Rate region boundary of the Z-interference channel with improper signaling

Christian Lameiro*, Member, IEEE, Ignacio Santamaría†, Senior Member, IEEE, and Peter J. Schreier*, Senior Member, IEEE

Abstract

This paper provides a complete characterization of the boundary of the achievable rate region, also called the Pareto boundary, of the Z interference channel (Z-IC), when interference is treated as noise and users employ an improper Gaussian signaling scheme. By considering the augmented complex formulation, we describe the Pareto boundary by three interference regimes, for which different closed-form expressions for the optimal transmit power and circularity coefficient (i.e., degree of impropriety) are provided. These regimes are defined through thresholds on the ratio between the gain of the interfering channel and the gain of the direct channel of the interfering user. The simplicity of the obtained characterization permits drawing interesting insights into when and how improper signaling outperforms proper signaling in the single-antenna Z-IC. We provide an in-depth discussion on the interference regimes and the relationship between the degrees of impropriety of both users.

Keywords

Improper signaling, Z interference channel, Pareto boundary.

I. INTRODUCTION

It is widely known that proper Gaussian signals are capacity-achieving in different wireless communication networks, such as the point-to-point, broadcast and multiple-access channels. Because of that, the use of such a signaling scheme is generally assumed in the study of

* C. Lameiro and P. J. Schreier are with the Signal & System Theory Group, Universität Paderborn, Germany (email: {christian.lameiro, peter.schreier}@sst.upb.de).
† I. Santamaría is with the Department of Communications Engineering, University of Cantabria, Spain (e-mail: nacho@gtas.dicom.unican.es).
multiuser wireless networks. The capacity-achieving property of proper signaling stems from the maximum entropy theorem, which states that the entropy of a random variable is maximized for a proper Gaussian distribution [1]. However, in networks where interference presents the major limiting factor, proper Gaussian signaling has recently been shown to be suboptimal. Specifically, improper Gaussian signaling, also known as asymmetric complex signaling, has been proved to outperform proper signaling in different interference networks [2]–[18].

An improper complex random variable differs from its proper counterpart in that its real and imaginary parts are correlated or have unequal variance, or, in other words, the random variable is correlated with its complex conjugate [19]. Such signals arise naturally in communications, e.g., due to gain imbalance between the in-phase and in-quadrature branches, or due to the use of specific digital modulations, such as binary phase shift keying (BPSK) or Gaussian minimum shift keying (GMSK). Whenever the receive signal is improper, a widely linear operation, which is linear in both the random variable and its complex conjugate, must be performed in order to fully exploit the correlation between the signal and its complex conjugate [19]–[21]. The design of such widely linear receivers has been extensively studied in the literature (see, e.g., [22]–[25] and references therein). However, the transmission of improper signals to handle interference more effectively is a rather new line of research.

The first study on the benefits of improper signaling for interference management was carried out in the 3-user interference channel (IC) [2]. That work showed an improvement in terms of degrees-of-freedom (DoF), which represent the maximum number of interference-free streams and characterize the asymptotic sum-capacity. Similar DoF results were derived for the 4-user IC [3], the 3-user multiple-input multiple-output (MIMO) IC [4], the interfering broadcast channel [5], and the MIMO X-channel [6]. Nevertheless, the performance improvements due to improper signaling are not only linked to an increase in the achievable DoF, but also to a general improvement in the achievable rates in interference-limited networks. The optimal rate region boundary for maximally improper transmissions (i.e., full correlation or power imbalance between real and imaginary parts) was derived for the 2-user IC in [7], showing substantial improvements over proper signaling. Additionally, [8] proposed a suboptimal design of the improper transmit parameters, which outperforms the proper and the maximally improper scheme. A similar suboptimal design was also proposed in [9] for the $K$-user multiple-input single-output (MISO) IC. Improper signaling in the IC has also been applied to reduce the symbol error rate
[10], and as a mixed improper/proper approach in the MIMO-IC [11]. In addition to the IC, the use of improper Gaussian signaling has also been shown beneficial for other multiuser scenarios, such as the broadcast channel with linear precoding [12], relay-assisted communications [13], or underlay cognitive radio networks [14]–[16].

A particular case of the 2-user IC is the Z-IC, also known as one-sided IC [26]. The difference with respect to the 2-user IC is the fact that only one of the receivers is affected by interference. The capacity region of the Z-IC is only known in the strong and very strong interference regimes [26], [27], and it is achievable by a non-linear operation at the receiver. Nevertheless, it is more convenient from a practical viewpoint to perform linear (or widely-linear) operations while treating the interference as noise, so that the complexity can be reduced. Restricted to linear operations, improper signaling presents a useful tool to improve the performance over proper signaling. Improper signaling for the Z-IC has recently been considered in [17], where the sum-rate maximizing scheme is derived in closed form. To that end, [17] considered the so-called real-composite model, where complex signals are regarded as real signals of double dimension. However, despite some remarkable efforts [28], this model is still not as insightful as the augmented complex model, which works with the signal and its complex conjugate. For example, the circularity coefficient, which measures the degree of impropriety, is a quantity easily derived in the augmented complex formulation, but is much more difficult to express through its real-composite counterpart.

In this work, we adopt the augmented complex model to provide a complete and insightful characterization of the optimal rate region boundary, called the Pareto boundary, of the Z-IC, when users may transmit improper Gaussian signals, assuming that interference is treated as noise. Our main contributions are summarized next.

- We extend the results of [17], where only one point of the rate region boundary is derived, to provide a complete characterization of the Pareto optimal boundary in closed form. We show that the rate region boundary can be described by three interference regimes, depending on the ratio between the gain of the interfering link and that of the direct link of the interfering user. We denote these regimes as strong, moderate and weak interference, which slightly differ from their information-theoretic definition [27], [29], and account for the behavior of the improper signaling scheme depending on the relative level of interference.
By adopting the augmented complex formulation, we provide, for each point of the boundary, closed-form expressions for the transmit powers and circularity coefficients, which are a direct measure of the degree of impropriety of the transmit signals. This permits insightful conclusions and a full assessment of the improvements of improper signaling over proper signaling in the single-antenna Z-IC. Thus, we analyze how the degree of impropriety affects the rate in the different interference regimes, and we investigate the conditions that must be fulfilled for improper signaling to outperform proper signaling. The connection between the optimal circularity coefficients of both users and a further in-depth discussion of our characterization is also provided.

The rest of the paper is organized as follows. Section II provides some preliminaries of improper random variables and describes the system model. The characterization of the rate region boundary is derived in Section III and a discussion on the results is presented in Section IV along with several numerical examples illustrating our findings. Finally, Section V concludes the paper.

II. System Model

A. Preliminaries of improper complex random variables

In this section we provide some definitions and results for improper random variables that will be used throughout the paper. For a comprehensive treatment of the subject, we refer the reader to [19].

The complementary variance of a zero-mean complex random variable $x$ is defined as $\tilde{\sigma}^2 = E[x^2]$, where $E[\cdot]$ is the expectation operator. If $\tilde{\sigma}^2 = 0$, then $x$ is called proper, otherwise improper. Furthermore, $\sigma^2$ and $\tilde{\sigma}^2$ are a valid pair of variance and complementary variance if and only if $\sigma^2 \geq 0$ and $|\tilde{\sigma}^2| \leq \sigma^2$.

