The Construction and Analysis of Compact and Noncompact Schemes

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Abstract. In this paper, several families of schemes are obtained based on the idea of n\textsuperscript{th} order polynomial. Each family has different schemes with different types and orders. Also, the error terms, which determine the order of the scheme, of each scheme in all families are computed using the same methodology. Traditional finite different schemes beside backward, central, and forward compact schemes at each family are introduced. This work proposes a clear and simple method of constructing finite difference schemes, and it presents the flexibility and the properties of compact schemes. Beside the feature of the high order accuracy, which are gained by compact schemes without increasing the width of points set, compact schemes achieve better spectrum resolution compared to the traditional noncompact ones. Additionally, comparing the numerical dissipation of many schemes illustrates the favor of compact schemes when using problems with high frequency. Finally, there is an effort to solve and compare solutions of some standard problems from Computational Fluid Dynamics (CFD) using compact and noncompact schemes.

1. Introduction

Most disciplines of science and engineering rely on numerical schemes to solve differential equations that represent problems from these fields. According to the properties and the requirements of each problem, wide sets of explicit and implicit numerical schemes have been constructed. In explicit schemes, the dependent variables result from all known coefficients while they are represented by equations in implicit schemes. Hence, iteration techniques and matrices should be accompanied with the implicit methods in order to get a solution. In computational fluid dynamics (CFD), many types of schemes have been developed. One approach is constructing ENO (Essentially Non-Oscillatory) schemes, which are in the form of cell averages [1]. Although the advantages of these schemes, they are considered of low order and costly schemes for multidimensional problems. To reduce the cost and obtain conservative schemes, the flux version of ENO schemes was introduced [2]. To gain high order schemes, new approach of implicit compact schemes has been introduced [6].

Although to the large effort of computational work with programing complexity that are required by the implicit schemes, their needs are still essential because of allowing for large sizes of time-step. For example, the governing equations in CFD are non-linear with large number of unknown variables, so the implicitly approach is mostly used with iteration techniques as in [5] and [7]. In deriving schemes, various types polynomials have been used, see [3],[4], and [8]. In this work, six families of compact and non-compact schemes are derived in section 2 by using simple n\textsuperscript{th} order polynomial. Section 3 contains the numerical results of selected schemes when they are applied to some standard problems.

2. Construction method for compact and non-compact schemes

In the finite difference approach, an expression of linear combination of a given set of nodes, which represent the function values, is used to approximate the derivative of the function. Let h be the mesh size of a uniformly space such that

\[ h = x_i - x_{i-1} \quad \text{for} \quad 1 \leq i \leq N \]

\( f_i \) represents the value of \( f(x_i) \) while \( f'_i \) represents the approximation of the first derivative of \( f(x_i) \) at the node \( i \). This approximation depends on the values of the function \( f \) at the nodes located near \( i \). In
general, when approximating the first derivative \( f'_i \) at the node \( i \), the finite difference schemes can be written in the form:

\[
a_if_{i-2} + a_2f_{i-1} + a_3f_{i+1} + a_4f_{i+2} = \frac{b_1f_{i-2} + b_2f_{i-1} + b_3f_{i+1} + b_4f_{i+2}}{h}
\]

(1)

From the Taylor series expansion at each node, the relations and the values of the coefficients \( a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, \) and \( b_5 \) can be determined. When the coefficients \( a_1, a_2, a_3, \) and \( a_4 \) in the left-hand side of (1) are zeros, we have the traditional finite different schemes. If at least one of them is not zero, we get the compact finite difference schemes.

2.1. Family of schemes using \( f_{i-2}, f_{i-1}, \) and \( f_i \)

From (1), the schemes in this family have the following form:

\[
a_if_{i-2} + a_2f_{i-1} + a_3f_{i+1} + a_4f_{i+2} = \frac{b_1f_{i-2} + b_2f_{i-1} + b_3f_i}{h}
\]

(2)

From Taylor series expansion, the relations between the coefficients \( a_1, a_2, a_3, a_4, b_1, b_2, \) and \( b_3 \) can be determined as in [6]. In this paper, the idea of \( n \)th order polynomial

\[
P_n(x) = \sum_{k=0}^{n} c_kx^k
\]

