Switching of quantum synchronization in coupled optomechanical oscillators

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Abstract

We explore the phenomenon of quantum phase synchronization in two optomechanical oscillators, coupled either bidirectionally or unidirectionally to each other. We first show that irrespective of the configuration of the optomechanical oscillators, synchronization can be achieved, with a finite degree of quantum correlation. However, while looking at the variation of the synchronization against the frequency detuning of the two oscillators, we observe a profound effect of the directionality of the optical coupling. For instance, we find that when the two optomechanical cavities exchange photons bidirectionally, synchronization traces the classic Arnold tongue. Whereas, for the unidirectional configuration, synchronization exhibits a novel blockade-like behavior where finite detuning favors synchronization. We also observe a strong connection between synchronization blockade and synchronization phase transition.

1. Introduction

Recent experimental demonstration of quantum phase synchronization in spin-1 atoms [1] has enthused tremendous interest in exploration of quantum synchronization in various other quatum systems [2–6]. In particular, hybrid quantum optomechanical and electromechanical systems are getting lots of attention owing to its significance in quantum technologies [7–9].

The phenomenon of synchronization describes the ability of a group of self-oscillators to spontaneously adjust their intrinsic rhythms to oscillate in unison [10]. Its first scientific observation could be traced back to the early 17th century, when Huygens studied the synchronous motion of two maritime pendulum clocks [11]. Since then, synchronization has been observed in a wide variety of physical, biological, chemical and social systems [12–17]. Spontaneous synchronization is also of great technological importance, as it finds applications in high-precision clocks [18] and information processing [19].

While classical nonlinear dynamical systems stand as an excellent paradigm for synchronization, there has been attempts to observe analogous phenomena in their quantum counterparts. van der Pol (vdP) oscillators [20–22], Kerr-anharmonic oscillators [23], atomic ensembles [24], ions [25] and spin systems [1, 26–28] are a few quantum models where synchronization has been observed and thoroughly analyzed. A cavity optomechanical system [29, 30] is another suitable platform for the study of synchronization in micro- and nano-mechanical oscillators. A key advantage of such a system is the ability to couple high frequency mechanical oscillators to one or more electromagnetic fields inside a resonant cavity. As this coupling is inherently nonlinear, the resultant classical dynamics can undergo limit-cycle oscillations, often referred to as optomechanical self-oscillations. Theoretical studies on optomechanical synchronization initially focused only on the classical realm of such self-oscillators, where the Kuramoto-type model is considered to be the most effective one [31, 32]. While on the experimental side, synchronization has been demonstrated using optically coupled micro-disk resonators [33] and optical racetrack cavities with integrated mechanical oscillators [34]. Recently, synchronization has been theoretically studied in networks of such cavities [35, 36] and experimentally achieved using seven micro-disk oscillators sharing a common optical field [37]. Apart from these classically
inspired investigations, there has always been a growing interest to explore synchronization deep in the quantum regime. A remarkable proposal for producing stronger degrees of quantum synchronization by invoking a squeezed drive in a quantum vdP oscillator has been reported [38]. Also, taking thermal noise into account, a quantum noise-driven bistable regime in optomechanical synchronization has been predicted [39]. Particular to optomechanical arrays, it has already been pointed out that under weak intercellular interaction, quantum noise can give rise to unsynchronous motion even for identical mechanical oscillators [40]. Instability and chaotic dynamics of synchronization have also been studied in coupled optomechanical systems [41]. Notably, a fascinating outcome of quantum synchronization is the observation of synchronization blockade, where identical self-oscillators are restricted from attaining maximum synchronization. This phenomenon was first observed in Kerr-anharmonic oscillators [42] and later has been realized in circuit QED [43], spin-1/2 [28] systems and recently, in optomechanical systems [44].

However, measuring quantum synchronization is still a challenging task [45]. Also, there has been numerous efforts over the past decade to connect the onset of quantum synchronization and the generation of quantum correlations [22, 25, 26, 46–48]. A connection between classical synchronization and persistent entanglement in isolated quantum system has been demonstrated [49]. Moreover, mutual information has been suggested as a purely information-theoretic measure of quantum synchronization [50]. Besides, a quantum synchronization measure relying on the distance to limit-cycle dynamics has also been proposed recently [51].

