ON GEOMETRY AND SYMMETRY OF KEPLER SYSTEMS. I.

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ABSTRACT. We study the Kepler metrics on Kepler manifolds from the point of view of Sasakian geometry and Hessian geometry. This establishes a link between the problem of classical gravity and the modern geometric methods in the study of AdS/CFT correspondence in string theory.

1. INTRODUCTION

This is the first of a series of papers in which we make a revisit to the geometry and symmetry of Kepler problem with the vantage point of view of quantum gravity, supergravity, and string theory.

Time and again in history, geometry and symmetry have been used to study problems in gravity, in particular, the two-body problem, aka the Kepler problem. This tradition goes back to the ancient Greeks who regarded the circles and spheres as the geometric objects with perfect symmetries and used them to build their models of the universe. Geometric objects with discrete symmetries, such as the five Platonic solids, were assigned by Plato to earth, air, water, fire, and a heavenly fifth element. In Mysterium Cosmographicum, published in 1596, Kepler proposed a model of the solar system by relating the five extraterrestrial planets known at that time to the five Platonic solids. This work attracted the attention of Tycho Brahe who offered Kepler a job as an assistant and hence also access to his empirical data, based on which Kepler formulated his second law in 1602, first law in 1605, and third law in 1618. With the introduction of elliptic orbits, the Kepler system seemed to lose the connection with the circular symmetry. In Newton’s Principia published in 1687, Euclidean geometry was used to establish the connection between the inverse square law of the universal gravity and Kepler’s laws. With the advance of calculus, the Kepler system became one of the first examples of integrable systems, i.e., those solvable by quadrature, and geometry and symmetry seemed to lose its relevance to the Kepler problem.

An interesting attempt in the nineteenth century to resurrect geometry and symmetry in the study of the Kepler system in the nineteenth century is Hamilton’s work [22] on hodograph published in 1846. He showed that the velocity vector of the Kepler system moves on a circle. Only in the twentieth century, geometry and symmetry gradually regained their positions in the study of gravity. First of all, Einstein’s general relativity in 1915 is a geometric theory of gravity; secondly, Nöther’s theorem published in 1917 revealed the connection between symmetry and conservation laws. Progresses were made first in the quantum Kepler system, e.g., the quantum mechanical system of the hydrogen atom, where hidden symmetry was first discovered. In 1926, one year before Schrödinger equation was discovered, Pauli [37] solved the problem of finding the spectrum of the hydrogen atom using an \( o(4) = o(3) \oplus o(3) \)-symmetry generated by the angular-momentum operators and the
Laplace-Runge-Lenz operators. In 1935, Fock [17] gave a geometric interpretation of the appearance of \( o(4) \) by working in the momentum space by a Fourier transform followed by an embedding of \( \mathbb{R}^3 \) into \( S^3 \) by stereographic projection. After thirty years, in the 1960s, the \( o(4) \)-symmetry of the quantum Kepler system was extended to the \( o(4,1) \)- and \( o(4,2) \)-symmetry, see e.g. Bander and Itzykson [5] and Györgyi [21] and the references therein. In Bander-Itzykson [5] the results were generalized to quantum Kepler system in arbitrary dimension. Such studies of the symmetry of quantum Kepler system also leads to the study of classical Kepler system, for example in a 1965 paper in which Bacry [3] tried to construct the \( o(4,1) \)-symmetry of the quantum Kepler system he started with the classical Kepler system. In another direction, attentions were shifted from infinitesimal symmetry described by Lie algebras to global symmetry described by Lie group action. In the 1966 paper Bacry-Ruegg-Souriau [4], the global study of the hidden symmetry of the Kepler system was initiated. Quantum aspects of the Kepler problem was discussed in a book by Englefield [16] published in 1972.

Since the 1970s modern differential geometry was applied to Kepler problem. First Riemannian geometry was applied by Moser [35] to give a classical mechanics version of Fock’s result. He applied the idea of regularization of many-body system developed by e.g. Levi-Civita [29], Sundman [36], Siegel [39], Kustaanheimo-Stiefel [27] to relate the solutions of negative energy of the Kepler system to geodesic flow on the 3-sphere, and to identify the phase space of orbits of negative energy of the Kepler system to \( K = T^+ S^3 = T^+ S^3 - S^3 \). Moser’s results were generalized to the case of nonnegative energy by Belbruno [6] and Osipov [36]. For an exposition, see Milnor [34]. See also Ligon-Schaaf [30] and Heckman-de Laat [23].

The manifold \( K \) was called the Kepler manifold by Souriau who as a pioneer of symplectic geometry contributed to the introduction or the development of many important concepts, such as the moment map, the coadjoint action and the coadjoint orbits, geometric quantization, and he gave a classification of the homogeneous symplectic manifolds, known as the Kirillov-Kostant-Souriau theorem. Many of these developments can be found in his book [42] first published in 1970. He applied these developments in symplectic geometry to the Kepler problem in two papers, Souriau [40] and Souriau [41], where \( K \) was shown to be a coadjoint orbit of \( SO(4,2) \), and a complex structure of \( K \) was revealed: \( K \) is diffeomorphic to the conifold in \( \mathbb{C}^4 \) with the conifold point removed. The work of Moser and Souriau were nicely presented in a section in the 1977 book by Guillemin and Sternberg [19]. In a remarkable book published in 1990 [20], these authors revisited the Kepler problem by adding more insights. They established the connections of the Kepler problem to other theories in mathematics and physics, such as Howe dual pairs, orbit method of representations, Penrose’s twistor theory, etc. More precisely, they showed that the Kepler manifold can be realized either as a coadjoint orbit of the first factor or as a reduction with respect to the second one in the Howe pairs \( (SO(2,4), SL(2,\mathbb{R})) \) in \( Sp(4,\mathbb{R}) \) and \( (U(2,2), U(1)) \) in \( Sp(8,\mathbb{R}) \). The orbit approach leads to representation of the Poincaré group, while Marsden-Weinstein symplectic reduction in stages identifies \( K \) with the space of forward null geodesics on the conformal completion of the Minkowski space, and hence it is natural to discuss different models of the universe whose symmetries are subgroups of \( SO(2,4) \) and also Kostant’s model based on the group \( SO(4,4) \). For later work in this direction, see e.g. Keane-Barrett [25], Keane-Barrett-Simons [26] and Keane [24]. In 2004
Cordani [11] published a comprehensive book on Kepler problem, which treated both the classical and quantum aspects, together with perturbation theory. For more recent work on Kepler problem see e.g. Cariglia [9].

In the 1960s there have appeared some generalizations of the Kepler problem called the MICZ-Kepler problem [33, 47]. Meng discovered the generalizations to higher dimensions and their connections to Euclidean Jordan algebra, symmetric cones of tube type, and minimal representations. For a survey, see Meng [32]. Even though we will not discuss them in this paper, it is interesting to generalize our results in this direction.

From the above brief sketch of some aspects in the development of the Kepler problem, it is clear that most of the researches in the literature focus on the classical mechanics of the Kepler system or quantum mechanics of the quantum Kepler system. The purpose of this series of papers is to study the geometry and symmetry of the classical Kepler system by some techniques from general relativity, quantum field theory, and string theory. Our motivation is to test some important ideas in string theory in the classical setting of the Kepler system. They include: connection with the theory of integrable hierarchy, Calabi-Yau spaces and mirror symmetry, AdS/CFT correspondence.

Since its early days, string theory was proposed as a candidate theory for the unification of gravity and the standard model of other fundamental forces. The explosive development of string theory in the past thirty years or so has brought forward a remarkable supply of new ideas and new techniques. Our modest goal of this work is to apply a sample of ideas and techniques developed in string theory to revisit the Kepler problem in the hope of while testing the modern developments to a classical problem, new insights can be gained for both of them. As it turns out, such consideration leads us to many connections of the Kepler system with many different mathematical objects overlooked in the literature. In this paper, we will make connections to Sasakian geometry and Hessian geometry. In later parts of the series, we will relate to Lie sphere geometry and integrable systems.

Our point of departure from the geometry of the Kepler problem established by Moser and Souriau in the 1970s is to shift attentions to the Kähler metrics on the Kepler manifolds. It was observed by Rawnsley [38] in 1977 that the complex structure and the symplectic structure on the Kepler manifolds are compatible, leading to Kähler metrics on the Kepler manifolds. We will refer to such metrics as Kepler metrics.

In this paper we will present two types of results for Kepler metrics: Relationship with Sasakian geometry and relationship with Hessian geometry. First, we will show that the Kepler metrics are Sasaki metric on the cotangent bundles of the round spheres, and they are related to Sasakian metrics on the unit conormal bundle of the round spheres. Secondly, we will focus on Kepler metrics on Kepler n-manifold with n = 2 and 3. In these cases, we will use the symmetry of the metrics to present explicit constructions of these metrics by Calabi Ansatz. This will lead us a generalization of some work of Guillemin [18], Abreu [1], Donaldson [12] to the noncompact case. With the introduction of symplectic coordinates we will establish a connection of Kepler metrics for n = 2, 3 with Hessian geometry. We also present a treatment of Kähler Ricci-flat metrics on n-conifolds, resolved n-conifolds in the same fashion. We will propose a connection between such metrics using some special Kähler Ricci flow.
We arrange the rest of this paper as follows. In §2 we recall the Levi-Civita regularization and the Moser regularization of the Kepler problem, and introduce the Kepler manifold $K_n$. In §3 we recall the complex structures on $K_n$ introduced by Souriau. Some detailed discussions in the cases of $n = 2$ and $3$ are also presented. In §4 we first recall the Kepler metrics and their Kähler potentials, then we show that they are Sasaki metrics on the conormal bundles of the round spheres, and relate them to Sasaki metrics on the unit conormal bundles of the round spheres, which we show are Sasakian. In §5 we recall some explicit constructions of Kähler Ricci-flat metrics on the $n$-conifolds and the deformed $n$-conifolds. Starting in §6 we will relate some explicit constructions of Kähler metrics with symmetries with Hessian geometry. In §6 we treat the case of $U(n)$-symmetric Kähler metrics on $\mathbb{C}^n - \{0\}$. We specialize to the case of $U(n)$-symmetric Kähler Ricci-flat metrics in §7 and we further specialize to the case of Kepler metric on $K_2$ and Eguchi-Hanson spaces in §8. Kepler metric on $K_3$, Kähler Ricci-flat metrics on 3-conifold and resolved 3-conifold are studied from the point of view of Kähler metrics with symmetries and Hessian geometry in §9 and §10 in various ways. Finally, in §11 we summarize our results and propose some generalizations.

2. Regularizations of Kepler System and Kepler Manifolds

A first key step towards the global study of the geometry and symmetry of the Kepler system is regularization. This was first introduced by Levi-Civita [29] more than a hundred years ago in his research on restricted three-body problem. The idea was used by Sundman [46] a few years later to solve the three-body problem by power series. The idea of regularization was further developed by Siegel [39] and Kustaanheimo-Stiefel [27], and it was used by Moser [35] to study the Kepler problem in arbitrary dimensions.

In this Section we recall the Levi-Civita regularization of the two-dimensional Kepler problem and the Moser regularization of $n$-dimensional Kepler problem. For details, see e.g. Moser [35], Souriau [40], Guillemin-Sternberg [20] and Cordani [11], and the references therein.

2.1. Levi-Civita regularization of two-dimensional Kepler system. Let us recall the Levi-Civita regularization of the two-dimensional Kepler system:

\[
\begin{align*}
\frac{d^2 x}{dt^2} &= \frac{x}{(x^2 + y^2)^{3/2}}, \\
\frac{d^2 y}{dt^2} &= -\frac{y}{(x^2 + y^2)^{3/2}}.
\end{align*}
\] (1)

One first rewrites it as a Hamiltonian system:

\[
\begin{align*}
\frac{dx}{dt} &= p = \{H, x\}, \\
\frac{dp}{dt} &= -\frac{x}{\rho^2} = \{H, p\}, \\
\frac{dy}{dt} &= q = \{H, q\}, \\
\frac{dq}{dt} &= -\frac{y}{\rho^2} = \{H, q\},
\end{align*}
\] (2)

where $H = \frac{1}{2}(p^2 + q^2) - \frac{1}{\rho}$, $\rho = (x^2 + y^2)^{1/2}$, and the Poisson bracket is defined by

\[
\{f, g\} := \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial q}.
\] (4)

In other words, the phase space of the two-dimensional Kepler problem is

\[
\{(x, y, p, q) \in \mathbb{R}^4 \mid (x, y) \neq (0, 0)\} = (\mathbb{R}^2 - \{0\}) \times \mathbb{R}^2,
\] (5)
which is endowed with a symplectic form
\[ \omega = dp \wedge dx + dq \wedge dy. \]

To remove the singularity of the system, one first introduces the fictitious time \( \tau \) such that
\[ \frac{d\tau}{dt} = \frac{1}{\rho}, \]
and make the following change of variables:
\[ x + iy = (\xi + i\eta)^2, \quad p - iq = \frac{\omega - i\chi}{2(\xi + i\eta)}, \]
then the two-dimensional system becomes:
\[ \frac{d}{d\tau}(\xi + i\eta) = \frac{1}{4}(\omega + i\chi), \]
\[ \frac{d}{d\tau}(\omega + i\chi) = \frac{1}{4} \omega^2 + \chi^2 - \frac{2}{\xi - i\eta}. \]

Next, note that
\[ H = \frac{1}{2} |p + iq|^2 - \frac{1}{|x + iy|} = \frac{1}{4} |\omega + i\chi|^2 - \frac{1}{|\xi + i\eta|^2}. \]
Hence by conservation of energy, one can restrict to the energy surface \( H = E \), where one has
\[ |\omega + i\chi|^2 = 8(1 + E|\xi + i\eta|^2), \]
and so the equation of motion becomes:
\[ \frac{d}{d\tau}(\xi + i\eta) = \frac{1}{4}(\omega + i\xi), \]
\[ \frac{d}{d\tau}(\omega + i\chi) = 2E \cdot (\xi + i\eta). \]

Now we restrict to the case of \( E < 0 \) and convert the above three equations into the following form:
\[ \frac{d}{d\tilde{\tau}}(\xi + i\eta) = (\tilde{\omega} + i\tilde{\xi}), \]
\[ \frac{d}{d\tilde{\tau}}(\tilde{\omega} + i\tilde{\chi}) = 2E \cdot (\tilde{\xi} + i\tilde{\eta}), \]
\[ |\tilde{\xi} + i\tilde{\eta}|^2 + |\tilde{\omega} + i\tilde{\chi}|^2 = -E. \]
by a further change of variable:
\[ \tilde{\tau} = \sqrt{-\frac{E}{2}} \tau, \quad \tilde{\xi} + i\tilde{\eta} = -E(\xi + i\eta), \quad \tilde{\omega} + i\tilde{\chi} = \sqrt{-\frac{E}{8}}(\omega + i\chi). \]

2.2. The Moser regularization. Moser [35] extended the inverse map of the stereographic projection \( \mathbb{R}^n \to \mathbb{R}^{n+1} \) defined by
\[ y = (y_1, \ldots, y_n) \mapsto (x_0 = \frac{|\tilde{y}|^2 - 1}{|\tilde{y}|^2 + 1}, \frac{2y_1}{|\tilde{y}|^2 + 1}, \ldots, \frac{2y_n}{|\tilde{y}|^2 + 1}), \]
to a map \( T^*\mathbb{R}^n \to T^*\mathbb{R}^{n+1} \)
\[ (\tilde{y}, \tilde{\eta}) \mapsto (\tilde{x}, \tilde{\xi}), \]
called the Moser map, where \( \vec{\xi} = (\xi_0, \ldots, \xi_n) \) is defined by:

\[
\xi_0 = \vec{\eta} \cdot \vec{y}, \quad \xi_j = \frac{|\vec{y}|^2 + 1}{2} \cdot \eta_j - (\vec{\eta} \cdot \vec{y}) \cdot y_j, \quad j = 1, \ldots, n.
\] (21)

One can check that

\[
\vec{x} \cdot \vec{x} = 1, \quad \vec{x} \cdot \vec{\xi} = 0.
\] (22)

Furthermore,

\[
\vec{\xi} \cdot d\vec{x} = \vec{\eta} \cdot d\vec{y},
\] (23)

\[
\eta_j = (1 - x_0)\xi_j + \xi_0 x_j,
\] (24)

\[
|\vec{\xi}| = \frac{|\vec{y}|^2 + 1}{2} \cdot |\vec{\eta}|.
\] (25)

The Moser map defines an inclusion of \((\mathbb{R}^n - \{0\}) \times \mathbb{R}^n\) into the Kepler manifold:

\[
K_n := \{(\vec{x}, \vec{\xi}) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} - \{0\}) \mid \vec{x} \cdot \vec{x} = 1, \quad \vec{x} \cdot \vec{\xi} = 0\}.
\] (26)

This space is naturally identified with \(T^+ S^n := T^* S^n - S^n\).

One can check that \(\Phi = \frac{1}{2}|\vec{\xi}|^2|\vec{x}|^2\), restricted to \(T^* S^n\), generates the geodesic flow on \(T^* S^n\). Using the Moser map and the introduction of the fictitious time \(\tau\), the energy surface \(\Phi = \frac{1}{2}\) corresponds to \(H = -\frac{1}{2}\). Using the Lie scaling

\[
q \mapsto \lambda^2 q, \quad p \mapsto \lambda^{-1} p, \quad t \mapsto \lambda^3 t,
\] (27)

the energy surface \(\Phi = \frac{1}{2}\lambda^2\) corresponds to \(H = -\frac{1}{2\lambda^2}\).

3. Complex Structures on Kepler Manifolds

The reason that in last Section we single out the two-dimensional Kepler problem and Levi-Civita regularization before presenting the more general Moser regularization is because it naturally leads to a construction of a complex structure on \(K_2\). Complex structures by a different construction on general \(K_n\) were discovered by Souriau [40]. We will recall both of these constructions in this Section and show that they give a hypercomplex structure on \(K_2\), in particular, it is doubly covered by \(\mathbb{C}^2 - \{0\}\). We also recall the complex structure on \(K_3\).

3.1. Complex structures on \(K_2\). Let us now point out some algebraic geometry implicit in the process of Levi-Civita regularization. This leads us to the discussions of complex structures on \(K_n\) in the next subsection.

The original phase space of the two-dimensional Kepler problem is \((\mathbb{R}^2 - \{(0, 0)\}) \times \mathbb{R}^2\), with coordinates \((x, y, p, q)\). Using complex coordinates \((x + iy, p - iq)\) on this space, one can identify it with \(\mathbb{C}^* \times \mathbb{C}\). The Levi-Civita regularization suggests to introduce \(z_1, z_2\) such that:

\[
x + iy = z_1^2, \quad p - iq = \frac{z_2}{z_1}.
\] (28)

One can understand \((z_1, z_2)\) as linear coordinates on \(\mathbb{C}^2\) and \((x + iy, p - iq)\) as local coordinates on \(O_{\mathbb{P}^1}(-2)\). Indeed, we have the following diagram:

\[
\begin{array}{ccc}
O_{\mathbb{P}^1}(-1) & \overset{2:1}{\longrightarrow} & O_{\mathbb{P}^1}(-2) \\
\downarrow & & \downarrow \\
\mathbb{C}^2 & \longrightarrow & \mathbb{C}^2/\mathbb{Z}_2
\end{array}
\]
Let \((\alpha_1, \beta_1), (\alpha_2, \beta_2)\) be local coordinates on \(O_{\mathbb{P}^1}(-1)\) such that

\[
\alpha_2 = \frac{1}{\alpha_1}, \quad \beta_2 = \alpha_1 \beta_1,
\]

and \((\alpha_1, \gamma_1), (\alpha_2, \gamma_2)\) be local coordinates on \(O_{\mathbb{P}^1}(-2)\) such that

\[
\alpha_2 = \frac{1}{\alpha_1}, \quad \gamma_2 = \alpha_1^2 \gamma_1,
\]

then the upper horizontal map is given by:

\[
\gamma_1 = \beta_2^1, \quad \gamma_2 = \beta_2^2,
\]

and the left vertical map is given by

\[
z_1 = \beta_1 = \alpha_2 \beta_2, \quad z_2 = \alpha_1 \beta_1 = \beta_2.
\]

One can then see that

\[
\alpha_1 = \frac{z_2}{z_1}, \quad \gamma_1 = \frac{z_2}{z_1},
\]

and so one can get

\[
x + iy = \gamma_1, \quad p - iq = \alpha_1,
\]

and with this identification one identifies \(\mathbb{C}^* \times \mathbb{C}\) as a coordinate chart on the Kepler manifold \(K_2 = O_{\mathbb{P}^1}(-2) - \mathbb{P}^1\). For those points in \(K_2\) not in the image of inclusion of \(\mathbb{C}^* \times \mathbb{C}\), \(\alpha_2 = 0\) and so \(\alpha_1 = \infty\), and the regularization procedure enlarge the original phase space by including some points with infinite momentum, and for the solution space, add some orbits that correspond to collision solutions.

The Kepler manifold \(K_2\) is diffeomorphic to \(T^* S^2 = T^* S^2 - S^2\). The above discussion suggests us to regard \(S^2\) as \(\mathbb{P}^1\), then one can identify \(K_2\) as the holomorphic cotangent bundle of \(\mathbb{P}^1\) minus the zero section, i.e. \(O_{\mathbb{P}^1}(-2) - \mathbb{P}^1\). Another way to understand this space is to examine the vertical map on the right in (29). The quotient space \(\mathbb{C}^2/\mathbb{Z}_2\) can be understood as an affine variety in \(\mathbb{C}^3\), defined by the equation

\[
uv - w^2 = 0,
\]

and the lower horizontal map in (29) is given by

\[
(z_1, z_2) \mapsto (u = z_1^2, v = z_2^2, w = z_1 z_2).
\]

Define new coordinates \((w_1, w_2, w_3)\) on \(\mathbb{C}^3\) by:

\[
u = w_0 + iw_1, \quad v = w_0 - iw_1, \quad w_2 = iw,
\]

then one gets the equation of the two-dimensional conifold \(C_2\)

\[
w_0^2 + w_1^2 + w_2^2 = 0.
\]

The vertical map on the right in (29) is defined by:

\[
u = \gamma_1 = \alpha_2^2 \gamma_2, \quad v = \alpha_1 \gamma_1 = \alpha_2 \gamma_2, \quad w = \alpha_1^2 \gamma_1 = \gamma_2.
\]

Therefore, \(K_2\) can also be identified with \(C_2^*\), the two-dimensional conifold with the conifold point removed.
3.2. Complex structures on $K_n$. The introduction of the complex structures on $K_2$ in last subsection relies on the identification of $S^2$ with $\mathbb{P}^1$ and makes use of the complex structure on $S^2$, and hence it cannot be generalized to $K_n$. Nevertheless, the diffeomorphism from $K_2$ to $C_2^* := C_2 - \{0\}$ suggests a diffeomorphism of $K_n$ with $C_n^* := C_n - \{0\}$ first discovered by Sourian [40], where $C_n$ is the $n$-conifold in $\mathbb{C}^{n+1}$ defined by the equation:

$$w_0^2 + w_1^2 + \cdots + w_n^2 = 0.$$  
(39)

For $(w_0, w_1, \ldots, w_n) \in C_n - \{0\}$, write $w_j = u_j + iv_j, u_j, v_j \in \mathbb{R}, j = 0, 1, \ldots, n$. Then we have

$$u_0^2 + u_1^2 + \cdots + u_n^2 = v_0^2 + v_1^2 + \cdots + v_n^2,$$

$$u_0v_0 + u_1v_1 + \cdots + u_nv_n = 0.$$

Since $(w_0, w_1, \ldots, w_n) \neq 0$, none of $\vec{u} = (u_0, u_1, \ldots, u_n)$ and $\vec{v} = (v_0, v_1, \ldots, v_n)$ is zero. A diffeomorphism $\varphi : C_n - \{0\} \to K_n$ can be defined as follows:

$$\vec{w} = (w_0, w_1, \ldots, w_n) \mapsto (\vec{p}, \vec{q}) = \left(\frac{\vec{x}}{|\vec{x}|}, \vec{y}\right),$$

with its inverse map given by:

$$\vec{p}, \vec{q}) \mapsto \vec{w} = |\vec{q}| \cdot \vec{p} + i\vec{q}.$$  
(41)

3.3. Hypercomplex structure on $K_2$. In the last two subsections we have constructed two different complex structures on $K_2$. Let us write them down explicitly in the local coordinates $(x, y, p, q)$. For $J_1$ defined in [3,1] it is clear that

$$J_1 \frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad J_1 \frac{\partial}{\partial y} = -\frac{\partial}{\partial x}, \quad J_1 \frac{\partial}{\partial p} = -\frac{\partial}{\partial q}, \quad J_1 \frac{\partial}{\partial q} = \frac{\partial}{\partial p}.$$  

Using the diffeomorphism $\varphi$ defined in [3,2] we get a complex structure $J_2$. We first use the Moser map to get

$$\xi_0 = px + qy, \quad \xi_1 = \frac{p^2 + q^2 + 1}{2} x - (px + qx)p, \quad \xi_2 = \frac{p^2 + q^2 + 1}{2} y - (px + qy)q,$$

$$x_0 = \frac{p^2 + q^2 - 1}{p^2 + q^2 + 1}, \quad x_1 = \frac{2p}{p^2 + q^2 + 1}, \quad x_2 = \frac{2q}{p^2 + q^2 + 1}.$$  

and we have

$$|\xi| = \frac{p^2 + q^2 + 1}{2} \sqrt{x^2 + y^2},$$

so we get

$$w_0 = \sqrt{\frac{x^2 + y^2}{2}} (p^2 + q^2 - 1) + i(px + qy),$$

$$w_1 = \sqrt{\frac{x^2 + y^2}{2}} p + i\left(\frac{p^2 + q^2 + 1}{2} x - (px + qy)p\right),$$

$$w_2 = \sqrt{\frac{x^2 + y^2}{2}} q + i\left(\frac{p^2 + q^2 + 1}{2} y - (px + qy)q\right).$$
Indeed one can check that

\[ J_2 \frac{\partial}{\partial u_1} = \frac{\partial}{\partial v_1}, \quad J_2 \frac{\partial}{\partial u_2} = \frac{\partial}{\partial v_2} \]

One can check that

\[
J_2 \frac{\partial}{\partial x} = \frac{px - qy}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial y} + \frac{yp + xq}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial x} + \frac{(p^2 + q^2 + 1)}{2(x^2 + y^2)^{1/2}} \frac{\partial}{\partial p},
\]
\[
J_2 \frac{\partial}{\partial y} = \frac{yp + xq}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial x} - \frac{px - qy}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial y} - \frac{p^2 + q^2 + 1}{2(x^2 + y^2)^{1/2}} \frac{\partial}{\partial q},
\]
\[
J_2 \frac{\partial}{\partial p} = 2(x^2 + y^2)^{1/2} \frac{\partial}{\partial x} - \frac{px - qy}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial p} - \frac{yp + xq}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial q},
\]
\[
J_2 \frac{\partial}{\partial q} = 2(x^2 + y^2)^{1/2} \frac{\partial}{\partial y} - \frac{yp + xq}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial p} + \frac{px - qy}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial q}.
\]

One can define a third almost complex structure \( J_3 \) by

\[ J_3 = J_1 J_2. \]

Indeed one can check that

\[
J_3 \frac{\partial}{\partial x} = \frac{px - qy}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial y} - \frac{yp + xq}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial x} + \frac{(p^2 + q^2 + 1)}{2(x^2 + y^2)^{1/2}} \frac{\partial}{\partial q} = -J_2 \frac{\partial}{\partial y} = -J_2 J_1 \frac{\partial}{\partial x},
\]
\[
J_3 \frac{\partial}{\partial y} = \frac{yp + xq}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial x} + \frac{px - qy}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial y} - \frac{p^2 + q^2 + 1}{2(x^2 + y^2)^{1/2}} \frac{\partial}{\partial p} = J_2 \frac{\partial}{\partial x} = -J_2 J_1 \frac{\partial}{\partial y},
\]
\[
J_3 \frac{\partial}{\partial p} = 2(x^2 + y^2)^{1/2} \frac{\partial}{\partial x} + \frac{px - qy}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial p} - \frac{yp + xq}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial q} = J_2 \frac{\partial}{\partial q} = -J_2 J_1 \frac{\partial}{\partial p},
\]
\[
J_3 \frac{\partial}{\partial q} = -2(x^2 + y^2)^{1/2} \frac{\partial}{\partial x} + \frac{yp + xq}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial q} + \frac{px - qy}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial p} = -J_2 \frac{\partial}{\partial p} = -J_2 J_1 \frac{\partial}{\partial q},
\]

and from these identities it is easy to see that

\[ J_3 = -J_2 J_1, \quad J_3^2 = -1. \]

By the following Lemma, \( J_1, J_2, J_3 \) make \( K \) a hypercomplex manifold.

