Efficient cyclic permutations for qudits

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One of the main challenges in quantum technologies is the ability to control individual quantum systems. This task becomes increasingly difficult as the dimension of the system grows. Here we propose an efficient setup for cyclic permutations $X_d$ in $d$ dimensions, a major primitive for constructing arbitrary qudit gates. Using orbital angular momentum (OAM) states as a qudit, the simplest implementation of the $X_d$ gate in $d$ dimensions requires a single quantum sorter $S_d$ and two spiral phase plates (SPPs). We then extend this construction to a generalised $X_d(p)$ gate to perform a cyclic permutation of a set of $d$, equally spaced values $\{|0\rangle, |0 + p\rangle, \ldots, |0 + (d-1)p\rangle\} \mapsto \{|0\rangle, |0 + p\rangle, \ldots, |0\rangle\}$. We find efficient implementations for the generalised $X_d(p)$ gate in both Michelson (one sorter $S_d$, two SPPs) and Mach-Zehnder configurations (two sorters $S_d$, two SPPs). Remarkably, the number of spiral phase plates is independent of the qudit dimension $d$. Our architecture for $X_d$ and generalised $X_d(p)$ gate will enable complex quantum algorithms for qudits, for example quantum protocols using photonic OAM states.

I. INTRODUCTION

All successful technologies are based on harnessing a specific resource, such as energy, electricity or information. The ability to generate, control, transform and ultimately, find useful applications for quantum resources, is central to the development of quantum technologies [1–4].

Controlling the simplest quantum systems, the qubit, is relatively straightforward [5, 6]. We can achieve complete control over the 2-dimensional Hilbert space of a qubit with rotations generated by Pauli matrices $X$ and $Z$. The natural next step is to go to $d$-dimensional systems, or qudits. In this case we have the generalised Pauli matrices $X_d$ and $Z_d$. Progressing in this direction, we need to find physical implementations for qudits, together with the experimental ability to control them.

Orbital angular momentum (OAM) is one of the most used implementations for photonic qudits. Photon states $|\ell\rangle$ carry an OAM of $\ell h$, where $\ell = 0, \pm 1, \pm 2, \ldots$ is a theoretically unbounded integer. OAM states have a helical phase front, with $\ell \neq 0$ corresponding to the number of helices.

Photonic OAM states have been used in entanglement generation [7,8] and alignment-free quantum key distribution [9,11]. Thus OAM is attractive since it allows us to use a larger alphabet to transmit quantum information with a single photon. However, without the appropriate tools, a larger alphabet for encoding information has only a limited functionality. This brings us to the problem of how to implement efficiently the generalised Pauli operators $X_d$ and $Z_d$ for qudits [12].

For photonic OAM states, $Z_d$ can be implemented efficiently with Dove prisms. An open question is how to implement a cyclic permutation $X_d$ for any dimension $d$. Experimentally, cyclic $X_d$ gates for OAM states have been realised only for $d = 4$ [13] and $d = 5$ [14].

In this article we propose an efficient scheme to perform cyclic permutations $X_d$ for any set of $d$ consecutive states. Then we generalise it for cyclic permutations $X_d(p)$ of an arbitrary set of $d$, equally spaced states $\{|0\rangle, |0 + p\rangle, \ldots, |0 + (d-1)p\rangle\}$. For any dimension $d$, the minimal implementation of both $X_d$ and $X_d(p)$ requires a single sorter $S_d$ and two spiral phase plates. To arrive at this setup, we use quantum information methods and quantum network analysis. This approach has been employed previously to design a universal quantum sorter [15] and spin measuring devices [16,17].

We focus on OAM encoded qudits, as several experimental tools are already available [18–22]. Nevertheless, our scheme can be extended in principle to other degrees of freedom as well.

