Extended Kung–Traub Methods for Solving Equations with Applications

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Abstract: Kung and Traub (1974) proposed an iterative method for solving equations defined on the real line. The convergence order four was shown using Taylor expansions, requiring the existence of the fifth derivative not in this method. However, these hypotheses limit the utilization of it to functions that are at least five times differentiable, although the methods may converge. As far as we know, no semi-local convergence has been given in this setting. Our goal is to extend the applicability of this method in both the local and semi-local convergence case and in the more general setting of Banach space valued operators. Moreover, we use our idea of recurrent functions and conditions only on the first derivative and divided difference, which appear in the method. This idea can be used to extend other high convergence multipoint and multistep methods. Numerical experiments testing the convergence criteria complement this study.

Keywords: Kung–Traub method; Banach space; convergence criterion

1. Introduction

We consider approximating a solution \( x^* \) of equation

\[
F(x) = 0,
\]

where \( F : \Omega \subset V_1 \rightarrow V_2 \) is an operator acting between Banach spaces \( V_1 \) and \( V_2 \) with \( \Omega \neq \emptyset \). Kung and Traub, in [1], introduced a fourth-order iterative method for solving nonlinear equations on the real line. This method in Banach space is defined for \( n = 0, 1, 2, \ldots \) by

\[
\begin{align*}
  y_n &= x_n - F'(x_n)^{-1}F(x_n), \\
  x_{n+1} &= y_n - [y_n, x_n; F]^{-1}F'(x_n)[y_n, x_n; F]^{-1}F(y_n).
\end{align*}
\]

Here \([.,.; F] : \Omega \times \Omega \rightarrow L(V_1, V_2)\) is a divided difference of order one [2]. The convergence order was obtained using Taylor expansions and hypotheses on the derivative of \( F \) of order up to five. Note that the method involves also the derivative of order one, so the assumptions on the fifth derivative reduce the applicability of the method [1,3–5].

For example: Let \( V_1 = V_2 = \mathbb{R}, \Omega = [-0.5, 1.5] \). Define \( \lambda \) on \( \Omega \) by

\[
\lambda(t) = \begin{cases} 
  t^3 \log t^2 + t^5 - t^4 & \text{if } t \neq 0 \\
  0 & \text{if } t = 0.
\end{cases}
\]
Then, we have \( t_\ast = 1 \),
\[
\lambda'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.
\]

Obviously \( \lambda'''(t) \) is not bounded on \( \Omega \). Therefore, the convergence of method (2) is not guaranteed by the analysis in [1]. In order to avoid Taylor series expansions but still obtain the fourth order of convergence for method (2), we use the computational order of convergence and the approximate computational order of convergence, which do not require more than one derivative (see Remark 1.2b).

In this paper, we introduce a majorant sequence and use our idea of recurrent functions to extend the applicability of method (2). Our analysis includes error bounds and results on uniqueness of \( x^\ast \) based on computable Lipschitz constants not given before in [1] and in other similar studies using Taylor series [3–13]. The advantages of the extended method include: Applications for solving nonlinear Banach space valued equations are not limited to systems of finite dimensional Euclidean space. Local convergence includes computable upper error bounds not given before. Moreover, the semi-local convergence not given before is proved. The motivation for writing this paper is the extension of the applicability of method (2), as already illustrated by the example. The novelty of the paper includes the extension of the convergence domain in both the local as well as the semi-local convergence case and the introduction of the recurrent functions proving technique, which can be used in other methods too [14–27].

The rest of the paper is set up as follows: In Section 2, we present results on majorizing sequences. Sections 3 and 4 contain the semi-local and local convergence, respectively, where in Section 5, the numerical experiments are presented. Concluding remarks are given in Section 6.

2. Majorizing Sequences

We present results on majorizing sequences.

**Definition 1.** Let \( \{u_n\} \) be a sequence in a Banach space. Then, a nondecreasing scalar sequence \( \{m_n\} \) is called majorizing for \( \{u_n\} \) if
\[
\|u_{n+1} - u_n\| \leq m_{n+1} - m_n \quad \text{for each } n = 0, 1, 2, \ldots .
\]

By this definition, we can use sequence \( \{m_n\} \) to study the convergence of \( \{u_n\} \).

Let \( \eta > 0, \ell > 0, \ell_i > 0, i = 0, 1, 2, \ldots , 5 \) be the given parameters. Define scalar sequences \( \{s_n\}, \{t_n\} \) for each \( n = 0, 1, 2, \ldots \) by
\[
\begin{align*}
t_0 & = 0, \\
s_0 & = \eta, \\
t_1 & = s_0 + \frac{\ell_0(s_0 - t_0)^2}{2(1 - \ell_0 t_0)^2}, \\
s_{n+1} & = t_{n+1} + \frac{\alpha_{n+1}}{1 - \ell_0 t_{n+1}}, \\
t_{n+1} & = s_{n+1} + \frac{\ell_4 t_{n+1}(s_{n+1} - t_{n+1})^2}{2(1 - \ell_1 (s_{n+1} + t_{n+1}))^2},
\end{align*}
\]
where \( \alpha_{n+1} = \left( \ell_3(t_{n+1} - t_n) + \frac{\ell_3(1 + \ell_1)(s_n + t_n)}{1 - \ell_0 t_n} \right)(t_{n+1} - s_n) \).