(3)

is used to find these relations, and the first non-disappeared coefficients \( (c_0, c_1, c_2, \) and \( c_n) \) of this polynomial determines the order of the scheme. To construct the schemes of this family, applying (3) in (2) results in:

\[
a_1 \sum_{k=1}^{n} kc_kx_i^{k-1} + a_2 \sum_{k=1}^{n} kc_kx_i^{k-1} + a_3 \sum_{k=1}^{n} kc_kx_i^{k-1} + a_4 \sum_{k=1}^{n} kc_kx_i^{k-1} + b_2 \sum_{k=0}^{n} c_k(x_i-1)^k + b_3 \sum_{k=0}^{n} c_kx_i^k = \frac{1}{h} \left(b_2 \sum_{k=0}^{n} c_k(x_i-2)^k + b_3 \sum_{k=0}^{n} c_kx_i^k\right)
\]

Assuming that \( x_i \) is located at the origin leads to:

\[
\sum_{k=1}^{n} kc_k[a_1(-2)^{k-1} + a_2(-1)^{k-1} + a_3 + a_4(2)^{k-1}] = \frac{1}{h} \left(\sum_{k=0}^{n} c_k[b_1(-2)^k + b_2(-1)^k + b_3]\right)
\]

(4)

Matching both sides of (4) gives the schemes of this family, which are of second, third, fourth, and fifth order schemes as illustrated in TABLE 1 below.

| Scheme   | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( b_1 \) | \( b_2 \) | \( b_3 \) | Error terms                          |
|----------|----------|----------|----------|----------|----------|----------|----------|-------------------------------------|
| F1BFD2   | 0        | 0        | 0        | 0        | 1        | -2       | 3        | \( 2(c_3 - 3c_4 + 7c_5 - 15c_6) \) |
| F1BC3    | 0        | 0        | 0        | 0        | -1       | -2       | 5        | \( 2(c_4 - 4c_5 + 11c_6 - 26c_7) \) |
| F1BC4    | 1        | 4        | 0        | 0        | -3       | 0        | 3        | \( 4(c_5 - 6c_6 - 23c_7) \)       |
| F1CC4    | 0        | 17       | -1       | 14       | 0        | -3       | 12       | 27       | \( -2(10c_5 - 27c_6 + 74c_7) \)   |
| F1BC5    | 5        | 19       | 1        | -24      | 0        | -11      | 19       | \( -11c_4 + 26c_7 - 93c_8 \)     |
| F1FC5    | 0        | 413      | -23      | 456      | 5        | -1       | 29       | 251      | \( \frac{1}{38}(281c_6 - 92c_7 + 1806c_8) \) |
| F1CC6    | 281      | 257      | -13      | 147      | 11       | -165     | -60      | 405      | \( 4(170c_7 - 405c_8 + 1896c_9) \) |

TABLE 1: Family of schemes using \( f_{i-2}, f_{i-1}, \) and \( f_i \)
The schemes in TABLE 1 above are named according to family, type, and order. For example, the scheme F1BFD2 stands for Family 1 - Backward Finite Difference – order 2. Also, the scheme F1BC3 stands for Family 1 - Backward Compact – order 3. Additionally, \[ \sum_{k=r+1}^{n} c_k (-1)^k \] represents the error terms, which point out the accuracy and order of the scheme by noticing the first coefficient \( c_{r+1} \). Similarly, the schemes of the other families can be derived as follows.

2.2. Family of schemes using \( f_{i-1}, f_i, \) and \( f_{i+1} \)

\[
a_1 f_{i-2} + a_2 f_{i-1} + f_i + a_3 f_{i+1} + a_4 f_{i+2} = \frac{b_2 f_{i-1} + b_3 f_i + b_4 f_{i+1}}{h}
\]

