Bicommutant categories from conformal nets

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Abstract

We prove that the category of solitons of a finite index conformal net is a bicommutant category, and that its Drinfel’d center is the category of representations of the conformal net. In the special case of a chiral WZW conformal net with finite index, the second result specializes to the statement that the Drinfel’d center of the category of representations of the based loop group is equivalent to the category of representations of the free loop group. These results were announced in [Hen15].

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1 Introduction and statement of results

In [Hen15], we made the announcement that, at least for $G = SU(n)$, the Drinfel’d center of the category of locally normal\(^1\) representations of the based loop group is equivalent, as a braided tensor category, to the category of locally normal representations of the free loop group:

$$Z(\text{Rep}^k(\Omega G)) \cong \text{Rep}^k(LG). \quad (1)$$

One of the main goals of this paper is to establish the above relation (see Theorem 1.1 for a precise statement).

It should be noted that the representation theory of based loop groups had not been considered before. The mere fact that the fusion product makes sense for these representations is, in itself, remarkable.

The broader relevance of the above result comes from topological quantum field theory (TQFT), specifically from Chern-Simons theory. There are two main classes of topological quantum field theories in dimension three: theories of Turaev-Viro type, associated to fusion categories [TV92, BW96], and theories of Reshetikhin-Turaev type, associated to modular tensor categories [RT91, BK01] (Chern-Simons theories are of the latter kind). Since the groundbreaking work of Jacob Lurie on the classification of extended TQFTs [Lur09], it has been an important question to determine which theories fit into that formalism; a theory for which that is the case is said to “extend down to points”. It is broadly accepted (even though this has not yet been proven) that theories of Turaev-Viro type extend down to points [DSPS13, Wra10]. On the other hand, for a typical Reshetikhin-Turaev theory, it was generally thought that this should not be possible (the results in [DMNO13, §5.5] can be interpreted as a no-go theorem — see [Hen15, Rem. 5] for a discussion).

The theory of bicommutant categories (which still needs to be developed) promises to achieve two things. First, it shows that, contrary to general expectations, Reshetikhin-Turaev theories do seem to extend down to points (at least the ones coming from conformal nets). Second, and more importantly, it puts Turaev-Viro theories and Reshetikhin-Turaev theories on an equal footing, by providing a unified language that applies to both of them. The expected relations are summarised in the following diagram:

\[
\begin{array}{ccc}
\{ \text{Unitary fusion category} \} & \overset{1}{\longrightarrow} & \{ \text{Bicommutant categories} \} \\
\text{Turaev–Viro construction}\(^2\) & \text{\quad Conformal nets} \}
\end{array}
\]

\[
\begin{array}{c}
\{ \text{Extended 3-dim. TQFTs} \}
\end{array}
\]

\[
\begin{array}{c}
\{ \text{Reshetikhin–Turaev construction}\(^3\) \text{ applied to } \text{Rep}_k(A) \}
\end{array}
\]

The arrow labelled 1 was constructed in our earlier paper [HP17]. The arrow labelled 2

\(^1\)Local normality is a technical condition which might be equivalent to the positivity of the energy [Hen17b, Conj. 22 & 34].

\(^2\)The Turaev-Viro construction requires the choice of a pivotal structure on the fusion category. A unitary fusion category admits a canonical pivotal structure [ENO05, Prop. 8.23].

\(^3\)The Reshetikhin-Turaev construction requires the choice of a pivotal structure on the fusion category.
is the content of the present paper (see Corollary 1.8 below for a precise statement). The arrow labelled 3 is still conjectural and is only expected to exist when the bicommutant category satisfies certain finiteness conditions (ensuring that it is fully dualisable).

1.1 Motivations from Chern-Simons theory

By the celebrated cobordism hypothesis [BD95, Lur09], a topological field theory is entirely determined by its value on a point. The present line of research was motivated by the quest for a mathematical object that one may reasonably declare to be the value of Chern–Simons theory on a point.

Given a compact connected Lie group \( G \), with classifying space \( BG \), let \( H^4_+(BG, \mathbb{Z}) \) be the subset of elements \( k \in H^4(BG, \mathbb{Z}) \) whose image under the Chern–Weil homomorphism

\[
H^4(BG, \mathbb{Z}) \to \text{Sym}^2(\mathfrak{g}^*)^G
\]

are positive definite metrics \( \langle \cdot, \cdot \rangle_k \) on \( \mathfrak{g} \). By [Hen16, Thm. 6], the map (2) is injective and the image of \( H^4_+(BG, \mathbb{Z}) \) under that map is, up to a scalar, the set of invariant metrics on \( \mathfrak{g} \) such that \( \|X\|^2 \in \mathbb{Z} \) for all \( X \in \{X \in \mathfrak{g} : \exp(X) = e\} \).

In our earlier paper [Hen16], given \( G \) and \( k \in H^4_+(BG, \mathbb{Z}) \) as above, we constructed a vertex operator algebra \( V_{G,k} \) and a chiral conformal net \( A_{G,k} \), called the chiral WZW vertex algebra and the chiral WZW conformal net, respectively.\(^4\) A bijective correspondence was established in [CKLW15] between a certain class of unitary vertex algebras and a certain class of chiral conformal nets. We conjecture that \( V_{G,k} \) and \( A_{G,k} \) map to each other under that correspondence, and that there is an equivalence of modular tensor categories

\[
\text{Rep}^f(V_{G,k}) \cong \text{Rep}^f(A_{G,k}).
\]

Here, \( \text{Rep}_f \) denotes the category of representations which are finite direct sums of irreducible ones. Assuming the above conjectures, we define \( \text{Rep}^k(LG) \), the modular tensor category of positive energy representations of the loop group \( LG \) at level \( k \), to be the category \( \text{Rep}^f(V_{G,k}) \), equivalently \( \text{Rep}^f(A_{G,k}) \).

Let \( \text{CS}_{G,k} \) be the Chern–Simons theory associated to the gauge group \( G \) and the level \( k \) [DW90, Wit89]. This is a 3-dimensional topological field theory with action functional given, up to a scalar, by:\(^5\)

\[
S = \int \langle A \wedge dA \rangle_k + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle_k.
\]

In [Hen15], we argued that a necessary condition for a tensor category \( T \) to be the value of \( \text{CS}_{G,k} \) on a point is for its Drinfel’d center \( Z(T) \) to be braided equivalent to \( \text{Rep}^k(LG) \), or possibly \( \text{Rep}^k(LG) \) (see Section 1.3 for a definition of the Drinfel’d center). We proposed the category \( \text{Rep}^k(\Omega G) \) of locally normal representations of the

\(^3\)For this construction to work, one needs to assume that the conformal net \( A \) has finite index, so that \( \text{Rep}_f(A) \) is modular—see Remark 1.2.

\(^4\)Earlier references on these models include [DLM96],[Li01],[FGK88],[MS89, §2],[FSS96, §6].

\(^5\)When \( G \) is not simply connected, one cannot use the formula (3) to define the action. See [CJM+05, DW90, FSS15] for ways to overcome this difficulty.
based loop group as a candidate for the value of Chern–Simons theory on a point (see [FHLT10, Wra10] for previous work in that direction), and offered the relation (1) as evidence for our claim.

For the remainder of this section, let us commit to the following definitions:

\[
\text{Rep}_k^f(LG) := \text{Rep}_f(A_{G,k}) \quad \text{Rep}_k^f(LG) := \text{Rep}(A_{G,k}).
\]

(4)

Let us also define \(\text{Rep}_k^f(\Omega G)\) to be the category of solitons of \(A_{G,k}\) (see Definition 1.4, in the next section). We call it the category of locally normal representations of the based loop group\(^6\)

It is widely believed that the chiral WZW conformal nets \(A_{G,k}\) satisfy a certain finiteness condition called finite index, or complete rationality (see Section 1.4 for a definition). This property is known to hold for \(G = SU(n)\) [Was98, Xu00], and in a few other cases.

**Theorem 1.1.** Let \(\text{Rep}_k^f(LG)\) be as in (4). If \(A_{G,k}\) has finite index, then

\[Z(\text{Rep}_k^f(\Omega G)) \cong \text{Rep}_k^f(LG).\]

**Proof.** This is a special case of Theorem A, in Section 1.4. □

**Remark 1.2.** If a conformal net \(A\) has finite index, then \(\text{Rep}_f(A)\) is a modular tensor category, and \(\text{Rep}(A) = \text{Hilb} \otimes_{\text{Vec}} \text{Rep}_f(A)\) [KLM01, Cor. 37][BDH17, Thm. 3.9].\(^7\) The latter implies that every object of \(\text{Rep}(A)\) is a (potentially infinite) direct sum of simple objects.

**Remark 1.3.** When \(G\) is not simply connected, we presently do not know, in general, whether the vertex algebra \(V_{G,k}\) is unitary. When \(G \neq SU(n)\), it is not known whether \(A_{G,k}\) is completely rational or whether \(\text{Rep}_f(A_{G,k})\) is modular, except in some isolated cases. Even when \(G = SU(n)\), where it is known that \(\text{Rep}_f(A_{G,k})\) is modular, it is not known whether \(\text{Rep}_f(V_{G,k}) \cong \text{Rep}_f(A_{G,k})\) as (modular) tensor categories, except when \(n = 2\) [Hen15, §3]. Establishing the above properties are important open problems.

### 1.2 Representations and solitons

Conformal nets [BDH15, Def.1.1] are functors \(A : \text{INT} \to \text{VN}\) from the category of intervals (an interval is a manifold diffeomorphic to \([0,1]\)) to the category of von Neumann algebras (see Definition 3.1 for the axioms that such a functor should satisfy).

Let \(S^1 := \{z \in \mathbb{C} : |z| = 1\}\) be the standard circle. A representation of a conformal net consists of a Hilbert space \(H\) and a collection of compatible actions

\[\rho_I : A(I) \to \mathcal{B}(H)\]

of the algebras \(A(I)\), where \(I\) ranges over all subintervals of \(S^1\). We write \(\text{Rep}(A)\) for the category of representations of \(A\) whose underlying Hilbert space is separable.

---

\(^6\)This category is equivalent to the version of \(\text{Rep}_k^f(\Omega G)\) defined in [Hen15, §4] ([Hen17b, Thm. 31]).

\(^7\)The braiding on \(\text{Rep}(A)\) defined in [BDH17, Sec. 3B] has not been compared to the one in [KLM01]. We can therefore not exclude the possibility that, when \(\mu(A) < \infty\), the category \(\text{Rep}(A)\) has two distinct modular structures. The braided structure used in Theorem A is the one used in [KLM01].
Throughout this work, all Hilbert spaces are assumed to be separable. This will be important for the results in Section 4.3 to hold, see Remark 4.17.

The monoidal structure on $\text{Rep}(\mathcal{A})$ is defined as follows. Let $H$ and $K$ be representations. Let $I_+$ be the upper half of $S^1$, and let $I_-$ be its lower half. Precomposing the left action of $\mathcal{A}(I_+)$ on $H$ by the map

$$\mathcal{A}(z \mapsto \bar{z} : I_- \to I_+) : \mathcal{A}(I_-)^{\text{op}} \to \mathcal{A}(I_+)$$

(5)
yields a right action of $\mathcal{A}(I_-)$ on $H$. We let

$$H \boxtimes K := H \boxtimes_{\mathcal{A}(I_-)} K.$$  (6)

Here, the symbol $\boxtimes$ denotes Connes’ relative tensor product (see Section 2.3 for a definition). The algebra $\mathcal{A}(I_-)$ acts on $K$ in the usual way, and it acts on $H$ on the right as described above.

The left actions of $\mathcal{A}(I_-)$ on $H$ and of $\mathcal{A}(I_+)$ on $K$ induce corresponding actions on $H \boxtimes K$. For every interval $I \subset S^1$, the actions of $\mathcal{A}(I \cap I_-)$ and $\mathcal{A}(I \cap I_+)$ on $H \boxtimes K$ extend to an action

$$\rho_I : \mathcal{A}(I) \to B(H \boxtimes K).$$

Together, these equip $H \boxtimes K$ with the structure of a representation. We refer the reader to Section 3.2 for more details. There is also a braiding on $\text{Rep}(\mathcal{A})$, discussed in Section 3.3.

A soliton of a conformal net is something akin to a representation [BE98, Kaw02, LR95, LX04] (the usage of the term ‘soliton’ in algebraic quantum field theory goes back to at least Frö76):

**Definition 1.4 ([LX04, §3.0.1]).** A soliton of $\mathcal{A}$ is a Hilbert space (always assumed separable) equipped with compatible actions of the algebras $\mathcal{A}(I)$, where $I$ ranges over all subintervals of the standard circle whose interior does not contain the base point $1 \in S^1$. We write $T_{\mathcal{A}}$ for the category of solitons of $\mathcal{A}$.

Equivalently, a soliton is a Hilbert space equipped with compatible actions of all the algebras $\mathcal{A}(I)$ as $I$ ranges over all subintervals $I \subset S^1_{\text{cut}}$, where $S^1_{\text{cut}}$ is the manifold obtained from the standard circle by removing its base point and replacing it by two points:

$$S^1 : \quad \bigcirc \quad \quad S^1_{\text{cut}} : \quad \bigcirc \bigcirc$$

Further down, we sometimes write $T^+_{\mathcal{A}}$ in place of $T_{\mathcal{A}}$, for reasons that will become clear later on.

