Super Ricci flow for disjoint unions
Sajjad Lakzian, Michael Munn

May 5, 2014

Abstract

In this paper we consider compact, Riemannian manifolds $M_1, M_2$ each equipped with a one-parameter family of metrics $g_1(t), g_2(t)$ satisfying the Ricci flow equation. Motivated by a characterization of the super Ricci flow developed by McCann-Topping in [11], we introduce the notion of a super Ricci flow for a family of distance metrics defined on the disjoint union $M_1 \sqcup M_2$. In particular, we show such a super Ricci flow property holds provided the distance function between points in $M_1$ and $M_2$ evolves by the heat equation. We also discuss possible applications and examples.

1 Introduction

For $i = 1, 2$, let $M_i$ be a compact $n$-dimensional Riemannian manifold equipped with a smooth family of metrics $g_i(t)$ satisfying the Ricci flow equation introduced by Hamilton [9]

\[
\frac{\partial g_i(t)}{\partial t} = -2 \text{Ric}(g_i(t)),
\]

for $t \in [0, T_i)$. The short-time existence and uniqueness of solutions was demonstrated in [9] and we denote $T = \min(T_1, T_2)$. In this note, we consider the disjoint union $M_1 \sqcup M_2$ equipped with a one-parameter family of metrics $D^t$, for $t \in [0, T)$, so that $(M_1 \sqcup M_2, D^t)$ is a complete, compact metric space whose metric is compatible with the evolving metrics $g_i(t)$; i.e. for $i = 1, 2$

\[
D^t|_{M_i} = d_{g_i(t)},
\]

where $d_g$ denotes the distance metric induced by the Riemannian metric $g$. Following [11], we generalize the characterization of super Ricci flow solutions for an individual family of smooth metrics, say for $(M_1, g_1(t))$ or $(M_2, g_2(t))$, to the family of metric spaces $(M_1 \sqcup M_2, D^t)$ as follows

Definition 1.1. With $M_1$ and $M_2$ as above, a family of metrics $D^t$ on $M_1 \sqcup M_2$, for $t \in [0, T)$, is called a super Ricci flow of the disjoint union $M_1 \sqcup M_2$ provided whenever $0 < a < b < T$ and $u(x, t) : M_1 \sqcup M_2 \times (a, b) \rightarrow \mathbb{R}$ is a solution to the heat equation on $M_1 \sqcup M_2$, then

\[
\text{Lip}(u, t) := \sup_{x \neq y \in M_1 \sqcup M_2} \frac{|u(x, t) - u(y, t)|}{D^t(x, y)}
\]

is non-increasing in $t$. (1.3)

In Section 3 we recall work of von Renesse-Sturm [17] to clarify precisely the Laplacian we are using on $M_1 \sqcup M_2$ and exactly what it means for $u(x, t)$ to satisfy the heat equation for such a disconnected space (see Definition 3.4 and the discussion therein).
Furthermore, we show that,

**Theorem 1.2.** For \( i = 1, 2 \), let \( M_i \) be a compact, oriented \( n \)-dimensional manifold equipped with a smooth family of metrics \( g_i(t) \) satisfying the Ricci flow equation (1.1) for \( t \in [0, T_i) \) and let \( T = \min(T_1, T_2) \). Consider the family of metric spaces \((M_1 \sqcup M_2, D^t)\) and suppose that for \( t \in (0, T) \),

\[
\frac{\partial}{\partial t} D^t(x, y) \geq \Delta_{M_1 \times M_2} D^t(x, y), \quad \text{for } x \in M_1, y \in M_2, \tag{1.4}
\]

where \( \Delta_{M_1 \times M_2} \) denotes the Laplacian on \((M_1, g_1(t)) \times (M_2, g_2(t))\). Then the family of metrics \( D^t \) is a super Ricci flow of \( M_1 \sqcup M_2 \).

**Remark 1.3.** The statement of Theorem 1.2 can be phrased slightly more generally in that \((M_1, g_1(t))\) and \((M_2, g_2(t))\) need only be supersolutions to the Ricci flow equation; i.e., \( g_i(t) \) are super Ricci flows (see Definition 3.2) on \( M_i, i = 1, 2 \). Indeed, the proof requires only this slightly weaker assumption.

**Remark 1.4.** Note that condition (1.4) alone isn’t enough to guarantee that the family of distance functions \( D^t \) between \( M_1 \) and \( M_2 \) remain distance functions for all \( t \). This is because it is possible that the triangle inequality may fail at certain times \( t > 0 \), particularly if either \( M_1 \) or \( M_2 \) has highly negative sectional curvature. However, in the statement of Theorem 1.2 we implicitly restrict our attention to only those families \( D^t \) which in fact are distance functions. In Section 2 we give simple constructions which verify that the class of such distance functions on \( M_1 \sqcup M_2 \) is nonempty.

**Acknowledgements.** This research was sponsored in part by the National Science Foundation Grants OISE #0754379 and DMS #1006059. We would like to thank the Graduate Center at CUNY where part of this work was completed and Prof. Sorberri for her interest and support. In addition, MM would like to thank the Mathematics Institute of the University of Warwick and Peter Topping for their hospitality as this work was completed. SL also thanks Dan Knopf for helpful discussions during his visit to U Texas at Austin.

**1.1 Motivation**

We now say a few words of context for Theorem 1.2 and give possible perspectives for considering such a family of metric spaces \((M_1 \sqcup M_2, D^t)\).

A primary advantage of Theorem 1.2 is that the nature of condition (1.4) is purely metric and gives a sufficient condition for a family of distance metrics on the set \( M_1 \sqcup M_2 \) to evolve in a way that is compatible with the smooth evolution of the Ricci flow for the Riemannian metrics on \( M_1 \) and \( M_2 \). This metric perspective allows for a more broad description of solutions to the Ricci flow (or in this case, super solutions to the Ricci flow) which can persist through the development of singularities provided one has knowledge of the metric after the singular time.

Given \( M^n \) a compact, \( n \)-dimensional Riemannian manifold and \( g(t) \) a family of smooth metrics evolving by (1.1), we say a finite time singularity develops at time \( T \) if this family cannot be extended beyond \( T < \infty \). Standard long time existence theorems imply that such finite time singularities develop if and only if the Riemann curvature tensor \( Rm \) blows up as \( t \searrow T \); i.e.

\[
\limsup_{t \searrow T} \sup_{x \in M} |Rm(x, t)|_{g(t)} = \infty. \tag{1.5}
\]
Sesum [13] improved this by showing that a finite-time singularity occurs if and only if
\[ \limsup_{t \to T} \sup_{x \in M} |\text{Ric}(x, t)|_{g(t)} = \infty. \] (1.6)
In some sense, the formation of such singularities is a ‘typical’ property of the Ricci flow. Indeed, it follows from the parabolic maximum principle that if the scalar curvature \( R \) satisfies \( R \geq \alpha > 0 \) at time \( t = 0 \), then a finite time singularity must develop for \( T \leq \frac{n}{2\alpha} \).

