A NONCONVEX TRUNCATED REGULARIZATION AND BOX-CONSTRAINED MODEL FOR CT RECONSTRUCTION

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(Communicated by Jin Keun Seo)

Abstract. X-ray computed tomography has been a useful technology in cancer detection and radiation therapy. However, high radiation dose during CT scans may increase the underlying risk of healthy organs. Usually, sparse-view X-ray projection is an effective method to reduce radiation. In this paper, we propose a constrained nonconvex truncated regularization model for this low-dose CT reconstruction. It preserves sharp edges very well. Although this model is quite complicated to analyze, we establish two useful theoretical results for its minimizers. Motivated by them, an iterative support shrinking algorithm is introduced. To handle more nondifferentiable points of the regularization function except zero point, we use a general proximally linearization technique at them, which is helpful to implement our algorithm. For this algorithm, we prove the convergence of the objective function, and give a lower bound theory of the iterative sequence. Numerical experiments and comparisons demonstrate that our model with the proposed algorithm performs good for low-dose CT reconstruction.

1. Introduction. In the past several decades, X-ray computed tomography (CT) has been one important nondestructive testing technologies [5], and is widely used for disease detection and radiation therapy. Mathematically, the imaging model of CT system can be approximated as a discrete linear inverse problem

\[ g = Au + \varepsilon, \]

where \( A \in \mathbb{R}^{N \times N} \) is the projection matrix, \( g \in \mathbb{R}^N \) is the observed projection data, \( u \in \mathbb{R}^N \) is the vector form of a \( n \times n \) clean image (\( N = n^2 \)) and \( \varepsilon \in \mathbb{R}^N \) is the random Gaussian noise. Many methods such as filtered back projection (FBP), algebraic reconstruction techniques (ART) and simultaneous algebraic reconstruction techniques (SART) [26, 19, 1, 23] usually need high doses of radiation to produce good-quality diagnostic images. However, high radiation dose may increase the underlying risk of cancer [15]. To reduce the radiation, a direct strategy is to impose few number of projections to patient, i.e., adopting low-dose CT. For this low-dose
CT, since $N \ll N$, it is a challenge to reconstruct a good image $u$ from $g$. According to compressive sensing theory, sparsity-based regularization models are very suit to handle the under-determined linear system. In this paper, we focus on proposing an appropriate sparsity-based method for low-dose CT reconstruction.

By exploiting the sparse prior of CT images, total variation (TV)-based models were studied \[38, 22, 29, 11, 28, 53, 24, 14, 33\], which are approximately sparse in image gradient transform \[36\]. Another popular regularization technique is the wavelet frame based approaches \[35, 3, 21, 17, 13, 12\]. Other regularization models \[39, 34, 25, 52, 49\], such as statistical based methods, have also been introduced.

Recently, some studies \[31, 48, 51, 45, 2\] indicate that nonconvex and nonsmooth models are able to enhance the sparsity of images and obtain better restored results. Motivated by them, nonconvex regularization models were introduced in \[44, 16, 6\] for CT image reconstruction problem. As we know, $\ell_p (0 < p < 1)$ quasi-norm of image gradient is an important nonconvex regularization term, which keeps neat edges well \[32, 20, 48\]. However, it usually leads to non-Lipschitz at zero point, which brings challenges for the design of algorithms. To overcome this difficulty, many algorithms based on smoothing approximation strategies were introduced in \[20, 8, 27\], while there is a non-smoothing technique \[46\].

A recent paper \[42\] showed that most convex and nonconvex regularizers suffer from a contrast reduction effect at image edges. To preserve contrasts, a truncated strategy has been introduced in \[31, 42\], based on which another class of regularizers were constructed in \[18\] by introducing extra parameters. In particular, the truncated model in \[42\] makes the commonly used nonconvex regularizers flat on $(\tau, +\infty)$ for some positive $\tau$ to recover the blurred signal and image. Some theoretical analyses in 1D case demonstrate the good contrast-preserving ability. However, compared to models without truncation, the direct truncated models in \[31, 42\] will produce more nondifferentiable points, which bring more difficulties for designing algorithms. For example, \[31\] does not present a corresponding algorithm for its nonconvex truncated model consisting of at least two nondifferentiable points. The authors in \[42\] just use the alternating direction method of multipliers (ADMM) to solve their truncated model, where the algorithm will yield oscillation phenomenon and even have no convergence of the objective function.

In this paper, we propose a nonconvex and nonsmooth truncated regularization with box-constrained model for CT reconstruction. Our model consists of a nonconvex truncated regularizer of image gradient, which can preserve not only image edges but also its contrasts. Moreover, we introduce the very useful box constraints, because numerous numerical experiments in \[7, 9\] have shown that constrained models can achieve reasonable improvement compared to unconstrained ones in qualitatively and quantitatively. To handle multiple nondifferentiable points produced by the truncated regularization, we extend the previous iterative support shrinking algorithm \[46\] with a general proximal linearization technique. Some theoretical results for the proposed algorithm are also obtained, such as the convergence of the objective function and a lower bound theory of the iterative sequence. To summarize, the contributions of the paper are listed as follows:

- We propose a new constrained nonconvex truncated model for structure and contrast preservation in CT reconstruction.
- We show two useful theoretical results for the local minimizers of our model. They help us to design an effective algorithm, which adopts a general linearization technique to handle nondifferentiable points of the truncated regularization function.
- The convergence of the objective function and a lower bound theory for the nonzero entries of gradients of the iterative sequence are provided.
Experimental results demonstrate that our model with the proposed algorithm achieves better CT reconstruction compared to some existing regularization methods.

The paper is organized as follows. In section 2, we propose our model with some basic assumptions, definitions and remarks. In subsection 2.3, we prove two theoretical results for local minimizers of our model. In section 3, we present an effective numerical algorithm with some theoretical properties. Section 4 lists some numerical experiments and comparisons. Finally, we conclude the paper in section 5.

2. The proposed model and theoretical analyses.

2.1. Basic notation. For an image, let \( J = \{(i_x, i_y) : 1 \leq i_x, i_y \leq n\} \) and \( J = \{1, 2, \ldots, N\} \) respectively denote the index sets of its matrix form \( U \in \mathbb{R}^{n \times n} \) and vector form \( u \in \mathbb{R}^N \) \((N = n^2)\). Obviously, there is a one-to-one mapping between \( J \) and \( J \), i.e., \( J \rightarrow J: i \mapsto (i_x, i_y) \), where

\[
\begin{align*}
    i_x &= \left\lfloor \frac{i}{n} \right\rfloor + 1, \quad i_y = \text{mod}(i, n), \quad \text{mod}(i, n) \neq 0, \\
    i_x &= \left\lfloor \frac{i}{n} \right\rfloor, \quad i_y = n, \quad \text{mod}(i, n) = 0.
\end{align*}
\]

At a pixel \( i \), we can define the discrete gradient operator

\[ d_i = \begin{pmatrix} d^{\text{f}}_i \\ d^{\text{v}}_i \end{pmatrix} \in \mathbb{R}^{2 \times N}, \]

where \( d_i^\text{h}, d_i^\text{v} \in \mathbb{R}^{1 \times N} \) and \( d_i^\text{f}, d_i^\text{v} \) respectively denote the discrete horizontal forward difference operator and vertical forward difference operator [43]. The Euclidean norm is denoted as \( \| \cdot \| \). For a real-valued matrix \( A \), \( A^T \) is its transpose, and \( \| A \| \) is its spectral norm.

2.2. The model. To recover \( u \) from \( g \), we propose the following regularization model:

\[
\min \limits_{u \in \mathbb{R}^N} \quad F(u) := \sum \limits_{i \in J} \varphi(|d_i u|) + \frac{\alpha}{2} \| Au - g \|^2
\]

\[
\text{s.t.} \quad l_1 e \leq u \leq l_2 e,
\]

where \( \varphi(t) : [0, +\infty) \rightarrow [0, +\infty) \) is a potential function; \( |d_i u| = \sqrt{(d^{\text{f}}_i u)^2 + (d^{\text{v}}_i u)^2} \); \( \alpha > 0 \) is a model parameter; \( l_1 \) and \( l_2 \) are respectively nonnegative lower and upper parameters and \( e = (1, 1, \ldots, 1)^T \in \mathbb{R}^N \). Denote \( \mathcal{U} = \{ u : l_1 e \leq u \leq l_2 e \} \).

