1. Introduction and notations

Let $X$ be a (connected and reduced) complex space. We recall that $X$ is said to be strongly $q$-pseudoconvex in the sense of Andreotti-Grauert [1] if there exists a compact subset $K$ and a smooth function $\varphi : X \to \mathbb{R}$, $\varphi \geq 0$, which is strongly $q$-plurisubharmonic on $X \setminus K$ and such that

a) for every $b > \max_K \varphi$ the subset

$$B_b = \{ x \in X : \varphi(x) < b \}$$

is relatively compact in $X$.

If $K = \emptyset$, $X$ is said to be $q$-complete. We remark that, for a space, being 1-complete is equivalent to being Stein. Replacing the condition a) by

a') for every $\min_X \varphi < a < \min_K \varphi$ and every $b > \max_K \varphi$ the subset

$$C_{a,b} = \{ x \in X : a < \varphi(x) < b \}$$

is relatively compact in $X$.
we obtain the notion of \( q \)-corona (see [1], [2]).

A \( q \)-corona is said to be complete whenever \( K = \emptyset \).

The cohomology of coherent sheaves defined on \( q \)-coronae was studied in [2].

In this paper we will focus on semi \( q \)-coronae, which are defined as follows.

Consider a strongly \( q \)-pseudoconvex space (or, more generally, a \( q \)-corona) \( X \), and a smooth function \( \varphi : X \to \mathbb{R} \) displaying the \( q \)-pseudoconvexity of \( X \). Let \( C_{a,b} \) be a \( q \)-corona of \( X \) and let \( h : X \to \mathbb{R} \) be a pluriharmonic function (i.e. locally the real part of a holomorphic function) such that: \( K \cap \{ h = 0 \} = \emptyset \), \( \{ h = 0 \} \) and the boundaries of \( B_b \) and \( B_a \) are \( \mathbb{C} \)-transversal. A connected component \( C_{a,b}^+ \) of \( C_{a,b} \setminus \{ h = 0 \} \) is, by definition, a semi \( q \)-corona.

At the origin of the interest for domains whose boundary contains a Levi flat part there is an extension theorem for CR-functions proved in [11] (see also [8], [10], [13]).

A more general type of semi \( q \)-corona is obtained by replacing the zero set of \( h \) with a Levi flat hypersurface \( H \).

In both cases the semi \( q \)-coronae are differences \( B_b^+ \setminus \overline{B}_a^+ \) where \( B_b^+, B_a^+ \) are strongly \( q \)-pseudoconvex spaces. Indeed, the function \( \psi = -\log h^2 \) (respectively \( \psi = -\log \delta_H(z) \), where \( \delta_H(z) \) is the distance of \( z \) from \( H \)) is plurisubharmonic in \( W \setminus \{ h = 0 \} \) (respectively in \( W \setminus H \)) where \( W \) is a neighbourhood of \( B_b \cap \{ h = 0 \} \) (respectively of \( B_b \cap H \)). Let \( \chi : \mathbb{R} \to \mathbb{R} \) be an increasing convex function such that \( \chi \circ \varphi > \psi \) on a neighbourhood of \( B_b \setminus W \). The function \( \Phi = \sup (\chi \circ \varphi, \psi) + \varphi \) is an exhaustion function for \( B_b \setminus \{ h = 0 \} \) (respectively for \( B_b \setminus H \)) and it is strongly \( q \)-plurisubharmonic in \( B_b \setminus (\{ h = 0 \} \cup K) \) (respectively in \( B_b \setminus H \cup K \)).

Results on the cohomology of coherent sheaves on semi \( q \)-coronae were obtained by the authors in [12] under the hypothesis that they are defined on the larger set \( B_b \).

The aim of this paper is to give a generalization for coherent sheaves \( \mathcal{F} \) defined only on the semi \( q \)-corona. For the sake of simplicity we restrict ourselves to the case of smooth semi 1-coronae.

Following Andreotti-Grauert (see [1]), given a semi 1-corona

\[
C_{a,b}^+ = C_{a,b} \cap \{ h > 0 \},
\]

where \( h \) is pluriharmonic, and a coherent sheaf \( \mathcal{F} \) on \( C_{a,b}^+ \) we consider the strongly plurisubharmonic functions \( P_\varepsilon(z) = \varepsilon|z|^2 - h, \varepsilon > 0 \), and an exhaustion of \( C_{a,b}^+ \) by the following relatively compact domains

\[
C_\varepsilon^+ = \{ z \in \mathbb{C}^n : P_\varepsilon(z) < 0 \} \bigcap \overline{C_{a+b-\varepsilon}}.
\]

The idea is to prove for the domains \( C_\varepsilon^+ \) a bump lemma and an approximation theorem as in the classical case of coronae. Here the situation is more complicated because of the presence of a non-empty
pseudoconvex-pseudoconcave part in the boundary of each $C_a^+$. In order to circumvent this difficulty, we work with the closed sets $\overline{C}_{\varepsilon}^+$ using in a crucial way a regularity result on the $\overline{\partial}$-equation due to Laurent-Thiébaut and Leiterer (see Section 3). This enables us to prove the following results: assume that $\text{depth} \mathcal{F}_z \geq 3$ for $z$ near to the pseudo-concave part of the boundary of $C_{a,b}^+$; then

1) if $\varepsilon$ is sufficiently small and $\varepsilon' < \varepsilon$ is near $\varepsilon$

$$H^1(\overline{C}_{\varepsilon'}, \mathcal{F}) \simeq H^1(\overline{C}_{\varepsilon}, \mathcal{F})$$

2) the cohomology spaces $H^1(\overline{C}_{\varepsilon}^+, \mathcal{F})$ are finite dimensional.

(see Lemma 3.3, Lemma 3.9 and Proposition 3.10).

Thus the function

$$d(\varepsilon) = \dim_C H^1(\overline{C}_{\varepsilon}^+, \mathcal{F})$$

is piecewise constant, but, in general, it could have frequently a “jump-discontinuity” and it could happen that $d(\varepsilon) \to +\infty$ (see Remark 3.2). Nevertheless, the isomorphism 1) allows us to prove in the last section:

1) the fact that Oka-Cartan-Serre Theorem A holds in semi 1-coronae for sheaves which satisfy the condition of Theorem 3.1 (see Theorem 4.3);

2) an extension theorem for analytic subsets (see Corollary 4.4).

It is worth noticing that an extension theorem for codimension one analytic subsets of a non-singular semi 1-corona was proved in [12] and for higher codimensions, using different methods based on Harvey-Lawson’s theorem [7], by Della Sala and the first author in [4].