In this paper, we investigate the phenomenon of quantum synchronization in optically coupled optomechanical oscillators. This specific choice of exploiting the optical coupling as opposed to the mechanical one is primarily motivated by the recent theoretical investigation [40, 52, 53], as well as the experimental demonstrations [33, 54] of synchronization in coupled optomechanical oscillators. Our work throws light on the behaviour of quantum phase synchronization for variation in coupling paradigms and mechanical detunings and highlight the ability to perform controlled switching between quantum phase-locked conditions of coupled oscillators.

We begin by discussing the physical models, followed by the derivation of the quantum Langevin equations (QLEs) describing their full dissipative dynamics in section 2. Section 3 outlines the measures of quantum synchronization and correlations. In section 4, we present the results with a detailed discussion and finally summarize our work in section 5.

2. Models and dynamics

Let us first consider two optomechanical oscillators (j = L(left), R(right)), each of which contains one optical mode of frequency $\omega_{lj}$ and one mechanical mode of frequency $\omega_{mj}$. The generic Hamiltonian [30] ($\hbar = 1$) that describes these individual oscillators is given by $H_j = \omega_{lj}a_j^{\dagger}a_j + \omega_{mj}b_j^{\dagger}b_j - g_j a_j^{\dagger} a_j (b_j^{\dagger} + b_j)$, where $a_j, b_j$ are the annihilation operators of the optical and mechanical modes, respectively, and $g_j$ is the usual optomechanical coupling rate.

Now, as depicted in figure 1, we take into account two distinct topologies in which these two optomechanical oscillators can interact. To begin with, we consider the bidirectional configuration (figure 1(a)) where both these optomechanical cavities are allowed to exchange photons in a reversible manner. Such an interaction can be mimicked by a Hamiltonian of the form, $H_{coup} = -\lambda (a_L^{\dagger} a_R + a_R^{\dagger} a_L)$, with $\lambda$ being any arbitrary coupling strength. Also, to drive the optical cavities, two separate laser sources are used, each characterized by an amplitude $A_i$ and frequency $\omega_i$. Taking the dissipative effects into account, and in a frame rotating at $\omega_i$ the QLEs

Figure 1. Schematic diagrams of (a) bidirectional and (b) unidirectional topologies. Each optomechanical oscillator comprises of an optical mode and a mechanical mode and couples optically to one another.
can be written as
\[ a_L = \left[i \Delta_L^0 + ig_L \left(b_L^\dagger + b_L\right) - \kappa_L \right] a_L + \lambda a_R + \frac{\sqrt{2} \kappa_L}{\sqrt{\kappa_L}} a_L^m, \]  
\[ a_R = \left[i \Delta_R^0 + ig_R \left(b_R^\dagger + b_R\right) - \kappa_R \right] a_R + \lambda a_L + \frac{\sqrt{2} \kappa_R}{\sqrt{\kappa_R}} a_R^m, \]  
\[ b_j = -i(\omega_{mj} + \gamma_j) b_j + ig_j a_j^\dagger a_j + \frac{\sqrt{2} \gamma_j}{\sqrt{\gamma_j}} b_j^m. \]  

Here, \( \Delta_j^0 = \omega_j - \omega_{mj} \) are the input optical detunings, \( \kappa_j(\gamma_j) \) are the optical (mechanical) mode damping rates and \( a_j^m(b_j^m) \) are the optical (mechanical) bath operators. These operators are considered to be zero-mean Gaussian fields, satisfying the standard correlation functions [55],
\[ \langle a_j^m(t) a_j^m(t') \rangle = \delta_j^0 \delta_j^0 \delta(t - t') \]  
and
\[ \langle b_j^m(t) b_j^m(t') \rangle = (2\eta_{mj} + 1) \delta_j^0 \delta_j^0 \delta(t - t') \].

We now proceed to the second configuration, illustrated in figure 1 (b), where the two optomechanical oscillators are arranged in a forward-fed manner. This configuration is quite different from the one described above, since photons are allowed to leave the left cavity and enter the right one but not the converse. Such cascaded geometry can be modelled using [56–58] the cavity configuration is quite different from the one described above, since photons are allowed to leave the left cavity and enter the right one but not the converse. Such cascaded geometry can be modelled using [56–58] and the equation of motion pertaining to the CM then reads as,
\[ \dot{V}(t) = A(t) V(t) + V(t) A(t)^\dagger(t) + D, \]  
where \( D \) is the matrix of noise correlations. We now proceed to the measures of quantum synchronization and correlation.

