GEOMETRIC CORRECTION FOR HYDRODYNAMIC LIMIT OF STEADY BOLTZMANN EQUATION

LEI WU

ABSTRACT. We consider a steady Boltzmann equation in a unit plate with non-standard Maxwellian boundary. The classical theory intends to approximate its solution by the sum of the interior solution and the Knudsen layer in the hydrodynamic limit. We construct a counterexample to show its invalidity and present a different boundary layer expansion with geometric correction.

Keywords: $\epsilon$-Milne problem, Boundary layer, Geometric correction.

1. Introduction

1.1. Problem Formulation. We consider steady Boltzmann equation for $F^{\epsilon}(\vec{x}, \vec{v})$ in a two-dimensional unit disk $\Omega = \{ \vec{x} = (x_1, x_2) : |\vec{x}| \leq 1 \}$ with velocity $\Sigma = \{ \vec{v} = (v_1, v_2) \in \mathbb{R}^2 \}$ as

\[
\begin{align*}
\epsilon \vec{v} \cdot \nabla F^{\epsilon} &= Q[F^{\epsilon}, F^{\epsilon}] \text{ in } \Omega \times \mathbb{R}^2, \\
F^{\epsilon}(\vec{x}_0, \vec{v}) &= P^{\epsilon}[F^{\epsilon}](\vec{x}_0, \vec{v}) \text{ for } \vec{x}_0 \in \partial \Omega \text{ and } \vec{n}(\vec{x}_0) \cdot \vec{v} < 0,
\end{align*}
\]

where

\[
P^{\epsilon}[F^{\epsilon}](\vec{x}_0, \vec{v}) = \mu_b^{\epsilon}(\vec{x}_0, \vec{v}) \int_{\vec{n}(\vec{x}_0) \cdot \vec{v}^* > 0} F^{\epsilon}(\vec{x}_0, \vec{v}^*) | \vec{n}(\vec{x}_0) \cdot \vec{v}^* | d\vec{v}^*,
\]

for some $\mu_b^{\epsilon}$, $\vec{n}_b^\epsilon$ and $\theta_b^\epsilon$ satisfying

\[
\int_{\vec{n}(\vec{x}_0) \cdot \vec{v} > 0} \mu_b^{\epsilon}(\vec{x}_0, \vec{v}) | \vec{n}(\vec{x}_0) \cdot \vec{v} | d\vec{v} = 1,
\]

$\vec{n}(\vec{x}_0)$ is the outward normal vector at $\vec{x}_0$ and the Knudsen number $\epsilon$ satisfies $0 < \epsilon << 1$. Here we have

\[
Q[F, G] = \int_{\mathbb{R}^2} \int_{S^1} B(\vec{\omega}, |\vec{u} - \vec{v}|) \left(F(\vec{u})G(\vec{v}) - F(\vec{u})G(\vec{v})\right) d\vec{\omega} d\vec{u},
\]

with

\[
\vec{u}_* = \vec{u} + \vec{\omega} \left((\vec{v} - \vec{u}) \cdot \vec{\omega}\right) , \quad \vec{v}_* = \vec{v} - \vec{\omega} \left((\vec{v} - \vec{u}) \cdot \vec{\omega}\right),
\]

and the hard-sphere collision kernel

\[
B(\vec{\omega}, |\vec{u} - \vec{v}|) = b_0 |\vec{u} - \vec{v}| |\cos \phi|,
\]

for positive constant $b_0$ related to the size of ball, $|\vec{\omega} \cdot (\vec{v} - \vec{u})| = |\vec{v} - \vec{u}| \cos \phi$ and $0 \leq \phi \leq \pi/2$. We intend to study the behavior of $F^{\epsilon}$ as $\epsilon \to 0$.

Based on the flow direction, we can divide the boundary $\gamma = \{ (\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial \Omega \}$ into the in-flow boundary $\gamma_-$, the out-flow boundary $\gamma_+$, and the grazing set $\gamma_0$ as

\[
\begin{align*}
\gamma_- &= \{ (\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial \Omega, \vec{v} \cdot \vec{n}(\vec{x}_0) < 0 \}, \\
\gamma_+ &= \{ (\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial \Omega, \vec{v} \cdot \vec{n}(\vec{x}_0) > 0 \}, \\
\gamma_0 &= \{ (\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial \Omega, \vec{v} \cdot \vec{n}(\vec{x}_0) = 0 \}.
\end{align*}
\]
We denote the standard Maxwellian as
\[ \mu(v) = \frac{1}{2\pi} \exp\left(-\frac{|v|^2}{2}\right). \]

Assume boundary Maxwellian
\[ \mu^e(x_0, v) = \mu(v) + \sum_{k=1}^{\infty} c^k \mu_k(x_0, v), \]
and satisfies
\[ \left| \langle v \rangle^\theta e^{\zeta |v|^2} \frac{\mu^e - \mu}{\sqrt{\mu}} \right|_{L^\infty} \leq C_0 \epsilon, \]
for \(0 < \zeta \leq 1/4\) and \(\theta \geq 3\), where \(C_0 > 0\) is sufficiently small. We naturally have
\[ \int_{\tilde{n}(x_0) \cdot \tilde{v} > 0} \mu_k(x_0, v) |\tilde{n}(x_0) \cdot \tilde{v}| \, d\tilde{v} = 0 \quad \text{for} \quad k \geq 1. \]

The solution \(F^e\) can be expressed as a perturbation of the standard Maxwellian
\[ F^e(x, v) = M \mu + \sqrt{\mu} f^e. \]
for some constant \(M\) with normalization condition.
\[ \int_{\Omega} \int_{\mathbb{R}^2} f^e(x, v) \sqrt{\mu(v)} \, dv \, dx = 0. \]

Then \(f^e\) satisfies the equation
\[ \left\{ \begin{array}{l}
\epsilon \tilde{v} \cdot \nabla_x f^e + \mathcal{L}[f^e] = \Gamma[f^e, f^e], \\
f^e(x, \tilde{v}) = \mathcal{P}^*[f^e](\bar{x}_0, \tilde{v}) \quad \text{for} \quad \bar{n} \cdot \bar{v} < 0 \quad \text{and} \quad \bar{x}_0 \in \partial \Omega,
\end{array} \right. \]
where
\[ \Gamma[f^e, f^e] = \frac{1}{\sqrt{\mu}} Q[\sqrt{\mu} f^e, \sqrt{\mu} f^e], \]
\[ \mathcal{L}[f^e] = -\frac{2}{\sqrt{\mu}} Q[\mu, \sqrt{\mu} f^e] = \nu(v) f^e - K[f^e], \]
\[ \nu(v) = \int_{\mathbb{R}^2} \int_{S^1} B(\bar{v} - \bar{u}, \bar{\omega}) \mu(\bar{u}) d\bar{\omega} d\bar{u}, \]
\[ = \pi^2 b_0 \left( \frac{2 |\bar{v}|}{|\bar{v}|} + \frac{1}{|\bar{v}|} \right) \int_0^{\bar{v}} e^{-z^2} \, dz + e^{-|\bar{v}|^2}, \]
\[ K[f^e] = K_2[f^e] - K_1[f^e] \]
\[ = \int_{\mathbb{R}^2} k(\bar{v}, \bar{u}) f^e(\bar{u}) d\bar{u} = \int_{\mathbb{R}^2} k_2(\bar{v}, \bar{u}) f^e(\bar{u}) d\bar{u} - \int_{\mathbb{R}^2} k_1(\bar{v}, \bar{u}) f^e(\bar{u}) d\bar{u}, \]
\[ K_1[f^e] = \sqrt{\mu(\bar{v})} \int_{\mathbb{R}^2} \int_{S^1} B(\bar{v} - \bar{u}, \bar{\omega}) \sqrt{\mu(\bar{u})} f^e(\bar{u}) d\bar{\omega} d\bar{u}, \]
\[ K_2[f^e] = \int_{\mathbb{R}^2} \int_{S^1} B(\bar{v} - \bar{u}, \bar{\omega}) \sqrt{\mu(\bar{u})} \left( \sqrt{\mu(\bar{v})} f^e(\bar{u}_*) + \sqrt{\mu(\bar{u}_*)} f^e(\bar{v}_*) \right) d\bar{\omega} d\bar{u}, \]
\[ k_1(u, v) = \pi b_0 |\bar{u} - \bar{v}| \exp\left(-\frac{1}{2} |\bar{u}|^2 - \frac{1}{2} |\bar{v}|^2\right), \]
\[ k_2(u, v) = \frac{2\pi b_0}{|\bar{u} - \bar{v}|} \exp\left(-\frac{1}{4} |\bar{u} - \bar{v}|^2 - \frac{1}{4} \left(\frac{|\bar{u}|^2 - |\bar{v}|^2}{|\bar{u} - \bar{v}|^2}\right)^2\right), \]
and

\[ P^\varepsilon[f^\varepsilon](\bar{x}_0, \bar{\varepsilon}) = \frac{\mu_b^\varepsilon(\bar{x}_0, \bar{\varepsilon})}{\sqrt{\mu(\bar{v})}} \int_{\bar{n}(\bar{x}_0) \cdot \bar{v}^\varepsilon > 0} \sqrt{\mu(\bar{v}^\varepsilon)} f^\varepsilon(\bar{x}_0, \bar{v}^\varepsilon) |\bar{n}(\bar{x}_0) \cdot \bar{v}^\varepsilon| d\bar{v}^\varepsilon + \frac{\mu_b^\varepsilon(\bar{x}_0, \bar{\varepsilon}) - \mu(\bar{\varepsilon})}{\sqrt{\mu(\bar{v})}}. \]

### 1.2. Main Theorem.

**Theorem 1.1.** For given \( \mu_b^\varepsilon > 0 \) satisfying (1.12) and (1.13), we have

\[
\left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} \left( f^\varepsilon - \epsilon (F_1^\varepsilon + \mathcal{F}_1^\varepsilon) \right) \right\|_{L^\infty} = O(\varepsilon^2),
\]

for \( 0 \leq \zeta \leq 1/4 \) and \( \vartheta \geq 3 \), where the interior solution \( F_1^\varepsilon \) is defined in (2.92) and boundary layer \( \mathcal{F}_1^\varepsilon \) is defined in (2.91).

**Theorem 1.2.** For given \( \mu_b^\varepsilon > 0 \) satisfying (1.12) and (1.13), there exists a non-negative solution \( F^\varepsilon = M\mu + \sqrt{\mu} f^\varepsilon \) with constant \( M \) to the steady Boltzmann equation (1.1) satisfying the normalization condition (1.16) such that for \( 0 \leq \zeta \leq 1/4 \) and \( \vartheta \geq 3 \),

\[
\left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} f^\varepsilon \right\|_{L^\infty} + \left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} g^\varepsilon \right\|_{L^\infty} \leq C\varepsilon,
\]

for some constant \( C > 0 \). If \( M\mu + \sqrt{\mu} g^\varepsilon \) is another solution satisfying the normalization condition (1.16) such that

\[
\left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} f^\varepsilon \right\|_{L^\infty} + \left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} g^\varepsilon \right\|_{L^\infty} \leq C\varepsilon,
\]

then \( f^\varepsilon = g^\varepsilon \).

**Theorem 1.3.** For given \( \mu_b^\varepsilon > 0 \) satisfying (1.12) and (1.13) with

\[
\frac{\mu_1}{\sqrt{\mu}} = v_\varphi e^{-(v_\varphi^2 - 1) - M v_\varphi^2} = h(v_\varphi, v_\varphi),
\]

where \( v_\varphi \) and \( v_\varphi^2 \) are defined as in (2.68) and we take \( M \) sufficiently large such that

\[
h(0, 1) = 1,
\]

\[
|h|_{L^2} << 1,
\]

there exists \( C > 0 \) such that

\[
\|f^\varepsilon - \epsilon(F_1 + F_1)\|_{L^\infty} \geq C\varepsilon,
\]

where the interior solution \( F_1 \) is defined in (2.63) and boundary layer \( \mathcal{F}_1 \) is defined in (2.61).
2. Asymptotic Analysis

2.1. Interior Expansion. We define the interior expansion

\[ F \sim \sum_{k=1}^{\infty} \epsilon^k F_k(\vec{x}, \vec{v}). \]

Plugging it into the equation \( \text{(1.17)} \) and comparing the order of \( \epsilon \), we obtain

\[ \mathcal{L}[F_1] = 0, \]

\[ \mathcal{L}[F_2] = -\vec{v} \cdot \nabla \vec{x} F_1 + \Gamma[F_1, F_1], \]

\[ \mathcal{L}[F_3] = -\vec{v} \cdot \nabla \vec{x} F_2 + \Gamma[F_1, F_2] + \Gamma[F_2, F_1], \]

\[ \ldots \]

\[ \mathcal{L}[F_k] = -\vec{v} \cdot \nabla \vec{x} F_{k-1} + \sum_{i=1}^{k-1} \Gamma[F_i, F_{k-i}]. \]

Based on the analysis in [1, 2], each \( F_k \) consists of three parts:

\[ F_k(\vec{x}, \vec{v}) = A_k(\vec{x}, \vec{v}) + B_k(\vec{x}, \vec{v}) + C_k(\vec{x}, \vec{v}), \]

where

\[ A_k(\vec{x}, \vec{v}) = \sqrt{\mu} \left( A_{k,0}(\vec{x}) + 2A_{k,1}(\vec{x})v_1 + 2A_{k,2}(\vec{x})v_2 + A_{k,3}(\vec{x}) \left( |\vec{v}|^2 - 1 \right) \right), \]

\[ B_k(\vec{x}, \vec{v}) = \sqrt{\mu} \left( B_{k,0}(\vec{x}) + 2B_{k,1}(\vec{x})v_1 + 2B_{k,2}(\vec{x})v_2 + B_{k,3}(\vec{x}) \left( |\vec{v}|^2 - 1 \right) \right), \]

with \( B_k \) depending on \( A_{s,i} \) in \( 1 \leq s \leq k - 1 \) and \( i = 0, 1, 2, 3 \) as

\[ B_{k,0} = 0, \]

\[ B_{k,1} = 2 \sum_{i=1}^{k-1} A_{i,0} A_{k-i,1}, \]

\[ B_{k,2} = 2 \sum_{i=1}^{k-1} A_{i,0} A_{k-i,2}, \]

\[ B_{k,3} = \sum_{i=1}^{k-1} \left( A_{i,0} A_{k-i,3} + A_{i,1} A_{k-i,1} + A_{i,2} A_{k-i,2} \right) + \sum_{j=1}^{k-1-i} A_{i,0} (A_{j,1} A_{k-i-j,1} + A_{j,2} A_{k-i-j,2}), \]

and \( C_k(\vec{x}, \vec{v}) \) satisfies

\[ \int_{\mathbb{R}^2} \sqrt{\mu(\vec{v})} C_k(\vec{x}, \vec{v}) \left( \frac{1}{|\vec{v}|^2} \right) d\vec{v} = 0, \]

with

\[ \mathcal{L}[C_k] = -\vec{v} \cdot \nabla \vec{x} F_{k-1} + \sum_{i=1}^{k-1} \Gamma[F_i, F_{k-i}]. \]

which can be solved explicitly at any fixed \( \vec{x} \). Hence, we only need to determine the relations satisfied by \( A_k \). For convenience, we define

\[ A_k = \sqrt{\mu} \left( \rho_k + 2u_{k,1} v_1 + 2u_{k,2} v_2 + \theta_k \left( |\vec{v}|^2 - 1 \right) \right), \]
Then the analysis in [1,2] shows that $A_k$ satisfies the equations as follows:

0\textsuperscript{th} order equations:

\begin{align}
(2.16) & \quad P_1 - (\rho_1 + \theta_1) = 0, \\
(2.17) & \quad \nabla_x P_1 = 0,
\end{align}

1\textsuperscript{st} order equations:

\begin{align}
(2.18) & \quad P_2 - (\rho_2 + \theta_2 + \rho_1 \theta_1) = 0, \\
(2.19) & \quad \bar{u} \cdot \nabla_x \bar{u}_1 - \frac{1}{2} \gamma_1 \Delta_x \bar{u}_1 + \frac{1}{2} \nabla_x P_2 = 0, \\
(2.20) & \quad \nabla_x \cdot \bar{u}_1 = 0, \\
(2.21) & \quad \bar{u}_1 \cdot \nabla_x \theta_1 - \frac{1}{2} \gamma_2 \Delta_x \theta_1 = 0,
\end{align}

$k$\textsuperscript{th} order equations:

\begin{align}
(2.22) & \quad P_{k+1} - \left( \rho_{k+1} + \theta_{k+1} + \sum_{i=1}^{k+1-i} \rho_i \theta_{k+1-i} \right) = 0, \\
(2.23) & \quad \sum_{i=1}^{k} \bar{u}_i \cdot \nabla_x \bar{u}_{k+1-i} - \frac{1}{2} \gamma_1 \Delta_x \bar{u}_k + \frac{1}{2} \nabla_x P_{k+1} = G_{k,1}, \\
(2.24) & \quad \nabla_x \cdot \bar{u}_k = G_{k,2}, \\
(2.25) & \quad \sum_{i=1}^{k} \bar{u}_i \cdot \nabla_x \theta_{k+1-i} - \frac{1}{2} \gamma_2 \Delta_x \theta_k = G_{k,3},
\end{align}

where

\begin{equation}
G_{k,j} = G_{k,j} \bar{r}, \bar{v}; \rho_1, \ldots, \rho_{k-1}; \theta_1, \ldots, \theta_{k-1}; \bar{u}_1, \ldots, \bar{u}_{k-1}; \nabla_x (\rho_k + \theta_k),
\end{equation}

is explicit functions depending on lower order terms, and $\gamma_1$ and $\gamma_2$ are two positive constants. In order to determine the boundary condition for $\bar{u}_k$, $\theta_k$ and $\rho_k$, we have to define the boundary layer expansion.

2.2. Milne Expansion. The formulation of boundary layer in [1,2] can be generalized as follows:

Substitution 1:
Define the substitution into polar coordinate $f^\ast(x_1, x_2, \bar{v}) \rightarrow f^\ast(r, \phi, \bar{v})$ with $(r, \phi, \bar{v}) \in [0, 1) \times [-\pi, \pi) \times \mathbb{R}^2$ as

\begin{equation}
\begin{cases}
x_1 = r \cos \phi, \\
x_2 = r \sin \phi, \\
\bar{v} = \bar{v}.
\end{cases}
\end{equation}

The equation (1.17) can be rewritten as

\begin{equation}
\begin{cases}
- e(\bar{v} \cdot \bar{n}) \frac{\partial f^\ast}{\partial r} + \frac{\epsilon}{r} (\bar{v} \cdot \hat{\bar{r}}) \frac{\partial f^\ast}{\partial \phi} + \mathcal{L}[f^\ast] = \Gamma[f^\ast, f^\ast], \\
f^\ast(1, \phi, \bar{v}) = \mathcal{P}^\ast[f^\ast](1, \phi, \bar{v}) \quad \text{for} \quad \bar{v} \cdot \bar{n}(\phi) < 0,
\end{cases}
\end{equation}

where

\begin{equation}
\mathcal{P}^\ast[f^\ast](1, \phi, \bar{v}) = \frac{\mu^\ast(\phi, \bar{v})}{\sqrt{\mu(\bar{v})}} \int_{\bar{v} \cdot \bar{n}(\phi) > 0} \sqrt{\mu(\bar{v}^\ast)} f^\ast(1, \phi, \bar{v}^\ast) |\bar{n}(\phi) \cdot \bar{v}^\ast| d\bar{v}^\ast + \frac{\mu^\ast(\phi, \bar{v}) - \mu(\bar{v})}{\sqrt{\mu(\bar{v})}},
\end{equation}

for $\bar{n}$ the outer normal vector and $\hat{\bar{r}}$ the tangential vector on $\partial \Omega$. 

Substitution 2:
We further perform the scaling substitution $f^r(r, \phi, \vec{v}) \rightarrow f^r(\eta, \phi, \vec{v})$ with $(\eta, \phi, \vec{v}) \in [0, 1/\epsilon) \times [-\pi, \pi) \times \mathbb{R}^2$ as

\[
\begin{align*}
\eta &= (1-r)/\epsilon, \\
\phi &= \phi, \\
\vec{v} &= \vec{v},
\end{align*}
\]

which implies

\[
\frac{\partial f^r}{\partial r} = -\frac{1}{\epsilon} \frac{\partial f^r}{\partial \eta}.
\]

Then the equation (1.17) in $(\eta, \phi, \vec{v})$ becomes

\[
\begin{align*}
\left\{ \begin{array}{l}
(\vec{v} \cdot \vec{n}) \frac{\partial f^r}{\partial \eta} + \frac{\epsilon}{1 - \epsilon \eta} (\vec{v} \cdot \vec{n}) \frac{\partial f^r}{\partial \phi} + \mathcal{L}[f^r] = \Gamma[f^r, f^r], \\
f^r(0, \phi, \vec{v}) = \mathcal{P}^r[f^r](0, \phi, \vec{v}) \text{ for } \vec{v} \cdot \vec{n}(\phi) > 0,
\end{array} \right.
\]

where

\[
\mathcal{P}^r[f^r](0, \phi, \vec{v}) = \frac{\mu(\phi, \vec{v})}{\sqrt{\mu(\vec{v})}} \int_{\vec{v}(\phi) > 0} \sqrt{\mu(\vec{v})} f^r(0, \phi, \vec{v}^*) |\vec{n}(\phi) \cdot \vec{v}^*| \, d\vec{v}^* + \frac{\mu(\phi, \vec{v})}{\sqrt{\mu(\vec{v})}}.
\]

We define the boundary layer expansion

\[
\mathcal{F} \sim \sum_{k=1}^{\infty} \epsilon^k \mathcal{F}_k(\eta, \phi, \vec{v}),
\]

where $\mathcal{F}_k$ can be determined by plugging it into the equation (2.32) and comparing the order of $\epsilon$. Thus in a neighborhood of the boundary, we have

\[
\begin{align*}
(\vec{v} \cdot \vec{n}) \frac{\partial \mathcal{F}_1}{\partial \eta} + \mathcal{L}[\mathcal{F}_1] &= 0, \\
(\vec{v} \cdot \vec{n}) \frac{\partial \mathcal{F}_2}{\partial \eta} + \mathcal{L}[\mathcal{F}_2] &= -\frac{1}{1 - \epsilon \eta} (\vec{v} \cdot \vec{n}) \frac{\partial \mathcal{F}_1}{\partial \phi} + \Gamma[\mathcal{F}_1, \mathcal{F}_1] + 2 \Gamma[\mathcal{F}_1, \mathcal{F}_1], \\
&\ldots \\
(\vec{v} \cdot \vec{n}) \frac{\partial \mathcal{F}_k}{\partial \eta} + \mathcal{L}[\mathcal{F}_k] &= -\frac{1}{1 - \epsilon \eta} (\vec{v} \cdot \vec{n}) \frac{\partial \mathcal{F}_{k-1}}{\partial \phi} + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i, \mathcal{F}_{k-i}] + 2 \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i, \mathcal{F}_{k-i}].
\end{align*}
\]

The bridge between the interior solution and boundary layer is the boundary condition

\[
f^r(\bar{x}_0, \vec{v}) = \mathcal{P}^r[f^r](\bar{x}_0, \vec{v}),
\]

where

\[
\mathcal{P}^r[f^r](\bar{x}_0, \vec{v}) = \frac{\mu(\bar{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}} \int_{\vec{n}(ar{x}_0) \cdot \vec{v}^* > 0} \sqrt{\mu(\vec{v}^*)} f^r(\bar{x}_0, \vec{v}^*) |\vec{n}(\bar{x}_0) \cdot \vec{v}^*| \, d\vec{v}^* + \frac{\mu(\bar{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}}.
\]

Plugging the combined expansion

\[
f^r \sim \sum_{k=1}^{\infty} \epsilon^k (\mathcal{F}_k + \mathcal{F}_k^*),
\]

into the boundary condition yields

\[
\sum_{k=1}^{\infty} \epsilon^k (\mathcal{F}_k + \mathcal{F}_k^*) = \frac{\mu(\bar{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}} \int_{\vec{n}(\bar{x}_0) \cdot \vec{v}^* > 0} \sum_{k=1}^{\infty} \epsilon^k (\mathcal{F}_k + \mathcal{F}_k^*) |\vec{n}(\bar{x}_0) \cdot \vec{v}^*| \, d\vec{v}^* + \frac{\mu(\bar{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}}.
\]

Comparing the order of $\epsilon$, we obtain

\begin{align}
\mathcal{F}_1 + \mathcal{F}_1 &= \sqrt{\mu(\vec{v})} \int_{\vec{n}(\vec{x}_0) \cdot \vec{v}^* > 0} \sqrt{\mu(\vec{v}^*)}(\mathcal{F}_1 + \mathcal{F}_1) \left| \vec{n}(\vec{x}_0) \cdot \vec{v}^* \right| d\vec{v}^* + \frac{\mu_1(\vec{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}}, \\
\mathcal{F}_2 + \mathcal{F}_2 &= \sqrt{\mu(\vec{v})} \int_{\vec{n}(\vec{x}_0) \cdot \vec{v}^* > 0} \sqrt{\mu(\vec{v}^*)}(\mathcal{F}_2 + \mathcal{F}_2) \left| \vec{n}(\vec{x}_0) \cdot \vec{v}^* \right| d\vec{v}^* \\
&\quad + \frac{\mu_1(\vec{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}} \int_{\vec{n}(\vec{x}_0) \cdot \vec{v}^* > 0} \sqrt{\mu(\vec{v}^*)}(\mathcal{F}_1 + \mathcal{F}_1) \left| \vec{n}(\vec{x}_0) \cdot \vec{v}^* \right| d\vec{v}^* + \frac{\mu_2(\vec{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}}, \\
&\quad \ldots
\end{align}

\begin{align}
\mathcal{F}_k + \mathcal{F}_k &= \sqrt{\mu(\vec{v})} \int_{\vec{n}(\vec{x}_0) \cdot \vec{v}^* > 0} \sqrt{\mu(\vec{v}^*)}(\mathcal{F}_k + \mathcal{F}_k) \left| \vec{n}(\vec{x}_0) \cdot \vec{v}^* \right| d\vec{v}^* \\
&\quad + \sum_{i=1}^{k-1} \left( \frac{\mu_{k-i}(\vec{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}} \int_{\vec{n}(\vec{x}_0) \cdot \vec{v}^* > 0} \sqrt{\mu(\vec{v}^*)}(\mathcal{F}_i + \mathcal{F}_i) \left| \vec{n}(\vec{x}_0) \cdot \vec{v}^* \right| d\vec{v}^* \right) + \frac{\mu_k(\vec{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}}.
\end{align}

Define

\begin{align}
P[f](\vec{x}_0, \vec{v}) &= \sqrt{\mu(\vec{v})} \int_{\vec{n}(\vec{x}_0) \cdot \vec{v}^* > 0} \sqrt{\mu(\vec{v}^*)}f(\vec{x}_0, \vec{v}^*) \left| \vec{n}(\vec{x}_0) \cdot \vec{v}^* \right| d\vec{v}^*.
\end{align}

Then we have

\begin{align}
(\mathcal{F}_1 + \mathcal{F}_1) &= P[\mathcal{F}_1 + \mathcal{F}_1] + \frac{\mu_1(\vec{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}}, \\
(\mathcal{F}_2 + \mathcal{F}_2) &= P[\mathcal{F}_2 + \mathcal{F}_2] + \frac{\mu_1(\vec{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}} \int_{\vec{n}(\vec{x}_0) \cdot \vec{v}^* > 0} \sqrt{\mu(\vec{v}^*)}(\mathcal{F}_1 + \mathcal{F}_1) \left| \vec{n}(\vec{x}_0) \cdot \vec{v}^* \right| d\vec{v}^* + \frac{\mu_2(\vec{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}}, \\
&\quad \ldots
\end{align}

\begin{align}
(\mathcal{F}_k + \mathcal{F}_k) &= P[\mathcal{F}_k + \mathcal{F}_k] + \sum_{i=1}^{k-1} \left( \frac{\mu_{k-i}(\vec{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}} \int_{\vec{n}(\vec{x}_0) \cdot \vec{v}^* > 0} \sqrt{\mu(\vec{v}^*)}(\mathcal{F}_i + \mathcal{F}_i) \left| \vec{n}(\vec{x}_0) \cdot \vec{v}^* \right| d\vec{v}^* \right) + \frac{\mu_k(\vec{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}}.
\end{align}

This is the boundary conditions $\mathcal{F}_k$ and $\mathcal{F}_k$ need to satisfy.

### 2.3. Matching of Interior Expansion and Milne Expansion

We divide this process into several steps:

**Step 1: Milne Problem.**

We solve the Milne problem \[1723\] for $g_k$ with the in-flow boundary data

\begin{align}
h_k &= -\left( (B_k + C_k) - P[B_k + C_k] \right) \\
&\quad + \sum_{i=1}^{k-1} \left( \frac{\mu_{k-i}(\vec{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}} \int_{\vec{n}(\vec{x}_0) \cdot \vec{v}^* > 0} \sqrt{\mu(\vec{v}^*)}(\mathcal{F}_i + \mathcal{F}_i) \left| \vec{n}(\vec{x}_0) \cdot \vec{v}^* \right| d\vec{v}^* \right) + \frac{\mu_k(\vec{x}_0, \vec{v})}{\sqrt{\mu(\vec{v})}}.
\end{align}

and source term

\begin{align}
S_k &= -\psi(\sqrt{\eta}) (\vec{v} \cdot \vec{v}^*) \partial \mathcal{F}_k^{-1} \phi + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i, \mathcal{F}_{k-i}] + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_{i}, \mathcal{F}_{k-i}].
\end{align}

Based on Theorem \[1714\] there exist modified boundary data

\begin{align}
\tilde{h}_k &= \sqrt{\mu(\vec{v})} \left( \tilde{D}_{k,0} + \tilde{D}_{k,1} v_1 + \tilde{D}_{k,2} v_2 + \tilde{D}_{k,3} |\vec{v}|^2 \right),
\end{align}

where \( \tilde{D}_{k,i} \) are constants determined by the boundary conditions.
such that the problem (4.236) is well-posed.

Step 2: Definition of Interior Solution and Boundary Layer.
Assume the cut-off function ψ and ψ0 are defined as

\[
\psi(\mu) = \begin{cases} 
1 & 0 \leq \mu \leq 1/2, \\
0 & 3/4 \leq \mu \leq \infty.
\end{cases}
\]

\[
\psi_0(\mu) = \begin{cases} 
1 & 0 \leq \mu \leq 1/4, \\
0 & 3/8 \leq \mu \leq \infty.
\end{cases}
\]

Define \( \mathcal{F}_k = G_k \cdot \psi_0(\sqrt{\eta}) \), where \( G_k \) the solution of Milne problem (4.236). We have

\[
\lim_{\eta \to 0} \mathcal{F}_k(\eta, \phi, \tilde{v}) = 0.
\]

Then define

\[
A_{k,0} = \tilde{D}_{k,0}, \\
A_{k,1} = \tilde{D}_{k,1}, \\
A_{k,2} = \tilde{D}_{k,2}, \\
A_{k,3} = \tilde{D}_{k,3}.
\]

This determines \( A_{k,1}, A_{k,2} \) and \( A_{k,3} \). Now we already show

\[
\mathcal{J} + \mathcal{F}_k = \mathcal{P}[\mathcal{F}_k + \mathcal{F}_k] + \sum_{i=1}^{k-1} \frac{\mu_{k-i}(\tilde{x}_0, \tilde{v})}{\sqrt{\mu(\tilde{v})}} \int_{n(\tilde{x}_0) \cdot \tilde{v}^* > 0} \sqrt{\mu(\tilde{v}^*)}(\mathcal{F}_i + \mathcal{F}_i) d^n(v^*) + \frac{\mu_k(\tilde{x}_0, \tilde{v})}{\sqrt{\mu(\tilde{v})}}
\]

Based on Lemma 4.1 and our construction, we have

\[
\mathcal{P}[A_k + \mathcal{F}_k] = 0.
\]

Hence, boundary condition is satisfied.

Step 3: Normalization Adjustment.
If \( \mathcal{F}_k + \mathcal{F}_k \) satisfies the boundary condition, then \( \mathcal{F}_k + \mathcal{F}_k + C \) for any constant \( C \) is also acceptable. Also, from the interior relation, \( \rho_k \) can be determined uniquely up to a constant. Then we require the normalization condition as

\[
\int_\Omega \int_{\mathbb{R}^2} \sqrt{\mu(\mathcal{F}_k + \mathcal{F}_k)} d\tilde{v} d\tilde{x} = 0.
\]

where we can adjust the value of \( \rho_k \). Hence, all the data are uniquely determined.

In particular, when \( k = 1 \), \( \mathcal{F}_1 \) satisfies

\[
\begin{cases}
\mathcal{F}_1(\eta, \phi, \tilde{v}) = G_1(\eta, \phi, \tilde{v}) \cdot \psi_0(\sqrt{\eta}) \\
\mathcal{G}_1(0, \phi, \tilde{v}) = h_1(\phi, \tilde{v}) - \tilde{h}_1(\phi, \tilde{v}) \text{ for } \tilde{n}(\phi) \cdot \tilde{v} > 0,
\end{cases}
\]

\[
\int_{\mathbb{R}^2} (\tilde{n}(\phi) \cdot \tilde{v}) \sqrt{\mu(\mathcal{G}_1(0, \phi, \tilde{v}))} d\tilde{v} = \int_{(\tilde{n}(\phi) \cdot \tilde{v}) > 0} (\tilde{n}(\phi) \cdot \tilde{v}) \sqrt{\mu} h_1(\phi, \tilde{v}) d\tilde{v} - \int_{\mathbb{R}^2} (\tilde{n}(\phi) \cdot \tilde{v}) \sqrt{\mu} \tilde{h}_1(\phi, \tilde{v}) d\tilde{v},
\]

\[
\lim_{\eta \to \infty} \mathcal{G}_1(\eta, \phi, \tilde{v}) = 0,
\]

with

\[
\tilde{h}_1 = \sqrt{\mu(\tilde{v})} \left( \tilde{D}_{1,0} + \tilde{D}_{1,1} v_1 + \tilde{D}_{1,2} v_2 + \tilde{D}_{1,3} |\tilde{v}|^2 \right).
\]
and \( F_1 \) satisfies

\[
F_1 = \sqrt{\mu} \left( \rho_1 + 2u_{1,1}v_1 + 2u_{1,2}v_2 + \theta_1 \left( |\vec{v}|^2 - 1 \right) \right),
\]

with

\[
\begin{aligned}
\nabla_x (\rho_1 + \theta_1) &= 0, \\
\vec{u} \cdot \nabla_x \vec{u}_1 - \frac{1}{2} \gamma_1 \Delta_x \vec{u}_1 + \frac{1}{2} \nabla_x P_2 &= 0, \\
\nabla_x \vec{u}_1 &= 0, \\
\vec{u}_1 \cdot \nabla_x \theta_1 - \frac{1}{2} \gamma_2 \Delta_x \theta_1 &= 0, \\
u_{1,1}(x_0) &= \tilde{D}_{1,1}(x_0), \\
u_{1,2}(x_0) &= \tilde{D}_{1,2}(x_0), \\
\theta_{1}(x_0) &= \tilde{D}_{1,3}(x_0),
\end{aligned}
\]  

(2.64)

and the normalization condition

\[
\int_\Omega \int_{\mathbb{R}^2} \sqrt{\mu} (F_1 + \mathcal{F}_1) d\vec{d} \vec{e} = 0.
\]

(2.65)

The analysis in [1, 2] anticipates this process can be generalized to arbitrary \( k \). However, in order to show the hydrodynamic limit, we have to expand to \( k = 2 \) at least. Therefore, based on Theorem 4.14, we require \( S_2 \in L^\infty \) to obtain a well-posed \( \mathcal{F}_2 \), i.e. we need

\[
\frac{\partial \mathcal{F}_1}{\partial \phi} \in L^\infty,
\]

(2.66)

which further requires

\[
\frac{\partial \mathcal{F}_1}{\partial \eta} \in L^\infty.
\]

(2.67)

Theorem 6.1 states that for certain boundary data \( \mu_0^\varepsilon \), this is invalid. Hence, this formulation breaks down.