We assume that \( \varphi(t) \) has some properties:

**Assumption 1.** (a) \( \varphi(t) : [0, +\infty) \rightarrow [0, +\infty) \) is \( C^0 \), concave and increasing, with \( \varphi(0) = 0 \); 
(b) \( \varphi(t) \) is \( C^1 \) on \((0, +\infty) \setminus M \) \((0 \notin M)\), where \( M \) is a finite set of points (possibly empty) such that if \( t \in M \) then \( \varphi'(t^-) > \varphi'(t^+) \); 
(c) \( \varphi'(t) = 0 \), when \( t \in S_M = \{ t : t > \max (M) \} \); \( \varphi'(t) > 0 \), when \( t \in (0, +\infty) \setminus (M \cup S_M) \).

Assumption 1(c) clearly shows that 0 is a strict minimizer of \( \varphi(t) \). Three different \( \varphi(t) \) are shown in Fig. 1. The set \( M = \{0.5\} \) in Fig. 1(a) and Fig. 1(b), while \( M = \{1, 2\} \) in Fig. 1(c). Note that, Assumption 1(c) shows that the potential function \( \varphi(t) \) is flat on \((\max(M), +\infty)\). Here, the value \( \tau := \max(M) \) is related to edge contrasts of images. That is, if \( |d_i u| \) at the edge is equal or larger than \( \tau \), our regularization will not have increasing penalization on \( |d_i u| \). Although the exact
relationship between image contrasts and the truncated scheme can not be easily obtained for general cases, the truncated regularizers can give perfect restorations with contrast preserving for some interesting examples, see [30, 42, 18].

To conveniently analyze our box-constrained model \( (P) \), we give the following index sets, some of which are inspired by [9, 40].

**Definition 2.1.** For any \( u \in \mathcal{U} \), we define some index sets

1. \( I(u) = \{ i \in J : l_1 < u_i < l_2 \} \).
2. \( B(u) = J \setminus I(u) \).
3. \( \Omega_1(u) = \{ i \in J : \| d_i u \| \neq 0 \} \).
4. \( \Omega_{1,M}(u) = \{ i \in \Omega_1(u) : \| d_i u \| = t, \text{ for } t \in M \} \).
5. \( \Omega_0(u) = \{ i \in J : \| d_i u \| = 0 \} \).

To understand these index sets above, we explain them in Appendix 6.1 using a simple example. Moreover, according to the introduced index sets, we use \( (d_i^T)_{I(u)} \) and \( (d_i^T)_{I(u)} \) to respectively denote the subvectors whose entries lie in the position of \( d_i^T \) and \( d_i^T \) indexed by \( I(u) \). Similarly, we define \( (d_i^T)_{B(u)}, (d_i^T)_{B(u)}, u_{I(u)} \) and \( u_{B(u)} \). Then, we have

\[
2. (d_i)_{I(u)} = \begin{pmatrix} (d_i^T)_{I(u)} \\ (d_i^T)_{I(u)} \end{pmatrix}, \quad (d_i)_{B(u)} = \begin{pmatrix} (d_i^T)_{B(u)} \\ (d_i^T)_{B(u)} \end{pmatrix},
\]

\[
d_i^T u = (d_i^T)_{I(u)} u_{I(u)} + (d_i^T)_{B(u)} u_{B(u)}, \quad d_i^T u = (d_i^T)_{I(u)} u_{I(u)} + (d_i^T)_{B(u)} u_{B(u)}.
\]

These representations of \( d_i^T u \) and \( d_i^T u \) will be used in the following.

### 2.3. Theoretical analyses of our model

Since \( F(u) \) is \( C^0 \) defined on a compact set in (1), there exists a solution for \( (P) \). In general, a global minimizer is difficult to obtain. Here, we assume that \( u^* \) is a local minimizer. Denote \( I^* = I(u^*), B^* = B(u^*), \Omega_1^* = \Omega_1(u^*), \Omega_{1,M}^* = \Omega_{1,M}(u^*), \Omega_0^* = \Omega_0(u^*), \) as the simplified index sets in Definition 2.1. Let \( A_{I^*} = (a_k)_{k \in I^*} \) and \( A_{B^*} = (a_k)_{k \in B^*} \) be submatrices of \( A = [a_1, a_2, \cdots, a_N] \). Moreover, \( u^* \) is rearranged as \( (u^*_I, u^*_B) \), where \( u^*_I \) is in \( \mathbb{R}^{|I^*|} \), \( u^*_B \) is in \( \mathbb{R}^{|B^*|} \) and \( Au^* = A_{I^*} u^*_I + A_{B^*} u^*_B \). Define

\[
f^* = g - A_{B^*} u^*_B^*,
\]

\[
t_i^{r,s} = (d_i^T)_{B^*} u_{B^*}^s, \quad t_i^{b,s} = (d_i^T)_{B^*} u_{B^*}^s, \quad \forall i \in J.
\]

---

**Figure 1.** Three different \( \varphi(t) \). (a) \( \varphi(t) = t, \ 0 \leq t \leq 0.5; \ \varphi(t) = 0.5, \ t > 0.5. \) (b) \( \varphi(t) = t^{0.5}, \ 0 \leq t \leq 0.5; \ \varphi(t) = 0.5^{0.5}, \ t > 0.5. \) (c) \( \varphi(t) = \log(t + 1), \ 0 \leq t \leq 1; \ \varphi(t) = \log(t^{0.5} + 1), \ 1 < t \leq 2; \ \varphi(t) = \log(2^{0.5} + 1), \ t > 2. \)**
Let $I_0^* := \Omega_0^* \cap I^*$ and $B_0^* := \Omega_0^* \cap B^*$. If $i \in I_0^*$, then $t_i^{x,*} = t_i^{y,*} = 0$ by Remark 1(f); if $i \in B_0^*$, then $t_i^{x,*} = (d_i^0)_{I^*}u_i^* + (d_i^0)_Bu_i^* = (d_i^0)u_i^* = 0$ and $t_i^{y,*} = (d_i^0)_{I^*}u_i^* + (d_i^0)_Bu_i^* = (d_i^0)u_i^* = 0$ by Remark 1(e). Thus,

$$
(3)
$$

Due to $(d_i)_{I^*} - 0$ for $\forall i \in B_0^*$ by Remark 1(e), it holds that $(d_i)_{I^*} - 0 = 0$ for all $i \in \Omega_0^*$ when $z \in \{ z \in \mathbb{R}^{d^0} : (d_i)_{I^*} - 0, \forall i \in I_0^* \}$. Then, we construct a new minimization problem:

$$
\min_{z \in \mathbb{R}^{d^0}} \quad R(z) := \sum_{i \in \Omega_0^*} \phi((d_i^0)_{I^*}z + t_i^{x,*})^2 + \frac{\gamma}{2} \| A_{I^*}z - f^* \|^2 \text{s.t.} \quad (d_i)_{I^*} - 0, \forall i \in I_0^*.
$$

(4)

We can verify that $z^* = u_i^*$. is a local minimizer of the problem (R) by contradiction as [40].

Based on the problem (R), we give an important result about $\|d_i u^*\|$. Although the proof is motivated by [31], no constraint was considered in it.

**Theorem 2.2.** Given $u^*$ as a local minimizer of $F(u)$ in (1). If $\|d_i u^*\| > 0$, then $\|d_i u^*\| \notin M$.

For readability, we put the detailed proof of Theorem 2.2 in Appendix 6.2.

In our model (1), $\phi(t)$ can be either a Lipschitz function or a non-Lipschitz function. However, non-Lipschitz $\phi(t)$ is more helpful to keep image edges compared to Lipschitz $\phi(t)$. In the following, we focus on the non-Lipschitz $\phi(t)$ satisfying $\phi'(t)|_{0+} = +\infty$. As discussed in [32, 10], it is usually difficult to design algorithms for non-Lipschitz models. Thus, we give a theorem useful to the design of our algorithm.

**Theorem 2.3.** If a local minimizer $u^*$ is sufficiently near to a given point $\hat{u}$, then $\Omega_1(u^*) \subseteq \Omega_1(\hat{u})$.