II. THE PROPOSAL

A. Cyclic $X_d$ gate

We now introduce our setup for performing the cyclic gate. Let $\mathcal{H}_d$ be the Hilbert space of a qudit, $\dim \mathcal{H}_d = d$ and let $\{|j\rangle\}_{j=0}^{d-1}$ be an orthonormal basis of $\mathcal{H}_d$. The generalised Pauli operators $X_d$, $Z_d$ are defined as:

$$X_d: |j\rangle \mapsto |j \oplus 1\rangle \quad (1)$$

$$Z_d: |j\rangle \mapsto e^{i\theta}|j\rangle \quad (2)$$

with $\oplus$ addition mod $d$ and $\theta = e^{2\pi i/d}$ a root of unity of order $d$. The gate $X_d$ performs a cyclic permutation of the basis states, i.e., maps the set $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ to $\{|1\rangle, |2\rangle, \ldots, |0\rangle\}$.

Our scheme for the $X_d$ gate is shown in Fig.1. The main element of our proposal is a $d$-dimensional sorter $S_d$ introduced in Ref. [13,23]. A quantum sorter $S_d$ is a device which directs an incoming particle into different outputs (i.e., sorts) according to the value of an internal degree of freedom $\Sigma$. In the following we take $\Sigma$ to be orbital angular momentum (OAM). Nevertheless, the setup is general and can be implemented for other variables as well, like wavelength [15] or radial quantum number [24,25].

The quantum sorter $S_d$ is formally equivalent to a controlled-$X_d$ gate between the degree of freedom we want to sort (OAM, $\Sigma$ etc) and spatial modes $m$, see Fig. 2.

$$S_d := C(X_d): |i\rangle_{OAM}|j\rangle_m \mapsto |i\rangle_{OAM}|j \oplus i\rangle_m$$  (3)
where $|i\rangle_{OAM}$, $|j\rangle_m$ are OAM and mode qudits, respectively. Thus a photon in OAM state $|i\rangle$ incident on port (mode) 0 will exit on port $(i \mod d)$ with unit probability.

Apart from the sorter $S_d$, another ingredient are spiral phase plates (SPPs) of order $n$. The action of the SPP on OAM states is:

$$SPP(n) : |i\rangle_{OAM} \mapsto |i + n\rangle_{OAM}$$

with $n \in \mathbb{Z}$ integer. This transformation adds (or subtracts) $n$ units of OAM. Since this is normal addition, it shifts the whole $\mathbb{Z}$ axis by $n$ units.

We now discuss how the $X_d$ gate in Fig. 1 works. The first SPP adds +1 to all OAM states. Then the sorter $S_d$ directs each OAM state $|i\rangle_{OAM}$ to the corresponding output $|i \mod d\rangle_m$, eq. (5). Since sorting on modes is done modulo $d$, the state $|d\rangle_{OAM}$ will exit on mode 0. Consequently, only the state on mode 0 needs to be shifted by $-d$; in terms of quantum networks, this is equivalent to a controlled-SPP($-d$) gate, with the control on the mode $k = 0$ (open circle on control qudit in Fig. 1). After this operation the states from all spatial modes are recombined on mode 0 by the gate $C(X_d)^{-1}$, which is nothing else but a sorter run in reverse $S_d^\dagger$. This decouples the OAM and mode qudits, such that the final state is factorised and the photon always exits on mode 0 with unit probability. Thus the gate in Fig. 1 performs the following sequence:

$$|i\rangle_{OAM}|0\rangle_m \xrightarrow{+1} |i + 1\rangle_{OAM}|0\rangle_m \xrightarrow{S_d} |i + 1\rangle_{OAM}|i \oplus 1\rangle_m \xrightarrow{-d} |i \oplus 1\rangle_{OAM}|i \oplus 1\rangle_m \xrightarrow{S_d^{-1}} |i \oplus 1\rangle_{OAM}|0\rangle_m$$

(5)

Since the ancilla is decoupled after the gate, an arbitrary superposition of OAM states transforms under the cyclic gate $X_d$ as

$$\alpha_0|0\rangle + \alpha_1|1\rangle + \ldots + \alpha_{d-1}|d-1\rangle \rightarrow \alpha_0|0\rangle + \alpha_1|1\rangle + \ldots + \alpha_{d-2}|d-1\rangle$$

(6)

Consequently, our scheme can be used in general quantum algorithms.

