Square Turning Maps and their Compactifications

Richard Evan Schwartz *

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Abstract

In this paper we introduce some infinite rectangle exchange transformations which are based on the simultaneous turning of the squares within a sequence of square grids. We will show that such noncompact systems have higher dimensional dynamical compactifications. In good cases, these compactifications are polytope exchange transformations based on pairs of Euclidean lattices. In each dimension $8m + 4$ there is a $4m + 2$ dimensional family of them. Here $m = 0, 1, 2, ...$ The case $m = 0$, which we studied in depth in [S1], has close connections to the $E_4$ Weyl group and the $(2, 4, \infty)$ hyperbolic triangle group.

1 Introduction

1.1 Background

A piecewise isometry is a map defined on a typically polyhedral subset of Euclidean space. The domain is partitioned into smaller polyhedra in such a way that the map is defined, and an isometry, when restricted to the interior of each polyhedron in the partition. (One could make a similar definition for piecewise affine maps.) These maps often have a special beauty and combinatorial feel to them, but currently they are very far from being understood. In particular, there is a scarcity of examples above dimension 1, and especially above dimension 2.

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The simplest examples of piecewise isometries are 1-dimensional interval exchange transformations. See [K], [R], [Y], [Z] for an important but small sample of this large field. The paper [H] is an early paper on rectangle exchange transformations. The papers [AE], [AG], [AKT], [Go], [Low1], [Low2], [LVK], [T], all treat closely related sets of systems with $k$-fold symmetry for $k$ typically equal to 5, 7, or 8. Some definitive theoretical work concerning the (zero) entropy of such maps is done in [GH1], [GH2], and [B]. My monographs [S1] and [S2] have additional references.

The purpose of this paper is to study the dynamics of maps defined in terms of square grids. Given some $s > 0$ and some $z \in C$, let $G_{s,z}$ be the (usual) infinite grid of squares in the plane such that $z$ is a vertex of the grid and the sides of the squares are parallel to $s$ and $is$. In particular, the squares have side length $s$. Let $R_{s,z}$ denote the map which rotates each square of $G_{s,z}$ clockwise by $\pi/2$ radians. The map $R_{s,z}$ is only defined on the interiors of the squares of $G_{s,z}$.

The map $R_{s,z}$ is not very interesting from a dynamical perspective, because $R_{s,z}^4$ is the identity map. However, the compositions of the form

$$R_{s,z} := R_{s_1,z_1} \circ \ldots \circ R_{s_n,z_n}$$

(1)

can have quite intricate behavior. Here $S = \{s_1, \ldots, s_n\}$ and $Z = \{z_1, \ldots, z_n\}$ are the data the the composition. When $n$ is divisible by 4, the map $R_{s,z}$ is a piecewise translation. More specifically, $R_{s,z}$ is an infinite rectangle exchange transformation in this case.

One nice feature of (planar) piecewise translations is that they define a natural family of convex polygons in the plane. When $R = R_{s,z}$ is a piecewise translation, every periodic point $p$ of $R$ is contained in a maximal open convex polygon $I_p$, called a periodic island, such that $R$ is defined entirely on $I_p$ and periodic there. Since $R$ is a rectangle exchange, the polygon $I_p$ is in fact an open rectangle.

Probably the two most basic questions one can ask about the maps in Equation (1) are about their periodic points and about their unbounded orbits. In this paper we will talk mainly about periodic points, though a few of our results have to do with the existence of unbounded orbits. Our interest in understanding these basic questions for a special case is what led us to the idea of compactifying the systems.
1.2 A Motivating Example

The original motivation behind this paper was to understand the dynamics of the maps

\[ A_s = (R_{1,0}R_{s,0})^2 = R_{S,Z}, \quad S = \{1, s, 1, s\}, \quad Z = \{0, 0, 0, 0\}. \quad (2) \]

In [S1], we called \( A_s \) the alternating grid system because we generate the dynamics by alternately turning the squares in one grid and in the other.

Let \( Q_s \) denote the square of \( G_{s,0} \) whose bottom left vertex is the origin. Figure 1.1 shows how \( Q_s \) is tiled by the periodic islands for \( s = 36/31 \). The continued fraction expansion of \( s \) is 0 : 1 : 1 : 2 : 3 : 1 : 2.

![Figure 1.1: Periodic island tiling of \( Q_s \) for \( s = 36/61 \).](image)

If we study the blue islands in the picture, we see that they encode the continued fraction expansion of \( s \). The red islands suggest “blemishes”. It seems that these red islands should be subdivided into smaller tiles in order to make the overall pattern perfect.
Figure 1.2 shows the same kind of picture for the parameter $s = 55/89$. Here, the C.F.E. is $0 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 2$.

![Figure 1.2: Periodic island tiling of $Q_s$ for $s = 55/89$.](image)

Pictures such as these suggest

**Conjecture 1.1** Let $s \in \mathbb{R}_+$. Almost every point of $A_s$ is periodic. Inside $Q_s$ one can find a sequence of periodic islands, alternately sharing edges with the two coordinate axes, which encodes the continued fraction expansion of $s$.

The difficulty in proving Conjecture 1.1 is that the periodic orbits corresponding to small islands seem to be quite long and also to have huge diameters. (The mysterious red islands are also an obstacle to our understanding.) This leads one to the idea of trying to compactify the system, in order to bring all the orbits into view, so to speak. We took this approach (with partial success) in [S1] for $A_s$ and here we will do it more generally.
1.3 Dynamical Results

The alternating grid system is an example of a composition from Equation 1 which just involves 2 grids. For the purpose of stating some results, we formalize the idea of such compositions. Also, for convenience, we always take the parameter $s$ to be irrational. In the rational case, all the orbits are periodic. There is a lot to say about the rational case, but we will not say it here.

Let $G$ be the free product $(\mathbb{Z}/4) \ast (\mathbb{Z}/4)$. Let $A_1$ and $A_2$ be the usual generators of $G$. Let $G^0 \subset G$ denote the index 4 subgroup consisting of words whose total exponent is divisible by 4. The word $(A_1A_2)^2$, which corresponds to the alternating grid system, is an example of a word in $G^0$.

Given $S = \{s_1, s_2\}$ and $Z = \{z_1, z_2\}$, we have a representation of $G$ into $\text{map}(C)$ which sends $A_j$ to $R_{s_j, z_j}$. By scaling and translating the plane, an operation which does not effect the dynamics, it suffices to consider the case when $s_1 = 1$ and $z_1 = 0$. We set $s = s_2$ and $z = z_2$. We let $g_{s, z}$ denote the image of $g \in G$ under our representation. When $g \in G^0$, the map $g_{s, z}$ is a piecewise translation.

There is an auxiliary homomorphism $H : G \to \text{isom}(C)$ such that $H(A_1)$ rotates by $\pi/2$ radians counterclockwise around the origin and $H(A_2)$ rotates by $\pi/2$ radians counterclockwise about some other point. We usually take this other point to be 1. The homomorphism $H$ does not depend on the parameters $s$ and $z$. We say that the word $g \in G^0$ is a stationary word if $H(g)$ is the identity. We say that $g \in G^0$ is a drifter if $H(g)$ is not the identity map. The set of stationary words is the kernel of $H$, an infinite index subgroup of $G^0$.

There is a big difference between the stationary words and the drifters.

**Theorem 1.2** Suppose that $g \in G^0$ is a stationary word, and $s$ is irrational, and $z$ is arbitrary. Then $g_{s, z}$ has a positive density set of fixed points. In particular, $g_{s, z}$ has infinitely many periodic islands of period 1.

**Theorem 1.3** Let $g \in G^0$ be a drifter and let $s$ be irrational. For any $N > 0$ and any $z$, the map $g_{s, z}$ has only finitely many periodic islands of period less than $n$. Moreover, for all but countably many choices of $z$, the map $g_{s, z}$ has no periodic points at all.

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1Somewhat later on, we will find it more useful to set $z_1 = (1 + i)/2$, to that the origin is the center of a square of the first grid, rather than a vertex of the grid.
The word $g = (A_1A_2)^2$ is a stationary word. In this case, the results in [S1] give us an improved version of Theorem 1.2.

**Theorem 1.4** Let $g = (A_1A_2)^2$. Let $s \in \mathbb{R}_+$ be irrational and let $z \in \mathbb{C}$ be arbitrary. There is an unbounded sequence $\{p_k\}$ such that a positive density set of points in $\mathbb{C}$ are periodic with period $p_k$ with respect to $g_{s,z}$.

Theorem 1.4 takes a step in the direction of Conjecture 1.1. We will deduce Theorem 1.4 from some results in [S1], and we will give a self-contained proof for the parameter $s = \sqrt{2}/2$.

The same circle of techniques used to prove Theorem 1.4 also proves

**Theorem 1.5** Let $g = (A_1A_2)^2$. Let $s \in \mathbb{R}_+$ be irrational and let $z \in \mathbb{C}$ be arbitrary. Then for any $N$ there are orbits having diameter greater than $N$. Moreover, there exists uncountably many choices of $z$ such that $g_{s,z}$ has unbounded orbits.

Again, we will sketch the proof of Theorem 1.5 in general, and give a self-contained proof for the parameter $s = \sqrt{2}/2$.

### 1.4 The Compactifications

Our proofs of the dynamical results mentioned in the previous section rely on the construction of compactifications for the square turning systems. Given $s \in \mathbb{R}_+$ and $z \in \mathbb{C}$, we get a group action of $G = (\mathbb{Z}/4) \star (\mathbb{Z}/4)$ on $\mathbb{C}$, as discussed in the previous section. Our first result below gives a compactification of this group action.

Let $\mathbb{Z}[i]$ be the Gaussian integers. Geometrically, $(\mathbb{Z}[i])^4$ is just the square grid in $\mathbb{C}^4$. Let $T^4 = C^2/(\mathbb{Z}[i]^2)$ be the usual 4-dimensional square torus. We define $\Psi = \Psi_s : \mathbb{C} \to T^4$ by the equation

$$\Psi(x + iy) = (z, z/s) \mod (\mathbb{Z}[i]^2).$$

When $s$ is irrational $\Psi$ is injective on $\mathbb{C}$ and $\Psi(\mathbb{C})$ is dense in $T^4$. Note that $\Psi$ depends on $s$, but we often suppress this dependence from our notation.

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2We often find it more convenient to work in $\mathbb{C}^4$ rather than $\mathbb{R}^{2k}$, even though sometimes we will make the identification of $\mathbb{C}^4$ with $\mathbb{R}^{2k}$. 3
Theorem 1.6 (Compactification) Suppose that $s$ is irrational. There is a piecewise affine action $\hat{G}_s$ of $G$ on $T^4$ having the following properties.

1. For any $z \in C$ there is some translation $\hat{\tau}$ of $T^4$ so that $\hat{\tau} \circ \Psi$ conjugates the action of $G_s$ on $C$ to the action of $\hat{G}_s$ on $\hat{\tau} \circ \Psi(C)$.

2. For each $g \in G$, the element $\hat{g}_s$ acts on $T^4$ in such a way that its linear part $L_{g,s}$ is independent of the point where it is evaluated.

3. When $g \in G^0$ is stationary, $L_{g,s}$ is the identity, so that $\hat{g}_s$ is a polytope exchange transformation.

4. When $g \in G^0$ is a drifter, $L_{g,s}$ is a nontrivial complex linear parabolic. The fixed space of $L_{g,s}$ is parallel to $\Psi(C)$.

Here is how we will deduce the dynamical results from the Compactification Theorem

- When $g$ is stationary, the element $\hat{g}_s$ fixes a nontrivial polytope in $T^4$. The set $\hat{\tau} \circ \Psi(C)$ intersects this polytope in a positive density set. This is how we will prove Theorem 1.2.

- When $g$ is a drifter, the nontrivial parabolic nature of the action of $\hat{g}_s$ produces a local shear which destroys the periodic points. This is how we will prove Theorem 1.3.

- When $g = (A_1 A_2)^2$ the map $\hat{g}_s$ has a compact invariant 2-dimensional slice, which we call an octagonal PET. In [S1] we showed that this invariant slice admits a renormalization scheme. This renormalization scheme produces periodic points of arbitrarily high order and also aperiodic points. We will deduce Theorems 1.4 and 1.5 by studying how $\hat{\tau}(C)$ interacts with the orbits produced by the renormalization scheme.

Independent of the dynamical consequences, one might wonder about the geometry of the compactifications produced by the Compactification Theorem. In some cases, we can give a nice answer. In §4 we will define what we mean by a double lattice PET. These are special polytope exchange transformations which are defined in terms of a pair of Euclidean lattices and a pair of fundamental domains for those lattices. We first defined these maps in [S3], but [S1] has a more thorough account. The work in [S3] is what led us to the present paper.
Theorem 1.7 Let \( S = \{s_1, \ldots, s_n\} \) and \( Z = \{z_1, \ldots, z_n\} \), where \( n \) is congruent to 2 mod 4. Let \( R = R_{S,Z}^2 \). Then there is a double lattice PET \((X, \hat{R})\), whose domain \( X \subset \mathbb{R}^{2n} \) is a parallelotope, and an injective map \( \Psi : C \to X \) which conjugates \((R, C)\) to the restriction of \( \hat{R} \) to \( \Psi(C) \). The closure of \( \Psi(C) \) is a finite invariant union of \( 2k \)-dimensional convex polytopes, where \( k \) is the dimension of the \( \mathbb{Q} \)-vector space \( \mathbb{Q}[s_1, \ldots, s_n] \).

The nicest case of our construction is when there are no rational relations amongst \( s_1, \ldots, s_n \). In this case, \( \Psi(C) \) is dense in \( X \), and \((X, \hat{R})\) is a compactification of \((C, R)\) in the most basic sense. Near the other extreme, we can take periodic sequences \( S = \{1, s, 1, s, \ldots\} \) and \( Z = \{0, z, 0, z, \ldots\} \), where \( s \) is irrational. In this case, we get the following corollary, which refers to the torus action produced by the Compactification Theorem.

Corollary 1.8 Suppose that \( g \in G_0 \) has the form \( g = h^2 \) where \( h \) has word length \( n = 4m + 2 \). Then the action of \( \tilde{g}_s \) on \( T^4 \) is conjugate, by a piecewise translation, to the restriction of a \((2n)\)-dimensional double lattice PET to a finite invariant union of \( 4 \)-dimensional convex polytopes.

The double lattice PETs from Theorem 1.7 occur in dimensions of the form \( 8m + 4 \) for \( m = 0, 1, 2, \ldots \). In dimension \( 8m + 4 \) there is a \( 4m + 2 \) dimensional family. \(^3\) We will describe our examples explicitly for \( m > 0 \) in §5, and for \( m = 0 \) in §6.

For \( m = 0 \), the examples have a special beauty; they are related to the \( E_4 \)-Weyl group and to the \((2, 4, \infty)\) hyperbolic triangle group. Our monograph \([S1]\) is devoted to explaining the structure associated to a 2-dimensional invariant slice of the compactification. For the purposes of explaining Theorems 1.4 and 1.5 we will re-derive some of the structure here. Here is a conjecture which encapsulates some of the connection to the \( E_4 \) Weyl group.

Conjecture 1.9 When \( g = (A_1 A_2)^2 \) and \( s \) is irrational, the associated double lattice PET from Theorem 1.8 has an almost everywhere defined invariant tiling \( \hat{T}_s \) by polytopes. Each polytope in \( \hat{T}_s \) has order 192 symmetry coming from the action of the \( E_4 \) Weyl group. For any \( z \in C \) there is a leaf \( L_z \) of the invariant foliation such that \( L_z \cap \hat{T}_s \) is a refinement of the tiling of \( C \) by periodic islands of \( g_s \).

