REGULARITY AND EXTREMALITY OF QUASICONFORMAL
HOMEOMORPHISMS ON CR 3-MANIFOLDS

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ABSTRACT. This paper first studies the regularity of conformal homeomorphisms on smooth locally embeddable strongly pseudoconvex CR manifolds. Then moduli of curve families are used to estimate the maximal dilatations of quasiconformal homeomorphisms. On certain CR 3-manifolds, namely, CR circle bundles over flat tori, extremal quasiconformal homeomorphisms in some homotopy classes are constructed. These extremal mappings have similar behaviors to Teichmüller mappings on Riemann surfaces.

1. Introduction

A contact manifold $M$ is a manifold of odd dimension with a non-integrable distribution $HM$ of tangent hyperplanes. A Cauchy-Riemann (CR) manifold is a contact manifold $M$ endowed a complex structure on the contact bundle $HM$. Two CR manifolds are equivalent if there is a homeomorphism between them which preserves both contact and CR structures. Generally, between any two CR structures assigned on the same contact manifold, there may be no such so-called CR homeomorphism between them. Therefore we consider those homeomorphisms between CR manifolds which preserve the underlying contact structures and distort the CR structures boundedly. They are called quasiconformal homeomorphisms. In a class of homeomorphisms between two CR manifolds, an extremal mapping is a quasiconformal homeomorphisms which distorts the CR structures in a minimal way. This paper studies regularity of quasiconformal homeomorphisms and extremal quasiconformal homeomorphisms on smooth strongly pseudoconvex CR manifolds.

The notion of quasiconformal homeomorphisms is a new tool to study CR structures as initiated by Korányi and Reimann. In this paper, we use an analytic definition of quasiconformal homeomorphisms given in [12] which is a generalization of the one given by Korányi and Reimann in [5]. We restrict ourselves to the 3-dimensional case here not only because the notion of quasiconformality is not invariant under CR transformations in higher dimensional cases, but also because 3-dimensional CR structures are among the most interesting objects in the theory of CR manifolds. We refer to [7] for details about the second point.

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Korányi and Reimann proved that $C^4$ conformal homeomorphisms on Heisenberg groups must be smooth and CR (Theorem 8, [6]). By applying a regularity theorem of weak CR mappings of Pinchuk and Tsyganov, we generalize Korányi and Reimann’s result to that a conformal homeomorphism $f$ between two smooth, strongly pseudoconvex, locally embeddable CR manifolds must be smooth and CR, if $f$ has $L^1_{loc}$ horizontal derivatives (Theorem 2.3). Hence between such CR manifolds, quasiconformal homeomorphisms with this weak regularity are actually “quasi-CR”.

To study the extremality of quasiconformal homeomorphisms is a global problem. But our analytic definition of quasiconformality is proposed infinitesimally. Therefore we need some global notion to describe the quasiconformality. The one best fitting our later developments is the notion of moduli of curve families. We prove that a $C^2$ diffeomorphism is quasiconformal if and only if it preserves moduli of certain curve families up to a fixed bounded multiple (Theorem 3.3). On the other hand, a homeomorphism satisfying this property is absolutely continuous on lines (ACL) (Theorem 3.4).

Between CR circle bundles over flat tori, we construct extremal quasiconformal homeomorphisms in certain homotopy classes (Theorem 4.2). There are two transversal Legendrian foliations such that the extremal homeomorphism constructed preserves these two foliations. More precisely, it is a stretching by a constant factor along leaves of one foliation and a compressing by the same factor along leaves of another foliation. This behavior is analogous to those of Teichmüller mappings on Riemann surfaces (see [13] or [1]). The generator $T$ of the circle action is transversal to the contact bundle. In this transversal direction, the extremal mappings are equivariant under the circle action.

But on an arbitrary CR 3-manifold with a transversal free circle action, an extremal quasiconformal homeomorphism is not necessarily equivariant under the circle action. Such CR manifolds are constructed in [12] so that no extremal quasiconformal mapping between them is equivariant.

This work is heavily influenced by the theory of quasiconformal homeomorphisms on Riemann surfaces and Teichmüller theory. Numerous proofs in this paper are motivated by the proofs of the analogous facts on Riemann surfaces. For example, the construction of the extremal homeomorphisms made in Theorem 4.2, one of the main results of this paper, can find its root in the classical Grötzsch’s theorem which is proved by a length-area argument [9]. Teichmüller generalized this result to closed Riemann surfaces, in particular tori, by an ergodic version of the length-area argument [13]. The notion of modulus of a curve family is a formalism of the length-area (volume) argument. In section 4, we reformulate Teichmüller’s method to the CR setting by computing moduli of some special families of curves and successfully find the extremal quasiconformal homeomorphisms in certain homotopy classes of mappings between CR circle bundles over flat tori.

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2. Regularity of conformal homeomorphisms

For $j = 1, 2$, let $M_j$ be a smooth strongly pseudoconvex CR 3-manifolds. The contact
bundles $HM_j$ is assumed to be smooth and orientable, that is, there exists smooth global 1-form $\eta_j$ on $M_j$, which is called a contact form, so that $HM_j = \text{Ker} \eta_j$. Let $J_j : HM_j \to HM_j$ denote the CR structure on $M_j$. $H^{0,1}M_j \triangleq \{ X + iJ_j X \mid X \in HM_j \} \subset \mathbb{C} \otimes HM_j$ is the (0,1) tangent bundle on $M_j$. $\wedge^{1,0}M_j \triangleq \{ \text{linear functional } \psi : \mathbb{C} \otimes HM_j \to \mathbb{C} \mid \psi(J_jX) = i\psi(X), \text{ for } X \in HM_j \}.$

A mapping $f : M_1 \to M_2$ is said to be absolutely continuous on lines (ACL) if for any open set with a smooth contact fibration, $f$ is absolutely continuous along all fibers of this fibration except a subfamily of measure zero. Here a subfamily of fibers of the fibration is said to have measure zero if intersections of these fibers with any transversal regular surface has measure zero on the surface (see [12]).

**Definition 2.1.** A homeomorphism $f : M_1 \to M_2$ is said to be $K$-quasiconformal for a finite constant $K \geq 1$ if

(i) $f$ is ACL;

(ii) $f$ is differentiable almost everywhere, and its differential $f_*$ preserves the contact structures, i.e., $f_*(H_qM_1) \subset H_{f(q)}M_2$, for $q \in M_1$ where $f$ has differential.

(iii) for norms $| \cdot |_1$ and $| \cdot |_2$ defined by any Hermitian metrics on $HM_1$ and $HM_2$ respectively,

\[
K(f)(q) = \max_{X \in H_qM_1, |X|_1 = 1} \frac{|f_*(X)|_2}{\min_{X \in H_qM_1, |X|_1 = 1} |f_*(X)|_2} \leq K < \infty,
\]

for almost all $q \in M_1$. Here $K = \text{ess sup } K(f)(q)$ is called the maximal dilatation of $f$. $f$ is conformal if $K(f) = 1$. $K(f) = \infty$ if $f$ is not $K$-quasiconformal for any finite $K \geq 1$.

A mapping $f : M_1 \to M_2$ which is differentiable almost everywhere is said to have $L^p_{loc}$ horizontal derivatives for $p \geq 1$ if for any smooth function $h : M_2 \to \mathbb{R}$ and any smooth local section $X$ of $HM_1$ on an open set $U \subset \subset M_1$, the function $X(h \circ f)$ which is defined almost everywhere on $U$ is in $L^p(U)$.

