Les Houches Lectures on
Effective Field Theories for
Nuclear and (some) Atomic Physics

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Dedicated to the memory of Professor Cécile DeWitt-Morette

Abstract

These lectures are a pedagogical — not comprehensive — introduction to the applications of effective field theory in the context of nuclear and atomic physics. A common feature of these applications is the interplay between nonperturbative physics (needed at leading order to produce nonrelativistic bound states and resonances) and controlled perturbative corrections (crucial for predictive power). The essential ideas are illustrated with the simplest nuclear EFT, Pionless EFT, which contains only contact interactions and, with minor changes, can be adapted to certain atomic systems. This EFT exploits the two-body unitarity limit, where renormalization leads to discrete scale invariance in systems of three and more bodies. Remarkably complex structures then arise from very simple leading-order interactions. Some of the challenges and rewards of including long-range forces — pion exchange in Chiral EFT for nuclear systems or Van der Waals forces between atoms — are briefly described.

1Professor DeWitt-Morette was a towering figure in mathematical physics who founded the Les Houches School. I had the privilege of her guidance while studying stochastic systems in my early graduate-student days at the University of Texas. Like other great theorists, she did not look at physics from the perspective of a specific field, but strove to build a consistent view of nature. Besides an accomplished scientist, she was passionate about science, kind and supportive. Near the end of my Ph.D. she recommended me to come to Les Houches, but it took me a whole quarter of a century to get here. It is a sad twist of fate that, when writing this dedication, I found out I had to add “the memory of” to it.
1 Introduction

For us humans, who unlike frogs are born basically smaller versions of ourselves at the time we have babies, one of the most natural transformations of the physical world is that of scale, where all distances change by a common numerical factor. Despite its familiarity, scale invariance is usually not directly useful in physics — we do not particularly resemble stars or molecules. As we embark on an exploration of a physical problem, the first task is always to identify its relevant scales.  

The existence of characteristic scales means that our resolution when probing a system is important. Tracking changes in resolution is the task of the renormalization group (RG). Once a resolution is chosen, we have at least two momentum scales to contend with, that of the physics of interest (call it \( M_{lo} \)) and that of the physics at much shorter distances (call it \( M_{hi} \gg M_{lo} \)). Effective field theory (EFT) is the formalism to exploit this separation of scales and expand observable quantities in powers of \( M_{lo}/M_{hi} \) without making assumptions about the short-range dynamics, other than its symmetries \cite{1}. Schematically, the \( T \) matrix for momentum \( Q \sim M_{lo} \) (from which the \( S \) matrix and other observables can be obtained) can be written as

\[
T(Q) = \sum_{\nu=0}^{\infty} \left( \frac{M_{lo}}{M_{hi}} \right)^{\nu} F_{\nu} \left( \frac{Q}{M_{lo}} ; \gamma^{(\nu)} \right) \equiv \sum_{\nu=0}^{\infty} T^{(\nu)}(Q),
\]

where each \( F_{\nu} \) is a calculable function parametrized by a finite set of “low-energy constants” (LECs) or “Wilson coefficients” \( \{ \gamma^{(\nu)} \} \). Clearly this is a paradigm to tackle any physical system, as the variety of topics in this school attests to. As far as we know, nature is but a pile of EFTs.  

In these two lectures we will see how the paradigm fares when facing a very traditional field, nuclear physics, where it is hard to come up with something that has not been tried before, usually without success. The strong interactions encapsulated in QCD produce non-trivial structures at the most fundamental level we can study today. Understanding how hadrons and their bound states — nuclei — arise remains an open problem in the Standard Model of particle physics, which hampers our ability to make predictions about processes involving new physics and astrophysical reactions.  

It has been around 25 years since Weinberg \cite{2,3} and Rho \cite{4} proposed that EFT could reproduce much of what was known in low-energy nuclear physics, while at the same time explaining some of its mysteries. Weinberg had earlier articulated the EFT paradigm \cite{5} and was interested in general ways to set up the electroweak symmetry-breaking sector of the Standard Model \cite{1,6}. Nuclear forces from pion exchange naturally come to mind when pondering about how Goldstone bosons couple to matter. The hope was that one would be able to formulate a renormalizable theory of nuclear interactions, overcoming the obstacles faced since the 1950s. “Chiral potentials” constructed according

\footnote{And, for convenience, simple units. Since I am interested in problems that are rooted in quantum mechanics and relativity, I use throughout these lectures natural units where Planck’s constant and the speed of light are \( h = c = 1 \). Then distance and time have the same units as inverse momentum, energy and mass. (To convert to more conventional units, use \( hc \simeq 200 \text{ MeV fm} \).}
to Weinberg’s suggestion have now become the favorite input to “ab initio” methods to calculate nuclear structure and reactions. In traditional nuclear physics, the *initio* is a nonrelativistic potential among nucleons, yet the remarkable recent progress in Lattice QCD \[7\] means that soon the starting point will be QCD itself \[8, 9\].

Unfortunately these chiral potentials produce scattering amplitudes that do not respect RG invariance. In the process of discovering and fixing this problem, other nuclear EFTs have been formulated, which apply to various energy regimes. Weinberg’s basic insights have survived but building nuclear EFT turned out to be much more interesting than he anticipated. Some of the issues and advantages of explicit pion exchange are mentioned at the end of these notes, but I cannot possibly cover all the twists and turns of this story, not even all nuclear EFTs. I will focus instead on the simplest one, Pionless EFT, and refer to others only occasionally. Pionless EFT is so simple, in fact, that with small changes it can be applied to systems of cold, neutral atoms as well. These lectures serve also as a (somewhat idiosyncratic) introduction to some of the physics that reinvigorated the atomic field over roughly the same time frame. I will make no attempt to provide an extensive coverage of the literature; only papers that are best suited to make specific points are cited.

Pionless EFT is sufficient for a sample taste of nuclear EFTs. In contrast to many of the other lectures in the school, here we shall deal with a situation where the leading order (LO) of the $M_{lo}/M_{hi}$ expansion must be non-perturbative in order to generate poles of the $S$ matrix: the bound states and resonances that we identify as nuclei (or molecules). The combination of non-perturbative LO and perturbative corrections — relative $O(M_{lo}/M_{hi})$ at next-to-leading order (NLO), relative $O(M_{lo}^2/M_{hi}^2)$ at $N^2$LO, etc. — is at the core of the beauty of the nuclear and atomic applications of EFT. Pionless EFT is the “poster EFT” to describe such combination. There is much regularity in the properties of nuclear and atomic bound states and resonances, and Pionless EFT captures a class of these regularities that sometimes goes by the name of “Efimov physics”. In a magical paper almost half-a-century ago \[10\], Efimov showed that for certain nonrelativistic systems, if a two-body bound state lives on the verge of non-existence, then a geometric tower of three-body bound states exists, with the ground state potentially quite deep. We now know that this phenomenon is not limited to the three-body system, but reverberates through larger clusters and even “infinite” matter. As we will see, scale invariance is the key to understand this sort of structure.

## 2 Some nuclear and atomic scales

A cursory look at the Particle Data Book \[11\] shows a bewildering variety of hadrons, which nevertheless fall into isospin multiplets containing various charge states with approximately the same mass. When made out of light quarks, they have masses in the 1, 2 GeV range, such as the proton and the neutron that can be paired in an isospin doublet of mass $m_N \simeq 940$ MeV. We can infer that QCD has a characteristic scale $M_{QCD} \sim 1$ GeV associated with its nonperturbative dynamics.
The one clear exception is the isospin triplet of light pions, with mass \( m_\pi \simeq 140 \text{ MeV} \ll M_{\text{QCD}} \). This low mass has long been understood as the result of the spontaneous breaking of an approximate \( \text{SU}(2)_L \times \text{SU}(2)_R \sim \text{SO}(4) \) chiral symmetry of independent rotations in the space of two flavors for left- and right-handed quarks. Because the diagonal subgroup \( \text{SU}(2)_{L+R} \sim \text{SO}(3) \) of isospin rotations remains unbroken, the three Goldstone bosons in the coset space \( \text{SO}(4)/\text{SO}(3) \sim S^3 \) can be identified with the three pions. Their interactions are governed by a dimensionful parameter, the pion decay constant \( f_\pi \simeq 92 \text{ MeV} \sim M_{\text{QCD}}/(4\pi) \), which is the radius of this “chiral sphere”. Pions are not exactly massless because chiral symmetry is explicitly broken by the quark masses. From perturbation theory one expects \( m_\pi^2 \sim M_{\text{QCD}}\bar{m} \), which is in the right ballpark if the average quark mass \( \bar{m} \) is a few MeV. Isospin is broken by the down-up quark mass difference \( \varepsilon \bar{m} \), where \( \varepsilon \sim 1/3 \) \cite{12}, which gives rise to small splittings among isospin multiplets. Of course, isospin is also broken explicitly by electromagnetic interactions. All this has been understood for quite some time and is discussed in some detail in Pich’s lectures \cite{6}.

What might be somewhat surprising is that once quarks hug tight into nucleons and pions with intrinsic sizes \( \sim M_{\text{QCD}}^{-1} \simeq 0.2 \text{ fm} \), nucleons form nuclei of much larger size and feebler binding. One proton and one neutron bind into an isospin singlet — the simplest nucleus, the deuteron \( (^2\text{H}) \) — with total spin \( S = 1 \) and a binding energy \( B_d \simeq 2.2 \text{ MeV} \). When \( S = 0 \), proton and neutron are part, with two neutrons and two protons, of an isospin-triplet virtual state at \( B_d^* \simeq 0.08 \text{ MeV} \). (A bound state is a pole of the \( S \) matrix with positive imaginary momentum, while a virtual state is a pole with negative imaginary momentum — both have negative energy \( -B \). A resonance consists of a pair of poles in the lower half of the complex-momentum plane with opposite, nonzero real parts.) As shown in the leftmost column of Table 1, the binding energy per particle \( B_A/A \) increases for the trinucleon isodoublet with \( S = 1/2 \) — triton \( (^3\text{H}) \) and a slightly less-bound helion \( (^3\text{He}) \) — and even more for the isospin-singlet alpha particle \( (^4\text{He}) \) with \( S = 0 \). The exclusion principle makes it harder for more nucleons to be together and \( B_A/A \) first decreases, but then increases again (on average) till it reaches a maximum of about 8 MeV for \( A = 56 \), decreasing slowly beyond that. Table 1 also shows the value of the binding energy per particle for nuclear matter, \( \lim_{A \to \infty} (B_A/A) \equiv b_\infty \). Nuclear matter is an idealized system without surface or electroweak interactions, defined by extrapolation from heavy nuclei via the “liquid-drop” relation \( B_A/A - b_\infty \propto A^{-1/3} \). Consonant with such “saturation”, the typical nucleus size is \( \sim r_{\text{nuc}}A^{1/3} \), with \( r_{\text{nuc}} \simeq 1.2 \text{ fm} \) — but it can be much larger for more loosely bound nuclei, such as light nuclei and “halo nuclei”, which have a cloud of loose nucleons orbiting a more tightly bound core. (The typical example is \(^6\text{He} \), which is thought to be essentially two neutrons around a \(^4\text{He} \) core.) Halo EFT \cite{13, 14} provides a description of this type of state similar to Pionless EFT, but with the core as an additional degree of freedom.

This disparity between QCD and nuclear scales tells us several things. First, nucleons are relatively far apart inside nuclei and retain their identity. Second, they move slowly, that is, are approximately nonrelativistic. Third, their interactions must come from the exchange of the lightest color singlets — the pions, giving rise to a force of range \( \sim m_\pi^{-1} \simeq 3 \)
Table 1: Ground-state (first-excited) binding energies per particle $B_A/A$ ($B_{A^*}/A$) of selected light nuclei and $^4$He atomic clusters, in units of the three-body binding energy per particle, where $B_3 = 8.48$ MeV and $B_3 = 0.1265$ K, respectively. (To convert between K and eV, use $k = 8.6 \cdot 10^{-5}$ eV K$^{-1}$.) In the nuclear case, an entry corresponds to the deepest isobar state with the respective nucleon number $A$, a parenthesis indicating a virtual state. Entries in the left column are experimental. Entries in the middle and right columns are from Pionless EFT at LO: from Refs. [15, 16, 9, 17] away from unitarity and from Ref. [18] at unitarity. For $^4$He atomic clusters, the left column shows the results [19, 20, 21] of calculations with phenomenological $^4$He-$^4$He potentials. Entries in the middle and right columns are from Pionless EFT at LO: from Refs. [22, 23, 24] away from unitarity and from Refs. [22, 25, 26] at unitarity. For simplicity I do not indicate quantities used in the construction of the interactions nor show error estimates, which can be found in the original references. For some entries similar numbers exist from other references.

| $3B_A/(AB_3)$ | nucleons | $^4$He atoms |
|---------------|-----------|--------------|
| $A$ | experiment | LO EFT | unitarity | potential | LO EFT | unitarity |
| 2 | 0.39 | 0.39 | 0 | 0.0156 | 0.0152 | 0 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3* | (0.17) | (0.19) | 0.0019 | 0.0180 | 0.0175 | 0.0019 |
| 4 | 2.50 | 2.6 | 3.5 | 3.3 | 3.2 | 3.46 |
| 4* | 0.71 | 0.90 | 0.75 | 0.755 | 0.759 | 0.752 |
| 5 | 1.94 | ? | ? | 6.2 | 5.7 | 6.3 |
| 6 | 1.89 | 1.4 | ? | 9.2 | 8.2 | 8.9 |
| ... | ... | ... | ... | ... | ... | ... |
| 16 | 2.82 | 2.5 | ? | ? | ? | 27.4 |
| ... | ... | ... | ... | ... | ... | ... |
| $\rightarrow \infty$ | 5.7 | ? | ? | 180 | ? | 90 |
1.4 fm — plus complicated mechanisms of shorter range \( \sim M_{QCD}^{-1} \). Nuclei should thus be described by an EFT with nonrelativistic nucleons and pions (and possibly the lightest nucleon excitations) subject to approximate chiral symmetry — this is Chiral EFT, whose restriction to \( A = 0, 1 \) is Chiral Perturbation Theory (ChPT) \([5, 6]\). Moreover, in the long distances relevant for light nuclei, even pion exchange can be considered a short-range interaction. This is the regime of Pionless EFT, where the only explicit degrees of freedom are nonrelativistic nucleons.

The situation is not totally dissimilar for some (neutral) atoms, like the boson \( ^4\text{He} \). Instead of QCD binding quarks into hadrons, QED binds a nucleus and \( Z \) electrons into atoms with energies \( E_{\text{at}} \sim -(Z \alpha)^2 m_e \), where \( \alpha \approx 1/137 \) is the fine-structure constant and \( m_e \) the electron mass. The atoms themselves form much more loosely bound molecules, through the exchange of (at least) two photons and shorter-range interactions. The former gives rise to the Van der Waals potential \( \sim -l_{\text{vdW}}^4/(2 \mu r^6) \), where \( \mu \) is the reduced mass and \( l_{\text{vdW}} \) is the “Van der Waals length”, which depends on what kind of atom we are considering. For a nice compilation of values, as well as a discussion of long-range interactions, see Ref. \([27]\). For certain types of atoms, clusters have sizes that are significantly larger than \( l_{\text{vdW}} \); in these cases, the short-range interactions dominate and we can, up to a point, treat the system as one with short-range forces only. This Contact EFT is analogous to Pionless EFT, just with a field for the atom substituted for the nucleon’s.

\( ^4\text{He} \) is a particularly interesting atom because a macroscopic sample remains liquid at zero temperature and exhibits the remarkable property of superfluidity. The \( ^4\text{He} \) dimer has been measured to have an average separation \( \langle r \rangle = 52(4) \) Å \([28]\), which is an order of magnitude larger than the corresponding \( l_{\text{vdW}} \approx 5.4 \) Å \([29, 30]\). Experimental numbers exist for the dimer \([31, 32]\) and excited-trimer \([33]\) binding energies, which confirm that they are loosely bound. Sophisticated \( ^4\text{He} \)-\( ^4\text{He} \) potentials have been developed over the years, which are consistent with a variety of experimental data and allow for the prediction of the energies of larger clusters. The results for one of these potentials — dubbed the “LM2M2 potential” \([34]\) — are shown in the left \( ^4\text{He} \) column of Table \([1]\). Other potentials give similar results. Apart from a huge difference in overall scale, the numbers for \( A \leq 4 \) atoms have some qualitative similarity to those for nucleons. The most obvious difference is the lack of exclusion principle, which translates into a monotonic increase in binding energy, which starts approximately as \( (N - 2)^2 B_3 \) \([24]\). Just as for nucleons, though, the interaction saturates to a constant binding energy per particle, the value of which was calculated with another potential — the “HFDHE2 potential” \([35]\) — in Refs. \([19, 36]\). Similarly to nuclei, cluster size first decreases, then starts to increase till it settles into an \( \sim r_{\text{at}} A^{1/3} \) behavior, with \( r_{\text{at}} \approx 2.2 \) Å \([36]\).

These two families of systems — nuclei and \( ^4\text{He} \) atomic clusters — are peculiar for their large sizes compared to the interaction range \( R \). A rough estimate of the characteristic momentum \( Q_A \) of the particles in the bound state is obtained by assuming that every one of the \( A \) particles of mass \( m \) contributes the same energy \( Q_A^2/(2m) \) to \( B_A \):

\[
Q_A \sim \sqrt{2mB_A/A}.
\]

(This estimate reflects the correct location of the bound state in relative momentum for
$A = 2$ and gives a finite $Q_A$ in the limit $A \to \infty$.) For nucleons, we find $Q_3 \sim 70$ MeV, while for $^{4}$He atoms $Q_3 \sim (12 \text{ Å})^{-1}$, each one about half of the corresponding $R^{-1}$. That is, particles in the three-body bound state are separated by a distance about twice as large as the range of the interaction. How can this be? Classically, the size of an orbit is given by the range of the force. Thus, these are intrinsically quantum-mechanical systems, which hold a few surprises in store for us.

3 EFT of short-range forces

In the class of systems sometimes referred to as “quantum halos”, we are interested in the $S$ matrix for processes with a typical external momentum $Q \ll M_{hi}$, where the EFT breakdown scale $M_{hi}$ is related to the inverse of the force range, $R^{-1} = m_{\pi}$ or $l_{vdW}^{-1}$, as the case may be. The few-body structures we want to describe are characterized by momenta $Q \sim Q_3$, so I will use $Q_3$ as a proxy for $M_{lo}$. The idea is to construct an expansion of the form (1) from the most general Lagrangian (density) allowed by the symmetries supported by the relevant degrees of freedom, exploiting the “folk theorem” that the resulting quantum field theory will then generate the most general $S$ matrix allowed by the same symmetries [5].

3.1 Degrees of freedom

In systems whose sizes are larger than the range $R$ of the force, the constituent particles are not able to resolve details of the potential. They feel the interaction as a contact: the potential can be represented by a Dirac delta function and its derivatives. The only degrees of freedom we need to consider are the constituent particles themselves — there is no need to account explicitly for other particles whose exchanges are responsible for the specific form of the potential.

Since in the cases of interest here the particle mass $m \gtrsim M_{hi}$, a nonrelativistic expansion must hold, which yields a much simpler field theory than in the relativistic case. Because it takes $\gtrsim 2m$ in energy to create a particle-antiparticle pair, there is no need to include antiparticles — unless we are interested in processes that include antiparticles to start with, or in processes that involve particle- (baryon- and/or lepton-) number violation. We can exploit this simplification by a convenient choice of field. (I remind you that fields are not directly observable and in an EFT all choices that represent the same degrees of freedom are equivalent [37, 38].) It is sufficient to employ a field $\psi_a(x)$ that only annihilates particles,

$$\psi_a(x) \equiv \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{e^{-ip \cdot x}}{2p^0} u_s(\bar{p}) a_{\bar{p}}$$

(3)

where $a_{\bar{p}}$ is the annihilation operator for a particle of momentum $\bar{p} \equiv \vec{k}$, $u_s(\bar{p})$ carries information about the spin, and $p^0 = \sqrt{\vec{p}^2 + m^2} = m + \bar{p}^2/(2m) - \bar{p}^4/(8m^3) + \ldots \equiv m + k^0$ (I am sorry, I am using the “wrong”, West Coast metric here: $p \cdot x = p^0 t - \vec{p} \cdot \vec{r}$.) Propagation is only forward in time, and it is represented in Feynman diagrams by a line going up
(when time is represented as increasing upwards, as I do here). The absence of pair creation means that, if particle number is conserved, these lines go through the diagram. The theory breaks into separate sectors, each with a given number of particles $A$. Life is made tremendously easier by the fact that we can deal in turn with sectors of increasing $A$ without simultaneously having to worry about the feedback from larger number of particles.

