A NOVEL APPROACH TO DISTURBANCE REJECTION IN CONSTRAINED MODEL PREDICTIVE CONTROL

By

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Abstract

This thesis is concerned with the rejection of time-varying disturbances in linear model predictive control of discrete-time systems. In the literature, disturbances are widely rejected by using velocity models, disturbance model with observer approach or a scheme that combines the compensation of a disturbance observer and the feedback regulation of MPC. Contrary to the widely used methods, the technique proposed in this research utilises the increment model of plants, with the assumption of fast-changing disturbances, to formulate a control law to reject the varying-disturbances. The uniqueness of the method stems from the compensation of the disturbance magnitude and rate of change. By proposing a cost function where the increment form of the system disturbance is taken as an optimisation variable, a control signal that is a function of a computed optimal disturbance increment is formulated to ensure that the plant is driven according to the minimisation of the cost function. The degree of freedom introduced by using the optimal disturbance in the control law was exploited to introduce the estimated disturbance increment into the control signal such that it is always in opposition to the external disturbance increment. Moreover, the proposed cost function provides a weighting matrix that can be used to manipulate the impacts of the exogenous disturbances on the response of the system. To estimate the unmeasurable disturbances, a combined state and disturbance observer is designed based on a convex optimisation stated in terms of an $\mathcal{H}_2$-minimisation problem. Simulations of three different systems are used to show the benefits of the proposed algorithm when compared with conventional offset-free MPC techniques. The results demonstrated that the proposed scheme can give significantly improved output tracking and regulation. Additionally, the simulation study showed that the novel technique is particularly more tolerant of actuator saturation.
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Notations

General

$x, y, z,...$ Constants or variables that may be scalar or vector-valued.

$x^*$ Optimal value of variable $x$ i.e. if $x^*$ denotes a feasible solution of the function $f(x)$, then $f(x^*) \leq f(x) \forall x$ in the feasible set.

$X \prec 0$ Matrix $X$ is a negative definite

$X \succ 0$ Matrix $X$ is a positive definite

$X \preceq 0$ Matrix $X$ is negative semi-definite

$X \succeq 0$ Matrix $X$ is positive semi-definite

$(\cdot)^T$ Transpose of vector or matrix $(\cdot)$

$\text{tr}(A)$ The trace of a square matrix $A$

$\star$ The entry $\star$ of a matrix denotes the transpose of the corresponding symmetric element

$I_n$ Identity matrix of size $n \times n$

$\mathbb{N}$ Set of natural numbers

$\mathbb{R}$ Set of real numbers

$\mathbb{R}^n$ Shows an $n$-dimensional real vector.

$\mathbb{R}^{n \times m}$ Shows a real matrix of size $n \times m$

$n_x$ The dimension of variable $x$ i.e. $n_x = \dim(x)$

$\|x\|_Q^2 = x^TQx$

$\mathcal{T}_{xy}$ Transfer function matrix from $y$ to $x$
\[ \Gamma_x \] The matrix on the left-hand side of the inequality constraint defined on variable \( x \)

\[ b_x \] The column vector on the right-hand side of the inequality constraint defined on variable \( x \)

\( A^{-1} \) The inverse of matrix \( A \)

\( \nabla f(x) \) Gradient of function \( f(x) \). That is, \( \nabla f(x) = (f'(x))^T \), where \( f'(x) \) denotes the first derivative of \( f(x) \). \( \hat{x} \) Denotes the estimate of variable \( x \).

\( \bar{x} \) Represents an augmentation of variable \( x \) over an horizon at any given time step \( k \). That is, \( \bar{x} = [x^T_t, x^T_{t+1}, \ldots, x^T_{t+n}]^T \), where \( k = \{t, \ldots, t+n\} \) and \( n \) may be equal to \( N \) or \( N_u - 1 \)

\( x_{ss} \) Represents the steady-state value of variable \( x \)

\( A \triangleq B \) The definition of \( A \) is \( B \)

**Specific**

\( x_k \) State of a dynamic system at time step \( k \)

\( u_k \) Manipulated variable or control input at time step \( k \)

\( z_k \) Dynamic system performance output (or controlled variable) at time step \( k \)

\( y_k \) Measurement output vector at time step \( k \)

\( r_k \) Reference point for the controlled variable \( z_k \) at sample \( k \)

\( d_k \) External time-changing input disturbance at time step \( k \)

\( e_k \) Error in controlled variable at sample \( k \), \( e_k = r_k - z_k \)

\( \mu_k \) Control input increment at time step \( k \), \( \mu_k = u_k - u_{k-1} \)

\( \psi_k \) Modified control input increment that is dependent on disturbance increment, \( \psi_k = \mu_k + \lambda_k \)

\( \lambda_k \) Component of \( \psi_k \) which incorporates optimal disturbance increment and the estimated disturbance increment

\( \delta_k \) External input disturbance increment at time step \( k \), \( \delta_k = d_k - d_{k-1} \)

\( \sigma_k \) Dynamic state increment at time step \( k \), \( \sigma_k = x_k - x_{k-1} \)

\( N \) Prediction horizon

\( N_u \) Control horizon

\( \zeta_k \) The estimated signal at time-step \( k \)
$u_k$ State of the dynamic input filter at time-step $k$

$w_k$ Filter external input disturbance at time-step $k$

$\xi_k$ State of the integrated disturbance and state estimator at time-step $k$

$\varepsilon_k$ Weighted estimated signal error, $\varepsilon_k = W(\zeta_k - \hat{\zeta}_k)$, where $W$ is a weighting matrix typically taken to be of a diagonal form

$T_s$ Sampling time (period)

$T_c$ Computational time of the optimal solver
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Chapter 1

Introduction

1.1 Overview

Model predictive control (MPC) has achieved enormous success in the control of industrial processes and has shown dominant influence in the direction of researches in the field of feedback control in recent years [71, 66]. The strategy has been widely studied [39, 30, 4, 64] and it is widely accepted because of its numerous merits, which include its ability to effectively control multi-input, multi-output systems, including those with non-minimum or unstable dynamics and long time-delays [10]. MPC also provides a systematic way of handling system constraints, which contributed to the quick acceptance of the control strategy by industry [31].

However, system disturbances are common and they tend to result in deteriorated tracking of reference points since the efficient performance of MPC is highly reliant on future prediction whose quality is dependent on the plant model. Hence, a model that is distorted as a result of input disturbances that are not considered in the controller design would inevitably impair the effective performance of MPC, which makes it expedient to consider disturbance information for systems subjected to external disturbances or model mismatch.

This thesis provides a novel approach to reject exogenous system disturbances affecting constrained linear time-invariant (LTI) systems. The proposed controller is also able to handle situations where there is a model mismatch because the velocity form of the linear models is used in the controller design. In this thesis, the external disturbances are assumed to be unmeasurable, making it expedient to employ an observer for the estimation of unmeasured disturbances. Therefore, a combined state and disturbance observer design that uses an anti-stable input filter is also proposed.

The rest of this chapter is structured as follows. The next section provides background to MPC as a receding horizon strategy. The motivation behind this research is given in Section 1.3 and
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Section 1.4 provides the thesis outline and the contribution given by each chapter.

1.2 Background

1.2.1 Model Predictive Control

MPC refers to a range of optimal control strategies that explicitly utilises plant model to obtain control signal by the minimisation of a penalty (cost) function. MPC has been widely accepted and employed in the process industry [66, 26, 36]. Although computational burden is recognized as a major shortcoming of MPC, it is now being used in fast systems such as wind turbines [7, 21, 70, 41] and aerospace applications [38, 73] due to the continuous advancements in microelectronics technology. MPC can be linear or non-linear depending on the type of models or constraints used in the implementation of the controller.

This work focuses on discrete-time LTI models. Hence, the state-space model of the form

\[ x_{k+1} = A_p x_k + B_p u_k, \]
\[ z_k = C_z x_k, \]

is generally considered, where \( x_k \in \mathbb{R}^{n_x} \) is the system state vector, \( u_k \in \mathbb{R}^{n_u} \) is the control vector, \( z_k \in \mathbb{R}^{n_z} \) is the controlled/manipulated output vector. \( A_p \in \mathbb{R}^{n_x \times n_x}, B_p \in \mathbb{R}^{n_x \times n_u} \) and \( C_z \in \mathbb{R}^{n_z \times n_x} \) are the system matrices. Fundamentally, the MPC problem is an infinite optimisation problem [60]. The conversion of this infinite horizon quadratic program (QP) to a finite horizon QP has been widely studied along with stability analysis [63, 40, 61]. Generally, the MPC formulation for the system (1.1) can then be represented by the finite convex optimisation problem

\[
\begin{align*}
\min_{u_t, \ldots, u_{t+N-1}} & \quad J = \frac{1}{2} \| x_{t+N} \|_S^2 + \frac{1}{2} \sum_{k=0}^{N-1} (\| x_{t+k} \|_Q^2 + \| u_{t+k} \|_R^2) \\
\text{Subject to:} & \\
& \quad x_{t+k+1} = A_p x_{t+k} + B_p u_{t+k} \\
& \quad x_k \in \mathbb{X} = \{ x : x_{\text{min}} \leq x \leq x_{\text{max}} \}, \forall k > 0 \\
& \quad z_k \in \mathbb{Z} = \{ z : z_{\text{min}} \leq z \leq z_{\text{max}} \}, \forall k > 0 \\
& \quad u_k \in \mathbb{U} = \{ u : u_{\text{min}} \leq u \leq u_{\text{max}} \}, \forall k > 0 \\
& \quad \mu_k \in \mathbb{V} = \{ \mu : \mu_{\text{min}} \leq \mu \leq \mu_{\text{max}} \}, \forall k > 0 \\
& \quad \mu_k = u_k - u_{k-1} \text{ is the control signal increment or rate and } N \in \mathbb{N} \text{ is the prediction horizon.}
\end{align*}
\]

The current state \( x_t \), is updated as time progresses and it is needed at the beginning of the prediction
The upper and lower bounds defined by the feasible sets $X, U$ and $V$ may be respectively implemented as hard constraints on the system states, inputs and input rates which must be fulfilled at all times. However, the constraints defined by set $Z$ on the outputs are generally considered as soft constraints in this thesis. A soft constraint imply that the boundary condition may be mildly violated when the solver finds it necessary. It has been argued [60] that soft output constraints are reasonable from an engineering point of view because the bounds defined by set $Z$ may only imply a desired range of operation. Furthermore, it is possible to implement a tighter constraint $T$ on the terminal state $x_{t+N}$ such that $T \subset X$. $Q \succeq 0$, $S \succeq 0$ and $R \succ 0$ are symmetric weighting matrices. The terminal state weighting matrix $S$, is usually taken as the solution to the discrete-time algebraic Riccati equation (DARE) given by

$$S = A^T S A - A^T S B (B^T S B + R)^{-1} B^T S A + Q,$$

(1.3)

to ensure nominal stability of the closed-loop system [15]. In general, the region defined by the constraints on the system is assumed to be convex and includes the origin as an interior-point so that it acts as a feasible stationary point for the system. The optimisation problem (1.2) is solved in every time step to determine the control input vector $\{u_t \cdots u_t+N-1\}$, which is applied to the system to obtain desired performance and this could be to drive the system to a outputs as close as possible to a set-point signal. In the strategy described above, the set-point must be known throughout the prediction horizon. The technique described, however, results in an open-loop control as no feedback is included. The disadvantage of such a control scheme is that if a change in the reference occurs or a disturbance affects the system as time progresses, the controller will not be able to account for them as the computed $N$ control moves have already been applied. Hence, the desired performance cannot be obtained. The solution to this problem is presented in the next subsection.

### 1.2.2 Receding Horizon Strategy

As already stated, the control signal is computed over the horizon and thus contains $u_t$, $u_{t+1}$, $\ldots$, $u_{t+N-1}$. The receding horizon principle dictates that only the first control sequence i.e $u_t$ is applied to the system. All other control inputs are discarded and the QP is re-solved to obtain another control sequence where only the first sample is also used. The repetition of this procedure is performed in real-time to give the receding horizon control law. The receding horizon control (RHC) results in a standard state feedback control law as detailed in [78]. Hence, the use of a receding horizon strategy in MPC guarantees closed-loop control using the open-loop plant model. The generic receding horizon control that has been described can be summarised as given in Algorithm 1.
**Algorithm 1**: Receding Horizon Control Strategy

1. Obtain the current system state $x_t$ either by measurement or observation.
2. Apply only the first optimal control input vector $u_t$ to the system (1.1).
3. Set $t \leftarrow t + 1$ and return to step 1.

### 1.2.3 Tracking Problem

In a control problem where output tracking is desired, the cost function (1.2) must be modified to achieve best tracking performance especially for non-zero references. The aim in tracking control is to formulate the MPC problem such that at steady state

$$u_k = u_{ss}, \quad y_k = y_{ss} \quad (1.4)$$

can be realised, where $u_{ss}$ and $y_{ss}$ are desired constant control and output, which will be non-zeros for non-zero set-points at steady state. Note that these steady-state values may vary depending on whether the set-point changes during the run-time. Hence, the objective function which would give a well-posed optimisation problem to achieve this goal (1.5) will be of the form

$$\min_{\mu_t, \ldots, \mu_{t+N-1}} J = \frac{1}{2} \|e_{t+N}\|^2_S + \frac{1}{2} \sum_{k=0}^{N-1} (\|e_{t+k}\|^2_Q + \|\mu_{t+k}\|^2_R) \quad (1.5)$$

where $e_k$ is the tracking error signal. Given the cost function (1.5), it is easy to see that (1.4) can be achieved since at steady-state, $\mu_k = 0$ and $e_k = 0$ are realisable without forcing either the control $u_k$ or output $y_k$ to zeros.

As already noted, in practical applications of control systems, model mismatch is almost inevitable and disturbances are common and they tend to result in deteriorated tracking of reference points since the efficient performance of MPC is highly reliant on future prediction whose quality is dependent on the plant model. These external disturbances may be measurable or unmeasurable disturbances. The approach used in handling the disturbances is dependent on the choice of the individual from a range of methods that can be used in MPC.

### 1.3 Motivation

In control systems design, feedback plays a salient role in the drive to cope with unavoidable mathematical modeling errors and to reduce the negative impacts of external disturbances. To aid the articulation of the motivation behind this research, consider an aircraft in flight. It is well known that turbulence and wind gust is detrimental to the performance of a controller designed
for its flight control when due consideration is not given to the external disturbances in the control system design. As a result, different control schemes have been proposed [69, 73, 74, 3] for effective closed-loop performance even in the presence of wind gust and/or uncertainties. Furthermore, physical limitations on the control input and its rate (which may even be further tightened due to a fault), the constraints on the states and outputs depending on the parameters of interest in the control design may also adversely affect the controller performance. Moreover, it is also important to consider performance objectives such as the minimisation of fuel consumption.

Motivated by the above application and other systems that may operate in similar conditions, this dissertation focuses on the control of linear time-invariant (LTI) systems which have all or some of the features that are summarized as follows:

- Varying external input disturbances
- Model mismatch
- Physical system parameter constraints
- Required performance objective

MPC provides a ‘natural’ way of handling system constraints and performance objectives since it relies on the online solution of an optimisation problem. Two well-known techniques for the rejection of constant or slowly varying disturbances and/or model mismatch are the so-called disturbance model-based MPC and the use of velocity (or increment) form of linear models in the formulation of MPC law. The fact that MPC, inherently, can be used to address the listed issues remain the author’s greatest motivation to investigate how existing disturbance rejection techniques can be improved in the presence of relatively fast-varying disturbances. Moreover, disturbances are not always malevolent to the control. However, the methods are effective when disturbance increment in every time step is negligible (i.e. the external disturbances are constant or slowly-changing). This limitation of the existing approaches raised the following important questions that this thesis aims to address:

- What role can a posed MPC optimization problem play in ensuring effective disturbance rejection?
- Is it possible for a control designer to influence (even to a limited extent) how unmeasured external disturbances affect a given system?
- How can an already existing algorithm be reformulated to improve on the rejection of time-varying disturbances?
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1.4 Delimitation

This thesis is restricted to discrete-time systems and MPC design, which implies that the control inputs are considered constants between sampling time. Specifically, linear time-invariant systems represented in state-space, generally, of the form (1.1) are considered. However, external disturbances \( d_k \) will be introduced into the system to alter the true dynamics of the system to test how well a controller can reject it. In doing this, measurement noise will not be considered in the investigation. Furthermore, the optimization problem formulation will be restricted to the use of a quadratic cost function and all constraints are assumed to be linear. This ensures that the QP is convex and a local minimum is indeed the global minimum. No attempt will be made to provide a theoretical stability analysis of the proposed controller as stability tests will rely solely on simulations. Moreover, stability analysis remains a challenge even in the established MPC algorithm.

1.5 Thesis Outline and Contributions Summary

Chapter 2 presents a review of the most commonly used linear MPC strategies for disturbance rejection, that is, the disturbance model and increment model approaches. The equivalence of the different techniques is discussed based on the results presented by [47] which showed that the increment form of MPC without output delay is not an alternative approach to disturbance rejection but a particular form of the disturbance model and observer. It was shown that the results do not exactly apply to the increment form with output delay in the augmented state. However, it was then proven that the increment form with output is equivalent to an alternative choice of the disturbance model and observer and this was validated using a simulation example. The simulation of the simple system was also employed to show that both increment forms (with and without output delay) may generally provide similar responses, which is not far-fetched since different disturbance models are now known to be equivalent.

In Chapter 3, the new method that is based on the use of increment models for effective disturbance rejection is presented. The augmentation with output delay is employed to form the augmented state. In contrast to the assumption [15] that the external disturbances are constant or slowly-varying, it is assumed that they may change quickly within the sampling period. Hence, the disturbance increment is non-zero and needs to be accounted for. To effectively reject this disturbance, a cost function which has disturbance increment as a variable is proposed. This makes it possible to influence the impact of the disturbance variation on the system response by chosen an appropriate weighting matrix. Furthermore, the optimisation problem is formulated in such a way to return an optimal disturbance and control increments. The additional freedom of control that is introduced by the optimal disturbance is exploited to include an estimate of the exogenous disturbance increment in the control signal such that the estimated disturbance increment is always
in opposition to the actual disturbance increment. Consequently, helping to reduce the effects of the external disturbances. Thereafter, the constrained optimisation problem is then formulated. In the second part of the chapter, an integrated state and disturbance observer with an anti-stable input filter is proposed for the estimation of unmeasured system states and disturbances. The observer design problem is synthesised as an $H_2$ minimisation problem, which can readily be solved to obtain the observer gain matrix.

