HIGHER-CATEGORICAL COMBINATORICS OF CONFIGURATION SPACES OF EUCLIDEAN SPACE

ANNA CEPEK

Abstract. We approach manifold topology by examining configurations of finite subsets of manifolds within the homotopy-theoretic context of $\infty$-categories by way of stratified spaces. Through these higher categorical means, we identify the homotopy types of such configuration spaces in the case of Euclidean space in terms of the category $\Theta_n$.

1. Introduction

Configuration spaces (of finite subsets of topological spaces) play fruitful roles across algebraic topology, geometric and differential topology, and applied topology. See [22] for a lovely exposition. For example, the embedding calculus of Goodwillie-Weiss uses them as fundamental building blocks to study differential topology [34, 17]. More basically, the homotopy type of a fixed configuration

2010 Mathematics Subject Classification. Primary 55R80. Secondary 57N80, 18B30, 55P60.

Key words and phrases. Configuration spaces. Ran space. Stratified spaces. Exit-path categories. $\mathcal{E}_n$-algebras. $\infty$-categories. Localizations. The category $\Theta_n$. Wreath product.

This work was supported by NSF award 1507704, IBS-R003-D1 and NSF-RTG grant DMS-2039316.
space often detects subtle topological features of the background manifold. For example, in [23], Salvatore-Longoni identify two homotopy equivalent Lens spaces of distinct homeomorphism type whose configuration spaces of just two points are not homotopy equivalent.

We are motivated by Ayala-Francis’s factorization homology which uses configuration spaces to define manifold invariants within the homotopy-theoretic framework of $\infty$-categories [2]. More specifically, through the theory of stratified spaces and exit-path $\infty$-categories thereof [24, 6, 4], we codify the homotopy type of configuration spaces of finite subsets of manifolds. Our main result identifies such configuration spaces of Euclidean space $\mathbb{R}^n$ combinatorially in terms of the category $\Theta_n$. Joyal introduced the ‘theta-categories’ $\Theta_n$ to present the first model of $(\infty,n)$-categories [20]. This was a direct generalization of his theory of quasi-categories [19] as $\Theta_1$ is defined to be the simplex category $\Delta$. We will use Berger’s definition of $\Theta_n$ as the n-fold wreath product of $\Delta$ with itself [9].

The relationship between configuration spaces of $\mathbb{R}^n$ and the category $\Theta_n$ is a natural one, as Ayala-Hepworth demonstrate in [7]. Using only basic techniques in homotopy theory, they show that the ordered configuration space of a fixed number of points in $\mathbb{R}^n$ is homotopy equivalent to the classifying space of a certain subcategory of $\Theta_n$ (endowed with some extra structure). At work here is that the Fox-Neuwirth cells of the configuration space of $\mathbb{R}^n$ [15] are contractible [16] and form a partially ordered set.

The main result of this paper is both a generalization and an enhancement of [7]. Our main tool is an $\infty$-category which codifies the homotopy type of all the unordered configurations (as cardinality varies) of a fixed manifold (nonempty, connected, smooth) at once, including the empty configuration. It also codifies the maps between the configuration spaces of different cardinalities by witnessing anticollision and vanishing of points on the manifold. In this framework the relationship between the configuration space of $\mathbb{R}^n$ and the category $\Theta_n$ is most naturally phrased as a localization of $\infty$-categories. Heuristically this means that upon formally inverting certain morphisms of $\Theta_n$ we obtain our $\infty$-category of configurations of points in $\mathbb{R}^n$. Essentially this localization yields all of Ayala-Hepworth’s homotopy equivalences (as cardinality varies) at once, in the unordered setting.

The advantage of codifying [7] in a higher-categorical framework is that it yields a direct connection between $E_n$-algebras and $(\infty,n)$-categories as the authors suggest. We give more details about this in §2. Additionally, our $\infty$-category relates to factorization algebras [8, 24, 13] and we conjecture that it admits the structure of an $\infty$-operad.

1.1. Stratified spaces. Before stating our main result, we review some definitions. Unless otherwise stated, let $M$ denote a connected, nonempty, smooth manifold. Those familiar with stratified spaces may want to proceed to §1.2, taking note of examples (1.1.4) and (1.1.5).

**Definition 1.1.1.** For a nonempty, smooth manifold $M$, the configuration space of finite subsets of $r$ points in $M$ is defined to be the subspace

$$\text{Conf}_r(M) := \{(x_1, ..., x_r) \in M^r \mid x_i \neq x_j \text{ if } i \neq j\}$$

of $M^r$ equipped with the subspace topology.

There is an evident action of $\Sigma_r$ on $\text{Conf}_r(M)$ given by permuting the order of the indexing. The resulting quotient space

$$\text{Conf}_r(M)_{\Sigma_r} := \text{Conf}_r(M)/\Sigma_r = \{S \subset M \mid |S| = r\}$$

is called the unordered configuration space of finite subsets of $r$ points in $M$. As cardinality varies, these spaces naturally organize together into one space. Following that of Beilinson and Drinfeld in §3.4.1 of [8], we define this space as follows.

**Definition 1.1.2.** The Ran space of $M$ is the topological space whose underlying set is

$$\text{Ran}(M) := \{S \subset M \mid S \text{ is finite and nonempty}\}$$
and whose topology is the finest for which the maps
\[ M^r \to \text{Ran}(M) \]
given by \((m_1, \ldots, m_r) \mapsto \{m_1, \ldots, m_r\}\) are continuous for all \(r \geq 1\).\(^1\)

Famously, the Ran space of a connected manifold is weakly contractable \([12]\) (see \([8]\) for a pleas-
antly simple proof). However, we can stratify the Ran space according to cardinality, thereby encoding
the unordered configuration spaces as the underlying strata. The following is Definition A.5.1 of \([24]\) and Definition 2.1.10 of \([6]\).

**Definition 1.1.3.**

- (Topological structure on a poset.) Let \(P\) be a partially ordered set. We equip \(P\) with the
topology that defines \(U \subset P\) to be open if and only if it is *closed upwards:* that is, if \(a \in U\),
then every \(b \geq a\) is also in \(U\).

- A *topologically stratified space* \(X \xrightarrow{\sigma} P\) is a paracompact, Hausdorff topological space \(X\)
together with a poset \(P\) and a continuous map \(\sigma\) such that for each \(p \in P\), the fiber over \(p\)
is nonempty and connected.

The fiber over \(p \in P\) is called the *\(p\)-stratum*, which we denote by \(X_p\).

- A *stratified map* from \((X \to P)\) to \((Y \to Q)\) is a continuous map \(X \xrightarrow{f} Y\) such that
\[ X \xrightarrow{f} Y \]
\[ \downarrow \quad \downarrow \]
\[ P \longrightarrow Q \]
is a commutative diagram of topological spaces.

We will denote a stratified space \(X \xrightarrow{S} P\) by its underlying topological space \(X\), if we expect \(S\)
and \(P\) to be understood. The following example of a stratified space from \([6]\) is essential in this
work.

**Example 1.1.4.** The standard stratification of the topological \(k\)-simplex
\[
\Delta^k := \{(t_0, \ldots, t_k) \in [0,1]^{k+1} \mid \Sigma_{i=0}^k t_i = 1\} \to [k] := \{0 < \cdots < k\}
\]
is given by
\[
(t_0, \ldots, t_k) \mapsto \max\{i \mid t_i \neq 0\}.
\]
This stratification for \(\Delta^0, \Delta^1,\) and \(\Delta^2\) is depicted below, from left to right, respectively:

![Stratification of \(\Delta^2\)](image)

**Example 1.1.5.** The Ran space of a connected manifold \(M\) admits a stratification over the natural
numbers by cardinality. Namely, the stratified space
\[ \text{Ran}(M) \to \mathbb{N} \]
assigns to each point \(S \subset M\) in \(\text{Ran}(M)\) the cardinality of the underlying set \(S\).
Observe that for \(r \in \mathbb{N}\), the \(r\)-stratum of \(\text{Ran}(M)\) is precisely the unordered configuration space
\[ \text{Conf}_r(M)_{\Sigma_r}. \]

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\(^1\)There are other natural topologies that one can consider for the Ran space. See \([11]\) for an exposition.
1.2. Exit-path ∞-categories. We recover the homotopy type of the unordered configuration spaces arising as the strata of the Ran space by applying the following construction.

**Definition 1.2.1.** For a stratified space $X \xrightarrow{i} P$, the exit-path ∞-category of $X$, denoted $\text{Exit}(X)$, is the simplicial set whose value on $[k]$ is the set

$$\{\Delta^k \xrightarrow{\sigma} X \mid \sigma \text{ is a stratified map}\}.$$ 

Informally, an object of the exit-path ∞-category is a point in $X$ and a morphism is an ‘exit-path’ in $X$, i.e., paths that are allowed to witness points moving from a stratum $X_p$ to a stratum $X_q$ if $p \leq q$, but not if $p > q$. In other words, allowable paths agree with the ordering of the poset $P$. Moreover, these paths must exit immediately - this is due to the standard stratification of the topological 1-simplex $\Delta^1$ (1.1.4). Indeed, each morphism, as a stratified map from $\Delta^1$ to $X$, must respect the underlying stratifications of each space. Likewise, 2-morphisms are certain kinds of homotopies between paths in $X$ governed by the standard stratification of $\Delta^2$ (1.1.4) respecting the stratification of $X$.

**Remark 1.2.2.** Definition 1.2.1 is a generalization of MacPherson’s ‘exit-path category’ construction, which was introduced as a way to classify constructible sheaves on stratified spaces (locally constant on each stratum) [33]. Our definition is inspired by and is equivalent to Lurie’s Definition A.6.2 of [24] which also classifies constructible sheaves on nice enough stratified spaces (Theorem A.9.3, [24]). Likewise, it is equivalent to the opposite of the ‘enter-path ∞-category’ (Definition 1.1.5, [6]) and the ‘exit-path ∞-category’ (Definition 3.3.1, [4]) for even nicer stratified spaces.

In the case of the Ran space, an object of $\text{Exit}(\text{Ran}(M))$ is a finite, nonempty subset of $M$ and a morphism is a path in $\text{Ran}(M)$ that is allowed to witness anticolision of points in $M$, but not collision. Furthermore, the invertible morphisms are all those paths that stay in a single stratum. Thus, the homotopy types of the unordered configuration spaces are encoded as the maximal ∞-subgroupoid of $\text{Exit}(\text{Ran}(M))$.

**Remark 1.2.3.** The simplicial set $\text{Exit}(X)$ is guaranteed to be an ∞-category if the stratified space $X$ is ‘conically stratified’ (Theorem A.6.4, [24]). However, the Ran space $\text{Ran}(M)$ under the topology of (1.1.2) is not conical (though it is under a courser topology c.f. Theorem 4.12, [11]). In §7.1 we show that $\text{Exit}(\text{Ran}(M))$ is an ∞-category by other means.

1.2.1. The exit-path ∞-category of the unital Ran space. We directly construct an ∞-category $\text{Exit}(\text{Ran}^u(M))$, containing $\text{Exit}(\text{Ran}(M))$ as an ∞-subcategory, in order to encode the empty configuration. The empty configuration is the only additional object and the morphisms witness the vanishing of points in $M$ in addition to anticolision of points.

More explicitly, $\text{Exit}(\text{Ran}^u(M))$ is the simplicial space in which an object is a finite, possibly empty subset $S$ of $M$, and a morphism from $S \subseteq M$ to $T \subseteq M$ is a map between sets $T \to S$ together with an injection $(S \amalg_{T \times \{0\}} (T \times \Delta^1)) \hookrightarrow M \times \Delta^1$ over $\Delta^1$. We formally define the exit-path ∞-category of the unital Ran space in §7.

**Example 1.2.4.** Consider the map of sets $T = \{t, t', t''\} \to S = \{s, s', s''\}$ given by $t, t' \mapsto s$ and $t'' \mapsto s'$. The coproduct $S \amalg_{T \times \{0\}} (T \times \Delta^1)$ (over $\Delta^1$) is depicted by Figure 1. Any embedding of Figure 1 into $M \times \Delta^1$ over $\Delta^1$ is a morphism in $\text{Exit}(\text{Ran}^u(M))$. In particular, since the point $s''$ vanishes it is a degeneracy morphism. Also note that the point $s$ anticollides into two points.

Any morphism in $\text{Exit}(\text{Ran}^u(M))$ given as an embedding from Figure 1 into $M \times \Delta^1$ (over $\Delta^1$) is a degeneracy morphism, since the point $s''$ vanishes. Also note that the point $s$ anticollides into two points.

**Remark 1.2.5.** There are a few natural and well-used topologies to consider for the unital Ran space $\text{Ran}(M) \amalg 0$ [11]. However, none of these topologies on the unital Ran space of $\mathbb{R}^n$ (after taking the exit-path ∞-category) have a natural relationship with the category $\Theta_n$. Further, the topology on the unital Ran space of $\mathbb{R}^n$ that would yield a relationship with $\Theta_n$ does not admit
the structure of a topologically stratified space, preventing the use of Definition 1.2.1. The $\infty$-category $\text{Exit}(\text{Ran}^n(M))$ codifies the desired topology on (what would be) the unital Ran space and by constructing it directly we bypass the actual input of a stratified space.

Remark 1.2.6. An object similar to $\text{Exit}(\text{Ran}^n(M))$ called a configuration category is given in [1] and further developed in [10].

1.3. Main results. The purpose of this paper is to provide a higher-categorical, combinatorial identification of the configuration spaces of $\mathbb{R}^n$, codified by the $\infty$-category $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$, in terms of the category $\Theta_n$. Before stating our main result, we review the categories $\Theta_n$ in terms of certain kinds of trees [9].

Definition 1.3.1.

(1) A level tree is a finite, rooted tree, the root of which is a choice of vertex thereof. The choice of root uniquely determines a direction to each edge such that there is a unique directed path from each vertex to the root.

(2) For each vertex, we may equip the set of edges directed toward the vertex with a linear order. A tree is called a planar level tree if such an order is specified with respect to each vertex.

(3) A vertex is at level $i$ if the directed path from the vertex to the root counts $i$ edges.

(4) A tree has height $n$ if the maximum level of all the vertices is $n$.

(5) A vertex is a leaf if it has no edges directed towards it.

(6) A planar level tree of height $n$ is healthy if all of its leaves are at level $n$.

We will say tree to mean finite rooted planar level tree. If a tree is not healthy, we call it unhealthy.

The following are depictions of trees, where the root is at the bottom of each diagram; on the left, $T_1$ is a healthy tree of height one, in the middle, $T_2$ and $T_2'$ are a healthy trees of height two, and on the far right, $T_{2h}$ is a unhealthy tree of height two.

In particular, the set of leaves of a healthy tree of height $n$ naturally emits an $n$-order. Formally, an $n$-order on a finite set $S$ is a sequence of surjective maps

$$S \overset{\sigma_{n-1}}{\rightarrow} S_{n-1} \rightarrow \cdots \overset{\sigma_1}{\rightarrow} S_1$$
among finite sets together with a linear order on $S_i$ and a linear order on each fiber of each map in the sequence; that is, for each $1 \leq i \leq n - 1$, a linear order on $\sigma_i^{-1}(s)$ for each $s \in S_i$. A 1-order on a set $S$, then, is just a linear order on $S$.

A healthy tree $T$ of height $n$ naturally encodes an $n$-order on its set of leaves as follows. The $i$th set of the sequence is the set of vertices at level $i$. The assignment from the set of vertices at level $i$ to the set of vertices at level $i - 1$ is canonical, given by assigning to each vertex at level $i$ the vertex at level $i - 1$ that is adjacent to it. Such a sequence of maps, then, is surjective precisely because $T$ is healthy. The planar condition (2) of Definition 1.3.1 fixes the set of incoming edges of each vertex with a linear order; this condition induces a linear order on each fiber in the sequence and on the set of vertices at level 1. As a convention, we will fix these linear orders to be read from left to right on trees.

For now we will informally regard the objects of the category $\Theta_n$ as trees of height $n$. In these terms however, the morphisms do not have a nice combinatorial description. Thus we forgo their full definition until §6 and for now only indicate what values certain classes of them take in finite pointed sets. Indeed, the assembly functor on $\Theta_n$ (induced by Segal’s gamma functor) takes values in the opposite of Segal’s category of finite pointed sets by sending an object to its set of leaves disjoint union a basepoint. A morphism in $\Theta_n$ is called active if its value in finite pointed sets is active in the sense of [24], meaning that the preimage of the basepoint consists only of the basepoint. An active morphism is called exiting if its value in finite pointed sets is surjective.

Our main results articulates the sense in which the subcategories of $\Theta_n$ of active morphisms $\Theta_n^{\text{act}}$ and of exiting morphisms $\Theta_n^{\text{exit}}$ give combinatorial descriptions of the configuration space of $\mathbb{R}^n$ codified by the $\infty$-categories $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ and $\text{Exit}(\text{Ran}(\mathbb{R}^n))$, respectively.

**Theorem 1.3.2** (Theorem 7.2.1). For each $n \geq 1$ there is a localization of $\infty$-categories

$$G_n : \Theta_n^{\text{act}} \longrightarrow \text{Exit}(\text{Ran}^u(\mathbb{R}^n))$$

from the subcategory of $\Theta_n$ consisting of active morphisms.

As a consequence we recover an infinity-categorical generalization of [7].

**Corollary 1.3.3** (Corollary 10.0.2). For each $n \geq 1$ there is a localization of $\infty$-categories

$$\Theta_n^{\text{exit}} \longrightarrow \text{Exit}(\text{Ran}(\mathbb{R}^n))$$

from the subcategory of $\Theta_n$ consisting of healthy trees and exiting morphisms.

An immediate consequence of (1.3.2) is that the wreath product decomposition of $\Theta_2$ induces a likewise decomposition of $\text{Exit}(\text{Ran}^u(\mathbb{R}^2))$.

**Corollary 1.3.4** (Corollary 9.3.3). There is a localization of $\infty$-categories

$$\text{Exit}(\text{Ran}^u(\mathbb{R})) \cdot \text{Exit}(\text{Ran}^u(\mathbb{R})) \longrightarrow \text{Exit}(\text{Ran}^u(\mathbb{R}^2)).$$

**Remark 1.3.5.** Corollary 1.3.4 is related to Dwyer-Hess-Knudsen’s decomposition of the configuration space of a product of parallelizable manifolds into the factors according to the Boardman-Vogt tensor product [14].

We conjecture natural generalizations of (9.3.3).

**Conjecture 1.3.6.** For $n \geq 2$ there is a localization of $\infty$-categories

$$\text{Exit}(\text{Ran}^u(\mathbb{R})) \cdot \text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1})) \longrightarrow \text{Exit}(\text{Ran}^u(\mathbb{R}^n)).$$

**Conjecture 1.3.7.** For connected, parallelizable manifolds $M$ and $N$ there is a localization of $\infty$-categories

$$\text{Exit}(\text{Ran}^u(M)) \cdot \text{Exit}(\text{Ran}^u(N)) \longrightarrow \text{Exit}(\text{Ran}^u(M \times N)).$$

**Remark 1.3.8.** The $\infty$-category $\text{Exit}(\text{Ran}(\mathbb{R}^n))$ does not admit a wreath product decomposition because the subcategory $\Theta_n^{\text{exit}}$ does not admit a wreath product decomposition.
1.3.1. Intuition for the main results. With the following informal discussion, we hope to reveal some of the natural intuition of the localizations of Theorem 1.3.2 and Corollary 1.3.3. Heuristically, a localization of an ∞-category C is obtained by taking formal inverses of a class of morphisms of C which contains all of the isomorphisms. For a formal definition see 9.0.1. Roughly speaking, (1.3.2) states that by inverting all of the morphisms of $\Theta^n_{\text{exit}}$ that are sent to isomorphisms by $G_n$, there results an ∞-category equivalent to $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$.

Note that the functor in (1.3.3) is a restriction of $G_n$ to the ∞subcategory $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ of $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$. Thus the following discussion also applies to (1.3.3) except when otherwise stated.

When $n = 1$, the localization of (1.3.2) is trivial, which is to say, it is an equivalence. Indeed, recall that the category $\Theta_1 := \Delta$ has as its objects finite, linearly ordered sets. These are represented by trees of height 1, e.g., $T_1(2)$. The functor $G_1$ identifies each subset of $\mathbb{R}$ with its underlying linearly ordered set (induced by fixing an orientation on $\mathbb{R}$). For example, any two point subset of $\mathbb{R}$ is identified with $T_1(2)$. More generally, any $k$ point subset of $\mathbb{R}$ is identified with the tree of height 1 with $k$ leaves. Since the space of $k$ point subsets of $\mathbb{R}$ (unordered) is contractable, this assignment of objects is well-defined. Note that the empty configuration is identified with the empty tree; on the level of objects, this is the only difference between (1.3.2) and (1.3.3).

For $n > 1$, (1.3.2) is also straightforward on the level of objects since finite subsets of $\mathbb{R}^n$ naturally inherit the $n$-order from $\mathbb{R}^n$ given by the sequence of projection maps

$$S \twoheadrightarrow \text{pr}_{<n}(S) \twoheadrightarrow \cdots \twoheadrightarrow \text{pr}_{<2}(S)$$

each of which projects off the last coordinate. (The linear order on the base set and on each fiber is canonical after fixing an orientation on $\mathbb{R}^n$.) For example, when $n = 2$ the trees $T_2$ and $T_2^\text{sh}(2)$ are assigned to the left and right configurations (up to isomorphism), respectively, depicted in Figure 3, where $\mathbb{R}^2$ is given the standard basis (i.e., the horizontal axis is the first coordinate of $\mathbb{R}^2$).

![Figure 3. Left: Image of $T_2$ and $T_2^\text{sh}$; Right: Image of $T_2'$](image)

Next we consider the tree $T_2^\text{sh}(2)$, which only applies to (1.3.2) because it is not an object of $\Theta^n_{\text{exit}}$ as it is unhealthy. The tree $T_2^\text{sh}(2)$ is assigned to the subset depicted on the left of (3). In general, the functor $G_n$ canonically assigns each tree $T$ to the subset of $\mathbb{R}^n$ which admits the $n$-order encoded by the leaves of $T$, which is unique up to isomorphism. Furthermore, the contractibility of the Fox-Neuwirth cells implies that this assignment is well-defined.

Note that the only isomorphisms in $\Theta_n$ are automorphisms. Thus the trees $T_2$, $T_2'$ and $T_2^\text{sh}(2)$ are not isomorphic in $\Theta_2^\text{act}$. However, the images of these trees under $G_2$, namely the objects of $\text{Exit}(\text{Ran}^n(\mathbb{R}^2))$ depicted in (3), are isomorphic since they are both in the 2-stratum. Thus it is that $T_2$, $T_2'$, and $T_2^\text{sh}$ all become isomorphic after $\Theta_2^\text{act}$ is localized. More generally, since subsets of the same cardinality are isomorphic objects in $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$, the functor $G_n$ inverts all the morphisms of $\Theta_n$ that induce bijections between their sets of leaves in the category of finite pointed sets.

1.4. Approach. The key idea behind Theorem 1.3.2 is that the exit-path ∞-category construction, which is actually a functor from stratified spaces to ∞-categories, carries refinements of stratified spaces (Definition 7.1.4) to localizations of their exit-path ∞-categories (Theorem 3.3.12, [4]). However, because the construction of $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ does not actually have the input of a stratified space (c.f. Remark 1.2.5), we likewise directly construct an ∞-category $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ to interpolate between $\Theta_n$ and $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$. Just as the $[p]$-spaces of $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$ heuristically encode
the stratification of the unital Ran space of \( \mathbb{R}^n \) by cardinality, the \([p]\)-spaces of \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) heuristically encode a refinement of the unital Ran space of \( \mathbb{R}^n \) given by cardinality and by remembering coordinate coincidence, i.e., the Fox-Neuwirth cells. We show that \( \Theta_n^{\text{act}} \) is equivalent to \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \), and then we prove that \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) localizes to \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) by using Theorem 9.0.7 of [25], which gives a method for identifying localizations in favorable cases.

Much of the proof of Corollary 1.3.3 is extrapolated from the proof of Theorem 1.3.2, since both the domain \( \Theta_n^{\text{ext}} \) and codomain \( \text{Exit}(\text{Ran}(\mathbb{R}^n)) \) are \( \infty \)-subcategories (respectively) of the domain \( \Theta_n^{\text{act}} \) and codomain \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) of (1.3.2).

2. Motivation and conjectures

This work stands as a new perspective on a bridge that is conjectured to exist between configuration spaces of finite subsets of \( \mathbb{R}^n \) and \((\infty,n)\)-categories. In particular, our work gives an approximation to the folklore conjecture that relates \( \mathcal{E}_n \)-algebras to \((\infty,n)\)-categories. Since we have not found this conjecture explicitly stated anywhere, we will articulate it here. Let \( \mathcal{V} \) be a symmetric monoidal \( \infty \)-category. Consider the \( \infty \)-category \( \text{Alg}_{\mathcal{E}_n}(\mathcal{V}) \) of \( \mathcal{E}_n \)-algebras in \( \mathcal{V} \). Consider the \( \infty \)-category \( \text{Cat}_{(\infty,n)}(\mathcal{V}) \) of \( \mathcal{V} \)-enriched \((\infty,n)\)-categories. Consider the object \( \mathbb{I} \in \text{Cat}_{(\infty,n)}(\mathcal{V}) \) whose underlying \((\infty,n-1)\)-category \( \mathbb{I}_{\leq n} \simeq * \) is terminal, and whose \( \text{nEnd}_{\mathbb{I}}(*) = \mathbb{I} \in \mathcal{V} \).

Say a morphism \( A \xrightarrow{F} B \) between \((\infty,n)\)-categories is \( n \)-connected if, for each \( 0 \leq k \leq n \), each diagram among \((\infty,n)\)-categories

\[
\begin{array}{ccc}
\partial c_k & \longrightarrow & A \\
\downarrow & & \downarrow F \\
c_k & \longrightarrow & \mathbb{I}
\end{array}
\]

admits a filler. Here, \( c_k \) is the \((\infty,n)\)-category corepresenting the space of \( k \)-morphisms, and \( \partial c_k \rightarrow c_k \) is the inclusion of its maximal \((\infty,k-1)\)-subcategory.

**Conjecture 2.0.1.** Let \( \mathcal{V} \) be a symmetric monoidal \( \infty \)-category. There is a fully faithful left adjoint

\[
\Theta_n^{\text{act}} : \text{Alg}_{\mathcal{E}_n}(\mathcal{V}) \rightarrow \text{Cat}_{(\infty,n)}(\mathcal{V})^{\mathcal{V}}
\]

the image consists of those morphisms \( \mathbb{I} \rightarrow \mathcal{C} \) between \( \mathcal{V} \)-enriched \((\infty,n)\)-categories whose underlying morphism \( * \rightarrow \mathcal{C}_{\leq n} \) between \((\infty,n-1)\)-categories is \( (n-1) \)-connected.

If we specialize to the case that \( (\mathcal{V}, \otimes) = (\text{Spaces}, \times) \), our results Theorem 1.3.2 and Corollary 1.3.3 give an approximation to this conjecture. Let us review some key results from the literature before we explain how.

The collection of unordered configuration spaces of finite subsets of \( \mathbb{R}^n \) exhibit the algebraic structure of the \( \mathcal{E}_n \)-operad. Lurie identifies this relationship as a fully faithful embedding

\[
\text{Alg}_{\mathcal{E}_n}(\text{Spaces}) \rightarrow \text{coShv}_{\text{Spaces}}^{\text{bl}}(\text{Ran}(\mathbb{R}^n))
\]

from \( \mathcal{E}_n \)-algebras valued in spaces to constructible cosheaves (locally constant on each stratum) on the Ran space of \( \mathbb{R}^n \) (Theorem 5.5.4.10 [24]).

Corollary 3.3.11 of [4] yields an equivalence of \( \infty \)-categories

\[
\text{Shv}_{\text{Spaces}}^{\text{bl}}(\text{Ran}(\mathbb{R}^n)) \simeq \text{Fun}(\text{Exit}(\text{Ran}(\mathbb{R}^n)), \text{Spaces})
\]

between space-valued constructible sheaves on \( \text{Ran}(\mathbb{R}^n) \) and space-valued functors on \( \text{Exit}(\text{Ran}(\mathbb{R}^n)) \). Thus, there is a fully faithful embedding

\[
\text{Alg}_{\mathcal{E}_n}(\text{Spaces}) \rightarrow \text{Fun}(\text{Exit}(\text{Ran}(\mathbb{R}^n)))^{\text{op}}, \text{Spaces}
\]

We conjecture that this functor factors through space-valued functors on \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n))^{\text{op}} \).

Lastly, Theorem 1.3.2 and Corollary 1.3.3 induce fully faithful functors from space-valued functors on \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) and \( \text{Exit}(\text{Ran}(\mathbb{R}^n)) \) respectively, to space-valued functors out of the appropriate subcategories of \( \Theta_n^{\text{act}} \). These functors fit into the following diagram as indicated.
We are presently working on generalizing Theorem 1.3.2 to show that the category $\Theta_n$ localizes to an $\infty$-category, denoted $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))^{\text{op}}$, which codifies pointed configurations as its objects and in addition to the morphisms of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$, morphisms that witness appearances of points; such morphisms correspond with the non-active morphisms of $\Theta_n$. This generalization would yield the closest approximation of $\mathbb{B}^n$.

3. Use of $\infty$-categories

In this paper, we use $(\infty, 1)$-categories, or simply, `$\infty$-categories', to package the homotopy type of configuration spaces. There are many models for $\infty$-categories; notable for the scope of this paper is the work of Lurie in [24] on the theory of quasi-categories and the work of Rezk in [30] on the theory of complete Segal spaces.

Quasicategories. We use Joyal’s quasi-category model of $\infty$-categories from [19], wherein a quasi-category is defined to be a simplicial set, that is, a functor from the opposite of the simplex category $\Delta^{\text{op}}$ (Definition 6.0.1) to the category of sets Set, that satisfies a certain condition called the inner-horn filling condition. For the definition of this condition, together with other basic notions regarding quasi-categories see Rezk’s friendly exposition [31].

Complete Segal spaces. We use Rezk’s complete Segal spaces to model $\infty$-categories [30]. The $\infty$-category of spaces $\text{Spaces}$ is the localization of the category of topological spaces that admit a CW structure and continuous maps thereof, localized on (weak) homotopy equivalences. We call its objects spaces, i.e., CW complexes. A complete Segal space is a simplicial space, that is, a functor from the opposite of the simplex category $\Delta^{\text{op}}$ to the $\infty$-category of spaces $\text{Spaces}$ that satisfies the completeness and Segal conditions (Definition 7.1.1). We refer to a point in the $[p]$-space of such a functor as a $[p]$-point. For any additional information about complete Segal spaces, see [30].

The nerve functor. There is a construction which takes an ordinary category $\mathcal{C}$ and produces an $\infty$-category $\text{NC}$ called the (ordinary) nerve of $\mathcal{C}$. This construction is explicated by a fully faithful functor from the category of categories to the category of simplicial sets, through which each category is carried to a quasi-category. In light of the fully faithfulness of this functor, we refer to an ordinary category as an $\infty$-category without any reference to its nerve, whenever appropriate within the context. For a definition of the nerve, see Definition 3.1 in [31].
Model independence. In this paper, we work model independently, which, by the work of Joyal and Tierney, is a valid approach, since quasi-categories are shown to be equivalent to complete Segal spaces [21]. Model independence is exercised in this paper, for example, in that the hom-\(\infty\)-groupoid with fixed source and target of a quasi-category is equivalent to a space (i.e., CW complex) by way of the equivalence between quasi-categories and complete Segal spaces [21]. Throughout this work, we are liberal with our use of model independence and typically do not give forewarning of its implementation. Additionally, we frequently construct \(\infty\)-categories by taking finite limits of \(\infty\)-categories, an allowable maneuver precisely because the category of complete Segal spaces (or any other equivalent model) admits all finite limits and colimits [32].

