Discrete spectral triples and their symmetries

Mario Paschke*
Andrzej Sitarz†

Universität, 55130 Mainz, Germany

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Abstract: We classify 0-dimensional spectral triples over complex and real algebras and provide some general statements about their differential structure. We investigate also whether such spectral triples admit a symmetry arising from the Hopf algebra structure of the finite algebra. We discuss examples of commutative algebras and group algebras.

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1 Introduction

Recently Connes [1] has proposed a definition of spectral triples, which is supposed to extend the notion of the Riemannian geometry to a noncommutative setup. This could be a natural geometrical framework for theories of fundamental interactions and gravity at a small distance scale and possibly it could open a way towards quantum gravity.

One of the promising applications to fundamental particle physics is an attempt to explain the particle content and interaction pattern in the Standard Model. In the spectral triple in question the finite algebra $M_3(\mathbb{C}) \oplus \mathbb{H} \oplus \mathbb{C}$ plays the significant role. Here we shall not discuss this particular physical application.

In this paper we derive the general structure of 0-dimensional spectral triples and of the Dirac operator for complex and real semisimple algebras. Moreover, assuming that such algebras are Hopf algebras we discuss possible symmetry structures. As examples we present $\mathbb{C}^n$ commutative algebra with discrete group structure as well as group algebras, particularly $\mathbb{C}S_3$ for the permutation group $S_3$.

2 Complex Spectral Triples

Let $\mathcal{A}$ be a finite dimensional semi-simple algebra over the field of complex numbers $\mathbb{C}$. Such an algebra is isomorphic to $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ for some non-zero integers $n_1, \ldots, n_k$ (see [3], pp.29-36).

A spectral triple for a finite algebra means a finite Hilbert space $\mathcal{H}$, a faithful representation $\pi : \mathcal{A} \to B(\mathcal{H})$ over the field $\mathbb{C}$, a reality structure $J$, such that $J$ is an antilinear isometry of $\mathcal{H}$, $J^2 = 1$ and $\pi^0 = J \circ \pi^* \circ J$ is a representation of the opposite algebra $\mathcal{A}^0$, which commutes with $\pi$:

$$[\pi^0(a), \pi(b)] = 0, \quad \forall a, b \in \mathcal{A}$$

We assume that there exists a grading $\gamma \in B(\mathcal{H})$, such that $\gamma^* = \gamma$, $\gamma^2 = 1$, $J\gamma = \gamma J$, $\gamma \pi(a) = \pi(a) \gamma$ and $\gamma$ is equal to $\pi(c)$ for some Hochschild cocycle $c \in Z_0(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0)$.

We begin with some simple observations.

\footnote{For a detailed discussion of the Standard Model spectral triple see our forthcoming paper [2].}

\footnote{$C^*$ algebras are semisimple, thus we consider only them. Of course, nonsemisimple algebras (grassman variables, for instance) may also appear in physical models. Whether one can successfully construct a corresponding theory for them remains an interesting problem.}
Observation 1. $\mathcal{H}$ is a bimodule over $\mathcal{A}$. If $\xi \in \mathcal{H}$ then we have: $a\xi = \pi(a)\xi$ and $\xi a = \pi^0(a)\xi$.

We can now use the structure of the finite algebra $\mathcal{A}$ to learn more about the Hilbert space of the spectral triple.

Lemma 1. Let $P_i$ be the element of $\mathcal{A}$ with identity matrix in the $i$-th entry and zeroes elsewhere. Let $H_{ij}$ be $P_i\mathcal{H}P_j$. Then $\mathcal{H}$ is a direct sum of all $H_{ij}$:

$$\mathcal{H} = \bigoplus_{i,j} H_{ij}$$

and if for every $a \in \mathcal{A}$ we denote $P_i a$ ($= a P_i$) as $a_i$ we have:

$$(a\xi)_{ij} = a_i \xi_{ij}, \quad (\xi a)_{ij} = \xi_{ij} a_j$$

The proof is a simple calculation and we skip it.

Lemma 2. The Hilbert subspace $H_{ij}$ has the following form:

$$H_{ij} = C^{n_i} \otimes C^{r_{ij}} \otimes C^{n_j}$$

where $r_{ij}$ are some integers, $r_{ij} \geq 0$. We take $H_{ij} = 0$ iff $r_{ij} = 0$. An element $a$ acts on $H_{ij}$ from the left as $a_i \otimes 1 \otimes 1$ and from the right as $1 \otimes 1 \otimes a^T_j$, where $^T$ denotes the matrix transposition.

From the previous lemma we know that $H_{ij}$ is the representation space for $M_{n_i}(\mathbb{C})$ and for the opposite algebra of $M_{n_j}(\mathbb{C})$. Because these representations commute, the structure of $H_{ij}$ must have the above form.

Lemma 3. The grading operator $\gamma$ is $\pm 1$ when restricted to the subspace $H_{ij}$.

Proof: Since $\gamma$ commutes both with the elements of $\mathcal{A}$ and $J$, and is self-adjoint, we conclude that it also commutes with the right multiplication on $\mathcal{H}$:

$$a(\gamma \xi)b = \gamma(a \xi b)$$

Hence, $\gamma H_{ij} \subset H_{ij}$. When restricted to $H_{ij}$, $\gamma$ is an isometry, which commutes with the left multiplication by matrices from $M_{n_i}(\mathbb{C})$ and right multiplication by matrices from $M_{n_j}(\mathbb{C})$. Therefore it must be of the form:

$$\gamma_{ij} : H_{ij} \to H_{ij}, \quad \gamma = 1_{n_i} \otimes \Gamma_{ij} \otimes 1_{n_j},$$
where $\Gamma_{ij}$ is an arbitrary self-adjoint matrix, such that $\Gamma_{ij}^2 = 1$ acting on $\mathbb{C}^{r_{ij}}$. However, if we take into account that $\gamma$ is an image of a Hochschild cocycle, i.e., there exists an element $c$ of $\mathcal{A} \otimes \mathcal{A}^0$ such that $\pi(c) = \gamma$, we see that $\Gamma_{ij}$ must be proportional to the identity matrix, so it leaves only the possibility for $+1$ or $-1$.

**Lemma 4.** The reality $J$ maps $H_{ij}$ onto $H_{ji}$, therefore $r_{ij}$ must be equal to $r_{ji}$.

**Proof.** Let us take $\xi \in H_{ij}$. There exists $v \in \mathcal{H}$ such that $\xi = P_i v P_j$. Next, calculate $J\xi$, writing all expressions carefully:

$$J\xi = J P_i v P_j = J \pi(P_i)(J \pi(P_j)J) v = (JP_iJP_j)(Jv) = P_j(Jv) P_i \in H_{ji}.$$

Of course, since $J^2 = 1$ we must have $r_{ij} = r_{ji}$ so that the dimensions of both Hilbert spaces are equal.

