q_{i+1}R_{i+1} &= Z_{i+1}
\end{aligned}
\end{equation}

where $Q_1 \in \mathbb{C}^{n \times s}$ is a starting orthonormal matrix comprising the initial approximations of the desired eigenvectors. A detailed convergence analysis of this iteration can be found in [2]. It is found that the convergence is properly understood in terms of invariant subspaces. Specifically, under mild assumptions it is proved that the angle between the subspaces generated by the columns of $Q_i$ and the invariant subspace associated with the $s$ largest-magnitude generalized eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s \in \mathbb{C}$, with $|\lambda_1| \geq \ldots \geq |\lambda_s| > |\lambda_{s+1}| \geq \ldots \geq |\lambda_n|$, tends to zero as $O((|\lambda_{n+1}|/|\lambda_s|)^i)$ for $i$ going to infinity. An effective stopping criterion is the following

\begin{equation}
\|E_i\| = \|(I - Q_{i-1}Q_i^*)^{-1}Q_i\| < \tau,
\end{equation}

where $\tau$ is the desired tolerance on the residual. Observe that this quantity measures the distance between the subspaces $S_{i-1} = \text{span}\{Q_{i-1}\}$ and $S_i = \text{span}\{Q_i\}$, in fact $(I - Q_{i-1}Q_i^*)^{-1}Q_i$, can be taken as a measure of the angle between $S_{i-1}$ and $S_i$. Note moreover that

$$E_i^*E_i = Q_i^*(I - Q_{i-1}Q_i^*)^{-1}(I - Q_{i-1}Q_i^*)Q_i = Q_i^*(I - Q_{i-1}Q_i^*)Q_i = I_s - W^*W;$$

where $W = Q_{i-1}^*Q_i$. Then $\|E_i\|^2 = 1 - \sigma_{\text{min}}^2(W)$. At convergence we expect $S_{i-1} \equiv S_i$ then $Q_i = Q_{i-1}U$ for an $s \times s$ unitary matrix $U$, then $\sigma_i(W) = \sigma_i(U) = 1$ for $i = 1, \ldots, s$.

Assume now we are given a pencil $A - \lambda B$ with $A$ invertible and that we would like to compute the $s$ smallest-magnitude generalized eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s \in \mathbb{C}$ with $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_s| < |\lambda_{s+1}| \leq \cdots \leq |\lambda_n|$. Inverse orthogonal iterations can be used to approximate the desired eigenvalues. Starting with a set of $s$ orthogonal vectors stored in matrix $Q_0 \in \mathbb{C}^{n \times s}$, we compute the sequences

\begin{equation}
\begin{aligned}
AZ_i &= BQ_{i-1} \\
Q_iR_i &= Z_i
\end{aligned}
\end{equation}

Direct and inverse orthogonal iterations (2.1),(2.3) can be carried out by solving the associated linear systems, but such an approach is prone to numerical instabilities due to the conditioning of the resulting coefficient matrices. A more accurate way to perform these schemes is using the QR factorization of the matrices involved. In particular for the inverse iteration (2.3) one may proceed at each step as follows:

1. Compute the full QR factorization of $QR_R := BQ_{i-1}$;
2. Compute the full RQ factorization of $RLQ_L := Q_R^*A$;
3. Determine $Q_i$ such that $Z_i = Q_iR_i$ is the solution of $RLQ_LZ_i = R_R$. The set of orthogonal vectors satisfying the linear system is such that $Q_i^* = Q_L(1: s, :)$.

In the next subsection we introduce a suitable factorization of modified unitary matrices which makes possible to realize this QR-based process in an efficient way. Since (2.1) can be implemented similarly by interchanging the role of the matrices $A$ and $B$ in the sequel we refer to orthogonal iteration as the scheme (2.3).

2.2. Fast compressed representations of modified unitary matrices.
In this section we introduce a suitable compressed factorization of unitary plus rank-$k$ matrices which can be exploited for the design of fast orthogonal iterations according to the QR-based process described above. See [11, 12] for additional theoretical results on this factorization.

We denote by $\mathcal{U}_k$ the set of unitary-plus-rank-$k$ matrices, that is, $A \in \mathcal{U}_k$ if and only if there exists a unitary matrix $V$ and two skinny matrices $X, Y \in \mathbb{C}^{n \times k}$ such that $A = V + XY^*$. A key role is played by generalized Hessenberg factors.

**Definition 2.1.** A matrix $R \in \mathbb{C}^{m \times m}$ is called $k$-upper Hessenberg if $r_{ij} = 0$ when $i > j + k$. Similarly, $L$ is called $k$-lower Hessenberg if $l_{ij} = 0$ when $j > i + k$. In addition, when $R$ is $k$-upper Hessenberg ($L$ is $k$-lower Hessenberg) and the outermost entries are non-zero, that is, $r_{j+k,j} \neq 0$ ($l_{j,j+k} \neq 0$), $1 \leq j \leq m-k$, then the matrix is called proper. A matrix which is simultaneously $k$-lower and $k$-upper Hessenberg is called $k$-banded.

Note that for $k=1$ a Hessenberg matrix is proper if and only if it is unreduced. Also, a $k$-upper Hessenberg matrix $R \in \mathbb{C}^{m \times m}$ is proper if and only if $\det(R(k+1: m, 1:m-k)) \neq 0$. Similarly a $k$-lower Hessenberg matrix $L$ is proper if and only if $\det(L(1:m-k, k+1:m)) \neq 0$. To make the presentation easier when possible, we use the letter $R$ to denote unitary generalized upper Hessenberg matrices, and the letter $L$ for unitary generalized lower Hessenberg matrices.

Note that $k$-lower (upper) Hessenberg matrices can be obtained as the product of $k$ matrices with the lower (upper) Hessenberg structure, and that unitary block Hessenberg matrices with blocks of size $k$ are (non-proper) $k$-Hessenberg matrices.

In the following we will work with Givens rotations acting on two consecutive rows and columns. In particular we will denote by $G_i = I_{i-1} \oplus G_i \oplus I_{n-i-1}$ the $n \times n$ unitary matrix where $G_i$ is a $2 \times 2$ complex Givens rotation of the form $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ such that $|c|^2 + s^2 = 1$, with $s \in \mathbb{R}, s \geq 0$. The subscript index $i$ indicates the active part of the matrix $G_i$. In the case $G_i = I_2$ we say that $G_i$ is a trivial rotation.

**Definition 2.2.** Given a unitary matrix $U \in \mathbb{C}^{n \times n}$, we say that $U$ has a data-sparse representation if can be expressed as the product of $O(n)$ Givens matrices of the form $G_i$ described before, possibly multiplied by a phase matrix.

Note that the definition 2.2 includes unitary (generalized) Hessenberg defined in 2.1, CMV-matrices [14], and other zig-zag pattern [37], as well as the product of a constant number of these structures.

Next lemma shows how the product between data-sparse unitary terms can be factorized swapping the role of the two factors.

**Lemma 2.3.** Let $R \in \mathbb{C}^{n \times n}$ be a unitary $k$-upper Hessenberg matrix and let $U$ a unitary matrix. Then there exist two unitary matrices $V$ and $S$ such that $RU = VS$ where $S$ is $k$-upper Hessenberg and $V = \begin{bmatrix} I_k & \bar{V} \end{bmatrix}$. Similarly, let $L$ be a unitary $k$-lower Hessenberg matrix and let $U$ be a unitary matrix. Then there exist two unitary matrices $V$ and $M$ such that $LU = VM$ where $M$ is $k$-lower Hessenberg and $V = \begin{bmatrix} \bar{V} & I_k \end{bmatrix}$.