4. Linear overview

§6. We define the wreath product of categories, the assembly functor, the category \(\Theta_n\) and its subcategory of active morphisms.

§7. This section is devoted to the \(\infty\)-category \(\text{Exit}(\text{Ran}^n(M))\). After defining it as a simplicial space, we prove that it is a complete Segal space. We conclude this section by restating our main result (1.3.2) in its full generality as Theorem 7.2.1, which states that \(\Theta^\text{act}_n\) localizes to \(\text{Exit}(\text{Ran}^n(\mathbb{R}^n))\).

§8. This section is the first of two parts of the proof of Theorem 7.2.1. We define the \(\infty\)-category \(\text{Exit}(\text{Ran}^n(\mathbb{R}^n))\), which interpolates between \(\Theta^\text{act}_n\) and \(\text{Exit}(\text{Ran}^n(\mathbb{R}^n))\), and then we show that it is a complete Segal space. The main result of this section is Lemma 8.2.2, which states that there is an equivalence of \(\infty\)-categories \(\text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \simeq \Theta^\text{act}_n\).

§9. This section is the second part of the proof of Theorem 1.3.2 and contains the technical heart of this paper. The main result of this section is Lemma 9.0.3, which states that the natural forgetful functor from \(\text{Exit}(\text{Ran}^n(\mathbb{R}^n))\) to \(\text{Exit}(\text{Ran}^n(\mathbb{R}^n))\) is a localization. Our argument is built around Theorem 9.0.7 from [25], which identifies localizations of \(\infty\)-categories in favorable cases. We implement (9.0.7) by proving two lemmas, (9.0.8) and (9.0.9).

Lemma 9.0.8 naturally organizes into two lemmas, (9.1.1) and (9.2.1). Central to both of these is the technical result that the configuration space \(\text{Conf}_r(\mathbb{R}^n)_{\Sigma_r}\) stratified by the Fox-Neuwirth cells (9.1.14) is conically smooth (9.1.18). Because of this, we gain hands-on access to \(\text{Conf}_r(\mathbb{R}^n)_{\Sigma_r}\) as the stratified space operating in the background of the \(\infty\)-subcategory \(\text{Exit}(\text{Conf}_r(\mathbb{R}^n))\) of \(\text{Exit}(\text{Ran}^n(\mathbb{R}^n))\), something we do not have with \(\text{Exit}(\text{Ran}^n(\mathbb{R}^n))\).

§10. This section is devoted proving Corollary 1.3.3. First we define the \(\infty\)-category \(\text{Exit}(\text{Ran}(M))\) and the subcategory \(\Theta^\text{ext}_n\). Then we restate (1.3.3) in full generality as Corollary 10.0.2. Our method of proof mainly extrapolates the proof of Theorem 7.2.1, in that it too is built from Theorem 9.0.7 of [25].

5. Acknowledgements

I would like to thank David Ayala for sharing his time, energy and ideas, contributing significantly to my fantastic time in Bozeman. Additionally, I would like to thank Dan Perry, Dev Sinha, Ben Moldstad, Peter May, Damien Lejay, Ben Knudsen and Greg Friedman for helpful conversations.

6. The categories \(\Theta_n\)

The purpose of this section is to define \(\Theta_n\) as the \(n\)-fold wreath product of the simplex category \(\Delta\) with itself [9]. Along the way, we define the wreath product of categories, Segal’s assembly functors, and active morphisms.

Definition 6.0.1. The simplex category \(\Delta\) is the category in which an object is a nonempty, finite, linearly ordered set and in which a morphism is a non-decreasing map of sets. Composition is composition of maps between sets.
For each object $S$ of $\Delta$, there is a unique non-negative integer $p$ such that $S$ is canonically isomorphic to the linearly ordered set $[p] := \{ 0 < \cdots < p \}$. We call $[p]$ the $p$-simplex and will henceforth refer to the objects of $\Delta$ as $p$-simplices.

The category of posets $\text{Poset}$ has an evident fully faithfully embedding into the category of categories $\text{Cat}$. In light of this fully faithful functor, we refer to $[p]$ as either a linearly ordered set or as the category whose objects are $\{ 0, 1, \ldots, p \}$ and in which there is a unique morphism from $i$ to $j$ precisely when $i \leq j$, and no morphism otherwise.

**Definition 6.0.2.** The category of pointed, finite sets $\text{Fin}_*$ is the category in which an object is a finite, pointed set and a morphism is a pointed map; composition is evident.

**Notation 6.0.3.** Given a finite set $S$, let $S_*$ denote the finite, pointed set $S \sqcup \{ * \}$.

**Definition 6.0.4.** The wreath product $\text{Fin}_* \wr \mathcal{D}$ for an arbitrary category $\mathcal{D}$ is the category defined as follows: An object is a symbol $S(d_s)$ where $S$ is a finite set and $(d_s)_{s \in S}$ is a tuple of objects in $\mathcal{D}$ indexed by $S$. A morphism $S(d_s) \to T(e_t)$ consists of a pair of data:

i) A morphism $S_* \xrightarrow{\delta} T_*$ in $\text{Fin}_*$

ii) For each pair $(s \in S, t \in T)$ such that $\delta(s) = t$, a morphism $d_s \xrightarrow{\delta_s} e_t$ in $\mathcal{D}$.

Composition is given by composition in $\text{Fin}_*$ and $\mathcal{D}$.

**Observation 6.0.5.** There is a forgetful functor $\text{Fin}_* \wr \mathcal{D} \to \text{Fin}_*$ given by $S(d_s) \mapsto S_*$; its value on morphisms is evident.

**Definition 6.0.6.** Given a category $\mathcal{C} \to \text{Fin}_*$ over the category of based finite sets and a category $\mathcal{D}$, the wreath product $\mathcal{C} \wr \mathcal{D}$ is the pullback of categories

\[
\begin{array}{ccc}
\mathcal{C} \wr \mathcal{D} & \xrightarrow{\cdot} & \text{Fin}_* \wr \mathcal{D} \\
\downarrow & & \downarrow \\
\mathcal{C} & \to & \text{Fin}_*
\end{array}
\]

where the vertical arrow on the right is the forgetful functor from Observation 6.0.5.

We take advantage of the previous observation and define Joyal’s category $\Theta_n$ over $\text{Fin}_*^{op}$ inductively as the $n$-fold wreath product of the simplex category $\Delta$ with itself.

**Definition 6.0.7.** The assembly functor

\[\text{Fin}_* \wr \text{Fin}_* \xrightarrow{\lambda} \text{Fin}_*\]

is given by the wedge sum. Explicitly, the value of $\nu$ on an object $S((T_s)_s)$ is the wedge sum $\bigvee_{s \in S} (T_s)_s$. Its value on a morphism $S((T_s)_s) \to S'((T'_s)_s)$ given by

\[
\begin{array}{ccc}
S_* & \xrightarrow{\delta} & S'_* \\
\downarrow & & \downarrow \\
\bigvee_{s \in S} (T_s)_s & \to & \bigvee_{s' \in S'} (T'_s)_s
\end{array}
\]

is defined by $t \in T_s \mapsto \delta_{ss'}(t)$ for every pair $(s, s')$ such that $\delta(s) = s'$.

**Definition 6.0.8.** The simplicial circle is the functor

\[\Delta \xrightarrow{\gamma} \text{Fin}_*^{op}\]

the value of which on an object $[p]$ is the quotient morphism set $\Delta([p], [1])//\{ \{ 0 \}, \{ 1 \} \}$, where $\{ i \}$ denotes the constant map at $i$; $\{ 0 \} \sim \{ 1 \}$ is the evident basepoint of the image. The value of $\gamma$ on a morphism $[p] \xrightarrow{\delta} [q]$ is precomposition with $f$. 

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Observation 6.0.9. The map induced by $\gamma$ between each hom-set is injective. This observation comes down to the fact that on morphisms $\gamma$ is given by precomposition and composition is unique in categories.

Observation 6.0.10. There is an evident isomorphism $\gamma([p]) \cong \{1, \ldots, p\}$ in $\text{Fin}_*$. Let $\nu_j$ denote the morphism that assigns each $0 \leq i \leq j - 1$ to 0 and each $j \leq i \leq p$ to 1 (i.e., a unique composite of degeneracy maps). The value of $\nu_j$ under the isomorphism is $j$. Its assignment on morphisms, then, is evident.

Terminology 6.0.11. In light of the previous observation, we will freely refer to a non-basepoint value in the pointed set $\gamma([p])$ by $j$ for some $1 \leq j \leq p$.

Definition 6.0.12. For each integer $n \geq 1$, the categories $\Theta_n$ are defined inductively by setting

$$\Theta_1 := \Delta \quad \text{and} \quad \Theta_n := \Theta_1 \wr \Theta_{n-1},$$

where the assembly functors $\Theta_n \to \text{Fin}^{\text{op}}$ are also defined inductively by setting

$$\gamma_1 := \gamma \quad \text{and} \quad \gamma_n := \nu \circ (\gamma_1 \wr \gamma_{n-1}).$$

Recall that a planar level tree of height $n$ is a finite rooted tree with an $n$-order on its set of leaves (Definition 1.3.1). These trees, in fact, naturally describe the objects of $\Theta_n$, as we will explain in the following observation.

Observation 6.0.13 (4.5 in [7]). Finite rooted planar level trees of height $n$ naturally describe the objects of $\Theta_n$ as follows: When $n = 1$, the object $[p]$ corresponds to the tree of height 1 that has $p$ leaves. For $n > 1$, the object $[p](T_1, \ldots, T_p)$ of $\Theta_n$ corresponds to the tree described as follows: The tree which has $p$ vertices at level 1; the $i$th vertex at level 1 (according to the linear order) is the root of the tree which corresponds to $T_i$, i.e., it is the subtree consisting of those vertices for which the unique directed path from each one to the root intersects the $i$th vertex at level 1. In terms of trees, then, the assembly functor $\gamma_n$ assigns a tree to its set of leaves.

In this paper, we will use this description of the objects of $\Theta_n$ whenever convenient. In fact, we often prefer it since it makes the objects of $\Theta_n$ so accessible.

Example 6.0.14. The object $[3](\{1\}, \{3\}, \{0\})$ in $\Theta_2$, corresponds to the following planar level tree of height 2.

![Planar Level Tree](image)

Remark 6.0.15. Alternatively (but equivalently), Rezk defined the category $\Theta_n$ as the full subcategory of the category of strict $n$-categories, $\text{Cat}_n$, in which an object is a pasting diagram ([29]). For example, the object $[3](\{1\}, \{3\}, \{0\})$ in $\Theta_2$ corresponds to the pasting diagram

![Pasting Diagram](image)

6.1. Active morphisms of $\Theta_n$.

Definition 6.1.1. Fin is the category in which an object is a finite set and a morphism is a map of sets; composition is evident.
**Definition 6.1.2.** Given a category \( \mathcal{C} \xrightarrow{F} \text{Fin}_* \) over pointed, finite sets, a morphism \( \sigma \) in \( \mathcal{C} \) is *active* if \( (F(\sigma))^{-1}(\{*\}) = \{*\} \). The subcategory \( \mathcal{C}_{\text{act}} \) of \( \mathcal{C} \) is defined to be the pullback

\[
\begin{array}{c}
\mathcal{C}_{\text{act}} \\
\downarrow j \\
\text{Fin} \\
\end{array}
\quad \begin{array}{c}
\downarrow F \\
\text{Fin}_* \\
\end{array}
\]

For a category \( \mathcal{D} \), there is a monomorphism \( \text{Fin}_! \mathcal{D} \to \text{Fin}_* \mathcal{D} \) and thus, the category \( \text{Fin}_! \mathcal{D} \) has an explicit description of objects and morphisms similar to that of \( \text{Fin}_* \mathcal{D} \) as given in Definition 6.0.4. We will use this description of \( \text{Fin}_! \mathcal{D} \) whenever convenient. Additionally, we would like to point out that given a category \( \mathcal{C} \xrightarrow{F} \text{Fin}_* \) over pointed, finite sets, \( F \) restricts to a functor \( \mathcal{C}_{\text{act}} \to \text{Fin}_* \) which, by definition, factors through \( \text{Fin}_* \). Evidently then, the category \( \mathcal{C}_{\text{act}} \text{Fin} \) is canonically isomorphic to \( \mathcal{C}_{\text{act}} \text{Fin}_* \).

The following alternative definition of \( \Theta_{\text{act}}^n \) coincides with that of Definition 6.1.2.

**Definition 6.1.3.** For each integer \( n \geq 1 \), the categories \( \Theta_{\text{act}}^n \) are the subcategories of \( \Theta_n \) defined inductively by setting

\[ \Theta_{\text{act}}^1 := \Delta_{\text{act}} \]

and defining \( \Theta_{\text{act}}^n := \Theta_{\text{act}}^1 \wr \Theta_{\text{act}}^{n-1} \), i.e., the pullback

\[
\begin{array}{c}
\Theta_{\text{act}}^n \\
\downarrow j \\
\text{Fin}_! \Theta_{\text{act}}^{n-1} \\
\end{array}
\quad \begin{array}{c}
\downarrow \text{tr}_{n-1} \\
\text{Fin}^{\text{op}} \\
\end{array}
\]

The benefit of this particular formulation of \( \Theta_{\text{act}}^n \) is that it makes the following facts straightforward.

**Observation 6.1.4.** Because the wreath product is associative, equivalently \( \Theta_{\text{act}}^n := \Theta_{\text{act}}^{n-1} \wr \Theta_{\text{act}}^1 \).

**Observation 6.1.5.** For each \( n \), there is a natural forgetful functor

\[ \Theta_{\text{act}}^n \xrightarrow{\text{tr}} \Theta_{\text{act}}^{n-1}. \]

The value on an object \( T_{n-1}([m_k]) \in \Theta_{\text{act}}^{n-1} \wr \Theta_{\text{act}}^1 =: \Theta_{\text{act}}^n \) is \( T_{n-1} \).

Let \( T_{n-1}([m_k]) \xrightarrow{\sigma} W_{n-1}([p_l]) \) be a morphism in \( \Theta_{\text{act}}^n \) defined by

i) a morphism \( T_{n-1} \xrightarrow{\sigma'} W_{n-1} \) in \( \Theta_{\text{act}}^{n-1} \) and,

ii) a morphism \( [m_k] \xrightarrow{\alpha_k} [p_l] \) in \( \Theta_{\text{act}}^1 \) for each pair \( (k,l) \) such that \( \gamma_{n-1}(l) = k \).

The value of \( \sigma \) under \( \text{tr} \) is \( \sigma' \).

**Example 6.1.6.** Let \( T \) be the object of \( \Theta_3 \) depicted as the far left tree in the figure below. We depict two iterations of the truncation functor \( \text{tr} \) on \( T \):

![Diagram](image)

**Notation 6.1.7.** For each \( 1 \leq i \leq n-1 \), denote the \( (n-i) \)-fold composite of the truncation functor \( \text{tr} \) by \( \text{tr}_i : \Theta_{\text{act}}^n \to \Theta_{\text{act}}^i \).
**Observation 6.1.8.** For each $1 \leq i \leq n - 1$, there is a natural transformation from $\Theta^\text{act}_n \xrightarrow{\gamma_n} \text{Fin}^{\text{op}}$ to the composite $\gamma_i \circ \text{tr}_i$,

\[
\begin{array}{ccc}
\Theta^\text{act}_n & \xrightarrow{\gamma_i} & \text{Fin}^{\text{op}} \\
\downarrow \text{tr}_i & & \\
\Theta^\text{act}_i & \xrightarrow{\gamma_n} & \text{Fin}^{\text{op}}.
\end{array}
\]

For each tree $T$, the natural transformation $\epsilon$ is given by the natural map $\gamma_n(T) \xrightarrow{\epsilon_T} \gamma_i(\text{tr}_i(T))$ from the leaves of $T$ to the leaves of the truncation of $T$ to height $i$, the assignment of which is the evident one given by the structure of the tree $T$. It is straightforward to verify that $\epsilon$ does indeed define a natural transformation.

**Definition 6.1.9.** Given a category $\mathcal{C}$, we define $\text{Fun}(\{1 < \cdots < n\}, \mathcal{C})$ to be the category in which an object is a functor $\{1 < \cdots < n\} \to \mathcal{C}$ which selects out a sequence of composable morphisms in $\mathcal{C}$:

\[
c_1 \to c_2 \to \cdots \to c_n
\]

and a morphism from $c_1 \to c_2 \to \cdots \to c_n$ to $d_1 \to d_2 \to \cdots \to d_n$ is a commutative diagram in $\mathcal{C}$:

\[
\begin{array}{ccc}
c_1 & \xrightarrow{f} & d_1 \\
\downarrow & & \downarrow \\
c_2 & \xrightarrow{f} & d_2 \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
c_n & \xrightarrow{f} & d_n.
\end{array}
\]

Composition is evident.

**Observation 6.1.10.** We use Observation 6.1.8 to define the natural functor

\[
\Theta^\text{act}_n \xrightarrow{\tau_n} \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})
\]

the value of which on an object $T$ is the functor which selects out the sequence of composable maps of sets

\[
\gamma_n(T) \xrightarrow{\epsilon_T} \gamma_{n-1}(\text{tr}(T)) \xrightarrow{\epsilon_n(T)} \gamma_{n-2}(\text{tr}_{n-2}(T)) \to \cdots \to \gamma_1(\text{tr}_1(T))
\]

and on a morphism $T \xrightarrow{f} S$ is the diagram of finite sets

\[
\begin{array}{ccc}
\gamma_n(S) & \xrightarrow{\gamma_n(f)} & \gamma_n(T) \\
\downarrow \epsilon_S & & \downarrow \epsilon_T \\
\gamma_{n-1}(\text{tr}_{n-1}(S)) & \xrightarrow{\gamma_{n-1}(\text{tr}_{n-1}(f))} & \gamma_{n-1}(\text{tr}_{n-1}(T)) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\gamma_1(\text{tr}_1(S)) & \xrightarrow{\gamma_1(\text{tr}_1(f))} & \gamma_1(\text{tr}_1(T))
\end{array}
\]

which is guaranteed to commute in $\text{Fin}$ because the downward arrows in the diagram are given by the natural transformation $\epsilon$ from Observation 6.1.8.
7. The $\infty$-category $\text{Exit}(\text{Ran}^{u}(M))$

We introduce the main avatar of this work, an $\infty$-category that encodes the configuration spaces of $\mathbb{R}^n$.

Definition 7.0.1 (6.6.12 in [4]). The reversed cylinder of a map between finite sets $T \to S$ is $\text{cylr}(T \to S) := S \amalg_{T \times \{0\}} T \times \Delta^1$.

More generally, the reversed cylinder of a composable sequence of maps between finite sets $S_p \to S_{p-1} \to \cdots \to S_0$ is $\text{cylr}(S_p \to S_{p-1} \to \cdots \to S_0) := S_0 \amalg_{S_1 \times \{0\}} S_1 \amalg_{S_2 \times \Delta^1} \cdots \amalg_{S_p \times \Delta^{p-1}} S_p \times \Delta^p$.

For an example of the cylinder construction for an explicit map of sets, see Example 1.2.4.

Observation 7.0.2. Recall that by definition the nerve of $\text{Fin}^{\text{op}}$ is a presheaf on $\Delta$. Thus, we may define the $\infty$-category $\Delta$ slice over $\text{Fin}^{\text{op}}$ as the pullback of $\infty$-categories $\Delta \downarrow_{\text{Fin}^{\text{op}}} \text{PShv}(\Delta)$ where $\text{PShv}(\Delta)$ is the category of presheaves on the simplex category.

Definition 7.0.3. For a smooth, connected manifold $M$, the exit-path $\infty$-category of the unital Ran space of $M$, $\text{Exit}(\text{Ran}^{u}(M))$, is the simplicial space over $\text{Fin}^{\text{op}}$ representing the presheaf on $\Delta \downarrow_{\text{Fin}^{\text{op}}} \text{PShv}(\Delta)$ whose value on an object $[p] \xrightarrow{<\sigma> \in \text{Fin}^{\text{op}}} S_p \to \cdots \to S_0$, is the space of embeddings $\text{cylr}(\sigma) \hookrightarrow M \times \Delta^p$ over $\Delta^p$ equipped with the compact-open topology; the structure maps are evident.

Observation 7.0.4. Explicitly, an object of $\text{Exit}(\text{Ran}^{u}(M))$ in the fiber over the finite (possibly empty) set $S$ is an embedding $S \hookrightarrow M$.

A morphism from $S \hookrightarrow M$ to $T \overset{d}{\to} M$ over the map of finite sets $T \overset{c}{\to} S$ is an embedding $\text{cylr}(T \overset{c}{\to} S) \overset{E}{\hookrightarrow} M \times \Delta^1$ over $\Delta^1$ such that $E|_S = e$ and $E|_{T \times \{1\}} = d$.

Observation 7.0.5. There is a natural forgetful functor $\text{Exit}(\text{Ran}^{u}(M)) \xrightarrow{\phi} \text{Fin}^{\text{op}}$ the value of which on an object $S \hookrightarrow M$ is $S$ and on a morphism $\text{cylr}(J \overset{c}{\to} S) \hookrightarrow M \times \Delta^1$ is $\sigma$.

Recall Example 1.2.4, wherein we gave an example of the cylinder of a fixed map of finite sets. Any embedding of this cylinder (depicted by (1)) into $M \times \Delta^1$ over their projections to $\Delta^1$ is an example of a morphism in $\text{Exit}(\text{Ran}^{u}(M))$. Heuristically, we can think of such a morphism simply as a collection of paths in $M$ parameterized by $\Delta^1$ each of which is pairwise disjoint in $M$ for all $0 < t \leq 1$ in $\Delta^1$ (i.e., ‘anticollision of points’ is allowed) together with some points which ‘disappear’ after $t = 0 \in \Delta^1$. Such a morphism, however, is not a path in the Ran space of $M$ (Definition 1.1.2) precisely because the map of sets $T \to S$ in Example 1.2.4 is not surjective. We are led to observe the following characterization of the exit-path $\infty$-category of the Ran space of $M$ relative to the exit-path $\infty$-category of the unital Ran space of $M$. 


Observation 7.0.6. There is a natural forgetful functor
\[ \text{Exit} (\text{Ran}(M)) \xrightarrow{\phi} \text{Fin}^{\text{op}} \]
whose value on an object \( S \subset M \) is the underlying set \( S \) and whose value on a morphism \( \Delta^1 \xrightarrow{f} \text{Ran}(M) \) is the map of finite sets \( f(\{1\}) \to f(\{0\}) \) given by the reverse of the path in \( M \). Note that, in particular, the image is a surjection. With the functor \( \phi \) in mind then, observe that a point \( \Delta^k \xrightarrow{f} \text{Ran}(M) \) in the set of \([k]\)-values of \( \text{Exit}(\text{Ran}(M)) \) is precisely an embedding
\[ \text{cylr}(S_k \to \cdots \to S_0) \hookrightarrow M \times \Delta^k \]
over \( \Delta^k \) of the reverse cylinder of the sequence of surjective maps \( S_k \to \cdots \to S_0 \) defined to be the image of \( f \) under \( \phi \). Thus, the exit-path \( \infty \)-category of the Ran space of \( M \) is a sub simplicial space of the exit-path \( \infty \)-category of the unital Ran space of \( M \), described as the following pullback of simplicial spaces:
\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}(M)) & \xleftarrow{\phi} & \text{Exit}(\text{Ran}^u(M)) \\
\downarrow & & \downarrow \\
(\text{Fin}_{\text{surj}})^{\text{op}} & \xleftarrow{\phi} & \text{Fin}_{\text{surj}}^{\text{op}}
\end{array}
\]
where \( \text{Fin}_{\text{surj}}^{\text{op}} \) is the subcategory of finite sets consisting of nonempty sets and all those morphisms that are surjections.

7.1. Proving \( \text{Exit}(\text{Ran}^u(M)) \) is an \( \infty \)-category. This subsection is devoted to proving the technical result that for a connected manifold \( M \), \( \text{Exit}(\text{Ran}^u(M)) \) is a complete Segal space.

Definition 7.1.1 (after [30]). A simplicial space \( \Delta^{\text{op}} \xrightarrow{F} \text{Spaces} \) is a complete Segal space if it satisfies the following two conditions:

i) (Segal Condition) For each \( p > 1 \), the diagram of spaces
\[
\begin{array}{ccc}
F[p] & \xrightarrow{\phi} & F\{p-1 < p\} \\
\downarrow & & \downarrow \\
F\{0 < \cdots < p-1\} & \xrightarrow{\phi} & F\{p-1\}
\end{array}
\]
is a pullback.

ii) (Completeness Condition) The diagram of spaces
\[
\begin{array}{ccc}
F(*) & \xleftarrow{\phi} & F[3] \\
\downarrow & & \downarrow \\
F\{0 < 2\} & \xleftarrow{\phi} & F\{1 < 3\}
\end{array}
\]
is a limit.

Proposition 7.1.2. The simplicial space \( \text{Exit}(\text{Ran}^u(M)) \) satisfies the Segal and completeness conditions.

Corollary 7.1.3. The simplicial space \( \text{Exit}(\text{Ran}(M)) \) satisfies the Segal and completeness conditions.
Proof. In Observation 7.0.6, we observed the pullback diagram among simplicial spaces

\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}(M)) & \longrightarrow & \text{Exit}(\text{Ran}^u(M)) \\
\downarrow & & \downarrow \\
(\text{Fin}^\text{surj})^\text{op} & \longrightarrow & \text{Fin}^\text{op}
\end{array}
\]

The result follows because the full ∞-subcategory of simplicial spaces consisting of the complete Segal spaces is closed under the formation of pullbacks.

We allow ourselves, in this subsection, to freely use notation and results from [4]. The idea for this proof is to witness the simplicial space \( \text{Exit}(\text{Ran}^u(M)) \) as one derived through formal constructions among complete Segal spaces from a complete Segal space \( \mathcal{B} \), defined in §6 of [4].

Namely, the simplicial space \( \mathcal{B} : \Delta^{\text{op}} \to \text{Spaces} \) is that for which the value on \( [p] \) is the moduli space of constructible bundles over \( \Delta^p \) with its standard stratification (see Example 1.1.4). The simplicial structure maps are implemented by base change of constructible bundles. Section §6 of [4] is devoted to the proof that this simplicial space satisfies the Segal and completeness conditions, which is to say \( \mathcal{B} \) is an ∞-category.

So the space of objects in \( \mathcal{B} \) is the moduli space of constructible bundles over \( \Delta^0 = * \), which is simply the moduli space of stratified spaces. So an object of \( \mathcal{B} \) is simply a stratified space. In particular, a finite set is an example of an object in \( \mathcal{B} \), and a smooth manifold is an example of an object in \( \mathcal{B} \) as well. Lemma 6.3.11 of [4] uses the reverse cylinder (Definition 7.0.1) construction to construct a fully faithful functor

\[ \text{Fin}^{\text{op}}_* \longrightarrow \mathcal{B} \]

whose image consists of finite sets. In particular, there is a composite monomorphism between ∞-categories:

\[ \text{Fin}^{\text{op}}_* \xrightarrow{(-)_+} \text{Fin}^{\text{op}}_* \longrightarrow \mathcal{B} \]

Note that a \([p]\)-point \( X \to \Delta^p \) of \( \mathcal{B} \) factors through this monomorphism (4) if and only if \( X \to \Delta^p \) is a finite proper constructible bundle.

Lemma 3.31 of [5] constructs, for each dimension \( k \), the \( k \)-skeleton functor

\[ \text{sk}_k : \mathcal{B} \longrightarrow \mathcal{B} \]

Explicitly, the value on a stratified space \( X \) is the proper constructible stratified subspace \( \text{sk}_k(X) \subset X \) that is the union of the strata whose dimension is at most \( k \). The value of \( \text{sk}_k \) on a \([p]\)-point \( X \to \Delta^p \) of \( \mathcal{B} \) is the constructible bundle

\[ \text{sk}^{\text{fib}}_k(X) \longrightarrow \Delta^p \]

which is the fiberwise \( k \)-skeleton of \( X \): the union of those strata of \( X \) whose projection to \( \Delta^p \) have fiber-dimension at most \( k \).

Note, then, that \( \text{sk}_0 \) factors through \( \text{Fin}_* \):

\[ \text{sk}_0 : \mathcal{B} \longrightarrow \text{Fin}^{\text{op}}_* \longrightarrow \mathcal{B} \]

Consider the ∞-category \( \text{Strat}^{\text{ref}} \) underlying the topological category in which an object is a stratified space and the space of morphisms is that of refinements.

**Definition 7.1.4** (3.6.1 in [6]). A map of stratified spaces \( (X \to P) \xrightarrow{f} (Y \to Q) \) is a refinement if \( f \) is a homeomorphism between the underlying topological spaces, and if for each \( p \in P \) the restriction of \( f \) to each stratum \( X_p \) is an embedding into \( Y \).

Section §6.6 of [4] constructs the open cylinder functor between ∞-categories

\[ \text{Cylo} : \text{Strat}^{\text{ref}} \longrightarrow \mathcal{B} \]
which is an equivalence on spaces of objects. Theorem 6.6.15 of [4] verifies that this functor is a
monomorphism. So each refinement between stratified spaces defines a morphism in \( \mathcal{B} \). As a
matter of notation, a morphism in \( \mathcal{B} \) that is in the image of this functor is called a refinement; the \( \infty \)-category of refinement arrows in \( \mathcal{B} \) is the full \( \infty \)-subcategory
\[\text{Ar}^{\text{ref}}(\mathcal{B}) \subset \text{Ar}(\mathcal{B})\]
consisting of the refinements arrows. Evaluation at source-target defines a functor
\[(\text{ev}_s, \text{ev}_t): \text{Ar}^{\text{ref}}(\mathcal{B}) \to \mathcal{B} \times \mathcal{B} .\]
Denote the pullback \( \infty \)-category:
\[\text{Ref}^0(M) \]
Unpacking this definition (and using the open cylinder construction of [4] referenced above in (4)
\( \mathcal{R} \) is a simplicial space whose value on \([p] \in \Delta\) is the moduli space of
- constructible bundles
  \[X \to \Delta^p\]
  for which the \((n-1)\)-skeleton of each fiber of which is a finite set,
- together with a refinement
  \[X \xrightarrow{\text{refinement}} M \times \Delta^p\]
on \( \Delta^p \).
We will denote such a \([p]\)-point of \( \mathcal{R} \) simply as \((X, M)\). Example 2.1.7 of [6] shows
that the product of stratified spaces is naturally a stratified space. We consider \( M \times \Delta^p \) as a product
stratified space, where \( M \) is trivially stratified over the poset with a singleton, and \( \Delta^p \) is given the
standard stratification (see Example 1.1.4).

Remark 7.1.5. Informally, an object in \( \mathcal{R} \) is a refinement of \( M \) in which the \((n-1)\)-
skeleton of the domain is a finite set, and a morphism in \( \mathcal{R} \) is a path of such refinements of \( M \)
witnessing anti-collision of strata and disappearances of strata.

Observation 7.1.6. There is a natural forgetful functor
\[\mathcal{R}: \mathcal{R} \to \text{Fin}^{\text{op}}\]
to the opposite of the category of finite sets, the value of which on an object \( X \to M \) is the
underlying set of the \((n-1)\)-skeleton of \( X \), and on a morphism \( X \to M \times \Delta^1 \) is the canonical
assignment between sets, from the \((n-1)\)-skeleton of the fiber of \( X \to \Delta^1 \) over \( \{1\} \in \Delta^1 \) to the
\((n-1)\)-skeleton of the fiber over \( \{0\} \in \Delta^1 \), implemented by taking connected components of the
\((n-1)\)-skeleton of \( X \). In other words, taking connected components of the fiberwise \((n-1)\)-
skeleton of \( X \) induces a canonical assignment of sets
\[\text{sk}^{\text{fib}}_{n-1}(X_{|1}) \to \text{sk}^{\text{fib}}_{n-1}(X_{|0})\]
from the \((n-1)\)-skeleton of the target of \( X \) to the \((n-1)\)-skeleton of the source of \( X \).

