Dense Eulerian Graphs are (1, 3)-Choosable

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Abstract
A graph \( G \) is total weight \((k, k')\)-choosable if for any total list assignment \( L \) which assigns to each vertex \( v \) a set \( L(v) \) of \( k \) real numbers, and each edge \( e \) a set \( L(e) \) of \( k' \) real numbers, there is a proper total \( L \)-weighting, i.e., a mapping \( f: V(G) \cup E(G) \to \mathbb{R} \) such that for each \( z \in V(G) \cup E(G) \), \( f(z) \in L(z) \), and for each edge \( uv \) of \( G \), \( \sum_{e \in E(u)} f(e) + f(u) \neq \sum_{e \in E(v)} f(e) + f(v) \). This paper proves that if \( G \) decomposes into complete graphs of odd order, then \( G \) is total weight \((1, 3)\)-choosable. As a consequence, every Eulerian graph \( G \) of large order and with minimum degree at least \( 0.91|V(G)| \) is total weight \((1, 3)\)-choosable. We also prove that any graph \( G \) with minimum degree at least \( 0.999|V(G)| \) and sufficiently large order is total weight \((1, 4)\)-choosable.

Mathematics Subject Classifications: 05C15, 05C72

1 Introduction
Assume \( G = (V, E) \) is a graph with vertex set \( V = \{1, 2, \ldots, n\} \). Each edge \( e \in E \) of \( G \) is a 2-subset \( e = \{i, j\} \) of \( V \). For \( i \in V \), we denote by \( E(i) \) the set of edges incident to \( i \). A total weighting of \( G \) is a mapping \( \phi: V \cup E \to \mathbb{R} \). A total weighting \( \phi \) is proper if for any edge \( \{i, j\} \in E \),

\[
\sum_{e \in E(i)} \phi(e) + \phi(i) \neq \sum_{e \in E(j)} \phi(e) + \phi(j).
\]

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A proper total weighting $\phi$ with $\phi(i) = 0$ for all vertices $i$ is also called a vertex coloring edge weighting. A vertex coloring edge weighting of $G$ using weights $\{1, 2, \ldots, k\}$ is called a vertex coloring $k$-edge weighting. Note that if $G$ has an isolated edge, then $G$ does not admit a vertex coloring edge weighting. We say a graph is nice if it does not contain any isolated edge.

Karoński, Łuczak and Thomason [11] conjectured that every nice graph has a vertex coloring 3-edge weighting. This conjecture received considerable attention [1, 2, 9, 10, 14, 15, 19], and it is known as the 1-2-3 conjecture. The best result on 1-2-3 conjecture so far was obtained by Kalkowski, Karoński and Pfender [10], who proved that every nice graph has a vertex coloring 5-edge weighting.

The list version of edge weighting of graphs was introduced by Bartnicki, Grytczuk and Niwczyk [5]. The list version of total weighting of graphs was introduced independently by Przybyło and Woźniak in [12] and by Wong and Zhu in [17]. Let $\psi : V \cup E \to \mathbb{N}^+$. A $\psi$-list assignment of $G$ is a mapping $L$ which assigns to $z \in V \cup E$ a set $L(z)$ of $\psi(z)$ real numbers. Given a total list assignment $L$, a proper $L$-total weighting is a proper total weighting $\phi$ such that $\phi(z) \in L(z)$ for all $z \in V \cup E$. We say $G$ is total weight $\psi$-choosable ($\psi$-choosable for short) if for any $\psi$-list assignment $L$, there is a proper $L$-total weighting of $G$. We say $G$ is total weight $(k, k')$-choosable ($(k, k')$-choosable for short) if $G$ is $\psi$-total weight choosable, where $\psi(i) = k$ for $i \in V(G)$ and $\psi(e) = k'$ for $e \in E(G)$.

The list version of edge weighting also received a lot of attention [5, 6, 7, 8, 13, 14, 16, 17, 18, 20]. As strengthenings of the 1-2-3 conjecture, it was conjectured in [17] that every nice graph is $(1, 3)$-choosable. A weaker conjecture was also proposed in [17], which asserts that there is a constant $k$ such that every nice graph is $(1, k)$-choosable. This weaker conjecture was recently confirmed by Cao [6], who proved that every nice graph is $(1, 17)$-choosable. This result was improved in [20], where it was shown that every nice graph is $(1, 5)$-choosable.