| Scheme   | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) | Error terms                          |
|----------|--------|--------|--------|--------|--------|--------|--------|--------------------------------------|
| F2CFD2   | 0      | 1      | 0      | 0      | \( \frac{1}{2} \) | 0      | \( \frac{1}{2} \) | \( -c_3-c_5-c_7 \)                   |
| F2BC3    | 0      | \( \frac{1}{2} \) | 0      | 0      | \( -\frac{3}{4} \) | 1      | \( \frac{1}{4} \) | \( -c_4-c_5-2c_6+2c_7 \)           |
| F2FC3    | 0      | 0      | \( \frac{1}{2} \) | 0      | \( -\frac{3}{4} \) | -1     | \( \frac{3}{4} \) | \( c_4+c_5+2c_6+2c_7 \)           |
| F2CC4    | 0      | \( \frac{1}{4} \) | 1      | \( \frac{1}{4} \) | 0      | \( \frac{3}{4} \) | \( \frac{3}{4} \) | \( c_5+2c_7+3c_9 \)                |
| F2BC5    | \( -\frac{1}{57} \) | \( \frac{8}{19} \) | 10     | \( \frac{57}{57} \) | 0      | \( -\frac{1}{19} \) | \( \frac{8}{19} \) | \( \frac{11}{19} \) \( (11c_6-25c_7+78c_8) \) |
| F2FC5    | 0      | \( \frac{10}{57} \) | \( \frac{8}{19} \) | \( -\frac{1}{57} \) | \( \frac{11}{19} \) | \( -\frac{8}{19} \) | 1        | \( -\frac{4}{19} \) \( (11c_6+25c_7+78c_8) \) |
| F2CC6    | \( -\frac{1}{114} \) | \( \frac{17}{57} \) | \( \frac{17}{57} \) | \( \frac{114}{114} \) | 0      | \( -\frac{15}{19} \) | \( \frac{15}{19} \) | \( -\frac{4}{19} \) \( (25c_7+174c_9) \) |

**TABLE 2: Family of schemes using \( f_{i-1}, f_i, \) and \( f_{i+1} \)**

2.3. Family of schemes using \( f_i, f_{i+1}, \) and \( f_{i+2} \)

\[
a_1 f_{i-2} + a_2 f_{i-1} + f_i + a_3 f_{i+1} + a_4 f_{i+2} = \frac{b_3 f_i + b_4 f_{i+1} + b_5 f_{i+2}}{h}
\]

| Scheme   | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( b_3 \) | \( b_4 \) | \( b_5 \) | Error terms                          |
|----------|--------|--------|--------|--------|--------|--------|--------|--------------------------------------|
| F3FFD2   | 0      | 0      | 0      | 0      | \( -\frac{3}{2} \) | 2      | \( \frac{1}{2} \) | \( 2(c_3+3c_4+7c_5+15c_6) \)       |
| F3FC3    | 0      | 0      | 0      | 2      | \( -\frac{5}{2} \) | 2      | \( \frac{1}{2} \) | \( -2(c_4+4c_5+11c_6+26c_7) \)     |
| F3CC4    | 0      | \( -\frac{1}{14} \) | 1    | \( \frac{17}{14} \) | 0      | \( -\frac{27}{14} \) | \( \frac{12}{7} \) | \( \frac{3}{14} \) \( (10c_5+27c_6+74c_7) \) |
| F3FC4    | 0      | 0      | 4      | 1      | \( -3 \) | 0      | 3      | \( 4(c_5+6c_6+3c_7) \)            |
| F3BC5    | \( \frac{5}{228} \) | \( -\frac{23}{152} \) | \( \frac{413}{456} \) | 0      | \( \frac{251}{152} \) | \( \frac{29}{19} \) | \( \frac{1}{8} \) | \( -\frac{1}{38} \) \( (281c_6+92c_7+1806c_8) \) |
| F3FC5    | 0      | \( -\frac{1}{24} \) | \( \frac{19}{8} \) | 5      | \( -\frac{19}{8} \) | 1      | \( \frac{11}{8} \) | \( \frac{8}{11} \) \( c_6+26c_7+93c_8 \) |
| F3CC6    | 11     | \( -\frac{13}{147} \) | \( \frac{257}{147} \) | \( \frac{281}{147} \) | \( 405 \) | 60 | \( 165 \) | \( 196 \) \( (170c_7+405c_8+1896c_9) \) |
TABLE 3: Family of schemes using \( f_i, f_{i+1}, \) and \( f_{i+2} \)