2. REMARKS ON THE PROOFS OF THEOREMS IN [12]

Let $X$ be a complex space. For every coherent sheaf $\mathcal{F}$ on $X$ and every subset $A$ of $X$ we set

$$p(A; \mathcal{F}) = \inf_{x \in A} \text{depth}(\mathcal{F}_x)$$

$$p(A) = p(A; \mathcal{O}).$$

Let $C = C_{a,b}$ be a $q$-corona of $X$. All the results in [12] on finite and/or vanishing cohomology results for $q$-coronae and semi $q$-coronae are obtained using Andreotti-Grauert methods. They consist of two main points

i) the bump lemma;

ii) for every corona $C_{a', b'} \subseteq C$ there exist a corona $C_{a'+\varepsilon, b'+\varepsilon} \subseteq C$, $\varepsilon > 0$ such that the homomorphism

$$H^r(C_{a'-\varepsilon, b'+\varepsilon}, \mathcal{F}) \to H^r(C_{a', b'}, \mathcal{F})$$

is bijective for $q \leq r \leq p(C; \mathcal{F})$.  

As a matter of fact the method of proof shows that the condition on the depth is needed only in $C_{a,a'}$, i.e. the homomorphism
\[ H^r(C_{a'-\varepsilon}, B_{-\varepsilon}^+, F) \rightarrow H^r(C', F) \]
is bijective for $q \leq r \leq p(C_{a,a'}; F)$.

Let $X$ be a strongly $q$-pseudoconvex space (respectively $X \subset \mathbb{C}^n$ be a strongly $q$-pseudoconvex open set) and $H = \{h = 0\}$ where $h$ is pluriharmonic in $X$ (respectively $H$ Levi-flat), and $C = C_{a,b} = B_b \setminus \overline{B_a}$ a $q$-corona. We can suppose that $B_b \setminus H$ has two connected components, $B^+$ and $B^-$, and we define $C^+ = B^+ \cap C$, $C^- = B^- \cap C$.

From the above remark we derive the following improvements of Theorem 1, Corollary 2 and Theorem 3 in [12].

**Theorem 2.1.** Let $F \in \text{Coh}(B_b)$. Then the image of the homomorphism
\[ H^r(B^+_b, F) \oplus H^r(C, F) \rightarrow H^r(C^+, F) \]
(all closures are taken in $B_b$), defined by $(\xi \oplus \eta) \mapsto \xi_{|C^+} - \eta_{|C^+}$ has finite codimension provided that $q - 1 \leq r \leq p(B_{a}; F) - q - 2$.

**Corollary 2.2.** If $K \cap H = \emptyset$, under the same assumption of Theorem 2.1
\[ \dim C^+ H^r(C^+, F) < \infty \]
for $q \leq r \leq p(B_{a}; F) - q - 2$.

**Theorem 2.3.** If $B^+_{a}$ is a $q$-complete space, then
\[ H^r(C, F) \sim H^r(C^+, F) \]
for $q \leq r \leq p(B_{a}; F) - q - 2$, and the homomorphism
\[ H^{q-1}(B^+_b, F) \oplus H^{q-1}(C, F) \rightarrow H^{q-1}(C^+, F) \]
is surjective for $p(B_{a}; F) \geq 2q + 1$.

If $B^+_b$ is a 1-complete space and $p(B_{a}; F) \geq 3$, then
\[ H^0(B^+_b, F) \sim H^0(C^+, F). \]

This implies the following. Let $C_1 = B_{b_1} \setminus B_{a_1} \subset C_2 = B_{c_2} \setminus B_{a_2}$. Then
\[ H^r(C^+_1, M_{\{x\}}, F) \sim H^r(C^+_2, F), \]
for $q \leq r \leq p(B_{a_1}; F)$.

In particular, if $x \in C_2 \setminus B_{a_1}$ and $M_{\{x\}}$ denotes the sheaf of ideals of $\{x\}$, then
\[ H^r(C^+_2, M_{\{x\}}, F) \sim H^r(C^+_1, F), \]
for $q \leq r \leq p(B_{a_1}; F)$.
3. An isomorphism theorem for semi 1-coronae

Our aim is to give a generalization of the above results for sheaves defined only on the semi $q$-coronae, i.e. for the case when the “hole” is real. For the sake of simplicity will consider only complete 1-coronae in $\mathbb{C}^n$ with $n \geq 3$. So we consider connected 1-coronae of the form

$$C = \{ z \in \mathbb{C}^n : 0 < \varphi(z) < 1 \} \subset \mathbb{C}^n,$$

where $\varphi : \mathbb{C}^n \to \mathbb{R}$ is a smooth strongly plurisubharmonic function in a Stein neighborhood $U$ of $\{ 0 \leq \varphi \leq 1 \}$, $d\varphi \neq 0$ on $\varphi = 0, 1$. Let $h$ be a pluriharmonic function on $U$ and $H$ the zero set of $h$. We assume that $H$ is smooth and transversal to the hypersurfaces $\{ \varphi = 0 \}$, $\{ \varphi = 1 \}$, that $U \setminus H$ has two connected components $U^\pm$ and $h > 0$ on $U^+$. For $0 < a < b < 1$ we set

$$B_b = \{ z \in U : \varphi < b \}, \quad B_b^+ = B_b \cap U^+,$$

$$C_{a,b} = (B_b \setminus \overline{B_a}), \quad C_{a,b}^+ = C_{a,b} \cap U^+.$$

Let $P_\varepsilon(z) = \varepsilon |z|^2 - h$; then there is $\varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the hypersurfaces $\{ \varphi = \varepsilon \}$, $\{ \varphi = 1 - \varepsilon \}$ meet $\{ P_\varepsilon = 0 \}$ transversally. Finally we define the following subsets (which are locally 1-convex, 1-concave, see [9] and Remark 3.1 below)

$$C_{\varepsilon}^+ = \{ z \in \mathbb{C}^n : P_\varepsilon(z) \leq 0 \} \cap C_{\varepsilon,1-\varepsilon}.$$

We want to prove the following

**Theorem 3.1.** Let $C^+$ be a semi 1-corona in $\mathbb{C}^n$. Then for every $\varepsilon \in (0, \varepsilon_0)$ there exists $\varepsilon' \in [0, \varepsilon)$ such that for every $\mathcal{F} \in \text{Coh}(C^+)$ satisfying

$$\{ z \in C^+ : \text{depth}(\mathcal{F}_z) < 3 \} \cap B_{\varepsilon_0} = \emptyset,$$

and every $\varepsilon' \in (\varepsilon, \varepsilon)$ the homomorphism

$$H^1(C_{\varepsilon'}, \mathcal{F}) \longrightarrow H^1(C_{\varepsilon}^+, \mathcal{F})$$

is an isomorphism.

