Optimal with Respect to Accuracy Algorithms for Calculation of Multidimensional Weakly Singular Integrals and Applications to Calculation of Capacitances of Conductors of Arbitrary Shapes

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Abstract: Cubature formulas, asymptotically optimal with respect to accuracy, are derived for calculating multidimensional weakly singular integrals. They are used for developing a universal code for calculating capacitances of conductors of arbitrary shapes.

1. Introduction

Optimal with respect to accuracy methods for calculating singular integrals are being actively developed presently. They represent an important field of computational mathematics. Asymptotically optimal and optimal with respect to order (to accuracy and to complexity) algorithms for calculating singular integrals on closed and open contours, and multidimensional singular integrals have been constructed in [1-3] on Hölder and Sobolev classes of functions.

In constructing optimal with respect to accuracy methods for calculating one-dimensional, bisingular and multidimensional singular integrals, a general method, proposed in monograph [1], was used. This method can be applied not only to singular integrals but also to other integrals with moving singularities.

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This method allows one to construct several asymptotically optimal and optimal with respect to order and to accuracy algorithms for calculating hypersingular integrals [4], the Poisson and Cauchy type integrals [5], and multidimensional Cauchy type integrals [6].

Although multidimensional weakly singular integrals are used in many applications, optimal methods for calculating these integrals are not developed.

An exception is the book [1], where asymptotically optimal with respect to accuracy methods for calculating integrals of the form

$$\int_0^{2\pi} \int_0^{2\pi} f(\sigma_1, \sigma_2) \left| \operatorname{ctg} \frac{\sigma_1 - s_1}{2} \right|^{\gamma_1} \left| \operatorname{ctg} \frac{\sigma_2 - s_2}{2} \right|^{\gamma_2} d\sigma_1 d\sigma_2,$$

are constructed on H"older and Sobolev classes.

Thus, the development of optimal methods for calculating multidimensional weakly singular integrals is an actual problem. Construction of efficient cubature formulas for calculating weakly singular integrals for calculating capacitances of conductors of arbitrary shapes by iterative methods proposed in [7] is very important in many applications, for example, in wave scattering by small bodies of arbitrary shapes and in antenna theory. A bibliography on methods for calculating capacitances and polarizability tensors is contained in [7].

In this paper the method proposed in [1] is generalized to multidimensional weakly singular integrals. As a result the analogs of the basic results for singular integrals, obtained earlier, are obtained for weakly singular integrals. Moreover, we study the applications of optimal with respect to order cubature formulas for calculating weakly singular integrals on Lyapunov surfaces. Our results are used for constructing an universal code for calculating capacitances and polarizability tensors of bodies of arbitrary shapes.

This paper consists of two parts.

In the first part of the paper optimal methods for calculating integrals of the types:

$$Kf \equiv \int_0^{2\pi} \int_0^{2\pi} \frac{f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2}{\left( \sin^2 \left( \frac{\sigma_1 - s_1}{2} \right) + \sin^2 \left( \frac{\sigma_2 - s_2}{2} \right) \right)^\lambda}, \quad 0 \leq s_1, s_2 \leq 2\pi; \quad (1.1)$$

and

$$Tf \equiv \int_{-1}^{1} \int_{-1}^{1} \frac{f(\tau_1, \tau_2) d\tau_1 d\tau_2}{\left( (\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 \right)^\lambda}, \quad -1 \leq t_1, t_2 \leq 1, \quad 0 < \lambda < 1, \quad (1.2)$$

are constructed on H"older and Sobolev classes of functions.

Our results for integrals (1.1) can be generalized to the integrals with other periodic kernels and functions. The development of cubature formulas for integrals (1.1) is of considerable interest because the results are applicable to integrals with weakly singular kernels defined on closed Lyapunov surfaces.
It will be clear from our arguments, that the results can be generalized to \(l\)-dimensional integrals, \(l = 3, 4, \ldots\).

The second part of this paper is deals with the iterative methods for calculating capacitances of conductors of arbitrary shapes. A general numerical method for calculating these capacitances is developed, and the results of numerical tests are given.

2. Definitions of optimality.

Various definitions of optimality of numerical methods and a detailed bibliography can be found in [8,9,10]. Let us recall the definitions of algorithms, optimal with respect to accuracy, for calculating weakly singular integrals.

Consider the quadrature rule

\[
Tf = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \sum_{p_1}^{p_1} \sum_{p_2}^{p_2} p_{k_1 k_2 l_1 l_2} (t_1, t_2) f^{(l_1 l_2)} (x_{k_1}, y_{k_2}) +
\]

\[
+ R_{n_1 n_2} (f; p_{k_1 k_2 l_1 l_2}; x_{k_1}, y_{k_2}; t_1, t_2),
\]

where coefficients \(p_{k_1 k_2 l_1 l_2}(t_1, t_2)\) and nodes \((x_{k_1}, y_{k_2})\) are arbitrary. Here

\[
f^{(l_1 l_2)} (s_1, s_2) = \partial^{l_1 + l_2} f (s_1, s_2) / \partial s_1^{l_1} \partial s_2^{l_2}.
\]

The error of quadrature rule (2.1) on the class \(\Psi\) is defined as

\[
R_{n_1 n_2} (f; p_{k_1 k_2 l_1 l_2}; x_{k_1}, y_{k_2}) = \sup_{(t_1, t_2) \in [-1, 1]^2} |R_{n_1 n_2} (f; p_{k_1 k_2 l_1 l_2}; x_{k_1}, y_{k_2}; t_1, t_2)|.
\]

The error of quadrature rule (2.1) on the class \(\Psi\) is defined as

\[
R_{n_1 n_2} (\Psi; p_{k_1 k_2 l_1 l_2}; x_{k_1}, y_{k_2}) = \sup_{f \in \Psi} R_{n_1 n_2} (f; p_{k_1 k_2 l_1 l_2}; x_{k_1}, y_{k_2}).
\]

Define a functional

\[
\zeta_{n_1 n_2} (\Psi) = \inf_{p_{k_1 k_2 l_1 l_2}} R_{n_1 n_2} (\Psi; p_{k_1 k_2 l_1 l_2}; x_{k_1}, y_{k_2}).
\]

The quadrature rule with the coefficients \(p_{k_1 k_2 l_1 l_2}^{*}\) and the nodes \((x_{k_1}^{*}, y_{k_2}^{*})\) is optimal, asymptotically optimal, optimal with respect to order on the class \(\Psi\) among all quadrature rules of type (2.1) provided that:

\[
\frac{R_{n_1 n_2} (\Psi; p_{k_1 k_2 l_1 l_2}^{*}; x_{k_1}^{*}, y_{k_2}^{*})}{\zeta_{n_1 n_2} (\Psi)} = 1, \sim 1, \asymp 1.
\]

The symbol \(\alpha \asymp \beta\) means \(A \alpha \leq \beta \leq B \alpha\), where \(0 < A, B < \infty\).

Consider the quadrature rule

\[
Tf = \sum_{k=1}^{n} p_k (t_1, t_2) f (M_k) +
\]

\[
+ R_n (f; p_k; M_k; t_1, t_2),
\]

where coefficients \(p_k(t_1, t_2)\) and nodes \((M_k)\) are arbitrary.
The error of quadrature rule (2.2) is defined as

\[ R_n(f; p_k; M_k) = \sup_{(t_1, t_2) \in [-1, 1]^2} |R_n(f; p_k; M_k; t_1, t_2)|. \]

The error of quadrature rule (2.2) on the class \( \Psi \) is defined as

\[ R_n(\Psi; p_k; M_k) = \sup_{f \in \Psi} R_n(f, p_k; M_k). \]

Define a functional

\[ \zeta_n(\Psi) = \inf_{p_k; M_k} R_n(\Psi; p_k; M_k). \]

The quadrature rule with the coefficients \( p_k^* \) and the nodes \( (M_k^*) \) is optimal, asymptotically optimal, optimal with respect to order on the class \( \Psi \) among all quadrature rules of type (2.2) provided that:

\[ \frac{R_n(\Psi; p_k^*; M_k^*)}{\zeta_n(\Psi)} = 1, \sim 1, \asymp 1. \]

By \( R_{n1,n2}(\Psi) \) the error of optimal cubature formulas on the class \( \Psi \) is defined. It is obvious that \( R_{n1,n2}(\Psi) = \zeta_{n1,n2}(\Psi) \).

3. Classes of functions

In this section, we list several classes of functions which are used below. Some definitions are from [11,12].

A function \( f \) is defined on \( A=[a,b] \) or \( A=K \), where \( K \) is a unit circle, satisfies the Hölder conditions with constant \( M \) and exponent \( \alpha \), or belongs to the class \( H_\alpha(M), M > 0, 0 \leq \alpha \leq 1, \) if \( |f(x') - f(x'')| \leq M|x' - x''|^\alpha \) for any \( x', x'' \in A \).

Class \( W_\alpha(M) \), where \( \omega(h) \) is a modulus of continuity, consists of all functions \( f \in C(A) \) with the property \( |f(x_1) - f(x_2)| \leq M\omega(|x_1 - x_2|), x_1, x_2 \in A \).

Class \( W^r(M) \) consists of functions \( f \in C(A) \) which have continuous derivatives \( f', f'', \ldots, f^{(r-1)} \) on \( A \), a piecewise derivative \( f^{(r)} \) on \( A \) satisfying \( \max_{x \in [a,b]} |f^{(r)}(x)| \leq M \).

Class \( W_p^r(M) \), \( r = 1, 2, \ldots, 1 \leq p \leq \infty \), consists of functions \( f(t) \), defined on a segment \([a,b]\) or one \( A = K \), that have continuous derivatives \( f', f'', \ldots, f^{(r-1)} \), and an integrable derivative \( f^{(r)} \) such that

\[ \left[ \int_A |f^{(r)}(x)|^p dx \right]^{1/p} \leq M. \]

Class \( W^r_\alpha(M) \), \( r = 1, 2, \ldots, 0 < \alpha \leq 1 \), consists of functions \( f(t) \), defined on a segment \([a,b]\) or one \( A = K \), that have continuous derivatives \( f', f'', \ldots, f^{(r)} \), such that

\[ |f^{(r)}(x_1) - f^{(r)}(x_2)| \leq M|x_1 - x_2|^{\alpha}. \]

A function \( f(x_1, x_2, \ldots, x_l), l = 2, 3, \ldots, \) defined on \( A = [a_1, b_2; a_2, b_2; \ldots; a_l, b_l] \) or \( A = K_1 \times K_2 \times \cdots \times K_l \), where \( K_i = 1, 2, \ldots, l, \) are unit circles, satisfying Hölder conditions...
with constant $M$ and exponent $\alpha_i, i = 1, 2, \ldots, l$, or belongs to the class $H_{\alpha_1, \ldots, \alpha_l}(M), M > 0, 0 \leq \alpha \leq 1, i = 1, 2, \ldots, l$, if

$$|f(x_1, x_2, \ldots, x_l) - f(y_1, y_2, \ldots, y_l)| \leq M(|x_1 - y_1|^{\alpha_1} + \cdots + |x_l - y_l|^{\alpha_l}).$$

Let $\omega, \omega_i$, where $i = 1, 2, \ldots, l, l = 1, 2, \ldots,$ be moduli of continuity.

Class $H_{\omega_1, \ldots, \omega_l}(M)$, consists of all functions $f \in C(A), A = [a_1, b_2; a_2, b_2; \ldots; a_l, b_l]$ or $A = K_1 \times K_2 \times \cdots \times K_l$, with the property

$$|f(x_1, x_2, \ldots, x_l) - f(y_1, y_2, \ldots, y_l)| \leq M(\omega_1(|x_1 - y_1|) + \cdots + \omega_l(|x_l - y_l|)).$$

Let $H^j_\omega(A), j = 1, 2, 3, A = [a_1, b_2; a_2, b_2; \ldots; a_l, b_l]$ or $A = K_1 \times K_2 \times \cdots \times K_l, l = 2, 3, \ldots,$ be the class of functions $f(x_1, x_2, \ldots, x_l)$ defined on $A$ and such that

$$|f(x) - f(y)| \leq \omega(x, y), j = 1, 2, 3,$$

where $x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l), \rho_1(x, y) = \max_{1 \leq i \leq l}(|x_i - y_i|), \rho_2(x, y) = \sum_{i=1}^l |x_i - y_i|, \rho_3(x, y) = \left(\sum_{i=1}^l |x_i - y_i|^2\right)^{1/2}$.

Let $H^\alpha_\omega(A), j = 1, 2, 3, A = [a_1, b_2; a_2, b_2; \ldots; a_l, b_l]$ or $A = K_1 \times K_2 \times \cdots \times K_l, l = 2, 3, \ldots,$ be the class of functions $f(x_1, x_2, \ldots, x_l)$ defined on $A$ and such that

$$|f(x) - f(y)| \leq (\rho_j(x, y))^{\alpha}, j = 1, 2, 3.$$

More general is the class $H^\alpha_\rho_j(A), j = 1, 2, 3$. It consists of all functions $f(x)$ which can be represented as $f(x) = \rho(x)g(x)$, where $g(x) \in H^\alpha_j(A), j = 1, 2, 3$, and $\rho(x)$ is a nonnegative weight function.

Let $Z^\alpha_j(A), j = 1, 2, 3$, be the class of functions $f(x_1, x_2, \ldots, x_l)$ defined on $A$ and satisfying

$$|f(x) + f(y) - 2f((x + y)/2)| \leq \omega(x, y)/2, j = 1, 2, 3.$$

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$$|f(x) + f(y) - 2f((x + y)/2)| \leq (\rho_j(x, y)/2)^{\alpha}, j = 1, 2, 3.$$

Class $Z^\alpha_\rho_j(A), j = 1, 2, 3$, consists of all functions $f(x)$ which can be represented as $f(x) = \rho(x)g(x)$, where $g(x) \in Z^\alpha_j(A), j = 1, 2, 3$, and $\rho(x)$ is a nonnegative weight function.

Let $W^{r_1, \ldots, r_l}(M), l = 1, 2, \ldots$, be the class of functions $f(x_1, x_2, \ldots, x_l)$ defined on a domain $A$, which have continuous partial derivatives

$$\partial^{|v|}f(x_1, \ldots, x_l)/\partial x_1^{r_1} \cdots \partial x_l^{r_l}, 0 \leq |v| \leq r - 1, |v| = v_1 + \cdots + v_l, r_i \geq v_i \geq 0, i = 1, 2, \ldots, l, r = r_1 + \cdots + r_l$$
and all piece-continuous derivatives of order $r$, satisfying $\|\partial^r f(x_1, \ldots, x_l)/\partial x_1^{r_1} \cdots \partial x_l^{r_l}\|C \leq M$ and $\|\partial^i f(0, \ldots, 0, x_i, 0, \ldots, 0)/\partial x_i^{r_i}\|C \leq M, \ i = 1, \ldots, l$.  

5
Let $W_p^{r_1,\ldots,r_l}(M), l = 1, 2, \ldots, 1 \leq p \leq \infty$ be the class of functions $f(x_1, x_2, \ldots, x_l)$, defined on a domain $A = [a_1, b_1; \ldots a_l, b_l]$, with continuous partial derivatives

$$
\partial^{|v|} f(x_1, \ldots, x_l)/\partial x_1^{v_1} \ldots \partial x_l^{v_l}, 0 \leq |v| \leq r - 1, |v| = v_1 + \cdots + v_l, r_i \geq v_i \geq 0, i = 1, 2, \ldots, l, r = r_1 + \cdots + r_l, \text{ and all derivatives of order } r, \text{ satisfying}
$$

$$
\|\partial^r f(x_1, \ldots, x_l)/\partial x_1^{v_1} \partial x_2^{v_2} \ldots \partial x_l^{v_l}\|_{L_p(A)} \leq M, \\
\|\partial^{r_1+v_2+\cdots+v_l} f(x_1, 0, \ldots, 0)/\partial x_1^{v_1} \partial x_2^{v_2} \ldots \partial x_l^{v_l}\|_{L_p([a_1, b_1])} \leq M, |v_2| + \cdots + |v_l| \leq r - r_1 - 1;
$$

and so on.

Let be $A = [a_1, b_2; a_2, b_2; \ldots; a_l, b_l]$ or $A = K_1 \times K_2 \times \ldots \times K_l$. Let $C^r(M)$ be the class of functions $f(x_1, x_2, \ldots, x_l)$ which are defined in $A$ and which have continuous partial derivatives of order $r$. Partial derivatives of order $r$ satisfy the conditions

$$
\|\partial^{|v|} f(x_1, \ldots, x_l)/\partial x_1^{v_1} \ldots \partial x_l^{v_l}\|_{C} \leq M
$$

for any $v = (v_1, \ldots, v_l)$, where $v_i \geq 0, i = 1, 2, \ldots, l$ are integer and $\sum_{i=1}^l v_i = r$.