2.4. \( \varepsilon \)-Milne Expansion. In order to overcome the difficulty in Milne expansion, we introduce one more change of variables.

Substitution 3:

We further define the velocity decomposition with respect to the normal and tangential directions \( f^\varepsilon(\eta, \phi, v_1, v_2) \to f^\varepsilon(\eta, \phi, \eta, v_\eta, v_\phi) \) with \( (\eta, \phi, \eta, v_\eta, v_\phi) \in [0, 1/\varepsilon) \times [-\pi, \pi] \times \mathbb{R}^2 \)

\[
\begin{aligned}
\eta &= \eta, \\
\phi &= \phi, \\
v_\eta &= v_1 \cos \phi + v_2 \sin \phi, \\
v_\phi &= -v_1 \sin \phi + v_2 \cos \phi.
\end{aligned}
\]  

(2.68)

Denote \( \vec{v} = (v_\eta, v_\phi) \). Then the equation (1.17) can be rewritten as

\[
\begin{aligned}
\frac{\partial f^\varepsilon}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \left( -v_\phi \frac{\partial f^\varepsilon}{\partial \phi} + v_2 \frac{\partial f^\varepsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial f^\varepsilon}{\partial v_\phi} \right) + L[f^\varepsilon] &= \Gamma[f^\varepsilon, f^\varepsilon], \\
f^\varepsilon(0, \phi, \vec{v}) &= \mathcal{P}^\varepsilon[f^\varepsilon](0, \phi, \vec{v}) \quad \text{for} \quad v_\eta > 0,
\end{aligned}
\]

(2.69)

where

\[
\mathcal{P}^\varepsilon[f^\varepsilon](0, \phi, \vec{v}) = \frac{\mu_0^\varepsilon(\phi, \vec{v})}{\sqrt{\mu(\vec{v})}} \int_{v_\eta > 0} \sqrt{\mu(\vec{v}^*)} f^\varepsilon(0, \phi, \vec{v}^*) v_\eta^* d\vec{v}^*.
\]

(2.70)

We define the boundary layer expansion

\[
\mathcal{F}^\varepsilon \sim \sum_{k=1}^{\infty} \epsilon^k \mathcal{F}_k^\varepsilon(\eta, \phi, \vec{v}),
\]

(2.71)
where $F_k^i$ can be determined by plugging it into the equation (2.69) and comparing the order of $\epsilon$. In a neighborhood of the boundary, we have

\begin{align}
(2.72) \quad & v_0 \frac{\partial F_1^i}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \left( v_0^2 \frac{\partial F_1^i}{\partial v_0} - v_0 v_\phi \frac{\partial F_1^i}{\partial v_\phi} \right) + \mathcal{L}[F_1^i] = 0, \\
(2.73) \quad & v_0 \frac{\partial F_2^i}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \left( v_0^2 \frac{\partial F_2^i}{\partial v_0} - v_0 v_\phi \frac{\partial F_2^i}{\partial v_\phi} \right) + \mathcal{L}[F_2^i] = -\frac{1}{1 - \epsilon \eta} v_\phi \frac{\partial F_1^i}{\partial \phi} \\
& \quad \quad \quad + \Gamma[F_1^i, F_1^i] + 2 \Gamma[F_1^i, F_1^i], \\
(2.74) \quad & v_0 \frac{\partial F_k^i}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \left( v_0^2 \frac{\partial F_k^i}{\partial v_0} - v_0 v_\phi \frac{\partial F_k^i}{\partial v_\phi} \right) + \mathcal{L}[F_k^i] = -\frac{1}{1 - \epsilon \eta} v_\phi \frac{\partial F_{k-1}^i}{\partial \phi} \\
& \quad \quad \quad + \sum_{i=1}^{k-1} \Gamma[F_i^i, F_{k-1}^i] + 2 \sum_{i=1}^{k-1} \Gamma[F_i^i, F_{k-1}^i].
\end{align}

Similar to the Milne expansion, we define the interior expansion

\begin{equation}
(2.75) \quad F^* \sim \sum_{k=1}^{\infty} \epsilon^k F_k^i(x, \bar{v}).
\end{equation}

Then $F_k^i$ satisfies the same equation as $F_k$ with different boundary data.

Similar to the boundary layer expansion of Milne problem, define

\begin{equation}
(2.76) \quad \mathcal{P}[f](\bar{x}_0, \bar{v}) = \sqrt{\mu(\bar{v})} \int_{v_\phi > 0} \sqrt{\mu(\bar{v}^*)} f(\bar{x}_0, \bar{v}^*) |v_\phi^*| d\bar{v}^*.
\end{equation}

Then we have

\begin{equation}
(2.77) \quad (F_1^i + F_1^i) = \mathcal{P}[F_1^i + F_1^i] + \frac{\mu_1(\bar{x}_0, \bar{v})}{\sqrt{\mu(\bar{v})}},
\end{equation}

\begin{equation}
(2.78) \quad (F_2^i + F_2^i) = \mathcal{P}[F_2^i + F_2^i] + \frac{\mu_1(\bar{x}_0, \bar{v})}{\sqrt{\mu(\bar{v})}} \int_{v_\phi > 0} \sqrt{\mu(\bar{v}^*)} (F_1^i + F_1^i) |v_\phi^*| d\bar{v}^* + \frac{\mu_2(\bar{x}_0, \bar{v})}{\sqrt{\mu(\bar{v})}},
\end{equation}

\begin{equation}
(2.79) \quad (F_k^i + F_k^i) = \mathcal{P}[F_k^i + F_k^i] + \sum_{i=1}^{k-1} \left( \frac{\mu_{k-1}(\bar{x}_0, \bar{v})}{\sqrt{\mu(\bar{v})}} \int_{v_\phi > 0} \sqrt{\mu(\bar{v}^*)} (F_1^i + F_1^i) |v_\phi^*| d\bar{v}^* \right) + \frac{\mu_k(\bar{x}_0, \bar{v})}{\sqrt{\mu(\bar{v})}}.
\end{equation}

This is the boundary conditions $F_k^i$ and $F_k^i$ need to satisfy.

2.5. **Matching of Interior Expansion and $\epsilon$-Milne Expansion.** We divide this process into several steps:

**Step 1: $\epsilon$-Milne Problem.**

We solve the $\epsilon$-Milne problem (4.11) for $q_k^i$ with the in-flow boundary data

\begin{equation}
(2.80) \quad h_k^i = - \left( (B_k^i + C_k^i) - \mathcal{P}[B_k^i + C_k^i] \right) + \sum_{i=1}^{k-1} \left( \frac{\mu_{k-1}(\bar{x}_0, \bar{v})}{\sqrt{\mu(\bar{v})}} \int_{\bar{n} \cdot \bar{v}^* > 0} \sqrt{\mu(\bar{v}^*)} (F_1^i + F_1^i) |\bar{n} \cdot \bar{v}^*| d\bar{v}^* \right) + \frac{\mu_k(\bar{x}_0, \bar{v})}{\sqrt{\mu(\bar{v})}}.
\end{equation}
and source term

\[
S_k^e = -\psi(\sqrt{\eta}) \frac{1}{1 - \epsilon_\eta} v_\phi \left( \frac{\partial \mathcal{F}_{k-1}^e}{\partial \phi} + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i^e, \mathcal{F}_{k-i}^e] + 2 \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i^e, \mathcal{F}_{k-i}^e] \right). 
\]

Based on Theorem 4.14 there exist modified boundary data

\[
\hat{h}_k = \sqrt{\mu(\bar{v})} \left( \bar{D}_{k,0}^e + \bar{D}_{k,1}^e v_\eta + \bar{D}_{k,2}^e v_\phi + \bar{D}_{k,3}^e |\bar{v}|^2 \right),
\]

such that the problem (4.8) is well-posed.

Step 2: Definition of Interior Solution and Boundary Layer with Geometric Correction.

Define \( \mathcal{F}_k^e = \mathcal{G}_k^e \cdot \psi_0 \), where \( \mathcal{G}_k^e \) the solution of the \( \epsilon \)-Milne problem (4.8). We have

\[
(2.82) \quad \lim_{\eta \to 0} \mathcal{F}_k^e(\eta, \phi, \bar{v}) = 0.
\]

(2.83) \quad \begin{align*}
A_{k,0}^e &= \bar{D}_{k,0}^e, \\
A_{k,1}^e &= \bar{D}_{k,1}^e \cos \phi - \bar{D}_{k,2}^e \sin \phi, \\
A_{k,2}^e &= \bar{D}_{k,1}^e \sin \phi + \bar{D}_{k,2}^e \cos \phi, \\
A_{k,3}^e &= \bar{D}_{k,3}^e. 
\end{align*}

This determines \( A_{k,1}^e, A_{k,2}^e \) and \( A_{k,3}^e \). Now we already show

\[
(2.87) \quad (\mathcal{F}_k^e + \mathcal{F}_k^e) = \mathcal{P}[\mathcal{F}_k^e + \mathcal{F}_k^e] + \sum_{i=1}^{k-1} \left( \frac{\mu_{k-i}(\bar{x}_0, \bar{v})}{\sqrt{\mu(\bar{v})}} \int_{\bar{v}_\eta > 0} \sqrt{\mu(\bar{v})} (\mathcal{F}_i^e + \mathcal{F}_i^e) |\bar{v}_\eta| d\bar{v}_\eta \right) + \frac{\mu_k(\bar{x}_0, \bar{v})}{\sqrt{\mu(\bar{v})}}.
\]

Based on Lemma 4.1 and our construction, we have

\[
(2.88) \quad \mathcal{P}[A_{k}^e + \mathcal{F}_k^e] = 0.
\]

Hence, boundary condition is satisfied.

Step 3: Normalization Adjustment.

If \( \mathcal{F}_k^e + \mathcal{F}_k^e \) satisfies the boundary condition, then \( \mathcal{F}_k^e + \mathcal{F}_k^e + C \) for any constant \( C \) is also acceptable. Also, from the interior relation, \( \rho_k^e \) can be determined uniquely up to a constant. Then we require the normalization condition as

\[
(2.89) \quad \int_{\Omega} \int_{\mathbb{R}^2} \sqrt{\mu(\mathcal{F}_k^e + \mathcal{F}_k^e)} d\bar{v} d\bar{\varepsilon} = 0.
\]

where we can adjust the value of \( \rho_k^e \). Hence, all the data are uniquely determined.

This process can be applied to arbitrary \( k \) since now \( \phi \) derivative is well-posed in \( L^\infty \). In particular, when \( k = 1 \), \( \mathcal{F}_1^e \) satisfies

\[
(2.90) \quad \begin{cases}
\begin{align*}
v_\eta \frac{\partial \mathcal{G}_1^e}{\partial \eta} + G(\epsilon; \eta) \left( v_\phi \frac{\partial \mathcal{G}_1^e}{\partial \eta} - v_\eta v_\phi \frac{\partial \mathcal{G}_1^e}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{G}_1^e] &= 0, \\
\mathcal{G}_1^e(0, \phi, \bar{v}) &= h^e(\phi, \bar{v}) - \hat{h}^e(\phi, \bar{v}) \quad \text{for} \quad v_\eta > 0, \\
\int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \mathcal{G}_1^e(0, \phi, \bar{v}) d\bar{v} &= \int_{v_\eta > 0} v_\eta \sqrt{\mu} h^e(\phi, \bar{v}) d\bar{v} \\
&\quad - \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} h^e(\phi, \bar{v}) d\bar{v}, \\
\lim_{\eta \to \infty} \mathcal{G}_1^e(\eta, \phi, \bar{v}) &= 0,
\end{align*}
\end{cases}
\]
with
\begin{equation}
\hat{h}_1' = \sqrt{\mu(v)} \left( \tilde{D}_{1,0}^\epsilon + \tilde{D}_{1,1}^\epsilon v_\eta + \tilde{D}_{1,2}^\epsilon v_\phi + \tilde{D}_{1,3}^\epsilon \|v\|^2 \right),
\end{equation}
and \( \mathcal{F}_1^\epsilon \) satisfies
\begin{equation}
\mathcal{F}_1^\epsilon = \sqrt{\mu} \left( \rho_1^\epsilon + 2u_{1,1}^\epsilon v_1 + 2u_{1,2}^\epsilon v_2 + \theta_1^\epsilon (|v|^2 - 1) \right),
\end{equation}
with
\begin{equation}
\begin{cases}
\nabla_x (\rho_1^\epsilon + \theta_1^\epsilon) = 0, \\
u_1^\epsilon \cdot \nabla_x u_1^\epsilon - \frac{1}{2} \gamma_1 \Delta_x u_1^\epsilon + \frac{1}{2} \nabla_x P_2^\epsilon = 0, \\
abla_x \cdot u_1^\epsilon = 0, \\
u_1^\epsilon \cdot \nabla_x \theta_1^\epsilon - \frac{1}{2} \gamma_2 \Delta_x \theta_1^\epsilon = 0, \\
u_{1,1}^\epsilon (\tilde{x}_0) = \tilde{D}_{1,1}^\epsilon (\tilde{x}_0) \cos \phi - \tilde{D}_{1,2}^\epsilon (\tilde{x}_0) \sin \phi, \\
u_{1,2}^\epsilon (\tilde{x}_0) = \tilde{D}_{1,1}^\epsilon (\tilde{x}_0) \sin \phi + \tilde{D}_{1,2}^\epsilon (\tilde{x}_0) \cos \phi, \\
\theta_1^\epsilon (\tilde{x}_0) = \tilde{D}_{1,3}^\epsilon (\tilde{x}_0),
\end{cases}
\end{equation}
and the normalization condition
\begin{equation}
\int_{\Omega} \int_{\mathbb{R}^2} \sqrt{\mu} (\mathcal{F}_1^\epsilon + \mathcal{F}_1^\epsilon') d\tilde{x} d\tilde{\varepsilon} = 0.
\end{equation}
3. LINEARIZED STEADY BOLTZMANN EQUATION

We consider the linearized steady Boltzmann equation

\[
\begin{align*}
\epsilon \vec{v} \cdot \nabla_x f + L[f] &= S(\vec{x}, \vec{v}) \quad \text{in } \Omega, \\
f(\vec{x}_0, \vec{v}) - P[f](\vec{x}_0, \vec{v}) &= h(\vec{x}_0, \vec{v}) \quad \text{for } \vec{x}_0 \in \partial \Omega \text{ and } \vec{v} \cdot \vec{n} < 0,
\end{align*}
\]

provided the compatibility condition

\[
\int_{\Omega} \int_{\mathbb{R}^2} S(\vec{x}, \vec{v}) \sqrt{\mu} \, d\vec{v} \, d\vec{x} = 0, \quad \int_{\gamma^{-}} h(\vec{x}, \vec{v}) \sqrt{\mu} \, d\gamma = 0.
\]

It is easy to see if \( f \) is a solution to (3.1), then \( f + C \sqrt{\mu} \) is also a solution for arbitrary \( C \in \mathbb{R} \). Hence, we require the solution should satisfy the normalization condition

\[
\int_{\Omega} \int_{\mathbb{R}^2} f(\vec{x}, \vec{v}) \sqrt{\mu} \, d\vec{v} \, d\vec{x} = 0.
\]

Let \( \langle \cdot, \cdot \rangle \) be the standard \( L^2 \) inner product in \( \Omega \times \mathbb{R}^2 \). We define the \( L^p \) and \( L^\infty \) norms in \( \Omega \times \mathbb{R}^2 \) as usual:

\[
\| f \|_{L^p} = \left( \int_{\Omega} \int_{\mathbb{R}^2} |f(\vec{x}, \vec{v})|^p \, d\vec{v} \, d\vec{x} \right)^{1/p},
\]

\[
\| f \|_{L^\infty} = \sup_{(\vec{x}, \vec{v}) \in \Omega \times \mathbb{R}^2} |f(\vec{x}, \vec{v})|.
\]

Define \( d\gamma = |\vec{v} \cdot \vec{n}| \, d\vec{x} \, d\vec{v} \) on the boundary \( \partial \Omega \times \mathbb{R}^2 \) for \( \omega \) as the curve measure. Define the \( L^p \) and \( L^\infty \) norms on the boundary as follows:

\[
|f|_{L^p} = \left( \int_{\gamma} |f(\vec{x}, \vec{v})|^p \, d\gamma \right)^{1/p},
\]

\[
|f|_{L^p_{\pm}} = \left( \int_{\gamma_{\pm}} |f(\vec{x}, \vec{v})|^p \, d\gamma \right)^{1/p},
\]

\[
|f|_{L^\infty} = \sup_{(\vec{x}, \vec{v}) \in \gamma} |f(\vec{x}, \vec{v})|,
\]

\[
|f|_{L^\infty_{\pm}} = \sup_{(\vec{x}, \vec{v}) \in \gamma_{\pm}} |f(\vec{x}, \vec{v})|.
\]

Also, we define

\[
\| f \|_{L^2_{\pm}} = \| \sqrt{\mu} f \|_{L^2}.
\]

Denote the Japanese bracket as

\[
\langle \vec{v} \rangle = \sqrt{1 + |\vec{v}|^2}
\]

Define the kernel operator \( P \) as

\[
P[f] = \sqrt{\mu} \left( a_f(t, \vec{x}) + \vec{v} \cdot \vec{b}_f(t, \vec{x}) + \frac{|\vec{v}|^2 - 2}{2} c_f(t, \vec{x}) \right),
\]

and the non-kernel operator \( \mathbb{I} - P \) as

\[
(\mathbb{I} - P)[f] = f - P[f].
\]

with

\[
\int_{\mathbb{R}^2} (\mathbb{I} - P)[f] \left( \begin{array}{c} \vec{v} \\ |\vec{v}|^2 \end{array} \right) \, d\vec{v} = 0
\]

Our analysis is based on the ideas in \[3, 7\].
3.1. Preliminaries.

Lemma 3.1. Define the near-grazing set of $\gamma_+$ or $\gamma_-$ as

$$
\gamma_{\pm}^\delta = \left\{ (\bar{x}, \bar{v}) \in \gamma_{\pm} : |\bar{v}(\bar{x}) \cdot \bar{v}| \leq \delta \text{ or } |\bar{v}| \geq \frac{1}{\delta} \text{ or } |\bar{v}| \leq \delta \right\}.
$$

Then

$$
|f1_{\gamma_+ \setminus \gamma_{\pm}^\delta}|_{L^1} \leq C(\delta) \left( \|f\|_{L^1} + \|\bar{v} \cdot \nabla_x f\|_{L^1} \right).
$$

Proof. See the proof of [3, Lemma 2.1].

Lemma 3.2. (Green’s Identity) Assume $f(\bar{x}, \bar{v}), \; g(\bar{x}, \bar{v}) \in L^2(\Omega \times \mathbb{R}^2)$ and $\bar{v} \cdot \nabla_x f, \; \bar{v} \cdot \nabla_x g \in L^2(\Omega \times \mathbb{R}^2)$ with $f, \; g \in L^2(\gamma)$. Then

$$
\iint_{\Omega \times \mathbb{R}^2} ((\bar{v} \cdot \nabla_x f)g + (\bar{v} \cdot \nabla_x g)f) d\bar{x} d\bar{v} = \int_{\gamma_+} fg d\gamma - \int_{\gamma_-} fg d\gamma.
$$

Proof. See the proof of [3, Lemma 2.2].

Lemma 3.3. For any $\lambda > 0$, there exists a unique solution $f_\lambda(\bar{x}, \bar{v}) \in L^\infty(\Omega \times \mathbb{R}^2)$ to the equation

$$
\begin{cases}
\lambda f_\lambda + e\bar{v} \cdot \nabla_x f_\lambda &= S(\bar{x}, \bar{v}) \text{ in } \Omega, \\
f_\lambda(\bar{x}_0, \bar{v}) &= h(\bar{x}_0, \bar{v}) \text{ for } \bar{x}_0 \in \partial \Omega \text{ and } \bar{v} \cdot \bar{v} < 0,
\end{cases}
$$

such that

$$
\left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} f_\lambda \right\|_{L^\infty} \leq C \left( \frac{1}{\lambda^2} \left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} S \right\|_{L^\infty} + \left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} h \right\|_{L^\infty} \right),
$$

for all $\theta \geq 0$, $\zeta \geq 0$, and

$$
\left\| f_\lambda \right\|_{L^2}^2 + \frac{\zeta}{\lambda} \left\| f_\lambda \right\|_{L^2}^2 \leq C \left( \frac{1}{\lambda^2} \|S\|_{L^2}^2 + \frac{\zeta}{\lambda} \|h\|_{L^2}^2 \right).
$$

Proof. The characteristics $(X(s), V(s))$ of the equation (3.18) which goes through $(\bar{x}, \bar{v})$ is defined by

$$
\begin{cases}
(X(0), V(0)) &= (\bar{x}, \bar{v}) \\
\frac{dX(s)}{ds} &= eV(s), \\
\frac{dV(s)}{ds} &= 0,
\end{cases}
$$

which implies

$$
\begin{cases}
X(s) &= \bar{x} + \epsilon s\bar{v} \\
V(s) &= \bar{v}
\end{cases}
$$

Define the backward exit time $t_b(\bar{x}, \bar{v})$ and backward exit position $\bar{x}_b(\bar{x}, \bar{v})$ as

$$
\begin{align*}
t_b(\bar{x}, \bar{v}) &= \inf\{ t > 0 : \bar{x} - \epsilon t\bar{v} \notin \Omega \}, \\
\bar{x}_b(\bar{x}, \bar{v}) &= \bar{x} - \epsilon t_b(\bar{x}, \bar{v}) \bar{v} \notin \Omega.
\end{align*}
$$

Hence, we can rewrite the equation (3.18) along the characteristics as

$$
\langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} f_\lambda(\bar{x}, \bar{v}) = \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} h(\bar{x}_b, \bar{v}) e^{-\lambda t_b} + \int_0^{t_b} \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} S(\bar{x}_b + \epsilon s\bar{v}, \bar{v}) e^{-\lambda(t_b-s)} ds.
$$

Then we can naturally estimate

$$
\left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} f_\lambda \right\|_{L^\infty} \leq e^{-\lambda t_b} \left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} h \right\|_{L^\infty} + \frac{1 - e^{-\lambda t_b}}{\lambda} \left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} S \right\|_{L^\infty}
\leq \left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} h \right\|_{L^\infty} + \frac{1}{\lambda} \left\| \langle \bar{v} \rangle^\theta e^{\xi |\bar{v}|^2} S \right\|_{L^\infty},
$$
which further implies
\begin{equation}
\left| \langle \iota \rangle^\theta \, e^\iota \langle \iota \rangle^2 f_\lambda \right|_{L^\infty} \leq \left| \langle \iota \rangle^\theta \, e^\iota \langle \iota \rangle^2 \eta \right|_{L^\infty} + \frac{1}{\lambda} \left\| \langle \iota \rangle^\theta \, e^\iota \langle \iota \rangle^2 S \right\|_{L^\infty}.
\end{equation}

Since \(f_\lambda\) can be explicitly traced back to the boundary data, the existence naturally follows from above estimate. The uniqueness and \(L^2\) estimates follow from Green’s identity and \(\|f_\lambda\|_{L^2} \leq C \left\| \langle \iota \rangle^\theta \, e^\iota \langle \iota \rangle^2 f_\lambda \right\|_{L^\infty} \).

3.2. \(L^2\) Estimates of Linearized Steady Boltzmann Equation.

**Lemma 3.4.** For any \(\lambda > 0, m > 0\), and for any integer \(j > 0\), there exists a unique solution \(f_{\lambda,m,j}(\bar{x}, \bar{v}) \in L^2(\Omega \times \mathbb{R}^2)\) to the equation
\begin{equation}
\left\{ \begin{array}{l}
\lambda f_{\lambda,m,j} + \epsilon \bar{v} \cdot \nabla_x f_{\lambda,m,j} + \mathcal{L}_m[f_{\lambda,m,j}] = S(\bar{x}, \bar{v}) \text{ in } \Omega, \\
F_{\lambda,m}(\bar{x}_0, \bar{v}) - \left(1 - \frac{1}{j}\right) \mathcal{P}[f_{\lambda,m,j}] = h(\bar{x}_0, \bar{v}) \text{ for } \bar{x}_0 \in \partial \Omega \text{ and } \bar{v} \cdot \hat{n} < 0,
\end{array} \right.
\end{equation}
with \(\mathcal{L}_m\) the linearized Boltzmann operator corresponding to the cut-off cross section \(B_m = \min\{B, m\}\). Moreover, uniformly in \(j\), the solution satisfies
\begin{equation}
\|f_{\lambda,m,j}\|_{L^2} + |f_{\lambda,m,j}|_{L^2} \leq C(\lambda, m, j) \left(\|S\|_{L^2} + |h|_{L^2}\right).
\end{equation}
Finally, the limit \(f_{\lambda,m,j} \to f_{\lambda,m}\) as \(j \to \infty\) exists and solves uniquely the equation
\begin{equation}
\left\{ \begin{array}{l}
\lambda f_{\lambda,m} + \epsilon \bar{v} \cdot \nabla_x f_{\lambda,m} + \mathcal{L}_m[f_{\lambda,m}] = S(\bar{x}, \bar{v}) \text{ in } \Omega, \\
F_{\lambda,m}(\bar{x}_0, \bar{v}) - \mathcal{P}[f_{\lambda,m}] = h(\bar{x}_0, \bar{v}) \text{ for } \bar{x}_0 \in \partial \Omega \text{ and } \bar{v} \cdot \hat{n} < 0,
\end{array} \right.
\end{equation}
satisfying
\begin{equation}
\|f_{\lambda,m}\|_{L^2} + |f_{\lambda,m}|_{L^2} \leq C(\lambda, m) \left(\|S\|_{L^2} + |h|_{L^2}\right).
\end{equation}

**Proof.** We divide the proof into several steps:

Step 1: Definition of iteration. Denote \(E_m = \nu_m - K_m\). For any \(j\), we define the iteration in \(l\): \(f_{\lambda,m,j}^0 = 0\) and for \(l \geq 0\),
\begin{equation}
\left\{ \begin{array}{l}
\lambda f_{\lambda,m,j}^{l+1} + \epsilon \bar{v} \cdot \nabla_x f_{\lambda,m,j}^{l+1} + (1 + M)\nu_m f_{\lambda,m,j}^{l+1} = S(\bar{x}, \bar{v}) - (K_m + M\nu_m)[f_{\lambda,m,j}^l], \\
f_{\lambda,m,j}^{l+1}(\bar{x}_0, \bar{v}) - \left(1 - \frac{1}{j}\right) \mathcal{P}[f_{\lambda,m,j}^l] = h(\bar{x}_0, \bar{v}) \text{ for } \bar{x}_0 \in \partial \Omega \text{ and } \bar{v} \cdot \hat{n} < 0,
\end{array} \right.
\end{equation}
where \(M > 0\) is a fixed real number to be determined later. Since
\begin{equation}
\|(K_m + M\nu_m)[f_{\lambda,m,j}^l]\|_{L^2} \leq C(m, M) \|f_{\lambda,m,j}^l\|_{L^2},
\end{equation}
\begin{equation}
\|\mathcal{P}[f_{\lambda,m,j}^l]\|_{L^2} \leq |f_{\lambda,m,j}^l|_{L^2},
\end{equation}
Lemma 3.3 implies \(f_{\lambda,m,j}^l \in L^2(\Omega \times \mathbb{R}^2)\) are well-defined for \(l \geq 0\). However, we cannot directly obtain the existence of limit \(f_{\lambda,m,j}^l\) as \(l \to \infty\).

Step 2: The limit \(l \to \infty\). Based on Green’s identity in Lemma 3.2, we have
\begin{equation}
\lambda \left| f_{\lambda,m,j}^{l+1} \right|_{L^2}^2 + \frac{\epsilon}{2} \left| f_{\lambda,m,j}^{l+1} \right|_{L^2}^2 + \left(1 + M\right)\nu_m f_{\lambda,m,j}^{l+1}, f_{\lambda,m,j}^{l+1} \right\rangle = \left( (K_m + M\nu_m)[f_{\lambda,m,j}^l], f_{\lambda,m,j}^{l+1} \right) + \frac{\epsilon}{2} \left| f_{\lambda,m,j}^{l+1} \right|_{L^2}^2 + \left( f_{\lambda,m,j}^{l+1}, S \right).
\end{equation}
Since $L_m = \nu_m - K_m$ is a non-negative symmetric operator, we can always find $M$ sufficiently large such that $K_m + M\nu_m$ is also a non-negative operator. Then we deduce

\begin{equation}
\langle (K_m + M\nu_m)[f^{l+1}_{\lambda,m,j}], f^{l+1}_{\lambda,m,j} \rangle 
\leq \sqrt{\langle (K_m + M\nu_m)[f^{l}_{\lambda,m,j}], f^{l}_{\lambda,m,j} \rangle} \sqrt{\langle (K_m + M\nu_m)[f^{l+1}_{\lambda,m,j+1}], f^{l+1}_{\lambda,m,j} \rangle}
\end{equation}

\begin{align*}
&\leq \frac{1}{2} \left( \langle (K_m + M\nu_m)[f^{l}_{\lambda,m,j}], f^{l}_{\lambda,m,j} \rangle + \langle (K_m + M\nu_m)[f^{l+1}_{\lambda,m,j+1}], f^{l+1}_{\lambda,m,j} \rangle \right) \\
&\leq \frac{1}{2} \left( \langle (1 + M)\nu_m[f^{l}_{\lambda,m,j}], f^{l}_{\lambda,m,j} \rangle + \langle (1 + M)\nu_m[f^{l+1}_{\lambda,m,j+1}], f^{l+1}_{\lambda,m,j} \rangle \right).
\end{align*}

Moreover, we have

\begin{equation}
\left| \left( 1 - \frac{1}{j} \right) P[f^{l}_{\lambda,m,j}] + h \right|_{L^2}^2 \leq \left| \left( 1 - \frac{1}{j} \right) P[f^{l}_{\lambda,m,j}] \right|_{L^2}^2 + \frac{1}{2j^2} \left| P[f^{l}_{\lambda,m,j}] \right|_{L^2}^2 + (1 + j^2) \left| h \right|_{L^2}^2.
\end{equation}

Considering the fact

\begin{equation}
\left| P[f^{l}_{\lambda,m,j}] \right|_{L^2}^2 \leq \left| f^{l}_{\lambda,m,j} \right|_{L^2}^2,
\end{equation}

\begin{equation}
\langle (1 + M)\nu_m[f^{l}_{\lambda,m,j}], f^{l}_{\lambda,m,j} \rangle \leq (1 + M)m \left| f^{l}_{\lambda,m,j} \right|_{L^2}^2,
\end{equation}

we obtain

\begin{align*}
&\frac{\lambda}{(1 + M)m} + 1 \left( 1 + M \right)\nu_m[f^{l+1}_{\lambda,m,j}], f^{l+1}_{\lambda,m,j}] + \frac{\epsilon}{2} \left| f^{l+1}_{\lambda,m,j} \right|_{L^2}^2 \\
&\leq \frac{1}{2} \left( \langle (1 + M)\nu_m[f^{l}_{\lambda,m,j}], f^{l}_{\lambda,m,j} \rangle + \langle (1 + M)\nu_m[f^{l+1}_{\lambda,m,j}, f^{l+1}_{\lambda,m,j} \rangle \right) + \frac{\epsilon}{2} \left( 1 - \frac{2}{j} + \frac{3}{j^2} \right) \left| f^{l}_{\lambda,m,j} \right|_{L^2}^2 \\
&\quad + \frac{\epsilon}{2} \left( 1 + j^2 \right) \left| h \right|_{L^2}^2 + \frac{\lambda}{2(1 + M)m} \left( \langle 1 + M \rangle\nu_m[f^{l+1}_{\lambda,m,j}, f^{l+1}_{\lambda,m,j} \rangle + (1 + M)m \left| f^{l+1}_{\lambda,m,j} \right|_{L^2}^2 \right).
\end{align*}

Since

\begin{equation}
\frac{\lambda}{(1 + M)m} + 1 - \frac{2}{j} + \frac{3}{j^2} < 1,
\end{equation}

\begin{equation}
\frac{\lambda}{1 + (1 + M)m} > \frac{1}{2},
\end{equation}

by iteration over $l$, for

\begin{equation}
C_1(\lambda, m, j) = \max \left\{ \frac{1}{1 + (1 + M)m}, 1 - \frac{2}{j} + \frac{3}{j^2} \right\} < 1,
\end{equation}

\begin{equation}
C_2(\lambda, m, j) = \frac{1}{1 + \frac{\lambda}{(1 + M)m}} \max \left\{ \frac{(1 + M)m}{\lambda}, (1 + j^2) \right\} > 0,
\end{equation}

we have

\begin{align*}
&\epsilon \left| f^{l+1}_{\lambda,m,j} \right|_{L^2}^2 + \left( 1 + \frac{\lambda}{(1 + M)m} \right) \left( \langle 1 + M \rangle\nu_m[f^{l+1}_{\lambda,m,j}, f^{l+1}_{\lambda,m,j} \rangle \\
&\quad \leq C_1(\lambda, m, j) \left( \epsilon \left| f^{l}_{\lambda,m,j} \right|_{L^2}^2 + \left( 1 + \frac{\lambda}{(1 + M)m} \right) \langle 1 + M \rangle\nu_m[f^{l}_{\lambda,m,j}, f^{l}_{\lambda,m,j} \rangle \\
&\quad + C_2(\lambda, m, j) \left( \left| S \right|_{L^2}^2 + \epsilon \left| h \right|_{L^2}^2 \right). \right.
\end{align*}

Taking the difference of $f^{l+1}_{\lambda,m,j} - f^{l}_{\lambda,m,j}$, we conclude that $f^{l}_{\lambda,m,j}$ is a Cauchy sequence. We take $l \to \infty$ to obtain $f_{\lambda,m,j}$ as a solution to the equation \ref{eq:3.28} satisfying

\begin{equation}
\epsilon \left| f^{l}_{\lambda,m,j} \right|_{L^2}^2 + \left( 1 + \frac{\lambda}{(1 + M)m} \right) \langle 1 + M \rangle\nu_m[f^{l}_{\lambda,m,j}, f^{l}_{\lambda,m,j} \rangle \leq \frac{C_2(\lambda, m, j)}{1 - C_1(\lambda, m, j)} \left( \left| S \right|_{L^2}^2 + \epsilon \left| h \right|_{L^2}^2 \right).
\end{equation}
However, the estimate is not uniform in $j \to \infty$.