The main ingredients for the proof are the bump lemma and a density theorem as in Andreotti-Grauert [1]. In order to treat points belonging to the pseudoconvex-pseudoconcave part of the boundary we work with closed bumps using the following result due to Laurent-Thiébaut and Leiterer (see [9, Proposition 7.5]):

**Proposition 3.2.** Let $D \subset \mathbb{C}^n$ be a 1-concave, 1-convex domain of order 1 of special type, and suppose that $n \geq 3$. If $f$ is a continuous $(n, r)$-form in some neighborhood $U_{\overline{D}}$ of $\overline{D}$, $1 \leq r \leq n-2$, such that $\overline{\partial f} = 0$ in $U_{\overline{D}}$, then there exists a form $u \in \bigcap_{\varepsilon > 0} C_{n,r-1}^{1/2-\varepsilon}(\overline{D})$ such that $\overline{\partial u} = f$ in $D$. 
Remark 3.1. Proposition 7.5 in [9] is much more general, but we state it this way, since the semi 1-coronae we consider are locally 1-concave, 1-convex domain of order 1 of special type, i.e. they are locally biholomorphic to the set-difference of two convex domains.

The proof of Theorem 3.1 is a consequence of several intermediate results.

3.1. Bump lemma: surjectivity of cohomology. With the same notations as above let \( D = C_\varepsilon^+ \), \( 0 < \varepsilon < \varepsilon_0 \) where \( \varepsilon_0 < b \) is so chosen that for all \( \varepsilon \in (0, \varepsilon_0) \) the hypersurfaces \( \{ \varphi = \varepsilon \}, \{ \varphi = 1 - \varepsilon \} \) are \( C \)-transversal to \( \{ P_\varepsilon = 0 \} \). Let \( \Gamma_1, \Gamma_2 \) be respectively the pseudoconvex and the pseudoconcave part of the boundary \( bD \) of \( D \). Thus \( bD = \Gamma_1 \cup \Gamma_2 \) and \( \Gamma_2 \) is contained in the smooth hypersurface \( \{ \varphi = \varepsilon \} \).

Lemma 3.3 (bump lemma). There exists a finite open covering \( \mathcal{U} \) of \( bD \), \( \mathcal{U} = \{ U_j \}_{1 \leq j \leq m} \), and compact subsets \( D_1, \ldots, D_m \) of \( C^+ \) such that
\[
(1) \quad \overline{D} = \overline{D}_0 \subset \overline{D}_1 \subset \cdots \subset \overline{D}_m;
\]
\[
(2) \quad D \subset D_m;
\]
\[
(3) \quad \overline{D}_j \setminus \overline{D}_{j-1} \subset \overline{U}_j \text{ for } 1 \leq j \leq m;
\]
\[
(4) \quad \text{if } F \in \text{Coh}(C^+) \text{ then } H^r(\overline{U}_j \cap \overline{D}_k, F) = 0 \text{ for every } j, k \text{ and } 1 \leq r \leq p(D; \overline{D}, F) - 2.
\]
Moreover, the family of the coverings \( \mathcal{U} \) as above is cofinal in the family of all finite coverings of \( bD \).

Proof. If \( z^0 \in \Gamma_1 \cup \Gamma_2 \) i.e. \( z^0 \) is a point of pseudoconvexity or pseudoconcavity we argue as in the proof of the classical Andreotti-Grauert bump lemma.

Assume that \( z^0 \in \Gamma_1 \cap \Gamma_2 \). There exists a a sufficiently small closed ball \( B \) of positive radius, centered at \( z^0 \) and a biholomorphism on \( \Phi : B \to \Phi(B) \) which transform \( B \cap \{ \varphi \geq \varepsilon \} \) and \( B \cap \{ P_\varepsilon \geq 0 \} \) respectively in a strictly concave and strictly convex set. We may also assume that \( F \mid \overline{B} \) has a homological resolution
\[
0 \to O^{\oplus n} \to \cdots \to O^{\oplus 1} \to F \to 0 \tag{2}
\]
with \( n-k \geq 3 \). Choose a smooth function \( g \in C_0^\infty(B) \) such that \( g \geq 0 \) and \( g(z^0) \neq 0 \) and a positive number \( \lambda \) such that the closed domains
\[
\overline{B}_1 = \{ \varphi - \varepsilon - \lambda \rho \leq 0 \} \cap \overline{B}, \overline{B}_2 = \{ P_\varepsilon + \lambda \rho \leq 0 \} \cap \overline{B}
\]
are respectively strictly concave and strictly convex and contain \( z^0 \) as an interior point. Set \( \overline{B}_3 = \overline{B}_1 \cap \overline{B}_2 \) and \( \overline{D}_1 = \overline{C}_\lambda^+ \cup \overline{B}_3; z^0 \) is an interior point of \( \overline{D}_1 \) and \( b\overline{B}_1 \setminus b\overline{B}_2 \in B \). By construction \( \overline{D} \setminus \overline{D}_1 \cap \overline{B} = \overline{D} \cap \overline{B} \).
and $D \cap B$ is an intersection of two strictly convex domains with smooth boundaries thus applying Proposition 3.2 we obtain
\[ H^r(\overline{D} \cap \overline{B}, \mathcal{O}) = \{0\} \]
for $1 \leq r \leq n - 2$ and consequently, in view of (2), the vanishing
(3) \[ H^r(\overline{D} \cap \overline{B}, \mathcal{F}) = \{0\}. \]
Iterating this procedure we get the conclusion. □

**Proposition 3.4.** For every $\varepsilon \in (0, \varepsilon_0)$ there exists $\varepsilon' < \varepsilon$ such that the homomorphism
\[ H^r(\overline{C}_\varepsilon^+, \mathcal{F}) \longrightarrow H^r(\overline{C}_\varepsilon^+, \mathcal{F}) \]
is onto for $1 \leq r \leq p(\overline{C}_\varepsilon^+; \mathcal{F}) - 2$.

**Proof.** Keeping the notations of Lemma 3.3 we apply the Mayer-Vietoris exact sequence for closed sets to $D_1 = \overline{D} \cup (\overline{D} \cap \overline{B})$. We get
\[ \cdots \longrightarrow H^r(\overline{D}_1, \mathcal{F}) \longrightarrow H^r(\overline{D}, \mathcal{F}) \oplus H^r(\overline{D} \cap \overline{B}, \mathcal{F}) \longrightarrow H^r(\overline{D} \cap \overline{D}_1 \cap \overline{B}, \mathcal{F}) \longrightarrow \cdots \]
thus in view of (3) the homomorphism
\[ H^r(\overline{D}_1, \mathcal{F}) \longrightarrow H^r(\overline{D}, \mathcal{F}) \]
is onto for $1 \leq r \leq n - 2$. By induction, we obtain that the homomorphism
\[ H^r(\overline{D}_m, \mathcal{F}) \longrightarrow H^r(\overline{D}, \mathcal{F}) \]
is onto for $1 \leq r \leq p(\overline{C}_\varepsilon^+; \mathcal{F}) - 2$. Since $\overline{C}_\varepsilon^+ \subset D_m$ if $\varepsilon' < \varepsilon$ is near $\varepsilon$ one has $\overline{C}_\varepsilon^+ \subset C_{\varepsilon'}^+ \Subset D_m$, whence the homomorphism
\[ H^r(\overline{C}_{\varepsilon'}^+, \mathcal{F}) \longrightarrow H^r(\overline{C}_\varepsilon^+, \mathcal{F}) \]
is onto for $1 \leq r \leq p(\overline{C}_{\varepsilon'}^+; \mathcal{F}) - 2$. In particular, the canonical homomorphism
(4) \[ H^r(C_{\varepsilon'}^+, \mathcal{F}) \overset{\delta}{\longrightarrow} H^r(\overline{C}_\varepsilon^+, \mathcal{F}) \]
is onto for $1 \leq r \leq p(\overline{C}_{\varepsilon'}^+; \mathcal{F}) - 2$. □