To produce an $M_{\text{lo}}/M_{\text{hi}}$ expansion, we expect some form of derivative expansion in the action. Yet derivatives of $\psi_a(x)$ contain the large $m$ coming from the trivial factor $\exp(-imt)$ in the evolution of the field. In the particle’s rest frame, where the particle’s four-velocity $v^\mu$, $v^2 = 1$, is $(1, \vec{0})$ it is convenient to remove this factor by defining a new “heavy field” [39]

$$\psi_h(x) \equiv e^{imv \cdot x}\psi_a(x).$$

The evolution of the new field is governed by the kinetic energy $k^0 \ll m$, since particles will exchange small three-momenta. Instead of the $-ip_\mu$ that contains the large $m$, $\partial_\mu \psi_h(x)$ gives $-ik_\mu$ in momentum space.

For cases where antiparticles are present, a conjugate field can be introduced similarly. Of course the possibility of pair annihilation when both particles and antiparticles are present allows for large momenta, and the applicability of a $Q/m$ expansion is limited to times prior to annihilation.

### 3.2 Symmetries

So far, no sign has been seen of violation of Lorentz invariance or the product CPT of charge conjugation (C), parity (P), and time reversal (T). For simplicity I will also neglect the small effects that arise from the violation of P, T, and baryon and lepton numbers, although they are part of the EFT. All these symmetries restrict the form of the terms allowed in the action. For example, the transformation associated with particle number is an arbitrary phase change

$$\psi_h(x) \rightarrow e^{i\alpha} \psi_h(x),$$

and particle-number conservation implies that $\psi_h$ enters the Lagrangian in combination with $\psi_h^\dagger$.

For a heavy field, the symmetry whose implementation is least obvious is Lorentz invariance. To start with, note that the definition of $\psi_h$ can be made in other frames where $v^\mu$ is not necessarily $(1, \vec{0})$. We introduce the residual momentum $k^\mu$ through $p^\mu \equiv mv^\mu + k^\mu$. The residual momentum is only constrained by $k^2 = -2mv \cdot k \ll m^2$. There is freedom to consider a different residual momentum $k^\mu - q^\mu$ with $q^2 = -2mv \cdot q \ll m^2$, and a simultaneous relabeling of the velocity, $v^\mu \rightarrow v^\mu + q^\mu/m$. (You can check that $(v + q/m)^2 = 1$). Thus the theory must be invariant under this “reparametrization invariance” (RPI) [40], when the field transforms as

$$\psi_h(x) \rightarrow e^{iq \cdot x} D(v + q/m, v) \psi_h(x),$$

7
where $D$ is determined by the representation of the Lorentz group encoded in $u$. Because of the phase, $\partial_\mu \psi_h(x)$ is not covariant under reparametrization. As standard in such cases, we can introduce a “reparametrization covariant derivative”

$$D_\mu \psi_h(x) \equiv [\partial_\mu - imv_\mu] \psi_h(x) \to e^{iq^x} D(v + q/m, v) D_\mu \psi_h(x). \quad (7)$$

We account for Lorentz invariance by properly contracting Lorentz indices in the Lagrangian as usual, but using the reparametrization covariant derivative.

As an example, take the kinetic terms of a scalar field, whose RPI form is

$$L_{\text{kin}} = (D^\mu \psi_h)^\dagger D_\mu \psi_h - m^2 \psi_h^\dagger \psi_h = 2m \psi_h^\dagger \left[ iv \cdot \partial - \partial^2 / (2m) \right] \psi_h. \quad (8)$$

The large mass term has disappeared, as desired, except for an overall normalization. The remaining terms resemble the inverse of the usual nonrelativistic propagator, but contain also $\partial_0^2 \psi$. To bring the propagator to the usual form, we can define yet another field, which is canonically normalized:

$$\psi(x) \equiv \frac{1}{\sqrt{2m}} \left[ 1 - \frac{i}{4m} v \cdot \partial + \ldots \right] \psi_h(x). \quad (9)$$

Substitution into Eq. (8) leads to

$$L_{\text{kin}} = \psi^\dagger \left\{ iv \cdot \partial + \frac{1}{2m} \left[ (v \cdot \partial)^2 - \partial^2 \right] + \ldots \right\} \psi. \quad (10)$$

Now in the rest frame the term in square brackets reduces to the usual $\vec{\nabla}^2$.

**Exercise:** Consider the higher orders in $1/m$ in the “…” of Eq. (9). Show that an appropriate field choice makes the next term in the “…” of Eq. (10) equal to $\vec{\nabla}^4 / (8m^3)$ in the particle’s rest frame, in line with $k^0 = \vec{k}^2 / (2m) - \vec{k}^4 / (8m^3) + \ldots$

The same procedure can be followed for interaction terms. As the number of derivatives increase, we have to contend with two or more time derivatives. In this case, there is a nontrivial relation between the time derivative of the field and the field’s conjugate momentum. As a result the interaction Hamiltonian is not simply minus the interaction Lagrangian, but contains additional, Lorentz noncovariant terms. It is no problem to include these interactions, as other noncovariant pieces arise in covariant perturbation theory from contractions involving derivatives, and in a time-ordered formalism from its inherent noncovariance. One can show explicitly [41] that the sum of diagrams contributing to any given process is indeed covariant. Alternatively, one can integrate over the field’s momentum in the path integral arriving at an effective Lagrangian, which is covariant but has additional terms compared to the classical Lagrangian we started from [42].

For heavy fields these complications can be avoided altogether by including in the “…” of Eq. (9) terms with additional fields, designed to remove $\partial_0 \psi$ from interactions as well.

Accounting for a nonzero spin manifest in the $D(v + q/m, v)$ in Eq. (6) is a bit more complicated. To represent only particles, the fields obey constraints. For example, of
the four components of a Dirac spinor only two are associated with the two spin states of a particle with \( S = 1/2 \). In the rest frame one can project onto the two upper (in the Dirac representation) components with the projector \((1 + \gamma^0)/2\) and thereby use a Pauli spinor, the gamma matrices reducing to the spin operator \((0, \vec{\sigma}/2)\). In a generic frame the two “large” components can be selected by the constraint \((1 - \vec{v}/v)\psi = 0\). Spin properties are encoded in \( S^{\mu} = i\gamma_5\sigma^{\mu\nu}\nu_\nu/2 \), which satisfies \( S \cdot \nu = 0 \) and \( S^2 = -3/4 \), as a little gamma-matrix algebra shows. The “reparametrization covariant spin” that appears in interactions is then \( \Sigma^{\mu} = -\gamma_5\sigma^{\mu\nu}\nu_\nu/(2m) \), which satisfies \( \psi^\dagger \Sigma \cdot D \psi = 0 \) and \( \psi^\dagger \Sigma^2 \psi = 3\psi^\dagger D^2 \psi/(4m^2) \).

At the end of the day, we simply end up with a Lagrangian that complies with Lorentz invariance in a \( Q/m \) expansion. Of course other, perhaps simpler, ways exist to implement the constraints of Lorentz invariance, for example an explicit reduction of a fully relativistic theory. Heavy fields with RPI are, however, more EFT-like, because they do not rely on detailed knowledge of the theory at \( Q \sim m \). This is particularly relevant in nuclear physics, where, if fully relativistic nucleons are to be considered, we need also to include mesons heavier than the pion, for which no systematic \( Q/M_{hi} \) expansion is known. RPI is thus well suited to the expansion in \( Q/M_{hi} \) we seek. More details about the heavy field formalism can be found in the lectures by Mannel at this school [43]. In the following, for simplicity, I will drop the interactions that are explicitly spin-dependent and work in the rest frame.

### 3.3 The action

The most general action for nonrelativistic particles of mass \( m \) interacting under these symmetries through short-range forces can therefore be written as

\[
S = \int d^4x \mathcal{L} = \int dt \int d^3r \left\{ \psi^\dagger \left( 2im \frac{\partial}{\partial t} + \vec{\nabla}^2 + \ldots \right) \psi - 4\pi \left[ C_0 (\psi^\dagger \psi)^2 + C_2 (\psi^\dagger \psi) \left( \psi^\dagger \vec{\nabla}^2 \psi + H.c. \right) + \ldots \right] - \frac{(4\pi)^2}{3} D_0 (\psi^\dagger \psi)^3 + \ldots \right\}, \tag{11}
\]

where \( C_{0,2}, D_0, \text{etc.} \) are the LECs and “…” include terms with more derivatives and/or fields. Beware that I chose a normalization of the LECs that is not normally used in the literature, but is very convenient for the subsequent discussion of orders of magnitude and scale invariance. With this choice, the mass appears in the combination \( t/m \), and in observables together with the energy \( E \) as \( mE \equiv k^2 + \ldots \). As a consequence, binding energies are all \( \propto 1/m \). This is at the root of a choice of units frequently made in atomic physics: \( m = 1 \). I prefer to keep \( m \) explicit instead.

In writing Eq. (11) I neglected spin projections in the interaction terms. For fermions, we can use Fierz reordering, which encodes the exclusion principle, to reduce the number of independent terms. For a two-state fermion (e.g. a neutron with spin up or down, in the absence of protons), the \( C_0 \) interaction operates only when the two particles are in
different states. Similarly, the $D_0$ interaction, which requires three particles at the same point, vanishes. For more-state fermions, more than one $C_0$- and/or $D_0$-type interaction is possible, depending on the symmetries. For example, because a nucleon consists of four states — isospin up (proton $p$) and down (neutron $n$), each with two spin states — four $C_0$-type interactions exist, corresponding to the four $S$-wave channels: the isospin-singlet $pn$ channel with spin $S = 1$ — the deuteron channel — and the $pp$, $pn$, and $nn$ interactions in the isospin-triplet channels with $S = 0$. In situations where isospin is a good approximate symmetry, these interactions are reduced in a first approximation to two, the one operating in the $^3S_1$ channel, and a single one in $^1S_0$. (We commonly use the spectroscopic notation $2S + 1L_J$ where $S$ is the spin, $L$ is the orbital angular momentum, and $J$ the total angular momentum.) Similarly, there is a single $D_0$-type interaction, operating in the nucleon-deuteron ($Nd$) $^2S_{1/2}$ wave — the triton/helion channel. (In the $^4S_{3/2}$ channel there are at least two protons or two neutrons in the same spin state.)

Exercise: Show that Fermi statistics and isospin symmetry imply that two nucleons interact in two $S$ waves: $i)$ $^3S_1$, which is symmetric in spin and antisymmetric in isospin, and thus involves an $np$ pair; and $ii)$ $^1S_0$, which is antisymmetric in spin and symmetric in isospin, and is split by isospin-violating interactions into $pp$, $pn$, and $nn$. Analyze the two $S$ waves for the $Nd$ system, $^4S_{3/2}$ and $^2S_{1/2}$, similarly.

I have not considered electromagnetic interactions explicitly in Eq. (11). The associated $U(1)_e$ gauge symmetry can be introduced in the usual way, that is, by requiring that all interactions be built out of the electromagnetic covariant derivative and the field strength. In the neutral-atom case, the covariant derivative is just the usual derivative, but electromagnetic interactions still proceed through the coupling of the atom to the field strength. The longest-range effects arise from two-photon exchange, the most important being the Van der Waals force. By remaining at momenta below $l_{vdW}^{-1}$, dimensional analysis shows that the non-analytic contributions from two-photon exchange enter only at $O(Q^3)$, so at lower orders we can pretend to have only short-range interactions. In the nuclear case, where the proton is charged, one-photon exchange leads to long-range interactions, the most important being the well-known Coulomb force. There is further isospin violation from shorter distances stemming from “hard” photons and from the quark mass difference. The associated isospin violation splits various charge states. For nuclear ground states one can argue that isospin-breaking effects are subleading and bring no fundamental changes to the discussion below.

To keep the notation lean, I will ignore spin-isospin complications in the formulas that follow, and pretend that only one short-range interaction of each type is important. I will simply remark on the changes that take place when $S > 0$.

### 3.4 Renormalization-group invariance and power counting

As in any EFT, the interactions in the action (11) are local and require regularization, that is, the introduction of a method to suppress explicit high-momentum, or equivalently short-distance, modes. There is an infinite number of ways of doing this. What is common
to all methods is the presence of a parameter with dimensions of mass, which I will denote \( \Lambda \). In perturbation theory, frequently the cleanest method is dimensional regularization because it keeps no terms that go as negative powers of \( \Lambda \). (In most subtractions it keeps also no positive powers.) However, dimensional regularization is difficult (impossible?) to apply in generic nonperturbative contexts, where we are restricted (at least for now) to momentum or distance regulators. The simplest examples of these regulators are, respectively, a sharp momentum cutoff \( \Lambda \) and a minimum length, or lattice spacing, \( \Lambda^{-1} \). More generally, one can use a function of the momentum \( \vec{p} \) which vanishes smoothly as \( |\vec{p}| \) increases beyond \( \Lambda \), say a Gaussian function of \( |\vec{p}|/\Lambda \) with unit coefficient, or a smooth representation of a delta function in coordinate space \( \vec{r} \), say a Gaussian function of \( |\vec{r}|\Lambda/2 \) with a coefficient that scales as \( \Lambda^3 \). From here on I will talk mostly about a momentum cutoff, with only occasional remarks about dimensional regularization.

The effects of the high-momentum modes must, of course, reappear in the LECs, which are thus dependent on \( \Lambda \). Renormalization is the process of ensuring that this dependence is such that observables are independent of the arbitrary regulator, or

\[
\frac{dT(Q)}{d\Lambda} = 0. \tag{12}
\]

That this must be possible is a consequence of the uncertainty principle coupled to our accounting of all interactions allowed by symmetries: modes of momentum \( \gtrsim \Lambda \) can be absorbed in interactions of range \( \lesssim \Lambda^{-1} \) with the same symmetries. If \( \Lambda \gtrsim M_{\text{hi}} \), this adds no extra limitation to the EFT. If \( \Lambda \lesssim M_{\text{hi}} \), we cannot apply the EFT all the way to its physical breakdown scale.

A central role is played in EFT by “power counting”, that is, a rule that organizes the infinite sequence of interactions according to their relevance to observables by relating the counting index \( \nu \) in Eq. (1) to the properties of the various terms in the action (number of derivatives, fields, etc.). To preserve the insensitivity to the regulating procedure, we want each truncation of Eq. (1),

\[
T^{[\nu]}(Q) \equiv \sum_{\nu=0}^{\nu} T^{(\nu)}(Q) \tag{13}
\]

to satisfy

\[
\frac{\Lambda}{T^{[\nu]}(Q)} \frac{dT^{[\nu]}(Q)}{d\Lambda} = \mathcal{O}
\left( \frac{Q}{\Lambda} \right). \tag{14}
\]

In this way, the regulator does not increase the truncation error, as long as \( \Lambda \gtrsim M_{\text{hi}} \). This places a constraint on the power counting: that it contains enough interactions at each order to remove non-negative powers of \( \Lambda \) from observables. If we do not include all interactions needed to ensure Eq. (14), observables are sensitive to the arbitrary regulator — not only its dimensionful parameter but also its form — and there is no guarantee that results reflect the low-energy limit of the underlying theory. The canonical dimension of a LEC suggests a lower bound on its magnitude, when we use \( M_{\text{hi}} \) to make it dimensionless and \( \mathcal{O}(1) \). However, as we are going to see, our problem involves significant departures
from this simple dimensional analysis, in which case RG invariance in the form (14) offers particularly useful guidance regarding the LEC sizes. Regularization schemes — such as dimensional regularization with minimal subtraction — that kill positive powers of $\Lambda$ are not the most useful in this context, because they might lead you to overlook the need for a LEC at the order under consideration. (A physically relevant example is given in Ref. [13].)

This approach is a generalization of the old concept of “renormalizable theory”, where renormalization is achieved by a finite number of parameters and at the end $\Lambda \to \infty$ is taken. As Weinberg is fond of saying (see, e.g., Ref. [46]), “nonrenormalizable theories” are just as renormalizable as “renormalizable theories”. Now we only need a finite number of parameters at each order and $\Lambda \gtrsim M_{\text{hi}}$. Each of the LECs obeys an RG equation that tells us how it depends on $\Lambda$. A generic observable will look like

$$O^{[\bar{\nu}]}(\Lambda) = O^{[\bar{\nu}]}(\infty) \left[1 + \alpha^{[\bar{\nu}]}_{O} \frac{M_{\text{lo}}}{\Lambda} + \beta^{[\bar{\nu}]}_{O} \frac{M_{\text{lo}}^2}{\Lambda^2} + \ldots\right], \quad (15)$$

with $\alpha^{[\bar{\nu}]}_{O}$, $\beta^{[\bar{\nu}]}_{O}$, etc. numbers of $O(1)$. Values for a finite number of observables are used to determine the LECs — for these observables $\alpha^{[\bar{\nu}]}_{O} = \beta^{[\bar{\nu}]}_{O} = \ldots = 0$. For other observables, a nonzero $\alpha^{(\bar{\nu})}_{O}$ indicates the existence of at least one interaction at next order, since it generates a term of size $O(M_{\text{lo}}/M_{\text{hi}})$ when $\Lambda \sim M_{\text{hi}}$. There is no need to take $\Lambda \to \infty$ for the theory contains in any case errors of $O(M_{\text{lo}}/M_{\text{hi}})$ from the truncation in the action. But regulator cutoff variation from $\sim M_{\text{hi}}$ to much larger values does usually provide an estimate of the truncation error. Regularization schemes that kill negative powers of $\Lambda$, such as dimensional regularization, deprive you of this tool.

This newer view of renormalization arose from the combination of the traditional view with the more intuitive “Wilsonian RG”. The latter is usually applied to a (e.g. condensed-matter) system where the underlying theory is known, and the effective theory is constructed by explicitly reducing $\Lambda$ starting from $M_{\text{hi}}$. In this process all interactions allowed by symmetries are generated, and of course depend on $\Lambda$ by construction. For example, even if there is an underlying potential that is mostly two-body, its expansion at large distances will contain higher-body components, stemming from successive two-body encounters at unresolved distances and times. (In most situations these higher-body forces are relatively small, but not always, as we will see below.) In the Wilsonian RG, $\Lambda$ marks the highest on-shell momentum to which we can apply the EFT. As $\Lambda$ is decreased it eventually crosses a physical scale $M'_{\text{hi}}$ where the EFT needs to be reorganized, for example due to the emergence of new degrees of freedom (say, a Goldstone boson or another type of low-energy collective effect). If one extends this new EFT to $\Lambda$ above $M'_{\text{hi}}$, we are in the situation I described earlier: real momenta up to $M'_{\text{hi}}$ can be considered as $\Lambda$ cuts off virtual momenta only. Thus, while intuitive, there is no need, in fact no advantage, in keeping the regulator parameter $\Lambda$ below the physical breakdown scale we are interested in. This is particularly true when the underlying theory is unknown (as for the Standard Model) or known but hard to solve explicitly (as for QCD at low energies).

Beware that applications of this more general EFT implementation of the RG have been muddled by the multiple uses of the word “cutoff”. When the Wilsonian RG is favored,
typically “cutoff” is used for the regulator, which also limits the range applicability of the EFT. In contrast, in modern particle physics where dimensional regularization is almost exclusively employed, “cutoff” is often used for the physical breakdown scale. I will try to consistently distinguish the regulator cutoff $\Lambda$, which is not physical, and the physical breakdown scale $M_{hi}$.