Lemma 7.1.7. There is a canonical equivalence between simplicial spaces over \( \text{Fin}^{\text{op}} \): 
\[\mathcal{R} \simeq \text{Exit}(\text{Ran}^\theta(M))\]
Proof. A rightward morphism is implemented by, for each \([ p ] \in \Delta\), the assignment,

\[
\left( X \xrightarrow{\text{ref}} M \times \Delta^p \right) \mapsto \left( \text{sk}_{n-1}^\text{fib}(X) \hookrightarrow X \rightarrow M \times \Delta^p \right),
\]

whose value is the embedding over \(\Delta^p\) from the fiberwise \((n-1)\)-skeleton, which maps to \(\Delta^p\) as a finite proper constructible bundle. A leftward morphism is implemented by, for each \([ p ] \in \Delta\), the assignment,

\[
\left( \text{Cylr}(\sigma) \hookrightarrow M \times \Delta^p \right) \mapsto \left( \left( \text{Cylr}(\sigma) \subset M \times \Delta^p \right) \xrightarrow{\text{ref}} M \times \Delta^p \right),
\]

whose value is the coarsest refinement of \(M \times \Delta^p\) for which the embedding from \(\text{Cylr}(\sigma)\) is a proper and constructible. (Such a refinement exists because the image of this embedding is, by definition, a properly embedded stratified subspace.)

It is straightforward to verify that these two assignments are mutually inverse to one another, and further, that they are both over \(\text{Fin}^{\text{op}}\). Furthermore, it is evident that the structure maps are equivalent.

\[\square\]

\textit{Proof of Proposition 7.1.2.} Being an \(\infty\)-category, the simplicial space \(\text{Ref}^0(M)\) satisfies the Segal and completeness conditions. Through the equivalence of Lemma 7.1.7, then so does the simplicial space \(\text{Exit}(\text{Ran}^u(R^n))\).

\[\square\]

7.2. Statement of the main result Theorem 7.2.1. We restate our main result, Theorem 1.3.2, in its full generality.

\textbf{Theorem 7.2.1.} For \(n \geq 1\) there is a localization of \(\infty\)-categories

\[
\Theta_n^{\text{act}} \rightarrow \text{Exit}(\text{Ran}^u(R^n))
\]

over \(\text{Fin}^{\text{op}}\) from the subcategory \(\Theta_n^{\text{act}}\) of active morphisms of \(\Theta_n\), to the exit-path \(\infty\)-category of the unital Ran space of \(R^n\).

8. Part 1 of the proof of Theorem 7.2.1: Interpolating between \(\Theta_n^{\text{act}}\) and \(\text{Exit}(\text{Ran}^u(R^n))\)

In this section we introduce the \(\infty\)-category \(\text{Exit}(\text{Ran}^u(R^n))\) to interpolate between \(\Theta_n^{\text{act}}\) and \(\text{Exit}(\text{Ran}^u(R^n))\). The main goal is to show that \(\text{Exit}(\text{Ran}^u(R^n))\) is equivalent to \(\Theta_n^{\text{act}}\).

\textbf{Observation 8.0.1.} Recall that by definition the nerve of \(\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})\) is a presheaf on \(\Delta\). Thus, we may define the \(\infty\)-category \(\Delta\) slice over \(\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})\) as the following pullback

\[
\Delta/\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}}) \xrightarrow{\text{Yoneda}} \text{PShv}(\Delta) \xrightarrow{\text{Yoneda}} \text{PShv}(\Delta)
\]

where \(\text{PShv}(\Delta)\) is the category of presheaves on the simplex category.

\textbf{Definition 8.0.2.} The exit-path \(\infty\)-category of the projective unital Ran space of \(R^n\) \(\text{Exit}(\text{Ran}^u(R^n))\) is the simplicial space over \(\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})\) representing the presheaf on \(\Delta/\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})\) whose value on an object \([p] \rightarrow \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})\)
which selects a diagram of finite sets

\[
\begin{array}{c}
\sigma_n : S_n^p \longrightarrow \cdots \longrightarrow S_n^0 \\
\downarrow \\
\vdots \\
\downarrow \\
\sigma_1 : S_1^p \longrightarrow \cdots \longrightarrow S_1^0
\end{array}
\]

is the space of compatible embeddings

\[
\begin{array}{c}
\text{cylr}(\sigma_n) \xrightarrow{E_n} \mathbb{R}^n \times \Delta^p \\
\downarrow \\
\text{cylr}(\sigma_{n-1}) \xrightarrow{E_{n-1}} \mathbb{R}^{n-1} \times \Delta^p \\
\downarrow \\
\vdots \\
\downarrow \\
\text{cylr}(\sigma_1) \xrightarrow{E_1} \mathbb{R} \times \Delta^p
\end{array}
\]

where each embedding is over \( \Delta^p \) and the downward arrows on the lefthand side are induced by the downward arrows of (7). This embedding space is given the compact-open topology; the structure maps are evident. Observation 8.0.5 defines a canonical functor \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \rightarrow \text{Fin}^{op} \).

**Observation 8.0.3.** Explicitly, an object of \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) over the sequence of finite sets,

\[
S_n \xrightarrow{\tau_{n-1}} S_{n-1} \xrightarrow{\tau_{n-2}} \cdots \rightarrow S_1
\]

is a sequence of embeddings,

\[
\begin{array}{c}
S_n \xrightarrow{\tau_{n-1}} S_{n-1} \xrightarrow{\tau_{n-2}} \cdots \rightarrow S_1 \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
S_1 \xrightarrow{\tau_{n-1}} S_{n-1} \xrightarrow{\tau_{n-2}} \cdots \rightarrow S_1
\end{array}
\]
When the context is clear, we denote (10) by $S \xhookrightarrow{e} \mathbb{R}^n$ or just $e$. A morphism from $S \xhookrightarrow{e} \mathbb{R}^n$ to $T \xhookrightarrow{d} \mathbb{R}^n$ over the diagram of finite sets,

$$
\begin{array}{ccc}
T_n & \xrightarrow{\sigma_n} & S_n \\
\downarrow{\omega_{n-1}} & & \downarrow{\tau_{n-1}} \\
T_{n-1} & \xrightarrow{\sigma_{n-1}} & S_{n-1} \\
\downarrow{\omega_{n-2}} & & \downarrow{\tau_{n-2}} \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
T_1 & \xrightarrow{\sigma_1} & S_1
\end{array}
$$

(11)
is a sequence of embeddings,

$$
\begin{array}{ccc}
cylr(\sigma_n) & \xhookrightarrow{E_n} & \mathbb{R}^n \times \Delta^1 \\
\downarrow & & \downarrow{pr_{\leq n} \times id_{\Delta^1}} \\
cylr(\sigma_{n-1}) & \xhookrightarrow{E_{n-1}} & \mathbb{R}^{n-1} \times \Delta^1 \\
\downarrow & & \downarrow{pr_{\leq n-1} \times id_{\Delta^1}} \\
\vdots & & \vdots \\
cylr(\sigma_1) & \xhookrightarrow{E_1} & \mathbb{R} \times \Delta^1
\end{array}
$$

(12)

over $\Delta^1$ such that $E_{i|S_i} = e_i$ and $E_{i|T_i \times \{1\}} = d_i$, for each $1 \leq i \leq n$. When the context is clear, we denote (12) by $cylr(\sigma) \xhookrightarrow{E} \mathbb{R}^n \times \Delta^1$ or $E$.

Heuristically, a morphism in the exit-path $\infty$-category of the projective unital Ran space of $\mathbb{R}^n$ consists of a morphism in $\text{Exit}(\text{Ran}^u(\mathbb{R}^1))$ for each $1 \leq i \leq n$, the collection of which are compatible under, but not limited to projection.

**Notation 8.0.4.** We denote a point in the $[p]$-space of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ over (11) by

$$
cylr(\sigma) \xhookrightarrow{E} \mathbb{R}^n \times \Delta^p.
$$

**Observation 8.0.5.** For each $1 \leq i \leq n$, there is a natural forgetful functor to finite sets

$$
\phi_i : \text{Exit}(\text{Ran}^u(\mathbb{R}^1)) \longrightarrow \text{Fin}^{op}
$$

that forgets all but the set data at the $\mathbb{R}^i$ level. Its value on an object $S \xhookrightarrow{e} \mathbb{R}^n$ is $S_i$ and on a morphism

$$
cylr(T \xrightarrow{\sigma} S) \xhookrightarrow{E} \mathbb{R}^n \times \Delta^1
$$

from $S \hookrightarrow \mathbb{R}^n$ to $T \hookrightarrow \mathbb{R}^n$ is the map of finite sets

$$
T_i \xrightarrow{\sigma_i} S_i.
$$

The collection of functors $\{\phi_i\}_{i=1}^n$ naturally compile to name the canonical functor

$$
\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \xrightarrow{\Phi_n} \text{Fun}(\{1 < \cdots n\}, \text{Fin}^{op})
$$
which just remembers the underlying set data, i.e., its value on an object $S \hookrightarrow \mathbb{R}^n$ is the functor which selects out the composable sequence of maps of finite sets $S$ and its value on a morphism $\text{cylr}(T \xrightarrow{\sigma} S) \xrightarrow{E_n} \mathbb{R}^n \times \Delta^1$ is the commutative diagram of finite sets $T \xrightarrow{\sigma} S$.

**Observation 8.0.6.** There is a natural forgetful functor

$$F : \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \longrightarrow \text{Exit}(\text{Ran}^u(\mathbb{R}))$$

over $\text{Fin}^{op}$ induced by the functor from $\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{op})$ to $\text{Fin}^{op}$ that evaluates on $\{n\}$.

The value of a $[p]$-value $\text{cylr}(\sigma) \xrightarrow{E_n} \mathbb{R}^n \times \Delta^p$ over (7) is the embedding of $E_n$ at the $\mathbb{R}^n$ level $\text{cylr}(\sigma_n) \xrightarrow{E_n} \mathbb{R}^n \times \Delta^p$ over $\sigma_n : S^n \rightarrow \cdots \rightarrow S^0$.

**Observation 8.0.7.** There is a natural forgetful functor

$$\rho : \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \longrightarrow \text{Exit}(\text{Ran}^u(\mathbb{R}))$$

that forgets all but the first coordinate data. The image of a $[p]$-value $\text{cylr}(\sigma) \xrightarrow{E_n} \mathbb{R}^n \times \Delta^p$ over (7) under $\rho$ is

$$\text{cylr}(\sigma_1) \xrightarrow{E_1} \mathbb{R} \times \Delta^p$$

defined over $\sigma_1 : S^1 \rightarrow \cdots \rightarrow S^0$.

**Observation 8.0.8.** There is a natural functor

$$\pi : \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \longrightarrow \text{Fin}^{op} \text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1}))$$

the value of which on an object $S \xrightarrow{\xi} \mathbb{R}^n$ is

$$S_1(\xi_s) \xrightarrow{\xi_s} \mathbb{R}^{n-1}$$

where for each $s \in S_1$, $(S_s) \xrightarrow{\xi_s} \mathbb{R}^{n-1}$ denotes the object of $\text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1}))$ determined by the compatible sequence of embeddings of pullbacks over $s$:

$$
\begin{array}{ccc}
(S_s) & \xrightarrow{(S_n)_s} & S_n \xrightarrow{\tau_n-1} \mathbb{R}^n \\
\downarrow^{\tau_n} & & \downarrow^{\text{pr}_{<n}} \\
(S_{n-1})_s & \xrightarrow{(S_{n-1})_s} & S_{n-1} \xrightarrow{\tau_{n-1}} \mathbb{R}^{n-1} \\
\downarrow^{\tau_{n-2}} & & \downarrow^{\text{pr}_{<n-1}} \\
\vdots & \vdots & \vdots \\
(S_2)_s & \xrightarrow{(S_2)_s} & S_2 \xrightarrow{\tau_1} \mathbb{R}^2 \\
\downarrow^{\tau_2} & & \downarrow^{\text{pr}_{<2}} \\
\{s\} & \xrightarrow{s} & S_1 \xrightarrow{e_1} \mathbb{R}. \\
\end{array}
$$

(13)

Note that for each $1 \leq i \leq n$, each embedding $(S_i)_s \hookrightarrow S_i \hookrightarrow \mathbb{R}^i$ agrees on its first coordinate and thus, canonically factors through $\mathbb{R}^{i-1}$, which in particular means that (13) yields an object of $\text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1}))$.

The value of $\pi$ on a morphism $\text{cylr}(\sigma) \xrightarrow{E_n} \mathbb{R}^n \times \Delta^1$ is:

i. the morphism $T_1 \xrightarrow{\sigma_1} S_1$ in $\text{Fin}$
ii. for each pair \( r \in T_1, s \in S_1 \) such that \( \sigma_1(r) = s \), the morphism

\[
\begin{array}{ccc}
cyl_r(\sigma_n)|_{\sigma_1 \mapsto s} & \xrightarrow{\partial_j} & cyl(\sigma_n) \xrightarrow{\partial_{\leq n}} \mathbb{R}^n \\
\downarrow & & \downarrow pr_{<n} \\
cyl(\sigma_{n-1})|_{\sigma_1 \mapsto s} & \xrightarrow{\partial_j} & cyl(\sigma_{n-1}) \xrightarrow{\partial_{\leq n-1}} \mathbb{R}^{n-1} \\
\downarrow & & \downarrow pr_{<n-1} \\
\vdots & & \vdots \\
cyl(\sigma_2)|_{\sigma_1 \mapsto s} & \xrightarrow{\partial_j} & cyl(\sigma_2) \xrightarrow{\partial_{\leq 2}} \mathbb{R}^2 \\
\downarrow & & \downarrow pr_{<2} \\
cyl(\sigma_1) \simeq \Delta^1 & \xrightarrow{\partial_j} & cyl(\sigma_1) \xrightarrow{\partial_{\leq 1}} \mathbb{R} \\
\end{array}
\]

in \( \text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1})) \), where for each \( 1 \leq i \leq n \), each embedding

\[\text{cyl}(\sigma_i)|_{\sigma_1 \mapsto s} \hookrightarrow \mathbb{R}^i \times \Delta^1\]

canonically factors through \( \mathbb{R}^{i-1} \times \Delta^1 \) and thus, (14) yields a morphism in \( \text{Exit}(\text{Ran}^u(\mathbb{R}^{n-1})) \).

8.1. Proving \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) is an \( \infty \)-category. This subsection is devoted to proving that the simplicial space \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) is a complete Segal space. We build directly off of Section §7.1, wherein we showed that \( \text{Exit}(\text{Ran}^u(M)) \) is a complete Segal space. Our approach was to witness the simplicial space \( \text{Exit}(\text{Ran}^u(M)) \) as one derived through formal constructions among complete Segal spaces from the complete Segal space \( \text{Bun} \), defined in §6 of [4]. Our approach in this subsection is similar in that we show that \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \), too, can be derived through formal constructions among complete Segal spaces from \( \text{Bun} \).

**Proposition 8.1.1.** The simplicial space \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) satisfies the Segal and completeness conditions.

We build off of §7.1, and freely use notation and results from [4]. Recall the \( \infty \)-category \( \text{Ref}^0(\mathbb{R}) \) defined as the pullback in (6). Heuristically, an object is a refinement of \( \mathbb{R} \) for which the 0-skeleton of the domain is a finite set, and a morphism is a path of such refinements of \( \mathbb{R} \) witnessing anti-collision of strata and disappearances of strata. Observe that \( \text{Ref}^0(\mathbb{R}) \simeq \text{Ar}^\text{ref}(\text{Bun})_{||} \) is equivalent to the \( \infty \)-category of refinement arrows in \( \text{Bun} \) with the target fixed as the trivially stratified space \( \mathbb{R} \). This is because every refinement of \( \mathbb{R} \) has as its 0-skeleton, a finite (possibly empty) set. The equivalence is given by the functor from \( \text{Ar}^\text{ref}(\text{Bun})_{||} \) to \( \text{Ref}^0(\mathbb{R}) \) which forgets the target. Let \( \text{Ref}(\mathbb{R}^n) \) denote the \( \infty \)-category of refinement arrows in \( \text{Bun} \) with the target fixed as \( \mathbb{R}^n \) equipped with the trivial stratification. Explicitly, \( \text{Ref}(\mathbb{R}^n) \) is the simplicial space whose value on \( [p] \in \Delta \) is the moduli space of

- constructible bundles

\[ Y \to \Delta^p \]

- together with a refinement

\[ Y \xrightarrow{\text{refinement}} \mathbb{R}^n \times \Delta^p \]

over \( \Delta^p \).

We will denote such a \([p]\)-point of \( \text{Ref}(\mathbb{R}^n) \) simply as \( (Y \xrightarrow{\text{ref}} \mathbb{R}^n \times \Delta^p) \). Note that \( \mathbb{R}^n \times \Delta^p \) is stratified as a product stratified space, where \( \mathbb{R}^n \) is trivially stratified over the poset consisting of a singleton, and \( \Delta^p \) is given the standard stratification, defined in Example 1.1.4.
For $n \geq 2$, define the functor

$$F_n : \text{Ref}^0(\mathbb{R}) \to \text{Ref}(\mathbb{R}^n)$$

from refinements of $\mathbb{R}$ to refinements of $\mathbb{R}^n$ as follows: The value of a $[p]$-point $(X \xrightarrow{\text{ref}} \mathbb{R} \times \Delta^p)$ under $F_n$ is the refinement of $\mathbb{R}^n \times \Delta^p$ defined as the pullback of stratified spaces

$$F_n(X) \xrightarrow{\alpha} \mathbb{R}^n \times \Delta^p$$

(15)

$$F_n(X) \xrightarrow{\text{ref}} \mathbb{R} \times \Delta^p.$$  

It is straightforward to check that the value $F_n(X)$ is a refinement of $\mathbb{R}^n \times \Delta^p$ by virtue of it being a pullback. Explicitly, $(F_n(X) \to \mathbb{R}^n \times \Delta^p)$ is

$$(\text{ref} \circ \text{pr}_{\leq 2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}^0_{\text{fib}}(X)) \subset \mathbb{R}^n \times \Delta^p \xrightarrow{\text{ref}} \mathbb{R}^n \times \Delta^p$$

which denotes the coarsest refinement of $\mathbb{R}^n \times \Delta^p$ for which the embedding from $$(\text{ref} \circ \text{pr}_{\leq 2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}^0_{\text{fib}}(X))$$ is proper and constructible. As a technicality (that we will use later) define $F_1$ to be the identity on $\text{Ref}^0(\mathbb{R})$. The following figure is a sketch of the values of an object and a morphism under $F_2 : \text{Ref}^0(\mathbb{R}) \to \text{Ref}(\mathbb{R}^2)$.

![Diagram](image)

**Figure 4.** The values of a $[0]$-point and a $[1]$-point in $\text{Ref}^0(\mathbb{R})$ under $F_2$.

Denote the pullback $\infty$-category

$$\overline{\text{Ref}}(\mathbb{R}^n) \xrightarrow{\text{target}} \text{Ar}^\text{ref}(\text{Ref}(\mathbb{R}^n))$$

(16)

$$\text{Ref}^0(\mathbb{R}) \xrightarrow{F_n} \text{Ref}(\mathbb{R}^n)$$

where $\text{Ar}^\text{ref}(\text{Ref}(\mathbb{R}^n))$ is the $\infty$-category of refinement arrows of $\text{Ref}(\mathbb{R}^n)$; that is, it is the full $\infty$-subcategory of the $\infty$-category of arrows of $\text{Ref}(\mathbb{R}^n)$ consisting of the refinement arrows.

We employ the open cylinder construction of [4] (previously referenced in §7.1) to determine that $\overline{\text{Ref}}(\mathbb{R}^n)$ is the following simplicial space: Its value on $[p] \in \Delta$ is the moduli space of
• pairs of constructible bundles
  \(((X \to \Delta^p), (Y \to \Delta^p))\)
  
  • together with a pair of refinements of stratified spaces
  \(((X \xrightarrow{\text{refinement}} \mathbb{R} \times \Delta^p), (Y \xrightarrow{\text{refinement}} F_n(X)))\)
  each of which is over \(\Delta^p\).

To keep in mind that \(Y\) is, in particular, a refinement of \(\mathbb{R}^n \times \Delta^p\), we denote such a \([p]\)-point of \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) by

\[
Y \xrightarrow{\text{ref}} F_n(X) \xrightarrow{\text{ref}} \mathbb{R}^n \times \Delta^p
\]  

The following figure is a sketch of an object \((p = 0)\) and a morphism \((p = 1)\) in \(\widetilde{\text{Ref}}(\mathbb{R}^2)\).

\[\text{Figure 5. A } [0]\text{-point and a } [1]\text{-point in } \widetilde{\text{Ref}}(\mathbb{R}^2).\]
For each such \([p]\)-point of \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) as in (17) above, there is a canonical stratified map of stratified spaces from \(Y\) to \(X\) defined to be the composite of the stratified maps \(Y \xrightarrow{\text{ref}} F_n(X) \xrightarrow{\alpha} X\), where \(\alpha\) is the stratified map in (15), which is given by virtue of \(F_n(X)\) being defined as a pullback. The map of underlying topological spaces is projection onto the first Euclidean coordinate product with the identity on \(\Delta^p\). We denote this stratified projection map \(Y \xrightarrow{\text{pr}\times^2} X\).

**Definition 8.1.2.** For \(n \geq 2\), the \(\infty\)-category \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) is defined inductively on \(n\):
- \(\widetilde{\text{Ref}}(\mathbb{R}^2)\) is the full \(\infty\)-subcategory of \(\widetilde{\text{Ref}}(\mathbb{R}^2)\) consisting of those objects
  \[
  Y \xrightarrow{\text{ref}} F_2(X) \quad \xrightarrow{\text{ref}} \quad \mathbb{R}^2
  \]
  such that
  i. the \((n-1)\)-skeleton of the open cylinder of \((Y \xrightarrow{\text{ref}} F_2(X))\) is a refinement morphism in \(\text{Bun}\).
  ii. the fiber of the stratified projection map \(Y \xrightarrow{\text{pr}\times^2} X\) over each point in the 0-skeleton of \(X\) is an object in \(\widetilde{\text{Ref}}(\mathbb{R}^{n-1})\).

Explicitly, \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) is the simplicial space whose value on \([p] \in \Delta\) is the moduli space of
- pairs of constructible bundles
  \[
  ((X \rightarrow \Delta^p), (Y \rightarrow \Delta^p))
  \]
- together with a pair of refinements among stratified spaces
  \[
  ((X \xrightarrow{\text{refinement}} \mathbb{R} \times \Delta^p), (Y \xrightarrow{\text{refinement}} F_n(X)))
  \]
  each of which is over \(\Delta^p\)
satisfying the conditions
  i. the fiberwise \((n-1)\)-skeleton of the open cylinder of \((Y \xrightarrow{\text{ref}} F_n(X))\) is a refinement morphism in \(\text{Bun}\).
  ii. the fiber of the stratified projection map \(Y \xrightarrow{\text{pr}\times^2} X\) over each point in the 0-skeleton of \(X\) is an object in \(\widetilde{\text{Ref}}(\mathbb{R}^{n-1})\).

We will denote such a \([p]\)-point in \(\widetilde{\text{Ref}}(\mathbb{R}^n)\) as
\[
Y \xrightarrow{\text{ref}} F_n(X) \quad \xrightarrow{\text{ref}} \quad \mathbb{R}^n \times \Delta^p
\]
The following figure is a sketch of an object \((p = 0)\) and a morphism \((p = 1)\) in \(\widetilde{\text{Ref}}(\mathbb{R}^2)\).
Lemma 8.1.3. For each integer \( n \geq 1 \) and \( 1 \leq k \leq n - 1 \), there is a canonical functor
\[
pr_k : \tilde{\text{Ref}}(\mathbb{R}^n) \rightarrow \tilde{\text{Ref}}(\mathbb{R}^k)
\]
induced by projection from \( \mathbb{R}^n \) onto the first \( k \)-coordinates. The functor is given by, for each \( [p] \in \Delta \), assigning to the \([p]\)-point
\[
Y \xrightarrow{\text{ref}} F_n(X)
\]
the refinement
\[
(pr_{<k+1} \times Id_{\Delta^p}(sk_{n-k}^\text{fib}(Y)) \subset \cdots \subset pr_{<k+1} \times Id_{\Delta^p}(sk_{n-1}^\text{fib}(Y)) \subset \mathbb{R}^k \times \Delta^p) \xrightarrow{\text{ref}} F_k(X)
\]
the value of which is the coarsest refinement of \( \mathbb{R}^k \times \Delta^p \) for which the embeddings from \( pr_{<k+1}(sk_i^\text{fib}(Y)) \) into \( \mathbb{R}^k \times \Delta^p \) for each \( n - k \leq i \leq n - 1 \) are proper and constructible.
We denote such a value by

\[
\begin{array}{c}
Y_k \\
\downarrow \text{ref} \\
\mathbb{R}^k \times \Delta^p
\end{array}
\xrightarrow{\text{ref}}
\begin{array}{c}
F_k(X) \\
\downarrow \text{ref} \\
\mathbb{R}^k \times \Delta^p
\end{array}
\]

Before we proceed with the proof, we provide a sketch of the values of an object in \(\tilde{\text{Ref}}(\mathbb{R}^3)\) under \(pr_2\) and \(pr_1\) in the following figure.

![Figure 7. The values of a \([0]\)-point of \(\tilde{\text{Ref}}(\mathbb{R}^3)\) under \(pr_2\) and \(pr_1\).](image)

**Proof of Lemma 8.1.3.** We proceed by induction on \(k\).

\([k = 1]\) : First, recall the definition of \(\tilde{\text{Ref}}(\mathbb{R}^n)\) as the pullback in (16). Observe that the target \(\text{Ref}^0(\mathbb{R})\) of the leftmost vertical functor in (16) is, by definition, \(\text{Ref}(\mathbb{R})\). We claim that \(pr_1\) is precisely this functor upon restricting the domain to \(\tilde{\text{Ref}}(\mathbb{R}^n)\). To verify this claim, first note that the functor in (16), by virtue of \(\tilde{\text{Ref}}(\mathbb{R}^n)\) being a pullback, is given by the assignment, for each \([p] \in \Delta\),

\[
\begin{array}{c}
Y \\
\downarrow \text{ref} \\
\mathbb{R}^n \times \Delta^p
\end{array}
\xrightarrow{\text{ref}}
\begin{array}{c}
F_n(X) \\
\downarrow \text{ref} \\
\mathbb{R}^n \times \Delta^p
\end{array}
\]

Therefore, we must show that the value of such a \([p]\)-point under \(pr_1\),

\[
\begin{array}{c}
Y_1 \\
\downarrow \text{ref} \\
\mathbb{R} \times \Delta^p
\end{array}
\xrightarrow{\text{ref}}
\begin{array}{c}
F_1(X) \\
\downarrow \text{ref} \\
\mathbb{R} \times \Delta^p
\end{array}
\]

is \((X \xrightarrow{\text{ref}} \mathbb{R} \times \Delta^p)\). Recall that (as a technicality) \(F_1\) was previously defined to be the identity on \(\text{Ref}^0(\mathbb{R})\). Thus, we need to verify that \(Y_1\) and \(X\) are equivalent refinements of \(\mathbb{R} \times \Delta^p\). By virtue of being a \([p]\)-point of \(\tilde{\text{Ref}}(\mathbb{R}^n)\), the fiberwise \((n - 1)\)-skeleton of \(Y\) refines the fiberwise \((n - 1)\)-skeleton of \(F_n(X)\). This means that, in particular, the underlying topological spaces of the \((n - 1)\)-skeletons of \(Y\) and \(F_n(X)\) are homeomorphic. Thus, their projections onto the first Euclidean coordinate product with the identity on \(\Delta^p\)

\[
(18) \quad \text{pr}_{<2} \times \text{Id}_{\Delta^p}(\text{sk}_{n-1}^{\text{fib}}(Y)) \cong \text{pr}_{<2}(\text{sk}_{n-1}^{\text{fib}}(F_n(X)))
\]

are homeomorphic subspaces of \(\mathbb{R} \times \Delta^p\) over \(\Delta^p\). This simply means that the fibers of each over the same point in \(\Delta^p\) have the same cardinality.

Previously, we observed that explicitly \(F_n(X)\) is the coarsest refinement

\[
((\text{pr}_{<2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}_{n-1}^{\text{fib}}(X)) \subset \mathbb{R}^n \times \Delta^p)
\]
of $\mathbb{R}^n \times \Delta^p$ for which the embedding from $(\text{pr}_{<2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}^{\text{fib}}_0(X))$ is proper and constructible. This means that the fiberwise $(n - 1)$-skeleton of $F_n(X)$ is $(\text{pr}_{<2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}^{\text{fib}}_0(X))$. Thus, the projection of the $(n - 1)$-skeleton onto the first Euclidean coordinate product with the identity on $\Delta^p$ is simply the fiberwise 0-skeleton of $X$, i.e.,

$$\text{pr}_{<2} \times \text{Id}_{\Delta^p}(\text{sk}^{\text{fib}}_{n-1}(F_n(X))) = \text{sk}^{\text{fib}}_0(X).$$

Therefore, through the previous equivalence (18), the fiberwise 0-skeleton of $X$ is homeomorphic to $\text{pr}_{<2} \times \text{Id}_{\Delta^p}(\text{sk}^{\text{fib}}_{n-1}(Y))$. Upon making the (somewhat trivial) observation that $(X \xrightarrow{\text{ref}} \mathbb{R} \times \Delta^p)$ has the explicit description as the coarsest refinement $(\text{sk}^{\text{fib}}_0(X) \subset \mathbb{R} \times \Delta^p)$ of $\mathbb{R} \times \Delta^p$ for which the embedding from the fiberwise 0-skeleton of $X$ is proper and constructible, we conclude that $Y_1 := (\text{pr}_{<2}(\text{sk}^{\text{fib}}_{n-1}(Y) \subset \mathbb{R} \times \Delta^p))$ is equivalent to $X$ as a refinement of $\mathbb{R} \times \Delta^p$.

**[General case]**: We need to check that the value

$$Y_k \xrightarrow{\text{ref}} F_k(X) \quad \xrightarrow{\text{ref}} \quad \Delta^p$$

is in fact a $[p]$-value in $\overline{\text{Ref}}(\mathbb{R}^k)$. Thus, we must verify three things:

(a) $Y_k$ refines $F_k(X)$,

(b) the $(k - 1)$-skeleton of the open cylinder of $(Y_k \xrightarrow{\text{ref}} F_k(X))$ is a refinement, and

(c) the fiber of the stratified projection map $Y_k \xrightarrow{\text{pr}_{<2}} X$ over each point in the 0-skeleton of $X$ is an object in $\overline{\text{Ref}}(\mathbb{R}^{k-1})$.

