CONVERGENCE OF TEICHMÜLLER DEFORMATIONS IN THE
UNIVERSAL TEICHMÜLLER SPACE

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Abstract. Let \( \varphi : \mathbb{D} \to \mathbb{C} \) be an integrable holomorphic function on the unit disk \( \mathbb{D} \) and \( D_\varphi : \mathbb{D} \to T(\mathbb{D}) \) the Teichmüller disk in the universal Teichmüller space \( T(\mathbb{D}) \). For a positive \( t \) it is known that \( D_\varphi(t) \to [\mu_\varphi] \in PML_b(\mathbb{D}) \) as \( t \to 1 \), where \( \mu_\varphi \) is a bounded measured lamination representing a point on the Thurston boundary of \( T(\mathbb{D}) \). We extend this result by showing that \( D_\varphi : \mathbb{D} \to T(\mathbb{D}) \) extends as a continuous map from the closed disk \( \overline{\mathbb{D}} \) to the Thurston bordification. In addition, we prove that the rate of convergence of \( D_\varphi(\lambda) \) when \( \lambda \to e^{i\theta} \) is independent of the type of the approach to \( e^{i\theta} \in \partial \mathbb{D} \).

1. Introduction

Let \( \mathbb{D} \) be the unit disk equipped with the hyperbolic metric and \( f : \mathbb{D} \to \mathbb{D} \) a quasiconformal map. Then \( f \) extends by continuity to a quasisymmetric map \( h : S^1 \to S^1 \) of the unit circle \( S^1 \). Conversely, a quasisymmetric map \( h : S^1 \to S^1 \) extends to a quasiconformal map of \( \mathbb{D} \) and there are infinitely many such extensions (See \([5],[12]\)).

The universal Teichmüller space \( T(\mathbb{D}) \) consists of all quasiconformal maps \( f : \mathbb{D} \to \mathbb{D} \) up to an equivalence relation (See \([11],[5]\)). Namely, two quasiconformal maps \( f_1, f_2 : \mathbb{D} \to \mathbb{D} \) are equivalent if there exists a conformal map \( c : \mathbb{D} \to \mathbb{D} \) such that \( f_2^{-1} \circ c \circ f_1 \) extends by continuity to the identity on \( S^1 \). We will use an equivalent definition (See \([11],[5]\)):

\[
T(\mathbb{D}) = \{ h : S^1 \to S^1 : h \text{ is quasisymmetric and fixes } 1, i, -1 \}.
\]

The Thurston boundary of the universal Teichmüller space \( T(\mathbb{D}) \) is identified with the space \( PML_b(\mathbb{D}) \) of projective bounded measured laminations on \( \mathbb{D} \) (See \([2],[3],[17]\)). We describe the closure of Teichmüller disks in the Thurston bordification \( T(\mathbb{D}) \cup PML_b(\mathbb{D}) \) of the universal Teichmüller space \( T(\mathbb{D}) \). In particular, if a sequence in the parameter of the Teichmüller disk converges to a point on the unit circle the corresponding sequence in \( T(\mathbb{D}) \) converges to a unique point in \( PML_b(\mathbb{D}) \) independently of the type of approach (e.g. along a geodesic, along a horocycle or even outside any horoball).

Given an integrable holomorphic quadratic differential \( \varphi \) on \( \mathbb{D} \), it is known that the corresponding Teichmüller geodesic of \( T(\mathbb{D}) \) has a unique limit point on Thurston boundary.
of $T(\mathbb{D})$ (See [6], [7], [8]). The limit point $[\mu_\varphi] \in PML_b(\mathbb{D})$ is the projective class of the geodesic straightening of the horizontal foliation of $\varphi$ scaled by the reciprocal of the length of the horizontal leaves (See [6]).

Namely, $\mu_\varphi$ is the measured lamination on $\mathbb{D}$ which is given by the transverse measure

$$\int_I \frac{1}{l(z)} \, dy$$

for transverse arc $I$ to the vertical foliation of $\varphi$, where $l(z)$ is the length of the horizontal trajectory through $z = x + yi \in I$ and $dy$ is the differential of the vertical displacement in the natural parameter $z = x + yi$ of $\varphi$. For an integrable holomorphic quadratic differential $\varphi$ on $\mathbb{D}$, the Teichmüller disk is a holomorphic disk in $T(\mathbb{D})$ given by the family of Beltrami differentials $\{\lambda \varphi / |\varphi|\}_{\lambda \in \mathbb{D}}$. Let $D_\varphi(\lambda)$ be the Teichmüller equivalence class associated to the Beltrami differential $\lambda \varphi / |\varphi|$. We will prove the following, which is a generalization of a result in [6].

**Theorem 1.1** (Teichmüller deformation has the limit). As $\lambda \to 1 \in \partial \mathbb{D}$, $D_\varphi(\lambda)$ converges to the projective class of $\mu_\varphi$ in the Thurston boundary $PML_b(\mathbb{D})$ of $T(\mathbb{D})$. Furthermore, the Teichmüller disk $D_\varphi : \mathbb{D} \to T(\mathbb{D})$ is extended as an injective, continuous map from the closed unit disk $\overline{\mathbb{D}}$ to the Thurston bordification $T(\mathbb{D}) \cup PML_b(\mathbb{D})$.