Step 3: The limit $j \to \infty$.

By Green’s identity in Lemma 3.2, we have

\begin{equation}
\lambda \|f_{\lambda,m,j}\|_{L^2}^2 + \langle \mathcal{L}_m[f_{\lambda,m,j}], f_{\lambda,m,j} \rangle + \frac{\epsilon}{2} \|f_{\lambda,m,j}\|_{L^2}^2 = \frac{\epsilon}{2} \left(1 - \frac{1}{j}\right) \mathcal{P}[f_{\lambda,m,j}] + h \|f_{\lambda,m,j}\|_{L^2}^2 + \langle f_{\lambda,m,j}, S \rangle.
\end{equation}

We estimate for any $\eta > 0$,

\begin{equation}
\frac{\epsilon}{2} \left(1 - \frac{1}{j}\right) \mathcal{P}[f_{\lambda,m,j}] + h \|f_{\lambda,m,j}\|_{L^2}^2 = \frac{\epsilon}{2} \left(1 - \frac{1}{j}\right)^2 \|\mathcal{P}[f_{\lambda,m,j}]\|_{L^2}^2 + \frac{\epsilon}{2} |h|_{L^2}^2 + \epsilon \left(1 - \frac{1}{j}\right) \int_{\Omega^{-}} \mathcal{P}[f_{\lambda,m,j}] h d\gamma
\end{equation}

\begin{equation}
\leq \frac{\epsilon}{2} \|\mathcal{P}[f_{\lambda,m,j}]\|_{L^2}^2 + \left(1 + \frac{1}{2\eta}\right) \frac{\epsilon}{2} |h|_{L^2}^2 + \eta \frac{\epsilon}{2} \|\mathcal{P}[f_{\lambda,m,j}]\|_{L^2}^2.
\end{equation}

From

\begin{equation}
\langle f_{\lambda,m,j}, S \rangle \leq \frac{\lambda}{2} \|f_{\lambda,m,j}\|_{L^2}^2 + \frac{1}{2\lambda} \|S\|_{L^2}^2,
\end{equation}

and the spectral gap of $\mathcal{L}_m$, which is actually

\begin{equation}
\langle \mathcal{L}_m[f_{\lambda,m,j}], f_{\lambda,m,j} \rangle \geq C_3 \|\mathcal{P}[f_{\lambda,m,j}]\|_{L^2}^2,
\end{equation}

we obtain

\begin{equation}
\lambda \|f_{\lambda,m,j}\|_{L^2}^2 + 2C_3 \|\mathcal{P}[f_{\lambda,m,j}]\|_{L^2}^2 + \epsilon \|\mathcal{P}[f_{\lambda,m,j}]\|_{L^2}^2
\end{equation}

\begin{equation}
\leq C_4(\eta, \lambda) \left(\|S\|_{L^2}^2 + \epsilon |h|_{L^2}^2\right) + \eta \epsilon \|\mathcal{P}[f_{\lambda,m,j}]\|_{L^2}^2,
\end{equation}

due to

\begin{equation}
\|1 - \mathcal{P}\|f_{\lambda,m,j}\|_{L^2}^2 = \int_{\gamma^+} \left(f_{\lambda,m,j}^2 - 2 \mathcal{P}[f_{\lambda,m,j}] + \left(\mathcal{P}[f_{\lambda,m,j}]^2\right)\right) |\vec{n} \cdot \vec{v}| \, d\vec{v} \, d\vec{x}
\end{equation}

\begin{equation}
= \int_{\Omega} \int_{\vec{v} > 0} f_{\lambda,m,j}^2 |\vec{n} \cdot \vec{v}| \, d\vec{v} \, d\vec{x} - 2 \int_{\Omega} \int_{\vec{v} > 0} f_{\lambda,m,j} \sqrt{\mu(\vec{v})} z_\gamma(\vec{x}) |\vec{n} \cdot \vec{v}| \, d\vec{v} \, d\vec{x}
\end{equation}

\begin{equation}
+ \int_{\Omega} \int_{\vec{v} > 0} \mu(\vec{v}) z_\gamma^2(\vec{x}) |\vec{n} \cdot \vec{v}| \, d\vec{v} \, d\vec{x}
\end{equation}

\begin{equation}
= \int_{\Omega} \left(\int_{\vec{v} > 0} f_{\lambda,m,j}^2 |\vec{n} \cdot \vec{v}| \, d\vec{v} - 2 z_\gamma(\vec{x}) \int_{\vec{v} > 0} f_{\lambda,m,j} \sqrt{\mu(\vec{v})} |\vec{n} \cdot \vec{v}| \, d\vec{v}
\end{equation}

\begin{equation}
+ z_\gamma^2(\vec{x}) \int_{\vec{v} > 0} \mu(\vec{v}) |\vec{n} \cdot \vec{v}| \, d\vec{v}\right) \, d\vec{x}
\end{equation}

\begin{equation}
= \int_{\Omega} \left(\int_{\vec{v} > 0} f_{\lambda,m,j}^2 |\vec{n} \cdot \vec{v}| \, d\vec{v} - 2 z_\gamma^2(\vec{x}) \int_{\vec{v} > 0} \mu(\vec{v}) |\vec{n} \cdot \vec{v}| \, d\vec{v}\right) \, d\vec{x}
\end{equation}

\begin{equation}
\leq \|f_{\lambda,m,j}\|_{L^2}^2 - \|\mathcal{P}[f_{\lambda,m,j}]\|_{L^2}^2
\end{equation}

\begin{equation}
\leq \|f_{\lambda,m,j}\|_{L^2}^2 - \|\mathcal{P}[f_{\lambda,m,j}]\|_{L^2}^2,
\end{equation}

where

\begin{equation}
\mathcal{P} f_{\lambda,m,j} = \sqrt{\mu(\vec{v})} \int_{\vec{v} > 0} \sqrt{\mu(\vec{v})} f_{\lambda,m,j}(\vec{x}, \vec{v}) |\vec{n} \cdot \vec{v}| \, d\vec{v} = \sqrt{\mu(\vec{v})} z_\gamma(\vec{x}),
\end{equation}

\begin{equation}
\int_{\vec{v} > 0} \mu(\vec{v}) |\vec{n} \cdot \vec{v}| \, d\vec{v} = 1,
\end{equation}
and
\[(3.55)\quad C_4(\eta, \lambda) = \max \left\{ \frac{1}{2\lambda}, 1 + \frac{1}{2\eta} \right\}.\]

On the other hand, multiplying \(f_{\lambda,m,j}\) on both sides of the equation \[(3.30)\], we have
\[(3.56)\quad \varepsilon \bar{v} \cdot \nabla_x (f_{\lambda,m,j})^2 = -2\lambda (f_{\lambda,m,j})^2 - 2 f_{\lambda,m,j} \mathcal{L}_m [f_{\lambda,m,j}] + 2 f_{\lambda,m,j} S.\]

Taking absolute value and integrating \[(3.56)\] over \(\Omega \times \mathbb{R}^2\), from \[(3.51)\] we deduce
\[(3.57)\quad \left\| \bar{v} \cdot \nabla_x (f_{\lambda,m,j}) \right\|^2_1 \leq \frac{1 + \lambda}{\epsilon} \left( \|f_{\lambda,m,j}\|_{L^2}^2 + \| (I - \mathcal{P}[f_{\lambda,m,j}]) \|_{L^2}^2 + \| S \|_{L^2}^2 \right) \leq \frac{1 + \lambda}{\epsilon} C_4(\eta, \lambda) \left( \|S\|_{L^2}^2 + |h|_{L^2}^2 \right) + (1 + \lambda) \eta |\mathcal{P}[f_{\lambda,m,j}]|_{L^2}^2.\]

Hence, by Lemma \[3.1\] for any \(\gamma \setminus \gamma'\) away from \(\gamma_0\), we have
\[(3.58)\quad \left| 1_{\gamma \setminus \gamma'} f_{\lambda,m,j} \right|_2 \leq C_5(\delta) \left( \frac{1 + \lambda}{\epsilon} C_4(\eta, \lambda) \left( \|S\|_{L^2}^2 + |h|_{L^2}^2 \right) + (1 + \lambda) \eta |\mathcal{P}[f_{\lambda,m,j}]|_{L^2}^2 \right).\]

Based on the definition, we can rewrite \(\mathcal{P} f = z_{\gamma}(\bar{x}) \sqrt{\nu}\) for a suitable function \(z_{\gamma}(\bar{x})\) and from
\[(3.59)\quad \left| \mathcal{P}[1_{\gamma \setminus \gamma'} f_{\lambda,m,j}] \right|_2 \leq \left| 1_{\gamma \setminus \gamma'} f_{\lambda,m,j} \right|_2 < \infty,\]

for \(\delta'\) small, we deduce
\[(3.60)\quad \left| \mathcal{P}[1_{\gamma \setminus \gamma'} f_{\lambda,m,j}] \right|_2 = \int_{\partial \Omega} \left| z_{\gamma}(\bar{x}) \right|^2 \left( \int_{|\bar{v}(\bar{x})| = \delta', \delta' \leq |\bar{\nu}| \leq \frac{1}{\delta'}} \mu(\bar{v}) |\bar{n}(\bar{x}) \cdot \bar{v}| \, d\bar{v} \right) \, d\bar{x} \geq \frac{1}{2} \left( \int_{\partial \Omega} \left| z_{\gamma}(\bar{x}) \right|^2 \, d\bar{x} \right) \left( \int_{\mathbb{R}^2} \mu(\bar{v}) |\bar{n}(\bar{x}) \cdot \bar{v}| \, d\bar{v} \right) = \frac{1}{2} |\mathcal{P}[f_{\lambda,m,j}]|_2^2,\]

where we utilize the fact that
\[(3.61)\quad \int_{|\bar{v}(\bar{x})| = \delta'} \mu(\bar{v}) |\bar{n}(\bar{x}) \cdot \bar{v}| \, d\bar{v} \leq C_6 \delta',\]
\[(3.62)\quad \int_{|\bar{v}| \leq \delta' \text{ or } |\bar{v}| \geq \frac{1}{\delta'}} \mu(\bar{v}) |\bar{n}(\bar{x}) \cdot \bar{v}| \, d\bar{v} \leq C_6 \delta'.\]

Therefore, we conclude
\[(3.63)\quad \frac{1}{2} |\mathcal{P}[f_{\lambda,m,j}]|_2^2 - |(1 - \mathcal{P})[f_{\lambda,m,j}]|_2^2 \leq \left| \mathcal{P}[1_{\gamma \setminus \gamma'} f_{\lambda,m,j}] \right|_2^2 - \left| (1 - \mathcal{P})[1_{\gamma \setminus \gamma'} f_{\lambda,m,j}] \right|_2^2 \leq C_7 \left| 1_{\gamma \setminus \gamma'} f_{\lambda,m,j} \right|_2^2 \leq C_7 C_5(\delta) \left( \frac{1 + \lambda}{\epsilon} C_4(\eta, \lambda) \left( \|S\|_{L^2}^2 + |h|_{L^2}^2 \right) + (1 + \lambda) \eta |\mathcal{P}[f_{\lambda,m,j}]|_{L^2}^2 \right).\]

Adding \(2 \times (3.51)\) to \(\epsilon \times (3.63)\), we obtain
\[(3.64)\quad 2\lambda \|f_{\lambda,m,j}\|_{L^2}^2 + 4C_3 \| (I - \mathcal{P})[f_{\lambda,m,j}] \|_{L^2}^2 + \epsilon |(1 - \mathcal{P})[f_{\lambda,m,j}]|_{L^2}^2 + \frac{1}{2} |\mathcal{P}[f_{\lambda,m,j}]|_2^2 \leq 2C_7 C_5(\delta) \left( C_4(\eta, \lambda) \left( 1 + \frac{1 + \lambda}{\epsilon} \right) \left( \|S\|_{L^2}^2 + \epsilon |h|_{L^2}^2 \right) + \eta (1 + \lambda) |\mathcal{P}[f_{\lambda,m,j}]|_{L^2}^2 \right).\]

Taking \(\eta\) sufficiently small and pushing the weak limit \(j \to \infty\), we complete the proof. However, the estimate is not uniform in \(\lambda \to 0\). \(\Box\)
Lemma 3.5. Assume condition \((3.2)\) holds. Then the solution \(f_{\lambda,m}\) to the equation \((3.30)\) satisfies the estimate
\[
\epsilon \|P[f_{\lambda,m}]\|_{L^2} \leq C \left( \epsilon |(1 - P)[f_{\lambda,m}]|_{L^2} + \|(1 - P)[f_{\lambda,m}]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2} \right),
\]
for \(0 \leq \lambda \leq \epsilon << 1\) with
\[
\int_{\Omega} \int_{\mathbb{R}^2} f_{\lambda,m}(\vec{x}, \vec{v}) \sqrt{\mu d\vec{v} d\vec{x}} = 0.
\]

Proof. Applying Green’s identity in Lemma \(3.2\) to the solution of the equation \((3.30)\). Then for any \(\psi \in L^2(\Omega \times \mathbb{R}^2)\) satisfying \(\nabla \cdot \nabla_x \psi \in L^2(\Omega \times \mathbb{R}^2)\) and \(\psi \in L^2(\gamma)\), we have
\[
\int_{\Omega \times \mathbb{R}^2} \lambda f_{\lambda,m} \psi + \epsilon \int_{\gamma^+} f_{\lambda,m} \psi d\gamma - \epsilon \int_{\gamma^-} f_{\lambda,m} \psi d\gamma - \epsilon \int_{\Omega \times \mathbb{R}^2} (\vec{v} \cdot \nabla_x \psi) f_{\lambda,m} = - \int_{\Omega \times \mathbb{R}^2} \psi (1 - P)[f_{\lambda,m}] + \int_{\Omega \times \mathbb{R}^2} S \psi.
\]

Since
\[
\mathbb{P}[f] = \sqrt{\mu} \left( a + \vec{v} \cdot \vec{b} + |\vec{v}|^2 - \frac{2}{c} \right),
\]
our goal is to choose a particular test function \(\psi\) to estimate \(a, \vec{b}\) and \(c\).

Step 1: Estimates of \(c\).
We choose the test function
\[
\psi = \psi_c = \sqrt{\mu(\vec{v})} \left( |\vec{v}|^2 - \beta_c \right) (\vec{v} \cdot \nabla_x \phi_c(\vec{x})),
\]
where
\[
\left\{ \begin{array}{l}
-\Delta_x \phi_c(\vec{x}) = c(\vec{x}) \text{ in } \Omega, \\
\phi_c = 0 \text{ on } \partial \Omega,
\end{array} \right.
\]
and \(\beta_c\) is a real number to be determined later. Based on the standard elliptic estimates, we have
\[
\|\phi_c\|_{H^2} \leq C \|c\|_{L^2}.
\]

With the choice of \((3.69)\), the right-hand side (RHS) of \((3.67)\) is bounded by
\[
\text{RHS} \leq C \|c\|_{L^2} \left( \|(1 - P)[f_{\lambda,m}]\|_{L^2} + \|S\|_{L^2} \right).
\]

We have
\[
\vec{v} \cdot \nabla_x \psi_c = \sqrt{\mu(\vec{v})} \sum_{i,j=1}^2 \left( |\vec{v}|^2 - \beta_c \right) v_i v_j \partial_{ij} \phi_c(\vec{x}),
\]
so the left-hand side (LHS) of \((3.67)\) takes the form
\[
\text{LHS} = \lambda \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\vec{v})} \left( |\vec{v}|^2 - \beta_c \right) \left( \sum_{i=1}^2 v_i \partial_i \phi_c \right) \\
+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\vec{v})} \left( |\vec{v}|^2 - \beta_c \right) \left( \sum_{i=1}^2 v_i \partial_i \phi_c \right) (\vec{v} \cdot \vec{v}) \\
- \epsilon \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\vec{v})} \left( |\vec{v}|^2 - \beta_c \right) \left( \sum_{i,j=1}^2 v_i v_j \partial_{ij} \phi_c \right).
\]

We decompose
\[
(f_{\lambda,m})_{\gamma} = P f_{\lambda,m} + 1_{\gamma^+} (1 - P)[f_{\lambda,m}] + 1_{\gamma^-} h \text{ on } \gamma,
\]
\[
f_{\lambda,m} = \sqrt{\mu(\vec{v})} \left( a + \vec{v} \cdot \vec{b} + \frac{|\vec{v}|^2 - 2}{2} c \right) + (1 - P)[f_{\lambda,m}] \text{ in } \Omega \times \mathbb{R}^2.
\]
Note that the operator $\mathbb{P}$ and $1 - \mathbb{P}$ are defined independent of cut-off parameter $m$. We will choose $\beta_c$ such that

\begin{equation}
\int_{\mathbb{R}^2} \sqrt{\mu(\vec{v})} \left( |\vec{v}|^2 - \beta_c \right) v_i^2 \, d\vec{v} = 0 \quad \text{for} \quad i = 1, 2.
\end{equation}

Since $\mu(\vec{v})$ takes the form

\begin{equation}
\mu(\vec{v}) = C \exp \left( -\frac{|\vec{v}|^2}{2} \right),
\end{equation}

this $\beta_c$ can always be achieved. Now substitute (3.75) and (3.76) into (3.74). Then based on this choice of $\beta_c$ and oddness in $\vec{v}$, there is no $\mathbb{P}[f_{\lambda,m}]$ contribution in the first term, no $\mathbb{P}f_{\lambda,m}$ contribution in the second term, and no $a$ contribution in the third term of (3.74). Since $b$ contribution and the off-diagonal $c$ contribution in the third term of (3.74) also vanish due to the oddness in $\vec{v}$, we can simplify (3.74) into

\begin{equation}
\text{LHS} = \lambda \int_{\Omega \times \mathbb{R}^2} (1 - \mathbb{P})[f_{\lambda,m}] \sqrt{\mu(\vec{v})} \left( |\vec{v}|^2 - \beta_c \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_c \right) + \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\mathcal{y}_+} (1 - \mathbb{P})[f_{\lambda,m}] \sqrt{\mu(\vec{v})} \left( |\vec{v}|^2 - \beta_c \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_c \right) (\vec{n} \cdot \vec{v})
\end{equation}

\begin{equation}
+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\mathcal{y}_-} h \sqrt{\mu(\vec{v})} \left( |\vec{v}|^2 - \beta_c \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_c \right) (\vec{n} \cdot \vec{v})
\end{equation}

\begin{equation}
- \epsilon \sum_{i=1}^{2} \int_{\mathbb{R}^2} \mu(\vec{v}) |v_i|^2 \left( |\vec{v}|^2 - \beta_c \right) \frac{1}{2} |\vec{v}|^2 - 2 \frac{1}{2} \partial \Omega \partial_i \phi_c (x) d\vec{x}
\end{equation}

\begin{equation}
- \epsilon \int_{\Omega \times \mathbb{R}^2} (1 - \mathbb{P})[f_{\lambda,m}] \sqrt{\mu(\vec{v})} \left( |\vec{v}|^2 - \beta_c \right) \left( \sum_{i,j=1}^{2} v_i v_j \partial_{ij} \phi_c \right).
\end{equation}

Since

\begin{equation}
\int_{\mathbb{R}^2} \mu(\vec{v}) |v_i|^2 \left( |\vec{v}|^2 - \beta_c \right) \frac{1}{2} |\vec{v}|^2 - 2 \frac{1}{2} \, d\vec{v} = C,
\end{equation}

we have

\begin{equation}
\epsilon \left| \int_{\Omega} \Delta_x \phi_c (\vec{x}) c(\vec{x}) d\vec{x} \right| \leq C \|c\|_{L^2} \left( \epsilon |(1 - \mathbb{P})[f_{\lambda,m}]|_{L^2}^2 + (1 + \epsilon + \lambda) \|((1 - \mathbb{P})[f_{\lambda,m}]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2} \right),
\end{equation}

where we have used the elliptic estimates and the trace estimate: $|\nabla_x \phi_c|_{L^2} \leq C \|\phi_c\|_{H^2} \leq C \|\phi_c\|_{L^2}$. Since $-\Delta_x \phi_c = c$, we know

\begin{equation}
\epsilon \|c\|_{L^2}^2 \leq C \|c\|_{L^2} \left( \epsilon |(1 - \mathbb{P})[f_{\lambda,m}]|_{L^2}^2 + (1 + \epsilon + \lambda) \|((1 - \mathbb{P})[f_{\lambda,m}]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2} \right),
\end{equation}

which further implies

\begin{equation}
\epsilon \|c\|_{L^2} \leq C \epsilon |(1 - \mathbb{P})[f_{\lambda,m}]|_{L^2}^2 + (1 + \epsilon + \lambda) \|((1 - \mathbb{P})[f_{\lambda,m}]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2} \right).
\end{equation}

Step 2: Estimates of $\vec{b}$.

Step 2 - Phase 1: Estimates of $(\partial_{ij} \Delta_x^{-1} b_j)b_i$ for $i, j = 1, 2$.

We choose the test function

\begin{equation}
\psi = \psi_{b_i}^{i,j} = \sqrt{\mu(\vec{v})} \left( v_i^2 - \beta_b \right) \partial_j \phi_b^i,
\end{equation}

where

\begin{equation}
\left\{ \begin{array}{ll}
-\Delta_x \phi_b^i (\vec{x}) & = b_j (\vec{x}) \text{ in } \Omega, \\
\phi_b^i & = 0 \text{ on } \partial \Omega,
\end{array} \right.
\end{equation}
and $\beta_b$ is a real number to be determined later. Based on the standard elliptic estimates, we have

$$\|\phi_b^l\|_{H^2} \leq C \|\delta\|_{L^2},$$

With the choice of (3.84), the right-hand side (RHS) of (3.67) is bounded by

$$\text{RHS} \leq C \|\delta\|_{L^2} \left( \|I - P\| \|f_{\lambda,m}\|_{L^2} + \|S\|_{L^2} \right).$$

Hence, the left-hand side (LHS) of (3.67) takes the form

$$\text{LHS} = \lambda \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\mathbf{v})} \left( v_i^2 - \beta_b \right) \partial_j \phi_b^l$$

$$+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\mathbf{v})} \left( v_i^2 - \beta_b \right) \partial_j \phi_b^l (\mathbf{n} \cdot \mathbf{v})$$

$$- \epsilon \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\mathbf{v})} \left( v_i^2 - \beta_b \right) \left( \sum_{i=1}^{2} v_i \partial_j \phi_b^l \right).$$

Now substitute (3.75) and (3.76) into (3.88). Then based on the oddness in $\mathbf{v}$, there is no $\delta$ contribution in the first term, no $P [f_{\lambda,m}]$ contribution in the second term, and no $a$ and $c$ contribution in the third term of (3.88). We can simplify (3.88) into

$$\text{LHS} = \lambda \int_{\Omega \times \mathbb{R}^2} (I - P) [f_{\lambda,m}] \sqrt{\mu(\mathbf{v})} \left( v_i^2 - \beta_b \right) \partial_j \phi_b^l$$

$$+ \lambda \int_{\Omega \times \mathbb{R}^2} a(\mathbf{x}) \mu(\mathbf{v}) \left( v_i^2 - \beta_b \right) \partial_j \phi_b^l$$

$$+ \lambda \int_{\Omega \times \mathbb{R}^2} c(\mathbf{x}) \mu(\mathbf{v}) \frac{|\mathbf{v}|^2}{2} \left( v_i^2 - \beta_b \right) \partial_j \phi_b^l$$

$$+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma+} (1 - P) [f_{\lambda,m}] \sqrt{\mu(\mathbf{v})} \left( v_i^2 - \beta_b \right) \partial_j \phi_b^l (\mathbf{n} \cdot \mathbf{v})$$

$$+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma-} h \sqrt{\mu(\mathbf{v})} \left( v_i^2 - \beta_b \right) \partial_j \phi_b^l (\mathbf{n} \cdot \mathbf{v})$$

$$- \epsilon \sum_{l=1}^{2} \int_{\Omega \times \mathbb{R}^2} \mu(\mathbf{v}) v_i^2 \left( v_i^2 - \beta_b \right) \partial_j \phi_b^l (\mathbf{x} b_l)$$

$$- \epsilon \sum_{l=1}^{2} \int_{\Omega \times \mathbb{R}^2} (I - P) [f_{\lambda,m}] \sqrt{\mu(\mathbf{v})} \left( v_i^2 - \beta_b \right) v_l \partial_j \phi_b^l.$$

We will choose $\beta_b$ such that

$$\int_{\mathbb{R}^2} \mu(\mathbf{v}) \left(|v_i|^2 - \beta_b\right) d\mathbf{v} = 0 \quad \text{for} \quad i = 1, 2.$$

Since $\mu(\mathbf{v})$ takes the form

$$\mu(\mathbf{v}) = C \exp \left( -\frac{|\mathbf{v}|^2}{2} \right),$$

this $\beta_b$ can always be achieved. Based on this choice of $\beta_b$, we have

$$\lambda \int_{\Omega \times \mathbb{R}^2} a f_{\lambda,m} \mu(\mathbf{v}) \left( v_i^2 - \beta_b \right) \partial_j \phi_b^l = 0.$$

For such $\beta_b$ and any $i \neq l$, we can directly compute

$$\int_{\mathbb{R}^2} \mu(\mathbf{v}) \left(|v_i|^2 - \beta_b\right) v_i^2 d\mathbf{v} = 0,$$

$$\int_{\mathbb{R}^2} \mu(\mathbf{v}) \left(|v_i|^2 - \beta_b\right) v_l^2 d\mathbf{v} = C \neq 0.$$
Hence, the left-hand side (LHS) of (3.67) takes the form

\[ \sum_{i=1}^{2} \int_{\Omega \times \mathbb{R}^2} \mu(\bar{v}) v_i^2 \left( v_i^2 - \beta_b \right) \partial_{\bar{v}} \phi_b^i(\bar{x}) b_i \]

and

\[ \lambda \int_{\Omega \times \mathbb{R}^2} c(\bar{x}) \mu(\bar{v}) \left( |\bar{v}|^2 - \frac{2}{\lambda} (v_i^2 - \beta_b) \partial_{\bar{v}} \phi_b^i \right) = \lambda \int_{\Omega \times \mathbb{R}^2} c(\bar{x}) \mu(\bar{v}) \left( |\bar{v}|^2 - \frac{2}{\lambda} (v_i^2 - \beta_b) \partial_{\bar{v}} \phi_b^i \right). \]

Hence, by (3.83), we may estimate

\[ \varepsilon \int_{\Omega} \left( \int_{\Omega \times \mathbb{R}^2} \mu(\bar{v}) |\bar{v}|^2 \right) \left( \left( 1 - \mathcal{P} \right)[f_{\lambda,m}]_{L^2} + (1 + \varepsilon + \lambda) \left( \mathcal{P} + \mathcal{P} f_{\lambda,m} \right) \right) \]

Step 2 - Phase 2: Estimates of \((\partial_{\bar{v}} \Delta_{\bar{x}}^{-1} b_i) b_i\) for \(i \neq j\).

We choose the test function

\[ \psi = \sqrt{\mu(\bar{v})} |\bar{v}|^2 v_i v_j \partial_{\bar{v}} \phi_b^i \quad \text{for} \quad i \neq j. \]

The right-hand side (RHS) of (3.67) is still bounded by

\[ \text{RHS} \leq C \left( \left( 1 - \mathcal{P} \right)[f_{\lambda,m}]_{L^2} + \| S \|_{L^2} \right). \]

Hence, the left-hand side (LHS) of (3.67) takes the form

\[ \text{LHS} = \lambda \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\bar{v})} |\bar{v}|^2 v_i v_j \partial_{\bar{v}} \phi_b^i \]

\[ + \varepsilon \int_{\partial \Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\bar{v})} |\bar{v}|^2 v_i v_j \partial_{\bar{v}} \phi_b^i \]

Now substitute (3.75) and (3.76) into (3.100). Then based on the oddness in \(\bar{v}\), there is no \(\mathcal{P} f_{\lambda,m}\) contribution in the first term, no \(\mathcal{P} f_{\lambda,m}\) contribution in the second term, and no \(a\) and \(c\) contribution in the third term of (3.100). We can simplify (3.100) into

\[ \text{LHS} = \lambda \int_{\Omega \times \mathbb{R}^2} (1 - \mathcal{P})[f_{\lambda,m}] \sqrt{\mu(\bar{v})} |\bar{v}|^2 v_i v_j \partial_{\bar{v}} \phi_b^i \]

\[ + \varepsilon \int_{\partial \Omega \times \mathbb{R}^2} \mathbf{1}_{\gamma} (1 - \mathcal{P})[f_{\lambda,m}] \sqrt{\mu(\bar{v})} |\bar{v}|^2 v_i v_j \partial_{\bar{v}} \phi_b^i (\bar{n} \cdot \bar{v}) \]

\[ + \varepsilon \int_{\partial \Omega \times \mathbb{R}^2} \mathbf{1}_{\gamma} h \sqrt{\mu(\bar{v})} |\bar{v}|^2 v_i v_j \partial_{\bar{v}} \phi_b^i (\bar{n} \cdot \bar{v}) \]

\[ - \varepsilon \int_{\Omega \times \mathbb{R}^2} \mu(\bar{v}) |\bar{v}|^2 v_i^2 v_j^2 (\partial_{\bar{v}} \phi_b^i (\bar{x}) b_j + \partial_{\bar{v}} \phi_b^i (\bar{x}) b_i) \]

\[ - \varepsilon \sum_{l=1}^{2} \int_{\Omega \times \mathbb{R}^2} (1 - \mathcal{P})[f_{\lambda,m}] \sqrt{\mu(\bar{v})} |\bar{v}|^2 v_i v_j v_l \partial_{\bar{v}} \phi_b^l. \]
Then we deduce

\[
\epsilon \int_{\Omega \times \mathbb{R}^2} \mu(\vec{v}) |v|^2 v_i^2 v_j^2 (\partial_{ij} \phi_a^b(\vec{x})) b_j + \partial_{ij} \phi_a^b(\vec{x}) b_i = C \left( \int_{\Omega} (\partial_{ij} \Delta_x^{-1} b_i)b_j + \int_{\Omega} (\partial_{ij} \Delta_x^{-1} b_i)b_i \right).
\]

Hence, we may estimate for \( i \neq j \),

\[
\epsilon \int_{\Omega} (\partial_{ij} \Delta_x^{-1} b_i)b_i \leq C \left\| \vec{b} \right\|_{L^2} \left( \epsilon |(1 - \mathcal{P})[f_{\lambda,m}]|_{L^2} + (1 + \epsilon + \lambda) \left\| (I - \mathcal{P})[f_{\lambda,m}] \right\|_{L^2} \right) + \epsilon \left\| \vec{S} \right\|_{L^2} + \epsilon \left| h \right|_{L^2} + C \epsilon \left( \int_{\Omega} (\partial_{ij} \Delta_x^{-1} b_i)b_i \right).
\]

Moreover, by (3.101), for \( i = j, 1, 2 \),

\[
\epsilon \int_{\Omega} (\partial_{ij} \Delta_x^{-1} b_j)b_j \leq C \left\| \vec{b} \right\|_{L^2} \left( \epsilon |(1 - \mathcal{P})[f_{\lambda,m}]|_{L^2} + (1 + \epsilon + \lambda) \left\| (I - \mathcal{P})[f_{\lambda,m}] \right\|_{L^2} \right) + \epsilon \left\| \vec{S} \right\|_{L^2} + \epsilon \left| h \right|_{L^2} + (1 + \lambda) \left\| S \right\|_{L^2} + (\epsilon + \lambda) \left| h \right|_{L^2}.
\]

Step 2 - Phase 3: Synthesis.

Summarizing (3.103) and (3.104), we may sum up over \( j = 1, 2 \) to obtain, for any \( i = 1, 2 \),

\[
\epsilon \| b_i \|^2_{L^2} \leq C \left\| \vec{b} \right\|_{L^2} \left( \epsilon |(1 - \mathcal{P})[f_{\lambda,m}]|_{L^2} + (1 + \epsilon + \lambda) \left\| (I - \mathcal{P})[f_{\lambda,m}] \right\|_{L^2} \right) + (1 + \lambda) \left\| S \right\|_{L^2} + (\epsilon + \lambda) \left| h \right|_{L^2},
\]

which further implies

\[
\epsilon \left\| \vec{b} \right\|_{L^2} \leq C \left( \epsilon |(1 - \mathcal{P})[f_{\lambda,m}]|_{L^2} + (1 + \epsilon + \lambda) \left\| (I - \mathcal{P})[f_{\lambda,m}] \right\|_{L^2} + (1 + \lambda) \left\| S \right\|_{L^2} + (\epsilon + \lambda) \left| h \right|_{L^2} \right).
\]

Step 3: Estimates of \( a \).