**Lemma 3.1.** Suppose that \( J_1 \) and \( J_2 \) are two integrable almost complex structures on a manifold \( M \), and if \( J_3 = J_1 J_2 \) satisfies \( J_3 = -J_2 J_1 \) and \( J_3^2 = -1 \), then \( J_3 \) is also integrable.

**Proof.** By Newlander-Nirenberg theorem, an almost complex structure \( J \) is integrable iff its Nijenhuis tensor vanishes, i.e.,

\[ N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0 \]

for any vector fields \( X, Y \) on \( M \). By conditions we have

\[ N_{J_1}(X, Y) = [X, Y] + J_1[J_1 X, Y] + J_1[X, J_1 Y] - [J_1 X, J_1 Y] = 0, \]
\[ N_{J_2}(X, Y) = [X, Y] + J_2[J_2 X, Y] + J_2[X, J_2 Y] - [J_2 X, J_2 Y] = 0. \]

One can derive that

\[ N_{J_3}(X, Y) = N_{J_3}(J_1 X, J_2 Y) \]
by the following computations:

\[ N_{J_3}(X, Y) = [X, Y] + J_3[J_3X, Y] + J_3[X, J_3Y] - [J_3X, J_3Y] \]
\[ = [X, Y] + J_1J_2[J_1J_2X, Y] + J_1J_2[X, J_1J_2Y] - [J_1J_2X, J_1J_2Y] \]
\[ = [X, Y] + J_1J_2[J_1J_2X, Y] + J_1J_2[X, J_1J_2Y] - ([J_2X, J_2Y] + J_1[J_1J_2X, J_1J_2Y] + J_1J_2[X, J_1J_2Y]) \]
\[ = [X, Y] - [J_2X, J_2Y] + J_1J_2[J_1J_2X, Y] + J_1J_2[X, J_1J_2Y] + J_1[J_2J_1X, J_2J_1Y] + J_1[J_2X, J_2J_1Y] \]
\[ = [X, Y] - [J_2X, J_2Y] + J_1J_2[J_1J_2X, Y] + J_1J_2[X, J_1J_2Y] \]
\[ = J_1([J_1X, Y] + J_2[J_2J_1X, Y] + J_2[J_1X, J_2Y]) \]
\[ = J_1([J_1X, Y] + J_2[J_2J_1X, Y] + J_2[X, J_2J_1Y]) \]
\[ = [X, Y] - [J_2X, J_2Y] + J_1([J_1X, Y] + J_2[J_1X, J_2Y]) \]
\[ = J_1([J_1X, Y] + J_2[J_2J_1X, Y] + J_2[J_1X, J_2Y]) \]
\[ = [J_1X, J_1Y] - [J_2X, J_2Y] + J_1J_2[J_1X, J_2Y] + J_1J_2[J_2X, J_1Y] \]
\[ = N_{J_3}(J_1X, J_1Y). \]

In the same fashion, one gets

\[ N_{J_3}(X, Y) = N_{J_3}(J_2X, J_2Y). \]

Combining (40) and (47), one gets:

\[ N_{J_3}(X, Y) = N_{J_3}(J_3X, J_3Y). \]

Note \( N_{J_3}(J_3X, J_3Y) = -N_{J_3}(X, Y) \), so we get

\[ N_{J_3}(X, Y) = 0. \]

Let us now understand the hypercomplex structure on \( K_2 \) from (28). This formula gives a 2 : 1 covering map from \( \mathbb{R}^4 - \{0\} \) to \( K_2 \). We will call this map the Levi-Civita map. Each identification of \( \mathbb{R}^4 \) with the space \( \mathbb{H} \) of quaternion numbers gives \( \mathbb{R}^4 \) a hypercomplex structure.

**Proposition 3.1.** Let \( \xi, \eta, \omega, \chi \) be linear coordinates on \( \mathbb{R}^4 \), consider the 2 : 1 map from \( \mathbb{R}^4 - \{0\} \rightarrow K_2 \) defined in the \((x, y, p, q)\)-coordinate patch by:

\[ x + iy = (\xi + i\eta)^2, \quad p - iq = \frac{\omega - i\chi}{\xi + i\eta}. \]

then the hypercomplex structure \( J_1, J_2, J_3 \) corresponds to the following hypercomplex structure on \( \mathbb{R}^4 \):

\[ J_1 \partial \xi = \partial \eta, \quad J_1 \partial \eta = -\partial \xi, \quad J_1 \partial \omega = -\partial \chi, \quad J_1 \partial \chi = \partial \omega, \]
\[ J_2 \partial \xi = -\partial \omega, \quad J_2 \partial \eta = -\partial \chi, \quad J_2 \partial \omega = \partial \xi, \quad J_2 \partial \chi = \partial \eta, \]
\[ J_3 \partial \xi = \partial \chi, \quad J_3 \partial \eta = -\partial \omega, \quad J_3 \partial \omega = \partial \eta, \quad J_3 \partial \chi = -\partial \xi. \]

**Proof.** First we have

\[ x = \xi^2 - \eta^2, \quad y = 2\xi\eta, \quad p = \frac{\xi\omega - \eta\chi}{\xi^2 + \eta^2}, \quad q = \frac{\xi\chi + \eta\omega}{\xi^2 + \eta^2}. \]
It follows that
\[
\begin{align*}
\partial_\xi &= 2\xi \partial_x + 2\eta \partial_y + \frac{-\omega \xi^2 + \omega \eta^2 + 2\xi \eta \chi}{(\xi^2 + \eta^2)^2} \partial_p - \frac{\chi \xi^2 - \chi \eta^2 + 2\xi \eta \omega}{(\xi^2 + \eta^2)^2} \partial_q, \\
\partial_\eta &= -2\eta \partial_x + 2\xi \partial_y - \frac{\chi \xi^2 - \chi \eta^2 + 2\xi \eta \omega}{(\xi^2 + \eta^2)^2} \partial_p - \frac{-\omega \xi^2 + \omega \eta^2 + 2\xi \eta \chi}{(\xi^2 + \eta^2)^2} \partial_q, \\
\partial_\omega &= \frac{\xi}{\xi^2 + \eta^2} \partial_p + \frac{\eta}{\xi^2 + \eta^2} \partial_q, \\
\partial_\chi &= \frac{-\eta}{\xi^2 + \eta^2} \partial_p + \frac{\xi}{\xi^2 + \eta^2} \partial_q.
\end{align*}
\]

It is then straightforward to check (51). To check (52), it suffices to note:
\[
J_2 \partial_\omega = \frac{-\eta}{\xi^2 + \eta^2} J_2 \partial_p + \frac{\xi}{\xi^2 + \eta^2} J_2 \partial_q
\]
\[
= \frac{\eta}{\xi^2 + \eta^2} \left(2(x^2 + y^2)^{1/2} \frac{\partial}{\partial x} - \frac{px - qy}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial p} - \frac{py + qx}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial q} \right)
\]
\[
+ \frac{\xi}{\xi^2 + \eta^2} \left(2(x^2 + y^2)^{1/2} \frac{\partial}{\partial y} - \frac{y_p + xq}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial p} + \frac{xp - yq}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial q} \right)
\]
\[
= 2\xi \partial_x + 2\eta \partial_y - \frac{-\omega \xi^2 + \omega \eta^2 + 2\chi \eta \chi}{(\xi^2 + \eta^2)^2} \partial_p - \frac{\chi \xi^2 - \chi \eta^2 + 2\xi \eta \omega}{(\xi^2 + \eta^2)^2} \partial_q
\]
\[
= \partial_\xi,
\]
\[
J_2 \partial_\chi = \frac{-\eta}{\xi^2 + \eta^2} J_2 \partial_p + \frac{\xi}{\xi^2 + \eta^2} J_2 \partial_q
\]
\[
= \frac{-\eta}{\xi^2 + \eta^2} \left(2(x^2 + y^2)^{1/2} \frac{\partial}{\partial x} - \frac{px - qy}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial p} - \frac{py + qx}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial q} \right)
\]
\[
+ \frac{\xi}{\xi^2 + \eta^2} \left(2(x^2 + y^2)^{1/2} \frac{\partial}{\partial y} - \frac{y_p + xq}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial p} + \frac{xp - yq}{(x^2 + y^2)^{1/2}} \frac{\partial}{\partial q} \right)
\]
\[
= -2\eta \partial_x + 2\xi \partial_y - \frac{-wx^2 - wy^2 + 2xyz}{2(x^2 + y^2)^2} \partial_p - \frac{-xz^2 + 2y^2 + 2xyw}{2(x^2 + y^2)^2} \partial_q
\]
\[
= \partial_\eta.
\]

Finally (53) follows from (51) and (52).

3.4. **Kepler manifold $K_3$ as a complex manifold.** In this subsection we recall some well-known resolutions of the conifold singularity of $C_3$. Since $K_3 \cong C_3 - \{0\}$, one can use the resolutions to describe $K_3$.

Blow up $\mathbb{C}^3$ at the origin, and consider the strict transform of the quadric conifold $C_3$. The singular point by the smooth quadric in $\mathbb{P}^3$
\[
\{(w_0 : w_1 : w_2 : w_3) \in \mathbb{P}^3 \mid w_0^2 + w_1^2 + w_2^2 + w_3^2 = 0\},
\]
which is a copy of $\mathbb{P}^1 \times \mathbb{P}^1$. The total space of strict transform is isomorphic to the total space $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$, therefore one gets an isomorphism
\[
K_3 \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) - \mathbb{P}^1 \times \mathbb{P}^1.
\]
For later use, we need to explicit write down such an isomorphism. First let
\[
\begin{align*}
z_1 &= w_0 + iw_1, \\
z_2 &= w_0 - iw_1, \\
z_3 &= iw_2 + w_3, \\
z_4 &= iw_2 - w_3.
\end{align*}
\]
Then one has
\begin{equation}
(57) \quad z_1 z_2 = z_3 z_4.
\end{equation}
On \( K_3 \), all of \( z_1, \ldots, z_n \) are not zero, and one can define four local coordinate patches with local coordinates as follows:
\begin{align}
(58) \quad & \alpha_{1,1} = \frac{z_3}{z_1}, \quad \alpha_{1,2} = \frac{z_4}{z_1}, \quad \beta_1 = \frac{z_1}{z_1} \\
(59) \quad & \alpha_{2,1} = \frac{z_4}{z_2}, \quad \alpha_{2,2} = \frac{z_4}{z_2}, \quad \beta_2 = \frac{z_2}{z_2} \\
(60) \quad & \alpha_{3,1} = \frac{z_1}{z_3}, \quad \alpha_{3,2} = \frac{z_2}{z_3}, \quad \beta_3 = \frac{z_3}{z_3} \\
(61) \quad & \alpha_{4,1} = \frac{z_2}{z_4}, \quad \alpha_{4,2} = \frac{z_1}{z_4}, \quad \beta_4 = \frac{z_4}{z_4}.
\end{align}
These exactly correspond to four local coordinate patches on \( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \). This resolution is not a minimal one. There are two different minimal resolutions obtained from this one blowing down along each copy of \( \mathbb{P}^1 \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \), and they are said to be related to each other by flop. Correspondingly, there are two different ways to get an isomorphism of the form:
\begin{equation}
(62) \quad K_3 \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) - \mathbb{P}^1.
\end{equation}
Indeed, one can define the following local coordinate patches on \( K_3 \):
\begin{align}
(63) \quad & \alpha_1 = \frac{z_3}{z_1}, \quad \beta_{1,1} = \frac{z_1}{z_1} \\
(64) \quad & \alpha_2 = \frac{z_4}{z_2}, \quad \beta_{2,1} = \frac{z_3}{z_3}
\end{align}
they correspond to two local coordinate patches on \( \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \), or one can take alternatively:
\begin{align}
(65) \quad & \alpha_3 = \frac{z_1}{z_3}, \quad \beta_{3,2} = \frac{z_2}{z_2} \\
(66) \quad & \alpha_4 = \frac{z_2}{z_4}, \quad \beta_{4,1} = \frac{z_4}{z_4}.
\end{align}

For later use, we need the following:

**Proposition 3.2.** On \( K_3 \) the following identities hold:
\begin{equation}
(67) \quad \sum_{j=0}^{3} |w_j|^2 = \frac{1}{2} \sum_{j=1}^{4} |z_j|^2,
\end{equation}
for \( k = 1, \ldots, 4 \),
\begin{equation}
(68) \quad (1 + |\alpha_k|^2)(|\beta_{k,1}|^2 + |\beta_{k,2}|^2) = (1 + |\alpha_{k,1}|^2)(1 + |\alpha_{k,2}|^2)|\beta_k|^2 = \sum_{j=1}^{4} |z_j|^2.
\end{equation}

**Proof.** Write \( w_j \) and \( z_j \) in terms of their real parts and imaginary parts: \( w_j = u_j + \sqrt{-1} v_j, \ z_j = x_j + \sqrt{-1} y_j \). By (50),
\begin{align}
x_1 &= u_0 - v_1, \quad y_1 = v_0 + u_1, \quad x_2 = u_0 + v_1, \quad y_2 = v_0 - u_1, \\
x_3 &= -v_2 + u_3, \quad y_3 = u_2 + v_3, \quad x_4 = -v_2 - u_3, \quad y_4 = u_2 - v_3,
\end{align}
therefore,
\[
\sum_{j=1}^{4} |z_j|^2 = \sum_{j=1}^{4} (x_j^2 + y_j^2) = \sum_{j=0}^{3} (2u_j^2 + 2v_j^2) = 2 \sum_{j=0}^{3} |w_j|^2.
\]
This proves the first formula. The second formula follows easily from \([57]\). □

4. Kepler Metrics on Kepler Manifolds

In this Section we first recall that the symplectic structures on the Kepler manifolds are compatible with their complex structures, hence one obtains Kähler structures on the Kepler manifolds. We will call these metrics the Kepler metrics and show that they are Sasaki metrics on the conormal bundles of the round spheres. We will also relate the Kepler metrics to Sasaki metrics on the unit conormal bundles of the round spheres, which we show are Sasakian.

4.1. Symplectic structure on \(K_2\). The symplectic form on the original phase space \((\mathbb{R}^2 - \{(0, 0)\}) \times \mathbb{R}^2\) is

\[
\omega = dp \wedge dx + dq \wedge dy.
\]
In the complex coordinate \((\alpha_1, \gamma_1)\) it is the real part of the holomorphic volume form

\[
\Omega = d\alpha_1 \wedge d\gamma_1.
\]
This form is defined everywhere on \(\mathcal{O}_{p_1}(-2)\) and is nowhere vanishing, indeed, in the other coordinate patch with local coordinate \((\alpha_2, \gamma_2)\), one has

\[
\Omega = -d\alpha_2 \wedge d\gamma_2.
\]
By taking the real part of \(\Omega\), one gets a symplectic form, still denoted by \(\omega\), on the whole space of \(K_2\). By an easy computation we get

**Proposition 4.1.** The pullback to \(\mathbb{R}^4\) of symplectic form \(\omega\) on \(K_2\) is

\[
\varphi_C^* \omega = 2d\omega \wedge d\xi + 2d\chi \wedge d\eta.
\]

4.2. Hyperkähler metric on Kepler manifold \(K_2\). On \(K_2\) define \(g\) by

\[
g(X, Y) = \omega(X, J_2 Y).
\]
In the local coordinate patch with local coordinates \((x, y, p, q)\) we have:

\[
g = \frac{p^2 + q^2 + 1}{2(x^2 + y^2)^{1/2}} dx^2 + \frac{2(px - qy)}{(x^2 + y^2)^{1/2}} dx dp + \frac{2(py + qx)}{(x^2 + y^2)^{1/2}} dx dq + \frac{p^2 + q^2 + 1}{2(x^2 + y^2)^{1/2}} dy^2 + \frac{2(py + qx)}{(x^2 + y^2)^{1/2}} dy dp - \frac{2(px - qy)}{(x^2 + y^2)^{1/2}} dy dq + 2(x^2 + y^2)^{1/2} dp^2 + 2(x^2 + y^2)^{1/2} dq^2.
\]
This is a Riemannian metric compatible with \(J_2\), hence it defines a Kähler metric in this patch, hence a Kähler metric on the whole \(K_2\) since this coordinate patch is dense in \(K_2\).

**Proposition 4.2.** The Riemannian metric \(g\) is a hyperkähler metric with respect to \(J_2, J_3, J_1\).
Proof. For $a = 1, 2, 3$, let

$$\omega_{Ja}(X, Y) = g(J_a X, Y).$$

In the $(x, y, p, q)$ coordinate patch,

$$\omega_1 = \frac{p^2 + q^2 + 1}{2(x^2 + y^2)^{1/2}} dx \wedge dy + \frac{py + qx}{(x^2 + y^2)^{1/2}} dx \wedge dp - \frac{px - qy}{(x^2 + y^2)^{1/2}} dx \wedge dq,$$

$$\omega_2 = \omega = dp \wedge dx + dq \wedge dy,$n$$

$$\omega_3 = dx \wedge dq - dy \wedge dp.$$

One can easily see that

$$d\omega_1 = 0,$$

$$(75) \quad \omega_2 + i\omega_3 = d(p - iq) \wedge d(x + iy) = \Omega.$$

This finishes the proof. \qed

It is easy to see that:

$$\omega_1 = \frac{i}{2} \left( \frac{|\alpha_1|^2 + 1}{2|\gamma_1|} d\gamma_1 \wedge d\bar{\gamma}_1 + \frac{\alpha_1 \bar{\gamma}_1}{|\gamma_1|} d\gamma_1 \wedge d\bar{\alpha}_1 - \frac{\bar{\alpha}_1 \gamma_1}{|\gamma_1|} d\bar{\gamma}_1 \wedge d\alpha_1 + 2|\gamma_1| \alpha_1 \wedge d\bar{\alpha}_1 \right),$$

$$= i\partial \bar{\partial}(|\gamma_1|(|\alpha_1|^2 + 1)).$$

Note we have

$$|\gamma_1|(|\alpha_1|^2 + 1) = |\gamma_2|(|\alpha_2|^2 + 1),$$

this defines a global function on $K_2$, which is the Kähler potential for the Kähler form $\omega_1$.

We now pull back everything to $\mathbb{R}^4 - \{0\}$ by the Levi-Civita map. We take the complex coordinates with respect to $J_2$ to be

$$z_1 = \omega + i\xi, \quad z_2 = \chi + i\eta.$$

Then the Kähler form becomes

$$i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2),$$

and the Kähler potential becomes

$$|z_1|^2 + |z_2|^2.$$

4.3. Kähler metrics and Kähler potentials on Kepler Manifolds $K_n$. Recall that the Moser map satisfies the identity (19), so after taking exterior differentials on both sides of this identity one has

$$\sum_{j=0}^{n} dx_j \wedge d\xi_j = \sum_{j=1}^{n} dy_j \wedge d\eta_j.$$

This defines a symplectic structure on $K_n$. As first noted by Rawnsley [35], $\omega$ is the Kähler form of a Riemannian metric with respect to the complex structure $J = J_2$ using the diffeomorphism $K_n \cong C_n^*$, and

$$\omega = 2i\partial \bar{\partial}|\bar{u}|,$$
where $|\vec{u}| = (u_0^2 + \cdots + u_n^2)^{1/2}$. These facts can be checked as follows. We will first work on $\mathbb{C}^{n+1}$ and then restrict to $\mathbb{C}_n^*$. On $\mathbb{C}^{n+1}$ one has

$$\omega = \sum_{j=0}^n d{x_j} \wedge d{\xi_j} = \sum_{j=0}^n d \left( \sum_{k=0}^n \frac{u_j}{(\sum_{k=0}^n u_k^2)^{1/2}} \right) \wedge dv_j,$$

and so (82) can be checked directly, and using the fact that $J\partial_{u_j} = \partial_{v_j}$, $J\partial_{v_j} = -\partial_{u_j}$, $j = 0, \ldots, n$, (84) one defines $g$ by (73):

$$g = \frac{1}{(\sum_{k=0}^n u_k^2)^{1/2}} \sum_{j=0}^n du_j^2 - \frac{1}{(\sum_{k=0}^n u_k^2)^{3/2}} \left( \sum_{j=0}^n u_j du_j \right)^2$$

$$+ \frac{1}{(\sum_{k=0}^n u_k^2)^{1/2}} \sum_{j=0}^n dv_j^2 - \frac{1}{(\sum_{k=0}^n u_k^2)^{3/2}} \left( \sum_{j=0}^n u_j dv_j \right)^2.$$

This is not positive definite on $\mathbb{C}^{n+1}$, and we have to restrict $g$ to $\mathbb{C}_n^*$. Let $u = |\vec{u}| = (u_0^2 + u_1^2 + \cdots + u_n^2)^{1/2}$, and set

$$u_j = u\hat{u}_j, \quad v_j = u\hat{v}_j, \quad j = 0, 1, \ldots, n,$$

then we have:

$$\sum_{j=0}^n \hat{u}_j^2 = \sum_{j=0}^n \hat{v}_j^2 = 1,$$

$$\sum_{j=0}^n \hat{u}_j \hat{v}_j = 0.$$

and also

$$\sum_{j=0}^n du_j^2 = \sum_{j=0}^n (d(u\hat{u}_j))^2 = \sum_{j=0}^n (\hat{u}_j du + u d\hat{u}_j)^2$$

$$= \sum_{j=0}^n \hat{u}_j^2 du^2 + 2udu \sum_{j=0}^n \hat{u}_j d\hat{u}_j + u^2 \sum_{j=0}^n d\hat{u}_j^2$$

$$= du^2 + u^2 \sum_{j=0}^n d\hat{u}_j^2,$$

$$\sum_{j=0}^n dv_j^2 = \sum_{j=0}^n (d(u\hat{v}_j))^2 = \sum_{j=0}^n (\hat{v}_j du + u d\hat{v}_j)^2$$

$$= \sum_{j=0}^n \hat{v}_j^2 du^2 + 2udu \sum_{j=0}^n \hat{v}_j d\hat{v}_j + u^2 \sum_{j=0}^n d\hat{v}_j^2$$

$$= du^2 + u^2 \sum_{j=0}^n d\hat{v}_j^2.$$
\[
\begin{align*}
\sum_{j=0}^{n} u_j d u_j &= u d u, \\
\sum_{j=0}^{n} u_j d v_j &= \sum_{j=0}^{n} u \hat{u}_j d (\hat{u}_j) = \sum_{j=0}^{n} u \hat{u}_j \hat{v}_j d u + \sum_{j=0}^{n} u^2 \hat{u}_j \hat{v}_j d v_j \\
&= u^2 \sum_{j=0}^{n} \hat{u}_j \hat{v}_j.
\end{align*}
\]

It follows that
\[
g = \frac{1}{u} (d u^2 + u^2 \sum_{j=0}^{n} d \hat{u}_j^2) - \frac{1}{u^3} (u d u)^2 \\
+ \frac{1}{u} \sum_{j=0}^{n} (d u^2 + u^2 \sum_{j=0}^{n} d \hat{v}_j^2) - \frac{1}{u^3} \left( u^2 \sum_{j=0}^{n} \hat{u}_j \hat{v}_j \right)^2 \\
= \frac{1}{u} d u^2 + u \left( \sum_{j=0}^{n} d \hat{u}_j^2 + \sum_{j=0}^{n} d \hat{v}_j^2 - \left( \sum_{j=0}^{n} \hat{u}_j \hat{v}_j \right)^2 \right).
\]

In the following two subsections we will show that
\[
(89) \quad h = \sum_{j=0}^{n} \hat{u}_j^2 + \sum_{j=0}^{n} \hat{v}_j^2 - \left( \sum_{j=0}^{n} \hat{u}_j \hat{v}_j \right)^2 = \sum_{j=0}^{n} \hat{u}_j^2 + \sum_{j=0}^{n} \hat{v}_j^2 - \left( \sum_{j=0}^{n} \hat{v}_j \hat{u}_j \right)^2
\]
is the Sasaki metric on the unit conormal bundle of \( S^n \).

Let \( u = \frac{\sqrt{2}}{2} \), since we have
\[
(90) \quad g = \frac{1}{\sqrt{2}} \left( d r^2 + \sqrt{2} h \right)
\]
is a Kähler metric, \( h \) is Sasakian (cf. Sparks [43]).

We note that in the same way one can define Kähler metrics on deformed conifolds.

