Implications of Spontaneous Glitches in the
Mass and Angular Momentum in Kerr Space–Time

C. Barrabès†‡ and G. F. Bressange†*

† Physics Department, UPRES A 6083 du CNRS,
Université de Tours, 37200 France;
‡ Département d’Astrophysique Relativiste et Cosmologie,
UPR 176 du CNRS, Observatoire de Paris,
92190 Meudon, France

and

P. A. Hogan**

Mathematical Physics Department,
University College Dublin,
Belfield, Dublin 4, Ireland

PACS numbers: 04.30.+x, 04.20Jb

The outward–pointing principal null direction of the Schwarzschild Riemann tensor is null hypersurface–forming. If the Schwarzschild mass spontaneously jumps across one such hypersurface then the hypersurface is the history

* barrabes@celfi.phys.univ-tours.fr; bressang@celfi.phys.univ-tours.fr
** phogan@ollamh.ucd.ie
of an outgoing light–like shell. The outward–pointing principal null direction of
the Kerr Riemann tensor is asymptotically (in the neighbourhood of future null
infinity) null hypersurface–forming. If the Kerr parameters of mass and angular
momentum spontaneously jump across one such asymptotic hypersurface then
the asymptotic hypersurface is shown to be the history of an outgoing light–like
shell and a wire singularity–free spherical impulsive gravitational wave.
1. Introduction

In a recent paper [1] the authors studied the physical consequences of abrupt changes occurring spontaneously in the multipole moments of a static axially symmetric isolated gravitating body. The conclusion was that a disturbance propagates with the speed of light away from the source which when analysed near future null infinity is shown to consist of a spherical outgoing light–like shell accompanied by a spherical impulsive gravitational wave. Motivated by the well–known phenomenon of glitches observed in pulsars [2] we examine in this paper the physical implications of glitches in the mass and angular momentum associated with the source of the Kerr space–time. We find that in the neighbourhood of future null infinity a disturbance consisting of a spherical light–like shell and a spherical impulsive gravitational wave can be identified. If the Kerr angular momentum vanishes (the Schwarzschild special case) then the gravitational wave does not exist. If the angular momentum glitch includes an abrupt change in the direction of the angular momentum then the gravitational wave has the maximum two degrees of freedom of polarisation. The spherical impulsive gravitational waves appearing in this paper (section 3 below) and in [1] are the only examples of such waves known to the authors which are free of unphysical directional (or wire) singularities (see section 1 of [1] where this is discussed).

To study the physical properties of the disturbances mentioned above we use the Barrabès–Israel (BI) theory of light–like shells and impulsive waves [3].
This theory is easily accessible and a useful description of part of it is also available in [1] where in particular the identification of the gravitational wave and the light–like shell, when both exist, is given explicitly. To re–derive the results stated in the present paper the reader must be familiar with the BI theory. For readers who wish to work through the calculations based on the BI theory we give some intermediate steps in the Appendix. The consequences of such calculations can be easily followed independently however in the main body of this paper.

To simplify the presentation and introduce our approach we present first in section 2 the well–known (see [3], for example) Schwarzschild example in which the mass of the source spontaneously undergoes an abrupt but finite change. This is also useful as a special case of the corresponding Kerr example, which is the main point of the paper, given in section 3. This is followed by a brief discussion of our results in section 4.

2. The Schwarzschild Example

Consider Schwarzschild space–time with line–element

\[ ds^2 = -\frac{2r^2 d\zeta d\bar{\zeta}}{(1 + \frac{1}{2}\zeta \bar{\zeta})^2} + 2du \, dr + \left(1 - \frac{2m}{r}\right) du^2. \]  