**Remark 1.2.** Almost all horizontal leaves of $\varphi$ have exactly two limit points on $S^1$ (See [18]). The measured lamination $\mu_\varphi$ has support on the geodesic lamination of $\mathbb{D}$ obtained by replacing the leaves of $\varphi$ with geodesics (for the hyperbolic metric) of $\mathbb{D}$ that have the same endpoints as horizontal leaves. The transverse measure of $\mu_\varphi$ is given by the above integral and notice that (unlike in the case of closed surfaces) it is not simply the vertical transverse measure induced by $\varphi$. In fact, almost all horizontal leaves of $\varphi$ have finite length (in the metric $|\sqrt{\varphi}(\zeta) d\zeta|$) and the transverse measure is scaled by the reciprocal of the length of the horizontal leaves thus giving us a new type of limit when compared to closed surfaces (See [6]). The projective class of $\mu_\varphi$ is the unique accumulation point on the Thurston boundary of a sequence in the Teichmüller disks corresponding to the Teichmüller deformations $D_\varphi(\lambda)$ as $\lambda \to 1$.

Theorem 1.1 suggests that the behaviour of the Teichmüller disks in $T(\mathbb{D})$ is “tame” when compared to the behavior of those in the finite dimensional Teichmüller space. Namely, when $S$ is a closed Riemann surface of genus at least two, Masur [15] proved that the Teichmüller geodesics in $T(S)$ corresponding to holomorphic quadratic differentials with uniquely ergodic horizontal foliations have a unique limit point on the Thurston boundary of $T(S)$. However, when the horizontal foliation of a holomorphic quadratic differential on $S$ is not uniquely ergodic the corresponding Teichmüller geodesic can have more than one limit point on the Thurston boundary of $T(S)$ (See [14], [4], [13]). As related topic, for a closed Riemann surface $S$ of genus at least two and a holomorphic quadratic differential on $S$ whose horizontal foliation is uniquely ergodic, Jiang-Su [9] and Alberge [1] proved that the corresponding horocyclic path has a unique limit point on the the Thurston boundary.
of $T(S)$ which is the projective class of the horizontal foliation. However, to the authors knowledge, the other cases (e.g. the convergence to 1 outside any horoball based at 1) are still not understood for compact surfaces.

A geodesic current on $\mathbb{D}$ is a positive Radon measure on the space of geodesics of $\mathbb{D}$ (See [2], [3], [17]. See §2). The Liouville map $\mathcal{L}$ maps the universal Teichmüller space $T(\mathbb{D})$ into the space of geodesic currents by taking the pull-back of the Liouville measure $L$ on the space of geodesics of $\mathbb{D}$ (See [2], [3], [17]. See §2). The universal Teichmüller space $T(\mathbb{D})$ is homeomorphic to its image $L(T(\mathbb{D}))$ inside the space of geodesic currents (See [2] and [3]).

By definition, the Thurston boundary of the universal Teichmüller space $T(\mathbb{D})$ consists of limit points in the space of projective geodesic currents of the projectivization of $L(T(\mathbb{D}))$.

Let $h_\lambda : S^1 \to S^1$ be the quasisymmetric map fixing 1, $i$ and $-1$ that represents the Teichmüller deformation $D_\varphi(\lambda)$. Since $L(h_\lambda) \to \infty$ in the space of geodesic currents as $\lambda \to 1$, it is natural to consider the rate of convergence to the infinity.

**Theorem 1.3** (Asymptotics of Teichmüller deformation). Let $h_\lambda : S^1 \to S^1$ be the quasisymmetric map that represents $D_\varphi(\lambda)$ in $T(\mathbb{D})$. Then, as $\lambda \to 1$,

$$\frac{1 - |\lambda|}{2\pi} L(h_\lambda) \to \mu_\varphi.$$

2. Thurston boundary of the universal Teichmüller space

Denote by $\mathbb{D} = \{ z : |z| < 1 \}$ the unit disk equipped with the hyperbolic metric $\frac{2|dz|}{1-|z|^2}$. Each oriented hyperbolic geodesic is uniquely determined by an ordered pair of distinct endpoints, the initial point and end point of the geodesic. Therefore, the space of all oriented (hyperbolic) geodesics of $\mathbb{D}$ is identified with $(S^1 \times S^1) - \text{diag}$, where $\text{diag}$ is the diagonal of $S^1 \times S^1$ and $S^1 = \partial \mathbb{D}$. The space of geodesics contains a unique (up to a positive multiple) positive Borel measure of full support which is invariant under the isometries of $\mathbb{D}$, called the Liouville measure. It is defined by

$$L(A) = \int_A \frac{d\alpha d\beta}{|e^{i\alpha} - e^{i\beta}|^2}$$

for any Borel set $A \subset (S^1 \times S^1) - \text{diag}$. For a box of geodesics $[a,b] \times [c,d]$, with $a,b,c,d \in S^1$ given in the counterclockwise order, the Liouville measure is given by (See Bonahon [2])

$$L([a,b] \times [c,d]) = \log \frac{(c-a)(d-b)}{(d-a)(c-b)}.$$

The universal Teichmüller space $T(\mathbb{D})$ consists of all quasisymmetric maps $h : S^1 \to S^1$ that fix 1, $i$ and $-1$. A geodesic current on $\mathbb{D}$ is a positive Borel measure on the space of geodesics $(S^1 \times S^1) - \text{diag}$. Bonahon [2] introduced an embedding of the Teichmüller space $T(S)$ into the space of geodesic currents equipped with the weak* topology via the Liouville map, where $S$ is a closed surface of genus at least two. Moreover, $T(\mathbb{D})$ and $T(X)$, for $X$
any Riemann surface, embeds into the space of geodesic currents when equipped with the uniform weak* topology (See [3], [17]). The Liouville map

\[ \mathcal{L} : \mathcal{T}(\mathbb{D}) \to \mathcal{G}(\mathbb{D}) \]

is defined by

\[ \mathcal{L}(h) = h_*(L) \]

where \( h : S^1 \to S^1 \) is quasisymmetric and \( (h^{-1})_*(L) \) is the pull-back of the Liouville measure \( L \) (i.e. the push-forward of \( L \) by \( h^{-1} \)).