As a result, the study of the formation of singularities remains an intensely studied aspect of the Ricci flow and geometric evolution equations in general.

Angenent-Knopf were the first to give examples of finite time singularities for compact manifolds [1, 2], although certain constructions did exist for local singularities on non-compact manifolds [13, 8]. Specifically, Angenent-Knopf examined the behavior of the metrics on topological spheres \( S^{n+1} \) evolving by the Ricci flow and showed that when the initial metric \( g_0 \) is sufficiently pinched, the Ricci flow will develop a neck-pinching singularity.

A *neck-pinching singularity* is a special kind of local Type I singularity and (except for the round sphere shrinking to a point) is arguably the best known and simplest example of a finite-time singularity that can develop through the Ricci flow. More precisely, a solution \( (M^{n+1}, g(t)) \) of the Ricci flow develops a *neck pinch* at time \( T < \infty \) if there exists a time-dependent family of proper open subsets \( N(t) \subset M^{n+1} \) and diffeomorphisms \( \phi_t : \mathbb{R} \times S^n \to N(t) \) such that \( g(t) \) remains regular on \( M^{n+1} \setminus N(t) \) and the pullback \( \phi_t^* \left( g(t)_{|N(t)} \right) \) on \( \mathbb{R} \times S^n \) approaches the “shrinking cylinder” soliton metric
\[ ds^2 + 2(n-1)(T-t)g_{can} \]
in \( C^\infty_{loc} \) as \( t \searrow T \), where \( g_{can} \) denotes the round metric on the unit sphere \( S^n \). In [1], the authors show how these neck pinch singularities arise for a class of rotationally symmetric initial metrics on \( S^{n+1} \). In [2], they derive detailed asymptotics of the profile of the solution near the singularity as well as comparable asymptotics for fully general neck pinches whose initial metric need not be rotationally symmetric.

Later in [3], Angenent-Caputo-Knopf extended this work by constructing smooth forward evolutions of the Ricci flow starting from initial singular metrics which arise from rotationally symmetric neck pinches on \( S^{n+1} \) by passing to the limit of a sequence of Ricci flows with surgery. Together [1, 2, 3] provide a framework (albeit in the restrictive context of rotational symmetry) for developing the notion of a ‘canonically defined Ricci flow through singularities’ as conjectured by Perelman in [12]. Up to this point, continuing a solution of the Ricci flow past a singular time \( T < \infty \) required surgery and a series of carefully made choices so that certain crucial estimates remain bounded through the flow. A complete ‘canonical Ricci flow through singularities’ would avoid these arbitrary choices and would be broad enough to address all types of singularities that arise in the Ricci flow.

Returning now to the current paper, our motivation follows from this work of Angenent-Knopf and Angenent-Caputo-Knopf, though our result allows for application in a more general context. Since the smooth forward evolution described in [3] performs a topological surgery on \( S^{n+1} \) at the singular time \( t = T \), all future times will consist of two disjoint smooth Ricci flows on a pair of manifolds. Furthermore, although the metric \( g(t) \) is no longer a smooth Riemannian metric at the singular time \( t = T \), the space \( S^{n+1} \) does retain the structure of a metric space with distance metric denoted \( d_T \) arising from the convergence of the distance metrics \( d_t \) on \( (S^{n+1}, g(t)) \) through the evolution. As metric spaces, these spaces converge in the Gromov-Hausdorff sense as well,
\[ \lim_{t \to T} d_{GH} \left( (S^{n+1}, d_t), (S^{n+1}, d_T) \right) = 0. \] (1.7)
Our Theorem 1.2 gives a metric context in which to frame the evolution of the Ricci flow for \( t > T \), after this singularity develops.

The remainder of this paper is organized as follows. In Section 2, we give some simple examples of metric constructions for the super Ricci flow for disjoint unions of two smooth Riemannian manifolds. In particular, we consider the situation when \( M_1 \cong M_2 \) and consider the case of the flat torus and the round sphere. In Section 3, we recall the characterization of the super Ricci flow given by McCann-Topping for compact Riemannian manifolds which motivates our Definition 1.1. Also, we recall a construction of von Renesse-Sturm [17] and use a generalization of the Trotter-Chernov product formula for time dependent operators to describe solutions to the heat equation on the disjoint union \( M_1 \sqcup M_2 \). With these definitions and context in place, we then prove Theorem 1.2 in Section 4 and give implications.

2 Examples

To better illustrate the content of Theorem 1.2, we mention a few simple examples. In general, for \((M_1, g_1(t))\) and \((M_2, g_2(t))\) as in Section 1, a family of distance metrics \( D^t \) on \( M_1 \sqcup M_2 \) for \( t \in [0, T) \) is a family of non-negative functions

\[
D^t : M_1 \sqcup M_2 \times M_1 \sqcup M_2 \rightarrow \mathbb{R}
\]

such that the following properties hold. For \( a, b, c \in M_1 \sqcup M_2 \), and all \( t \in [0, T) \),

- \( D^t(a, b) = 0 \) if and only if \( a = b \)
- \( D^t(a, b) = D^t(b, a) \)
- \( D^t(a, b) \leq D^t(a, c) + D^t(c, b) \)

Thus, we require these properties to hold implicitly in the statement of Theorem 1.2.

Note, however, that the metric \( D^t \) is not an intrinsic distance as \( M_1 \sqcup M_2 \) is disconnected.

Consider the case where \((M_1, g_1(0)) \cong (M_2, g_2(0))\) and thus \( g_1(t) = g_2(t) \) for all \( t \) satisfying (1.1) by uniqueness. Set \( D^t \) on \( M_1 \sqcup M_2 \) to be

\[
D^t(a, b) = \begin{cases} 
  d_{g_i(t)}(a, b), & \text{if } a, b \in M_i \\
  \sqrt{L^2(t) + d_{g_i(t)}^2(\phi(a), b)}, & \text{if } a \in M_1, b \in M_2 \text{ or } a \in M_2, b \in M_1,
\end{cases}
\]

where \( \phi : M_1 \rightarrow M_2 \) is the identity map and \( L(t) \) depends only on \( t \). Note that each of the properties for \( D^t \) to be a distance function hold naturally in this construction.

