The Zariski topology-graph of modules over commutative rings II

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Abstract
Let $M$ be a module over a commutative ring $R$. In this paper, we continue our study about the Zariski topology-graph $G(\tau_T)$ which was introduced in Ansari-Toroghy et al. (Commun Algebra 42:3283–3296, 2014). For a non-empty subset $T$ of $\text{Spec}(M)$, we obtain useful characterizations for those modules $M$ for which $G(\tau_T)$ is a bipartite graph. Also, we prove that if $G(\tau_T)$ is a tree, then $G(\tau_T)$ is a star graph. Moreover, we study coloring of Zariski topology-graphs and investigate the interplay between $\chi(G(\tau_T))$ and $\omega(G(\tau_T))$.

Mathematics Subject Classification
13C13 · 13C99 · 05C75

1 Introduction
Throughout this paper, $R$ is a commutative ring with a non-zero identity and $M$ is a unital $R$-module. By $N \leq M$ (resp. $N < M$) we mean that $N$ is a submodule (resp. proper submodule) of $M$.

Define $(N:RM)$ or simply $(N : M) = \{ r \in R \mid rM \subseteq N \}$ for any $N \leq M$. We denote $((0) : M)$ by $\text{Ann}_R(M)$ or simply $\text{Ann}(M)$. $M$ is said to be faithful if $\text{Ann}(M) = (0)$.

Let $N, K \leq M$. Then the product of $N$ and $K$, denoted by $NK$, is defined by $(N : M)(K : M)$ (see [3]).

A prime submodule of $M$ is a submodule $P \neq M$ such that whenever $re \in P$ for some $r \in R$ and $e \in M$, we have $r \in (P : M)$ or $e \in P$ [10].

The prime spectrum of $M$ is the set of all prime submodules of $M$ and denoted by $\text{Spec}(M)$.

If $N$ is a submodule of $M$, then $V(N) = \{ P \in \text{Spec}(M) \mid (P : M) \supseteq (N : M) \}$ [11].

The Zariski topology on $X = \text{Spec}(M)$ is the topology $\tau_M$ described by taking the set $Z(M) = \{ V(N) \mid N \text{ is a submodule of } M \}$ as the set of closed sets of $\text{Spec}(M)$ [11].

A topological space $X$ is irreducible if for any decomposition $X = X_1 \cup X_2$ with closed subsets $X_i$ of $X$ with $i = 1, 2$, we have $X = X_1$ or $X = X_2$.

There are many papers on assigning graphs to rings or modules (see, for example, [1,5,6,9]). In [4], the present authors introduced and studied the graph $G(\tau_T)$ and $AG(M)$, called the Zariski topology-graph and the annihilating-submodule graph, respectively.

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Let $T$ be a non-empty subset of $\text{Spec}(M)$. The Zariski topology-graph $G(\tau_T)$ is an undirected graph with vertices $V(G(\tau_T))=\{N < M\}$ there exists $K < M$ such that $V(N) \cup V(K) = T$ and $V(N), V(K) \neq T$ and distinct vertices $N$ and $L$ are adjacent if and only if $V(N) \cup V(L) = T$ (see [4, Definition 2.3]).

$AG(M)$ is an undirected graph with vertices $V(AG(M))=\{N \leq M\}$ there exists $(0) \neq K < M$ with $NK = (0)$. In this graph, distinct vertices $N, L \in V(AG(M))$ are adjacent if and only if $NL = (0)$. Let $AG(M)^*$ be the subgraph of $AG(M)$ with vertices $V(AG(M)^*)=\{N < M$ with $(N : M) \neq \text{Ann}(M)\}$ there exists a submodule $K < M$ with $(K : M) \neq \text{Ann}(M)$ and $NK = (0)$. By [4, Theorem 3.4], one conclude that $AG(M)^*$ is a connected subgraph.

If $\text{Spec}(M) \neq \emptyset$, the mapping $\psi: \text{Spec}(M) \to \text{Spec}(R/\text{Ann}(M))$ such that $\psi(P) = (P : M)/\text{Ann}(M)$ for every $P \in \text{Spec}(M)$, is called the natural map of $\text{Spec}(M)$ [11].

The prime radical $\sqrt{N}$ is defined to be the intersection of all prime submodules of $M$ containing $N$, and in case $N$ is not contained in any prime submodule, $\sqrt{N}$ is defined to be $M$ [10].

We recall that $N < M$ is said to be a semiprime submodule of $M$ if for every ideal $I$ of $R$ and every submodule $K$ of $M$ with $I^2K \subseteq N$ implies that $IK \subseteq N$. Further $M$ is called a semiprime module if $(0) \subseteq M$ is a semiprime submodule. Every intersection of prime submodules is a semiprime submodule (see [17]).

The notations $\text{Nil}(R)$, $\text{Min}(M)$, and $\text{Min}(T)$ will denote the set of all nilpotent elements of $R$ and the set of all minimal prime submodules of $M$, and the set of minimal members of $T$, respectively.

A clique of a graph is a complete subgraph and the supremum of the sizes of cliques in $G$, denoted by $\omega(G)$, is called the clique number of $G$. Let $\chi(G)$ denote the chromatic number of the graph $G$, that is, the minimal number of colors needed to color the vertices of $G$ so that no two adjacent vertices have the same color. Obviously $\chi(G) \geq \omega(G)$.

In this article, we continue our studying about $G(\tau_T)$ and $AG(M)$ and we try to relate the combinatorial properties of the above mentioned graphs to the algebraic properties of $M$.

In Sect. 2 of this paper, we state some properties related to the Zariski topology-graph that are basic or needed in the later sections. In Sect. 3, we study the bipartite Zariski topology-graphs of modules over commutative rings (see Proposition 3.1). Also, we prove that if $G(\tau_T)$ is a tree, then $G(\tau_T)$ is a star graph (see Theorem 3.5). In Sect. 4, we study coloring of the Zariski topology-graph of modules and investigate the interplay between $\chi(G(\tau_T))$ and $\omega(G(\tau_T))$. We show that under condition over minimal submodules of $M/\bigcap_{P \in T}P : M)M$, we have $\omega(G(\tau_T)) = \chi(G(\tau_T))$ (see Theorem 4.1). Moreover, we investigate some relations between the existence of cycles in the Zariski topology-graph of a cyclic module and the number of its minimal members of $T$ (see Proposition 4.10).

Let us introduce some graphical notions and denotations that are used in what follows: A graph $G$ is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set of vertices, $V(G)$, a set $E(G)$ of edges, and an incident function $\psi_G$ that associates an unordered pair of distinct vertices with each edge. The edge $e$ joins $x$ and $y$ if $\psi_G(e) = \{x, y\}$, and we say $x$ and $y$ are adjacent. A path in a graph $G$ is a finite sequence of vertices $\{x_0, x_1, \ldots, x_n\}$, where $x_{i-1}$ and $x_i$ are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for existing an edge between $x_{i-1}$ and $x_i$.