(a) Recall that $F_n(X)$ is the coarsest refinement $((\text{pr}_{<2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}^{\text{fib}}_0(X)) \subset \mathbb{R}^n \times \Delta^p)$ of $\mathbb{R}^n \times \Delta^p$ for which the embedding from $(\text{pr}_{<2} \times \text{Id}_{\Delta^p})^{-1}(\text{sk}^{\text{fib}}_0(X))$ is proper and constructible. Projection of $F_n(X)$ onto its first $k$ Euclidean coordinates (product with the identity on $\Delta^p$) yields an explicit description

$$F_k(X) = (\text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}^{\text{fib}}_{n-1}(F_n(X)))) \subset \mathbb{R}^k \times \Delta^p$$

which denotes the coarsest refinement of $\mathbb{R}^k \times \Delta^p$ for which the embedding from $\text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}^{\text{fib}}_{n-1}(F_n(X)))$ is proper and constructible. By definition of a $[p]$-point of $\overline{\text{Ref}}(\mathbb{R}^n)$,

$$Y \xrightarrow{\text{ref}} F_n(X) \quad \xrightarrow{\text{ref}} \quad \mathbb{R}^n \times \Delta^p$$

satisfies that the fiberwise $(n - 1)$-skeleton of $Y$ is a refinement of the fiberwise $(n - 1)$-skeleton of $F_n(X)$. In particular then, the underlying topological spaces of the $(n - 1)$-skeletons are homeomorphic. Thus, so are their projections onto the first $k$ Euclidean coordinates (product with the identity on $\Delta^p$), i.e.,

$$\text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}^{\text{fib}}_{n-1}(Y)) \cong \text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}^{\text{fib}}_{n-1}(F_n(X))).$$

By definition, $Y_k$ refines the coarsest refinement $((\text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}^{\text{fib}}_{n-1}(Y)) \subset \mathbb{R}^k \times \Delta^p)$ of $\mathbb{R}^k \times \Delta^p$ for which the embedding from $\text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}^{\text{fib}}_{n-1}(Y))$ is proper and constructible. Thus, through the equivalences (20) and (21) above, we conclude that $Y_k$ refines $F_k(X)$.

(b) By definition, the fiberwise $(k - 1)$-skeleton of $Y_k$ is a refinement of $\text{pr}_{<k+1} \times \text{Id}_{\Delta^p}(\text{sk}^{\text{fib}}_{n-1}(Y))$. Through the equivalences (20) and (21) above, then, the fiberwise $(k - 1)$-skeleton of $Y_k$ refines the fiberwise $(k - 1)$-skeleton of $F_k(X)$. That is to say, the refinement $(Y_k \xrightarrow{\text{ref}} F_k(X))$ of stratified
spaces restricts to a refinement \((\text{sk}_{k-1}(Y_k) \xrightarrow{\text{ref}} \text{sk}_{k-1}(F_k(X)))\) of stratified spaces between the \((k-1)\)-skeletons. Thus, the \((k-1)\)-skeleton of the open cylinder of \((Y_k \xrightarrow{\text{ref}} F_k(X))\) is a refinement morphism in \(\mathcal{B}un\).

(c) The fiber of \(Y_k \xrightarrow{\text{pr}} X\) over a point in the 0-skeleton of \(X\) can, by definition of \(Y_k\), be described in terms of projection of the fiber of \(Y \xrightarrow{\text{pr}} X\) over the same point as follows: Let \(x\) be a point in the 0-skeleton of \(X\) and let \(Y_x\) denote the fiber over \(x\) in \(Y\). The fiber over \(x\) in \(Y_k\) is

\[
(Y_k)_x := (\text{pr}_{<k} \times \text{Id}_{\Delta^p}(\text{sk}_n^{\text{fib}}(Y_x))) \subset \cdots \subset \text{pr}_{<k} \times \text{Id}_{\Delta^p}(\text{sk}_{n-2}^{\text{fib}}(Y_x)) \subset \mathbb{R}^{k-1}
\]

which denotes the coarsest refinement of \(\mathbb{R}^{k-1}\) for which the embedding from \(\text{pr}_{<k} \times \text{Id}_{\Delta^p}(\text{sk}_i(Y_x))\) into \(\mathbb{R}^{k-1}\), for each \(n-k \leq i \leq n-2\), is proper and constructible. By definition, \((Y_k)_x\) is the value of \(Y_x\) under the projection functor \(\overline{\text{Ref}}(\mathbb{R}^{n-1}) \xrightarrow{\text{pr}_{k-1}} \overline{\text{Ref}}(\mathbb{R}^{k-1})\), which exists by the inductive step, and therefore identifies \((Y_k)_x\) as an object in \(\overline{\text{Ref}}(\mathbb{R}^{k-1})\), as desired.

\[\square\]

**Observation 8.1.4.** There is a natural functor

\[
\overline{\text{Ref}}(\mathbb{R}^n) \rightarrow \text{Fun}(\{1 \ldots < n\}, \text{Fin}^{\text{op}})
\]

the value of which on an object

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{ref}} & F_n(X) \\
& \text{ref} \downarrow & \text{ref} \\
& \mathbb{R}^n & \\
\end{array}
\]

is the functor from \(\{1 \ldots < n\}\) to \(\text{Fin}^{\text{op}}\) that selects out the sequence

\[
\text{sk}_0^{\text{fib}}(Y) \xrightarrow{\text{pr}_{<n}} \text{sk}_0^{\text{fib}}(Y_{n-1}) \rightarrow \cdots \xrightarrow{\text{pr}_{<2}} \text{sk}_0^{\text{fib}}(X)
\]

of finite sets.

The value on a morphism

\[
\begin{array}{ccc}
Y' & \xrightarrow{\text{ref}} & F_n(X) \\
& \text{ref} \downarrow & \text{ref} \\
& \mathbb{R}^n \times \Delta^1 & \\
\end{array}
\]

is given by selecting out the diagram

\[
\begin{array}{ccc}
\text{sk}_0^{\text{fib}}(Y_{[n]}) & \rightarrow & \text{sk}_0^{\text{fib}}(Y_{[n-1]}) \\
\text{sk}_0^{\text{fib}}((Y_{n-1})_{[n-1]}) & \rightarrow & \text{sk}_0^{\text{fib}}((Y_{n-1})_{n-1}) \\
\text{sk}_0^{\text{fib}}((X)_{[n-1]}) & \rightarrow & \text{sk}_0^{\text{fib}}((X)_{[n-1]}) \\
\vdots & & \vdots \\
\text{sk}_0^{\text{fib}}((X)_{[1]}) & \rightarrow & \text{sk}_0^{\text{fib}}((X)_{[1]}) \\
\end{array}
\]

among finite sets, where, for each \(1 \leq k \leq n\), the horizontal arrow is from the \((0)\)-skeleton of the fiber of \(Y_k \rightarrow \Delta^1\) over \(\{1\} \in \Delta^1\) to the \((0)\)-skeleton of the fiber of \(Y_k \rightarrow \Delta^p\) over \(\{0\} \in \Delta^1\), and is a canonical map of sets implemented by taking connected components of the fiberwise \((0)\)-skeleton of \(X\).
**Lemma 8.1.5.** There is a canonical equivalence between simplicial spaces over \( \text{Fun}(\{1 \cdots n\}, \text{Fin}^{\text{op}}) \):

\[
\tilde{\text{Ref}}(\mathbb{R}^n) \simeq \text{Exit}(\text{Ran}^u(\mathbb{R}^n)).
\]

**Proof.** A rightward morphism is implemented by, for each \([p] \in \Delta\), the assignment,

\[
\begin{array}{cccccc}
\text{sk}_0(Y) & \rightarrow & Y & \rightarrow & \mathbb{R}^n \times \Delta^p \\
\downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} \\
F_n(X) & \rightarrow & \text{sk}_0(Y_{n-1}) & \rightarrow & \mathbb{R}^{n-1} \times \Delta^p \\
\downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} \\
\mathbb{R}^n \times \Delta^p & \rightarrow & \cdots & \rightarrow & \mathbb{R} \times \Delta^p
\end{array}
\]

whose value is the sequence of embeddings over \( \Delta^p \), each of which is from the fiberwise \((0)\)-skeleton of the value \( Y_k \) of \( Y \) under the functor \( \text{pr}_k \) (8.1.3), which maps to \( \Delta^p \) as a finite proper constructible bundle.

A leftward morphism is given by assigning to each \([p]\)-point

\[
\begin{array}{cccccc}
\text{cyl}_r(\sigma_n) & \rightarrow & \mathbb{R}^n \times \Delta^p \\
\downarrow \text{id} & & \downarrow \text{id} \\
\text{cyl}_r(\sigma_{n-1}) & \rightarrow & \mathbb{R}^{n-1} \times \Delta^p \\
\downarrow \text{id} & & \downarrow \text{id} \\
\vdots & & \vdots \\
\text{cyl}_r(\sigma_1) & \rightarrow & \mathbb{R} \times \Delta^p
\end{array}
\]

the refinement

\[
(\text{cyl}_r(\sigma_n) \subset (\text{pr}_{<n} \times \text{id}_{\Delta^p})^{-1}(\text{cyl}_r(\sigma_{n-1})) \subset \cdots \subset (\text{pr}_{<2} \times \text{id}_{\Delta^p})^{-1}(\text{cyl}_r(\sigma_1)) \subset \mathbb{R}^n \times \Delta^p)
\]

whose value is the coarsest refinement of \( \mathbb{R}^n \times \Delta^p \) for which the embeddings from \( \text{cyl}_r(\sigma_n) \) and each prevalue \( (\text{pr}_{<i} \times \text{id}_{\Delta^p})^{-1}(\text{cyl}_r(\sigma_{i-1})) \) for \( 2 \leq i \leq n \) are proper and constructible. (Such a refinement exists because the value of each embedding is, by definition, a properly embedded stratified subspace.)

It is straightforward to verify that these two assignments are mutually inverse to one another, and further, that they are over \( \text{Fun}(\{1 \cdots n\}, \text{Fin}^{\text{op}}) \). Furthermore, it is evident that the structure maps are equivalent.

\[ \square \]

**Proof of Proposition 8.1.1.** The simplicial space \( \tilde{\text{Ref}}(\mathbb{R}^n) \) is a pullback of complete Segal spaces and is therefore also a complete Segal space, since the full \( \infty \)-subcategory of simplicial spaces consisting of the complete Segal spaces is closed under formation of pullbacks. By virtue of \( \tilde{\text{Ref}}(\mathbb{R}^n) \) being a full sub-simplicial space of \( \tilde{\text{Ref}}(\mathbb{R}^n) \), it too satisfies the Segal and completeness conditions. Through the equivalence of Lemma 8.1.5, then so does the simplicial space \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \).

\[ \square \]
8.2. Proving $\Theta^\text{act}_n$ is equivalent to $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$. The main goal of this section is to prove Lemma 8.2.2, which states that the following functor is an equivalence of $\infty$-categories.

**Lemma 8.2.1.** For $n \geq 1$, there is a functor $G_n : \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \to \Theta^\text{act}_n$ over $\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})$, from the exit-path $\infty$-category of the projective unital Ran space of $\mathbb{R}^n$ to the subcategory of active morphisms of $\Theta_n$.

**Proof.** We proceed by induction on $n$.

[$n = 1$]: We seek to define a functor $G_1$ over $\text{Fin}^{\text{op}}$:

$$
\begin{array}{ccc}
\text{Exit}(\text{Ran}^u(\mathbb{R})) & \xrightarrow{G_1} & \Delta^\text{act} \\
\phi_1 & \downarrow & \downarrow \gamma_1 \\
\text{Fin}^{\text{op}}.
\end{array}
$$

A functor from an $\infty$-category to the nerve of a (small) category is completely determined by its assignment on objects, morphisms and the requirement that composition is respected. This is due to the fact that the nerve of a small category is completely determined by its values on $[i]$ for $0 \leq i \leq 2$. See the proof of Lemma 3.5 in [18] for more details. Thus, since $\Delta^\text{act}$ is an ordinary category, we will simply define $G_1$ on objects and morphisms and check that composition is respected.

Let $S \hookrightarrow \mathbb{R}$ be an object in $\text{Exit}(\text{Ran}^u(\mathbb{R}))$. The value of $G_1$ on $e$ is the linearly ordered set of connected components of the complement of $e(S)$ in $\mathbb{R}$

$$
G_1 : e \mapsto \pi_0(\mathbb{R} - e(S))
$$

the linear order of which is inherited from the linear order on $\mathbb{R}$.

Let $\text{cylr}(T \xrightarrow{\sigma} S) \hookrightarrow E \xrightarrow{E} \mathbb{R} \times \Delta^1$ be a morphism from $S \xrightarrow{c} \mathbb{R}$ to $T \xrightarrow{\sigma} \mathbb{R}$ in $\text{Exit}(\text{Ran}^u(\mathbb{R}))$. Let $C_E$ denote the compliment of the image of the embedding of $E$,

$$
C_E := (\mathbb{R} \times \Delta^1) - E(\text{cylr}(\sigma)).
$$

Before we name the value of $G_1$ on $E$, we make three observations:

1. Consider the inclusion $\iota_1 : (\mathbb{R} - d(T)) \hookrightarrow C_E$ given by $x \mapsto (x, \{1\})$. Taking connected components induces an inclusion of sets

$$
\pi_0(\iota_1) : \pi_0(\mathbb{R} - d(T)) \hookrightarrow \pi_0(C_E).
$$

It is easy to see that $\pi_0(\iota_1)$ is, in particular, a bijection. We denote its inverse $\pi_0(\iota_1)^{-1}$.

2. Taking connected components of the inclusion $\iota_0 : (\mathbb{R} - e(S)) \hookrightarrow C_E$ given by $x \mapsto (x, \{0\})$ induces a map between sets

$$
\pi_0(\iota_0) : \pi_0(\mathbb{R} - e(S)) \hookrightarrow \pi_0(C_E).
$$

Note that $\pi_0(\iota_0)$ is not necessarily injective nor surjective because $\sigma$ is not necessarily injective nor surjective.

3. $\pi_0(\iota_1)$ determines a linear order on $\pi_0(C_E)$ and thus, $\pi_0(C_E)$ is an object in $\Delta$.

Then, the value of $G_1$ on $\text{cylr}(T \xrightarrow{\sigma} S) \hookrightarrow E \xrightarrow{E} \mathbb{R} \times \Delta^1$ is the composite

$$
\begin{array}{ccc}
\pi_0(\mathbb{R} - e(S)) & \xrightarrow{\pi_0(\iota_0)} & \pi_0(C_E) \\
\downarrow G_1(E) & & \downarrow \pi_0(\iota_1)^{-1} \\
\pi_0(\mathbb{R} - d(T)) & & \pi_0(C_E)
\end{array}
$$

in $\Delta^\text{act}$. It must be checked that $G_1(E)$ is linear and active. We do this by verifying that each morphism in the composite is linear and active. $\pi_0(\iota_1)^{-1}$ is a linear map because it defines the
linear order of \(\pi_\omega(C_E)\). Bijectivity of \(\pi_\omega(\iota_1)^{-1}\) implies that it is active. Similarly, it is easy to see that \(\pi_\omega(\iota_0)\) is order-preserving and sends unbounded components to unbounded components thereby being active.

Next, we will show that \(G_1\) respects composition by showing that the diagram of \(\infty\)-categories (23) commutes on the level of objects and morphisms:

\[
\begin{array}{c}
\text{Exit} \left( \text{Ran}^\omega(\mathbb{R}) \right) \xrightarrow{G_1} \Delta^{\text{act}} \xrightarrow{\gamma_1} \text{Fin}^{\text{op}}.
\end{array}
\]

Indeed, if (23) commutes, then faithfulness of \(\gamma_1\) together with functoriality of \(\phi_1\) guarantee that \(G_1\) respects composition. More precisely, let

\[
\begin{array}{ccc}
a & \xrightarrow{f} & b \\
\downarrow{h} & & \downarrow{g} \\
c & \xrightarrow{j} & d
\end{array}
\]

denote a commutative triangle in \(\text{Exit} \left( \text{Ran}^\omega(\mathbb{R}) \right)\). We will show that if (23) commutes, then \(G_1\) carries the composite \(h\) in (24) to \(G_1(g) \circ G_1(f)\). First, note that functoriality of \(\phi_1\) implies that \(\phi_1(h) = \phi_1(g) \circ \phi_1(f)\). Commutativity of (23) guarantees that the morphisms

\[
\gamma_1(G_1(h)), \gamma_1(G_1(f)) \text{ and } \gamma_1(G_1(g))
\]

are equivalent (upto composition with canonical isomorphisms) to

\[
\phi_1(h), \phi_1(f) \text{ and } \phi_1(g)
\]

respectively, in \(\text{Fin}\). Thus,

\[
\gamma_1(G_1(h)) = \gamma_1(G_1(g)) \circ \gamma_1(G_1(f)) = \gamma_1(G_1(g) \circ G_1(f)).
\]

Then, faithfulness of \(\gamma_1\) guarantees that \(G_1(h) = G_1(g) \circ G_1(f)\), as desired.

Now, we verify commutativity of (23) on objects and morphisms. Let \(S \xrightarrow{c} \mathbb{R}\) be an object in \(\text{Exit} \left( \text{Ran}^\omega(\mathbb{R}) \right)\). There is a canonical bijection of sets

\[
\gamma_1(G_1(e)) \xrightarrow{\cong} \phi_1(e) := S
\]

in \(\text{Fin}\) given by

\[
(\pi_\omega(\mathbb{R} - e(S)) \xrightarrow{\alpha} [1]) \mapsto \inf\{x \in \Pi_{U \in \alpha^{-1}(1)} U\}
\]

verifying commutativity of (23) on objects.

Let \(cylr(T \xrightarrow{c} S) \xrightarrow{E} \mathbb{R} \times \Delta^1\) be a morphism from \(S \xrightarrow{c} \mathbb{R}\) to \(T \xrightarrow{d} \mathbb{R}\) in \(\text{Exit} \left( \text{Ran}^\omega(\mathbb{R}) \right)\). We consider the canonical bijections of the source and target of \(\gamma_1(G_1(E))\), and the corresponding composite, \(\alpha\), in \(\text{Fin}\) from \(T\) to \(S\):

\[
\begin{array}{c}
\gamma_1(G_1(d)) \xrightarrow{\gamma_1(G_1(E))} \gamma_1(G_1(e)) \\
\uparrow{\cong} & & \uparrow{\cong} \\
T & \xrightarrow{\alpha} & S.
\end{array}
\]

By definition, the value of \(\alpha\) on \(r \in T\) is

\[
\alpha(r) := \inf\{x \in \Pi_{U \in S'} U\},
\]

where \(S' := \{U \in \pi_0(\mathbb{R} - e(S))| \inf\{y \in G_1(E)(U)\} \geq r\}\). The composite \(\alpha\) agrees with \(\phi_1(E) := \sigma\), as desired. Indeed, if \(U \in S'\), then

\[
\inf\{x \in U\} = \sigma(r) \text{ or } \inf\{x \in U\} = \sigma(r'),
\]

for some \(r' > r\). But \(\sigma(r') \geq \sigma(r)\) whenever \(r' > r\), which implies

\[
\inf\{x \in U_{U \in S'} U\} = \sigma(r).
\]
In summary, we have just shown that (23) commutes on objects and morphisms, which, as previously argued, implies that \( G_1 \) respects composition. Therefore, \( G_1 \) is a functor, and moreover is defined naturally over \( \text{Fin}^{\text{op}} \).

[General case]: In the inductive step, we assume the existence of a functor over \( \text{Fun}\left(\{1 < \cdots < n-1\}, \text{Fin}^{\text{op}}\right) \)

\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}^{u}(\mathbb{R}^{n-1})) & \xrightarrow{G_{n-1}} & \Theta_{n-1}^{\text{act}} \\
\phi_{n-1} & \downarrow & \tau_{n-1} \\
\text{Fun}(\{1 < \cdots < n-1\}, \text{Fin}^{\text{op}}) & \xrightarrow{} & \\
\end{array}
\]

(27)

In particular, this implies that \( G_{n-1} \) is over \( \text{Fin}^{\text{op}} \) for each \( 1 \leq i \leq n-1 \); i.e., the following diagram commutes

\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}^{u}(\mathbb{R}^{n-1})) & \xrightarrow{G_{n-1}} & \Theta_{n-1}^{\text{act}} \\
\phi_{i} & \downarrow & \tau_{i} \\
\text{Fin}^{\text{op}}(\Theta_{n-1}^{\text{act}}) & \xrightarrow{} & \\
\end{array}
\]

for each \( 1 \leq i \leq n-1 \), where, recall that \( \tau_{i} \) denotes the \((n-1-i)\)-fold self-composite of the truncation map \( \tau \) defined in Observation 6.1.5.

We define the functor \( \text{Exit}(\text{Ran}^{u}(\mathbb{R}^{n})) \xrightarrow{G_n} \Theta_n^{\text{act}} \) by defining \( \Psi \) and \( \Gamma \) such that

\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}^{u}(\mathbb{R}^{n})) & \xrightarrow{G_n} & \Theta_n^{\text{act}} \\
\phi_{n} & \downarrow & \gamma_{n} \\
\text{Fin}^{\text{op}}(\Theta_{n-1}^{\text{act}}) & \xrightarrow{} & \\
\end{array}
\]

\( \Gamma \) is defined to be the composite of the forgetful functor \( \rho \) followed by \( G_1 \circ \rho \), where \( \rho \) is the functor defined in Observation 8.0.7 which forgets all but the first coordinate data.

\( \Psi \) is defined to be the composite of the functor \( \text{Exit}(\text{Ran}^{u}(\mathbb{R}^{n})) \xrightarrow{\pi} \text{Fin}^{\text{op}} \circ \text{Exit}(\text{Ran}^{u}(\mathbb{R}^{n-1})) \)

which was defined in Observation 8.0.8, followed by the functor \( \text{Fin}^{\text{op}} \circ \text{Exit}(\text{Ran}^{u}(\mathbb{R}^{n-1})) \rightarrow \text{Fin}^{\text{op}} \circ \Theta_{n-1}^{\text{act}} \)

determined by the identity on \( \text{Fin}^{\text{op}} \) and the functor \( G_{n-1} \) given by the inductive step. Thus, \( G_n \) is a well-defined functor.

Unwinding the above definition of \( G_n \), an inductive description of \( G_n \) is apparent. We explicate this inductive description on objects and morphisms: Let \( S \xrightarrow{\xi} \mathbb{R}^{n} \) be an object in \( \text{Exit}(\text{Ran}^{u}(\mathbb{R}^{n})) \).

Its value under \( G_n \) is inductively defined as

\[
G_n(S \xrightarrow{\xi} \mathbb{R}^{n}) := G_1(S_1 \xrightarrow{\xi_1} \mathbb{R})(G_{n-1}((S)_s \xrightarrow{\xi_s} \mathbb{R}^{n-1}))
\]

where \( (S)_s \xrightarrow{\xi_s} \mathbb{R}^{n-1} \) denotes the object of \( \text{Exit}(\text{Ran}^{u}(\mathbb{R}^{n-1})) \) determined by (13).

Let \( \text{cylr}(T \xrightarrow{d} S) \xrightarrow{\xi} \mathbb{R}^{n} \times \Delta^1 \) be a morphism from \( S \xrightarrow{\xi} \mathbb{R}^{n} \) to \( T \xrightarrow{d} \mathbb{R}^{n} \) in \( \text{Exit}(\text{Ran}^{u}(\mathbb{R}^{n})) \). Its value under \( G_n \) is inductively defined by:
i) the morphism \( G_1(S_1 \hookrightarrow \mathbb{R}) \) \( \Phi_i^{cylr(\sigma_1)}(d_1 \times \Delta^1) \rightarrow G_1(T_1 \hookrightarrow \mathbb{R}) \) in \( \Delta^{act} \)

ii) for each pair \((t \in T_1, s \in S_1)\) such that \(\sigma_1(t) = s\), the morphism given by the image of (14) under \(G_{n-1} \in \Theta_{n-1}^{act}\).

Next, we will show that for each \(1 \leq i \leq n\),

\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}^n(\mathbb{R})) & \xrightarrow{G_n} & \Theta_n^{act} \\
\downarrow & & \downarrow \gamma_i \\
\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{op}) & \phi_i & \Phi_n \\
\end{array}
\]

(28)

For the cases \(1 \leq i \leq n - 1\), this diagram follows by the inductive step wherein we assume commutativity of (27). For the remaining case, \(i = n\), we use the inductive definitions of \(G_n\) and \(\gamma_n\) in terms of \(G_1\) and \(G_{n-1}\), and \(\gamma_1\) and \(\gamma_{n-1}\), respectively. Then, indeed, in employing the commutativity of (23) and (27) for \(i = n - 1\), we see that for the case \(i = n\), (28) must commute. Through Observation 8.0.5 wherein the functor \(\Phi_n\) was defined in terms of \(\phi_i\) for \(1 \leq i \leq n\), commutativity of this diagram for each \(1 \leq i \leq n\) compiles to prove that \(G_n\) is over \(\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{op})\),

\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}^n(\mathbb{R})) & \xrightarrow{G_n} & \Theta_n^{act} \\
\downarrow & & \downarrow \gamma_n \\
\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{op}) & \phi_n & \Phi_n \\
\end{array}
\]

\[\Box\]

**Lemma 8.2.2.** For each \(n \geq 1\), the functor \(G_n : \text{Exit}(\text{Ran}^n(\mathbb{R})) \xrightarrow{\sim} \Theta_n^{act}\) over \(\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{op})\) from Lemma 8.2.1 is an equivalence of \(\infty\)-categories.

**Proof.** We proceed by induction on \(n\).

\([n = 1]\) : We will show that \(G_1\) is essentially surjective and fully faithful; the former follows easily:

Let \([p] \in \Delta^{act}\). Define the set \(T_p := \{1,2,\ldots,p\}\) together with the object \(T_p \hookrightarrow \mathbb{R}\) in \(\text{Exit}(\text{Ran}^p(\mathbb{R}))\), given by \(i \mapsto i\). Then, \([p]\) is isomorphic to \(G_1(T_p) := \pi_0(\mathbb{R} - d(T_p))\) in \(\Delta\), with the isomorphism given by \(i \mapsto \lfloor i + \frac{1}{2} \rfloor\).

Fix a pair of objects \(S \hookrightarrow \mathbb{R}\) and \(T \hookrightarrow \mathbb{R}\) in \(\text{Exit}(\text{Ran}^n(\mathbb{R}))\). Showing fully faithfulness of \(G_1\) amounts to showing that the map induced by \(G_1\) between corresponding hom-spaces

\[
\text{Hom}_{\text{Exit}(\text{Ran}^n(\mathbb{R}))}(e, d) \xrightarrow{G_1} \text{Hom}_{\Delta^{act}}(\pi_0(\mathbb{R} - e(S)), \pi_0(\mathbb{R} - d(T)))
\]

is a surjection on connected components with contractible fibers.

Fix a morphism \(\pi_0(\mathbb{R} - e(S)) \xrightarrow{\sim} \pi_0(\mathbb{R} - d(T))\) in \(\Delta^{act}\). Any morphism

\[
\text{cylr}(T) \xrightarrow{\gamma_1} S \xrightarrow{E} \mathbb{R} \times \Delta^1
\]

in \(\text{Hom}_{\text{Exit}(\text{Ran}^n(\mathbb{R}))}(e, d)\) is in the fiber of \(G_1\) over \(\varphi\). Indeed, we observed in Observation 6.0.9 that \(\gamma_1\) is injective on hom-sets. Thus, commutativity of (23) guarantees that \(E\) is in the fiber of \(G_1\) over \(\varphi\). Hence, (29) is a surjection on connected components.

The fiber of (29) over \(\varphi\) is the topological space of embeddings \(\text{cylr}(\gamma_1 \circ \varphi) \xrightarrow{E} \mathbb{R} \times \Delta^1\) over \(\Delta^1\) such that \(E|_S = e\) and \(E|_{\{0\} \times \{1\}} = d\), which we denote by

\[
G_1^{-1}(\varphi) \cong \text{Emb}_{\Delta^1}(\text{cylr}(\gamma_1(\varphi)), \mathbb{R} \times \Delta^1)
\]
under the compact-open topology. We will show that this space is contractible. Fix an embedding \( \tilde{E} \) in the fiber of \( G_1 \) over \( \varphi \). Let \( \mathbb{S}^k \xrightarrow{\psi} G^{-1}_1(\varphi) \) be continuous and based at \( \tilde{E} \). We construct a null-homotopy of \( \psi \). For each \( z \in \mathbb{S}^n \), denote the image of \( z \) under \( \psi \) by \( \psi_z \). The straight-line homotopy, \( H_z \), from \( \psi_z \) to \( \tilde{E} \) defined by

\[
H_z(x, t) = (1 - t)\psi_z(x) + t\tilde{E}(x)
\]

defines a path from \( \psi_z \) to \( \tilde{E} \) in \( G^{-1}_1(\varphi) \). For each \( z \in \mathbb{S}^k \), we let each path \( H_z \) run simultaneously to name a null-homotopy of \( \psi \) to the constant path at \( \{ \tilde{E} \} \). Explicitly, the null-homotopy \( \mathbb{S}^k \times [0, 1] \rightarrow G^{-1}_1(\varphi) \) is given by \( (z, t) \mapsto H_z(-, t) \).

[General case]: We will show that \( G_n \) is essentially surjective and fully faithful. Let \( [k]|T_s \) be an object in \( \Theta_{n^2}^{\text{act}} \). Because \( G_1 \) is essentially surjective, we may choose an object of \( \text{Exit}(\text{Ran}^n(\mathbb{R})) \) that is in the fiber of \( G_1 \) over \( [k] \):

\[
(1, \ldots, k) \xrightarrow{e} \mathbb{R}.
\]

Likewise, by essential surjectivity of \( G_{n-1} \), for each \( s \in \{1, \ldots, k\} \), we may choose an object of \( \text{Exit}(\text{Ran}^n(\mathbb{R}^{n-1})) \) that is in the fiber of \( G_{n-1} \) over \( T_s \):

\[
\begin{array}{c}
(S_{n-1})_s \xrightarrow{(e_{n-1})_s} \mathbb{R}^{n-1} \\
\downarrow \quad \downarrow \\
(S_{n-2})_s \xrightarrow{(e_{n-2})_s} \mathbb{R}^{n-2} \\
\downarrow \quad \downarrow \\
(S_{n-3})_s \xrightarrow{(e_{n-3})_s} \mathbb{R}^{n-3} \\
\downarrow \quad \downarrow \\
\vdots \quad \vdots \\
\downarrow \quad \downarrow \\
(S_1)_s \xrightarrow{(e_1)_s} \mathbb{R}.
\end{array}
\]

The choices (31) and (32) for each \( s \), uniquely determine an object of \( \text{Exit}(\text{Ran}^n(\mathbb{R})) \) that is in the fiber of \( G_n \) over \([k]|T_s\):

\[
\begin{array}{c}
\Pi_{1 \leq s \leq k} (S_{n-1})_s \xrightarrow{\Pi(e(s)) \times (e_{n-1})} \mathbb{R} \times \mathbb{R}^{n-1} \\
\downarrow \quad \downarrow \\
\Pi_{1 \leq s \leq k} (S_{n-2})_s \xrightarrow{\Pi(e(s)) \times (e_{n-2})} \mathbb{R} \times \mathbb{R}^{n-2} \\
\downarrow \quad \downarrow \\
\Pi_{1 \leq s \leq k} (S_{n-3})_s \xrightarrow{\Pi(e(s)) \times (e_{n-3})} \mathbb{R} \times \mathbb{R}^{n-3} \\
\downarrow \quad \downarrow \\
\vdots \quad \vdots \\
\downarrow \quad \downarrow \\
\Pi_{1 \leq s \leq k} (S_1)_s \xrightarrow{\Pi(e(s)) \times (e_1)} \mathbb{R} \times \mathbb{R} \\
\downarrow \quad \downarrow \\
\{1, \ldots, k\} \xrightarrow{e} \mathbb{R}
\end{array}
\]

where each map defined in terms of a coproduct is indexed over \( 1 \leq s \leq k \), and \( \{e(s)\} \) and \( \{s\} \) denote the constant maps at \( e(s) \) and \( s \), respectively.

