\textbf{α-STABLE RANDOM WALK HAS MASSIVE THORNS}

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\begin{abstract}
We introduce and study a class of random walks defined on the integer lattice \(\mathbb{Z}^d\) – a discrete space and time counterpart of the symmetric α-stable process in \(\mathbb{R}^d\). When \(0 < \alpha < 2\) any coordinate axis in \(\mathbb{Z}^d, d \geq 3\), is a non-massive set whereas any cone is massive. We provide a necessary and sufficient condition for the thorn to be a massive set.
\end{abstract}

1. Introduction

\textbf{Motivating questions.} This paper is motivated by the following two closely related questions.

1. Assuming that the probability \(\phi\) on the group \(\mathbb{Z}^d\) is symmetric and its support generates the whole \(\mathbb{Z}^d\), what is the possible decay of the Green function

\[ G(x) = \sum_{n \geq 0} \phi^{(n)}(x) \]

as \(x\) tend to infinity?

2. If \(\phi\) is as above, which sets are massive with respect to the random walk driven by \(\phi\)?

Recall that the answer to the first question is known when \(\phi\) is symmetric, has finite second moment and \(d \geq 3\). Indeed, it is proved in Spitzer [13] (see also Saloff-Coste and Hebisch [6] for the treatment of general finitely generated groups) that \(G(x) \sim c(\phi) \|x\|^{2-d}\) at infinity. When the second moment of \(\phi\) is infinite but \(\phi\) belongs to the domain of attraction of the \(\alpha\)-stable law with \(d/2 < \alpha < \min\{d, 2\}\), \(G(x) \sim c(\phi)\|x\|^{\alpha-d}\)"
at infinity, where $l$ is an appropriately chosen slowly varying function, see Williamson [14]. However, there are many symmetric probabilities $\phi$ for which the behaviour of the Green function $G$ at infinity is not known.

In the present paper we use discrete subordination, a natural technique developed in Bendikov and Saloff-Coste [2] that produces interesting examples of probabilities $\phi$ for which one can estimate the behaviour of the Green function $G$ at infinity. This in turn allows us to describe massiveness of some interesting classes of infinite sets. For instance, we give necessary and sufficient conditions for the thorn to be a massive set, see Section 4. Massiveness of thorns for the simple random walk in $\mathbb{Z}^d$, $d \geq 4$, was studied in the celebrated paper of Itô and McKean [7].

The main idea behind this technique is the well-known idea of subordination in the context of continuous time Markov semigroups but the applications we have in mind require some adjustments and variations. The results we obtain shed some light on the questions formulated above. The present paper is concerned with examples when $\phi$ has neither finite support nor finite second moment.

Subordinated random walks. In the case of continuous time Markov processes, subordination is a well-known and useful procedure of obtaining new process from an original process. The new process may differ very much from the original process, but the properties of this new process can be understood in terms of the original process. The best known application of this concept is obtaining the symmetric stable process from the Brownian motion. See e.g. Bendikov [1].

From a probabilistic point of view, a new process $(Y_t)_{t>0}$ is obtained from the original process $(X_t)_{t>0}$ by setting $Y_t = X_{\varsigma_t}$, where the "subordinator" $(\varsigma_t)_{t>0}$ is a Lévy process taking values in $(0, \infty)$ and independent of $(X_t)_{t>0}$. See e.g. Feller [5, Section X.7].

From an analytical point of view, the transition function $h_\varsigma(t, x, B)$ of the new process is obtained as a time average of the transition function $h(t, x, B)$ of the original process, that is,

$$h_\varsigma(t, x, B) = \int_0^\infty h(s, x, B) d\nu_\varsigma(s).$$
In this formula \( \nu_t(s) \) is the distribution function of the random variable \( \varsigma_t \).

Subordination was first introduced by Bochner in the context of semigroup theory. See [5, footnote, p. 347].

Ignoring technical details, the minus infinitesimal generator \( B \) of the process \( (Y_t)_{t>0} \) is a function of the minus infinitesimal generator \( A \) of the process \( (X_t)_{t>0} \), that is, \( B = \psi(A) \). See Jacob [3, Chapters 3 & 4] for a detailed discussion.

A discrete time version of subordination in which the functional calculus equation \( B = \psi(A) \) serves as the defining starting point has been considered by Bendikov and Saloff-Coste in [2]. Given a probability \( \phi \) on \( \mathbb{Z}^d \) consider the random walk \( X = \{X(n)\}_{n \geq 0} \) driven by \( \phi \). In its simplest form, discrete subordination is the consideration of a probability \( \Phi \) defined as a convex linear combination of the convolution powers \( \phi^{(n)} \). That is,

\[
\Phi = \sum_{n \geq 1} c_n \phi^{(n)},
\]

where \( c_n \geq 0 \) and \( \sum_{n \geq 1} c_n = 1 \). We easily find that

\[
\Phi^{(n)} = \sum_{k \geq n} \left( \sum_{k_1 + \ldots + k_n = k} \prod_{i=1}^{n} c_{k_i} \right) \phi^{(k)}.
\]

The probabilistic interpretation is as follows: let \( (R_i) \) be a sequence of i.i.d. integer valued random variables, which are independent of \( X \) and such that \( \mathbb{P}(R_i = k) = c_k \). Set \( \tau_n = R_1 + \ldots + R_n \), then

\[
\mathbb{P}(\tau_n = k) = \sum_{k_1 + \ldots + k_n = k} \prod_{i=1}^{n} c_{k_i}
\]

and \( \Phi^{(k)} \) is the law of \( Y(k) = X(\tau_k) \).

The other way to introduce the notion of discrete subordination is to use Markov generators. Let \( P \) be the operator of convolution by \( \phi \). The operator \( L = I - P \) may be considered as minus the Markov generator of the associated random walk. For a proper function \( \psi \) we want to define a "subordinated" random walk with Markov generator \( -\psi(L) \). The appropriate class of functions is the class of Bernstein functions, see the book Schilling, Song and Vondraček [12].