4 Two-body system

Now let us see how this EFT works in the simplest case, the two-body system. Were we considering photons explicitly, we could look at the electromagnetic properties of one particle, such as form factors (accessible through electron scattering) or polarizabilities (through Compton scattering). But the corresponding amplitudes are purely perturbative (except possibly around special kinematic points), and my goal here is to illustrate the more challenging issue of building a power counting when a subset of interactions has to be treated nonperturbatively. That is the case when the $T$ matrix has low-energy poles. In particular, I want to tackle the situation relevant to the systems discussed in Sec. 2: a shallow two-body bound or virtual state.

4.1 Amplitude

Because the EFT splits into sectors of different $A$, we can focus on $A = 2$ without worrying, as we do in relativistic theories, about interactions with more than four fields in the action (11). Of the one-body terms, all we need is the propagator for a particle of four-momentum $l$,

$$iD_1(l^0, \vec{l}) = iD_1^{(0)}(l^0, \vec{l}) + iD_1^{(0)}(l^0, \vec{l}) \frac{i\vec{t}^4}{8m^3} iD_1^{(0)}(l^0, \vec{l}) + \ldots, \quad (16)$$

where

$$iD_1^{(0)}(l^0, \vec{l}) = \frac{i}{l^0 - \vec{l}^2/(2m) + i\epsilon}, \quad (17)$$

with $\epsilon > 0$, is the LO propagator. Because they are suppressed by $\mathcal{O}(Q^2/m^2)$, relativistic corrections come at $N^2$LO or higher depending on how we decide to count $m$ relative to $M_{hi}$. There is not much consensus about the most efficient scheme to do this, but the issue does not appear up to NLO, which is sufficient for these lectures. Likewise, for an on-shell particle of momentum $\vec{p}$, to this order we need only the first term in the expansion of the energy, $E = k^2/(2m) - k^4/(8m^3) + \ldots$, where $k = |\vec{p}|$. Note that I did keep the recoil term $\propto 1/m$ in the denominator of Eq. (17). Were we considering light probes (such as photons) with momenta $\mathcal{O}(Q)$, they would deposit energies $\mathcal{O}(Q)$ onto the virtual particles. In this case, we could also expand the propagator (17) in powers of $Q/m$, the leading term being the static propagator $i/(l^0 + i\epsilon)$. Instead, here particles start off with $E = \mathcal{O}(Q^2/m)$ and remain nearly on-shell: $l^0 = \mathcal{O}(Q^2/m)$ and thus comparable to $\vec{l}^2/(2m)$. The propagator at LO is not static. This is the first indication that the power
counting for processes involving two or more heavy particles is different from that for the simpler one-body processes.

The simplest of the four-field (or in nonrelativistic parlance, two-body) interactions in the action \( (11) \) is the \( C_0 \) term, which is represented by a vertex with four legs. It gives a momentum-independent tree-level contribution to the two-body \( T \) matrix,

\[
T_2^{(0,0)}(\Lambda) = -\frac{4\pi}{m} C_0^{(0)}(\Lambda) \equiv -V_2^{(0)}(\Lambda),
\]

where \( C_0^{(0)}(\Lambda) \) denotes the LO part of \( C_0 \). Things are more interesting at one-loop level, where two \( C_0 \) vertices are connected by two nonrelativistic propagators \( (17) \). In the center-of-mass system, where one incoming particle has four-momentum \( (p_0^0, \vec{p}) \) and the other \( (p_0^0, -\vec{p}) \),

\[
T_2^{(0,1)}(k; \Lambda) = i \left( \frac{4\pi i}{m} C_0^{(0)}(\Lambda) \right)^2 \int \frac{d^3 l}{(2\pi)^3} \int \frac{d l_0}{2\pi} D_1^{(0)}(l_0 + p_0^0, \vec{l} + \vec{p}) D_1^{(0)}(-l_0 + p_0^0, -\vec{l} - \vec{p})
\]

\[
= \left( \frac{4\pi}{m} C_0^{(0)}(\Lambda) \right)^2 \int \frac{d^3 l}{(2\pi)^3} \frac{m}{l^2 - k^2 - i\epsilon}.
\]

Here, I first integrated over the 0th component of the loop momentum, as it is standard in nonrelativistic theories. We can do this by contour integration: closing the contour in the upper plane, we pick a contribution from the residue of the pole at \( p_0^0 - (\vec{l} + \vec{p})^2/(2m) + i\epsilon \); closing on the lower plane, a contribution from the other pole, which gives, of course, the same result. As first noticed in this context by Weinberg [2, 3], had we neglected recoil in the one-body propagator \( (17) \), we would have faced a pinched singularity at the origin. The reflection of this is the appearance of the large \( m \) in the numerator of Eq. \( (19) \), where I additionally relabeled \( \vec{l} \rightarrow \vec{l} - \vec{p} \). This form of the one-loop amplitude should come as no surprise: we simply have an integration over the virtual three-momentum of the standard Schrödinger propagator.

In the form \( (19) \), it is clear that the integral would diverge without regularization. The most intuitive regularization consists of a “non-local” regulator that depends separately on the incoming and outgoing nucleon momenta: a function \( F(l/\Lambda) \) with the properties that \( F(0) = 1 \) (to preserve the physics at low momentum) and \( F(x \rightarrow \infty) = 0 \) (to kill high momenta):

\[
\int \frac{d^3 l}{(2\pi)^3} \frac{m}{l^2 - k^2 - i\epsilon} \equiv \frac{m}{2\pi^2} \int_0^\infty dl \frac{l^2}{l^2 - k^2 - i\epsilon} F(l/\Lambda) \equiv I_1(k; \Lambda)
\]

The simplest choice is a step function, \( F(x) = \theta(1 - x) \):

\[
I_1(k; \Lambda) = \frac{m}{2\pi^2} \int_0^\Lambda dl \frac{l^2}{l^2 - k^2 - i\epsilon} = \frac{m}{2\pi^2} \left( \Lambda + \frac{k^2}{2} \int_{-\Lambda}^\Lambda dl \frac{1}{l + k + i\epsilon} \frac{1}{l - k - i\epsilon} \right) = \frac{m}{4\pi} \left( \frac{2\Lambda + ik}{\pi \Lambda} + \mathcal{O}(k^4/\Lambda^3) \right),
\]
where I redefined $\epsilon \rightarrow 2k\epsilon > 0$ and again used contour integration, now in the complex-$k$ plane. More generally,

$$I_1(k; \Lambda) = \frac{m}{4\pi} \left( \theta_1 \Lambda + i k + \theta_{-1} \frac{k^2}{\Lambda} + \ldots \right),$$  

where $\theta_{1-n}$, $n = 0, 1, \ldots$, are numbers that depend on the specific form of the regulator [47].

**Exercise:** Show that $\theta_{1-n} = 0$ in dimensional regularization with minimal subtraction.

While a subtraction of the pole in two spatial dimensions (“power divergence subtraction”, or PDS) [48, 49] retains the linear divergence, subtracting instead all poles leads to a result identical to that of a momentum-cutoff regulator [50].

We then arrive at

$$T_2^{(0,1)}(k; \Lambda) = \frac{4\pi}{m} C_0^{(0)}(\Lambda) \left( \theta_1 \Lambda + i k + \theta_{-1} \frac{k^2}{\Lambda} + \ldots \right).$$  

(23)

As we will show shortly, the regulator-dependent terms can be eliminated. As in any EFT, the meaningful loop term is the non-analytic (in energy) $ik$, the “unitarity term”. Relative to the tree-level [18], it is $O(C_0 Q)$. This is to be contrasted with the analogous situation in ChPT [6], where the one-loop diagrams are down with respect to the tree by $Q^2/(4\pi f)^2$ [5, 51]. The difference is that, whereas the relativistic loop gives $Q^2/(4\pi)^2$, the nonrelativistic loop gives $mQ/(4\pi)$. There are two enhancements for heavy particle processes: an infrared enhancement by $m$ [21, 3] and an “angular” enhancement of $4\pi$ [52, 48, 49, 47]. We can think of $mQ/(4\pi)$ as arising from the simple rules:

- heavy particle propagator $\rightarrow m/Q^2$,
- loop integral $\rightarrow Q^5/(4\pi m)$,
- derivative $\rightarrow Q$.

(24)

If we make the dimensional guess $C_0^{(0)} = O(M_{hi}^{-1})$, we see that the loop is suppressed by one order of the expansion parameter, $O(Q/M_{hi})$. The theory is perturbative and any pole of $T_2$ can only have binding momentum of $O(M_{hi})$ or higher, which is outside the EFT. In order to accommodate a shallow pole with $Q_2 = O(M_{lo})$, we must assume instead that $C_0^{(0)} = O(M_{lo}^{-1})$. I will discuss this assumption in Sec. 5, for now, let us see what it implies.

Under this assumption the one-loop diagram is $O(Q/M_{lo})$ compared to the tree diagram. It is not difficult to see that the $n$-loop diagram $T_2^{(0,n)}$ is proportional to the $n$th power of one-loop diagram and its magnitude is $O(Q^n/M_{lo}^n)$ compared to the tree. For $Q \gtrsim M_{lo}$, the whole geometric series must be resummed,

$$T_2^{(0)}(k; \Lambda) \equiv \sum_{n=0}^{\infty} T_2^{(0,n)}(k; \Lambda) = - \frac{4\pi}{m} \left( \frac{1}{C_0^{(0)}(\Lambda)} + \theta_1 \Lambda + i k + \theta_{-1} \frac{k^2}{\Lambda} + \ldots \right)^{-1}. $$  

(25)

15
Figure 1: Two-body $T$ matrix in Pionless EFT at LO, $T_2^{(0)}$. Solid lines denote particle propagation up, Eq. (17). The dotted vertex stands for the $C_0^{(0)}$ contact interaction, Eq. (18). The first equality represents Eq. (25); the second, Eq. (26).

This LO amplitude is shown in Fig. 1. Even though we derived it by explicitly summing up the individual diagrams, it can be obtained directly from an integral equation, the Lippmann-Schwinger (LS) equation, also shown in Fig. 1. Quite generally, in a way that also applies to other EFTs for heavy particles, we can define the potential $V$ as the sum of diagrams that cannot be split by cutting only heavy particle lines [2, 3]. In Pionless EFT $V$ reduces to the sum of all tree diagrams, but in Chiral EFT it includes also loop diagrams with pions, as we will discuss in Sec. 9. Apart from a sign, the potential gives the $T$ matrix in first Born approximation, one-loop amplitude diagrams represent the second Born approximation, and so on. In our case here, the LO potential is given by Eq. (18) and the LS equation is, after integrating over the 0th component of the loop momentum,

$$T_2^{(0)}(k, \Lambda) = -V_2^{(0)}(\Lambda) - \int \frac{d^3l}{(2\pi)^3} T_2^{(0)}(k, \Lambda) \frac{m}{l^2 - k^2 - i\epsilon} V_2^{(0)}(\Lambda).$$

More generally the $T$ matrix depends on the incoming $\vec{p}$ and outgoing $\vec{p}'$ relative momenta, as well as the energy. The LS equation then involves the “half-off-shell” $T$ matrix, since one of the momenta is integrated over. In contrast, our LS equation [26] with a non-local or “separable” regulator can be solved easily with the Ansatz that $T_2^{(0)}$ depends only on the energy: by taking $T_2^{(0)}(k, \Lambda)$ and $V_2^{(0)}(\Lambda)$ out of the integral in the right-hand side and combining this term with the left-hand side, we obtain Eq. (25) directly from Eqs. (26) and (18). When the regulator is chosen to be “local” or “non-separable” — a function solely of the momentum transfer $\vec{p}' - \vec{p}$, which translates into a function of the spatial coordinate $\vec{r}$ — the loops do not form a simple geometric series and we are usually forced to solve Eq. (26) numerically.

Equation (25) still contains explicit $\Lambda$ dependence which, if not controlled, will lead to regulator dependence in observables. For example, there is a pole in Eq. (25) at imaginary momentum, or equivalently negative energy; if $C_0^{(0)}$ were $\Lambda$ independent, then the corresponding binding energy would be $\propto \Lambda^2/m$. To avoid such a disaster, the $\Lambda$
dependence of $C^{(0)}_0(\Lambda)$ must cancel the term linear in $\Lambda$,

$$\frac{1}{C^{(0)}_0(\Lambda)} = -\theta_1 \Lambda + \frac{1}{C^{(0)}_{0R}} \left( 1 + \frac{\zeta_{-1}}{C^{(0)}_{0R} \Lambda} + \ldots \right),$$

(27)

with $C^{(0)}_{0R}$ — the renormalized LEC — a constant to be determined from experimental data or from matching to the underlying theory, and $\zeta_{-1-2n}$, $n = 0, 1, \ldots$, a set of arbitrary numbers related to the choice of input. In terms of the renormalized LEC the LO amplitude becomes

$$T_2^{(0)}(k; \Lambda) = -\frac{4\pi}{m} \left[ \frac{1}{C^{(0)}_{0R}} + ik + \frac{\zeta_{-1}}{C^{(0)}_{0R} \Lambda} \left( \frac{\zeta_{-1}^2}{C^{(0)}_{0R}^2} + \theta_{-1} k^2 \right) + \ldots \right]^{-1}.$$  

(28)

Now the LO amplitude has a physical pole at $k = i\kappa^{(0)}$, where the binding momentum $\kappa^{(0)} = 1/C^{(0)}_{0R} + \ldots$. This is a bound state if $C^{(0)}_{0R} > 0$ and a virtual state if $C^{(0)}_{0R} < 0$. In either case the binding energy is

$$B_2^{(0)}(\Lambda) = \frac{2}{m C^{(0)}_{0R}^2} \left[ 1 + \frac{2}{C^{(0)}_{0R} \Lambda} (\zeta_{-1} - \theta_{-1}) + \ldots \right].$$

(29)

The renormalized LEC is obtained from one datum. For example, we could use the value of the amplitude at a chosen momentum $k = \mu$: in order to have $T_2^{(0)}(\mu; \Lambda) = T_2^{(0)}(\mu)$ $\Lambda$ independent, we take $\zeta_{-1} = -\theta_{-1} C^{(0)}_{0R} \mu^2$ etc., and the renormalized LEC is fixed by $C^{(0)}_{0R} = -[4\pi/(m T_2(\mu)) + i\mu]^{-1}$. If we choose $\mu = i\sqrt{m B_2}$, then $B_2^{(0)}(\Lambda) = B_2$ is $\Lambda$ independent, but other observables, such as the amplitude at zero momentum, will have residual regulator dependence. For other choices of $\mu$, the binding energy contains nonvanishing $\Lambda^{-1}$ terms. We might as well choose to fit $C^{(0)}_{0R}$ to several low-energy data simultaneously, each with a certain weight, in which case $C^{(0)}_{0R}$ does not necessarily depend on a single fixed momentum $\mu$.

Regardless of the choice of input observable(s), we have eliminated the dangerous regulator dependence: the amplitude (28) satisfies (14), only negative powers of $\Lambda$ appear in observables, and they can be made arbitrarily small by increasing $\Lambda$. The bare parameter $C^{(0)}_0(\Lambda)$ has disappeared as well. Its size does not matter; it is of course the surviving, renormalized LEC that has size $C^{(0)}_{0R} = O(M^{-1}_{lo})$, giving $T_2^{(0)}(k; \Lambda) = O(4\pi/(m M_{lo}))$ and $\kappa^{(0)} = O(M_{lo}) \ll M_{hi}$. Equation (29), for example, is exactly of the generic form (15). Renormalization at LO has been completed: the simple contact interaction is renormalizable at the two-body level.

Now we can describe the physics of the shallow state in a controlled expansion. Let us see how this works at NLO. We expect nonvanishing NLO corrections on the basis that the residual-regulator effects in observables such as Eq. (29) can be as large as $O(M_{lo}/M_{hi})$. Indeed, we can write the LO amplitude as

$$T_2^{(0)}(k; \Lambda) = -\frac{4\pi}{m} \left( \frac{1}{C^{(0)}_{0R}} + ik \right)^{-1} + \delta T_2^{(1)}(k; \Lambda) + \ldots,$$  

(30)
Figure 2: Two-body $T$ matrix in Pionless EFT at NLO, $T_2^{(1)}$. The circled vertex stands for the $C_0^{(1)}$ and $C_2^{(1)}$ contact interactions, Eq. (32).

where

$$\delta T_2^{(1)}(k; \Lambda) \equiv \frac{m}{4\pi\Lambda} \left[ T_2^{(0)}(k; \infty) \right]^2 \left( \frac{\zeta - 1}{C_{0R}} + \theta - 1 k^2 \right) \tag{31}$$

has the size of an NLO term: $\delta T_2^{(1)}(k; M_{hi}) = \mathcal{O}(4\pi/(mM_{hi}))$. The form of this correction suggests that the NLO interaction that removes such a residual regulator dependence is the $C_2$ term in Eq. (11), plus a perturbative correction to $C_0$. At tree level,

$$T_2^{(1,0)}(\vec{p}', \vec{p}; \Lambda) = -\frac{4\pi}{m} \left( C_0^{(1)}(\Lambda) - C_2^{(1)}(\Lambda) \frac{\vec{p}'^2 + \vec{p}^2}{2} \right) = -V_2^{(1)}(\vec{p}', \vec{p}; \Lambda), \tag{32}$$

where I denoted the NLO parts of $C_{0,2}$ by $C_{0,2}^{(1)}(\Lambda)$. We expect $T_2^{(1,0)}(\vec{p}', \vec{p}; M_{hi}) \sim \delta T_2^{(1)}(k; M_{hi})$, which is a perturbative correction to the LO amplitude.

However, adding a loop connecting $V_2^{(1)}$ to $T_2^{(0)}$ through a Schrödinger propagator gives a contribution of the same size. Thus, at NLO we have only one vertex representing NLO interactions, but it is dressed in all possible ways by LO interactions — see Fig. 2. In the language of the LS equation, Eq. (32) is an NLO correction to the potential. The subleading potentials are solved in what is called distorted-wave Born approximation, which differs from the ordinary Born approximation in that the LO is not just a plane wave.

Exercise: Show that when the external particles are on-shell, $|\vec{p}'| = |\vec{p}| = k$, the amplitude in Fig. 2 is

$$T_2^{(1)}(k; \Lambda) = -\frac{m}{4\pi} \left( \frac{T_2^{(0)}(k; \Lambda)}{C_0^{(0)}(\Lambda)} \right)^2 \left[ C_0^{(1)}(\Lambda) - \left( -\frac{4}{\pi} \theta_3 \Lambda^3 C_0^{(0)}(\Lambda) + k^2 \right) C_2^{(1)}(\Lambda) + \ldots \right], \tag{33}$$

where $\theta_3$ is a regulator-dependent number.

Combining Eqs. (33) and (31), we can write the full NLO amplitude as

$$T_2^{(1)}(k; \Lambda) + \delta T_2^{(1)}(k; \Lambda) = -\frac{m}{4\pi} \left( \frac{T_2^{(0)}(k; \infty)}{C_0^{(0)}(\Lambda)} \right)^2 \left[ C_0^{(1)}(\Lambda) - C_2^{(1)}(\Lambda + \ldots \right] \tag{34}$$
if we impose that the NLO LECs be given by

$$\frac{C_{(1)}(0)}{C_{(0)}^{(0)2}(\Lambda)} = \frac{C_{(1)}(0)}{C_{(0)}^{(0)2}} - \frac{\theta_1}{\Lambda} + \ldots, \quad (35)$$

$$\frac{C_{(1)}(0)}{C_{(0)}^{(0)2}(\Lambda)} = -\frac{4}{\pi} \theta_3 \Lambda^3 \frac{C_{(1)}(0)}{C_{(0)}^{(0)2}(\Lambda)} + \frac{C_{(0)}^{(1)}}{C_{(0)}^{(0)2}} + \frac{\zeta_{-1}}{C_{(0)}^{(0)2}} + \ldots, \quad (36)$$

in terms of the renormalized LECs $C_{(1)}^{(0)2} = O((M_{lo}^2 M_{hi})^{-1})$. We have succeeded in renormalizing the NLO amplitude.

