T-dualization of Gödel string cosmologies via Poisson-Lie T-duality approach

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ABSTRACT: Using the homogeneous Gödel spacetimes we find some new solutions for the field equations of bosonic string effective action up to first order in $\alpha'$ including both dilaton and axion fields. We then discuss in detail the (non-)Abelian T-dualization of Gödel string cosmologies via Poisson-Lie (PL) T-duality approach. In studying Abelian T-duality of the models we get seven dual models in such a way that they are constructed by one-, two- and three-dimensional Abelian Lie groups acting freely on the target space manifold. The results of our study show that the Abelian T-dual models are, under some of the special conditions, self-dual. Afterwards, non-Abelian duals of the Gödel spacetimes are constructed by two- and three-dimensional non-Abelian Lie groups such as $A_2$, $A_2 \oplus A_1$ and $SL(2,\mathbb{R})$. It is shown that the non-Abelian dual models are conformally invariant to the order $\alpha'$ only for the case of $\beta = 1$. Following that, the PL self-duality of $AdS_3 \times \mathbb{R}$ space ($\beta = 1$ case) is discussed.

KEYWORDS: Conformal and W Symmetry, Sigma Models, String Duality

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1 Introduction

One of the most interesting cosmological solutions to Einstein’s field equations is Gödel spacetime \[1\] which constituted (and still constitutes) a considerable motivation to the investigation of solutions more complex than those treated until then. It was obtained \[2\] some new solutions of string theory, including terms up to the first order in the inverse string tension \(\alpha'\) for the homogeneous Gödel spacetimes. Recently, it has been shown that
four-dimensional Gödel universe can be embedded in string theory. The corresponding
Lagrangians to the Einstein-Maxwell-Axion, Einstein-Proca-Axion and Freedman-Schwarz
$SU(2) \times SU(2)$ gauged supergravity theories admit the Gödel metrics as solutions, all
involving only the fundamental matter fields [3]. In Ref. [2], to find a class of Gödel
universes without Closed Timelike Curves (CTC’s) within the framework of low-energy
effective string theory, it has been considered a convenient ansatz for both dilaton and
axion fields in an orthonormal frame. In the present work, we obtain the other forms of
solutions of equations (but not in an orthonormal frame) for the two-loop beta-function
including the Gödel spacetimes, field strength and dilaton field. We furthermore get new
solutions by considering a simpler form of Gödel metrics, and then focus on finding target
space duals of the solutions.

Target space duality (T-duality) is a very important symmetry of string theory which
was originally defined for a string theory $\sigma$-model where the backgrounds of model have
an Abelian group of isometries [4, 5]. T-duality is a peculiar feature of strings, since it
provides a method for relating seemingly inequivalent string theories, and allows to build
new string backgrounds which could not be addressed otherwise. The theory of Abelian
T-duality is well understood and had been the subject of much research (for a review see
e.g. [6]). Then, the basic duality procedure could be generalized to the case where the
original $\sigma$-model had a non-Abelian group of isometries [7] (further work in this direction
was carried out in [8–13]). Indeed, it has been shown that non-Abelian T-duality is a valid
symmetry of the string theory, as in the Abelian case. There has been a new interest in
non-Abelian T-duality, which was ignited by [14], that provided the transformation rule
for the Ramond-Ramond fields under the non-Abelian T-duality. It was then extended to
$\sigma$-models with nonvanishing Ramond fluxes, thus allowing to search for new supergravity
solutions [15–19]. Klimčík and Ševera proposed a generalization of T-duality, or the so-
called PL T-duality [20, 21], which allows the duality to be performed on a target space
without isometries. Afterwards, PL T-duality transformations could be generalized to the
Lie supergroups [22] as well as supermanifolds [23], in such a way that super PL symmetry
of the WZW models based on some of the Lie supergroups of superdimension (2|2) was
studied [24, 25]. Of course, in the Lie groups case, PL symmetry of the WZW models was
already been studied in Ref. [26] (see also [27–29]). Recently, PL T-duality also appears as
an important tool in the study of integrable models and their deformations [30–36].

The main purpose of this paper is to study the (non-)Abelian T-dualization of the
Gödel string cosmologies via PL T-duality approach in the presence of spectator fields.
Lately, using this approach we have found new dual solutions for some of the gravitational
and string backgrounds such as BTZ black hole [37] and the WZW models based on the Lie
groups $H_4$ and $GL(2, \mathbb{R})$ [38]. As explained above, we obtain that the Abelian T-duals of the
Gödel metrics by using PL T-duality approach and then testing the conformal invariance
conditions of the duals up to two-loop order. The two-loop $\sigma$-model corrections to the
Abelian T-duality map were obtained by Kaloper and Meissner (KM) in [39]. They had
used the effective action approach by focusing on backgrounds which have a single Abelian
isometry. For one case of our Abelian dual solutions we obtain the same result from KM
approach (see appendix A).
This paper is organized as follows. After the introduction section, Sec. 2 reviews the conformal invariance conditions of the \( \sigma \)-model up to the first order in \( \alpha' \). We start Sec. 3 by introducing the Gödel metrics and then discuss the solutions of two-loop beta-function equations possessing the Gödel spacetimes. A short review of PL T-dual \( \sigma \)-models construction in the presence of spectator fields is presented in Sec. 4, where necessary formulas are summarized. In Sec. 5, we study the Abelian T-dualization of the Gödel spacetimes via the PL T-duality approach. The non-Abelian duals of the Gödel spacetimes constructed by two- and three-dimensional non-Abelian Lie groups are given in Sec. 6. In this section, we also study the PL self-duality of the \( AdS_3 \times \mathbb{R} \) space. Some concluding remarks are given in the last section. The study of Abelian T-duality of the Gödel spacetimes using KM approach, when the duality is implemented by a shift of the \( z \) coordinate, is left to appendix A.

2 Two-loop conformal invariance conditions of the bosonic string \( \sigma \)-model

A bosonic string propagating on a non-trivial background can be described by the well-known \( \sigma \)-model defined on a two-dimensional curved surface \( \Sigma \) in \( d \) spacetime dimensions with metric \( G_{MN} \), antisymmetric tensor field \( B_{MN} \) (axion field) and dilaton field \( \Phi \)

\[
S = \frac{1}{4\pi \alpha'} \int_{\Sigma} d\tau d\sigma \sqrt{h} \left[ h^{\alpha\beta} G_{MN}(X) + \epsilon^{\alpha\beta} B_{MN}(X) \right] \partial_\alpha X^M \partial_\beta X^N + \frac{1}{8\pi} \int_{\Sigma} d\tau d\sigma \ R^{(2)} \Phi(X),
\]

(2.1)

where \( \sigma^a = (\tau, \sigma) \) are the string worldsheet coordinates, and \( X^M (M = 1, ..., d) \) are coordinates in spacetime. \( h_{\alpha\beta} \) and \( R^{(2)} \) are the induced metric and curvature scalar on the string worldsheet, respectively. \( \epsilon^{\alpha\beta} \) is an antisymmetric tensor on the worldsheet and \( h = \det h_{\alpha\beta} \).

The dimensionful coupling constant \( \alpha' \) turns out to be the inverse string tension.

Since we are considering bosonic string theory, there is only one more massless degree of freedom of the string, namely the dilaton \( \Phi \). This gives a contribution to the action in the form of the second term of (2.1). This term breaks Weyl invariance on a classical level as do the one-loop corrections to \( G \) and \( B \).

In the \( \sigma \)-model context, the conformal invariance conditions of the \( \sigma \)-model (2.1) are provided by the vanishing of the beta-function equations [40]. In order for the fields \( (G, B, \Phi) \) to provide a consistent string background at low-energy up to the two-loop order (the first order in \( \alpha' \)) they must satisfy the following equations [41–45]

\[
\begin{align*}
R_{MN} - H_{MN}^2 + \nabla_M \nabla_N \Phi + \frac{1}{2\alpha'} \left[ R_{MPQR} R_{N}^{PQR} + 2 R_{MPQN} H_{P}^{QR} \right. \\
+ 2 R_{PQR(M} H_{N)}^{RS} H_{S}^{PQ} + \frac{1}{3} (\nabla_M H_{PQR}) (\nabla_N H_{PQR}) - (\nabla_P H_{RSM}) (\nabla_P H_{RNS}) \\
+ 2 H_{MPQ} H_{NHS} H_{RP}^{TS} H_{T}^{RP} + 2 H_{MPQ} H_{NR}^{Q} H_{P}^{RP} \left. \right] + O(\alpha'^2) &= 0, \\
\left. \right| (2.2) \\
\nabla_P H_{PMN} - (\nabla_P \Phi) H_{MPN} + \alpha' \left[ \nabla_P H_{MPN}^{RS} \left[ \right. \right. \\
\nabla_P H_{MPN}^{RS} - (\nabla_P H_{RNM}) H_{MPN}^{PR} \\
\left. \right| + O(\alpha'^2) &= 0,
\right| (2.3)
\end{align*}
\]
\[2\Lambda + \nabla^2 \Phi' - (\nabla \Phi')^2 + \frac{2}{3} H^2 - \alpha' \left[ \frac{1}{4} \mathcal{R}_{MNS} \mathcal{R}^{MNRS} - \frac{1}{3} (\nabla_M H_{NRS})(\nabla^M H^{NRS}) \right. \\
- \frac{1}{2} H_{MNP} H^{RS} R_{MNRS} - \mathcal{R}^{MN}_{MN} H^2_{MN} + \frac{3}{8} H^2_{MN} H^{2MN} \\
+ \frac{5}{6} H_{MNP} H^M_{RS} H^N_{QR} H^{PSQ} \right] + \mathcal{O}(\alpha'^2) = 0, \tag{2.4}\]

where \(H_{MNP}\) defined by \(H_{MNP} = 1/2(\partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN})\) is the field strength of the field \(B_{MN}\). We have used the conventional notations \(H^2_{MN} = H_{MPQ}H^{PQ} N\), \(H^2 = H_{MNP}H^{MNP}\), \(H^2_{MN} = H^{MPQ}H^N_{PQ}\) and \((\nabla \Phi)^2 = \partial_M \Phi \partial^M \Phi\). \(\mathcal{R}_{MN}\) and \(\mathcal{R}_{MNPS}\) are the Ricci tensor and Riemann tensor field of the metric \(G_{MN}\), respectively. Moreover, in Eq. (2.4), \(\Phi' = \Phi + \alpha' q H^2\) for some coefficient \(q\) [44], and \(\Lambda\) is a cosmological constant. In string theory, the \(\Lambda\) is related to the dimension of spacetime, \(d\), and the inverse string tension, \(\alpha'\), whereas in this paper it is, in some cases, treated as a free parameter. We note that round brackets denote the symmetric part on the indicated indices whereas square brackets denote the antisymmetric part.

On the other hand, the conditions for conformal invariance (Eqs. (2.2)-(2.4)) can be interpreted as field equations for \(G_{MN}, B_{MN}\) and \(\Phi\) of the string effective action [45]. As shown in Ref. [44], in \(d=26\) (where \(\Lambda = 0\)), the string effective action up to the first order in \(\alpha'\) is given by

\[S_{eff} = \int d^4 X \sqrt{-G} \ e^{-\Phi} \left[ \mathcal{R} - \frac{1}{3} H^2 + (\nabla \Phi)^2 + \alpha' \left[ \frac{1}{4} \mathcal{R}_{MNPQ} \mathcal{R}^{MNPS} - \frac{1}{2} \mathcal{R}_{MNPS} H^{MPS} H^P_N \right. \right. \\
+ \frac{1}{6} H_{MNP} H^{PQR} R^{NR}_{RQ} H^M_{SQ} + \frac{1}{2} H^2_{MN} H^{2MN} - \mathcal{R}_{MN} \nabla^M \Phi \nabla^N \Phi \\
- \frac{1}{6} H^2 (\nabla \Phi)^2 + \frac{1}{3} \nabla^M \Phi \nabla_M H^2 - \frac{1}{6} \mathcal{R} H^2 + \frac{1}{2} \mathcal{R} (\nabla \Phi)^2 + \frac{1}{2} (\nabla \Phi)^2 \nabla^2 \Phi \\
\left. - \mathcal{R}_{MN} \mathcal{R}^{MN} + \frac{1}{36} (H^2)^2 + \frac{1}{4} \mathcal{R}^2 - \frac{1}{4} ((\nabla \Phi)^2)^2 \right] \right] + \mathcal{O}(\alpha'^2). \tag{2.5}\]

As announced in the introduction, the Gödel spacetimes can be considered as exact solutions in string theory for the full \(\mathcal{O}(\alpha')\) action including both dilaton and axion fields [2]. In Ref. [2], it has been considered a convenient ansatz for both dilaton and axion fields in an orthonormal frame. In the next section, we obtain the other forms of solutions for the two-loop beta-function equations including the Gödel spacetimes, the field strength \(H\) and dilaton \(\Phi\) in such a way that we do not work in an orthonormal frame.

