Progressive Differences Convolutional Low-Density Parity-Check Codes

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Abstract

We present a new family of low-density parity-check (LDPC) convolutional codes that can be designed using ordered sets of progressive differences. We study their properties and define a subset of codes in this class that have some desirable features, such as fixed minimum distance and Tanner graphs without short cycles. The design approach we propose ensures that these properties are guaranteed independently of the code rate. This makes these codes of interest in many practical applications, particularly when high rate codes are needed for saving bandwidth. We provide some examples of coded transmission schemes exploiting this new class of codes.

Index Terms

Convolutional codes, LDPC codes, progressive differences.

I. INTRODUCTION

An increasing interest has been devoted, in recent years, to low-density parity-check (LDPC) convolutional codes, which have revitalized the field of classical convolutional coded transmission schemes by exploiting the powerful error correcting performance of LDPC codes.

Traditional convolutional codes [1] are among the most important classes of channel codes and have found many applications in past and present communications systems, especially when low latency is needed. Furthermore, convolutional codes are the main components of turbo codes [2], that deeply changed the scenario of forward error correction.

Convolutional codes are usually designed by optimizing the generator polynomial and the associated matrix $G$. The goal is to keep the number of memory elements in the encoder shift registers (that is, the constraint length) as small as possible while maximizing the free distance [3], [4]. In this way, good codes with limited decoding complexity can be designed.

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This work was supported in part by the MIUR project “ESCAPADE” (Grant RBFR105NLC) under the “FIRB - Futuro in Ricerca 2010” funding program.
The advent of LDPC codes [5], [6] changed this point of view, because the computational complexity of LDPC decoding performed over Tanner graphs with belief propagation algorithms is mostly influenced by the density of symbols 1 in the parity-check matrix $H$. Adopting low-density parity-check matrices and avoiding short cycles in the Tanner graph ensures to achieve good performance and low decoding complexity.

In the literature, two main methods for designing LDPC convolutional codes have been proposed [7], [8] and have originated a number of variants and improvements (see [9] and the references therein). Both approaches start from LDPC block codes and rearrange their parity-check matrices through suitable unwrapping procedures to obtain convolutional codes.

In this paper, we present a method to design LDPC convolutional codes having very simple structured matrices (thus yielding very low encoding and decoding complexity) and, at the same time, good distance properties and no short cycles in their Tanner graphs. According to previous approaches, high constraint lengths and time-varying codes might be required to achieve good performance. Moreover, the minimum distance of the LDPC convolutional codes so obtained is usually unknown. On the contrary, our aim is to keep the constraint length as small as possible and to design time-invariant codes, in order to limit complexity. In addition, the proposed method allows designing codes with known minimum distance, which does not depend on the code rate.

The proposed codes are based on ordered sets of progressive differences as separations between symbols 1 in their parity-check matrix columns. For this reason, we call these codes progressive differences convolutional (PDC) LDPC codes.

The organization of the paper is as follows: in Section II, we define the new codes and study their properties; in Section III, we report some examples of code design according to the proposed approach; finally, Section IV is devoted to concluding remarks.

II. DEFINITION AND PROPERTIES OF PDC-LDPC CODES

As for LDPC convolutional codes, each PDC-LDPC code is defined by a semi-infinite binary parity-check matrix $H_{\text{conv}}$ [9]. For classical LDPC convolutional codes, $H_{\text{conv}}$ is an all-zero matrix except for a set of binary submatrices $H_i(t)$, $t \in \mathbb{Z}_{\geq 0}$, $i = 0, 1, \ldots, L_h - 1^1$, which form a staircase from top left towards bottom right within $H_{\text{conv}}$ [9]. Each matrix $H_i(t)$ has size $(a - b) \times a$ and the design rate of the code is, therefore, $R_d = b/a$. When $t_1 \neq t_2 \Rightarrow H_i(t_1) \neq H_i(t_2)$, the convolutional LDPC code is said to be time-varying; otherwise, if $H_i(t)$ does not depend on $t$, then the code is said to be time-invariant. The structure of $H_{\text{conv}}$ for a time-invariant LDPC convolutional code is shown in (1), where $0$ represents the all-zero $(a - b) \times a$ matrix.

