A normalization factor was missing in the definition of the transition matrix in equation (2). The corrected equation is

\[ T_{fi} = N_{PW,V}^k \langle \psi_f | V | \psi_i \rangle, \]

(2)

where \( N_{PW} = -(2\pi)^2 \) for a non-vortex plane wave projectile and \( N_V = -(2\pi)^3 \) for a vortex electron projectile. This missing normalization factor propagates through to equations (10), (13)–(14), (16)–(17), and (20). The corrected versions of these equations are

\[ T_{fi}^{V,J}(\vec{q}, \vec{b}) = \frac{(-i)^j}{(2\pi)} \int_0^{2\pi} d\phi_k \, e^{i\phi_k} T_{fi}^{PW}(\vec{q}) e^{-i\vec{k} \cdot \vec{b}}, \]

(10)

\[ \frac{d^3\sigma}{d\Omega_d d\Omega_e dE_e} = \mu_{pa}^{H^f} \lambda \frac{k_i k_e}{(2\pi)^3} \int d\phi_k |T_{fi}^{PW}(\vec{q})|^2. \]

(13)

\[ T_{fi}^{PW} = -\frac{1}{(2\pi)^3} \int d\vec{r}_1 d\vec{r}_2 e^{i\vec{k} \cdot \vec{r}_1} \frac{e^{-i\vec{k} \cdot \vec{r}_2}}{(2\pi)^3/2} \Phi(\vec{r}_2), \]

(14)

\[ T_{fi}^{PW} = -\frac{2}{q^2} \int \frac{e^{-i\vec{k} \cdot \vec{r}_2}}{(2\pi)^3} (-1 + e^{i\vec{k} \cdot \vec{r}_2}) \Phi(\vec{r}_2) d\vec{r}_2^2 \]

(16)

\[ T_{fi}^{PW}_{\perp} = -4\frac{\sqrt{2}}{\pi q^2} \frac{1}{(1 + k_e^2)^2} \frac{1}{(1 + |\vec{q} - \vec{k}_e|^2)^2} \]

(17)

\[ T_{fi}^{PW} = -\frac{2}{q^2} \sum_{\lambda=0}^{\infty} (2\lambda + 1) (-i)^\lambda \int j_{\lambda}(k_e \vec{r}_2) P_{\lambda}(k_e \cdot \vec{r}_2) \]

(20)

Equations (A2)–(A8) in the appendix are also affected by the normalization factor. The corrected versions are

\[ T_{fi}^{V,J}(\vec{q}, \vec{b}) = -(2\pi)^{3/2} \int \chi_f^*(\vec{r}_1) \chi_i^*(\vec{r}_2) V_{\lambda}(\vec{r}_1 - \vec{b}) \times \Phi(\vec{r}_2) d\vec{r}_1 d\vec{r}_2, \]

(A2)

\[ T_{fi}^{V,J}(\vec{q}, \vec{b}) = -\frac{1}{2\pi} \int \chi_f^*(\vec{r}_1) \chi_i^*(\vec{r}_2) V e^{i\vec{k} \cdot \vec{a}_{\kappa,l}} (\vec{k}) \times \delta(k_{\perp} - \kappa) e^{-i\vec{k} \cdot \vec{b}} \times T_{fi}^{PW}(\vec{q}) d^2k_{\perp}, \]

(A3)

\[ |T_{fi}^{V,J}(\vec{q}, \vec{b})|^2 = \frac{1}{(2\pi)^3} \int d^3k_{\perp} d^2k_{\perp} e^{i\phi_k - \phi_{\kappa}} \delta(k_{\perp} - k) \times T_{fi}^{PW}(\vec{q}) d^2k_{\perp}, \]

(A4)

\[ \int |T_{fi}^{V,J}(\vec{q}, \vec{b})|^2 d^2b = \frac{1}{\kappa^2} \int d^3k_{\perp} d^2k_{\perp} e^{i\phi_k - \phi_{\kappa}} \times \delta(k_{\perp} - \kappa) T_{fi}^{PW}(\vec{q}) d^2k_{\perp}, \]

(A5)

\[ \int |T_{fi}^{V,J}(\vec{q}, \vec{b})|^2 d^2b = \frac{1}{\kappa^2} \int d^3k_{\perp} |T_{fi}^{PW}(\vec{q})|^2 \delta(k_{\perp} - k)^2, \]

(A6)

\[ \frac{d^3\sigma}{d\Omega_d d\Omega_e dE_e} = \mu_{pa}^{H^f} \lambda \frac{k_i k_e}{(2\pi)^3} \int d\phi_k |T_{fi}^{PW}(\vec{q})|^2. \]

(A8)
Figure 1. FDCS for ionization of hydrogen by plane wave (black line) and on-axis electron vortex beam projectiles with opening angles of 1 mrad (red dotted line), 10 mrad (blue dash dot line), and 100 mrad (green dash dot dot line). The EVB’s OAM is $1\hbar$ and the projectile scattering angle is 1 mrad.

Figure 2. Same as figure 1, but with a projectile scattering angle of 10 mrad.
Figure 3. Same as figure 1, but with a projectile scattering angle of 100 mrad.

Figure 4. Same as figure 1, but with an OAM = 2ℏ.
Figure 5. Same as figure 2, but with an OAM $\hbar$. 

Figure 6. Same as figure 3, but with an OAM $2\hbar$. 