Here \( u = \) constant are future–directed null hypersurfaces (null–cones) generated by the geodesic integral curves of the null vector field \( \partial/\partial r \). This vector field is also the outward–pointing principal null direction of the Riemann tensor of
the space–time. We wish to consider this space–time undergoing a spontaneous abrupt change in the mass $m$ of the source across one of the outward null hypersurfaces $u = 0$ (say) and then ask: what are the physical properties of $u = 0$? To do this we imagine the space–time divided into two halves $M^+$ corresponding to $u > 0$ and $M^−$ corresponding to $u < 0$ both with boundary $u = 0$ and then re–attaching the halves on $u = 0$ preserving, with the identity map, the induced line–element on $u = 0$:

$$dl^2 = −r^2(dθ^2 + \sin^2 \theta dϕ^2) . \tag{2.2}$$

We denote the resulting space–time by $M^−∪M^+$. For the space–time $M^−∪M^+$ described above there is a stress–energy tensor concentrated on $u = 0$ of the form

$$T^{μν} = S^{μν} δ(u) , \tag{2.3}$$

with $x^μ = (θ, φ, r, u)$ and $δ$ is the Dirac delta function. We refer to $S^{μν}$ as the surface stress–energy tensor of the light–like shell with history $u = 0$ (see [3]). The normal to $u = 0$ is the null vector with components $n^μ$ given via the 1–form

$$n_μ dx^μ = du . \tag{2.4}$$

The BI theory [3] gives

$$16\pi S^{μν} = −\frac{4[m]}{r^2} n^μ n^ν , \tag{2.5}$$

where $[m]$ is the finite jump in the mass $m$ across $u = 0$. This means that there is no stress in the out–going light–like shell (as might be expected because the
shell is spherical and expanding) and the surface energy–density of the shell measured by a radially moving observer (discussed in [3]) is a positive multiple of
\[ \sigma = -\frac{[m]}{4\pi r^2}, \]  
(1.6)
and so it is natural to assume that $[m] < 0$ for an outgoing shell. Thus we conclude that the space–time $M^- \cup M^+$ describes a Schwarzschild gravitational field (described by the space–time $M^-$) with an expanding spherical light–like shell propagating through it leaving behind a Schwarzschild field described by $M^+$ and with mass reduced compared to that of $M^-$. 

In general in the type of situation described here (subdivision and re–attachment, or ”cut and paste”, of a space–time on a null hypersurface) the space–time $M^- \cup M^+$ has a Weyl conformal curvature tensor containing a delta function term singular on the null hypersurface and composed of a matter part (which is non–zero provided the stress in the shell is anisotropic) and a part describing an impulsive gravitational wave (see [1] where this is explicitly demonstrated and [3] for the calculation of these terms). However on account of the spherical symmetry of the Schwarzschild example above the Weyl tensor of $M^- \cup M^+$ vanishes identically in this case. Thus in particular the shell above is unaccompanied by an impulsive gravitational wave.
3. The Kerr Example

We consider here an analogous situation in the Kerr space–time to that considered in the Schwarzschild space–time in the previous section. We begin with a form of the Kerr space–time which makes it easy to identify the outgoing principal null direction of the Riemann tensor and which specialises to (2.1) when the Kerr angular momentum parameter is put to zero. In addition it will be interesting not only to consider spontaneous changes in the magnitude of the Kerr angular momentum but also to include spontaneous changes in the direction of the angular momentum. We thus want to use a form of the Kerr solution which involves the mass parameter and three components of the angular momentum per unit mass. One such form can readily be obtained by first noting that the Kerr solution with mass $m$ and angular momentum per unit mass $A$ may be written in Kerr’s [4] original coordinates $(\zeta, \bar{\zeta}, r, u)$ [with the simple replacement, as in (2.1), of the polar angles $(\theta, \phi)$ with the complex coordinate $\zeta = \sqrt{2} e^{i\phi} \tan \theta/2$ and its complex conjugate $\bar{\zeta}$] in the form

$$ds^2 = -2 \frac{(r^2 + P^2)}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} d\zeta d\bar{\zeta} + 2 d\Sigma \left( dr - iP_\zeta d\zeta + iP_{\bar{\zeta}} d\bar{\zeta} + S d\Sigma \right),$$  