A mapping $f : M_1 \to M_2$ is said to have $L^p_{loc}$ weak horizontal derivatives for some $p \geq 1$ if for any smooth function $h : M_2 \to \mathbb{R}$ and any open set $U \subset \subset M_1$ with a smooth local section $X \neq 0$ of $HM_1$ on $U$, there exists a function $g \in L^p_{loc}(M_1)$ so that

\[
\int_U X \phi \cdot (h \circ f) \, dv_1 = - \int_U g \phi \, dv_1
\]

for all $\phi \in C_0^\infty(U)$, where $dv_1$ is a smooth volume form on $M_1$. Certainly, the function $g$ depends on the choice of $dv_1$.

For the proof of the next theorem, we fix a norm $| \cdot |$ on $HM_1$. A regular curve on a contact manifold is called Legendrian if it is tangent to the contact structure. For any smooth Legendrian curve $\gamma : I \to M_1$ with an interval $I \in \mathbb{R}$ and a function $g$ defined on an open neighborhood of $\gamma$, define the line integral

\[
\int_\gamma g = \int_I g(\gamma(t))|\gamma'(t)| \, dt.
\]
Theorem 2.2. A mapping $f : M_1 \to M_2$ is ACL and has $L^p_{\text{loc}}$ horizontal derivatives for some $p \geq 1$ if and only if $f$ has $L^p_{\text{loc}}$ weak horizontal derivatives.

Proof. First assume that the homeomorphism $f : M_1 \to M_2$ is ACL and has $L^p_{\text{loc}}$ horizontal derivatives. Let $h : M_2 \to \mathbb{R}$ be a smooth function and $U \subset \subset M_1$ be any open set with a smooth section $X \not\equiv 0$ of $HM_1$ on it.

We can assume that the trajectories $\Gamma = \{\gamma\}$ of $X$ form a contact fibration of $U$ by shrinking $U$ appropriately. Let $\Gamma_1$ be the subfamily of those $\gamma \in \Gamma$ along which $f$ is absolutely continuous. Then $\Gamma \setminus \Gamma_1$ has measure zero. Along $\gamma \in \Gamma_1$, $h \circ f$ is absolutely continuous, so $X(h \circ f)$ exists almost everywhere on $\gamma$ and

$$\int_{\gamma} X\phi \cdot (h \circ f) + \int_{\gamma} \phi \cdot X(h \circ f) = \int_{\gamma} X(\phi \cdot h \circ f) = 0, \quad \forall \phi \in C_0^\infty(U).$$

We have topological and differential structures on $\Gamma$ such that the natural projection $p : U \to \Gamma$ is open and smooth. Then $\Gamma$ becomes a smooth surface. Let $t$ be the parameter of the flow generated by $X$ and $\omega$ be any area form on $\Gamma$. Then $dv_1 \triangleq p^* \omega \wedge dt$ is a volume form of $U$. Integrating the expressions in (2.3) against $\omega$ with respect to $\gamma \in \Gamma_1$, then by the ACL property of $f$, local $L^p$ integrability of $X(h \circ f)$ and Fubini’s theorem, we obtain

$$\int_U X\phi \cdot (h \circ f) dv_1 = -\int_U \phi \cdot X(h \circ f) dv_1.$$  

So the weak derivative of $h \circ f$ in the $X$ direction is given by $X(h \circ f) \in L^p_{\text{loc}}$.

Conversely, assume $f$ has $L^p_{\text{loc}}$ weak horizontal derivatives. For any open set $U \subset \subset M_1$ and a smooth contact fibration $\Gamma$ of $U$, let $X$ be the nonzero horizontal vector field on $U$ so that $X_q$, for any $q \in U$, is the tangent vector at $q$ of the fiber $\gamma \in \Gamma$ passing through $q$. Let $B \subset \subset U$ be an open set with the coordinate system $\{(x,y,t) \mid a_1 < x < a_2, b_1 < y < b_2, c_1 < t < c_2\}$, here $t$ is the parameter of the flow generated by $X$, i.e., $X = \frac{\partial}{\partial t}$.

For any smooth function $h$ on $M_1$, $h \circ f$ has $L^p(U)$ weak derivative in the direction $X$. Denote it by $\psi \in L^p(U)$. Hence there exists a sequence of $C^1$ functions $g_n$ on $B$ such that $g_n$ converges to $h \circ f$ uniformly in $B$ and $Xg_n$ converges to $\psi$ in $L^p(B)$. Let $B_{x,y,t} = (a_1,x) \times (b_1,y) \times (c_1,t)$, $R_{x,y} = (a_1,x) \times (b_1,y)$.

$$\int_{B_{x,y,t}} Xg_n(u,v,w) dudvdw = \int_{R_{x,y}} (g_n(u,v,t) - g_n(u,v,c_1)) dudv.$$  

Hence by taking limits, we have

$$\int_{B_{x,y,t}} \psi(u,v,w) dudvdw = \int_{R_{x,y}} ((h \circ f)(u,v,t) - (h \circ f)(u,v,c_1)) dudv.$$  

Let $\{t_n\}$ be a countable dense set of $(c_1,c_2)$. (2.6) implies for each $t_n$, there exists a set $E_n \subset R_{a_2,b_2}$ so that $R_{a_2,b_2} \setminus E_n$ is of measure zero and

$$\int_{c_1}^{t_n} \psi(x,y,w)dw = (h \circ f)(x,y,t_n) - (h \circ f)(x,y,c_1), \quad \forall (x,y) \in E_n.$$
Then (2.7) is true for all \((x, y) \in E \triangleq \cap E_n\) and all \(t_n\). Hence by continuity of both sides in \(t\),

\[
\int_{c_1}^{t} \psi(x, y, w)dw = (h \circ f)(x, y, t) - (h \circ f)(x, y, c_1), \quad \forall (x, y) \in E, \ t \in (c_1, c_2).
\]

So \((h \circ f)(x, y, t)\) is absolutely continuous in \(t\) for \((x, y) \in E\). Note \(R_{a_2, b_2} \setminus E\) has measure zero. So \(f\) is ACL since \(B \subset U\) is an arbitrary rectangular coordinate chart and \(U \subset M_1\) is arbitrary in \(M_1\). Moreover, \(X(h \circ f)\) exists almost everywhere and \(X(h \circ f) = \psi \in L^p(U)\) on \(U\). So \(f\) has \(L^p_{loc}\) horizontal derivatives. \(\square\)

A homeomorphism \(f : M_1 \to M_2\) is said to be weakly CR if for any smooth CR function \(h : M_2 \to \mathbb{C}\), open set \(U \subset M_1\), \(\phi \in C^\infty_0(U)\) and \(Z \in H^{0,1}M_1\), \(\int_U (h \circ f) \cdot Z\phi = 0\).

**Theorem 2.3.** Assume \(M_1\) and \(M_2\) are two smooth, strongly pseudoconvex, locally embeddable CR 3-manifolds, \(f : M_1 \to M_2\) is a conformal homeomorphism with \(L^1_{loc}\) horizontal derivatives. Then \(f\) is smooth and CR.