### 3. Quantum synchronization and correlation measures

Considering the CV description of the mechanical oscillators, we employ Mari’s criteria [45] for quantum phase synchronization. This measure essentially relies on the phase shift operator of the two oscillators, defined with respect to a frame rotating at the phases of the individual classical trajectories. A figure of merit can be written as
\[ S_p(t) = \left( \langle \delta p'_\phi(t) - \delta p'_R(t) \rangle \right)^{-1}, \]

where \( \delta p'_\phi(t) = \delta p \cos \phi - \delta q \sin \phi \) are the phase fluctuation operators, with the phases defined as \( \phi(t) = \arg(\langle b(t) \rangle) \).

For a direct estimation of equation (5), we extract the CM pertaining to the two mechanical oscillators. Such a bipartite system is commonly expressed as

\[ V^{(2)} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}. \]

where \( A, B \) and \( C \) are 2 \times 2 block matrices. The rotation can be accomplished by performing an unitary transformation \( V^{(2)} = U(t) V^{(2)} U(t)^\dagger \), where \( U(t) = ((R\phi_1(t), \ X), (Z, R\phi_2(t))) \), \( Z \) being the two-dimensional null matrix and

\[ R\phi(t) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}, \]

is the two-dimensional rotation matrix. Then, in terms of the CM elements, equation (5) can be expressed as \( S_p(t) = (V^{(2)}_{22}(t) + V^{(2)}_{44}(t) - 2 \times V^{(2)}_{42}(t))^{-1} \).

To measure the degree of quantum correlation between the two oscillators, we take Gaussian quantum discord into account [61]. Following the standard expression of \( V^{(2)} \), we extract the following four symplectic invariants \( I_1 = \det[A], I_2 = \det[B], I_3 = \det[C] \) and \( I_4 = \det[V^{(2)}] \), and compute the two symplectic eigenvalues \( \nu = 2^{-1/2}(\Sigma(V) + \sqrt{\Sigma(V)^2 - 4I_4^2})^{1/2} \) with \( \Sigma(V) = I_1 + I_2 + 2I_4 \). Gaussian quantum discord could then be evaluated as

\[ D_{GQD} = f(\sqrt{W}) - f(\nu^+ - \nu^-) + f(\sqrt{W}) \]

where, the function \( f \) is defined as \( f(x) \equiv (x + \frac{1}{2}) \ln(x + \frac{1}{2}) - (x - \frac{1}{2}) \ln(x - \frac{1}{2}) \) and,

\[ W = \begin{cases} \frac{2I_4 + \sqrt{4I_4^2 + (4I_4 - 1)(4I_4 - I_2)}}{(4I_4 - 1)} & \text{if } \frac{4(I_4 - I_2)^2}{(I_4 + 4I_4)(1 + 4I_4)I_2^2} \leq 1, \\ I_2I_4 + I_2 - \sqrt{(I_2I_4 + I_4 - I_2)^2 - 4I_4I_2} & \text{otherwise.} \end{cases} \]