We choose the test function

\[
\psi = \psi_a = \sqrt{\mu(\vec{v})} \left( |\vec{v}|^2 - \beta_a \right) (\vec{v} \cdot \nabla_x \phi_a(\vec{x})),
\]

where

\[
\begin{aligned}
-\Delta_x \phi_a(\vec{x}) &= a(\vec{x}) \text{ in } \Omega, \\
\partial_{ni} \phi_a &= 0 \text{ on } \partial \Omega,
\end{aligned}
\]

and \( \beta_a \) is a real number to be determined later. Based on the standard elliptic estimates with

\[
\int_{\Omega} a(\vec{x}) d\vec{x} = \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m}(\vec{x}, \vec{v}) d\vec{v} d\vec{x} = 0,
\]

we have

\[
\left\| \phi_a \right\|_{H^2} \leq C \| a \|_{L^2}.
\]

With the choice of (3.107), the right-hand side (RHS) of (3.67) is bounded by

\[
\text{RHS} \leq C \| a \|_{L^2} \left( \left\| (I - \mathcal{P})[f_{\lambda,m}] \right\|_{L^2} + \left\| S \right\|_{L^2} \right).
\]

We have

\[
\vec{v} \cdot \nabla_x \psi_a = \sqrt{\mu(\vec{v})} \sum_{i,j=1}^2 \left( |\vec{v}|^2 - \beta_a \right) v_i v_j \partial_{ij} \phi_a(\vec{x}),
\]
so the left-hand side (LHS) of \(3.67\) takes the form

\[
(3.113) \quad \text{LHS} = \lambda \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\bar{v})} \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) \\
+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\bar{v})} \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) (\vec{n} \cdot \vec{v}) \\
- \epsilon \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\bar{v})} \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i,j=1}^{2} v_i v_j \partial_{ij} \phi_a \right).
\]

We will choose \(\beta_a\) such that

\[
(3.114) \quad \int_{\mathbb{R}^2} \sqrt{\mu(\bar{v})} \left( |\bar{v}|^2 - \beta_a \right) \frac{|\bar{v}|^2 - 2}{2} v_i^2 d\bar{v} = 0 \quad \text{for} \quad i = 1, 2.
\]

Since

\[
(3.115) \quad \int_{\mathbb{R}^2} \sqrt{\mu(\bar{v})} \left( |\bar{v}|^2 - \beta_a \right) \frac{|\bar{v}|^2 - 2}{2} v_i^2 d\bar{v} \neq 0,
\]

this \(\beta_a\) can always be achieved. Now substitute \(3.73\) and \(3.76\) into \(3.113\). Then based on this choice of \(\beta_a\) and oddness in \(\bar{v}\), there is no \(a\) and \(c\) contribution in the first term, and no \(\bar{b}\) and \(c\) contribution in the third term of \(3.113\). Since \(\bar{b}\) contribution and the off-diagonal \(c\) contribution in the third term of \(3.113\) also vanish due to the oddness in \(\bar{v}\), we can simplify \(3.113\) into

\[
(3.116) \quad \text{LHS} = \lambda \int_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathbb{P}) [f_{\lambda,m}] \sqrt{\mu(\bar{v})} \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) \\
+ \lambda \int_{\Omega \times \mathbb{R}^2} \mu(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} b_i v_i^2 \partial_i \phi_a \right) \\
+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} \mathcal{P} [f_{\lambda,m}] \sqrt{\mu(\bar{v})} \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) (\vec{n} \cdot \vec{v}) \\
+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} \mathbf{1}_{\gamma+} (1 - \mathcal{P}) [f_{\lambda,m}] \sqrt{\mu(\bar{v})} \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) (\vec{n} \cdot \vec{v}) \\
+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} \mathbf{1}_{\gamma-} \sqrt{\mu(\bar{v})} \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) (\vec{n} \cdot \vec{v}) \\
- \sum_{i=1}^{2} \epsilon \int_{\mathbb{R}^2} \mu(\bar{v}) |v_i|^2 \left( |\bar{v}|^2 - \beta_a \right) d\bar{v} \int_{\Omega} a(\vec{x}) \partial_i \phi_a(\vec{x}) d\vec{x} \\
- \epsilon \int_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathbb{P}) [f_{\lambda,m}] \sqrt{\mu(\bar{v})} \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i,j=1}^{2} v_i v_j \partial_{ij} \phi_a \right) .
\]

We make an orthogonal decomposition on the boundary

\[
\bar{v} = (\vec{n} \cdot \vec{v}) \vec{n} + (\vec{v}_\perp) = v_n \vec{n} + (\vec{v}_\perp).
\]

Then the contribution of \(\mathcal{P} [f_{\lambda,m}] = z_\gamma(\vec{x}) \sqrt{\mu(\bar{v})}\) is

\[
(3.118) \quad \epsilon \int_{\partial \Omega \times \mathbb{R}^2} \mathcal{P} [f_{\lambda,m}] \sqrt{\mu(\bar{v})} \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) (\vec{n} \cdot \vec{v}) \\
= \epsilon \int_{\partial \Omega \times \mathbb{R}^2} \mu(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) v_n \vec{v} \cdot \nabla_x \phi_a z_\gamma(\vec{x}) \\
= \epsilon \int_{\partial \Omega \times \mathbb{R}^2} \mu(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) v_n^2 \frac{\partial \phi_a}{\partial n} z_\gamma(\vec{x}) + \epsilon \int_{\partial \Omega \times \mathbb{R}^2} \mu(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) v_n \vec{v}_\perp \cdot \nabla_x \phi_a z_\gamma(\vec{x}).
\]
Based on the definition of $\phi_\alpha$ and the oddness of $v_n\vec{v}_\perp$, we know the contribution of $\mathcal{P}[f_{\lambda,m}]$ in the second term of (3.113) vanishes. Since

$$\int_{\mathbb{R}^2} \sqrt{\mu(\vec{v})} |v| \left( |\vec{v}|^2 - \beta_\alpha \right) \, d\vec{v} = C,$$

we have

$$-\epsilon \int_{\Omega} \Delta_x \phi_\alpha(\vec{x}) a(\vec{x}) \, d\vec{x}$$

$$\leq C \|a\|_{L^2} \left( \epsilon \| (1 - \mathcal{P}) [f_{\lambda,m}] \|_{L^2_+} + (1 + \epsilon + \lambda) \| (I - \mathcal{P}) [f_{\lambda,m}] \|_{L^2} + \| S \|_{L^2} + \epsilon \| h \|_{L^2} + \lambda \| h \|_{L^2} \right).$$

Since $-\Delta_x \phi_\alpha = a$, by (3.106), we know (3.121)

$$\epsilon \| a \|_{L^2} \leq C \| a \|_{L^2} \left( (\epsilon + \lambda) \| (1 - \mathcal{P}) [f_{\lambda,m}] \|_{L^2_+} + (1 + \epsilon + \lambda) \| (I - \mathcal{P}) [f_{\lambda,m}] \|_{L^2} + (1 + \lambda) \| S \|_{L^2} + (\epsilon + \lambda) \| h \|_{L^2} \right).$$

which further implies

$$\epsilon \| a \|_{L^2} \leq C \left( (\epsilon + \lambda) \| (1 - \mathcal{P}) [f_{\lambda,m}] \|_{L^2_+} + (1 + \epsilon + \lambda) \| (I - \mathcal{P}) [f_{\lambda,m}] \|_{L^2} + (1 + \lambda) \| S \|_{L^2} + (\epsilon + \lambda) \| h \|_{L^2} \right).$$

Step 4: Synthesis. Collecting (3.83), (3.106) and (3.122), we deduce

$$\epsilon \| \mathbb{P}[f_{\lambda,m}] \|_{L^2}$$

$$\leq C \left( (\epsilon + \lambda) \| (1 - \mathcal{P}) [f_{\lambda,m}] \|_{L^2_+} + (1 + \epsilon + \lambda) \| (I - \mathcal{P}) [f_{\lambda,m}] \|_{L^2} + (1 + \lambda) \| S \|_{L^2} + (\epsilon + \lambda) \| h \|_{L^2} \right).$$

This completes our proof. □

**Lemma 3.6.** Assume condition (3.14) holds. Then there exists a unique solution $f \in L^2(\Omega \times \mathbb{R}^2)$ to the equation (3.1) that satisfies the normalization condition (3.3) and the estimate

$$\| f \|_{L^2} + |f|_{L^2_+} \leq C \left( \frac{1}{\epsilon^2} \| S \|_{L^2} + \frac{1}{\epsilon^{1/2}} \| h \|_{L^2_-} \right).$$

**Proof.** We keep $m$ fixed and take $\lambda \to 0$ for $f_{\lambda,m} \to f_m$ in Lemma 3.3 with the uniform estimate in Lemma 3.5. Then the compatibility condition (3.2) implies

$$\lambda \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m}(\vec{x}, \vec{v}) \, d\vec{v} \, d\vec{x} = 0.$$

Therefore, we naturally have

$$\lambda \int_{\Omega \times \mathbb{R}^2} f_m(\vec{x}, \vec{v}) \, d\vec{v} \, d\vec{x} = 0.$$

We square on both sides of (3.65) to obtain

$$\epsilon^2 \| \mathbb{P}[f_{\lambda,m}] \|_{L^2} \leq C \left( \epsilon^2 \| (1 - \mathcal{P}) [f_{\lambda,m}] \|_{L^2_+} + \| (I - \mathcal{P}) [f_{\lambda,m}] \|_{L^2} + \| S \|_{L^2} + \epsilon^2 \| h \|_{L^2_-} \right).$$

On the other hand, Green’s identity implies

$$\lambda \| f_{\lambda,m} \|_{L^2}^2 + \langle S_m f_{\lambda,m}, f_{\lambda,m} \rangle + \frac{1}{2} \| f_{\lambda,m} \|_{L^2_+}^2 = \frac{1}{2} \epsilon \| \mathcal{P}[f_{\lambda,m}] \|_{L^2}^2 + \frac{1}{2} \epsilon \| h \|_{L^2_-}^2 + \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} S.$$

We deduce from the spectral gap of $\mathcal{L}_m$,

$$\lambda \| f_{\lambda,m} \|_{L^2}^2 + \| (I - \mathcal{P}) [f_{\lambda,m}] \|_{L^2} + \epsilon \| f_{\lambda,m} \|_{L^2_+} \leq \eta \epsilon \| \mathcal{P}[f_{\lambda,m}] \|_{L^2}^2 + \frac{1}{\eta} \| h \|_{L^2_-}^2 + \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} S.$$
From the argument of (3.60) and the equation (3.30), since $\mathcal{L}_m = \mathcal{L}_m(\mathbb{I} - \mathbb{P})$, we have

$$
(3.130) \quad |\mathcal{P}[f_{\lambda,m}]|_{L^2}^2 \leq C \left( \| \hat{v} \cdot \nabla_x (f_{\lambda,m})^2 \|_{L^1} + \| f_{\lambda,m} \|_{L^2} \right) 
\leq C \left( \frac{\lambda}{\epsilon} + 1 \right) \| f_{\lambda,m} \|_{L^2}^2 + C \left( \| (\mathbb{I} - \mathbb{P})[f_{\lambda,m}] \|_{L^2}^2 + \| S \|_{L^2}^2 \right).
$$

Plugging (3.130) into (3.129) with $\eta$ sufficiently small to absorb into the left-hand side, we obtain

$$
(3.131) \quad \lambda \| f_{\lambda,m} \|_{L^2}^2 + \| (\mathbb{I} - \mathbb{P})[f_{\lambda,m}] \|_{L^2}^2 + \epsilon \| (1 - \mathbb{P})[f_{\lambda,m}] \|_{L^2}^2 \leq C \left( \epsilon \| h \|_{L^2} + \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} S + \| S \|_{L^2} \right),
$$

which further implies

$$
(3.132) \quad \| (\mathbb{I} - \mathbb{P})[f_{\lambda,m}] \|_{L^2}^2 + \epsilon \| (1 - \mathbb{P})[f_{\lambda,m}] \|_{L^2}^2 \leq C \left( \epsilon \| h \|_{L^2} + \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} S + \| S \|_{L^2} \right).
$$

Multiplying a small constant on both sides of (3.131) and adding to (3.132), we obtain

$$
(3.133) \quad \epsilon^2 \| f_{\lambda,m} \|_{L^2}^2 + \epsilon \| (1 - \mathbb{P})[f_{\lambda,m}] \|_{L^2}^2 \leq C \left( \epsilon \| h \|_{L^2}^2 + \int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} S + \| S \|_{L^2} \right).
$$

Since

$$
\int_{\Omega \times \mathbb{R}^2} f_{\lambda,m} S \leq C \epsilon^2 \| f_{\lambda,m} \|_{L^2}^2 + \frac{1}{4\epsilon^2} \| S \|_{L^2}^2,
$$

for $C$ sufficiently small, we have

$$
(3.135) \quad \epsilon^2 \| f_{\lambda,m} \|_{L^2}^2 + \epsilon \| (1 - \mathbb{P})[f_{\lambda,m}] \|_{L^2}^2 \leq C \left( \epsilon \| h \|_{L^2}^2 + \frac{1}{\epsilon^2} \| S \|_{L^2} \right).
$$

Hence, we deduce

$$
(3.136) \quad \| f_{\lambda,m} \|_{L^2} \leq \frac{1}{\epsilon^2} \| S \|_{L^2} + \frac{1}{\epsilon^{1/2}} \| h \|_{L^2}.
$$

Then based on (3.128), we have

$$
(3.137) \quad \| f_{\lambda,m} \|_{L^2} \leq C \left( \frac{1}{\epsilon^{3/2}} \| S \|_{L^2} + \frac{1}{\epsilon^{1/2}} \| h \|_{L^2} \right).
$$

This is a uniform estimate in $\lambda$, so we can obtain a weak solution $f_{\lambda,m} \to f_m$ with the same estimate (3.137). Moreover, we have

$$
(3.138) \quad \left\{ \begin{array}{l}
\lambda(f_{\lambda,m} - f_m) + \epsilon \hat{v} \cdot \nabla_x (f_{\lambda,m} - f_m) + \mathcal{L}_m[f_{\lambda,m} - f_m] = \lambda f_m, \\
(f_{\lambda,m} - f_m)(x_0, \hat{v}) = \mathcal{P}[f_{\lambda,m} - f_m].
\end{array} \right.
$$

Then we have the estimate

$$
(3.139) \quad \| f_{\lambda,m} - f_m \|_{L^2} \leq C \frac{\lambda}{\epsilon^2} \| f_m \|_{L^2}.
$$

Hence, $f_{\lambda,m} \to f_m$ strongly in $L^2(\Omega \times \mathbb{R}^2)$ as $\lambda \to 0$. Then we can take the limit $f_m \to f$ as $m \to \infty$. By a diagonal process, there exists a unique weak solution such that $f_m \to f$ weakly in $L^2(\Omega \times \mathbb{R}^2)$. Then the weak lower semi-continuity implies $f$ satisfies the same estimate (3.137).

**Theorem 3.7.** Assume condition (3.2) holds. Then there exists a unique solution $f \in L^2(\Omega \times \mathbb{R}^2)$ to the equation (3.1) that satisfies the normalization condition (3.3) and the estimate

$$
(3.140) \quad \| f \|_{L^2} + \| f \|_{L^2}^2 \leq C \left( \frac{1}{\epsilon^2} \| \mathbb{P}[S] \|_{L^2}^2 + \frac{1}{\epsilon} \| (\mathbb{I} - \mathbb{P})[S] \|_{L^2}^2 + \frac{1}{\epsilon^{1/2}} \| h \|_{L^2} \right).
$$

Proof. Based on the proof of Lemma 3.6 and the fact

$$
(3.141) \quad \| S \|_{L^2} = \| \mathbb{P}[S] \|_{L^2}^2 + \| (\mathbb{I} - \mathbb{P})[S] \|_{L^2}^2,
$$

this is obvious.
3.3. $L^\infty$ Estimates of Linearized Steady Boltzmann Equation.

**Definition 3.8.** (Stochastic Cycle) For a fixed point $(\vec{x}, \vec{v})$ with $(\vec{x}, \vec{v}) \notin \gamma_0$, let $(t_0, \vec{x}_0, \vec{v}_0) = (0, \vec{x}, \vec{v})$. For $\vec{v}_{k+1}$ such that $\vec{v}_{k+1} \cdot \vec{n}(\vec{x}_{k+1}) > 0$, define the $(k+1)$-component of the back-time cycle as

$$
(t_{k+1}, \vec{x}_{k+1}, \vec{v}_{k+1}) = (t_k + t_b(\vec{x}_k, \vec{v}_k), \vec{x}_b(\vec{x}_k, \vec{v}_k), \vec{v}_{k+1}),
$$

where

$$
t_b(\vec{x}, \vec{v}) = \inf\{t > 0 : \vec{x} - t\vec{v} \notin \Omega\},
$$

$$
x_b(\vec{x}, \vec{v}) = \vec{x} - t_b(\vec{x}, \vec{v})\vec{v} \notin \Omega.
$$

Set

$$
X_{cl}(s; \vec{x}, \vec{v}) = \sum_k 1_{\{t_k \leq s < t_{k+1}\}} (\vec{x}_k - \epsilon(t_k - s)\vec{v}_k),
$$

$$
V_{cl}(s; \vec{x}, \vec{v}) = \sum_k 1_{\{t_k \leq s < t_{k+1}\}} \vec{v}_k.
$$

Define $V_k = \{\vec{v} \in \mathbb{R}^2 : \vec{v} \cdot \vec{n}(\vec{x}_k) > 0\}$, and let the iterated integral for $k \geq 2$ be defined as

$$
\int_{\prod_{j=1}^{k-1} V_j} \prod_{j=1}^{k-1} d\sigma_j = \int_{V_1} \left( \int_{V_{k-1}} \ldots \left( \int_{V_1} d\sigma_{k-1} \right) \ldots d\sigma_1 \right)
$$

where $d\sigma_j = |\vec{n}(\vec{x}_j) \cdot \vec{v}| d\vec{v}$ is a probability measure. We define a weight function scaled with parameter $\varrho$,

$$
w_{\varrho}(\vec{v}) = w_{\varrho, \beta, \zeta}(\vec{v}) = (1 + \varrho^2 |\vec{v}|^2)^{\frac{\beta}{2}},
$$

and

$$\tilde{w}_{\varrho}(\vec{v}) = \frac{1}{\sqrt{\mu(\vec{v}) w_{\varrho}(\vec{v})}} = \frac{e^{(\frac{1}{4} - \zeta)|\vec{v}|^2}}{\sqrt{2\pi}(1 + \varrho^2 |\vec{v}|^2)^{\frac{\beta}{2}}}.
$$

**Lemma 3.9.** For $T > 0$ sufficiently large, there exists constants $C_1, C_2 > 0$ independent of $T$, such that for $k = C_1 T^{3/4}$, and $(\vec{x}, \vec{v}) \in \times \Omega \times \mathbb{R}^2$,

$$
\int_{\prod_{j=1}^{k-1} V_j} 1_{\{t_k(\vec{x}, \vec{v}, \vec{v}_1, ..., \vec{v}_{k-1}) < T\}} \prod_{j=1}^{k-1} d\sigma_j \leq \left(\frac{1}{2}\right)^{C_2 T^{3/4}}.
$$

We also have, for $\beta > 2$,

$$
\int_{\prod_{j=1}^{k-1} V_j} \tilde{w}_{\varrho}(\vec{v}) \prod_{j=1}^{k-1} d\sigma_j \leq \frac{C(\beta, \zeta)}{\varrho^3},
$$

$$
\int_{\prod_{j=1}^{k-1} V_j} 1_{\{t_k(\vec{x}, \vec{v}, \vec{v}_1, ..., \vec{v}_{k-1}) < T\}} \tilde{w}_{\varrho}(\vec{v}) \prod_{j=1}^{k-1} d\sigma_j \leq \frac{C(\beta, \zeta)}{\varrho^3} \left(\frac{1}{2}\right)^{C_2 T^{3/4}}.
$$

**Proof.** The proof of (3.150) can be found in the proof of \[\text{Lemma 4.1},\] so we only present a brief explanation of (3.151) and (3.152). We directly estimate

$$
\int_{\prod_{j=1}^{k-1} V_j} \tilde{w}_{\varrho}(\vec{v}) \prod_{j=1}^{k-1} d\sigma_j \leq \int_{V_j} \tilde{w}_{\varrho}(\vec{v}) d\sigma_j \leq \frac{1}{\sqrt{2\pi}} \int_{u_t > 0} v_t e^{-\left(\frac{1}{4} + \zeta\right)|u_t|^2} \frac{u_t^3}{\left(1 + |u_t|^2\right)^{\frac{\beta}{2}}} \, du_t \leq \frac{1}{\sqrt{2\pi}} \int_{u_t > 0} \frac{C(\zeta) u_t^3}{\left(1 + |u_t|^2\right)^{\frac{\beta}{2}}} \, du_t \leq \frac{C(\beta, \zeta)}{\varrho^3},
$$
where $\vec{u} = \varrho \vec{v}$ and $\beta > 2$. This completes the proof of \([3.151]\). Since $v_l^2 \mu^2(\vec{v}) \leq C v_l \mu(\vec{v})$ with $v_l > 0$ and some uniform constant $C > 0$, we have

\begin{equation}
(3.154) \quad \int_{\Pi_{j=1}^{k} V_j} 1_{\{t_k(x, \vec{v}, \vec{v}_1, \ldots, \vec{v}_{k-1}) < T\}} \tilde{w}_\varrho(\vec{v}) \prod_{j=1}^{k-1} d\sigma_j
\end{equation}

\begin{align*}
&\leq \left( \int_{\Pi_{j=1}^{k} V_j} 1_{\{t_k(x, \vec{v}, \vec{v}_1, \ldots, \vec{v}_{k-1}) < T\}} \prod_{j=1}^{k-1} d\sigma_j \right)^{1/2}
&\quad \times \left( \int_{\Pi_{j=1}^{k} V_j} \tilde{w}_\varrho^2(\vec{v}) \prod_{j=1}^{k-1} d\sigma_j \right)^{1/2}
&\leq \left( \frac{1}{2} \right)^{C_2 T^{5/4}} \frac{C_2 T^{5/4}}{\sqrt{2\pi}} \int_{\vec{v}_l > 0} \frac{v_l e^{-2\varrho |\vec{v}|^2}}{(1 + \varrho^2 |\vec{v}|^2)^2} d\vec{v}
&\leq \left( \frac{1}{2} \right)^{C_2 T^{5/4}} \frac{C(\beta, \zeta)}{\varrho^3}.
\end{align*}

This completes the proof of \([3.152]\). \hfill \Box

**Lemma 3.10.** We have

\begin{equation}
(3.155) \quad |k(\vec{v}, \vec{v}')| \leq C \left( |\vec{v} - \vec{v}'| + \frac{1}{|\vec{v} - \vec{v}'|} \right) \exp \left( -\frac{1}{8} \frac{|\vec{v} - \vec{v}'|^2}{|\vec{v} - \vec{v}'|^4} - \frac{1}{8} \frac{|\vec{v} - \vec{v}'|^2}{|\vec{v} - \vec{v}'|^2} \right).
\end{equation}

Let $0 \leq \zeta \leq 1/4$. Then there exists $0 \leq C_1(\zeta) < 1$ and $C_2(\zeta) > 0$ such that for $0 \leq \delta \leq C_1(\zeta)$,

\begin{equation}
(3.156) \quad \int_{\mathbb{R}^2} \left( |\vec{v} - \vec{v}'| + \frac{1}{|\vec{v} - \vec{v}'|} \right) \exp \left( -\frac{1 - \delta}{8} \frac{|\vec{v} - \vec{v}'|^2}{|\vec{v} - \vec{v}'|^2} - \frac{1 - \delta}{8} \frac{|\vec{v} - \vec{v}'|^2}{|\vec{v} - \vec{v}'|^2} \right) g(\vec{v}) e^{\zeta |\vec{v}|^2} d\vec{v} \leq \frac{C_2(\zeta)}{1 + |\vec{v}|}.
\end{equation}

**Proof.** See [7] Lemma 3]. \hfill \Box

**Lemma 3.11.** Assume condition \([3.2]\) holds. Then there exists a unique solution $f \in L^\infty(\Omega \times \mathbb{R}^2)$ to the equation \([3.7]\) that satisfies the normalization condition \([3.3]\) and the estimate for $\beta > 2$, $0 \leq \zeta \leq 1/4$ and $v \geq 3$,

\begin{equation}
(3.157) \quad \|w_\varrho f\|_{L^\infty} + |w_\varrho f|_{L^\infty} \leq C \left( \frac{1}{e^\delta} \|w_\varrho S\|_{L^\infty} + \frac{1}{e^{3/2}} |w_\varrho h|_{L^\infty} \right).
\end{equation}

**Proof.** We divide the proof into several steps:

**Step 1: Mild formulation.**

Denote

\begin{equation}
(3.158) \quad g = w_\varrho f,
\end{equation}

\begin{equation}
(3.159) \quad K_{w_\varrho}[g] = w_\varrho K \left[ \frac{1}{w_\varrho} g \right] = \int_{\mathbb{R}^2} k_{w_\varrho}(\vec{v}, \vec{v}') g(\vec{v}') d\vec{v}'.
\end{equation}

We can rewrite the solution of the equation \([3.1]\) along the stochastic cycle by Duhamel’s principle as

\begin{equation}
(3.160) \quad g(\vec{x}, \vec{v}) = w_\varrho(\vec{v}) h(\vec{x} - c t_1 \vec{v}, \vec{v}) e^{-\nu(\vec{v}) t_1} + \int_0^{t_1} w_\varrho S(\vec{x} - c (t_1 - s) \vec{v}, \vec{v}) e^{-\nu(\vec{v})(t_1 - s)} ds
\end{equation}

\begin{equation}
+ \int_0^{t_1} K_{w_\varrho}[g(\vec{x} - c (t_1 - s) \vec{v}, \vec{v})] e^{-\nu(\vec{v})(t_1 - s)} ds + \frac{e^{-\nu(\vec{v}) t_1}}{w_\varrho(\vec{v})} \int_{V_j} g(\vec{x}_1, \vec{v}_1) \tilde{w}_\varrho(\vec{v}_1) d\sigma_1.
\end{equation}
We may further rewrite the equation [3.161] along the stochastic cycle by applying Duhamel’s principle \( k \) times as

\[
g(\bar{x}, \bar{v}) = w_\rho(\bar{v})h(\bar{x} - c t_1 \bar{v}, \bar{v})e^{-\nu(\bar{v}) t_1} + \int_0^{t_1} w_\rho(\bar{v})S(\bar{x} - c(t_1 - s)\bar{v}, \bar{v})e^{-\nu(\bar{v})(t_1 - s)}ds
\]

\[
+ \int_0^{t_1} K_{w_\rho(\bar{v})}[g](\bar{x} - c(t_1 - s)\bar{v}, \bar{v})e^{-\nu(\bar{v})(t_1 - s)}ds
\]

\[
+ \frac{1}{w_\rho(\bar{v})} \sum_{j=1}^{k-1} \int_{\Pi_{j=1}^{k-1} v_j} G\bar{w}_\rho(\bar{v}_j) \left( \prod_{j=1}^{k} e^{-\nu(\bar{v}_j) (t_{j+1} - t_j)} d\sigma_j \right)
\]

\[
+ \frac{1}{w_\rho(\bar{v})} \int_{\Pi_{j=1}^{k-1} v_j} P_{w_\rho(\bar{v}_{k-1})}[g](\bar{x}_k, \bar{v}_{k-1})\tilde{w}_\rho(\bar{v}_{k-1}) \left( \prod_{j=1}^{k} e^{-\nu(\bar{v}_j) (t_{j+1} - t_j)} d\sigma_j \right),
\]

where

\[
\tilde{G} = h(\bar{x}_l - c t_{l+1} \bar{v}_l, \bar{v}_l)w_\rho(\bar{v}_l) + \int_0^{t_l} S(\bar{x}_l - c(t_{l+1} - s)\bar{v}_l, \bar{v}_l)w_\rho(\bar{v}_l)e^{\nu(\bar{v}_l) s}ds
\]

\[
+ \int_0^{t_l} \left( K_{w_\rho(\bar{v}_l)}[g](\bar{x}_l - c(t_{l+1} - s)\bar{v}_l, \bar{v}_l)e^{\nu(\bar{v}_l) s} \right) ds.
\]

The right-hand side of [3.161] can be divided into several parts, which will be estimated in the following.

Step 2: Estimates of source terms and boundary terms.

We set \( k = CT^{5/4} \) and take absolute value on both sides of [3.161]. Then all the terms in [3.161] related to the source term \( S \) and boundary term \( h \) can be bounded as

\[
\text{Part 1} \leq C \left( |w_\rho h|_{L^\infty} + \|w_\rho S\|_{L^\infty} \right)
\]

due to Lemma 3.9 and

\[
\frac{1}{\bar{w}_\rho} \leq C(\beta, \zeta) g^3.
\]

The last term in [3.161] can be decomposed as follows:

\[
\text{Part 2} = \frac{1}{\bar{w}_\rho(\bar{v})} \int_{\Pi_{j=1}^{k-1} v_j} P_{w_\rho(\bar{v}_{k-1})}[g](\bar{x}_k, \bar{v}_{k-1})\tilde{w}_\rho(\bar{v}_{k-1}) \left( \prod_{j=1}^{k} e^{-\nu(\bar{v}_j) (t_{j+1} - t_j)} d\sigma_j \right)
\]

\[
= \frac{1}{\bar{w}_\rho(\bar{v})} \int_{\Pi_{j=1}^{k-1} v_j} 1\{t_k \leq T\} P_{w_\rho(\bar{v}_{k-1})}[g](\bar{x}_k, \bar{v}_{k-1})\tilde{w}_\rho(\bar{v}_{k-1}) \left( \prod_{j=1}^{k} e^{-\nu(\bar{v}_j) (t_{j+1} - t_j)} d\sigma_j \right)
\]

\[
+ \frac{1}{\bar{w}_\rho(\bar{v})} \int_{\Pi_{j=1}^{k-1} v_j} 1\{t_k > T\} P_{w_\rho(\bar{v}_{k-1})}[g](\bar{x}_k, \bar{v}_{k-1})\tilde{w}_\rho(\bar{v}_{k-1}) \left( \prod_{j=1}^{k} e^{-\nu(\bar{v}_j) (t_{j+1} - t_j)} d\sigma_j \right).
\]

Based on Lemma 3.9 we have

\[
\left| \frac{1}{\bar{w}_\rho(\bar{v})} \int_{\Pi_{j=1}^{k-1} v_j} 1\{t_k \leq T\} P_{w_\rho(\bar{v}_{k-1})}[g](\bar{x}_k, \bar{v}_{k-1})\tilde{w}_\rho(\bar{v}_{k-1}) \left( \prod_{j=1}^{k} e^{-\nu(\bar{v}_j) (t_{j+1} - t_j)} d\sigma_j \right) \right|
\]

\[
\leq C(\beta, \zeta) g^{\beta - 3} \left( \frac{1}{2} \right) C_2 T^{5/4} \|g\|_{L^\infty}.
\]
Based on Lemma 3.9 and $\nu_0(1 + |\vec{v}|) \leq \nu(\vec{v}) \leq \nu_1(1 + |\vec{v}|)$, we obtain

$$
\left| \frac{1}{\hat{w}_g(\vec{v})} \int_{\Pi_{j=1}^{T_j} V_j} 1\{t_k \geq T\} \mathcal{P}_{\hat{w}_g(\vec{v}_{k-1})}[g](\vec{x}_k, \vec{v}_{k-1}) \hat{w}_g(\vec{v}_{k-1}) \left( \prod_{j=1}^{k-1} e^{-\nu(\vec{v}_j)(t_j+1-t_j)} d\sigma_j \right) \right|
\leq e^{-\omega T} C(\beta) \|g\|_{L_\infty} .
$$

Taking $T$ sufficiently large, we get

$$
\text{Part 2} \leq \delta \|g\|_{L_\infty} .
$$

for $\delta$ arbitrarily small.

Step 3: Estimates of $K_{w_\rho}$ terms.

Collecting all above estimates, we have

$$
\left| g(\vec{x}, \vec{v}) \right|
\leq A + \left| \int_0^T K_{w_\rho(\vec{v})}[g](\vec{x} - \epsilon(t_1 - s)\vec{v}, \vec{v}) e^{-\nu(\vec{v})(t_1-s)} ds \right|
+ \left| \int \left( \int_{\Pi_{j=1}^{T_j} V_j} \hat{w}_g(\vec{v}_j) \left( \prod_{j=1}^{l} e^{-\nu(\vec{v}_j)(t_j+1-t_j)} d\sigma_j \right) \nu(\vec{v}_j) \right) .
$$

where

$$
A = \text{Part 1} + \text{Part 2} \leq C \left( \|w_\rho h\|_{L_\infty} + \|w_\rho S\|_{L_\infty} \right) + \delta \|g\|_{L_\infty} .
$$

Define the back-time stochastic cycle from $(s, X_{cl}(s; \vec{x}, \vec{v}), \vec{v}')$ as $(t'_1, \vec{x}'_1, \vec{v}'_1)$. Then we can rewrite $K_{w_\rho}$ along the stochastic cycle as

$$
\left| K_{w_\rho(\vec{v})}[g](\vec{x} - \epsilon(t_1 - s)\vec{v}, \vec{v}) \right|
= \int_{\mathbb{R}^2} k_{w_\rho(\vec{v})}(\vec{v}, \vec{v}') g(X_{cl}, \vec{v}') d\vec{v}'
= \int_{\mathbb{R}^2} \int_0^{t'_1} k_{w_\rho(\vec{v})}(\vec{v}, \vec{v}') K_{w_\rho(\vec{v})}[g](X_{cl} - \epsilon(t'_1 - r)\vec{v}', \vec{v}') e^{-\nu(\vec{v}')(t'_1-r)} dr d\vec{v}'
+ \int_{\mathbb{R}^2} \left( \int_{\Pi_{j=1}^{T_j} V_j} \hat{w}_g(\vec{v}_j) \left( \prod_{j=1}^{l} e^{-\nu(\vec{v}_j)(t_j+1-t_j)} d\sigma_j \right) \nu(\vec{v}_j) \right) .
$$

I + II + III.
Part III can be directly estimated as

$$III \leq C \left( |w_p h|_{L_{\infty}} + \|w_p S\|_{L_{\infty}} + \delta \|g\|_{L_{\infty}} \right).$$

We can divide $I$ into four cases:

$$I = I_1 + I_2 + I_3 + I_4.$$ 

Case I: $|\vec{v}| \geq N$.

Based on Lemma 3.11 with $\delta = 0$, we have

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} k_{w_p}(\vec{v}, \vec{v}')k_{w_p}(\vec{v}', \vec{v}'')d\vec{v}'d\vec{v}'' \leq \frac{C}{1 + |\vec{v}|} \leq \frac{C}{N}.$$ 

Hence, we get

$$I_1 \leq \frac{C}{N}\|g\|_{L_{\infty}}.$$ 

Case II: $|\vec{v}| \leq N$, $|\vec{v}'| \geq 2N$, or $|\vec{v}'| \leq 2N$, $|\vec{v}''| \geq 3N$.

