Flavor-Dependence and Higher Orders of Gauge-Independent Solutions in Strong Coupling Gauge Theory

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Abstract

The fermion flavor $N_f$ dependence of non-perturbative solutions in the strong coupling phase of the gauge theory is reexamined based on the interrelation between the inversion method and the Schwinger-Dyson equation approach. Especially we point out that the apparent discrepancy on the value of the critical coupling in QED will be resolved by taking into account the higher order corrections which inevitably lead to the flavor-dependence. In the quenched QED, we conclude that the gauge-independent critical point $\alpha_c = 2\pi/3$ obtained by the inversion method to the lowest order will be reduced to the result $\alpha_c = \pi/3$ of the Schwinger-Dyson equation in the infinite order limit, but its convergence is quite slow. This is shown by adding the chiral-invariant four-fermion interaction.

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1 Introduction

To study the dynamical symmetry breaking, we need some non-perturbative tool. In recent several years, the Schwinger-Dyson (SD) equation has played such a role and much effort is devoted to solving the SD equation for the fermion propagator in gauge theories. In QED$_4$ with $N_f$ fermion flavors, QED$_4[N_f]$, there exists a critical gauge coupling $\alpha_c$ for $\alpha$, above which ($\alpha > \alpha_c$) the chiral symmetry is spontaneously broken, i.e., $\langle \bar{\psi} \psi \rangle \neq 0$ in the absence of bare fermion mass. Such non-perturbative solutions of the SD equation were able to reveal many interesting features of the strong coupling phase in gauge theories, see e.g., [1, 2] for review. However this approach has the following drawbacks.

1. systematic improvement: It is rather difficult to incorporate the higher order effect systematically. In contrast to the perturbation theory, it is not necessarily clear which terms should be taken account of to the next order. Even if this may be possible, it is quite difficult to actually solve it without drastic approximations. In essence, the SD equation for the fermion propagator should be solved simultaneously with those for the photon propagator and the vertex function and such an attempt has been done in a restricted case with a bare vertex [3]. However the ad hoc approximation for the vertex function and the photon propagator inevitably truncates this series of hierarchy.

2. gauge invariance: Any gauge-independent result has not been obtained so far. Almost all the analyses of the SD equation have been done under the choice of the Landau gauge. However, qualitatively different and pathological results are obtained, once the other gauges than the Landau gauge are chosen [4]. This implies that the adopted approximation breaks the gauge invariance which should hold in gauge theories. In other gauges than the Landau gauge, we must take into account the wavefunction renormalization for the fermion, even in the quenched approximation [5]. This enforces us to consider the vertex correction in the light of the Ward-Takahashi (WT) identity. Indeed, many physicists have tried to recover the gauge-invariance by modifying the vertex so as to satisfy the WT identity. However the WT identity does not uniquely determine the vertex function of the SD equation in the non-perturbative sense [6], in contrast with the perturbation theory. We must impose further conditions or requirements; absence of the kinematical singularity, multiplicative renormalizability, agreement with the perturbation theory in the weak-coupling limit and so on. In spite of these trials, however, completely gauge-independent results have not yet been obtained in the framework of the SD equation. See references [7, 8, 9, 10] for the most recent result.
The inversion method \cite{12} is a generalization of the Legendre transformation in the effective-action formalism \cite{11} and enables us to calculate the order parameter in the symmetry-breaking non-perturbative phase in a perturbative way. The order parameter in the effective action is uniquely determined by the form of the source term in the action $S_J$ from which we start, while in the inversion method it can be chosen more freely providing that it coincides with the original action when the source term vanishes. In the inversion method, we can extract the non-perturbative solution by perturbative calculations. This method allows us to perform the systematic improvement \cite{13}. By using this method, the strong-coupling phase of QED is studied in a gauge-invariant manner \cite{14, 15}. Accordingly the inversion method is very useful and has remarkable features.

