Vaidya Solution in General Covariant Hořava-Lifshitz Gravity with the Minimum Coupling and without Projectability: Infrared Limit

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In this paper, we have studied nonstationary radiative spherically symmetric spacetime, in general covariant theory \((U(1)\) extension) of the Hořava-Lifshitz gravity with the minimum coupling, in the post-newtonian approximation (PPN), without the projectability condition and in the infrared limit. The Newtonian prepotential \(\varphi\) was assumed null. We have shown that there is not the analogue of the Vaidya’s solution in the Hořava-Lifshitz Theory (HLT) with the minimum coupling, as we know in the General Relativity Theory (GRT).

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I. INTRODUCTION

Since its publication, the Hořava-Lifshitz gravity theory (HLT) \([1, 2]\) has draw a lot of interest primarily due to its prospects to solve the perturbative non-renormalizability of quantized standard Einstein gravity at the expense of breaking relativistic invariance at very high energies, while restoring it at low energies. After that, further deep connections of the Hořava-Lifshitz formalism to the condensed matter physics were discovered within the framework of the gauge/gravity duality (holography).

Unfortunately, the original Hořava formulation was contaminated by a number of serious problems which have caused the development of various non-trivial modifications. One of the promising latter modifications is the one without the so called projectability condition and with an additional Abelian gauge symmetry, which in particular solves the problem with the unphysical scalar graviton. Due to the extensive works in this area, we suggest to the reader the references \([3]-[36]\).

In order to establish the physical relevance of any modification of the Hořava-Lifshitz gravity it is very important to check that the latter reproduce the known physically feasible properties of the standard Einstein’s General Relativity (EGR) at low energies. Within this condition, we have checked whether the non-projectable version of the Hořava-Lifshitz gravity with the extra U(1) gauge symmetry contains in the low energy limit solutions of Vaidya-type \([39]\). The answer was negative and, therefore, led us to the important conclusion that in order to establish consistency with the EGR at low energies the gauge field associated with the extra U(1) gauge symmetry of the enlarged non-projectable Hořava-Lifshitz gravity should have some interaction with the pure radiation matter generating the Vaidya spacetime geometry.

In a recent paper, Lin et al. (2014) \([37]\), have proposed a universal coupling between the gravity and matter in the framework of the Hořava-Lifshitz theory of gravity with an extra U(1) symmetry for both the projectable and non-projectable cases. Then, using this universal coupling they have studied the PPN approximations and they have obtained the PPN parameters in terms of the coupling constants of the theory.

In this present work, using the results of Lin et al. (2014) \([37]\), we have studied nonstationary radiative spherically symmetric spacetime, in the general covariant theory \((U(1)\) extension) of the Hořava-Lifshitz gravity with a minimum coupling \([37]\), in the PPN approximation, without the projectability condition and in the infrared limit. We will analyze if the Vaidya’s spacetime can be described as a null radiation fluid in the general covariant HLT of gravity without the projectability condition \([23, 24]\). In Section II we present a brief introduction to the HLT with the minimum coupling \([37]\) considered here and we present the field equations of the HLT modified. In Section III we show the Vaidya’s spacetime, expressed in ADM decomposition \([38]\). In Section IV we present the HLT equations in the PPN approximation, for the infrared limit. In Section V we analyze all the possible solution for the HLT field equations. In Section VI we discuss the results. Finally, in the Appendix A we present some quantities necessary in the HLT field equations with minimum coupling and without projectability.
II. GENERAL COVARIANT HOŘAVA-LIFSHITZ GRAVITY WITH MINIMUM COUPLING AND WITHOUT PROJECTABILITY

In this section, we shall give a very brief introduction to the general covariant HLT gravity with the minimum coupling and without the projectability condition. For detail, we refer readers to [23, 24, 37].

The Arnowitt-Deser-Misner (ADM) form is given by

$$ds^2 = -N^2 dt^2 + g_{ij} (dx^i + N^i dt) (dx^j + N^j dt),$$

$$i, j = 1, 2, 3,$$  \(1\)

where the nonprojectability condition imposes that $N = N(t, x^i)$.

In the work of Lin et al. (2014) [37] it is proposed that, in the IR limit, it is possible to have matter fields universally couple to the ADM components through the transformations

$$\tilde{N} = FN, \quad \tilde{N}^i = N^i + Ng^i j \nabla_j \varphi,$$

$$\tilde{g}_{ij} = \Omega^2 g_{ij},$$  \(2\)

with

$$F = 1 - a_1 \sigma, \quad \Omega = 1 - a_2 \sigma,$$  \(3\)

where

$$\sigma = \frac{A - a}{N},$$

$$A = -\dot{\varphi} + N^i \nabla_i \varphi + \frac{1}{2} N (\nabla^i \varphi) (\nabla_i \varphi),$$  \(4\)

and where $A$ and $\varphi$ are the the gauge field and the Newtonian prepotential, respectively, and $a_1$ and $a_2$ are two arbitrary coupling constants. Note that by setting the first terms in $F$ and $\Omega$ to unity, we have used the freedom to rescale the units of time and space. We also have

$$\tilde{N}_i = \Omega^2 (N_i + N \nabla_i \varphi), \quad \tilde{g}^{ij} = \Omega^{-2} g^{ij}.$$  \(5\)

Considering the exposed before, the matter action can be written as

$$S_m = \int dt dx \tilde{N} \sqrt{\tilde{g}} \tilde{L}_m \left( \tilde{N}, \tilde{N}_i; \tilde{g}_{ij}; \psi_n \right),$$

where $\psi_n$ collectively stands for matter fields. One can then define the matter stress-energy in the ADM decomposition, with the minimum coupling.