**Proof.** Let us partition $R$ and $U$ as follows

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$
where \(R_{21}\) is an upper triangular matrix of size \(n - k\), and \(U_{11}\) is square of size \(n - k\). Multiplying \(R\) and \(U\) and imposing the conditions on the blocks of the product \(VS\) we get \(S_{11} = R_{11}U_{11} + R_{12}U_{21}, S_{12} = R_{11}U_{22}\). Moreover, since \(S\) should be a \(k\)-upper Hessenberg matrix, we have that \(S_{12}\) should be triangular. Hence \(V\) and \(S_{21}\) can be computed as the \(Q\) and \(R\) factor of the QR factorization of \(R_{21}U_{11} + R_{22}U_{21}\). Finally we set \(S_{22} = V^*(R_{21}U_{12} + R_{22}U_{22})\). Using the same technique we prove that there exist \(V\) and \(M\) such that \(LU = VM\).

**Definition 2.4.** The (lower) staircase of a matrix \(A = (a_{i,j}) \in \mathbb{C}^{n \times n}\) is the sequence \(m_j(A), 1 \leq j \leq n\), defined as follows

\[
m_0(A) = 0, \quad m_j(A) = \max\{m_{j-1}(A), \max_{i < j} \{i : a_{i,j} \neq 0\}\}\]

The sequence \(m_j(A)\) allows to represent the zero pattern of a matrix, in particular to identify zero sub-blocks in the matrix, in fact for each \(1 \leq j \leq n\), it holds \(A(m_j(A)+1 : n, 1 : j) = 0\). We note that proper \(k\)-upper Hessenberg matrices have \(m_j(A) = j + k\) for \(j = 1, \ldots, n - k\), and \(m_j(A) = n\) for \(j = n - k + 1, \ldots, n\).

**Lemma 2.5.** Let \(A \in \mathbb{C}^{n \times n}\) be a matrix with staircase described by the sequence \(\{m_j(A)\}, 1 \leq j \leq n\) and let \(T \in \mathbb{C}^{n \times n}\) be a non-singular upper triangular matrix, we have \(m_j(TA) = m_j(AT) = m_j(A)\) for \(1 \leq j \leq n\).

**Proof.** Let \(B = TA\). We have \(b_{ij} = \sum_{s=0}^{\infty} t_{is} a_{sj}\). Because of the staircase profile of \(A\) we have \(a_{sj} = 0\), for \(s > m_j(A)\), hence \(b_{ij} = 0\) for \(i > m_j(A)\), implying that \(m_j(B) \leq m_j(A)\). To prove the equality of the staircase profile of \(B\) and \(A\) consider the entry \(b_{m_j(A),j} = t_{m_j(A),m_j(A)} a_{m_j(A),j}\). If \(a_{m_j(A),j} \neq 0\) we conclude that \(m_j(B) = m_j(A)\), however it may happen that \(a_{m_j(A),j} = 0\), but from the definition of staircase profile we know that there exists an index \(s, s < j\) such that \(a_{m_j(A),s} \neq 0\), and \(m_s(A) = m_j(A)\). Hence \(b_{m_j(A),s} = r_{m_j(A),m_j(A)} a_{m_j(A),s} \neq 0\), implying that \(m_j(B) = m_s(A) = m_j(A)\). The proof that \(m_j(AT) = m_j(A)\) can be carried on with a similar technique.

Any unitary matrix of size \(n\) can be factorized as the product of at most \(n - 1\) unitary upper Hessenberg matrices\(^1\), that is \(U = R_{n-1}R_{n-2} \ldots R_1 D\), where each \(R_i = \prod_{j=i}^{n-1} G_j\) and \(D\) is a phase matrix. To describe the representation and the algorithm we use a pictorial representation already introduced in several papers (compare with [4] and the references given therein). Specifically, the action of a Givens rotation acting on two consecutive rows of the matrix is depicted as \(\uparrow\downarrow\). Then a chain of ascending two-pointed arrows as below

\[
\begin{array}{c}
\uparrow \uparrow \\
\downarrow \downarrow \\
\uparrow \uparrow
\end{array}
\]

represents a unitary upper Hessenberg matrix (in the case of size 8). Some of the rotations may be identities (trivial rotations), and we might omit them in the picture.

\(^1\)The argument still holds if we take lower unitary Hessenberg in place of the upper Hessenberg matrices.
For example, in the above definition of $H_i$ we only have non-trivial rotations $G_j$ for $j \geq i$, while the representation of $H_3$ is

$$
\begin{bmatrix}
1 & & & & & & \\
\times & \times & \times & \times & \times & & \\
\times & \times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \times & \\
\end{bmatrix} = G_3 G_4 \cdots G_7
$$

Givens transformations can also interact with each other by means of the fusion or the turnover operations (see [38], pp.112-115). The fusion operation will be depicted as $\Rightarrow$ and consists of the concatenation of two Givens transformations acting on the same rows. The result is a Givens rotation multiplied possibly by a $2 \times 2$ phase matrix. The turnover operation allows to rearrange the order of some Givens transformations (see [38]).

Graphically we will depict this rearrangement of Givens transformations as follows:

```
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\downarrow
\end{array}
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\downarrow
\end{array}
\end{array}
```

or

```
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\downarrow
\end{array}
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\downarrow
\end{array}
\end{array}
```

Note that if the Givens transformations involved in turnover operations are all non-trivial also the resulting three new matrices are non-trivial (see [6]).

In this paper we are interested in the computation of a few eigenvalues of matrices belonging to $U_k$ by means of the orthogonal iteration schemes outlined in Subsection 2.1. We will represent these matrices in the so-called LFR format, a factorization introduced in [11, 12].

**Definition 2.6.** We say that a matrix $A \in \mathbb{C}^{n \times n}$, $A \in U_k$ is represented in the LFR format if $(L, F, R)$ are matrices such that:

1. $A = LFR$;
2. $L \in \mathbb{C}^{n \times n}$ is a unitary $k$-lower Hessenberg matrix;
3. $R \in \mathbb{C}^{n \times n}$ is a unitary $k$-upper Hessenberg matrix;
4. $F = U + EZ^* \in \mathbb{C}^{n \times n}$ is a unitary plus rank-$k$ matrix, where $U$ is a block diagonal unitary matrix of the form $[I_k \hat{U}]$, with $\hat{U}$ unitary, $E = [I_k, 0]^T$ and $Z \in \mathbb{C}^{n \times k}$.

Any matrix in $U_k$ can be brought in the LFR format as follows. Let $A \in U_k$, such that $A = V + XY^*$, then $L$ is a $k$-lower unitary Hessenberg such that $L^*X = \begin{bmatrix} T_k \\
0
\end{bmatrix}$, where $T_k$ is upper triangular. Then $A = L(L^*V + T_k Y^*).$ Using Lemma 2.3 we can rewrite $L^*V = UR$, where $R$ is unitary $k$-upper Hessenberg and $U = \begin{bmatrix} I_k \\
U
\end{bmatrix}$ with
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\( \hat{U} \) unitary. Bringing \( R \) on the right we get our factorization, where \( F = U + E Z^* \) and \( Z = RY T^*_k \).

The LFR format of modified unitary matrices is the key tool to developing fast and accurate adaptations of the orthogonal iteration schemes. This will be the subject of the next section.

3. Fast Adaptations of the Orthogonal Iterations.

As underlined in Subsection 2.1, to compute the next orthogonal vectors approximating a basis of the invariant subspace we need to compute the QR decomposition of \( BQ_i \) and the RQ decomposition of \( Q_k^*A \).

3.1. RQ and QR factorization of unitary plus low rank matrices.

To recognize the triangular factors from the LFR decomposition of \( A \) and \( B \) it is useful to embed the matrices of the pencil into larger matrices obtained edging the matrices with \( k \) additional rows and columns. Next theorem explains how we can get such a larger matrices still maintaining the unitary plus rank-\( k \) structure.

