The Dantzig selector: Recovery of Signal via $\ell_1 - \alpha \ell_2$ Minimization

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Abstract

In the paper, we proposed the Dantzig selector based on the $\ell_1 - \alpha \ell_2$ ($0 < \alpha \leq 1$) minimization for the signal recovery. In the Dantzig selector, the constraint $\|A^\top (b - Ax)\|_\infty \leq \eta$ for some small constant $\eta > 0$ means the columns of $A$ has very weakly correlated with the error vector $e = Ax - b$. First, recovery guarantees based on the restricted isometry property (RIP) are established for signals. Next, we propose the effective algorithm to solve the proposed Dantzig selector. Last, we illustrate the proposed model and algorithm by extensive numerical experiments for the recovery of signals in the cases of Gaussian, impulsive and uniform noise. And the performance of the proposed Dantzig selector is better than that of the existing methods.

Key Words and Phrases. Dantzig selector, $\ell_1 - \alpha \ell_2$ minimization, Sparse signal recovery, Restricted isometry property.

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1 Introduction

1.1 Signal Recovery

We consider the linear regression model

$$b = Ax + e,$$

where $b \in \mathbb{R}^m$ are available measurements, the matrix $A \in \mathbb{R}^{m \times n}$ ($m \ll n$) models the linear measurement process, $x \in \mathbb{R}^n$ is unknown signal and $e \in \mathbb{R}^m$ is a vector of measurement errors. To reconstruct $x$, the most intuitive approach is to find the sparsest signal in the set of feasible solutions, that is, one solves the $\ell_0$ minimization problem:

$$\min_{x \in \mathbb{R}^n} \|x\|_0 \text{ subject to } b - Ax \in \mathcal{B},$$

where $\|x\|_0$ (it usually is called the $\ell_0$ norm of $x$, but is not a norm) denotes the number of nonzero coordinates of $x$, and $\mathcal{B}$ is a bounded set determined by the error structure.

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However, this problem (1.2) is NP-hard and thus computationally infeasible in high-dimensional background.

The underdetermined problem (1.1) puts forward both theoretical and computational challenges at the interface of statistics and optimization (see, e.g., [15, 38, 58]). In the linear regression model, the so-called Dantzig selector [8] was proposed to perform variable selection and model fitting. Its formulation model is

$$\min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{subject to} \quad \|A^T(b - Ax)\|_\infty \leq \eta$$

where $\eta \geq 0$ is a tuning or penalty parameter. In [8], the performance of Dantzig selector was analyzed theoretically by deriving sharp nonasymptotic bounds on the error of estimated coefficients in the $\ell_2$ norm.

In Dantzig selector, the constraint $\|A^T(b - Ax)\|_\infty \leq \eta$ implies that the correlation between the residual vector $e = Ax - b$ and the columns of $A$ is small for the small penalty parameter $\eta$. Moreover, the constraint can be viewed as a data fitting term and it does not force the residual $e = Ax - b$ like the $\ell_2$-bounded Gaussian noise. The Dantzig selector has a wide range of potential applications, especially in statistics. In Fig 1 and Table 1, we present a graphical illustration for Gaussian, impulsive and uniform noises. And we show their distributions and probability density functions (PDF) as following.

1. Gaussian Distribution: $e \sim \mathcal{N}(0, \sigma^2) \in \mathbb{R}^{m \times 1}$. The noise is usually modeled by the $\ell_2$ norm, i.e., $\|Ax - b\|_2$ with $b = Ax + e$. The probability density function $p$ of $e$ is

$$p(e) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{e^2}{2\sigma^2}\right)$$

where $\sigma$ is standard deviation.

2. Distribution of impulsive noise: the distribution is Symmetric $\tilde{\alpha}$-stable ($S\tilde{\alpha}S$) distribution, which has been used to model impulsive noise in [43, 44, 46, 47]. The noise is usually modeled by the $\ell_1$ norm, i.e., $\|Ax - b\|_1$ with $b = Ax + e$. Although one cannot analytically present the probability density function for a general stable distribution, its characteristic function of a zero-location $S\tilde{\alpha}S$ distribution can be expressed as

$$\phi(\omega) = \exp(i\tilde{\delta}\omega - \gamma\tilde{\alpha}|\omega|^\tilde{\alpha})$$

where $\tilde{\delta} \in (-\infty, \infty)$ is the location parameter, $\gamma \in (0, \infty)$ is the scale parameter, and $\tilde{\alpha}$ is the characteristic exponent measuring the thickness of the distributional tail with $\tilde{\alpha} \in (0, 2]$. If the value of $\tilde{\alpha}$ is smaller, then the tail of the $S\tilde{\alpha}S$ distribution is thicker and consequently the noise is more impulsive.

3. Uniform Distribution: $e \sim \mathcal{U}(-\varsigma, \varsigma) \in \mathbb{R}^{m \times 1}$. Its probability density function is

$$p(e) = \begin{cases} \frac{1}{(2\varsigma)^m}, & \text{if } -\varsigma \leq e_j \leq \varsigma, \\ 0, & \text{otherwise.} \end{cases}$$

The noise is usually modeled by the $\ell_\infty$ norm, i.e., $\|Ax - b\|_\infty$ with $b = Ax + e$. The $\ell_\infty$ minimization problem arises in curve fitting [50], optimal control of partial differential equations [13], image compression [2, 51, 59]. More results about the uniform noise, see [12, 49, 57].
Figure 1: Probability density function (PDF) for Gaussian, impulsive and uniform noises. Left: Gaussian noise with $\sigma = 10^{-2}, 5 \times 10^{-2}$; Middle: S\(\tilde{\alpha}\) type impulsive noise with $\tilde{\alpha} = 1, \tilde{\delta} = 0, \gamma = 10^{-2}, 5 \times 10^{-3}$; Right: Uniform noise with $\varsigma = 10^{-1}, 5 \times 10^{-1}$.

In Table 1, we display the average of $\|A^T e\|_\infty$ over 1000 repeated tests, where $e$ is a noisy vector and $A$ is measurement matrix. Here, let $A$ be Gaussian matrix or the oversampled partial DCT matrix.

(1) The random Gaussian matrix $A \in \mathbb{R}^{m \times n}$ satisfies $A_i \sim \mathcal{N}(0, \frac{1}{m})$, $i = 1, \ldots, n$. The random Gaussian matrix is of particular interest in the practical and theoretical research. It has been a very active area of recent research in signal processing \[14, 41\] and image processing \[25, 34\]. The random matrix $A$ has small coherence and RIP constants (see Definition (1)) with high probability \[7, 10\]. The coherence of a matrix $A$ in \[16\] is the maximum absolute value of the cross-correlations between the columns of $A$, namely,

$$\mu(A) := \max_{i \neq j} \frac{|\langle A_i, A_j \rangle|}{\|A_i\|_2 \|A_j\|_2}.$$

(2) The randomly oversampled partial DCT matrix $A \in \mathbb{R}^{m \times n}$ satisfies

$$A_i = \cos\left(\frac{2\pi \xi_j}{F}\right) \sqrt{\frac{1}{m}}, \quad i = 1, \ldots, n,$$

where $\xi \in \mathbb{R}^m \sim \mathcal{U}([0, 1]^m)$ which means $\xi$ uniformly and independently distributes in $[0, 1]^m$, and $F \in \mathbb{N}$ is the refinement factor. Actually, it is the real part of the random partial Fourier matrix analyzed in \[19\]. The number $F$ is closely related to the conditioning of $A$ in the sense that $\mu(A)$ tends to get larger as $F$ increases. For example, for $A \in \mathbb{R}^{32 \times 640}$, $\mu(A) \approx 0.97$ when $F = 5$, and $\mu(A)$ easily exceeds 0.99 when $F = 10$. The over-sampled DCT matrices are derived from the problem of spectral estimation \[19, 29\] in signal processing, and radar imagining \[19, 21\] and surface scattering \[20\] in image processing.

Table 1 shows $\|A^T(Ax - b)\|_\infty$, which measures the correlation between the noisy vector $e = Ax - b$ and the columns of $A$, where $e$ is Gaussian, impulsive and uniform noise. It works efficiently for the three type noises (see to Section 6), which are different from that of the $\ell_2$ norm $\|Ax - b\|_2$, the $\ell_1$ norm $\|Ax - b\|_1$ and the $\ell_\infty$ norm $\|Ax - b\|_\infty$. These norms only work efficiently for their corresponding noises, i.e., the $\ell_2$ norm $\|Ax - b\|_2$ is only valid for Gaussian noise, the $\ell_1$ norm $\|Ax - b\|_1$ is only valid for impulsive noise, and the $\ell_\infty$ norm $\|Ax - b\|_\infty$ is only valid for bounded noise.
Table 1: The average of $\|A^T e\|_\infty$ over $10^4$ repeated tests. Let $e = Ax - b$, $A$ be Gaussian matrix with $n = 64, m = 256$ and the oversampled DCT matrix with $n = 64, m = 256, F = 10$. Here we take Gaussian noise with noise levels $\sigma = 10^{-2}, 5 \times 10^{-2}$; $S\alpha S$ type impulsive noise with $\tilde{\alpha} = 1, \tilde{\delta} = 0$ and $\gamma = 5 \times 10^{-3}, 10^{-2}$; uniform noise with noise levels $\varsigma = 10^{-1}, 5 \times 10^{-1}$.

\begin{tabular}{|c|c|c|c|}
\hline
Noise Type & Parameter & Gaussian Matrix & Oversampled DCT \\
\hline
Gaussian & $\sigma = 1e-2$ & 0.0295 & 0.0270 \\
 & $\sigma = 5e-2$ & 0.1476 & 0.1350 \\
\hline
Impulsive & $\gamma = 5e-3$ & 0.6936 & 0.3610 \\
 & $\gamma = 1e-2$ & 1.2604 & 1.397 \\
\hline
Uniform & $\varsigma = 1e-1$ & 0.1710 & 0.1571 \\
 & $\varsigma = 5e-1$ & 0.8563 & 0.7850 \\
\hline
\end{tabular}

Authors in [42] proposed the Least Absolute Shrinkage and Selector Operator (Lasso) as follows

$$\min_{x \in \mathbb{R}^n} \lambda \|x\|_1 + \frac{1}{2} \|Ax - b\|_2^2,$$  \hspace{1cm} (1.4)

where $\lambda > 0$ is a parameter to balance the data fidelity term $\frac{1}{2} \|Ax - b\|_2^2$ and the objective function $\|x\|_1$. In some sense, Lasso estimator and Dantzig selector exhibit similar behavior. Essentially, the Dantzig selector model (1.3) is a linear program while the Lasso model (1.4) is a quadratic program. For an extensive study on the relation between the Dantzig selector and Lasso, we refer to a series of discussion papers which have been published in “The Annals of Statistics”, e.g., [1, 3, 9, 17, 22, 39, 40]. Readers also can refer to [18, Chapter 8].

Many effective algorithms have been developed to solve Dantzig selector. For example, Candès et.al. in [5] apply primal-dual algorithm, Wang et.al. in [45] use Linear ADMM, and Lu et.al. in [35] solve Dantzig selector on account of ADMM. Chatterjee et.al. in [11] also proposed a Generalized Dantzig Selector (GDS) and solved it by ADMM.

1.2 Contributions

In this paper, we introduce the following $\ell_1 - \alpha \ell_2$ minimization problem:

$$\min_{x \in \mathbb{R}^n} \|x\|_1 - \alpha \|x\|_2 \text{ subject to } \|A^T (b - Ax)\|_\infty \leq \eta$$ \hspace{1cm} (1.5)

for some constant $\eta \geq 0$. Denote (1.5) as $\ell_1 - \alpha \ell_2$-DS. When $\eta = 0$, note that (1.5) could not reduce to

$$\min_{x \in \mathbb{R}^n} \|x\|_1 - \alpha \|x\|_2 \text{ subject to } Ax = b,$$ \hspace{1cm} (1.6)

which means that the $\ell_1 - \alpha \ell_2$-DS (1.5) is different from the minimization problems such as the $\ell_0$ minimization (1.2) with the constraint term $\|Ax - b\|_p \leq \eta$ for $0 < p \leq 2$. In fact, $\eta = 0$ means that $A$ is orthogonal with $Ax = b$, i.e., $A^T (b - Ax) = 0$. 

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Similarly, the nonconvex \( \ell_1 - \alpha \ell_2 \) \((0 < \alpha \leq 1)\) minimization method was introduced in [30, 31] to recover \( x \in \mathbb{R}^n \)
\[
\min_{x \in \mathbb{R}^n} \|x\|_1 - \alpha \|x\|_2 \quad \text{subject to} \quad b - Ax \in B.
\]
Clearly, the method (1.7) with \( \alpha = 1 \) reduces to the \( \ell_1 - \ell_2 \) minimization method [32, 55]. Specifically, Lou et al. [32] and Yin et al. [55] respectively studied the \( \ell_1 - \ell_2 \) minimization under \( B = \{0\} \) and \( B = \{e : \|e\|_2 \leq \eta\} \), respectively. They obtained sufficient conditions based on RIP for the recovery of \( x \) from (1.1) via the \( \ell_1 - \ell_2 \) minimization method. To solve (1.7), they proposed the unconstrained problem:
\[
\min_{x \in \mathbb{R}^n} \lambda(\|x\|_1 - \|x\|_2) + \frac{1}{2}\|Ax - b\|_2^2,
\]
and an effective algorithm based on the different of convex algorithm (DCA) to solve (1.8). Several numerical examples in [32, 55] have demonstrated that the \( \ell_1 - \ell_2 \) minimization consistently outperforms the \( \ell_1 \) minimization and the \( \ell_p \) minimization in [27] when the measurement matrix \( A \) is highly coherent. In addition, the metric \( \ell_1 - \ell_2 \) has shown advantages in various applications such as signal processing [24, 28, 48], point source super-resolution [33], image restoration [23, 28, 34], matrix completion [36], uncertainty quantification [26, 54] and phase retrieval [52, 56].

