Unified theory of quantum many-particle systems

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Abstract

Using eigen-functional bosonization method, we study quantum many-particle systems, and show that the quantum many-particle problems end in to solve the differential equation of the phase fields which represent the particle correlation strength. Thus, the physical properties of these systems are completely determined by the differential equation of the phase fields. We mainly focus on the study of D-dimensional electron gas with/without transverse gauge fields, two-dimensional electron gas under an external magnetic field, D-dimensional boson systems, a D-dimensional Heisenberg model and a one-band Hubbard model on a square lattice, and give their exact (accurate for Heisenberg model) functional expressions of the ground state energy and action, and the eigen-functional wave functions of the fermions/bosons. With them, we can calculate a variety of correlation functions of the systems, such as single particle Green’s functions and their ground state wave functions. In present theoretical framework, we can unify represent the Landau Fermi liquid, non-Fermi liquid ($D \geq 2$) and Tomonaga-Luttinger liquid.
I. Introduction

Quantum many-particle systems are the main topics of the condensed matter physics, in which strongly correlated electron systems are the most interesting and hard problems, such as heavy fermion systems, high Tc cuprate superconductors, fractional quantum Hall effects, and some one-dimensional interacting fermion systems. In general, the quantum many-particle (fermion) systems can be divided into two categories, one is represented by the Landau Fermi liquid theory [1,2], and may be called as weakly correlated systems, and another one is represented by non-Fermi liquid theory, such as the Tomonaga-Luttinger liquid theory [3–5] and marginal Fermi liquid theory [6,7], and may be called as strongly correlated fermion systems. Thus there explicitly exists a key parameter hidden in the quantum many-particle systems, which represents the fermion/boson correlation which is produced by the particle interaction. However, the traditional perturbation theories, such as Hartree-Fock approximation, mean field theory, random-phase approximation (RPA) and renormalization group theory, are starting from the bare Green’s function to perturbatively treat the interaction (potential) terms of the particles by taking some low order Feynman diagrams, i.e., exactly treating the kinetic energy term and perturbatively treating the particle interaction. In these methods, there does not exist a parameter used to represent the particle correlation which completely determines the physical properties of the strongly correlated systems. Thus, the previous perturbation methods are successful in treating the weakly correlated systems, but using them to treat the strongly correlated systems, one can meet some serious problems, because in these systems it is difficult to find a suitable small quantity as a perturbation expansion parameter.

It becomes clear that if one wants to unifiably represent the weakly and strongly correlated systems, one must find a well-defined parameter which can represent the particle correlation strength. Due to the particle correlation is produced by the particle interaction, it is natural that we are starting from the interaction (potential) terms to exactly or perturbatively treat the kinetic energy term of the particles, i.e., we first exactly treat
the interaction terms, then with them we exactly or perturbatively treat the kinetic energy term. In this way, the basic ingredient of the systems is particle density field rather than particle field operators, like usual bosonization method in treating one-dimensional interacting fermion systems. In Ref. [8], we proposed an eigen-functional bosonization method, in which there naturally appears a key parameter called the phase field, and with it we can unifiably represent the weakly and strongly correlated electron gases, i.e., the Landau Fermi liquid theory and the non-Fermi liquid theory can be unified under our eigen-functional bosonization theory.

According to the Hohenberg-Kohn theorem [9], the ground state energy of the quantum many-particle systems is uniquely determined by their ground state particle density. This theorem does not tell us how to construct the ground state energy by the particle density, but it clearly tells us that the ground state energy can be exactly represented by the particle density. However, in the Kohn-Sham scheme [10,11], we can only approximately obtain the expression of the ground state energy as a functional of the ground state particle density, because in this scheme the exchange correlation energy term is unclear. Only for small exchange correlation energy can one use the Kohn-Sham scheme to obtain reliable results, i.e., one cannot use it to study the strongly correlated systems, where the exchange correlation energy is large. Thus it is very desirable to write out an exact expression of the ground state energy of the quantum many-particle systems as a functional of the particle density. In fact, in usual bosonization representation of one-dimensional interacting fermion systems [12,13], we learn how to construct the Hamiltonian by the fermion density operators, where we use the fermion density field to uniquely represent the kinetic energy and the fermion field operators. With the eigen-functional bosonization theory, we can treat any quantum many-particle system by using the fermion/boson density field to represent the kinetic energy and the eigen-functional wave functions of the fermions/bosons. In Ref. [14], we demonstrated how to use the particle density field to exactly represent the ground state energy of the quantum many-particle systems.

In this paper, we shall give a detail description of the eigen-functional bosonization
method, and apply it to quantum many-particle systems in continuous coordinate space and some lattice models, such as the electron gases with or without transverse gauge fields, the two-dimensional electron gas under an external magnetic field, the interacting boson systems, the Heisenberg model and one-band Hubbard model. We clearly show that the problems of these systems end in to solve the differential equation of the phase fields which completely controls the physical properties of the systems, thus the quantum many-particle systems can be unifiably represented by the eigen-functional bosonization theory. In section II, we give the exact expressions of the ground state energy as a functional of the particle density field for different quantum many-particle systems. These expressions are universal, i.e., they are valid not only for weak fermion/boson interactions, but also for strong fermion/boson interactions. (1). we give the exact expression of the ground state energy as a functional of the electron density field for electron gases with and without transverse gauge fields, respectively; (2). we give the exact functional expression of the ground state energy for interacting boson systems. (3). we give the exact functional expression of the ground state energy of one-band Hubbard model on a square lattice, and the approximate functional expression of the ground state energy for the Heisenberg model. In section III, we focus on to calculate the action of the quantum many-particle systems. (1). we give the exact functional expression of the action for the electron gas with the transverse gauge fields, and the eigen-functional wave functions of the electrons. Under linearization approximation (only keeping the linear terms in solving the differential equation of the phase fields), we obtain its effective action which is the same as that obtained by usual random-phase approximation (RPA). (2). we give the exact functional expression of the action of the one-band Hubbard model on the square lattice, and the eigen-functional wave functions of the electrons. In section IV, we calculate the ground state wave function of the interacting boson systems, which is very similar to the correlated basis functions used in the study of the liquid $^4$He. We also approximately calculate the ground state wave function of the two-dimensional electron gas under the external magnetic field, which has a little similarity with Laughlin’s trial wave-functions \cite{15,16}. In section V, by calculating the correlation function of the phase fields, we show
that the imaginary part of the phase fields does represent the particle correlation strength. We also demonstrate that if the non-Fermi liquid behavior of the systems is derived from the electron Coulomb interactions, the correlation function of the phase fields has a log-type expression; If their non-Fermi liquid behavior is derived from the transverse gauge fields, the correlation function of the phase fields has a power-law form, \( t^\alpha, \alpha < 1 \). Under our present theoretical framework, the Landau Fermi liquid, non-Fermi liquid and Tomonaga-Luttinger liquid can be unifiably represented. We give some discussions and our conclusion in section VI.

II. Exact functional expression of the ground state energy

In this section, we shall give the exact expression of the ground state energy as a functional of the particle density field for a variety of quantum many-particle systems. In general, a quantum many-particle system can be described by the Hamiltonian, \( H_T = H_0 + H_{\text{ex}} + H_{\text{spin}} \), where \( H_0 \) presents the kinetic energy and the density-density interactions of the particles, \( H_{\text{ex}} \) presents the interactions between the system and external fields, and \( H_{\text{spin}} \) presents the spin-spin interactions among the particles, which is usually written out on lattice sites. In the following we consider the systems with these different particle interactions.

A. Electron gas

For simplicity, we first consider an electron gas with the Hamiltonian (here omitting the spin label),

\[
H_0 = \psi^\dagger(x) \left( \frac{\hat{p}^2}{2m} - \mu \right) \psi(x) + \frac{1}{2} \int d^Dy v(x - y)\rho(y)\rho(x)
\]

where \( \hat{p} = -i\hbar \nabla, \rho(x) = \psi^\dagger(x)\psi(x) \) is the electron density operator, and \( D \) the dimensions of the system. In usual perturbation theories, such as Hartree-Fock approximation, Random-Phase approximation (RPA) and renormalization group theory, one mainly focuses on how to
perturbatively treat the density-density interaction (four-fermion interaction) term, and how to get more accurate results by considering some special Feynman diagrams. However, these methods do not always work for low-dimensional strongly correlated systems. According to the Hohenberg-Kohn theorem and the bosonization representation of one-dimensional fermion systems, we know that the ground state energy and/or the Hamiltonian can be uniquely represented by the ground state particle density, thus we can also study this system by perturbatively or exactly treating the kinetic energy term of the electrons. In order to calculate the ground state energy, we introduce a Lagrangian multiplier (or Hubbard-Stratonovich field) $\phi(x)$ which takes $\rho(x) = \psi^\dagger(x)\psi(x)$ as a constraint condition, then the Hamiltonian (1) can be re-written as,

$$H_0[\phi, \rho] = \psi^\dagger(x) \left( \frac{\hat{p}^2}{2m} - \mu + \phi(x) \right) \psi(x) - \phi(x)\rho(x) + \frac{1}{2} \int d^Dy v(x-y) \rho(y)\rho(x)$$

(2)

where $\phi(x)$ and $\rho(x)$ are independent boson fields, and the Hamiltonian $H_0[\phi, \rho]$ only has the quadratic term of the electron field operator $\psi(x)$. It is clear that the kinetic energy and potential energy of the electrons are completely separated, and the electrons move in the "effective potential" $\phi(x)$. Only in this way (four-fermion interaction replaced by the "effective potential"), we can exactly treat the kinetic energy term. We would like to point out that after introducing the boson field $\phi(x)$, the Hilbert space of the system is enlarged, because the electron density field $\rho(x)$ and the electron field operator now become independent of each other, and some unphysical electron density field $\rho(x) \neq \psi^\dagger(x)\psi(x)$ also appears in the Hamiltonian (2). To only keep the physical electron density field $\rho(x)$, we must add a constraint condition to the system (see below) $\delta E[\phi, \rho]/\delta \phi(x) = 0$. In analogy to the eigen-functional bosonization theory (3), here we solve the eigen-functional equation of the "propagator" operator (in correspondence with below $\hat{M}(x, t)$ which is a real propagator operator) $\hat{M}(x) = \hat{p}^2/(2m) - \mu + \phi(x)$,

$$\left( \frac{\hat{p}^2}{2m} - \mu + \phi(x) \right) \Psi_k(x, [\phi]) = E_k[\phi] \Psi_k(x, [\phi])$$

(3)
where $\Psi_k(x, [\phi])$ are the eigen-functional wave functions of the electrons for a definite boson field $\phi(x)$. Using the Helmann-Feynman theorem, we have the following expression of the eigen-values,

$$E_k[\phi] = \varepsilon(k) + \Sigma_k[\phi]$$

$$\Sigma_k[\phi] = \int_0^1 d\xi \int d^Dx \phi(x) |\Psi_k(x, [\xi\phi])|^2$$

where $\varepsilon(k) = (\hbar k)^2/(2m) - \mu$. The eigen-functionals $\Psi_k(x, [\phi])$ have the following exact expression,

$$\Psi_k(x, [\xi\phi]) = \frac{A_k}{L^{D/2}} e^{ik \cdot x} e^{iQ_k(x, \xi)}$$

where $A_k$ is the normalization constant, and the phase fields $Q_k(x, \xi)$ satisfy usual (static) Eikonal equation,

$$\left(\frac{\hat{p}^2}{2m} + \frac{\hbar}{m} \hat{p} \cdot \hat{\mathbf{p}}\right) Q_k(x, \xi) + \left[\hat{p} Q_k(x, \xi)\right]^2 + \xi \phi(x) = \Sigma_k[\phi]$$

It is worthily noted that the eigen-functionals are composed of two parts, one represents the free electron, and another one represents the contributions from other electrons by the interaction potential $v(x - y)$, therefore it is clear that the eigen-functionals are the eigen-functional wave functions of the interacting electrons corresponding to the definite boson field $\phi(x)$. With them, we can construct the ground state wave-function, and calculate a variety of correlation functions of the system by taking a functional average over the boson field $\phi(x)$.