**Proof.** Theorem 2.2 shows $\|d_i u^*\| \notin M$, which implies $R(z)$ being $C^1$ on a neighborhood of $z^*$. Then, the proof procedure is similar to the proof of Theorem 3.6 in [40].

Theorem 2.3 implies that if a local minimizer $u^*$ is sufficiently near to a given point $\hat{u}$, then the support would not expand from $\hat{u}$ to $u^*$. This observation is important and inspires us to design the algorithm.

3. **Numerical algorithm and its properties.**

3.1. **The algorithm.** Motivated by Theorem 2.3, we propose an iterative support shrinking strategy to solve (1) as [46, 40] for other optimization models. Based on an approximate solution $u^k$ in the $k$th iteration, $u^{k+1}$ is computed from the objective function with a constraint on the gradient support of the optimization variable as a subset of the gradient support of $u^k$

$$
(F_k)
$$

$$
\begin{cases}
\min_{u \in \mathbb{R}^N} \quad F_k(u) := \sum_{i \in \Omega_k^*} \phi(\|d_i u\|) + \frac{\gamma}{2} \| Au - g \|^2,
\quad \\
\text{s.t.} \quad d_i u = 0, \quad \forall i \in \Omega_k^*,
\quad l_1 e \leq u \leq l_2 e,
\end{cases}
$$

Inverse Problems and Imaging Volume 14, No. 5 (2020), 867–890
where $\Omega_0^k = \Omega_0(u^k)$ and $\Omega_1^k = \Omega_1(u^k)$.

To handle concave function $\varphi(\|d_i u\|)$ in $(\mathcal{F}_k)$, a practical approach is to construct a surrogate function by its first order Taylor approximation. Indeed, existing references like [46, 50] applied this strategy to linearize concave penalties differentiable over $(0, +\infty)$. However, here, our truncated regularization function has multiple non-differentiable points like those in $M$. A good thing is that we can adopt a general linearization method at non-differentiable points belonging to the set $M$; see the weight $\hat{w}_i^{k+1}$ in (5). This guarantees that the approximating linear function is above the original truncated regularization function, and serves as a surrogate function. Moreover, together with a proximal idea, we obtain the following iterative support shrinking algorithm with proximal general linearization (ISSAPGL).

**ISSAPGL: iterative support shrinking algorithm with proximal general linearization**

1. Input $\alpha, A, g, \rho > 0$. Initialize $u^0$ and $w^0 = 1$.
2. For each $k = 0, 1, \ldots$,
   (a) compute $u^{k+1}$ by solving
   \[
   \begin{aligned}
   \min_{u \in \mathbb{R}^N} & \quad H_k(u) := \sum_{i \in \Omega_1^k} w_i^k \|d_i u\| + \frac{\alpha}{2}\|Au - g\|^2 + \frac{\rho}{2}\|u - u^k\|^2, \\
   \text{s.t.} & \quad d_i u = 0, \quad \forall i \in \Omega_0^k, \\
   & \quad l_1 e \leq u \leq l_2 e,
   \end{aligned}
   \]
   where $\Omega_0^k = \Omega_0(u^k)$ and $\Omega_1^k = \Omega_1(u^k)$.
   (b) update $w_i^{k+1}$ as
   \[
   w_i^{k+1} = \begin{cases} 
   \varphi'(\|d_i u^{k+1}\|) & \text{for } i \in \Omega_1^{k+1} \setminus \Omega_1^{k+1,M}, \\
   \hat{w}_i^{k+1} & \text{for } i \in \Omega_1^{k+1,M},
   \end{cases}
   \]
   where $\Omega_1^{k+1,M} = \{i \in \Omega_1^{k+1} : \|d_i u^{k+1}\| = t, \text{ for } t \in M\}$ and $\hat{w}_i^{k+1} \in [\varphi'(\|d_i u^{k+1}\|^{-}), \varphi'(\|d_i u^{k+1}\|^{+})]$.

3.2. Some properties of ISSAPGL. In this subsection, some properties of the proposed ISSAPGL are given. We first verify that the iterative sequence $\{F(u^k)\}$ satisfies a sufficient decrease property, which shows the convergence of the objective function sequence. Then, we establish a lower bound theory for $\|d_i u^k\|$.

**Proposition 1.** *(Sufficient decrease property)*  The sequence $\{F(u^k)\}$ is a nonincreasing sequence and satisfies

\[
\frac{\rho}{2}\|u^{k+1} - u^k\|^2 \leq F(u^k) - F(u^{k+1}), \quad \forall k \geq 0.
\]

**Proof.** According to Assumption 1(a), we have

\[
\varphi(\|d_i u\|) \leq \varphi(\|d_i u^k\|) + w_i^k(\|d_i u\| - \|d_i u^k\|), \quad \forall i \in \Omega_1^k,
\]
where \( w^k_i \) is defined in ISSAPGL. Then,

\[
F_k(u) + \frac{\rho}{2} \| u - u^k \|^2 = \sum_{i = 1}^{\Omega_0^K} \varphi(\|d_i u^k\|) + \frac{\alpha}{2} \| Au - g \|^2 + \frac{\rho}{2} \| u - u^k \|^2 \leq \sum_{i = 1}^{\Omega_0^K} \varphi(\|d_i u^k\|) + \frac{\alpha}{2} \| Au - g \|^2 + \frac{\rho}{2} \| u - u^k \|^2.
\]

(7)

\[
= H_k(u) + \sum_{i \in \Omega_0^K} \left( \varphi(\|d_i u^k\|) - w^k_i \|d_i u^k\|) \right).
\]

Since \( F(u^{k+1}) = F_k(u^{k+1}) \) from \( \varphi(0) = 0 \), one can obtain

\[
F(u^{k+1}) + \frac{\rho}{2} \| u^{k+1} - u^k \|^2 = F_k(u^{k+1}) + \frac{\rho}{2} \| u^{k+1} - u^k \|^2
\]

[ by (7) ] \[ \leq H_k(u^{k+1}) + \sum_{i \in \Omega_0^K} \left( \varphi(\|d_i u^k\|) - w^k_i \|d_i u^k\|) \right)
\]

\[
\leq H_k(u^k) + \sum_{i \in \Omega_0^K} \left( \varphi(\|d_i u^k\|) - w^k_i \|d_i u^k\|) \right)
\]

\[
= \sum_{i \in \Omega_0^K} w^k_i \|d_i u^k\| + \frac{\alpha}{2} \| Au^k - g \|^2 + \sum_{i \in \Omega_0^K} \left( \varphi(\|d_i u^k\|) - w^k_i \|d_i u^k\|) \right)
\]

[ by \( d_i u^k = 0, \forall i \in \Omega_0^K \) ] \[ = \frac{\alpha}{2} \| Au^k - g \|^2 + \sum_{i \in \Omega_0^K} \varphi(\|d_i u^k\|)
\]

\[
= F(u^k),
\]

which proves (6).

\[ \square \]

**Corollary 1.** The sequence \( \{F(u^k)\} \) converges. Moreover, the sequence \( \{u^k\} \) satisfies

\[ \lim_{k \to \infty} \| u^{k+1} - u^k \| = 0. \]

**Proof.** It is straightforward from Proposition 1 that \( \{F(u^k)\} \) is not only nonincreasing but also bounded. Thus, it converges as \( k \to \infty. \) Accordingly, \( \lim_{k \to \infty} \| u^{k+1} - u^k \| = 0 \) from (6).

\[ \square \]

Now, we establish a lower bound theory for \( \|d_i u^k\| \), which shows that ISSAPGL has the ability of keeping image edges.

From the monotonicity and boundedness of \( \{\Omega_0^k\}_{k=0}^\infty \) in ISSAPGL, this sequence converges in a finite number of steps as in [46, 40], i.e., \( \exists K, \) such that \( \Omega_0^K = \Omega_0^k, \forall k \geq K \).