Given a graph $G$ and a family of graphs $H$, we say that $G$ has an $H$-decomposition, if the edges of $G$ can be partitioned into the edge sets of copies of graphs from $H$. In particular, a triangle decomposition of $G$ is a partition of $E(G)$ into triangles, and for a given graph $H$, an $H$-decomposition of $G$ partitions $E(G)$ into subsets, each inducing a copy of $H$. The following is the main result of this paper.

**Theorem 1.** If $E(G)$ can be decomposed into cliques of odd order, then $G$ is $(1, 3)$-choosable.

As a consequence of Theorem 1, we prove the following result.

**Theorem 2.** If $G$ is an $n$-vertex Eulerian graph with minimum degree at least $0.91n$ and $n$ is sufficiently large, then $G$ is $(1, 3)$-choosable.

In [19], Zhong confirmed the 1-2-3 conjecture for graphs that can be edge-decomposed into cliques of order at least 3. As a consequence of this result, it was proved in [19] that the 1-2-3 conjecture holds for every $n$-vertex graph with minimum degree at least $0.99985n$, where $n$ is sufficiently large.
Our result is the list version of Zhong’s result, but with one degree restriction: \( E(G) \) needs to be decomposed into complete graphs of odd order. Hence we can only show that dense Eulerian graphs are \((1,3)\)-choosable. For general dense graphs, we prove the following result:

**Theorem 3.** If \( G \) is an \( n \)-vertex graph with minimum degree at least \( 0.999n \) and \( n \) is sufficiently large, then \( G \) is \((1,4)\)-choosable.

## 2 Some preliminaries

The proofs of Theorems 1, 2 and 3 use tools that were introduced in [6] and were further developed in [20]. In this section, we introduce some definitions and present a result from [6] that will be used in this paper.

Given a graph \( G = (V, E) \), let

\[
\hat{P}_G(\{x_z : z \in V \cup E\}) = \prod_{(i,j) \in E, i < j} \left( \left( \sum_{e \in E(i)} x_e + x_i \right) - \left( \sum_{e \in E(j)} x_e + x_j \right) \right).
\]

Assign a real number \( \phi(z) \) to each variable \( x_z \), and view \( \phi(z) \) as the weight of \( z \). Let \( \hat{P}_G(\phi) \) be the evaluation of the polynomial at \( x_z = \phi(z) \), \( z \in V \cup E \). Then \( \phi \) is a proper total weighting of \( G \) if and only if \( \hat{P}_G(\phi) \neq 0 \). Thus the problem of finding a proper \( L \)-total weighting of \( G \) (for a given total list assignment \( L \)) is equivalent to finding a non-zero point of the polynomial \( \hat{P}_G(\{x_z : z \in V \cup E\}) \) in the grid \( \prod_{z \in V \cup E} L(z) \).

Combinatorial Nullstellensatz [3] gives a sufficient condition for the polynomial \( \hat{P}_G(\{x_z : z \in V \cup E\}) \) has a non-zero point in the grid \( \prod_{z \in V \cup E} L(z) \): If some non-vanishing (i.e., with non-zero coefficient) highest degree monomial \( \prod_{z \in V \cup E} x_z^{K(z)} \) in the expansion of \( \hat{P}_G(\{x_z : z \in V \cup E\}) \) satisfies \( K(z) \leq |L(z)| - 1 \) for \( z \in V \cup E \), then \( \hat{P}_G(\{x_z : z \in V \cup E\}) \) has a non-zero point in the grid \( \prod_{z \in V \cup E} L(z) \).

We denote by \( \mathbb{N} \) the set of non-negative integers. To prove a graph \( G = (V, E) \) is \((1,k)\)-choosable, it suffices to show that for some \( K : V \cup E \to \mathbb{N} \) such that \( K(v) = 0 \) and \( K(e) \leq k - 1 \), and the monomial \( \prod_{z \in V \cup E} x_z^{K(z)} \) has non-zero coefficient in the expansion of \( \hat{P}_G(\{x_z : z \in V \cup E\}) \).