Now letting \( d_t \) denote \( d_{g_i(t)} \) where there is no confusion since \( g_1(t) = g_2(t) \) and considering \( d_t \) and \( D^t \) as maps on \( M_1 \times M_2 \), we can relate \( \Delta_{M_1 \times M_2} D^t \) to \( \Delta_{M_1 \times M_2} d_t \). Computing in local coordinates we have

\[
\Delta(D^t)^2 = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j (D^t)^2 \right) = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} 2 D^t \partial_j D^t \right) = 2 g^{ij} \partial_i D^t \partial_j D^t + 2 D^t \Delta D^t;
\]

where \( i = 1, \ldots, n \) for \( n \) the dimension of the manifold.

4
and, directly we find $\Delta (d^t)^2 = \Delta \left( L^2(t) + d_t^2 \right) = \Delta d_t^2 = 2 |\nabla d_t|^2 + 2 d_t \Delta d_t$, since in this simple example we assume that $L(t)$ depends only on $t$. Thus,

$$d_t \Delta_{M^t \times M^t} d_t = \frac{1}{2} \Delta_{M^t \times M^t} (d^t)^2 - \left| \nabla_{M^t \times M^t} d_t \right|_{g(t)}^2,$$

(2.6)

Therefore,

$$\Delta_{M^t \times M^t} = \frac{d_t}{dt} \Delta_{M^t \times M^t} d_t + \frac{\left| \nabla_{M^t \times M^t} d_t \right|_{g(t)}^2}{D^t} - \frac{\left( d_t \right)^2}{D^t} g_{ij} \partial_t d_t \partial_j d_t.$$

(2.7)

Noting that $\partial_t D^t = \frac{dt}{dt} \partial_t d_t$, thus, we can further simplify the last term in the expression above to give

$$\Delta_{M^t \times M^t} d_t = \frac{d_t}{dt} \Delta_{M^t \times M^t} d_t + \frac{\left| \nabla_{M^t \times M^t} d_t \right|_{g(t)}^2}{D^t} - \frac{\left( d_t \right)^2}{D^t} g_{ij} \partial_t d_t \partial_j d_t.$$

(2.8)

Furthermore, since $\frac{\partial}{\partial t} D^t = \frac{1}{D^t} \left( L\left( \frac{\partial}{\partial t} D^t \right) + d_t \frac{\partial d_t}{\partial t} \right)$ and using (2.9), the inequality (1.4) can be written

$$L(t) \frac{\partial L}{\partial t} + d_t \frac{\partial d_t}{\partial t} \geq d_t \Delta_{M^t \times M^t} d_t + \left| \nabla_{M^t \times M^t} d_t \right|_{g(t)}^2 - \frac{\left( d_t \right)^2}{D^t} g_{ij} \partial_t d_t \partial_j d_t,$$

(2.9)

which simplifies as

$$L(t) \frac{\partial L}{\partial t} + d_t \frac{\partial d_t}{\partial t} \geq d_t \Delta_{M^t \times M^t} d_t + \left( 1 - \left( \frac{d_t}{D^t} \right)^2 \right) \left| \nabla_{M^t \times M^t} d_t \right|_{g(t)}^2;$$

(2.10)

where we used the fact that, for a local basis of tangent vectors $\partial_i$ on $M_1 \times M_2$,

$$\left| \nabla_{M^t \times M^t} d_t \right|_{g(t)}^2 = g_t \left( \nabla_{M^t \times M^t} d_t, \nabla_{M^t \times M^t} d_t \right)$$

(2.11)

(2.12)

To make this construction more explicit, consider

**Example 2.1.** $M_1 \equiv M_2 \equiv \text{the flat torus } \mathbb{T}^2$.

Let $(M_1, g_1(t)) \equiv (M_2, g_2(t)) \equiv (\mathbb{R}^2 / \mathbb{Z}^2, g_{\mathbb{T}^2}) \cong \left( S^1 \times S^1, \left( \frac{1}{\pi} \right)^2 dx^2 + \left( \frac{1}{\pi} \right)^2 dy^2 \right)$. Since $\text{Ric}_{\mathbb{T}^2} \equiv 0$, the flat torus is a stationary point for the Ricci flow and thus, $g_t(t) \equiv g_{\mathbb{T}^2}$, for all $t$, and $i = 1, 2$. Define a family of metrics $D^t : M_1 \sqcup M_2 \times M_1 \sqcup M_2 \to \mathbb{R}^\geq 0$ by setting

$$D^t(a, b) = \begin{cases} 
 d_{\mathbb{T}^2}(a, b), & \text{if } a, b \in M_1 \\
 \sqrt{L^2(t) + d_{\mathbb{T}^2}^2(\phi(a), b)}, & \text{if } a \in M_1, b \in M_2 \text{ or } a \in M_2, b \in M_1.
\end{cases}$$

(2.13)

(2.14)

where $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ is the identity map. To interpret (2.14), we consider $D^t$ and $d_t$ as functions on $\mathbb{T}^2 \times \mathbb{T}^2$ with canonical metric

$$g_{\mathbb{T}^2 \times \mathbb{T}^2} = g_{\mathbb{T}^2} \times g_{\mathbb{T}^2} = \left( \frac{1}{2\pi} \right)^2 \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 \right].$$

(2.15)

(2.16)
Furthermore, since the metrics are stationary, the Laplacian $\Delta_{M^1 \times M^2_t}$ is independent of $t$ and $dt = d$ so that, inside the cut locus,

\[
\Delta_{T^2 \times T^2} d(a, b) = \Delta_{T^2} d(a, b) + \Delta_{T^2} d(a, b) \big|_a = \left( \frac{2(2\pi)^2}{d_b(a)} + \frac{2(2\pi)^2}{d_a(b)} \right); \tag{2.17}
\]

and thus,

\[
d_t \Delta_{M^1 \times M^2_t} d_t = d_t \cdot \Delta_{T^2 \times T^2} d_t = d_t \cdot \frac{2(2\pi)^2}{d} = 2(2\pi)^2. \tag{2.18}
\]

Also, since $g^{ij}_t = g^{ij}$, we have

\[
g^{ij}_t \partial_i d_t \partial_j d_t = g^{ij} \partial_i d \partial_j d = (2\pi)^2 \sum_{i=1}^{4} (\partial_i d)^2 = 2(2\pi)^2. \tag{2.19}
\]