Thus, the set \( \Omega_0^K \) in ISSAPGL is unchanged when \( k \) is sufficiently large. Accordingly, when \( k \geq K, \) \( (H_k) \) can be represented as

\[
(\mathcal{H}_K) \left\{ \min_{u \in \mathbb{R}^N} \mathcal{H}_K(u) := \sum_{i \in \Omega_0^K} \frac{w^k_i}{\Omega_0^K} \|d_i u\| + \frac{\alpha}{2} \| Au - g \|^2 + \frac{\rho}{2} \| u - u^k \|^2 + \chi_{[\ell_1, \ell_2]}(u), \right.
\]

\[
\text{s.t. } d_i u = 0, \quad \forall i \in \Omega_0^K.
\]

Since \( d_i u = 0 \) is just a linear constraint for \( \forall i \in \Omega_0^K \) in \( (\mathcal{H}_K) \), we can eliminate these constraints of \( (\mathcal{H}_K) \) by constructing \( E_{\omega} \) and \( u_{\omega} \) such that \( u = E_{\omega} u_{\omega} \) as [46]. Thus,
(\overline{\Omega}_k) becomes
\begin{equation}
\min_{u_\omega \in \mathbb{R}^{N_\omega}} \overline{\Omega}_k^{\omega}(u_\omega) := \sum_{i \in \Omega_k^{\omega}} w^k_i d_i(E_\omega u_\omega) + \frac{\alpha}{2} \|A(E_\omega u_\omega) - g\|^2 \\
+ \frac{\rho}{2} \|(E_\omega u_\omega) - u^k\|^2 + \lambda \chi_{[1,t_2]}(E_\omega u_\omega).
\end{equation}

Once \(u^{k+1}\) solves (\(\overline{\Omega}_k\)), then \(u^{k+1}_\omega\) solves (\(\overline{\Omega}_k^{\omega}\)).

For \(u^{k+1}_\omega\), we introduce some index sets, which will be used in the following proof.

**Definition 3.1.** For any \(u^{k+1}_\omega\), we define the following index sets
\begin{enumerate}[(a)]
\item The index set of \(u^{k+1}_\omega\) is denoted as \(J_k^{\omega}\).
\item \(I_k^{\omega}(u^{k+1}_\omega) = \{i \in J_k^{\omega} : l_1 < (u^{k+1}_\omega)_i < l_2\}\).
\item \(B_k^{\omega}(u^{k+1}_\omega) = J_k^{\omega} \setminus I_k^{\omega}(u^{k+1}_\omega)\).
\end{enumerate}

**Theorem 3.2.** There exist an integer \(\tilde{K} \geq K\) and a constant \(\eta > 0\) such that
\[
\|d_i u^{k+1}\| > \eta, \quad \forall k \geq \tilde{K}, \forall i \in \Omega^{\tilde{K}}.
\]

**Proof.** To overcome the difficulties from the last term in (\(\overline{\Omega}_k^{\omega}\)) (box constraints), we first construct a new optimization problem as in [40].

For \(u^{k+1}\) being a solution of (\(\overline{\Omega}_k\)), index sets in Definitions 2.1, 3.1 and 6.1 are simplified as \(I^{k+1} = I(u^{k+1}), B^{k+1} = B(u^{k+1}), \Omega^{k+1} = \Omega, L^{k+1} = L_1(u^{k+1}), B_1^{k+1} = B_1(u^{k+1}), t^0_1 = t_0(u^{k+1}), B_0^{k+1} = B_0(u^{k+1}), J^{k+1} = J_\omega(u^{k+1}), I^{k+1}_\omega = I_\omega(u^{k+1}), B^{k+1}_\omega = B_\omega(u^{k+1}).\) Let \(A_{I^{k+1}} = (a_i)_{i \in I^{k+1}}\) and \(A_{B^{k+1}} = (a_i)_{i \in B^{k+1}}\) are two submatrices of \(A\). Accordingly, we represent \(u^{k+1}\) as \((u_I^{k+1}, u_{B^{k+1}})\), and \(u^{k+1}_\omega\) as ((\(u_I^{k+1}\)_\omega, \(u_{B^{k+1}}\)_\omega)). Because each row of \(E_\omega\) only has one nonzero element 1 and \(I^{k+1} \cap B^{k+1} = \emptyset\), we can take two submatrices \((E_\omega)_{I^{k+1}} \in \mathbb{R}^{[I^{k+1} \times N_\omega]}\) and \((E_\omega)_{B^{k+1}} \in \mathbb{R}^{[B^{k+1} \times N_\omega]}\) from \(E_\omega\) such that \(u_I^{k+1} = ((E_\omega)_{I^{k+1}}(u^{k+1}_I))_{I^{k+1}}\) and \(u_{B^{k+1}} = ((E_\omega)_{B^{k+1}}(u^{k+1}_B))_{B^{k+1}}\).

Define the following problem:
\begin{equation}
\min_{r_\omega \in \mathbb{R}^{[I^{k+1} \times 1]}} \overline{\mathcal{R}}_k^{\omega}(r_\omega) := \sum_{i \in \Omega^{k+1}} w_i^{k+1} h_i(r_\omega) + \frac{\alpha}{2} \|A_{I^{k+1}}((E_\omega)_{I^{k+1}} r_\omega) - f^{k+1}\|^2 \\
+ \frac{\rho}{2} \|((E_\omega)_{I^{k+1}} r_I) - u^{k+1}\|^2 \\
+ \lambda \chi_{[1,t_2]}(((E_\omega)_{I^{k+1}} r_\omega; u^{k+1}_{B^{k+1}}))
\end{equation}

s.t. \(r_\omega \in \mathcal{K}(I^{k+1}_0)\),

where
\[
h_i(r_\omega) = \sqrt{((d_i^{k+1})_{I^{k+1}}((E_\omega)_{I^{k+1}} r_I) + t_i^{k+1}u^{k+1}_{I^{k+1}})^2 + ((d_i^{k+1})_{B^{k+1}}((E_\omega)_{B^{k+1}} r_B) + t_i^{k+1}u^{k+1}_{B^{k+1}})^2},
\]
\[
f^{k+1} = g - A_{B^{k+1}} u^{k+1}_{B^{k+1}}, \quad t_i^{k+1} = (d_i^{k+1})_{B^{k+1}} u^{k+1}_{B^{k+1}}, \quad t_i^{k+1} = (d_i^{k+1})_{B^{k+1}} u^{k+1}_{B^{k+1}}, \quad \forall i \in J,
\]
and \(\mathcal{K}(I^{k+1}_0) := \{r_\omega \in \mathbb{R}^{[I^{k+1}_0 \times 1]} : (d_i)_{I^{k+1}}(((E_\omega)_{I^{k+1}} r_I) = 0, \forall i \in I^{k+1}_0\}.\)

By a similar argument for (3), we have \(t_i^{k+1} = 0, \forall i \in \Omega^{K},\) and we can also verify \((u^{k+1}_I)_{I^{k+1}}\) being a local minimizer of (9) as the discussion for (4).
Compared to $\nabla \tilde{R}_k^\omega$ in (8), the first-order necessary condition of $\tilde{R}_k^\omega$ is easier to be obtained, because the indicator function in $\tilde{R}_k^\omega(r_\omega)$ is defined on an open set. Choosing $r_\omega \in K(L_0^{k+1})$, we use the first-order necessary condition at $(u_\omega^{k+1})_{l_k+1}$ for $\tilde{R}_k^\omega(r_\omega)$ as

$$0 = \left\langle \nabla \tilde{R}_k^\omega((u_\omega^{k+1})_{l_k+1}), r_\omega \right\rangle$$

$$= \sum_{i \in \Omega_k^\omega} w_i^k \left\langle \frac{d_i u^{k+1}}{\|d_i u^{k+1}\|}, (d_i)_{l_k+1}, ((E_\omega)_{l_k+1}, r_\omega) \right\rangle$$

$$+ \alpha \left\langle A u^{k+1} - g, A_{l_k+1}, (E_\omega)_{l_k+1}, r_\omega \right\rangle + \rho \left\langle u^{k+1} - u^k, (E_\omega)_{l_k+1}, r_\omega \right\rangle.$$

Let $\bar{z} = (E_\omega)_{l_k+1}, r_\omega$. The equation above is reformulated as

$$\sum_{i \in \Omega_k^\omega} w_i^k \left\langle \frac{d_i u^{k+1}}{\|d_i u^{k+1}\|}, (d_i)_{l_k+1}, \bar{z} \right\rangle$$

$$= -\alpha \left\langle A u^{k+1} - g, A_{l_k+1}, \bar{z} \right\rangle - \rho \left\langle u^{k+1} - u^k, \bar{z} \right\rangle$$

$$\leq \alpha (\|A\| \|u^{k+1}\| + \|g\|) \|A_{l_k+1}\| \|\bar{z}\| + \rho \|u^{k+1} - u^k\| \|\bar{z}\|$$

$$\leq \alpha (\|A\| \|u^{k+1}\| + \|g\|) \|A\| \|\bar{z}\| + \rho \|u^{k+1} - u^k\| \|\bar{z}\|.$$

Since $\{u^k\}$ is bounded, there exists $\zeta > 0$, which is independent of $k$ such that (10)

$$\zeta \|\bar{z}\| \geq \sum_{i \in \Omega_k^\omega} w_i^k \left\langle \frac{d_i u^{k+1}}{\|d_i u^{k+1}\|}, (d_i)_{l_k+1}, \bar{z} \right\rangle = \frac{1}{b} \sum_{i \in L_{l_k+1}^1} w_i^k \left\langle \frac{d_i u^{k+1}}{\|d_i u^{k+1}\|}, (d_i)_{l_k+1}, \bar{z} \right\rangle.$$

Based on (10), we prove the conclusion by induction.