The monoidal structure on $T_{\mathcal{A}}$ is defined in the same way as the one of $\text{Rep}(\mathcal{A})$. Given two solitons $H$ and $K$, we consider the right action $\mathcal{A}(I_-)^{\text{op}} \to B(H)$ given as the composite of the map (5) with the left action $\mathcal{A}(I_+) \to B(H)$, and we let

$$H \boxtimes K := H \boxtimes_{\mathcal{A}(I_-)} K.$$  (6)

*Here, we use the convention $\mathcal{A}(I_1 \sqcup I_2) := \mathcal{A}(I_1) \bar{\otimes} \mathcal{A}(I_2)$, where $\bar{\otimes}$ denotes the spatial tensor product, to define the value of $\mathcal{A}$ on disjoint unions of intervals.*
The left actions of $A(I_-)$ on $H$ and of $A(I_+)$ on $K$ induce corresponding actions on $H \boxtimes K$. Finally, for any interval $I \subset S^1$, $1 \notin I$, the actions of $A(I \cap I_-)$ and $A(I \cap I_+)$ extend to an action
\[ \rho_I : A(I) \to B(H \boxtimes K). \]
The details of this construction can be found in Section 4.1.

Remark 1.5. We remind the reader that, by definition, when $A = A_{G,k}$, the category of solitons agrees with the category $\text{Rep}_k^b(\Omega G)$ of locally normal representations of the based loop at level $k$.

### 1.3 Bicommutant categories

Bicommutant categories are higher categorical analogs of von Neumann algebras. They are obtained by replacing the algebra $B(H)$, in the definition of a von Neumann algebra, by the tensor category $\text{Bim}(R)$ of all bimodules over a hyperfinite factor.

Let $R$ be a hyperfinite factor, and let $\text{Bim}(R)$ be its category of bimodules, equipped with the monoidal structure given by Connes’ relative tensor product (we insist that all Hilbert spaces be separable). The category $\text{Bim}(R)$ admits an antilinear involution at the level of objects (the conjugate of a bimodule) and a second involution at the level of morphisms (the adjoint of a linear map). Together, these two involutions equip this category with the structure of a bi-involutive tensor category (Definition 2.3).

A bicommutant category is a particular kind of bi-involutive tensor category. Given a bi-involutive functor $\iota : T \to B$ between bi-involutive tensor categories, one may consider the commutant $Z_B(T)$ of $T$ inside $B$. The objects of $Z_B(T)$ are pairs $(X,e)$ with $X \in B$ and $e = (e_Y)_{Y \in T}$ a unitary half-braiding $e_Y : X \otimes \iota(Y) \to \iota(Y) \otimes X$, natural in $Y$, and subject to the ‘hexagon’ axiom $e_{Y_1 \otimes Y_2} = (id_{\iota(Y_1)} \otimes e_{Y_1}) \circ (e_{Y_1} \otimes id_{\iota(Y_2)})$ (see Section 2.1 for more details). The category $Z_B(T)$ is again bi-involutive, and is equipped with a bi-involutive functor $(X,e) \mapsto X$ to $B$:
\[ T \to B \leftarrow Z_B(T). \]

The Drinfel’d center is a special case of the above notion:

**Definition 1.6.** The Drinfel’d center $Z(T)$ of a bi-involutive tensor category $T$ is the commutant of $T$ inside itself.

The Drinfel’d center of a bi-involutive tensor category is braided and bi-involutive.

When $B = \text{Bim}(R)$, we write $C' := Z_{\text{Bim}(R)}(T)$ for the commutant of $T$ inside $\text{Bim}(R)$. There is an obvious ‘inclusion’ functor $T \to T''$ from any category to its bicommutant which sends an object $Y \in T$ to the object $(\iota(Y), e')$, with half-braiding $e'$ given by $e'_{(X,e)} := e_Y^{-1}$ for $(X,e) \in C'$.

**Definition 1.7.** A bicommutant category is a bi-involutive tensor category $T$ for which there exists a hyperfinite factor $R$ and a bi-involutive functor $T \to \text{Bim}(R)$ such that the inclusion functor $T \to T''$ is an equivalence of (bi-involutive tensor) categories.
The category of solitons of a conformal net is bi-involutive in the following way. Given \( H \in T_A \), with actions \( \rho_I : A(I) \to B(H) \) for \( I \subseteq S^1_{\text{cut}} \), its conjugate \( \overline{H} \) is the complex conjugate Hilbert space equipped with the actions

\[
A(I) \xrightarrow{A(I)^{\text{op}}} A(\overline{I}) \xrightarrow{\rho_I} B(H) = B(\overline{H}).
\]

Here, \( \overline{I} \) denotes the image of \( I \subset S^1 \) under the complex conjugation map \( S^1 \to S^1 \). The conjugation operation on \( T_A \) squares to the identity, and satisfies \( \overline{H \boxtimes K} \cong \overline{K} \boxtimes \overline{H} \).

Given a conformal net \( A \), set \( R := A(I_{-}) \). Then there is an obvious fully faithful bi-involutive functor

\[
T_A \to \text{Bim}(R).
\]

It sends a soliton \( H \) to the \( R-R \)-bimodule with left action given by the usual left action of \( A(I_{-}) \) on \( H \), and right action given by the left action of \( A(I_{+}) \) precomposed by the map (5). One of our main results (Corollary 1.8) is that when \( A \) has finite index, the above functor exhibits \( T_A \) as a bicommutant category.

1.4 Main results

Recall that \( T_A = T^+_A \) is the category whose objects are Hilbert spaces equipped with compatible actions of the algebras \( A(I) \), for \( I \subset S^1 \), \( 1 \notin \overline{I} \).

Let \( T_{\overline{A}} \) denote the category whose objects are Hilbert spaces equipped with compatible actions of \( A(I) \), for \( I \subset S^1 \), \( -1 \notin \overline{I} \). Letting \( R := A(I_{-}) \), the same formulas (6) and (7) endow \( T_{\overline{A}} \) with the structure of a bi-involutive tensor category, and we have a bi-involutive functor

\[
T_{\overline{A}} \to \text{Bim}(R).
\]

**Theorem A.** Let \( A \) be a conformal net with finite index and let \( R := A(I_{-}) \). Let \( T_A = T^+_A \) be its category of solitons, with canonical inclusion \( T^+_A \to \text{Bim}(R) \) as in (8). Then:

- The canonical map \((T^+_A)' \to \text{Bim}(R)\) is fully faithful and we have \((T^+_A)' = T_{\overline{A}}\).
- The canonical map \((T^-_A)' \to \text{Bim}(R)\) is fully faithful and we have \((T^-_A)' = T^+_A\).
- The Drinfel’d center of \( T^+_A \) is equivalent to \( \text{Rep}(A) \) as a braided bi-involutive tensor category.

**Corollary 1.8.** If \( A \) is a conformal net with finite index, then \( T_A \) is a bicommutant category.

**Remark 1.9.** The main theorem in [Hen15, §5] is stated as an equivalence of balanced tensor categories (a balanced tensor category is a braided tensor categories with twists [JS91]). When \( X \) is a dualizable object, the twist \( \theta_X : X \to X \) is expressible in terms of the braiding and the dagger structure as \( \theta_X := (\text{ev}_X \otimes \text{id})(\text{id} \otimes \beta_{X,X})(\text{ev}_X^* \otimes \text{id}) \), where \( \text{ev}_X : X \otimes X \to 1 \) and \( \text{coev}_X : 1 \to X \otimes X \) are solutions to the normalized duality equations [HP17, §2.2]. This can then be extended to arbitrary objects by additivity (see Remark 1.2).

**Remark 1.10.** If we do not assume that \( A \) has finite index, then we can still define the tensor functor \( T^-_A \to (T^+_A)' \) and the braided tensor functor \( \text{Rep}(A) \to Z(T_A) \), but we do not know whether they are equivalences.
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2 Bicommutant categories

Bicommutant categories are higher categorical analogs of von Neumann algebras. They were introduced in [Hen15], and the first examples were constructed in [HP17].

Let $R$ be a hyperfinite factor, and let $R\text{-Mod}$ be the category of $R$-modules whose underlying Hilbert space is separable. We think of $R\text{-Mod}$ as a higher categorical analog of an infinite dimensional Hilbert space. Our slogan is: von Neumann algebras act on Hilbert spaces; bicommutant categories act on categories like $R\text{-Mod}$.

In this context, the higher categorical analog of $B(H)$ is the tensor category $\text{End}(R\text{-Mod})$ of completely additive endofunctors of $R\text{-Mod}$ (see [Lur11, Lecture 21] or [BDH16, §B.VIII] for a definition of completely additive functors). The latter is equivalent to the tensor category $\text{Bim}(R)$ of all $R\text{-}R$-bimodules.

Recall that a von Neumann algebra is an algebra which admits a map to $B(H)$ such that the natural inclusion $A \to A''$ into its bicommutant is an isomorphism:

$$ A \to B(H) \quad A = A''. $$

Analogously, a bicommutant category is a tensor category $T$ which admits a bi-involutive functor to $\text{Bim}(R)$ such that the natural inclusion functor $T \to T''$ of $T$ into its bicommutant is an equivalence of categories:

$$ T \to \text{Bim}(R) \quad T \cong T''. $$

2.1 The commutant of a tensor category

Let $T$ be a tensor category. The Drinfel’d center $\hat{Z}(T)$ of $T$ is the category whose objects are pairs $(X,e)$, where $X$ is an object of $T$ and $e = (e_Y : X \otimes Y \to Y \otimes X)_{Y \in T}$ is a family of isomorphisms called a half-braiding. The half-braiding is required to be natural in $Y$, and to make the following diagram\(^8\) commute for every $Y, Z \in T$:

$$ \begin{array}{c}
Y \otimes X \otimes Z \\
\downarrow^{e_Y \otimes \text{id}_Z} \\
X \otimes Y \otimes Z \\
\downarrow^{\text{id}_Y \otimes e_Z} \\
Y \otimes Z \otimes X.
\end{array} \tag{9} $$

A morphism $(X^1, e^1) \to (X^2, e^2)$ in the Drinfel’d center is a morphism $f : X^1 \to X^2$ in $T$ such that $(\text{id}_Y \otimes f) \circ e^1_Y = e^2_Y \circ (f \otimes \text{id}_Y)$ for every $Y \in T$. The tensor product

\(^8\)Here, we have suppressed associators for brevity. By adopting this simplified notation, we do not mean to imply that our tensor categories are strict.
of two objects of $\hat{Z}(T)$ is given by $(X^1, e^1) \otimes (X^2, e^2) := (X^1 \otimes X^2, e^{12})$ with $e^{12} := (e_Y \otimes \text{id}_{X^2}) \circ (\text{id}_{X^1} \otimes e_Y)$. Finally, $\hat{Z}(T)$ is equipped with a braiding

$$\beta : (X^1, e^1) \otimes (X^2, e^2) \cong (X^2, e^2) \otimes (X^1, e^1)$$

given by $e_{X^2}^1$. Basic references include [JS91, Maj91, M"ug03].

The above definition can be relativized to the case when $T$ is a subcategory of some bigger tensor category $B$ (or, more generally, when $T$ is equipped with a functor $\iota : T \to B$, not necessarily an inclusion).

**Definition 2.1.** Let $\iota : T \to B$ be a tensor functor between tensor categories. The commutant $\hat{Z}_B(T)$ of $T$ inside $B$ is the category whose objects are pairs $(X, e)$, where $X$ is an object of $B$ and

$$e = (e_Y : X \otimes \iota Y \to \iota Y \otimes X)_{Y \in T}$$

is a collection of isomorphisms, called a half-braiding. The half-braiding is required to be natural in $Y$, and to satisfy the following analog of (9) for every $Y, Z \in T$:

$$\begin{array}{ccc}
X \otimes \iota Y \otimes \iota Z & \xrightarrow{e_Y \otimes \text{id}_{\iota Z}} & \iota Y \otimes \iota Z \otimes X \\
\xrightarrow{\text{id}_X \otimes e_Y} & & \xrightarrow{\text{id}_X \otimes e_Y} \\
X \otimes \iota (Y \otimes Z) & \xrightarrow{e_Y \otimes e_Z} & \iota (Y \otimes Z) \otimes X.
\end{array}$$

A morphism $(X^1, e^1) \to (X^2, e^2)$ in $\hat{Z}_B(T)$ is a morphism $f : X^1 \to X^2$ in $B$ satisfying $(\text{id}_{\iota Y} \otimes f) \circ e^1_Y = e^2_Y \circ (f \otimes \text{id}_{\iota Y})$ for every $Y \in B$.

The tensor product in $\hat{Z}_B(T)$ is given by the same formula as for the Drinfel’d center:

$$(X^1, e^1) \otimes (X^2, e^2) := (X^1 \otimes X^2, e^{12}),$$

$e^{12} := (e_Y^1 \otimes \text{id}_{X^2}) \circ (\text{id}_{X^1} \otimes e_Y^2)$.

Finally, there is a tensor functor $\hat{Z}_B(T) \to B$ given by $(X, e) \mapsto X$.

In the presence of dagger structures, the definitions of Drinfel’d center and of commutant of a tensor category inside another tensor category can be modified by insisting that the half-braidings be unitary. We reserve the notations $Z(T)$ and of $Z_B(T)$ for the unitary versions.