PDC-LDPC codes, here proposed, are a family of time-invariant LDPC convolutional codes with $b = (a - 1)$ and $R_d = (a - 1)/a$. Therefore, each $H_i$ is a row vector with length $a$. A block column formed by $L_h$ submatrices $H_0, H_1, \ldots, H_{L_h - 1}$ is denoted by $H^T$ and coincides with the transpose of the $a \times L_h$ syndrome former matrix, $H_s$. It follows that the syndrome former constraint length is equal to $v_s = L_h \cdot a$. This value should be kept as small as possible.

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1Note that $L_h$ measures the memory order of the syndrome former and it has the same meaning of $m_s + 1$ in [7], [9].
Lemma II.1

In PDC-LDPC codes, the matrix $H^T_s$ has columns with fixed Hamming weight $w$. The first row is the all-one vector, while the other $L_h - 1$ rows have Hamming weight 0 or 1. The matrix $H^T_s$ can be defined by $a$ ordered sets of $w - 1$ values each. The $i$-th of these sets is denoted as $D_i = \{d_{i,1}, d_{i,2}, \ldots, d_{i,w-1}\}$, $i = 1, 2, \ldots, a$. Each value $d_{i,j}$, with $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, w - 1$, is the difference between the row positions of the $(j+1)$-th and the $j$-th symbol within the $i$-th column of the matrix $H^T_s$. For each ordered set $D_i$, the set $S_i$, $i = 1, 2, \ldots, a$, contains all the elements in $D_i$ and all the sums of two or more consecutive elements in $D_i$. We also define the multisets $D = D_1 \uplus D_2 \uplus \ldots \uplus D_a$ and $S = S_1 \uplus S_2 \uplus \ldots \uplus S_a$, where $\uplus$ denotes the multiset sum.

In the following, we recall some propositions that are known in coding theory (proofs are omitted for the sake of brevity) and demonstrate some lemmas and theorems on the properties of PDC-LDPC codes.

Lemma II.1 Given $D_1, D_2, \ldots, D_a$ and the corresponding $S_1, S_2, \ldots, S_a$, if $S$ does not contain duplicates then the Tanner graph of the associated PDC-LDPC code is free of length-4 cycles.

Proof: A length-4 cycle in a Tanner graph corresponds to two columns in the associated parity-check matrix having two symbols 1 at the same positions. This configuration cannot occur when $S$ does not contain duplicates, that is, $S_1, S_2, \ldots, S_a$ are disjoint. Hence, $H^T_s$ is free of length-4 cycles. Following (1), $H_{conv}$ consists of copies of $H^T_s$ that are progressively shifted down by one row from each other. Therefore, the absence of duplicates in $S$ is sufficient to ensure that also $H_{conv}$ is free of length-4 cycles.

It is well known that the absence of length-4 cycles in the Tanner graph of a code is a necessary condition for achieving good performance under belief propagation decoding. A PDC-LDPC code with this feature can be easily designed by choosing at random the elements of $D_1, D_2, \ldots, D_a$ and checking that the condition imposed by Lemma II.1 is satisfied.

Provided that length-4 cycles are avoided, longer length cycles may obviously remain in the Tanner graph of PDC-LDPC codes. In particular, a length-6 cycle appears if and only if, given three elements $s_{i,l} \in S_i$, $s_{j,m} \in S_j$ and $s_{k,t} \in S_k$, they verify the condition $s_{i,l} + s_{j,m} = s_{k,t}$. 