A graph $H$ is a subgraph of $G$, if $V(H) \subseteq V(G), E(H) \subseteq E(G)$, and $\psi_H$ is the restriction of $\psi_G$ to $E(H)$. A bipartite graph is a graph whose vertices can be divided into two disjoint sets $U$ and $V$ such that every edge connects a vertex in $U$ to one in $V$; that is, $U$ and $V$ are each independent sets and complete bipartite graph on $n$ and $m$ vertices, denoted by $K_{n,m}$, where $V$ and $U$ are of size $n$ and $m$, respectively, and $E(G)$ connects every vertex in $V$ with all vertices in $U$. Note that a graph $K_{1,m}$ is called a star graph and the vertex in the singleton partition is called the center of the graph. For some $U \subseteq V(G)$, we denote by $N(U)$, the set of all vertices of $G \setminus U$ adjacent to at least one vertex of $U$. For every vertex $v \in V(G)$, the size of $N(v)$ is denoted by $\text{deg}(v)$. If all the vertices of $G$ have the same degree $k$, then $G$ is called $k$-regular, or simply regular. We denote by $C_n$ a cycle of order $n$. Let $G$ and $G'$ be two graphs. A graph homomorphism from $G$ to $G'$ is a mapping $\phi: V(G) \to V(G')$ such that for every edge $\{u, v\}$ of $G$, $\{\phi(u), \phi(v)\}$ is an edge of $G'$. A retract of $G$ is a subgraph $H$ of $G$ such that there exists a homomorphism $\phi: G \to H$ such that $\phi(x) = x$, for every vertex $x$ of $H$. The homomorphism $\phi$ is called the retract (graph) homomorphism (see [13]).

Throughout the rest of this paper, we denote by $T$ a non-empty subset of $\text{Spec}(M)$, $F := \bigcap_{P \in T}P$, $Q := (F : M)M$, $\tilde{M} := M/Q$, $\tilde{N} := \tilde{N}/Q$, $\tilde{m} := m + Q$, and $\tilde{I} := \tilde{I}/(Q : M)$, where $N$ is a submodule of $M$ containing $Q, m \in M$, and $I$ is an ideal of $R$ containing $(Q : M)$. 
2 Auxiliary results

In this section, we provide some properties related to the Zariski topology-graph that are basic or needed in the sequel.

**Remark 2.1** Let \( N \) be a submodule of \( M \). Set \( V^*(N) := \{ P \in \text{Spec}(M) \mid P \supseteq N \} \). By [4, Remark 2.2], for submodules \( N \) and \( K \) of \( M \), we have

\[
V(N) \cup V(K) = V(N \cap K) = V(NK) = V^*(NK).
\]

By [4, Remark 2.5], we have \( T \) is a closed subset of \( \text{Spec}(M) \) if and only if \( T = V(F) \) and \( G(\tau_T) \neq \emptyset \) if and only if \( T = V(F) \) and \( T \) is not irreducible. So if \( N \) and \( K \) are adjacent in \( G(\tau_T) \), then \( V^*(NK) = V^*(Q) \) and hence \( \sqrt{NK} = F \). Therefore, \( F \subseteq \sqrt{(N : M)M} \) and \( F \subseteq \sqrt{(K : M)M} \).

**Lemma 2.2** (See [2, Proposition 7.6]) Let \( R_1, R_2, \ldots, R_n \) be non-zero ideals of \( R \). Then the following statements are equivalent:

(a) \( R = R_1 \oplus \cdots \oplus R_n \);
(b) As an abelian group \( R \) is the direct sum of \( R_1, \ldots, R_n \);
(c) There exist pairwise orthogonal idempotents \( e_1, \ldots, e_n \) with \( 1 = e_1 + \ldots + e_n \), and \( R_i = Re_i, i = 1, \ldots, n \).

**Proposition 2.3** Suppose that \( e \) is an idempotent element of \( R \). We have the following statements.

(a) \( R = R_1 \oplus R_2 \), where \( R_1 = eR \) and \( R_2 = (1 - e)R \).
(b) \( M = M_1 \oplus M_2 \), where \( M_1 = eM \) and \( M_2 = (1-e)M \).
(c) For every submodule \( N \) of \( M \), \( N = N_1 \oplus N_2 \) such that \( N_1 \) is an \( R_1 \)-module \( M_1 \), \( N_2 \) is an \( R_2 \)-module \( M_2 \), and \( (N : M) = (N_1 : M_1) \oplus (N_2 : M_2) \).
(d) For submodules \( N \) and \( K \) of \( M \), \( NK = N_1K_1 \oplus N_2K_2 \), \( N \cap K = N_1K_1 \oplus N_2K_2 \) such that \( N = N_1 \oplus N_2 \) and \( K = K_1 \oplus K_2 \).
(e) Prime submodules of \( M \) are \( P \oplus M_2 \) and \( M_1 \oplus Q \), where \( P \) and \( Q \) are prime submodules of \( M_1 \) and \( M_2 \), respectively.
(f) For submodule \( N \) of \( M \), we have \( \sqrt{N} = \sqrt{N_1} \oplus \sqrt{N_2} = \sqrt{N_1} \oplus \sqrt{N_2} \), where \( N = N_1 \oplus N_2 \).

**Proof** This is clear. \( \square \)

An ideal \( I \subseteq R \) is said to be nil if \( I \) consists of nilpotent elements.

**Lemma 2.4** (See [15, Theorem 21.28]) Let \( I \) be a nil ideal in \( R \) and \( u \in R \) such that \( u + I \) is an idempotent in \( R/I \). Then there exists an idempotent \( e \) in \( uR \) such that \( e - u \in I \).

**Lemma 2.5** (See [5, Lemma 2.4]) Let \( N \) be a minimal submodule of \( M \) and let \( \text{Ann}(M) \) be a nil ideal. Then we have \( N^2 = (0) \) or \( N = eM \) for some idempotent \( e \in R \).

We note that \( M \) is said to be primeful if either \( M = (0) \) or \( M \neq (0) \) and the natural map of \( \text{Spec}(M) \) is surjective (see [12]).

**Proposition 2.6** We have the following statements.

(a) If \( N, L \) are adjacent in \( G(\tau_T) \), then \( \sqrt{(N : M)M} \mid F \) and \( \sqrt{(L : M)M} \mid F \) are adjacent in \( AG(M/F) \).
(b) If \( M \) is a primeful module and \( N, L \) are adjacent in \( G(\tau_T) \), then \( \sqrt{N} \mid F \) and \( \sqrt{L} \mid F \) are adjacent in \( AG(M/F) \).