The electron field operators $\psi^\dagger(x)$ and $\psi(x)$ can be represented as, respectively,

$$\psi^\dagger(x) = \sum_k \Psi_k^\ast(x, [\phi]) \hat{c}_k^\dagger$$

$$\psi(x) = \sum_k \Psi_k(x, [\phi]) \hat{c}_k$$

where $\hat{c}_k^\dagger$ ($\hat{c}_k$) is the electron’s creation (annihilation) operator with momentum $\hbar k$. With equations (3) and (7), we can obtain the exact expression of ground state energy as a functional of the boson field $\phi(x)$ and the electron density $\rho(x)$.
\[ E[\phi, \rho] = \int d^D x < H_0 [\phi, \rho] > = \sum_k \theta(-E_k[\phi])E_k[\phi] - \int d^D x \phi(x)\rho(x) + \frac{1}{2} \int d^D x d^D y v(x-y)\rho(x)\rho(y) \quad (8) \]

where the boson field \( \phi(x) \) is determined by the condition \( \delta E[\phi, \rho]/\delta \phi(x) = 0 \), which leads to the equation,

\[ \sum_k \theta(-E_k[\phi]) \left(G_k(x) + \int d^D y \phi(y) \frac{\delta G_k(y)}{\delta \phi(x)}\right) = \rho(x) \quad (9) \]

where \( G_k(x) = \int_0^1 d\xi e^{2Q_k^R(x,\xi)} / \int d^D x e^{2Q_k^R(x,\xi)} \) and \( Q_k(x,\xi) = Q_k^R(x,\xi) + iQ_k^I(x,\xi) \). This equation shows that the boson field \( \phi(x) \) is the functional of the electron density field \( \rho(x) \).

The chemical potential is determined by the constraint equation,

\[ \sum_k \theta(-E_k[\phi]) = N \quad (10) \]

where \( N \) is the total electron number. With equation (10), the ground state energy can be represented as another form, in which it is only the functional of the electron density field \( (\phi(x) \) is the functional of \( \rho(x) \) determined by (10)),

\[ E_g[\rho] = E_0 + \frac{1}{2} \int d^D x d^D y \left(v(x-y)\rho(x)\rho(y) - 2\phi(x)\phi(y) \sum_k \theta(-E_k[\phi]) \frac{\delta G_k(y)}{\delta \phi(x)}\right) \quad (11) \]

where \( E_0 = \sum_k \theta(-E_k[\phi])\epsilon(k) \), and the self-energy can be written as a simple form, \( \Sigma_k[\phi] = \int d^D x \phi(x)G_k(x) \). The equations (6), (9), (10) and (11) give the exact expression of the ground state energy as a functional of the electron density, we can numerically calculate it by taking \( \delta E_g[\rho]/\delta \rho(x) = 0 \). These equations have more advantages than that in the Kohn-Sham scheme, because that, 1) the expression of the ground state energy as the functional of the electron density is exact, with it we can exactly determine the ground state energy and the ground state electron density; 2) with these equations, we can easily estimate the contributions from high order terms, and obtain enough accurate results we hoped for some special considerations; 3) Using these equations to calculate the ground state energy and the ground state electron density, we may need much less computer time than that in the Kohn-Sham scheme, because here we do not need to self-consistently solve
the eigen-equation of single electron. The ground state wave-function of the system can be obtained by the eigen-functionals, however, we cannot write it as a simple form, because the phase field $Q_k(x, \xi = 1)$ is the function of the momentum $k$.

The expression of the ground state energy (11) is valid not only for Landau Fermi liquid (weak correlation systems), but also for non-Fermi liquid (strongly correlated systems), because it is universal for weak and strong electron interactions. The phase field $Q_k(x, \xi)$ is a key parameter for unifiably representing the Landau Fermi liquid and non-Fermi liquid. Its imaginary part represents the electron correlation strength, and its real part ($D \geq 2$) only contributes to the ground state energy and the action of the systems, this can be clearly seen in the expression of the ground state energy (11).

B. Electron motion under an external magnetic field

We now consider the electron motion under an external magnetic field $\mathbf{B} = (0, 0, B)$, where the response of the system to the magnetic field cannot be written as a simple density form. It is well-known that for enough strong magnetic field, in low temperature limit a two-dimensional electron gas shows the fractional quantum Hall effects due to the Coulomb interaction of the electrons. Here we only give the exact expression of the ground state energy of this system, and do not compare it with that obtained by Laughlin’s trial wave-functions [15], because if so it needs more detail numerical calculations. Under the magnetic field, the Hamiltonian (2) becomes,

$$H[\phi, \rho] = \psi^\dagger(x) \left( \frac{1}{2m} (\hat{p} - \frac{e}{c} A)^2 - \mu + \phi(x) \right) \psi(x)$$

$$- \phi(x) \rho(x) + \frac{1}{2} \int d^2y v(x - y) \rho(y) \rho(x)$$

(12)

where the gauge field $\mathbf{A} = (-yB/2, xB/2, 0)$, and for simplicity we take $D = 2$. The eigen-functional equation corresponding to equation (11) reads,

$$\left( \frac{1}{2m} (\hat{p} - \frac{e}{c} A)^2 - \mu + \phi(x) \right) \Psi_{nl}(x, [\phi]) = E_{nl}[\phi] \Psi_{nl}(x, [\phi])$$

(13)
and the eigen-values are

\[ E_{nl}[\phi] = \epsilon_n + \Sigma_{nl}[\phi] \]

\[ \Sigma_{nl}[\phi] = \int_0^1 d\xi \int d^2x \phi(x) |\Psi_{nl}(x, [\xi \phi])|^2 \]  \hspace{1cm} (14)

where \( \epsilon_n = \hbar \omega_0 (n + 1/2) - \mu, \) \( n = 0, 1, 2, \ldots \), \( \omega_0 = eB/(mc) \) is the cyclotron frequency, and \( \Psi_{nl}(x, [\phi]) \) are the eigen-wave functions of the electrons under the external magnetic field \( B \) for a definite boson field \( \phi(x) \). It is well-known that at \( \phi(x) = 0 \) the eigen-equation (13) has exact solutions,

\[ \Psi_{nl}(x, [0]) = \psi_{nl}(x) = a_{nl} \left( z - \frac{\partial}{\partial z^*}\right)^l \left( z^* - \frac{\partial}{\partial z}\right)^n e^{-zz^*} \]  \hspace{1cm} (15)

where \( l = 0, 1, 2, \ldots, L_{\text{max}} = \Phi/\Phi_0, \Phi = BS \) is the total flux, \( \Phi_0 = 2\pi\hbar c/e \) is the flux quantum, \( z = (x - iy)/(2l_B), \) \( z^* = (x + iy)/(2l_B), \) \( l_B = (hc/(eB))^{1/2} \) is the magnetic length, and \( a_{nl} \) a normalization constant. Therefore, the eigen-functionals \( \Psi_{nl}(x, [\phi]) \) have the following exact form,

\[ \Psi_{nl}(x, [\xi \phi]) = A_{nl} \psi_{nl}(x) e^{Q_{nl}(x,\xi)} \]  \hspace{1cm} (16)

where \( A_{nl} \) is a normalization constant, and the phase field \( Q_{nl}(x,\xi) \) satisfy (static) Eikonal-type equation,

\[ \left( \frac{\mathbf{p}^2}{2m} + \frac{e}{mc} \mathbf{A} + \mathbf{a}_{nl} \cdot \mathbf{p} \right) Q_{nl}(x,\xi) + \frac{[\mathbf{p} Q_{nl}(x,\xi)]^2}{2m} + \xi \phi(x) = \Sigma_{nl}[\phi] \]  \hspace{1cm} (17)

where \( \mathbf{a}_{nl} = \frac{1}{m} \mathbf{p} \ln \psi_{nl}(x) \).

Following the above same procedures, we have the exact expression of the ground state energy as the electron density field \( \rho(x) \),

\[ E_g[\rho] = E_0 + \frac{1}{2} \int d^2x d^2y \left( v(x - y)\rho(x)\rho(y) - 2\phi(x)\phi(y) \sum_{nl} \theta(-E_{nl}[\phi]) \frac{\delta G_{nl}(y)}{\delta \phi(x)} \right) \]  \hspace{1cm} (18)

where \( E_0 = \sum_{nl} \theta(-E_{nl}[\phi]) \epsilon_n \), and the boson field \( \phi(x) \) is the functional of the electron density \( \rho(x) \), and determined by the equation,

\[ \sum_{nl} \theta(-E_{nl}[\phi]) \left( G_{nl}(x) + \int d^2y \phi(y) \frac{\delta G_{nl}(y)}{\delta \phi(x)} \right) = \rho(x) \]  \hspace{1cm} (19)
where \( G_{nl}(x) = \int_0^1 d\xi |\psi_{nl}(x)|^2 e^{2Q_{nl}(x,\xi)} / \int d^2 x |\psi_{nl}(x)|^2 e^{2Q_{nl}(x,\xi)} \), and the self-energy can written as a simple form, \( \Sigma_{nl}[\phi] = \int d^2 x \phi(x) G_{nl}(x) \). The chemical potential is determined by the constraint condition,

\[
\sum_{nl} \theta(-E_{nl}[\phi]) = N
\]  

(20)

where \( N \) is the total electron number. The equations (17), (18), (19) and (20) can be used to exactly determine the ground state energy and the ground state electron density of the system by taking \( \delta E_g[\rho]/\delta \rho(x) = 0 \). However, the expression of the ground state energy (18) has a great advantages comparing with the Laughlin’s trial wave-functions: 1). it is a microscopic theory expression. 2). it shows that the odd- and even-denominator’s fractional quantum Hall states have the same expression of the ground state energy. 3). it can exactly determine the ground state electron density. 4). it is very simple to study the fractional quantum Hall effects taken place in higher Landau levels \((n \geq 1)\).

C. Electron gas with transverse gauge interaction

We consider a two-dimensional electron gas represented by the Hamiltonian with transverse gauge fields \( \mathbf{A}(x) \) \((\nabla \cdot \mathbf{A}(x) = 0)\),

\[
H = \psi^\dagger(x) \left[ \frac{1}{2m} (\hat{p}-g\mathbf{A})^2 - \mu \right] \psi(x) + \frac{1}{2} \int d^2 y \nu(x - y) \rho(x) \rho(y)
\]  

(21)

It is well-known that this system has the following low energy behavior, (1) as \( \nu(x - y) = 0 \), it shows the non-Fermi liquid behavior; (2) as \( \mathbf{A}(x) = 0 \) and \( \nu(x) \sim \ln |x| \), it also shows the non-Fermi liquid behavior. Thus the system belongs to the non-Fermi liquid. Introducing the Lagrangian multiplier \( \phi(x) \), we can re-write it as,

\[
H[\phi, \rho] = \psi^\dagger(x) \left[ \frac{1}{2m} (\hat{p}-g\mathbf{A})^2 - \mu + \phi(x) \right] \psi(x)
\]

\[
-\phi(x) \rho(x) + \frac{1}{2} \int d^2 y \nu(x - y) \rho(x) \rho(y)
\]  

(22)