Fix a pair of objects \( T \xrightarrow{\delta} \mathbb{R}^n \) and \( S \xrightarrow{\epsilon} \mathbb{R}^n \) in \( \text{Exit}(\text{Ran}^n(\mathbb{R})) \). We will show fully faithfulness of \( G_n \) by showing that the map induced by \( G_n \) between hom-spaces

\[
\text{Hom}_{\text{Exit}(\text{Ran}^n(\mathbb{R}))}(\epsilon, \delta) \xrightarrow{G_n} \text{Hom}_{\Theta_n^{\text{act}}}(G_n(\epsilon), G_n(\delta))
\]
is a surjection on connected components with contractible fibers.

Fix a morphism 
\[ G_n(e) \rightarrow G_n(d) \]
in \( \Theta_{n}^{\text{act}} \). Using the inductive description of \( G_n \), \( \varphi \) is given by:

i) a morphism \( G_1(S_1 \xrightarrow{e_1} \mathbb{R}) \rightarrow G_1(T_1 \xrightarrow{d_1} \mathbb{R}) \) in \( \Delta_{\text{act}} \)

\[ \xymatrix{ G_{n-1}((S)_{s} \ar[r]^-{\varphi_{r}} & \mathbb{R}^{n-1}) \ar[d]^-{pr_{\leq n-1} \times \text{id}_{\Delta^1}} \ar[r]^-{\delta_{(T)r}} & G_{n-1}((T)_{r} \ar[r]^-{d_{(T)r}} & \mathbb{R}^{n-1}) } \]

Using the basecase and inductive step, we define a morphism that is in the fiber of \( G_n \) over \( \varphi \): By fullness of \( G_1 \), we may choose a morphism in the fiber of \( G_1 \) over \( \varphi_1 \),

\[ \text{cyl}_{r}(\gamma_1(\varphi_1)) \rightarrow \mathbb{R} \times \Delta^1 \]

which is defined over the map of finite sets \( T_1 \rightarrow S_1 \).

By fullness of \( G_n \) as assumed in the inductive step, for each pair \((r \in T_1, s \in S_1)\) such that \( \gamma_1(\varphi_1(r)) = s \), we may choose a morphism in the fiber of \( G_{n-1} \) over \( \varphi_{r} \),

\[ \xymatrix{ \text{cyl}_{r}(\gamma_{n-1} \circ \varphi_{r}) \ar[r]^-{(E_{n})_{r}} & \mathbb{R}^{n-1} \times \Delta^1 } \]

Note that (27) guarantees that (36) must be defined over the diagram of finite sets,

\[ \xymatrix{ (T_n)_{r} \ar[r]^-{\gamma_{n-1} \circ \varphi_{r}} & (S_n)_{s} \ar[d]^-{r_{n-1}_*} \ar[d]^-{r_{n-1}_*} \\
\omega_{n-1}_* \ar[d] & (T_{n-1})_{r} \ar[r]^-{\gamma_{n-2} \circ \text{tr}_{n-1} \circ \varphi_{r}} & (S_{n-1})_{s} \ar[d]^-{r_{n-2}_*} \ar[d]^-{r_{n-2}_*} \\
\omega_{n-2}_* \ar[d] & \vdots & \vdots \ar[d] \\
(T_2)_{r} \ar[r]^-{\gamma_1 \circ \text{tr}_1 \circ \varphi_{r}} & (S_2)_{s}. } \]
Using (35) and (36), we define a morphism in the fiber of $G_n$ over $\varphi$:

\[
\text{cyl}(\Pi_{r \in T_1} \gamma_{n-1} \circ \varphi_r) \xleftarrow{\Pi(E_1(r)) \times (E_r)} (\mathbb{R} \times \mathbb{R}^{n-1}) \times \Delta^1
\]

\[
\text{cyl}(\Pi_{r \in T_1} \gamma_{n-2} \circ \text{tr}_{n-2} \circ \varphi_r) \xleftarrow{\Pi(E_1(r)) \times (E_{n-1})_r} (\mathbb{R} \times \mathbb{R}^{n-2}) \times \Delta^1
\]

(38)

\[
dots
\]

\[
\text{cyl}(\Pi_{r \in T_1} \gamma_1 \circ \text{tr}_1 \circ \varphi_r) \xleftarrow{\Pi(E_1(r)) \times (E_2)_r} (\mathbb{R} \times \mathbb{R}) \times \Delta^1
\]

\[
\text{cyl}(\gamma_1(\varphi_1)) \xleftarrow{E_1} \mathbb{R} \times \Delta^1
\]

where $\{r\}$ and $\{E_1(r)\}$ denote the constant map at $r$ and $E_1(r)$, respectively, and (38) is defined over the diagram of finite sets,

\[
T_n = \Pi_{r \in T_1} (T_n)_r \xrightarrow{\Pi(\gamma_{n-1} \circ \varphi_r)} \Pi_{s \in S_1} (S_n)_s = S_n
\]

\[
T_{n-1} = \Pi_{r \in T_1} (T_{n-1})_r \xrightarrow{\Pi(\gamma_{n-2} \circ \text{tr}_{n-2} \circ \varphi_r)} \Pi_{s \in S_1} (S_{n-1})_s = S_{n-1}
\]

(39)

\[
T_2 = \Pi_{r \in T_1} (T_2)_r \xrightarrow{\Pi(\gamma_1 \circ \text{tr}_1 \circ \varphi_r)} \Pi_{s \in S_1} (S_2)_s = S_2
\]

\[
T_1 = \Pi_{r \in T_1} r \xrightarrow{\gamma_1 \circ \varphi_1} \Pi_{s \in S_1} s = S_1.
\]

Lastly, we will show that each fiber of (34) is contractible. The fiber of $G_n$ in (34) over $\varphi$ is, under the compact-open topology, the topological space of compatible embeddings

\[
\text{cyl}(\gamma_n \circ \varphi) \xleftarrow{E_n} \mathbb{R}^n \times \Delta^1
\]

\[
\text{cyl}(\gamma_{n-1} \circ \text{tr}_{n-1} \circ \varphi) \xleftarrow{E_{n-1}} \mathbb{R}^{n-1} \times \Delta^1
\]

(40)

\[
\cdots 
\]

\[
\text{cyl}(\gamma_1 \circ \text{tr}_1 \circ \varphi) \xleftarrow{E_1} \mathbb{R} \times \Delta^1
\]
over $\Delta^1$ over $\Delta^1$ such that $E|_{S_n} = e_n$ and $E|_{T_n \times \{1\}} = d_n$. Note that (40) guarantees that each morphism in $G_n^{-1}(\varphi)$ is defined over the diagram of finite sets

\[
\begin{array}{ccc}
T_n & \xrightarrow{\gamma_n \circ \varphi} & S_n \\
\downarrow{\omega_n} & & \downarrow{\tau_n} \\
T_{n-1} & \xrightarrow{\gamma'_{n-1} \circ \text{tr}_{n-1} \circ \varphi} & S_{n-1} \\
\downarrow{\omega_{n-2}} & & \downarrow{\tau_{n-2}} \\
& \vdots & \\
T_1 & \xrightarrow{\gamma_1 \circ \text{tr}_1 \circ \varphi} & S_1.
\end{array}
\]

(41)

Fix an embedding $E$ in the fiber of $G_n$ over $\varphi$. Let $S^k \xrightarrow{\varphi} G_n^{-1}(\varphi)$ be continuous and based at $E$. We construct a null-homotopy of $\psi$: For each $z \in S^n$, denote the image of $z$ under $\psi$ by $\psi_z$. The straight-line homotopy, $H_z$, from $\psi_z$ to $E$ defined by

$$H_z(x, t) = (1 - t)\psi_z(x) + tE(x)$$

names a path from $\psi_z$ to $E$ in $G_n^{-1}(\varphi)$. For each $z \in S^k$, we let each path $H_z$ run simultaneously to name a null-homotopy of $\psi$ to the constant path at $\{E\}$. Explicitly, the null-homotopy $S^k \times [0, 1] \to G_n^{-1}(\varphi)$ is given by $(z, t) \mapsto H_z(-, t)$.

$\square$

9. Part 2 of the proof of Theorem 7.2.1: Exit($\text{Ran}^n(\mathbb{R}^n)$) localizes to Exit($\text{Ran}^n(\mathbb{R}^n)$)

The focus of this section is to prove Lemma 9.0.3, which states that the natural forgetful functor from Exit($\text{Ran}^n(\mathbb{R}^n)$) to Exit($\text{Ran}^n(\mathbb{R}^n)$) which forgets all but the $n$-dimensional data (8.0.6) is a localization. Together with Lemma 8.2.2 then, this proves our main result Theorem 7.2.1, that $\Theta_n^{\text{ext}}$ localizes to Exit($\text{Ran}^n(\mathbb{R}^n)$).

**Definition 9.0.1.** Let $\mathcal{C}$ be an $\infty$-category and let $W$ be an $\infty$-subcategory of $\mathcal{C}$ which contains the maximal $\infty$-subgroupoid $\mathcal{C}^\infty$ of $\mathcal{C}$. The localization of $\mathcal{C}$ on $W$ is an $\infty$-category $\mathcal{C}[W^{-1}]$ and a functor $\mathcal{C} \xrightarrow{L} \mathcal{C}[W^{-1}]$ satisfying the following universal property: For any $\infty$-category $\mathcal{D}$, any functor $F$ from $\mathcal{C}$ to $\mathcal{D}$ uniquely factors through $L$ if and only if $F$ maps each morphism in $W$ to an isomorphism in $\mathcal{D}$; otherwise, there is no filler, as indicated by the following diagram:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow{L} & \ddots & \downarrow{\exists! \text{ or } \emptyset} \\
\mathcal{C}[W^{-1}],
\end{array}
\]

Heuristically, a localization $\mathcal{C}[W^{-1}]$ is the $\infty$-category that results after formally inverting all of the morphisms in $W \subset \mathcal{C}$ . We define the localizing $\infty$-subcategory of our main result next.

**Definition 9.0.2.** Let $W_n$ to be the $\infty$-subcategory of Exit($\text{Ran}^n(\mathbb{R}^n)$) defined to be the pullback

\[
\begin{array}{ccc}
W_n & \xleftarrow{\cdot} & \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \\
\downarrow & & \downarrow{\Phi_n} \\
\text{Fun}^{n\text{-bij}}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}}) & \longrightarrow & \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})
\end{array}
\]

where $\text{Fun}^{n\text{-bij}}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})$ is the subcategory of $\text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})$ consisting of all the same objects and only those morphisms whose value under evaluation at $n$ is a bijection; $\Phi_n$ is
the forgetful functor from Observation 8.0.5 that simply remembers the underlying data of sets at each level $1 \leq i \leq n$.

**Lemma 9.0.3.** The forgetful functor from Observation 8.0.6
\[ \mathcal{F} : \text{Exit}(\text{Ran}^\alpha([\mathbb{R}^n])) \rightarrow \text{Exit}(\text{Ran}^\alpha([\mathbb{R}^n])) \]
is a localization of $\infty$-categories on the $\infty$-subcategory $W_n$ defined in Definition 9.0.2.

In other words, there is a natural equivalence of $\infty$-categories $\text{Exit}(\text{Ran}^\alpha([\mathbb{R}^n]))[W_n^{-1}] \simeq \text{Exit}(\text{Ran}^\alpha([\mathbb{R}^n]))$.

**Notation 9.0.4.** Whenever convenient, we will abuse notation and refer to $W_n$ as the image of $W_n$ in $\Theta_n^{\text{set}}$ under the functor from (8.2.1) which was shown to be an equivalence in Lemma 8.2.2.

Heuristically, $W_n$ has the same objects as $\text{Exit}(\text{Ran}^\alpha([\mathbb{R}^n]))$ and all those morphisms whose values under $\phi_n$ from Observation 8.0.5 are bijections. Intuitively then, $\text{Exit}(\text{Ran}^\alpha([\mathbb{R}^n]))$ localizing on $W_n$ to $\text{Exit}(\text{Ran}^\alpha([\mathbb{R}^n]))$ is no surprise. Indeed, in formally declaring all those morphisms in $W_n$ to be isomorphisms, we forget the restriction by coordinate coincidence which defines morphisms in $\text{Exit}(\text{Ran}^\alpha([\mathbb{R}^n]))$ and only remember cardinality, which is the defining restriction of morphisms in $\text{Exit}(\text{Ran}^\alpha([\mathbb{R}^n]))$. However intuitive, our procedure for showing Lemma 9.0.3 is in fact quite technical, built around Theorem 9.0.7 from [26]. First, we define the following notion, which will be used in the statement of Theorem 9.0.7.

**Definition 9.0.5.** Given an $\infty$-category $\mathcal{C}$ and an $\infty$-subcategory $W \hookrightarrow \mathcal{C}$, $\text{Fun}^W([p], \mathcal{C})$ is defined to be the pullback of $\infty$-categories
\[
\begin{array}{c}
\text{Fun}^W([p], \mathcal{C}) \\
\downarrow \quad \downarrow \\
\text{Fun}([p]^{-}, W) & \text{Fun}([p]^{-}, \mathcal{C})
\end{array}
\]
where $[p]^{-}$ denotes the underlying maximal $\infty$-subgroupoid of $[p]$.

**Observation 9.0.6.** In the case $p = 0$, $\text{Fun}^W([0], \mathcal{C})$ is equivalent to $W$. Indeed, an object is a functor $[0] \rightarrow \mathcal{C}$ selects out an object of $W$, which is precisely an object of $\mathcal{C}$; a morphism is a natural transformation between any two such functors, which is precisely determined by a morphism in $W$.

A similar examination of the $p = 1$ case identifies that $\text{Fun}^W([1], \mathcal{C})$ is the $\infty$-category whose objects are objects of $\mathcal{C}$ and whose morphisms are all those natural transformations given by morphisms in $W$, i.e., a morphism from $c \rightarrow d$ in $\mathcal{C}$ to $c' \rightarrow d'$ in $\mathcal{C}$ is a commutative square in $\mathcal{C}$
\[
\begin{array}{c}
c \\
\downarrow \\
d
\end{array} \quad \begin{array}{c}
c' \\
\downarrow \\
d'
\end{array}
\]
such that both horizontal arrows are morphisms in $W$. In general, a $[p]$-point in $\text{Fun}^W([1], \mathcal{C})$ is a commutative diagram in $\mathcal{C}$ of the shape $[p] \times [1]$ such that the two $p$-simplicies $[p] \cong [p] \times \{0\}$ and $[p] \cong [p] \times \{1\}$ must be $[p]$-points of the $\infty$-subcategory $W$.

The following theorem identifies localization of $\infty$-categories in favorable cases. It will be our route for identifying the localization of Lemma 9.0.3.

**Theorem 9.0.7 (3.8 in [26]).** For an $\infty$-category $\mathcal{C}$ and an $\infty$-subcategory containing the maximal $\infty$-subgroupoid of $\mathcal{C}$, $\mathcal{C}^{-} \subset W \subset \mathcal{C}$, if the classifying space $\mathcal{B}\text{Fun}^W([\bullet], \mathcal{C})$ is a complete Segal space, then it is equivalent as a simplicial space to the localization of $\mathcal{C}$ on $W$,
\[
\mathcal{B}\text{Fun}^W([\bullet], \mathcal{C}) \simeq \mathcal{C}[W^{-1}].
\]
Ultimately, we show that the $\infty$-category $\mathcal{B}Fun^W\left(\{\bullet\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right)$ is equivalent to $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$. Then, through Theorem 9.0.7, we will have proven Lemma 9.0.3. Our approach is dependent on the next two lemmas.

**Lemma 9.0.8.** For $p = 0, 1$, there is an equivalence of spaces

$$\mathcal{B}Fun^W\left(\{p\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right) \simeq \text{Hom}_{\text{Cat}_\infty}\left(\{p\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right)$$

between the classifying space of the $\infty$-category $\text{Fun}^W\left(\{p\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right)$ and the hom-space in $\infty$-categories from $\{p\}$ to the exit-path $\infty$-category of the unital Ran space of $\mathbb{R}^n$.

**Lemma 9.0.9.** The classifying space $\mathcal{B}Fun^W\left(\{\bullet\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right)$ is a complete Segal space.

Let us explain how it is that Lemma 9.0.3 follows from these two lemmas; we will give a formal proof at the end of this section. First, note that Lemma 9.0.9 verifies the hypothesis of Theorem 9.0.7. In particular then, the classifying space $\mathcal{B}Fun^W\left(\{\bullet\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right)$ is determined by its values on $[0]$ and $[1]$ because it satisfies the Segal condition (Definition 7.1.1). Through Theorem 9.0.7, Lemma 9.0.8 identifies the space of objects and morphisms of the localization $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))|W_n$ as the space of objects and morphisms of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$, respectively. The Segal condition, then, implies the desired result, that the localization $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))|W_n^{-1}$ is equivalent to the $\infty$-category $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$.

We organize our proofs of Lemma 9.0.8 and Lemma 9.0.9 as follows. Lemma 9.0.8 naturally decomposes into its two cases, $p = 0$ and $p = 1$; we make each case into a lemma, each of which is stated and proven in the subsequent subsections §9.1 and §9.2. And lastly, §9.3 is devoted to the proof of Lemma 9.0.9.

9.1. **Identifying the space of objects of the localization of Lemma 9.0.3.** This subsection is devoted to proving the $p = 0$ case of Lemma 9.0.8. In light of Observation 9.0.6, where we observed that $\text{Fun}^W([0], \mathcal{C}) \simeq W$, we rephrase this case as the following lemma:

**Lemma 9.1.1** (Lemma 9.0.8; $p = 0$). There is an equivalence of spaces

$$\mathcal{B}W_n \simeq \text{Exit}(\text{Ran}^u(\mathbb{R}^n))$$

from the classifying space of the $\infty$-subcategory $W_n$ of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ to the maximal $\infty$-subgroupoid of the exit-path $\infty$-category of the unital Ran space of $\mathbb{R}^n$.

To prove Lemma 9.1.1, we will first prove Lemma 9.1.3, which states that there is an adjunction between $W_n$ and the subcategory of $W_n$ consisting of healthy trees. Explicitly, this subcategory is defined as follows.

**Definition 9.1.2.** $W_n^{\text{ht}}$ is the subcategory of $W_n$ defined to be the pullback

$$
\begin{array}{ccc}
W_n^{\text{ht}} & \longrightarrow & W_n \\
\downarrow & & \downarrow \\
\text{Fun}^{n,\text{bij}}\left(\{1 < \cdots < n\}, (\text{Fin}_{\text{surj}})^{\text{op}}\right) & \longrightarrow & \text{Fun}^{n,\text{bij}}\left(\{1 < \cdots < n\}, \text{Fin}_{\text{op}}\right)
\end{array}
$$

of categories.

Informally, the category $W_n^{\text{ht}}$ is the full subcategory of $W_n$ consisting of all those objects of $\Theta_n$ that are healthy trees.

**Lemma 9.1.3.** The inclusion functor $W_n^{\text{ht}} \hookrightarrow W_n$ is a right adjoint.

A left adjoint to the inclusion functor is given by forgetting all of the data associated to the ‘unhealthy’ parts of the trees in $W_n$. We construct such a functor as follows.
**Construction 9.1.4** (The Pruning Functor, $P_n$). For each $n \geq 1$, we define a canonical functor

$$P_n : W_n \rightarrow W_{n}^{\text{ht}}.$$ 

For $n = 1$, $P_1 := \text{Id}_{W_1}$ since $W_1 = W_{1}^{\text{ht}}$.

For $n \geq 2$, we define $P_n$ inductively. First, for each object $T = [p](T_i) \in W_n$ for $n \geq 2$, define the sub-linearly ordered set

$$N_T := \left\{ 0 = i_0 < i_1 < \cdots < i_k \mid i_j \in \{1, \ldots, p\} \forall 1 \leq j \leq k, T_i = \emptyset \iff \exists 1 \leq j \leq k \text{ s.t. } i = i_j \right\} \subset [p].$$

For the case $n = 2$, we define

$$P_2 : T = [p](\{q_i\}) \mapsto N_T(\{q_{i_j}\})$$

on objects; the value of a morphism under $P_2$ is given by restriction of that morphism to $N_T$. $P_2$ respects composition because restriction respects composition.

For general $n$, we define

$$P_n : T = [p](T_i) \mapsto N_T(P_{n-1}(T_{i_j}))$$

on objects; the value of a morphism under $P_n$ is again determined by restriction of that morphism to $N_T$ together with $P_{n-1}$. Composition is preserved by $P_n$ because restriction and $P_{n-1}$ both respect composition.

**Proof of Lemma 9.1.3.** We use Lemma 2.17 from [3] which states that for a functor $C \rightarrow D$ of $\infty$-categories, $F$ is a right adjoint if and only if for each object $d \in D$, the $\infty$-undercategory $C/d$ has an initial object, and verify that for each $T \in W_n$, the undercategory $W_{n}^{\text{ht}}T$ has an initial object. Recall that the $\infty$-undercategory $C/d$ is defined as the pullback of $\infty$-categories

$$C \downarrow \bigg\uparrow ^{\text{frgt}} \downarrow \bigg\downarrow \rightarrow \uparrow F \downarrow \rightarrow D$$

Fix an object $T \in W_n$. To define the initial object of $W_{n}^{\text{ht}}T$, we will define a morphism $T \xrightarrow{\alpha_T} P_n(T)$ in $W_n$ such that for any morphism $T \xrightarrow{f} S$ in $W_n$ to a healthy tree $S$, there is a unique (up to isomorphism) factorization through $\alpha_T$:

$$T \xrightarrow{\alpha_T} P_n(T) \xrightarrow{f} S$$

in $W_n$.

We define $\alpha_T$ as follows: For $n = 1$, $T = P_1(T)$ for each $T \in W_1$ and thus we define $\alpha_T := \text{Id}_T$.

To define $\alpha_T$ for $T \in W_n$ for $n \geq 2$, we proceed by induction. Fix an object $T \in W_2$. Define $\alpha_T : T = [p](\{q_i\}) \rightarrow N_T(\{q_{i_j}\})$ by

i) $[p] \rightarrow N_T$ is given by the assignment $i \mapsto \begin{cases} i_j, & \text{if } \exists 0 \leq j \leq k - 1 \text{ s.t. } i_j \leq i < i_{j+1} \\ i_k, & \text{if } i \geq i_k. \end{cases}$

ii) For each pair $(i, i_j)$ such that $i = i_j$, $[q_i] \xrightarrow{\text{Id}} [q_{i_j}]$.

For the general case, fix an object $T \in W_n$. Define $\alpha_T : T = [p](T_i) \rightarrow N_T(P_{n-1}(T_{i_j}))$ by

i) $[p] \rightarrow N_T$ is the same as i) for $n = 2$.

ii) For each pair $(i, i_j)$ such that $i = i_j$, $T_i \xrightarrow{\alpha_{T_i}} P_{n-1}(T_{i_j})$, where $\alpha_{T_i}$ is guaranteed by the inductive step.
Next, we observe that by design each morphism $T \xrightarrow{f} S$ in $W_n$ to a healthy tree $S\xrightarrow{f} W_n$ factors through $\alpha_T$ via $P_n(f)$:

$$
\begin{array}{ccc}
T & \xrightarrow{f} & S \\
\downarrow{\alpha_T} & & \downarrow{P_n(f)} \\
& & P_n(T).
\end{array}
$$

Further, up to isomorphism $P_n(f)$ is the unique such filler.

We have just verified that for each fixed object $T \in W_n$, the initial object of $W_n^{ht}/$ is $(P_n(T), T \xrightarrow{\alpha_T} P_n(T))$.

\[ \square \]

9.1.1. The space of configurations of $r$ unordered points in $\mathbb{R}^n$. We have just proven the first of the two key facts that we will use to prove Lemma 9.1.1. The other key fact is to identify the classifying space of the following $\infty$-category.

**Definition 9.1.5.** For each $r \geq 1$, $W_n^{ht}(r)$ is the $\infty$-subcategory of $W_n^{ht}$ defined to be the pullback of $\infty$-categories

$$
\begin{array}{ccc}
W_n^{ht}(r) & \xleftarrow{\iota} & W_n^{ht} \\
\downarrow & & \downarrow \\
\text{Fun}^n_{r-bij}\left(\{1 < \cdots < n\}, (\text{Fin}^{\text{surj}})^{\text{op}}\right) & \xhookrightarrow{\text{Fun}^n_{\text{bij}}} & \text{Fun}^n_{\text{bij}}\left(\{1 < \cdots < n\}, (\text{Fin}^{\text{surj}})^{\text{op}}\right).
\end{array}
$$

where $\text{Fun}^n_{r-bij}\left(\{1 < \cdots < n\}, (\text{Fin}^{\text{surj}})^{\text{op}}\right)$ is the full subcategory of $\text{Fun}^n_{\text{bij}}\left(\{1 < \cdots < n\}, (\text{Fin}^{\text{surj}})^{\text{op}}\right)$ in which the value of an object upon evaluation at $n$ has cardinality $r$. We define $W_n^{ht}(0)$ to be the terminal $\infty$-category.

Heuristically, an object of $W_n^{ht}(r)$ is a configuration of $r$ distinct points in $\mathbb{R}^n$ and a morphism is a path in the unordered configuration space of $r$ points in $\mathbb{R}^n$ such that for each $0 < k < n$, the projection of the path onto its first $k$-coordinates may witness ‘anti-collision’ of points, but not collision. In other words, configurations are allowed to move from having more coordinate coincidences to less, but not vice-versa.

**Observation 9.1.6.** The underlying data of sets of a $[p]$-point in $W_n^{ht}(r)$ is restricted to be bijective between sets of cardinality $r$. This allows us to significantly simplify the description of a $[p]$-point. Recall the definition (8) of a $[p]$-point in $\text{Exit}(\text{Ran}^n(\mathbb{R}^n))$. We may simplify the description for those in $W_n^{ht}(r)$ in two ways.

First, the embedding at dimension $n$ will determine all of the embeddings of the lower dimensions by taking its projection. This is due to the fact that all of our maps between sets are, in particular, surjective. Thus, it suffices to only describe the $n$-dimensional embedding data.

Second, the reverse cylinder construction on a sequence of $p$ bijections is trivial in that it is simply the product of the set with the topological $p$-simplex.

Thus it is that the data of a $[p]$-point in $W_n^{ht}(r)$ may be described as an embedding

$$
R \times \Delta^p \hookrightarrow \mathbb{R}^n \times \Delta^p
$$

over $\Delta^p$ for a set $R$ of cardinality $r$ whose projection onto the first $k$-coordinates for each $1 \leq k < n$ may witness anti-collision but not collision.

**Observation 9.1.7.** By definition, the $\infty$-category $W_n^{ht}$ (Definition 9.0.2) is canonically equivalent to the coproduct of $\infty$-categories $\coprod_{r \geq 0} W_n^{ht}(r)$. 

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To identify the classifying space of $W^h_t(r)$ we will define a canonically equivalent $\infty$-category in order to gain access to a result which will make quick work of identifying its classifying space. We begin with a slight detour to review some of the basic concepts of [7]. First recall Definition 1.3.1 of planar level rooted trees. The following is a poset on these trees.

**Definition 9.1.8** (The poset of $n$-orderings on $A$ [7]).

- An $n$-ordering on $A$ is a pair $(S, \sigma)$ consisting of a healthy planar level tree $S$ of height $n$ and a bijection $\sigma$ between $A$ and level-$n$ leaves of $S$. We will usually leave the $\sigma$ implicit and denote an $n$-ordering by simply $S$.
- The leaves of $S$ inherit a canonical linear order, which induces a linear ordering on $A$ that we denote by $\leq_S$.
- Given $a, b \in A$, the branching level of $a$ and $b$, denoted $b_S(a, b)$, is defined to be the level of the vertex at which the directed paths from $a$ and $b$ to the root first meet.
- The poset of $n$-orderings on $A$, denoted $nOrd(A)$, is the poset of $n$-orderings on $A$. A morphism $S \to T$ exists if and only if the following condition, called the branching condition, holds: for all $a, b \in A$, $b_T(a, b) \leq b_S(a, b)$, with equality only if the orderings of $a$ and $b$ under $<_T$ and $<_S$ agree.

**Definition 9.1.9** (Lemma 13 [7]). For an injection $\phi : A \hookrightarrow \mathbb{R}^n$, define $S_\phi$ to be a tree in $nOrd(A)$ with the following properties:

- The linear order on the set of leaves labeled by $A$ is given by the lexicographic order on $A$ in $\mathbb{R}^n$.
- For all $a, b \in A$, the branching number $b(a, b)$ is the largest integer $i$ such that $\phi(a)_i = \phi(b)_i$.

**Example 9.1.10.** We draw a tree $S_\phi$ associated to an injection $\phi : A = \{1, 2, 3, 4\} \hookrightarrow \mathbb{R}^2$:

![Figure 8](image)

Figure 8. The image of $\phi$ on the left; $S_\phi$ on the right.

We will stratify the space of configurations of $A$-labeled points in Euclidean space over the poset $nOrd(A)$.

**Lemma 9.1.11.** The map of sets $\nu : Conf_A(\mathbb{R}^n) \longrightarrow nOrd(A)$ given by

$$(A \hookrightarrow \mathbb{R}^n) \mapsto S_\phi,$$

where $S_\phi$ is defined in (9.1.9) is a topologically stratified space.

**Proof.** In Lemma 13 of [7] they show that $S_\phi$ is unique. Thus $\nu$ is well defined. To show continuity, we will show that the preimage of a closed subset is closed. Take the downward closure of $S_\phi$ in $nOrd(A)$ and consider its preimage $Conf_A(\mathbb{R}^n) \leq S_\phi \subset Conf_A(\mathbb{R}^n)$. Following Definition 11 of [7], the preimage consists of all those injections $\phi : A \hookrightarrow \mathbb{R}^n$ with the following properties: For each pair $a, b \in A$ with $a <_{S_\phi} b$,

- $\phi(a)_i = \phi(b)_i$ for all $i = 1, \ldots, b_{S_\phi}(a, b)$,
- $\phi(a)_i \leq \phi(b)_i$ for $i = b_{S_\phi}(a, b) + 1$.

From this description, it is evident that the preimage is closed in $Conf_A(\mathbb{R}^n)$.

□
The strata of \( \nu : \text{Conf}_A(\mathbb{R}^n) \to n\text{Ord}(A) \) are the Fox-Neuwirth cells of the configuration space, introduced by Fox and Neuwirth in [15]. We refer to this as the Fox-Neuwirth stratification. Our goal is to prove Theorem 9.1.13, which states that the Fox-Neuwirth stratification of the configuration space is Whitney stratified.

**Definition 9.1.12 (Thom and Mather; Nocerta and Volpe).**

- (Whitney condition in Euclidean space.) Let \( X, Y \) be smooth submanifolds of \( \mathbb{R}^n \), and let \( y \in Y \) be a point. The pair \((X, Y)\) is said to satisfy Whitney’s Condition B at \( y \) if for any two sequence of points \((x_i) \subset X \) and \((y_i) \subset Y \) both converging to \( y \) such that the sequence of tangent planes \( T_{x_i}X \) converges to some vector space \( \tau \) in the \( r \)-Grassmannian of \( \mathbb{R}^n \) and the sequence of secant lines \( x_iy_i \) converges to some line \( l \) in the \( 1 \)-Grassmannian of \( \mathbb{R}^n \), then \( l \subset \tau \).

- (Whitney stratification in Euclidean space.) A topological stratification \( X \to P \) of a subset of Euclidean space is a Whitney stratification if each stratum of \( X \) is a smooth submanifold of Euclidean space and Whitney’s condition B is satisfied at every point in each pair of strata.

**Theorem 9.1.13.** The Fox-Neuwirth stratification \( \nu : \text{Conf}_A(\mathbb{R}^n) \to n\text{Ord}(A) \) is a Whitney stratification.