By definition, the Thurston boundary of \( \mathcal{T}(\mathbb{D}) \) is the set of all boundary points of the image of \( \mathcal{L}(\mathcal{T}(\mathbb{D})) \) in the projective geodesic currents \( \mathcal{P}\mathcal{G}(\mathbb{D}) \) equipped with the quotient of the uniform weak* topology. It turns out that the Thurston boundary consists of all projective bounded measured laminations \( \mathcal{PML}_b(\mathbb{D}) \) (See [3], [17]).

Given an integrable holomorphic function \( \varphi : \mathbb{D} \to \mathbb{C} \), the corresponding Teichmüller geodesic is given by the equivalence class of quasiconformal maps

\[ g_t(z) = x + \frac{t}{t - 1}yi \]

where \( 0 \leq t < 1 \) and \( z = \int_{\varphi} \sqrt{\varphi(\zeta)}d\zeta \) is the natural parameter for \( \mathbb{D} \) defined by \( \varphi \). Let \( k_t : S^1 \to S^1 \) be a quasisymmetric map representing the equivalence class of \( g_t \). Notice that the maximal dilatation \( K(g_t) \) of \( g_t \) is equal to \( \frac{1+t}{1-t} \) when \( 0 \leq t < 1 \).

Let \( \mu_{\varphi} \) be measured lamination on \( \mathbb{D} \) whose support is the closure of the set of (hyperbolic) geodesics which have the same endpoints on \( S^1 \) as horizontal trajectories of \( \varphi \), i.e the leaves of \( \mu_{\varphi} \) are obtained by straightening the horizontal trajectories of \( \varphi \). Since almost all horizontal trajectories have two distinct limit points on \( S^1 \) (See [18]), it follows that to almost every horizontal trajectory there corresponds a hyperbolic geodesic in the support of \( \mu_{\varphi} \).

We define the \( \mu_{\varphi} \)-measure of a box of geodesics \([a, b] \times [c, d]\) as follows. If no horizontal trajectories have one endpoint in \([a, b]\) and another endpoint in \([c, d]\) then \( \mu_{\varphi}([a, b] \times [c, d]) = 0 \). In general, consider the set of all horizontal trajectories that have one endpoint in \([a, b]\) and another endpoint in \([c, d]\). Let \( \{J_k\}_k \) be at most countable collection of vertical arcs of \( \varphi \) such that every horizontal trajectory with one endpoint in \([a, b]\) and another endpoint in \([c, d]\) intersects exactly one \( J_k \), and no other horizontal trajectory of \( \varphi \) intersects any \( J_k \). We define

\[ \mu_{\varphi}([a, b] \times [c, d]) := \sum_{k=1}^{\infty} \int_{J'_k} \frac{Im(\sqrt{\varphi(\zeta)}d\zeta)}{l(\zeta)} = \sum_{k} \int_{J'_k} \frac{dy}{l(z)} \]

where \( J'_k \) is the image of \( J_k \) under the canonical coordinates \( z = x + ti \) corresponding to \( \varphi \) and \( l(\zeta) \) is the \( \varphi \)-length of the horizontal trajectory through \( \zeta \), and similar for \( l(z) \).

Then (See [6])

\[ \lim_{t \to 1} K(g_t)^{-1}\mathcal{L}(k_t) = \mu_{\varphi} \]
in the weak* topology on the geodesic currents $\mathcal{G}$, where $\mu_\varphi$ is the above measured lamination of $\mathbb{D}$. In general, an example showed that the above convergence does not hold for the uniform weak* topology (See [6]).

3. Modulus of a curve family

Let $\Gamma$ be a family of curves in $\mathbb{C}$. A metric $\rho(z)|dz|$, where $\rho(z) \geq 0$ and measurable, is said to be allowable for $\Gamma$ if for every $\gamma \in \Gamma$ we have

$$\int_{\gamma} \rho(z)|dz| \geq 1.$$ 

The modulus of a curve family $\Gamma$ is given by

$$\text{mod}(\Gamma) = \inf_{\rho} \int_{\mathbb{C}} \rho^2(z)dxdy$$

where the infimum is over all allowable metrics $\rho(z)|dz|$ for the curve family $\Gamma$.

The following properties of the modulus of families of curves are standard:

1. If $\Gamma_1 \subset \Gamma_2$ then $\text{mod}(\Gamma_1) \leq \text{mod}(\Gamma_2)$.
2. $\text{mod}(\bigcup_{i=1}^{\infty} \Gamma_i) \leq \sum_{i=1}^{\infty} \text{mod}(\Gamma_i)$.
3. If every $\gamma_2 \in \Gamma_2$ contains some $\gamma_1 \in \Gamma_1$ as a subcurve then $\text{mod}(\Gamma_1) \geq \text{mod}(\Gamma_2)$.

4. Liouville measure of boxes and modulus of curves

In [7], Hakobyan and the second author observed that the Liouville measure and the modulus of a curve family are asymptotic to each other.

Lemma 4.1 (See [7]). Let $(a, b, c, d)$ be a quadruple of points on $\mathbb{S}^1$ in the counterclockwise order. Let $\Gamma_{[a,b] \times [c,d]}$ consist of all differentiable curves $\gamma$ in $\mathbb{D}$ which connect $[a, b] \subset \mathbb{S}^1$ with $[c, d] \subset \mathbb{S}^1$. Then

$$\text{mod}(\Gamma_{[a,b] \times [c,d]}) - \frac{1}{\pi} L([a, b] \times [c, d]) - \frac{2}{\pi} \log 4 \to 0$$

as $\text{mod}(\Gamma_{[a,b] \times [c,d]}) \to \infty$, where $L$ is the Liouville measure.