Thus, the total action of the theory can be written as,

$$S = \zeta^2 \int dt dx \sqrt{g} N \left( \mathcal{L}_K - \mathcal{L}_V + \mathcal{L}_A + \mathcal{L}_\varphi + \mathcal{L}_S + \frac{1}{\zeta^2} \mathcal{L}_M \right),$$  \(7\)

where $g = \det(g_{ij}), N$ is given in the equation \(1\) and

$$\mathcal{L}_K = K_{ij} K^{ij} - \lambda K^2,$$

$$\mathcal{L}_V = \gamma_0 \zeta^2 - \left( \beta_0 a_i a^i - \gamma_1 R \right) + \frac{1}{\zeta^2} \left( \gamma_2 R^2 + \gamma_3 R_{ij} R^{ij} \right)$$

$$+ \frac{1}{\zeta^2} \left[ \beta_1 (a_i a^i)^2 + \beta_2 (a^i a_i)^2 + \beta_3 (a_i a^i) a^2_j \right]$$

$$+ \beta_4 a^i a^j a^i + \beta_5 (a_i a^i) R + \beta_6 a_i a^i R^{ij} + \beta_7 R a^i_{\lambda i}$$

$$+ \frac{1}{\zeta^2} \left[ \gamma_5 C_{ij} C^{ij} + \beta_8 (\Delta a)^2 \right],$$

$$\mathcal{L}_A = \frac{A}{\tilde{N}} (2 \Lambda_0 - R),$$

$$\mathcal{L}_\varphi = \varphi \mathcal{G}^{ij} \left( 2 K_{ij} + \nabla_i \nabla_j \varphi + a_i \nabla_j \varphi \right)$$

$$+(1 - \lambda) \left[ (\Delta \varphi + a_i \nabla^i \varphi)^2 + 2 (\Delta \varphi + a_i \nabla^i \varphi) K \right]$$

$$+ \frac{1}{3} \mathcal{G}^{ijk} \left[ 4 (\nabla_i \nabla_j \varphi) a_k (\nabla_k \varphi) \right.$$

$$+ 5 (a_i \nabla_j \varphi) a_k (\nabla_k \varphi) + 2 (\nabla_i \varphi) a_j (\nabla_k \varphi)$$

$$+ 6 K_{ij} a_k (\nabla_k \varphi) \right],$$

$$\mathcal{L}_S = \sigma a_S,$$  \(8\)

where

$$a_S = \sigma_1 a_i a^i + \sigma_2 a^i,$$  \(9\)

where $G$ denotes the Newtonian constant, $\mathcal{L}_M$ is the Lagrangian of matter fields, $\mathcal{G}^{ijk} = g^{ij} g^{jk} - g^{ij} g^{jk}$ [31]. Here $\Delta \equiv g^{ij} \nabla_i \nabla_j, \Lambda_0$ is the cosmological constant, and all the coefficients, $\beta_n$ and $\gamma_n$, are dimensionless and arbitrary. Note that if $a_1 = a_2 = \sigma_1 = \sigma_2 = 0$ we recover the HLT without any coupling with the matter [39].

In order to be consistent with observations in the infrared limit [37], we assume that

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8 = \beta_9 = 0,$$  \(11\)

$$\gamma_0 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = \gamma_8 = \gamma_9 = 0,$$  \(12\)

implying that

$$\Lambda_g = \frac{1}{2} \zeta^2 \gamma_0 = 0.$$  \(13\)

For the PPN approximation in minimum coupling theory, we have

$$\beta_0 = -2 (\gamma_1 + 1),$$  \(14\)

$$a_1 = a_2 = 0,$$  \(15\)
\[ \sigma_1 = 0, \]  
\[ \sigma_2 = 4(1 - a_1) = 4. \]  
(16)  
(17)

Then, for the choice of the parameters above, we have
\[ \hat{N} = N, \]  
\[ \hat{N}^i = N^i, \]  
(18)  
(19)

and
\[ \hat{g}_{ij} = g_{ij}. \]  
(20)

\[ C_{ij} \] denotes the Cotton tensor, defined by
\[ C^{ijkl} = \frac{\varepsilon^{ijkl}}{\sqrt{g}} \nabla_k \left( R^j_i - \frac{1}{4} R \delta^j_i \right), \]  
(21)

with \( \varepsilon^{123} = 1 \). Using the Bianchi identities, one can show that \( C_{ij} C^{ij} \) can be written in terms of the five independent sixth-order derivative terms in the form
\[ C_{ij} C^{ij} = \frac{1}{2} R^3 - \frac{5}{2} R R_{ij} R^{ij} + 3 R^2 R_k^k R^i_i + \frac{3}{8} R \Delta R \]
\[ + \left( \nabla_i R_{jk} \right) \left( \nabla^j R^k_i \right) + \nabla_k G^k, \]  
(22)

where
\[ G^k = \frac{1}{2} R^{jl} \nabla_j R - R_{ij} \nabla_j R^i_j - \frac{3}{8} R \nabla^k R. \]  
(23)

The Ricci and Riemann tensors \( R_{ij} \) and \( R^{ijkl} \) all refer to the 3-metric \( g_{ij} \), with \( R_{ij} = R^k_{ikj} \) and
\[ R_{ijkl} = g_{ik} R_{jl} + g_{jl} R_{ik} - g_{jk} R_{il} - g_{il} R_{jk} \]
\[ - \frac{1}{2} \left( g_{ik} g_{jl} - g_{il} g_{jk} \right) R, \]  
\[ K_{ij} = \frac{1}{2N} \left( -g_{ij} + \nabla_i N_j + \nabla_j N_i \right), \]
\[ G_{ij} = R_{ij} - \frac{1}{2} \hat{g}_{ij} R + \hat{\Lambda}_g, \]
\[ a_i = \frac{N_i}{N}, \quad a_{ij} = \nabla_j a_i, \]  
(24)

where \( N_i \) is defined in the ADM form of the metric \( g_{ij} \), given by equation (11).

The variations of the action \( S \) with respect to \( N \) and \( N^i \) give rise to the Hamiltonian and momentum constraints,
\[ \mathcal{L}_K + \mathcal{L}_V + F_V - F_\varphi - F_\lambda + \mathcal{H}_S = 8\pi G J^i, \]  
(25)

\[ M^i_S + \nabla_j \left\{ \pi^{ij} - (1 - \lambda) g^{ij} (\nabla^2 \varphi + a_k \nabla^k \varphi) \right. \]
\[ - \varphi G^{ij} - \hat{g}^{ijkl} a_l \nabla_k \varphi \left. \right\} = 8\pi G J^i, \]  
(26)

where
\[ \mathcal{H}_S = \frac{2\sigma_1}{N} \nabla_i \left[ a_i (A - A) \right] - \frac{\sigma_2}{N} \nabla^2 (A - A) \]
\[ + \frac{1}{2} \nabla_j \varphi \nabla^j \varphi, \]
\[ M^i_S = - \frac{1}{2} a_S \nabla^i \varphi, \]
\[ J^i = - N \delta \mathcal{L}_M, \quad J^i = \frac{2}{\delta N} \frac{\delta (N \mathcal{L}_M)}{\delta N}, \]
\[ \pi^{ij} = - K^{ij} + \lambda K g^{ij}, \]  
(27)

with and \( F_V, F_\varphi \) and \( F_\lambda \) are given in the Appendix A.