Figure 7. FDCS for ionization of hydrogen by plane wave (black line) and electron vortex beam projectiles averaged over impact parameter with opening angles of 1 mrad (red dotted line), 10 mrad (blue dash dot line), and 100 mrad (green dash dot dot line). The projectile scattering angle is 1 mrad. Multiplicative factors shown are for the data in the same row (i.e. panels (a) and (d) have the same multipliers).

Figure 8. Same as figure 7 but the scattering angle is 10 mrad.
Figure 9. Same as figure 7 but the scattering angle is 100 mrad.

Figure 10. FDCS and transition amplitudes for ionization of hydrogen by plane wave (black line) and electron vortex beam projectiles with opening angles of 1 mrad (red lines) and 10 mrad (blue lines). Results are shown for individual partial waves of the ionized electron (rows 1–3) and all partial waves (row 4). The incident projectile energy is 1 keV, the ionized electron energy is 20 eV, the scattering angle is 1 mrad, and the EVB’s OAM is 1\(\hbar\). In column 1, the FDCS for EVB ionization have been multiplied by 2000 (\(\alpha = 1\) mrad) and 40 (\(\alpha = 10\) mrad); in columns 2 and 3, the amplitudes for \(\alpha = 1\) mrad have been multiplied by 10. The plane wave amplitudes and FDCS are absolute. The FDCS and amplitudes for \(l = 0\) are constant with respect to ionized electron angle and their values are listed in table 1.
The normalization correction to equation (2) alters the magnitude of the fully differential cross sections in the figures and table. Additionally, we found minor errors in labeled multiplicative factors shown in figures 5–9 and 11. The corrected figures and table are shown below with updated labels and captions. The only conclusion that is affected by these corrections is the comparison of the magnitudes for fully differential cross sections averaged over impact parameters (figures 7–9). The corrected results show that the magnitude of the FDCS for electron vortex beam projectiles averaged over impact parameters is similar to that of plane wave projectiles.

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Ionization of hydrogen by electron vortex beam

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Abstract
Optical vortex beams have an extensive history in terms of both theory and experiment, but only recently have electron vortex beams been proposed and realized. The possible applications of these matter vortex waves are numerous, but a fundamental understanding of their interactions with atoms and molecules has not yet been developed. In this work, fully differential cross sections for fast \((e, 2e)\) collisions using electron vortex projectiles with small amounts of quantized orbital angular momentum are presented. A comparison is made with the fully differential cross sections using plane wave projectiles and a detailed study of angular momentum transfer is included. Results indicate that ionization by electron vortex beam projectiles is less likely than for plane wave projectiles, and for the special case of aligned collisions, the angular momentum of the incident electron is transferred directly to the ionized electron.

Keywords: ionization, electron vortex beam, angular momentum

(Some figures may appear in colour only in the online journal)

1. Introduction

Vortex beams are freely propagating beams characterized by their non-zero orbital angular momentum (OAM) around the propagation direction and phase singularity at the center of the vortex. Their quantized topological structure with spiraling wave fronts has been widely studied in optical contexts beginning with Nye and Berry in 1974 [1]. In the optical case, vortex beams are used extensively in applications such as optical tweezers [2, 3], microscopy [4, 5], micromanipulation [6], astronomy [7], and many others. In these applications, the transfer of OAM allows for the control and manipulation of atoms and molecules on the nanoscale. Only recently it was demonstrated that similar vortex beams could be generated with electrons [8]. Since then, several experimental groups have produced electron vortex beams (EVBs) using various methods [9–12], but applications of EVBs are still in the speculative state.

Clearly, the possibilities for EVBs are numerous, and all involve their interaction with matter at the atomic scale. Unfortunately, very little is known about how EVBs interact with individual atoms, and because the experimental generation of EVBs has only recently become possible, there are no experimental results yet for collisions between EVBs and individual atoms. Also, very little theoretical work exists on this topic, with only a handful of theoretical studies to date for EVB collisions with hydrogen atoms [13–17]. If EVBs are to be used for any of the above applications, it is crucial to understand how electrons with non-zero OAM interact with atoms on a fundamental level.

To date, most of the theoretical work for collisions between EVBs and individual atoms has been performed by the group of van Boxem, Partoens, and Verbeeck [14, 15], in which they used the First Born Approximation (FBA) to study potential scattering and inelastic excitation collisions. In [14], Rutherford potential scattering was examined for incident Bessel beams scattering from a Yukawa potential, and it was shown that for non-zero angular momentum, the cross section along the beam direction was zero, indicating a vortex wave is emitted from the scattering center. In a follow-up study, the group expanded upon the Rutherford scattering model to look at excitation of hydrogen by EVB projectiles [15], and a set of selection rules was derived showing that OAM is transferred directly from the projectile to the atom. Both of these studies assumed targets located on the...
beam axis, but as [16] showed, the transverse structure of the EVB needs to be included by averaging over possible impact parameters. Unfortunately, this averaging causes any OAM information to be washed out due to interference between OAM modes. However, from a theoretical perspective, the study of on-axis collisions does provide some insight into OAM transfer mechanisms. Several other studies have also examined OAM transfer to atoms, molecules, and thin films [18–21].

