On a special value of the Ruelle L-function

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Abstract

Let $X$ be a a complete hyperbolic threefold of a finite volume with only one cusp. For a unitary local system $\rho$ of rank one on $X$, one may associate the Ruelle L-function $R_{\rho}(z)$. Suppose the restriction of $\rho$ to the cusp is nontrivial. We will show that the Ruelle L-function has a pole at the origin whose order is equal to $-2 \dim H^1(X, \rho)$. Moreover we will prove if $\dim H^1(X, \rho)$ is zero $R_{\rho}(0)$ is equal to the square of the Franz-Reidemeister torsion of $(X, \rho)$.

1 Introduction

In [10] we have shown that a geometric analog of the Iwasawa conjecture holds for the Ruelle L-function and the twisted Alexander invariant.

More precisely let $\Gamma$ be a torsion free cofinite discrete subgroup of $PSL_2(\mathbb{C})$. It acts on the three dimensional Poincaré upper half space

$$\mathbb{H}^3 = \{(x, y, r) \mid x, y \in \mathbb{R}, r > 0\}$$

endowed with a metric

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2},$$

whose sectional curvature $\equiv -1$. Let $X$ be the quotient, which is a complete hyperbolic threefold of finite volume. We will assume that it has only one cusp. Let $\rho$ be a unitary character of $\Gamma$. It defines a unitary local system on $X$ of rank one, which will be denoted by the same symbol. By the one to one correspondence between the set of loxodromic conjugacy classes of $\Gamma$ and one of closed geodesics of $X$, the Ruelle L-function is defined as

$$R_{\rho}(z) = \prod_{\gamma} \det[1 - \rho(\gamma) e^{-z(\gamma)}],$$

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where $\gamma$ runs through primitive closed geodesics. Here $z$ is a complex number and $l(\gamma)$ is the length of $\gamma$. It is known $R_\rho(z)$ is absolutely convergent if $\text{Re } z$ is sufficiently large. Suppose the restriction $\rho|_{\Gamma_\infty}$ of $\rho$ to the fundamental group $\Gamma_\infty$ of the cusp is nontrivial. In [10] we have shown that $R_\rho(z)$ is meromorphically continued on the whole plane and that

$$\text{ord}_{z=0} R_\rho(z) = -2 h^1(\rho),$$

where $h^1(\rho)$ is the dimension of $H^1(X, \rho)$.

Let us assume there is a surjective homomorphism from $\Gamma$ to $\mathbb{Z}$ and $X_\infty$ the corresponding infinite cyclic covering of $X$. Moreover suppose that the dimensions of all of $H_*(X_\infty, \mathbb{C})$ and $H_*(X_\infty, \rho)$ are finite. Let $g$ be a generator of the infinite cyclic group. Then the twisted Alexander invariant $A^*_X(\rho)$ is defined to be an alternating product of characteristic polynomials of the action of $g$ on $H_*(X_\infty, \rho)$. (See [9] for the precise definition.) In [9] we have prove that

$$\text{ord}_{z=0} R_\rho(z) \geq 2 \text{ord}_{t=1} A^*_X(\rho),$$

and that if $h^1(\rho)$ is zero, $R_\rho(z)$ and $A^*_X(\rho)(t)$ does not vanish at $z = 0$ and $t = 1$, respectively. It should be natural to compare their values. In fact we will prove the following theorem.

**Theorem 1.1.** Suppose that $\rho|_{\Gamma_\infty}$ is nontrivial and that $h^1(\rho)$ vanishes. Then we have

$$R_\rho(0) = \tau_X(\rho)^2,$$

where $\tau_X(\rho)$ is the Reidemeister torsion of $X$ and $\rho$.

If the manifold is compact, the corresponding result has been already proved by Fried ([4]).