For all $i \in \Omega_1^K$, we sort $\|d_i u^{k+1}\|$ as $\|d_{i_k} u^{k+1}\| \geq \|d_{i_{k-1}} u^{k+1}\| \geq \cdots \geq \|d_{i_1} u^{k+1}\| > 0$ with $\hat{r} = \#(\Omega_1^K) \leq N$. We define $L_1^{k+1}$ as a set consisting leaping pixels [47, 40] with respect to $u^{k+1}$. First, we establish a lower bound for $\|d_{i_k} u^{k+1}\|$. We tell $\theta_1$. The equation above is reformulated as

$$\sum_{i \in \Omega_k^\omega} w_i^k \left\langle \frac{d_i u^{k+1}}{\|d_i u^{k+1}\|}, (d_i)_{l_k+1}, \bar{z} \right\rangle$$

$$= \frac{1}{b} \sum_{i \in L_{l_k+1}^1} w_i^k \left\langle \frac{d_i u^{k+1}}{\|d_i u^{k+1}\|}, (d_i)_{l_k+1}, \bar{z} \right\rangle.$$ 

with $\|\bar{z}\| \leq \frac{N(N+1)(2N+1)}{6}$ and $\|\langle d_i \rangle\| \|\bar{z}\| \leq 8N^2$, then $\bar{z} = \bar{z}$ in (10), we obtain

$$\zeta \sqrt{\Gamma} > \zeta \|\bar{z}\| \geq \sum_{i \in L_{l_k+1}^1} w_i^k \left\langle \frac{d_i u^{k+1}}{\|d_i u^{k+1}\|}, (d_i)_{l_k+1}, \bar{z} \right\rangle \geq w_{i_k} \frac{\|d_{i_k} u^{k+1}\|}{\|d_{i_k} u^{k+1}\|} = w_{i_k}.$$ 

Only one of the following two cases holds:

Case3a: if $i_k \in \Omega_1^K$, then $\|d_{i_k} u^{k+1}\| \leq M$. Define $0 < \eta_1 < \frac{1}{2} \min \{t : t \in M\}.$ Accordingly, $\|d_{i_k} u^{k+1}\| > 2 \eta_1$. Based on Corollary 1, $\|d_{i_k} u^{k+1} - d_{i_k} u^{k+1}\| \leq \|d_{i_k} ||u^k - u^{k+1}\| \rightarrow 0$, which implies that there exists $K \geq K$, such that $\|d_{i_k} u^{k+1}\| > \eta_1$ for $\forall k > K$. 

Case3b: if $i_k \notin \Omega_1^K$, then $d_{i_k} u^{k+1} = \varphi(||d_{i_k} u^k||) < \zeta \sqrt{\Gamma}$, and we define

$$\eta_1 = \frac{1}{2} \inf \left\{ t > 0 : \varphi(t) = \zeta \sqrt{\Gamma} \right\} > 0,$$ 

$$\tilde{R}_k^\omega$$ 

$$\sum_{i \in \Omega_k^\omega} w_i^k \left\langle \frac{d_i u^{k+1}}{\|d_i u^{k+1}\|}, (d_i)_{l_k+1}, \bar{z} \right\rangle$$ 

$$= \frac{1}{b} \sum_{i \in L_{l_k+1}^1} w_i^k \left\langle \frac{d_i u^{k+1}}{\|d_i u^{k+1}\|}, (d_i)_{l_k+1}, \bar{z} \right\rangle.$$ 

$$\zeta \sqrt{\Gamma} > \zeta \|\bar{z}\| \geq \sum_{i \in L_{l_k+1}^1} w_i^k \left\langle \frac{d_i u^{k+1}}{\|d_i u^{k+1}\|}, (d_i)_{l_k+1}, \bar{z} \right\rangle \geq w_{i_k} \frac{\|d_{i_k} u^{k+1}\|}{\|d_{i_k} u^{k+1}\|} = w_{i_k}.$$
which is independent of $k$. Since $\varphi'$ is decreasing from Assumption 1(a), $\|d_i u^k\| > 2\eta_1$. Due to $\|d_i u^k - d_i u^{k+1}\| \leq \|d_i\||u^k - u^{k+1}\| \to 0$, there exists $K_1 > K$, such that $\|d_i u^{k+1}\| > \eta_1$ for $\forall k > K_1$.

Assume that there exist $K_{s-1}$ and $\eta_{s-1}$, such that $\|d_i u^{k+1}\| > \eta_{s-1}$ and $\|d_i u^k\| > \eta_{s-1}$, for any $i \in \{i_1, i_2, \cdots, i_{s-1}\}$ with $1 < i_s < i_r$. Note that, the constant $\eta_{s-1}$ is independent of $k$. Next, our task is to find a lower bound for $\|d_i u^{k+1}\|$.

Case 1: if $\|d_i u^{k+1}\| \in M$, then we define $0 < \eta_s < \min\{t : t \in M\}$.

Case 2: if $\|d_i u^{k+1}\| \notin M$ and $\|d_i u^{k+1}\| > \frac{2 - \lambda}{N}$, then we define $\eta_s = \frac{2 - \lambda}{N}$.

Case 3: if $\|d_i u^{k+1}\| \notin M$ and $\|d_i u^{k+1}\| \leq \frac{2 - \lambda}{N}$, then we can similarly select a $\pi^*$ satisfying

$$ (d_i)_{k+1}^* = \frac{1}{\|d_i u^{k+1}\|} d_i u^{k+1}, \forall i \in (\Omega_{s-1}^k \cup \Omega_i^K) \setminus E_{s-1}^{k+1}, $$

where $E_{s-1}^{k+1} = \{i_1, i_2, \cdots, i_{s-1}\}$. Define $E_{s-1}^{k+1} = E_{s-1}^{k+1} \cap L_{s-1}^{k+1}$ and let $\pi = \pi^*$ in (10), we have

$$ \sqrt{\lambda} > \|\pi^*\| \geq \sum_{i \in E_{s-1}^{k+1}} w_i \|d_i u^{k+1}\| - \sum_{i \in E_{s-1}^{k+1}} w_i \langle d_i u^{k+1}, (d_i)_{k+1}^* \rangle $$

$$ \geq w_{i_s} \|d_i u^{k+1}\| - \sum_{i \in E_{s-1}^{k+1}} w_i \|d_i u^{k+1}\| \|d_i u^{k+1}\| \|d_i u^{k+1}\| $$

$$ > w_{i_s}^k - 2\sqrt{2N} \sum_{i \in E_{s-1}^{k+1}} w_i^k $$

Thus,

$$ w_{i_s}^k < \sqrt{\lambda} + 2\sqrt{2N} \sum_{i \in E_{s-1}^{k+1}} \varphi'\|d_i u^k\| < \sqrt{\lambda} + 2\sqrt{2N} \varphi'(\eta_{s-1}). $$

Similarly, only one of the following two cases holds:

Case 3a: if $i_s \in \Omega_{s-1}^k \cap M$, then $|d_i u^k| \in M$. Let $0 < \eta_s < \frac{1}{2} \min\{t : t \in M\}$. Since $\|d_i u^k - d_i u^{k+1}\| \leq \|d_i\||u^k - u^{k+1}\| \to 0$, there exists $K_s > K_{s-1}$, such that $\|d_i u^{k+1}\| > \eta_s$ for $\forall k > K_s$.