(3.1)

where

$$P = A \left( \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \right), \quad S = \frac{1}{2} - \frac{mr}{r^2 + P^2},$$  

(3.2)

and the 1–form $d\Sigma$ is given by

$$d\Sigma = du + iP_\zeta d\zeta - iP_{\bar{\zeta}} d\bar{\zeta},$$  

(3.3)
with \( P_\zeta = \partial P/\partial \zeta \). The rotation

\[
\zeta \rightarrow \frac{\sqrt{2} \sin \frac{\theta_1}{2} - \zeta \cos \frac{\theta_1}{2}}{\cos \frac{\theta_1}{2} + \frac{\zeta}{\sqrt{2}} \sin \frac{\theta_1}{2}},
\]

(3.4)

where \( \theta_1, \phi_1 \) are constants, leaves the form of (3.1) invariant with \( P \) replaced by

\[
P = \frac{a}{\sqrt{2}} \left( \frac{\zeta + \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \right) + \frac{b}{i \sqrt{2}} \left( \frac{\zeta - \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \right) + c \left( 1 - \frac{1}{2} \zeta \bar{\zeta} \right),
\]

(3.5)

where

\[
a = A \sin \theta_1 \cos \phi_1, \quad b = A \sin \theta_1 \sin \phi_1, \quad c = A \cos \theta_1.
\]

(3.6)

We thus obtain the Kerr solution with mass \( m \) and angular momentum 3–vector \( J = (ma, mb, mc) \) having the same magnitude but a different direction than the initial one \( J = (0, 0, mA) \). Restoring the polar coordinates \((\theta, \phi)\) as above we arrive at the line–element [5]

\[
ds^2 = -\rho^2 (d\theta^2 + \sin^2 \theta 
\]

\[
d\phi^2) + 2 d\Sigma (dr - N d\theta - M \sin \theta d\phi + S d\Sigma),
\]

(3.7)

with

\[
\rho^2 = r^2 + P^2, \quad P = (a \cos \phi + b \sin \phi) \sin \theta + c \cos \theta, \quad \rho^2 = r^2 + P^2,
\]

(3.8)

\[
d\Sigma = du + N d\theta + M \sin \theta d\phi, \quad S = \frac{1}{2} - \frac{mr}{\rho^2},
\]

(3.9)

and

\[
N = -a \sin \phi + b \cos \phi,
\]

(3.10a)

\[
M = -(a \cos \phi + b \sin \phi) \cos \theta + c \sin \theta.
\]

(3.10b)
We note from (3.10) that

\[ E^2 \equiv M^2 + N^2 = m^{-2} \left( |J|^2 - (n \cdot J)^2 \right), \tag{3.11} \]

where the unit 3–vector \( n \) is given by \( n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). Also the outward–pointing principal null direction of the Riemann tensor is tangent to the vector field \( \partial / \partial r \) or equivalently is given via the (non–exact) 1–form \( d\Sigma \) in (3.9). When this line element is written in a Kerr-Schild form in which the flat background is expressed in rectangular cartesian coordinates and time it can be shown - see [5] in which this is described in detail- that in the linear approximation it takes the form of the line element of the spacetime outside the history of a slowly rotating sphere of mass \( m \) and angular momentum \( J = (ma, mb, mc) \) in exactly the same form as Kerr showed -see [4]- that his original form approximates the line element of the spacetime outside the history of such a sphere of mass \( m \) and angular momentum \( J = (0, 0, mA) \).

