Every complete atomic Boolean algebra is the congruence lattice of a cBCK-algebra

C. Matthew Evans

Abstract. Given a complete atomic Boolean algebra, we show there is a commutative BCK-algebra whose congruence lattice is that Boolean algebra. As a corollary we show that, for any discrete topological space, there is a commutative BCK-algebra whose prime spectrum is that space.

Mathematics Subject Classification. 06F35, 08A30, 06E99.

Keywords. BCK-algebra, ideal lattice, Boolean algebra.

1. Introduction

The class of BCK-algebras was introduced in 1966 by Imai and Iséki [6] as the algebraic semantics for a non-classical logic having only implication. While this class of algebras is not a variety [16], many subclasses are; for example, the subclass of commutative BCK-algebras forms a variety.

As with any algebra of logic, the ideal theory plays an important role in their study. For a commutative BCK-algebra $A$, it is known that $\text{Con}(A) \cong \text{Id}(A)$ [17], where $\text{Id}(A)$ is the lattice of ideals. It is also known that $\text{Id}(A)$ is a distributive lattice for any BCK-algebra $A$ [11].

In [4], the author proved that each of the following five types of distributive lattice is the congruence lattice of a commutative BCK-algebra:

1. any finite chain,
2. any countably infinite chain isomorphic to $\mathbb{Z}_{\leq 0} \cup \{-\infty\}$,
(3) any finite subdirectly irreducible distributive p-algebra,
(4) any finite Boolean algebra,
(5) any distributive lattice \( D \) such that \( \text{MI}(D) \cong T^\partial \), as posets, for some finite rooted tree \( T \), where \( T^\partial \) is the order dual of \( T \).

Additionally, any finite product of lattices where each factor lies in one of the above classes is the congruence lattice of a commutative BCK-algebra. For example, every divisor lattice is the congruence lattice of a commutative BCK-algebra since it is a finite product of finite chains. We note also that \( \text{Con}(A) \) is Boolean for any finite cBCK-algebra \( A \) \[12\].

The purpose of this paper is to extend (4) above by showing that every complete atomic Boolean algebra is the congruence lattice of a commutative BCK-algebra. The next several paragraphs provide the necessary definitions and background material. In Section 2 we define an algebra denoted \( \mathcal{FS} \), characterize its ideals and prime ideals, and prove the main theorem of the paper. In the closing paragraphs, we present an application of the main theorem: any discrete topological space is the spectrum of a commutative BCK-algebra.

**Definition 1.1.** A commutative BCK-algebra (or cBCK-algebra) is an algebra \( A = \langle A; \cdot, 0 \rangle \) of type \( (2, 0) \) such that

\begin{align*}
\text{(BCK1)} & \quad (x \cdot y) \cdot z = (x \cdot z) \cdot y \\
\text{(BCK2)} & \quad x \cdot (x \cdot y) = y \cdot (y \cdot x) \\
\text{(BCK3)} & \quad x \cdot x = 0 \\
\text{(BCK4)} & \quad x \cdot 0 = x
\end{align*}

for all \( x, y, z \in A \).

On any cBCK-algebra \( A \) we define a partial order by: \( x \leq y \) if and only if \( x \cdot y = 0 \), and it can be shown that \( 0 \cdot x = 0 \) for all \( x \in A \); so 0 is the smallest element of \( A \) with respect to \( \leq \). The term operation \( x \land y := y \cdot (y \cdot x) \) is the greatest lower bound of \( x \) and \( y \), and \( A \) is a semilattice with respect to \( \land \). Note that the identity (BCK2) tells us \( x \land y = y \land x \). We also have \( x \cdot y \leq x \) with equality if and only if \( x \land y = 0 \).

For proofs of these, as well as other elementary properties of cBCK-algebras, we point the reader to \[7\], \[8\], \[9\], \[12\], \[14\], \[15\], and \[17\].

**Definition 1.2.** An ideal of a BCK-algebra \( A \) is a subset \( I \subseteq A \) such that

\begin{align*}
\text{(1)} & \quad 0 \in I \\
\text{(2)} & \quad x \cdot y \in I \text{ and } y \in I \text{ implies } x \in I.
\end{align*}
We denote the lattice of ideals of \( A \) by \( \text{Id}(A) \), and point out that \( \text{Id}(A) \cong \text{Con}(A) \), see [1] or [17]. Of course \( \{0\} \) and \( A \) are always ideals of \( A \), and if these are the only ideals we say \( A \) is simple.