Recall that a function \( \psi: (0, \infty) \to \mathbb{R} \) is called a Bernstein function if it is non-negative and \((-1)^{n-1}\psi^{(n)}(\theta) \geq 0, n = 1, 2, \ldots\). Each Bernstein
function $\psi$ has the following representation

\begin{equation}
\psi(\theta) = a + b\theta + \int_{(0,\infty)} (1 - e^{-\theta s}) \, d\nu(s),
\end{equation}

for some constants $a, b \geq 0$ and some measure $\nu$ on $(0, \infty)$ such that

$$\int_{(0,\infty)} \min\{1, s\} \, d\nu(s) < \infty.$$

**Proposition 1.1.** [2, Proposition 2.3] Assume that $\psi$ is a Bernstein function with its representation (1.1), such that $\psi(0) = 0$, $\psi(1) = 1$ and set

\begin{equation}
c(\psi, 1) = b + \int_{(0,\infty)} te^{-t} \, d\nu(t),
\end{equation}

\begin{equation}
c(\psi, n) = \frac{1}{n!} \int_{(0,\infty)} t^n e^{-t} \, d\nu(t), \quad n > 1.
\end{equation}

Let $\phi$ be a probability on $\mathbb{Z}^d$. Let $P$ be the operator of convolution by $\phi$ and set

\begin{equation}
P_\psi = I - \psi(I - P).
\end{equation}

Then $P_\psi$ is the convolution by a probability $\Phi$ defined as

\begin{equation}
\Phi = \sum_{n \geq 1} c(\psi, n) \phi(n).
\end{equation}

**Definition 1.2.** Let $X = \{X(n)\}_{n \geq 0}$ be the random walk driven by $\phi$. The random walk with the transition operator $P_\psi$ defined at (1.3) will be called the $\psi$-subordinated random walk and will be denoted by $X_\psi = \{X_\psi(n)\}_{n \geq 0}$. When $\psi(\lambda) = \lambda^{\alpha/2}$, $0 < \alpha \leq 2$, and $X = S$ is the simple random walk in $\mathbb{Z}^d$, we call $X_\psi$ the $\alpha$-stable random walk and denote it by $S_\alpha$.

It is straightforward to show that the increments of $S_\alpha$ belong to the domain of attraction of the $\alpha$-stable law. This fact justifies the name "$\alpha$-stable random walk" given in the Definition 1.2.

**Notation.** For any two non-negative functions $f$ and $g$, $f(r) \sim g(r)$ at $a$ means that $\lim_{r \to a} f(r)/g(r) = 1$, $f(x) = O(g(x))$ if $f(x) \leq C g(x)$, for some constant $C > 0$, and $f(x) \asymp g(x)$ if $f(x) = O(g(x))$ and $g(x) = O(f(x))$.

**2. Green function asymptotic**

Let $X(n) := S(n)$ be the simple random walk in $\mathbb{Z}^d$, $d \geq 3$. In this paragraph we study asymptotic behaviour of the Green function $G_\psi$ of the $\psi$-subordinated random walk $S_\psi$. In the course of study we will use the
following technical assumption: \( \psi \) is a complete Bernstein function having some special behaviour at 0, which will be specified later.

Recall that a function \( \psi : (0, \infty) \to \mathbb{R} \) is a complete Bernstein function if there exists a Bernstein function \( g \) such that \( \psi(\lambda) = \lambda^2 Lg(\lambda) \), where \( L \) stands for the Laplace transform.

A function \( f : (0, 1] \to \mathbb{R}^+ \) varies regularly of index \( \beta \) at 0 if the following equality holds

\[
\lim_{x \to 0} \frac{f(\lambda x)}{f(x)} = \lambda^\beta, \quad \text{for all} \ \lambda > 1.
\]

When \( \beta = 0 \), we say that \( f \) varies slowly at 0. Any regularly varying function of index \( \beta \) is of the form \( f(x) = x^\beta l(x) \), where \( l \) is a slowly varying function. For instance, each of the following functions vary regularly at 0 of index \( \beta \): \( x^\beta (\log 1/x)^\delta \), \( x^\beta \exp\{\log 1/x\}^\delta \), \( 0 < \delta < 1 \), etc.

**Theorem 2.1.** [2, Theorem 2.5.] Assume that \( \psi_1 : (0, \infty) \to (0, \infty) \) is smooth, increasing and tend to 0 at 0. Assume further that the functions

\[
x \mapsto \psi_1(x) \quad \text{and} \quad x \mapsto x\psi_1'(x)
\]

vary regularly at 0. Then there exists a complete Bernstein function \( \psi \) such that \( \psi(0) = 0 \), \( \psi(1) = 1 \) and

\[
\psi \sim \gamma \psi_1, \quad \psi' \sim \gamma \psi'_1 \quad \text{at} \ 0,
\]

for some constant \( \gamma \).

**Assumption 2.2.** Let \( S_\psi \) be the subordinated random walk. We assume that \( \psi \) is a complete Bernstein function satisfying

\[
\psi(\lambda) \sim \lambda^{\alpha/2}/l(1/\lambda) \quad \text{at} \ 0,
\]

where \( l(1/\lambda) \) varies slowly and \( 0 < \alpha < 2 \).

Let \( p(k, x) \) be the \( k \)-step transition probability associated with the simple random walk \( S \) starting at 0. Let \( (R_i) \) be the sequence of i.i.d. integer valued random variables independent of \( S \) and with the law \( \mathbb{P}(R_i = k) = c(\psi, k) \), where \( c(\psi, k) \) are defined at (1.2). Let \( \tau_n = R_1 + \ldots + R_n \). Let \( p_\psi(n, x) \) be the \( n \)-step transition probability of the random walk \( S_\psi \) started at 0. By Proposition (1.1) (1.3),

\[
p_\psi(n, x) = \sum_{k \geq n} p(k, x) \mathbb{P}(\tau_n = k).
\]
Hence

\[ G_\psi(x) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} p(k, x) \mathbb{P}(\tau_n = k). \]

Define a sequence \( C(k), k \geq 1 \), as

\[ C(k) = \sum_{n=1}^{\infty} \mathbb{P}(\tau_n = k). \]

**Lemma 2.3.** The following asymptotic relation holds

\[ C(k) \sim \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} k^{\frac{\alpha}{2} - 1} \text{ as } k \to \infty. \]  

(2.2)

**Proof.** Setting \( C(x) = \sum_{k=0}^{\infty} C(k) 1_{[k,k+1)}(x) \) and \( M(x) = \int_0^x C(t)dt \). We compute the Laplace transform \( \mathcal{L}(M) \) of the function \( M \).

\[
\int_0^\infty e^{-\lambda x} dM(x) = \int_0^\infty e^{-\lambda x} C(x) dx = \sum_{k=0}^{\infty} \frac{C(k)}{\lambda} (e^{-\lambda k} - e^{-\lambda(k+1)}) \\
\sim \sum_{k=0}^{\infty} C(k) e^{-\lambda k} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} e^{-\lambda k} \mathbb{P}(\tau_n = k) \\
= \sum_{n=1}^{\infty} (\mathbb{E}(e^{-\lambda \tau_1}))^n = \frac{1}{1 - \mathbb{E}(e^{-\lambda \tau_1})} - 1.
\]