Variations of \( S \) with respect to \( \varphi \) and \( A \) yield, respectively,
\[ \frac{1}{2} G^{ij} \left( 2K_{ij} + \nabla_i \nabla_j \varphi + a_i \nabla_j \varphi \right) \]
\[ + \frac{1}{2N} \left\{ G^{ij} \nabla_j \nabla_i (N \varphi) - g^{ij} \nabla_j (N \varphi a_i) \right\} \]
\[ - \frac{1}{N} G^{ijkl} \left\{ \nabla_i (a_l) N K_{ij} \right\} + \frac{2}{3} \nabla_i (a_l) N \nabla_j \nabla_k \varphi \]
\[ - \frac{2}{3} \nabla_i \nabla_j (N a_l \nabla_k \varphi) + \frac{5}{2} \nabla_j (N a_l a_k \nabla_i \varphi) \]
\[ + \frac{2}{3} \nabla_j (N a_{ik} \nabla_i \varphi) \right\} + \Sigma_S \]
\[ + \frac{1 - \lambda}{N} \left\{ \nabla^2 \left[ N (\nabla^2 \varphi + a_k \nabla^k \varphi) \right] \right. \]
\[ - \nabla^i \left[ (N^k + N \nabla^k \varphi) a_i \right] \]
\[ + \nabla^2 (N K) - \nabla^i (N K a_i) \right\} = 8\pi G J^i, \]  
(28)

where
\[ \Sigma_S = - \frac{1}{2N} \left\{ \frac{1}{\sqrt{g}} \frac{\partial}{\partial t} \left[ \sqrt{g} a_S \right] \right. \]
\[ - \nabla_k \left[ \left( N^k + N \nabla^k \varphi \right) a_S \right] \} \]  
(29)

and
\[ R - 2\Lambda_g - a_S = 8\pi G J_A, \]  
(30)

where
\[ J_\varphi = - \frac{\delta \mathcal{L}_M}{\delta \varphi}, \quad J_A = 2 \frac{\delta (N \mathcal{L}_M)}{\delta A}. \]  
(31)

On the other hand, the variation of \( S \) with respect to \( g_{ij} \) yields the dynamical equations,
\[ \frac{1}{\sqrt{g} N} \frac{\partial}{\partial t} \left( \sqrt{g} \pi^{ij} \right) + 2 (K^{ik} K^j_k - \lambda K g^{ij}) \]
\[ - \frac{1}{2} \pi^{ij} K_K + \frac{1}{N} \nabla_k (a_k N^j + \pi^{ij} N^i - \pi^{ij} N^k) \]
\[ + F_{ij} - F_{ij}^S - \frac{1}{2} g^{ij} L_S + F^i_a \frac{1}{2} g^{ij} L_A + F^{ij}_\varphi \]
\[ - \frac{1}{N} (A R^{ij} + g^{ij} \nabla^2 A - \nabla^j \nabla^i A) = 8\pi G r^{ij}, \]  
(32)
From equation (37), we immediately obtain

\[ r_{ij} = \frac{2}{\sqrt{g} N} \frac{\delta (\sqrt{-g} N L M)}{\delta g_{ij}} \],

(33)

and \( F^{ij}, F_S^{ij}, F_a^{ij} \) and \( F_c^{ij} \) are given in the Appendix A.

III. VAIĐYA’S SPACETIME

The Vaidya’s spacetime with an ingoing null dust usually written in the form \[ 32 \],

\[ ds^2 = - \left( 1 - \frac{2m(v)}{r} \right) dv^2 + 2dvdr + r^2d\Omega^2, \]

(34)

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \), and the corresponding energy-momentum tensor is given by

\[ T_{\mu\nu} = \rho(v, r) l_\mu l_\nu, \]

(35)

with

\[ \rho = \frac{2}{r^2} \frac{dm}{dv}, \quad l_\mu = -\delta_\mu^v. \]

(36)

Hereinafter, the Newtonian prepotential \( \varphi \) is assumed null and \( G = c = 1 \).

Introducing a time-like coordinate \( t \) via the relation, \( v = 2(t + r) \), the metric \[ 33 \] can be cast in the form,

\[ ds^2 = -\frac{r}{M} dt^2 + \frac{4M}{r} \left[ dr + \left( 1 - \frac{r}{2M} \right) dt \right]^2 + r^2d\Omega^2, \]

(37)

where

\[ M \equiv M(V) = 2m(v), \quad V \equiv t + r. \]

(38)

From equation \[ 37 \], we immediately obtain

\[ N = \sqrt{\frac{r}{M}}, \quad N^i = \left( 1 - \frac{r}{2M} \right) \delta_i^t, \]

\[ g_{rr} = \frac{4M}{r}, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2 \theta, \]

(39)

and

\[ N_i = g_{ij} N^j = -2 \left( 1 - \frac{2M}{r} \right) \delta_i^t, \]

\[ g^{rr} = \frac{r}{4M}, \quad g^{\theta\theta} = \frac{1}{r^2}, \quad g^{\phi\phi} = \frac{1}{r^2 \sin^2 \theta}, \]

(40)

\[ \rho = \frac{M^*}{2r^2}, \quad l_\mu = -2 \left( \delta_\mu^t + \delta_\mu^r \right), \]

(41)

where \( M^* \equiv dM/V \).

Since \( M = M(V) \), introducing another independent variable, \( U = t - r \), we can find that

\[ M' = \dot{M} = \frac{1}{2} M^*, \]

(42)

since \( dM/V)/dU = 0 \).

Then, we find that the non null metric components are

\[ (4) g_{tt} = - \left( N^2 - N_i N^i \right) = -\frac{4}{r} (M - r), \]

\[ (4) g_{tt} = N_i = \frac{2}{r} (2M - r) \delta_i^r, \]

\[ (4) g_{rr} = g_{rr} = \frac{4M}{r}, \]

\[ (4) g_{\theta\theta} = g_{\theta\theta} = \frac{r^2}{r^2}, \]

\[ (4) g_{\phi\phi} = g_{\phi\phi} = r^2 \sin^2 \theta, \]

(43)

\[ (4) g_{tt}^{(4)} = -\frac{1}{N^2} = -\frac{M}{r}, \]

\[ (4) g_{rr}^{(4)} = \frac{1}{r^2}, \]

\[ (4) g_{\theta\theta}^{(4)} = \frac{1}{r^2 \sin^2 \theta}. \]

(44)

For the projection tensor the non null components are

\[ (4) h_{tt}^{(4)} = 1 - \frac{r}{2M}, \]

\[ (4) h_{rr}^{(4)} = 1, \]

\[ (4) h_{\theta\theta}^{(4)} = 1, \]

\[ (4) h_{\phi\phi}^{(4)} = 1. \]

(45)