Then by solving the eigen-functional equation of the ”propagator” operator \( \hat{M}(x) = (\hat{p}-g\mathbf{A})^2/(2m) - \mu + \phi(x) \),
\[
\left( \frac{1}{2m}(\hat{p} - gA)^2 - \mu + \phi(x) \right) \Psi_k(x, [\phi]) = E_k[\phi] \Psi_k(x, [\phi])
\] (23)

we can obtain the exact functional expression of the ground state energy. The solutions of this equation are that,

\[
\Psi_k(x, [\xi_\phi]) = \frac{A_k}{L} e^{ik \cdot x} e^{iQ_k(x, \xi)}
\]

\[
E_k[\phi, A] = \epsilon_k + \Sigma_k[\phi, A]
\] (24)

\[
\Sigma_k[\phi, A] = \int_0^1 d\xi \int d^2x \Psi_k^*(x, [\xi_\phi]) \phi(x) - \frac{g}{2m} A \cdot (2\hat{p} - gA) \Psi_k(x, [\xi_\phi])
\]

where \( \epsilon_k = (\hbar k)^2/(2m) - \mu \), and \( A_k \) is a normalization constant. The phase fields \( Q_k(x, \xi) \) satisfy the (static) Eikonal-type equation,

\[
\left( \frac{\hat{p}^2}{2m} + \left( \frac{\hbar}{m} k - \frac{\xi_\phi}{m} A(x) \right) \cdot \hat{p} \right) Q_k(x, \xi) + \frac{[\hat{p}Q_k(x, \xi)]^2}{2m} + \frac{\xi (gA(x))^2}{2m} - \frac{\hbar \xi g}{m} k \cdot A(x) + \xi \phi(x) = \Sigma_k[\phi, A]
\] (25)

and are the functional of the boson field \( \phi(x) \) and the gauge fields \( A(x) \). In general, this equation cannot be analytically solved but a series solution. Using the eigen-functionals \( \Psi_k(x, [\phi]) \) to represent the electron field operators \( \psi(x) \) and \( \psi^\dagger(x) \), we can obtain the exact expression of ground state energy as a functional of the boson field \( \phi(x) \), the gauge field \( A(x) \) and the electron density \( \rho(x) \),

\[
E[\phi, A, \rho] = \int d^2x < H[\phi, \rho] >
\]

\[
= \sum_k \theta(-E_k[\phi, A]) E_k[\phi, A] - \int d^2x \phi(x) \rho(x) + \frac{1}{2} \int d^2x d^2y v(x - y) \rho(x) \rho(y)
\] (26)

where the boson field \( \phi(x) \) is determined by the condition \( \delta E[\phi, A, \rho] / \delta \phi(x) = 0 \), which leads to the equation,

\[
\sum_k \theta(-E_k[\phi, A]) \left( G_k(x) + \int d^2y [\phi(y) \frac{\delta G_k(y)}{\delta \phi(x)} - \frac{g}{2m} G_k(y) A(y) \frac{\delta \left( \hat{p}Q_k(y) \right)}{\delta \phi(x)}] \right) = \rho(x)
\] (27)

where \( G_k(x) = \int_0^1 d\xi e^{2Q_k(x, \xi)} / \int d^2xe^{2Q_k(x, \xi)} \), and \( Q_k(x, \xi) = Q_k^R(x, \xi) + iQ_k^I(x, \xi) \). The gauge fields \( A(x) \) are determined by the conditions \( J_i(x) = \delta E[\phi, A, \rho] / \delta A_i(x) = 0, i = 1, 2, \ldots \)
which means that the current induced by the gauge fields is zero. The ground state energy $E[\phi, A, \rho]$ has the similar functional expression to that of two-dimensional electron gas without gauge fields. However, they have different phase field dependence, this can be clearly seen from their self-energy expressions $\Sigma_k[\phi]$ and $\Sigma_k[\phi, A]$, respectively. The gauge fields produce the term $\Psi_k^*(x, [\xi \phi]) A(x) \cdot \hat{p} \Psi_k(x, [\xi \phi]) = |\Psi_k(x, [\xi \phi])|^2 A(x) \cdot (\hbar k + \hat{p}Q_k(x, \xi))$ in the self-energy $\Sigma_k[\phi, A]$, thus in this case the imaginary part of the phase fields may contribute to the ground state energy. We would like to point that due to the phase fields $Q_k(x, \xi)$ are complex, to keep the eigen-values $E_k[\phi, A]$ being real, we must have an additional constraint condition on the gauge fields $A(x) \cdot \hat{p}Q_k^R(x, \xi) = 0$. Thus we have three equations to determine two independent transverse gauge fields. If these equations are independent, we only have the solution $A(x) = 0$, i.e., the pure transverse gauge fields do not contribute to the ground state energy. If they are not independent each other, we may have non-zero solution, and the transverse gauge fields contribute to the ground state energy.

D. spin coupling systems

Spin coupling systems are strongly correlated systems and generally represented by the lattice Hamiltonians. For simplicity, we only consider a D-dimensional spin-1/2 Heisenberg model,

$$H = J \sum_{<ij>} \hat{S}_i \cdot \hat{S}_j$$

(28)

where $<ij>$ indicates the summation over the nearest neighbor sites. This spin-spin interaction represents the four-fermion/boson interaction, and is difficult to exactly treat it by usual perturbation theory. Here we only give a way for treating this kind of systems, which may give more useful and accurate informations than previous perturbation methods.

In general, the spin operator (spin-1/2) $\hat{S}_i$ can be represented by the fermion operators $\hat{S}_i = \frac{1}{2} \hat{f}^\dagger_{i\sigma} \sigma_{\alpha \beta} \hat{f}_{i\beta}$, and the Hamiltonian can be re-written as $-\frac{1}{2} \sum_{<ij>, \sigma, \sigma'} \hat{f}^\dagger_{i\sigma} \hat{f}_{j\sigma} \hat{f}^\dagger_{j\sigma'} \hat{f}_{i\sigma'}$, or $\sum_{<ij>} (\hat{f}^\dagger_{i\uparrow} \hat{f}_{j\downarrow} - \hat{f}^\dagger_{i\downarrow} \hat{f}_{j\uparrow})(\hat{f}_{j\downarrow} \hat{f}_{i\uparrow} - \hat{f}_{j\uparrow} \hat{f}_{i\downarrow})$, thus we can write the Hamiltonian (28) as another form,
\[ H = -\frac{J}{2} \sum_{<ij>} \hat{\chi}^\dagger_{ij} \hat{\chi}_{ij} + J \sum_{<ij>} \hat{\mathcal{D}}^\dagger_{ij} \hat{\mathcal{D}}_{ij} + J \sum_{<ij>} \hat{S}_i \cdot \hat{S}_j + \sum_i \lambda_i \cdot (\hat{S}_i - \frac{1}{2} \hat{f}^\dagger_{i\alpha} \sigma_{\alpha\beta} \hat{f}_{i\beta}) \]  

(29)

where \( \hat{\chi}^\dagger_{ij} = \hat{f}^\dagger_{i\sigma} \hat{f}_{j\sigma} \), and \( \hat{D}_{ij} = \hat{f}_{j\downarrow} \hat{f}_{i\uparrow} - \hat{f}_{j\uparrow} \hat{f}_{i\downarrow} \). The Lagrangian multiplier \( \lambda_i \) take the equation \( \hat{S}_i = \frac{1}{2} \hat{f}^\dagger_{i\alpha} \sigma_{\alpha\beta} \hat{f}_{i\beta} \) as the constraint condition adding to the Hamiltonian (29), and the fermion operators \( \hat{f}_{i\sigma} \) and the spin operator \( \hat{S}_i \) are independent each other. In order to treat the four-fermion interactions in the Hamiltonian (29), we can further introduce the Lagrangian multipliers \( \phi'_{ij} \) and \( \Delta_{ij} \) to decouple these interactions, and have the following form,

\[ H = -\frac{J}{2} \sum_{<ij>} \hat{\chi}^\dagger_{ij} \hat{\chi}_{ij} + J \sum_{<ij>} \hat{\mathcal{D}}^\dagger_{ij} \hat{\mathcal{D}}_{ij} + J \sum_{<ij>} \hat{S}_i \cdot \hat{S}_j + \sum_i \lambda_i \cdot (\hat{S}_i - \frac{1}{2} \hat{f}^\dagger_{i\alpha} \sigma_{\alpha\beta} \hat{f}_{i\beta}) + \sum_{<ij>} [\phi'_{ij}(\hat{\chi}^\dagger_{ij} - \hat{f}^\dagger_{i\sigma} \hat{f}_{j\sigma}) + \text{h.c.}] + \sum_{<ij>} [\Delta_{ij}(\hat{\mathcal{D}}_{ij} - \hat{f}_{j\downarrow} \hat{f}_{i\uparrow} + \hat{f}_{j\uparrow} \hat{f}_{i\downarrow}) + \text{h.c.}] \]  

(30)

which is completely equivalent to the Hamiltonian (28). In fact, if we integrate out the boson fields \( \hat{\chi}_{ij} \) and \( \hat{D}_{ij} \), the Hamiltonian (30) becomes usual one [17], and the Lagrangian multipliers \( \phi'_{ij} \) and \( \Delta_{ij} \) become usual Hubbard-Stratonovich fields.

Now the Hamiltonian (30) only includes the quadratic terms of the fermion operators, and in principle, we can exactly calculate the ground state energy. However, in the lattice sites, it is hard to obtain the eigen-functionals of the fermions and the ground state energy by present method, we need to mapping this lattice Hamiltonian into a continuous one. To this end, we take the following approximations,

\[
\begin{align*}
< \hat{\chi}_{ij} > &= a\chi(\frac{x_i + x_j}{2}), & < \hat{\mathcal{D}}_{ij} > &= aD(\frac{x_i + x_j}{2}) \\
\phi'_{ij} &= \phi'(\frac{x_i + x_j}{2}), & \Delta_{ij} &= \Delta(\frac{x_i + x_j}{2}) \\
\phi'_{ij} \hat{f}^\dagger_{i\sigma} \hat{f}_{j\sigma} &= \phi'(\frac{x_i + x_j}{2}) \hat{f}^\dagger_{i\sigma} \hat{f}_{j\sigma} \approx \gamma \hat{f}^\dagger_{i\sigma} \hat{f}_{j\sigma} + \phi(\frac{x_i + x_j}{2}) \hat{f}^\dagger_{i\sigma} \hat{f}_{i\sigma}
\end{align*}
\]

(31)

where \( a \) is the lattice constant, and \( \phi'(\frac{x_i + x_j}{2}) = \gamma + \phi(\frac{x_i + x_j}{2}) \), and obtain the continuous Hamiltonian,
\[ H = \sum_k \varepsilon(k) \hat{f}^\dagger_{\sigma k} \hat{f}_{\sigma k} - \int d^D x [\phi(x) + \phi^*(x)] \hat{f}^\dagger_\sigma(x) \hat{f}_\sigma(x) + \frac{1}{2} \int d^D x \lambda(x) \cdot \hat{f}^\dagger_\alpha(x) \sigma_{\alpha \beta} \hat{f}_\beta(x) \\
+ \int d^D x \{2\Delta(x) \hat{f}_\uparrow(x) \hat{f}_\downarrow(x) - 2\Delta^*(x) \hat{f}^\dagger_\uparrow(x) \hat{f}^\dagger_\downarrow(x) + \lambda(x)(\hat{f}^\dagger_\sigma(x) \hat{f}_\sigma(x) - 1)\} \\
+ \int d^D x [aJ|D(x)|^2 - \frac{a}{2}|\chi(x)|^2] + \int d^D x [\phi'(x) \chi(x) + \Delta(x)D(x) + \text{h.c.}] + H_S \]

where \( \varepsilon(k) = -2\gamma \sum_k \cos(ak) \), \( H_S = J \sum_{<ij>} \hat{S}_i \cdot \hat{S}_j + \sum_i \lambda_i \cdot \hat{S}_i \), \( \hat{f}_\sigma \) is a Lagrangian multiplier which adds the constraint condition at each lattice site \( \hat{f}^\dagger_\sigma \hat{f}_\sigma = 1 \). The parameter \( \gamma \) can be approximately determined by usual mean field theory as taking \( \phi(x) = 0 \), where the parameter \( \Delta(x) = \Delta_0 \) can also be determined by its self-consistent equation. In the continuous limit, the Hamiltonian \( H_S \) is reduced into one of usual non-linear \( \sigma \) model under an "external magnetic field" \( \lambda(x) \).