However it has not attracted so much attention of most physicists who have been studying the dynamical symmetry breaking. For one thing, the relationship between the inversion method and the SD equation approach was not necessarily made clear so far; for another thing, the SD equation approach is more direct to obtain the physical quantities, such as the fermion mass and the pion decay constant.

Moreover there exists an apparently contradicting result on the critical coupling already in QED between the inversion method and the SD equation approach.

In this letter we show that there is no contradiction between the result of the inversion method and the SD equation approach. Our systematic treatment of the solution of the SD equation and its comparison with the inversion result suggests that the correct gauge-independent critical coupling in the quenched limit should be given by $\alpha_c = \pi/3$.

\section{SD equation and inversion method}

First we recall the result of the SD equation. The SD equation is also derived from the inversion method as follows. After introducing the source term,

$$S_J = \int d^4 x \int d^4 y J(x, y) \bar{\psi}(x) \psi(y),$$

into the original action of (euclidean) QED\[4][N_f]:

$$S_{QED} = \int d^4 x \mathcal{L}_{QED},$$

$$\mathcal{L}_{QED} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}^a \gamma_\mu \gamma_5 \psi^a A^\mu, \quad (a = 1, ..., N_f) \quad (2)$$
the inversion process with respect to the general bilocal source $J(x, y)$ yields, to the lowest order $O(e^2_0)$, the equation:

$$J(p) = - S(p)^{-1} + S_0(p)^{-1} - e_0^2 \int \frac{d^4q}{(2\pi)^4} \gamma_\mu S(q) \gamma_\nu D_0^{\mu\nu}(q-p),$$  \hspace{1cm} (3)

where $J(p)$ is the Fourier transformation of $J(x-y)$, and $S_0(p)$ and $D_0^{\mu\nu}(k)$ denote the bare fermion propagator and the bare photon propagator, respectively. By putting $J = 0$, the quenched ladder SD equation for the fermion propagator $S(p)$ is obtained. The quenched ladder SD equation in the Landau gauge \[16\] leads to the critical coupling $^1$

$$\alpha_c(0) = \frac{\pi}{3} = 1.041975 \cdots.$$  \hspace{1cm} (4)

On the other hand, the lowest order inversion method applied to the chiral condensates $\langle \bar{\psi}\psi \rangle$ of QED$_4[N_f = 1]$ leads to the gauge-independent critical point $^2$

$$\alpha_c = \frac{2\pi}{3} = 2.094395 \cdots,$$  \hspace{1cm} (5)

as shown in \[14\]. The critical point $\alpha_c$ depends in general on the number of fermion flavors $N_f$, $\alpha_c(N_f)$ \[17, 18, 20\]. To the lowest order inversion, however, the critical coupling $\alpha_c$ of QED$_4[N_f]$ is independent of the flavor number of fermion, $N_f$, as pointed out in \[15\].

The critical coupling eq. (5) is twice as large as the result eq. (4) of the quenched ladder SD equation eq. (3) in the Landau gauge. Then the result eq. (5) seems to contradict with the inversion scheme. Because the result eq. (5) follows from a special case, i.e., the uniform source $J$ with $J(x, y) = J\delta^4(x-y)$ in the same inversion procedure in which case we have the relation after inversion:

$$J = \frac{4\pi^2}{N_f\Lambda_f^2} \left[1 - \frac{\alpha}{\alpha_c}\right] \langle \bar{\psi}\psi \rangle$$