For \( n \in \mathbb{N}_0 \), define the notation \( x \cdot y^n \) recursively as

\[
x \cdot y^0 = x \\
x \cdot y^n = (x \cdot y^{n-1}) \cdot y,
\]

and note that this gives a decreasing sequence

\[
x \cdot y^0 \geq x \cdot y^1 \geq x \cdot y^2 \geq \cdots \geq x \cdot y^n \geq \cdots.
\]

If the underlying poset of a cBCK-algebra is totally ordered, we will call it a cBCK-chain. A cBCK-chain \( A \) is simple if and only if, for any \( x, y \in A \) with \( y \neq 0 \), there exists \( n \in \mathbb{N} \) such that \( x \cdot y^n = 0 \); see [13] or [4] for a proof. For example, the non-negative reals \( \mathbb{R}^+ \) become a simple cBCK-chain under the operation \( x \cdot y = \max\{x-y, 0\} \). More generally, the positive cone of any Archimedean group can be viewed as a cBCK-algebra, and with this structure it is a simple cBCK-chain. For examples of non-simple cBCK-chains, see Examples 3.13 and 3.14 of [4].

For \( S \subseteq A \), the smallest ideal of \( A \) containing \( S \) is the ideal generated by \( S \), which we denote by \( (S) \). If \( S = \{a\} \), we will write \( (a) \) rather than \( \{a\} \). In [7], Iséki and Tanaka showed that \( x \in (S) \) if and only if there exist \( s_1, \ldots, s_n \in S \) such that

\[
((\cdots((x \cdot s_1) \cdot s_2) \cdots) \cdot s_{n-1}) \cdot s_n = 0.
\]

One can show \( I \cup J = (I \cup J) \) for \( I, J \in \text{Id}(A) \).

**Definition 1.3.** A proper ideal \( P \) of \( A \) is prime if \( x \land y \in P \) implies \( x \in P \) or \( y \in P \).

Let \( X(A) \) denote the set of prime ideals of \( A \).

2. The Main Theorem

Let \( X \) be any set and \( T \) a simple cBCK-chain. Put

\[
\mathcal{F}(X,T) = \{ f : X \to T \},
\]

the set of all functions from \( X \) to \( T \). When \( X \) and \( T \) are understood, we will write \( \mathcal{F} \) instead of \( \mathcal{F}(X,T) \). The set \( \mathcal{F} \) becomes a cBCK-algebra with pointwise operation \( (f \cdot g)(x) = f(x) \cdot_T g(x) \), where the zero element is the zero function \( 0 \). A partial proof is given in the author’s PhD thesis [3], though it omits some of the more tedious details. This algebra is a generalization of Example 1 from [8].
will use \( \leq \) to denote the order on both \( \mathcal{F} \) and \( \mathbf{T} \), and it will always be clear from context.

For \( f \in \mathcal{F} \), define the support of \( f \) to be
\[
\text{supp}(f) = \{ x \in X \mid f(x) \neq 0 \} = \{ x \in X \mid f(x) > 0 \}.
\]
We say that \( f \) has finite support if the support of \( f \) is a finite set. Let \( \mathcal{FS}(X, \mathbf{T}) \) be the set of functions \( f : X \to \mathbf{T} \) with finite support; when \( X \) and \( \mathbf{T} \) are understood we will write \( \mathcal{FS} \).

**Lemma 2.1.** \( \mathcal{FS} \) is a cBCK-subalgebra of \( \mathcal{F} \).

**Proof.** Take \( f, g \in \mathcal{FS} \). We claim that \( \text{supp}(f \cdot g) \subseteq \text{supp}(f) \), from which the claim then follows.