**Proof** (a) First, we see easily that for any submodule \( N \) of \( M \), \( V(N) = V(\sqrt{(N : M)M}) \). Suppose that \( N \) and \( L \) are adjacent in \( G(\tau_T) \) so that \( V(N) \cup V(L) = T \). Then we have \( V^*(\sqrt{(N : M)M} \sqrt{(L : M)M}) = T \). It follows that \( \sqrt{(N : M)M} \sqrt{(L : M)M} \subseteq F \) (see Remark 2.1). Also by Remark 2.1, \( F \subseteq \sqrt{(N : M)M} \) and \( F \subseteq \sqrt{(L : M)M} \). Therefore, \( \sqrt{(N : M)M} \mid F \) and \( \sqrt{(L : M)M} \mid F \) are adjacent in \( AG(M/F) \).

(b) This is clear by [4, Corollary 4.5]. \( \square \)

**Remark 2.7** The Proposition 2.6(a) extends [4, Theorem 4.4].

**Lemma 2.8** Assume that \( T \) is a closed subset of \( \text{Spec}(M) \). Then \( AG(M) \) is isomorphic with a subgraph of \( G(\tau_T) \). In particular, \( AG(M/F) \) is isomorphic with an induced subgraph of \( G(\tau_T) \).
First, we show that the descending chain of non-trivial submodules should be finite.

**Lemma 2.9** If \( \tilde{M} \) is a faithful module and \( T \) is a closed subset of \( \text{Spec}(M) \), then \( G(\tau \text{Spec}(M)) \) and \( AG(M)^* \) are the same.

**Proof** \( \tilde{M} \) is a faithful module and \( T \) is a closed subset of \( \text{Spec}(M) \) so that \( T = \text{Spec}(M) \). If \( G(\tau \text{Spec}(M)) \neq \emptyset \), then there exist non-trivial submodules \( N \) and \( K \) of \( M \) which are adjacent in \( G(\tau \text{Spec}(M)) \). Hence \( V(NK) = \text{Spec}(M) \) which implies that \( NK = \emptyset \) so that \( AG(M)^* \neq \emptyset \). By Lemma 2.8, \( AG(M)^* \) is isomorphic with a subgraph of \( G(\tau \text{Spec}(M)) \). One can see that the vertex map \( \phi : V(G(\tau \text{Spec}(M))) \rightarrow V(AG(M)^*) \), defined by \( N \rightarrow N \) is an isomorphism.

Recall that \( \Delta(G(\tau_T)) \) is the maximum degree of \( G(\tau_T) \) and the length of an \( R \)-module \( M \), is denoted by \( l_R(M) \).

**Lemma 2.10** Let every nontrivial submodule of \( M \) be a vertex in \( G(\tau_T) \). If \( \Delta(G(\tau_T)) < \infty \), then \( l_R(M) \leq \Delta(G(\tau_T)) + 1 \). Also, every non-trivial submodule of \( M \) has finitely many submodules.

**Proof** First, we show that the descending chain of non-trivial submodules \( K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots \) terminates. Since \( G(\tau_T) \) is connected, there exists a submodule \( N \) such that \( V(N) \cup V(K_1) = T \). Hence for each \( i, i \geq 1, V(N) \cup V(K_i) = T \) and so \( \text{deg}(N) = \infty \), a contradiction. Next, let \( N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \) be an ascending chain of non-trivial submodules of \( M \). Since \( G(\tau_T) \) is connected, there exists a submodule \( K \) such that \( V(K) \cup V(N_{\Delta+1}) = T \), where \( \Delta = \Delta(G(\tau_T)) \). Hence \( V(K) \cup V(N_i) = T \) for each \( 1 \leq i \leq \Delta + 1 \). Thus \( \text{deg}(K) \geq \Delta + 1 \), a contradiction. It follows that \( l_R(M) \leq \Delta + 1 \). For the proof of the last assertion, let \( N \) be a non-trivial submodule of \( M \). Since \( G(\tau_T) \) is connected, there exists a submodule \( K \) such that \( V(N) \cup V(K) = T \). Hence for every submodule \( N' \) of \( N \), \( V(N') \cup V(K) = T \). As \( \Delta < \infty \), the number of submodules of \( N \) should be finite. \( \square \)

**Theorem 2.11** Suppose that \( \tilde{M} \) is a multiplication module and \( G(\tau_T) \neq \emptyset \). If \( G(\tau_T) \) has acc (resp. dcc) on vertices, then \( \tilde{M} \) is a Noetherian (resp. an Artinian) module.

**Proof** Suppose that \( G(\tau_T) \) has acc (resp. dcc) on vertices. By [4, Remark 2.6], \( F \) is not a prime submodule of \( M \) and hence there exist \( r \in R \) and \( m \in M \) such that \( rm \in F \) but \( m \notin F \) and \( r \notin (F : M) \). Now \( \tilde{M} \cong M/(\hat{0} : \hat{M} r) \). Further, \( \tilde{M} \) and \( (\hat{0} : \hat{M} r) \) are vertices in \( AG(M) = AG(M)^* \) (\( M \) is a multiplication module) because \( (\hat{0} : \hat{M} r)(\tilde{r}M) = ((\hat{0} : \hat{M} r)(\tilde{r}M) \mid M) \cap \tilde{M} \subseteq \tilde{r}M((\hat{0} : \hat{M} r) \mid M) \cap \tilde{r}M = 0 \). Then by Lemma 2.8, \( |N| \tilde{N} \leq \hat{M}, \hat{N} \leq \tilde{r}M \cap N \leq \hat{M}, \hat{N} \leq (\hat{0} : \hat{M} r) \leq V(G(\tau_T)) \). It follows that the \( \tilde{M} \)-modules \( \tilde{r}M \) and \( (\hat{0} : \hat{M} r) \) have acc (resp. dcc) on submodules. Since \( \tilde{M} \cong M/(\hat{0} : \hat{M} r) \), \( M \) has acc on submodules and the proof is completed. \( \square \)

### 3 Zariski topology-graph of modules

First, in this section we give the more notation to be used throughout the remainder of this article. Suppose that \( e \neq 0, 1 \) is an idempotent element of \( R \). Let \( M_1 := eM, M_2 := (1-e)M, T_1 := \{ p \in \text{Spec}(M_1) \mid |p| = 1 \} \}, T_2 := \{ p \in \text{Spec}(M_2) \mid |p| = 2 \} \}. F_1 := \cap p \in T_1 F_1, F_2 := \cap p \in T_2 F_2, Q_1 := (F_1 : M_1), Q_2 := (F_2 : M_2), M_1 := eM = eM/1, and M_2 := (e-1)M = (e-1)M/Q_2. Consequently, we have, \( Q = Q_1 \cap Q_2 \), where \( Q = (\cap p \in T P : M) \) and \( M \cong M_1 \oplus M_2 \)

We recall that a submodule \( N \) of \( M \) is a prime \( R \)-module if and only if it is a prime \( R/\text{Ann}(M) \)-module (see [16, Result 1.2]).
Proposition 3.1 Suppose that $\bar{M}$ does not have a non-zero submodule $\bar{F} \neq \bar{N}$ with $V(N) = T$ and $\text{Ann}(\bar{M})$ is a nil ideal. Then the following statements hold.

