Conformal symmetry
of the coupled Chern-Simons and
gauged nonlinear Schrödinger equations

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Abstract.
The non-relativistic conformal symmetry found by Jackiw and Pi for the coupled Chern-Simons and gauged nonlinear Schrödinger equations in the plane is derived in a non-relativistic Kaluza-Klein framework.

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1. Introduction

Recently, Jackiw and Pi [1] found that the gauged, planar non-linear Schrödinger equation
\begin{equation}
\text{(1.1)} \quad i \partial_t \Psi = \left\{ -\frac{1}{2m} (\bar{\nabla} - ie \bar{A})^2 + e A^0 - \Lambda \Psi^* \Psi \right\} \Psi,
\end{equation}
where \((A^0, \bar{A})\) is an electromagnetic vector potential and \(\Lambda = \text{const.}\), admits a ‘non-relativistic conformal’ symmetry [2]. The electromagnetic field and the current \((\rho, \vec{J})\),
\begin{equation}
\text{(1.2)} \quad \rho = \Psi^* \Psi, \quad \vec{J} = \frac{1}{2im} \left[ \Psi^* \bar{D} \Psi - \Psi (\bar{D} \Psi)^* \right],
\end{equation}
are assumed to satisfy the field-current identity
\begin{equation}
\text{(1.3)} \quad B = \epsilon^{ij} \partial_i A^j = -\frac{e}{\kappa} \rho, \quad E^i = -\partial_i A^0 - \partial_t A^i = \frac{e}{\kappa} \epsilon^{ij} J^j,
\end{equation}
\((i, j = 1, 2)\), rather than the conventional Maxwell equations. For a special value of \(\Lambda\) these equations admit static solutions [1], interpreted as Chern-Simons vortices.

In this Letter, we investigate non-relativistic Chern-Simons theory in a Kaluza-Klein-type framework. The clue is that non-relativistic space-time in \(2+1\) dimensions is conveniently viewed as the quotient of a \((3 + 1)\)-dimensional Lorentz manifold by the integral curves of a covariantly constant, lightlike vector field [3]. Then the non-relativistic conformal symmetries found by Jackiw and Pi are deduced from those — relativistic — which arise in extended space.

The peculiar aspect of our approach is that we mainly work with the field equations, since we have not yet been able to lift completely the 3-dimensional variational approach [1] to 4 dimensions.

Although our theory is formulated in full generality, the only example we present here is the one in Ref. [1], reproduced by reduction from Minkowski space. However, the Chern-Simons vortices in external harmonic and uniform magnetic fields found in Ref. [4] fit also into our framework. This will be explained in a forthcoming publication.

2. Chern-Simons theory in Bargmann space

Let \((M, g)\) denote a Lorentz 4-manifold of signature \((-+,+,+,+\)) which is endowed with a complete, covariantly constant and lightlike vector field \(\xi\). Following our previous terminology [3], we call such a manifold a Bargmann space. Then the quotient, \(Q\), of \(M\) by the integral curves of \(\xi\) is a \((2+1)\)-dimensional manifold which carries a Newton-Cartan structure, i.e. the geometric structure of non-relativistic space-time and gravitation. An adapted coordinate system on \(M\) is provided by \((t, x, y, s)\) where \((t, x, y)\) are coordinates on \(Q\) and \(\xi = \partial_s\). The couple \((x, y)\) yields position coordinates and \(t\) is non-relativistic absolute time, \(g_{\mu\nu} \xi^\nu = \partial_\mu t\).
Consider first a $U(1)$ vector-potential $a_\mu dx^\mu$ on $M$, denote by $f = \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu$ its field strength, $f_{\mu\nu} = 2 \partial_\mu a_\nu$, and let $j^\mu$ be some given current on $M$. Let us now postulate on $M$ the field-current identity (FCI)

\[ \kappa f_{\mu\nu} = e \sqrt{-g} \epsilon_{\mu \nu \rho \sigma} \xi^\rho j^\sigma, \]