By $\Psi$ we denote the set of periodic functions of the class $\Psi$.

It is known [13] that Lyapunov spheres are defined as regions bounded by a finite number of closed surfaces satisfying the three Lyapunov conditions:

1. At each point of the surface a tangent plane (and, therefore, a normal) exist.

2. If $\Theta$ is the angle between the normals at the points $m_1$ and $m_2$, and $r$ is the distance between these points, then

$$
\Theta < Ar^\lambda, \quad 0 < \lambda \leq 1,
$$

where $A$ and $\lambda$ are positive numbers which do not depend on $m_1$ and $m_2$.

3. For all points of the surface, a number $d > 0$ exists such that there is exactly one point at which a straight line, parallel to the normal at the surface point $m$, intersects the surface inside a sphere of radius $d$ centered at $m$.

Let $S$ be a Lyapunov sphere, and $N$ be the external normal to this sphere. We introduce a local system of Cartesian coordinates $(\chi, \eta, \zeta)$, whose origin is located at an arbitrary point $m_0$ of $S$, the $\zeta$ axis is directed along the normal $N_0$ at the point $m_0$, and the $\chi$ and $\eta$ axes lie in the tangential plane. Within a sufficiently small neighborhood of $m_0$, the equation of the surface $S$ in the local coordinates $(\chi, \eta, \zeta)$ has the form

$$
\zeta = F(\chi, \eta).
$$

**Definition 4.1.** [13] The surface $S$ belongs to the class $L_k(B, \alpha)$ if $F(\chi, \eta) \in W_k^{\alpha}(B)$, and the constants $B$ and $\alpha$ do not depend on the choice of the point $m_0$. 

6
4. Auxiliary statements.

We need the following known facts from the theory of quadrature and cubature formulas. These facts can be found, for example, in [11],[12],[14], [15].

**Lemma 4.1.** Let $\Psi_1$ be the class of functions $W_p^r(1), 1, 2, \ldots, 1 \leq p \leq \infty, 0 \leq t \leq 1, f(t) \in \Psi_1$, and the quadrature rule

$$\int_0^1 f(t)dt = \sum_{k=1}^{n} p_k f(t_k) + R_n(f)$$

is exact on all the polynomials of order up to $p - 1$, and has error $R_n(\Psi_1)$ on the class $\Psi_1$. Let $\Psi_2$ be the class of functions $W_p^r(1), r = 1, 2, \ldots, 1 \leq p \leq \infty, a \leq x \leq b$, and $g(x) \in W_p^r(1)$. Then the quadrature rule

$$\int_a^b g(x)dx = \sum_{k=1}^{n} p_k g(a + (b - a)t_k) + R_n(g)$$

has error $R_n(\Psi_2)$ on the class of functions $\Psi_2$ and

$$R_n(\Psi_2) = (b-a)^{r+1-1/p}R_n(\Psi_1).$$

**Theorem 4.1.** [11] Among quadrature formulas

$$\int_0^1 f(x)dx = \sum_{k=1}^{m} \sum_{l=0}^{\rho} p_{kl} f^{(l)}(x_k) + R(f) \equiv L(f) + R(f)$$

the best formula for the class $W_p^r(1)$ ($1 \leq p \leq \infty$) with $\rho = r - 1$ and $r = 1, 2, \ldots$, or $\rho = r - 2$ and $r = 2, 4, 6, \ldots$, is the unique formula defined by the following nodes $x^*_k$ and coefficients $p^*_kl$:

$$x^*_k = h(2(k-1) + [R_{rq}(1)]^{1/r}), \quad k = 1, 2, \ldots, m,$$

$$p^*_kl = (-1)^{l} p^*_{ml} = h^{-l+1} \left\{ \frac{(-1)^{l}}{(l+1)!} [R_{rq}(1)]^{(l+1)/r} - \frac{1}{r!} R_{rq}^{(r-1-1)}(1) \right\},$$

$$(l = 0, 1, \ldots, \rho), \quad p^*_{k,2v} = \frac{2h^{2v+1}}{r!} R_{rq}^{(r-2v-1)}(1), \quad \left( k = 2, 3, \ldots, m - 1; \quad v = 0, 1, \ldots, \left[ \frac{r-1}{2} \right] \right),$$

$$p^*_{k,2v+1} = 0 \left( k = 2, 3, \ldots, m - 1; \quad v = 0, 1, \ldots, \left[ \frac{r-2}{2} \right] \right), \quad h = 2^{-1}(m-1+[R_{rq}(1)]^{1/r})^{-1},$$

and $R_{rq}(t)$ is the Chebyshev polynomial $t^r + \sum_{i=0}^{r-1} \beta_i t^i$, deviating least from zero in the norm $L_q(-1,1)$, where $p^{-1} + q^{-1} = 1$. Here

$$\zeta_n[W_p^r(1)] = R_n[W_p^r(1)] = \frac{R_{rq}(1)}{2^r r! \sqrt{rq} \sqrt{(m-1+[R_{rq}(1)]^{1/r})}},$$

Let a function $f(x, y)$ be given on a rectangle $D = [a, b; c, d]$. Consider the cubature formula

$$\iint_{D} f(x,y) dxdy = \sum_{k=1}^{m} \sum_{i=1}^{n} p_{ki} f(x_k, y_i) + R_{mn}(f), \quad (4.1)$$
defined by a vector \((X, Y, P)\) of a nodes \(a \leq x_1 < x_2 < \cdots < x_m \leq b, \ c \leq y_1 < y_2 < \cdots < y_n \leq d, \) and coefficients \(p_{ki}\).

**Theorem 4.2** [11]. Among all quadrature formulas of the form of (4.1) the formula

\[
\iint_D f(x,y)dx\,dy = 4hxq\sum_{k=1}^m\sum_{i=1}^n f(a + (2k-1)h, c + (2i-1)q) + R_{mn}(f),
\]

where \(h = \frac{b-a}{2m}, \ q = \frac{d-c}{2n} \) is optimal on the classes \(H_{\omega_1,\omega_2}(D)\) and \(H_3^\alpha(D)\). In addition

\[
R_{mn}[H_{\omega_1,\omega_2}(D)] = 4mn[q \int_0^h \omega_1(t)dt + q \int_0^q \omega_2(t)dt];
\]

\[
R_{mn}[H_3^\alpha(D)] = 4mn \int_0^q \int_0^h \omega(\sqrt{t^2 + \tau^2})dtd\tau.
\]

Consider the cubature formulas of the form:

\[
\iint_D p(x,y)f(x,y)dx\,dy = \sum_{k=1}^N p_k f(M_k) + R(f), \quad (4.2)
\]

where \(p(x,y)\) is a nonnegative and bounded on \(D\) function, \(p_k, M_k(M_k \in D)\) are coefficients and nodes.

**Theorem 4.3** [11]. Let \(p(x,y)\) be a nonnegative bounded weight function. If \(R_N[H_{p,j}^\alpha(D)]\) and \(R_N[Z_{p,j}^\alpha(D)]\), where \(j = 1, 2, 3,\) and \(0 < \alpha \leq 1,\) are the errors of optimal formulas as (4.2) on the classes \(H_{p,j}^\alpha(D)\) and \(Z_{p,j}^\alpha(D),\) respectively, then

\[
\lim_{N \to \infty} N^{\alpha/2} R_N[H_{p,j}^\alpha(D)] = \lim_{N \to \infty} N^{\alpha/2} R_N[Z_{p,j}^\alpha(D)] = D_j \quad \left[ \int_D (p(x,y))^{2/(2+\alpha)}dx\,dy \right]^{(2+\alpha)/\alpha}, \quad j = 1, 2, 3,
\]

where \(D_1 = \frac{12}{2+\alpha} \left( \frac{2+\alpha}{2\sqrt{3}} \right)^{(2+\alpha)/\alpha} \frac{\pi^6}{6} \int_0^{\pi/2} \frac{dx}{\cos^2 x}, \ D_2 = 2^{1-\alpha}/(2 + \alpha),\) and \(D_3 = 2^{1-\alpha/2}/(2 + \alpha).\)

If \(j = 2,\) then the conclusion holds for \(n\)-dimensional cubature formulas.

**Remark.** Theorem 4.2 is generalized to the case of unbounded weights \(p(x,y)\) in [2].

In this paper we will use the result of S.A. Smolyak (see [16]):

**Lemma 4.4** . Let \(H\) be a linear metric space, \(F\) be a convex centrally symmetric set with center of symmetry \(\theta\) at the origin, and \(L(f), l_1(f), \ldots, l_N(f),\) be some linear functionals. Let \(S(l_1(f), \ldots, l_N(f))\) be some method for calculating the functional \(L(f)\) using functionals \((l_1(f), \ldots, l_N(f)),\) and \(S\) be the set of all such methods. Then the numbers \(D_1, \ldots, D_N\) exist such that

\[
\sup_{f \in F} |L(f) - \sum_{k=1}^N D_k l_k(f)| = \inf_S \sup_{f \in F} |L(f) - S(l_1(f), \ldots, l_N(f))|, \quad (4.3)
\]
This means that among the best methods of calculating functional \( L(f) \)
\[
L(f) \approx S(l_1(f), \ldots, l_N(f))
\] (4.4)
there is a linear method.

**Proof.** Let us associate with each \( f \in F \) a point \((L(f), l_1(f), \ldots, l_N(f))\). Let \( Y \) be a set of all such points \((y_0, \ldots, y_N)\) for \( f \in F \).

From our assumptions, it follows that \( Y \) is a closed centrally symmetric set with the center of symmetry at the origin.

Let 
\[
D_0 = \sup_{(y_0, 0, \ldots, 0) \in Y} y_0,
\]
and consider the case \( D_0 \neq 0 \) and \( D_0 \neq \infty \).

Draw the support plane for the set \( Y \) through the point \((D_0, 0, \ldots, 0)\):
\[
C_0(y_0 - D_0) + \sum_{j=1}^{N} C_j y_j = 0.
\]

One can choose a support plane for which \( C_0 \neq 0 \). Since \( Y \) is centrally symmetric with respect to the origin, the plane
\[
C_0(y_0 + D_0) + \sum_{j=1}^{N} C_j y_j = 0
\]
is also a support plane for \( Y \), and \( Y \) lies between these two planes.

Hence, we have for the points of \( Y \) the inequality:
\[
|y_0 - \sum_{j=1}^{N} D_j y_i| \leq D_0, \quad D_j = -C_j/C_0.
\]

The definition of \( y_i \) implies
\[
\sup_{f \in F} |L(f) - \sum_{j=1}^{N} D_j l_j(f)| \leq D_0. \quad (4.5)
\]

Let \( f_0 \) be an element \( F \) corresponding the point \((D_0, 0, \ldots, 0)\). Then \( S(l_1(\pm f_0), \ldots, l_N(\pm f_0)) = S(0, \ldots, 0) \), and \( L(f_0) - L(-f_0) = 2D_0 \). Therefore either for \( f_0 \) or for \(-f_0\) the error in (4.3) is not less than \( D_0 \). This and (4.4) imply that the right-hand side in (4.3) is not less than the left-hand one. But the right-hand side of (4.3) can not be less than the left-hand side of (4.3) because a set of methods \( S \) for calculating functional (4.3) contains a set of linear methods. Lemma 4.4 is proved.

**Corollary.** Among all functions for which the optimal method for calculating \( L(t) \) has the greatest error for a given set of functionals, there exists a function satisfying the conditions \( l_1(f) = \cdots = l_N(f) = 0 \). It follows from the proof that such a function is the function \( f_0 \).

**5. Optimal methods for calculating integrals of the form (1.1).**
5.1. Lower bounds for the functionals $\zeta_{nm}$ and $\zeta_{N}$.

In this Section we derive lower bounds for the functionals $\zeta_{nm}$ and $\zeta_{N}$, defined in Section 2, for calculating integrals (1.1) by the cubature formulas

\[ Kf = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \sum_{l_1=0}^{\rho_1} \sum_{l_2=0}^{\rho_2} p_{k_1 k_2 l_1 l_2} (s_1, s_2) f^{(l_1, l_2)} (x_{k_1}, x_{k_2}) + R_{n_1 n_2} (f; p_{k_1 k_2 l_1 l_2}; x_{k_1}, x_{k_2}; s_1, s_2), \]  

(5.1)

and

\[ Kf = \sum_{k=1}^{N} p_k (s_1, s_2) f(M_k) + R_N (f; p_k; M_k; s_1, s_2) \]  

(5.2)

on Hölder and Sobolev classes.

**Theorem 5.1.** Let $\Psi = H_{\omega_1, \omega_2} (D)$ or $\Psi = H_{\omega_3} (D)$, and calculate integral (1.1) by formula (5.1) with $\rho_1 = \rho_2 = 0$. Then the inequality

\[ \zeta_{n_1 n_2} [\Psi] \geq \frac{\gamma}{\pi^2} n_1 n_2 [q \int_0^h \omega_1 (t) dt + h \int_0^q \omega_2 (t) dt], \]

where $q = \frac{\pi}{n_2}$, $h = \frac{\pi}{n_1}$, and

\[ \gamma := \int_0^{2 \pi} \int_0^{2 \pi} \frac{ds_1 ds_2}{(\sin^2 (s_1/2) + \sin^2 (s_2/2))^\lambda} \]  

(5.1′)

is valid.

**Corollary.** Let $\Psi = H_{\alpha \alpha} (D)$ or $\Psi = H_{\alpha 3} (D)$, and calculate integral (1.1) by formula (5.1) with $n_1 = n_2 = n$ and $\rho_1 = \rho_2 = 0$. Then the inequality

\[ \zeta_{nn} [\Psi] \geq \frac{2 \gamma \pi^n}{(1 + \alpha)n^\alpha} \]

is valid.

**Proof of Theorem 5.1.** Denote by $\psi (s_1, s_2)$ a nonnegative function belonging to the class $H_{\omega_1 \omega_2} (1)$ and vanishing at the nodes $(x_{k_1}, x_{k_2})$, $1 \leq k_1 \leq n_1$, $1 \leq k_2 \leq n_2$.

One has:

\[ R_{n_1 n_2} (\psi; p_{k_1 k_2}; x_{k_1}, x_{k_2}) \geq \]

\[ \int_0^{2 \pi} \int_0^{2 \pi} \left( \int_0^{2 \pi} \psi (s_1, s_2) \frac{ds_1 ds_2}{\sin^2 ((s_1 - s_2)/2) + \sin^2 ((s_2 - s_2)/2)} \right) \]  

\[ = \frac{1}{4 \pi^2} \int_0^{2 \pi} \int_0^{2 \pi} \psi (s_1, s_2) \left( \int_0^{2 \pi} \frac{ds_1 ds_2}{\sin^2 ((s_1 - s_1)/2) + \sin^2 ((s_2 - s_2)/2)} \right) \]  

\[ = \frac{1}{4 \pi^2} \int_0^{2 \pi} \int_0^{2 \pi} \psi (s_1, s_2) \frac{ds_1 ds_2}{\sin^2 (s_1/2) + \sin^2 (s_2/2)} \int_0^{2 \pi} \psi (s_1, s_2) ds_1 ds_2. \]  

(5.3)
From Lemma 4.4 and Theorem 4.2 one concludes that the following inequality

$$R_{n_1 n_2}(\Psi; p_{k_1 k_2}; x_{k_1}, x_{k_2}) \geq \frac{\gamma}{\pi^2} n_1 n_2 \left[ q \int_0^h \omega_1(t) dt + h \int_0^q \omega_2(t) dt \right],$$

holds for arbitrary weights $p_{k_1 k_2}$ and nodes $(x_{k_1}, x_{k_2})$ and

$$\zeta_{n_1 n_2}(\Psi) \geq \frac{\gamma}{\pi^2} n_1 n_2 \left[ q \int_0^h \omega_1(t) dt + h \int_0^q \omega_2(t) dt \right].$$

Theorem 5.1 is proved. ■

**Theorem 5.2.** Let $\Psi = H_i^\alpha$ or $\Psi = Z_i^\alpha$, $i = 1, 2, 3$, and calculate the integral $Kf$ by cubature formula (5.2). Then

$$\zeta_N[H_i^\alpha] = 2\zeta_N[Z_i^\alpha] = (1 + o(1))\gamma(4\pi^2)^{2/\alpha} D_1 N^{-\alpha/2},$$

where $D_1 = \frac{12}{2+\alpha} \left( \frac{\alpha+2}{2\sqrt{3}} \right)^{(\alpha+2)/2} \frac{\pi^6}{\cos^{1+\alpha} \alpha}$, $D_2 = \frac{2}{2\pi(2+\alpha)}$, and $D_3 = \frac{2^{1-\alpha/2}}{2+\alpha}$.