We can further simplify the Hamiltonian (32) by taking \( \varepsilon(k) = \frac{(nk)^2}{2m} - \mu \), where \( m \sim 1/\gamma \) and \( \mu \) is the chemical potential of the system. In general, due to \( \Delta_0 \neq 0 \), there appears a gap in the excitation spectrum of the fermions, the dynamic term of the vector fields \( \lambda(x) \) produced by the fermions is very small and can be neglected in the low energy (temperature) region \( k_B T \ll \Delta_0 \), therefore, we can simply take \( \lambda(x) = 0 \). We now briefly consider the influence of this approximation on the spin field. As \( J > 0 \), the system has a long-range or short-range antiferromagnetic correlation, thus the spin field operator can be written as \( \hat{S}_i \approx (-1)^i S \Omega(x_i) + a\mathbf{L}(x_i) \), where \( \Omega(x_i) \cdot \mathbf{L}(x_i) = 0 \) and \( |\Omega(x_i)|^2 = 1 \). In the continuous limit the coupling term becomes \( \sum_i \lambda_i \cdot \hat{S}_i \rightarrow \frac{S}{L^{D/2}} \sum_q \lambda(q + \pi/a) \cdot \Omega(q) \), thus only the fermion-hole excitations with large momentum \( \sim \pi/a \) contribute to this coupling term. However, due to \( \Delta_0 \neq 0 \), these excitations are very small and can be safely neglected in the low energy limit. Therefore, the Hamiltonian (32) can be approximately separated into two parts, \( H = H_f + H_S \), where \( H_f \) mainly represents the high energy \( \sim \Delta_0 \) behavior, and \( H_S \) mainly represents the low energy \( \ll \Delta_0 \) behavior of the system. Here we only consider the contribution of the Hamiltonian \( H_f \) to the ground state energy, where the Hamiltonian \( H_f \) reads,

\[ H = \int d^D x \hat{f}^\dagger_\sigma(x) \left\{ \frac{|\mathbf{p}|^2}{2m} - \mu + \lambda(x) - [\phi(x) + \phi^*(x)] \right\} \hat{f}_\sigma(x) \]
\[+2 \int d^Dx [\Delta(x)\hat{F}_1(x)\hat{f}_1(x) - \Delta^*(x)\hat{f}_1^\dagger(x)\hat{F}_1^\dagger(x)] \]
\[+ \int d^Dx [\frac{2}{aJ}|\phi'(x)|^2 - \frac{1}{aJ}|\Delta(x)|^2 - \lambda(x)]\]

Due to appearing the pairing parameter \(\Delta(x) = -aJ < \hat{f}_1^\dagger(x)\hat{f}_1^\dagger(x) >\), the "propagator" operator \(\hat{M}(x)\) of the fermions is a \(2 \times 2\) matrix, and can be written as \(\hat{M}(x) = \hat{M}_0 + \Phi(x)\), where

\[
\hat{M}_0 = \begin{pmatrix}
\frac{p^2}{2m} - \mu & -2\Delta_0 \\
-2\Delta_0 & -\frac{p^2}{2m} + \mu
\end{pmatrix}
\]

\[
\Phi(x) = \begin{pmatrix}
\lambda(x) - [\phi(x) + \phi^*(x)] & -2\Delta^*(x) \\
-2\Delta^*(x) & -\lambda(x) + [\phi(x) + \phi^*(x)]
\end{pmatrix}
\]

where \(\mu = \mu - \lambda_0\), \(\lambda(x) = \Delta(x) - \Delta_0\), \(\lambda(x) = \lambda(x) - \lambda_0\), and the pairing parameter \(\Delta_0\) can be determined by its self-consistent equation as taking \(\phi(x) = \Delta(x) = \lambda(x) = 0\). The eigen-functional equation of the "propagator" operator \(\hat{M}(x)\) reads,

\[
\hat{M}(x)\Psi_k^{(i)}(x, [\Phi]) = E_k^{(i)}[\Phi]\Psi_k^{(i)}(x, [\Phi])
\]

where the eigen-values \(E_k^{(\pm)}[\Phi] = \pm E_k + \Sigma_k^{(\pm)}[\Phi]\), \(E_k = \sqrt{\varepsilon^2(k) + 4\Delta_0^2}\), and the self-energy \(\Sigma_k^{(\pm)}[\Phi] = \int_0^1 d\xi \int d^Dx \Psi_k^{(\pm)*}(x, [\xi\Phi])\Phi(x)\Psi_k^{(\pm)}(x, [\xi\Phi])\). The eigen-functionals \(\Psi_k^{(\pm)}(x, [\xi\Phi])\) can be generally written as,

\[
\Psi_k^{(+)}(x, [\xi\Phi]) = \frac{A_k^{(+)}}{L^{D/2}} \begin{pmatrix}
u_k e^{Q_k^{(+)}}(x, \xi) \\
-v_k e^{Q_k^{(+)}}(x, \xi)
\end{pmatrix} e^{ik \cdot x}
\]

\[
\Psi_k^{(-)}(x, [\xi\Phi]) = \frac{A_k^{(-)}}{L^{D/2}} \begin{pmatrix}
u_k e^{Q_k^{(-)}}(x, \xi) \\
u_k e^{Q_k^{(-)}}(x, \xi)
\end{pmatrix} e^{ik \cdot x}
\]

where \(u_k = (1/\sqrt{2})(1+\varepsilon(k)/E_k)^{1/2}\), \(v_k = (1/\sqrt{2})(1-\varepsilon(k)/E_k)^{1/2}\), and \(A_k^{(\pm)}\) are the normalization constants. To calculate the ground state energy, we only consider the eigen-functionals \(\Psi_k^{(-)}(x, [\xi\Phi])\) which represent the states occupied by the fermions at zero temperature, and have the differential equations of the phase fields \(Q_k^{(-)}(x, \xi)\) and \(\overline{Q}_k^{(-)}(x, \xi)\),
\[
\left( \frac{\hat{p}^2}{2m} + \frac{\hbar \mathbf{k} \cdot \hat{p}}{m} \right) Q_k^{(-)} + \frac{[\hat{p} Q_k^{(-)}]^2}{2m} + \xi \Phi(x) + \frac{2u_k}{v_k} (\Delta_0 - \Delta^*(x)) e^{\overline{Q}_k^{(-)} - Q_k^{(-)}} = \Sigma_k^{(-)}[\Phi] \]
\[
\left( -\frac{\hat{p}^2}{2m} - \frac{\hbar \mathbf{k} \cdot \hat{p}}{m} \right) \overline{Q}_k^{(-)} - \frac{[\hat{p} Q_k^{(-)}]^2}{2m} - \xi \Phi(x) + \frac{2u_k}{v_k} (\Delta_0 - \Delta(x)) e^{Q_k^{(-)} - \overline{Q}_k^{(-)}} = \Sigma_k^{(-)}[\Phi] \quad (37)
\]

where the pairing parameter \( \Delta(x) \) is determined by its self-consistent equation \( \Delta(x) = -aJ \langle \hat{f}_1^\dagger(x) \hat{f}_1^\dagger(x) \rangle \). The functional expression of the ground state energy can be written as a simple form,

\[
E[\lambda, \phi, \Delta] = \sum_k \theta(-E_k^{(-)}[\Phi]) E_k^{(-)}[\Phi] + \int d^D x \left[ \frac{2}{aJ} |\phi'(x)|^2 - \frac{1}{aJ} |\Delta(x)|^2 - \lambda(x) \right] \quad (38)
\]

If we can exactly solve the differential equations of the phase fields (37), we can exactly calculate the self-energy \( \Sigma_k^{(-)}[\Phi] \) by its definition, then we obtain the ground state energy contributed by the fermions by taking the constraint conditions, \( \delta E[\lambda, \phi, \Delta]/\delta \lambda(x) = \delta E[\lambda, \phi, \Delta]/\delta \phi(x) = \delta E[\lambda, \phi, \Delta]/\delta \Delta(x) = 0 \).

E. One-band Hubbard model

We now consider the one-band Hubbard model on a square lattice, and give the exact expression of the ground state energy, which is very similar to that of the two-dimensional electron gas in the continuous coordinates. The Hamiltonian of the Hubbard model on the square lattice reads,

\[
H = -t \sum_{<ij>,\sigma} \left( \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + \hat{c}_{j\sigma}^\dagger \hat{c}_{i\sigma} \right) - \sum_{i\sigma} \mu \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \quad (39)
\]

where \( <ij> \) indicates the summation over the nearest neighbor sites, and \( U \) is the on-site Coulomb interaction. To decouple the four-fermion interaction, we introduce the Lagrangian multipliers \( \phi_{i\sigma} \) which take \( n_{i\sigma} = \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma} \) as the constraints to the system, and re-write the Hamiltonian (39) as,

\[
H[\phi, n] = -t \sum_{<ij>,\sigma} \left( \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + \hat{c}_{j\sigma}^\dagger \hat{c}_{i\sigma} \right) - \sum_{i\sigma} \mu \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma} - \sum_{i\sigma} \phi_{i\sigma} (n_{i\sigma} - \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma}) + U \sum_i n_{i\uparrow} n_{i\downarrow} \quad (40)
\]

On the lattice site coordinates, the "propagator" operator of the electrons now can be written as a \( N \times N \) matrix,
(\(\hat{M}_{ij}\)) \(\hat{M}_{ij}^{(0)} = (\hat{M}_{ij}^{(0)} + \phi_{i\sigma}\delta_{ij})\) \(\tag{41}\)

where \(\hat{M}_{ij}^{(0)} = -t(\gamma_{ij} + \gamma_{ji}) - \mu\delta_{ij}, \gamma_{ij} = 1 \text{ for } j = i \pm 1, \text{ and } \gamma_{ij} = 0 \text{ for other } j.\) The eigen-functional equation of \(\hat{M}_{\sigma}\) reads,

\[
\hat{M}_{\sigma}\Psi_{k\sigma}[\phi] = E_{k\sigma}[\phi]\Psi_{k\sigma}[\phi]\]

\(\tag{42}\)

where the eigen-values \(E_{k\sigma}[\phi] = \varepsilon_k + \Sigma_{k\sigma}[\phi],\) \(\varepsilon_k = -2t(\cos(ak_x) + \cos(ak_y)) - \mu,\) the self-energy \(\Sigma_{k\sigma}[\phi] = \int_0^1 d\xi \Psi_{k\sigma}^\dagger[\xi\phi]\phi_{\sigma}\Psi_{k\sigma}[\xi\phi],\) and \(\phi_{\sigma} = (\phi_{i\sigma}\delta_{ij})\) is a \(N \times N\) matrix, in which only the diagonal terms are non-zero. In general, the eigen-functionals \(\Psi_{k\sigma}[\phi]\) can be written as a simple form,

\[
\Psi_{k\sigma}[\xi\phi] = \frac{A_k}{L} \begin{pmatrix}
e^{i\mathbf{k} \cdot \mathbf{x}_1} e^{Q_{k\sigma}(x_1, \xi)} \\
\vdots \\
e^{i\mathbf{k} \cdot \mathbf{x}_N} e^{Q_{k\sigma}(x_N, \xi)}
\end{pmatrix}\]

\(\tag{43}\)

where \(A_k\) is the normalization constant. With the equation (42), the phase fields \(Q_{k\sigma}(x_i, \xi)\) satisfy the following equation,

\[
\gamma_k(1 - e^{-Q_{k\sigma}(x_1, \xi) + Q_{k\sigma}(x_{i+1}, \xi)}) + \gamma_k^*(1 - e^{-Q_{k\sigma}(x_1, \xi) + Q_{k\sigma}(x_{i-1}, \xi)}) + \xi\phi_{i\sigma} = \Sigma_{k\sigma}[\phi]\]

\(\tag{44}\)

where \(\gamma_k = t(e^{iak_x} + e^{iak_y}),\) and the self-energy \(\Sigma_{k\sigma}[\phi]\) is self-consistently determined by its definition. The electron operators \(\hat{c}_{i\sigma}\) can be represented by the eigen-functionals \(\Psi_{k\sigma}[\phi]\) which are the eigen-functional wave functions of the electrons for a definite boson field \(\phi_{i\sigma},\)

\[
\hat{c}_{i\sigma} = \frac{1}{L} \sum_k A_k e^{i\mathbf{k} \cdot \mathbf{x}_i} e^{Q_{k\sigma}(x_i, \xi)} \hat{c}_{k\sigma}\]

\(\tag{45}\)

then we obtain the exact expression of the ground state energy as a functional of the boson fields \(\phi_{i\sigma}\) and the electron density fields ,

\[
E[\phi, n] = \langle H[\phi, n] \rangle
= \sum_{k\sigma} \theta(-E_{k\sigma}[\phi]) E_{k\sigma}[\phi] - \sum_{i\sigma} \phi_{i\sigma} n_{i\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}\]

\(\tag{46}\)

where the chemical potential of the system is determined by the constraint condition,
\[
\sum_{k\sigma} \theta(-E_{k\sigma}[\phi]) = N_e
\] (47)

where \(N_e\) is the total electron number. By taking \(\delta E[\phi, n]/\delta \phi_{i\sigma} = \delta E[\phi, n]/\delta n_{i\sigma} = 0\), we can exactly calculate the ground state energy and the ground state electron density of the one-band Hubbard model by using the equations (44), (46) and (47). We hope that our present results can provide a powerful and exact method to study the strongly correlated electron systems represented by the Hubbard model, and can give some important criterions by numerical calculations for analytical calculations.