The circularity coefficient of a complex random variable $x$ is defined as the absolute value of the quotient of its complementary variance and its variance, i.e.,

$$\kappa = \frac{|\tilde{\sigma}^2|}{\sigma^2}. \quad (1)$$

The circularity coefficient satisfies $0 \leq \kappa \leq 1$. If $\kappa = 0$, then $x$ is proper, otherwise improper. If $\kappa = 1$ we call $x$ maximally improper.
B. System description

We consider the Z-IC with single-antenna users and no symbols extensions. Denoting by $h_{ij}$ the channel response between transmitter $j$ and receiver $i$, the signal at each receiver can be modeled by

$$y_1 = h_{11}s_1 + h_{12}s_2 + n_1,$$

$$y_2 = h_{22}s_2 + n_2,$$

where $s_i$ and $n_i$ are the transmitted signal and noise of the $i$th user, respectively, as depicted in Fig. 1. The additive white Gaussian noise (AWGN) has variance $\sigma^2$, whereas the transmitted signals are assumed to be distributed as improper Gaussian random variables with variance $E[|s_i|^2] = p_i$ and complementary variance $E[s_i^2] = \tilde{p}_i$. Thus, the rate achieved by each user, as a function of the design parameters $p_i$ and $\tilde{p}_i$, $i = 1, 2$, is given by [8]

$$R_1 (p_1, \tilde{p}_1, p_2, \tilde{p}_2) = \log_2 \left( 1 + \frac{p_1|h_{11}|^2}{\sigma^2 + p_2|h_{12}|^2} \right) + \frac{1}{2} \log_2 \left( 1 - |c_{y_1}|^{-2} |\tilde{c}_{y_1}|^2 \right)$$

$$R_2 (p_1, \tilde{p}_1, p_2, \tilde{p}_2) = \log_2 \left( 1 + \frac{p_2|h_{22}|^2}{\sigma^2} \right) + \frac{1}{2} \log_2 \left( 1 - |c_{y_2}|^{-2} |\tilde{c}_{y_2}|^2 \right),$$
where

\[ c_{y1} = p_1 |h_{11}|^2 + p_2 |h_{12}|^2 + \sigma^2, \]  
\[ c_{y2} = p_2 |h_{22}|^2 + \sigma^2, \]  
\[ c_{y1} = \tilde{p}_1 h_{11}^2 + \tilde{p}_2 h_{12}^2, \]  
\[ c_{y2} = \tilde{p}_2 h_{22}^2, \]  
\[ c_{z1} = p_2 |h_{12}|^2 + \sigma^2, \]  
\[ c_{z2} = \sigma^2, \]  
\[ \tilde{c}_{z1} = \tilde{p}_2 h_{12}^2, \]  
\[ \tilde{c}_{z2} = 0, \]

are the variances of the received signals,

are the complementary variances of the received signals,

are the variances of the interference-plus-noise signals, and

the complementary variances of the interference-plus-noise signals. Notice that the rate of user 2
does not directly depend on \( p_1 \) and \( \tilde{p}_1 \), but implicitly through their impact on the optimal value
of \( p_2 \) and \( \tilde{p}_2 \). Because of that and in order to keep an homogeneous notation, we express \( R_2 \)
as a function of the four parameters in (5). Assuming that the power budget of the \( i \)th user is \( P_i \),
the achievable rate region with improper Gaussian signaling is then the union of all achievable
rate tuples, i.e.,

\[ \mathcal{R} = \bigcup_{0 \leq p_i \leq P_i, |\tilde{p}_i| \leq p_i} (R_1(p_1, \tilde{p}_1, p_2, \tilde{p_2}), R_2(p_1, \tilde{p}_1, p_2, \tilde{p_2})) . \]  

III. Pareto Boundary of the Rate Region

The boundary of the rate region described by (14), which is called the Pareto boundary,
is comprised of all rate pairs \((R_1, R_2)\) such that \((R'_1, R_2)\) and \((R_1, R'_2)\), with \( R'_1 > R_1 \) and
\( R'_2 > R_2 \), are not achievable. In this section we characterize this boundary by deriving the
optimal transmission parameters, \( p_i \) and \( \tilde{p}_i \), \( i = 1, 2 \), that achieve each point of the boundary.

\[ ^1 \text{For the sake of clarity, we omit the dependence of } R_1 \text{ and } R_2 \text{ on the design parameters when it is self-evident or not relevant.} \]
First, we notice that, since user 1 does not interfere with user 2, its optimal transmit strategy maximizes its own achievable rate. Consequently, its transmit power must be maximized, which implies $p_1 = P_1$. Second, according to [19], $p_i$ and $\tilde{p}_i$ are a valid pair of variance and complementary variance if and only if $p_i \geq 0$ and $|\tilde{p}_i| \leq p_i$. Consequently, the complementary variance can be expressed as $\tilde{p}_i = p_i \kappa_i e^{j\phi_i}$, where $\kappa_i$ is the circularity coefficient, which measures the degree of impropriety. Hence, $|\tilde{p}_i| \leq p_i$ is equivalent to $0 \leq \kappa_i \leq 1$. With these considerations, $R_1$ can then be expressed as

$$R_1 = \frac{1}{2} \log_2 \left[ \frac{(p_2|h_{12}|^2 + P_1|h_{11}|^2 + \sigma^2)^2}{(p_2|h_{12}|^2 + \sigma^2)^2 - |p_2e^{j\phi_2}\kappa_2 h_{121}|^2} \right] - \frac{|p_2e^{j\phi_2}\kappa_2 h_{121}^* + P_1e^{j\phi_1}\kappa_1 h_{111}|^2}{(p_2|h_{12}|^2 + \sigma^2)^2 - |p_2e^{j\phi_2}\kappa_2 h_{121}|^2} \right].$$

(15)

Through (15) it is clear that $R_1$ is maximized when $|p_2e^{j\phi_2}\kappa_2 h_{121}^* + P_1e^{j\phi_1}\kappa_1 h_{111}|^2$ is minimized, which yields

$$\kappa_1 = \min \left( \frac{p_2h_{121}^*}{P_1|h_{111}|^2}, 1 \right),$$

(16)

$$\phi_1 = \arg \left[ (h_{111}^*)^2 h_{121} e^{j\phi_2} \right] + \pi.$$  

(17)

From (16) we observe that, if $\kappa_2 = 0$, i.e., user 2 transmits a proper signal, then user 1 must also transmit a proper signal by setting $\kappa_1 = 0$. Similarly, if user 2 transmits an improper signal ($\kappa_2 > 0$), then the signal transmitted by user 1 must also be improper. According to (17), the difference between the phases of the complementary variances of the desired and interference signals at receiver 1 is $\pi$. Such a phase difference can be interpreted by looking at the joint distribution of the real and imaginary parts of the desired signal and interference at receiver 1. The contours of their distribution are ellipses whose major axes are rotated by $\pi/2$ with respect to each other [19], so that the signal and interference power are concentrated along orthogonal dimensions.