**Lemma 1.** Suppose:
\[
\ell_1 \eta < 1,
\]
for each \( n = 0, 1, 2, \ldots \),
\[
t_{n+1} < \frac{1}{\ell_0}
\]
and
\[
s_{n+1} + t_{n+1} < \frac{1}{\ell_1}.
\]
Then, sequences \( \{s_n\}, \{t_n\} \) are nondecreasing, bounded from above by \( \frac{1}{\ell} \) and as such they converge to their unique least upper bound \( t^* \in [\eta, \frac{1}{\ell}] \). Moreover, the following hold for each \( n = 0, 1, 2, \ldots \)

\[ t_n \leq s_n \leq t_{n+1}. \]

**Proof.** It follows from the definition of sequence \( \{s_n\}, \{t_n\} \) and hypotheses (5)–(7).

**Remark 1.** Hypotheses (6) and (7) are verified only in special cases. That is why we introduce stronger hypotheses implying those of Lemma 1 but not necessarily vice versa.

It is convenient for us to define sequences of functions and functions on the interval \( M = [0,1) \) for each \( n = 1, 2, \ldots \) as follows:

\[
\begin{align*}
    f_n(t) &= \ell_5(t^{2n} + t^{2n-1})\eta + \ell_1 t^{2n-1} \eta \\
           &\quad + \ell_1 \eta(t^{2n} + 2(1 + t + \ldots + t^{2n-1}))\eta^2 \\
           &\quad + \ell_0 \eta(t^{2n} + 2(1 + t + \ldots + t^{2n-1})) - \ell_2^2 \eta^2 - 1, \\
    f(t) &= a_7 t^7 + a_6 t^6 + a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0, \\
    f_\infty(t) &= -\left(1 - \frac{\ell_0 \eta}{1-t}\right)^2
\end{align*}
\]

\[
\begin{align*}
    g_n(t) &= \frac{\ell^4 \eta^2}{2} t^{2n+2}(1 + t + \ldots t^{2n+1}) \\
           &\quad + 2\ell_1 \eta(t^{2n+2} + 2(1 + t + \ldots + t^{2n+1})) \\
           &\quad - \ell_1^2 \eta^2(t^{2n+2} + 2(1 + t + \ldots + t^{2n+1}))^2 - 1, \\
    g(t) &= \frac{\ell^4}{2} \eta(t^7 + t^6 + t^5 + t^4 - t - 1) + 2\ell_1 \eta(1 + t)^2 \\
           &\quad + \ell_1^2 \eta(1 + t^2)(\frac{4}{1-t} + 3t^5 + 2t^4 + t^6),
\end{align*}
\]

and

\[
g_\infty(t) = -\left(1 - \frac{2\ell_1 \eta}{1-t}\right)^2,
\]

where

\[
\begin{align*}
    a_0 &= -(\ell_1 + \ell_5 + 2\ell_1 \ell_2 \eta), \\
    a_1 &= -(\ell_5 + 2\ell_1 \ell_2 \eta), \\
    a_2 &= \ell_5 + \ell_1 + \ell_1 \ell_2 \eta + \ell_5, \\
    a_3 &= \ell_0 + \ell_5 + 2\ell_1 \ell_2 \eta, \\
    a_4 &= \ell_0 + 2\ell_1 \ell_2 \eta, \\
    a_5 &= \ell_0 + 2\ell_1 \ell_2 \eta, \\
    a_6 &= 2\ell_1 \ell_2 \eta \\
    a_7 &= \ell_1 \ell_2 \eta.
\end{align*}
\]
By these definitions we have

\[ f(0) = -(\ell_1 + \ell_5 + 2\ell_1\ell_2\eta) < 0, \]
\[ f(1) = 2(2\ell_0 + 3\ell_1\ell_2\eta) > 0, \]
\[ g(0) = -\frac{\ell\ell_4\eta}{2} < 0 \]

and

\[ g(t) \to +\infty \text{ as } t \to 1^{-}. \]

It then follows by the intermediate value theorem that functions \( f \) and \( g \) have zeros in the interval \((0, 1)\). Denote the smallest such zero by \( b_1 \) and \( b_2 \), respectively. Moreover, we have for each \( t \in M \)

\[ f_\infty(t) \leq 0 \quad (8) \]

and \( g_\infty(t) \leq 0. \quad (9) \)

Furthermore, define scalar sequences \( \{\gamma_n\} \) and \( \{\delta_n\} \) by

\[ \gamma_n = \frac{\ell_3(t_{n+1} - t_n)(1 - \ell_0 t_n) + \ell_4(1 + \ell_1(s_n + t_n))(s_n - t_n)}{(1 - \ell_0 t_n)(1 - \ell_0 t_{n+1})} \]

and

\[ \delta_n = \frac{\ell_4 t_{n+1}(s_{n+1} - t_{n+1})}{291 - \ell_4(t_{n+1} + s_{n+1})^2}. \]

\[ \mu_0 = \max\{\gamma_0, \delta_0\}, \mu_1 = \min\{b_1, b_2\}. \quad (10) \]

Next, we present a second auxiliary result on majorizing sequences.

