A 2-COMPACT GROUP AS A SPETS

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Abstract. In [Mal98] Malle introduced spetses which are mysterious objects with non-real Weyl groups. In algebraic topology, a $p$-compact group $X$ is a space which is a homotopy-theoretic $p$-local analogue of a compact Lie group. A connected $p$-compact group $X$ is determined by its root datum which in turn determines its Weyl group $W_X$. In this article we give strong numerical evidence for a connection between these two subjects by considering the case when $X$ is the exotic 2-compact group $DI(4)$ constructed by Dwyer–Wilkerson and $W_X$ is the complex reflection group $G_{24} \cong GL_3(2) \times C_2$. Inspired by results in Deligne–Lusztig theory for classical groups, if $q$ is an odd prime power we propose a set $\text{Irr}(X(q))$ of ‘ordinary irreducible characters’ associated to the space $X(q)$ of homotopy fixed points under the unstable Adams operation $\psi^q$. Notably $\text{Irr}(X(q))$ includes the set of unipotent characters associated to $G_{24}$ constructed by Broué, Malle and Michel from the Hecke algebra of $G_{24}$ using the theory of spetses. By regarding $X(q)$ as the classifying space of a Benson–Solomon fusion system $\text{Sol}(q)$ we formulate and prove an analogue of Robinson’s ordinary weight conjecture that the number of characters of defect $d$ in $\text{Irr}(X(q))$ can be counted locally.

1. Introduction

Let $k$ be an algebraically closed field of characteristic $p$. In [KLLS18, Section 4], the authors introduce the concept of an $\mathcal{F}$-compatible family associated to a saturated fusion system $\mathcal{F}$ on a finite $p$-group $S$. This is a family $\alpha = (\alpha_Q)_{Q \in \mathcal{F}^c}$ of cohomology classes $\alpha_Q \in H^2(\text{Out}_\mathcal{F}(Q), k^\times)$ satisfying certain compatibility conditions, where $\mathcal{F}^c$ denotes the set of all $\mathcal{F}$-centric subgroups. The motivation for considering such families comes from block theory where, if $B$ is a block of $kG$ for some finite group $G$, there is a compatible family of Külshammer–Puig classes $\alpha$ associated to the fusion system $\mathcal{F}$ of $B$ on its defect group. In this situation, we say that the pair $(\mathcal{F}, \alpha)$ realizes $B$. Conjectures relating the local and global properties of $B$ involve $\alpha$ on the local side, and for this reason it becomes possible to make several conjectures for arbitrary pairs $(\mathcal{F}, \alpha)$ (see [KLLS18, Section 2]).

In a separate paper together with Lynd [LS17], the author shows that, when $\mathcal{F}$ is a Benson–Solomon fusion system $\text{Sol}(q)$, any compatible family associated to $\mathcal{F}$ is trivial. Moreover the set of all $\mathcal{F}$-centric radical subgroups and their $\mathcal{F}$-outer automorphism groups is listed, and used to show that the number of weights associated to $\mathcal{F}$ is 12, independently of $q$. Here we prove an analogous result (Theorem 2.2) for the ordinary weights associated to $\text{Sol}(q)$ and its close relative, the 2-fusion system $\mathcal{H}$ of $\text{Spin}_7(q)$. Precisely, we show that for each $d \geq 0$ the number of 2-weights of defect $d$ can be expressed as a polynomial in the 2-part of $q^2 - 1$. As a consequence, for the principal 2-block $B$ of $\text{Spin}_7(q)$ we prove Robinson’s Ordinary Weight Conjecture that these numbers count characters of defect $d$ in $B$.

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There is no block with fusion system $\text{Sol}(q)$ (see [Kes06, Cra11, Theorem 9.34]), but it seems one can still construct a global object and conjecture a local-global correspondence in which $\text{Sol}(q)$ features on the local side. Via the ‘obvious’ generalizations of certain results in Deligne–Lusztig theory it is possible to associate a set of irreducible characters to $\text{Sol}(q)$ with a distinguished subset of unipotent characters being exactly those constructed using the Broué–Malle–Michel theory of spetses [BMM99]. In Section 6 we state and prove an analogue of Robinson’s Weight Conjecture that the number of such characters is given purely in terms of the fusion system. Finally, we verify several of the weight conjectures considered in [KLLS18, Section 2] for $\text{Sol}(q)$.

2. Main results

Let $\mathcal{F}$ be a saturated fusion system, $\alpha$ be an $\mathcal{F}$-compatible family and $d \geq 0$ be a non-negative integer. Following [KLLS18, Section 2], for any $\mathcal{F}$-centric subgroup $P$ of $S$ let $\mathcal{N}_P$ be the set of non-empty normal chains $\sigma$ of $p$-subgroups of $\text{Out}_F(P)$ starting at the trivial subgroup; that is, chains of the form

$$\sigma = (1 = X_0 < X_1 < \cdots < X_m)$$

with the property that $X_i$ is normal in $X_m$ for $0 \leq i \leq m$. We set $|\sigma| = m$, and call $m$ the length of $\sigma$. We also define

$$\text{Irr}^d(P) := \{ \mu \in \text{Irr}(P) \mid v_p(|P|/\mu(1)) = d \},$$

the set of ordinary irreducible characters of $P$ of defect $d$ and define $\mathcal{W}^d_P = \mathcal{N}_P \times \text{Irr}^d(P)$. The obvious actions of the group $\text{Out}_F(P)$ on $\mathcal{N}_P$ and $\text{Irr}^d(P)$ yield an action on $\mathcal{W}^d_P$. We denote by $I(\sigma, \mu)$ the stabilisers in $\text{Out}_F(P)$ under these actions, where $(\sigma, \mu) \in \mathcal{W}^d_P$ and set

$$w_P(\mathcal{F}, \alpha, d) := \sum_{\sigma \in \mathcal{N}_P/\text{Out}_F(P)} (-1)^{|\sigma|} \sum_{\mu \in \text{Irr}^d(P)/I(\sigma)} z(k_{\alpha}I(\sigma, \mu)).$$

Finally, we set

$$m(\mathcal{F}, \alpha, d) := \sum_{P \in \mathcal{F}/\mathcal{F}} w_P(\mathcal{F}, \alpha, d) \quad \text{and} \quad m(\mathcal{F}, \alpha) := \sum_{d \geq 0} m(\mathcal{F}, \alpha, d).$$

By [KLLS18, Lemma 7.5], the quantities $m(\mathcal{F}, \alpha, d)$ and $m(\mathcal{F}, \alpha)$ remain unchanged on restricting the sums to isomorphism classes of $\mathcal{F}$-centric radical subgroups. If $d \geq 0$ is an integer and $B$ is a block of $kG$ for some finite group $G$, we write $k_d(B)$ for the number of irreducible characters of $G$ of defect $d$ in the block $B$. The relevance of the invariant $m(\mathcal{F}, \alpha, d)$ is the following:

**Conjecture 2.1** (Robinson). Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$, and let $\alpha$ be an $\mathcal{F}$-compatible family. Suppose that $(\mathcal{F}, \alpha)$ realizes a block $B$ of $kG$ for some finite group $G$. Then

$$m(\mathcal{F}, \alpha, d) = k_d(B).$$

We fix some notation which will be used until Section 6. Let $q$ be a (fixed) odd prime power, define $l := v_2(q^2 - 1) - 3$ and set $x := 2^l$. Notice that $l \geq 0$ since $8 \mid q^2 - 1$. Moreover set:

1. $H$ equal to $\text{Spin}_7(q)$ and $S$ equal to a (fixed) Sylow 2-subgroup of $H$;
Table 1. \(m(D, 0, d)\) for \(D \in \{H, F\}\) and all \(d \geq 0\)

| \(d\) | \(m(H, 0, d)\) | \(m(F, 0, d)\) |
|-------|----------------|----------------|
| 4     | 2              | 2              |
| \(l + 5\) | \(2(x - 1)\) | 0              |
| \(l + 6\) | \(2x + 7\)   | \(2x - 1\) |
| \(l + 7\) | 4              | 4              |
| \(2l + 5\) | \(3x(x - 1)\) | \(x(x - 1)\) |
| \(2l + 6\) | \(12x\)       | \(4x\)        |
| \(3l + 6\) | \(\frac{1}{3}x^3 - 3x^2 + \frac{11}{3}x - 1\) | \(\frac{4}{21}x^3 - x^2 + \frac{11}{3}x - \frac{20}{7}\) |
| \(3l + 7\) | \(12x^2 - 8x + 2\) | \(4x^2 - 6x + 4\) |
| \(3l + 8\) | \(14x - 4\) | \(6x - 4\) |
| \(3l + 9\) | \(8x + 4\) | \(8x + 4\) |
| \(3l + 10\) | 16             | 16             |
| \(m(D, 0)\) | \(\frac{4}{3}x^3 + 12x^2 + \frac{92}{3}x + 28\) | \(\frac{4}{21}x^3 + 4x^2 + \frac{50}{3}x + \frac{155}{7}\) |

(2) \(H = H(q) = F_{S}(H)\) equal to the 2-fusion system of \(H\) on \(S\);
(3) \(F = F(q) := \text{Sol}(q)\), a Benson–Solomon fusion system on \(S\) which contains \(H\) (see \cite{LS17}.)

In \cite{LS17} Theorem 1.1 it was shown that \(\lim_{S(F^c)} A_{F}^{2} = 0\), which means that there are no non-trivial \(F\)-compatible families associated to \(F\). Here is our first main result:

**Theorem 2.2.** Let \(D \in \{H, F\}\) and \(d \geq 0\). \(m(D, 0, d)\) is expressible as a rational polynomial in \(x\). Moreover its precise values are listed in Table 1 when \(m(D, 0, d)\) is non-zero.

As a byproduct of our computations, we obtain:

**Theorem 2.3.** For all odd prime powers \(q\), Conjecture 2.1 holds for the principal 2-block of \(\text{Spin}_{7}(q)\).

We now explain why Theorem 2.2 is surprising when \(D = F\). In \cite{LS17}, the local structure of \(D\) is determined by treating the case \(l = 0\) separately from the generic case \(l > 0\) because the structure of the centric radical subgroups and their automorphism groups does not admit a uniform treatment (see \cite{LS17}, Tables 1,4.] As a result we must handle the computation of \(m(D, 0, d)\) in the case \(l = 0\) separately. On the other hand when a polynomial we obtain for \(l > 0\) is specialised to the case \(l = 0\) we always get the correct answer! When \(D = H\), this phenomenon is explained by the fact that Theorem 2.3 equates \(m(D, 0, d)\) with a polynomial obtained using character counting methods in Deligne–Lusztig theory (see Section 5) which do not distinguish between the cases \(l = 0\) and \(l > 0\).