**Observation 2.** If we denote by $\gamma_{ij}$ the sign of the grading on $H_{ij}$ then $\gamma_{ij} = \gamma_{ji}$.

This is a trivial consequence of the commutation relation $[J, \gamma] = 0$.

We can now introduce $q_{ij} = r_{ij}\gamma_{ij}$, which, later on, turns out to be the intersection form for our spectral triple. So far, we know that $q_{ij}$ is a symmetric matrix with integer entries.

**Lemma 5.** There exists a basis of $H_{ij}, H_{ji}$ of the form $H_{ij} \ni v = v_i \otimes v_{ij} \otimes v_j$ such that

$$Jv = v_j \otimes v_{ji} \otimes v_i \in H_{ji}. \quad (1)$$

**Proof:** We begin with the $i \neq j$ case. First of all, let us fix an orthonormal basis of the spaces $H_{ij}$ and $H_{ji}$ of the form $e_1 \otimes e_2 \otimes e_3$ and define $\tilde{J}$ as an antilinear isometry operator, which exchanges the first and the third element of the tensor product:

$$\tilde{J}(e_1 \otimes e_2 \otimes e_3) = e_3 \otimes e_2 \otimes e_1$$

Then $\tilde{J}$ satisfies all the axioms for the spectral triple, i.e., $\tilde{J}^2 = 1$ and $a^0 = \tilde{J}a \tilde{J}$.

Next, observe that $J \circ \tilde{J}$ is a linear invertible map, such that:

$$J \circ \tilde{J}a = aJ \circ \tilde{J} \quad (2)$$
and

\[ J \circ \bar{J}a^0 = a^0 J \circ \bar{J} \]

We prove only the first of the two:

\[
J \circ \bar{J} = J \circ \bar{J}aJ^2 = J(a^\dagger)^0 \bar{J} = J^2aJ \circ \bar{J} = aJ \circ \bar{J}.
\]

Therefore \( J \circ \bar{J} \) commutes both with the left and the right multiplication, so it has to be of the form: \( 1 \otimes \bar{l} \otimes 1 \), where \( \bar{l} \) is a linear isometry \( \bar{l} : C^{\tau ij} \rightarrow C^{\tau ji} \). Thus \( J = (1 \otimes \bar{l}^{-1} \otimes 1) \circ \bar{J} \) and the only possible nontrivial part of \( J \) comes from \( \bar{l} \).

In the last step we have to consider the cases \( j = i \) and \( j \neq i \) separately. The latter one is simpler, we just chose any orthonormal basis \( e_a \) of \( C^{\tau ij} \) (same as in the definition of \( \bar{J} \)) and then \( \bar{l}^{-1}e_a \) constitutes a basis of \( C^{\tau ji} \). Take as \( v_{ij} \) just the vectors of the chosen basis and \( \bar{v}_{ji} \) their image under \( \bar{l}^{-1} \). Then the property (1) follows immediately.

A more subtle situation occurs in the \( i = j \) case. Here, however, we can use another simple argument. Take \( v \in C^{\tau ii} \). Then either \( \bar{l}^{-1}v \) is proportional to \( v \) or is linearly independent from it. In the latter case it is sufficient to consider \( v_{ii} \) to be \( v + \bar{l}^{-1}v \) and \( i(v - \bar{l}^{-1}v) \). In the first case \( \bar{l}^{-1}v = e^{i\phi}v \) and then \( e^{i\phi}v \) will do as \( v_{ii} \). Since our space is finite dimensional and \( \bar{l} \) is an isometry we can carry on with this procedure until we find a complete basis of \( H_{ii} \) satisfying our requirement.

### 2.1 The Dirac operator

Up to this point we have constructed the basic geometry of a finite spectral triple and in the next step we shall introduce the Dirac operator, which gives the differential and metric structures.

The Dirac operator \( D \) of a finite spectral triple is a self-adjoint linear operator on \( H \), which commutes with \( J \): \( DJ = JD \), anticommutes with \( \gamma \): \( D\gamma = -\gamma D \) and satisfies the following relation:

\[
[[D, \pi(a)], \pi^0(b)] = 0, \quad \forall a, b \in \mathcal{A}.
\] (2)

**Observation 3.** Let \( D_{ij,kl} = P_{ij}DP_{kl} \), where \( P_{ij} \) is the projection operator on \( H_{ij} \); \( (P_{ij}v = P_{i}vP_{j} \) ). Then the following is true:

1. \( D_{ij,kl} : H_{kl} \rightarrow H_{ij} \) is a linear map, which is zero unless \( i_{ij}\gamma_{kl} < 0 \) (i.e., they are of opposite sign)
2. \( D^\dagger_{ij,kl} = D_{kl,ij} \). This follows easily from the requirement \( D = D^\dagger \).
3. $D_{ij,kl} = JD_{ji,lk}J$. This follows from the relation $[D, J] = 0$ and $J^2 = id$. If we choose the basis of the spaces $H_{ij}$ so that the operator $J$ has a preferred form $Jv = \pm v$, the above condition can be simplified to $D_{ij,kl} = D_{ji,lk}^*$, where * means the complex conjugation of matrix elements of $D_{ji,lk}$ in the preferred basis.

The strongest restriction for the Dirac operator follows from the commutation property (2).

**Lemma 6.** The Dirac operator component $D_{ij,kl}$ vanishes unless $i = k$ or $j = l$. In either of these cases $D$ must commute with the corresponding action of the $A$ or $A^0$, respectively.

**Proof:** Let us rewrite the condition (2) for $v \in H_{kl}$:

$$D_{ij,kl}(a_kvb_l) - a_iD_{ij,kl}(vb_l) - (D_{ij,kl}(a_kv))b_j + a_i(D_{ij,kl}v)b_j = 0$$

Choosing $b = P_j j \neq l$ we get that $D_{ij,kl} = 0$ unless $i = k$ and, for $i = k$, $D_{ij,il}a_i = a_iD_{ij,il}$ for every $a_i \in A$.

Similarly, if we choose $a = P_i i \neq k$ we get that $D_{ij,kl} = 0$ unless $j = l$ and additionally $D_{ij,kj}(vb_j) = (D_{ij,kj}v)b_j$.

Of course, the case $i = k$ and $j = l$ is excluded by the requirement that $D$ acts only between spaces of different grading.

### 2.2 Differential algebra

The Dirac operator $D$ provides us with both the metric and the differential properties on the finite space corresponding to the algebra $A$. We shall now concentrate on some general properties of the differential structure.