### 2.2 Bi-involutive tensor categories

A **dagger category** is a linear category over $\mathbb{C}$ equipped with an antilinear map $* : \text{Hom}(X, Y) \to \text{Hom}(Y, X)$ which satisfies $f^{**} = f$ and $(f \circ g)^* = g^* \circ f^*$. An invertible morphism in a dagger category is called **unitary** if $f^* = f^{-1}$.

**Definition 2.2.** A **dagger tensor category** is a dagger category equipped with a monoidal structure whose associator and unitor isomorphisms are unitary, and which satisfies $(f \otimes g)^* = f^* \otimes g^*$.
A dagger functor $F$ between dagger tensor categories is a dagger tensor functor if it comes along with a unitary natural transformation $\mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ and a unitary $i : 1 \to F(1)$ such that $\mu_{X,Y \otimes Z} \circ (\text{id}_F(X) \otimes \mu_{Y,Z}) = \mu_{X \otimes Y,Z} \circ (\mu_{X,Y} \otimes \text{id}_F(Z))$ and $\mu_{1,X} \circ (i \otimes \text{id}_F(X)) = \text{id}_{F(X)} = \mu_{X,1} \circ (\text{id}_F(X) \otimes i)$.

A dagger functor between dagger tensor categories is a dagger anti-tensor functor if it comes along with a unitary natural transformation $\nu_{X,Y} : F(X) \otimes F(Y) \to F(Y \otimes X)$ and a unitary $j : 1 \to F(1)$ such that $\nu_{X,Z \otimes Y} \circ (\text{id}_F(X) \otimes \nu_{Y,Z}) = \nu_{Y \otimes X,Z} \circ (\nu_{X,Y} \otimes \text{id}_F(Z))$ and $\nu_{1,X} \circ (j \otimes \text{id}_F(X)) = \text{id}_{F(X)} = \nu_{X,1} \circ (\text{id}_F(X) \otimes j)$.

Bi-involutive tensor categories are dagger tensor categories equipped with a second involution, denoted $X \mapsto \overline{X}$, which is a dagger anti-tensor functor:

**Definition 2.3.** A bi-involutive tensor category is a dagger tensor category $T$ equipped with a covariant anti-linear dagger anti-tensor functor

$$\tau : T \to T$$

called the conjugate. The structure data of this anti-tensor functor are denoted

$$\nu_{X,Y} : \overline{X} \otimes Y \xrightarrow{\sim} Y \otimes \overline{X} \quad \text{and} \quad j : 1 \to \overline{1}.$$  

This functor is involutive, meaning that for every $X \in T$, we are given unitary natural isomorphisms $\varphi_X : X \to \overline{X}$ satisfying $\varphi_X = \overline{\varphi_X}$. Finally, we require the compatibility conditions $\varphi_1 = \overline{j \circ j}$ and $\varphi_{X \otimes Y} = \overline{\nu_{Y,X}} \circ \nu_{X,Y} \circ (\varphi_X \otimes \varphi_Y)$.

**Definition 2.4.** A dagger tensor functor $F$ between bi-involutive tensor categories is called a bi-involutive functor if it comes equipped with a unitary natural transformation

$$\gamma_X : F(X) \to \overline{F(X)}$$

satisfying $\gamma_X = \overline{\nu_{X^{-1}}} \circ \varphi_{F(X)} \circ F(\varphi_X)^{-1}$, $\gamma_1 = \overline{i \circ j \circ i^{-1} \circ F(j)^{-1}}$, and $\gamma_{X \otimes Y} = \overline{\mu_{X,Y}} \circ \nu_{F(Y),F(X)} \circ (\gamma_Y \otimes \gamma_X) \circ \overline{\nu_{X,Y}} \circ F(\nu_{X,Y})^{-1}$.

The prototypical example of a bi-involutive category is the category $\text{Bim}(R)$ of all bimodules over a von Neumann algebra $R$.

Given a bi-involutive functor $T \to B$ between bi-involutive tensor categories, we let $Z_B(T)$ be the full subcategory of the category described in Definition 2.1 where the half-braiding (10) are unitary. This category has the advantage of being, once again, a bi-involutive tensor category. The dagger structure is inherited from $B$, and the conjugate of an object $(X, e) \in Z_B(T)$ is given by $(\overline{X}, e')$, with

$$e' : X \otimes Y \xrightarrow{\text{id} \otimes \varphi_Y} X \otimes \overline{Y} \xrightarrow{\nu_{X,Y}} \overline{Y} \otimes X \xrightarrow{\overline{\varphi}_X^{-1}} X \otimes \overline{Y} \xrightarrow{\overline{\nu}_{Y,X}^{-1}} \overline{Y} \otimes X \xrightarrow{\varphi^{-1}_Y \circ \text{id}} Y \otimes X.$$  

The Drinfel’d center $Z(T)$ of a bi-involutive tensor category $T$ is the commutant of $T$ inside itself.

**Remark 2.5.** The categories $Z_B(T)$ and $\hat{Z}_B(T)$ need not, in general, be equivalent. However, in the cases studied in this paper, they will turn out equivalent (see Remark 4.23).
2.3 Bim(\(R\))

Let \( R \) be a hyperfinite factor, and let Bim(\(R\)) be the category of all \(R\)-\(R\)-bimodules whose underlying Hilbert spaces is separable. It is a dagger category by means of the operation that sends a bimodule map to its adjoint.

Let \( L^2(R) \) be the non-commutative \(L^2\)-space of \(R\) [Haa75, Yam92] (for any faithful state \( \phi : R \to \mathbb{C} \), there is a canonical identification between \(L^2R\) and the GNS Hilbert space associated to \( \phi \) [Tak03, Def IX.1.18]). By Tomita-Takesaki theory, this Hilbert space is equipped with two actions of \(R\) that are each other’s commutants, and an antilinear involution \( J \) that satisfies \( J(x^*y) = y^*J(x)x^* \).

The tensor structure

\[ \boxtimes_R : \text{Bim}(R) \times \text{Bim}(R) \to \text{Bim}(R). \] (12)

on Bim(\(R\)) is known as Connes fusion, or relative tensor product. The bimodule \( L^2(R) \) is the unit object for that operation.

Given two bimodules \( H \) and \( K \), their fusion \( H \boxtimes_R K \) is the completion of \( \text{Hom}_{\text{Mod-R}}(L^2R, H) \otimes_R K \) with respect to the inner product \( \langle \varphi \otimes \xi, \psi \otimes \eta \rangle := \langle \lambda^{-1}(\psi^* \circ \varphi)(\xi), \eta \rangle \), where \( \lambda : R \to \text{End}_{\text{Mod-R}}(L^2R) \) denotes the left action of \(R\) on its \(L^2\)-space. The left action of \(R\) on \(H\) and the right action of \(R\) on \(K\) equip \(H \boxtimes_R K\), once again, with the structure of a bimodule. The Connes fusion can be equivalently described as a completion of \( H \otimes_R \text{Hom}_{R-\text{Mod}}(L^2R, K) \), or \( \text{Hom}_{\text{Mod-R}}(L^2R, H) \otimes_R L^2R \otimes_R \text{Hom}_{R-\text{Mod}}(L^2R, K) \).

Basic references include [BDH14] [Con94, V.B.δ] [Sau83] [Tho11].

Remark 2.6. Using the last description of the fusion, and the fact that \( L^2(R) \cong L^2(R^{op}) \), we see that there is a canonical isomorphism \( H \boxtimes_R K \cong K \boxtimes_{R^{op}} H \).

Given \( H \in \text{Bim}(R) \), its complex conjugate \( \overline{H} \) is a bimodule by means of the actions \( a\overline{\xi}b := b^*\overline{\xi}a^* \). We call it the conjugate bimodule. This operation comes with canonical isomorphisms

\[ \nu : \overline{H} \boxtimes_R K \to K \boxtimes_{R^{op}} \overline{H} \quad \text{and} \quad j : L^2(R) \to L^2(R) \]

reviewed in [HP17, §2.4].

All together, these operations endow Bim(\(R\)) with the structure of a bi-involutive tensor category. By definition, a bicommutant category is a bi-involutive tensor category \(T\) for which there exists a hyperfinite factor \(R\) and a bi-involutive functor \( T \to \text{Bim}(R) \) such that the inclusion functor \( T \to T'' = Z_{\text{Bim}(R)}(Z_{\text{Bim}(R)}(T)) \) is an equivalence of categories.

3 Conformal nets

In this section, we recall the definition of conformal net from [BDH15], along with the notion of representation of a conformal net, the fusion product

\[ \boxtimes : \text{Rep}(A) \times \text{Rep}(A) \to \text{Rep}(A), \]

and the braiding of representations \( \beta_{H,K} : H \boxtimes K \to K \boxtimes H \).
3.1 Coordinate free conformal nets

Let us define an interval to be an oriented manifold diffeomorphic to \([0, 1]\). We write \(\text{Diff}_+(I)\) for the group of orientation preserving diffeomorphisms of an interval \(I\). Let \(\text{INT}\) be the category whose objects are intervals and whose morphisms are embeddings, not necessarily orientation preserving, and let \(\mathcal{VN}\) be the category whose objects are von Neumann algebras and whose morphisms are normal maps which are either \(*\)-algebra homomorphisms or \(*\)-algebra anti-homomorphisms.

**Definition 3.1 ([BDH15, Def. 1.1])**. A conformal net is a covariant functor \(\mathcal{A}: \text{INT} \to \mathcal{VN}\) from the category of oriented intervals and embeddings to the category of von Neumann algebras. It sends orientation-preserving embeddings to injective homomorphisms and orientation-reversing embeddings to injective antihomomorphisms. Moreover, for any intervals \(I\) and \(J\), the natural map \(\text{Hom}_{\text{INT}}(I,J) \to \text{Hom}_{\mathcal{VN}}(\mathcal{A}(I),\mathcal{A}(J))\) should be continuous for the \(C^\infty\) topology on \(\text{Hom}_{\text{INT}}(I,J)\) and Haagerup’s \(u\)-topology\(^{10}\) on \(\text{Hom}_{\mathcal{VN}}(\mathcal{A}(I),\mathcal{A}(J))\). In addition to that, a conformal net should satisfy the following five axioms:

1. **Locality**: If \(I, J \subset K\) have disjoint interiors, then \(\mathcal{A}(I)\) and \(\mathcal{A}(J)\) are commuting subalgebras of \(\mathcal{A}(K)\).

2. **Strong additivity**: If \(K = I \cup J\), then \(\mathcal{A}(K)\) is generated as a von Neumann algebra by its two subalgebras: \(\mathcal{A}(K) = \mathcal{A}(I) \vee \mathcal{A}(J)\).

3. **Split property**: If \(I, J \subset K\) are disjoint, then the natural map from the algebraic tensor product \(\mathcal{A}(I) \otimes_{\text{alg}} \mathcal{A}(J) \to \mathcal{A}(K)\) extends to a map from the spatial tensor product \(\mathcal{A}(I) \bar{\otimes} \mathcal{A}(J) \to \mathcal{A}(K)\).

4. **Inner covariance**: If \(\varphi \in \text{Diff}_+(I)\) restricts to the identity in a neighbourhood of the boundary of \(I\), then \(\mathcal{A}(\varphi) : \mathcal{A}(I) \to \mathcal{A}(I)\) is an inner automorphism.

5. **Vacuum sector**: Let \(J \subseteq I\) contain the boundary point \(p \in \partial I\), and let \(\tilde{J}\) denote \(J\) with reversed orientation; \(\mathcal{A}(J)\) acts on \(L^2(\mathcal{A}(I))\) via the left action of \(\mathcal{A}(I)\), and \(\mathcal{A}(\tilde{J}) \cong \mathcal{A}(J)^{\text{opp}}\) acts on \(L^2(\mathcal{A}(I))\) via the right action of \(\mathcal{A}(I)\). In that case, we require that the action of \(\mathcal{A}(J) \otimes_{\text{alg}} \mathcal{A}(\tilde{J})\) on \(L^2(\mathcal{A}(I))\) extends to an action of \(\mathcal{A}(J \cup_p \tilde{J})\)^{11}:

\[
\begin{array}{ccc}
\mathcal{A}(J) \otimes_{\text{alg}} \mathcal{A}(\tilde{J}) & \longrightarrow & \mathcal{B}(L^2(\mathcal{A}(I))) \\
\mathcal{A}(J \cup_p \tilde{J}) & \downarrow & \\
\end{array}
\]

A conformal net \(\mathcal{A}\) is called **irreducible** if the algebras \(\mathcal{A}(I)\) are factors. We will always assume that our conformal nets are irreducible.

**Remark 3.2.** For any interval \(I\), the identity map \(\bar{I} \to I\) (which is orientation reversing) induces an isomorphism \(\mathcal{A}(\bar{I}) \cong \mathcal{A}(I)^{\text{opp}}\). This was used above, in the formulation of the vacuum sector axiom.

---

\(^{10}\)Topology of pointwise convergence on the preduals.