**Theorem 3.1.** Let \( A \in \mathbb{C}^{n \times n}, A \in U_k \), then it is always possible to construct a matrix of size \( m = n + k \), \( \hat{A} \in U_k \) of size \( m = n + k \) such that

\[
\hat{A} = \begin{bmatrix} A & B \\ 0_{kn} & 0_{kk} \end{bmatrix} \text{ for a suitable } B.
\]

The unitary part of \( \hat{A} \) can be described with additional \( nk \) Givens rotations respect to the representation of the unitary part of \( A \).

**Proof.** Let \( A = V + XY^* \), with \( V \) unitary. We assume that \( Y \in \mathbb{C}^{n \times n} \) has orthogonal columns otherwise we compute the economy size QR factorization of \( Y \) and we set \( Y = Q \) and \( X = XR^* \). Set \( B = VY \), and consider the matrices

\[
\hat{V} = \begin{bmatrix} V - VYY^* & B \\ Y^* & 0_k \end{bmatrix}, \quad \hat{X} = \begin{bmatrix} X + B \\ -I_k \end{bmatrix}, \quad \hat{Y} = \begin{bmatrix} Y \\ 0_k \end{bmatrix}.
\]

We can prove that \( \hat{V} \) is unitary by direct substitution. The last \( k \) rows of \( \hat{A} = \hat{V} + \hat{X} \hat{Y}^* \) are null.

Matrix \( V \) can be factorized as product of unitary factors which are related to the original players of \( A \), namely \( V, X \) and \( Y \). In particular

\[
\hat{V} = \begin{bmatrix} V \\ I_k \end{bmatrix} S \begin{bmatrix} -I_k \\ I_n \end{bmatrix} S^*,
\]

where \( S \) is a \( k \)-lower Hessenberg matrix such that

\[
S^* \begin{bmatrix} Y \\ -I_k \end{bmatrix} = \begin{bmatrix} \sqrt{2}I_k \\ 0 \end{bmatrix}.
\]

Such a \( S \) always exists and is proper (see Lemma 3 in [12]).

Note that the LFR format of \( \hat{A} \) is such that \( L \) is proper, since \( \hat{X} \) has the last \( k \) rows equal to \( -I_k \) (see [12] Lemma 3).

**Theorem 3.2.** Let \( L, R \in \mathbb{C}^{m \times m}, m = n + k \), be two unitary matrices, where \( L \) is a proper unitary \( k \)-lower Hessenberg matrix and \( R \) is a proper unitary \( k \)-upper Hessenberg matrix. Let \( U \) be a block diagonal unitary matrix of the form

\[
U = \begin{bmatrix} I_k \\ U \end{bmatrix},
\]

\[
\hat{A} = \begin{bmatrix} A & B \\ 0_{kn} & 0_{kk} \end{bmatrix} \text{ for a suitable } B.
\]
with $\tilde{U}$ $n \times n$ unitary. Let $F$ be the unitary plus rank--$k$ matrix defined as $F = U + EZ^*$ with $Z \in \mathbb{C}^{m \times k}$. Suppose that the matrix $\hat{A} = LFR$ satisfies the block structure in (3.1). Then $A = A(1 : n, 1 : n)$ is nonsingular and has the same staircase profile as $\hat{U}$.

**Proof.** Since $L$ is unitary, we have $L^* \hat{A} = FR$. Let partition $L$ and $R$ as follows

$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$ with $L_{12}$ a $n \times n$ lower triangular matrix, and similarly partition $R$ in such a way $R_{21}$ is $n \times n$ upper triangular. Then we get

$L_{12}^* A = \tilde{U} R_{21}$.

For Lemma 2.5 we know that $L_{12}^* A$ has the same staircase profile as $A$, and $\tilde{U} R_{21}$ has the same staircase profile as $\hat{U}$.

Next lemma helps us recognizing triangular matrices in the LFR format.

**Lemma 3.3.** If $\hat{A} = L(I + E Z^*)R$ satisfies the block structure in (3.1), then $\hat{A}$ is upper triangular.

**Proof.** We have $L_{21}^* A = R_{21}$. Because $L$ is proper, the triangular block $L_{12}^*$ is nonsingular and $A = (L_{12}^*)^{-1} R_{21}$. Hence $A$ is upper triangular because is the product of upper triangular factors. $\hat{A}$ is upper triangular as well because is obtained padding with zeros (3.1).

We now give an algorithmic interpretation of Lemma 2.3. A pictorial interpretation of the lemma is given in Figure 3.1, where we omit the phase factors that are possibly present in the general case.

Starting from Figure 3.1 we can describe an algorithm for the “swap” of two unitary terms. In fact, we can obtain the new Givens rotations in the factors $V$ and $S$ simply applying repeatedly fusion and turnover operations as described by the algorithm in Figure 3.2. We formalize the algorithm as if the Givens involved in the swap were all non-trivial. The algorithm has a cost $O(nk)$ only when $U$ admits a data-sparse representation. Matrix $U$ can be generally factorized as the product of at most $\ell \leq n - 1$ unitary upper or lower Hessenberg matrices. In this case the overall cost is $O(nk\ell)$. In procedure **SwapRU** we choose to factorize $U$ as the product of lower
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Procedure SwapRU

Input: \(R = R^{(k)} R^{(k-1)} \ldots R^{(1)}\), with \(R^{(i)} = G^{(i)}_1 G^{(i)}_2 \ldots G^{(i)}_{n-1}\), 
\[U = L^{(1)} \ldots L^{(\ell)},\] 
where \(\ell \leq n - 1\) and \(L^{(i)} = \Gamma^{(i)}_{n-1} \ldots \Gamma^{(i)}_i\)

for \(i = 1, \ldots, k\), set \(S^{(i)} = R^{(i)}\)
for \(i = 1: \ell\)
    for \(j = n - 1: -1: n - k\) \{ These Givens are fused \}
        if \(\Gamma^{(i)}_j \neq I_2\) \{ only non trivial rotations are removed \}
            after possible turnovers, apply a fusion with the last Givens in \(S^{(n-j)}\)
        endif
    endfor
for \(j = n - k - 1: -1: i\)
    if \(\Gamma^{(i)}_j \neq I_2\) \{ only non trivial rotations are removed \}
        \(k\) turnover between Givens in \(S\) acting on rows \(j: n-1\) and \(\Gamma^{(i)}_j\).
        The result is a Givens rotation \(\tilde{\Gamma}^{(i)}_{j+k}\).
    endif
endfor
endfor

Output:
\(S = S^{(k)} S^{(k-1)} \ldots S^{(1)}\), \(V = \tilde{L}^{(1)} \ldots \tilde{L}^{(\ell)}\),
\(\tilde{L}^{(i)} = \tilde{\Gamma}^{(i)}_{n-1} \ldots \tilde{\Gamma}^{(i)}_{i+k}\).

Figure 3.2. Procedure to swap an upper and a lower generalized Hessenberg matrices.

Hessenberg factors, but we can obtain a similar algorithm expressing \(U\) in terms of upper Hessenberg factors, and consider the worst case \(\ell = n - 1\). At step \(i - 1\) we have removed the first \(i - 1\) chains of ascending Givens rotations from \(U\), and the situation is the following

\[RU = V^{(1)} V^{(2)} \ldots V^{(i-1)} \Sigma_i L^{(i)} \ldots L^{(n-1)},\]

where \(\Sigma_i\) is an intermediate \(k\)-upper Hessenberg which is transformed by the turnover and fusion operations. In particular \(\Sigma_1 = R\) and \(\Sigma_{n-1} = S\).