**Proof.** A simple linear algebra argument shows that at a point \(q \in M_1\) where \(f\) is differentiable

\[
K(f)(q) = \frac{1 + |\mu(q)|}{1 - |\mu(q)|} \quad \text{with} \quad |\mu(q)| = \left|\frac{\langle f^*\psi_2, Z\rangle}{\langle f^*\psi_2, Z\rangle}\right|(q),
\]

for any nonzero \(Z \in H^{0,1}M_1\) and nonzero \(\psi_2 \in \land^{1,0}M_2\).

Then \(K(f) = 1\) implies that if \(f\) is differentiable at \(q \in M_1\), \(\langle f^*\psi_2, Z\rangle = 0\) for any \(Z \in H^{0,1}qM_1\) and \(\psi_2 \in \land^{1,0}_{f(q)}M_2\). In other words, \(f\) preserves the CR structures at the points where it is defined. Thus \(Z(h \circ f) = 0\) for any CR function \(h\) on \(M_2\) and \(Z \in H^{0,1}M_2\). Theorem 2.2 says that \(f\) is weak CR. \(M_1\) and \(M_2\) are locally embeddable implies they are locally embeddable into \(\mathbb{C}^2\) as hypersurfaces. A theorem of Pinchuk and Tsyganov (Theorem 2, [10]) asserts that such \(f\) must be smooth, hence CR. \(\square\)

### 3. Moduli of curve families

Let \(M\) be a smooth, compact, contact 3-manifold. We always assume \(HM\) is smooth and oriented. A sub-Riemannian metric on \(M\) with respect to \(HM\) is a smooth positive definite quadratic form on \(HM\). Fix a sub-Riemannian metric on \(M\) with respect to \(HM\) momentarily, and denote by \(|\cdot|\) the corresponding norm on \(HM\). For general theory of sub-Riemannian geometry, we refer to [11] and [2].

The sub-Riemannian metric on \(M\) can be extended to a Riemannian metric on \(M\) canonically as follows. Let \(\omega\) be the oriented area form on \(HM\) with respect to the sub-Riemannian metric. Then there exists a unique contact form \(\eta\) so that \(d\eta|_{HM} = \omega\). Let \(T\) be the characteristic vector field of \(\eta\), namely, \(T\) is the unique vector field satisfying that \(T^*_\eta = 1\) and \(T_\eta d\eta = 0\). Declaring \(T\) is a unit vector orthogonal to \(HM\), we obtain a Riemannian metric
which is called the canonical extension of the sub-Riemannian metric. The positive volume form of this Riemannian metric is \( dv \triangleq d\eta \wedge \eta \).

A curve \( \gamma : I_\gamma \to M \) with an interval \( I_\gamma \subset \mathbb{R} \) is called locally rectifiable if \( \gamma \) is absolutely continuous and \( \gamma'(t) \) is tangent to \( HM \) for almost all \( t \in I_\gamma \). \( \gamma \) is called rectifiable if \( \gamma \) is locally rectifiable and the length

\[
(3.1) \quad l(\gamma) \triangleq \int_{I_\gamma} |\gamma'(t)| \, dt < \infty.
\]

We set \( l(\gamma) = \infty \) if \( \gamma \) is not rectifiable. For a locally rectifiable curve \( \gamma \) and a non-negative Borel-measurable function \( \sigma \) on \( M \), define the line integral

\[
(3.2) \quad \int_\gamma \sigma = \int_{I_\gamma} \sigma(\gamma(t)) |\gamma'(t)| \, dt.
\]

**Definition 3.1.** Let \( \Gamma \) be a family of curves \( \gamma : I_\gamma \to M \). An admissible measure for \( \Gamma \) is a Borel-measurable function \( \sigma : M \to \mathbb{R} \) so that \( \sigma \geq 0 \) and \( \int_\gamma \sigma \geq 1 \), for all locally rectifiable \( \gamma \in \Gamma \). Denote the set of admissible measures for \( \Gamma \) by \( A(\Gamma) \). The modulus of \( \Gamma \) is defined by

\[
(3.3) \quad \text{Mod}_M(\Gamma) = \inf_{\sigma \in A(\Gamma)} \int_M \sigma^4 \, dv.
\]

**Remark.** (1) It is easy to see that if two sub-Riemannian metrics on \( M \) with respect to \( HM \) define the same conformal structure on \( HM \), then they give the same value to \( \text{Mod}_M(\Gamma) \).

(2) If \( \Gamma_r \subset \Gamma \) consisting of all locally rectifiable curves of \( \Gamma \), then \( \text{Mod}_M(\Gamma_r) = \text{Mod}_M(\Gamma) \).

The following proposition shows that modulus, regarded as a measure of (more precisely, locally rectifiable) curve families, generalizes the concept of measure zero used in the definition of ACL property. Thereafter if a property holds for all curves in a family \( \Gamma \) except a subfamily with zero modulus, we say this property is true for almost all curves in \( \Gamma \).

**Proposition 3.2.** Let \( U \) be an open set of \( M \), \( \Gamma \) a contact fibration of \( U \), \( \Gamma_1 \subset \Gamma \). Then \( \Gamma_1 \) has measure zero if and only if \( \text{Mod}_M(\Gamma_1) = 0 \).

**Proof.** Without loss of generality, we assume \( U \) is a domain of the coordinator system \( \{(x,y,t) \mid a_1 < x < a_2, b_1 < y < b_2, c_1 < t < c_2\} \) and \( X = \frac{\partial}{\partial t} \) is tangent to \( \Gamma \). Let \( E \subset (a_1,a_2) \times (b_1,b_2) \) so that \( \Gamma_1 = \{ \text{curves } t \mapsto (x,y,t) \mid (x,y) \in E \} \).

If \( \Gamma_1 \) has measure zero, then \( \int_E dx dy = 0 \). Notice

\[
(3.4) \quad \sigma_0 = \begin{cases} \frac{1}{c_2 - c_1}, & \text{when } (x,y) \in E, t \in (c_1,c_2), \\ 0, & \text{otherwise}, \end{cases} \in A(\Gamma_1).
\]

Therefore

\[
(3.5) \quad \text{Mod}_M(\Gamma_1) \leq c \int_U \sigma_0^4 \, dx dy dt = 0,
\]
where $c$ is a constant upper bound of the Jacobian $J$ on $U$ with $J \, dx \, dy \, dt = dv$.

If $\text{Mod}_M(\Gamma_1) = 0$, then for any $\sigma \in \mathcal{A}(\Gamma_1)$ with $\sigma = 0$ outside $U$, $\gamma \in \Gamma_1$,

$$1 \leq (\int_\gamma \sigma)^4 \leq (\int_\gamma \sigma^4)(\int_\gamma 1)^3 = (c_2 - c_1)^3 \int_\gamma \sigma^4.$$  

Taking the integral over $E$,

$$\int_E dx \, dy \leq c'(c_2 - c_1)^3 \int_M \sigma^4.$$  

Hence

$$\int_E dx \, dy \leq c'(c_2 - c_1)^3 \text{Mod}_M(\Gamma_1) = 0,$$

that is, $\Gamma_1$ has measure zero. □

Let $M_1$ and $M_2$ be two compact, smooth, strongly pseudoconvex CR 3-manifolds with smooth contact form $\eta_1$ and $\eta_2$ respectively. Here the roles of sub-Riemannian metrics on $HM_1$ and $HM_2$ are played by Hermitian metrics with respect to the CR structures $J_1$ and $J_2$ respectively.