\(^{11}\)Here, \(J \cup_p \tilde{J}\) is equipped with any smooth structure extending the given smooth structures on \(J\) and \(\tilde{J}\), and for which the orientation-reversing involution that exchanges \(J\) and \(\tilde{J}\) is smooth.
Let $S^1 := \{ z \in \mathbb{C} : |z| = 1 \}$. A representation of $A$ consists of a Hilbert space $H$ (always assumed separable) and a collection of homomorphisms $\rho_I : A(I) \to B(H)$ for every interval $I \subset S^1$, subject to the compatibility condition $\rho_I|A(J) = \rho_J$ whenever $J \subset I$.

The vacuum representation, or vacuum sector, is a representation of $A$ on the Hilbert space $H_0 = H_0^A := L^2(A(\bigcup))$.

The algebra $A(\bigcup)$ acts via the usual left multiplication on its $L^2$-space. The algebra $A(b) := A(\bigcup) \to A(\bigcup)^{op}$, where $b(z) = \bar{z}$, followed by right multiplication of $A(\bigcup)$ on its $L^2$-space. The vacuum sector axiom then ensures that those two actions uniquely extend to actions of $A(I)$ for every $I \subset S^1$ [BDH15, §1.1b].

Given an interval $I \subset S^1$, let $I' := S^1 \setminus \hat{I}$ denote the complement of the interior of $I$. The representation $H_0$ satisfies the important property of Haag duality [BDH15, Prop 1.18]:

$$\rho_I(A(I')) = \rho_I(A(I))'.$$

We note that Definition 3.1 is rigged in such a way so as to have Haag duality essentially built into it. Using the classical definition of conformal nets, Haag duality is an important theorem [GF93, §II.2][Lon08, Thm 6.2.3][BSM90].

Recall that $PSU(1,1)$ is the group of Möbius transformations, acting on $S^1$ by $(a, b) : z \mapsto \frac{az + b}{\bar{b}z + \bar{a}}$ for $|a|^2 - |b|^2 = 1$. Then the vacuum sector admits a continuous representation

$$PSU(1,1) \to U(H_0)$$

$$\varphi \mapsto u_{\varphi}$$

which satisfies the covariance property $A(\varphi)(a) = u_{\varphi}au_{\varphi}^*$ for every $I \subset S^1$ and $a \in A(I)$ [BDH15, Thm 2.13]. Let $r_t = \left( \begin{smallmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{smallmatrix} \right) \in PSU(1,1)$ denote rotation by $t$, and let $R_t = u_{r_t}$ be its image under the above homomorphism. The unbounded self-adjoint operator $L_0 := -i \frac{d}{dt} \bigg|_{t=0} R_t$ is called the energy operator; it generates the subgroup of rotations in the sense that $R_t = e^{itL_0}$. We call a conformal net chiral if the energy operator $L_0$ has positive spectrum and the $PSU(1,1)$-invariant subspace $H_0$ is one dimensional (equivalently, the subspace invariant under all the $R_t$’s is one dimensional).

Remark 3.3. Our definition of conformal net (Definition 3.1) is different from the one usually encountered in the literature. If a conformal net in the sense of [GF93, Lon08] satisfies the additional assumptions of strong additivity and diffeomorphism covariance\textsuperscript{12}, then it induces a conformal net in the sense of Definition 3.1 [BDH15, Prop 4.9]. Conversely, a conformal net (in the sense of Definition 3.1) which is chiral induces a conformal net in the classical sense by restricting it to the circle. This establishes a bijective correspondence between chiral conformal nets in the sense described above, and conformal nets in the sense of [GF93, Lon08] subject to the additional assumptions of strong additivity and diffeomorphism covariance.

\textsuperscript{12}The split property was recently shown to be a consequence of diffeomorphism covariance [MTW16].
3.2 Fusion of representations

The standard monoidal structure

\[ \boxtimes : \text{Rep}(\mathcal{A}) \times \text{Rep}(\mathcal{A}) \to \text{Rep}(\mathcal{A}) \]  \tag{15}  

on the category of representations of a conformal net is called *fusion*. There are two main approaches for defining that operation [GF93, Sec. IV.2] and [Was98, Sec. 30] (see [Con94, Prop V.B.δ.17] for the equivalence between the two). We will follow the latter.

Let \( I_0 \) be a fixed `standard interval` (in Section 1.2, this was taken to be the lower half of \( S^1 \), but \( I_0 = [0,1] \) is also a pleasant choice), and let \( R := \mathcal{A}(I_0) \) be the value of our conformal net on that standard interval. A representation \( H \in \text{Rep}(\mathcal{A}) \) has commuting left actions of the algebras

\[ \mathcal{A}(\leftarrow) \cong R \quad \text{and} \quad \mathcal{A}(\rightarrow) \cong R^{\text{op}}, \]  \tag{16} 

so we get commuting left and right actions of \( R \), making \( H \) into an \( R-R \)-bimodule. The above construction gives a functor \( \text{Rep}(\mathcal{A}) \to \text{Bim}(R) \). We will see later, in Section 4.1, that this functor is fully faithful (Lemma 4.1), and that its image is closed under the operation (12) of Connes fusion (Lemma 4.4). This will allow us to define the fusion product on \( \text{Rep}(\mathcal{A}) \) as the restriction of the corresponding operation on \( \text{Bim}(R) \).

Remark 3.4. In our situation of interest, there is an alternative way of defining Connes fusion that avoids \( L^2 \)-spaces and Tomita-Takesaki theory, and avoids the parametrisations (16). It is defined directly as an operation on the category of \( \mathcal{A}(\leftarrow)\mathcal{A}(\rightarrow)^{\text{op}} \)-bimodules, equivalently, \( \mathcal{A}(\leftarrow)\mathcal{A}(\rightarrow) \)-bimodules. All that we use is the fact that there is a distinguished bimodule \( H_0 \) with the property that the actions of \( A := \mathcal{A}(\leftarrow) \) and of \( B := \mathcal{A}(\rightarrow) \) are each other’s commutants (Haag duality). The fusion of \( H \) and \( K \) is the completion of

\[ \text{Hom}_B(H_0,H) \otimes_A K \]

under the inner product \( \langle \varphi \otimes \xi, \psi \otimes \eta \rangle := \langle (\psi^* \varphi) \xi, \eta \rangle \), where \( \psi^* \varphi \in \text{Hom}_B(H_0, H_0) \) is identified with an element of \( A \). Equivalently, the fusion is the completion of \( H \otimes_B \text{Hom}_A(H_0, K) \), or the completion of \( \text{Hom}_B(H_0, H) \otimes_A H_0 \otimes_B \text{Hom}_A(H_0, K) \).

There is also a `coordinate free’ version of the operation of fusion, that goes as follows. Recall that a representation of a conformal net is a Hilbert space equipped with compatible actions of the algebras \( \mathcal{A}(I) \) for all the subintervals of the standard circle. More generally, for any circle \( S \) (a *circle* is an oriented 1-manifold diffeomorphic to \( S^1 \)) there is a notion of \( S \)-sector of \( \mathcal{A} \) that generalises that of a representation [BDH15, Def 1.7]:

Definition 3.5. Let \( \mathcal{A} \) be a conformal net. An \( S \)-sector of \( \mathcal{A} \) is a Hilbert space \( H \) and a collection of homomorphisms

\[ \rho_I : \mathcal{A}(I) \to \text{B}(H), \quad I \subset S \]
subject to the compatibility condition $\rho_I|_{\mathcal{A}(J)} = \rho_J$ whenever $J \subset I$. We write $\text{Sect}_S(\mathcal{A})$ for the category of $S$-sectors of $\mathcal{A}$.

The category $\text{Sect}_S(\mathcal{A})$ contains a distinguished object $H_0(S, \mathcal{A})$, well defined up to non-canonical isomorphism, called the vacuum sector.\footnote{Since $H_0(S, \mathcal{A})$ is only well defined up to non-canonical isomorphism, it would be more correct to say a vacuum sector as opposed to the vacuum sector.} By definition [BDH15, Def 1.17], for every interval $I \subset S$ and every orientation reversing involution $j : S \to S$ that fixes $\partial I$, the vacuum sector $H_0(S, \mathcal{A})$ is isomorphic to $L^2(\mathcal{A}(I))$ via an isomorphism

$$v : H_0(S, \mathcal{A}) \to L^2(\mathcal{A}(I)),$$

well defined up to phase, that intertwines the two left actions of $\mathcal{A}(I)$ and satisfies $v(\mathcal{A}(j)(x)\xi) = (v(\xi))x$ for all $x \in \mathcal{A}(I)$ and $\xi \in L^2(\mathcal{A}(I))$.

Let $\Theta$ be any theta-graph, and let $S_1, S_2, S_3$ be its three circle subgraphs, oriented as follows:

\[
\Theta : \begin{array}{c}
\bullet \\
\circ \\
\bullet
\end{array} \quad S_1 : \begin{array}{c}
\bullet \\
\circ \\
\bullet
\end{array} \quad S_2 : \begin{array}{c}
\bullet \\
\circ \\
\bullet
\end{array} \quad S_3 : \begin{array}{c}
\bullet \\
\circ \\
\bullet
\end{array}
\]

The circles $S_1, S_2, S_3$ are equipped with smooth structures which are compatible in the following sense: around each of the trivalent vertices of $\Theta$, it is possible to pick a neighbourhood $Y \subset \Theta$, $(Y \cong \bigstar)$ and local coordinates $\{f_i : [0, \varepsilon[ \to Y\}_{i=1,2,3}$ of the three legs of $Y$ so that for each pair $i, j \in \{1,2,3\}$ of distinct indices, the map $(-f_i) \cup f_j : ]-\varepsilon, \varepsilon[ \to Y$ is smooth when viewed as a map to the relevant circle. Let $I := S_1 \cap S_2$ be the central interval, equipped with the orientation inherited from $I_2$. Then Connes fusion along $\mathcal{A}(I)$ defines a functor [BDH15, Def 1.31]:

$$\boxtimes_{\mathcal{A}(I)} : \text{Sect}_{S_1}(\mathcal{A}) \times \text{Sect}_{S_2}(\mathcal{A}) \to \text{Sect}_{S_3}(\mathcal{A}).$$

We denote the fusion of representations graphically by:

$$H \boxtimes K = H \boxtimes_{\mathcal{A}(\leftarrow)} K = \begin{array}{c}
\bigcirc \\
K \\
H
\end{array}$$

\subsection{The braiding}

In the literature on algebraic quantum field theory, the braiding on $\text{Rep}(\mathcal{A})$ is usually defined as follows [FRS89, §2][Lon89, §7][GF93, Def 4.16]. First of all, the objects of $\text{Rep}(\mathcal{A})$ are represented by localised endomorphisms of the $*$-algebra

$$\mathfrak{A} := \colim_{I \subset \mathbb{R}} \mathcal{A}(I).$$

Here, a $*$-algebra endomorphism $\rho : \mathfrak{A} \to \mathfrak{A}$ is said to be localised in a bounded region $O \subset \mathbb{R}$ if it acts as the identity on the subalgebras $\mathcal{A}(I) \subset \mathfrak{A}$ for every $I$ disjoint from $O$.\footnote{13Since $H_0(S, \mathcal{A})$ is only well defined up to non-canonical isomorphism, it would be more correct to say a vacuum sector as opposed to the vacuum sector.}
(We refer the reader to [GF93, §IV] for an explanation of the bijective correspondence between non-zero representations of $\mathcal{A}$ and localised endomorphisms of $\mathfrak{A}$.)

Given localised endomorphisms $\rho_1$ and $\rho_2$, one picks unitaries $U_1, U_2 \in \mathfrak{A}$ such that $\hat{\rho}_1 := \text{Ad}(U_1)\rho_1$ is localised in a region which is to the right of the region where $\hat{\rho}_2 := \text{Ad}(U_2)\rho_2$ is localised. The element

$$\varepsilon := \rho_2(U_1)^{-1}U_2^{-1}\rho_1(U_2)$$

is then an intertwiner from $\rho_1\rho_2$ to $\rho_2\rho_1$ (it satisfies $\varepsilon\rho_1\rho_2(x) = \rho_2\rho_1(x)\varepsilon$), which is called the braiding of $\rho_1$ and $\rho_2$. It is independent of the choice of unitaries $U_1$ and $U_2$, provided $\text{Ad}(U_1)\rho_1$ is localised to the right of the localisation region of $\text{Ad}(U_2)\rho_2$.

Here, the intertwining property is best explained by noting that $\varepsilon$ is a composite of intertwiners

$$\rho_1\rho_2 \xrightarrow{\rho_1(U_1)} \rho_1\hat{\rho}_2 \xrightarrow{U_1} \hat{\rho}_1\hat{\rho}_2 = \hat{\rho}_2\hat{\rho}_1 \xrightarrow{U_2^{-1}} \rho_2\hat{\rho}_1 \xrightarrow{\rho_2(U_1)^{-1}} \rho_2\rho_1.$$

We now adapt the above definition\(^{14}\) of the braiding to the case of the fusion product (18). Given a representation $H$ and an element $x \in \mathcal{A}(I)$ for some interval $I \subset S^1$, we write $\rho^H(x)$ for the action of $x$ on $H$. Let $I_n := \{e^{2\pi i \theta} : \theta \in \left[\frac{n-1}{4}, \frac{n}{4}\right]\}$ for $n = 1, 2, 3, 4$:

Let us adopt the notations $I_{123} := I_1 \cup I_2 \cup I_3$, $I_{234} := I_2 \cup I_3 \cup I_4$, $I_{34} = I_3 \cup I_4$, etc. By definition, the fusion of representations is given by $H \boxtimes K = H \boxtimes_{\mathcal{A}(I_{34})} K$.