\(^3\)There is a certain redundancy, in the sense that the data \( \{\lambda s_1, \ldots, \lambda s_n\} \) leads to a system which is conjugate to the system defined by \( \{s_1, \ldots, s_n\} \). So, perhaps it is more accurate to say that there is a \( 4m + 1 \) dimensional family of examples in dimension \( 8m + 4 \).
Referring to the discussion surrounding Figures 1.1 and 1.2, the refinement we mention would be the result of suitably subdividing all the “red islands” in the tiling, so as to improve the picture. Just so that this conjecture doesn’t seem completely off the wall, we will give a proof for the portions of the tilings associated to fixed point sets.

1.5 Organization

This paper is organized as follows.

- In §2, we will prove the Compactification Theorem, Theorem 1.6.
- In §3 we prove Theorems 1.2 and 1.3.
- In §4 we prove Theorem 1.8.
- In §5 we will give an explicit description of the PETs produced by Theorem 1.8 in dimensions 4, 12, 20, 28, ....
- In §6 we will consider in detail the 4-dimensional case from §5. We prove Theorems 1.4 and 1.5 from Theorem 1.8 and a result from [S1], Theorem 6.2. At the end of §6 we will prove the special case of Conjecture 1.9 corresponding to the fixed point set.
- In §7 we will roughly sketch the proof of Theorem 6.2 and we will give a self-contained proof (modulo a calculation) for $s = \sqrt{2}/2$.
- In §8 we discuss connections to the $E_4$ Weyl group and prove the special case of Conjecture 1.9 corresponding to the fixed point set.

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2 The Compactification Theorem

2.1 The Main Construction

Let \( T^{2n} \) denote the usual \( 2n \)-dimensional torus in \( C^n \). We think of a fundamental domain for \( T^{2n} \) as the unit cube centered at the origin in \( C^n \). We let \( Q^{2n} \) denote the interior of this cube.

Given \( s \), define

\[
  z = \frac{s + is}{2}.
\]

(4)

The grid with parameters \( (s, z) \) is such that the origin the center of one of the squares. Let \( R_s \) denote the corresponding grid map.

Given a finite sequence \( s_1, ..., s_n \), we set \( R_k = R_{s_k} \) and consider the composition

\[
  g = R_n \circ ... \circ R_1.
\]

(5)

We define \( \Psi : C \to T^{2n} \) by the map

\[
  \Psi(z) = \left( \frac{z}{s_1}, ..., \frac{z}{s_n} \right) \mod (Z[i])^n
\]

(6)

The image \( \Psi(C) \) is dense in a linear subspace of \( T^{2n} \). The dimension of this subspace, which we will discuss in more detail below depends on the number of rational relations between \( s_1, ..., s_n \).

We define \( \hat{R}_1 \) by permuting the coordinates and interchanging the roles of \( s_1 \) with \( s_j \). For instance,

\[
  \hat{R}_1(z_1, ..., z_n) = (iz_1, z_2, ..., z_n) + (-1+i)z_1\left( 0, \frac{s_1}{s_2}, ..., \frac{s_1}{s_n} \right) \mod (Z[i])^n.
\]

(7)

This map is the identity on \( \{0\} \times C^{n-1} \) and locally affine on the set \( Q^2 \times T^{2n-2} \). It is not possible to extend \( \hat{R}_1 \) to all of \( T^{2n} \), but this does not bother us. The set \( (\partial Q^2) \times T^{2n-2} \), considered as a subset of \( T^{2n} \), is a union of 2 flat tori of dimension \( 2n - 1 \). This is the singular set for \( \hat{R}_1 \).

We define \( \hat{R}_j \) by permuting the coordinates and interchanging the roles of \( s_1 \) with \( s_j \). For instance,

\[
  \hat{R}_2(z_1, ..., z_n) = (z_1, iz_2, ..., z_n) + (-1+i)z_2\left( \frac{s_2}{s_1}, 0, ..., \frac{s_2}{s_n} \right) \mod (Z[i])^n.
\]

(8)

The map \( \hat{R}_j \) is locally affine on the product of an open square and a torus of dimension \( 2n - 2 \).
Lemma 2.1 \( \hat{R}_j \circ \Psi = \Psi \circ R_j \) on the domain of \( R_j \).

Proof: By symmetry, it suffices to consider the case of \( R_1 \). Notice that our lemma is true for the sequence \( \lambda s_1, ..., \lambda s_n \) if and only if it is true for the original sequence. For this reason, it suffices to consider the case when \( s_1 = 1 \). The domain of \( R_1 \) is the infinite grid of open unit squares, one of which, namely \( Q^2 \), is centered at the origin.

We first check that the equation holds in a neighborhood of 0. For \( |z| \) sufficiently small, we compute

\[
\Psi \circ R_1(z) = \Psi(iz) = \left( iz, \frac{iz}{s_2}, ..., \frac{iz}{s_n} \right).
\]  

On the other hand

\[
\hat{R}_1(\Psi(z)) = \hat{R}_1\left( z, \frac{z}{s_2}, ..., \frac{z}{s_n} \right) = \left( iz, \frac{z}{s_2}, ..., \frac{z}{s_n} \right) + (-1 + i)z\left( 0, \frac{1}{s_2}, ..., \frac{1}{s_n} \right)
\]  

One can see that the two expressions are equal.

Next, we observe that \( \Psi(Q^2) \subset Q^2 \times T^{2n-2} \). The restrictions of \( R_1 \) and \( \Psi \) to \( Q^2 \) are affine and the map \( \hat{R}_1 \) is affine on \( Q^2 \times T^{2n-2} \). Therefore, the check we have already made implies that our lemma holds true on \( Q^2 \).

Suppose we knew that \( \Psi \circ R_1(z) = \hat{R}_1 \circ \Psi(z) \) for some Gaussian integer \( z \).

The differentials of \( \Psi \circ R_1 \) and \( \hat{R}_1 \circ \Psi \) at \( z \) are the same as the differentials of these maps at 0. Since they already agree at 0, they agree at \( z \) as well. This implies that our basic equation holds in a neighborhood of \( z \). Note finally that \( \Psi \) maps the unit square \( Q_z^2 \) centered at \( z \) into \( Q^2 \times T^{2n-1} \). The same continuation principle now implies that our lemma holds on \( Q_z^2 \). So, to finish the proof, we just have to check the basic equation on Gaussian integers.

When \( z \) is a Gaussian integer, we have \( R_1(z) = z \). So, we just have to prove that \( \hat{R}_1 \) fixes \( \Psi(z) \). But \( \Psi(z) \subset \{0\} \times T^{2n-2} \), and such points are fixed by \( \hat{R}_1 \). ♠

We define

\[
\hat{g} = \hat{R}_n \circ ... \circ \hat{R}_1.
\]  

An immediate consequence of our previous result is that

\[
\Psi \circ \hat{g} = \hat{g} \circ \Psi
\]  

wherever all maps are defined.
2.2 Statement 1 of Theorem 1.6

Theorem 1.6 concerns the case \( n = 2 \) in the construction above. In this case, we normalize so that \( s_1 = 1 \) and \( s_2 = s \). We will first prove Statement 1 for the choice of \( z \) in Equation 4. In this case, Equation 11 gives us the desired group action on \( T^4 \). When \( z = 0 \), we take the translation \( \hat{\tau} \) in Theorem 1.6 to be the identity. Equation 12 gives the desired conjugacy between the two group actions. This is Statement 1.

Now we explain how things work for other choices of \( z \). Note that the grids corresponding to the parameters \( z \) and \( z + sm + isn \) are the same, for any \( m, n \in \mathbb{Z} \). Hence, it suffices to prove Statement 1 for one representative of each point in \( C \) in the torus

\[
T^2_s = C/\Lambda, \quad \Lambda = \{sm + isn \mid m, n \in \mathbb{Z}\}. \quad (13)
\]

We first show that it suffices to consider a dense set of points in the torus, and then we give an argument which covers such a dense set of points.

**Lemma 2.2** Suppose Statement 1 of the Compactification Theorem holds for a dense set of points in \( T^2_s \). Then Statement 1 of the Compactification Theorem holds for all points in \( C \).

**Proof:** Let \( z_\infty \in T^2_s \) be some point, and let \( \{z_n\} \) be a sequence of points in our dense set which converges to \( z_\infty \). Let \( \hat{\tau}_n \) be the corresponding set of translations of \( T^4 \). Since the group of translations of \( T^4 \) is compact, we may pass to a subsequence to that \( \hat{\tau}_n \) converges to a translation \( \hat{\tau} \). For any \( n \), we have the equation

\[
\Psi_n \circ G_{s,n} = \hat{G}_s \circ \Psi_n, \quad \Psi_n = \hat{\tau}_n \circ \Psi. \quad (14)
\]

Here \( G_{s,n} \) is the group action corresponding to \( z_n \). Since \( \Psi_n \) is an injection, Equation 14 is equivalent to Statement 1 of the Compactification Theorem for the parameter \( z_n \).

Consider what happens as \( n \to \infty \). For any given element \( g \in G \), the maps defined by Equation 14 converge uniformly on compact sets to the corresponding maps defined in terms of the limit parameter \( z \). Hence, Equation 14 holds as well for the \( n = \infty \). But then Statement 1 of the Compactification Theorem holds for \( z_\infty \). ♠
Let $z_{00}$ be as in Equation \[4\]. Consider the points
\[z_{mn} = z_{00} + m + in, \quad m, n \in \mathbb{Z}.\] (15)

What makes these points special is that the group action defined in terms of $z_{mn}$ is conjugate to the group action defined in terms of $z_{00}$. It is not generally true that the group actions defined in terms of different choices of $z$ are conjugate.

When $s$ is irrational, the set of representatives of the points $\{z_{mn}\}$ in $T_s^2$ is dense. So, it suffices to prove Statement 1 of the Compactification Theorem for the points $z_{mn}$.

**Lemma 2.3** Statement 1 of the Compactification Theorem holds for $z_{mn}$.

**Proof:** Let $G_{s,m,n}$ be the group action corresponding to the point $z_{mn}$. Let $\tau_{mn}$ be the translation of $C$ which carries $z_{00}$ to $z_{mn}$. By construction, $\tau_{mn}$ conjugates $G_{s,0,0}$ to $G_{s,m,n}$. More specifically,
\[\tau \circ G_{s,m,n} \circ \tau^{-1} = G_{s,0,0}.\] (16)

Since $\Psi$ is locally affine, there is some translation $\hat{\tau}_{mn}$ such that
\[\Psi \circ \tau_{mn} = \hat{\tau}_{mn} \circ \Psi.\] (17)

For ease of notation we make the following abbreviations.
\[\tau = \tau_{mn}, \quad \hat{\tau} = \hat{\tau}_{mn}, \quad G = G_{s,0,0}, \quad G' = G_{s,m,n}, \quad \hat{G} = \hat{G}_s.\] (18)

Combining Equations (16) and (18) with the special case of Statement 1 we have already proved, we have
\[\hat{\tau} \Psi G' \tau^{-1} = \Psi \tau G' \tau^{-1} = \Psi G = \hat{G} \Psi.\] (19)

Hence
\[\hat{\tau} \Psi G' = \hat{G} \Psi \tau = \hat{G} \hat{\tau} \Psi.\] (20)

This last equation says that $\hat{\tau}_{m,n} \circ \Psi$ is a semi-conjugacy between $G_{s,m,n}$ and $\hat{G}_s$. This equivalent to Statement 1 for $z_{mn}$. ♠
2.3 The Rest of Theorem 1.6

Statement 2 of Theorem 1.6 is immediate. The linear parts of the maps \( \hat{R}_1 \) and \( \hat{R}_2 \) are independent of the point where they are evaluated, and so the same goes for any word in these generators.

Statements 3 and 4 of the Compactification Theorem are purely statements of linear algebra. The linear parts of \( \hat{R}_1 \) and \( \hat{R}_2 \) are given by

\[
L_1 = \begin{bmatrix} i & 0 \\ -1/s + i/s & 1 \end{bmatrix} \quad L_2 = \begin{bmatrix} 1 & -s + is \\ 0 & i \end{bmatrix}
\]  

(21)

We introduce the matrix

\[
E = \begin{bmatrix} s & 0 \\ 1 & 1 \end{bmatrix}.
\]  

(22)

\( E \) is a basis of eigenvectors of \( L_1 \). Setting \( M_k = E^{-1}L_kE \), we have

\[
M_1 = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} i & -1 + i \\ 0 & 1 \end{bmatrix}.
\]  

(23)

As long as \( s \neq 0 \), the matrix \( E \) is nonsingular. Notice that \( M_1 \) and \( M_2 \) do not depend on \( s \). So, the conjugacy class of some word in \( L_1 \) and \( L_2 \) does not depend of \( s \).

A calculation shows that \( M_1 \) and \( M_2 \) both preserve \( \Pi = C \times \{-1\} \). Moreover, \( M_1 \) rotates \( \Pi \) by \( \pi/2 \) counterclockwise about the point \((0, -1)\) and \( M_2 \) does the same thing about the point \((1, -1)\). Therefore, the auxiliary homomorphism \( H \) discussed in connection with Theorem 1.6 can be interpreted as the map which carries \( A_k \) (a generator of \( G = \mathbb{Z}/4 \ast \mathbb{Z}/4 \)) to \( M_k|\Pi \) for \( k = 1, 2 \). In short

\[
H(A_k) = M_k|\Pi, \quad k = 1, 2.
\]  

(24)

So, \( g \) is a stationary word if and only if the corresponding word in \( M_1 \) and \( M_2 \) acts trivially on \( \Pi \). More formally, let \( g \in G^0 \). Let \( \mu_g \) be the word in \( M_1 \) and \( M_2 \) corresponding to \( g \). By construction,

\[
E^{-1}L_gE = \mu_g.
\]  

(25)

Here \( L_g \) is the linear part of \( \hat{g} \).

Note that \( g \in G_0 \) forces \( \mu_g \) to be the identity on \( C \times \{0\} \). Suppose \( H(g) \) is trivial. Then \( \mu_g \) is the identity on both \( C \times \{0\} \) and \( C \times \{-1\} \). Since \( \mu_g \) is complex linear, this forces \( \mu_g \) to be the identity. Hence \( L_g \) is the identity as well. This proves Statement 3.
We use the same notation for Statement 4. If $H(g)$ is not the identity, then the nontrivial action of $\mu_g$ on $\Pi$ forces $\mu_g$ to be a nontrivial parabolic. Since $\mu_g$ is complex linear and acts as the identity on $C \times \{0\}$, it must be the case that $\mu_g$ is a parabolic of real rank 2, when interpreted as acting on $R^4$. The same goes for $L_g$, which is conjugate to $\mu_g$ over $C$ and a fortiori over $R$. This proves Statement 4.
3 Existence of Periodic Points

3.1 Good Partitions

We fix some drifter $g \in G^0$ and consider $\hat{g}_s$ acting on $T^4$. Recall that the unit cube $Q^4$ is our fundamental domain for $T^4$. Recall also that $\hat{g}_s$ has an invariant foliation $F = F_s$ that is parallel to $\Phi_s(C)$.

We say that a a small polytope is a convex polytope $P \subset Q^4$ such that no face of $P$ contains an open subset of a plane in $F$. We also insist that $g(P) \subset Q^4$. In other words, neither $P$ nor $g(P)$ is allowed to cross the boundary of $Q^4$. Given a piecewise affine map $h : T^4 \to T^4$, we say that a partition $\Pi$ of $T^4$ is good for $h$ if $\Pi$ consists of small polytopes and $h$ is defined and affine on the interior of each one.

**Lemma 3.1** There is a good partition for $\hat{g}_s$.

**Proof:** We suppress the dependence on $s$ from our argument. From the construction in §2.1, we see that $\hat{R}_1$ and $\hat{R}_2$ are defined on the complement of a finite union of flat 3-tori. These 3-tori are transverse to our foliation. By removing additional 3-dimensional flats, including the boundary of the unit cube, we can find a partition which is simultaneously good for all the words $\hat{R}_j^i$ where $i = 1, 2$ and $j = \pm 1$.