Then, it can be shown that

\[ n_\mu = N \delta_\mu^t = \sqrt{\frac{r}{M}}, \]

\[ \rho = \frac{1}{4M} \left[ K_\mu^\nu - N^\nu N^\mu \right] = \sqrt{\frac{r}{M}} \left[ -\delta_\mu^r + \left( 1 - \frac{r}{2M} \right) \delta_\mu^t \right], \]

(46)

\[ h_\nu^{(4)} = \frac{4}{3} g_\nu^{(4)} + n_\nu, \]

\[ h_\nu^{(4)} = 1, \]

\[ h_\theta^{(4)} = 1, \]

\[ h_\phi^{(4)} = 1. \]

(47)
\[ J_t = T_{\mu\nu}n^\nu h^\mu_t \]
\[ = \frac{1}{r^2} \sqrt{\frac{M}{r}} \frac{dM}{dr} \delta_{ij}^\tau, \]
\[ = \frac{1}{r^2} \sqrt{\frac{M}{r}} (\dot{M} + M') \delta_{ij}^\tau, \] 
\[ \tau_{ij} = T_{\mu\nu}h^\mu_i h^\nu_j \]
\[ = \frac{8M}{r^3} \frac{dM}{dr} \delta_{ij}^\tau \]
\[ = \frac{8M}{r^3} (\dot{M} + M') \delta_{ij}^\tau, \]

where the prime and dot denotes the partial differentiation in relation to the coordinate \( r \) and \( t \), respectively.

### IV. INFRARED LIMIT

In the infrared limit we must have

\[ J_t = -2\rho. \]  
(51)

Besides, hereinafter, we have assumed that \( \lambda = 1 \).

Thus we have

\[ K_{rr} = \frac{2M'Mr + M'r^2 - 2\dot{M}Mr - 2M^2 - Mr}{\sqrt{r/MM'}}, \]
\[ K_{\theta\theta} = \frac{r(2M - r)}{2\sqrt{r/MM'}}, \]
\[ K_{\phi\phi} = \frac{\sin^2 \theta (2M - r)}{2\sqrt{r/MM'}}, \]
\[ K = \frac{2M'Mr + M'r^2 - 2\dot{M}Mr + 6M^2 - 5Mr}{4\sqrt{r/MM'^2}}, \]
\[ R_{rr} = \frac{M'r - M}{Mr^2}, \]
\[ R_{\theta\theta} = \frac{M'r^2 + 8M^2 - 3Mr}{8M^2}, \]
\[ R_{\phi\phi} = \frac{\sin^2 \theta M'r^2 + 8M^2 - 3Mr}{8M^2}, \]
\[ R = \frac{M'r^2 + 4M^2 - 2Mr}{2M^2r^2}, \]  
(52)

and

\[ \mathcal{L}_K = \frac{1}{16M^3r^3} \times [(\lambda - 1)(-4M'^2M^{2,2} - 4M^{2}2M^3 - M'2M^r + 8\dot{M}M'M^{2,2} + 4\dot{M}M'M^3 + 9M'^22M^{2,2} - 4\dot{M}M'^2M^2 - 36M^4 + 28M^r + 2M'M^3) - (3\lambda + 1)M'\dot{r}(M' - M) + \lambda (8M'M^3 - 16\dot{M}M^2M^2 + 32M^3r - 16M'^2r)], \]  
(54)

\[ \mathcal{L}_V = \frac{1}{2M^2r^2} [M'r^2\gamma_1 + 4M^2 - 2Mr] \]  
(55)

\[ F_V = \frac{1}{8M^3r} \times \left[ 4M''M'^3M^2 - 7M'^2M^r - 7M^2r^2 + 14M'Mr\gamma_1 + 14M'Mr - 7M^2\gamma_1 - 7M^2 \right], \]  
(56)

\[ \mathcal{H}_S = \frac{1}{8M^3r^3} \times \left[ \sqrt{\frac{r}{M}} A'M'^2(M'^r - 5M) - 8M''M^2r^4 + 6M'M'^3r^3 + 32M'M'^2M^r - 70M'^2M^r - 14\gamma_1 + \lambda M'^2M^2r^4 + (3\gamma_1 + 10\lambda)M^3M^3r^3 + 4(\gamma_1 + \gamma_3)M'^4r^2 + 2(\gamma_1 + \gamma_3)M^2r^4 - 5(\gamma_1 + \gamma_3)M'^2M^r - 13M'^2M^r - 24M'^2M^2r^2 + 8M'^3M^3 - 48M'^3r + 26M'^2r^2 + 32M^3r\gamma_1 - 30M^4r^2\gamma_1 - 120M^4 + 12M^3r \right], \]  
(57)

From equation (56) we have

\[ H = \mathcal{L}_K + \mathcal{L}_V + F_V + \mathcal{H}_S = 8\pi J_t = \frac{1}{16M^3r^3} \times [4(\lambda - 1)(-4M'^2M^{2,2} - 4M'^2M^3 - M'^2M^r + 8\dot{M}M'M^{2,2} + 4\dot{M}M'M^3 + 9M'^22M^{2,2} - 4\dot{M}M'^2M^2 - 36M^4 + 28M^r + 2M'M^3) - (3\lambda + 1)M'\dot{r}(M' - M) + \lambda (8M'M^3 - 16\dot{M}M^2M^2 + 32M^3r - 16M'^2r)], \]  
(58)

\[ J_r = \frac{1}{8\sqrt{r/MM'^2}} \times [(\lambda - 1)(-4M'^2M^{2,2} - 4M'^2M^3 - 2M'^2M^r + 3M'^2M^2 + 4\dot{M}M'^2M^2 - 4M'^2M^r + 6M'Mr^2 + 5M^2r - 18M^3) + \dot{M}M^3r(2\lambda + 6)], \]  
(59)

\[ J_A = \frac{1}{4M^3r^2} [2M''M^3 - 3M'^2r^3 + 8M'M^2r^2 + 8M^3 - 7M^2r], \]  
(60)

\[ J_{\phi} = \frac{1}{64\sqrt{r/MM'^3}} \times \]
\[
\left\{ (\lambda - 1)[16\dot{M}'M^2r^3(M - M') + 4M^2r^4(-2M''Mr - M''r + 2M'Mr) + 2M''Mr^4(10M'M + 11M'r - 2M'M) + M^2r^4(-10M'M - 21M'r + 10M\dot{M}) - 4M\dot{M}'M^2r^3] + \sqrt{\frac{r}{M}}[16M''Mr^4(2M - r) - M''M'Mr^4(104r - 176M) + M''Mr^2r^3(80M - 56r) + M''Mr^4(168M - 108r) + M^2r^3r(180r - 216M) - 36M'M^2r^2(2M + r) + 12M^3r(10M - 3r)] + \lambda(-28M''Mr^3 + 26M''Mr^2 + 22M'M^2r^2 + 51M'^2Mr^4 + 30M'M^3r^2 - 15M'Mr^2 + 6M^4r^2 + 18M^3r - 15M^3r^2 - 16M''Mr^3 + 12M''Mr^3 + 34M'\dot{M}^2r^4 + 24M'M^2r^4 + 2M^2Mr^4 - 71M'^2Mr^4 - 32M'Mr^4 - 48M'M^3r^3 + 26M'^3r^3 + 59M'M^2r^3 + 32\dot{M}^4r - 2\dot{M}^3r^2 + 32M^3 + 24M^4r^2 + 14M^4r - 9M^3r^2\}. \right.
\]