$$H_{conv} = \begin{bmatrix} H_0 & 0 & 0 & \cdots \\ H_1 & H_0 & 0 & \cdots \\ H_2 & H_1 & H_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$ (1)
Lemma II.2 For a PDC-LDPC code free of length-4 cycles in the Tanner graph, the syndrome former matrix $H^T_s$ has a number of rows

$$L_h \geq 1 + \frac{aw(w-1)}{2}.$$ 

Proof: The cardinality of $D_i$ is $|D_i| = w - i$, $i = 1, 2, \ldots, a$. Therefore, the number of possible sums of $j$ consecutive elements of $D_i$ is $w - j$, $j = 2, 3, \ldots, w - 1$. It follows that the cardinality of $S_i$ is

$$|S_i| = \sum_{j=1}^{w-1} (w - j) = \frac{w(w-1)}{2},$$

with $i = 1, 2, \ldots, a$, and the cardinality of $S$ is $|S| = a |S_i|$. Since all the elements of $S$ must be different by Lemma II.1, it follows that $\max\{S\} \geq |S|$ and $L_h \geq 1 + \max\{S\} \geq 1 + |S| = 1 + a |S_i|$, where $\max\{S\}$ is the maximum in $S$.

Lemma II.2 establishes a lower bound on the value of $L_h$ that can be approached or even reached by actual codes. In practical applications, it is important to keep $L_h$ as low as possible, to reduce the syndrome former memory [9]. On the other hand, we will see in the following that an increased value of $L_h$ can be the price to pay for improving other characteristics of the code, like its minimum distance.

Definition II.1 We define a uniform PDC-LDPC code as the code having $D_i$ ($i = 1, 2, \ldots, a$) with elements:

$$d_{i,2m+1} = 4^m[2(a-i)+1],$$
$$d_{i,2m+2} = 4^m(4i-2),$$

where $m = 0, 1, \ldots, \lfloor \frac{w-2}{2} \rfloor$, $w \geq 2$. Note that each $D_i$ has $w - 1$ elements; hence, (2) must be used to find up to $w - 1$ values for each $i$.

It is easy to verify that a uniform PDC-LDPC code satisfies the hypothesis of Lemma II.1, so its Tanner graph is free of length-4 cycles. An example of $H_s$ is reported in (3), for a uniform PDC-LDPC code with $a = 4$ and $w = 4$.

Next we provide some results on the minimum distance of PDC-LDPC codes and uniform PDC-LDPC codes, in particular.

Proposition II.1 ([10]) A code having a regular parity-check matrix with column weight $w = 2$ and minimum cycle length $g$ in its Tanner graph has minimum distance $d_{\min} = g/2$.

$$H_s = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

(3)
It follows from Proposition II.1 that a PDC-LDPC code with \( w = 2 \) that satisfies the hypothesis of Lemma II.1 has minimum distance \( d_{\text{min}} \geq 3 \), since length-4 cycles are avoided while length-6 cycles may still exist. On the other hand, for uniform PDC-LDPC codes the following Theorem holds.

**Theorem II.1** A uniform PDC-LDPC code with \( w = 2 \) has cycles with length \( \geq 8 \) in the Tanner graph and minimum distance \( d_{\text{min}} \geq 4 \).

**Proof:** Starting from the definition of uniform PDC-LDPC codes, it is easy to verify that, for any three elements \( s_{i,l} \in S_i \), \( s_{j,m} \in S_j \) and \( s_{k,t} \in S_k \), the condition \( s_{i,l} + s_{j,m} = s_{k,t} \) never occurs. Therefore, the length of the cycles in the Tanner graph of a uniform PDC-LDPC code with \( w = 2 \) is \( \geq 8 \). It follows from Proposition II.1 that the minimum distance is \( d_{\text{min}} \geq 4 \).

**Example II.1** Let us consider the PDC-LDPC code with \( a = 4 \), \( w = 2 \) and \( d_{1,1} = 4 \), \( d_{2,1} = 3 \), \( d_{3,1} = 2 \), \( d_{4,1} = 1 \). It is easy to verify that it has minimum distance 3. If we instead consider the uniform PDC-LDPC code with the same parameters, it has \( d_{1,1} = 7 \), \( d_{2,1} = 5 \), \( d_{3,1} = 3 \), \( d_{4,1} = 1 \) and it can be easily verified that the minimum distance is increased to 4. On the other hand, the considered PDC-LDPC code has \( L_h = 5 \), which meets the lower bound given by Lemma II.2, while the corresponding uniform PDC-LDPC code has \( L_h = 8 \). Therefore, the improvement in the minimum distance is paid in terms of the syndrome former memory.