**Proof.** The proof of Theorem 5.2 is similar to the proof of Theorem 5.1, with some difference is in the estimation of the integral $\int_0^{2\pi} \int_0^{2\pi} \psi(s_1, s_2) ds_1 ds_2$, where the function $\psi(s_1, s_2)$ belongs to the class $H_i^\alpha$ (or $Z_i^\alpha$), is nonnegative in the domain $D = [0, 2\pi]^2$, and vanishes at $N$ nodes $M_k$, $k = 1, 2, \ldots, N$.

Using Lemma 4.4 and Theorem 4.3, one checks that the inequalities

$$\inf_{M_k \psi \in H_i^\alpha, \psi(M_k) = 0} \sup_{0 \leq \psi(s_1, s_2) \leq 1} \int_0^{2\pi} \int_0^{2\pi} \psi(s_1, s_2) ds_1 ds_2 = (1 + o(1)) D_1 (4\pi^2)^{(2+\alpha)/\alpha} N^{-\alpha/2},$$

$$\inf_{M_k \psi \in Z_i^\alpha, \psi(M_k) = 0} \sup_{0 \leq \psi(s_1, s_2) \leq 1} \int_0^{2\pi} \int_0^{2\pi} \psi(s_1, s_2) ds_1 ds_2 = (1 + o(1)) \frac{1}{2} D_2 (4\pi^2)^{(2+\alpha)/\alpha} N^{-\alpha/2}$$

hold for arbitrary $M_k \in D$, $k = 1, 2, \ldots, N$.

Substituting these values into inequality (5.3), we complete the proof of Theorem. ■

**Theorem 5.3.** Let $\Psi = \tilde{C}_x^\alpha(1)$, and calculate the integral $Kf$ by formula (5.1) with $\rho_1 = \rho_2 = 0$, and $n_1 = n_2 = n$. Then

$$\zeta_{n_1 n_2}[\Psi] \geq (1 + o(1)) \frac{2\gamma K_r}{n^2},$$

where $K_r$ is the Favard constant.

**Proof.** Let

$$\psi(s_1, s_2) = \psi_1(s_1) + \psi_2(s_2),$$

where $0 \leq \psi_1(s) \in W^r(1)$ vanishes at the nodes $x_k$, $k = 1, 2, \ldots, n$, and $0 \leq \psi_2(s) \in W^r(1)$ vanishes at the nodes $y_k$, $k = 1, 2, \ldots, n$.

According to [11], for arbitrary nodes $x_k$, $k = 1, 2, \ldots, n$ one has:

$$\int_0^{2\pi} \psi_i(s) ds \geq \frac{2\pi K_r}{n^r}, \quad i = 1, 2.$$
Thus, the inequality
\[
\int_0^{2\pi} \int_0^{2\pi} \psi(s_1, s_2) ds_1 ds_2 \geq \frac{8\pi^2 K_r}{n^r}
\]
holds for arbitrary nodes \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\).

The conclusion of Theorem 5.3 follows from this inequality and from (5.3).

**Theorem 5.4.** Let \(\Psi = W_p^r (1), r = 1, 2, \ldots, 1 \leq p \leq \infty,\) and calculate the integral \(K f\) by formula (5.1) with \(\rho_1 = \rho_2 = r - 1\) and \(n_1 = n_2 = n.\) Then

\[
\zeta_{nn}[\Psi] \geq (1 + o(1)) \frac{2^{1/q} \pi^{r-1/p} R_{pq}(1)}{r!(rq + 1)^{1/q}(n - 1 + [R_{pq}(1)]^{1/r})^r} \gamma,
\]

where \(R_{pq}(t)\) is a polynomial of degree \(r\), least deviating from zero in \(L_q([-1, 1]).\)

**Proof.** Let \(L = [\frac{n}{\log n}]\). Take an additional set of nodes \((\xi_k, \xi_l), \xi_k = \frac{2\pi k}{L}, k, l = 0, 1, \ldots, L - 1.\) By \((v_i, w_j), i, j = 0, 1, \ldots, N - 1, N = n + L,\) denote the union of the sets \((x_k, y_l)\) and \((\xi, \xi_j).\) Let \(\psi(s_1, s_2) = \psi_1(s_1) + \psi_2(s_2),\) where \(\psi_1(s) \in W_p^r (1)\) vanishes with its derivatives up to the order \(r - 1\) at the nodes \(v_i, i = 0, 1, \ldots, N - 1,\) and \(\psi_2(s) \in W_p^r (1)\) vanishes with its derivatives up to order \(r - 1\) at the nodes \(w_j, j = 0, 1, \ldots, N - 1.\) Assume \(\int_{v_i} \psi_1(s) ds > 0, i = 0, 1, \ldots, N - 1,\) and \(\int_{w_j} \psi_2(s) ds > 0, j = 0, 1, \ldots, N - 1,\) where \(v_N = 2\pi\) and \(w_N = 2\pi.\)

Let
\[
h(s_1, s_2, \sigma_1, \sigma_2) := \begin{cases} 0, & \text{if } (\sigma_1, \sigma_2) = (s_1, s_2), \\ \frac{1}{(\sin^2(\sigma_1 - s_1)/2 + \sin^2(\sigma_2 - s_2)/2)^{r}}, & \text{otherwise,} \end{cases}
\]

\[
\psi^+(s_1, s_2) = \begin{cases} \psi(s_1, s_2), & \text{if } \psi(s_1, s_2) \geq 0, \\ 0, & \text{if } \psi(s_1, s_2) < 0, \end{cases}
\]

\[
\psi^-(s_1, s_2) = \begin{cases} 0, & \text{if } \psi(s_1, s_2) \geq 0, \\ -\psi(s_1, s_2), & \text{if } \psi(s_1, s_2) < 0. \end{cases}
\]

For each value \((\xi_i, \xi_j), i, j = 0, 1, \ldots, N - 1,\) we have:

\[
\int_0^{2\pi} \int_0^{2\pi} h(\xi_i, \xi_j, \sigma_1, \sigma_2) \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 =
\]

\[
= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \xi_{k+1} \xi_{l+1} \int_{\xi_l}^{2\pi} \int_{\xi_k}^{2\pi} h(\xi_i, \xi_j, \sigma_1, \sigma_2) \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 =
\]

\[
= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \xi_{k+1} \xi_{l+1} \int_{\xi_l}^{2\pi} \int_{\xi_k}^{2\pi} h(\xi_i, \xi_j, \sigma_1, \sigma_2) \psi^+(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 -
\]

\[
- \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \xi_{k+1} \xi_{l+1} \int_{\xi_l}^{2\pi} \int_{\xi_k}^{2\pi} h(\xi_i, \xi_j, \sigma_1, \sigma_2) \psi^-(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \geq
\]

12
\[
\begin{align*}
&\geq \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j+1}^{j+[(N_2-1)/2]} h(\xi_i, \xi_{j+1}, \xi_{k+1}) \int \int \psi^+ (\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 + \\
&+ \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j-[(N_2-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l+1}) \int \int \psi^+ (\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 + \\
&+ \sum_{k=i-[(N_1-1)/2]}^{i-1} \sum_{l=j+1}^{j-[(N_2-1)/2]} h(\xi_i, \xi_j, \xi_{k-1}, \xi_{l+1}) \int \int \psi^+ (\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 - \\
&- \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j+1}^{j+[(N_2-1)/2]} h(\xi_i, \xi_j, \xi_k, \xi_l) \int \int \psi^- (\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 - \\
&- \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j-[(N_2-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_k, \xi_l) \int \int \psi^- (\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 - \\
&- \sum_{k=i-[(N_1-1)/2]}^{i-1} \sum_{l=j+1}^{j-[(N_2-1)/2]} h(\xi_i, \xi_j, \xi_k, \xi_l) \int \int \psi^- (\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 - \\
&- \sum_{k=i-[(N_1-1)/2]}^{i-1} \sum_{l=j-[(N_2-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_k, \xi_l) \int \int \psi^- (\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 = \\
&= \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j+1}^{j+[(N_2-1)/2]} h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l+1}) \int \int \psi (\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 + \\
&+ \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j-[(N_2-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l-1}) \int \int \psi (\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 + \\
&+ \sum_{k=i-[(N_1-1)/2]}^{i-1} \sum_{l=j+1}^{j+[(N_2-1)/2]} h(\xi_i, \xi_j, \xi_{k-1}, \xi_{l+1}) \int \int \psi (\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 + \\
&+ \sum_{k=i-[(N_1-1)/2]}^{i-1} \sum_{l=j-[(N_2-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_{k-1}, \xi_{l-1}) \int \int \psi (\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 - \\
&- \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j+1}^{j+[(N_2-1)/2]} (h(\xi_i, \xi_j, \xi_k, \xi_l) - h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l+1})) \times \\
&\times \int \int \psi^- (\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 -
\end{align*}
\]
where we have used the fact that the functions vanish with derivatives up to order \( r \).

Let us estimate the integral

\[
\int_{\xi_{k+1}}^{\xi_{l+1}} \int_{\xi_{l}}^{\xi_{k+1}} \psi^{-}(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 -
\]

\[
\int_{\xi_{k}}^{\xi_{l}} \int_{\xi_{l}}^{\xi_{k+1}} \psi^{-}(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 -
\]

\[
\int_{\xi_{k-1}}^{\xi_{l-1}} \int_{\xi_{l-1}}^{\xi_{k+1}} \psi^{-}(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 =
\]

\[
= J_1 + J_2 + J_3 + J_4 + I_1 + I_2 + I_3 + I_4.
\]

Let us estimate the integral

\[
\left| \int_{\xi_{k}}^{\xi_{l}} \int_{\xi_{l}}^{\xi_{k+1}} \psi^{-}(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right| \leq \int_{\xi_{k}}^{\xi_{l}} \int_{\xi_{l}}^{\xi_{k+1}} |\psi^{-}(\sigma_1, \sigma_2)| d\sigma_1 d\sigma_2 \
\]

\[
\leq \int_{\xi_{k}}^{\xi_{l}} \int_{\xi_{l}}^{\xi_{k+1}} |\psi(\sigma_1, \sigma_2)| d\sigma_1 d\sigma_2 \leq (\xi_{l+1} - \xi_{l}) \int_{\xi_{k}}^{\xi_{l+1}} |\psi_1(\sigma)| d\sigma + (\xi_{k+1} - \xi_{k}) \int_{\xi_{k}}^{\xi_{l+1}} |\psi_2(\sigma)| d\sigma \leq
\]

\[
\leq 2 \left( \frac{2\pi}{L} \right)^{r+2} \frac{1}{r!},
\]

where we have used the fact that the functions \( \psi_1(s) \) and \( \psi_2(s) \) on the segments \([\xi_k, \xi_{k+1}]\) and \([\xi_{l}, \xi_{l+1}]\) vanish with derivatives up to order \( r - 1 \).

Now let us estimate the sum:

\[
\sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j-[(N_2-1)/2]}^{j-1} \left| h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l+1}) - h(\xi_i, \xi_j, \xi_k, \xi_l) \right| =
\]

\[
= \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j+1}^{j+[(N_1-1)/2]} \left| \frac{1}{\left( \frac{\sin^2 \frac{\pi(k+1-i)}{L} + \sin^2 \frac{\pi(l+1-j)}{L}}{L} \right)^{\lambda}} - \frac{1}{\left( \frac{\sin^2 \frac{2\pi(k-i)}{L} + \sin^2 \frac{2\pi(l-j)}{L}}{L} \right)^{\lambda}} \right| \leq
\]

14
\[
\begin{align*}
&\leq \frac{c}{L} \sum_{k=i+1}^{i+[(L-1)/2]} \sum_{l=j+1}^{i+[(L-1)/2]} \frac{1}{\left(\sin^{2}\frac{\pi(k-i)}{L} + \sin^{2}\frac{\pi(l-j)}{L}\right)^{1+\lambda}} \left| \frac{k-i}{L} + \frac{l-j}{L} \right| \leq \\
&\leq \frac{c}{L} \sum_{k=i+1}^{i+[(L-1)/2]} \sum_{l=j+1}^{i+[(L-1)/2]} \frac{L^{2+2\lambda}}{((k-i)^2 + (l-j)^2)^{1+\lambda} L} \left( k-i \right) + \left( l-j \right) \leq \\
&\leq c(L)^{2\lambda} \left( \sum_{t=1}^{[(L-1)/2]} \frac{1}{t^{2\lambda}} + \sum_{k=1}^{[(L-1)/2]} \frac{1}{k^{2\lambda}} \right) \leq \\
&\leq c(L)^{2\lambda} \begin{cases} 
L^{1-2\lambda} & \text{if } \lambda < \frac{1}{2}, \\
\log L & \text{if } \lambda = \frac{1}{2}, \\
1 & \text{if } \lambda > \frac{1}{2}.
\end{cases}
\end{align*}
\]

By \(c > 0\) various estimation constants are denoted. Thus

\[
I_1 = o\left(\frac{1}{n^r}\right).
\]

The expressions \(I_2, I_3,\) and \(I_4\) are estimated similarly.

From the definition of the function \(\psi(s_1, s_2)\) it follows that the error of cubature formula (5.1) for \(s_1 = \xi_i, s_2 = \xi_j\) can be estimated as follows:

\[
R(\psi, \xi_i, \xi_j) = \int_0^{2\pi} \int_0^{2\pi} \psi(\sigma_1, \sigma_2) h(\xi_i, \xi_j, \sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \geq c \left( \frac{1}{n^r} \right) + \\
+ \sum_{k=i+1}^{i+[(L-1)/2]} \sum_{l=j+1}^{i+[(L-1)/2]} h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l+1}) \int_{\xi_k}^{\xi_i} \int_{\xi_l}^{\xi_{l+1}} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 + \\
+ \sum_{k=i+1}^{i+[(L-1)/2]} \sum_{l=j-[(L-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l-1}) \int_{\xi_k}^{\xi_i} \int_{\xi_{l-1}}^{\xi_{l+1}} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 + \\
+ \sum_{k=i-[(L-1)/2]}^{i-1} \sum_{l=j+1}^{i+[(L-1)/2]} h(\xi_i, \xi_j, \xi_{k-1}, \xi_{l+1}) \int_{\xi_{k-1}}^{\xi_i} \int_{\xi_l}^{\xi_{l+1}} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 + \\
+ \sum_{k=i-[(L-1)/2]}^{i-1} \sum_{l=j-[(L-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_{k-1}, \xi_{l-1}) \int_{\xi_{k-1}}^{\xi_i} \int_{\xi_{l-1}}^{\xi_{l+1}} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2.
\]

Averaging the above inequality over \(i\) and \(j\), one gets:

\[
R_{nn}[\Psi] \geq \sup_{\psi \in \Psi} \max_{i,j} R_{nn}(\psi, \xi_i, \xi_j) \geq \\
\geq \frac{1}{L^2} \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \left[ \sum_{k=i+1}^{i+[(L-1)/2]} \sum_{l=j+1}^{i+[(L-1)/2]} h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l+1}) \int_{\xi_k}^{\xi_i} \int_{\xi_l}^{\xi_{l+1}} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 + \right.
\]

15
where the following relation was used:

\[ k + 4 + L_k^2 - L_k^2 = \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} h(\xi_i, \xi_j, \xi_k, \xi_l) \int \int \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 + \]

\[ \geq o\left(\frac{1}{n'}\right) + \]

\[ + \frac{1}{L^2} \left[ \sum_{k=i+1}^{L-1} \sum_{l=j+1}^{L-1} h(\xi_k, \xi_l) \int \int \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} h(\xi_i, \xi_j, \xi_k, \xi_l) + \right. \]

\[ + \sum_{k=i+1}^{L-1} \sum_{l=j+1}^{L-1} h(\xi_k, \xi_l) \int \int \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} h(\xi_i, \xi_j, \xi_k, \xi_l) + \]

\[ + \sum_{k=i+1}^{L-1} \sum_{l=j+1}^{L-1} h(\xi_k, \xi_l) \int \int \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} h(\xi_i, \xi_j, \xi_k, \xi_l) = \]

\[ = o\left(\frac{1}{n^2}\right) + \frac{1}{4\pi^2} \int \int \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \times \]

\[ \times \left( \int \int \frac{d\sigma_1 d\sigma_2}{(\sin^2(\sigma_1/2) + \sin^2(\sigma_2/2))^\lambda} + O\left(\left(\frac{\log n}{n}\right)^{2-2\lambda}\right) \right), \quad (5.4) \]

where the following relation was used:

\[ \frac{4\pi^2}{L^2} \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} h(\xi_i, \xi_j, \xi_k, \xi_l) = O\left(\frac{\log n}{n}\right) + \int \int \frac{d\sigma_1 d\sigma_2}{(\sin^2(\sigma_1/2) + \sin^2(\sigma_2/2))^\lambda}. \]

Without loss of generality one may assume \( k = 1, l = 1 \) in the previous equation. Let us estimate

\[ U_0 = \left| \frac{4\pi^2}{L^2} \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} h(\xi_i, \xi_j, 0, 0) - \int \int \frac{d\sigma_1 d\sigma_2}{(\sin^2(\sigma_1/2) + \sin^2(\sigma_2/2))^\lambda} \right| \leq \]

\[ \leq \left| \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \int \int \frac{1}{(\sin^2((\xi_i)/2) + \sin^2((\xi_j)/2))^\lambda} \right| - \]

16
\[
- \frac{1}{(\sin^2(\sigma_1/2) + \sin^2(\sigma_2/2))^{\lambda}} d\sigma_1 d\sigma_2 + \\
+ \left| \int_0^{\xi_1} \int_0^{\xi_2} \frac{1}{(\sin^2(\sigma_1/2) + \sin^2(\sigma_2/2))^{\lambda}} d\sigma_1 d\sigma_2 \right| = u_1 + u_2,
\]
where \(\sum \sum'\) means summation over \((i, j) \neq (0, 0)\).