Now \(F\) is also the fusion system of a 2-local finite group with classifying space \(BX(q)\), where \(X = DI(4)\) is the space constructed by Dwyer–Wilkerson \cite{DW93} and \(X(q)\) denotes
the space of homotopy fixed points under the unstable Adams operation $\psi^q$ acting on $X$ (see Section 6.1). For this reason, we propose that the existence of the polynomials in Table 1 can be explained by regarding DI(4) as a spets which is a pseudo reductive algebraic group with a non-real Weyl group $G_{24}$. Thus we define the set $\text{Irr}^u(X(q))$ of unipotent characters to be those given in [BMM14]. What about the remaining characters? If $B$ denotes the principal 2-block of $H$, a result of Cabanes–Enguehard in Deligne–Lusztig theory (Proposition 5.2) states that $\text{Irr}(B)$ is a union $\bigcup_s E(H, s)$ of Lusztig series taken over conjugacy class representatives of 2-elements in the Langlands dual group $H^* := \text{Aut}(\text{PSp}_6(q))$. Here, $E(H, s)$ is in bijection with the set of unipotent characters of the centralizer $C_{H^*}(s)$ (see the discussion which precedes Proposition 5.2.) Treating $X(q)$ like a Langlands self-dual finite group of Lie-type, and using the fact that centralizers of non-trivial 2-elements in $F$ are contained in $H$ we are led to define

\begin{equation}
E(X(q), s) := \begin{cases} 
\text{Irr}^u(X(q)) & \text{if } s = 1; \\
\mathcal{E}(H, s) & \text{otherwise},
\end{cases}
\end{equation}

where $s$ runs over a complete set of (fully $F$-centralized) $F$-conjugacy class representatives. We then set $k_d(X(q))$ equal to the number of characters of defect $d$ in $\text{Irr}(X(q))$ and prove that the following analogue of Robinson's Conjecture holds for the space $X(q)$:

**Theorem 2.4.** For each value of $d$ listed in Table 1 we have $m(F(q), 0, d) = k_d(X(q))$.

We prove Theorem 2.4 in Section 6.2. Note that the restriction on $d$ is necessary. Indeed six of the twenty-two spetsial characters in $\text{Irr}^u(X(q))$ have defect 0 (possibly corresponding to ‘other blocks’ associated to $X(q)$).

Finally, we use Theorem 2.2 to give further evidence towards the various conjectures considered in [KLLS18, Section 2]. Recall the definition of the invariant $k(D, \alpha)$ as given in [KLLS18, Section 1]. One of the main results in [KLLS18] is that Alperin’s Weight Conjecture implies $k(D, \alpha) = m(D, \alpha)$.

**Theorem 2.5.** Let $D \in \{H, F\}$ and assume that $k(D, 0) = m(D, 0)$. Then [KLLS18] Conjectures 2.1, 2.5, 2.8, 2.9 and 2.10 are true for the pair $(D, 0)$.

**Structure of the paper.** For $D \in \{H, F\}$ in Section 3 we provide a combinatorial description of the action of $\text{Out}_P(P)$ on $\text{Irr}(P)$ for each $P \in D^r$ a $D$-centric radical subgroup. This is used to prove Corollary 3.9 which asserts that when $l > 0$ the quantity $w_P(D, 0, d)$ is a rational polynomial in $x$ for each $d \geq 0$. In Section 4 we explicitly calculate these polynomials using a computer and deduce Theorem 2.2. In Section 5 we prove Theorems 2.3 and 2.5. In Section 6 we briefly introduce spetses and $p$-compact groups before proving Theorem 2.4.

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3. Characters of centric radical subgroups

For \( D \in \{ \mathcal{H}, \mathcal{F} \} \) a complete classification of the \( D \)-centric radical subgroups of \( S \) and their automorphism groups is one of the main results in [LS17]. While we do not repeat that classification here we emphasise that:

\[
\text{the notation used to describe } D^c \text{ is identical to that in [LS17] Sections 2 and 3].}
\]

The aforementioned classification is understood using certain group \( K \) which contains \( S \) as a Sylow 2-subgroup. This group represents the ‘difference’ between \( \mathcal{H} \) and \( \mathcal{F} \) in the sense that \( \mathcal{F} = \langle \mathcal{H}, K \rangle_S \) is the smallest fusion system on \( S \) generated by morphisms in \( \mathcal{H} \) and \( K = F_S(K) \). The following fact concerning \( K \) will be needed in Section 4:

**Proposition 3.1.** Let \( D \in \{ \mathcal{H}, \mathcal{F} \} \) and \( P \in D^c \) be a \( D \)-centric radical subgroup. Then either \( \text{Out}_D(P) \leq \text{Out}_K(P) \) (with index at most 3) or else \( P \in \{ R_{17}, R_1^{17}, R_{1^72}, CS(E/Z), A, CS(E) \} \).

**Proof.** See [LS17, Tables 1,3,4]. \( \square \)

One useful observation is that the elements of \( D^c \) satisfying \( \text{Out}_D(P) \leq \text{Out}_K(P) \) are groups containing a normal subgroup isomorphic to a central product of three generalised quaternion groups of index at most 2. The characters of such groups have an elementary description by virtue of the following result. If \( G_0 \leq G \) are groups, we let \( \text{Aut}(G,G_0) \) denote the set of automorphisms of \( G \) which fix \( G_0 \).

**Theorem 3.2** (Method of little groups). Let \( G \) be a group and \( G_0 \) be a normal subgroup of \( G \), and define \( \Theta := \{ \text{Irr}(G_0)/G_0 \} \) to be set of orbit representatives for the action of \( G \) on \( \text{Irr}(G_0) \). Suppose that each \( \theta \in \text{Irr}(G_0) \) extends to a character \( \hat{\theta} \in \text{Irr}(I_G(\theta)) \) (so \( \hat{\theta}|_{G_0} = \theta \)) where \( I_G(\theta) \) denotes the inertia subgroup. The following hold:

1. There is a bijection
   \[
   \Phi : \{ (\theta, \beta) \mid \theta \in \Theta \text{ and } \beta \in \text{Irr}(I_G(\theta)/G_0) \} \longrightarrow \text{Irr}(G)
   \]
   which sends a pair \( (\theta, \beta) \) to the character \( \hat{\beta}(\hat{\theta})^G \), where \( \hat{\theta} \in \text{Irr}(I_G(\theta)) \) is some (any) character which extends \( \theta \).

2. \( \Phi \) is \( \text{Aut}(G,G_0) \)-equivariant in the sense that for each pair \( (\theta, \beta) \) as above and element \( \alpha \in \text{Aut}(G,G_0) \) the following hold:
   (i) \( I_G(\theta) \alpha = I_G(\theta^\alpha) \);
   (ii) there exists an extension \( \hat{\beta}^\alpha := \hat{\theta}(\theta^\alpha) \in \text{Irr}(I_G(\theta^\alpha)) \) of \( \theta^\alpha \); and
   (iii) we have, \( \Phi((\theta^\alpha, \beta^\alpha)) = \Phi(\theta, \beta)^\alpha \).

**Proof.** (1) is proven in [LS17, Theorem 4.2] and follows from [CR90 11.5]. If \( (\theta, \beta) \) and \( \alpha \) are as in (2), then for each \( g \in I_G(\theta) \) we have \( \theta^{\alpha \circ \theta \alpha} = \theta^{\alpha \circ \alpha} = \theta^\alpha \) from which we conclude that \( g \alpha \in I_G(\theta^\alpha) \) and (i) holds. This shows that \( \hat{\theta}^\alpha \in \text{Irr}(I_G(\theta^\alpha)) \) extends \( \theta^\alpha \) and (ii) also holds. Finally, if \( T = [G/I_G(\theta)] \) is a transversal then for each \( g \in G \),

\[
(\hat{\theta}(\beta)^G(g) \sum_{t \in T} \hat{\theta}(t \alpha)^{-1} g \alpha t \alpha^{-1} g \alpha t \alpha) = \sum_{t \in T} \hat{\theta}(t^{-1} g \alpha t)^{-1} g \alpha t = (\hat{\theta}(\beta)^G)^G(g),
\]

since \( T \alpha \) is a transversal for \( I_G(\theta^\alpha) \) in \( G \) by (i), proving (iii). \( \square \)
When applying Theorem 3.2 we will require a precise description of the irreducible characters of a generalized quaternion group. This is provided by Lemma 3.4 below using the following standard character-theoretic fact:

**Lemma 3.3.** Let $P$ be a non-abelian $p$-group with an abelian subgroup $P_0$ of index $p$. Let $Z_0$ be a subgroup of order $p$ in $P' \cap P_0 \cap Z(P)$. Then every irreducible character of $P$ is either:

1. the lift of an irreducible character of $P/Z_0$; or
2. $\psi^P$ with $\psi \in \text{Irr}(P_0)$ and $Z_0 \nsubseteq \ker(\psi)$

Moreover there are $p^{n-2} - p^{n-3}$ characters of the second type, where $|P| = p^n$.

**Proof.** See [JL01, Theorem 26.4].

**Lemma 3.4.** Let $R := \langle a, b \mid a^{2^{i+2}} = b^4 = 1, a^{2^i+1} = b^2, b^{-1}ab = a^{-1} \rangle$ be a generalized quaternion group of order $2^{i+3}$ and let $\omega \in \mathbb{C}$ be a primitive root of unity of order $2^{i+2}$. The following hold:

1. $R$ has $2^{i+1}+3$ conjugacy classes with a complete set of representatives given by $\{b, ab\} \cup \{a^i \mid i = 0, 1, \ldots, 2^{i+1} \}$.
2. Every non-linear irreducible character of $R$ is of degree 2 and given by

$$
\psi_{u,t}(g) := \begin{cases} 
0 & \text{if } g \notin \langle a \rangle; \\
\omega^{2^i-ut} + \omega^{-2^i-ti} & \text{if } g = a^t, \text{ some } 0 \leq i \leq 2^{i+2}-1.
\end{cases}
$$

for some $0 \leq u \leq l$ and $t = 1, 3, \ldots, 2^{u+1}-1$.

In particular, $|\{\chi \in \text{Irr}(R) \mid Z(R) \nsubseteq \ker(\chi)\}| = 2^l$.

**Proof.** Part (1) is proved in [LS17]. By Lemma 3.3, every character is either the lift of an irreducible character of $R/Z(R)$ or of the form $\psi^R$ where $\psi \in \text{Irr}(\langle a \rangle)$ and $Z(R) \nsubseteq \ker(\psi)$.