**Theorem 3.3.** A $C^2$ homeomorphism $f : M_1 \to M_2$ is $K$-quasiconformal for a constant $K > 1$ if and only if for any family $\Gamma$ of $C^1$ Legendrian curves on $M_1$

$$\frac{1}{K^2} \text{Mod}_{M_1}(\Gamma) \leq \text{Mod}_{M_2}(f(\Gamma)) \leq K^2 \text{Mod}_{M_1}(\Gamma),$$

where $f(\Gamma) = \{f(\gamma) \mid \gamma \in \Gamma\}$.

**Proof.** Assume $f : M_1 \to M_2$ is $K$-quasiconformal. Then $f$ is contact, i.e., for contact forms $\eta_1$ and $\eta_2$ on $M_1$ and $M_2$ respectively, $f^* \eta_2 = \lambda \eta_1$ with a $C^1$ function $\lambda$ on $M_1$. Then

$$f^*(d\eta_2) = d\lambda \wedge \eta_1 + \lambda \, d\eta_1.$$  

Therefore

$$f^*(d\eta_2|_{HM_2}) = \lambda \, d\eta_1|_{HM_1}.$$  

For $j = 1, 2$, the Levi form $L_j$ on $M_j$ is a bilinear form on $HM_j$ defined by

$$L_j(X, Y) = d\eta_j(X, J_j Y), \quad \text{for } X, Y \in HM_j.
By replacing $\eta_j$ by $-\eta_j$, if necessary, we can always assume that $L_j$ is positive definite. Hence $L_j$ is a Hermitian form on $HM_j$. With respect to the Levi forms on $M_1$ and $M_2$, we define

$$
\lambda_1(q) = \max_{Y \in H_q M_1, |Y|_1 = 1} |f^*_2(Y)|_2, \\
\lambda_2(q) = \min_{Y \in H_q M_1, |Y|_1 = 1} |f^*_2(Y)|_2.
$$

Then (3.11) implies that $|\lambda| = \lambda_1 \lambda_2$. Moreover, (3.10) implies that

$$
f^*(d\eta_2 \wedge \eta_2) = \lambda^2 d\eta_1 \wedge \eta_1.
$$

Thus the Jacobian of $f$ with respect to the volume forms $dv_1 = d\eta_1 \wedge \eta_1$ on $M_1$ and $dv_2 = d\eta_2 \wedge \eta_2$ on $M_2$ is $J(f) = (\lambda_1 \lambda_2)^2$.

For $\sigma_2 \in A(f(\Gamma))$,

$$
\int_{f(\gamma)} \sigma_2 = \int_{I_\gamma} |f^*_2(\gamma'(t))|_2 \sigma_2(f(\gamma(t))) dt
\leq \int_{I_\gamma} \lambda_1 |\gamma'(t)|_1 \sigma_2(f(\gamma(t))) dt
= \int_{\gamma} \lambda_1 \cdot \sigma_2 \circ f.
$$

Hence $\sigma_2 \in A(f(\Gamma))$ implies that $\lambda_1 \cdot \sigma_2 \circ f \in A(\Gamma)$. On the other hand,

$$
\int_{f(\gamma)} \sigma_2 \geq \int_{I_\gamma} \lambda_2 |\gamma'(t)|_1 \sigma_2(f(\gamma(t))) dt
= \int_{\gamma} \lambda_2 \cdot \sigma_2 \circ f.
$$

Hence $\sigma_1 \in A(\Gamma)$ implies $(\sigma_1/\lambda_2) \circ f^{-1} \in A(f(\Gamma))$. Therefore,

$$
\text{Mod}_{M_2}(f(\Gamma)) = \inf_{\sigma_2 \in A(f(\Gamma))} \int_{M_2} \sigma_2^4 dv_2
\leq \inf_{\sigma_1 \in A(\Gamma)} \int_{M_2} ((\sigma_1/\lambda_2) \circ f^{-1})^4 dv_2
= \inf_{\sigma_1 \in A(\Gamma)} \int_{M_1} \sigma_1^4 \cdot (\lambda_1 \lambda_2)^2 dv_1
\leq K^2 \text{Mod}_{M_1}(\Gamma).
$$
\[
\text{Mod}_{M_1}(\Gamma) = \inf_{\sigma_1 \in A(\Gamma)} \int_{M_1} \sigma_1^4 \, dv_1 \\
\leq \inf_{\sigma_2 \in A(f(\Gamma))} \int_{M_1} (\lambda_1 \cdot \sigma_2 \circ f)^4 \, dv_1 \\
= \inf_{\sigma_2 \in A(f(\Gamma))} \int_{M_2} \frac{(\lambda_1 \cdot f^{-1} \cdot \sigma_2)^4}{(\lambda_1 \lambda_2)^2} \, dv_2 \\
\leq K^2 \text{Mod}_{M_2}(f(\Gamma)).
\]

(3.18)

Assume that \( f \) satisfies the inequalities (3.9). Then \( f \) must preserve the contact structures. Otherwise, there will be a point \( q \in M_1 \) with a tangent vector \( X_q \in H_q M_1 \) so that \( f_*(X_q) \notin H_{f(q)} M_2 \). \( f \) is \( C^1 \), so there is a neighborhood \( U \) of \( q \) with a nonzero smooth section \( X \) of \( H M_1 \) on \( U \) so that \( f_*(X_q') \) is not contact for each \( q' \in U \). Let \( \Gamma \) be the family of trajectories of \( X \), and shrink \( U \) appropriately such that \( \text{Mod}_{M_1}(\Gamma) \neq 0 \). But each curve in \( f(\Gamma) \) is not Legendrian everywhere, hence not locally rectifiable. So \( \text{Mod}_{M_2}(f(\Gamma)) = 0 \) by Definition 3.1. Then \( f \) cannot satisfy (3.9) for this family \( \Gamma \). This contradiction shows that \( f \) must be contact.

If \( f \) is not \( K \)-quasiconformal, there exists an open set \( U \subset M_1 \) so that \( \lambda_1 / \lambda_2 \geq K + \epsilon \), for some \( \epsilon > 0 \). Let \( X \) be the vector field on \( U \) so that \( |X|_1 = 1 \), \( |f_*(X)|_2 = \lambda_2 \). Let \( \Gamma \) be the family of trajectories of \( X \) in \( U \). Then for any integrable function \( \sigma_2 \) on \( f(U) \),

\[
\int_{f(\gamma)} \sigma_2 = \int_{\gamma} \sigma_2 \circ f \cdot |f_*(X)|_2 = \int_{\gamma} \lambda_2 \cdot \sigma_2 \circ f.
\]

So \( \sigma_2 \in A(f(\Gamma)) \) if and only if \( \sigma_1 \triangleq \lambda_2 \cdot \sigma_2 \circ f \in A(\Gamma) \). Hence

\[
\text{Mod}_{M_2}(f(\Gamma)) = \inf_{\sigma_2 \in A(f(\Gamma))} \int_{M_2} \sigma_2^4 \, dv_2 \\
= \inf_{\sigma_1 \in A(\Gamma)} \int_{f(U)} \left( \frac{\sigma_1}{\lambda_2} \circ f^{-1} \right)^4 \, dv_2 \\
= \inf_{\sigma_1 \in A(\Gamma)} \int_{U} \left( \frac{\sigma_1}{\lambda_2} \right)^4 \, dv_2 \\
= \inf_{\sigma_1 \in A(\Gamma)} \int_{U} \left( \frac{\lambda_1}{\lambda_2} \right)^2 \sigma_1^4 \, dv_1 \\
\geq (K + \epsilon)^2 \text{Mod}_{M_1}(\Gamma).
\]