By analogy with the Schwarzschild example in section 2 we might expect that a spontaneous abrupt change in the parameters \( \{m, a, b, c\} \) will result in a disturbance propagating through space–time along the out–going principal null direction of the Kerr Riemann tensor. As this vector field is not surface–forming for all values of \( r \) we cannot use the BI theory to study the disturbance for all \( r \). However for large \( r \), specifically if \( O \left( r^{-2} \right) \)–terms are neglected, then \( \partial / \partial r \) is tangent to null hypersurfaces \( u = \text{constant} \). The normal \( n^\mu \) to \( u = \text{constant} \), given via the 1–form \( n_\mu dx^\mu = du \), satisfies \( g_{\mu\nu} n^\mu n^\nu = O \left( r^{-2} \right) \). In this approximation \( u = \text{constant} \) are portions of future null–cones having the
integral curves of \( \partial / \partial r \) as generators with \( r \) an affine parameter along them. These generators are, in the approximation under consideration, shear–free null geodesics with expansion \( r^{-1} \) and the induced line–element on \( u = \text{constant} \) is given approximately by

\[
dl^2 = -r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) .\tag{3.12}
\]

This follows from (3.7) in which the ratio of the neglected terms to the retained terms in the induced line–element is \( O\left(r^{-2}\right) \). Thus if the disturbance is propagating in the direction of \( \partial / \partial r \) then for sufficiently large values of \( r \) a front is formed (a null hypersurface is formed in space–time) with history \( u = 0 \) (say). We now assume that across the null portion of \( u = 0 \) a spontaneous jump in the parameters \( \{m, a, b, c\} \) occurs from values \( \{m, a, b, c\} \) to the past \( (u < 0) \) of this null portion of \( u = 0 \) to values \( \{m_+, a_+, b_+, c_+\} \) to the future \( (u > 0) \) of this null portion of \( u = 0 \), and that the regions of space–time, \( M^+(u > 0) \) and \( M^-(u < 0) \), on either side of the null part of \( u = 0 \) join on this null part with the identity map and thus preserving the induced line–element (3.12). We can now use the BI theory to study the physical properties of the null part of \( u = 0 \) by calculating the surface stress–energy tensor there and by calculating the delta function term (the coefficient of \( \delta(u) \)) in the Weyl conformal curvature tensor. The results of our calculations for this Kerr example will naturally be a generalisation of those for the Schwarzschild example in section 2. Again there is a stress–energy tensor of the form (2.3) on the null part of \( u = 0 \) with surface
stress–energy described by the tensor $S^{\mu\nu}$ which in this case has components:

$$16\pi S^{13} = \frac{[N]}{r^3} + O(r^{-4}) ,$$

$$16\pi S^{23} = \frac{[M] \csc \theta}{r^3} + O(r^{-4}) ,$$

$$16\pi S^{33} = -\frac{4[m]}{r^2} + \frac{2[E^2]}{r^3} + O(r^{-4}) ,$$

with all other components small of order $r^{-5}$. Here as before the square brackets denote the jump across the null part of $u = 0$ of the quantities contained therein. $N, M, E^2$ are given by (3.10) and (3.11) and jump because the Kerr parameters jump. By (3.13) the null part of $u = 0$ is the history of an outgoing light–like shell. By (3.13a, b) there is an anisotropic stress in the shell (on account of the jump in the Kerr angular momentum per unit mass). By (3.13c) the surface energy density of the shell measured by a radially moving observer is a positive multiple of

$$\sigma = -\frac{1}{4\pi r^2} \left( [m] - \frac{[E^2]}{2r} + O(r^{-2}) \right) .$$

This is the generalisation of (2.6) and $\sigma > 0$ implies $[m] < 0$ once again. It is interesting to note that an expanding light–like shell sandwiched between two Reissner–Nordstrom space–times with different masses and charges (the charged version of the Schwarzschild example in section 2) has an exact surface energy density given by (3.14), without the error term, with $[E^2]$ replaced by $[\epsilon^2]$, the jump in the square of the charge across the history of the shell.