From the dynamical equation \eqref{eq:1}, we have
\[
D^{\tau\tau} = 8\pi r^{\phi\phi} = \frac{d^{\tau\tau}}{128\sqrt{\frac{r}{M}MM^4r^2}},
\]
where
\[
d^{\tau\tau} = \sqrt{\frac{r}{M}}\left[4(\lambda - 1)\left(\dot{M}'M^2r^2 - 4\dot{M}^2M^3r^2 - 27M^4 + M''Mr^2(4M^2r^2 - 3) + 4\dot{M}'M^2r^2 - 3M'
\dot{M}Mr^3 - 3M^2Mr^2(M + r^2) + 6M'M^2r^2 + 2M^2Mr^2 + 116Mr^2 + M^2r^4(5\lambda + 2\gamma_1 - 3) + 8\dot{M}^3r(7\lambda + 5) - 2\dot{M}M^3r(2\gamma_1 - 1 + 3) + 7M^2r^2(3 - 2\gamma_1 - 5\lambda) + 4\dot{M}^3r(8\gamma_1 - 21) - 4M^2r(2M'M + \dot{M}r)(\lambda + 3) + 8AM^2r(4M - r) - 16\dot{M}'M^2r^3.\right]\]
\[
D^{\phi\phi} = 8\pi r^{\phi\phi} = \frac{d^{\phi\phi}}{32\sqrt{\frac{r}{M}MM^3r^3}},
\]
where
\[
d^{\phi\phi} = \sqrt{\frac{r}{M}}\left[4(\lambda - 1)\left(M^2Mr^2(M + r) - \dot{M}M'Mr^2(2M + r) + 10M^3r(\dot{M} - M') - 2M'M^2r^2 + \dot{M}^2M^2r^2 + \dot{M}M^2r^2 + 9M^4 + M^3r + \lambda(32\dot{M}'M^3r^2 - 16\dot{M}'M^3r^2 - 7M^2r^4) + 4M'M^2r^4(\lambda + \gamma_1) - 3\dot{M}'M^3r^2 - 2M'Mr^4(11\gamma_1 + 10\gamma_1 - 5M^2r^2(\gamma_1 - 2\gamma_1 + 1)] - 8\dot{M}^2Mr^2 + 4\dot{M}'Mr^4(3M' - 3M) + 4AM^2r^2(M'r - M'), \right. \]
\[\]
\begin{equation}
D^{\phi\phi} = 8\pi r^{\phi\phi} = D^{\phi\phi}\sin^2 \theta. \tag{66}
\end{equation}

V. POSSIBLE SOLUTION

We are looking for a HLT solution which is equivalent to the Vaidya’s solution in GRT. In the minimum coupling theory, we can see that \(J_A = 0\), since \(\mathcal{N}\) does not depend on \(A\) (since \(a_1 = a_2 = 0\)) in the Vaidya’s space-time [see equation (4.11) in \cite{37}]. Then, since \(J_A = 0\), from equation \(66\) we have
\[2M^3M'' - 3r^3M''r^2 + 8Mr^2M' + 8M^3 - 7M^2r = 0. \tag{67}\]

Using the equations \(U = t - r\) and \(V = t + r\) and the fact that \(M = M(V)\), we can write that
\[M' = \dot{M}' = \frac{1}{2}M' = \frac{1}{2}\frac{dM}{dV}, \tag{68}\]
and
\[M'' = \frac{1}{4}\frac{d^2M}{dV^2}. \tag{69}\]

Using these equations we can rewrite equation the \(67\) as
\[\frac{1}{16}
M(V - U)^3 \frac{d^2M}{dV^2} - \frac{3}{32}(V - U)^3 \left(\frac{dM}{dV}\right)^2 + M(V - U)^2 \frac{dM}{dV} + 8M^3 - \frac{7}{2}M^2(V - U) = 0. \tag{70}\]

Deriving the equation \(70\) three times in relation to \(U\) we get
\[-\frac{3}{8}M \frac{d^2M}{dV^2} + \frac{9}{16} \left(\frac{dM}{dV}\right)^2 = 0, \tag{71}\]
whose solution is
\[M(V) = \frac{4}{(c_1V + c_2)^2}, \tag{72}\]
where \(c_1\) and \(c_2\) are constants. Substituting equation \(72\) into \(70\) we get
\[6c_1(V - U)^3 - 6c_2^2(c_1V + c_2)^2(V - U)^3 - 32c_1(c_1V + c_2)^2(V - U)^2 + 512(c_1V + c_2) - 56(c_1V + c_2)^2(V - U) = 0. \tag{73}\]

This equation can not be identically null since \(c_1V + c_2 \neq 0\).
VI. CONCLUSION

In this paper, we have analyzed nonstationary radiative spherically symmetric spacetime, in general covariant theory of Hořava-Lifshitz gravity with the minimum coupling, without the projectability condition, in the PPN approximation and in the infrared limit. The Newtonian prepotential $\varphi$ was assumed null.

In our previous paper [39] we have concluded that it does not exist a Vaidya’s solution, as we know in the GRT. The unknowing coupling with the matter was the reason for that result.

Now we can conclude at the present work that, using the minimal coupling with matter, in the way proposed in [37], there is not a solution in the HLT like the solution of Vaidya, as we know in GRT. This result can suggest that this theory needs to be treated more carefully as a generalization of the GRT. Therefore, there is not any guarantee that starting from a given well-defined metric in the GRT, this will lead to a solution in the HLT, whose infrared limit coincides with the corresponding solution of the GRT.

