Fast Green Function Evaluation for Method of Moment

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Abstract—In this letter, an approach to accelerate the matrix filling in method of moment (MOM) is proposed. Based on the fact that the Green function is dependent on the Euclidean distance between the source and the observation points, we constructed an efficient adaptive one-dimensional interpolation approach to fast calculate the Green function and \( \text{Exp} \) function values. In the proposed method, several adaptive interpolation tables are constructed based on the maximum and minimum distance between any two integration points with local refinement near zero function values to obtain relatively uniform error rate in the overall computational domain. An efficient approach to obtain the sampling points used in the interpolation phase is carefully designed. Then, any function values could be efficiently calculated through a linear interpolation method for \( \text{Exp} \) and a Lagrange polynomial interpolation method for the Green function. In addition, the error bound of the proposed method is rigorously investigated. The proposed method could be quite easily integrated into the available MOM codes for the phase is carefully designed. Then, any function values could be efficiently calculated through a linear interpolation method for \( \text{Exp} \) and a Lagrange polynomial interpolation method for the Green function. In addition, the error bound of the proposed method is rigorously investigated. The proposed method could be quite easily integrated into the available MOM codes for different integration equation (IE) formulations, like the electrical field integral equation (EFIE), magnetic field integral equation (MFIE), the combined field integral equation (CFIE), the Poggio-Miller-Chan-Harrington-Wu-Tsai (PMCHWT), with few efforts. Comprehensive numerical experiments validate its accuracy and efficiency through several IE formulations. Results show that over 20% efficiency improvement could be achieved without sacrificing the accuracy.

Index Terms—Green function, interpolation, local refinement, method of moment

I. INTRODUCTION

The method of moment (MOM) is widely used to extract composite parameters of large-scale integrated circuits (ICs) [1], solve electrically large, multiscale scattering problems [2] due to its unknowns residing on the interface of different homogenous media. Therefore, the count of unknowns is much smaller than that of other differential equation based methods, like the finite-difference time-domain (FDTD) method [3], the finite element method (FEM) [4], which require volumetric discretization.

However, a dense and full impedance matrix in MOM is constructed. Investigations found that matrix filling in MOM is quite time-consuming and is one of the bottlenecks to prohibit MOM from solving practical problems, which is directly related to calculate the Green function, \( \text{Exp}(−jkr)/r, \) and \( \text{Exp}(−jkr) \). When constructing the impedance matrix, one could directly evaluate those analytical expressions. However, it’s reported that it is time-consuming to calculate \( \text{Exp} \) function, especially for complex numbers [5], which is the exact scenario in MOM for the time-harmonic electromagnetic simulations.

In this paper, we show that direct evaluation of those analytical formulations is not optimal since a large number of time-consuming and redundant evaluation of \( \text{Exp} \) function may exist. We proposed an efficient adaptive interpolation approach to fast evaluate \( \text{Exp}(−jkr)/r \) and \( \text{Exp}(−jkr) \). Through careful design of interpolation tables and an efficient approach to retrieve the sampling points, we could efficiently calculate \( \text{Exp}(−jkr)/r \) and \( \text{Exp}(−jkr) \) values with controllable accuracy and significant performance improvement.

This paper is organized as follows. In Section II, some key observations related to the integral identities used in MOM are presented and the proposed approach is illustrated in details. In Section III, its error bound is rigorously analyzed. In Section IV, its accuracy and efficiency are comprehensively investigated through numerous experiments. At last, we draw some conclusions in Section V.

II. METHOD

A. Observations of Integral Identities Used in MOM

To efficiently construct the impedance matrix in MOM, several types of integrand are involved. In this paper, singular cancellation technique along with angular transformation [6] is used to demonstrate the proposed method. However, it could be easily extended to other integration techniques without any difficulties. The following four integrals are required to be evaluated [6].

\[
I_a = \int_S \frac{e^{-jkr}}{R} d\mathbf{r}'
= \frac{1}{-ik} \sum_{i=1}^{3} \int_{x_i^-}^{x_i^+} \frac{h_i}{h_i^2 + x^2} (e^{-jkr} - e^{-jkd}) dx
\]

(1)

\[
I_b = \int_S \rho \frac{e^{-jkr}}{R} d\mathbf{r}'
= \frac{1}{-ik} \sum_{i=1}^{3} \hat{u}_i \int_{x_i^-}^{x_i^+} e^{-jkr} dx
\]

(2)

\[
I_c = \int_S \frac{e^{-jkr}(1 + jkr)}{R} d\mathbf{r}'
= F \sum_{i=1}^{3} \int_{x_i^-}^{x_i^+} \frac{h_i}{h_i^2 + x^2} \left( \frac{e^{-jkr} - e^{-jdk}}{d} \right) dx
\]

(3)
\[ I_d = - \int_S \frac{e^{-jKR}(1+jkR)}{R} \, dr' \]
\[ = F \sum_{i=1}^{3} \int_{x_i}^{x_i} \frac{e^{-jkr}}{R} \, dx \]

where \( k \) is the wavenumber, \( d \) is the normal distance from the evaluation point to the source triangle plane, \( h \) is the distance from the projection point of the evaluation point on the source plane to the integration line, \( x \) is the angular coordinate at the integral line.