We produce a good partition for any word in $\hat{R}_1$ and $\hat{R}_2$ by induction on the length of the word. Suppose that $g = \hat{R}_1 h$ and $\Pi_h = \{H_1, ..., H_n\}$ is a good partition for $h$. Let $\{A_1, ..., A_m\}$ be a good partition for $\hat{R}_1$. We define

$$G_k = \bigcup_{i=1}^{m} h^{-1}(h(H_k) \cap A_i) \quad k = 1, ..., n. \quad (26)$$

Then $G_k$ is a partition of $H_k$ into finitely many small polytopes such that $g$ is defined on each one. If $B$ is some polytope in this partition, then $g(B) \subset \hat{R}_1(A_i)$ for some $i$. Hence $g(B)$ is also small. The union of all the polyhedra in $G_k$ as $k$ ranges from 1 to $n$, gives the desired partition for $g$. A similar construction works for $\hat{R}_j^i h$ for the other relevant choices of $i$ and $j$. ♠

For use in the next chapter, we record the following corollary of our construction above.

**Corollary 3.2** For any $n$, there is a good partition $\Pi_n$ for $\hat{g}_s^n$. Moreover we can choose these partitions so that $\Pi_n$ is a refinement of $\Pi_m$ when $n > m$. 

16
3.2 Proof of Theorem 1.3

Let $g \in G^0$ be a drifter. We first show that $\hat{g}_{s,z}$ has only finitely many periodic islands of period less than $N$. Since there are only finitely many positive integers less than $N$, it suffices to prove instead that $\hat{g}_{s,z}$ has only finitely many fixed points of period $N$. Replacing $\hat{g}_{s,z}$ by $g^N_{s,z}$, it suffices to prove the result when $N = 1$.

That is, we want to prove that $g_{s,z}$ has only finitely many fixed islands. By fixed island, we mean a periodic island corresponding to a point of period 1. Any two fixed islands are either identical or have disjoint interiors. We will prove the result for $z = (s + is)/2$, as in Equation 4. The general case has the same proof, except that the map $\hat{\tau} \circ \Phi$ is used in place of the map $\Phi$, for some suitable translation $\hat{\tau}$. For ease of notation we set $g = \hat{g}_{s,z}$.

Let $\Pi$ be a good partition for $\hat{g}$. By the Compactification Theorem, $\hat{g}$ preserves $F$ and the restriction of $\hat{g}$ to a suitable leaf of the foliation $F$ is conjugate to the action of $g$ on $C$.

Say that a good disk is an open polygon of the form $D = X \cap P$, where $X$ is a 2-plane parallel to the foliation $F$. We say that $D$ is fixed if $\hat{g}$ fixes $D$ pointwise.

Lemma 3.3 If $D$ is a good disk but not a fixed good disk, then $\hat{g}$ has no fixed points in $D$.

Proof: Recall that $D$ is parallel to the eigenspace of the linear part of $\hat{g}$, and the eigenvalues are all 1. Hence, the restriction of $g$ to $D$ is a translation. This means that $D$ is either a fixed disk or $g$ fixes no points of $D$ at all. ♠

Now we come to the key structural result.

Lemma 3.4 Each small polytope of $\Pi$ contains at most one fixed good disk. Hence, the set of fixed points of $\hat{g}$ is contained in finitely many good disks.

Proof: Suppose that $\hat{g}$ fixes two good disks in a good polytope $P$. The restriction of $\hat{g}$ to the interior of $P$ is affine. Let $A$ be the affine map which extends $\hat{g}|P$. By construction, $A$ is the identity on two parallel 2-planes. But then $A$ is the identity on a certain 3-dimensional subspace. But this is impossible, because the linear part of $A$ is a parabolic of real rank 2. ♠
Let \( X \) denote the set of fixed points of \( g \). Let \( X' \subset X \) denote these points \( x \) such that \( \Phi(x) \) lies in the interior of some small polytope of the good partition. Since the faces of these small polytopes are transverse to the invariant foliation, we see that \( X' \) is dense in \( X \). In particular, every fixed island contains points of \( X' \).

Let \( Y \) denote the set of fixed good disks. Since \( \Psi_s : C \to T^4 \) is injective and since all the fixed points of \( \hat{g} \) lie in fixed good disks, \( \Psi_s \) induces a map from \( X' \) into \( Y \).

**Lemma 3.5** Suppose that \( p, q \in X' \). If \( \Psi \) maps \( p \) and \( q \) into the same fixed good disk, then \( p \) and \( q \) lie in the same fixed island.

**Proof:** Let \( I_p \) and \( I_q \) be the fixed islands containing \( p \) and \( q \) respectively. Either \( I_p = I_q \) or these sets have disjoint interiors. If \( \Psi(p) \) and \( \Psi(q) \) lie in the same good fixed disk \( D \), then \( \Psi \) maps the line segment \( pq \) into \( D \). But then \( \hat{g} \) fixes every point of \( \Psi(pq) \). Hence \( \hat{g} \) is entirely defined, and the identity, on \( pq \). But then \( pq \subset I_p \cap I_q \). This forces \( I_p = I_q \). ♠

If \( g \) had infinitely many fixed islands, then the previous result would give us infinitely many fixed good disks. This is a contradiction. Hence \( g \) only has finitely many fixed islands. As we mentioned above, this completes the proof that, for any \( N \), the map \( g = g_{s,z} \) has only finitely many periodic islands of period less than \( N \). This is the first statement of Theorem 1.3.

Now we prove Statement 2 of Theorem 1.3 which says that \( g_{s,z} \) has periodic points only for countably many choices of \( z \). Using the same trick as for Statement 1, it suffices to prove this result for fixed points.

Let \( \hat{\tau}_z \) be the translation of \( T^4 \) such that \( \hat{\tau} \circ \Psi_s \) conjugates the action of \( g_{s,z} \) on \( C \) to the action of \( \hat{g}_s \) on

\[
C_z = \hat{\tau}_z \circ \Psi_s(C).
\]

Here \( C_z \subset T^4 \) is one of the leaves of the invariant foliation. Note that the map \( C \to C_z \) is a bijection. Here is the key lemma for our result.

**Lemma 3.6** If \( g_{s,z} \) has a fixed point, then \( C_z \) is one of finitely many leaves of the invariant foliation \( F \).

**Proof:** There are only finitely many fixed good disks. Hence there are only finitely many leaves of \( F \) which contain good fixed disks. Call these leaves...
special. If $g_{s,z}$ has a fixed point then $C_z$ contains an fixed good disk and hence is special. ♠

Again, we call a leaf in the invariant foliation special if it contains a fixed good disk. The next result shows that there are only countable many choices of $w$ such that $C_w$ is a special leaf. This finishes the proof that there are only countably many choices of $z$ for which $g_{s,z}$ has a fixed point.

**Lemma 3.7** For each $z \in C$, there are only countably many choices of points $w \in C$ such that $C_z = C_w$.

**Proof:** Suppose that $C_z = C_w$. We can interpret the translation by $\hat{\tau}_z - \hat{\tau}_w$ as a translation of $C$ which conjugates the action of $G_{s,z}$ to the action of $G_{s,w}$. Call this translation $\tau$.

Since $\tau$ conjugates the rotation $R_1$ to the rotation $R_1$, we see that $\tau$ must be translation by some integer lattice vector. In particular, $\tau$ preserves the grid $G_1$. At the same time, $\tau(G_{s,z}) = G_{s,w}$. Hence, there are only countably many choices for $G_{s,w}$. But only countably many choices of $w$ lead to the same choice of $G_{s,w}$. Hence, there are only countable many choices of $w$ which lead to one of the countably many possible choices of grid. ♠

### 3.3 Proof of Theorem 1.2

Now suppose that $g \in G^0$ is a stationary word. Inspecting the maps $\hat{R}_1$ and $\hat{R}_2$ which define the group action $\hat{G}_s$, we see that both elements fix the origin $O_4 \in T^4$ and both elements are defined in a neighborhood of $O_4$. Hence, for any $g \in G$, the word $\hat{g}_s$ fixes $O_4$ and is defined in a neighborhood of $O_4$.

When $g \in G^0$ is a stationary word, $\hat{g}_s$ is a piecewise translation. In this case, there is some nontrivial and maximal open polytope $P$ which contains $O_4$ in its interior such that $\hat{g}_s$ fixes every point of $P$.

We choose $z \in C$ and let $\Psi = \tau \circ \Psi_s$ be the map from the Compactification Theorem. Let $P' = \Psi^{-1}(P)$. Since $\Psi$ is locally affine and $\Psi(C)$ is dense in $T^4$, we see that $P'$ has positive density. In fact, the density of $P'$ is just the volume of $P$. But $g_{s,z}$ fixes every point of $P'$. Hence $g_{s,z}$ has a positive density set of fixed points. Since $P'$ has positive density, $P'$ is unbounded. Since every fixed island is compact, there must be infinitely many fixed islands. This completes the proof of Theorem 1.2.
3.4 A Generalization

It seems worth mentioning a generalization of some of the results in this chapter. Let \( \hat{g} \) be a piecewise affine map of the flat torus \( T^n \). We say that an \textit{invariant foliation} is a foliation \( F \) such that \( \hat{g} \) preserves every leaf of \( F \). We also require that \( \hat{g} \) is defined almost everywhere on every leaf of \( F \).

**Theorem 3.8** Suppose \( \hat{g} \) is a piecewise affine map of \( T^n \) having a flat, dense, invariant \( k \)-dimensional foliation \( F \). Suppose that the linear part \( L \) of \( \hat{g} \) is independent of the point of evaluation and is a parabolic of real rank \( k \) whose real eigenspace is tangent to \( F \). Then there are only finitely many leaves of \( F \) containing fixed points of \( \hat{g} \) and only countably many leaves containing periodic points.

The proof of Theorem 3.8 is almost the same as what we have done for Theorem 1.3. Mainly, we will point out the differences. First of all, the first statement in Theorem 3.8 can be applied to powers of \( \hat{g} \). Hence, the first statement of Theorem 3.8 implies the second statement. Thus, it suffices to prove that only finitely many leaves of \( F \) contain fixed points of \( \hat{g} \).

The same argument as in Lemma 3.1 shows that \( T^n \) has a good partition. Here we use the fact that \( \hat{g} \) is almost everywhere defined on every leaf of \( F \). This guarantees that the faces of the polyhedra in the initial partition for \( \hat{g} \) do not contain open subsets of the leaves of \( F \).

There are two kinds of fixed points of \( \hat{g} \), those contained in the interiors of the small polytopes of the good partition, and those contained in the faces of these polytopes. Call these fixed points of the first kind and second kind respectively.

**Lemma 3.9** Any fixed point of the second kind is contained in the same leaf as some fixed point of the first kind.

**Proof:** Let \( p \in T^n \) be a fixed point of the second kind. If \( p \) lies in a face of the good partition, it means that \( \hat{g} \) is actually defined on \( p \). Since \( \hat{p} \) is a parabolic of real rank \( k \) whose real eigenspace is parallel to \( F \), we see that \( \hat{g} \) fixes an entire \( k \)-disk \( \Delta \) containing \( p \). By assumption, \( \Delta \) is not contained in the union of the boundaries of the small polytopes. Hence \( \Delta \) contains fixed points of the first kind. But \( \Delta \) lies in a single leaf of \( F \). \( \spadesuit \)
In light of the previous result, we just have to prove that the fixed points of the first kind are contained in finitely many leaves of $F$. Lemma 3.4 now goes through, almost word for word, to show that $\hat{g}$ has only finitely many fixed good disks, and that these fixed good disks contain all the fixed points of the first kind. These finitely many fixed good disks lie inside finitely many leaves of $F$. 
4 Nature of the Compactifications

4.1 Double Lattice PETs

In this chapter we prove Theorem 1.7. For starters, we define what we mean by a double lattice PET.

The data for a double lattice PET is a quadruple $(X_1, X_2, \Lambda_1, \Lambda_2)$, where

- $X_1$ and $X_2$ are polytopes in $\mathbb{R}^n$.
- $\Lambda_1$ and $\Lambda_2$ are lattices in $\mathbb{R}^n$.
- $X_i$ is a fundamental domain for $\Lambda_j$ for all possible $i, j \in \{1, 2\}$.

To say that $\Lambda$ is a lattice is to say that there is some affine isomorphism $T$ of $\mathbb{R}^n$ such that $\Lambda = T(\mathbb{Z}^n)$. To say that $X$ is a fundamental domain for $\Lambda$ is to say that the orbit $\Lambda(X)$ tiles $\mathbb{R}^n$: The translates have pairwise disjoint interiors and the union of the translates is a covering.

In all our examples, the polytopes $X_1$ and $X_2$ are parallelotopes centered at the origin.

There is a natural map $f_{ij}: X_i \to X_j$ defined as follows. Given $p \in X_i$, there is generically a unique vector $\lambda_p \in \Lambda_j$ such that $p + \lambda_p \in \Lambda_j$. We define

$$f_{ij}(p) = p + \lambda_p.$$  \hfill (28)

The maps $f_{11}$ and $f_{22}$ are the identity; we do not care about these maps. The maps $f_{12}$ and $f_{21}$ are the ones of interest to us. These maps are piecewise translations. The composition

$$f = f_{21} \circ f_{12}: X_1 \to X_1$$ \hfill (29)

is a polytope exchange transformation (PET) having $X_1$ as a domain. We call the system $(X_1, f)$ a double lattice PET.

We needed to break symmetry in order to define the maps $f_{ij}$. We could equally well define the maps $g_{ij}: X_i \to X_j$ as follows. For $p \in X_i$ there is generically a unique vector $\mu_p \in \Lambda_i$ such that $p + \mu_i \in X_j$. We can then define $g_{ij}(p) = p + \mu_p$. It is easy to see that $g_{ij} = f_{ji}^{-1}$. Hence $f^{-1} = g_{12}g_{21}$.

At first it might seem difficult to produce quadruples satisfying the necessary conditions. However, in [S3] we showed that essentially all polygonal outer billiards systems have compactifications which are double lattice PETs. In this chapter we will see that the construction in the previous chapter leads naturally to double lattice PETs as well.
4.2 The Linear Part

We continue with the notation from §2.1 As a start on the proof of Theorem 1.7, we analyze the linear part $L_g$ of $\hat{g}$ in case $g$ is a word of length $n = 4m+2$. Here is our main result.

**Lemma 4.1** $L_g$ is an involution whose $(-1)$-eigenspace is 2-dimensional and whose $(+1)$-eigenspace is $(2n - 2)$-dimensional.

We prove Lemma 4.1 through a series of smaller results. We also note that the calculations done in §5.4 give an alternate proof of Lemma 4.1.

**Lemma 4.2** The $(-1)$ eigenspace of $L_g$ has real dimension at least 2.

**Proof:** The action of $\hat{g}$ on $\Psi(C)$ is conjugate to the action of $g$ on $C$. Since $g$ has length $4n_2$, the linear part of $g$ is rotation by $\pi$. Hence, the same goes for $\hat{g}$. This implies that $L_g$ preserves the complex line through the origin and parallel to $\Psi(C)$. In other words, considered as a real matrix $L_g$ has a $(-1)$-eigenspace which is at least 2 dimensional. ♠

To finish the proof of Lemma 4.1 we will produce a $(2n - 2)$-dimensional subspace on which $L_g$ is the identity. This will finish the proof. We will first consider the case when the numbers 1, $s_2, ..., s_n$ have no rational relations amongst them. Once we take care of this case, we will deduce the general case by a limiting argument. When there are no rational relations amongst the numbers 1, $s_2, ..., s_n$, the image $\Psi(C)$ is dense in $T^{2n}$.

We say that a subset $\Sigma \subset C$ is a net if there is some $K$ such that every point of $C$ is within $K$ of some point of $\Sigma$. If $U$ is any open subset of $T^{2n}$, the inverse image $\Psi^{-1}(U)$ is a net in $C$.

**Lemma 4.3** Let $\epsilon > 0$ be given. If $U$ is sufficiently small then every point of $\Psi^{-1}(U)$ is within $\epsilon$ of a fixed point of $g$.