4.4. Reformulation of \( h \) by the Moser map. We now understand \( h \) by the Moser map. We take \( x_j = \hat{u}_j \) and \( \xi_j = \hat{v}_j \),
\[
(91) \quad h = \left( \frac{d |y|^2 - 1}{|y|^2 + 1} \right)^2 + \sum_{j=1}^{n} \left( \frac{2y^j}{|y|^2 + 1} \right)^2 \\
+ (d (\bar{\eta} \cdot y))^2 + \sum_{j=1}^{n} \left( \frac{d \left( |y|^2 + 1 \right) \cdot \eta_j - (\bar{\eta} \cdot y) \cdot y_j}{2} \right)^2 \\
- \left( \frac{|y|^2 - 1}{|y|^2 + 1} d (\bar{\eta} \cdot y) + \sum_{j=1}^{n} \frac{2y^j}{|y|^2 + 1} (d \frac{|y|^2 + 1}{2} \cdot \eta_j - (\bar{\eta} \cdot y) \cdot y_j) \right)^2.
\]

The first line on the right-hand side of (91) can be simplified to be
\[
I = \frac{4 \sum_{j=1}^{n} dy^j}{(\sum_{j=1}^{n} y_j^2 + 1)^2}.
\]
The third line on the right-hand side of (91) can be simplified as follows:

\[ III = - \left( \frac{|\vec{y}|^2 - 1}{|\vec{y}|^2 + 1} d(\vec{n} \cdot \vec{y}) \right. \]
\[ + \sum_{j=1}^{n} \frac{2y_j}{|\vec{y}|^2 + 1} \left( \frac{|\vec{y}|^2 + 1}{2} d\eta_j + \frac{1}{2} \eta_j d|\vec{y}|^2 - y_j d(\vec{n} \cdot \vec{y}) - (\vec{n} \cdot \vec{y}) d\eta_j \right) \right) \left. \right)^2 \]
\[ = - \left( \frac{|\vec{y}|^2 - 1}{|\vec{y}|^2 + 1} d(\vec{n} \cdot \vec{y}) \right. \]
\[ + \sum_{j=1}^{n} \eta_j d\eta_j + \frac{|\vec{y}|^2 - 1}{|\vec{y}|^2 + 1} d|\vec{y}|^2 \right)^2 \]
\[ = - \left( -d(\vec{n} \cdot \vec{y}) + \sum_{j=1}^{n} \eta_j d\eta_j \right)^2 = \left( \sum_{j=1}^{n} \eta_j d\eta_j \right)^2 \].

The second line on the right-hand side of (91) is more complicated and will be treated in steps as follows: First we expand the square in the second term:

\[ II = (d(\vec{n} \cdot \vec{y}))^2 + \sum_{j=1}^{n} \left( \frac{|\vec{y}|^2 + 1}{2} d\eta_j + \frac{1}{2} \eta_j d|\vec{y}|^2 - y_j d(\vec{n} \cdot \vec{y}) - (\vec{n} \cdot \vec{y}) d\eta_j \right) \left. \right)^2 \]
\[ = (d(\vec{n} \cdot \vec{y}))^2 + \left( \frac{1 + |\vec{y}|^2}{4} \sum_{j=1}^{n} \eta_j^2 + \frac{1}{4} |\eta|^2 (d|\vec{y}|^2)^2 + |\vec{y}|^2 (d(\vec{n} \cdot \vec{y}))^2 \right. \]
\[ + \left( \vec{n} \cdot \vec{y} \right)^2 \left. \right) \left( \sum_{j=1}^{n} d\eta_j \right) + \frac{|\vec{y}|^2 + 1}{4} d|\eta|^2 d|\vec{y}|^2 - (|\vec{y}|^2 + 1) \sum_{j=1}^{n} y_j d\eta_j \cdot d(\vec{n} \cdot \vec{y}) \]
\[ - (|\vec{y}|^2 + 1) \cdot (\vec{n} \cdot \vec{y}) \cdot \sum_{j=1}^{n} d\eta_j \cdot d|\vec{y}|^2 - (\vec{n} \cdot \vec{y}) \cdot d|\vec{y}|^2 \cdot d(\vec{n} \cdot \vec{y}) \]
\[ - (\vec{n} \cdot \vec{y}) \cdot \sum_{j=1}^{n} \eta_j d\eta_j \cdot d|\vec{y}|^2 - (\vec{n} \cdot \vec{y}) \cdot d(\vec{n} \cdot \vec{y}) d|\vec{y}|^2. \]

Next we simplify the above expression as follows:

\[ II = \left( \frac{1 + |\vec{y}|^2}{4} \sum_{j=1}^{n} d\eta_j^2 + \frac{1}{4} |\eta|^2 (d|\vec{y}|^2)^2 + (1 + |\vec{y}|^2) (d(\vec{n} \cdot \vec{y}))^2 \right. \]
\[ + \left( \vec{n} \cdot \vec{y} \right)^2 \sum_{j=1}^{n} d\eta_j^2 + \frac{|\vec{y}|^2 + 1}{4} d|\eta|^2 d|\vec{y}|^2 - (|\vec{y}|^2 + 1) \cdot \sum_{j=1}^{n} y_j d\eta_j \cdot d(\vec{n} \cdot \vec{y}) \]
\[ - (|\vec{y}|^2 + 1) \cdot (\vec{n} \cdot \vec{y}) \cdot \sum_{j=1}^{n} d\eta_j \cdot d|\vec{y}|^2 - (\vec{n} \cdot \vec{y}) \cdot \sum_{j=1}^{n} \eta_j d\eta_j \cdot d|\vec{y}|^2 \]
\[ = A + B + C, \]

where we have split the terms according to their types:

\[ A = \frac{1 + |\vec{y}|^2}{4} \sum_{j=1}^{n} d\eta_j^2, \]
\[ B = \frac{|\vec{y}|^2 + 1}{4} \cdot d|\eta|^2 d|\vec{y}|^2 + (|\vec{y}|^2 + 1) \cdot \sum_{j=1}^{n} \eta_j dy_j \cdot \sum_{k=1}^{n} y_k d\eta_k \]

\[ - (|\vec{y}|^2 + 1) \cdot (\vec{\eta} \cdot \vec{y}) \cdot \sum_{j=1}^{n} d\eta_j dy_j \]

\[ = (|\vec{y}|^2 + 1) \sum_{j=1}^{n} d\eta_j \cdot \left( \sum_{k=1}^{n} (\eta_j y_k + \eta_k y_j) dy_k - (\vec{\eta} \cdot \vec{\eta}) dy_j \right). \]

\[ C = \frac{1}{4} |\eta|^2 (d|\vec{y}|^2)^2 + (\vec{\eta} \cdot \vec{y}) \cdot \sum_{j=1}^{n} dy_j^2 \]

\[ + (|\vec{y}|^2 + 1) \cdot \left( \sum_{j=1}^{n} \eta_j dy_j \right)^2 - (\vec{\eta} \cdot \vec{y}) \cdot \sum_{j=1}^{n} \eta_j dy_j \cdot d|\vec{y}|^2 \]

\[ = \sum_{j=1}^{n} \left( - (\vec{\eta} \cdot \vec{y}) dy_j + \sum_{k=1}^{n} (y_k \eta_j + y_j \eta_k) dy_k \right)^2 + \left( \sum_{j=1}^{n} \eta_j dy_j \right)^2. \]

Combining the above terms together, we get:

\[ h = \frac{4 \sum_{j=1}^{n} dy_j^2}{(\sum_{j=1}^{n} y_j^2 + 1)^2} + \frac{(1 + |\vec{y}|^2)^2}{4} \sum_{j=1}^{n} d\eta_j^2 \]

\[ + (|\vec{y}|^2 + 1) \sum_{j=1}^{n} d\eta_j \cdot \left( \sum_{k=1}^{n} (\eta_j y_k + \eta_k y_j) dy_k - (\vec{\eta} \cdot \vec{\eta}) dy_j \right) \]

\[ + \sum_{j=1}^{n} \left( - (\vec{\eta} \cdot \vec{y}) dy_j + \sum_{k=1}^{n} (y_k \eta_j + y_j \eta_k) dy_k \right)^2 \]

\[ = \frac{4 \sum_{j=1}^{n} dy_j^2}{(\sum_{j=1}^{n} y_j^2 + 1)^2} \]

\[ + \frac{(1 + |\vec{y}|^2)^2}{4} \sum_{j=1}^{n} d\eta_j + \frac{2}{(|\vec{y}|^2 + 1)} \left( \sum_{k=1}^{n} (\eta_j y_k + \eta_k y_j) dy_k - (\vec{\eta} \cdot \vec{\eta}) dy_j \right)^2. \]

To summarize we have proved the following:

**Proposition 4.3.** On \( T^*S^n \), the expression \( h \) defined in (91) can be expressed in local coordinates \( \{y_i, \eta_i\} \) as follows:

\[ h = \frac{4 \sum_{j=1}^{n} dy_j^2}{(\sum_{j=1}^{n} y_j^2 + 1)^2} + \frac{(1 + |\vec{y}|^2)^2}{4} \sum_{j=1}^{n} D\eta_j, \]

where

\[ D\eta_j = d\eta_j + \frac{2}{(|\vec{y}|^2 + 1)} \left( \sum_{k=1}^{n} (\eta_j y_k + \eta_k y_j) dy_k - (\vec{\eta} \cdot \vec{\eta}) dy_j \right). \]

4.5. **Sasaki metric.** To under \( h \) in last Proposition let us recall the *Sasaki metric* on the tangent bundle of a Riemannian manifold \( (M^n, g) \). Suppose in a local coordinate patch \( U \) with local coordinates \( \{x^i\} \) the Riemannian metric of \( M \) is
Using the Riemannian metric one can define an isomorphism
\begin{equation}
\frac{d\sigma}{dx^i} = \Gamma^k_{ij} dx^j \otimes \frac{d}{dx^k}.
\end{equation}
Denote by $\nabla$ the Levi-Civita connection of $(M, g)$,
\begin{equation}
\nabla \frac{\partial}{\partial x^i} = \Gamma^k_{ij} dx^j \otimes \frac{d}{dx^k}.
\end{equation}
Then
\begin{equation}
\nabla(\frac{\partial}{\partial x^i}) = dv^i \otimes \frac{\partial}{\partial x^i} + v^i \nabla \frac{\partial}{\partial x^i} = Dv^i \otimes \frac{\partial}{\partial x^i},
\end{equation}
where $Dv^i$ means the covariant differential of $v^i$, i.e.
\begin{equation}
Dv^i = dv^i + \Gamma^i_{jk} v^j dx^k.
\end{equation}
Similarly, we have
\begin{equation}
\nabla(p_i dx^i) = dp_i \otimes dx^i + p_i \nabla dx^i = Dp_j \otimes dx^j,
\end{equation}
where $Dp_i$ means the covariant differential of $p_i$, i.e.
\begin{equation}
Dp_j = dp_j - \Gamma^i_{jk} p_i dx^k.
\end{equation}
The Sasaki metric of $TM$ is given in $\pi^{-1}(U)$ by the quadratic form
\begin{equation}
d\sigma^2 = g_{jk}(x) dx^j dx^k + g_{jk}(x) Dv^j Dv^k.
\end{equation}
It can also be written as:
\begin{equation}
d\sigma^2 = g_{ij}(x) dx^i dx^j + g_{ij}(x) \Gamma^i_{ab} \Gamma^j_{cd} v^a v^c dx^b dx^d
+ 2g_{ij}(x) \Gamma^i_{ab} v^a v^c dx^b dv^d + g_{ij}(x) dv^i dv^j.
\end{equation}
Using the Riemannian metric one can define an isomorphism $\pi: T^* M \rightarrow TM$ by:
\begin{equation}
p_{ij} dx^j \mapsto v^i \frac{\partial}{\partial x^i} = p_j g^{ij} \frac{\partial}{\partial x^i}.
\end{equation}
Using this isomorphism one can pull back the Sasaki metric to $T^* M$. Since we have
\begin{equation}
v^i = g^{ij} p_j,
\end{equation}
it follows that
\begin{align*}
d\sigma^2 &= g_{ij} dx^i dx^j + g_{ij} \Gamma^i_{ab} \Gamma^j_{cd} p_k g^{cl} p_l dx^b dx^d
+ 2g_{ij} \Gamma^i_{ab} p_k dx^b d(g^{jl} p_l) + g_{ij} d(g^{ik} p_k) d(g^{jl} p_l)
= g_{ij} dx^i dx^j + g_{ij} \Gamma^i_{ab} \Gamma^j_{cd} p_k g^{cl} p_l dx^b dx^d
+ 2g_{ij} \Gamma^i_{ab} p_k dx^b g^{jl} dp_l + 2g_{ij} \Gamma^i_{ab} p_k dx^b p_l d g^{jl}
+ g_{ij} p_k p_l d g^{ik} d g^{jl} + 2g_{ij} g^{jl} p_k p_l + g_{ij} g^{ik} g^{jl} p_k p_l
g_{ij} dx^i dx^j + g_{ij} (\Gamma^i_{ab} g^{ak} dx^b + d g^{ik}) (\Gamma^j_{cd} g^{cl} dx^d + d g^{jl}) p_k p_l
+ 2(\Gamma^i_{ab} g^{ak} dx^b + d g^{ik}) p_k dp_l + g^{kl} dp_k dp_l.
\end{align*}
Now note we have the following computations:

\[ \Gamma^i_{ab} g^{ak} \, dx^b + dg^{ik} \]

\[ = \frac{1}{2} g^{ij} (g_{ij,b} + g_{b,j,a} - g_{ab,j}) g^{ak} \, dx^b - g^{ij} \cdot g_{ij,b} \, dx^b \cdot g^{ik} \]

\[ = \frac{1}{2} g^{ij} g_{aj,b} g^{ak} \, dx^b + \frac{1}{2} g^{ij} g_{b,j,a} g^{ak} \, dx^b - \frac{1}{2} g^{ij} g_{ab,j} g^{ak} \, dx^b - g^{ij} \cdot g_{ij,b} \, dx^b \cdot g^{ik} \]

\[ = -\frac{1}{2} g^{ij} g_{aj,b} g^{ak} \, dx^b + \frac{1}{2} g^{ij} g_{b,j,a} g^{ak} \, dx^b - \frac{1}{2} g^{ij} g_{ab,j} g^{ak} \, dx^b \]

\[ = -g^{ij} \Gamma^k_{jkl} \, dx^b. \]

It follows that

\[ (104) \quad d\sigma^2 = g_{jk}dx^jdx^k + g^{kl}Dp_kDp_l. \]

Indeed, we have:

\[ d\sigma^2 = g_{ij}dx^i \, dx^j + g_{ij}g^{is}\Gamma^k_{si}dx^b \cdot g^{js}\Gamma^l_{js}dx^c p_kp_l + 2(\Gamma^{ij}_{ab} g^{ak} \, dx^b + dg^{ik})p_kdp_b + g^{kl}dp_k dp_l \]

\[ = g_{ij}dx^i \, dx^j + g^{kl}\Gamma^i_{kab}p_i \Gamma^j_{b}p_j \, dx^a \, dx^b - 2g^{ij}\Gamma^l_{jk}dx^b p_k dp_l + g^{kl}dp_k dp_l \]

\[ = g_{ij}dx^i \, dx^j + g^{ij}Dp_iDp_j. \]

Now we let

\[ (105) \quad g_{ij} = \frac{4}{(1 + |\vec{x}|^2)^2}\delta_{ij}, \quad i, j = 1, \ldots, n, \]

where \( |\vec{x}| = \sum_{j=1}^{n} (x^j)^2 \). Then one has

\[ (106) \quad Dp_j = dp_j - \frac{2\vec{x} \cdot \vec{p}}{1 + |\vec{x}|^2} \, dx^j + \sum_k \frac{2(x^k p_j + x^j p_k)}{1 + |\vec{x}|^2} \, dx^k. \]

where \( \vec{x} \cdot \vec{p} = \sum_i x^i p_i \).

By comparing with Proposition we have

**Proposition 4.4.** On \( T^*S^n \), the expression \( h \) defined in (11) is the Sasaki metric on the round metric on \( S^n \).

4.6. **Ricci curvature of the Kepler metric \( g \) on \( K_n \).** In the above we have shown that the \( g \) on \( K_n \) is Kähler, now we show it is also Ricci-flat. Recall

\[ (107) \quad \omega = 2i\partial\bar{\partial}u, \]

where

\[ u = (w_0^2 + \cdots + w_n^2)^{1/2} = \frac{1}{\sqrt{2}}(|w_0|^2 + |w_1|^2 + \cdots + |w_n|^2)^{1/2}. \]

Since we have

\[ (108) \quad \partial|\vec{w}| = \frac{1}{2|\vec{w}|} \sum_{j=0}^{n} \vec{w}_j \, dw_j, \quad \bar{\partial}|\vec{w}| = \frac{1}{2|\vec{w}|} \sum_{k=0}^{n} w_k \, d\bar{w}_k, \]

after taking differential again, one gets

\[ (109) \quad \omega = \frac{\sqrt{2}}{2|\vec{w}|} \sum_{j=0}^{n} dw_j \wedge d\bar{w}_j - \frac{\sqrt{2}}{4|\vec{w}|^3} \sum_{j=0}^{n} \vec{w}_j \, dw_j \wedge \sum_{k=0}^{n} w_k \, d\bar{w}_k. \]
On the local coordinate patch on $C_n^*$ with $w_1, \ldots, w_n$ as local coordinates, we have
\begin{equation}
 w_0 = \pm i (w_1^2 + \cdots + w_n^2)^{1/2},
\end{equation}
and so we have
\begin{equation}
 \partial w_0 = - \frac{\sum_{j=1}^n w_j dw_j}{w_0}, \quad \bar{\partial} \bar{w}_0 = - \frac{\sum_{k=1}^n \bar{w}_k d\bar{w}_k}{\bar{w}_0}.
\end{equation}

Now the Kähler form is given by:
\begin{equation}
 \omega = \frac{\sqrt{2i}}{2|w|} \sum_{j=1}^n dw_j \wedge d\bar{w}_j + \frac{\sqrt{2i}}{4|w|^2} \left( \sum_{j=1}^n \bar{w}_j dw_j - \bar{w}_0 \sum_{j=1}^n w_j dw_j \right) \wedge \left( \sum_{k=1}^n \bar{z}_k d\bar{w}_k - w_0 \sum_{k=1}^n \bar{w}_k d\bar{w}_k \right)
\end{equation}
\begin{equation}
 = \frac{\sqrt{2i}}{2|w|} \sum_{j=1}^n dw_j \wedge d\bar{w}_j + \frac{\sqrt{2i}}{4|w|^2} \sum_{j=1}^n w_j dw_j \wedge \sum_{k=1}^n \bar{w}_k d\bar{w}_k
\end{equation}
\begin{equation}
 - \frac{\sqrt{2i}}{4|w|^2} \sum_{j=1}^n (w_0 \bar{w}_j - \bar{w}_0 w_j) dw_j \sum_{k=1}^n (\bar{w}_0 w_k - w_0 \bar{w}_k) d\bar{w}_k.
\end{equation}

Write $\omega = \omega_j dw_i \wedge dw_j$, one can get:
\begin{equation}
 \det(\omega_{ij}) = \left( \frac{\sqrt{2i}}{2} \right)^n \frac{1}{2 \sum_{j=1}^n w_j^2 \left( \sum_{j=1}^n w_j^2 \right) + \sum_{j=1}^n |w_j|^2 (n-2)/2},
\end{equation}
using the following easily proved identity:
\begin{equation}
 \det(\delta_{ij} + w_i \bar{w}_j + x z_i \bar{z}_j)_{i,j=1,\ldots,n}
\end{equation}
\begin{equation}
 = (1 + \sum_{j=1}^n |w_j|^2) (1 + x \sum_{k=1}^n |z_k|^2 - x \sum_{j=1}^n \bar{z}_j w_j)^2.
\end{equation}

The Ricci form of $\omega$ is
\begin{equation}
 \rho = -i \partial \bar{\partial} \log \det(\omega_{ij}) = - \frac{n-2}{2} \partial \bar{\partial} \log \left( \sum_{j=1}^n w_j^2 \sum_{j=1}^n |w_j|^2 \right).
\end{equation}

In particular, when $n = 2$, $\rho = 0$.

5. Kähler Ricci-Flat Metric on Conifolds and Deformed Conifolds

In last Section we have seen that the Kepler metrics on the Kepler manifolds $K_n$ are in general not Kähler Ricci-flat. In this Section we show that it is possible to construct natural Kähler Ricci-flat metrics on conifold and the deformed conifold.

5.1. Some Kähler Ricci-flat metrics on the conifold. Let us consider a Kähler potential of the form:
\begin{equation}
 K = f(t)
\end{equation}
on $C_n^*$, where $t$ is defined by:
\begin{equation}
 t = \sum_{j=0}^n |w_j|^2.
\end{equation}
Since we have
\begin{equation}
\partial t = \sum_{j=1}^{n} \bar{w}_j dw_j - \frac{\bar{w}_0}{w_0} \sum_{j=1}^{n} w_j dw_j = \frac{1}{w_0} \sum_{j=1}^{n} (w_0 \bar{w}_j - \bar{w}_0 w_j) dw_j,
\end{equation}
\begin{equation}
\bar{\partial} t = \sum_{j=1}^{n} w_j d\bar{w}_j - \frac{w_0}{\bar{w}_0} \sum_{k=1}^{n} \bar{w}_k d\bar{w}_k = \frac{1}{\bar{w}_0} \sum_{j=1}^{n} (\bar{w}_0 w_j - w_0 \bar{w}_j) d\bar{w}_j,
\end{equation}
and
\begin{equation}
\partial \bar{\partial} t = \sum_{j=1}^{n} dw_j \wedge d\bar{w}_j + \frac{1}{|w_0|^2} \sum_{j=1}^{n} w_j dw_j \wedge \sum_{k=1}^{n} \bar{w}_k d\bar{w}_k,
\end{equation}
the Kähler form associated with $K$ is then
\[
\omega_K = i \partial \bar{\partial} t = i f'(t) \partial \bar{\partial} t + i f''(t) \partial t \wedge \bar{\partial} t
\]
\[
= i f'(t) \left( \sum_{j=1}^{n} dw_j \wedge d\bar{w}_j + \frac{1}{|w_0|^2} \sum_{j=1}^{n} w_j dw_j \wedge \sum_{k=1}^{n} \bar{w}_k d\bar{w}_k \right)
\]
\[
+ i f''(t) \cdot \frac{1}{|w_0|^2} \sum_{j=1}^{n} (w_0 \bar{w}_j - \bar{w}_0 w_j) dw_j \wedge \sum_{j=1}^{n} (\bar{w}_0 w_j - w_0 \bar{w}_j) d\bar{w}_j.
\]

By the formula \ref{112}, the determinant of the associated Hermitian matrix is:
\[
\prod_{n}(f'(t))^n \cdot \left(1 + \frac{1}{|w_0|^2} \sum_{j=1}^{n} |w_j|^2\right) \cdot \left(1 + \frac{f''(t)}{f'(t)} \frac{1}{|w_0|^2} \sum_{j=1}^{n} |w_0 w_j - w_0 \bar{w}_j|^2\right)
\]
\[
- \frac{f''(t)}{f'(t)} \frac{1}{|w_0|^2} \left| \sum_{j=1}^{n} w_j (w_0 w_j - w_0 \bar{w}_j) \right|^2
\]
\[
= \prod_{n}(f'(t))^n \cdot \frac{1}{|w_0|^2} \left| \sum_{j=0}^{n} |w_j|^2\right| \cdot \left(1 + \frac{f''(t)}{f'(t)} \cdot 2 \sum_{j=0}^{n} |w_j|^2\right) - \frac{f''(t)}{f'(t)} \left(\sum_{j=0}^{n} |w_j|^2\right)^2
\]
\[
= \prod_{n} \frac{1}{|w_0|^2} \left[ t(f'(t))^n + t^2(f'(t))^{n-1} f''(t) \right].
\]

In the above we have used the following computations:
\begin{equation}
\prod_{j=1}^{n} |w_0 w_j - w_0 \bar{w}_j|^2 = \prod_{j=1}^{n} (2|w_0|^2 |w_j|^2 - w_0^2 \bar{w}_j^2 - \bar{w}_0^2 w_j^2) = 2|w_0|^2 \sum_{j=1}^{n} |w_j|^2,
\end{equation}
\begin{equation}
\prod_{j=1}^{n} w_j (w_0 w_j - w_0 \bar{w}_j) = w_0 \prod_{j=1}^{n} |w_j|^2 - w_0 \sum_{j=1}^{n} |w_j|^2 = -w_0 \sum_{j=0}^{n} |w_j|^2,
\end{equation}
and in these equalities we use the equation
\begin{equation}
\sum_{j=0}^{n} |w_0|^2 = 0.
\end{equation}

The Ricci form of $\omega_K$ is
\begin{equation}
\rho_K = -i \partial \bar{\partial} \log \left[ t(f'(t))^n + t^2(f'(t))^{n-1} f''(t) \right].
\end{equation}

To get Kähler Ricci-flat metric, it suffices to have
\begin{equation}
t(f'(t))^n + t^2(f'(t))^{n-1} f''(t) = c
\end{equation}
Let $y(t) = t\phi'(t)$,

\[(124) \qquad y^{n-1}y' = ct^{n-2}\]

Integrating once,

\[(125) \qquad y^n = \frac{nc}{n-1}t^{n-1} + c_1.\]

Rewrite it as follows:

\[(126) \qquad f'(t) = \frac{1}{t} \left( \frac{nc}{n-1} t^{n-1} + c_1 \right)^{1/n},\]

and so

\[(127) \qquad f(t) = \int \frac{1}{t} \left( \frac{nc}{n-1} t^{n-1} + c_1 \right)^{1/n} dt.\]

In particular, $f(t) = Ct^{(n-1)/n}$ is a solution.