Since $R/Z(R)$ is generalized quaternion, and characters of $Q_8$ are well-known we can calculate the degree 2 characters inductively. Every character of $\langle a \rangle$ is given by $\varphi_t(a^i) = \omega^{ti}$ for some $0 \leq t \leq 2^{i+2}-1$, and one easily shows that:

(a) $Z(R) \nsubseteq \ker(\varphi_t) \iff t \equiv 1 \mod 2$;
(b) $\varphi_t^R = \varphi_s^R \iff s + t \equiv 0 \mod 2^{i+2}$.

Thus, apart from the characters obtained by lifting $R/Z(R)$ we obtain $2^{i+1} - 2^i = 2^i$ distinct characters by inducing from $\langle a \rangle$ given by

$$
\psi_{l,t}(g) := \begin{cases} 
0 & \text{if } g \notin \langle a \rangle; \\
\omega^{ti} + \omega^{-ti} & \text{if } g = a^t, \text{ some } 0 \leq i \leq 2^{i+2}-1.
\end{cases}
$$

for each $t = 1, 3, \ldots, 2^{i+1}-1$. (2) now follows by induction.

The following example is illustrative of our general approach to calculating the integers $w_P(D, 0, d)$. It is also included to emphasise that many of the computer computations in this paper replace lengthy and unenlightening by-hand calculations. Recall from the introduction that $x = 2^i$.

**Example 3.5.** Let $U$ be as in [LS17, Section 2.6], so that $C_S(U) \in \mathcal{F}^{cr}$. For all $l > 0$ and $P = C_S(U)$ we have

$$
w_P(\mathcal{F}, 0, 3l + 6) = -\frac{1}{3}(8x^3 - 6x^2 - 8x + 9).$$
Proof. Set \( d := 3l + 6 \). Since \( v_2(|P|) = 3l + 9 \), we must consider characters of degree 8. By [LS17, Notation 3.3], \( P = (R_1 \times R_2 \times R_3)(c)/Z \) where \( R_i \cong Q_{2l+3} \). \( c \) interchanges the two classes of subgroups isomorphic with \( Q_8 \) in each component and \( Z = \langle z_1, z_2, z_3 \rangle \) with \( \langle z_i \rangle = Z(R_i) \) for each \( i \). By Theorem 3.2, for each \( \chi \in \text{Irr}^d(P) \), either:

(a) \( \chi = (\theta_1 \otimes \theta_2 \otimes \theta_3)^P \), for some \( \theta_1 \otimes \theta_2 \otimes \theta_3 \in \text{Irr}(R_1 \times R_2 \times R_3) \) of degree 4 which is moved by \( c \) and whose kernel contains \( Z \); or

(b) \( \chi \) is one of two constituents of \( (\theta_1 \otimes \theta_2 \otimes \theta_3)^P \) for some \( \theta_1 \otimes \theta_2 \otimes \theta_3 \in \text{Irr}(R_1 \times R_2 \times R_3) \) of degree 8 which is fixed by \( c \) and whose kernel contains \( Z \).

From Lemma 3.4 we see that \( c \) fixes all degree 2 characters of the \( R_i \) and interchanges two of the four linear characters in each component. Write \( \text{Irr}_{(a)}^d(P) \) and \( \text{Irr}_{(b)}^d(P) \) for the sets of characters in cases (a) and (b) respectively. By Theorem 3.2(2) the action of \( \text{Out}_F(P) \cong S_3 \) on \( \text{Irr}^{3d+6}(P) \) is described by permutation of the tensor factors in the corresponding character of \( R_1 \times R_2 \times R_3 \), and this action preserves the decomposition \( \text{Irr}^d(P) = \text{Irr}_{(a)}^d(P) \sqcup \text{Irr}_{(b)}^d(P) \) described above. Let \( [C_\tau] \in \text{Out}_F(P) \) be the class of \( C_\tau \in \text{Aut}_F(P) \) (where \( \tau \) is the element defined in [LS17, Notation 2.11]) and set \( \sigma_1 := (1) \) and \( \sigma_2 := (1 < [C_\tau]) \). Thus \( \{\sigma_1, \sigma_2\} \) is a complete set of \( \text{Out}_F(P) \)-representatives for elements of \( N_P \). We split the sum

\[
\text{w}_P(F, 0, d) = \sum_{\sigma \in N_P/\text{Out}_F(P)} (-1)^{[\sigma]} \sum_{\chi \in \text{Irr}^d(P)/I(\sigma)} z(kI(\sigma, \chi))
\]

\[
= \sum_{\chi \in \text{Irr}^d(P)/I(\sigma_1)} z(kI(\sigma_1, \chi)) - \sum_{\chi \in \text{Irr}^d(P)/I(\sigma_2)} z(kI(\sigma_2, \chi)),
\]

and then for each \( i = 1, 2 \) we have

\[
\sum_{\chi \in \text{Irr}^d(P)/I(\sigma_i)} z(kI(\sigma_i, \chi)) = \sum_{\chi \in \text{Irr}_{(a)}^d(P)/I(\sigma_i)} z(kI(\sigma_i, \chi)) + \sum_{\chi \in \text{Irr}_{(b)}^d(P)/I(\sigma_i)} z(kI(\sigma_i, \chi)),
\]

which gives a total of four sums to compute. The following notation will be helpful when enumerating characters whose kernel contains \( Z \):

\[
\text{Irr}_0(R_i) := \{\mu \in \text{Irr}(R_i), \mu(1) = 2, Z(R_i) \leq \ker(\mu)\}; \text{ and } \text{Irr}_1(R_i) := \{\nu \in \text{Irr}(R_i), \nu(1) = 2, Z(R_i) \not\leq \ker(\nu)\}.
\]

for each \( 1 \leq i \leq 3 \). By Lemma 3.4, \( |\text{Irr}_0(R_i)| = x - 1 \) and \( |\text{Irr}_0(R_i)| = x \).

Case 1: \( \chi \in \text{Irr}_{(a)}^d(P)/I(\sigma_1) \). Let \( \rho \) be a linear character of \( R_3 \) which is moved by \( c \).

In Table 2 we list orbit representatives \( \theta \in \text{Irr}(R_1 \times R_2 \times R_3) \) satisfying the conditions of (a) according to their ‘type’, together with the number of such representatives, and their \( I(\sigma_1) \)-stabilizers. From the table, we obtain a (positive) contribution to \( \text{w}_P(F, 0, 3l + 6) \) of

\[
\binom{x - 1}{2} + \binom{x}{2}
\]

from characters of types 1 and 3.

Case 2: \( \chi \in \text{Irr}_{(a)}^d(P)/I(\sigma_2) \). We have \( I(\sigma_1) = \langle [C_\tau] \rangle \) and the negative contribution to
Table 2. $I(\sigma_1)$-orbits of characters in case (a) and their stabilizers

| Type | $\theta$ | Conditions | $\#$ | $\text{stab}_{I(\sigma_1)}(\chi)$ |
|------|-----------|-------------|------|----------------------------------|
| 1    | $\mu_1 \otimes \mu_2 \otimes \rho$ | $\mu_i \in \text{Irr}_0(R_1), \mu_1 \neq \mu_2, \left(\frac{x-1}{2}\right)$ | 1    |                                   |
| 2    | $\mu_1 \otimes \mu_2 \otimes \rho$ | $\mu_i \in \text{Irr}_0(R_1), \mu_1 = \mu_2, \ x - 1$ | $\langle c_r \rangle$ |                                   |
| 3    | $\nu_1 \otimes \nu_2 \otimes \rho$ | $\nu_i \in \text{Irr}_1(R_1), \nu_1 \neq \nu_2, \left(\frac{x}{2}\right)$ | 1    |                                   |
| 4    | $\nu_1 \otimes \nu_2 \otimes \rho$ | $\nu_i \in \text{Irr}_1(R_1), \nu_1 = \nu_2, \ x$ | $\langle c_r \rangle$ |                                   |

Table 3. $I(\sigma_2)$-orbits of characters in case (b) and their stabilizers

| Type | $\theta$ | Conditions | $\#$ | $\text{stab}_{I(\sigma_2)}(\chi)$ |
|------|-----------|-------------|------|----------------------------------|
| 1    | $\nu_1 \otimes \nu_2 \otimes \mu_3$ | $\nu_i \in \text{Irr}_1(R_1), \mu_3 \in \text{Irr}_0(R_3), \nu_1 \neq \nu_2$ | $x(x-1)^2$ | 1                               |
| 2    | $\nu_1 \otimes \nu_2 \otimes \mu_3$ | $\nu_i \in \text{Irr}_1(R_1), \mu_3 \in \text{Irr}_0(R_3), \nu_1 = \nu_2$ | $2x(x-1)$ | $\langle c_r \rangle$           |
| 3    | $\nu_1 \otimes \mu_2 \otimes \nu_3$ | $\nu_i \in \text{Irr}_1(R_4), \mu_2 \in \text{Irr}_0(R_2)$ | $2x^2(x-1)$ | 1                               |
| 4    | $\mu_1 \otimes \mu_2 \otimes \mu_3$ | $\mu_i \in \text{Irr}_0(R_1), \mu_1 = \mu_2$ | $(x-1)^2$ | $\langle c_r \rangle$           |
| 5    | $\mu_1 \otimes \mu_2 \otimes \mu_3$ | $\mu_i \in \text{Irr}_0(R_1), \mu_1 \neq \mu_2$ | $(x-1)^2(x-2)$ | 1                               |

$w_P(\mathcal{F}, 0, 3l + 6)$ in this case is

$$\left(\frac{x - 1}{2}\right) + \left(\frac{x}{2}\right) + (x - 1)^2 + x^2.$$ 

Case 3: $\chi \in \text{Irr}^d(P)/I(\sigma_2)$. Table 3 lists the possible orbit representatives satisfying (b), where we see that a negative contribution to $w_P(\mathcal{F}, 0, 3l + 6)$ of

$$x(x-1)^2 + 2x^2(x-1) + (x-1)^2(x-2)$$

is provided by representatives of types 1, 3 and 5.

Case 4: $\chi \in \text{Irr}^d(P)/I(\sigma_1)$. Here we obtain a contribution of

$$x(x-1)^2 + \frac{(x-1)(x-2)(x-3)}{3} + 2(x-1).$$

Combining Cases 1 to 4 we have $w_P(\mathcal{F}, 0, 3l + 6)$ is equal to

$$-\left((x-1)^2 + x^2\right) + \frac{(x-1)(x-2)(x-3)}{3} + 2(x-1) - 2x^2(x-1) - (x-1)^2(x-2)$$

$$= -\frac{1}{3}(8x^3 - 6x^2 - 8x + 9),$$

whence the result.