(a) If there exists a vertex of $G(\tau_T)$ which is adjacent to every other vertex, then $\bar{M}_1$ is a simple module and $\bar{M}_2$ is a prime module for some idempotent element $e \in R$.

(b) If $\bar{M}_1$ and $\bar{M}_2$ are prime modules for some idempotent element $e \in R$, then $G(\tau_T)$ is a complete bipartite graph.

Proof • (a) Suppose that $N$ is adjacent to every other vertex of $G(\tau_T)$. Since $V(N) = V((N : M)M)$, we have $N = (N : M)M$ and hence $V(N) = V'(N)$. Thus, $N = \sqrt{N}$ because $V(N) = V(\sqrt{N})$. We claim that $\bar{N}$ is a minimal submodule of $M$. Let $Q \subseteq K \subseteq N$. If $V(K) \neq T$, then $K$ is adjacent to $N$ and hence $V(K) = T$, a contradiction. So $\bar{N}$ is a minimal submodule of $M$. We have $(\bar{N})^2 \neq (0)$ because $V(N) \neq T$. Then Lemma 2.5 implies that $M \cong \bar{e}M \oplus (e - 1)\bar{M}$ for some idempotent element $e$ of $R$. Without loss of generality we may assume that $M_1 \oplus Q_2$ is adjacent to every other vertex. Since $V(F_1 \oplus Q_2) = V(Q_1 \oplus F_2) = T$, the assumption of theorem implies that $F = Q$. We claim that $M_1$ is a simple module and $\bar{M}_2$ is a prime module. Let $Q_1 \subseteq K < M_1$. We have $V(K \oplus Q_2) \neq T$ because $Q_1 \oplus Q_2 \subseteq K \oplus Q_2$. Since $V(K \oplus Q_2) \cup V(Q_1 \oplus M_2) = T$, we have $K \oplus Q_2$ is a vertex and hence is adjacent to $M_1 \oplus Q_2$. Therefore $V(K \oplus Q_2) \cup V(M_1 \oplus Q_2) = V(K \oplus Q_2)$, a contradiction. It implies that $M_1$ is a simple module. Now, we show that $M_2$ is a prime module. It is enough to show that it is a prime $R/(Q_2 : M_2)$-module. Otherwise, $I \bar{K} = (0)$, where $(Q_2 : M_2) \subseteq I < R$ and $Q_2 \subseteq K < M_2$. It follows that $V(M_1 \oplus K) \cup V(Q_1 \oplus I \bar{M}_2) = V(Q_1 \oplus K \bar{M}_2) = T$ because $K \bar{M}_2 \subseteq I \bar{K} \subseteq Q_2$, (note that $(Q_2 : M_2) \subseteq (K : M_2)$ and $(Q_2 : M_2) \subseteq I$). Therefore, $M_1 \oplus K$ is a vertex and hence is adjacent to $M_1 \oplus Q_2$. So $V(M_1 \oplus K) \cup V(M_1 \oplus Q_2) = T = V(M_1 \oplus Q_2)$, a contradiction (note that $M_1 \oplus K$ is properly containing $Q_1 \oplus Q_2)$.

• (b) Assume that $N_1 \oplus N_2$ is adjacent to $K_1 \oplus K_2$. One can see that $\sqrt{N_1}K_1 \oplus \sqrt{N_2}K_2 = \sqrt{Q_1} \oplus \sqrt{Q_2}$. It implies that $\sqrt{(K_1 : M_1)M_1} \cap (N_1 : M_1)M_1 = (0)$ and $(N_1 : M_1)M_1 \cap (K_2 : M_2)M_2 = (0)$. Since $M_1$ and $M_2$ are prime modules, $(\sqrt{(K_1 : M_1)M_1} : M_1) = (Q_1 : M_1)$ or $(\sqrt{(N_1 : M_1)M_1} : M_1) = Q_1$ and $(\sqrt{(K_2 : M_2)M_2} : M_2) = (Q_2 : M_2)$ or $(\sqrt{(N_2 : M_2)M_2} : M_2) = Q_2$. Therefore $G(\tau_T)$ is a complete bipartite graph with two parts $U$ and $V$ such that $N \subseteq U$ if and only if $V(N) = V(M_1 \oplus Q_2)$ and $K \subseteq V$ if and only if $V(K) = V(Q_1 \oplus M_2)$.

Corollary 3.2 Let $\bar{M}$ be a faithful module which does not have a non-zero submodule $\bar{F} \neq \bar{N}$ with $V(N) = T$. Then the following statements are equivalent.

(a) There is a vertex of $G(\tau_{\text{Spec}(M)})$ which is adjacent to every other vertex of $G(\tau_{\text{Spec}(M)})$.

(b) $G(\tau_{\text{Spec}(M)})$ is a star graph.

(c) $M = \bar{F} \oplus D$, where $\bar{F}$ is a simple module and $D$ is a prime module.

Proof (a) $\Rightarrow$ (b) Let $\bar{M}$ be a faithful module. Then $Q = (0)$ and we have $T = \text{Spec}(M)$. By Proposition 3.1, $M_1 \oplus M_2$, where $M_1$ is a simple module and $M_2$ is a prime module. Then every non-zero submodule of $M$ of the form $M_1 \oplus N_2$ and $(0) \oplus N_2$, where $N_2$ is a non-zero submodule of $M_2$. We show that non of the submodules of the form $(0) \oplus N_2$ can be adjacent to each other. Assume that $(0) \oplus N_2$ and $(0) \oplus K_2$ are adjacent in $G(\tau_{\text{Spec}(M)})$, where $(0) \neq N_2 \leq M_2$ and $(0) \neq K_2 \leq M_2$. Since $(0)$ is a prime submodule of $M_2$, by Remark 2.1, we have $N_2K_2 = (0)$. Hence $V((0) \oplus N_2) = \text{Spec}(M)$ or $V((0) \oplus K_2) = \text{Spec}(M)$, a contradiction. Similarly, we can not have any vertex of the form $M_1 \oplus N_2$, where $N_2$ is a non-zero proper submodule of $M_2$. Now it is easy to see that $M_1 \oplus (0)$ is adjacent to every other vertex and so $G(\tau_{\text{Spec}(M)})$ is a star graph.

(b) $\Rightarrow$ (c) This follows by Proposition 3.1(a).