The studies discussed above represent the majority of the theoretical work that has been performed for collisions involving EVB projectiles. Clearly this leaves large gaps in our understanding of the interaction of EVBs with even simple atomic targets. Ionizing collisions using EVB projectiles represent a much broader field of study due to the additional degrees of freedom offered by a second free electron after the collision. Here, we study ionization of atomic hydrogen using EVB projectiles and specifically explore angular momentum transfer to the ionized electron in on-axis collisions. We also calculate FDCS averaged over impact parameter and show that the theoretical approach is easily generalized to more complex targets and more sophisticated models than the FBA. Atomic units are used throughout unless otherwise noted.

2. Theory

Because very little work exists for collisions between EVBs and atoms, we present here fully differential cross sections (FDCS) using the first Born approximation (FBA) for ionization of hydrogen. In an FBA model, the free electrons are modeled as plane waves, however, in the case of EVBs, the incident projectile is a vortex beam, such as a Bessel beam. In a perturbative model such as the FBA, the FDCS is proportional to the square of the transition matrix $T_{ji}$

$$\frac{d^4\sigma}{d\Omega_1 d\Omega_2 dE_2} = \mu_{pu}^2 \mu_{ie}^2 \frac{k_f k_e}{k_i} |T_{ji}|^2,$$

with

$$T_{ji} = \langle \Psi_f | V | \Psi_i \rangle.$$  

Here $\mu_{pu}$ is the reduced mass of the projectile and target atom, $\mu_{ie}$ is the reduced mass of the proton and the ionized electron,

![Figure 1. FDCS for ionization of hydrogen by plane wave (black line) and on-axis electron vortex beam projectiles with opening angles of 1 mrad (red dotted line), 10 mrad (blue dash dot line), and 100 mrad (green dash dot dot line). The EVB’s OAM is $1\hbar$ and the projectile scattering angle is 1 mrad.](image-url)
\[ \vec{k}_f \] is the momentum of the scattered projectile, \( \vec{k}_e \) is the momentum of the ionized electron, and \( \vec{k}_i \) is the momentum of the incident projectile. Insertion of complete sets of position states into equation (2) results in an integral over all position space for each of the particles in the collision. For EVBs, it is most convenient to use cylindrical coordinates for the projectile and spherical coordinates for the atomic electron. Then, the projectile momenta can be written in terms of their respective longitudinal and transverse components as \( \vec{k}_i = k_{i,\perp} \hat{\rho}_i + k_{i,z} \hat{\zeta}_i \), \( \vec{k}_f = k_{f,\perp} \hat{\rho}_f + k_{f,z} \hat{\zeta}_f \). We consider here what is traditionally referred to as coplanar geometry, in which the final projectile and ionized electron momentum lie in the same plane.

The initial state wave function is a product of the incident projectile Bessel beam wave function \( \chi_{\ell,i}(\vec{r}_1) \) and the target hydrogen atom wave function \( \Phi(\vec{r}_2) \)

\[ \Psi_i = \chi_{\ell,i}(\vec{r}_1) \Phi(\vec{r}_2). \]  

(3)

Like the plane wave, the Bessel beam extends throughout all space in the transverse direction. However, unlike the plane wave, it is not uniform in the transverse direction, but has a phase singularity at a particular point in space. If we define the \( z \)-axis to be along the longitudinal incident momentum direction with the target atom located on this axis, we can then define an impact parameter \( \vec{b} \) that describes the incident beam’s transverse orientation relative to the target atom. For the special case of \( |\vec{b}| = 0 \), the phase singularity of the vortex beam is centered on the \( z \)-axis and the beam’s OAM axis intersects the center of the atom. The Bessel beam is then the free particle solution to the Schrödinger equation in cylindrical coordinates and is given by

\[ \chi_{\ell,i}(\vec{r}) = \frac{e^{i\varphi_i}}{2\pi} J_l(k_{l,\rho} \rho) e^{i\Delta \zeta_i z}, \]  

(4)

where \( \varphi_i \) is the azimuthal position coordinate for the incident projectile, \( l \) is the quantized OAM of the incident projectile, and \( J_l(k_{l,\rho}) \) is the Bessel function. The longitudinal and transverse projectile momentum appear explicitly in this equation, as does the quantized OAM \( l \). This Bessel beam can be rewritten as a superposition of plane waves [15] such that

\[ \chi_{\ell,i}(\vec{r}) = \frac{(-i)^l}{(2\pi)^2} \int_0^{2\pi} d\varphi_i e^{i\varphi_i} e^{i\Delta \zeta_i z}, \]  

(5)
where $\phi_{\text{ii}}$ is the azimuthal angle of the incident projectile momentum. In any laboratory experiment, it would be impractically difficult to achieve exact alignment of the projectile beam with the target. Therefore, it is necessary to consider Bessel beam projectiles with non-zero impact parameters. This can be achieved by acting on the incident centered Bessel beam with the translation operator, and yields a more generalized version of equation (5) such that

$$\chi_{\ell_f,\ell_i}(\vec{r}_1 - \vec{b}) = \frac{(-i)^l}{(2\pi)^2} \int_0^{2\pi} d\phi \frac{e^{i\phi}}{r_{\text{ii}}} e^{i\ell_f \phi} r_{\text{ii}} e^{-i\ell_i \phi}.$$  

Then the average cross section can be found by integrating over all possible impact parameters to achieve an average cross section, as shown in section 2.1.