At the step \(i\) we pass the rotations in \(L^{(i)}\), from right to left. The bottom \(k\) Givens of each \(L^{(i)}\) are fused with the Givens in the last rows of \(\Sigma_i\), so that the shape of \(\tilde{V}\) reproduces the shape of the Givens rotations in the blue top triangle of \(U\).

Similarly to the procedure SwapRU we can design a procedure SwapLU to factorize the product between a unitary \(k\)-lower Hessenberg matrix \(L\) and a unitary matrix \(U\) as the product of a unitary factor \(V = \begin{bmatrix} \tilde{V} & I_k \end{bmatrix}\) and a unitary \(k\)-lower Hessenberg matrix \(M\). Note that from these two swapping procedures we can obtain also new factorizations when multiplying on the left a \(k\)-lower or \(k\)-upper Hessenberg unitary matrix, that is \(U^* R^* = (RU)^* = (VS)^* = S^* V^*,\) and \(U^* L^* = (LU)^* = (VM)^* = M^* V^*.\) We will denote the analog procedures as SwapUL and SwapUR keeping in mind that the matrices involved are unitary, and that we denote generalized lower Hessenberg matrices using the letter \(L\) and generalized upper Hessenberg matrices using the letter \(R\).
From the LFR format of $\hat{A}$ we can easily get the QR and RQ factorization of $\hat{A}$. This procedure requires $O(nk\ell)$ flops where $\ell$ is the number of Hessenberg unitary factors in $U$. Let $\hat{A} = L(U + EZ^*)R$ as in Definition 2.6. Swapping $L$ and $U$ according with Lemma 2.3, i.e. $LU = Q\hat{L}$, we have that $\hat{L}(I + EZ^*)R$ is upper triangular for Lemma 3.3. Since $Q$ is unitary, we have a QR decomposition of $A$. Similarly swapping $U$ and $R$ in such a way $UR = \hat{R}Q$ we get an RQ decomposition of $A$ where the triangular factor is $L(I + EZ^*)\hat{R}$, with $Z = UZ$, and the unitary factor is $Q$. The proof is straightforward since, again from Lemma 3.3, we have that $L(I + E, Z^*)\hat{R}$ is upper triangular.

3.2. The algorithm.

In this section we describe the orthogonal iterations on a pencil $(A, B)$ where $A$ and $B$ are unitary-plus-low-rank matrices. For the sake of readability we assume $A, B \in U_k$, even if situation where the low-rank part of $A$ and $B$ do not have the same rank is possible: in that case we assume that $k$ is the maximum between the values of the rank part in $A$ and $B$.

We will assume that $A$ and $B$ have been embedded in larger pencil $(\hat{A}, \hat{B})$ as described in Theorem 3.1. Since $\det(\hat{A} - \lambda\hat{B}) = 0$ for all $\lambda$, the pencil is singular and the $k$ new eigenvalues introduced with the embedding are indeterminate: Matlab returns “NaN” as eigenvalues in these cases. However, thanks to the block triangular structure of $\hat{A}$ and $\hat{B}$ the other eigenvalues coincide with those of the original pencil $(A, B)$. To guarantee that the orthogonal iterations on $(\hat{A}, \hat{B})$ do not converge to an invariant subspace corresponding with an indeterminate eigenvalue it is sufficient to start with a bunch of orthogonal vectors of the kind $\hat{Q}_0 = \begin{bmatrix} Q_0 \\ 0_k \end{bmatrix}$. This guarantees that at each step we still have that $Q_i$ has the last $k$ rows equal to zero and therefore the iterative process is basically applied to the smaller pencil.

We assume that $A$ and $B$ are in LFR format and we describe how we can carry out an orthogonal iteration using only the turnover and fusion operations. The key ingredient for the algorithm is the SwapUR procedure and its variants as described in Subsection 3.1. In fact, working with the pencil and with the LFR factorization, the orthogonal iterations can be reformulated as follows. Let

$$\hat{A} = L_A(U_A + E, Z_A^*)R_A, \quad \hat{B} = L_B(U_B + E, Z_B^*)R_B$$

be the LFR decomposition of $\hat{A}$ and $\hat{B}$. The $s$ starting orthogonal vectors in $\hat{Q}_0 \in \mathbb{C}^{N \times s}, N = n + k$, as well as all the intermediate orthogonal vectors $Q_i$, can be represented as the product of $s$ sequences of ascending Givens rotations. In fact, the columns of $\hat{Q}_0$ can be always be completed to an orthogonal basis $\{q_0, q_1, \ldots, q_N\}$ such that $[q_0, q_1, \ldots, q_N]$ is a $k$-lower Hessenberg matrix (see [12]).

As we see in the algorithm in Figure 3.3, the procedure boils down to the description of the two procedures $\text{MoveSequencesLeft}$ and $\text{MoveSequencesRight}$ that should be described in terms of the LFR representation.

| MoveSequencesLeft |
|-------------------|
| **Input:** $\hat{B} = Q_B \hat{L}_B(I + E, \hat{Z}_B^*)R_B, Q_0$ |
| $[Q_0, \mathcal{R}_B] = \text{SwapRU}(R_B, Q_0)$ |
| $[\mathcal{P}, \mathcal{L}_B] = \text{SwapLU}(\hat{L}_B, Q_0)$ |
| $[Q_L, \mathcal{Q}_B] = \text{SwapUL}(Q_B, \mathcal{P})$ |
| **Output:** $Q_L$ |
Orthogonal iterations on structured pencils

Orthogonal Iterations
Input: LFR representation of $\hat{A}$ and $\hat{B}$, $\hat{Q}_0$, tolerance $\tau$, maxiter

Represent $\hat{Q}_0$ with $s$ sequences of ascending Givens rotations
Using $\text{SwapUR}(U_A, R_A)$ compute the RQ decomposition of $\hat{A} = (L_A(I + E\hat{Z}^*_A)\hat{R})Q_A$
Using $\text{SwapLU}(L_B, U_B)$ compute the QR decomposition of $\hat{B} = Q_B(\hat{B}(I + E\hat{Z}^*_B)R_B)$
while $i < \text{maxiter}$ and $\|E_i\| > \tau$

$Q_L := \text{MoveSequencesLeft}(B, \hat{Q}_i)$
$Q_R := \text{MoveSequencesRight}(Q_L^*, A)$
$Q_{i+1} := Q_R$
$E_i := (I - Q_{i-1}Q_{i-1}^*)Q_i$
i := i + 1

endwhile
As = $Q_i^*AQ_i$, Bs = $Q_i^*BQ_i$ (As, Bs) is a pencil of dimension $s \times s.$

Output: $\text{eig}(As, Bs)$

Figure 3.3. Inverse orthogonal iterations described in terms of the LFR representation of the pencil.

MoveSequencesRight
Input: $Q_L$, $\hat{A} = (L_A(I + E\hat{Z}^*_A)\hat{R})Q_A$,
$[L_A, Q_L^*] = \text{SwapRU}(Q_L^*, L_A)$
$[R_A, P_R^*] = \text{SwapUR}(R_A, \hat{R})$
$[Q_A, Q_R^*] = \text{SwapRU}(P_R^*, Q_A)$
Output: $Q_R$.

3.3. Measure of Backward stability.

Suppose that our orthogonal iterations method has reached a numerically invariant subspace spanned by the $s$ orthogonal columns of matrix $Q$. To measure the backward stability we analyze the quantity

$$\text{back}_s = \sqrt{2}\sigma_{s+1}(|AQ, BQ|) / \|\|A, B\|\|.$$ 

This quantity is an upper bound to the usual backward stability measure. Indeed, we seek $\Delta_A$ and $\Delta_B$ such that

$$(A + \Delta_A)Q = (B + \Delta_B)Q\Lambda,$$

for a suitable invertible $\Lambda$. Since $A$ is invertible we may also suppose that $BQ$ is of maximum rank $s$. We say that the algorithm is backward stable if

$$\|\|\Delta_A, \Delta_B\|\| / \|\|A, B\|\| \approx O(\epsilon),$$

where $\epsilon$ is the machine precision.