2.4. Family of schemes using \( f_{i-2}, f_{i-1}, f_i, \) and \( f_{i+1} \)

\[
a_1f_{i-2} + a_2f_{i-1} + a_3f_i + a_4f_{i+1} = \frac{b_1f_{i-2} + b_2f_{i-1} + b_3f_i + b_4f_{i+1}}{h}
\]

| Scheme | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) | Error terms |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|-------------|
| F4BFD3 | 0      | 0      | 0      | 0      | \( \frac{1}{6} \) | -1     | \( \frac{1}{2} \) | \( \frac{1}{3} \) | -2(\( c_4 - 2c_5 + 5c_6 - 10c_7 \)) |
| F4BC4  | 0      | 1      | 0      | 0      | -\( \frac{1}{6} \) | -\( \frac{3}{2} \) | \( \frac{3}{2} \) | \( \frac{1}{6} \) | -2(\( c_5 - 3c_6 + 8c_7 \)) |
| F4FC4  | 0      | 0      | \( \frac{1}{3} \) | 0      | \( \frac{1}{18} \) | -\( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{17}{18} \) | 2(\( c_5 - c_6 + 4c_7 \)) |
| F4BC5  | \( \frac{1}{5} \) | 2      | 0      | 0      | -\( \frac{10}{9} \) | -1     | 2     | \( \frac{1}{5} \) | -4(\( c_5 - 5c_7 + 18c_8 \)) |
| F4CC5  | 0      | \( \frac{1}{2} \) | \( \frac{1}{6} \) | 0      | -\( \frac{1}{18} \) | -1     | \( \frac{1}{2} \) | \( \frac{5}{9} \) | 2(\( c_6 - 2c_7 + 6c_8 \)) |
| F4BC6  | \( \frac{1}{5} \) | 1      | \( \frac{1}{9} \) | 0      | -\( \frac{11}{27} \) | -1     | 1     | \( \frac{11}{27} \) | 4(\( c_7 - 4c_8 + 14c_9 \)) |
| F4FC6  | 0      | 41     | 35     | -1     | -\( \frac{11}{123} \) | \( \frac{33}{123} \) | \( \frac{33}{123} \) | \( \frac{281}{369} \) | -\( \frac{4}{41} \)(47\( c_7 - 12c_8 + 306c_9 \)) |
| F4CC7  | 47     | 23     | 19     | -1     | -25    | 10     | 25    | 170    | -\( \frac{100}{11} \)\( c_8 + 16c_9 \) |

TABLE 4: Family of schemes using \( f_{i-2}, f_{i-1}, f_i \) and \( f_{i+1} \)

2.5. Family of schemes using \( f_{i-1}, f_i, f_{i+1}, \) and \( f_{i+2} \)

\[
a_1f_{i-2} + a_2f_{i-1} + a_3f'_i + a_4f_{i+1} + a_4f'_{i+2} = \frac{b_2f_{i-1} + b_3f_i + b_4f_{i+1} + b_5f_{i+2}}{h}
\]

| Scheme | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) | \( b_5 \) | Error terms |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|-------------|
| F5FFD3 | 0      | 0      | 0      | 0      | -\( \frac{1}{3} \) | -\( \frac{1}{2} \) | 1      | -\( \frac{1}{6} \) | 2(\( c_4 + 2c_5 + 5c_6 + 10c_7 \)) |
| F5BC4  | 0      | \( \frac{1}{3} \) | 0      | 0      | -\( \frac{17}{18} \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | -\( \frac{1}{18} \) | 2(\( c_5 + c_6 + 4c_7 \)) |
| F5FC4  | 0      | 0      | 1      | 0      | -\( \frac{1}{6} \) | -\( \frac{3}{2} \) | \( \frac{3}{2} \) | \( \frac{1}{6} \) | -2(\( c_5 + 3c_6 + 8c_7 \)) |
| F5CC5  | 0      | \( \frac{1}{7} \) | \( \frac{1}{2} \) | 0      | -\( \frac{5}{9} \) | -\( \frac{1}{2} \) | 1     | \( \frac{1}{18} \) | -2(\( c_6 + 2c_7 + 6c_8 \)) |
| F5FC5  | 0      | 0      | 2      | \( \frac{1}{3} \) | -\( \frac{1}{9} \) | -2     | 1     | \( \frac{10}{9} \) | 4(\( c_6 + 5c_7 + 18c_8 \)) |
| F5BC6  | -\( \frac{1}{123} \) | 35     | 43     | 0      | 281    | 3      | 33    | \( \frac{33}{123} \) | -\( \frac{4}{41} \)(47\( c_7 - 12c_8 + 306c_9 \)) |
| F5FC6  | 0      | \( \frac{1}{9} \) | 1      | \( \frac{1}{9} \) | -\( \frac{11}{27} \) | -1     | 1     | \( \frac{11}{27} \) | 4(\( c_7 - 4c_8 + 14c_9 \)) |
| F5CC7  | -\( \frac{1}{264} \) | 19     | 23     | 47     | -\( \frac{170}{297} \) | 25     | 10    | \( \frac{100}{11} \)\( c_8 + 16c_9 \) |
TABLE 5: Family of schemes using \(f_{i-1}, f_i, f_{i+1}\) and \(f_{i+2}\)