**Lemma 9.1.14.** Let the cardinality of \( A \) be \( k \). Let \( S \) be a tree in \( n\text{Ord}(A) \) and \( x \) be a configuration in the \( S \)-stratum \( \text{Conf}_A(\mathbb{R}^n)_S \).

1. The tangent space \( T_x\text{Conf}_A(\mathbb{R}^n) \) to \( x \) along the \( S \)-stratum is canonically identified with Euclidean space \( \mathbb{R}^{r_S} \subset \mathbb{R}^{nk} \) for some integer \( r_S \leq nk \).

2. There is a canonical closed embedding

\[
\text{Conf}_A(\mathbb{R}^n)_{\leq S} \hookrightarrow T_x\text{Conf}_A(\mathbb{R}^n)
\]

of the fiber over the downward closure of \( S \) in \( n\text{Ord}(A) \) into the tangent space at the point \( x \) in the \( S \)-stratum.

**Proof.** Let \( S \) be a tree in \( n\text{Ord}(A) \) and let \( x \in \text{Conf}_A(\mathbb{R}^n)_S \) be a point in the \( S \)-stratum. We can describe the \( S \)-stratum explicitly as all those injections \( \phi : A \to \mathbb{R}^n \) with the following properties: For each pair \( a, b \in A \) with \( a <_S b \),

- \( \phi(a)_i = \phi(b)_i \) for all \( i = 1, \ldots, b_S(a, b) \),
- \( \phi(a)_i < \phi(b)_i \) for \( i = b_S(a, b) + 1 \).

This evidently describes an open subset of some Euclidean space \( \mathbb{R}^{r_S} \subset \mathbb{R}^{nk} \) whose dimension \( r_S \) is determined by the branching numbers of each pair \( a <_S b \). Thus the tangent space \( T_x\text{Conf}_A(\mathbb{R}^n)_S \) to the point \( x \) in the \( S \)-stratum is canonically identified with \( \mathbb{R}^{r_S} \), verifying (1).

Observe that there is an equality

\[
\text{Conf}_A(\mathbb{R}^n)_S = \text{Conf}_A(\mathbb{R}^n)_{\leq S}
\]

between the closure of the \( S \)-stratum and the fiber \( \text{Conf}_A(\mathbb{R}^n)_{\leq S} \) over the downward closure of \( S \) in \( n\text{Ord}(A) \). This equality is evident from their explicit descriptions - see the proof of Lemma 9.1.11 for the description of \( \text{Conf}_A(\mathbb{R}^n)_{\leq S} \). In particular this means that the downward closure \( \text{Conf}_A(\mathbb{R}^n)_{\leq S} \) is a closed subset of \( \mathbb{R}^{r_S} \). The canonical equivalence \( T_x\text{Conf}_A(\mathbb{R}^n) = \mathbb{R}^{r_S} \) yields a canonical closed embedding \( \text{Conf}_A(\mathbb{R}^n)_{\leq S} \hookrightarrow T_x\text{Conf}_A(\mathbb{R}^n) \), which verifies (2).

**Observation 9.1.15.** Let \( V \) be a vector subspace of a finite dimensional vector space \( W \). The limit of a convergent sequence of vector subspaces in \( V \) (in the \( r \)-Grassmannian of \( V \)) will be a subspace of \( V \) (as opposed to being in \( V^\perp \)).

**Proof of Theorem 9.1.13.** In the proof of Lemma 9.1.14 we identified that each stratum is an open subset of \( \mathbb{R}^r \subset \mathbb{R}^{nk} \) for some \( r \) depending on the stratum. Hence each stratum is a smooth submanifold of \( \mathbb{R}^{nk} \). We will verify Whitney’s condition B: Let \( S \) and \( T \) be trees in \( n\text{Ord}(A) \) with \( T < S \), \( (x_i) \subset \text{Conf}_A(\mathbb{R}^n)_S \) and \( (y_i) \subset \text{Conf}_A(\mathbb{R}^n)_T \) be sequences in the \( S \)- and \( T \)-strata converging to a
point \( y \in \text{Conf}_A(\mathbb{R}^n)_T \) in the \( T \)-stratum such that the sequence of tangent spaces \( T_x, \text{Conf}_A(\mathbb{R}^n)_S \) is convergent and the sequence of secant lines \( x_iy_i \) converges to a line \( l \) in the 1-Grassmannian.

The assumption \( T < S \) implies that \( \text{Conf}_A(\mathbb{R}^n)_{T} \subset \text{Conf}_A(\mathbb{R}^n)_{S} \). Thus the canonical closed embedding \( \text{Conf}_A(\mathbb{R}^n)_{\leq S} \hookrightarrow T_x, \text{Conf}_A(\mathbb{R}^n)_S \) from Lemma 9.1.14 tells us that \( y_i \in T_x, \text{Conf}_A(\mathbb{R}^n)_S \) for each \( i \). Moreover, \( y_i \in T_x, \text{Conf}_A(\mathbb{R}^n)_S \) implies that each secant line \( x_iy_i \) is in \( T_x, \text{Conf}_A(\mathbb{R}^n)_S \).

Translating our tangent spaces \( T_x, \text{Conf}_A(\mathbb{R}^n)_S \) to the origin via their canonical identifications with \( \mathbb{R}^{r_S} \) from Lemma 9.1.14 yields the situation of Observation 9.1.15: The sequence of secant lines \( x_iy_i \) is a sequence of linear subspaces in \( \mathbb{R}^{r_S} \) converging to some \( l \), which must also be in \( \mathbb{R}^{r_S} \). Lastly, the sequence of tangent spaces to the \( x_i \) is the constant sequence of \( \mathbb{R}^{r_S} \) which must converge to \( \mathbb{R}^{r_S} \). This verifies Whitney’s condition B at any point in any stratum of \( \nu : \text{Conf}_A(\mathbb{R}^n) \rightarrow n\text{Ord}(A) \), which implies that \( \nu \) is a Whitney stratified space.

We will now consider the unordered configuration spaces of Euclidean space and argue that Theorem 9.1.13 implies that they too are Whitney stratified.

**Observation 9.1.16.** The action of \( \Sigma_A \) on the configuration space given by permuting the labels of the points extends to an action on the Fox-Neuwirth stratification of the configuration space by permuting the leaves in \( n\text{Ord}(A) \). We quotient out by this action as indicated in the following diagram

\[
\begin{array}{ccc}
\text{Conf}_A(\mathbb{R}^n) & \xrightarrow{\nu} & n\text{Ord}(A) \\
\downarrow /\Sigma_A & & \downarrow /\Sigma_A \\
\text{Conf}_A(\mathbb{R}^n)_{\Sigma_A} & \xrightarrow{\nu_{\Sigma_A}} & n\text{Ord}(A)_{\Sigma_A}
\end{array}
\]

to obtain the Fox-Neuwirth stratification of the unordered configuration space.

The quotient map from the ordered to the unordered configuration space is a smooth covering space and is thus a local diffeomorphism. Since the Whitney condition is local, the following corollary to Theorem 9.1.13 is immediate.

**Corollary 9.1.17.** The unordered Fox-Neuwirth stratification \( \nu_{\Sigma_A} : \text{Conf}_A(\mathbb{R}^n)_{\Sigma_A} \rightarrow n\text{Ord}(A)_{\Sigma_A} \) is a Whitney stratification.

In [27] the authors prove that Whitney stratifications are conically smooth (Theorem 2.7). Thus we have the following corollary to Theorem 9.1.13.

**Corollary 9.1.18.** The unordered Fox-Neuwirth stratification \( \nu_{\Sigma_A} : \text{Conf}_A(\mathbb{R}^n)_{\Sigma_A} \rightarrow n\text{Ord}(A)_{\Sigma_A} \) is a conically smooth stratification.

In Definition 3.3.1 of [4] they define the exit-path \( \infty \)-category functor \( \text{Exit} \) from conically smooth stratified spaces to \( \infty \)-categories. We will consider the value of this functor on the Fox-Neuwirth stratification of the configuration space, denoted \( \text{Exit}(\text{Conf}_A(\mathbb{R}^n)_{\Sigma_A}) \), and show that it is equivalent to the \( \infty \)-category \( W^\text{bht}_n(r) \) (9.1.5). Afterwards, we can finally prove Lemma 9.1.11 by making use of a result in [6] which identifies the classifying space of values of the exit-path \( \infty \)-category.

**Lemma 9.1.19.** There is a canonical equivalence between \( \infty \)-categories

\[
W^\text{bht}_n(r) \simeq \text{Exit}(\text{Conf}_r(\mathbb{R}^n)_{\Sigma_r})
\]

over \( (\text{Fin}^{\text{surj}})^{\text{op}} \).

**Proof.** Both \( W^\text{bht}_n(r) \) and \( \text{Exit}(\text{Conf}_r(\mathbb{R}^n)_{\Sigma_r}) \) are modeled as complete Segal spaces. Thus they are completely determined by their \([0]\) and \([1]\) values. Therefore it suffices to show an equivalence between the simplicial data at \([0]\) and \([1]\), which is nearly immediate from their definitions.

On the level of objects this is in fact immediate - the objects of both are points in the underlying stratified space, which are configurations of \( r \) unordered points in \( \mathbb{R}^n \).
On the level of morphisms, we recall the definitions of each. From (9.1.6), we observed that a morphism in $W^{\text{hlt}}_r(n)$ is an embedding

$$R \times \Delta^1 \hookrightarrow \mathbb{R}^n \times \Delta^1$$

over $\Delta^1$ and for a set $R$ of cardinality $r$, whose projections may witness anti-collision, but not collision. This restriction (which we note is actually part of the data of the definition) about anti-colliding projections is precisely encoded by the poset $n\text{Ord}(r)_{\Sigma_r}$ in the Fox-Neuwirth stratification of $\text{Conf}^r(\mathbb{R}^n)_{\Sigma_r}$. Now recall the definition of the exit-path $\infty$-category functor from Definition 3.3.1 of [4]. The value $\text{Exit}(X)$ of a conically smooth stratified space $X$ is a simplicial space assigning $[1]$ to the hom-space $\text{Strat}(\Delta^1, X)$ of the $\infty$-category of conically smooth stratified spaces. Here, $\Delta^1$ has the standard stratification defined in (1.1.4) over $[1]$. Thus, a $[1]$-point of $\text{Exit}(\text{Conf}^r(\mathbb{R}^n)_{\Sigma_r})$ is a stratified map from $\Delta^1$ to $\text{Conf}^r(\mathbb{R}^n)_{\Sigma_r}$, a.k.a. a path in $\text{Conf}^r(\mathbb{R}^n)_{\Sigma_r}$ that is allowed to take a configuration from having more coordinate coincidence to less, but not vice-versa. The projection of such a path would witness anti-collision of points, but not collision. This identifies the $[1]$-points. The equivalence between the structure maps are evident. □

**Notation 9.1.20.** For the remainder of the paper, we let $r$ denote the set $\{1, ..., r\}$.

**Proof of Lemma 9.1.1.** Corollary 2.1.28 in [25] states that an adjunction between $\infty$-categories yields an equivalence between their classifying spaces. We apply this result to the adjunction from Lemma 9.1.3 to obtain an equivalence of the classifying spaces,

$$\mathcal{B}W_n \simeq \mathcal{B}W^{\text{hlt}}_n.$$  

In Observation 9.1.7, we saw that by definition $W^{\text{hlt}}_n$ is equivalent to the coproduct $\Pi_{r \geq 0} W^{\text{hlt}}_n(r)$. Then, in Lemma 9.1.19, we proved an equivalence of $\infty$-categories between $\text{Exit}(\text{Conf}^r(\mathbb{R}^n)_{\Sigma_r})$ and $W^{\text{hlt}}_n(r)$. Taking the classifying space of these two equivalences yields the equivalence of spaces

$$\mathcal{B}W^{\text{hlt}}_n \simeq \Pi_{r \geq 0} \text{Exit}(\text{Conf}^r(\mathbb{R}^n)_{\Sigma_r}).$$

In Corollary 9.1.18 we proved that the Fox-Neuwirth stratification of the unordered configuration space $\text{Conf}^r(\mathbb{R}^n)_{\Sigma_r} \to n\text{Ord}(r)_{\Sigma_r}$ is conically smooth. In [6] Corollary 1.2.7 states that the classifying space of the exit-path $\infty$-category of a conically smooth stratified space is homotopy equivalent to the topological space underlying the stratified space. Thus we have the following equivalence of spaces:

$$\Pi_{r \geq 0} \text{Exit}(\text{Conf}^r(\mathbb{R}^n)_{\Sigma_r}) \simeq \Pi_{r \geq 0} \text{Conf}^r(\mathbb{R}^n)_{\Sigma_r}.$$  

Lastly, $\Pi_{r \geq 0} \text{Conf}^r(\mathbb{R}^n)_{\Sigma_r}$ is, by definition, equivalent to $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$, the maximal $\infty$-subgroupoid of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$. □

**9.2. Identifying the space of morphisms of the localization of Lemma 9.0.3.** By proving Lemma 9.1.1, we have identified that the space of objects of the localization $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))[W^{-1}_n]$ is equivalent to the space of objects of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$. This subsection is devoted to identifying the space of morphisms by proving the $p = 1$ case of Lemma 9.0.8; we restate it as follows.

**Lemma 9.2.1 (Lemma 9.0.8; $p = 1$).** There is an equivalence of spaces

$$\mathcal{B}\text{Fun}^{W^\omega}(\{1\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \simeq \text{mor}(\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))$$

induced by the forgetful functor $\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \to \text{Exit}(\text{Ran}^u(\mathbb{R}^n))$.

The core idea of the proof is the following: Both spaces of the equivalence in Lemma 9.2.1 naturally assemble as the fibrations depicted in (44) below. We will use the induced long exact sequence of homotopy groups to show a weak homotopy equivalence of total spaces by showing a
homotopy equivalence between the base spaces and between the fibers. By the Whitehead Theorem then, we obtain a homotopy equivalence of the total spaces since they are CW complexes.

\[
\begin{align*}
\mathcal{B}\text{Fun}^W_n\left([1], \text{Exit}(\text{Ran}^n(\mathbb{R}^n))\right) & \xrightarrow{\text{frg}t} \text{mor}\left(\text{Exit}(\text{Ran}^n(\mathbb{R}^n))\right) \\
\mathcal{B}\text{Fun}^W_n\left([0], \text{Exit}(\text{Ran}^n(\mathbb{R}^n))\right) & \xrightarrow{\text{mor}} \text{Exit}(\text{Ran}^n(\mathbb{R}^n))
\end{align*}
\]

(44)

Since we already showed that the base spaces are equivalent in Lemma 9.1.1, our work lies in showing an equivalence on the level of fibers. We identify the fibers throughout the course of three lemmas: Lemma 9.2.7 identifies the fibers of \(ev_0\), and Lemma 9.2.8 and Lemma 9.2.11 identify the fibers of \(\text{Bev}_0\). Each lemma is technical, relying on the concept of a Cartesian fibration. In the next subsection, we follow [3] to recall this technical machinery and further tailor it to the situation at hand, towards the goal of proving Lemma 9.2.7, Lemma 9.2.8, and Lemma 9.2.11.

9.2.1. Cartesian fibrations.

**Definition 9.2.2** (2.1 in [3]). Let \(\mathcal{E} \xrightarrow{\pi} \mathcal{B}\) be a functor between \(\infty\)-categories. A morphism \(c_1 \xrightarrow{e \xrightarrow{f} e'} \mathcal{E}\) is \(\pi\)-Cartesian if the diagram of \(\infty\)-overcategories

\[
\begin{array}{c}
\mathcal{E}_e \\
\downarrow \pi \\
\mathcal{B}_{/\pi(e)}
\end{array}
\xrightarrow{\phi_\pi} 
\begin{array}{c}
\mathcal{E}_{e'} \\
\downarrow \pi \\
\mathcal{B}_{/\pi(e')}
\end{array}
\]

is a pullback. If \(\pi\) is a Cartesian fibration if for every solid square

\[
\begin{array}{c}
* \\
\downarrow (t) \\
\mathcal{C}_1
\end{array}
\xrightarrow{\pi} 
\begin{array}{c}
\mathcal{E} \\
\downarrow \pi \\
\mathcal{B}
\end{array}
\]

there is a \(\pi\)-Cartesian filler.

**Observation 9.2.3.** The map

\[
\text{Fun}^W_n\left([1], \text{Exit}(\text{Ran}^n(\mathbb{R}^n))\right) \xrightarrow{\text{ev}_0} \text{Fun}^W_n\left([0], \text{Exit}(\text{Ran}^n(\mathbb{R}^n))\right)
\]

is a Cartesian fibration. The proof is straightforward, using Example 2.5 in [3], wherein it is shown that for an \(\infty\)-category \(\mathcal{C}\), the functor given by evaluation at 0, \(\text{Fun}([1], \mathcal{C}) \xrightarrow{\text{ev}_0} \text{Fun}([0], \mathcal{C})\) is a Cartesian fibration.

**Observation 9.2.4.** Let \(\mathcal{E} \xrightarrow{\pi} \mathcal{B}\) be a Cartesian fibration. For each object \(b \in \mathcal{B}\) there is a canonical inclusion \(\pi^{-1}(b) \hookrightarrow \mathcal{E}^{b/}\) from the fiber of \(\pi\) over \(b\) to the undercategory of \(\mathcal{E}\) under \(b\). Its value on an object \(e \in \pi^{-1}(b)\) such that \(b \cong \pi(e)\) is the equivalence \(b \xrightarrow{\cong} \pi(e)\). Its value on a morphism \(e \xrightarrow{f} e'\) is \(\pi(f)\).

**Definition 9.2.5** (Cartesian Monodromy Functor). Let \(\mathcal{E} \xrightarrow{\pi} \mathcal{B}\) be a Cartesian fibration. For each morphism \(b \xrightarrow{f} b'\) in \(\mathcal{B}\), the induced Cartesian monodromy functor \(f^* : \pi^{-1}(b') \rightarrow \pi^{-1}(b)\) from the fiber over \(b'\) to the fiber over \(b\) is defined to be the threefold composite

\[
\begin{array}{c}
\pi^{-1}(b') \\
\downarrow \\
\mathcal{E}^{b'}/
\end{array}
\xrightarrow{-\circ f} 
\begin{array}{c}
\pi^{-1}(b) \\
\uparrow \mu \\
\mathcal{E}^{b}/
\end{array}
\]
where $\mu$ is right adjoint to the inclusion functor $\pi^{-1}(b) \hookrightarrow \mathcal{E}^b$, the existence of which is guaranteed by Lemma 2.20 in [3].

**Observation 9.2.6.** Given a diagram of $\infty$-categories

\[
\begin{array}{c}
\mathcal{E} \\
\downarrow \pi
\end{array} \xymatrix{ \pi^{-1}(b') \ar[r]^-{\alpha^*} \ar[d] & \pi^{-1}(b) \ar[d] \\
\pi'^{-1}(F(b')) \ar[r]^-{(\alpha')^*} & \pi'^{-1}(F(b)).}
\]

in which $\pi$ and $\pi'$ are Cartesian fibrations, for each morphism $b \xrightarrow{\alpha} b'$ in $\mathcal{B}$, $G$ carries the induced monodromy functor $\alpha^*$ to the monodromy functor induced by $F(\alpha)$,

\[
\pi^{-1}(b') \xrightarrow{\alpha^*} \pi^{-1}(b)
\]

Further, if (45) is a pullback of $\infty$-categories, then the downward vertical arrows of (46) are equivalences.

We are now equipped with the technical machinery needed to prove Lemma 9.2.1. The proof relies on the upcoming three lemmas. For the remainder of this paper, we implement the following notational changes:

- **Exit($\text{Ran}^u(\mathbb{R}^n)$):** We denote an object $S \xrightarrow{\xi} \mathbb{R}^n$ by $S$, by which we mean the image of $S$ in $\mathbb{R}^n$ under the embedding $\xi$.

- **Exit($\text{Ran}^u(\mathbb{R}^n)$):** We denote an object $S \xrightarrow{\xi} \mathbb{R}^n$ by $S = S_n \to \cdots \to S_1$, by which we mean the images of $S_i$ under $e_i$ for each $1 \leq i \leq n$ together with the coordinate projection data given by the sequence of maps of finite sets $S_n \to \cdots \to S_1$.

We denote a morphism $\text{cyl}(S' \xrightarrow{\xi'} S) \xrightarrow{\xi} \mathbb{R}^n \times \Delta^1$ from $S$ to $S'$ by simply an arrow $S \to S'$.

The first lemma more generally identifies the fiber of the functor on the righthand side of (44) for any manifold $M$.

**Lemma 9.2.7.** For a smooth connected manifold $M$, the fiber of the map of spaces

\[
\text{mor}\left(\text{Exit}(\text{Ran}^u(M))\right) \xrightarrow{\text{mor}} \text{Exit}(\text{Ran}^u(M))
\]

from the space of morphisms of $\text{Exit}(\text{Ran}^u(M))$ to the maximal $\infty$-subgroupoid of $\text{Exit}(\text{Ran}^u(M))$ over an object $S \subset M$ of $\text{Exit}(\text{Ran}^u(M))$ is the product

\[
\Pi_{s \in S} \text{Exit}(\text{Ran}^u(T_sM))
\]

indexed over $s \in S$ of the maximal $\infty$-subgroupoid of the exit-path $\infty$-category of the unital Ran space of the tangent space $T_sM$ of $M$ at $s \in S$.

**Proof.** Fix $S \subset M$. We will identify the fiber over $S$ by constructing an explicit homotopy equivalence

\[
\Pi_{s \in S} \Pi_{r \geq 0} \text{Conf}_r(T_sM)_{\Sigma_r} \xrightarrow{\sim} \text{ev}_0(S).
\]

We remind the reader that the underlying maximal $\infty$-subgroupoid of $\text{Exit}(\text{Ran}^u(T_sM))$ is equivalent to the coproduct of configuration spaces: $\text{Exit}(\text{Ran}^u(T_sM)) \xrightarrow{\sim} \Pi_{s \in S} \text{Conf}_r(T_sM)_{\Sigma_r}$. For each $s \in S$, choose a smooth open embedding $T_sM \hookrightarrow M$ which carries the origin to $s$ such that the images $\{U_s\}_{s \in S}$ are disjoint. These open embeddings induce homeomorphisms $\text{Conf}_r(T_sM)_{\Sigma_r} \cong \text{Conf}_r(U_s)_{\Sigma_r}$, which in turn identify homeomorphisms

\[
\Pi_{s \in S} \Pi_{r \geq 0} \text{Conf}_r(T_sM)_{\Sigma_r} \cong \Pi_{s \in S} \Pi_{r \geq 0} \text{Conf}_r(U_s)_{\Sigma_r} \cong \Pi_{r \geq 0} \text{Conf}_r\left(\Pi_{s \in S} U_s\right)_{\Sigma_r}
\]

(47)
where the second homeomorphism is canonical because the images \( \{U_s\} \) are disjoint and the empty configuration is allowed. The inclusions \( \Pi s \in S U_s \hookrightarrow M \) determine an inclusion
\[
(48) \quad \Pi r \geq 0 \text{Conf}_s \left( \Pi s \in S U_s \right)_{\Sigma_r} \hookrightarrow \Pi r \geq 0 \text{Conf}_r(M)_{\Sigma_r}.
\]
Composing (47) and (48) results in the top horizontal map in the following diagram
\[
(49) \quad \Pi s \in S \Pi r \geq 0 \text{Conf}_r(T_s M)_{\Sigma_r} \longrightarrow \Pi r \geq 0 \text{Conf}_r(M)_{\Sigma_r}
\]
\[\text{mor} \text{Exit} \text{ Ran}^u(M) \]

This injection factors through the space of morphisms of \( \text{Exit} \text{ Ran}^u(M) \) by taking the straight-line paths from the origin of each \( T_s M \) to the given configuration in the image \( U_s \) - let us define the factorization \( \text{straight} \) explicitly.

Consider a configuration \( (R_s \subset T_s M)_{s \in S} \). For each \( s \in S \) define
\[
\gamma_s : \bar{C}(R_s) \rightarrow T_s M \times \Delta^1
\]
over \( \Delta^1 \) from the closed cone on the underlying set \( R_s \) to be the map which carries the cone point to the origin of \( T_s M \) at \( t = 0 \in \Delta^1 \) and which carries each \( \{r\} \times \Delta^1 \) to the straight-line path from 0 to \( r \in R_s \). Define the map \( \text{straight} \) in (49) to assign to each configuration \( (R_s \subset T_s M)_{s \in S} \) the morphism in \( \text{Exit} \text{ Ran}^u(M) \) determined by the composites
\[
\bar{C}(R_s) \overset{\gamma_s}{\longrightarrow} T_s M \times \Delta^1 \cong U_s \times \Delta^1 \hookrightarrow M \times \Delta^1
\]
for each \( s \in S \).

By construction, the functor \( \text{straight} \) in (49) factors through the fiber \( \text{ev}^{-1}_0(S) \)
\[
\Pi s \in S \Pi r \geq 0 \text{Conf}_r(T_s M)_{\Sigma_r} \overset{\text{straight}}{\longrightarrow} \text{mor} \text{Exit} \text{ Ran}^u(M) \]
\[\text{Str} \quad \overset{\gamma}{\longrightarrow} \text{ev}^{-1}_0(S).
\]
We will show that the factorization \( \text{Str} \) is a homotopy equivalence by showing that for any commutative square
\[
(50) \quad S^n \quad \rightarrow \Pi s \in S \Pi r \geq 0 \text{Conf}_r(T_s M)_{\Sigma_r} \quad \rightarrow \text{ev}^{-1}_0(S)
\]
\[\text{Str} \quad \overset{\gamma}{\rightarrow} \]
\[\mathbb{D}^{n+1} \rightarrow \text{ev}^{-1}_0(S)
\]
there exists a filler making each triangle commute up to homotopy. Recall that a point in \( \text{ev}^{-1}_0(S) \subset \text{mor} \text{Exit} \text{ Ran}^u(M) \) is the data of a finite proper constructible bundle
\[
\text{cyl}r(T \rightarrow S) \rightarrow \Delta^1
\]
together with an embedding
\[
\text{cyl}r(T \rightarrow S) \hookrightarrow M \times \Delta^1
\]
over \( \Delta^1 \) for some finite set \( T \) and some map \( T \rightarrow S \). (Explicitly, such a description is due to the equivalence \( \text{Exit} \text{ Ran}^u(M) \) \( \cong \) \( \text{Ref}^0(M) \) of Lemma \( 7.1.7 \).) Then, heuristically, a map \( \mathbb{D}^{n+1} \rightarrow \text{ev}^{-1}_0(S) \) is a movie of embeddings \( \text{cyl}r(T \rightarrow S) \hookrightarrow M \times \Delta^1 \) parameterized by \( \mathbb{D}^{n+1} \). Explicitly, this means that a map \( \mathbb{D}^{n+1} \rightarrow \text{ev}^{-1}_0(S) \) is the data of a finite proper constructible bundle
\[
\tilde{K} \rightarrow \mathbb{D}^{n+1} \times \Delta^1
\]

together with an embedding

\[ \tilde{K} \longrightarrow \mathbb{D}^{n+1} \times \Delta^1 \times M \]

\[ \downarrow \]

\[ \mathbb{D}^{n+1} \times \Delta^1 \]

that is constant at \( S \) over \( \mathbb{D}^{n+1} \times \Delta^{(0)} \). For ease of notation, identify \( \Delta^1 \cong [0,1] \) such that \( \Delta^{(0)} = 0 \). Compactness of \( \mathbb{D}^{n+1} \) ensures that there exists \( t \in (0,1] \) such that \( \tilde{K}|_{[0,t]} \) factors through \( \Pi_{s \in S} U_s \subset M \). Further, taking the restriction of \( \tilde{K}|_{[0,1]} \) along \( S^n \to \mathbb{D}^{n+1} \) and scaling by this \( t \) extends to a commutative square (50) for the case that \( M \) is a finite disjoint union of Euclidean spaces. By virtue of the coproduct, we may reduce yet further to the case of a single Euclidean space; i.e., the case that \( S = \{ s \} \) is a singleton. Without loss of generality, choose \( s \) to be the origin. So, showing that the map given by taking straight-line paths from the origin together with an embedding

\[ \Pi_{s \in S} \Pi_{r \geq 0} \text{Conf}_r(T_s M)_{\Sigma_r} \xrightarrow{\text{Str}} \text{ev}^{-1}_0(S) \]

is a homotopy equivalence has been reduced to showing that the same map

(51)

\[ \Pi_{r \geq 0} \text{Conf}_r(\mathbb{R}^n)_{\Sigma_r} \xrightarrow{\text{Str}} \text{ev}^{-1}_0(\{0\}) \]

in the case of \( M = \mathbb{R}^n \) and \( S = \{0\} \) is a homotopy equivalence. In fact, we will explicitly describe such a homotopy equivalence. A map in the other direction

(52)

\[ \text{ev}^{-1}_0(\{0\}) \longrightarrow \Pi_{r \geq 0} \text{Conf}_r(\mathbb{R}^n)_{\Sigma_r} \]

is defined as follows. A point in \( \text{ev}^{-1}_0(\{0\}) \) is a conically smooth embedding over \([0,1]\)

(53)

\[ \tilde{C}(I) \hookrightarrow \mathbb{R}^n \times [0,1] \]

from the closed cone of some finite set \( I \in \text{Fin} \) that carries the cone point to the origin. By conical smoothness, the blowup of such an embedding at the cone point is a smooth map

\[ \text{Bl}(\tilde{C}(I)) = I \times [0,1] \longrightarrow \mathbb{R}^n \]

which is an embedding over \((0,1]\) and which carries each \( \{i\} \times [0,1] \) to a smooth path \( \gamma_i \) in \( \mathbb{R}^n \) from the origin, with non-zero derivative at the origin. The collection of these derivatives determine the map (52). Explicitly, the point (53) is sent to the collection of non-zero initial velocity vectors \( \langle d\gamma_i(0) \subset \mathbb{R}^n \rangle \) associated to it; note that the image names a point in \( \text{Conf}_1(I)(\mathbb{R}^n)_{\Sigma_r} \).

To see that (51) and (52) yield a homotopy equivalence, first note that the composite of (51) followed by (52) is the identity on \( \Pi_{r \geq 0} \text{Conf}_r(\mathbb{R}^n)_{\Sigma_r} \). We describe a homotopy from the identity on \( \text{ev}^{-1}_0(S) \) to the other composite as follows. Let \( \tilde{C}(I) \to \mathbb{R}^n \times \Delta^1 \) be a point in \( \text{ev}^{-1}_0(\{0\}) \). By virtue of approximating smooth functions by piecewise linear functions, for each \( i \in I \), there is a \( t \in [0,1] \) such that the \( i \)-th path in \( \gamma \) projects orthogonally onto the straight ray from the origin in the direction of the initial velocity vector of that path. Choose the minimum such \( t \) over all \( i \in I \). Straight-line homotopy from each path in \( \gamma \) restricted to \([0,t]\) to the straight ray given by path’s initial velocity vector yields the desired homotopy from the identity on \( \text{ev}^{-1}_0(\{0\}) \) to the composite of (52) followed by (51).

\[ \square \]

Applying Lemma 9.2.7 to the case of \( M = \mathbb{R}^n \), we have identified the fiber of \( \text{ev}_0 \) in (44) as desired. The next step is to identify the fiber of the lefthand functor \( \text{Be}_0 \) in (44) as equivalent to that of the righthand functor. The next lemma is the first of two steps towards this - it identifies the classifying space of the fiber of the map from \( \text{Fun}^{W_u}(\{1\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \) to \( \text{Fun}^{W_u}(\{0\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \) given by evaluation at 0.