### 3 Gödel spacetimes as solutions in string theory for the full \(\mathcal{O}(\alpha')\) action

Among the known exact solutions of Einstein field equations, the Gödel and Gödel-type metrics [1] play a special role. It was shown within the usual general relativity that these solutions describe rotating string cosmologies, and allow for the existence of CTC’s. It is a well-known result that all Gödel-type metrics, i.e., homogeneous spacetimes exhibiting vorticity, characterized by \(\Omega\), and a given value of \(m\) parameter can be rewritten in cylindrical coordinates \((\hat{t}, \hat{r}, \hat{\varphi}, \hat{z})\) as

\[ds^2 = -d\hat{t}^2 - 2C(\hat{r}) \ d\hat{t} d\hat{\varphi} + G(\hat{r}) \ d\hat{\varphi}^2 + d\hat{r}^2 + d\hat{z}^2, \tag{3.1}\]
where the functions $C(\hat{r})$ and $G(\hat{r})$ must obey the relations

$$C(\hat{r}) = \frac{4\Omega}{m^2} \sinh^2\left(\frac{m\hat{r}}{2}\right),$$

$$G(\hat{r}) = \frac{4}{m^2} \sinh^2\left(\frac{m\hat{r}}{2}\right) \left[1 + (1 - \frac{4\Omega^2}{m^2}) \sinh^2\left(\frac{m\hat{r}}{2}\right)\right].$$

(3.2)

We note that $m^2 = 2\Omega^2$ is a particular case of the hyperbolic class which corresponds to the original Gödel solution [1]. An interesting aspect of Gödel-type solutions is the possibility for existence of CTC’s. The existence of CTC’s, which allows for violation of causality, depends upon the sign of the metric function $G(\hat{r})$. Indeed, from Eqs. (3.1) and (3.2) one finds that the circles defined by $\hat{t} = t_0, \hat{r} = r_0, \hat{\phi} \in [0, 2\pi], \hat{z} = z_0$ become CTC’s whenever $G(\hat{r}) < 0$ [46]. In fact, the range, $m^2 \geq 4\Omega^2$, does not present CTC’s.

Below we discuss the solutions of the two-loop beta-function equations (2.2)-(2.4) possessing the Gödel spacetimes in a coordinate base $(\hat{t}, \hat{r}, \hat{\phi}, \hat{z})$. Our solutions are, in general, classified into two special classes:

**Class I:** In this class of solutions the field strength and dilaton field are, respectively, given by

$$H = E \sinh(m\hat{r})d\hat{t} \wedge d\hat{r} \wedge d\hat{\phi}, \quad \Phi = f\hat{z} + b,$$

(3.3)

for some constants $E, f, b$. The equations (2.2)-(2.4) together with the fields given by Eq. (3.3) possess a Gödel solution with the metric (3.1) if the following conditions hold between the constants $m, \Omega, E, f, \alpha'$ and $\Lambda$:

(i) The first constraint to satisfy the field equations (2.2)-(2.4) with the metric (3.1) and the fields (3.3) is that

$$m^2 = 4\Omega^2, \quad \alpha' = \frac{1}{\Omega^2(1 - 12E^2)}, \quad f^2 = 2\Lambda - \Omega^2 \frac{(176E^4 - 56E^2 + 3)}{(1 - 12E^2)}.$$

(3.4)

This confirms that it is possible to obtain a Gödel solution with no CTC’s. By noting the relation (3.4), the field strength $H$ depending on $E$ can vanish. Then, the second relation of (3.4) gives the velocity of rotation of the Gödel universe in terms of the inverse string tension; moreover, there is another constraint, $f^2 = 2\Lambda - 3\Omega^2$, which shows that the cosmological term has to be positive. This particular case, $E = 0$, is in agreement with those of Ref. [2].

(i') Similar to the case (i) we have $m^2 = 4\Omega^2$ with no CTC’s. In this case, the value of $E$ is fixed to be $E^2 = 1/4$, and $f^2 = 2\Lambda - 4\Omega^2(1 + 2\alpha'\Omega^2)$. This case of solutions can also satisfy the Eqs. (2.2)-(2.4) up to zeroth order in $\alpha'$ provided that $f^2 = 2(\Lambda - 2\Omega^2)$. This solution is also in agreement with those of Ref. [2].

(ii) In this case, the relation between the constants is given by

$$E^2 = 1 - \frac{3\Omega^2}{m^2}, \quad \alpha' = \frac{1 - E^2}{\Omega^2(1 - 10E^2)}, \quad \frac{f^2}{2} = \Lambda - \Omega^2 \frac{(1 - 4E^2)}{(1 - E^2)}.$$

(3.5)
The first relation requires that $m^2 > 3\Omega^2$ for $E \neq 0$. The range of $3\Omega^2 \leq m^2 < 4\Omega^2$ allows CTC’s. When $m^2 = 3\Omega^2$, the field strength vanishes, then, $\alpha' = 1/\Omega^2$ and $\Lambda > f^2/2$.

(iii) The last case refers to the following relation between the constants $m, \Omega, E, \alpha'$, and $\Lambda$

$$E^2 = \frac{\Omega^2}{m^2}, \quad \alpha' = \frac{E^2}{\Omega^2(1 - 6E^2)}, \quad f^2 = 2\Lambda - \frac{\Omega^2(1 - 4E^2)^2}{E^2(1 - 6E^2)}. \quad (3.6)$$

From the first relation of (3.6) one can easily deduce that the field strength must not vanish if a solution is to exist and act as a source of rotation. In this situation by fixing $E^2$ to $1/2$ we then have the original Gödel solution\(^1\) for Eqs. (2.2)-(2.4).

**Class II:** The corresponding field strength to this class of solutions is given by

$$H = E \sinh(m\hat{r})d\hat{t} \wedge d\hat{r} \wedge d\hat{\phi} + F \sinh(m\hat{r})d\hat{r} \wedge d\hat{z} \wedge d\hat{\phi}, \quad (3.7)$$

for some constants $E, F$. In this case, dilaton field is assumed to be constant, $\Phi = b$. Using these, the equations (2.2)-(2.4) with the metric (3.1) are satisfied if the following relation hold between the constants $m, \Omega, E, F, \alpha'$ and $\Lambda$:

$$E^2 = \frac{216\Omega^4 - (108m^2 + 48\Lambda)\Omega^2 + (17m^2 + 6\Lambda)m^2 + (m^2 - 8\Omega^2)\Gamma}{88m^4\Omega^2},$$

$$\alpha' = \frac{(m^2 - 8\Omega^2)}{2m^2\Omega^2(11 + 6E^2) - 3m^4 - 44\Omega^4},$$

$$F^2 = \frac{(m^2 - 4\Omega^2)[3\Omega^2 - m^2(1 - E^2)]}{m^2(m^2 - 8\Omega^2)}, \quad (3.8)$$

where $\Gamma = \sqrt{80\Omega^4 - (104m^2 - 160\Lambda)\Omega^2 + (25m^2 - 60\Lambda)m^2 + 36\Omega^2}$.

Since our main aim in the present work is to study the T-dualization of the Gödel spacetimes we write down the metric (3.1) in a simpler form. To this end, one uses the coordinate transformation

$$\sqrt{\beta} \ t = \left\{ \frac{\sqrt{\beta}}{l} \ \hat{t} - \hat{\phi} + 2 \ \arctan \left[ e^{-m\hat{r}} \ \tan \left( \frac{\hat{\phi}}{2} \right) \right] \right\},$$

$$r = \cosh(m\hat{r}) + \cos \hat{\phi} \sinh(m\hat{r}),$$

$$\sqrt{\beta} \ r\phi = \sin \hat{\phi} \ \sinh(m\hat{r}),$$

$$z = \frac{\hat{z}}{l}, \quad (3.9)$$

to obtain

$$ds^2 = l^2 \left[ -dt^2 + \frac{dr^2}{r^2} + (\beta - 1)r^2 \ d\phi^2 - 2r \ dt d\phi + dz^2 \right], \quad (3.10)$$

\(^1\)In this case, one gets $\alpha' = -\frac{1}{4\Omega^2}$. In string theory, the resulting spectrum of the bosonic string contains a finite number of massless and infinitely many massive excitations with mass $\frac{\text{mass}}{\text{mass}} = \frac{n}{2}$ with $n \in N$. Among the states there are also tachyons with mass $\frac{\text{mass}}{\text{mass}} < 0$ implying that the vacuum is unstable. This is unavoidable in the bosonic string.
such that the following relations must be held between the constants $l, \beta, m$ and $\Omega$:

$$l = \frac{1}{m}, \quad \beta = \frac{m^2}{4\Omega^2}. \quad (3.11)$$

Here, the condition of the existence of CTC’s, $m^2 < 4\Omega^2$, is induced on the range of the parameter $\beta$. In fact, for the range $\beta \geq 1$ we do not encounter CTC’s. In this case, the original Gödel metric [1] is recovered when we take $\beta = 1/2$. As it is seen, the metric (3.10) is a direct product of $\mathbb{R}$ associated with the coordinate $z$ and the three-dimensional metric of $(t, \varphi, r)$. For the case of $\beta = 1$ one may use the following transformation

$$dt = \rho(d\tau - dx), \quad r = \rho, \quad \varphi = x, \quad (3.12)$$

then, the metric becomes

$$ds^2 = l^2 \left[-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2} + \rho^2 dx^2 + dz^2\right], \quad (3.13)$$

so when $\beta = 1$, the metric (3.10) is locally $AdS_3 \times \mathbb{R}$ space.

In order to obtain new solutions we now solve the field equations (2.2)-(2.4) for the metric (3.10). Here, the forms of our solutions including the field strength and dilaton field are given by two special classes A and B:

**Class A:** In this class of solutions the field strength $H$ and dilaton field $\Phi$ are given by

$$H = E dt \wedge dr \wedge d\varphi, \quad \Phi = fz + b, \quad (3.14)$$

for some constants $E, f, b$. The field equations (2.2)-(2.4) are then satisfied with the metric (3.10) together with the fields (3.14) if the following four conditions held between the constants $l, \beta, E, \alpha', \Lambda$:

1. The first condition is devoted to a special condition on the parameter $\beta$, and that is $\beta = 1$. The rest of constants are then related to each other in the following way:

$$\alpha' = \frac{4l^6}{l^4 - 12E^2}, \quad f^2 = \frac{176E^4 + 8l^4(12\Lambda - 7)E^2 + l^8(3 - 8\Lambda)}{4l^4(12E^2 - l^4)}. \quad (3.15)$$

This result confirms that it is possible to have a solution corresponding to zero field strength, that is, $E = 0$. Then, it is followed that $\alpha' = 4l^2$ and $f^2 = 2l^2\Lambda - 3/4$.

1' In addition to the first condition, we have another solution corresponding to $\beta = 1$ and nonzero field strength $E^2 = l^4/4$, so that the relation between other constants may be expressed as $f^2 = -1 + 2l^2\Lambda - \alpha'/(2l^2)$. This case of solutions can also satisfy the field equations (2.2)-(2.4) up to zeroth order in $\alpha'$ in a way that we must have $f^2 = -1 + 2l^2\Lambda$.

2. The field equations (2.2) and (2.3) are also fulfilled if the values of $E$ and $\alpha'$ can now be expressed in terms of the parameter $\beta$

$$E^2 = \frac{l^4}{4}(4\beta - 3), \quad \alpha' = \frac{2\beta l^2}{5 - 6\beta}. \quad (3.16)$$
This is an interesting case in which the field strength depends on the parameter $\beta$ of the metric. From the relation (3.16) one can easily deduce that $\beta \in [3/4, +\infty) - \{5/6\}$. We note that in this case, the range of $\beta \in [3/4, 1) - \{5/6\}$ allows CTC’s. The relation between other constants can be obtained from Eq. (2.4), which gives, $f^2 = 2(1 + \xi^2 \Lambda - 1/\beta)$.

(3) The last condition refers to a Gödel solution with a nonzero field strength, $E^2 = t^4/4$. From Eqs. (2.2) and (2.3) the value of $\alpha'$ is expressed in terms of the parameter $\beta$, obtaining

$$\alpha' = \frac{2\xi^2}{2\beta - 3}. \quad (3.17)$$

Finally after using Eq. (2.4), we obtain the relation

$$\frac{f^2}{2} = \frac{\beta^2(2\xi^2 \Lambda - 1) + \beta(2 - 3\xi^2 \Lambda) - 1}{\beta(2\beta - 3)}. \quad (3.18)$$

This is only case of the class A which is valid for $\beta = 1/2$. Putting $\beta = 1/2$, we then get $\alpha' = -t^2/2$. Also, from Eq. (3.18) it is followed that $\Lambda \geq -1/(4t^2)$, which shows that the cosmological term can be considered to be negative.