Our general strategy is to reduce the computation of $w_P(\mathcal{D}, 0, d)$ to a few small values of $l$ by proving that it is a polynomial in $x$. This is achieved in Corollary 3.9 below. We are
most reliant on this approach in cases where we do not have an explicit description of the action of $\text{Out}_D(P)$ on $\text{Irr}(P)$ such as when $D = F$, $P = C_S(E) \in F^{cr}$ (as in [LS17, Section 2.6]) and $\text{Out}_D(P) = \text{GL}_3(2)$. Here, the following lemma is relevant:

**Lemma 3.6.** For all $k \geq 1$ there exists an embedding $G_0 \hookrightarrow \text{GL}(V) = \text{GL}_3(\mathbb{Z}/2^k)$ with $G_0 := \text{GL}_3(2)$ and $V := (\mathbb{Z}/2^k \mathbb{Z})^3$. For each subgroup $W \leq G_0$, let $O$ be the set of orbits for the action of $W$ on $V$ (or $V^*$). For each $W_0 \leq W$, the integer $|\{\alpha \in O \mid \text{stab}_W(\alpha) = W_0\}|$ can be expressed as a rational polynomial in $x$ of degree at most 3.

**Proof.** By [DW93, Theorem 4.1], we have an embedding $G_0 \hookrightarrow \text{GL}_3(\mathbb{Z}_2)$ where $\mathbb{Z}_2$ denotes the 2-adic numbers. Let $D$ be the kernel under reduction mod $2^k$, which fits into an exact sequence

$$0 \rightarrow D \rightarrow \text{GL}_3(\mathbb{Z}_2) \rightarrow \text{GL}_3(\mathbb{Z}_2/2^k \mathbb{Z}_2) \rightarrow 1,$$

where $\text{GL}_3(\mathbb{Z}_2/2^k \mathbb{Z}_2) \cong \text{GL}_3(\mathbb{Z}/2^k)$. Since $G$ is simple, $G \cap D = 1$ and the first statement holds. Now let $V, W$ and $O$ be as in the lemma. For each $W_0 \leq W$, define

$$O_{W_0} := \{\alpha \in O \mid \text{stab}_W(\alpha) = W_0\} \text{ and } P_{W_0}(W) := \{X \leq W \mid W_0 \leq X\}.$$

Clearly,

$$(3.1) \quad |O_{W_0}| = \frac{1}{|W : W_0|} \cdot |C_V(W_0)\setminus \bigcup_{X \in P_{W_0}(W)} C_V(X)|.$$

By inclusion-exclusion, we have

$$\left| \bigcup_{X \in P_{W_0}(W)} C_V(X) \right| = \sum_{X \in P_{W_0}(W)} |C_V(X)| - \sum_{\{X,Y\} \subset P_{W_0}(W)} |C_V(X) \cap C_V(Y)| \pm \cdots \pm \left| \bigcap_{X \in P_{W_0}(W)} C_V(X) \right|$$

is a rational polynomial in $x$ of degree at most 3, so the same is true of $|O_{W_0}|$ by (3.1). Replacing $V$ with $V^*$ yields the analogous result for $V^*$ and the lemma is proved. \(\square\)

It will be helpful to distinguish centric radical subgroups by their cardinality:

**Definition 3.7.** For $D \in \{H, F\}$, a centric radical subgroup $P \in D^{cr}$ has type $a$ if for some $b \in \mathbb{Z}$ we have $v_2(|P|) = al + b$.

**Proposition 3.8.** Suppose that $l > 0$, $D \in \{H, F\}$ and $P \in D^{cr}$ has type $a$. For any $d \geq 0$ and $W \leq \text{Out}_D(Q)$ let $O$ be the set of orbits for the action of $W$ on $\text{Irr}^d(P)$. For each $W_0 \leq W$, the integer $|\{\alpha \in O \mid \text{stab}_W(\alpha) = W_0\}|$ can be expressed as a rational polynomial in $x$ of degree at most $a$.

**Proof.** Let $D, P, a, W, W_0$ and $O$ be as in the statement of the Proposition and set $O_{W_0} := \{\alpha \in O \mid \text{stab}_W(\alpha) = W_0\}$. We may assume that $a > 0$ since otherwise the result holds trivially.

Let $R_0 \leq S$ be defined as in [LS17, Notation 2.12] so that $R_0$ is the product of three generalized quaternion groups each of order $2^{l+3}$, and set $P_0 := P \cap R_0$. Suppose first that $P_0 = X_1 X_2 X_3$ is a product of three quaternion groups, and let $k$ be the number of these isomorphic with $Q_8$. Note that $|P : P_0| \leq 2$. By Theorem 3.2 there is a bijection

$$\Phi : \{(\theta, \beta) \mid \theta \in \text{Irr}^{d_0}(P_0)/W \text{ and } \beta \in \text{Irr}^{d_1}(I_P(\theta)/P_0)\} \rightarrow O,$$
and it is easily seen that \( d_0 + d_1 = d \). \( \Phi \) restricts to a bijection

\[
\Phi_{W_0} : \{(\theta, \beta) \mid I_W(\theta, \beta) = W_0, \theta \in \text{Irr}^{d_0}(P_0)/W, \beta \in \text{Irr}^{d_1}(I_P(\theta)/P_0)\} \rightarrow \mathcal{O}_{W_0}.
\]

Since \( \text{Irr}(P_0) = \{\chi_1 \otimes \chi_2 \otimes \chi_3 \mid \chi_i \in \text{Irr}(X_i)\} \) the action of \( W \) on \( \text{Irr}(P_0) \) is determined by the action on \( P_0 \). Thus we see that, for each \( d_0 \geq 0 \), the number of \( W \)-orbits of elements of \( \text{Irr}^{d_0}(P_0) \) with stabilizer equal to \( W_0 \leq W \) is a sum of binomial coefficients in \( x \) (as in Example 3.3) and \( |\mathcal{O}_{W_0}| \) is a rational polynomial in \( x \) of degree at most \( 3 - k = a \). Next suppose that \( P \in \{C_S(E/Z), C_S(E)\} \) so that \( P \) contains the torus \( T \) as defined in [LS17, Section 2.5]. Note that in either case there is \( P_0 \leq P \) such that \( P = T \rtimes P_0 \) and \( P \) has type 3. Now by Theorem 3.2, \( |\mathcal{O}_{W_0}| \) is a rational polynomial in \( x \) if the same is true of \( |\{\alpha \in \text{Irr}(T)/W \mid \text{stab}_W(\alpha) = W_0\}| \), and the result follows from Lemma 3.6.

Finally suppose that \( P = R_{1+2} \). Then \( P \) is isomorphic to a group \((Y_1 \times Y_2 \times Y_3)/Z\) where \( Y_1 = Q_8, Y_2 = D_8, Y_3 = Q_{2^{i+3}} \) and \( Z := \langle (z_1, z_2, 1), (z_1, 1, z_3) \rangle \) where \((z_i) = Z(Y_i) \) for \( i = 1, 2, 3 \). Hence \( P \) is a central product \( 2_1^{1+4} \times Q_{2^{i+3}} \), and \( \text{Out}_D(P) = \text{Out}(2_1^{1+4}) = S_5 \). Then \( \chi = \chi_1 \otimes \chi_2 \otimes \chi_3 \in \text{Irr}(Y_1 \times Y_2 \times Y_3) \) has \( Z \leq \ker(\chi) \) then either \( \chi_i(1) = 1 \) for all \( i \) or else \( \chi_i(1) = 2 \) for one or three values of \( i \), one of which is 3. In this latter case either \( \chi(1) = 2 \) and there are \( x - 1 \) choices for \( \chi_3 \) or else \( \chi(1) = 8 \) and there are \( x \) choices. Since \( W \) acts trivially on \( \text{Irr}(Y_3) \), we see that for all values of \( d \), \( |\mathcal{O}_{W_0}| \) is a polynomial in \( x \) determined by the action of \( \text{Out}_D(P) \) on the image of \( Y_1 \times Y_2 \) in \( P \).

\[
\corollary 3.9. \text{Suppose that } l > 0, \mathcal{D} \in \{\mathcal{H}, \mathcal{F}\} \text{ and } P \in \mathcal{D}^{cr} \text{ has type } a. \text{ Then for each } d \geq 0, w_P(\mathcal{D}, 0, d) \text{ is a rational polynomial in } x \text{ of degree at most } a.\]

\textbf{Proof.} Let \( \mathcal{D} \) and \( P \) be as in the statement and set \( G := \text{Out}_\mathcal{F}(P) \). From Section 2 we have

\[
(3.2) \quad w_P(\mathcal{D}, 0, d) := \sum_{\sigma \in \mathcal{N}_P/G} (-1)^{|\sigma|} \sum_{\mu \in \text{Irr}^d(P)/I(\sigma)} z(k_0 I(\sigma, \mu)).
\]

Fix \( \sigma \in \mathcal{N}_P \) and let \( \mathcal{O} \) be the set of orbits for the action of \( W := I(\sigma) \) on \( \text{Irr}^d(P) \). For each \( W_0 \leq W \), write \( \mathcal{O}_{W_0} := \{\alpha \in \mathcal{O} \mid \text{stab}_W(\alpha) = W_0\} \). We have

\[
\sum_{\chi \in \text{Irr}^d(P)/W} z(k I(\sigma, \chi)) = \sum_{W_0 \leq W} z(k W_0) \times |\mathcal{O}_{W_0}|
\]

is a rational polynomial in \( x \) of degree at most \( a \) by Proposition 3.8. Hence by (3.2) the same is true of \( w_P(\mathcal{D}, 0, d) \). \( \square \)

\section{Proof of Theorem 2.2}

In order to apply Corollary 3.9, we need computable representations of elements of \( \mathcal{D}^{cr} \) and their automorphism groups. If \( P \in \mathcal{D}^{cr} \) then a convenient means of constructing \( \text{Out}_\mathcal{D}(P) \) is to first construct a model \( M_P \) for \( N_P(P) \) (see [AKO11, Theorem 5.10]). By Proposition 3.1 such a model is provided, in a large number of cases, by the group \( N_K(P) \). To calculate these models we work with an explicit 6-dimensional representation of the group \( K \) which we now describe. Following [LS17, Section 2.4], under the natural inclusion of \( \text{SL}_2(q) \) into \( \text{SL}_2(q^2) \) let \( N := N_{\text{SL}_2(q^2)}(\text{SL}_2(q)) \) be its normalizer. Form the wreath product \( W := N \wr S_3 \) and let \( N_0 := N_1 \times N_2 \times N_3 \) and \( X = S_3 = \langle \tau, \gamma \rangle \) be the base and acting group, where \( \tau \) and
\( \gamma \) act like \((1, 2)\) and \((2, 3)\) respectively. The natural representations \( \rho_i : N_i \hookrightarrow \text{SL}_2(q^2) \) induce a representation \( \rho \) of \( W \) given by

\[
N_0 \mapsto \begin{pmatrix}
\rho_1(N_1) & 0 & 0 \\
0 & \rho_2(N_2) & 0 \\
0 & 0 & \rho_3(N_3)
\end{pmatrix}, \quad \tau \mapsto \begin{pmatrix}
0 & I_2 & 0 \\
I_2 & 0 & 0 \\
0 & 0 & I_2
\end{pmatrix}, \quad \gamma \mapsto \begin{pmatrix}
I_2 & 0 & 0 \\
0 & I_2 & 0 \\
0 & 0 & I_2
\end{pmatrix},
\]

where \( I_2 \) denotes the \( 2 \times 2 \) identity matrix. This leads to a representation of \( K := \hat{K}/Z(\hat{K}) \), where \( \hat{K} := O^2(N_0)C_{N_0}(X)X \leq W \) and \( Z(\hat{K}) = \langle (-1,-1,-1) \rangle \leq N \). Combining the above discussion with Proposition 3.1 we have proved the following:

**Lemma 4.1.** Let \( D \in \{ H, F \} \) and \( P \in D^\text{cr} \) be a \( D \)-centric radical subgroup. If \( \text{Out}_D(P) \leq \text{Out}_K(P) \) then there exists a faithful 6-dimensional representation of \( N_K(P) \) over \( \mathbb{F}_{q^2} \).