The BI theory enables us to calculate the coefficient of $\delta(u)$ in the Weyl conformal curvature tensor for the re–attached space–time. This coefficient in
general splits [1, 3] into a matter part, which is present if there is anisotropic stress in the shell (as there is in the Kerr example), and a wave part describing an impulsive gravitational wave accompanying the light–like shell. To display the components of this coefficient it is convenient to introduce the asymptotically null tetrad given via the 1–forms $du, dr + S du$ (with $S$ given in (3.9)), $(\sqrt{2})^{-1} \, r (d\theta + i \sin \theta \, d\phi)$ and its complex conjugate. This tetrad is asymptotically parallel transported along the integral curves of $\partial/\partial r$. By this we mean that the components on the tetrad of the covariant derivatives of the tetrad vectors in the direction of $\partial/\partial r$ are small of order $r^{-2}$. Denoting the Newman–Penrose components on this tetrad of the matter part of the coefficient of $\delta(u)$ by $^{M}\Psi_A \ (A = 0, 1, 2, 3, 4)$ and those of the wave part of this coefficient of $\delta(u)$ by $^{W}\Psi_A$, we find for the matter part -see [1]

\[ ^{M}\Psi_0 = O \left(r^{-5}\right), \quad ^{M}\Psi_1 = O \left(r^{-4}\right), \quad ^{M}\Psi_2 = O \left(r^{-3}\right), \quad (3.15a) \]

\[ ^{M}\Psi_3 = -\frac{1}{4\sqrt{2}r^2} \left[N - iM\right] + O \left(r^{-3}\right), \quad (3.15b) \]

\[ ^{M}\Psi_4 = O \left(r^{-3}\right), \quad (3.15c) \]

and for the wave part all Newman–Penrose components vanish with the exception of

\[ ^{W}\Psi_4 = \frac{1}{4r^4} \left[m \left(N - iM\right)^2\right] + O \left(r^{-5}\right). \quad (3.16) \]

Here again the square brackets denote the jump across the null part of $u = 0$ of the quantity contained therein.

4. Discussion
We first notice that $M\Psi_A$ is predominantly Type III in the Petrov classification with $n^\mu$ as degenerate principal null direction. The presence of $M\Psi_A$ is due to the presence of anisotropic stress in the light–like shell (see (3.13a, b)) which is a consequence of the non–vanishing Kerr angular momentum parameters in this case. $W\Psi_A$ is Type N in the Petrov classification with $n^\mu$ as four–fold degenerate principal null direction. This means that the shell is accompanied by a spherical impulsive gravitational wave whose presence is again due to the non–vanishing Kerr angular momentum parameters. Since both $N$ and $M$ are smooth bounded functions of $\theta, \phi$ for $0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$ neither (3.15) nor (3.16) possess line–singularities.

It is interesting to note that the predominant radial dependence of $M\Psi_A$ and $W\Psi_A$ (which is $O(r^{-2})$ for $M\Psi_A$ and $O(r^{-4})$ for $W\Psi_A$) is the same for $W\Psi_A$ as in our earlier paper [1]. This is because the light–like shell and the gravitational wave share the same null hypersurface history $u = 0$ in space–time and are therefore in direct competition with each other. Hence it is no surprise that the matter part is more dominant than the wave part.

Finally we see that if the angular momentum 3–vector $\mathbf{J}$ introduced after (3.6) had for $u < 0$, $\mathbf{J} = (0,0,mc)$ and for $u > 0$, $\mathbf{J} = (0,0,m_+c_+)$ then $N^+ = N = 0$ and the spherical impulsive wave with amplitude (3.16) has one degree of freedom of polarisation. Adding a change of direction to this change of magnitude of the angular momentum clearly adds the extra degree of freedom to the gravitational wave.
References

[1] C. Barrabès, G. F. Bressange and P. A. Hogan, Phys. Rev. D55, 3477 (1997).

[2] C. R. Kitchin, Stars, Nebulae and Interstellar Medium (Adam Hilger, Boston 1987), p.147.

[3] C. Barrabès and W. Israel, Phys. Rev. D43, 1129 (1991).

[4] R. P. Kerr, Phys. Rev. Lett. 11, 237 (1963).

[5] P. A. Hogan, Phys. Lett. A60, 161 (1977).
Appendix

Useful Formulas for Sections 2 and 3

For readers who are familiar with the BI theory and wish to derive the results stated in sections 2 and 3 we list here the results of some useful intermediate calculations. These apply to the Kerr example of section 3. They all specialise to the Schwarzschild example of section 2 when the Kerr angular momentum parameters are put to zero.