In this paper, the main contributions are as followings:

(i) We show sufficient conditions under RIP frame for the recovery of the signal \( x \) from (1.1) via the \( \ell_1 - \alpha \ell_2 \)-DS (1.5).

(ii) We propose an unconstrained penalty problem (5.1) to solve the \( \ell_1 - \alpha \ell_2 \)-DS (1.5), and develop an effective algorithm based on ADMM to solve the proposed unconstrained penalty problem.

(iii) We present numerical experiments for the recover signal in the cases of Gaussian, impulsive and uniform noises to illustrate the performance of the \( \ell_1 - \alpha \ell_2 \)DS. As far as we know, this is the first paper which explore the performances of Dantzig selector under different noises.

1.3 Organization and Notations

The rest of the paper is organized as follows. We recall some definitions and lemmas in Section 2. In Section 3, the theoretical results based on \((\ell_2, \ell_1)\)-RIP frame are showed for the signal recovery via the \( \ell_1 - \alpha \ell_2 \) minimization (1.5). We show sufficient conditions for the stable recovery of signal under \((\ell_2, \ell_2)\)-RIP (i.e., classical RIP) frame via the \( \ell_1 - \alpha \ell_2 \) minimization (1.5) in Section 4. Effective algorithms to solve (1.5) is developed in Section 5. In Section 6, numerical results for sparse signals is given. Section 7 presents a conclusion.

Throughout the paper, we use the following basic notations. Denote the positive integer set by \( \mathbb{Z}_+ \). Let \( |S| \) be the number of entries in the set \( S \). Let \( \lceil t \rceil \) be the nearest integer greater than or equal to \( t \). For any positive integer \( n \), let \( \llbracket 1, n \rrbracket \) be the set \( \{1, \ldots, n\} \). For \( x \in \mathbb{R}^n \), denote \( x_{\text{max}(s)} \) as the vector \( x \) with all but the largest \( s \) entries in absolute value set to zero, and \( x_{-\text{max}(s)} = x - x_{\text{max}(s)} \). Let \( x_S \) be the vector equal to \( x \) on \( S \) and to
zero on $S^c$. Let $\|x\|_{\alpha,1-2}$ be $\|x\|_1 - \alpha \|x\|_2$. Especially, when $\alpha = 1$, denote $\|x\|_{\alpha,1-2}$ with $\|x\|_{1-2}$. And we denote $n \times n$ identity matrix by $I_n$ and zeros matrix by $O$. And we denote the transpose of matrix $A$ by $A^T$. Use the phrase “s-sparse vector” to refer to vectors of sparsity at most $s$. We use boldfaced letter denote matrix or vector. The $A \succeq O$ (resp., $A > O$) represents that the matrix $A$ is positive semidefinite (resp. positive definite) and denote the set of all positive semidefinite (resp. positive definite) matrices of size $n$ by $S^n_+$ (resp. $S^n_{++}$). Given $A \succeq O$ of size $n$, we define $\langle u, v \rangle_A := u^T A v$ and $\|u\|_A := \sqrt{\langle u, u \rangle_A}$ for vectors $u, v \in \mathbb{R}^n$. For a positive definite matrix $A$, $\langle \cdot, \cdot \rangle_A$ and $\| \cdot \|_A$ define an inner product and norm on $\mathbb{R}^n$ respectively, which become the standard inner product $\langle \cdot, \cdot \rangle$ and Euclidean norm $\| \cdot \|_2$ respectively when $A$ is the identity matrix $I$.

## 2 Preliminaries

In this section, we recall some significant definitions and lemmas in order to characterize the recovery guarantees of the $\ell_1 - \alpha \ell_2$-DS (1.5) for the signal $x$ recovery. The following definition of restricted ($\ell_2, \ell_p$)-isometry property is introduced in [28].

**Definition 1.** For $0 < p \leq 1$ or $p = 2$, $s \in \mathbb{Z}_+$, we define the restricted $\ell_2/\ell_p$ isometry constant pair $(\delta^b_s, \delta^{ub}_s)$ of order $s$ with respect to the measurement matrix $A \in \mathbb{R}^{m \times n}$ as the smallest numbers $\delta^b_s$ and $\delta^{ub}_s$ such that

\[ (1 - \delta^b_s)\|x\|_2^p \leq \|Ax\|_2^p \leq (1 + \delta^{ub}_s)\|x\|_2^p, \tag{2.1} \]

holds for all $s$-sparse signals $x$. We say that $A$ satisfies the $(\ell_2, \ell_p)$-RIP if $\delta^b_s$ and $\delta^{ub}_s$ are small for reasonably large $s$.

**Remark 1.** When $\delta^b_s = \delta^{ub}_s = \delta_s$ and $p = 1$, the $(\ell_2, \ell_p)$-RIP in Definition 1 reduces to the definition of the $\ell_1$-RIP [10]

\[ (1 - \delta_s)\|x\|_2 \leq \|Ax\|_1 \leq (1 + \delta_s)\|x\|_2. \tag{2.2} \]

In addition, Gaussian matrix satisfies the $(\ell_2, \ell_1)$-RIP with high probability.

When $\delta^b_s = \delta^{ub}_s = \delta_s$ and $p = 2$, the $(\ell_2, \ell_p)$-RIP in Definition 1 is the classic RIP in [7, 6].

**Definition 2.** The matrix $A \in \mathbb{R}^{m \times n}$ satisfies the $(\ell_2, \ell_2)$-RIP of order $s$ with constant $\delta_s \in [0, 1)$ if

\[ (1 - \delta_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_s)\|x\|_2^2 \tag{2.3} \]

holds for all $s$-sparse vectors $x \in \mathbb{R}^n$, i.e., $\|x\|_0 \leq s$, where $s$ is an integer. The smallest constant $\delta_1^k$ is called as the restricted isometry constant (RIC). When $s$ is not an integer, we define $\delta_s$ as $\delta_{\lceil s \rceil}$.

Here, we show a lemma from the proof of [54, Theorem 3.3], which is a modified cone constraint inequality for $\ell_1 - \alpha \ell_2$. 

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Lemma 1. For any vectors \( \mathbf{x}, \hat{\mathbf{x}} \in \mathbb{R}^n \), let \( \mathbf{h} = \hat{\mathbf{x}} - \mathbf{x} \). Assume that \( \|\hat{\mathbf{x}}\|_{\alpha, 1-2} \leq \|\mathbf{x}\|_{\alpha, 1-2} \). Then

\[
\begin{align*}
\|h_{-\max(s)}\|_1 &\leq \|h_{\max(s)}\|_1 + 2\|x_{-\max(s)}\|_1 + \alpha\|h\|_2, \\
\|h_{-\max(s)}\|_1 - \alpha\|h_{-\max(s)}\|_2 &\leq \|h_{\max(s)}\|_1 + 2\|x_{-\max(s)}\|_1 + \alpha\|h_{\max(s)}\|_2.
\end{align*}
\]

(2.4)

Essentially, when \( \mathbf{x} \) is \( s \)-sparse, one has

\[
\begin{align*}
\|h_{-\max(s)}\|_1 &\leq \|h_{\max(s)}\|_1 + \alpha\|h\|_2, \\
\|h_{-\max(s)}\|_1 - \alpha\|h_{-\max(s)}\|_2 &\leq \|h_{\max(s)}\|_1 + \alpha\|h_{\max(s)}\|_2.
\end{align*}
\]

(2.6)

(2.7)

The following lemma is the fundamental properties of \( \|\mathbf{x}\|_1 - \alpha\|\mathbf{x}\|_2 \) with \( 0 \leq \alpha \leq 1 \). The item (a) is a generalization of [55, Lemma 2.1 (a)] and item (b) is trivial. It will be frequently used in our proofs.

Lemma 2. For any \( \mathbf{x} \in \mathbb{R}^n \), the following statements hold:

(a) For \( 0 \leq \alpha \leq 1 \), let \( T = \text{supp}(\mathbf{x}) \) and \( \|\mathbf{x}\|_0 = s \), then

\[
(s - \alpha \sqrt{s}) \min_{j \in T} |x_j| \leq \|\mathbf{x}\|_1 - \alpha\|\mathbf{x}\|_2 \leq (\sqrt{s} - \alpha)\|\mathbf{x}\|_2.
\]

(2.8)

(b) Let \( S, S_1, S_2 \subseteq [n] \) satisfy \( S = S_1 \cup S_2 \) and \( S_1 \cap S_2 = \emptyset \), then

\[
\|x_{S_1}\|_1 - \alpha\|x_{S_1}\|_2 + \|x_{S_2}\|_1 - \alpha\|x_{S_2}\|_2 \leq \|x_S\|_1 - \alpha\|x_S\|_2.
\]

(2.9)

3 Stable Recovery Under the (L2, L1)-RIP Frame

In this section, we will give a sufficient condition based on \((\ell_2, \ell_1)\)-RIP for the stable recovery of \( \ell_1 - \alpha \ell_2 \)-DS (1.5).

3.1 Auxiliary Lemmas Under (L2, L1)-RIP Frame

Before showing sufficient conditions based on \((\ell_2, \ell_1)\)-RIP of the \( \ell_1 - \alpha \ell_2 \)-DS (1.5) for the recovery of signals, we first develop an auxiliary lemma.

Lemma 3. Assume that \( \|\hat{\mathbf{x}}\|_{\alpha, 1-2} \leq \|\mathbf{x}\|_{\alpha, 1-2} \). Let \( \mathbf{h} = \hat{\mathbf{x}} - \mathbf{x} \), \( T_{01} = T_0 \cup T_1 \), where \( T_0 = \text{supp}(h_{\max(s)}) \) and \( T_1 \) be the index set of the \( k \) \( \in \mathbb{Z}_+ \) largest entries of \( h_{-\max(s)} \). Then

\[
\|h_{T_{01}}\|_2 \leq \frac{1}{2\sqrt{t}} \left( \|h_{T_{01}}\|_2 + \frac{2\|\mathbf{x}_{-\max(s)}\|_1 + \alpha\|\mathbf{h}\|_2}{\sqrt{s}} \right),
\]

and

\[
\|\mathbf{h}\|_2 \leq \left( 1 + \frac{1}{2\sqrt{t}} \right)\|h_{T_{01}}\|_2 + \frac{1}{2\sqrt{t}} \frac{2\|\mathbf{x}_{-\max(s)}\|_1}{\sqrt{s}} + \frac{1}{2\sqrt{t}} \frac{\alpha\|\mathbf{h}\|_2}{\sqrt{s}},
\]

where \( t = \frac{k}{s} \).
Proof. By the fact that \( \| h \|_2 = \sqrt{\| h_{T_0} \|_2^2 + \| h_{T_0^c} \|_2^2} \), we need to estimate the upper bound of \( \| h_{T_0^c} \|_2 \). Without loss of generality, we assume that \( |h_1| \geq \cdots \geq |h_s| \geq |h_{s+1}| \geq \cdots \geq |h_n| \) with \( k = ts \in \mathbb{Z}_+ \). Then,

\[
\| h_{T_0^c} \|_2 \leq \sqrt{\| h_{T_0^c} \|_1 \| h_{T_0^c} \|_\infty} \overset{(a)}{\leq} \sqrt{\left( \| h_{T_0} \|_1 - \sum_{j \in T_0} |h_j| \right) |h_{s+k}|}
\]

\[
\overset{(b)}{\leq} \sqrt{|h_{T_0} \|_1 - k|h_{s+k}|} |h_{s+k}| = \sqrt{-k \left( |h_{s+k}| - \frac{\| h_{T_0} \|_1}{2k} \right)^2 + \frac{\| h_{T_0} \|_1^2}{4k}}
\]

\[
\overset{(c)}{\leq} \frac{\| h_{T_0} \|_1}{2\sqrt{k}} \left( \| h_{T_0^c} \|_1 + 2\| x_{max} \|_1 + \alpha \| h \|_2 \right)
\]

\[
\overset{(d)}{\leq} \frac{1}{2\sqrt{k}} \left( \| h_{T_0^c} \|_2 + \frac{2\| x_{max} \|_1 + \alpha \| h \|_2}{\sqrt{s}} \right)
\]

\[
\overset{(e)}{\leq} \frac{1}{2\sqrt{t}} \left( \| h_{T_0^c} \|_2 + \frac{2\| x_{max} \|_1 + \alpha \| h \|_2}{\sqrt{s}} \right),
\]

(3.1)

where (a) and (b) are from \( T_0 = T_0 \cup T_1, |T_1| = k \) and the assumption \( |h_1| \geq \cdots \geq |h_s| \geq |h_{s+1}| \geq \cdots \geq |h_{s+k}| \geq \cdots \geq |h_n| \), (c) follows from Lemma 1, (d) is due to \( \| h_{max} \|_1 \leq \sqrt{s}\| h_{max} \|_2, T_0 = \text{supp}(h_{max}) \) and \( T_0^c = T_0 \cup T_1 \), and (e) follows from \( k = ts \in \mathbb{Z}_+ \).