**Proof:** We will crucially use the fact that the linear part of $g$ is rotation by $\pi$. Consider a point $z \in C$ such that $\Psi(z)$ lies within $\delta$ of the origin. Then $z$ is very nearly the center of all the grids used to define the maps $R_1, ..., R_n$. Were $z$ the center of all these grids, $g$ would locally be a rotation by $\pi$ about $z$. As it is, $g$ is a small perturbation of a rotation by $\pi$ about $z$, and the differential $dg$ is still rotation by $\pi$. But, in this situation, $g$ has a fixed point $z'$ very close to $z$. The distance $|z - z'|$ only depends on $\delta$ and $n$. ♠
Corollary 4.4 Let $U$ be any open set containing the origin in $\mathbb{C}^n$. The set of fixed points of $g$ mapping into $U$ is a net in $\mathbb{C}$.

We choose some small open set $U$ as in the corollary, and let $H$ denote the smallest linear subspace containing all the points of the set $\Psi(\Sigma)$. By construction, $\hat{g}$ fixes every point of $\Psi(\Sigma)$. Since these points span $H$, we see that $\hat{g}$ is the identity on $H$. We just have to prove that $H$ has dimension $2n - 2$.

Lemma 4.5 $H$ has dimension at least $2n - 2$.

Proof: Suppose that $H$ has dimension $d < 2n - 2$. Given any point $p \in \Sigma$, there is some small disk $\Delta$ so that the restriction of $g$ to $\Delta$ is rotation by $\pi$. The radius of $\Delta$ can be chosen to be uniformly large. Call this radius $\rho$. The image $\Psi(\Delta)$ is an isometric disk centered at a point of $H$ and parallel to $\Psi(\mathbb{C})$.

Let $\Sigma'$ denote the $\rho$-tubular neighborhood of $\Sigma$. By construction, $\Psi(\Sigma')$ is a union of disks, all parallel to $\Psi(\mathbb{C})$, and all centered at points of $H$. But this implies that the closure of $K$ of $\Psi(\Sigma')$ has dimension at most $d + 2 < 2n$.

On the other hand, since $\Sigma$ is a net, $\Sigma'$ is a set of positive density in $\mathbb{C}$. Given the affine nature of $\Psi$ and the fact that $\Psi(\mathbb{C})$ is dense in $\mathbb{T}^n$, we see that $K$ must have positive volume – i.e. $(2n)$-dimensional Lebesgue measure. This is impossible if $\dim(K) < 2n$. This contradiction shows that $\dim(H) \geq 2n - 2$. ♠

We have produced a subspace of dimension at least $2n - 2$ which is fixed by $\hat{g}$. This means that the linear part $L_g$ has a $(+1)$-eigenspace of dimension at least $2n - 2$. This is all we needed for Lemma 4.1 in the arational case.

In case there are some rational relations between the numbers $1, s_2, \ldots, s_n$, we can perturb the numbers slightly to get a new sequence with no rational relations. Hence, in the general case, the linear part of $\hat{g}$ has a sequence of approximations by linear maps which all have $(+1)$-eigenspaces of dimension $2n - 2$. But then the dimension of the $(+1)$-eigenspace of the limiting map must be at least $2n - 2$. This completes the proof of Lemma 4.1 in the general case.

We mention the obvious corollary of Lemma 4.1

Corollary 4.6 Suppose that $g = h^2$ where $h$ is a word of length $4m + 2$. Then the linear part of $\hat{g}$ is the identity. Hence $(\mathbb{T}^{2n}, \hat{g})$ is a PET.
4.3 A Picture

Before we continue our analysis in the next section, we show a picture that the reader should keep in mind throughout the discussion.

The square shown in Figure 4.1 is meant to be the torus $T^2$. The opposite sides are meant to be identified. The thick line $\Omega_1$ is really a circle and the thick line $\Omega_2$ is really a line segment. The set $\Omega_1 \cup \Omega_2$ is meant to be a toy version of the singular sets for the map $\hat{h}$ we consider in the next section. The complement of the singular set is a parallelogram – i.e. a set affinely equivalent to $Q^2$ – embedded in $T^2$ in a funny way.

![Figure 4.1: A parallelogram sitting inside a torus.](image)

The set $\Omega_1$ is a 1-torus sitting inside $T^2$. The set $\Omega_2$ is obtained from a horizontal line by applying a locally affine map defined in the annulus $D_1 = T^2 - \Omega_1$. This locally affine map does not extend to $\Omega_1$; it is a kind of partial Dehn twist. The universal cover $\Delta_1$ of $D_1$ is an infinite strip. This strip is a low dimensional version of the slabs $\Delta_k$ we consider in our proof below.
4.4 The Singular Set

We continue with the case where $g = h^2$ and $h$ has length $n = 4m + 2$. Let $\Sigma_j \subset T^{2n}$ denote the singular set of $\hat{R}_j$. Recall that $\Sigma_j$ is a union of 2 codimension 1 tori. Let $\Omega_1 = \Sigma_1$ and then

$$\Omega_k = \hat{R}_1^{-1} \circ \ldots \circ \hat{R}_{k-1}^{-1}(\Sigma_k), \quad k = 2, \ldots, n.$$  \hfill (30)

It follows from induction that $\hat{h}$ is entirely defined on $T^{2n} - \Omega$, $\Omega = \bigcup_{j=1}^{n} \Omega_j$. \hfill (31)

Recall that $Q^k$ is the open unit $k$-dimensional cube centered at the origin. The goal of this section is to prove the following result.

**Lemma 4.7** $T^{2n} - \Omega$ is affinely equivalent to $Q^{2n}$.

We will prove Lemma 4.7 through a series of smaller results.

Say that a singular hyperplane is a hyperplane parallel to some $\Omega_j$. Each $\Omega_j$ supplies 2 singular hyperplanes, so there are $2n$ singular hyperplanes in total.

**Lemma 4.8** The $2n$ singular hyperplanes are linearly independent in the sense that their normal vectors form a real basis for $R^{2n}$.

**Proof:** We use complex notation. Let $e_1, \ldots, e_6$ denote the standard basis vectors for $C^6$. Let $L_j$ denote the linear part of $\hat{R}_j$. The hyperplanes associated to $\Omega_1$ are

$$(e_1) \perp, \ (ie_1) \perp, \ L_1^{-1}((e_2) \perp), \ L_1^{-1}((ie_2) \perp), \ L_1^{-1}L_2^{-1}((e_3) \perp), \ L_1^{-1}L_2^{-1}((ie_3) \perp), \ldots$$ \hfill (32)

The corresponding normals are given by

$$e_1, \ ie_1, \ L_1^t(e_2), \ L_1^t(ie_2), \ L_1^tL_2^t(e_3), \ L_1^tL_2^t(ie_3), \ldots$$ \hfill (33)

An easy calculation shows that the $k$th column of $L_1^t \ldots L_{k-1}^t$ has a 1 in the $(kk)$th position and (0)s below. This implies that the normals in Equation (33) are linearly independent. ♠

26
We will prove Lemma 4.7 by induction. Define
\[ D_k = T^{2n} - (\Omega_1 \cup \ldots \cup \Omega_k). \] (34)
Our final goal is to show that \( D_n \) is affinely equivalent to \( Q^{2n} \). Our induction step will be that \( D_k \) is affinely equivalent to \( Q^{2k} \times C^{n-k} \). We have already seen that this statement is true for \( k = 1 \). We will suppose that the statement is true for some choice of \( k \) and then prove it for \( k + 1 \).

Define
\[ \hat{h}_k = \hat{R}_k \circ \ldots \circ \hat{R}_1. \] (35)
The map \( \hat{h}_k \) is defined on \( D_k \). We have projection
\[ \pi : C^n \to T^{2n}. \] (36)
Let \( L_k \) be the linear part of \( \hat{h}_k \). Let \( \Delta_k \subset C^n \) denote the universal cover of \( D_k \). We think of \( \Delta_k \) as being one connected component of the preimage \( \pi^{-1}(D_k) \). The set \( \Delta_k \) is affinely equivalent to the “slab” \( Q^{2k} \times C^{n-k} \).

We have a commuting square
\[
\begin{array}{ccc}
\Delta_k & \xrightarrow{L_k} & C^n \\
\downarrow \pi & & \downarrow \pi \\
D_k & \xrightarrow{\hat{h}_k} & T^{2n}
\end{array}
\] (37)
The set \( \pi^{-1}(\Sigma_{k+1}) \) consists of two infinite families of parallel hyperplanes. We call these infinite families \( A_{k+1} \) and \( B_{k+1} \). The two families are transverse to each other. By Lemma 4.8, the hyperplanes in \( L_k^{-1}(A_{k+1}) \) and \( L_k^{-1}(B_{k+1}) \) are transverse to \( \partial \Delta_k \).

The intersection \( \Delta_k \cap L_k^{-1}(A_{k+1}) \) is an infinite parallel family of sets, each of which is affinely equivalent to \( Q^{2k} \times C^{n-k-1} \). The same goes for \( D_k \cap L_k^{-1}(B_{k+1}) \). The projection \( \pi \) carries these sets to \( \Omega_{k+1} \).

Let \( \Delta_{k+1} \) be some connected component of
\[ \Delta_k - L_k^{-1}(A \cup B). \]
The set \( \Delta_{k+1} \) is affinely equivalent to \( Q^{2k+2} \times C^{2n-2k-2} \). Informally, the sets \( L_k^{-1}(A_{k+1}) \) and \( L_k^{-1}(B_{k+1}) \) chop up two of the noncompact directions into compact pieces. The map \( \pi : \Delta_{k+1} \to D_{k+1} \) is the universal covering map. From this picture we see that \( D_{k+1} \) is affinely equivalent to \( Q^{2k+2} \times T^{2n-2k-2} \). This completes the induction step.
4.5 The Double Lattice PET

Now we start enhancing the notation from the previous section. Define

$$\Lambda_1 = (C[i])^n.$$  \hspace{1cm} (38)

Let $\pi : C^n \to C^n/\Lambda_1$ be projection, as above.

Since $T^{2n} - \Omega$ is affinely equivalent to the contractible set $Q^{2n}$, we have a parallelotope $X_1$ and a continuous local inverse

$$\pi^{-1} : T^{2n} - \Omega \to X_1 \subset C^n.$$  \hspace{1cm} (39)

The parallelotope $X_1$ is just the image of $T^{2n} - \Omega$ under the lift $\pi^{-1}$. Since $\pi$ is injective on $X_1$ and $\pi(X_1)$ has full measure in $T^{2n}$, we see that $X_1$ is a fundamental domain for $\Lambda_1$.

Lemma 4.9 $\hat{h} = \pi \circ L_h \circ \pi^{-1}$ on $T^{2n} - \Omega$.

Proof: Both maps agree in a neighborhood of the origin and are locally affine and entirely defined on the contractible domain in question. Hence, these two maps agree everywhere. ♠

Now we introduce another lattice and another parallelotope. Recall that the linear part $L_h$ of $\hat{h}$ is an affine involution. We define

$$X_2 = L_h(X_1), \quad \Lambda_2 = L_h(\Lambda_1).$$  \hspace{1cm} (40)

Now we have specified all the data for a double lattice PET. We just have to check the conditions.

Lemma 4.10 $X_2$ is a fundamental domain for $\Lambda_1$.

Proof: We already know that $\hat{h} = \pi \circ L_h \circ \pi^{-1}$ on $T^{2n} - \Omega$. But this means that the map $\pi \circ L_h : X_1 \to T^{2n}$ is injective and has dense image. But this means that $\pi : L_h(X_1) \to T^{2n}$ is injective and has dense image. Hence $L_h(X_1)$ is a fundamental domain for $\Lambda_1$. But $L_h(X_1) = X_2$. ♠

Since $L_h$ is an involution, we see that $X_1$ is also a fundamental domain for $\Lambda_2$. In short, the data $(X_1, X_1, \Lambda_1, \Lambda_2)$ define a double lattice PET. We denote this PET by $(X, f)$. Here $X = X_1$ is the domain.
Lemma 4.11  The locally affine isomorphism $\pi : X_1 \to T^{2n}-\Omega$ conjugates the system $(X, f)$ to the system $(T^{2n}, \hat{g})$.

Proof: We first want to understand the map

$$F = \pi^{-1} \circ \pi : X_2 \to X_1.$$  

For any $p \in X_2$, the points $p$ and $F(p)$ differ by an element of $\Lambda_1$. Thus, the map $F_1 : X_2 \to X_1$ is just the map $f_{21}$ discussed in §4.1. In short,

$$f_{21} = \pi^{-1} \circ \pi. \quad (41)$$

Since $L_h$ is an involution which interchanges the roles of $X_1$ and $X_2$, and also the roles of $\Lambda_1$ and $\Lambda_2$, we have

$$f_{12} = L_h \circ \pi^{-1} \circ \pi \circ L_h. \quad (42)$$

Now we put these equations together.

$$f =$$

$$f_{21} \circ f_{12} =$$

$$\pi^{-1} \circ \pi \circ L_h \circ \pi^{-1} \circ \pi \circ L_h = 1$$

$$\pi^{-1} \circ \pi \circ L_h \circ \pi^{-1} \circ \pi \circ L_h \circ (\pi^{-1} \circ \pi) =$$

$$\pi^{-1} \circ (\pi \circ L_h \circ \pi^{-1}) \circ (\pi \circ L_h \circ (\pi^{-1}) \circ \pi = 2$$

$$\pi^{-1} \circ \tilde{h} \circ \hat{h} \circ \pi =$$

$$\pi^{-1} \circ \hat{g} \circ \pi. \quad (43)$$

Equality 1 comes from the fact that $\pi^{-1} \circ \pi$ is the identity on $X = X_1$. Equality 2 is two applications of Lemma 4.9. In short, $f = \pi^{-1} \circ \hat{g} \circ \pi$. ♠

It is worth emphasizing that the domain for $\pi^{-1}$ is $T^{2n}-\Omega$, though we think of $\pi$ as giving a piecewise isometric conjugacy between a system in $T^{2n}$ and a system in $X_1$. Since our maps are not everywhere defined, the difference in topology has no meaning here.

29
4.6 The Invariant Slice

Consider the composition

$$\Xi = \pi^{-1} \circ \Psi : C \to X_1.$$  \hspace{1cm} (44)

$\Xi$ is not defined on the set $\Psi^{-1}(\Omega)$. This set is a countable collection of line segments. The fact that $\Xi$ is not defined at these points does not bother us. These are the points where the original planar system $g_{s,z}$ is not defined. Putting together the results above, we see that $\Xi$ conjugates the action of $g_{s,z}$ on $C$ to the action of the double lattice PET $(X, f)$ on $\Xi(C)$.

Now let $k$ denote the dimension of the $Q$-vector space $Q[s_1, ..., s_n]$. Looking at the map $\Psi$, we see that $\Psi(C)$ is a $(2k)$-dimensional linear subspace of $T^{2m}$. The intersection

$$\hat{C} = \text{closure}(\Psi(C)) \cap (T^{2n} - \Omega)$$  \hspace{1cm} (45)

is a finite invariant union of $(2k)$-dimensional open polytopes. The set $\pi^{-1}(\hat{C})$ is likewise an invariant finite union of convex $(2k)$-dimensional polytopes in $X_1$. At the same time, this set is the closure of $\Xi(C)$ in $X_1$. This proves Theorem 1.7.

4.7 Proof of Corollary 1.8

It only remains to reconcile our compactification here with the one from Theorem 1.6: recall that $n = 4m + 2$. In case we have the sequence $1, s, 1, s, ..., 1, s$, the construction here is identical to the construction given in §2, except that we are repeating the coordinates $2m + 1$ times. In other words, the diagonal embedding $C^2 \to C^m$ carries the compactification produced by Theorem 1.6 to the system $(\hat{T}^{2n}, \hat{g})$. The composition of $\pi^{-1}$ with the diagonal embedding gives the desired conjugacy.
5 A Concrete Family of PETs

5.1 Generalities

We will work in \( C^n \). We can specify a double lattice PET \((X_1, X_2, \Lambda_1, \Lambda_2)\) by a quadruple of \( n \times n \) matrices \((\chi_1, \chi_2, L_1, L_2)\), where

- \( X_j = \chi_j(Q^{2n}) \).
- \( \Lambda_j \) is the \( Z[i] \)-span of the columns of \( L_j \).