If $\text{mod}(h_t(\Gamma_{[a,b] \times [c,d]})) \to \infty$ as $t \to \infty$, then Lemma 4.1 implies

$$\lim_{t \to \infty} \frac{\text{mod}(h_t(\Gamma_{[a,b] \times [c,d]}))}{L(h_t)([a, b] \times [c, d])} \to \frac{1}{\pi};$$

(1)

Moreover, $\text{mod}(h_t(\Gamma_{[a,b] \times [c,d]}))$ is bounded if and only if $L(h_t)([a, b] \times [c, d])$ is bounded.
5. **Proof of Theorem 1.1**

5.1. **Convergence of Liouville measures.** We identify (by a Möbius map) the unit disk $\mathbb{D}$ with the right half-plane $\mathbb{H}_{>0} = \{w \in \mathbb{C} \mid \text{Re}(w) > 0\}$ such that the positive radius in $\mathbb{D}$ is mapped to the real axis in $\mathbb{H}_{>0}$ and $1 \in S^1 = \partial \mathbb{D}$ is mapped to $\infty \in \partial \mathbb{H}_{>0}$. Then, Theorem 1.1 is rephrased as follows: For $s + ti \in \mathbb{H}_{>0}$, we consider the quasiconformal mapping $f_{s+ti}$ on $\mathbb{D}$ with the Beltrami differential

$$\frac{1 - (s + ti) \varphi}{1 + (s + ti) |\varphi|}.$$ 

In the natural parameter $z = x + yi = \int_\ast \sqrt{\varphi}(\zeta) d\zeta$, the corresponding quasiconformal mappings are

$$f_{s+ti}(z) = \frac{1}{s}x - \frac{t}{s}y + yi$$

for $s + ti \in \mathbb{H}_{>0}$. We will prove

**Theorem 5.1.** Let $\{f_{s+ti}\}_{s+ti \in \mathbb{H}_{>0}}$ be the Teichmüller deformations corresponding to an integrable holomorphic quadratic differential (i.e. an integrable function) $\varphi$ on the unit disk $\mathbb{D}$. Then

$$\frac{1}{K(f_{s+ti})} f^*_{s+ti}(L) \to \mu_\varphi$$

in the weak* sense as $s + |t| \to \infty$.

Indeed, Theorem 5.1 implies that $D_\varphi(\lambda) \to [\mu_\varphi]$ in the Thurston bordification. We define the extension of the Teichmüller disk $D_\varphi$ to the closed disk $\overline{\mathbb{D}}$ by setting

$$D_\varphi(\lambda) = [\mu_{\lambda \varphi}] \quad (\lambda \in S^1).$$

Let $\{\lambda_n\}_n \subset \mathbb{D}$ be a sequence converging to $\lambda_0 \in S^1$. Since $D_{\lambda_0 \varphi}(\lambda) = D_\varphi(\lambda_0 \lambda)$, from Theorem 5.1 we conclude $D_\varphi(\lambda_n) \to [\mu_{\lambda_0 \varphi}]$ as $n \to \infty$ in the Thurston bordification.

The continuity of the extension also follows from Theorem 5.1 and the diagonal argument. Indeed, let $\{s_n + t_n i\}_n$ be a sequence in $\mathbb{H} - \{\infty\}$ with $s_n + t_n i \to \infty$. We may assume that all $t_n = 0$. From the above discussion, the limit measure at $s_n$ is $\mu_{\varphi_n}$ where $\varphi_n = (s_n + i)/(s_n - i) \varphi$. For $[a, b] \times [c, d] \subset (S^1 \times S^1) - \text{diag}$ and $\epsilon > 0$, we take $t_n > 0$ for any $n$, with

$$\left| \frac{1}{K(f_{s_n+t_n i})} f^*_{s_n+t_n i}(L)([a, b] \times [c, d]) - \mu_{\varphi_n}([a, b] \times [c, d]) \right| < \epsilon$$

and $t_n \to 0$ where $\varphi_n = (s_n + i)/(s_n - i) \varphi$. Since $s_n + |t_n| \to \infty$, we obtain the convergence $\mu_{\varphi_n}([a, b] \times [c, d]) \to \mu_\varphi([a, b] \times [c, d])$ as $n \to \infty$. Thus, $\mu_{\varphi_n}$ converges to $\mu_\varphi$ in the weak* sense.
5.2. Proof of Theorem 5.1. An elementary computation gives the Beltrami coefficient of $f_{s+ti}(z)$ to be

$$
\mu(z) = \mu_{s+ti}(z) = \frac{1 - (s + ti)}{1 + (s + ti)}.
$$

Since $s^2 + t^2 \to \infty$ as $s + |t| \to \infty$, the maximal dilatation $K(f_{s+ti})$ of $f_{s+it}$ behaves

$$(2)\quad K(f_{s+ti}) = \frac{\sqrt{(1+s)^2 + t^2} + \sqrt{(1-s)^2 + t^2}}{\sqrt{(1+s)^2 + t^2} - \sqrt{(1-s)^2 + t^2}}$$

$$\quad = (\sqrt{(1+s)^2 + t^2} + \sqrt{(1-s)^2 + t^2})^2 = \frac{s^2 + t^2}{s} (1 + o(1))$$

as $s + |t| \to \infty$.