### 5.2. Some Kähler Ricci-flat metrics on the deformed conifold.

It is remarkable that similar computations can be done on the deformed conifold $\hat{C}_n(a)$ defined by the equation

\[(128) \qquad \sum_{j=0}^n w_j^2 = a,\]

where $a \in \mathbb{C}^* = \mathbb{C} - \{0\}$. We need to modify (119) and (120):

\[
\sum_{j=1}^n |\bar{w}_0 w_j - w_0 \bar{w}_j|^2 = \sum_{j=1}^n (2|w_0|^2 |w_j|^2 - w_0^2 \bar{w}_j^2 - \bar{w}_j^2 w_0^2)
\]

\[= 2|w_0|^2 \sum_{j=0}^n |w_j|^2 - (a \bar{w}_0^2 + \bar{a} w_0^2),\]

\[
\sum_{j=1}^n w_j (\bar{w}_0 w_j - w_0 \bar{w}_j) = \bar{w}_0 \sum_{j=1}^n w_j^2 - w_0 \sum_{j=1}^n |w_j|^2
\]

\[= a \bar{w}_0 - w_0 \sum_{j=0}^n |w_j|^2.\]
Now let \( i^n(f'(t))^n \cdot \left[ \left( 1 + \frac{1}{|w_0|^2} \sum_{j=1}^n |w_j|^2 \right) \cdot \left( 1 + \frac{f''(t)}{f'(t)} \frac{1}{|w_0|^2} \sum_{j=1}^n |\bar{w}_j w_j - w_0 \bar{w}_j|^2 \right) \right] \)

\[- \frac{f''(t)}{f'(t)} \frac{1}{|w_0|^4} \left| \sum_{j=1}^n w_j (\bar{w}_j w_j - w_0 \bar{w}_j) \right|^2 \]

\[= i^n(f'(t))^n \cdot \frac{1}{|w_0|^2} \sum_{j=0}^n |w_j|^2 \cdot \left( 1 + \frac{f''(t)}{f'(t)} \frac{1}{|w_0|^2} \left( 2|w_0|^2 \sum_{j=0}^n |w_j|^2 - (a \bar{w}_0^2 + \bar{a} w_0^2) \right) \right) \]

\[- \frac{f''(t)}{f'(t)} \frac{1}{|w_0|^4} \left| a \bar{w}_0 - w_0 \sum_{j=0}^n |w_j|^2 \right|^2 \]

\[= i^n(f'(t))^n \cdot \frac{1}{|w_0|^2} \left[ t \cdot \left( 1 + \frac{f''(t)}{f'(t)} \cdot \left( 2t - \frac{(a \bar{w}_0^2 + \bar{a} w_0^2)}{|w_0|^2} \right) \right) \right] \]

\[- \frac{f''(t)}{f'(t)} \frac{1}{|w_0|^4} \left( |a|^2 |w_0|^2 - t(a \bar{w}_0^2 + \bar{a} w_0^2) + t^2 |w_0|^2 \right) \]

\[= i^n \cdot \frac{1}{|w_0|^2} \left[ t(f'(t))^n + (t^2 - |a|^2)(f'(t))^{n-1} f''(t) \right]. \]

Therefore, to get a Kähler Ricci-flat metric, it suffices to have

\[(129) \quad t(f'(t))^n + (t^2 - |a|^2)(f'(t))^{n-1} f''(t) = c. \]

When \( a = 1 \), this was obtained by Stenzel [14]. Let us modify his solution for this case to general \( a \in \mathbb{C}^* \) as follows. Let \( t = |a|x \), and \( g(x) := f(t) \). Then since \( g'(x) = |a|f'(t) \), \( g''(x) = |a|^2 f''(t) \),

\[(130) \quad x(g'(x))^n + (x^2 - 1)(g'(x))^{n-1} g''(x) = c|a|^{n-1}. \]

Now let \( x = \cosh w \), and \( h(w) := g(x) \). Then one has

\[(131) \quad \frac{d}{dw}(h'(w))^n = n|a|^{n-1}(\sinh w)^{n-1}. \]

5.3. Sasakian-Einstein metric. One can redo the calculations in Section 5.1 in the fashion of Section [13]. I.e., one first works on \( \mathbb{C}^{n+1} \), then restricts to \( \mathbb{C}^n \). Then

\[\omega_K = \sqrt{-1} f'(t) \sum_{j=0}^n dw_j \wedge d\bar{w}_j + \sqrt{-1} f''(t) \sum_{j=0}^n \bar{w}_j dw_j \wedge \sum_{k=0}^n w_k d\bar{w}_k. \]

The Riemannian metric is

\[g_K = 2f'(t) \sum_{j=0}^n (du_j^2 + dv_j^2) + 2f''(t) \left( \sum_{j=0}^n (u_j du_j + v_j dv_j) \right)^2 \]

\[+ 2f''(t) \left( \sum_{j=0}^n (u_j dv_j - v_j du_j) \right)^2. \]
By the same computations as in Section 4.3,

\[ g_K = 2f'(t) \cdot \left( 2du^2 + u^2 \sum_{j=0}^{n} (d\hat{u}_j^2 + d\hat{v}_j^2) \right) \]

\[ + 8f''(t)u^2d^2 + 8f'(t)u^4 \left( \sum_{j=0}^{n} \hat{u}_j \hat{v}_j \right)^2. \]

(133)

Note \( u^2 = \frac{1}{2} t \), so

\[ g_K = \left( f'(t) \frac{2}{2t} + \frac{f''(t)}{2} \right) \cdot dt^2 + tf'(t) \sum_{j=0}^{n} (d\hat{u}_j^2 + d\hat{v}_j^2) \]

\[ + 2tf''(t) \left( \sum_{j=0}^{n} \hat{u}_j \hat{v}_j \right)^2. \]

(134)

When \( f(t) = Ct^{(n-1)/n} \),

\[ g_K = C \frac{(n-1)^2}{2n^2} t^{-(n+1)/ndt^2} \]

\[ + C n^{-1} t^{(n-1)/n} \left( \sum_{j=0}^{n} (d\hat{u}_j^2 + d\hat{v}_j^2) - \frac{2}{n} \left( \sum_{j=0}^{n} \hat{u}_j \hat{v}_j \right)^2 \right). \]

(135)

Take \( C = \frac{2n^2}{(n-1)^2} \) and \( r = \sqrt{\frac{2n^2}{(n-1)^2}} \),

\[ g_K = dr^2 + r^2 \left( \sum_{j=0}^{n} (d\hat{u}_j^2 + d\hat{v}_j^2) - \frac{2}{n} \left( \sum_{j=0}^{n} \hat{u}_j \hat{v}_j \right)^2 \right). \]

(136)

As a corollary, this shows that

\[ \sum_{j=0}^{n} (d\hat{u}_j^2 + d\hat{v}_j^2) - \frac{2}{n} \left( \sum_{j=0}^{n} \hat{u}_j \hat{v}_j \right)^2 \]

defines a Sasakian-Einstein metric \([7]\) on the unit conormal bundle of the round sphere \( S^n \).

5.4. A special Kähler-Ricci flow. Consider Kähler-Ricci flow of the form

\[ \frac{d}{ds} \omega_K = \rho_K + \lambda \cdot \omega_K, \]

where \( \lambda \) is some constant, and

\[ \omega_K = \sqrt{-1} \partial \bar{\partial} f_s(t), \]

\[ \rho_K = -i \partial \bar{\partial} \log \left[ t(f'_s(t))^n + t^2(f'_s(t))^{n-1}f''_s(t) \right]. \]

for some family \( \{f_s(t)\} \) of functions in \( t \), parameterized by \( s \). One can consider the following flow in the space of Kähler potentials:

\[ \frac{d}{ds} f_s(t) = - \log \left[ t(f'_s(t))^n + t^2(f'_s(t))^{n-1}f''_s(t) \right] + \lambda f_s(t) + C_1. \]

(138)
6. $U(n)$-Symmetric Kähler Metrics and Hessian Geometry

Let us summarize what we have discussed so far in this paper: (1) Regularizations of the Kepler systems lead to the symplectic manifolds $K_n$. (2) $K_n$ have natural complex structures which can be used to relate them to the $n$-conifolds $C_n$. (3) The complex structures and symplectic structures on $K_n$ are compatible, leading to the Kepler metrics. (4) The Kepler metrics have Kähler potential $\frac{1}{\sqrt{2}}|\vec{w}|$, and this is also the Kähler potential for some Kähler metrics on the deformed conifolds. (5) There are Kähler Ricci-flat metrics on the $K_n$-folds and the deformed conifolds with Kähler potentials of the form $f(|\vec{w}|^2)$, in particular

\[ C|\vec{w}|^2(n-1)/n \]

defines a Kähler Ricci-flat metric on $K_n$. Furthermore, there are several ways to understand $K_n$: (a) holomorphically equivalent to $\mathcal{O}_{P^1}(-1) \oplus \mathcal{O}_{P^1}(-1) - P^1$; (b) holomorphically equivalent to $\mathcal{O}_{P^1 \times P^1}(-1,-1) - P^1 \times P^1$; (c) holomorphically equivalent to $C_3^\ast$. Similarly, there are several ways to understand $K_2$: (a) there is a 2 : 1 covering by $\mathbb{C}^2$; (b) holomorphically equivalent to $\mathcal{O}_{P^1}(-2) - P^1$; (c) holomorphically equivalent to $C_2^\ast$. In the preceding Sections we have discussed the Kähler metrics on $K_2$ from the third point of view, and considered Kähler metrics on the two-dimensional deformed conifolds. In this Section we will take the first two points of view. This will lead to a discussion of the Kähler metrics on the two-dimensional resolved conifold. We will take a more general point of view and follow the approach of construction of Kähler metrics with $U(n)$-symmetry for $n \geq 2$ developed in Duan-Zhou [13, 14], generalizing a construction by LeBrun [28]. This will lead us to the application of symplectic coordinates and Hessian geometry in the noncompact case.

6.1. A simple example. Consider the flat Kähler metric on $\mathbb{C}$:

(139) $\omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} u = \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$,

where $u = |z|^2$. Let $z = re^{i\theta}$, then the Riemannian metric can be written in the following form:

(140) $g = (dr)^2 + r^2 (d\theta)^2 = \frac{(dy)^2}{4y} + y \cdot (d\theta)^2$,

where a new local coordinate $y$ is introduced instead of $r$:

(141) $y = r^2$

One also introduces:

(142) $\psi = y (\ln y - 1)$,

and a dual coordinate $y^\vee$ by

(143) $y^\vee = \frac{\partial \psi}{\partial y} = \ln y$.

It is clear that:

(144) $y = e^{y^\vee}$.

The Legendre transform of $\psi$ is given by:

(145) $\psi^\vee = yy^\vee - \psi = y \frac{\partial \psi}{\partial y} - \psi$.

Then

(146) $\psi^\vee = y = e^{y^\vee}$. 
Now the Kähler form and the Riemannian metric can be rewritten as:

\begin{align}
\omega &= \frac{1}{2} dy \wedge d\theta, \\
g &= \frac{1}{4} \frac{\partial^2 \psi}{\partial y \partial \bar{y}} (dy)^2 + \frac{\partial^2 \psi}{\partial y^\vee \partial y^\vee} (d\theta)^2.
\end{align}

In the rest of this paper, we will generalize this example in various ways and use them to study the Kepler metrics.

6.2. Symplectic coordinates. On \( \mathbb{C}^n - \{0\} \) with linear coordinates \( z_1, \ldots, z_n \) we consider \( U(n) \)-symmetric Kähler metrics. Take the Kähler potential to be a function of the form \( \phi(u) \), where \( u = |z_1|^2 + \cdots + |z_n|^2 \). Then the Kähler form is the \((1,1)\)-form given by:

\begin{equation}
\omega = \sqrt{-1} \partial \bar{\partial} \phi(u) = \sqrt{-1} (\phi'(u) \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i + \phi''(u) \sum_{i=1}^{n} \bar{z}_i dz_i \wedge \sum_{j=1}^{n} z_j d\bar{z}_j).
\end{equation}

Let \( z_i = r_i e^{\sqrt{-1} \theta_i} \), one can rewrite \( \omega \) as:

\begin{equation}
\omega = 2\phi'(u) \sum_{i=1}^{n} r_idr_i \wedge d\theta_i + 2\phi''(u) \sum_{i=1}^{n} r_idr_i \wedge \sum_{j=1}^{n} r_j^2 d\theta_j.
\end{equation}

Let \( T^n \) be the torus subgroup of \( U(n) \) consisting of diagonal unitary matrices. Restricting the \( U(n) \)-action to \( T^n \), one gets a Hamiltonian action with moment map:

\begin{equation}
(y_1, \ldots, y_n) = (|z_1|^2 \phi'(u), \ldots, |z_n|^2 \phi'(u)).
\end{equation}

It is easy to see that

\begin{equation}
\omega = \sum_{i=1}^{n} dy_i \wedge d\theta_i.
\end{equation}

Therefore, \((y_i, \theta_i)\) are action-angle variables, or following Abreu [1] and Donaldson in the compact toric Kähler case, they will be called the symplectic coordinates.

6.3. Kähler potential in symplectic coordinates. Let

\begin{equation}
y := y_1 + \cdots + y_n.
\end{equation}

It is clear that:

\begin{equation}
y = u \cdot \phi'(u).
\end{equation}

This was introduced by LeBrun [25]. See also [13, 14]. Integrating the above differential equation, we get the Kähler potential \( \phi \) as a function of \( u \):

\begin{equation}
\phi = \int \frac{y(u)}{u} du.
\end{equation}

Now under suitable conditions \( u \) is a function of \( y \), and so is \( \phi \). We have:

\begin{equation}
\frac{d\phi}{dy} = \frac{y}{u} \frac{du}{dy} = y \frac{d \log u}{dy},
\end{equation}

and so after integration:

\begin{equation}
\phi = \int y \log u = y \log u - \int \log(u) du.
\end{equation}
This formula expresses the Kähler potential in terms of the symplectic coordinates.

6.4. Riemannian metric in symplectic coordinates. Now let us express the Riemannian metric also in terms of the symplectic coordinates. The Hermitian metric is given by

\begin{equation}
(158) \quad h = 2 \phi'(u) \sum_{i=1}^{n} dz_i \otimes d\bar{z}_i + 2 \phi''(u) \sum_{i=1}^{n} \bar{z}_i dz_i \otimes \sum_{j=1}^{n} z_j d\bar{z}_j,
\end{equation}

and so the Riemannian metric is

\begin{equation}
(159) \quad g = 2 \phi'(u) \sum_{i=1}^{n} ((dr_i)^2 + r_i^2 (d\theta_i)^2) + 2 \phi''(u) \sum_{i,j=1}^{n} r_i r_j (dr_i dr_j + r_i r_j d\theta_i d\theta_j).
\end{equation}

Theorem 6.1. In symplectic coordinates the Riemannian metric \( g \) takes the following form:

\begin{equation}
(160) \quad g = \sum_{i,j=1}^{n} \frac{1}{2} G_{ij} dy_i dy_j + 2 G^{ij} d\theta_i d\theta_j,
\end{equation}

where the coefficients \( G_{ij} \) and \( G^{ij} \) are given by:

\begin{equation}
(161) \quad G_{ij} = \frac{\delta_{ij}}{y_i} - \frac{1}{y} \frac{du}{dy},
\end{equation}

\begin{equation}
(162) \quad G^{ij} = y_i \delta_{ij} + \left( \frac{u}{y^2 \frac{du}{dy}} - \frac{1}{y} \right) y_i y_j.
\end{equation}

Furthermore, the matrices \((G_{ij})\) and \((G^{ij})\) are inverse to each other.

Proof. Note we have

\begin{equation}
(163) \quad r_i = \sqrt{\frac{y_i}{\phi'(u)}}.
\end{equation}

So the Riemannian metric can be written as

\[ g = \frac{1}{2} \sum_{i=1}^{n} y_i \left( \frac{dy_i}{y_i} - \frac{\phi''(u)}{\phi'(u)} \frac{du}{dy} \right)^2 + \frac{1}{2} \phi''(u) (du)^2
\]

\[ + 2 \sum_{i=1}^{n} y_i (d\theta_i)^2 + 2 \phi''(u) \left( \sum_{i=1}^{n} y_i d\theta_i \right)^2. \]

We first rewrite \( \frac{\phi''(u)}{\phi'(u)} du \) and \( \frac{\phi''(u)}{\phi'(u)} \) as follows. Taking logarithmic differential of \( (154) \) one can get:

\begin{equation}
(164) \quad \frac{\phi''(u)}{\phi'(u)} du = \left( \frac{1}{y} - \frac{1}{u} \frac{du}{dy} \right) dy.
\end{equation}

We also have:

\[ \phi''(u) = \frac{1}{u} \cdot u \phi''(u) = \frac{1}{u} ((u \phi'(u))' - \phi'(u)) = \frac{1}{u} \left( \frac{dy}{du} - \frac{y}{u} \right) = \frac{1}{u} \frac{dy}{du} - \frac{y}{u^2}. \]
\[
\phi''(u) = \frac{u \cdot (u \phi''(u))}{(u \phi'(u))^2} = \frac{u \left( \frac{du}{dy} - \frac{u}{y} \right)}{y^2} = \frac{u}{y^2 \frac{du}{dy} - 1},
\]

\[
g = \frac{1}{2} \sum_{i=1}^{n} y_i \left( \frac{dy_i}{y_i} - \left( \frac{1}{y} - \frac{1}{u} \frac{du}{dy} \right) dy \right)^2 + \frac{1}{2} \left( \frac{1}{u \frac{du}{dy} - \frac{u}{y}} \right) \left( \frac{dy}{du} \right)^2
\]

\[
= \frac{1}{2} \sum_{i=1}^{n} \left( \frac{dy_i}{y_i} \right)^2 - \sum_{i=1}^{n} \left( \frac{1}{y} - \frac{1}{u} \frac{du}{dy} \right) dy + \frac{1}{2} \sum_{i=1}^{n} y_i \left( \frac{1}{y} - \frac{1}{u} \frac{du}{dy} \right) \left( \frac{dy}{du} \right)^2
\]

\[
+ \frac{1}{2} \left( \frac{u \frac{du}{dy} - \frac{dy}{du}}{y} \right) \left( \frac{dy}{du} \right)^2
\]

\[
+ 2 \sum_{i=1}^{n} y_i (d\theta_i)^2 + 2 \left( \frac{u}{y^2 \frac{du}{dy} - \frac{u}{y}} \right) \left( \sum_{i=1}^{n} y_i d\theta_i \right)^2.
\]

Using the fact that \( y = \sum_{i=1}^{n} y_i \), one can make a further simplification:

\[
g = \frac{1}{2} \sum_{i=1}^{n} \left( \frac{dy_i}{y_i} \right)^2 + \sum_{i=1}^{n} \left( \frac{1}{y} - \frac{1}{u} \frac{du}{dy} \right) \left( \sum_{i=1}^{n} y_i d\theta_i \right)^2
\]

This proves the first statement. The second statement can be proved as follows:

\[
\sum_{j=1}^{n} G_{ij} G_{jk} = \sum_{j=1}^{n} \left( \frac{\delta_{ij}}{y_i} - \frac{1}{y} + \frac{1}{u} \frac{du}{dy} \right) \left( \frac{dy_j}{y_j} \right) \int \left( \delta_{jk} + \frac{\frac{u}{y^2 \frac{du}{dy} - \frac{u}{y}}}{y_j} y_j y_k - \frac{1}{y} + \frac{1}{u} \frac{du}{dy} \right) \left( \sum_{j=1}^{n} y_j y_k \right)
\]

\[
= \delta_{ik} + \left( \frac{\frac{u}{y^2 \frac{du}{dy} - \frac{u}{y}}}{y} \right) y_k - \frac{1}{y} + \frac{1}{u} \frac{du}{dy} y_k
\]

\[
+ \left( \frac{1}{y} + \frac{1}{u} \frac{du}{dy} \right) \left( \frac{\frac{u}{y^2 \frac{du}{dy} - \frac{u}{y}}}{y} \right) \sum_{j=1}^{n} y_j y_k
\]

\[
= \delta_{ik}.
\]

6.5. The complex potential in symplectic coordinates. It is clear that

\[
(165) \quad G_{ij} = \frac{\partial^2 \psi}{\partial y_i \partial y_j}
\]

for the function \( \psi \) defined by:

\[
(166) \quad \psi = \sum_{i=1}^{n} y_i (\ln y_i - 1) - y(\ln y - 1) + \int \log u(y) dy.
\]

The function \( \psi \) is called the complex potential because

\[
(167) \quad \frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2 \psi}{\partial y_i \partial y_j} dy_j + \sqrt{-1} d\theta_i = \frac{dz_i}{z_i}
\]
is of type $(1, 0)$. Indeed, we have
\[ \frac{dz_i}{z_i} = \frac{dr_i}{r_i} + \sqrt{-1} d\theta_i = \frac{1}{2} \left( \frac{dy_i}{y_i} \cdot \frac{\phi''(u)}{\phi'(u)} du \right) + \sqrt{-1} d\theta_i \]
\[ = \frac{1}{2} \left( \frac{dy_i}{y_i} - \frac{dy}{y} + \frac{du}{u} \right) + \sqrt{-1} d\theta_i \]
\[ = \frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2 \psi}{\partial y_i \partial y_j} dy_j + \sqrt{-1} d\theta_i. \]

6.6. **Legendre transform.** Introduce the dual local coordinates $y_i^\vee$ by
\[ (168) \quad y_i^\vee = \frac{\partial \psi}{\partial y_i}, \]
and introduce a dual potential function:
\[ (169) \quad \psi^\vee = \sum_{i=1}^{n} y_i \frac{\partial \psi}{\partial y_i} - \psi \]
By (166) we get:
\[ (170) \quad y_i^\vee = \log y_i - \log y + \log u(y) = 2 \log r_i, \]
and
\[ (171) \quad \psi^\vee = y \ln u(y) - \int \ln u(y) dy = \phi. \]

**Theorem 6.2.** The Riemannian metric $g$ satisfies:
\[ (172) \quad g = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \psi}{\partial y_i \partial y_j} dy_i dy_j + 2 \sum_{i,j=1}^{n} \frac{\partial^2 \psi^\vee}{\partial y_i^\vee \partial y_j^\vee} d\theta_i d\theta_j. \]
In particular,
\[ (173) \quad G^{ij} = \frac{\partial^2 \psi^\vee}{\partial y_i^\vee \partial y_j^\vee} = \frac{\partial^2 \phi}{\partial y_i^\vee \partial y_j^\vee}. \]

**Proof.** We start with:
\[ \psi = \sum_{j=1}^{n} y_j y_j^\vee - \psi^\vee \]
and take $\frac{\partial}{\partial y_i}$ on both sides to get:
\[ \frac{\partial \psi}{\partial y_i} = y_i^\vee + \sum_{j=1}^{n} y_j \frac{\partial y_j^\vee}{\partial y_i} - \sum_{j=1}^{n} \frac{\partial \psi^\vee}{\partial y_j^\vee} \frac{\partial y_j^\vee}{\partial y_i} \]
\[ = \frac{\partial \psi}{\partial y_i} + \sum_{j=1}^{n} \left( y_j - \frac{\partial \psi^\vee}{\partial y_j^\vee} \right) \frac{\partial^2 \psi}{\partial y_i \partial y_j}. \]
Hence
\[ \sum_{j=1}^{n} \left( y_j - \frac{\partial \psi^\vee}{\partial y_j^\vee} \right) \frac{\partial^2 \psi}{\partial y_i \partial y_j} = 0. \]
Since the matrix \( (\frac{\partial^2 \psi}{\partial y_i \partial y_j})_{i,j=1,\ldots,n} \) is invertible, we have:

\[
(174) \quad y_j = \frac{\partial \psi^\vee}{\partial y_j^\vee}.
\]

Take \( \frac{\partial}{\partial y_i^\vee} \) on both sides:

\[
\frac{\partial y_j}{\partial y_i^\vee} = \frac{\partial^2 \psi^\vee}{\partial y_i^\vee \partial y_j^\vee}.
\]

In other words, the matrix \( (\frac{\partial^2 \psi^\vee}{\partial y_i^\vee \partial y_j^\vee})_{i,j=1,\ldots,n} \) is the matrix \( (\frac{\partial y_j}{\partial y_i^\vee})_{i,j=1,\ldots,n} \), hence it is the inverse matrix of \( (\frac{\partial y_i}{\partial y_j})_{i,j=1,\ldots,n} = (\frac{\partial^2 \psi}{\partial y_i \partial y_j})_{i,j=1,\ldots,n} \), \( \square \)

6.7. **SYZ mirror construction.** Inspired by the Strominger-Yau-Zaslow \[45\] construction, introduce:

\[
(175) \quad g^\vee = n \sum_{i,j=1}^n \left( \frac{\partial^2 \psi^\vee}{\partial y_i^\vee \partial y_j^\vee} dy_i^\vee dy_j^\vee + \frac{\partial^2 \psi}{\partial y_i \partial y_j} d\theta_i^\vee d\theta_j^\vee \right).
\]

Since we have:

\[
(176) \quad \sum_{i,j=1}^n \frac{\partial^2 \psi}{\partial y_i \partial y_j} dy_i dy_j = \sum_{i,j=1}^n \frac{\partial^2 \psi^\vee}{\partial y_i^\vee \partial y_j^\vee} dy_i^\vee dy_j^\vee,
\]

therefore,

\[
(177) \quad g^\vee = \sum_{i,j=1}^n \left( \frac{\partial^2 \psi}{\partial y_i \partial y_j} dy_i dy_j + \frac{\partial^2 \psi^\vee}{\partial y_i^\vee \partial y_j^\vee} d\theta_i^\vee d\theta_j^\vee \right).
\]

This has Kähler potential \( \psi \) and complex potential \( \psi^\vee \). And one can check that

\[
(178) \quad \omega^\vee = \sum_{i=1}^n dy_i^\vee \wedge d\theta_i^\vee.
\]

6.8. **Ricci form in symplectic coordinates.** It is not hard to see that

\[
(179) \quad \frac{\omega^n}{n!} = \sqrt{-1} y'(u) \left( \frac{y(u)}{u} \right)^{n-1} \prod_{i=1}^n dz_i \wedge d\bar{z}_i.
\]

Indeed, because of the \( U(n) \)-symmetry one can restrict to the \( z_1 \)-axis, where the Kähler form can be written as:

\[
(180) \quad \omega = \sqrt{-1} (y'(u)dz_1 \wedge d\bar{z}_1 + \frac{y(u)}{u} \sum_{i=2}^n dz_i \wedge d\bar{z}_i).
\]

Now if \( \omega \) is nondegenerate, by \[179\], the Ricci form is given by

\[
(181) \quad \rho = -\sqrt{-1} \bar{\partial} \partial \Phi(u),
\]

where

\[
(182) \quad \Phi = \log \left[ y'(u) \left( \frac{y(u)}{u} \right)^{n-1} \right].
\]

Similar to \[180\], one has:

\[
(183) \quad \rho = -2\Phi'(u) \sum_{i=1}^n r_i dr_i \wedge d\theta_i - 2\Phi''(u) \sum_{i=1}^n r_i dr_i \wedge \sum_{j=1}^n r_j^2 d\theta_j.
\]
Comparing (183) with (150) it is clear that $\rho = \lambda \omega$ for some constant $\lambda$ if and only if

$$\Phi'(u) = -\lambda \phi'(u).$$

This equation can be solved by quadrature [13]:

$$\int \frac{y^{n-1} dy}{n+1} = \ln u.$$

**Proposition 6.1.** The Ricci form $\rho$ can be expressed in symplectic coordinates by the following formula:

$$\rho = -\sum_{i=1}^{n} \left( \frac{u \Phi'(u)}{y} - \frac{y_{i}}{y} \right) \wedge d\theta_{i}.$$