4. Results and discussions

In figure 2, we present the simulation of quantum phase synchronization \( (S_p) \) between the two mechanical oscillators. In both configurations, it is observed that after an initial transient, the system settles down to a periodic steady-state with a significant degree of synchronization. However, the length of this transient varies between the two configurations, with the unidirectional configuration taking longer times to reach synchronization. To understand this, let us rewrite the rotated phase quadratures defined in equation (5), as \( \delta p'_\phi(t) = i(\delta b\ e^{i\phi(t)} - \delta b\ e^{-i\phi(t)})/\sqrt{2}, \) where \( \delta b \) are the linearized quantum mechanical operators and \( \phi \) are the classical phases of the limit-cycle dynamics (figure 3). It can be seen here that, (i) for \( S_p \) to have a physical significance, the mechanical mode expectation values \( (\langle b(t) \rangle = \bar{b} + r(t)e^{i\phi(t)} \) must attain limit-cycles \( (r(t) > 0) \) in the classical dynamics, and (ii) since \( S_p \) is defined in the frame of the limit-cycle dynamics, it must contain signatures.
of the classical nonlinear dynamics together with the quantum signatures. Moreover, classical nonlinearity enters the linearized QLEs via the effective optomechanical coupling strengths and the effective detunings (refer A for explicit relations). These nonlinear effects, together with the asymmetry associated with the intrinsic unidirectionality of the second configuration, delays the sympathetic synchronization between its mechanical oscillators. We further find that the achieved quantum phase-synchronous motion is strongly associated with a finite degree of Gaussian quantum discord. This hints at a relation between quantum synchronization and quantum correlations. However, it is worth mentioning that even though the systems are quantum mechanically correlated \( (D_G > 0) \), we did not find any regime of finite entanglement, characterized by the non-zero logarithmic negativity \( \rho \), in the chosen parameter regime.

We next proceed to study the dependence of synchronization against the variation of frequency detuning between the two mechanical oscillators \( \delta = (\omega_{mR} - \omega_{mL}) \). Figure 4 depicts the time-averaged asymptotic values of quantum phase synchronization \( \langle S_p \rangle \) as functions of frequency detuning and coupling parameters, respectively for the bi- and uni-directional configurations. We observe that in the bidirectional coupling, synchronization traces the classic 'Arnold tongue', yielding a range of frequency detunings \( \delta \) and their corresponding coupling strengths \( \lambda \) for which synchronization is achievable. Not surprisingly, here one finds the maximum degree of synchronization always around the resonance condition \( \delta = 0 \), i.e., for \( \omega_{mL} = \omega_{mR} \).

However, exercising the same on the unidirectional configuration reveals a very intriguing outcome. Here, we find that the degree of quantum synchronization \( \langle S_p \rangle \) is suppressed at the resonance while reaching its maximum at a finite frequency detuning. Such phenomenon of 'synchronization blockade' was first observed in Kerr-anharmonic oscillators [42] where detuning favors synchronization. Although such an intrinsic
nonlinearity is absent in our system, we try to explain in the following why one can still expect synchronization blockade particularly to the unidirectional configuration. To this end, let us consider the asymptotic mechanical field amplitudes given as $\langle b_R \rangle = g_1 |\langle a_L \rangle|^2/(i \omega_m + \gamma)$. It is apparent that these amplitudes increase with increasing intracavity field amplitudes ($|\langle a_L \rangle|$) and decrease for higher mechanical frequency ($\omega_m$). For simplicity, let us now consider the scenario where the mechanical oscillators are identical, i.e., $\omega_mL = \omega_mR$ (and $\gamma_mL = \gamma_mR$). Then, for bidirectional exchange of cavity photons, the two oscillators are likely to acquire proportionate amplitudes and therefore tend to synchronize naturally. Whereas for unidirectional transfer of cavity photons, the amplitude of the right cavity field always surpasses the left one, resulting in higher $|\langle b_R \rangle|$ values. Therefore, in this case, detuned oscillators, (with $\omega_mR > \omega_mL$), are the ones that acquire proportionate field amplitudes and are more suitable for synchronization than identical ones. To verify this analytically, we calculate the phonon number difference for either case in the linearized quantum regime (refer analytical expressions in B). We find that the phonon numbers acquire equal magnitudes for identical oscillators in the bidirectional configuration, whereas, for the unidirectional configuration, the difference does not vanish at the resonance condition. Rather, corresponding non-zero detunings that give equivalent number of phonons largely depend on $\eta$.