From Proposition 3.4 we derive

**Proposition 3.5.** For every $\varepsilon \in (0, \varepsilon_0)$ there exists an $\theta < \varepsilon$ such that for every $\varepsilon' \in [\theta, \varepsilon)$ the homomorphism
(5) \[ H^r(C_{\varepsilon'}^+, \mathcal{F}) \overset{\delta}{\longrightarrow} H^r(\overline{C}_\varepsilon^+, \mathcal{F}) \]
is onto for $1 \leq r \leq p(\overline{C}_{\varepsilon'}^+; \mathcal{F}) - 2$. 


Proof. We fix $\varepsilon_0$ as in Lemma 3.3. Let $\Lambda$ be the (non-empty) set of the positive numbers $\varepsilon' < \varepsilon$ such that the homomorphism (4) is onto and $\overline{\varpi} = \inf \Lambda$. It follows (cfr. [1, Lemma pag. 241] for closed subsets) that the homomorphism (5) is onto. $\square$

A second consequence of Proposition 3.4 is the following finiteness theorem.

**Theorem 3.6.** Under the conditions of Theorem 3.1, there exists $\varepsilon_1 \leq \varepsilon_0$ such that

$$\dim_{\mathbb{C}} H^1(\overline{C}_{\varepsilon}^+, \mathcal{F}) < +\infty.$$ 

for every $\varepsilon \in (0, \varepsilon_1)$.

Proof. We first observe the following. Let $\Omega \subset \mathbb{C}^n$ be a domain, $K \subset \Omega$ a compact subset. It is known that $\mathcal{F}(\Omega)$ is a Fréchet space. The space $\mathcal{F}(K)$ is an $\mathcal{LF}$-space i.e. a direct limit of Fréchet spaces and its topology is complete (cfr. [6, pag. 315]). Moreover, the restriction

$$\mathcal{F}(\Omega) \xrightarrow{\delta} \mathcal{F}(K)$$

is a compact map i.e. there exists a neighbourhood $U$ of the origin in $\mathcal{F}(\Omega)$ such that $\delta(U)$ is a compact subset of $\mathcal{F}(K)$. This is a consequence of the following well known fact: if $\Omega'$ is a relatively compact subdomain of $\Omega$ then the restriction $\mathcal{F}(\Omega) \rightarrow \mathcal{F}(\Omega')$ is a compact map. Take $\varepsilon_0$ as in Lemma 3.3. The proof is similar to that of Théorème 11 in [1] taking into account the following facts:

1) Leray theorem for acyclic closed coverings (see Théorème 5.2.4 and Corollaire in [5]).

2) the theorem of L. Schwartz on compact perturbations $u + v$ of a surjective linear operator $u : E \rightarrow F$ where $E$ is a Fréchet (see [6, Corollaire 1]).

$\square$

We remark that, up to some modifications in the technical details of the proof, the finiteness result holds for all cohomology groups:

**Theorem 3.7.** Under the conditions of Theorem 3.1 there exists $\varepsilon_1 \leq \varepsilon_0$ such that

$$\dim_{\mathbb{C}} H^r(\overline{C}_{\varepsilon}^+, \mathcal{F}) < +\infty,$$

for every $\varepsilon \in (0, \varepsilon_1)$ and $1 \leq r \leq p(\overline{C}_{\varepsilon}^+; \mathcal{F}) - 2$.

3.2. Approximation. This subsection is devoted to approximation by global sections.
Lemma 3.8. Let \( \text{depth}(F_z) \geq 4 \) for every \( z \in \{ \varphi = \varepsilon \} \), \( \varepsilon \in (0, \varepsilon_0) \). Then, for every \( z^0 \in bC_{\varepsilon}^z \) there exists a closed neighbourhood \( \overline{U} \) of \( z^0 \) such that the homomorphism
\[
H^0(\overline{U}, F) \to H^0(\overline{U} \cap C_{\varepsilon}^+, F)
\]
is dense image.

Proof. This is known if \( z^0 \in \Gamma_1 \cup \Gamma_2 \) i.e. when \( z^0 \) is a point of pseudoconvexity or pseudoconcavity (see [1]), thus we may assume that \( z^0 \in \Gamma_1 \cap \Gamma_2 \). First we consider the case \( F = \mathcal{O} \). We may suppose that there exists a sufficiently small closed ball \( B \) of positive radius, centered at \( z^0 \) such that \( B \cap \{ \varphi \geq \varepsilon \} \) and \( B \cap \{ P_{\varepsilon} \leq 0 \} \) respectively are strictly concave and strictly convex (again, locally, up to a biholomorphism).

Take a real hyperplane with equation \( l = 0 \) such that \( z^0 \in \{ l > 0 \} \) and \( \{ l = 0 \} \cap \{ \varphi \leq \varepsilon \} \) respectively are strictly concave and strictly convex (again, locally, up to a biholomorphism).

Let \( \psi = \alpha \varphi - \varepsilon + \beta l \), \( \alpha, \beta \) positive real numbers; \( \psi \) is strongly plurisubharmonic. For \( \alpha, \beta \) sufficiently small the hypersurface \( \{ \psi = 0 \} \cap \{ l < 0 \} \) is a portion of a compact smooth hypersurface which bounds a domain \( D \subset B \). Set
\[
\overline{V} = \{ P_{\varepsilon} \leq 0 \} \cap \overline{D}, \overline{W} = \overline{D} \setminus \{ \varphi < \varepsilon \}
\]
and \( \overline{U} = \overline{V} \cap \overline{W} \). We are going to prove that \( H^1(\overline{V} \cup \overline{W}, \mathcal{O}) = 0 \).

Let \( R = \overline{D} \setminus (\overline{V} \cup \overline{W}) \). Since \( \overline{D} \) is a Stein compact, from the exact sequence of cohomology relative to the closed subspace \( \overline{V} \cup \overline{W} \) we get the isomorphism
\[
H^r(\overline{V} \cup \overline{W}, \mathcal{O}) \simeq H^{r+1}_c(R, \mathcal{O})
\]
for \( r \leq n-2 \). \( R \) is an open subset of \( S = \overline{D} \cap \{ \varphi < \varepsilon \} \). Set \( R' = S \setminus R \).