The final state wave function is a product of the scattered projectile wave function $\chi_{\ell_f,\ell_i}(\vec{r}_1)$ and the ionized electron wave function $\chi_{\ell_i}(\vec{r}_2)$

$$\Psi_f = \chi_{\ell_f,\ell_i}(\vec{r}_1) \chi_{\ell_i}(\vec{r}_2).$$  

The perturbation $V_i$ is the Coulomb interaction between the projectile and target atom, which is given by

$$V_i = -\frac{1}{r_{\text{ii}}} + \frac{1}{r_{\text{ii}2}}.$$  

for an electron incident on a hydrogen atom.

As in [15], we assume that the scattered projectile leaves the collision as a plane wave given by

$$\chi_{\ell_i,\ell_i}(\vec{r}_i) = \frac{e^{i\ell_i \vec{k}}}{(2\pi)^{1/2}}.$$  

Combining the equations above allows us to write the vortex transition amplitude in terms of the plane wave transition amplitude, which is easily calculated and well-known

$$T_{\ell_f,\ell_i}^{V}(\ell_i, \vec{b}) = \left(-\frac{i}{(2\pi)^2}\right) \int_0^{2\pi} d\phi \frac{e^{i\phi}}{r_{\text{ii}}} T_{\ell_f}^{PW}(\ell_i, \vec{b}) e^{-i\ell_i \phi},$$  

where $T_{\ell_f}^{PW}(\ell_i)$ is the transition amplitude for an incident plane wave scattering on a hydrogen atom. A major advantage of equation (10) is that because the vortex beam transition amplitude is written in terms of the plane wave transition amplitude, it
is easily generalized to more complex targets or more sophisticated models. While the derivation of equation (10) requires that the scattered projectile be a plane wave, it has no such requirement for the treatment of the ionized electron or the target atom. Thus, one can calculate $T_{fi}^{PW}$ using any theoretical technique or any target atom and simply insert the result into equation (10) to find the EVB amplitude. One only needs to take care of the calculation of the ‘momentum transfer’ $\mathbf{q} = \mathbf{k}_i - \mathbf{k}_f$, which in the case of an incident vortex beam must be written in terms of its parallel and perpendicular components. Keeping close track of the azimuthal angle for each of the momenta, the magnitude of the momentum transfer can be written as

$$q^2 = k_i^2 + k_f^2 - 2k_i \cdot k_f - 2k_{i\perp} k_{f\perp} \cos(\phi_{ki} - \phi_{kf}).$$

where $\phi_{ki} = 0$ for the coplanar geometry used here.

This expression for momentum transfer combined with the Bessel beam expression as a superposition of plane waves makes it clear that there is no longer a single momentum transfer vector because of the dependence on $\phi_{ki}$ [15]. In the case of either excitation of hydrogen or ionization of hydrogen where the ionized electron is treated as plane wave, evaluation of $T_{fi}^{PW}$ can be performed analytically. Then, equation (10) is evaluated numerically. Note that if the OAM of the incident projectile is zero and its momentum vector is oriented along the $z$-axis, the momentum transfer reduces to the standard form and evaluation of equation (10) results in the appropriate plane wave transition amplitude.

### 2.1. Average over impact parameters

The transition matrix of equation (10) yields the amplitude for ionization by a vortex beam with a specific impact parameter $\mathbf{b}$. In order to realistically compare with experiment, one needs to consider an average of the cross section over all possible impact parameters [16] such that

$$\frac{d^3\sigma}{d\Omega_d d\Omega_2 d\mathcal{E}_2} \propto \lim_{b_{max} \to \infty} \int_0^{b_{max}} \int_0^{2\pi} |T_{fi}^{V,\ell}(\mathbf{q}, \mathbf{b})|^2 d\mathbf{b}. \quad (12)$$

Evaluation of the integral over impact parameter yields

$$\frac{d^3\sigma}{d\Omega_d d\Omega_2 d\mathcal{E}_2} = \mu_{pa} \mu_{pi} \frac{k_f k_e}{k_{i\perp}} \int d\phi_{k_i} |T_{fi}^{PW}(\mathbf{q})|^2. \quad (13)$$

It is interesting to note that this expression does not depend on the OAM of the incident beam. This is because as the impact parameter increases, additional OAM terms contribute to the cross section and their interference causes the

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**Figure 4.** Same as figure 1, but with an OAM $= 2\hbar$. 

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![Figure 4](image-url)
average over impact parameter to be independent of OAM. This can be seen if one uses the Bessel addition theorem to displace the beam \[14, 15\].

### 2.2. Ionization from ground state

For ionization from the ground state, the plane wave transition amplitude is

\[
T_p^{PW} = \frac{1}{(2\pi)^3} \int d\vec{r}_1 d\vec{r}_2 e^{i\vec{q} \cdot \vec{r}_2} \frac{e^{-i\vec{k}_f \cdot \vec{r}_1}}{(2\pi)^{3/2}} V_i(\vec{r}_2),
\]  

(14)

where the ground state hydrogen wave function is given by

\[
\Phi(\vec{r}_2) = \frac{e^{-r_2}}{\sqrt{\pi}}.
\]  

(15)

Evaluation of the integral \(\vec{r}_1\) in equation (14) yields

\[
T_p^{PW-1s} = \frac{\sqrt{2}}{\pi^3 q^2} \left[ \frac{-1}{(1 + k_f^2)^2} + \frac{1}{(1 + |\vec{q} - \vec{k}_e|^2)^2} \right],
\]  

(17)

where again \(\vec{q}\) is written in terms of its parallel and perpendicular components and

\[
|\vec{q} - \vec{k}_e|^2 = q^2 + k_e^2 - 2k_e [k_{e\perp} \cos(\phi_{f\perp} - \phi_{e\perp}) - k_{e\perp} \cos(\phi_{f\perp} - \phi_{e\perp})] - 2k_e q_z^2,
\]  

(18)

with \(\phi_{e\perp} = 0\) or \(\pi\) for the coplanar geometry used here, depending on the plane of the ejected electron.