**Proposition 5.2.** Under the above notation

$$\limsup_{s+|t|\to\infty} \frac{1}{s + \frac{s^2}{a}} \mod(f_{s+ti}(\Gamma_{[a,b] \times [c,d]})) \leq \mu_\varphi([a, b] \times [c, d]).$$

**Proof.** Let $B = [a, b] \times [c, d] \subset (S^1 \times S^1) - \text{diag}$ be a fixed box of geodesics. Let $\nu = \mathfrak{H}(\sqrt{\varphi(\zeta)}d\zeta) = dy$ be the horizontal foliation of $\varphi$. Note that $\nu$ is a measured foliation of $\mathbb{D}$ while $\mu_\varphi$ is a measured geodesic lamination of $\mathbb{D}$.

Let $\Gamma_B$ be the family of all Jordan curves that connect $[a, b]$ to $[c, d]$ inside $\mathbb{D}$. Let $\delta > 0$ be fixed. Define $\Gamma_{\geq \delta}$ to be the family of all $\gamma \in \Gamma_B$ such that $\nu(\gamma) \geq \delta$, and define $\Gamma_{< \delta}$ to be all $\gamma \in \Gamma_B$ such that $\nu(\gamma) < \delta$ (See [6]). Since $f_{s+ti}$ does not change the $y$-coordinate in the canonical parameter of $\varphi$, we have that curves in $\Gamma_{\geq \delta}$ are mapped onto curves in $\Gamma_{s+ti}(B)$ and curves in $\Gamma_{< \delta}$ are mapped onto curves in $\Gamma_{s+ti}(B)$.

Define $\rho_\delta(z') = \frac{1}{\delta}$ in the canonical parameter $z' = f_{s+ti}(z) = x' + y'i$ of the terminal quadratic differential on $f_{s+ti}(\mathbb{D})$ corresponding to $\varphi$ and the map $f_{s+ti}$. Since $l_\rho(\gamma) := \int_\gamma \rho(\gamma)|dz'| \geq \int_\gamma \frac{1}{\delta}dy'$, we have

$$\quad l_\rho(\gamma) \geq \frac{1}{\delta} \delta = 1$$

for all $\gamma \in f_{s+ti}(\Gamma_{\geq \delta}) = \Gamma_{s+ti}(B)$. Thus $\rho_\delta$ is allowable for the family $f_{s+ti}(\Gamma_{\geq \delta}) = \Gamma_{s+ti}(B)$ and by the definition of the modulus $\mod(f_{s+ti}(\Gamma_{\geq \delta})) \leq \frac{1}{\delta^2} \int_\mathbb{D} |\varphi(\zeta)|d\xi d\eta$. We obtain

$$(3)\quad \lim_{s+|t|\to\infty} \frac{1}{s + \frac{s^2}{a}} \mod(f_{s+ti}(\Gamma_{\geq \delta})) \leq \lim_{s+|t|\to\infty} \frac{1}{s + \frac{s^2}{a}} \frac{1}{\delta^2} \int_\mathbb{D} |\varphi(\zeta)|d\xi d\eta = 0.$$
From (2) and the quasi-invariance of the modulus we obtain

\[ \lim_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} \text{mod}(f_{s+ti}(\Gamma^\delta_B)) \leq \lim_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} K(f_{s+ti}) \text{mod}(\Gamma^\delta_B) = \text{mod}(\Gamma^\delta_B). \]

If all points on \( S^1 \) are on a finite \( \varphi \)-distance from a point in \( \mathbb{D} \), then as \( \delta \to 0 \), \( \Gamma^\delta_B \) converges to the set \( |\nu_B| \) of horizontal trajectories of \( \varphi \) that have one endpoint in \( [a, b] \) and the other endpoint in \( [c, d] \) (See [6]). Here, a sequence of rectifiable curves \( \gamma_n \) converges to a curve \( \gamma \) if there is a uniformly Lipschitz parametrizations of all curves by the same interval so that \( \gamma_n \) converge uniformly to \( \gamma \) as functions. A sequence \( \Gamma_n \) of families of rectifiable curves has limit \( \Gamma \) if \( \Gamma \) consists of all curves \( \gamma \) such that there is a sequence \( \gamma_n \in \Gamma_n \) with \( \gamma_n \) converges to \( \gamma \) in the above sense. By applying Keith’s theorem (See [10] and [6]) we have

\[ \text{mod}(\Gamma^\delta_B) \to \text{mod}(|\nu_B|) \]

as \( \delta \to 0 \). Keith’s theorem is stated for compact metric spaces and in the metric induced by an integrable holomorphic quadratic differentials some points of \( S^1 \) could be on infinite distance from the interior points. Note that even when some points of \( S^1 \) are on infinite \( \varphi \)-distance the formula (5) holds (See [6, proof of Theorem 1.4]).

Therefore by (4) and (3) we obtain

\[
\limsup_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} \text{mod}(f_{s+ti}(\Gamma_B)) \leq \limsup_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} \text{mod}(f_{s+ti}(\Gamma^\delta_B)) \\
+ \limsup_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} \text{mod}(f_{s+ti}(\Gamma^{\geq \delta}_B)) \\
= \text{mod}(\Gamma^\delta_B).
\]

Since the left hand side of the above inequality does not depend on \( \delta \) and the right hand side converges to \( \text{mod}(|\nu_B|) \) as \( \delta \to 0 \), we obtained

\[
\limsup_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} \text{mod}(f_{s+ti}(\Gamma_B)) \leq \text{mod}(|\nu_B|).
\]