(3.19)

Therefore, (3.9) implies that

\[
(K + \epsilon)^2 \text{Mod}_{M_1}(\Gamma) \leq K^2 \text{Mod}_{M_1}(\Gamma).
\]

(3.20)

But we can always choose \( U \) such that \( \text{Mod}_{M_1}(\Gamma) \neq 0 \). So (3.9) cannot be true for such \( \Gamma \). Hence \( \lambda_1 / \lambda_2 \leq K \) on \( M_1 \), i.e., \( f \) is \( K \)-quasiconformal. \( \square \)

We like to know if we can use the conclusion of Theorem 3.3 to define the quasiconformality for a homeomorphism \( f : M_1 \rightarrow M_2 \). The rest of this section gives some partial solutions to this problem.
Theorem 3.4. If $f : M_1 \to M_2$ is a homeomorphism so that for a constant $K \geq 1$ and any curve family $\Gamma$ which forms a smooth contact fibration of an open set in $M_1$

\begin{equation}
\text{Mod}_{M_1}(\Gamma) \leq K^2 \text{Mod}_{M_2}(f(\Gamma)),
\end{equation}

then $f$ is ACL.

Proof. Let $U \subset M_1$ be an open set with a smooth contact fibration $\Gamma$, $X \neq 0$ be a horizontal vector field on $U$ tangent to $\Gamma$. By replacing $X$ by $X/|X|$, we can assume $|X| = 1$. Shrink $U$, if necessary, so that there is a smooth surface $S \subset U$ which intersects each fiber of $\Gamma$ transversally once and only once. Parametrize fibers $\gamma \in \Gamma$ by $t$ so that $X = \frac{\partial}{\partial t}$ and $\gamma(0) \in S$. $p : U \to S$ is the natural projection given by $\gamma(t) \mapsto \gamma(0)$.

Recall that Hermitian metrics on $HM_1$ and $HM_2$ can be extended canonically to Riemannian metrics on $M_1$ and $M_2$ respectively. Restricting the Riemannian metric on $M_1$ to $S$ makes $S$ a Riemannian 2-manifold. Let $\omega$ be the area form on $S$, and $A_\omega(E) \triangleq \int_E \omega$ the $\omega$-area of a measurable set $E \subset S$. Define a set function $F$ for measurable set $E \subset S$ by letting $F(E) = \text{vol}(f(p^{-1}(E)))$, where vol refers to the Riemannian volume on $M_2$. Then Lebesgue’s theorem (Theorem 23.5, [14]) asserts that $F$ has finite derivatives at all points in $S_1$ with respect to $\omega$-area, for a subset $S_1 \subset S$ with $A_\omega(S \setminus S_1) = 0$. We want to prove $f$ is absolutely continuous along the fibers of $\Gamma$ passing through points in $S_1$.

For $q \in S_1$, let $D_r \subset S \cap U$ be a disc centered at $q$ with radius $r$, $q_j : I \to U$ be the fiber of $\gamma$ passing through $q$. Take any sequence $t_1', t_1'', t_2', t_2'', ..., t_k', t_k'' \in I$ such that

\begin{equation}
t_1' < t_1'' < t_2' < t_2'' < ... < t_k' < t_k''.
\end{equation}

Let $\Delta t_j \triangleq t_j'' - t_j'$. When $\max_{1 \leq j \leq k} \Delta t_j$ is small enough,

\begin{equation}
d_{r,j} \leq \Delta t_j \leq 2 d_{r,j}
\end{equation}

where $d_{r,j}$ is the sub-Riemannian distance between the set $B_{r,j}' \triangleq \{\gamma(t_j') \mid \gamma \in \Gamma, \gamma(0) \in D_r\}$ and the set $B_{r,j}'' \triangleq \{\gamma(t_j'') \mid \gamma \in \Gamma, \gamma(0) \in D_r\}$. This is true because fibers of $\Gamma$ are Legendrian and the geodesic with respect to the sub-Riemannian metric tangent to $\gamma$ at $\gamma(t_j')$ is locally a length minimizing curve (Theorem 5.4, [11]). For $j = 1, 2, ..., k$, denote

$$R_{r,j} \triangleq \{\text{points } \gamma(t) \mid \gamma \in \Gamma, \gamma(0) \in D_r, t_j' \leq t \leq t_j''\}$$

and let

$$\Gamma_{r,j} \triangleq \{\text{curves } \gamma_j : t \mapsto \gamma(t) \mid \gamma \in \Gamma, \gamma(0) \in D_r, t_j' \leq t \leq t_j''\},$$

a contact fibration of $R_{r,j}$. Then by (3.22),

\begin{equation}
\text{Mod}_{M_1}(\Gamma_{r,j}) \leq K^2 \text{Mod}_{M_2}(f(\Gamma_{r,j})).
\end{equation}
Next we use length-volume argument to give an estimate for \( \text{Mod}_{M_1}(\Gamma_{r,j}) \). For \( \sigma \in A(\Gamma_{r,j}) \) and \( \gamma_j \in \Gamma_{r,j} \),

\[
1 \leq (\int_{\gamma_j} \sigma)^4 \leq (\int_{\gamma_j} \sigma^4)(\int_{\gamma_j} 1)^3 = (\Delta t_j)^3 \int_{\gamma_j} \sigma^4.
\]

(3.26)

Integrating each term against \( \omega \) over \( D_r \), then

\[
\int A_\omega(D_r) \leq c(\Delta t_j)^3 \int_{R_{r,j}} \sigma^4 dv_1.
\]

(3.27)

where \( c \) is constant, \( dv_1 \) is the volume form of the Riemannian metric on \( M_1 \). By (3.3) and (3.24),

\[
\int A_\omega(D_r) \leq 8c d_{r,j}^3 \text{Mod}_{M_1}(\Gamma_{r,j}).
\]

(3.28)

On the other hand, let \( \delta_{r,j} \) be the sub-Riemannian distance between \( f(B_{r,j}') \) and \( f(B_{r,j}'') \). Then \( 1/\delta_{r,j} \in A(f(\Gamma_{r,j})) \). Thus

\[
\text{Mod}_{M_2}(f(\Gamma_{r,j})) \leq \frac{1}{\delta_{r,j}^4} \text{vol}(f(R_{r,j})).
\]

(3.29)

Combining (3.25), (3.28) and (3.29), we have

\[
(A_\omega(D_r))^{\frac{1}{4}} \delta_{r,j} \leq (8cK^2)^{\frac{1}{4}} d_{r,j}^{\frac{3}{4}} \text{vol}(f(R_{r,j}))^{\frac{1}{4}}.
\]

(3.30)

Summing (3.30) over \( j \),

\[
(A_\omega(D_r))^{\frac{1}{4}} \sum_{j=1}^{k} \delta_{r,j} \leq (8cK^2)^{\frac{1}{4}} \sum_{j=1}^{k} d_{r,j}^{\frac{3}{4}} \text{vol}(f(R_{r,j}))^{\frac{1}{4}}.
\]