Notice this implies either $|\vec{v} - \vec{v}'| \geq N$ or $|\vec{v}' - \vec{v}''| \geq N$. Hence, either of the valid correspondingly:

$$|k_{w_p}(\vec{v}, \vec{v}')| \leq \frac{\|k_{w_p}(\vec{v}, \vec{v}')e^{\frac{1}{2}|\vec{v} - \vec{v}'|^2}|}{N^2},$$

$$|k_{w_p}(\vec{v}')e^{\frac{1}{2}|\vec{v}' - \vec{v}''|^2}| \leq \frac{\|k_{w_p}(\vec{v}', \vec{v}'')e^{\frac{1}{2}|\vec{v}''|^2}|}{N^2}.$$ 

Then based on Lemma 3.10

$$\int_{\mathbb{R}^2} \left| k_{w_p}(\vec{v}, \vec{v}') e^{\frac{1}{2}|\vec{v} - \vec{v}'|^2} \right| d\vec{v}' < \infty,$$

$$\int_{\mathbb{R}^2} \left| k_{w_p}(\vec{v}') e^{\frac{1}{2}|\vec{v}' - \vec{v}''|^2} \right| d\vec{v}'' < \infty.$$ 

Hence, we have

$$I_2 \leq Ce^{-\frac{1}{2}N^2}\|g\|_{L_{\infty}}.$$ 

Case III: $t' - r \leq \delta$ and $|\vec{v}| \leq N$, $|\vec{v}'| \leq 2N$, $|\vec{v}''| \leq 3N$.

In this case, since the integral is restricted in a very short interval, there is a small contribution as

$$I_3 \leq C\delta \|g\|_{L_{\infty}}.$$ 

Case IV: $t' - r \geq \delta$ and $|\vec{v}| \leq N$, $|\vec{v}'| \leq 2N$, $|\vec{v}''| \leq 3N$.

Note that all the other possibilities of $\vec{v}$, $\vec{v}'$, and $\vec{v}''$ have been included in Case II. Since $k_{w_p}(\vec{v}, \vec{v}')$ has possible integrable singularity of $1/|\vec{v} - \vec{v}'|$, we can introduce $k_N(\vec{v}, \vec{v}')$ smooth with compact support such that

$$\sup_{|p| \leq 3N} \int_{|\vec{v}'| \leq 3N} \left| k_N(p, \vec{v}') - k_{w_p}(p, \vec{v}') \right| d\vec{v}' \leq \frac{1}{N}.$$ 

Then we can split

$$k_{w_p}(\vec{v}, \vec{v}')k_{w_p}(\vec{v}', \vec{v}'') = k_N(\vec{v}, \vec{v}')k_N(\vec{v}', \vec{v}'') + \left( k_{w_p}(\vec{v}, \vec{v}') - k_N(\vec{v}, \vec{v}') \right)k_{w_p}(\vec{v}', \vec{v}'') + \left( k_{w_p}(\vec{v}', \vec{v}'') - k_N(\vec{v}', \vec{v}'') \right)k_N(\vec{v}, \vec{v}').$$

This means we further split $I_4$ into

$$I_4 = I_{4,1} + I_{4,2} + I_{4,3}.$$
Based on the estimate (3.182), we have
\[ I_{4,2} \leq \frac{C}{N} \| g \|_{L^\infty}, \]
\[ I_{4,3} \leq \frac{C}{N} \| g \|_{L^\infty}. \]
Therefore, the only remaining term is \( I_{4,1} \). Note we always have \( X_{cl} - \epsilon(t'_1 - r)\bar{v} \in \Omega \). Hence, we define the change of variable \( \bar{y} = (y_1, y_2) = X_{cl} - \epsilon(t'_1 - r)\bar{v} \) such that
\[ \left| \frac{d\bar{y}}{d\bar{v}} \right| = \begin{vmatrix} \epsilon(t'_1 - r) & 0 \\ 0 & \epsilon(t'_1 - r) \end{vmatrix} = \epsilon^2(t'_1 - r) \geq \epsilon^2 \delta^2. \]
Since \( k_\Omega \) is bounded and \( |\bar{v}''| \leq 3N \), we estimate
\[
I_{4,1} \leq C \left| \int_{|\bar{v}'| \leq 2N} \int_{|\bar{v}''| \leq 3N} \int_{0}^{t'_1} 1_{\{X_{cl} - \epsilon(t'_1 - r)\bar{v} \in \Omega\}} g(X_{cl} - \epsilon(t'_1 - r)\bar{v}, \bar{v}'') \, dr \, d\bar{v} \, d\bar{v}'' \right|
\leq \frac{C}{\epsilon \delta} \left\| g(\bar{y}, \bar{v}'') \right\|_{L^2}
= \frac{C}{\epsilon \delta} \left\| g \right\|_{L^2}.
\]
Summarize all above in Case IV, we obtain
\[ I_4 \leq \frac{C}{N} \| g \|_{L^\infty} + \frac{C}{\epsilon \delta} \left\| g \right\|_{L^2}.
\]
Therefore, we already prove
\[ I \leq \left( Ce^{-\frac{\delta}{C}} N^2 + \frac{C}{N} + \delta \right) \| g \|_{L^\infty} + \frac{C}{\epsilon \delta} \left\| g \right\|_{L^2}.
\]
Choosing \( \delta \) sufficiently small and then taking \( N \) sufficiently large, we have
\[ I \leq C \delta \| g \|_{L^\infty} + \frac{C}{\epsilon \delta} \left\| g \right\|_{L^2}.
\]
A similar technique can justify
\[ II \leq C \delta \| g \|_{L^\infty} + \frac{C}{\epsilon \delta} \left\| g \right\|_{L^2}.
\]
Hence, this yields the \( K_{w_e,g} \) estimates in (3.171) and (3.161).

Step 4: Synthesis.
Collecting all above, based on the mild formulation (3.101) we have shown
\[ \| g \|_{L^\infty} \leq C \delta \| g \|_{L^\infty} + \frac{C}{\epsilon \delta} \left\| g \right\|_{L^2} + C \| w_e S \|_{L^\infty} + C \| w_e h \|_{L^\infty}. \]
Taking \( \delta \) sufficiently small, based on Lemma 3.6 we obtain
\[ \| g \|_{L^\infty} \leq C \left( \frac{1}{\epsilon^3} \left\| S \right\|_{L^2} + \frac{1}{\epsilon^{5/2}} \left| h \right|_{L^2} + \| w_e S \|_{L^\infty} + \| w_e h \|_{L^\infty} \right). \]
due to \( f = g/w_e \). Then this completes the proof. \( \square \)
Lemma 3.12. Assume condition (3.2) holds. Then there exists a unique solution \( f \in L^\infty(\Omega \times \mathbb{R}^2) \) to the equation (3.1) that satisfies the normalization condition (3.3) and the estimate for \( \vartheta \geq 3 \) and \( 0 \leq \zeta \leq 1/4 \),

\[
\left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} f \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S \right\|_{L^\infty} + \frac{1}{\epsilon^{3/2}} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h \right\|_{L^\infty} \leq C \left( \frac{1}{\epsilon^3} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S \right\|_{L^\infty} + \frac{1}{\epsilon^{3/2}} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h \right\|_{L^\infty} \right).
\]

Proof. It is easy to see the relation for \( \vartheta = \beta \), we have

\[
C_1 \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} \leq w_\beta \leq C \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2},
\]

for some constant \( C_1, C_2 > 0 \). Then Lemma 3.11 naturally implies the desired result. \( \square \)

Theorem 3.13. Assume condition (3.2) holds. Then there exists a unique solution \( f \in L^\infty(\Omega \times \mathbb{R}^2) \) to the equation (3.1) that satisfies the normalization condition (3.3) and the estimate for \( \vartheta \geq 3 \) and \( 0 \leq \zeta \leq 1/4 \),

\[
\left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} f \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S \right\|_{L^\infty} + \frac{1}{\epsilon^{3/2}} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h \right\|_{L^\infty} \leq C \left( \frac{1}{\epsilon^3} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S \right\|_{L^\infty} + \frac{1}{\epsilon^{3/2}} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h \right\|_{L^\infty} \right).
\]

Proof. Based on Theorem 3.7 and the proof of Lemma 3.11 this is obvious. \( \square \)
4. $\epsilon$-Milne Problem

We consider the $\epsilon$-Milne problem for $g'(\eta, \phi, \vec{v})$ in the domain $(\eta, \phi, \vec{v}) \in [0, \infty) \times [-\pi, \pi] \times \mathbb{R}^2$

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{\partial g'}{\partial \eta} + G(\epsilon; \eta) \left( v_\phi^2 \frac{\partial g'}{\partial v_\phi} - v_\eta v_\phi \frac{\partial g'}{\partial v_\phi} \right) + \mathcal{L}[g'] & = S'(\eta, \phi, \vec{v}), \\
g'(0, \phi, \vec{v}) & = h'(\phi, \vec{v}) \quad \text{for} \quad v_\eta > 0,
\end{array} \right.
\end{align*}
\]

where

\[
g'_\infty(\phi, \vec{v}) = \sqrt{\mu} \left( D'_0(\phi) + D'_1(\phi)v_\eta + D'_2(\phi)v_\phi + D'_3(\phi) |\vec{v}|^2 \right),
\]

\[
G(\epsilon; \eta) = -\frac{\epsilon \psi(\epsilon^{1/2} \eta^{1/2})}{1 - \epsilon \eta},
\]

\[
\psi(z) = \left\{ \begin{array}{ll}
1 & 0 \leq z \leq 1/2, \\
0 & 3/4 \leq z \leq \infty,
\end{array} \right.
\]

such that we can define the well-posed $\epsilon$-Milne problem for $G'(\eta, \phi, \vec{v})$ in the domain $(\eta, \phi, \vec{v}) \in [0, \infty) \times [-\pi, \pi] \times \mathbb{R}^2$

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{\partial G'}{\partial \eta} + G(\epsilon; \eta) \left( v_\phi^2 \frac{\partial G'}{\partial v_\phi} - v_\eta v_\phi \frac{\partial G'}{\partial v_\phi} \right) + \mathcal{L}[G'] & = S'(\eta, \phi, \vec{v}), \\
G'(0, \phi, \vec{v}) & = h'(\phi, \vec{v}) - \hat{h}'(\phi, \vec{v}) \quad \text{for} \quad v_\eta > 0,
\end{array} \right.
\end{align*}
\]

\[
\int_{\mathbb{R}^2} v_\eta \sqrt{\mu} G'(0, \phi, \vec{v}) d\vec{v} = \int_{v_\eta > 0} v_\eta \sqrt{\mu} h'(\phi, \vec{v}) d\vec{v} - \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \hat{h}'(\phi, \vec{v}) d\vec{v},
\]

\[
\lim_{\eta \to \infty} G'(\eta, \phi, \vec{v}) = 0.
\]

**Lemma 4.1.** $G'(\eta, \phi, \vec{v})$ satisfies

\[
P[G' + \hat{h}'](0, \phi, \vec{v}) = 0.
\]

**Proof.** This special mass flux implies

\[
\int_{\mathbb{R}^2} v_\eta \sqrt{\mu} (G' + \hat{h}') (\phi, \vec{v}) d\vec{v} = \int_{v_\eta > 0} v_\eta \sqrt{\mu} h'(\phi, \vec{v}) d\vec{v},
\]
and also
\begin{align}
\int_{\mathbb{R}^2} v_\eta \sqrt{\mu} (G^r + \tilde{h}^r)(\phi, \tilde{v}) d\tilde{v} \\
= \int_{v_\eta > 0} v_\eta \sqrt{\mu} (G^r + \tilde{h}^r)(\phi, \tilde{v}) d\tilde{v} + \int_{v_\eta < 0} v_\eta \sqrt{\mu} (G^r + \tilde{h}^r)(\phi, \tilde{v}) d\tilde{v},
\end{align}
which further yields
\begin{align}
\int_{v_\eta < 0} v_\eta \sqrt{\mu} (G^r + \tilde{h}^r)(\phi, \tilde{v}) d\tilde{v} = 0.
\end{align}

Hence, our result naturally follows. \qed

For notational simplicity, we omit \( \epsilon \) and \( \phi \) dependence in \( g^r \) and \( G^r \) in this section. The same convention also applies to \( H(\epsilon; \eta) \), \( W(\epsilon; \eta) \), \( S^r(\eta, \phi, \tilde{v}) \) and \( \tilde{h}^r(\phi, \tilde{v}) \). However, our estimates are uniform in \( \epsilon \) and \( \phi \). Our analysis is based on the ideas in [4, 5, 6].

In this section, we introduce some special notations to describe the norms in the space \((\eta, \tilde{v}) \in [0, \infty) \times \mathbb{R}^2\). Define the \( L^2 \) norm as follows:
\begin{align}
\|f(\eta)\|_{L^2} &= \left( \int_{\mathbb{R}^2} |f(\eta, \tilde{v})|^2 d\tilde{v} \right)^{1/2}, \\
\|f\|_{L^2, L^2} &= \left( \int_{0}^{\infty} \int_{\mathbb{R}^2} |f(\eta, \tilde{v})|^2 d\tilde{v} d\eta \right)^{1/2}.
\end{align}

Define the inner product in \( \tilde{v} \) space
\begin{align}
\langle f, g \rangle(\eta) &= \int_{\mathbb{R}^2} f(\eta, \tilde{v}) g(\eta, \tilde{v}) d\tilde{v}.
\end{align}

Define the \( L^\infty \) norm as follows:
\begin{align}
\|f(\eta)\|_{L^\infty, \phi} &= \sup_{\tilde{v} \in \mathbb{R}^2} \langle \tilde{v} \rangle^\phi e^{\zeta|\tilde{v}|^2} |f(\eta, \tilde{v})|, \\
\|f\|_{L^\infty, \phi} &= \sup_{(\eta, \tilde{v}) \in [0, \infty) \times \mathbb{R}^2} \langle \tilde{v} \rangle^\phi e^{\zeta|\tilde{v}|^2} |f(\eta, \tilde{v})|, \\
\|f\|_{L^\infty, L^\infty} &= \sup_{\eta \in [0, \infty)} \left( \int_{\mathbb{R}^2} e^{2\zeta|\tilde{v}|^2} |f(\eta, \tilde{v})|^2 d\tilde{v} \right)^{1/2}.
\end{align}

Since the boundary data \( h(\tilde{v}) \) is only defined on \( v_\eta > 0 \), we naturally extend above definitions on this half-domain as follows:
\begin{align}
\|h\|_{L^2} &= \left( \int_{v_\eta > 0} |h(\tilde{v})|^2 d\tilde{v} \right)^{1/2}, \\
\|h\|_{L^\infty, \phi} &= \sup_{v_\eta > 0} \langle \tilde{v} \rangle^\phi e^{\zeta|\tilde{v}|^2} |h(\tilde{v})|.
\end{align}

Define the mass flux
\begin{align}
m_f[g] &= \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g(0, \tilde{v}) e^{-W(\eta)} d\tilde{v}.
\end{align}

Since the kernel of operator \( \mathcal{L} \) is \( \mathcal{N} = \sqrt{\mu} \{ 1, \tilde{v}, |\tilde{v}|^2 \} = \{ \psi_0, \psi_1, \psi_2, \psi_3 \} \), we can decompose the solution as
\begin{align}
g &= w_g + q_g,
\end{align}
where
\begin{align}
q_g &= \sqrt{\mu} (q_0 + q_1 v_\eta + q_2 v_\phi + q_3 |\tilde{v}|^2),
\end{align}
and
\[(4.25)\]
\[w_g \in \mathcal{N}^\perp.\]
When there is no confusion, we will simply write \(g = w + q\).

**Lemma 4.2.** We have the estimates
\[(4.26)\]
\[\nu_0(1 + |\bar{v}|) \leq \nu(\bar{v}) \leq \nu_1(1 + |\bar{v}|),\]
\[(4.27)\]
\[\langle g', \mathcal{L}[g'] \rangle(\eta) \geq C \|\sqrt{\nu}w_g(\eta)\|_{L^2}^2,\]
for \(\nu_0, \nu_1\) and \(C\) positive constants.

**Proof.** See [8, Chapter 3]. \(\square\)

### 4.1. \(L^2\) Estimates.

#### 4.1.1. \(L^2\) Estimates in a finite slab.

We first consider the case with zero source term in a finite slab \([0, L] \times \mathbb{R}^2\)
\[(4.28)\]
\[\begin{cases}
\nu_\eta \frac{\partial g^L}{\partial \eta} + G(\eta) \left( \nu_\phi \frac{\partial g^L}{\partial \phi} - \nu_\eta v_\phi \frac{\partial g^L}{\partial v_\phi} \right) + \mathcal{L}[g^L] = 0, \\
g^L(0, \bar{v}) = h(\bar{v}) \text{ for } v_\eta > 0, \\
g^L(L, R[\bar{v}]) = g^L(\bar{v}),
\end{cases}\]
where
\[(4.29)\]
\[R[v_\eta, v_\phi] = (-v_\eta, v_\phi).\]
Similarly, we can decompose \(g^L\) as
\[(4.30)\]
\[g^L = w^L + g^L.\]

**Lemma 4.3.** There exists a solution of the equation (4.28) satisfying the estimates
\[(4.31)\]
\[\int_0^L \|\sqrt{\nu}w^L(\eta)\|_{L^2}^2 \, d\eta \leq C,\]
\[(4.32)\]
\[\|q^L(\eta)\|_{L^2}^2 \leq C \left( 1 + \eta + \|\sqrt{\nu}w^L(\eta)\|_{L^2}^2 \right),\]
where \(C\) is a constant independent of \(L\).

**Proof.** We divide the proof into several steps:

**Step 1: Estimate of \(w^L\).**

Multiplying \(g^L\) on both sides of (4.28) and integrating over \(\bar{v} \in \mathbb{R}^2\), we have
\[(4.33)\]
\[\frac{1}{2} \frac{d}{d\eta} \langle v_\eta g^L, g^L \rangle + \frac{1}{2} G(\eta) \langle v_\eta g^L, g^L \rangle = -(w^L, \mathcal{L}[w^L]).\]

Define
\[(4.34)\]
\[\alpha(\eta) = \frac{1}{2} \langle v_\eta g^L, g^L \rangle(\eta),\]
then we have
\[(4.35)\]
\[\alpha(\eta) = \alpha(L) \exp \left( \int_\eta^L G(y) \, dy \right) + \int_\eta^L \exp \left( - \int_\eta^y G(z) \, dz \right) \left( \langle w^L, \mathcal{L}[w^L] \rangle(y) \right) \, dy,\]
\[(4.36)\]
\[\alpha(\eta) = \alpha(0) \exp \left( - \int_0^\eta G(y) \, dy \right) + \int_0^\eta \exp \left( \int_y^\eta G(z) \, dz \right) \left( - \langle w^L, \mathcal{L}[w^L] \rangle(y) \right) \, dy.\]

Since \(\alpha(L) = 0\), (4.35) implies
\[(4.37)\]
\[\alpha(\eta) \geq 0.\]

By
\[(4.38)\]
\[\alpha(0) = \frac{1}{2} \int_{v_\eta > 0} v_\eta (g^L)^2(0) \, d\bar{v} + \frac{1}{2} \int_{v_\eta < 0} v_\eta (g^L)^2(0) \, d\bar{v} \leq \frac{1}{2} \int_{v_\eta > 0} v_\eta h^2 \, d\bar{v} \leq C,\]
and (4.36), we obtain
\begin{equation}
\alpha(\eta) \leq C.
\end{equation}
Hence, (4.35) and (4.39) lead to
\begin{equation}
\int_0^L \exp \left( - \int_0^\eta G(z) \, dz \right) \left( \langle w^L, \mathcal{L}[w^L] \rangle(y) \right) \, dy \leq C,
\end{equation}
which, by Lemma 4.2, further yields
\begin{equation}
\int_0^L \left\| \sqrt{\tau} w^L(\eta) \right\|_{L^1}^2 \, d\eta \leq C
\end{equation}

Step 2: Estimate of $q^L$.

Multiplying $v_\eta \psi_j$ with $j \neq 1$ and integrating over $\bar{u} \in \mathbb{R}^2$, we obtain
\begin{equation}
\frac{d}{d\eta} \langle v^2_\eta \psi_j, g^L \rangle + G(\eta) \left( \langle v_\eta \psi_j, v^2_\eta \frac{\partial g^L}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g^L}{\partial v_\phi} \rangle = - \langle v_\eta \psi_j, \mathcal{L}[w^L] \rangle \right).
\end{equation}
Define $\tilde{q}^L = q^L - q^L_1$ and
\begin{align}
\beta_j(\eta) &= \langle v^2_\eta \psi_j, \tilde{q}^L \rangle(\eta), \\
\beta(\eta) &= \left( \beta_0(\eta), \beta_1(\eta), \beta_2(\eta), \beta_3(\eta) \right)^T.
\end{align}

By definition, $\beta_1 = 0$. For $j \neq 1$, we have
\begin{equation}
\frac{d}{d\eta} \beta_j = G(\eta) \left( \frac{\partial}{\partial v_\eta} (v_\eta v^2_\eta \psi_j) - \frac{\partial}{\partial v_\phi} (v^2_\eta v_\phi \psi_j), q^L_1 \psi_1 + w^L \right) - \langle v_\eta \psi_j, \mathcal{L}[w^L] \rangle - \frac{d}{d\eta} \langle v^2_\eta \psi_j, w^L \rangle.
\end{equation}
Put $\tilde{q}^L_i = \delta_{ij} q^L_i$. Then we can write
\begin{equation}
\left\langle \frac{\partial}{\partial v_\eta} (v_\eta v^2_\eta \psi_j) - \frac{\partial}{\partial v_\phi} (v^2_\eta v_\phi \psi_j), \tilde{q}^L \right\rangle(\eta) = \sum_i B_{ji} \tilde{q}^L_i(\eta),
\end{equation}
where
\begin{equation}
B_{ji} = \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v^2_\eta \psi_j) - \frac{\partial}{\partial v_\phi} (v^2_\eta v_\phi \psi_j), \psi_i \right\rangle.
\end{equation}

Moreover,
\begin{equation}
\beta_j(\eta) = \sum_k A_{jk} \tilde{q}^L_k(\eta),
\end{equation}
where
\begin{equation}
A_{jk} = \langle v^2_\eta \psi_j, \psi_k \rangle,
\end{equation}
is a non-singular matrix such that we can express back
\begin{equation}
\tilde{q}^L_j(\eta) = \sum_k A^{-1}_{jk} \beta_k(\eta).
\end{equation}

Hence, (4.45) can be rewritten as
\begin{equation}
\frac{d}{d\eta} \beta_j = G(BA^{-1})_{ji} \beta_i + D_j,
\end{equation}
where
\begin{equation}
D_j = G(\eta) \left( \frac{\partial}{\partial v_\eta} (v_\eta v^2_\eta \psi_j) - \frac{\partial}{\partial v_\phi} (v^2_\eta v_\phi \psi_j), q^L_1 \psi_1 + w^L \right) - \langle v_\eta \psi_j, \mathcal{L}[w^L] \rangle - \frac{d}{d\eta} \langle v^2_\eta \psi_j, w^L \rangle.
\end{equation}

We can solve for $\beta$ as
\begin{equation}
\beta(\eta) = \exp \left( - W(\eta)BA^{-1} \right) \theta - \zeta(\eta) + \int_0^\eta \exp \left( (W(\eta) - W(y))BA^{-1} \right) Z(y) \, dy,
\end{equation}
where
\( \theta_j = \langle v_n^2 \psi_j, g^L \rangle (0), \quad j \neq 1, \)
\( \zeta_j(\eta) = \langle v_n^2 \psi_j, w^L \rangle (\eta), \)
and
\( Z = D + \frac{d}{d\eta} + G(BA^{-1})\zeta. \)

We have used the fact
\( \int_0^\eta \exp \left( (W(\eta) - W(y))BA^{-1} \right) \frac{d\zeta}{dy} dy = \zeta(\eta) - \exp \left( - W(\eta)BA^{-1} \right) \zeta(0) - \int_0^\eta G(BA^{-1})\zeta(y) dy. \)

Hence, using the boundedness of \( W(\eta) \) and \( BA^{-1} \), we have
\( |\beta(\eta)| \leq C |\theta| + C \int_0^\eta |D(y)| dy + C \int_0^\eta \| \sqrt{\mu}w^L(y) \|_{L^2} dy. \)

By Cauchy’s inequality, we obtain
\( |D(\eta)| \leq C \left( \| \sqrt{\mu}g^L(\eta) \|_{L^2} + \| q^L(\eta)\psi_1 \|_{L^2} \right). \)

Multiplying \( \sqrt{\mu} \) on both sides of (4.28) and integrating over \( \bar{v} \in \mathbb{R}^2 \), we have
\( \frac{d}{d\eta} \langle \sqrt{\mu}v_\eta, g^L \rangle = G(\eta) \left( \frac{\partial}{\partial v_\eta} (v_\eta^2) - \frac{\partial}{\partial v_\phi} (v_\eta v_\phi), \sqrt{\mu}g^L \right) = -G \langle \sqrt{\mu}v_\eta, g^L \rangle. \)

Since \( q^L_1(L) = 0 \), we have
\( q^L_1(\eta) = 0. \)

Also,
\( \langle v_n^2 \psi_j, g^L \rangle (0) \leq C \langle |v_n| g^L(0), g^L(0) \rangle^{1/2} \langle |v_n|^3, \psi_j^2 \rangle^{1/2}, \)
\( \langle |v_n| g^L(0), g^L(0) \rangle = \int_{v_n > 0} \mu v_n h^2 - \int_{v_n < 0} \mu v_n (g^L(0))^2. \)

Since
\( \int_{v_n > 0} \mu v_n h^2 + \int_{v_n < 0} \mu v_n (g^L(0))^2 = 2\alpha(0) \geq 0, \)
we have
\( \theta_j \leq C. \)

In conclusion, we have
\( |\beta(\eta)| \leq C \left( 1 + \| \sqrt{\mu}w^L(\eta) \|_{L^2} + \int_0^\eta \| \sqrt{\mu}w^L(y) \|_{L^2} dy \right), \)
which further implies
\( |q^L(\eta)| \leq C \left( 1 + \| \sqrt{\mu}w^L(\eta) \|_{L^2} + \int_0^\eta \| \sqrt{\mu}w^L(y) \|_{L^2} dy \right). \)

An application of Cauchy’s inequality leads to our desired result.

Step 3: Orthogonal Properties.
In the equation (4.28), multiplying \( \sqrt{\mu} \) on both sides and integrating over \( \bar{v} \in \mathbb{R}^2 \), we have
\( \frac{d}{d\eta} \langle \sqrt{\mu}v_\eta, g^L \rangle = G(\eta) \left( \frac{\partial}{\partial v_\eta} (v_\eta^2) - \frac{\partial}{\partial v_\phi} (v_\eta v_\phi), \sqrt{\mu}g^L \right) = -G \langle \sqrt{\mu}v_\eta, g^L \rangle. \)

Since \( \langle \sqrt{\mu}v_\eta, g^L \rangle (L) = 0 \), we have
\( \langle \sqrt{\mu}v_\eta, g^L \rangle (\eta) = 0. \)
Introducing $\hat{\psi}_i = \psi_i$ and $\hat{\psi}_3 = \psi_3 - 4$. It is easy to check that

\begin{align}
\langle v_\eta \hat{\psi}_i, q^L \rangle &= 0 \quad i \neq 1, \\
\langle v_\eta q^L, q^L \rangle &= 0.
\end{align}

Multiplying $\hat{\psi}_i$ for $i = 2, 3$ on both sides of (4.28) and integrating over $\vec{v} \in \mathbb{R}^2$, we have

\begin{equation}
\frac{d}{d\eta} \langle v_\eta \hat{\psi}_i, w^L \rangle = -CG \langle v_\eta \hat{\psi}_i, w^L \rangle.
\end{equation}

Since $\langle v_\eta \hat{\psi}_i, w^L(L) \rangle = 0$, then we have

\begin{equation}
I_i = \langle v_\eta \hat{\psi}_i, w^L(\eta) \rangle = 0.
\end{equation}

\begin{proof}
\end{proof}

\noindent 4.1.2. $L^2$ Estimates in an infinite slab. We consider the case with zero source term and zero mass flux in an infinite slab

\begin{equation}
\begin{cases}
v_\eta \frac{\partial g}{\partial \eta} + G(\eta) \left( v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) + \mathcal{L}[g] = 0, \\
g(0, \vec{\nu}) = h(\vec{\nu}) \text{ for } v_\eta > 0, \\
\int_{\mathbb{R}^2} v_\eta \sqrt{\nu} g(0, \vec{\nu}) d\vec{\nu} = 0 \\
\lim_{\eta \to \infty} g(\eta, \vec{\nu}) = g_\infty(\vec{\nu}).
\end{cases}
\end{equation}

\begin{lemma}
There exists a unique solution of the equation (4.74) satisfying the estimate

\begin{align}
\|\sqrt{\nu}w\|_{L^2 L^2} &\leq C, \\
\|q_{i, \infty}\| &\leq C, \\
\|q - q_\infty\|_{L^2 L^2} &\leq C,
\end{align}

where $q_\infty = \sum_{i=0}^3 q_{i, \infty} \psi_i$ and the orthogonal properties:

\begin{equation}
\langle v_\eta \psi_i, w \rangle = 0 \quad i \neq 1.
\end{equation}

\end{lemma}

\begin{proof}
We divide the proof into several steps:

Step 1: Weak convergence and estimate of $w$.

We can extend the solution $g^L$ by passing $L \to \infty$. Hence, we can always take weakly subsequence

\begin{align}
q_i^L(\eta) &\to q_i(\eta) \text{ in } L^2_{\text{loc}}([0, \infty)), \\
w^L &\to w \text{ in } L^2_{\text{loc}}([0, \infty), L^2(\mathbb{R}^2)).
\end{align}

Therefore,

\begin{equation}
g = \sum_{i=0}^3 q_i \psi_i + w,
\end{equation}

is a weak solution of the equation (4.74). Also, by the weak lower semi-continuity, the estimate of $w$ is obvious. Also, we can show the orthogonal properties when $L \to \infty$.

Step 2: Estimate of $q_\infty$.

It is easy to see

\begin{equation}
q_1(\eta) = m_f[g] = 0,
\end{equation}

so we do not need to bother with it. Then multiplying $\mathcal{L}^{-1}[v_\eta \hat{\psi}_i]$ for $i = 2, 3$ on both sides of (4.74) and integrating over $\vec{v}$, we get

\begin{equation}
\frac{d}{d\eta} \left( \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], v_\eta g \right) + G(\eta) \left( \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], \left( v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) \right) = -\langle \hat{\psi}_i v_\eta, w \rangle.
\end{equation}
where
\[
(4.84) \quad \left\langle \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], \mathcal{L}[w] \right\rangle = \left\langle \hat{\psi}_i v_\eta, w \right\rangle.
\]
Based on the orthogonal property, we have
\[
(4.85) \quad \left\langle \hat{\psi}_i v_\eta, w \right\rangle = 0.
\]
Therefore, we have
\[
(4.86) \quad \frac{d}{d\eta} \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], g \right\rangle + G(\eta) \left\langle \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], \left( v_\eta^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) \right\rangle = 0.
\]
Since \(v_\eta \in \mathcal{N}\) and \(\mathcal{L}^{-1}[\hat{\psi}_i v_\eta] \in \mathcal{N}^{-1}\), we have
\[
(4.87) \quad \left\langle v_\eta, \mathcal{L}^{-1}[\hat{\psi}_i v_\eta] \right\rangle = 0.
\]
For \(k, i = 2, 3\), put
\[
N_{i,k} = \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], \psi_k \right\rangle,
\]
\[
(4.89) \quad P_{i,k} = \left\langle \left( v_\eta^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], \psi_k \right\rangle.
\]
Thus,
\[
(4.90) \quad \Omega_i = \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], q \right\rangle = \sum_{k=2}^3 N_{i,k} q_k(\eta),
\]
and
\[
(4.91) \quad \left\langle \left( v_\eta^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], q \right\rangle = \sum_{k=2}^3 P_{i,k} q_k(\eta).
\]
Since matrix \(N\) is invertible, from (4.89) and integration by parts, we have for \(i = 2, 3\),
\[
(4.92) \quad \frac{d\Omega_i}{d\eta} = - \frac{d}{d\eta} \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], w \right\rangle + \sum_{k=2}^3 G(\eta)(PN^{-1})_{ik} \Omega_k \ + G(\eta) \left\langle \left( v_\eta^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], w \right\rangle.
\]
Denote
\[
(4.93) \quad \Omega'_i = \exp \left( V(\eta)PN^{-1} \right) \Omega_i.
\]
We can solve
\[
(4.94) \quad \Omega'_i(\eta) = \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], g \right\rangle(0) - \exp \left( W(\eta)PN^{-1} \right) \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], g \right\rangle(\eta) \\
+ \int_0^\eta \exp \left( W(y)PN^{-1} \right) G(y) \left( \left\langle \left( v_\eta^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], w \right\rangle(y) \\
+ \sum_{k=2}^3 (PN^{-1})_{ik} \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], w \right\rangle(y) \right) dy.
\]
We can show
\[
(4.95) \quad \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], g \right\rangle(0) < \infty.
\]
Since \(w \in L^2([0, \infty) \times \mathbb{R}^2)\), then it implies that
\[
(4.96) \quad \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], w \right\rangle(\eta) \to 0 \text{ as } \eta \to \infty.
\]
Considering \( W(\eta) \) and \( PN^{-1} \) are bounded, and \( G(\eta) \in L^2 \), we conclude the limit
\[
\lim_{\eta \to \infty} \Omega'_i(\eta)
\]
eexists, which is denoted as \( \Omega'_{i,\infty} \). Then we can define
\[
q_{i,\infty} = N^{-1} \exp \left( V(\infty)PN^{-1} \right) \Omega'_{i,\infty} \quad \text{for} \quad k = 2, 3.
\]
Finally, we consider \( q_0 \). Multiplying \( v_\eta \) on both sides of (4.74) and integrating over \( \bar{w} \), we obtain
\[
\frac{d}{d\eta} \langle v_\eta g, v_\eta \rangle = -G(\eta) \left( v_\eta^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) = G(\eta) \langle g, v_\eta^2 - v_\eta^2 \rangle.
\]
Then integrating over \( \eta \), we obtain
\[
\langle v_\eta g, v_\eta \rangle (\eta) = \langle v_\eta g, v_\eta \rangle (0) + \int_0^\eta G(y) \langle g, v_\phi^2 - v_\phi^2 \rangle (y) dy.
\]
It is easy to see the right-hand side of (4.100) only depends on \( w \) and \( q_i \), for \( i = 1, 2, 3 \). Hence, we can define
\[
\beta_{\infty} = \lim_{\eta \to \infty} \langle v_\eta g, v_\eta \rangle (\eta).
\]
Hence, the remaining \( q_{0,\infty} \) can be defined based on \( \beta_{\infty} \) and \( q_{i,\infty} \) for \( i = 1, 2, 3 \).