(c) $\Rightarrow$ (a) Assume that $M = \bar{F} \oplus D$, where $\bar{F}$ is a simple module and $D$ is a prime module. Using the Proposition 3.1 (b), $G(\tau_{\text{Spec}(M)})$ is a complete bipartite graph with two parts $U$ and $V$ such that $N \subseteq U$ if and only if $V(N) = V(\bar{F} \oplus (0))$ and $K \subseteq V$ if and only if $V(K) = V((0) \oplus D)$. We claim that $|U| = 1$. Otherwise, $V(\bar{F} \oplus (0)) = V(N \oplus K)$, where $N = (0)$ or $N = \bar{F}$ and $(0) \neq K < D$. Therefore $V(N \oplus K) \cup V((0) \oplus D) = \text{Spec}(M)$ and hence $V((0) \oplus K) = \text{Spec}(M)$ that is a contradiction with our assumption. So $\bar{F} \oplus (0)$ is adjacent to every other vertex of $G(\tau_{\text{Spec}(M)})$.

Lemma 3.3 Let $e \in R$ be an idempotent element of $R$ and suppose that $\bar{M}$ does not have a non-zero submodule $\bar{F} \neq \bar{N}$ with $V(N) = T$. If $G(\tau_T)$ is a triangle-free graph, then both $\bar{M}_1$ and $\bar{M}_2$ are prime $R$-modules. Moreover, if $G(\tau_T)$ has no cycle, then $\bar{M}_1$ is a simple module and $\bar{M}_2$ is a prime module.
Proof First recall that if $\tilde{M}$ does not have a non-zero submodule $\overline{F} \neq \overline{N}$ with $V(N) = T$, then $F = Q$ because $V(F_1 \oplus Q_2) = V(Q_1 \oplus F_2) = T$. Without loss of generality, we can assume that $M_1$ is not a prime module. Then $IK = (0)$, where $(Q_1 : M_1) \subseteq I < R$ and $Q_1 \subseteq K \subseteq M_1$. It follows that $Q_1 \oplus M_2, K \oplus Q_2$, and $IM_1 \oplus Q_2$ form a triangle in $G(\tau_2)$, a contradiction (note that $V(K \oplus Q_2) \cup V(\{M_1 \oplus Q_2\}) = V(K(M_1) \oplus Q_2) = T$). Thus $IM_1 \neq K$. Otherwise, $V(K \oplus Q_3) = V(K^2 \oplus Q_2) = V(K(M_1) \oplus Q_2) = T$, a contradiction. So both $M_1$ and $M_2$ are prime-$R$ modules. Now suppose that $G(\tau_2)$ has no cycle. If none of $M_1$ and $M_2$ is a simple module, then we choose non-trivial submodules $N_i$ in $M_i$ for some $i = 1, 2$. So $Q_1 \oplus Q_2, Q_1 \oplus N_2, M_1 \oplus Q_2$, and $Q_1 \oplus M_2$ form a cycle, a contradiction. □

Corollary 3.4 Assume that $M$ is a multiplication module or a primeful module, $\text{Ann}(M)$ is a nil ideal, and $M$ does not have a non-zero submodule $\overline{F} \neq \overline{N}$ with $V(N) = T$. Then $G(\tau_2)$ is a star graph if and only if $\tilde{M}$ is a simple module and $M_2$ is a prime module for some idempotent $e \in R$.

Proof The necessity is clear by Proposition 3.1(a). For the converse, assume that $\tilde{M} = M_1 \oplus \tilde{M}_2$, where $\tilde{M}_1$ is a simple module and $M_2$ is a prime for some idempotent $e \in R$. Using the Proposition 3.1(b), $G(\tau_2)$ is a complete bipartite graph with two parts $U$ and $V$ such that $N \in U$ if and only if $V(N) = V(M_1 \oplus Q_2)$ and $K \in V$ if and only if $V(K) = V(Q_1 \oplus M_2)$. We claim that $|U| = 1$. Otherwise, $V(M_1 \oplus Q_2) = V(N_\emptyset \oplus N_\emptyset)$, where $N_1 \notin M_1$ and $N_2 \notin M_2$. If $N_1 \notin M_1$, then $\sqrt{N_1 : M_1} \cap M_1 = 1$, a contradiction (note that if $M$ is a multiplication module or a primeful module, then $\sqrt{N : M} \cap M = \emptyset$, where $N \in M$). If $N_2 \notin Q_2$, then $V(Q_1 \oplus N_2) = T$, a contradiction. So $G(\tau_2)$ is a star graph. □

Theorem 3.5 If $G(\tau_2)$ is a tree, then $G(\tau_2)$ is a star graph.

Proof Suppose that $G(\tau_2)$ is not a star graph. Then $G(\tau_2)$ has at least four vertices. Obviously, there are two adjacent vertices $L$ and $K$ of $G(\tau_2)$ such that $|N(L) \setminus \{K\}| \geq 1$ and $|N(K) \setminus \{L\}| \geq 1$. Let $N(L) \setminus \{K\} = \{L_i\}_{i \in \Lambda}$ and $N(K) \setminus \{L\} = \{K_j\}_{j \in \Gamma}$. Since $G(\tau_2)$ is a tree, we have $N(L) \cap N(K) = \emptyset$. By [4, Theorem 2.10], $\text{diam}(G(\tau_2)) \leq 3$. So every edge of $G(\tau_2)$ is of the form $(L, K), \{L, K_i\} \in \{K, K_j\}$, for some $i \in \Lambda$ and $j \in \Gamma$. Now, Pick $p \in \Lambda$ and $q \in \Gamma$. Since $G(\tau_2)$ is a tree, $L_pK_q$ is a vertex of $G(\tau_2)$. If $L_pK_q = L_u$ for some $u \in \Lambda$, then $V(LK_u) = T$, a contradiction. If $L_pK_q = L_v$ for some $v \in \Gamma$, then $V(LK_v) = T$, a contradiction. If $L_pK_q = L$ or $L_pK_q = K$, then $V(L^2) = T$ or $V(K^2) = T$, respectively, and then $V(L) = T$ or $V(K^2) = T$, a contradiction. So the claim is proved. □

Theorem 3.6 Let $R$ be an Artinian ring, $M$ be a multiplication or a primeful module, and suppose that $\tilde{M}$ does not have a non-zero submodule $\overline{F} \neq \overline{N}$ with $V(N) = T$. If $G(\tau_2)$ is a bipartite graph, then $|T| = 2$ and $G(\tau_2) \cong K_2$.