Class B: Class B of solutions is devoted to a constant dilaton field $\Phi = b$ and the field strength

$$H = E \ dt \wedge dr \wedge d\varphi + N dr \wedge d\varphi \wedge dz, \quad (3.19)$$

for some constants $E, N$, together with the metric (3.10). The equations of motion (2.2)-(2.4) are then fulfilled in this general case. From Eqs. (2.2) and (2.3) we obtain

$$\alpha' = \frac{4\beta b^6 (\beta - 2)}{t^4 (12\beta^2 - 22\beta + 11) - 12E^2},$$

$$N^2 = \frac{(\beta - 1)[4E^2 + t^4 (3 - 4\beta)]}{4(\beta - 2)}. \quad (3.20)$$

Finally, Eq. (2.4) is satisfied if

$$E^2 = \frac{t^4}{44} [34\beta^2 - 54\beta + 12\beta^2 \Lambda (\beta - 2) + 27 + 2(\beta - 2)\Xi], \quad (3.21)$$

where $\Xi = \sqrt{\beta^2(5 - 6t^2 \Lambda)^2 + 2\beta(20t^2 \Lambda - 13) + 5}$. We have thus obtained some new solutions for the field equations of bosonic string effective action up to the first order in $\alpha'$ in the forms of classes A and B. These solutions will be useful in studying the T-dualization of Gödel string cosmologies. We shall address this problem in Secs. 5 and 6.

4 A short review of PL T-duality with spectators

In this section, we recall the main features of PL T-duality transformations at the level of the $\sigma$-model. For the description of PL T-duality, we need to introduce the Drinfeld double group $D$ [47], which by definition has a pair of maximally isotropic subgroups $G$ and $\tilde{G}$.
corresponding to the subalgebras \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \), respectively. The generators of \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \) are denoted, respectively, \( T_a \) and \( \tilde{T}^a \), \( a = 1, \ldots, \dim G \) and satisfy the commutation relations

\[
[T_a, T_b] = f^{c}_{ab} T_c, \quad [\tilde{T}^a, \tilde{T}^b] = \tilde{f}^{ab}_{c} \tilde{T}^c,
\]

\[
[T_a, \tilde{T}^b] = \tilde{J}^{bc}_a T_c + f^b_{ca} \tilde{T}^c.
\] (4.1)

The Lie algebra structure defined by (4.1) is called the Drinfeld double \( \mathcal{D} \). The structure constants \( f^{c}_{ab} \) and \( \tilde{f}^{ab}_{c} \) are subject to the Jacobi identities and the following mixed Jacobi identities

\[
f^{a}_{bc} \tilde{J}^{de} = f^{d}_{ac} \tilde{J}^{ae}_b + f^{e}_{ba} \tilde{J}^{da}_c + f^{e}_{ac} \tilde{J}^{da}_b.
\] (4.2)

In addition, the Drinfeld double \( \mathcal{D} \) is equipped with an invariant inner product \( <\cdot,\cdot> \) with the following properties

\[
<T_a, \tilde{T}^b> = \delta_a^b,
\]

\[
<T_a, T_b> = <\tilde{T}^a, \tilde{T}^b> = 0.
\] (4.3)

Let us now consider a \( d \)-dimensional manifold \( M \) and some coordinates \( X^M = (x^\mu, y^i) \) on it, where \( x^\mu \), \( \mu = 1, \ldots, \dim G \) stand for the coordinates of Lie group \( G \) acting freely from right on \( M \) and \( y^i \), \( i = 1, \ldots, d - \dim G \) are the coordinates labeling the orbit \( O \) of \( G \) in the target space \( M \). We note that the coordinates \( y^i \) do not participate in the PL T-duality transformations and are therefore called spectators [27].

Before proceeding to write the \( \sigma \)-models let us introduce the components of the right-invariant Maurer-Cartan forms \( (g^{-1}\partial_\alpha g)^a \equiv R^a_\alpha = \partial_\alpha x^\mu R^a_\mu \), where \( g \) is an element of the Lie group \( G \) corresponding to the Lie algebra \( \mathcal{G} \). For notational convenience we will also use \( R^a_\alpha = \partial_\alpha y^i \). In order to define \( \sigma \)-models with PL T-duality symmetry, it is convenient to define matrices \( a(g), b(g) \) and \( \Pi(g) \) in the following way

\[
g^{-1}T^a = a^b_a (g) T_b,
\]

\[
g^{-1}\tilde{T}^a = b^{ab}_a (g) T_b + (a^{-1})^b_a (g) \tilde{T}^b,
\]

\[
\Pi^{ab}(g) = b^{ac}_a (g) (a^{-1})^c_b (g).
\] (4.4)

Then, the original \( \sigma \)-model is defined by the action [20, 21, 27]

\[
S = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ E_{ab} R^b_+ R^a_- + \phi^{(1)}_{\alpha} R^a_+ \partial_\alpha y^i + \phi^{(2)}_{\beta} \partial_\beta y^j R^a_- + \phi_i \partial_i y^j \partial_- y^j \right] - \frac{1}{4\pi} \int d\sigma^+ d\sigma^- R^{(2)} F(X).
\] (4.5)

Here, we have used the standard light-cone variables on the worldsheet, \( \sigma^\pm = (\tau \pm \sigma)/2 \) together with \( \partial_\pm = \partial_\tau \pm \partial_\sigma \). The backgrounds appearing in this action are given in matrix notation by [27]

\[
E = (E_0^{-1} + \Pi)^{-1},
\] (4.6)

\[
\phi^{(1)} = E E_0^{-1} F^{(1)},
\] (4.7)

\[
\phi^{(2)} = F^{(2)} E_0^{-1} E,
\] (4.8)

\[
\phi = F - F^{(2)} \Pi E E_0^{-1} F^{(1)}.
\] (4.9)
The matrices \((E_0, F, F^{(1)}, F^{(2)})\) are all functions of the variables \(y^i\) only.

Similarly we consider another \(\sigma\)-model for the \(d\) field variables \(\tilde{X}^M = (\tilde{x}^\mu, y^i)\), where \(\tilde{x}^\mu, \mu = 1, 2, ..., \dim \tilde{G}\) parameterize an element \(\tilde{g}\) of a Lie group \(\tilde{G}\), whose dimension is, however, equal to that of \(G\). The rest of the variables are the same \(y^i\) used in (4.5). Accordingly, we introduce the components of the right-invariant Maurer-Cartan forms 
\[
\tilde{g}^{-1}\partial_{\pm} \tilde{g} = \partial_{\pm} \tilde{x}^\mu \tilde{R}_{\mu a}
\]
on the Lie group \(\tilde{G}\). The corresponding \(\sigma\)-model has the form
\[
\tilde{S} = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ \tilde{E}^{ab}_{\pm} \tilde{R}_{\pm a} \tilde{R}_{-b} + \tilde{\Phi}^{(1)b}_i \tilde{R}_{\pm a} \partial_- y^i + \tilde{\Phi}^{(2)b}_i \partial_+ y^i \tilde{R}_{-a} + \tilde{\Phi}_a^b \partial_+ y^i \partial_- y^j \right] - \frac{1}{4\pi} \int d\sigma^+ d\sigma^- \tilde{R}^{(2)} \tilde{\Phi}(\tilde{X}). \tag{4.10}
\]
The backgrounds of the dual theory are related to those of the original one by [20, 27]
\[
\tilde{E} = (E_0 + \tilde{\Pi})^{-1}, \tag{4.11}
\]
\[
\tilde{\phi}^{(1)} = \tilde{E} F^{(1)}, \tag{4.12}
\]
\[
\tilde{\phi}^{(2)} = -F^{(2)} \tilde{E}, \tag{4.13}
\]
\[
\tilde{\phi} = F - F^{(2)} \tilde{E} F^{(1)}, \tag{4.14}
\]
where \(\tilde{\Pi}\) is defined as in (4.4) by replacing untilded quantities with tilded ones.

Notice that the transformations for both the dilatons \(\Phi\) and \(\tilde{\phi}\) of the actions (4.5) and (4.10), which have been obtained in Ref. [48] by quantum considerations, are given by
\[
\Phi = \Phi_0 + \log(\det E) - \log(\det E_0), \tag{4.15}
\]
\[
\tilde{\phi} = \Phi_0 + \log(\det \tilde{E}), \tag{4.16}
\]
where \(\Phi_0\) is just a function of \(y^i\). These transformations were based on a regularization of a functional determinant in the path integral formulation of PL duality. This duality is a canonical transformation and two \(\sigma\)-models related by PL duality are, therefore, equivalent at the classical level [49]. The quantum equivalence of \(\sigma\)-models related by PL T-duality transformations at the level of the one-loop string effective action has been analyzed in [50]. There, it has been shown that relations (4.15) and (4.16) only hold at the one-loop level for both \(\sigma\)-models admitting PL duality if the traces of the structure constants of each Lie algebra constituting the Drinfeld double are zero. As mentioned in the introduction section, in the case of Abelian T-duality, the two-loop \(\sigma\)-model corrections were obtained by KM in [39]. There, one can find the duality transformation of the dilaton field at the two-loop level. We will test one of our results by applying KM T-duality rules.

Let us turn into the actions (4.5) and (4.10). These actions correspond to PL T-dual \(\sigma\)-models [20, 21]. If the group \(G(\tilde{G})\) besides having free action on the manifold \(\mathcal{M}(\hat{\mathcal{M}})\), acts transitively on it, then the corresponding manifold \(\mathcal{M}(\hat{\mathcal{M}})\) will be the same as the group \(G(\tilde{G})\). In this case only the first term appears in the actions (4.5) and (4.10). The T-duality transformations are said to be Abelian [4, 5] or non-Abelian [7, 11] according to the nature of the Lie algebra formed by the generators of the isometries. Notice that:
• If both the Lie groups $G$ and $\tilde{G}$ become the isometry groups of the manifolds $M$ and $\tilde{M}$, respectively, namely, both $G$ and $\tilde{G}$ are chosen to be Abelian groups ($\tilde{f}_{abc} = f_{abc} = 0$), then we get $\Pi(g) = \Pi(\tilde{g}) = 0$, recovering thus the standard Abelian duality.

• In case of non-Abelian T-duality the former represents group of symmetries of the original $\sigma$-model, while the latter is Abelian ($\tilde{f}_{abc} = 0$).

• Furthermore, there are Drinfeld doubles where both $G$ and $\tilde{G}$ are non-Abelian. In such a case the symmetry of the original model is replaced with the so-called PL symmetry (or generalized symmetry), and the full PL T-duality transformation [20, 21] applies. We now wish to apply the above discussions to study the Abelian T-dualization of the Gödel string cosmologies in the next section.

5 Abelian T-dualization of Gödel string cosmologies

In order to study Abelian T-duality in Buscher’s construction [4, 5] one starts with a manifold $M$ with metric $G_{MN}$, antisymmetric tensor $B_{MN}$ and dilaton field $\Phi$, and requires the metric to admit at least one continuous Abelian isometry leaving invariant the action of $\sigma$-model constructed out of $(G, B, \Phi)$. As announced in the introduction, PL T-duality proposed by Klimčik and Ševera is a generalization of Abelian and non-Abelian T-dualities. It was then shown that Buscher’s duality transformations can be obtained from the PL T-duality approach by a convenient choice of the spectator-dependent matrices [51] (see also [37]). In this section, we obtain all possible Abelian T-duals of the Gödel string cosmologies by using the approach of PL T-duality in the presence of spectators. In this regard, the Lie groups $G$ and $\tilde{G}$ acting freely on the target space manifolds $M \approx O \times G$ and $\tilde{M} \approx O \times \tilde{G}$, respectively, can be considered to be Abelian Lie groups of dimensions one, two and three. In all cases, both structures $\Pi(g)$ and $\Pi(\tilde{g})$ are vanished as mentioned above.

5.1 Abelian T-duality with one-dimensional Abelian Lie group

5.1.1 Dualizing with respect to the coordinate $\varphi$

Here we assume that target space $M \approx O \times G$ is defined by coordinates $X^M = (x; r, t, z)$. The coordinates of orbit $O$ are represented by $y^i = (r, t, z)$, while $x$ is the coordinate of one-dimensional Abelian Lie group $G = A_1$ parameterizing by element $g = e^{x^T}$. In order to write the original $\sigma$-model on the manifold $M$ we need to determine the model couplings. Let us choose the spectator-dependent matrices in the following form\footnote{In the context of PL T-duality, the matrix $E_0$ must be invertible. Moreover, with regard to the first relation of (5.2) the dual solution will be valid for all values of $\beta$ except for 1.}

$$E_{0ab} = C_0 r^2, \quad F_{a_1}^{(1)} = \begin{pmatrix} F_{11}^{(1)} & F_{12}^{(1)} & f_3(r, t, z) \end{pmatrix},$$

$$F_{ij} = \begin{pmatrix} \frac{e^{2x}}{2} & f_1(r, t, z) & f_2(r, t, z) \\ -f_1(r, t, z) & -l^2 & F_{23} \\ -f_2(r, t, z) & -F_{23} & l^2 \end{pmatrix}, \quad F_{a_1}^{(2)} = -\begin{pmatrix} F_{11}^{(1)} & 2\gamma_0 l^2 r \\ F_{12}^{(1)} & f_3(r, t, z) \end{pmatrix}.$$ (5.1)
where \( C_0 \) is a nonzero constant and

\[
\gamma_0^2 = \frac{C_0}{l^2(\beta - 1)},
\]

\[
F^{(1)}_{11} = h_1(r) + \int \left[ \partial_r f_3(r, t, z) - 2N\gamma_0 \right] dz + \int \left[ l^2 \gamma_0 (1 - \frac{2E}{l^2}) + \partial_r h_2(r, t) \right] dt,
\]

\[
F_{12}^{(1)} = h_2(r, t) + \int \partial_t f_3(r, t, z) dz,
\]

\[
F_{23} = h_3(t, z) + \int \left[ \partial_t f_3(r, t, z) - \partial_z f_1(r, t, z) \right] dr,
\]

(5.2)

in which \( h_i \)'s and \( f_i \)'s are some arbitrary functions that may depend on some coordinates \( r, t \) and \( z \). Henceforth, the constants \( E, N, l \) and \( \beta \) are the same ones introduced in Sec. 3.