**Proposition II.2** Regular LDPC codes with odd column weight \( w \) have even minimum distance and all their codewords have even weight.

**Proposition II.3** Regular LDPC codes with column weight \( w = 3 \), free of length-4 cycles in their Tanner graph, have minimum distance \( d_{\text{min}} \geq 4 \).

**Theorem II.2** A uniform PDC-LDPC code with \( w = 3 \) has minimum distance \( d_{\text{min}} \geq 6 \).

**Proof:** By Proposition II.3, a uniform PDC-LDPC code with \( w = 3 \) has minimum distance \( d_{\text{min}} \geq 4 \). It is easy to prove that, in order to have \( d_{\text{min}} = 4 \), each column in a group of four columns summing into the all-zero vector must be included in two distinct length-6 cycles involving other two columns within the same group. It can be verified that, for uniform PDC-LDPC codes, this never occurs, so \( d_{\text{min}} \geq 5 \). By Proposition II.2, a uniform PDC-LDPC code with \( w = 3 \) can only have even weight codewords, so it must be \( d_{\text{min}} \geq 6 \).

**Example II.2** Let us consider the PDC-LDPC code with \( a = 3 \) and \( w = 3 \), defined by: \( d_{1,1} = 5 \), \( d_{2,1} = 3 \), \( d_{3,1} = 1 \), \( d_{1,2} = 4 \), \( d_{2,2} = 7 \), \( d_{3,2} = 12 \). It can be easily verified that it has \( d_{\text{min}} = 4 \). In fact, \( d_{2,1} + d_{3,1} = d_{1,2} \) and \( d_{1,1} + d_{2,2} = d_{3,2} \), so each column of \( H^T \) is involved in two distinct length-6 cycles, that is a sufficient condition for having \( d_{\text{min}} = 4 \). If we instead consider the uniform PDC-LDPC code with the same parameters, characterized by \( d_{1,1} = 5 \), \( d_{2,1} = 3 \), \( d_{3,1} = 1 \), \( d_{1,2} = 2 \), \( d_{2,2} = 6 \), \( d_{3,2} = 10 \), such condition is no longer satisfied and the code has \( d_{\text{min}} = 6 \).
**Theorem II.3** A PDC-LDPC code with tail-biting termination and block length \( n \) has minimum distance \( d_{\min} \leq 2w \) and minimum distance multiplicity \( A_{d_{\min}} \approx \left( \frac{a}{a} \right) \cdot \frac{n}{a} \).

**Proof:** Let us consider the parity-check matrix \( H \) of a tail-biting PDC-LDPC code with parameters \( a \) and \( w \). Let us reorder the columns of \( H \) by taking first those at positions \( 1, a + 1, 2a + 1, \ldots \), then those at positions \( 2, a + 2, 2a + 2, \ldots \), and so on. We obtain the parity-check matrix \( H^c \) of an equivalent code, in the form of a single row of \( a \) circulant blocks: \( H^c \left[ H^c_1 | H^c_2 | \ldots | H^c_a \right] \), each with size \( n/a \) and row and column weight \( w \). For parity-check matrices in this form, it is shown in [11] that \( d_{\min} \leq 2w \) and \( A_{d_{\min}} \approx \left( \frac{a}{a} \right) \cdot \frac{n}{a} \).

**III. CODE EXAMPLES**

The main advantages of uniform PDC-LDPC codes with respect to other time-invariant LDPC convolutional codes are the guaranteed minimum distance and the very small memory requirements. An important aspect is that the minimum distance value, though not being very high, is guaranteed independently of the code rate, which allows designing very high rate codes with known distance. In order to assess these benefits, we compare PDC-LDPC codes with codes obtained through the unwrapping procedure proposed in [7], which is known to provide very good results.