The jump $\gamma_{\mu\nu}$ in the transverse extrinsic curvature across the null part of $u = 0$ is given by

$$\gamma_{11} = -2[m] + \frac{[E^2]}{r^2} + O \left( r^{-2} \right), \quad (1)$$

$$\gamma_{22} = \gamma_{11} \sin^2 \theta, \quad (2)$$

$$\gamma_{12} = -\frac{[mMN]}{r^2} \sin \theta + O \left( r^{-3} \right), \quad (3)$$

$$\gamma_{13} = -\frac{[N]}{r} + O \left( r^{-2} \right), \quad (4)$$

$$\gamma_{23} = -\frac{[M]}{r} \sin \theta + O \left( r^{-2} \right), \quad (5)$$

$$\gamma_{33} \equiv 0, \quad \gamma_{\mu 4} \equiv 0. \quad (6)$$

As well as the $O \left( r^{-2} \right)$ leading term in $\gamma_{12}$ given in (3) we require the $O \left( r^{-2} \right)$ leading term in $\gamma_{11} - \gamma_{22} \csc^2 \theta$. This is neatly given along with (3) by

$$\gamma_{11} - \gamma_{22} \csc^2 \theta - 2 \gamma_{12} \csc \theta = -\frac{1}{r^2} \left[ m \left( N - iM \right)^2 \right] + O \left( r^{-3} \right). \quad (7)$$

The stress–energy tensor $S_{\mu\nu}$ of the shell in terms of $\gamma_{\mu\nu}$ is [3]

$$16\pi \eta^{-1} S_{\mu\nu} = 2\gamma^{(\mu}_{\nu}) - \gamma n^\mu n^\nu - \gamma^\dagger g^{\mu\nu} - q^{\mu\nu}, \quad (8)$$
where in the present case $\eta = 1 + O(r^{-4})$,

$$\gamma^\mu = \gamma^{\mu\nu} n_\nu, \quad \gamma^\dagger = \gamma^{\mu} n_\mu, \quad \gamma = g^{\mu\nu} \gamma_{\mu\nu},$$  \hspace{1cm} (9)

and

$$q^{\mu\nu} = \epsilon (\gamma^{\mu\nu} - \gamma g^{\mu\nu}),$$  \hspace{1cm} (10)

with $\epsilon = g_{\mu\nu} n^\mu n^\nu$. In the Kerr case $\epsilon = O(r^{-2})$ ($\epsilon = 0$ in the Schwarzschild case) and

$$q^{11} = O(r^{-6}), \quad q^{12} = O(r^{-7}), \quad q^{22} = O(r^{-6}),$$  \hspace{1cm} (11)

$$q^{13} = O(r^{-5}), \quad q^{23} = O(r^{-5}), \quad q^{33} = O(r^{-4}),$$  \hspace{1cm} (12)

so that the $q^{\mu\nu}$ term in (8) is absorbed into the $O(r^{-4})$ error in (3.13). This ensures that the accuracy given in (3.13) is the optimum consistent with $u=0$ being approximately null (in the sense that $\epsilon = O(r^{-2})$).

The delta function term in the Weyl tensor is calculated from [3]

$$C^{\kappa\lambda}_{\mu\nu} = \left\{ 2\eta n^{|\kappa|^{(\kappa} n_{|\nu|]} - 16\pi \delta^{[\kappa}_{|\mu|} S^{\lambda]}_{|\nu|] + \frac{8\pi}{3} S^{\alpha}_{\mu|\nu|} \delta^{\kappa\lambda}_{\mu\nu} \right\} \delta(u).$$  \hspace{1cm} (13)

Care must be taken in identifying the wave part of the coefficient of $\delta(u)$ here. It is not given by the first term in (13) but is contained in the first term (the reader must consult [1] to see this clearly).