By (3.1), one have

\[
\| h \|_2 = \sqrt{\| h_{T_0} \|_2^2 + \| h_{T_0^c} \|_2^2}
\]

\[
\leq \sqrt{\| h_{T_0} \|_2^2 + \frac{1}{4t} \left( \| h_{T_0^c} \|_2 + \frac{2\| x_{max} \|_1 + \alpha \| h \|_2}{\sqrt{s}} \right)^2}
\]

\[
\leq \left( 1 + \frac{1}{2\sqrt{t}} \right) \| h_{T_0} \|_2 + \frac{2\| x_{max} \|_1 + \alpha \| h \|_2}{\sqrt{s}} + \frac{1}{2\sqrt{t}} \frac{\alpha \| h \|_2}{\sqrt{s}},
\]

(3.2)

where the last inequality is due to the basic inequality \( \sqrt{a^2 + b^2} \leq a + b \) for \( a, b \geq 0 \).

\[\Box\]

Moreover, we recall a vital lemma, which describes the lower bound of \( \| A(\hat{x} - x) \|_1 \). It plays an important role in the proof of the main result based on \((\ell_2, \ell_1)\)-RIP frame.

**Lemma 4.** ([28, Lemma 2.6]) Assume that \( \| \hat{x} \|_{\alpha, 1-2} \leq \| x \|_{\alpha, 1-2} \). Let \( h = \hat{x} - x, T_0 = \text{supp}(h_{max}), T_1 \) be the index set of the \( k \in \mathbb{Z}_+ \) largest entries of \( h_{max} \) and \( T_0 = T_0 \cup T_1 \), the matrix \( A \) satisfies the \((\ell_2, \ell_1)\)-RIP condition of \( k + s \) order. Then

\[
\| Ah \|_1 \geq \rho_k \| h_{T_0} \|_2 - \frac{2(1 + \delta_k^{lb}) \| x_{max} \|_1}{\sqrt{k} - \alpha},
\]

(3.3)

where

\[
\rho_k = 1 - \delta_k^{lb} \frac{(1 + \delta_k^{lb})}{a(s, k; \alpha)}
\]

(3.4)

and \( a(s, k; \alpha) = \frac{\sqrt{s - k}}{s + \alpha} \).
3.2 Main Result Based on (L2, L1)-RIP Frame

Now, we consider the recovery of signals from (1.1) with $\|A^\top e\|_\infty \leq \eta$ via the $\ell_1 - \alpha \ell_2$-DS (1.5).

**Theorem 1.** Consider $b = Ax + e$ with $\|A^\top e\|_\infty \leq \eta$. For some $s \in [[1, n]]$ and $0 < \alpha \leq 1$, let $t > 0$ such that $ts \in \mathbb{Z}^+$, $a(s, ts; \alpha) = \frac{\sqrt{ts} - \alpha}{\sqrt{s} + \alpha} > 2$ and $b(t, ts; \alpha) = \frac{8(2\sqrt{ts} - \alpha)}{17\alpha(2\sqrt{t+1})} > 1$ satisfying $a(s, ts; \alpha)b(t, ts; \alpha) < a(s, ts; \alpha) + b(t, ts; \alpha)$. Let $\hat{x}^{DS}$ be the minimizer of the $\ell_1 - \alpha \ell_2$-DS (1.5). If the measurement matrix $A$ satisfies the $(\ell_2, \ell_1)$-RIP condition with

$$(b(t, ts; \alpha) + 1)\delta^{ub}_{ts} + a(s, ts; \alpha)b(t, ts; \alpha)\delta^{sh}_{(t+1)s} < a(s, ts; \alpha)b(t, ts; \alpha) - b(t, ts; \alpha) - 1,$$  

then

$$\|\hat{x}^{DS} - x\|_2 \leq \sqrt{\delta} \frac{2\|x_{-\text{max}(s)}\|_1}{\sqrt{s} - \alpha\tau} + \frac{2(2\sqrt{t+1})(1 + \delta^{ub}_{ts}) + a(s, ts; \alpha)\rho_{ts}m}{\tau(\sqrt{s} - 2\tau)(1 + \delta^{ub}_{ts})\rho_{ts}^2} \eta$$

where \(\tau = \frac{1}{2\sqrt{t}}\left(\frac{17(2\sqrt{t+1})(1 + \delta^{ub}_{ts})}{8a(s, ts; \alpha)\rho_{ts}} + 1\right)\) and \(\rho_{ts} = 1 - \delta^{ub}_{(t+1)s} - \frac{1 + \delta^{ub}_{ts}}{a(s, ts; \alpha)}\).

**Remark 2.** The conditions in Theorem 1 seem strict. In fact, these conditions can be satisfied. For example, for \(\alpha = 1\), if we take \(t = 16\), then

$$a(s, ts; \alpha) = \frac{4\sqrt{s} - 1}{\sqrt{s} + 1} =: a(s), \quad b(t, ts; \alpha) = \frac{8(8\sqrt{s} - 1)}{153} =: b(s).$$

If we restrict $7 \leq s \leq 14$, we can check that $a(s) > 2$, $b(s) > 1$ and $a(s)b(s) < a(s) + b(s)$. Therefore, $(\ell_2, \ell_1)$-RIP condition (3.5) can be formulated as

$$(b(s) + 1)\delta^{ub}_{ts} + a(s)b(s)\delta^{sh}_{(t+1)s} < a(s)b(s) - b(s) - 1.$$  

And if we take $\delta^{sh}_{s} = \delta^{ub}_{s}$ in Remark 1, then condition (3.5) can be simplified as

$$\delta_{17s} < \frac{192s - 305\sqrt{s} - 137}{320s + 113\sqrt{s} + 153}.$$  

**Proof.** Our proof is motivated by the proof of [4, Lemma 7.9 in Supplement]. Take $h = \hat{x}^{DS} - x$. Since $\hat{x}^{DS}$ is the minimizer of (1.5), which implies $\|\hat{x}^{DS}\|_{\alpha,1-2} \leq \|x\|_{\alpha,1-2}$ and $\|A^\top(b - A\hat{x}^{DS})\|_\infty \leq \eta$. Then, by (2.5) in Lemma 1, we have

$$\|h_{-\text{max}(s)}\|_1 \leq \|h_{\text{max}(s)}\|_1 + 2\|x_{-\text{max}(s)}\|_1 + \alpha\|h\|_2.$$  

From the facts $\|A^\top z\|_\infty = \|A^\top(b - Ax)\|_\infty \leq \eta$ and $\|A^\top(b - A\hat{x}^{DS})\|_\infty \leq \eta$, we have the following tube constraint inequality

$$\|A^\top Ah\|_\infty = \|A^\top(A\hat{x}^{DS} - Ax)\|_\infty.$$
\[
\leq \|A^T (Ax^{DS} - b)\|_\infty + \|A^T (b - Ax)\|_\infty \\
\leq \eta + \eta = 2\eta.
\] (3.7)

Let \( T_0 = \text{supp}(h_{\max(s)}) \). First, we partition \( T_0^c = [[1, n]] \backslash T_0 \) as

\[
T_0^c = \bigcup_{j=1}^J T_j,
\]

where \( T_1 \) is the index set of the \( ts \in \mathbb{Z}_+ \) largest entries of \( h_{-\max(s)} \), \( T_2 \) is the index set of the next \( ts \in \mathbb{Z}_+ \) largest entries of \( h_{-\max(s)} \), and so on. Notice that the last index set \( T_J \) may contain less \( ts \in \mathbb{Z}_+ \) elements. Similarly, let \( T_0 = T_0 \cup T_1 \). Thus, by \( A \) satisfies the \((\ell_2, \ell_1)\)-RIP condition of \((t+1)s\) order, and Lemma 4 with \( k = ts \), one obtains a lower bound of \( \|Ah\|_1 \)

\[
\|Ah\|_1 \geq \rho_{ts} \|h_{T_0}\|_2 - (1 + \delta^{lb}_{ts}) \frac{2\|x_{-\max(s)}\|_1}{\sqrt{ts - \alpha}},
\]

(3.8)

where

\[
\rho_{ts} = 1 - \delta^{lb}_{ts} - \frac{(1 + \delta^{lb}_{ts})}{a(s, ts; \alpha)}\]

with \( a(s, ts; \alpha) = \frac{\sqrt{\alpha}}{\sqrt{s+\alpha}} > 1 \). Furthermore,

\[
1 - \delta^{lb}_{(t+1)s} - \frac{1 + \delta^{lb}_{ts}}{a(s, ts; \alpha)} > 1 - \delta^{lb}_{(t+1)s} - \frac{(1+b(s, ts; \alpha))(1+\delta^{lb}_{ts})}{a(s, ts; \alpha)} > 0
\]

where the first and second inequalities are from \( b(s, ts; \alpha) = \frac{b(2\sqrt{\alpha})}{(2\sqrt{\alpha})(1 + \alpha)} > 0 \) with \( 0 < \alpha \leq 1 \) and (3.5), respectively.

Next, we estimate the upper bound of \( \|Ah\|_1 \). By Cauchy-Schwartz inequality, we have

\[
\|Ah\|_1 \leq \sqrt{m} \|Ah\|_2 = \sqrt{m} \langle Ah, Ah \rangle^{1/2}
\]

\[
= \sqrt{m} \langle A^T Ah, h \rangle^{1/2} \leq \sqrt{m} \sqrt{\|A^T Ah\|_\infty \|h\|_1}
\]

\[
= \sqrt{m} \sqrt{\|A^T Ah\|_\infty ( \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1 )}
\]

\[
\overset{(a)}{\leq} \sqrt{m} \sqrt{2\eta (2\|h_{T_0}\|_1 + 2\|x_{-\max(s)}\|_1 + \alpha \|h\|_2)}
\]

\[
\overset{(b)}{\leq} \sqrt{2m\sqrt{s}} \eta \left( 2\|h_{T_0}\|_2 + \frac{2\|x_{-\max(s)}\|_1 + \alpha \|h\|_2}{\sqrt{s}} \right)
\]

(3.9)

where (a) is from \( T_0 = \text{supp}(h_{\max(s)}) \), (3.6) and (3.7), (b) is due to \( T_0 = T_0 \cup T_1 \) and \( \|h_{T_0}\|_1 \leq \sqrt{s} \|h_{T_0}\|_2 \) with \( \|T_0\| \leq s \).

Combining (3.8) with (3.9), we have

\[
\rho_{ts} \|h_{T_0}\|_2 - \frac{(1 + \delta^{lb}_{ts})\sqrt{s}}{\sqrt{ts - \alpha}} \frac{2\|x_{-\max(s)}\|_1}{\sqrt{s}}
\]

\[
\leq \sqrt{2m\sqrt{s}} \eta \left( 2\|h_{T_0}\|_2 + \frac{2\|x_{-\max(s)}\|_1 + \alpha \|h\|_2}{\sqrt{s}} \right).
\]

(3.10)
To estimate $\|h_{T_0}\|_2$ from (3.10), we consider the following two cases.