As \( K(v) = 0 \) for all \( v \in V \), the monomials in concern are of the form \( \prod_{e \in E} x_e^{K(e)} \). Such monomials have the same coefficient in the expansions of \( \hat{P}_G(\{x_z : z \in V \cup E\}) \) and

\[
P_G(\{x_e : e \in E\}) = \prod_{(i,j) \in E, i < j} \left( \sum_{e \in E(i)} x_e - \sum_{e \in E(j)} x_e \right).
\]

We denote by \( \mathbb{N}^E \) the set of mappings \( K : E \to \mathbb{N} \). Let

\[
\mathbb{N}^E_m = \{ K \in \mathbb{N}^E : \sum_{e \in E} K(e) = m \}, \quad \mathbb{N}^E_{(b^-)} = \{ K \in \mathbb{N}^E : K(e) \leq b, \forall e \in E \}.
\]
For $K \in \mathbb{N}^E$, let
\[ x^K = \prod_{e \in E} x^K_e, \quad K! = \prod_{e \in E} K(e)!. \]
Denote the coefficient of the monomial $x^K$ in the expansion of $P_G$ by $\text{co}(x^K, P_G)$.

For a positive integer $b$, to prove that $G = (V, E)$ is $(1, b + 1)$-choosable, it suffices to show that $\text{co}(x^K, P_G) \neq 0$ for some $K \in \mathbb{N}^E_{(b-)}$. For this purpose, we use a formula given in [6] for the calculation of $\text{co}(x^K, P_G)$.

For $m, n \in \mathbb{N}$, let $C[x_1, x_2, \ldots, x_n]_m$ be the vector space of homogeneous polynomials of degree $m$ in variables $x_1, \ldots, x_n$ over the field $\mathbb{C}$ of complex numbers.

Assume $|E| = m$. Consider the vector space of homogeneous polynomials of degree $m$ in $\mathbb{C}[x_e : e \in E]$. For $f, g \in \mathbb{C}[x_e : e \in E]$, we define the inner product of $f$ and $g$ as
\[ \langle f, g \rangle = \sum_{K \in \mathbb{N}^E_m} K! \text{co}(x^K, f) \overline{\text{co}(x^K, g)}. \]

The following lemma was proved in [6].

**Lemma 4.** Assume $G = (V, E)$, $|E| = m$ and $K \in \mathbb{N}^E_m$. Let
\[ Q_E = \prod_{\{i, j\} \in E, i < j} (x_i - x_j), \quad H^K_E = \prod_{\{i, j\} \in E, i < j} (x_i + x_j)^{K(e)}. \]
Then
\[ \text{co}(x^K, P_G) = \frac{1}{K!} \langle Q_E, H^K_E \rangle. \]

**Definition 5.** For $K \in \mathbb{N}^E$, let $W^K_{E, m}$ be the complex linear space spanned by
\[ \{ H^K'_{E, m} : K' \leq K, K' \in \mathbb{N}^E_m \}. \]

It is obvious that there exists $K' \in \mathbb{N}^E_m$ such that $K' \leq K$ and $\langle Q_E, H^K'_E \rangle \neq 0$ if and only if there exists $F \in W^K_{E, m}$ such that $\langle Q_E, F \rangle \neq 0$. Thus we have the following corollary.

**Corollary 6.** If $K \in \mathbb{N}^E_{(b-)}$ and there exists $F \in W^K_{E, m}$ such that $\langle Q_E, F \rangle \neq 0$, then $G$ is $(1, b + 1)$-choosable.

## 3 Proofs of Theorems 1, 2, 3

The following lemma is an easy observation, but it is the key tool for proving the main results of this paper.

**Lemma 7.** If $Q_E \in W^K_{E, m}$ for some $K \in \mathbb{N}^E_{(b-)}$, then $G$ is $(1, b + 1)$-choosable.

**Proof.** Assume $Q_E \in W^K_{E, m}$. As $Q_E \neq 0$, we have $\langle Q_E, Q_E \rangle > 0$. By Corollary 6, $G$ is $(1, b + 1)$-choosable. \qed
As an example, consider a triangle $T$ with vertex set $\{i, j, k\}$. By definition, $Q_E = (x_i - x_j)(x_j - x_k)(x_i - x_k)$. To prove that $Q_E \in W_{E,3}^K$, it suffices to express each of the three factors of $Q_E$, $(x_i - x_j)$, $(x_j - x_k)$ and $(x_i - x_k)$, as a linear combination of $(x_i + x_j), (x_j + x_k), (x_i + x_k)$, and for each edge $e$, say for $e = \{i, j\}$, the term $(x_i + x_j)$ occurs in at most $K(e)$ of such linear combinations. We can write $Q_E$ as

$$Q_E = ((x_i + x_k) - (x_j + x_k))((x_i + x_j) - (x_i + x_k))((x_i + x_j) - (x_j + x_k)).$$