**Step 3: Estimate of \( q - q_{\infty} \).**

Define \( \tilde{g} = g - q_{\infty} \). Then \( \tilde{g} \) satisfies the equation
\[
\langle v_\eta \tilde{g}, v_\eta \rangle (\eta) = \langle v_\eta, g \rangle (\eta) = \langle v_\eta \tilde{\psi}_1, \tilde{g} \rangle (\eta) = \langle v_\eta \tilde{\psi}_1, g \rangle (\eta) = 0 \quad \text{for} \quad i = 2, 3.
\]
Then multiplying \( \mathcal{L}^{-1} [\tilde{\psi}_i v_\eta] \) for \( i = 2, 3 \) on both sides of (4.102) and integrating over \( \bar{w} \), we get
\[
\frac{d}{d\eta} \langle \mathcal{L}^{-1} [\tilde{\psi}_i v_\eta], v_\eta \tilde{g} \rangle + G(\eta) \left( \mathcal{L}^{-1} [\tilde{\psi}_i v_\eta], \langle v_\phi^2 \frac{\partial \tilde{g}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \tilde{g}}{\partial v_\phi} \rangle \right) = -\langle \tilde{\psi}_i v_\eta, \tilde{w} \rangle,
\]
where
\[
\langle \mathcal{L}^{-1} [\tilde{\psi}_i v_\eta], \mathcal{L} [\tilde{w}] \rangle = \langle \tilde{\psi}_i v_\eta, \tilde{w} \rangle.
\]
Based on the orthogonal property, we have
\[
\langle \tilde{\psi}_i v_\eta, \tilde{w} \rangle = 0.
\]
Therefore, we have
\[
\frac{d}{d\eta} \langle v_\eta \mathcal{L}^{-1} [\tilde{\psi}_i v_\eta], \tilde{g} \rangle + G(\eta) \left( \mathcal{L}^{-1} [\tilde{\psi}_i v_\eta], \langle v_\phi^2 \frac{\partial \tilde{g}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \tilde{g}}{\partial v_\phi} \rangle \right) = 0.
\]
Since \( v_\eta \in \mathcal{N} \) and \( \mathcal{L}^{-1} [\tilde{\psi}_i v_\eta] \in \mathcal{N}^\perp \), we have
\[
\langle v_\eta, \mathcal{L}^{-1} [\tilde{\psi}_i v_\eta] \rangle = 0.
\]
For $k, i = 2, 3$, put

\begin{align}
N_{i,k} &= \left\langle v_\eta L^{-1}[^\tilde{v}_iv_\eta], \psi_k \right\rangle, \\
P_{i,k} &= \left\langle \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) L^{-1}[^\tilde{v}_iv_\eta], \psi_k \right\rangle.
\end{align}

Thus,

\begin{align}
\tilde{\Omega}_i &= \left\langle v_\eta L^{-1}[^\tilde{v}_iv_\eta], \tilde{q} \right\rangle = \sum_{k=2}^{3} N_{i,k} \tilde{q}_k(\eta),
\end{align}

and

\begin{align}
\left\langle \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) L^{-1}[^\tilde{v}_iv_\eta], \tilde{w} \right\rangle &= \sum_{k=2}^{3} P_{i,k} \tilde{q}_k(\eta).
\end{align}

Since matrix $N$ is invertible, from (4.109) and integration by parts, we have for $i = 2, 3$,

\begin{align}
\frac{d\tilde{\Omega}_i}{d\eta} &= -\frac{d}{d\eta} \left\langle v_\eta L^{-1}[^\tilde{v}_iv_\eta], \tilde{w} \right\rangle (\eta) \\
&\quad + \sum_{k=2}^{3} G(\eta)(PN^{-1})_{ik} \tilde{\Omega}_k + G(\eta) \left\langle \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) L^{-1}[^\tilde{v}_iv_\eta], \tilde{w} \right\rangle (y).
\end{align}

Integrating over $[\eta, \infty)$, we obtain

\begin{align}
\tilde{\Omega}_i(\eta) &= -\left\langle v_\eta L^{-1}[^\tilde{v}_iv_\eta], \tilde{w} \right\rangle (\eta) \\
&\quad + \int_{\eta}^{\infty} \exp \left( \left( -W(x) + W(y) \right) PN^{-1} \right) G(y) \left\langle \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) L^{-1}[^\tilde{v}_iv_\eta], \tilde{w} \right\rangle (y) \\
&\quad + \sum_{k=2}^{3} (PN^{-1})_{ik} \left\langle v_\eta L^{-1}[^\tilde{v}_iv_\eta], \tilde{w} \right\rangle (y) dy.
\end{align}

where we have used the fact $\tilde{\Omega}_i(\eta) \to 0$ as $\eta \to \infty$ for $i = 2, 3$. Then we naturally have

\begin{align}
|q_i(\eta) - q_i, \infty| \leq C \left( \| \sqrt{\nu} w(\eta) \|_{L^2} + \int_{\eta}^{\infty} \| \sqrt{\nu} w(y) \|_{L^2} |G(y)| dy \right) \quad \text{for} \quad i = 2, 3.
\end{align}

Multiplying $\sqrt{\nu} v_\eta$ on both sides of (4.102) and integrating over $\tilde{\nu}$, we get

\begin{align}
\frac{d}{d\eta} \left( \tilde{q}_0 + 4 \tilde{q}_3 + \langle v_\eta^2 \sqrt{\nu}, \tilde{w} \rangle \right) &= -2G\tilde{q}_3 + G \langle 1 - v_\eta^2, \sqrt{\nu} \tilde{w} \rangle.
\end{align}

Integrating over $[\eta, \infty)$, we have

\begin{align}
\tilde{q}_0(\eta) &= -4\tilde{q}_3(\eta) - \langle v_\eta^2, \sqrt{\nu} \tilde{w} \rangle (\eta) - \int_{\eta}^{\infty} 2G(y)\tilde{q}_3(y) dy + \int_{\eta}^{\infty} G(y) \langle 1 - v_\eta^2, \sqrt{\nu} \tilde{w}(y) \rangle dy.
\end{align}

Hence, it is easy to derive

\begin{align}
|q_0(\eta) - q_{0, \infty} + 2(W - W_\infty)q_4, \infty| \\
&\leq C \left( \| \sqrt{\nu} w(\eta) \|_{L^2} + |q_3(\eta) - q_3, \infty| + \int_{\eta}^{\infty} \left( \| \sqrt{\nu} w(y) \|_{L^2} + |q_3(y) - q_3, \infty| \right) |G(y)| dy \right).
\end{align}

We can directly verify

\begin{align}
\int_{0}^{\infty} \int_{\eta}^{\infty} G^2(y) dy d\eta \leq C.
\end{align}
Then we can obtain

\begin{align}
\int_0^\infty \left( \int_\eta^\infty \| v w(y) \|_{L^1} \, dy \right)^2 \, d\eta & \leq C \int_0^\infty \int_\eta^\infty G^2(y) \, dy \, d\eta \leq C, \\
\int_0^\infty \left( \int_\eta^\infty |q_3(y) - q_{3,\infty}| \, dy \right)^2 \, d\eta & \leq C \| q_3 - q_{3,\infty} \|_{L^2 L^2}.
\end{align}

It is obvious that

\begin{equation}
\| q_0 - q_{0,\infty} + 2(W - W_{\infty})q_{3,\infty} \|_{L^2 L^2} \leq C.
\end{equation}

Therefore, we have

\begin{equation}
\| q - q_{\infty} \|_{L^2 L^2} \leq C.
\end{equation}

Step 4: Uniqueness.

If \( g_1 \) and \( g_2 \) are two solutions of (4.74), define \( g' = g_1 - g_2 \). Then \( g' \) satisfies the equation

\begin{equation}
\begin{cases}
\frac{\partial g'}{\partial \eta} + G(\eta) \left( v_\phi \frac{\partial g'}{\partial v_\phi} - v_{\psi} \frac{\partial g'}{\partial v_{\psi}} \right) + \mathcal{L}[g'] = 0, \\
g'(0, \vec{v}) = 0 \text{ for } \eta > 0, \\
\int_{\mathbb{R}^2} v_\eta \sqrt{\nu} g'(0, \vec{v}) \, d\vec{v} = 0, \\
\lim_{\eta \to \infty} g'(\eta, \vec{v}) = g'_{\infty},
\end{cases}
\end{equation}

Define the linearized entropy as

\begin{equation}
H[g'](\eta) = \langle v_\eta g', g' \rangle (\eta).
\end{equation}

Multiplying \( g' \) on both sides of (4.120) and integrating over \( \vec{v} \), we get

\begin{equation}
\frac{1}{2} \frac{d}{d\eta} \langle v_\eta g', g' \rangle = -e^{-W(\eta)} \langle w_{g'}, \mathcal{L} w_{g'} \rangle - \frac{1}{2} G(\eta) \langle v_\eta g', g' \rangle.
\end{equation}

Hence, we have

\begin{equation}
\frac{1}{2} \frac{d}{d\eta} \left( e^{W} \langle v_\eta g', g' \rangle \right) = -\langle w_{g'}, \mathcal{L} w_{g'} \rangle,
\end{equation}

which implies \( e^{W} H[g'] \) is decreasing. Furthermore, we have

\begin{equation}
e^{W(\eta)} H[g'](\eta) = H[g'](0) - \int_0^\eta \langle w_{g'}, \mathcal{L} w_{g'} \rangle (y) \, dy < \infty.
\end{equation}

Hence, we can take a subsequence such that \( \| \sqrt{\nu} w_{g'}(\eta_n) \|_{L^2} \) goes to zero. Then we can always assume \( q_{g'}(\eta_n) \) goes to \( q_{g',\infty} \). Therefore, we have

\begin{equation}
e^{W(\eta_n)} H[g'](\eta_n) \to \langle v_\eta q_{g',\infty}, q_{g',\infty} \rangle.
\end{equation}

Since \( m_f[g'] = 0 \), we naturally obtain

\begin{equation}
e^{W(\eta_n)} H[g'](\eta_n) \to 0 \text{ as } \eta_n \to \infty.
\end{equation}

Hence, we have

\begin{equation}
e^{W(\eta)} H[g'](\eta) \geq 0,
\end{equation}

and

\begin{equation}
e^{W(\eta)} H[g'](\eta) \to 0 \text{ as } \eta \to \infty.
\end{equation}

In (4.129), integrating over \([0, \infty)\), we achieve

\begin{equation}
- \int_{\nu_\eta < 0} v_\eta (g')^2(0) \, d\vec{v} + \int_0^\infty \langle w_{g'}, \mathcal{L} w_{g'} \rangle (\eta) \, d\eta = \int_{\nu_\eta > 0} v_\eta (g')^2(0) \, d\vec{v} = 0.
\end{equation}
Hence, we have

\begin{equation}
\int_{v_{\eta}<0} v_{\eta}(g')^2(0)d\vec{w} = \int_0^\infty \langle w_{g'}, Lw_{g'} \rangle (\eta)d\eta = 0,
\end{equation}

which implies \( g'(0) = 0 \) and \( w_{g'} = 0 \). Hence, \( g' = q_{g'} \) and satisfies

\begin{equation}
\int_{v_{\eta}<0} v_{\eta}\frac{\partial g'}{\partial \eta} + G(\eta) \left( v_{\phi}^2 \frac{\partial g'}{\partial v_{\phi}} - v_{\eta}v_{\phi} \frac{\partial g'}{\partial v_{\phi}} \right) = 0.
\end{equation}

\( m_f[g'] = 0 \) implies \( q_{g',1} = 0 \). Therefore, multiplying \( v_{\eta}\psi_i \) for \( i \neq 1 \) on both sides of (4.137) and integrating over \( \vec{v} \), we obtain a linear system on \( q_{g',k} \) for \( k \neq 1 \), which possesses a unique solution zero. This means \( g' = 0 \). The solution is unique. \( \square \)

\subsection{L^{2} Estimates with general source term and non-vanishing mass flux.}

\begin{equation}
\begin{cases}
\frac{\partial g}{\partial \eta} + G(\eta) \left( v_{\phi}^2 \frac{\partial g}{\partial v_{\phi}} - v_{\eta}v_{\phi} \frac{\partial g}{\partial v_{\phi}} \right) + \mathcal{L}[g] = S, \\
g(0, \vec{v}) = h(\vec{v}) \text{ for } v_{\eta} > 0, \\
\int_{\mathbb{R}^2} v_{\eta}g(0, \vec{v})\sqrt{\mu}d\vec{w} = m_f[g], \\
\lim_{\eta \to \infty} g(\eta, \vec{v}) = g(\infty)(\vec{v}).
\end{cases}
\end{equation}

\textbf{Lemma 4.5. There exists a unique solution of the equation (4.138) satisfying the estimate}

\begin{align}
\|\sqrt{\nu}w\|_{L^2L^2} &\leq C, \\
|q_{k,\infty}| &\leq C, \\
\|q - q_{\infty}\|_{L^2L^2} &\leq C.
\end{align}

\textbf{Proof.} For the non-vanishing mass flux problem, we can see the function \( g' = g - m_f[g]\sqrt{\mu}e^{-\eta}v_{\eta} \) satisfies the \( \epsilon \)-Milne problem with zero mass flux with the source term

\begin{equation}
S' = S + m_f[g]\sqrt{\mu}\left(v_{\eta}^2 - G(\eta)v_{\phi}^2\right)e^{-\eta},
\end{equation}

and the boundary data

\begin{equation}
h' = h - m_f[g]\sqrt{\mu}v_{\eta}.
\end{equation}

Therefore, we only need to consider the case with general source term and zero mass flux. However, if \( \mathcal{L}[S] \neq 0 \), the mass flux is not conserved when \( \eta \) changes. The construction of solutions can be divided into several steps:

\textbf{Step 1: Decomposition of the source term.} We rewrite the source term as

\begin{equation}
S = S_Q + S_W,
\end{equation}

where \( S_Q \in \mathcal{N} \) is the kernel part and \( S_W = S - S_Q \in \mathcal{N}' \).

\textbf{Step 2: Construction of \( g_1 \).} We first solve the problem with source term \( S_W \) as

\begin{equation}
\begin{cases}
\frac{v_{\eta}\partial g_1}{\partial \eta} + G(\eta) \left( v_{\phi}^2 \frac{\partial g_1}{\partial v_{\phi}} - v_{\eta}v_{\phi} \frac{\partial g_1}{\partial v_{\phi}} \right) + \mathcal{L}[g_1] = S_W, \\
g_1(0, \vec{v}) = h(\vec{v}) \text{ for } v_{\eta} > 0, \\
\int_{\mathbb{R}^2} v_{\eta}g_1(0, \vec{v})\sqrt{\mu}d\vec{w} = 0, \\
\lim_{\eta \to \infty} g_1(\eta, \vec{v}) = g_{1,\infty}(\vec{v}).
\end{cases}
\end{equation}

In this case, we apply the same techniques as the analysis of \( S = 0 \) case. All the results can be generalized in a natural way. Hence, we know \( g_1 \) is well-defined.
Step 3: Construction of $g_2$.
We try to find a function $g_2$ such that
\begin{equation}
\mathcal{L} \left[ v_\eta \frac{\partial g_2}{\partial \eta} + G(\eta) \left( v_\phi^2 \frac{\partial g_2}{\partial v_\phi} - v_\eta v_\phi \frac{\partial g_2}{\partial v_\phi} \right) + S_Q \right] = 0.
\end{equation}
Consider
\begin{equation}
S_Q = \sqrt{\mu} \left( a(\eta) + b(\eta) \cdot \bar{v} + c(\eta) |\bar{v}|^2 \right).
\end{equation}
We make an ansatz that
\begin{equation}
g_2 = \sqrt{\mu} \left( A(\eta)v_\eta + B_1(\eta) + B_2(\eta)v_\phi v_\eta + C(\eta)v_\eta |\bar{v}|^2 \right).
\end{equation}
Plugging this ansatz into the equation (4.146), since the kernel is four-dimensional, we obtain four linear ordinary differential equations which can be solved explicitly. Hence, we can naturally obtain $g_2$. Furthermore, $g_2$ decreases exponentially with respect to $\eta$ when $S_Q$ decays and the initial data are taken properly.

Step 4: Construction of $g_3$.
We may directly verify
\begin{equation}
\mathcal{L} \left[ v_\eta \frac{\partial g_2}{\partial \eta} + G(\eta) \left( v_\phi^2 \frac{\partial g_2}{\partial v_\phi} - v_\eta v_\phi \frac{\partial g_2}{\partial v_\phi} \right) + \mathcal{L}[g_2] + S_Q \right] = 0
\end{equation}
Then we may define $g_3$ as the solution of the equation
\begin{equation}
\begin{cases}
v_\eta \frac{\partial g_3}{\partial \eta} + G(\eta) \left( v_\phi^2 \frac{\partial g_3}{\partial v_\phi} - v_\eta v_\phi \frac{\partial g_3}{\partial v_\phi} \right) + \mathcal{L}[g_3] = v_\eta \frac{\partial g_2}{\partial \eta} + G(\eta) \left( v_\phi^2 \frac{\partial g_2}{\partial v_\phi} - v_\eta v_\phi \frac{\partial g_2}{\partial v_\phi} \right) + \mathcal{L}[g_2] + S_Q, \\
g_3(0, \bar{v}) = -g_2(0, \bar{v}) \text{ for } v_\eta > 0, \\
\int_{\mathbb{R}^2} v_\eta g_3(0, \bar{v}) \sqrt{\mu} d\bar{v} = 0, \\
\lim_{\eta \to \infty} g_3(\eta, \bar{v}) = g_{3,\infty}(\bar{v}).
\end{cases}
\end{equation}
We can obtain $g_3$ is well-posed.

Step 5: Construction of $g_4$.
We may directly verify $g_4 = g_2 + g_3$ satisfies the equation
\begin{equation}
\begin{cases}
v_\eta \frac{\partial g_4}{\partial \eta} + G(\eta) \left( v_\phi^2 \frac{\partial g_4}{\partial v_\phi} - v_\eta v_\phi \frac{\partial g_4}{\partial v_\phi} \right) + \mathcal{L}[g_4] = S_Q, \\
g_4(0, \bar{v}) = h(\bar{v}) \text{ for } v_\eta > 0, \\
\int_{\mathbb{R}^2} v_\eta g_4(0, \bar{v}) \sqrt{\mu} d\bar{v} = 0, \\
\lim_{\eta \to \infty} g_4(\eta, \bar{v}) = g_{4,\infty}(\bar{v}).
\end{cases}
\end{equation}
In summary, we know $g = g_1 + g_4$ satisfies the equation (4.138) and is well-posed.

Lemma 4.6. Assume (4.5) and (4.6) hold. There exists a unique solution $g(\eta, \bar{v})$ to the $\epsilon$-Milne problem (4.1) satisfying
\begin{equation}
\|g - g_\infty\|_{L^2 L^2} \leq C.
\end{equation}
Proof. Taking $g_\infty = q_\infty$, we can naturally obtain the desired result.

Theorem 4.7. Assume (4.5) and (4.6) hold. There exists $\tilde{h}$ satisfying the condition (4.7) such that there exists a unique solution $G(\eta, \bar{v})$ to the $\epsilon$-Milne problem (4.8) satisfying
\begin{equation}
\|G\|_{L^2 L^2} \leq C.
\end{equation}
Proof. The key part is the construction of $\hat{h}$. We intend to find proper $\hat{h}$ such that the equation

$$\begin{align}
\begin{cases}
v_\eta \frac{\partial \tilde{g}}{\partial \eta} + G(\eta) \left(v_\phi \frac{\partial \tilde{g}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \tilde{g}}{\partial v_\phi}\right) + \mathcal{L}[\tilde{g}] & = 0, \\
\int_{\mathbb{R}^2} v_\eta \sqrt{\mu \tilde{g}(0, \vec{v})} d\vec{v} & = \int_{\mathbb{R}^2} v_\eta \sqrt{\mu \hat{h}(\vec{v})} d\vec{v}, \\
\lim_{\eta \to \infty} \tilde{g}(\eta, \vec{v}) & = \hat{g}(0, \vec{v}) \text{ for } v_\eta > 0,
\end{cases}
\end{align}$$

(4.154)

for $\tilde{g}(\eta, \vec{v})$ is well-posed, where $g_\infty \in \mathcal{N}$ is given a priori. The coefficient for $\sqrt{\mu} v_\eta$ in $\hat{h}$ is completed determined by $g_{1, \infty}$, so we do not need to bother with this. Since $\hat{h}$ and $\hat{g}_\infty$ both belong to the kernel space, then consider the three dimensional linear transform $L$ in $\psi_0, \psi_2, \psi_3$. It is easy to see for $\hat{h} = \sqrt{\mu}$ and $\tilde{h} = \sqrt{\mu} |\vec{v}|^2$, $L$ is identity map. The main obstacle is when $\hat{h} = \sqrt{\mu} v_\phi$. Define $\tilde{g} = \tilde{g} - \sqrt{\mu} v_\phi$. Then $\tilde{g}$ satisfies the equation

$$\begin{align}
\begin{cases}
v_\eta \frac{\partial \tilde{g}}{\partial \eta} + G(\eta) \left(v_\phi \frac{\partial \tilde{g}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \tilde{g}}{\partial v_\phi}\right) + \mathcal{L}[\tilde{g}] & = G(\eta) \sqrt{\mu} \tilde{g} v_\phi, \\
\int_{\mathbb{R}^2} v_\eta \sqrt{\mu \tilde{g}(0, \vec{v})} d\vec{v} & = 0 \text{ for } v_\eta > 0, \\
\lim_{\eta \to \infty} \tilde{g}(\eta, \vec{v}) & = \tilde{g}_\infty(\vec{v}).
\end{cases}
\end{align}$$

(4.155)

Based on Lemma 4.6 since $L^1$ and $L^2$ norm of $G$ can be sufficiently small as $\epsilon \to 0$, we know $|q_{\tilde{g}, \infty}|$ is also sufficiently small. This means the transform $L$ is the identity map with an sufficiently small perturbation as $\epsilon \to 0$ when $\hat{h} = \sqrt{\mu} v_\phi$. Hence, $L$ is an invertible transform. Therefore, we can always find $\tilde{h}$ such that $\tilde{g}_\infty = g_{\infty}$, which is desired. Then by Lemma 4.6 and superposition property of equation (4.1) and (4.130), when define $G^* = g^* - \tilde{g}$, the theorem naturally follows. \qed

4.2. $L^\infty$ Estimates.

4.2.1. Mild formulation in a finite slab. Consider the $\epsilon$-transport problem for $g^L(\eta, \vec{v})$ in a finite slab

$$\begin{align}
\begin{cases}
v_\eta \frac{\partial g^L}{\partial \eta} + G(\eta) \left(v_\phi^L \frac{\partial g^L}{\partial v_\eta} - v_\eta v_\phi^L \frac{\partial g^L}{\partial v_\phi}\right) + v g^L & = Q(\eta, \vec{v}), \\
g^L(0, \vec{v}) & = h(\vec{v}) \text{ for } v_\eta > 0, \\
g^L(L, R(\vec{v})) & = g^L(\vec{v}),
\end{cases}
\end{align}$$

(4.156)

We define the characteristics as starting from $(\eta(0), v_\eta(0), v_\phi(0))$ as $(\eta(\eta), v_\eta(\eta), v_\phi(\eta))$ defined by

$$\begin{align}
\eta(\eta) & = \eta, \\
v_\eta^2(\eta) + v_\phi^2(\eta) & = C_1, \\
v_\phi(\eta) e^{-W(\eta)} & = C_2,
\end{align}$$

(4.157) \hspace{1cm} (4.158) \hspace{1cm} (4.159)

where $C_1$ and $C_2$ are two constants depending on the starting point. Along the characteristics, the equation (4.1) can be rewritten as

$$v_\eta \frac{\partial g}{\partial \eta} + v g = Q.$$

(4.160)

Define the energy

$$E(\eta, \vec{v}) = v_\eta^2(\eta) + v_\phi^2(\eta).$$

(4.161)
For $E \geq v_0^2$, define

\begin{align}
(4.162) \quad v'_\phi(\eta, \vec{v}; \eta') &= v_0 e^{W(\eta') - W(\eta)}, \\
(4.163) \quad v'_v(\eta, \vec{v}; \eta') &= \sqrt{E - v_0^2(\eta, \vec{v}; \eta')}, \\
(4.164) \quad \vec{v}'(\eta, \eta') &= (v'_\phi(\eta, \vec{v}; \eta'), v_\phi(\eta, \vec{v}; \eta')), \\
(4.165) \quad R[\vec{v}'(\eta, \eta')] &= (-v'_\phi(\eta, \vec{v}; \eta'), v_\phi(\eta, \vec{v}; \eta')).
\end{align}

Moreover, define an implicit function $\eta^+(\eta, \vec{v})$ by the equation

\begin{equation}
E(\eta, \vec{v}) = v_0^2(\eta, \vec{v}; \eta^+).
\end{equation}

We know $\eta^+$ is the intersection point of the characteristics passing through $(\eta, \vec{v})$ and the axis $v_\eta = 0$. Finally put

\begin{align}
(4.166) \quad G_{\eta, \eta'} &= \int_{\eta'}^{\eta} \frac{\nu(\vec{v}'(\eta, y))}{v'_v(\eta, \vec{v}, y)} dy, \\
(4.167) \quad R[G_{\eta, \eta'}] &= \int_{\eta'}^{\eta} \frac{\nu[R(\vec{v}'(\eta, y))]}{v'_v(\eta, \vec{v}, y)} dy.
\end{align}

We can rewrite the solution to the equation \((4.159)\) along the characteristics as follows:

**Case I:**

For $v_\eta > 0$,

\begin{equation}
(4.169) \quad g^L(\eta, \vec{v}) = h(\vec{v}'(\eta, \vec{v}; 0)) \exp(-G_{\eta, 0}) + \int_0^{\eta} \frac{Q(\eta', \vec{v}(\eta, \vec{v}; \eta'))}{v'_v(\eta, \vec{v}, \eta')} \exp(-G_{\eta, \eta'}) d\eta'.
\end{equation}

**Case II:**

For $v_\eta < 0$ and $|E(\eta, \vec{v})| \geq v_0^2(\eta, \vec{v}; L)$,

\begin{equation}
(4.170) \quad g^L(\eta, \vec{v}) = h(\vec{v}'(\eta, \vec{v}; 0)) \exp(-G_{L, 0} - R[G_{L, \eta}])
\end{equation}

\begin{align*}
&+ \left( \int_0^{\eta} \frac{Q(\eta', \vec{v}(\eta, \vec{v}; \eta'))}{v'_v(\eta, \vec{v}, \eta')} \exp(-G_{L, \eta'} - R[G_{L, \eta}]) d\eta' \\
&+ \int_{\eta}^{L} \frac{Q(\eta', R[\vec{v}(\eta, \vec{v}; \eta')])}{v'_v(\eta, \vec{v}, \eta')} \exp(R[G_{\eta, \eta'}]) d\eta' \right).
\end{align*}

**Case III:**

For $v_\eta < 0$ and $|E(\eta, \vec{v})| \leq v_0^2(\eta, \vec{v}; L)$,

\begin{equation}
(4.171) \quad g^L(\eta, \vec{v}) = h(\vec{v}'(\eta, \vec{v}; 0)) \exp(-G_{\eta+, 0} - R[G_{\eta+, \eta}])
\end{equation}

\begin{align*}
&+ \left( \int_0^{\eta^+} \frac{Q(\eta', \vec{v}(\eta, \vec{v}; \eta'))}{v'_v(\eta, \vec{v}, \eta')} \exp(-G_{\eta+, \eta'} - R[G_{\eta+, \eta}]) d\eta' \\
&+ \int_{\eta}^{\eta^+} \frac{Q(\eta', R[\vec{v}(\eta, \vec{v}; \eta')])}{v'_v(\eta, \vec{v}, \eta')} \exp(R[G_{\eta, \eta'}]) d\eta' \right).
\end{align*}

4.2.2. **Mild formulation in an infinite slab.** Consider the $\epsilon$-transport problem for $g(\eta, \vec{v})$ in an infinite slab

\begin{equation}
\left\{ \begin{aligned}
\frac{v_\eta}{\partial \eta} + G(\eta) \left( v_0^2 \frac{\partial g}{\partial v_0} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) + \nu g &= Q(\eta, \vec{v}), \\
g(0, \vec{v}) &= h(\vec{v}) \text{ for } v_\eta > 0,
\end{aligned} \right.
\end{equation}

We can define the solution via taking limit $L \to \infty$ in \((4.169), (4.170)\) and \((4.171)\) as follows:

\begin{equation}
(4.173) \quad g(\eta, \vec{v}) = \mathcal{A}[h(\vec{v})] + \mathcal{T}[Q(\eta, \vec{v})],
\end{equation}
where

Case I:
For $v_\eta > 0$,

\begin{align}
\mathcal{A}[h(\tilde{r})] &= h(\tilde{r}(\eta, \tilde{v}; 0)) \exp(-G_{\eta, 0}), \\
\mathcal{T}[Q(\eta, \tilde{v})] &= \int^{\eta}_{0} \frac{Q(\eta', R[\tilde{r}(\eta, \tilde{v}; \eta')])}{v_\eta'(\eta, \tilde{v}, \eta')} \exp(-G_{\eta, \eta'}) \, d\eta'.
\end{align}

Case II:
For $v_\eta < 0$ and $\frac{E(\eta, \tilde{v})}{v_\eta(\eta, \tilde{v}; \infty)} \geq v_\eta'(\eta, \tilde{v}; \infty)$,

\begin{align}
\mathcal{A}[h(\tilde{r})] &= 0, \\
\mathcal{T}[Q(\eta, \tilde{v})] &= \int^{\infty}_{\eta} \frac{Q(\eta', R[\tilde{r}(\eta, \tilde{v}; \eta')])}{v_\eta'(\eta, \tilde{v}, \eta')} \exp(R[\eta, \eta']) \, d\eta'.
\end{align}

Case III:
For $v_\eta < 0$ and $\frac{E(\eta, \tilde{v})}{v_\eta(\eta, \tilde{v}; \infty)} \leq v_\eta'(\eta, \tilde{v}; \infty)$,

\begin{align}
\mathcal{A}[h(\tilde{r})] &= h(\tilde{r}(\eta, \tilde{v}; 0)) \exp(-G_{\eta^+, 0} - R[\eta^+, \eta]), \\
\mathcal{T}[Q(\eta, \tilde{v})] &= \left( \int^{\eta^+}_{\eta} \frac{Q(\eta', R[\tilde{r}(\eta, \tilde{v}; \eta')])}{v_\eta'(\eta, \tilde{v}, \eta')} \exp(-G_{\eta^+, \eta'}) \, d\eta' \right) + \int^{\eta^+}_{\eta} \frac{Q(\eta', R[\tilde{r}(\eta, \tilde{v}; \eta')])}{v_\eta'(\eta, \tilde{v}, \eta')} \exp(R[\eta, \eta']) \, d\eta'.
\end{align}

Notice that

\begin{align}
\lim_{L \to \infty} \exp(-G_{\eta, \eta}) = 0,
\end{align}

for $v_\eta < 0$ and $\frac{E(\eta, \tilde{v})}{v_\eta(\eta, \tilde{v}; \infty)}$. Hence, above derivation is valid. In order to achieve the estimate of $g$, we need to control $\mathcal{T}[Q]$ and $\mathcal{A}[h]$.

4.2.3. Preliminaries.

**Lemma 4.8.** There is a positive $0 < \beta < \nu_0$ such that for any $\tilde{v} \geq 0$ and $0 \leq \zeta \leq 1/4$,

\begin{align}
\|e^{\beta \eta} \mathcal{A}[h]\|_{L^\infty_{\tilde{v}, \zeta}} \leq C \|h\|_{L^\infty_{\tilde{v}, \zeta}}.
\end{align}

**Proof.** Based on Lemma 4.2, we know

\begin{align}
\frac{\nu(\tilde{r}(\eta, y))}{v_\eta'(\eta, \tilde{v}, y)} &\geq \nu_0, \\
\frac{\nu(R[\tilde{r}(\eta, y)])}{v_\eta'(\eta, \tilde{v}, y)} &\geq \nu_0.
\end{align}

It follows that

\begin{align}
\exp(-G_{\eta, 0}) &\leq e^{-\beta \eta}, \\
\exp(-G_{\eta^+, 0} - R[\eta^+, \eta]) &\leq e^{-\beta \eta}.
\end{align}

Then our results are obvious.

**Lemma 4.9.** For any integer $\tilde{v} \geq 0$, $0 \leq \zeta \leq 1/4$ and $0 \leq \beta \leq \nu_0/2$, there is a constant $C$ such that

\begin{align}
\|\mathcal{T}[Q]\|_{L^\infty_{\tilde{v}, \zeta}} \leq C \left\| \frac{Q}{\nu} \right\|_{L^\infty_{\tilde{v}, \zeta}}.
\end{align}
Moreover, we have

\[ \| e^{\frac{\beta}{\nu} T[Q]} \|_{L^\infty L_{x,z}^\infty} \leq C \| e^{\frac{\beta q}{\nu} Q} \|_{L^\infty L_{x,z}^\infty}. \]

**Proof.** The first inequality is a special case of the second one, so we only need to prove the second inequality. For \( v_\eta > 0 \) case, we have

\[ \beta(\eta - \eta') - G_{\eta, \eta'} \leq \beta(\eta - \eta') - \frac{\nu_0(\eta - \eta')}{2} = -\frac{G_{\eta, \eta'}}{2}. \]

It is natural that

\[ \int_0^\eta \frac{\nu(\vec{v}(\eta, \eta'))}{v_\eta'(\eta, \vec{v}, \eta')} \exp(\beta(\eta - \eta') - G_{\eta, \eta'})d\eta' \leq \int_0^\infty \exp\left(-\frac{z}{2}\right)dz = 2. \]

Then we estimate

\[ \left| \langle \vec{v} \rangle^\theta e^{\frac{\beta}{\nu} Q} e^{\frac{\beta q}{\nu} T[Q]} \right| \leq e^{\frac{\beta}{\nu} \int_0^\eta \langle \vec{v} \rangle^\theta e^{\frac{\beta}{\nu} Q} \left| Q(\eta', \vec{v}(\eta, \vec{v}, \eta')) \right| \frac{v_\eta'(\eta, \vec{v}, \eta')}{v_\eta'(\eta, \vec{v}, \eta')} \exp(-G_{\eta, \eta'})d\eta' \]

\[ \leq C \left\| e^{\frac{\beta q}{\nu} Q} \right\|_{L^\infty L_{x,z}^\infty} \left\| e^{\frac{\beta q}{\nu} T[Q]} \right\|_{L^\infty L_{x,z}^\infty} \leq C \left\| e^{\frac{\beta q}{\nu} Q} \right\|_{L^\infty L_{x,z}^\infty}. \]

The \( v_\eta < 0 \) case can be proved in a similar fashion. \( \square \)

**Lemma 4.10.** For any \( \delta > 0, \theta \geq 3 \) and \( 0 \leq \zeta \leq 1/4 \), there is a constant \( C(\delta) \) such that

\[ \| T[Q] \|_{L^\infty L^2} \leq C(\delta) \| \nu^{-1/2} Q \|_{L^2 L^2} + \delta \| Q \|_{L^\infty L_{x,z}^\infty}. \]

**Proof.** We divide the proof into several cases:

Case I:
For \( v_\eta > 0 \),

\[ T[Q(\eta, \vec{v})] = \int_0^\eta \frac{Q(\eta', \vec{v}(\eta, \vec{v}, \eta'))}{v_\eta'(\eta, \vec{v}, \eta')} \exp(-G_{\eta, \eta'})d\eta'. \]

We need to estimate

\[ \int_{\mathbb{R}^2} e^{2\zeta|\vec{s}|^2} \left( \int_0^\eta \frac{Q(\eta', \vec{v}(\eta, \vec{v}, \eta'))}{v_\eta'(\eta, \vec{v}, \eta')} \exp(-G_{\eta, \eta'})d\eta' \right)^2 d\vec{v}. \]

Assume \( m > 0 \) is sufficiently small, \( M > 0 \) is sufficiently large and \( \sigma > 0 \) is sufficiently small which will be determined in the following. We can split the integral into the following parts

\[ I = I_1 + I_2 + I_3 + I_4. \]

Case I - Type I: \( \chi_1: M \leq v_\eta'(\eta, \vec{v}, \eta') \) or \( M \leq v_\phi'(\eta, \vec{v}, \eta') \).