**Proof.** Similar to the computations for the Riemannian metric we have:

$$\rho = -\Phi'(u) \sum_{i=1}^{n} \frac{y_{i}}{\phi'(u)} \left( \frac{dy_{i}}{y} - \frac{\phi'(u)}{\phi''(u)} du \right) \wedge d\theta_{i}$$

$$= -\Phi'(u) \sum_{i=1}^{n} \left( \frac{u y_{i}}{y} \frac{dy_{i}}{y} - \left( \frac{1}{y} - \frac{1}{u} \frac{du}{dy} \right) dy \right) \wedge d\theta_{i}$$

$$= -\frac{u \Phi'(u)}{y} \sum_{i=1}^{n} dy_{i} \wedge d\theta_{i} + \left( \frac{u \Phi'(u)}{y^{2}} - \Phi'(u) \frac{du}{dy} \right) dy \wedge \sum_{i=1}^{n} d\theta_{i}$$

$$= -\sum_{i=1}^{n} \left( \frac{u \Phi'(u)}{y} - \frac{y_{i}}{y} \right) \wedge d\theta_{i}.$$

Note we have:

$$\frac{u \Phi'(u)}{y} = \frac{u}{y} \frac{d\Phi(u)}{du} = \frac{u}{y} \frac{du}{dy} \log \left[ \frac{1}{u} \left( \frac{y(u)}{u} \right)^{n-1} \right]$$

$$= -\frac{u \frac{du}{dy}}{y} + \frac{(n-1) u}{y^{2}} \frac{du}{dy} - \frac{n-1}{y}.$$

**6.9. A special Kähler-Ricci flow.** Consider Kähler-Ricci flow of the form

$$\frac{d}{dt} \omega = \rho + \lambda \cdot \omega,$$

where $\lambda$ is some constant. Since in our case we have

$$\omega = \sqrt{-1} \partial \bar{\partial} \phi(u),$$

(189)

$$\rho = -\sqrt{-1} \partial \bar{\partial} \Phi(u),$$
one can consider the following flow in the space of Kähler potentials:

\begin{equation}
\frac{d}{dt}\phi = -(\Phi + \lambda \phi + C_1).
\end{equation}

By (182) this is just:

\begin{equation}
\frac{d}{dt}\phi = -\log \left[ (\phi' + u\phi'') \phi^{n-1} \right] + \lambda \phi + C_1.
\end{equation}

6.10. Scalar curvature. The scalar curvature $R$ of $\omega$ is given by:

\begin{equation}
\rho \wedge \omega^{n-1} = R \omega^n.
\end{equation}

By (152) and (186), one gets:

\begin{equation}
R = -\frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left( \frac{u\Phi'(u)}{y} \cdot y_i \right).
\end{equation}

From this one can derive other formula for the scalar curvature in the literature. First of all, the right-hand side of this formula can be rewritten as follows:

\begin{equation}
nR = -\frac{d^2}{dy^2} \left( \frac{u\Phi'(u)}{y} \right) - \frac{n-1}{n} \frac{u u'}{y}.
\end{equation}

This was derived in [14] in the following way. Along the $z_1$-axis we have

\begin{equation}
\rho = -\sqrt{-1} ((u\Phi'(u))'dz_1 \wedge d\bar{z}_1 + \Phi'(u) \sum_{i=2}^{n} dz_i \wedge d\bar{z}_i).
\end{equation}

By (180), (179) and (195), we get

\begin{equation}
(u\Phi')' \left( \frac{y}{u} \right)^{n-1} + (n-1)\Phi' \left( \frac{y}{u} \right)^{n-2} y' = -nRy' \left( \frac{y}{u} \right)^{n-1}.
\end{equation}

This equation can be integrated to get the following:

**Proposition 6.2.** ([14]) A pseudo-Kähler form as in (149) has a constant scalar curvature $R$ if and only if

\begin{equation}
\int \frac{y^{n-1}dy}{-\frac{R}{n+1} y^{n+1} + y^n + C_1 y + C_2} = \ln u.
\end{equation}

By the definition of $\Phi$,

\begin{equation}
\Phi = \log \left[ y'(u) \left( \frac{y(u)}{u} \right)^{n-1} \right].
\end{equation}

we get

\begin{equation}
\Phi'(u) = \frac{y''(u) y'(u)}{y'(u)^2} + (n-1) \frac{y'(u)}{y(u)} - \frac{n-1}{u}.
\end{equation}

The right-hand side can be rewritten as a function in $y$:

\begin{equation}
\Phi'(u) = -\frac{\partial^2}{\partial y^2} \left( \frac{y}{u} \right)^2 + \frac{n-1}{y} \frac{u}{y} - \frac{n-1}{u}.
\end{equation}
It follows that:
\[
R = - \frac{du}{dy} \Phi'(u) - u \frac{d}{dy} \Phi'(u) - (n - 1) \frac{u \Phi'(u)}{y} \\
= \frac{\partial^2 u}{\partial y^2} + \frac{(n - 1)(n - 2)}{y} + \frac{2(n - 1)u \partial^2 u}{2 y^2 \partial y^2} - \frac{(n - 2)(n - 1)u}{y^2 \partial y^2} \\
+ \frac{y \partial^3 u}{\partial y^3} - \frac{2u(\partial^2 u)}{(\partial y)^2}.
\]

(200)

Now we generalize Abreu’s formula \[1, 12\] in the compact case to the situation of this Section:

**Proposition 6.3.** For the Kähler metric considered in this Section, the formulas for scalar curvature hold:

\[
R = - \sum_{i,j=1}^{n} \frac{\partial^2 G^{ij}}{\partial y_i \partial y_j} = - \sum_{i,j=1}^{n} \frac{\partial^4 \phi}{\partial y_i^2 \partial y_j^2}.
\]

**Proof.** To prove the first equality, recall that

\[
G^{ij} = y_i \delta_{ij} + \left( \frac{u}{y^2 \partial u} - \frac{1}{y} \right) y_i y_j.
\]

Hence we have:

\[
\sum_{i=1}^{n} \frac{\partial G^{ij}}{\partial y_i} = \sum_{i=1}^{n} \left[ \delta_{ij} + \left( \frac{u}{y^2 \partial u} - \frac{1}{y} \right) (y_j + \delta_{ij} y_i) \right]
\]

\[
+ \left( \frac{\partial u}{\partial u} \right) - y^2 \frac{\partial^2 u}{\partial y^2} + \frac{u}{y^2 \partial u} \frac{\partial^2 u}{\partial y^2} + \frac{1}{y^2} y_i y_j \right]
\]

\[
= 1 + (n + 1) \left( \frac{u}{y^2 \partial u} - \frac{1}{y} \right) y_j + \left( \frac{2}{y} - \frac{2u}{y^2 \partial u} - \frac{u}{y(\partial u)^2} \right) y_j
\]

\[
= 1 + \left( \frac{n - 1}{y} + \frac{(n - 1)u}{y^2 \partial u} - \frac{u}{y(\partial u)^2} \right) y_j.
\]

Furthermore,

\[
- \sum_{i,j=1}^{n} \frac{\partial^2 G^{ij}}{\partial y_i \partial y_j} = - \sum_{j=1}^{n} \left[ \left( - \frac{n - 1}{y} + \frac{(n - 1)u}{y^2 \partial u} - \frac{u}{y(\partial u)^2} \right) \right]
\]

\[
+ \left( \frac{n - 1}{y} + \frac{(n - 1)u}{y^2 \partial u} - \frac{2(n - 1)u}{y^3 \partial u} - \frac{(n - 1)u}{y^2 \partial u} \right)
\]

\[
- \left( \frac{\partial u}{\partial u} \right) - \frac{2u}{y^2 \partial u} \frac{\partial^2 u}{\partial u} + \frac{2u}{y^2 (\partial u)^2} \frac{\partial^2 u}{\partial u} - \frac{u}{y^2 (\partial u)^2} \frac{\partial^3 u}{\partial u} \right) y_j
\]

\[
= \left( \frac{n - 1}{y} + \frac{(n - 1)(n - 2) u}{y^2 \partial u} + \frac{2(n - 1) u}{y(\partial u)^2} \right) \frac{\partial^2 u}{\partial u} \frac{\partial^3 u}{\partial u} - \frac{u}{y^2 (\partial u)^2} \frac{\partial^3 u}{\partial u} \right) y_j
\]

\[
+ \frac{\partial^2 u}{\partial u} \frac{\partial^3 u}{\partial u} - \frac{2u}{y^2 (\partial u)^2} \frac{\partial^2 u}{\partial u} + \frac{u}{y^3 (\partial u)^2} \frac{\partial^3 u}{\partial u}.
\]
The right-hand side of the last equality equals to $R$ by (201). This proves the first equation in (201), and the second equation follows from (173).

6.11. Derivations of Ricci curvature and scalar curvature in Hessian geometry. In §6.8 and §6.10 we have derived formulas for Ricci curvature and the scalar curvature, respectively, in the context of $U(n)$-symmetric Kähler metrics. In this Subsection, we will derive the corresponding formulas in the context of Hessian geometry.

Suppose that $\psi$ is a convex function on a domain $\Omega \subset \mathbb{R}^n$ endowed with linear coordinates $\{y_i\}$. Introduce the dual local coordinates $y^i$ by

$$y^i = \frac{\partial \psi}{\partial y_i},$$

and introduce a dual potential function:

$$\psi^\vee = \sum_{i=1}^n y_i \frac{\partial \psi}{\partial y_i} - \psi.$$

Define a Riemannian metric $g$ by:

$$g = \frac{1}{2} \sum_{i,j=1}^n G_{ij} dy_i dy_j + 2 \sum_{i,j=1}^n G^{ij} d\theta_i d\theta_j,$$

where the coefficients are defined by:

$$G_{ij} = \frac{\partial^2 \psi}{\partial y_i \partial y_j}, \quad G^{ij} = \frac{\partial^2 \psi^\vee}{\partial y^i \partial y^j}.$$

One can check that the matrices $(G_{ij})$ and $(G^{ij})$ are inverse to each other, and

$$g = 2 \sum_{i,j=1}^n G^{ij} \left(\frac{1}{4} dy^i dy^j + d\theta_i d\theta_j\right).$$

Introduce complex coordinates $w_i$:

$$w_i = \frac{1}{2} y^i + \sqrt{-1} \theta_i.$$

Then $g$ is a Kähler metric with Kähler form:

$$\omega = \sqrt{-1} \sum_{i,j=1}^n G^{ij} dw_i \wedge d\bar{w}_j = \sqrt{-1} \sum_{i,j=1}^n \frac{\partial^2 \psi^\vee}{\partial y^i \partial y^j} dw_i \wedge d\bar{w}_j.$$

Since

$$\frac{\partial}{\partial \bar{w}_i} = -\frac{1}{2} \left(2 \frac{\partial}{\partial y^i} - \sqrt{-1} \frac{\partial}{\partial \theta_i}\right), \quad \frac{\partial}{\partial w_i} = \frac{1}{2} \left(2 \frac{\partial}{\partial y^i} + \sqrt{-1} \frac{\partial}{\partial \theta_i}\right),$$

we have

$$\omega = \sqrt{-1} \sum_{i,j=1}^n \frac{\partial^2 \psi^\vee}{\partial w_i \partial \bar{w}_j} dw_i \wedge d\bar{w}_j.$$
Therefore, its Ricci form

\[ \rho = -\sqrt{-1} \sum_{i,j=1}^{n} \frac{\partial^2 \log \det(G^{ij})}{\partial w_i \partial \bar{w}_j} dw_i \wedge d\bar{w}_j \]

(211)

\[ = -\sqrt{-1} \sum_{i,j=1}^{n} \frac{\partial^2 \log \det(G^{ij})}{\partial y_i^\vee \partial y_j^\vee} dw_i \wedge d\bar{w}_j. \]

The scalar curvature \( R \) is then

(212)

\[ R = -\sqrt{-1} \sum_{i,j=1}^{n} G_{ij} \frac{\partial^2 \log \det(G^{ij})}{\partial y_i^\vee \partial y_j^\vee}. \]

By the computations in Abreu [1],

(213)

\[ R = -\sum_{i,j=1}^{n} \frac{\partial^2 G^{ij}}{\partial y_i \partial y_j}. \]

6.12. Kähler-Ricci flow in Hessian geometry.

One can consider Kähler-Ricci flow of the form (187) with \( \omega \) and \( \rho \) given by (208) and (211). This leads us to the following flow:

(214)

\[ \frac{d}{dt} \psi^\vee = -\log \det \left( \frac{\partial^2 \psi^\vee}{\partial y_i^\vee \partial y_j^\vee} \right) + \lambda \psi^\vee. \]

Dually, one can consider the flow for the complex potential:

(215)

\[ \frac{d}{dt} \psi = -\log \det \left( \frac{\partial^2 \psi}{\partial y_i \partial y_j} \right) + \lambda \psi. \]

7. The Case of \( U(n) \)-Symmetric Kähler Ricci-Flat Metrics

In this Section we apply the results developed in last Section to the case of some \( U(n) \)-symmetric Kähler Ricci-flat metrics on \( \mathcal{O}_{p=-1}(-n) \) (see e.g Duan-Zhou [13]). We will first work on \( \mathbb{C}^n - \{0\} \), then consider the extension from \( (\mathbb{C}^n - \{0\})/\mathbb{Z}_n \). When \( n = 2 \), one recovers the Kepler metric on \( K_2 \) and the Eguchi-Hanson metric.

7.1. \( U(n) \)-symmetric Kähler Ricci-flat metrics. If \( \lambda = 0 \) i.e. \( \rho = 0 \) in (185) then we get:

(216)

\[ y(u)^n = C_3 u^n + C_4 \]

for some constants \( C_3, C_4 \). When \( C_3 = 1 \) and \( C_4 > 0 \), write \( C_4 = b^n \) for some \( b > 0 \), then we have \( y(u) = (u^n + b^n)^{1/n} \) and so the Kähler potential is given by:

\[ \phi(u) = \int \frac{(u^n + b^n)^{1/n}}{u} du. \]

It satisfies

(217)

\[ \phi'(u) = \frac{(u^n + b^n)^{1/n}}{u}, \quad \phi''(u) = -\frac{b^n}{u^2(u^n + b^n)^{(n-1)/n}}. \]
and so the Kähler form is given by:

\[
\omega = \sqrt{-1}(\phi'(u)\partial \bar{u} + \phi''(u)\partial u \wedge \bar{\partial}u)
\]

\[
= \sqrt{-1}\left(\frac{(u^n + b^n)^{1/n}}{u}(dz^1 \wedge d\bar{z}^1 + \cdots + dz^n \wedge d\bar{z}^n) - \frac{b^n}{u^2(u^n + b^n)^{(n-1)/n}}(\bar{z}^1dz^1 + \cdots + \bar{z}^ndz^n) \wedge (z^1d\bar{z}^1 + \cdots + z^nd\bar{z}^n)\right).
\]

When \(b \to 0\), we recover the flat metric on \(\mathbb{C}^n\):

\[
\omega = \sqrt{-1}\sum_{i=1}^{n} dz^i \wedge d\bar{z}^i.
\]

By (151) and (217), the moment map of the diagonal torus subgroup of \(U(n)\) is given by

\[(y_1, \ldots, y_n) = \left(\left|z_1\right|^2\frac{(u^n + b^n)^{1/n}}{u}, \ldots, \left|z_n\right|^2\frac{(u^n + b^n)^{1/n}}{u}\right).
\]

The image of the moment map is the convex body:

\[(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_j \geq 0, \ j = 1, \ldots, n, \ y_1 + \cdots + y_n \geq b\].

When \(b > 0\), the convex body has \(n\) vertex points. When \(b = 0\), the convex body is a simplex.

By (154) and (217), we now have

\[y = (u^n + b^n)^{1/n},\]

and so

\[u = (y^n - b^n)^{1/n}.
\]

Hence by (157),

\[\phi = \int y^n \frac{y^n - b^n}{y^n - b^n} dy = \frac{b}{n} \sum_{j=0}^{n-1} \xi_j \ln(y - \xi_j b) + C.
\]

And by (150), the complex potential is:

\[\psi = \sum_{i=1}^{n} y_i(\ln y_i - 1) - y(\ln y - 1) + \frac{1}{n} \sum_{j=0}^{n-1} (y - b\xi_j^n)\log(y - b\xi_j^n) - 1 - C.
\]

It is interesting to compare this formula with the formula of Guillemin [18] in the case of compact toric manifolds.

The dual local coordinates \(y_i^\vee\) are then

\[y_i^\vee = \frac{\partial \psi}{\partial y_i} = \ln y_i - \ln y + \frac{1}{n} \sum_{j=0}^{n-1} \log(y - b\xi_j^n) = \ln \left(\frac{y_i(y^n - b^n)^{1/n}}{y}\right),\]

and from this we find:

\[y = \left(\sum_{j=1}^{n} e^{y_j^\vee} \right)^{1/n}.
\]
\[ y_i = \frac{e^{\psi}}{\sum_{j=1}^{n} e^{\psi_j}} \left( \left( \sum_{i=1}^{n} e^{\psi_i} \right)^n + b^n \right)^{1/n}. \]

It follows that \( \psi^\vee = \phi \) can be written in terms of \( y_i^\vee \) as follows:

\[ \psi^\vee = \frac{b}{n} \sum_{j=0}^{n-1} \frac{\xi_j}{\ln \left( \left( \sum_{i=1}^{n} e^{\psi_i} \right)^n + b^n \right)} + C. \]

One checks that

\[
\frac{\partial \psi^\vee}{\partial y_k} = \frac{b}{n} \sum_{j=0}^{n-1} \frac{\xi_j}{\ln \left( \left( \sum_{i=1}^{n} e^{\psi_i} \right)^n + b^n \right)} - \xi_j b
\]

\[
= \frac{b}{n} \sum_{j=0}^{n-1} \left( \sum_{i=1}^{n} e^{\psi_i} \right)^{n-1} e^{\psi_k}
\]

\[
= y_k.
\]

By Theorem 6.1 and Theorem 6.2, the Riemannian metric \( g \) takes the following form:

\[
g = \sum_{i,j=1}^{n} \left( \frac{1}{2} G_{ij} dy_i dy_j + 2 G_i^i d\theta_i d\theta_j \right)
\]

where the coefficients \( G_{ij} \) and \( G_i^i \) are now given by:

\[
G_{ij} = \frac{\delta_{ij}}{y_i} - \frac{y_i^{n-1}}{y^n - b^n} = \frac{\delta_{ij}}{y_i} + \frac{b^n}{y(y^n - b^n)}.
\]

\[
G_i^i = y_i \delta_{ij} + \left( \frac{y^n - b^n}{y^{n+1}} \right) y_i y_j = y_i \delta_{ij} - \frac{b^n}{y^{n-1}} y_i y_j.
\]

Their determinants are

\[
det(G_{ij}) = \frac{y^n}{y_1 \cdots y_n (y^n - b^n)},
\]

\[
det(G_i^i) = \frac{y_1 \cdots y_n (y^n - b^n)}{y^n} = \exp(y_1^\vee + \cdots + y_n^\vee).
\]

By (211), the Ricci form for \( g \) is

\[
\rho = -\sqrt{-1} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial y_i \partial y_j} \left( y_1^\vee + \cdots + y_n^\vee \right) \cdot dw_i \wedge d\bar{w}_j = 0.
\]
Let $g^\vee$ be the metric defined by:

$$g^\vee = \sum_{i,j=1}^{n} \left( \frac{\partial^2 \varphi^\vee}{\partial y_i \partial y_j} dy_i dy_j + \frac{\partial^2 \varphi^\vee}{\partial \varphi_i \partial \varphi_j} d\varphi_i d\varphi_j \right).$$

Then its Ricci form is

$$\rho^\vee = -\sqrt{-1} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial y_i \partial y_j} \log \left( \frac{y^n}{y_1 \cdots y_n (y^n - b^n)} \right) \cdot dw_i \wedge d\bar{w}_j.$$ 

Clearly, $g^\vee$ is not Ricci-flat. To make $g^\vee$ Ricci-flat in the literature of SYZ Conjecture it was proposed to quantum correct the complex structure $J^\vee$ (see e.g. [2, 10]).

**7.2. Quotient by $\mathbb{Z}_n$.** The above family of metrics on $\mathbb{C}^n \setminus \{0\}$ is invariant under the $\mathbb{Z}/n\mathbb{Z}$-action:

$$(z^1, \ldots, z^n) \mapsto (e^{2\pi i/n} z^1, \ldots, e^{2\pi i/n} z^n).$$

Since the quotient space $(\mathbb{C}^n \setminus \{0\})/(\mathbb{Z}/n\mathbb{Z})$ can be identified with $\mathcal{O}_{\mathbb{P}^{n-1}}(-n) - \mathbb{P}^{n-1}$, one obtains a family of Kähler Ricci-flat metrics on the latter space. To get explicit expressions, make the following change of variables:

$$(228) \quad z^1 = v^1/n, \quad z^2 = v^1/n w^2, \quad \ldots \quad z^n = v^1/n w^n. $$

Equivalently, 

$$(229) \quad v = (z^1)^n, \quad w^2 = \frac{z^2}{z^1}, \quad \ldots \quad w^n = \frac{z^n}{z^1}. $$

Then $\omega$ becomes

$$\hat{\omega}_b = \sqrt{-1} \left( \frac{(1 + |w|^2)^n}{n^2 |v|^2 (1 + |w|^2)^n + b^n (n-1)/n} dv \wedge d\bar{v} + \sum_{i=2}^{n} \frac{\bar{w}_i (1 + |w|^2)^{n-1}}{n |v|^2 (1 + |w|^2)^n + b^n (n-1)/n} dv \wedge d\bar{w}_i \right. $$

$$- \sum_{i=2}^{n} \frac{v \bar{w}_i (1 + |w|^2)^{n-1}}{n |v|^2 (1 + |w|^2)^n + b^n (n-1)/n} d\bar{v} \wedge dw_i $$

$$+ \frac{(|v|^2 (1 + |w|^2)^n + b^n)^{1/n}}{(1 + |w|^2)^n} \sum_{i=2}^{n} dw^i \wedge d\bar{w}_i $$

$$- \frac{b^n}{(1 + |w|^2)^2 (|v|^2 (1 + |w|^2)^n + b^n)^{n-1}/n} \sum_{i=2}^{n} \bar{w}_i dw^i \wedge \sum_{j=2}^{n} w^j d\bar{w}_j \right).$$

When $b > 0$, $\hat{\omega}_b$ defines a Kähler metric on the total space of the bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$. This family of metrics is the Calabi metrics [S] on $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$. It is well-known that when $n = 2$, this is the Eguchi-Hanson metric [E]. We will also explicitly check this in [S].

When $b \to 0$, $\hat{\omega}_b$ becomes:

$$\hat{\omega}_0 = \sqrt{-1} \left( \frac{(1 + |w|^2)^n}{n^2 |v|^2 (2n-1)/n} dv \wedge d\bar{v} + \sum_{i=2}^{n} \frac{\bar{w}_i (1 + |w|^2)^{n-1}}{n |v|^2 (2n-1)/n} dv \wedge d\bar{w}_i $$

$$- \sum_{i=2}^{n} \frac{v \bar{w}_i (1 + |w|^2)^{n-1}}{n |v|^2 (2n-1)/n} d\bar{v} \wedge dw_i $$

$$+ \frac{|v|^2 (1 + |w|^2)^n + b^n)^{1/n}}{(1 + |w|^2)^n} \sum_{i=2}^{n} dw^i \wedge d\bar{w}_i $$

$$- \frac{b^n}{(1 + |w|^2)^2 (|v|^2 (1 + |w|^2)^n + b^n)^{(n-1)/n} \sum_{i=2}^{n} \bar{w}_i dw^i \wedge \sum_{j=2}^{n} w^j d\bar{w}_j \right).$$
When $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$ is blown down to $\mathbb{C}^n/\mathbb{Z}_n$, $\hat{\omega}_0$ is transformed to the orbifold flat Kähler metric.

Let $T \subset U(n)$ be the group of diagonal $n \times n$ unitary matrices. It acts naturally on $\mathbb{C}^n$:

$$(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot (z_1, \ldots, z_n) = (e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n).$$

This action induces an action on $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$, in local coordinates $(v, w^2, \ldots, w^n)$, it is given by:

$$(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot (v, w^2, \ldots, w^n) = (e^{ni\theta_1}v, e^{i(\theta_2-\theta_1)}w^2, \ldots, e^{i(\theta_n-\theta_1)}w^n).$$

Make the following change of coordinates:

$$\alpha_1 = n\theta_1, \quad \alpha_j = \theta_j - \theta_1, \quad j = 2, \ldots, n.$$ 

This suggests a different torus action, defined in local coordinates $(v, w^2, \ldots, w^n)$ by:

$$(e^{i\alpha_1}, \ldots, e^{i\alpha_n}) \cdot (v, w^2, \ldots, w^n) = (e^{i\alpha_1}v, e^{i\alpha_2}w^2, \ldots, e^{i\alpha_n}w^n).$$

It has the following moment map:

$$(x_1, \ldots, x_n) = (ny_1, y_2 - y_1, \ldots, y_n - y_n),$$

whose image is

$$(230) \quad x_1 \geq 0, \quad x_j + \frac{1}{n}x_1 \geq 0, \quad j = 2, \ldots, n, \quad x_1 + \cdots + x_n \geq a.$$ 

Denote by $x_i^\vee$ the dual coordinates:

$$(231) \quad x_i^\vee = \frac{\partial \psi}{\partial x_i}.$$ 

Then we have

$$(232) \quad x_1^\vee = \frac{1}{n} \sum_{i=1}^n y_i^\vee, \quad x_j^\vee = y_j^\vee, \quad j = 2, \ldots, n.$$ 

8. Kepler Metric on $K_2$ and Eguchi-Hanson Metrics by Calabi Ansatz

In this section, we will rederive the Kepler metric on $K_2$ and the Eguchi-Hanson metric by applying Calabi Ansatz on $\mathcal{O}_{\mathbb{P}^1}(-2)$. This will lead us to a discussion of the Kepler metric on $K_3$ in the next two sections.