(3.31)

So by (3.24),

\[
\sum_{j=1}^{k} \delta_{r,j} \leq (8cK^2)^{\frac{1}{4}} \left( \sum_{j=1}^{k} \Delta t_j \right)^{\frac{3}{4}} \left( \frac{F(D_r)}{A_\omega(D_r)} \right)^{\frac{1}{4}}.
\]

(3.32)
Letting $r \to 0$,

\begin{equation}
\sum_{j=1}^{k} \delta_j \leq \left(8cK^2 F'(q)\right)^{\frac{1}{2}} \left(\sum_{j=1}^{k} \Delta t_j\right)^{\frac{3}{4}},
\end{equation}

where $\delta_j \equiv \lim_{r \to 0} \delta_{r,j}$, i.e., the sub-Riemannian distance between $f(\gamma_q(t))$ and $f(\gamma_q(t+1))$. Hence $f$ is absolutely continuous along $\gamma_q$, for $q \in S_1$. So $f$ is ACL. □

**Corollary 3.5.** If a homeomorphism $f : M_1 \to M_2$ satisfies (3.22), then $f$ has horizontal derivatives almost everywhere on $M_1$ in the sense that for any open set $U \subset M_1$ with a smooth section $X$ of $HM_1$ on $U$ and any smooth function on $f(U)$, $X(h \circ f)$ exists almost everywhere on $U$.

### 4. Extremal Quasiconformal Homeomorphisms

In this section we will construct CR 3-manifolds $M_1$, $M_2$ and a quasiconformal diffeomorphism $f_0 : M_1 \to M_2$ such that $K(f_0) \leq K(f)$, for any $C^2$ quasiconformal homeomorphism $f : M_1 \to M_2$ which is homotopic to $f_0$. $M_1$ to be constructed is the quotient of the 3-dimensional Heisenberg group over a lattice. So we start with the Heisenberg group.

The 3-dimensional Heisenberg group $H^3$ is the space $\mathbb{R}^3$ endowed with the group structure defined by

\begin{equation}
(x, y, t) (u, v, s) = (x + u, y + v, t + s + 2yu - 2xv).
\end{equation}

The standard contact structure on $H^3$ is given by the contact form

\begin{equation}
\tilde{\eta} = -\frac{1}{2} y dx + \frac{1}{2} x dy + \frac{1}{4} dt.
\end{equation}

The contact bundle $H^3$ is $\text{Ker} \tilde{\eta}$ has two global sections

\begin{equation}
\tilde{X} = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad \tilde{Y} = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}
\end{equation}

which span $H^3$ everywhere and are invariant under the left group translation. The standard CR structure on $H^3$ is

\begin{equation}
\tilde{J} : H^3 \to H^3, \quad \tilde{X} \mapsto \tilde{Y}, \tilde{Y} \mapsto -\tilde{X}.
\end{equation}

Note that $d\tilde{\eta} = dx \wedge dy$, thus the sub-Riemannian metric on $H^3$ determined by the area form $d\tilde{\eta}$ and the complex structure $\tilde{J}$, i.e., Levi form, is the one making $\{\tilde{X}, \tilde{Y}\}$ orthonormal. The canonical Riemannian extension of this metric is the Euclidean metric on $\mathbb{R}^3$.

Next we study the geodesics on $H^3$ with respect to this sub-Riemannian metric. Since the metric is invariant under the left group translation, thus it suffices to study geodesics joining the origin $O = (0, 0, 0)$ and a generic point $q$. Denote by $p$ the projection from $H^3$ to the horizontal plane $P \triangleq \{t = 0\}$. It is easy to see that if $\gamma$ is a rectifiable curve in $H^3$ with respect to the sub-Riemannian metric, $p(\gamma)$ is rectifiable with respect to the Euclidean metric on $\{t = 0\}$, and the respective lengths of $\gamma$ and $p(\gamma)$ coincide. The characterization of the geodesics given in the next theorem was first given by Korányi by studying Euler-Lagrange equations [3]. The following proof, which is due to Lempert [8], is a geometric one.
**Proposition 4.1.** On $\mathbb{H}^3$, a minimal geodesic to connect the origin $O$ and $q \in \mathbb{H}^3$ is on either a straight line if $q \in P$, or a helix whose projection to $P$ is a circle if $q \notin P$.

**Proof.** The conclusion is obvious if $q \in P$. Next we assume $q \notin P$. We first consider the case $p(q) = O$. i.e., $q = (0, 0, t)$ for some $t \neq 0$. Let

$$\gamma : [0, l] \to \mathbb{H}^3, \quad s \mapsto (x(s), y(s), t(s))$$

be any oriented rectifiable curve joining $O$ and $q$. Then

$$-2yx' + 2xy' + t' = 0,$$

almost everywhere on $\gamma$ by (4.2). Thus

$$t = \int_0^l t' \, ds = \int_{p(\gamma)} 2ydx - 2xdy = -4 \int \Omega \wedge dy,$$

where $\Omega$ is the 2-chain on $P$ so that $\partial \Omega = p(\gamma)$. Note length of $\gamma$ with respect to the sub-Riemannian metric is equal to the length of $p(\gamma)$ on $P$ with respect to Euclidean metric. Obviously, the length of $p(\gamma)$ is the minimal only when $\Omega$ is a simply connected domain with fixed area $|t|/4$. Furthermore, $p(\gamma)$ must be a circle if $\gamma$ is a length minimizing geodesic joining $O$ and $q$, by the isoperimetric property on the Euclidean plane.

If $p(q) \neq O$, the above reasoning with some slight modifications can be applied. For instance, $\Omega$ is a 2-chain bounded by the union of $p(\gamma)$ and the straight line segment from $p(q)$ to $O$. \[\square\]

Denote $A = (1, 0, 0), B = (\sigma, \tau, 0), C = (0, 0, 1) \in \mathbb{H}^3$ with $\tau \neq 0$. Define $\Gamma = \{n_1A + n_2B + n_3C | n_1, n_2, n_3 \in \mathbb{Z}\}$. The Euclidean translations by elements of $\Gamma$ define a $\Gamma$-action. The contact structure and CR structure on $\mathbb{H}^3$ are invariant under this $\Gamma$-action. We consider the quotient space $M_1 \triangleq \mathbb{H}^3/\Gamma$. $M_1$ has a contact structure with the contact form $\eta$ satisfying $\pi^*\eta = \tilde{\eta}$ and a CR structure $J_1$ inherited from $\tilde{J}$. Hence $M_1$ becomes a CR manifold. Let $\pi : \mathbb{H}^3 \to M_1$ denote the natural projection and $X = \pi_*\tilde{X}, Y = \pi_*\tilde{Y}$. Then with respect to the sub-Riemannian metric uniquely determined by $\eta$ and $J_1$, $\{X, Y\}$ is orthonormal. Note also $\overline{Z}_1 = \frac{1}{2}(X + iY) \in T^{0,1}M_1$. There is a canonical $\mathbb{R}$-action on $\mathbb{H}^3$ defined by

$$\pi(x, y, t) \mapsto (x, y, t + t'), \quad \text{for } t' \in \mathbb{R}.$$
and CR structures on $M_1$. We call such a bundle $M_1 \xrightarrow{\pi} S_1$ a CR circle bundle over a flat torus.