Inserting (5.1) into Eqs. (4.6)-(4.9) and then employing (4.5), the original \( \sigma \)-model on the manifold \( \mathcal{M} \) can be derived. Then corresponding metric and antisymmetric tensor may be expressed as

\[
ds^2 = l^2 \left[ -dt^2 + \frac{dr}{r^2} + (\beta - 1)\gamma_0^2 dx^2 - 2r\gamma_0 dt dx + dz^2 \right],
\]

(5.3)

\[
B = f_1(r, t, z) dr \wedge dt + f_2(r, t, z) dr \wedge dz + F_{23} dt \wedge dz + f_3(r, t, z) dx \wedge dz + F^{(1)}_{11} dx \wedge dr + (\gamma_0 l^2 r + F_{12}^{(1)}) dx \wedge dt.
\]

(5.4)

Carrying out the coordinate transformation \( \varphi = \gamma_0 x \) we see that (5.3) is nothing but the Gödel metric (3.10). Furthermore, one concludes that the field strength corresponding to the \( B \)-field (5.4) is the same as (3.19). Thus, one verifies the field equations (2.2)-(2.4) for the metric (5.3) and the \( B \)-field (5.4) with a constant dilaton field.

The dual background is obtained from a \( \sigma \)-model which is constructed on a 1 + 3-dimensional manifold \( \bar{\mathcal{M}} \approx O \times \bar{G} \) where \( \bar{G} \) is the same as the \( A_1 \) Lie group. Since duality is performed on the \( x \), we parameterize \( \bar{G} \) by the coordinate \( \tilde{x} \). Before proceeding to construct the dual background, let us consider a simpler form of the spectator-dependent matrices (5.1). Indeed, one may take \( h_1 = h_3 = f_1 = f_2 = 0 \), \( h_2 = (2E - l^2 r) \), \( f_3 = 2N \) and \( C_0 = l^2 (\beta - 1) \). Imposing these conditions on (5.1) and then utilizing Eqs. (4.11)-(4.14) and also (4.10), the metric and antisymmetric tensor field \( \bar{B} \) of the dual model can be cast in the forms

\[
ds^2 = \frac{l^2 dr^2}{r^2} + \frac{1}{(\beta - 1)} \left[ \left( \frac{4E^2}{l^2} - \beta l^2 \right) dt^2 + \frac{d\tilde{x}^2}{l^2 r^2} + \frac{4N^2}{l^2} + l^2 (\beta - 1) \right] dz^2
\]

\[
+ \frac{8EN}{l^2} dt dz + \frac{4E}{l^2 r} d\tilde{x} dt + \frac{4N}{l^2 r} d\tilde{x} dz,
\]

(5.5)

\[
\bar{B} = \frac{1}{(\beta - 1)} \left( \frac{1}{r} dt \wedge d\tilde{x} + 2N dt \wedge dz \right).
\]

(5.6)

One quickly deduces that the only nonzero component of the field strength corresponding to the \( \bar{B} \)-field (5.6) is

\[
\bar{H}_{ztr} = \frac{1}{2r^2 (\beta - 1)}.
\]

(5.7)
Let us now clarify the spacetime structure and conformal invariance conditions of the dual model. When \( N = 0 \), the metric (5.5) and the field strength (5.7) satisfy the field equations (2.2)-(2.4) with a dilaton field in the form of \( \tilde{\Phi} = \tilde{f}z + b \) provided that the constants \( E, l, \tilde{f}, \alpha' \) and \( \beta \) satisfy the following relations

\[
E^2 = \frac{l^4}{4}, \quad \alpha' = \frac{2l^2\beta}{2\beta - 3}, \quad \tilde{f}^2 = \frac{2\beta^2(2\tilde{\Lambda}l^2 - 1) - 2\beta(3\tilde{\Lambda}l^2 - 2) - 2}{\beta(2\beta - 3)}. \tag{5.8}
\]

Imposing conditions \( N = 0 \) and \( E^2 = l^4/4 \) on the metric (5.5) and then performing the coordinate transformation \( r \to 1/r, \tilde{x} \to -l^2(\beta - 1)\varphi \) one concludes that it is nothing but the Gödel metric (3.10) with condition \( \beta \neq \{1, 3/2\} \). This means that the model is self-dual. Also when \( N = 0 \) we have another solution in which relationship between the constants are given by

\[
E^2 = \frac{l^4(4\beta - 1)}{12}, \quad \alpha' = \frac{6l^2\beta}{2\beta - 5}, \quad \tilde{f}^2 = \frac{2}{3\beta}(1 + \beta(3\tilde{\Lambda}l^2 - 1)). \tag{5.9}
\]

In addition to the above, for the metric (5.5) and the field strength (5.7) one verifies the field equations (2.2)-(2.4) with a constant dilaton field when \( N \) differs from zero. In this way we must have

\[
E^2 = \frac{l^4(2\beta^2 - 2\beta + 1)}{4}, \quad \alpha' = \frac{2l^2\beta}{2 - 3\beta}, \quad \tilde{\Lambda} = \frac{\beta - 1)^2}{4l^2\beta(2 - 3\beta)}. \tag{5.10}
\]

5.1.2 Dualizing with respect to the coordinate \( t \)

In this case the coordinates of the target manifold \( \mathcal{M} \) are denoted by \( (t; r, \varphi, z) \) wherein \( t \) is the coordinate of the Lie group \( G \) which the duality is performed on, while \( (r, \varphi, z) \) are the coordinate representations of the orbit \( O \). If we choose the coupling matrices as

\[
F^{+1}_{a_j} = \begin{pmatrix} 0 & -r(2E + l^2) & 0 \end{pmatrix}, \quad F^{+2}_{ib} = \begin{pmatrix} 0 \\ r(2E - l^2) \\ 0 \end{pmatrix}, \quad F_{ij} = \begin{pmatrix} \frac{l^2}{l^2} & 0 & 0 \\ 0 & (\beta - 1)l^2r^2 & 2N_r \\ 0 & -2N_r & l^2 \end{pmatrix}, \tag{5.11}
\]

and \( E_{0}^{+ab} = -l^2 \), then using the formulae (4.6)-(4.9) and also (4.5) one finds that the metric and field strength corresponding to the original \( \sigma \)-model take the same form as (3.10) and (3.19), respectively. It is then straightforward to compute the corresponding dual spacetime. They are read

\[
\tilde{d}s^2 = \frac{l^2dr^2}{r^2} + l^2dz^2 - \frac{1}{l^2}d\tilde{x}^2 + (\beta l^2 - \frac{4E^2}{l^2})r^2d\varphi^2 + \frac{4E}{l^2}rd\tilde{x}d\varphi, \\
\tilde{H} = -\frac{1}{2}d\varphi \wedge d\varphi + \tilde{d}r \wedge d\varphi \wedge \tilde{d}x + \tilde{N}dr \wedge d\varphi, \quad \tilde{d}z, \tag{5.12}
\]

\(^3\) We recall that in the Abelian case with \( U(1) \) duality group and another \( U(1) \) co-duality group, the target space is a one-dimensional circle; in other words, we have here the standard \( R \to 1/R \) duality [4, 5].
where $\tilde{x}$ is the dualized coordinate of the dual manifold. The $\beta = 1$ case of this background satisfies the field equations (2.2)-(2.4) with the same dilaton field $\Phi = \tilde{f} z + b$ provided that

$$E^2 = \frac{l^4}{4}, \quad N = 0, \quad \tilde{f}^2 = -\frac{\alpha'}{2l^2} + 2l^2\tilde{\Lambda} - 1, \quad (5.13)$$

for each $\alpha'$. We note that the background (5.12), unlike the dual background in the preceding case, can be even conformally invariant up to the zeroth order in $\alpha'$. Actually, by imposing the conditions $2E = -l^2$ and $N = 0$ on the dual background (5.12) and then by making use of the transformation $\tilde{x} \rightarrow l^2 t$, we obtain the same forms as (3.10) and (3.14), that is, the model is self-dual.

We can also show that the dual background is conformally invariant up to first order in $\alpha'$ when $N$ differs from zero. In this way, if we consider

$$E^2 = \frac{l^4}{4}, \quad N^2 = \frac{l^4(\beta - 1)^2}{2 - \beta}, \quad \alpha' = \frac{2l^2\beta(2 - \beta)}{6\beta^2 - 11\beta + 4},$$

$$\tilde{\Lambda} = \frac{1}{\beta l^2} \left( \frac{-6\beta^4 + 18\beta^3 - 18\beta^2 + 7\beta - 1}{6\beta^3 - 23\beta^2 + 26\beta - 8} \right), \quad (5.14)$$

then the field equations (2.2)-(2.4) for (5.12) are satisfied with a constant dilaton.

5.1.3 Dualizing with respect to coordinate $z$

In what follows we shall dualize the Gödel metrics on the coordinate $z$. To this end, we choose the coupling matrices in the following form

$$F^+_a(x) = \begin{pmatrix} 0 & -2Nr & 0 \\ 0 & 2Nr & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F^+_b(x) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad F_i = \begin{pmatrix} \tilde{x}^2 & 0 & 0 \\ 0 & \frac{1}{l^2} & l^2 \\ 0 & l^2 & \frac{1}{l^2} \end{pmatrix}, \quad (5.15)$$

and $E_{0a}^+ = l^2$. Then, by utilizing (4.5) one obtains the background of the model in the same forms as (3.10) and (3.14). The corresponding dual background can be cast in the form

$$ds^2 = \frac{l^2 dr^2}{r^2} - l^2 dt^2 + \frac{1}{l^2} d\tilde{x}^2 + ((\beta - 1)l^2 + \frac{4N^2}{l^2}) r^2 d\varphi^2 - \frac{4N}{l^2} r d\tilde{x} d\varphi - 2r l^2 dt d\varphi,$$

$$H = E dr \wedge d\varphi \wedge dt, \quad (5.16)$$

where $\tilde{x}$ is the dualized coordinate of the dual manifold. Analogously, this background along with a constant dilaton satisfy the field equations (2.2)-(2.4) with the following condition

$$E^2 = \frac{\beta^2 l^4}{\beta + 1}, \quad N^2 = \frac{l^4}{4}, \quad \alpha' = -(\frac{\beta + 1)^2}{3\beta}, \quad \tilde{\Lambda} = \frac{5 - 6\beta}{6l^2(\beta + 1)}. \quad (5.17)$$

Also, there is an additional solution with the dilaton field $\Phi = \tilde{f} \tilde{x} + b$ provided that

$$E^2 = \frac{l^4}{4}(4\beta - 3), \quad N = 0, \quad \alpha' = \frac{2\beta l^2}{5 - 6\beta}, \quad \tilde{f}^2 = \frac{2}{\beta} + 2(\tilde{\Lambda} l^2 + 1). \quad (5.18)$$

In addition to this, by taking into account the conditions $N = 0$ and $E = l^2$ one can show that under the transformation $t' = \tilde{x}/l^2$, $r' = r$, $\varphi' = \varphi$, $\tilde{x}' = l^2 z$, the background
(5.16) turns into (5.12). In this situation we are faced with self-duality again. Henceforth, we use the \((t', r', \varphi', \tilde{x}')\) as the coordinates of the background (5.16). It should be noted that here and hereafter one must use the coordinate transformations on both the metric and corresponding \(\tilde{B}\)-field to \(\tilde{H}\) of the models.