It is easy to check that for each edge, say for $e = \{i, j\}$, the term $(x_i + x_j)$ occurs in two of the linear combinations. Thus $Q_E \in W_{E,3}^K$, where $K(e) = 2$ for each edge $e$ of $T$.

A path of length $k$ in $G$ connecting $i$ and $j$ is a sequence of distinct vertices $P = (i_0, i_1, \ldots, i_k)$ such that $i_0 = i$, $i_k = j$ and $\{i_l, i_{l+1}\} \in E$ for $l = 0, 1, \ldots, k - 1$.

**Definition 8.** Assume $G = (V, E)$ is a graph. A **path covering family** of $G$ is a family of paths $\mathcal{P} = \{P_e : e \in E\}$, where for each edge $e = \{i, j\} \in E$, $P_e$ is an even length path connecting $i$ and $j$.

For a subgraph $H$ of $G$, $K_H \in \mathbb{N}^E$ is the characteristic function of $E(H)$, i.e., $K_H(e) = 1$ if $e \in E(H)$ and $K_H(e) = 0$ otherwise. For a family $\mathcal{F}$ of subgraphs of $G$,

$$K_\mathcal{F} = \sum_{H \in \mathcal{F}} K_H.$$

Observe that if $F_i \in W_{E,m_i}^{K_i}$ for $i = 1, 2, \ldots, t$, then $\prod_{i=1}^t F_i \in W_{E,\sum_{i=1}^t m_i}^{\sum_{i=1}^t K_i}$.

**Lemma 9.** If $G$ has a path covering family $\mathcal{P}$ with $K_\mathcal{P}(e) \leq b$ for each edge $e$, then $G$ is $(1, b + 1)$-choosable.

**Proof.** Assume $\mathcal{P}$ is a path covering family with $K_\mathcal{P}(e) \leq b$ for each edge $e$. For each edge $e = \{i, j\}$ of $G$, let $P_e = (i_0, i_1, \ldots, i_{2k_e})$ be the even length path in $\mathcal{P}$ connecting $i$ and $j$, i.e., $i_0 = i$ and $i_{2k_e} = j$. Then

$$x_i - x_j = \sum_{l=0}^{2k_e-1} (-1)^l (x_{i_l} + x_{i_{l+1}}) \in W_{E,1}^{K_P}.$$

Hence

$$Q_E = \prod_{\{i,j\} \in E} (x_i - x_j) \in W_{E,m}^{K_\mathcal{P}}.$$

Since $K_\mathcal{P}(e) \leq b$ for each edge $e$, we have $Q_E \in W_{E,m}^K$ for some $K \in \mathbb{N}^E_{(b^-)}$. By Lemma 7, $G$ is $(1, b + 1)$-choosable. $\square$

The following lemma follows easily from the definitions and its proof is omitted.
Lemma 10. If $G$ decomposes into graphs $H_1, H_2, \ldots, H_n$, and each $H_i$ has a path covering family $\mathcal{P}_i$ with $F_{E(H_i)} \in W_{\mathbb{N}(b-)}^E$ and $K_i \in \mathbb{N}(b-)$, then $\mathcal{P} = \bigcup_{i=1}^n \mathcal{P}_i$ is a path covering family of $G$ and $K_\mathcal{P} \in W_{\mathbb{N}(b-)}^E$ for $K = \sum_{i=1}^n K_i \in \mathbb{N}(b-)$.

Proof of Theorem 1. By Lemmas 9 and 10, it suffices to show that each complete graph $K_n$ of odd order has a path covering family $\mathcal{P}$ with $K_\mathcal{P} \in \mathbb{N}(b-)$. Assume $K_n$ has vertex set $\{1, 2, \ldots, n\}$.