**Case I:**

$$\rho_s \|h_{T_0}\|_2 - \frac{(1 + \delta_t^{ub}) \sqrt{s} 2 \|x_{-\text{max}(s)}\|_1}{\sqrt{s}} < 0,$$

i.e.,

$$\|h_{T_0}\|_2 < \frac{(1 + \delta_t^{ub}) \sqrt{s} 2 \|x_{-\text{max}(s)}\|_1}{(\sqrt{s} - \alpha) \rho_s \sqrt{s}}.$$  \hspace{1cm} (3.11)

**Case II:**

$$\rho_s \|h_{T_0}\|_2 - \frac{(1 + \delta_t^{ub}) \sqrt{s} 2 \|x_{-\text{max}(s)}\|_1}{\sqrt{s}} \geq 0,$$

which implies

$$\|h_{T_0}\|_2 \geq \frac{(1 + \delta_t^{ub}) \sqrt{s} 2 \|x_{-\text{max}(s)}\|_1}{(\sqrt{s} - \alpha) \rho_s \sqrt{s}},$$

then the inequality (3.10) is equivalent to

$$\left(\rho_s \|h_{T_0}\|_2 - \frac{(1 + \delta_t^{ub}) \sqrt{s} 2 \|x_{-\text{max}(s)}\|_1}{\sqrt{s}} \right)^2 \leq 2m \sqrt{s}\eta \left(2 \|h_{T_0}\|_2 + \frac{2 \|x_{-\text{max}(s)}\|_1 + \alpha \|h\|_2}{\sqrt{s}} \right).$$ \hspace{1cm} (3.12)

Let $X = \|h_{T_0}\|_2$ and $Y = \frac{2 \|x_{-\text{max}(s)}\|_1 + \alpha \|h\|_2}{\sqrt{s}}$. By $\frac{2 \|x_{-\text{max}(s)}\|_1}{\sqrt{s}} \leq Y$, to guarantee that (3.12) holds, it suffices to show

$$\rho_s^2 X^2 - \left(\frac{2 \rho_s (1 + \delta_t^{ub}) \sqrt{s} Y + 4m \sqrt{s}\eta}{\sqrt{s} - \alpha} \right) X - 2m \sqrt{s}\eta Y \leq 0. \hspace{1cm} (3.13)$$

For the one-variable quadratic inequality $aZ^2 - bZ - c \leq 0$ with the constants $a, b, c > 0$ and $Z \geq 0$, there is the fact that

$$Z \leq \frac{b + \sqrt{b^2 + 4ac}}{2a} \leq \frac{b + \sqrt{c}}{a}.$$  \hspace{1cm} Hence,

$$X \leq \frac{2 \rho_s (1 + \delta_t^{ub}) \sqrt{s} Y + 4m \sqrt{s}\eta}{\rho_s^2 \sqrt{s} - \alpha} + \sqrt{\frac{2m \sqrt{s}\eta \varepsilon Y}{\rho_s^2 \varepsilon}} \rho_s^2 \varepsilon \eta + \frac{1}{\sqrt{s} - \alpha} \left(\frac{2 \rho_s (1 + \delta_t^{ub}) \sqrt{s} Y + 4m \sqrt{s}\eta}{\rho_s^2 \varepsilon} \right) \eta + \left(4 + \frac{1}{\varepsilon} \right) \frac{m \sqrt{s}}{\rho_s^2 \eta},$$  \hspace{1cm} (3.14)

where $\varepsilon > 0$ is to be determined later. Here the second inequality comes from the basis inequality $\sqrt{|a||b|} \leq (|a| + |b|)/2$. Therefore

$$\|h_{T_0}\|_2 \leq \left(\frac{2 \rho_s (1 + \delta_t^{ub}) \sqrt{s} + \frac{\varepsilon}{2} \sqrt{s}}{\rho_s^2 \sqrt{s} - \alpha} \right) \frac{2 \|x_{-\text{max}(s)}\|_1 + \alpha \|h\|_2}{\sqrt{s}} + \left(4 + \frac{1}{\varepsilon} \right) \frac{m \sqrt{s}}{\rho_s^2 \eta}. \hspace{1cm} (3.15)$$
Note that
\[
\left( \frac{2(1 + \delta_{ub}^{ts}) \sqrt{s}}{\sqrt{t} s - \alpha} \right) \rho_{ts} \frac{\varepsilon}{2} Y + \left( 4 + \frac{1}{\varepsilon} \right) \frac{m \sqrt{s}}{\rho_{ts}^2} \eta
\]
\[
= \left( \frac{2(1 + \delta_{ub}^{ts}) \sqrt{s}}{\sqrt{t} s - \alpha} \right) \rho_{ts} \frac{\varepsilon}{2} 2\|x_{-\max(s)}\| + \left( 4 + \frac{1}{\varepsilon} \right) \frac{m \sqrt{s}}{\rho_{ts}^2} \eta
\]
\[
\geq \left( \frac{1 + \delta_{ub}^{ts}) \sqrt{s}}{\sqrt{t} s - \alpha} \right) \rho_{ts} \frac{\varepsilon}{2} 2\|x_{-\max(s)}\| \sqrt{s},
\]
Therefore combining the estimation (3.11) in Case I and the estimation (3.15) in Case II, one has (3.15) holds for both cases.

By Lemma 3, ones have
\[
\|h\| \leq \left( 1 + \frac{1}{2 \sqrt{t}} \right) \|h_{T_01}\| + \frac{1}{2 \sqrt{t}} 2\|x_{-\max(s)}\| \|h\| \sqrt{s} + \alpha \|h\| \sqrt{s}.
\]
(3.16)

Substituting (3.15) into (3.16), ones obtain
\[
\|h\| \leq \left( 1 + \frac{1}{2 \sqrt{t}} \right) \left( \left( 4 + \frac{1}{\varepsilon} \right) \frac{m \sqrt{s}}{\rho_{ts}^2} \eta + \left( \frac{2(1 + \delta_{ub}^{ts}) \sqrt{s}}{\sqrt{t} s - \alpha} \right) \rho_{ts} \frac{\varepsilon}{2} \right)
\]
\[
+ \frac{1}{2 \sqrt{t}} 2\|x_{-\max(s)}\| \sqrt{s} + \alpha \|h\| \sqrt{s}
\]
\[
\leq \frac{1}{2 \sqrt{t}} \left( (2\sqrt{t} + 1) \left( \frac{2(\sqrt{s} + \alpha)(1 + \delta_{ub}^{ts})}{\sqrt{t} s - \alpha} \right) \rho_{ts} \frac{\varepsilon}{2} + 1 \right) \|h\| \sqrt{s}
\]
\[
+ \frac{\alpha}{2 \sqrt{t}} \left( (2\sqrt{t} + 1) \left( \frac{2(\sqrt{s} + \alpha)(1 + \delta_{ub}^{ts})}{\sqrt{t} s - \alpha} \right) \rho_{ts} \frac{\varepsilon}{2} + 1 \right) \|h\| \sqrt{s}
\]
\[
+ \left( 4 + \frac{1}{\varepsilon} \right) \frac{m \sqrt{s}}{\rho_{ts}^2} \eta
\]
\[
= \tau \frac{2\|x_{-\max(s)}\| \|h\| \sqrt{s}}{\sqrt{s}} + \alpha \tau \frac{\|h\| \sqrt{s}}{\sqrt{s}} + \left( 4 + \frac{1}{\varepsilon} \right) \frac{(2\sqrt{t} + 1)m \sqrt{s}}{2 \sqrt{t} \rho_{ts}^2} \eta,
\]
(3.17)
where the last equality is from
\[
\tau = \frac{1}{2 \sqrt{t}} \left( 2\sqrt{t} + 1 \right) \left( \frac{2(1 + \delta_{ub}^{ts})}{\alpha(s, t, \alpha) \rho_{ts}} \right) \rho_{ts} \frac{\varepsilon}{2} + 1 \right)
\]
with
\[
\varepsilon = \frac{(1 + \delta_{ub}^{ts})}{4 \alpha(s, t, \alpha) \rho_{ts}}.
\]
Then
\[
\frac{\alpha \tau}{\sqrt{s}} = \frac{1}{2 \sqrt{t}} \left( (2\sqrt{t} + 1) \left( \frac{17(1 + \delta_{ub}^{ts})}{8 \alpha(s, t, \alpha) \rho_{ts,t}} + 1 \right) \right) \frac{\alpha}{\sqrt{s}} < 1,
\]
(3.18)
where the inequality is from (3.5). In fact,
\[
\tau - \frac{\sqrt{s}}{\alpha} = \frac{1}{2\sqrt{t}} \left( (2\sqrt{t} + 1) \frac{17(1 + \delta_{ts}^{ub})}{8a(s, ts; \alpha)\rho_{ts}} + 1 \right) - \frac{\sqrt{s}}{\alpha}
\]
\[
= \frac{17(2\sqrt{t} + 1)}{16\sqrt{t}a(s, ts; \alpha)\rho_{ts}} \left( 1 + \delta_{ts}^{ub} - \left( \frac{2\sqrt{ts}}{\alpha} - 1 \right) \frac{8}{17(2\sqrt{t} + 1)}a(s, ts; \alpha)\rho_{ts} \right)
\]
\[
= \frac{17(2\sqrt{t} + 1)}{16\sqrt{t}a(s, ts; \alpha)\rho_{ts}} \left( 1 + \delta_{ts}^{ub} - a(s, t; \alpha)b(s, ts; \alpha)\rho_{ts} \right),
\]
where
\[
b(t, ts; \alpha) = \left( \frac{2\sqrt{ts}}{\alpha} - 1 \right) \frac{8}{17(2\sqrt{t} + 1)} = \frac{8(2\sqrt{ts} - \alpha)}{17\alpha(2\sqrt{t} + 1)}.
\]
Then, by
\[
\rho_{ts} = 1 - \frac{\delta_{(t+1)s}^{lb}}{a(s, ts; \alpha)},
\]
one has that
\[
\tau - \frac{\sqrt{s}}{\alpha} = \frac{17(2\sqrt{t} + 1)}{16\sqrt{t}a(s, t; \alpha)\rho_{ts}} \left( b(t, ts; \alpha) + 1 \right) \delta_{ts}^{ub} + a(s, ts; \alpha)b(t, ts; \alpha)\delta_{(t+1)s}^{lb}
\]
\[
- \left( a(s, ts; \alpha)b(t, ts; \alpha) - b(t, ts; \alpha) - 1 \right) < 0.
\]
where the inequality is due to (3.5). Therefore, combing with (3.17) and the fact \(\tau - \frac{\sqrt{s}}{\alpha} < 0\), one has that
\[
\|h\|_2 \leq \frac{\sqrt{s}}{\sqrt{s} - aT} \left( \frac{2\|x_{-\max(s)}\|_1}{\sqrt{s}} \right)
\]
\[
+ \frac{2(2\sqrt{t} + 1)(1 + \delta_{ts}^{ub}) + a(s, ts; \alpha)\rho_{ts})m}{\sqrt{t}(\sqrt{s} - aT)(1 + \delta_{ts}^{ub})\rho_{ts}^2} \eta,
\]
which finishes the proof of Theorem 1.

\[\square\]

4 Stable Recovery Under (L2, L2)-RIP Frame

In the section, we develop sufficient conditions based on the high order (\(\ell_2, \ell_2\))-RIP frame of the \(\ell_1 - \alpha\ell_2\)-DS (1.5) for the signal recovery applying the technique of the convex combination.

4.1 Auxiliary Lemmas Under (L2, L2)-RIP Frame

We first give two auxiliary results under (\(\ell_2, \ell_2\))-RIP frame. The following lemma describes a convex combination of sparse vectors for any point based on \(\| \cdot \|_1 - \| \cdot \|_2\). It is developed for the analysis of the constrained \(\ell_1 - \alpha\ell_2\) minimization and establish improved high order RIP conditions for the signal recovery.
Lemma 5. [23, Lemma 2.2] Let a vector $\nu \in \mathbb{R}^n$ satisfy $\|\nu\|_\infty \leq \theta$, where $\alpha$ is a positive constant. Suppose $\|\nu\|_1 - 2 \leq (s - \sqrt{s})\theta$ with a positive integer $s$ and $s \leq |\text{supp}(\nu)|$. Then $\nu$ can be represented as a convex combination of $s$-sparse vectors $u^{(i)}$, i.e.,

$$\nu = \sum_{i=1}^{N} \lambda_i u^{(i)},$$

where $N$ is a positive integer,

$$0 < \lambda_i \leq 1, \quad \sum_{i=1}^{N} \lambda_i = 1,$$ (4.2)

$$\text{supp}(u^{(i)}) \subseteq \text{supp}(\nu), \quad \|u^{(i)}\|_0 \leq s, \quad \|u^{(i)}\|_\infty \leq \left(1 + \frac{\sqrt{2}}{2}\right)\theta,$$ (4.3)

and

$$\sum_{i=1}^{N} \lambda_i \|u^{(i)}\|^2_2 \leq \left[\left(1 + \frac{\sqrt{2}}{2}\right)^2 (s - \sqrt{s}) + 1\right]\theta^2.$$ (4.4)

Combining the above lemma with Lemma 1, we next introduce the following new lemma which will play a crucial role in establishing the recovery condition based on the $L_2$-RIP frame.

Lemma 6. Assume that $\|\hat{x}\|_{\alpha,1-2} \leq \|x\|_{\alpha,1-2}$. Let $h = \hat{x} - x$ and

$$\chi = \frac{\sqrt{s} + \alpha \|h_{\text{max}(s)}\|_2}{\sqrt{s} - 1} + \frac{2}{s - \sqrt{s}} \|x_{-\text{max}(s)}\|_1$$ (4.5)

where $s \geq 2$ is a positive integer. And define two index sets

$$W_1 = \left\{ i : |h_{-\text{max}(s)}(i)| > \frac{\chi}{t - 1} \right\},$$ (4.6)

and

$$W_2 = \left\{ i : |h_{-\text{max}(s)}(i)| \leq \frac{\chi}{t - 1} \right\},$$ (4.7)

where $t = 2$ or $t \geq 3$. Then the vector $h_{W_2}$ can be represented as a convex combination of $\lfloor ts \rfloor - s - W_1$-sparse vectors $u^{(i)}$, i.e.,

$$h_{W_2} = \sum_{i=1}^{N} \lambda_i u^{(i)},$$ (4.8)

where $N$ is a positive integer. And

$$\sum_{i=1}^{N} \lambda_i \|u^{(i)}\|^2_2 \leq \left(1 + \frac{\sqrt{2}}{2}\right)^2 \left[\lfloor ts \rfloor - s - \sqrt{\lfloor ts \rfloor - s}\right] + 1 \chi^2.$$ (4.9)
Proof. From $\|\hat{x}\|_{\alpha,1-2} \leq \|x\|_{\alpha,1-2}$ and (2.5) in Lemma 1, it follows that

\[
\|h_{-\max(s)}\|_1 - \|h_{-\max(s)}\|_2 \leq \|h_{-\max(s)}\|_1 - \alpha \|h_{-\max(s)}\|_2 \\
\leq \|h_{\max(s)}\|_1 + 2\|x_{-\max(s)}\|_1 + \alpha \|h_{\max(s)}\|_2 \\
\leq (\sqrt{s} + \alpha) \|h_{\max(s)}\|_2 + 2\|x_{-\max(s)}\|_1,
\]

(4.10)

where the first inequality comes from $0 < \alpha \leq 1$, and the last inequality is because of $\|h_{\max(s)}\|_1 \leq \sqrt{s} \|h_{\max(s)}\|_2$.