Here \( Q^{2n} = (Q^2)^n \), where \( Q^2 \) is the unit square centered at the origin in \( C \).

There is some ambiguity in our choice of matrices. Let us call two vectors \( V \) and \( V' \) equivalent if either \( V' = \omega V \) or \( V' = \overline{\omega} V \). Here \( \omega \) is some 4th root of unity and \( \overline{\omega} \) is the complex conjugate of \( \omega \). Typically there are 8 vectors in each equivalence class. We say that two matrices \( M \) and \( M' \) are equivalent if each column of \( M \) is equivalent to the corresponding column of \( M' \). If we replace the matrix \( \chi_j \) by an equivalent matrix \( \chi_j' \) then we still recover \( X_j \). Likewise, if we replace the matrix \( L_j \) by \( L_j' \) we still recover \( \Lambda_j \).

Here is a criterion which will help us verify that the matrices we list give rise to double lattice PETs.

**Lemma 5.1** Suppose that \( \chi \) and \( L \) are \( n \times n \) matrices. Let \( X = \chi(Q^{2n}) \) and let \( \Lambda \) be the \( Z[i] \)-span of the columns of \( L \). Then \( X \) is a fundamental domain for \( \Lambda \) provided that \( \chi^{-1}L \) is a triangular and has 4-th roots of unity along the diagonal.

**Proof:** \( X \) is a fundamental domain for \( \Lambda \) if and only if \( T(X) \) is a fundamental domain for \( T(\Lambda) \). Here \( T \) is any invertible linear transformation of \( R^{2n} \approx C^n \). In particular, this is true for \( T = \chi^{-1} \). In other words, it suffices to consider the case when \( \chi \) is the identity matrix and \( X = Q^{2n} \). Replacing \( L \) by an equivalent matrix, we can assume that \( L \) has 1s along the diagonal. But \( Q^{2n} \) is indeed a fundamental domain for a lattice whose defining matrix is triangular and has 1s along the diagonal. ♠

We call \((\chi_1, \chi_2, L_1, L_2)\) a special system if \( \chi_i^{-1}L_j \) is a triangular matrix with 4th roots along the diagonal for each pair \((i, j) \in \{1, 2\}\). The following corollary produces an \( n \)-parameter family of double lattice PETs from a special system.

---

4Not all double lattice PETs can be specified this way, but the ones here can be.
Corollary 5.2 Let $D$ be any nonsingular diagonal matrix. If $(\chi_1, \chi_2, L_1, L_2)$ is a special system, then $(\chi_1 D, \chi_2 D, L_1 D, L_2 D)$ is the data for a double lattice PET.

Proof: We compute

$$(\chi_iD)^{-1}(L_jD) = D^{-1}(\chi_i^{-1}L_j)D = D^{-1}TD = T'.$$

(46)

Here $T$ is the triangular matrix guaranteed by the hypotheses, and $T'$ is the conjugate matrix. $T'$ is also triangular, and has 4th roots of unity along the diagonals. So, for all indices $i, j$, the parallelootope $\chi_i D(Q^{2n})$ is a fundamental domain for the lattice defined by $L_j D$. ♠

Remark: We could produce an example of a special system by taking 4 upper triangular matrices (or lower triangular matrices). However, this would lead to a fairly trivial double lattice PET. The map would essentially be a parabolic linear transformation.

5.2 An Explicit Example

We discovered the construction in this section by working out the details of the compactifications described in the previous section. We will present the construction first and then identify it with what we did in the previous chapter. Our construction works for $n = 2, 6, 10, 14, \ldots$. We introduce matrices

$$\chi_\pm = \begin{bmatrix} \pm i & \pm i & \pm i & \pm i & \ldots \\ +i & -1 & 0 & 0 & \ldots \\ 0 & +i & -1 & 0 & \ldots \\ 0 & 0 & +i & -1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

(47)

$$L_\pm = \begin{bmatrix} \mp 1 & \pm i & \pm i & \mp i & \ldots \\ +1 & -1 & 0 & 0 & \ldots \\ 0 & +1 & -1 & 0 & \ldots \\ 0 & 0 & +1 & -1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

(48)

The first row for $L_\pm$ repeats every 4 entries. Each entry in this first row is $(-i)$ times the preceding element. Just to be clear, the top left entry of $L_+$ is $(-1)$. We set $\chi_1 = \chi_+$ and $\chi_2 = \chi_-$ and $L_1 = L_+$ and $L_2 = L_-$. 

32
Now we will verify that \((\chi_1, \chi_2, L_1, L_2)\) is a special system. Letting \(J\) be the diagonal matrix with entries \((-1, +1, +1, +1, \ldots)\) we see that \(\chi_2 = I\chi_1\) and \(L_2 = IL_1\). We compute

\[
\chi_2^{-1}L_1 = (I\chi_1)^{-1}L_2 = \chi_1^{-1}I^{-1}L_1 = \chi_1^{-1}IL_1 = \chi_1^{-1}L_2.
\]

(49)

Similarly,

\[
\chi_2^{-1}L_2 = \chi_1^{-1}L_1.
\]

(50)

For this reason, we just have to check the conditions for \(\chi_1^{-1}L_1\) and \(\chi_1^{-1}L_2\).

A routine calculation verifies that

\[
L_1 = \chi_1 \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
-1 + i & 1 & 0 & 0 & 0 & 0 & \ldots \\
-1 - i & -1 + i & 1 & 0 & 0 & 0 & \ldots \\
+1 - i & -1 - i & -1 + i & 1 & 0 & 0 & \ldots \\
+1 + i & +1 - i & -1 - i & -1 + i & 1 & 0 & \ldots \\
-1 + i & +1 + i & +1 - i & -1 - i & -1 + i & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]

(51)

The matrix listed is such that entry in the lower triangle is \((-i)\) times the entry directly below it. Going down a column, the pattern has period 4. A similar calculation verifies that

\[
L_2 = \chi_1 \begin{bmatrix}
-i & -1 + i & +1 + i & +1 - i & -1 - i & -1 + i & \ldots \\
0 & -i & -1 + i & +1 + i & +1 - i & -1 - i & \ldots \\
0 & 0 & -i & -1 + i & +1 + i & +1 - i & \ldots \\
0 & 0 & 0 & -i & -1 + i & +1 + i & \ldots \\
0 & 0 & 0 & 0 & -i & -1 + i & \ldots \\
0 & 0 & 0 & 0 & 0 & -i & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]

(52)

The matrix listed is such that each entry in the upper triangle is \((i)\) times the entry immediately to its right. Going across a row, the pattern has period 4.

Thus, \((\chi_1, \chi_2, L_1, L_2)\) is a special system. If \(S = \{s_1, \ldots, s_n\}\), we let \(D_S\) be the diagonal matrix whose diagonal entries are \(s_1, \ldots, s_n\). We will see that the double lattice PET specified by \((\chi_1D_S, \chi_2D_S, L_1D_S, L_2D_S)\) is conjugate to the one produced by Theorem 1.7.
5.3 The Invariant Foliation

Choose some $D_S$. Let $X_j = \chi_j D_S(Q^{2n})$ and let $\Lambda_j$ be the lattice which is the $\mathbb{Z}[i]$ span of the columns of $L_j D_S$. Here we discuss some symmetries of the double lattice PET $(X_1, X_2, \Lambda_1, \Lambda_2)$. Let $J$ be the involution mentioned in the previous section. By construction, the action of $J$ swaps $X_1$ and $X_2$, and simultaneously swaps $\Lambda_1$ and $\Lambda_2$.

Considering $J$ as a matrix acting on $\mathbb{R}^{2n}$, this map has a 2-dimensional $(-1)$-eigenspace and an $(2n - 2)$-dimensional $(+1)$-eigenspace. This is just like the map $L_g$ considered in connection with Theorem 1.7. The $(-1)$-eigenspace of $J$ defines a complex line foliation $\mathcal{F}$ of $X_1$ (and $X_2$). The leaves of this foliation are parallel to $\mathbb{C} \times 0^{n-1}$.

**Lemma 5.3** The foliation $\mathcal{F}$ is invariant under the action of the double lattice PET.

**Proof:** Let $\Pi = \mathbb{C} \times 0^{n-1}$. Since the matrices defining $\Lambda_1$ and $\Lambda_2$ agree below the first row, we see that the two sets $\Lambda_1(\Pi)$ and $\Lambda_2(\Pi)$ are identical. But this translates into the statement that the leaf of $\mathcal{F}$ through the origin is preserved by the double lattice PET. Finally, if one of the leaves is preserved, then all the leaves are preserved. ♠

Let $\mathcal{L}$ denote the leaf of $\mathcal{F}$ through the origin. The double lattice PET preserves $\mathcal{L}$ and induces an action on this real 2-dimensional space. Thus, even without knowing that we have simply recreated the compactifications from Theorem 1.7, we can see that the double lattice PETs here have associated planar actions. In the next section, we will identify these planar actions with the square turning maps from Theorem 1.7.

5.4 Connection to Theorem 1.7

We will work out the case $n = 6$ explicitly. This is a representative case. The cases $n = 10, 14, 18, ...$ follow the same pattern. We will treat the case $n = 2$ separately, and from a different point of view, in the next chapter. In our discussion, we flip back and forth between $\mathbb{C}^6$ and $\mathbb{R}^{12}$ using the identification $(z_1, ..., z_6) \leftrightarrow (x_1, y_1, ..., x_6, y_6)$. Our matrices are defined over $\mathbb{C}$, but $Q^{12} = [-1/2, 1/2]^{12}$ is defined over $\mathbb{R}$. Let

$$s_{ij} = \frac{s_i}{s_j}. \quad (53)$$
Referring to the construction in §4, the matrix $L_g$ turns out to be
\[
\begin{pmatrix}
-i & -(1+i)s_{21} & (1+i)s_{31} & (1-i)s_{41} & -(1-i)s_{51} & -(1+i)s_{61} \\
(1-i)s_{12} & i & (1+i)s_{32} & (1-i)s_{42} & -(1-i)s_{52} & -(1+i)s_{62} \\
(1-i)s_{13} & -(1+i)s_{23} & 2+i & (1-i)s_{43} & -(1-i)s_{53} & -(1+i)s_{63} \\
(1-i)s_{14} & -(1+i)s_{24} & (1+i)s_{34} & 2-i & -(1-i)s_{54} & -(1+i)s_{64} \\
(1-i)s_{15} & -(1+i)s_{25} & (1+i)s_{35} & -(1-i)s_{45} & -i & -(1+i)s_{65} \\
(1-i)s_{16} & -(1+i)s_{26} & (1+i)s_{36} & (1-i)s_{46} & -(1-i)s_{56} & i
\end{pmatrix}
\]
(54)

In general, the pattern is 4-periodic, except for a suitable shift in the indices of $s_{ij}$.

The matrices $I_6$ and $L_g$ represent the two lattices $\Lambda_1$ and $\Lambda_2$. Here $I_6$ is the identity matrix. We seek a matrix $M_1$ which represents $X_1$ in the sense that $X_1 = M_1(Q^{12})$. Let $e_1, ..., e_6$ be the standard basis vectors. Looking at the proof of Lemma 4.8, we see that the first row of $M_1^{-1}$ is $e_1$ and for $k > 1$ the $k$th row of $M_1^{-1}$ is
\[
L_1^k \circ \cdots \circ L_1^{k-1}(e_k).
\]
(55)

Using this formula to compute $M_1^{-1}$, and then taking inverses, we see that $M_1$ is the matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
(1-i)s_{12} & 1 & 0 & 0 & 0 & 0 \\
(1-i)s_{13} & (1-i)s_{23} & 1 & 0 & 0 & 0 \\
(1-i)s_{14} & (1-i)s_{24} & (1-i)s_{34} & 1 & 0 & 0 \\
(1-i)s_{15} & (1-i)s_{25} & (1-i)s_{35} & (1-i)s_{45} & 1 & 0 \\
(1-i)s_{16} & (1-i)s_{26} & (1-i)s_{36} & (1-i)s_{46} & (1-i)s_{56} & 1
\end{pmatrix}
\]
(56)

Let $M_2 = L_g M$. The matrix data for our double lattice PET is given by $(M_1, M_2, I_6, L_g)$.

Now we change coordinates. Let $D_S$ and $\chi_1$ be as in §5.2. Let $A = \chi_1 D_S$. We compute
\[
AI_6 = \chi_1 D_S, \quad AL_g = \chi_2 D_S, \quad AM_1 = L_1 D_S, \quad AM_2 = L_2 D_S.
\]
(57)

Thus, the double lattice PET $(A(X_1), A(X_2), A(\Lambda_1), A(\Lambda_2))$ is precisely the one constructed in §5.2. Note finally that
\[
I := AL_g A^{-1} = J,
\]
(58)

where $J$ is the involution discussed in §5.3. This, $A$ carries the invariant foliation for $(X_1, X_2, \Lambda_1, \Lambda_2)$ to the invariant foliation discussed in §5.3.
5.5 Extra Symmetry

There is one additional symmetry we mention, though we will not need this symmetry for any purpose. The matrix

\[
L'_1 = \begin{bmatrix}
-1 & +1 & +1 & +1 & \\
-1 & -i & 0 & 0 & \\
0 & +i & -1 & 0 & \\
0 & 0 & +1 & +i & \\
& & & & \\
\end{bmatrix}
\] (59)

is equivalent to \(L_1\) and defines the same lattices. We compute that

\[
\chi_1 = \begin{bmatrix}
+i & 0 & 0 & 0 & 0 \\
0 & -i & 0 & 0 & 0 \\
0 & 0 & +1 & 0 & 0 \\
0 & 0 & 0 & +i & 0 \\
0 & 0 & 0 & 0 & -1 \\
& & & & \\
\end{bmatrix} L'_1.
\] (60)

Let \(A\) be the matrix listed. We have \(A_{33} = iA_{22}\) and \(A_{44} = iA_{33}\), etc. The entry \(A_{11}\) is special. Thus, if we exclude the top left entry of \(A\), the pattern along the diagonal has period 4.

Let \([X_j]\) denote the lattice generated by the sides of \(X_j\). Algebraically, \([X_j]\) is just the \(\mathbb{Z}[i]\)-span of the columns of \(\chi_j D_S\). Geometrically, Equation 60 says that there is an order 4 isometry \(J_{11}\) which carries \([X_1]\) to \(\Lambda_1\). More generally, there is an order 4 isometry \(J_{ij}\) which carries \([X_i]\) to \(\Lambda_j\) for any pair of indices. The case \(n = 2\), which we will treat specially in the next chapter, has even more symmetry. In this case, we can find a single order 4 isometry \(J\) which has the action \([X_1] \rightarrow \Lambda_1 \rightarrow [X_2] \rightarrow \Lambda_2\).

We wonder if we can replace \(L_1\) and \(L_2\) by different matrices so as to arrange a similar situation in higher dimensions. The most natural thing would be to let \(L'_\pm\) denote the matrix obtained from \(\chi_{\pm}\) by multiplying the top row by \(i\). In this case, we could take \(J(z_1, z_2, \ldots, z_n) = (iz_1, z_2, \ldots, z_n)\). However, the quadruple \((\chi_1, \chi_2, L'_1, L'_2)\) turns out not to be a special system, and for for random choices of \(\{s_1, s_2, s_3, s_4, s_5, s_6\}\) we saw that the parallelootope \(\chi_1 D_S(Q^{12})\) is not a fundamental domain for the lattice defined by \(L'_1 D_S\). So, this attempt does not work. We mention this because we think that our construction is the simplest possible one which will work.
6 The Octagonal PETs

One should view this chapter as an elaboration of the case $n = 2$ from the previous chapter. Here we will take a different point of view. We are not sure if the cases $n = 6, 10, 14, \ldots$ can be treated in the same way we treat the case $n = 2$ here. This chapter mostly repeats material from [S1]. In this chapter, we use parameters $\{1, s\}$ rather than $\{s_1, s_2\}$. This is our habit for the 2-grid systems.