**Lemma 9.2.8.** The classifying space of the fiber of the functor

\[ \text{Fun}^{W_u}(\{1\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \xrightarrow{\text{ev}_0} \text{Fun}^{W_u}(\{0\}, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \]

51
over an object \( g := S_n \rightarrow S_{n-1} \rightarrow \cdots \rightarrow S_1 \) is the product space
\[
Π_{s \in S_n} \text{Exit}(\text{ran}^n(T_nR^n))
\]
indexed by \( S_n \) of the maximal \( \infty \)-subgroupoid of the exit-path \( \infty \)-category of the unital Ran space of the tangent space of \( R^n \) at \( s \in S_n \).

**Proof.** Fix an object \( g := (S_n \rightarrow S_{n-1} \rightarrow \cdots \rightarrow S_1) \in \text{Fun}(\{0\}, \text{Exit}(\text{ran}^n(R^n))) \). Our procedure for identifying the fiber over \( g \) has two steps. First recall the Fox-Neuwirth stratification of the unordered configuration space \( ν_{Σ_r}: \text{Conf}_r(R^n)_{Σ_r} \rightarrow n\text{Ord}(r)_{Σ_r} \) from (9.1.14). Step 1 is to show that there exists a refinement of the stratified space
\[
Π ν_{Σ_r}: Π s ∈ S_n, Π r ≥ 0 \text{Conf}_r(T_nR^n)_{Σ_r} \rightarrow Π s ∈ S_n Π r ≥ 0 n\text{Ord}(r)_{Σ_r}
\]
where the stratification is given by taking the product of the coproduct of the Fox-Neuwirth stratification of the configuration space of the tangent space of \( R^n \) at \( s \in S_n \). Note that products of stratified spaces emit a natural stratification - see Example 2.1.7 in [6] for details. Step 2 is to show that there exists an adjunction between the exit-path \( \infty \)-category of that refinement and the fiber of \( ev_0 \) over \( g \). Though it is not obvious, together these two steps yield the result; we will explain how at the end of the proof.

Step 1: Just like in the proof of Lemma 9.2.7, for each \( s \in S_n \), choose a smooth open embedding
\[
T_nR^n \hookrightarrow R^n
\]
which carries the origin to \( s \) such that the images \( \{U_s\}_{s ∈ S_n} \) are disjoint. The inclusions \( Π s ∈ S_n U_s \hookrightarrow R^n \) determine an inclusion of the configuration spaces naming an inclusion of the substratified space
\[
\text{Conf}_r\left( Π s ∈ S_n U_s \right)_{Σ_r} \hookrightarrow \text{Conf}_r(R^n)_{Σ_r} \xrightarrow{ν_{Σ_r}} n\text{Ord}(r)_{Σ_r}
\]
consisting of configurations of \( r \) unordered points in the images \( Π s ∈ S_n U_s \subset R^n \) stratified by the Fox-Neuwirth cells. Taking the coproduct over cardinality determines the substratified space
\[
Π r ≥ 0 \text{Conf}_r(Π s ∈ S_n U_s)_{Σ_r} \hookrightarrow Π r ≥ 0 \text{Conf}_r(R^n)_{Σ_r} \xrightarrow{ν_{Σ_r}} Π r ≥ 0 n\text{Ord}(r)_{Σ_r}
\]
consisting of all finite subsets in the images \( Π s ∈ S_n U_s \hookrightarrow R^n \) stratified by the Fox-Neuwirth cells.

The open embeddings (55) induce homeomorphisms \( \text{Conf}_r(T_nR^n)_{Σ_r} \xrightarrow{≃} \text{Conf}_r(U_s)_{Σ_r} \), which in turn identify the topological spaces
\[
Π r ≥ 0 \text{Conf}_r(Π s ∈ S_n U_s)_{Σ_r} \xrightarrow{≃} Π s ∈ S_n Π r ≥ 0 \text{Conf}_r(U_s)_{Σ_r} \cong Π r ≥ 0 \text{Conf}_r\left( Π s ∈ S_n U_s \right)_{Σ_r}
\]
where the last equivalence is canonical because the subsets \( U_s \) are disjoint and we have the empty configuration. The composite of these homeomorphisms (57) is a refinement (c.f. Definition 7.1.4) of stratified spaces, as indicated in the following diagram:
\[
\begin{array}{ccc}
Π r ≥ 0 \text{Conf}_r(Π s ∈ S_n U_s)_{Σ_r} & \xrightarrow{\text{refine}} & Π r ≥ 0 \text{Conf}_r(T_nR^n)_{Σ_r} \\
\downarrow & & \downarrow \\
Π r ≥ 0 n\text{Ord}(r)_{Σ_r} & \longrightarrow & Π r ≥ 0 n\text{Ord}(r)_{Σ_r}
\end{array}
\]
Let us explain further. The left-hand downward arrow is the Fox-Neuwirth stratification \( ν_{Σ_r} \) restricted to configurations in the images \( Π s ∈ S_n U_s \subset R^n \). This stratification remembers coordinate coincidence between any points, even those in distinct \( U_s \) subsets. The right-hand downward arrow is also determined by the Fox-Neuwirth stratification. However, it only remembers the coordinate coincidence between points in the same \( U_s \) subset. Any coordinate coincidence between points in distinct \( U_s \) subsets is ignored. Thus, this map of stratified spaces forgets information, meaning the map between posets is a many-to-one map. Therefore, since the top horizontal arrow is a homeomorphism, each stratum of the domain in (58) is embedded into a stratum of the codomain, which verifies that this map is indeed a refinement.
Step 2: We will show that there is an adjunction between the exit-path $\infty$-category of the stratified space $\Pi_{r \geq 0} \text{Conf}_r (\Pi_{s \in S_n} U_s)_{\Sigma_r}$ and the fiber $e^*_0 \mathcal{S} \subset \text{Fun}^W_n \left([1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right)$. First, we apply the Exit functor (3.3.1 in [4]) to the inclusion of stratified spaces (56) to obtain the top horizontal functor in the following diagram

$$\Pi_{r \geq 0} \text{Exit} \left(\text{Conf}_r \left(\Pi_{s \in S_n} U_s\right)_{\Sigma_r}\right) \xrightarrow{\text{Exit}(56)} \Pi_{r \geq 0} \text{Exit} \left(\text{Conf}_r (\mathbb{R}^n)_{\Sigma_r}\right)$$

(59)

In particular, this functor factors through $\text{Fun}^W_n \left([1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right)$ by taking the straight-line paths from each $s \in S_n$ - the same idea as that of (49) in the proof of Lemma 9.2.7. We will define the value of straight on a $[p]$-point. We will use the model of $W^\text{ht}(r)$ as a simplicial space (inherited from $\text{Exit}(\text{Ran}(\mathbb{R}^n))$) to describe $[p]$-points in $\text{Exit} \left(\text{Conf}_r \left(\Pi_{s \in S_n} U_s\right)_{\Sigma_r}\right)$, taking advantage of the equivalence from (9.1.19). Also note that the $\infty$-category $\text{Fun}^W_n \left([1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right)$ inherits a model as a simplicial space from that of $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$. Recall from Observation 9.1.6 that a $[p]$-point of $\text{Exit} \left(\text{Conf}_r \left(\Pi_{s \in S_n} U_s\right)_{\Sigma_r}\right)$ is an embedding

$$R \times \Delta^p \hookrightarrow \Pi_{s \in S_n} U_s \times \Delta^p$$

(60)

over $\Delta^p$ for a set $R$ of cardinality $r$. Consider the natural decomposition $R = \Pi_{s \in S_n} R_s$ into the subsets of its image contained in each $U_s$. The $[p]$-point (60) can be described as a collection of disjoint embeddings

$$R_s \times \Delta^p \xrightarrow{E_s} U_s \times \Delta^p$$

(61)

one for each $s \in S_n$. The value of this $[p]$-point when $p = 0$ is described by the factorization (49) in the proof of Lemma 9.2.7 using the cone construction, but the same concept applies for any $p$; let us make this explicit. For each $s \in S_n$ define

$$\bar{C}(R_s \times \Delta^p) \xrightarrow{E_s} U_s \times \Delta^{p+1}$$

(62)

over $\Delta^{p+1}$ from the closed cone of the product $R_s \times \Delta^p$ to be the map which carries the cone point to $s \in U_s$ at $t = 0 \in \Delta^{p+1}$ and which carries each $\{r\} \times \{t\} \times \Delta^1$, for $r \in R_s$ and $t \in \Delta^p$, to the straight-line path from $s$ to the point $E_t(r, t)$. The collection of these embeddings (62) for $s \in S_n$ determines a $[p+1]$-point in $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ whose data at $t = 0 \in \Delta^{p+1}$ is the object $\mathcal{S}$. Moreover, this $[p+1]$-point determines the value of (60) under the functor straight; let us explain further. Recall from Observation 9.0.6 that a $[p]$-point in $\text{Fun}^W_n \left([1], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))\right)$ is a commutative diagram in $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ of the shape $[p] \times [1]$ such that the two $p$-simplicies $[p] \cong [p] \times \{0\}$ and $[p] \cong [p] \times \{1\}$ must be $[p]$-points of the $\infty$-subcategory $W_n$. By taking the identity on $\mathcal{S}$ the appropriate number of times, we can build the $[p]$-point (62) of $\text{Exit}(\mathbb{R}^n))$ into a commutative diagram of the shape $[p] \times [1]$. All of the morphisms $\mathcal{S} \to R$ given by straight-line paths in (60) comprise the “$[p]$-thickened walking arrow” of the diagram; namely, all those arrows $\{i\} \times [1]$ for each $0 \leq i \leq p$. These are not necessarily in $W_n$. The arrows of $[p] \times \{0\}$ are all the identity on $\mathcal{S}$, creating a $[p]$-point in $W_n$. Lastly, the $p$-simplex $[p] \times \{0\}$ is precisely the $[p]$-point (60) of $\text{Exit} \left(\text{Conf}_r (\mathbb{R}^n)_{\Sigma_r}\right)$. It, too, is in $W_n$ because $\text{Exit} \left(\text{Conf}_r (\mathbb{R}^n)_{\Sigma_r}\right)$ embeds fully faithfully into $W_n$ by the equivalences in (9.1.7) and (9.1.19): $\Pi_{r \geq 0} \text{Exit} \left(\text{Conf}_r (\mathbb{R}^n)_{\Sigma_r}\right) \simeq \Pi_{r \geq 0} \text{W}^\text{ht}(r) \simeq W_n^\text{ht} \hookrightarrow W_n$. 

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By construction, the functor \textit{straight} in (59) factors through the fiber \(ev_0^{-1}(\Sigma)\)

\[
\Pi_{r \geq 0} \text{Exit} \left( \text{Conf}_r \left( \Pi_{s \in S_n} U_s \right)_{\Sigma_r} \right) \xrightarrow{\text{straight}} \text{Fun}^{W_n} \left( [1], \text{Exit} (\text{Ran}^u(\mathbb{R}^n)) \right) \xrightarrow{\text{Str}} ev_0^{-1}(\Sigma)
\]

We claim that \(\text{Str}\) is a right adjoint functor. This follows from Lemma 9.1.3. Indeed, observe that in the following commutative diagram of \(\infty\)-categories

\[
\Pi_{r \geq 0} \text{Exit} \left( \text{Conf}_r \left( \Pi_{s \in S_n} U_s \right)_{\Sigma_r} \right) \xrightarrow{\text{Str}} ev_0^{-1}(\Sigma)
\]

both vertical arrows are fully faithful. In Lemma 9.1.3, we showed that the bottom horizontal arrow in (63) is a right adjoint. Thus it is that \(\text{Str}\) is a right adjoint functor.

Corollary 2.1.28 of [25] states that the classifying spaces of adjoint \(\infty\)-categories are equivalent. Applying this result yields the equivalence

\[
\mathcal{B} ev_0^{-1}(\Sigma) \simeq \mathcal{B} \Pi_{r \geq 0} \text{Exit} \left( \text{Conf}_r \left( \Pi_{s \in S_n} U_s \right)_{\Sigma_r} \right).
\]

Moreover, Corollary 1.2.7 of [6] identifies the classifying space of a value under the \text{Exit} functor as its underlying topological space. Recall the refinement \(\Pi_{r \geq 0} \text{Conf}_r (\Pi_{s \in S_n} U_s)_{\Sigma_r} \to \Pi_{s \in S_n} \Pi_{r \geq 0} \text{Conf}_r (T_s \mathbb{R}^n)_{\Sigma_r}\) from (58). Refinements always have the same underlying topological space as that which it refines. Thus, the topological space underlying \(\Pi_{r \geq 0} \text{Conf}_r (\Pi_{s \in S_n} U_s)_{\Sigma_r}\) is \(\Pi_{s \in S_n} \Pi_{r \geq 0} \text{Conf}_r (T_s \mathbb{R}^n)_{\Sigma_r}\). Applying Corollary 1.2.7 then, we have the equivalence

\[
\mathcal{B} \Pi_{r \geq 0} \text{Exit} \left( \text{Conf}_r \left( \Pi_{s \in S_n} U_s \right)_{\Sigma_r} \right) \simeq \Pi_{s \in S_n} \Pi_{r \geq 0} \text{Conf}_r (T_s \mathbb{R}^n)_{\Sigma_r},
\]

which together with (64) yields the desired equivalence, since for each \(s \in S_n\), the maximal \(\infty\)-subgroupoid is identified \(\text{Exit} (\text{Ran}^u(T_s \mathbb{R}^n)) \sim \Pi_{r \geq 0} \text{Conf}_r (T_s \mathbb{R}^n)_{\Sigma_r}\).

Let us briefly review: We seek to show an equivalence on the level of fibers in (44). We have just shown that the classifying space of the fiber of

\[
ev_0 : \text{Fun}^{W_n} \left( [1], \text{Exit} (\text{Ran}^u(\mathbb{R}^n)) \right) \to \text{Fun}^{W_n} \left( [0], \text{Exit} (\text{Ran}^u(\mathbb{R}^n)) \right)
\]

over \(\Sigma\) is equivalent to the fiber of

\[
ev_0 : \text{mor} \left( \text{Exit} (\text{Ran}^u(\mathbb{R}^n)) \right) \to \text{Exit} (\text{Ran}^u(\mathbb{R}^n))
\]

over \(S\), as identified in Lemma 9.2.7. Thus, we have yet to show that this fiber over \(S\) is equivalent to the fiber of

\[
\mathcal{B} \text{ev}_0 : \mathcal{B} \text{Fun}^{W_n} \left( [1], \text{Exit} (\text{Ran}^u(\mathbb{R}^n)) \right) \to \mathcal{B} \text{Fun}^{W_n} \left( [0], \text{Exit} (\text{Ran}^u(\mathbb{R}^n)) \right)
\]

over \(\Sigma\), since, in general, the classifying space of a fiber is not inherently equivalent to the fiber of the map induced between classifying spaces. That is to say, for a functor of \(\infty\)-categories \(\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{D}\), the classifying space \(\mathcal{B}(\mathcal{F}^{-1}d)\) of the fiber of \(\mathcal{F}\) over \(d \in \mathcal{D}\) is not necessarily equivalent to the homotopy fiber \((\mathcal{B}\mathcal{F})^{-1}(d)\) of the map induced between the classifying spaces \(\mathcal{B}\mathcal{C} \xrightarrow{\mathcal{B}\mathcal{F}} \mathcal{B}\mathcal{D}\) over \(d \in \mathcal{B}\mathcal{D}\). Quillen’s Theorem B identifies the homotopy fibers of the map induced between classifying spaces in favorable cases.
Theorem 9.2.9 (Quillen’s Theorem B). Given a functor between ∞-categories \( C \xrightarrow{F} D \), if each morphism \( d \xrightarrow{\xi} d' \) in \( D \) induces a weak equivalence \( B(\xi) \xrightarrow{\simeq} B(d'/d) \) between the classifying spaces of the induced ∞-undercategories, then \( B(d'/d) \) is the homotopy fiber of \( B F \) over \( d \) and thus, \( B(\xi) \xrightarrow{\simeq} B C \xrightarrow{B F} B D \) is a fiber sequence.

Remark 9.2.10. Quillen originally proved Theorem B in [28] for categories. Theorem 5.16 in [3] generalizes the result for ∞-categories, which is the statement of Quillen’s Theorem B given above.

We apply Quillen’s Theorem B for the situation at hand and identify the homotopy fibers of \( B e v_0 \) as follows.

Lemma 9.2.11. The fiber of the map of spaces

\[
B \text{Fun}^W_n \left( [1], \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) \right) \xrightarrow{B e v_0} B \text{Fun}^W_n \left( [0], \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) \right)
\]

over an object \( \mathcal{S} = S_n \to \cdots \to S_1 \) is equivalent to the classifying space of the fiber of \( e v_0 \) over \( \mathcal{S} \).

\[
\left( B e v_0 \right)^{-1} (\mathcal{S}) \simeq B \left( e v_0^{-1} (\mathcal{S}) \right).
\]

Proof. Fix a morphism \( \mathcal{S} \xrightarrow{\alpha} \mathcal{S}' \) in \( \text{Fun}^W_n \left( [0], \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) \right) \). Recall that the induced monodromy functor \( \alpha^* \) is defined as the composite

\[
\xymatrix{ ev_0^{-1} (\mathcal{S}') \ar[r]^-{\alpha^*} \ar[d] & ev_0^{-1} (\mathcal{S}) \ar[d]^-{\mu} \\
\text{Fun}^W_n \left( [1], \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) \right) \ar[r]^-{\mathcal{S}'} & \text{Fun}^W_n \left( [1], \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) \right) \ar[r]^-{\mathcal{S}'} &}
\]

where \( \mu \) is right adjoint to inclusion (which exists by Lemma 2.20 of [3]). The diagram induced upon taking classifying spaces

\[
\xymatrix{ B e v_0^{-1} (\mathcal{S}') \ar[r]^-{B \alpha^*} \ar[d]^-{\simeq} & B e v_0^{-1} (\mathcal{S}) \ar[d]^-{\simeq} \\
B \text{Fun}^W_n \left( [1], \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) \right) \ar[r]^-{B \mathcal{S}'} & B \text{Fun}^W_n \left( [1], \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) \right) \ar[r]^-{\mathcal{S}'} &}
\]

yields the vertical arrows to be equivalences by Corollary 2.1.28 of [25].

We seek to show that \( B \alpha^* \) is an equivalence. Indeed, our desired equivalence

\[
\left( B e v_0 \right)^{-1} (\mathcal{S}) \simeq B \left( e v_0^{-1} (\mathcal{S}) \right)
\]

will follow by Quillen’s Theorem B; let us explain further. If \( \alpha^* \) is an equivalence, then, according to (65), \( B (- \circ \alpha) \) is an equivalence. By Quillen’s Theorem B, \( B (- \circ \alpha) \) being an equivalence yields an equivalence

\[
\left( B e v_0 \right)^{-1} (\mathcal{S}) \simeq B \text{Fun}^W_n \left( [1], \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) \right) \mathcal{S}'.
\]

The upward vertical arrow in (65) is an equivalence between \( B \text{Fun}^W_n \left( [1], \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) \right) \mathcal{S}' \) and \( B e v_0^{-1} (\mathcal{S}) \). Thus, \( \left( B e v_0 \right)^{-1} (\mathcal{S}) \) is equivalent to \( B e v_0^{-1} (\mathcal{S}) \).

First, consider the diagram

\[
\xymatrix{ \text{Fun}^W_n \left( [1], \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) \right) \ar[r]^-{\text{frgt}} \ar[d]_{e v_0} & \text{mor} \left( \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) \right) \ar[d]^{e v_0} \\
\text{Fun}^W_n \left( [0], \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) \right) \ar[r]^-{\simeq} & \text{Exit} \left( \text{Ran}^u \left( \mathbb{R}^n \right) \right) }
\]

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in which each vertical arrow is a Cartesian fibration. By Observation 9.2.6, the induced monodromy functor $\alpha^*$ is carried by the forgetful functor to the induced monodromy functor of the image of $\alpha$ under the forgetful functor, frgt($\alpha$)∗:

$$\begin{array}{cccc}
ev_0^{-1}(S') & \xrightarrow{\alpha^*} & ev_0^{-1}(S) \\
\downarrow_{\text{frgt}} & & \downarrow_{\text{frgt}} \\
ev_0^{-1}(S'_n) & \xrightarrow{\text{frgt}(\alpha)^*} & ev_0^{-1}(S_n).
\end{array}$$

(67)

Observe that frgt($\alpha$)∗ is an equivalence precisely because the image of the morphism $\alpha$ under the forgetful functor, $S_n \xrightarrow{\alpha_n} S'_n$, is a bijection. We apply the universal property of localization to the canonical localization $ev_0^{-1}(S) \to Bev_0^{-1}(S)$ to obtain

$$\begin{array}{cccc}
ev_0^{-1}(S) & \xrightarrow{\text{frgt}} & ev_0^{-1}(S) \simeq \Pi_{s \in S_n} \text{Exit}(\text{Ran}^u(T_s\mathbb{R}^n)_{\Sigma_s}) \sim \\
\downarrow & & \text{frgt} \quad \sim \\
Bev_0^{-1}(S) & \simeq \Pi_{s \in S_n} \text{Exit}(\text{Ran}^u(T_s\mathbb{R}^n)_{\Sigma_s}) \sim
\end{array}$$

(68)

and observe that such a filler must be an equivalence. We paste (67) and (68) for $S$ and $S'$ together to see that $\mathcal{B}\alpha^*$ is an equivalence:

$$\begin{array}{cccc}
ev_0^{-1}(S') & \xrightarrow{\alpha^*} & ev_0^{-1}(S) \\
\downarrow_{\text{frgt}} & & \downarrow_{\text{frgt}} \\
ev_0^{-1}(S'_n) & \xrightarrow{\sim} & Bev_0^{-1}(S'_n) \xrightarrow{\sim} Bev_0^{-1}(S) \\
\text{loc} & & \text{loc} & & \text{frgt} \\
\text{frgt} & & \text{frgt} & & \text{frgt}
\end{array}$$

According to (65), the map $\mathcal{B}\alpha^*$ being an equivalence implies that the map $\mathcal{B}(- \circ \alpha)$ is an equivalence. Thus, by Quillen’s Theorem B, the fiber of $Bev_0$ over $S$ is identified as $\mathcal{B}\text{Fun}^W_n([-1, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))])$, which, according to (65), is equivalent to $Bev_0^{-1}(S)$.

□

With Lemma 9.2.7, Lemma 9.2.8, and Lemma 9.2.11, we have just identified that the map induced between the fibers of 44 is an equivalence. Lemma 9.2.1, which states that the map induced by the forgetful functor $\mathcal{B}\text{Fun}^W_n([-1, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))]) \to \text{mor}(\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))$ is an equivalence, follows almost immediately. The proof is as follows.

Proof of Lemma 9.2.1. We first recall (44):

$$\begin{array}{cccc}
\mathcal{B}\text{Fun}^W_n([-1, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))]) & \xrightarrow{\text{frgt}} & \text{mor}(\text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \\
\downarrow_{\text{ev}_0} & & \downarrow_{\text{ev}_0} \\
\mathcal{B}\text{Fun}^W_n([0, \text{Exit}(\text{Ran}^u(\mathbb{R}^n))]) \xrightarrow{\sim} \text{Exit}(\text{Ran}^u(\mathbb{R}^n))
\end{array}$$

(69)

The equivalence in Lemma 9.1.1,

$$\mathcal{B}W_n \simeq \text{Exit}(\text{Ran}^u(\mathbb{R}^n))$$
is induced by the forgetful functor from \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) to \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \); this is precisely the map in (69), since the map between total spaces is induced by the forgetful functor, and the domain \( \mathcal{B}\text{Fun}^W_n([0], \text{Exit}(\text{Ran}^n(\mathbb{R}^n))) \) is inherently equivalent to \( \mathcal{B}W_n \).

In Lemma 9.2.11, through Lemma 9.2.11, we identified the fiber of the vertical map on the left of (69), \( \mathcal{B}\text{ev}_0 \), over \( S \in \mathcal{B}\text{Fun}^W_n([0], \text{Exit}(\text{Ran}^n(\mathbb{R}^n))) \) as the product \( \prod_{s \in S_n} \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \), where \( T_s \mathbb{R}^n \) is the tangent space of \( \mathbb{R}^n \) at \( s \in S_n \subset \mathbb{R}^n \). In Lemma 9.2.7, we identified the fiber of the map on the right of (69), \( \text{ev}_0 \), over the image \( S_n \in \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) as the product \( \prod_{s \in S_n} \text{Exit}(\text{Ran}^n(T_s \mathbb{R}^n)) \).

Thus, the induced long exact sequence in homotopy of each fibration induces a weak homotopy equivalence between the total spaces, which is, in fact, a homotopy equivalence since the total spaces are CW complexes. □

We have finally proven Lemma 9.0.8, the first of two lemmas that we will use to prove the main result of this section, Lemma 9.0.3.

9.3. **Proving Lemma 9.0.9.** The main goal of this subsection is to prove Lemma 9.0.9 which states that the simplicial space \( \mathcal{B}\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^n(\mathbb{R}^n))) \) is a complete Segal space (7.1.1). The proof is technical, also using Cartesian fibrations and Quillen’s Theorem B, and builds off of arguments developed in §9.2. Before we can prove Lemma 9.0.9, we need the following lemma.

**Lemma 9.3.1.** Given a pullback of \( \infty \)-categories

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{G} & \mathcal{E}' \\
\pi \downarrow & & \downarrow \pi' \\
\mathcal{B} & \xrightarrow{F} & \mathcal{B}'
\end{array}
\]

in which \( \pi' \) is a Cartesian fibration, if \( \pi' \) satsifies Quillen’s Theorem B, then so does \( \pi \).

**Proof.** Let \( b \xrightarrow{f} b' \) be a morphism in \( \mathcal{B} \). Lemma 2.8 of [3] states that Cartesian fibrations are closed under base change. Thus, \( \pi' \) being a Cartesian fibration and (70) being a pullback implies that \( \pi \) is a Cartesian fibration. By Observation 9.2.6 then, \( G \) carries the induced monodromy functor \( f^* \) to the induced monodromy functor \( F(f)^* \) and yields equivalences between the fibers:

\[
\begin{array}{ccc}
\pi^{-1}(b') & \xrightarrow{f^*} & \pi^{-1}(b) \\
\cong \downarrow & & \downarrow \cong \\
\pi'^{-1}(F(b')) & \xrightarrow{F(f)^*} & \pi'^{-1}(F(b))
\end{array}
\]

Combining this diagram with the definition of the monodromy functor yields

\[
\begin{array}{ccc}
\mathcal{E}^{b'} & \xrightarrow{- \circ f} & \mathcal{E}^{b} \\
\pi^{-1}(b') & \xrightarrow{f^*} & \pi^{-1}(b) \\
\cong \downarrow & & \downarrow \cong \\
\pi'^{-1}(F(b')) & \xrightarrow{F(f)^*} & \pi'^{-1}(F(b)) \\
\mathcal{E}^{F(b')} & \xrightarrow{- \circ F(f)} & \mathcal{E}^{F(b)}
\end{array}
\]

The diagram induced upon taking classifying spaces yields the desired result. Indeed, \( \pi' \) satsifying Quillen’s Theorem B implies \( \mathcal{B}(- \circ F(f)) \) is a equivalence and thus, each horizontal arrow resulting
between classifying spaces is an equivalence, which in particular means $\mathcal{B}(- \circ f)$ is an equivalence:

\[
\begin{array}{c}
\mathcal{B}E_b/ \\
\simeq \downarrow \quad \simeq \downarrow \\
\mathcal{B}\pi^{-1}(b') \xrightarrow{\mathcal{B}f^*} \mathcal{B}\pi^{-1}(b) \\
\simeq \downarrow \quad \simeq \downarrow \\
\mathcal{B}\pi'^{-1}(F(b')) \xrightarrow{\mathcal{B}(\sigma F)(f)^*} \mathcal{B}\pi'^{-1}(F(b)) \\
\simeq \downarrow \quad \simeq \downarrow \\
\mathcal{B}E^{F(b)}/ \xrightarrow{\mathcal{B}(\sigma F)(f)} \mathcal{B}E^{F(b)}/. \\
\end{array}
\]

**Observation 9.3.2.** The following diagram of $\infty$-categories is a pullback:

\[
\begin{array}{c}
\mathcal{B}\Fun^W_n \left( [p], \Exit(\Ran^u(R^n)) \right) \\
\downarrow \quad \downarrow \\
\mathcal{B}\Fun^W_n \left( \{0 < \cdots < p-1\}, \Exit(\Ran^u(R^n)) \right) \xrightarrow{\tau} \mathcal{B}\Fun^W_n \left( \{1 - p < p\}, \Exit(\Ran^u(R^n)) \right). \\
\end{array}
\]

(72)

Indeed, for an $\infty$-category $\mathcal{C}$, $\Fun(\bullet, \mathcal{C})$ satisfies the Segal condition, i.e., for each $p \geq 2$, the diagram obtained by replacing $\Fun^W_n \left( [p], \Exit(\Ran^u(R^n)) \right)$ with $\Fun(\bullet, \mathcal{C})$ in (72) is pullback. Using this, it is straightforward to show (72) is pullback.

We now prove Lemma 9.0.9 which states that the simplicial space $\mathcal{B}\Fun^W_n \left( \bullet, \Exit(\Ran^u(R^n)) \right)$ is a complete Segal space.

**Proof of Lemma 9.0.9.** First, we will show that $\mathcal{B}\Fun^W_n \left( \bullet, \Exit(\Ran^u(R^n)) \right)$ satisfies the Segal condition. Consider the diagram of spaces obtained by taking the classifying spaces of (72)

\[
\begin{array}{c}
\mathcal{B}\Fun^W_n \left( [p], \Exit(\Ran^u(R^n)) \right) \\
\downarrow \quad \downarrow \\
\mathcal{B}\Fun^W_n \left( \{0 < \cdots < p-1\}, \Exit(\Ran^u(R^n)) \right) \xrightarrow{\mathcal{B}\tau} \mathcal{B}\Fun^W_n \left( \{p - 1 < p\}, \Exit(\Ran^u(R^n)) \right). \\
\end{array}
\]

(73)

To show that this diagram is a pullback, we will show that the map induced between fibers of (73) is an equivalence. By Observation 9.2.3, the functor $s$ in (72) is a Cartesian fibration. Further, in the proof of Lemma 9.2.11, we showed that $s$ satisfies Quillen’s Theorem B. Thus, (72) satisfies the hypothesis’ of Lemma 9.3.1 and we identify the fibers of $\mathcal{B}\sigma$ and $\mathcal{B}s$ over the objects

$S_0 \to \cdots \to S_{p-1}$ in $\mathcal{B}\Fun^W_n \left( \{0 < \cdots < p-1\}, \Exit(\Ran^u(R^n)) \right)$

and

$S_{p-1}$ in $\mathcal{B}\Fun^W_n \left( \{p - 1\}, \Exit(\Ran^u(R^n)) \right)$

respectively, as the classifying spaces of the fibers of $\sigma$ and $s$ over $S_0 \to \cdots \to S_{p-1}$ and $S_{p-1}$, respectively:

$$(\mathcal{B}\sigma)^{-1}(S_0 \to \cdots \to S_{p-1}) \simeq \mathcal{B}\sigma^{-1}(S_0 \to \cdots \to S_{p-1})$$

and

$$(\mathcal{B}s)^{-1}(S_{p-1}) \simeq \mathcal{B}s^{-1}(S_{p-1}).$$
Therefore, because (72) being a pullback implies an equivalence between fibers induced by \( \tau \)
\[
\tau : \sigma^{-1}(S_0 \to \cdots \to S_{p-1}) \cong s^{-1}(S_{p-1})
\]
there results an equivalence between fibers of (73) given by \( \mathcal{B}\tau \)
\[
(\mathcal{B}\sigma)^{-1}(S_0 \to \cdots \to S_{p-1}) \cong \mathcal{B}\sigma^{-1}(S_0 \to \cdots \to S_{p-1}) \cong \mathcal{B}s^{-1}(S_{p-1}) \cong (\mathcal{B}s)^{-1}(S_{p-1})
\]
which verifies that (73) is a pullback.