**Lemma 2.** Suppose that there exists \( \mu \) such that

\[ \mu_0 \leq \mu \leq \mu_1 < 1 - 2\ell_1\eta \quad (11) \]

and (5) holds. Then, sequences \( \{s_n\}, \{t_n\} \) are well defined, nondecreasing, bounded from above by \( t^* = \frac{\eta}{1-\eta} \), and as such they converge to their unique least upper bound \( t^* \in [\eta, t^*] \). Moreover, the following estimates hold for each \( n = 1, 2, \ldots \)

\[ 0 \leq t_{n+1} - s_n \leq \mu(s_n - t_n) \leq \mu^{2n+1}\eta, \quad (12) \]

\[ 0 \leq s_{n+1} - t_n \leq \mu(t_n - s_{n-1}) \leq \mu^{2n}\eta, \quad (13) \]

and

\[ 0 \leq t_n - s_n \leq t_{n+1}. \quad (14) \]

**Proof.** Estimates (12)–(14) hold if

\[ 0 \leq \gamma_k \leq \mu, \quad (15) \]

\[ 0 \leq \delta_k \leq \mu, \quad (16) \]

and

\[ t_k \leq s_k \leq t_{k+1}, \quad (17) \]

are true for \( k = 0, 1, 2, \ldots \). Notice that by the definition of \( s_0, t_1 \) and (5), we have \( s_0 \leq t_1 \). We also have (15)–(17), which hold for \( k = 0 \) by (11). Suppose that estimates (15) and (16) hold for \( k = 1, 2, \ldots, n \). Then, we obtain
function $f$ nondecreasing. Estimates (15) holds if we instead show for $\ell$ or $g$

Function $f$ need a relationship between two consecutive functions $f_k$. It follows by the induction hypotheses and (17) that sequences \( \{s_k\} \) and \( \{t_k\} \) are nondecreasing. Estimates (15) holds if we instead show for $\ell_5 = \ell_3(1 - \ell_0\ell_2)$ that

\[
\ell_5\left(\mu^{2k+3} + \mu^{2k}\right)\eta + \ell_1\eta\mu^{2k} + \ell_2\eta^2\mu^{2k}\left(\frac{1 - \mu^{2k+1}}{1 - \mu} + \frac{1 - \mu^{2k}}{1 - \mu}\right)
\]

\[
- \mu \left(1 - \ell_0 \left(1 - \frac{\mu^2}{1 - \mu} + \frac{1 - \mu^{2k+2}}{1 - \mu}\right)\eta\right)
\]

\[
+ \ell_0^2 (1 - \mu^{2k})(1 - \mu^{2k+2})\eta^2
\]

\[
\leq 0
\]

or

\[
f_k(t) \leq 0 \text{ for } t = \mu.
\]

We need a relationship between two consecutive functions $f_k$. By the definition of function $f_k$, we can write in turn, by adding and subtracting $f_k$

\[
f_{k+1}(t) = f_k(t) + \ell_5(t^{2k+2} + t^{2k+1} - t^2 - t^{2k})\eta + \ell_1\eta(t^{2k+1} - t^{2k-1})
\]

\[
+ \ell_2\eta^2 t^{2k-1}[2(t + t + \ldots + (t^{2k+3}) + (t + t + \ldots + (t^{2k+3}))]
\]

\[
- ((t + t + \ldots + t^{2k}) + (t + t + \ldots + t^{2k}))
\]

\[
+ \ell_0\eta[(1 + t + \ldots + t^{2k+1}) + (1 + t + \ldots + t^{2k+3})]
\]

\[
- ((1 + t + \ldots + t^{2k+1}) + (1 + t + \ldots + t^{2k}))
\]

\[
\leq f_k(t) + \ell_5(t^3 + t^2 - t - 1) + \ell_1(t^2 - 1)
\]

\[
+ (t^2 + t^2 + 2t^2 + 2t^3 + 2t^2 + t^2 - 2)\ell_1\ell_2\eta
\]

\[
+ \ell_0(t^2 + t^2 + t^2 + t^2)\eta^2
\]

\[
= f_k(t) + f(t)t^{2k-1}\eta,
\]

where we used $t^k \leq t, k = 1, 2, \ldots$, since $t \in (0, 1)$. Define $f_\infty(t) = \lim_{k \to \infty} f_k(t)$.

Then, we can show instead of (18) that

\[
f_\infty(\mu) \leq 0,
\]

which is true by (8). Set $c_k = t^{2k+2}(1 + t + \ldots + t^{2k})$ and $d_k = t^{2k+2} + 2(1 + t + \ldots + t^{2k+1})$. As in (15), estimate (16) holds if

\[
g_k(t) \leq 0 \text{ for } t = \mu.
\]

Function $g_k(t)$ can be written as

\[
g_k(t) = \frac{\ell_4\eta^2}{2} t^{2k+2}c_n + 2\ell_1\eta d_n - \ell_1^2\eta^2 d_n^2 - 1.
\]
Then, we again need a relationship between two consecutive functions $g_k$. Notice that
\[
c_{k+1} - c_k = t^{2k+4}(1 + t + \ldots + t^{2k+3}) - t^{2k+2}(1 + t + \ldots + t^{2k+1})
= t^{2k+2}(-1 - t + t^{2k+2} + t^{2k+3} + t^{2k+4} + t^{2k+5}),
\]
\[
d_{k+1} - d_k = (1 + t + \ldots + t^{2k+3}) + (1 + t + \ldots + t^{2k+4})
= t^{2k+2} + 2t^{2k+3} + t^{2k+4},
\]
and
\[
d_{k+1} - d_k = 4(1 + t + \ldots + t^{2k+1}) + 3t^{2k+2} + 2t^{2k+3} + t^{2k+4}.
\]
By adding and substracting $g_k$ from $g_{k+1}$ we obtain
\[
g_{k+1}(t) = g_k(t) + \frac{\ell_4\eta^2}{2}t^{2k+2}(t^{2k+5} + t^{2k+4} + t^{2k+3} + t^{2k+2} - t - 1)
+ 2\ell_1\eta(t^{2k+2} + 2t^{2k+3} + t^{2k+4}) + \ell_2\eta^2(t^2_{k+1} - d_k^2)
\leq g_k(t) + g(t)t^{2k+2}\eta.
\]
Define $g_\infty(t) = \lim_{k \to \infty} g_k(t)$. Then, we can show instead of (21) that
\[
g_\infty(t) \leq 0,
\]
which is true by (11). The induction for estimates (15)–(17) is completed. Hence, sequences $\{s_n\}, \{t_n\}$ are nondecreasing, bounded from above by $t^*$ so they converge to $t^*$. \qed