By the use of the Beurling’s criteria (See [7]), we conclude that the metric \( \frac{dv}{(\zeta)} \) in the canonical coordinates \( z = x + yi \) of \( \varphi \) is extremal for the family of curves \( |\nu^B| \). Consequently, we get \( \text{mod}(|\nu^B|) = \mu{\varphi}(B) \) and the proof is finished. \( \square \)

We also need a converse inequality whose proof is our main contribution. When the convergence is only along the Teichm"uller geodesic this inequality follows essentially by Beurling’s criteria (See [6]) while the proof when the convergence is along an arbitrary sequence in \( D_\varphi \) is more substantial. We first prove a special case of the converse inequality in the following lemma.
Lemma 5.3. Let $R = [0, a] \times [0, b]$ and $A \subset [0, b]$ be a measurable subset of a possibly positive Lebesgue measure $m(A)$. Let $\Gamma$ be the family of curves in $R$ that connects $\{0\} \times ([0, b] \setminus A)$ and $\{a\} \times ([0, b] \setminus A)$. Then

$$\lim_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} \text{mod}(f_{s+ti}(\Gamma)) \geq \frac{b - 2m(A)}{a}.$$ 

Proof. From the large right angled triangle in Figure 2 we get

$$\tan \alpha = \frac{b}{|t|} = \frac{s}{|t|}.$$ 

Let $h$ be the Euclidean length of the orthogonal arcs to the two slanted sides of $f_{s+ti}(R)$. Then the small right-angled triangle gives

$$\sin \alpha = \frac{h}{s/|t|}.$$ 

The above two equalities give

$$h = \frac{a}{s} \frac{s}{|t|} = \frac{a}{\sqrt{s^2 + t^2}}.$$ 

Let $l$ be the length of the subarc of the slated side of $f_{s+ti}(R)$ that consists of endpoints of the arcs of length $h$ orthogonal to both slanted sides. Then we have

$$l = \sqrt{b^2 + \frac{t^2}{s^2}b^2} - \cos \alpha \frac{a}{s} = \frac{b}{\sqrt{s^2 + t^2}} - \frac{1}{\sqrt{1 + \frac{s^2}{t^2}}} \frac{a}{s}.$$
which gives

\[ l = \frac{b(s + \frac{t^2}{s}) - a}{\sqrt{s^2 + t^2}}. \]

The family of curves \( f_{s+t_i}(\Gamma) \) contains a subfamily \( \Gamma_{s+t_i}' \) of all line segments that are orthogonal at both ends to the slanted boundary sides of \( f_{s+t_i}(R) \) except the line segments that have at least one endpoint in \( f_{s+t_i}((\{0\} \times \{\alpha\}) \cup (\{a\} \times A)) \). The total Lebesgue measure of the set of endpoints of \( \Gamma_{s+t_i}' \) is at least \( l - 2m(A)\sqrt{1 + \frac{t^2}{s^2}} \). This gives

\[ \text{mod}(f_{s+t_i}(\Gamma)) \geq \text{mod}(\Gamma_{s+t_i}') \geq \frac{l - 2m(A)\sqrt{1 + \frac{t^2}{s^2}}}{h}. \]

Using the above estimates we obtain

\[
\text{mod}(f_{s+t_i}(\Gamma)) \geq \left[ \frac{b(s + \frac{t^2}{s}) - a}{\sqrt{s^2 + t^2}} - 2m(A)\sqrt{1 + \frac{t^2}{s^2}} \right] : \frac{a}{\sqrt{s^2 + t^2}} = \frac{b}{a} \frac{s + \frac{t^2}{s}}{s - \frac{|t|}{s}} - \frac{2m(A)}{a} \frac{s^2 + t^2}{s}
\]

and

\[
\liminf_{s+|t| \to \infty} \text{mod}(f_{s+t_i}(\Gamma)) \geq \liminf_{s+|t| \to \infty} \left[ \frac{b}{a} - \frac{|t|}{s(s + \frac{t^2}{s})} - \frac{2m(A)}{a} \right].
\]

Note that \( \limsup_{s+|t| \to \infty} \frac{|t|}{s(s + \frac{t^2}{s})} = \limsup_{s+|t| \to \infty} \frac{|t|}{s^2 + t^2} = 0 \) and we obtain

\[
\liminf_{s+|t| \to \infty} \frac{\text{mod}(f_{s+t_i}(\Gamma))}{s + \frac{t^2}{s}} \geq \frac{b - 2m(A)}{a}.
\]

\[ \square \]

We consider the following situation. Let \( D = \{ z = x + yi | y \in [0, s_0], h_2(y) < x < h_1(y) \} \) where \( h_1 \) is lower semicontinuous and \( h_2 \) upper semicontinuous function such that \( h_1(y) > c_1 > 0 \) and \( h_2(y) < c_2 < 0 \). Then \( D \) is a domain in \( \mathbb{C} \) and we further assume that \( D \) has finite Lebesgue area. Denote by \( \Gamma_D \) the family of curves that connects the graphs of \( h_1 \) and \( h_2 \) inside \( D \). Let \( R = [a, b] \times [0, s_0] \) be a rectangle that contains the graphs of \( h_1 \) and \( h_2 \) over \([0, s_0] - A \), where \( A \subset [0, s_0] \) is Lebesgue measurable and \( 2m(A) < s_0 \). Then we prove

**Lemma 5.4.** Under the above notation,

\[
\liminf_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} \text{mod}(f_{s+t_i}(\Gamma_D)) \geq \frac{s_0 - 2m(A)}{b - a}.
\]

**Proof.** Let \( D_{s+t_i} := f_{s+t_i}(D) \) and \( R_{s+t_i} := f_{s+t_i}(R) \). Let \( \Gamma_{s+t_i}^\perp \) be the family of curves that consists of orthogonal segments to the slanted sides of the parallelogram \( R_{s+t_i} \) such that the
endpoints of each \( \gamma \in \Gamma_{s+ti}^\perp \) do not belong to \( \{a\} \times A \cup \{b\} \times A \). We claim that each \( \gamma \in \Gamma_{s+ti}^\perp \) contains a subcurve in \( f_{s+ti}(\Gamma_D) \).