Let us estimate \(u_1\) and \(u_2\). One has:

\[
u_1 \leq \left| \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \int_{\xi_i}^{\xi_{i+1}} \int_{\xi_j}^{\xi_{j+1}} \frac{1}{(\sin^2(\sigma_1/2) + \sin^2(\sigma_2/2))^{\lambda}} d\sigma_1 d\sigma_2 \right| - \\
- \left| \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \int_{\xi_i}^{\xi_{i+1}} \int_{\xi_j}^{\xi_{j+1}} \frac{1}{(\sin^2(\xi_1/2) + \sin^2(\xi_2/2))^{\lambda}} d\sigma_1 d\sigma_2 \right| + \\
+ \left| \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \int_{\xi_i}^{\xi_{i+1}} \int_{\xi_j}^{\xi_{j+1}} \frac{1}{(\sin^2(\xi_1/2) + \sin^2(\xi_2/2))^{\lambda}} d\sigma_1 d\sigma_2 \right|
\]

\[
= u_{11} + u_{12}.
\]

The expressions \(u_{11}\) and \(u_{12}\) can be estimated similarly. Let us estimate \(u_{11}\):

\[
u_{11} \leq \frac{c}{L^4} \sum_{i=0}^{L} \sum_{j=0}^{L} \int_{\xi_i}^{\xi_{i+1}} \int_{\xi_j}^{\xi_{j+1}} \frac{1}{(\sin^2(\xi_1/2) + \sin^2(\xi_2/2))^{1+\lambda}} \leq \\
\leq \frac{c}{L^{2-2\lambda}} \sum_{i=0}^{L} \sum_{j=0}^{L} \frac{1}{(j^2 + j^2)^{1+\lambda}} \leq c \left( \frac{L^2}{L^{2-2\lambda}} \right)
\]

where \(c > 0\) stands for various estimation constants. Hence

\[
u_1 \leq \frac{c}{L^{2-2\lambda}}.
\]

Let us estimate \(u_2\):

\[
u_2 = \left| \int_0^{\xi_1} \int_0^{\xi_2} \frac{1}{(\sin^2(\sigma_1/2) + \sin^2(\sigma_2/2))^{\lambda}} d\sigma_1 d\sigma_2 \right| \leq \\
\leq c \int_0^{\xi_1} \int_0^{\xi_2} \frac{1}{(\sigma_1^2 + \sigma_2^2)^{\lambda}} d\sigma_1 d\sigma_2.
\]

Using polar coordinates, one gets:

\[
u_2 \leq c \int_0^{1/L} \int_0^{2\pi} \frac{1}{\rho^{2\lambda-1}} d\rho d\phi \leq \frac{c}{L^{2-2\lambda}}.
\]

17
Thus:

\[ U_0 \leq \frac{c}{L^2-2\lambda}. \]

From Lemmas 4.4, 4.1, and Theorem 4.1 it follows that

\[ \int_0^{2\pi} \psi_1(\sigma_1)d\sigma_1 \geq \frac{(1 + o(1))(2\pi)^{r+1/q} R_{rq}(1)}{2^{r+1}(rq + 1)^{1/q}(n - 1 + [R_{rq}(1)]^{1/r})}. \]  

(5.5)

where \( R_{rq}(t) \) is a polynomial of degree \( r \), least deviating from zero in \( L_q([-1, 1]) \).

Theorem 5.4 follows from inequalities (5.4) and (5.5).

5.2. Optimal cubature formulas for calculating integrals as (1.1).

Hölder class of functions.

Let \( x_k := 2k\pi/n, k = 0, 1, \ldots, n, \Delta_{kl} = [x_k, x_{k+1}, x_l, x_{l+1}], k, l = 0, 1, \ldots, n - 1, x_k' = (x_{k+1} + x_k)/2, k = 0, 1, \ldots, n - 1, \) and \( (s_1, s_2) \in \Delta_{ij}, i, j = 0, 1, \ldots, n - 1. \)

Calculate the integral \( Kf \) by the formula:

\[ Kf = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} f(x_k', x_l') \int_0^{2\pi} \int_0^{2\pi} \frac{d\sigma_1 d\sigma_2}{(\sin^2 \frac{\sigma - \sigma'}{2} + \sin^2 \frac{\sigma - \sigma'}{2})^\lambda} + R_{nm}. \]  

(5.6)

Theorem 5.5. Let \( \Psi = H_{\alpha \alpha}(D), 0 < \alpha < 1. \) Among all cubature formulas (5.1) with \( \rho_1 = \rho_2 = 0, \) formula (5.6), which has the error

\[ R_{nm}[\Psi] = \frac{(2 + o(1))\gamma}{1 + \alpha} \left( \frac{\pi}{n} \right)^\alpha, \]

is asymptotically optimal. Here \( \gamma \) is defined in (5.1’).

Proof. Using the periodicity of the integrand, we estimate the error of cubature formula (5.6) as follows:

\[ |R_{nm}| \leq \left| \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \int_{\Delta_{kl}} \left[ \frac{f(\sigma_1, \sigma_2) - f(x_k', x_l')}{(\sin^2 \frac{\sigma_1 - \sigma_2}{2} + \sin^2 \frac{\sigma_1 - \sigma_2}{2})^\lambda} - \frac{f(x_k', x_l') - f(x_k', x_l')}{(\sin^2 \frac{\sigma_1 - \sigma_1}{2} + \sin^2 \frac{\sigma_1 - \sigma_1}{2})^\lambda} \right] d\sigma_1 d\sigma_2 \right| \]

\[ \leq \left| \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \int_{\Delta_{kl}} \frac{f(\sigma_1, \sigma_2) - f(x_k', x_l')}{(\sin^2 \frac{\sigma_1 - \sigma_2}{2} + \sin^2 \frac{\sigma_1 - \sigma_2}{2})^\lambda} d\sigma_1 d\sigma_2 \right| + \]

\[ + \left| \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \int_{\Delta_{kl}} (f(x_k', x_l') - f(x_k', x_l')) \times \]

\[ \times \left[ \frac{1}{(\sin^2 \frac{\sigma_1 - \sigma_1}{2} + \sin^2 \frac{\sigma_1 - \sigma_1}{2})^\lambda} - \frac{1}{(\sin^2 \frac{\sigma_1 - \sigma_1}{2} + \sin^2 \frac{\sigma_1 - \sigma_1}{2})^\lambda} \right] d\sigma_1 d\sigma_2 \right| = \]
Let us estimate each of the sums $r_1$ and $r_2$ separately. One has:

$$r_1 \leq \left| \sum_{k=M}^{i+M} \sum_{l=M}^{j+M} \int \int_{\Delta_{kl}} \frac{f(\sigma_1, \sigma_2) - f(x'_k, x'_l)}{(\sin^2 \frac{\sigma_1 - s_1}{2} + \sin^2 \frac{\sigma_2 - s_2}{2})^{\lambda}} d\sigma_1 d\sigma_2 \right| +$$

$$+ \left| \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \int \int_{\Delta_{kl}} \frac{f(\sigma_1, \sigma_2) - f(x'_k, x'_l)}{(\sin^2 \frac{\sigma_1 - s_1}{2} + \sin^2 \frac{\sigma_2 - s_2}{2})^{\lambda}} d\sigma_1 d\sigma_2 \right| = r_{11} + r_{12},$$

where $\sum \sum'$ means summation over $(k, l)$ such that $\Delta_{kl} \notin \Delta^*$, $\Delta^* = [x_{i-M}, x_{i+M+1}; x_{j-M}, x_{j+M+1}], M = \lfloor \ln n \rfloor$.

Furthermore

$$r_{11} \leq \frac{c}{n^\alpha} \int \int_{\Delta^*} \frac{d\sigma_1 d\sigma_2}{(\sin^2 \frac{\sigma_1 - s_1}{2} + \sin^2 \frac{\sigma_2 - s_2}{2})^{\lambda}} \leq$$

$$\leq \frac{c}{n^\alpha} \frac{2 \pi M/n}{2 \pi} \leq \frac{c \log n}{n^{\alpha + 2 - 2 \lambda}} = o \left( \frac{1}{n^\alpha} \right).$$

Estimating $r_{12}$, one can assume without loss of generality $(i, j) = (0, 0)$, and get:

$$r_{12} \leq 4 \int_0^{\pi/n} \int_0^{\pi/n} (\omega_1(\sigma_1) + \omega_2(\sigma_2)) d\sigma_1 d\sigma_2 \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} h_{kl}(s_1, s_2, \sigma_1, \sigma_2) \leq$$

$$\leq 4 \int_0^{\pi/n} \int_0^{\pi/n} (\sigma_1^\alpha + \sigma_2^\alpha) d\sigma_1 d\sigma_2 \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} h_{kl}(s_1, s_2, \sigma_1, \sigma_2) \leq$$

$$\leq \frac{8}{1 + \alpha} \left( \frac{\pi}{n} \right)^{2+\alpha} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} h_{kl}(s_1, s_2, \sigma_1, \sigma_2) \leq$$

$$\leq \frac{1 + o(1)}{1 + \alpha} 2 \left( \frac{\pi}{n} \right)^{\alpha} \int_0^{\pi/2} \int_0^{\pi/2} \frac{d\sigma_1 d\sigma_2}{(\sin^2 \frac{\sigma_1}{2} + \sin^2 \frac{\sigma_2}{2})^{\lambda}}.$$

Here

$$h_{kl}(s_1, s_2; \sigma_1, \sigma_2) = \sup_{(\sigma_1, \sigma_2) \in \Delta_{kl}} h(s_1, s_2; \sigma_1, \sigma_2).$$

Combining the estimates of $r_{11}$ and $r_{12}$, one gets:

$$r_1 \leq \frac{1 + o(1)}{1 + \alpha} 2 \left( \frac{\pi}{n} \right)^{\alpha} \gamma$$

Let us estimate $r_2$. To this end we estimate the difference

$$r_2(k, l) = \int \int_{\Delta_{kl}} |f(x'_k, x'_l) - f(x'_i, x'_j)| \times$$
First, we estimate

\[ r_2(i, j) \leq \frac{c}{n^\alpha} \int \int_{\Delta_{ij}} \left| \frac{1}{\sin^2 \frac{\sigma_1 - \sigma_i}{2} + \sin^2 \frac{\sigma_2 - \sigma_j}{2}} - \frac{1}{\sin^2 \frac{\sigma_1 - \sigma_i'}{2} + \sin^2 \frac{\sigma_2 - \sigma_j'}{2}} \right| d\sigma_1 d\sigma_2 \leq \frac{c}{n^{2+\alpha-2\lambda}}. \]

The value \( r_2(k, l) \) is estimated similarly for \(|k - i| \leq 3\) and \(|l - j| \leq 3\).

Let us estimate \( r_2(k, l) \) for other values of \( k \) and \( l \).

One has:

\[ r_2(k, l) = \int \int_{\Delta_{kl}} |f(x'_k, x'_l) - f(x_i, x_j)| \times \]

\[ \times \left| \frac{1}{\sin^2 \frac{\sigma_1 - \sigma_i}{2} + \sin^2 \frac{\sigma_2 - \sigma_j}{2}} - \frac{1}{\sin^2 \frac{\sigma_1 - \sigma_i'}{2} + \sin^2 \frac{\sigma_2 - \sigma_j'}{2}} \right| d\sigma_1 d\sigma_2 \leq \]

\[ \leq \frac{c}{n} \int \int_{\Delta_{kl}} \left[ |x'_k - x'_l|^\alpha + |x_i - x_j|^\alpha \right] \left[ \left( \frac{|k - i|}{n} \right) + \left( \frac{|l - j|}{n} \right) \right] \times \]

\[ \times \left| \frac{1}{\sin^2 \frac{\sigma_1 - \sigma_i'}{2} + \theta \sin^2 \frac{\sigma_2 - \sigma_j'}{2}} - \frac{1}{\sin^2 \frac{\sigma_1 - \sigma_i}{2} + \theta \sin^2 \frac{\sigma_2 - \sigma_j}{2}} \right| \leq \]

\[ \leq \frac{c}{n^\alpha} \left( \left( \frac{|k - i|}{n} \right)^\alpha + \left( \frac{|l - j|}{n} \right)^\alpha \right) \left( \left( \frac{|k - i|}{n} \right) + \left( \frac{|l - j|}{n} \right) \right) \frac{n^2}{|k - i|^2 + |l - j|^2}^{1+\lambda} \]

\[ \leq \frac{c}{n^{\alpha+2-2\lambda}} \left( \frac{|k - i| + |l - j|}{|k - i|^2 + |l - j|^2} \right)^{1+\lambda} \leq \]

\[ \leq \frac{c}{n^{\alpha+2-2\lambda}} \left( \frac{|k - i|^2 + |l - j|^2}{(1+\alpha)/2} \right)^{1+\lambda} \leq \]

\[ \leq \frac{c}{n^{\alpha+2-2\lambda}} \frac{1}{(1/2-\alpha/2)\lambda}. \]

To estimate \( r_2 \), one sums up the last expression over \( k \) and \( l \). Without loss of generality assume \((i, j) = (0, 0)\). Then

\[ r_2 \leq \frac{c}{n^{\alpha+2-2\lambda}} \left( 16 + 4 \sum_{k=0}^{[n/2]+1} \sum_{l=0}^{[n/2]+1} \frac{1}{(k^2 + l^2)^{\lambda+1/2-\alpha/2}} \right), \]

where \( \sum \sum' \) means summation over \( k \) and \( l \) such that \( k > 3 \) or \( l > 3 \).
One has:

\[ \sum_{k=0}^{[n/2]+1} \sum_{l=0}^{[n/2]+1} \frac{1}{(k^2 + l^2)\lambda + 1/2 - \alpha/2} \leq \]

\[ \leq A \left[ \sum_{k=3}^{[n/2]+1} \frac{1}{k^{2\lambda+1-\alpha}} + \sum_{k=3}^{[n/2]+1} \frac{1}{l^{2\lambda+1-\alpha}} \right] \leq \]

\[ \leq A \begin{cases} 1, & \text{if } 2\lambda - \alpha > 1; \\
\log n, & \text{if } 2\lambda - \alpha = 1; \\
n^{1-2\lambda+\alpha}, & \text{if } 2\lambda - \alpha < 1. \end{cases} \]

Hence

\[ r_2 \leq A \begin{cases} n^{-(\alpha+2-2\lambda)}, & \text{if } 2\lambda - \alpha > 1; \\
n^{-1}\log n, & \text{if } 2\lambda - \alpha = 1; \\
n^{-1}, & \text{if } 2\lambda - \alpha < 1. \end{cases} \]

Thus, if \( \alpha < 1 \), then

\[ r_2 \leq o(n^{-\alpha}). \]

Combining the estimates of \( r_1 \) and \( r_2 \), one gets:

\[ R_{mn}[\Psi] \leq \gamma \frac{(2 + o(1))}{1 + \alpha} \left( \frac{\pi}{n} \right)^{\alpha}. \]

Theorem 5.5 follows from the comparison of this inequality with the lower bound of the value \( \zeta_{mn}[\mathcal{H}_{\alpha,\alpha}(D)] \), mentioned in the Corollary to Theorem 5.1. ■

Remark. If \( \alpha = 1 \), the cubature formula (5.6) is optimal with respect to order.

The proof of Theorem 5.5 yields also the following result:

\textbf{Theorem 5.5'. Let } \Psi = \mathcal{H}_{\alpha,\alpha}(D), 0 < \alpha \leq 1. \text{ Among all possible cubature formulas (5.1) with } \rho_1 = \rho_2 = 0, \text{ formula}

\[ Kf = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} f(x'_k, x'_l) \int \int_{\Delta_{kl}} \frac{d\sigma_1 d\sigma_2}{(\sin^2 (\frac{x'_k}{2}) + \sin^2 (\frac{x'_l}{2}))^\lambda} + R_{mn}, \]

which has the error

\[ R_{mn}[\Psi] = \frac{(2 + o(1))\gamma}{1 + \alpha} \left( \frac{\pi}{n} \right)^{\alpha}, \]

is asymptotically optimal.