We claim that

\[ \mathbb{E}(e^{-\lambda \tau_1}) = 1 - \psi(1 - e^{-\lambda}). \]

Indeed, by Proposition (1.2), we have

\[ \mathbb{E}(e^{-\lambda \tau_1}) = \sum_{k=1}^{\infty} e^{-\lambda k} c(\psi, k) \]

Using (1.1) and the fact that \( \psi(1) = 1 \) we obtain

\[ 1 - \psi(1 - e^{-\lambda}) = 1 - b(1 - e^{-\lambda}) - \int_{(0,\infty)} (1 - e^{-t(1-e^{-\lambda})}) \, d\nu(t) \\
= 1 - \left(b + \int_{(0,\infty)} (1 - e^{-t}) \, d\nu(t)\right) + be^{-\lambda} \\
+ \int_{(0,\infty)} e^{-t} \sum_{n=1}^{\infty} \frac{t^n e^{-n\lambda}}{n!} \, d\nu(t) \\
= be^{-\lambda} + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\int_{0}^{\infty} e^{-t} t^n \, d\nu(t)\right) e^{-\lambda n} = \sum_{n=1}^{\infty} c(\psi, n) e^{-\lambda n}, \]
as desired. It follows that

\[ \mathcal{L}M(\lambda) = \frac{1}{\psi(1 - e^{-\lambda})} - 1, \]

or equivalently, thanks to Assumption (2.1),

\[ \mathcal{L}M(\lambda) \sim \lambda^{-\alpha/2} l(1/\lambda) \quad \text{at 0.} \]

By the Karamata theorem [4, Theorem 1.7.1],

\[ M(x) \sim \frac{1}{\Gamma(1 + \frac{\alpha}{2})} x^{\frac{\alpha}{2}} l(x) \quad \text{as} \quad x \to \infty. \]

Applying the Monotone Density Theorem [4, Theorem 1.7.2] we obtain

\[ C(x) \sim \frac{\alpha}{2\Gamma(1 + \frac{\alpha}{2})} x^{\frac{\alpha}{2} - 1} l(x) \quad \text{as} \quad x \to \infty. \]

The proof is finished. \( \square \)

**Theorem 2.4.** Let \( G_\psi \) be the Green function of the subordinated random walk \( S_\psi \), then

\[ G_\psi(x) \sim \frac{C_{d,\alpha}}{\|x\|^d \psi(1/\|x\|^2)} \quad \text{as} \quad \|x\| \to \infty, \]

where

\[ C_{d,\alpha} = \left( \frac{d}{2} \right)^{\alpha/2} \frac{\pi^{-d/2}}{\Gamma\left(\frac{\alpha}{2}\right)} \Gamma\left(\frac{d - \alpha}{2}\right). \]

**Proof.** Remember that \( p(k, x) \) is the \( k \)-step transition probability of the simple random walk started at 0. Since \( p(k, x) = 0 \) for \( k < \frac{\|x\|}{\sqrt{d}} \), we have

\[ G_\psi(x) = \sum_{k \geq \|x\| / \sqrt{d}} C(k) p(k, x) \]

\[ = \sum_{k \geq \|x\|^2 / d} C(k) p(k, x) + \sum_{\|x\|^2 / d \leq k \leq \frac{\|x\|^2}{2}} C(k) p(k, x), \]

where \( A > 1 \) is a constant which will be specified later. For \( n \) large enough we set

\[ q(n, x) = 2\left( \frac{d}{2\pi n} \right)^{\frac{d}{2}} e^{-d\|x\|^2 / (4n)} \quad \text{and} \quad E(n, x) = p(n, x) - q(n, x). \]

By [9] Theorem 1.2.1,\[ (2.3) \quad |E(n, x)| \leq \frac{1}{\|x\|^2} \cdot O\left(\frac{1}{n^{\frac{d}{2}}}\right) .\]
Writing $I_1$ in the form,

$$I_1 = \underbrace{\sum_{k > \frac{\|x\|^2}{A}} C(k) q(k, x)}_{I_{11}} + \underbrace{\sum_{k > \frac{\|x\|^2}{A}} C(k) E(k, x)}_{I_{12}}$$

and using (2.2) and (2.3) we obtain

$$I_{12} \leq \frac{c}{\Gamma\left(\frac{\alpha}{2}\right)} \sum_{k > \frac{\|x\|^2}{A}} k^{\alpha/2 - 1} l(k) \frac{k^{-d/2}}{\|x\|^2} \exp\left\{\frac{-d\|x\|^2}{2k}\right\}$$

$$\sim \frac{c\|x\|^{-2}}{\Gamma\left(\frac{\alpha}{2}\right)} \int_{\frac{\|x\|^2}{A}}^{\infty} t^{\alpha/2-d/2-1} l(t) \, dt \quad \text{as } x \to \infty,$$

for some constant $c > 0$. By [4, Proposition 1.5.10],

$$\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_{\frac{\|x\|^2}{A}}^{\infty} t^{\alpha/2-d/2-1} l(t) \, dt \sim \frac{d - \alpha}{2} A^{\frac{\alpha - d}{2}} \|x\|^{\alpha + d} l(\|x\|^2) \quad \text{as } x \to \infty.$$

It follows that

$$\lim_{\|x\| \to \infty} \frac{\|x\|^{d-\alpha}}{l(\|x\|^2)} I_{12} = 0.$$

Similarly, when $\|x\| \to \infty$,

$$I_{11} = \frac{2}{\Gamma\left(\frac{\alpha}{2}\right)} \frac{(d/2)^{d/2}}{\Gamma\left(\frac{\alpha}{2}\right)} \sum_{k > \frac{\|x\|^2}{A}} k^{\alpha/2 - 1} l(k) \frac{k^{-d/2}}{\|x\|^2} \exp\left\{-\frac{d\|x\|^2}{2k}\right\}$$

$$\sim \frac{2}{\Gamma\left(\frac{\alpha}{2}\right)} \frac{(d/2)^{d/2}}{\Gamma\left(\frac{\alpha}{2}\right)} \int_{\|x\|^2/A}^{\infty} t^{\alpha/2-d/2-1} \exp\left\{\frac{-d\|x\|^2}{2t}\right\} l(t) \, dt.$$