By Lemma 4.12 we have

\[ |\vec{v}(\eta, \vec{v}, \eta')| + 1 \leq C(\vec{v}(\eta, \vec{v}, \eta')). \]

Then for \( \theta \geq 3 \), since \( |\vec{v}| \) is conserved along the characteristics, we have

\[ I_1 \leq C \| Q \|_{L^\infty L_{x,z}^\infty} \int_{\mathbb{R}^2} \chi_1 \left( \int_0^\eta \frac{1}{v_\eta'(\eta, \vec{v}, \eta')} \exp(-G_{\eta, \eta'})d\eta' \right)^2 d\vec{v} \]

\[ \leq \frac{C}{M^\theta} \| Q \|_{L^\infty L_{x,z}^\infty} \int_{\mathbb{R}^2} \frac{1}{v_\eta'(\eta, \vec{v}, \eta')} \left( \int_0^\eta \exp(-G_{\eta, \eta'})d\eta' \right)^2 d\vec{v} \]

\[ \leq \frac{C}{M^\theta} \| Q \|_{L^\infty L_{x,z}^\infty}. \]
Therefore, the integral domain for (4.202) \( \eta \) for

by Cauchy’s inequality, we have (4.201) \( \chi \)

Collecting all four types, we have

Taking

Then after substitution, the integral is not from zero, but from \(-\sigma/m\). Hence, we have

Case I - Type III: \( \chi_3 \): \( 0 \leq v''_\eta(\eta, \vec{v}, \eta') \leq m \), \( v'_\phi(\eta, \vec{v}, \eta') \leq M \) and \( \eta - \eta' \geq \sigma \). In this case, we know

Thus, \( G_{\eta, \eta'} \geq \sigma/m \).

Then after substitution, the integral is not from zero, but from \(-\sigma/m\). Hence, we have

Case I - Type IV: \( \chi_4 \): \( 0 \leq v''_\eta(\eta, \vec{v}, \eta') \leq m \), \( v'_\phi(\eta, \vec{v}, \eta') \leq M \) and \( \eta - \eta' \leq \sigma \).

For \( \eta' \leq \eta \) and \( \eta - \eta' \leq \sigma \), we have

\( v_\eta \leq C v''_\eta(\eta, \vec{v}, \eta') \leq C(m + \sigma) \).

Therefore, the integral domain for \( v_\eta \) is very small. We have the estimate

Collecting all four types, we have

Taking \( M \) sufficiently large, \( \sigma \) sufficiently small and \( m << \sigma \), this is the desired result.
Case II:
For \( v_\eta < 0 \) and \( |E(\eta, \bar{v})| \geq v'_\eta(\eta, \bar{v}; \infty) \),
\[
(4.205) \quad T[Q(\eta, \bar{v})] = \int_\eta^\infty \frac{Q(\eta', R[\bar{v}(\eta, \bar{v}; \eta')])}{v'_\eta(\eta, \bar{v}, \eta')} \exp(R[G_{\eta, \eta'}]) d\eta'.
\]
We need to estimate
\[
(4.206) \quad \int_{\mathbb{R}^2} e^{2\zeta|\tilde{v}|^2} \left( \int_\eta^\infty \frac{Q(\eta', R[\bar{v}(\eta, \bar{v}; \eta')])}{v'_\eta(\eta, \bar{v}, \eta')} \exp(R[G_{\eta, \eta'}]) d\eta' \right)^2 d\tilde{v}.
\]
We can split the integral into the following types:
(4.207) \( II = II_1 + II_2 + II_3 \).

Case II - Type I: \( \chi_1: M \leq v'_\eta(\eta, \bar{v}, \eta') \) or \( M \leq v'_\phi(\eta, \bar{v}, \eta') \).
Similar to Case I - Type I, we have
\[
(4.208) \quad II_1 \leq C \|Q\|_{L^{\infty, L^{\infty}_{\bar{v}, \eta}}}, \quad \int_{\mathbb{R}^2} \chi_1 \left( \int_\eta^\infty \frac{1}{\bar{v}(\eta', \bar{v}, \eta')} \exp(R[G_{\eta, \eta'}]) d\eta' \right)^2 d\tilde{v}
\]
\[
\leq C \frac{1}{M^2} \|Q\|_{L^{\infty, L^{\infty}_{\bar{v}, \eta}}}.
\]

Case II - Type II: \( \chi_2: m \leq v'_\eta(\eta, \bar{v}, \eta') \leq M \) and \( v'_\phi(\eta, \bar{v}, \eta') \leq M \).
Similar to Case I - Type II, by Cauchy’s inequality, we have
\[
(4.209) \quad II_2 \leq C \frac{e^{4\zeta M^2}}{m} \int_\eta^\infty \frac{Q^2(\eta, \bar{v}(\eta, \bar{v}, \eta')) d\eta'}{\nu(\eta, \bar{v}, \eta')} \int_\eta^\infty \exp(2R[G_{\eta, \eta'}]) d\eta' \leq C \frac{e^{4\zeta M^2}}{m} \left\| \nu^{-1/2}Q \right\|_{L^2 L^2}.
\]

Case I - Type III: \( \chi_3: 0 \leq v'_\eta(\eta, \bar{v}, \eta') \leq m \) and \( v'_\phi(\eta, \bar{v}, \eta') \leq M \).
In this case, we can directly verify the fact
\[
(4.210) \quad v_\eta \leq v'_\eta(\eta, \bar{v}, \eta')
\]
for \( \eta \leq \eta' \). Then we know the integral of \( v_\eta \) is always in a small domain. Similar to Case I - Type IV, we have the estimate
\[
(4.211) \quad II_3 \leq C m \|Q\|_{L^{\infty, L^{\infty}_{\bar{v}, \eta}}}.
\]
Hence, collecting all three types, we obtain
\[
(4.212) \quad II \leq C \frac{e^{4\zeta M^2}}{m} \left\| \nu^{-1/2}Q \right\|_{L^2 L^2} + C \left( \frac{1}{M^2} + m \right) \|Q\|_{L^{\infty, L^{\infty}_{\bar{v}, \eta}}}.
\]

Taking \( M \) sufficiently large and \( m \) sufficiently small, this is the desired result.

Case III:
For \( v_\eta < 0 \) and \( |E(\eta, \bar{v})| \leq v'_\eta(\eta, \bar{v}; \infty) \),
\[
(4.213) \quad T[Q(\eta, \bar{v})] = \left( \int_0^{\eta'_{+}} \frac{Q(\eta', \bar{v}(\eta, \bar{v}; \eta'))}{\nu(\bar{v}(\eta, \bar{v}; \eta'))} \exp(-G_{\eta^+, \eta'} - R[G_{\eta^+, \eta'}]) d\eta' \right)
\]
\[
+ \int_\eta^{\eta'_{+}} \frac{Q(\eta', R[\bar{v}(\eta, \bar{v}; \eta')])}{\nu(\bar{v}(\eta, \bar{v}; \eta'))} \exp(R[G_{\eta, \eta'}]) d\eta'.
\]
This is a combination of Case I and Case II, so it naturally holds. \( \square \)
4.2.4. **Estimates of \(\epsilon\)-Milne problem.**

**Lemma 4.11.** Assume \((4.5)\) and \((4.6)\) hold. The solution \(g(\eta, \vec{v})\) to the \(\epsilon\)-Milne problem \((4.1)\) satisfies for \(\delta \geq 3\) and \(0 \leq \zeta \leq 1/4\),

\[
\|g - g_\infty\|_{L^\infty_{\delta, \zeta}} \leq C + C\|g - g_\infty\|_{L^2_{L^2}}.
\]

**Proof.** Define \(u = g - g_\infty\). Then \(u\) satisfies the equation

\[
\begin{aligned}
&v_\eta \frac{\partial u}{\partial \eta} + G(\eta) \left( v_\phi \frac{\partial u}{\partial v_\eta} - v_\eta v_\phi \frac{\partial u}{\partial v_\phi} \right) + \mathcal{L}[u] = S(\eta, \vec{v}) + g_{2, \infty} G(\eta) \sqrt{v_\eta} v_\phi = \tilde{S}, \\
u(0, \vec{v}) &= (h - g_\infty)(\vec{v}) = p(\vec{v}) \text{ for } v_\eta > 0,
\end{aligned}
\]

\[
\int_{\mathbb{R}^2} v_\eta \sqrt{\mu} u(0, \vec{v}) d\vec{v} = \int_{v_\eta > 0} v_\eta \sqrt{\mu} h'(\phi, \vec{v}) d\vec{v} - \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g_{\infty}^2(\vec{v}) d\vec{v},
\]

\[
\lim_{\eta \to \infty} u(\eta, \vec{v}) = 0,
\]

\(u = A[p] + T[K[u] + \tilde{S}]\) Based on Lemma \[4.10\] we have

\[
\begin{aligned}
\|u - A[p]\|_{L^\infty_{L^2}} &\leq C(\delta) \left( \|v^{-1/2} K[u]\|_{L^2_{L^2}} + \|v^{-1/2} \tilde{S}\|_{L^2_{L^2}} \right) + \delta \left( \|K[u]\|_{L^\infty_{L^2, \zeta}} + \|\tilde{S}\|_{L^\infty_{L^\infty}} \right) \\
&\leq C(\delta) \left( \|u\|_{L^2_{L^2}} + \|\tilde{S}\|_{L^2_{L^2}} \right) + \delta \left( \|K[u]\|_{L^\infty_{L^\infty, \zeta}} + \|\tilde{S}\|_{L^\infty_{L^\infty}} \right),
\end{aligned}
\]

where we can directly verify

\[
\begin{aligned}
\|v^{-1/2} K[u]\|_{L^2_{L^2}} &\leq \|u\|_{L^2_{L^2}}, \\
\|v^{-1/2} \tilde{S}\|_{L^2_{L^2}} &\leq \|\tilde{S}\|_{L^2_{L^2}}.
\end{aligned}
\]

In \[8\] Lemma 3.3.1, it is shown that

\[
\begin{aligned}
\|K[u]\|_{L^\infty_{L^\infty, \zeta}} &\leq \|u\|_{L^\infty_{L^\infty, \zeta}}, \\
\|K[u]\|_{L^\infty_{L^\infty, \zeta}} &\leq \|u\|_{L^2_{L^2}}.
\end{aligned}
\]
Since \( u = A[p] + T[K[u] + \tilde{S}] \), for \( \epsilon \) and \( \delta \) sufficiently small, we can estimate

\[
\|u\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \leq C \left( \|T[K[u]]\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} + \|T[S]\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} + \|A[p]\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \right)
\]

\[
\leq C \left( \|K[u]\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} + \|S\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} + \|A[p]\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \right)
\]

\[
\leq C \left( \|u\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} + \|S\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} + \|A[p]\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \right)
\]

\[
\leq \ldots
\]

\[
\leq C \left( \|K[u]\|_{L^\infty L^{\infty}_{0, \zeta}} + \|S\|_{L^\infty L^{\infty}_{0, \zeta}} + \|A[p]\|_{L^\infty L^{\infty}_{0, \zeta}} \right)
\]

\[
\leq C \left( \|u\|_{L^\infty L^{2}_{0, \zeta}} + \|S\|_{L^\infty L^{\infty}_{0, \zeta}} + \|A[p]\|_{L^\infty L^{\infty}_{0, \zeta}} \right)
\]

\[
\leq C(\delta) \left( \|v^{-1/2}K[u]\|_{L^2 L^2} + \|v^{-1/2} \tilde{S}\|_{L^2 L^2} \right) + \delta \left( \|K[u]\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} + \|\tilde{S}\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \right)
\]

\[
+ C \left( \|S\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} + \|A[p]\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \right).
\]

Therefore, absorbing \( \delta \|K[u]\|_{L^\infty L^{\infty}_{0, \zeta}} \) into the right-hand side of the second inequality, we have

\[
\|u\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \leq C \left( \|K[u]\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} + \|S\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} + \|A[p]\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \right)
\]

\[
\leq C \left( \|u\|_{L^\infty L^{2}_{\vartheta, \zeta}} + \|S\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} + \|p\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \right).
\]

Then our result naturally follows. \( \square \)

**Lemma 4.12.** Assume (4.3) and (4.6) hold. There exists a unique solution \( g(\eta, \bar{\nu}) \) to the \( \epsilon \)-Milne problem (4.4) satisfying for \( \theta \geq 3 \) and \( 0 \leq \zeta \leq 1/4 \),

\[
\|g - g_{\infty}\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \leq C.
\]

**Proof.** Based on Lemma 4.11, this is obvious. \( \square \)

**Theorem 4.13.** Assume (4.3) and (4.6) hold. There exists a unique solution \( G(\eta, \bar{\nu}) \) to the \( \epsilon \)-Milne problem (4.8) satisfying for \( \theta \geq 3 \) and \( 0 \leq \zeta \leq 1/4 \),

\[
\|G\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \leq C.
\]

**Proof.** Based on Theorem 4.12 and Lemma 4.12, this is obvious. \( \square \)

### 4.3. Exponential Decay.

**Theorem 4.14.** Assume (4.3) and (4.6) hold. For sufficiently small \( K_0 \), there exists a unique solution \( \tilde{G}(\eta, \bar{\nu}) \) to the \( \epsilon \)-Milne problem (4.8) satisfying for \( \theta \geq 3 \) and \( 0 \leq \zeta \leq 1/4 \),

\[
\|e^{K_0 \eta}G\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \leq C.
\]

**Proof.** Define \( U = e^{K_0 \eta}G \). Then \( U \) satisfies the equation

\[
\begin{cases}
\frac{\partial U}{\partial \eta} + G(\eta) \left( v_\eta \frac{\partial U}{\partial \bar{\nu}} - v_\nu v_\eta \frac{\partial U}{\partial \nu_\eta} \right) + \mathcal{L}[U] = e^{K_0 \eta}S(\eta, \bar{\nu}) + K_0 v_\eta U, \\
U(0, \bar{\nu}) = e^{K_0 \eta}(h - \bar{h})(\bar{\nu}) \text{ for } v_\eta > 0,
\end{cases}
\]

\[
\int_{\mathbb{R}^2} v_\eta \sqrt{\mu U}(0, \bar{\nu}) d\bar{\nu} = -\int_{\mathbb{R}^2} \sqrt{\mu h}(\bar{\nu}) d\bar{\nu},
\]

\[
\lim_{\eta \to \infty} U(\eta, \bar{\nu}) = 0,
\]

\[
\|e^{K_0 \eta}G\|_{L^\infty L^{\infty}_{\vartheta, \zeta}} \leq C.
\]
We divide the proof into several steps:

Step 1: $L^2$ Estimates for $S = 0$ and $m_f[U] = 0$.

The orthogonal property implies
\begin{equation}
\langle v_\eta G, G \rangle = \langle v_\eta w_\varphi, w_\varphi \rangle.
\end{equation}

Multiplying $e^{2K_0\eta}$ on both sides of (4.138) and integrating over $\vec{u}$, we obtain
\begin{equation}
\frac{1}{2} \frac{d}{d\eta} e^{2K_0\eta - W(\eta)} \langle v_\eta w_\varphi, w_\varphi \rangle = e^{2K_0\eta - W(\eta)} \langle S, w_\varphi \rangle.
\end{equation}

Since
\begin{equation}
\langle L[w_\varphi], w_\varphi \rangle \geq \nu_0 \langle (1 + |\vec{\varphi}|)w_\varphi, w_\varphi \rangle,
\end{equation}
if $K_0$ sufficiently small, we have
\begin{equation}
\langle L[w_\varphi], w_\varphi \rangle - \langle K_0 v_\eta w_\varphi, w_\varphi \rangle \geq C \langle w_\varphi, w_\varphi \rangle.
\end{equation}

Then by a similar argument as in Lemma 4.3, we can show
\begin{equation}
\int_0^\infty e^{2K_0\eta} \langle \nu w_\varphi, w_\varphi \rangle (\eta) d\eta \leq C.
\end{equation}

Then we have
\begin{equation}
\int_0^\infty e^{2K_0\eta} \nu \int_{\mathbb{R}^2} G^2(\eta, \vec{\varphi}) d\vec{\varphi} d\eta
\end{equation}
\begin{align*}
&\leq \int_0^\infty e^{2K_0\eta} \left| q_{G,0}(\eta) - q_{G,0,\infty} + 2 \left( W(\eta) - W_\infty \right) q_{G,3,\infty} \right| d\eta \\
&+ \sum_{k=2}^3 \int_0^\infty e^{2K_0\eta} \left| q_{G,k}(\eta) - q_{G,k,\infty} \right|^2 d\eta + \int_0^\infty e^{2K_0\eta} \langle \nu w_\varphi, w_\varphi \rangle (\eta) d\eta \\
&\leq C \int_0^\infty e^{2K_0\eta} \langle \nu w_\varphi, w_\varphi \rangle (\eta) d\eta + C \int_0^\infty e^{2K_0\eta} \| S \|_{L^2}(\eta) d\eta \\
&+ C \int_0^\infty e^{2K_0\eta} \left( \int_0^\infty G(y) \| \sqrt{\nu w_\varphi} \|_{L^2}(y) dy \right)^2 d\eta \\
&\leq C + C \int_0^\infty e^{2K_0\eta} \langle \nu w_\varphi, w_\varphi \rangle (\eta) d\eta \cdot \int_0^\infty \int_0^\infty e^{2K_0(\eta - y)} G^2(y) dy d\eta \\
&\leq C.
\end{align*}

This shows
\begin{equation}
\| U \|_{L^2 L^2} < C.
\end{equation}

Step 2: $L^2$ Estimates for general source term and mass flux.

We may follow the idea of Lemma 4.4. Note that all the auxiliary functions we construct decays exponentially.

Hence, the result naturally follows.

Step 3: $L^\infty$ Estimates.

By a similar argument of Lemma 4.11 for $K_0$ sufficiently small, we can show
\begin{equation}
\| U \|_{L^\infty L^\infty_{\theta,\zeta}} \leq C \left( \| U \|_{L^2 L^2} + \| e^{K_0 \eta} S \|_{L^\infty L^\infty_{\theta,\zeta}} + \| K_0 \nu_\eta U \|_{L^\infty L^\infty_{\theta,\zeta}} + \| h \|_{L^\infty_{\theta,\zeta}} \right).
\end{equation}

Then for $K_0$ small, absorbing $K_0$ term into the left-hand side, we naturally obtain the result.
Remark 4.15. Taking $G = 0$, we consider the Milne problem

\[
\begin{align*}
\frac{(\vec{n} \cdot \vec{v})}{\eta} \frac{\partial g}{\partial \eta} + \mathcal{L}[g] &= S(\eta, \phi, \vec{v}), \\
g(0, \phi, \vec{v}) &= h(\phi, \vec{v}) \text{ for } v_\eta > 0, \\
\int_{\mathbb{R}^2} (\vec{n} \cdot \vec{v}) \sqrt{\mu} g(0, \phi, \vec{v}) d\vec{v} &= \int_{\vec{n} \cdot \vec{v} > 0} (\vec{n} \cdot \vec{v}) \sqrt{\mu} h(\phi, \vec{v}) d\vec{v}, \\
\lim_{\eta \to \infty} g(\eta, \phi, \vec{v}) &= g_\infty(\phi, \vec{v}).
\end{align*}
\]

(4.234)

Then we can obtain the similar result as in Theorem 4.14, i.e. there exists

\[
\tilde{h}(\phi, \vec{v}) = \sqrt{\mu} \left( \tilde{D}_0(\phi) + \tilde{D}_1(\phi) v_1 + \tilde{D}_2(\phi) v_2 + \tilde{D}_3(\phi) |\vec{v}|^2 \right),
\]

(4.235)

such that the Milne problem for $\mathcal{G}(\eta, \phi, \vec{v})$ in the domain $(\eta, \phi, \vec{v}) \in [0, \infty) \times [-\pi, \pi) \times \mathbb{R}^2$

\[
\begin{align*}
(\vec{n} \cdot \vec{v}) \frac{\partial G}{\partial \eta} + \mathcal{L}[G] &= S(\eta, \phi, \vec{v}), \\
G(0, \phi, \vec{v}) &= h(\phi, \vec{v}) - \tilde{h}(\phi, \vec{v}) \text{ for } \vec{n} \cdot \vec{v} > 0, \\
\int_{\mathbb{R}^2} (\vec{n} \cdot \vec{v}) \sqrt{\mu} G(0, \phi, \vec{v}) d\vec{v} &= \int_{\vec{n} \cdot \vec{v} > 0} (\vec{n} \cdot \vec{v}) \sqrt{\mu} h(\phi, \vec{v}) d\vec{v} - \int_{\mathbb{R}^2} (\vec{n} \cdot \vec{v}) \sqrt{\mu} \tilde{h}(\phi, \vec{v}) d\vec{v}, \\
\lim_{\eta \to \infty} G(\eta, \phi, \vec{v}) &= 0.
\end{align*}
\]

(4.236)

is well-posed in $L^\infty$ and decays exponentially.
5. Main Theorem

5.1. Validity of the Hilbert Expansion.

**Theorem 5.1.** For given \( \mu_\varepsilon > 0 \) satisfying (1.12) and (1.13), we have

\[
\left\| (\bar{v})^\vartheta e^{\varepsilon|\bar{v}|^2} \left( f^\varepsilon - \varepsilon (\mathcal{F}_1 + \mathcal{F}_2) \right) \right\|_{L^\infty} = O(\varepsilon^2),
\]

for \( 0 \leq \zeta \leq 1/4 \) and \( \vartheta \geq 3 \).

**Proof.** We divide the proof into several steps:

Step 1: Remainder definitions.

We combine the interior solution and boundary layer as follows:

\[
f^\varepsilon \sim \sum_{k=1}^{\infty} \varepsilon^k \mathcal{F}_k^\varepsilon + \sum_{k=1}^{\infty} \varepsilon^k \mathcal{F}_k^\varepsilon.
\]

Define the remainder as

\[
R_N = \frac{1}{\varepsilon^3} \left( f^\varepsilon - \sum_{k=1}^{N} \varepsilon^k \mathcal{F}_k^\varepsilon - \sum_{k=1}^{N} \varepsilon^k \mathcal{F}_k^\varepsilon \right) = \frac{1}{\varepsilon^3} \left( f^\varepsilon - Q_N - \mathcal{Q}_N \right),
\]

where

\[
Q_N = \sum_{k=1}^{N} \varepsilon^k \mathcal{F}_k^\varepsilon,
\]

\[
\mathcal{Q}_N = \sum_{k=1}^{N} \varepsilon^k \mathcal{F}_k^\varepsilon.
\]

Noting the equation (2.45) is equivalent to the equation (1.17), we write \( \mathcal{L} \) to denote the linearized Boltzmann operator as follows:

\[
\mathcal{L}[f] = \varepsilon \bar{v} \cdot \nabla_x u + \mathcal{L}[f] = v_\eta \frac{\partial f}{\partial \eta} - v_\phi \frac{\partial f}{\partial \vartheta} + v_\vartheta \frac{\partial f}{\partial v_\vartheta} - v_\eta v_\phi \frac{\partial f}{\partial v_\vartheta} + \mathcal{L}[f].
\]

Step 2: Estimates of \( \mathcal{L}[R_N] \).

The interior contribution can be estimated as

\[
\mathcal{L}[Q_N] = \varepsilon \bar{v} \cdot \nabla_x Q_N + \mathcal{L}[Q_N] = \sum_{k=1}^{N} \varepsilon^k \left( \varepsilon \bar{v} \cdot \nabla_x \mathcal{F}_k^\varepsilon + \mathcal{L}[\mathcal{F}_k^\varepsilon] \right)
\]

\[
= \varepsilon \mathcal{L}[\mathcal{F}_1^\varepsilon] + \sum_{k=2}^{N} \varepsilon^k \left( \bar{v} \cdot \nabla_x \mathcal{F}_{k-1}^\varepsilon + \mathcal{L}[\mathcal{F}_k^\varepsilon] \right) + \varepsilon^{N+1} \bar{v} \cdot \nabla_x \mathcal{F}_N^\varepsilon
\]

\[
= \varepsilon^{N+1} \bar{v} \cdot \nabla_x \mathcal{F}_N^\varepsilon + \sum_{1 \leq i, j \leq N} \varepsilon^{i+j} \Gamma[\mathcal{F}_i^\varepsilon, \mathcal{F}_j^\varepsilon].
\]
The boundary layer solution is $\mathcal{F}_N = G_k \cdot \psi_0$ where $G_k$ solves the $\epsilon$-Milne problem. Notice $\psi_0 \psi = \psi_0$, so the boundary layer contribution can be estimated as

\[
(5.8) \quad \mathcal{L}[\mathcal{Q}_N] = v_0 \frac{\partial \mathcal{Q}_N}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \left( - v_0 \frac{\partial \mathcal{Q}_N}{\partial \phi} + v_0^2 \frac{\partial \mathcal{Q}_N}{\partial v_0} - v_0 v_\phi \frac{\partial \mathcal{Q}_N}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{Q}_N] 
\]

\[
= \sum_{k=1}^{N} \epsilon^k \left( \frac{\partial \mathcal{Q}_k}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \left( - v_0 \frac{\partial \mathcal{Q}_k}{\partial \phi} + v_0^2 \frac{\partial \mathcal{Q}_k}{\partial v_0} - v_0 v_\phi \frac{\partial \mathcal{Q}_k}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{Q}_k] \right) 
\]

\[
= \sum_{k=1}^{N} \epsilon^k \left( v_0 \frac{\partial G_k}{\partial \eta} - \frac{\epsilon v_0}{1 - \epsilon \eta} \left( v_0 \frac{\partial G_k}{\partial \psi} - v_\psi v_\phi \frac{\partial G_k}{\partial v_\phi} \right) + \mathcal{L}[G_k] \right) 
\]

\[
= \sum_{k=1}^{N} \epsilon^k v_0 \frac{\partial G_k}{\partial \eta} + \sum_{k=1}^{N} \epsilon^k \left( v_0 \frac{\partial G_k}{\partial \eta} - \frac{\epsilon v_0}{1 - \epsilon \eta} \left( v_0 \frac{\partial G_k}{\partial \psi} - v_\psi v_\phi \frac{\partial G_k}{\partial v_\phi} \right) + \mathcal{L}[G_k] \right) 
\]

\[
= \sum_{k=1}^{N} \epsilon^k v_0 \frac{\partial G_k}{\partial \eta} + \sum_{k=1}^{N} \epsilon^k \left( v_0 \frac{\partial G_k}{\partial \eta} - \frac{\epsilon v_0}{1 - \epsilon \eta} \left( v_0 \frac{\partial G_k}{\partial \psi} - v_\psi v_\phi \frac{\partial G_k}{\partial v_\phi} \right) + \mathcal{L}[G_k] \right) 
\]

Note that for any $f, g \in L^2$,

\[
(5.9) \quad P[\Gamma(f, g)] = 0. 
\]

Since

\[
(5.10) \quad \mathcal{L}[f] = \Gamma[f, f'], 
\]

then we can naturally obtain

\[
(5.11) \quad \mathcal{L}[R N] = \frac{1}{\epsilon^3} \mathcal{L}[f - \mathcal{Q}_N - \mathcal{Q}_N] = \frac{1}{\epsilon^3} \mathcal{L}[f'] - \frac{1}{\epsilon^3} \mathcal{L}[\mathcal{Q}_N] - \frac{1}{\epsilon^3} \mathcal{L}[\mathcal{Q}_N] 
\]

\[
= \frac{1}{\epsilon^3} \Gamma[\mathcal{Q}_N + \mathcal{Q}_N + \epsilon^3 \mathcal{R}_N, \mathcal{Q}_N + \mathcal{Q}_N + \epsilon^3 \mathcal{R}_N] 
\]

\[
- \sum_{1 \leq i, j, k \leq N} \epsilon^{i+j-3} \Gamma[\mathcal{F}_{i'}, \mathcal{F}_{j'}] - \psi_0 \sum_{1 \leq i, j \leq N} \epsilon^{i+j-3} \left( \Gamma[\mathcal{F}_{i'}, \mathcal{F}_{j'}] + 2 \Gamma[\mathcal{F}_{i'}, \mathcal{F}_{j'}] \right) 
\]

\[
- \epsilon^{N-2} \nabla_x \mathcal{F}_N - \sum_{k=1}^{N} \epsilon^{k-3} v_\eta \frac{\partial G_k}{\partial \phi} = \epsilon^{N-2} \psi_0 \frac{\partial G_k}{\partial \phi} 
\]

\[
= \epsilon^3 \Gamma[R_N, R_N] + 2 \Gamma[R_N, \mathcal{Q}_N + \mathcal{Q}_N] 
\]

\[
- \sum_{1 \leq i, j \leq N} \epsilon^{i+j-3} \Gamma[\mathcal{F}_{i'}, \mathcal{F}_{j'}] - \psi_0 \sum_{1 \leq i, j \leq N} \epsilon^{i+j-3} \left( \Gamma[\mathcal{F}_{i'}, \mathcal{F}_{j'}] + 2 \Gamma[\mathcal{F}_{i'}, \mathcal{F}_{j'}] \right) 
\]

\[
- \epsilon^{N-2} \nabla_x \mathcal{F}_N - \sum_{k=1}^{N} \epsilon^{k-3} v_\eta \frac{\partial G_k}{\partial \phi} = \epsilon^{N-2} \psi_0 \frac{\partial G_k}{\partial \phi}. 
\]
Based on (5.9), we may directly verify that the compatibility condition \( \text{(5.2)} \) holds.

Step 3: Estimates of \( R_N - \mathcal{P}[R_N] \).

Since

\[
(5.12) \quad f^* - \mathcal{P}[f^*] = \frac{\mu_0^* - \mu}{\sqrt{\mu}} \int_{\tilde{n}(\tilde{x}_0) \cdot \tilde{v}^* > 0} \sqrt{\mu(\tilde{v}^*)} f^*(\tilde{x}_0, \tilde{v}^*) \left| \tilde{n}(\tilde{x}_0) \cdot \tilde{v}^* \right| \, d\tilde{v}^* + \frac{\mu_0^*(\tilde{x}_0, \tilde{v}) - \mu(\tilde{v})}{\sqrt{\mu(\tilde{v})}},
\]

we have

\[
(5.13) \quad R_N - \mathcal{P}[R_N] = \frac{1}{\epsilon^3}(f^* - \mathcal{Q}_N - \mathcal{L}_N) - \frac{1}{\epsilon^3}\mathcal{P}[f^* - \mathcal{Q}_N - \mathcal{L}_N] = \frac{\mu_0^* - \mu}{\sqrt{\mu}} \int_{\tilde{n}(\tilde{x}_0) \cdot \tilde{v}^* > 0} \sqrt{\mu(\tilde{v}^*)} R_N(\tilde{x}_0, \tilde{v}^*) \left| \tilde{n}(\tilde{x}_0) \cdot \tilde{v}^* \right| \, d\tilde{v}^* \\
+ \sum_{1 \leq i \leq N} \epsilon^{i+j} \frac{\mu_i}{\sqrt{\mu}} \int_{\tilde{n}(\tilde{x}_0) \cdot \tilde{v}^* > 0} \sqrt{\mu(\tilde{v}^*)} (\mathcal{F}_j^i + \mathcal{F}_j^i)(\tilde{x}_0, \tilde{v}^*) \left| \tilde{n}(\tilde{x}_0) \cdot \tilde{v}^* \right| \, d\tilde{v}^* \\
+ \frac{1}{\epsilon^3} \sum_{i=1}^N \epsilon^i \mu_i \int_{\tilde{n}(\tilde{x}_0) \cdot \tilde{v}^* > 0} \sqrt{\mu(\tilde{v}^*)} \left( \sum_{i=1}^N \epsilon^i (\mathcal{F}_i^* + \mathcal{F}_i^*) \right)(\tilde{x}_0, \tilde{v}^*) \left| \tilde{n}(\tilde{x}_0) \cdot \tilde{v}^* \right| \, d\tilde{v}^* \\
+ \frac{1}{\epsilon^3} \sum_{i=1}^N \epsilon^i \mu_i \frac{\mu_0^* - \mu}{\sqrt{\mu}}.
\]

Based on (5.9), we may directly verify that the compatibility condition \( \text{(5.2)} \) holds.

Step 4: Estimates of \( R_N \).