Take \( x \in \text{supp}(f \cdot g) \). Then \( (f \cdot g)(x) = f(x) \cdot_{\mathbf{T}} g(x) > 0 \). But we know \( f(x) \geq f(x) \cdot_{\mathbf{T}} g(x) > 0 \), and so \( x \in \text{supp}(f) \). Hence \( \text{supp}(f \cdot g) \subseteq \text{supp}(f) \). \( \square \)

Consider the relation \( R \subseteq \mathcal{FS} \times X \) defined by
\[
R = \{ (f, x) \in \mathcal{FS} \times X \mid f(x) = 0 \}.
\]
This relation induces a Galois connection:
- for \( \mathcal{G} \subseteq \mathcal{FS} \), put \( P(\mathcal{G}) = \{ x \in X \mid g(x) = 0 \text{ for all } g \in \mathcal{G} \} \)
- for \( S \subseteq X \), put \( V(S) = \{ f \in \mathcal{F} \mid f(s) = 0 \text{ for all } s \in S \} \).

For a singleton \( \{x\} \subseteq X \), we will write \( V(x) \) for \( V(\{x\}) \). We note this is an antitone Galois connection: if \( S \subseteq T \subseteq X \), then certainly \( V(T) \subseteq V(S) \) since any function vanishing on \( T \) must also vanish on \( S \). Similarly, \( P(-) \) is order-reversing.

**Theorem 2.2.** A subset \( I \subseteq \mathcal{FS} \) is an ideal if and only if \( I = VP(I) \). In particular, every ideal of \( \mathcal{FS} \) has the form \( V(S) \) for some \( S \subseteq X \).

**Proof.** First we show that \( V(S) \) is an ideal for any \( S \subseteq X \). That \( 0 \in V(S) \) is clear. Suppose \( f \cdot g \in V(S) \) and \( g \in V(S) \). If we had \( f(s) > 0 \) for some \( s \in S \), then \( (f \cdot g)(s) > 0 \) since \( g(s) = 0 \), but this contradicts the fact that \( f \cdot g \in V(S) \). So we must have \( f(s) = 0 \) for all \( s \in S \), and thus \( f \in V(S) \). Hence, \( V(S) \) is an ideal of \( \mathcal{FS} \). Thus, if \( I = VP(I) \), we see that \( I \) is an ideal.

On the other hand, suppose \( I \) is an ideal of \( \mathcal{FS} \). Since \( V(-) \) and \( P(-) \) are a Galois connection, we know \( I \subseteq VP(I) \). For the other inclusion, take \( f \in VP(I) \) with \( f \neq 0 \). We know that \( \text{supp}(f) \) is finite, so enumerate the elements \( \text{supp}(f) = \{ x_1, x_2, \ldots, x_k \} \). For each \( x_i \in \text{supp}(f) \), the fact that \( f(x_i) \neq 0 \) tells us \( x_i \notin PVP(I) = P(I) \). Hence, for each \( i = 1, \ldots, k \), there is an element \( g_i \in I \) such that \( g_i(x_i) \neq 0 \). We claim that
\[
(\cdots ((f \cdot g_1^{n_1}) \cdot g_2^{n_2}) \cdots \cdot g_k^{n_k - 1}) \cdot g_k^{n_k} = 0
\]
Proposition 2.3. An ideal \( \mathfrak{a} \) does not matter. Finally, for any \( f, g \), we note also that by (BCK1), the order in which the prime ideals are proper. If \( |S| \geq 2 \), take \( x, y \in S \) with \( x \neq y \). Let \( f, g \in \mathcal{F}S \) such that \( f(x) \neq 0 \) but \( f(s) = 0 \) for all \( s \in S \setminus \{ x \} \), while \( g(y) \neq 0 \) but \( g(s) = 0 \) for all \( s \in S \setminus \{ y \} \). Then \( (f \wedge g)(s) = 0 \) for all suitable \( n_1, n_2, \ldots, n_k \in \mathbb{N} \).

Since T is a simple cBCK-chain, for each \( i \) there is \( n_i \in \mathbb{N} \) such that
\[
(f \cdot g_i^{n_i})(x_i) = f(x_i) \cdot T (g_i(x_i))^{n_i} = 0.
\]
Now, it may the case that \( g_i(x_j) \neq 0 \) for \( i \neq j \), but
\[
(f \cdot g_i^{n_i})(x_j) \leq f(x_j),
\]
and hence
\[
((f \cdot g_i^{n_i}) \cdot g_j^{n_j})(x_j) \leq (f \cdot g_j^{n_j})(x_j) = 0.
\]
We note also that by (BCK1), the order in which the \( g_i \)’s are applied does not matter. Finally, for any \( z \in X \setminus \text{supp}(f) \) we have \( (f \cdot h)(z) = 0 \) for any \( h \in \mathcal{F}S \), so the function \( (\cdots ((f \cdot g_1^{n_1}) \cdot g_2^{n_2}) \cdots \cdot g_k^{n_k}) = 0 \in I \).