2.6. Family of schemes using \(f_{i-2}, f_{i-1}, f_i, f_{i+1}\), and \(f_{i+2}\)

\[
a_1f'_{i-2} + a_2f'_{i-1} + f'_i + a_3f'_{i+1} + a_4f'_{i+2} = \frac{b_1f_{i-2} + b_2f_{i-1} + b_3f_i + b_4f_{i+1} + b_5f_{i+2}}{h}
\]

| Scheme | \(a_1\) | \(a_2\) | \(a_3\) | \(a_4\) | \(b_1\) | \(b_2\) | \(b_3\) | \(b_4\) | \(b_5\) | Error terms |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|-------------|
| F6CFD4 | 0      | 0      | 0      | 0      | 1/12   | -2/3   | 0      | 2      | -1/12  | \(4(c_5 + 5c_7 + 21c_9)\) |
| F6BC5  | 0      | 2/3    | 0      | 0      | -1/12  | -11/9  | 1      | 1      | -1/36  | \(4(c_6 - c_7 + 6c_8)\) |
| F6FC5  | 2/3    | 0      | 0      | 1/36   | -1/3   | -1     | 11/9   | 1      | -1/12  | \(-4(c_8 + c_9 + 6c_8)\) |
| F6BC6  | 1/6    | 4/3    | 0      | 0      | -43/72 | -10/3  | 3      | 2      | -1/72  | \(8(c_7 - 3c_8 + 12c_9)\) |
| F6CC6  | 0      | 1/3    | 1/3    | 0      | -1/36  | -7/9   | 0      | 7      | -1/36  | \(-4(c_7 + 6c_8 + 27c_{11})\) |
| F6FC6  | 0      | 3/6    | 1/3    | 1      | -1/36  | -7/9   | 0      | 7      | -1/36  | \(8(c_7 + 3c_8 + 12c_9)\) |
| F6BC7  | 1/18   | 2/5    | 2/5    | 0      | -47/216| -8/7   | 1      | 16/72  | 1      | \(-8(c_8 - 2c_9 + 10c_{10})\) |
| F6FC7  | 0      | 2/9    | 2/3    | 1/18   | -1/72  | -16/7  | -1     | 8      | 47/216 | \(8(c_8 + 2c_9 + 10c_{10})\) |
| F6CC8  | 1/36   | 4/9    | 4/9    | 1/36   | 216/27 | 20/27  | 0      | 20/27  | 216/27 | \(16(c_9 + 10c_{11} + 67c_{13})\) |

TABLE 6: Family of schemes using \(f_{i-2}, f_{i-1}, f_i, f_{i+1}\), and \(f_{i+2}\)

3. Numerical Tests over Linear Wave Equation

3.1. Comparison of Order of Accuracy

The one-dimensional linear wave equation

\[u_t + u_x = 0, \quad -1 < x < 1\]

is solved using different schemes at time \(t=1\) with initial function

\[u(x, 0) = \sin(\pi x)\]

to verify the order of accuracy of some schemes by computing and comparing errors as shown in tables (7-9). For time discretization, Runge-Kutta method of order 6 is applied, see [2].

| N  | \(L_\infty\) error | \(L_\infty\) order | \(L_\infty\) error | \(L_\infty\) order | \(L_\infty\) error | \(L_\infty\) order | \(L_\infty\) error | \(L_\infty\) order | \(L_\infty\) error | \(L_\infty\) order |
|----|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| 10 | 4.96E-03           | 3.26E-04           | 3.07E-05           | 4.04E-06           | 5.95E-07           |
| 20 | 3.21E-04           | 1.02E-05           | 4.63E-07           | 6.051              | 3.13E-08           | 7.013              | 2.19E-09           | 8.084              |
| 40 | 2.02E-05           | 3.987              | 7.17E-09           | 6.013              | 2.42E-10           | 7.017              | 8.44E-12           | 8.021              |
| 80 | 1.27E-06           | 3.997              | 9.96E-09           | 5.000              | 1.12E-10           | 6.003              | 1.88E-12           | 7.003              | 3.51E-14           | 7.911              |
| 160| 7.93E-08           | 3.11E-10           | 1.75E-12           | 5.997              | 1.54E-14           | 6.933              | 1.47E-16           | 7.894              |
Table 7. $L_\infty$ errors of the numerical solution from F6CFD4, F6BC5, F6CC6, F6FC7, and F6CC8