Case 3b: if $i_s \not\in \Omega_{s-1}^k$, then $w_{i_s}^k = \varphi'\|d_i u^k\| < \sqrt{\lambda} + 2\sqrt{2N} \varphi'(\eta_{s-1})$. We define

$$ \eta_s = \frac{1}{2} \inf\{t > 0 : \varphi'(t) = \sqrt{\lambda} + 2\sqrt{2N} \varphi'(\eta_{s-1})\} > 0, $$

which is independent of $k$. Thus, there exists $K_s > K_{s-1}$, such that $\|d_i u^{k+1}\| > \eta_s$ for $\forall k > K_s$.

From Case 3a and Case 3b, by choosing $\eta_s = \min\{\eta_s, \eta_{s-1}\}$, we obtain $\|d_i u^{k+1}\| > \eta_s$.

According to the inductive above, there exist $\tilde{K} > K_r$ and $\eta = \eta_{\tilde{r}}$, such that $\|d_i u^{k+1}\| > \eta$, for $\forall k \geq K_r, \forall i \in \Omega_{\tilde{r}}^K$. This completes the proof. \hfill \Box

3.3. The implementation details of ISSAPGL. Since $(H_k)$ in ISSAPGL is a strongly convex optimization problem, it has a unique optimal solution. We use the idea of ADMM [43, 7] to solve it. At first, we introduce an auxiliary variable $v$, where

$$ v_i = d_i u, \text{ for } i \in \Omega_i^k. $$
Consequently, \((\mathcal{H}_k)\) is equivalent to
\[
\min_{u,v} \sum_{i \in \Omega_k^1} w_i^k \|v_i\| + \alpha \|Au - g\|^2 + \rho \|u - u^k\|^2 + \mathcal{X}_{[t_1,t_2]}(u),
\]  
subject to
\[
d_i u = 0, \quad \forall i \in \Omega_0^k,
\]
\[
v_i = d_i u, \quad \forall i \in \Omega_k^1,
\]
with the indicator function \(\mathcal{X}_{[t_1,t_2]}(u)\) given by
\[
\mathcal{X}_{[t_1,t_2]}(u) = \begin{cases} 0 & 0 \leq u \leq t_2 e, \\ \infty & \text{otherwise.} \end{cases}
\]

The augmented Lagrangian functional for the optimization problem (13) is
\[
\mathcal{L}(u,v; \lambda) = \sum_{i \in \Omega_k^1} w_i^k \|v_i\| + \sum_{i \in \Omega_k^1} \langle \lambda_i, d_i u - v_i \rangle + \frac{\rho}{2} \sum_{i \in \Omega_k^1} \|d_i u - v_i\|^2 + \sum_{i \in \Omega_k^1} \langle \lambda_i, d_i u \rangle + \frac{\rho}{2} \|u - u^k\|^2 + \mathcal{X}_{[t_1,t_2]}(u),
\]
where \(\lambda\) is a Lagrange multiplier and \(r\) is a positive constant. The corresponding ADMM procedure for solving (13) is shown in the following algorithm.

**ADMM:** the alternating direction method of multipliers for solving (13) \((\mathcal{H}_k\) in (2a))

1. Input \(u^k, w^k, \Omega_k^1, \Omega_k^2, \alpha, \lambda, r\).
   Initialize \(u_0^k = 0, v_0^k = 0, \lambda_0^k = 0\).
2. For \(k = 0, 1, 2, \cdots\), compute
   \[
   u^{k+1} = \arg \min_u \mathcal{L}(u,v^k,\lambda^k); \quad v^{k+1} = \arg \min_v \mathcal{L}(u^{k+1},v^k,\lambda^k); \quad \lambda_i^{k+1} = \lambda_i^k + r d_i u^{k+1} - v_i^{k+1},
   \]
3. If a termination criterion is not met, then go to Step 1. Otherwise, output \(u^{k+1} = u^{k+1}\).

We now solve the two subproblems in the above algorithm.

1. the \(u\)-subproblem (15): Let \(v^{k,k} = (v^{k,k}, v^{k,k})\), where \(v^{k,k}_j = 0, \forall j \in \Omega_0^k\). We can obtain a good structure of (15) given by
   \[
u^{k,k+1} = \arg \min_u \left\{ \sum_{i \in J} \langle \lambda_i^{k,k}, d_i u \rangle + \frac{\rho}{2} \sum_{i \in J} \|d_i u\|^2 - r \sum_{i \in J} \langle d_i u, v_i^{k,k} \rangle \right. \]
   \[+ \frac{\alpha}{2} \|Au - g\|^2 + \frac{\rho}{2} \|u - u^k\|^2 + \mathcal{X}_{[t_1,t_2]}(u) \}.
   \]
   To solve these types of the constrained quadratic programming problems, we apply the nonmonotone projected gradient method proposed in [4, 51].
2. the \(v\)-subproblem (16): (16) can be simplified as
   \[
v^{k,k+1} = \arg \min_v \left\{ \sum_{i \in \Omega_k^1} w_i^k \|v_i\| + \frac{\rho}{2} \sum_{i \in \Omega_k^1} \|v_i - (d_i u^{k,k+1} + \frac{\lambda_i^{k,k}}{r})\|^2 \right\}.
   \]
It is clear that the optimal value of $v_i^{k,k+1}$ uses shrinkage operators as follows

$$(17) \quad v_i^{k,k+1} = \text{shrink}(d_i u_i^{k,k+1} + \frac{\lambda_i^{k,k+1}}{r} w_i^{k,k+1}),$$

where

$$\text{shrink}(a, b) = \frac{a}{\|a\|} \times \max(\|a\| - b, 0).$$

4. **Experimental results.** In this section, we show the performance of our model in CT reconstruction compared with two existing popular methods, TV based model [22] and the tight frame wavelet based model [51]. We respectively denote them as TV and TW-$\ell_0$. We implemented TV model by ADMM, while downloaded the code of TW-$\ell_0$ from the authors’ homepage. In all numerical examples, the quality of image recovery is measured by peak signal to noise ratio (PSNR) and structural similarity index (SSIM) [41].

All experiments are conducted in MATLAB R2016a on a desktop with Intel Core i5 CPU at 3.3 GHz and 8 GB of memory.

We use the well-known “Shepp-Logan” phantom and NURBS-based cardiac-torso (NCAT) phantom [37] as our test images. They are shown in Figure 2, where the range of image values is $[0, 1]$. Throughout our experiments, the projection matrix $A$ is based on parallel-beam scanning with different numbers of angles and $\varepsilon$ is generated from a zero mean Gaussian distribution with three noise levels: $\sigma = 0.005\|g\|_\infty$, $0.01\|g\|_\infty$ and $0.02\|g\|_\infty$, respectively.

In (1), we use the Lipschitz case $\varphi(t) = \begin{cases} \ln(at + 1), & t < \tau, \\ \ln(at\tau + 1), & \text{otherwise,} \end{cases}$ with $a > 0$ and the non-Lipschitz case $\varphi(t) = \begin{cases} t^p, & t < \tau, \\ \tau^p, & \text{otherwise,} \end{cases}$ with $p \in (0, 1)$. Our model with the two potential functions are respectively denoted as trunc-LN and trunc-$\ell_p$.