Applying [4, Proposition 4.1.2] we obtain

$$I_{11} \sim \left(\frac{d}{2}\right)^{\alpha/2} \frac{\pi^{-d/2}}{\Gamma\left(\frac{\alpha}{2}\right)} \|x\|^{\alpha-d} l(\|x\|^2) \int_{0}^{Ad/2} s^{d/2-\alpha/2-1} e^{-s} \, ds,$$

It follows that

$$\lim_{\|x\| \to \infty} \frac{\|x\|^{d-\alpha}}{l(\|x\|^2)} I_{11} = \left(\frac{d}{2}\right)^{\alpha/2} \frac{\pi^{-d/2}}{\Gamma\left(\frac{\alpha}{2}\right)} \int_{0}^{dA/2} s^{d/2-\alpha/2-1} e^{-s} \, ds := C_1(A).$$
To estimate $I_2$ we use the Gaussian upper bound from [6, Theorem 2.1],

\[ I_2 \leq \frac{c_1}{\Gamma\left(\frac{d}{2}\right)} \sum_{\|x\| \leq k \leq \frac{\|x\|^2}{\lambda}} k^{\alpha/2 - 1} l(k) k^{-d/2} \exp\left\{ -\frac{\|x\|^2}{c_2 k} \right\} \]

\[ \sim \frac{c_1}{\Gamma\left(\frac{d}{2}\right)} \int_{\|x\|}^{\frac{\|x\|^2}{\lambda}} t^{\alpha/2 - d/2 - 1} \exp\left\{ -\frac{\|x\|^2}{c_2 t} \right\} l(t) \, dt \]

\[ = \frac{c_1}{c_2^{\alpha/2 - d/2} \Gamma\left(\frac{d}{2}\right)} \|x\|^{\alpha - d} \int_{\frac{\|x\|}{c_2}}^{\frac{\|x\|^2}{\lambda c_2}} s^{d/2 - \alpha/2 - 1} e^{-s} l\left(\frac{\|x\|^2}{c_2 s}\right) \, ds \]

\[ \leq \frac{c_1}{c_2^{\alpha/2 - d/2} \Gamma\left(\frac{d}{2}\right)} \|x\|^{\alpha - d} \int_{\frac{\|x\|}{c_2}}^{\infty} s^{d/2 - \alpha/2 - 1} e^{-s} l\left(\frac{\|x\|^2}{c_2 s}\right) \, ds, \]

for some constants $c_1, c_2 > 0$. It follows that

\[ \limsup_{\|x\| \to \infty} \frac{\|x\|^{d-\alpha}}{l(\|x\|^2)} I_2 \leq \frac{c_1}{c_2^{\alpha/2 - d/2} \Gamma\left(\frac{d}{2}\right)} \int_{\frac{\|x\|}{c_2}}^{\infty} s^{d/2 - \alpha/2 - 1} e^{-s} \, ds := C_2(A). \]

All the above shows that

\[ \limsup_{\|x\| \to \infty} \frac{\|x\|^{d-\alpha}}{l(\|x\|^2)} G_\psi(x) \leq C_1(A) + C_2(A) \]

and

\[ \liminf_{\|x\| \to \infty} \frac{\|x\|^{d-\alpha}}{l(\|x\|^2)} G_\psi(x) \geq C_1(A). \]

Finally, $C_2(A)$ tend to 0 as $A \to \infty$ and

\[ \lim_{A \to \infty} C_1(A) = \left(\frac{d}{2}\right)^{\alpha/2} \frac{\pi^{-d/2}}{\Gamma\left(\frac{d}{2}\right)} \Gamma\left(\frac{d - \alpha}{2}\right). \]

The proof is finished.

\[ \square \]

**Remark 2.5.** More generally, assuming that

\[ \psi(\theta) \asymp \theta^{\alpha/2} / l(1/\theta) \quad \text{at 0}, \]

and following the same line of reasons as above we obtain

\[ G_\psi(x) \asymp \|x\|^{\alpha-d} l(\|x\|^2) \quad \text{at } \infty. \]

**Example 2.6.** Let $\psi(\lambda) \sim \lambda^{\alpha/2} \log^{-\beta}(1/\lambda)$ at 0, then

\[ G_\psi(x) \sim C_{d,\alpha} \|x\|^{\alpha-d} \log^{\beta}(\|x\|) \quad \text{at } \infty. \]
3. Massive sets

**Basic definitions.** Let $X = \{X(n)\}_{n \geq 0}$ be a transient random walk on $\mathbb{Z}^d$. Let $B$ be a proper subset of $\mathbb{Z}^d$ and $p_B$ the hitting probability of $B$. The set $B$ is called *massive* if $p_B(x) = 1$ for all $x \in \mathbb{Z}^d$ and *non-massive* otherwise.

Let $\pi_B(x)$ be the probability that the random walk $X$ starting from $x$ visits the set $B$ infinitely many times. The set $B$ is massive if and only if $\pi_B \equiv 1$; for non-massive $B$, $\pi_B$ is identically 0.

Let $G(x, y)$ be the Green function of $X$. In general, the function $p_B$ is excessive, whence it can be written in the form

$$p_B = G\phi_B + \pi_B.$$ 

When $B$ is a non-massive set, i.e. $\pi_B \equiv 0$, $p_B$ is a potential. It is called the *equilibrium potential* of $B$, respectively $\phi_B$ - the *equilibrium distribution*. When $B$ is non-massive, the *capacity* of $B$ is defined as

$$\text{Cap}(B) = \sum_{y \in B} \phi_B(y).$$

The quantity $\text{Cap}(B)$ can be also computed as

$$\text{Cap}(B) = \sup \left\{ \sum_{y \in B} \phi(y) : \phi \in \Xi_B \right\},$$

where

$$\Xi_B = \{ \phi \geq 0 : \text{supp} \phi \subset B \text{ and } G\phi \leq 1 \}.$$ 

For all of this we refer to Spitzer [13, Chapter VI].

**Test of massiveness.** Assume that the Green function $G(x)$ is of the form:

$$G(x) = \frac{a(x)}{\chi(||x||)}, \quad x \neq 0,$$

where $\chi$ is a non-decreasing function satisfying the doubling condition

$$\chi(2\theta) \leq C\chi(\theta), \quad \text{for all } \theta > 0 \text{ and some } C > 1,$$

and $1 \leq a(x) \leq 2$ uniformly in $x$.