5.2 Abelian T-duality with two-dimensional Abelian Lie group

5.2.1 Dualizing with respect to both coordinates \((t, \varphi)\)

In this subsection we shall perform the dualizing on both the coordinates \(t\) and \(\varphi\). That is, the Lie group \(G\) of the target manifold \(\mathcal{M} \approx O \times G\) is parameterized by \(g = e^{tT_1}e^{\varphi T_2}\), wherein \((T_1, T_2)\) are the basis of the Abelian Lie algebra \(G\) of \(G\). So the coordinates of the orbit \(O\) are, in this case, represented by \((r, z)\). In this respect, one may choose the background matrices in the following forms

\[
E^+_{0\ ab} = \begin{pmatrix}
-\ell^2 & -r(2E^2 + \ell^2) \\
r(2E^2 - \ell^2) & (\beta - 1)\ell^2 r^2
\end{pmatrix}, \\
F^{+ (1)}_{\ a \ j} = \begin{pmatrix}
0 & 0 \\
0 & 2Nr
\end{pmatrix}, \\
F^{+ (2)}_{\ b \ i} = \begin{pmatrix}
\ell^2 & 0 \\
0 & 0
\end{pmatrix}, \\
F_{ij} = \begin{pmatrix}
\ell^2 r^2 & 0 \\
0 & r^2
\end{pmatrix}. 
\]

(5.19)

Hence, the background of the original \(\sigma\)-model implies the same metric (3.10) and field strength (3.19). The corresponding elements to the dual model can be obtained by making use of relations (5.19) and (4.11)-(4.14). They are then read

\[
ds^2 = \frac{\ell^2 dr^2}{r^2} + \frac{1}{4E^2 - \beta \ell^4} \left[ (\beta - 1)\ell^2 d\tilde{x}_1^2 - \frac{\ell^2}{r^2} d\tilde{x}_2^2 + (4E^2 - 4N^2 - \beta \ell^4)\ell^2 dz^2 + \frac{2\ell^2}{r} d\tilde{x}_1 d\tilde{x}_2 + 4N\ell^2 (d\tilde{x}_1 dz - \frac{1}{r} d\tilde{x}_2 dz) \right],
\]

\[
\tilde{H} = -\frac{E}{r^2 (4E^2 - \beta \ell^4)} d\tilde{x}_1 \wedge d\tilde{x}_2 \wedge dr,
\]

(5.20)

where \((\tilde{x}_1, \tilde{x}_2)\) are the dualized coordinates of the dual manifold. One can check that the dual background including (5.20) along with a constant dilaton field is conformally invariant up to the first order in \(\alpha'\) provided that

\[
E^2 = \frac{\ell^4}{4}, \\
N^2 = \frac{(1 - \beta)(\beta^2 + 2\beta - 3)\ell^4}{3\beta^2 + 7\beta - 6}, \\
\alpha' = \frac{2\beta(3\beta - 2)\ell^2}{2\beta^2 + \beta - 4},
\]

(5.21)

The above solution is valid for every \(\beta \in (0, 2/3)\). Moreover, by taking into consideration \(N = 0\) for both the backgrounds (5.16) and (5.20) we see that under the transformation

\[
t' = \sqrt{\frac{\beta - 1}{\beta \ell^4 - 4E^2}} \tilde{x}_1, \\
r' = \frac{1}{r\sqrt{(\beta - 1)(\beta \ell^4 - 4E^2)}}, \\
\varphi' = \tilde{x}_2, \\
\tilde{x}' = \ell^2 z,
\]

(5.22)

they can be turned into each other.
5.2.2 Dualizing with respect to both coordinates \((t, z)\)

When the dualizing is performed on the coordinates \((t, z)\), it is more appropriate to choose the spectator-dependent matrices as follows:

\[
E_{0\ ab}^+ = \begin{pmatrix} -l^2 & 0 \\ 0 & l^2 \end{pmatrix}, \quad F_{aj}^{+(1)} = \begin{pmatrix} 0 & -r(2E + l^2) \\ 0 & -2Nr \end{pmatrix}, \quad F_{ib}^{+(2)} = \begin{pmatrix} 0 \\ r(2E - l^2) \end{pmatrix}, \quad F_{ij} = \begin{pmatrix} \frac{l^2}{r^2} & 0 \\ 0 & -l^2 \end{pmatrix}.
\]

(5.23)

With regard to this choice, we arrive at familiar results, i.e., Eqs. (3.10) and (3.19), and thus one applies formulae (4.11)-(4.14) and (5.23) to obtain the dual background in the following form

\[
d\tilde{s}^2 = \frac{l^2 dr^2}{r^2} + \frac{1}{l^2} \left[ -d\tilde{x}_1^2 + d\tilde{x}_2^2 - (4E^2 - 4N^2 - \beta l^4)r^2 d\varphi^2 + 4r(E d\tilde{x}_1 d\varphi - Nd\tilde{x}_2 d\varphi) \right],
\]

\[\tilde{H} = -\frac{1}{2} d\tilde{x}_1 \wedge dr \wedge d\varphi.\]

(5.24)

Analogously, \((\tilde{x}_1, \tilde{x}_2)\) are the dualized coordinates. A constant dilaton field guarantees the conformal invariance of the dual background up to the first order in \(\alpha'\) if the constants \(E, N, l, \beta, \tilde{\Lambda}\) and \(\alpha'\) satisfy the following relations

\[
E^2 = 2l^4, \quad N^2 = \frac{l^4}{24} (\Theta - 7\beta + 50), \quad \alpha' = \frac{24\beta l^2}{7\beta - \Theta - 20},
\]

\[
\tilde{\Lambda} = \frac{-59\beta^2 + (236 + 5\Theta)\beta + 20\Theta - 110}{24\beta l^2(\Theta - 7\beta + 20)},
\]

(5.25)

where \(\Theta = \sqrt{\beta^2 - 16\beta + 100}\). When the constant \(N\) goes to zero and also \(E = l^2/2\) it is followed that under the transformation \(t' = -\tilde{x}_1/l^2, \ r' = r, \ \varphi' = \varphi, \ \tilde{x}_1' = \tilde{x}_2,\) the background (5.24) turns into (5.16).

5.2.3 Dualizing with respect to both coordinates \((\varphi, z)\)

It is also possible to perform the dualizing on both the coordinates \(\varphi\) and \(z\). In this way, both the metric (3.10) and field strength (3.19) may be yielded from the original \(\sigma\)-model if one considers

\[
E_{0\ ab}^+ = \begin{pmatrix} l^2 & -2Nr \\ 2Nr & (\beta - 1)r^2l^2 \end{pmatrix}, \quad F_{aj}^{+(1)} = \begin{pmatrix} 0 & 0 \\ 0 & r(2E - l^2) \end{pmatrix}, \quad F_{ib}^{+(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad F_{ij} = \begin{pmatrix} \frac{l^2}{r^2} & 0 \\ 0 & -l^2 \end{pmatrix}.
\]

(5.26)
Similar to the preceding cases one obtains the corresponding dual \( \sigma \)-model. If the dualized coordinates are considered to be \((\tilde{x}_1, \tilde{x}_2)\), then we have

\[
ds^2 = \frac{l^2 dr^2}{r^2} + \frac{1}{4N^2 + (\beta - 1)t^4} \left[ (\beta - 1)t^2 d\tilde{x}_1^2 + \frac{l^2}{r^2} d\tilde{x}_2^2 + (4E^2 - 4N^2 - \beta t^4)l^2 dt^2 \right. \\
- \left. 4l^2 (N d\tilde{x}_1 dt - \frac{E}{r} d\tilde{x}_2 dt) \right], \\
\hat{H} = \frac{1}{r^2 [4N^2 + (\beta - 1)t^4]} \left( \frac{l^4}{2} dr \wedge d\tilde{x}_2 \wedge dt - N dr \wedge d\tilde{x}_1 \wedge d\tilde{x}_2 \right). 
\]

Looking at the field equations (2.2)-(2.4) one verifies the conformal invariance conditions of the dual background (5.27) with a constant dilaton field. It’s worth mentioning that under the transformation \( t' = -t, \ r' = 1/(\beta - 1)t^2 \), \( \varphi' = \tilde{x}_2, \ \tilde{x}' = \tilde{x}_1 \), the background (5.27) can be reduced to (5.16); of course when \( N = 0 \) and \( E = l^2/2 \).

### 5.3 Abelian T-duality with three-dimensional Abelian Lie group

There is a possibility that we can perform the dualizing on the coordinates \( x^\mu = (z, t, \varphi) \). In fact, these are the coordinates of the Abelian Lie group of the target space. Choosing the appropriate spectator-dependent matrices in the forms

\[
E^+_{0 \ ab} = \begin{pmatrix} \frac{l^2}{r^2} & 0 & -2N r \\ 0 & -l^2 & -(l^2 + 2E)r \\ 2Nr & -(l^2 - 2E)r & (\beta - 1)t^2r^2 \end{pmatrix}, \quad F^{(1)}_{a j} = 0, \quad F^{(2)}_{ib} = 0, \quad F_{ij} = \frac{l^2}{r^2},
\]

one gets the coupling matrices of the original \( \sigma \)-model. Then, by making use of (4.5) the metric and field strength of the model are obtained to be of the same forms as (3.10) and (3.19), respectively. Accordingly, the dual background can be cast in the form

\[
ds^2 = \frac{l^2 dr^2}{r^2} - \frac{1}{4(E^2 - N^2) - \beta t^4} \left[ (\beta t^2 - \frac{4E^2}{l^2})d\tilde{x}_1^2 - (\beta - 1)t^2 + \frac{4N^2}{l^2} d\tilde{x}_2^2 \\
- \frac{l^2}{r^2} d\tilde{x}_3^2 + \frac{8E N}{l^2} d\tilde{x}_1 d\tilde{x}_2 - \frac{2l^2}{r} d\tilde{x}_2 d\tilde{x}_3 \right], \\
\hat{H} = -\frac{1}{r^2 (4(E^2 - N^2) - \beta t^4)} \left[ E dr \wedge d\tilde{x}_2 \wedge d\tilde{x}_1 - N dr \wedge d\tilde{x}_3 \wedge d\tilde{x}_1 \right],
\]

where \((\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)\) are the dualized coordinates. For the \( \beta = 1/2 \) case of the background (5.29), one can easily show that the field equations (2.2)-(2.4) are satisfied with a constant dilaton provided that

\[
E^2 = \frac{l^4}{8} - N^2, \quad \alpha' = 4l^2, \quad \Lambda = \frac{3}{8l^2}.
\]

Putting \( N = 0 \) in both the metric and corresponding \( B \)-field of the backgrounds (5.16) and (5.29), and then utilizing the transformation

\[
t' = \sqrt{\frac{\beta - 1}{\beta l^4 - 4E^2}} \tilde{x}_2, \quad r' = \frac{1}{r}, \quad \varphi' = \frac{\tilde{x}_3}{\sqrt{(\beta - 1)(\beta l^4 - 4E^2)}}, \quad \tilde{x}' = \frac{\tilde{x}}{l^2},
\]

- 17 –
one concludes that these backgrounds are equivalent together.

In summary, we obtained seven Abelian duals for the Gödel spacetimes and showed that they were equivalent together when \( N \) went to zero, and in some instances, one must consider \( E = l^2/2 \); moreover, the models were, under these particular conditions, self-dual.

Before closing this section, we note the fact that our Abelian dual models were obtained by the PL T-duality transformations at the classical level. Hence, the study carried out in this section raises the question whether the Abelian dual solutions can be obtained by making use of the \( \alpha' \)-corrected T-duality rules derived by KM [39]. In order to answer this question, we reobtain the dual background (5.16), where the dualizing is performed by a shift of the \( z \) coordinate, from KM T-duality rules (appendix A). In the next section, we will study the non-Abelian T-duality of the Gödel spacetimes using the PL T-duality approach.

6 Non-Abelian T-dualization of Gödel string cosmologies

We now wish to apply the discussion of Sec. 4 in order to study the non-Abelian T-dualization of the Gödel spacetimes. In this section, we explicitly construct some of the PL T-dual \( \sigma \)-models on \( 2+2 \)- and \( 3+1 \)-dimensional target manifolds. In any case, the metrics of the original \( \sigma \)-models represent the Gödel spacetimes. From the analysis of Sec. 3, it is followed that the backgrounds of the models are conformally invariant up to two-loop order such that they satisfy Eqs. (2.2)-(2.4). We clarify the corresponding dual spacetimes structure, as well as the conformal invariance conditions of them. In a particular case of the models, we study the PL self-duality of \( AdS_3 \times \mathbb{R} \) space.

6.1 Non-Abelian T-duality from a \( 2+2 \)-dimensional manifold with the \( A_2 \) Lie group

As discussed in Ref. [38], the \( 2+2 \)-dimensional manifold \( \mathcal{M} \approx O \times G \) with two-dimensional real non-Abelian Lie group \( G = A_2 \) is wealthy. We shall obtain the Gödel spacetimes from a T-dualizable \( \sigma \)-model constructed out of the \( 2+2 \)-dimensional manifold \( \mathcal{M} \approx O \times G \) where \( G \) acts freely on \( \mathcal{M} \), while \( O \) is the orbit of \( G \) in \( \mathcal{M} \). In this way, the non-Abelian T-duality of the Gödel spacetimes is studied here. The dual Lie group \( \tilde{G} \) acting freely on the dual manifold \( \tilde{\mathcal{M}} \approx O \times \tilde{G} \) is considered to be two-dimensional Abelian Lie group \( 2A_1 \). The Lie algebras of the Lie groups \( G \) and \( \tilde{G} \) are denoted by \( \mathcal{A}_2 \) and \( 2\mathcal{A}_1 \), respectively. As mentioned in Sec. 4, having Drinfeld doubles one can construct PL T-dual \( \sigma \)-models on them. The four-dimensional Lie algebra of the Drinfeld double \( (\mathcal{A}_2, 2\mathcal{A}_1) \) is defined by the following nonzero commutation relations:

\[
[T_1, T_2] = T_2, \quad [T_1, \tilde{T}_2] = -\tilde{T}_2, \quad [T_2, \tilde{T}_2] = \tilde{T}_1,
\]

(6.1)

where \( (T_1, T_2) \) and \( (\tilde{T}_1, \tilde{T}_2) \) are the basis of \( \mathcal{A}_2 \) and \( 2\mathcal{A}_1 \), respectively.