Since we know the absolute value of $\tau_X(\rho)$ is equal to a product of $|A^*_X(\rho)(1)|$ and a certain positive constant $\delta_\rho$ which can be computed explicitly ([9 Theorem 3.4]), we have

$$|R_\rho(0)| = (\delta_\rho \cdot |A^*_X(\rho)(1)|)^2.$$

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## 2 Laplace-Mellin transform

We define the Laplace transform of a function $f$ on $\mathbb{R}$ to be

$$L(f)(z) = \int_0^{\infty} e^{-tz} f(t) \frac{dt}{t},$$

if the RHS is absolutely convergent.
Lemma 2.1. Let \( l \) be a positive number and suppose \( z > 0 \). Then
\[
L\left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{z^2}{4t}}\right)(z) = \frac{e^{-lz}}{l}.
\]

Proof. Taking a derivative of
\[
\int_0^\infty \exp\left(-t^2 - \frac{x^2}{t^2}\right) dt = \frac{\sqrt{\pi}}{2} e^{-2x}
\]
with respect to \( x \), we have
\[
x \int_0^\infty \frac{1}{t^2} \exp\left(-t^2 - \frac{x^2}{t^2}\right) dt = \frac{\sqrt{\pi}}{2} e^{-2x}.
\]
Let \( \alpha \) be a positive number. If we make a change of variables:
\[
t \rightarrow \sqrt{\alpha} t,
\]
we will obtain
\[
\int_0^\infty t^{-\frac{3}{2}} \exp\left(-t^2 - \frac{x^2}{t^2}\right) dt = \frac{\sqrt{\pi \alpha}}{x} e^{-2x}.
\]
Now (1) and a simple computation will show the desired identity.

We also define the Laplace-Mellin transform of \( f \) to be
\[
\mathcal{L}(f)(s, z) = \int_0^\infty e^{-tz^2} t^{s-1} f(t) dt,
\]
for sufficiently large real numbers \( z \) and \( s \) if the RHS is absolutely convergent. Suppose that \( \mathcal{L}(f)(s, z) \) is continued to a meromorphic function on an open domain \( U \) of \( \mathbb{C}^2 \) which contains
\[
\{(s, z) \mid s, z \in \mathbb{R}, s, z >> 0, \},
\]
and that its polar set \( P_{\mathcal{L}(f)(s, z)} \) does not contain
\[
U_{0,z} = U \cap \mathbb{C}_{0,z},
\]
where \( \mathbb{C}_{0,z} = \{(0, z) \mid z \in \mathbb{C}\} \). Then we define the Laplace transform \( L(f)(z) \) on \( U_{0,z} \) to be
\[
L(f)(z) = \mathcal{L}(f)(0, z).
\]

For a nonnegative integer \( k \), let us consider a function:
\[
p_k(t) = \int_0^\infty e^{-tz^2} x^{2k} dx.
\]

3
Lemma 2.2. For \( z > 0 \) and \( s > \frac{1}{2} + k \), the Laplace-Mellin transform of \( p_k \) is

\[
L(p_k)(s, z) = \frac{\sqrt{\pi} C_k}{2} z^{1+2k-2s} \Gamma(s - \frac{1}{2} - k),
\]

which is defined over \( \{(s, z) \in \mathbb{C}^2 | s \in \mathbb{C}, -\pi < \text{Im } z < \pi \} \). Here we put

\[
C_0 = 1
\]

and

\[
C_k = \prod_{m=0}^{k-1} \left( m + \frac{1}{2} \right)
\]

for \( k \geq 1 \).

Proof. Let \( t \) be a positive number. Take the \( k \)-times derivative of

\[
\int_{-\infty}^{\infty} e^{-tx^2} dx = \sqrt{\pi} t^{-\frac{1}{2}}
\]

with respect to \( t \), we obtain

\[
\int_{-\infty}^{\infty} x^{2k} e^{-tx^2} dx = \frac{\sqrt{\pi} C_k}{2} t^{-\frac{1}{2} - k}. \tag{2}
\]

Then we compute:

\[
L(p_k)(s, z) = \int_0^\infty \frac{dt}{t} t^s e^{-tx^2} \int_{-\infty}^{\infty} x^{2k} e^{-tx^2} dx
\]

\[
= \frac{\sqrt{\pi} C_k}{2} \int_0^\infty e^{-tx^2} t^{s-\frac{1}{2} - k} \frac{dt}{t}
\]

\[
= \frac{\sqrt{\pi} C_k}{2} z^{1+2k-2s} \Gamma(s - \frac{1}{2} - k).
\]

\[
\square
\]

Corollary 2.1. For a nonnegative integer \( k \),

\[
L(p_k)(z) = \frac{\sqrt{\pi} C_k}{2} \Gamma(-\frac{1}{2} - k) z^{1+2k}.
\]

Note that this is defined over the whole \( z \)-plane.