There is only one new physical parameter at NLO, $C_{(1)}^{(0)2}$, which is related to the energy dependence of the amplitude. The LEC $C_{(0)}^{(1)}$ is introduced for convenience only, so as to allow us to keep the observable used to fix $C_{(0)}^{(0)}(\Lambda)$ unchanged. For example, if we want to keep the energy-independent part of the amplitude unchanged, we take $C_{(1)}^{(0)} = 0$, but more generally $C_{(0)}^{(1)}$ depends on $C_{(0)}^{(0)}$ and $C_{(1)}^{(0)}$. In any case, the leading regulator dependence from Eq. (30) gets replaced by the physical effect from $C_{(1)}^{(0)2}$. In particular, $\zeta_{-1}$ disappeared, showing that the different choices of input observable at LO are an NLO effect. Similarly, the freedom we have now to fix $C_{(1)}^{(0)2}$ is a higher-order effect.

To see how $C_{(1)}^{(0)2}$ affects the bound or virtual state it is convenient to write the amplitude up to NLO as

$$T_{(1)}^{[1]}(k; \Lambda) = T_{(0)}(k; \Lambda) \left[ 1 - m C_{(1)}^{(0)2} \right] \left( C_{(0)}^{(0)} - C_{(1)}^{(0)2} k^2 \right) + \ldots \right]. \quad (37)$$

Now the binding energy becomes

$$B_{(1)}^{[1]}(\Lambda) = \frac{1}{m C_{(0)}^{(0)2}} \left[ 1 - 2 \left( \frac{C_{(1)}^{(0)}}{C_{(0)}^{(0)}} + \frac{C_{(1)}^{(0)2}}{C_{(0)}^{(0)}} \right) + \ldots \right]. \quad (38)$$

If we choose to fit $B_{(2)}$ at LO and want it unchanged at NLO, then we choose $C_{(0)}^{(1)} = -\frac{C_{(1)}^{(0)2}}{C_{(0)}^{(0)2}}$. If we choose to fit two other observables at NLO — say the energy-independent part of the amplitude and its linear dependence on the energy — the binding energy will have a physical shift $O(M_{lo}/M_{hi})$ instead of the regulator dependence $O(M_{lo}/\Lambda)$ in Eq. (29). The remaining regulator dependence in the “…” is $O((M_{lo}^2)/(\Lambda M_{hi}))$ and no larger than $N^2$LO as long as $\Lambda \gtrsim M_{hi}$: the binding energy is predicted to a better precision.

The procedure can be generalized to higher orders \[47, 53\]. At $N^2$LO, for example, we have to consider two insertions of $C_2$ and one of a four-derivative, four-field operator, both acting on $S$ waves only. At the same order in the two-nucleon case there is also a
tensor operator that mixes $^3S_1$ and $^3D_1$ [48, 49, 53]. The first contributions to higher-wave phase shifts enter at N$^3$LO via two-derivative four-field operators. The simple rule [52, 48, 49, 47, 53] is that under the assumption that $M_{lo}$ does not contaminate other waves — an operator gets an $(M_{hi}/M_{lo})^n$ enhancement over dimensional analysis, where $n = 1, 2$ is the number of $S$ waves it connects. For example, since $C_2$ connects $S$ to $S$ waves, $C^{(1)}_{2R} = O((M_{hi}^2/M_{lo})^{-1})$ instead of $C^{(1)}_{2R} = O(M_{hi}^{-3})$.

### 4.2 Connection to the effective-range expansion

The form of Eq. (37) should be familiar from scattering in quantum mechanics. Recall that $S$ waves should dominate at low energies for their lack of centrifugal barrier, and that the corresponding phase shift $\delta_2(k)$ is defined in terms of the $S$ matrix as $S_2(k) = \exp(2i\delta_2(k))$. Thus,

$$T_2(k) = -\frac{2\pi i}{mk} [S_2(k) - 1] = -\frac{4\pi}{m} [-k \cot \delta_2(k) + ik]^{-1}. \quad (39)$$

At very low energies, the effective-range expansion (ERE) [54] is known to hold,

$$k \cot \delta_2(k) = -1/a_2 + r_2 k^2/2 - P_2^3 k^4/4 + \ldots, \quad (40)$$

where $a_2$, $r_2$, $P_2$, $\ldots$ are real parameters — respectively, the scattering length, effective range, shape parameter, etc., all with dimensions of distance. Comparison with Eq. (37) gives

$$a_2 = C^{(0)}_{0R} + C^{(1)}_{0R} + \ldots = O(M_{lo}^{-1}), \quad r_2 = -2 \frac{C^{(1)}_{2R}}{C^{(0)}_{0R}} + \ldots = O(M_{hi}^{-1}), \quad \ldots \quad (41)$$

The scattering length is just the amplitude at zero energy and fixes $C^{(0)}_{0R}$ at LO if we choose the renormalization scale $\mu = 0$ in the discussion below Eq. (29). The effective range provides a simple, explicit example of the usefulness of a high regulator cutoff. Because we chose a regulator for which we could iterate $C^{(0)}_{0R}$ analytically, we could separate $\delta T^{(1)}_2(k; \Lambda)$ from $T^{(0)}_2(k; \infty)$ in Eq. (30). Since $\Lambda$ is not physical, we might as well take the amplitude at a given order to be its $\Lambda \to \infty$ limit, as in old times. Alas, in a numerical calculation — which is necessary for this EFT with a local regulator that depends on the transferred momentum, or for $A \geq 3$, or for other EFTs — this cannot be done. An LO calculation will have subleading pieces buried in them. In the case considered here, an effective range $-\theta_1 \Lambda^{-1}$ is induced. For the error of the LO calculation not to exceed its truncation error, which is $\pm O(M_{hi}^{-1})$ for $r_2$, we have to take $\Lambda \geq M_{hi}$. Even though the theory is renormalizable at LO, a “Wilsonian cutoff” $\Lambda \lesssim M_{hi}$ would lead, in the absence of an explicit integration of modes in a known underlying theory, to excessive errors. Moreover, the magnitude of the change in the LO amplitude upon variation in $\Lambda$ from $M_{hi}$, when $|\theta_1| \Lambda^{-1} = O(M_{hi}^{-1})$, to much larger values, when $\theta_1 \Lambda^{-1} \simeq 0$, gives an estimate of the actual size of $r_2$ and thus of the LO error.

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Given the direct relation between renormalized LECs and ERE parameters in Eq. (41), the latter are often used as input data in Pionless EFT. Once the LECs present at each order are determined from an equal number of data points, the full phase shifts can be predicted within the truncation error, as long as $\Lambda \gtrsim M_{hi}$. As an example, the phase shifts in the $^3S_1$ two-nucleon system at lowest orders obtained in Ref. [53] are shown in Fig. 3.

You might be disappointed that, after so much work, we found that Pionless EFT in the two-body system is essentially just the ERE, known for 70 years. In fact, one can show [47] that this EFT is also equivalent to even older approaches: short-distance boundary conditions [56, 57] and Fermi’s pseudopotential [58]. The EFT can be seen as a derivation of these older results, once EFT’s general framework is deployed to this particular class of nonrelativistic systems with short-range interactions. As such, it is the proverbial cannon to kill a fly. However, a cannon can kill more. The calculation above can be generalized (numerically) to Chiral EFT, where it provides a guide for much more complicated renormalization. Moreover, Pionless EFT applies to systems which are outside the scope of the ERE. For example, in the nuclear case we can look at the electromagnetic properties of the deuteron [53], thanks to the inclusion of consistent one- and two-nucleon electromagnetic operators. I will instead describe how Pionless EFT allows us to generalize the ERE to more than two particles, where amazing new structures arise.
5 Fine tuning, unitarity, and scale invariance

Before we consider more bodies, a few remarks about the EFT in the two-body system.

One would naively expect all ERE parameters to be comparable to the force range \( R \) and indeed, for most parameter values of most finite-range potentials, that is what one finds, with bound states or resonances that are not particularly shallow. In this case, for \( k \ll R^{-1} \) each term in the ERE is larger than the next and the corresponding EFT is purely perturbative. But by dialing one or more potential parameters one can make \( |a_2| \gg R \) with other ERE parameters still of \( O(R) \). For \( k \ll |a_2|^{-1} \), the EFT is still perturbative but as we have just seen, if we want to continue to describe physics of momentum up to \( R^{-1} \), we have to resum (only!) \( C_0^{(0)} \). By fine tuning we can reach the extreme point, the "unitarity limit" where \( |a_2| \to \infty \) and \( T_2(k) \) takes (up to range corrections) its maximum value \( \propto k^{-1} \).

Exercise: Consider a three-dimensional spherical well with dimensionless depth \( \alpha \),

\[
V_2(\vec{r}) = -\frac{\alpha^2}{mR^2} \theta(R - r),
\]

(42)

Solve the Schrödinger equation for the \( S \) wave in the usual way, \( i.e. \), by matching inside and outside solutions. Obtain \( \delta_2(k) \) from the outside solution (asymptotically \( \psi_2(r) \propto |\sin(kr + \delta_2(k))|/r \)) and show that

\[
T_2(k) = -\frac{2\pi i}{mk} \left[ e^{-2ikR} \frac{\sqrt{\alpha^2 + (kR)^2 \cot \alpha}}{\sqrt{\alpha^2 + (kR)^2 \cot \alpha + (kR)^2 - i k R}} - 1 \right],
\]

(43)

and thus

\[
a_2 = R \left( 1 - \frac{\tan \alpha}{\alpha} \right), \quad r_2 = R \left( 1 - \frac{R}{\alpha a_2} - \frac{R^2}{3 a_2^2} \right), \quad \ldots
\]

(44)

For generic values of \( \alpha \), we see that \( a_2 \sim r_2 \sim \ldots \sim R \). However, when \( \alpha \simeq (2n+1)\pi/2 \equiv \alpha_c \), while still \( r_2 \sim \ldots \sim R \), \( a_2 \simeq R/|\alpha_c| \alpha - \alpha_c | \gg R \). For \( \alpha \) just below \( \alpha_c \), there is a shallow virtual state; as the attraction increases past \( \alpha_c \), a shallow bound state appears. In this example, \( M_{hi} \sim 1/R \) and \( M_{lo} \sim \alpha_c | \alpha - \alpha_c |/R \).

As we saw in the previous section, we incorporate such fine tuning in the EFT by allowing the LECs to scale with the small \( M_{lo} \). Although I used the methods of quantum field theory (the proverbial cannon), we can also use standard quantum mechanics, supplemented by regularization and renormalization. The LO interaction at tree level, Eq. (18), is just a constant in momentum space, which in coordinate space is a delta function. The appearance of the Schrödinger propagator at one loop, Eq. (19), reveals that the iteration of this interaction is equivalent to solving the Schrödinger equation with a delta-function potential. In fact, one can show on general grounds that the LS equation is equivalent to the Schrödinger equation. Even if we prefer to solve the Schrödinger equation, regularization and renormalization are still necessary. For example, the LO potential in coordinate space can be written as

\[
V_2^{(0)}(\vec{r}; \Lambda) = \frac{4\pi}{m} C_0^{(0)}(\Lambda) \delta^{(3)}(\vec{r}),
\]

(45)
where $\delta^{(3)}_{\Lambda}(\vec{r})$ is a regularization of the three-dimensional delta function, that is, a smearing over distances $r \lesssim \Lambda^{-1}$ with $\lim_{\Lambda \to \infty} \delta^{(3)}_{\Lambda}(\vec{r}) = \delta^{(3)}(\vec{r})$. Whatever the underlying potential is, we can use, say, the spherical well as such regularization. In this guise, the range of the well functions as a regulator parameter, $\Lambda \equiv 1/R$, and $\alpha = \alpha(R)$ can be adjusted so that $a_2$ in Eq. (44) is $R$ independent and reproduces a given experimental value. Another popular regularization for its analytical simplicity is a delta-shell potential. For many-body calculations a Gaussian regularization is particularly convenient for its smoothness.

**Exercise:** Solve the Schrödinger equation for the potential (45). *Hint:* Fourier transform to momentum space and choose a sharp momentum regulator. Show that the LO results of the previous section are reproduced and, in addition, the negative-energy wavefunction is

$$\psi^{(0)}_2(r) \propto \frac{1}{r} \exp(-r/a_2)$$  \hspace{1cm} (46)

as $\Lambda \to \infty$. Thus $a_2$, which is basically the scattering amplitude at $k = 0$, is a measure of the size of the system. A real bound (virtual) state has a (non-)normalizable wavefunction and corresponds to $a_2 > 0$ ($a_2 < 0$).

The bare LECs change with the regulator parameter $\Lambda$. At LO, renormalization requires Eq. (27). This is the solution of the RG equation

$$\mu \frac{d}{d\mu} \left( \mu C^{(0)}_0(\mu) \right) = \mu C^{(0)}_0(\mu) \left( 1 + \mu C^{(0)}_0(\mu) \right),$$  \hspace{1cm} (47)

with $\mu \equiv \theta_1 \Lambda$. This equation admits two fixed points: a “trivial” $\mu C^{(0)}_0(\mu) = 0$ and the “non-trivial” $\mu C^{(0)}_0(\mu) = -1 \ [3]$. When $\mu \ll |a_2|^{-1}$, we are near the trivial point where perturbation theory holds. On the other hand, for $\mu \gg |a_2|^{-1},$

$$C^{(0)}_0(\Lambda; C^{(0)}_{0R}) = -\frac{1}{\theta_1 \Lambda} \left[ 1 + \frac{1}{\theta_1 \Lambda C^{(0)}_{0R}} + \mathcal{O} \left( \frac{1}{\Lambda^2 C^{(0)}_{0R}} \right) \right],$$  \hspace{1cm} (48)

the flow is close to the non-trivial fixed point, and all diagrams containing only this vertex should be resummed. The unitarity limit corresponds to the non-trivial fixed point,

$$C^{(0)}_0(\Lambda; \infty) = -\frac{1}{\theta_1 \Lambda}. \hspace{1cm} (49)$$

The fine tuning needed to produce the unitarity limit can be carried out experimentally for cold atoms through the mechanism of Feshbach resonances [59] — this was one of the reasons for the explosion of interest in these systems. The mechanism works when the system has two coupled channels with different spins and thresholds — “open” and “closed” channels — and the relative position of a bound state in the closed channel can be changed by an external magnetic field. In the open channel, the scattering length varies and diverges as the energy of the bound state crosses the open threshold. A short-range EFT for this situation is discussed in Ref. [60]. In contrast, the $^4$He dimer just happens to be close to the unitarity limit even in the absence of a magnetic field. The scattering length
and effective range calculated with the LM2M2 potential are \[ a_2 \approx 100 \text{ Å} \approx 18 l_{vdW} \] and \( r_2 \approx 7.3 \text{ Å} \approx 1.3 l_{vdW} \), with similar values for other sophisticated potentials — even though, of course, \( a_2 \) is very sensitive to potential details because of fine tuning.

Nucleons are not as close to unitarity: for np in the \( ^3S_1 \) (deuteron) channel, \( a_{2,S=1} \approx 5.4 \text{ fm} \approx 3.8 m_\pi^{-1} \) and \( r_{2,S=1} \approx 1.8 \text{ fm} \approx 1.3 m_\pi^{-1} \). However, in the \( ^1S_0 \) channel, where there is a shallow virtual state, the relative magnitudes of np parameters are not very different from atomic \(^4\text{He} \), \( a_{2,S=0} \approx -23.7 \text{ fm} \approx -17 m_\pi^{-1} \) and \( r_{2,S=0} \approx 2.7 \text{ fm} \approx 1.9 m_\pi^{-1} \). In QCD, the only free parameters are the quark masses, and we can imagine alternative worlds where the interactions are fundamentally unchanged but explicit chiral-symmetry breaking is larger and the range of the pion-exchange force is, consequently, smaller. As the quark masses change, so do nuclear binding energies. Because heavier quarks are easier to evolve in imaginary time in a four-dimensional space-time lattice, Lattice QCD has provided so far only “alternative facts” about light nuclei [7]. The situation is still in flux, with different methods of signal extraction leading to contradictory results, but in the majority of calculations it seems that nuclei at larger \( \tilde{m} \) are more bound versions of their counterparts in our world [62]. At large quark masses, where there is no Chiral EFT, Pionless EFT offers the only viable description of these nuclei. One can take few-body observables calculated in Lattice QCD as input to Pionless EFT, thus bypassing the need for experimental data, and use Pionless EFT to calculate the structure of heavier nuclei [8, 9]. A possible scenario [63] for quark-mass variation is one in which the deuteron and the \( ^1S_0 \) virtual state become, respectively, unbound below and bound above, but near, the physical point. If this is the case, then the mechanism of fine tuning in QCD is parallel to that of Feshbach resonances for atoms, with the magnetic field replaced by the quark masses.

Even when \(|a_2| \gg R\) is finite, as for nucleons at physical quark masses and \(^4\text{He} \) atoms, the unitarity limit is useful: in the “unitarity window” \(|a_2|^{-1} \ll k \ll R^{-1} \), \( T_2(k) \) is close to the maximum value \( \propto k^{-1} \):

\[
T_2(|a_2|^{-1} \ll k \ll R^{-1}) = \frac{4\pi}{m} (ik)^{-1} \left[ 1 + O(k M_{hi}^{-1}, k \Lambda^{-1}, (ka_2)^{-1}) \right].
\]

When we retain only the first term, there is no dimensionful parameter other than \( k \) itself. The vanishing of the binding energy then is a reflection of scale invariance. Under a change of scales [64] with parameter \( \alpha > 0 \),

\[
r \to \alpha r, \quad t/m \to \alpha^2 t/m, \quad \Lambda \to \alpha^{-1} \Lambda, \quad \psi \to \alpha^{-3/2} \psi,
\]

the first two terms in Eq. (11) — the nucleon bilinear and the \( C^{(0)}_0 \) contact interaction — are invariant on account of Eq. (49). Under a scale change, \( mE \to \alpha^{-2} mE \), but in the unitarity limit there is no scale, so \( B_2 = 0 \). In this limit the \( A = 2 \) system is also conformally invariant [65].

Away from the unitarity limit, scale symmetry is explicitly broken by the dimensionful parameter \( C^{(0)}_{0R} = a_2 \) in Eq. (48). At subleading orders scale symmetry is also broken by \( M_{hi} \) in the form of the higher ERE parameters. The dependence of \( B_2 \) on dimensionful parameters may be determined with the “spurion field” method [66], which is
designed to exploit the consequences of an approximate symmetry. The idea is that if under scale invariance these parameters changed according to their canonical dimension, then the system would remain invariant. For example, if $a_2$ changed to $\alpha a_2$, then the first two terms in Eq. (11) would still be invariant. In that case, the energy after the transformation should equal the transformed energy: $B_2(\alpha a_2) = \alpha^{-2} B_2(a_2)$. This implies $B_2(a_2) \propto (ma_2^2)^{-1}$, see Eq. (29). Now, since $a_2$ is actually fixed, $B_2(a_2)$ reflects the specific way in which $a_2$ breaks scale invariance. In this particular case the spurion method is just dimensional analysis, since by allowing $a_2$ to vary we are changing all dimensionful quantities appearing to this order according to their (inverse mass) dimension. And, of course, this relation was obtained earlier directly from Eq. (28), but the spurion method illustrates how considerations of symmetry underlie dynamical results.

The message is that we are dealing with fine-tuned systems, where for $A = 2$ we are close to the non-trivial fixed point associated with unitarity and scale invariance. There are significant departures from naive dimensional analysis, but renormalization provides a useful guide to infer the corresponding enhancements.

6 Three-body system

Whatever the reason for fine tuning, one can ask what structures it produces. The first surprise comes when $A = 3$.