Indeed, since the graphs of \( h_1 \) and \( h_2 \) over \([0, s_0] - A\) are in \( R \) it follows that both endpoints of \( \gamma \) are outside \( D_{s+ti} \). On the other hand \( \gamma \) intersects \( D_{s+ti} \) and does not intersect the real axis or the line parallel to the real axis through point \( s_0i \). This implies that \( \gamma \) intersects the images under \( f_{s+ti} \) of both graphs of \( h_1 \) and \( h_2 \) over \([0, s_0] \). Therefore there is a subsegment \( \gamma' \) of \( \gamma \) that is in \( f_{s+ti}(\Gamma_D) \).

By the monotonicity of the module (See §3, property 3.) we have

\[
\mod(f_{s+ti}(\Gamma_D)) \geq \mod(\Gamma_{s+ti}^\perp).
\]

The family of curves \( \Gamma_{s+ti} \) is used in the proof of Lemma 5.3 to obtain the lower bound. Therefore the lower bound in Lemma 5.3 implies

\[
\liminf_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} \mod(f_{s+ti}(\Gamma_D)) \geq \frac{s_0 - 2m(A)}{b - a}.
\]

We are ready to prove the lower bound for the general case.

**Proposition 5.5.** Under the above notation we have

\[
\liminf_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} \mod(f_{s+ti}(\Gamma_B)) \geq \mu_\varphi([a, b] \times [c, d]),
\]

for all boxes of geodesics \([a, b] \times [c, d] \subset (S^1 \times S^1) - \text{diag}\).

**Proof.** Consider the set of all horizontal trajectories of \( \varphi \) that connect \([a, b] \) to \([c, d] \). They are divided into at most countably many horizontal strips \( \{S_\omega\}_{\omega=1}^\infty \) such that \( S_\omega \) and \( S_{\omega'} \) can have in common at most one of their boundary horizontal trajectories for \( \omega \neq \omega' \). Let \( \beta_\omega \) be a segment of the vertical trajectory that defines \( S_\omega \) (See Strebel [18]). Denote by \( \Gamma_\omega \) the family of curves in the strip \( S_\omega \) that connects the vertical sides of \( S_\omega \). Then \( \Gamma_\omega \subset \Gamma_B \) and by monotonicity and additivity

\[
\mod(f_{s+ti}(\Gamma_B)) \geq \sum_\omega \mod(f_{s+ti}(\Gamma_\omega)).
\]

We fix \( \omega \) and estimate \( \mod(f_{s+ti}(\Gamma_\omega)) \). Let \( D \) be the image of \( S_\omega \) in the natural parameter such that \( \beta_\omega \) is mapped onto the interval \([0, s_0] \) of the \( y \)-axis. Then there exist \( h_1, h_2 : [0, s_0] \to \mathbb{R} \cup \{-\infty, \infty\} \) such that

\[
D = \{x + yi \in \mathbb{C} \mid 0 < y < s_0, h_2(y) < x < h_1(y)\},
\]

\( h_1(y) > 0 > h_2(y) \) for all \( y \in [0, s_0] \) and that \( \liminf_{y \to y_0} h_1(y) \geq h_1(y_0) \) and \( \limsup_{y \to y_0} h_2(y) \leq h_2(y_0) \) for all \( y_0 \in [0, s_0] \). The function \( h_1 \) is lower semicontinuous and the function \( h_2 \) is upper semicontinuous. Thus \( h_1 \) has a minimum \( c_1 > 0 \) and \( h_2 \) has a maximum \( c_2 < 0 \) on \([0, s_0] \).
In particular, both \( h_1 : [0, s_0] \to [c_1, \infty] \) and \( h_2 : [0, s_0] \to (-\infty, c_2] \) are Lebesgue measurable and thus so are \( 1/h_1 : [0, s_0] \to [0, 1/c_1] \) and \( 1/h_2 : [0, s_0] \to [1/c_2, 0] \). By a corollary to Lusin’s theorem applied to \( 1/h_1 \) and \( 1/h_2 \) (See Rudin [16] page 56), there exist sequences of continuous functions \( 1/g_n : [0, s_0] \to [0, 1/c_1] \) and \( 1/f_n : [0, s_0] \to [1/c_2, 0] \) such that \( 1/g_n(y) \to 1/h_1(y) \) and \( 1/f_n(y) \to 1/h_2(y) \) as \( n \to \infty \) for a.a. \( y \in [0, s_0] \). Since \( g_n(y) - f_n(y) \geq c_1 - c_2 \) (again by Lusin’s theorem), Lebesgue Dominated Convergence Theorem implies that

\[
\int_B \frac{dy}{g_n(y) - f_n(y)} \to \int_B \frac{dy}{h_1(y) - h_2(y)}
\]

for any Lebesgue measurable set \( B \subset [0, s_0] \) as \( n \to \infty \).