To apply formula (5.6), one has to calculate the integrals

\[ I_{kl} = \int \int_{\Delta_{kl}} \frac{d\sigma_1 d\sigma_2}{(\sin^2 (\frac{\sigma_1 - x'_k}{2}) + \sin^2 (\frac{\sigma_2 - x'_l}{2}))^\lambda} \]

for \( k, l = 0, 1, \ldots, n - 1 \). Exact values of these integrals for arbitrary values \( \lambda \) are apparently unknown. Therefore the procedure of numerical calculation of integrals (5.7) should be given for practical application of formula (5.6).
Let $k = i$ and $l = j$. Then the integral $I_{ij}$ is replaced by the integral

$$p_{ij}^* = \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} \frac{d\sigma_1 d\sigma_2}{(sin^2 \frac{\sigma_1}{2} + sin^2 \frac{\sigma_2}{2})^\lambda + h}, \quad h > 0,$$

which can be calculated by cubature formulas (in particular, Gauss quadrature rule) with arbitrary degree of accuracy because the function

$$\frac{1}{(sin^2 \frac{\sigma_1}{2} + sin^2 \frac{\sigma_2}{2})^\lambda + h},$$

has derivatives up to arbitrary order. The choice of parameter $h$ is discussed in Section 8.

Let $k = i, l \neq j$, and

$$I_{il} = \frac{4\pi^2}{n^2} \left( \frac{sin^2 \frac{x_i'}{2} - x_j'}{2} \right)^{-\lambda} = p_{il}^*.$$

Let $k \neq i, l = j$, and

$$I_{kj} = \frac{4\pi^2}{n^2} \left( \frac{sin^2 \frac{x_k'}{2} - x_j'}{2} \right)^{-\lambda} = p_{kj}^*.$$

Let $k \neq i, l \neq j$, and

$$I_{kl} = \frac{4\pi^2}{n^2} \left( \frac{sin^2 \frac{x_k'}{2} - x_i'}{2} + sin^2 \frac{x_l'}{2} - x_j'}{2} \right)^{-\lambda} = p_{kl}^*.$$

The integral $Kf$ is calculated by the formula

$$Kf = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} p_{kl}^* f(x_k', y_l') + R_{nn}(f, p_{kl}^*, x_k', y_l'). \quad (5.8)$$

Formula (5.8) is not optimal since it is not exact on constant functions $f(x, y) = const.$ But one can estimate the error of this formula:

$$|R_{nn}(f, p_{kl}^*, x_k', y_l')| \leq M \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} |I_{kl} - p_{kl}^*| + R_{nn}(\Psi),$$

where $M = \max |f(x, y)|$.

The values $|I_{kl} - p_{kl}^*|$ are easily estimated, and one gets the conclusion of Theorem 5.5’.

**Classes of smooth functions**

**Theorem 5.6.** Assume $\varphi \in \tilde{W}^{r,r}(1).$ Let $\Psi = \tilde{W}^{r,r}(1)$, and calculate the integral $K\varphi$ by formula (5.1) with $\rho_1 = r - 1,$ $\rho_2 = r - 1,$ and $n_1 = n_2 = n.$ Then the cubature formula

$$K\varphi = \int_{0}^{2\pi} \int_{0}^{2\pi} \frac{\varphi_{mn}(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2}{(sin^2(\sigma_1 - s_1)/2 + sin^2(\sigma_2 - s_2)/2)^\lambda} + R_{nn}(\varphi) \quad (5.9)$$

is asymptotically optimal.

Before proving Theorem 5.6, let us describe the construction of the spline $\varphi_{mn}$. Let $x_k = 2k\pi/n, \quad k = 0, 1, \ldots, n.$ Divide the sides of the squares $\Omega = [0, 2\pi; 0, 2\pi]$ into $n$ equal parts. Denote by $\Delta_{kl}$ the rectangle $\Delta_{kl} = [2k\pi/n, 2(k+1)\pi/n; 2l\pi/n, 2(l+1)\pi/n], k, l = 0, 1, \ldots, n - 1.$ Let $(s_1, s_2) \in \Delta_{ij}.$
First we approximate \( \varphi(\sigma_1, \sigma_2) \) as a function of \( \sigma_2 \), and construct a spline \( \varphi_n(\sigma_1, \sigma_2) \) by the following rule. Let \( \sigma_1 \) be an arbitrary fixed number, \( 0 \leq \sigma_1 \leq 2\pi \). On the segments \([2k\pi/n, 2(k+1)\pi/n]\) for \( k \neq j-2, \ldots, j+1 \), one has:

\[
\varphi_n(\sigma_1, \sigma_2) = \sum_{l=0}^{n-1} \left[ \frac{\varphi^{(0,l)}(\sigma_1, 2k\pi/n)}{l!} (\sigma_2 - 2k\pi/n)^l + B_l \delta(l) (\sigma_1, (k+1)/n) \right],
\]

where

\[
\delta(\sigma_1, \sigma_2) := \varphi(\sigma_1, \sigma_2) - \sum_{l=0}^{r-1} \frac{\varphi^{(0,l)}(\sigma_1, 2k\pi/n)}{l!} (\sigma_2 - 2k\pi/n)^l.
\]

The coefficients \( B_l \) are defined by the equation

\[
(2(k+1)\pi/n - \sigma_2)^r - \sum_{l=0}^{r-1} \frac{B_l r!}{(r-l-1)!} \frac{2\pi}{n} (2\pi(k+1)/n - \sigma_2)^{r-l-1} = (-1)^r R_{r1} (2\pi(2k+1)/2n; \pi/n; \sigma_2),
\]

where \( R_{r1}(a, h, x) \) is a polynomial of degree \( r \), least deviating from zero in the norm of the space \( L \) on the segment \( [a-h, a+h] \). On the segment \([2\pi(j-2)/n, 2\pi(j+2)/n]\) the function \( \varphi_n(\sigma_1, \sigma_2) \) is defined by the partial sum of the Taylor series:

\[
\varphi_n(\sigma_1, \sigma_2) = \varphi(\sigma_1, 2\pi j/n) + \frac{\varphi^{(0,1)}(\sigma_1, 2\pi j/n)}{1!}(\sigma_2 - j/n) + \cdots + \frac{\varphi^{(0,r-1)}(\sigma_1, 2\pi j/n)}{(r-1)!}(\sigma_2 - 2\pi j/n)^{r-1}.
\]

We define the function \( \varphi_{mn}(\sigma_1, \sigma_2) \) by analogy with the function \( \varphi_n(\sigma_1, \sigma_2) \).

**Proof of Theorem 5.6.** Let \((s_1, s_2) \in \Delta_{ij}\). The error of formula (5.9) we estimate by the inequality

\[
|R_{mn}| \leq \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \left| \int_{\Delta_{kl}} \frac{\varphi(\sigma_1, \sigma_2) - \varphi_{mn}(\sigma_1, \sigma_2)}{\sin^2 \frac{\sigma_1 - s_1}{2} + \sin^2 \frac{\sigma_2 - s_2}{2}} d\sigma_1 d\sigma_2 \right| + \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \left| \int_{\Delta_{kl}} \frac{\varphi(\sigma_1, \sigma_2) - \varphi_{mn}(\sigma_1, \sigma_2)}{\sin^2 \frac{\sigma_1 - s_1}{2} + \sin^2 \frac{\sigma_2 - s_2}{2}} d\sigma_1 d\sigma_2 \right| = r_1 + r_2,
\]

where \( \sum_k \) means summation over \( (k, l) \) such that \( i - 1 \leq k \leq i + 1, 0 \leq l \leq n - 1 \) or \( 0 \leq k \leq n - 1, j - 1 \leq l \leq j + 1 \), and \( \sum_l \) means summation over the other values of \( (k, l) \).

Let us estimate each of the sums \( r_1 \) and \( r_2 \) separately. In addition without loss of generality assume that \( \int_{\Delta_{kl}} (\varphi(\sigma_1, \sigma_2) - \varphi_{mn}(\sigma_1, \sigma_2)) d\sigma_1 d\sigma_2 \geq 0 \). Then

\[
r_1 \leq \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \left| \varphi(\sigma_1, \sigma_2) - \varphi_{mn}(\sigma_1, \sigma_2) \right| \int_{\Delta_{kl}} \frac{d\sigma_1 d\sigma_2}{\sin^2 \frac{\sigma_1 - s_1}{2} + \sin^2 \frac{\sigma_2 - s_2}{2}} \leq \ldots
\]

23
\[ r_2 \leq 4 \sum_{k=i+2}^{i+1+[(n-1)/2]} \sum_{l=j+2}^{j+1+[(n-1)/2]} \frac{1}{\left( \sin^2 \frac{x_k-s_1}{2} + \sin^2 \frac{x_l-s_2}{2} \right)^\lambda} \int_{\Delta_{kl}} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 - 4 \sum_{k=i+2}^{i+1+[(n-1)/2]} \sum_{l=j+2}^{j+1+[(n-1)/2]} \left[ \frac{1}{\left( \sin^2 \frac{x_k-s_1}{2} + \sin^2 \frac{x_l-s_2}{2} \right)^\lambda} - \frac{1}{\left( \sin^2 \frac{x_{k+1}-s_1}{2} + \sin^2 \frac{x_{l+1}-s_2}{2} \right)^\lambda} \right]. \]

where \( \psi(\sigma_1, \sigma_2) = \varphi(\sigma_1, \sigma_2) - \varphi_{nn}(\sigma_1, \sigma_2) \),

\[ \psi^+(\sigma_1, \sigma_2) = \begin{cases} \psi(\sigma_1, \sigma_2), & \text{if } \psi(\sigma_1, \sigma_2) \geq 0 \\ 0, & \text{if } \psi(\sigma_1, \sigma_2) < 0 \end{cases} \]

\[ \psi^-(\sigma_1, \sigma_2) = \begin{cases} 0, & \text{if } \psi(\sigma_1, \sigma_2) \geq 0 \\ -\psi(\sigma_1, \sigma_2), & \text{if } \psi(\sigma_1, \sigma_2) < 0 \end{cases} \]

It is obvious

\[ 4 \sum_{k=i+2}^{i+1+[(n-1)/2]} \sum_{l=j+2}^{j+1+[(n-1)/2]} \frac{1}{\left( \sin^2 \frac{x_k-s_1}{2} + \sin^2 \frac{x_l-s_2}{2} \right)^\lambda} \leq 1 + o(1) \int_{0}^{2\pi} \int_{0}^{2\pi} \frac{d\sigma_1 d\sigma_2}{\left( \sin^2 \frac{\sigma_1}{2} + \sin^2 \frac{\sigma_2}{2} \right)^\lambda} \]

Let us estimate the integral

\[ i = \int_{\Delta_{kl}} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \leq \left| \int_{\Delta_{kl}} (\varphi(\sigma_1, \sigma_2) - \varphi_{nn}(\sigma_1, \sigma_2)) d\sigma_1 d\sigma_2 \right| + \left| \int_{\Delta_{kl}} (\varphi_n(\sigma_1, \sigma_2) - \varphi_{nn}(\sigma_1, \sigma_2)) d\sigma_1 d\sigma_2 \right| = i_1 + i_2. \]

Since the expressions \( i_1 \) and \( i_2 \) are estimated similarly, we estimate only \( i_1 \). One has:

\[ i_1 \leq \frac{2\pi}{n} \max_{s_1} \left| \int_{x_1}^{x_{l+1}} (\varphi(s_1, \sigma_2) - \varphi_n(s_1, \sigma_2)) d\sigma_2 \right|. \]

This integral is a continuous function of \( s_1 \), which attains its maximum at a point \( s^* \), and

\[ i_1 \leq \frac{2\pi}{n} \left| \int_{x_1}^{x_{l+1}} (\varphi(s^*, \sigma_2) - \varphi_n(s^*, \sigma_2)) d\sigma_2 \right| \leq \frac{2\pi}{r!n} \int_{x_1}^{x_{l+1}} \left| \varphi^{(0,r)}(s^*, \sigma_2) \right| (x_{l+1} - \sigma_2)^r. \]
respect to order, and easier to apply. These formulas will be less accurate than the ones in Theorem 5.3, but they will be optimal with respect to order, and easier to apply.

From inequalities (5.14) and (5.15) one gets:

\[ i \leq \frac{8}{(r + 1)!} R_{r1}(1) \]

and

\[ r_{21} \leq \frac{2 + o(1)}{(r + 1)!} \left( \frac{\pi}{n} \right)^r R_{r1}(1) \int_0^{2\pi} \int_0^{2\pi} \frac{d\sigma_1 d\sigma_2}{\left( \sin^2 \frac{\sigma_1}{2} + \sin^2 \frac{\sigma_2}{2} \right)^\lambda}. \] (5.16)

One has:

\[ r_{22} = o(n^{-r}). \] (5.17)

Estimate (5.17) follows from the inequalities:

\[ \left| \int_{\Delta_{kl}} \psi^-(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right| \leq \int_{\Delta_{kl}} \left| \psi(\sigma_1, \sigma_2) \right| d\sigma_1 d\sigma_2 = O(n^{-r-2}) \]

and

\[ \sum_{k=i+2}^{i+1+[n-1]/2} \sum_{l=j+2}^{j+1+[n-1]/2} \left| \frac{1}{\left( \sin^2 \frac{x_k-s_k}{2} + \sin^2 \frac{x_l-s_l}{2} \right)^\lambda} - \frac{1}{\left( \sin^2 \frac{x_{k+1}-s_k}{2} + \sin^2 \frac{x_{l+1}-s_l}{2} \right)^\lambda} \right| \leq An^{2\lambda} \sum_k \sum_l \frac{(k-i)+(l-j)}{((k-i)^2 + (l-j)^2)^{\lambda+1}} \leq \begin{cases} n, & \lambda < 1/2 \\ n \log n, & \lambda = 1/2 \\ n^{2\lambda}, & \lambda > 1/2. \end{cases} \]

The estimate

\[ R_{nn}(\Psi) \leq (1 + o(1)) \frac{2\pi^r R_{r1}(1)}{(r + 1)!(n - 1 + [R_{r1}(1)]^{1/r})^r} \gamma \]

follows from inequalities (5.10), (5.11), (5.16), and (5.17).

Theorem 5.5 follows from the comparison of the values \( \zeta_{nn}[\Psi] \) and \( R_{nn}[\Psi] \).

Let us construct cubature formulas for calculating integrals \( Kf \) on classes of functions \( W^{r\gamma}(1) \). These formulas will be less accurate than the ones in Theorem 5.3, but they will be optimal with respect to order, and easier to apply.
First, we investigate the smooth function

$$
\psi(t_1, t_2) = \frac{2\pi}{0} \frac{f(\tau_1, \tau_2)d\tau_1d\tau_2}{(\sin^2 \frac{\tau_1-t_1}{2} + \sin^2 \frac{\tau_2-t_2}{2})^\lambda}
$$

assuming $f(t_1, t_2) \in \mathcal{W}^{r,r}$. Changing the variables $\tau_1 = t_1 - t$, $\tau_2 = t_2 - t$, in the last integral, one gets:

$$
\psi(t_1, t_2) = \frac{2\pi}{0} \frac{f(\tau_1 + t_1, \tau_2 + t_2)d\tau_1d\tau_2}{(\sin^2 \frac{\tau_1}{2} + \sin^2 \frac{\tau_2}{2})^\lambda}
$$

Thus, $\psi(t_1, t_2) \in \mathcal{W}^{r,r}$.

**Remark.** It is known [9] that Kolmogorov and Babenko widths on the class of functions $\mathcal{W}^{r,r}(1)$ are equal to $\delta_n(\mathcal{W}^{r,r}(1)) \asymp d_n(\mathcal{W}^{r,r}(1), C) \asymp \frac{1}{n^{r/2}}$. Hence the recovery of the function $\psi(t_1, t_2)$ using $n$ functionals is not possible with accuracy greater than $O\left(\frac{1}{n^{r/2}}\right)$. More precise conclusions are obtained in Theorems 5.3 and 5.4.

Thus, for recovery of a function $\psi(t_1, t_2)$, $(t_1, t_2) \in [0, 2\pi]^2$ with the accuracy $O(n^{-r/2})$, it is sufficient to calculate the value of the function $\psi(t_1, t_2)$ at the nodes $(v_k, v_l)$, where $v_k = 2k\pi/N$, $k, l = 0, 1, \ldots, N$, and $N^2 = n$, and to use the local spline $\psi_N(t_1, t_2)$ of degree $r$ with respect to each variable.

Let us describe the construction of such spline.