Chapter 4 presents a comparative study between the proposed MPC in Chapter 3 and the previous approach of formulating increment form MPC (with output delay). In the comparative study, three different systems were considered with different input disturbance types and the integral of time-weighted absolute error was used as the performance measure to place emphasis on steady-state performance. The results showed that the proposed scheme can be superior to the conventional approach in the presence of the external disturbances and system constraints. More interestingly, the analysis of the time taken to solve the optimisation problem online showed that the proposed scheme did not lead to an increased computation cost despite the improved performances that were obtained.

Finally, Chapter 5 provides a discussion of the important features of the proposed controller of this thesis. An extensive concluding remark that discussed the important points from the literature review and the results obtained from the comparative study is then presented. The concluding remarks also attempted to answer the questions raised in the motivation behind this thesis and then, discussions of the interesting areas that may be considered in future researches are given.
Chapter 2

Disturbance Rejection in MPC: An Analytical Review

2.1 Introduction

In practical applications of control systems, disturbances are common and they tend to result in deteriorated tracking of reference points since the efficient performance of MPC is highly reliant on future prediction whose quality is dependent on the plant model. Hence, a model that is distorted as a result of input disturbances that are not considered in the controller design would inevitably impair the effective performance of MPC, which makes it expedient to consider disturbance information for systems subjected to external disturbances. These disturbances may be measurable or unmeasurable constant, slowly-varying or fast-changing disturbances.

In the area of handling measurable disturbances, [54] showed that Generalised Predictive Control (GPC) which implicitly incorporates disturbance compensation cannot always reject measurable disturbances even with accurate disturbance models and they proposed a tuning method to overcome this challenge. The proposed tuning method achieves desired reference tracking by using a reference filter and the method’s robustness was improved by implementing two degrees of freedom control strategy in a filtered Smith predictor based GPC. An MPC that uses Auto-regressive (AR) model for measurable disturbance prediction was proposed in [6]. The controller was tested on an olive oil mill and it was shown that disturbance rejection can be improved by the estimation technique. In [32], an MPC strategy that includes disturbance dynamics in the formulation of the MPC control problem was proposed and the developed controller was implemented on a permanent magnet synchronous motor (PMSM), where the measurable disturbance dynamics were included in the plant model. The proposed controller showed improvements in regulation over an integral
Unfortunately, it may be uneconomical or even technically impossible to measure disturbances in many applications. This may explain the large body of published works [42, 81, 52, 53, 18, 44, 20] on control schemes to improve controller performance in the presence of unmeasurable disturbances. In MPC strategy for systems represented in state-space, two approaches are generally used and they involve the use of disturbance model method and plant deviation model [18].

The first method involves the use of a disturbance model with an observer. A majority of the offset-free MPC schemes based on disturbance model achieves zero tracking error by the introduction of a constant output disturbance in the plant model [18, 13, 76]. The approach has been suggested in the control of a variety of systems subjected to unmeasured disturbances and these include the control of continuous stirred tank reactor [18, 79], quadruple-tank system [2], 3-φ inverter [25] and it was proposed in the efficient operation of an energy system [29]. General formulations of disturbance models for MPC with observers in linear state-space systems have been widely studied [43, 51, 37]. An analysis of the offset-free properties for square systems without integrating modes was first presented by [62], while [43] and [51] gave a more generalized analysis for the conditions of detectability of augmented systems.

However, the conventional approach of incorporating a constant output disturbance into the plant model fails to eliminate offset when the unmeasured disturbances access the process. To tackle this problem, [79] used the disturbance model approach and employed the Kalman filter to estimate the unmeasured disturbance and its impact on the output response. This made it possible to improve the dynamic matrix control (DMC) algorithm by using the feed-forward compensation strategy. Nonetheless, a major general challenge of offset-free MPC is that the controller may have to be specifically designed for a plant to achieve the desired performance coupled with the fact that it demands the separate design of a disturbance model.

The second method used to reject disturbances and to ensure offset-free output in the presence of model mismatch involves the use of velocity or increment form of linear models in the control law formulation [50, 77, 5]. In partial increment form, only the change in control input is used and the augmented state contains the actual system state and control signal. On the other hand, complete increment form utilizes the velocity form of both the input and states and the augmented model contains the state increment and system output. The aforementioned categorization does not include the approach [23] commonly used in robust MPC designs, where input and output increments are also used.

The use of disturbance models and increment form of MPC were considered to be completely different approaches to disturbance rejection until [47] presented an important result to show that the ‘convention’ complete increment model is indeed a particular form of the disturbance model and observer gains.

The rest of the review will be restricted to the aforementioned two methods of attaining off-set
2.2 DISTURBANCE MODEL AND OBSERVER METHOD

free control in the presence of external disturbances. The review will point out important results that have been presented in the literature, the salient issues to be considered in the design of the offset-free controllers and areas requiring further research, some of which this thesis aim to address, would be pointed out.

The remaining part of this chapter is organized as follows. In Section 2.2, the widely used disturbance model with observer will be discussed. In Section 2.3, the use of increment models will be presented - partial and complete schemes. Next, a discussion and analysis of the increment form equivalent disturbance model will be given in Section 2.4 and illustrative examples will be used to compare some of the discussed controllers in Section 2.5. Lastly, Section 2.6 will present a condensed summary of this literature review.

2.2 Disturbance Model and Observer Method

The disturbance model approach can be used to eliminate permanent offset by formulating it either as an input or output disturbance model. The output disturbance model is widely used in offset-free control in many industrial implementations of MPC; for instance, it is used in DMC [27, 12], QDMC [17] and IDCOM [72]. Consider a system affected by an external disturbance $d_k$ as follows

\[
x_{k+1} = A_p x_k + B_p u_k + B_d d_k,
\]
\[
y_k = C_y x_k + D_y d_k,
\]

where $y_k \in \mathbb{R}^{n_y}$ is the measurement vector, $d_k \in \mathbb{R}^{n_d}$ is an exogenous unmeasurable system disturbance, which may be constant or slowly-varying. $A_p \in \mathbb{R}^{n_x \times n_x}$, $B_p \in \mathbb{R}^{n_x \times n_u}$ are system and control input matrices respectively. $B_d \in \mathbb{R}^{n_x \times n_d}$ is the disturbance input matrix in the state equation, and $C_y \in \mathbb{R}^{n_y \times n_x}$ and $D_y \in \mathbb{R}^{n_y \times n_d}$ are respectively the states and disturbances measurement matrices, selected in such a way to collect all measurable states and disturbances. The pairs $(A_p, B_p)$ and $(C_y, A_p)$ are assumed to be respectively stabilisable and detectable. To implement the MPC with output disturbance model, the augmentation used by [51] is presented here. Let the augmented state be defined as $\tau_k \triangleq \begin{bmatrix} x_k \\ d_k \end{bmatrix}$. Then, the augmented system equation can be given as

\[
\tau_k = \tilde{A} \tau_{k-1} + \tilde{B} u_{k-1},
\]

and the corresponding augmented system matrices are given as follows:

\[
\tilde{A} \triangleq \begin{bmatrix} A_p & B_d \\ 0 & I \end{bmatrix}, \quad \tilde{B} \triangleq \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \quad \tilde{C} \triangleq \begin{bmatrix} C_y & D_y \end{bmatrix}.
\]
They showed [51] that for the system (2.2) to be detectable, the system (2.1) must be detectable and the condition
\[
\text{rank} \begin{bmatrix} I - A_p & -B_d \\ C & D \end{bmatrix} = n_x + n_d
\] (2.4)
must be satisfied. Moreover, the dimension of the disturbance that will guarantee that (2.4) holds is given as \( n_d \leq n_y \). This also guarantees the existence of the pair \((B_d, D_y)\), which is the disturbance model in principle. Note that the control designer selects the disturbance model matrices without any consideration to the actual disturbance affecting the system. The satisfaction of (2.4) is sufficient to obtain offset-free response in the presence of model mismatch and/or external disturbance provided that the number of integrating disturbances used in the plant augmentation is equal to the measurement outputs, that is \( n_d = n_y \) [51].

Since not all states may be measurable and the disturbance needs to be estimated, an observer is needed in general. To implement an observer to obtain \( \hat{\tau}_k \), the filtered estimate of \( \tau_k \), let \( L_o = [L_x L_e] \) be the gain of the observer. Where \( L_x \) is the gain associated with the state \( x_k \), and \( L_e \) is the observer gain associated with the external input disturbance \( d_k \). Then, the estimate \( \hat{\tau}_k \) of the augmented state is given by the observer
\[
\hat{\tau}_k = \dot{A}\hat{\tau}_{k-1} + \dot{B}u_{k-1} + L_o(y_k - \hat{y}_k),
\] (2.5)
where \( (y_k - \hat{y}_k) \) is the output prediction error in time-step \( k \). Although earlier studies [35, 43, 51, 46] have alluded to the fact that different disturbance models \((B_d, D_y)\) give different closed-loop performance when external disturbances are present, [57] presented a very interesting result that established the equivalence of different disturbance models.

[49] validated the results in [51] and demonstrated that if the disturbance model is added only to the desired output (which may not represent all measured vectors), this could result in a steady-state error and closed-loop instability when a mismatch exists between the plant and the model. Nevertheless, it is possible to obtain a gain \((L_x, L_e)\) that eliminates steady-state offset in this condition but the gain would be dependent on the parameters of the penalty function [51, 57]. Hence, such practice is discouraged because a change in the parameters of the controller’s cost function would imply a retuning of the observer.

### 2.2.1 Combined Offset-free MPC

In general, the design of disturbance model \((B_d, D_y)\) is usually separated from the observer design in offset-free MPC. However, the choice of disturbance models must ensure that (2.4) holds which implies that the augmented model (2.2) is detectable and consequently, establishes whether a stable estimator exists or not. Generally, different disturbance models can be used and design of models that guarantee offset-free control are discussed in [51, 43] and the procedure for the design of the
2.3. INCREMENT MODEL-BASED MPC

estimator are presented. For the first time, an innovative procedure for the simultaneous design of the disturbance model and observer gain was proposed by [48].

In the combined disturbance model and observer design scheme, the general augmentation given in (2.2) is employed and they considered the discrete-time linear state-space model of the form

\[
\begin{align*}
x_{k+1} &= A_P x_k + B_P u_k + B_q q_k \\
y_k &= C_y x_k + D_q q_k
\end{align*}
\]

(2.6)

where \( q_k \), represents unmeasured signals lumping the effects of all unmodelled disturbances and causes of mismatch between the model \((A_P, B_P, C_Y)\) and the real plant. If the real plant is given by

\[
\begin{align*}
x_{k+1} &= f(x_k, u_k, d_k) \\
y_k &= g(x_k, d_k)
\end{align*}
\]

(2.7)

one can recover the system (2.6) from the actual plant by defining: \( B_q q_k = f(x_k, u_k, d_k) - A_P x_k - B_P u_k \), and \( D_q q_k = g(x_k, d_k) - C_Y x_k \). It was assumed that \( \text{dim}(q) = n_x + n_y \) and

\[
B_q = \begin{bmatrix} I_{n_x} & 0 \end{bmatrix}, \quad D_q = \begin{bmatrix} 0 & I_{n_y} \end{bmatrix}.
\]

With the assumption that the plant model (2.2) is detectable, the design problem was formulated to determine the disturbance model \((B_d, D_q)\) along with the observer gain \( L_o \) such that the static observer (2.5) makes the error in the predicted output converge to null for any asymptotically constant lumped disturbance \( q \). The aim was achieved by synthesizing a dynamic observer for the nominal system and it was then shown that this is equivalent to an integrating disturbance model and a static observer gain \( L_o \) for the augmented model when offset-free control is desired. To minimize the effects of the unmeasured disturbances on the system predicted output, the dynamic observer was designed by solving an appropriate \( H_\infty \) problem. The effectiveness of the approach was validated by simulating the CSTR process. More recently, the approach was used [82] in the control of diesel engine where experimental validation of the approach was presented.

2.3 Increment Model-based MPC

Increment Model-based MPC or simply increment form MPC involves the use of the deviation of the system variable(s) to introduce integral action in the closed-loop control. This could be achieved either by using only the control input increment in the MPC formulation or by using the increments of both the state and the control. In this thesis, the former method is termed partial increment form while the latter is referred to as complete increment form.
CHAPTER 2. DISTURBANCE REJECTION IN MPC: AN ANALYTICAL REVIEW

2.3.1 Partial Increment Form

In the rejection of disturbance in this scheme, the external disturbance \( d_k \) given in (2.1) is rejected via an ‘indirect’ means. The augmented state is formed by augmenting the actual system state with \( u_k = u_{k-1} + \mu_k \) to obtain

\[
\begin{bmatrix}
  x_{k+1} \\
  u_k
\end{bmatrix} =
\begin{bmatrix}
  A_p & B_p \\
  0 & I
\end{bmatrix}
\begin{bmatrix}
  x_k \\
  u_{k-1}
\end{bmatrix} +
\begin{bmatrix}
  B_p \\
  I
\end{bmatrix} \mu_k,
\]

(2.8a)

\[
y_k =
\begin{bmatrix}
  C_y & 0
\end{bmatrix}
\begin{bmatrix}
  x_k \\
  u_{k-1}
\end{bmatrix}.
\]

(2.8b)

The optimisation problem for the system (2.8) is formed using the objective function (1.5) such that the output error is given as \( e_k = [C_y \ 0] [x_{k-1} - r_k] \) if one assumes that the controlled variable is equal to the measurement vector. For a tracking problem, the QP seemed to be well posed since at steady-state, the combination \( y = r \) (set-point) and \( \mu = 0 \) are possible. If an observer is implemented for the system (2.8), it is easy to conclude that there is no need to estimate \( \hat{u}_k \) since it is the immediate past implemented control signal that can easily be saved. However, [18] showed that, this however, causes the partial increment form not to achieve offset-free control when a disturbance enters the system. In the work, it was shown that by obtaining the control estimate \( \hat{u}_k \) and using it in the prediction model instead of the actual control \( u_k \), the permanent offset at steady state is eliminated. Hence, to guarantee the elimination of the constant disturbance \( d_k \) affecting the system using this approach, the estimate of the control \( \hat{u}_k \) must be used in the prediction equation instead of the actual control signal. In principle, this seem to be counter-intuitive because it is difficult to relate with estimating a variable that was implemented in the immediate past time-step and this could lead the controller designer to make a wrong decision. The reader may refer to [18] for more details on how the offset is eliminated in this scheme.

2.3.2 Complete Increment Form - Conventional Approach

In the complete increment form of linear models described in this section, the increments in both the inputs and states are used \([50, 51, 77, 18, 78, 49]\) and the augmented state contains the state deviation and output of the same time step. Here, this method is referred to as conventional approach because it is the widely adopted method. In general, the disturbances are eliminated by obtaining the increment models with the assumption that the disturbances are constant or slowly varying \([50, 49]\). For convenience, let the state increment for any time instant \( k > 0 \) be defined as
\( \sigma_k \triangleq x_k - x_{k-1} \) and the augmented state be given by
\[
\pi_k = \begin{bmatrix} \sigma_k \\ y_k \end{bmatrix} .
\] (2.9)

Then the nominal model can be written as
\[
\pi_k = \bar{A} \pi_{k-1} + \bar{B} \mu_{k-1}
\] (2.10a)
\[
y_k = C_y \sigma_k + y_{k-1} = C_y A_p \sigma_{k-1} + C_y B_p \mu_{k-1} + y_{k-1}
\] (2.10b)
where
\[
\bar{A} \triangleq \begin{bmatrix} A_p & 0 \\ C_y A_p & I \end{bmatrix}, \quad \bar{B} \triangleq \begin{bmatrix} B_p \\ C_y B_p \end{bmatrix}
\]
\[
\bar{C} \triangleq \begin{bmatrix} 0 & I \end{bmatrix}.
\]

As in the previous case, the QP is formed using the cost (1.5) with \( e_k = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \sigma_k \\ y_k \end{bmatrix} - r_k \). The optimization problem is solved such that at steady-state, the error \( e_k = 0 \) and control increment \( \mu_k = 0 \). The formulation guarantees zero tracking error even in the presence of disturbances provided that \( \mu_k = 0 \) and \( \sigma_k = 0 \) holds at steady-state. It is important to assertively state that this method eliminates offset even when no estimator is used in the presence of disturbance \( d_k \) unlike the partial increment form where the use of an estimator to obtain the estimate \( \hat{u}_k \) is a requirement to achieve offset-free steady-state when \( d_k \) is present.

In the next section, the increment form with output delay will be presented. To show that it guarantees offset-free control in the face of disturbance and model mismatch just like this conventional method, it will be good to introduce the conditions that establish the offset-free property of the conventional approach. To do this, the approach used in [18] is employed and it is based on the assumption that not all system states are measurable. To proceed, it is re-iterated that \( y_k \) is measured but the state increment \( \sigma_k \) may not be measurable. Hence, an observer is generally needed to obtain the estimate of the unmeasured components of \( \sigma_k \). Then, the observer to be designed for the model (2.10) has a gain matrix defined as
\[
L_v \triangleq \begin{bmatrix} L_\sigma \\ L_y \end{bmatrix},
\] (2.11)

which makes it convenient to construct a general observer equation as
\[
\begin{bmatrix} \hat{\sigma}_k \\ \hat{y}_k \end{bmatrix} = \begin{bmatrix} A_p & 0 \\ C_y A_p & I \end{bmatrix} \begin{bmatrix} \hat{\sigma}_{k-1} \\ \hat{y}_{k-1} \end{bmatrix} + \begin{bmatrix} B_p \\ C_y B_p \end{bmatrix} \mu_{k-1} + \begin{bmatrix} L_\sigma \\ L_y \end{bmatrix} [y_k - \hat{y}_k].
\] (2.12)
At steady-state, $y_k = y_{k-1}$ for a tracking problem. From equation \([2.12]\), if the condition $\mu_k = 0$ and $\sigma_k = 0 \ \forall \ k$, hold in the steady-state condition, it is obvious that $y_k = \hat{y}_k$ is achieved as long as either $L_\sigma$ or $L_y$ is of full-rank. The prediction of the system output by the controller will depend on the estimated output. Since one can guarantee that the estimate reaches the actual output at steady-state, the elimination of steady-state error by this scheme is guaranteed even in the presence of model mismatch and external disturbance.