We fix \( n \) and divide the interval \([0, s_0]\) into \( p \) subintervals \( I_k = [\frac{k-1}{p}s_0, \frac{k}{p}s_0) \) for \( k = 1, 2, \ldots, p \). Define \( s^1_p(y) = \max_{y \in I_k} g_n \) and \( s^2_p(y) = \min_{y \in I_k} f_n \) for \( y \in I_k \). The two step functions satisfy \( s^1_p \geq g_n \) and \( s^2_p \leq f_n \) on \([0, s_0]\). Since \( 1/(g_n - f_n) \) is bounded and continuous on \([0, s_0]\), it follows that

\[
\int_B \frac{1}{s^1_p(y) - s^2_p(y)} dy \to \int_B \frac{1}{g_n(y) - f_n(y)} dy
\]

for any Lebesgue measurable \( B \subset [0, s_0] \) as \( p \to \infty \).

Denote by \( A_n \subset [0, s_0] \) the set of all \( y \) such that either \( g_n(y) \neq h_1(y) \) or \( f_n(y) \neq h_2(y) \). The Lebesgue measure \( m(A_n) \) is going to zero as \( n \to \infty \) by Lusin’s theorem. Let \( D_p \) denote the domain between the graphs of \( s^1_p \) and \( s^2_p \). Since \( s^j_p \) for \( j = 1, 2 \) each have \( p \) steps we conclude that \( D_p \) is the union of \( p \) rectangles \( D_{p,k} \) for \( k = 1, 2, \ldots, p \) whose vertical sides are the steps of \( s^j_p \) for \( j = 1, 2 \). Let \( A_{n,k} = A_n \cap I_k \).

We consider the curve family \( (\Gamma^{p,k}_{s+t})^\perp \) that consists of all Euclidean segments orthogonal at both endpoints to the slanted sides of the parallelogram \( f_{s+t}(D_{p,k}) \) such that the endpoints do not belong \( f_{s+t}[\{(s^1_p(y), y)\} \cup \{(s^2_p(y), y)\} | y \in A_{n,k}] \). By Lemma 5.1 we get that

\[
\liminf_{s+|t| \to \infty} \frac{1}{s + \frac{r^2}{s}} \text{mod}(f_{s+t}(\Gamma_{\omega})) \geq \frac{s_0 - 2m(A_{n,k})}{s^1_p(y_k) - s^2_p(y_k)}
\]

where \( y_k \in I_k \) is arbitrary. Since \( (\Gamma^{p,k}_{s+t})^\perp \) for \( k = 1, 2, \ldots, p \) are pairwise disjoint and each curve in \( (\Gamma^{p,k}_{s+t})^\perp \) contains a curve in \( f_{s+t}(\Gamma_{\omega}) \), we get

\[
\liminf_{s+|t| \to \infty} \frac{1}{s + \frac{r^2}{s}} \text{mod}(f_{s+t}(\Gamma_{\omega})) \geq \sum_{k=1}^{p} \frac{m(I_k)}{s^1_p(y_k) - s^2_p(y_k)} - 2 \sum_{k=1}^{p} \frac{m(A_{n,k})}{s^1_p(y_k) - s^2_p(y_k)}
\]

which gives

\[
\liminf_{s+|t| \to \infty} \frac{1}{s + \frac{r^2}{s}} \text{mod}(f_{s+t}(\Gamma_{\omega})) \geq \int_{[0, s_0]} \frac{dy}{s^1_p(y) - s^2_p(y)} - 2 \int_{A_n} \frac{dy}{s^1_p(y) - s^2_p(y)}.
\]
By letting \( p \to \infty \) in the above inequality we obtain
\[
\liminf_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} \text{mod}(f_{s+ti}(\Gamma_\omega)) \geq \int_{[0,s_0]} \frac{dy}{g_n(y) - f_n(y)} + 2 \int_{A_n} \frac{dy}{g_n(y) - f_n(y)}.
\]
Since \( \int_{A_n} \frac{dy}{g_n(y) - f_n(y)} \geq \frac{m(A_n)}{c_2 - c_1} \to 0 \) as \( n \to \infty \), by letting \( n \to \infty \) the above inequality gives
\[
\liminf_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} \text{mod}(f_{s+ti}(\Gamma_\omega)) \geq \int_{[0,s_0]} \frac{dy}{h_1(y) - h_2(y)} = \mu_\varphi(\beta_\omega).
\]
The proposition follows by summing the above inequality over all \( \omega \).

Since \( \mu_\varphi \) has no atoms (See [6]), Propositions 5.2 and 5.5 imply the weak* convergence of \( D_\lambda \) to \( \mu_\varphi \) as \( \lambda \to 1 \) which is the first statement of Theorem 1.1. The injectivity of the extension of \( D_\varphi \) to the unit circle \( S^1 \) follows by [6, Theorem 2].

5.3. Proof of Theorem 1.3. If \( B = [a,b] \times [c,d] \) is a box of geodesics, in the previous section we already established that \( \lim_{s+|t| \to \infty} \frac{1}{s + \frac{t^2}{s}} \text{mod}(f_{s+ti}(\Gamma_B)) = \mu_\varphi(B) \). By Lemma 4.1 it is enough to prove that \( \lim_{\lambda \to 1} \frac{1-|\lambda|}{2} \text{mod}(\Gamma h_\lambda(B)) = \mu_\varphi(B) \) for any box of geodesics \( B = [a,b] \times [c,d] \).

We map \( \mathbb{H}_{>0} \) onto the unit disk \( \mathbb{D} \) by a Möbius map \( \lambda(\zeta) = \frac{\zeta - 1}{\zeta + 1} \) which gives \( \zeta = \frac{1 + \lambda}{1 - \lambda} \) and
\[
|1 + \lambda|^2 = \frac{1}{1 - |\lambda|^2}
\]
and since \( \frac{|1 + \lambda|^2}{1 + |\lambda|^2} \to 2 \) as \( \lambda \to 1 \) we obtain the desired conclusion.

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