In order to write the action of the \( \sigma \)-models (4.5) and (4.10) on the \( 2+2 \)-dimensional manifolds \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) we need to calculate the components of the right-invariant Maurer-Cartan forms \( R^a_\pm \) and \( \tilde{R}_{\pm a} \) on the Lie groups \( \mathcal{A}_2 \) and \( 2\mathcal{A}_1 \), respectively. To this end, we
parametrize elements of the $A_2$ and $2A_1$ as

$$g = e^{x_1 T_1} e^{x_2 T_2}, \quad \tilde{g} = e^{\tilde{x}_1 \tilde{T}_1} e^{\tilde{x}_2 \tilde{T}_2},$$

(6.2)

where $(x_1, x_2)$ and $(\tilde{x}_1, \tilde{x}_2)$ stand for the coordinates of the Lie groups $A_2$ and $2A_1$, respectively. $R^4_\pm$’s and $\tilde{R}_{\pm a}$’s are then obtained to be of the forms

$$R^1_\pm = \partial_\pm x_1, \quad R^2_\pm = e^{x_1} \partial_\pm x_2,$$

(6.3)

$$\tilde{R}_{\pm 1} = \partial_\pm \tilde{x}_1, \quad \tilde{R}_{\pm 2} = \partial_\pm \tilde{x}_2.$$  

(6.4)

For our purpose it is also necessary to compute the matrices $\Pi^{ab}(g)$ and $\tilde{\Pi}_{ab}(\tilde{g})$. Using Eqs. (6.1) and (6.2) and also applying Eq. (4.4) for both the Lie groups $A_2$ and $2A_1$, we get

$$\Pi^{ab}(g) = 0, \quad \tilde{\Pi}_{ab}(\tilde{g}) = \begin{pmatrix} 0 & -\tilde{x}_2 \\ \tilde{x}_2 & 0 \end{pmatrix}. \quad (6.5)$$

Let us now choose the spectator-dependent background matrices as

$$E^{+}_{0 \cdot ab} = \begin{pmatrix} l^2 & 0 \\ 0 & l^2 (\beta - 1) \end{pmatrix}, \quad F_{a\cdot j}^{+1} = \begin{pmatrix} 0 & 0 \\ 0 & 2E - l^2 \end{pmatrix},$$

$$F^{+2}_{ib} = \begin{pmatrix} 0 & -(2E + l^2) \\ 0 & -2N \end{pmatrix}, \quad F_{ij} = \begin{pmatrix} -l^2 & 0 \\ 0 & l^2 \end{pmatrix}. \quad (6.6)$$

Because of the invertibility of the matrix $E^{+}_{0 \cdot ab}$, the parameter $\beta$ must be here considered to be different from one. Inserting the relations (6.6) into Eqs. (4.6)-(4.9) and then utilizing formulas (6.3) and (6.5) together with (4.5), the original $\sigma$-model is worked out to be

$$S = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ l^2 \partial_+ x_1 \partial_- x_1 + l^2 (\beta - 1) e^{2x_1} \partial_+ x_2 \partial_- x_2 + (2E - l^2) e^{x_1} \partial_+ x_2 \partial_- y_1 \\
- (2E + l^2) e^{x_1} \partial_+ y_1 \partial_- x_2 + 2N e^{x_1} \partial_+ x_2 \partial_- y_2 - \partial_+ y_2 \partial_- x_2 \right] - \frac{1}{4\pi} \int d\sigma^+ d\sigma^- R^{(2)} \Phi(X), \quad (6.7)$$

where $y^i = (y_1, y_2)$ are the coordinates of the orbit $O$ of $G$ in manifold $M$. To be more specific, we use the coordinate transformation

$$e^{x_1} = r, \quad x_2 = \varphi, \quad y_1 = t, \quad y_2 = z. \quad (6.8)$$

Then, by identifying the action (6.7) with the $\sigma$-model of the form (2.1), the corresponding line element and antisymmetric tensor can be, respectively, cast in the forms

$$ds^2 = l^2 \left[ -dt^2 + \frac{dr^2}{r^2} + (\beta - 1) r^2 d\varphi^2 - 2r dt d\varphi + dz^2 \right],$$

$$B = 2Erd\varphi \wedge dt + 2Nrd\varphi \wedge dz. \quad (6.9)$$

The line element (6.9) is nothing but the Gödel metric (3.10), and the field strength corresponding to the $B$-field (6.10) is easily obtained to be the same form as (3.19). As discussed
in Sec. 3 (class B of solutions), the metric (6.9) and field strength (3.19) together with a constant dilaton field satisfy the two-loop beta-function equations (2.2)-(2.4). Therefore, the $\sigma$-model (6.7) is conformally invariant up to the first order in $\alpha'$.

In order to construct the dual $\sigma$-model corresponding to the model (6.7) we use the action (4.10). The dual coupling matrices can be obtained by inserting (6.5) and (6.6) into (4.11)-(4.14). Accordingly, they are read

$$\tilde{E}^{ab} = \frac{1}{\Delta} \left( l^2(\beta - 1) \tilde{x}_2^2 - \tilde{x}_2^2 \right)$$

\[ \tilde{\phi}_{ij} = \left( -l^2 \left[ 1 - \frac{1}{2\Delta} \left( 4E^2 - l^4 \right) \right] \frac{2Nl^2}{2\Delta} \frac{2E + l^2}{l^2(1 + 4N^2)} \right) \]

\[ \tilde{\phi}_{j}^{(1)} = \frac{1}{\Delta} \left( \frac{2l^2(\beta - 1)\tilde{x}_2^2}{2\Delta l^2} \right) , \tilde{\phi}_{i}^{(2)} = \frac{1}{\Delta} \left( \frac{-(2E + l^2)\tilde{x}_2^2}{-2Nl^2} \right) , (6.11) \]

where $\Delta = \tilde{x}_2^2 + (\beta - 1)l^4$. Finally, one can write down the action of dual $\sigma$-model in the following form

$$\tilde{S} = \frac{1}{2} \int d\sigma^+ d\sigma^- \left\{ -l^2 \left( 1 - \frac{4E^2 - l^4}{\Delta} \right) \partial_+ t \partial_- t + l^2 \left( 1 + \frac{4N^2}{\Delta} \right) \partial_+ z \partial_- z \right. $$

\[ + \frac{1}{\Delta} \left[ l^2(\beta - 1) \partial_+ \tilde{x}_2 \partial_- \tilde{x}_2 + l^2 \partial_+ \tilde{x}_2 \partial_- \tilde{x}_2 + \tilde{x}_2 \partial_+ \tilde{x}_2 \partial_- \tilde{x}_2 - \partial_+ \tilde{x}_2 \partial_- \tilde{x}_2 \right] \]

\[ + 2N \tilde{x}_2 \partial_+ \tilde{x}_2 \partial_- z - \partial_+ z \partial_- \tilde{x}_2 + 2N l^2 \left( \partial_+ \tilde{x}_2 \partial_- z + \partial_+ z \partial_- \tilde{x}_2 \right) + \left( 2E - l^2 \right)(\tilde{x}_2 \partial_+ \tilde{x}_2 \partial_- t \]

\[ + l^2 \partial_+ \tilde{x}_2 \partial_- t + 2N l^2 \partial_+ \tilde{x}_2 \partial_- t + \left( 2E + l^2 \right)(-\tilde{x}_2 \partial_+ \partial_- \tilde{x}_2 + l^2 \partial_+ \partial_- \tilde{x}_2 + 2N l^2 \partial_+ \partial_- \tilde{x}_2) \right\} \]

\[ - \frac{1}{4\pi} \int d\sigma^+ d\sigma^- R^{(2)}(\tilde{X}). \] (6.12)

Comparing the above action with the $\sigma$-model action of the form (2.1), the line element and the tensor field $\tilde{B}$ take the following forms

$$d\tilde{s}^2 = \frac{1}{\Delta} \left[ -l^2(\tilde{x}_2^2 + \beta t^4 - 4E^2)dt^2 + l^2(\tilde{x}_2^2 + (\beta - 1)t^4 + 4N^2)dz^2 + l^2(\beta - 1)dx_1^2 \right. $$

\[ + l^2 \tilde{x}_2 dx_1 \partial_+ d\tilde{x}_2 + 2E l^2 dx_2 d\tilde{x}_1 \partial_+ dt + 4N l^2 dx_2 dz + 8E N l^2 dz dt \right] \]

\[ \tilde{B} = \frac{1}{\Delta} \left[ \tilde{x}_2 dx_1 \wedge d\tilde{x}_2 + 2E \tilde{x}_2 dx_1 \wedge dt + 2N \tilde{x}_2 dx_1 \wedge dz - l^2 d\tilde{x}_2 \wedge dt - 2N l^4 dz \wedge dt \right]. (6.14) \]

In order to enhance and clarify the structure of the dual spacetime, we can test whether there are true singularities by calculating the scalar curvature, which is

$$\tilde{R} = \frac{1}{2\beta l^2 \Delta^2} \left[ (1 - 4\beta)\tilde{x}_2^2 + \left( 4N l^2 (1 - 20\beta) + 2(\beta - 1)l^4 + 4E^2 \right) \tilde{x}_2^2 \right. $$

\[ - 32E^2 l^4(\beta - 1)^2 + 4N l^4(\beta - 1)(1 + 8\beta) + l^8(\beta - 1)^2(1 + 4\beta) \]. (6.15)\]

As it is seen from the formulas (6.13) and (6.15), the regions $\tilde{x}_2 = \pm l^2 \sqrt{(1 - \beta)}$ are true curvature singularities. Therefore, the dual metric has true singularities for the range $0 < \beta < 1$. The results indicate that the corresponding dual spacetime to the Gödel solutions allowing CTC’s has true singularities, since the metric (3.10) allows CTC’s for the range $0 < \beta < 1$ as mentioned in Sec. 3.
In order to investigate the conformal invariance conditions of the dual model up to the first order in \( \alpha' \), we look at the two-loop beta-function equations. Before proceeding to this, one easily gets that the field strength corresponding to the \( \tilde{B} \)-field (6.14) is

\[
\tilde{H} = \frac{1}{\Delta^2} \left[ \mathbb{E} (\tilde{x}_2^2 - (\beta - 1)l^4) dx_1 \wedge dx_2 \wedge dt + \mathbb{N}(\tilde{x}_2^2 - (\beta - 1)l^4) dx_1 \wedge dx_2 \wedge dz - 2\mathbb{N}l^4 \tilde{x}_2 dx_2 \wedge dt \wedge dz \right].
\]

(6.16)

Now by solving the field equations (2.2)-(2.4) for the metric (6.13) and the field strength (6.16) one concludes that there is no suitable dilaton field to satisfy these equations. In the next subsection, we study the non-Abelian T-duality of the Gödel spacetimes by making use of a T-dual \( \sigma \)-model constructing on a 3 + 1-dimensional manifold with the \( A_2 \oplus A_1 \) Lie group. It is then shown that the dual background can be conformally invariant up to the first order in \( \alpha' \).