To efficiently evaluate (1), we obviously need to fast calculate three exponential functions
\[ e^{-jkr}, \frac{e^{-jkr}}{r}, \frac{e^{-jkd}}{d}. \] (5)

It is intrinsic to evaluate (1) through (5) since they are analytical expressions and we could obtain the function values easily. However, as stated in (5), it is quite time-consuming for exponential function evaluation since numerous numerical operations are required to obtain convergence results. Unfortunately, even worse redundant calculations of those functions may exist when we fill the impedance matrix in MOM. To better understand this problem, we could roughly estimate the count of function calls to construct the impedance matrix as follows.

To obtain the quantitative analysis, we make the following assumptions. The structures are first discretized into \( N \) planar triangles. \( n \)-point numerical integration at each edge of the source triangle to evaluate the inner surface integration is used and three lines overall for each triangle are required. \( m \)-point numerical integration on average for the outer surface integration is considered. Therefore, the total count of exponential function calls approximately equal to \( 3mnN^2 \) to generate the impedance matrix. It is easy to see that for a large number \( N \) the count is extremely large because it depends on the square of \( N \). When acceleration techniques, like MLFMP [2], pFFT [1] and AIM [1], are used, the total count of function calls is approximately reduced to \( 3mnNC \), where \( C \) is the averaged count of triangles in the near field region and we only consider the near-field interaction calculation. In those scenarios, in which the electrically large or multiscale structures are often involved, \( N \) itself is an extremely large number. Therefore, we still make a large number of function calls to evaluate (5) which is inevitably time-consuming. Further carefully investigations could find that redundant function calls possibly exist in the direct evaluation method since (5) are functions of distance between the field point or its project point to the source point.

In the following subsection, an adaptive interpolation method along with an efficient approach to retrieve sampling points is proposed to fast calculate (5).

B. Efficient Evaluations of Exp Functions

To efficient evaluate values for the expotential type functions with complex numbers, in the proposed method we should first determine sampling intervals. As shown in Fig. 1 (a) and (b), real and imaginary parts of \( \text{Exp}(-jkr) \) and \( \text{Exp}(-jkr)/r \) show different oscillating behaviors, which implies that special attention must be paid to selecting appropriate sampling points. \( \text{Exp}(-jkr) \) is a periodic function without any singularity near the origin. Therefore, we could use a relatively uniform sampling interval and a linear interpolation method to obtain accurate interpolated values. For \( \text{Exp}(-jkr)/r \), we start to sample it at the minimum distance \( R_{\text{min}} \) near the origin since there is a singularity point where it shows large function variation. Large interpolation error could occur if the linear interpolation method is still used. In the proposed method, we select the Langrage polynomial method to interpolate \( \text{Exp}(-jkr)/r \). We first uniformly sample the function using the intervals defined as follows
\[ t = \frac{\lambda_0}{10^3 \sqrt{\varepsilon_r \mu_r}} \sim \frac{\lambda_0}{10^3 \varepsilon_r \mu_r} \] (6)
where \( \lambda_0, \varepsilon_r, \mu_r \) denote the wavelength in free space, relative permittivity and relative permeability of residing homogenous medium, respectively. As stated in (6), to obtain acceptable accuracy of function values, we use \( 10^3 \sim 10^4 \) sampling points per wavelength due to the error possibly accumulating in the integration phase when we evaluate double surface integrals to construct the impedance matrix. Since we only require to construct one-dimensional interpolation, the computational resource consumptions in terms of memory usage and CPU time could be ignored compared with those in the simulation even if we use such a dense sampling rate.

However, various numerical experiments reveal that quite large relative interpolation error will occur when the desired function values are small near the zero function values. We proposed the following strategy to keep the error uniformly distributed in the whole computational domain.

(a) Local refinement sampling is used near the zero function values. When sampling location is near zero function values, a local refinement to use smaller sampling interval is used.

(b) Zero function values is forced to be sampled. If we miss zero function values in (a), we directly sample zero function values when constructing the interpolation tables. One must pay special attention to avoiding that two sampling points are too close, which could keep the hash array introduced in the next subsection with reasonable dimension.

Therefore, the proposed method could efficiently utilize the memory and construct an optimal interpolation tables. It is a balance between the accuracy and computational cost.

With the definition of the sampling points, several tables are efficiently constructed through (5) without any redundant
calculations. Therefore, we could significantly improve the efficiency of matrix filling in MOM.