Proof At first we recall that if $G(\tau_2) \neq \emptyset$, then $|E(G(\tau_2))| \geq 1$. Assume that $G(\tau_2)$ is a bipartite graph. Therefore $G(\tau_2)$ is not empty. Show that $R$ cannot be a local ring. Otherwise, $m$ is the unique maximal ideal of $R$ and hence is the unique prime ideal. Then [14, Corollary 2.11] implies that $mM$ is the only prime submodule of $M$ so that $G(\tau_2) = \emptyset$, a contradiction. Hence by [8, Theorem 8.7], $R = R_1 \oplus \ldots \oplus R_n$, where $R_i$ is an Artinian local ring for $i = 1, \ldots, n$ and $n \geq 2$. By Lemma 2.2 and Proposition 2.3, since $G(\tau_2)$ is a bipartite graph, we have $n = 2$ and hence $\tilde{M} \cong M_1 \oplus M_2$ for some idempotent $e \in R$ (for example, if $n = 3$, then $M_1 \oplus Q_2 \oplus Q_3, Q_1 \oplus M_2 \oplus Q_3$, and $Q_1 \oplus Q_2 \oplus M_3$ form a triangle that is a contradiction). By Lemma 3.3, $M_1$ and $M_2$ are prime modules. Then it is easy to see that $M_1$ and $M_2$ are vector spaces over $R/\text{Ann}(M_1)$ and $R/\text{Ann}(M_2)$, respectively and so are semisimple $R$-modules. Since $G(\tau_2)$ is a bipartite graph, $M_1$ and $M_2$ are simple $R$-modules. A Similar argument as we did in proof of Corollary 3.4 implies that $T = \{M_1 \oplus Q_2, Q_1 \oplus M_2\}$ and $G(\tau_2) \cong K_2$. □

Proposition 3.7 Assume that $M$ is a multiplication module, $\text{Ann}(M)$ is a nil ideal, and $\tilde{M}$ does not have a non-zero submodule $\overline{F} \neq \overline{N}$ with $V(N) = T$.

(a) If $G(\tau_2)$ is a finite bipartite graph, then $|T| = 2$ and $G(\tau_2) \cong K_2$.

(b) If $G(\tau_2)$ is a regular graph of finite degree, then $|T| = 2$ and $G(\tau_2) \cong K_2$.

Proof (a) By Theorem 2.11, $\tilde{M}$ is an Artinian and Noetherian module so that $R/\text{Ann}(\tilde{M})$ is an Artinian ring. A similar arguments in Theorem 3.6 says that, $R/\text{Ann}(M)$ is a non-local ring. So by [8, Theorem 8.7] and Lemma 2.2, there exist pairwise orthogonal idempotents modulo $\text{Ann}(M)$. By Lemma 2.4, $M \cong M_1 \oplus M_2$, for some idempotent $e \in R$. Now, the proof that $G(\tau_2) \cong K_2$ is similar to the proof of Theorem 3.6.

(b) We may assume that $G(\tau_2)$ is not empty. So $F$ is not a prime submodule by [4, Remark 2.6] and hence there exist $r \in R$ and $m \in M$ such that $rm \in F$ but $m \notin F$ and $r \notin (F : M)$. A similar manner in proof of
Theorem 2.11, shows that if the set of \( R \)-submodules of \( \overline{\mathcal{M}} \) (resp. \( (0 : \overline{\mathcal{M}} r) \)) is infinite, then \( (0 : \overline{\mathcal{M}} r) \) (resp. \( \overline{\mathcal{M}} \)) has infinite degree, a contradiction. Thus \( \overline{\mathcal{M}} \) and \( (0 : \overline{\mathcal{M}} r) \) have finite length so that \( M \) has a finite length. Therefore \( R / \text{Ann}(\overline{\mathcal{M}}) \) is an Artinian ring. As in the proof of part (a), \( \overline{\mathcal{M}} \cong \overline{M}_1 \oplus \overline{M}_2 \) for some idempotent \( e \in R \). If \( \overline{M}_1 \) has one non-trivial submodule \( \overline{N} \), then \( \deg(Q_1 \oplus M_2) > \deg(N \oplus M_2) \) (we note that by [6, Proposition 2.5], \( N \overline{K} = (0) \) for some \( (0) \neq \overline{K} < M_1 \)) and this contradicts the regularity of \( G(\tau_T) \). Hence \( \overline{M}_1 \) is a simple module. Similarly, \( \overline{M}_2 \) is a simple module. Finally a similar argument as we have seen in Theorem 3.6 gives \( G(\tau_T) \cong K_2 \).

\[ \square \]

4 Coloring of the Zariski-topology graph of modules

The purpose of this section is to study the coloring of the Zariski topology-graph of modules and investigate the interplay between \( \chi(G(\tau_T)) \) and \( \omega(G(\tau_T)) \). We note that since \( E(G(\tau_T)) \geq 1 \) when \( G(\tau_T) \neq \emptyset \), then \( \chi(G(\tau_T)) \geq 2 \).

**Theorem 4.1** Let \( \overline{\mathcal{M}} \) be an Artinian module such that for every minimal submodule \( \overline{N} \) of \( \overline{\mathcal{M}} \), \( N \) is a vertex in \( G(\tau_T) \). Then \( \omega(G(\tau_T)) = \chi(G(\tau_T)) \).

**Proof** \( \overline{\mathcal{M}} \) is Artinian, so it contains a minimal submodule. Since for every minimal submodule \( \overline{N} \) of \( \overline{\mathcal{M}} \), \( N \) is a vertex in \( G(\tau_T) \), we have \( V(N) \neq T \). Also, \( N \cap L = \{ \} \) where \( N \) and \( L \) are minimal submodules of \( \mathcal{M} \). It follows that \( N \) and \( L \) are adjacent in \( G(\tau_T) \), where \( \overline{N} \) and \( \overline{L} \) are minimal submodules of \( \mathcal{M} \). First, suppose that \( \mathcal{M} \) has infinitely many minimal submodules. Then \( \omega(G(\tau_T)) = \infty \) and there is nothing to prove. Next, assume that \( \mathcal{M} \) has \( k \) minimal submodules, where \( k \) is finite. We conclude that \( \chi(G(\tau_T)) = k = \omega(G(\tau_T)) \).

Obviously, \( \omega(G(\tau_T)) \geq k \). If possible, assume that \( \omega(G(\tau_T)) > k \). Let \( \Sigma = \{ N_j \}_{j \in I} \), where \( |I| = \omega(G(\tau_T)) \) be a maximum clique in \( G(\tau_T) \). As for every \( N_i \in \Sigma \), \( \sqrt{\langle N_j : M \rangle} \) contains a minimal submodule, there exists a minimal submodule \( \overline{K} \) and submodules \( N_i \) and \( N_j \) in \( \Sigma \), such that \( \overline{K} \subseteq \sqrt{\langle N_j : M \rangle} \cap \sqrt{\langle N_j : M \rangle} \), and hence \( V(K) = T \), a contradiction. Hence \( \omega(G(\tau_T)) = k \). Next, we claim that \( G(\tau_T) \) is \( k \)-colorable. In order to prove, put \( A = \{ K_1, \ldots, K_k \} \) be the set of all minimal submodules of \( \mathcal{M} \). Now, we define a coloring \( f \) on \( G(\tau_T) \) by setting \( f(N) = \min \{ t | K_t \subseteq \sqrt{\langle N : M \rangle} \} \) for every vertex \( N \) of \( G(\tau_T) \). Let \( N \) and \( L \) be adjacent in \( G(\tau_T) \) and \( f(N) = f(L) = j \). Thus \( K_j \subseteq \sqrt{\langle N : M \rangle} \cap \sqrt{\langle L : M \rangle} \), a contradiction. It implies that \( f \) is a proper \( k \) coloring of \( G(\tau_T) \) and hence \( \chi(G(\tau_T)) \leq k = \omega(G(\tau_T)) \), as desired. \( \square \)

**Theorem 4.2** Assume that \( \overline{\mathcal{M}} \) is a faithful module. Then the following statements are equivalent.