Lastly, since $\nabla ^{T^2 \times T^2} d = g^{ij} \partial_i d \partial_j d = (2\pi)^2 \delta^{ij} \partial_i d \partial_j d = \sum_{i=1}^{4} (2\pi)^2 \partial_i d \partial_i$, we get

\[
\left| \nabla ^{M^1 \times M^2_t} d_t \right|_{g(t)}^2 = \left| \nabla ^{T^2 \times T^2} d \right|^2 = g^{ij} \left( \nabla ^{T^2 \times T^2} d \right)^i \left( \nabla ^{T^2 \times T^2} d \right)^j \tag{2.20}
\]

\[
= (1/2\pi)^2 \delta^{ij} \left( \nabla ^{T^2 \times T^2} d \right)^i \left( \nabla ^{T^2 \times T^2} d \right)^j \tag{2.21}
\]

\[
= \sum_{i=1}^{4} (1/2\pi)^2 \left( \nabla ^{T^2 \times T^2} d \right)^2 \tag{2.22}
\]

\[
= \sum_{i=1}^{4} (1/2\pi)^2 \left( \nabla ^{T^2 \times T^2} d \right)^2 \tag{2.23}
\]

\[
= \sum_{i=1}^{4} (2\pi)^2 (\partial_i d)^2 = \left| \nabla^1 d \right|^2 + \left| \nabla^2 d \right|^2 = 2(2\pi)^2. \tag{2.24}
\]

Therefore, in this setting (2.11) becomes

\[
L(t) \frac{\partial L}{\partial t} \geq 4(2\pi)^2 - 2(2\pi)^2 \left( \frac{d}{D} \right)^2. \tag{2.25}
\]

Roughly estimating $0 \leq d/D \leq 1$, we can take $L(t) = 2\pi \sqrt{8t + L^2(0)}$ so that $L \frac{\partial L}{\partial t} = 4(2\pi)^2$ which satisfies (2.25). Naturally, any $L(t)$ with growth larger than $t^{1/2}$ would also satisfy condition (1.4) as well and give another family of distance metrics satisfying the super Ricci flow on the disjoint union.

**Remark 2.2.** In general, for $M_1 \cong M_2 \cong (\mathbb{R}^n/Z^n, g_{\text{can}})$ we would have $L(t) = 2\pi \sqrt{4nt + L^2(0)}$.

**Example 2.3.** $M_1 \cong M_2 \cong \text{the round sphere } S^2$.

Let $(M_1, g_1(t)) \cong (M_2, g_2(t)) \cong (S^2, g_{\text{can}})$, the unit 2-sphere with its canonical round metric. For $i = 1, 2$ we have $\text{Ric}(g_i(t)) = g_{can}$ so the metrics on $M_i$ evolving by $L(t)$ satisfy $g_i(t) = (1 - 2t) g_{can}$, for $t \in [0, \frac{1}{2})$. We will often write $S^2_t$ to denote $(S^2, (1 - 2t) g_{can})$ and $d$ for the distance metric induced by $g_{can}$.

As before, any family of distance metrics $D^t$ on $M_1 \cup M_2$ must satisfy $D^t|_{M_i} = d_{g_i(t)}$ and for $a \in M_1, b \in M_2$, we take

\[
D^t(a, b) = \sqrt{L^2(t) + d^2_t(\phi(a), b)}. \tag{2.26}
\]
where, as before, $\phi : S^2 \to S^2$ is the identity map and $d_t$ denotes the distance metric on $S^2$. That is to say, $d_t^2 = d^2_{g_t(t)} = (1 - 2t) d^2_{g_t(0)} = (1 - 2t) d^2$.

Furthermore, in this setting we have

$$d_t \frac{\partial d_t}{\partial t} = \sqrt{1 - 2t} \frac{\partial}{\partial t} \sqrt{1 - 2t d_t} = -d^2$$

and, since $\Delta_{S^2 \times S^2} = \frac{1}{2} \Delta_{S^2} + \frac{1}{2} \Delta_{S^2}$, we have

$$d_t \Delta_{S^2 \times S^2} d_t = \sqrt{1 - 2t} \Delta_{S^2 \times S^2} \sqrt{1 - 2t d_t} = (1 - 2t d) \Delta_{S^2 \times S^2} d$$

$$= d \Delta_{S^2 \times S^2} d$$

$$= d \Delta^1 d + d \Delta^2 d$$

$$= 2d \cot d,$$

where in the third line

$$\Delta_{S^2} d(a, b) = \Delta^1 d(x, y) \big|_{x=a, y=b} = \Delta^1 d(x, b) \big|_{x=a} = \cot d_b(a) = \cot d(a, b).$$

Also, we have

$$\left| \nabla_{S^2 \times S^2}^2 d_{g(t)} \right|^2 = (1 - 2t) \left| \nabla_{S^2 \times S^2}^2 d_{\text{can}} \right|^2 = \left| \nabla_{S^2 \times S^2}^2 d_{\text{can}} \right|^2 - \left| \nabla^1 d_{\text{can}} \right|^2 + \left| \nabla^2 d_{\text{can}} \right|^2;$$

thus, the expression for $L(t)$ in this setting can be written as

$$L(t) \frac{\partial L}{\partial t} - d^2 \geq 2d \cot d + \left(1 - \left( \frac{d_t}{d^2} \right)^2 \right) \left( \left| \nabla^1 d_{\text{can}} \right|^2 + \left| \nabla^2 d_{\text{can}} \right|^2 \right).$$

Keeping in mind $0 \leq d \leq \pi$, any $L(t)$ which satisfies (2.35) for all $t \in [0, 1/2)$ gives a suitable distance metric on $S^2 \times S^2$. This can also be extended to higher dimensional spheres in the obvious way.

**Remark 2.4.** A variation of this construction can be used for $M_1$ and $M_2$ which are only assumed to be homeomorphic. In the definition of $D'$ given in (2.2), take, for $a \in M_1, b \in M_2$

$$\min \left( \inf_{\phi} \sqrt{L^2(t) + d^2_{g_t(t)}(\phi(a), b)}, \inf_{\phi} \sqrt{L^2(t) + d^2_{g_t(t)}(a, \phi^{-1}(b))} \right),$$

where the infimum is taken over all homeomorphisms $\phi : M_1 \to M_2$ and, as before, $L(t)$ depends only on $t$.

### 3 Background

As we hope to make clear, our current results tie together a progression of ideas which originated with a 2005 paper by M. von Renesse and K.T. Sturm [17], although its true origins can be recognized in earlier work of Bakry-Emery [4], Cordero-Erausquin, McCann, Schmuckenschlager [6, 5] and others.
3.1 Metric characterizations of Ricci curvature lower bounds and the Ricci flow

In [17], von Renesse-Sturm characterize uniform lower Ricci curvature bounds of smooth Riemannian manifolds \((M^n, g)\) using various convexity properties of the entropy as well as transportation inequalities of volume measures, heat kernels, and gradient estimates of the heat semigroup on \(M^n\). In fact, the metric nature of the ideas presented in that paper introduced into the literature a discussion of so called “synthetic” definitions of Ricci curvature lower bounds which do not rely on the underlying smooth structure of the manifold and thus lend themselves to spaces lacking that smooth structure, such as metric measure spaces, Alexandrov spaces, or general metric spaces.