We now deal with the elements of \( D^\text{cr} \) to which Lemma 4.1 does not apply. Firstly if \( P = R_{17} \) or \( P = R'_{17} \) we take the 7-dimensional representations of \( M_P \) provided by taking the normalizer of \( P \) in \( \text{Spin}_7(3) \) or \( \text{Spin}_7(9) \) respectively. If \( P = R_{152} \) we use the description of \( P \) as a central product \( 2^{1+4} \times Q_{2^{1+3}} \) as in the proof of Proposition 3.8 to construct \( M_P \) as a semi-direct product \( P \rtimes \text{GO}_1^2(2) \). When \( P = A \), it is well-known that \( M_P \) is the unique non-split extension of \( \text{GL}_4(2) \) by \( \mathbb{F}_2^4 \) (see [Cra11, Theorem 9.15].) This leaves the groups \( C_S(E) \) and \( C_S(E/Z) \) for which models are constructed inductively using the following result:

**Lemma 4.2.** For all \( l \geq 0 \) a model \( M_P \) for \( P = C_S(E) \) in \( F \) can be constructed as a semi-direct product \( T \rtimes G \), where \( T = \langle \mathbb{Z}/2^{l+2}, 3 \rangle \), \( G = G_0 \times \langle -I_3 \rangle \leq \text{GL}_3(\mathbb{Z}/2^{l+2}) \) and \( G_0 \cong \text{GL}_3(2) \). If \( P = C_S(E/Z) \) then \( M_P \) can be constructed as a maximal subgroup of \( M_{C_S(E)} \) with \( M_P/T \cong C_2 \times S_4 \).

**Proof.** As noted in the proof of Lemma 3.6, the inclusion \( G_0 := \text{GL}_3(2) \hookrightarrow \text{GL}_3(\mathbb{Z}/2^k) \) of [DW93, Theorem 4.1] gives rise to an embedding \( G_0 \hookrightarrow \text{GL}_3(\mathbb{Z}/2^k) \) for each \( k \geq 0 \). Hence we construct \( G_0 \) inductively as a group isomorphic to \( \text{GL}_3(2) \) in the preimage (isomorphic to \( \text{GL}_3(\mathbb{Z}/4) \)) under the map \( \text{GL}_3(\mathbb{Z}/2^{k+1}) \to \text{GL}_3(\mathbb{Z}/2^k) \) given by reduction modulo 2. Since \( C_S(E) \) is isomorphic to a semidirect product \( T \rtimes C_2 \), where the \( C_2 \) factor acts by inversion, we can construct \( M_P \) as in the statement of the theorem. By saturation, \( \text{Out}_F(C_S(E/Z)) \) can be identified with \( N_{\text{Out}_F(C_S(E))}(\text{Out}_C(S(E/Z))/(C_S(E)) \cong S_4 \) whence the last statement.

We can now prove:

**Theorem 4.3.** For \( l > 0 \), \( D \in \{ H, F \} \) and \( P \in D^\text{cr} \) the polynomials \( w_P(D, 0, d) \) are listed in Table 4 according to the type of \( P \).

**Proof.** Fix \( d \geq 0 \) and let \( D \in \{ H, F \} \) and \( P \in D^\text{cr} \) have type \( a \) for some \( 0 \leq a \leq 3 \). By Corollary 3.9, \( w_P(D, 0, d) \) is determined by the values it takes when \( 1 \leq l \leq a + 1 \). We calculate these values using MAGMA [BCP97]. Assume we are given a model \( M_P \) for \( N_D(P) \) as constructed above and let \( G \) denote the image in \( \text{Out}(P) \) of \( M_P \) under the natural map \( M_P \to \text{Out}(P) \). By [KLLS18, Lemma 4.13], \( N_P \) can be replaced with the set \( \mathcal{E}_P \) of elementary abelian chains in the definition of \( w_P(D, 0, d) \). The computation of \( w_P(D, 0, d) \) is broken down into the following stages:

1. Calculate the set \( \mathcal{E}_P \) of non-empty elementary abelian chains of \( p \)-subgroups of \( G \).
(2) For each $\sigma \in \mathcal{E}_P$, determine $I(\sigma) \leq G$, and the set of conjugates $\sigma^G$; list a set of representatives $\mathcal{E}_P/G$ for $G$-conjugacy classes of chains.

(3) For each representative $\sigma \in \mathcal{E}_P/G$ consider the action of $I(\sigma)$ on $\text{Irr}^d(P)$.

(4) For each $I(\sigma)$-orbit $\mu \in \text{Irr}^d(P)$, determine $C_{I(\sigma)}(\mu)$ and calculate $z(kC_{I(\sigma)}(\mu))$.

We briefly describe the MAGMA implementation. We consider the poset of all elementary abelian subgroups of $G$ to construct $\mathcal{E}_P$ as in (1). $I(\sigma)$ is calculated as a subgroup of the normalizer in $G$ of the largest element in $\sigma$, and then $\sigma^G$ is calculated using a transversal $[G/I(\sigma)]$. We use built-in commands to construct the $I(\sigma)$-set $\text{Irr}^d(P)$. Finally, $z(kC_{I(\sigma)}(\mu))$ is determined using the character table of $C_{I(\sigma)}(\mu)$ by applying [LS17] Theorem 4.1.

\[\square\]

Proof of Theorem 2.2 When $l > 0$, the polynomials in Table 1 are given by summing the appropriate columns in Table 4. It remains to check that for each $d \geq 0 \text{ m}(\mathcal{D}, 0, d)$ evaluates to the correct number when $l = 0$. As in the proof of Table 4.3 we calculate $w_P(\mathcal{D}, 0, d)$ for each $P \in \mathcal{D}^e$ by constructing an appropriate model $M_P$ and then following steps (1)-(4). Models are constructed as follows: if $P = C_S(E)$ or $P = C_S(E/Z)$ we use Lemma 4.2 if $P = C_S(U)$, then $M_P := N_{K}(P)$; and in all remaining cases $M_P = N_{H}(P)$ where $H = \text{Spin}_r(3)$. The results of this calculation are presented in Table 5 and the result follows immediately by comparison with Table 1.

\[\square\]

5. Verification of some conjectures

In this section, we prove Theorems 2.3 and 2.5 beginning with the latter. We need a specific fact concerning $S$:

Proposition 5.1. The following statements hold:

1. The number of conjugacy classes of $S$ is equal to $4x^3 + 15x^2 + 24x + 18$.
2. The number of conjugacy classes of $[S, S]$ is equal to $16x^3 + 12x$.

Proof. (1) follows by summing the entries in the first row of Table 4. By Theorem 3.2, the number of conjugacy classes of $[S, S]$ is a polynomial in $x$ of degree at most 3, and using this we calculate that it is equal to $16x^3 + 12x$.

\[\square\]

Proof of Theorem 2.3. Let $\mathcal{D} \in \{\mathcal{H}, \mathcal{F}\}$ and suppose that $\text{m}(\mathcal{D}, 0) = k(\mathcal{D}, 0)$. Inspecting the list of conjectures given in the theorem, we require:

1. $\text{k}(\mathcal{D}, 0) \leq |S|$;
2. $\text{m}(\mathcal{D}, 0, d) \geq 0$ for each positive integer $d$;
3. $\text{m}(\mathcal{D}, 0, d) \neq 0$ for some $d \neq v_2(|S|)$;
4. $\min \{r : \text{Irr}^{d-r}(S) \neq 0\} = \min \{r : \text{m}(\mathcal{D}, 0, d-r) \neq 0\}$, where $d = v_2(|S|)$;
5. $\text{k}(\mathcal{D}, 0)/\text{m}(\mathcal{D}, 0, d)$ is at most the number of conjugacy classes of $[S, S]$ for each $d \geq 0$;
6. $\text{k}(\mathcal{D}, 0)/\text{w}(\mathcal{D}, 0)$ is at most the number of conjugacy classes of $S$.

All points follow from Table 1 where in (5) and (6) we also invoke Proposition 5.1 and the fact that $\text{w}(\mathcal{D}, 0) = 12$ ([LS17] Theorem 1.1)]. This completes the proof.

\[\square\]

We now turn our attention to Theorem 2.3. We briefly survey the background from Deligne–Lusztig theory we shall need. Let $G = G^F$ be a finite reductive group of characteristic $r$. To a pair $(T, \theta)$ where $T$ is an $F$-stable maximal torus of $G$ and $\theta \in \text{Irr}(T^F)$
A notable property of these characters is that for every \( \chi \in \text{Irr}(G) \) we have \( \langle \chi, R_T^G(\theta) \rangle \neq 0 \) for some pair \((T, \theta)\) as above. Letting \( 1 \in \text{Irr}(T^F) \) denote the trivial character, we say that \( \chi \in \text{Irr}(G) \) is unipotent if \( \langle \chi, R_T^G(1) \rangle \neq 0 \) for some \( T \) and we denote by \( \text{Irr}^u(G) \) the set of all unipotent characters.

