Hamiltonian Formulation of Jackiw–Pi
3-Dimensional Gauge Theories

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Abstract

A 3-dimensional non-abelian gauge theory was proposed by Jackiw and Pi to create mass for the gauge fields. However, the quadratic action obtained by switching off the non-abelian interactions possesses more gauge symmetries than the original one, causing some difficulties in quantization. Jackiw and Pi proposed another action by introducing new fields, whose gauge symmetries are consistent with the quadratic part. It is shown that all of these theories have the same number of physical degrees of freedom in the hamiltonian framework. Hence, as far as the physical states are considered there is no inconsistency. Nevertheless, perturbation expansion is still problematic. To cure this we propose to modify one of the constraints of the non-abelian theory without altering neither its canonical hamiltonian nor the number of physical states.
There are some different approaches to generate mass for the gauge fields of non-abelian gauge theories in 3–dimensions in terms of the gap equation

$$\Pi(p^2) \big|_{p^2=m^2} = m^2,$$

where, $\Pi$ is the transverse vacuum polarization tensor. This equation is studied for different actions up to one loop level leading to some different results\[1\]–\[3\]. However, considering higher loops seems to be essential\[4\],\[5\].

In the approach of Jackiw and Pi\[3\] one deals with actions whose quantization in lagrangian formalism exhibits some uncommon features. Quadratic action possesses more gauge symmetries than the non-abelian one. Thus, perturbative expansion of the latter is not well defined. To cure this in Ref.\[3\] another action is proposed to the cost of introducing new fields. We will show that considered as constrained hamiltonian systems there is no inconsistency between these actions: the number of physical states is the same.

Although all of them possess the same number of physical states, hamiltonian quantization of non-abelian case is still problematic. To overcome this difficulty, we propose to modify one of the original constraints of the non-abelian theory by making use of gauge fixing conditions of quadratic action, without altering its canonical hamiltonian and number of physical states.

Jackiw and Pi proposed the action\[3\]

$$S = \int d^3x \left[ -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{4} G^{a\mu
u} G^{a\mu
u} + \frac{m}{2} \epsilon^{\mu\nu\rho} F^a_{\mu\nu} \phi^a_{\rho} \right],$$

(1)

where the group index $a = 1, \cdots, N$, and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \alpha f^{abc} A^b_\mu A^c_\nu,$$

$$G^{a\mu
u} = (D_\mu \phi_\nu)^a - (D_\nu \phi_\mu)^a.$$  

(2)  

(3)

Covariant derivative is given in terms of the structure constants $f^{abc}$ as

$$D^a_\mu = \delta^a \partial_\mu + \alpha f^{acb} A^c_\mu.$$

(4)  

is invariant under the gauge transformations

$$\delta_1 A^a_\mu = (D_\mu \theta)^a, \quad \delta_1 \phi^a_\mu = f^{abc} \phi^b_\mu \theta^c.  

(5)$$

However, when the coupling is switched off ($\alpha = 0$), (1) yields the quadratic action

$$\mathcal{S}_f \equiv S(\alpha = 0)$$

which is invariant under two different types of abelian gauge transformations

$$\delta_{f1} A_\mu = \partial_\mu \theta, \quad \delta_{f1} \phi_\mu = 0,$$

$$\delta_{f2} A_\mu = 0, \quad \delta_{f2} \phi_\mu = \partial_\mu \xi.$$  

(6)  

(7)
Obviously, (11) is not invariant under the non-abelian generalization of (7):

\[ \delta_2 A_\mu^a = 0, \quad \delta_2 \phi_\mu^a = (D_\mu \xi)^a. \]  

(8)

Because of this, perturbation quantization of the full (non-abelian) theory (11) is not straightforward. When (11) is used in Green functions generating functional or in partition function, there is only need of gauge fixing terms for its gauge symmetries (4). But, the propagators will be calculated in terms of the quadratic action (5), which still possesses the gauge symmetry (7). i.e. gauge fixing of the non-abelian action will not be sufficient to eliminate the redundant fields in (5) which is essential to define finite propagators.

A general quantization procedure of the theories whose gauge symmetries in the quadratic and the full cases are not consistent is not available yet.