For a set $B$ define the following sequence of sets

$$B_k = \{ x \in B : 2^k \leq ||x|| < 2^{k+1} \}, \quad k = 0, 1, \ldots.$$

**Theorem 3.1.** A set $B$ is non-massive if and only if

$$\sum_{k=0}^{\infty} \frac{\text{Cap}(B_k)}{\chi(2^k)} < \infty.$$
To prove this statement, crucial in fact in our study, we use the assumptions (3.1) and (3.2) and follow step by step the classical proof of Spitzer [13, Section 26, T1].

**Example 3.2.** Let \( S_\alpha, 0 < \alpha \leq 2 \) be the \( \alpha \)-stable random walk as defined in Section 1. Assume first that \( \alpha = 2 \), i.e. \( S_\alpha = S_2 \) is the simple random walk. The set \( B = \mathbb{Z}_+ \times \{0\} \times \{0\} \) is massive. Moreover, its proper subset \( \mathcal{P} \times \{0\} \times \{0\} \), where \( \mathcal{P} \) is the set of primes, is massive, see [7], [10].

Let \( 0 < \alpha < 2 \). We claim that the set \( B = \mathbb{Z}_+ \times \{0\} \times \{0\} \) is non-massive. To prove the statement we apply Theorem 3.1 with \( \chi(\theta) = \theta^{d-\alpha} \). Let \( \cap(B_k) \) be the cardinality of \( B_k \). Since \( \cap(B_k) \leq |B_k| \), we have

\[
\sum_{k=0}^{\infty} \frac{\cap(B_k)}{\chi(2^{k+1})} \leq \sum_{k=0}^{\infty} \frac{|B_k|}{2^{(k+1)(3-\alpha)}} \\
\leq \sum_{k=0}^{\infty} \sum_{n: (n,0,0) \in B_k} \frac{1}{n^{3-\alpha}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3-\alpha}} < \infty,
\]

as was claimed.

**Example 3.3.** Let \( B \) be the hyperplane \( \{ x \in \mathbb{Z}^d : x_1 = 0 \} \). We claim that

(i) If \( 0 < \alpha < 1 \), then \( B \) is a non-massive set with respect to \( S_\alpha \);

(ii) If \( 1 \leq \alpha \leq 2 \), then \( B \) is a massive set with respect to \( S_\alpha \).

Let \( s_\alpha(n) \) be the projection of \( S_\alpha(n) \) on the \( x_1 \)-axis. Evidently the set \( B \) is \( S_\alpha \)-massive if and only if the random walk \( \{s_\alpha(n)\} \) is recurrent. The characteristic function of the random variable \( S_\alpha(1) \) is

\[
H_\alpha(\theta) = 1 - \left(1 - \frac{1}{d} \sum_{j=1}^{d} \cos \theta_j \right)^{\alpha/2}, \quad \theta \in \mathbb{R}^d.
\]

It follows that the characteristic function \( h_\alpha(\xi) \) of \( s_\alpha(1) \) is

\[
h_\alpha(\xi) = 1 - d^{-\alpha/2}(1 - \cos \xi)^{\alpha/2}, \quad \xi \in \mathbb{R}.
\]

Let \( p(n) \) be the probability of return to 0 in \( n \) steps defined by the random walk \( \{s_\alpha(n)\} \), then taking the inverse Fourier transform we obtain

\[
p(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (h_\alpha(\xi))^{n} d\xi.
\]

It follows that

\[
\sum_{n \geq 0} p(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\xi}{1 - h_\alpha(\xi)} \asymp \int_{0}^{1} \frac{d\xi}{\xi^{\alpha}} < \infty
\]
if and only if $0 < \alpha < 1$. By the well known criterion of transience, $s_\alpha(n)$ is transient.

4. **THORNS**

For $x = (x_1, \ldots, x_{d-1}, x_d)$ we set $x' = (x_1, \ldots, x_{d-1})$ and write $x = (x', x_d)$. The thorn $\mathcal{T}$ is defined as

$$\mathcal{T} = \{(x', x_d) \in \mathbb{Z}^d : \|x'\| \leq t(x_d), x_d \geq 1\},$$

where $t(n)$ is a non-decreasing sequence of positive numbers.

We study $S_\alpha$-massiveness of $\mathcal{T}$. When $d = 3$, the thorn $\mathcal{T}$ is $S_2$-massive, because the straight line is $S_2$-massive. Thus when $\alpha = 2$ we will assume that $d \geq 4$, whereas when $0 < \alpha < 2$ we consider $d \geq 3$.

The classical case $\alpha = 2$ and $d \geq 4$ was studied in Itô and McKean [7]. Theorem 4.4 below complements the result of Itô and McKean when $0 < \alpha < 2$ and $d \geq 3$.

By $\text{Cap}_\alpha(B)$ we denote the $S_\alpha$-capacity of the set $B \subset \mathbb{Z}^d$, whereas $\widetilde{\text{Cap}}_\alpha(A)$ stands for the capacity of the set $A \subset \mathbb{R}^d$, associated with the symmetric $\alpha$-stable process.

**Proposition 4.1.** Assume that

$$\limsup_{n \to \infty} \frac{t(n)}{n} = \delta > 0,$$

then the thorn $\mathcal{T}$ is $S_\alpha$-massive for any $0 < \alpha \leq 2$ and $d \geq 3$.

**Proof.** By the assumption, $t(2^n)/2^n > \delta/2$ for infinitely many $n$. For such $n$ consider the following sets

$$\mathcal{T}_n = \mathcal{T} \cap \{x \in \mathbb{Z}^d : 2^n \leq \|x\| < 2^{n+1}\}. (4.1)$$

Let $B_n$ be the ball of radius $\delta 2^{n-1}$ centered at $(0, \ldots, 0, 3 \cdot 2^{n-1})$, see Figure 1. Since $\delta 2^{n-1} < t(2^n)$, we have $B_n \subset \mathcal{T}_n$, whence

$$\text{Cap}_\alpha(\mathcal{T}_n) \geq \text{Cap}_\alpha(B_n).$$

By the inequality (5.7), Section 5, for some $c > 0$,

$$\text{Cap}_\alpha(B_n) \geq c 2^{(n-1)(d-\alpha)}.$$

It follows that

$$\sum_{n \geq 0} \frac{\text{Cap}_\alpha(\mathcal{T}_n)}{2^{n(d-\alpha)}} = \infty.$$

By Theorem 3.1 the thorn $\mathcal{T}$ is massive. □
Remark 4.2. Using the capacity bounds given in Corollary 5.3, Section 5, and following the same line of reasons as above, we prove that the thorn $T$ as in Proposition 4.1 is $S_\psi$-massive for any $\psi$ satisfying Assumption (2.1). $S_\psi$-massiveness under the assumption (4.2) is a more delicate question which is, to our best knowledge, open at present in such a great generality.