### 2.3. Partial wave amplitudes

Because one of the mechanisms we are interested in is the OAM transfer from the projectile to the ionized electron, it is advantageous to evaluate equation (16) numerically.

***Figure 5.*** Same as figure 2, but with an OAM = 2\(h\).
allows for a straightforward insertion of a partial wave expansion for the ionized electron plane wave

\[ e^{-i\mathbf{k}_e \cdot \mathbf{r}_2} = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=0}^{\infty} (2\lambda + 1)(-i)^\lambda \mathbf{j}_\lambda(\mathbf{k}_e \cdot \mathbf{r}_2)P_\lambda(\mathbf{k}_e \cdot \mathbf{r}_2), \]

where \( j_\lambda(\mathbf{k}_e \cdot \mathbf{r}_2) \) is the spherical Bessel function, \( P_\lambda(\mathbf{k}_e \cdot \mathbf{r}_2) \) is the Legendre polynomial, and \( \lambda \) corresponds to the partial wave with angular momentum \( \lambda \hbar \). Then, equation (16) becomes

\[ T_{\Phi}^{PW} = \frac{4\pi}{(2\pi)^{3/2}} \sum_{\lambda=0}^{\infty} (2\lambda + 1)(-i)^\lambda \int j_\lambda(\mathbf{k}_e \cdot \mathbf{r}_2)P_\lambda(\mathbf{k}_e \cdot \mathbf{r}_2)(-1 + e^{i\mathbf{k}_s \cdot \mathbf{r}_2})\Phi(\mathbf{r}_2)d\mathbf{r}_2. \]  

From this, the transition amplitudes for individual partial waves can be calculated, which show the relative importance of the amount of angular momentum transferred to the ionized electron. Insertion of equation (20) into the vortex transition amplitude of equation (10) then explicitly demonstrates the transfer of the quantized angular momentum of the incident vortex beam to the ionized electron.

### 3. Results

#### 3.1. Plane wave versus EVB

The use of EVBs as projectiles results in a larger kinematical parameter space than traditional plane wave (e, 2e) collisions. In addition to the standard kinematical parameters of incident energy, ionized electron energy, and projectile scattering angle for EVBs, there are also the parameters of opening angle \( \alpha = \tan^{-1} \frac{k_{e2}}{k_{ic}} \) and incident OAM. In order to conduct a comprehensive study of the effect of these parameters, we explore a range of parameter space within acceptable limitations of the FBA and experimental limitations of EVB generation.

We begin by calculating the FDCS for incident plane waves and the special case of on-axis vortex beam electrons with energies of 500 and 1000 eV. Typical energies of experimental EVBs are on the order of a few hundred eV [8, 9, 11], and our use of incident energies of 500–1000 eV ensures that the perturbative FBA is valid. Specifically, the perturbation parameter \( \eta = \left| \frac{e}{\mathbf{k}_e} \right| \) should be less than unity for the FBA to be valid, and the energies chosen here correspond
to \( \eta = 0.16 \) (500 eV) and \( \eta = 0.12 \) (1000 eV). Three ejected electron energies of 20, 50, and 100 eV are used, again keeping in mind that the use of a plane wave to model the ejected electron requires larger energies. From (e, 2e) studies, it is known that the differences between a plane wave treatment and other more sophisticated close coupling, R-Matrix, and distorted wave treatments of ionized electrons are minimized for energies greater than 20 eV [22–24]. Additionally, the FBA model does not include electron exchange, but for the kinematics chosen here, the exchange amplitude is at least one order of magnitude smaller than the direct ionization amplitude. Therefore, it can safely be neglected.

Typically, plane wave (e, 2e) collisions involve scattering angles of a few degrees, depending on the incident projectile energies. We have chosen to use three scattering angles of 1, 10, and 100 mrad (0.0573°, 0.573°, and 5.73°). Finally, we study the effect of the EVB parameters by examining projectiles with one or two units of OAM, as well as opening angles of 1, 10, and 100 mrad. We note that EVBs with angular momentum values up to a few hundred \( \hbar \) have been produced [10], but for the sake of clarity we limit the scope of this work to small OAM values.

Figures 1–3 show the FDCS for incident plane wave and on-axis EVB projectiles with OAM of one for the range of kinematical values described above, and figures 4–6 show the corresponding FDCS for on-axis EVBs with incident OAM of two. Although the basics of (e, 2e) FDCS are well-known, it is worth mentioning some relevant trends observable in the plane wave FDCS in order to compare the FDCS for EVB projectiles. The plane wave FBA predicts a forward binary and backward recoil peak oriented along the positive and negative linear momentum transfer directions. As the scattering angle increases or the ionized electron energy decreases, the direction of the momentum transfer moves away from the beam direction. Also, as the ionized electron energy increases, the magnitude of the FDCS decreases. Analysis of the EVB results in figures 1–6 yields the following global trends. Like the plane wave FDCS, the EVB FDCS show a two peak structure. However, at the smallest scattering angle, there are no longer forward and backward peaks, but instead
two forward peaks are located symmetrically to either side of the \( z \)-axis. At the largest scattering angle, the binary and recoil peak structure returns, but the peaks are no longer located at the plane wave linear momentum transfer direction. In fact, in most cases, a minimum is observed at or near the plane wave linear momentum transfer direction. Because the incident vortex beam with \( |l| > 0 \) has a zero on the axis, the overlap of the incident wave function with the target is reduced, leading to smaller cross sections. Also, because the EVB can be written as a superposition of tilted plane wave states, no single momentum transfer can be defined, but rather a ring of momentum transfer vectors exists. With the incident momentum now having a transverse component, the ‘momentum transfer’ direction will be shifted from its plane wave value.