Let $\varphi_1 : I_{341} \rightarrow I_{341}$ be a diffeomorphism that sends $1$ to $-i$ and is the identity near the boundary of that interval. Given two representations $H, K \in \text{Rep}(\mathcal{A})$, we define

$$H \boxtimes_{\mathcal{A}(I_{34})} (\varphi_1 K) := H \boxtimes_{\mathcal{A}(I_{34})} (\varphi_1 K),$$

where $\varphi_1 K$ is the Hilbert space $K$ with action of $\mathcal{A}(I_{34})$ twisted by $\mathcal{A}(\varphi_1) : \mathcal{A}(I_{34}) \rightarrow \mathcal{A}(I_{34})$. We equip $H \boxtimes_{\varphi_1} K$ with the following actions of $\mathcal{A}(I_{412})$ and of $\mathcal{A}(I_3)$. The algebra $\mathcal{A}(I_{412})$ acts on $H \boxtimes_{\varphi_1} K$ by means of its usual action on $K$. The algebra $\mathcal{A}(I_3)$ acts on $H \boxtimes_{\varphi_1} K$ by first applying $\mathcal{A}(\varphi_1)^{-1} : \mathcal{A}(I_3) \rightarrow \mathcal{A}(I_{341})$ and then using the action of $\mathcal{A}(I_{34})$ on $H$. We will see later that these actions extend, by strong additivity, to the structure of a representation on $H \boxtimes_{\varphi_1} K$.

Pick a unitary $u_1 \in \mathcal{A}(I_{341})$ such that $\text{Ad}(u_1) = \mathcal{A}(\varphi_1)$ (Definition 3.1.iv).

**Lemma 3.6.** The isomorphism

$$U_1 = U_1^{(H,K)} := (\text{id}_{H \boxtimes K} \rho^K(u_1)) \circ \rho^{H \boxtimes K}(u_1)^{-1} : H \boxtimes K \rightarrow H \boxtimes_{\varphi_1} K$$

intertwines the actions of $\mathcal{A}(I_3)$ and of $\mathcal{A}(I_{412})$.

\(^{14}\)Note that we also have $\varepsilon = U_2^{-1}\hat{\rho}_2(U_1)^{-1}\hat{\rho}_1(U_2)U_1$. It is that second formula which most closely resembles our working definition (24) of the braiding.
Proof. We write \( \varphi, u, \) and \( U \) in place of \( \varphi_1, u_1, \) and \( U_1. \) For \( x \in A(I_3), \) we have:

\[
U \circ \rho^{H \boxtimes K}(x) = (\text{id}_H \boxtimes \rho^K(u)) \circ \rho^{H \boxtimes K}(u^{-1}x)
\]

\[
= (\text{id}_H \boxtimes \rho^K(u)) \circ \rho^{H \boxtimes K}(A(\varphi)^{-1}(x)u^{-1})
\]

\[
= (\text{id}_H \boxtimes \rho^K(u)) \circ \rho^{H \boxtimes K}(A(\varphi)^{-1}(x)) \circ \rho^{H \boxtimes K}(u^{-1})
\]

\[
= (\text{id}_H \boxtimes \rho^K(u)) \circ (\rho^H(\varphi)^{-1}(x) \boxtimes \text{id}_K) \circ (\text{id}_H \boxtimes \rho^K(u)) \circ \rho^{H \boxtimes K}(u^{-1})
\]

\[
= \rho^{H \boxtimes \varphi}(K)(x) \circ U.
\]

For \( x \in A(I_{412}), \) we have:

\[
U \circ \rho^{H \boxtimes K}(x) = (\text{id}_H \boxtimes \rho^K(u)) \circ \rho^{H \boxtimes K}(u^{-1}x)
\]

\[
= (\text{id}_H \boxtimes \rho^K(u)) \circ \rho^{H \boxtimes K}(A(\varphi)^{-1}(x)u^{-1})
\]

\[
= (\text{id}_H \boxtimes \rho^K(u)) \circ \rho^{H \boxtimes K}(A(\varphi)^{-1}(x)) \circ \rho^{H \boxtimes K}(u^{-1})
\]

\[
= (\text{id}_H \boxtimes \rho^K(u)) \circ (\rho^H(\varphi)^{-1}(x)) \circ \rho^{H \boxtimes K}(u^{-1})
\]

\[
= (\text{id}_H \boxtimes \rho^K(u)) \circ \rho^{H \boxtimes K}(u^{-1})
\]

\[
= \rho^{H \boxtimes \varphi}(K)(x) \circ U.
\]

\[
\square
\]

Corollary 3.7. The actions of \( A(I_3) \) and \( A(I_{412}) \) on \( H \boxtimes \varphi K \) endow it with the structure of an object of \( \text{Rep}(\mathcal{A}). \) The map (21) is an isomorphism of representations.

Let us now consider a diffeomorphism \( \varphi_2 : I_{234} \to I_{234} \) that sends \(-1\) to \(-i\) and is the identity near the boundary, and let \( u_2 \in A(I_{341}) \) be such that \( \text{Ad}(u_2) = A(\varphi_2). \) We can then define \( H \boxtimes \varphi_2 K \) analogously to (20), and we have an isomorphism of representations

\[
U_2 = U_2^{(H,K)} := (\text{id}_H \boxtimes \rho^K(u_2)) \circ \rho^{H \boxtimes K}(u_2)^{-1} : H \boxtimes K \to H \boxtimes \varphi_2 K.
\]

We are now ready to translate the classical definition of the braiding into the language of Connes fusion:

Definition 3.8. Given two representations \( H \) and \( K, \) the braiding isomorphism

\[
\beta_{H,K} : H \boxtimes K \to K \boxtimes H
\]

is the composite

\[
\beta_{H,K} : H \boxtimes K \cong H \boxtimes K \boxtimes H_0
\]

\[
\xrightarrow{U_1 \boxtimes \text{id}} H \boxtimes \varphi_1 K \boxtimes H_0
\]

\[
\xrightarrow{\text{id} \boxtimes U_2} H \boxtimes \varphi_1 \left( K \boxtimes \varphi_2 H_0 \right) \cong K \boxtimes \varphi_2 \left( H \boxtimes \varphi_1 H_0 \right)
\]

\[
\xrightarrow{\text{id} \boxtimes U_1^*} \left( K \boxtimes \varphi_2 \right) (H \boxtimes H_0)
\]

\[
\xrightarrow{U_2^* \boxtimes \text{id}} K \boxtimes \left( H \boxtimes H_0 \right) \cong K \boxtimes H,
\]

where \( H_0 \) denotes the vacuum sector of the conformal net. The middle isomorphism is explained below.
Pictorially, we like to represent the isomorphisms (24) as the following sequence of moves:

\[
\beta_{H,K} : \begin{array}{c}
\begin{array}{c}
K \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H_0 \\
\downarrow \quad K \\
\end{array}
\begin{array}{c}
H \\
\downarrow \quad K \\
\end{array}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H \\
\downarrow \quad K \\
\end{array}
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
K \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H \\
\downarrow \quad K \\
\end{array}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H \\
\downarrow \quad K \\
\end{array}
\end{array}
\begin{array}{c}
\cong \\
\cong \\
\cong \\
\cong \\
\cong \\
\cong \\
\end{array}
\begin{array}{c}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
K \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H \\
\downarrow \quad K \\
\end{array}
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
K \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H_0 \\
\downarrow \quad H \\
\end{array}
\begin{array}{c}
H \\
\downarrow \quad K \\
\end{array}
\end{array}
\begin{array}{c}
\cong \\
\cong \\
\cong \\
\cong \\
\cong \\
\cong \\
\end{array}
\end{array}
\end{array}
\]

The isomorphism \( H \boxtimes \varphi_1 (K \boxtimes \varphi_2 H_0) \cong K \boxtimes \varphi_2 (H \boxtimes \varphi_1 H_0) \) which occurs in (24) requires some explanation. Recall that for a right module \( X \) and a left module \( Y \) there is symmetry isomorphism \( s : X_R Y \cong Y_R \text{op} X \) (Remark 2.6). The isomorphism in the middle of (24) is the composite

\[
\begin{align*}
H \boxtimes \varphi_1 (K \boxtimes \varphi_2 H_0) & \xrightarrow{a} \varphi_1 (K \boxtimes \varphi_2 H_0) \boxtimes_R H \\
& \xrightarrow{\text{id}_{\boxtimes} \circ \varphi_2} K \boxtimes \varphi_2 (H \boxtimes \varphi_1 H_0),
\end{align*}
\]

where the arrow labelled \( a \) is the associator of Connes fusion.

**Remark 3.9.** At this point, it is not clear whether the braiding (24) depends on the choice of diffeomorphisms \( \varphi_1 \) and \( \varphi_2 \), or whether it depends on our convention to add the vacuum sector on the top as opposed to the bottom. In Section 4.4, we will show that it is independent of all these choices, by using the fact that it extends to the case when \( H \) and \( K \) are solitons (\( H \in T_A^-, K \in T_A^+ \), see Corollary 4.22).

**Lemma 3.10.** Let \( H, K, \) and \( L \) be representations. Then the following diagram is commutative:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
H \boxtimes (K \boxtimes L) \\
\downarrow \quad a
\end{array}
\begin{array}{c}
\begin{array}{c}
(H \boxtimes K) \boxtimes L \\
\downarrow \quad U_{1}^{(H,K) \boxtimes \text{Id} L}
\end{array}
\begin{array}{c}
\begin{array}{c}
(H \boxtimes (K \boxtimes L)) \boxtimes \varphi_1 (K) \boxtimes L \\
\downarrow \quad a
\end{array}
\begin{array}{c}
\begin{array}{c}
(H \boxtimes (K \boxtimes L)) \boxtimes \varphi_1 (K) \boxtimes L.
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

A corresponding property holds for \( U_2 \).

**Proof.** The maps involved only depend on \( L \) as an \( \mathcal{A}(I_{341}) \)-module. Let \( M := \{ L \in \mathcal{A}(I_{341}) \text{-Mod} : (26) \text{ holds} \} \). That category contains the vacuum sector \( H_0 \) as, in that case, the horizontal arrows in (26) can be both identified with the map \( U_{1}^{(H,K)} : H \boxtimes K \rightarrow H \boxtimes \varphi_1 K \). The category \( M \) is closed under taking direct sums and taking direct summands. \( H_0 \) generates \( \mathcal{A}(I_{341}) \text{-Mod} \) under those operations. Therefore \( M = \mathcal{A}(I_{341}) \text{-Mod} \).

**Lemma 3.11.** Let \( H, K, \) and \( L \) be representations. Then the following diagram is commutative:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
(H \boxtimes K) \boxtimes \varphi_1 L \\
\downarrow \quad U_{1}^{(H \boxtimes K) \boxtimes \varphi_1 L}
\end{array}
\begin{array}{c}
\begin{array}{c}
(H \boxtimes K) \boxtimes \varphi_1 (K \boxtimes L) \\
\downarrow \quad a
\end{array}
\begin{array}{c}
\begin{array}{c}
(H \boxtimes K) \boxtimes \varphi_1 (K \boxtimes L).
\end{array}
\end{array}
\end{array}
\end{align*}
\]

A corresponding property holds for \( U_2 \).
Proof. By definition,
\[
(id_H \boxtimes U_1^{(K,L)}) \circ U_1^{(H,K \boxtimes L)} = \ id_H \boxtimes (id_K \boxtimes \rho^L(u_1)) \circ \rho^{K \boxtimes L}(u_1)^{-1} \circ ((id_H \boxtimes \rho^{K \boxtimes L}(u_1)) \circ \rho^{H \boxtimes K \boxtimes L}(u_1)^{-1})\]
\[
= (id_{H \boxtimes K} \boxtimes \rho^L(u_1)) \circ \rho^{H \boxtimes K \boxtimes L}(u_1)^{-1}
\]
(27)
\[
= U_1^{(H \boxtimes K,L)}.
\]

Proposition 3.12. The isomorphism (24) satisfies the ‘hexagon’ axioms

\[
\beta_{H,K;L} = (id_K \boxtimes \beta_{H,L})(\beta_{H,K} \boxtimes id_L) \text{ and } \beta_{H;L,K} = (\beta_{H,L} \boxtimes id_K)(id_H \boxtimes \beta_{K,L})
\]
(we omit the associators for brevity).

Proof. We only prove the first axiom. To keep notations short, we drop the symbol \( \boxtimes \), we write \( H_1 \) for \( H \boxtimes \phi_1 K \), we write \( H_2 \) for \( H \boxtimes \phi_2 K \), and we omit the vacuum sector \( H_0 \). The definition (24) of the braiding then becomes:

\[
\beta_{H,K} : HK \rightarrow H_1K \rightarrow H_1K_2 \rightarrow K_2H_1 \rightarrow K_2H \rightarrow KH,
\]

where the isomorphism \( H_1K_2 \rightarrow K_2H_1 \) is the map constructed in (25). Consider the following diagram, where all expressions are associated to the right unless otherwise indicated (for example, \( HKL \) stands for \( H \boxtimes (K \boxtimes (L \boxtimes H_0)) \), \( H_1K_2L_2 \) stands for \( H \boxtimes \phi_1(K \boxtimes \phi_2(L \boxtimes \phi_2 H_0)) \), and \( H_1(KL)_2 \) stands for \( H \boxtimes \phi_1((K \boxtimes L) \boxtimes \phi_2 H_0) \):

The arrows labelled \( a \) involve associators.