Now we observe the following. With the optimal choice of $\phi_1$, given by (17), the effect of $\phi_2$ is compensated at receiver 1, thus the achievable rate of user 1 is independent of the specific value of $\phi_2$. Furthermore, since user 2 is not affected by interference, $\phi_2$ has also no impact on its achievable rate. Hence, without loss of generality, we can take $\phi_2 = 0$. With these considerations, the design parameters are reduced to the transmit power and circularity coefficient of user 2, $p_2$. 


where $R_1(p_2, \kappa_2) = \left\{ \begin{array}{ll} \frac{1}{2} \log_2 \left[ \frac{(p_2|h_{21}|^2 + p_1|h_{31}|^2 + \kappa_2)^2}{1 + p_2|h_{21}|^2 \left( \frac{p_2|h_{21}|^2}{\sigma^2} (1 - \kappa_2^2) + 2 \right)} \right] & \text{if } \kappa_1 < 1 \\ \frac{1}{2} \log_2 \left[ 1 + \frac{2 p_2|h_{21}|^2}{\sigma^2} \left( \frac{p_2|h_{21}|^2}{\sigma^2} (1 + \kappa_2) + 1 \right) \right] & \text{if } \kappa_1 = 1 \end{array} \right.$, \hspace{1cm} \text{(18)}$

and $R_2(p_2, \kappa_2) = \frac{1}{2} \log_2 \left[ 1 + \frac{p_2|h_{22}|^2}{\sigma^2} \left( \frac{p_2|h_{22}|^2}{\sigma^2} (1 - \kappa_2^2) + 2 \right) \right]$. \hspace{1cm} \text{(19)}$

Fig. 2. Example of the transmit power and power constraints of user 2, for $P_1 = 20$, $P_2 = 10$, $\sigma^2 = 1$, $\alpha = 0.7$, $h_{11} = 1$ and $h_{12} = 0.5$. The shaded area is the set of possible transmit powers.

and $\kappa_2$, respectively. After some manipulations of (15) and (5), the achievable rates of user 1 and user 2, as a function of the design parameters, are respectively given by (18) and (19), and the achievable rate region defined in (14) can then be expressed as

$$\mathcal{R} = \bigcup_{0 \leq p_2 \leq P_2, \ 0 \leq \kappa_2 \leq 1} (R_1(p_2, \kappa_2), R_2(p_2, \kappa_2)) \hspace{1cm} \text{(20)}$$
In order to characterize the boundary of the region defined in (20), we notice that the achievable rate of user 1 is bounded as

\[ 0 \leq R_1(p_2, \kappa_2) \leq \log_2 \left( 1 + \frac{P_1|h_11|^2}{\sigma^2} \right) . \]

(21)

Hence, for each achievable rate of user 1, the corresponding Pareto optimal point is given by the one maximizing the rate of user 2, \( R_2(p_2, \kappa_2) \). It is worth pointing out that the first user can only approach the lower bound in the left-hand side of (21) by reducing its transmit power. That is, once the second user chooses its transmit parameters as \( p_2 = P_2 \) and \( \kappa_2 = 0 \) (and thus it transmits at its maximum rate) we have that \( R_1(P_2, 0) = R_1^{\text{min}} > 0 \). \( R_1^{\text{min}} \) corresponds to the minimum achievable rate of the first user when \( p_1 = P_1 \). Therefore, in order to obtain all the other boundary points between \( R_1 = R_1^{\text{min}} \) and \( R_1 = 0 \), the transmit power of the first user must be reduced, which does not have any impact on the rate achieved by user 2 (since it is already transmitting at its maximum rate). With this remark in mind we can safely continue with our initial assumption \( p_1 = P_1 \) and cast the computation of the Pareto optimal points as following optimization problem

\[ \mathcal{P} : \begin{array}{ll}
\text{maximize} & R_2(p_2, \kappa_2) , \\
\text{subject to} & 0 \leq p_2 \leq P_2 , \\
& 0 \leq \kappa_2 \leq 1 , \\
& R_1(p_2, \kappa_2) \geq \alpha \log_2 \left( 1 + \frac{P_1|h_11|^2}{\sigma^2} \right) ,
\end{array} \]

(22)

for a given \( \alpha \in [0, 1] \). Thus, we can compute every point of the rate region boundary by varying \( \alpha \) between 0 and 1 and solving problem \( \mathcal{P} \).

The set of constraints of problem \( \mathcal{P} \), which defines the feasibility set of our design parameters, is comprised of two constraints affecting the design parameters independently, namely, the power budget constraint and the bounds on the circularity coefficient, and an additional one that jointly constrains \( p_2 \) and \( \kappa_2 \). The latter expresses a rate constraint on user 1, so that a specific point of the region boundary, determined by \( \alpha \), is computed. For a given \( \kappa_2 \), this constraint essentially limits the transmit power of user 2, \( p_2 \). Consequently, we can rewrite it in a more convenient form as

\[ R_1(p_2, \kappa_2) \geq \alpha \log_2 \left( 1 + \frac{P_1|h_11|^2}{\sigma^2} \right) \Leftrightarrow p_2 \leq q(\kappa_2) , \]

(23)
\[
q(\kappa_2) = \begin{cases} 
\frac{\sigma^2}{|h_{12}|^2 (1 - \frac{\gamma_2 R + 1}{\gamma_2 R})} \left\{ \gamma_2 R \left( \frac{\gamma_2 R}{\gamma_2 R} - 1 \right) - 1 + \left[ \frac{\gamma_2 R}{\gamma_2 R} - 1 \right]^2 + \left( 1 - \frac{\gamma_2 R + 1}{\gamma_2 R} \right) \frac{(\gamma + 1)^2 - (\gamma_2 R + 1)}{\gamma_2 R} \right\}^{\frac{1}{2}} & \text{if } \kappa_1 < 1 \\
\frac{\sigma^2}{|h_{12}|^2 (1 - \kappa_2^2)} \left\{ \gamma_2 R (1 + \kappa_2) - 1 + \left[ 1 - \gamma_2 R (1 + \kappa_2) \right]^2 - (1 - \kappa_2^2) \frac{\gamma_2 R - 2\gamma}{\gamma_2 R} \right\}^{\frac{1}{2}} & \text{if } \kappa_1 = 1 
\end{cases}
\]

with \( q(\kappa_2) \) given by (24), where we have defined \( \gamma = \frac{P_1 |h_{11}|^2}{\sigma^2} \), \( \gamma_x = 2^x - 1 \) and \( \bar{R} = \alpha \log_2 \left( 1 + \frac{P_1 |h_{11}|^2}{\sigma^2} \right) \).