Again, by the cohomology with compact supports relative to the closed subspace \( R' = S \setminus R \) we get the exact sequence of groups
\[
\cdots \to H^r_c(S, \mathcal{O}) \to H^r_c(R', \mathcal{O}) \to H^{r+1}_c(R, \mathcal{O}) \to H^{r+1}_c(S, \mathcal{O}) \to \cdots.
\]
Since \( S \) and \( R' \) have a fundamental system of Stein neighbourhoods (see [14]) and \( n \geq 3 \), we have
\[
H^r_c(S, \mathcal{O}) = H^r_c(R', \mathcal{O}) = 0
\]
for \( 1 \leq r \leq n-2 \) and consequently \( H^r_c(R, \mathcal{O}) = 0 \) for \( 1 \leq r \leq n-2 \). In view of the isomorphism (6) we obtain
\[
H^r(\overline{V} \cup \overline{W}, \mathcal{O}) = 0
\]
for \( 1 \leq r \leq n-2 \). In particular, since \( n \geq 3 \), (6) implies that
\[
H^1(\overline{V} \cup \overline{W}, \mathcal{O}) = 0,
\]
thus that every function \( f \in \mathcal{O}(\overline{U}) \) is a difference of two functions \( f_1 - f_2 \) where \( f_1 \in \mathcal{O}(\overline{V}) \), \( f_2 \in \mathcal{O}(\overline{W}) \). Since \( \overline{V} \) is Runge in \( \overline{D} \) there exists a sequence of holomorphic functions \( f_{\nu} \in \mathcal{O}(\overline{D}) \) such that \( f_{\nu} \to f_1 \) in \( \mathcal{O}(\overline{V}) \). Moreover, by the extension theorem in [11] the function
\( f_2 \) extends holomorphically to \( W \cap \{ l \leq 0 \} \). Choose a smooth function \( \varrho \in C_0^\infty(D) \) such that \( \varrho \geq 0 \) and \( \varrho(z^0) \neq 0 \) and a positive number \( \lambda \) such that the closed domains
\[
\overline{D}_1 = \{ \varphi - \varepsilon + \lambda \varrho \leq 0 \} \cap \overline{D}, \quad \overline{D}_2 = \{ P_x - \lambda \varrho \leq 0 \} \cap \overline{D}
\]
are respectively strongly pseudoconcave and strongly pseudoconvex, both contain \( z^0 \) as an interior point, \( \partial D_1 \setminus \{ \varphi = \varepsilon \} \cap D \) is relatively compact in \( D \cap \{ l > 0 \} \) and \( \partial D_2 \setminus \{ P_x = 0 \} \) is relatively compact in \( D \). Then we define \( \overline{U} = \overline{D}_1 \cap \overline{D}_2 \).

Observe that, by construction, Proposition 3.2 applies, thus \( H^r(U \cap C^+, \mathcal{O}) = 0 \) for \( 1 \leq r \leq n - 2 \).

In the general case, since \( D \) is Stein, we have an exact sequence
\[
0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{O}^q \overset{\beta}{\longrightarrow} \mathcal{F} \longrightarrow 0.
\]

Consider the following commutative diagram of continuous maps
\[
\begin{array}{ccc}
H^0(U, \mathcal{O}^q) & \overset{\alpha}{\longrightarrow} & H^0(U, \mathcal{F}) \\
r \downarrow & & \downarrow r \\
H^0(U \cap C^+_\varepsilon, \mathcal{O}^q) & \overset{\beta}{\longrightarrow} & H^1(U \cap C^+_\varepsilon, \mathcal{F})
\end{array}
\]
where \( r \) denotes the natural restriction. Then, since \( \text{depth}(\mathcal{F}_z) \geq 4 \) for every \( z \in D \), we have \( \text{depth}(\mathcal{H}_z) \geq 5 \) for every \( z \in D \). Again by Proposition 7.5 in [9] we have \( H^1(U \cap C^+_\varepsilon, \mathcal{F}) = 0 \) whence the homomorphism
\[
H^0(U \cap C^+_\varepsilon, \mathcal{O}^q) \longrightarrow H^0(U \cap C^+_\varepsilon, \mathcal{F})
\]
is onto. Let \( \sigma \in H^0(U \cap C^+_\varepsilon, \mathcal{F}) \) and \( N \) a neighbourhood of \( \sigma \). Let \( g \in H^0(U, \mathcal{O}^q) \) such that \( \beta(g) = \sigma \). Since the homomorphism
\[
H^0(U, \mathcal{O}^q) \longrightarrow H^0(U \cap C^+_\varepsilon, \mathcal{O}^q)
\]
is dense image there exists \( h \in H^0(U, \mathcal{O}^q) \) such that \( r(h) \in \beta^{-1}(N) \). Then \( r(\alpha(h)) \in N \) with \( \alpha(h) \in H^0(U, \mathcal{F}) \). This shows that the homomorphism
\[
H^0(U, \mathcal{O}^q) \longrightarrow H^0(U \cap C^+_\varepsilon, \mathcal{O}^q)
\]
is dense image. \( \square \)

**Lemma 3.9.** Let \( \mathcal{F} \) and \( \varepsilon_0 \) be as in Lemma 3.8. Then for every \( \varepsilon \in (0, \varepsilon_0) \) there exists \( \varepsilon_2 \in (0, \varepsilon) \) such that for every \( \varepsilon' \in (\varepsilon_2, \varepsilon) \) the homomorphism
\[
H^0(C^+_{\varepsilon'}, \mathcal{F}) \longrightarrow H^0(C^+_{\varepsilon}, \mathcal{F})
\]
is dense image.
Proof. With the notations of Lemma 3.3 we have
\[ D = C_0^d, \quad D_1 = D \cup B, \quad \overline{D}_1 = \overline{D} \cup (D_1 \cap \overline{B}) \]
and we set \( \overline{V} = \overline{D}_1 \cap \overline{B} \). In view of Lemma 3.8 we may assume that
the homomorphism \( H^0(\overline{V}, \mathcal{F}) \rightarrow H^0(\overline{V} \cap D, \mathcal{F}) \)
is dense image. Moreover, \( H^1(\overline{V}, \mathcal{F}) = 0 \). Let \( \mathcal{U} \) be the closed covering \( \{ \overline{D}, \overline{V} \} \) of \( \overline{D}_1 \), \( Z^1(\mathcal{U}, \mathcal{F}) \) and \( B^1(\mathcal{U}, \mathcal{F}) \) respectively the space of cocycles and coboundaries of \( \mathcal{U} \) with values in \( \mathcal{F} \). Since \( H^1(\mathcal{U}, \mathcal{F}) \) is a subgroup of \( H^1(\overline{D}_1, \mathcal{F}) \) which is of finite dimension (cfr. Theorem 3.6) we have
\[ \dim C^1 \mathcal{H}^1(\mathcal{U}, \mathcal{F}) < +\infty. \]
It follows that \( H^1(\mathcal{U}, \mathcal{F}) \) is of finite dimension in the \( \mathcal{L} \mathcal{F} \)-space \( Z^1(\mathcal{U}, \mathcal{F}) \), thus an \( \mathcal{L} \mathcal{F} \)-space for the induced topology. Moreover, in view of the Banach open mapping theorem the surjective map
\[ H^0(\overline{D}, \mathcal{F}) \oplus H^0(\overline{V}, \mathcal{F}) \rightarrow B^1(\mathcal{U}, \mathcal{F}) \]
given by \( s \oplus \sigma \mapsto s|_{\overline{D}} - \sigma|_{\overline{D}} \) is a topological homomorphism.