$(\mu, \nu, \rho, \sigma = 0, \ldots, 3)$. It follows that $f_{\mu\nu} \xi^\nu = 0$ and since $f$ is closed, $\partial_\mu f_{\nu\rho} = 0$, $f$ is the lift from the $(2 + 1)$-dimensional space-time, $Q$, of a closed 2-form $F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$, $(\alpha, \beta = 0, 1, 2)$. With no loss of generality, $(a_\mu)$ can be chosen to be the lift from $Q$ of a vector potential $(A_\alpha) = (A_0, \vec{A})$ for $F$. Since $\xi$ is covariantly constant, Eq. (2.1) implies also that the Lie derivative of the current with respect to $\xi$ is proportional to $\xi$; the four-current $(j^\mu)$ projects therefore into a three-current, $(J^\alpha) = (\rho, \vec{J})$, on $Q$. Observe, finally, that the determinant $g = \text{det}(g_{\mu\nu})$ is $\xi$-invariant so that the contraction of the four-volume element by the covariantly constant vector $\xi$, namely $\sqrt{-g} \epsilon_{\mu \nu \rho \sigma} \xi^\sigma$, projects to a preferred volume element on Galilean space-time, $Q$. Therefore, Eq. (2.1) itself descends to the quotient to yield

\[ \kappa F_{\alpha\beta} = -e \sqrt{-g} \epsilon_{\alpha \beta \gamma} J^\gamma, \]

$(\alpha, \beta, \gamma = 0, 1, 2)$, which constitutes the Galilean version of the Chern-Simons equations.

Let, for example, $M$ be Minkowski space with its flat metric $dx^2 + dy^2 + 2dt ds$ where $t$ and $s$ are light-cone coordinates. Putting $\xi = \partial_s$ we plainly get a Bargmann structure over Galilean space-time. The only components of the electromagnetic field are $E^x = F_{xt}$, $E^y = F_{yt}$ and $B = F_{xy}$, which satisfy $\vec{\nabla} \times \vec{E} + \partial_t B = 0$ since $F$ is closed. Then the projected FCI (2.2) is readily seen to reduce to that in Ref. [1], Eq. (1.3). Note that Jackiw and Pi take their field-current identity from relativistic Chern-Simons theory although their system is non-relativistic.

Consider next a massless relativistic scalar field $\psi$ on $(M, g)$ coupled to the gauge field $a_\mu$ and described by the gauged, non-linear wave equation (NLWE)

\[ \left\{ D_\mu D^\mu - \frac{R}{6} + \lambda \psi^* \psi \right\} \psi = 0, \]

where $\lambda = \text{const.}$, $D_\mu = \nabla_\mu - ie a_\mu$ and $\nabla$ is the metric-covariant derivative; we have also included $R$, the scalar curvature of $g_{\mu\nu}$, for later convenience. The mass, $m$, is then introduced by requiring that the matter field satisfy the equivariance condition

\[ \xi^\mu D_\mu \psi = im \psi. \]

Due to our choice of gauge, $\Psi = e^{-imts} \psi$ is thus a function on $Q$ and Eq. (2.3) can be expressed as an equation on $Q$. This equation looks in general quite complicated and is
not reproduced here. In the absence of a gauge field and for \( \lambda = 0 \) it provides a covariant Schrödinger equation on Newton-Cartan space-time [5,3]. In the flat case, however, things simplify: the NLWE (2.3) reduces precisely to the non-linear Schrödinger equation (1.1) with \( \Lambda = \lambda/2m \).

Our system of equations becomes coupled by requiring, in addition to Eqs (2.1,3,4), that the current be related to the scalar field according to

\[
(2.5) \quad j^\mu = \frac{1}{2mi} \left[ \psi^*(D^\mu \psi) - \psi(D^\mu \psi)^* \right].
\]

We note that Eq. (2.3) implies that \( j^\mu \) is conserved, \( \nabla_\mu j^\mu = 0 \). The system (2.1,3–5) projects, in the flat case, to Eqs (1.1–3) on Galilean space-time.