We have designed two LDPC convolutional codes with rate \( R = 7/8 \) by applying the method described in [7]. With the meaning of the parameters as in [7], these two codes correspond to \( j = 3, k = 24, m = 73, a = 7, b = 8 \) and \( j = 4, k = 32, m = 193, a = 8, b = 81 \), respectively. For these codes, the value of \( j \) also coincides with the weight of the parity-check matrix columns. We have chosen the smallest values of \( m \) and, hence, \( m_s \) that can be found for these two pairs of \( j \) and \( k \). Thus, for these parameters, the two codes have the minimum constraint length achievable through the approach in [7]. The values of the memory order and the constraint length are \( m_s = 71, v_s = 1728 \) for the first code, and \( m_s = 191, v_s = 6144 \) for the second code. The minimum distance is 6 and 8, respectively. We have compared them with two uniform PDC-LDPC codes having the same code rate, parity-check matrix column weight and minimum distance. For the two uniform PDC-LDPC codes, the constraint length is, respectively, \( v_s = 256 \) and \( v_s = 624 \), which are about 7 and 10 times less than those of the codes designed following [7].

The performance of these codes has been assessed by simulating transmission over the additive white Gaussian noise channel, using binary phase shift keying modulation. The unterminated LDPC convolutional codes have been decoded by performing the sum-product algorithm with log-likelihood ratios on sufficiently long blocks. The simulation results, reported in Fig. 1, evidence that the codes from [7] have a better waterfall performance, due to their graph properties, but then their curves approach those of uniform PDC-LDPC codes and tend to the same slope. For \( w = 3 \), in particular, the performance gap between the two designs becomes very small in the error floor region. Hence, uniform PDC-LDPC codes are able to greatly reduce the memory requirements by assuring a moderate performance loss. This conclusion would be even more evident for higher rate codes, while for lower code rates the approach in [7] achieves excellent performance.
Fig. 1. Comparison between the solution proposed in [7] and uniform PDC-LDPC codes for the same code rate ($R = 7/8$) and column weight ($w = 3, 4$).

Owing to their high code rate and short constraint length, PDC-LDPC codes can be efficiently employed in concatenated schemes. We have considered the serial concatenation (SC) of a uniform PDC-LDPC outer code and a multiple serially-concatenated multiple-parity-check (M-SC-MPC) inner code [12]. Both they are LDPC codes; hence, the overall concatenated code is an LDPC code as well, and it has been designed in such a way as to avoid short cycles in its Tanner graph. We have focused on a set of parameters used in the IEEE 802.16e standard [13] and designed two SC codes with $n = 1632$ and rates $R \simeq 2/3, 3/4$. In both cases, the outer uniform PDC-LDPC code has $a = 7$ and $w = 2$. Concerning termination, the $R \simeq 3/4$ code is a tail-biting code, while the $R \simeq 2/3$ code uses a staircase termination block [14], which is often used for designing systematic LDPC codes and produces a negligible performance loss. The simulation results are reported in Fig. 2: The performance of the considered SC codes is very close to that of the standard QC-LDPC codes [13] in the waterfall region, and can be even better in the error floor region.

IV. CONCLUSION

We have proposed a new family of LDPC convolutional codes, named PDC-LDPC codes, and studied their properties. We have defined a special class of these codes, named uniform PDC-LDPC codes, that make the code design easier and ensure some desirable features, like an increased minimum distance.

PDC-LDPC codes are time-invariant in $H$ and have two main advantages: i) very small syndrome former memory and ii) guaranteed minimum distance, independently of the code rate. This allows designing good codes with high rate.

These codes have also the desirable feature that any matrix obtained as a segment of $H_{\text{conv}}$ has full-rank. So, an efficient encoding procedure can be implemented, based on $H_{\text{conv}}$ [8]. Finally, though not used in this paper,
a systematic generator matrix for uniform PDC-LDPC codes can be easily found, thus in turn enabling the use of very simple encoding circuits.

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