As a result, we can equivalently state problem \( \mathcal{P} \) as

\[
\mathcal{P}: \text{ maximize } R_2 (p_2, \kappa_2) ,
\]
subject to

\[
0 \leq p_2 \leq \min \left[ q(\kappa_2), P_2 \right] ,
\]
\[
0 \leq \kappa_2 \leq 1 .
\]

For the sake of illustration, we plot in Fig. 2 an example of the different constraints affecting the transmit power of user 2, namely, \( q(\kappa_2) \) and \( P_2 \). Obviously, \( q(\kappa_2) \) is increasing in \( \kappa_2 \), since an interference with a higher degree of impropriety is less harmful, hence user 1 tolerates a higher amount of interference power without reducing its achievable rate. To achieve the global maximum of problem \( \mathcal{P} \), it is clear that we must set \( p_2(\kappa_2) = \min[q(\kappa_2), P_2] \), where we have explicitly expressed its dependence on \( \kappa_2 \). Consequently, the number of design parameters is reduced to one, \( \kappa_2 \), and \( \mathcal{P} \) is further simplified to

\[
\mathcal{P}: \text{ maximize } R_2 (\kappa_2) ,
\]
subject to

\[
0 \leq \kappa_2 \leq 1 ,
\]

where \( R_2(\kappa_2) \) can now be expressed as

\[
R_2 (\kappa_2) = \frac{1}{2} \log_2 \left\{ 1 + \frac{p_2(\kappa_2) |h_{22}|^2}{\sigma^2} \times \left[ \frac{p_2(\kappa_2) |h_{22}|^2}{\sigma^2} (1 - \kappa_2^2) + 2 \right] \right\} .
\]

(25)

Notice that, by expressing \( p_2 \) as a function of \( \kappa_2 \), \( R_2(\kappa_2) \) is now also a function of \( \kappa_2 \) only. That is, the key task now is to determine the optimal circularity coefficient of the second user, or, in other words, the degree of impropriety of its transmit signal such that its achievable rate, given by (25), is maximized.
In the forthcoming lines we will analyze when $R_2(\kappa_2)$ is maximized by a strictly improper signal, i.e., $\kappa_2 > 0$, and the optimal value of $\kappa_2$ in those cases. That is, we want to determine the conditions that must be fulfilled for improper signaling to outperform conventional proper signaling. We start by presenting the following lemma, which will be useful for deriving our main result.

**Lemma 1.** Let $q(1) \leq P_2$, and assume that there exists $\kappa_0$ such that $\frac{\partial R_2(\kappa_2)}{\partial \kappa_2} \bigg|_{\kappa_2=\kappa_0} > 0$ for all $\kappa_2 > \kappa_0$.

**Proof:** Please refer to Appendix A.

Lemma 1 leads to the following key result.

**Lemma 2.** Let $q(1) \leq P_2$. Then, there exists $\kappa_0$ such that $R_2(\kappa_2) \geq R_2(0)$ for all $\kappa_2 \geq \kappa_0$ if and only if

$$\frac{|h_{12}|^2}{|h_{22}|^2} \geq \frac{(\gamma_R - \gamma)^2}{(\gamma_{2R} - \gamma_R)(\sqrt{\gamma_{2R} - 2\gamma} - \gamma_R)^2}.$$ \hspace{1cm} (26)

Furthermore, if (26) holds, $R_2(\kappa_2)$ increases monotonically in the interval $[\kappa_0, 1]$.

**Proof:** As in the proof of Lemma 1 we have to consider the case $\kappa_1 < 1$. By setting the condition $R_2(1) \geq R_2(0)$ and solving for $\frac{|h_{12}|^2}{|h_{22}|^2}$, we obtain that the rate achieved by maximally improper signaling is equal to or greater than that of proper signaling if and only if (26) holds. By Lemma 1 we also know that, if improper signaling is beneficial for $\kappa_2 = \kappa_0$, then the rate improvement will be strictly positive for $\kappa_2 > \kappa_0$. Therefore, when (26) holds, there must be $\kappa_0$ such that $R_2(\kappa_0) = R_2(0)$ and $R_2(\kappa_2) > R_2(0)$ for all $\kappa_2 > \kappa_0$, which concludes the proof.

Lemma 2 determines a threshold that defines the regime under which improper signaling is optimal provided that the power budget is sufficiently high. That is, if $q(1) \leq P_2$, maximally improper signaling improves the rate compared to proper signaling if and only if (26) holds. However, if the power budget constraint is active, i.e., $q(1) > P_2$, condition (26) does not ensure the optimality of improper signaling since the achievable rate does not increase monotonically for $\kappa_2 > 0$ but only for $\kappa_2 > \kappa_0$. Another interesting point to remark is the fact that condition (26) depends on $\sqrt{\gamma_{2R} - 2\gamma}$. This threshold would then have a non-negative imaginary part if $\gamma_{2R} < 2\gamma$, which corresponds to $\tilde{R} < \frac{1}{2} \log_2(1 + 2\frac{P_1|h_{11}|^2}{\sigma^2})$, i.e., when user 1 can achieve $\tilde{R}$ with a maximally improper signal. Nevertheless, such a case violates condition $q(1) \leq P_2$, so
that Lemma 2 is not applicable in this case making the threshold in the right-hand side of (26) well-defined. The aforementioned condition is not fulfilled since the first user tolerates an infinite amount of a maximally improper interference along the orthogonal direction for $\gamma_2 R < 2 \gamma$. That is, $\lim_{\kappa_2 \to 1} q(\kappa_2) = \infty$, which clearly violates $q(1) \leq P_2$. This intuition permits in fact dividing the boundary into two regions, as described in the next lemma.

**Lemma 3.** The boundary of the rate region defined in (20) can be described by the following two regions:

- **Power-limited region:** When $2 \gamma \geq \gamma_2 R$, the rate of user 2 is asymptotically limited by its power budget, i.e.,
  \[
  \lim_{P_2 \to \infty} \max_{0 \leq \kappa_2 \leq 1} R_2(\kappa_2) = \infty ,
  \]  
  where $R_2(\kappa_2)$ is given by (19).

- **Interference-limited region:** When $2 \gamma < \gamma_2 R$, the rate of user 2 is asymptotically limited by interference, i.e.,
  \[
  \lim_{P_2 \to \infty} \max_{0 \leq \kappa_2 \leq 1} R_2(\kappa_2) < \infty .
  \]  

**Proof:** In the power-limited region we have $2 \gamma \geq \gamma_2 R$, which implies $\bar{R} \leq \frac{1}{2} \log_2(1 + 2 \frac{P_1 |h_{11}|^2}{\sigma^2})$. In words, this means that user 1 can achieve its rate with a maximally improper signal, i.e., with $\kappa_1 = 1$. Thus, the interference along the unused dimension does not have any impact on its achievable rate, which yields (27). When $2 \gamma < \gamma_2 R$, user 1 must choose a circularity coefficient strictly lower than one to achieve the desired rate, and hence the tolerated interference is finite for all values of $\kappa_2$. As a result, the transmit power of the second user is eventually limited by $q(\kappa_2)$ as $P_2$ grows, thus yielding (28) and concluding the proof.

With all these ingredients, we can derive a complete characterization of the optimality of improper signaling for this scenario, and, consequently, of the Pareto optimal region. Our main result is formally presented in the following theorem.