Now, for each \( g \in G \) we have a Jordan decomposition \( g = us = su \) of \( g \) into its semisimple and unipotent parts \( s \) and \( u \) respectively. Let \( G^s \) denote the reductive group dual to \( G \), set \( G^s = (G^u)^F \) and assume that \( C_{G^s}(s) \) is connected for each semisimple element \( s \in G^s \). By Jordan’s decomposition of characters there is a partition \( \text{Irr}(G) = \bigcup_s \mathcal{E}(G, s) \) where \( s \) runs over a set of \( G^s \)-conjugacy classes of semisimple elements of \( G^s \). Moreover each series \( \mathcal{E}(G, s) \) is in bijection with the set of unipotent characters \( \text{Irr}^u(C_{G^s}(s)) \). Under this bijection, for each \( \lambda \in \text{Irr}^u(C_{G^s}(s)) \), we denote by \( \chi_{s, \lambda} \) the corresponding element of \( \text{Irr}(G) \). By [DM91, Remark 13.24], we have:

\[
\chi_{s, \lambda}(1) = |G^s : C_{G^s}(s)| \theta(s) \lambda(1),
\]

Here is the main result we shall need:

**Proposition 5.2.** Suppose that \( G = \text{Spin}_r(q) = B_3(q) \) and let \( B \) be the principal 2-block of \( G \). Then \( \text{Irr}(B) = \bigcup_s \mathcal{E}(G, s) \), where \( s \) runs over a complete set \( G \)-conjugacy class representatives of 2-elements of the dual group \( G^* = \text{Aut}(\text{PSp}_6(q)) = C_3(q) \).

**Proof.** See [CE04, Theorem 21.14]. \( \square \)

We can now prove Theorem 2.3.

**Proof of Theorem 2.3.** We use the explicit enumeration of the characters in the principal 2-block of \( \text{Spin}_r(q) = B_3(q) \) provided by Proposition 5.2. The 2-elements in \( C_3(q) \), and the degrees of unipotent characters of their centralizers are calculated using CHEVIE [GHL+96]. The relevant information is summarised in Tables 6 and 7 for \( q \equiv 1 \mod 4 \) and \( q \equiv 3 \mod 4 \) respectively. Precisely, the 2-elements \( s \) are enumerated according to the types of their centralizers, with the number of classes of a given type being given in the third column. For each \( s \) we list \( |G^s : C_{G^s}(s)| \) and the degrees of the unipotent characters of \( C_{G^s}(s) \) in the fourth and fifth columns in terms of cyclotomic polynomials \( \phi_n := \phi_n(q) \). This information is combined to list, in the final column, the numbers \( v_2(\lambda(1)|G^s : C_{G^s}(s)|) \) for each \( \lambda \in \text{Irr}^u(C_{G^s}(s)) \) in terms of \( l \). Note in particular that

\[
v_2(q - 1) = \begin{cases} 
  l + 2 & \text{if } q \equiv 1 \mod 4 \\
  1 & \text{if } q \equiv 3 \mod 4
\end{cases} \quad \text{and} \quad v_2(q + 1) = \begin{cases} 
  1 & \text{if } q \equiv 1 \mod 4 \\
  l + 2 & \text{if } q \equiv 3 \mod 4
\end{cases}.
\]

It is now easy to check from the tables that the number of characters of defect \( d \) coincides with the integer \( m(H, 0, d) \) listed in Table 1 and Theorem 2.3 follows. \( \square \)

6. Local-global correspondences for exotic fusion systems

6.1. \( p \)-compact groups. A connected \( p \)-compact group is is a triple \((X, BX, e)\) where \( X \) is a connected space with \( H^*(X; \mathbb{F}_p) \) finite, \( BX \) is a pointed \( p \)-complete space, and \( e : X \to \Omega BX \) is a weak homotopy equivalence. It is a deep theorem in homotopy theory that connected \( p \)-compact groups are in 1-1 correspondence with \( \mathbb{Z}_p \) root data (see [Gro10, Theorem 2.3].) When \( p \) is odd a root datum is completely determined by the corresponding
finite $\mathbb{Z}_p$-reflection group (see [AGMV08]); when $p = 2$, there is a unique exotic 2-compact group DI(4) first constructed by Dwyer–Wilkerson [DW93] determined by the $\mathbb{Z}_2$-reflection group $G_{24} \cong \text{GL}_3(2) \times 2$ (see [AG09]). If $p$ is odd, $X$ is a simply connected $p$-compact group and $q$ is prime power (prime to $p$), then by [BM07, Theorem A] the space $X(q)$ of homotopy fixed points under the unstable Adams operation $\psi^q$ on $X$ is the classifying space of a saturated fusion system $F(q)$. These are the fusion systems of the so-called Chevalley $p$-local finite groups. When $p = 2$, the fusion systems $\text{Sol}(q)$ are also obtained in this way, by taking $X$ to be the 2-compact group DI(4) (see [LO02], [LO05]). The above discussion motivates the following generalization of Theorem 2.2 to all primes $p$:

**Conjecture 6.1.** Let $X$ be a (simply) connected $p$-compact group, $q$ be a prime power prime to $p$ and $F(q)$ be the saturated $p$-fusion system associated to the space of homotopy fixed points $X(q)$ of the unstable Adams operation $\psi^q$. Let $\alpha$ be a compatible Külshammer–Puig family associated to $F(q)$. Then for each $d \geq 0$, $\mathfrak{m}(F(q),\alpha,d)$ is a rational polynomial in $p^l$ where $l = v_p(1 - q^k)$ and

$$k := \begin{cases} \text{ord}_p(q) & \text{if } p > 2; \\ 2 & \text{if } p = 2, \end{cases}$$

where $\text{ord}_p(q)$ the order of $q$ mod $p$.

When $(F(q),\alpha)$ is realised by a block $B$ of a finite group of Lie type of characteristic $q$, one expects Conjecture 6.1 to follow from Robinson’s Conjecture 2.1 and a result in Deligne–Lusztig theory along the lines of Proposition 5.2. When $(F(q),\alpha)$ is exotic, the possibilities for $F(q)$ are listed in [BM07], and Conjecture 6.1 will be handled for these cases in future work.

### 6.2. Spetses

Using the results of the present paper we give some evidence in support of Conjecture 6.1 using the Broué–Malle–Michel theory of spetses [BMM99]. That is, we treat the topological space $X := \text{DI}(4)$ as if it were a connected reductive algebraic group over $\mathbb{F}_q$ with Weyl group isomorphic to $G_{24}$, and consider its 2-local structure. A spets has a naturally defined set of unipotent characters determined by the corresponding $\Phi$-cyclotomic Hecke algebra (see [BMR98]). We set $\text{Irr}^u(X(q))$ equal to this set of twenty-two characters, the degrees of which are listed in [BMM14, Section A.9]. For the remaining ‘spetsial characters’ of $X(q)$ we take inspiration from Deligne–Lusztig theory or, more precisely, Proposition 5.2. Our starting point is the fact that $C_F(s) \subseteq H$ for each non-trivial fully $F$-centralized element $s \in S$. Hence by [AKO11, Lemma I.1.2] $C_F(s) = C_H(s)$ is a saturated fusion system realized by $C_H(s)$. Since $X$ is the unique exotic 2-compact group with $W_X = G_{24}$, we treat $X$ as if it were Langlands self-dual, and define a ‘Lusztig series’ $\mathcal{E}(X(q),s)$ and a set $\text{Irr}(X(q))$ of irreducible characters exactly as in (2.1). This motivates Theorem 2.4 which we now prove:

**Proof of Theorem 2.4** We require a complete set of $F$-conjugacy class representatives of elements of $S$. We first obtain such a set for $H$ using CHEVIE. We then determine which classes are fused in $F$. If $x,y \in T$ are fully $F$-centralized then $x$ and $y$ are $F$-conjugate if and only if they are $\text{Aut}_F(T)$-conjugate. If $G_x$ realises $C_F(x)$ then the Weyl group of $G_x$ is isomorphic to $\text{Stab}_{\text{Aut}_F(T)}(x)$. We use MAGMA to enumerate orbits on $T$ under the action of $\text{Aut}_F(T)$ for $0 \leq l \leq 3$ from which we determine polynomial expressions for the number of class representatives in $T$ with a given type of centralizer. The classes of elements outside
of $T$ are distinguished according to the type of their centralizer which is $A_1(q^2)$, $A_1(q)$ or $A_0(q)$. For notational convenience write $|X(q) : C_{X(q)}(s)|$ for the index $|H : C_H(s)|$ and set $\text{Irr}^u(C_{X(q)}(s)) := \text{Irr}^u(C_H(s))$ whenever $s \neq 1$. Tables 8 and 9 list the unipotent degrees of elements of $\bigcup_s \mathcal{E}(X(q), s)$ together with their 2-adic valuations. By comparison with Table 11 we observe that for each $d > 0$, $m(F, 0, d)$ is exactly the number of characters in $\bigcup_s \mathcal{E}(X(q), s)$ of defect $d$, as needed.
Table 4. $w_D(D, 0, d)$ for $D \in \{H, F\}$ and all $d \geq 0$ when $l > 0$

| $P$   | $D$   | $a$ | $al+4$ | $al+5$ | $al+6$ | $al+7$ | $al+8$ | $al+9$ | $al+10$ |
|-------|-------|-----|--------|--------|--------|--------|--------|--------|---------|
| $S$   | $H,F$ | 3   |  -     |  -     | 4      | 2      | 6      | 4      | 16      |
| $C_S(U)$ | $F$   | 3   |  -     |  -     | $-\frac{1}{4}(8x^3 - 6x^2 - 8x + 9)$ | $-2(4x^2 - x - 1)$ | $-8x$ | 0     | -       |
| $C_S(E/Z)$ | $H,F$ | 3   |  -     |  -     | $-\frac{5}{4}x^3 + 2x^2 + \frac{1}{4}x - 1$ | $-8x^2 + 8x$ | $-16x + 8$ | $-8$ | -       |
| $C_S(E)$ | $F$   | 3   |  -     |  -     | $\frac{21}{4}x^3 - \frac{8}{4}x + \frac{8}{4}$ | 0     |  -     | -     | -       |
| $R_1R_2Q_3(\tau)$ | $H,F$ | 2   |  -     | $x(x - 1)$ | 4      | 0 | 0 | 0 | -       |
| $Q_1Q_2R_3$ | $H$   | 2   |  -     | $2x(x - 1)$ | 8      | 0 | 0 | -     | -       |
| $Q_1Q_2R_3$ | $H,F$ | 1   |  -     | 0     | 0     | 2      | 8     | -     | -       |
| $Q_1Q_2Q'_3$ | $H$   | 1   |  -     | $x - 1$ | 4     | 0     | 0 | -     | -       |
| $Q_1Q_2Q'_3$ | $H,F$ | 1   |  -     | 0     | 0     | 2      | 8     | -     | -       |
| $Q_1Q_2R_3(\tau)$ | $H,F$ | 1   |  -     | 0     | 0     | 2      | 8     | -     | -       |
| $Q_1Q'_2R_3(\tau)$ | $H,F$ | 1   |  -     | $2x - 1$ | 6     | -8    | 0    | -     | -       |
| $R_1\nu_2$ | $H,F$ | 1   | 0     | 0     | $-2(x - 1)$ | -8 | - | -     | -       |
| $Q_1Q_2Q_3$ | $H,F$ | 0   |  -     | 0     | 0     | 0     | 0     | -     | -       |
| $Q'_1Q_2Q_3$ | $H,F$ | 0   |  -     | 0     | 0     | 0     | 0     | -     | -       |
| $Q_1Q_2Q'_3$ | $H$   | 0   |  -     | 0     | -     | -     | 0     | -     | -       |
| $Q_1Q_2Q'_3(\tau)$ | $H,F$ | 0   |  -     | 0     | 0     | 0     | 0     | -     | -       |
| $Q_1Q_2Q'_3(\tau)$ | $H,F$ | 0   |  -     | 0     | 0     | 0     | 0     | -     | -       |
| $R_1\gamma$ | $H,F$ | 0   | 0     | 0     | 0     | 0     | 0     | -     | -       |
| $R'_1\gamma$ | $H,F$ | 0   | 2     | 0     | 0     | -     | -     | -     | -       |
| $A$   | $F$   | 0   | 0     | 0     | -     | -     | -     | -     | -       |
Table 5. \( w_P(D, 0, d) \) for \( D \in \{ \mathcal{H}, \mathcal{F} \} \) and all \( d \geq 0 \) when \( l = 0 \)