We move to develop a sparse decomposition for $h_{-\max(s)}$ by Lemma 5. Now, we derive that $h_{-\max(s)}$ satisfies conditions in Lemma 5.

By (4.10), we have that

\[
\|h_{-\max(s)}\|_1 - \|h_{-\max(s)}\|_2 \leq \|h_{-\max(s)}\|_1 - \alpha \|h_{-\max(s)}\|_2 \\
\leq (s - \sqrt{s}) \left( \frac{\sqrt{s} + \alpha \|h_{\max(s)}\|_2}{\sqrt{s} - 1} + \frac{2}{s - \sqrt{s}} \|x_{-\max(s)}\|_1 \right) \\
= (s - \sqrt{s}) \chi,
\]

(4.11)

where the equality is from the definition of $\chi$ in (4.5). Using the fact that $\frac{\sqrt{s} + \alpha}{\sqrt{s} - 1} > 1$, one has that

\[
\|h_{-\max(s)}\|_{\infty} \leq \frac{\|h_{\max(s)}\|_1}{s} \leq \frac{\|h_{\max(s)}\|_2}{\sqrt{s}} \leq \frac{\sqrt{s} + \alpha \|h_{\max(s)}\|_2}{\sqrt{s} - 1} \leq \chi,
\]

(4.12)

where the last inequality is due to the definition of $\chi$ in (4.5).

Next, we prove results in the lemma. The following two cases need to be considered.

\(i\) $t = 2$: By (4.12) and the definition of $W_1$ in (4.6), it is clear that $W_1 = \emptyset$. Then proving (4.8) and (4.9) are respectively equivalent to showing

\[
h_{-\max(s)} = \sum_{i=1}^{N} \lambda_i u^{(i)},
\]

(4.13)

and

\[
\sum_{i=1}^{N} \lambda_i \|u^{(i)}\|_2^2 \leq \left( \left( 1 + \frac{\sqrt{s}}{2} \right)^2 (s - \sqrt{s}) + 1 \right) \chi^2,
\]

(4.14)

where $0 < \lambda_i \leq 1$, $\sum_{i=1}^{N} \lambda_i = 1$, and $\|u^{(i)}\|_0 \leq s$. By (4.11), (4.12) and Lemma 5, it is clear that (4.13) and (4.14) hold. Thus we completes the proofs of (4.8) and (4.9) for $t = 2$.

\(ii\) $t \geq 3$: We prove (4.8) and (4.9) by applying Lemma 5. Then, we first consider upper bounds of $\|h_{W_2}\|_{\infty}$ and $\|h_{W_2}\|_1 - \|h_{W_2}\|_2$. By definitions of $W_1$ and $W_2$, one has

\[
W_1 \cap W_2 = \emptyset, \quad h_{-\max(s)} = h_{W_1} + h_{W_2}
\]

and

\[
\|h_{W_2}\|_{\infty} \leq \frac{\chi}{t - 1},
\]

(4.15)
We next establish a upper bound of $\|h_{W_2}\|_1 - \|h_{W_2}\|_2$. From $h_{\cdot \max(s)} = h_{W_1} + h_{W_2}$, it follows that

$$
\|h_{W_2}\|_1 - \|h_{W_2}\|_2 = \|h_{\cdot \max(s)} - h_{W_1}\|_1 - \|h_{\cdot \max(s)} - h_{W_1}\|_2
\overset{(a)}{=} \|h_{\cdot \max(s)}\|_1 - \|h_{W_1}\|_1 - \|h_{\cdot \max(s)} - h_{W_1}\|_2
\overset{(b)}{\leq} \|h_{\cdot \max(s)}\|_1 - \|h_{\cdot \max(s)}\|_2 - (\|h_{W_1}\|_1 - \|h_{W_1}\|_2)
\overset{(c)}{\leq} (s - \sqrt{s})\chi - (\|h_{W_1}\|_1 - \|h_{W_1}\|_2),
$$

where (a), (b) and (c) follow from $W_1 \subseteq \supp(h_{\cdot \max(s)})$, the triangle inequality on $\| \cdot \|_2$, and (4.11), respectively. Then, establishing a upper bound of $\|h_{W_2}\|_1 - \|h_{W_2}\|_2$ is equivalent to showing the lower bound of $\|h_{W_1}\|_1 - \|h_{W_1}\|_2$. By Lemma 2 (b) with $S_1 = W_1$ and $S_2 = W_2$, we have that

$$
\|h_{\cdot \max(s)}\|_1 - \|h_{\cdot \max(s)}\|_2 \geq (\|h_{W_1}\|_1 - \|h_{W_1}\|_2) + (\|h_{W_2}\|_1 - \|h_{W_2}\|_2)
\overset{(1)}{\geq} (\|h_{W_1}\|_1 - \|h_{W_1}\|_2)
\overset{(2)}{\geq} (\|h_{W_1}\|_1 - \sqrt{|W_1|})\min_{i \in W_1} |h_{W_1}(i)|
\overset{(3)}{\geq} (\|h_{W_1}\|_1 - \sqrt{|W_1|})\frac{\chi}{t - 1},
$$

where (1) and (2) follow from (2.8) in Lemma 2 (a) and the definition of $W_1$ in (4.6), respectively. Combining the above inequality with (4.16), we have

$$
\|h_{W_2}\|_1 - \|h_{W_2}\|_2 \leq \left((s(t - 1)|W_1| - (\sqrt{s}(t - 1) - \sqrt{|W_1|})\right) \frac{\chi}{t - 1},
$$

(4.17)

Inspired by Lemma 5, (4.15) and (4.18), we explore that

$$
\|h_{W_2}\|_1 - \|h_{W_2}\|_2 \leq (s(t - 1)|W_1| - \sqrt{s}(t - 1) - \sqrt{|W_1|}) \frac{\chi}{t - 1},
$$

(4.19)

In fact, based on (4.18), (4.19) follows from

(i) $|W_1| < s(t - 1)$,
(ii) $\sqrt{s(t - 1)} - |W_1| \leq \sqrt{s}(t - 1) - \sqrt{|W_1|}$

for $t \geq 3$ and $s \geq 2$. We first prove item (i). Observe that

$$
(\|W_1\|_1 - \sqrt{|W_1|})\frac{\chi}{t - 1} \leq \|h_{\cdot \max(s)}\|_1 - \|h_{\cdot \max(s)}\|_2
\leq (s(t - 1) - \sqrt{s}(t - 1))\frac{\chi}{t - 1}
$$

where the equalities follow from (4.17) and (4.11), respectively. Furthermore, since $t \geq 3$, which implies $\sqrt{s}(t - 1) > \sqrt{s}(t - 1)$, it is clear that

$$
|W_1| - \sqrt{|W_1|} < s(t - 1) - \sqrt{s}(t - 1),
$$
meaning
\[ |W_1| < s(t - 1), \quad (4.20) \]
since the function \( g(x) = x - \sqrt{x} \) is increasing monotony for \( x \geq 1 \) when \( |W_1| \geq 1 \), and \( s(t - 1) \geq 2s \geq 4 \) using \( t \geq 3 \) and \( s \geq 2 \) when \( |W_1| = 0 \).

Now, we turn to show item (ii). We observe that the second-order function
\[
 f(\sqrt{|W_1|}) = (\sqrt{s(t - 1) - |W_1|})^2 - (\sqrt{s(t - 1) - \sqrt{|W_1|}})^2
 = -2|W_1| + 2\sqrt{s}|W_1|(t - 1) + s(t - 1)(2 - t) \leq 0,
\]
which implies item (ii), i.e., \( \text{Lemma 5} \), since its discriminant \( \Delta = 4s(t - 1)^2 + 8s(t - 1)(2 - t) = 4s(t - 1)(-t + 3) \leq 0 \) under \( t \geq 3 \).

Then, from (4.15), (4.19) and Lemma 5 with \( \theta = \frac{\chi}{t - 1} \), \( \nu = h_{W_2} \) and \( s = \lceil ts \rceil - s - |W_1| \), where \( s \geq 1 \) since \( W_1 > s(t - 1) \), it follows that
\[
 h_{W_2} = \sum_{i=1}^{N} \lambda_i u^{(i)}, \quad (4.21)
\]
where \( N \) is certain positive integer, \( 0 < \lambda_i \leq 1 \) with \( \sum_{i=1}^{N} \lambda_i = 1 \). And using (4.4) in Lemma 5, we have
\[
 \sum_{i=1}^{N} \lambda_i \|u^{(i)}\|_2^2 \leq \left[ \left( 1 + \frac{\sqrt{2}}{2} \right)^2 \lceil ts \rceil - s - |W_1| - \sqrt{\lceil ts \rceil - s - |W_1|} + 1 \right] \left( \frac{\chi}{t - 1} \right)^2
 \leq \left[ \left( 1 + \frac{\sqrt{2}}{2} \right)^2 \lceil ts \rceil - s - \sqrt{\lceil ts \rceil - s} + 1 \right] \left( \frac{\chi}{t - 1} \right)^2
 = \frac{\left( 1 + \frac{\sqrt{2}}{2} \right)^2 \left( \lceil ts \rceil - s - \sqrt{\lceil ts \rceil - s} + 1 \right) \chi^2}{(t - 1)^2}, \quad (4.22)
\]
where the second inequality follows from the decreasing monotony of the function \( g(x) = x - \sqrt{x} \) for \( x \geq 1 \) and \( \lceil ts \rceil - s - |W_1| \geq 1 \). We complete the proof. 

4.2 Main Result Under (L2, L2)-RIP Frame

Now, we show the stable recovery under \((\ell_2, \ell_2)\)-RIP frame.

**Theorem 2.** Consider \( b = Ax + e \) with \( \|A^T e\|_\infty \leq \eta \). Let \( \hat{x}^{DS} \) be the minimizer of the \( \ell_1 - \alpha \ell_2\)-DS (1.5). If the measurement matrix \( A \) satisfies
\[
 \delta_{ts} < \frac{1}{\sqrt{(\sqrt{\pi + \alpha})^2 \left( \frac{1 + \sqrt{2}}{s(t - 1)^2 (\sqrt{\pi - 1})^2} \right) (ts - s - \sqrt{ts - s} + 1) + 1}}, \quad (4.23)
\]
for some \( t \geq 3 \) or \( t = 2 \), where \( s \geq 2 \) is an positive integer. Then
\[
 \|\hat{x}^{DS} - x\|_2 \leq \left( 1 + \sqrt{\frac{\alpha + \sqrt{s}}{\sqrt{s}} + \frac{\alpha^2}{4s} + \frac{\alpha + \sqrt{2}}{2\sqrt{s}}} \right) \frac{2(1 - \mu)\mu \sqrt{\lceil ts \rceil}}{\mu - \mu^2 - (\mu - 1)^2 \delta_{ts} \eta}
\]
\[
+ \left( \sqrt{s} + \frac{\alpha \sqrt{s} + s + \alpha^2}{4} + \frac{\alpha + \sqrt{2}}{2} \right) \left( \frac{2 \delta_{ts}(1 - 2\mu)}{\sqrt{s} + \alpha} (\mu - \mu^2 - (\mu - 1)^2 \delta_{ts}) \right) + \sqrt{\frac{2s}{2}} \parallel x - \max(s) \parallel_1 \right) \\
\frac{\sqrt{s}}{\sqrt{s}},
\]

where
\[
\mu = \frac{1}{\sqrt{(\sqrt{s} + \alpha)^2 \left( \frac{(s - \sqrt{s})}{s} \right) + 1}}.
\]

**Remark 3.** When \( t = 2 \), the condition (4.23) reduces to
\[
\delta_{2s} < \frac{1}{\sqrt{1 + \frac{(\sqrt{s} + \alpha)^2 \left( \frac{(s - \sqrt{s})}{s} \right) + 1}{s(\sqrt{s} + \alpha)^2 + 1}}}
\]

It is clearly weaker than the following condition in [23, Theorem 3.4]
\[
\delta_{2s} < \frac{1}{\sqrt{1 + \frac{(\sqrt{s} + \alpha)^2 \left( \frac{(s - \sqrt{s})}{s} \right) + 1}{s(\sqrt{s} + \alpha)^2 + 1}}}
\]

for the \( \ell_1 - \alpha \ell_2 \)-DS (1.5) with \( \alpha = 1 \), because of \( 0 < \alpha \leq 1 \).