We state here only a small part the results in [17] which are relevant to our later discussion. First a bit of notation: Let \((M^n, g)\) be a smooth, connected, complete Riemannian manifold of dimension \(n\). Denoting the heat kernel on \(M^n\) by \(p_t(x, y)\) one can define the operators \(p_t: C^\infty_c(M) \to C^\infty(M)\) and \(p_t: L^2 \to L^2(M)\) by \(f \mapsto p_t f(x) := \int_M p_t(x, y)f(y) \, d\text{Vol}(y)\). They prove

**Theorem 3.1.** (von Renesse-Sturm, [17]) For any smooth, complete Riemannian manifold \((M^n, g)\) \(\text{Ric}_M \geq 0\) if and only if for all bounded \(f \in C^{Lip}(M)\) and all \(t > 0\),

\[
\text{Lip}(p_t f) \leq \text{Lip}(f)
\]

Later McCann-Topping [11] took a dynamic approach and reinterpreted the work of von Renesse-Sturm in relation to a metric evolving by the Ricci flow. Specifically, they characterize super solutions of the Ricci flow on \(M^n\) by the contractivity of mass diffusions backwards in time \(t\). We refer to a super solutions of the Ricci flow as a super Ricci flow. That is

**Definition 3.2.** (McCann-Topping; c.f. [11], Definition 1). For a compact, oriented \(n\)-dimensional manifold, a super Ricci flow is a smooth family \(g(t)\) of metrics on \(M\), \(t \in [0, T]\), such that at each \(t \in (0, T)\) and each point on \(M\), one has

\[
\frac{\partial g}{\partial t} + 2 \text{Ric}(g(t)) \geq 0.
\]  

(3.1)

In addition, and more closely related to our purposes, they prove the following

**Theorem 3.3.** (McCann-Topping; c.f. [11], Theorem 2). Let \(M^n\) be a compact, Riemannian manifold of dimension \(n\). A smooth one-parameter family of metrics for \(t \in [0, T]\) is a super Ricci flow if and only if whenever \(0 < a < b < T\) and \(f: M \times (a, b) \to \mathbb{R}\) is a solution to \(\frac{\partial f}{\partial t} = \Delta_{g(t)} f\), then

\[
\text{Lip}(f, t) := \sup_{x \neq y} \frac{|f(x, t) - f(y, t)|}{d(x, y, t)}
\]

is non-increasing in \(t\).

The quantity \(\text{Lip}(f, t)\) is the Lipschitz constant of \(f(\cdot, t)\) evaluated using the metric \(g(t)\). It is precisely this characterization which we use to define the notion of a super Ricci flow for the disjoint union of two evolving Riemannian manifolds. However, we must first make sense of the local representation for the heat kernel on \(M_1 \sqcup M_2\) in order to describe what it means for a function \(u(x, t)\) on \(M_1 \sqcup M_2\) to solve the heat equation.
3.2 Heat kernel operators from the metric and measure

In [17], von Renesse-Sturm focus on smooth, connected complete \( n \)-dimensional Riemannian manifolds \( M \) and characterize a uniform lower Ricci curvature bound of \( M \) using, among other things, heat kernels and transportation inequalities for uniform distribution measures on distance spheres in \( M \). One striking advantage of these characterizations is that they depend only on the metric and measure of the underlying smooth Riemannian manifold and thus allow for a notion of a Ricci curvature lower bound depending solely this basic, non-smooth data. In fact, these characterizations ultimately led to the current definitions of Ricci curvature for arbitrary metric measure spaces introduced independently by Lott-Villani and Sturm [10, 15, 16]. We recall now the original discussion of von Renesse-Sturm.

Following the comment at the end of Section 1 of [17], one can view a smooth, connected Riemannian manifold \((M, g)\) as a separable metric measure space \((M, d, \mu)\) and define a family of Markov operators \(\sigma_r\) acting on the set of bounded Borel measurable functions by

\[
\sigma_r f(x) = \int_M f(y) \, d\sigma_{r,x}(y),
\]

where the measure \(\sigma_{r,x}\) is defined as

\[
\sigma_{r,x}(A) := \frac{\Vol_g(A \cap \partial B(x, r))}{\Vol_g(\partial B(x, r))}, \quad A \in \mathcal{B}(M). \tag{3.2}
\]

Here \(B(x, r)\) denotes the ball of radius \( r \) centered at \( x \). By the Arzela-Ascoli theorem and applying the Trotter-Chernov product formula [7], there exist a subsequence such that for all bounded \( f \in C^{\text{Lip}}(M) \) the limit

\[
p_t f(x) := \lim_{j \to \infty} \left( \sigma_{r, \frac{\sqrt{2\pi}t}{j}} \right)^j f(x)
\]

exists and converges uniformly in \( x \in M \) and locally uniformly in \( t \geq 0 \) for all bounded \( f \in C^{\text{Lip}}(M, g) \).

3.3 Constructing a heat kernel on \( M_1 \sqcup M_2 \)

Now we return the dynamic situation and consider a single smooth manifold evolving by the Ricci flow. Take \( g(t) \) a family of metrics on \( M \) satisfying (1.1) for \( t \in [0, T), \ T > 0 \). At each time \( t \), just as in (3.2), define the normalized Riemannian uniform distribution on spheres centered at \( x \in (M, g(t)) \) of radius \( r > 0 \) by

\[
\sigma^t_{r,x}(A) := \frac{\mathcal{H}^{n-1}(A \cap \partial B^t(x, r))}{\mathcal{H}^{n-1}(\partial B^t(x, r))}, \quad A \in \mathcal{B}(M), \tag{3.4}
\]

where \( B^t(x, r) \) denotes the ball of radius \( r \) centered at \( x \) with respect to the fixed metric \( g(t) \). As before, we have a family of Markov operators \(\sigma^t_r\) on the set of bounded Borel-measurable functions \((M, g(t))\) defined above replacing \(\sigma_r\) by \(\sigma^t_r\) and integrating over \((M, g(t))\). Just as before, we have (for a subsequence)

\[
\left( \sigma_{r, \frac{\sqrt{2\pi}t}{j}} \right)^j f(x) \to p_t^\ast f(x) = e^{t\Delta_{g(t)}} f(x)
\]

uniformly in \( x \in (M, g(t)) \) and locally uniformly in \( t \geq 0 \) for all bounded \( f \in C^{\text{Lip}}(M, g(t)) \).
Consider now the entire time where the Ricci flow is defined for $M$; i.e. $M \times [0, T)$. Let $B$ denote the Banach space $c^{\text{Lip}}(M^n, g(t))$ with the sup-norm and $\mathcal{L}(B)$ the space of bounded linear operators on $B$. For each $t$, consider functions $F_t : [0, \infty) \to \mathcal{L}(B)$ where

$$F_t(t) = e^{t\Delta g(t)}. \quad (3.6)$$

Note that $F_t(0) = Id$ for every $t \in [0, T)$ and for any $f \in B$

$$F_t'(0)f = \lim_{t \downarrow 0} \frac{F_t(t)f - f}{t} = \lim_{t \downarrow 0} \frac{e^{t\Delta g(t)}f - f}{t} = \Delta g(t)f.$$  