Since we know \( \mathcal{L}[R_N] \) and \( R_N - \mathcal{P}[R_N] \), we can estimate \( R_N \). \( R_N \) satisfies the equation

\[
(5.14) \quad \begin{cases}
\epsilon \tilde{v} \cdot \nabla \tilde{x} R_N + \mathcal{L}[R_N] = \epsilon^3 \Gamma[R_N, R_N] + 2 \Gamma[R_N, \mathcal{Q}_N + \mathcal{L}_N] + S_N \quad \text{in} \quad \Omega, \\
R_N(\tilde{x}_0, \tilde{v}) - \mathcal{P}[R_N](\tilde{x}_0, \tilde{v}) = \frac{\mu_0^* - \mu}{\sqrt{\mu}} \int_{\tilde{n}(\tilde{x}_0) \cdot \tilde{v}^* > 0} \sqrt{\mu(\tilde{v}^*)} R_N(\tilde{x}_0, \tilde{v}^*) \left| \tilde{n} \cdot \tilde{v}^* \right| \, d\tilde{v}^* + h_N \quad \text{for} \quad \tilde{v} \cdot \tilde{n} < 0 \quad \text{and} \quad \tilde{x}_0 \in \partial \Omega,
\end{cases}
\]

where

\[
S_N = - \sum_{1 \leq i,j \leq N} \epsilon^{i+j-3} \Gamma[\mathcal{F}_i^*, \mathcal{F}_j^*] - \psi_0 \sum_{1 \leq i,j \leq N} \epsilon^{i+j-3} \left( \Gamma[\mathcal{F}_i^*, \mathcal{F}_j^*] + 2 \Gamma[\mathcal{F}_i^*, \mathcal{F}_j^*] \right) - \epsilon^{N-2} \tilde{v} \cdot \nabla \mathcal{F}_N - \sum_{k=1}^N \epsilon^{k-3} \psi_0 \frac{\partial \psi_0}{\partial \eta} \frac{\partial \mathcal{G}_N^*}{\partial \phi} - \epsilon^{N-2} \psi_0 \frac{1 - \epsilon \eta \psi_0}{1 - \epsilon \eta \psi_0} \frac{\partial \mathcal{G}_N^*}{\partial \phi},
\]

and

\[
(5.15) \quad h_N = \sum_{1 \leq i,j \leq N} \epsilon^{i+j} \frac{\mu_i}{\sqrt{\mu}} \int_{\tilde{n}(\tilde{x}_0) \cdot \tilde{v}^* > 0} \sqrt{\mu(\tilde{v}^*)} (\mathcal{F}_j^i + \mathcal{F}_j^i)(\tilde{x}_0, \tilde{v}^*) \left| \tilde{n} \cdot \tilde{v}^* \right| \, d\tilde{v}^* \\
+ \frac{1}{\epsilon^3} \sum_{i=1}^N \epsilon^i \mu_i \int_{\tilde{n}(\tilde{x}_0) \cdot \tilde{v}^* > 0} \sqrt{\mu(\tilde{v}^*)} \left( \sum_{i=1}^N \epsilon^i (\mathcal{F}_i^* + \mathcal{F}_i^*) \right)(\tilde{x}_0, \tilde{v}^*) \left| \tilde{n} \cdot \tilde{v}^* \right| \, d\tilde{v}^* \\
+ \frac{1}{\epsilon^3} \sum_{i=1}^N \epsilon^i \mu_i \frac{\mu_0^* - \mu}{\sqrt{\mu}}.
\]
By the classical estimate of steady Navier-Stokes-type equations, exponential decay of $\mathcal{F}_s^\epsilon$ and \([5.9]\), we can directly verify

\begin{align}
(5.16) & \quad \int_{\Omega} \int_{\mathbb{R}^2} S_N(\bar{x}, \bar{v}) \sqrt{\mu} d\bar{v} d\bar{x} = 0, \\
(5.17) & \quad \int_{\gamma_s} h_N(\bar{x}, \bar{v}) \sqrt{\mu} d\gamma = 0,
\end{align}

and

\begin{align}
(5.18) & \quad \left\| \langle \bar{v} \rangle^\theta e^{\bar{c} |\bar{v}|^2} S_N \right\|_{L^\infty} \leq C e^{N-2}, \\
(5.19) & \quad \left\| \langle \bar{v} \rangle^\theta e^{\bar{c} |\bar{v}|^2} h_N \right\|_{L^\infty} \leq C e^{N-2}.
\end{align}

Also, considering the normalization condition of $f^\star$, we know $R_N$ satisfies

\begin{equation}
(5.20) \quad \int_{\Omega} \int_{\mathbb{R}^2} R_N(\bar{x}, \bar{v}) \sqrt{\mu} d\bar{v} d\bar{x} = 0.
\end{equation}

Based on Theorem \([3.7]\) we have

\begin{equation}
(5.21) \quad \|R_N\|_{L^2} + |R_N|_{L^2_\perp} \leq C \left( \frac{1}{\epsilon^2} \|S_N\|_{L^2} + \frac{1}{\epsilon} \|e^{3\Gamma} [R_N, R_N]\|_{L^2} + \frac{1}{\epsilon} \|2\Gamma [R_N, Q_N + \mathcal{O}_N]\|_{L^2}ight.
\end{equation}

\begin{equation}
\left. + \frac{1}{\epsilon^{1/2}} \frac{\mu_0 - \mu}{\sqrt{\mu}} \int_{\mathbb{R}^2} \frac{1}{\sqrt{|\bar{v}|^2}} \left( |\bar{v}| \right) R_N(\bar{x}, \bar{v}) |\bar{n} \cdot \bar{v}| d\bar{v} \right) + \frac{1}{\epsilon^{1/2}} |h_N|_{L^2_\perp}.
\end{equation}

Then since

\begin{equation}
(5.22) \quad \left\| \langle \bar{v} \rangle^\theta e^{\bar{c} |\bar{v}|^2} (\mu_0^\epsilon - \mu) \right\|_{L^\infty} \leq C_0 \epsilon,
\end{equation}

for $C_0$ is sufficiently small, we deduce $\mu_1$ is sufficiently small. Hence, small boundary data naturally yields

\begin{equation}
(5.23) \quad \left\| \langle \bar{v} \rangle^\theta e^{\bar{c} |\bar{v}|^2} (Q_N + \mathcal{O}_N) \right\|_{L^\infty} \leq C_0 C \epsilon,
\end{equation}

which further implies

\begin{equation}
(5.24) \quad \frac{1}{\epsilon} \|2\Gamma [R_N, Q_N + \mathcal{O}_N]\|_{L^2} \leq \frac{C}{\epsilon} \|R_N\|_{L^2} \left\| \langle \bar{v} \rangle^\theta e^{\bar{c} |\bar{v}|^2} (Q_N + \mathcal{O}_N) \right\|_{L^\infty} \leq \delta \|R_N\|_{L^2},
\end{equation}

and

\begin{equation}
(5.25) \quad \frac{1}{\epsilon^{1/2}} \frac{\mu_0 - \mu}{\sqrt{\mu}} \int_{\mathbb{R}^2} \frac{1}{\sqrt{|\bar{v}|^2}} \left( |\bar{v}| \right) R_N(\bar{x}, \bar{v}) |\bar{n} \cdot \bar{v}| d\bar{v} \leq \delta |R_N|_{L^2_\perp},
\end{equation}

for some small $\delta > 0$. Hence, absorbing them into the left-hand side of \((5.21)\) yields

\begin{equation}
(5.26) \quad \|R_N\|_{L^2} + |R_N|_{L^2_\perp} \leq C \left( \frac{1}{\epsilon^2} \|S_N\|_{L^2} + \epsilon^2 \|\Gamma [R_N, R_N]\|_{L^2} + \frac{1}{\epsilon^{1/2}} |h_N|_{L^2_\perp} \right).
\end{equation}
Based on Theorem 3.13 we have

\[
(5.27) \quad \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} R_N \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} R_N \right\|_{L^\infty}
\]

\[
\leq C \left( \frac{\epsilon}{\epsilon} \left\| R_N \right\|_{L^2} + \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} S_N \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} h_N \right\|_{L^\infty}
\]

\[
+ \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} \Gamma[R_N, R_N] \right\|_{L^\infty} + \left\| 2 \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} \Gamma[R_N, Q_N + \mathcal{L}_N] \right\|_{L^\infty}
\]

\[
+ \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} \mathcal{L}_N \left( \frac{\mu \pi - \mu}{\sqrt{\pi}} \int_{\vec{v} \cdot \vec{v} > 0} \sqrt{\mu(x \cdot \vec{v})} R_N(x_0, \vec{v}) \, |\vec{v} \cdot \vec{v}| \, d\vec{v} \right) \right\|_{L^\infty}
\]

\[
\leq C \left( \epsilon \left\| \Gamma[R_N, R_N] \right\|_{L^2} + \frac{1}{\epsilon^2} \left\| S_N \right\|_{L^2} + \frac{1}{\epsilon^2} \left\| h_N \right\|_{L^2} + \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} S_N \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} h_N \right\|_{L^\infty}
\]

\[
+ \epsilon^3 \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} \Gamma[R_N, R_N] \right\|_{L^\infty} + \left\| 2 \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} \Gamma[R_N, Q_N + \mathcal{L}_N] \right\|_{L^\infty} + \epsilon \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} R_N \right\|_{L^\infty}
\]

\[
\leq C \left( \epsilon \left\| \Gamma[R_N, R_N] \right\|_{L^2} + \epsilon^3 \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} \Gamma[R_N, R_N] \right\|_{L^\infty} + \epsilon \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} R_N \right\|_{L^\infty} + \epsilon^{N-5} \right).
\]

Moreover, we can directly estimate

\[
(5.28) \quad \left\| \epsilon \Gamma[R_N, R_N] \right\|_{L^2} \leq C \epsilon \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} R_N \right\|_{L^\infty}
\]

\[
(5.29) \quad \epsilon^3 \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} \Gamma[R_N, R_N] \right\|_{L^\infty} \leq C \epsilon^3 \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} R_N \right\|_{L^\infty}.
\]

Then if \( N \geq 4 \), we obtain

\[
(5.30) \quad \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} R_N \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} R_N \right\|_{L^\infty} \leq C \epsilon^{N-5} + C \epsilon \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} R_N \right\|_{L^\infty},
\]

which further implies

\[
(5.31) \quad \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} R_N \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} R_N \right\|_{L^\infty} \leq \frac{C}{\epsilon},
\]

for \( \epsilon \) sufficiently small. This means we have shown

\[
(5.32) \quad \frac{1}{\epsilon^3} \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} \left( f^* - \sum_{k=1}^{N} \epsilon^k \mathcal{F}_k^* \right) \right\|_{L^\infty} = O \left( \frac{1}{\epsilon} \right),
\]

which naturally leads to the desired result.

\[
\square
\]

5.2. Well-Posedness of Boltzmann Equation.

Theorem 5.2. For given \( \mu^* > 0 \) satisfying (1.12) and (1.13), there exists a non-negative solution \( F^* = M \mu + \sqrt{\pi} g^* \) with constant \( M \) to the steady Boltzmann equation (1.7) satisfying the normalization condition (1.10) such that for \( 0 \leq \zeta \leq 1/4 \) and \( \theta \geq 3 \),

\[
(5.33) \quad \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} f^* \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} f^* \right\|_{L^\infty} \leq C \epsilon.
\]

If \( M \mu + \sqrt{\pi} g^* \) is another solution satisfying the normalization condition (1.10) such that

\[
(5.34) \quad \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} g^* \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^\theta \, e^{\zeta|\vec{v}|^2} g^* \right\|_{L^\infty} \leq C \epsilon,
\]

then \( f^* = g^* \).

Proof. Based on the proof of Theorem 5.2 we know \( R_N \) exists and is well-posed in \( L^\infty \). Hence, combining the estimate of steady Navier-Stokes-type equations and \( \epsilon \)-Milne problem, we have

\[
(5.35) \quad f^* = e^3 R_N + Q_N + \mathcal{L}_N,
\]

exists and is well-posed. The uniqueness follows from a standard argument.

\[
\square
\]
6. Counterexamples

In this section, we will construct some counterexamples to the classical boundary layer analysis in \[1,2\]. Our boundary data is given by \( \mu_1/\sqrt{\eta} \) which satisfies the compatibility condition

\[
\int_{v_\eta > 0} \mu_1(\phi, \bar{v}) |v_\eta| \, d\bar{v} = 0.
\]

6.1. Counterexample on Normal Derivative.

**Theorem 6.1.** For the Milne problem

\[
\begin{align*}
  v_\eta \frac{\partial g}{\partial \eta} + \mathcal{L}[g] &= 0, \\
  g(0,\bar{v}) &= h(\bar{v}) \quad \text{for} \quad v_\eta > 0, \\
  \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g(0,\bar{v}) \, d\bar{v} &= m_f, \\
  \lim_{\eta \to \infty} g(\eta,\bar{v}) &= g_\infty(\bar{v}).
\end{align*}
\]

for some constant \( m_f \) as mass flux, with

\[
h = v_\phi e^{-(v_\phi^2 - 1) - Mt_\eta^2},
\]

where \( v_\eta \) and \( v_\phi \) are defined as in \(2.68\) and we take \( M \) sufficiently large such that

\[
h(0,1) = 1, \\
|h|_{L^2} << 1,
\]

then we have

\[
\left\| \frac{\partial g}{\partial \eta} \right\|_{L^\infty} \notin L^\infty([0,\infty) \times \mathbb{R}^2).
\]

**Proof.** We divide the proof into several steps: We first assume \( \partial_\eta g \in L^\infty([0,\infty) \times \mathbb{R}^2) \) and then show it can lead to a contradiction.

Step 1: Definition of trace.

It is easy to see \( \partial_\eta g \) satisfies the Milne problem

\[
\begin{align*}
  v_\eta \frac{\partial (\partial_\eta g)}{\partial \eta} + \mathcal{L}[\partial_\eta g] &= 0.
\end{align*}
\]

Since \( k(\bar{u}, \bar{v}) = k_2(\bar{u}, \bar{v}) - k_1(\bar{u}, \bar{v}) \) is in \( L^1 \) with respect to \( \bar{u} \) uniformly in \( \bar{v} \), then we have \( K[\partial_\eta g] \in L^\infty([0,\infty) \times \mathbb{R}^2) \). For fixed \( N > 0, \nu(\bar{v}) \) is bounded in the domain \( S = \{|\bar{v}| \leq N\} \). Hence, we have \( \nu(\bar{v}) \partial_\eta g \in L^\infty([0,\infty) \times S) \), which further implies \( \mathcal{L}[\partial_\eta g] \in L^\infty([0,\infty) \times S) \). Therefore, by a standard cut-off argument and Ukai’s trace theorem, we deduce \( \partial_\eta g(0) \in L^\infty(S) \) is well-defined.

However, we can define the trace of \( \partial_\eta g \) in another fashion. For any \( \nu_\eta \neq 0 \), since we have \( \nu(\bar{v}) \in L^\infty([0,\infty) \times S) \) as well as \( K[g] \in L^\infty([0,\infty) \times S) \), by the Milne problem \(6.2\), it is naturally to define for \( \eta > 0 \)

\[
\partial_\eta g(\eta,\bar{v}) = \frac{K[g](\eta,\bar{v}) - \nu g(\eta,\bar{v})}{v_\eta}.
\]

Since \( \partial_\eta g \in L^\infty([0,\infty) \times S) \), we know \( g \) is continuous with respect to \( \eta \) for a.e. \( \bar{v} \). Taking \( \eta \to 0 \) defines the trace for \( \partial_\eta g \) at \( (0,\bar{v}) \)

\[
\partial_\eta g(0,\bar{v}) = \frac{K[g](0,\bar{v}) - \nu g(0,\bar{v})}{v_\eta}.
\]

Since the grazing set \( \{\bar{v}: v_\eta = 0\} \) is zero-measured on the boundary \( \eta = 0 \), then we have the trace of \( \partial_\eta g \) is a.e. well-defined.
By the uniqueness of trace of $\partial_\nu g$, above two types of traces must coincide with each other a.e.. Then we may combine them both and obtain $\partial_\nu g(0, \tilde{\nu}) \in L^\infty(S)$ is a.e. well-defined and satisfies the formula

$$
\partial_\nu g(0, \tilde{\nu}) = \frac{K[g](0, \tilde{\nu}) - \nu g(0, \tilde{\nu})}{v_\eta}.
$$

Step 2: Limiting Process.
Therefore, we may consider the limiting process

$$
\lim_{\tilde{\nu} \to (0,1)} \frac{\partial g}{\partial \eta}(0, \tilde{\nu}) = \frac{K[g](0, \tilde{\nu}) - \nu g(0, \tilde{\nu})}{v_\eta}.
$$

Based on [8, Lemma 3.3.1], we can take $M$ sufficiently large such that

$$
\|K[g](0)\|_{L^\infty_{\nu,0}} \leq \|K[g]\|_{L^\infty_{\nu,0}} \leq C\|g\|_{L^\infty_{\mu,\delta}}.
$$

By Lemma 4.10, we have

$$
\|g\|_{L^\infty_{\mu,\delta}} \leq C(\delta)\|g\|_{L^2_{\mu,\delta}} + \delta\|g\|_{L^\infty_{\mu,\delta}},
$$

for $\delta > 0$ sufficiently small and $\theta \geq 3$. With Lemma 4.4, we know

$$
\|g\|_{L^\infty_{\mu,\delta}} \leq C\|\mu\|_{L^2}.
$$

Combining this with Theorem 4.7 and Theorem 4.13, we know

$$
\|g\|_{L^\infty_{\mu,\delta}} \leq C\|h\|_{L^2}.
$$

Taking $\delta$ sufficiently small, and then taking $M$ sufficiently large, we have

$$
\|g\|_{L^\infty_{\mu,\delta}} \leq \|g\|_{L^\infty_{\mu,\delta}} < 1.
$$

On the other hand, we can see

$$
\nu(0,1)g(0,0,1) \geq Ch(0,1) \geq C_0 > 0,
$$

for some positive constant $C_0$. Therefore, we have shown

$$
\frac{\partial g}{\partial \eta} = -\frac{\mathcal{L}[g](0)}{v_\eta} \to \infty,
$$

which contradicts our assumption that $\partial_\nu g(0, \tilde{\nu}) \in L^\infty(S)$.

\[
\Box
\]

6.2. **Counterexample on Difference Equation.** Define $\mathcal{W} = \mathcal{F}_1 + \mathcal{F}_1$ and $\mathcal{W} = \mathcal{F}_1 + \mathcal{F}_1$. Consider the $\epsilon$-Milne problem

$$
\begin{cases}
\epsilon \frac{\partial \mathcal{W}^\epsilon}{\partial \eta} + G(\epsilon; \eta)\left(\nu_0^2 \frac{\partial \mathcal{W}^\epsilon}{\partial \nu_0} - \nu_\eta \nu_\phi \frac{\partial \mathcal{W}^\epsilon}{\partial \nu_\phi}\right) + \mathcal{L}[\mathcal{W}^\epsilon] = 0, \\
\mathcal{W}^\epsilon(0, \nu, \nu_\phi) = h(\nu_\eta, \nu_\phi) \text{ for } \nu_\eta > 0,
\end{cases}
$$

and Milne problem

$$
\begin{cases}
\frac{\partial \mathcal{W}}{\partial \eta} + \mathcal{L}[\mathcal{W}] = 0, \\
\mathcal{W}(0, \nu, \nu_\phi) = h(\nu_\eta, \nu_\phi) \text{ for } \nu_\eta > 0,
\end{cases}
$$

and

$$
\begin{cases}
\int_{\mathbb{R}^2} \nu_\eta \sqrt{\mu} \mathcal{W}(0, \nu, \nu_\phi) d\nu_\eta d\nu_\phi = 0, \\
\lim_{\eta \to \infty} \mathcal{W}(\eta, \nu, \nu_\phi) = \mathcal{W}_\infty(\nu, \nu_\phi),
\end{cases}
$$

and

$$
\begin{cases}
\int_{\mathbb{R}^2} \nu_\eta \sqrt{\mu} \mathcal{W}(0, \nu, \nu_\phi) d\nu_\eta d\nu_\phi = 0, \\
\lim_{\eta \to \infty} \mathcal{W}(\eta, \nu, \nu_\phi) = \mathcal{W}_\infty(\nu, \nu_\phi),
\end{cases}
$$
Note that $\mathcal{W}^\varepsilon$ actually satisfies an $\varepsilon$-Milne problem with non-trivial source term. However, based on the proof of Theorem 6.1, this source term will add a $O(\varepsilon)$ perturbation to $\mathcal{W}^\varepsilon$, so we can omit it and concentrate on above simpler form.

**Theorem 6.2.** For

\begin{equation}
 h = v_\beta e^{-(\nu_\beta^2 - 1) - Mv_\beta^2},
\end{equation}

where we take $M$ sufficiently large such that

\begin{equation}
 h(0, 1) = 1,
\end{equation}

\begin{equation}
 |h|_{L^2} << 1,
\end{equation}

we have

\begin{equation}
 \|\mathcal{W}^\varepsilon - \mathcal{W}\|_{L^\infty L^0, 0} \geq C > 0,
\end{equation}

for some $C > 0$.

**Proof.** We divide the proof into several steps:

Step 1: Continuity of $K[\mathcal{W}^\varepsilon]$ and $K[\mathcal{W}]$ at $\eta = 0$.

For any $R_0 > r_0 > 0$ and $\vec{u} = (u_\eta, u_\psi)$, we have

\begin{equation}
 \left| K[\mathcal{W}](0, \vec{v}) - K[\mathcal{W}](\eta, \vec{w}) \right|
 \leq \int_{u_\eta \leq r_0} |k(\vec{u}, \vec{w})| |\mathcal{W}(0, \vec{u}) - \mathcal{W}(\eta, \vec{u})| d\vec{u} + \int_{u_\eta \geq R_0} |k(\vec{u}, \vec{w})| |\mathcal{W}(0, \vec{u}) - \mathcal{W}(\eta, \vec{u})| d\vec{u}.
\end{equation}

Since we know $\mathcal{W} \in L^\infty([0, \infty) \times \mathbb{R}^2)$, then for any $\delta > 0$ we can take $r_0$ sufficiently small such that

\begin{equation}
 \int_{u_\eta \leq r_0} |k(\vec{u}, \vec{w})| |\mathcal{W}(0, \vec{u}) - \mathcal{W}(\eta, \vec{u})| d\vec{u} \leq C \int_{u_\eta \leq r_0} |k(\vec{u}, \vec{w})| d\vec{u} \leq \frac{\delta}{3}.
\end{equation}

Since we know

\begin{equation}
 \left\| (\vec{v}^\theta e^{\frac{1}{2}(\eta^2)} (\mathcal{W} - \mathcal{W}_\infty) \right\|_{L^\infty} \leq C < \infty,
\end{equation}

then there exists a $R_0 > 0$, such that for $u_\eta \geq R_0$,

\begin{equation}
 |\mathcal{W}(\eta, \vec{u})| \leq \delta,
\end{equation}

where $\delta$ is sufficiently small. Therefore, we have

\begin{equation}
 \int_{u_\eta \geq R_0} |k(\vec{u}, \vec{w})| |\mathcal{W}(0, \vec{u}) - \mathcal{W}(\eta, \vec{u})| d\vec{u} \leq 2 \delta \int_{u_\eta \geq R_0} |k(\vec{u}, \vec{w})| d\vec{u} \leq \frac{\delta}{3}.
\end{equation}

For fixed $r_0$ and $R_0$ satisfying above requirement, we estimate the integral on $r_0 \leq u_\eta \leq R_0$. By Ukai’s trace theorem, we have $\mathcal{W}(0, \vec{w})$ is well-defined and

\begin{equation}
 \partial_\eta \mathcal{W}(0, \vec{u}) = \frac{K[\mathcal{W}](0, \vec{u}) - \nu(\vec{u}) \mathcal{W}(0, \vec{u})}{u_\eta}.
\end{equation}

The in $r_0 \leq u_\eta \leq R_0$, $\partial_\eta \mathcal{W}$ is bounded, which implies $\mathcal{W}(\eta, \vec{w})$ is uniformly continuous at $\eta = 0$. Then there exists a $\eta_0$ such that for $0 \leq \eta \leq \eta_0$,

\begin{equation}
 \int_{r_0 \leq u_\eta \leq R_0} |k(\vec{u}, \vec{w})| |\mathcal{W}(0, \vec{u}) - \mathcal{W}(\eta, \vec{u})| d\vec{u} \leq C\delta \int_{r_0 \leq u_\eta \leq R_0} |k(\vec{u}, \vec{w})| d\vec{u} \leq \frac{\delta}{3}.
\end{equation}

In summary, we have shown for any $\delta > 0$, there exists a $\eta_0 > 0$ such that for any $0 \leq \eta \leq \eta_0$ and fixed $\vec{w}$,

\begin{equation}
 \left| K[\mathcal{W}](0, \vec{w}) - K[\mathcal{W}](\eta, \vec{w}) \right| \leq \delta.
\end{equation}

Therefore, $K[\mathcal{W}]$ is continuous at $\eta = 0$. A similar argument can be implemented to $\mathcal{W}^\varepsilon$. It is easy to see above estimate is uniform in $\vec{v}$ since $L^1$ estimate of $k(\vec{u}, \vec{w})$ in $\vec{u}$ is uniform with respect to $\vec{v}$. Also, it is obvious to see $K$ is continuous with respect to $\vec{v}$ at $\eta = 0$. 
Step 2: Milne formulation.
We consider the solution at a specific point \( \eta = n \epsilon \), \( v_\eta = \epsilon \) and \( v_\phi = \sqrt{1 - \epsilon^2} \) for some fixed \( n > 0 \). The solution along the characteristics can be rewritten as follows:

\[
W(\eta, \phi, \epsilon, \sqrt{1 - \epsilon^2}) = \int_0^{n \epsilon} e^{-\frac{\nu(1)}{\nu(1) + \epsilon}} d\kappa,
\]

where \( \nu(1) \) denote the value of \( \nu(\mathbf{v}) \) at \( |\mathbf{v}| = 1 \) and we have the conserved energy along the characteristics

\[
E(\eta, v_\eta, v_\phi) = v_\phi e^{-W(\eta)},
\]
in which \((0, \epsilon, \sqrt{1 - \epsilon^2})\) and \((\zeta, v_\eta(\zeta), \sqrt{1 - v_\eta^2(\zeta)})\) are in the same characteristics of \((n \epsilon, \epsilon, \sqrt{1 - \epsilon^2})\).

Step 3: Estimates of \((6.35)\).
We turn to the Milne problem for \( W \). We have the natural estimate

\[
\int_0^{n \epsilon} e^{-\frac{\nu(1)}{\nu(1) + \epsilon}} d\kappa = e^{-\nu(1)} \int_0^{n \epsilon} e^{-\frac{\nu(1)}{\nu(1) + \epsilon}} d\kappa = e^{-\nu(1)} \int_0^{n \epsilon} e^{\nu(1)} d\zeta = \frac{1}{\nu(1)} \left( 1 - e^{-\nu(1)} \right).
\]

Then for \( 0 < \epsilon \leq \eta \), we have \( |K[W](0, 0, 1) - K[W](\kappa, \epsilon, \sqrt{1 - \epsilon^2})| \leq \delta + \epsilon \), which implies

\[
\int_0^{n \epsilon} e^{-\frac{\nu(1)}{\nu(1) + \epsilon}} K[W](\kappa, \epsilon, \sqrt{1 - \epsilon^2}) d\kappa = \int_0^{n \epsilon} e^{-\frac{\nu(1)}{\nu(1) + \epsilon}} K[W](0, 0, 1) d\kappa + O(\delta) + O(\epsilon)
\]

For the boundary data term, it is easy to see

\[
h(\epsilon, \sqrt{1 - \epsilon^2}) e^{-\frac{\nu(1)}{\nu(1) + \epsilon}} = e^{-\nu(1)} h(\epsilon, \sqrt{1 - \epsilon^2}).
\]

In summary, we have

\[
W(n \epsilon, \epsilon, \sqrt{1 - \epsilon^2}) = \frac{1}{\nu(1)} (1 - e^{-\nu(1)}) K[W](0, 0, 1) + e^{-\nu(1)} h(0, 1) + O(\delta) + O(\epsilon).
\]

Step 4: Estimates of \((6.36)\).
We consider the \( \epsilon \)-Milne problem for \( W^\epsilon \). For \( \epsilon << 1 \) sufficiently small, \( \psi(\epsilon) = 1 \). Then we may estimate

\[
v_\phi(\zeta)e^{-W(\zeta)} = \sqrt{1 - \epsilon^2} e^{-W(n \epsilon)},
\]

which implies

\[
v_\phi(\zeta) = \frac{1 - n \epsilon^2}{1 - \epsilon \zeta} \sqrt{1 - \epsilon^2},
\]

and hence

\[
v_\eta(\zeta) = \sqrt{1 - v_\phi^2(\zeta)} = \sqrt{\frac{\epsilon (n \epsilon - \zeta)(2 - \epsilon \zeta - n \epsilon^2)}{(1 - \epsilon \zeta)^2 (1 - \epsilon^2) + \epsilon^2}}.
\]

For \( \zeta \in [0, \epsilon] \) and \( n \epsilon \) sufficiently small, by Taylor’s expansion, we have

\[
1 - \epsilon \zeta = 1 + o(\epsilon),
\]

\[
2 - \epsilon \zeta - n \epsilon^2 = 2 + o(\epsilon).
\]

Hence, we have

\[
v_\eta(\zeta) = \sqrt{\epsilon(1 + 2n \epsilon - 2 \zeta) + o(\epsilon^2)}.
\]
Since $\sqrt{\epsilon + 2n\epsilon - 2\zeta} = O(\epsilon)$, we can further estimate

$$
(6.48) \quad \frac{1}{v_\eta(\zeta)} = \frac{1}{\sqrt{\epsilon + 2n\epsilon - 2\zeta}} + o(1)
$$

and

$$
(6.49) \quad - \int_\kappa^{n\kappa} \frac{\nu(1)}{v_\eta(\zeta)} d\zeta = \nu(1) \sqrt{\epsilon} + o(\epsilon) = \nu(1) \left(1 - \frac{\epsilon + 2n\epsilon - 2\kappa}{\epsilon}\right) + o(\epsilon).
$$

Then we can easily derive the integral estimate

$$
(6.50) \quad \int_0^{n\kappa} e^{-\int_\kappa^{n\kappa} \frac{\nu(1)}{v_\eta(\zeta)} d\zeta} \frac{1}{v_\eta(\kappa)} d\kappa = e^{\nu(1)} \int_0^{n\kappa} e^{-\nu(1)\sqrt{\epsilon + 2n\epsilon - 2\kappa} - \frac{1}{\sqrt{\epsilon + 2n\epsilon - 2\kappa}}} d\kappa + o(\epsilon)
$$

Then for $0 < \epsilon \leq \eta_0$, we have $|K[\mathcal{W}^\epsilon](0, 0, 1) - K[\mathcal{W}^\epsilon](\kappa, v_\eta(\kappa), v_\phi(\epsilon))| \leq \delta$, which implies

$$
(6.51) \quad \int_0^{n\kappa} e^{-\int_\kappa^{n\kappa} \frac{\nu(1)}{v_\eta(\zeta)} d\zeta} \frac{1}{v_\eta(\kappa)} K[\mathcal{W}^\epsilon](\kappa, v_\eta(\kappa), v_\phi(\epsilon)) d\kappa = \int_0^{n\kappa} e^{-\int_\kappa^{n\kappa} \frac{\nu(1)}{v_\eta(\zeta)} d\zeta} \frac{1}{v_\eta(\kappa)} K[\mathcal{W}^\epsilon](0, 0, 1) d\kappa + O(\delta) + O(\epsilon)
$$

For the boundary data term, since $h(v_\eta, v_\phi)$ is $C^1$, a similar argument shows

$$
(6.52) \quad h(\epsilon_0, \sqrt{1 - \epsilon_0^2}) e^{-\int_0^{\epsilon_0} \frac{\nu(1)}{v_\eta(\zeta)} d\zeta} = e^{\nu(1)\sqrt{1 - \epsilon_0^2}} h(\sqrt{1 + 2n\epsilon}, \sqrt{1 - (1 + 2n)\epsilon^2}) + O(\epsilon).
$$

Therefore, we have

$$
(6.53) \quad \mathcal{W}^\epsilon(n\epsilon, \epsilon) = \frac{1}{\nu(1)} \left(1 - e^{\nu(1)\sqrt{1 + 2n\epsilon}}\right) K[\mathcal{W}^\epsilon](0, 0, 1) + e^{\nu(1)\sqrt{1 + 2n\epsilon}} h(0, 1) + O(\epsilon) + O(\delta).
$$

Step 5: Estimate of Difference.

Collecting all above, we can estimate the value at point $\eta = n\epsilon, v_\eta = \epsilon$ and $v_\phi = \sqrt{1 - \epsilon^2}$ as

$$
(6.54) \quad \mathcal{W}^\epsilon(n\epsilon, \epsilon, \sqrt{1 - \epsilon^2}) = \frac{1}{\nu(1)} \left(1 - e^{\nu(1)\sqrt{1 + 2n\epsilon}}\right) K[\mathcal{W}^\epsilon](0, 0, 1) + e^{\nu(1)\sqrt{1 + 2n\epsilon}} h(0, 1) + O(\epsilon) + O(\delta),
$$

and

$$
(6.55) \quad \mathcal{W}^\epsilon(n\epsilon, \epsilon, \sqrt{1 - \epsilon^2}) = \frac{1}{\nu(1)} \left(1 - e^{-\nu(1)n}\right) K[\mathcal{W}^\epsilon](0, 0, 1) + e^{-\nu(1)n} h(0, 1) + O(\epsilon) + O(\delta).
$$

By our assumptions on $h$ and a similar argument as in the proof of Theorem 6.1, we know $|K[\mathcal{W}^\epsilon](0, 0, 1)| < < 1$ and $|K[\mathcal{W}^\epsilon](0, 0, 1)| < < 1$. However, $h(0, 1) = 1$. Since $n$ is arbitrary and $e^{\nu(1)\sqrt{1 + 2n\epsilon}} \neq e^{-\nu(1)n}$, we have

$$
(6.56) \quad \left| \mathcal{W}^\epsilon(n\epsilon, \epsilon, \sqrt{1 - \epsilon^2} - \mathcal{W}^\epsilon(\epsilon, n\epsilon, \sqrt{1 - \epsilon^2}) \right| \geq C > 0,
$$

which further implies

$$
(6.57) \quad \left\| \mathcal{W}^\epsilon - \mathcal{W}^\epsilon \right\|_{L^\infty L^2_{0,0}} \geq C > 0.
$$

□
REFERENCES

[1] Sone, Yoshio; Kinetic theory and fluid dynamics. Modeling and Simulation in Science, Engineering and Technology. Birkhauser Boston, Inc., Boston, MA, 2002.
[2] Sone, Yoshio; Molecular gas dynamics. Theory, techniques, and applications. Modeling and Simulation in Science, Engineering and Technology. Birkhauser Boston, Inc., Boston, MA, 2007.
[3] Esposito, R.; Guo, Y.; Kim, C.; Marra, R.; Non-isothermal boundary in the Boltzmann theory and Fourier law. Comm. Math. Phys. 323 (2013), no. 1, 177-239.
[4] Arkeryd, Leif; Esposito, Raffaele; Marra, Rossana; Nouri, Anne; Ghost effect by curvature in planar Couette flow. Kinet. Relat. Models 4 (2011), no. 1, 109-138.
[5] Cercignani, Carlo; Marra, R.; Esposito, R.; The Milne problem with a force term. Transport Theory Statist. Phys. 27 (1998), no. 1, 1-33.
[6] Yang, Xiongfeng; Asymptotic behavior on the Milne problem with a force term. J. Differential Equations 252 (2012), no. 9, 4656-4678.
[7] Guo, Yan; Decay and continuity of the Boltzmann equation in bounded domains. Arch. Ration. Mech. Anal. 197 (2010), no. 3, 713-809.
[8] Glassey, Robert T.; The Cauchy problem in kinetic theory. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.

Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA

E-mail address, L. Wu: Lei.Wu@brown.edu