**Remark 4.** When \( s \geq 2 \) and \( \alpha = 1 \), by the condition (4.23), one has
\[
\delta_{ts} < \frac{1}{\sqrt{1 + \frac{(\sqrt{s} + \alpha)^2 \left( \frac{(t_s - s - \sqrt{t_s - s})}{s(t_s - \sqrt{t_s - s}) + 1} \right)}{2(s(\sqrt{s} + \alpha)^2 + 1)}}}
\]

It is weaker than [23, the condition (3.30)]
\[
\delta_{ts} < \frac{1}{\sqrt{1 + \frac{(\sqrt{s} + \alpha)^2 \left( \frac{(t_s - s - \sqrt{t_s - s})}{s(t_s - \sqrt{t_s - s}) + 1} \right)}{2(s(\sqrt{s} + \alpha)^2 + 1)}}}
\]

since \( \sqrt{t_s - s} > 1 \) and \( \left( \frac{2}{2 + \sqrt{2}} \right)^2 < 1 \) implying
\[
(t_s - s - \sqrt{t_s - s} + \left( \frac{2}{2 + \sqrt{2}} \right)^2)^2 < 1.
\]
Furthermore, the condition (4.23) with \( 0 < \alpha < 1 \) is weaker than (4.25).
Proof. Our proof is motivated by the proof of [23, Theorem 3.1]. Since $\hat{x}^{DS}$ is the minimizer of (1.5), $\|\hat{x}^{DS}\|_{\infty;1-2} \leq \|x\|_{\infty;1-2}$ and $\|A^\top (b - A\hat{x}^{DS})\|_{\infty} \leq \eta$.

Take $h = \hat{x}^{DS} - x$. On the one hand, using $\|A^\top e\|_{\infty} = \|A^\top (b - Ax)\|_{\infty} \leq \eta$ and $\|A^\top (b - A\hat{x}^{DS})\|_{\infty} \leq \eta$, we have the following tube constraint inequality

\[
\|A^\top Ah\|_{\infty} = \|A^\top (A\hat{x}^{DS} - Ax)\|_{\infty} \\
\leq \|A^\top (A\hat{x}^{DS} - b)\|_{\infty} + \|A^\top (b - Ax)\|_{\infty} \\
\leq \eta + \eta = 2\eta. \tag{4.26}
\]

On the other hand, using $\|\hat{x}^{DS}\|_{\infty;1-2} \leq \|x\|_{\infty;1-2}$ and Lemma 6, one has that

\[
h_{W_2} = \sum_{i=1}^{N} \lambda_i u^{(i)}, \tag{4.27}
\]

and

\[
\sum_{i=1}^{N} \lambda_i \|u^{(i)}\|^2 \leq \frac{(1 + \frac{\sqrt{s}}{2})^2 \left(\lceil ts \rceil - s - \sqrt{\lceil ts \rceil - s} + 1\right)}{(t - 1)^2} \chi^2,
\]

where $0 < \lambda_i \leq 1$, $\sum_{i=1}^{N} \lambda_i = 1$, $u^{(i)} (1 \leq i \leq N)$ are $([ts] - s - W_1)$-sparse vectors and $W_1$, $W_2$, $\chi$ are respectively defined in (4.6), (4.7) and (4.5). By the definition of $\chi$ in (4.5), we get

\[
\sum_{i=1}^{N} \lambda_i \|u^{(i)}\|^2 \leq \frac{\left(\frac{\sqrt{s}}{2}\right)^2 \left(\lceil ts \rceil - s - \sqrt{\lceil ts \rceil - s} + 1\right)}{(t - 1)^2} \left(\frac{\sqrt{s} + \alpha}{s(s - 1)}\right) \|h_{\max(s)}\|_2^2 \\
+ \frac{4(s + \alpha)}{(s - 1)^2} \|h_{\max(s)}\|_2 \|x_{-\max(s)}\|_1 + \frac{4}{s(s - 1)} \|x_{-\max(s)}\|_1^2 \\
= \frac{1 - 2\mu}{\mu^2} \left(\|h_{\max(s)}\|_2^2 + \frac{4\|h_{\max(s)}\|_2 \|x_{-\max(s)}\|_1}{\sqrt{s} + \alpha} + \frac{4\|x_{-\max(s)}\|_1^2}{(\sqrt{s} + \alpha)^2}\right), \tag{4.28}
\]

where the equality is from the definition of $\mu$ in (4.24).

Next, we develop the inequality on $\|h_{\max(s)} + h_{W_1}\|_2$ to estimate an upper bound of $\|h_{\max(s)} + h_{W_1}\|_2$ by the following identity.

\[
\sum_{i=1}^{N} \lambda_i \|A(h_{\max(s)} + h_{W_1} + \mu u^{(i)})\|^2 \\
= \sum_{i=1}^{N} \lambda_i \left\|A \left(\frac{1}{2} - \mu \right)(h_{\max(s)} + h_{W_1}) - \frac{1}{2} \mu u^{(i)} + \mu h\right\|^2. \tag{4.29}
\]

We show an upper bound on the right-hand side (RHS) and a lower bound on the left-hand side (LHS) for the identity (4.29) by following techniques for deriving [23, the inequality (3.49) and (3.52)], respectively. They are not hard to check that

\[
\text{LHS} \geq (1 - \delta_{[ts]}) \sum_{i=1}^{N} \lambda_i \|h_{\max(s)} + h_{W_1} + \mu u^{(i)}\|^2
\]

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\[ = \frac{1 - \delta_{[ts]}}{4} \left( \| h_{\max(s)} + h_{W_1} \|^2 + \mu^2 \sum_{i=1}^N \lambda_i \| u^{(i)} \|^2 \right), \]

where the inequality is due to that \( h_{\max(s)} + h_{W_1} + \mu u^{(i)} \) is \( ts \)-sparse, and the equality follows from \( \sum_{i=1}^N \lambda_i = 1 \), supp(\( u^{(i)} \)) \( \subseteq \) supp(\( h_{W_2} \)) \( \subseteq \) supp(\( h_{\max(s)} \)) and \( W_1 \cap W_2 = \emptyset \). For RHS of the identity (4.29), we have that

**RHS**

\[ = \sum_{i=1}^N \lambda_i \left\| A \left( \frac{1}{2} - \mu \right) (h_{\max(s)} + h_{W_1}) - \frac{1}{2} \mu u^{(i)} \right\|_2^2 + (1 - \mu) \mu \langle A (h_{\max(s)} + h_{W_1}), A h \rangle \]

\[ \leq (1 + \delta_{[ts]}) \sum_{i=1}^N \lambda_i \left\| \left( \frac{1}{2} - \mu \right) (h_{\max(s)} + h_{W_1}) - \frac{1}{2} \mu u^{(i)} \right\|_2^2 + (1 - \mu) \mu \langle h_{\max(s)} + h_{W_1}, A^\top A h \rangle \]

\[ \leq (1 + \delta_{[ts]}) \left( \frac{1}{2} - \mu \right)^2 \| h_{\max(s)} + h_{W_1} \|^2 + \frac{\mu^2}{4} \sum_{i=1}^N \lambda_i \| u^{(i)} \|^2 \]

\[ + 2(1 - \mu) \mu \sqrt{\|ts\|} \| h_{\max(s)} + h_{W_1} \|_2 \geq 0. \]

By applying (4.28), the definition of \( \chi \) in (4.5) and \( \| h_{\max(s)} \|_2 \leq \| h_{\max(s)} + h_{W_1} \| _2 \), the above inequality reduces to

\[ \left[ \mu^2 - \mu + (\mu - 1)^2 \delta_{[ts]} \right] \| h_{\max(s)} + h_{W_1} \|^2 + \left[ 2(1 - \mu) \mu \sqrt{\|ts\|} \eta \right. \]

\[ + \frac{2 \delta_{[ts]} (1 - 2 \mu) \| x_{\max(s)} \|_1}{\sqrt{s + \alpha}} \| h_{\max(s)} + h_{W_1} \|_2 + \left. \frac{2 \delta_{[ts]} (1 - 2 \mu)}{\sqrt{s + \alpha}} \| x_{\max(s)} \|_1 \right] \geq 0, \]

which is a second-order inequality for \( \| h_{\max(s)} + h_{W_1} \|_2 \). Note that

\[ \delta_{[ts]} = \begin{cases} \delta_{ts}, & ts \text{ is an integer} \\ \delta_{[ts]}, & ts \text{ is not an integer} \end{cases} \]

and

\[ h(x) = \frac{1}{\sqrt{(\sqrt{s + \alpha})^2 \left( \frac{1 + \sqrt{s}}{2} (x - \sqrt{s}) \right) s(t-1)^2 (\sqrt{s-1})^2} + 1 \]

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is a decrease function for \( x \geq \frac{\sqrt{s}}{2} \). Thus, our condition (4.23) on \( \delta_t \) with \( t = 2 \) or \( t \geq 3 \) and \( s \geq 2 \) can guarantee that

\[
\delta_t < \frac{\mu}{1 - \mu - \frac{1}{\sqrt{s} + \alpha}} \leq \frac{1}{(\sqrt{s} + \alpha)^2} \left[ \frac{1}{s(t-1)^2(\sqrt{s}-1)^2} \right] + 1
\]  

holds. By solving the above second-order inequality under (4.30) we get

\[
\| h_{\max}(s) + h_W \|_2 \leq \frac{2(1 - \mu)\sqrt{|t|s}|\eta + \left( \frac{2\delta_t(1-2\mu)}{\sqrt{s+\alpha}} + \frac{2\delta_t(1-2\mu)(\mu - \mu^2 - (\mu - 1)^2\delta_t)}{\sqrt{s+\alpha}} \right) \| x_{\max}(s) \|_1}{\mu - \mu^2 - (\mu - 1)^2\delta_t},
\]

where \( z \leq \alpha (\sqrt{s} + \alpha) \) satisfying second-order inequality \( az^2 - bz - c \leq 0 \) with \( a, b, c > 0 \).

Following the argument in [23, Step 2] and (4.10), we can express an upper bound \( \| h_{\max}(s) \|_2 \). First, by estimation (2.4) in Lemma 1 and \( \| h_{\max}(s) \|_\infty \leq \| h_{\max}(s) \|_1/s \leq \| h_{\max}(s) \|_2/\sqrt{s} \), we know

\[
\| h_{\max}(s) \|_2^2 \leq \| h_{\max}(s) \|_1 \| h_{\max}(s) \|_\infty \leq \left( (\sqrt{s} + \alpha) \| h_{\max}(s) \|_2 + \alpha \| h_{\max}(s) \|_2 + 2 \| x_{\max}(s) \|_1 \right) \| h_{\max}(s) \|_2 \sqrt{s} \]
\[
= \frac{\alpha \| h_{\max}(s) \|_2}{\sqrt{s}} \| h_{\max}(s) \|_2 + \sqrt{s + \alpha} \| h_{\max}(s) \|_2 + \frac{2 | x_{\max}(s) \|_1 | h_{\max}(s) \|_2}{\sqrt{s}}.
\]

Therefore

\[
\left( \frac{\| h_{\max}(s) \|_2 - \alpha \| h_{\max}(s) \|_2}{2\sqrt{s}} \right) \leq \left( \frac{\alpha^2}{4s} + \frac{\alpha + \sqrt{s}}{\sqrt{s}} \right) \| h_{\max}(s) \|_2 + \frac{2 \| x_{\max}(s) \|_1 | h_{\max}(s) \|_2}{\sqrt{s}}
\]

which implies

\[
\| h_{\max}(s) \|_2^2 \leq \left( \frac{\alpha + \sqrt{s}}{\sqrt{s}} + \frac{\alpha^2}{4s} + \frac{\alpha + \sqrt{s}}{2\sqrt{s}} \right) \| h_{\max}(s) \|_2 + \sqrt{s} - \frac{2 | x_{\max}(s) \|_1 | h_{\max}(s) \|_2}{\sqrt{s}} \leq \left( \frac{\alpha + \sqrt{s}}{\sqrt{s}} + \frac{\alpha^2}{4s} + \frac{\alpha + \sqrt{s}}{2\sqrt{s}} \right) \| h_{\max}(s) \|_2 + \sqrt{s} - \frac{2 | x_{\max}(s) \|_1 | h_{\max}(s) \|_2}{\sqrt{s}}
\]

where the second inequality comes from the basic inequality \( 2|a||b| \leq (|a| + |b|)^2 \).

Therefore, from the fact \( \| h \|_2 = \sqrt{\| h_{\max}(s) \|_2^2 + \| h_{\max}(s) \|_2^2} \) and the above estimation (4.32), it follows that

\[
\| h \|_2 \leq \sqrt{\| h_{\max}(s) \|_2^2 + \left( \frac{\alpha + \sqrt{s}}{\sqrt{s}} + \frac{\alpha^2}{4s} + \frac{\alpha + \sqrt{s}}{2\sqrt{s}} \right) \| h_{\max}(s) \|_2 + \frac{\sqrt{2}}{2} \| x_{\max}(s) \|_1}^2
\]
\[
\left(1 + \sqrt{\frac{\alpha + \sqrt{s}}{\sqrt{s}} + \frac{\alpha^2}{4s} + \frac{\alpha + \sqrt{2}}{2\sqrt{s}}} \right) \|h_{\text{max}}(s)\|_2 + \frac{\sqrt{2}}{2} \|x_{\text{max}}(s)\|_1
\]

\[
\leq \left(1 + \sqrt{\frac{\alpha + \sqrt{s}}{\sqrt{s}} + \frac{\alpha^2}{4s} + \frac{\alpha + \sqrt{2}}{2\sqrt{s}}} \right) \frac{2(1 - \mu)\sqrt{\mu|ts|}}{\mu - \mu^2 - (\mu - 1)^2\delta ts} + \frac{2\delta ts(1 - 2\mu)}{(\sqrt{s} + \alpha)(\mu - \mu^2 - (\mu - 1)^2\delta ts)} + \frac{\sqrt{2s}}{2} \|x_{\text{max}}(s)\|_1,
\]

where the last inequality is due to (4.31). Therefore, we complete the proof.

\[\square\]

## 5 Effective Algorithm for L1-L2-DS

In the section, we present an effective algorithm to solve the \(\ell_1 - \alpha\ell_2\)-DS (1.5). Based on the fact that Dantzig selector and Lasso estimator exhibit similar behavior, we propose an unconstraint penalty problem as follows

\[
\min_{x, y \in B^\infty(\eta)} \lambda(\|x\|_1 - \alpha\|x\|_2) + \frac{1}{2} \|A^T Ax - y - A^T b\|_2^2,
\]

(5.1)

where \(\lambda > 0\) is the regularized parameter.