$^1$The value $\alpha_c(\Lambda)$ of the critical coupling obtained from the quenched ladder SD equation for the fermion propagator $S(p) = [A(p^2)\gamma^\mu p_\mu - B(p^2)]^{-1}$ depends on both the gauge parameter $\xi$ and the ultraviolet cutoff $\Lambda$ \[4, 9\]. For $\xi \leq -3$ there is no nontrivial solution: $B(x) \equiv 0$. For $0 > \xi > -3$, $\alpha_c(\Lambda)$ exists, but $\alpha_c(\Lambda) \downarrow 0$ as $\Lambda \uparrow \infty$. In the Landau gauge $\xi = 0$, $\alpha_c(\Lambda) \rightarrow \pi/3$ as $\Lambda \uparrow \infty$. For $\xi > 0$, $\alpha_c(\Lambda) \rightarrow \alpha_c^\infty > \pi/3$ as $\Lambda \uparrow \infty$ and $\alpha_c^\infty$ increases monotonically in $\xi$. Note that $A(p^2) > 1$ for $\xi > 0$ and $0 < A(p^2) < 1$ for $\xi < 0$. The quenched ladder SD equation with the bare vertex is consistent with the WT identity only in the Landau gauge where $A(p^2) \equiv 1$.

$^2$This value is obtained if we put the same ultraviolet cutoff for fermion momentum $\Lambda_f$ and photon one $\Lambda_p$ in $\alpha_c = 2\pi/(3\eta)$ with $\eta = \Lambda_p^2/\Lambda_f^2$ \[14\]. Regularization independence of the critical coupling will be discussed in a forthcoming paper in detail.
Rather the inversion result eq. (5) is in good agreement with that of the unquenched ladder SD equation (in the Landau gauge) for one fermion flavor $N_f=1$ \cite{17, 18, 19, 20, 21}: \[ \alpha_c(1) = 2.00, \tag{7} \]
in the presence of the 1-loop vacuum polarization for the photon propagator \[ D_{\mu\nu}(k) = \frac{1}{k^2 [1 + \Pi^{(1)}(k^2)]} \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right), \tag{8} \]
where $\Lambda$ is the ultraviolet cutoff.

We clarify these points based on the systematic treatment of the SD equation \cite{18, 22, 23}.

### 3 Analysis

All the above statements will be made clear if we include the chiral-invariant four-fermion interaction \cite{24} (gauged Nambu–Jona-Lasinio model \cite{25}) as \[ \mathcal{L}_{4F} = \frac{1}{2} G_0 [(\bar{\psi}^a \psi^a)^2 - (\bar{\psi}^a \gamma_5 \psi^a)^2], \quad (a = 1, \ldots, N_f), \tag{9} \]
and consider the critical line in the two-dimensional space of coupling constants $(\epsilon_0, G_0)$ instead of the critical point. We define the dimensionless four-fermion coupling $g$ by $g = \frac{N_f G_0 \Lambda^2}{4\pi^2}$, so that the critical four-fermion coupling constant $g_c$ is kept fixed: $g_c = 1$ for arbitrary $N_f$ (in the chain approximation).

In the previous paper \cite{15} the inversion method was applied to the gauged Nambu–Jona-Lasinio (NJL) model which reduces to QED by switching off the chiral-invariant four-fermion interaction. \footnote{Within the framework of the SD equation, an approach to recover the gauge invariance approximately was tried for the gauged NJL model in \cite{26}.} To the lowest order, we have obtained the gauge-independent results:

\[ s_{\pi N} + \frac{64\pi^6}{\eta N_f^3 \Lambda^8} (\bar{\psi}\psi)^3 \ln^2 \left( \frac{16\pi^4}{N_f^2 \Lambda^4 \Lambda^2_f} (\bar{\psi}\psi)^2 \right) + \mathcal{O}(\langle \bar{\psi}\psi \rangle^3 \ln \langle \bar{\psi}\psi \rangle^2). \tag{6} \]
1. the critical line separating the spontaneous chiral-symmetry breaking phase from the chiral symmetric one,

2. the mean-field value $1/2$ for the critical exponent of the chiral order parameter $\langle \bar{\psi}\psi \rangle$,

3. the large anomalous dimension $\gamma_m = 2$ for the composite operator $\bar{\psi}\psi$ on the whole critical line.

These results are in good agreement with the previous results of the SD equation which includes the 1-loop vacuum polarization to the photon propagator in QED [22, 23]. However the critical line obtained from the SD equation deviates from the inversion result in the region of the strong gauge coupling, $\alpha = g_0^2/4\pi > 4$ where the bare coupling of the four-fermion interaction is negative [23]. This indicates necessity of incorporating the higher orders in the inversion calculation.