Jackiw and Pi proposed to enlarge the space of states by introducing the new fields \( \rho^a \) and to deal with the action

\[ S_g = \int d^3x \left[ -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu} - \frac{1}{4} (G^{a\mu\nu} + f^{abc} F_{b\mu\nu} \rho^c) (G_{a\mu\nu} + f^{abc} F_{b\mu\nu} \rho^c) + \frac{m}{2} \epsilon^{\mu\nu\rho} F_{a\mu\nu} \phi_\rho^a \right], \]

(9)

which is invariant under both of the gauge transformations (4) and (8) supplemented by

\[ \delta_1 \rho^a = f^{abc} \rho^b \theta^c, \quad \delta_2 \rho^a = -\xi^a. \]

(10)

We would like to analyze the above mentioned actions in terms of the hamiltonian methods.

Let us first deal with the quadratic case (5). By using the definition of canonical momenta

\[ \pi_\mu^a \equiv \frac{\delta S}{\delta A^{a\mu}}; \quad P_\mu^a \equiv \frac{\delta S}{\delta \phi^{a\mu}}, \]

(11)

we obtain the primary constraints

\[ \pi_0^a = 0, \quad P_0^a = 0, \]

(12)

and the canonical hamiltonian

\[ H_f = \int d^2x \left[ \frac{1}{2} (\pi_i^a)^2 + \frac{1}{2} (P_i^a)^2 - m \epsilon_{ij} \pi^{ai} \phi^{aj} + \frac{1}{2} m^2 (\phi_i^a)^2 + \frac{1}{2} (\partial_i A_j^a - \partial_j A_i^a)^2 
\right.
\]

\[ + \frac{1}{2} (\partial_i \phi_j^a - \partial_j \phi_i^a)^2 - A_0^a \psi_{f1}^a - \phi_0^a \psi_{f2}^a, \]

(13)

where we use the metric \( \eta_{\mu\nu} = \text{diag}(-1,1,1) \) and the definitions

\[ \psi_{f1}^a \equiv \partial^i \pi_i^a, \]

(14)

\[ \psi_{f2}^a \equiv \partial_i P_i^a + m \epsilon^{ij} \partial_i A_j^a. \]

(15)

Obviously, here the group index \( a \) is for \( N \) copies of \( U(1) \).
Time evolution of classical observables will be given by the extended hamiltonian

\[ H'_f = H_f + \int d^2x \left[ \kappa^a_1 \pi^a_0 + \kappa^a_2 P^a_0 \right]. \]

The primary constraints (12) should be preserved in time\(^1\) on the constraint surface defined by vanishing of the related constraints (denoted by \(\approx\)):

\[ \dot{\pi}^a_0(x) = \{\pi^a_0(x), H'_f\} \approx 0; \quad \dot{P}^a_0(x) = \{P^a_0(x), H'_f\} \approx 0. \]

These lead to the secondary constraints

\[ \psi^a_{f1}(x) = 0; \quad \psi^a_{f2}(x) = 0, \] (16)

which are conserved in time:

\[ \{\psi^a_{f1}(x), H'_f\} \approx 0, \] (17)
\[ \{\psi^a_{f2}(x), H'_f\} \approx 0. \] (18)

Thus, there is no more constraint. Moreover, all of the constraints (12) and (16) give vanishing Poisson brackets among themselves.

In the non-abelian case (1) the canonical hamiltonian is

\[ H = \int d^2x \left[ \frac{1}{2} (\pi^a_i)^2 + \frac{1}{2} (P^a_i)^2 - m_\epsilon_{ij} \pi^{ai} \phi^{aj} + \frac{1}{2} m^2 (\phi^a_i)^2 + \frac{1}{2} F^{aij} F_{aij}^a \right. \]
\[ \left. + \frac{1}{2} G^{aij} G_{aij}^a - A^a_0 \tilde{\psi}^a_1 - \phi^a_0 \tilde{\psi}^a_2 \right], \] (19)

where

\[ \tilde{\psi}^a_1 \equiv (D_i \pi^a_i) + \alpha f^{abc} \phi^b_k P^c_i; \] (20)
\[ \tilde{\psi}^a_2 \equiv (D_i P^a_i) + \frac{m}{2} \epsilon^{ij} F_{ij}^a. \] (21)