Various interpolation methods, like the linear interpolation method and the Lagrange interpolation method [9], are available to efficiently interpolate the continuous function values for \( \text{Exp}(-jkr) \) and \( \text{Exp}(-jkr)/r \). They are widely used in the multilayer Green function evaluations [7][10] since its spatial analytical expression is unavailable. We proposed to use the linear interpolation method to interpolate \( \text{Exp}(-jkr) \) and the Lagrange polynomial method for \( \text{Exp}(-jkr)/r \). Therefore, the interpolated function values could be expressed as

\[
 f(r) = \sum_{j=0}^{n} f(r_j) P_{n,j},
\]

where \( r \) is the desired point location, \( r_j \) is the sampling location, \( n \) is the number of sampling points in the local interpolation, \( P_{n,j} \) is the linear interpolation function or the Lagrange interpolation polynomial of degree \( n \).

### C. Efficient Searching Elements in the Sampling Table

The dimension of sampling tables is usually quite large for electrically large or multiscale structures. If we naively search elements one-by-one in the sampling tables required by the interpolation methods, the performance will be severely deteriorated and even worse than that of original analytical method. We proposed the following efficient approach to mitigate the problem.

The hash-table method is designed to fast obtain the desired elements in a large container [11]. We use the hash method to map the distance \( r \) to the index of sampling point in the sampling tables. The following many-to-one hash function is proposed

\[
 [R - R_{\text{min}}] \Delta R_{\text{min}}
\]

where \( R \) is the location of desired function values, \( R_{\text{min}} \) is the minimum sampling location, \( \Delta R_{\text{min}} \) is the minimum sampling interval, \([\cdot]\) denotes the integer part of the corresponding float number. It could be found that once the desired \( r \) is obtained we could fast locate the sampling point in the tables through (8). On the implementation aspect, one could simply construct a one dimensional array with the dimension of \( [R_{\text{max}} - R_{\text{min}}]/(\Delta R_{\text{min}}) \), where \( R_{\text{max}} \) is the maximum sampling location. We first fill the array with the values of sampling locations using (5) and then other positions are recursively filled by its previous elements starting at the lower bound of the hash array.

### III. Error Estimates

In this session, we will analyze the error bound for the proposed method with the linear and Lagrange polynomial interpolation method for \( \text{Exp}(-jkr) \) and \( \text{Exp}(-jkr)/r \).

#### A. Error Estimates for Linear Interpolation

Since \( \text{Exp}(-jkr) \) and \( \text{Exp}(-jkr)/r \) are infinite smooth, the \( n \)-th derivative of these two functions are always available. The error bound for the polynomial interpolations for complex number could be easily derived according to the Mean Value theorem similar in [9] as follows.

\[
 \max_{r \in [a,b]} |f(r) - f_n(r)| \leq \max_{r \in [a,b]} |\omega_{n+1}(r)| \frac{\max_{r \in [a,b]} |f^{n+1}(\xi_r)|}{(n+1)!},
\]

where

\[
 \omega_{n+1}(r) = \prod_{j=0}^{n} (r - r_j),
\]

\( f(r) \) is the complex function value, \( f^n(r) \) is the interpolated function value in \([a,b]\), \( f^{n+1} \) is the \( n \)-th derivative of \( f \).

For the uniform sampling points, denoted as

\[
 r_i = a + \frac{i}{n} (b - a), i = 0, 1, ..., n,
\]

we could arrive at the analytical error bound with some mathematical manipulations based on [9] as follows

\[
 \max_{r \in [a,b]} |f(r) - f_n(r)| \leq \left( \frac{b - a}{n} \right)^{n+1} \max_{r \in [a,b]} |f^{n+1}(\xi_r)| \frac{4(n+1)}{n(n+1)!}. 
\]

It is obvious that the error decreases as we increase the interpolation order or use smaller interpolation intervals. This is the reason why we locally refine the intervals near zero function values to obtain uniform relative error bound in the whole simulation domain. Therefore, to balance memory cost and accuracy, we use the uniform sampling far away from zero function values and non-uniform sampling near zero function values in the proposed method.

#### B. Error Estimates for Lagrange Interpolation

It is easy to derive the error bound for the Lagrange interpolation through the Generalized Rolle’s theorem [9] as follows

\[
 \max_{r \in [a,b]} |f(r) - f_n(r)| \leq \prod_{j=0}^{n} |r - r_j| \frac{|f^{n+1}(\xi_r)|}{(n+1)!}. 
\]

The general remarks for (13) are the same as that of (12). The Lagrange interpolation shows better performance compared with the linear interpolation for highly oscillating kernel. However, the Lagrange method is more time-consuming compared with the linear interpolation. As stated before, we proposed the linear interpolation for \( \text{Exp}(-jkr) \) and the Lagrange interpolation method for \( \text{Exp}(-jkr)/r \) to balance the accuracy and efficiency.