Thus, by applying a generalization of the Trotter-Chernov product formula (19, Main Theorem) to the time-dependent operators of (3.6), for any function $u : M \times (0, T) \to \mathbb{R}$ solving the initial value problem

$$\begin{cases}
\frac{d}{dt}u(x, t) = \Delta g(t)u(x, t) \\
u(x, 0) = f(x),
\end{cases} \quad (3.7)$$

for which there exists a corresponding one-parameter family of bounded linear operators $U(t, 0)_{0 \leq t \leq T}$ in $B$ such that $u(x, t) = U(t, 0)f(x)$, it follows that for all $0 \leq t \leq T$ we have

$$U(t, 0) = \lim_{m \to \infty} \prod_{m = m - 1}^0 F_{\frac{t}{m}} \left( \frac{t}{m} \right) = \lim_{m \to \infty} \prod_{m = m - 1}^0 e^{\frac{t}{m} \Delta g(\frac{t}{m})} \quad (3.8)$$

with convergence of the limit in the strong operator topology of $\mathcal{L}(B)$. Combining this with (3.6) we can further write, for any $f \in B$,

$$u(x, t) = U(t, 0)f(x) = \lim_{m \to \infty} \prod_{m = m - 1}^0 \lim_{j \to \infty} \left( \sigma_t \frac{1}{\sqrt{m^2}} \right)^j f(x). \quad (3.9)$$

Naturally, as we saw earlier, this description gives a metric measure characterization of solutions to the heat equation on the evolving manifold $(M, g(t))$.

Finally, we turn our attention to the situation of the current paper and use the characterization above to describe solutions for the heat equation on $M_1 \sqcup M_2$. Note that the description in (3.6) is locally defined and thus allows for generalization to the disjoint union $M_1 \sqcup M_2$. Indeed, as $j \to \infty$ the operators $\sigma_t \frac{1}{\sqrt{m^2}}$ are ultimately restricted to individual components $M_1$ or $M_2$ of $M_1 \sqcup M_2$ depending on whether $x \in M_1$ or $x \in M_2$ (resp.). Motivated by the discussion above we define

**Definition 3.4.** Let $(M_i, g_i(t))$, for $i = 1, 2$, be compact Riemannian manifolds supporting smooth families of metrics satisfying the Ricci flow equation given by (1.1) for $t \in [0, T_1)$. Also, let $D^i$ be a family of distance functions on $M_1 \sqcup M_2$ so that each $t \in [0, \min(T_1, T_2))$ we have $(M_1 \sqcup M_2, D^i)$ is a complete, compact metric space compatible with the family of metrics $g_i(t)$ on $M_i$ resp.; i.e. for $i = 1, 2$, 

$$D^i|_{M_i} = \sigma_{g_i(t)}, \quad (3.10)$$

and such that

$$\frac{\partial}{\partial t} D^i(x, y) \geq \Delta_{M_1 \times M_2} D^i(x, y), \quad \text{for } x \in M_1, y \in M_2. \quad (3.11)$$

A function $u : M_1 \sqcup M_2 \times (0, T) \to \mathbb{R}$ is said to solve the initial value problem (3.6) on $M_1 \sqcup M_2$ for $f \in C^{\text{Lip}}(M_1 \sqcup M_2, D^i)$, provided

$$u(x, t) = \lim_{m \to \infty} \prod_{m = m - 1}^0 \lim_{j \to \infty} \left( \sigma_t \frac{1}{\sqrt{m^2}} \right)^j f(x). \quad (3.12)$$
Note that

Lemma 3.5. Let \((M_i, g_i(t))\), for \(i = 1, 2\), and \((M_1 \sqcup M_2, D')\) be as above and suppose \(D^0(x, y) > 0\) for all \(x \in M_1, y \in M_2\). A function \(u : M_1 \sqcup M_2 \times (0, T) \to \mathbb{R}\) solves the initial value problem (3.7) on \(M_1 \sqcup M_2\) if and only if \(u|_{M_i}\) and satisfies smooth heat equation on \(M_i\), for \(i = 1, 2\).

Proof. First, note that if \(D^0(x, y) > 0\) for \(x \in M_1, y \in M_2\) at the initial time \(t = 0\), then by the maximum principle (see, for example, Theorem 3.1.1 of [18]) we have

\[ D^t(x, y) > 0, \text{ for all } t > 0 \text{ and } x \in M_1, y \in M_2. \]

For a fixed \(t\), it follows that the measures \(\sigma^t_{r,x}\) when defined on \(M_1 \sqcup M_2\) agree with \(\sigma^t_{r,M_i}|_{M_i}\) for \(x \in M_i\) provided \(r \) is taken small enough: namely \(r < \inf_{x \in M_1, y \in M_2} D^t(x, y)\). Thus, for \(j\) large enough it follows that

\[ \sigma^t_{\sqrt{m^2 t}} = \sigma^t_{\sqrt{m^2 t}}|_{M_i}. \] (3.13)

Now for \(u : M_1 \sqcup M_2 \times (0, T) \to \mathbb{R}\) which satisfies the IVP given in (3.7) we have that

\[ u(x, t)|_{M_i} = \lim_{m \to \infty} \prod_{i=m-1}^0 \lim_{j \to \infty} \left( \sigma^t_{\sqrt{m^2 t}}|_{M_i} \right)^j f(x). \] (3.14)

As pointed out in the discussion above, for a smooth Riemannian manifold \(M_1\) whose heat kernel is denoted by \(p_t(x, y)\), since \(p_t f(x) = p_t f(x)\), we have

\[ p_t f(x) = \lim_{j \to \infty} \left( \sigma^t_{\sqrt{2nt/j}} \right)^j f(x). \]

Thus, we can write using the notation as before where \(F_t(y) = e^{t\Delta g(t)} = p_t^t\),

\[ u(x, t)|_{M_i} = \lim_{m \to \infty} \prod_{i=m-1}^0 p_{\sqrt{m^2 t}} f(x) = \lim_{m \to \infty} \prod_{i=m-1}^0 F_{\sqrt{m^2 t}} f(x) = U(t, 0) f(x). \] (3.15)

Thus, by the generalized Trotter product formula and [8], it follows that \(u(x, t)|_{M_i}\) solves the heat equation on \(M_1\). In precisely the same way, we verify that \(u|_{M_2}\) also satisfies the heat equation on \(M_2\).