Next we study the case

$$\limsup_{n \to \infty} \frac{t(n)}{n} = 0. \tag{4.2}$$

Our reasons are based on the criterion of massiveness given in Theorem 3.1 but require more advanced tools than those in the proof of Proposition 4.1. Define a cylinder $\mathcal{F}_L$ as follows

$$\mathcal{F}_L = \{(x', x_d) \in \mathbb{R}^d : \|x'\|^2 \leq 1, 0 < x_d \leq L\}.$$

Proposition 4.3. There exist constants $c_0, c_1 > 0$ which depend only on $d$ and $\alpha$ such that the following inequality holds

$$c_0 L \leq \widetilde{\text{Cap}}_\alpha(\mathcal{F}_L) \leq c_1 L, \quad L \geq 1.$$

Proof. Indeed, for the upper bound we write $L = k + m,$ where $k$ is an integer such that $0 < k \leq L$ and $0 \leq m < 1.$ Then, for some $c_1 > 0,$

$$\widetilde{\text{Cap}}_\alpha(\mathcal{F}_L) \leq k \widetilde{\text{Cap}}_\alpha(\mathcal{F}_1) + \widetilde{\text{Cap}}_\alpha(\mathcal{F}_m) \leq c_1 L.$$
To obtain the lower bound we define the following sets

\[ D_i = \{(x', x_d) \in \mathbb{R}^d : \|x'\|^2 \leq 1, i - 1 \leq x_d \leq i\}, \quad i \geq 1. \]

Let \( \mu_i \) be the equilibrium measure of \( D_i \), i.e. \( \mu_i(D_i) = \widetilde{\text{Cap}}_\alpha(D_i) \). We have

\[ \widetilde{G}_\alpha \mu_{i+1}(x) = \widetilde{G}_\alpha \mu_1(x - ie_d), \]

where \( \widetilde{G}_\alpha \) is the Green function associated with the symmetric \( \alpha \)-stable process in \( \mathbb{R}^d \) and \( e_d = (0, 0, \ldots, 1) \). Without loss of generality we can assume that \( L \) is an integer number. Define the following measure

\[ \sigma = \mu_1 + \ldots + \mu_L. \]

Clearly \( \sigma(\mathbb{R}^d) = L \widetilde{\text{Cap}}_\alpha(D_1) \). We claim that

\[ (4.3) \quad \widetilde{G}_\alpha \sigma \leq K < \infty. \]

Indeed, we have \( G_\alpha \mu_1(x) \leq 1 \), for all \( x \), and

\[ \lim_{\|x\| \to \infty} \|x\|^{d-\alpha} \widetilde{G}_\alpha \mu_1(x) < C, \]

for some constant \( C > 0 \). It follows that

\[ \sum_{i>0} \widetilde{G}_\alpha \mu_1(x - ie_d) \leq C \sum_{i>0} \|x - ie_d\|^{d-\alpha} \wedge 1. \]

Observe that the series above converges uniformly in \( x \) which proves the claim. The inequality (4.3) in turn implies the lower bound

\[ \widetilde{\text{Cap}}_\alpha(F_L) \geq \sigma(F_L)/K = L/K \widetilde{\text{Cap}}_\alpha(D_1). \]

The proof is finished. \( \square \)

Define the following sets

\[ \mathcal{F}^-_n = \{(x', x_d) \in \mathbb{R}^d : \|x'\|^2 \leq t(2^n)^2, \frac{4}{3} 2^n \leq x_d < \frac{3}{4} 2^{n+1}\}; \]

\[ \mathcal{F}^+_n = \{(x', x_d) \in \mathbb{R}^d : \|x'\|^2 \leq t(2^{n+1})^2, \frac{3}{4} 2^n \leq x_d < \frac{4}{3} 2^{n+1}\}; \]

\[ F^-_n = \mathcal{F}^-_n \cap \mathbb{Z}^d \quad \text{and} \quad F^+_n = \mathcal{F}^+_n \cap \mathbb{Z}^d. \]

Let \( Q(b) \) be the cube \([0,1]^d\) centered at \( b \). For any set \( B \subset \mathbb{Z}^d \), we denote by \( \widetilde{B} \) the subset of \( \mathbb{R}^d \) defined as

\[ (4.4) \quad \widetilde{B} = \bigcup_{b \in B} Q(b). \]
Theorem 4.4. Under the assumption (4.2), the thorn $T$ is $S_\alpha$-massive if and only if the series
\begin{equation}
\sum_{n>0} \left( \frac{t(2^n)}{2^n} \right)^{d-\alpha-1}
\end{equation}
diverges.

Before embarking on the proof of Theorem 4.4 we illustrate the statement by the following example. Consider the thorn $T$ with $t(n) = n / (\log(1+n))^\beta$, $\beta > 0$. Then $T$ is $S_\alpha$-massive if and only if $\beta \leq 1/(d-\alpha-1)$.

Proof. Assume that the series (4.5) is convergent. Show that the set $T$ is non-massive. For any compact set $A \subset \mathbb{R}^d$ and for any $s > 0$ the following scaling property holds
\begin{equation}
\widetilde{\operatorname{Cap}}_\alpha(sA) = s^{d-\alpha} \widetilde{\operatorname{Cap}}_\alpha(A),
\end{equation}
see e.g. Sato [11, Example 42.17]. Using Lemma 4.3, the assumption (4.2) and the equation (4.6) we have
\begin{align*}
\widetilde{\operatorname{Cap}}_\alpha(F_n^+) &= \widetilde{\operatorname{Cap}}_\alpha \left( t(2^{n+1}) \cdot F_n^+ / t(2^{n+1}) \right) \\
&= t(2^{n+1})^{d-\alpha} \cdot \widetilde{\operatorname{Cap}}_\alpha \left( F_n^+ / t(2^{n+1}) \right) \\
&\leq c_1 t(2^{n+1})^{d-\alpha} \cdot 2^{n+1} \cdot \left( \frac{4}{3} 2^{n+1} - \frac{3}{4} 2^n \right) \\
&\leq c_2 t(2^{n+1})^{d-\alpha-1} \cdot 2^{n+1},
\end{align*}
for some $c_1, c_2 > 0$. Let $T_n$ be as in (4.1). Since $T_n \subset F_n^+$, see Figure 2,
\begin{equation}
\operatorname{Cap}_\alpha(T_n) \leq \operatorname{Cap}_\alpha(F_n^+).
\end{equation}
By Theorem 5.2, Section 5,
\begin{equation}
c_3 \widetilde{\operatorname{Cap}}_\alpha(F_n^+) \leq \operatorname{Cap}_\alpha(F_n^+) \leq c_4 \widetilde{\operatorname{Cap}}_\alpha(F_n^+),
\end{equation}
for some $c_3, c_4 > 0$. Using again Lemma 4.3 we obtain
\begin{equation}
c_5 \widetilde{\operatorname{Cap}}_\alpha(F_n^+) \leq \widetilde{\operatorname{Cap}}_\alpha(F_n^+) \leq c_6 \widetilde{\operatorname{Cap}}_\alpha(F_n^+),
\end{equation}
for some $c_5, c_6 > 0$. All the above shows that
\begin{equation*}
\sum_{n>0} \frac{\operatorname{Cap}_\alpha(T_n)}{2^{n(d-\alpha)}} \leq c_7 \sum_{n>0} \left( \frac{t(2^{n+1})}{2^{n+1}} \right)^{d-\alpha-1} < \infty,
\end{equation*}
as desired.