**Theorem 1.** Let $q(0) < P_2$ and

\[
\mu_1(\alpha) = 1 - \frac{\gamma}{\gamma_2 R - \gamma},
\]

\[
\mu_2(\alpha) = \frac{(\gamma_R - \gamma)^2}{(\gamma_2 R - \gamma_R)(\sqrt{\gamma_2 R - 2 \gamma} - \gamma_R)^2}.
\]
The optimal circularity coefficient, \( \kappa^*_2 \), that maximizes \( R_2(\kappa_2) \), can be characterized in terms of \( \frac{|h_{12}|^2}{|h_{22}|^2} \) by the following operation regimes:

**Interference-limited region:**

- **Strong interference.** If \( \frac{|h_{12}|^2}{|h_{22}|^2} \geq \mu_1(\alpha) \):
  \[
  \kappa^*_2 = \begin{cases} 
  1 & \text{if } q(1) \leq P_2 \\
  \kappa_{\text{max}} & \text{otherwise}
  \end{cases},
  \tag{31}
  \]
  where \( \kappa_{\text{max}} \) satisfies \( q(\kappa_{\text{max}}) = P_2 \).

- **Moderate interference.** If \( \mu_1(\alpha) > \frac{|h_{12}|^2}{|h_{22}|^2} > \mu_2(\alpha) \):
  \[
  \kappa^*_2 = \begin{cases} 
  1 & \text{if } q(1) \leq P_2 \\
  \kappa_{\text{max}} & \text{if } \kappa_{\text{max}} > \kappa_0 \\
  0 & \text{otherwise}
  \end{cases},
  \tag{32}
  \]
  where \( \kappa_{\text{max}} \) and \( \kappa_0 \) satisfy \( q(\kappa_{\text{max}}) = P_2 \) and \( R_2(\kappa_0) = R_2(0) \).

- **Weak interference.** If \( \mu_2(\alpha) \geq \frac{|h_{12}|^2}{|h_{22}|^2} \):
  \[
  \kappa^*_2 = 0.
  \tag{33}
  \]

**Power-limited region:**

- **Strong interference.** If \( \frac{|h_{12}|^2}{|h_{22}|^2} \geq \mu_1(\alpha) \):
  \[
  \kappa^*_2 = \begin{cases} 
  1 & \text{if } 2\gamma = \gamma_{2R} \text{ and } P_2|h_{12}|^2 \geq P_1|h_{11}|^2 \\
  \kappa_{\text{max}} & \text{otherwise}
  \end{cases},
  \tag{34}
  \]
  where \( \kappa_{\text{max}} \) satisfies \( q(\kappa_{\text{max}}) = P_2 \).

- **Moderate/Weak interference.** If \( \mu_1(\alpha) > \frac{|h_{12}|^2}{|h_{22}|^2} \):
  \[
  \kappa^*_2 = \begin{cases} 
  1 & \text{if } 2\gamma = \gamma_{2R} \text{ and } P_2|h_{12}|^2 \geq P_1|h_{11}|^2 \\
  \kappa_{\text{max}} & \text{if } \kappa_{\text{max}} > \kappa_0 \\
  0 & \text{otherwise}
  \end{cases},
  \tag{35}
  \]
  where \( \kappa_{\text{max}} \) and \( \kappa_0 \) satisfy \( q(\kappa_{\text{max}}) = P_2 \) and \( R_2(\kappa_0) = R_2(0) \).

**Proof:** Please refer to Appendix [B].

**Remark:** We would like to point out that the definition of strong, moderate and weak interference in Theorem [1] is made for the sake of exposition and does not adhere to the information-theoretic definition of such interference regimes (see [27], [29]). Hence, the definitions that
we provide in Theorem 1 account for the system behavior when we are restricted to Gaussian signaling and treating interference as noise. Notice also that the thresholds depend on $\alpha$, hence the interference regimes are relative to the rate of the first user. The intuition behind this is that the slope of the logarithm is lower the higher its argument is. As a result, a fixed interference power yields a higher decrease in achievable rate when the signal-to-noise ratio (SNR) is low. In our scenario, this means that the interference is more significant when $R_1$ is low (more details will be given in the next section), which explains the dependence of the interference regimes on $R_1$.

IV. DISCUSSION AND NUMERICAL EXAMPLES

This section provides a discussion on the derived characterization along with some numerical examples illustrating the most remarkable features of improper signaling in the Z-IC. Afterwards, the connection to related works in the literature is presented.

A. Optimal strategies

1) Optimality of proper signaling: As pointed out at the beginning of Section III, if proper signaling is the optimal strategy for one of the users, then it is also optimal for the other one. This means that any point of the region boundary is achieved by either both users employing proper signaling or improper signaling, but no mixed strategies.

2) Maximally improper signaling for both users is optimal, at most, at one boundary point: It can be noticed that there is at most one boundary point where both users simultaneously transmit a maximally improper signal, i.e., $\kappa_1 = \kappa_2 = 1$ is fulfilled for no more than one Pareto optimal point. This is due to the fact that, if the rate constraint (22) can be fulfilled for $\kappa_1 = 1$ (which corresponds to the case $2\gamma \geq \gamma_2\bar{R}$, i.e., the power-limited region), then user 1 tolerates an infinite amount of maximally improper interference along the orthogonal direction (see second equation in (24)). However, user 2 may only increase its rate by increasing $\kappa_2$ if it is operating below its power budget, since the only purpose of increasing $\kappa_2$ is to increase $p_2$ as well. Because of this and because $q(\kappa_2)$ is a continuous function for $2\gamma > \gamma_2\bar{R}$, setting $\kappa_1 = \kappa_2 = 1$ is always suboptimal when $2\gamma > \gamma_2\bar{R}$. However, this reasoning is not applicable when $2\gamma = \gamma_2\bar{R}$, since $q(\kappa_2)$ turns into a non-continuous function. In this case, both users choose their transmit signals as maximally improper if $P_2|h_{12}|^2 \geq P_1|h_{11}|^2$ (see Theorem 1). That is, both users transmit
maximally improper signals at the boundary point for which \( R_1 = \frac{1}{2} \log_2 (1 + 2 \frac{P_1 |h_{11}|^2}{\sigma^2}) \) only if \( P_2 |h_{12}|^2 \geq P_1 |h_{11}|^2 \).

To illustrate this property, we provide two simulation examples. Firstly, we consider the channel coefficients \( h_{11} = 0.94 - 0.35j, h_{22} = 0.98 - 0.21j \) and \( h_{12} = 0.45 - 1.34j \), so \( |h_{11}|^2 = |h_{22}|^2 = 1 \) and \( |h_{12}|^2 = 2 \). The power budgets and noise variance are respectively set to \( P_1 = P_2 = 10 \) and \( \sigma^2 = 1 \). Figure 3 shows the optimal circularity coefficients and the transmit power of user 2. Notice that \( P_2 |h_{12}|^2 \geq P_1 |h_{11}|^2 \) holds and because of that both users transmit maximally improper signals for \( R_1 = 2.20 \) b/s/Hz. Secondly, we consider \( |h_{11}|^2 = |h_{22}|^2 = 1 \) and \( |h_{12}|^2 = 0.8 \). In this case \( P_2 |h_{12}|^2 < P_1 |h_{11}|^2 \), hence it is expected that there are no boundary points that are achieved by both users transmitting maximally improper signals. This can be observed in Fig. 4 which depicts the dependency of the circularity coefficients and transmit power on the rate, \( R_1 \). At the boundary point for \( R_1 = 2.20 \) b/s/Hz, user 1 chooses the transmit signal as maximally improper, but the optimal circularity coefficient of user 2 equals 0.8.
3) Relationship between $\kappa_1$ and $\kappa_2$: Expression (16) also permits drawing insightful conclusions about the relationship between the circularity coefficients of both users. According to Theorem 1, $0 < \kappa_2 < 1$ implies $p_2 = P_2$. Hence, $\kappa_1 < \kappa_2$ holds if $P_2|h_{12}|^2 < P_1|h_{11}|^2$, i.e., when the signal-to-interference ratio (SIR) is greater than one. In such a case, it can be noticed that the signal transmitted by the first user is never chosen as maximally improper at any point of the Pareto boundary. This behavior can be clearly observed in Fig. 4. On the other hand, when $P_2|h_{12}|^2 \geq P_1|h_{11}|^2$, or, alternatively, when the SIR is equal to or lower than one, $\kappa_1 \geq \kappa_2$ holds whenever $0 < \kappa_2 < 1$, as can be seen in Fig. 3.