**Observation 4.** The representation $\pi$ of $A$ extends to the universal differential algebra $\Omega_uA$, by:

$$\pi(a_0da_1d_2\cdots da_n) = \pi(a_0)[D, \pi(a_1)]\cdots[D, \pi(a_n)].$$

The differential calculus is usually constructed as follows: first, $\Omega^1(A)$ is the quotient of $\Omega^1_uA$ by the kernel of $\pi$. Therefore the bimodule $\Omega^1(A)$ is isomorphic to $\pi(\Omega^1_u(A))$. Higher order forms are obtained by taking the quotient of the universal forms of a given order by the ideal $\Omega^0_uA \cap (\ker \pi \cup d\ker \pi$).

Let us state some general results.
Lemma 7. The first order differential calculus is inner, there exists an one-form:

\[ \xi = \sum_{i \neq j} P_i dP_j, \]  

such that for every \( a \in \mathcal{A} \) one has: \( da = [\xi, a] \). The Dirac operator \( D = \pi(\xi) + J\pi(\xi)J \).

Proof: Of course, from the definition we have \( \pi(da) = [D, \pi(a)] \), so one has to show:

\[ \sum_{i \neq j} [\pi(P_i) [D, \pi(P_j)], \pi(a)] = [D, \pi(a)]. \]

Let us compute \( \xi_{ij,kl} = P_{ij} \pi(\xi) P_{kl} \). For any \( h \in H_{kl} \) we have:

\[
(p(\xi)h)_{ij} = \left( \sum_{s \neq t} P_s [D, P_t] h \right)_{ij} = \sum_{s \neq t} (P_s DP_t h)_{ij} = (P_i DP_k h)_{ij} = \delta_{jl} D_{ij,kj} h.
\]

Note that:

\[ \xi = \sum_{i \neq j} P_i dP_j = -\sum_i P_i dP_i. \]

The relation \( D = \pi(\xi) + J\pi(\xi)J \) follows immediately from the properties of \( J \) and \( D \), additionally \( [J\xi, a] = 0 \) for every element \( a \in \mathcal{A} \), so that the property \( da = [D, a] = [\xi, a] \) holds.\[\square\]

Observation 5. \( \pi(\xi) \) is a selfadjoint operator, thus the first order calculus has a natural involution map such that \( d \circ \ast = -(\ast \circ d) \) if we define \( \xi^* = \xi \).

The one-form \( \xi \) has some more interesting properties, which make the task of determining the differential structure easier. Before we discuss the construction of \( \Omega^2(\mathcal{A}) \) let us make another useful remark.

Observation 6. The bimodule \( \Omega^1(\mathcal{A}) \) has no center, if \( a\omega = \omega a \) for every element \( a \in \mathcal{A} \) then \( \omega = 0 \).

Proof: First, let us observe that for every idempotent \( e = e^2 \), we have \( e de e = 0 \). This follows from differentiating \( 0 = e^2 - e \) and multiplying the result by \( e \) from the left and from the right. Therefore \( P_i dP_i P_i = P_i \xi P_i = 0 \).
Every element of $\Omega^1(A)$ could be written as a finite sum of the type $\omega = \sum b_a \xi c_a$. Suppose that it commutes with every $a \in A$, in particular with $P_i$. Then if we multiply both sides by $P_i$, since $P_i^2 = P_i$ we get $P_i \omega P_i = P_i \omega$. However $P_i \omega P_i = 0$ as $P_i$ commute with elements of the algebra and $P_i \xi P_i = 0$. So $P_i \omega = 0$ for every $i$, and therefore $\omega = 0$. ■

**Lemma 8.** $d\xi = \xi \xi + \sum P_i \xi \xi P_i$

*Proof:*

\[
 d\xi = -\sum_i dP_i dP_i = -\sum_i [\xi, P_i][\xi, P_i] = \xi \xi + \sum_i P_i \xi \xi P_i. 
\]

The above equality holds only within the differential algebra, which means that all products of one-forms in the line above must be calculated using the product in $\Omega^2(A)$. We have not determined it yet (in a general situation it is a difficult technical calculation), however, using the above relations we will be able to say something about it.

**Lemma 9.** Let $\Xi \in \Omega^1(A) \otimes_A \Omega^1(A)$ be

\[
 \Xi = \sum_i P_i \xi \otimes_A \xi P_i. 
\] (4)

Let $J_1 = \ker \pi$ (where $\pi$ on $\Omega^1(A) \otimes_A \Omega^1(A)$ is defined in the usual way: $\pi(\omega_1 \otimes_A \omega_2) = \pi(\omega_1) \pi(\omega_2)$), $J_2$ be the subbimodule generated by commutators $[a, \Xi]$ for all $a \in A$ and finally let $J_3$ be the subbimodule generated by all elements of the form $\sum_i a_i (\xi \otimes_A \xi - \Xi) b_i$ for all $a_i, b_i$ such that $\sum_i a_i b_i = 0$.

Then $J = J_1 \oplus J_2 \oplus J_3$ is a subbimodule of $\Omega^1(A) \otimes_A \Omega^1(A)$ and $\Omega^2(A) = \Omega^1(A) \otimes_A \Omega^1(A) / J$.

*Proof:* We know that every element of $\Omega^1(A)$ can be written as a finite sum of $\omega = a \xi b$. One easily finds $d\omega$:

\[
 d\omega = \xi \omega + \omega \xi + a(\tilde{\Xi} - \xi \xi) b, 
\] (5)

and, in a special case, when $\omega$ is of the form $da = \xi a - a \xi$, we get:

\[
 d^2 a = \tilde{\Xi} a - a \tilde{\Xi}, 
\] (6)

where $\tilde{\Xi}$ denotes an element in $\Omega^2(A)$ which is the image of $\Xi$. Now, we clearly see that division by $J_2$ guarantees that $d$ is well-defined ($d0 = 0$) and $d^2 \equiv 0$ if and only if we quotient out $J_3$. 

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Note, that in order to obtain a differential algebra alone we do not need to include \( J_1 \) as the quotient. However, we proceed so in order to be in agreement with the standard definition of the differential algebra for spectral triples.

Before we start considering an interesting simplification, let us say a bit more about the interpretation of \( \xi \) and \( \Xi \). The image of the latter (under \( \pi \)) is an operator which links pairs of subspaces of \( \mathcal{H} \), which have the same first index \( i \). Clearly, it means that the Dirac operator “connects” \( H_{ik} \rightarrow H_{pk} \rightarrow H_{ik} \), which is always the case, as \( \xi \) is selfadjoint (for every map \( H_{ik} \rightarrow H_{pk} \) there exists a conjugate map \( H_{pk} \rightarrow H_{ik} \)). Now, \( \pi(\Xi - \xi \xi) \) is a set of maps between subspaces of \( \mathcal{H} \) with different first index, so it does not vanish if and only if \( \xi \) connects \( H_{ij} \rightarrow H_{kj} \rightarrow H_{pj} \) for some \( p \neq i \) and the composition of these maps does not vanish.