3 Selberg trace formula and Laplace transforms of orbital integrals

Let \( \Omega^j_X(\rho) \) be a vector bundle of \( j \)-forms on \( X \) twisted by \( \rho \) and the space of its square integrable sections will be denoted by \( L^2(X, \Omega^j_X(\rho)) \). The positive
Hodge Laplacian has the selfadjoint extension to $L^2(X, \Omega_X^j(\rho))$, which will be denoted by $\Delta$. Note that the Hodge star operator induces an isomorphism of Hilbert spaces:

$$L^2(X, \Omega_X^j(\rho)) \simeq L^2(X, \Omega_X^{3-j}(\rho)), \quad j = 0, 1,$$

which commutes with $\Delta$.

Since $\rho|_{\Gamma_\infty}$ is nontrivial we know that the spectrum of $\Delta$ consists of only eigenvalues and the Selberg trace formula becomes (See §4 of [10]):

$$\text{Tr}[e^{-t\Delta} | L^2(X, \Omega_X^j(\rho))] = I_j(t) + H_j(t) + U_j(t).$$

Here $I_j(t)$, $H_j(t)$ and $U_j(t)$ are the identity, the hyperbolic and the unipotent term, respectively. In this section we will compute their Laplace transform.

1. **The hyperbolic term**

   Let $A$ be a split Cartan subgroup of $G = \text{PSL}_2(\mathbb{C})$. The Lie algebras of $G$ and $A$ will be denoted by $\mathfrak{g}$ and $\mathfrak{a}$, respectively. The choice of $A$ determines a positive root $\alpha$ of $\mathfrak{g}$ and let $H$ be an element of $\mathfrak{a}$ satisfying $\alpha(H) = 1$.

   If we exponentiate a linear isomorphism:

   $$\mathbb{R} \xrightarrow{h} \mathfrak{a}, \quad h(t) = tH,$$

   we know $A$ is isomorphic to the multiplicative group of positive real numbers $\mathbb{R}_+$ and will identify them.

   Let $K \simeq SO_3(\mathbb{R})$ be the maximal compact subgroup. According to the Iwasawa decomposition $G = KAN$ an element $g$ of $G$ can be written as

   $$g = k(g) a(g) n(g).$$

   Let $M$ be the centralizer of $A$ in $K$, which is isomorphic to $SO_2(\mathbb{R})$. It determines a paraboloic subgroup:

   $$P = MAN.$$

   Let $\Gamma_h$ be the set of conjugacy classes of loxidromic elements of $\Gamma$. Since there is a natural bijection between closed geodesics of $X$ and $\Gamma_h$, we may identify them. Thus an element $\gamma$ of $\Gamma_h$ is written as

   $$\gamma = \gamma_0^\mu(\gamma),$$

   where $\gamma_0$ is a primitive closed geodesic and $\mu(\gamma)$ is a positive integer, which will be referred as the multiplicity. The length of $\gamma \in \Gamma_h$ will be denoted by $l(\gamma)$ and let $\Gamma_{h,\text{prim}}$ be the set of primitive closed geodesics.
Using the Langlands decomposition (4), $\gamma \in \Gamma_h$ may be written as
\[ g\gamma g^{-1} = m(\gamma) \cdot a(\gamma) \in MA \]
for a certain $g \in G$. Here $m(\gamma)$ is nothing but the holonomy of a parallel transformation along $\gamma$. Note that elements of $GL_2(\mathbb{R})$:
\[ A^u(\gamma) = e^{l(\gamma)}m(\gamma), \quad A^s(\gamma) = e^{-l(\gamma)}m(\gamma) \]
describe an unstable or a stable action of the linear Poincaré map, respectively.