Furthermore, suppose some function \(u(x, t)\) defined on \(M_1 \sqcup M_2\) when restricted to either \(M_i\) satisfies the heat equation on that component. Again by (3.13) it follows that \(u(x, t)\) satisfies the IVP on the disjoint union \(M_1 \sqcup M_2\).

4 Proof of Theorem 1.2 and consequences

Proof. (Theorem 1.2). With \(M_i\) as above, let \(u_i : M_i \times (0, T) \to \mathbb{R}\) be solutions to \(\frac{\partial u_i}{\partial t} = \Delta_{g_i(t)} u_i, i = 1, 2\). Consider the disjoint union \(M_1 \sqcup M_2\) and define a function \(u : M_1 \sqcup M_2 \times (0, T) \to \mathbb{R}\) by

\[ u(x, t) = \begin{cases} u_1(x, t), & \text{when } x \in M_1 \\ u_2(x, t), & \text{when } x \in M_2. \end{cases} \] (4.1)
Recall, by assumption
\[ D^t(m_1, m_2) > 0, \quad \text{for all} \quad m_1 \in M_1, m_2 \in M_2, t > 0, \quad (4.2) \]
so by Lemma 3.3 the function \( u(x, t) \) satisfies the heat equation on \( M_1 \sqcup M_2 \). Note that
for any \( t \in [0, T] \), there exists \( p, q \in (M_1 \sqcup M_2, D^t) \) such that
\[ \text{Lip}(u, t) = \frac{|u(p, t) - u(q, t)|}{D^t(p, q)}. \quad (4.3) \]
Clearly, if \( p, q \in M_1 \), fixed, then by Theorem 3.3 of Topping-McCann, the property that
\( \text{Lip}(u, t) \) is non-increasing as a function of \( t \) is equivalent to \( g_i(t) \) being a solution to
the super Ricci flow. Thus, we are done since each \( (M_i, g_i(t)) \) in fact solves \( \text{11} \) by
assumption and so obviously \( \text{11} \). Therefore, we focus on the case when the Lipschitz
constant of \( u \) is achieved by a point in \( M_1 \) and a point in \( M_2 \).

Fix \( t \in (0, T) \). Without loss of generality, assume the value of \( \text{Lip}(u, t) \) is attained by
the points \( p \in M_1, q \in M_2 \). In a neighborhood sufficiently near \( (p, q) \in M_1 \times M_2 \), we may
also assume (without loss of generality) that \( u_1(x, t) - u_2(y, t) \geq 0 \) so that the function
on \( M_1 \times M_2 \) given by
\[ (x, y) \mapsto \frac{u_1(x, t) - u_2(y, t)}{D^t(x, y)} \]
is nonnegative and has an absolute maximum at the point \( (p, q) \). Therefore,
\[ \nabla \left( \frac{u_1(x, t) - u_2(y, t)}{D^t(x, y)} \right) \bigg|_{(p, q)} = 0, \quad (4.4) \]
and
\[ \Delta \left( \frac{u_1(x, t) - u_2(y, t)}{D^t(x, y)} \right) \bigg|_{(p, q)} \leq 0. \quad (4.5) \]
Furthermore, for points \( x, y \in M_1 \sqcup M_2 \) sufficiently close to \( p \in M_1 \) and \( q \in M_2 \) (resp.)
it follows from \( \text{11} \) that \( u_1(x, t) - u_2(y, t) = u|_{M_1}(x, t) - u|_{M_2}(y, t) = u(x, t) - u(y, t) \).

To simplify notation, set \( \pi(x, y, t) = u_1(x, t) - u_2(y, t) \). From \( \text{11} \) we have
\[ \nabla \left( \frac{\pi}{D^t} \right) = \frac{D^t \nabla \pi - \pi \nabla D^t}{(D^t)^2} = 0, \quad (4.6) \]
and thus
\[ \pi \nabla D^t = D^t \nabla \pi. \quad (4.7) \]
To evaluate \( \text{11} \), note that
\[ \nabla^2 \left( \frac{\pi}{D^t} \right) = \frac{(D^t)^2 (\nabla D^t \nabla \pi + D^t \nabla^2 \pi - \nabla \pi \nabla D^t - \pi \nabla^2 D^t) - 2 \nabla D^t \nabla D^t (D^t \nabla \pi - \pi \nabla D^t)}{(D^t)^2} \quad (4.8) \]
\[ = \frac{\nabla^2 \pi - \frac{\nabla \pi D^t}{(D^t)^2} - \frac{\nabla D^t \nabla \pi}{(D^t)^2} - \frac{\nabla \pi \nabla D^t}{(D^t)^2} + \frac{2 \pi \nabla D^t \nabla D^t}{(D^t)^2}}{(D^t)^2}; \quad (4.9) \]
and, therefore
\[ \Delta \left( \frac{\pi}{D^t} \right) = \text{tr} \nabla^2 \left( \frac{\pi}{D^t} \right) \quad \quad (4.10) \]
\[ = \text{tr} \frac{(D^t)^2 (\nabla D^t \nabla \pi + D^t \nabla^2 \pi - \nabla \pi \nabla D^t - \pi \nabla^2 D^t) - 2 \nabla D^t \nabla D^t (D^t \nabla \pi - \pi \nabla D^t)}{(D^t)^2} \quad (4.11) \]
\[ = \frac{\Delta \pi - \frac{\nabla \pi D^t}{(D^t)^2} - 2 \frac{(\nabla D^t) \nabla \pi}{(D^t)^2} + \frac{2 \pi \nabla D^t \nabla D^t}{(D^t)^2}}{(D^t)^2}. \quad (4.12) \]
where we used \( \text{(4.7)} \) to evaluate in the last term. Furthermore, using \( \text{(4.7)} \) to write
\[
\nabla u = \nabla D_t D_t,
\]
we have
\[
2 \langle \nabla D_t, \nabla u, \rangle (D_t)^2 = \langle \nabla D_t, u \nabla D_t D_t \rangle (D_t)^2 = 2 \frac{\nabla |\nabla D_t|^2}{(D_t)^2}
\]
which implies
\[
\Delta \left( \frac{\nabla D_t}{D_t} \right) \bigg|_{(p,q)} = \Delta \frac{\nabla D_t}{D_t} - \frac{\nabla \Delta D_t}{(D_t)^2}. \tag{4.14}
\]
So, by \( \text{(4.5)} \), it follows that at \((p,q)\)
\[
\frac{\Delta \nabla D_t}{D_t} \leq \frac{\nabla \Delta D_t}{(D_t)^2}; \tag{4.15}
\]
or, equivalently,
\[
\Delta \nabla (p,q) \leq \frac{\nabla(p,q)}{D_t(p,q)} \Delta D_t(p,q). \tag{4.16}
\]
By assumption, \( \frac{\partial}{\partial t} D_t \geq \Delta D_t \), and since \( \frac{\nabla D_t}{D_t} \geq 0 \) we get
\[
\Delta \nabla \leq \frac{\nabla \partial D_t}{D_t} \bigg|_{(p,q)}, \tag{4.17}
\]
and, thus, since \( \nabla : (M_1 \times M_2) \times (0,T) \to \mathbb{R} \) solves the heat equation by Lemma 3.5
\[
\frac{\partial \nabla}{\partial t} \leq \frac{\nabla \partial D_t}{D_t}. \tag{4.18}
\]
Finally, note that
\[
\frac{\partial}{\partial t} \text{Lip}(u,t) = \frac{\partial}{\partial t} \sup_{x \neq y \in M_1 \cup M_2} \frac{|u(x,t) - u(y,t)|}{D_t(x,y)} \leq \sup_{x \neq y \in M_1 \cup M_2} \frac{D_t \frac{\partial u}{\partial t} - \nabla \partial D_t}{(D_t(x,y))^2}. \tag{4.19}
\]
Since \( \text{(4.18)} \) holds for any pair of points which achieve the Lipschitz constant, it follows that \( \frac{\partial}{\partial t} \text{Lip}(u,t) \leq 0 \) and thus we have \( \text{Lip}(u,t) \) is decreasing as a function of \( t \) and we are done.