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VII. APPENDIX A: DEFINITION OF $F^{ij}$, $F_S^{ij}$, $F_a^{ij}$, $F_\varphi^{ij}$ AND $F_S^{ij}$

The quantities $F^{ij}$, $F_S^{ij}$, $F_a^{ij}$, $F_\varphi^{ij}$ and $F_S^{ij}$ are given by

$$F^{ij} = \frac{1}{\sqrt{gN}} \frac{\delta(-\sqrt{g}NC^S_{ij})}{\delta g^{ij}} = \sum_{s=0}^{\xi} \zeta_s^{m}(F_s)^{ij},$$

$$F_S^{ij} = -\sigma \left( \sigma_1 a^i a^j + \sigma_2 a^{ij} \right) + \frac{a_s}{2} \left[ (\nabla^i \varphi)(\nabla^j \varphi) + 2N^{ij}(\nabla^i \varphi)^2 \right]$$

$$+ \frac{\sigma_2}{N} \nabla^i[a^j](A - A) - g^{ij} \frac{\sigma_2}{2N} \nabla^k[a^j](A - A),$$

$$F_a^{ij} = \frac{1}{\sqrt{gN}} \frac{\delta(-\sqrt{g}NC^S_{ij})}{\delta g^{ij}} = \sum_{s=0}^{\xi} \zeta_s^{m}(F_s)^{ij},$$

$$F_\varphi^{ij} = \frac{1}{\sqrt{gN}} \frac{\delta(-\sqrt{g}NC^S_{ij})}{\delta g^{ij}} = \sum_{s=0}^{\xi} \zeta_s^{m}(F_s)^{ij},$$

$$F_S^{ij} = -\sigma \left( \sigma_1 a^i a^j + \sigma_2 a^{ij} \right) + \frac{a_s}{2} \left[ (\nabla^i \varphi)(\nabla^j \varphi) + 2N^{ij}(\nabla^i \varphi)^2 \right]$$

$$+ \frac{\sigma_2}{N} \nabla^i[a^j](A - A) - g^{ij} \frac{\sigma_2}{2N} \nabla^k[a^j](A - A),$$

with

$$\zeta_s = \left( \gamma_0, \frac{\gamma_1, \gamma_2, \gamma_3, 1}{2}, \frac{1}{2}, \frac{3}{7}, \frac{7}{15}, \frac{5}{2}, \frac{1}{2}, \frac{1}{2} \right),$$

$$n_s = (2, 0, -2, -2, -4, -4, -4, -4, -4, -4),$$

$$m_s = (0, -2, -2, -2, -2, -2, -2, -2, -2, -2),$$

$$\mu_s = (2, 1, 1, 2, 4, 5, 2, \frac{5}{3}, \frac{2}{3}, 1 - \lambda, 2, 2 \lambda).$$

Thus, $F_V$, $F_\varphi$ and $F_\lambda$ are given, respectively, by

$$F_V = \beta_0(2a^i + a_i a^i) - \frac{\beta_0}{2} \left[ 3(a_i a^i)^2 + 4\nabla_i(a_k a^k a^i) \right]$$

$$+ \frac{\beta_0}{2} \left[ (a_i a^i)^2 + 2N \nabla^2(N a_k^i) \right]$$

$$+ \frac{\beta_0}{2} \left[ (a_i a^i)^2 - 2 \nabla_i(a_j a^j) + \frac{1}{N} \nabla^2(N a_i a^i) \right]$$

$$+ \frac{\beta_0}{\zeta^2} \left[ a_i a^i \nabla_j \nabla_i(N a^{ij}) \right]$$

$$+ \frac{\beta_0}{\zeta^2} \left[ -R(a_i a^i) - 2\nabla_i(R a^i) \right]$$

$$+ \frac{\beta_0}{\zeta^2} \left[ a_i a_j R^{ij} - \nabla_i(a_j R^{ij}) - \nabla_j(a_i R^{ij}) \right]$$

$$+ \frac{\beta_0}{\zeta^2} \left[ R a^i + \frac{N}{2} \nabla^2(N R) \right]$$

$$+ \frac{\beta_0}{\zeta^2} \left( (a_i a^i)^2 - \frac{2}{N} \nabla^i[\Delta(N a_i)] \right),$$

$$F_\varphi = -\varphi^{ij} \nabla_i \varphi \nabla_j \varphi - \frac{2}{N} \varphi^{ijkl} \nabla_i(N K_{ij} \nabla_k \varphi),$$

$$- \frac{4}{3} \varphi^{ijkl} \nabla_i(N k_{ij} \nabla_k \varphi),$$

$$- \frac{4}{3} \varphi^{ijkl} \nabla_i(N k_{ij} \nabla_k \varphi).$$
\[
F_\lambda = (1 - \lambda) \left\{ (\nabla^2 \varphi + a_i \nabla^i \varphi)^2 - \frac{2}{N} \nabla_i (N K \nabla^i \varphi) - \frac{2}{N^2} \nabla_i [N(\nabla^2 \varphi + a_i \nabla^i \varphi) \nabla^i \varphi] \right\},
\]

\[
(F_{0})_{ij} = -\frac{1}{2} g_{ij},
\]

\[
(F_{1})_{ij} = R_{ij} - \frac{1}{2} R g_{ij} + \frac{1}{N} (g_{ij} \nabla^2 N - \nabla_j \nabla^i N),
\]

\[
(F_{2})_{ij} = -\frac{1}{2} g_{ij} R^2 + 2 R R_{ij} + \frac{2}{N} \left[ g_{ij} \nabla^2 (NR) - \nabla_j \nabla_i (NR) \right],
\]

\[
(F_{3})_{ij} = -\frac{1}{2} g_{ij} R_{mn} R^{mn} + 2 R R_{ij} R^k_k + \frac{1}{N} \left[ -2 \nabla_k \nabla_i (NR^k_j) + \nabla^2 (NR_{ij}) + g_{ij} \nabla_m \nabla_n (NR^{mn}) \right],
\]

\[
(F_{4})_{ij} = -\frac{1}{2} \left[ g_{ij} \nabla^2 (N R_{mn} R^{mn}) - \nabla_j \nabla_i (N R_{mn} R^{mn}) + \nabla^2 (N R R_{ij}) + g_{ij} \nabla_m \nabla_n (N R R^{mn}) - 2 \nabla_m \nabla_i (N R^k_j R^k_l) \right],
\]

\[
(F_{5})_{ij} = -\frac{1}{2} g_{ij} R R_{mn} R^{mn} + R_{ij} R_{mn} R^{mn} + 2 R R_{ij} R^k_k + \frac{1}{2} \left[ g_{ij} \nabla^2 (N R_{mn} R^{mn}) - \nabla_j \nabla_i (N R_{mn} R^{mn}) + \nabla^2 (N R R_{ij}) + g_{ij} \nabla_m \nabla_n (N R R^{mn}) - 2 \nabla_m \nabla_i (N R^k_j R^k_l) \right],
\]