In the proposed algorithm, we set $\rho = 0.1$ and $r = 40$, and the stop condition is $\|u_i^{k+1} - u_i^k\|_\infty < 10^{-3}$. In terms of $a$, $p$ and $\tau$ in our model, many experiments for the “Shepp-Logan” image with 36 projections corrupted by $0.01\|g\|_\infty$ Gaussian noise show that the reasonable range of $a$ is $[3, 6]$, of $p$ is $[0.3, 0.7]$ and of $\tau$ is $[0.4, 0.7]$; see Figure 3. For the sake of convenience, we set $a = 5$, $p = 0.5$ and $\tau = 0.5$ in our all experiments. Thus, we only need to tune the model parameter $a$ in our method, which is dependent on the noise level.
In the following, we conduct different experiments with various projection numbers: \( N_p = 18, 36 \) and 72 under different noise levels: 0.005\( \|g\|_\infty \), 0.01\( \|g\|_\infty \) and 0.02\( \|g\|_\infty \). For a fair comparison, the parameters of the compared methods are also tuned to achieve the best PSNR and SSIM. In Figures 4 and 5, we exhibit the reconstructions under three projection numbers with noise level \( \sigma = 0.01 \|g\|_\infty \). In terms of visual inspection, one can see that our proposed model can recover piecewise constant images better than the other two models. In contrast, the TV-based model produces false edges in homogeneous parts. Although the non-convex TW-\( \ell_0 \) and our method have a stronger tendency to connect the three different ellipses (located at the bottom of the Logan phantom) compared to TV when \( N_p = 18 \) and 36, our method with trunc-\( \ell_p \) has the similar result with TV in recovering them when \( N_p = 72 \). Moreover, from zoom-in views of Figure 6, the edges reconstructed by TV are not preserved well and seem to be unclear. Although the TW-\( \ell_0 \) model can reconstruct these neat edges and behaves well in homogeneous regions, it yields some abnormal points on edges; see the results in Figure 5. This phenomenon also can be observed in Figure 6. In contrast, our model can preserve this edge structure very well, and obtain the best results qualitatively and quantitatively. More comparisons in terms of PSNR and SSIM with different numbers of projection angles and noise levels are shown in Tables 1 and 2. It is clear that the proposed method outperforms the TV-based and tight wavelet frame-based models.

To further visualize the difference between these methods, Figure 7 shows the residual errors of the reconstruction results in Figures 4 and 5, i.e., \(|u - u_0|\) with \( u \) being the reconstruction image and \( u_0 \) being the ground truth. We observe that our model with the proposed algorithm obtains the smallest errors among compared methods. Moreover, Figure 8 lists the 60th row and 80th row of the results restored by different methods for “Shepp-Logan”. One can also see that our non-Lipschitz trunc-\( \ell_p \) model can match the ground truth very well. Figure 9 shows the values \( F(u_k) \) of trunc-LN and trunc-\( \ell_p \) versus the iteration number for “Shepp-Logan” image with 36 projections. We see that when the iteration number increases and is larger than 10, \( F(u_k) \) is decreasing and converges. This verifies the theoretical results in Proposition (1).

Figure 10 shows the reconstructions for a real data of the cheat (Case courtesy of Dr Andrew Dixon, Radiopaedia.org, rID: 36676). Note that it is noiseless testing.
Figure 4. CT reconstruction comparisons for “Shepp-Logan” image with \( \sigma = 0.01 \| g \|_{\infty} \). The first, second and third rows are the reconstructed results when \( N = 18, 36 \) and 72, respectively. The PSNR and SSIM values are attached in the brackets.

| \( N_p \) | noise level \( \| g \|_{\infty} \) | TV PSNR/SSIM | TW-\( \ell_0 \) PSNR/SSIM | trunc-LN PSNR/SSIM | trunc-\( \ell_p \) PSNR/SSIM |
|---|---|---|---|---|---|
| 18 | 0.005 & 36.28/0.9688 & 39.31/0.9729 & 43.36/0.9902 & 43.66/0.9924 |
| | 0.01 & 31.88/0.9159 & 34.67/0.9729 & 39.19/0.9800 & 38.24/0.9794 |
| | 0.02 & 27.52/0.8389 & 29.54/0.9460 & 32.08/0.9507 & 31.53/0.9561 |
| 36 | 0.005 & 39.49/0.9860 & 44.52/0.9941 & 46.99/0.9959 & 49.52/0.9991 |
| | 0.01 & 34.45/0.9564 & 39.84/0.9860 & 42.80/0.9897 & 42.87/0.9910 |
| | 0.02 & 29.71/0.8972 & 33.56/0.9695 & 38.09/0.9776 & 37.43/0.9769 |
| 72 | 0.005 & 43.27/0.9916 & 49.93/0.9984 & 50.20/0.9980 & 52.92/0.9989 |
| | 0.01 & 37.80/0.9779 & 43.95/0.9942 & 45.61/0.9954 & 46.41/0.9954 |
| | 0.02 & 32.54/0.9393 & 37.36/0.9758 & 41.07/0.9862 & 40.58/0.9868 |

Table 1. More quantitative comparisons of reconstruction on “Shepp-Logan” in terms of PSNR and SSIM.
Table 2. More quantitative comparisons of reconstruction on “NCAT” in terms of PSNR and SSIM.
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Figure 6. The first row shows the ground truth. The second and third rows show the zoom-in views of the “Shepp-Logan” and “NCAT” image reconstructions with 36 projections.

with projection number $N = 60$. In order to compare the difference, we enlarge two regions of interest (the red and green squares). One can see that the TV based model can achieve the highest PSNR among them, but it has some artifacts in the homogenous region, as shown in the amplifications of red square. Considering the texture part of vessel (the green square), our proposed trunc-LN and trunc-$\ell_p$ models can preserve this structure as well as TV, better than TW-$\ell_0$.

5. Conclusion. In this paper, we proposed a nonconvex truncated regularization with box-constrained model for low-dose CT reconstruction. It combined the sparse recovery property of nonconvex potential functions and the contrast preservation property of truncation strategy. By showing two theoretical results for the local minimizers of our model, an iterative support shrinking algorithm was presented. Then, a general linearization technique was introduced to handle nondifferentiable points. For the proposed algorithm, we verified its sufficient decrease property of the objective function and provided a lower bound theory of the iterative sequence. Among all compared approaches, our model with the proposed algorithm performs quite well in visual and quantitative results.
Figure 7. The residual errors of the reconstructions by three methods with 36 projections for “Shepp-Logan” and “NCAT”.

\[ \text{TV} \quad \text{TW-} \ell_0 \quad \text{trunc-LN} \quad \text{trunc-} \ell_p \]
Figure 8. The reconstruction comparisons of 60th and 80th rows of “Shepp-Logan” image for the three methods with 36 projections.

(a) 60th row

(b) 80th row

Figure 9. The values $F(u_k)$ versus the iteration number for “Shepp-Logan” image with 36 projections.

(a) trunc-LN

(b) trunc-$\ell_p$

Since $\varphi(||d_i u||)$ in our model is not subdifferentially regular at these non-differential points in $M$, it is difficult to analyze the convergence of the iterative sequence. Our future work is to improve the current algorithm to guarantee its convergence.
More index sets and some facts. We further introduce the following index sets used in our proof and give a simple example to explain them.

**Definition 6.1.** For any $u \in \mathcal{U}$, we define more index sets

1. $B_1(u) = \{i \in \Omega_1(u) \cap B(u) : u_{i+1} \in B(u) \text{ and } u_{i+n} \in B(u)\}$.
2. $L_1(u) = \Omega_1(u) \setminus B_1(u)$.
3. $B_0(u) = \Omega_0(u) \cap B(u)$.
4. $I_0(u) = \Omega_0(u) \setminus B_0(u)$.

Here, the subset $B_1(u)$ of $\Omega_1(u)$ includes pixels equal to $l_1$ or $l_2$ with its left and right pixels being $l_1$ or $l_2$. Similarly, the subset $B_0(u)$ of $\Omega_0(u)$ consists of pixels equal to $l_1$ or $l_2$ with its left and right pixels being the same as itself. A simple example with the periodic boundary condition is shown in Fig. 11 to explain these index sets in Definitions 2.1 and 6.1. Here, $l_1 = 0$, $l_2 = 1$, $t = \sqrt{\frac{13}{4}} u$ is

\[ u = [0, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]^T, \]

and $(\|d_i\|)_{i \in J}$ is $[\sqrt{2}, \sqrt{\frac{13}{4}}, 0, \sqrt{\frac{13}{4}}, \frac{3}{4}, \sqrt{\frac{13}{4}}, \sqrt{\frac{13}{4}}, 0, \frac{3}{4}, \sqrt{\frac{13}{4}}, \frac{3}{4}, \sqrt{\frac{13}{4}}, 0, \frac{3}{4}, \sqrt{\frac{13}{4}}, \frac{3}{4}, \sqrt{\frac{13}{4}}]^T$. Then,

1. $I(u) = \{3, 4, 6, 7, 8, 11, 12, 14, 15, 16\}$; $B(u) = \{1, 2, 5, 9, 10, 13\}$;
2. $\Omega_1(u) = \{1, 2, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16\}$; $\Omega_{1, M}(u) = \{2, 4, 10\}$;
3. $B_1(u) = \{1\}$; $L_1(u) = \{2, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16\}$;
4. $\Omega_0(u) = \{3, 9, 11\}$; $B_0(u) = \{9\}$; $I_0(u) = \{3, 11\}$.