Let us consider the SVD decomposition of $[AQ, BQ]$. We expect that this matrix has $\sigma_{s+1}$ small since in floating point arithmetic $AQ = BQ\hat{\Lambda}$, with $\Lambda$ invertible. We
have
\[ [AQ, BQ] = [U_1, U_2] \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix} \begin{bmatrix} V_{11}^* \\ V_{12} \\ V_{21} \\ V_{22} \end{bmatrix}, \quad \Sigma_1 \in \mathbb{R}^{s \times s}. \]
We find \( AQ = U_1 \Sigma_1 V_{11}^* + U_2 \Sigma_2 V_{12}^* \) and \( BQ = U_1 \Sigma_1 V_{21}^* + U_2 \Sigma_2 V_{22}^* \). Moreover the \( s \times s \) matrix \( Q^* B^* BQ = V_{21} \Sigma_1^* V_{21}^* + V_{22} \Sigma_2^* V_{22}^* \) is invertible, then we have
\[ V_{21} \Sigma_1^* V_{21} = Q^* B^* BQ(I - (Q^* (B^* B)Q)^{-1} V_{22} \Sigma_2^* V_{22}^*). \]
Consider now the matrix \( I - (Q^* (B^* B)Q)^{-1} V_{22} \Sigma_2^* V_{22}^* \), which is invertible if
\[ \|(Q^* (B^* B)Q)^{-1} V_{22} \Sigma_2^* V_{22}^*\| < 1. \]
This shows that under this assumption \( V_{21} \) is invertible as well.

Consider now the equality \( \Delta_A Q - \Delta_B QA = -AQ + BQA \). Rewriting in terms of the SVD factors we get
\[ \Delta_A Q - \Delta_B QA = U_1 \Sigma_1 (-V_{11}^* + V_{21}^*) + U_2 \Sigma_2 (-V_{12}^* + V_{22}^*). \]
Since \( V_{21} \) is invertible, among the infinite \( s \times s \) matrices \( \Lambda \) we can chose \( \Lambda = V_{21}^{-1} V_{11}^* \), so that
\[ \Delta_A Q - \Delta_B QA = -U_2 \Sigma_2 V_{11}^* + U_2 \Sigma_2 V_{22}^*. \]
We can then set \( \Delta_A = -U_2 \Sigma_2 V_{11}^* Q^* \) and \( \Delta_B = U_2 \Sigma_2 V_{22}^* Q^* \), and it holds \( \|\Delta_A\| \leq \sigma_{s+1}([AQ, BQ]) \) and \( \|\Delta_B\| \leq \sigma_{s+1}([AQ, BQ]) \). Finally, we conclude that
\[ \|\Delta_A, \Delta_B\|^2 = \rho(\Delta_A^* \Delta_A + \Delta_B^* \Delta_B) = \|\Delta_A^* \Delta_A + \Delta_B^* \Delta_B\| \leq (\|\Delta_A\|^2 + \|\Delta_B\|^2) \leq 2(\sigma_{s+1}([AQ, BQ])^2). \]

4. Numerical results.

We perform several tests using nonlinear matrix functions \( T(\lambda) \in \mathbb{C}^{k \times k} \). For matrix polynomials we consider the companion linearization while for non-polynomial matrix functions, we first approximate the matrix function with polynomials of different degrees which are then linearized in pencils \((A, B)\) with \( A, B \in U_k \).

In all cases, when the pencil is built, our method performs the inverse orthogonal iterations as defined in the Algorithm Orthogonal Iterations in Figure 3.3 until an invariant subspace is revealed. Then the corresponding eigenvalues \( \tilde{\lambda}_i \) are computed applying the Matlab \texttt{eig} function to the \( s \times s \) pencil \((A_s, B_s)\) determined as the restriction of \( A \) and \( B \) to the subspace spanned by the columns of \( Q_s \), i.e. the generalized Rayleigh quotients of \( A \) and \( B \).

As a measure of the forward error we consider
\[ \text{err}_T(i) = \frac{\|T(\tilde{\lambda}_i)v\|_2}{\|T(\tilde{\lambda}_i)\|_2 \|v\|_2}, \]
where \( v \) is the \( k \)-th right singular vector of \( T(\tilde{\lambda}_i) \). In practice we compute
\[ \text{err}_T(i) = \frac{\sigma_k}{\sigma_1}, \]
where \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \) are the singular values of \( T(\tilde{\lambda}_i) \). As a measure of convergence of the orthogonal iterations we consider instead
\[ \text{averr}_T = \frac{1}{s} \sum_{i=1}^{s} |\tilde{\lambda}_i - \mu_i|, \]
where $\mu_i$ are the “exact” eigenvalues of the pencil $(A, B)$ obtained with Matlab $\text{eig}$. Note that in the general non-linear case averr$_p$ refers to the average error with respect to the zeros of the approximating polynomial while err$_T(i)$ is the total error with respect to the zeros of the nonlinear function $T(\lambda)$. Hence, small values of averr$_p$ guarantee the good behavior of the inverse orthogonal iterations on the pencil, while err$_T(i)$ measures also the quality of the approximation of the nonlinear function with the matrix polynomial.

We tested our method on some matrix polynomials of degree greater than 2 from the NLEVP collection [8] using the companion linearization. Table 4.1 summarizes the results. For the polynomial plasma_drift we repeated the experiment twice by setting the stopping criteria in (2.2) to a tolerance of $1.0e-4$, and the maximum number of iterations to 450, respectively. This is a very challenging problem for any eigensolver [25] with several eigenvalues of high multiplicity and/or clustered around zero. In this case the estimated backward error is not so significant since our measure of backward stability assumes that we are at convergence and we have identified an invariant subspace. In [25] the authors proposed a variation of the Jacobi-Davidson method for computing several eigenpairs of the polynomial eigenvalue problem. For the plasma_drift problem they fixed a residual threshold of $1.0e-2$ and within 200 iterations they were able to compute the approximations of the 19 eigenvalues closer to the origin. The performance of our method for the approximation of the same set of eigenvalues is examined in Table 4.1. Regarding the other tests, by comparison with those reported in [5] where a structured version of the QZ method is employed, we see that for the orr_sommerfeld problem we get a higher backward stability. However, we are estimating only 2 or 4 eigenvalues while the QZ allows to approximate all the spectrum and, moreover, differently from [5] our error analysis assumes an uniform bound for the norm of the perturbation of $A$ and $B$. The accuracy of the computed eigenvalues is in accordance with the conditioning estimates. Our method on the orr_sommerfeld problem has a backward stability similar to that reported in [15] where a balanced version of the Sakurai-Sugiura method with Rayleigh-Ritz projection was presented. For the butterfly problem our method achieves a higher backward stability w.r.t. [15]. The number of iterations in relative_pose_5pt agrees with the separation ratio of the eigenvalues. For the other tests there are remarkable differences in the number of iterations depending on the sensibility of our stopping criterion (2.2) used in Algorithm 

| name              | deg | k  | s  | $|\lambda_k|/|\lambda_{k+1}|$  | averr$_p$ | it  | back$_a$ |
|-------------------|-----|----|----|-----------------|-----------|-----|----------|
| butterfly         | 4   | 64 | 4  | 9.53e-01        | 5.45e-14  | 654 | 5.01e-15 |
| orr_sommerfeld    | 4   | 64 | 2  | 9.95e-01        | 5.75e-06  | 22  | 1.92e-18 |
| orr_sommerfeld    | 4   | 64 | 4  | 9.91e-01        | 8.33e-06  | 28  | 1.92e-18 |
| plasma_drift      | 3   | 128| 19 | 9.98e-01        | 6.60e-02  | 6   | 7.38e-06 |
| plasma_drift      | 4   | 128| 19 | 9.98e-01        | 5.85e-04  | 450 | 7.92e-08 |
| relative_pose_5pt | 3   | 10 | 4  | 3.20e-01        | 1.99e-14  | 33  | 1.34e-15 |

Matrix polynomials of degree 3 and 4 from the NLEVP collection.