3. Semi-Local Convergence

Let $U(x_0, r) = \{x \in V_1 : \|x - x_0\| < r, r > 0\}$ and $U[x_0, r] = \{x \in V_1 : \|x - x_0\| \leq r, r > 0\}$. The semi-local convergence analysis of method (2) uses conditions (H1)–(H4).

Suppose:

(H1) There exists $x_0 \in \Omega$ and $\eta \geq 0$ such that $F'(x_0)^{-1} \in L(V_2, V_1)$ and
\[
\|F'(x_0)^{-1}F(x_0)\| \leq \eta.
\]

(H2) For each $x \in \Omega$
\[
\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq \ell_0\|x - x_0\|.
\]

Set $\Omega_0 = U[x_0, \frac{1}{\ell_0}] \cap \Omega$.

(H3) For each $x, y \in \Omega_0$, the following holds
\[
\|F'(x_0)^{-1}(F'(y) - F'(x))\| \leq \ell\|y - x\|,
\]
\[
\|F'(x_0)^{-1}([y, x; F] - F'(x_0))\| \leq \ell_1(\|y - x_0\| + \|x - x_0\|),
\]
\[
F'(x_0)^{-1}([y, x; F] - F'(x))\| \leq \ell_2\|y - x\|
\]
\[
\|F'(x_0)^{-1}([z, y; F] - [y, x; F])\| \leq \ell_3(\|z - y\| + \|y - x\|).
\]
and
\[
\|F'(x_0)^{-1}F'(x)\| \leq \ell_4\|x - x_0\|.
\]

(H4) $U[x_0, t^*] \subset \Omega$.

Then, we can show the main semi-local convergence result for method (2).

**Theorem 1.** Suppose that conditions (H1)–(H4) hold. Then, sequence $\{x_n\}$ generated by method (2) is well defined in $U[x_0, t^*]$, remain in $U[x_0, t^*]$ for each $n = 0, 1, 2, \ldots$ and converge to a solution $x_* \in U[x_0, t^*]$ of equation $F(x) = 0$, so that
\[\|x_n - x_0\| \leq t^* = t_0.\]

**Proof.** Assertions

(A) \[\|y_k - x_k\| \leq s_k - t_k\]

(B) \[\|x_{k+1} - y_k\| \leq t_{k+1} - s_k\]

shall be proven using induction on \(k\). It follows from the first substep of method (2) that

\[\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = s_0 - t_0 = s_0 \leq t^*.\]

Hence, (A) is true and \(y_0 \in U[x_0, t^*]\). We can write by the first substep of method (2) for \(n = 0\) and (H2)

\[F(y_0) = F(y_0) - F(x_0) - F'(x_0)(y_0 - x_0),\]

so

\[\|F'(x_0)^{-1}F(y_0)\| \leq \frac{\ell_0}{2}\|y_0 - x_0\|^2 \leq \frac{\ell}{2}(s_0 - t_0)^2.\]

Next, we show the invertibility of linear operator \([y_0, x_0; F]\). Indeed, we have by (H2) that

\[\|\|F(x_0)^{-1}([y_0, x_0; F] - F'(x_0))\| \leq \ell_1(\|y_0 - x_0\| + \|x_0 - x_0\|) \leq \ell_1(s_0 - t_0) < 1,\]

so by the Banach lemma on linear invertible operators [20], \([y_0, x_0; F]^{-1}\) exists,

\[\|\|y_0, x_0; F\|^{-1}F(x_0)\| \leq \frac{1}{1 - \ell_1\|y_0 - x_0\|} \leq \frac{1}{1 - \ell_1(s_0 - t_0)} \tag{22}\]

and iterate \(x_1\) is well defined by the second substep of method (2) for \(n = 0\). We can write

\[x_1 - y_0 = -\|y_0, x_0; F|^{-1}F'(x_0)\|y_0, x_0; F|^{-1}F(y_0)\]

leading to

\[\|x_1 - y_0\| \leq \|\|y_0, x_0; F|^{-1}F'(x_0)\| \times \|y_0, x_0; F|^{-1}F(x_0)\| \|F'(x_0)^{-1}F(y_0)\| \leq \frac{\ell(s_0 - t_0)^2}{2(1 - \ell_1s_0)^2} = t_1 - s_0,\]

showing (B). We also obtain

\[\|x_1 - x_0\| \leq \|x_0 - y_0\| + \|y_0 - x_0\| \leq t_1 - s_0 + s_0 - t_0 = t_1 \leq t^*,\]

so \(x_1 \in U[x_0, t^*]\). Suppose that (A) and (B) hold, \(y_k, x_{k+1} \in U[x_0, t^*]\) and \(F'(x_k)^{-1}, \|y_k, x_{k}; F\|^{-1}\) exist for each \(k = 1, 2, \ldots\) We shall show they hold for \(k = n + 1\). By the second substep of method (2), we can write, in turn