Assume for simplicity that $M := N/r$ is an integer, and cover the domain $[0, 2\pi]^2$ with the squares $\Delta_{kl} = [w_k, w_l]$, $k, l = 0, 1, \ldots, M - 1$, here $w_k = 2k\pi/M$, $k = 0, \ldots, M$. Approximate the function $\psi(t_1, t_2)$ in each domain $\Delta_{kl}$ by the interpolation polynomial $\psi_N(t_1, t_2, \Delta_{kl})$ constructed on the nodes $(x^k_i, x^l_j)$, $i, j = 0, 1, \ldots, r$, $x^k_i = w_k + \frac{2r\pi}{Mr^i}$, $i = 0, 1, \ldots, r$.

Denote the local spline, which is defined by the polynomials $\psi_N(t_1, t_2, \Delta_{kl})$, by $\psi_N(t_1, t_2)$.

If the values $\psi(v_k, v_l)$ are calculated by formula (5.9) with the accuracy $O(n^{-r/2})$, then

$$
\|\psi(t_1, t_2) - \psi_N(t_1, t_2)\|_C \leq O(n^{-r/2}).
$$

Therefore the spline $\psi_N(t_1, t_2)$ is optimal with respect to order, and a method for recovery of the function $\psi(t_1, t_2)$, which has the error $O(n^{-r/2})$ (in the sup–norm) is constructed.

6. Optimal methods for calculating integrals of the form $Tf$.

**Theorem 6.1.** Let $\Psi = H_{\alpha\alpha}(D)$, and calculate the integral $Tf$ by formula (2.1) with $n_1 = n_2 = n$ and $\rho_1 = \rho_2 = 0$. Then the estimate:

$$
\zeta_{nn}[\Psi] \gtrsim \frac{(1 + o(1))}{(1 + \alpha)\pi^\alpha} \int_{-1}^{1} \int_{-1}^{1} \frac{dt_1 dt_2}{(\tau^2 + t^2)^{\lambda}}
$$

(6.1)
Proof. Let \( n \) be an integer number, \( L = [n/\log n] \). Let \( v_k := -1 + 2k/L, k = 0, 1, \ldots, L \). By \((\xi_k, \eta_l)\) we denote a set which is the union of nodes \((x_i, y_j), i, j = 1, 2, \ldots, n\) of formula (2.1) and the nodes \((v_i, v_j), i, j = 1, 2, \ldots, L\). Let \( \Delta_{kl} = [v_k, v_{k+1}; v_l, v_{l+1}], k, l = 0, 1, \ldots, L - 1 \). Let \( 0 \leq \psi(t_1, t_2) \in H_{aa}(D) \), where \( D = [-1, 1]^2 \), vanishing at the nodes \((\xi_k, \eta_l), k, l = 0, 1, \ldots, N\), where \( N = n + L \).

Consider the integrals

\[
(T\psi)(v_i, v_j) = \int_{-1}^{1} \int_{-1}^{1} \frac{\psi(\tau_1, \tau_2)d\tau_1 d\tau_2}{((\tau_1 - v_i)^2 + (\tau_2 - v_j)^2)\lambda} = 
\]

\[
= \left( \sum_{k=1}^{L-1} \sum_{l=j}^{L-1} + \sum_{k=0}^{i-1} \sum_{l=j}^{L-1} + \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} + \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} \right) \times 
\]

\[
\times \int_{\Delta_{kl}} \int_{\Delta_{kl}} \frac{\psi(\tau_1, \tau_2)d\tau_1 d\tau_2}{((\tau_1 - v_i)^2 + (\tau_2 - v_j)^2)\lambda} \geq 
\]

\[
\geq \sum_{k=0}^{L-i-1} \sum_{l=0}^{L-j-1} \left( \frac{L}{2} \right)^{2\lambda} \frac{1}{((k+1)^2 + (l+1)^2)\lambda} \int_{\Delta_{k+l+1}} \int_{\Delta_{k+l+1}} \psi(\tau_1, \tau_2)d\tau_1 d\tau_2 + 
\]

\[
+ \sum_{k=0}^{L-i-1} \sum_{l=0}^{j-1} \left( \frac{L}{2} \right)^{2\lambda} \frac{1}{((k+1)^2 + (l+1)^2)\lambda} \int_{\Delta_{k+l+1}} \int_{\Delta_{k+l+1}} \psi(\tau_1, \tau_2)d\tau_1 d\tau_2 + 
\]

\[
+ \sum_{k=0}^{i-1} \sum_{l=0}^{L-j-1} \left( \frac{L}{2} \right)^{2\lambda} \frac{1}{((k+1)^2 + (l+1)^2)\lambda} \int_{\Delta_{k+l+1}} \int_{\Delta_{k+l+1}} \psi(\tau_1, \tau_2)d\tau_1 d\tau_2 + 
\]

\[
+ \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} \left( \frac{L}{2} \right)^{2\lambda} \frac{1}{((k+1)^2 + (l+1)^2)\lambda} \int_{\Delta_{k+l+1}} \int_{\Delta_{k+l+1}} \psi(\tau_1, \tau_2)d\tau_1 d\tau_2 = 
\]

\[
= \sum_{k=0}^{L-i} \sum_{l=0}^{L-j} \left( \frac{L}{2} \right)^{2\lambda} \frac{U(L - i - 1 - k)U(L - j - 1 - l)}{((k+1)^2 + (l+1)^2)\lambda} \int_{\Delta_{k+l+1}} \int_{\Delta_{k+l+1}} \psi(\tau_1, \tau_2)d\tau_1 d\tau_2 + 
\]

\[
+ \sum_{k=0}^{L-i-1} \sum_{l=0}^{j-1} \left( \frac{L}{2} \right)^{2\lambda} \frac{U(i - 1 - k)U(j - 1 - l)}{((k+1)^2 + (l+1)^2)\lambda} \int_{\Delta_{k+l+1}} \int_{\Delta_{k+l+1}} \psi(\tau_1, \tau_2)d\tau_1 d\tau_2 + 
\]

\[
+ \sum_{k=0}^{i-1} \sum_{l=0}^{L-j-1} \left( \frac{L}{2} \right)^{2\lambda} \frac{U(i - 1 - k)U(L - j - 1 - l)}{((k+1)^2 + (l+1)^2)\lambda} \int_{\Delta_{k+l+1}} \int_{\Delta_{k+l+1}} \psi(\tau_1, \tau_2)d\tau_1 d\tau_2 + 
\]

\[
+ \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} \left( \frac{L}{2} \right)^{2\lambda} \frac{U(i - 1 - k)U(j - 1 - l)}{((k+1)^2 + (l+1)^2)\lambda} \int_{\Delta_{k+l+1}} \int_{\Delta_{k+l+1}} \psi(\tau_1, \tau_2)d\tau_1 d\tau_2. 
\]

Here \( U(k) = 1 \) for \( k \geq 0 \), and \( U(k) = 0 \) for \( k < 0 \).
Averaging the above inequality over all \( i \) and \( j \), \( i, j = 0, 1, \ldots, L - 1 \), one gets:

\[
R_{nn}(\Psi, \eta_l; x_k, y_l) \geq \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} T(\psi)(\xi_i, \eta_j) \geq
\]

\[
\frac{1}{L^2} \sum_{k=0}^{L-1} \sum_{l=0}^{L-1} \frac{1}{((k+1)^2 + (l+1)^2)^{\lambda}} \left[ \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} U(L - i - 1 - k) \times \right.
\]

\[
\times U(L - j - 1 - l) \int_{\Delta_{k+i,l+j}} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 +
\]

\[
+ \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} U(L - i - 1 - k) U(j - 1 - l) \int_{\Delta_{k+i,j-l-1}} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 +
\]

\[
+ \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} U(i - 1 - k) U(L - j - 1 - l) \int_{\Delta_{i-j-1,l-j}} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 +
\]

\[
+ \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} U(i - 1 - k) U(j - 1 - l) \int_{\Delta_{i-j-1,j-l-1}} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 \geq
\]

\[
\frac{1}{L^2-2\lambda^22^{2\lambda}} \sum_{k=0}^{L-1} \sum_{l=0}^{L-1} \frac{1}{((k+1)^2 + (l+1)^2)^{\lambda}} \left[ \int_{v_k}^{1} \int_{v_l}^{1} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 +
\]

\[
+ \int_{v_k}^{1} \int_{-1}^{-1} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 + \int_{v_l}^{1} \int_{-1}^{-1} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 + \int_{-1}^{-1} \int_{-1}^{-1} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 \right] \geq
\]

\[
\frac{1}{L^2-2\lambda^22^{2\lambda}} \sum_{k=0}^{L-1} \sum_{l=0}^{L-1} \frac{1}{((k+1)^2 + (l+1)^2)^{\lambda}} \int_{-1}^{-1} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2. \quad (6.2)
\]

From inequality (6.2) it follows that

\[
\zeta_{nn}[H_{\alpha\alpha}(D)] \geq (1 + o(1)) \frac{1}{L^2-2\lambda^22^{2\lambda}} \sum_{k=0}^{L-1} \sum_{l=1}^{L-1} \frac{1}{(k^2 + l^2)^{\lambda}} \int_{-1}^{-1} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 =
\]

\[
= \frac{1 + o(1)}{4} \int_{-1}^{-1} \int_{-1}^{-1} \frac{dt_1 dt_2}{(t_1^2 + t_2^2)^{\lambda}} \int_{-1}^{-1} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2. \quad (6.3)
\]

From Theorem 4.2 and Lemma 4.4 it follows that the inequality

\[
\int_{-1}^{-1} \int_{-1}^{-1} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 \geq \frac{4}{1 + \alpha n^{2\lambda}} \frac{1}{1 + \alpha n^{2\lambda}} \quad (6.4)
\]

is valid for an arbitrary vector of the weights and the nodes \((X, Y, P)\) on the class \(H_{\alpha\alpha}(D)\).

Theorem 6.1 follows from inequalities (6.3) and (6.4).
**Theorem 6.2.** Let $Ψ = C_2^r(1)$, and calculate the integral $Tf$ by formula (2.1) with $ρ_1 = ρ_2 = 0$. If $n_1 = n_2 = n$, then
\[
ζ_{nn}[Ψ] ≥ (1 + o(1)) \frac{2K_r}{(πn)^r} \int_{-1}^{1} \int_{-1}^{1} ds_1 ds_2 \left( \frac{ds_1 ds_2}{(s_1^2 + s_2^2)^{\lambda}} \right),
\]
where $K_r$ is the Favard constant.

**Proof.** Let
\[
ψ(s_1, s_2) = ψ_1(s_1) + ψ_2(s_2),
\]
where $0 ≤ ψ_1(s) ∈ W^r(1)$, vanishes at the nodes $x_k, k = 1, 2, \ldots, n$, and $0 ≤ ψ_2(s) ∈ W^r(1)$ vanishes at the nodes $y_k, k = 1, 2, \ldots, n$.

For arbitrary nodes $x_k, k = 1, 2, \ldots, n$, one has (see [11]):
\[
\int_{-1}^{1} ψ_i(s) ds ≥ \frac{2K_r}{(πn)^r}, \quad i = 1, 2.
\]
Thus the inequality
\[
\int_{-1}^{1} \int_{-1}^{1} ψ(s_1, s_2) ds_1 ds_2 ≥ \frac{8K_r}{(πn)^r}
\]
holds for arbitrary nodes $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$.

Theorem 6.2 follows from this estimate and inequality (6.3).

**Theorem 6.3.** Let $Ψ = W_p^{r,r}(1), r = 1, 2, \ldots, 1 ≤ p ≤ ∞$, and calculate the integral $Tf$ by formula (2.1) with $ρ_1 = ρ_2 = r - 1$ and $n_1 = n_2 = n$. Then the estimate
\[
ζ_{nn}[Ψ] ≥ (1 + o(1)) \frac{2^{1/r} R_{rq}(1)}{r!(rq + 1)^{1/s}(n - 1 + |R_{rq}(1)|^{1/r})^r} \int_{-1}^{1} \int_{-1}^{1} ds_1 ds_2 \left( \frac{ds_1 ds_2}{(s_1^2 + s_2^2)^{\lambda}} \right),
\]
holds, where $R_{rq}(t)$ is a polynomial of degree $r$, least deviating from zero in $L_q([-1, 1])$.

**Proof.** Let $L = [n/\log n]$. Consider the nodes $(v_k, v_l), v_k = \frac{2k}{L}, k, l = 0, 1, \ldots, L - 1$. By $(ξ_i, η_j), i, j = 0, 1, \ldots, N - 1, N = n + L$ denote the union of the nodes $(x_k, y_l)$ and $(ξ_i, ξ_j)$. Let $ψ(s_1, s_2) = ψ_1(s_1) + ψ_2(s_2)$, where $0 ≤ ψ_1(s) ∈ W_p^r(1)$ vanishes with its derivatives up to order $r - 1$ at the nodes $ξ_i, i = 0, 1, \ldots, N - 1$, and $0 ≤ ψ_2(s) ∈ W_p^r(1)$ vanishes with its derivatives up to order $r - 1$ at the nodes $η_j, j = 0, 1, \ldots, N - 1$. Assume that $\int_{v_i}^{v_{i+1}} ψ_1(s) ds > 0, \quad i = 0, 1, \ldots, N - 1$, and $\int_{w_j}^{w_{j+1}} ψ_2(s) ds > 0, \quad j = 0, 1, \ldots, N - 1$.

Using the argument similar to the one in the proof of Theorem 6.1, one gets:
\[
ζ_{nn}(Ψ, pkl; v_k, v_l) ≥ \frac{1}{L^2} \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} T(ψ)(v_i, v_j) ≥ \frac{1}{L^2 - 2λq_2\lambda} \sum_{k=0}^{L-1} \sum_{l=0}^{L-1} \frac{1}{((k + 1)^2 + (l + 1)^2)^{\lambda}} \int_{-1}^{1} \int_{-1}^{1} ψ(τ_1, τ_2) dτ_1 dτ_2 =
\]

29
\[ = \frac{1 + o(1)}{4} \int_{-1}^{1} \int_{-1}^{1} \frac{dt_1 dt_2}{(t_1^2 + t_2^2)^\lambda} \int \int \psi(\tau_1, \tau_2) d\tau_1 d\tau_2. \]  

(6.6)

From Theorem 4.1 and lemma 4.4 it follows that the inequality

\[ \int \int \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 \geq (1 + o(1)) \frac{2^{2+1/q} R_{rq}(1)}{r!(rq + 1)^{1/q} (n - 1 + [R_{rq}(1)]^{1/q})^r} \]  

(6.7)

is valid for arbitrary weights and the nodes \((X, Y, P)\) on the class \(H_{\alpha}(D)\).

Theorem 6.3 follows from inequalities (6.6)- (6.7). ■

**Cubature formulas.**

Let us construct a cubature formula for calculating the integral \(Tf\) on the Hölder class \(H_{\alpha}(D)\). Let \(x_k = -1 + 2k/n, k = 0, 1, \ldots, n, x'_k = (x_{k+1} + x_k)/2, k = 0, 1, \ldots, n - 1, and \Delta_{kl} = [x_k, x_{k+1}; x_l, x_{l+1}], k, l = 0, 1, \ldots, n - 1.\)

Calculate the integral \(Tf\) by the formula

\[ Tf = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} f(x'_k, x'_l) \int \int_{\Delta_{kl}} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} + R_{nn}(f). \]  

(6.8)

Consider another cubature formula for calculating the integral \(Tf\). Let \((t_1, t_2) \in \Delta_{ij}\). By \(\Delta_\ast\) denote the union of the square \(\Delta_{ij}\) and of those squares \(\Delta_{kl}\) which have common points with the \(\Delta_{ij}\). Consider the formula

\[ Tf = f(x'_i, x'_j) \int \int_{\Delta_\ast} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} + \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} f(x'_k, x'_l) \int \int_{\Delta_{kl}} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} + R_{nn}(f), \]  

(6.9)

where \(\sum \sum'\) means summation over the squares which do not belong to \(\Delta_\ast\).

**Theorem 6.4.** Among all cubature formulas (2.1) with \(p_1 = p_2 = 0\) and \(n_1 = n_2 = n, formula (6.8), with the error estimate (6.1), is asymptotically optimal.

**Remark.** Similar statement holds for formula (6.9).

**Proof of the Theorem 6.4.** Let us estimate errors of formulas (6.8) and (6.9).

The error of formula (6.8) can be estimated as follows:

\[ |R_{nn}(f)| \leq \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \int \int_{\Delta_{kl}} \frac{|f(\tau_1, \tau_2) - f(x'_i, x'_j)|}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} d\tau_1 d\tau_2 + \]  

\[ + \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \int \int_{\Delta_{kl}} \frac{|f(\tau_1, \tau_2) - f(x'_i, x'_j)|}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} d\tau_1 d\tau_2 = r_1 + r_2. \]  

(6.10)

where \(\sum \sum'\) means summation over \(k\) and \(l\) such that the squares \(\Delta_{kl}\) belong to \(\Delta_\ast\), and \(\sum \sum''\) means summation over the other squares.
Let us estimate \( r_1 \) and \( r_2 \):

\[
\begin{align*}
\quad r_1 & \leq \frac{2}{n^\alpha}\int_{\Delta_k} \int_{\Delta_{kl}} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} \leq \frac{c}{n^{2\lambda + \alpha}} = o(n^{-\alpha}); \quad (6.11) \\
\quad r_2 & \leq \frac{4}{1 + \alpha n^{2+\alpha}} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} n' h(\Delta_{kl}). \quad (6.12)
\end{align*}
\]

Here \( h(\Delta_{kl}) \) denotes the maximum value of the function \( ((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^{-\lambda} \) in the square \( \Delta_{kl} \).