From the SD equation, the critical line of the gauged NJL model is shown to take the following form [22, 23].

$$g = \frac{1 + \sum_{n=1}^{\infty} a_n / z_0^n}{1 + \sum_{n=1}^{\infty} b_n / z_0^n},$$ (10)

where

$$a_1 = -\sigma^2 - 1,$$

$$a_2 = \frac{1}{2}\sigma^2(\sigma^2 - 2\sigma + 1),$$

$$b_1 = -(\sigma - 1)^2,$$

$$a_{n(\geq 3)} = \frac{(-1)^n}{n!}\sigma^2[P_n(\sigma) + n] \prod_{i=2}^{n-1} P_i(\sigma),$$

$$b_{n(\geq 2)} = \frac{(-1)^n}{n!}(\sigma - n)^2 \prod_{i=2}^{n} P_i(\sigma),$$ (11)

and

$$P_i(\sigma) := \sigma^2 - 2(i-1)\sigma + (i-1)(i-2),$$ (12)

with

$$\sigma = \frac{9}{4N_f}, \quad z_0 := \frac{3\pi}{N_f\alpha}.$$ (13)
This is derived from the asymptotic solution of the SD equation \[18\] with an expansion parameter \(z_0^{-1}\). Up to the order \(\mathcal{O}(z_0^{-10})\), the critical line is written as

\[
g = 1 - 2\lambda - \left(1 - \frac{2}{\sigma}\right)\lambda^2 - 2 \left(1 - \frac{3}{\sigma} + \frac{2}{\sigma^2}\right)\lambda^3 \\
- \left(5 - \frac{22}{\sigma} + \frac{30}{\sigma^2} - \frac{12}{\sigma^3}\right)\lambda^4 - 2 \left(1 - \frac{2}{\sigma}\right)^2 \left(7 - \frac{14}{\sigma} + \frac{6}{\sigma^2}\right)\lambda^5 \\
-2 \left(1 - \frac{2}{\sigma}\right)^2 \left(21 - \frac{79}{\sigma} + \frac{90}{\sigma^2} - \frac{30}{\sigma^3}\right)\lambda^6 \\
-4 \left(1 - \frac{2}{\sigma}\right)^2 \left(33 - \frac{187}{\sigma} + \frac{379}{\sigma^2} - \frac{318}{\sigma^3} + \frac{90}{\sigma^4}\right)\lambda^7 \\
- \left(1 - \frac{2}{\sigma}\right)^2 \left(429 - \frac{3304}{\sigma} + \frac{9960}{\sigma^2} - \frac{14508}{\sigma^3} + \frac{9996}{\sigma^4} - \frac{2520}{\sigma^5}\right)\lambda^8 \\
-2 \left(1 - \frac{2}{\sigma}\right)^2 \left(715 - \frac{7048}{\sigma} + \frac{28760}{\sigma^2} - \frac{61756}{\sigma^3} + \frac{72708}{\sigma^4}\right)\lambda^9 \\
-\frac{43608}{\sigma^5} + \frac{10080}{\sigma^6}\lambda^9 \\
-2 \left(1 - \frac{2}{\sigma}\right)^2 \left(2431 - \frac{29479}{\sigma} + \frac{153724}{\sigma^2} - \frac{445060}{\sigma^3} + \frac{766920}{\sigma^4}\right)\lambda^{10} \\
+\mathcal{O}\left(\lambda^{11}\right),
\]  

(14)

where

\[
\lambda := \frac{\sigma}{z_0} = \frac{3\alpha}{4\pi}.
\]  

(15)

Now this expression allows us to take the quenched limit \(N_f \to 0\):

\[
g = 1 - 2\lambda - \lambda^2 - 2\lambda^3 - 5\lambda^4 - 14\lambda^5 \\
-42\lambda^6 - 132\lambda^7 - 429\lambda^8 - 1430\lambda^9 - 4862\lambda^{10} - \mathcal{O}(\lambda^{11}),
\]  