Before wrapping up the discussion on this MPC scheme, it is pertinent to highlight the following important points:

- It is usual to employ a deadbeat observer for the observer gain matrix component that is associated with $y_k$ i.e. $L_y = I$, because it is measurable. Therefore, $y_k = \hat{y}_k$ holds for any time $k$.

- Offset-free control is guaranteed for the system because the estimate $\hat{y}_k$ reaches the actual plant output at steady-state $y_k$, which implies that the output prediction is unbiased at steady-state.

### 2.3.3 Complete Increment Form with Output Delay

In opposition to the conventional approach of formulating MPC using complete increment models, \([15]\) proposed a scheme where the current state deviation is augmented with the previous output to obtain the augmented system state. Hence, the prediction equation utilizes the previous output information. To proceed, let the augmented state be defined as $\varsigma_k \triangleq \begin{bmatrix} \sigma_k \\ y_k-1 \end{bmatrix}$. Then, one can write the equation of the augmented model in the compact form

$$\varsigma_k = \hat{A}\varsigma_{k-1} + \hat{B}\mu_{k-1}, \quad (2.13a)$$

$$y_k = \hat{C}\varsigma_k. \quad (2.13b)$$

The corresponding augmented system matrices are given as follows:

$$\hat{A} \triangleq \begin{bmatrix} A_y & 0 \\ C_y & I \end{bmatrix}, \quad \hat{B} \triangleq \begin{bmatrix} B_y \\ 0 \end{bmatrix}, \quad \hat{C} \triangleq \begin{bmatrix} C_y & I \end{bmatrix}.$$

The proof used to show that conventional increment form achieves offset-free control can readily be extended to this scheme. If an observer where to be designed for the model \([2.13]\), it must also be shown that the integral mode introduced by the increment model into the observer guarantees that the output estimate $\hat{y}_k$ attains the actual output of the system. To see this, let the observer
2.4. INCREMENT FORM EQUIVALENTS MODELS

Gain be $L_w = \begin{bmatrix} L_x \\ L_y \end{bmatrix}$ and it becomes convenient to write a general observer equation as

$$
\dot{\varsigma}_k = \tilde{A} \varsigma_k - 1 + \tilde{B} \mu_k - 1 + L_w (y_k - \hat{y}_k),
$$

(2.14a)

$$
\hat{y}_k = C \tilde{\sigma}_k + \hat{y}_{k-1}.
$$

(2.14b)

Given the condition that at steady-state $\sigma_k = 0$ and $\mu_k = 0$, and recalling that for a tracking problem, $y_k = y_{ss}$ and $\hat{y}_k = \hat{y}_{ss}$ $\forall$ $k$, where $y_{ss}$ and $\hat{y}_{ss}$ are respectively the true plant output and the estimated output at steady-state. The stationary observer relation can be written explicitly as

$$
\begin{bmatrix} 0 \\ \hat{y}_{ss} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{y}_{ss} \end{bmatrix} + \begin{bmatrix} L_s \\ L_y \end{bmatrix} \begin{bmatrix} y_{ss} - \begin{bmatrix} C_y & I \end{bmatrix} [0] \end{bmatrix},
$$

which implies $0 = L_s [y_{ss} - \hat{y}_{ss}]$ and $\hat{y}_{ss} = \hat{y}_{ss} + L_y [y_{ss} - \hat{y}_{ss}]$. Hence, if either $L_s$ or $L_y$ has a full rank, the following holds

$$
y_{ss} = \hat{y}_{ss}
$$

(2.15)

Therefore, an off-set free control is also ensured provided that the conditions $\sigma_k = 0$ and $\mu_k = 0$ are satisfied at steady-state. At this point, it is pertinent to mention that in practical applications, the integral modes introduced by the increment forms that were described above all cause the controlled system to lose its open-loop stability [18] and the usual way of achieving stability in MPC, which is by taking $N = \infty$ [63] leads to an unbounded objective function. However, this problem can be solved [65, 45] by introducing some constraint conditions and panic variables into the online optimization, which ensure that the integrating mode goes to zero at the end of the control horizon, $N_u$. By using this approach, closed-loop stability can be shown [18] to be guaranteed by the objective function (i.e the objective function can be shown to be a Liapunov function) and the offset-free property of the increment forms of MPC are preserved. An alternative approach to guarantee nominal stability of the closed-loop system is what has already been alluded to and it involves choosing a weighting matrix for the terminal state that is the solution of DARE given in (1.3).

2.4 Increment Form Equivalents Models

In the past, the use of disturbance models with observers and deviation models have been considered to be completely different techniques for disturbance rejection in MPC. The results presented by [47], however, showed that the conventional deviation model-based MPC is indeed a particular case of the disturbance model and observer approach. This made it clear that it is no longer appropriate to consider the methods as alternative techniques but ‘simply as particular choices of the general approach’ [47].
2.4.1 Motivation for the Comparative Study

In the design of increment models, the type described in Subsection 2.3.2 is widely used. This thesis aims to use the augmentation with output delay given in Subsection 2.3.3. Apparently, no significant difference exists between the formulations except that the previous output is used in the augmented state in the latter approach. However, the need to show that the results presented by [47] can also be extended to the increment form with output delay motivated this study. This is particularly important in order to assert that the increment form with output delay is part of the general approach.

In the following subsections, the important highlights of the findings given by [47] will be summarised and an attempt will then be made to extend the results to the velocity form with output delay.

2.4.2 Increment form Without Output Delay

Here, the particular disturbance model that is equivalent to the conventional increment form of offset-free MPC will be presented and it will be shown that this choice of disturbance model and observer gains preserve stabilizability and detectability of the augmented system observer.

**Theorem 2.4.1.** [49] The increment model (2.10) and the observer (2.12) with a stable output deadbeat observer gain such that $L_v = \begin{bmatrix} L_\sigma \\ \mathbf{I} \end{bmatrix}$, is equivalent to a specific form of of the disturbance model and observer gains given as

$$B_d = L_\sigma, \quad D_y = I - C_y L_\sigma, \quad L_x = L_\sigma, \quad L_e = I.$$  \hspace{1cm} (2.16)

In choosing a disturbance model and observer gains, it is pertinent to ensure that the detectability of the original system is preserved i.e the condition (2.4) is fulfilled. Then, it becomes essential to show that the disturbance model and observer gains (2.16) ensures that the condition holds true.

**Proposition 2.4.2.** [47] The choice of the disturbance model, $B_d = L_\sigma, D_y = I - C_y L_\sigma$ ensures that the detectability condition (2.4) holds, provided that $L_\sigma$ is chosen such that $(A_p - L_\sigma C_y A_p)$ is stable.

To proceed, it is necessary to establish that the choice of the disturbance model and observer gains (2.16) does not lead to loss of asymptotic stability of the augmented system. Indeed, the augmented systems (2.5) and (2.12) are stable as long as the choice of the observer gain $L_\sigma$ stabilizes the unaugmented system.

**Proposition 2.4.3.** [47] Consider the augmented system (2.2) and observer (2.5) with matrices given by (2.16), and the gain $L_x = L_\sigma$ is selected such that $(A_p - L_\sigma C_y A_p)$ is stable. Then, the augmented matrix $(\bar{A} - L_\sigma \bar{C} \bar{A})$ of the designed observer is stable.
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Proposition 2.4.4. Consider the augmented system \((2.10)\) and observer \((2.12)\) with matrices given by \((2.16)\), and the gain \(L_s\) is selected such that \((A_p - L_s C_y A_p)\) is stable. Then, the augmented observer matrix \((\hat{A} - L_s \hat{C} \hat{A})\) is stable.

Remark 1. The above results are very important as they guarantee that the stability of the augmented observer system matrices, \((\hat{A} - L_o \hat{C} \hat{A})\) and \((\hat{A} - L_o \hat{C} \hat{A})\), are solely dependent on the stability of unaugmented system gain matrix \((A_p - L_s C_y A_p)\). Hence, the choice of the disturbance model and observer gain does not impose eigenvalues that may cause the system to become unstable.

2.4.3 Increment Form with Output Delay

In this section, the relationship between the velocity form with output delay and the disturbance model and observer gains given by \((2.16)\) is investigated.

To proceed, using the gain matrix \(L_w = \left[ L_s^T \right]\), expand \((2.14a)\) to get \(\hat{\sigma}_k\) and substitute \((2.14b)\) into the result to obtain

\[
\hat{\sigma}_k = A_p \hat{\sigma}_{k-1} + B_p \mu_{k-1} + L_s (y_k - C \hat{\sigma}_k - \hat{y}_{k-1}). \tag{2.17}
\]

By noting that \(y_{k-1} = \hat{y}_{k-1}\) because of the deadbeat output observer and substituting the uncorrected estimate \(\hat{\sigma}_k = A_p \hat{\sigma}_{k-1} + B_p \mu_{k-1}\) into the left hand side of \((2.17)\) one obtains

\[
\hat{\sigma}_k = A_p \hat{\sigma}_{k-1} + B_p \mu_{k-1} + L_s (y_k - C_y A_p \hat{\sigma}_{k-1} - C_y B_p \mu_{k-1} - y_{k-1}),
\]

\[= (A_p - L_s C_y A_p) \hat{\sigma}_{k-1} + (B_p - L_s C_y B_p) \mu_{k-1} + L_s (y_k - y_{k-1}). \tag{2.18}
\]

Based on \((2.18)\), it is easy to hastily conclude that this form is also equivalent to \((2.16)\) provided that \(L_s = L_o\). However, this conclusion cannot be fully substantiated without showing that the augmented observer matrix \((\hat{A} - L_w \hat{C} \hat{A})\) is asymptotically stable. This can quickly be investigated as follows:

\[
(\hat{A} - L_w \hat{C} \hat{A}) = \begin{bmatrix} A_p & 0 \\ C_y & I \end{bmatrix} - \begin{bmatrix} L_s & C_y \\ I & I \end{bmatrix} \begin{bmatrix} A_p & 0 \\ C_y & I \end{bmatrix},
\]

\[= \begin{bmatrix} A_p & 0 \\ C_y & I \end{bmatrix} - \begin{bmatrix} L_s C_y A_p + L_o C_y & L_o \\ C_y A_p + C_y & I \end{bmatrix}, \tag{2.19}
\]

From \((2.19)\), it is indeed obvious that the eigenvalues of the observer gain matrix \((\hat{A} - L_w \hat{C} \hat{A})\) is not necessarily the same as that of the unaugmented matrix \((A_p - L_o C_y A_p)\). This implies that one cannot guarantee the stability of the augmented observer matrix \((\hat{A} - L_w \hat{C} \hat{A})\) by simply ensuring that the unaugmented system \((A_p - L_s C_y A_p)\) is stable. Hence, it would be inaccurate to
conclude that the increment form with output delay is equivalent to the disturbance model (2.16). Nonetheless, it is possible to show that an alternative choice of the disturbance model and observer is equivalent to this form of complete increment model.

**Theorem 2.4.5.** The increment model (2.13) and the observer (2.14) with a stable output deadbeat observer gain such that \( L_w = \begin{bmatrix} L_s \\ I \end{bmatrix} \), is equivalent to a specific form of the disturbance model and observer gains given as

\[
B_d = L_s, \quad D_y = I, \quad L_x = L_s, \quad L_e = I. \tag{2.20}
\]

**Proof.** To proceed, let (2.14a) be re-written in the form

\[
\hat{\varsigma}_k = \tilde{A}\hat{\varsigma}_{k-1} + \tilde{B}\mu_{k-1} + L_w(y_{k-1} - \hat{y}_{k-1}). \tag{2.21}
\]

By substituting \( \hat{y}_{k-1} = \hat{\mathcal{C}}\hat{\sigma}_{k-1} + y_{k-2} \) into (2.21), one can obtain the equation of the estimated state increment as

\[
\hat{\sigma}_k = A\hat{\sigma}_{k-1} + B_p\mu_{k-1} + L_s(y_{k-1} - \hat{\mathcal{C}}\hat{\sigma}_{k-1} - y_{k-2}) \tag{2.22}
\]

Following similar procedure, one can conveniently write (2.5) as

\[
\hat{\tau}_k = \tilde{A}\hat{\tau}_{k-1} + \tilde{B}u_{k-1} + L_o(y_{k-1} - \hat{y}_{k-1}), \tag{2.23}
\]

By expanding (2.23), the estimated state equation is given by

\[
\hat{x}_k = A_p\hat{x}_{k-1} + B_pu_{k-1} + L_s(y_{k-1} - C_y\hat{x}_{k-1}). \tag{2.24}
\]

If the above equation is re-written for the time step \( k - 1 \) and the resulting equation is then subtracted from (2.24), the following can be obtained

\[
\hat{\sigma}_k = (A_p - L_sC_y)\hat{\sigma}_{k-1} + B_p\mu_{k-1} + L_s(y_{k-1} - y_{k-2}). \tag{2.25}
\]

The comparison of (2.25) and (2.22) completes the proof.

It is pertinent to ensure that the detectability of the original system is preserved i.e, the condition (2.4) is fulfilled by the choice of disturbance model and observer gains. Hence, the author will now show that the disturbance model in (2.20) ensures that the condition holds true.

**Proposition 2.4.6.** The choice of the disturbance model, \( B_d = L_s, D_y = I \) ensures that the detectability condition (2.4) holds, provided that \( L_s \) is chosen such that \( (A_p - L_sC_y) \) is Hurwitz.
2.4. INCREMENT FORM EQUIVALENTS MODELS

**Proof.** To show that condition (2.4) is satisfied, consider the system

\[
\begin{bmatrix}
I - A_p & -L_{\sigma} \\
C_y & I
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

(2.26)

which is equivalent to the equations

\[
(I - A_p)x - L_s y = 0,
\]

(2.27a)

\[
C_y x + y = 0.
\]

(2.27b)

By solving (2.27b) for \(y\) and substituting the result into (2.27a), one obtains

\[
(A_p - L_s C_y - I)x = 0 \implies x = 0
\]

(2.28)

Equation (2.28) holds since \((A_p - L_s C_y)\) is assumed to be stable, which guarantees that \((A_p - L_s C_y - I)\) is invertible. Lastly, by substituting \(x = 0\) into (2.27b), one readily obtains \(y = 0\). Therefore, the system (2.26) has a unique solution \([x \ y] = [0 \ 0]\), which completes the proof.

It is also essential to show the conditions under which the augmented observers (2.21) and (2.23) are asymptotically stable given the disturbance model and observer (2.20).

**Proposition 2.4.7.** Consider the augmented system observer (2.21) with the gains \(L_y = I\) and \(L_s\) that is selected such that \((A_p - L_s C_y)\) is stable. Then, the augmented observer matrix \((\tilde{A} - L_w \tilde{C})\) is stable.

**Proof.** This can directly be shown by direct substitution and simplification as follows:

\[
(\tilde{A} - L_w \tilde{C}) = \begin{bmatrix}
A_p & 0 \\
C_y & I
\end{bmatrix} - \begin{bmatrix}
L_s \\
I
\end{bmatrix} \begin{bmatrix}
C_y & I
\end{bmatrix},
\]

(2.29)

It is obvious from (2.29) that the eigenvalues of the augmented observer matrix \((\tilde{A} - L_w \tilde{C})\) has the same eigenvalues as \((A_p - L_s C_y)\) and \(n_y\) zero eigenvalues at the origin. This, therefore, completes the proof since \((A_p - L_s C_y)\) is assumed to be stable.

**Proposition 2.4.8.** Consider the augmented system observer (2.23) with matrices given by (2.20), and the gain \(L_x = L_s\) is selected such that \((A_p - L_s C_y)\) is stable. Then, the augmented matrix \((\tilde{A} - L_o \tilde{C})\) of the designed observer is stable.
Proof. This can directly be shown as follows:

\[
(\ddot{A} - L_\omega \ddot{C}) = \begin{bmatrix} A_p & B_d \\ 0 & I \end{bmatrix} - \begin{bmatrix} L_x \\ L_e \end{bmatrix} \begin{bmatrix} C_y & D_y \end{bmatrix}.
\] (2.30)

Based on (2.20), the above becomes

\[
(\ddot{A} - L_\omega \ddot{C}) = \begin{bmatrix} A_p & L_s \\ 0 & I \end{bmatrix} - \begin{bmatrix} L_s & \end{bmatrix} \begin{bmatrix} C_y & I \end{bmatrix},
\]

\[
= \begin{bmatrix} A_p - L_s C_y & 0 \\ -C_y & 0 \end{bmatrix}.
\] (2.31)

It is obvious from (2.31) that the eigenvalues of the augmented observer matrix \((\ddot{A} - L_\omega \ddot{C})\) has the same eigenvalues as the unaugmented system \((A_p - L_s C_y)\) along with \(n_y\) eigenvalues at the origin. This completes the proof since \((A_p - L_s C_y)\) is assumed to be stable.

It has been shown, theoretically, that the two forms of complete increment models can be used to achieve offset-free control and that they both have different equivalent disturbance models that satisfy the conditions for offset-free control. It will be interesting to compare the performances of the complete increment form with output delay and its equivalent disturbance model with observer (2.20). The reader may refer to [47] to see the simulation result that was used to demonstrate the equivalence of the disturbance model (2.16) and the complete (conventional) increment form. However, in the comparison presented in this work, the author does not consider measurement noise. Since the proposed controller of this thesis that will be presented in a later chapter for improved disturbance rejection is based on deviation models, an illustrative example will also be presented to compare the performance of the two formulations of the complete increment model in the presence of external disturbance. In the next section, the simulation study will be presented to help in the final analysis of the findings of this review.

2.5 Illustrative Examples

As a means to effectively illustrate and summarise the findings of this literature review and also put the objective of this thesis into perspective, the simulation of a simple second-order system will be used in this simulation study. For the sake of simplicity, regulation problem will be considered in the examples.
2.5. ILLUSTRATIVE EXAMPLES

2.5.1 Example 1

The aim of this example is to validate using a simulated system, the Theorem 2.4.5 that was developed in this review. To achieve this, consider the discrete-time state-space model of a second-order system given by

\[
A_p = \begin{bmatrix}
0.9074 & 0.0899 \\
0.0899 & 0.8555
\end{bmatrix}, \quad B_p = \begin{bmatrix}
6.1827 & 0.3047 \\
0.3047 & 6.0068
\end{bmatrix},
\]

\[
B_d = \begin{bmatrix}
1 & 0 \\
0 & 1.5
\end{bmatrix}, \quad C_y = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

The outputs of the model which are the two states of the system are to be regulated at zero in the presence of unmeasured exogenous system disturbance \(d\), shown in Figure 2.1. Please note that \(d\) is a vector with \(n_d = 2\), but the two entries are equal to the disturbance given in the figure. The following MPC algorithms are compared:

- MPC-1 is the MPC algorithm based on disturbance model and observer matrices given in (2.20), where the identity matrices are \((2 \times 2)\) matrices. The observer gain \(L_s = L_x\) is designed such that the poles are placed at the location \((0.098, 0.079)\).