An interesting situation occurs when \( \pi(\xi \otimes_A \xi) = \pi(\Xi) \), so that \( J_2 \subset J_1 \).

2.2.1 Inner calculi for spectral triples.

**Lemma 10.** When \( \pi(\xi \otimes_A \xi) = \pi(\Xi) \) then \( \tilde{\Xi} = \xi \xi \) and the calculus remains inner in \( \Omega^2(A) \):

\[
d\omega = \xi \omega + \omega \xi,
\]

for any one-form \( \omega \). The submodule \( J \) contains the kernel of \( \pi \) and all commutators \([a, \xi \otimes_A \xi] \).

This happens, for instance, if \( \pi(\xi)\pi(\xi) \) is in \( \pi(A) \). In addition, if it commutes with \( \pi(A) \) then \( J = \ker \pi \) and the structure of \( \Omega^2(A) \) simplifies considerably.

The next lemma makes these observations more precise:

**Lemma 11.** If \( \pi(\xi)\pi(\xi) \) commutes with \( \pi(Z(A)) \), where \( Z(A) \) is the center of \( A \) then \( \pi(\xi)\pi(\xi) = \pi(\Xi) \). The converse is also true.

**Proof:** Of course, if \( \pi(\xi)\pi(\xi) \) commutes with \( \pi(Z(A)) \) it commutes with every \( \pi(P_i) \). Using \( \sum_i P_i \equiv 1 \) we get the required identity. For the converse, let us assume that there exists \( i \) such that \( \pi(P_i) \) does not commute with \( \pi(\xi)\pi(\xi) \) and let the commutator be \( \rho \). Then it is easy to verify that \( \pi(\xi)\pi(\xi) - \pi(\Xi) = \rho \).

It is quite interesting to characterize the space, in which \( \pi(\xi)\pi(\xi) \) takes its values for this particular situation. We already know that it commutes with \( \pi(A^0) \). Since additionally it has to commute with the center of \( \pi(A) \) the only possibility is that it is diagonal (i.e., when restricted to \( H_{ij} \) maps
it to itself), and using the decomposition into tensor products presented in section 2, we find that it is $T_{ij} \otimes \text{id}$ on each of the subspaces $H_{ij}$.

If the calculus (at least up to $\Omega^2(A)$) is inner, we can immediately make an interesting observation about the gauge potentials.

**Observation 7.** If $H$ is a selfadjoint one-form and $F(H) = dH + HH$ is its curvature form, then for the inner calculus, as described above, $F(-2\xi) = 0$.

This follows from a simple calculation. Of course, we have presented here only the formal solution of the $F(H) = 0$ equation in $\Omega^2(A)$. It does not guarantee that this solution is unique, it could happen (for instance, if we take a tensor product of the spectral triple in question with a triple corresponding to a manifold) that there are far more solutions or even that the whole subbimodule $\Omega^2(A)$ for the discrete part vanishes.

### 2.2.2 Commutative discrete spectral triples.

A lot more can be said about the commutative case $A = \mathbb{C}^n$. The generators of the algebra can be identified with the projection operators $P_i$ introduced earlier. The structure of $\Omega^1(A)$ is completely determined by the elements $P_i dP_j$, $i \neq j$ (using $\sum_i dP_i = 0$). In our representation we have:

$$\pi(P_i dP_j)h = P_i D P_j h - P_i \delta_{ij} D h,$$

which for the $kl$-component becomes:

$$(\pi(P_i dP_j)h)_{kl} = \delta_{ik} D_{klp} h_{jp} - \delta_{ik} \delta_{ij} D_{klp} h_{ps} = \ldots,$$

which after taking into account the properties of $D$, becomes:

$$\ldots = \delta_{ik} D_{kljl} h_{jl} - \delta_{ik} \delta_{ij} D_{klpl} h_{pl}.$$

Since we are interested in the case $i \neq j$ we obtain that for $P_i dP_j \neq 0$ it is sufficient that there exists $l$ such that $D_{ijjl} \neq 0$.

Let us now investigate $\Omega^2(A)$, which would be built from the bimodule $\Omega^2_u(A)$ after we quotient out the subbimodule generated by ker $\pi \cup d$ker $\pi$.

First, we show the representation action of the basis of $\Omega^2_u(A)$, $e_{kij} = P_k dP_i dP_j$, $k \neq i$ and $i \neq j$ (again, to see that this is the basis we use the identity $\sum_i dP_i = 0$).

**Observation 8.** For $i, j, k$ such that $k \neq i$ and $i \neq j$ $\pi(e_{kij})$ is a collection of maps $H_{jl} \mapsto H_{kl}$, each of the form:

$$\pi(e_{kij}) = D_{klij} D_{ijjl},$$
Next we shall show the structure of $d\ker \pi$.

Let us take a one-form $\rho$ in $\ker \pi$. Of course, it is sufficient to take a generator: $\pi(P_i dP_j) = 0$ for some $i \neq j$.

Let us calculate $\pi(dP_i dP_j)$. After little calculation we get the following result:

$$\pi(dP_i dP_j) h = -P_i D^2 P_j h, \ \text{if} \ P_i dP_j \in \ker \pi, i \neq j.$$ \hfill (8)

Rewriting all that in terms of the basis of $\Omega^2_u$ we obtain that for each $i \neq j$ if $\pi(P_i dP_j) = 0$ we have the following generator added to the junk:

$$\sum_{i \neq s \neq j} P_i dP_s dP_j$$

An easy way to visualize the construction of the differential structure coming from the commutative spectral triple is by way of graphs. Imagine we have $n$ vertices, and each vertex is split into $n$ points. The nonzero entries of the Dirac operator shall correspond to arrows linking two points of two different vertices or within the same vertex (of course, due to the reality structure there exists a link between the two).

If we look at the differential forms, the fine structure of the vertices is less important.

Now, as for the one forms, we know that $P_i dP_j, i \neq j$ exists if there is at least one arrow connecting vertex $j$ with $i$. For a two-form $P_k dP_i dP_j$ there must exist links $j \rightarrow i \rightarrow k$. However, if there is no direct link $j \rightarrow k$ (note that due to chirality this is always the case) then the sum of all forms $\sum P_k dP_i dP_j$ over $i, i \neq k$ and $i \neq j$ vanishes. This bears a resemblance to some general structure of differential geometry on graphs [8].