8.1. Calabi Ansatz on $\mathcal{O}_{\mathbb{P}^1}(-1)$. Consider local coordinates $(z, w)$, $(\tilde{z}, \tilde{w})$ on $\mathcal{O}_{\mathbb{P}^1}(-1)$ related to each other by:

$$\tilde{z} = \frac{1}{z}, \quad \tilde{w} = zw, \quad z = \frac{1}{\tilde{z}}, \quad w = \frac{\tilde{z}}{\tilde{w}}.$$ 

Define an Hermitian metric on $\mathcal{O}_{\mathbb{P}^1}(-1)$ by

$$(233) \quad u = v^2 = |w|^2(1 + |z|^2).$$ 

Consider a Kähler metric of the form:

$$(234) \quad \omega = a\pi^*\omega_{\mathbb{P}^1} + \sqrt{-1}\partial\bar{\partial}\phi(u) = \sqrt{-1}a\partial\bar{\partial}\log(1 + |z|^2) + \sqrt{-1}\partial\bar{\partial}\phi(u).$$
By a computation similar to that in \cite{22}, we have
\[
\omega = \sqrt{-1} \left\{ \frac{1}{1 + |z|^2} (a + y + u|z|^2y') dz \wedge d\bar{z} + y' \bar{w} \cdot dz \wedge d\bar{w} + y' z \bar{w} \cdot dw \wedge d\bar{z} + y' \cdot (1 + |z|^2) dw \wedge d\bar{w} \right\},
\]
where $y(u) = u\phi'(u)$. It follows that
\[
\Phi = \frac{\omega^2 / 2!}{(\sqrt{-1})^2 dz \wedge d\bar{z} \wedge dw \wedge d\bar{w}} = \frac{1}{1 + |z|^2} \left( a + y + u|z|^2y' \right) \cdot y' - \frac{u|z|^2}{1 + |z|^2} (y')^2 = \frac{(a + y)y'}{1 + |z|^2}.
\]
Since we have
\[
\partial \bar{\partial} \log \Phi = - \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} + \frac{y''}{y'} \partial \bar{\partial} u + \frac{y'y'' - (y'')^2}{(y')^2} \partial u \wedge \partial u
\]
\[
+ \frac{y'}{a + y} \partial \bar{\partial} u + \frac{y''(a + y) - (y')^2}{(a + y)^2} \partial u \wedge \bar{\partial} u
\]
\[
= - \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}
\]
\[
+ (\log((a + y)y'))' \cdot |w|^2 dz \wedge d\bar{z} + z \bar{w} dw \wedge d\bar{z} + \bar{z} wdz \wedge d\bar{z} + (1 + |z|^2) dw \wedge d\bar{w}
\]
\[
+ (\log((a + y)y'))'' \cdot (|w|^4|z|^2 dz \wedge d\bar{z} + u \bar{z} wdz \wedge d\bar{z} + u z \bar{w} dw \wedge d\bar{z} + u(1 + |z|^2) dw \wedge d\bar{w}),
\]
to get a Kähler Ricci-flat metric, we need all the coefficients of $\partial \bar{\partial} \Phi$ to vanish. It suffices to solve the following system of two equations:
\[
- \frac{1}{(1 + |z|^2)^2} + (\log((a + y)y'))' \cdot |w|^2 + (\log((a + y)y'))'' \cdot |w|^4|z|^2 = 0,
\]
\[
(\log((a + y)y'))' + (\log((a + y)y'))'' \cdot u = 0.
\]
From the second equation
\[
\tag{235}
(\log((a + y)y'))' \cdot u = c,
\]
plug this into the first equation:
\[
- \frac{1}{(1 + |z|^2)^2} + \frac{c|w|^2}{u} - \frac{c \cdot |w|^4|z|^2}{u^2} = 0.
\]
From this one gets $c = 1$, and so
\[
\tag{236}
(\log(y'(a + y))')' = \frac{1}{u},
\]
hence we can follow the following steps to get the solutions:
\[
\tag{237}
(a + y)y' = C_1 u,
\]
\[
\tag{238}
y = -a \pm \sqrt{C_1 u^2 + C_2},
\]
\[
\tag{239}
\phi(u) = \int \frac{-a \pm \sqrt{C_1 u^2 + C_2}}{u} du.
\]
We will take the plus sign and set \( C_1 = 1, \, C_2 = b^2 \). Then the Kähler form becomes:
\[
\omega = \sqrt{-1} \left\{ \frac{1}{(1 + |z|^2)^2} \left( \sqrt{u^2 + b^2} + u|z|^2 \cdot \frac{u}{\sqrt{u^2 + b^2}} \right) dz \wedge d\bar{z} \\
+ \frac{u}{\sqrt{u^2 + b^2}} \bar{z}w \cdot dz \wedge d\bar{w} + \frac{u}{\sqrt{u^2 + b^2}} z\bar{w} \cdot dw \wedge d\bar{z} \\
+ \frac{u}{\sqrt{u^2 + b^2}} \cdot (1 + |z|^2) dw \wedge d\bar{w} \right\}.
\]

This metric degenerate along the zero section on \( \mathcal{O}_\mathbb{P}^1(-1) \), so we shift to \( \mathcal{O}_\mathbb{P}^1(-1) \).

This can be achieved by taking \( v = w^2 \), \( \tilde{v} = \tilde{w}^2 \).

Then the Kähler form becomes:
\[
\omega = \sqrt{-1} \left\{ \frac{1}{(1 + |z|^2)^2} \left( \sqrt{|v|^2(1 + |z|^2)^2} + b^2 + \frac{|v|^2(1 + |z|^2)^2|z|^2}{\sqrt{|v|^2(1 + |z|^2)^2} + b^2} \right) dz \wedge d\bar{z} \\
+ \frac{1 + |z|^2}{\sqrt{|v|^2(1 + |z|^2)^2} + b^2} \bar{z}v \cdot dz \wedge d\bar{v} + \frac{1 + |z|^2}{\sqrt{|v|^2(1 + |z|^2)^2} + b^2} z\bar{v} \cdot dv \wedge d\bar{z} \\
+ \frac{1 + |z|^2}{4\sqrt{|v|^2(1 + |z|^2)^2} + b^2} \cdot dv \wedge d\bar{v} \right\}.
\]

This matches with the metric \( \tilde{\omega}_b \) for \( n = 2 \) in \( \mathbb{L} \).

8.2. Polar coordinates. Use the stereographic projection to get:
\[
(240) \quad x^0 = \frac{1 - |z|^2}{|z|^2 + 1}, \quad x^1 + \sqrt{-1}x^2 = \frac{2z}{|z|^2 + 1},
\]
and
\[
(241) \quad z = \frac{x^1 + \sqrt{-1}x^2}{x^0 + 1}.
\]

In terms of the Euler angles,
\[
(242) \quad x^0 = \cos \theta, \quad x^1 = \sin \theta \cos \varphi, \quad x^2 = \sin \theta \sin \varphi.
\]

It follows that
\[
(243) \quad z = e^{i\varphi} \tan \left( \frac{\theta}{2} \right),
\]
and so we can find from
\[
(244) \quad 1 + |z|^2 = \frac{1}{\cos^2 \left( \frac{\theta}{2} \right)},
\]
and
\[
(245) \quad dz = ie^{i\varphi} \tan \left( \frac{\theta}{2} \right) d\phi + \frac{1}{2 \cos^2 \left( \frac{\theta}{2} \right)} e^{i\varphi} d\theta
\]
the following formula:
\[
(246) \quad \frac{|dz|^2}{(|z|^2 + 1)^2} = \frac{1}{4} (d\theta^2 + \sin^2 \theta d\phi^2).
\]
Let
\[
(247) \quad w = r \cos \left( \frac{\theta}{2} \right) \cdot e^{i\beta}.
\]
Its differential is given by:

\begin{equation}
(248) \quad dw = r \cos\left(\frac{\theta}{2}\right) \cdot e^{i\beta} \left( \frac{dr}{r} - \frac{1}{2} \tan\left(\frac{\theta}{2}\right) d\theta + i d\beta \right).
\end{equation}

Now the Riemannian metric associated to (231) is given by:

\begin{align*}
g &= \frac{1}{(1 + |z|^2)^2} \left( a + y(u) + |z|^2 uy'(u) \right) |dz|^2 \\
&\quad + 2(1 + |z|^2)y'(u) \cdot \Re(\bar{w}zdwd\bar{z}) \\
&\quad + y'(u) \cdot \frac{u}{|w|^2} \cdot |dw|^2.
\end{align*}

It can be rewritten as:

\begin{equation}
(249) \quad g = \frac{a + y(u)}{(1 + |z|^2)^2} |dz|^2 + uy'(u) \cdot |\gamma|^2,
\end{equation}

where

\begin{equation}
(250) \quad \gamma = \frac{\bar{z}}{1 + |z|^2} dz + \frac{\bar{w}}{|w|^2} dw.
\end{equation}

It is easy to find

\begin{equation}
(251) \quad \gamma = \frac{dr}{r} + i(\beta + \sin^2(\frac{\theta}{2}) d\varphi),
\end{equation}

so we get:

\begin{equation}
\begin{aligned}
g &= y'(u) dr^2 + \frac{1}{4} \left( a + y(u) \right) \left( d\theta^2 + \sin^2(\frac{\theta}{2}) d\varphi^2 \right) + r^2 y'(u) \cdot (d\beta + \sin^2(\frac{\theta}{2}) d\varphi)^2.
\end{aligned}
\end{equation}

When

\begin{equation}
(252) \quad y = -a + \sqrt{u^2 + b^2},
\end{equation}

the metric is

\begin{equation}
\begin{aligned}
g &= \frac{r^2}{\sqrt{r^4 + b^2}} dr^2 + \frac{\sqrt{r^4 + b^2}}{4} \left( d\theta^2 + \sin^2(\frac{\theta}{2}) d\varphi^2 \right) + \frac{r^4}{\sqrt{r^4 + b^2}} \cdot (d\beta + \sin^2(\frac{\theta}{2}) d\varphi)^2.
\end{aligned}
\end{equation}

Let \( \psi = 2\beta - \varphi \), one can rewrite it as

\begin{equation}
\begin{aligned}
g &= \frac{r^2}{\sqrt{r^4 + b^2}} dr^2 + \frac{\sqrt{r^4 + b^2}}{4} \left( d\theta^2 + \sin^2(\theta) d\varphi^2 \right) + \frac{r^4}{4\sqrt{r^4 + b^2}} \cdot (d\psi + \cos(\theta) d\varphi)^2.
\end{aligned}
\end{equation}

Finally, letting \( s = (r^4 + b^2)^{1/4} \),

\begin{equation}
(253) \quad g = \left( 1 - \frac{b^2}{s^4} \right)^{-1} \frac{s^2}{4} \left( 1 - \frac{b^2}{s^4} \right) (d\psi + \cos(\theta) d\varphi)^2 + \frac{s^2}{4} (d\theta^2 + \sin^2(\theta) d\varphi^2).
\end{equation}

This is the standard form of the Euguchi-Hanson metric \[15\].

When \( b = 0 \),

\begin{equation}
(254) \quad g = dr^2 + \frac{r^2}{4} ((d\theta^2 + \sin^2(\theta) d\varphi^2) + (d\psi + \cos(\theta) d\varphi)^2).
\end{equation}

This recovers the Kepler metric on \( K_2 \), and its shows that the standard metric on \( \mathbb{R}P^3 \) up to a constant is a Sasaki-Einstein metric.
9. Kepler Metric on 3-Conifold and Related Metrics by Calabi Ansatz on $O_{P^1}(-1) \oplus O_{P^1}(-1)$

In this Section we will study the Kepler metric on $K_3$ and some related metrics from the point of view of both $\S8$ and $\S6$.

9.1. Calabi Ansatz on $O_{P^1}(-1)\oplus O_{P^1}(-1)$. Consider local coordinates $(z, w_1, w_2)$ and $(\tilde{z}, \tilde{w}_1, \tilde{w}_2)$ on the total space of the rank two vector bundle $O_{P^1}(-1) \oplus O_{P^1}(-1)$ related to each other by the following formulas:

$$
\tilde{z} = \frac{1}{z}, \quad \tilde{w}_j = zw_j, \\
z = \frac{1}{\tilde{z}}, \quad w_j = \tilde{z}\tilde{w}_j.
$$

Define an Hermitian metric by setting the Hermitian norm square of the element in the vector bundle with local coordinate $(z, w_1, w_2)$ to be:

$$
u = r^2 = |w|^2(1 + |z|^2),
$$

where $|w|^2 = \sum_{j=1}^2 |w_j|^2$. As a special case of a construction of Calabi [8], we consider Kähler metrics of the form:

$$
\omega_a = a\pi^*\omega_{P^1} + \sqrt{-1} \frac{\partial \bar{\partial} \phi(u)}{(1 + |z|^2)^2} + \sqrt{-1} \partial \bar{\partial} \phi(u) \cdot \partial \bar{\partial} u + \sqrt{-1} \phi''(u) \cdot \partial \bar{\partial} u,
$$

where $\phi$ is a suitable function in $u$. Since we have

$$
\partial u = |w|^2 \tilde{z}dz + (1 + |z|^2) \sum_{j=1}^2 \tilde{w}_j dw_j,
$$

$$
\bar{\partial} u = |w|^2 zd\tilde{z} + (1 + |z|^2) \sum_{j=1}^2 w_j d\tilde{w}_j,
$$

$$
\partial u \wedge \bar{\partial} u = |w|^4|z|^2 dz \wedge d\tilde{z} + u \sum_{j=1}^2 \tilde{w}_j dz \wedge d\tilde{w}_j + u \sum_{j=1}^2 z\tilde{w}_j dw_j \wedge d\tilde{z} + (1 + |z|^2) \sum_{j,k} \tilde{w}_j w_k dw_j \wedge d\tilde{w}_k,
$$

$$
\partial \bar{\partial} u = |w|^2 dz \wedge d\tilde{z} + \sum_{j=1}^2 z\tilde{w}_j dw_j \wedge d\tilde{z} + \sum_{j=1}^2 \tilde{z} w_j dz \wedge d\tilde{w}_j + \sum_{j=1}^2 (1 + |z|^2) dw_j \wedge d\tilde{w}_j,
$$
one has:
\[
\omega_a = \sqrt{-1} \left\{ \left( \frac{a}{(1 + |z|^2)^2} + \phi'(u) \cdot |w|^2 + \phi''(u) \cdot |w|^2 |z|^2 \right) dz \wedge d\bar{z} \right.
\]
\[
+ \sum_{j=1}^2 \left( \phi'(u) + u \phi''(u) \right) \left( \bar{z} w_j \cdot dz \wedge d\bar{w}_j + z w_j \cdot dw_j \wedge d\bar{z} \right)
\]
\[
+ \sum_{j,k=1}^2 \left( \phi'(u) \cdot (1 + |z|^2) \delta_{jk} + \phi''(u) \cdot (1 + |z|^2)^2 \bar{w}_j w_k \right) dw_j \wedge d\bar{w}_k \right\}
\]  
(257)

9.2. Symplectic coordinates. Similar to (152) we have:

**Proposition 9.1.** Let \( z = r_0 e^{\sqrt{-1} \theta_0} \), \( w_i = r_i e^{\sqrt{-1} \theta_i} \), then

\[
\omega_a = \sum_{j=0}^2 dy_j \wedge d\theta_j,
\]

where \( y_0, y_1, y_2 \) are defined by:

\[
y_0 = \frac{-a}{1 + r_0^2} + r_0^2 (r_1^2 + r_2^2) \cdot \phi'((1 + r_0^2)(r_1^2 + r_2^2)),
\]

\[
y_j = r_j^2 (1 + r_0^2) \cdot \phi'((1 + r_0^2)(r_1^2 + r_2^2)), \quad j = 1, 2.
\]

**Proof.** We first check that:

\[
\omega_a = \left[ \frac{2ar_0 dr_0}{(1 + r_0^2)^2} + 2\phi'(u) \cdot \left( r_1^2 + r_2^2 \right) r_0 dr_0 + \sum_{j=1}^2 r_0^2 r_j dr_j \right]
\]
\[
+ 2\phi''(u) \cdot \left( r_1^2 + r_2^2 \right) r_0^2 dr_0 \left( 1 + r_0^2 \right) r_j^2 r_j dr_j + \sum_{j=1}^2 2\phi'(u) \cdot \left( r_1^2 + r_2^2 \right) r_0^2 dr_0 
\]
\[
+ \sum_{j=1}^2 \left[ 2\phi'(u) \cdot \left( r_1^2 + r_2^2 \right) r_0^2 dr_0 \left( 1 + r_0^2 \right) r_j^2 r_j dr_j \right]
\]
\[
+ 2\phi''(u) \cdot \left( 1 + r_0^2 \right) r_1^2 r_2^2 r_0 dr_0 + (1 + r_0^2)^2 \sum_{k=1}^2 r_1^2 r_2^2 r_k r_k dr_k \right] \wedge d\theta_j.
\]

From this the Proposition can be easily checked. \( \square \)

Define a \( T^3 \)-action by

\[
(t_0, t_1, t_2) \cdot (z, w_1, w_2) = (t_0 z, t_1 w_1, t_2 w_2),
\]
\[
(t_0, t_1, t_2) \cdot (\tilde{z}, \tilde{w}_1, \tilde{w}_2) = (t_0^{-1} \tilde{z}, t_0 t_1 \tilde{w}_1, t_0 t_2 \tilde{w}_2).
\]

They are generated by the following holomorphic vector fields:

\[
X_0 = z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} + w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2},
\]
\[
X_1 = w_1 \frac{\partial}{\partial w_1} = \tilde{w}_1 \frac{\partial}{\partial \tilde{w}_1},
\]
\[
X_2 = w_2 \frac{\partial}{\partial w_2} = \tilde{w}_2 \frac{\partial}{\partial \tilde{w}_2}.
\]
One can check that 
\begin{equation}
 i \chi_j \omega_a = \sqrt{-1} \partial y_j,
\end{equation}
for \( j = 0, 1, 2 \).

### 9.3. Kähler potential in symplectic coordinates.

Let 
\begin{equation}
y := y_1 + y_2.
\end{equation}

By (260) it is clear that: 
\begin{equation}
y = u \cdot \phi'(u).
\end{equation}

Then one can generalize §6.3 almost verbatim.

### 9.4. Riemannian metric in symplectic coordinates.

The Riemannian metric \( g_a \) associated with \( \omega_a \) can be explicitly written down as follows:

\begin{align*}
g_a &= \frac{2a(dr_0^2 + r_0^2 d\theta_0^2)}{(1 + r_0^2)^2} + 2\phi'(u) \cdot \left( (r_1^2 + r_2^2)(dr_0^2 + r_0^2 d\theta_0^2) \right) \\
&\quad + 2 \sum_{j=1}^2 r_0 r_j (dr_0 dr_j + r_0 d\theta_0 r_j d\theta_j) + \sum_{j=1}^2 (1 + r_0^2)(dr_j^2 + r_j^2 d\theta_j^2) \\
&\quad + 2\phi''(u) \cdot \left( (r_1^2 + r_2^2)^2 r_0^2 (dr_0^2 + r_0^2 d\theta_0^2) \right) \\
&\quad + 2(1 + r_0^2)(r_1^2 + r_2^2) \sum_{j=1}^2 r_0 r_j (dr_j dr_0 + r_j d\theta_j r_0 d\theta_0) \\
&\quad + (1 + r_0^2)^2 \sum_{j,k=1}^2 r_j r_k (dr_j dr_k + r_j r_k d\theta_j d\theta_k) \\ 
\end{align*}

**Theorem 9.1.** In symplectic coordinates the Riemannian metric \( g_a \) takes the following form:

\begin{equation}
g_a = \sum_{i,j=1}^n \left( \frac{1}{2} G_{ij} dy_i dy_j + 2 G^{ij} d\theta_i d\theta_j \right),
\end{equation}

where the coefficients \( G_{ij} \) and \( G^{ij} \) will be given in the proof. Furthermore, the matrices \( (G_{ij}) \) and \( (G^{ij}) \) are inverse to each other.

**Proof.** We separate the terms with \( dr_i \)'s and those with \( d\theta_j \)'s in the above expression for \( g_a \) to get:

\begin{equation}
g_a = g_a^r + g_a^\theta,
\end{equation}
where \( g_a^r \) and \( g_a^\theta \) are given by:

\[
g_a^r = \frac{2adr_0^2}{(1 + r_0^2)^2} + 2\phi'(u) \cdot \left((r_1^2 + r_2^2)dr_0^2 + 2 \sum_{j=1}^{2} r_0 r_j dr_0 dr_j + \frac{2}{2} (1 + r_0^2) dr_j^2\right)
+ 2\phi''(u) \cdot \left((r_1^2 + r_2^2)dr_0^2 + (1 + r_0^2) \sum_{j=1}^{2} r_j dr_j\right)^2,
\]

\[
g_a^\theta = \frac{2ar_0^2d\theta_0^2}{(1 + r_0^2)^2} + 2\phi'(u) \cdot \left((r_1^2 + r_2^2)r_0^2d\theta_0^2 + 2 \sum_{j=1}^{2} r_0^2 \theta_0 r_j d\theta_j + \frac{2}{2} (1 + r_0^2) r_j^2 d\theta_j\right)
+ 2\phi''(u) \cdot \left((r_1^2 + r_2^2)r_0^2d\theta_0 + (1 + r_0^2) \sum_{j=1}^{2} r_j^2 d\theta_j\right)^2.
\]

From (269) and (260) one can get:

\[
r_0 = \sqrt{\frac{y_0 + a}{y_1 + y_2 - y_0}},
\]

(266)

\[
r_j = \sqrt{\frac{y_j(y_1 + y_2 - y_0)}{\phi'(u) \cdot (y_1 + y_2 + a)}}, \quad j = 1, 2.
\]

With these identities one can compute \( g_a^r \) and \( g_a^\theta \). For \( g_a^\theta \) we first get:

\[
g_a^\theta = 2 \left(\frac{a(y - y_0)(y_0 + a)}{(y + a)^2} + \frac{(y_0 + a)y}{y + a} + \frac{\phi''(u)}{\phi'(u)^2} \cdot \left(\frac{(y_0 + a)y}{y + a}\right)^2\right) d\theta_0^2
+ 4 \sum_{j=1}^{2} \left(\frac{(y_0 + a)y_j}{y + a} + \frac{\phi''(u)}{\phi'(u)^2} \cdot \frac{(y_0 + a)y_j}{y + a}\right) d\theta_0 d\theta_j
+ 2 \sum_{j=1}^{2} y_j d\theta_j^2 + 2 \frac{\phi''(u)}{\phi'(u)^2} \left(\sum_{j=1}^{2} y_j d\theta_j\right)^2.
\]

We also have

\[
\phi'(u) = \frac{u}{y},
\]

(268)

\[
\frac{\phi''(u)}{\phi'(u)^2} = \frac{u}{y^2 \frac{du}{dy}} - \frac{1}{y},
\]

so we get:

\[
g_a^\theta = 2 \left(\frac{(y - y_0)(y_0 + a)}{y + a} + \frac{(y_0 + a)^2 u}{(y + a)^2} \frac{du}{dy}\right) d\theta_0^2
+ 4 \sum_{j=1}^{2} \left(\frac{(y_0 + a)y_j}{y(y + a)} \frac{u}{\frac{du}{dy}}\right) d\theta_0 d\theta_j
+ 2 \sum_{j,k=1}^{2} (-1)^j k \frac{y_1 y_2}{y} d\theta_j d\theta_k + 2 \frac{u}{y^2 \frac{du}{dy}} \left(\sum_{j=1}^{2} y_j d\theta_j\right)^2
= 2 \sum_{j,k} G^{jk} d\theta_j d\theta_k.
\]
Denote by \((G_{jk})\) the inverse matrix of \((G^{jk})\). By a computation we have found the following remarkably simple expressions for \(G_{jk}\):

\[
G_{00} = \frac{1}{y_0 + a} + \frac{1}{y - y_0} = \frac{y + a}{(y_0 + a)(y - y_0)},
\]
\[
G_{01} = G_{10} = G_{02} = G_{20} = -\frac{1}{y - y_0},
\]
\[
G_{11} = \frac{d u}{d y} \frac{1}{u} + \frac{1}{y - y_0} - \frac{1}{y} - \frac{1}{y + a} + \frac{1}{y_1},
\]
\[
G_{22} = \frac{d u}{d y} \frac{1}{u} + \frac{1}{y - y_0} - \frac{1}{y} - \frac{1}{y + a} + \frac{1}{y_2},
\]
\[
G_{12} = G_{21} = \frac{d u}{d y} \frac{1}{u} + \frac{1}{y - y_0} - \frac{1}{y} - \frac{1}{y + a}.
\]

It is clear that

\[
g + a^r = 2 \sum_{i,j=0} G^{ij} d \log r_i \cdot d \log r_j.
\]

By a long and tedious computation we check that:

\[
g^r_a = 2 \sum_{j,k=0}^2 G_{jk} dy_j dy_k.
\]

This completes the proof. \qed

9.5. **Hessian geometry.** Now as in \(\S 6\) we are in the regime of Hessian geometry and many results in that Section can be applied. It is easy to see that

\[
G_{ij} = \frac{\partial^2 \psi}{\partial y_i \partial y_j}
\]

for the function \(\psi\) defined by:

\[
\psi = (y - y_0)(\ln(y - y_0) - 1) + (y_0 + a)(\ln(y_0 + a) - 1) + \sum_{i=1}^2 y_i(\ln y_i - 1)
\]
\[
- y(\ln y - 1) - (y + a)(\ln(y + a) - 1) + \int \log u(y)dy.
\]

One can check that for \(j = 0, 1, 2\),

\[
\frac{dw_j}{w_j} = \frac{dr_j}{r_j} + \sqrt{-1} d\theta_j = \frac{1}{2} \sum_{k=0}^2 \frac{\partial^2 \psi}{\partial y_j \partial y_k} dy_k + \sqrt{-1} d\theta_j,
\]

where \(w_0 = z\).

The dual local coordinates \(y_i^\vee\) are given by

\[
y_0^\vee = \ln(y_0 + a) - \ln(y - y_0) = 2 \ln r_0,
\]
\[
y_i^\vee = \ln(y - y_0) + \ln(y_i) - \ln(y) - \ln(y + a) + \ln(u(y)) = 2 \ln r_i,
\]

for \(i = 1, 2\), and dual potential function:

\[
\psi^\vee = y \ln u(y) - \int \ln u(y)dy - a \ln \frac{y_0 + a}{y + a} = \phi - a \ln \frac{r_0^2}{1 + r_0^2}.
\]
And the proof of Theorem 6.2 works in this case too.