\[F(x_{n+1}) = F(x_{n+1}) - F(y_n) - \|y_n, x_{n}; F|F'(x_n)^{-1}[y_n, x_{n}; F](x_{n+1} - y_n)\]

\[= \|\|x_{n+1}, y_{n}; F| - \|y_n, x_{n}; F|F'(x_n)^{-1}[y_n, x_{n}; F](x_{n+1} - y_n)\|

\[= \|\|x_{n+1}, y_{n}; F| - \|y_n, x_{n}; F|F'(x_n)^{-1}[y_n, x_{n}; F](x_{n+1} - y_n)\|

\[= \|\|x_{n+1}, y_{n}; F| - \|y_n, x_{n}; F|F'(x_n)^{-1}[y_n, x_{n}; F](x_{n+1} - y_n)\|

\[= \|\|x_{n+1}, y_{n}; F| - \|y_n, x_{n}; F|F'(x_n)^{-1}[y_n, x_{n}; F](x_{n+1} - y_n)\|

\[= \|\|x_{n+1}, y_{n}; F| - \|y_n, x_{n}; F|F'(x_n)^{-1}[y_n, x_{n}; F](x_{n+1} - y_n)\|

\[= \|\|x_{n+1}, y_{n}; F| - \|y_n, x_{n}; F|F'(x_n)^{-1}[y_n, x_{n}; F](x_{n+1} - y_n)\|.\]
Then, by conditions (H3) and the induction hypotheses, in turn, we obtain that

\[
\|F'(x_0)^{-1}F(x_{n+1})\| \leq \left[ \ell_3\left( \|x_{n+1} - y_n\| + \|y_n - x_n\| \right) + (1 + \ell_1(\|y_n - x_0\| + \|x_n - x_0\|)) \right] \times \frac{\ell_2(\|y_n - x_n\|)}{1 - \ell_0\|x_n - x_0\|} \|x_{n+1} - y_n\|
\]

\[
\leq \left[ \ell_3(l_{n+1} - l_n) + \frac{(1 + \ell_1(s_n + l_n))\ell_2(s_n - l_n)}{1 - \ell_0l_n} \right](l_{n+1} - s_n)
\]

\[
= \alpha_{n+1}.
\]

We must show \(F'(x_{n+1})\) is invertible. Indeed, we have by (H2)

\[
\|F'(x_0)^{-1}(F'(x_{n+1}) - F'(x_0))\| \leq l_0\|x_{n+1} - x_0\| \leq l_0l_{n+1} < 1,
\]

so

\[
\|F'(x_{n+1})^{-1}F'(x_0)\| \leq \frac{1}{1 - l_0l_{n+1}}.
\]

Hence, we obtain by method (2) and the two preceding estimates that

\[
\|y_{n+1} - x_{n+1}\| \leq \|F'(x_{n+1})^{-1}F'(x_{n+1})\|\|F'(x_0)^{-1}F(x_{n+1})\|
\]

\[
\leq \frac{\alpha_{n+1}}{1 - l_0l_{n+1}} = s_{n+1} - l_{n+1},
\]

showing (A_k) for \(k = n + 1\). We also obtain

\[
\|y_{n+1} - x_0\| \leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_0\|
\]

\[
\leq s_{n+1} - l_{n+1} + l_{n+1} - l_0 = s_{n+1} \leq t^*,
\]

so \(y_{n+1} \in U[x_0, t^*]\). In view of the first substep of method (2), we can write

\[
F(y_{n+1}) = F(y_{n+1}) - F(x_{n+1}) + F'(x_{n+1})(y_{n+1} - x_{n+1}),
\]

leading to

\[
\|F'(x_0)^{-1}F(y_{n+1})\| \leq \frac{\ell}{2}\|y_{n+1} - x_{n+1}\|^2 \leq \frac{\ell}{2}(s_{n+1} - l_{n+1})^2
\]

so

\[
\|x_{n+2} - y_{n+1}\| \leq \|\{y_{n+1}, y_{n+1}; F\}^{-1}F'(x_0)\|^2\|F'(x_0)^{-1}F'(x_{n+1})\|\|F'(x_0)^{-1}F(y_{n+1})\|
\]

\[
\leq \frac{\ell l_{n+1}(s_{n+1} - l_{n+1})^2}{2(1 - \ell_1(s_{n+1} + l_{n+1}))^2}
\]

\[
= l_{n+2} - s_{n+1},
\]

showing (B_k) for \(k = n + 1\). Moreover, we obtain

\[
\|x_{n+2} - x_0\| \leq \|x_{n+2} - y_{n+1}\| + \|y_{n+1} - x_0\|
\]

\[
\leq l_{n+2} - s_{n+1} + s_{n+1} - l_0 = l_{n+2} \leq t^*,
\]

and

\[
\|x_{n+1} - x_0\| \leq \|x_{n+1} - y_n\| + \|y_n - x_0\|
\]

\[
\leq l_{n+1} - s_n + s_n - l_n.
\]

Hence, we deduce \(x_{n+2} \in U[x_0, t^*]\) and sequence \(\{t_n\}\) is Cauchy in a Banach space \(V_1\). Hence, it converges to some \(x^* \in U[x_0, t^*]\). By letting \(n \rightarrow \infty\) in the estimate

\[
\|F'(x_0)^{-1}F(x_{n+1})\| \leq \alpha_{n+1},
\]
and the continuity of $F$, we obtain $F'(x^*) = 0$. □

Concerning the uniqueness of the solution $x_*$ we have:

**Proposition 1.** Suppose: There exists
(i) A simple solution $x^*$ of equation $F(x) = 0$.
and
(ii) $s \geq t^*$ such that

$$\ell_0(s + t^*) < 2.$$

Set $\Omega_1 = U[x_0, s] \cap \Omega$. Then, the only solution of equation $F(x) = 0$ in the region $\Omega_1$ is $x^*$.