2.2.3 The metric

On the algebra $\mathcal{A}$ we have a natural trace coming from the representation on the Hilbert space $\mathcal{H}$. Since our representation extends naturally to the differential algebra, such a trace gives a generic scalar product on the space of one-forms. In particular, for the generators of the commutative algebra (which is the easiest example) we have:

$$(P_i dP_j, P_k dP_l) = \text{Tr} (\pi^*(P_i dP_j) \pi(P_k dP_l))$$

$$= \delta_{ik} \delta_{jl} \sum_p \text{Tr}(D_{l p k p} D_{k p l p}).$$
The last term of the expression is a selfadjoint map from $H_{lp}$ onto itself, therefore the scalar product is positive. All generating one-forms $P_i dP_j$ are orthogonal to each other and the norm of each is fixed by the elements of the Dirac operator. Of course, should the norm be zero then it is obvious that $\forall \rho D_{ip}d\rho = 0$ and hence the one-form $P_i dP_j$ vanishes.

The above choice is not unique, and one may consider its deformation of the following type:

$$(\omega, \rho)_Z = \text{Tr} \left( Z \pi(\omega) \pi(\rho)^\dagger \right),$$

where $Z$ is an operator, for which it is sufficient to assume that $Z$ is self-adjoint, and, to guarantee the gauge invariance, $\pi(U)Z\pi(U)^\dagger = Z$ for every unitary $U$ in the algebra.

The reason for introducing such a scaling may follow from the requirement that the trace has some additional symmetry (for instance we might fix it to recover the Haar measure on the algebra, when it has a Hopf algebra structure).

Of course, changing the trace would have significant physical consequences. For details of the construction and application to the Standard Model, see the forthcoming paper [3].

### 2.3 Finite spectral triples with real algebras.

So far we have discussed finite spectral triples with algebras over $\mathbb{C}$. Since the possible application of noncommutative geometry in high energy physics uses rather a real algebra, we shall now briefly discuss the above construction assuming that $\mathcal{A}$ is over the field of real numbers.

**Lemma 12 (See [4]).** The simple finite algebras over $\mathbb{R}$ are of the type $M_n(F)$, where $F$ can be $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Therefore every semi-simple algebra over $\mathbb{R}$ can be decomposed as:

$$\mathcal{A} = \bigoplus_i M_n(F_i), \quad (9)$$

where each $F_i$ could be the field of real, complex or quaternionic numbers.

We shall now investigate irreducible representations of $\mathcal{A}$ on a complex Hilbert space. If one considers $M_n(\mathbb{R})$ and $M_n(\mathbb{H})$ as real algebras, then each of these has only one irreducible representation on $\mathbb{C}$ (for quaternions the conjugate representation is equivalent to the fundamental one). In the case of $M_n(\mathbb{C})$ we have two inequivalent irreducible representations:

$$\pi(a) = a \quad \bar{\pi}(a) = \bar{a}$$
where $\bar{a}$ means the involution of matrix elements.

How does the description of our spectral triple change? Apparently for $M_n(\mathbb{R})$ and $M_n(\mathbb{H})$ there is no change at all, whereas for $M_n(\mathbb{C})$ we have to take into account the existence of these two inequivalent representations:

**Lemma 13.** If $F_i = \mathbb{C}$ then the Hilbert subspace $H_{ij}$ decomposes into $H_{ij}$ and $\bar{H}_{ij}$, with the representation of $M_n(F_i)$ by $m$ and $\bar{m}$, respectively.

Of course, an analogous decomposition takes place if one takes into account the right action of the algebra on $\mathcal{H}$.

Until now, there has not been much change apart from the additional splitting of the Hilbert space. All other results remain unchanged, in particular, $\gamma$ cannot be different for $H_{ij}$ and $\bar{H}_{ij}$.

If we look at the restrictions for the Dirac operator, which come from (2) we see that the constraints on allowed values of $D$ also remain unchanged, however, one should add some more, which arise from the fact that one cannot mix the fundamental and the conjugated representation. To see this explicitly, in the proof of Lemma 6 one should replace $b = P_j$ by $b = z P_j$ for any complex number $z$.

Generally speaking, the derivation of the constraint which we had for the complex case, could be repeated separately for each of the real situations. Therefore, we may summarize the result by stating that $D$ does not mix two inequivalent representations of the same algebra.

### 2.4 The intersection form.

The intersection form in $K$-theory is a map $K_*(A) \times K_*(A) \to \mathbb{Z}$ and its invertibility (Poincaré duality) is fundamental for a characterization of homotopy types of spaces, which possess the structure of a smooth manifold.

For finite complex spectral triples the intersection form, evaluated on generators of the $K_0$-group, $e, f$, is $[1]$:

$$\langle e, f \rangle = \langle D, e \otimes f^o \rangle,$$

and, generally, the pairing $\langle D, E \rangle$ for a projector $E$ means:

$$\langle D, E \rangle = \dim \ker_{\mathcal{H}_R}(ED^+E) - \dim \ker(ED^+E)$$

where $\mathcal{H}_L = \frac{1}{2}(1 + \gamma)\mathcal{H}$, $\mathcal{H}_R = \frac{1}{2}(1 - \gamma)\mathcal{H}$ and $D^+ = \frac{1}{4}(1 - \gamma)D(1 + \gamma)$.

Since for matrix algebras the group $K_1$ is trivial (any element can be deformed to 1), the only nontrivial part is $K_0$. For $M_n(\mathbb{C})$ all projectors are equivalent (within $M_\infty(M_n(\mathbb{C}))$) to a diagonal matrix with 1 in the first
diagonal entry and zeroes elsewhere, which we shall call $e$. Of course, for a direct sum of $N$-simple algebras the corresponding $K$-theory will have $N$ independent generators, each coming from the simple component of the sum.

Let us calculate the $\langle e_i, e_j \rangle$ entry. First, we have to determine the dimensions of Hilbert subspaces appearing in the above formulae:

\[
\dim e_i e^0_j \mathcal{H}_L = \begin{cases} r_{ij} & \text{if } \gamma_{ij} = -1 \\ 0 & \text{if } \gamma_{ij} = 1 \end{cases} \quad (10)
\]

\[
\dim e_i e^0_j \mathcal{H}_R = \begin{cases} r_{ij} & \text{if } \gamma_{ij} = 1 \\ 0 & \text{if } \gamma_{ij} = -1 \end{cases} \quad (11)
\]

Now, observe that out of $e_i e^0_j \mathcal{H}_L$ and $e_i e^0_j \mathcal{H}_R$ one must always be empty, so the $e_i e^0_j D^+ e_i e^0_j$ operator, which acts between them has either empty domain or empty target space. Thus, the index is independent of $D$, and reads:

\[
\langle e_i, e_j \rangle = \gamma_{ij} r_{ij} = q_{ij}
\]

so we recover the matrix $q$ as our intersection form.

