Closing a Loophole in the Case Against the Counterfactual Usage of the ABL Rule

R.E. Kastner*

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Department of Philosophy
University of Maryland
College Park, MD 20742 USA.

Abstract

A currently discussed interpretation of quantum theory, Time-Symmetrized Quantum Theory, makes certain claims about the properties of systems between pre- and post-selection measurements. These claims are based on a counterfactual usage of the Aharonov-Bergmann-Lebowitz (ABL) rule for calculating the probabilities of measurement outcomes between such measurements. It has been argued by several authors that the counterfactual usage of the ABL rule is, in general, incorrect. This paper examines what might appear to be a loophole in those arguments and shows that this apparent loophole cannot be used to support a counterfactual interpretation of the ABL rule. It is noted that the invalidity of the counterfactual usage of the ABL rule implies that the characterization of those outcomes receiving probability 1 in a counterfactual application of the rule as ‘elements of reality’ is, in general, unfounded.

1. Introduction.

Time-Symmetrized Quantum Theory (TSQT) is an interpretation of quantum theory in which it is argued that information propagating in a time-reversed direction from future measurements can provide ontological information about appropriately selected systems—information which is not taken into account in the standard time-asymmetric
interpretations. Ensembles of such systems are referred to as ‘pre- and post-selected ensembles’. These are defined not only by the usual pre-selection measurement at some time \( t_1 \), yielding the state \( |\psi_1(t_1)\rangle \), but also by a post-selection measurement at time \( t_2 > t_1 \), yielding the state \( |\psi_2(t_2)\rangle \). These two states then comprise a two-state or ‘generalized state’, \( \Psi = \langle \psi_2(t_2)|\psi_1(t_1) \rangle \) (Aharonov and Vaidman 1991).

TSQT advocates, most recently L. Vaidman, have made certain claims about the properties of pre- and post-selected systems based on a rule derived by Aharonov, Bergmann, and Lebowitz (1964), commonly known as ‘the ABL rule’. The ABL rule gives the probability of outcome \( c_j \) out of the set of possible outcomes \( \{c_k\} \) for a measurement of nondegenerate observable \( C \) occurring between pre- and post-selection measurements:

\[
P_{ABL}(c_j) = \frac{|\langle \psi_2(t_2)|c_j\rangle|^2|\langle c_j|\psi_1(t_1)\rangle|^2}{\sum_k |\langle \psi_2(t_2)|c_k\rangle|^2|\langle c_k|\psi_1(t_1)\rangle|^2} \tag{1}
\]

However, it has recently been argued by the author and others that certain of these claims are based on a counterfactual interpretation of the ABL rule, which is, in general, incorrect; and that, therefore, these claims cannot be maintained. Specifically, the counterfactual usage allows the outcome \( c_j \) to be an eigenvalue of an observable that has not actually been measured in the pre- and post-selection of the system in question.

Kastner (1998) gives a quantitative characterization of the two possible readings of the ABL rule, non-counterfactual (which is the original, correct usage) and counterfactual (the incorrect usage). It is shown therein that the counterfactual usage involves a fundamental co tenability problem and that, in general, counterfactual statements based on this usage are false.

However, in terms of Lewis’ theory of counterfactuals (1973), that argument implicitly assumed a similarity relation (or at least a class of similarity relations) obtaining between possible worlds. Therefore, one possible avenue for evading the conclusion of the argument is to challenge the similarity relation (SR); i.e., to find a different SR for which the

\[1\text{cf. Vaidman (1996b) and (1998)}
\[2\text{These } \{c_k\} \text{ are, of course, the eigenvalues of } C.
\[3\text{cf. Sharp and Shanks (1993), Cohen (1995), Miller (1996), Kastner (1998)}\]
aforementioned cotenability problem does not arise. The purpose of this paper is to identify the required class of similarity relations and to show that any such SR will be one that makes the counterfactual statement irrelevant or inapplicable to the actual world.