3. Non-relativistic conformal symmetries

A conformal transformation \( \varphi : M \to M \) is such that the pulled-back metric is of the form \( \varphi^* g = \Omega^2 g \) for some strictly positive function \( \Omega \). Such transformations give rise to the conformal group of \((M, g)\). If \( \varphi \) is also required to preserve the vertical vector \( \xi \), namely \( \varphi^* \xi = \xi \), then it is easy to prove that \( \Omega \) is a function of time, \( t \), alone. This latter condition entails that \( \varphi \) defines a transformation on the quotient, \( Q \). These transformations form a subgroup of the conformal group we call the generalized Schrödinger group. For example, the conformal transformations of (compactified) Minkowski space \( M \) form the group \( O(4,2) \); those which also preserve the lightlike vector \( \xi = \partial_s \) yield a 9-dimensional subgroup identified with the Schrödinger group in 2 + 1 dimensions [6]. The infinitesimal action of this group on Minkowski space \( M \) is given by the vector field

\[
(3.1) \quad (X^\mu) = \begin{pmatrix}
-\chi t^2 - \delta t - \epsilon \\
R \bar{\vec{x}} - \left( \frac{1}{2} \delta + \chi \frac{t}{2} \right) \bar{\vec{x}} + t \bar{\vec{\beta}} + \bar{\eta} \\
\frac{1}{2} \chi r^2 - \bar{\beta} \cdot \bar{x} + \eta
\end{pmatrix}
\]

where \( r = |\bar{x}| \) and \( R \in \text{so}(2), \bar{\beta}, \bar{\eta} \in \mathbb{R}^2, \epsilon, \chi, \delta, \eta \in \mathbb{R} \), interpreted as rotation, boost, space translation, time translation, expansion, dilatation and vertical translation.

Now, let us determine which conformal transformations act as symmetries. By a ‘symmetry’ we mean here a transformation which takes a solution of some system of equations into another solution of this same system.

Consider first the massless wave equation (2.3) on extended space-time \((M, g)\) with \( a_\mu \) some given vector potential, and let \( \varphi \) be a conformal transformation, viz \( \varphi^* g = \Omega^2 g \). Expanding the term \( D_\mu D^\mu \psi \) and using the transformation law of the scalar curvature, \( \varphi^* R = \Omega^{-2} [R - 6\Omega^{-1} \nabla_\mu \nabla^\mu \Omega] \), a term-by-term inspection shows that the NLWE (2.3) goes into

\[
(3.2) \quad \Omega^{-3} \left\{ \tilde{D}_\mu \tilde{D}^\mu - \frac{R}{6} + \frac{\lambda}{6} \tilde{\psi}^* \tilde{\psi} \right\} \tilde{\psi} = 0
\]
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 upon setting
\[(3.3) \quad \tilde{\psi} = \Omega \varphi^* \psi, \quad \tilde{a}_\mu = (\varphi^* a)_\mu \quad \text{and} \quad \tilde{D}_\mu = \nabla_\mu - ie\tilde{a}_\mu. \]

Hence, if \(\psi\) and \(a_\mu\) solve the NLWE (2.3), the same is true for \(\tilde{\psi}\) and \(\tilde{a}_\mu\): the full conformal group is, indeed, a symmetry for the NLWE. Our result generalizes the statement saying that the ungauged non-linear Klein-Gordon equation on \(n\)-dimensional Minkowski space, \{\(\Box + \lambda (\psi^* \psi)^p\)^\(p\)\(\psi = 0\)\}, is conformally invariant precisely for \(p = 2/(n - 2)\) [7].

Let us now suppose that \(a_\mu\) satisfies the FCI (2.1). This latter is manifestly not invariant under arbitrary conformal transformations since it involves the vector field \(\xi\). Using \(\varphi^* \sqrt{-g} = \Omega^4 \sqrt{-\tilde{g}}\) one proves, however, that if \(\varphi\) is assumed \(\xi\)-preserving, then \(\tilde{f}_{\mu\nu} = (\varphi^* f)_{\mu\nu}\) satisfies the FCI (2.1) with the new current
\[(3.4) \quad \tilde{\eta}^\mu = \Omega^4 (\varphi^* j)^\mu. \]

The FCI reduces hence the symmetry to the Schrödinger subgroup.

Remembering that \(\Omega\) is thus a function of time only, we see that the \(\xi\)-preserving conformal transformations also preserve the equivariance condition (2.4), \(\tilde{D}_\xi \tilde{\psi} = im\tilde{\psi}\).