If the signal transmitted by user 1 is chosen as maximally improper for some points of the boundary, it will remain unchanged as $\kappa_2 \to 1$, or, in other words, as $\gamma_2 R \to \gamma$. This is because $\kappa_2 < 1$ implies $p_2 = P_2$, so that, by (16), the first user will not decrease its circularity coefficient. However, once $\kappa_2$ equals 1, which corresponds to the point $2\gamma = \gamma_2 R$, the degree of impropriety of the first user will then start decreasing with $R_1$, since these rates are not achievable for $\kappa_1 = 1$. 

Fig. 4. Dependency of the optimal circularity coefficients and the transmit power $p_2$ on $R_1$ for $|h_{11}|^2 = |h_{22}|^2 = 1$ and $|h_{12}|^2 = 0.8$. 
B. Properties of the Pareto boundary

1) Transition between power-limited and interference-limited regions: An interesting feature of this scenario is that there may be abrupt changes in the achievable rate of user 2 when we shift from one region to the other, and which are due to a drop in the transmit power of user 2. This can be explained as follows. In the power-limited region the transmit power always equals the power budget, i.e., \( p_2 = P_2 \). However, in the interference-limited region the transmit power is dominated by the function \( q(\kappa_2) \) when the power budget exceeds the value of that function, i.e., when \( P_2 > q(1) \). Hence, by using (16), it can be easily seen that \( \lim_{\gamma_2 R \to 2\gamma} q_2(1) = \frac{P_1|h_{11}|^2}{|h_{12}|^2} \), so that there may be a power drop, i.e., a discontinuity in the transmit power of user 2 as a function of \( R_1 \), from \( p_2 = P_2 \) to \( p_2 = \frac{P_1|h_{11}|^2}{|h_{12}|^2} \) and whenever \( P_2 > \frac{P_1|h_{11}|^2}{|h_{12}|^2} \). That is, the lower the SIR, the more prominent the power drop is, whereas no drop will be observed when the SIR is equal to or greater than 1. This discontinuity on the transmit power implies a similar drop on the achievable rate of user 2. Nevertheless, the power drop can be eliminated by just reducing \( p_2 \) at the transition point, i.e., at \( 2\gamma = \gamma_2 R \). At that point of the boundary, both users may be transmitting non-interfering maximally improper signals, so reducing \( p_2 \) translates into a vertical line on the rate region boundary. This is illustrated in Figs. 5 and 6, where the Pareto boundary of the rate region is depicted for the previous considered examples, i.e., \( |h_{11}|^2 = |h_{22}|^2 = 1, |h_{12}|^2 = 2; \) and \( |h_{11}|^2 = |h_{22}|^2 = 1, |h_{12}|^2 = 0.8 \), respectively. We also depict in the figures the rate region boundary for proper signaling.

Figure 5 corresponds to a scenario where the SIR is below one, therefore we can observe the vertical line of the achievable rate region boundary when we move from one region to the other. As aforementioned explained, this is the result of a drop in the transmit power of user 2, \( p_2 \), what can be observed in Fig. 3. Specifically, the maximum transmit power in the interference limited region equals \( p_2 = \frac{P_1|h_{11}|^2}{|h_{12}|^2} = 5 \), so there is a power drop from \( p_2 = P_2 = 10 \) to \( p_2 = 5 \), and the vertical line in Fig. 5 is just obtained by varying \( p_2 \) from 10 to 5 at \( 2\gamma = \gamma_2 R \). We would like to remark that the pointy corner observed in the Pareto boundary in Fig. 5 corresponds, in this case, to the transition between the two regions. This is because \( P_2 > q(1) \) in the entire interference-limited region. However, a system operating in this region is asymptotically limited by interference, hence it will be limited by its power budget whenever \( P_2 < q(1) \). In such a case, the pointy corner in the boundary will shift to the right. Figure 5 also shows the enlargement of
Fig. 5. Optimal rate region boundary for proper and improper transmissions and a specific channel realization. In this example, $|h_{11}|^2 = |h_{22}|^2 = 1$ and $|h_{12}|^2 = 2$.

the rate region due to improper signaling. In this example, the interference level is significant (actually, since $|h_{12}|^2 > |h_{22}|^2$ the system operates in the strong interference regime for all boundary points), so the achievable rate region by improper signaling is substantially larger than that of proper signaling.

The scenario corresponding to Fig. 6, however, presents an SIR greater than one. Because of that, the transition from the power-limited to the interference-limited regions is smoother as there is no drop in the transmit power (see Fig. 4). In this scenario the condition for strong interference is not always met, and actually we can observe in Fig. 4 a transition to the weak interference regime at approximately $R_1 = 2.9$ b/s/Hz. At that point we can also notice a small power drop, since the transmitter jumps from a transmit power $p_2 = q(1)$ to $p_2 = q(0)$.

2) Remarks on the interference regimes: By Theorem 1 it is clear that, if $|h_{12}|^2 \geq |h_{22}|^2$, the condition for strong interference is met at every point of the rate region boundary (since the corresponding threshold is always equal to or lower than 1). Consequently, the rate of the second
user increases monotonically with $\kappa_2$, which must then be chosen as the maximum possible value allowed by the available power budget. This case also corresponds to the information-theoretic definition of strong interference, where the optimal strategy at the receiver is to decode and subtract the interference. That is, when the information-theoretic condition for strong interference is met, the optimal strategy for the second user is improper signaling for all boundary points. On the other hand, when $|h_{12}|^2 < |h_{22}|^2$, the system may still be in the strong interference regime (according to our definition in Theorem 1) at some points of the boundary. Moreover, at those points where the first threshold is equal to or lower than 0, i.e., when $\gamma \geq \gamma_2 \bar{R} - \gamma_{\bar{R}}$, the condition for strong interference is met independently of the channel gains.\footnote{It is worth stressing at this point that in Theorem 1 and, consequently, in this discussion, we are assuming that $q(0) < P_2$, i.e., proper signaling does not permit maximum power transmission. Clearly, if $q(0) \geq P_2$, transmitting proper signals is optimal, since increasing $\kappa_2$ does not permit an increase in the transmit power.}

When improper signaling is the preferred strategy, there are two possible strategies: maximally
improper signaling (i.e., $\kappa_2 = 1$) with $p_2 \leq P_2$, or maximum power transmission (i.e., $p_2 = P_2$) with $\kappa_2 \leq 1$. This happens in the strong and moderate interference regimes, where managing interference is beneficial in terms of achievable rates. Thus, the strong and moderate interference regimes as defined in Theorem 1 conceptually correspond to their information-theoretic definition, but not in a one-to-one fashion.