For any constant $K \geq 1$, define a new CR structure $M_2$ on the contact structure of $M_1$ by declaring that

$$(4.9) \quad Z_2 = \frac{1}{2}(\sqrt{K}X + \frac{i}{\sqrt{K}}Y) = \frac{K+1}{4\sqrt{K}}(Z_1 + \frac{K-1}{K+1}Z_1)$$

is an (0,1) tangent vector. Obviously, the identity mapping $f_0 : M_1 \to M_2$ is quasiconformal and $K(f_0)(q) = K$ for all $q \in M_1$.

**Theorem 4.2.** Let $f : M_1 \to M_2$ be a $C^2$ homeomorphism homotopic to $f_0 : M_1 \to M_2$. Then $K(f) \geq K$.

Note the CR structure on $M_2$ is also invariant under the circle action. So if $S_2$ denotes the same smooth torus as $S_1$, but endowed with the complex structure induced from CR structure on $M_2$, then $M_2 \xrightarrow{\pi} S_2$ is also a CR circle bundle.

**Lemma 4.3 (Strichartz).** For any $q$ in a sub-Riemannian manifold $M$, there is an $\epsilon > 0$ so that if $q_1 \in M$ with $d(q_1, q) \leq \epsilon$, there exists a length minimizing curve joining $q_1$ and $q$.

This is Lemma 3.2 in [11], a proof was given there. If $\alpha$ is a curve, denote by $[\alpha]$ the homotopy class with fixed end points of $\alpha$. Let $l(\alpha)$ be the length of $\alpha$ with respect to the sub-Riemannian metric on $M$.

**Lemma 4.4.** Let $\alpha : [0, 1] \to M$ be a curve on a compact sub-Riemannian manifold to connect two points $q_1$ and $q_2$, $\inf l(\beta)$ be taken over all $\beta \in [\alpha]$. Then this infimum is attained at a rectifiable curve $\hat{\alpha} \in [\alpha]$.

**Proof.** Let $\epsilon_q$ be the maximal $\epsilon > 0$ for $q \in M$ determined by Lemma 4.3. Since $M$ is compact, $\inf_{q \in M} \epsilon_q > 0$. Take any $r$ so that $0 < 3r < \inf_{q \in M} \epsilon_q$. Cover $M$ with finitely many closed $r$-balls $B(q_j, r)$, $j = 1, 2, ..., k$, with respect to the sub-Riemannian metric. Note any two points in the same $B(q_j, r)$ can be joined by a length minimizing curve within $B(q_j, 3r)$.

Let $L = \inf l(\beta)$ over all $\beta \in [\alpha]$. Let $\beta_n \in [\alpha]$ be curves such that $\lim_{n \to \infty} l(\beta_n) = L$. Divide each $\beta$ by ordered points $q_0 = q_1, p_{n1}, p_{n2}, ..., p_{nN} = q_2 \in \beta_n$ into ordered subcurves $\tau_{n1}, \tau_{n2}, ..., \tau_{nN}$ so that each $\tau_{nj} \subset B(q_{m_{nj}}, r)$ for some integer $m_{nj}$ with $1 \leq m_{nj} \leq k$. Note this $N$ is uniform for all $n$ by appropriately choosing $\beta_n$.

Let $\sigma_{nj}$ be a length minimizing curve in $B(q_{m_{nj}}, 3r)$ to join the end points of $\sigma_{nj}$ and $\sigma_n = \sigma_{n1}\sigma_{n2}...\sigma_{nN}$. Then since $B(q_{m_{nj}}, 3r)$ is simply connected, $\sigma_{nj} \in [\tau_{nj}]$, and hence $\sigma_n \in [\beta_n] = [\alpha]$. Furthermore

$$(4.10) \quad L \leq l(\sigma_n) \leq \sum_{j=1}^{N} l(\sigma_{nj}) \leq \sum_{j=1}^{N} l(\tau_{nj}) = l(\beta_n).$$

Therefore $\lim_{n \to \infty} l(\sigma_n) = L$. 

Since $M$ is compact, there is a sequence $\{n_j\} \subset \mathbb{N}$ so that for each $j = 1, 2, \ldots, N-1$, $p_{n_j}$ is convergent to some point $p_j$, when $n_j \to \infty$. Note $p_{j-1}, p_j \in B(q_{m_j}, r)$ for some integer $m_j$ with $1 \leq m_j \leq k$. Connect $p_{j-1}, p_j$ by a length minimizing curve $\alpha_j \subset B(q_{m_j}, 3r)$ and let $\tilde{\alpha} = \alpha_1 \alpha_2 \ldots \alpha_N$. Then $\tilde{\alpha} \in [\alpha]$ and

$$l(\tilde{\alpha}) = \lim_{n_i \to \infty} l(\sigma_{n_i}) = L. \quad \Box$$

The next two lemmas are the key properties to establish the extremality of $f_0$ described by Theorem 4.2.

**Lemma 4.5.** The flow of diffeomorphisms $h_t$ generated by $X$ on $M_1$ preserve the volume form $dv_1 = d\eta \wedge \eta$.

*Proof.* This is obvious since the the flow of diffeomorphisms $\tilde{h}_s$ on $H^3$ generated by $\tilde{X}$ are given by

$$\tilde{h}_s : (x, y, t) \mapsto (x + s, y, t + 2sy), \; s \in \mathbb{R}$$

which preserve the volume form $d\tilde{\eta} \wedge \tilde{\eta} = dx \wedge dy \wedge dt. \quad \Box$

We define a sub-Riemannian metric on $M_2$ so that $\{\sqrt{K}X, \frac{Y}{\sqrt{K}}\}$ is orthonormal. Note this sub-Riemannian metric induces the same CR structure on $M_2$. Denote the corresponding curve length by $l_2$.

**Lemma 4.6.** Let $\gamma : \mathbb{R} \to M_2$ be an integral line of the vector field $\sqrt{K}X$. Then for $[a, b] \subset \mathbb{R}$ and with respect to the sub-Riemannian metric on $M_2$, the curve $\gamma|_{[a,b]}$ is the unique length minimizing curve in its homotopy class of curves with end points $\gamma(a)$ and $\gamma(b)$.