From Definitions 2.1 and 6.1, we can similarly obtain the following facts as [40].

![CT reconstruction comparisons](image-url)
For $u \in \mathcal{U}$, let $I(u), B(u), \Omega_1(u), B_1(u), L_1(u), \Omega_0(u), B_0(u), I_0(u)$ be the index sets defined in Definitions 2.1 and 6.1. Then, the following statements hold:

(a) $L_1(u) \cap B_1(u) = \emptyset$ and $L_1(u) \cup B_1(u) = \Omega_1(u)$;
(b) $(d^1_x)_I(u) = 0$ and $(d^0_x)_I(u) = 0$, for any $i \in B_1(u)$;
(c) $(d^2_x)_I(u) \neq 0$ or $(d^0_y)_I(u) \neq 0$, for any $i \in L_1(u)$;
(d) $I_0(u) \cup B_0(u) = \emptyset$ and $I_0(u) \cup B_0(u) = \Omega_0(u)$;
(e) $(d^1_x)_I(u) = 0$ and $(d^0_y)_I(u) = 0$, for any $i \in B_0(u)$;
(f) $(d^2_x)_B(u) = 0$ and $(d^0_y)_B(u) = 0$, for any $i \in I_0(u)$.

6.2. The proof of theorem 2.2.

Proof. Recall the right-side and left-side derivatives of $R(\cdot)$ at $z$ in the direction of $v$:

$$d^+_v R(z) = \lim_{t \to 0^+} \frac{R(z + tv) - R(z)}{t},$$

and

$$d^-_v R(z) = \lim_{t \to 0^+} \frac{R(z - tv) - R(z)}{-t}.$$  

If $R(\cdot)$ is differentiable at $z$, then $d^+_v R(z) = d^-_v R(z) = \langle \nabla R(z), v \rangle$.

By the first-order necessary condition at the local minimizer $z^*$ of $(R)$, we have

(18) $d^-_v R(z^*) \leq d^+_v R(z^*), \forall v \in \mathcal{K}(I^*_0) = \{z \in \mathbb{R}^{|I'|} : (d_i)_I \cdot z = 0, \forall i \in I^*_0\}$.

To compute $d^+_v R(z^*)$ and $d^-_v R(z^*)$, we first analyze the first term of $R(z^*)$ in (4)

$$\sum_{i \in \Omega_1^*} \varphi(\sqrt{(d^1_i)_I \cdot z^* + t^i_x}^2 + (d^0_i)_I \cdot z^* + t^i_y)^2)$$

$$= (\sum_{i \in \Omega_1^*} + \sum_{i \in \Omega_1^* \cup \Omega_2^*}) \varphi(\sqrt{(d^1_i)_I \cdot z^* + t^i_x}^2 + (d^0_i)_I \cdot z^* + t^i_y)^2).$$
If \( i \in \Omega_1 \setminus \Omega_{1,M} \), then \( d_i^+ \varphi(z^*) = d_i^- \varphi(z^*) = \langle \nabla \varphi(z^*), v \rangle \). In detail, we compute

\[
\langle \nabla \varphi(z^*), v \rangle = \sum_{i \in \Omega_1 \setminus \Omega_{1,M}} \left\langle \nabla \varphi(\sqrt{(d_i^+)^2 + (d_i^-)^2}), v \right\rangle \\
= \sum_{i \in \Omega_1 \setminus \Omega_{1,M}} \left\langle \varphi'(||d_i^+||)\frac{d_i^+}{||d_i^+||}, (d_i^+) \right\rangle \\
= \sum_{i \in \Omega_1 \setminus \Omega_{1,M}} \varphi'(||d_i^+||) \left\langle \frac{d_i^+}{||d_i^+||}, (d_i^+) \right\rangle.
\]

If \( i \in \Omega_{1,M} \), then, with \( v = z^* \), the right-side and left-side derivatives of \( \varphi(\cdot) \) at \( z^* \) are

\[
d_i^\pm \varphi(\sqrt{(d_i^\pm)^2 + (d_i^\mp)^2}) = \lim_{t \to 0^+} \varphi(\sqrt{(d_i^\pm u^* \pm t(d_i^\pm)^2 + (d_i^\mp u^* \pm t(d_i^\mp)^2)} - \varphi(||d_i^\pm u^||) \\
= \lim_{t \to 0^+} \varphi(\sqrt{2||d_i u^*||^2 + 2\beta_t t + \beta_2 t^2}) - \varphi(||d_i u^||) \\
= \lim_{t \to 0^+} \varphi(||d_i u^*|| + \frac{\beta_1 t}{||d_i u^||} + \frac{\beta_2 t^2}{2||d_i u^||}) - \varphi(||d_i u^||) \\
= \lim_{t \to 0^+} \varphi(||d_i u^*|| \pm \frac{\beta_1 t}{||d_i u^||} + \frac{\beta_2 t^2}{2||d_i u^||}) - \varphi(||d_i u^||) \\
+ \lim_{t \to 0^+} \frac{\varphi(||d_i u^*|| \pm \frac{\beta_1 t}{||d_i u^||} + \frac{\beta_2 t^2}{2||d_i u^||}) - \varphi(||d_i u^||)}{t} \\
= \lim_{t \to 0^+} \frac{\varphi(||d_i u^*|| \pm \frac{\beta_1 t}{||d_i u^||} + \frac{\beta_2 t^2}{2||d_i u^||})}{t} - \varphi(||d_i u^||) \\
= \lim_{\tau \to 0^+} \frac{\varphi(||d_i u^*|| \pm \tau) - \varphi(||d_i u^||)}{\tau} = \varphi'(||d_i u^*||) \frac{||d_i u^||}{\beta_1}
\]

where \( \beta_1 = d_i^\pm u^*(d_i^\pm)^2 + d_i^\mp u^*(d_i^\mp)^2 \) and \( \beta_2 = ||d_i||^2 \).
Based on the discussion above, when $v = z^*$, the first-order necessary condition in (18) produces

$$d_{z^*}^* R(z^*) = \alpha \langle A_{z^*} z^* - f^*, A_{z^*} z^* \rangle + \sum_{i \in \Omega_1 \setminus \Omega_1^*} \varphi'(\|d_i u^*\|) \left\langle \frac{d_i u^*}{\|d_i u^*\|}, (d_i)_{(z^*)} \right\rangle$$

$$+ \sum_{i \in \Omega_1^*} \varphi'\left(\|d_i u^*\| - \frac{\|d_i u^*\|}{\beta_1}\right) \left\langle \frac{d_i u^*}{\|d_i u^*\|}, (d_i)_{(z^*)} \right\rangle$$

$$\leq \alpha \langle A_{z^*} z^* - f^*, A_{z^*} z^* \rangle + \sum_{i \in \Omega_1 \setminus \Omega_1^*} \varphi'(\|d_i u^*\|) \left\langle \frac{d_i u^*}{\|d_i u^*\|}, (d_i)_{(z^*)} \right\rangle$$

$$+ \sum_{i \in \Omega_1^*} \varphi'\left(\|d_i u^*\| + \frac{\|d_i u^*\|}{\beta_1}\right) = d_{z^*}^* R(z^*),$$

which leads to

$$\sum_{i \in \Omega_1 \setminus \Omega_1^*} \varphi'(\|d_i u^*\| - \frac{\|d_i u^*\|}{\beta_1}) \leq \sum_{i \in \Omega_1^*} \varphi'\left(\|d_i u^*\| + \frac{\|d_i u^*\|}{\beta_1}\right).$$

From Assumption 1(b), this inequality is impossible to be true. Thus, we have $\Omega_1^* = \emptyset$, which completes the proof.

\[\square\]

**Acknowledgement.** This work was supported by Zhejiang Provincial Natural Science Foundation of China (No. LQ20A010007) the National Natural Science Foundation of China (NSFC) (Nos. 11871035, 11531013, 11971138), Recruitment Program of Global Young Expert, and Postdoctoral Science Foundation of China (No. 2019M651002). We thank the anonymous referees for valuable comments and suggestions, which significantly improved the content of the paper.

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Received January 2020; revised May 2020.

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