**Proof.** Let $\bar{x} \in \Omega_1$ with $F(\bar{x}) = 0$. Set $T = \int_0^1 F'(\bar{x} + \theta(x^* - \bar{x}))d\theta$. Then, by (H2) and (ii), we obtain

$$\|F'(x_0)^{-1}(T - F'(x_0))\| \leq \int_0^1 \varphi_0((1 - \theta)\|\bar{x} - x_0\| + \theta\|x^* - x_0\|)d\theta$$

$$\leq \frac{\ell_0}{2}(s + t^*) < 1,$$

leading to $\bar{x} = x^*$, where we used the identity $T(x^* - \bar{x}) = F(x^*) - F(\bar{x}) = 0 - 0 = 0$ and the invertability of $T$. □

4. Local Convergence

Let $L, L_j, j = 0, 1, 2, 3, 4$ be positive parameters. Set $S = [0, \frac{1}{L_0}]$. Define function $\psi_1$ on the interval $S = [0, \frac{1}{L_0})$ by

$$\psi_1(t) = \frac{Lt}{2(1 - L_0t)}.$$

Then, parameter $r_1$ is defined by

$$r_1 = \frac{2}{2L_0 + L},$$

solves equation

$$\psi_1(t) - 1 = 0.$$

Moreover, define functions $q, p$ on interval $S$ by

$$q(t) = L_0\psi_1(t)t - 1 \quad \text{and} \quad p(t) = L_1(1 + \psi_1(t))t - 1.$$

Suppose that equations

$$q(t) = 0, \quad p(t) = 0$$

have smallest solutions $r_q, r_p \in S - \{0\}$. Set $S_0 = [0, r_0)$, where $r_0 = \min\{r_q, r_p\}$. Define function $\psi_2$ on $S_0$ by

$$\psi_2(t) = \frac{L}{2} \frac{\psi_1^2(t)t}{1 - L_0\psi_1(t)t} + \frac{L_3L_4(1 + \psi_1(t))\psi_1(t)t}{(1 - L_0\psi_1(t)t)(1 - L_1(1 + \psi_1(t))t)} + \frac{L_2L_4(1 + \psi_1(t))\psi_1(t)t}{(1 - L_1(1 + \psi_1(t)))^2}.$$

Suppose that equation

$$\psi_2(t) = 0$$
has the smallest solution $r_2 \in S_0 - \{0\}$. We shall prove that

$$r = \min\{r_i\}, \ i = 1, 2$$  \hfill (24)

is a convergence radius for method (2). Set $S_1 = [0, r)$. By these definitions, we have that for each $t \in S_1$

$$0 \leq L_0 t < 1,$$  \hfill (25)

$$0 \leq q(t) < 1,$$  \hfill (26)

$$0 \leq p(t) < 1,$$  \hfill (27)

and

$$0 \leq \varphi_i(t) < 1.$$  \hfill (28)

As in the semi-local convergence case we develop the following conditions (C1)–(C4).

Suppose:

(C1) $x^*$ is a simple solution of equation $F(x) = 0$.

(C2) For each $x \in \Omega$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0 \|x - x_0\|.$$  \hfill (29)

Set $\Omega_1 = U(x^*, \frac{1}{L_0}) \cap \Omega$.

(C3) For each $x, y \in \Omega_1$

$$\|F'(x^*)^{-1}(F'(y) - F'(x))\| \leq L_1 \|y - x\|,$$

$$\|F'(x^*)^{-1}([y, x; F] - F'(x^*))\| \leq L_1 (\|y - x^*\| + \|x - x^*\|),$$

$$\|F'(x^*)^{-1}([y, x; F] - F'(x))\| \leq L_2 \|y - x\|,$$

and

$$\|F'(x^*)^{-1}(y - F'(x))\| \leq L_3 \|y - x\|.$$  \hfill (30)

(C4) $U[x^*, r] \subset \Omega$.

Then, we can show the local convergence result for method (2).

**Theorem 2.** Under conditions (C1)–(C4) further suppose that $x_0 \in U(x^*, r) - \{x^*\}$. Then, sequence $\{x_n\}$ generated by method (2) is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to $x^*$ so that

$$\|y_n - x^*\| \leq \varphi_1(\|x_n - x^*\|) \|x_n - x^*\| \leq \|x_n - x^*\| < r$$  \hfill (29)

and

$$|x_{n+1} - x^*| \leq \varphi_2(\|x_n - x^*\|) \|x_n - x^*\| \leq \|x_n - x^*\|.$$  \hfill (30)

**Proof.** Let $z \in U(x^*, r) - \{x^*\}$. Using (C1), (C2), (24) and (25) we obtain

$$\|F'(x^*)^{-1}(F'(z) - F'(x^*))\| \leq L_0 \|z - x^*\| \leq L_0 r < 1,$$

so $F'(z)$ is invertible with

$$\|F'(z)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0 \|z - x^*\|}.$$  \hfill (31)