For real algebras the calculation is slightly more complicated. First, one has to take into account the form of the projectors for quaternions, which results in the doubling of the corresponding entries in the intersection form. Moreover, for real algebras the group $K_1$ is nontrivial, as $K_1(M_n(\mathbb{R})) = \mathbb{Z}_2$, however, this still does not change the intersection form if one does not take into account torsion in $K$-theory.

3 Hopf algebra symmetries.

Spectral triples are expected to be an extension of the notion of Riemannian manifolds to noncommutative geometry. Similarly as in the standard differential geometry we may attempt to explore symmetries of such manifolds. We expect, however, that the correct idea of a symmetry should be extended from a group to a Hopf algebra (for a good introduction to Hopf algebras and quantum groups we refer the reader to [5]). Another potentially interesting point is the speculation [1, 6, 7] that the finite algebra of the Standard Model originates from the $q$-deformation of the Spin structure and may possess some Hopf algebra symmetry, for details see also [6].

We shall try to answer the question whether spectral triples admit (and if so, to what extent) a symmetry understood as a Hopf algebra symmetry. Of course, the natural thing is to use the Haar measure as a trace on the algebra and to extend it to the whole representation space. Another possibility is to
demand that differential structures of the spectral triple are also symmetric, for instance, that they are left-covariant or bicovariant. Furthermore, the Hilbert space may be investigated for the existence of a comodule structure. Last not least, we may turn our attention to the tensor product of representations (this is allowed only if a coproduct on the algebra exists), which physically has the meaning of constructing composite states. In this paper we concentrate our efforts on the first two aspects of symmetries, leaving the others for future investigations.

3.1 Discrete group structure on $\mathbb{C}^n$ algebra

One of the simplest examples of real spectral triples are those based on a finite-dimensional commutative algebra $\mathbb{C}^n$ understood as function algebras on a discrete $n$-point space. The general construction scheme was presented by Connes [1] and falls into the general classification of complex spectral triples presented in the previous section.

Suppose that we assume the discrete space to have the structure of a finite group and demand that the first-order differential calculus generated by $D$ is bicovariant.

First, let us recall the basics of the bicovariant differential structure for finite groups (details in [10]). The calculus is generated by left-covariant forms:

$$\chi^g = \sum_h e_{gh} de_g$$

where $e_g$ is a Kronecker-delta function: 1 at $g$ and zero elsewhere. The right coaction on them is:

$$\Delta_R\chi^g = \sum_h \chi^{hg^{-1}} \otimes e_h$$

Using the spectral triple approach, we may calculate the $\chi^g$ forms as operators on our Hilbert space. It appears that

$$(\chi^g h)_{ij} = D_{ij,(ig^{-1})j} h_{(ig^{-1})j}.$$  

Because of the properties of $D$ we see that it is only possible that $j = l$ and therefore finally:

$$(\chi^g h)_{ij} = D_{ij,(ig^{-1})j} h_{(ig^{-1})j};$$

so it is a collection of operators from $H_{(ig^{-1})j} \to H_{ij}$. Of course, if $\gamma_{ij} \gamma_{(ig^{-1})j} = 1$ they all vanish because $D$ acts only between spaces of different chirality.

What we have obtained so far is the representation of left-invariant forms as operators on our Hilbert space. We see that all $\chi^g$, which do not vanish are linearly independent (since they act between different Hilbert subspaces).
If we demand that the calculus is bicovariant we immediately see that if for any $g$, $\chi^g$ vanishes, then for every $h$ $\chi^{gh^{-1}}$ must vanish as well. Of course, for commutative groups such a condition is void, however, for nonabelian ones it leads to certain restrictions on possible forms of $D$, which we make precise in the following lemma:

**Lemma 14.** If the spectral-triple calculus on $\mathbb{C}^n$ with a structure of the discrete group $G$ is bicovariant then:

- if $\chi^g \neq 0$ then $\forall i \in G \exists j \in G : D_{ij,(ig)j} \neq 0$
- if $\chi^g = 0$ then $\forall i \in G \forall j \in G : D_{ij,(ig)j} = 0$

The proof is an immediate consequence of previous observations.

### 3.1.1 Function algebra on $S_3$.

Consider as an example the 6-dimensional space $S_3$. We shall check what restrictions on the spectral triple and the Dirac operator are set by the requirement that the corresponding calculus be bicovariant.

First, let us take the universal calculus. It appears that there exists a spectral triple which gives such a calculus. The simplest example is given by the following intersection form:

\[
\begin{pmatrix}
-1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 \\
\end{pmatrix}
\]

and is 36-dimensional! It is quite easy to see that this must be the case - for each $\chi^g$ not to vanish, the Dirac operator must link (for every $i$) some $H_{(ig^{-1})j}$ with $H_{ij}$. For the intersection matrix it means that for every row there always exists an entry which has an opposite sign to an entry in the same column of another row. It is easy to verify that such a matrix is nondegenerate so that the Poincaré duality axiom is fulfilled. Of course, to get the universal calculus, all possible Dirac operator elements must not vanish.

A more interesting case is when we require that the calculus is the lowest-dimensional bicovariant. For $S_3$ such a calculus is generated by $\chi^{ab}$ and $\chi^{ba}$, where $a, b$ denote the generators of $S_3$. 

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If we order the elements of $S_3$ as $a, b, c, 1, ab, ba$, the example of a spectral triple which gives the desired calculus is defined by:

$$
\begin{pmatrix}
1 & 1 & -1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}.
$$

Again, all the possible elements of the Dirac operator (components which act between spaces of different chirality) must not vanish. Of course if we slightly change the intersection form (introduce, for instance, other nonzero entries in the intersection matrix) the bicovariance (2-dimensional) shall also be preserved provided that the additional allowed Dirac operator components vanish!

For physical interpretation this would mean that the symmetry requires some of the fermion particles in the model to be massless, whereas in the spectral triple presented above all mass matrices must be nonzero.

Noncommutative finite algebras cannot carry such a simple group structure and therefore we have to look for nonabelian Hopf algebras. This in itself is an interesting topic and very little is known about the general classification of such (semisimple) objects. The easiest examples come from group algebras and as we try to make our examples comprehensible, we shall use the simplest one of them, the group algebra $\mathbb{C}S_3$.

### 3.2 Group algebra $S_3$

We shall begin here with some generalities on Hopf and bicovariant differential structures on group algebras and then we shall present the interesting example of $S_3$.