Then, Lemma 9.0.8 extends to an equivalence of spaces
\[
\mathcal{B}\text{Fun}^W_n([p], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \cong \text{Hom}_{\mathcal{C}at_{\infty}}([p], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))
\]
for each \( p \geq 0 \), since \( \mathcal{B}\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \) satisfying the Segal condition means that its values on \([0]\) and \([1]\) determine all of its higher \([p]\) values. This, in particular, implies that \( \mathcal{B}\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \) is a complete Segal space, since \( \text{Hom}_{\mathcal{C}at_{\infty}}([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \) is a complete Segal space precisely because \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) is a complete Segal space.

We have just verified the hypothesis of Theorem 9.0.7, which means that the localization of \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) on \( W_n \) is equivalent to the simplicial space \( \text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \). Using Lemma 9.0.8, we now prove Lemma 9.0.3, which states that the forgetful functor localizes \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) on \( W_n \) to \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \).

**Proof of Lemma 9.0.3.** In the proof of the previous Lemma 9.0.9, we showed
\[
\mathcal{B}\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \cong \text{Hom}_{\mathcal{C}at_{\infty}}([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))
\]
which, in particular means that the hypothesis of Theorem 9.0.7 is satisfied. Thus, by Theorem 9.0.7,
\[
\text{Exit}(\text{Ran}^u(\mathbb{R}^n))[W_n^{-1}] \cong \mathcal{B}\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))).
\]
Then, by the equivalence (74), we have an equivalence of simplicial spaces
\[
\text{Exit}(\text{Ran}^u(\mathbb{R}^n))[W_n^{-1}] \cong \text{Hom}_{\mathcal{C}at_{\infty}}([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n)))
\]
which establishes that \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) localizes on \( W_n \) to \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \).

Note that this localization is given by the forgetful functor from \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) to \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) because our identification of \( \mathcal{B}\text{Fun}^W_n([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \) with \( \text{Hom}_{\mathcal{C}at_{\infty}}([\bullet], \text{Exit}(\text{Ran}^u(\mathbb{R}^n))) \) was induced by the forgetful functor; this was proven in Lemma 9.2.1.

Lastly, we will show this localization is over \( \text{Fin}^{op} \). In Observation 8.0.6, we observed that the forgetful functor from \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) to \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) is naturally over \( \text{Fin}^{op} \) by just remembering the data of underlying sets at the \( \mathbb{R}^n \) level. Then, by the universal property of localization, we have:

\[
\text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \xrightarrow{\text{frkt}} \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \cong \text{Exit}(\text{Ran}^u(\mathbb{R}^n))[W_n^{-1}]
\]

The unique existence of such a filler is guaranteed because each morphism in \( W_n \) gets carried to isomorphisms in \( \text{Fin}^{op} \) under \( \phi_n \). Thus, we see that the forgetful functor from \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) to \( \text{Exit}(\text{Ran}^u(\mathbb{R}^n)) \) yields a localization over \( \text{Fin}^{op} \).

\( \square \)
This concludes the proof of Lemma 9.0.3 thereby finishing the proof of our main result, Theorem 7.2.1, that the category \( \Theta_n^{\text{act}} \) localizes to the \( \infty \)-category \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \). An immediate corollary to Theorem 7.2.1 is that the wreath product decomposition of \( \Theta_2 \) induces a likewise decomposition of \( \text{Exit}(\text{Ran}^n(\mathbb{R}^2)) \).

**Corollary 9.3.3.** There is a localization of \( \infty \)-categories

\[
\text{Exit}(\text{Ran}^n(\mathbb{R})) \downarrow \text{Exit}(\text{Ran}^n(\mathbb{R})) \to \text{Exit}(\text{Ran}^n(\mathbb{R}^2))
\]

from the two fold wreath product of the exit-path \( \infty \)-category of the unital Ran space of \( \mathbb{R} \) with itself and the exit-path \( \infty \)-category of the unital Ran space of \( \mathbb{R}^2 \).

**10. A corollary to Theorem 7.2.1: \( \Theta_n^{\text{exit}} \) localizes to \( \text{Exit}(\text{Ran}(\mathbb{R}^n)) \)**

In this section, we identify the exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \) as a localization of a certain subcategory of \( \Theta_n^{\text{act}} \). This result specializes the localization of Theorem 7.2.1 to a localization of the \( \infty \)-subcategory \( \text{Exit}(\text{Ran}(\mathbb{R}^n)) \) of \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) by restricting to this subcategory.

Following Definition 1.2.1, the exit-path \( \infty \)-category of the Ran space of a smooth connected manifold \( M \) is the simplicial set whose value on a simplex \( [k] \in \Delta \) is

\[
\{ \Delta_k \xrightarrow{f} \text{Ran}(M) \mid f \text{ is a stratified map} \}
\]

the set of continuous, stratified maps from the topological \( k \)-simplex stratified over \([k]\) to the Ran space of \( M \) stratified over \( \mathbb{N} \). So, heuristically, an object of \( \text{Exit}(\text{Ran}(M)) \) is a finite subset \( S \subset M \) and a morphism is a path in \( \text{Ran}(M) \) that witnesses anticollision of points.

In Observation 7.0.6, we observed that for a connected manifold \( M \) \( \text{Exit}(\text{Ran}(M)) \) is equivalently the sub simplicial space of \( \text{Exit}(\text{Ran}^n(M)) \) defined as the pullback of simplicial spaces

\[
\begin{array}{ccc}
\text{Exit}(\text{Ran}(M)) & \xrightarrow{\phi} & \text{Exit}(\text{Ran}^n(M)) \\
\downarrow & & \downarrow \\
\text{Fin}^{\text{surj}}_{\not\emptyset}^{\text{op}} & \xrightarrow{\sigma} & \text{Fin}^{\text{op}}
\end{array}
\]

where \( \text{Fin}^{\text{surj}}_{\not\emptyset}^{\text{op}} \) is the subcategory of finite sets consisting of nonempty sets and all those morphisms that are surjections. In light of this observation, we deduced that \( \text{Exit}(\text{Ran}(M)) \) is an \( \infty \)-category by virtue of it being a pullback of \( \infty \)-categories (Corollary 7.1.3).

The main result of this section identifies the exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \) as a localization of the following subcategory of \( \Theta_n^{\text{act}} \).

**Definition 10.0.1.** The category \( \Theta_n^{\text{exit}} \) is the subcategory of \( \Theta_n^{\text{act}} \) defined as the pullback

\[
\begin{array}{ccc}
\Theta_n^{\text{exit}} & \xrightarrow{\tau} & \Theta_n^{\text{act}} \\
\downarrow & & \downarrow \\
\text{Fun}(\{1 < \cdots < n\}, (\text{Fin}^{\text{surj}}_{\not\emptyset}^{\text{op}})) & \xrightarrow{\gamma} & \text{Fun}(\{1 < \cdots < n\}, \text{Fin}^{\text{op}})
\end{array}
\]

where we recall the functor \( \tau \) from Observation 6.1.10 (defined by the truncation functor \( \text{tr}_i \) and \( \gamma_i \)).

Heuristically, \( \Theta_n^{\text{exit}} \) consists of healthy trees as its objects and all those morphisms that induce surjections between the sets of leaves. We state the main result of this section, which articulates the sense in which the exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \) is identified combinatorially in terms of \( \Theta_n \).

**Corollary 10.0.2.** There is a localization

\[
\Theta_n^{\text{exit}} \to \text{Exit}(\text{Ran}(\mathbb{R}^n))
\]

over \( (\text{Fin}^{\text{surj}}_{\not\emptyset}^{\text{op}}) \) from the subcategory \( \Theta_n^{\text{exit}} \) of \( \Theta_n \) to the exit-path \( \infty \)-category of the Ran space of \( \mathbb{R}^n \).
The localizing subcategory of $\Theta_n^{\text{exit}}$ is nearly equivalent to the subcategory $W_n^{\text{ht}}$ of $\Theta_n^{\text{act}}$ from Definition 9.1.2, which consists of healthy trees as its objects and all those morphisms in $\Theta_n$ that induce bijections on the sets of leaves under $\gamma_n$ to $\text{Fin}$. Note that here we are considering $W_n^{\text{ht}}$ as a subset of $\Theta_n^{\text{act},\text{ht}}$ in light of the equivalence $\text{Exit}(\text{Ran}^n(\mathbb{N})) \simeq \Theta_n^{\text{act},\text{ht}}$ of Lemma 8.2.2. We implement a slight abuse of notation as we will denote the localizing subcategory of Corollary 10.0.2 as $W_n^{\text{ht}}$ as well, which is distinct from $W_n^{\text{ht}} \subset \Theta_n^{\text{ht}}$ only in that it does not have the empty tree as an object. We will always clarify which category $W_n^{\text{ht}}$ we mean by indicating contextually which category it is a subcategory of, namely, $W_n^{\text{ht}} \subset \Theta_n^{\text{exit}}$ or $W_n^{\text{ht}} \subset \Theta_n^{\text{act},\text{ht}}$. Formally, we define the localizing subcategory of Corollary 10.0.2 as follows.

**Definition 10.0.3.** The subcategory $W_n^{\text{ht}}$ of $\Theta_n^{\text{exit}}$ is defined to be the pullback

$$
\begin{array}{ccc}
W_n^{\text{ht}} & \longrightarrow & W_n^{\text{ht}} \\
\downarrow & & \downarrow \\
\Theta_n^{\text{exit}} & \longrightarrow & \Theta_n^{\text{act},\text{ht}}
\end{array}
$$

of categories.

Heuristically then, $W_n^{\text{ht}} \subset \Theta_n^{\text{exit}}$ has all those nonempty, healthy trees of $\Theta_n$ as its objects and only those morphisms that induce bijections between the sets of leaves.

The proof of Corollary 10.0.2 is similar to that of Lemma 9.0.3 in that we identify the localization of $\Theta_n^{\text{exit}}$ on $W_n^{\text{ht}}$ using Theorem 9.0.7, which states that if the simplicial space $\mathcal{B}\text{Fun}^n_W([\bullet], \Theta_n^{\text{exit}})$ is a complete Segal space, then it is equivalent to the localization of $\Theta_n^{\text{exit}}$ on $W_n^{\text{ht}}$. We do this by extrapolating the argument of Lemma 9.0.3, using the fact that the domain and codomain of Corollary 10.0.2 are sub-$\infty$-categories of the domain and codomain of the localization of Lemma Theorem 7.2.1.

First, we compile the following lemma, which follows (not immediately) from Lemma 9.1.3 wherein we showed that there is an adjunction between $W_n^{\text{ht}}$ and $W_n$.

**Lemma 10.0.4.** For each $p \geq 0$, the inclusion functor between $\infty$-categories

$$
\text{Fun}^n_W([p], \Theta_n^{\text{act},\text{ht}}) \hookrightarrow \text{Fun}^n_W([p], \Theta_n^{\text{act}})
$$

induces an between their classifying spaces

$$
\mathcal{B}\text{Fun}^n_W([p], \Theta_n^{\text{act},\text{ht}}) \xrightarrow{\simeq} \mathcal{B}\text{Fun}^n_W([p], \Theta_n^{\text{act}}).
$$

**Proof.** First, observe that we can describe the subcategory $W_n^{\text{ht}}$ of $W_n$ as the following pullback of categories over $\Theta_n^{\text{act},\text{ht}}$:

$$
\begin{array}{ccc}
W_n^{\text{ht}} & \longrightarrow & W_n \\
\downarrow & & \downarrow \\
\Theta_n^{\text{act},\text{ht}} & \longrightarrow & \Theta_n^{\text{act}}
\end{array}
$$

where we recall that $W_n$ consists of all the same objects as $\Theta_n^{\text{act}}$ and all those morphisms that induce bijections on the sets of leaves, and $W_n^{\text{ht}}$ is the full subcategory consisting of only those trees that are healthy.

In Lemma 9.1.3, we showed that the inclusion functor $W_n^{\text{ht}} \hookrightarrow W_n$ is a right adjoint. The reader may observe that no where in the proof did we use that the morphisms of $W_n^{\text{ht}}$ and $W_n$ induce bijection between their sets of leaves. Thus, Lemma 9.1.3 immediately extends to an adjunction between $\Theta_n^{\text{act},\text{ht}}$ and $\Theta_n^{\text{act}}$ whose right adjoint is given by inclusion. Further, observe that the unit transformation of this adjunction is given by morphisms in $W_n$. Indeed, for each tree $T \in \Theta_n^{\text{act}}$, the morphism assigned to $T$ by the unit is $T \xrightarrow{\epsilon_T} P_n(T)$, which, in particular, induces a bijection on the leaves, and is thus in $W_n$. In identifying that the unit of the right adjoint $\Theta_n^{\text{act},\text{ht}} \hookrightarrow \Theta_n^{\text{act}}$ is given
by morphisms in $W_n$, we may extend this adjunction to an adjunction between $\Fun^\text{Whlt}_n([p], \Theta_n^{\text{act, hlt}})$ and $\Fun^W_n([p], \Theta_n^{\text{act}})$ whose right adjoint is inclusion.

Corollary 2.1.28 in [25] states that the classifying space of an adjunction is an equivalence of spaces. Thus, upon taking the classifying space of the right adjoint

$$\Fun^\text{Whlt}_n([p], \Theta_n^{\text{act, hlt}}) \hookrightarrow \Fun^W_n([p], \Theta_n^{\text{act}}),$$

there results the desired equivalence of spaces. □

We will compile one more lemma before commencing with the proof of Corollary 10.0.2. First, note that the inclusion of the subcategory $\Theta_n^{\text{exit}} \hookrightarrow \Theta_n^{\text{act, hlt}}$ together with the induced inclusion of the respective subcategories $W_n^{\text{hlt}} \hookrightarrow W_n^{\text{h}}$ guarantees that for each $p \geq 0$, the induced map

$$\Fun^W_n([p], \Theta_n^{\text{exit}}) \hookrightarrow \Fun^W_n([p], \Theta_n^{\text{act, hlt}})$$

is also an inclusion of an $\infty$-subcategory. Towards the proof of Corollary 10.0.2, we need that the map induced between the classifying spaces of this inclusion is, in particular, a monomorphism. This is articulated by the following lemma.

**Lemma 10.0.5.** For each $p \geq 0$, the inclusion functor

$$\Fun^\text{Whlt}_n([p], \Theta_n^{\text{exit}}) \hookrightarrow \Fun^W_n([p], \Theta_n^{\text{act, hlt}})$$

induces a monomorphism between classifying spaces

$$\mathcal{B}\Fun^\text{Whlt}_n([p], \Theta_n^{\text{exit}}) \hookrightarrow \mathcal{B}\Fun^W_n([p], \Theta_n^{\text{act, hlt}}).$$

Towards the proof of this lemma, we record a technical result, Lemma 7.1.2, which involves the following notion.

**Definition 10.0.6.** A functor $\mathcal{C} \to \mathcal{D}$ between $\infty$-categories is an inclusion of a cofactor if there is an $\infty$-category $\mathcal{E}$ and an equivalence between $\infty$-categories under $\mathcal{C}$:

$$\mathcal{C} \amalg \mathcal{E} \cong \mathcal{D}.$$

**Lemma 10.0.7.** A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is an inclusion of a cofactor if and only if $F$ is a monomorphism and for each solid commutative square

(75)

for either $\nu := (0)$ or $\nu := (1)$, there exists a filler.

**Proof.** First, notice that if $F$ is a monomorphism and (75) is satisfied (with the two possible lifts), then $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is fully faithful. Consider the full $\infty$-subcategory $\mathcal{E} \subset \mathcal{D}$ consisting of those objects that are not isomorphic to objects in the image of $\mathcal{C} \to \mathcal{D}$. Consider the canonical functor

$$\mathcal{C} \amalg \mathcal{E} \to \mathcal{D},$$

which is canonically under $\mathcal{C}$. By design, this functor is essentially surjective, and fully faithful. This established the implication that $F$ being a monomorphism and satisfying (75) implies $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is an inclusion of a cofactor.

We now show the converse. Suppose there is an $\infty$-category $\mathcal{E}$ together with an equivalence $\mathcal{C} \amalg \mathcal{E} \simeq \mathcal{D}$ under $\mathcal{C}$. Consider a solid diagram

(75)

The functor $[0] \xrightarrow{\nu} [1]$ has the feature that every object in $[1]$ admits a morphism to or from an object in the image of $\nu$. It follows that there is a unique filler, as desired. □
Proof of Lemma 10.0.7. First, we will verify that the functor

\[
\text{Fun}^\text{Wh}([p], \Theta_n^\text{exit}) \hookrightarrow \text{Fun}^\text{Wh}([p], \Theta_n^\text{act, hit})
\]

is an inclusion of a cofactor; that is, we will show that this functor is a monomorphism and satisfies (75).

First, note that because \( \text{Fun}^\text{Wh}([p], \Theta_n^\text{exit}) \hookrightarrow \text{Fun}^\text{Wh}([p], \Theta_n^\text{act, hit}) \) is an inclusion of \( \infty \)-categories, it is in particular a monomorphism.

Similar to Observation 9.3.2, it is straightforward to verify that for each \( p \geq 0 \) the following diagram of \( \infty \)-categories

\[
\begin{array}{ccc}
\text{Fun}^\text{Wh}([p], \Theta_n^\text{exit}) & \hookrightarrow & \text{Fun}^\text{Wh}([1-p], \Theta_n^\text{exit}) \\
\downarrow & & \downarrow \\
\text{Fun}^\text{Wh}([0 \cdots p-1], \Theta_n^\text{exit}) & \hookrightarrow & \text{Fun}^\text{Wh}([p-1], \Theta_n^\text{exit}).
\end{array}
\]

(76)

is pullback, as is the diagram obtained by just replacing \( \Theta_n^\text{exit} \) with \( \Theta_n^\text{act, hit} \). Thus, to show that (75) for our situation, namely

\[
\begin{array}{ccc}
[0] & \longrightarrow & \text{Fun}^\text{Wh}([p], \Theta_n^\text{exit}) \\
\downarrow & & \downarrow \\
[1] & \longrightarrow & \text{Fun}^\text{Wh}([p], \Theta_n^\text{act, hit})
\end{array}
\]

(77)

is satisfied for all \( p \geq 0 \), it suffices to show for the cases for \( p = 0, 1 \). Indeed, it is straightforward to verify that the cases \( p = 0, 1 \) imply each \( p \geq 0 \) case upon applying the universal property of pullback from (76) and the diagram obtained by replacing \( \Theta_n^\text{exit} \) with \( \Theta_n^\text{act, hit} \) in (72).

Both cases, \( p = 0, 1 \), come down to the following observation: Each morphism in \( W_n^\text{hit} \) between objects in \( \Theta_n^\text{act, hit} \) is a morphism in \( \Theta_n^\text{exit} \). The root reason for this is that surjections enjoy the ‘2 out of 3’ property; that is, for any commutative triangle of morphisms among sets in which two of the objects in \( \Theta \) is a morphism in \( \Theta_n^\text{exit} \). The third map is necessarily a surjection as well. For our situation, any morphism \( T \overset{f}{\to} T' \) in \( W_n^\text{hit} \) yields a bijection between the sets of leaves, \( \gamma_n(f) : \gamma_n(T') \xrightarrow{\sim} \gamma_n(T) \). For any \( 1 \leq i \leq n-1 \), in applying the natural transformation \( \epsilon \) from Observation 6.1.8, whose value on \( T \) is the natural map between sets of leaves \( \gamma_n(T) \overset{\epsilon}{\to} \gamma_i(\text{tr}_i(T)) \) induced by the structure of \( T \), we obtain the following diagram among sets:

\[
\begin{array}{ccc}
\gamma_n(T') & \overset{\cong}{\longrightarrow} & \gamma_n(T) \\
\downarrow & & \downarrow \\
\gamma_i(\text{tr}_i(T')) & \overset{\gamma_i(\text{tr}_i(f))}{\longrightarrow} & \gamma_i(\text{tr}_i(T)).
\end{array}
\]

Observe that because \( T \) and \( T' \) are healthy trees, both \( \epsilon_T \) and \( \epsilon_{T'} \) are surjections. Then, by the ‘2 out of 3’ property, \( \gamma_i(\text{tr}_i(T')) \overset{\gamma_i(\text{tr}_i(f))}{\longrightarrow} \gamma_i(\text{tr}_i(T)) \) is a surjection. Such a surjection at each \( i \) guarantees that the image of \( f \) under the functor \( \Theta_n^\text{act, hit} \to \text{Fun}([1 \cdots < n], \text{Fin}^\text{op}) \) lands in \( \text{Fun}(\{1 \cdots < n\}, (\text{Fin}^\text{sur})^\text{op}) \) and is thus a morphism in \( \Theta_n^\text{exit} \). Using this observation, we now verify (77) for the cases \( p = 0, 1 \):

For \( p = 0 \), the desired lift in

\[
\begin{array}{ccc}
[0] & \longrightarrow & W_n^\text{hit} \\
\downarrow & & \downarrow \\
[1] & \longrightarrow & W_n^\text{hit}
\end{array}
\]

\[
\begin{array}{ccc}
\{T\} & \longrightarrow & W_n^\text{hit} \\
\downarrow & & \downarrow \\
\{\langle T \rangle \} & \longrightarrow & W_n^\text{hit}
\end{array}
\]
is given by selecting out the morphism $T \xrightarrow{f} T'$, which is in $\Theta_n^{\exit}$ because each morphism in $W_n^{\exit}$ between objects in $\Theta_n^{\act,\exit}$ is a morphism in $\Theta_n^{\exit}$, as previously discussed. A similar argument yields a lift for the square whose downward arrow on the left is $\langle 1 \rangle$.

For $p = 1$, the desired lift in

$$
\begin{array}{ccc}
[0] & \longrightarrow & \Fun(W_n^{\exit}([1], \Theta_n^{\exit})) \\
(0) & \downarrow & \downarrow \\
[1] & \xrightarrow{\alpha} & \Fun(W_n^{\exit}([1], \Theta_n^{\act,\exit}))
\end{array}
$$

is again given by $\alpha$, which is straightforward to check upon applying the fact discussed above, that each morphism in $W_n^{\exit}$ between objects in $\Theta_n^{\act,\exit}$ is a morphism in $\Theta_n^{\exit}$. A similar argument applies for the square whose downward arrow on the left is $\langle 1 \rangle$.

Thus, $\Fun(W_n^{\exit}([p], \Theta_n^{\exit})) \hookrightarrow \Fun(W_n([p], \Theta_n^{\act,\exit}))$ is an inclusion of a cofactor, meaning the target is equivalent to a coproduct, one term of which is the source. Thus, because the classifying space respects colimits, the induced map between classifying spaces $\BB \Fun(W_n([p], \Theta_n^{\exit})) \hookrightarrow \BB \Fun(W_n([p], \Theta_n^{\act,\exit}))$ is, in particular, still a monomorphism.

Finally, we are equipped to prove Corollary 10.0.2 by showing that the category $\Theta_n^{\exit}$ localizes on $W_n^{\exit}$ to the exit-path $\infty$-category of the Ran space of $\mathbb{R}^n$.

**Proof.** Corollary 10.0.2 Similar to the proof of Lemma 9.0.3, we will use Theorem 9.0.7 that if the classifying space of $\Fun(W_n^{\exit}([\bullet], \Theta_n^{\exit}))$ is a complete Segal space, then it is equivalent to the localization $\Theta_n^{\exit}$ about $W_n^{\exit}$. First, we will show that there is an equivalence of simplicial spaces from the classifying space of $\Fun(W_n^{\exit}([\bullet], \Theta_n^{\exit}))$ to $\Hom_{\Cat_{\infty}}([\bullet], \Exit(Ran(\mathbb{R}^n)))$. To do this, we use the following diagram of simplicial spaces

$$
\begin{align*}
\BB \Fun(W_n^{\exit}([\bullet], \Theta_n^{\exit})) & \xrightarrow{\sim} \BB \Fun(W_n^{\exit}([\bullet], \Theta_n^{\act,\exit})) \\
\Hom_{\Cat_{\infty}}([\bullet], \Exit(Ran(\mathbb{R}^n))) & \hookrightarrow \Hom_{\Cat_{\infty}}([\bullet], \Exit(Ran(\mathbb{R}^n))).
\end{align*}
$$

This diagram needs some explanation. The top horizontal arrow on the left is a monomorphism by Lemma 10.0.5. We showed the top horizontal arrow on the left to be an equivalence in Lemma 10.0.4. Because $\Exit(Ran(\mathbb{R}^n))$ is a $\infty$-subcategory of $\Exit(Ran^u(\mathbb{R}^n))$, the bottom horizontal arrow is a monomorphism. We showed the downward equivalence in the proof of Lemma 9.0.9.

Lastly, to define the induced downward functor on the left of (78), first recall Observation 7.0.6, where we witnessed that the exit-path $\infty$-category of the Ran space of $\mathbb{R}^n$ is naturally an $\infty$-subcategory of the exit-path $\infty$-category of the unital Ran space of $\mathbb{R}^n$, given as a pullback over surjective finite sets. Then, the induced downward arrow in (78) from $\BB \Fun(W_n^{\exit}([\bullet], \Theta_n^{\exit}))$ to $\Hom_{\Cat_{\infty}}([\bullet], \Exit(Ran(\mathbb{R}^n)))$ is induced by the unique (up to a contractible space of choices) functor given by the universal property of pullback in the following diagram of $\infty$-categories:
is a surjection on path components. Let cylr$((I,T)\rightarrow E)\rightarrow \mathbb{R}^n\times \Delta^1$ be a point in the target. Recall that from (79), $\kappa$ is determined by $\Theta_n^{\text{exit}}\rightarrow \Theta_n^{\text{act}}\simeq \text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ over $\text{Fun}\left(\{1,\ldots,n\},(\text{Fin}^{\text{sur}})^{\text{op}}\right)$.

Thus, we will identify a point in the fiber over $E$ under $\kappa$ by identifying a morphism in $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ over $(\text{Fin}^{\text{sur}})^{\text{op}}$. Such a morphism is precisely obtained by naming the projection data of $E$, namely:

$$\text{Fun}\left(\{1,\ldots,n\},(\text{Fin}^{\text{sur}})^{\text{op}}\right) \xrightarrow{\text{ev}_n} \text{Fin}^{\text{op}}$$

where we note that the top, back horizontal functor is the localization from Theorem 7.2.1, and the square on the right wall is the pullback that we just observed in (3). Also note that we apply the universal property of the classifying space to ensure that the unique functor in (79) from $\Theta_n^{\text{act}}$ to $\text{Exit}(\text{Ran}(\mathbb{R}^n))$ induces a functor from $\mathcal{B}\text{Fun}_{W_n}^h((\bullet),\Theta_n^{\text{exit}})$ to $\text{Hom}_{\text{Cat}_{\ast/\ast}}\left((\bullet),\text{Exit}(\text{Ran}(\mathbb{R}^n))\right)$ in (78).

We wish to show that this induced functor

$$\mathcal{B}\text{Fun}_{W_n}^h((\bullet),\Theta_n^{\text{exit}}) \xrightarrow{\kappa} \text{Hom}_{\text{Cat}_{\ast/\ast}}\left((\bullet),\text{Exit}(\text{Ran}(\mathbb{R}^n))\right)$$

in (78) is an equivalence. First, observe that monomorphisms enjoy the ‘2 out of 3’ property (by Observation 5.4 in [5]) and thus, $\kappa$ is a monomorphism.

All that remains to be shown then, in showing that $\kappa$ is an equivalence of simplicial spaces, is to show that $\kappa$ induces a surjection on path components between each space given by the value on $[p]$,

$$\mathcal{B}\text{Fun}_{W_n}^h([p],\Theta_n^{\text{exit}}) \xrightarrow{\kappa} \text{Hom}_{\text{Cat}_{\ast/\ast}}\left([p],\text{Exit}(\text{Ran}(\mathbb{R}^n))\right).$$

Recall in the proof of Lemma 9.0.9 where we show that $\mathcal{B}\text{Fun}_{W_n}^h((\bullet),\text{Exit}(\text{Ran}^u(\mathbb{R}^n)))$ satisfies the Segal condition. Observe that the same argument applies to $\mathcal{B}\text{Fun}_{W_n}^h((\bullet),\Theta_n^{\text{exit}})$ to show that it, too, satisfies the Segal condition. Thus, to show $\kappa$ is a surjection on path components, it suffices to show it for the cases $p = 0, 1$.

For the case $p = 0$, we wish to show that the map of spaces

$$\mathcal{B}\text{W}_n^{\text{ht}} \xrightarrow{\kappa} \Pi_{r \geq 1}\text{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$$

is a surjection on path components. Indeed, this follows by Lemma 9.1.1, wherein we showed a homotopy equivalence between the classifying space of the subcategory $W_n^{\text{ht}}$ of $\Theta_n^{\text{act},\text{ht}}$ and the coproduct $\Pi_{r \geq 0}\text{Conf}_r(\mathbb{R}^n)_{\Sigma_r}$; the only difference here is that $r = 0$ is allowed.

For the case $p = 1$, we wish to show that the map of spaces

$$\mathcal{B}\text{Fun}_{W_n}^h([1],\Theta_n^{\text{exit}}) \xrightarrow{\kappa} \text{mor}\left(\text{Exit}(\text{Ran}(\mathbb{R}^n))\right)$$

is a surjection on path components. Let $\text{cylr}(S^1\hookrightarrow T)\rightarrow E\rightarrow \mathbb{R}^n\times \Delta^1$ be a point in the target. Recall that from (79), $\kappa$ is determined by $\Theta_n^{\text{exit}}\hookrightarrow \Theta_n^{\text{act}}\simeq \text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ over $\text{Fun}\left(\{1,\cdot\cdot\cdot,n\},(\text{Fin}^{\text{sur}})^{\text{op}}\right)$.

Thus, we will identify a point in the fiber over $E$ under $\kappa$ by identifying a morphism in $\text{Exit}(\text{Ran}^u(\mathbb{R}^n))$ over $(\text{Fin}^{\text{sur}})^{\text{op}}$. Such a morphism is precisely obtained by naming the projection data of $E$, namely:
\[
\begin{array}{ccc}
\text{cylr}(S \xrightarrow{f} T) & \xrightarrow{E} & \mathbb{R}^n \times \Delta^1 \\
\text{pr}_{<n} & \downarrow & \downarrow \\
\text{cylr}(\text{pr}_{<n}(S) \to \text{pr}_{<n}(T)) & \xrightarrow{\cdot} & \mathbb{R}^{n-1} \times \Delta^1 \\
\vdots & \downarrow & \vdots \\
\text{cylr}(\text{pr}_1(S) \to \text{pr}_1(T)) & \xrightarrow{\cdot} & \mathbb{R} \times \Delta^1 \\
\end{array}
\]

the value of which under the functor \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \to \text{Fun}(\{1 \leq \cdots \leq n\}, \text{Fin}^\text{op}) \) factors through \( \text{Fun}(\{1 \leq \cdots \leq n\}, \text{Fin}^\text{surj}^\text{op}) \) precisely because \( \text{Exit}(\text{Ran}^n(\mathbb{R}^n)) \) is naturally over \( \text{Fun}(\{1 \leq \cdots \leq n\}, \text{Fin}^\text{surj}^\text{op}) \), which implies that the localization, too, is over \( \text{Fin}^\text{surj}^\text{op} \).

\[\square\]

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