One reads \( \beta_{H,K} \boxtimes id_L \) along the top left (this is a consequence of Lemma 3.10 given our convention that all expressions are associated to the right), one reads \( id_K \boxtimes \beta_{H,L} \) along the top right, and one reads \( \beta_{H,K;L} \) along the bottom. In order to show that the desired equation \( \beta_{H,K;L} = (id_K \boxtimes \beta_{H,L})(\beta_{H,K} \boxtimes id_L) \) holds, it is therefore enough to argue that each individual cell in the above diagram is commutative. The commutativity of the cells marked by a little star is the content of Lemma 3.11. The other cells are easily seen to be commutative. \(\square\)
4 Solitons

Let $\mathcal{A}$ be a conformal net. Recall that a soliton is a Hilbert space (always assumed separable) equipped with compatible actions of the algebras $\mathcal{A}(I)$, for all subintervals of $S^1$ whose interior does not contain the point 1. We write $T_\mathcal{A} = T^+\mathcal{A}$ for the category of solitons. The category $T^-\mathcal{A}$ is defined similarly, using the point $-1$ instead of the point 1. We also call the elements of $T^-\mathcal{A}$ solitons, when this creates no confusion.

Our first goal is to identify the categories $T^+\mathcal{A}$ and $T^-\mathcal{A}$ with subcategories of $\text{Bim}(\mathcal{R})$.

4.1 Solitons as bimodules

Let $I_1, \ldots, I_4$ be the subintervals of $S^1$ depicted in (19), and let us adopt the same notations as in the previous section: $I_{12} = I_1 \cup I_2$, $I_{23} = I_2 \cup I_3$, etc. Let $I_0$ be the standard interval which we use to parametrize the lower and upper halves of the standard circle, as in (16). Let $\mathcal{A}$ be a conformal net, and let $\mathcal{R} := \mathcal{A}(I_0)$.

The parametrizations $I_0 \to I_{34}$ and $\bar{I}_0 \to I_{12}$ induce an equivalence of categories between the category $\text{Bim}(\mathcal{R})$ and the category whose objects are separable Hilbert spaces equipped with commuting left actions of the algebras $\mathcal{A}(I_{12})$ and $\mathcal{A}(I_{34})$. This allows us to identify $T^+\mathcal{A}$, $T^-\mathcal{A}$, and $\text{Rep}(\mathcal{A})$ with subcategories of $\text{Bim}(\mathcal{R})$:

\[
\iota^+: T^+\mathcal{A} \to \text{Bim}(\mathcal{R}) \\
\iota^-: T^-\mathcal{A} \to \text{Bim}(\mathcal{R}) \\
\iota: \text{Rep}(\mathcal{A}) \to \text{Bim}(\mathcal{R}).
\]

Lemma 4.1. The functors $\iota^+$, $\iota^-$, and $\iota$ are fully faithful.

Proof. The functors $\iota^+$, $\iota^-$, and $\iota$ are clearly faithful. We prove that they are full. Let $H, K \in T^+\mathcal{A}$ (respectively $H, K \in T^-\mathcal{A}$, respectively $H, K \in \text{Rep}(\mathcal{A})$), and let $f: H \to K$ be a morphism in $\text{Bim}(\mathcal{R})$. By definition, $f$ commutes with the actions of $\mathcal{A}(I_{12})$ and of $\mathcal{A}(I_{34})$. Let $I \subset S^1$ be an interval which does not contain 1 in its interior (respectively an interval such that $-1 \notin \bar{I}$, respectively any subinterval of $S^1$). By assumption, $f$ commutes with $\mathcal{A}(I \cap I_{12})$ and $\mathcal{A}(I \cap I_{34})$. By strong additivity (Definition 3.1.ii), these two algebras generate a dense subalgebra of $\mathcal{A}(I)$. So $f$ commutes with $\mathcal{A}(I)$. This being true for any $I$, $f$ is a morphism in $T^+\mathcal{A}$ (respectively a morphism in $T^-\mathcal{A}$, respectively a morphism in $\text{Rep}(\mathcal{A})$).

For a Hilbert space equipped with commuting left actions of $\mathcal{A}(I_{12})$ and of $\mathcal{A}(I_{34})$, consider the following properties:

(a) The actions of $\mathcal{A}(I_2)$ and $\mathcal{A}(I_3)$ extend to an action of $\mathcal{A}(I_{23})$.

(b) The actions of $\mathcal{A}(I_4)$ and $\mathcal{A}(I_1)$ extend to an action of $\mathcal{A}(I_{41})$.

By using the parametrizations (16), we may treat (a) and (b) as conditions on $\mathcal{R}$-$\mathcal{R}$-bimodules.
**Lemma 4.2.** The essential images of the functors \( \iota^+ \), \( \iota^- \), and \( \iota \) are given by:

\[
\begin{align*}
\text{Im}(\iota^+) &= \{ H \in \text{Bim}(R) \mid \text{condition (a) holds} \}, \\
\text{Im}(\iota^-) &= \{ H \in \text{Bim}(R) \mid \text{condition (b) holds} \}, \\
\text{Im}(\iota) &= \{ H \in \text{Bim}(R) \mid \text{both (a) and (b) hold} \}.
\end{align*}
\]

In order to establish this lemma, we will need the following technical result:

**Lemma 4.3.** Let \( M \) be a connected 1-manifold (either a circle or an interval), and let \( \{ I_i \subset M \}_{i \in I} \) be a collection of intervals that satisfy

\[
\bigcup_{i \in I} I_i = M \quad \text{and} \quad \bigcup_{i \in I} \hat{I}_i = \hat{M}.
\]

Let \( H \) be a Hilbert space equipped with actions \( \rho_i : \mathcal{A}(I_i) \to B(H) \) for \( i \in I \) which are compatible in the sense that:

1. \( \rho_i |_{\mathcal{A}(I_i \cap I_j)} = \rho_j |_{\mathcal{A}(I_i \cap I_j)} : \mathcal{A}(I_i \cap I_j) \to B(H) \).
2. For every \( j, k \in I \) and every intervals \( J \subset I_j \), \( K \subset I_k \) with disjoint interiors, the algebras \( \rho_j(\mathcal{A}(J)) \) and \( \rho_k(\mathcal{A}(K)) \) commute.

Then for every interval \( I \not\subset M \), the actions

\[
\rho_i |_{\mathcal{A}(I \cap I_i)} : \mathcal{A}(I \cap I_i) \to B(H)
\]

extend uniquely to an action of \( \mathcal{A}(I) \) on \( H \).

**Proof.** The case when \( M \) is a circle was proved in [BDH15, Lem. 1.9]. The case when \( M \) is an interval was proved in [Hen17a, Lem. 4]. (And the two proofs are essentially the same.) \( \square \)

**Proof of Lemma 4.2.** Clearly, every \( H \in T^+(\mathcal{A}) \) satisfies (a), every \( H \in T^-(\mathcal{A}) \) satisfies (b), and every \( H \in \text{Rep}(\mathcal{A}) \) satisfies both.

Let \( S^1_{\text{cut}} \) be the manifold described in Section 1.2. If a bimodule \( H \in \text{Bim}(R) \) satisfies condition (a), then we may apply Lemma 4.3 with \( M = S^1_{\text{cut}} \). The Hilbert space \( H \) admits actions of the algebras \( \mathcal{A}(I) \) for all \( I \not\subset S^1_{\text{cut}}, \) and is therefore a soliton. The argument for \( T^-(\mathcal{A}) \) is identical.

If a bimodule \( H \in \text{Bim}(R) \) satisfies both (a) and (b), then we can apply Lemma 4.3 with \( M = S^1 \). The Hilbert space \( H \) admits actions of the algebras \( \mathcal{A}(I) \) for all \( I \not\subset S^1 \), and is therefore a representation of \( \mathcal{A} \).

**Lemma 4.4.** The subcategories \( T^+_\mathcal{A}, T^-_{\mathcal{A}}, \) and \( \text{Rep}(\mathcal{A}) \) of \( \text{Bim}(R) \) are closed under Connes fusion.

**Proof.** In view of Lemma 4.2, it is enough to show that the properties (a) and (b) are preserved under fusion. By symmetry, it is enough to treat just one of them.

Let \( H \) and \( K \) be bimodules that satisfy property (a). Then, by [BDH15, Cor. 1.29], the actions of \( \mathcal{A}(I_2) \) on \( K \) and \( \mathcal{A}(I_3) \) on \( H \) extend to an action of \( \mathcal{A}(I_{23}) \) on \( H \boxtimes K \). That is, \( H \boxtimes K \) satisfies (a). (The intervals which were denoted by \( I, I_l, I_r, \) and \( I_l \odot I_r \) in [BDH15, Cor. 1.29] correspond to \( I_0, I_{123}, I_{234}, \) and \( I_{23}, \) respectively.) \( \square \)
4.2 The braiding between $T^-_A$ and $T^+_A$

Given two solitons $H \in T^-_A$ and $K \in T^+_A$, we can use the inclusions (28) to define their fusion $H \boxtimes K \in \text{Bim}(R)$. The goal of this section is to extend the braiding on $\text{Rep}(A)$ to a braiding

$$\beta_{H,K} : H \boxtimes K \rightarrow K \boxtimes H$$

which is defined for all $H \in T^-_A$ and $K \in T^+_A$.

Let $\varphi_1 : I_{341} \rightarrow I_{341}$ and $H \boxtimes^{\varphi_1} K := H \boxtimes_R^{\varphi_1} K$ be as in Section 3.3. Here, as before, $\varphi_1 K$ denotes the Hilbert space $K$ with action of $A(I_{34})$ twisted by $A(\varphi_1) : A(I_{34}) \rightarrow A(I_3)$. We equip $H \boxtimes^{\varphi_1} K$ with the following actions of the algebras $A(I_{12})$, $A(I_3)$, and $A(I_1)$. The algebras $A(I_{12})$ and $A(I_4)$ act on $H \boxtimes^{\varphi_1} K$ by their usual action on $K$. The algebra $A(I_3)$ acts by first applying $A(\varphi_1)^{-1} : A(I_3) \rightarrow A(I_{34})$ and then using the usual action of $A(I_{34})$ on $H$. We find it useful to represent the Hilbert spaces $H \boxtimes K$ and $H \boxtimes^{\varphi_1} K$ by the following pictures:

$$H \boxtimes K = \begin{array}{c} \text{K} \\ \text{H} \end{array} \quad H \boxtimes^{\varphi_1} K = \begin{array}{c} \text{K} \\ \text{H} \end{array}$$

Here, the little star is a reminder that $H$ and $K$ are solitons, as opposed to representations. We will see later, in Corollary 4.10, that the actions of $A(I_{34})$ and $A(I_4)$ on $H \boxtimes^{\varphi_1} K$ extend, by strong additivity, to an action of $A(I_{341})$.

Let $u_1 \in A(I_{341})$ be such that $\text{Ad}(u_1) = \varphi_1$. The unitary

$$U_1^{(H,K)} = (\text{id}_H \boxtimes \rho^K(u_1)) \circ \rho^{H \boxtimes K}(u_1)^{-1} : H \boxtimes K \rightarrow H \boxtimes^{\varphi_1} K$$

(29)

that was used in the definition (24) of the braiding no longer makes sense when $H$ and $K$ are solitons, because the actions of $A(I_{34})$ and $A(I_1)$ on $H \boxtimes K$ might not extend to an action of $A(I_{341})$. We circumvent this difficulty by a trick that is based on the following lemma:

**Lemma 4.5** ([BDH16, Lem. B.24]). Let $R$ be a factor, let $A$ be any von Neumann algebra, and let

$$F,G : R\text{-Mod} \rightarrow A\text{-Mod}$$

be completely additive functors. Let $M \subset R\text{-Mod}$ be a full subcategory with only one object, which is not the zero object.

Then a natural transformation $\tau : F \rightarrow G$ is entirely determined by its restriction to $M$. Conversely, any natural transformation $F|_M \rightarrow G|_M$ extends to a natural transformation $F \rightarrow G$.

**Proof.** By complete additivity, $\tau|_M$ determines $\tau$ on the subcategory of $R$-modules which are direct sums of the object of $M$. Every $R$-module is a direct summand of one of the above form, so $\tau$ is determined on all of $R\text{-Mod}$. (The proof is even simpler when $R$ is a type III factor as, in that case, $M$ is equivalent to the subcategory of all non-zero modules.) □
Fix $H \in T^+_A$ and consider the functors $H \boxtimes -$ and $H \boxtimes \varphi_1 -$ from $\text{Rep}(\mathcal{A})$ to the category $C$ whose objects are Hilbert spaces equipped with commuting actions of the algebras $\mathcal{A}(I_3)$ and $\mathcal{A}(I_4)$. The definition (29) of $U_1^{(H,K)} : H \boxtimes K \to H \boxtimes \varphi_1 K$ makes sense in that context, so we get a natural transformation

$$U_1^{(H,-)} : \text{Rep}(\mathcal{A}) \boxtimes C$$

from

$$K \mapsto H \boxtimes K \quad \text{to} \quad K \mapsto H \boxtimes \varphi_1 K.$$

We now observe that $H \boxtimes -$ and $H \boxtimes \varphi_1 -$ also make sense as functors from $\mathcal{A}(I_{34})\text{-Mod}$ to $C$. Let $M \subset \mathcal{A}(I_{34})\text{-Mod}$ be the full subcategory consisting of only the vacuum Hilbert space.