Like the plane wave FDCS, the magnitude of the EVB FDCS decreases with increasing scattering angle and ionized electron energy. For the smallest scattering angle, the EVB peak magnitude and locations are nearly symmetric about \( \theta_0 = 180^\circ \), but this symmetry is broken as the scattering angle increases. At the largest scattering angle, the binary peak is generally larger in magnitude and almost no recoil peak is observed. The two peaks that are present are both located in the forward direction on either side of the \( z \)-axis. While most of these trends hold for an incident OAM of two (figures 4–6), the symmetry of peak location and magnitude does not exist in this case. Also, the overall magnitude of the EVB FDCS for OAM of two is 2–8 orders of magnitude smaller than the plane wave FDCS and 1–5 orders of magnitude smaller than the FDCS for OAM = 1. This indicates that ionization by on-axis EVB projectiles is most likely to occur for small quanta of OAM and large opening angle, and may only rarely occur for large angular momentum or small opening angle. While the FDCS for ionization by on-axis EVB is much smaller than the FDCS for ionization by plane wave, the EVB cross sections here are roughly the same order of magnitude as those for on-axis excitation of hydrogen by EVB [15].

3.2. Average over impact parameters

As mentioned above, it is highly unlikely that an experiment could be performed for purely on-axis collisions. For a more accurate idea of the likelihood of ionization by EVB compared to that of plane wave projectiles, it is necessary to average over impact parameters. Unfortunately, this means a loss of OAM information. The information that remains in the
average cross sections is the opening angle dependence, as well as information about how EVB FDCS compare to those of plane wave projectiles. Figures 7–9 show the FDCS for ionization by EVB projectiles averaged over all impact parameters compared to the FDCS for plane wave projectiles. With the exception of the lowest energy ionized electron, the FDCS for EVB projectiles are two to four orders of magnitude smaller than their plane wave counterparts. This indicates that the likelihood of ionization by EVB is much less than for plane wave projectiles, even when an ensemble of impact parameters is included. Additionally, with the exception of $\alpha = 100$ mrad, the shape of the averaged FDCS is quite similar to that of the plane wave projectiles with peaks located near the positive and negative plane wave linear momentum transfer directions. This is a stark contrast to the FDCS on-axis projectiles, in which the peaks were located to either side of the $z$-axis.

Also, the FDCS averaged over impact parameter decrease as opening angle increases. Even though the on-axis FDCS are largest for large opening angles, this is only one special case in the average over impact parameters. The observation that the average FDCS is smallest for large opening angles then implies that off-axis contributions must be small in this case. In contrast, for small opening angles, the off-axis contributions are likely more important.

### 3.3. Angular momentum transfer

Despite the small cross sections and the special case of on-axis alignment, these collisions nevertheless present an opportunity to study OAM transfer. As was shown in [15], for excitation of hydrogen by on-axis EVBs, the scattering amplitudes were strongly influenced by OAM transfer. In our case, EVBs with discrete amounts of OAM combined with the partial wave amplitude analysis described in section 2.3 provide a unique opportunity to learn about OAM transfer to the ionized electron.

Figures 10 and 11 show the individual partial wave amplitudes and FDCS for plane wave and EVB projectiles with $E_i = 1$ keV, $E_e = 20$ eV, $\theta_i = 1$ mrad, and OAM = 1 $\hbar$ or 2 $\hbar$. Only the $l = 1, 2$, and 3 amplitudes are shown because these are the three that contribute significantly to the shape and magnitude of the total amplitude. The $\lambda = 0$ amplitudes are constant with the values listed in table 1. These kinematics were chosen for more detailed study...
because they have the largest FDCS of the kinematical parameters used here.

The partial wave analysis shows some interesting features and we again begin with some observations of the plane wave amplitudes. From figure 10 and equation (17), it is obvious that the plane wave amplitude is purely real and the magnitude of the individual partial wave amplitudes decreases with increasing \( l \). For a plane wave projectile, the largest amplitude is for \( l = 1 \), indicating that the ionized electron is most likely to leave the collision with one unit of angular momentum. However, the \( l = 1 \) FDCS shows equal magnitude binary and recoil peaks and the correct magnitudes are only obtained when the \( l = 2 \) amplitude is included. The binary peak amplitudes constructively interfere, while the recoil peak amplitudes destructively interfere to give the resulting large binary and small recoil peaks in the FDCS. The number of peaks in the plane wave amplitudes and FDCS correspond to the appropriate shapes for s, p, d, f, etc waves, and all plane wave partial wave amplitudes are symmetric about the momentum transfer direction.