(16)

with \(\lambda\) being kept fixed, although both \(\sigma\) and \(z_0\) have singular \(N_f\)-dependence. Therefore we are lead to conclude that the critical line eq. \((14)\) has the correct quenched limit \[\] [27]:

\[
g = \frac{1}{4} \left(1 + \sqrt{1 - \lambda/\lambda_c}\right)^2, \quad \lambda_c = \frac{3\alpha_c}{4\pi} = \frac{1}{4},
\]  

(17)

\footnote{Similar argument can be done for the QCD-like gauged NJL model, as already pointed out in [28, 2].}
since eq. (17) is expanded as
\[ g = 1 - 2\lambda - \sum_{n=2}^{\infty} \frac{\prod_{k=1}^{n-1} (4k - 2)}{n!} \lambda^n. \] (18)

To the lowest order \( \mathcal{O}(\alpha) \), the critical line eq. (14) is given by
\[ g = 1 - \frac{3\alpha}{2\pi}. \] (19)

This is independent of the fermion flavor and completely agrees with the lowest order inversion result [15]. This implies the critical coupling \( \alpha_c = \frac{2\pi}{3} \) in pure QED \( (g = 0) \) which is nothing but the result of Ukita, Komachiya and Fukuda [14].

The complete agreement of the SD result eq. (19) with the inversion result in the lowest order \( \mathcal{O}(\alpha) \) is somewhat surprising. Because in the analytical treatment of the SD equation the standard approximation \( \Pi^{(1)}((p - q)^2) \approx \Pi^{(1)}(\text{max}\{p^2, q^2\}) \) is adopted in order to perform the angular integration and to conclude no wavefunction renormalization in the Landau gauge as in the quenched case, while there is no approximation in the inversion method to this order. However the critical coupling is insensitive to this approximation [19], although numerical calculation of the SD equation leads to slightly different value under different approximations [29]. For \( N_f = 1 \), our result shows the critical point up to the order \( \mathcal{O}(\alpha^{10}) \) (see Table 1)
\[ \alpha_c(1) = \frac{\pi}{3} \times 1.90942 = 1.9995. \] (20)

In order to obtain \( N_f \)-dependent result, we must take into account at least the next order \( \mathcal{O}(\alpha^2) \). Actually, the critical line to the order \( \mathcal{O}(\alpha^2) \) is given by
\[ g = 1 - \frac{3\alpha}{2\pi} - \frac{9}{16} \left( 1 - \frac{8}{9} N_f \right) \left( \frac{\alpha}{\pi} \right)^2. \] (21)

From the viewpoint of the inversion, the different \( N_f \)-dependence does appear in the next order from a vacuum diagram with two fermion loops, see Figure 1 of [14]. It is worth remarking that the numerical result of the SD equation, \( \alpha_c = 2.00 \) for \( N_f = 1 \) [17], is obtained from the above equation by solving the second order algebraic equation: \( 0 = 1 - 2\lambda - (1 - \frac{8}{9} N_f)\lambda^2 \), which has indeed a positive solution at \( \alpha_c = 2.03924 \) for \( N_f = 1 \). The ratio of the second order term to the first order one is \((1 - \frac{8}{9} N_f)/4 = 1/36 = 0.0277 \) for \( N_f = 1 \) at \( \alpha = 2\pi/3 \). This is why the lowest order result eq. (5) is very close to the value eq. (7). Therefore this coincidence for \( N_f = 1 \) is an accidental one due to the negligible second order term. However, the coefficient of the \( \mathcal{O}(\alpha^2) \) term in eq. (21) changes the sign for \( N_f > 9/8 \). Hence \( g \searrow +\infty \) as
\(\alpha \to +\infty\), which contradicts with the SD equation result: \(g \downarrow -\infty\) as \(\alpha \to +\infty\) \[23, 30\]. Nevertheless the critical line eq. \[21\] can give good estimate at least for not so large \(\alpha\): \(\alpha \leq \mathcal{O}(\alpha_c)\). Actually the result in the case of \(N_f = 2\) should be compared with the value \(\alpha_c(2) = 2.5\) obtained from the numerical solution of the SD equation \[17, 18, 19\].