Conversely, assume that the series (4.5) is divergent. Show that the set $T$ is massive. Applying Lemma 4.3, the assumption (4.2) and the equation
\(\text{We have }
\tilde{\text{Cap}}_{\alpha}(\mathcal{F}^{-}_n) = \text{Cap}_{\alpha}(t(2^n) \cdot \mathcal{F}^{-}_n / t(2^n))
\]
\[
= t(2^n)^{d-\alpha} \cdot \text{Cap}_{\alpha}(\mathcal{F}^{-}_n / t(2^n))
\]
\[
\geq c'_1 t(2^n)^{d-\alpha} \cdot t(2^n)^{-1} \cdot \left(\frac{3}{4}2^{n+1} - \frac{4}{3}2^n\right)
\]
\[
\geq c'_2 t(2^n)^{d-\alpha-1}2^{n+1},
\]
for some \(c'_1, c'_2 > 0\). Since \(F^{-}_n \subset \mathcal{T}_n\), see Figure 2,
\[
\text{Cap}_{\alpha}(\mathcal{T}_n) \geq \text{Cap}_{\alpha}(F^{-}_n).
\]
Similarly to (4.7) and (4.8) we get
\[c'_3 \tilde{\text{Cap}}_{\alpha}(\tilde{\mathcal{F}}^{-}_n) \leq \text{Cap}_{\alpha}(F^{-}_n) \leq c'_4 \tilde{\text{Cap}}_{\alpha}(\tilde{\mathcal{F}}^{-}_n)
\]
and
\[c'_5 \tilde{\text{Cap}}_{\alpha}(\mathcal{F}^{-}_n) \leq \tilde{\text{Cap}}_{\alpha}(\tilde{\mathcal{F}}^{-}_n) \leq c'_6 \tilde{\text{Cap}}_{\alpha}(\mathcal{F}^{-}_n),
\]
for some constants \(c'_3, c'_4, c'_5, c'_6 > 0\). Thus, at last,
\[
\sum_{n>0} \frac{\text{Cap}_{\alpha}(\mathcal{T}_n)}{2^{n(d-\alpha)}} \geq c'_7 \sum_{n>0} \left(\frac{t(2^n)}{2^n}\right)^{d-\alpha-1} = \infty,
\]
as desired. \(\square\)

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Two cylinders inscribed in and circumscribed around the thorn.}
\end{figure}
5. Two Comparisons.

Let $B_\psi$ be a Lévy process in $\mathbb{R}^d$ obtained by subordination of the Brownian motion $B$. Let $S_\psi$ be the random walk obtained by subordination of the simple random walk $S$. Let $\widetilde{G}_\psi(x)$ (resp. $G_\psi$) be the Green function of $B_\psi$ (resp. $S_\psi$). Assume that the corresponding Bernstein function $\psi$ satisfies Assumption (2.1). Since $\psi$ is a complete Bernstein function, the potential measure $U$ associated with the corresponding subordinator has a monotone density $u(t)$, see e.g. [3, Chapter V, Corollary 5.3].

Proposition 5.1. The function $\widetilde{G}_\psi(x)$ has the following asymptotic at infinity

$$\widetilde{G}_\psi(x) \sim \frac{\widetilde{C}_{d,\alpha}}{\|x\|^d \psi(1/\|x\|^2)},$$

where

$$\widetilde{C}_{d,\alpha} = \frac{\Gamma((d - \alpha)/2)}{2^{\alpha} \cdot \pi^{d/2} \cdot \Gamma(\alpha/2)}.$$

In particular, at infinity,

$$\widetilde{G}_\psi(x) \sim (2/d)^{\alpha/2} G_\psi(x).$$

Proof. Since $\mathcal{L}(U)(\lambda) = 1/\psi(\lambda)$, the Karamata theorem implies that the density function $u(t)$ satisfies

$$u(t) \sim \frac{1}{\Gamma(\alpha/2)} t^{\alpha/2 - 1} l(t) \quad \text{at } \infty.$$

Recall that, by definition,

$$\widetilde{G}_\psi(x) = \int_0^\infty (4\pi t)^{-d/2} \exp\left\{-\frac{\|x\|^2}{4t}\right\} u(t) \, dt,$$

whence, as $\|x\| \to \infty$

$$\widetilde{G}_\psi(x) = 4^{-1} \pi^{-d/2} \|x\|^{2-d} \int_0^\infty s^{d/2-2} e^{-s} u\left(\frac{\|x\|^2}{4s}\right) ds$$

$$\sim 4^{-1} \pi^{-d/2} \|x\|^{2-d} \int_0^\infty s^{d/2-2} e^{-s} u(\|x\|^2) \left(\frac{1}{4s}\right)^{\alpha/2 - 1} ds$$

$$= 2^{-\alpha} \pi^{-d/2} \|x\|^{2-d} u(\|x\|^2) \int_0^\infty s^{d/2-\alpha/2 - 1} e^{-s} ds$$

$$\sim \frac{\Gamma\left(\frac{d - \alpha}{2}\right)}{2^{\alpha} \cdot \pi^{d/2} \cdot \Gamma(\alpha/2)} \|x\|^{\alpha - d} l(\|x\|^2).$$