To get further insights into the behaviour of synchronization, we first take the Pearson correlation factor into account. Considering the two dimensionless momentum fluctuation operators $\delta p_L(t)$ and $\delta p_R(t)$, the Pearson correlation measure reads as $C = \langle \delta p_L(t) \delta p_R(t) \rangle / \sqrt{\langle \delta p_L^2 \rangle \langle \delta p_R^2 \rangle}$. While $C \sim 1 (-1)$ characterizes an in-phase (anti-phase) evolution, $C \sim 0$ refers to the absence of any correlation. Figure 5 (left panel) depicts the variation of $C$ with respect to the frequency detuning $\delta$, for a fixed transmission loss parameter $\eta = 0.75$. It can be seen that in the region where the synchronization gets blocked, the two oscillators undergo successive phase transitions, from in-phase to anti-phase and anti-phase to in-phase oscillations, respectively. This is followed by a sharp rise of synchronization at $\delta = 0.039 \omega_m$, where the two oscillators again switches from a perfectly ($C \sim 1$) synchronized state to a moderately in-phase oscillating state. To account for the stability of the synchronized (and the anti-synchronized) states, we estimate the transverse Lyapunov exponents (TLEs) [32] for the two difference modes of the mechanical oscillators (right panel of figure 4). We find that the largest nontrivial TLE remains negative till $\delta = 0.039 \omega_m$. Post $\delta = 0.039 \omega_m$, although in-phase synchronization continues, the TLE first attains positive values and then quickly switches to the negative ones, marking a transition from unstable to stable synchronization. We would further like to add here that the signature of synchronization blockade and synchronization phase transition could also be observed for higher number of thermal phonons. This makes our proposed scheme more feasible to observe the same experimentally.

5. Conclusion

In conclusion, we have systematically investigated the phenomenon of quantum phase synchronization in two coupled optomechanical oscillators, with bidirectional and unidirectional topology. We found that irrespective of the coupling topologies, when synchronization builds up, the two oscillators also become quantum mechanically correlated. In our study of the dependence of synchronization against the variation of the frequency detuning, we observed two distinctive outcomes. Firstly, for the bidirectional configuration, the synchronization is always found to be maximum around the resonance condition, yielding a classic tongue-like pattern. However, in the unidirectional one, synchronization does not attain maximum values for identical oscillators, rather it peaks around a finite detuning of the mechanical oscillator frequencies. Secondly, we found that for the unidirectional configuration, the two oscillators undergo synchronization transitions, from in-phase to anti-phase as well as from stable to unstable synchronizations. Our work throws light on the ability to switch...
between degrees of synchronization by either changing the directionality of coupling or the amount of mechanical detuning. Moreover, reduction of phase-noise in highly synchronized systems can be a great resource in time-keeping applications. It can also act as a characteristic measure of the effect of fluctuation noises and coherent laser amplitudes on the stability of limit-cycle oscillators, which has applications in quantum communication and in building multi-oscillator quantum networks. Further, synchronization is of great importance for the development of quantum memories required for quantum computation. Overall, our work sheds light on the influence of different coupling topologies on quantum synchronization.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Linearized quantum dynamics

In this section, we provide the detailed expressions for the drift and noise-correlation matrices. For the bidirectional configuration, the drift matrix \( A \) can be derived as

\[
A = \begin{pmatrix}
-\kappa_L -\Delta_L -2G_{Lm} & 0 & 0 & -\lambda & 0 & 0 \\
\Delta_L -\kappa_L & 0 & -2G_{Lm} & \lambda & 0 & 0 \\
0 & 0 & -\gamma_L & \omega_{mL} & 0 & 0 & 0 \\
2G_{Lm} & 2G_{Lm} & -\omega_{mL} & -\gamma_L & 0 & 0 & 0 \\
0 & -\lambda & 0 & 0 & -\kappa_R -\Delta_R -2G_{Rm} & 0 & 0 \\
\lambda & 0 & 0 & 0 & \Delta_R -\kappa_R & 2G_{Rm} & 0 \\
0 & 0 & 0 & 0 & 0 & -\gamma_R & \omega_{mR} \\
0 & 0 & 0 & 0 & 0 & 2G_{Rm} & 2G_{Rm} & -\omega_{mR} & -\gamma_R \\
\end{pmatrix},
\]

where we have used \( g_i = g_i \langle \alpha_i \rangle = G_{Lm} + i G_{Rm} \) as the effective optomechanical coupling rate. The noise correlation matrix becomes

\[
D = \text{Diag}[\kappa_L, \kappa_L, \gamma_{Lm}, \gamma_{Lm}, \kappa_R, \gamma_{Rm}, \gamma_{Rm}, \gamma_{Rm}],
\]

where \( \gamma_{Lm} = \gamma (2n_{th} + 1) \).