#### 3.2.1 Generalities on group algebras.

The group algebra $\mathbb{C}G$ has the natural structure of a Hopf algebra with the following coproduct, counit and antipode:

$$
\Delta g = g \otimes g, \quad \epsilon(g) = 1, \quad Sg = g^{-1}
$$

The adjoint coaction is trivial:

$$
ad(g) = g \otimes 1
$$
The Haar invariant measure \( \mu \) is just \( \mu(g) = 0 \) for \( g \) different from the neutral element of the group.

Finally let us state some observations on the differential calculi.

**Observation 9.** The differential calculi on group algebras are always inner, i.e., there exists a one-form \( \chi \), such that \( dg = [g, \chi] \)

It is sufficient to take:

\[
\chi = -\frac{1}{N_G} \left( \sum_h h^{-1}dh \right).
\]

Then:

\[
[g, \chi] = -\frac{1}{N_G} \sum_h (gh^{-1} dh - h^{-1}dh g)
= -\frac{1}{N_G} \sum_h (gh^{-1} dh - h^{-1}d(hg) - dg)
= \frac{1}{N_G} \sum_h dg = dg
\]

**Observation 10.** Since \( C^*G \) is cocommutative every left-covariant calculus is bicovariant. There exists a 1:1 correspondence between bicovariant calculi and representations of the group \( G \)

The first remark is obvious, so let us concentrate on the latter. Suppose we have a representation of \( G \) on some vector space \( V \). We may define \( \Omega_1 \) as a right-module \( C^*G \otimes V \), with the multiplication from the right by elements of \( A \) only on the first component of the tensor product. Then the following rule (it is sufficient to define it only for the generators) makes \( \Omega_1 \) a bimodule:

\[
h(g \otimes v) = (hg) \otimes (\pi(h)v).
\]

Of course, \( \Omega_1 \) is a bicovariant bimodule, with all the elements of type \( 1 \otimes v \) being left and right invariant. We already know that every differential calculus is inner and, moreover, it is easy to check that the one-form \( \chi \) is left and right invariant Therefore, to construct the calculus, it is sufficient to choose a \( \chi = 1 \otimes v \in 1 \otimes V \) and define \( dg = [g, \chi] \). It remains to show that \( \text{Im} d \) generates \( \Omega_1 \). This is equivalent to verifying whether \( \{v - \pi(h)v \}, h \in G \) generates the whole space \( V \). Note that the representation \( \pi \) could be linearly extended on the whole group algebra \( C^*G \). Then the latter requirement tells us that the action of the kernel of the counit on \( v \) spans the whole space \( V \).

The proof of the converse (every bicovariant calculus provides us with a representation of the group) is obvious.
Observation 11. The central part of $\Omega^1(A)$ consists of all one-forms of the type: $\omega = \sum_g g \otimes v_g$, where $v_g \in V$ are vectors satisfying for every $g, h \in G$:

$$\pi(h)v_g = v_{hgh^{-1}}.$$ \hspace{1cm} (12)

Of course, this condition may be void for some representation of the group $G$ (in particular for a nontrivial one-dimensional representation).

**Proof:** Every one form can be written as $\omega = \sum g \otimes v_g$ for some $v_g \in V$. If we require that $h\omega = \omega h$ for every $h \in G$ then it is easy to verify that (12) follows.\hfill ■

As an example we shall consider the smallest noncommutative group algebra $\mathbb{C}S_3$.

### 3.2.2 The group algebra $\mathbb{C}S_3$

The group $S_3$ has two generators $a, b$ with the following multiplication rules:

$$a^2 = e, b^2 = e, aba = bab$$

It is convenient to name the element $aba = c$, $c^2 = e$ and use the rules for multiplication between $a, b, c$:

$$ab = bc, \quad ac = ba, \quad bc = ca$$

As an algebra $\mathbb{C}S_3$ is isomorphic to the algebra $M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$. We shall present the form of this isomorphism $i$ for the generators $a, b$:

$$i(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i(b) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$  

The Haar measure on the Hopf algebra is defined as a linear map, which satisfies:

$$1\mu(f) = (\text{id} \otimes \mu)\Delta f = (\mu \otimes \text{id})\Delta f.$$

On any group algebra there exists only one such measure, which is $\mu(g) = 0, g \neq 0$ and $\mu(1) = 1$ (normalized). For our 4-dimensional representation ($M \in M_2(\mathbb{C}), p, q \in \mathbb{C}$) it is given by:
Before we start constructing spectral triples let us have a look at the covariant differential calculi, constructed according to the general scheme. Therefore, first we have to give the representations of the group $S_3$:

$$\mu \left( \begin{pmatrix} M \\ p \\ q \end{pmatrix} \right) = \frac{1}{3} \text{Tr} M + \frac{1}{6} p + \frac{1}{6} q$$

| dimension | $\pi(a)$ | $\pi(b)$ |
|-----------|---------|---------|
| 1*        | 1       | 1       |
| 1         | $-1$    | $-1$    |
| 2         | \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) | \( \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \) |

where $1^*$ is the trivial representation. Every representation of higher dimension is a direct sum of them.

- **1-dimensional calculus** For the one-dimensional calculus we take the representation 1 (the trivial one gives $d \equiv 0$).

There is only one generating form $\chi$ and the bimodule rules follow from the one-dimensional representation of $S_3$. Since there is only one such representation, $a = b = -1$ we get:

$$a\chi = -\chi a, \quad b\chi = -\chi b.$$ 

and the external derivative is

$$da = a\chi, \quad db = b\chi, \quad dc = c\chi,$$

notice that:

$$d(ab) = 0, \quad d(ba) = 0.$$ 

Such a calculus is extendible to a higher-order calculus with $d^2 = 0$ provided that we set $d\chi = -\chi\chi$.

This calculus has an interesting splitting property:

**Observation 12.** Let $A = M_2(C)$ and $B = C \oplus C$, $CS_3 = A \oplus B$. Then the split short exact sequence of algebras:

$$0 \to A \to CS_3 \to B \to 0$$
extends to a split short exact sequence of differential modules:

\[0 \to \Omega(A) \to \Omega(CS_3) \to \Omega(B) \to 0\]

where \(\Omega(CS_3)\) is the one-dimensional bicovariant calculus, and \(\Omega(A)\), \(\Omega(B)\) are its restriction to the corresponding subalgebras.

This tells us immediately that no spectral triple data could generate this calculus. Indeed, since the Dirac operator \(D\) can act only between spaces of different left (or right) representation of the algebra, such a splitting of differential bimodules is not possible.

**two-dimensional calculus** As we have pointed out there exists only one nontrivial two-dimensional representation of \(S_3\) (here nontrivial means that there exists a vector \(v\) such that \((\ker \epsilon)v\) spans the whole representation space. For instance, this does not hold for \(1 \oplus 1\)). Various possibilities for the choice of \(v\) would correspond to differential calculi related by automorphisms of the group algebra.