We must mention the behavior of the series in more higher orders. For \(N_f > 0\), it is known \[22, 2\] that the series eq. \(\text{(14)}\) is not convergent and at most strong asymptotic series (hence Borel summable) \[31\], while in the quenched case the series eq. \(\text{(14)}\) converges to eq. \(\text{(17)}\) as \(N \to \infty\). Therefore the critical coupling constant \(\alpha_c\) approaches to the optimal value at a certain \(N\) and then deviates from it, which can be seen in Table 1. However, for large \(N_f\) the asymptotic series fails to give good estimate on the critical coupling, since the expansion parameter \(z_0^{-1}\) should be small: \(z_0^{-1} = N_f \alpha / (3\pi) < 1\), as already pointed out \[18\].

In Table 1, it is demonstrated that all order results are needed in order to recover the correct critical coupling \(\alpha_c(0) = \pi / 3\) of the quenched limit from the unquenched SD equation, since the convergence to \(\alpha_c(0) = \pi / 3\) is very slow; \(\alpha_c(0) = (\pi / 3) \times 1.00957\) to the order \(\mathcal{O}(\alpha^{500})\).

## 4 Conclusion

For QED\(_4\)[\(N_f\)], it has been shown that the flavor \(N_f\)-independent critical value \(\alpha_c = 2\pi / 3\) is a common value to the "lowest order" \(\mathcal{O}(\alpha)\) in the SD equation approach as well as the inversion method. The key point lies in the fact that the \(N_f\)-dependence appears in the next order \(\mathcal{O}(\alpha^2)\) for the first time. Therefore it is merely an accidental coincidence that the lowest order inversion result \(\alpha_c = 2\pi / 3\) is nearly equal to the value \(\alpha_c = 2.00\) obtained from the unquenched ladder SD equation for \(N_f = 1\).

The quenched case, \(N_f = 0\), can be extrapolated from the expression eq. \(\text{(14)}\) which we have obtained from the SD equation. However, the convergence to \(\alpha_c(0) = \pi / 3\) is very slow as demonstrated in Table 1. Therefore, in order to obtain the quenched result eq. \(\text{(4)}\) in the inversion method, we need to include all the orders of vacuum diagrams with only one fermion loop (with no further internal fermion loop). This subtlety seems to reflect the essential singularity in the Miransky scaling \[16\] in the quenched QED. To obtain such a correct \textit{gauge-independent} quenched result, we have to sum up all the order results of the inversion.

To really confirm our claims, anyway it is necessary to perform the calculation to higher orders in the inversion. It is easily shown that the gauge-independence

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5The "order" of the result obtained from the SD equation is defined in correspondence with that of the inversion method, see eq. \(\text{(14)}\).
is guaranteed also in the higher orders. Then it is important to study the next order \( O(\alpha^2) \) of the inversion method in order to determine whether or not the gauge-independent inversion result really agrees with the SD equation result in the Landau gauge eq. (21). Details on this point will be reported in a subsequent paper.

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Table 1. Critical coupling constant $\alpha_c$ in unit of $\pi/3$ in QED$_t[N_f]$, obtained by solving the algebraic equation eq. (14) $g = f(\alpha) = 0$ up to the order $O(\alpha^N)$. To the lowest order $O(\alpha)$, $\alpha_c = 2\pi/3$ irrespective of the fermion flavor $N_f$. In the quenched case ($N_f = 0$) the critical coupling approaches the value, $\alpha_c = \pi/3$ for large $N$. The symbol $-$ stands for no real positive solution to the algebraic equation $f(\alpha) = 0$ and $*$ implies that we have no available data.