For $\gamma \in \Gamma_h$ we set
\[ \Delta(\gamma) = \det[I_2 - A^s(\gamma)] \]
and
\[ a_0(\gamma) = \frac{\rho(\gamma) \cdot l(\gamma_0)}{\Delta(\gamma)}, \quad a_1(\gamma) = \frac{\rho(\gamma) \cdot \text{Tr}[m(\gamma)] \cdot l(\gamma_0)}{\Delta(\gamma)}. \]

Now Theorem 2 of [4] shows the hyperbolic terms are given by
\[ H_0(t) = H_0(t), \quad H_1(t) = H_0(t) + H_1(t), \]
where
\[ H_0(t) = \sum_{\gamma \in \Gamma_h} \frac{a_0(\gamma)}{\sqrt{4\pi t}} \exp\left[-\left(\frac{l(\gamma)^2}{4t} + t + l(\gamma)\right)\right], \]
and
\[ H_1(t) = \sum_{\gamma \in \Gamma_h} \frac{a_1(\gamma)}{\sqrt{4\pi t}} \exp\left[-\left(\frac{l(\gamma)^2}{4t} + l(\gamma)\right)\right]. \]

We will explain a relation between these hyperbolic terms and the Ruelle L-function

For $j = 0, 1$ we set
\[ S_j(z) = \exp\left[-\sum_{\gamma \in \Gamma_h} \frac{a_j(\gamma)}{l(\gamma)} e^{-z(l(\gamma))}\right]. \]

Then the formula (RS) of [4] shows
\[ R_\rho(z) = \frac{S_0(z)S_0(z + 2)}{S_1(z + 1)}. \]

Using Lemma 2.1, a simple computation implies the following lemma.

**Lemma 3.1.** (a) $L(H_1)(z) = -\log S_1(z + 1)$.
(b) $L(e^t H_0)(z) = -\log S_0(z + 1)$. 

6
Thus we have proved the following proposition.

**Proposition 3.1.**

\[
\log R_\rho(0) = L(H_1)(0) - L(e^t H_0)(-1) - L(e^t H_0)(1)
\]

2. **The identity term** In §6 of [11] we have computed the identity terms to be:

\[ I_0(t) = I_0(t), \quad I_1(t) = I_0(t) + I_1(t), \]

where

\[
I_0(t) = \text{vol}(X) \int_{-\infty}^{\infty} e^{-t(x^2+1)}x^2 dx,
\]

and

\[
I_1(t) = 2\text{vol}(X) \int_{-\infty}^{\infty} e^{-tx^2}(x^2 + 1) dx.
\]

**Lemma 2.2** implies

\[
\mathcal{L}(e^t I_0)(s, z) = \frac{\sqrt{\pi}}{4} \text{vol}(X) z^{3-2s} \Gamma(s - \frac{3}{2}).
\]

Also the identity

\[
\Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3}
\]

implies

\[
L(e^t I_0)(z) = \frac{\pi}{3} \text{vol}(X) z^3.
\]

By the same computation, we will have

\[
\mathcal{L}(I_1)(s, z) = \frac{\sqrt{\pi}}{2} \text{vol}(X) (z^{3-2s} \Gamma(s - \frac{3}{2}) + 2z^{1-2s} \Gamma(s - \frac{1}{2})),
\]

and

\[
L(I_1)(z) = 2\pi \text{vol}(X) \left(\frac{z^3}{3} - z\right).
\]

Thus we have proved

**Proposition 3.2.**

\[
L(I_1)(0) - L(e^t I_0)(-1) - L(e^t I_0)(1) = 0.
\]

3. **The unipotent term**

We put

\[
U_0(t) = U_0(t), \quad U_1(t) = U_1(t) - U_0(t).
\]

In **Proposition 7.1** of [11], we have proved the following fact.
Fact 3.1.  (a) 
\[ U_0(t) = C_{\rho, \Gamma} e^{-t} \int_{-\infty}^{\infty} e^{-tx^2} dx, \]

(b) 
\[ U_1(t) = 2C_{\rho, \Gamma} \int_{-\infty}^{\infty} e^{-tx^2} dx. \]

where \( C_{\rho, \Gamma} \) is a constant determined by \( \Gamma \) and \( \rho \).