| \( P \) | \( D \) | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| \( S \) | \( \mathcal{H}, \mathcal{F} \) | – | – | 1 | 6 | 18 | 20 | 16 |
| \( Q(\tau') \) | \( \mathcal{H}, \mathcal{F} \) | – | – | 1 | 4 | –8 | 0 | – |
| \( C_S(E/Z) \) | \( \mathcal{H}, \mathcal{F} \) | – | – | 0 | 0 | –8 | –8 | – |
| \( Q(\tau) \) | \( \mathcal{H}, \mathcal{F} \) | – | – | 4 | 8 | –8 | 0 | – |
| \( Q \) | \( \mathcal{H}, \mathcal{F} \) | – | – | 16, 0 | – | 16 | – | – |
| \( R_{17} \) | \( \mathcal{H}, \mathcal{F} \) | 0 | – | – | –8 | – | – | – |
| \( R'_{17} \) | \( \mathcal{H}, \mathcal{F} \) | 2 | – | 0 | – | – | – | – |
| \( A \) | \( \mathcal{F} \) | 0 | – | – | – | – | – | – |
| \( C_S(E) \) | \( \mathcal{F} \) | – | – | 0 | 0 | – | – | – |
| \( C_S(U) \) | \( \mathcal{F} \) | – | – | –1 | –4 | –8 | 0 | – |
| \( m(D, 0, d) \) | \( \mathcal{H}, \mathcal{F} \) | 2 | 0 | 22, 5 | 10, 6 | 10, 2 | 12 | 16 |
Table 6. Degrees of irreducible characters in the principal 2-block of $\text{Spin}_7(q)$, $q \equiv 1 \mod 4$

| type of $C_{G^*}(s)$ | # classes | $|G^* : C_{G^*}(s)|$ | $\lambda(1), \lambda \in \text{Irr}^w(C_{G^*}(s))$ | $v_2(\lambda(1)|G^* : C_{G^*}(s)|), \lambda \in \text{Irr}^w(C_{G^*}(s))$ |
|-----------------------|-----------|------------------|---------------------------------|---------------------------------|
| $C_3(q)$              | 1         | 1                | $\frac{1}{2}q^4\phi_4\phi_6, q^6\phi_3\phi_6, q^2\phi_3\phi_4, q^2, \frac{1}{2}q\phi_3\phi_4, q^2\phi_3\phi_6, \frac{1}{2}q\phi_3\phi_4, \frac{1}{2}q^2\phi_3\phi_4, q^2\phi_3\phi_6, \frac{1}{2}q\phi_3\phi_4, \frac{1}{2}q^2\phi_3\phi_4, q^2\phi_3\phi_6$ | $0, 0, 0, 0, 1, 0, 0, 0, 0, 2l + 3, 2l + 3$ |
| $C_2(q) + A_1(q)$     | 1         | $q^4\phi_3\phi_6$ | $1, \frac{1}{2}q\phi_4, \frac{1}{2}q^2\phi_4, \frac{1}{2}q^3\phi_4, \frac{1}{2}q^4\phi_4, q, \frac{1}{2}q^2\phi_4, \frac{1}{2}q^3\phi_4, \frac{1}{2}q^4\phi_4, q^3$ | $0, 2l + 3, 1, 0, 0, 0, 0, 2l + 3, 1, 0, 0, 0$ |
| $C_2(q)$              | $2x - 1$  | $q^7\phi_2\phi_3\phi_4\phi_6$ | $1, \frac{1}{2}q\phi_4, \frac{1}{2}q^2\phi_4, \frac{1}{2}q^3\phi_4, \frac{1}{2}q^4\phi_4, q, \frac{1}{2}q^2\phi_4, \frac{1}{2}q^3\phi_4, \frac{1}{2}q^4\phi_4, q^3$ | $0, 2l + 3, 1, 0, 0, 0, 0, 2l + 3, 1, 0, 0, 0$ |
| $A_1(q) + A_1(q)$     | $2x - 1$  | $q^7\phi_2\phi_3\phi_4\phi_6$ | $1, q, q^2$ | $2, 2, 2, 2$ |
| $A_1(q) + A_1(q)$     | $x$       | $q^7\phi_2\phi_3\phi_4\phi_6$ | $1, q, q^2$ | $2, 2, 2, 2$ |
| $(A_1(q) + A_1(q))$.2 | $1$       | $\frac{1}{2}q^7\phi_2\phi_3\phi_4\phi_6$ | $1, q^2, q^2, 2q$ | $1, 1, 1, 2$ |
| $A_1(q)$.2            | $1$       | $\frac{1}{2}q^7\phi_2\phi_3\phi_4\phi_6$ | $1, q^2$ | $2l + 4, 2l + 4, 2l + 4, 2l + 4$ |
| $A_1(q)$              | $2x - 1$  | $q^6\phi_2\phi_3\phi_4\phi_6$ | $1, q\phi_2, q^3$ | $3, 4, 3$ |
| $A_2(q)$.2            | $1$       | $\frac{1}{2}q^6\phi_2\phi_3\phi_4\phi_6$ | $1, q, q\phi_2, q\phi_2, q^2, q^3$ | $2, 2, 3, 3, 2, 2$ |
| $2A_2(q)$.2           | $1$       | $\frac{1}{2}q^6\phi_2\phi_3\phi_4\phi_6$ | $1, q\phi_1, q\phi_1, q^3$ | $2l + 4, 2l + 4, 2l + 4, 2l + 4, 2l + 4$ |
| $A_1(q)$              | $2x^2 - 3x + 1$ | $q^8\phi_3^2\phi_4\phi_6\phi_6$ | $1, q$ | $3, 3$ |
| $A_1(q)$              | $x$       | $q^8\phi_1\phi_2\phi_3\phi_4\phi_6$ | $1, q$ | $l + 4, l + 4$ |
| $A_1(q)$              | $4x^2 - 4x + 1$ | $q^8\phi_2^2\phi_3\phi_4\phi_6$ | $1, q$ | $3, 3$ |
| $A_1(q)$              | $x$       | $q^8\phi_1\phi_2\phi_3\phi_4\phi_6$ | $1, q$ | $l + 4, l + 4$ |
| $A_1(q)$              | $x$       | $q^8\phi_1\phi_2\phi_3\phi_4\phi_6$ | $1, q$ | $l + 4, l + 4$ |
| $A_0(q)$              | $\frac{1}{2}x^3 - 3x^2 + 2x$ | $q^9\phi_3^2\phi_4\phi_6\phi_6$ | $1$ | $4$ |
| $A_0(q)$              | $x^2 - x$ | $q^9\phi_1\phi_2^2\phi_3\phi_4\phi_6$ | $1$ | $l + 5$ |
| $A_0(q)$              | $2x^2 - 2x$ | $q^9\phi_1\phi_2\phi_3^2\phi_4\phi_6$ | $1$ | $l + 5$ |
| $A_0(q)$.2            | $x - 1$  | $\frac{1}{2}q^9\phi_1\phi_2^2\phi_3\phi_4\phi_6$ | $1, 1$ | $3, 3$ |
| $A_0(q)$.2            | $x$       | $\frac{1}{2}q^9\phi_1\phi_2\phi_3^2\phi_4\phi_6$ | $1, 1$ | $l + 4, l + 4$ |
| $A_0(q)$.2            | $x$       | $\frac{1}{2}q^9\phi_1\phi_2\phi_3\phi_4\phi_6$ | $1, 1$ | $l + 4, l + 4$ |
| $A_0(q)$.2            | $x - 1$  | $\frac{1}{2}q^9\phi_1\phi_2\phi_3\phi_4\phi_6$ | $1, 1$ | $2l + 5, 2l + 5$ |
Table 7. Degrees of irreducible characters in the principal 2-block of Spin₇(q), q ≡ 3 mod 4