F. Boson systems

For a boson system, such as the liquid \(^4\)He, at zero temperature it has Bose-Einstein condensation, and the bosons only occupy the state of the momentum \(k = 0\), thus the equation (10) is trivial, and the ground state energy can be written as,

\[
E_g[\rho] = \frac{1}{2} \int d^D x d^D y [v(x - y)\rho(x)\rho(y) - 2\delta(x - y)\phi(x)\rho(x)]
\] (48)

where we have taken \(\mu = \Sigma_0[\phi]\). The equation (9) reduces into,

\[
G_0(x) + \int d^D y \phi(y) \frac{\delta G_0(y)}{\delta \phi(x)} = \frac{1}{N} \rho(x)
\] (49)

It is noted that the last term in (48) is the contributions of kinetic energy of the bosons, and in general it may be non-zero for interacting boson systems. We can also give the exact functional expression of the ground state energy of some trapped boson systems, where the procedures are very similar that we study the two-dimensional electron gas under the external magnetic field: (1). we first solve the eigen-functional equation of the "propagator" operator \(\hat{M}(x)\) of the bosons under some external field and/or constraints, and obtain its eigen-values and the differential equations of the phase fields; (2). using the eigen-functionals to represent the boson field operators, we obtain the exact expression of the ground state energy as a functional of the Lagrangian multiplier boson field \(\phi(x)\) and the boson density field \(\rho(x)\).
III. Exact functional expression of the action

In this section, we mainly focus on to study two kinds of systems, one is a D-dimensional electron gas with transverse gauge fields, and another one is the one-band Hubbard model on a square lattice. In these two cases, we shall give the exact functional expression of the action and the eigen-functional wave functions of the electrons. Under linearization approximation (only keeping the linear terms in solving the differential equation of the phase fields), we give the effective action which is equivalent to that obtained by usual RPA method. With them, we can calculate the excitation spectrum and the variety of correlation functions of the systems, and study their low energy behavior.

A. Electron gas with transverse gauge field

In general, we consider the system with the Hamiltonian (omitting spin label $\sigma$ of the electron operators $c_k$)

$$H_0 = \psi^\dagger(x) \left( \frac{1}{2m}(\hat{p} - gA)^2 - \mu \right) \psi(x) + \frac{1}{2} \int d^Dy v(x-y)\rho(y)\rho(x)$$

(50)

where $A(x)$ ($\nabla \cdot A(x) = 0$) are the transverse gauge fields. The last term represents usual electron’s Coulomb interaction (four-fermion interaction). Introducing the Hubbard-Stratonovich (HS) field $\phi(x,t)$ to decouple this four-fermion interaction term, and using the standard path integral method [18], we have the action of the system,

$$S[\phi, A] = \int dt \int d^Dx \left\{ \Psi^\dagger(x,t) \left[ i\hbar \frac{\partial}{\partial t} + \mu - \frac{1}{2m}(\hat{p} - gA)^2 + \phi(x,t) \right] \Psi(x,t) \right\}$$

$$+ \frac{1}{2TLD} \sum_{q,\Omega} \frac{1}{v(q)} \phi(-q, -\Omega)\phi(q, \Omega)$$

(51)

where $\Psi(x,t)$ is the electron field, and only quadratic term of $\Psi(x,t)$ appears in the action. Therefore, the integration of the electron field becomes a standard Gaussian form. It is worthily noted that after introducing the boson field $\phi(x,t)$, the Hilbert space of the system
is enlarged, and the unphysical electron density field \( \rho(x, t) \neq \Psi(x, t) \Psi(x, t) \) appears in the action \([51]\). To erase the unphysical electron density field, we must take the functional average over the boson field \( \phi(x, t) \) in calculating a variety of correlation functions and response functions of the system to external fields.

After integrating out \( \Psi(x, t) \), the contribution of “kinetic” energy of the electrons to the action is \(-i \text{Tr} \ln(\hat{M})\), where \( \hat{M}(x, t) = i \hbar \partial_t + \mu - (\hat{p} - gA)^2/(2m) + \phi(x, t) \) is the propagator operator of the electrons. Using the formula,

\[
\text{Tr} \ln(\hat{M}) = \text{Tr} \ln(i \hbar \partial_t + \mu - \hat{p}^2/(2m)) + \int_0^1 d\xi \int dtdP x\phi(x, t)G(x, t; x', t', [\xi \phi])\bigg|_{t' \to t, x' \to x}. \tag{52}
\]

where \( \hat{M}(x, t)G(x, t; x', t', [\phi]) = \delta(x - x')\delta(t - t') \), and neglecting the constant term, we have the following expression of the action,

\[
S[\phi, A] = -i \int_0^1 d\xi \int dtdP x\phi(x, t)G(x, t; x', t', [\xi \phi])\bigg|_{t' \to t, x' \to x} + \frac{1}{2TL^3} \sum_{q, \Omega} \frac{1}{v(q)} \phi(-q, -\Omega)\phi(q, \Omega) \tag{53}
\]

The eigen-functional equation of the propagator operator \( \hat{M}(x, t) \) reads,

\[
\left[i \hbar \frac{\partial}{\partial t} + \mu - \frac{1}{2m}(\hat{p} - gA)^2 + \phi(x, t)\right] \Psi_{k\omega}(x, t, [\phi]) = E_{k\omega}^{[\phi]} \Psi_{k\omega}(x, t, [\phi]) \tag{54}
\]

where the eigen-values \( E_{k\omega}^{[\phi]} = \hbar \omega - \varepsilon(k) + \Sigma_k[\phi], \varepsilon(k) = (\hbar k)^2/(2m) - \mu \), and the self-energy \( \Sigma_k[\phi] = \int_0^1 d\xi \int dtdP x\Psi^*_k(x, [\xi \phi])[\phi(x) + \frac{g}{2m}A \cdot (2\hat{p} - gA)]\Psi_k(x, [\xi \phi]) \) is a regular function, and independent of \( \omega \) (see below equation \([56]\)). Using the orthogonality and completeness of the eigen-functionals \( \Psi_{k\omega}(x, t, [\phi]) \), the Green’s functionals \( G(x, t; x', t', [\phi]) \) can be represented by these eigen-functionals,

\[
G(x, t; x', t', [\xi \phi]) = \sum_{k\omega} \frac{1}{E_{k\omega}^{[\phi]}} \Psi_{k\omega}(x, t, [\xi \phi])\Psi_{k\omega}^*(x', t', [\xi \phi]) \tag{55}
\]

The eigen-functionals \( \Psi_{k\omega}(x, t, [\phi]) \) are the eigen-functional wave functions of the electrons for a definite HS field \( \phi(x, t) \), and have the following exact expressions,
\[ \Psi_{k\omega}(x, t, [\xi\phi]) = A_k \left( \frac{1}{T L^D} \right)^{1/2} e^{Q_k(x, t, \xi)} e^{ik \cdot x - i(\omega + \Sigma_k[\phi]) t} \] (56)

where \( A_k \) is the normalization constant, and \( T \to \infty \) is the time length of the system (for finite \( T \), the \( \Psi_{k\omega}(x, t, [\xi\phi]) \) satisfy the boundary condition \( \Psi_{k\omega}(x, t + T, [\xi\phi]) = -\Psi_{k\omega}(x, t, [\xi\phi]) \)). These eigen-functionals are composed of two parts, one represents the free electrons, and another one represents the correlation of the electrons produced by the electron interaction. Thus we can formally write

\[ \Psi_{k\omega}(x, t, [\xi\phi]) = \psi_{k\omega}(x, t) A_k e^{Q_k(x, t, \xi)} e^{-i\Sigma_k[\phi] t} \] (57)

where \( \psi_{k\omega}(x, t) \) are the wave-functions of the free electrons. The phase fields \( Q_k(x, t, \xi) \) satisfy the usual Eikonal-type equation with the condition \( Q_k(x, t, \xi) = 0 \) as \( \phi(x, t) = 0 \) and \( A(x, t) = 0 \),

\[ \left( i\hbar \frac{\partial}{\partial t} - \frac{\hat{p}^2}{2m} - \left( \frac{\hbar^2}{m} \mathbf{k} - \frac{\xi g}{2m} \mathbf{A} \right) \cdot \hat{p} \right) Q_k(x, t, \xi) - \frac{[\hat{p}Q_k(x, t, \xi)]^2}{2m} - \frac{\xi (gA)^2}{2m} + \frac{\hbar \xi g}{m} \mathbf{k} \cdot \mathbf{A} + \xi \phi(x, t) = 0 \] (58)

which can be exactly solved by a series expansion of the HS field \( \phi(x, t) \) and the transverse gauge fields \( A(x, t) \) and/or by computer calculations [19, 20]. It is noted that the \( \phi(x, t) \) and \( A(x, t) \) dependence of the eigen-functionals \( \Psi(x, t, [\xi\phi]) \) is completely determined by the phase fields \( Q_k(x, t, \xi) \).