Primary constraints are still given by (12). So that, the extended hamiltonian is

\[ H' = H + \int d^2x \left[ \kappa^a_1 \pi^a_0 + \kappa^a_2 P^a_0 \right]. \]

Conservation of the primary constraints (12) in time leads to the secondary constraints

\[ \tilde{\psi}^a_1 = 0; \quad \tilde{\psi}^a_2 = 0. \] (22)

\(^1\text{We deal with the equal time Poisson brackets}

\[ \{O(x), K(y)\} = \int d^2z \left[ \frac{\delta O(x)}{\delta p^l(z)} \frac{\delta K(y)}{\delta q^l(z)} - \frac{\delta O(x)}{\delta q^l(z)} \frac{\delta K(y)}{\delta p^l(z)} \right] \]

where \(O, K\) are some classical observables and \(q_I = (A_\mu, \phi_\mu), \ p^I = (\pi^\mu, P^\mu)\).
Poisson brackets satisfied by the secondary constraints $\tilde{\psi}^a_1$, $\tilde{\psi}^a_2$ are

\[
\{\tilde{\psi}^a_1(x), \tilde{\psi}^b_1(y)\} = -\alpha f^{abc} \tilde{\psi}^c_1(x) \delta(x - y),
\]
\[
\{\tilde{\psi}^a_1(x), \tilde{\psi}^b_2(y)\} = -\alpha f^{abc} \tilde{\psi}^c_2(x) \delta(x - y),
\]
\[
\{\tilde{\psi}^a_2(x), \tilde{\psi}^b_2(y)\} = 0.
\]

By making use of these relations one can show that

\[
\{\tilde{\psi}^a_1(x), H'\} \approx 0,
\]
\[
\{\tilde{\psi}^a_2(x), H'\} \approx -\tilde{\psi}^a_3(x),
\]

where

\[
\tilde{\psi}^a_3 = \alpha f^{abc}[F^b_{ij} G^{cij} - P^b_i (\pi^{ic} - m e^{ij} \phi^c_j)].
\]

Hence, conservation of the secondary constraints yields the new constraints

\[
\tilde{\psi}^a_3 = 0.
\]

Obviously, $\pi^a_0$ and $P^a_0$ give vanishing Poisson brackets with the other constraints $\tilde{\psi}^a_{1,2,3}$. However,

\[
\{\tilde{\psi}^a_1(x), \tilde{\psi}^b_3(y)\} = -\alpha f^{abc} \tilde{\psi}^c_3(x) \delta(x - y),
\]
\[
\{\tilde{\psi}^a_2(x), \tilde{\psi}^b_3(y)\} = \alpha^2 f^{acd} f^{bde} [P^d_i P^e_d + F^d_{ij} F^d_{ij}] \delta(x - y).
\]

Thus, by denoting the canonical hamiltonian (extended hamiltonian evaluated on the constraint surface) by $H_0$, the condition that the constraints $\tilde{\psi}^a_3(x)$ should be conserved in time

\[
\{\tilde{\psi}^a_3(x), H'\} = \{\tilde{\psi}^a_3(x), H_0\} - \int d^2y \phi^b_0(y) \{\tilde{\psi}^a_3(x), \tilde{\psi}^b_3(y)\} \approx 0,
\]

will yield a solution for $\phi^a_0$, as far as we exclude the configurations which make the right hand side of (31) vanishing. Thus, by accepting that the right hand side of (31) is non-vanishing we conclude that there is no more constraint. Although (31) may lead to a non-local $\phi^a_0(x)$, for the functional integrals the relevant hamiltonian is the one evaluated on the constraint surface where the term including $\phi^a_0(x)$ is absent.