9.6. **Kepler metric on $K_3$ lifted to $O_p^1(-1) \oplus O_p^1(-1)$**. By (82) and Proposition [82], the Kepler metric on $K_3$ is of the type (259) with $a = 0$ and $\phi(u) = u^{1/2}$. Since $y = \frac{1}{2}u^{1/2}$, we have $u = \frac{1}{4}y^2$. In this case the moment map is given by:

\[
y_0 = \frac{r_0^2(r_1^2 + r_2^2)}{2\sqrt{(1 + r_0^2)(r_1^2 + r_2^2)}},
\]

\[
y_j = \frac{r_j^2(1 + r_0^2)}{2\sqrt{(1 + r_0^2)(r_1^2 + r_2^2)}}, \quad j = 1, 2.
\]

It has the following inverse map:

\[
r_0 = \sqrt{\frac{y_0}{y_1 + y_2 - y_0}},
\]

\[
r_j = 2\sqrt{\frac{y_j(y_1 + y_2 - y_0)}{(y_1 + y_2)}}, \quad j = 1, 2.
\]

The image of the moment map is the convex cone generated by $(1, 1, 0), (1, 0, 1), (0, 1, 0), (0, 0, 1)$. This cone is bounded by four planes:

\[
y_j = 0, \quad j = 0, 1, 2, \quad y_1 + y_2 - y_0 = 0.
\]

One can check that

\[
\psi = (y_1 + y_2 - y_0)(\ln(y_1 + y_2 - y_0) - 1) + y_0(\ln y_0 - 1) + \sum_{i=1}^2 y_i(\ln y_i - 1).
\]

The dual local coordinates $y_i^\vee$ are given by

\[
y_0^\vee = \ln(y_0) - \ln(y_1 + y_2 - y_0),
\]

\[
y_i^\vee = \ln(y_1 + y_2 - y_0) + \ln(y_i), \quad i = 1, 2,
\]

and the dual potential function is:

\[
\psi^\vee = \phi = 2y.
\]

9.7. **Kähler Ricci-flat metric on $K_3$**. In this case, $a = 0$, $\phi(u) = \frac{3}{2}u^{2/3}$. Since $y = u^{2/3}$, we have $u = y^{3/2}$. In this case the moment map is given by:

\[
y_0 = \frac{r_0^2(r_1^2 + r_2^2)}{\sqrt{(1 + r_0^2)(r_1^2 + r_2^2)}},
\]

\[
y_j = \frac{r_j^2(1 + r_0^2)}{\sqrt{(1 + r_0^2)(r_1^2 + r_2^2)}}, \quad j = 1, 2.
\]

It has the following inverse map:

\[
r_0 = \sqrt{\frac{y_0}{y_1 + y_2 - y_0}},
\]

\[
r_j = \sqrt{\frac{y_j(y_1 + y_2 - y_0)}{(y_1 + y_2)}}, \quad j = 1, 2.
\]

The image of the moment map is the convex cone generated by $(y_0, y_1, y_2) = (1, 1, 0), (1, 0, 1), (0, 1, 0), (0, 0, 1)$. This cone is bounded by four planes:

\[
y_j \geq 0, \quad j = 0, 1, 2, \quad y_1 + y_2 - y_0 \geq 0.
\]
One can check that
\[
\psi = (y_1 + y_2 - y_0) \ln(y_1 + y_2 - y_0) - 1 + y_0 \ln(y_0) - 1 + \sum_{i=1}^{2} y_i (\ln y_i - 1)
\]
(283)
\[-\frac{1}{2} (y_1 + y_2) (\ln(y_1 + y_2) - 1).
\]
The dual local coordinates \( y_i^\vee \) are given by
\[
y_0^\vee = \ln(y_0) - \ln(y_1 + y_2 - y_0),
\]
(284)
\[y_i^\vee = \ln(y_1 + y_2 - y_0) + \ln(y_i) - \frac{1}{2} \ln(y_1 + y_2), \quad i = 1, 2,
\]
and dual potential function:
\[
\psi^\vee = \phi = \frac{3}{2} y.
\]

9.8. \textbf{Kähler Ricci-flat metric on the resolved 3-conifold.} Write \( \omega_\alpha = \sqrt{-1} \sum_{j,k=0}^{2} \omega_{j,k} d\bar{w}_j \wedge d\bar{w}_k \), where \( w_0 = z \). Let \( \Phi = \det(\omega_{j,k})_{j,k=0,2} \). It is easy to find:
\[
\Phi = \frac{(a + u\varphi'(u))(u\varphi'(u))'}{1 + |z|^2} \cdot \varphi'(u) \cdot (1 + |z|^2)
\]
\[= (a + y) y' (1 + |z|^2)^{-1} |w|^{-2}.
\]
It is easy to see that
\[
\partial \bar{\partial} \log \Phi = -\frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} - (n - 1) \left( \frac{dw_j \wedge d\bar{w}_j}{|w|^2} - \frac{\sum_{j,k} \bar{w}_j w_k dw_j \wedge d\bar{w}_k}{|w|^4} \right)
\]
\[+ \left( \ln((a + y) y' y'^{-1})' \cdot \partial \bar{\partial} u + (\ln((a + y) y' y'^{-1}))'' \cdot \bar{\partial} u \wedge \partial u \right).
\]
when \( \rho = -\sqrt{-1} \partial \bar{\partial} \log \Phi = 0 \), the coefficients of \( v \) all vanish:
\[
\frac{-1}{(1 + |z|^2)^2} + (\ln((a + y) y' y'))' \cdot |w|^2 + (\ln((a + y) y' y'))'' \cdot |w|^4 |z|^2 = 0,
\]
\[(\ln((a + y) y' y'))' + (\ln((a + y) y'))'' \cdot u = 0,
\]
\[(\ln((a + y) y' y'))'(1 + |z|^2) \delta_{jk} + (\ln((a + y) y' y'))'' \cdot (1 + |z|^2)^2 \bar{w}_j w_k
\]
\[= \frac{\delta_{jk}}{|w|^2} - \frac{\bar{w}_j w_k}{|w|^4}.
\]
In the third equation we take \( j \neq k \), then we get:
\[
(\ln((a + y) y' y'))'' = -\frac{1}{u^2}.
\]
(287)
and so from the second equation
\[
(\ln((a + y) y' y'))' = \frac{1}{u},
\]
(288)
It follows that
\[
(a + y) y' y' = c u,
\]
(289)
After integration:
\[
\frac{a}{2} y'^2 + \frac{y^3}{3} = \frac{c}{2} u^2 + c_1.
\]
(290)
Take $c = \frac{2}{3}$ and $c_1 = 0$, and one can find by solving a cubic equation:

\begin{equation}
3a \frac{y^2}{2} + y^3 = u^2.
\end{equation}

Since $a > 0$, by the Inverse Function Theorem, when $y > 0$, $y$ is a smooth function in $u$. Let us now analyse the behavior of $y$ and $y'(u)$ as $u \to 0$. Clearly from (291),

\begin{equation}
y(u) \sim \sqrt{\frac{2}{3a}} u - \frac{2u^2}{9a^2}.
\end{equation}

and so from (289),

\begin{equation}
 y'(u) \sim \sqrt{\frac{2}{3a}} - \frac{4u}{9a^2}.
\end{equation}

It follows that

\begin{equation}
\phi'(u) \sim \sqrt{\frac{2}{3a}} - \frac{2u}{9a^2}, \quad \phi''(u) \sim -\frac{2}{9a^2}.
\end{equation}

Therefore, as $u \to 0$,

\begin{equation}
\omega \sim \sqrt{-1} \left\{ \left( \frac{a}{1 + |z|^2} \right)^2 + \sqrt{\frac{2}{3a}} \cdot |w|^2 - \frac{2}{9a^2} \cdot |w|^4 |z|^2 \right\} dz \wedge d\bar{z}
\end{equation}

\begin{equation}
+ \sum_{j=1}^{2} \sqrt{\frac{2}{3a}} \left( \bar{z}w_j \cdot dz \wedge d\bar{w}_j + z\bar{w}_j \cdot dw_j \wedge d\bar{z} \right)
\end{equation}

\begin{equation}
+ \sum_{j,k=1}^{2} \left( \sqrt{\frac{2}{3a}} \cdot (1 + |z|^2) \delta_{jk} - \frac{2}{9a^2} \cdot (1 + |z|^2)^2 \bar{w}_j w_k \right) dw_j \wedge d\bar{w}_k \right\}.
\end{equation}

Clearly it can be extended smoothly over the zero section given by $w = 0$.

It is actually possible in this case to find explicit expression of $\phi$ in terms of $y$. By (296),

\begin{equation}
u = \left( \frac{3a}{2} y^2 + y^3 \right)^{1/2} = \sqrt{y + \frac{3a}{2}},
\end{equation}

therefore,

\begin{equation}
\phi = y \ln(u(y)) - \int \ln(u(y)) dy = \frac{3y}{2} - \frac{3a}{4} \ln(y + \frac{3a}{2}) + C.
\end{equation}

This can be also derived as follows. We get from (296) the following two identities:

\begin{equation}
\frac{du}{dy} = \sqrt{y + \frac{3a}{2}} + \frac{y}{2 \sqrt{y + \frac{3a}{2}}},
\end{equation}

\begin{equation}
\frac{d\phi}{du} = \frac{\phi'(u)}{u} = \frac{y}{\sqrt{y + \frac{3a}{2}}}.
\end{equation}

It follows that

\begin{equation}
\frac{d\phi}{dy} = \frac{d\phi}{du} \frac{du}{dy} = 1 + \frac{y}{2(y + \frac{3a}{2})}.
\end{equation}

Therefore, (297) can be derived after integration.
By (259) and (260), the moment map is:

\[ y_0 = \frac{a}{1 + r_0^2} + \frac{r_0^2}{1 + r_0^2} y, \]
\[ y_j = \frac{r_j}{r_1 + r_2^2} y, \quad j = 1, 2. \]

We have:

\[ r_0 = \sqrt{\frac{y_0 + a}{y_1 + y_2 - y_0}}, \]
\[ r_j = \sqrt{\frac{y_j (y_1 + y_2 - y_0) (y + 3a)}{(y_1 + y_2 + a)}}, \quad j = 1, 2. \]

The image of the moment map is the convex domain bounded by four planes:

\[ y_j \geq 0, \quad j = 1, 2, \quad y_0 \geq -a, \quad y_1 + y_2 - y_0 \geq 0. \]

This convex domain has five edges, the four rays \( \mathbb{R}_{\geq 0}(1, 1, 0), \mathbb{R}_{\geq 0}(1, 0, 1), \mathbb{R}_{\geq 0}(-a, 1, 0), \mathbb{R}_{\geq 0}(-a, 0, 1) \), and an interval \( \{(-at, 0, 0) \mid 1 \leq t \leq 1\} \). By (273),

\[ \psi = (y - y_0)(\ln(y - y_0) - 1) + (y_0 + a)(\ln(y_0 + a) - 1) + \sum_{i=1}^{2} y_i (\ln y_i - 1) \]
\[ - (y + a)(\ln(y + a) - 1) + \frac{1}{2} (y + \frac{3a}{2})(\ln(y + \frac{3a}{2}) - 1). \]

The dual local coordinates \( y_i^\vee \) are given by

\[ y_0^\vee = \ln(y_0 + a) - \ln(y - y_0) = 2 \ln r_0, \]
\[ y_i^\vee = \ln(y - y_0) + \ln(y_i) - \ln(y + a) - \frac{1}{2} \ln(y + \frac{3a}{2}) = 2 \log r_i, \]

for \( i = 1, 2 \), and one can check that:

\[ \psi^\vee = \phi - a \ln \frac{r_0^2}{1 + r_0^2}. \]

### 10. Kepler Metric on 3-Conifold and Resolved 3-Conifold by Calabi Ansatz on \( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \)

In this Section we repeat the steps in last Section for another way to understand \( K_3 \). It turns out to give us some natural way to understand the flop transformation of the resolved conifold, and it also leads to Kähler Ricci-flat metrics on \( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2) = \kappa_{\mathbb{P}^1 \times \mathbb{P}^1} \), the total space of the canonical line bundle of \( \mathbb{P}^1 \times \mathbb{P}^1 \).

**10.1. Calabi Ansatz on \( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \).** We will work with the local coordinates \((z_1, z_2, w)\) on \( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \). They are related to other local coordinates, denoted by \((\hat{z}_1, \hat{z}_2, \hat{w}), (\hat{z}_1, \hat{z}_2, \hat{w}), (\hat{z}_1, \hat{z}_2, \hat{w})\) in the following fashion:

\[ \hat{z}_1 = \frac{1}{z_1}, \quad \hat{z}_2 = z_2, \quad \hat{w} = z_1 w, \]
\[ \check{z}_1 = z_1, \quad \check{z}_2 = \frac{1}{z_2}, \quad \check{w} = z_2 w, \]
\[ \hat{z}_1 = \frac{1}{z_1}, \quad \hat{z}_2 = \frac{1}{z_2}, \quad \hat{w} = z_1 z_2 w. \]
Define an Hermitian metric by

\[ u = r^2 = |w|^2(1 + |z_1|^2)(1 + |z_2|^2). \]

We consider Kähler metrics of the form:

\[ \omega_{a_1, a_2} = a_1 \pi_1^* \omega_{\mathbb{P}^1} + a_2 \pi_2^* \omega_{\mathbb{P}^1} + \sqrt{-1} \partial \bar{\partial} \phi(u) \]

\[ \quad = \sqrt{-1} a_1 \partial \bar{\partial} \log(1 + |z_1|^2) + \sqrt{-1} a_2 \bar{\partial} \partial \log(1 + |z_2|^2) + \sqrt{-1} \bar{\partial} \partial \phi(u). \]

### 10.2. Symplectic coordinates.

One finds

\[ \partial u = |w|^2(1 + |z_2|^2) \bar{z}_1 dz_1 + |w|^2(1 + |z_1|^2) \bar{z}_2 dz_2 + (1 + |z_1|^2)(1 + |z_2|^2) \bar{w} dw, \]

\[ \bar{\partial} u = |w|^2(1 + |z_2|^2) z_1 d\bar{z}_1 + |w|^2(1 + |z_1|^2) z_2 d\bar{z}_2 + (1 + |z_1|^2)(1 + |z_2|^2) w d\bar{w}, \]

\[ \partial \bar{\partial} u = |w|^2(1 + |z_2|^2) d\bar{z}_1 \wedge d\bar{z}_1 + |w|^2(1 + |z_1|^2) d\bar{z}_2 \wedge d\bar{z}_2 + \frac{u}{|w|^2} dw \wedge d\bar{w} \]

\[ + |w|^2(\bar{z}_1 z_2 dz_1 \wedge d\bar{z}_2 + \bar{z}_2 z_1 dz_2 \wedge d\bar{z}_1) \]

\[ + (1 + |z_1|^2)(\bar{w} z_1 dw \wedge d\bar{z}_1 + w \bar{z}_1 dz_1 \wedge d\bar{w}) \]

\[ + (1 + |z_2|^2)(\bar{w} z_2 dw \wedge d\bar{z}_2 + w \bar{z}_2 dz_2 \wedge d\bar{w}), \]

\[ \partial u \wedge \bar{\partial} u = |w|^4(1 + |z_2|^2)^2 |z_1|^2 dz_1 \wedge d\bar{z}_1 + |w|^4(1 + |z_1|^2)^2 |z_2|^2 dz_2 \wedge d\bar{z}_2 \]

\[ + |w|^2 u(\bar{z}_1 z_2 dz_1 \wedge d\bar{z}_2 + \bar{z}_2 z_1 dz_2 \wedge d\bar{z}_1) \]

\[ + u(1 + |z_1|^2)(\bar{w} z_1 dw \wedge d\bar{z}_1 + z_1 \bar{w} dw \wedge d\bar{z}_1) \]

\[ + u(1 + |z_2|^2)(\bar{w} z_2 dw \wedge d\bar{z}_2 + z_2 \bar{w} dw \wedge d\bar{z}_2) + \frac{u^2}{|w|^2} dw \wedge d\bar{w}. \]

Therefore, since we have

\[ \omega_{a_1, a_2} = \sqrt{-1} a_1 \pi_1^* \omega_{\mathbb{P}^1} + \sqrt{-1} a_2 \pi_2^* \omega_{\mathbb{P}^1} + \sqrt{-1} \phi'(u) \cdot \partial \bar{\partial} u + \sqrt{-1} \phi''(u) \cdot \partial u \wedge \bar{\partial} u, \]

\[ \omega_{a_1, a_2} = \sqrt{-1} \left( \sum_{j=1}^{2} \frac{1}{(1 + |z_j|^2)^2} \left( a_j + \phi'(u) \cdot u(1 + |z_j|^2) + \phi''(u) \cdot u^2 |z_j|^2 \right) dz_j \wedge d\bar{z}_j \right. \]

\[ + |w|^2(\phi'(u) + u \phi''(u)) \cdot (\bar{z}_1 z_2 dz_1 \wedge d\bar{z}_2 + \bar{z}_2 z_1 dz_2 \wedge d\bar{z}_1) \]

\[ + (1 + |z_1|^2)(\phi'(u) + u \phi''(u)) (\bar{w} z_1 dw \wedge d\bar{z}_1 + w \bar{z}_1 dz_1 \wedge d\bar{w}) \]

\[ + (1 + |z_2|^2)(\phi'(u) + u \phi''(u)) (\bar{w} z_2 dw \wedge d\bar{z}_2 + w \bar{z}_2 dz_2 \wedge d\bar{w}) \]

\[ \left. + \left( \phi'(u) \cdot \frac{u}{|w|^2} + \phi''(u) \cdot \frac{u^2}{|w|^2} \right) dw \wedge d\bar{w} \right). \]
Write $z_j = r_j e^{iθ_j}$, $j = 1, 2$, $w = r_3 e^{iθ_3}$.

$$\omega_{a_1, a_2} = 2 \left\{ \frac{a_1}{(1 + r_1^2)^2} + r_3^2 (1 + r_2^2) φ'(u) + r_1^2 (1 + r_2^2)^2 r_3^4 φ''(u) \right\} r_1 dr_1 \wedge dθ_1 + \left( \frac{a_2}{1 + r_2^2} + r_3^2 (1 + r_1^2) φ'(u) + r_2^2 (1 + r_1^2)^2 r_3^4 φ''(u) \right) r_2 dr_2 \wedge dθ_2 + r_3^2 (φ'(u) + w φ''(u)) \cdot r_1 r_2 (dr_1 \wedge r_2 dθ_2 + dr_2 \wedge r_1 dθ_1) + (1 + r_2^2) (φ'(u) + w φ''(u)) \cdot r_1 r_3 (dr_1 \wedge r_3 dθ_3 + dr_3 \wedge r_1 dθ_1) + \left( φ'(u) + w φ''(u) \right) \cdot r_2 r_3 (dr_2 \wedge r_3 dθ_3 + dr_3 \wedge r_2 dθ_2) + (φ'(u) + w φ''(u)) \cdot (1 + r_1^2) (1 + r_2^2) \cdot r_3 dr_3 \wedge dθ_3 \right\}.
$$

From this one can easily check that

$$(307) \quad \omega = dy_1 \wedge dθ_1 + dy_2 \wedge dθ_2 + dy_3 \wedge dθ_3,$$

where $y_1, y_2, y_3$ are defined by:

$$(308) \quad y_1 = -\frac{a_1}{1 + r_1^2} + r_1^2 r_3^2 (1 + r_2^2) \cdot φ'(1 + r_1^2) (1 + r_2^2) r_3^2),
$$

$$(309) \quad y_2 = -\frac{a_2}{1 + r_2^2} + r_2^2 r_3^2 (1 + r_1^2) \cdot φ'(1 + r_1^2) (1 + r_2^2) r_3^2),
$$

$$(310) \quad y_3 = (1 + r_1^2) (1 + r_2^2) r_3^2 \cdot φ'(1 + r_1^2) (1 + r_2^2) r_3^2).$$

These functions generate $T^3$-action on $O_{P_1, x P_1} (-1, -1)$ defined in local coordinates $(z_1, z_2, z_3)$ by:

$$(311) \quad (e^{iα_1}, e^{iα_2}, e^{iα_3}) \cdot (z_1, z_2, w) = (e^{iα_1} z_1, e^{iα_2} z_2, e^{iα_3} w).$$

Let $y := y_3$. Then

$$(312) \quad y = u \cdot φ'(u)$$

and one can express $φ$ as a function in $y$.

10.3. Riemannian metric in symplectic coordinates and Hessian geometry. The Riemannian metric $g_u$ associated with $ω_u$ can be explicitly written down as follows:

$$g_{a_1, a_2} = 2 \left\{ \frac{a_1}{(1 + r_1^2)^2} + r_3^2 (1 + r_2^2) φ'(u) + r_1^2 (1 + r_2^2)^2 r_3^4 φ''(u) \right\} (dr_1^2 + r_1^2 dθ_1^2) + \left( \frac{a_2}{1 + r_2^2} + r_3^2 (1 + r_1^2) φ'(u) + r_2^2 (1 + r_1^2)^2 r_3^4 φ''(u) \right) (dr_2^2 + r_2^2 dθ_2^2) + 2r_3^2 (φ'(u) + w φ''(u)) \cdot r_1 r_2 (dr_1 dr_2 + +r_1 dθ_1 r_2 dθ_2) + 2(1 + r_2^2) (φ'(u) + w φ''(u)) \cdot r_1 r_3 (dr_1 dr_3 + r_1 dθ_1 r_3 dθ_3) + 2(1 + r_2^2) (φ'(u) + w φ''(u)) \cdot r_2 r_3 (dr_2 dr_3 + r_2 dθ_2 r_3 dθ_3) + (φ'(u) + w φ''(u)) \cdot (1 + r_1^2) (1 + r_2^2) \cdot (dr_3^2 + r_3^2 dθ_3^2) \right\}.
$$

**Theorem 10.1.** In symplectic coordinates the Riemannian metric $g$ takes the following form:

$$g_{a_1, a_2} = \sum_{i,j=1}^n \left( \frac{1}{2} G_{ij} dy_i dy_j + 2G_{ij} dθ_i dθ_j \right),$$

$$(313)$$
where the coefficients $G_{ij}$ and $G^{ij}$ will be given in the proof. Furthermore, the matrices $(G_{ij})$ and $(G^{ij})$ are inverse to each other.

**Proof.** We separate the terms with $dr_i$'s and those with $d\theta_j$'s in the above expression for $g_a$ to get:

\begin{equation}
(314) \quad g_{a_1,a_2} = g^r_{a_1,a_2} + g^\theta_{a_1,a_2},
\end{equation}

where $g^r_a$ and $g^\theta_a$ are given by:

\begin{align*}
 g^r_{a_1,a_2} &= 2 \left\{ \left( \frac{a_1}{1 + r_1^2} + r_3^2(1 + r_2^2)\phi'(u) + r_2^2(1 + r_2^2)^2r_3^4\phi''(u) \right) dr_1^2 \\
 &\quad + \left( \frac{a_2}{(1 + r_1^2)^2} + r_3^2(1 + r_1^2)\phi'(u) + r_2^2(1 + r_2^2)^2r_3^4\phi''(u) \right) dr_2^2 \\
 &\quad + 2r_3^2(\phi'(u) + u\phi''(u)) \cdot r_1r_2dr_1dr_2 \\
 &\quad + 2(1 + r_2^2)(\phi'(u) + u\phi''(u)) \cdot r_1r_3dr_1dr_3 \\
 &\quad + 2(1 + r_2^2)(\phi'(u) + u\phi''(u)) \cdot r_2r_3dr_2dr_3 \\
 &\quad + (\phi'(u) + u\phi''(u)) \cdot (1 + r_1^2)(1 + r_2^2) \cdot dr_3^2 \right\},
\end{align*}

\begin{align*}
 g^\theta_{a_1,a_2} &= 2 \left\{ \left( \frac{a_1}{1 + r_1^2} + r_3^2(1 + r_2^2)\phi'(u) + r_2^2(1 + r_2^2)^2r_3^4\phi''(u) \right) r_1^2 d\theta_1^2 \\
 &\quad + \left( \frac{a_2}{(1 + r_1^2)^2} + r_3^2(1 + r_1^2)\phi'(u) + r_2^2(1 + r_2^2)^2r_3^4\phi''(u) \right) r_2^2 d\theta_2^2 \\
 &\quad + 2r_3^2(\phi'(u) + u\phi''(u)) \cdot r_1^2 r_3 d\theta_1 d\theta_2 \\
 &\quad + 2(1 + r_2^2)(\phi'(u) + u\phi''(u)) \cdot r_1 r_3^2 d\theta_1 d\theta_3 \\
 &\quad + 2(1 + r_2^2)(\phi'(u) + u\phi''(u)) \cdot r_2^2 r_3^2 d\theta_2 d\theta_3 \\
 &\quad + (\phi'(u) + u\phi''(u)) \cdot (1 + r_1^2)(1 + r_2^2) r_3^2 d\theta_3^2 \right\}.
\end{align*}

From (308) to (310) one can get:

\begin{equation}
(315) \quad r_j = \sqrt{\frac{y_j + a_j}{y - y_j}}, \quad j = 1, 2,
\end{equation}

\begin{equation}
(316) \quad r_3 = \sqrt{\frac{y(u - y_1)(y - y_2)}{\phi'(u) \cdot (y + a_1)(y + a_2)}}.
\end{equation}

With these identities one can compute $g^r_a$ and $g^\theta_a$. For $g^\theta_a$ we get:

\begin{equation}
(317) \quad g^\theta_{a_1,a_2} = 2 \sum_{j=1}^{2} \frac{(y - y_j)(y_j + a_j)}{y + a_j} d\theta_j^2 + 2 \frac{u}{dy} \left( \sum_{j=1}^{3} \frac{y_j + a_j}{y + a_j} d\theta_j \right)^2.
\end{equation}
Denote by \((G_{jk})\) the inverse matrix of \((G^{jk})\). By a computation we have found the following remarkably simple expressions for \(G_{jk}\):

\[
G_{11} = \frac{1}{y_1 + a_1} + \frac{1}{y - y_1}, \quad G_{22} = \frac{1}{y_2 + a_2} + \frac{1}{y - y_2},
\]

\[G_{12} = G_{21} = 0,\]

\[
G_{13} = G_{31} = -\frac{1}{y - y_1}, \quad G_{23} = G_{32} = -\frac{1}{y - y_2}, \quad G_{33} = \frac{du}{u} + \frac{1}{y - y_1} - \frac{1}{y + a_1} + \frac{1}{y - y_2} - \frac{1}{y + a_2}.
\]

It is clear that

\[
g_{a_1,a_2}^r = 2 \sum_{i,j=1}^{3} G_{ij} \frac{dr_i}{r_1} \frac{dr_j}{r_2}.
\]

So we have:

\[
g_{a_1,a_2}^r = 2 \sum_{j=1}^{2} \frac{(y_j - y)(y_j + a_j)}{y + a_j}(d \log r_j)^2 + 2 \frac{u}{dy} \left( \sum_{j=1}^{3} \frac{y_j + a_j}{y + a_3} d \log r_j \right)^2.
\]

By a straightforward computation we check that:

\[
g_{a_1,a_2}^r = \frac{1}{2} \sum_{j,k=0}^{3} G_{jk} dy_j dy_k.
\]

This completes the proof. \(\square\)

Again as in §6 we are in the regime of Hessian geometry. It is easy to see that

\[
G_{ij} = \frac{\partial^2 \psi}{\partial y_i \partial y_j}
\]

for the function \(\psi\) defined by:

\[
\psi = \sum_{j=1}^{2} \left( (y_j - y)(\ln(y - y_j) - 1) + (y_j + a_j)(\ln(y_j + a_j) - 1) \right)
\]

\[
- (y + a_j)(\ln(y + a_j) - 1) \right) + \int \log u(y) dy.
\]

One can check that for \(i = 1, 2, 3,\)

\[
\frac{dz_i}{z_i} = \frac{1}{2} \sum_{j=1}^{3} \frac{\partial^2 \psi}{\partial y_i \partial y_j} dy_j + \sqrt{-1} d\theta_i,
\]

where \(z_3 = w\).