Thus, \( V P(I) \subseteq I \) and we have \( I = VP(I) \) as claimed.

Before the next result, we note briefly that
\[
(f \wedge g)(x) = f(x) \wedge T g(x).
\]
To see this, take \( f, g \in \mathcal{F} \) and \( x \in X \). Without loss of generality, we may assume \( f(x) \leq g(x) \). This implies that \( (f \cdot g)(x) = 0 \) and \( f(x) \wedge T g(x) = f(x) \). Then
\[
(f \wedge g)(x) = (f \cdot (f \cdot g))(x) = f(x) \cdot T (f \cdot g)(x)
\]
\[
= f(x) \cdot 0
\]
\[
= f(x)
\]
\[
= f(x) \wedge T g(x).
\]

Proposition 2.3. An ideal \( J \) of \( \mathcal{F}S \) is prime if and only if \( J = V(x) \) for some \( x \in X \).

Proof. We first show any ideal of the form \( V(x) \) is prime. Suppose \( f \wedge g \in V(x) \). Then \( (f \wedge g)(x) = f(x) \wedge T g(x) = 0 \), which implies either \( f(x) = 0 \) or \( g(x) = 0 \) since \( T \) is totally ordered. Thus, either \( f \in V(x) \) or \( g \in V(x) \), and we see \( V(x) \) is a prime ideal.

Conversely, suppose \( J \) is a prime ideal. Since \( J \) is an ideal, we know from Theorem 2.2 that \( J = V(S) \) for some subset \( S \subseteq X \). If \( S = \emptyset \) — that is, \( |S| = 0 \) — then \( V(S) = \mathcal{F}S \), a contradiction since prime ideals are proper. If \( |S| \geq 2 \), take \( x, y \in S \) with \( x \neq y \). Let \( f, g \in \mathcal{F}S \) such that \( f(x) \neq 0 \) but \( f(s) = 0 \) for all \( s \in S \setminus \{ x \} \), while \( g(y) \neq 0 \) but \( g(s) = 0 \) for all \( s \in S \setminus \{ y \} \). Then \( (f \wedge g)(s) = 0 \) for all
s ∈ S, so f ∧ g ∈ V(S). Yet f, g /∈ V(S), and J = V(S) is not prime, a contradiction. Hence, we must have |S| = 1, meaning J is of the form V(x) for some x ∈ X.

**Lemma 2.4.** The map $V : \mathcal{P}(X) \rightarrow \text{Id}(\mathcal{FS})$ is bijective, where $\mathcal{P}(X)$ is the powerset of X.

**Proof.** The map $V$ is surjective by Theorem 2.2.

To see $V$ is injective, let $Y, Z \subseteq X$ with $Y \neq Z$. Without loss of generality, there is some $y \in Y \setminus Z$. Pick some non-zero $t \in T$, and define $f : X \rightarrow T$ by $f(x) = 0$ for all $x \in X \setminus \{y\}$ and $f(y) = t$. Then certainly $f \in V(Z)$ but $f \notin V(Y)$, and hence $V(Y) \neq V(Z)$. □

**Lemma 2.5.** Let $Y, Z \subseteq X$. Then

$V(Y) \cap V(Z) = V(Y \cup Z)$ and $V(Y) \cup V(Z) = V(Y \cap Z)$.

**Proof.** We begin with the first equality. Since $Y, Z \subseteq Y \cup Z$, we have $V(Y \cup Z) \subseteq V(Y), V(Z)$ since $V$ is order-reversing, and hence $V(Y \cup Z) \subseteq V(Y) \cap V(Z)$. For the other inclusion, simply note that any function vanishing on both $Y$ and $Z$ necessarily vanishes on $Y \cup Z$.