\[
(F_{6})_{ij} = -\frac{1}{2} g_{ij} R_{i}^m R^{i}_m + 3 R^{mn} R_{mi} R_{nj} + \frac{3}{2} \left[ g_{ij} \nabla_m \nabla_n (N R_{mn} R^{mn}) + \nabla^2 (N R_{mi} R^{mn}) - 2 \nabla_m \nabla_i (N R_{ij} R^{mn}) \right],
\]

\[
(F_{7})_{ij} = -\frac{1}{2} g_{ij} R^2 + R_{ij} R^2 + R \nabla_j \nabla_i R + \frac{1}{2} \left[ g_{ij} \nabla^2 (N R^2) - \nabla_j \nabla_i (N R^2) \right] + R_{ij} \nabla^2 (NR) + g_{ij} \nabla^3 (NR) - \nabla_j \nabla_i (\nabla^2 (NR)) - \nabla_j (N R \nabla_i R) + \frac{1}{2} g_{ij} \nabla_k (N R \nabla^k R),
\]

\[
(F{k})_{ij} = -\frac{1}{2} g_{ij} (\nabla_m R_{mj})^2 + 2 \nabla_m R^{i}_{m} \nabla_{m} R_{nj} + \nabla_i R^{mn} \nabla_j R_{mn} + \frac{1}{N} \left[ 2 \nabla_n \nabla_i (N R^{mn} R_{nj}) - \nabla^2 \nabla_m (N R^{mn} R_{ij}) - g_{ij} \nabla_m \nabla_p (N R^{mp} R_{nj}) - 2 \nabla_m (N R_{li} R^{mn} R_{nj}) - 2 \nabla_n (N R_{li} R^{mp} R_{nj}) \right] + \nabla_j (N a_{k} A^k) R_{ij},
\]

\[
(F_n)_{ij} = -\frac{1}{2} g_{ij} a_{k} A^k + \frac{1}{2} (a_{k} R_{kl} j) \nabla_i R + a_{l} (R_{ij} k) \nabla^k R + \frac{1}{2} \left[ a_{l} R^{km} \nabla_m R_{kj} + a_{j} R^{km} \nabla_m R_{ki} \right] + \frac{3}{8} [R \nabla_k (Na_{k} R_{ij}) R_{ij}]
\]

\[
+ g_{ij} \nabla^2 \left[ R \nabla_k (Na_{k} R_{ij}) \right] \nabla_j \nabla_i \left[ R \nabla_k (Na_{k} R_{ij}) \right] \]

\[
+ \frac{1}{4 N} \left[ -\frac{1}{2} \nabla^m \left[ \nabla_i (Na_{k} R_{ij}) \nabla_m R + \nabla_i (R_{ij} R) N a_{m} \right] \right.
\]

\[
+ \nabla^2 (Na_{k} R_{ij} R) + g_{ij} \nabla^m \nabla^m (N a_{m} R_{ij}) R^{m}_{ij} + \nabla^{m} \left[ \nabla_i (R_{ij} R^{m}_{ij}) N a_{k} + \nabla_i (R_{ij} R^{m}_{ij} R a_{k}) \right] - 2 \nabla^2 (Na_{k} R_{ij} R^{m}_{ij}) - g_{ij} R^{m} \nabla^m (N a_{k} \nabla_{m} R_{ij})
\]

\[
- \nabla^{m} \left[ \nabla_i \nabla_{p} \left( Na_{k} R_{ij} R^{m}_{ij} + N a_{m} R^{p}_{ij} \right) \right]
\]

\[
+ \nabla_{j} \nabla_{p} \left( Na_{k} R_{ij} R^{m}_{ij} + N a_{m} R^{p}_{ij} \right)
\]

\[
+ \nabla^2 \nabla_{p} \left( Na_{k} R_{ij} R^{m}_{ij} + N a_{m} R^{p}_{ij} \right)
\]

\[
+ 2 \nabla_{j} \nabla_{p} \left( Na_{k} R_{ij} R^{m}_{ij} \right)
\]

\[
+ 2 g_{ij} \nabla^{m} \nabla^{n} \nabla^{p} \left( Na_{k} R_{ij} R^{m}_{ij} R^{n}_{ij} R^{p}_{ij} \right)
\]

\[
(81)
\]
\[
(F^g_{ij})_{ij} = -\frac{1}{N} \left[ \nabla^k (2Na_i(a_ja_k) - Na_{ij}a_k) \right],
\]
\[
(F^w_{ij})_{ij} = -\frac{1}{2}g_{ij}(a_k a^k) R + a_i a_j R + a^k a_k R_{ij}
+ \frac{1}{N} \left[ g_{ij} \nabla^2 (Na_k a^k) - \nabla_i \nabla_j (Na_k a^k) \right],
\]
\[
(F^\omega_{ij})_{ij} = -\frac{1}{2}g_{ij} a_m a_n R_{mn} + 2a^m R_{m(i)aj}
- \frac{1}{2N} \left[ 2\nabla^k \nabla_{(i)}(a_j) Na_k - \nabla^2 (Na_i a_j) \right]
- g_{ij} \nabla^m \nabla^n (Na_m a_n),
\]
\[
(F^\phi_{ij})_{ij} = -\frac{1}{2}g_{ij} Ra_k + a_k R_{ij} + Ra_{ij}
+ \frac{1}{N} \left[ g_{ij} \nabla^2 (Na_k) - \nabla_i \nabla_j (Na_k) \right]
- \nabla_{(i)} (NRa_{j}) + \frac{1}{2}g_{ij} \nabla^k (NRa_k),
\]
\[
(F^\varphi_{ij})_{ij} = -\frac{1}{2}d_{ij}(\Delta a_k)^2 + (\Delta a_i)(\Delta a_j) + 2\Delta a^k \nabla_{(i)} \nabla_j a_k
+ \frac{1}{N} \left[ \nabla_k [a_i \nabla^k (N\Delta a_j)] + a_i \nabla_j (N\Delta a_k) \right]
- a^k \nabla_{(i)} (N\Delta a_j) + g_{ij} Na^\beta \Delta a_\beta - Na_{ij} \Delta a^k]
- 2\nabla_{(i)} (Na_{jk} \Delta a^k),
\] (82)
\[
(F^\varphi_{ij})_{ij} = -\frac{1}{2}g_{ij} \varphi G_{mn} K_{mn}
+ \frac{1}{2\sqrt{g}} N \partial_i(\sqrt{g} \varphi g_{ij}) - 2\varphi K_{(i} R_{j)\nu}
+ \frac{1}{2} \varphi (KR_{ij} + K_{ij} R - 2K_{ij} \Lambda_g)
+ \frac{1}{2N} \left\{ G_{ij} \nabla^k (\varphi N_k) - 2g_{ik} \nabla^k (N\varphi_j) \right\}
+ g_{ij} \nabla^2 (N\varphi K) - \nabla_i \nabla_j (N\varphi K)
+ 2\nabla^k \nabla_{(i} (K_{j)k} N)\varphi)
- \nabla^2 (N\varphi K_{ij}) - g_{ij} \nabla^\alpha \nabla^\beta (N\varphi K_{\alpha\beta}) \right\},
\]
\[
(F^\omega_{ij})_{ij} = -\frac{1}{2}g_{ij} \varphi G_{mn} \nabla_m \nabla_n \varphi
- 2\varphi \nabla_{(i} \nabla^k R_{j)k} + \frac{1}{2} \varphi (R - 2\Lambda_g) \nabla_i \nabla_j \varphi
- \frac{1}{N} \left\{ - \frac{1}{2} (R_{ij} + g_{ij} \nabla^2 - \nabla_i \nabla_j) (N\varphi \nabla^2 \varphi) \right\}
- \nabla_k \nabla_{(i} (N\varphi \nabla^k \nabla_j \varphi) + \frac{1}{2} \nabla^2 (N\varphi \nabla_i \nabla_j \varphi)
+ \frac{g_{ij}}{2} \nabla^\alpha \nabla^\beta (N\varphi \nabla_{\alpha} \nabla_{\beta} \varphi)
+ g_{k(i} \nabla^k (N\varphi \nabla_{j)} \varphi)
\]
\[
(F^\varphi_{ij})_{ij} = -\frac{1}{2}g_{ij} \varphi G_{mn} a_m \nabla_n \varphi
- \varphi (a_i R_{j)k} \nabla^k \varphi + a^k R_{k(i} \nabla_j \varphi)
+ \frac{1}{2} (R - 2\Lambda_g) \varphi a_i \nabla_j \varphi