Finally, the weak interference regime defined in Theorem 1 explains the cases where managing the interference by means of improper signaling strictly decreases the performance independently of the power budget. Hence, it contains its information-theoretic counterpart, for which treating interference as noise is optimal.

As an example, we plot in Fig. 7 the achievable rate of user 2 as a function of $|h_{22}|^2 |h_{12}|^2$, for $P_1 = 10$, $h_{11} = 1$, $\sigma^2 = 1$ and $\alpha = 0.65$, which corresponds to $R_1 = 2.25$ b/s/Hz, i.e., the interference-limited region. Also, we assume $p_2(1) < P_2$, i.e., the power budget constraint of the second user is never active. The shaded area in the figure illustrates the range of achievable rates by varying $\kappa_2$. The thresholds defining the three interference regimes are also depicted, where $\mu_1(\alpha)$ and $\mu_2(\alpha)$ are defined in Theorem 1. As can be observed, the minimum and maximum rate is achieved by proper and maximally improper signaling, respectively, in the strong interference regime. In the moderate interference regime, however, the performance of proper signaling approaches that of maximally improper, and both schemes achieve the same rate when $|h_{22}|^2 |h_{12}|^2$ equals the second threshold. From that point onward, which corresponds to the weak interference regime, proper signaling achieves the highest rate, which is in agreement with our discussion.

3) Behavior of the thresholds: A closer look at the thresholds that delimit the different interference regimes provides some insights into their behavior at the different points of the Pareto boundary. Let us first analyze the relationship between both thresholds, $\mu_1(\alpha)$ and $\mu_2(\alpha)$, since it determines the size of the moderate interference regime. As previously discussed, $\mu_2(\alpha)$ is defined for $2\gamma \leq \gamma_2 \bar{R}$. Let $\alpha_0$ be such that $2\gamma = \gamma_2 \bar{R}$, i.e., the transition from the power-limited

\footnote{For the sake of clarity, we have dropped the dependence of the thresholds with $\alpha$ in Fig. 7}
to the interference-limited region. At this point we have

\[
\mu_1(\alpha_0) = \frac{1}{2} \frac{\gamma_R}{\gamma_R + 1} = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{1 + 2\gamma}} \right),
\]

(36)

\[
\mu_2(\alpha_0) = \frac{1}{4} \frac{\gamma_R}{\gamma_R + 1} = \frac{1}{4} \left( 1 - \frac{1}{\sqrt{1 + 2\gamma}} \right).
\]

(37)

That is, \( \mu_1(\alpha_0) = 2\mu_2(\alpha_0) \). Furthermore, as \( \gamma \) increases, \( \mu_1(\alpha_0) \) and \( \mu_2(\alpha_0) \) approach \( \frac{1}{2} \) and \( \frac{1}{4} \), respectively. Thus, in the power-limited region, \( |h_{12}|^2 \geq \frac{1}{2} |h_{22}|^2 \) is a sufficient condition to be under strong interference. Similarly, if \( |h_{12}|^2 \leq \frac{1}{4} |h_{22}|^2 \), the system will be under weak interference almost in the entire interference-limited region.

As the rate of the first user increases, it can also be shown that the difference between both thresholds diminishes. Thus, when \( \alpha = 1 \), i.e., when the first user achieves its maximum rate, the thresholds become

\[
\mu_1(1) = \mu_2(1) = \frac{\gamma}{\gamma + 1}.
\]

(38)

That is, the moderate interference regime shrinks as the rate of the first user increases, thus
becoming almost non-existent for high values of $R_1$. Furthermore, the thresholds approach 1 as $\gamma$ increases, which means that the conditions to operate under strong and moderate interference become more stringent as the SNR of the first users grows.

These properties are illustrated in Fig. 8 which shows the value of the two thresholds $\mu_1(\alpha)$ and $\mu_2(\alpha)$ for different values of $\gamma = \frac{P|h_{11}|^2}{\sigma^2}$. The shaded area represents the moderate interference regime defined in Theorem 1. The solid circles indicate the starting point of $\mu_2(\alpha)$, i.e., $\mu_2(\alpha_0)$ (since it is defined only for $2\gamma \leq \gamma_2 \bar{R}$ and, consequently, for $\alpha \geq \alpha_0$). It can be noticed that the wider the shadowed areas are the lower the thresholds. This means that the moderate interference regime constitutes a larger portion of the Pareto boundary when $|h_{12}|^2$ is small, i.e., the weaker the relative level of interference. In such a case, the second user is likely to be able to transmit a proper signal at maximum power, and hence the improvement of improper signaling will be negligible in most cases. Improper signaling will be more interesting for high values of $|h_{12}|^2$, in which case both thresholds approach one another, as previously discussed, and the moderate interference regime shrinks.

C. Relationship to previous work

Finally, we would like to connect our results to related works in the literature. In our previous work [14], we studied a similar scenario in the context of underlay cognitive radio. In that work, we considered the Z-IC with the restriction that the first user transmits only proper signals, i.e., $\kappa_1 = 0$. A characterization of the maximum achievable rate of user 2 was derived in terms of a threshold in $|h_{12}|^2$. In [14], only two interference regimes were shown to exist, corresponding to the strong and weak interference in Theorem 1. Also, the threshold derived in [14] is given by $1 - \frac{\alpha}{\gamma_2 \bar{R}}$, which is strictly higher that the one obtained for the general Z-IC (see the strong interference regime in Theorem 1). This is in agreement with the fact that, if we let the first user optimize its circularity coefficient, the rate achieved by the second user can only increase.

The Z-IC was also considered in [17], and the transmit strategy that maximizes the sum-rate is derived in closed form based on the real-composite model. Although such a model is usually more convenient from an optimization point-of-view, it is not as insightful as the augmented complex model since some of the features of the improper signal are not easily captured. This is the case for the degree of impropriety, which is elegantly explained by the circularity coefficient. We would like to stress that in [17] only one point of the rate region is characterized, whereas in this
work we completely characterize the boundary of the rate region. Furthermore, since we consider
the augmented complex model, we provide closed-form formulas for the circularity coefficients,
thus providing a more insightful description of how improper signaling behaves in this scenario.
Nevertheless, some of the conclusions drawn in [17] fall within our characterization of the rate
region boundary. By looking at the structure of the sum-rate maximizing transmit strategies in
[17, Eq. (31)] we observe that improper signaling is chosen when $|h_{12}|^2 > |h_{22}|^2$. This condition
belongs to the strong interference regime defined in Theorem 1 and the solution presented in
[17, Eq. (31)] can be seen as a special case of (31) and (34). Furthermore, according to [17],
when improper signaling is preferred for the sum-rate maximization, the signal transmitted by
one of the users is chosen as maximally improper, and the selection of the user depends on
whether or not the condition $P_2|h_{12}|^2 \leq P_1|h_{11}|^2$ holds. This is in agreement with our previous
discussion, where we pointed out that the circularity coefficient of the first user is always equal

\[\gamma = 100\]

\[\gamma = 10\]

\[\gamma = 1\]

\[\alpha\]

\[\mu_1(\alpha)\]

\[\mu_2(\alpha)\]