- MPC-2 is the complete increment form with output delay. The observer gain matrix \(L_s\) is as described above.
The prediction horizon is chosen to be \( N = 50 \) and the weighting matrices of the controllers are chosen as follows: \( Q = I_2 \), \( R = 0.1I_2 \) and \( S \) is taken as the solution to DARE (1.3). The result of the comparative study is presented in Figure 2.2 where MPC-1 and MPC-2 ensure the removal of permanent offset-set in the presence of the varying disturbances and there is hardly any difference between the responses given by both controllers. Both controllers gave the same root mean square (RMS) values for the outputs \( y_1 \) and \( y_2 \) which are 0.0193 and 0.0302 respectively.

### 2.5.2 Example 2

In this case, the same system used in Example 1 is also considered and the following controllers are compared:

- **MPC-1** is a none offset-free algorithm. Specifically, the partial increment form is used where the actual control \( u_k \) is used in the prediction model. Recall that to guarantee offset-free steady-state using this approach, the estimate \( \hat{u}_k \) should be used instead.

- **MPC-2** is the complete increment form with output delay.

- **MPC-3** is based on the conventional complete increment form without output delay.

The result of the comparative study is presented in Figure 2.3 and as expected, MPC-1 is unable to eliminate offset when the varying disturbance entered the system. MPC-2 and MPC-3, on the other hand, ensured the removal of permanent offset-set in the presence of the disturbance and the responses of both controllers is similar. Indeed, both controllers gave the same RMS values for the outputs \( y_1 \) and \( y_2 \) given as 0.0265 and 0.0397 respectively. Hence both methods can effectively be used to eliminate constant disturbances. The similar performance of MPC-2 and MPC-3 is not far-fetched even though both approaches lead to different choices of disturbance model and observer because it has been proven [57] that different disturbance models are equivalent.

### 2.6 Review Highlights and Concluding Remarks

The rejection of exogenous disturbances as well as coping with model mismatch in MPC can generally be done by using augmented states with disturbance models. In systems represented in state-space where the former approach is used, an output correction term is calculated based on the difference between the actual system output and the predicted output. Although the disturbance models used to achieve offset-free control can be modeled in different ways, the issue of what choice of disturbance model is more effective is no longer of interest to researchers because of the important results that were presented by [57] to show the equivalence of the different choices of the disturbance model.
2.6. REVIEW HIGHLIGHTS AND CONCLUDING REMARKS

Figure 2.2: Example 1: closed-loop output response of the two MPC formulations in the presence of external disturbance \( d \), confirms their equivalence. System outputs (top) and controls (bottom).
Figure 2.3: Example 2: closed-loop output response of the different MPC formulation in the presence of external disturbance $d$. System outputs (top) and controls (bottom).
The ‘so-called’ increment models are also used, where the velocity form of the system model is used to implement a control law in order to introduce integral action into the MPC algorithm. The partial increment form augments the system state with the control signal \( u_k = \mu_k + u_{k-1} \), to introduce integral action. However, the integral action introduced in this way cannot guarantee offset-free control when there is a model mismatch or external input disturbance. Hence, an observer is required to obtain the control signal estimate \( \hat{u}_k \) along with the state estimate \( \hat{x}_k \). Unlike the partial increment form, the complete increment forms guarantee the rejection of external disturbances (and plant mismatch) without using any estimator.

The complete increment form was considered an alternative form to disturbance rejection until [49] presented some results that showed that the conventional complete increment form is indeed equivalent to specific choices of the disturbance model and observer. This review extended the results to the increment form with output delay. It was shown that the increment form with output delay is also equivalent to a different choice of the disturbance models and observer and this was substantiated using a simulated system. Hence, the two forms of formulating the augmented system when complete velocity models are used in MPC are indeed particular choices of different disturbance models and observers. It can, therefore, be concluded that the complete increment forms are specific forms of the general approach. A simulation example was also used to show that, generally, the complete increment forms will give similar responses in the presence of an external input disturbance.

Furthermore, it is well known that the increment form of MPC results in an increased state dimension which increases the computational cost of MPC. However, it eliminates the need to compute the steady-state targets which remain a competitive advantage of the approach over the disturbance model. Moreover, the disturbance model approach dictates [43] that a separate design of the disturbance model is performed as well as imposing the need to estimate the disturbance states which are not required in the complete increment forms. Besides, a controller based on the disturbance model technique may need to be tuned or designed for a specific plant [51, 37]. Hence, the author opines that it is expedient for researchers to reconsider these approaches in terms of simplicity, superiority, and suitability for general applications.

Based on the approaches presented in this chapter, one can make an important conclusion that for offset-free control to be achieved in MPC, the optimisation problem has to be correctly formulated and the augmented system must be formed such that it ensures unbiased output prediction in the presence of external disturbances and model mismatch. To understand the concept of ensuring that the objective function is well-posed, assume that the cost (1.2) is used in conjunction with partial increment form. At steady-state, the values of \( u_k \) and \( x_k \) will become minimum (possibly zeros) which obviously are not desired if the system outputs were to track non-zero set-points.

It is re-iterated that the approaches described in this review are well suited only for constant or slowly varying disturbances. To address this issue, the complete increment forms seem to provide
a natural way of achieving this goal because one can readily assume that the disturbances are not constant, thereby leaving the dynamics of the disturbance increment in the controller formulation. If one can device a means to further minimize the impacts of the disturbance increment, it may lead to improved performance. Motivated by the benefit that could result from minimizing the impacts of the non-zero disturbance increment, this thesis aims to develop a novel approach to improve relatively fast disturbance rejection. To achieve this, the author notes that the way an optimization problem is posed plays a crucial role in the rejection of disturbances. The developed controller and a new estimator that uses an anti-stable input filter will be presented in the next chapter.
Chapter 3

Proposed Disturbance Rejection
MPC

3.1 Introduction

This chapter presents a novel approach to disturbance rejection in MPC based on complete increment form of plant models. In standard approaches of obtaining increment models, the disturbances are eliminated by obtaining the increment models with the assumption that the disturbances are constant or slowly varying \[14, 15, 49\] or left in the model with other assumptions such as the disturbance is a zero-mean white noise \[77\]. In reality, it is possible to have external disturbances that vary such that the disturbance increment is non-zero and this could pose serious challenges to the control of a system by MPC. Even though this increment may be small, it would become a serious problem as the prediction horizon increases due to the accumulation of the error introduced by the disturbance increment in the prediction model. Therefore, the distorted model would inevitably impair the effective performance of MPC, which makes it expedient to consider the disturbance increment information in the controller design.

Motivated by the need to address the challenge that could be posed by disturbances with non-zero increment in MPC based on increment models, a novel approach is proposed to handle the system disturbances in order to improve on the rejection of external disturbances which may be measured or estimated. Here, the increment form with output delay is used. However, it is important to assert that the proposed controller is not-discriminatory in the manner in which the increment models are augmented \(i.e\). it can also be used with the conventional complete increment form. The algorithm is based on a novel cost function that includes disturbance increment as an optimisation variable.
Furthermore, the modified cost function is used to create additional freedom of control by obtaining an 'optimal disturbance' which allows for the manipulation of the control signal to further improve system disturbance rejection using estimated or measured disturbances. In this dissertation, however, it is assumed that all disturbances are unmeasurable; therefore, a combined state and disturbance observer design that uses an anti-stable input filter is also proposed. The proposed dynamic observer is designed by solving an appropriate discrete-time $H_2$–optimisation problem with the aim of minimizing the effects of the exogenous, unmeasurable disturbances on the output predictions. The operation of the proposed controller is depicted in Figure 3.1 where the outputs of a model-based optimizer are the ‘optimal disturbance’ increment $\delta$ and optimal control increment $\mu$. Unlike the estimated states that are fed back into the model-based observer, the disturbance estimates are inputted into the control input calculator which gives a control signal $u$ that is a function of the two outputs from the optimizer and the (increment) disturbance estimates.

The remaining part of this chapter is organized as follows. In Section 3.2, the problem that is aimed at addressing is defined. In Section 3.3, the proposed algorithm is developed by deriving the needed constrained optimization problem that must be solved in every time step to obtain the required control. The formulation of all physical constraints on the controlled system into a single inequality is presented in Section 3.4. Section 3.5 presents the observer design with an anti-stable input filter and a summary of the proposed MPC algorithm is presented in Section 3.6.
3.2 Problem Definition

Consider a discrete-time linear time-invariant system governed by the state-space model

\[ x_{k+1} = A_p x_k + B_p u_k + B_d d_k, \]
\[ z_k = C_z x_k + D_z d_k, \]
\[ y_k = C_y x_k + D_y d_k, \]

(3.1)

where \( z_k \in \mathbb{R}^{n_z} \) is the controlled output vector. In Chapter 2, it was assumed that the measurement vector \( y_k \in \mathbb{R}^{n_y} \) also represents the controlled output just for the sake of simplicity. In general, this is not the case as more measurements may be obtained than one needs to control. Here, it is assumed that \( d_k \in \mathbb{R}^{n_d} \) is an exogenous unmeasurable system disturbance, which can be relatively fast-varying i.e. it has a non-zero increment. The matrix \( C_z \in \mathbb{R}^{n_z \times n_z} \) and \( D_z \in \mathbb{R}^{n_z \times n_d} \) are matrices of appropriate dimensions. The pairs \((A_p, B_p)\) and \((A_p, C_y)\) are assumed to be respectively stabilisable and detectable.

The objective of this chapter is to design an MPC using the velocity form of the system model (3.1) that reduces the impact of the unmeasurable input disturbances by minimizing a quadratic cost function of the form

\[
\min_{\hat{\mu}, \hat{\delta}, \hat{e}} J(\hat{\mu}, \hat{\delta}, \hat{e})
\]

subject to: \( (1.2c-1.2f) \),

(3.2a)

(3.2b)

where \( \hat{\mu}, \hat{\delta} \) and \( \hat{e} \) are respectively the control increment, disturbance increment and the output error defined over an entire horizon, which will be fully described in due course.

The inclusion of the disturbance term in the cost function makes it possible to introduce an additional degree of freedom for the control that can help in the manipulation of disturbance impacts. If at any time \( k \), \( \mu_k^* \) and \( \delta_k^* \) are the required optimal solution of the QP such that they both respectively represent the first component vectors of \( \hat{\mu} \) and \( \hat{\delta} \), the aim is to obtain a controller that utilizes both \( \mu_k^* \) and \( \delta_k^* \) to ensure that the system is driven according to the minimization problem. Through the additional control variable \( \delta_k^* \), an objective is to incorporate the disturbance estimate \( \hat{\delta}_k \) into the closed-loop control of the system to mitigate the negative consequences of the actual disturbance increment \( \delta_k \). Hence, the control input deviation in every iteration \( k \) can be defined by the function

\[
\psi_k = f(\mu_k^*, \delta_k^*, \hat{\delta}_k),
\]

(3.3)

where \( \psi_k \) is the modified control signal deviation from which the actual control signal \( u_k \) for the state-space model (3.1) can be obtained as

\[
u_k = \psi_k + u_{k-1}, \]

(3.4)
where \( u_{k-1} \) denotes the previous control signal which is initialised as zero, that is, \( u_{-1} = 0 \) for \( k = 0 \).

Since the proposed method of this note relies on the information of the disturbances, the second task is to introduce an optimal observer to estimate the states that are not measured and unmeasurable disturbances by constructing an estimated signal of the form

\[
\zeta_k = Ex_k + Fd_k,
\]

where \( \zeta_k \in \mathbb{R}^{n_e} \) is the signal to be estimated. \( E \) and \( F \) are chosen in such a way that they collect all system states and disturbances to be estimated. To characterise the input disturbances to obtain a good estimate, an input filter whose dynamics is to be constructed as

\[
\begin{align*}
v_{k+1} &= A_i v_k + B_i w_k, \\
d_k &= C_i v_k + D_i w_k,
\end{align*}
\]

is proposed, where \( v_k \in \mathbb{R}^{n_i} \) denotes the state vector of the input filter and \( w_k \in \mathbb{R}^{n_w} \) is an external disturbance input. The filter dynamics does not necessarily have to be stable and by merging the stable dynamics of the filter to that of the controlled plant, the unstable part can always be left as exo-system. Furthermore, by formulating an appropriate dynamic estimator, an observer-based solution is desired such that its gain \( L \) is obtained from the solution of an \( \mathcal{H}_2 \) minimization problem.

### 3.3 Novel MPC Design

For convenience, we re-write the plant model (3.1) to implement the proposed MPC as follows:

\[
\begin{align*}
x_{k+1} &= A_p x_k + B_p u_k + B_d d_k, \\
z_k &= C_z x_k + D_z d_k.
\end{align*}
\]

The use of deviation models in MPC algorithm formulation is mainly due to the fact that it helps eliminate constant disturbances or disturbances that can be approximated as constant in addition to the introduction of integral action. Contrary to the assumption in the literature that the input disturbance \( d_k \) is constant such that its deviation \( \delta_k = 0 \), it is assumed that the deviation of the disturbance \( \delta_k \neq 0 \), which means that \( d_k \) may be relatively fast varying with time. This assumption enables the use of either measured or estimated disturbances in the control law formulation. Given the plant (3.7), let the increment form of the disturbance vector be defined as

\[
\delta_k \triangleq d_k - d_{k-1}.
\]
The deviation models of the state and output equation of (3.7) can be written as

\[ \sigma_{k+1} = A_p \sigma_k + B_p \mu_k + B_d \delta_k, \]
\[ z_k = C_z \sigma_k + z_{k-1} + D_z \delta_k. \]  

Equation (3.9) can be written more elegantly as an augmented system given below.

\[
\begin{bmatrix}
\sigma_{k+1} \\
z_k
\end{bmatrix} =
\begin{bmatrix}
A_p & 0 \\
C_z & I
\end{bmatrix}
\begin{bmatrix}
\sigma_k \\
z_{k-1}
\end{bmatrix} +
\begin{bmatrix}
B_p \\
0
\end{bmatrix} \mu_k +
\begin{bmatrix}
B_d \\
D_z
\end{bmatrix} \delta_k,
\]

\[
z_k =
\begin{bmatrix}
C_z \\
I
\end{bmatrix}
\begin{bmatrix}
\sigma_k \\
z_{k-1}
\end{bmatrix} +
\begin{bmatrix}
0
\end{bmatrix} \delta_k.
\]

Hence, in a compact form, one can obtain the augmented system state-space model using the definitions which have been given as follows:

\[ \varsigma_{k+1} = \tilde{A} \varsigma_k + \tilde{B} \mu_k + \tilde{B}_d \delta_k \]
\[ z_k = \tilde{C} \varsigma_k + \tilde{D} \delta_k. \]  

In conventional methods of formulating MPC, the term \( \delta_k \) in the augmented system (3.12) is seen as a disturbance which needs to be eliminated. However, it can be argued that not all disturbances are malevolent to the control of a system; in fact, they may sometimes be helpful to the control. For instance, if the disturbance vector \( \delta_k \) affects the system in such a way that it drives the system states \( \varsigma_{k+1} \) in the same direction as the control signal \( \mu_k \), it will reduce the control effort required to drive the states to the desired reference.

As previously noted, MPC based on linear velocity model is lucrative because of its simplicity and effectiveness in eliminating constant or slowly changing disturbances. However, if one needs to design an MPC based on velocity models that utilises disturbance measurements or estimations to further improve the performance in the presence of persistent varying disturbances, there will be a need to use a modified algorithm such that the disturbance increment is not eliminated by assuming that \( \delta_k \neq 0 \). As a way to mitigate the impacts of \( \delta_k \) when it is significant, to the best of the author’s knowledge, no attempt has been made to utilize a control input that is a function of an ‘optimal disturbance’, which is obtained by introducing an additional term into the optimization variable to mimic the system disturbance.

In this thesis, the disturbance increment \( \delta_k \) is used to the control’s advantage by considering its manipulated form as a control variable which when properly formulated, can give an additional tuning parameter to improve the controller’s performance in the presence of exogenous disturbances.
Therefore, the addition of the system disturbance increment $\delta_k$ into the cost function of MPC is proposed to enable the implementation of the new scheme and the modified cost function is given as

$$
J = \frac{1}{2} \| e_{t+N} \|_S^2 + \frac{1}{2} \sum_{k=0}^{N-1} \| e_{t+k} \|_Q^2 + \frac{1}{2} \sum_{k=0}^{N_u-1} (\| \mu_{t+k} \|_R^2 + \| \delta_{t+k} \|_P^2) + \frac{1}{2} \rho_1 \epsilon^2 + \rho_2 \epsilon,
$$

(3.13)

where the tracking error signal, $e_k \triangleq r_k - z_k$ and $r$ is the reference signal, $P \succ 0$ is a symmetric weighting matrix and $N_u \in \mathbb{N}$ is the control horizon. The parameter $\epsilon > 0$, a panic variable, is introduced to facilitate the implementation of soft-output constraints as this is reasonable from an engineering point of view because they may only imply a desired range of operation and $\rho_1$, $\rho_2$ are positive weighting scalars of $\epsilon$.

Remark 2. In the form given above, the cost function would be minimized with respect to both $\mu_k$ and $\delta_k$ since the contribution of $\epsilon$ is almost always negligible. Consequently, the optimal solution of the cost function $J$ may not give $\mu_k^*$ that can be used to obtain the control signal as $u_k = \mu_k^* + u_{k-1}$ that would drive the system (3.7) to the desired state. Hence, it is necessary to device a means to use both $\mu_k^*$ and $\delta_k^*$ to achieve the desired control. It is pertinent to emphasize here that there is no control over the actual system disturbance $\delta_k$ but it is included in the cost function for two main reasons. First, to create a weighting parameter that can be used to adjust, to an extent, the manner in which the disturbance affects the system’s response. Secondly, and more importantly, to introduce a freedom of control that can be manipulated to help improve on the rejection of the external disturbance increment form $\delta_k$.