2. A (Bad) Similarity Relation That Works.

The type of counterfactual claim being made in the context of pre- and post-selected systems is as follows. Imagine a system K pre-selected in state $|a\rangle$ at time $t_1$ and post-selected in state $|b\rangle$ at time $t_2$. We now consider possible measurements that might have been made at an intermediate time $t$, $t_1 < t < t_2$, but were not, in fact, made. The counterfactual claim is:

A: “If we had performed a measurement of observable $C$ on system K at time $t$, the probability of finding outcome $c_j$ would have been as given by the ABL rule.”

Statement A can be represented by the proposition:

$$P \boxrightarrow{\text{Q}} (2),$$

where $P$=“Observable $C$ is measured” and $Q$=“The probability of outcome $c_j$ is as given by the ABL rule”, and the symbol ‘$\boxrightarrow{\text{ ‘}}$’ denotes the counterfactual connective.

Recall that Lewis’ theory of counterfactuals involves defining a similarity relation $\$ over a set of possible worlds $\$i$, in which $i$ denotes the actual world. $\$ determines what kinds of worlds will be considered as ‘closer’ or more ‘distant’ in terms of similarity to the actual world $i$. These worlds fall into concentric spheres $\zeta_i^{(k)}$ which are nested in such a way that the smallest sphere, $\zeta_i^{(1)}$, contains the actual world $i$ and worlds more similar to $i$ than are any worlds in progressively larger spheres but which are not not also in $\zeta_i^{(1)}$. The greater the value of $k$, the less similar to $i$ are those worlds in $\zeta_i^{(k)}$ that are not also in $\zeta_i^{(k-1)}$. (See Figure 1.)
Further, a proposition $\psi$ is defined to be ‘cotenable’ with another proposition $\phi$ at $i$ iff $\psi$ holds throughout some $\phi$-permitting sphere in $S_i$. That is, given a sphere $\zeta_i^{(k)}$ in which $\phi$ holds for at least one possible world, $\psi$ holds at all worlds in $\zeta_i^{(k)}$. (See Figure 2.)
Now, according to Lewis’ theory, a proposition of the form $P \rightarrow Q$ is true at $i$ iff $P$ and some auxiliary premise $X$, cotenable with $P$ at $i$, logically imply $Q$ (Lewis 1973, p. 57). It was shown in Kastner (1998) that the TSQT counterfactual claim (2) fails to fulfill this condition because the necessary auxiliary premise—essentially the requirement that the system in question have the same generalized state as in world $i$—is not cotenable with the antecedent $P$ under a natural similarity relation. (See Figure 3.)
The SR shown in Figure 3 is one in which it is assumed that the occurrence of a measurement at time $t$, denoted by the proposition $P$, affects the possible outcomes of the post-selection measurement at time $t_2$. Therefore, a system $K$ that has generalized state $\Psi$ in $i$, where $\neg P$ holds, might not necessarily have that same state in another world $m$ in which $P$ holds. In Figure 3, the proposition $T$ states that system $K$ has the same generalized state as in $i$. In the second sphere $\zeta_2^{(2)}$, the occurrence of possible measurements at time $t$ ($P$ stating that one particular kind of measurement has been performed) results in an ambiguous result for the post-selection state for $K$ and therefore there may be possible worlds in this sphere in which $T$ holds and other worlds in which it does not. Since, in order for $T$ to be cotenable with $P$, it must hold throughout a $P$-permitting sphere—i.e. at all worlds in that sphere—proposition $T$ is not cotenable with

Figure 3. A natural similarity relation equivalent to that presented in Kastner (1998).
P under this similarity relation.

It was also shown in Kastner (1998) that Vaidman’s definition of a ‘closest possible world’ \( j \) (equivalently a choice of $ for which the counterfactual usage of the ABL rule was correct) was untenable because that closest possible world did not, in general, exist. That definition, presented in Vaidman (1996a), stipulated that the closest possible world was one in which all measurements, excluding the intermediate measurement asserted by P, have the same outcomes as in \( i \).

However, there is a way to define a ‘closest possible world’ \( j \), equivalently a similarity relation, in such a way that \( j \) in general does exist and so that there is, at least formally, no problem with cotenability as described in Kastner (1998). Call this similarity relation \( Z \), and define it as follows:

Given a system \( K \) in the actual world \( i \) (where no intervening measurement has been performed, i.e. \( \neg P \) holds) with two-state vector \( \Psi = \langle b || a \rangle \), \( j \) is the world in which the following conjunction holds:

\[
P \& T \tag{3}
\]

where \( T \), as in the preceding discussion of Figure 3, is the proposition: “\( K \) has two-state vector \( \Psi \).”