### 6.2 Non-Abelian T-duality from a 3 + 1-dimensional manifold with the \( A_2 \oplus A_1 \) Lie group

There is a possibility that we can also get the Gödel spacetimes from a T-dualizable \( \sigma \)-model constructing on a 3 + 1-dimensional manifold \( \mathcal{M} \approx O \times G \), in which \( G \) is three-dimensional decomposable Lie group \( A_2 \oplus A_1 \) acting freely on \( \mathcal{M} \), while the orbit \( O \) is, here, a one-dimensional space with time coordinate \( y' = \{ t \} \). The dual model is constructed on manifold \( \mathcal{M} \approx O \times \tilde{G} \), where \( \tilde{G} \) is three-dimensional Abelian Lie group 3A1. The Lie algebra of the semi-Abelian double \( (A_2 \oplus A_1, 3A_1) \) is generated by the generators \( (T_a, \tilde{T}^a) \) with the following Lie brackets [52–54]

\[
[T_1, T_2] = T_2, \quad [T_1, \tilde{T}^2] = -\tilde{T}^2, \quad [T_2, \tilde{T}^2] = \tilde{T}^1, \quad [T_3, \cdot] = 0, \quad [\tilde{T}^1, \cdot] = 0, \quad [\tilde{T}^3, \cdot] = 0.
\]

(6.17)

Taking a convenient element of the Lie group \( A_2 \oplus A_1 \),

\[
g = e^{x_1 T_1} e^{x_2 T_2} e^{x_3 T_3},
\]

(6.18)

where \( (x_1, x_2, x_3) \) stand for the coordinates of \( A_2 \oplus A_1 \), we immediately find that \( R^1_\pm = \partial_\pm x_1 \), \( R^2_\pm = x_1 \partial_\pm x_2 \) and \( R^3_\pm = \partial_\pm x_3 \). In addition, we parameterize the Lie group 3A1 with coordinates \( (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \) so that its element is defined as in (6.18) by replacing untilded quantities with tilded ones. Hence, using (4.4) with and without tilded quantities we obtain

\[
\Pi^{ab}(g) = 0, \quad \tilde{\Pi}_{ab}(\tilde{g}) = \begin{pmatrix} 0 & -\tilde{x}_2 & 0 \\ \tilde{x}_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

(6.19)

The coupling matrices of the original \( \sigma \)-model are, here, obtained by considering the spectator-dependent background matrices

\[
F^{+}_{0,ab} = \begin{pmatrix} l^2 & s \\ -s (\beta - 1)l^2 & 2N \end{pmatrix}^T, \quad F^{+(1)}_{a,b} = \begin{pmatrix} 0 & 0 \\ 2N - l^2 & 0 \end{pmatrix}, \quad F^{+(2)}_{i,b} = \begin{pmatrix} 0 & -(2N + l^2) \\ 0 & 0 \end{pmatrix},
\]

(6.20)

\(^4\)Notice that the Lie algebra \( A_2 \oplus A_1 \) of the Lie group \( A_2 \oplus A_1 \) is isomorphic to the Lie algebra of Bianchi type III.
and $F_{ij} = -l^2$ for some constant $s$. Inserting (6.19) and (6.20) into Eqs. (4.6)-(4.9), and using (4.5), then we obtain the original $\sigma$-model in the following form

$$S = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ l^2 \partial_+ x_1 \partial_- x_1 + l^2 (\beta - 1) e^{2x_1} \partial_+ x_2 \partial_- x_2 + (2E - l^2) e^{x_1} \partial_+ x_2 \partial_- t 
- (2E + l^2) e^{x_1} \partial_+ t \partial_- x_2 + 2Ne^{x_1} (\partial_+ x_2 \partial_- x_3 - \partial_+ x_3 \partial_- x_2) 
- l^2 \partial_+ t \partial_- t + l^2 \partial_+ x_3 \partial_- x_3 + se^{x_1} (\partial_+ x_1 \partial_- x_2 - \partial_+ x_2 \partial_- x_1) \right]$$

$$- \frac{1}{4\pi} \int d\sigma^+ d\sigma^- R^{(2)} \Phi(X). \quad (6.21)$$

Utilizing the coordinate transformation $e^{x_1} = r, x_2 = \varphi, x_3 = z$, the metric of the model becomes the same as the Gödel metric (3.10), and $B$-field is given by

$$B = 2Erd\varphi \wedge dt + 2Nrd\varphi \wedge dz + sdr \wedge d\varphi, \quad (6.22)$$

such that the corresponding field strength takes the same form as (3.19). Hence the model (6.21) is conformally invariant up to the first order in $\alpha'$ as mentioned in Sec. 3, in such a way that the dilaton field is obtained to be constant. In order to obtain the dual $\sigma$-model for (6.21), we use the action (4.10). The dual coupling matrices can be obtained by inserting (6.19) and (6.20) into (4.11)-(4.14). Finally, the dual background is read

$$d\tilde{s}^2 = \frac{1}{\Delta} \left\{ -l^2 [(s - \tilde{x}_2)^2 + \beta t^4 + 4(N^2 - E^2)] dt^2 + \frac{1}{l^2} \left[ 4N^2 + t^4 (\beta - 1) \right] d\tilde{x}_1^2 
+ l^2 d\tilde{x}_2^2 + \frac{1}{l^2} \left[ (s - \tilde{x}_2)^2 + t^4 (\beta - 1) \right] d\tilde{x}_3^2 - 4Nl^2 d\tilde{x}_3 dt + 4E l^2 d\tilde{x}_2 dt 
+ 2l^2 (s - \tilde{x}_2) d\tilde{x}_1 dt + \frac{4}{l^2} N(s - \tilde{x}_2) d\tilde{x}_1 d\tilde{x}_3 \right\}, \quad (6.23)$$

$$\tilde{B} = \frac{1}{\Delta} \left\{ (\tilde{x}_2 - s) d\tilde{x}_1 \wedge d\tilde{x}_2 + 2E (\tilde{x}_2 - s) d\tilde{x}_1 \wedge dt - 2N d\tilde{x}_2 \wedge d\tilde{x}_3 
- t^4 d\tilde{x}_2 \wedge dt + 4En d\tilde{x}_3 \wedge dt \right\}, \quad (6.24)$$

where $\Delta = (s - \tilde{x}_2)^2 + 4N^2 + (\beta - 1) t^4$. Before proceeding to investigate the conformal invariance conditions of the dual model up to the first order in $\alpha'$, we get that the only nonzero components of the field strength corresponding to the $\tilde{B}$-field (6.24) are

$$\tilde{H}_{x_1 x_2 t} = \frac{E}{\Delta^2} \left[ (s - \tilde{x}_2)^2 - 4N^2 - (\beta - 1)t^4 \right], \quad \tilde{H}_{x_2 x_3 t} = \frac{4En(s - \tilde{x}_2)}{\Delta^2}. \quad (6.25)$$

Then, one verifies the field equations (2.2)-(2.4) for the metric (6.23) and the field strength (6.25) only with the following conditions

$$E^2 = \frac{t^4}{4}, \quad N = 0, \quad \beta = 1. \quad (6.26)$$

In this way, the resulting dilaton field is $\tilde{\Phi} = \tilde{f} \tilde{x}_3 + b$ for which $\tilde{f} = \pm (4t^4 \Lambda - 2l^2 - \alpha')^2 / \sqrt{2} l^3$ for each $\alpha'$. To enhance and clarify the structure of the dual spacetime we first impose the
condition (6.26) on the dual solution (Eqs. (6.23) and (6.24)), then by shifting \( \tilde{x}_2 \) to \( \tilde{x}_2 + s \) we get

\[
d\tilde{s}^2 = -l^2 dt^2 + \frac{1}{l^2} d\tilde{x}_2^2 + \frac{l^2}{x_2} d\tilde{x}_2^2 - \frac{2l^2}{x_2} (d\tilde{x}_1 dt - \frac{l^2}{x_2} d\tilde{x}_2 dt),
\]

(6.27)

\[
\tilde{B} = \frac{1}{\tilde{x}_2} \left[ d\tilde{x}_1 \wedge d\tilde{x}_2 + l^2 d\tilde{x}_1 \wedge dt - \frac{l^4}{\tilde{x}_2^2} d\tilde{x}_2 \wedge dt \right].
\]

(6.28)

Here we have used \( E = l^2/2 \). To have a better understanding of the structure of metric (6.27), we use the coordinate transformation

\[
\tilde{x}_1 = - l^2 \ln r + \varphi, \quad \tilde{x}_2 = \frac{1}{r}, \quad \tilde{x}_3 = l^2 z.
\]

(6.29)

Accordingly, the metric (6.27) turns into the case of \( \beta = 1 \) of the Gödel metric (3.10). Also, after performing the transformation (6.29) on the solutions, the only nonzero component of the field strength corresponding to the \( \tilde{B} \)-field (6.28) becomes \( \tilde{H}_{tr} = l^2/2 \). This result is in agreement with the case \((1')\) of the solutions of class A. Thus, we showed that the \( AdS_3 \times \mathbb{R} \) space does remain invariant under the non-Abelian T-duality transformation, that is, the model is PL self-dual. If one uses the condition (6.26) with \( E = -l^2/2 \), then, they will obtain similar results.

6.3 Non-Abelian T-duality from a 3+1-dimensional manifold with the \( SL(2, \mathbb{R}) \) Lie group

In the preceding subsection, we studied the non-Abelian T-duality of the Gödel spacetimes by applying the 3+1-dimensional manifold \( \mathcal{M} \approx O \times G \) with \( G = A_2 \oplus A_1 \). In addition, we can only derive the case of \( \beta = 1 \) of the Gödel metrics from a 3+1-dimensional manifold with the \( SL(2, \mathbb{R}) \) Lie group and then obtain the corresponding dual spacetime. In this way, both original and dual \( \sigma \)-models are constructed on semi-Abelian double \((sl(2, \mathbb{R}), 3A_1)\). The Lie algebra of double \((sl(2, \mathbb{R}), 3A_1)\) is defined by nonzero Lie brackets as [52–54],

\[
[T_1, T_2] = T_2, \quad [T_1, T_3] = - T_3, \quad [T_2, T_3] = 2T_1, \quad [T_1, \tilde{T}^2] = - \tilde{T}^2,
\]

\[
[T_2, \tilde{T}^3] = \tilde{T}^3, \quad [T_2, \tilde{T}^1] = - 2\tilde{T}^3, \quad [T_2, \tilde{T}^2] = \tilde{T}^1, \quad [T_3, \tilde{T}^3] = - \tilde{T}^1,
\]

\[
[T_3, \tilde{T}^1] = - 2\tilde{T}^2.
\]

(6.30)

where \((T_1, T_2, T_3)\) and \((\tilde{T}^1, \tilde{T}^2, \tilde{T}^3)\) are the basis of the Lie algebras \( sl(2, \mathbb{R}) \) and \( 3A_1 \), respectively. We note that the double \((sl(2, \mathbb{R}), 3A_1)\) has the vanishing trace in the adjoint representations. In such a situation, at the one-loop level a conformally invariant \( \sigma \)-model leads, under the PL duality, to a dual theory with the same property [50]. Now we parametrize an element of \( SL(2, \mathbb{R}) \) as

\[
g = e^{x_2 T_2} e^{x_1 T_1} e^{x_3 T_3},
\]

(6.31)

\footnote{Notice that the Gödel metric (3.10) with the condition \( \beta = 1 \) represents the \( AdS_3 \times \mathbb{R} \) space locally as shown in Eq. (3.13).}
where \((x_1, x_2, x_3)\) are the coordinates of \(SL(2, \mathbb{R})\). Then one gets the corresponding one-forms components in the following way

\[
R^1_\pm = \partial_\pm x_1 + 2e^{-x_1}x_2\partial_\pm x_3,
\]
\[
R^2_\pm = -x_2\partial_\pm x_1 + \partial_\pm x_2 - x_2^2e^{-x_1}\partial_\pm x_3,
\]
\[
R^3_\pm = e^{-x_1}\partial_\pm x_3.
\] (6.32)

Since the dual Lie group is considered to be Abelian, hence, by using (4.4) and (6.30) it is followed that \(\Pi(g) = 0\). Using the above results and also choosing the spectator-dependent matrices in the form

\[
E_{0 \, ab}^+ = \begin{pmatrix} \ell^2/4 & 0 & 0 \\ 0 & 0 & \ell^2/2 + E \\ 0 & \ell^2/2 - E & 0 \end{pmatrix}, \quad F_{a\, j}^{(1)} = 0, \quad F_{d\, b}^{(2)} = 0, \quad F_{ij} = \ell^2,
\] (6.33)

we obtain the background of the original \(\sigma\)-model. It is given by

\[
ds^2 = \ell^2 \left( \frac{1}{4} dx_1^2 + e^{-x_1} dx_2 dx_3 + dz^2 \right),
\] (6.34)
\[
B = E e^{-x_1} \left( -x_2 dx_1 \wedge dx_3 + dx_2 \wedge dx_3 \right),
\] (6.35)

where \(z\) stands for the coordinate of the orbit \(O\) of \(G\). The metric (6.34) can be written as \(ds^2 = ds^2_{AdS_3} + \ell^2 dz^2\). In order to get more insight of this metric one may use the following transformation

\[
e^{x_1/2} = \frac{1}{\rho}, \quad x_2 = \frac{1}{\ell}(\tau - x), \quad x_3 = -\ell(\tau + x),
\] (6.36)

then, the metric becomes the same as the \(AdS_3 \times \mathbb{R}\) space given by (3.13). As explained in Sec. 3, the metric (3.13) is locally equivalent to the case of \(\beta = 1\) of the Gödel metric (3.10). In addition, one easily gets that the field strength corresponding to the \(B\)-field (6.35) is zero. One immediately verifies the field equations (2.2)-(2.4) for the metric (6.34) and zero field strength together with the dilaton field \(\Phi = fz + b\) for which \(f = \pm (2\ell^2\Lambda - 3)^{1/2}\). Moreover, to satisfy the field equations we must have a coupling constant in the form of \(\alpha' = \ell^2\). Also, we have checked that the model constructed on the double \((sl(2, \mathbb{R}), 3A_1)\) can't be conformally invariant at the one-loop order. Therefore, according to Ref. [50] we don't expect to have a conformally invariant dual theory at the one-loop order.