Substituting equation (56) into equation (55), the Green’s functionals can be written as,

\[ G(x, t; x', t', [\xi\phi]) = \frac{i}{L^D} \sum_k \theta(\Sigma_k[\phi] - \epsilon(k)) | A_k |^2 e^{i \mathbf{k} \cdot (x - x') - i \epsilon(k)(t - t')} e^{Q_k(x, t, \xi) + Q_k^*(x', t', \xi)} \] (59)

In order to calculate the Green’s functionals \( G(x, t; x', t', [\xi\phi]) \) as \( t' \to t \), we take the following regular procedure,

\[ G(x, t; x', t', [\xi\phi]) \big|_{t' \to t} = \frac{1}{2} \lim_{\eta \to 0^+} \left( G(x, t; x - \eta, t, [\xi\phi]) + G(x, t; x + \eta, t, [\xi\phi]) \right) \] (60)

then we obtain the following exact functional expression of the action,
can calculate a variety of correlation functions, such as the single electron Green’s function, \( \phi \) by taking the functional average on the boson field \( \phi \). We can obtain the following effective action (omitting constant terms),
\[
S[\phi, A] = \frac{1}{2} \int dt d^Dx \left[ F(x, x - \eta, t) + F(x, x + \eta, t) \right]_{\eta \to 0^+} + \frac{1}{2TL^D} \sum_{q, \Omega} \frac{1}{v(q)} \phi(-q, -\Omega) \phi(q, \Omega)
\]
(61)
\[
F(x, x', t) = T \int_0^1 d\xi \sum_k \theta(\Sigma_k[\phi] - \varepsilon(k)) \frac{\phi(x, t) e^{i(k(x-x') e Q_k(x,t,\xi) + Q_k^*(x',t,\xi)}}{\int dt d^Dx e^{Q_k(x,t,\xi) + Q_k^*(x',t,\xi)}}
\]
It becomes very clear that this kind of problems end in to solve the differential equation (58) of the phase fields \( Q_k(x, t, \xi) \), which can be solved by a series expansion of the boson field \( \phi(x, t) \) and the transverse gauge fields \( A(x, t) \). With this action and the eigen-functionals, we can calculate a variety of correlation functions, such as the single electron Green’s function, by taking the functional average on the boson field \( \phi(x, t) \) and the transverse gauge fields \( A(x, t) \),
\[
G(x - x', t - t') = i < G(x, t; x', t', [\phi]) >_{\phi, A} = i \left[ \prod D\phi \prod D\phi^\dagger \right] \frac{DAG(x, t; x', t', [\phi]) e^{iS[\phi, A]}}{\int \prod D\phi \prod D\phi^\dagger e^{iS[\phi, A]}}
\]
(62)
In general, it is difficult to exactly calculate the action, in order to study the low energy behavior of the system, we can approximately calculate it. After taking some approximations, we can obtain the following effective action (omitting constant terms),
\[
S[\phi, A] = \frac{1}{2TL^D} \sum_{q, \Omega} \frac{\phi(-q, -\Omega) \phi(q, \Omega)}{v(q)} + \frac{1}{2} \int dt d^Dx \phi(x, t) [F_1(x, t, \delta) + F_2(x, t)]_{\eta \to 0}
\]
(63)
where \( F_1(x, t, \eta) = -(1/(2\pi)^D) \int d^Dk \theta(-\varepsilon(k)) \sin(k \cdot \eta) \nabla Q^\dagger_k(x, t) \), \( F_2(x, t) = (2/(2\pi)^D) \int d^Dk \theta(-\varepsilon(k)) Q^R_k(x, t) \). We have taken \( \theta(\Sigma_k[\phi] - \varepsilon(k)) \approx \theta(-\varepsilon(k)) \) and \( e^{Q_k(x,t,\xi)+Q_k^*(x',t,\xi)} \approx 1 + Q_k(x, t, \xi) + Q_k^*(x', t, \xi) \), and written the phase fields as \( Q_k(x, t, \xi = 1) = Q_k(x, t) = Q^R_k(x, t) + iQ^I_k(x, t) \). It is worthily noted that the contribution of the "kinetic" energy of the electrons to the action is composed of two parts, one is from the real phase field \( Q^R_k(x, t) \), and another one is from the imaginary phase field \( Q^I_k(x, t) \). Due to the Fermi surface structure of the system, the momentum integral in \( F_1(x, t, \eta) \) can be written as
\[
\int d^Dk = S_{D-1} \int d|k| |k|^{D-1} \int_0^\pi d\theta (\sin \theta)^{D-2}, \text{ where } S_D = 2\pi^{D/2}/\Gamma(D/2). \text{ As } D \geq 2, \text{ the}
\]
integration of \(\sin(\mathbf{k} \cdot \eta)\) is regular, thus as \(\eta \to 0\) the function \(F_1(x, t, \eta) = 0\), the imaginary phase field \(Q_k^I(x, t)\) does not contribute to the action. Only at \(D = 1\), it has contribution to the action, where the real phase field \(Q_k^R(x, t)\) is zero (see below). This property is independent of the electron interaction, it is completely determined by the Fermi surface structure of the system.

For a 1D interacting electron gas (taking \(A(x, t) = 0\)), the electron energy spectrum near its Fermi level \(\pm k_F\) can be written as, \(\varepsilon(k) = \pm v_F k\), where \(v_F\) is the Fermi velocity. The branch \(\varepsilon(k) = v_F k\) represents the right-moving electrons, and the branch \(\varepsilon(k) = -v_F k\) the left-moving electrons. In general, the electron interaction term reads, 
\[(1/L) \sum_q v(q) \rho_R(q) \rho_L(-q),\]
where \(\rho_R(q)\) and \(\rho_L(q)\) are the right- and left-moving electron densities, respectively, and \(v(q) \equiv V\), a constant. To decouple this four-fermion interaction, we introduce two HS fields \(\phi_R(L)(x, t)\), and have two sets of wave-functions of the right- and left-moving electrons for the definite \(\phi_R(L)(x, t)\), respectively,
\[
\Psi_{R(L)k\omega}(x, t, [\xi \phi]) = \left(\frac{1}{TL}\right)^{1/2} e^{Q_{R(L)}(x,t,\xi)} e^{ikx-i(\omega-\Sigma_{R(L)}[\phi])t} (64)
\]
where \(\Sigma_{R(L)}[\phi]\) is a regular quantity, and independent of \(k\) and \(\omega\). The phase fields \(Q_{R(L)}(x, t, \xi)\) are independent of \(k\), and satisfy the simplified Eikonal equation,
\[
\left(i \frac{\partial}{\partial t} \pm iv_F \frac{\partial}{\partial x}\right) Q_{R(L)}(x, t, \xi) - \xi \phi_{R(L)}(x, t) = 0 (65)
\]
These linear differential equations can be easily solved, and the phase fields \(Q_{R(L)}(x, t, \xi = 1) = Q_{R(L)}(x, t)\) are imaginary because the HS fields \(\phi_{R(L)}(x, t)\) are real. The action (61) becomes the following simple form [22],
\[
S[\phi] = \frac{1}{T L} \sum_{q, \Omega} \left[ \frac{1}{4 \pi q - \Omega} |\phi_R|^2 + \frac{1}{4 \pi q + \Omega} |\phi_L|^2 + \frac{1}{V} \phi_R(-q, -\Omega) \phi_L(q, \Omega) \right] (66)
\]
where \(\phi_{R(L)} = \phi_{R(L)}(q, \Omega)\). It is worthily noted that the imaginary phase fields \(Q_{R(L)}(x, t)\) not only determine the electron correlation, but also contribute to the action. This is qualitatively different from that in 2D and 3D electron gases, where the imaginary part of the phase field \(Q_k(x, t)\) does not contribute to the action due to their Fermi surface structures.
As $D \geq 2$, for simplicity, we can solve the Eikonal equation by neglecting the quadratic term $(\nabla Q_k)^2$. This approximation is reasonable because for the long-range Coulomb interaction, only the states near the Fermi surface with momentum $q < q_c$ ($q_c \ll k_F$) are important in the low energy regime, and for the smooth function $Q_k(x,t)$, this quadratic term is proportional to $(q_c/k_F)^2 \sim 0$. Under this approximation, we can obtain the effective action,

$$S_{\text{eff.}} = \frac{1}{2TL^D} \sum_{q,\Omega} \left[ \left( \frac{1}{v(q)} - \chi(q, \Omega) \right) |\phi(q, \Omega)|^2 + \Pi_{ij}(q, \Omega) A_i(-q, -\Omega) A_j(q, \Omega) \right]$$

where $\chi(q, \Omega)$ is usual Lindhard function, and $\Pi_{ij}(q, \Omega) = (i\gamma \Omega/q + q^2)(\delta_{ij} - q_i q_j/q^2)$ the propagator of the transverse gauge fields $A_i(q, \Omega)$ produced by the electron-hole excitations. In fact, the above approximation is equivalent to usual random-phase approximation (RPA). However, at present framework, it gives more useful and important informations than usual RPA method, because we have the well-defined phase fields, and can use their imaginary part to determine the electron correlation.

B. One-band Hubbard model on a square lattice

In the above subsection, we have considered the D-dimensional electron gas with the transverse gauge fields in the continuous coordinate space, now for simplicity we consider an one-band Hubbard model on a square lattice, but this method can be easily extended to other lattice models. The one-band Hubbard model on a square lattice is presented by the Hamiltonian,

$$H = -t \sum_{\langle ij \rangle, \sigma} (\hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + \hat{c}_{j\sigma}^\dagger \hat{c}_{i\sigma}) - \mu \sum_{i\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

Introducing the Lagrangian multiplier fields $\phi_{i\sigma}(t)$, we have the following action,

$$S[\phi, n] = \sum_{\langle ij \rangle, \sigma} \int dt \hat{c}_{i\sigma}^\dagger(t) M_{ij\sigma}(t) \hat{c}_{j\sigma}(t) - \sum_{i\sigma} \int dt [n_{i\sigma}(t) \phi_{i\sigma}(t) + \frac{1}{2} U n_{i\uparrow}(t) n_{i\downarrow}(t)]$$
where $M_{ij\sigma}(t) = \delta_{ij}\left[i\partial_t + \mu + \phi_{i\sigma}(t)\right] + t(\gamma_{ij} + \gamma_{ji})$ is a $N \times N$ matrix, where $\gamma_{ij} = 1$ for $j = i \pm 1$, and $\gamma_{ij} = 0$ for other $j$. After integrating out the electron fields $c_{i\sigma}(t)$, we obtain the following action,

$$S[\phi, n] = -i Tr \ln (M_{ij\sigma}(t)) - \sum_{i\sigma} \int dt [n_{i\sigma}(t)\phi_{i\sigma}(t) + \frac{1}{2}Un_{i\uparrow}(t)n_{i\downarrow}(t)]$$  \hspace{1cm} (70)

Using the formula,

$$-i Tr \ln (M_{ij\sigma}(t)) = -i Tr \ln \left(M_{ij}^{(0)}\right) - i \sum_{i\sigma} \int_0^1 d\xi \int dt \phi_{i\sigma}(t)G_{\sigma}(x_i, t; x_j, t', [\xi])\big|_{\xi' = \xi}$$

where $M_{ij}^{(0)} = \delta_{ij}(i\partial_t + \mu) + t(\gamma_{ij} + \gamma_{ji})$, and $M_{ij\sigma}(t)G_{\sigma}(x_j, t; x_t, t', [\xi]) = \delta_{il}\delta(t - t')$, this action can be re-written as (neglecting the constant term),

$$S[\phi, n] = -i \sum_{i\sigma} \int_0^1 d\xi \int dt \phi_{i\sigma}(t)G_{\sigma}(x_i, t; x_j, t', [\xi])\big|_{\xi' = \xi}$$

$$- \sum_{i\sigma} \int dt [n_{i\sigma}(t)\phi_{i\sigma}(t) + \frac{1}{2}Un_{i\uparrow}(t)n_{i\downarrow}(t)]$$  \hspace{1cm} (71)

The eigen-functional equation of the matrix operator $M_{ij\sigma}(t)$ reads,

$$M_{\sigma}(t)\Psi_{k\omega\sigma}(t, [\phi]) = E_{k\omega\sigma}[\phi]\Psi_{k\omega\sigma}(t, [\phi])$$  \hspace{1cm} (72)

where the eigen-values $E_{k\omega\sigma}[\phi] = \omega - \varepsilon_k + \Sigma_{k\sigma}[\phi]$, $\varepsilon_k = -2t[\cos(ak_x) + \cos(ak_y)] - \mu$, and $\Sigma_{k\sigma}[\phi] = \int_0^1 d\xi \int dt \Psi^\dagger_{k\omega\sigma}(t, [\phi])\phi_{\sigma}(t)\Psi_{k\omega\sigma}(t, [\phi])$, where $\phi_{\sigma}(t) = (\phi_{i\sigma}(t)\delta_{ij})$ is a $N \times N$ matrix.

