GEODESIC COMPLETENESS OF DIAGONAL $G_2$ METRICS

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In this talk a sufficient condition for a diagonal orthogonally transitive cylindrical $G_2$ metric to be geodesically complete is given. The condition is weak enough to comprise all known diagonal perfect fluid cosmological models that are non-singular.

1 Introduction

The interest on non-singular cosmological models has been triggered by the appearance of Senovilla’s cylindrical solution of Einstein’s equations. However, whereas it is easy to check the regularity of the curvature invariants, it is usually cumbersome to determine whether a spacetime is geodesically complete.

Therefore it would be appealing to have a sufficient condition on the metric coefficients that could easy to check in order to settle the issue.

In this talk we provide a condition that is not too restrictive in the sense that all known non-singular diagonal cylindrical perfect fluid models are comprised in it.

2 Geodesic equations

From the beginning we shall restrict ourselves to diagonal cylindrical orthogonally transitive models. The metric can be written as,

$$ds^2 = e^{2g(t,r)} \{ -dt^2 + dr^2 \} + \rho^2(t,r)e^{2f(t,r)}d\phi^2 + e^{-2f(t,r)}dz^2,$$

in a coordinate patch where the time and radial coordinates are isotropic and the angular and axial coordinates are adapted to the Killing fields. The usual ranges for the cylindrical coordinates are assumed,

$$-\infty < t, z \leq \infty, \quad 0 < r \leq \infty, \quad 0 < \phi < 2\pi.$$

The metric functions will be taken to be $C^2$ in order to have a well defined Riemann curvature and we shall also assume that there is an axis in the spacetime. The coordinates are chosen so that it is located on $r = 0$. 

1
Since there is a two-dimensional group of isometries, the order of two of the geodesic equations,
\[ \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \]  
(3)
can be lowered by the introduction of constants of motion, \( L \) and \( P \),
\[ L = \rho^{-2}(t, r)e^{-2f(t, r)} \dot{\phi}, \]  
(4)
\[ P = e^{2f(t, r)} \dot{z}, \]  
(5)
and the other equations can be written in a compact way as,
\[ \left\{ e^{2g(t, r)} \dot{t} \right\}_t - \frac{e^{-2g(t, r)}}{2} \left\{ e^{2g(t, r)} \left[ \delta + P^2 e^{2f(t, r)} + L^2 e^{-2f(t, r)} \right] \right\}_t = 0, \]  
(6)
\[ \left\{ e^{2g(t, r)} \dot{r} \right\}_r + \frac{e^{-2g(t, r)}}{2} \left\{ e^{2g(t, r)} \left[ \delta + P^2 e^{2f(t, r)} + L^2 e^{-2f(t, r)} \right] \right\}_r = 0, \]  
(7)
where the dot stands for derivation with respect to the affine parameter.

Finally, the equation that determines the affine parametrization is,
\[ \delta = e^{2g(t, r)} \left\{ \dot{t}^2 - \dot{r}^2 \right\} - L^2 \rho^{-2}(t, r)e^{-2f(t, r)} - P^2 e^{2f(t, r)}, \]  
(8)
where \( \delta \) is zero for null, one for timelike and minus one for spacelike geodesics, that is, it is non-negative for causal geodesics.

The previous system of equations can be reduced to a first order one in an efficient way by making use of hyperbolic functions of a function \( \xi \), that is related to the radial speed along the geodesic,
\[ \dot{t}(t, r) = e^{-2g(t, r)} F(t, r) \cosh \xi(t, r), \]  
(9)
\[ \dot{r}(t, r) = e^{-2g(t, r)} F(t, r) \sinh \xi(t, r), \]  
(10)
\[ \dot{\xi}(t, r) = -e^{-2g(t, r)} \left\{ F_t(t, r) \sinh \xi(t, r) + F_r(t, r) \cosh \xi(t, r) \right\}, \]  
(11)
\[ F(t, r) = e^{g(t,r)} \sqrt{\delta + \frac{L^2 e^{-2f(t,r)}}{\rho^2(t,r)} + P^2 e^{2f(t,r)}}, \]  

(12)

for future-pointing geodesics.

In order to have a similar expression for past-pointing geodesics one has only to reverse the sign of the time derivatives and of \( \dot{t} \).

If we impose on this system of equations conditions in order to prevent arbitrarily large growth of the time and radial coordinate for finite affine parameter we get the following theorem,

**Theorem:** A cylindrically symmetric diagonal metric in the form of Eq. 1 with \( C^2 \) metric coefficients \( f, g, \rho \) is future causally geodesically complete if the following conditions are fulfilled for large values of \( t \).

- For \( \dot{r} > 0 \),
  \[ \begin{align*}
  g_r + g_t & \quad (g - f - \ln \rho)_r + (g - f - \ln \rho)_t \\
  (g + f)_r + (g + f)_t & > 0,
  \end{align*} \]
  
  (13)

- For \( \dot{r} < 0 \),
  \[ \begin{align*}
  \delta & \quad \frac{g_r}{(g - f - \ln \rho)_r + (g - f - \ln \rho)_t} \\
  \left( g - f \right)_r + \left( g + f \right)_t & > 0,
  \end{align*} \]
  
  (14)

positive or at most of the same order as the respective terms in the previous equations.

- For \( \dot{r} > 0 \),
  \[ \delta \{g_t - g_r\} + L^2 e^{-2f} \rho^2 \left\{ (g - f - \ln \rho)_t - (g - f - \ln \rho)_r \right\} + \\
  P^2 e^{2f} \left\{ (g + f)_t - (g + f)_r \right\} > 0, \]
  
  (15)

- For \( \dot{r} < 0 \),
  \[ \delta g_r + L^2 e^{-2f} \rho^2 \left( g_r - f_r - \frac{\partial f}{\partial \rho} \right) + P^2 e^{2f} \left( g_r + f_r \right), \]
  
  (16)

negative or at most of the same order as the term in the previous equation.
There must be constants $a$, $b$, such that,

$$\begin{align*}
2g(t,r) & \\
g(t,r) + f(t,r) + \ln \rho & \\
g(t,r) - f(t,r) &
\end{align*} \geq - \ln |t + a| + b. \quad (17)
$$

A similar theorem is obtained for past-pointing geodesics just reversing the sign of the time derivatives and expressing the conditions for small values of $t$ instead of for large values.

If the conditions in this theorem are fulfilled, the spacetime is globally hyperbolic, since every null geodesic intersects once and only once every hypersurface $t = \text{const.}$.

3 Discussion

The only known non-singular cylindrical diagonal perfect fluid cosmological models that are known to us are those in $^1,^4,^5,^6$. It is easy to check that all of them fulfill the conditions stated in the theorem. Therefore it cannot be considered too restrictive.

A similar condition is being prepared for non-diagonal models and it is expected to be published soon $^7$.

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