For the second equality, begin by noting that $Y \cap Z \subseteq Y, Z$, and so $V(Y), V(Z) \subseteq V(Y \cap Z)$. This implies $V(Y) \cup V(Z) \subseteq V(Y \cap Z)$. For the opposite inclusion, take $f \in V(Y \cap Z)$ and let $\text{supp}(f) = \{x_1, x_2, \ldots, x_k\} \subseteq Y^c \cup Z^c$. Pick a non-zero element $t \in T$, and for $i = 1, \ldots, k$ define $g_i : X \rightarrow T$ by

$g_i(x) \begin{cases} 
t & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases} \ .$

If $x_i \in Y^c$ then $g_i \in V(Y)$, and if $x_i \in Z^c$ then $g_i \in V(Z)$. So $\{g_1, g_2, \ldots, g_k\} \subseteq V(Y) \cup V(Z)$.

If we now mimic the steps taken in Theorem 2.2 we have $(\cdots ((f \cdot g_1^{n_1}) \cdot g_2^{n_2}) \cdots g_{k-1}^{n_{k-1}}) \cdot g_k^{n_k} = 0$. Since each $g_i \in V(Y) \cup V(Z)$ we have $f \in (V(Y) \cup V(Z)) = V(Y) \cap V(Z)$ by □. Thus, $V(Y \cap Z) \subseteq V(Y) \cap V(Z)$ and the equality follows. □

**Theorem 2.6.** The ideal lattice $\text{Id}(\mathcal{FS})$ is a Boolean algebra. In particular, $\text{Id}(\mathcal{FS}) \cong \mathcal{P}(X)$.

**Proof.** We know $\text{Id}(\mathcal{FS})$ is a bounded distributive lattice, so all that remains is to show we have complements in $\text{Id}(\mathcal{FS})$. Given an ideal
\( I = V(S) \), its complement in \( \text{Id}(\mathcal{F} \mathcal{S}) \) is the obvious natural candidate \( V(S^c) \), which follows from Lemma 2.5:

\[
V(S) \cap V(S^c) = V(S \cup S^c) = V(X) = \{0\} \\
V(S) \lor V(S^c) = V(S \cap S^c) = V(\varnothing) = \mathcal{F} \mathcal{S}.
\]

Thus, \( \text{Id}(\mathcal{F} \mathcal{S}) \) is a Boolean algebra.

We saw in Lemma 2.4 that \( V : \mathcal{P}(X) \to \text{Id}(\mathcal{F} \mathcal{S}) \) is bijective, and the preceding paragraph shows that \( V \) sends complements in \( \mathcal{P}(X) \) to complements in \( \text{Id}(\mathcal{F} \mathcal{S}) \). Lemma 2.5 shows that \( V \) sends meets to joins and vice versa. So \( V \) is a Boolean anti-isomorphism, but Boolean algebras are self-dual. Hence, \( \text{Id}(\mathcal{F} \mathcal{S}) \cong \mathcal{P}(X) \). □

**Theorem 2.7.** Every complete atomic Boolean algebra is the congruence lattice of a cBCK-algebra.

**Proof.** If \( \mathbb{B} \) is a complete atomic Boolean algebra, then \( \mathbb{B} \cong \mathcal{P}(X) \) for some set \( X \). But then

\[
\mathbb{B} \cong \mathcal{P}(X) \cong \text{Id}(\mathcal{F} \mathcal{S}(X, +)) \cong \text{Con}(\mathcal{F} \mathcal{S}(X, +)).
\]

□

As an application of this theorem, we show that any discrete topological space is the prime spectrum of a cBCK-algebra. We first remind the reader of some definitions.

Let \( \mathbf{A} \) be a cBCK-algebra. For \( S \subseteq \mathbf{A} \), define

\[
\sigma(S) = \{ P \in X(\mathbf{A}) \mid S \nsubseteq P \}.
\]

The collection \( \mathcal{J}(\mathbf{A}) = \{ \sigma(I) \mid I \in \text{Id}(\mathbf{A}) \} \) is a topology on \( X(\mathbf{A}) \). There are several proofs of this in the literature, but we point the reader to the paper [2], in which it is also shown that \( \mathcal{J}(\mathbf{A}) \cong \text{Id}(\mathbf{A}) \) as lattices.

**Definition 2.8.** The space \( (X(\mathbf{A}), \mathcal{J}(\mathbf{A})) \) is the (prime) spectrum of \( \mathbf{A} \).