| type of C₂⁻(s) | # classes | |G* : C₂⁻(s)| |λ(1), λ ∈ Irₙ⁺(C₂⁻(s))| |v₂(λ(1)|G* : C₂⁻(s)|), λ ∈ Irₙ⁺(C₂⁻(s)) |
|----------------|-----------|----------------------------|------------------|-------------------------------------------------|
| C₃(q)          | 1         | 1, q                        | 1/4q²φ₁φ₂φ₃φ₄, 1/4q⁻²φ₂φ₃φ₄, 1/4q⁻²φ₂φ₃φ₄, 1/4q⁻²φ₂φ₃φ₄, 1/4q⁻²φ₂φ₃φ₄, 1/4q⁻²φ₂φ₃φ₄, 1/4q⁻²φ₂φ₃φ₄, 1/4q⁻²φ₂φ₃φ₄ | 0, 0, 0, 0, 2f + 3, 0, 0, 2f + 3, 0, 0, 1, 1 |
| C₂(q) + A₁(q)  | 1         | q²φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈      | 1/2 q⁻²φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈, 1/2 q⁻²φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈, 1/2 q⁻²φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈, 1/2 q⁻²φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈ | 0, 1, 2f + 3, 0, 0, 0, 1, 2f + 3, 0, 0, 0 |
| C₂(q)          | 2x - 1    | q⁵φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈      | 1, q, q, q²         | 2, 2, 2, 2                                       |
| A₁(q) + A₁(q)  | 2x - 1    | q⁵φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈      | 1, q, q, q²         | 2, 2, 2, 2                                       |
| A₁(q²)         | x         | q⁷φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈      | 1, q²              | l + 4, l + 4                                    |
| (A₁(q) + A₁(q))₂ | 1         | 1/2 q⁷φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈ | 1, 1, q², q², 2q   | 2f + 4, 2f + 4, 2f + 4, 2f + 4, l + 4           |
| A₁(q²),₂       | 1         | 1/2 q⁷φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈ | 1, 1, q², q²       | 2f + 4, 2f + 4, 2f + 4, 2f + 4, l + 4           |
| ²A₂(q)         | 2x - 1    | q⁶φ₁²φ₂φ₃φ₄φ₅φ₆φ₇φ₈      | 1, qφ₁, q³         | 3, 4, 3                                         |
| ²A₂(q),₂       | 1         | q²φ₁²φ₂φ₃φ₄φ₅φ₆φ₇φ₈      | 1, 1, qφ₁, qφ₁, q³ | 2, 2, 3, 3, 2, 2                                |
| A₂(q),₂        | 1         | q²φ₁²φ₂φ₃φ₄φ₅φ₆φ₇φ₈      | 1, 1, qφ₂, qφ₂, qφ₂, q³ | 2f + 4, 2f + 4, 2f + 4, 2f + 4, 2f + 4 |
| A₁(q)          | 2x² - 3x + 1 | q⁸φ₁²φ₂φ₃φ₄φ₅φ₆φ₇φ₈ | 1, q, q         | 3, 3                                            |
| A₁(q)          | x         | q⁸φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈φ₈    | 1, q              | l + 4, l + 4                                    |
| A₁(q)          | 4x² - 4x + 1 | q⁸φ₁²φ₂φ₃φ₄φ₅φ₆φ₇φ₈ | 1, q            | 3, 3                                            |
| A₁(q)          | x         | q⁸φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈φ₈    | 1, q              | l + 4, l + 4                                    |
| A₁(q)          | x         | q⁸φ₁φ₂φ₃φ₄φ₅φ₆φ₇φ₈φ₈    | 1, q              | l + 4, l + 4                                    |
| A₀(q)          | 1/4x³ - 3x² + 4x | q⁹φ₁³φ₂φ₃φ₄φ₅φ₆φ₇φ₈φ₈ | 1                | 4                                               |
| A₀(q)          | x² - x    | q⁹φ₁³φ₂φ₃φ₄φ₅φ₆φ₇φ₈φ₈    | 1                | l + 5                                          |
| A₀(q)          | 2x² - 2x  | q⁹φ₁³φ₂φ₃φ₄φ₅φ₆φ₇φ₈φ₈    | 1                | l + 5                                          |
| A₀(q),₂        | x - 1     | 1/2 q⁹φ₁³φ₂φ₃φ₄φ₅φ₆φ₇φ₈φ₈ | 1, 1             | 3, 3                                           |
| A₀(q),₂        | x         | 1/2 q⁹φ₁³φ₂φ₃φ₄φ₅φ₆φ₇φ₈φ₈ | 1, 1             | l + 4, l + 4                                    |
| A₀(q),₂        | x - 1     | 1/2 q⁹φ₁³φ₂φ₃φ₄φ₅φ₆φ₇φ₈φ₈ | 1, 1             | l + 4, l + 4                                    |
| A₀(q),₂        | x         | 1/2 q⁹φ₁³φ₂φ₃φ₄φ₅φ₆φ₇φ₈φ₈ | 1, 1             | 2f + 5, 2f + 5                                 |
| Type of \(C_{\mathbf{X}(q)}(s)\) | # Classes | \(\mathbf{X}(q) : C_{\mathbf{X}(q)}(s)\) | \(\lambda(1), \lambda \in \text{Irr}^n(C_{\mathbf{X}(q)}(s))\) | \(v_2(\lambda(1)|\mathbf{X}(q) : C_{\mathbf{X}(q)}(s)|)\) |
|-----------------|-----------|-----------------|-----------------|-----------------|
| \(\mathbf{X}(q)\) | 1 | 1 | 1, \(\frac{q}{2}q_3\phi_3\phi_3\phi_6\), \(\frac{q}{2}q_3\phi_3\phi_3\phi_6\phi_7\phi_{14}\), \(\frac{q}{2}q_3\phi_3\phi_3\phi_6\phi_8\phi_9\), \(\frac{q}{2}q_3\phi_3\phi_3\phi_6\phi_8\phi_9\phi_14\), | 0, 0, 0, 1, 2l + 3, 3l + 10, 3l + 6, 3l + 10, 3l + 10, 3l + 10, 3l + 10, 0 |
| \(B_3(q)\) | 1 | 1 | \(\frac{q}{2}q_3\phi_3\phi_3\phi_6\), \(\frac{q}{2}q_3\phi_3\phi_3\phi_6\phi_7\phi_{14}\), \(\frac{q}{2}q_3\phi_3\phi_3\phi_6\phi_8\phi_9\), \(\frac{q}{2}q_3\phi_3\phi_3\phi_6\phi_8\phi_9\phi_14\), | 0, 0, 0, 0, 1, 0, 1, 0, 0, 2l + 3, 2l + 3 |
| \(\mathbf{C}_2(q)\) | 2x - 1 | \(q^6\phi_2\phi_3\phi_6\) | \(1, \frac{1}{3}q_3\phi_3^2, \frac{1}{3}q_3\phi_3^2, \frac{1}{3}q_3\phi_3^2, \frac{1}{3}q_3\phi_3^2, q^4\) | 1, 2l + 4, 2, 1, 1, 1 |
| \(\mathbf{A}_3(q)\) | 1 | \(q^6\phi_2\phi_3\phi_6\) | \(q^6, q^6\phi_2^3, q^6\phi_2^3, q^6\phi_2^3, q^6\phi_2^3, 1\) | 1, 1, 2, 1, 1 |
| \(\mathbf{A}_2(q)\) | 2x - 2 | \(q^6\phi_2^3\phi_4\phi_6\) | \(1, q\phi_2, q^3\) | 3, 4, 3 |
| \(\mathbf{A}_1(q)\) | 2x - 5x + 3 | \(q^6\phi_2^3\phi_3\phi_4\phi_6\) | \(1, q\) | 3, 3 |
| \(\mathbf{A}_1(q) + \mathbf{A}_1(q)\) | x - 1 | \(q^7\phi_2\phi_3\phi_4\phi_6\) | \(1, q, q^2\) | 2, 2, 2 |
| \(\mathbf{A}_0(q)\) | \(\frac{1}{3}x^3 - x^2 + \frac{2}{3}x - \frac{2}{3}\) | \(q^6\phi_2^3\phi_3\phi_4\phi_6\) | \(1\) | 4 |
| \(\mathbf{A}_1(q^2)\) | x | \(q^7\phi_1\phi_2^3\phi_3\phi_6\) | \(1, q^2\) | \(l + 4, l + 4\) |
| \(\mathbf{A}_1(q)\) | x | \(q^6\phi_1^2\phi_2\phi_3\phi_4\phi_6\) | \(1, q\) | \(l + 4, l + 4\) |
| \(\mathbf{A}_0(q)\) | \(x^2 - x\) | \(q^6\phi_1^2\phi_2^3\phi_3\phi_6\) | \(1\) | \(l + 5\) |
Table 9. Degrees of irreducible characters in Irr($X(q)$), $q \equiv 3 \mod 4$

| type of $C_{X(q)}(s)$ | $\#$ classes | $|X(q) : C_{X(q)}(s)|$ | $\lambda(1), \lambda \in \text{Irr}^u(C_{X(q)}(s))$ | $v_2(\lambda(1)|X(q) : C_{X(q)}(s)|)$ |
|------------------------|----------------|---------------------|-----------------------------------|---------------------------------|
| $X(q)$                 | 1              | 1                   | $1, \frac{1}{14}q^6\phi_3\phi_4\phi_0\tau_{14}, \frac{1}{7}q^4\phi_3\phi_4\phi_0\tau_{14}, \frac{1}{7}q^2\phi_3\phi_4\phi_0\tau_{14}$, $\frac{1}{7}q\phi_3\phi_4\phi_0\tau_{14}$, $\phi_3\phi_4\phi_0\tau_{14}$ | $0, 0, 0, 2l + 3, 1, 3l + 10, 3l + 10, 3l + 10, 0, 0, 0, 2l + 3, 1, 3l + 10, 3l + 10, 3l + 10, 0$ |
| $B_3(q)$               | 1              | 1                   | $\frac{1}{7}q^6\phi_3\phi_4\phi_0\tau_{14}, \frac{1}{7}q^4\phi_3\phi_4\phi_0\tau_{14}, \frac{1}{7}q^2\phi_3\phi_4\phi_0\tau_{14}, \frac{1}{7}q\phi_3\phi_4\phi_0\tau_{14}$ | $0, 0, 0, 2l + 3, 0, 0, 2l + 3, 0, 1, 1$ |
| $C_2(q)$               | $2x - 1$       | $q^6\phi_1\phi_3\phi_0$ | $1, \frac{1}{q}\phi_1, \frac{1}{q}\phi_2, \frac{1}{q}\phi_3, \frac{1}{q}\phi_4, q^4$ | $1, 2, 2l + 4, 1, 1, 1$ |
| $^2A_3(q)$             | 1              | $q^3\phi_1\phi_3$   | $q^6, q^4\phi_3, q^2\phi_4, q_0, 1$ | $1, 1, 1, 1$ |
| $^2A_2(q)$             | $2x - 2$       | $q^6\phi_1^2\phi_3\phi_0$ | $1, q\phi_1, q^2$ | $3, 4, 3$ |
| $A_1(q)$               | $2x^2 - 5x + 3$| $q^6\phi_1^2\phi_3\phi_4\phi_6$ | $1, q$ | $3, 3$ |
| $A_1(q) + A_1(q)$      | $x$            | $q^7\phi_1\phi_3\phi_4\phi_0$ | $1, q, q^2$ | $2, 2, 2$ |
| $A_0(q)$               | $\frac{1}{q^2}x^3 - x^2 + \frac{1}{q}x - \frac{1}{q}$ | $q^6\phi_1^2\phi_3\phi_4\phi_6$ | $1$ | $4$ |
| $A_1(q^2)$             | $x$            | $q^7\phi_1^2\phi_3\phi_4\phi_0$ | $1, q^2$ | $l + 4, l + 4$ |
| $A_1(q)$               | $x$            | $q^6\phi_1^2\phi_3\phi_4\phi_0$ | $1, q$ | $l + 4, l + 4$ |
| $A_0(q)$               | $x^2 - x$      | $q^6\phi_1^2\phi_3\phi_4\phi_0$ | $1$ | $l + 5$ |
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