In terms of Lewis’ system of concentric spheres, \( Z \) is the configuration in which all worlds in which \( P \& T \) holds are assigned to the smallest sphere in which \( P \) holds. In other words, \( T \) is the auxiliary proposition that, together with \( P \), implies the consequent \( Q \). Since \( T \) holds throughout a \( P \)-permitting sphere, it is cotenable with \( P \) under \( Z \) (see figure 4).
Figure 4. The similarity relation $Z$ that attempts to exploit the loophole.

This is almost the same as Vaidman’s definition for a closest possible world, except that here the only measurement outcome (after time $t$) that is required to be the same as in $i$ is the post-selection measurement outcome for system $K$ (rather than requiring that all measurement outcomes be the same). In general, we should always be able to find such a world (barring cases in which the post-selection state $|b\rangle$ is orthogonal to the state resulting from the measurement at time $t$).

Similarity relation $Z$ sidesteps the cotenability problem because, at least for world $j$, the intervening measurement is simply stipulated not to change the background conditions (i.e., the pre- and post-selected states), asserted to hold at $j$ by the proposition $T$.

This may be what Vaidman has in mind in his alternative definition for a time-
symmetrized counterfactual, which involves ‘fixing’ the pre- and post-selected states (Vaidman 1998). It was argued in Kastner (1998) that this definition is untenable because there is no way to accomplish this fixing requirement. This difficulty is related to what is wrong with the choice of Z as similarity relation.

3. Similarity defined in terms of likelihood.

In choosing a similarity relation relative to the actual world \( i \), we have to be loyal to the physical and probabilistic structure of the actual world \( i \). What has to be argued in order to justify similarity relation \( Z \) is that worlds assigned to the sphere \( \zeta_{i}^{(2)} \) in which \( T \) holds are really more similar to \( i \) than worlds assigned to \( \zeta_{i}^{(3)} \) in which \( \neg T \) holds. We should therefore be able to give arguments justifying the characterization of the ‘closest possible world’ \( j \) as more similar to \( i \) than some other P-world \( j' \) in which system K has a different post-selection outcome than in \( i \).

The tempting argument to make here is that \( j \) is objectively more similar to \( i \) than is \( j' \) in virtue of K’s having the same post-selection outcome in \( j \) as in \( i \); i.e., the fact that \( T \) holds at \( j \) but not at \( j' \). However, as Lewis has pointed out (1973, pp. 52-53), what is relevant in deciding whether the holding of \( T \) at any particular world makes that world closer to \( i \) than some other world in which \( T \) does not hold, is the comparative possibility of \( T \) with respect to \( i \), rather than the superficial similarity of world \( j \) to world \( i \) owing to the truth of \( T \) in both worlds.

The term ‘comparative possibility’ is the notion that some propositions are more possible than others given the relevant conditions obtaining in the actual world \( i \). This can be seen as an intuitively sensible measure of similarity of worlds, as Lewis shows via this illustration: “It is more possible for a dog to talk than for a stone to talk, since some worlds with talking dogs are more like our world than is any world with talking stones.”

One way to quantify this notion of comparative possibility of propositions is through the likelihood of a given apparent point of similarity, say \( X \), conditional on some proposition of interest \( P \) not holding in \( i \), and relevant background conditions in \( i \). In order for some world \( j \) to qualify as more similar to \( i \) than some other world \( j' \) based on the coincidence of \( X \) in worlds \( i \) and \( j \) but not in world \( j' \), the likelihood of \( X \) given \( P \) (and background conditions) must be sufficiently high for \( j \) to be considered more similar to \( i \) than \( j' \).
conditions B) must be greater than the likelihood of \( \neg X \) given \( P \& B \). Thus, we require:

\[
\text{Prob}(X| P\&B) > \text{Prob}(\neg X| P\&B).
\] (4)