**Lemma 4.6.** The map $U_1^{(H,H_0)} : H \boxtimes H_0 \to H \boxtimes \varphi_1 H_0$ defined in (29) is a natural transformation $M \boxtimes C$.

**Proof.** By Haag duality, $\text{End}_M(H_0) = \text{End}_{\mathcal{A}(I_{34})\text{-Mod}}(H_0) = \mathcal{A}(I_{12})$. For every endomorphism $x \in \mathcal{A}(I_{12})$ of $H_0$, we need to show that the diagram

$$
\begin{array}{ccc}
H \boxtimes H_0 & \xrightarrow{U_1^{(H,H_0)}} & H \boxtimes \varphi_1 H_0 \\
\text{id} \boxtimes x & & \text{id} \boxtimes \varphi_1 x \\
H \boxtimes H_0 & \xrightarrow{U_1^{(H,H_0)}} & H \boxtimes \varphi_1 H_0
\end{array}
$$

commutes. That computation was performed in (22). \qed

The category $C$ is of the form $\mathcal{A}\text{-Mod}$ for some von Neumann algebra ([Gui66, §8]). By Lemma 4.5, we therefore get:

**Corollary 4.7.** There exists a unique natural transformation

$$U_1^{(H,-)} : \mathcal{A}(I_{34})\text{-Mod} \boxtimes C$$

whose value on the vacuum sector $H_0 \in \mathcal{A}(I_{34})\text{-Mod}$ is given by the map (29).

We now have two definitions of $U_1^{(H,-)}$ that we need to reconcile:

**Lemma 4.8.** Let $H \in T^+_A$ and $K \in \mathcal{A}(I_{34})\text{-Mod}$. Then the map $U_1^{(H,K)} : H \boxtimes K \to H \boxtimes \varphi_1 K$ defined by (29) agrees with the one given by Corollary 4.7.

**Proof.** Let $M \subset \mathcal{A}(I_{34})\text{-Mod}$ be the subcategory on which the two definitions of $U_1^{(H,K)}$ agree. By definition, $M$ contains the vacuum sector. Since $M$ is closed under direct sums and direct summands and the vacuum sector generates $\mathcal{A}(I_{34})\text{-Mod}$ under those operations, $M = \mathcal{A}(I_{34})\text{-Mod}$. \qed

We now restrict the natural transformation (30) along the functor $T^+_A \to \mathcal{A}(I_{34})\text{-Mod}$, to get a natural transformation $U_1^{(H,-)} : T^+_A \boxtimes C$ from $H \boxtimes -$ to $H \boxtimes \varphi_1 -$. 23
Lemma 4.9. Let $H \in T^-_A$ and $K \in T^+_A$ be solitons. Then the map
\[ U_1^{(H,K)} : H \boxtimes K \to H \boxtimes^{\varphi_1} K \tag{31} \]
defined above intertwines the actions of $A(I_{12})$, $A(I_3)$, and $A(I_4)$.

Proof. The map $(31)$ intertwines the actions of $A(I_3)$ and $A(I_4)$ because it is a morphism in $C$. Recall from (30) that $U_1^{(H,-)}$ is natural with respect to all morphisms of $A(I_{34})$-modules. So the map $U_1^{(H,K)} : H \boxtimes K \to H \boxtimes^{\varphi_1} K$ intertwines the two actions of $\text{End}_{A(I_{34})}$-$\text{Mod}(K)$. The actions of $A(I_{12})$ on the source and on the target of $(31)$ factor through the aforementioned actions of $\text{End}_{A(I_{34})}$-$\text{Mod}(K)$. The map $(31)$ therefore intertwines the actions of $A(I_{12})$. \hfill \Box

Corollary 4.10. The actions of $A(I_3)$ and of $A(I_4)$ on $H \boxtimes^{\varphi_1} K$ extend, by strong additivity, to an action of $A(I_{34})$.

Given two solitons $H \in T^-_A$ and $K \in T^+_A$, we have upgraded $H \boxtimes^{\varphi_1} K$ to an object of $\text{Bim}(R)$, and we have made sense of the unitary $U_1^{(H,K)} : H \boxtimes K \to H \boxtimes^{\varphi_1} K$. Similarly, given a diffeomorphism $\varphi_2 : I_{34} \to I_{34}$ as in Section 3.3, we can define $K \boxtimes^{\varphi_2} H$ and make sense of $U_2^{(K,H)} : K \boxtimes H \to K \boxtimes^{\varphi_2} H$.

Definition 4.11. Let $H \in T^-_A$ and $K \in T^+_A$ be solitons. The braiding isomorphism $\beta_{H,K} : H \boxtimes K \to K \boxtimes H$ is the composite
\[ \beta_{H,K} : \begin{array}{rcl}
H \boxtimes K & \cong & H \boxtimes K \boxtimes H_0 \\
\xrightarrow{U_1 \boxtimes \text{id}} & & H \boxtimes^{\varphi_1} K \boxtimes H_0 \\
\xrightarrow{\text{id} \boxtimes U_2} & & H \boxtimes^{\varphi_1} (K \boxtimes^{\varphi_2} H_0) \cong K \boxtimes^{\varphi_2} (H \boxtimes^{\varphi_1} H_0) \\
\xrightarrow{\text{id} \boxtimes U_1^{-1}} & & K \boxtimes^{\varphi_2} (H \boxtimes H_0) \\
\xrightarrow{U_2^{-1} \boxtimes \text{id}} & & K \boxtimes (H \boxtimes H_0) \cong K \boxtimes H,
\end{array} \tag{32} \]
where $H_0$ denotes the vacuum sector.

We represent this isomorphism graphically as follows:
\[ \beta_{H,K} : \begin{array}{c}
K \\
H
\end{array} \cong \begin{array}{c}
K \\
H
\end{array} \boxtimes \begin{array}{c}
H_0 \\
H
\end{array} \boxtimes \begin{array}{c}
H_0 \\
H
\end{array} \boxtimes \begin{array}{c}
H_0 \\
H
\end{array} \boxtimes \begin{array}{c}
H_0 \\
H
\end{array} \boxtimes \begin{array}{c}
H_0 \\
H
\end{array} \cong \begin{array}{c}
H \\
K
\end{array}
\]

We have the following analogs of Lemma 3.10 and Lemma 3.11:

Lemma 4.12. Let $H \in T^-_A$ and $K, L \in T^+_A$ be solitons. Then the following diagram is commutative:
\[ \begin{array}{ccc}
(H \boxtimes K) \boxtimes L & \xrightarrow{U_1^{(H,K)} \boxtimes \text{id}_L} & (H \boxtimes^{\varphi_1} K) \boxtimes L \\
\alpha & & \alpha \\
H \boxtimes (K \boxtimes L) & \xrightarrow{U_1^{(H,K,L)}} & H \boxtimes^{\varphi_1} (K \boxtimes L).
\end{array} \tag{33} \]
A similarly diagram holds for $U_2$. 24
Proof. By Corollary 4.7, the maps in the above diagram only depend on \( L \) as an \( \mathcal{A}(I_{34}) \)-module. We can therefore proceed as in Lemma 3.10. Let \( M := \{ L \in \mathcal{A}(I_{34})-\text{Mod} : (33) \text{ holds}\} \). If \( L = H_0 \), then the horizontal arrows in (33) can be both identified with \( U_1^{(H,K)} \). So \( M \) contains the vacuum sector. So \( M \) is all of \( \mathcal{A}(I_{34})-\text{Mod} \).

**Lemma 4.13.** Let \( H, K \in T_A^- \), and \( L \in T_A^+ \) be solitons. Then the following diagram is commutative:

\[
\begin{array}{ccc}
H \boxtimes K \boxtimes L & \xrightarrow{U_1^{(H,K;L)}} & H \boxtimes \varphi_1 (K \boxtimes L) \\
\downarrow{U_1^{(H;K,L)}} & & \downarrow{\text{id}_H \boxtimes U_1^{(K,L)}} \\
(H \boxtimes K) \boxtimes \varphi_1 L & \xrightarrow{\alpha} & H \boxtimes \varphi_1 (K \boxtimes \varphi_1 L),
\end{array}
\]

where the top horizontal arrow is the one constructed in (30). A similarly diagram holds for \( U_2 \).

**Proof.** Once again, the maps in (34) only depend on \( L \) as an \( \mathcal{A}(I_{34}) \)-module. Let \( M := \{ L \in \mathcal{A}(I_{34})-\text{Mod} : (34) \text{ holds}\} \). By Lemma 4.8, we may use the computation (27) to deduce that \( M \) contains the vacuum sector. As before, \( M \) is closed under taking direct sums and direct summands, so \( M \) is all of \( \mathcal{A}(I_{34})-\text{Mod} \).

Finally, we have:

**Proposition 4.14.** The braiding isomorphism (32) satisfies the two ‘hexagon’ axioms

\[
\beta_{H,K;L} = (\text{id}_K \boxtimes \beta_{H,L})(\beta_{H,K} \boxtimes \text{id}_L) \quad \text{and} \quad \beta_{H;K,L} = (\beta_{H,L} \boxtimes \text{id}_K)(\text{id}_H \boxtimes \beta_{K,L})
\]

(once again, we omit the associators for brevity).

**Proof.** The proof of Proposition 3.12 applies word for word (use Lemma 4.12 in place of Lemma 3.10, and Lemma 4.13 in place of Lemma 3.11).

We will show later, in Proposition 4.21, that there exists a unique braiding \( \beta : T_A^- \times T_A^+ \boxtimes \text{Bim}(R) \) that satisfies the hexagon axiom \( \beta_{H,K;L} = (\text{id}_K \boxtimes \beta_{H,L})(\beta_{H,K} \boxtimes \text{id}_L) \). As a consequence, the braiding (32) is independent of the various choices that we made (e.g., the choice of diffeomorphisms \( \varphi_1 \) and \( \varphi_2 \)).

### 4.3 The absorbing object

In this section, we recall the results of our earlier paper [Hen17a], according to which the category of solitons admits an absorbing object. This is the only place where the condition that \( \mathcal{A} \) has finite index is needed. We start by recalling the definition of an absorbing object:
**Definition 4.15.** An object $\Omega$ of a tensor category $(T, \otimes)$ is called absorbing if it is non-zero and satisfies

$$(X \neq 0) \Rightarrow (X \otimes \Omega \cong \Omega \cong \Omega \otimes X) \quad \forall X \in T.$$ 

If $T$ admits a conjugation (in particular, if $T$ is bi-involutive), then $\Omega \in T$ is absorbing if and only if it is non-zero and satisfies $X \otimes \Omega \cong \Omega$ for every $X \neq 0$ (see the comments after [HP17, Def. 5.3]).

Consider the following manifold (an equilateral triangle):

$$\Delta := \begin{array}{c}
\end{array}$$

equipped with the smooth structure given by constant speed parametrization. We call the upper left side of this triangle $\Delta_+$, the lower left side $\Delta_-$, and the right side $\Delta_{\text{free}}$. Let $\Omega := H_0(\Delta, \mathcal{A})$ be the vacuum sector of $\mathcal{A}$ associated to $\Delta$, let $S^1_{\text{cut}}$ be as in Section 1.2, and let

$$\varphi_\Delta : S^1_{\text{cut}} \to \Delta_- \cup \Delta_+$$

be the constant speed parametrization that sends the lower half of $S^1_{\text{cut}}$ to $\Delta_-$ and the upper half of $S^1_{\text{cut}}$ to $\Delta_+$. We use the diffeomorphism $\varphi_\Delta$ to pull back the action of $\mathcal{A}(\Delta_- \cup \Delta_+)$ on $\Omega$ to an action of $\mathcal{A}(S^1_{\text{cut}})$, and thus endow $\Omega$ with the structure of a soliton. Note that, by Haag duality,

$$\text{End}_{T_A}(\Omega) = \mathcal{A}(\Delta_{\text{free}}).$$

The following important result was proven in [Hen17a, Thm. 9]:

**Proposition 4.16.** If $\mathcal{A}$ is a conformal net with finite index, then the object $\Omega \in T_A$ is absorbing.

**Remark 4.17.** It is for the above proposition to hold that it was important to insist that all Hilbert spaces be separable. If we allow Hilbert spaces of arbitrarily large cardinalities, then the tensor category $T_A$ does not have an absorbing object.

**Remark 4.18.** In the absence of the finite index condition, we do not know whether $\Omega$ is absorbing.

Absorbing objects are important because they control half-braidings:

**Proposition 4.19** ([HP17, Prop. 5.9]). Let $T$ be a category equipped with a tensor functor to Bim($R$). Let $\Omega \in T$ be an absorbing object, and let $(X, e)$ be in $T'$. Then $e$ is completely determined by its value on $\Omega$.