The various equations have been treated separately so far. Their consistency follows from showing, using \((\varphi^* D \psi)_\mu = \Omega^{-1} (\tilde{D}_\mu \tilde{\psi} - \Omega^{-1} \nabla_\mu \Omega \tilde{\psi})\), that the current \(\tilde{\eta}^\mu\) obtained by replacing \(\psi\) and \(D_\mu \psi\) by \(\tilde{\psi}\) and \(\tilde{D}_\mu \tilde{\psi}\) in Eq. (2.5), does verify Eq. (3.4). We conclude that \(\xi\)-preserving conformal transformations of \(M\) are indeed symmetries of the reduced system. This generalizes the result known for the ungauged non-linear Schrödinger equation [8]. Note that the value \(p = 1\) in the last term \(|\Psi|^p\Psi\) of (1.1) corresponds to the one dictated by relativistic conformal invariance of the above mentioned non-linear Klein-Gordon equation in \(n = 4\) dimensions.

4. Conserved quantities

The traditional form of Noether’s theorem uses an action principle while here we only have field equations. We propose therefore a mixed approach motivated by Souriau’s viewpoint [9]. Let us start with the ‘partial’ action on Bargmann space \(M\)
\[(4.1) \quad S = \frac{1}{2m} \int_M \left\{ (D_\mu \psi)^* D^\mu \psi + \frac{R}{6} |\psi|^2 - \lambda \frac{\mu}{2} |\psi|^4 \right\} \sqrt{-g} d^4 x, \]
with \(D_\mu = \nabla_\mu - ie a_\mu\) for some vector potential \(a_\mu\).

Variation with respect to \(\psi^*\) yields the NLWE (2.3). However, no equation for \(a_\mu\) is obtained because the action (4.1) contains no kinetic term for the gauge field. We have therefore to add the FCI (2.1) as an extra condition. The variational derivative \(-\delta S/\delta a_\mu\) yields nevertheless the current \(e j^\mu\) in Eq. (2.5), which is automatically conserved,
\( \nabla_\mu j^\mu = 0 \), as a consequence of the invariance of the action \( S \) with respect to \( U(1) \) gauge transformations. Similarly, the variational derivative \( 2 \delta S/\delta g^{\mu\nu} \) yields, after a tedious calculation, the energy-momentum tensor \( \vartheta_{\mu\nu} \) given by

\[
3m \vartheta_{\mu\nu} = (D_\mu \psi)^* D_\nu \psi + D_\mu \psi (D_\nu \psi)^* - \frac{1}{2} (\psi^* D_\mu D_\nu \psi + \psi (D_\mu D_\nu \psi)^*) \\
+ \frac{1}{2} |\psi|^2 \left( R_{\mu\nu} - \frac{R}{6} g_{\mu\nu} \right) - \frac{1}{2} g_{\mu\nu} (D^\rho \psi)^* D_\rho \psi - \frac{\lambda}{4} g_{\mu\nu} |\psi|^4.
\]

Note the strange coefficient 3, the Ricci tensor \( R_{\mu\nu} \) and the weird term \( R/6 \).

Using the transformation law of the scalar curvature and of the covariant derivative, we see that a conformal transformation \( \varphi \) replaces the integrand in the ‘partial’ action (4.1) by

\[
\left\{ (\bar{D}_\mu \bar{\psi})^* \bar{D}_\mu \bar{\psi} + \frac{R}{6} |\bar{\psi}|^2 - \frac{\lambda}{2} |\bar{\psi}|^4 \right\} \sqrt{-g} - \nabla_\mu (|\bar{\psi}|^2 \Omega^{-1} \nabla^\mu \Omega) \sqrt{-g},
\]

with the notation (3.3), proving again the conformal symmetry of the NLWE (2.3) from a more conventional viewpoint. It follows from the conformal invariance (or directly from Eqs (4.2) and (2.3)) that \( \vartheta_{\mu\nu} \) is traceless, \( \vartheta_{\mu\mu} = 0 \). Being the variational derivative with respect to a symmetric tensor, \( \vartheta_{\mu\nu} \) is plainly symmetric. Finally, the invariance of the action (4.1) with respect to diffeomorphisms \([9]\) implies that \( \vartheta_{\mu\nu} \) satisfies the relation

\[
\nabla_\mu \vartheta^{\mu\nu} + ej_\mu f^{\mu\nu} = 0,
\]

where \( j^\mu \) is the current found above. But \( j_\mu f^{\mu\nu} \) vanishes because of the FCI (2.1), yielding

\[
\nabla_\mu \vartheta^{\mu\nu} = 0.
\]

We conclude that \( \vartheta_{\mu\nu} \) is a conserved, symmetric and traceless energy-momentum tensor.