Remark 3. If the optimisation problem needs to be solved by a solver, the exogenous disturbance increment $\delta_k$ is used for warm start by the solver along with the previously computed control increment. Despite the fact that the previous computed optimal disturbance increment is not used, a global minimum will always be guaranteed because the optimisation problem is convex. Hence, the control $\mu_k^*$ and disturbance $\delta_k^*$ that minimises the cost function can always be obtained.
3.3. NOVEL MPC DESIGN

3.3.1 Prediction Models

The future prediction of system states and outputs is crucial for the effective performance of MPC. The future states of the system can be obtained by using the computed control sequence, given the information of the current state. For convenience, let the following augmented vectors be defined:

\[
\bar{\mu} \triangleq [\mu^T_t, \mu^T_{t+1}, \ldots, \mu^T_{t+N_u-1}]^T, \\
\bar{\varsigma} \triangleq [\varsigma^T_{t+1}, \varsigma^T_{t+2}, \ldots, \varsigma^T_{t+N}]^T, \\
\bar{z} \triangleq [z^T_{t+1}, z^T_{t+2}, \ldots, z^T_{t+N}]^T, \\
\bar{\delta} \triangleq [\delta^T_t, \delta^T_{t+1}, \ldots, \delta^T_{t+N_u-1}]^T.
\] (3.14)

Given the prediction horizon \(N\), control horizon \(N_u\), the current state \(\varsigma_t\) and the disturbance increment \(\delta_k\) where \(k \in \{t, t+1, \ldots, t+N\}\), the augmented state equation (3.12) can be recursively solved as

\[
\varsigma_{t+1} = \tilde{A}\varsigma_t + \tilde{B}\mu_t + \tilde{B}_d\delta_t, \\
\varsigma_{t+2} = \tilde{A}^2\varsigma_t + \tilde{A}\tilde{B}\mu_t + \tilde{A}\tilde{B}_d\mu_{t+1} + \tilde{A}\tilde{B}_d\delta_t + \tilde{B}_d\delta_{t+1}, \\
\vdots
\]

\[
\varsigma_{t+N} = \tilde{A}^N\varsigma_t + \begin{bmatrix} \tilde{A}^{N-1}\tilde{B} & \tilde{A}^{N-2}\tilde{B} & \ldots & \tilde{A}\tilde{B} & \tilde{B} \end{bmatrix} \begin{bmatrix} \mu_t \\ \mu_{t+1} \\ \vdots \\ \mu_{t+N_u-1} \\ \delta_t \\ \delta_{t+1} \\ \vdots \\ \delta_{t+N_u-1} \end{bmatrix}.
\] (3.15)

Equation (3.15) can be re-written more elegantly in a compact form to give the state prediction model as

\[
\bar{\varsigma} = \mathcal{G}\bar{\mu} + \mathcal{H}\varsigma_t + \mathcal{E}\bar{\delta},
\] (3.16)

where

\[
\mathcal{G} = \begin{bmatrix} \tilde{B} & 0 & \ldots & 0 \\
\tilde{A}\tilde{B} & \tilde{B} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{A}^{N-1}\tilde{B} & \tilde{A}^{N-2}\tilde{B} & \ldots & \tilde{A}^{N-N_u}\tilde{B} \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} \tilde{A} \\
\tilde{A}^2 \\
\vdots \\
\tilde{A}^N \end{bmatrix}.
\]
\[ E = \begin{bmatrix} \hat{B}_d & 0 & \cdots & 0 \\ \hat{A}\hat{B}_d & \hat{B}_d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{A}^{N-1}\hat{B}_d & \hat{A}^{N-2}\hat{B}_d & \cdots & \hat{A}^{N-N_d}\hat{B}_d \end{bmatrix}. \]

Similarly, the output prediction model can be given by

\[
\bar{z} = Z\bar{\varsigma} + Y\bar{\delta},
\]

where

\[
Z = \begin{bmatrix} \hat{C} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \hat{C} \end{bmatrix}, \quad Y = \begin{bmatrix} \hat{D} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \hat{D} \end{bmatrix}.
\]

By substituting the state prediction model (3.16) into (3.17), one can obtain the output prediction model as

\[
\bar{z} = M\bar{\mu} + N\bar{\varsigma} + F\bar{\delta},
\]

where

\[
M = \begin{bmatrix} \hat{C}\hat{B} & 0 & \cdots & 0 \\ \hat{C}\hat{A}\hat{B} & \hat{C}\hat{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{C}\hat{A}^{N-1}\hat{B} & \hat{C}\hat{A}^{N-2}\hat{B} & \cdots & \hat{C}\hat{A}^{N-N_d}\hat{B} \end{bmatrix}, \quad N = \begin{bmatrix} \hat{C}\hat{A} \\ \vdots \\ \hat{C}\hat{A}^{N} \end{bmatrix},
\]

\[
F = \begin{bmatrix} \hat{C}\hat{B}_d + \hat{D} & 0 & \cdots & 0 \\ \hat{C}\hat{A}\hat{B}_d & \hat{C}\hat{B}_d + \hat{D} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{C}\hat{A}^{N-1}\hat{B}_d & \hat{C}\hat{A}^{N-2}\hat{B}_d & \cdots & \hat{C}\hat{A}^{N-N_d}\hat{B}_d + \hat{D} \end{bmatrix}.
\]

### 3.3.2 Standard QP formulation

The QP that needs to be solved at every sampling instant to implement the receding horizon control can be stated in terms of the state prediction model as

\[
\min_{\bar{\mu}, \bar{\delta}} \quad \text{(3.13)},
\]

subject to: \(3.16\), \(1.2c[1.2f]\).

However, standard MPC can be viewed as an optimal control problem in which the equality constraint (3.16) is eliminated. In the proposed scheme, this can be readily achieved by the substi-
tution of (3.16) via the output error $e_k$ into (3.13). To proceed, let the error in the system’s output at any time step $k$ be given as

$$e_k = r_k - \tilde{C} \varsigma_k - \tilde{D} \delta_k.$$  

(3.20)

By substituting (3.20) into (3.13) and eliminating the constant terms which do not have influence on the optimization, the cost function is given as

$$J = \frac{1}{2} \left\{ \tilde{C}^T \tilde{S} \tilde{C} \varsigma_{t+N} + \sum_{k=0}^{N-1} \tilde{C}^T \tilde{Q} \tilde{C} \varsigma_{t+k} \right\} + \frac{1}{2} \sum_{k=0}^{N-1} \delta_{t+k}^T (\tilde{D} \tilde{Q} \tilde{D} + P) \delta_{t+k}$$

$$- \left\{ \tilde{r}^T_{t+N} \tilde{S} \varsigma_{t+N} + \sum_{k=0}^{N-1} \tilde{r}^T_{t+k} \tilde{Q} \tilde{C} \varsigma_{t+k} \right\} - \sum_{k=0}^{N-1} \tilde{r}^T_{t+k} \tilde{Q} \tilde{D} \delta_{t+k}$$

$$+ \sum_{k=0}^{N-1} \delta_{t+k}^T \tilde{D} \tilde{C} \varsigma_{t+k} + \sum_{k=0}^{N-1} \mu_{t+k}^T R \mu_{t+k} + \frac{1}{2} \rho_1 \epsilon^2 + \rho_2 \epsilon.$$  

(3.21)

In the light of the definitions (3.14), the objective function (3.21) can be expressed as

$$J = \frac{1}{2} \varsigma^T \tilde{Q} \varsigma + \frac{1}{2} \delta^T S \delta - \tilde{r}^T \tilde{T} \varsigma - \tilde{r}^T U \tilde{S} + \tilde{\delta}^T V \tilde{\varsigma}$$

$$+ \frac{1}{2} \tilde{\mu}^T R \tilde{\mu} + \frac{1}{2} \rho_1 \epsilon^2 + \rho_2 \epsilon,$$  

(3.22)

where

$$\tilde{r} = [r_{t+1}^T, r_{t+2}^T, \ldots, r_{t+N}^T]^T,$$

$$\mathcal{R} = \begin{bmatrix} R & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R \end{bmatrix}, \quad \mathcal{T} = \begin{bmatrix} Q \tilde{C} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & Q \tilde{C} & 0 \\ 0 & \cdots & 0 & S \tilde{C} \end{bmatrix},$$

$$\mathcal{Q} = \begin{bmatrix} \tilde{C}^T \tilde{Q} \tilde{C} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \tilde{C}^T \tilde{Q} \tilde{C} & 0 \\ 0 & \cdots & 0 & \tilde{C}^T S \tilde{C} \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} \tilde{D} \tilde{Q} \tilde{D} + P & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \tilde{D} \tilde{Q} \tilde{D} + P \end{bmatrix},$$

$$\mathcal{V} = \begin{bmatrix} \tilde{D}^T \tilde{Q} \tilde{C} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & \tilde{D}^T \tilde{Q} \tilde{C} & 0 \\ 0 & \cdots & 0 & \tilde{D}^T S \tilde{C} \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} \tilde{Q} \tilde{D} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \tilde{Q} \tilde{D} & 0 \\ 0 & \cdots & 0 & \tilde{S} \tilde{D} \end{bmatrix}. $$
Similarly, by substituting (3.16) into (3.22) and eliminating the constant terms, the objective function can be written in a compact form as

$$J = \frac{1}{2} \eta^T H \eta + f^T \eta,$$  
(3.23)

where

$$\eta = \begin{bmatrix} \bar{\mu}^T & \bar{\delta}^T & \epsilon^T \end{bmatrix}^T,$$

$$H = \begin{bmatrix} \mathcal{G}^T \mathcal{Q} \mathcal{G} + \mathcal{R} & 0 & 0 \\ (\mathcal{E}^T \mathcal{Q} \mathcal{G} + \mathcal{V} \mathcal{G}) & (\mathcal{E}^T \mathcal{Q} \mathcal{E} + \mathcal{S} + \mathcal{V} \mathcal{E} + \mathcal{E}^T \mathcal{V}) & 0 \\ 0 & 0 & \rho_1 \end{bmatrix},$$

$$f^T = \begin{bmatrix} \varsigma^T & r^T \end{bmatrix} \Delta_1 \begin{bmatrix} \varsigma^T & r^T \end{bmatrix} \Delta_2 \begin{bmatrix} \varsigma^T & r^T \end{bmatrix} \rho_2,$$

and

$$\Delta_1 = \begin{bmatrix} \mathcal{H}^T \mathcal{Q} \mathcal{G} \\ -\mathcal{T} \mathcal{G} \end{bmatrix},$$

$$\Delta_2 = \begin{bmatrix} \mathcal{H}^T \mathcal{Q} \mathcal{E} + \mathcal{H}^T \mathcal{V} \mathcal{E} \\ -\mathcal{T} \mathcal{E}^T - \mathcal{U} \end{bmatrix}.$$

Therefore, one can now re-state the optimization problem that needs to be solved at any sampling instant $k$ as follows:

$$\min_{\eta} J(\eta) = \frac{1}{2} \eta^T H \eta + f^T \eta,$$  
(3.24)

where $H$ is the Hessian matrix and the vector $f$, must be computed online at every iteration as it contains terms that must be updated such as $\varsigma_1$. For an unconstrained optimization problem, the optimal solution $\eta^*$ that minimizes the objective function (3.24) is obtained by zeroing the gradient as follows:

$$\nabla J(\eta) = H \eta + f = 0.$$  
(3.25)

Hence, for every sampling instant, the controller computes

$$\eta^* = -H^{-1} f,$$  
(3.26)

where $\eta^* = \begin{bmatrix} \bar{\mu}^* \\ \bar{\delta}^* \\ \epsilon^* \end{bmatrix}.$

It is convenient to obtain $\mu_k^*$ in every iteration $k$ from the calculated $\eta^*$, where only the first vector component of $\bar{\mu}^*$ is extracted based on the receding horizon principle as follows:

$$\mu_k^* = \begin{bmatrix} I_{n_u} & 0_{n_u \times (N_u - 1)n_u + N_u \times n_d + 1} \end{bmatrix} \eta^*.$$  
(3.27)

Therefore, the control $u_k^*$ which corresponds to $\mu_k^*$ that is used to control the system (3.7) in
standard MPC is given as
\[ u_k^* = \mu_k^* + u_{k-1}. \] (3.28)

In order to utilize the optimal disturbance \( \delta_k^* \) along with \( \mu_k^* \), the first vector component of \( \tilde{\delta}^* \) is extracted as follows
\[ \delta_k^* = \begin{bmatrix} 0_{n_d \times (N_u-1)n_d+1} \\ I_{n_d} \end{bmatrix} \eta^*. \] (3.29)

To utilize the optimal disturbance increment \( \delta_k^* \), the control signal increment that is obtained in every time step \( k \) is re-defined as
\[ \psi_k = \mu_k^* + \lambda_k, \] (3.30)
where \( \psi_k \) is the new control signal increment to be applied to the system (3.12), \( \lambda_k \) represents a component of the control signal \( \psi_k \) that is dependent on \( \delta_k^* \) and it gives an additional freedom of control. The controlled augmented deviation model (3.12) is then given as
\[ \varsigma_{k+1} = \tilde{A}\varsigma_k + \tilde{B}\psi_k + \tilde{B}d\delta_k. \] (3.31)

To incorporate \( \delta_k^* \) into the control while ensuring that the output error \( e_k \) is minimized and the effects of the disturbance increment \( \delta_k \) is reduced, it must be ensured that
\[ \tilde{B}\lambda_k = B_d(\delta_k^* - \hat{\delta}_k) \] (3.32)
is satisfied for all \( k \geq 0 \). Note that \( \hat{\delta}_k \) can be obtained either by measurement or estimation of the disturbance signal \( d_k \). If the estimation of disturbance is used, it is important to use a good estimate of the actual disturbance in order to ensure that the effects of the disturbance increment is adequately reduced. In general, \( \tilde{B} \) is not invertible but one can readily obtain \( \lambda_k \) as
\[ \lambda_k = (\tilde{B}^T\tilde{B})^{-1}\tilde{B}^T\tilde{B}d(\delta_k^* - \hat{\delta}_k). \] (3.33)

Therefore, the stage is set to define the optimal control signal to be applied to the system (3.7) in every time step as
\[ u_k^* = \mu_k^* + \lambda_k + u_{k-1}. \] (3.34)

**Remark 4.** In general, the inverse \( (\tilde{B}^T\tilde{B})^{-1} \) is not an issue because \( \tilde{B} \) is a tall matrix of \( (n+p) \) columns and \( m \) rows where \( (n+p) > m \). And in well defined systems, \( \text{rank}(\tilde{B}) = m \) so that \( \tilde{B}^T\tilde{B} \) is always an \( m \times m \) matrix with \( \text{rank}(\tilde{B}^T\tilde{B}) = m \), which means that an inverse will always exist. However, if the inverse is not applicable one can always use the pseudo inverse of the matrix.

In summary, given that a system (3.7), which is affected by an exogenous disturbance \( d_k \) needs to be controlled to obtain the desired output, the control signal to be applied to the plant is given by (3.34). Based on the definition given in (3.32), the use of \( \psi_k \) essentially entails the combined
use of the $\mu^*_k$ and $\delta^*_k$ to ensure that the error $e_k$ is minimized according to the cost function (3.13). In addition, the definition is formulated such that the effect of the disturbance increment $\delta_k$ is mitigated by $\hat{\delta}_k$ in the augmented increment form (3.12) of the system, thereby, helping to further minimize the impact of the external disturbance.

The solution to the optimisation problem that has been given is for the unconstrained case. However, system constraints that must also be fulfilled in every time $k$ by the controller are common; hence, the handling of systems constraints are discussed in the next section.

3.4 System Constraints

In physical systems, it is common to have physical constraints. They could be in the form of maximum and minimum inputs that can be supplied, a limitation on its change rate or even on the system states. Hence, there is a need to present constraint conditions in terms of linear matrix inequalities (LMIs) for the system outputs, states, controls and the rate of change in controls. The constraints defined in (1.2c - 1.2f) on a plant is generally formulated as an LMI of the form

$$\Gamma \eta \leq b,$$

where $\Gamma$ and $b$ are matrix and vector respectively which will be explicitly defined later in this section. $\Gamma$ is computed offline whereas $b$ must be computed online as it depends on parameters that need to be updated (which may be measured or estimated) during run-time. Owing to the structure defined in (3.35), it is convenient to solve the constrained optimization problem as a QP as discussed in references such as [8] and [34]. Each LMI is formulated in the subsequent subsections.