Another set of experiments have dealt with root-finding for a nonlinear matrix function. Consider the holomorphic nonlinear matrix-valued function $T : \Omega \rightarrow \mathbb{C}^{k \times k}$, with $\Omega \subset \mathbb{C}$ a connected and open set and let $\lambda_i$ be the $i-th$ exact eigenvalue of $T$, that
is $T(\lambda)v = 0$, with $v$ a corresponding eigenvector. Computing an approximation of $\lambda_i$ can be accomplished as follows. We first approximate the nonlinear function with a matrix polynomial of a given degree $d$ interpolating on the roots of unity or on the Chebyshev points. With these choices of points we have theoretical results [10, 17] about the uniform convergence of the interpolating polynomials to the nonlinear function inside the unit disk. Let $P_d(z)$ be the approximating polynomial of degree $d$, then we may consider suitable linearizations which give us a unitary plus-low-rank pencil $(A, B)$. The eigenvalues of this pencil provide an approximation of the zeros of $T(z)$ inside $\Omega$.

Specifically, we tested our algorithm on the companion linearization, and on the unit diagonal plus-low-rank linearization obtained from the companion linearization applying a block-Fourier transform which diagonalizes the unitary part of $A$ (see [10] for more details). The same structure can also be obtained starting from an “arrowed linearization” similar to the one proposed in [1] where the interpolating polynomial is written in the Lagrange basis. Combining the different linearizations with the different choices of the nodes (roots of unity, roots of unity plus the origin, Chebyshev nodes) the following cases are treated:

- Companion linearization on the roots of unity (denoted in the tables by “CL”)
- Companion linearization on the roots of unity plus the origin (denoted in the tables by “CL0’”)
- Companion linearization on the Chebyshev roots (denoted in the tables by “T’”)
- Diagonal linearization on the roots of unity (denoted in the tables by “DL”)

We note that in the diagonal linearization the diagonal factor contains the interpolation nodes, hence to guarantee that this factor is unitary we can choose as interpolation nodes only the roots of unity but not the other choices of nodes considered in the companion linearization such as the Chebyshev points or the origin.

We tested several non-linear matrix-valued functions found in the literature. Below is a description of these functions.

**Time-delay equation** [13]. The matrix function is $T(z) = z + T_0 + T_1 \exp(6z - 1)$ with

$$T_0 = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}; \quad T_1 = \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}.$$  

This function has three eigenvalues inside the unit circle.

**Model of cancer growth** [7]. The matrix function is $T(z) = z - A_0 - A_1 \exp(-rz)$, where

$$A_0 = \begin{bmatrix} -\mu_1 & 0 & 0 \\ 2b_1 & -\mu_2 & b_Q \\ 0 & \mu_Q & -(b_Q + \mu_G) \end{bmatrix}, \quad A_1 = \exp(-\mu_2 r) \begin{bmatrix} 2b_1 & 0 & b_Q \\ -2b_1 & 0 & -b_Q \\ 0 & 0 & 0 \end{bmatrix}.$$  

The parameters are chosen as suggested in [7] by setting $r = 5; b_1 = 0.13; b_Q = 0.2; \mu_1 = 0.28; \mu_0 = 0.11; \mu_Q = 0.02; \mu_G = 0.0001, \mu_2 = \mu_0 + \mu_Q$. We refer to [7] for the physical meaning of the constants and for the description of the model. This function has three eigenvalues inside the unit circle.

**Neutral functional differential equation** [18]. The function is scalar $t(z) = -1 + 0.5z + z^2 + h z^2 \exp(\tau z)$. The case $h = -0.824650483736655, \tau = 6.74469732735569$ is analyzed in [27] corresponding to a Hopf bifurcation point. This function has three eigenvalues inside the unit circle.

**Spectral abscissa optimization** [30]. The function is $T(z) = zI_3 - A - B \exp(-z\tau)$
with \( \tau = 5 \), \( B = bq^T \) and

\[
A = \begin{bmatrix}
-0.08 & -0.03 & 0.2 \\
0.2 & -0.04 & -0.005 \\
-0.06 & 0.2 & -0.07
\end{bmatrix},
\quad
b = \begin{bmatrix}
-0.1 \\
-0.2 \\
0.1
\end{bmatrix},
\quad
q = \begin{bmatrix}
0.47121273 \\
0.50372106 \\
0.60231834
\end{bmatrix}.
\]

Abscissa optimization techniques favor multiple roots and clustered eigenvalues with potential numerical difficulties. This function has 4 eigenvalues inside the unit circle.

**Hadeler problem** [8]. The matrix function is \( T(z) = (\exp(z) - 1)A_2 + z^2A_1 - \alpha A_0 \) where \( A_0, A_1, A_2 \in \mathbb{R}^{k \times k} \), and \( V = \text{ones}(k,1) \cdot [1 : k] \). \( A_0 = \alpha I_k \), \( A_1 = k \cdot I_k + 1/(V + V') \), \( A_2 = (k + 1 - \max(V, V')) \cdot [(1 : k) \cdot [1 : k]] \). In our experiments we set \( k = 8 \) and \( \alpha = 100 \). Ruhe [31] proved that the problem has \( k \) real and positive eigenvalues, in particular two of them are \( 0 < \lambda < 1 \), and hence lie inside the unit circle.

**Vibrating string** [8,34]. The model refers to a string of unit length clamped at one end, while the other one is free but is loaded with a mass \( m \) attached by an elastic spring of stiffness \( k_p \). Assuming \( m = 1 \) and discretizing the differential equation one gets the non linear eigenvalue problem \( F(z)v = 0 \), where \( F(z) = A - Bz + k_p C \frac{z}{z-k_p} \) is rational, \( A, B, C \in \mathbb{R}^{k \times k} \), \( k_p = 0.01, h = 1/k \),

\[
A = \begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
2 & -1 & 1
\end{bmatrix},
\quad
B = \begin{bmatrix}
4 & 1 & 0 \\
1 & 4 & 1 \\
0 & 1 & 2
\end{bmatrix},
\quad
C = e_k e_k^T.
\]

The function \( F(z) : \Omega \rightarrow \mathbb{C}^{3 \times 3} \) from [3] is defined as follows:

\[
F(z) = \begin{bmatrix}
2e^z + \cos(z) - 14 & (z^2 - 1)\sin(z) + (2e^z + 14)\cos(z) & 2e^z - 14 \\
(z + 3)(e^z - 7) & \sin(z) + (z + 3)(e^z - 7)\cos(z) & (z + 3)(e^z - 7) \\
e^z - 7 & (e^z - 7)\cos(z) & e^z - 7
\end{bmatrix}.
\]

This function has six real known eigenvalues \( \{ \pm \pi, \pm \pi/2, 0, \log(7) \} \). We applied the transformation \( z \rightarrow 4z + 1 \) to bring five of the six eigenvalues inside the unit disk. With this transformation we do not get an approximation of the eigenvalue \( -\pi \) which after the translation is not inside the unit disk.