We now prove Noether’s theorem in our framework. The vector field

\[
k^\mu = \vartheta^\mu X^\nu,
\]

(\( \mu, \nu = 0, \ldots, 3 \)), is a conserved current, \( \nabla_\mu k^\mu = 0 \), for any conformal vector field \( X^\nu \). Indeed, \( \nabla_\mu (\vartheta^\mu X^\nu) = (\nabla_\mu \vartheta^\mu) X^\nu + \frac{1}{2} \vartheta^{\mu\nu} L_X g_{\mu\nu} = 0 \) using (4.5) and \( \vartheta^\mu = 0 \).

So far, we have associated conserved four-currents to each infinitesimal conformal transformation of \((M, g)\). Let us henceforth assume that \( X \) be also \( \xi \)-preserving and proceed to derive conserved quantities in \( 2+1 \) dimensions. Firstly, observe that the current \( (k^\mu) \) is \( \xi \)-invariant because so is \( S \), and hence \( \vartheta \), as well as \( X \). Therefore, it projects into a three-current \( (K^\alpha) = (K^0, \vec{K}) \) on ordinary space-time, \( Q \). But \( \nabla_s k^s = 0 \) because \( \xi = \partial_s \) is covariantly constant. The projected current is thus also conserved,

\[
\nabla_\alpha K^\alpha = 0,
\]
where \( \nabla \) is, here, the (non-metric) Newton-Cartan covariant derivative on \( Q \) obtained by projecting the metric covariant derivative of \((M, g, \xi)\) \cite{3}. The Bargmann metric, \( g_{\mu \nu} \), canonically induces a Riemannian metric, \( \gamma_{ij} \), on each 2-surface \( \Sigma_t \) \((t = \text{const.})\) of \( Q \). The quantities \( Q_X = \int_{\Sigma_t} K^0 \sqrt{\gamma} \, d^2 \vec{x} \) are thus conserved provided all currents vanish at infinity. Now \( K^0 = \xi_\nu k^\nu \), hence the quantity

\[
Q_X = \int_{\Sigma_t} \partial_{\mu \nu} X^\mu \xi^\nu \sqrt{\gamma} \, d^2 \vec{x},
\]

does not depend on time \( t \) for each \( \xi \)-preserving conformal vector field \( X \). The conserved quantities are conveniently calculated using

\[
\partial_{\mu \nu} \xi^\nu = \frac{1}{2i} [\psi^* (D_\mu \psi) - \psi (D_\mu \psi)^*] - \frac{1}{6m} \xi_\mu \left( \frac{R}{6} |\psi|^2 + (D^\nu \psi)^* D_\nu \psi + \frac{\lambda}{2} |\psi|^4 \right),
\]

obtained from Eq. (4.2) by the equivariance condition (2.4). In the flat case, for example, a conserved quantity is associated to each generator in Eq. (3.1) of the planar Schrödinger group. Using the equivariance and the field equations we find

\[
\begin{align*}
\mathcal{H} &= \int \left[ \frac{1}{2m} (\bar{D}\Psi)^* \cdot \bar{D}\Psi - \frac{\Lambda}{2} (\Psi^* \Psi)^2 \right] \, d^2 \vec{x} \quad \text{energy} \\
\vec{P} &= \int \vec{P} \, d^2 \vec{x} = \int \frac{1}{2i} \left[ \Psi^* (\bar{D} \Psi) - (\bar{D} \Psi)^* \Psi \right] \, d^2 \vec{x} \quad \text{momentum} \\
\mathcal{J} &= \int \vec{x} \times \vec{P} \, d^2 \vec{x} \quad \text{angular momentum} \\
\mathcal{G} &= t \vec{P} - m \int \vec{x} (\Psi^* \Psi) \, d^2 \vec{x} \quad \text{boost} \\
\mathcal{D} &= t \mathcal{H} - \frac{1}{2} \int \vec{x} \cdot \vec{P} \, d^2 \vec{x} \quad \text{dilatation} \\
\mathcal{K} &= t^2 \mathcal{H} - 2t \mathcal{D} + \frac{m}{2} \int r^2 (\Psi^* \Psi) \, d^2 \vec{x} \quad \text{expansion} \\
\mathcal{M} &= m \int \Psi^* \Psi \, d^2 \vec{x} \quad \text{mass}
\end{align*}
\]

where \( \Psi = e^{-ims} \psi \). Note that the ‘vortex number’ of Ref. [1] is associated here to the vertical Killing vector \( \xi \), namely \( \mathcal{M} = \mathcal{Q}_\xi \).