Iterate $y_0$ is well defined from (31) for $z = x_0$ and the first substep of method (2). Using (24), (28) (for $i = 1$), (31) (for $z = x_0$) and (C3), we obtain
\[
\|y_0 - x^*\| \leq \left\| F'(x_0)^{-1}F'(x^*) \right\| \\
\times \left\| \int_0^1 F'(x^*)^{-1}(F'(x_0 + \theta(x_0 - x^*)) - F'(x_0))d\theta(x_0 - x^*) \right\|
\]
\[
\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)}
\]
\[
\leq \theta_1(\|x_0 - x^*\|)\|x_0 - x^*\|
\]
\[
\leq \|x_0 - x^*\| < r,
\]
showing (29) for \(n = 0\) and \(y_0 \in U(x^*, r)\). Next, we shall show that \([u, v; F]^{-1} \in L(V_2, V_1)\) for \(u, v \in U(x^*, r)\). Indeed, by (24), (26), (C3) and (32) we have
\[
\|F'(x^*)^{-1}[y_0, x_0; F] - F'(x^*)\| \leq L_1(\|y_0 - x^*\| + \|x_0 - x^*\|)
\]
\[
\leq L_1(\theta_1(\|x_0 - x^*\|) + 1)\|x_0 - x^*\|
\]
\[
= p(\|x_0 - x^*\|) \leq p(r) < 1,
\]
so
\[
\|F'(x^*)^{-1}[y_0, x_0; F] - F'(x^*)\| \leq \frac{1}{1 - p(\|x_0 - x^*\|)}.
\]
We also have that (11) holds for \(z = y_0\). Hence, iterate \(x_1\) is well defined by the second substep of method (2). Then, we can write in turn that
\[
x_1 - x^* = y_0 - x^* - F'(y_0)^{-1}F(y_0) + (F'(y_0)^{-1} - [y_0, x_0; F]^{-1}F'(x_0)) [y_0, x_0; F]^{-1}F(y_0).
\]
However, we obtain
\[
F'(y_0)^{-1} - [y_0, x_0; F]^{-1}F'(x_0) = F'(y_0)^{-1}(1 - F'(y_0)[y_0, x_0; F]^{-1}F'(x_0)) [y_0, x_0; F]^{-1}
\]
\[
= F'(y_0)^{-1}([y_0, x_0; F] - F'(y_0)) + F'(y_0) - F'(y_0)[y_0, x_0; F]^{-1}F'(x_0)) [y_0, x_0; F]^{-1}
\]
and
\[
[y_0, x_0; F] - F'(y_0) - F'(y_0) = ([y_0, x_0; F] - F'(y_0))
\]
\[
+ F'(y_0)[y_0, x_0; F]^{-1}([y_0, x_0; F] - F'(x_0)),
\]
so
\[
x_1 - x^* = y_0 - x^* - F'(y_0)^{-1}F(y_0)
\]
\[
+ F'(y_0)^{-1}([y_0, x_0; F] - F'(y_0)) [y_0, x_0; F]^{-1}F(y_0)
\]
\[
+ [y_0, x_0; F]^{-1}([y_0, x_0; F] - F'(x_0)) [y_0, x_0; F]^{-1}F(y_0).
\]
(34)
In view of (24), (28) (for \(i = 2\), (31) (for \(z = x_0, y_0\)) and (32)–(34), we obtain, in turn,
\[
\|x_1 - x^*\| \leq \frac{L\|y_0 - x^*\|^2}{2(1 - L_0\|y_0 - x^*\|)}
\]
\[
+ \frac{L_3\|y_0 - x_0\|L_4\|y_0 - x^*\|}{2(1 - L_0\|y_0 - x^*\|)(1 - L_1(\|y_0 - x^*\| + \|x_0 - x^*\|))}
\]
\[
+ \frac{L_2\|y_0 - x_0\|L_4\|y_0 - x^*\|}{2(1 - L_1(\|y_0 - x^*\| + \|x_0 - x^*\|))}
\]
\[
\leq \theta_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|,
\]
(35)
There exists a simple solution \( x^* \). Notice that preceding calculations, respectively, to complete the induction for estimates (29) and (30). Furthermore, from the estimate
\[
\|x_{j+1} - x^*\| \leq c\|x_j - x^*\|, \tag{36}
\]
where \( c = \psi_2(\|x_0 - x^*\|) \in [0, 1] \), we conclude \( \lim_{j \to \infty} x_j = x^* \) and \( x_{j+1} \in U(x^*, r) \). \( \square \)

**Remark 2.** (a) The value \( r_1 \) was given by us in [6] for the radius of convergence for Newton’s method. It then follows from (24) that
\[
r < r_1. \tag{37}
\]
Hence, the radius of convergence \( r \) for method (2) cannot be larger than Newton’s. Notice that the radius of convergence given independently by Rheinboldt [7] and Traub [8] is \( r = \frac{2}{\pi}r \), where \( K \) is the Lipschitz constant on \( \Omega \). We also have \( r \leq r \), since \( L_0 \leq K \) and \( L \leq K \).

(b) We compute the computational order of convergence (COC) defined by
\[
\text{COC} = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)
\]
or the approximate computational order of convergence (ACOC)
\[
\text{ACOC} = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).
\]

Then, we obtain in practice the convergence order and avoid the existence of the higher order Fréchet derivatives for operator \( F \).

Next, we present a uniqueness of the solution result.