**Proof.** Let $Y$ be a non-zero object of $T$. Since $e$ is a half-braiding, we have a commutative diagram

$$
\begin{array}{c}
\begin{array}{ccc}
Y \boxtimes X \boxtimes \Omega & \xrightarrow{e_Y \boxtimes \text{id}_\Omega} & Y \boxtimes \Omega \boxtimes X \\
\xrightarrow{\text{id}_Y \boxtimes e_\Omega} & & \\
X \boxtimes Y \boxtimes \Omega & \xrightarrow{e_Y \boxtimes \Omega} & Y \boxtimes \Omega \boxtimes X.
\end{array}
\end{array}
$$
Pick an isomorphism \( \phi : Y \boxtimes \Omega \to \Omega \). The following square is commutative

\[
\begin{array}{ccc}
X \boxtimes (Y \boxtimes \Omega) & \xrightarrow{\epsilon_Y \boxtimes \Omega} & (Y \boxtimes \Omega) \boxtimes X \\
\downarrow \text{id}_X \boxtimes \phi & & \downarrow \phi \boxtimes \text{id}_X \\
X \boxtimes \Omega & \xrightarrow{\epsilon_\Omega} & \Omega \boxtimes X
\end{array}
\]

and so we get an equation \( \epsilon_Y \boxtimes \text{id}_\Omega = (\text{id}_Y \boxtimes \epsilon^{-1}_\Omega) \circ (\phi^{-1} \boxtimes \text{id}_X) \circ \epsilon_\Omega \circ (\text{id}_X \boxtimes \phi) \). In particular, we see that \( \epsilon_Y \boxtimes \text{id}_\Omega \) is completely determined by \( \epsilon_\Omega \). Since \( - \boxtimes \Omega \) is a faithful functor, \( \epsilon_Y \) is completely determined by \( \epsilon_Y \boxtimes \text{id}_\Omega \). Putting these two facts together, we see that \( \epsilon_Y \) is completely determined by \( \epsilon_\Omega \).

\[\square\]

### 4.4 The Drinfel’d center

This section is devoted to the proof of our main theorem:

**Theorem 4.20.** If \( \mathcal{A} \) is a conformal net with finite index. Then the canonical map \((T_\mathcal{A}^+)’ \to \text{Bim}(R)\) is fully faithful, and \((T_\mathcal{A}^+)’ = T_\mathcal{A}^--\).

The statements in Theorem A are easy consequences of Theorem 4.20. The second bullet point in Theorem A is obtained by exchanging the roles of \( T_\mathcal{A}^+ \) and \( T_\mathcal{A}^- \), and the third bullet point is the following computation:

\[
Z(T_\mathcal{A}^+) = Z_{T_\mathcal{A}^+}(T_\mathcal{A}^+) = Z_{\text{Bim}(R)}(T_\mathcal{A}^+) \cap T_\mathcal{A}^+ = T_\mathcal{A}^- \cap T_\mathcal{A}^+ = \text{Rep}(\mathcal{A}),
\]

where we have used Lemma 4.2 for the last equality. We note that Lemma 4.1 (according to which \( T_\mathcal{A}^- \), \( T_\mathcal{A}^+ \), and \( \text{Rep}(\mathcal{A}) \) are full subcategories of \( \text{Bim}(R) \)) has been used implicitly here, in the usage of the symbol \( \cap \).

**Proposition 4.21.** An object \( H \in \text{Bim}(R) \) admits at most one half-braiding with \( T_\mathcal{A} \):

\[
e = (e_K : H \boxtimes K \to K \boxtimes H)_{K \in T_\mathcal{A}}.
\]

**Proof.** Let \( e \) and \( e' \) be half-braidings of \( H \) with \( T_\mathcal{A} \). We wish to show that \( e = e' \). By Proposition 4.16 and Proposition 4.19, it is enough to show that \( e_\Omega = e'_\Omega \). Consider the following \( R-R \)-bimodule map:

\[
e'_\Omega \circ e^{-1}_\Omega : \Omega \boxtimes_R H \to \Omega \boxtimes_R H.
\]

By the naturality of \( e \) and of \( e' \), this map is equivariant for the actions of \( \text{End}_{T_\mathcal{A}}(\Omega) = \mathcal{A}(\Delta_{\text{free}}) \). We may therefore treat (35) as a map of \( \mathcal{A}(\Delta_- \cup \Delta_{\text{free}})-R \)-bimodules.

By Haag duality, \( \Omega \) is an invertible \( \mathcal{A}(\Delta_- \cup \Delta_{\text{free}})-R \)-bimodule [BDH14, Prop. 3.10] (here, \( R \) is identified with \( \mathcal{A}(\Delta_-)^{\text{op}} \)). So there exists an invertible \( R-R \)-bimodule map \( u : H \to H \) that satisfies

\[
e'_\Omega \circ e^{-1}_\Omega = \text{id}_\Omega \boxtimes u.
\]

27
Pick an isomorphism $\omega : \Omega \boxtimes \Omega \to \Omega$ in $T_A$ and consider the following diagram:

The middle pentagon commutes because $e$ is a half-braiding. The outer pentagon commutes by the corresponding property of $e'$. All the quadrilaterals are visibly commutative. It follows that

$$
\text{id}_\Omega \boxtimes u \boxtimes \text{id}_\Omega = \text{id}_{\Omega \boxtimes H \boxtimes \Omega}.
$$

The functors $\Omega \boxtimes -$ and $- \boxtimes \Omega$ being faithful, we conclude that $u = \text{id}_H$. \hfill \Box

**Corollary 4.22.** The braiding

$$
\beta : T_A^- \times T_A^+ \boxtimes \text{Bim}(R)
$$

defined in (32) is independent of the choices of diffeomorphisms $\varphi_1$ and $\varphi_2$. The same holds true for its restriction (24) to $\text{Rep}(A)$.

**Proof of Theorem 4.20.** The half-braiding constructed in Section 4.2 provides a functor

$$
T_A^- \to (T_A^+)' \tag{36}
$$

that fits in a diagram

We need to show that the functor (36) is an equivalence of categories. It is clearly faithful as $T_A^- \to \text{Bim}(R)$ and $(T_A^+)' \to \text{Bim}(R)$ are both faithful functors. Recall from Lemma 4.1 that the functor $T_A^- \to \text{Bim}(R)$ is fully faithful. In order to check that the functor (36) is an equivalence of categories, it is therefore enough to show that:

- The functor $(T_A^+)' \to \text{Bim}(R)$ is full (and thus fully faithful).
- For every object $Y \in (T_A^+)'$, there exists an object $X \in T_A^-$ and an isomorphism between their images in $\text{Bim}(R)$.

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We start by the second item. By Lemma 4.2, it is enough to check that for every object \((H, e) \in \mathcal{T}_A^+\), the actions of \(A(I_4)\) and \(A(I_1)\) extend to an action of \(A(I_{41})\) on \(H\).

Recall that \(\Omega := H_0(\Delta, \mathcal{A})\), and recall the definition of \(\psi_\Delta : S^1_{\text{cut}} \to \Delta_- \cup \Delta_+\) from the previous section. Let \(j : \Delta \to \Delta\) be an orientation reversing involution that satisfies

\[
j(\Delta_-) = \Delta_+ \cup \Delta_{\text{free}} \quad \quad j(\Delta_+) = \varphi_\Delta(I_3) \quad \quad j(\Delta_{\text{free}}) = \varphi_\Delta(I_4),
\]

and is length-preserving in a neighbourhood of the vertex \(\Delta_- \cap \Delta_{\text{free}}\) of \(\Delta\). Recall from (17) that there is a unitary isomorphism \(v : \Omega \to L^2(\mathcal{A}(\Delta_-))\) that intertwines the actions of \(\mathcal{A}(\Delta_-)\) and satisfies

\[
v(A(j)(x) \xi) = (v(\xi))x \quad \text{for all} \quad x \in \mathcal{A}(\Delta_-) \quad \text{and} \quad \xi \in \Omega.
\]

Let us write \(b : S^1 \to S^1\) for the complex conjugation map \(z \mapsto \bar{z}\). Then the isomorphism \(f : \Omega \to L^2(\mathcal{A}(I_{34})) \cong L^2(R)\) intertwines the left actions of \(\mathcal{A}(I_{34})\), and satisfies

\[
f(A(j \circ \varphi_\Delta \circ b)(x) \cdot \xi) = x \cdot (f(\xi))
\]

for all \(x \in \mathcal{A}(I_{12})\) and \(\xi \in \Omega\).

Recall that our goal is to show that the actions of \(A(I_4)\) and \(A(I_1)\) extend to an action of \(A(I_{41})\) on \(H\). Consider the isomorphism

\[
e_\Omega : H \otimes \Omega \to \Omega \otimes H
\]

provided by the half-braiding. It is a homomorphism of \(R-R\)-bimodules. By the naturality axiom of half-braidings, it is also equivariant with respect to the actions of \(\text{End}_{\mathcal{T}_A^+}(\Omega) = \mathcal{A}(\Delta_{\text{free}})\) on \(H \otimes \Omega\) and on \(\Omega \otimes H\). We now consider the composite:

\[
F : \Omega \otimes H \xrightarrow{e_\Omega^{-1}} H \otimes \Omega \xrightarrow{\text{id} \otimes f} H \otimes L^2 R \cong H.
\]

It is equivariant with respect to the left actions of \(A(I_{34})\), and intertwines the action of \(\mathcal{A}(\Delta_{\text{free}})\) on \(\Omega \otimes H\) with the following action on \(H\):

\[
\mathcal{A}(\Delta_{\text{free}}) \xrightarrow{A(j \circ \varphi_\Delta \circ b)^{-1}} \mathcal{A}(I_1) \to B(H).
\]

We represent the isomorphism (37) graphically as follows:

\[
\begin{array}{c}
\begin{picture}(100,100)
\put(50,50){\circle{50}}
\put(50,50){\vector(0,-1){50}}
\put(50,50){\vector(-1,-1){50}}
\put(0,0){\framebox(100,100){}}
\put(0,0){\framebox(50,50){}}
\put(0,0){\framebox(50,50){}}
\put(0,0){\framebox(50,50){}}
\end{picture}
\end{array}
\]

The little stars are there to indicate that \(H\) is a priori a mere bimodule, as opposed to a soliton or a representation.

Let us write

\[
\rho_H^i : \mathcal{A}(I_i) \to B(H) \quad \text{and} \quad \rho_\Omega^i : \mathcal{A}(I_i) \to B(\Omega)
\]

29
for the actions of $A(I_i)$ on $H$ and on $\Omega$, and let us write $\alpha$ for the following action of $A(I_1)$ on $\Omega$:

$$\alpha : \mathcal{A}(I_1) \xrightarrow{A(j \circ \varphi \circ b)} \mathcal{A}(\Delta_{\text{free}}) \to B(\Omega).$$

By construction, the map (37) satisfies

$$F \circ (\rho^O_i(x) \boxtimes \text{id}_H) = \rho^H_i(x) \circ F \quad \forall x \in \mathcal{A}(I_4)$$

$$F \circ (\alpha(x) \boxtimes \text{id}_H) = \rho^H_i(x) \circ F \quad \forall x \in \mathcal{A}(I_1).$$

(38)

Since $j$ is an isometry in a neighbourhood of $\Delta_- \cap \Delta_{\text{free}}$, the maps $\varphi_{\Delta} : I_4 \to \Delta$ and $j \circ \varphi_{\Delta} \circ b : I_1 \to \Delta$ extend to a smooth map

$$\varphi_{\Delta} \cup (j \circ \varphi_{\Delta} \circ b) : I_{14} \to \Delta.$$ 

The actions $\rho^O_i : \mathcal{A}(I_4) \to B(\Omega)$ and $\alpha : \mathcal{A}(I_1) \to B(\Omega)$ therefore extend to an action of $A(I_{14})$ on $\Omega$. By the intertwining properties (38) of $F$, the actions $\rho^H_i : \mathcal{A}(I_4) \to B(H)$ and $\rho^H_1 : \mathcal{A}(I_1) \to B(H)$ therefore also extend to an action of $A(I_{14})$ on $H$. This finishes the proof of the second item in the bullet list.

We now turn our attention to the first item in the list. Let us write

$$s : T^-_A \to (T^+_A)'$$

for the functor (36). Let $(H_1, e_1)$ and $(H_2, e_2)$ be objects of $(T_A^+)'$, and let $f : H_1 \to H_2$ be a morphism between their images in $\text{Bim}(R)$. In the first half of the proof, we learned that $H_1$ and $H_2$ are in fact objects of $T^-_A$. By Lemma 4.1, $f$ is a morphism in $T^-_A$. By Proposition 4.21, $s(H_1) = (H_1, e_1)$ and $s(H_2) = (H_2, e_2)$. Therefore,

$$s(f) : (H_1, e_1) \to (H_2, e_2)$$

is a morphism in $(T^+_A)'$. Now, by construction, $s(f)$ maps to $f$ under the forgetful map $(T^+_A)' \to \text{Bim}(R)$.

\[\square\]

**Remark 4.23.** The arguments in the proofs of Proposition 4.21 and of Theorem 4.20 never used the fact that the half-braidings are unitary (it just so happens that every half-braiding with $T_A^+$ is unitary). The non-unitary version of Theorem A therefore also holds:

$$\hat{Z}_{\text{Bim}(R)}(T^+_A) \cong T^-_A, \quad \hat{Z}_{\text{Bim}(R)}(T^-_A) \cong T^+_A, \quad \hat{Z}(T^+_A) \cong \text{Rep}(A).$$

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