We find the optimal solution of (5.1) using the alternating direction method of multipliers (ADMM) algorithm. First, splitting the term \(\|x\|_1 - \alpha\|x\|_2\) and letting \(B = A^T A, c = A^T b\), one gets an equivalent problem of (5.1):

\[
\min_{x, w, y \in B^\infty(\eta)} \lambda(\|w\|_1 - \alpha\|w\|_2) + \frac{1}{2} \|Bx - y - c\|_2^2,
\]

s. t. \(x - w = 0\).

(5.2)

The augmented Lagrangian function of (5.2) is

\[
\mathcal{L}_\beta(x, y, w; z) = \lambda(\|w\|_1 - \alpha\|w\|_2) + \frac{1}{2} \|Bx - y - c\|_2^2 + \beta \|x - w\|_2^2 + \langle z, x - w \rangle,
\]

(5.3)

where \(z\) is the Lagrangian multiplier. Given \((x^0, y^0, w^0; z^0)\), iterations for (5.3) based on the ideas of ADMM are

\[
\begin{align*}
\left\{\begin{array}{l}
w^{k+1} = \arg\min_w \mathcal{L}_\beta(x^k, y^k, w; z^k), \\
(x^{k+1}, y^{k+1}) = \arg\min_{x, y} \mathcal{L}_\beta(x, y, w^{k+1}; z^k), \\
z^{k+1} = z^k + \beta(x^{k+1} - w^{k+1}).
\end{array}\right.
\]

\[
\left\{\begin{array}{l}
(w^{k+1} = \arg\min_w \mathcal{L}_\beta(x^k, y^k, w; z^k), \\
(x^{k+1}, y^{k+1}) = \arg\min_{x, y} \mathcal{L}_\beta(x, y, w^{k+1}; z^k), \\
z^{k+1} = z^k + \beta(x^{k+1} - w^{k+1}).
\end{array}\right.
\]

(5.4)

Before solving the \(w\)-subproblem in (5.4), we first recall results for a proximal operator. In [30, Proposition 7.1] and [31, Section 2], proximal operator

\[
\arg\min_x \frac{1}{2}\|x - b\|_2^2 + (\mu_1\|x\|_1 - \mu_2\|x\|_2), \quad \mu_1 \geq \mu_2 > 0
\]

(5.5)
has an explicit formula for $x$, denoting $\text{Prox}_{\mu_1 \ell_1 - \mu_2 \ell_2}(b)$. The minimization in (5.4) respecting to $w$ has the following closed-form solution

$$w^{k+1} = \text{Prox}_{\frac{\alpha}{\beta} \ell_1 - \alpha \ell_2}(x^k + \frac{z^k}{\beta}).$$

(5.6)

Next, we turn our attention to the $(x, y)$-subproblem. The $x$-subproblem has closed-form solution as follows

$$x^{k+1} = (B^T B + \beta I)^{-1} \left( B^T (y^k + c) + \beta (w^{k+1} - \frac{z^k}{\beta}) \right),$$

(5.7)

where the inverse of $B^T B + \beta I$ is computed by Woodbury matrix identity. The $y$-subproblem also has closed-form solution as follows

$$y^{k+1} = \text{Proj}_{B^\infty(\eta)}(Bx^k - c),$$

(5.8)

where $\text{Proj}_{B^\infty(\eta)}(x)$ is a projection on the ball $B^\infty(\eta)$, i.e.,

$$\text{Proj}_{B^\infty(\eta)}(x)_j = \min(\max(x_j, -\eta), \eta), \quad j = 1, \ldots, n.$$

On account of the above discussions, the effective Algorithm to approximately solve (5.2) is summarized as Algorithm 1.

---

**Algorithm 1: ADMM for solving (5.2)**

**Input:** $A$, $b$, $\eta$, $0 < \alpha \leq 1$, $\lambda$, $\beta$.

**Initials:** $(x, y, w; z) = (x^0, y^0, w^0; z^0)$, $k = 0$.

**Circulate** Step 1–Step 4 until “some stopping criterion is satisfied”:

**Step 1:** Compute $w^{k+1}$ by (5.6).

**Step 2:** Compute $x^{k+1}, y^{k+1}$ by (5.7) and (5.8), respectively.

**Step 3:** Update dual variables

$$z^{k+1} = z^k + \beta(x^{k+1} - w^{k+1})$$

**Step 4:** Update $k$ to $k + 1.

**Output:** $x^k$.

**Remark 5.** In Algorithm 1, $\lambda, \alpha, \eta$ are model parameters satisfying $\lambda > 0$, $0 < \alpha \leq 1$ and $\eta > 0$, and $\beta > 0$ is the regularized parameter in ADMM algorithm.

---

### 6 Numerical Experiments

In the section, we present numerical experiments for the recovery of sparse signals to demonstrate the performance of $\ell_1 - \alpha \ell_2$-DS (1.5).

In our experiments, our method $\ell_1 - \alpha \ell_2$-DS (1.5) is compared with $\ell_1$-DS (1.3) implemented by linear ADMM [45], and the $\ell_p$-DS (6.1) as follows

$$\min_{x \in \mathbb{R}^n} \|x\|_p \quad \text{subject to} \quad \|A^\top (b - Ax)\|_\infty \leq \eta$$

(6.1)
where $0 < p < 1$. Similarly, an effective algorithm for solving (6.1) can be developed based on Algorithm 1. We only need

$$w^{k+1} = \text{Prox}_{\frac{1}{\beta}p} \left( x^k + \frac{z^k}{\beta} \right)$$

instead of (5.6) for Algorithm 1, where the proximal operator

$$\text{Prox}_{\mu \ell_p} (b) := \arg \min_x \frac{1}{2} \| x - b \|_2^2 + \mu \| x \|_p^p, \quad \mu > 0$$

has an explicit formula for $x$ in [53, 37], denoting $\text{Prox}_{\mu \ell_p} (b)$.

And we apply the proposed Algorithm 1 for the $\ell_1 - \alpha \ell_2$-DS (1.5) to reconstruct sparse signals in the cases of Gaussian, Symmetric $\tilde{\alpha}$-stable ($S\tilde{\alpha}S$) and uniform noises, which have been defined in Section 1.1.

In our experiments, we test two measurement matrices defined in Subsection 1.1, which have different coherence. Let $x^0 \in \mathbb{R}^n$ be a simulated $s$-sparse signal, where the support of $x^0$ is a random index set and the $s$ non-zero entries obey the Gaussian distribution $\mathcal{N}(0, 1)$. In addition, the signal $x^0$ is normalized to have a unit energy value. Let $\hat{x}$ be the estimation of $x^0$ via each solver.

Each provided result is an average over 100 independent tests. All experiments are performed under Windows Vista Premium and MATLAB v9.1 (R2016b) running on a Huawei laptop-qolkaflg with an Intel(R) Core(TM)i5-8250U CPU at 1.8 GHz and 8195MB RAM of memory.

We take a self-adapting strategy to update the parameter $\alpha$. The initial value was $\alpha^0 = 0.1$ and then it was adjusted iteratively by the strategy

$$\alpha^{k+1} = \begin{cases}    \alpha^k, & \text{if } \text{mod}(k, 5) \neq 0, \\    \min\{1.5\alpha^k, 1\}, & \text{if } \text{mod}(k, 5) = 0. \end{cases} \quad (6.2)$$

Recall that $\alpha \leq 1$ is required to ensure $\| x \|_{\alpha, 1-2} \geq 0$ for any $x$. The above strategy of choosing $\alpha$ clearly satisfies this condition.

### 6.1 Observations with Gaussian Noise

First, the measurement matrix $A$ is Gaussian matrix. We follow the method in [8] to generate Gaussian matrix $A$ whose columns all have the unit norm. More specifically, we first generated an $m \times n$ matrix with independent Gaussian entries and then normalized each column with the unit norm. After that, we randomly choose a sample set $S$ with cardinality $s$. Then, the coefficient vector $x^0$ was generated by

$$x^0_i = \begin{cases}    \xi_i (1 + |c_i|), & i \in S, \\    0, & \text{otherwise}, \end{cases} \quad (6.3)$$

where $\xi_i \in \mathcal{U}(-1, 1)$ (i.e., the uniform distribution on the interval $(-1, 1)$) and $c_i \sim \mathcal{N}(0, 1)$. Finally, the vector of observations $b$ was generated by $b = Ax + e$ with $e \sim \mathcal{N}(0, \sigma^2)$.

To compare with the $\ell_1$-DS in [5] and $\ell_p$-DS (6.1), we test the same cases of $\sigma$, i.e., $\sigma = 0.01$ and $\sigma = 0.05$, with $(n, p, s) = (72i, 256i, 8i)$ for $i = 1, 2, 3$. Here the coherence
Table 2: Numerical results for Gaussian matrix with unit column norms.

| i   | Algorithms           | $\sigma = 0.01$ | $\sigma = 0.05$ |
|-----|----------------------|-----------------|-----------------|
|     | Time(s) $\rho^2$ $\rho^2_{\text{orign}}$ | Time(s) $\rho^2$ $\rho^2_{\text{orign}}$ |
| i = 1 | $\ell_1$-DS          | 0.11 1.24 17.54 | 0.11 1.12 20.51 |
|      | $\ell_{0.9}$-DS      | 1.15 1.07 6.57  | 1.26 1.13 14.87 |
|      | $\ell_{0.5}$-DS      | 1.16 1.07 8.55  | 1.24 1.10 14.41 |
|      | $\ell_{0.1}$-DS      | 1.36 1.17 6.53  | 1.11 1.06 13.00 |
|      | $\ell_1 - \alpha \ell_2$-DS | 0.53 1.03 5.15 | 0.51 1.02 9.10 |
| i = 2 | $\ell_1$-DS          | 0.34 1.18 21.65 | 0.26 1.09 20.91 |
|      | $\ell_{0.9}$-DS      | 7.12 1.10 20.43 | 7.09 1.10 19.97 |
|      | $\ell_{0.5}$-DS      | 7.61 1.05 16.41 | 6.10 1.04 18.96 |
|      | $\ell_{0.1}$-DS      | 7.93 1.13 17.24 | 7.48 1.11 21.19 |
|      | $\ell_1 - \alpha \ell_2$-DS | 2.28 1.04 8.32 | 4.16 1.02 7.53 |
| i = 3 | $\ell_1$-DS          | 1.02 1.16 24.61 | 1.52 1.13 24.59 |
|      | $\ell_{0.9}$-DS      | 23.23 1.15 22.05 | 13.95 1.12 21.45 |
|      | $\ell_{0.5}$-DS      | 23.39 1.14 23.05 | 13.94 1.11 21.56 |
|      | $\ell_{0.1}$-DS      | 22.54 1.11 23.15 | 14.04 1.18 22.59 |
|      | $\ell_1 - \alpha \ell_2$-DS | 4.38 1.09 8.24 | 10.22 1.06 8.52 |

$\mu(A)$ decreases from 0.50 to 0.25. For each case, we generated ten different problems and reported the average performance. As in [8], the quality of the Dantzig selector is measured by

$$
\rho^2_{\text{orign}} = \frac{\sum_j |\hat{x}_j - x_0^j|^2}{\sum_j \min\{(x_0^j)^2, \sigma^2\}}, \quad \rho^2 = \frac{\sum_j |\tilde{x}_j - x_0^j|^2}{\sum_j \min\{(x_0^j)^2, \sigma^2\}},
$$

(6.4)

where $\hat{x}$ denotes the Dantzig selector via solving (1.3), (1.5) and (6.1), and $\tilde{x}$ is the corresponding refined Dantzig selector after the bias-removing two-stage procedure in [8]. In [35], $\rho^2_{\text{orign}}$ and $\rho^2$ are named as the preprocessing and postprocessing errors, respectively. Note that $\rho^2_{\text{orign}}$ and $\rho^2$ are two measurements on the performance of the Dantzig selector. Obviously, we are pursuing better selectors which have smaller values of them.

From Table 2, we repeat the numerical performance $\ell_1$-DS, $\ell_p$-DS and the proposed $\ell_1 - \alpha \ell_2$-DS for $\sigma = 0.01, 0.05$. We display the average values of $\rho^2_{\text{orign}}, \rho^2$, the number of iterations (Iter), and the computing time in seconds (“Time (s)”) over 100 independent trials. The data in Table 2 show the efficiency of the proposed $\ell_1 - \alpha \ell_2$-DS. We can see that the $\ell_1$-DS needs the least time, following by the propose method. However, our method has the best performance in terms of the value of $\rho^2_{\text{orign}}$ and $\rho^2$.