A similar transformation for unidirectional configuration gives us

\[
A = \begin{pmatrix}
-\kappa_L -\Delta_L -2G_{Lm} & 0 & 0 & 0 & 0 & 0 \\
\Delta_L -\kappa_L & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\gamma_L & \omega_{mL} & 0 & 0 & 0 \\
2G_{Lm} & 2G_{Lm} & -\omega_{mL} & -\gamma_L & 0 & 0 & 0 \\
-2\mu & 0 & 0 & 0 & -\kappa_R -\Delta_R -2G_{Rm} & 0 & 0 \\
0 & -2\mu & 0 & 0 & \Delta_R -\kappa_R & 2G_{Rm} & 0 \\
0 & 0 & 0 & 0 & 0 & -\gamma_R & \omega_{mR} \\
0 & 0 & 0 & 0 & 0 & 2G_{Rm} & 2G_{Rm} & -\omega_{mR} & -\gamma_R \\
\end{pmatrix},
\]

as the drift matrix with \( \mu = \sqrt{\kappa_L \kappa_R} \) and the noise correlation matrix as

\[
D = \begin{pmatrix}
\kappa_L & 0 & 0 & 0 & \mu & 0 & 0 \\
0 & \kappa_L & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & \gamma_{Lm} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma_{Lm} & 0 & 0 & 0 \\
\mu & 0 & 0 & 0 & \kappa_R & 0 & 0 \\
0 & \mu & 0 & 0 & 0 & \kappa_R & 0 \\
0 & 0 & 0 & 0 & \gamma_{Rm} & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_{Rm} \\
\end{pmatrix},
\]
Appendix B. Effective dynamics of the mechanical oscillators

In order to have a clearer understanding on the dynamics of the mechanical oscillators, we resort to the technique of adiabatic elimination of the cavity fields. To do so, we first switch to the interaction picture with respect to the left mechanical oscillator, in the bidirectional configuration the system dynamics is then given by

\[ \dot{a}_L = -\kappa_L \dot{a}_L + iG_L \dot{b}_L + i\lambda \dot{a}_L + \sqrt{2\kappa_L} \hat{a}_\text{in}^L, \]

\[ \dot{a}_R = -\kappa_R \dot{a}_R + iG_R \dot{b}_R + i\lambda \dot{a}_L + \sqrt{2\kappa_R} \hat{a}_\text{in}^R, \]

\[ \dot{b}_L = -\gamma_L \dot{b}_L + iG_L \hat{a}_L^\dagger + \sqrt{2\gamma} \hat{b}_\text{in}^L, \]

\[ \dot{b}_R = -\gamma_R \dot{b}_R - i\eta \dot{b}_L + iG_R \hat{a}_L^\dagger + \sqrt{2\gamma} \hat{b}_\text{in}^R, \]

where we assume \( \chi = \chi_\text{L} = \chi_\text{R} = \gamma = \gamma_L = \gamma_R \). Now, eliminating the cavity modes and considering nearly identical cavities with \( \kappa_L = \kappa_R = \kappa \), \( \gamma_L = \gamma_R = \gamma \), \( \lambda = \lambda_L = \lambda_R \), we obtain the rate equations for the mechanical modes as,

\[ \dot{b}_L = (\Gamma_L - \gamma) \dot{b}_L - i\eta \chi \hat{a}_L^\dagger + \frac{\chi_\text{R}}{\kappa} \hat{a}_L^\dagger + \sqrt{2\gamma} \hat{b}_\text{in}^L, \]

\[ \dot{b}_R = (\Gamma_R - \gamma) \dot{b}_R - i\eta \chi \hat{a}_L^\dagger + \frac{\chi_\text{R}}{\kappa} \hat{a}_\text{in}^R + \sqrt{2\gamma} \hat{b}_\text{in}^R, \]

where \( \Gamma_i = \kappa |g_i| (a_i^\dagger)^2/(\lambda^2 + \kappa^2) \) and \( \chi = \lambda g_L g_R (a_i^\dagger)^2 / (\lambda^2 + \kappa^2) \). From these, one can estimate the rate of change of the difference in phonon numbers (assuming \( \Gamma_i \gg \gamma \)), as

\[ \frac{d}{dt} \langle \hat{b}_L^\dagger \hat{b}_L \rangle = -4 \text{Im} \left[ \chi \hat{a}_L^\dagger \hat{a}_L^\dagger \right] + 2 \Gamma_L \langle \hat{b}_L^\dagger \hat{b}_R \rangle - 2 \Gamma_L \langle \hat{b}_L^\dagger \hat{b}_L \rangle. \]