For all these problems we computed the interpolating polynomials over the roots of unity, the roots of unity plus the origin or the Chebyshev points in the range \([-1,1]\) of different degrees, and we compared the performance of our algorithm on the different linearizations. Linearizations based on interpolating at the roots of unity (plus the origin) perform very similarly with negligible differences in convergence and accuracy. The linearization using the Chebyshev points can suffer of numerical instabilities for large degrees of the interpolating polynomial due to the transformation of the polynomial basis. For the sake of brevity we report here only the best result obtained using the polynomial with lower degree which guarantees the best performance in terms of forward error \( err_T(i) \). Ask the authors for the complete set of results. In Table 4.2 are summarized the results for the nonlinear functions considered. In Table 4.3 are reported the complete results for the Function in [3].

We underline that the algorithm, accordingly with the measure of backward stability in Section 3.3, behaves as a backward stable method in every case. The number of iterations needed to meet the stopping condition which was set to \( 1.0e - 14 \) reflects the rate of convergence of the orthogonal iterations which depends on the ratio
| name          | m(deg) | s | $|\lambda_s|/|\lambda_{s+1}|$ | error_T(1) | error_T(s) | averr_p | it | back_s |
|--------------|--------|---|-----------------|------------|------------|---------|----|--------|
| Time-del     | CL(64) | 3 | 5.25e-01        | 5.74e-16   | 2.91e-14   | 7.56e-14 | 45 | 1.95e-15 |
| Cancer       | CL(32) | 2 | 5.18e-01        | 4.22e-16   | 4.78e-15   | 5.76e-15 | 48 | 1.91e-15 |
| Neutral      | CL(64) | 2 | 4.02e-01        | 6.72e-13   | 9.86e-13   | 1.63e-13 | 33 | 5.33e-16 |
| Spec-abs     | T(32)  | 4 | 1.02e-01        | 1.66e-16   | 1.16e-16   | 2.12e-09 | 24 | 4.34e-16 |
| Hadeler      | DL(32) | 2 | 6.35e-01        | 5.58e-16   | 1.65e-14   | 7.30e-15 | 87 | 3.14e-15 |
| Vib-str.     | CL(32) | 1 | 5.10e-01        | 5.24e-15   | -          | 4.27e-14 | 64 | 4.64e-16 |

Table 4.2

Best results for the non-linear non-polynomial matrix functions.

$|\lambda_s|/|\lambda_{s+1}|$. The values averr_p measure the effectiveness of orthogonal iterations to approximate the eigenvalues of the pencil and of course are affected by the conditioning of the problem. When Chebyshev points are used as interpolation nodes sometimes the pencil obtained is seriously ill conditioned and hence both Matlab eig and our algorithm return inaccurate results. On the contrary, when working with the roots of unity the coefficients of the interpolating polynomial in the monomial basis are computed by means of an FFT which is very stable.

In general, for sufficiently large values of the degree we get a very good approximation of the eigenvalues inside the unit disk. We can compare these results with those reported in [10] and [5] where respectively a QR and QZ method were employed to compute all the eigenvalues of the matrix/pencil. We see that our results are comparable with those obtained in the literature but we need less operations. In fact the algorithms based on QR or QZ need $O(d^2 k^3)$ flops, while here we need $O(d k^2 s \text{it})$ flops. In general the number of iterations does not depend on the size of the problem, but only by on the ratio $|\lambda_s|/|\lambda_{s+1}|$, so the cost of the orthogonal iterations can be asymptotically lower and we do not have any advantage in computing all the eigenvalues since only $s$ of them are reliable because the polynomial is a good approximation of the non-linear function only inside the unit disk. Comparing the results in Table 4.3 with those reported in paper [3] we see that our results are much better, in particular when using the method CL, that is the approximation of the non-linear function with the interpolating polynomial over the roots of unity combined with the companion linearization. In particular using the same degree (d = 64) as in [3] we get a results with 5 more digits of precision respect to the results reported in [3].

5. Conclusions and Future Work.

In this paper we have presented a fast and backward stable subspace tracker for block companion forms using orthogonal iterations. The proposed method exploits the properties of a suitable data-sparse factorization of the matrix involving unitary factors. The method can be extended to more generally perturbed unitary matrices and it can incorporate the acceleration techniques based on the updated computation of Ritz eigenvalues and eigenvectors [2]. The design of fast adaptations using adaptive shifting techniques such as the ones proposed in [26] is an ongoing research project.

REFERENCES

[1] A. AMIRASLANI, R. M. CORLESS, AND P. LANCASTER, Linearization of matrix polynomials expressed in polynomial bases, IMA J. Numer. Anal., 29 (2009), pp. 141–157, https://doi.org/10.1093/imanum/drm051; http://dx.doi.org/10.1093/imanum/drm051.
Orthogonal iterations on structured pencils

| m  | deg | \(|\lambda_s|/|\lambda_{s+1}|\) | err\(_T(1)\) | err\(_T(2)\) | err\(_T(3)\) | aver\(_p\) | it | back\(_5\) |
|----|-----|---------------------------------|--------------|-------------|-------------|-----------|----|--------|
| CL | 16  | 6.77e-01                        | 3.68e-07     | 1.52e-06    | 3.57e-06    | 1.32e-12  | 81 | 3.23e-15 |
| CL0| 16  | 6.77e-01                        | 2.99e-12     | 2.29e-10    | 2.64e-10    | 1.21e-12  | 86 | 3.89e-15 |
| T  | 16  | 7.10e-01                        | 5.25e-08     | 1.89e-07    | 2.66e-06    | 4.14e-12  | 82 | 4.31e-15 |
| DL | 32  | 6.92e-01                        | 5.25e-08     | 3.62e-07    | 5.24e-06    | 1.10e-12  | 86 | 3.87e-15 |
| CL | 32  | 6.92e-01                        | 1.76e-16     | 5.40e-07    | 2.64e-06    | 4.14e-12  | 82 | 4.31e-15 |
| T  | 32  | 6.92e-01                        | 6.87e-16     | 2.85e-10    | 5.85e-14    | 1.94e-12  | 83 | 4.06e-15 |
| DL | 32  | 6.92e-01                        | 9.74e-16     | 1.69e-15    | 2.19e-15    | 1.08e-13  | 101| 2.95e-15 |
| CL | 64  | 6.92e-01                        | 2.59e-17     | 4.26e-17    | 1.14e-16    | 2.65e-11  | 83 | 3.31e-15 |
| CL0| 64  | 6.92e-01                        | 5.40e-15     | 6.25e-15    | 8.61e-15    | 1.56e-13  | 101| 2.45e-15 |

Function \(F(z)\) in (4.1). For the inverse orthogonal iterations we used \(s = 5\), but we show the results only for the first three eigenvalues (corresponding to the value \(\pi/2, \log(7)\) and 0). The other remaining two roots are approximated just as well.