Jackiw and Pi obtain the same conserved quantities by constructing a non-symmetric energy-momentum tensor \( T^{\alpha \beta} \), \((\alpha, \beta = 0, 1, 2)\), which satisfies the unusual trace condition \( T^{ij} \delta_{ij} = 2T^{00} \). The \( T^{00}, T^{ij} \) (resp. \( T^{0j}, T^{ij} \)) components describe the density and flux of the energy (resp. momentum). Compared to the \( K^0 \) and \( K^i \) of time (resp. space)
translations, we find
\begin{equation}
\begin{aligned}
T^{00} &= -\vartheta^0 + \frac{1}{6m} \Delta \rho, \\
T^{i0} &= -\vartheta^i - \frac{1}{6m} \partial_i \partial_t \rho, \\
T^{0j} &= \vartheta^j, \\
T^{ij} &= \vartheta^i + \frac{1}{3m} (\delta^i_j \Delta - \partial^i \partial_j) \rho,
\end{aligned}
\end{equation}

where $\Delta$ denotes the planar Laplace operator and $\rho = |\Psi|^2$. The time components differ in surface terms and yield, therefore, the same conserved quantities. The lack of symmetry, $T^{0i} \neq T^{i0}$, is manifest since $\vartheta^0 = \vartheta^i \neq -\vartheta^0 - \partial_i \partial_t \rho/6m$. However, $T^{ij} = T^{ji}$ because $\vartheta_{\mu\nu}$ is symmetric. The trace condition of $T^{\alpha\beta}$ comes from the tracelessness of $\vartheta_{\nu}$ and $\vartheta^3 = \vartheta^0$. Continuity equations, e.g. $\partial_\alpha T^{\alpha0} = 0$, follow from Eq. (4.5).

5. Generalizations

The FCI (2.1) can easily be generalized by including a Maxwell-type term,
\begin{equation}
\sqrt{-g} \epsilon_{\mu\nu\rho\sigma} \xi^\rho \nabla_\tau f^{\tau\sigma} + \kappa f_{\mu\nu} = \epsilon \sqrt{-g} \epsilon_{\mu\nu\rho\sigma} \xi^\rho j^\sigma.
\end{equation}

This equation descends, in the same way as before, to the quotient, $Q$. In the Minkowski case, for example, we still get $\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0$ from $\partial_{[\mu} f_{\nu\rho]} = 0$ while (1.3) is now generalized to
\begin{equation}
\kappa B = -e\rho, \quad \partial_\alpha B + \kappa E_\alpha = e \epsilon_{ij} J^j.
\end{equation}

Note that the equation for $B$ (Gauss’ law) retains the same form it had without the Maxwell term because $\xi$ is null. The other equation generalizes Ampère’s equation in the ‘magnetic-type’ Galilean electromagnetism [10].

Since, in 4 dimensions, the Maxwell term $\nabla_\tau f^{\tau\sigma}$ goes into $\Omega^{-4} \nabla_\tau \tilde{f}^{\tau\sigma}$ under a conformal transformation, the new FCI (5.1) is also invariant with respect to $\xi$-preserving conformal transformations if the current changes as in (3.4). This happens e.g. if we identify the so far unspecified $j^\mu$ with the right-hand side of Eq. (2.5). Then Eqs (2.3–5) and (5.1) form a coupled system of field equations invariant under the $\xi$-preserving conformal transformations, i.e. the generalized Schrödinger transformations.