3.4.1 Output Constraints

Consider the constraint defined on a system output below

$$z_{\min} \leq z_k \leq z_{\max} \quad \forall \ k \in \{t + 1, \ldots, t + N\}.$$  

By introducing the positive panic variable $\epsilon$ to enable the implementation of a soft output constraint, one obtains

$$\bar{z}_{\min} - \epsilon \bar{v}_{\min} \leq \bar{z} \leq \bar{z}_{\max} + \epsilon \bar{v}_{\max},$$  

where $\Gamma$ and $b$ are matrix and vector respectively which will be explicitly defined later in this section. $\Gamma$ is computed offline whereas $b$ must be computed online as it depends on parameters that need to be updated (which may be measured or estimated) during run-time. Owing to the structure defined in (3.35), it is convenient to solve the constrained optimization problem as a QP as discussed in references such as [8] and [34]. Each LMI is formulated in the subsequent subsections.
which is the constraint condition that needs to be satisfied over the entire prediction horizon \( N \). Here,

\[
\bar{z}_{\text{min}} = \begin{bmatrix} z_{\text{min}} \\ \vdots \\ z_{\text{min}} \end{bmatrix} \in \mathbb{R}^{Nn}, \quad \bar{z}_{\text{max}} = \begin{bmatrix} z_{\text{max}} \\ \vdots \\ z_{\text{max}} \end{bmatrix} \in \mathbb{R}^{Nn}.
\]

\[
\bar{v}_{\text{min}} = \begin{bmatrix} v_{\text{min}} \\ \vdots \\ v_{\text{min}} \end{bmatrix} \in \mathbb{R}^{Nn}, \quad \bar{v}_{\text{max}} = \begin{bmatrix} v_{\text{max}} \\ \vdots \\ v_{\text{max}} \end{bmatrix} \in \mathbb{R}^{Nn}.
\]

Substituting (3.18) into (3.37) and simplifying the resulting inequality one can obtain the output constraint LMI as

\[
\begin{bmatrix} \mathcal{M} & \mathcal{F} & -\bar{v}_{\text{max}} \\ -\mathcal{M} & -\mathcal{F} & -\bar{v}_{\text{min}} \end{bmatrix} \begin{bmatrix} \bar{z} \\ \delta \\ \epsilon \end{bmatrix} \leq \begin{bmatrix} \bar{z}_{\text{max}} \\ -\bar{z}_{\text{min}} \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} N \zeta_t. \quad (3.38)
\]

### 3.4.2 State Constraints

Let the upper and lower bounds on the system states be given by

\[
x_{\text{min}} \leq x_k \leq x_{\text{max}} \quad \forall \ k \in \{t + 1, \ldots, t + N\}. \quad (3.39)
\]

Given the current state \( x_t \), then the next system state is given by \( x_{t+1} = \sigma_{t+1} + x_t \), from which the following can be obtained:

\[
\bar{x} = \bar{I}_{n_x} x_t + \mathcal{L}_1 \bar{\sigma}, \quad (3.40)
\]

where

\[
\bar{\sigma} = \begin{bmatrix} \sigma_{t+1} \\ \sigma_{t+2} \\ \vdots \\ \sigma_{t+N} \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} x_{t+1} \\ x_{t+2} \\ \vdots \\ x_{t+N} \end{bmatrix}, \quad \bar{I}_{n_x} = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} \in \mathbb{R}^{Nn_x \times n_x},
\]

\[
\mathcal{L}_1 = \begin{bmatrix} I & 0 & \cdots & 0 \\ I & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \cdots & I \end{bmatrix} \in \mathbb{R}^{N(n_x \times n_x)}.
\]

Also, the constraint condition that must be fulfilled over the prediction horizon \( N \) by the system state can be given as

\[
\bar{x}_{\text{min}} \leq \bar{x} \leq \bar{x}_{\text{max}}, \quad (3.41)
\]
where

\[ \bar{x}_{\min} = \begin{bmatrix} x_{\min} \\ \vdots \\ x_{\min} \end{bmatrix} \in \mathbb{R}^{N_n x}, \quad \bar{x}_{\max} = \begin{bmatrix} x_{\max} \\ \vdots \\ x_{\max} \end{bmatrix} \in \mathbb{R}^{N_n x}. \]

Recall that, \( \sigma_k = [I \ 0] \varsigma_k \), from which a relationship between the terms over the entire prediction horizon \( N \) can be obtained as follows

\[ \bar{\sigma} = L_2 \varsigma, \quad (3.42) \]

where

\[ L_2 = \begin{bmatrix} [I \ 0] & 0 \\ \vdots & \ddots \\ 0 & [I \ 0] \end{bmatrix} \in \mathbb{R}^{N(n_x \times (n_x + n_z))}. \]

By substituting (3.40) into (3.41) and in the light of (3.16) and (3.42), the state constraint LMI over the prediction horizon \( N \) is given as

\[ \Gamma_x \eta \leq b_x, \quad (3.43) \]

where

\[ \Gamma_x = \begin{bmatrix} \mathcal{L}_1 \mathcal{L}_2 \mathcal{G} & \mathcal{L}_1 \mathcal{L}_2 \mathcal{E} & 0 \\ -\mathcal{L}_1 \mathcal{L}_2 \mathcal{G} & -\mathcal{L}_1 \mathcal{L}_2 \mathcal{E} & 0 \end{bmatrix}, \]

\[ b_x = \begin{bmatrix} \bar{x}_{\max} - \bar{x}_{\min} \\ -\bar{x}_{\min} \end{bmatrix} - \begin{bmatrix} \hat{I}_{n_x} \\ -\hat{I}_{n_x} \end{bmatrix} x_t - \begin{bmatrix} I \\ -I \end{bmatrix} \mathcal{L}_1 \mathcal{L}_2 \mathcal{H}_{\varsigma t}. \]

### 3.4.3 Input Constraints

Let the upper and lower limit of the control input be given as

\[ u_{\min} \leq u_k \leq u_{\max} \quad \forall \ k \in \{t, \ldots, t + N_u - 1\}. \quad (3.44) \]

Recall that the control signal at any given time \( t \) is given by

\[ u_t = \mu_t + \lambda_t + u_{t-1}. \quad (3.45) \]

By defining \( \Lambda \triangleq (\hat{B}^T \hat{B})^{-1} \hat{B}^T \hat{B}_d \), one can readily re-write (3.33) as

\[ \lambda_t = \Lambda \delta_t - \Lambda \hat{\delta}_t. \quad (3.46) \]
Substituting (3.46) into (3.45) and noting that $\mu_t$ is indeed $\mu^*_t$ which can be obtained from the solution of the QP as described in (3.27), one obtains

$$u_t = u_{t-1} + \mu^*_t + \Lambda \delta^*_t - \Lambda \delta_t.$$  
(3.47)

The above can be expressed over the control horizon $N_u$ by using the definition in (3.14) and eliminating the ($\ast$) symbol for the sake of simplicity to obtain

$$\bar{u} = \mathcal{L}_3 \bar{\mu} + \mathcal{L}_4 \bar{\delta} + \tilde{I}_{n_u} u_{t-1} - \tilde{I}_{n_u} \Lambda \delta_t,$$  
(3.48)

where

$$\bar{u} \triangleq \begin{bmatrix} u^T_t, u^T_{t+1}, \ldots, u^T_{t+N_u-1} \end{bmatrix}^T,$$

$$\mathcal{L}_3 = \begin{bmatrix} I & 0 & \cdots & 0 \\ I & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \cdots & I \end{bmatrix} \in \mathbb{R}^{N_u (n_u \times n_u)},$$

$$\tilde{I}_{n_u} = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} \in \mathbb{R}^{n_u n_u \times n_u}, \quad \mathcal{L}_4 = \begin{bmatrix} \Lambda & 0 & \cdots & 0 \\ \Lambda & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda & \Lambda & \cdots & \Lambda \end{bmatrix}.$$

The equation (3.44) can be re-written more compactly as

$$\bar{u}_{\min} \leq \bar{u} \leq \bar{u}_{\max},$$  
(3.49)

where

$$\bar{u}_{\min} = \begin{bmatrix} u_{\min} \\ \vdots \\ u_{\min} \end{bmatrix} \in \mathbb{R}^{N_u n_u}, \quad \bar{u}_{\max} = \begin{bmatrix} u_{\max} \\ \vdots \\ u_{\max} \end{bmatrix} \in \mathbb{R}^{N_u n_u}.$$

By substituting (3.48) into (3.49), the constraint on the input can be shown to be given in terms of the optimization variable $\eta$ as

$$\Gamma_u \eta \leq b_u,$$  
(3.50)

where

$$\Gamma_u = \begin{bmatrix} \mathcal{L}_3 & \mathcal{L}_4 & 0 \\ -\mathcal{L}_3 & -\mathcal{L}_4 & 0 \end{bmatrix}.$$
\[ b_u = \begin{bmatrix} \bar{u}_{\text{max}} \\ -\bar{u}_{\text{min}} \end{bmatrix} - \begin{bmatrix} \bar{I}_{\text{n}_u} \\ -\bar{I}_{\text{n}_u} \end{bmatrix} u_{t-1} + \begin{bmatrix} \bar{I}_{\text{n}_u} \\ -\bar{I}_{\text{n}_u} \end{bmatrix} \Lambda \delta_t. \]

### 3.4.4 Rate of Input Constraints

The constraints on the input rate must be implemented on \( \psi_k \), which is now the actual control input increment due to the introduction of \( \lambda_k \). Let the constraint be given as

\[ \psi_{\text{min}} \leq \psi_k \leq \psi_{\text{max}} \quad \forall \ k \in \{t, \ldots, t + N_u - 1\}. \tag{3.51} \]

At any time \( t \), the constraint above can be written in the compact form

\[ \bar{\psi}_{\text{min}} \leq \bar{\psi} \leq \bar{\psi}_{\text{max}}, \tag{3.52} \]

where

\[ \bar{\psi}_{\text{min}} = \begin{bmatrix} \psi_{\text{min}} \\ \vdots \\ \psi_{\text{min}} \end{bmatrix} \in \mathbb{R}^{N_u n_u}, \quad \bar{\psi}_{\text{max}} = \begin{bmatrix} \psi_{\text{max}} \\ \vdots \\ \psi_{\text{max}} \end{bmatrix} \in \mathbb{R}^{N_u n_u}. \]

Since \( \psi_t \) can be given as \( \psi_t = \mu_t + \Lambda \delta_t - \Lambda \hat{\delta}_t \), which can conveniently be expressed over the control horizon \( N_u \) in the form

\[ \bar{\psi} = \mathcal{L}_5 \mu + \mathcal{L}_6 \delta - \bar{I}_{\text{n}_u} \Lambda \hat{\delta}_t, \tag{3.53} \]

where

\[ \mathcal{L}_5 = \begin{bmatrix} I & 0 \\ \vdots & \ddots \\ 0 & I \end{bmatrix}, \quad \mathcal{L}_6 = \begin{bmatrix} \Lambda & 0 \\ \vdots & \ddots \\ 0 & \Lambda \end{bmatrix}. \]

Note that \( \mathcal{L}_5 \) has the same size as \( \mathcal{L}_3 \). The substitution of (3.53) into (3.52) will give an inequality which can be simplified to obtain input increment LMI as follows:

\[ \begin{bmatrix} \mathcal{L}_5 & \mathcal{L}_6 & 0 \\ -\mathcal{L}_5 & -\mathcal{L}_6 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \delta \\ \epsilon \end{bmatrix} \leq \begin{bmatrix} \bar{\psi}_{\text{max}} \\ -\bar{\psi}_{\text{min}} \end{bmatrix} + \begin{bmatrix} \bar{I}_{\text{n}_u} \\ -\bar{I}_{\text{n}_u} \end{bmatrix} \Lambda \hat{\delta}_t. \tag{3.54} \]

Therefore, the matrix \( \Gamma \) and vector \( b \) in the generalised constraint defined in (3.35) are given as follows:
3.5 Observer Design

Consider the discrete-time plant (3.1) along with \( \zeta_k \), the signal to be estimated:

\[
\zeta_k = Ex_k + Fd_k.
\] (3.57)

Assume that the closed-loop system is stable because it is assumed to be controlled by the MPC controller proposed in the previous section. For convenience, the equations of the dynamic input filter (3.6) is re-written, which is assumed to generate the disturbance as follows:

\[
v_{k+1} = A_i v_k + B_i w_k,
\]

\[
d_k = C_i v_k + D_i w_k.
\] (3.58)

Generally, the disturbance filter can be unstable, which means that the eigenvalues of \( A_i \) can be on or outside the unit circle i.e. if \(|\lambda_j(A_i)|\) represents the \( j \)’th eigenvalue of \( A_i \) we have

\[
|\lambda_j(A_i)| \geq 1, \forall j.
\] (3.59)

Nevertheless, the stable dynamics of the filter can always be merged with that of the plant while the anti-stable part is left as an exo-system.
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Remark 5. The measurement vector \( y_k \) is formulated such that the estimator is able to use the measured states as well as the measurable disturbance. If the measured states are denoted as \( C_1 x_k \), and \( D_1 d_k \) represents the measurable disturbance, the measurement equation would be constructed as

\[
y_k = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ D_1 \end{bmatrix} d_k.
\]  

(3.60)

Remark 6. The signal \( \zeta_k \) would be constructed such that it collects all the signals that need to be estimated. For instance, if all system states and the input disturbances need to be estimated, \( \zeta_k \) will be constructed as

\[
\zeta_k = \begin{bmatrix} I \\ 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ I \end{bmatrix} d_k.
\]  

(3.61)

It is important to note that the estimation of all system states and input disturbance would likely lead to poor performance unless the disturbance is characterised properly. Indeed, the input filter is a tool to properly characterise the disturbance signal. A filter can be used to characterise the disturbance vector in a number of ways. It would usually be convenient to include a low-pass component in the input filter, whose bandwidth is to be decided based on the knowledge about the disturbance signals. Alternatively, one might use such a filter at the output to generate a \( \zeta \) signal that in fact represents the component of the signal to be estimated in the frequency band of interest (rather than the signal as it is). In both cases, the filter dynamics can be merged into the plant to arrive at a formulation of the estimation problem as in this dissertation with a suitably constructed \( \zeta \) signal.

Remark 7. The standard estimator design challenge can be formulated such that the filter has no dynamics. That is, \( A_i, B_i, C_i \) are void and \( D_i = I \).

The aim of the design of the estimator is to effectively utilize the measurement vector \( y_k \) to obtain reliable estimates of \( \zeta_k \). This dynamic estimator is given in the form

\[
\begin{align*}
\xi_{k+1} &= A_e \xi_k + B_e y_k, \\
\hat{\zeta}_k &= C_e \xi_k + D_e y_k,
\end{align*}
\]  

(3.62)

where \( \hat{\zeta} \in \mathbb{R}^{n_e} \) denotes the estimates of \( \zeta_k \) and \( A_e, B_e, C_e \) and \( D_e \) are the observer matrices to be found in order to realise the estimator.

The estimator is to be designed in a way to minimize the prediction error in some appropriate norm. It is more convenient to use a scaled version of the error signal as

\[
\varepsilon_k = W(\zeta_k - \hat{\zeta}_k),
\]  

(3.63)

where \( W \) is the weighting matrix typically chosen to be of a diagonal form. It enables the adjustment
3.5. OBSERVER DESIGN

of the relative emphasis on the components of the error signal.

Remark 8. In general, it would be more convenient to minimize a filtered version of the prediction error to emphasize the frequency band of interest. One can derive a solution for such a general formulation as well if the estimator is not restricted to a particular structure. The general formulation is not considered in this work since the proposed solution relies on an observer-based estimator. If one still wants to emphasize a certain frequency band with an observer-based estimator, this can be done simply by replacing the \( \xi \) signal with a filtered version of itself. Such a modification would mean that the interest is in estimating the component of \( \xi \) in a certain frequency band of interest (rather than the signal itself).

Based on the above, the estimator design problem can now be stated as follows: given the controlled closed-loop (or stable) plant and an anti-stable input filter \((3.58)\), the aim is to design an estimator \((3.62)\) such that the transfer matrix \( T_{\varepsilon w} \) from \( w \) to \( \varepsilon \) is stable and it has an upper bound \( \gamma \) defined in some appropriate norm.

Remark 9. It is interesting to observe that the disturbance filter can be introduced artificially to the problem (and more conveniently so in a discrete-time setting) even when it is not considered in the original problem formulation. Consider the discrete plant \((3.57)\) for which the aim is to design an estimator \((3.62)\). Since no exo-system exist because \( A_i, B_i, C_i \) are all void and \( D_i = I \), we would typically consider minimising a particular norm of the transfer matrix \( T_{\varepsilon d} \) from \( d \) to \( \varepsilon \). Alternatively, one can artificially introduce an input filter as follows:

\[
\begin{align*}
    d_{k+1} & = I \cdot d_k + I \cdot (d_{k+1} - d_k), \\
    d_k & = I \cdot v_k + 0 \cdot w_k.
\end{align*}
\]

(3.64)

Then, it is possible to consider minimising a preferred norm of \( T_{\varepsilon w} \) with the input filter \((3.64)\). The introduced input filter helps to characterise the disturbances as a low-pass type. Therefore, it is convenient to write a relationship between \( T_{\varepsilon d} \) and \( T_{\varepsilon w} \) as follows:

\[
T_{\varepsilon w}(z) = \frac{1}{z-1} T_{\varepsilon d}(z),
\]

(3.65)

where \( \frac{1}{z-1} I \) serves as a low-pass type filter that has infinite emphasis on zero frequency (since it is marginally stable with poles at 1). Since the weighting filter is unstable, the synthesis will force \( T_{\varepsilon d}(z) \) to have transmission zeros at \( z = 1 \) so that \( T_{\varepsilon w}(z) \) \( T_{\varepsilon w} \) can be stable. Consequently, we will have \( T_{\varepsilon w}(e^{j\omega})|_{\omega=0} \), thus attenuating the constant parts of the disturbances. If the disturbances have dominant time-varying components, it might be preferable to use a stable filter whose bandwidth is chosen in a way to appropriately characterise the disturbances.
3.5.1 Problem Solution

In this thesis, an observer-based solution to the estimation problem (defined in the previous section) with an anti-stable disturbance filter is presented. To implement the observer based solution, the controlled plant (3.1) dynamics is merged with that of the input filter (3.64). In this fashion, the dynamics of the extended plant can be expressed as follows:

\[
\begin{bmatrix}
  x_{k+1} \\
  y_{k+1} \\
  \vartheta_{k+1}
\end{bmatrix} =
\begin{bmatrix}
  A_p & B_d C_i & 0 \\
  0 & A_t & 0 \\
  A_o & B_o & B_i
\end{bmatrix}
\begin{bmatrix}
  x_k \\
  y_k \\
  u_k
\end{bmatrix} +
\begin{bmatrix}
  B_p \\
  0 \\
  B_o
\end{bmatrix} w_k,
\]

\[y_k =
\begin{bmatrix}
  C_y & D_y C_i & 0 \\
  0 & C_o & D_o \\
  C_o & 0 & F_o
\end{bmatrix}
\begin{bmatrix}
  \vartheta_k \\
  \vartheta_k \\
  \vartheta_k
\end{bmatrix} +
\begin{bmatrix}
  D_y D_i & 0 \\
  0 & D_o \\
  0 & 0
\end{bmatrix} w_k,
\]

\[\zeta_k =
\begin{bmatrix}
  E & FC_i & 0 \\
  0 & F_c & 0 \\
  E_o & F_o & F_o
\end{bmatrix}
\begin{bmatrix}
  \vartheta_k \\
  \vartheta_k \\
  \vartheta_k
\end{bmatrix} +
\begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix} w_k.
\]

(3.66)

The observer to estimate the state and the output signals of the extended plant model (3.66) can be constructed as,

\[\hat{\vartheta}_{k+1} = A_o \hat{\vartheta}_k + B_o u_k - L (y_k - \hat{y}_k),\]

\[\hat{y}_k = C_o \hat{\vartheta}_k,\]

\[\hat{\zeta}_k = E_o \hat{\vartheta}_k,\]

(3.67)

where the matrix \(L\) is the observer gain matrix to be computed. It is important to emphasize at this point that the observer-based estimator corresponds to the choice of the realization matrices in a specific way as follows

\[
\begin{bmatrix}
  A_e & B_e \\
  C_e & D_e
\end{bmatrix} =
\begin{bmatrix}
  A_o + L C_o & -L \\
  E_o & 0
\end{bmatrix}.
\]

(3.68)

The state estimation error is computed as \(\epsilon_k \triangleq \vartheta_k - \hat{\vartheta}_k\). Therefore, the evolution of \(\epsilon\) and \(\varepsilon\) is given as,

\[\epsilon_{k+1} = (A_o + L C_o) \epsilon_k + (B_o + L D_o) w_k,\]

\[\varepsilon_k = \begin{bmatrix} W & 0 \end{bmatrix} \epsilon_k + \begin{bmatrix} W E_o & W E_o \end{bmatrix} w_k.
\]

(3.69)

Based on this representation of the error dynamics, which is basically a realisation of \(\mathcal{T}_{\epsilon w}\), one can easily arrive at LMI conditions that ensure the stability of a specified norm bound on \(\mathcal{T}_{\epsilon w}\).
3.5.2 $H_2$ Synthesis

The matrix inequality conditions for $\|T_{ew}\|_2 < \gamma$ can be expressed \[^67\] in discrete-time as follows:

$$
\begin{align*}
\text{tr}(Z) &< \gamma, \\
\begin{bmatrix}
X & * & * \\
0 & \gamma I & * \\
XA & AB & X
\end{bmatrix} &> 0,
\begin{bmatrix}
X & 0 & C^T \\
0 & \gamma I & D^T \\
C & D & Z
\end{bmatrix} &> 0.
\end{align*}
$$

(3.70)

By introducing $XL = M$, one can obtain an LMI condition with $X = X$ as follows:

$$
\begin{align*}
\text{tr}(Z) &< \gamma, \\
\begin{bmatrix}
X & * & * \\
0 & \gamma I & * \\
XA_o + MC_o & XB_o + MD_o & X
\end{bmatrix} &> 0,
\begin{bmatrix}
X & * & * \\
0 & \gamma I & * \\
WE_o & WF_o & Z
\end{bmatrix} &> 0.
\end{align*}
$$

(3.71)

Then, the observer gain can be computed from the solution of this LMI problem as $L = X^{-1}M$ for the minimum achievable $\gamma$ which satisfy (3.71).