The eigen-functionals $\Psi_{k\omega\sigma}(t, [\phi])$ have the following general expression,

$$\Psi_{k\omega\sigma}(t, [\xi\phi]) = A_k \left(\frac{1}{TL^2}\right)^{1/2} \begin{pmatrix} e^{ik_x x_1}e^{Q_{k\sigma}(x_1, t, \xi)} \\ \vdots \\ e^{ik_x x_N}e^{Q_{k\sigma}(x_N, t, \xi)} \end{pmatrix} e^{-i(\omega + \Sigma_{k\sigma}[\phi])t}$$  \hspace{1cm} (73)

where $A_k$ is a normalization constant, and the phase fields $Q_{k\sigma}(x_i, t, \xi)$ satisfy the following differential equation,

$$i\partial_t Q_{k\sigma}(x_i, t, \xi) + \xi\phi_{i\sigma}(t) - \gamma_k(1 - e^{-Q_{k\sigma}(x_i, t, \xi)+Q_{k\sigma}(x_{i+1}, t, \xi)})$$

$$-\gamma_k^* (1 - e^{-Q_{k\sigma}(x_i, t, \xi)+Q_{k\sigma}(x_{i-1}, t, \xi)}) = 0$$  \hspace{1cm} (74)

where $\gamma_k = t(e^{iak_x} + e^{iak_y})$. Using the orthogonality and completeness of the eigen-functionals $\Psi_{k\omega\sigma}(t, [\xi\phi])$, the Green’s functionals can be written as,
\begin{equation}
G_{\sigma}(x_i, t; x_j, t', [\xi \phi]) = \frac{i}{L^2} \sum_k \theta(\Sigma_k[\phi] - \varepsilon(k)) |A_k|^2 e^{i k \cdot (x_i - x_j) - i \varepsilon(k)(t - t')} \times e^{Q_{k\sigma}(x_i, t, \xi) + Q_{k\sigma}^*(x_j, t', \xi)}
\end{equation}

then the action of the system reads,

\begin{equation}
S[\phi, n] = \frac{1}{2} \sum_{i\sigma} \int dt \left[ F_{\sigma}(x_i, x_i - \eta, t) + F_{\sigma}(x_i, x_i + \eta, t) \right]_{\eta \to 0^+} - \sum_{i\sigma} \int dt \left[ n_{i\sigma}(t) \phi_{i\sigma}(t) + \frac{1}{2} U n_{i\uparrow}(t) n_{i\downarrow}(t) \right] \tag{76}
\end{equation}

\begin{equation}
F_{\sigma}(x_i, x_j, t) = T \int_0^1 d\xi \sum_k \theta(\Sigma_k[\phi] - \varepsilon(k)) \frac{\phi(x_i, t) e^{i k \cdot (x_i - x_j)} e^{Q_{k\sigma}(x_i, t, \xi) + Q_{k\sigma}^*(x_j, t, \xi)}}{\sum_i \int dt e^{Q_{k\sigma}(x_i, t, \xi) + Q_{k\sigma}^*(x_j, t, \xi)}}
\end{equation}

It is clear that this action is very similar to that of the electron gas in the continuous coordinate space, the difference between them is that the phase fields \( Q_{k\sigma}(x_i, t, \xi) \) and \( Q_k(x, t, \xi) \) satisfy different differential equations (74) and (58), respectively. With equations (73), (74) and (76), we can completely determine the low energy behavior of the one-band Hubbard model on the square lattice, at least we can do that by numerical calculations, because we believe that with the help of these equations we may need much less computer time than previous methods for the large lattice site numbers.

Now there is a common consensus that the physical properties of the high Tc cuprate superconductivity is determined by the 2D strongly correlated electron gas in their copper-oxide plane(s) which can be approximately represented by the one-band Hubbard model \[7]. If this is true, our present results can provide enough informations to determine the physical properties of the high Tc cuprate superconductivity, because using equation (46), we can exactly determine their ground state energy and ground state electron density field, and using equations (73) and (76), we can calculate their variety of correlation functions and response functions to external fields.

IV. Ground state wave function of the systems

In principle, we can calculate the ground state wave function of any systems by using the eigen-functional. For boson systems, at zero temperature they have the Bose-Einstein
condensation, and the bosons occupy the state with the momentum \( k = 0 \), and have the
eigen-functional wave function \( \Psi_0(x, [\phi]) \). However, for fermion systems, the phase fields
may depend on the momentum \( k \) of the fermions, it becomes very difficult to calculate their
ground state wave functions. In general, the ground state wave function of the fermion
system can be written as,

\[
\Psi(x_1, x_2, ..., x_N) = < \| \Psi_{k_i}(x, [\phi]) \| >_\phi
\]

(77)

where \( k_N = k_F \) is the Fermi momentum, and the functional average over the HS boson field
\( \phi(x) \) can be done by taking \( Q_{k_i}(x_j) \equiv Q_{k_i}(x_j, t) \) and \( \phi(x_i) \equiv \phi(x_i, t) \). Here we only consider
the boson systems and the two-dimensional electron gas under an external magnetic field
there appears the fractional quantum Hall effect for enough strong magnetic field.

A. Boson systems

For boson systems, due to the condensation of the bosons, we have the following ground
state wave-function,

\[
\Psi(x_1, x_2, ..., x_N) = \left( \frac{A_0}{L^{D/2}} \right)^N < \Psi(x_1, x_2, ..., x_N, [\phi]) >_\phi
\]

\[
\Psi(x_1, x_2, ..., x_N, [\phi]) = e^{\sum_{i=1}^N Q_0(x_i, \xi=1)}
\]

(78)

where \( < ... >_\phi \) means the functional average over the boson field \( \phi(x) \). The functional
\( \Psi(x_1, x_2, ..., x_N, [\phi]) \) is very similar to the generalized London wave-function \([24]\), where
\( f(x_i) = \exp\{Q_0^R(x_i, \xi = 1)\} \) and \( S(x_i) = Q_0^I(x_i, \xi = 1) \). To calculate this functional average,
we need knowing the action of the systems. Without external fields and constraints, the
phase fields \( Q_0(x, t, \xi) \) satisfy the simple Eikonal equation with the condition \( Q_0(x, t, \xi) = 0 \)
as \( \phi(x, t) = 0 \),

\[
\left( i\hbar \frac{\partial}{\partial t} - \frac{\hat{p}^2}{2m} \right) Q_0(x, t, \xi) - \frac{[\hat{p}Q_0(x, t, \xi)]^2}{2m} + \xi \phi(x, t) = 0
\]

(79)

If we only keep the linear terms, we have the following simple solution,
\[ Q_0(x, t, \xi) = -\frac{\xi}{TLD} \sum_{q, \Omega} \phi(q, \Omega) \frac{\hbar \Omega}{\hbar \Omega - \epsilon_q} e^{iq \cdot x - i\Omega t} \]  

(80)

where \( \epsilon_q = \hbar^2 q^2 / (2m) \). Substituting it into (83), we obtain the following effective action with \( k = 0 \),

\[ S_0[\phi] = \frac{1}{TLD} \sum_{q, \Omega} \left( \frac{-N\epsilon_q}{(\hbar \Omega)^2 - \epsilon_q^2} + \frac{1}{2v(q)} \right) \phi(-q, -\Omega)\phi(q, \Omega) \]  

(81)

It is worthily noted that with this action we cannot obtain the correct excitation spectrum of the systems, because it is obtained by using the phase fields \( Q_0(x, t, \xi) \). In order to get the correct excitation spectrum of the bosons, we need knowing the action obtained by the phase fields \( Q_k(x, t, \xi) \). However, we can approximately use the action \( S_0[\phi] \) to calculate the functional average over the boson field \( \phi(x, t) \) in equation (78) by taking \( Q_0(x_i, \xi = 1) \equiv Q_0(x_i, t, \xi = 1) \). The detail expression form of the ground state wave function depends on the interaction potential \( v(q) \), but it can be generally written as,

\[ \Psi(x_1, x_2, ..., x_N) \sim e^{-\sum_{i<j} U(x_i - x_j)} \]  

(82)

where \( U(x_i - x_j) \) is uniquely determined by the interaction potential \( v(x_i - x_j) \). It is clearly seen that this ground state wave-function is uniquely determined by single effective potential function, and has the expression very similar to usual correlated basis functions [?] that are the type of wave-function most often employed in the study of the ground state properties of \(^4\)He.

B. Two-dimensional electron gas under an external magnetic field

The ground state wave function of this system cannot be written as a simple form even if for the lowest Landau level \( (n = 0) \), because the phase fields \( Q_{nl}(x, \xi = 1) \) depend on the quantum numbers \( n \) and \( l \). In the lowest Landau level, if we approximately take \( Q_{01}(x, \xi = 1) \sim Q_{00}(x) \), we can obtain the following expression of the ground state wave-function,
\[ \Psi(x_1, x_2, ..., x_N) = A \prod_{i=1}^{N} z_i \prod_{i>j}(z_i - z_j)e^{\sum_{i=1}^{N}|z_i|^2} < e^{\sum_{i=1}^{N} Q_{00}(x_i)} >_\phi \]  

\( (83) \)

where \( A \) is a normalization constant, and the factor \( \prod_{i>j}(z_i - z_j) \) guarantees the anti-commutation of the electrons. The last factor \( < \exp\{\sum_{i=1}^{N} Q_{00}(x_i)\} >_\phi \) is very similar to the ground state wave-function of the boson systems \( (82) \), and it is very clear that this factor is the contribution of the electron interactions. Using the procedure as above for the boson systems, we can obtain an approximate action by only keeping the linear terms in solving the differential equation of the phase field \( Q_{00}(x) \), then with this action we can calculate the functional average in \( (83) \). Only for long range Coulomb interaction \( v(q) = e^2/(4\pi q^2) \) can one have \( < \exp\{\sum_{i=1}^{N} Q_{00}(x_i)\} >_\phi \sim \prod_{i>j}|z_i - z_j|^\gamma \), where \( \gamma \) is a dimensionless constant, but at present approximation, it is hard to determine the relation between \( \gamma \) and the filling factor of the lowest Landau level. Thus this ground state wave function is different from the Laughlin’s trial wave-functions of the odd-denominator fractional quantum Hall states. However, we believe that with the equations of the ground state energy and the ground state wave function \( (18) \) and \( (83) \), we can obtain more important informations of the fractional quantum Hall effects than that by the Laughlin’s trial wave-functions, because they are directly derived from the microscopic theory.

V. Unified description of strongly and weakly correlated electron gases

In this section, we show that the strongly and weakly correlated electron gases can be unifiably represented under present theoretical framework. In fact, as shown in the section II and /or section III, the problems of the quantum many-particle systems end in to solve the differential equation of the phase fields \( Q_k(x, t, \xi) \). Thus the phase fields \( Q_k(x, t, \xi) \) are the key parameters hidden in the quantum many-particle systems, and completely determine their low energy behavior. It is natural that we can unifiably represent the quantum many-particle systems with the help of the phase fields \( Q_k(x, t, \xi) \).
It is simple to prove that the 1D interacting electron gas is a strongly correlated system even for very weak electron interaction $V \sim 0$. With the action (66), by simple calculation we can obtain the relations,

\[
\langle \psi_{R(L)}(x,t) \Psi^\dagger_{R(L)}(x', t, [\phi]) \rangle_\phi \sim \left( \frac{1}{L} \right)^{\alpha} e^{ik(x-x')}
\]

\[
\langle e^{iQ_{R(L)}(x,t)} e^{-iQ_{R(L)}(x', t)} \rangle_\phi \sim e^{-2\alpha \ln|x-x'|}, \quad |x - x'| \to \infty
\]

where $\alpha \sim (1 - V/(2\pi \hbar v_F))/2$ for $V \sim 0$, is the dimensionless coupling strength parameter, the $\psi_{R(L)}(x,t)$ are the wave-functions of the right(left)-moving free electrons, and $\langle \cdots \rangle_\phi$ means taking functional average over the HS fields $\phi_{R(L)}(x, t)$. The first equation presents the zero overlap between the eigen-functional wave functions $\Psi_{R(L)}(x, t, [\phi])$ of the interaction electrons and the wave functions of the free electrons ($V = 0$) as $L \to \infty$, thus the states of the interacting electron gas does not have one-to-one correspondence via adiabatic continuation with those of the free electron gas. The second equation presents the strong electron correlation, in which the electron correlation length is infinity even for weak electron interaction. Due to this strong electron correlation, the low energy excitation modes of the 1D interacting electron gas are those collective excitation modes, such as the charge and spin density waves, and there are not well-defined quasi-particles (holes) near the two Fermi levels $\pm k_F$. Thus the Tomonaga-Luttinger liquid theory is a strongly correlated theory, and is universal for 1D interacting electron gases.