For the action $S_\rho$ (31) primary constraints are

\[
\pi^a_0 = 0; ~ P^a_0 = 0; ~ \lambda^a = \frac{\delta S_\rho}{\delta \rho^a(x)} = 0.
\]

Hence, the extended hamiltonian reads

\[
H_g = H' + \int d^2x [\alpha f^{abc} \rho^a (m e^{ij} P^b_i \phi^c_j - P^b_i \pi^{ic} + G^b_{ij} F^{cij})
\]

\[
+ \frac{\alpha^2}{2} f^{acd} f^{bde} \rho^a \rho^b (P^d_i P^e_d + F^d_{ij} F^d_{ij}) + \kappa^a_3 \lambda^a].
\]
Vanishing of Poisson brackets (now including also derivatives with respect to \(\rho\) and \(\lambda\)) of the primary constraints (32) with the extended Hamiltonian \(H_g\) will yield as before
\[
\tilde{\psi}_1^a(x) = 0; \quad \tilde{\psi}_2^a(x) = 0.
\] (34)

Moreover, there are the following secondary constraints
\[
\psi_3^a \equiv \tilde{\psi}_3^a + \alpha f^{abc} f^{cbd'} \rho^b (P_{i} P_{d} + F_{ij} F_{d'ij}) = 0.
\] (35)

Because of the fact that
\[
\{\psi_3^a(x), \lambda^b(y)\} = -\alpha^2 f^{abc} f^{cbd'} (P_{d} P_{d'} + F_{ij} F_{d'ij}) \delta(x - y),
\] (36)

which is assumed to be non-vanishing, constraints are terminated:
\[
\{H_g, \psi_3^a(x)\} \approx 0
\]

will be satisfied by choosing \(\kappa_3^a(x)\) appropriately.

Let us deal with the following linear combination of the constraints
\[
\psi_1^a(x) \equiv \tilde{\psi}_1^a(x) + \alpha f^{abc} \rho^b(x) \lambda^c(x),
\] (37)
\[
\psi_2^a(x) \equiv \tilde{\psi}_2^a(x) + \lambda^a(x).
\] (38)

which satisfy the Poisson bracket relations
\[
\{\psi_1^a(x), \psi_1^b(y)\} = -\alpha f^{abc} \psi_1^c(x) \delta(x - y),
\] (39)
\[
\{\psi_1^a(x), \psi_2^b(y)\} = -\alpha f^{abc} \psi_2^c(x) \delta(x - y),
\] (40)
\[
\{\psi_1^a(x), \psi_2^b(y)\} = -\alpha f^{abc} \psi_2^c(x) \delta(x - y),
\] (41)
\[
\{\psi_1^a(x), \lambda^b(y)\} = -f^{abc} \lambda^c \delta(x - y),
\] (42)
\[
\{\psi_2^a(x), \psi_2^b(y)\} = 0,
\] (43)
\[
\{\psi_2^a(x), \psi_2^b(y)\} = 0.
\] (44)

One can show that the new constraints possess consistent equations of motion:
\[
\{\psi_1^a(x), H_g\} \approx 0,
\] (45)
\[
\{\psi_2^a(x), H_g\} \approx 0,
\] (46)

where, the constraint surface is defined in terms of the new set of constraints.

Let us classify the constraints a l’ a Dirac[6] to find number of physical degrees of freedom (at least in reduced phase space method). These are listed below for the three cases: quadratic given by (5), non-abelian given by (1), enlarged given by (9).
Hence, as far as the physical states are concerned there is no inconsistency between the three cases considered. There is no negative norm state in any of the Hilbert spaces in the quantum case.

Let us discuss the $\alpha = 0$ limit of the non-abelian and the enlarged cases.

i) non-abelian: $H|_{\alpha = 0} = H_f, \tilde{\psi}_1|_{\alpha = 0} = \psi_{f1}, \tilde{\psi}_2|_{\alpha = 0} = \psi_{f2}$ and $\tilde{\psi}_3|_{\alpha = 0} = 0$. So that, the second class constraints $\tilde{\psi}_2$ become first class. In principle, by making appropriate changes in the related hamiltonian, one can consider $\tilde{\psi}_2$ as first class constraints and $\tilde{\psi}_3$ as their gauge fixing (subsidiary) conditions. However, these will cease to be gauge fixing conditions for the quadratic case. In fact, this explains how the inconsistency of the gauge symmetries of the two cases arises in the lagrangian formalism, although the number of physical degrees of freedom is the same.