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[2] P. Arbenz, Lecture notes on solving large scale eigenvalue problems, 2016.
[3] J. Asakura, T. Sakurai, H. Tadano, T. Ikegami, and K. Kimura, A numerical method for nonlinear eigenvalue problems using contour integrals, SIAM Letters, 1 (2009), pp. 52–55, https://doi.org/10.14495/jsiaml.1.52.
[4] J. Aurentz, T. Mach, L. Robol, R. Vandebril, and D. S. Watkins, Core-chasing algorithms for the eigenvalue problem, Fundamentals of Algorithms, SIAM, 2018.
[5] J. Aurentz, T. Mach, L. Robol, R. Vandebril, and D. S. Watkins, Fast and backward stable computation of eigenvalues and eigenvectors of matrix polynomials, Math. Comp., 88 (2019), pp. 313–347, https://doi.org/10.1090/mcom/3338, https://doi.org/10.1090/mcom/3338.
[6] J. L. Aurentz, T. Mach, R. Vandebril, and D. S. Watkins, Fast and backward stable computation of roots of polynomials, SIAM Journal Matrix Ana. Appl., 36 (2015), pp. 942–973.
[7] M. V. Barbarossa, C. Kuttler, and J. Zinsl, Delay equations modeling the effects of phase-specific drugs and immunotherapy on proliferating tumor cells, Math. Biosci. Eng., 9 (2012), pp. 241–257, https://doi.org/10.3934/mbe.2012.9.241, https://doi.org/10.3934/mbe.2012.9.241.
[8] T. Betcke, N. J. Higham, V. Mehrmann, C. Schröder, and F. Tisseur, NLEVP: A collection of nonlinear eigenvalue problems, ACM Trans. Math. Softw., 39 (2013), pp. 7:1–7:28, https://doi.org/10.1145/2427023.2427024, http://doi.acm.org/10.1145/2427023.2427024.
[9] R. Bevilacqua, G. M. Del Corso, and L. Gemignani, A CMV-based eigensolver for companion matrices, SIAM J. Matrix Anal. Appl., 36 (2015), pp. 1046–1068, https://doi.org/10.1137/140978065, https://doi.org/10.1137/140978065.
[10] R. Bevilacqua, G. M. Del Corso, and L. Gemignani, A QR based approach for the nonlinear eigenvalue problem, Rendiconti Sem. Mat. Univ. Pol. Torino, 76 (2018), pp. 57–67.
[11] R. Bevilacqua, G. M. Del Corso, and L. Gemignani, Efficient reduction of compressed unitary plus low rank matrices to Hessenberg form, SIAM J. Matrix Anal. Appl., 41 (2020), pp. 984–1003, https://doi.org/10.1137/19M1280363, https://doi.org/10.1137/19M1280363.
[12] R. Bevilacqua, G. M. Del Corso, and L. Gemignani, Fast QR iterations for unitary plus low rank matrices, Numer. Math., 144 (2020), pp. 23–53, https://doi.org/10.1007/s00211-019-01080-4, https://doi.org/10.1007/s00211-019-01080-4.
[16] K. Cherifi and K. Hariche, Model reduction based on matrix polynomials, in 2018 IEEE Workshop on Advanced Robotics and its Social Impacts (ARSO), 2018, pp. 31–31, https://doi.org/10.1109/ARSO.2018.8625833.

[17] C. Effenberger and D. Kressner, Chebyshev interpolation for nonlinear eigenvalue problems, BIT, 52 (2012), pp. 933–951, https://doi.org/10.1007/s10543-012-0381-5, https://doi.org/10.1007/s10543-012-0381-5.

[18] K. Engelborghs, D. Roose, and T. Luzyanna, Bifurcation analysis of periodic solutions of neutral functional-differential equations: a case study, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 8 (1998), pp. 1889–1905, https://doi.org/10.1142/S0218124980001595, https://doi.org/10.1142/S0218124980001595.

[19] R. W. Freund, Model reduction methods based on Krylov subspaces, Acta Numer., 12 (2003), pp. 267–319, https://doi.org/10.1017/S0962492902000120, https://doi.org/10.1017/S0962492902000120.

[20] R. Galindo, Stabilisation of matrix polynomials, International Journal of Control, 88 (2015), pp. 1925–1932, https://doi.org/10.1080/00207179.2015.1012651.

[21] B. Gavin, A. Miedlar, and E. Polizzi, FEAST eigensolver for nonlinear eigenvalue problems, J. Comput. Sci., 27 (2018), pp. 107–117, https://doi.org/10.1016/j.jocs.2018.05.006, https://doi.org/10.1016/j.jocs.2018.05.006.

[22] Y. Gu and R. Ding, Observable state space realizations for multivariable systems, Comput. Math. Appl., 63 (2012), pp. 1389–1399, https://doi.org/10.1016/j.camwa.2012.01.024, https://doi.org/10.1016/j.camwa.2012.01.024.

[23] S. Gütel and F. Tisseur, The nonlinear eigenvalue problem, Acta Numer., 26 (2017), pp. 1–94, https://doi.org/10.1017/S0962492917000034, https://doi.org/10.1017/S0962492917000034.

[24] N. J. Higham, R. C. Li, and F. Tisseur, Backward error of polynomial eigenproblems solved by linearization, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 1218–1241, https://doi.org/10.1137/060663738, https://doi.org/10.1137/060663738.

[25] P. B. Hochstenbach, Computing several eigenvalues of nonlinear eigenvalue problems by selection, Calcolo, 57 (2020).

[26] H. J. Jung, M. C. Kim, and I. W. Lee, An improved subspace iteration method with shifting, Comput. & Structures, 70 (1999), pp. 625–633, https://doi.org/10.1016/S0045-7949(98)00201-6, https://doi.org/10.1016/S0045-7949(98)00201-6.

[27] P. Kravanja and M. Van Barel, Computing the zeros of analytic functions, vol. 1727 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2000, https://doi.org/10.1007/BFb0103927, https://doi.org/10.1007/BFb0103927.

[28] D. Kressner, A block Newton method for nonlinear eigenvalue problems, Numer. Math., 114 (2009), pp. 355–372, https://doi.org/10.1007/s00211-009-0259-x, https://doi.org/10.1007/s00211-009-0259-x.

[29] V. Mehrmann and H. Voss, Nonlinear eigenvalue problems: a challenge for modern eigenvalue methods, GAMM Mitt. Ges. Angew. Math. Mech., 27 (2004), pp. 121–152 (2005), https://doi.org/10.1002/gamm.201490007, https://doi.org/10.1002/gamm.201490007.

[30] W. Michiels, I. Boussaada, and S. I. Niculescu, An explicit formula for the splitting of multiple eigenvalues for nonlinear eigenvalue problems and connections with the linearization for the delay eigenvalue problem, SIAM J. Matrix Anal. Appl., 38 (2017), pp. 599–620, https://doi.org/10.1137/16M107774X, https://doi.org/10.1137/16M107774X.

[31] A. Ruhe, Algorithms for the nonlinear eigenvalue problem, SIAM J. Numer. Anal., 10 (1973), pp. 674–689, https://doi.org/10.1137/0710059, https://doi.org/10.1137/0710059.

[32] H. Rutishauser, Computational aspects of F. L. Bauer’s simultaneous iteration method, Numer. Math., 13 (1969), pp. 4–13, https://doi.org/10.1007/BF02165269, https://doi.org/10.1007/BF02165269.

[33] A. Sinap and W. Van Assche, Orthogonal matrix polynomials and applications, in Proceedings of the Sixth International Congress on Computational and Applied Mathematics (Leuven, 1994), vol. 66, 1996, pp. 27–52, https://doi.org/10.1016/0377-0427(95)00193-X, https://doi.org/10.1016/0377-0427(95)00193-X.

[34] S. I. Solov’ev, Preconditioned iterative methods for a class of nonlinear eigenvalue problems, Linear Algebra and its Applications, 415 (2006), pp. 210 – 229, https://doi.org/10.1016/j.laa.2005.03.034, http://www.sciencedirect.com/science/article/pii/S0024379505001576. Special Issue on Large Scale Linear and Nonlinear Eigenvalue Problems.

[35] P. Ströbchen, The recursive companion matrix root tracker, IEEE Transactions on Signal Processing, 45 (1997).
[36] M. Van Barel and P. Kravanja, *Nonlinear eigenvalue problems and contour integrals*, J. Comput. Appl. Math., 292 (2016), pp. 526–540, https://doi.org/10.1016/j.cam.2015.07.012, https://doi.org/10.1016/j.cam.2015.07.012.

[37] R. Vandebril, *Chasing bulges or rotations? A metamorphosis of the QR-algorithm*, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 217–247, https://doi.org/10.1137/100809167, http://dx.doi.org/10.1137/100809167.

[38] R. Vandebril, M. Van Barel, and N. Mastronardi, *Matrix computations and semiseparable matrices. Vol. II*, Johns Hopkins University Press, Baltimore, MD, 2008. Eigenvalue and singular value methods.