Many-body forces are not forbidden by any symmetry, and yet we are used to think of them as small. That is a consequence of their high dimensionality. From dimensional analysis we might expect them to be highly suppressed, e.g. $D_0 = O(M_\text{hi}^{-4})$. If this is the case, the properties of many-body systems are determined, to a very good approximation, by two-body interactions. The simplest connected diagram for three particles consists of an LO interaction between two particles (say 1 and 2), followed by propagation of one particle (say 2), and its interaction with the third particle. In the next simplest diagram, either particle 2 or 3 further propagates till it interacts with particle 1, giving rise to a loop. Using the power-counting rules (24), the expected size of the latter diagram relative to the former is

$$
O \left( \left(4\pi C_0^{(0)}/m\right)^3 (Q_3^2/(4\pi m)) (m/Q_3^2)^3 \right) = O(C_0^{(0)} Q_3). \quad (52)
$$

This counting extends straightforwardly to diagrams with more loops. For $Q_3 \gtrsim C_0^{(0)-1} = O(M_\text{hi})$, the arguments of Sec. 4 apply to any of the three two-body subsystems, meaning the LO interactions $C_0^{(0)}$ must resummed into the LO two-body $T$ matrix $T_2^{(0)}$. Subleading corrections are treated perturbatively. This argument applies also to the scattering of one particle on a two-body bound state, such as nd scattering or particle-dimer scattering in the atomic lingo. From the corresponding three-body $T$ matrix, $T_3$, one can find the three-body bound states. The issue now is, is $T_3$ properly renormalized up to $N^4$LO, when we naively expect the appearance of the first three-body force?
6.1 Auxiliary field

In the systems we want to describe, where there are shallow \( T \)-matrix poles, it is often times convenient to introduce auxiliary fields with the quantum numbers of these poles. They can be thought of as “composite” fields for the corresponding states, which are not essential but do simplify the description of the larger systems, particularly reactions involving the bound state.

Most useful is a field for the dimer, the “dimeron”, which I denote by \( d \), with the quantum number of the \( A = 2 \) pole — first introduced in this context in Ref. [67]. From the evolution of this field, whose mass is defined as \( 2m - \Delta \), \( 2m \) is removed by a field redefinition. The corresponding action is obtained by replacing the Lagrangian \( \mathcal{L} \) in Eq. (11) by

\[
\mathcal{L}_d = \psi^\dagger \left( i \frac{\partial}{\partial t} + \frac{\vec{\nabla}^2}{2m} + \ldots \right) \psi + d^\dagger \left[ \Delta + \sigma \left( i \frac{\partial}{\partial t} + \frac{\vec{\nabla}^2}{4m} + \ldots \right) \right] d
\]

\[
- \frac{g_0}{\sqrt{2}} (d^\dagger \psi \psi + \psi^\dagger \psi^\dagger d) - h_0 d^\dagger d \psi^\dagger \psi + \ldots ,
\]

where \( \sigma = \pm 1 \) and \( g_0, h_0, \ldots \) are LECs. In particle-dimer scattering with a relative momentum \( Q \lesssim Q_2 \), when the dimer cannot be broken up, a lower-energy, Halo EFT can be constructed where \( d \) is an “elementary” field. For that, take \( g_0 = 0 \) and \( \sigma = +1 \), with \( h_0 \) being the leading contact interaction between particle and dimer analogous to \( C_0 \) in Eq. (11). In the case we are interested in here, \( Q \gtrsim Q_2 \), the coupling \( g_0 \neq 0 \) to two particles ensures the composite nature of the dimeron field. The particle-particle interaction proceeds through the dimeron propagator, and \( h_0 \) represents a three-body force. Integrating out the \( d \) field brings back Eq. (11).

The power counting of Sec. 4 is reproduced if \( \Delta = \mathcal{O}(M_{lo}) \) and \( g_0 = \mathcal{O}(\sqrt{4\pi/m}) \). In this case the kinetic terms of the dimeron are subleading. The full dimeron propagator at LO is the sum of bubbles,

\[
iD_2^{(0)}(p^0, \vec{p}; \Lambda) = \frac{i}{\Delta^{(0)} + i\epsilon} \left[ 1 - \frac{g_0^2}{\Delta^{(0)} + i\epsilon} I_1(k; \Lambda) + \left( \frac{g_0^2}{\Delta^{(0)} + i\epsilon} I_1(k; \Lambda) \right)^2 + \ldots \right]
\]

\[
= \frac{i}{m g_0^2} \left[ \frac{4\pi}{m} \left( \frac{\Delta^{(0)}}{g_0^2} + I_1(k; \Lambda) \right) + i\epsilon \right]^{-1} ,
\]

and the NLO correction is

\[
iD_2^{(1)}(p^0, \vec{p}; \Lambda) = i \left( \Delta^{(1)} + \frac{\sigma^{(1)} k^2}{m} \right) \left( iD_2^{(0)}(p^0, \vec{p}; \Lambda) \right)^2 ,
\]

where in these expressions \( k \equiv \sqrt{m p^0 - \vec{p}^2} + i\epsilon \). The dimeron can be thought to represent the bound- or virtual-state propagator once we multiply numerator and denominator by \( \Delta^{(0)}/g_0^2 - I_1(k; \Lambda) \) in order to remove the square root of the energy from the denominator. Expanding around the pole one can obtain the residue, that is, the wavefunction
renormalization,
\[
Z_2^{-1} = \frac{\partial}{\partial p^0} \left( D_2(p^0, \vec{p}) \right)^{-1} \bigg|_{p_0 = -B_2}.
\] (56)

Attaching two external particle legs to the dimeron propagator multiplies it by \(-g_0^2\) and gives \(iT_2\). Equation (54) shows that there is only one parameter, \(\Delta^{(0)} / g_0^2\), at LO. This redundancy is frequently eliminated with a redefinition of the auxiliary field to make \(g_0 \equiv \sqrt{4\pi/m}\), (57) which elevates \(\sigma\) to a full-blown LEC \(\sigma(\Lambda)\) rather than a just sign. With this choice and the renormalization scale \(\mu = 0\) for simplicity, the \(T\) matrix has the forms (30) and (34) with
\[
\Delta_R^{(0)} \equiv \Delta^{(0)}(\Lambda) + \theta_1 \Lambda = \frac{1}{C_0^{(0)}} = \frac{1}{a_2}, \quad \Delta_R^{(1)} \equiv \Delta^{(1)}(\Lambda) = 0,
\] (58)
and
\[
\sigma_R^{(1)} \equiv \sigma^{(1)}(\Lambda) + \theta_2 \frac{m}{\Lambda} = \frac{mC_R^{(1)}}{C_0^{(0)}} = -\frac{m r_2}{2}.
\] (59)
Note that \(r_2 > 0\), as in most situations, requires \(\sigma < 0\), that is, \(d\) is a ghost field — and yet the two-body amplitude is perfectly fine. Renormalization is different than before, however, because the dimeron induces energy-dependent corrections instead of the momentum-dependent \(C_2\) corrections. Thus, no \(\Lambda^3\) divergence appears at NLO. In this case one can resum the NLO corrections without running into problems with the RG [47], contrary to the case of momentum-dependent corrections [69].

6.2 Amplitude

In terms of the auxiliary field, we can represent the scattering of a particle on a dimer at LO through the “one-particle” exchange diagrams shown in Fig. 4. The whole series of “pinball” diagrams with multiple such exchanges needs to be included on account of Eq. (52), giving rise to an integral equation known as the Skorniakov–Ter-Martirosian equation. In these diagrams the dimeron propagator is the LO propagator (54). At NLO, one includes one insertion of the NLO propagator (55) in all possible ways, and analogously for higher orders.

Let us work again in the center-of-mass frame, where at LO the incoming (outgoing) dimer has energy \(k^2/(4m) - B_2^{(0)} (k^2/(4m) - B_2^{(0)} + E')\) and momentum \(\vec{p} (\vec{p}')\), and the incoming (outgoing) particle has energy \(k^2/(2m) (k^2/(2m) - E')\) and momentum \(-\vec{p} (-\vec{p}')\). The total energy is \(E = 3k^2/(4m) - B_2^{(0)}\). We can take the initial state to be on-shell, \(|\vec{p}| = k\). For simplicity, we take the \(\Lambda \to \infty\) limit in the dimeron propagator. The integration over the 0th component of the loop momentum is similar to the one done in Sec. 4.1: we pick a pole from, say, the particle propagator, and are left with a three-dimensional integral involving the dimeron propagator. At this point we can set
Figure 4: Three-body $T$ matrix in Pionless EFT at LO, $T_3^{(0)}$. The filled double line represents the full dimeron propagator (54), when internal, or the corresponding wavefunction renormalization (56), when external. The two-particle–dimeron vertex stands for the $g_0$ interaction, Eq. (57), while the particle-dimeron contact is the three-body force $h_0^{(0)}$, Eq. (68).

$E' = (k^2 - \vec{p}'^2)/(2m)$, which holds when the final state is on-shell. With the choice (57), we find for the half-off-shell amputated amplitude

$$t_3(p', \vec{p}) = -v_3(p', \vec{p}) - \lambda \int \frac{d^3l}{(2\pi)^3} \frac{t_3(p', \vec{l}) v_3(\vec{l}, \vec{p})}{-1/a_2 + \sqrt{3l^2/4 - mE}}$$

(60)

where

$$v_3(p', \vec{p}) = \frac{8\pi}{mE - \vec{p}'^2 - \vec{p}^2 - \vec{p}' \cdot \vec{p}}$$

(61)

The above equation with $\lambda = 1$ is derived from the EFT for bosons [70, 22]. For three nucleons with total spin $S = 3/2$, the equation takes the same form but with $\lambda = -1/2$ [71, 72]. Instead, when $S = 1/2$ one finds a pair of coupled integral equations. In the ultraviolet (UV) limit where scattering length and binding energy can be discarded, these equations decouple [15] into a pair of equations like Eq. (60), one with $\lambda = 1$, the other with $\lambda = -1/2$. Thus, even though I consider here the single equation (60), the lessons learned from different values of $\lambda$ can be applied to nucleons in the triton channel as well.

Now, for simplicity we focus on the most important, $S$ wave. We can project on it by integrating over the angle between $\vec{p}'$ and $\vec{p}$. Performing the integration also over the angles in the loop integral, the equation simplifies to

$$t_{3,0}(p', k) = -v_{3,0}(p', k) - \frac{\lambda}{2\pi^2} \int_0^\Lambda dl l^2 \frac{t_{3,0}(p', l) v_{3,0}(l, k)}{-1/a_2 + \sqrt{3l^2/4 - mE}}$$

(62)
where I took a sharp (three-body) cutoff for definiteness and

\[ v_{3,0}(p', k) = \frac{4\pi}{p'k} \ln \left( \frac{p'^2 - p'k + k^2 - mE}{p'^2 + p'k + k^2 - mE} \right). \]  

(63)

The on-shell scattering amplitude is obtained by making \( p' = k \) and accounting for wave-function renormalization,

\[ T_{3,0}(k) = \sqrt{Z_{(0)}^2} t_{3,0}(k, k) \sqrt{Z_{(0)}^2}. \]  

(64)

As in the two-body case, the first step to solve the integral equation is to look at the UV region, \( p' \gg k \gtrsim 1/a_2 \), where the equation reduces to

\[ t_{3,0}(p' \gg k) = \frac{4\lambda}{\sqrt{3}\pi} \int_0^{\Lambda} \frac{dl}{p'} \ln \left( \frac{p'^2 + p'l + l^2}{p'^2 - p'l + l^2} \right) t_{3,0}(l \gg k). \]  

(65)

This equation is homogeneous so it cannot fix the overall normalization of \( t_{3,0}(p' \gg k) \), but it does determine the dependence on \( p' \) in the region \( \Lambda > p' \gg k \). Scale invariance [51] suggests the Ansatz \( t_{3,0}(p' \gg k) \propto p'^{-s+1} \), which works if \( s \) obeys

\[ \frac{8\lambda}{\sqrt{3}s \cos(\pi s/2)} = 1. \]  

(66)

This relation is analyzed in detail in Ref. [73]. Because of the additional inversion symmetry \( p' \rightarrow 1/p' \), the roots come in pairs. For \(-1/2 \leq \lambda \leq \lambda_c \equiv 3\sqrt{3}/(4\pi) \simeq 0.4135 \), the roots are real. The root with \( \Re(s) > -1 \) ensures that \( t_{3,0}(p' \gg k) \) goes to zero, in which case the amplitude has no essential sensitivity to the regulator and predictions about the three-body system can be made at LO. In particular, for three nucleons with \( S = 3/2 \), when \( \lambda = -1/2 \), one finds that \( t_{3,0}(p' \gg k) \propto p'^{-3.17} \), which is softer than the \( p'^{-2} \) behavior expected in perturbation theory from \( v_{3,0}(p' \gg k) \propto p'^{-2} \). The numerical solution of Eq. (62) gives a low-energy amplitude in good agreement with phenomenology, which improves at subleading orders [71, 72]. Because of the good UV behavior of the LO amplitude, one can resum higher-order terms to make calculations easier without jeopardizing RG invariance. As an example, the \( S = 3/2 \) ad scattering length \( a_{nd,S=3/2} = 5.09 + 0.89 + 0.35 + \ldots \) fm = 6.33 ± 0.05 fm [71], to be compared with the experimental value 6.35 ± 0.02 fm [74].

In contrast, for other \( \lambda \) values the solutions are complex, and for \( \lambda > \lambda_c \equiv 3\sqrt{3}/(4\pi) \) the roots are imaginary. In particular, for the bosonic case \( \lambda = 1 \) there is a pair of imaginary solutions \( s = \pm is_0 \), with \( s_0 \simeq 1.00624 \). The two solutions are equally acceptable (or actually unacceptable...) and lead to an asymptotic behavior of the half-off-shell amplitude of the form

\[ t_{3,0}(p' \gg k) \propto \cos \left( s_0 \ln(p'/\Lambda) + \delta \right), \]  

(67)

where \( \delta \) is a dimensionless, \( p' \)-independent number. A numerical solution of Eq. (62) confirms this oscillatory behavior with \( \delta = 0.76 \pm 0.01 [70, 22] \). Small changes in \( \Lambda \)
propagate to lower momenta and lead to dramatic changes in the observable \(t_{3,0}(k,k)\), but the changes are periodic. One can show that this solution, first found in Ref. [75], supports a sequence of bound states that appear with the same periodicity as \(\Lambda\) increases, with the binding energy of each state growing as \(\Lambda^2/m\) [76]. This solution is obviously unacceptable: the first two terms in Eq. (11) are not renormalizable beyond \(A = 2\).

How can we maintain RG invariance? The only possibility is a three-body force, and the one provided by \(h_0\) is the simplest. For this fix to work, this force has to be assumed to be LO, but even then it is not obvious that it can remove the regulator dependence when iterated. Upon including

\[
h_0^{(0)}(\Lambda) \equiv 8\pi \frac{H(\Lambda)}{\Lambda^2},
\]

where \(H(\Lambda)\) is dimensionless, we have additional diagrams, also shown in Fig. 4. The LO amplitudes \(t_3^{(0)}(p',p)\) and \(t_{3,0}^{(0)}(p',k)\) still satisfy Eqs. (60) and (62) but with, respectively,

\[
v_3(p',p) \rightarrow v_3(p',p) + h_0^{(0)}(\Lambda) \equiv v_3^{(0)}(p',p;\Lambda),
\]

\[
v_{3,0}(p',k) \rightarrow v_{3,0}(p',k) + h_0^{(0)}(\Lambda) \equiv v_{3,0}^{(0)}(p',k;\Lambda).
\]

The asymptotic equation (65) now becomes, for the physically relevant \(\lambda = 1\),

\[
t_{3,0}^{(0)}(p' \gg k) = \frac{4}{\sqrt{3\pi}} \int_0^{\Lambda} \frac{dl}{p'} \left[ \ln \left( \frac{p'^2 + p' l + l^2}{p'^2 - p' l + l^2} \right) - 2 \frac{p'l}{\Lambda^2} H(\Lambda) \right] t_{3,0}^{(0)}(l \gg k).
\]

Only for \(p' \sim \Lambda\) is the three-body force important. In the region \(p' \ll \Lambda\), the behavior (67) still holds, but now \(\delta\) is determined by \(H(\Lambda)\). We can define the physical, dimensionful parameter \(\Lambda_*\) through

\[
\delta(H(\Lambda)) = s_0 \ln(\Lambda/\Lambda_*).
\]

Since

\[
t_{3,0}^{(0)}(p' \gg k) \propto \cos(s_0 \ln(p'/\Lambda_*))
\]

is now essentially cutoff independent, so will the low-energy on-shell amplitude. \(\Lambda_*\) can then be determined from low-energy data or matching to the underlying theory. Again, numerical experimentation shows [70, 22] that this can indeed be achieved. One can also show that bound states now accrete periodically from below (that is, from very large binding energies) as the regulator cutoff becomes large enough to accommodate them. As \(\Lambda\) increases their binding energies approach constants. With the addition of the three-body force, the EFT is renormalizable at LO for \(A = 3\).

An approximate form can be obtained for \(H(\Lambda)\) by going back to Eq. (62) for two values of the regulator cutoff, \(\Lambda\) and \(\Lambda' > \Lambda\). Imposing that the two equations agree in the region \(\Lambda > p \gg k\), and making the approximation (73) for the \(\Lambda'\) solution also when \(p' \sim \Lambda'\), one finds [70, 22]

\[
H(\Lambda) \simeq c \frac{\sin(s_0 \ln(\Lambda/\Lambda_*) - \tan^{-1}(s_0^{-1}))}{\sin(s_0 \ln(\Lambda/\Lambda_*) + \tan^{-1}(s_0^{-1}))},
\]

30
Figure 5: Dimensionless three-body force $H$ in Pionless EFT at LO, Eq. (68), as a function of the regulator cutoff $\Lambda$ (in units of $a_2^{-1}$): numerical solution for $a_3 = 1.56 a_2$, dots; and Eq. (74) with $\Lambda^\ast = 19.5 a_2^{-1}$, solid line. Reprinted from Ref. [22] with permission from Elsevier.

where $c \approx 1$. $H(\Lambda)$ can also be extracted purely numerically by demanding that one low-energy datum (for example, the particle-dimer scattering length $a_3$) be reproduced at any value of $\Lambda$. The agreement between approximate and numerical results is shown in Fig. 5. The best fit gives $c = 0.879$ [77].

At tree level, the $h_0$ particle-dimeron interaction generates the three-particle force

$$D_0^{(0)}(\Lambda) = \frac{3h_0^{(0)}(\Lambda)}{4\pi\Delta^{(0)2}(\Lambda)} \propto \frac{1}{\Lambda^4} \frac{\sin(s_0 \ln(\Lambda/\Lambda^\ast) - \tan^{-1}(s_0^{-1}))}{\sin(s_0 \ln(\Lambda/\Lambda^\ast) + \tan^{-1}(s_0^{-1}))} \left[ 1 + \mathcal{O}\left((C_0 R \Lambda)^{-1}\right) \right]. \quad (75)$$

In coordinate space, the corresponding potential is

$$V_3^{(0)}(\vec{r}_{12}, \vec{r}_{23}; \Lambda) = \frac{(4\pi)^2}{m} D_0^{(0)}(\Lambda) \delta^{(3)}_\Lambda(\vec{r}_{12}) \delta^{(3)}_\Lambda(\vec{r}_{23}). \quad (76)$$

where $\vec{r}_{ij}$ is the position of particle $i$ with respect to particle $j$.

The argument above applies directly to bosons and indirectly to nucleons with $S = 1/2$. Therefore a three-nucleon force is needed for RG invariance [15], consistently with the fact that the $D_0$ force has a non-vanishing projection onto the $^2S_{1/2}$ channel. In channels with angular momentum $l > 0$ similar equations are obtained with the logarithm replaced by Legendre polynomials of the second kind [73]. An equation for $s$ analogous to (66) is obtained, involving a hypergeometric function. For both $\lambda = 1$ and $\lambda = -1/2$, $s \approx l + 1$ in fair agreement with the expectation from perturbation theory, which can be shown from the Legendre polynomials to be $t_{3,l}(p' \gg k) \propto p^{-(l+2)}$. There is therefore no need for additional three-body forces at LO.
The perturbative NLO correction that accounts for two-body range effects induces a finite change in the three-body system and does not require an additional three-body force for RG invariance \[22, 78\]; a correction \(D_0^{(1)}(\Lambda)\) is sufficient. A two-derivative three-body force does enter, however, at N\(^2\)LO \[79\]. Thus, while the LO three-body force is enhanced by \((M_{\text{hi}}/M_{\text{lo}})^4\) over simple dimensional analysis, three-body force corrections, which are amenable to perturbation theory, seem to be suppressed by the expected relative factors of \(M_{\text{lo}}/M_{\text{hi}}\).