Resources. Our implementation of the cyclic $X_d$ gate is very efficient: it requires two sorters $S_d$ and two SPPs (of order +1 and $-d$, respectively). This result is important: the number of SPPs is constant (two), and thus independent of the qudit dimension $d$. One can further simplify the scheme by using a Mach-Zehnder configuration, which needs only one sorter, see Fig. 4.

The sorter $S_d$ requires $d - 1$ phases $Z_d$ (on OAM) and two Fourier gates $F_d, F_d^\dagger$ on spatial modes [15]. Fig. 2. Since the Fourier gates $F_d, F_d^\dagger$ act only on spatial modes, the same construction can be used to sort different observables $\Sigma$ by inserting (between $F_d$ and $F_d^\dagger$) the appropriate phases $Z_d^k$ acting on the observable $\Sigma$.

Fourier gates for spatial modes can be implemented in several ways. In linear optics one can use beam-splitters and phase-shifters [26]. In integrated optics Fourier gates are used in arrayed waveguide gratings (AWG) [27] and are constructed from a single block of silicon. Commercially available AWGs (using Fourier gates) have tens to hundreds of spatial model (channels).

FIG. 1: Cyclic $X_d$ gate. The gate performs the transformation $|i\rangle \mapsto |i \oplus 1\rangle$ on a qudit. (a) Equivalent quantum network. Spatial modes are used as a qudit ancilla, which is factorised before and after the gate, i.e., starts and ends up in the state $|0\rangle_m$. There are two SPPs of order +1 and $-d$ (green) and two C($X_d$) gates (cyan). The SPP($-d$) acts only on mode 0 (open circle on the control mode qudit). (b) Implementation. Photons enter from the left and all OAM states are shifted by SPP(+1). The sorter $S_d$ redirects each state to the corresponding output. The state $|d\rangle_{OAM}$ exists on spatial mode 0 and after the SPP($-d$) becomes $|0\rangle_{OAM}$; all other OAM states $|j\rangle, j \neq d$, are left invariant. Finally, all states enter the inverse sorter $S_d^\dagger$ and end up in the same spatial mode $|0\rangle_m$.

FIG. 2: Universal quantum sorter $S_d$. Top: equivalent quantum network. The controlled-$X_d$ gate $C(X_d)$ is decomposed on Fourier gates $F, F^\dagger$ and a C($Z_d$) gate. Bottom: implementation as a multimode interferometer with path-dependent phase shifts $Z_d^k$ acting on the variable to be sorted $\Sigma$. 

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The gate is characterised by two parameters: the qudit dimension $D$, and a state direction. We define the generalised gate $X_X(p)$ on any set of $d$ consecutive OAM states with $-k$, then perform the usual $X_X$ gate, and finally shift back all states with $+k$, Fig. 2(b).

A second method to perform the cyclic $X_X$ gate on the set $\{|k\ell_0\}, |(k+1)\ell_0\}, ..., |(d-1)\ell_0\}$ is to first shift all OAM states with $-k$, then perform the usual $X_X$ gate, and finally shift back all states with $+k$, Fig. 2(a).

Importantly, the value of the shift $k$ is bounded by the dimension $d$ of the qudit and not by $\ell_0$, since $0 \leq k < d$. This is noteworthy – although OAM with large values $\ell = 10,010$ have been experimentally prepared [28], SPPs with such large values are difficult to manipulate. Therefore, for $\ell \gg d$ we need to shift the state only with a much smaller value $(\ell_0 \bmod d) < d$, irrespective of the magnitude of $\ell_0$. Thus the same device can be used to perform the cyclic $X_X$ gate on any set of $d$ consecutive OAM states. In this case the only change is the position of SPP($-d$) (for the first method), or adding two SPP($\pm k$) (for the second method).