If we choose the above matrix form of \(a\) and \(b\) and \(v = (1, -\sqrt{3})\) we arrive at the following relations:

\[
\begin{align*}
\chi^1a &= -a\chi^1, \\
\chi^2b &= -b\chi^2, \\
\chi^1b &= b(\chi^1 - \chi^2), \\
\chi^2a &= a(\chi^2 - \chi^1),
\end{align*}
\]

where \(\chi^1 = a\, da\) and \(\chi^2 = b\, db\).

The same relations expressed in terms of \(dg\) become

\[
(da)b = c(da) - a(db) \quad (db)a = c(db) - b(da)
\]

then \(dc = (da)ba + a(da)b + ab(da) = 0\).

All other two-dimensional differential calculi could be obtained through an automorphism of the algebra.

Note that for this calculus we may construct an one form in the center of \(\Omega^1(A)\). For instance, the vector \(av_a + bv_b - c(v_a + v_b)\), where \(v_a, v_b\) are eigenvectors (to eigenvalue 1) of \(\pi(a), \pi(b)\), respectively, gives such a form. Using \(\chi^a\) and \(\chi^b\) we can write it as

\[
(2a - b - c)\chi^2 + (2b - a - c)\chi^1,
\]

so, again this rules out the compatibility between bicovariance and spectral-triple calculus for \(CS_3\).
Our result extends also for higher dimensional calculi, as all higher order representations of $S_3$ contain the two-dimensional part (and, if not, they must be composed out of 1 and $1^*$) and therefore one may repeat the arguments.

Thus, we may state a simple no-go lemma:

**Lemma 15.** For $\mathbb{C}S_3$ the differential structure from the spectral triple construction cannot carry a bicovariance symmetry.

Knowing that we may now concentrate on spectral triples and the symmetry of the measure.

### 3.3 Spectral Triples for $\mathbb{C}S_3$

Following the general construction scheme presented earlier we shall now turn our attention to an example of low-dimensional spectral triples for the algebra $\mathbb{C}S_3$. We will be interested in the non-trivial case, which admits a non-zero Dirac operator.

It is easy to verify that the lowest dimensional spectral triple is given by the intersection form:

$$
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & -1 \\
0 & -1 & 1
\end{pmatrix}.
$$

The Hilbert space has dimension 7 (it is odd, as it contains a vector in the subspace $H_{33}$ such that $Jv = v!$).

Instead of the Dirac operator it is more convenient to use the one-form $\xi$. Since the Dirac operator $D$ can only have nonzero entries $D_{12,32}$ and $D_{23,33}$ (and, respectively, their hermitian conjugates and $J$-conjugates) we have the following $\xi$ (where the Hilbert spaces on which $\xi$ acts are ordered as follows: $H_{12}, H_{32}, H_{23}, H_{33}$):

$$
\pi(\xi) = 
\begin{pmatrix}
0 & 0 & x & 0 & 0 \\
0 & 0 & y & 0 & 0 \\
x^* & y^* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z \\
0 & 0 & 0 & z^* & 0
\end{pmatrix}
$$

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Observation 13. The operator $\pi(\xi)\pi(\xi)$ is in $\pi(A)$ provided that $xx^* + yy^* = zz^*$. In any case it commutes with the image of the center of the algebra and therefore it follows from Lemma 11 that $\pi(\xi)^2 = \pi(\Xi)$ and the calculus is inner:

$$\pi(\xi)^2 = \begin{pmatrix}
x x^* & x y^* & 0 & 0 & 0 \\
y x^* & y y^* & 0 & 0 & 0 \\
0 & 0 & x x^* + y y^* & 0 & 0 \\
0 & 0 & 0 & z z^* & 0 \\
0 & 0 & 0 & 0 & z z^*
\end{pmatrix}.$$ 

We shall finish the investigation by writing the measure operator, which gives the Haar measure on $\mathbb{C}S_3$.

Observation 14. If $Z = \frac{1}{3}P_1 + \frac{1}{18}P_2 + \frac{1}{12}P_3$, where $P_i$ denote the corresponding projections in the algebra, then for every $a \in A$

$$\text{Tr}_Z \pi(a) = Tr (\pi(Z)\pi(a)),$$

is the Haar measure on $\mathbb{C}S_3$.

One may now use this deformed scalar product on forms to calculate the Yang-Mills action. Note that $[D, \pi(Z)] \neq 0$ and the results would differ from the ones obtained using the generic scalar product. For physical models this would result in different mass relations between the gauge bosons in the theory based upon such a triple.

4 Conclusions

The world of noncommutative geometry is much bigger than that of classical geometry. In classical differential geometry, once given a smooth manifold we already know its differential bundle. Given a Lie group we have a natural notion of the Lie group action on differential structures. All this seems to be lost when we enter the noncommutative world. For a single algebra we have many choices of differential structures and there is no universal rule to tell us which one to choose. Symmetries, or at least group symmetries are in many cases no longer present or drastically reduced.

The notion of spectral triples is an attempt to bring some order into our understanding of noncommutative differential geometry, by specifying what structures must appear in the models to make them a real geometry.

What we investigate in this paper are the restrictions, that this order introduces to the realm of 0-dimensional geometry. Even more, we attempt
to answer the question whether the proposed structures admit and restrict the possible 0-dimensional noncommutative symmetries.

We classify finite spectral triples and the allowed Dirac operators. This, for instance, enables us to state that the Standard Model Dirac operator is not the most general one. In fact, one may consider an identical spectral triple with some additional components of the Dirac operator, which link left-handed antileptons with right-handed quarks. Of course, the assumption that such a component exists might have some deep consequences for the model, as it could potentially lead to the breaking of the $SU(3)$ strong symmetry. Another consequence (mentioned originally by D.Testard, see [11]) is that Poincaré duality enforces the absence of right-handed neutrinos. This is correct provided that we add a pair of them (particle and antiparticle). However, it is easy to verify that adding only one right-handed Majorana particle (we doubt whether the name neutrino would be justified) still does preserve the Poincaré duality. These and other observations concerning the Standard model will be the topic of our forthcoming paper [2].

The symmetries of spectral triples are a far more complicated problem. We have seen that only in some cases there exists an extension (on the differential level at least) of the Hopf algebra structure into the spectral triple theory. Whether this is a general pattern is difficult to say, however, we believe that symmetries can also be realized in the 0-dimensional case. The examples we have analyzed do not give a conclusive answer, what they suggest, however, is that the symmetry restrictions could be much stronger and of different type than in classical geometry.

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While preparing a final version of this paper we learned that similar results were obtained by T.Krajewski [12], whom we would also like to thank for comments and remarks.

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