By obtaining the gain $L$, the observer (3.67) can conveniently be implemented to estimate the unmeasured states and disturbances which completes the design of the observer-based MPC proposed in this thesis. The procedure to implement the proposed algorithm is summarized in Algorithm 2.

3.6 Summary

A novel model predictive control algorithm based on velocity form of linear state-space models has been proposed for disturbance rejection in constrained systems. Important highlights of the proposed scheme include a cost function that is a function of the external disturbance increment, the concept of optimal disturbance and a control signal that is dependent on the estimated disturbance increment. To obtain the estimate of external disturbance increment, a new observer that uses an anti-stable input filter is proposed. However, it is possible to use a stable input filter if there is a need to characterize the disturbance with a dominant time-varying component. The observer gain was obtained by solving an $H_2$ minimization problem. In the next chapter, several simulation examples will be presented to demonstrate the effectiveness of the proposed algorithm.
Algorithm 2: The proposed MPC with disturbance estimation

**Offline:** Solve the $H_2$ optimization problem (3.71) for minimum $\gamma$ to obtain the observer gain $L$.

**Online:** (For every iteration $k \geq 0$)

Step 1: Get the current state $\varsigma_k$ and disturbance estimate $\hat{d}_k$.

Using $\hat{d}_k$, obtain $\hat{\delta}_k$ based on the definitions given in (3.8).

Step 2: Solve the constrained optimization problem (3.56) to obtain $\eta^*$ with initial conditions of $\varsigma_k$ and $\hat{\delta}_k$.

Step 3: Obtain $\mu_k^*$ and $\delta_k^*$ as the first components in $\tilde{\mu}^*$ and $\tilde{\delta}^*$ respectively according to equations (3.27) and (3.29).

Step 4: Compute $\lambda_k$ using definition (3.33).

Step 5: From definition (3.30), calculate the control increment $\psi_k$.

Step 6: Determine the current control signal $u_k^*$ according to (3.34).

Step 7: Apply the control sequence $u_k^*$ to the system to obtain $x_{k+1}$. 

Step 8: Apply $u_k^*$ to (3.67) to obtain $\zeta_{k+1}$; extract the disturbance estimate $\hat{d}_{k+1}$ (with all states measured, $\hat{d}_{k+1} = \zeta_{k+1}$).

Step 9: Set $k \leftarrow k + 1$ and return to step 1.
Chapter 4

Implementation and Results

4.1 Introduction

In this chapter, some simulation examples are presented to illustrate the effectiveness of the proposed MPC in the presence of external disturbances and system constraints. To show the benefits of the proposed controller, its performance will be compared with the complete increment form with output delay [15]. The conventional complete increment form is not presented because the two increment forms have been shown to be equivalent since different choices of disturbance model and observer are indeed equivalent. In the first case study, both state and output feedback control problems were considered while the other two examples are based only on state feedback control. The unmeasured quantities were estimated using the integrated state and disturbance observer presented in the previous chapter. To obtain the observer gain $L$ in each of the example, the $\mathcal{H}_2$-optimization problem is solved by using the Yalmip toolbox [33] in MATLAB environment which was interfaced with the SeDuMi Solver.

4.2 Estimator and Performance Index

In the implementation of the proposed state and disturbance estimator, the input filter can be used as a shaping filter to reduce the effects of the unmeasurable disturbances when the frequency characteristics of the signal are known. The anti-stable input fitter is deployed in all the case studies presented here. It is pertinent to emphasize here that the anti-stable input filter is essentially an integrator and it is most suitable to emphasize on the DC component of the estimated signal or to ensure the elimination of steady-state error in the estimations. Its deployment in the simulated systems in this chapter is mainly to ensure the elimination of steady-state error. Therefore, the filter dynamics employed in this study is generally given as $A_i = B_i = C_i = I$ and $D_i = 0$. However,
this assumption may not give good estimation results when the disturbance signal has a wide band (i.e. consists of different frequency components). Hence, it would be more appropriate to design a suitable input filter in such a case. Furthermore, the matrices $D_y$ and $D_z$ are taken as zeros since it is assumed that all system disturbances are unmeasurable and that they directly affect only the states.

### 4.2.1 Integral Time-weighted Absolute Error

To numerically access the system output regulation given by both MPCs such that emphasis is placed on the steady-state error in the presence of disturbances, Integral Time-weighted Absolute Error (ITAE) performance measure is used. This measure is widely used by academicians to compare the performance of alternative control schemes. ITAE integrates the absolute error of system output over time and multiplies this by time, which results in the larger weighting of errors occurring at steady-state. ITAE is given mathematically as

$$\text{ITAE} = \sum_{k=1}^{n} k |r_k - z_k|, \quad (4.1)$$

where $r_k$ is the reference point for the controlled output $z_k$ at any time $k$.

### 4.3 Case Study: Inverted Pendulum

The mechanism discrete-time model of the inverted pendulum positioned like the one shown in Figure 4.1 is used in this example.

The states are respectively the cart’s position $\phi$ [m], angular displacement from the vertical (in the clockwise direction) $\theta$ [rad], the linear velocity of the cart’s position $\kappa$ [m/s] and its angular velocity $\omega$ [rad/s]. The control signal is an input force $u$ [N], applied to the cart and the variables to be controlled are the cart’s position and the angular displacement of the pendulum. The non-linear model obtained is linearized and discretized at $T_s = 0.1$s to obtain a discrete-time state-space model and it is assumed that the system is affected by $d$, an unmeasured exogenous disturbance. The model is given as

$$\begin{bmatrix}
\phi_{k+1} \\
\kappa_{k+1} \\
\theta_{k+1} \\
\omega_{k+1}
\end{bmatrix} = \begin{bmatrix}
1.0745 & 0.1025 & 0 & 0 \\
1.5079 & 1.0745 & 0 & 0 \\
-0.0248 & -0.0008 & 1 & 0.1 \\
-0.5026 & -0.0248 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\phi_k \\
\kappa_k \\
\theta_k \\
\omega_k
\end{bmatrix} + \begin{bmatrix}
-0.0025 \\
-0.0512 \\
0.0025 \\
0.0504
\end{bmatrix} u_k + \begin{bmatrix}
0.002 \\
0.09 \\
0.0015 \\
0.05
\end{bmatrix} d_k, \quad (4.2a)$$
4.3. CASE STUDY: INVERTED PENDULUM

The aim here is to control the system on the horizontal plane of length 1m such that the pendulum is vertically inverted i.e. $\theta = 0$ rad and $\phi = 0$ m, which represents the center point of the 1 m long plane. This control action is to be achieved in the presence of the disturbance $d$ shown together with its estimate in Figure 4.2.

It is assumed that all system states are measured, which implies that $C_y = I_4$ and one can, therefore, construct an estimated signal as

$$\zeta_{k+1} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} x_k + d_k.$$  \hfill (4.3)

Table 4.1: Inverted Pendulum: design parameters for the compared MPC controllers

| Parameter | $N$ | $N_u$ | $Q$   | $R$   | $P$    | $W$  |
|-----------|-----|-------|-------|-------|--------|------|
| Value     | 20  | 10    | $I_2$ | 0.5   | 1000   | 10   |

The observer gain matrix obtained is given in Appendix A, Subsection A.1.1 and the parameters...
CHAPTER 4. IMPLEMENTATION AND RESULTS

Figure 4.2: Inverted pendulum: Exogenous system disturbance $d$ and its estimate $\hat{d}$.

Table 4.2: Inverted Pendulum: comparison of the proposed and standard MPC

| Index | ITAE Proposed MPC | ITAE Standard MPC | % Improvement |
|-------|-------------------|-------------------|---------------|
| $\phi$ | 43.30             | 110.23            | 60.72         |
| $\theta$ | 277.56         | 558.57            | 50.31         |

The system output, input and input rates evolution in time are shown in Figure 4.3 where it can be seen that the proposed controller provides a significantly minimal oscillations in the presence of the unmeasured disturbances and system constraints despite using lesser input energy with a root mean square (RMS) value of 0.86 while the RMS of the input energy of the conventional MPC is 1.31. Although the disturbance estimation is shown in Figure 4.2 lags the actual input disturbance, the proposed MPC was still able to give over 50% improvements in the regulation of both system outputs. The ITAE values and performance improvements given by the proposed controller over standard MPC are presented in Table 4.2.

If one assumes that in the above example that only the outputs of the inverted pendulum are

\begin{equation}
|u_k| \leq 2.5, \quad |\phi_k| \leq 0.5 \text{ and } |\psi_k| \leq 0.52, \quad \forall \quad k \geq 0.
\end{equation}
Figure 4.3: State feedback control of the inverted pendulum: evolution of system outputs (top), controls (bottom left) and control signal rates (bottom right)
measurable, which makes it expedient to also estimate the other two states, one will then need to construct the signal to be estimated as

\[
\zeta_{k+1} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} x_k + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} d_k.
\] (4.5)

The obtained observer gain matrix in this case is given in Subsection A.1.2 in Appendix A and the estimated disturbance obtained here is similar to that shown in Figure 4.2. Therefore, the system outputs obtained and the estimated state are respectively shown in Figure 4.4a and 4.4b, where it can be observed that the state estimate is a good approximation of the actual states and the proposed scheme was still able to outperform the standard MPC. Indeed, there is no much difference between the plots given in the state feedback and output scenarios. However, in the output feedback case, the improvements in the outputs, \( \phi \) and \( \theta \) reduced to 42.36\% and 26.73\% respectively. Finally, it is necessary to note that the parameters used to implement the controllers in the state feedback case is also used in the output feedback scenario.
4.4 Case Study: Flight Control

In this example, the continuous-time model of an airplane taken from [9] is considered. The model corresponds to the longitudinal motion of a Boeing 747 aircraft cruising in level flight at a height of 40,000 ft and a velocity of 774 ft/s. Usually, it is desired to control the airspeed, i.e., velocity with respect to air and the climb rate. This can be achieved by appropriate manipulation of the elevator angle $u_1$ and the throttle force $u_2$.

The dynamics of the longitudinal motion of the aircraft can adequately be represented [9] by employing:

- $x_1$ that denotes the velocity in the aircraft’s longitudinal body axis,
- $x_2$ the velocity in the $y$–axis denoted by $x_2$, a component of the angular velocity represented by $x_3$, and
- $x_4$ which stands for the angular displacement between the horizontal and the aircraft’s longitudinal body axis.

The reader may refer to [9, 10] for more details on the dynamical model.

Although the original model is disturbance-free, an exogenous disturbance $d$ is artificially introduced into the model. Hence, the continuous-time state-space model that represents the perturbed model of the aircraft in level flight at an altitude of 40,000 ft with a velocity of 774 ft/s is given as

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
-0.003 & 0.039 & 0 & -0.322 \\
-0.065 & -0.319 & 7.7400 & 0 \\
0.020 & -0.101 & -0.429 & 0 \\
0 & 0 & 1.000 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 - x_{1,p} \\
x_2 - x_{2,p} \\
x_3 \\
x_4
\end{bmatrix}
+ 
\begin{bmatrix}
0.01 & 1.00 \\
-0.18 & -0.04 \\
-1.16 & 0.598 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
+ 
\begin{bmatrix}
0.01 \\
0.002 \\
0.005 \\
0.01
\end{bmatrix} d.
$$

The aim is to control the airspeed, $(x_1 - x_{1,p})$ and climb rate, $(7.74x_4 - x_2)$. The output equation is given by

$$
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 7.74
\end{bmatrix}
\begin{bmatrix}
x_1 - x_{1,p} \\
x_2 - x_{2,p} \\
x_3 \\
x_4
\end{bmatrix}
$$

where $x_{1,p}$ and $x_{2,p}$ denotes perturbations in the wind velocity components. The continuous-time state-space model is sampled at $T_s = 0.1s$ to obtain the discrete-time model. Since all states are assumed to be measured, $C_y = I_4$ and one can then construct the estimated signal as

$$
\zeta_{k+1} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} x_k + d_k.
$$

The controllers’ design parameters are given in Table 4.3 and it can be noticed that the weighting
CHAPTER 4. IMPLEMENTATION AND RESULTS

(a) The system’s outputs (top), controls (bottom left) and control signal rates (bottom right)

(b) The evolution of the actual states $x_2$ and $x_4$ with their estimates $\hat{x}_2$ and $\hat{x}_4$ are shown.

Figure 4.4: Output feedback control of the Inverted Pendulum

Table 4.3: Flight control: design parameters for the compared MPC controllers

| Parameter | $N$ | $N_u$ | $Q$ | $R$ | $P$ | $W$ |
|-----------|-----|-------|-----|-----|-----|-----|
| Value     | 150 | 2     | $I_2$ | $10I_2$ | 200 | 10 |
4.4. CASE STUDY: FLIGHT CONTROL

Figure 4.5: Aircraft sketch showing notations used for longitudinal motion control (After [10])

Figure 4.6: Flight Control: system disturbance $d$ and its estimates $\hat{d}$ are shown
matrix $R$ is 10 times the weight of $Q$ because it is usually desired to emphasize on the minimization of energy consumption in flight control. The observer gain matrix used to implement the estimator is given in Section A.2 in Appendix A. The system constraints are defined as follows:

$$|u_{1,k}| \leq 0.5, \ |u_{2,k}| \leq 1, \ |\psi_k| \leq 0.75, \ \forall \ k \geq 0.$$

(4.8)

The evolution of the aircraft outputs, inputs and input rates are shown in Figure 4.7, where the proposed controller provides a significantly better output regulation in the presence of the unmeasured disturbances and system constraints. The estimations of the disturbance $\hat{d}$ used in the implementation of the controller is shown in Figure 4.6 and it can be said to be a good estimate of the actual signal even though it has a lag of 0.1s. Moreover, this did not prevent the proposed scheme from improving disturbance rejection. Even though the RMS value of the input energy used by the proposed controller is about 0.94 as opposed to the 0.97 used by conventional MPC, it can be seen that the proposed MPC gives a significantly improved performance as depicted in Table 4.4. The output, $z_1$ of the standard MPC could not track the reference $r_1 = 5$ ft/s and $z_2$ exhibits greater oscillations around its reference during the run-time. Based on the results obtained, it is appropriate to state that the proposed scheme is fault-tolerant if one views the control signal saturation due to the constraints on them as a fault from the signal source.

Table 4.4: Flight control: comparison of the proposed and standard MPC

| Index | ITAE | ITAE       | % Improvement |
|-------|------|-----------|---------------|
| Output | Proposed MPC | Standard MPC |               |
| $z_1$ | 41617.00 | 212380.00 | 80.40         |
| $z_2$ | 72272.00 | 142190.00 | 49.17         |

4.5 Case Study: PMSM Control

In this case study, the non-linear model of the permanent magnet synchronous motor (PMSM) borrowed from [55] is considered. The equations of the PMSM are given as

$$\frac{di_d}{dt} = \frac{1}{L_d}(V_d - Ri_d + \omega_c L_q i_q),$$

$$\frac{di_q}{dt} = \frac{1}{L_q}(V_q - Ri_q - \omega_c L_d i_d - \omega_c \phi),$$

$$\frac{d\omega_c}{dt} = \frac{p}{J} \left( T_c - \frac{B_v}{p} \omega_c - T_l \right),$$

(4.9)

$$T_c = \frac{3}{2}(\phi i_q + (L_d - L_q) i_d i_q).$$
Figure 4.7: Flight control: evolution of system outputs (top), controls (middle) and control signal deviations (bottom)
where the parameters used in the model are defined in Table 4.5 and their corresponding values given.