(a) \( \chi(G(\tau_{\text{Spec}(\mathcal{M})})) = 2 \).
(b) \( G(\tau_{\text{Spec}(\mathcal{M})}) \) is a bipartite graph with two non-empty parts.
(c) \( G(\tau_{\text{Spec}(\mathcal{M})}) \) is a complete bipartite graph with two non-empty parts.
(d) Either \( R \) is a reduced ring with exactly two minimal prime ideals or \( G(\tau_{\text{Spec}(\mathcal{M})}) \) is a star graph with more than one vertex.

**Proof** By using Lemma 2.8, \( G(\tau_{\text{Spec}(\mathcal{M})}) \) and \( AG(M)^* \) are the same and so [5, Theorem 3.3] completes the proof. \( \square \)

**Lemma 4.3** Assume that \( T \) is a finite set. Then \( \chi(G(\tau_T)) \) is finite. In particular, \( \omega(G(\tau_T)) \) is finite.

**Proof** Suppose that \( T = \{ P_1, P_2, \ldots, P_k \} \) is a finite set of distinct prime submodules of \( M \). Define a coloring \( f(N) = \min \{ n \in \mathbb{N} | P_n \notin V(N) \} \), where \( N \) is a vertex of \( G(\tau_T) \). We can see that \( \chi(G(\tau_T)) \leq k \). \( \square \)

**Theorem 4.4** For every module \( M \), \( \omega(G(\tau_T)) = 2 \) if and only if \( \chi(G(\tau_T)) = 2 \). In particular, \( G(\tau_T) \) is bipartite if and only if \( G(\tau_T) \) is triangle-free.

**Proof** Let \( \omega(G(\tau_T)) = 2 \). On the contrary assume that \( G(\tau_T) \) is not bipartite. So \( G(\tau_T) \) contains an odd cycle. Suppose that \( C := N_1 - N_2 - \cdots - N_{2k+1} - N_1 \) is a shortest odd cycle in \( G(\tau_T) \) for some natural number \( k \). Clearly, \( k \geq 2 \). Since \( C \) is a shortest odd cycle in \( G(\tau_T) \), \( N_3N_{2k+1} \) is a vertex. Now consider the vertices \( N_1, N_2, \) and \( N_3N_{2k+1} \). If \( N_1 = N_3N_{2k+1} \), then \( V(N_1N_1) = T \). This implies that \( N_1 - N_4 - \cdots - N_{2k+1} - N_1 \) is an odd cycle, a contradiction. Thus \( N_1 \neq N_3N_{2k+1} \). If \( N_2 = N_3N_{2k+1} \), then we have \( C_3 = N_2 - N_3 - N_4 - N_2 \), again a contradiction. Hence \( N_2 \neq N_3N_{2k+1} \). It is easy to check \( N_1, N_2, \) and \( N_3N_{2k+1} \) form a triangle in \( G(\tau_T) \), a contradiction. The converse is clear. We note that empty graphs are bipartite graphs. \( \square \)
Corollary 4.5 Assume that $e \in R$ is an idempotent element and $\tilde{M}$ does not have a non-zero submodule $\mathcal{F} \neq N$ with $V(N) = T$. Then $G(\tau_T)$ is a complete bipartite graph if and only if $M_1$ and $M_2$ are prime modules.

Proof Assume that $G(\tau_T)$ is a complete bipartite graph. Therefore Theorem 4.4 states that $G(\tau_T)$ is a triangle-free graph. So Lemma 3.3 follows that $M_1$ and $M_2$ are prime modules. The conversely holds by Proposition 3.1(b). □

Remark 4.6 Assume that $S$ is a multiplicatively closed subset of $R$ such that $S \cap (\cup_{P \in T}(P : M)) = \emptyset$. Let $T_S = [S^{-1}P | P \in T]$. One can see that $V(N) = T$ if and only if $V(S^{-1}N) = T_S$, where $M$ is a finitely generated module.

Theorem 4.7 Let $S$ be a multiplicatively closed subset of $R$ defined as in Remark 4.6 and $M$ is a finitely generated module. Then $G(\tau_{T_S})$ is a retract of $G(\tau_T)$ and $\omega(G(\tau_{T_S})) = \omega(G(\tau_T))$.

Proof Consider a vertex map $\phi : V(G(\tau_T)) \rightarrow V(G(\tau_{T_S})), N \rightarrow N_S$. Clearly, $N_S \neq K_S$ implies that $\omega(G(\tau_{T_S})) \leq \omega(G(\tau_T))$. If $N \neq K$ and $V(N) \cup V(K) = T$, then $V(N_S) \cup V(K_S) = T_S$. Thus $\phi$ is surjective and hence $\omega(G(\tau_{T_S})) = \omega(G(\tau_T)).$ If $N \neq K$ and $V(N) \cup V(K) = T$, then we show that $N_S \neq K_S$. On the contrary suppose that $N_S = K_S$. Then $V(N_S^2) = V(N_SK_S) = V(N_S) \cup V(K_S) = T_S$ and so $V(N^2) = T$, a contradiction. This shows that the map $\phi$ is a graph homomorphism. Now, for any vertex $N_S$ of $G(\tau_{T_S})$, we can choose a fixed vertex $N$ of $G(\tau_T)$. Then $\phi$ is a retract (graph) homomorphism which clearly implies that $\omega(G(\tau_{T_S})) = \omega(G(\tau_T))$ under the assumption. □

Corollary 4.8 Let $S$ be a multiplicatively closed subset of $R$ defined as in Remark 4.6 and let $M$ be a finitely generated module. Then $\chi(AG(M_S)) = \chi(AG(M))$.

Corollary 4.9 Assume that $M$ is a semiprime module and $AG(M)^*$ does not have an infinite clique. Then $M$ is a faithful module and the last assertion follows directly from the proof of [5, Theorem 3.8 (b)].

Proof By [5, Theorem 3.8 (b)], $M$ is a faithful module and the last assertion follows directly from the proof of [5, Theorem 3.8 (b)]. □

Recall that the girth of a graph $G$ is the length of a shortest cycle in $G$ and denoted by $gr(G)$.

Proposition 4.10 Let $R$ be an Artinian ring, $\tilde{M}$ be a multiplication module, and let $T$ be a closed subset of $\text{Spec}(M)$. Then we have the following statements.