An example may serve to clarify this point. Suppose that in the actual world \( i \), I (along with 10 million others) enter the lottery at time \( t \) and do not win at time \( t' \). Now consider a world \( m \) in which I am the only entrant at time \( t \), but I still do not win the lottery at time \( t' \) in \( m \). Superficially, world \( m \) is much like \( i \) in that the outcomes at \( t' \) are the same (I lose in both worlds). But we would not consider world \( m \) to be close to \( i \) under any natural similarity relation, since according to the laws of world \( i \), the outcome of my losing (call it L) at time \( t' \) given that I am the only entrant is impossible (assuming the lottery managers aren’t cheating). In term of the likelihood criterion, the likelihood of L conditional on the proposition of interest in \( m \) which causes it to differ from \( i \) (i.e., the fact that I am the only entrant) and the relevant laws in \( i \) (i.e., the laws of probability) is zero. Thus the fact that the outcomes at time \( t' \) are the same in both worlds is a freak occurrence rather than a legitimate indicator of closeness of those worlds.

The above example is an extreme case in which outcome L at \( t' \) in world \( m \) would be impossible according to the laws in world \( i \). However, it may just be that the likelihood of the outcome or event under consideration as a criterion of similarity between worlds \( m \) and \( i \) is less than 1 according to the laws in \( i \); rather than being impossible, the event is not guaranteed or possibly even farfetched to some degree. The more the likelihood deviates from 1, the further is the associated world from \( i \) in terms of similarity.

Of particular importance is the case in which we are considering the counterfactual \( P \rightarrow Q \), and \( m \) is a P-world in which some background condition or auxiliary premise \( T \) holding in \( i \) also holds in \( m \), despite the fact that according to the laws in \( i \) (i.e., the background conditions B), the holding of P would make \( T \) less likely. Here again we have a superficial resemblance of worlds \( i \) and \( m \). However, the likelihood of the event \( T \) given \( P \) is less than 1. Therefore the occurrence of \( T \) in both worlds is not a valid indicator of closeness.

Now suppose that the likelihood of \( T \) given \( P \) and the laws holding in \( i \) is 1/2. Consider
another P-world $m'$ identical to $m$ except for the fact that $\neg T$ holds in $m'$. There is no basis for saying that $m$ is closer to $i$ than is $m'$ in virtue of the holding of $T$, because it was no more likely that $T$ than that $\neg T$. By the comparative possibility criterion, the worlds $m$ and $m'$ are equally close to $i$, and so belong in the same sphere.

This is exactly the kind of problem we have under $Z$ for the counterfactual $P \square \rightarrow Q$. There is no basis for saying that world $j$ is more similar to $i$ than world $j'$, i.e., that $j$ belongs in a smaller sphere than $j'$, if the likelihoods of $T$ and $\neg T$ given $P$ are the same. Indeed, if $\neg T$ were more likely than $T$ given $P$ and the laws holding in $i$, then $j'$ would belong in a smaller sphere than $j$.

Quite simply, choosing $Z$ as the similarity relation amounts to ‘stacking the deck.’ To change the earlier example a bit: it is like saying, “If I were to enter the lottery, I would win,” and defining the closest possible world as the one in which I am the only entrant. Under this similarity relation, the counterfactual is true; but it doesn’t tell me very much about the actual world.

4. Conclusion.

It has been argued that an apparent loophole in arguments in Kastner (1998) establishing that the counterfactual usage of the ABL rule is, in general, invalid, is not a genuine loophole. This is because the only way to exploit the loophole is to use a similarity relation over the set of possible worlds that renders the counterfactual statement irrelevant to the actual world. Thus the conclusion of that paper stands: the ABL rule cannot, in general, be used to calculate the probability of various outcomes of an intervening measurement on a pre- and post-selected system if that measurement was not actually performed on the given system. Therefore, it is erroneous to refer to outcomes having probability 1 in a counterfactual application of the ABL rule as being ‘elements of reality’ of a system, as is done in Vaidman (1996b) and (1998).

In his most recent paper, Vaidman (1998) suggests that criticisms of the counterfactual usage of the ABL rule arise from a misunderstanding of the role of time symmetry in counterfactual statements. However, it should be noted that the conclusions of this paper and of Kastner (1998) in no way depend on any assumption of time asymmetry.
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