| N   | F6CFD4 | F6BC5 | F6CC6 | F6FC7 | F6CC8 |
|-----|--------|-------|-------|-------|-------|
|     | $L_1$ error | $L_1$ order | $L_1$ error | $L_1$ order | $L_1$ error | $L_1$ order | $L_1$ error | $L_1$ order | $L_1$ error | $L_1$ order |
| 10  | 1.70E-02 | 1.29E-03 | 1.06E-04 | 1.63E-05 | 2.05E-06 |
| 20  | 1.17E-03 | 3.862  | 4.07E-05 | 4.988  | 1.69E-06 | 5.963  | 1.24E-07 | 7.038  | 8.01E-09 | 7.996  |
| 40  | 7.76E-05 | 3.917  | 1.27E-06 | 4.997  | 2.75E-08 | 5.942  | 9.65E-10 | 7.009  | 3.24E-11 | 7.951  |
| 80  | 4.97E-06 | 3.965  | 3.98E-08 | 4.999  | 4.38E-10 | 5.972  | 7.53E-12 | 7.002  | 1.33E-13 | 7.925  |
| 160 | 3.14E-07 | 3.984  | 1.25E-09 | 5.000  | 6.92E-12 | 5.986  | 5.99E-14 | 6.974  | 5.67E-16 | 7.876  |

Table 8. $L_1$ errors of the numerical solution from F6CFD4, F6BC5, F6CC6, F6FC7, and F6CC8

| N   | F6CFD4 | F6BC5 | F6CC6 | F6FC7 | F6CC8 |
|-----|--------|-------|-------|-------|-------|
|     | $L_2$ error | $L_2$ order | $L_2$ error | $L_2$ order | $L_2$ error | $L_2$ order | $L_2$ error | $L_2$ order | $L_2$ error | $L_2$ order |
| 10  | 7.86E-03 | 5.83E-04 | 4.87E-05 | 7.45E-06 | 9.43E-07 |
| 20  | 5.40E-04 | 3.864  | 1.81E-05 | 5.008  | 7.79E-07 | 5.966  | 5.55E-08 | 7.068  | 3.69E-09 | 7.999  |
| 40  | 3.50E-05 | 3.948  | 5.65E-07 | 5.002  | 1.24E-08 | 5.974  | 4.28E-10 | 7.017  | 1.46E-11 | 7.982  |
| 80  | 2.22E-06 | 3.978  | 1.77E-08 | 5.001  | 1.96E-10 | 5.984  | 3.34E-12 | 7.004  | 6.00E-14 | 7.927  |
| 160 | 1.40E-07 | 3.990  | 5.52E-10 | 5.000  | 3.08E-12 | 5.992  | 2.79E-14 | 6.902  | 2.67E-16 | 7.809  |

Table 9. $L_2$ errors of the numerical solution from F6CFD4, F6BC5, F6CC6, F6FC7, and F6CC8

3.2. Comparison of Dissipation

To test the numerical dissipation of F6CFD4, F6BC5, F6CC6, F6FC7, and F6CC8 schemes, the linear wave equation

$$u_t + u_x = 0$$

is also used with a high frequency initial condition

$$u(x, 0) = \cos (16x)$$

where $-\pi < x < \pi$

as illustrated in figures 1 and 2 below.
Figure 1. The exact solution compared with the numerical solutions of the above equation.

Figure 2. The enlarged portion of the selected part that indicated in Figure 1.

4. Conclusion

For the first derivative, compact and noncompact approximations, which belong to the family of finite difference schemes, are constructed in this paper. Based on the \( n \)th order polynomial, the approximations are derived by solving systems such that the number of equations is equivalent to the desired order. In addition, selected schemes are applied to wave equation with different initial conditions. Comparisons of orders are made and verified for these schemes with three types of errors. Furthermore, the dissipations of the schemes are compared to confirm the ability of compact schemes in capturing and resolving high frequency regions. For future work, this method can be applied to derive high order schemes that might be used to solve several cases of the 1-D or 2-D Euler equations.

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