6.1 The Reflection Lemma

The eigenlattice of a parallelotope is the lattice generated by the vectors parallel to the sides of the parallelotope. Clearly a parallelotope is the fundamental domain for its eigenlattice. A reflection in a face of the parallelotope $P$ is an order 2 linear isometry whose fixed set is a subspace parallel to a face of $P$. The face in question need not be a top-dimensional face.

Lemma 6.1 Let $P$ be a parallelotope and $\Lambda$ be its eigenlattice. Let $I$ be a reflection in a face of $P$. Then $P$ is a fundamental domain for $I(\Lambda)$.

Proof: Let $Q \subset R$ be spaces and $L$ a lattice. We call $Q$ an overdomain in $R$ for $L$ if, for any $p \in R$, there is some $\lambda \in L$ such that $p + \lambda \in Q$. Since the covolume of $I(\Lambda)$ equals the volume of $P$. It suffices to prove that $P$ is an overdomain in $R^n$ for $I(\Lambda)$.

Let $\Pi_+$ be the fixed space of $I$ and let $\Pi_-$ be the orthogonal space. By construction $\Pi_-$ is the $(-1)$ eigenspace of $I$. Let $\pi_-$ be orthogonal projection onto $\Pi_-$. Since multiplication by $(-1)$ is an automorphism of $\Lambda$ and $\Pi_-$ is the $(-1)$ eigenspace, we have $\pi_-(I(\Lambda)) = \pi_-(\Lambda)$. Clearly $\pi_-(P)$ is an overdomain in $\Pi_-$ for $\pi_-(\Lambda)$. Hence $\pi_-(P)$ is an overdomain in $\Pi_-$ for $\pi_-(I(\Lambda))$. Hence $Q = \pi_-(P)$ is an overdomain in $R^n$ for $I(\Lambda)$. So, we can find $\lambda_1 \in I(\Lambda)$ such that $p + \lambda_1 \in P \cap \Pi_+', where $\Pi_+' is some fiber of $\pi_-$. Since $\Pi_+$ is parallel to a face of $P$, and $P$ is a parallelotope, $P \cap \Pi_+' is a translate of $P \cap \Pi_+$. Call this the translation property.

Now, $P \cap \Pi_+$ is an overdomain in $\Pi_+$ for $\Lambda \cap \Pi_+$. But $I$ acts as the identity on $\Pi_+$. Hence $P \cap \Pi_+$ is an overdomain in $\Pi_+$ for $I(\Lambda) \cap \Pi_+$. By the translation property, $P \cap \Pi_+' is an overdomain in $Q \cap \Pi_+' for for $I(\Lambda) \cap \Pi_+$. Hence there is some $\lambda_2 \in I(\Lambda)$ such that $p + \lambda_1 + \lambda_2 \in P$. Hence $P$ is an overdomain in $R^n$ for $I(\Lambda)$
6.2 The Real Case

Figure 6.1 shows a scheme for a 2-dimensional double lattice PET. $X_1$ and $X_2$ are the origin-centered translates of $F_1$ and $F_2$ respectively. $\Lambda_1$ and $\Lambda_2$ are the eigenlattices respectively of $L_1$ and $L_2$.

Several applications of the Reflection Lemma show that $(X_1, X_2, \Lambda_1, \Lambda_2)$ really is the data for a double lattice PET. This double lattice PET depends on the parameter $s \in (0, 1)$, which determines the shapes of our parallelograms. We call our system $(X_s, f_s)$.

In the next chapter we will give an account of the following theorem, which we proved (among many other things) in [S1].

**Theorem 6.2** Let $s \in (0, 1)$ be irrational. Then almost every point of $(X_s, f_s)$ is periodic. The sequence of periods is unbounded, and the periodic islands are all semi-regular octagons or squares. For almost every $s \in (0, 1)$, the system $(X_s, f_s)$ has uncountably many aperiodic points.

**Remark:** It would seem that our last statement of Theorem 6.2 is not as strong as it might be. Since there are infinitely many periodic islands and $X_s$ is compact, there must be accumulation points of these islands. It would seem that these accumulation points are aperiodic. However, the issue is that $f_s$ might not be defined at any of these periodic points. We were not able to rule out this pathology for a measure-zero set of parameters, though we suspect aperiodic points exist for all irrational parameters.
6.3 The Complex Case

In this section, we will define a 1-parameter family of 4-dimensional lattice PETs \((X_s^C, f_s^C)\) which contains \((X_s, f_s)\) as a 2-dimensional invariant slice. Our notation is meant to suggest the idea that we produce these PETs by complexifying the octagonal PETs.

Before we start, we remind the reader that a \textit{real plane} in \(\mathbb{C}^2\) is a 2-plane \(\Pi\) such that \(\Pi\) and \(i\Pi\) are orthogonal. A \textit{complex line} in \(\mathbb{C}^2\) is a 2-plane \(\Pi\) such that \(i\Pi\) and \(\Pi\) are parallel A \textit{complex foliation} is a 2-dimensional foliation whose tangent planes are complex lines.

We let \(G\) denote the order 8 dihedral group generated by \(\rho_1\) and \(\rho_2\). Here

\[
\rho_1(z_1, z_2) = (-z_1, z_2), \quad \rho_2(z_1, z_2) = (z_2, z_1).
\]

The invariant subspaces of \(\rho_1\) and \(\rho_2\) respectively are the complex lines given by \(\{z_2 = 0\}\) and \(\{z_1 = z_2\}\). The group \(G\) acts isometrically on \(\mathbb{C}^2\). The restriction of \(G\) to \(\mathbb{R}^2\) gives the group of symmetries of Figure 6.1.

Let \(X_1^C\) be the parallelogram centered at the origin and spanned by the vectors

\[
(0, 2), \quad (0, 2i), \quad (s\zeta, s\zeta), \quad (s\bar{\zeta}, s\bar{\zeta}); \quad \zeta = 1 + i.
\]

The first two vectors lie in the complex line fixed by \(\rho_1\) and the second two vectors lie in the complex line fixed by \(\rho_2\). We let \(X_2^C = \rho_1 \circ \rho_2(X_1^C)\). We let \(\Lambda_1^C\) be the eigenlattice for \(\rho_1(X_1^C)\). We could equally well describe \(\Lambda_1^C\) as the eigenlattice for \(\rho_2(X_2^C)\). We let \(\Lambda_2^C\) be the eigenlattice for \(\rho_2(X_1^C)\). We could equally well describe \(\Lambda_1^C\) as the eigenlattice for \(\rho_1(X_2^C)\). Note that \(\Lambda_k^C \cap \mathbb{R}^2 = \Lambda_k\). Several applications of the Reflection Lemma show that \((X_1^C, X_2^C, \Lambda_1^C, \Lambda_2^C)\) is a double lattice PET.

By construction, we have

\[
X_k^C \cap \mathbb{R}^2 = X_k, \quad \Lambda_k^C \cap \mathbb{R}^2 = \Lambda_k, \quad k = 1, 2.
\]

Here \((X_1, X_2, \Lambda_1, \Lambda_2)\) are the data for the octagonal PET at the parameter \(s\). Hence, the octagonal PET \((X_s, f_s)\) is an invariant “real slice” of \((X_s^C, f_s^C)\).

The complex octagonal PETs have quite a bit of symmetry. Let \(\Gamma\) be the order 8 dihedral group generated by the elements

\[
\iota_1(z_1, z_2) = (\overline{z_1}, \overline{z_2}), \quad \iota_2(z_1, z_2) = (iz_1, iz_2).
\]
Each element of $\Gamma$ preserves each parallelootope and each lattice defined in connection with the complex octagonal PETs. Hence, $\Gamma$ acts as an order 8 group of symmetries of a complex octagonal PET.

There are 4 elements of $\Gamma$ which act as real reflections – i.e., they pointwise fix real planes in $\mathbb{C}^2$. The planes $\Pi_0 = \mathbb{R}^2$ and $\Pi_2 = i\mathbb{R}^2$ are two of the fixed planes. The other two fixed planes are $\Pi_1 = \zeta \mathbb{R}^2$ and $\Pi_3 = \overline{\zeta} \mathbb{R}^2$. More simply,

$$\Pi_k = \zeta^k \mathbb{R}^2, \quad k = 0, 1, 2, 3. \tag{65}$$

For instance, the map $\iota_2 \circ \iota_1$ fixes $\Pi_1$ pointwise. It turns out that

- $(X^{C_s}_s, f^C_s) \cap \Pi_0$ is a copy of the octagonal PET $(X_s, f_s)$.
- $(X^{C_s}_s, f^C_s) \cap \Pi_2$ is copy of the octagonal PET $(X_s, f_s)$.
- $(X^{C_s}_s, f^C_s) \cap \Pi_1$ is a copy of the octagonal PET $(X_s/2, f_s/2)$.
- $(X^{C_s}_s, f^C_s) \cap \Pi_3$ is a copy of the octagonal PET $(X_s/2, f_s/2)$.

Here, the word *copy* means *up to a similarity*. We already derived the first of these assertions above, and the second one is not hard to see from symmetry. We sketch a proof of the last two assertions in [S1, §5]. We will not use them here, but we point them out in order to highlight some of the beautiful symmetry of the complex octagonal PETs. Each complex octagonal PET contains copies of the real octagonal PETs at two different parameter values!

At this point, the reader might wonder whether the complex octagonal PET is somehow just a product of two of the octagonal PETs it contains, the slice in $\Pi_0$ and the perpendicular slice in $\Pi_2$. This is not the case. When $s = 1$, the two lattices $\Lambda_1$ and $\Lambda_2$ coincide, and both are equal to the beautiful $E_4$ lattice. On the other hand, the product lattice when $s = 1$ would just be $\mathbb{Z}^4$.

In the last section of this chapter, we will discuss more connections between the complex octagonal PETs and the $E_4$-lattice. All in all, one might say that the complex octagonal PETs relate to the real octagonal PETs sort of in the way that the $E_4$ lattice relates to $\mathbb{Z}^2$. 

40
6.4 Connection to Square Turning

Theorem 1.7 produces a double lattice PET for the word $g = (A_1A_2)^2$ and for any irrational parameter $s$. In [S1] we recognized this double lattice PET as the complex octagonal PET at parameter $s$. In this section we will repeat the arguments, through a little more tersely.

The case $n = 2$ in §5.4 gives rise to the matrix data $(M_1, M_2, I, L_g)$ for the double lattice PET.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 & -s & s \\ 1 & 0 & -s & -s \\ -1/s & -1/s & 0 & 1 \\ 1/s & -1/s & -1 & 0 \end{bmatrix}$$

$$L_g = \begin{bmatrix} -i & -s + is \\ -1/s - i/s & i \end{bmatrix} C = \begin{bmatrix} 0 & -1 & -s & s \\ 1 & 0 & -s & -s \\ -1/s & -1/s & 0 & 1 \\ 1/s & -1/s & -1 & 0 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 \\ 1/s - i/s & 1 \end{bmatrix} C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/s & -1/s & 1 & 0 \\ 1/s & 1/s & 0 & 1 \end{bmatrix}$$

$$M_2 = L_g M_1 = \begin{bmatrix} i & -s + is \\ 0 & i \end{bmatrix} C = \begin{bmatrix} 0 & 1 & -s & s \\ -1 & 0 & -s & -s \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$ (66)

Here we are writing each matrix in two ways, as a matrix over $C$, and as a matrix over $R$. We are using the identification

$$(x_1 + iy_1, x_2 + iy_2) \leftrightarrow (x_1, y_1, x_2, y_2).$$ (67)

Now we transform the picture by a suitable real linear transformation. We introduce the matrix

$$\gamma = \begin{bmatrix} -2 & 0 & +s & +s \\ 0 & 0 & +s & +s \\ 0 & +2 & -s & +s \\ 0 & 0 & -s & +s \end{bmatrix}$$ (68)
We compute

\[
\begin{align*}
\gamma M_1 &= \begin{bmatrix}
0 & 0 & +s & +s \\
+2 & 0 & +s & +s \\
0 & 0 & -s & +s \\
0 & +2 & -s & -s
\end{bmatrix} \\
\gamma M_2 &= \begin{bmatrix}
0 & -2 & +s & -s \\
0 & 0 & +s & +s \\
+2 & 0 & -s & +s \\
0 & 0 & -s & +s
\end{bmatrix} \\
\gamma I &= \begin{bmatrix}
-2 & 0 & +s & +s \\
0 & 0 & +s & +s \\
0 & +2 & -s & +s \\
0 & 0 & -s & +s
\end{bmatrix} \\
\gamma L_g &= \begin{bmatrix}
0 & 0 & +s & -s \\
0 & -2 & -s & +s \\
0 & 0 & +s & +s \\
+2 & 0 & -s & -s
\end{bmatrix}
\end{align*}
\]

\[(69)\]

\[
\gamma L_g \gamma^{-1} = \begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix}
\]

\[(70)\]

Let \((X'_1, X'_2, \Lambda'_1, \Lambda'_2)\) be the double lattice PET determined by the above matrices. To recognize this as the complex octagonal PET, we identify \(R^4\) with \(C^2\) in a different way:

\[
(x_1, x_2, y_1, y_2) \rightarrow (x_1 + i y_1, x_2 + i y_2).
\]

\[(71)\]

**Lemma 6.3** \(X'_1 = X'_1^C\).

**Proof:** With this identification, the sides of \(X'_1 = \gamma M_1(Q^4)\) are

\[
(0, 2), \quad (0, 2i), \quad (s\zeta, s\zeta), \quad (s\zeta, s\zeta).
\]

\[(72)\]

This is exactly \(X'_1^C\). ♠
Lemma 6.4 $X'_2 = X^C_2$.

**Proof:** The second parallelootope $X'_2 = \gamma M_2(Q^4)$ is the image of $X'_1$ under the map $\gamma L_g \gamma^{-1}$. Looking at Equation (70) we see that with our new identification of $C^2$ and $R^4$, the map $\gamma L_g \gamma^{-1}$ has the same action as the map $-i\rho_1 \circ \rho_2$. But then

$$X'_2 = -i\rho_2(\rho_1(X'_1)) = -i\rho_1\rho_2(X^C_1) = -iX^C_2 = X^C_2.$$ (73)

The last equality comes from the fact that multiplication by $(-i)$ is a symmetry of $X^C_2$. ♠

Lemma 6.5 $\Lambda'_1 = \Lambda^C_1$

**Proof:** The lattice $\Lambda'_1$ represented by $\gamma I_4$ is the $Z$ span of the vectors

$$(2, 0), \quad (2i, 0), \quad (s\overline{\zeta}, s\overline{\zeta}), \quad (s\zeta, s\zeta).$$ (74)

technically, the first vector we read off is $(-2, 0)$, but changing the sign has no effect on the lattice. Recall that $\rho_2(z_1, z_2) = (z_2, z_1)$. If we apply $\rho_2$ to the vectors listed in Equation (72) we get the vectors in Equation (74). But then $\Lambda'_1$ is the image under $\rho_2$ of the eigenlattice for $X^C_1$. Since $\Lambda^C_1$ has the same description, we see that $\Lambda'_1 = \Lambda^C_1$. ♠

Lemma 6.6 $\Lambda'_2 = \Lambda^C_2$.

**Proof:** This has the same proof as Lemma 6.4. ♠

Thus we see that the double lattice PETs produced by Theorem 1.7 in the lowest dimensional case are the complex octagonal PETs. In other words, the compactifications associated to the alternating grid systems are the complex octagonal PETs. In the next section we will deduce some dynamical consequences from this fact, and from our knowledge of the complex octagonal PETs.
6.5 Dynamical Consequences

Combining Theorem 1.7 with the explicit calculation in the previous section, we get the following corollary.