### 6.3 Bound states and correlations

The EFT produces a series of discrete bound states whose spacing depends on the two-body scattering length. The three-body binding momenta quickly exceed \(M_{\text{hi}}\), so that only a finite number of states \((\sim \ln(|a_2|/R)/\pi\) for an underlying potential of range \(R\) \[80, 15\]) are within the range of applicability of the EFT. For atomic \(^4\)He, for example, both the ground \[81\] and first-excited \[33\] states have been detected, with a ratio of binding energies of about 60, see Table 1. For nucleons only the triton (and helion, separated from triton only by small isospin-breaking effects) is observed, but there is a virtual nd state with a binding energy about 6 times smaller.

Because a single parameter emerges in the three-body force up to NLO, one expects correlations among these binding energies and phase shifts in channels not affected by the exclusion principle. The classic example is the Phillips line \[83\]: a line in the plane spanned by the triton binding energy \(B_t\) and the \(S = 1/2\) nd scattering length \(a_{nd,S=1/2}\). This correlation was first discovered empirically, as a line formed by points representing various phenomenological potentials, which describe two-nucleon data up to relatively high momenta. From the potential-model perspective, this line is a mystery: one would expect results to form an amorphous cluster around the experimental point. From the EFT point of view, instead, this line is indication that these potentials differ by one relevant parameter not determined by two-body physics. As \(\Lambda_\star\) is varied, the LO EFT also produces a line \[22, 15\], which lies close not only to the experimental point but also to the phenomenological line. At NLO the line position changes \[79\], approaching models and experiment, see Fig. 6. Taking the EFT error into account, the line is actually a band. This generalizes an earlier (regulator-dependent) explanation \[84\].

As one would expect \[22\], this property is generic and \(^4\)He potentials also fall on a Phillips line \[85\]. Other correlations can be understood similarly. This means that the various phenomenological potentials, with their many parameters and varied forms, are basically equivalent to the same EFT with different values of \(\Lambda_\star\). For \(A \geq 3\), Pionless EFT is definitely not just the ERE.

The proximity of the EFT Phillips line to the experimental point means that once one datum is used to determine \(\Lambda_\star\) at LO and NLO, other three-body data can be predicted or postdicted in agreement with experiment. For example, if we fit \(a_{nd,S=1/2}\), the triton binding energy \(B_t = 8.0 + 0.8 + \ldots\) MeV \[15, 78\], compared with the measured 8.48 MeV. Alternatively, one can use the experimental value of \(B_t\) as input. Agreement with phase shifts is good already at LO \[15\], improves at NLO \[78\], and improves further still at...
Figure 6: Phillips line in the plane of doublet $nd$ scattering length $a_3 \equiv a_{nd,S=1/2}$ (in fm) and triton binding energy $B_3 \equiv B_t$ (in MeV), in Pionless EFT: LO, (black) dotted line; and NLO, (red) dashed line. The dots represent a variety of nuclear potentials with the same two-nucleon scattering lengths and effective ranges [84]. The cross is the experimental result. Reprinted from Ref. [79] with permission from Elsevier.

N$^2$LO [79], where a second three-body parameter is needed as input. The three-body amplitude and various observables have been calculated fully perturbatively up to N$^2$LO for bosons in Refs. [86, 87] and for nucleons in Ref. [88]. Reviews of the state-of-the-art three-body calculations in Pionless EFT can be found in Refs. [89, 90].

### 7 Limit cycle and discrete scale invariance

The three-body force (75) has a surprising cutoff dependence. $H(\Lambda)$ is the solution of an unusual RG equation,

$$\mu \frac{d}{d\mu} H(\mu) \simeq 2 (1 + H^2(\mu))$$

and is log-periodic, taking the same value for $\Lambda$ and $\alpha_t \Lambda$, with

$$\alpha_t = e^{l\pi/s_0} \simeq (22.7)^l,$$

$l$ an integer. This is an RG limit cycle. The possibility of such a cycle in QCD had been conjectured [91], and the three-body system provided the first example in a field theory [92]. Not many such examples exist — for a short review, see Ref. [93].

This force appears at LO not only for small $|a_2^{-1}|$ but also in the unitarity limit. In this limit there is no $A = 2$ dimensionful parameter at LO, yet renormalization for $A = 3$ forces
on us a dimensionful parameter \( \Lambda \). This is an example of “dimensional transmutation”: the scale invariance present in the unitarity limit is “anomalously” broken. The limit cycle signals a remaining discrete scale invariance (DSI) \([70, 22, 15]\). Because of the characteristic dependence on \( \Lambda \) in Eq. (75), the three-body term in Eq. (11) is invariant under the transformation (51), but only for the discrete values \( \{78\} \). Other examples of the anomalous breaking of scale invariance and of DSI with its characteristic log-periodicity are discussed in Refs. \([94, 95]\), respectively.

The first consequence of the breaking of scale invariance is that \( \Lambda \) offers a dimensionful scale for binding energies. By dimensional analysis, the three-body binding energies can be written as

\[
B_{3;n} = \frac{\kappa^2}{m} \beta_{3;n}((a_2 \kappa_*)^{-1}) = \frac{\kappa^2}{m} \left[ \beta_{3;n}(0) + \frac{\beta'_{3;n}(0)}{a_2 \kappa_*} \right], \quad \beta_{3;0}(0) = 1, \tag{79}
\]

where \( n \geq 0 \) (\( n = 0 \) denoting the deepest state within the EFT), \( \beta_{3;n}((a_2 \kappa_*)^{-1}) \) are universal, dimensionless functions, and \( \beta_{3;n}(0), \beta'_{3;n}(0), \ldots \) are pure numbers arising from an expansion in \((a_2 \kappa_*)^{-1})\). Because \( \Lambda \) is only defined up to a factor \( \exp(n_\Lambda \pi/s_0) \), with \( n_\Lambda \) an integer, it was traded above by a fixed scale \( \kappa_* \) defined from the ground-state energy at unitarity:

\[
\ln(\kappa_*) = \ln(\beta \Lambda) \mod \pi/s_0, \tag{80}
\]

with \( \beta \approx 0.383 [96] \).

DSI manifests itself in the spectrum. The energy of a bound state after a discrete scale transformation should equal the transformed energy but not necessarily of the same level, so that

\[
\beta_{3;n+l}(0) = a_l^{-2} \beta_{3;n}(0) \quad \Leftrightarrow \quad \beta_{3;n}(0) = e^{-2n\pi/s_0} \beta_{3;0}(0). \tag{81}
\]

Thus discrete scale invariance leads to a geometric tower of states extending up to threshold, with successive states having a ratio of binding energies

\[
B_{3;n+1}/B_{3;n} = \exp(-2n\pi/s_0) \approx 1/515. \tag{82}
\]

This amazing structure was first predicted by Efimov [10] and its signals have now been seen in cold-atom systems around Feshbach resonances, see for example Refs. [97, 98].

Away from unitarity DSI is only an approximate symmetry, even at LO. Although the deep spectrum might be little affected, a finite \( a_2^{-1} \) distorts the spectrum in the infrared (IR). Using the spurion field method, the deviation from unitarity due to the two-body scattering length can be followed,

\[
\beta_{3;n+l}((a_l a_2 \kappa_*)^{-1}) = a_l^{-2} \beta_{3;n}((a_2 \kappa_*)^{-1}). \tag{83}
\]

This relation gives information about how Efimov’s tower evolves as \( |a_2^{-1}| \) grows. For example, taking a derivative and expanding in \((a_2 \kappa_*)^{-1}\), we see the leading effect of tower deformation:

\[
\beta'_{3;n+l}(0) = a_l^{-1} \beta'_{3;n}(0) \quad \Leftrightarrow \quad \beta'_{3;n}(0) = e^{-n\pi/s_0} \beta'_{3;0}(0), \tag{84}
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\[
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\]
where \( \beta_{3,0}'(0) \simeq 2.11 \) \cite{96}. Note that here the spurion method is not simply dimensional analysis because \( \Lambda \) is kept fixed. It instead tracks how the two-body scattering length explicitly breaks DSI. Equation (83) is only valid to the extent that the three-body force retains DSI except for \((a_2 \Lambda)^{-1}\) terms — that is, as long as Eq. (75) contains no \( a_2 \Lambda \) dependence, which would require in the spurion method that we scaled \( \Lambda \), as well.

As \( a_2^{-1} > 0 \) grows the two-body bound state moves away from threshold quadratically in \( a_2^{-1} \). Progressively more excited Efimov states fail to have energy below the particle-dimer threshold, disappearing as virtual states \cite{82, 17}. We have the counter-intuitive situation where fewer three-body states survive as the two-body attraction increases. For \( a_2^{-1} < 0 \), three-body bound states exist even though there are no two-body bound states — the system is said to be Borromean in reference to the coat of arms of the Borromeo family, which displays three rings with the property that, when one is removed, the other two are free. The properties of the Efimov spectrum are discussed in detail in Ref. \cite{96}.

NLO corrections from the two-body effective range can be handled similarly. Since Eq. (75) contains no \( r_2 \Lambda \) dependence, the coefficients of linear corrections should scale with \( \alpha \)\cite{99}, as can be easily verified with the spurion method. (However, an explicit calculation \cite{99} indicates that these coefficients vanish.) The generalization to higher orders is obvious.

In the nuclear case, the existence of a single three-body force at LO leads to an additional approximate symmetry, which is exact in the unitarity window: independent rotations in spin and isospin, which form the SU(4)\(_W\) group proposed by Wigner \cite{100} to explain some of the properties of heavier nuclei. Away from unitarity, the symmetry is broken by the difference in inverse scattering lengths between \( ^3S_1 \) and \( ^1S_0 \) channels, in ranges at NLO, \textit{etc}. The approximate SU(4)\(_W\) symmetry of this EFT was elaborated upon in Refs. \cite{101, 102}.

8 More-body systems

Let us summarize the EFT so far. At LO, the action is given by one-body kinetic, two-body \( C_0 \) and three-body \( D_0 \) terms in Eq. (11), and at NLO by the two-body \( C_2 \) term. Other interactions need to be accounted for at \( N^2\text{LO} \) — including another three-body force and, for nucleons, a two-body tensor force — and higher orders. At two-body unitarity, there is a single scale at LO, \( \Lambda \) (or equivalently the \( \kappa \) of Eq. (80)). The crucial issue now is whether higher-body forces appear at LO. If they do, new scales will be introduced in an essential, nonperturbative way. If they do not, all low-energy properties for \( A \geq 4 \) can be predicted at LO, and the issue becomes whether any of these higher-body forces show up at NLO, or at another relatively low order that causes sizable distortions to the LO predictions.

A difficulty we face in answering these questions is the complexity of \( A \geq 4 \) calculations. In perturbative EFTs, where there is no fine tuning to dramatically enhance LO, the size of interactions can be guessed by naive dimensional analysis \cite{51}. This rule is inferred by looking at the regulator dependence of loops in perturbation theory. In our case, we
need instead to look at the regulator dependence of an integral equation at LO and of the
distorted Born approximation in subleading orders, as we have just done for \( A = 3 \).

All \( A \geq 4 \) Pionless EFT calculations that I am aware of are based on the numerical
solution of (some version of) the many-body Schrödinger equation. At LO this is done
with the potential

\[
V^{(0)} = \sum_{\{ij\}} V^{(0)}_2(\vec{r}_{ij}; \Lambda) + \sum_{\{ijk\}} V^{(0)}_3(\vec{r}_{ij}, \vec{r}_{jk}; \Lambda), \tag{85}
\]

where \( \{ij\} \) and \( \{ijk\} \) denote doublets and triplets, respectively, while \( V^{(0)}_2 \) and \( V^{(0)}_3 \) are
given by Eqs. (45) and (76). If a many-body force is missing, many-body observables
will not be renormalized properly — they will fail to converge as the regulator cutoff \( \Lambda \)
increases. If there is one regulator for which lack of convergence is seen, renormalizability
is disproved.

A hand-waving argument suggests that higher-body forces are not needed at LO for
RG invariance. The two-body potential (45) is singular but \( C^{(0)}_0(\Lambda) \) in Eq. (48) has an
overall \( \Lambda^{-1} \) so that the potential scales at high momentum just as the kinetic repulsion.
Smaller terms \( \propto (C^{(0)}_0(\Lambda)^{-1} \) are adjusted to give enough attraction for the two-body state
to be slightly bound, or slightly virtual. When we embed the two-body potential in an
\( A \)-body system, the number \( A(A-1)/2 \) of doublets grows faster than the number \( A - 1 \)
of kinetic terms (one term goes into the kinetic energy of the center of mass), so the
system collapses \[76\]. An effectively repulsive three-body force — Eq. (76), which at
high momentum scales just as the two-body potential thanks to Eq. (75) — is needed to
keep \( A = 3 \) stable. Because the number \( A(A - 1)(A - 2)/6 \) of triplets grows even faster
than doublets, \( A \geq 4 \) systems should not collapse but have instead well-defined limits for
\( \Lambda \gtrsim M_{hi} \).

This argument, simplistic as it is, seems to be borne out by explicit calculations. The
pioneering \( A = 4 \) calculations of Refs. \[23, 103, 104\] have found convergence — at least
in the range of cutoff values examined — in the binding energies of the ground state for
bosons and nucleons, as well as of the first excited state for bosons. This result has been
confirmed several times afterwards with various regulators, for example Refs. \[103, 106, 24, 18, 9\]. Similarly, the ground-state energies of \( A = 6, 16 \) nucleons converge without
many-nucleon forces \[16, 9\]. The four spin-isospin nucleon states require five- and more-
body forces to include derivatives, which should suppress them. This exclusion-principle
suppression is absent for bosons, but calculations of \( A = 5, 6 \) ground-state energies \[24\]
showed no evidence of the need for those forces, either. In fact, binding energies show
just the behavior \[15\] expected from a properly renormalized order. Discussions found in
the literature regarding this issue are summarized in Ref. \[107\].

The absence of LO higher-body forces has fundamental implications for the physics
of \( A \geq 4 \) systems. One is that there are correlations among low-energy many-body
observables through \( \Lambda_\star \), similar to the Phillips line. The simplest example is the Tjon
line \[108\] in the plane of the four- and three-body ground-state energies, \( B_{4;0} \) and \( B_{3;0} \).
As with the Phillips line, this correlation was discovered empirically by plotting results
of phenomenological nuclear potentials. It also exists for \( ^4\text{He} \) atoms \[109\]. It materializes
in EFT as a variation of $\Lambda^*$ at fixed two-body input \[23, 103, 104\]. The EFT line at LO again is close to both the phenomenological line and the experimental point, suggesting the EFT might converge for $A = 4$ as well. There is at least a very large class of potentials that do not seem to have an extra, essential parameter introduced by a four-body force. The absence of higher-body forces at LO further implies the existence of “generalized Tjon lines” in the planes spanned by other ground-state energies, for example \[24\] $B_{5;0}$ or $B_{6;0}$ for $A = 5, 6$, and $B_{3;0}$. Again, such correlations had been discovered earlier in the context of potential models \[110, 111\].

Table 1 summarizes existing results for binding energies of nuclei and $^4\text{He}$ atoms at LO in EFT. For nuclei \[15, 16, 9, 17\], one can see agreement at the level expected from an expansion where one of the parameters is $r_2/a_2 \sim 30\%$ (in the $^3S_1$ channel). Results for atomic $^4\text{He}$ obtained with potential models \[19, 20, 21\] and with LO EFT \[22, 23, 24\] are also given in Table 1. Here the discrepancy is no larger than $\simeq 10\%$ for $A \leq 6$, which is surprising because an estimate of the binding momentum, Eq. (2), suggests that $Q_{\text{vdW}} \sim 1.3$. Perhaps Eq. (2) for $Q_A$ is an overestimate.

Until recently, calculations for $A \geq 4$ that went beyond LO included a resummation of the NLO two-body interaction. Although they show improved results over LO for $A = 4, 16, 40$ nuclei \[105, 112, 113\], they are limited to cutoff values $\Lambda < M_{\text{hi}}$. A test of RG invariance requires a perturbative treatment of subleading corrections, which was carried out for bosons at NLO in Ref. \[114\]. Surprisingly, a four-body force is necessary and sufficient for renormalization of the $A \geq 4$ energies. Once it is fixed to $B_{4;0}, B_{5;0}^{[1]}$ and $B_{6;0}^{[1]}$ come out well, strengthening the case that Pionless EFT converges better than expected. The limit of validity of Pionless EFT remains an open question.

Implications of the absence of other LO forces are even stronger at unitarity, where DSI is expected to hold: except for small corrections, all states within the validity of the EFT, \textit{i.e.}, those states that are insensitive to the details of physics at distances $r \lesssim R$, are fixed by a single parameter $\Lambda^*$. Reference \[104\] discovered that for bosons an $A = 3$ state spawns two $A = 4$ states, one very close to the $A = 3$ threshold and one about four times deeper. According to the accurate calculation of Ref. \[25\], for the two lowest $A = 4$ states at unitarity, $B_{4;0,0}/B_{3;0} \simeq 4.611$ and $B_{4;0,1}/B_{3;0} \simeq 1.002$. These states have been spotted in atomic systems \[115\].

Remarkably, potential-model calculations show that this doubling process continues with increasing number of \textit{bosons} \[116, 117, 118, 119\], so that for a given $A$ there are $2^{(A-3)}$ “interlocking” towers of states. Generalizing Eq. (79),

$$B_{A;n;\{i\}} = \frac{\kappa^*}{m} \beta_{A;n;\{i\}} \left( (a_2k^*)^{-1} \right)^2,$$

by labeling each state with the $A = 3$ ancestor state ($n$) and a set $\{i\} = \{i_1, i_2, ..., i_{A-3}\}$ tracking the doubling, with $i_j = 0 \ (1)$ denoting the lower (higher) state in the $j$-th doubling. Just as before, the dimensionless functions $\beta_{A;n;\{i\}}$ of $(a_2k^*)^{-1}$ reduce at unitarity to pure numbers $\beta_{A;n;\{i\}}(0)$, which obey

$$\beta_{A;n+\ell;\{i\}}(0) = \alpha^{-2}_\ell \beta_{A;n;\{i\}}(0) \leftrightarrow \beta_{A;n;\{i\}}(0) = e^{-2n\pi/s_0} \beta_{A;0;\{i\}}(0).$$

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Again the spurion method gives 
\[ \beta_{A;n+l;\{i\}} (\alpha_l a_2 \kappa^*-1) = \alpha_l^{-2} \beta_{A;n;\{i\}} (a_2 \kappa^*-1), \] (88)
with similar implications as for \( \beta = 3 \).

I am not aware of an explanation for the doubling, which has a topological interpretation \[120\], but the replicating towers are a reflection of the surviving DSI. For \( A \geq 4 \), all but the two lower states appear as resonances in the scattering of a particle on the \((A - 1)\)-particle ground state. Because of the tower structure, we can focus on these two lower states. The higher one is near the ground state of the system with one less particle,
\[ \beta_{A;0,0,\ldots,0,1}(0) \simeq \beta_{A-1;0,0,\ldots,0}(0), \] (89)
and thus can be thought of as a two-body system of a particle and an \((A - 1)\) cluster. This state and its cousins up the tower are examples of “halo states”, such as observed in halo nuclei — states that have a clusterized structure in which a certain number of “valence” particles orbits a tight cluster of the remaining particles.

The ground states close to unitarity get deeper and deeper as \( A \) increases. We can write
\[ \frac{B_A}{A} \equiv \frac{B_{A;0;\{0\}}}{A} \simeq \frac{3}{A} \beta_{A;0;\{0\}}(0) \frac{B_{3;0}}{3} \equiv \kappa_A \frac{B_3}{3}, \] (90)
where the set of numbers \( \kappa_A \) encapsulates the dynamical information about the ground states at unitarity. Relation (90) expresses the generalized Tjon lines \[24\] at unitarity.

The \( \kappa_A \) for \( A \leq 60 \) bosons have been calculated recently using Monte Carlo techniques to solve the Schrödinger equation \[26\], with selected results shown in Table 1. At small \( A \), \( \kappa_A \) is approximately linear in \( A \), as obtained earlier \[116, 118, 121, 122, 123, 24\], but eventually saturation sets in, where the growth tapers off — see Fig. 7. This change in behavior is fitted well by a “liquid-drop” formula,
\[ \kappa_A = \kappa \left(1 + \eta A^{-1/3} + \ldots\right), \] (91)
with \( \kappa = 90 \pm 10 \) and \( \eta = -1.7 \pm 0.3 \), respectively, the dimensionless “volume” and “surface” terms. The factor of \( \approx 90 \) is large but still well below the \( \approx 515 \) that provides an upper bound for the EFT breakdown scale.