Thus by Lemma 2.2, we obtain
\[ L(U_1)(s, z) = 2L(e^t U_0)(s, z) = \sqrt{\pi} C_{\rho, \Gamma} z^{1-2s} \Gamma(s - \frac{1}{2}), \]

which implies
\[ L(U_1)(z) = 2L(e^t U_0)(z) = -\pi C_{\rho, \Gamma} z. \]

Thus the following proposition is proved.

Proposition 3.3.
\[ L(U_1)(0) - L(e^t U_0)(-1) - L(e^t U_0)(1) = 0. \]

4 Laplace transform of the heat kernel and the analytic torsion

We set
\[ \delta_{0, \rho}(t) = \text{Trace}[e^{-t\Delta} | L^2(X, \Omega^0_X(\rho))], \]

and
\[ \delta_{1, \rho}(t) = \text{Trace}[e^{-t\Delta} | L^2(X, \Omega^1_X(\rho))] - \delta_{0, \rho}(t). \]

The nontriviality of \( \rho|_{\Gamma_{\infty}} \) implies \( H^0(X, \rho) = 0 \) and by the Zucker’s result ([11], see also the introduction of [5] and §2 of [10]), we have
\[ \text{Ker}[\Delta | L^2(X, \Omega^0_X(\rho))] = 0. \]

Let us assume \( h^1(\rho) \) vanishes. As we have shown [10] Lemma 2.1, this implies
\[ \text{Ker}[\Delta | L^2(X, \Omega^1_X(\rho))] = 0. \]

Thus there is positive constants \( c_j \) and \( A \) such that
\[ |\delta_j(t)| \leq c_0 e^{-c_1 t^2} \text{ for } t \geq A. \] (5)
Lemma 4.1. 1. $L(\delta_1)(s, z)$ is absolutely convergent for $\Re s > 0$ and $z > 0$ and is meromorphically continued on an open domain of $\mathbb{C}^2$ containing

$$\{(s, z) | s \in \mathbb{C}, z \in \mathbb{R}\}.$$

2. $L(\delta_0)(s, z)$ is absolutely convergent for $\Re s > 0$ and $z \geq 1$ is meromorphically continued on an open subset of $\mathbb{C}^2$ containing

$$\{(s, z) | s \in \mathbb{C}, z \geq 1\}.$$

Proof. Since a proof of the both statements are same, we will only prove the first. The absolutely convergence is clear from (4).

Let us write

$$L(\delta_1)(s, z) = L_{(0, A]}(\delta_1)(s, z) + L_{[A, \infty)}(\delta_1)(s, z),$$

where we put

$$L_{(0, A]}(\delta_1)(s, z) = \int_0^A e^{-tz^2}t^{-1} \delta_1(t)dt,$$

and

$$L_{[A, \infty)}(\delta_1)(s, z) = \int_A^\infty e^{-tz^2}t^{-1} \delta_1(t)dt.$$ 

(5) implies $L_{[A, \infty)}(\delta_1)(s, z)$ is defined on such an open subset. The computation of the previous section and the equation (2) show the orbital integrals have the following asymptotic expansion when $t \to 0$:

$$H_1(t) = \sum_{\gamma \in \Gamma} \frac{a_1(\gamma)}{\sqrt{4\pi t}} \exp[-\frac{(l(\gamma))^2}{4t} + l(\gamma)]$$

$$\sim a_0 e^{-\frac{\alpha_0}{t}},$$

$$I_1(t) = 2\text{vol}(X) \int_{-\infty}^\infty e^{-tx^2} (x^2 + 1)dx$$

$$\sim \beta_1 t^{-\frac{1}{2}} + \beta_0 t^{-\frac{1}{2}},$$

and

$$U_1(t) = 2C_{\rho, \Gamma} \int_{-\infty}^\infty e^{-tx^2} dx$$

$$\sim \gamma_0 t^{-\frac{1}{2}}.$$ 

Thus for a real number $z$, using the Selberg trace formula, we have an asymptotic
expansion:

\[
\mathcal{L}_{(0,A)}(\delta_1)(s, z) \sim \alpha_0 \int_0^A e^{-\frac{\pi t}{z}} e^{-t z^2} t^{s-1} dt \\
+ \beta_1 \int_0^A e^{-t z^2} t^{s-\frac{5}{2}} dt \\
+ \gamma' \int_0^A e^{-t z^2} t^{s-\frac{3}{2}} dt \\
\sim A_0 + \frac{A_1}{s - \frac{5}{2}} + \frac{A_2}{s - \frac{3}{2}}.
\]

where \(\alpha_0, \beta_i, \beta'_i, \gamma_i, \gamma'_i\) and \(A_i\) are constants. Now we have obtained the desired result.

\[\Box\]

If \(\text{Re } s\) is sufficiently large, the integral

\[
\mathcal{L}(\delta_1)(s, 0) = \int_0^\infty \delta_1(t) t^{s-1} dt
\]

is absolutely convergent and is nothing but the Mellin transform \(M(\delta_1)(s)\) of \(\delta_1\). Since by Lemma 3.1 of [10] we know \(L(\delta_1)\) is regular at \(z = 0\), Lemma 4.1 implies

\[
L(\delta_1)(0) = L(\delta_1)(0, 0) = M(\delta_1)(0).
\]

Using Lemma 3.2 of [10], the same argument will imply

\[
L(e^t \delta_0)(1) = M(\delta_0)(0).
\]

In order to compute \(L(e^t \delta_0)(-1)\), we prepare the following lemma.

Lemma 4.2. Let us put

\[
L_0(z) = L(e^t \delta_1)(z - 1).
\]

Then it satisfies a functional equation:

\[
L_0(1 + z) = L_0(1 - z).
\]

Proof. Let \(F\) be their difference:

\[
F(z) = L_0(1 + z) - L_0(1 - z).
\]

Lemma 3.2 of [10] shows

\[
F'(z) = L_0'(1 + z) + L_0'(1 - z) = 0,
\]

and therefore \(F\) is a constant. But since

\[
\lim_{z \to +\infty} L_0(z) = \lim_{z \to -\infty} L_0(z) = 0
\]

we know \(F = 0\).
In particular we have
\[ L_0(2) = L_0(0), \]
which implies
\[ L(e^t \delta_0)(1) = L(e^t \delta_0)(-1). \]
Thus we have proved the equation:
\[ M(\delta_1)(0) - 2M(\delta_0)(0) = L(\delta_1)(0) - L(e^t \delta_0)(-1) - L(e^t \delta_0)(1). \] (6)

Using Proposition 3.1, Proposition 3.2 and Proposition 3.3, the Selberg trace formula informs us the RHS is equal to \( \log R_\rho(0) \). Thus we have obtained
\[ \log R_\rho(0) = M(\delta_1)(0) - 2M(\delta_0)(0). \] (7)

Now let us recall the definition of the analytic torsion \( T_X(\rho) \) of \( (X, \rho) \) due to Ray and Singer [8] (See also [2] and [6]):
\[ \log T_X(\rho) = \frac{1}{2} \frac{d}{ds} \left[ \frac{1}{\Gamma(s)} \sum_{j=0}^{3} (-1)^{j+1} j \cdot M(\text{Trace}[e^{-t\Delta_X} | L^2(X, \Omega^j(\rho))]) (s) \right]_{s=0} \]

As we have seen the Mellin transform of the heat kernel on \( L^2(X, \Omega^j(\rho)) \), \( j = 0, 1 \) is regular at the origin and (3) implies
\[ \log T_X(\rho) = \frac{1}{2}(2M(\delta_0)(0) - M(\delta_1)(0)). \]

Thus we have obtained the following theorem.