In the same way, to construct out the dual \(\sigma\)-model on the manifold \(\tilde{M} \approx O \times \tilde{G}\) with the Lie group \(3A_1\) we parameterize the corresponding Lie group with coordinates \((\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)\) so that its element is defined as (6.31) by replacing untilded quantities with tilded ones. Utilizing relation (6.30) and also (4.4) for tilded quantities we get

\[
\tilde{\Pi}_{ab}(\tilde{g}) = \begin{pmatrix} 0 & -\tilde{x}_2 & \tilde{x}_3 \\ \tilde{x}_2 & 0 & -2\tilde{x}_1 \\ -\tilde{x}_3 & 2\tilde{x}_1 & 0 \end{pmatrix}.
\] (6.37)
Finally, by using (4.10) the metric and $\tilde{B}$-field of the dual model take the following forms

$$d\tilde{s}^2 = l^2 dz^2 + 4 \tilde{\Delta}' \left[ l^2 (d\tilde{x}'_1^2 + d\tilde{x}_2 d\tilde{x}_3) - \frac{4}{l^2} \left( 2\tilde{x}'_1 d\tilde{x}_1' + d(\tilde{x}_2 \tilde{x}_3) \right) \right]^2,$$

$$\tilde{B} = \frac{8}{\tilde{\Delta}} \left[ \tilde{x}_3 d\tilde{x}_1' \wedge d\tilde{x}_2 - \tilde{x}_2 d\tilde{x}_1' \wedge d\tilde{x}_3 + \tilde{x}_1' d\tilde{x}_2 \wedge d\tilde{x}_3 \right],$$

where $\tilde{\Delta} = l^4 - 16(\tilde{x}'_1^2 + \tilde{x}_2 \tilde{x}_3)$ and $\tilde{x}'_1 = E/2 - \tilde{x}_1$. For the $\tilde{B}$-field there is only a nonzero component of the field strength, obtaining

$$\tilde{H}_{\tilde{x}_1' \tilde{x}_2 \tilde{x}_3} = \frac{4}{\tilde{\Delta}} \left[ 3l^4 - 16(\tilde{x}'_1^2 + \tilde{x}_2 \tilde{x}_3) \right].$$

As mentioned above, since the background of the original $\sigma$-model doesn’t satisfy the one-loop beta-function equations, the dual background can’t be conformally invariant at the one-loop order.

7 Conclusion

We have obtained some new solutions for the field equations of bosonic string effective action up to first order in $\alpha'$, including the Gödel spacetimes, axion field and dilaton. Our results have shown that these solutions can be appropriate to study (non-)Abelian T-dualization of Gödel string cosmologies via PL T-duality approach. In studying Abelian duality of the Gödel spacetimes we have found seven dual models in a way that the models are constructed by one-, two- and three-dimensional Abelian Lie groups acting freely on the target space manifold. We have further shown that the Abelian T-dual models are, under some of the special conditions, self-dual. Most importantly, we have obtained the non-Abelian duals of the Gödel spacetimes. First, we have constructed the T-dual models on the four-dimensional manifold $\mathcal{M} \approx O \times G$ with two-dimensional non-Abelian Lie group and have shown that the metric of the dual model has true singularities for the range of $0 < \beta < 1$. In this case, the models are valid for all values of $\beta$ except for 1. Unfortunately, the dual model doesn’t satisfy the two-loop beta-function equations. We have then found other non-Abelian duals for the Gödel spacetimes by applying the $A_2 \oplus A_1$ and $SL(2, \mathbb{R})$ Lie groups. The dual model constructed on the semi-Abelian double $(A_2 \oplus A_1, 3A_1)$ is conformally invariant to the order $\alpha'$ only when the parameter $\beta$ goes to 1. In this way, it has shown that the case of $\beta = 1$ of the models as the $AdS_3 \times \mathbb{R}$ space is PL self-dual. Finally, we have shown that the model constructed by the double $(sl(2, \mathbb{R}), 3A_1)$ leads to the case of $\beta = 1$ of the Gödel metrics with zero field strength. Indeed, the model didn’t satisfy the one-loop beta-function equations. Because of the vanishing traces of the structure constants corresponding to the double $(sl(2, \mathbb{R}), 3A_1)$, the dual model couldn’t be also conformally invariant at the one-loop order.

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A The $\alpha'$-corrections from T-duality rules at two-loop

In this appendix, we shall show that the Abelian duals of the Gödel string cosmologies obtained by using the PL T-duality approach in Sec. 5 are in agreement with the T-duality rules at two-loop order derived from KM in [39]. As mentioned in the introduction section, the authors of Ref. [39] had obtained the two-loop $\sigma$-model corrections to the T-duality map in string theory by using the effective action approach. They had found the explicit rules at two-loop order derived from KM in [39]. As mentioned in the introduction section, they had obtained the two-loop $\sigma$-model corrections to the T-duality rules at two-loop order derived from KM in [39].

In this appendix, we shall show that the Abelian duals of the Gödel string cosmologies obtained by using the PL T-duality approach in Sec. 5 are in agreement with the T-duality rules at two-loop order derived from KM in [39].

In order to apply the T-duality rules of KM to our purpose, we will therefore first need to implement the field redefinitions to go from Hull and Townsend (HT) scheme [55] to that of KM. This work has been done in [55]. The field redefinitions that we will use are

\[
G_{MN}^{(HT)} = G_{MN}^{(KM)} + \alpha'(\mathcal{R}_{MN} - \frac{1}{2} H_{MN}^2),
\]

\[
B_{MN}^{(HT)} = B_{MN}^{(KM)} + \alpha'(\mathcal{H}_{MN} \nabla^P \Phi),
\]

\[
\Phi^{(HT)} = \Phi^{(KM)} + \alpha' \left( \frac{3}{32} H^2 + \frac{1}{8} \mathcal{R} - \frac{1}{2} (\nabla \Phi)^2 \right).
\]

Thus, the two-loop T-duality transformation equations in the KM scheme are given by [39]

\[
\tilde{\sigma} = -\sigma + \alpha'[(\nabla \sigma)^2 + \frac{1}{8}(e^{2\sigma} Z + e^{-2\sigma} T)],
\]

\[
\tilde{V}_\mu = W_\mu + \alpha'[W_{\mu\nu} \nabla^\nu \sigma + \frac{1}{4} h_{\mu\nu\rho} V^{\mu\nu} e^{2\sigma}],
\]

\[
\tilde{W}_\mu = V_\mu + \alpha'[V_{\mu\nu} \nabla^\nu \sigma - \frac{1}{4} h_{\mu\nu\rho} W^{\mu\nu} e^{-2\sigma}],
\]

\[
\tilde{b}_{\mu\nu} = b_{\mu\nu} - \alpha' \left[ V_{[\mu|\rho} W^{\rho]}_{\nu]} + (W_{[\mu|\rho} \nabla^\nu \sigma + \frac{1}{4} e^{2\sigma} h_{[\mu\rho\lambda]} V^{\rho\lambda}) V_{\nu]} \right]
\]

\[
+ (V_{[\mu|\rho} \nabla^\nu \sigma - \frac{1}{4} e^{-2\sigma} h_{[\mu\rho\lambda]} W^{\rho\lambda}) W_{\nu]} \right].
\]

\[\text{Notice that here we use the two-loop beta-function equations of the HT scheme which was used in [55].}\]
These transformations are written using the following definitions

\[ W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu, \quad (A.11) \]
\[ V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu, \quad (A.12) \]
\[ h_{\mu\nu\rho} = H_{\mu\nu\rho} - 3W_{[\mu\nu}V_{\rho]}, \quad (A.13) \]

In addition,

\[ Z_{\mu\nu} = V_{\mu\rho}V_\rho^\nu, \quad Z = Z_\mu^{\mu}, \quad (A.14) \]
\[ T_{\mu\nu} = W_{\mu\rho}W_\rho^\nu, \quad T = T_\mu^{\mu}. \quad (A.15) \]

All the lowering and raising of the indices will be done with respect to the reduced metric \( g_{\mu\nu} \). Below, we show that our result in the case of dualizing with respect to the \( z \) coordinate (carried out in subsection 5.1.3) can be also obtained by making use of the KM approach.

### A.1 Dualizing with respect to the \( z \) coordinate using the KM approach

Here we use the conditions for two-loop conformal invariance of the bosonic string \( \sigma \)-model in the HT scheme which was used in [55]. A solution for the two-loop beta-function equations in the HT scheme including the Gödel spacetimes and zero cosmological constant is given by

\[
\begin{align*}
 ds^{2}(HT) &= l^2\left[-dt^2 + \frac{dr^2}{r^2} + (\beta - 1)r^2\, d\varphi^2 - 2r\, dt d\varphi + dz^2\right], \\
 B^{(HT)} &= 4E r\, d\varphi \wedge dt, \\
 \Phi^{(HT)} &= \frac{1}{2}(fz + b),
\end{align*}
\]

where \( b \) is an arbitrary constant. The rest of the parameters obey the relation (5.18). The above solution is in agreement with ours in subsection 5.1.3. Now, one can use the field redefinitions in Eqs. (A.4)- (A.6) to write the solution (A.16) in the KM scheme. Then, comparing the obtained result with Eqs. (A.1)- (A.3) we obtain the components of the reduced metric \( g_{\mu\nu} \) as follows:

\[
\begin{align*}
 g^{(KM)}_{tt} &= -l^2 + \frac{\alpha'}{2\beta l^4}(8E^2 - l^4), \\
 g^{(KM)}_{rr} &= \frac{l^2}{r^2} - \frac{\alpha'}{2\beta l^4}(8E^2 - (2\beta - 1)l^4), \\
 g^{(KM)}_{t\varphi} &= -l^2 r + \frac{\alpha'}{2\beta l^4}(8E^2 - l^4)r, \\
 g^{(KM)}_{\varphi\varphi} &= (\beta - 1)l^2 r^2 - \frac{\alpha'}{2\beta l^4}(\beta - 1)(8E^2 - (2\beta + 1)l^4)r^2.
\end{align*}
\]

\[ - 27 - \]
Furthermore, the antisymmetric tensor field $b_{\mu \nu}$, the fields $V_\mu$ and $W_\mu$, the scalar $\sigma$ and dilaton $\Phi$ are, respectively, given by

\begin{align}
\mathcal{F}_{\mu \nu}^{(KM)} &= 4E r, \\
V_\mu &= W_\mu = 0, \\
\sigma^{(KM)} &= \frac{1}{2} \ln l^2, \\
\hat{\phi} &= \Phi^{(KM)} - \frac{1}{4} \ln l^2, \quad (A.18)
\end{align}

where

\begin{align}
\Phi^{(KM)} &= \frac{1}{2} (fz + b) - \frac{\alpha'}{16 \beta l^6} \left[ 36E^2 - (2\beta f^2 + 4\beta - 1)l^4 \right]. \quad (A.19)
\end{align}

Notice that the isometry we want to dualize is that the shift of the $z$ coordinate, i.e., $\bar{x} = z$.

Inserting Eqs. (A.18) and (A.19) into the two-loop T-duality transformation equations (A.7)- (A.10) one can get the dual solution in the following form

\begin{align}
\tilde{ds}^2 &= g^{(KM)}_{\mu \nu} dx^\mu dx^\nu + e^{2\tilde{\sigma}} d\tilde{x}^2, \\
\tilde{b}^{(KM)}_{\mu \nu} &= 4E r, \\
\tilde{V}_\mu &= \tilde{W}_\mu = 0, \\
\tilde{\Phi}^{(KM)} &= \frac{1}{2} (fz + b) - \frac{1}{2} \ln l^2 - \frac{\alpha'}{16 \beta l^6} \left[ 36E^2 - (2\beta f^2 + 4\beta - 1)l^4 \right]. \quad (A.20)
\end{align}

where

\begin{align}
\tilde{\sigma} &= -\sigma^{(KM)} = - \frac{1}{2} \ln l^2. \quad (A.21)
\end{align}

Finally, one can apply Eqs. (A.4)- (A.6) to write the dual solution (A.20) in the HT scheme. It is then read

\begin{align}
\tilde{ds}^2^{(HT)} &= l^2 \left[ -dt^2 + \frac{dr^2}{r^2} + (\beta - 1)r^2 d\varphi^2 - 2r dt d\varphi + d\varphi^2 \right] + O(\alpha'^2), \\
\tilde{B}^{(HT)} &= 4E r \, d\varphi \wedge dt, \\
\tilde{\Phi}^{(HT)} &= \frac{1}{2} (fz + b) - \frac{1}{2} \ln l^2 - \frac{\alpha'}{8l^2} (l^4 - 1)f^2 + O(\alpha'^2). \quad (A.22)
\end{align}

To go from the HT conventions [55] to those of ours one can send $\tilde{\Phi}^{(HT)} \rightarrow 2\Phi$ and $\tilde{H}^{(HT)} \rightarrow \frac{1}{2} H$. Then, the solution is nothing but the same result derived in subsection 5.1.3 when $N$ goes to zero.

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