Atomic \(^4\)He is close enough to unitarity to sustain two trimers, an excited tetramer not too far from the ground-state trimer, and a ground-state tetramer about 4.4 deeper than the ground-state trimer. As shown in Table 1, \( A = 5, 6 \) systems have energies close to unitarity values as well. And an equation of the type (91) also describes calculated ground-state energies \[36\], yielding \( \kappa \approx 180 \) and \( \eta \approx -2.7 \). The energy of the bulk is thus \( \sim 2 \) away from unitarity. It is possibly beyond an EFT approach \[124\].

For \( A > 4 \) multi-state fermions at unitarity the spectral pattern is not clear, as no calculations are available, but towers must also exist. For the ground states, Eq. (90) still holds, but with a different set of numbers \( \kappa_A \). The four-state fermion system, for example, reduces to a bosonic system \[125\], similarly to the three-nucleon system \[15\]. \( \kappa_4 \)
Figure 7: Binding energy per particle at unitarity scaled by the three-body binding energy per particle, $\kappa_N = 3E_N/(NE_3)$, as function of the number $N$ of bosons in a cluster. Pionless EFT results at LO for various regulator cutoffs are shown as open and filled (colored) symbols. Black triangles show potential-model results up to $N = 15$ from Ref. [116]. The (blue) solid line shows a fit to the $N \geq 30$ (blue) points, corresponding to the largest cutoff values, using the liquid-drop formula (91). Reprinted figure with permission from Ref. [26]. Copyright 2017 by the American Physical Society.

is then the same in both cases, but $\kappa_{A \geq 5}$ must differ on account of the exclusion principle and shell structure.

Although it can be seen from Table 1 that the two-nucleon system is not as close to unitarity as two $^4$He atoms, the first excited state of the triton is almost bound, the alpha particle has an excited level close to the triton, and the alpha-particle ground state is about 3 times deeper. We argued recently [18] that nuclei are close enough to unitarity for a perturbative expansion in $(a_2 \kappa_*)^{-1}$ to converge. Reference [18] shows, for example, that the Tjon line can be obtained in perturbation theory from the Tjon line at unitarity. It would be extremely gratifying if it turns out that one can indeed devise a theory of nuclear physics based on a single parameter, plus perturbative corrections. For more speculation along these lines, see Ref. [107].

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9 Long-range forces

Pionless EFT acquired a life of its own. But it was born to address the nonperturbative renormalization issues befalling Chiral EFT. Despite Pionless EFT’s successes, many feel Chiral EFT should be better suited to nuclear physics, where traditionally the pion has been thought to be an important ingredient. And indeed, the binding momenta estimated with Eq. (2) are not far below the pion mass. (How seriously one should take factors of 2 or 3 in Eq. (2), which are important in this comparison, remains unclear.) So let me end these lectures with a taste of the basic issues that confront Chiral EFT. These challenges are more basic than pion exchange, as they arise from the renormalization of singular interactions and affect also Van der Waals forces. But, since little has been done for the latter, my main focus here will be on pion exchange. Much more complete accounts of Chiral EFT can be found in various reviews, e.g., Refs. [126, 127, 128].

Chiral EFT is an EFT for momenta $Q \sim m_\pi \simeq 140$ MeV, when pions appear as explicit degrees of freedom. (One might want to include also the lowest-lying nucleon excitations such as the Delta isobar, which is separated in mass from the nucleon by only $m_\Delta - m_N \simeq 300$ MeV.) As pseudo-Goldstone bosons of chiral symmetry, pions can be included through a non-linear realization of SO(4), when the isospin doublet of nucleon fields $N$ transforms as under the SO(3) subgroup of isospin, but with “parameters” that depend on the pion fields. The isospin triplet of pion fields $\pi$ itself parametrizes the coset space $S^3$ of radius $f_\pi$, and always appear in the combination $\pi / f_\pi$. One defines chiral covariant derivatives of pion and nucleon (and maybe nucleon-excitation) fields, and constructs all interactions that are isospin symmetric, which then will be automatically chiral invariant. Quark masses break chiral symmetry (including isospin) as components of certain SO(4) vectors, so we add to the action all interactions that break the symmetry as tensor products of the corresponding vectors. This procedure ensures that chiral symmetry is broken in Chiral EFT just as in QCD. Details can be found in Ref. [129].

For our purposes here, the most important terms are a subset of the leading Lagrangian: pion kinetic and mass terms, and the dominant pion-nucleon interaction, that is,

$$\Delta L_\pi = \frac{1}{2} \left( \partial_\mu \pi \partial^\mu \pi - m_\pi^2 \pi^2 \right) + \frac{g_A}{2f_\pi} N^\dagger \left( \vec{S} \cdot \vec{\tau} \cdot \vec{\nabla} \pi \right) N + \ldots , \quad (92)$$

where $\tau$ are the Pauli matrices is isospin space and $g_A \simeq 1.27$ is a LEC, the axial-vector coupling constant. I relegate to “…” interactions with more fields, derivatives and powers of quark masses, including the chiral partners of the terms shown explicitly. The Chiral EFT Lagrangian consists of the Lagrangian in Eq. (11) (with $\psi \rightarrow N$) supplemented by its chiral partners and the additional terms in Eq. (92). However, since this is a different theory, the LECs of short-range interactions in Eq. (11) have new values: they run differently and their renormalized values are, in general, also different — we have to repeat the renormalization procedure to relate them to data in the presence of pions.

The EFT still splits into sectors of different nucleon number $A$, but now there is more interesting physics at $A = 0, 1$ (the domain of ChPT), some of which is covered in Pich’s lectures [6]. Still, the sectors with $A \geq 2$ are richer because they involve two types of loops.
the “reducible” loops we have seen already in the context of Pionless EFT, where we can separate a diagram in at least two parts by cutting horizontally through heavy particle lines only; and “irreducible” loops, where we must cut also through one or more pion lines. In the first type of loop, we pick a pole from one heavy particle propagators, the typical energy is \( \sim Q^2/m \), and the magnitude of the contribution can be estimated by the rule (24). In the second type of loop, the pion propagators give energies \( \sim Q \), so that nucleon propagators are in a first approximation static — recoil is suppressed by a relative \( \mathcal{O}(Q/m) \) and can be included perturbatively. For irreducible loops the power counting rules are, instead, those used in ChPT [5],

\[
\text{pion propagator} \rightarrow Q^{-2}, \\
\text{heavy particle propagator} \rightarrow Q^{-1}, \\
\text{loop integral} \rightarrow Q^4/(4\pi)^2, \\
\text{derivative, pion mass} \rightarrow Q. \\
\]

(93)

Only the second type of loop is present for \( A = 0, 1 \). An extra irreducible loop amounts to \( \mathcal{O}(Q^2/(4\pi f_\pi)^2) \) [5, 51], so an expansion is possible for \( Q \sim m_\pi \ll 4\pi f_\pi \). By demanding RG invariance, one can infer the scaling of LECs known as NDA [51]. For consistency, it requires \( 4\pi f_\pi = \mathcal{O}(M_{QCD}) = m_N \), so that both the loop and nonrelativistic expansions are part of the \( m_\pi/M_{QCD} \) expansion.

To isolate irreducible loops when \( A \geq 2 \), we define the potential as the sum of these irreducible loops. In addition to the contact interactions of Pionless EFT, we have also pion exchange. The power counting (93) indicates that one-pion exchange (OPE) between two nucleons is the dominant long-range potential, with corrections starting two orders down the \( M_{lo}/M_{hi} \) expansion [130]. A crucial difference with respect to \( A = 0, 1 \) is that the sum of irreducible loops does not generate the full \( T \) matrix, is not directly observable, and is not RG invariant in itself. The logic we follow for \( T \) is the same as for \( A = 0, 1 \), but we now have to infer the sizes of LECs taking into account the reducible loops, as we do in the absence of pions. Since these loops generate regulator dependence, the potential itself cannot be regulator independent.

Challenges start already at the level of OPE: in momentum space

\[
\Delta V_\pi(p', \bar{p}) = -\frac{4\pi}{m_N M_{NN}} \frac{\tau_1 \cdot \tau_2}{q^2 + m_\pi^2} \bar{\sigma}_1 \cdot \bar{q} \sigma_2 \cdot q + \ldots, \\
\]

(94)

and in coordinate space

\[
\Delta V_\pi(\hat{r}) = \frac{\tau_1 \cdot \tau_2}{m_N M_{NN}} \left\{ S_{12}(\hat{r}) \left[ 1 + m_\pi r + \frac{(m_\pi r)^2}{3} \right] e^{-m_\pi r} \right.

+ \frac{\bar{\sigma}_1 \cdot \bar{\sigma}_2}{3} \left( -4\pi \delta^{(3)}(\hat{r}) + \frac{m_\pi^2}{r} e^{-m_\pi r} \right) \left\} + \ldots \\
\]

(95)

Here the subscripts _1,2_ label nucleons, \( \bar{q} \equiv \bar{p} - p' \) is the transferred three-momentum and \( q^{02} \ll q^2 \) was neglected in the pion propagator since it is a higher-order effect. The
tensor operator, defined as $S_{12}(\hat{r}) \equiv 3\vec{\sigma}_1 \cdot \hat{r} \vec{\sigma}_2 \cdot \hat{r} - \vec{\sigma}_1 \cdot \vec{\sigma}_2$, can be shown to vanish for total spin $S = 0$. For $S = 1$ it mixes waves of $l = j \pm 1$, where it has one positive and one negative eigenvalue, except for $^3P_0$ where it is diagonal with a negative eigenvalue. It also acts on states with $l = j$, where it has a positive eigenvalue. The “tensor force”, which is highly singular, is attractive in some uncoupled waves like $^3P_0$ and $^3D_2$, and in one of the eigenchannels of each coupled wave. In contrast, $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ takes the value 3 or $-1$ when $S = 0$ or $S = 1$, respectively. (An analogous remark holds for $\vec{r}_1 \cdot \vec{r}_2$ and total isospin $I = 0, 1$.) This “central force” has two components: i) a contact term that can be absorbed in the two $C_0^{(0)}$ contact interactions; and ii) a long-range component with Yukawa form, which is attractive in isovector (isoscalar) channels for $S = 0 (S = 1)$. In the “…” one finds higher orders, which are more singular still.

In order to bring the numerical factor in Eq. (94) to the same form I used for contact interactions, I introduced

$$M_{NN} \equiv \frac{16\pi f_\pi^2}{g^2 m_N}$$

(96)

following Refs. [48, 49]. From the rule (24) one expects

$$\frac{\mathcal{O}((4\pi/(m_N M_{NN}))^2 (m_N Q/(4\pi)))}{\mathcal{O}(4\pi/(m_N M_{NN}))} = \mathcal{O}(Q/M_{NN})$$

(97)

for the magnitude of the ratio between the non-analytic part of once-iterated OPE and single OPE. By NDA, $M_{NN} = \mathcal{O}(f_\pi)$, but numerically $M_{NN} \simeq 290$ MeV, or about three times larger. There are also other numbers of $\mathcal{O}(1)$ floating around. If we take $M_{NN} = \mathcal{O}(M_{hi})$, then OPE is an NLO effect in an $m_\pi/M_{NN}$ expansion [48, 49]. In this case LO would be formally identical to LO in Pionless EFT, perhaps explaining why the latter seems to work well even for the alpha particle where the binding momentum (2) is not very small compared to $m_\pi$. More generally, one can show [48, 49] that in this case the same power counting of Pionless EFT applies, with the pion mass counting as a derivative. All pion exchanges are perturbative. Results at NLO show the expected improvement over LO, but unfortunately at $N^2$LO, where the first iteration of OPE appears, all hell breaks loose: in channels where the tensor force is attractive, no convergence is found for momenta $Q \gtrsim 100$ MeV [131].

The inference is that the purely numerical factors do not in general help convergence and the NDA estimate $M_{NN} = \mathcal{O}(f_\pi)$ is realistic. For quark masses such that $m_\pi \lesssim M_{NN}$, pions are perturbative for $Q \sim m_\pi$, but for larger masses one has to consider OPE as an LO potential. This has the virtue of providing a scale for binding momenta which is related to chiral symmetry: the amplitude is a series of the schematic form [126]

$$T_2^{(0)}(Q) \sim \frac{4\pi}{m_N M_{NN}} \left[1 + \frac{Q}{M_{NN}} + \left(\frac{Q}{M_{NN}}\right)^2 + \ldots\right] \sim \frac{4\pi}{m_N M_{NN}} \frac{1}{M_{NN} - Q}.$$  

(98)

allowing for a pole at $Q \sim M_{NN} = \mathcal{O}(f_\pi) \ll M_{QCD}$. The existence of shallow nuclei, except perhaps the very lightest where binding momenta are even smaller, is then tied to spontaneous chiral-symmetry breaking.
Taking OPE as LO means that we need to deal with its singular nature, as for delta functions in Pionless EFT. For large momentum, the potential (94) is not more singular than (18) and, on the surface, does not seem to offer further challenges. However, the more intricate angular dependence leads to an $r^{-3}$ behavior at short distances instead of $\delta^{(3)}(\vec{r})$, with contributions also to waves higher than $S$. Stronger singularities such as $r^{-5}$ appear at higher orders. Things remain similar to atomic systems, in that we need to deal with the $r^{-6}$ singularity of the Van der Waals interaction if we are interested in momenta $Q \sim l_{\text{vdW}}^{-1}$. And smaller but more singular components exist in this context as well [27].

The quantum mechanics of singular interactions has a long history [132], when it was more or less agreed that an attractive singular interaction in itself is not sufficient to define the solution of the Schrödinger equation. The reason is that for a potential of the type $-\alpha^2/(2\mu r^n)$, with $\mu$ the reduced mass and $n \geq 2$, the two allowed solutions both oscillate with decreasing amplitude as the distance decreases, in contrast with a regular potential for which there are two clearly distinct solutions — regular and irregular. As a consequence, there is an undetermined phase, just like in Eq. (67): the zero-energy $l = 0$ wavefunction, for example, can be shown to be given at small $r$ by

$$\psi(r) \propto r^{n/4-1} \cos \left( \frac{\alpha}{n/2-1} r^{1-n/2} + \delta_n \right) + \ldots$$

(99)

for $n > 2$. For $n = 2$, $\alpha r^{1-n/2}/(n/2-1) \to \sqrt{\alpha^2-1/4} \ln(r/R)$. Equation (99) is parallel to Eq. (67). (In fact, Efimov [10] first arrived at his geometric spectrum by consideration of the three-particle equation in coordinate space.) The phase $\delta_n$ can be fixed by a single counterterm regardless of the value of $n$ [133]. For example, with a spherical-well regularization of the corresponding delta function (see Sec. 5), progressively more unphysical bound states cross threshold as the regulator distance $R$ decreases — similarly to the Thomas collapse in the three-body system — unless the depth is adjusted to produce a value of the wavefunction at $R$ that gives the phase one wants. The phase in turn determines the low-energy properties of the scattering amplitude. Just like in the three-body case above, the LEC oscillates as $\Lambda \sim R^{-1}$ increases. However, only for $n = 2$, when the classical system is scale invariant, is the dependence periodic in $\ln \Lambda$. The RG analysis of singular potentials is discussed in Ref. [134]. From the EFT point of view, the quantum mechanics of singular potentials is just the renormalization of the LO amplitude.

For $l > 0$, the centrifugal barrier effectively suppresses the effect of the long-range potential on the amplitude by factors of $l^{-1}$. Only for lower waves does the long-range potential need to be iterated in the low-energy region where the EFT applies [135, 136, 138]. At subleading orders where the potential gets more singular, renormalization can still be carried out with further contact interactions [138], at least as long as corrections are treated in perturbation theory as done for Pionless EFT in Sec. 4.

In the nuclear case there are complications arising from the tensor and spin operators:

- The $^3S_1$ wave gets mixed with $^3D_1$ by the OPE tensor force. The tensor force has one attractive eigenvector, which is finite in the chiral limit, and the $C_0$ LEC in
this channel, expected on the basis of NDA, is sufficient for renormalization at LO [139, 63].

- In spin-triplet channels where the tensor force is repulsive, OPE can be iterated without RG problems [135]. In contrast, when it is attractive, RG invariance is destroyed by iteration [135, 140]. One expects an angular-momentum suppression similar to the one seen for a central force, so that beyond a critical angular momentum $l_{cr}$, OPE can be considered subleading. However, in lower waves like $^3S_1-^3D_1$ and $^3P_0$, one can argue [135, 136, 137] that OPE needs to be iterated in the low-energy region, in agreement with the findings of Ref. [131]. In these waves, additional contact interactions with derivatives, which would be expected by NDA only at higher orders, are necessary and sufficient for renormalization at LO [135].

- In the simplest spin-singlet channel, $^1S_0$, OPE takes the form of an attractive Yukawa interaction $\propto m_\pi^2$, which by itself generates a finite amplitude but is far from providing enough binding to explain the virtual state. A contact $C_0$ interaction must still be present at LO. However, the interference between the two interactions gives rise to a $m_\pi^2 \ln \Lambda$ term — in perturbation theory, it comes from a diagram where OPE takes place between two contact interactions, but the same regulator dependence is seen nonperturbatively [141]. Renormalization requires a chiral-symmetry-breaking contact interaction with LEC $m_\pi^2 D_2$ at LO [141]. Thus this contact interaction is also enhanced with respect to NDA, and there is no straightforward chiral expansion of the contact interactions.

- In all other spin-singlet channels, the absence of a contact interaction in LO means OPE can be iterated without RG problems [135]. However, as the angular momentum $l$ increases factors of $l^{-1}$ suppress its contribution, and OPE is really perturbative and thus subleading [142].

- Residual cutoff dependence indicates the need at NLO for a single two-nucleon correction from the $^1S_0$ $C_2$ contact interaction, treated in perturbation theory [143] — in the same way as in Pionless EFT (Sec. [4]).

- Since OPE changes the asymptotic behavior of the two-nucleon amplitude, it is not immediately obvious whether a three-body force is needed for renormalization at LO or perhaps NLO. Explicit calculations [135, 144] show it is not.

During the period while nuclear EFTs were being formulated, rapid progress was achieved in the development of methods to solve the Schrödinger equation for increasingly higher $A$ with a given potential. While they first used purely phenomenological potentials, eventually most calculations became based on Weinberg’s original suggestion [2, 3] to use “chiral potentials”, where: i) contact interactions are assumed to have sizes given by NDA; ii) the expansion of irreducible diagrams is truncated at a certain order; and iii) the truncated potential is treated exactly. The resulting amplitudes are not renormalizable and much work goes into finding the “best” regulator to fit data with.
In contrast, a properly renormalized EFT has only been explored beyond NLO in the two-nucleon sector where it has given promising results [145, 146, 147, 148, 143]. There is still much to learn about implementing corrections in perturbation theory. Despite its age, this is a field with plenty of open problems.

10 Conclusion

EFT is not only a tool for inferring new degrees of freedom and symmetries, but also for understanding the emergence of new structures. I hope to have given you a flavor of this latter aspect of EFT’s power in the context of nuclear and atomic EFTs. Through Pionless EFT, I described how renormalization in a nonperturbative setting can be very different from perturbation theory, yet sufficiently tractable for us to observe the emergence of new phenomena: the non-trivial fixed point of two-body unitarity, the limit cycle of three-body physics, and the description of larger structures from a single essential parameter. The remaining discrete scale invariance allows for many-body spectra reminiscent of Russian dolls, for ground states that saturate as the number of particles grows very large, and for a (quantum) liquid. How such a picture can be matched with Chiral EFT, where equally bizarre renormalization takes place, is a question for you to tackle.

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References

[1] A.V. Manohar, “Introduction to Effective Field Theories”, lectures at this school, arXiv:1804.05863 [hep-ph].

[2] S. Weinberg, Phys. Lett. B 251 (1990) 288.