**Theorem 4.1.** Suppose \( h^1(\rho) \) vanishes. Then
\[ R_\rho(0) = T_X(\rho)^2. \]

5  The theorem of Cheeger and Müller

For a positive number \( A \) let \( X_A \) the image of
\[ \mathbb{H}^3_A = \{ (x, y, r) \in \mathbb{H}^3 \mid r \leq A \}, \]
under the natural projection
\[ \mathbb{H}^3 \xrightarrow{\pi} X. \]

Let \( Y_A \) be the complement of the interior of \( X_A \). We take \( A \) sufficiently large so that the boundary \( M_A \) of \( X_A \) is a flat torus and that \( Y_A \) is diffeomorphic to a product of \( M_A \) with an interval \([A, \infty)\).
We will review the analytic torsion of \((X_A, \rho)\) with respect to the absolute boundary condition. Let \(\Omega_X(\rho)\rvert_{M_A}\) be the restriction \(\Omega_X(\rho)\) to \(M_A\). Its section \(\omega\) can be written as

\[
\omega = \omega_t + dr \wedge \omega_n,
\]

where \(\omega_t\) and \(\omega_n\) are sections of \(\Omega_{M_A}(\rho)\). We put \(P_a(\omega) = \omega_n\), and \(P_r(\omega) = \omega_t\).

The space of smooth \(j\)-forms on \(X_A\) (resp. \(Y_A\)) twisted by \(\rho\) satisfying the absolute (resp. relative) boundary condition is defined to be

\[
C^\infty_{\text{abs}}(X_A, \Omega_j(\rho)) = \{\omega \in C^\infty(X_A, \Omega_j(\rho)) \mid P_a(\omega) = P_a(d\omega) = 0\},
\]

(resp.

\[
C^\infty_{\text{rel}}(Y_A, \Omega_j(\rho)) = \{\omega \in C^\infty(Y_A, \Omega_j(\rho)) \mid P_r(\omega) = P_r(\delta \omega) = 0\},
\]

where \(\delta\) is the formal adjoint of \(d\). It is known that elements of each space satisfy the self-adjoint boundary condition ([3] (2.8)):

\[
\omega, \omega' \in C^\infty_{\text{abs}}(X_A, \Omega_j(\rho)) \Rightarrow (\Delta \omega, \omega') = (\omega, \Delta \omega'), \tag{8}
\]

\[
\eta, \eta' \in C^\infty_{\text{rel}}(Y_A, \Omega_j(\rho)) \Rightarrow (\Delta \eta, \eta') = (\eta, \Delta \eta'), \tag{9}
\]

For \(\omega \in C^\infty_{\text{abs}}(X_A, \Omega_j(\rho))\) we define \(\tilde{\omega} \in L^2(X, \Omega_j(\rho))\) to be

\[
\tilde{\omega}(x) = \begin{cases} 
\omega(x) & \text{if } x \in X_A \\
0 & \text{if } x \notin X_A.
\end{cases}
\]

In this way we may consider \(C^\infty_{\text{abs}}(X_A, \Omega_j(\rho))\) as a subspace of \(L^2(X, \Omega_j(\rho))\) and let \(L^2_{\text{abs}}(X_A, \Omega_j(\rho))\) be its closure. By the same procedure, we have a closed subspace \(L^2_{\text{rel}}(Y_A, \Omega_j(\rho))\). (8) and (9) implies the positive Laplacian has a self-adjoint extension \(\Delta_{X_A}\) and \(\Delta_{Y_A}\) on \(L^2_{\text{abs}}(X_A, \Omega_j(\rho))\) and \(L^2_{\text{rel}}(Y_A, \Omega_j(\rho))\), respectively. Moreover there is an orthogonal decomposition:

\[
L^2(X, \Omega_j(\rho)) = L^2_{\text{abs}}(X_A, \Omega_j(\rho)) \oplus L^2_{\text{rel}}(Y_A, \Omega_j(\rho)),
\]

which makes \(\Delta\) into a block form

\[
\Delta = \begin{pmatrix}
\Delta_{X_A,j} & 0 \\
0 & \Delta_{Y_A,j}
\end{pmatrix}.
\]

For a positive \(t\), the heat operator \(e^{-t\Delta_{X_A,j}}\) is in the trace class and the integral

\[
M(\text{Trace}[e^{-t\Delta_{X_A,j}}])(s) = \int_0^\infty t^{s-1}\text{Trace}[e^{-t\Delta_{X_A,j}}]dt
\]

12
is absolutely convergent for $\text{Re} \ s \gg 0$. Moreover it is meromorphically continued on the whole plane.