Fig. 8. Value of the two thresholds as a function of $\alpha$, for different values of $\gamma$. The shaded areas represent the moderate interference regime defined in Theorem 1.

\[\text{For the sake of conciseness, we do not reproduce this expression and refer the interested reader to [17].}\]
to or lower than that of the second user whenever the aforementioned condition is fulfilled. In such a case, the Pareto optimal point that corresponds to the sum-rate maximization is then the point that satisfies \( q(1) = P_2 \). Otherwise, if \( P_2|h_{12}|^2 > P_1|h_{11}|^2 \), the sum-rate is maximized when the first user transmits a maximally improper signal, and in this case the circularity coefficient of the second user is strictly lower than 1 except for \( R_1 = \frac{1}{2} \log_2(1 + 2\frac{P_1|h_{11}|^2}{\sigma^2}) \).

V. CONCLUSION

We have analyzed the benefits of improper signaling in the Z-IC. We have derived a complete and insightful characterization of the optimal rate region boundary, and the corresponding transmit powers and circularity coefficients in closed-form. The rate region has thereby been described by three interference regimes, under which the degree of impropriety affects the achievable rates differently. We have shown that the rate region can be substantially enlarged by using improper signaling, especially when the relative level of interference is high.

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APPENDIX A

PROOF OF LEMMA 1

When \( \kappa_1 = 1 \), \( q(\kappa_2) \) approaches \( \infty \) as \( \kappa_2 \to 1 \), which means that user 1 tolerates infinite interference power without affecting its rate when both users are transmitting maximally improper signals. Since the lemma assumes \( q(1) \leq P_2 \), we have to consider the case \( \kappa_1 < 1 \). Taking into account that

\[
\frac{\partial q(\kappa_2)}{\partial \kappa_2^2} = \frac{q(\kappa_2)^2|h_{12}|^2(\gamma_2 - 1)}{2q(\kappa_2)|h_{12}|^2[(\gamma_2 + 1)(1 - \kappa_2^2) - 1] + 2\sigma^2(\gamma_2 - \gamma)},
\]

we obtain

\[
\frac{\partial R_2(\kappa_2)}{\partial \kappa_2^2} \geq 0 \iff \frac{|h_{12}|^2}{|h_{22}|^2} \geq \frac{\gamma_2 - 1}{\gamma_2 + 1}.
\]
where $\bar{q}(\kappa_2) = \frac{q(\kappa_2)|h_{12}|^2}{\sigma^2}$, which does not depend on $h_{12}$ as can be seen from (24). Let $\kappa_0$ be such that the right-hand side of (40) holds with equality. Since $\frac{\partial q(\kappa_2)}{\partial \kappa_2} > 0$ for $q(\kappa_2) > 0$ (i.e., the transmit power can increase if the degree of impropriety also increases), $\bar{q}(\kappa_2)$ increases with $\kappa_2$ and, consequently, the right-hand side of (40) holds with strict inequality when $\kappa_2 > \kappa_0$. As a result, the derivative of $R_2(\kappa_2)$ is strictly positive whenever $\kappa_2 > \kappa_0$, which concludes the proof.

**APPENDIX B**

**PROOF OF THEOREM 1**

Let us first consider the interference-limited region, i.e. $2\gamma < \gamma_2 \bar{R}$. This condition implies $R > \frac{1}{2} \log_2(1 + 2 \frac{P_1|h_{11}|^2}{\sigma^2})$, i.e., the rate of user 1 is higher than that achievable by a maximally improper signal, which results in $\kappa_1 < 1$. By evaluating (40) at $\kappa_2 = 0$, we obtain that the rate slope is equal to or greater than zero if $\frac{|h_{12}|^2}{|h_{22}|^2} \geq 1 - \frac{\gamma}{\gamma_2 \bar{R} - \gamma R}$. By Lemma 1 this means that $R_2(\kappa_2)$ increases monotonically with $\kappa_2$ as long as $p(\kappa_2) \leq P_2$, i.e., improper signaling is always beneficial and thus we operate in the strong interference regime. Hence, if the power budget is sufficiently high, maximally improper signaling is optimal. Otherwise, maximum power transmission is performed by setting $\kappa_2$ such that $q(\kappa_2) = P_2$, which yields (31).

If the condition for strong interference is not met, Lemma 2 states that there may be a rate improvement if $\frac{|h_{12}|^2}{|h_{22}|^2} > \frac{(\gamma_R - \gamma)^2}{(\gamma_2 R - \gamma)(\sqrt{\gamma_2 R - 2\gamma} - \gamma_R)}$, thus we operate in the moderate interference regime. By Lemma 2 we know that improper signaling is beneficial in this regime only when $\kappa_2$ exceeds a given value, and, in that case, the achievable rate increases monotonically with $\kappa_2$. If the power budget is sufficiently high, maximally improper signaling is therefore optimal. Otherwise, we need to find the crossing point with the proper rate, i.e., $R_2(\kappa_0) = R_2(0)$. Consequently, improper is beneficial as long as the circularity coefficient needed for maximum power transmission exceeds $\kappa_0$, which yields (32).

If the conditions for strong and moderate interference are not met, proper signaling is the optimal strategy, which yields the weak interference regime and (33).

In the power-limited region, i.e., when $2\gamma \geq \gamma_2 \bar{R}$, $\bar{R} \leq \frac{1}{2} \log_2(1 + 2 \frac{P_1|h_{11}|^2}{\sigma^2})$ and hence user 1 can achieve its rate with a maximally improper signal. This means that $q(1) = \infty$ for $\kappa_1 = 1$, and, as a result, the weak interference regime does not exist in this case. For $2\gamma = \gamma_2 \bar{R}$, we notice that (24) is not a continuous function for $\kappa_1 = 1$, since, in that case, $q(\kappa_2 < 1) = 0$ and $q(\kappa_2 = 1) = \infty$. The intuition behind this behavior is that, when $2\gamma = \gamma_2 \bar{R}$, the first user achieves
its corresponding rate with a maximally improper signal, i.e., $\kappa_1 = 1$, only if the interference is orthogonal to the signal subspace, i.e., only when the second user also transmits a maximally improper signal. Consequently, $\kappa_2^\star = 1$ must hold if $2\gamma = \gamma_{2R}$ and $\kappa_1 = 1$. Furthermore, since $p_2 = P_2$ when $\kappa_2 = 1$, the condition $\kappa_1 = 1$ is equivalent, by (16), to $P_2|h_{12}|^2 \geq P_1|h_{11}|^2$, thus yielding the first case in (34) and (35). If $2\gamma > \gamma_{2R}$, $q(\kappa_2)$ is a continuous function, and the derivation of the remaining cases in (34) and (35) follows directly from the derivation corresponding to $2\gamma < \gamma_{2R}$, which concludes the proof.

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