For EVB projectiles with OAM = 1, the partial wave amplitudes are purely imaginary, while for OAM = 2, they are purely real. This is due to the \((-i)^l\) factor in the incident Bessel beam expression of equation (5). For both values of incident OAM, the EVB partial wave amplitudes are smaller than their plane wave counterparts, which is expected from the relative magnitudes of the FDCS. Unlike the plane wave partial wave amplitudes, which are symmetric about \( \theta_e = 180^\circ \), the EVB amplitudes for OAM = 1 are antisymmetric about \( \theta_e = 180^\circ \) with nodes at \( \theta_e = 0^\circ \) and \( 180^\circ \) and they sum to produce a total amplitude that is antisymmetric about \( \theta_e = 180^\circ \). The EVB amplitudes for OAM = 2 are symmetric for \( \lambda > 1 \), but antisymmetric for \( \lambda = 1 \). This difference in symmetries results in significant shape differences in the EVB FDCS for OAM = 1 compared with OAM = 2. Specifically, the two peaks in the FDCS for OAM = 1 are located symmetrically

Figure 10. FDCS and transition amplitudes for ionization of hydrogen by plane wave (black line) and electron vortex beam projectiles with opening angles of 1 mrad (red lines) and 10 mrad (blue lines). Results are shown for individual partial waves of the ionized electron (rows 1–3) and all partial waves (row 4). The incident projectile energy is 1 keV, the ionized electron energy is 20 eV, the scattering angle is 1 mrad, and the EVB’s OAM is \( \ell \). In column 1, the FDCS for EVB ionization have been multiplied by 2000 (\( \alpha = 1 \) mrad) and 40 (\( \alpha = 10 \) mrad); in columns 2 and 3, the amplitudes for \( \alpha = 1 \) mrad have been multiplied by 10. The plane wave amplitudes and FDCS are absolute. The FDCS and amplitudes for \( \lambda = 0 \) are constant with respect to ionized electron angle and their values are listed in table 1.
about the momentum transfer direction and with equal magnitude. The FDCS for OAM = 2 on the other hand have a much larger recoil peak than binary peak. This can be traced directly to interference of the individual partial wave amplitudes.

The FBA model employed here assumed that the scattered projectile was a plane wave. However, with the incident particle an EVB, the momentum transfer vector is no longer well-defined. It is then possible for the scattered projectile to

Figure 11. Same as figure 10, but with OAM = 2\(\hat{\lambda}\). In column 1, the FDCS for EVB ionization have been multiplied by \(4 \times 10^8\) (\(\alpha = 1\) mrad) and 4000 (\(\alpha = 10\) mrad); in column 2, the amplitudes have been multiplied by \(10^3\) (\(\alpha = 1\) mrad) and 100 (\(\alpha = 10\) mrad). The plane wave amplitudes and FDCS are absolute. The FDCS and amplitudes for \(\lambda = 0\) are constant with respect to ionized electron angle and their values are listed in table 1.
Table 1. Ejected electron partial wave FDCS and amplitudes in a.u. for $\lambda = 0$. The incident projectile energy is 1 keV, the ionized electron energy is 20 eV, and the scattering angle is 1 mrad.

| Plane wave | EVB OAM = $1\hbar \alpha = 1$ mrad | EVB OAM = $1\hbar \alpha = 10$ mrad | EVB OAM = $2\hbar \alpha = 1$ mrad | EVB OAM = $2\hbar \alpha = 10$ mrad |
|------------|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| FDCS (a.u.) | $1.5 \times 10^{-6}$ | $6.9 \times 10^{-15}$ | $7.3 \times 10^{-13}$ | $3.8 \times 10^{-24}$ |
| $Re(T_\phi)$ (a.u.) | $1.81 \times 10^{-3}$ | $-2.0 \times 10^{-17}$ | $-1.5 \times 10^{-16}$ | $1.8 \times 10^{-12}$ |
| $Im(T_\phi)$ (a.u.) | $4.4 \times 10^{-17}$ | $-7.6 \times 10^{-8}$ | $-7.6 \times 10^{-7}$ | $-4.5 \times 10^{-19}$ |
|                |                                   |                                   |                                   | $3.6 \times 10^{-18}$ |
have non-zero momentum transverse to its propagation direction [15], thereby retaining some or all of its initial OAM. However, as seen in figures 10 and 11, comparison of the EVB amplitudes for OAM = 1 and 2 shows that for OAM = 1, the \( \lambda = 1 \) amplitude is the largest, while for OAM = 2, the \( \lambda = 2 \) amplitude is the largest. This indicates that if ionization occurs, the incident EVB is most likely to transfer all of its angular momentum to the ionized electron. For off-axis collisions, a superposition of OAM components of the incident beam interact with the target, making direct determination of OAM transfer more complicated.

4. Conclusion

We have presented fully differential cross sections for ionization of hydrogen by EVB projectiles calculated within the FBA. A range of kinematical parameters were used in order to study the effect of incident energy, ionized electron energy, and scattering angle. The special case of on-axis alignment was studied and it was shown that the FDGS for the EVB projectiles are several orders of magnitude smaller than those of the plane wave projectiles. The shape of the FDGS for on-axis EVBs is also significantly altered from those of the plane wave projectile. In the case of plane waves, the FBA predicts binary and recoil peaks oriented along and opposite of the linear momentum transfer direction. For EVBs, no single linear momentum transfer direction can be defined, and the binary and recoil peaks in the FDGS are shifted from the plane wave predicted locations due to the projectile’s transverse momentum components. A minimum typically occurs near the linear momentum transfer direction.