ii) enlarged: $H_g|_{\alpha = 0} = H_f, \psi_1|_{\alpha = 0} = \psi_{f1}, \psi_2|_{\alpha = 0} = \psi_{f2} + \lambda$ and $\psi_3|_{\alpha = 0} = 0$. Now, $\lambda^a = 0$ are first class. In this case one can adopt the gauge fixing conditions $\chi_2^a = 0$ corresponding to the first class constraints $\tilde{\psi}_2$, yielding gauge fixing conditions for $\psi_{f2}$ in the $\alpha = 0$ limit. However, there are some other problems in the perturbation expansions:

Let us consider the functional integral

$$Z = \int dp[q] \exp \int d^3x(p_Aq_A - \mathcal{H}),$$

where $q_A$ indicate $A^a_{\mu}, \phi^a_\mu$, in the quadratic and non-abelian cases and also $\rho^a$ in the enlarged case and $\mathcal{H}$ is the related hamiltonian density. Let us separate the measure $dp[q]$ as

$$dp[q] = \mu_0 dp_A dq_A,$$

where $\mu_0$ is the part related to the first class constraints $\pi_0, P_0$, and one of $\psi_{f1}, \tilde{\psi}_1, \psi_1$ depending on the action and their subsidiary conditions, which do not cause any difficulty.

For the enlarged case the other part of the measure is

$$\mu_e = \delta(\psi_2)\delta(\chi_2)\delta(\psi_3)\delta(\lambda) \det \left[ \alpha^2 f^{abc} f^{cde} (P^d_i P^e_i + F^d_{ij} F^e_{ij}) \right] \det \{\psi_2, \chi_2\}.$$  

Thus, the $\alpha = 0$ limit is not well defined for the functional integral. By integrating over $\rho$ the measure will possess the term $\det^{1/2} \left[ \alpha^2 f^{abc} f^{cde} (P^d_i P^e_i + F^d_{ij} F^e_{ij}) \right]$ as it was announced in [3].
If one would like to obtain the non-abelian case from the enlarged one, gauge fixing conditions can be chosen \( \chi^a_2 = \rho^a \). After integrating over \( \rho \) and \( \lambda : H_g \to H \) and the related part of the measure will yield

\[
\mu_e = \delta(\tilde{\psi}_2) \delta(\tilde{\psi}_3) \det[\delta^{ab}\delta(0)] \det \left[ \alpha^2 f^{ace} f^{bde} (P^d_i P^d_i + F^d_{ij} F^{d'ij}) \right].
\]

The term \( \det[\delta(0)] \) can be absorbed by the normalization of the partition function.

A way to cure the non-abelian theory would be to consider a combination of the second class constraints as \( \Sigma^a(x) = \kappa^a_{2b}(x) \tilde{\psi}^b_2(x) + \kappa^a_{3b}(x) \tilde{\psi}^b_3(x) \), where \( \kappa^a_{23}(x) \) are defined to satisfy \( \{ \Sigma^a(x), \Sigma^b(y) \} \approx 0 \), where the constraint surface is defined by vanishing of the other constraints and \( \Sigma^a(x) = 0 \). Moreover, we should define a new hamiltonian \( \tilde{H} \) satisfying \( \{ \Sigma^a(x), \tilde{H} \} \approx 0 \) and \( \tilde{H} \approx H \). Once this is achieved one can adopt gauge fixing conditions which do not vanish for \( \alpha = 0 \). However, this may force to introduce some undesired non-local terms.

We propose another way of resolving the problem. We reinterpret the coordinate fields of the primary constraints \( P_0 = \pi_0 = 0 \) as Lagrange multipliers:

\[
A^a_0, \phi^a_0 \to \lambda^a_1, \lambda^a_2,
\]

and do not consider zero components of the fields any more.

In the quadratic case we may introduce the gauge fixing conditions

\[
\chi^a_{f1}(x) = 0, \quad \chi^a_{f2}(x) = 0,
\]

satisfying

\[
\det\{\psi^a_{f1}(x), \chi^b_{f1}(y)\} \neq 0, \quad \det\{\psi^a_{f2}(x), \chi^b_{f2}(y)\} \neq 0.
\]

The appropriate hamiltonian is given by \( H_f \) (13), after the replacement (49), where \( \lambda^a_1, \lambda^a_2 \) are defined such that

\[
\{H_f, \chi^a_{f1}(x)\} \approx 0, \quad \{H_f, \chi^a_{f2}(x)\} \approx 0.
\]

Here the constraint surface is defined in terms of the original constraints and the gauge fixing conditions.