This can be easily generalized to address additional components.

**Corollary 4.1.** For \( i = 1, 2, \ldots, k \), let \((M_i, g_i(t))\) be compact \( n \)-dimensional manifolds whose metrics \( g_i(t) \) satisfy \( \text{Lip}(\text{Lip}, \text{Ricci}) \) for \( t \in [0,T_i) \). Consider a family of metric spaces \((M_1 \sqcup M_2 \sqcup \cdots \sqcup M_k , D^t)\) for \( t \in (0,T) \), \( T = \min(T_1, T_2, \ldots, T_k) \) and suppose that \( D^t \) satisfies \( \text{Lip}(\text{Lip}, \text{Ricci}) \) for all \( x \in M_i, y \in M_j \) with \( i \neq j \), then the family of metrics \( D^t \) is a super Ricci flow of \( M_1 \sqcup M_2 \sqcup \cdots \sqcup M_k \).

Furthermore, considering \((M_1 \sqcup M_2 , D^t)\) as a family of metric spaces, the evolution inequality given in \( \text{(4.18)} \) also provides control on how the distance between \( M_1 \) and \( M_2 \) changes over time. Namely, if at the initial time \( t = 0 \) we have \( D^0(x,y) \geq c > 0 \), then \( D^t(x,y) \geq c \) for all \( t > 0 \). This follows from a direct application of the maximum principle.
References

[1] Angenent, S. and Knopf, D., An example of neck-pinching for Ricci flow on $S^{n+1}$. Math. Res. Lett. 11 (2004), no. 4, 493-518.

[2] Angenent, S. and Knopf, D., Precise asymptotics of the Ricci flow neckpinch. Comm. Anal. Geom. 15 (2007), no. 4, 773-844.

[3] Angenent, S., Caputo, C., and Knopf, D. Minimally invasive surgery for Ricci flow singularities

[4] Bakry, D. and Emery, M. Diffusions hypercontractives. In Séminaire de probabilités, XIX, 1983/84, volume 1123 of Lecture Notes in Math., pp 177–206. Springer, Berlin, 1985.

[5] Cordero-Erausquin, D., McCann, R., and Schmuckenschläger, M. Prékopa-Leindler type inequalities on Riemannian manifolds, Jacobi fields, and optimal transport Ann. Fac. Sci. Toulouse Math. (6) 15 (2006), no. 4, 613–635.

[6] Cordero-Erausquin, D., McCann, R., and Schmuckenschläger, M. A Riemannian interpolation inequality à la Borell, Brascamp and Lieb, Invent. Math. 146 (2001), no. 2, 219–257.

[7] Ethier, S.N. and Kurtz, T. G. Markov processes. Characterization and convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986.

[8] Feldman, M., Ilmanen, T. and Knopf, D., Rotationally symmetric shrinking and expanding gradient Kahler-Ricci solitons. J. Differential Geom. 65 (2003), no. 2, 169-209.

[9] Hamilton, R. Three-manifolds with positive Ricci curvature. J. Differential Geom. 17 (1982), no. 2, 255-306.

[10] Lott, J. and Villani, C., Ricci curvature for metric-measure spaces via optimal transport. Ann. of Math. (2) 169 (2009), no. 3, 903-991.

[11] Topping, P. and McCann, R., Ricci flow, entropy and optimal transportation. Amer. J. Math. 132 (2010), no. 3, 711-730.

[12] Perelman, G. The entropy formula for the Ricci flow and its geometric applications, arXiv:

[13] Sesum, N., Curvature tensor under the Ricci flow. Amer. J. Math. 127 (2005), no. 6, 1315-1324.

[14] Simon, M. A class of Riemannian manifolds that pinch when evolved by Ricci flow. Manuscripta Math. 101 (2000), no. 1, 89-114.

[15] Sturm, K.T. On the geometry of metric measure spaces. I. Acta Math. 196 (2006), no. 1, 65-131.

[16] Sturm, K.T. On the geometry of metric measure spaces. II. Acta Math. 196 (2006), no. 1, 133-177.

[17] Sturm, K.T. and von Renesse, M., Transport inequalities, gradient estimates, entropy, and Ricci curvature. Comm. Pure Appl. Math. 58 (2005), no. 7, 923-940.

[18] Topping, P. Lectures on the Ricci flow. London Mathematical Society Lecture Note Series, 325. Cambridge University Press, Cambridge, 2006.
[19] Vuillermot, P. A generalization of Chernoff's product formula for time-dependent operators. J. Funct. Anal. 259 (2010), no. 11, 2923-2938.

S. Lakzian
CUNY Graduate Center, 365 Fifth Ave, NY, NY 10016, USA
Email address: slakzian@gc.cuny.edu

M. Munn
University of Missouri, Columbia, MO 65201, USA
University of Warwick, Coventry, CV4 7AL, UK
Email address: munnm@missouri.edu