Table 4.5: PMSM model: parameters description and their corresponding values

| Nomenclature | Description                        | Magnitude with units |
|--------------|------------------------------------|----------------------|
| $J$          | PMSM moment of inertia             | 2.35 kg cm$^2$       |
| $B_v$        | Coefficient of viscosity           | $1.1 \times 10^{-4}$ |
| $L_{eq}, L_d$| Inductance of the $d-q$-axis       | 7.0 mH               |
| $R$          | Stator resistance                  | 2.98 $\Omega$        |
| $\phi$       | Rotor flux                         | 0.125 Wb             |
| $p$          | poles                              | 2                    |

The system states $i_d$ and $i_q$ denotes the stator currents in the $d-q$ frame and $\omega_e$ is the electrical speed, where the states, $i_d$ and $\omega_e$ are taken as the output of the system. The electrical speed is given by $\omega_e = p\omega_m$, where $\omega_m$ is the motor speed and $p$ is the number of poles. The inputs are the stator voltages $V_d$ and $V_q$ in the $d-q$ frame. For a surface-mounted PMSM, the electromagnetic torque $T_c$ is independent [11] of $i_d$ and thus, the following relationship holds

$$T_c = \frac{3}{2}p\phi i_q$$
(4.10)

In the control of the speed of PMSM, it is desired to maintain $i_d = 0$ for higher efficiency [56]. Then, a reference of 100 rad/s is used for $\omega_e$ in the simulation study. The major exogenous disturbance affecting the PMSM is the exogenous load torque $T_l$ applied on its shaft [32]. The continuous-time state-space model obtained by linearizing the non-linear model around the operating points $i_d = 0A$, $i_q = 0A$ and $\omega_e = 100$ rad/s is given below.

$$\begin{bmatrix}
\dot{i}_d \\
\dot{i}_q \\
\dot{\omega}_e
\end{bmatrix} =
\begin{bmatrix}
-425.7 & 200 & 0 \\
-200 & -425.7 & -17.9 \\
0 & 3191.5 & -0.5
\end{bmatrix}
\begin{bmatrix}
i_d \\
i_q \\
\omega_e
\end{bmatrix} +
\begin{bmatrix}
142.856 & 0 \\
0 & 142.86 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
V_d \\
V_q \\
-T_l
\end{bmatrix}$$
(4.11)

The model is sampled at $T_s = 0.01s$ to obtain the discrete-time model. All three states are assumed to be measured. Hence, $C_y = I_3$ and one can then conveniently construct the signal to be estimated as follows

$$\zeta_{k+1} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} x_k + d_k.$$  
(4.12)

The computed observer gain matrix is given in Section A.3 in Appendix A. Furthermore, the pa-
4.5. CASE STUDY: PMSM CONTROL

Figure 4.8: PMSM: System’s estimated $\hat{T}_l$ and actual $T_l$ input disturbances

Parameters used in the implemented controllers are presented in Table 4.6 and the system constraints to be fulfilled for every time $k$ are defined as follows:

$$|V_{d,k}| \leq 4, |V_{q,k}| \leq 20, |i_{d,k}| \leq 2, |i_{q,k}| \leq 4, |\omega_{e,k}| \leq 180, |\psi_{1,k}| \leq 2 \text{ and } |\psi_{2,k}| \leq 5,$$

(4.13)

and these constraints did not result in the saturation of the input signal as was evident in the previous examples primarily to show that improved performance may also be obtained even when input saturation does not occur.

Table 4.6: PMSM control: design parameters for the compared MPC controllers

| Parameter | $N$ | $N_u$ | $Q$ | $R$ | $P$ | $W$ |
|-----------|-----|-------|-----|-----|-----|-----|
| Value     | 10  | 2     | $10I_2$ | $I_2$ | $1 \times 10^6$ | 10 |

In this example, the RMS values of the control voltages used by both controllers are approximately 16.31V. Figure 4.9 shows the plots of the outputs, controls, and rate of change of controls of the PMSM system. Even though the estimated disturbance lags the actual disturbance with about 0.2s as shown in Figure 4.8, the proposed controller is still able to give improvements in the output regulation. The improvement in performance given by the proposed MPC is given in Table 4.7. It is important to note that whereas the output response of the proposed scheme settles almost immediately after responding to the variation in the external disturbance, the standard MPC exhibits oscillations before settling and this accounts for its larger ITAE values documented in the table.
Figure 4.9: PMSM: closed-loop output responses (top), control signals (middle) and control signal deviation (bottom)

Table 4.7: PMSM control: comparison of the proposed and standard MPC

| Index | ITAE Proposed MPC | ITAE Standard MPC | % Improvement |
|-------|-------------------|-------------------|---------------|
| $i_d$ | 585.10            | 753.26            | 22.32         |
| $\omega_e$ | 32304.00      | 37311.00          | 13.42         |
4.6. COMPUTATIONAL LOAD ANALYSIS

At this junction, it is pertinent to highlight that by taking the weighting matrix $P$ to be far larger than the weights presented in the above case studies, the response of the proposed MPC approximates that of standard MPC.

4.6 Computational Load Analysis

The issue of computational time has long been identified as a major setback to the use of MPC in systems requiring fast sampling rates as opposed to chemical processes where long sampling rates are generally required. As a result, many works have been focused on fast MPC algorithms \[84, 80, 22\] and these have been aided by the continuous improvements in computer technology, witnessed in recent years. Therefore, it is crucial to investigate the impact of the introduction of an additional variable (i.e. $\delta$) into MPC algorithm proposed.

The number of inequality and equality constraints, control and prediction horizon also influence the computational time of MPC. Hence, before proceeding to compare the computational burden of the controllers, it is necessary to emphasize here that the parameters, $N_u$, $N$ and the weighting matrices used in the implementation of the proposed and standard MPC are the same. Also, the number of inequality constraints are the same and no equality constraints were implemented (because the formulation of MPC leads to the elimination of all equality constraints). However, the QP of standard MPC has two optimisation variables, $\bar{u}$ and $\epsilon$, while the proposed scheme has three variables, $\bar{u}$, $\bar{\delta}$ and $\epsilon$. The inequality constraints imposed on the inverted pendulum, aircraft and PMSM model are respectively given in (4.4), (4.8) and (4.13). The specifications of the author’s computer used in this study include, a RAM of 8 GB, core-i5 processor with a speed of 1.8 GHz.

4.6.1 Inverted Pendulum

To compare the computational time of both controllers, the time taken by the \texttt{quadprog} solver in MATLAB to solve the optimization problem was recorded and stored using the \texttt{tic} and \texttt{toc} commands available in MATLAB. Only the elapsed time between when the \texttt{quadprog} solver started to solve the QPs and the time taken before it obtained a solution is recorded. In the rest of this work, $T_c$ will be used to denote the time taken to obtain a solution by \texttt{quadprog}.

Figure 4.10 shows that the proposed MPC and the standard MPC used a comparable computational time both during transient and at steady-state. The average computational time $T_c$ for the proposed and standard MPCs are respectively 11.04ms and 11.02ms. This further demonstrates the strength of the proposed scheme since the standard MPC gave a significantly poorer performance in the tracking of the outputs references.
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Figure 4.10: Inverted pendulum: a plot of the time taken $T_c$, by quadprog solver to solve the QPs of the proposed and standard MPCs against the simulation time $t$.

4.6.2 Flight control

The plot obtained using the similar procedure as described in Section 4.6.1 is also used for the airplane control case study and the plot obtained for the computational time $T_c$ for the two MPC schemes is shown below.

Aside from the larger initial value of $T_c$ obtained for the proposed scheme, it is obvious from Figure 4.11 that the proposed scheme leads to a reduced computational burden overall. The average value of the computational time for the proposed and standard MPC are respectively 120.90ms and 160.30ms. The merit of the proposed MPC is more evident in the plot at steady-state despite the fact that the proposed scheme tracked the set-points while standard MPC could not get the system output $z_1$ to its set-point.

4.6.3 Permanent Magnet Synchronous Motor

The plot of $T_c$ obtained for the PMSM control problem is presented in Figure 4.12. The results follow from the previous analysis that has been given. The $T_c$ values for the proposed and standard MPC is averaged at 15.12ms and 14.90ms respectively.

At this junction, it is worthy of note that only the inverted pendulum could be considered for real-time application given the parameters used in the simulations. This is so because the average computational time of the solver is significantly smaller than the sampling rate. The sampling time of the inverted pendulum is $T_s = 0.1s$ while the mean time taken to solve the optimisation
4.6. COMPUTATIONAL LOAD ANALYSIS

Figure 4.11: Flight control: a plot of the time taken $T_c$, by quadprog solver to solve the QPs of the proposed and standard MPCs against the simulation time $t$.

Figure 4.12: PMSM: a plot of the time taken $T_c$, by quadprog solver to solve the QPs of the proposed and standard MPCs versus the simulation time $t$. 
problems are 11.00ms and 11.40ms. The other examples required longer average time $T_e$ to solve the QP than the sampling period. Therefore, a high-speed computer (with better specifications than that of the author) or even dedicated computers may be required if the MPC controllers were to be implemented in real-time. Another alternative could be to consider a faster online optimisation algorithm. However, this discussion on real-time implementation is out of the scope of this dissertation and hence, would not be given further considerations.

4.7 Summary

Three different case studies based on three different systems have been presented in this chapter to demonstrate some of the merits of the proposed scheme over the conventional approach used in the formulation of deviation models. In all three systems, the proposed scheme showed significant improvements over the standard MPC and this was quantitatively evidenced by presenting the ITAE values for every output in the different case studies. Computation cost is a very key issue in MPC, which motivated the comparison of the controllers in terms of the actual time it took \texttt{quadprog} to compute the optimal solution. The results showed that the proposed scheme did not lead to increased computational time despite the enormous performance improvements it gave in the simulated examples. In the next chapter, the results obtained will be discussed in more detail and interesting issues arising from this novel approach will be summed up.
Chapter 5

Discussions and Conclusions

5.1 Introduction

This chapter presents the modest contributions of this thesis by providing concise discussions on important findings that were made. Furthermore, concluding remarks are given that also addressed the question raised in the motivational statement in Chapter 1. Lastly, some of the issues that may be interesting to investigate in the future are discussed.

5.2 Discussions of the Thesis Highlights

5.2.1 Cost Function

The cost function is probably the most important part of the formulation of the MPC problem. The novelty introduced in the rejection of disturbances is primarily made possible by modifying the conventional quadratic cost function used in MPC to (3.13). This resulted in three optimization variables (two when soft output constraints are not considered) as opposed to the two variables (or one variable when hard output constraints are used) in standard MPC formulation.

Nevertheless, an optimal solution is guaranteed as the convexity of the problem is preserved, which implies that a local minimum obtained also represents the global minimum. Since $\epsilon$ is usually a negligibly small parameter, the plant to be controlled is driven by using the other optimization variables, $\mu$ and $\delta$ from the solution to the QP. The additional freedom of control introduced by $\delta$ was duly exploited to derive the new MPC algorithm.
5.2.2 Control Signal

By assuming that the disturbance increment $\delta$ is non-zero as opposed to the previous methods where it is eliminated, the disturbance increment is left in the augmented plant (3.12). The augmented plant velocity model is then exploited to formulate a control signal for the original plant that is dependent on the computed optimal disturbance and control increments as well as the estimated disturbance increment. The estimated disturbance is included such that it opposes the actual disturbance increment in the model (3.12).

One may want to think that since the control signal is now made somewhat complicated, it may affect the existence of a feasible solution in constrained problems. However, the simulation study contradicts this point because the proposed scheme was able to provide optimal solutions within the region that was defined by the strict constraints imposed on the systems used in the case studies. In fact, the strength of the controller in working at the boundaries of the constraints gave it superior performance over the conventional MPC, which is more evident from the flight and inverted pendulum control problems.

5.2.3 Performance and Computation Load

By presenting three different simulation examples, it was shown that the proposed MPC gives commendable performance improvement in the presence of exogenous, unmeasurable disturbances and system constraints. The improvements in output regulation were recorded in both state and output feedback control examples. Moreover, the novel MPC can be said to result in a more fault-tolerant control if one views the saturation of the inputs (as in the examples) to be caused by a fault in the controlled system. Pertinent to mention is that the performance of the proposed controller is highly dependent on selecting an appropriate weighting matrix for the disturbance increment introduced into the cost function. The role played by the proposed integrated state and disturbance estimator in providing estimates of unmeasured states and disturbances was also crucial in achieving the improved performance.

To ensure that the computational load of the proposed scheme is reasonable, the control horizon is separated from the prediction horizon. The ‘disturbance horizon’ is taken to be equal to the control horizon, which is generally smaller than the prediction horizon. The comparison of the time taken to solve the QPs of the compared MPCs showed that the proposed scheme did not result in a computational burden increase. The separation of disturbance horizon from the control horizon may also be considered to further reduce the computational load of the proposed algorithm when the control horizon used in a given problem is quite long.
5.3 Concluding Remarks

The most widely deployed approaches to disturbance rejection in MPC were discussed and it was pointed out that the methods have the same setback of being effective for constant disturbances. It was shown that the complete velocity forms of MPC are specific choices of disturbance models and observers. Therefore, making it appropriate to refer to the velocity forms as particular cases of the general approach to disturbance rejection in MPC.

As a main contribution of this thesis, a novel approach to the rejection of external disturbances, which may be relatively fast-varying in constrained linear MPC was presented. The proposed controller of this dissertation is designed using a cost function that is a function of the control input deviation and the disturbance deviation. To ensure that the system was driven according to the minimization of the cost function, the computed optimal control and disturbance deviations are used in every time step. This approach provides an additional degree of freedom for the control signal, which was exploited to formulate a control input that is also a function of the estimated exogenous disturbance. By ensuring that the estimated disturbance increment is always in opposition to the actual disturbance increment, the effects of the exogenous disturbances are further mitigated.

Although the proposed scheme is suitable when the external disturbances are measured, it was assumed that all system disturbances are not measurable as this seems to be more practical in many engineering applications. Hence, a combined state and disturbance estimator that uses an anti-stable input filter was also presented. However, it is possible to use a stable input filter if there is a need to characterise a disturbance with a dominant time-varying component. The gain of the observer was obtained by solving a discrete-time $\mathcal{H}_2$ minimization problem.

To demonstrate the merits of the proposed scheme over conventional increment form of MPC, the simulation of three different systems were presented. In the first case study presented, the proposed scheme exhibited minimal oscillations before reaching steady-state. The superiority of the new MPC was demonstrated in both state and output feedback control of the inverted pendulum model used in the example. In the case of state feedback control, the integral time-weighted absolute error (ITAE) performance measure showed that the proposed approach gave above 50% improvement over conventional MPC based on deviation models.

The longitudinal motion control of an airplane given in the second case study, perhaps, gave the most interesting results where conventional MPC could not get both outputs of the system to the desired reference points. Moreover, the output that reached the level of the reference exhibited enormous continuous oscillations around the set-points. This is in direct contrast to the performance of the proposed scheme that tracked the output references with reduced oscillations.

The permanent magnet synchronous motor example demonstrated the superiority of the proposed scheme even when there is no meaningful saturation of the control input. All the improvements discussed above were provided by the proposed scheme without necessarily increasing the
computational cost of the conventional method. The effective performance of the proposed controller under tight constraints that resulted in the control signal and its rate saturation is noteworthy. This important feature provides a basis to argue that the proposed scheme has shown potential to be more effective in actuator saturation fault tolerant-control. Therefore, making the new algorithm a choice to be considered for systems where input saturation is likely to occur.

The stage is now set to address the important questions raised in the motivation behind this thesis. Firstly, the role played by appropriately posing an optimisation problem in ensuring the elimination of steady-state error was discussed in Chapter 2, where it was shown that a non-zero output tracking is impossible even in the absence of disturbances or plant mismatch if the optimisation problem is not well-posed. This motivated the introduction of the concept of ‘optimal disturbance’, which allowed the combined use of the increment form of disturbance and control since one would otherwise not drive the controlled plant in accordance with the proposed cost function which depends on both variables.

Secondly, the introduced weighting matrix of the disturbance increment in the new cost function made it possible to influence the manner in which the external varying disturbances affect the response of a system.

And lastly, the assumption that the external disturbances are not constant or slowly-varying, as opposed to the conventional approaches, served as a bedrock to developed the proposed algorithm. This was then exploited to introduce estimated disturbance increment into the controller that helped to further minimise the negative effects of the non-zero disturbance increment.

5.4 Future Works

5.4.1 Velocity form versus Disturbance Models

The use of disturbance models and observers is more widely used in practice than the increment form of MPC. However, the major setbacks that are associated with the use of disturbance models which include the need to compute steady-state targets, separate design of disturbance model and estimator, the chances of re-design or re-tuning of the controller when it needs to be used on a new plant, provides a sufficient basis to question the superiority of the approach. Apart from the fact that the velocity forms may eliminate the need for separate design of an observer, this is especially the case since an increase in computational cost which is a major consideration in the use of velocity forms that are particular cases of the disturbance model and observer approach may no longer be very significant due to the continuous advancements that have been witnessed in computer technology. Although the deviation models based MPCs are particular forms of the disturbance model approach, their performances can be studied in the presence of measurement noise and also in terms of tolerance to actuator saturation. Such a study will help put into perspective, the most
suitable method in terms of simplicity, superiority and suitability for generic applications.

5.4.2 Robust Model Predictive Control

The systems considered in this work are based on nominal models but in many practical systems, uncertainties are prevalent. It would be interesting to investigate the use of the proposed ‘optimal disturbance’ concept in robust MPC design for systems having uncertainties with a particular interest in the now common tube-based techniques [28, 59, 58, 19, 83, 1] and even LMI approaches. This, of course, would also demand the use of robust observers as well.

5.4.3 Real-time Implementation

Although the proposed scheme has shown very impressive performance in simulated systems, it would be interesting to see how well it performs in real applications. The study could be carried out on a laboratory set-up of a particular system of interest.
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Appendix A

A.1 Inverted Pendulum: Observer Gain Matrix

A.1.1 State Feedback Case

$$L = \begin{bmatrix} -1.0749 & -0.1195 & -0.0003 & -0.0094 \\ -1.5249 & -1.8382 & -0.0127 & -0.4243 \\ 0.0245 & -0.0119 & -1.0002 & -0.1071 \\ 0.4932 & -0.3995 & -0.0071 & -1.2357 \\ -0.1886 & -8.4856 & -0.1414 & -4.7142 \end{bmatrix}$$

A.1.2 Output Feedback Case

$$L = \begin{bmatrix} 9.4 & -21.4 \\ 202.9 & -397.4 \\ 6.4 & -13.7 \\ 110.9 & -212.6 \\ 1342.1 & -2455.9 \end{bmatrix}$$

A.2 Flight Control: Observer Gain Matrix

$$L = \begin{bmatrix} -1.4364 & -0.0912 & -0.2182 & -0.4045 \\ -0.0817 & -0.9823 & -0.7883 & -0.0874 \\ -0.2203 & -0.0340 & -1.0634 & -0.2183 \\ -0.4368 & -0.0868 & -0.3161 & -1.4367 \\ -43.6691 & -8.7338 & -21.8345 & -43.6691 \end{bmatrix}$$
A.3 PMSM model: Observer Gain Matrix

\[ L = \begin{bmatrix}
6.1956 & 0.0000 \\
-2.7852 & 0.1786 \\
56.9232 & -1.9953 \\
-0.1252 & 0.0117 \\
\end{bmatrix} \]