One has:

\[
\begin{align*}
\left| \int_{\Delta_{kl}} \int_{\Delta_{kl}} \frac{1}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} - h(\Delta_{kl}) \right| d\tau_1 d\tau_2 &= \\
\leq & \int_{\Delta_{kl}} \int_{\Delta_{kl}} \left| \frac{2\lambda(x_k - t_1 + q_1(\tau_1 - x_k))(\tau_1 - x_k)}{((x_k - t_1 + q_1(\tau_1 - x_k))^2 + (\tau_2 - t_2)^2)^\lambda + 1} \right| d\tau_1 d\tau_2 + \\
+ & \int_{\Delta_{kl}} \int_{\Delta_{kl}} \left| \frac{2\lambda(x_l - t_2 + q_2(\tau_2 - x_l))(\tau_2 - x_l)}{((x_l - t_2 + q_2(\tau_2 - x_l))^2 + (\tau_2 - t_2)^2)^\lambda + 1} \right| d\tau_1 d\tau_2 \leq \\
\leq & \int_{\Delta_{kl}} \int_{\Delta_{kl}} \frac{2\lambda(\tau_1 - x_k)}{((x_k - t_1 + q(\tau_1 - x_k))^2 + (\tau_2 - t_2)^2)^\lambda + 1/2} d\tau_1 d\tau_2 + \\
+ & \int_{\Delta_{kl}} \int_{\Delta_{kl}} \frac{2\lambda(\tau_2 - x_l)}{(\tau_1 - t_1)^2 + (x_l - t_2 + q_2(\tau_2 - x_l))^2)^\lambda + 1/2} d\tau_1 d\tau_2 \leq \\
\leq & \frac{2^4\lambda}{n^{2\lambda + 1}} \frac{n^{2\lambda + 1}}{(k^2 + l^2)\lambda + 1/2} = \frac{2^4\lambda}{n^{2-2\lambda + \alpha}} \frac{1}{(k^2 + l^2)\lambda + 1/2},
\end{align*}
\]

where it was assumed that \( k \geq i + 1 \), and \( l \geq j + 1 \). Estimates for the other combinations of \( k \) and \( l \) are similar. Thus:

\[
\begin{align*}
\quad r_2 & \leq \frac{1}{(1 + \alpha n^{2+\alpha})} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \int_{\Delta_{kl}} \int_{\Delta_{kl}} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} + \\
+ & \frac{1}{(1 + \alpha n^{2+\alpha})} \frac{2^3\lambda}{n^{2-2\lambda + \alpha}} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \frac{1}{(k^2 + l^2)\lambda + 1/2}. \quad (6.13)
\end{align*}
\]

Let us estimate the last term in the previous inequality.

One has:

\[
\begin{align*}
\sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \frac{1}{(k^2 + l^2)\lambda + 1/2} & \leq \sum_{k=-[n/2]}^{[n/2]} \sum_{l=-[n/2]}^{[n/2]} \frac{1}{(k^2 + l^2)\lambda + 1/2} \leq 
\end{align*}
\]

31
of points with integer-value coordinates, situated in the circle $R$ is asymptotically optimal. Here where

$$\sum_{k=0}^{\lambda} \sum_{l=0}^{n-1} \sum_{n=1}^{1-2\lambda} \lambda > 1/2$$

$$log n, \quad \lambda = 1/2$$

$$n^{1-2\lambda}, \quad \lambda < 1/2$$

where $\sum_{k=0}^{\lambda} \sum_{l=0}^{n-1} \sum_{n=1}^{1-2\lambda}$ means summation over $k$ and $l$, $(k,l) \neq (0,0)$.

In deriving (6.14) we have used the known result ([17], Theorem 56) which says that a number of points with integer-value coordinates, situated in the circle $x^2 + y^2 = r^2$, is equal to $\pi r^2 + O(r)$.

From inequalities (6.13) and (6.14) it follows that

$$r_2 \leq \frac{(1 + o(1))}{(1 + \alpha)n^\alpha} \sum_{k=0}^{\lambda} \sum_{l=0}^{n-1} \sum_{n=1}^{1-2\lambda} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda}.$$ 

This and (6.11) yield:

$$R_{nn}[H_{\alpha\alpha}(D)] = \frac{1 + o(1)}{(1 + \alpha)n^\alpha} \sup_{(t_1,t_2) \in D} \int_{-1}^{1} \int_{-1}^{1} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} \leq$$

$$\leq \frac{1 + o(1)}{(1 + \alpha)n^\alpha} \int_{-1}^{1} \int_{-1}^{1} \frac{d\tau_1 d\tau_2}{(\tau_1^2 + \tau_2^2)^\lambda}. \quad (6.15)$$

Theorem 6.4 follows from a comparison the estimates of $\zeta[H_{\alpha\alpha}(D)]$ and $R_{nn}[H_{\alpha\alpha}(D)]$. ■

Let us construct asymptotically optimal cubature formula for calculating integrals $Tf$ on the classes $W^{r,r}$. In the derivation of formula (5.9) the local spline $\varphi_n(t_1,t_2)$, approximating the function $\varphi(t_1,t_2)$ in the domain $[0,2\pi;0,2\pi]$, was constructed. A spline $f_{nn}(t_1,t_2)$, approximating the function $f(t_1,t_2)$ in the domain $[-1,1] \times [-1,1]$, can be constructed analogously. Calculate the integral $Tf$ by the formula

$$Tf = \int_{-1}^{1} \int_{-1}^{1} \frac{f_{nn}(\tau_1,\tau_2)d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} + R_{nn}(f). \quad (6.16)$$

**Theorem 6.5.** Let $\Psi = W^{r,r}(1), r = 1,2, \ldots$, and calculate the integral $Tf$ by formula (2.1) with $\rho_1 = \rho_2 = r - 1$, and $n_1 = n_2 = n$. Then cubature formula (6.16), which has the error

$$R_{nn}(\Psi) = (1 + o(1)) \frac{2R_{r1}(1)}{(r+1)! \{n - 1 + [R_{r1}(1)]^{1/r}\}} \int_{-1}^{1} \int_{-1}^{1} \frac{d\tau_1 d\tau_2}{(\tau_1^2 + \tau_2^2)^\lambda},$$

is asymptotically optimal. Here $R_{r1}(t)$ is a polynomial of degree $r$, least deviating from zero in $L^p([-1,1])$.

As in the proof of the Theorem 5.6 one gets the following estimate

$$R_{nn}(\Psi) = (1 + o(1)) \frac{2R_{r1}(1)}{(r+1)! \{n - 1 + [R_{r1}(1)]^{1/r}\}} \int_{-1}^{1} \int_{-1}^{1} \frac{d\tau_1 d\tau_2}{(\tau_1^2 + \tau_2^2)^\lambda},$$

Comparing this estimate with the estimate of $\zeta_{nn}[W^{r,r}(1)]$, from Theorem 6.3, one finishes the proof. ■

32
7. Calculation of weakly singular integrals on piecewise continuous surfaces.

In Sections 5 and 6 of the paper asymptotically optimal methods for calculating weakly singular integrals defined on the squares $[0, 2\pi]^2$ or $[-1, 1]^2$ were constructed.

It is of interest to study optimal methods for calculating weakly singular integrals on piecewise-Lyapunov surfaces.

Consider the integral

$$Jf = \int_\mathcal{G} f(\tau_1, \tau_2, \tau_3) dS \lambda, \quad t_1, t_2, t_3 \in \mathcal{G},$$

(7.1)

where $\mathcal{G}$ is a Lyapunov surface of class $L_s(B, \alpha)$.

We show that the results derived in Sections 5 and 6 can be partially generalized to the integrals (7.1).

Calculate integrals (7.1) by the formula:

$$Jf = \sum_{k=1}^{n} \sum_{|v|=0}^{p} p_{kv} f^{(v)}(M_k) + R_n(f, G, M_k, p_{kv}, t),$$

(7.2)

where $t = (t_1, t_2, t_3)$, $v = (v_1, v_2, v_3)$, $|v| = v_1 + v_2 + v_3$, $f^{(v)}(t_1, t_2, t_3) = \frac{\partial^v f}{\partial t_1^{v_1} \partial t_2^{v_2} \partial t_3^{v_3}}$.

The error of formula (7.2) is:

$$R_n(f, G, M_k, p_{kv}) = \sup_{t \in \mathcal{G}} |R_n(f, G, M_k, p_{kv}, t)|.$$

Assume $f \in \Psi_1$, and $G \in \Psi_2$. Then the error of formula (7.2) on the classes $\Psi_1$ and $\Psi_2$ is:

$$R_n(\Psi_1, \Psi_2, M_k, p_{kv}) = \sup_{f \in \Psi_1, G \in \Psi_2} R_n(f, G, M_k, p_{kv}).$$

Let

$$\zeta_n[\Psi_1, \Psi_2] := \inf_{M_k, p_{kv}} R_n(\Psi_1, \Psi_2, M_k, p_{kv}).$$

A cubature formula with nodes $M_k^*$ and weights $p_{kv}^*$ is called optimal, asymptotically optimal, optimal with respect to order on the class of functions $\Psi_1$ and surfaces $\Psi_2$, if

$$\frac{R_n(\Psi_1, \Psi_2, M_k^*, p_{kv}^*)}{\zeta_n[\Psi_1, \Psi_2]} = 1, \sim 1, \asymp 1,$$

respectively.

Let $\Psi_1 = H_\alpha(1)$, $0 < \alpha \leq 1$, and $\Psi_2 = L_1(B, \beta)$ $0 < \beta \leq 1$. Let us construct an optimal with respect to order method for calculating integrals (7.1) on the classes of functions $\Psi_1$ and surfaces $\Psi_2$. Let $S(G)$ be a "square" of the surface $G$. Divide the surface $G$ into $n$ parts $g_k$, $k = 1, 2, \ldots, n$, so that a "square" of each of the domains $g_k$ has the area of order $|S(G)|/n$, where $|S(G)|$ is the area of $S(G)$. We take a point $M_k$ in each of domains $g_k$ at the center of the domain $g_k$. 

33
Calculate integral (7.1) by the formula

\[
Jf = \sum_{k=1}^{n} f(M_k) \iint_{g_k} \frac{dS}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 + (\tau_3 - t_3)^2)^{\lambda}} + R_n(f, G). \quad (7.3)
\]

**Theorem 7.1.** Formula (7.3), has the error

\[ R_n(\Psi_1, \Psi_2) \propto n^{-\alpha/2}, \]

and is optimal with respect to order on the classes \( \Psi_1 = H_{\alpha}, \ 0 < \alpha \leq 1, \) and \( \Psi_2 = L_1(B, \beta) \ 0 < \beta \leq 1 \) among all formulas (7.2) with \( \rho = 0. \)

**Proof.** Assume for simplicity that the surface \( G \) is given by the equation \( z = \varphi(x, y), \ (x, y) \in G_0, \ \varphi(x, y) \geq 0. \) Let \( \varphi_x(x, y) := p, \ \varphi_y(x, y) := q. \) Write the integral \( Jf \) as

\[
Jf = \iint_{G_0} \frac{f(\tau_1, \tau_2, \varphi(\tau_1, \tau_2)) \sqrt{1 + p^2(\tau_1, \tau_2) + q^2(\tau_1, \tau_2)} d\tau_1 d\tau_2}{[(\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 + (\varphi(\tau_1, \tau_2) - \varphi(t_1, t_2))^2]^{\lambda/2}}. \quad (7.4)
\]

The function \( f(\tau_1, \tau_2, \varphi(\tau_1, \tau_2)) \) belongs to the Hölder class \( H_{\alpha} \) over \( G_0, \) and the function
\[
\sqrt{1 + p^2 + q^2}
\]
\[
\left[(\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 + (\varphi(\tau_1, \tau_2) - \varphi(t_1, t_2))^2\right]^{\lambda/2}
\]
is positive.

Let \( M_k = (m_k^1, m_k^2, m_k^3) \) be the nodes of cubature formula (7.2). Let \( \psi(\tau) := (d(\tau, \{M_k\}))^{\alpha}, \) where \( d(\tau, \{M_k\}) \) is the distance between the point \( \tau \) and the set of the nodes \( \{M_k\}, \) where the distance is measured along the geodesics of the surface \( G. \) This distance satisfies the Hölder condition \( H_{\alpha}(1). \) Hence the function \( \psi^*(\tau_1, \tau_2) = \psi(\tau_1, \tau_2, \varphi(\tau_1, \tau_2)) \) belongs to the Hölder class \( H_{\alpha}(A) \) and vanishes at the nodes \( (m_k^1, m_k^2), \ k = 1, 2, \ldots, n. \) Thus,

\[
\zeta_n(\Psi_1, \Psi_2) \geq \frac{1}{S(G_0)} \iint_{G_0} \iint_{G_0} \frac{\psi(\tau_1, \tau_2, \varphi(\tau_1, \tau_2)) \sqrt{1 + p^2 + q^2} d\tau_1 d\tau_2 d\tau_1 d\tau_2}{[(\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 + (\varphi(\tau_1, \tau_2) - \varphi(t_1, t_2))^2]^{\lambda}} \geq
\]

\[
\geq \frac{1}{S(G_0)} \iint_{G_0} \psi(\tau_1, \tau_2, \varphi(\tau_1, \tau_2)) d\tau_1 d\tau_2 \times
\]

\[
\times \min_{t} \iint_{G_0} \frac{\sqrt{1 + p^2 + q^2}}{[(\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 + (\varphi(\tau_1, \tau_2) - \varphi(t_1, t_2))^2]^{\lambda/2}} d\tau_1 d\tau_2 \geq
\]

\[
\geq \frac{A}{n^{\alpha/2}} \min_{t} \iint_{G} \frac{ds}{(r(t, \tau))^{\lambda}}.
\]

where \( S(G_0) \) is the "square" of the surface \( G_0. \)

Therefore the error of formula (7.3) is estimated by the inequality \( R_n \leq \frac{A}{n^{\alpha/2}}. \)

Theorem 7.1 is proved. ■

**Remark 1.** The method of decomposition of the domain \( G \) into smaller parts \( g_k, \ k = 1, 2, \ldots, n, \)

described below, is optimal with respect to order for classes of functions \( \Psi_1 = H_{\alpha}, \ 0 < \alpha \leq 1, \) and of surfaces \( \Psi_2 = L_0(B, \beta), \ 0 < \beta \leq 1 \) for \( \alpha \leq \beta. \)
Remark 2. From formula (7.4) it follows that if the function $f \in W^{r,r}(1)$ and the surface $G \in \mathcal{L}_s(B,\alpha)$, then the function $f(\tau_1,\tau_2,\varphi(\tau_1,\tau_2)) \in W^{u,v}(A)$, where $v = \min(r,s)$. Therefore, repeating the above arguments, one proves that the accuracy of calculation of integral (7.4) by cubature formulas using $n$ values of integrand function does not exceed $O(n^{-v/2})$.

From this remark it follows that if the surface $G$ consists of several parts, for example of surfaces $G_1$ and $G_2$ having common edge $L$, then it is necessary to calculate the integrals for the surface $G_1$ and the surface $G_2$ separately. If the surface $G$ is divided into smaller parts $g_k$, $k = 1, 2, \ldots, n$, the domains $g_k$, the curve $L$ passes inside of these domains, should be associated with the class of surfaces $L_0(B,1)$. In these domains the accuracy of calculation of the integral does not exceed than $O(n^{-k})$, where $n_k$ is the number of nodes of the cubature formula used in the domain $g_k$.

For this reason the cusps and the nodes, in which three or more domains $G_k$, which are parts of the domain $G$ touch each other, must belong to the boundaries of the covering domains $g_k$, $k = 1, 2, \ldots, n$.

The universal code for computing the capacitances, described in Section 9, is based on optimal with respect to order cubature formulas for calculating integrals on the classes of functions $H_\alpha$, $0 < \alpha \leq 1$ and of surfaces $L_0(B,\beta)$, $B = \text{const}$, $\alpha \leq \beta$, $\beta \leq 1$.

The algorithm constructed in Section 9 is optimal on this class of surfaces and does not require special treatment of edges and conical points of the surface.

When one studies cubature formulas on the classes $W^{r,r}(A)$, $r > 1$, and $L_s(B,\beta)$, $s \geq 1$, $0 \leq \beta \leq 1$, one has to develop a method to compute accurately the integrals in a neighborhood of the above singular points of the surface.