Put the $n$ vertices $\{1, 2, \ldots, n\}$ of $K_n$ equally spaced along the perimeter of a circle $C$. For an edge $e = \{i, j\}$ of $K_n$, denote by $[i, j]$ the interval of $C$ from $i$ to $j$ along the clockwise direction (containing both $i$ and $j$). Since $n$ is odd, exactly one of $[i, j]$ and $[j, i]$ contains an odd number of vertices of $K_n$. Let $t_{i,j}$ be the vertex that is in the center of the interval $[i, j]$ or $[j, i]$ that contains an odd number of vertices, and let $P_e = (i, t_{i,j}, j)$. Then $\mathcal{P} = \{P_e : e \in E(K_n)\}$ is a path covering family of $K_n$. For each edge $e = \{i, j\}$ of $K_n$, let $a_e = \{i, 2j - i\}$ and $b_e = \{j, 2i - j\}$ (where calculations are modulo $n$). It is easy to verify that $e$ is contained in $P_e$ if and only if $e' \in \{a_e, b_e\}$. So each edge of $K_n$ is contained in two paths in $\mathcal{P}$, i.e., $K_\mathcal{P}(e) = 2$ for each edge $e$ of $K_n$. This completes the proof of Theorem 1.

For a graph $G$, let $\gcd(G)$ be the largest integer dividing the degree of every vertex of $G$. We say that $G$ is $F$-divisible if $|E(G)|$ is divisible by $|E(F)|$ and $\gcd(G)$ is divisible by $\gcd(F)$.

The following result was proved in [4]:

Theorem 11. For every $\epsilon > 0$, there is an integer $n_0$ such that if $G$ is a triangle-divisible graph of order $n \geq n_0$ and minimum degree at least $(0.9 + \epsilon)n$, then $G$ has a triangle decomposition.

Proof of Theorem 2. Assume $G$ is an $n$-vertex Eulerian graph of minimum degree $\delta(G) > (0.9 + \epsilon)n$ with large enough $n$. By Theorem 1, it suffices to show that $G$ decomposes into complete graphs of odd order.

Assume $|E(G)| \equiv i \pmod{3}$, where $i \in \{0, 1, 2\}$. Let $H_1, \ldots, H_i$ be vertex disjoint 5-cliques in $G$. Then $G' = G - \bigcup_{j=1}^i E(H_j)$ is triangle divisible and $\delta(G') \geq \delta(G) - 4 \geq (0.9 + \epsilon)n$. By Theorem 11, $G'$ is triangle decomposible. Hence $G$ decomposes into complete graphs of odd order. This completes the proof of Theorem 2.

Lemma 12. Let $H = (V, E)$ be the graph shown in Figure 1. Then $H$ has a path covering family $\mathcal{P}$ with $K_\mathcal{P} \in \mathbb{N}(b-)$.
Proof. We denote by $T_1 = (1, 2, 4)$, $T_2 = (2, 3, 5)$ the two edge disjoint triangles in $H$. For each triangle $T_i$, let $P_i$ be the path covering family with $K_{P_i} \in \mathbb{N}_{E(T_i)}^{E}$. For the edge $e = \{1, 3\}$ which is not contained in the two triangles, let $P_e = (1, 2, 3)$. Then

$$P = \bigcup_{i=1}^{2} P_i \cup \{P_e\}$$

is a path covering family of $H$ with $K_P \in \mathbb{N}_{(3-)}^{E}$. This completes the proof of Lemma 12.

The following theorem was proved in [4]:

**Theorem 13.** For every $\epsilon > 0$, there is an integer $n_0$ such that if $G$ is an $H$-divisible graph of order $n \geq n_0$ and minimum degree at least $(1 - 1/t + \epsilon)n$, where $t = \max\{16\chi(H)^2(\chi(H) - 1)^2, |E(H)|\}$, then $G$ has an $H$-decomposition.

**Proof of Theorem 3.** Assume $G$ is a graph of sufficiently large order and with minimum degree $\delta(G) \geq 0.999|V(G)|$. If $|E(H)|$ divides $|E(G)|$, then $G$ decomposes into copies of $H$ and Theorem 3 follows from Lemma 9. Otherwise, the same argument as in the proof of Theorem 2 shows that $G$ can be decomposed into at most 12 copies of triangles and copies of $H$, and hence again Theorem 3 follows from Lemma 9.

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