Combining this result with that of Theorem 2.4 we obtain the claimed comparison of Green functions $\widetilde{G}_\psi$ and $G_\psi$. □
Let \( \widehat{Cap}_\psi(A) \) be the capacity of a set \( A \subset \mathbb{R}^d \) associated with the process \( B_\psi \). Recall that by definition
\[
\widehat{Cap}_\psi(A) = \sup \{ \mu(A) : \mu \in \mathcal{K}_A \},
\]
where \( \mathcal{K}_A \) is the class of measures supported by \( A \) and such that
\[
\widehat{G}_\psi \mu(\xi) = \int_A \widehat{G}_\psi(\xi - \eta) \mu(d\eta) \leq 1, \quad \text{for all } \xi \in \mathbb{R}^d.
\]
Let \( \text{Cap}_\psi(B) \) be the capacity of a set \( B \subset \mathbb{Z}^d \) associated with the process \( S_\psi \). Similarly
\[
\text{Cap}_\psi(B) = \sup \{ \sum_{y \in B} \phi(y) : \phi \in \Xi_B \},
\]
where
\[
\Xi_B = \{ \phi \geq 0 : \text{supp } \phi \subset B \text{ and } G_\psi \phi \leq 1 \}.
\]

**Theorem 5.2.** Let \( B \) be a bounded subset of \( \mathbb{Z}^d \). Let \( \widetilde{B} \) be defined at \( \text{(4.4)} \).
There exist constants \( c_1, c_2 > 0 \), which depend only on \( d \) and \( \psi \), and such that
\[
c_1 \widehat{Cap}_\psi(\widetilde{B}) \leq \text{Cap}_\psi(B) \leq c_2 \widehat{Cap}_\psi(\widetilde{B}).
\]

**Proof.** Let \( a, b \in B \) and \( Q(a) \) be the cube \([0,1]^d \) centered at \( a \in B \). Let \( d\eta \) be the Lebesgue measure in \( \mathbb{R}^d \). By Proposition 5.1 for \( a \neq b \) we can find a constant \( c_2 > 0 \) which does not depend on \( a \) and \( b \), and such that for \( \xi \in Q(a) \) and \( \eta \in Q(b) \),
\[
\int_{Q(b)} \widehat{G}_\psi(\xi - \eta) \, d\eta \leq c_2 G_\psi(a - b).
\]
(5.1)

Let \( E \) be the equilibrium distribution of \( B \) associated with the random walk \( S_\psi \). We define a new measure
\[
d\nu(\eta) = \sum_{b \in B} E(b) 1_{Q(b)}(\eta) \, d\eta.
\]
Using (5.1) we compute the potential \( \widehat{G}_\psi \nu \)
\[
\int_{\widetilde{B}} \widehat{G}_\psi(\xi - \eta) \, d\nu(\eta) = \sum_{b \in B} \int_{Q(b)} \widehat{G}_\psi(\xi - \eta) E(b) \, d\eta
\leq c_2 \sum_{b \in B} G_\psi(a - b) E(b) = c_2 G_\psi E(a) \leq 1.
\]
Thus, the measure \( c_2^{-1} \nu \) belongs to the class \( \mathcal{K}_{\widetilde{B}} \), therefore
\[
\widehat{Cap}_\psi(\widetilde{B}) \geq \frac{1}{c_2} \nu(\widetilde{B}).
\]
(5.2)
On the other hand
\begin{equation}
\nu(\tilde{B}) = \int_{\tilde{B}} d\nu(\eta) = \sum_{b \in B} \int_{Q(b)} E(b) \, d\eta = \sum_{b \in B} E(b) = \text{Cap}_\psi(B).
\end{equation}
Combining (5.2) and (5.3) we obtain
\[
\text{Cap}_\psi(B) = \nu(\tilde{B}) \leq c_2 \tilde{\text{Cap}}_\psi(\tilde{B}).
\]
Let \(a, b \in B\) be such that \(a \neq b\). Using again Proposition 5.1, we can choose \(c_1 > 0\), which does not depend on \(a\) and \(b\), such that for \(\xi \in Q(a), \eta \in Q(b)\),
\begin{equation}
\label{eq:5.4}
c_1 G_\psi(a - b) \leq \tilde{G}_\psi(\xi - \eta).
\end{equation}
Let \(\tilde{E}\) be the equilibrium measure of \(\tilde{B}\), i.e. \(\tilde{\text{Cap}}_\psi(\tilde{B}) = \tilde{E}(\tilde{B})\). Define a distribution \(\rho\) supported by the set \(B\) as
\[
\rho(b) = \tilde{E}(Q(b)), \quad b \in B.
\]
Let
\[
p = c_1 G_\psi \rho.
\]
Using (5.4) we get
\[
p(a) \leq \sum_{b \in B} \tilde{G}_\psi(\xi - \eta) \rho(b) \leq \int_{\tilde{B}} \tilde{G}_\psi(\xi - \eta) \, d\tilde{E}(\eta) \leq 1.
\]
It follows that \(c_1 \rho \in \Xi_B\), whence
\begin{equation}
\label{eq:5.5}
\text{Cap}_\psi(B) \geq c_1 \rho(B).
\end{equation}
Computing \(\rho(B)\) we obtain
\begin{equation}
\label{eq:5.6}
\rho(B) = \sum_{b \in B} \rho(b) = \sum_{b \in B} \int_{Q(b)} d\tilde{E}(\eta) = \int_{\tilde{B}} d\tilde{E}(\eta) = \tilde{E}(\tilde{B}).
\end{equation}
From (5.5) and (5.6) we deduce that
\[
\text{Cap}_\psi(B) \geq c_1 \rho(B) = c_1 \tilde{E}(\tilde{B}) = c_1 \tilde{\text{Cap}}_\psi(\tilde{B}).
\]
The proof is finished. \(\Box\)

**Corollary 5.3.** Let \(B(0, r) \subset \mathbb{Z}^d\) be a ball of radius \(r > 0\) centered at 0. Assume that \(\psi\) satisfies (2.1), then
\[
cr^d \psi(1/r^2) \leq \text{Cap}_\psi(B(0, r)) \leq Cr^d \psi(1/r^2),
\]
for some constants \(c, C > 0\). In particular,
\begin{equation}
\label{eq:5.7}
cr^{d-\alpha} \leq \text{Cap}_\alpha(B(0, r)) \leq Cr^{d-\alpha}.
\end{equation}
To prove the statement we use the capacity bounds of balls in $\mathbb{R}^d$ given in [3, Proposition 5.55] and then apply Theorem 5.2.

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