Let \( s \in H^0(\overline{D}, \mathcal{F}); \ s|_{\overline{V} \cap \overline{D}} \in B^1(\mathcal{U}, \mathcal{F}) \). By Lemma 3.8 there exists a generalized sequence \( \{ s_\nu \} \subseteq H^0(\overline{V}, \mathcal{F}) \) such that
\[ s_\nu|_{\overline{V} \cap \overline{D}} \rightarrow 0. \]
In view of Banach theorem there exist two generalized sequences \( \sigma^1_\nu \in H^0(\overline{D}, \mathcal{F}), \sigma^2_\nu \in H^0(\overline{V}, \mathcal{F}) \) such that
\[ \sigma^1_\nu|_{\overline{D} \cap \overline{V}} - \sigma^2_\nu|_{\overline{D} \cap \overline{V}} = s_\nu|_{\overline{D} \cap \overline{V}} - s|_{\overline{D} \cap \overline{V}}, \]
\[ \sigma^1_\nu \rightarrow 0, \ \sigma^2_\nu \rightarrow 0. \]
It follows that for every \( \nu \)
\[ \tilde{s}_\nu = \begin{cases} s - \sigma^1_\nu & \text{on } \overline{D} \\ s_\nu - \sigma^2_\nu & \text{on } \overline{V} \end{cases} \]
is a section of \( \mathcal{F} \) on \( \overline{D}_1 \) and that \( \tilde{s}_\nu \rightarrow s \). In order to ends the proof we apply this procedure a finite numbers of times. \( \square \)

As a corollary we get the following

**Proposition 3.10.** Let \( \mathcal{F} \) and \( \epsilon_0 \) be as in Theorem 3.7. Then for every \( \epsilon \in (0, \epsilon_0) \) there exists \( \epsilon_0' \in [0, \epsilon] \) such that for every \( \epsilon' \in (\epsilon_0, \epsilon] \) the homomorphism
\[ H^0(C^d_\epsilon, \mathcal{F}) \rightarrow H^0(\overline{C}_\epsilon^d, \mathcal{F}) \]
is dense image.
Proof. Let \( I \subset (0, \varepsilon_0) \) be the (non-empty) set of \( \varepsilon' < \varepsilon \) such that the homomorphism
\[
H^0(\overline{C}_\varepsilon^+, \mathcal{F}) \longrightarrow H^0(\overline{C}_\varepsilon^+, \mathcal{F})
\]
is dense image. Let \( \overline{\varepsilon} = \inf I \) and \( \{\varepsilon_{\nu}\} \) be a decreasing sequence with \( \varepsilon_{0} = \varepsilon, \varepsilon_{\nu} \to \overline{\varepsilon} \) and set \( F_{\nu} = H^0(\overline{C}_{\varepsilon_{\nu}}^+, \mathcal{F}) \). The topology of \( F_{\nu} \) can be defined by an increasing sequence \( \{p_j(\nu)\}_{j \in \mathbb{N}} \) of translation invariant seminorms. Let for \( \nu \geq 1 \)
\[
r_{\nu} : F_{\nu} \longrightarrow F_{\nu-1}
\]
be the restriction map; then
\[
H^0(\overline{C}_\varepsilon^+, \mathcal{F}) = \lim_{\nu \to \infty} F_{\nu}
\]
and denote \( \pi_{\nu} : H^0(\overline{C}_\varepsilon^+, \mathcal{F}) \to F_{\nu} \) the natural map. We have to show that \( \pi_0 \) is dense image.

Let \( s \in F_0 = H^0(\overline{C}_\varepsilon^+, \mathcal{F}) \) and \( N \) a neighbourhood of \( s_0 \). We may assume that
\[
N = \left\{ s \in F_0 : p_0^{(0)} (s - s_0) < \varepsilon \right\}.
\]
Since the maps \( r_{\nu} \) are continuous and dense image we can choose elements \( s_{\nu} \in F_{\nu} \), for \( \nu \geq 0 \), satisfying the following conditions:
\[
\begin{align*}
s_1 & \in F_1 & p_0^{(0)} (r_1(s_1) - s_0) & < \varepsilon/2 \\
s_2 & \in F_2 & p_0^{(1)} (r_2(s_2) - s_1) & < \varepsilon/2 \\
 & & p_1^{(0)} (r_1 r_2(s_2) - r_1(s_1)) & < \varepsilon/2^2 \\
s_3 & \in F_3 & p_0^{(2)} (r_3(s_3) - s_2) & < \varepsilon/2 \\
 & & p_1^{(1)} (r_2 r_3(s_3) - r_2(s_2)) & < \varepsilon/2^2 \\
 & & p_2^{(0)} (r_1 r_2 r_3(s_3) - r_1 r_2(s_2)) & < \varepsilon/2^3 \\
\end{align*}
\]
and so on. Then, for every \( \nu \in \mathbb{N} \), the series
\[
s_{\nu} + (r_{\nu+1}(s_{\nu+1}) - s_{\nu}) + (r_{\nu+1} r_{\nu+2}(s_{\nu+2}) - r_{\nu+1}(s_{\nu+1})) + \ldots
\]
is convergent in \( F_{\nu} \) and \( r_{\nu}(\sigma_\nu) = \sigma_{\nu-1} \). Hence \( \sigma = \{\sigma_\nu\}_{\nu \in \mathbb{N}} \) belongs to \( H^0(\overline{C}_\varepsilon^+, \mathcal{F}) \) and, by definition \( p_0^{(0)}(\sigma_0 - s_0) < \varepsilon \), i.e. \( \pi_0(\sigma_0) \in N \). \( \square \)

Proof of Theorem 3.11. The proof uses Corollary 3.4 and Lemma 3.8. With the notations of Lemma 3.3 we have
\[
\overline{D} = \overline{C}_\varepsilon^+, \overline{D}_1 = \overline{D} \cup \overline{B}, \overline{D}_1 = \overline{D} \cup (\overline{D}_1 \cap \overline{B}), \overline{D} \cap (\overline{D}_1 \cap \overline{B}) = \overline{D} \cap \overline{B}.
\]
We may assume that the homomorphism
\[
H^1(\overline{D}_1, \mathcal{F}) \longrightarrow H^1(\overline{D}, \mathcal{F})
\]
is onto and
\[
H^0(\overline{D}_1, \mathcal{F}) \longrightarrow H^0(\overline{D}, \mathcal{F})
\]
COHOMOLOGY OF SEMI 1-CORONAE

is dense image. Moreover, $H^1(\overline{B} \cap \overline{D_1}, \mathcal{F}) = 0$. Thus it is sufficient to show that the homomorphism

$$H^1(\overline{D_1}, \mathcal{F}) \to H^1(\overline{D}, \mathcal{F})$$

is injective.