For a 3D electron gas with long-range Coulomb interaction $v(q) = e^2/(4\pi q^2)$, by simple calculation, we have the relations,

\[
\langle \psi_{k_\omega}(x, t) \Psi^\dagger_{k_\omega}(x', t, [\phi]) \rangle_\phi \sim Z_{k} e^{ik(x-x')}
\]

\[
\langle e^{iQ_{I}(x,t)} e^{-iQ_{I}(x', t)} \rangle_\phi \sim e^{z_{I}(x-x')}
\]

where $Z_k$ is finite, and $z_{I}(x)$ is a smooth function. As $|x| \to \infty$, the function $z_{I}(x)$ goes to zero. The first equation presents that the eigen-functionals $\Psi_{k_\omega}(x, t, [\phi])$ have large overlap with the eigen-functions $\psi_{k_\omega}(x, t)$, and the second equation presents that there is only weak electron correlation even for long-range Coulomb interaction. Thus the fundamental
assumption of the Landau Fermi liquid theory is satisfied, and the Landau Fermi liquid theory is universal for 3D interacting electron gases.

For a 2D electron gas with the long-range Coulomb interaction \( v(q) = e^2/(4\pi q^2) \), the situation is different from that in the 3D electron gas. By simple calculation, we can obtain the relations,

\[
\begin{align*}
&\langle \psi_{k\omega}(x,t)\psi_{k\omega}^\dagger(x',t,[\phi]) \rangle_\phi \sim \left( \frac{1}{q_c L} \right)^\beta e^{ik(x-x')}
&\langle e^{iQ_k(x,t)}e^{-iQ_k(x',t)} \rangle_\phi \sim e^{-2\beta \ln(q_c|x-x'|)}, \quad |x-x'| \to \infty
\end{align*}
\]

where \( \beta = e^2/(2(4\pi)^2\omega_p) \) is the dimensionless coupling strength parameter, and \( \omega_p \) is the plasma frequency. In fact, \( q_c L \to \infty \), the first equation presents that the eigen-functionals \( \Psi_{k\omega}(x,t,[\phi]) \) has no (or infinitesimal) overlap with the eigen-functions \( \psi_{k\omega}(x,t) \), and the second equation presents the strong electron correlation. In this case, the fundamental assumption of the Landau Fermi liquid theory fails, and the system shows the non-Landau-Fermi liquid behavior. In the low energy limit, as \( \Omega > q \to 0 \), we have \( \chi(q,\Omega) \sim q^2/\Omega^2 \), then we obtain the propagator of the HS field \( \phi(x,t) \) by the action (67), \( G_\phi(q,\Omega) = v(q)/(1 - \omega_p^2/\Omega^2) \). Using this propagator to calculate the functional average over \( \phi(x,t) \), we obtain the results in (86). These results are consistent with that of Ref. [25] obtained by other method. The higher order terms neglected in solving equation (58) cannot alter these results because in low energy limit they are proportional to \( (q_c/k_F)^{2+n} \to 0, \quad n = 0, 1, 2, \ldots \), and can be safely neglected. It is worthy noted that for 2D and 3D electron gases with long-range Coulomb interactions we have the same \( G_\phi(q,\Omega) \) in which the screening effects are denoted by the factor \( 1/(1 - \omega_p^2/\Omega^2) \). However, the reason they show different low energy behavior is that the integral \( \int d^d q e^{iq\cdot x}/q^2 \) appearing in the functional average of equations (85) and (86) is divergent (log-type) for \( d = 2 \), while it is convergent for \( d = 3 \).

Even if the asymptotic behavior of the single electron Green’s function is similar to that of the 1D interacting electron gas (see, (84) and (86)), they originate from different physics. However, this difference can also be clearly seen from their effective actions of the density field. For 1D interacting electron gases, this action can be easily obtained by usual
bosonization method \[13,15\], where the density field has well-defined propagator (density wave). For the 2D electron gas with long-range Coulomb interaction, the effective action (87) can be written as with the density field (taking $A(x, t) = 0$),

$$S_{eff.}[\rho] = \frac{1}{2TL^2} \sum_{q, \Omega} \left( \frac{1}{\chi(q, \Omega)} - v(q) \right) |\rho(q, \Omega)|^2$$

(87)

There appears a gap in the excitation spectrum of the density field (plasmon excitations). It was shown \[26\] that as $e^2 \gg 4\pi$ the system can be described as a Landau Fermi liquid formed by chargeless quasi-particles which has vanishing wavefunction overlap with the bare electrons in the system. For short range and/or weak electron interaction, the 2D interacting electron gases would show the Landau Fermi liquid behavior in the low energy region.

Based on the above calculations, it becomes clear that if the non-Fermi liquid behavior of the electron gases is produced by the (long range) Coulomb interaction of the electrons, the correlation function of the imaginary phase fields $Q^I_k(x, t)$ must be a log-type. However, there exists another type correlation function of the imaginary phase fields $Q^I_k(x, t)$, which also induces the non-Fermi liquid behavior of the electron gases. Here we only consider a simple example of 2D electron gas with transverse gauge fields $A(x, t)$. Solving the differential equation of the phase fields $Q_k(x, t)$ (58) where only keeps the linear terms, and using the effective action (87), we can obtain single electron Green’s function,

$$G(x - x', t - t') = i < G(x, t; x', t'; [\phi]) >_A$$

$$= - \frac{1}{L^2} \sum_k \theta(-\varepsilon(k)) e^{ik(x-x')-i\varepsilon(k)(t-t')} e^{P^I_k(x-x', t-t')}$$

(88)

where $P^I_k(x, t)$ comes from the contribution of the imaginary phase fields $Q^I_k(x, t)$, and the contribution from the real phase fields $Q^R_k(x, t)$ can be neglected. Near the Fermi surface $k \sim k_F$, we have the relation,

$$P^I_{k_F}(0, t) \simeq -i^{1/3} a g^2 t^{1/3}$$

(89)

where $a = \frac{k_F}{4\pi m}(\chi/\gamma)^{1/3} \int dx (1 - \cos(x)) x^{-4/3}$. This result is basic same as that of Ref. \[27,28\]. Due to the power-law time dependence of the phase factor $P^I_{k_F}(0, t)$, the single
electron Green’s function shows a singular low energy dependence which violates the quasiparticle excitation assumption of the Landau Fermi liquid. Thus, the system shows non-Fermi liquid behavior. In this case, the correlation function of the imaginary phase fields $Q^I_k(x,t)$ has the power-law form $t^\alpha$, $\alpha < 1$, near the Fermi surface. For a 2D electron gas with long range Coulomb interaction and transverse gauge fields, the correlation function of the imaginary phase fields $Q^I_k(x,t)$ is composed of two parts, one is a log-type which comes from the contribution of the HS boson field $\phi(x,t)$, and another one has the power-law form $t^\alpha$, $\alpha < 1$, near the Fermi surface, which is from the contribution of the transverse gauge fields $A(x,t)$. Thus this system shows the non-Fermi liquid behavior in the low energy region.

VI. Discussion and conclusion

Applying previous perturbation methods to study strongly correlated systems, we may meet many serious problems, because in these systems due to the strong correlation among fermions/bosons, there does not exist a small quantity to be used as a perturbation expansion parameter. The most prominent character of the strongly correlated systems is the strong fermion/boson correlation, which completely controls their low energy behavior. This strong fermion/boson correlation is produced by the fermion/boson interactions, thus the key point of this kind problems is how to exactly or accurately treat the particle interaction (potential) terms. However, the traditional treatment of these systems is starting from a bare particle Green’s function to perturbatively treat the particle interaction (potential) terms, i.e., exactly treating the kinetic energy term and perturbatively treating the potential terms. Explicitly, this treatment is unsuccessful for the strongly correlated systems.

It is natural that one uses another way to treat the strongly correlated systems, i.e., exactly treating the interaction (potential) terms and exactly or perturbatively treating the kinetic energy term of the particles, such as usual bosonization method in treating 1D interacting fermion systems. Only in this way can one effectively treat the strong particle
correlation. However, to realize this idea, we may meet two problems, one is how to de-
couple four-particle interaction (potential) term, and another one is how to represent the
particle correlation strength. The former one can be done by introducing Lagrangian mul-
tiplier or Hubbard-Stratonovich boson fields, and the latter one can be naturally done by
introducing phase fields in solving the eigen-functional equation of the propagator operator
of the particles. It must be mentioned that after introducing the Lagrangian multiplier or
Hubbard-Stratonovich boson fields, the Hilbert space of the system is enlarged, and the
unphysical particle density field also appears in the Hamiltonian and/or action. To delete
these unphysical states or leave the original Hilbert space intact, we can add a constraint
condition to the functional ground state energy, such as $\delta E[\phi, \rho]/\delta \phi(x) = 0$, or take the
functional average over the Lagrangian multiplier $\phi(x, t)$.

Based on the above idea, we have established an unified theory of the quantum many-
particle systems, which is valid not only for weakly correlated fermion/boson systems, but
also for strongly correlated fermion/boson systems. This unified theory of the quantum
many-particle systems has three prominent characters, (1). it is completely founded on
the basic principles of quantum mechanics. (2). it has well-defined phase fields used to
represent the particle correlation strength. (3). it has an explicit and simple exact functional
expression of the ground state energy, the action and the eigen-functional wave functions of
the particles, and one can easily use them to effectively (or exactly) treat general quantum
many-particle problems, and can obtain any accurate result as one hoped by taking high
order corrections. In this unified theory, we have shown that the problems of the quantum
many-particle systems end in solving the differential equation of the phase fields $Q_k(x, \xi)$
and/or $Q_k(x, t, \xi)$, and given the exact expression of the ground state energy and action,
and the eigen-functional wave functions of the particles as the functional of the HS boson
fields (and/or transverse gauge fields) and the particle density fields. With them, we can
calculate a variety of correlation functions of the systems. It is surprising but natural that the
physical properties of the many-particle systems is completely controlled by the differential
equation of the phase fields $Q_k(x, t, \xi)$. The above results have been easily applied to the
systems under external fields and/or constraint conditions, such as trapped boson systems and 2D electron gas under the external magnetic field, and to the lattice models, such as the Heisenberg model and the one-band Hubbard model on the square lattice.

The key points of this unified theory are that: (1). we introduce the Lagrangian multiplier (or HS) field $\phi(x, t)$ which takes the particle density $\rho(x, t) = \psi^\dagger(x, t)\psi(x, t)$ as a constraint condition, so that the four-particle interaction term is decoupled and the action can only have the quadratic form of the particle (fermion/boson) fields. (2). by introducing the phase field $Q_k(x, t, \xi)$, which is a functional of the boson field $\phi(x, t)$ and determined by usual Eikonal-type equation, we can use it to completely represent the ”kinetic” energy of the systems. Thus the problems of the quantum many-particle systems end in to solve the differential equation of the phase fields $Q_k(x, t, \xi)$. (3). the phase field $Q_k(x, t, \xi)$ is a key parameter hidden in the quantum many-particle systems, its imaginary part represents the particle correlation strength, and its real part only contributes to the ground state energy and action. (4). we are able to use the particle density field $\rho(x)$ and $\rho(x, t)$ to exactly represent the ground state energy and action of the quantum many-particle systems, respectively.

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