(ii) Cyclic permutation for $d$, equally spaced values. Let $p \in \mathbb{N}^+$ be a positive integer. We now show how to implement a cyclic permutation with step $p$

$$\{0, |p\}, ..., |(d-1)p\} \mapsto \{|p\}, |2p\}, ..., |0\} \quad (7)$$

We define the generalised gate $X_X(p)$

$$X_X(p) : |jp\} \mapsto |(j+1)p\} \quad (8)$$

with $j = 0, ..., d-1$, $p \in \mathbb{N}^+$ and $\oplus$ addition mod $d$. The gate is characterised by two parameters: the qudit dimension $d$ and the step $p$ between two consecutive values; clearly $X_X(1) = X_X$.

The implementation of $X_X(p)$ uses the same architecture as before, but with a different sorter $S_1^{1/p}$: this directs each state $|jp\} on a separate mode $|j\}$. From the decomposition $S_1^{1/p} = E^iC(Z_1^{1/p})F$, it follows that we can implement $S_1^{1/p}$ with a setup similar to Fig. 2, but with different mode-dependent phases $Z_1^{1/p}$ inside the interferometer.

The setup for $X_X(p)$ is shown in Fig. 3. As before, SPP($p$) first shift all OAM states with $p$. The sorter $S_1^{1/p}$ separates the states according to the OAM values, $|jp\} \mapsto |jp\}$. Then SPP($-pd$) maps $|dp\} \mapsto |0\}$. After which the sorter $S_1^{1/p}$ combines back all states on mode $|0\}$. As before, we can implement a cyclic gate $X_X(p)$ on an arbitrary set of equally spaced OAM values $\{|\ell_0\}, \{|\ell_0 + p\}, ..., \{|\ell_0 + (d-1)p\}\}$ in two ways, Fig. 3:

(a) by moving SPP($-pd$) on mode $k$, with $k = \ell_0 \bmod d$; or (b) by using two SPP($\pm k$) before and after the sorters.

To summarise, for any dimension $d$ and initial state $\ell_0$, the generalised gate $X_X(p)$ requires only two sorters $S_1^{1/p}$, $S_1^{1/p}$ and two spiral phase plates SPP($p$), SPP($-pd$).

C. Simplification: Michelson setup

We can further simplify our scheme if we use a Michelson instead of the Mach-Zehnder interferometer. In this case we need only one sorter $S_1^{1/p}$, Fig. 4. The first part of the scheme is identical to the one discussed previously. The state $|dp\}$ exits on spatial mode $k$ and, after a reflection on SPP($-pd$), becomes $|0\}$. All other OAM states undergo a double reflection on the retro-reflector $R$ and remain unchanged. Finally, all states re-enter the sorters from the opposite direction, thus performing $S_1^{1/p}$, and end up in the same spatial mode $|0\}$. A circulator C separates the output from the input. Note that a spiral phase plate acts as its inverse if the photon enters from the opposite direction; thus in Fig. 4(b) we need only a single SPP($-k$).
III. CONCLUSIONS

The ability to control higher-dimensional quantum systems is essential for developing useful quantum technologies. Due to coherence constraints, efficiency will play a key role in the success of real-life quantum protocols. In this article we developed efficient implementations for cyclic permutations $X_d$ in $d$ dimensions, one of the building blocks for constructing arbitrary single-qudit gates. The scheme is deterministic, works at single-particle level and can be applied to arbitrary superpositions of qudit states. Although our focus has been on orbital angular momentum, the method is general and can be adapted to other degrees of freedom. This will require a sorter $S_d$ and shift gates (the equivalent of SPP) for the respective degree of freedom. Since a general scheme for a universal quantum sorter exists [15], a future challenge is to implement the cyclic gate $X_d$ for a particular degree of freedom $\Sigma$ is to find appropriate implementations for shift gates $|i\rangle_\Sigma \mapsto |i+n\rangle_\Sigma$.

A possible application of the generalised cyclic permutation $X_d(p)$ is quantum communication. There are QKD protocols with Fibonacci coding for key distribution [29, 30]. An open problem is to develop secure QKD protocols which use OAM states in arithmetic progression.

Note added. While finishing this article we became aware of a similar method for performing $X_d$ gates for OAM [51].

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