To determine the conserved quantities belonging to these symmetries we note that Eq. (5.1) can be written as
\begin{equation}
\kappa f_{\mu\nu} = \sqrt{-g} \epsilon_{\mu\nu\rho\sigma} \xi^\rho (e_j^\sigma - \nabla_\tau f^{\tau\sigma}) = \sqrt{-g} \epsilon_{\mu\nu\rho\sigma} \xi^\rho j^\sigma,
\end{equation}
i.e. as Eq. (2.1) where the new current, $\tilde{j}^\mu$, is the variational derivative $\tilde{j}^\mu = -\delta \tilde{S}/\delta a_\mu$ of the action obtained from the expression (4.1) by adding the standard Maxwell term
\begin{equation}
\tilde{S} = S - \frac{1}{4} \int_M f_{\mu\nu} f^{\mu\nu} \sqrt{-g} d^4x.
\end{equation}
We emphasize that Eq. (5.1) (or (5.3)) is not the variational equation for $a_\mu$ associated with $\hat{S}$ — which would require $\hat{j}^\mu$ to vanish — whereas Eq. (2.3) is indeed the variational equation for $\psi^*$. This shows that the 4-dimensional dynamics of the $a_\mu$ field we consider is significantly different from the conventional one. The variational derivative $\hat{\vartheta}_{\mu\nu} = 2 \delta \hat{S}/\delta g^{\mu\nu}$ yields the energy momentum tensor, $\hat{\vartheta}_{\mu\nu} = \vartheta_{\mu\nu} + \Theta_{\mu\nu}$, where $\vartheta_{\mu\nu}$ is given by Eq. (4.2) and $\Theta_{\mu\nu} = f_{\mu\rho} f^\rho_{\nu} + \frac{1}{4} g_{\mu\nu} f_{\rho\sigma} f^{\rho\sigma}$ is the standard energy momentum tensor for the Maxwell field. Again, $\hat{\vartheta}_{\mu\nu}$ is a symmetric, traceless and divergencefree tensor, since all our previous arguments to establish these properties for $\vartheta_{\mu\nu}$ apply just as well. As a consequence, conserved Noether quantities follow immediately from the expression (4.8) specialized to $\hat{\vartheta}$. For example, in the flat case, the only contribution of the Maxwell field to the nine Noether quantities lies in the Hamiltonian, $\hat{\mathcal{H}}$, since $\hat{\vartheta}_{\mu\nu} \xi^\nu = \vartheta_{\mu\nu} \xi^\nu + \frac{1}{4} f_{\rho\sigma} f^{\rho\sigma} \xi_\mu$ due to the fact that $f_{\mu\nu} \xi^\nu = 0$. In this case we have $f_{\rho\sigma} f^{\rho\sigma} = 2 B^2$ and remembering Eq. (5.2), we realize that the new Hamiltonian, $\hat{\mathcal{H}}$, comes with the replacement $\hat{\Lambda} = \Lambda + e^2/\kappa^2$ of the coefficient of the quartic term $|\Psi|^4$ in (4.10).

Let us mention for completeness that choosing $\xi$ covariantly constant and spacelike, $\xi_\mu \xi^\mu = 1$, the quotient of the Lorentz 4-manifold $M$ by the integral curves of $\xi$ yields a Lorentz 3-manifold of signature $(-, +, +)$. The FCI (2.1) turns out to project to the same equation (2.2), explaining why Jackiw and Pi could use it in their non-relativistic theory. The generalized FCI (5.1) would in turn project into

$$\nabla_\alpha F^{\alpha\gamma} - \frac{\kappa}{2} \sqrt{-g} \epsilon^{\alpha\beta\gamma} F_{\alpha\beta} = e J^\gamma,$$

a generalization of the usual curved-space inhomogeneous Maxwell equations. For Minkowski space, for example, writing the metric as $-dt^2 + dx^2 + dy^2 + dz^2$ and choosing $\xi = \partial_z$, we recover the relativistic expressions in Ref. [1], viz

$$\partial_1 E^i + \kappa B = -\epsilon_\rho, \quad \epsilon_{ij} \partial_1 E^j + \partial_1 B + \kappa E_i = e \epsilon_{ij} J^j,$$

to be compared with the non-relativistic expression in Eq. (5.2).

Reduction of the NLWE (2.3) yields now a gauged non-linear Klein-Gordon equation in $2 + 1$ dimensions for which, when coupled to the gauge field through a FCI (with or without Maxwell term), the $\xi$-preserving conformal transformations still act as symmetries. However, no conformal extension of isometries is now obtained as the $\xi$-preserving conformal transformations merely yield ($\mathbf{R}$-times) the Poincaré group in $2 + 1$ dimensions.

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