8. Calculation of weights of cubature formulas.

In calculating weakly singular integrals by cubature formulas (6.8) it is necessary to calculate integrals of the form of

$$J_{kl}(t_1,t_2) = \int_{\Delta_{kl}} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda}$$

for different values $(t_1,t_2) \in [-1,1]^2$.

Let be $(t_1,t_2) \in \Delta_{ij}$. Let us consider two possibilities: 1) the square $\Delta_{kl}$ and the square $\Delta_{ij}$ have nonempty intersection; 2) the square $\Delta_{kl}$ is does not have common points with the square $\Delta_{ij}$.

First consider the second case, when the function

$$\varphi(\tau_1,\tau_2) = \frac{1}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda},$$

is smooth. Here $(\tau_1, \tau_2) \in \Delta_{kl}$, and $(t_1, t_2) \in \Delta_{ij}$.

In this case one has:

$$\left| \frac{\partial^r \varphi(\tau_1,\tau_2)}{\partial \tau_1^r} \right| \leq \frac{r!2^{2r}}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^{\lambda+r/2}}$$
and, if the squares $\Delta_{kl}$ and $\Delta_{ij}$ do not have common points, one gets:

$$\left| \frac{\partial^r \varphi(\tau_1, \tau_2)}{\partial \tau_1^r} \right| \leq \frac{2^r r! n^{2\lambda + r}}{2^\lambda}.$$  

Similar estimates holds for partial derivative with respect to $\tau_2$.

Calculate the integral $J_{kl}(t_1, t_2)$ by the Gauss cubature formula:

$$J_{kl}(t_1, t_2) = \int_{\Delta_{kl}} P_{mm} \left[ \frac{1}{\left( (\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 \right)^{\lambda}} \right] d\tau_1 d\tau_2 + R_{mm}(\Delta_{kl}),$$

where $P_{mm} = P_m^{\tau_1} P_m^{\tau_2}$, $P_m^{\tau_i} (i = 1, 2)$ is the projection operator onto the set of interpolation polynomials of degree $m$ with nodes at the zeros of the Legendre polynomial, which maps the segment $[-1, 1]$ onto the segment $[x_k, x_{k+1}]$ for $i = 1$, and onto the segment $[x_l, x_{l+1}]$ for $i = 2$.

An integer $m$ is chosen so that $|R_{mm}| \leq n^{-2-\alpha}$ for cubature formulas on the Hölder class $H^{\alpha\alpha}$, and $|R_{mm}| \leq n^{-r-\alpha}$ for cubature formulas on the class $W^{rr}$.

This requirement is made because the error of calculation of the coefficients $J_{kl}(t_1, t_2)$ must not exceed the error of formula (6.5).

Using $r$ derivatives of the integrand in the error $R_{mm}(\Delta_{kl})$, one gets:

$$|R_{mm}(\Delta_{kl})| \leq \frac{B_r 2^r r!}{m^{r-1}} \left( \frac{2}{n} \right)^{2-2\lambda},$$

where $B_r$ is the constant appearing in Jackson’s theorems. It is known that the constant $B_r$ are bounded by a constant, denoted $b$, uniformly with respect to $r$. In the case of periodic functions $b = 1$ ([18]), and in the general case $b$ is apparently unknown.

If $r = 2$ and $m = B_r 2^r r! n^{2\lambda}$, then one gets the error estimate given for cubature formula (6.5).

Now, consider a method for calculating the integrals $J_{kl}(t_1, t_2)$ when the square $\Delta_{kl}$ has nonempty intersection with the square $\Delta_{ij}$. For definiteness we consider the calculation of the integral $J_{ij}(t_1, t_2)$ by the formula:

$$J_{ij}(t_1, t_2) = \int_{\Delta_{ij}} P_{mm} \left[ \frac{1}{\left( (\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 \right)^{\lambda} + h} \right] d\tau_1 d\tau_2 + R_{mm}(\Delta_{ij}),$$

where $h = \text{const} > 0$ will be specified below.

One has:

$$|R_{mm}(\Delta_{ij})| \leq h \int_{\Delta_{ij}} \frac{d\tau_1 d\tau_2}{\left( (\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 \right)^{\lambda} \left( (\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 \right)^{\lambda} + h} + \int_{\Delta_{ij}} D_{mm} \left[ \frac{1}{\left( (\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 \right)^{\lambda} + h} \right] d\tau_1 d\tau_2 = r_1 + r_2,$$
where \( D_{mm} = I - P_{mm} \), and \( I \) is an identity operator, and

\[
r_1 \leq h \int \frac{d\tau_1 d\tau_2}{\Delta_{ij} \left( (\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 \right)^{(\lambda + 1)/2} \left( ((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda + h \right)^{(1+\lambda)/2}} \leq
\]

\[
\leq \frac{2\pi}{1-\lambda} h^{(1-\lambda)/2} \left( \frac{2\pi}{n} \right)^{1-2\lambda}.
\] (8.1)

The function \( \frac{1}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda + h} \) is infinitely smooth. Using bounds for its first derivatives for \( \lambda \geq 1/2 \), one gets:

\[
r_2 \leq \frac{8\lambda B_1}{n^4 h^2 m}.
\] (8.2)

From inequality (8.1) it follows that for getting accuracy \( O(n^{-1-\alpha}) \) one has to have \( h = n^{-2(2\lambda+\alpha)/(1-\lambda)} \) and from inequality (8.2) it follows that one has to have

\[ m = \max([n^{(8\lambda+4\alpha)/(1-\lambda)+\alpha-3}], 1) \].

9. Iterative methods for calculating electrical capacitancies of conductors of arbitrary shapes.

Numerical methods for solving electrostatic problems, in particular, calculating capacitances of conductors of arbitrary shapes, are of practical interest in many applications. Electrostatic problems solvable in closed form are collected in [19,20,21]. Some of the problems were solved in closed form using integral equations, Wiener-Hopf and singular integral equations [22]. Electrostatistic problems for a finite circular hollow cylinder (tube) were studied in [23] by numerical methods. In [24] the variational methods of Ritz and Trefftz are discussed. Galerkin’s and other projection methods are studied in [25]. In practice these methods are time-consuming and variational methods in three-dimensional static problems probably have some advantages over the grid method. There exists a vast literature on calculation of the capacitances of perfect conductors [20,26]. In [20] there is a reference section which gives the capacitance of the conductors of certain shapes. In [26] and [27] a systematic exposition of variational methods for estimation of the capacitances is given. In [28] there are some programs for calculating the two-dimensional static fields using integral equations method. In the monograph [7] iterative methods for solving interior and exterior boundary value problems in electrostatics are proposed and mathematically justified. Upper and lower estimates for some functionals of electrostatic fields are obtained. Such functionals are the capacitances of perfect conductors and the polarizability tensors of bodies of arbitrary shape. These bodies are described by their dielectric permittivity, magnetic permeability and conductivity. They can be homogeneous or flaky. The main point is: these bodies have arbitrary geometrical shapes.

The methods, developed in [7], yield analytical formulas for calculation of the capacitances and polarizability tensors of bodies of arbitrary shapes with any given accuracy. Error estimates for these formulas are obtained in [7]. We give here the formulas for calculating the capacitances of the conductors of arbitrary shapes [7]:

37
\[ C^{(n)} = 4\pi \varepsilon_0 S^2 \left\{ \frac{(-1)^n}{(2\pi)^n} \int_{\Gamma} \int_{\Gamma} \frac{ds dt}{r_{st}} \int_{\Gamma} \cdots \int_{\Gamma} \psi(t, t_1) \ldots \psi(t_{n-1}, t_n) dt_1 \ldots dt_n \right\}^{-1} \]

where \( S \) is the surface area of the surface \( \Gamma \) of the conductor, \( \varepsilon_0 \) is the dielectric constant of the medium, \( r_{st} := |s - t| \), and \( \psi(t, s) := \frac{\partial}{\partial N_s} \frac{1}{r_{st}} \).

\[ C^{(0)} = \frac{4\pi \varepsilon_0 S^2}{J} \leq C, \quad J \equiv \int_{\Gamma} \int_{\Gamma} \frac{ds dt}{r_{st}}, \quad S = \text{meas} \Gamma. \]

It is proved in [7] that
\[ |C - C^{(n)}| \leq A q^n, \quad 0 < q < 1, \]
where \( A \) and \( q \) are constants which depend only on the geometry of \( \Gamma \).

We use these formulas are used to construct the computer code for calculating the capacitances of the conductors of arbitrary shapes.

It is proved in [7], that
\[ C^{(n)} = 4\pi \varepsilon_0 S^2 \left( \int_{\Gamma} \int_{\Gamma} r_{st}^{-1} \delta_n(t) dt ds \right)^{-1}, \quad (9.1) \]

where \( \delta_n \) is defined by the iterative process:
\[ \delta_{n+1} = -A \delta_n, \quad \delta_0 = 1, \quad \int_{\Gamma} \delta_n dt = S, \quad (9.2) \]

and \( A \) is defined by the formula:
\[ A \delta = \int_{\Gamma} \delta(t) \frac{\partial}{\partial N_s} \frac{1}{2\pi r_{st}} dt, \]
where \( N_s \) is the outer unit normal to \( \Gamma \) at the point \( s \).

To use iterative process (9.2), one has to calculate the weakly singular integral
\[ \frac{1}{2\pi} \int_{\Gamma} \delta(t) \frac{\partial}{\partial N_s} \frac{1}{r_{st}} dt. \quad (9.3) \]

Let us describe the construction of the cubature formula for calculating integral (9.3), assuming for simplicity that the domain \( G \), bounded by the surface \( \Gamma \), is convex. This assumption can be removed.

Let \( S \) be the inscribed in the conductor sphere of maximal radius \( r^* \), centered at the origin. Introduce the spherical coordinates system \( (r, \phi, \theta) \), and the set of the nodes \( (r^*, \phi_k, \theta_l) \), where \( \phi_k = 2k\pi/n, \quad k = 0, 1, \ldots, n, \quad \theta_l = \pi l/m, \quad l = 0, 1, \ldots, m \). Assume that \( m \) is even, and cover the sphere \( S \) with the spherical triangles \( \Delta_k, \quad k = 1, 2, \ldots, N, \quad N = 2n(m-1) \).

Let us describe the construction of the spherical triangles. For \( 0 \leq \Theta \leq \pi/m \) the triangles \( \Delta_k, k = 1, 2, \ldots, n \) have vertices \( (r^*, 0, 0), (r^*, \phi_{k-1}, \theta_1), (r^*, \phi_k, \theta_1), k = 1, 2, \ldots, n \).
For \( \theta_l \leq \theta \leq \theta_{l+1} \), \( l = 1, 2, \ldots, m/2 - 1 \), the triangles \( \Delta_k, k = n + 2n(l - 1) + j, \ 1 \leq j \leq 2n \) are constructed as follows. The rectangle \([0, 2\pi; \theta_l, \theta_{l+1}]\) is covered with the squares \( \Delta_{kl} = [\phi_k, \phi_{k+1}; \theta_l, \theta_{l+1}] \), \( k = 0, 1, \ldots, n - 1 \). Each of the squares \( \Delta_{kl} \) is divided into two equal triangles \( \Delta_{kl}^1 \) and \( \Delta_{kl}^2 \), \( k = 0, 1, \ldots, n - 1, \ l = 1, 2, \ldots, m/2 - 1 \). The spherical triangles \( \Delta_{kl}^1 \) and \( \Delta_{kl}^2 \), \( k = 0, 1, \ldots, n - 1, \ l = 1, 2, \ldots, m/2 - 1 \), are images of triangles \( \Delta_{kl}^1 \) and \( \Delta_{kl}^2 \) on the sphere \( S \).

As a result of these constructions the sphere \( S \) is approximated by the surface \( \Gamma \). It is known [19, 20] that the exact value of the capacitance of ellipsoid with equidistant from the vertices of the triangle \( \Delta_k, k = 1, 2, \ldots, N \). The points of intersection of these lines with the surface \( \Gamma \) are vertices of the triangle \( \Delta_k, k = 1, 2, \ldots, N \). As a result of these constructions the surface \( \Gamma \) is approximated by the surface \( \Gamma_N \) consisting of triangle \( \Delta_k, k = 1, 2, \ldots, N \), and integral (9.3) is approximated by the integral

\[
U(s) = \int_{\Gamma_N} \delta(t) \frac{1}{\partial N_s} \frac{1}{r_{st}} dt.
\] (9.4)

We fix each triangle \( \Delta_k, k = 1, 2, \ldots, N \), and associate with it a point \( \tau_k \in \Delta_k, k = 1, 2, \ldots, N \), equidistant from the vertices of the triangle \( \Delta_k, k = 1, 2, \ldots, N \). We calculate integral (9.4) at the points \( \tau_k, k = 1, 2, \ldots, N \), by the cubature formulas constructed in paragraphs 5-7 for the Hölder classes. After calculating the values \( U(\tau_k), k = 1, 2, \ldots, N \) by these cubature the integral

\[
\tilde{C}^{(1)} = -4\pi\varepsilon_0 S_N^2 \left( \int_{\Gamma_N} \int_{\Delta_k} \tilde{U}(t) dt ds \right)^{-1}
\]

is calculated, where \( \tilde{U}(t) = U(\tau_k) \) for \( t \in \Delta_k, k = 1, 2, \ldots, N \), \( S_N \) is area of the surface \( \Gamma_N \), \( \tilde{C}^{(1)} \) is approximation to the value of \( C^{(1)} \). The successive iterations are calculated analogously.

10. Numerical examples.

In this section the numerical results are given. As an example we calculated the capacitances of various ellipsoids, because for ellipsoids one knows ([19]) the analytical formula for the capacitance, which makes it possible to evaluate the accuracy of the numerical results. Consider the ellipsoid:

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.
\]

It is known [19, 20] that the exact value of the capacitance of ellipsoid with \( a = b \) is:

\[
C = \frac{4\pi\varepsilon_0}{\arccos(c/a)} \sqrt{(a^2 - c^2)}.
\]

Let \( a = b = 1 \), and \( \varepsilon_0 = 1 \). We have calculate the capacitance \( C \) for different values of the semiaxis \( c \). The results of the calculations are given in Table 1.

It is known ([7], p.43), that the capacitance of a metallic disc of radius \( a \) is \( C = 8a\varepsilon_0 \), and one can see from Table 1, that asymptotically, as \( c \to 0 \), this formula can be used practically for the ellipsoids with \( c \leq 0.001 \) with the error about 0.005.
| C   | n  | m  | N  | Exact value | Error  | Relative error | Calculation time |
|-----|----|----|----|-------------|--------|----------------|------------------|
| 0.9 | 40 | 30 | 2320| 12.144630   | -0.221200 | 0.018212       | 25 sec           |
| 0.5 | 40 | 30 | 2320| 10.392304   | -0.222042 | 0.021366       | 25 sec           |
| 0.1 | 40 | 30 | 2320| 8.5020638   | -0.301189 | 0.035425       | 25 sec           |
| 0.01| 40 | 30 | 2320| 8.050854    | 0.072132  | 0.008959       | 25 sec           |
| 0.001| 40 | 30 | 2320| 8.005092    | -0.821528 | 0.106374       | 25 sec           |
| 0.0001|40 | 30 | 2320| 8.000509    | -1.068178 | 0.133513       | 25 sec           |
| 0.9 | 50 | 40 | 3900| 12.144630   | -0.180510 | 0.014801       | 1 min 15 sec     |
| 0.5 | 50 | 40 | 3900| 10.392304   | -0.185642 | 0.017860       | 1 min 15 sec     |
| 0.1 | 50 | 40 | 3900| 8.5020638   | -0.288628 | 0.033947       | 1 min 15 sec     |
| 0.01| 50 | 40 | 3900| 8.050854    | -0.372047 | 0.046212       | 1 min 15 sec     |
| 0.001| 50 | 40 | 3900| 8.005092    | -0.586733 | 0.073295       | 1 min 15 sec     |
| 0.0001|50 | 40 | 3900| 8.000509    | -0.933288 | 0.116653       | 1 min 15 sec     |
| 0.9 | 60 | 50 | 5880| 12.144630   | -0.152009 | 0.012516       | 4 min            |
| 0.5 | 60 | 50 | 5880| 10.392304   | -0.160023 | 0.015391       | 4 min            |
| 0.1 | 60 | 50 | 5880| 8.5020638   | -0.283364 | 0.033328       | 4 min            |
| 0.01| 60 | 50 | 5880| 8.050854    | 0.532250  | 0.061110       | 4 min            |
| 0.001| 60 | 50 | 5880| 8.005092    | -0.391755 | 0.048939       | 4 min            |
| 0.0001|60 | 50 | 5880| 8.000509    | -0.880394 | 0.110042       | 4 min            |
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