The dual local coordinates \(y_i^\vee\) are given by

\[
y_j^\vee = \ln(y_j + a_j) - \ln(y - y_j) = 2 \ln r_j, \quad j = 1, 2,
\]

\[
y_3^\vee = \sum_{j=1}^{2} \left( \ln(y_j - y_j) - \ln(y + a_j) \right) + \ln(u(y)) = 2 \log r_3,
\]
and dual potential function:

\[
\psi = y \ln u(y) - \int \ln u(y) dy - \sum_{j=1}^{2} a_j \ln \frac{y_j + a_j}{y + a_j} = \phi - \sum_{j=1}^{2} a_j \ln \frac{r_j^2}{1 + r_j^2}.
\]

10.4. Riemannian metric in polar coordinates. As in [8,2] we introduce polar coordinates, one for each of \(z_j\) as follows. For convenience, we will repeat some of the computations there. Let

\[
x_j^0 = \frac{1 - |z_j|^2}{|z_j|^2 + 1}, \quad x_j^1 + \sqrt{-1}x_j^2 = \frac{2z}{|z_j|^2 + 1}.
\]

Then one has

\[
z_j = x_j^1 + \sqrt{-1}x_j^2.
\]

Let \(\theta_j\) and \(\phi_j\) be Euler angles such that

\[
x_j^0 = \cos \theta_j, \quad x_j^1 = \sin \theta_j \cos \phi_j, \quad x_j^2 = \sin \theta_j \sin \phi_j.
\]

Then one has

\[
z_j = e^{i\phi_j} \tan(\frac{\theta_j}{2}),
\]

and so we have

\[
1 + |z_j|^2 = \frac{1}{\cos^2(\frac{\theta_j}{2})}
\]

and

\[
dz_j = ie^{i\phi_j} \tan(\frac{\theta_j}{2}) d\phi_j + \frac{1}{2} e^{i\phi_j} \cos^2(\frac{\theta_j}{2}) d\theta_j.
\]

Therefore,

\[
\frac{|dz_j|^2}{(1 + |z_j|^2)^2} = \frac{1}{4} (d\theta_j^2 + \sin^2 \theta_j d\phi_j^2).
\]

Now let

\[
w = r \cos(\frac{\theta_1}{2}) \cos(\frac{\theta_2}{2}) \cdot e^{i\beta}.
\]

Its differential is given by:

\[
dw = r \cos(\frac{\theta_1}{2}) \cos(\frac{\theta_2}{2}) \cdot e^{i\beta} \left( \frac{dr}{r} - \frac{1}{2} \tan(\frac{\theta_1}{2}) d\theta_1 - \frac{1}{2} \tan(\frac{\theta_2}{2}) d\theta_2 + id\beta \right).
\]

It is easy to see that \(u = r^2\).

The Riemannian metric is given by:

\[
g_{a_1,a_2} = \sum_{j=1}^{2} \frac{1}{(1 + |z_j|^2)^2} \left( a_j + y(u) + |z_j|^2 u y'(u) \right) |dz_j|^2
\]

\[
+ 2 |w|^2 y'(u) \cdot \Re(\bar{z}_1 z_2 d\bar{z}_1 d\bar{z}_2)
\]

\[
+ 2(1 + |z_2|^2) y'(u) \cdot \Re(\bar{w} z_1 dw d\bar{z}_1)
\]

\[
+ 2(1 + |z_1|^2) y'(u) \cdot \Re(\bar{w} z_2 dw d\bar{z}_2)
\]

\[
+ y'(u) \cdot \frac{u}{|w|^2} \cdot |dw|^2.
\]
It can be rewritten as:

\[(332)\]
\[g_{a_1,a_2} = \sum_{j=1}^{2} \frac{a_j + y(u)}{1 + |z_j|^2} |dz_j|^2 + uy'(u) \cdot |\gamma|^2,\]

where

\[(333)\]
\[\gamma = \frac{\bar{z}_1}{1 + |z_1|^2} dz_1 + \frac{\bar{z}_2}{1 + |z_2|^2} dz_2 + \frac{\bar{w}}{|w|^2} dw.\]

It is easy to find

\[(334)\]
\[\gamma = \frac{dr}{r} + i(d\beta + \sin^2(\frac{\theta_1}{2})d\phi_1 + \sin^2(\frac{\theta_2}{2})d\phi_2),\]

so we get:

\[(335)\]
\[g_{a_1,a_2} = y'(u)dr^2 + \sum_{j=1}^{2} \left(\frac{1}{4} \left( a_j + y(u) \right) (d\theta_j^2 + \sin^2 \theta_j d\phi_j^2) \right) + uy'(u) \cdot \left( d\psi + \cos(\theta_1) d\phi_1 + \cos(\theta_2) d\phi_2 \right)^2.\]

Let \(\psi = \theta_1 + \theta_2 - 2\beta\),

\[(336)\]
\[g_{a_1,a_2} = y'(u)dr^2 + \sum_{j=1}^{2} \left(\frac{1}{4} \left( a_j + y(u) \right) (d\theta_j^2 + \sin^2 \theta_j d\phi_j^2) \right) + \frac{1}{4} uy'(u) \cdot \left( d\psi + \cos(\theta_1) d\phi_1 + \cos(\theta_2) d\phi_2 \right)^2.\]

10.5. **Kepler metric on \(K_3\) lifted to \(\mathbb{O}P_1 \times \mathbb{P}^1(-1, -1)\).** By \(\text{(32)}\) and Proposition \(\text{3.2}\), the Kepler metric on \(K_3\) when lifted to \(\mathbb{O}P_1 \times \mathbb{P}^1(-1, -1)\) is of the type \(\text{(306)}\) with \(a_1 = a_2 = 0\) and \(\phi(u) = u^{1/2}\). Again we have \(y = \frac{1}{2} u^{1/2}\) and \(u = \frac{1}{y} y^2\). In this case the moment map is given by:

\[y_1 = \frac{r_1^2}{2} \sqrt{1 + \frac{r_2^2}{1 + r_1^2}}, \quad y_2 = \frac{r_2^2}{2} \sqrt{1 + \frac{r_1^2}{1 + r_2^2}}, \quad y_3 = \frac{r_3^2}{2} \sqrt{1 + \frac{r_1^2}{1 + r_2^2}}.\]

It has the following inverse map:

\[r_j = \sqrt{\frac{y_j}{y - y_j}}, \quad j = 1, 2,\]

\[r_3 = \frac{2 \sqrt{(y - y_1)(y - y_2)}}{2}.\]

The image of the moment map is the convex cone bounded by four planes:

\[(336)\]
\[y_j \geq 0, \quad y \geq y_j, \quad j = 1, 2,\]

generated by \((1, 1, 0), (1, 0, 1), (1, 1, 1), (0, 0, 1)\).

One can check that

\[\psi = \sum_{j=1}^{2} \left( (y - y_j)(\ln(y - y_j) - 1) + y_j(\ln(y_j) - 1) \right).\]
The dual local coordinates $y_j^\nu$ are given by
\[
y_j^\nu = \ln(y_j + a_j) - \ln(y - y_j), \quad j = 1, 2,
\]
and dual potential function:
\[
\psi^\nu = 2y = \phi.
\]

10.6. **Kähler Ricci-flat metrics.** Rewrite $\omega$ in the following form:
\[
\omega_{a_1, a_2} = \sqrt{-1}\left\{\sum_{j=1}^{2} \frac{1}{(1 + |z_j|^2)^2} \left( a_j + y(u) + |z_j|^2 u y'(u) \right) dz_j \wedge d\bar{z}_j \right. \\
+ |w|^2 y'(u) \cdot (\bar{z}_1 z_2 d\bar{z}_1 \wedge d\bar{z}_2 + \bar{z}_2 z_1 d\bar{z}_2 \wedge d\bar{z}_1) \\
+ (1 + |z_2|^2) y'(u) \cdot (\bar{w} z_1 dw \wedge d\bar{z}_1 + \bar{w} z_2 dw \wedge d\bar{z}_2) \\
+ (1 + |z_1|^2) y'(u) \cdot (\bar{w} z_2 dw \wedge d\bar{z}_2 + \bar{w} z_2 dw \wedge d\bar{z}_1) \\
+ y'(u) \cdot \frac{u}{|w|^2} dw \wedge d\bar{w} \bigg\}.
\]

Write $\omega = \sqrt{-1} h_{a_b} dz_a d\bar{z}_b$, where $z_3 = w$, and let $\Phi = \det(h_{a_b})$. It is straightforward to find:
\[
\Phi = \frac{(a_1 + y)(a_2 + y)y'}{(1 + |z_1|^2)(1 + |z_2|^2)}.
\]

The Ricci form is $\rho = -\sqrt{-1} \partial \bar{\partial} \Phi$, and since
\[
\partial \bar{\partial} \log \Phi = -\frac{dz_1 \wedge d\bar{z}_1}{(1 + |z_1|^2)^2} - \frac{dz_2 \wedge d\bar{z}_2}{(1 + |z_2|^2)^2} \\
+ (\log((a_1 + y)(a_2 + y)y'))' \cdot \partial \bar{\partial} u + (\log((a_1 + y)(a_2 + y)y'))'' \cdot \partial u \wedge \bar{\partial} u,
\]
for all the coefficients to vanish we need:
\[
-1 + \alpha'(u) \cdot u(1 + |z_j|^2) + \alpha''(u) \cdot u^2 |z_j|^2 = 0, \quad j = 1, 2,
\]
\[
\alpha'(u) + u \alpha''(u) = 0,
\]
where $\alpha(u) = \log((a_1 + y)(a_2 + y)y')$. One easily finds that
\[
(337) \quad u \alpha'(u) = 1,
\]
It follows that
\[
(338) \quad (a_1 + y)(a_2 + y)y' = cu.
\]

After integration:
\[
(339) \quad \frac{y^3}{3} + \frac{a_1 + a_2}{2} y^2 + a_1 a_2 y - \frac{c}{2} u^2 - c_1 = 0.
\]

Take $c = \frac{2}{3}$ and $c_1 = 0$, one gets a cubic equation
\[
(340) \quad \frac{y^3}{3} + \frac{a_1 + a_2}{2} y^2 + a_1 a_2 y - \frac{1}{3} u^2 = 0.
\]
10.6.1. The case of $a_1 = a_2 = 0$. In this case $y^3 = u^2$, $y = u^{2/3}$, $\phi'(u) = u^{-1/3} = y^{-1/2}$. The moment map is given by:

$$
y_1 = r_1^2 r_3^2 \frac{(1 + r_2^2)}{(1 + r_2^2)} \cdot \left((1 + r_1^2)(1 + r_2^2) r_3^2\right)^{-1/3},$$

$$
y_2 = r_2^2 r_3^2 \frac{(1 + r_2^2)}{(1 + r_2^2)} \cdot \left((1 + r_1^2)(1 + r_2^2) r_3^2\right)^{-1/3},$$

$$
y_3 = ((1 + r_1^2)(1 + r_2^2) r_3^2)^{2/3}.
$$

We have

$$
r_j = \sqrt{\frac{y_j}{y - y_j}}, \quad j = 1, 2,$$

$$
r_3 = \frac{y - y_1 (y - y_2)}{y^{1/2}}.
$$

The image of the moment map is the same as in the case of Kepler metric on $K_3$. The complex potential function $\psi$ is defined by:

$$
\psi = \sum_{j=1}^{2} (y - y_j) (\ln(y - y_j) - 1) + y_j (\ln(y_j) - 1) - \frac{1}{2} y (\ln(y) - 1).
$$

The dual local coordinates $y_j^\vee$ are given by

$$
y_j^\vee = \ln(y_j) - \ln(y - y_j) = 2 \ln r_j, \quad j = 1, 2,$$

$$
y_3^\vee = \sum_{j=1}^{2} \ln(y - y_j) - \frac{1}{2} \ln(y) = 2 \log r_3,
$$

and dual potential function:

$$
\psi^\vee = \frac{3}{2} y = \phi.
$$

10.6.2. The case of one of $a_1, a_2$ is nonzero. Suppose that $a_1 = a$, $a_2 = 0$. We blow down $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$ along the copy of $\mathbb{P}^1$ parameterized by $z_1$. In local coordinates $(z_1, z_2, w)$, we make the following change of variables:

$$
z = z_1, \quad w_1 = w, \quad w_2 = wz.
$$

Since

$$
(|z|^2 + 1)(|w_1|^2 + |w_2|^2) = (|z_1|^2 + 1)(|z_2|^2 + 1)|w|^2,
$$

after blowing down we are in the situation of [10.1.5].

10.6.3. The case of $a_1 > 0$ and $a_2 > 0$. By (340),

$$
u = \left(y^3 + \frac{3(a_1 + a_2)}{2} y^2 + 3a_1 a_2 y\right)^{1/2}.
$$

We will rewrite as

$$
u = (y(y - \beta_+)(y - \beta_-))^{1/2},$$

where $\beta_\pm$ are given by the root formula:

$$
\beta_\pm = \frac{3(a_1 + a_2)}{4} + \frac{1}{2} \sqrt{3a_1 - a_2}(a_1 - 3a_2).
$$

Hence one finds the potential functions $\phi$ and $\psi$ explicitly as functions in $y$:

$$
\phi = \frac{3y}{2} + \frac{1}{2} \beta_+ \ln(y - \beta_+) + \frac{1}{2} \beta_- \ln(y - \beta_-),
$$

$$
\psi = \sum_{j=1}^{2} (y - y_j) (\ln(y - y_j) - 1) + y_j (\ln(y_j) - 1) - \frac{1}{2} y (\ln(y) - 1).
$$
\[ \psi = \sum_{j=1}^{2} \left( (y - y_j)(\ln(y - y_j) - 1) + (y_j + a_j)(\ln(y_j + a_j) - 1) \right) \]

\[ -(y + a_j)(\ln(y + a_j) - 1) + \frac{1}{2} y(\ln y - 1) \]

\[ + \frac{1}{2} (y - \beta_+)(\ln(y - \beta_+) - 1) + \frac{1}{2} (y - \beta_-)(\ln(y - \beta_-) - 1). \]

By (308)-(310), the moment map is given by:

\[ y_1 = -\frac{a_1}{1 + r_1^2} + \frac{r_1^2}{1 + r_1^2} y, \]

\[ y_2 = -\frac{a_2}{1 + r_2^2} + \frac{r_2^2}{1 + r_2^2} y, \]

\[ y_3 = y, \]

and we have

\[ r_j = \frac{y_j + a_j}{y - y_j}, \quad j = 1, 2, \]

\[ r_3 = \sqrt{\frac{y^3 + \frac{3(a_1 + a_2)}{2} y^2 + 3a_1a_2y}{(y + a_1)(y + a_2)}}. \]

The image of the moment map is the convex domain in \( \mathbb{R}^3 \) bounded by:

\[ (347) \quad y_j \geq -a_j, \quad y \geq y_j, \quad j = 1, 2, \quad y \geq 0. \]

Examining the above formula for \( \phi \) and \( \psi \), we see that they are not directly given by the boundary of the convex domain.

By the Inverse Function Theorem, when \( y > 0 \), \( y \) is a smooth function in \( u \). So far we have obtained smooth Kähler Ricci-flat metrics on \( O_{\mathbb{P}_1 \times \mathbb{P}_1}(-1, -1) \) away from the zero section. To examine the behavior of the metric along the zero section, as in §9.8 we analyse the behavior of \( y \) and \( y'(u) \) as \( u \to 0 \). From (3410),

\[ y(u) \sim \frac{u^2}{3a_1a_2} - \frac{1}{18} \frac{a_1 + a_2}{a_1^2a_2^2} u^4, \]

and so from (289),

\[ y'(u) \sim \frac{2u}{3a_1a_2} - \frac{2}{9} \frac{a_1 + a_2}{a_1^3a_2^3} u^3. \]
Therefore, as $u \to 0$,

$$
\omega_{a_1, a_2} \sim \sqrt{-1} \left\{ \sum_{j=1}^{2} \frac{1}{(1 + |z_j|^2)^2} \left( a_j + \frac{u^2}{3a_1a_2} |z_j|^2 u \cdot \frac{2u}{3a_1a_2} \right) dz_j \wedge d\bar{z}_j 
+ |w|^2 \frac{2u}{3a_1a_2} \cdot (\bar{z}_1 z_2 d\bar{z}_1 \wedge d\bar{z}_2 + \bar{z}_2 z_1 d\bar{z}_2 \wedge d\bar{z}_1)
+ (1 + |z_2|^2) \frac{2u}{3a_1a_2} \cdot (\bar{v} \bar{z}_1 d\bar{v} \wedge \bar{d} \bar{z}_1 + w z_1 d\bar{z}_1 \wedge d\bar{w})
+ (1 + |z_1|^2) \frac{2u}{3a_1a_2} \cdot (\bar{v} z_2 d\bar{v} \wedge d\bar{z}_2 + w z_2 d\bar{z}_2 \wedge d\bar{w})
+ \frac{2u}{3a_1a_2} \cdot \frac{u}{|w|^2} \cdot dw \wedge d\bar{w} \right\}.
$$

Along the zero section of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$ given in local coordinates by $w = 0$,

$$
\omega_{a_1, a_2} \sim \sqrt{-1} \left\{ \sum_{j=1}^{2} \frac{1}{(1 + |z_j|^2)^2} \left( a_j + \frac{u^2}{3a_1a_2} |z_j|^2 u \cdot \frac{2u}{3a_1a_2} \right) dz_j \wedge d\bar{z}_j \right\}
$$

is degenerate. To fix this problem, recall there is a $2 : 1$ map from $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$ to $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2)$. We will work with local coordinates $(z_1, z_2, v)$, $(\hat{z}_1, \hat{z}_2, \hat{v})$, $(\hat{\bar{z}}_1, \hat{\bar{z}}_2, \hat{\bar{v}})$ on $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2)$. They are related in the following way:

$$
\begin{align*}
\hat{z}_1 &= 1 \quad \hat{z}_2 = z_2, \quad \hat{\bar{v}} = \frac{1}{2} z_1 \bar{v}, \\
\hat{\bar{z}}_1 &= z_1, \quad \hat{\bar{z}}_2 = 1 \quad \hat{\bar{v}} = \frac{1}{2} \bar{z}_2 \bar{v}, \\
\hat{\bar{z}}_1 &= 1 \quad \hat{\bar{z}}_2 = 1, \quad \hat{\bar{v}} = \frac{1}{2} \bar{z}_1 \bar{v}.
\end{align*}
$$

The $2 : 1$ map is given locally by:

$$
(z_1, z_2, w) \mapsto (z_1, z_2, v = w^2).
$$

Since $w = \pm v^{1/2}$, $\omega_{a_1, a_2}$ is the pullback of a Kähler form $\hat{\omega}_{a_1, a_2}$ on $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2) - \mathbb{P}^1 \times \mathbb{P}^1$, and as $v \to 0$, we have

$$
\hat{\omega}_{a_1, a_2} \sim \sqrt{-1} \left\{ \sum_{j=1}^{2} \frac{1}{(1 + |z_j|^2)^2} \left( a_j + \frac{|v|^2}{3a_1a_2} \sum_{k=1}^{2} (1 + |z_k|^2)^2 \right) \right\}
\setminus
\begin{align*}
&+ |z_j|^2 \frac{2|v|^2}{3a_1a_2} \left( d\bar{z}_j \wedge d\bar{z}_j \right) \\
&+ 2|v|^2 \frac{\sum_{k=1}^{2} (1 + |z_k|^2)^2}{3a_1a_2} \left( \bar{z}_1 z_2 d\bar{z}_1 \wedge d\bar{z}_2 + \bar{z}_2 z_1 d\bar{z}_2 \wedge d\bar{z}_1 \right) \\
&+ (1 + |z_2|^2) \frac{\sum_{k=1}^{2} (1 + |z_k|^2)^2}{3a_1a_2} \left( \bar{v} \bar{z}_1 d\bar{v} \wedge d\bar{z}_1 + v \bar{z}_1 d\bar{z}_1 \wedge d\bar{v} \right) \\
&+ (1 + |z_1|^2) \frac{\sum_{k=1}^{2} (1 + |z_k|^2)^2}{3a_1a_2} \left( \bar{v} z_2 d\bar{v} \wedge d\bar{z}_2 + v z_2 d\bar{z}_2 \wedge d\bar{v} \right) \\
&+ \frac{\sum_{k=1}^{2} (1 + |z_k|^2)^2}{6a_1a_2} \cdot dv \wedge d\bar{v} \right\}.
\end{align*}
$$
Clearly it can be extended smoothly over the zero section given by \( v = 0 \). Therefore, \( \hat{\omega}_{a_1, a_2} \) gives a family of Kähler Ricci-flat metric on the total space of \( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2) \).

Now that we are working on \( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2) \), we need to modify the symplectic coordinates. Write \( z_j = \hat{r}_j e^{i\hat{\theta}_j}, j = 1, 2, \) \( v = \hat{r}_3 e^{i\hat{\theta}_3} \). Obviously, \( \hat{\theta}_1 = \theta_1, \hat{\theta} - 2 = \theta_2, \hat{\theta}_3 = 2\theta_3 \). Then

\[
\hat{\omega}_{a_1, a_2} = d\hat{y}_1 \wedge d\hat{\theta}_1 + d\hat{y}_2 \wedge d\hat{\theta}_2 + d\hat{y}_3 \wedge d\hat{\theta}_3,
\]

where \( \hat{y}_1, \hat{y}_2, \hat{y}_3 \) are defined by:

\[
\hat{y}_1 = y_1, \quad \hat{y}_2 = y_2, \quad \hat{y}_3 = \frac{1}{2}y_3.
\]

Therefore, the image of the moment map on \( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2) \) is the convex domain in \( \mathbb{R}^3 \) bounded by:

\[
\hat{y}_j \geq -a_j, \quad \hat{y}_3 \geq \frac{1}{2}y_j, \quad j = 1, 2, \quad \hat{y}_3 \geq 0.
\]

### 11. Summary

In this paper we have considered the Kepler manifolds \( K_n \). The following facts are well-known in the literature. As symplectic manifolds, \( K_n \cong T^*S^n - S^n \), with their natural symplectic structures; as complex manifolds, \( K_n \cong \{z_1^2 + \cdots + z_n^2 = 0\} - \{0\} \), and when \( n = 2, K_2 \cong (\mathbb{C}^2 - \{0\})/\mathbb{Z}_2 \cong \mathcal{O}_{\mathbb{P}^1}(-2) - \mathbb{P}^1 \) is a hypercomplex manifold, and when \( n = 3, K_3 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) - \mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) - \mathbb{P}^1 \times \mathbb{P}^1 \); the complex structures and the symplectic structures on \( K_n \) are compatible, giving rise to Kähler structures called Kepler metrics on \( K_n \). The Kepler metric has Kähler potential \( \frac{1}{2}(|z_1|^2 + \cdots + |z_n|^2)^{1/2} \), and this can be also used to define Kähler metrics on deformed conifolds.

Based on these facts, in this paper we have presented the following results. Kepler metrics are Sasaki metrics on the conormal bundles of the round spheres. The Kepler metrics are related to Sasaki metrics on the unit conormal bundles of the round spheres, which are Sasakian. When \( n = 2 \) and \( 3 \), the Kepler metrics can be studied using the Hessian geometry, arising in the setting of Kähler metrics with toric symmetries. When \( n = 3 \) we have the following diagram of related spaces:

\[
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2) & \xrightarrow{\pi_1} & \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \\
& \pi_2 & \\
\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & \xrightarrow{\pi_1} & \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \\
& \{z_1^2 + \cdots + z_4^2 = 0\} & \{z_1^2 + \cdots + z_4^2 = a\}
\end{array}
\]
and for $n = 2$ we have a similar diagram. We have explicit constructions of the pullback of Kepler metrics or some Kähler Ricci-flat metrics. We conjecture that these metrics are related to each other by special Kähler Ricci flow in a fashion similar to the one described in §6.12. Surprisingly, in all of our examples, the complex potential functions take the following form:

$$
\psi = \sum_{i=1}^{k} l_i \ln(l_i),
$$

where $l_i$ are some linear functions, some of which are defining functions of the boundary hyperplanes of the convex bodies arising as the image of the moment maps. This generalizes the formula in Guillemin [18], Abreu [1] and Donaldson [12]. We will verify this formula for more examples in subsequent work.

The motivation for our investigations is to establish a link between the Kepler problem in classical gravity and the modern geometric studies of string theory. The basic principle is to apply the intrinsic symmetry of the problem. In this paper we have related the geometry of Kepler problem to Sasakian geometry and Hessian geometry which have play important roles in the study of ADS/CFT correspondence (see e.g. Martelli-Sparks-Yau [31]). A big difference from string theory is that we are using Hessian geometry to directly produce the phase space of the dynamical system of the Kepler problem, while in string theory one first obtains some spaces in extra dimensions for compactifications. We will continue this line of research in later parts of this series of papers.

Acknowledgements. The research in this work is partially supported by NSFC grant 11661131005 for Russia-China Collaborations on Integrable Systems in Mathematical Physics and Differential Geometry. The author’s interest in the geometric aspects of classical dynamical systems was stimulated at Dynamics in Siberia 2017. He thanks the organizers and participants for the hospitality enjoyed at this conference. Some of the materials in this paper were lectured on in a graduate course in Tsinghua University in the spring semester of 2017. The author thanks the attendants of this course for sharing their enthusiasms. Finally, special thanks are due to Guowu Meng for introducing the author to the Kepler problem many years ago.

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