(a) If $S$ is a finite subset of $T$, then there exists a clique of size $|S|$ in $G(\tau_T)$.
(b) We have $\omega(G(\tau_T)) \geq |\text{Min}(T)|$ and if $|\text{Min}(T)| \geq 3$, then $\text{gr}(G(\tau_T)) = 3$.
(c) If $\sqrt{(0)} = (0)$, then $\chi(G(\tau_{\text{Spec}(M)})) = \omega(G(\tau_{\text{Spec}(M)})) = |\text{Min}(T)|$.

Proof (a) Let $R$ be an Artinian ring and let $\tilde{M}$ be a multiplication module. Then [14, Corollary 2.9] implies that $\tilde{M}$ is a cyclic module. We show that $T = \text{Min}(T)$. Suppose that $P_1 \subseteq P_2$, where $P_1, P_2 \in T$. Then $(P_1 : M) = (P_2 : M)$ because every prime ideal in $R$ is maximal. Since $\tilde{M}$ is multiplication, we have $P_1 = P_2$ and finally the proof is straightforward by the facts that $AG(M) = AG(M)^*$, [6, Theorem 3.6], and $AG(M)$ is isomorphic with a subgraph of $G(\tau_T)$ by Lemma 2.8.

(b) This is clear by item (a).
(c) If $|\text{Min}(T)| = \infty$, then by part (b), there is nothing to prove. Otherwise, [6, Theorem 3.8] implies that $AG(M)$ does not have an infinite clique. So $\tilde{M}$ is a faithful module by Corollary 4.9. Next, Lemma 2.8 says that $G(\tau_{\text{Spec}(M)})$ and $AG(M)^*$ are the same. Now the result follows by [6, Theorem 3.8]. □

Lemma 4.11 Assume that $\tilde{M}$ is a semiprime module. Then the following statements are equivalent.

(a) $\chi(G(\tau_{\text{Spec}(M)})))$ is finite.
(b) $\omega(G(\tau_{\text{Spec}(M)})))$ is finite.
(c) $G(\tau_{\text{Spec}(M)}))$ does not have an infinite clique.

Proof (a) $\implies$ (b) $\implies$ (c) is clear.

(c) $\implies$ (a) Suppose that $G(\tau_{\text{Spec}(M)}))$ does not have an infinite clique. By Lemma 2.8, $AG(\tilde{M})^*$ does not have an infinite clique and so by Corollary 4.9, there exists a finite number of prime submodules $P_1, \ldots, P_k$ of $M$ such that $(F : M) = (P_1 \cap \ldots \cap P_k : M)$. Define a coloring $f(N) = \text{min}\{n \in \mathbb{N} | P_n \notin V(N)\}$, where $N$ is a vertex of $G(\tau_T)$. Then we have $\chi(G(\tau_{\text{Spec}(M)}))) \leq k$. □
Corollary 4.12 Assume that \( AG(M/F)^* \) does not have an infinite clique. Then \( G(\tau_{\text{Spec}(M)}) \) and \( AG(M)^* \) are the same. Also, \( \chi(G(\tau_{\text{Spec}(M)})) \) is finite.

**Proof** Since \( M/F \) is a semiprime module, by Corollary 4.9, \( M/F \) is a faithful module and there exists a finite number of prime submodules \( P_1, \ldots, P_k \) of \( M \) such that \( (F : M) = (P_1 \cap \ldots \cap P_k : M) \). So the result follows by Lemma 2.8 and from the proof of (c) \( \implies \) (a) of Lemma 4.11. \( \square \)

We recall that \( M \) is said to be \( X \)-injective if either \( \text{Spec}(M) = \emptyset \) or the natural map of \( X = \text{Spec}(M) \) is injective (see [7]).

**Proposition 4.13** Suppose that \( \sqrt{(0)} = (0) \), for every minimal member \( P \) of \( \text{Spec}(M) \), \( (P : M) \) is a minimal ideal of \( R \), and \( M \) is an \( X \)-injective module. Then the following statements are equivalent.

(a) \( \chi(G(\tau_{\text{Spec}(M)})) \) is finite.
(b) \( \omega(G(\tau_{\text{Spec}(M)})) \) is finite.
(c) \( G(\tau_{\text{Spec}(M)}) \) does not have an infinite clique.
(d) \( \text{Min}(\text{Spec}(M)) \) is a finite set.

**Proof** (a) \( \implies \) (b) \( \implies \) (c) is clear.

(c) \( \implies \) (d) Suppose \( G(\tau_{\text{Spec}(M)}) \) does not have an infinite clique. By Lemma 2.8, \( AG(M)^* \) does not have an infinite clique and hence by Corollary 4.9, there exists a finite number of prime submodules \( P_1, \ldots, P_k \) of \( M \) such that \( (F : M) = (P_1 \cap P_2 \cap \ldots \cap P_k : M) \). By assumptions, one can see that \( \text{Min}(\text{Spec}(M)) \) is a finite set.

(d) \( \implies \) (a) Assume that \( \text{Min}(\text{Spec}(M)) \) is a finite set (equivalently, \( \tilde{M} \) has a finite number of minimal prime submodules) so that \( (F : M) = (P_1 \cap P_2 \cap \ldots \cap P_k : M) \), where \( \text{Min}(\text{Spec}(M)) = \{P_1, \ldots, P_k\} \). Define a coloring \( f(N) = \min\{n \in N | P_n \notin V(N)\} \), where \( N \) is a vertex of \( G(\tau_{\text{Spec}(M)}) \). Then we have \( \chi(G(\tau_{\text{Spec}(M)})) \leq k \). \( \square \)

**Example 4.14** If \( M \) is a faithfully flat \( R \)-module (for example, free modules), then \( pM \) is a \( p \)-prime submodule of \( M \), where \( p \) is a prime ideal of \( R \) by [10, Theorem 3]. So for every minimal prime submodule \( P \) of \( M \), \( (P : M) \) is a minimal ideal of \( R \).

**Proposition 4.15** Assume that \( \sqrt{(0)} = (0) \) and \( \tilde{M} \) is a faithful module. Then the following statements are equivalent.

(a) \( \chi(G(\tau_{\text{Spec}(M)})) \) is finite.
(b) \( \omega(G(\tau_{\text{Spec}(M)})) \) is finite.
(c) \( G(\tau_{\text{Spec}(M)}) \) does not have an infinite clique.
(d) \( R \) has a finite number of minimal prime ideals.
(e) \( \chi(G(\tau_{\text{Spec}(M)})) = \omega(G(\tau_{\text{Spec}(M)})) = |\text{Min}(R)| = k \), where \( k \) is finite.

**Proof** This is clear by Lemma 2.8, [5, Proposition 3.10], and [5, Corollary 3.11]. \( \square \)

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