**Proposition 2.** Suppose:
(a) There exists a simple solution \( x^* \) of equation \( F(x) = 0 \)
(b) There exists \( r \) such that
\[
r < \frac{2}{L_0}. \tag{38}
\]
Set \( \Omega_2 = \Omega \cap U[x^*, r] \). Then, the only solution of equation \( F(x) = 0 \) in the region \( \Omega_1 \) is \( x^* \).

**Proof.** Let \( b \in \Omega_2 \) with \( F(b) = 0 \). Set \( Q = \int_0^1 F'(x^* + \theta(b - x^*))d\theta \). Then, using (C1) and (38), we obtain
\[
\|F'(x^*)^{-1}(Q - F'(x^*))\| \leq \int_0^1 L_0\theta\|b - x^*\|d\theta
\leq \frac{L_0}{2}r < 1,
\]
leading to \( b = x^* \), since \( Q^{-1} \in L(V_2, V_1) \) and \( A(b - x^*) = F(b) - F(x^*) = 0 - 0 = 0 \). \( \square \)

5. Numerical Experiments

We provide some examples, with \( [x, y; F] = \int_0^1 F'(y + \theta(x - y))d\theta \).

**Example 1.** Define function
\[
\psi(x) = b_0x + b_1 + b_2 \sin b_3x, \quad x_0 = 0,
\]
where \( b_j, j = 0, 1, 2, 3 \) are parameters. Then, clearly for \( b_3 \) large and \( b_2 \) small, \( \frac{\ell}{\ell} \) can be small (arbitrarily). Notice that \( \frac{\ell}{\ell} \to 0 \).
Example 2. Consider $V_1 = V_2 = C[0,1], \Omega = U[0,1]$ and $Q : \Omega \rightarrow V_2$ defined by

$$Q(\psi)(x) = \varphi(x) - 5 \int_0^1 x \varphi(\theta)^3 d\theta.$$  (39)

We obtain

$$Q'(\psi(\xi))(x) = \xi(x) - 15 \int_0^1 x \varphi(\theta)^2 \xi(\theta) d\theta,$$

for each $\xi \in D$.

Then, since $x^* = 0$, conditions (C1)–(C4) are verified for $L_0 = 7.5, L = L_4 = K = 15,$ $L_1 = \frac{L}{2}, L_2 = L_3 = \frac{L}{2}$. Then, the radii are:

$$r = r_1 = 0.066667, r_2 = 0.109818, \text{and } \rho = \frac{2}{3K} = 0.0667.$$

Example 3. Consider the motion system

$$G_1'(x) = e^x, G_2'(y) = (e - 1)y + 1, G_3'(z) = 1$$

with $G_1(0) = G_2(0) = G_3(0) = 0$. Let $G = (G_1, G_2, G_3)$. Let $V_1 = V_2 = \mathbb{R}^3, \Omega = U(0,1), x^* = (0,0,0)^T$. Define function $G$ on $\Omega$ for $w = (x,y,z)^T$ by

$$G(w) = (e^x - 1, \frac{e - 1}{2}y^2 + y, z)^T.$$

Then, we obtain

$$G'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e - 1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

Hence, conditions (C1)–(C4) are verified for $L_0 = (e - 1), L = e^{\frac{1}{2\pi}} = L_4, L_1 = \frac{L}{2}, L_2 = L_3 = \frac{L}{3}, K = e$. Then, the radii are:

$$r = r_1 = 0.382692, r_2 = 0.417923 \text{ and } \rho = \frac{2}{3K} = 0.2453.$$

Example 4. Let $V_1, V_2$ and $\Omega$ be as in the Example 2. It is well-known that the boundary value problem [2]

$$\varphi(0) = 0, (1) = 1,$$
$$\varphi'' = -\varphi - \sigma \varphi^2$$

can be given as a Hammerstein-like nonlinear integral equation

$$\varphi(s) = s + \int_0^1 M(s,t)(\varphi^3(t) + \sigma \varphi^2(t)) dt$$

where $\sigma$ is a parameter. Then, define $F : \Omega \rightarrow V_2$ by

$$[F(x)](s) = x(s) - s - \int_0^1 M(s,t)(x^3(t) + \sigma x^2(t)) dt.$$  

Choose $x_0(s) = s$ and $\Omega = U(x_0, \rho_0)$. Then, clearly $U(x_0, \rho_0) \subset U(0, \rho_0 + 1)$, since $\|x_0\| = 1$. Suppose $2\sigma < 5$. Then, conditions (H1)–(H4) are verified for

$$\ell_0 = \frac{2\sigma + 3\rho_0 + 6}{8}, \quad \ell = \frac{\sigma + 6\rho_0 + 3}{4},$$
$$\ell_1 = \frac{\ell}{2}, \ell_2 = \frac{\ell}{2}, \ell_3 = \frac{\ell}{2} \text{ and } \mu = \frac{1 + \sigma}{5}. \text{ Notice that } \ell_0 < \ell_1.$$
In general the radius of convergence decreases, when the order increases. However, notice that in the local convergence Examples 2 and 3, the radii for the fourth-order method (2) compare favorably to the ones given in [7,8] for Newton’s (see $r$ and $\rho$).

6. Conclusions

The Kung–Traub method was revisited, and its applicability was extended in both the semi-local and local convergence case from the real to the Banach space setting. Our analysis includes error bounds and uniqueness on $x^*$ information not available before and under weak conditions. This idea is very general and can be used to extend the applicability of other methods.

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