Unlike their plane wave counterparts, EVBs carry discrete quantities of OAM and are characterized by their opening angle which relates transverse and longitudinal momenta. This OAM can be transferred to other particles during the collision process. By expanding the ionized electron wave function in terms of partial waves, we were able to determine that for on-axis collisions the OAM of the incident EVB is transferred directly to the ionized electron. The partial wave amplitudes also showed zero amplitude for ejected electrons to be found near the linear momentum transfer direction.

While study of on-axis collisions is of interest from a theoretical perspective, it presents a great challenge for comparison with experiment because exact alignment of the beam axis with the target is unlikely to occur. This necessitates an average of the cross section over impact parameters, which results in a loss of OAM dependence. In this case, the shape of the average FDGS more closely resemble that of the plane wave FDGS, but only for the lowest ionized electron energy and smallest opening angles are the magnitudes similar. For all other kinematical conditions, the EVB average FDGS are two to four orders of magnitude smaller than their plane wave counterparts, indicating that ionization by EVB is generally less likely than by plane wave projectile.

While the FBA is a simplistic model for (e, 2e) collisions, the results presented here are to our knowledge the first FDGS for ionization by EVB. The theoretical derivation for the EVB transition matrix can be written in terms of the plane wave transition matrix, making expansion to more sophisticated models and more complicated targets straightforward. We are currently expanding the FBA model to include a distorted wave treatment of the ionized electron so that smaller ejected electron energies can be studied in regimes where the FDGS are expected to be larger and more experimentally realizable. In addition, we are also generalizing the model to multi-electron targets.

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Appendix

To average the cross section over impact parameter, it is necessary to calculate the average of the transition matrix squared over all impact parameters

\[
\int |T_{fi}^{V,i}(\vec{q}, \vec{b})|^2 \, d\vec{b},
\]  

(A1)

where \( T_{fi}^{V,i}(\vec{q}, \vec{b}) \) from equation (10) can be written as

\[
T_{fi}^{V,i}(\vec{q}, \vec{b}) = \int \chi_j^*(\vec{r}_1) \chi_e^*(\vec{r}_2) V_\ell \chi_i(\vec{r}_1 - \vec{b}) \Phi(\vec{r}_2) \, d\vec{r}_1 \, d\vec{r}_2.
\]  

(A2)

Using equation (6) and the plane wave expansion for a vortex state [14] yields

\[
T_{fi}^{V,i}(\vec{q}, \vec{b}) = \frac{1}{(2\pi)^2} \int \chi_j^*(\vec{r}_1) \chi_e^*(\vec{r}_2) V_\ell \chi_i(\vec{r}_1 - \vec{b}) \Phi(\vec{r}_2) \, d\vec{r}_1 \, d\vec{r}_2 \, d^2 k_{i,i}.
\]  

(A3)

where \( a_{e,i}(k_{i,i}) = (-i)^e b^{(\delta(k_{i,i}-\kappa))/\kappa} \) and \( \kappa \) is the magnitude of the transverse momentum.

Identifying in equation (A3) the expression for the plane wave transition matrix of equation (14) leads to

\[
T_{fi}^{V,i}(\vec{q}, \vec{b}) = \frac{1}{(2\pi)^2} \int (-i)^e b^{(\delta(k_{i,i}-\kappa))/\kappa} e^{-\frac{i}{\hbar} \vec{q} \cdot \vec{k}_{i,i}} \, d\vec{k}_{i,i}.
\]  

(A4)

Taking the absolute square of equation (A4) gives

\[
|T_{fi}^{V,i}(\vec{q}, \vec{b})|^2 = \frac{1}{(2\pi)^4 \hbar^2} \int d^2 k_{i,i} \, d^2 \vec{q} \delta(k_{i,i} - \kappa) e^{\frac{i}{\hbar} (\delta(k_{i,i} - \kappa) - \kappa) \vec{q} \cdot \vec{k}_{i,i}} \times T_{fi}^{PW,\kappa}(\vec{q}) T_{fi}^{PW,\kappa}(\vec{q})
\]  

(A5)

where \( \vec{q} \) represents the momentum transfer vector using \( \vec{k}'_{i,i} \). Plugging equation (A5) into equation (A1) and performing
the integral over $d^2b$ results in
\[
\int |T_{fi}^{ij}(q, \theta)|^2 d^2b = \frac{1}{(2\pi)^3} \int d^2k \delta(k - \theta) \delta(k' - \kappa) T_{fi}^{PW}(q')
\]
\[
\times T_{fi}^{PW}(q) \delta(k - k', k_{i\perp} - k_{j\perp}).
\]  
(A6)

which after performing the integral over $d^2k_{i\perp}$ becomes
\[
\int |T_{fi}^{ij}(q, \theta)|^2 d^2b = \frac{1}{(2\pi)^3} \int d^2k \delta(k - \theta) \delta(k' - \kappa) T_{fi}^{PW}(q') \delta(k - k', k_{i\perp} - k_{j\perp})^2.
\]  
(A7)

Finally, using the expression for the delta function squared from [25], performing the integral over $k_{i\perp}$, and multiplying by the appropriate flux constants yields equation (13)
\[
\frac{d^3\sigma}{d\Omega_1 d\Omega_2 dE_2} = \mu_{pi}^2 \frac{k_{j\perp}}{k_{i\perp}} \int d\phi_k |T_{fi}^{PW}(q')|^2.
\]  
(A8)

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