*Proof.* Lifting the sub-Riemannian metric on $M_2$ to its universal covering $\mathbb{H}^3$, we obtain a new sub-Riemannian manifold $\mathbb{H}^3_2$. Let us call this new sub-Riemannian manifold $\mathbb{H}^3_2$. Then on $\mathbb{H}^3_2$, there is a characterization of geodesics similar to Proposition 4.1. The trajectories of $\sqrt{K}X$ are exactly those straight geodesics. Indeed, the sub-Riemannian metric is invariant under the $\mathbb{R}$-action (4.8), hence a rectifiable curve $\gamma$ in $\mathbb{H}^3_2$ has the same length with $p(\gamma)$ on $P_2$. Here $P_2$ is the plane $P$ endowed with the flat Riemannian metric so that $\sqrt{K} \frac{\partial}{\partial x}, \frac{1}{\sqrt{K}} \frac{\partial}{\partial y}$ are orthonormal. Therefore the $x$-axis, which is the trajectory of $\sqrt{K}X$ passing through $O$, is a geodesic without conjugate points on it. Note $\sqrt{K}X$ and the sub-Riemannian metric are invariant under the group left translation. Hence this lemma is true. $\Box$

*Proof of Theorem 4.2.* Let $f : M_1 \to M_2$ be an $C^2$ quasiconformal homeomorphism. For $q \in M_1$, let $\gamma_{q,a} : [-a, a] \to M_1$ be the integral line of $X$ so that $\gamma_{q,a}(0) = q$. Let $\Gamma_a = \{\gamma_{q,a} | q \in M_1\}$. Then by Theorem 3.4,

$$\text{Mod}_{M_1}(\Gamma_a) \leq K(f)^2 \text{Mod}_{M_2}(f(\Gamma_a)).$$
First note the length of $\gamma_{q,a}$ is $2a$, so $1/2a \in A(\Gamma_a)$ and

\[
\text{(4.14)} \quad \text{Mod}_{M_1}(\Gamma_a) \leq \int_{M_1} \left(\frac{1}{2a}\right)^4 d\nu_1 = \left(\frac{1}{2a}\right)^4 \text{vol}(M_1).
\]

For $\sigma \in A(\Gamma_a)$,

\[
\text{(4.15)} \quad 1 \leq \left(\int_{\gamma_{q,a}} \sigma\right)^4 \leq \left(\int_{\gamma_{q,a}} 1\right)^3 \left(\int_{\gamma_{q,a}} \sigma^4\right) = (2a)^3 \int_{\gamma_{q,a}} \sigma^4.
\]

That implies

\[
\text{(4.16)} \quad \frac{1}{(2a)^3} \leq \int_{\gamma_{q,a}} \sigma^4 = \int_{-a}^a \left(\sigma(\gamma_{q,a}(s))\right)^4 ds.
\]

Taking the integrals of both sides of (4.16) against $d\nu_1$ with respect to $q$ over $M_1$,

\[
\frac{1}{(2a)^3} \text{vol}(M_1) \leq \int_{M_1} \left(\int_{-a}^a \left(\sigma(\gamma_{q,a}(s))\right)^4 ds\right) d\nu_1
\]

\[
= \int_{-a}^a \left(\int_{M_1} \left(\sigma(\gamma_{q,a}(s))\right)^4 d\nu_1\right) ds
\]

\[
= 2a \int_{M_1} \sigma^4 d\nu_1.
\]

The last equality in (4.17) is due to Lemma 4.5, which implies that

\[
\text{(4.18)} \quad \int_{M_1} \left(\sigma(\gamma_{q,a}(s))\right)^4 d\nu_1 = \int_{M_1} \sigma^4 h_s^*(d\nu_1) = \int_{M_1} \sigma^4 d\nu_1
\]

is independent of $s$. Therefore

\[
\text{(4.19)} \quad \left(\frac{1}{2a}\right)^4 \text{vol}(M_1) \leq \int_{M_1} \sigma^4 d\nu_1, \quad \forall \sigma \in A(\Gamma_a).
\]

Combining (4.14) and (4.19), we get

\[
\text{(4.20)} \quad \text{Mod}_{M_1}(\Gamma_a) = \left(\frac{1}{2a}\right)^4 \text{vol}(M_1).
\]

Next we estimate $\text{Mod}_{M_2}(f(\Gamma_a))$ in (4.13) for a quasiconformal homeomorphism $f$ homotopic to $f_0 : M_1 \to M_2$. Since $f$ is homotopic to $f_0$, there is a continuous map $H : [0, 1] \times M_1 \to M_2$ so that

\[
\text{(4.21)} \quad H(0, q) = f_0(q), \quad H(1, q) = f(q), \quad \forall q \in M_1.
\]
Let \( \alpha_q : [0, 1] \to M_2 \) be a curve given by \( t \mapsto H(t, q) \). Define \( G(q) = \inf l_2(\beta) \) over all \( \beta \in [\alpha_q] \). Then by Lemma 4.4, \( G(q) \) is attained at a rectifiable curve \( \tilde{\alpha}_q \in [\alpha_q] \).

Now we prove that \( G(q) \) is continuous on \( M_1 \). When \( q_1 \) is close enough to \( q \) on \( M_1 \), \( f_0(q_1) \) is close enough to \( f_0(q) \) so that there exist a length minimizing curve \( \delta_1 \) joining \( f_0(q), f_0(q_1) \) according to Lemma 4.3. Let \( \delta_2 = f(\delta_1) \). Then

\[
\delta_1 \tilde{\alpha}_q, \delta_2^{-1} \in [\alpha_q], \quad \delta_1^{-1} \tilde{\alpha}_q \delta_2 \in [\alpha_{q_1}].
\]

This implies

\[
l_2(\tilde{\alpha}_q) \leq l_2(\delta_1) + l_2(\tilde{\alpha}_{q_1}) + l_2(\delta_2^{-1}),
\]

\[
l_2(\tilde{\alpha}_{q_1}) \leq l_2(\delta_1^{-1}) + l_2(\tilde{\alpha}_q) + l_2(\delta_2).
\]

In other words,

\[
|G(q_1) - G(q)| \leq l_2(\delta_1) + l_2(\delta_2).
\]

Then there exists \( A > 0 \) so that

\[
G(q) \leq A < \infty, \quad \forall q \in M_1,
\]

since \( M_1 \) is compact.

For a curve \( \gamma : [a, b] \to M_1 \), \( \tilde{\alpha}_\gamma(a) f(\gamma) \tilde{\alpha}_\gamma^{-1} \in [f_0(\gamma)] \). Then by Lemma 4.6 and (4.25),

\[
l_2(f_0(\gamma)) \leq l_2(\tilde{\alpha}_\gamma(a)) + l_2(f(\gamma)) + l_2(\tilde{\alpha}_\gamma^{-1}) \leq 2A + l_2(f(\gamma)).
\]

Applying (4.26) to \( \gamma = \gamma_{q,a} \in \Gamma_a \), we obtain

\[
2\sqrt{K}a \leq 2A + l_2(f(\gamma_{q,a})).
\]

Hence \( 1/(2\sqrt{K}a - 2A) \in A(f(\Gamma_a)) \). So we have the following estimate.

\[
\text{Mod}_{M_2}(f(\Gamma_a)) \leq \int_{M_2} \left( \frac{1}{2\sqrt{K}a - 2A} \right)^4 dv_2 = \left( \frac{1}{2\sqrt{K}a - 2A} \right)^4 \text{vol}(M_2).
\]

Note \( \text{vol}(M_1) = \text{vol}(M_2) \neq 0 \), so by (4.13), (4.20) and (4.28), we have

\[
(\sqrt{K} - \frac{A}{a})^2 \leq K(f).
\]

Letting \( a \to \infty \), we get that \( K \leq K(f) \). \( \square \)

**Remark.** In horizontal directions, the extremal quasiconformal homeomorphism \( f_0 \) behaves as a stretching by the constant factor \( \sqrt{K} \) along trajectories of \( X \) and a compressing by the same factor along trajectories of \( JX \). The generator \( T \) of the circle action is transversal to the contact bundle. In this transversal direction \( T \), \( f_0 \) is equivariant under the circle action since it is simply the identity mapping while \( M_1 \) and \( M_2 \) have the same circle action.
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