**Corollary 6.7** The complex octagonal PET \((X^C_s, f^C_s)\) has two invariant irrational orthogonal complex foliations, \(F_1\) and \(F_2\). Here

1. \(F_1\) is spanned by \((\zeta, \bar{\zeta})\) and \((\bar{\zeta}, -\zeta)\).
2. \(F_2\) is spanned by \((\zeta, \zeta)\) and \((\zeta, -\zeta)\).

For any \(z \in C\), there is a leaf \(L = L(s, z)\) of \(F_1\) (or of \(F_2\)) such that the restriction of \(f^C_s\) to \(L\) is conjugate to the action of \(g_{s,z}\) on \(C\). The map \(C \to L\) is a piecewise isometry relative to the Euclidean metric on \(C\) and the path metric on \(L\).

**Remarks:**

(i) We get a piecewise isometry from \(C\) to \(L\) in our corollary because \(X^C\) is a parallelopiped rather than a torus. Were we to glue together the opposite sides of \(X^C\), the map \(C \to L\) would be an isometry.

(ii) One surprising thing about our corollary is that the complex octagonal PETs have two invariant foliations, and the restriction of the map to a leaf in either foliation is conjugate to the square turning map associated to \((A_1A_2)^2\).

**Proof of Theorem 1.4:** Let \(s\) be some irrational parameter and let \(z \in C\) be arbitrary. Let \(L\) be as in Corollary 6.7. Let \(\{p_k\}\) be the unbounded sequence of periods associated to the real octagonal PET \((X_s, f_s)\), as guaranteed by Theorem 6.2. For any \(k\), there is some periodic island in \((X_s, f_s)\) of period \(p_k\). But \((X_s, f_s)\) is an invariant slice of \((X^C_s, f^C_s)\) and this latter system is a PET. Hence, the points of our 2-dimensional periodic island are contained in a 4-dimensional periodic island \(P\) of period \(p_k\).

The piecewise isometry from \(C\) to \(L\) induces a locally Euclidean measure to \(L\), relative to which \(L \cap P\) has positive density in \(L\). This density is exactly \(\text{vol}(P)/\text{vol}(X)\). Hence, a positive density set of points in the invariant leaf \(L\) have period \(p_k\). But then the same statement holds for points in \(C\), relative to the map \(g_{s,z}\). Since \(g_{s,z}\) has a positive density set of points of period \(p_k\), it has infinitely many \(p_k\)-periodic islands. ♠
Proof of Theorem 1.5: We keep the same notation as in the proof of Theorem 1.4. This time, we choose $s$ so that the octagonal PET $(X_s, f_s)$ has uncountable many aperiodic points.

A routine calculation shows that the plane spanned by $(\zeta, \zeta)$ and $(\zeta, -\bar{\zeta})$ is transverse to $\mathbb{R}^2$. Hence, the leaf $L$ intersects $\mathbb{R}^2$ in a countable collection of points. The same goes for any leaf of $F_1$. Hence there uncountably many leaves of $F_1$ which contain aperiodic points of $X \subset \mathbb{R}^2$, the domain for the real octagonal PET. As in Lemma 3.7, this means that there are uncountably many choices of $z \in C$ such that $L = L(s, z)$ contains an aperiodic point of the real octagonal PET.

Let $\omega : C \to L$ be the conjugacy guaranteed by Corollary 5.7. Let $z \in C$ be such that $L$ contains an aperiodic point $x \in L \cap X$. The orbit of $O(x)$ of $x$ is infinite, contained in $X$, and also contained in $L$. Let $x' = \omega^{-1}(x)$. By construction the orbit $O(x')$ under $g_{s,z}$ is infinite. Moreover $\omega$ carries $O(x')$ to $O(x)$. We want to see that $O(x')$ is unbounded.

If $B$ is any bounded subset of $C$, then $\omega(B)$ intersects $X$ in only finitely many points. This follows from the fact that $\omega$ is an isometry when $X^C$ is interpreted as a torus, and the image $L = \omega(C)$ is transverse to $X$. Since $\omega(B)$ can only contain finitely many points of $O(x)$, the bounded set $B$ cannot contain all of $O(x')$. Hence $O(x')$ is unbounded. ♠
7 Renormalization and its Self-Similarity

7.1 Renormalization

Theorem 6.2 derives from a renormalization scheme we found for the (real) octagonal PETs. Let \((X_s, f_s)\) be the octagonal PET at parameter \(s\). Given \(Y \subset X\) (and suppressing the parameter) let \(f|Y\) denote the first return map of \(f\) to \(Y\), assuming that this map is well-defined.

Define the map \(R : (0, 1) \rightarrow [0, 1)\) by the formula

\[
\begin{align*}
R(s) &= 1 - s \text{ if } s \in [1/2, 1). \\
R(s) &= 1/(2s) - \text{floor}(1/(2s)) \text{ if } s \in (0, 1/2).
\end{align*}
\]

For all but countably many choices of \(s\), we have \(t = R(s) > 0\). In all these cases, we prove the following result.

**Theorem 7.1** There are clean convex polygons \(Y_t \subset X_t\) and \(Z_s \subset X_s\) and a similarity \(\phi_s : Y_t \rightarrow Z_s\) such that

- \(\phi_s\) conjugates \(f_t|Y_t\) to \(f_s^{-1}|Z_s\).
- Every nontrivial orbit of \(f_t\) intersects \(Y_t\) and every nontrivial orbit of \(f_s\), except possibly for certain orbits of period 2, intersects \(Z_s\).

The map \(\phi_s\) is an isometry when \(s > 1/2\) and a contraction when \(s < 1/2\).

A polygon is clean if its boundary does not intersect any of the open periodic islands. We describe \(Y\) and \(Z\) explicitly in [S1].

For any given parameters \(s\) and \(t = R(s)\), the proof of Theorem 7.1 is a fairly easy and finite calculation. Indeed, the way we will establish Theorem 6.2 for the single parameter \(\sqrt{2}/2\) is just to appeal to a finite calculation like this. See below for details.

Establishing Theorem 7.1 for all parameters simultaneously is also a finite calculation, but it is much more involved. The idea is that we consider a 3-dimensional piecewise affine system whose 2-dimensional fibers are the octagonal PETs. We then prove by direct calculation a version of Theorem 7.1 for the 3-dimensional system and observe that the 3-dimensional result reduces to the 2-dimensional result above in each slice. All this is easier said than done, however.
7.2 Sketch of Theorem 6.2

When $s$ is irrational, the infinite sequence $\{R^n(s)\}$ exists. It follows almost immediately from Theorem 7.1 that the octagonal PET $(X_s, f_s)$ has infinitely many periodic islands when $s$ is irrational. But this system can only have finitely many periodic islands less than any given period. Hence, the sequence of periods of points in $(X_s, f_s)$ is unbounded. This gives the first statement of Theorem 6.2.

For the second statement, say that $p \in X_s$ is a limit point if every open neighborhood of $p$ contains infinitely many periodic islands. Say that $p$ is bad if the orbit of $p$ is undefined. Otherwise say that $p$ is good. Since $X_s$ is compact and there are infinitely many periodic islands, we know that there is at least one limit point. The tricky part is showing that there are some good limit points.

We prove the following two statements in [S1].

1. The set of limit points of $(X_s, f_s)$ has positive 1-dimensional Hausdorff measure for all irrational $s$.

2. The set of bad limit points has zero 1-dimensional Hausdorff measure for almost all $s$.

This leaves some good limit points for almost all $s$. We will sketch the proof of the Statement 1, because we especially like the argument, and because it is related to the discussion in §8.2.

Let $D_4$ be the order 8 dihedral symmetry group of the unit square $Q^2$. We say that a dihedral polygon is a polygon which, up to translation, has $D_4$ as its symmetry group. To be clear, if we were to rotate $Q^2$ by a typical angle, it would not be a dihedral polygon. The dihedral polygons are either squares or semi-regular octagons.

We check, for the parameters $s = 1/(2n)$, that all the periodic islands are dihedral polygons and right-angled isosceles triangles. We then use Theorem 7.1 and induction to deduce the same result for all rational parameters $s$. Moreover, we use the explicit scaling factors in Theorem 7.1 to show that the diameter of any triangular tile is at most $2^{2-k/4}$, where $k$ is the length of the $R$-orbit of $s$. This quantity decays exponentially with $k$. Taking a limit, we see that every periodic island in the irrational case is a dihedral polygon. Figure 7.1 shows a picture of the tiling for the rational parameter $s = 13/21$. The yellow triangular tiles are quite small.
There are 3 kinds of edges in a $D_4$ polygon: horizontal, vertical, and diagonal. Consider how a horizontal line $L$ intersects a dihedral polygon. Assuming that $L$ does not contain a vertex of $P$, then $L$ either intersects $P$ in two diagonal sides or in two vertical sides. Moreover, if two $D_4$ polygons are tangent along an edge, and $L$ intersects the interior of this edge, then $L$ intersects both polygons in the same kinds of edges, either horizontal or vertical.

In all cases, the island of period 1 – i.e., the fixed island – is just $X_1 \cap X_2$. When $s < 1/2$ this set is a square with sides parallel to the coordinate axes. In Figure 7.1, this is the half-shown big red square on the right. Consider the case $s < 1/2$ for ease of exposition. All the horizontal lines of the form $y = y_0 \in [-s, s]$ intersect $X_s$. If we throw out countably many choices of $y_0$, then the remaining lines do not contain vertices of periodic islands.

The horizontal line $L$ intersects the left edge of $X_s$ in a diagonal edge. On the other hand, $L$ intersects the fixed island in a vertical edge. Given what we have said about how $L$ interacts with semi-regular octagons, we see that $L$ cannot simply run through the interior of a finite union of periodic islands. That is, $L$ must contain a limit point. This means that the projection of the set of limit points onto the vertical axis contains an interval. Hence, the set of limit points has positive 1-dimensional Hausdorff measure.

The projection result is rather surprising. The set of limit points in Figure 7.2 below is a Cantor set. Nonetheless, its vertical projection contains an interval.
7.3 Self-Similar Examples

Figures 7.2 and 7.3 show the picture respectively for the two related parameters $s = \sqrt{2}/2$ and $s = \sqrt{2}/4$. Both pictures are self-similar and both figures arise as different slices of the complex octagonal PET $(X_s^C, f_s^C)$ at the parameter $s = \sqrt{2}/2$.

![Figure 7.2: The tiling for $s = \sqrt{2}/2 = 0 : 1 : 2 : 2 : 2 : 2 : ...$](image1)

![Figure 7.3: The tiling for $s = \sqrt{2}/4 = 0 : 2 : 1 : 4 : 1 : 4 : ...$](image2)

These particular cases are very similar to other systems which arise in this kind of dynamics. Figure 7.2 is locally isometric to the tiling produced by the main example in [AKT], and also to the tiling produced by outer billiards on the regular octagon. Figure 7.3 is locally isometric to one of the tilings produced by the Truchet tile system in [Hoo].
Figure 7.4 shows another example. Another one of our result from [S1] is that the periodic tiling of \((X_s, f_s)\) consists entirely of squares if and only if the continued fraction expansion of \(s\) has the form \(0 : a_1 : a_2 : a_3...\) with \(a_k\) even for all odd \(k\). This condition turns out to be equivalent to the condition that \(R^k(s) < 1/2\) for all \(k\). We call such parameters \textit{oddly even}.

**Figure 7.4**: The tiling for \(s = \sqrt{3}/2 - 1/2 =: 0 : 2 : 1 : 2 : 1 : 2...\)

In all three cases shown, the self-similar nature of the picture derives from the fact that \(s\) is a periodic point of the renormalization map \(R\). In these cases, a finite calculation shows that the system \((X_s, f_s)\) is renormalizable, and then one can deduce the structure of the tiling. We will explain this in somewhat more detail for the most familiar of the pictures, Figure 7.2.

In the cases shown here, the set of limit points is a fractal having Hausdorff dimension greater than 1. For instance, the Cantor set in Figure 7.2, corresponding to \(s = \sqrt{2}/2\), has dimension \(\log(3) / \log(1 + \sqrt{2})\). Since the bad set has Hausdorff dimension at most 1, there are always good limit points.

Now we describe something fairly amazing. One could say that the existence of Figure 7.4 implies the existence of some unbounded orbits for the square turning system \(g_{s, z}\) for \(s = \sqrt{3}/2 - 1/2\) and a suitable choice of \(z\). Let \(U\) be an unbounded orbit corresponding to one of these limit points. It seems that the rescaled limit of \(U\) is locally isometric to the fractal curve in Figure 7.4! Thus, the fractal in Figure 7.4 as a kind of bird’s eye view of a particular unbounded square-turning orbit. The same statement seems to hold for any oddly even parameter. For oddly even parameters, the square turning system seems to “implement” the fractal limit set of the associated octagonal PET. We have not yet tried for a proof.
7.4 More Details in one Case

Here we discuss the case \( s = \sqrt{2}/2 \) in more detail. We set \( f = f_s \), etc.

Figure 7.5 shows the partition of definition for \( f \) and Figure 7.6 shows the partition of definition for \( f^{-1} \). Each polygon in Figure 7.5 is translation equivalent to a unique polygon in Figure 7.6. The map \( f \) simply performs the translations which carry each polygon in Figure 7.5 to the corresponding polygon in Figure 7.6. Thus, one can see the action of \( f \) just by comparing the two figures.

\[ s = \sqrt{2}/2 \]

\[ f = f_s \]

\[ f^{-1} \]

Figure 7.5: The forward partition for \( s = \sqrt{2}/2 \).

Figure 7.6: The backward partition for \( s = \sqrt{2}/2 \).

Inspecting the figures, we see that the central octagon \( O_1 \) is the fixed island of \( f \) and that there are two smaller octagonal islands \( O_{21} \) and \( O_{22} \) of period 2, as shown in Figure 7.7.
There are 8 sets of interest to us. $X - O_1$ is a union of two kites $K_{11}$ and $K_{12}$ and $X - O_1 - O_2$ is a union of 6 smaller kites $K_{21}, \ldots, K_{26}$. There are two similarities carrying $K_{1j}$ to $K_{2k}$ for $j \in \{1, 2\}$ and $k \in \{1, \ldots, 6\}$. One can check with a finite calculation that each of these 12 maps conjugates $f|K_{1j}$ either to $f|K_{2k}$ or $f^{-1}|K_{2k}$. This property implies that the tiling by periodic islands is invariant under the action of these 12 maps. The rest of Figure 7.2 is then determined by this structure.

We have reduced the proof of Theorem 6.2 for the parameter $s = \sqrt{2}/2$, to 12 calculations. We can use symmetry to cut down on the amount of work we have to do. Let $K_{11}$ be the big kite on the right and let $K_{21}$ be the rightmost small kite. Let $K_{22}$ be the bottom kite on the right. The left edge of $K_{22}$ is the right edge of $O_1$.

Let $\rho_1$ denote reflection through the origin. Let $\rho_2$ denote reflection through the long diagonal of $X$. It follows from the definition of the octagonal PETs that $f$ commutes with $\rho_1$. Interchanging the roles played by the two sides of $X_1$ in the definition of the system, we see that $\rho_2$ conjugates $f$ to $f^{-1}$. Using this symmetry, we see that it suffices to check just 2 of the 12 conjugacies mentioned above, namely one of the maps $K_{11} \to K_{12}$ and one of the maps $K_{11} \to K_{22}$.

We will discuss the calculation for $K_{11} \to K_{21}$ in the next section. The calculation $K_{11} \to K_{22}$ can be done in the same way, though we did not actually make the second calculation. Rather, in [S1] we established a general symmetry which reduces the second calculation to the first one. See §8.2 for a discussion of this extra symmetry.
7.5 The Calculation

Define
\[ K_j = K_{j1} \cup \rho_1(K_{j1}), \quad j = 1, 2. \] (75)

Let \( \phi \) be the piecewise orientation-reversing similarity which carries the left (respectively right) half of \( K_1 \) to the left (respectively right) half of \( K_2 \). If one can verify that
\[ \phi^{-1} \circ (f^{-1}|K_2) \circ \phi = f|K_1 \] (76)
then the same relation holds with \( K_{j1} \) in place of \( K_j \).