Given a topological space \( Y \), let \( \mathcal{J}_Y \) denote the lattice of open subsets. We say \( Y \) is a spectral space if it is homeomorphic to the prime spectrum of a commutative ring. Hochster [5] proved that a space \( Y \) is a spectral space if and only if \( Y \) is compact, \( T_0 \), sober, and the compact open subsets form a basis that is a sublattice of \( \mathcal{J}_Y \). A generalized spectral space is a space satisfying the latter three of these conditions. Thus, a generalized spectral space which is compact is simply a spectral space.

Meng and Jun [10] proved that the spectrum of a bounded cBCK-algebra is a spectral space. The present author [4] showed that the spectrum of any cBCK-algebra is a locally compact generalized
spectral space, with compactness if and only if the algebra is finitely generated as an ideal.

In general, given a generalized spectral space $Y$, it is difficult to say whether or not that space is the spectrum of a cBCK-algebra. But it is known that $Y \simeq X(A)$ for some cBCK-algebra $A$ if and only if there is a lattice isomorphism $\mathcal{J}_Y \cong \text{Id}(A)$; see [4] for a proof.

**Corollary 2.9.** Let $Y$ be a discrete topological space. Then $Y$ is the spectrum of a cBCK-algebra.

**Proof.** Since $Y$ is a discrete space, the lattice of open subsets is $\mathcal{P}(Y)$, and so by Theorem 2.7 we have $\mathcal{J}_Y \cong \mathcal{P}(Y) \cong \text{Id}(\mathcal{FS}(Y, \mathbb{R}^+))$. Hence $Y \simeq X(\mathcal{FS}(Y, \mathbb{R}^+))$. \qed

**References**

[1] Ahsan, J., Thaheem, A.: On ideals in BCK-algebras. Mathematics Seminar Notes 5, 167–172 (1977)
[2] Aslam, M., Deeba, E., Thaheem, A.: On spectral properties of BCK-algebras. Math. Japonica 38(6), 1121–1128 (1993)
[3] Evans, C.M.: Spectral properties of commutative BCK-algebras. Ph.D. thesis, Binghamton University (2020)
[4] Evans, C.M.: Spectral properties of cBCK-algebras. Algebra Universalis (in press)
[5] Hochster, M.: Prime ideal structure in commutative rings. Trans. Amer. Math. Soc. 142, 43–60 (1969)
[6] Imai, Y., Iséki, K.: On axiom systems of propositional calculi, xiv. Proc. Japan Acad. Ser. A Math. Sci. 42(1), 19–22 (1966)
[7] Iséki, K., Tanaka, S.: Ideal theory of BCK-algebras. Math. Japonica 21, 351–366 (1976)
[8] Iséki, K., Tanaka, S.: An introduction to the theory of BCK-algebras. Math. Japonica 23, 1–26 (1978)
[9] Meng, J., Jun, Y.: BCK-Algebras. Kyung Moon Sa Co. (1994)
[10] Meng, J., Jun, Y.: The spectral space of MV-algebras is a Stone space. Sci. Math. Jpn. 1(2), 211–215 (1998)
[11] Palasinski, M.: On ideal and congruence lattices of BCK-algebras. Math. Japonica 26(5), 543-544 (1981)
[12] Romanowska, A., Traczyk, T.: On commutative BCK-algebras. Math. Japonica 25(5), 567–583 (1980)
[13] Romanowska, A., Traczyk, T.: Commutative BCK-algebras. Subdirectly irreducible algebras and varieties. Math. Japonica 27, 35–48 (1982)
[14] Tanaka, S.: On $\wedge$-commutative algebras. Mathematics Seminar Notes 3, 59–64 (1975)
[15] Traczyk, T.: On the variety of bounded commutative BCK-algebras. Math. Japonica 24(3), 283–292 (1979)

[16] Wronski, A.: BCK-algebras do not form a variety. Math. Japonica 28(2), 211–213 (1983)

[17] Yutani, H.: Quasi-commutative BCK-algebras and congruence relations. Mathematics Seminar Notes 5, 469–480 (1977)

C. Matthew Evans
Mathematics Department
Oberlin College
Oberlin, OH 44074
USA
URL: https://sites.google.com/view/mattevans
e-mail: mevans4@oberlin.edu