- \frac{1}{N} \left\{ - \frac{1}{2} (R_{ij} + g_{ij} \nabla^2 - \nabla_i \nabla_j) (N\varphi a^k \nabla_k \varphi) \right\}
- \frac{1}{2} \nabla^k \left[ \nabla_{(i} \nabla_j \varphi N\varphi) + \nabla_{(i} (a_j) \varphi N\varphi_k \varphi \right]
+ \frac{1}{2} \nabla^2 (N\varphi a_i \nabla_j \varphi)
+ \frac{g_{ij}}{2} \nabla^\alpha \nabla^\beta (N\varphi a_m \nabla_{\beta} \varphi) \right\},
\]
\[
(F^\varphi_{ij})_{ij} = -\frac{1}{2}g_{ij} \varphi G^{mnkl} K_{mn} a_k \nabla_l \varphi
+ \frac{1}{2\sqrt{g} N} \partial_i(\sqrt{g} \varphi g_{ij}) - 2\varphi K_{(i} R_{j)\nu}
+ \frac{1}{2N} \left\{ \nabla^2 (N\varphi a_k \nabla_i \nabla_j \varphi - a^k \nabla^k \nabla_{(i} \nabla_j \varphi) \right\}
+ \frac{1}{2N} \left\{ \nabla^2 (N\varphi a_k \nabla_i \nabla_j \varphi)
+ a_i \nabla_j \varphi a^k \nabla_k \varphi
+ a^k K_{(i} \nabla_j \varphi + a_i K_{j)k} \nabla^k \varphi
- K a_{(i} \nabla_j \varphi - K_{ij} a^k \nabla_k \varphi,
\]
\[
(F^\varphi_{ij})_{ij} = -\frac{1}{2}g_{ij} \varphi G^{mnkl} [a_k \nabla_l \varphi] [\nabla_m \nabla_n \varphi]
- a_i \nabla^k \nabla_j \varphi \nabla_k \varphi - a^k \nabla^k \nabla_{(i} \nabla_j \varphi)
+ a_i \nabla_j \varphi a^k \nabla_k \varphi
+ \frac{1}{2N} \left\{ \nabla^2 (N\varphi a_k \nabla_i \nabla_j \varphi)
+ a_i \nabla_j \varphi a^k \nabla_k \varphi
+ g_{ij} \nabla^\alpha \nabla^\beta (N\varphi a_k \nabla^\beta \varphi) \right\},
\]
\[
(F^\omega_{ij})_{ij} = -\frac{1}{2}g_{ij} \varphi G^{mnkl} \nabla_m \nabla_n \varphi
- \varphi a_k \nabla^k \nabla_{(i} \nabla_j \varphi
- \frac{1}{2} a^k \nabla^k \nabla_{(i} \nabla_j \varphi - \frac{1}{2N} \left\{ \nabla_{(i} (Na_{j)} \nabla_k \varphi \nabla^k \varphi) \right\}
+ \nabla^k (Na_{(i} \nabla_j \varphi \nabla_k \varphi)
+ g_{ij} \nabla^k (Na_k \nabla^m \varphi \nabla_m \varphi)
- \frac{1}{2} \nabla^k (Na_k \nabla_i \varphi \nabla_j \varphi)
\]
\[
(F^\varphi_{ij})_{ij} = -\frac{1}{2}g_{ij} \varphi G^{mnkl} \nabla_m \nabla_n \varphi
- \varphi (a_i R_{j)k} \nabla^k \varphi + a^k R_{k(i} \nabla_j \varphi)
- 2(\nabla^2 \varphi + a_k \nabla^k \varphi)(\nabla_i \nabla_j + a_i \nabla_j \varphi)
- \frac{1}{N} \left\{ - 2\nabla_{(j} [N\nabla_{(i} \varphi (\nabla^2 \varphi + a_k \nabla^k \varphi)]
\]
.
\[ (F^\varphi_{ij}) = \frac{1}{2} g_{ij} \left( N (\nabla^2 \varphi + a_k \nabla_k \varphi) \Delta \varphi \right) \],

\[ + \frac{1}{2} g_{ij} \left[ N_a (\nabla^2 \varphi + a_k \nabla_k \varphi) \right] \nabla_a \varphi \bigg] + \frac{1}{2} g_{ij} \left[ N (\nabla^2 \varphi + a_k \nabla_k \varphi) \right] N K \nabla_i \varphi \bigg] + \frac{1}{2} g_{ij} \left[ N (\nabla^2 \varphi + a_k \nabla_k \varphi) \right] \nabla_j \varphi \bigg] \right) \].

(83)
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