It turns out that there is a more precise relationship in this case. One can check that \( K_2 \) is an invariant set for \( f^{-3} \) and that \( \phi \) conjugates the action of \( f \) on \( K_1 \) to the action of \( f^{-3} \) on \( K_2 \). We close this chapter by sketching how one makes this verification.

Let \( \Pi_+ \) denote the forward partition for \( f \), shown in Figure 7.5. Say that a finite sequence of points \( p_1, ..., p_k \) is feasible if there are polygons \( P_1, ..., P_k \) in the partition \( \Pi_+ \) such that \( p_j \in P_j \) for all \( j \) and \( f_j(p_j) = p_{j+1} \) for \( j = 1, ..., k-1 \). Here \( f_j \) is the extension of \( f|P_j \) to the closure \( \overline{P_j} \). A finite portion of a genuine orbit is feasible, and so are the limits of such things. However, a feasible sequence may not be a portion of a well-defined orbit. We call another feasible sequence \( q_1, ..., q_k \) compatible with \( p_1, ..., p_k \) if \( q_j \in P_j \) for all \( j \). That is, both sequences visit the same sequence of partition polygons.

Here is a helpful lemma.

Lemma 7.2 (Definedness Criterion) Suppose that \( Q \) is a \( n \)-gon, with vertices \( p_{11}, ..., p_{n1} \). If there exist \( n \) mutually compatible sequences \( p_{j1}, ..., p_{jk} \) for \( j = 1, ..., n \), then \( f^k \) is well defined on all points on the interior of \( Q \).

Proof: Let \( P_1, ..., P_k \) be the sequence of polygons of \( \Pi_+ \) visited by our sequences. Let \( q_1 \) be a point in the interior of \( Q \). \( q_1 \) lies in the interior of \( P_1 \) by convexity. Hence \( f \) is defined on \( q_1 \) and \( q_2 = f(q_1) \in P_2 \). And so on. ♠

Let \( g \) be the map on the left hand side of Equation (76). Using the definedness criterion, with respect to the inverse map \( f^{-1} \) and the inverse partition \( \Pi_- \), we check that \( f^{-3} \) is well defined on every polygon of the form \( \phi(\Pi_+|K) \). But this means that both \( f \) and \( g \) are well defined (and hence translations) on the interior of each of the 12 polygons of \( \Pi_+ - O_1 \). But then it suffices to check Equation (76) on 12 points, one per polygon. We omit the details of these few calculations.
8 Connections to $E_4$

8.1 Three Connections

There is a big literature on things related to $E_4$. See, for instance [CS]. To match what we have done in previous chapters, we will scale the $E_4$ lattice so that its shortest vectors have length 2.

The $E_4$ lattice $\Lambda$ is the lattice

$$2\mathbb{Z}^4 \cup \mathbb{Z}_{\text{odd}}^4.$$  \hspace{1cm} (77)

Here $\mathbb{Z}_{\text{odd}}^4$ is the subset of vectors having all odd coordinates. Geometrically, $\Lambda$ is the union of vertices and centers of the cubes in the cubical grid of side length 2 in $\mathbb{R}^4$. What makes $\Lambda$ so symmetric is the geometric miracle that the distance from the center of a 4-dimensional cube to a vertex of the cube is the same as the side length of the cube.

The $E_4$-polytope $P$ is the convex hull of the set of 24 vectors of length 2 in $E_4$. Up to permuting the coordinates and/or multiplying some of the coordinates by $-1$, these vectors are all equivalent to $(2,0,0,0)$ or to $(1,1,1,1)$. The $E_4$ polytope is one of the 4-dimensional platonic solids. It enjoys 3 properties.

1. $P$ is regular. The symmetry group of $P$ acts transitively on the complete flags of $P$.

2. $P$ tiles space. The Voronoi cells of $\Lambda$ are all translates of the smaller copy $(1/2)P$.

3. $P$ is self dual. Each length-2 normal to a facet (i.e. codimension one face) of $P$ has the form $(\sqrt{2}, \sqrt{2}, 0, 0)$ up to signs and permutation. The convex hull $P^*$ of these unit normals is isometric to $P$.

No 3-dimensional platonic solid has all these properties at the same time. So, in a sense, $P$ is even more symmetric than the familiar 3-dimensional platonic solids.

The 24 facets of $P$ are regular octahedra. These 24 facets are parallel in pairs, and there are 12 codimension 1 subspaces such that any facet of $P$ is parallel to one of these subspaces. Let $H$ denote the collection of these subspaces. The $E_4$-Weyl group is the group $W$ generated by reflections in the members of $H$.  

54
We make all the same constructions for $P^*$ and we arrive at the group $W^*$ generated by reflections in the subspaces parallel to the facets of $P^*$. The groups $W$ and $W^*$ are conjugate. We prefer to work with $W^*$, but we will also consider pay attention to the subset $H$ of hyperplanes defined in connection with $W$.

The faces of $P$ have a 3-coloring such that parallel faces get the same color such that each monochrome subset of $H$ consists of 4 pairwise perpendicular subspaces. Concretely, we can write $H = H_1 \cup H_2 \cup H_3$ where these set have the following normal vectors.

1. $(-a, a, 0, 0), (0, 0, a, -a), (0, 0, a, a), (a, a, 0, 0)$.
2. $(0, a, 0, -a), (0, a, 0, a), (a, 0, -a, 0), (a, 0, a, 0)$.
3. $(a, 0, 0, a), (-a, 0, 0, a), (0, a, a, 0), (0, a, -a, 0)$.

Lemma 8.1 The action of $W^*$ preserves the coloring of the facets of $P$.

Proof: By symmetry, it suffices to check this for of the generators of $W^*$. One of the generators has the action $(x_1, x_2, x_3, x_4) \rightarrow (\gamma x_1)$ where $\gamma M_1$ is as in Equation 69. The matrix in question is

$$
\frac{1}{2} \begin{bmatrix}
-1 & 0 & 1/s & 1/s \\
1 & 0 & 0 & 0 \\
0 & -1 & -1/s & 1/s \\
0 & 1 & 0 & 0
\end{bmatrix}
$$

(78)

Let us now turn to the definition of the complex octagonal PETs. We will interpret these systems as living in $\mathbb{R}^4$, using the identification given in Equation 71. For ease of notation, we set $X_1 = X_1^C$, etc. Thus $X_1$ and $X_2$ are 4-dimensional real polytopes which depend on a parameter $s$, and $\Lambda_1$ and $\Lambda_2$ are lattices in $\mathbb{R}^4$ which also depend on $s$. We will mention 3 connections with $E_4$.

First Connection: As we already mentioned, we have $\Lambda_1 = \Lambda_2 = \Lambda$, the $E_4$ lattice, when $s = 1$. This is the first connection.

Second Connection: Let $M^*$ denote the inverse transpose of the matrix $M$. The normals to the facets of $X_1$ are the columns of $(\gamma M_1)^*$, where $\gamma M_1$ is as in Equation 69. The matrix in question is

$$
\frac{1}{2} \begin{bmatrix}
-1 & 0 & 1/s & 1/s \\
1 & 0 & 0 & 0 \\
0 & -1 & -1/s & 1/s \\
0 & 1 & 0 & 0
\end{bmatrix}
$$

(78)
Two of these normals appear on the list for $H_1$ and two of them appear on the list for $H_2$. Similarly, two of the normals to the parallelootope $X_2$ appear on $H_1$ and the other two appear on $H_2$. In other words, the 8 hyperplanes through the origin parallel to $X_1$ and $X_2$ are precisely the 8 hyperplanes in $H_1 \cup H_2$.

Let $A$ be a periodic island for some complex octagonal PET. By construction, each facet of $A$ is parallel to one of the facets of $X_1$ or one of the facets of $X_2$. Therefore, every face of $A$ is parallel to one of the hyperplanes in $H_1 \cup H_2$.

**Third Connection:** We say that a polytope $P$ is $E_4$-semiregular if $P$ is invariant under the symmetry group $W^*$. The fixed island for the complex octagonal PET is precisely $Y = X_1 \cap X_2$. Here we prove that $Y$ is $E_4$-semiregular. This fact establishes the portion of Conjecture 1.9 having to do with the fixed point set.

Referring to the discussion in §6.3, we see that $Y$ is invariant under the group of order 8 generated by the maps

$$(z_1, z_2) \rightarrow (\overline{z}_1, \overline{z}_2), \quad (z_1, z_2) \rightarrow (iz_1, iz_2).$$

(We find it convenient to use complex notation for the moment.) Moreover, the map

$$(z_1, z_2) \rightarrow (-z_2, z_1)$$

interchanges $X_1$ and $X_2$. All these maps generate an order 16 subgroup $W' \subset W^*$ of symmetries of $Y$.

Now, $Y$ has at most 16 sides, and these sides must come in parallel pairs. Furthermore, the sides of $Y$ are colored (say) red and blue, according as they are parallel to hyperplanes in $H_1$ or hyperplanes in $H_2$. The group $W'$ transitively permutes the hyperplanes in $H_1$ and also the hyperplanes in $H_2$. Therefore, $W'$ transitively permutes the pairs of red facets of $Y$ and also transitively permutes the pairs of blue facets.

But now we can say that all the red facets of $Y$ are the same distance from the origin, and all the blue facets of $Y$ are the same distance from the origin. Since these facets are parallel to the hyperplanes in $H_1$ and $H_2$, and $W^*$ preserves each of these sets of hyperplanes, we see that $W^*$ permutes the hyperplanes extending the red facets of $Y$. The same goes for the blue facets. But $Y$ is just the intersection of halfspaces bounded by these hyperplanes. Hence $W^*$ preserves $Y$. This completes the proof.
8.2 The Weyl Pseudogroup Action

After conjecturing that almost every point in a complex octagonal PET is periodic, one might be tempted to conjecture that every periodic island is $E_4$-semiregular. This, however, is not the case. While many of the periodic islands are $E_4$-semiregular, there are some anomalous tiles. We think that the existence of these anomalous tiles is related to the red islands discussed in connection with Figures 1.1 and 1.2.

In the same way that it appears that we can subdivide the red island in the plane to reveal a more symmetric pattern for the square turning map, we think that perhaps we can subdivide the asymmetric tiles in the complex octagonal PET to produce a finer tiling by $E_4$ semiregular polytopes. If this is really the case, then one might ask if there is another dynamical system which produces this finer tiling. In this section we are going to describe a new system which seems to produce a finer tiling and yet seems to be compatible with the complex octagonal PETs. This new system is not a mapping in the traditional sense, but rather a pseudogroup action.

In this section, we will do three things. First, we will describe what we mean by a pseudogroup action. Second, we will describe the specific pseudogroup action that is related to the complex octagonal PETs and give evidence for the connection. Finally, we will say a word about how we discovered this alternate system.

**Pseudogroup Actions and Orbits:** Let $X \subset \mathbb{R}^d$ be a compact domain and let $g_1, ..., g_n$ be some finite list of isometries. We insist that this list is symmetric with respect to inverses. In other words, an isometry appears on the list if and only if the inverse isometry appears on the list. We do not require that $g_k(X) = X$.

For each $k$, we define

$$X_k = g^{-1}(g_k(X) \cap X).$$

(79)

By definition, $X_k \subset X$ is the maximal subset of $X$ such that $g_k(X_k) \subset X$. We call a subset $Y \subset X$ stable if it has for following property. If $p \in Y \cap X_k$ then $g_k(p) \in Y$ as well. In other words, the pseudogroup action maps $Y$ into itself. We call $Y$ a pseudogroup orbit if $Y$ is stable and no proper subset of $Y$ is stable. In case all the isometries preserve $X$, a pseudogroup orbit corresponds with an orbit of the group generated by the isometries.
The Weyl Pseudogroup: Our constructions, as usual, depend on a parameter \( s \). Consider the matrix

\[
A = \frac{s}{2} \begin{bmatrix}
+1 & +1 & +1 & +1 \\
+1 & +1 & -1 & -1 \\
+1 & -1 & -1 & +1 \\
+1 & -1 & -1 & -1
\end{bmatrix}
\]

(80)

This matrix is a multiple of a Hadamard matrix. The rows and columns are orthogonal and all have the same length, namely \( s \).

Let \( \Pi \) denote the tiling of \( \mathbb{R}^4 \) by unit cubes, such that the origin is a vertex of one of the cubes. Let \( \Gamma_1 \) denote the infinite list of elements generated by reflections in the hyperplanes of \( \Pi \). By construction, \( \Gamma_1 \) is an infinite Bieberbach group. Let

\[
\Gamma_2 = A \circ \Gamma_1 \circ A^{-1}
\]

(81)

Then \( \Gamma_2 \) is the group generated by reflections in the hyperplanes of the smaller and rotated grid \( A(\Pi) \). No matter how \( s \) is chosen, there are only finitely many elements \( g \in \Gamma_1 \cup \Gamma_2 \) such that \( g(X) \cap X \neq \emptyset \). To be clear, the group generated by \( \Gamma_1 \cup \Gamma_2 \) is a dense group, and there are infinitely many elements of this group within every neighborhood of the identity. However, we are only allowed to take elements in \( \Gamma_1 \cup \Gamma_2 \).

We call \( g \in \Gamma_1 \cup \Gamma_2 \) good if either \( g(X) \cap X \neq \emptyset \) or \( g^{-1}(X) \cap X \neq \emptyset \). We call the pseudogroup generated by the good elements the Weyl pseudogroup.

To discuss the properties of the Weyl pseudogroup, we introduce some notation. Let \( O_p \) denote the orbit of a point \( p \) under the complex octagonal PET. Let \( O^*_p \) denote the pseudogroup orbit of \( p \). Let \( Q_p \) denote the periodic island about \( p \), with respect to the complex octagonal PET. Numerically we observe the following things.

1. For almost every \( p \), the orbit \( O^*_p \) is finite.
2. For almost every \( p \), there is a special point \( q \in Q_p \) such that \( O_q \subset O^*_q \).
   
   Generically, \( O^*_p \) is 192 times as large as \( O^*_q \).
3. If \( p \in \mathbb{R}^2 \) then it seems \( O^*_q \cap \mathbb{R}^2 = O_q \).

The third item is especially surprising because \( \mathbb{R}^2 \cap X \) is an invariant slice of the complex octagonal PET – it is, by definition, the real octagonal PET. On the other hand, \( \mathbb{R}^2 \cap X \) is not invariant for the Weyl pseudogroup.
The properties above might lead the reader to think that the Weyl pseudogroup is just a minor tweak of the complex octagonal PET. However this is not the case. The pseudogroup orbits tend to be much larger than the octagonal PET orbits, and indeed the ratio between the sizes is unbounded. Nonetheless, the three properties above suggest that there is an invariant tiling associated to the Weyl pseudogroup and that this tiling refines the periodic tiling associated to the complex octagonal PETs.

The linear parts of the isometries on the Weyl pseudogroup belong to the group $W^*$ mentioned above. This property leads us to believe that the tiles in the invariant tiling associated to the Weyl pseudogroup are all $E_4$-semiregular. We have yet to investigate this tiling computationally.

The Origins of this System: Looking at Figures 6.2-6.4, the reader can probably see that certain subsets of the tiling exhibit bilateral symmetry. For instance, the tilings are all “as symmetric as possible” about $x = k$ for $k = -1, 0, 1$. What this means is that the tiling is symmetric in the domain $\rho(X) \cap X$, where $\rho$ is a vertical reflection in one of the lines mentioned. This is the symmetry we mentioned at the end of §7.4.

We decided to look at the pseudogroup generated by all these “partial symmetries” of the tiling in the real case. Eventually we got the idea of complexifying the picture, and we arrived at the Weyl pseudogroup mentioned here.
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