[3] S. Weinberg, Nucl. Phys. B 363 (1991) 3.

[4] M. Rho, Phys. Rev. Lett. 66 (1991) 1275.

[5] S. Weinberg, Physica A 96 (1979) 327.
[6] A. Pich, “Effective Field Theory with Nambu-Goldstone Modes”, lectures at this school, [arXiv:1804.05664 [hep-ph]].

[7] S.R. Beane, W. Detmold, K. Orginos, and M.J. Savage, Prog. Part. Nucl. Phys. 66 (2011) 1 [arXiv:1004.2935 [hep-lat]].

[8] N. Barnea, L. Contessi, D. Gazit, F. Pederiva, and U. van Kolck, Phys. Rev. Lett. 114 (2015) 052501 [arXiv:1311.4966 [nucl-th]].

[9] L. Contessi, A. Lovato, F. Pederiva, A. Roggero, J. Kirscher, and U. van Kolck, Phys. Lett. B 772 (2017) 839 [arXiv:1701.06516 [nucl-th]].

[10] V. Efimov, Phys. Lett. 33B (1970) 563.

[11] M. Tanabashi et al. [Particle Data Group], Phys. Rev. D 98 (2018) 030001.

[12] S. Weinberg, Trans. New York Acad. Sci. 38 (1977) 185.

[13] C.A. Bertulani, H.-W. Hammer, and U. van Kolck, Nucl. Phys. A 712 (2002) 37 [nucl-th/0205063].

[14] P.F. Bedaque, H.-W. Hammer, and U. van Kolck, Phys. Lett. B 569 (2003) 159 [nucl-th/0304007].

[15] P.F. Bedaque, H.-W. Hammer, and U. van Kolck, Nucl. Phys. A 676 (2000) 357 [arXiv:nucl-th/9906032].

[16] I. Stetcu, B.R. Barrett, and U. van Kolck, Phys. Lett. B 653 (2007) 358 [nucl-th/0609023].

[17] G. Rupak, A. Vaghani, R. Higa, and U. van Kolck, [arXiv:1806.01999 [nucl-th]].

[18] S. König, H.W. Grießhammer, H.-W. Hammer, and U. van Kolck, Phys. Rev. Lett. 118 (2017) 202501, [arXiv:1607.04623 [nucl-th]].

[19] M.H. Kalos, M.A. Lee, P.A. Whitlock, and G.V. Chester, Phys. Rev. B 24 (1981) 115.

[20] D. Blume and C.H. Greene, J. Chem. Phys. 112 (2000) 8053.

[21] E. Hiyama and M. Kamimura, Phys. Rev. A 85 (2012) 022502 [arXiv:1111.4370 [physics.atom-ph]].

[22] P.F. Bedaque, H.-W. Hammer, and U. van Kolck, Nucl. Phys. A 646 (1999) 444 [nucl-th/9811046].

[23] L. Platter, H.-W. Hammer, and U.-G. Meißner, Phys. Rev. A 70 (2004) 052101 [cond-mat/0404313].

46
[24] B. Bazak, M. Eliyahu, and U. van Kolck, Phys. Rev. A 94 (2016) 052502 [arXiv:1607.01509 [cond-mat.quant-gas]].

[25] A. Deltuva, Phys. Rev. A 82 (2010) 040701 [arXiv:1009.1295 [physics.atm-clus]].

[26] J. Carlson, S. Gandolfi, U. van Kolck, and S.A. Vitiello, Phys. Rev. Lett. 119 (2017) 223002 [arXiv:1707.08546 [cond-mat.quant-gas]].

[27] A. Calle Cordón and E. Ruiz Arriola, Phys. Rev. A 81 (2010) 044701 [arXiv:0912.1714 [cond-mat.other]].

[28] R.E. Grisenti et al., Phys. Rev. Lett. 85 (2000) 2284.

[29] Z.-C. Yan, J.F. Babb, A. Dalgarno, and G.W.F. Drake, Phys. Rev. A 54 (1996) 2824 [atom-ph/9607002].

[30] J.-Y. Zhang, Z.-C. Yan, D. Vrinceanu, J.F. Babb, and H.R. Sadeghpour, Phys. Rev. A 74 (2006) 014704 [physics/0603232].

[31] W. Cencek, M. Przybytek, J. Komasa, J.B. Mehl, B. Jezierski, and K. Szalewicz, J. Chem. Phys. 136 (2012) 224303.

[32] S. Zeller et al., Proc. Natl. Acad. Sci. U.S.A. 113 (2016) 14651 [arXiv:1601.03247 [physics.atom-ph]].

[33] M. Kunitski et al., Science 348 (2015) 551 [arXiv:1512.02036 [physics.atom-clus]].

[34] R.A. Aziz and M.J. Slaman, J. Chem. Phys. 94 (1991) 8047.

[35] R.A. Aziz, V.P.S. Nain, J.S. Carley, W.L. Taylor, and G.T. McConville, J. Chem. Phys. 70 (1979) 4330.

[36] V.R. Pandharipande, J.G. Zabolitzky, S.C. Pieper, R.B. Wiringa, and U. Helmbrecht, Phys. Rev. Lett. 50 (1983) 1676.

[37] J.S.R. Chisholm, Nucl. Phys. 26 (1961) 469.

[38] S. Kamefuchi, L. O’Raifeartaigh, and A. Salam, Nucl. Phys. 28 (1961) 529.

[39] H. Georgi, Phys. Lett. B 240 (1990) 447.

[40] M.E. Luke and A.V. Manohar, Phys. Lett. B 286 (1992) 348 [hep-ph/9205228].

[41] T.D. Lee and C.N. Yang, Phys. Rev. 128 (1962) 885.

[42] A. Salam and J.A. Strathdee, Phys. Rev. D 2 (1970) 2869.

[43] T. Mannel, “Effective Field Theories for Heavy Quarks”, lectures at this school.
[44] S. König, H.W. Griesshammer, H.-W. Hammer, and U. van Kolck, J. Phys. G 43 (2016) 055106 [arXiv:1508.05085 [nucl-th]].

[45] S. König, J. Phys. G 44 (2017) 064007 [arXiv:1609.03163 [nucl-th]].

[46] S. Weinberg, Phys. Rev. Lett. 121 (2018) 220001.

[47] U. van Kolck, Nucl. Phys. A 645 (1999) 273 [arXiv:nucl-th/9808007].

[48] D.B. Kaplan, M.J. Savage, and M.B. Wise, Phys. Lett. B 424 (1998) 390 [nucl-th/9801034].

[49] D.B. Kaplan, M.J. Savage, and M.B. Wise, Nucl. Phys. B 534 (1998) 329 [nucl-th/9802075].

[50] D.R. Phillips, S.R. Beane, and M.C. Birse, J. Phys. A 32 (1999) 3397 [hep-th/9810049].

[51] A. Manohar and H. Georgi, Nucl. Phys. B 234 (1984) 189.

[52] U. van Kolck, Lect. Notes Phys. 513 (1998) 62 [hep-ph/9711222].

[53] J.W. Chen, G. Rupak, and M.J. Savage, Nucl. Phys. A 653 (1999) 386 [nucl-th/9902056].

[54] H.A. Bethe, Phys. Rev. 76 (1949) 38.

[55] V.G.J. Stoks, R.A.M. Klomp, C.P.F. Terheggen, and J.J. de Swart, Phys. Rev. C 49 (1994) 2950 [nucl-th/9406039].

[56] H.A. Bethe and R. Peierls, Proc. Roy. Soc. Lond. A 148 (1935) 146.

[57] H.A. Bethe and R. Peierls, Proc. Roy. Soc. Lond. A 149 (1935) 176.

[58] E. Fermi, Ric. Scientifica 7 (1936) 13.

[59] T. Köhler, K. Goral, and P.S. Julienne, Rev. Mod. Phys. 78 (2006) 1311 [cond-mat/0601420].

[60] T.D. Cohen, B.A. Gelman, and U. van Kolck, Phys. Lett. B 588 (2004) 57 [nucl-th/0402054].

[61] A.R. Janzen and R.A. Aziz, J. Chem. Phys. 103 (1995) 8626.

[62] M.L. Wagman, F. Winter, E. Chang, Z. Davoudi, W. Detmold, K. Orginos, M.J. Savage, and P.E. Shanahan, Phys. Rev. D 96 (2017) 114510 [arXiv:1706.06550 [hep-lat]].

[63] S.R. Beane, P.F. Bedaque, M.J. Savage, and U. van Kolck, Nucl. Phys. A 700 (2002) 377 [nucl-th/0104030].
[64] C.R. Hagen, Phys. Rev. D 5 (1972) 377.
[65] T. Mehen, I.W. Stewart, and M.B. Wise, Phys. Lett. B 474 (2000) 145 [hep-th/9910025].
[66] B. d’Espagnat and J. Prentki, Nuovo Cim. 3 (1956) 1045.
[67] D.B. Kaplan, Nucl. Phys. B 494 (1997) 471 [nucl-th/9610052].
[68] H.W. Grießhammer, Nucl. Phys. A 744 (2004) 192 [nucl-th/0404073].
[69] S.R. Beane, T.D. Cohen, and D.R. Phillips, Nucl. Phys. A 632 (1998) 445 [nucl-th/9709062].
[70] P.F. Bedaque, H.-W. Hammer, and U. van Kolck, Phys. Rev. Lett. 82 (1999) 463 [nucl-th/9809025].
[71] P.F. Bedaque and U. van Kolck, Phys. Lett. B 428 (1998) 221 [nucl-th/9710073].
[72] P.F. Bedaque, H.-W. Hammer, and U. van Kolck, Phys. Rev. C 58 (1998) R641 [nucl-th/9802057].
[73] H.W. Grießhammer, Nucl. Phys. A 760 (2005) 110 [nucl-th/0502039].
[74] W. Dilg, L. Koester, and W. Nistler, Phys. Lett. 36B (1971) 208.
[75] G.S. Danilov, Sov. Phys. JETP 13 (1961) 349.
[76] L.H. Thomas, Phys. Rev. 47 (1935) 903.
[77] E. Braaten, D. Kang, and L. Platter, Phys. Rev. Lett. 106 (2011) 153005 [arXiv:1101.2854 [cond-mat.quant-gas]].
[78] H.-W. Hammer and T. Mehen, Phys. Lett. B 516 (2001) 353 [nucl-th/0105072].
[79] P.F. Bedaque, G. Rupak, H.W. Grießhammer, and H.-W. Hammer, Nucl. Phys. A 714 (2003) 589 [nucl-th/0207034].
[80] R.D. Amado and J.V. Noble, Phys. Rev. D 5 (1972) 1992.
[81] W. Schöllkopf and J.P. Toennies, J. Chem. Phys. 104 (1996) 1155.
[82] S.K. Adhikari and L. Tomio, Phys. Rev. C 26 (1982) 83.
[83] A.C. Phillips, Nucl. Phys. A 107 (1968) 209.
[84] V. Efimov and E.G. Tkachenko, Few-Body Syst. 4 (1988) 71.
[85] V. Roudnev and M. Cavagnero, Phys. Rev. Lett. 108 (2012) 110402 [arXiv:1109.4656 [physics.atm-clus]].
[86] C. Ji, D.R. Phillips, and L. Platter, Annals Phys. 327 (2012) 1803 [arXiv:1106.3837 [nucl-th]].

[87] C. Ji and D.R. Phillips, Few-Body Syst. 54 (2013) 2317 [arXiv:1212.1845 [nucl-th]].

[88] J. Vanasse, Phys. Rev. C 88 (2013) 044001 [arXiv:1305.0283 [nucl-th]].

[89] C. Ji, Int. J. Mod. Phys. E 25 (2016) 1641003 [arXiv:1512.06114 [nucl-th]].

[90] J. Vanasse, Int. J. Mod. Phys. E 25 (2016) 1641002 [arXiv:1609.03086 [nucl-th]].

[91] K.G. Wilson, Phys. Rev. D 10 (1974) 2445.

[92] K.G. Wilson, Nucl. Phys. Proc. Suppl. 140 (2005) 3 [hep-lat/0412043].

[93] K.M. Bulycheva and A.S. Gorsky, Phys. Usp. 57 (2014) 171 [arXiv:1402.2431 [hep-th]].

[94] H.E. Camblong and C.R. Ordóñez, Phys. Rev. D 68 (2003) 125013 [hep-th/0303166].

[95] D. Sornette, Phys. Rept. 297 (1998) 239 [cond-mat/9707012 [cond-mat.stat-mech]].

[96] E. Braaten and H.-W. Hammer, Phys. Rept. 428 (2006) 259 [cond-mat/0410417].

[97] T. Kraemer et al., Nature 440 (2006) 315 [cond-mat/0512394].

[98] B. Huang et al., Phys. Rev. Lett. 112 (2014) 190401 [arXiv:1402.6161 [cond-mat.quant-gas]].

[99] L. Platter, C. Ji, and D.R. Phillips, Phys. Rev. A 79 (2009) 022702 [arXiv:0808.1230 [cond-mat.other]].

[100] E. Wigner, Phys. Rev. 51 (1937) 106.

[101] T. Mehen, I.W. Stewart, and M.B. Wise, Phys. Rev. Lett. 83 (1999) 931 [hep-ph/9902370].

[102] J. Vanasse and D.R. Phillips, Few-Body Syst. 58 (2017) 26 [arXiv:1607.08585 [nucl-th]].

[103] L. Platter, H.-W. Hammer, and U.-G. Meißner, Phys. Lett. B 607 (2005) 254 [nucl-th/0409040].

[104] H.-W. Hammer and L. Platter, Eur. Phys. J. A 32 (2007) 113 [nucl-th/0610105].

[105] J. Kirscher, H.W. Grießhammer, D. Shukla, and H.M. Hofmann, Eur. Phys. J. A 44 (2010) 239 [arXiv:0903.5538 [nucl-th]].

[106] J. Kirscher, N. Barnea, D. Gazit, F. Pederiva, and U. van Kolck, Phys. Rev. C 92 (2015) 054002 [arXiv:1506.09048 [nucl-th]].

50
[107] U. van Kolck, Few-Body Syst. 58 (2017) 112.
[108] J.A. Tjon, Phys. Lett. B 56 (1975) 217.
[109] S. Nakaichi, Y. Akaishi, H. Tanaka, and T.K. Lim, Phys. Lett. A 68 (1978) 36.
[110] S. Nakaichi, T.K. Lim, Y. Akaishi, and H. Tanaka, J. Chem. Phys. 71 (1979) 4430.
[111] T.K. Lim, S. Nakaichi, Y. Akaishi, and H. Tanaka, Phys. Rev. A 22 (1980) 28.
[112] V. Lensky, M.C. Birse, and N.R. Walet, Phys. Rev. C 94 (2016) 034003
arXiv:1605.03898 [nucl-th].
[113] A. Bansal, S. Binder, A. Ekström, G. Hagen, G.R. Jansen, and T. Papenbrock,
Phys. Rev. C 98 (2018) 054301 arXiv:1712.10246 [nucl-th].
[114] B. Bazak, J. Kirscher, S. König, M. Pavón Valderrama, N. Barnea, and U. van Kolck,
arXiv:1812.00387 [cond-mat.quant-gas].
[115] F. Ferlaino et al., Phys. Rev. Lett. 102 (2009) 140401 arXiv:0903.1276 [cond-mat.other].
[116] J. von Stecher, J. Phys. B 43 (2010) 101002 arXiv:0909.4056 [cond-mat.quant-gas].
[117] M. Gattobigio, A. Kievsky, and M. Viviani, Phys. Rev. A 84 (2011) 052503
arXiv:1106.3853 [physics.atm-clus].
[118] J. von Stecher, Phys. Rev. Lett. 107 (2011) 200402 arXiv:1106.2319 [cond-mat.quant-gas].
[119] M. Gattobigio, A. Kievsky, and M. Viviani, Phys. Rev. A 86 (2012) 042513
arXiv:1206.0854 [physics.atm-clus].
[120] Y. Horinouchi and M. Ueda, Phys. Rev. A 94 (2016) 050702 arXiv:1603.05328
[cond-mat.quant-gas].
[121] A.N. Nicholson, Phys. Rev. Lett. 109 (2012) 073003 arXiv:1202.4402 [cond-mat.quant-gas].
[122] A. Kievsky, N.K. Timofeyuk, and M. Gattobigio, Phys. Rev. A 90 (2014) 032504
arXiv:1405.2371 [cond-mat.quant-gas].
[123] Y. Yan and D. Blume, Phys. Rev. A 92 (2015) 033626 arXiv:1508.00081 [cond-mat.quant-gas].
[124] A. Kievsky, A. Polls, B. Juliá-Díaz, and N.K. Timofeyuk, Phys. Rev. A 96 (2017)
040501 arXiv:1707.05628 [cond-mat.quant-gas].
[125] L. Platter, Ph.D. dissertation, University of Bonn (2005).
[126] P.F. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. 52 (2002) 339 [nucl-th/0203055].

[127] E. Epelbaum, H.-W. Hammer, and U.-G. Meißner, Rev. Mod. Phys. 81 (2009) 1773 [arXiv:0811.1338 [nucl-th]].

[128] R. Machleidt and D.R. Entem, Phys. Rept. 503 (2011) 1 [arXiv:1105.2919 [nucl-th]].

[129] S. Weinberg, The Quantum Theory of Fields: Modern applications (Vol. 2), Cambridge University Press (2013).

[130] C. Ordóñez and U. van Kolck, Phys. Lett. B 291 (1992) 459.

[131] S. Fleming, T. Mehen, and I.W. Stewart, Nucl. Phys. A 677 (2000) 313 [nucl-th/9911001].

[132] W. Frank, D.J. Land, and R.M. Spector, Rev. Mod. Phys. 43 (1971) 36.

[133] S.R. Beane, P.F. Bedaque, L. Childress, A. Kryjevski, J. McGuire, and U. van Kolck, Phys. Rev. A 64 (2001) 042103 [quant-ph/0010073].

[134] M. Pavón Valderrama and E. Ruiz Arriola, Annals Phys. 323 (2008) 1037 [arXiv:0705.2952 [nucl-th]].

[135] A. Nogga, R.G.E. Timmermans, and U. van Kolck, Phys. Rev. C 72 (2005) 054006 [nucl-th/0506005].

[136] M.C. Birse, Phys. Rev. C 74 (2006) 014003 [nucl-th/0507077].

[137] S. Wu and B. Long, arXiv:1807.04407 [nucl-th].

[138] B. Long and U. van Kolck, Annals Phys. 323 (2008) 1304 [arXiv:0707.4325 [quant-ph]].

[139] T. Frederico, V.S. Timóteo, and L. Tomio, Nucl. Phys. A 653 (1999) 209 [nucl-th/9902052].

[140] M. Pavón Valderrama and E. Ruiz Arriola, Phys. Rev. C 74 (2006) 064004 [Erratum: Phys. Rev. C 75 (2007) 059905] [nucl-th/0507075].

[141] D.B. Kaplan, M.J. Savage, and M.B. Wise, Nucl. Phys. B 478 (1996) 629 [nucl-th/9605002].

[142] M. Pavón Valderrama, M. Sánchez Sánchez, C.-J. Yang, B. Long, J. Carbonell, and U. van Kolck, Phys. Rev. C 95 (2017) 054001 [arXiv:1611.10175 [nucl-th]].

[143] B. Long and C.-J. Yang, Phys. Rev. C 86 (2012) 024001 [arXiv:1202.4053 [nucl-th]].
[144] Y.-H. Song, R. Lazauskas, and U. van Kolck, Phys. Rev. C 96 (2017) 024002 [arXiv:1612.09090 [nucl-th]].

[145] M. Pavón Valderrama, Phys. Rev. C 83 (2011) 024003 [arXiv:0912.0699 [nucl-th]].

[146] M. Pavón Valderrama, Phys. Rev. C 84 (2011) 064002 [arXiv:1108.0872 [nucl-th]].

[147] B. Long and C.-J. Yang, Phys. Rev. C 84 (2011) 057001 [arXiv:1108.0985 [nucl-th]].

[148] B. Long and C.-J. Yang, Phys. Rev. C 85 (2012) 034002 [arXiv:1111.3993 [nucl-th]].