In the non-abelian case we need to introduce gauge fixing conditions for \( \tilde{\psi}^a_1(x) \):

\[
\tilde{\chi}^a_1(x) = 0,
\]

satisfying

\[
\det\{\tilde{\psi}^a_1(x), \tilde{\chi}^b_1(y)\} \neq 0.
\]

Moreover,

\[
\{H', \tilde{\chi}^a_1(x)\} \approx 0,
\]

will lead an equation for the related Lagrange multipliers \( \lambda^a_1(x) \). Obviously, \( \lambda^a_2(x) \) have already been fixed by the condition (31).
Instead of the original non-abelian one, we propose to deal with the constrained Hamiltonian system given by the following set of constraints

\[ \Phi_A \equiv (\tilde{\psi}_1, \tilde{\chi}_1, \tilde{\psi}_2, \tilde{\chi}_2) = 0, \tag{51} \]

where the modified constraints are

\[ \tilde{\chi}_2 \equiv \chi_f + \tilde{\psi}_3. \]

However, the Hamiltonian is still given by (19), with the replacement (49). Because of being second class \( \Phi_A \) satisfy

\[ \det \{ \Phi_A(x), \Phi_B(y) \} \neq 0. \]

So that, the Lagrange multipliers \( \lambda_1^a(x), \lambda_2^a(x) \) are given as solutions of the equations

\[ \{ H, \Phi_A(x) \} \approx 0. \]

Now the limit \( \alpha = 0 \) is well defined and the number of the physical states are unaltered. Obviously, the mass induced by this theory should be calculated to see if it is satisfactory. However, it is out of the scope of this paper.

Arnowitt and Deser studied a theory similar to (1) in 4–dimensions (obviously the last term in the action is absent) exhibiting the same features of gauge transformations. Unfortunately, on the contrary, in the case of Ref. (6) Hamiltonian approach will yield inconsistency between the numbers of physical states of quadratic and non-abelian theories. In 3–dimensions one of the gauge symmetries of the quadratic action is preserved after introducing the non-abelian terms. This manifest itself as secondary first class constraint, namely \( \tilde{\psi}_1 \), which comes from the condition \( \dot{P}_0 = 0 \). The other primary constraint \( \pi_0 = 0 \) leads to \( \tilde{\psi}_2 \). The latter is related to the gauge symmetry of the quadratic action which is broken in the non-abelian case and it leads to the constraint \( \tilde{\psi}_3 = 0 \). So that, the number of the physical states of the abelian and the non-abelian cases is the same. However, in the 4–dimensional analog all of the gauge symmetries of the quadratic action are broken after introducing the non-abelian terms. Thus, none of the secondary constraints is first class. Preserving them in time does not lead to any other constraint but dictate form of the Lagrange multipliers \( \lambda_1, \lambda_2 \).

\[ ^2 \text{I would like to thank R. Jackiw and S-Y. Pi for asking me to comment on this point.} \]
References

[1] G. Alexanian and V.P. Nair, Phys. Lett. B 352 (1995) 435;
    W. Buchmüller and O. Philipsen, Nucl. Phys. B 443 (1995) 47; Phys. Lett. B 397
    (1997) 112;
    D. Comelli and M. Pietroni, DFPD 97/TH/37, hep-ph/9708489.

[2] R. Jackiw and S-Y. Pi, Phys. Lett. B 368 (1996) 131.

[3] R. Jackiw and S-Y. Pi, Phys. Lett. B 403 (1996) 297.

[4] J.M. Cornwall, UCLA/97/TEP/12, hep-th/9710128.

[5] F. Eberlein, ISSS 0418-9833, hep-th/9804460.

[6] P.A.M. Dirac, Lectures in quantum mechanics (Academic Press, New York, 1965).

[7] R. Arnowitt and S. Deser, Nucl. Phys. 49 (1963) 133.