Next, we consider that the measurement matrix $A$ is oversampled partial DCT matrix. We use the average of the signal-to-noise ratio (SNR) in dB,

$$
\text{SNR}(\hat{x}, x_0) = 20 \log_{10} \frac{\|x_0\|_2}{\|\hat{x} - x_0\|_2},
$$

(6.5)

over 100 independent trials as our performance measure, where $\hat{x}$ is the reconstructed signal.
Table 3: The average of SNR over 100 independent trials for different algorithms, the oversampled DCT $A \in \mathbb{R}^{m \times n}$ with $m = 64, n = 256, F = 10$, the measurements corrupted by Gaussian noises with two different noise levels $\sigma$ and different sparsity.

| $\sigma$ | Alg.       | Sparsity | 1    | 2    | 4    | 6    | 8    | 10   | 12   |
|----------|------------|----------|------|------|------|------|------|------|------|
| $10^{-3}$| $\ell^1$-DS|          | 29.52| 26.91| 26.00| 24.01| 18.00| 16.00| 11.42|
|          | $\ell^{0.9}$-DS|       | 38.61| 37.43| 31.04| 25.75| 14.71| 10.50| 8.06  |
|          | $\ell^{0.5}$-DS|       | 38.02| 37.13| 31.41| 24.64| 16.69| 12.24| 7.52  |
|          | $\ell^{0.1}$-DS|       | 37.77| 36.38| 31.63| 20.99| 12.95| 5.77 | 3.57  |
|          | $\ell_1 - \alpha \ell_2$-DS| | 41.35| 36.74| 34.91| 32.13| 29.13| 26.23| 23.80 |
| $10^{-2}$| $\ell^1$-DS|          | 11.73| 9.78 | 9.45 | 8.75 | 6.08 | 4.64 | 5.13  |
|          | $\ell^{0.9}$-DS|       | 14.72| 14.61| 12.82| 10.37| 8.47 | 8.01 | 5.85  |
|          | $\ell^{0.5}$-DS|       | 15.77| 15.28| 14.37| 12.19| 9.42 | 7.01 | 4.70  |
|          | $\ell^{0.1}$-DS|       | 16.47| 16.39| 10.69| 10.09| 4.91 | 3.47 | 2.75  |
|          | $\ell_1 - \alpha \ell_2$-DS| | 24.68| 17.41| 13.89| 13.52| 11.26| 8.57 | 6.12  |

signal. We display SNR of different algorithms to recover sparse signals over 100 repeated trials for $m = 64, n = 256$ and different sparsity $s$. The oversampled partial DCT matrix $A \in \mathbb{R}^{m \times n}$ with $F = 10$ has high coherence with $\mu(A) > 0.99$. From the Table 3, we see that SNR of the proposed $\ell_1 - \alpha \ell_2$-DS are higher than that of $\ell_1$-DS, $\ell_p$-DS.

### 6.2 Observations with Impulsive Noise

In this subsection, we consider that the observation is corrupted by impulsive noise. And let the measurement matrix $A \in \mathbb{R}^{m \times n}$ with $m = 64, n = 256$ first be Gaussian matrix, which has coherence $0.45 < \mu(A) < 0.50$. Next, $A \in \mathbb{R}^{m \times n}$ with $m = 64, n = 256$ is the oversampled DCT matrix with $F = 10$, which has high coherence $\mu(A) > 0.99$.

Tables 4 and 5 present the average of SNR over 100 independent trials for the $\ell_1$-DS, $\ell_p$-DS ($0 < p \leq 1$) and the proposed $\ell_1 - \alpha \ell_2$-DS versus the sparsity $s$ in the SαS noise with $\tilde{\alpha} = 1$ (Cauchy noise), $\tilde{\delta} = 0$ and $\gamma = 10^{-4}, 10^{-3}$. Table 4 shows that the proposed $\ell_1 - \alpha \ell_2$-DS provides the best robust performance no matter the measurement matrix $A$ has small or high coherence.
Table 4: The average of SNR over 100 independent trials for different algorithms, Gaussian matrix \( \mathbf{A} \in \mathbb{R}^{m \times n} \) with \( m = 64 \), \( n = 256 \), the measurements corrupted by Cauchy noises with two different levels \( \gamma \), and different sparsity.

| \( \gamma \) | Alg. | Sparsity |
|-------------|------|----------|
| \( 10^{-4} \) | \( \ell_1\text{-DS} \) | 35.52    |
|             | \( \ell_{0.9}\text{-DS} \) | 39.00    |
|             | \( \ell_{0.5}\text{-DS} \) | 37.50    |
|             | \( \ell_{0.1}\text{-DS} \) | 38.93    |
|             | \( \ell_1 - \alpha \ell_2\text{-DS} \) | 46.41    |
| \( 10^{-3} \) | \( \ell_1\text{-DS} \) | 10.59    |
|             | \( \ell_{0.9}\text{-DS} \) | 18.83    |
|             | \( \ell_{0.5}\text{-DS} \) | 17.00    |
|             | \( \ell_{0.1}\text{-DS} \) | 19.13    |
|             | \( \ell_1 - \alpha \ell_2\text{-DS} \) | 24.26    |

### 6.3 Observations with Uniform Noise

In this subsection, we consider the observation is corrupted by uniform noise. And the the measurement matrix \( \mathbf{A} \) is same with the presented Gaussian and oversampled DCT in Subsection 6.2. We take noise levels \( \varsigma = 10^{-3}, 10^{-2} \) and display the average of SNR over 100 independent trials in Tables 6 and 7. These results show that the Dantzig selector also work efficiently for uniform noise, which similar as that of \( \ell_\infty \) constraint. It also implies that our proposed method is better than \( \ell_1\text{-DS} \) and \( \ell_p\text{-DS} \) for both Gaussian and oversamped DCT matrices.

### 7 Conclusions

In this paper, we consider the signal reconstruction under Dantzig selector constraint via \( \ell_1 - \alpha \ell_2 \) (0 < \( \alpha \) ≤ 1) minimization. First, we introduce the \( \ell_1 - \alpha \ell_2\text{-DS} \) (1.5) to recover signals \( \mathbf{x} \) from \( \mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e} \). Next, we show a sufficient condition based on \((\ell_2, \ell_1)\text{-RIP}\) to guarantee the stable recovery of signal \( \mathbf{x} \) from \( \mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e} \) via (1.5) (see Theorem 1). And Based on the high order classical RIP (i.e., \((\ell_2, \ell_2)\text{-RIP}\)), we develop a sufficient condition for the stable reconstruction of signals \( \mathbf{x} \) via (1.5) (see Theorem 2 applying the technique of the convex combination for \( \ell_1 - \ell_2\)). Last, we show an effective algorithm based on ADMM to solve the \( \ell_1 - \alpha \ell_2\text{-DS} \) (1.5) Furthermore, we present numerical experiments for the sparse signal reconstruction in the cases of Gaussian, impulsive and uniform noises. Results demonstrate the efficiency of \( \ell_1 - \alpha \ell_2\text{-DS} \), which is different from that of \( \ell_2, \ell_1 \) and \( \ell_\infty \) data fitting terms. What should point out is that, this is the first paper which explores the performances of Dantzig selector for different type noises. The proposed method also outperform than existing methods no matter the measurement matrix has high or small coherence.
Table 5: The average of SNR over 100 independent trials for different algorithms, the oversampled DCT matrix $A \in \mathbb{R}^{m \times n}$ with $m = 64, n = 256, F = 10$, the measurements corrupted by two different Cauchy noise levels $\gamma$ and different sparsity.

| $\gamma$ | Alg. | Sparisty | 1 | 2 | 4 | 6 | 8 | 10 | 12 |
|----------|------|----------|---|---|---|---|---|----|----|
| $10^{-4}$| $\ell_1$-DS | 31.90 | 30.25 | 29.19 | 27.55 | 26.48 | 21.94 | 16.85 |
|          | $\ell_{0.9}$-DS | 31.43 | 29.95 | 28.41 | 26.08 | 20.73 | 17.85 | 13.60 |
|          | $\ell_{0.5}$-DS | 35.76 | 29.28 | 25.56 | 23.48 | 16.71 | 14.14 | 7.86 |
|          | $\ell_{0.1}$-DS | 34.66 | 33.05 | 31.05 | 26.07 | 20.49 | 16.00 | 12.41 |
|          | $\ell_1 - \alpha \ell_2$-DS | 35.27 | 33.51 | 31.82 | 31.63 | 28.06 | 27.32 | 22.96 |
| $10^{-3}$| $\ell_1$-DS | 14.42 | 12.23 | 8.18 | 6.01 | 5.52 | 3.75 | 2.99 |
|          | $\ell_{0.9}$-DS | 15.86 | 13.72 | 12.34 | 10.12 | 9.47 | 7.19 | 4.97 |
|          | $\ell_{0.5}$-DS | 16.35 | 14.78 | 13.84 | 12.56 | 8.78 | 3.17 | 2.75 |
|          | $\ell_{0.1}$-DS | 18.97 | 14.11 | 10.83 | 8.72 | 6.18 | 4.41 | 3.26 |
|          | $\ell_1 - \alpha \ell_2$-DS | 19.32 | 13.61 | 10.06 | 8.29 | 5.36 | 5.35 | 3.44 |

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Table 6: The average of SNR over 100 independent trials for different algorithms, Gaussian matrix $A \in \mathbb{R}^{m \times n}$ with $m = 64, n = 256$, the measurements corrupted by two different uniform noise levels $\varsigma$ and different sparsity.

| $\varsigma$ | Alg.      | $m/n$ | 1    | 2    | 4    | 6    | 8    | 10   | 12   |
|---------|-----------|-------|------|------|------|------|------|------|------|
| $10^{-3}$ | $\ell_1$-DS | 42.64 | 39.97 | 39.17 | 38.50 | 38.29 | 35.29 | 31.34 |
|         | $\ell_{0.9}$-DS | 44.53 | 43.91 | 43.67 | 43.53 | 43.12 | 42.61 | 41.58 |
|         | $\ell_{0.5}$-DS | 48.06 | 47.79 | 46.55 | 46.33 | 45.89 | 40.87 | 29.98 |
|         | $\ell_{0.1}$-DS | 45.34 | 44.62 | 44.08 | 43.21 | 43.36 | 42.50 | 42.18 |
|         | $\ell_1 - \alpha \ell_2$-DS | 61.04 | 57.06 | 56.91 | 56.12 | 55.50 | 55.33 | 55.02 |
| $10^{-2}$ | $\ell_1$-DS | 24.35 | 24.18 | 23.76 | 23.54 | 23.09 | 21.85 | 17.75 |
|         | $\ell_{0.9}$-DS | 24.49 | 23.93 | 23.66 | 22.86 | 21.25 | 20.04 | 18.79 |
|         | $\ell_{0.5}$-DS | 25.38 | 24.00 | 22.10 | 19.43 | 17.71 | 16.61 | 16.55 |
|         | $\ell_{0.1}$-DS | 26.65 | 23.09 | 21.62 | 20.79 | 20.73 | 19.92 | 17.32 |
|         | $\ell_1 - \alpha \ell_2$-DS | 29.01 | 25.61 | 25.11 | 24.55 | 23.69 | 22.17 | 21.75 |

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Table 7: The average of SNR over 100 independent trials for different algorithms, the oversampled DCT matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m = 64, n = 256, F = 10$, the measurements corrupted by two different uniform noise levels $\varsigma$ and different sparsity.

| $\varsigma$ | Alg.          | $m/n$ | 1    | 2    | 4    | 6    | 8    | 10   | 12   |
|------------|---------------|-------|------|------|------|------|------|------|------|
| $10^{-3}$  | $\ell_1 - 1$-DS |       | 33.67| 31.92| 31.00| 27.81| 22.82| 18.02| 16.94|
|            | $\ell_{0.9}$-DS |       | 37.24| 36.99| 36.53| 30.95| 30.90| 27.03| 21.09|
|            | $\ell_{0.5}$-DS |       | 38.71| 37.08| 34.93| 33.32| 32.42| 24.17| 16.82|
|            | $\ell_{0.1}$-DS |       | 38.82| 37.71| 36.17| 34.21| 25.42| 23.51| 19.74|
|            | $\ell_1 - \alpha \ell_2$-DS | | 44.05| 39.18| 37.66| 36.10| 32.72| 26.38| 27.15|
| $10^{-2}$  | $\ell_1 - 1$-DS |       | 16.02| 15.71| 13.52| 9.77 | 9.56 | 7.64 | 7.06 |
|            | $\ell_{0.9}$-DS |       | 19.65| 19.14| 18.37| 16.87| 16.02| 14.28| 12.07|
|            | $\ell_{0.5}$-DS |       | 21.49| 21.61| 18.90| 16.66| 16.06| 14.78| 11.12|
|            | $\ell_{0.1}$-DS |       | 22.44| 22.81| 18.79| 17.45| 14.66| 12.77| 7.17 |
|            | $\ell_1 - \alpha \ell_2$-DS | | 26.81| 24.49| 24.22| 22.02| 21.14| 19.33| 17.31|

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