### IV. Numerical Experiments and Discussion

#### A. Accuracy of the proposed method

We first validate the accuracy of the proposed method with an interpolation problem. The configurations are set up as, \( R_{\text{max}} = 1 \text{ m} \), corresponding to one wavelength at 300 \text{ MHz}, \( R_{\text{min}} = 0.0001 \text{ m} \), the sampling interval far away from zero function value is \( t \), local refinement sampling interval is \( t/2 \), and local refinement range is \( 4t \). We interpolated \( 10^4 \) function values uniformly distributed in \([0.0001 1]\). The 2-point linear
method and Lagrange polynomial method of order 3 are used in our simulations. The maximum relative error is defined as

$$E_{\text{max}} = \max \left( \frac{|f(r) - f_i(r)|}{|f_i(r)|} \right).$$  \hspace{1cm} (14)

It is easy to see that in Fig. 2 (a) and (b), (c) and (d), the proposed local refinement method could approximately improve the accuracy up to two order for both the real and imaginary parts of \(\exp(-jkr)\) and \(\exp(-jkr)/r\). Compared Fig. 2 (a) and (b) with (c) and (d), the linear method shows similar performance in terms of the maximum relative error for \(\exp(-jkr)\) as that of Lagrange polynomial method. Since the linear method has higher efficiency than that of Lagrange, we use the linear method to interpolate \(\exp(-jkr)\) to achieve better efficiency in the proposed method.

As shown in Fig. 2, the Lagrange polynomial method shows significant better performance than the linear method for \(\exp(-jkr)/r\). Therefore, we use the Lagrange method for the Green function.

B. Performance of the Proposed Method Incorporated in MOM based on Various Formulations

To comprehensively test the performance of the proposed method incorporated in MOM, three scenarios, a sphere with the radii of 0.5 wavelength is considered for various integral formulations including the EFIE, the MFIE, the CFIE and the PMCHWT. For the EFIE, MFIE, CFIE, the sphere is perfectly electrical conductor (PEC) and homogenous dielectric structure with \(\varepsilon_r = 4\) for the PMCHWT. A \(x\)-polarized plane wave incidents along \(z\) axis. All numerical experiments are carried out on a workstation with an intel i7-7700 3.6 GHz CPU and 32 G memory and all in-house codes are running with a thread without exploring any the parallel computation to make fair comparisons.

In Fig. 3, we plot the relative error of all elements in the impedance matrix obtained from the EFIE involved \(L\) operator and the MFIE involved \(K\) operator \cite{12}. Other formulations in essence are various combination of the EFIE and MFIE, therefore, the performance of the proposed method in other formulations are similar. As shown in Fig. 3 (a) and (b), (c) and (d), the relative error of both the real and imaginary parts for the EFIE reaches \(10^{-5}\) and \(10^{-6}\). For the MFIE, the relative error is further decreased to \(10^{-6}\) and \(10^{-7}\). That implies the proposed method could provide an excellent level of elementwise accuracy. The \(L_2\) error of RCS for both EFIE and MFIE could reach \(10^{-6}\). Those results show excellent performance in terms of accuracy for the proposed method.

| Metric | EFIE | MFIE |
|--------|------|------|
| Analytical | 1914  | 1914  |
| Proposed  | 1914  | 1914  |

Table I shows the time cost for the EFIE and the MFIE incorporated with the analytical method and the proposed method. For the EFIE, matrix filling in the analytical method takes 85s. However, when incorporated with the proposed method, matrix filling only takes 61s with \(L_2\) error of RCS \(10^{-6}\). It shows 27% and 24 % efficiency improvement for the EFIE and the MFIE formulations, respectively, without sacrificing accuracy.

We perform a number of numerical experiments with the EFIE, MFIE, CFIE and PMCHWT formulations for different
structures. Due to limited page space, we only showed part of results. However, the proposed method shows quite good performance of both the accuracy and efficiency, which imply that the proposed method is general, robust and efficient to accelerate matrix filling in MOM.

V. CONCLUSIONS

In this paper, we proposed an efficient interpolation approach to accelerate matrix filling in MOM. In the proposed method, several interpolation tables are constructed with carefully consideration of local refinement near the zero function values. The linear method, Lagrange polynomial are used to interpolate $\exp(-jkr)$ and $\exp(-jkr)/r$, respectively. An efficient approach to obtain sampling points are also presented. Compared with the analytical method, it shows over 20% efficiency improvement without sacrificing accuracy. In all, the proposed method is simple, robust and efficient to accelerate matrix filling for MOM and applicable for almost all SIE formulations and it is quite easy to integrate the available MOM codes with only a few modifications.

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