Since $H^1(\overline{D_1} \cap \overline{B}, \mathcal{F}) = 0$ the Mayer-Vietoris exact sequence applied to $\overline{D_1} = \overline{D} \cup (\overline{D_1} \cap \overline{B})$ gives the exact sequence

$$H^0(\overline{D} \cap \overline{B}, \mathcal{F}) \xrightarrow{a} H^1(\overline{D_1}, \mathcal{F}) \xrightarrow{b} H^1(\overline{D}, \mathcal{F}).$$

Let $\xi \in \text{Ker } b = \text{Im } a$, $\xi = a(\eta)$ with $\eta \in H^0(\overline{D} \cap \overline{B}, \mathcal{F})$. By Lemma 3.8, $\eta$ is approximated by a sequence $\{\eta_\nu\} \subset H^0(\overline{D_1} \cap \overline{B}, \mathcal{F})$. Each $\eta_\nu$ is a 1-coboundary of the closed covering $\mathcal{U} = \{\overline{D}, \overline{D_1} \cap \overline{B}\}$ with values in $\mathcal{F}$ and such a space is closed in the space $Z^1(\mathcal{U}, \mathcal{F})$ of the 1-cocycles. This proves that $\eta$ is a 1-coboundary of $\{\mathcal{U}, \mathcal{F}\}$, whence $\xi = a(\eta) = 0$.

\[\square\]

Remark 3.2. In the full $q$-corona the cohomology of all coronae are isomorphic (see \[\Pi\]). Differently, in the semi 1-corona case the cohomology groups are isomorphic up to a critical $\varepsilon$, where the dimension of the cohomology spaces jumps, then they are again all isomorphic up to a second critical value, and so on. They must not be all isomorphic, nor they dimensions must be bounded.

4. Extension of coherent sheaves and analytic subsets

An interesting consequence is that on a semi 1-corona $C^+ = C^+_{0,1}$ Theorem A of Oka-Cartan-Serre holds for a coherent sheaf $\mathcal{F}$ satisfying the conditions of Theorem 3.1. We first prove the following

Lemma 4.1. Let $X$ be a complex space, $\mathcal{F} \in \text{Coh}(X)$ satisfying the following property: for every $x \in X$ there exists a subset $Y \not\ni x$ of $X$ such that:

i) $H^1(X, \mathcal{F}) \simeq H^1(Y, \mathcal{F})$

ii) if $\mathcal{M}_x$ denotes the ideal of $\{x\}$ the homomorphism

$$H^1(X, \mathcal{M}_x \mathcal{F}) \to H^1(Y, \mathcal{M}_x \mathcal{F})$$

is injective.

Then, for every $x \in X$ the space $H^0(X, \mathcal{F})$ of the global sections of $\mathcal{F}$ generates $\mathcal{F}_x$ over $\mathcal{O}_{X,x}$.

**Proof.** Let $x \in X$ and $Y$ satisfying the conditions of the lemma. Consider the exact sequence of sheaves

$$0 \to \mathcal{M}_x \mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{M}_x \mathcal{F} \to 0$$
and the associated diagram

\[
H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}/\mathcal{M}_x[\mathcal{F}]) \to H^1(X, \mathcal{M}_{[x]}[\mathcal{F}]) \xrightarrow{\delta} H^1(X, \mathcal{F})
\]

The homomorphism is injective by hypothesis and \(\beta\) is an isomorphism since \(\mathcal{M}_{[x]}|_Y \simeq \mathcal{F}|_Y\), thus \(\gamma\) is an isomorphism. It follows that \(\delta\) is injective and consequently that the homomorphism

\[
H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}/\mathcal{M}_x[\mathcal{F}]) \simeq \mathcal{F}/\mathcal{M}_x[\mathcal{F}]
\]

is onto. Then the Lemma of Nakayama implies that \(H^1(X, \mathcal{F}) \to F_x\) is onto and this proves the lemma. \(\square\)

Keeping the notations of the proof of Theorem 3.1, we deduce the following Corollary 4.2.

**Corollary 4.2.** Under the conditions of Theorem 3.1 for every compact subset

\(K \subset C_\epsilon^+ \cap \{\varphi > \epsilon\} \cap \{P_\epsilon < 0\}\)

there exist sections \(s_1, \ldots, s_k \in H^0(C_\epsilon^+, \mathcal{F})\) which generate \(\mathcal{F}_z\) for every \(z \in K\).

**Theorem 4.3.** Let \(C^+ = (B_1 \setminus B_0) \cap \{h \geq 0\}\) and \(\mathcal{F} \in \text{Coh}(C^+)\). If \(\text{depth}(\mathcal{F}_z) \geq 3\) on \(\{\varphi = 0\}\) then for every \(a > 0\) near 0 \(\mathcal{F}|_{B_1 \setminus \pi_a}\) extends on \(B_1 \cap \{h \geq 0\}\) by a coherent sheaf \(\mathcal{F}_a\).

**Proof.** With the usual notations choose \(\epsilon_0 \in (0, a),\) and \(c_0 > 0\) such that

i) \(\mathcal{F}\) is defined on the semi 1-hyperplane \((B_1 \setminus B_{-c}) \cap \{h > -c\}\)

ii) \(\{z \in B_1 : h(z) \geq c\} \subset \{z \in B_1 : P_\epsilon(z) < 0\}\)

iii) for every \(\epsilon \in (0, \epsilon_0),\) \(c \in (0, c_0)\) the hypersurfaces \(\{P_\epsilon = -c\},\)

\(P_\epsilon(z) = \epsilon|z|^2 - h\), meet the hypersurfaces \(\{\varphi = \epsilon\},\) \(\{\varphi = -\epsilon\}\)

transversally.