Let us investigate its behavior at the origin. As we have seen in [10] §4, the nontriviality of $\rho|_{\Gamma_{\infty}}$ implies the cuspidality of any element of $L^2_{\text{rel}}(Y_A, \Omega^j(\rho))$. Then the proof of [7] Proposition 4.9 (especially the equation (4.12)) shows the infimum of the set of spectrum of $\Delta_{Y_A,j}$ has the following lower bound:

$$\inf \sigma(\Delta_{Y_A,j}) > cA,$$

where $c$ is a positive constant. Thus the dimension of the kernel of $\Delta_X$ on $L^2(X, \Omega^j(\rho))$ and $\Delta_{X_A,j}$ are same. By the assumption the previous has the trivial kernel, so does $\Delta_{X_A,j}$. This implies $M(\text{Tr} e^{-t\Delta_{X_A,j}})(s)$ is regular at $s = 0$. Now the analytic torsion $T_{X_A}(\rho)$ of $(X_A, \rho)$ (with respect to the absolute boundary condition) is defined to be

$$\log T_{X_A}(\rho) = \frac{1}{2} \frac{d}{ds} \left[ \frac{1}{\Gamma(s)} \sum_{j=0}^{3} (-1)^{j+1} j \cdot M(\text{Tr} e^{-t\Delta_{X_A,j}})(s) \right]_{s=0}$$

Let $\tau_{X_A}(\rho)$ be the Reidemeister torsion of $(X_A, \rho)$. If we apply Theorem 1.1 of [3], we obtain

$$T_{X_A}(\rho) = \tau_{X_A}(\rho). \quad (11)$$

Here we have used the following fact. First of all, one can directly check that the second fundamental form of $M_A$ is zero and therefore the term $\phi$ in the theorem vanishes. Next since the dimension of $X_A$ is three, the Chern-Simon class defined by Bisumut and Zhang ([1]) also vanishes. Finally the index theorem inform us the Euler characteristic $\chi(M_A, \rho)$ is zero.

For sufficiently large $A$ and $A'$, $X_A$ and $X_{A'}$ are isomorphic as PL-manifolds and the PL-invariance of the Reidemeister torsion implies

$$\tau_{X_A}(\rho) = \tau_{X_{A'}}(\rho).$$

Thus the following definition makes sense:

$$\tau_X(\rho) = \lim_{A \to \infty} \tau_{X_A}(\rho). \quad (12)$$

Let us compare the analytic torsion of $(X_A, \rho)$ and $(X, \rho)$

**Proposition 5.1.**

$$T_X(\rho) = \lim_{A \to \infty} T_{X_A}(\rho).$$
Proof. For $\Re s >> 0$, Müller’s result cited before implies

$$M(\text{Trace}[e^{-t\Delta} | L^2(X, \Omega^j(\rho))])(s) = \int_0^\infty t^{s-1}\text{Trace}[e^{-t\Delta} | L^2(X, \Omega^j(\rho))]dt$$

$$= \lim_{A \to \infty} \int_0^\infty t^{s-1}\text{Trace}[e^{-t\Delta X_{A,j}}]dt$$

$$= \lim_{A \to \infty} M(\text{Trace}[e^{-t\Delta X_{A,j}}])(s),$$

which yields the identity as meromorphic functions on the whole plane:

$$M(\text{Trace}[e^{-t\Delta} | L^2(X, \Omega^j(\rho))])(s) = \lim_{A \to \infty} M(\text{Trace}[e^{-t\Delta X_{A,j}}])(s).$$

Now the desired identity will follow from the definition of the analytic torsion.

□

Now Theorem 4.1, (11), (12) and Proposition 5.1 implies the following theorem.

Theorem 5.1. Suppose that $\rho|_{\Gamma_{\infty}}$ is nontrivial and that $h^1(\rho)$ vanishes. Then

$$R_\rho(0) = \tau_X(\rho)^2.$$ 

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