To obtain the exact steady-state expressions, we rewrite the rate equations in terms of the rotated quadrature components satisfying \( \hat{b}_L = (\delta \hat{b}_L^\dagger + i\hat{b}_R^\dagger) / \sqrt{2} \), such that their correlation matrix can be written in the form of equation (4), with the new drift matrix,

\[ \tilde{A} = \begin{pmatrix} \Gamma_L - \gamma & 0 & -\chi_l & \chi_R \\ 0 & \Gamma_R - \gamma & -\chi_R & -\chi_l \\ \chi_l & \chi_R & \Gamma_R - \gamma & \delta \\ -\chi_R & \chi_l & -\delta & \Gamma_R - \gamma \end{pmatrix}, \]

where \( \chi_l \) and \( \chi_R \) are the real and imaginary parts of \( \chi \), and the new noise matrix, \( \tilde{D} = \text{Diag} \{ d_L, d_R, d_L^* \} \), where \( d_L = |a_\text{L}|^2 (1 + \lambda^2/\kappa^2) / 2 + \gamma (2n_\text{L} + 1) \) and \( d_R = |a_\text{R}|^2 (1 + \lambda^2/\kappa^2) / 2 + \gamma (2n_\text{R} + 1) \). A steady-state solution of the reduced correlation matrix leads us to write the phonon number difference as

\[ n_{\text{diff}} \approx \frac{\gamma \kappa (n_\text{L} + 1) (\Gamma_R - \Gamma_L) (1 + \zeta^2)}{\Gamma_L \Gamma_R (\lambda^2 + \kappa^2)(1 + \zeta^2)}, \]

where we assume \( \Gamma_i \gg \gamma \), with \( \zeta = \delta / (\Gamma_R + \Gamma_L) \). It apparently shows that the phonon number difference vanishes in the steady-state only when \( \Gamma_L = \Gamma_R (\langle \hat{a}_L^\dagger \hat{a}_L \rangle = \langle \hat{a}_R^\dagger \hat{a}_R \rangle) \).

Following the exact same procedure for the unidirectional configuration, one obtains the rate equation pertaining to the left and right mechanical modes as

\[ \dot{b}_L = (\Gamma_L - \gamma) \dot{b}_L + i\eta \chi \hat{a}_L^\dagger + \sqrt{2\gamma} \hat{b}_\text{in}^L, \]

\[ \dot{b}_R = (\Gamma_R - \gamma) \dot{b}_R - i\eta \hat{b}_L + \chi \hat{a}_L^\dagger + \sqrt{2\gamma} \hat{b}_\text{in}^R, \]

where \( \Gamma_i = |g_i| (a_i^\dagger)^2 / \kappa \), \( \chi = 2 \sqrt{\eta} g_L g_R (a_i^\dagger)^2 / \kappa \) and \( \eta = \sqrt{2/\kappa} g_L \langle a_i \rangle \). In this case, we obtain the corresponding rate of change in the phonon number difference as,

\[ \frac{d}{dt} \langle \hat{b}_L^\dagger \hat{b}_L \rangle = -2 \text{Re} \left[ \chi \hat{a}_L^\dagger \hat{a}_L^\dagger \right] + 2 \Gamma_L \langle \hat{b}_L^\dagger \hat{b}_R \rangle - 2 \Gamma_L \langle \hat{b}_L^\dagger \hat{b}_L \rangle, \]

and the steady-state phonon number difference (assuming \( \Gamma_i \gg \gamma \)) now read as

\[ n_{\text{diff}} \approx \frac{\gamma (n_\text{L} + 1) (\Gamma_R - \Gamma_L)}{\Gamma_L \Gamma_R (1 + \zeta^2)^2}, \]

Here, it can easily be inferred that even at zero detuning (\( \zeta = 0 \)), a finite value of \( \eta \) gives rise to constant phonon number difference.
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