Let \(Y_{\alpha, \beta}\) denote the semi 1-corona \(\{\alpha < \varphi < \beta\} \cap \{h > c\}\) with \(\alpha < \beta < \epsilon\). In view of Corollary 4.2 applied to the semi 1-corona \(Y_{\alpha, \beta}\)

there exist \(\alpha, \beta, \gamma \in (0, a)\) with \(\alpha < \beta < \gamma\) such that \(H^0(Y_{\alpha, \beta, \gamma}, \mathcal{F})\) generates \(\mathcal{F}\)

on \(K_{\beta, \gamma} = Y_{\beta, \gamma}^+ \cap \{h \geq 0\}\). Thus on \(K_{\beta, \gamma}\) there exists an exact sequence

\[
O^p \xrightarrow{\beta} \mathcal{F} \longrightarrow 0.
\]

Since, by hypothesis, \(\text{depth}(\mathcal{F}_z) \geq 3\) for every \(z \in K_{\beta, \gamma}\) we have

\[
\text{depth}(\text{Ker} \alpha) \geq 4
\]
on $K_{\beta,\gamma}$ (cfr. [3]). Again by Corollary 1.2 there exist $\beta_1, \gamma_1 \in (\beta, \gamma)$, $\beta_1 < \gamma_1$ and sections $\sigma_1, \ldots, \sigma_l$ on $K_{\beta_1,\gamma_1} = \overline{Y}_{\beta_1,\gamma_1} \cap \{ h \geq 0 \}$ which generate $(\ker \alpha)_z$, for every $z \in V$. Since $\ker \alpha$ is a subsheaf of $\mathcal{O}^p$, by the theorem in [11] the sections $\sigma_1, \ldots, \sigma_l$ extend holomorphically on
\[
\{ \varphi \leq \gamma_1 \} \cap \{ h \geq 0 \}
\]
and their extensions $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_l$ generate a coherent sheaf $\mathcal{H}$ on
\[
\{ \varphi \leq \gamma_1 \} \cap \{ h \geq 0 \}.
\]
Let $\tilde{\mathcal{F}}'_{a}$ be the sheaf defined by
\[
\tilde{\mathcal{F}}'_{a,z} = \begin{cases} 
\mathcal{F}_z & \text{for } z \in \{ \varphi > \gamma_1 \} \cap \{ h \geq 0 \} \\
\mathcal{O}_z/\mathcal{H}_z & \text{for } z \in \{ \varphi \leq \gamma_1 \} \cap \{ h \geq 0 \} ;
\end{cases}
\]
$\tilde{\mathcal{F}}'_{\epsilon}$ is a coherent sheaf on $B_1^+ \cap \{ y_n > \epsilon \}$ extending $\mathcal{F}$.  

**Corollary 4.4.** Let $X^+ = (B_1 \setminus \overline{B_0}) \cap \{ h > 0 \}$ and $Y$ an analytic subset of $X^+$ such that depth$(\mathcal{O}_{Y,z}) \geq 3$ for $z$ near $\{ \varphi = 0 \}$. Then $Y$ extends on $B_1 \cap \{ h \geq 0 \}$ by an analytic subset.

**Proof.** We apply Theorem 4.3 to $X^+ \cap \{ h \geq \epsilon \}$, where $\epsilon \sim 0$ is positive. Then, for $\nu \in \mathbb{N}$ there exists a coherent sheaf $\tilde{\mathcal{O}}^{(\nu)}_{Y'}$ on $B_1 \cap \{ h \geq 0 \}$ which extends $\mathcal{O}_{Y'}$; $\tilde{Y}^{(\nu)} = \text{supp } \tilde{\mathcal{O}}^{(\nu)}_{Y'}$ is an analytic subset extending $Y \cap (B_1 \setminus B_1/\nu) \cap \{ h \geq \epsilon \}$. In view of the strong pseudoconvexity of $bB_1/\nu$, the subset $F_{\nu} = \tilde{Y}^{(\nu)} \setminus \tilde{Y}^{(\nu+1)}$ is a finite set of points which is contained in $B_1/\nu$. Start by $\nu = 2$ and consider the first extension $\tilde{Y}^{(2)}$. Then $\tilde{Y}^{(2)} \setminus F_2 \cap (B_1/2 \setminus B_1/3)$ coincide with $Y$ on $(B_1 \setminus B_1/3)$ and so on. To handle different extensions depending on $\epsilon$ we argue in the same way.  

**References**

[1] Aldo Andreotti and Hans Grauert: *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France 90, 193–259 (1962).

[2] Aldo Andreotti and Giuseppe Tomassini: *A remark on the vanishing of certain cohomology groups*, Compositio Math. 21, 417–430 (1969).

[3] Henri Cartan: *Faisceaux analytiques cohérents*, Corso C.I.M.E., Villa Monastero, Como (1963).

[4] Giuseppe Della Sala and Alberto Saracco: *Semi-global extension of maximally complex submanifolds*, [arXiv: math.CV/0607747] (2006).

[5] Roger Godement: *Topologie algébrique et théorie des faisceaux*, Actualités Sci. Ind. No. 1252, Publ. Math. Univ. Strasbourg. No. 13, Hermann, Paris (1958).

[6] Alexandre Grothendieck: *Espaces vectoriels Topologiques* (1954).

[7] F. Reese Harvey and H. Blaine Lawson Jr.: *On boundaries of complex analytic varieties. I*, Ann. of Math. (2) 102, 223–290 (1975).

[8] Christine Laurent-Thiébaut: *Sur l’extension des fonctions CR dans une variété de Stein*, Ann. Mat. Pura Appl. 150, 141–151 (1988).
[9] Christine Laurent-Thiébaut and Jürgen Leiterer: Uniform estimates for the Cauchy-Riemann equation on $q$-concave wedges, Astérisque 217, 151–182 (1993). Colloque d’Analyse Complex et Géométrie (Marseille, 1992).

[10] Christine Laurent-Thiébaut and Egmont Porten: Analytic extension from non-pseudoconvex boundaries and $A(D)$-convexity, Ann. Inst. Fourier (Grenoble) 53, 847–857 (2003).

[11] Guido Lupacciolu and Giuseppe Tomassini: Un teorema di estensione per le CR-funzioni, Ann. Mat. Pura Appl. 137, 257–263 (1985).

[12] Alberto Saracco and Giuseppe Tomassini: Cohomology and extension problems for semi $q$-coronae, to appear in Math. Z. (on-line first: http://dx.doi.org/10.1007/s00209-006-0097-9) [arXiv math.CV/0503490] (2007).

[13] Edgar Lee Stout: Removable singularities for the boundary values of holomorphic functions, Several complex variables (Stockholm, 1987/1988), Math. Notes 38, Princeton Univ. Press, Princeton, NJ, 1993, 600–629.

[14] Giuseppe Tomassini: Sur les algèbres $A^0(D)$ et $A^\infty(D)$ d’un domaine pseudo-convexe non borné, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 10, 243–256 (1983).

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