On the hierarchical structure of Pareto critical sets

Bennet Gebken · Sebastian Peitz · Michael Dellnitz

Received: 19 July 2018 / Accepted: 2 January 2019 / Published online: 14 January 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
In this article we show that the boundary of the Pareto critical set of an unconstrained multiobjective optimization problem (MOP) consists of Pareto critical points of subproblems where only a subset of the set of objective functions is taken into account. If the Pareto critical set is completely described by its boundary (e.g., if we have more objective functions than dimensions in decision space), then this can be used to efficiently solve the MOP by solving a number of MOPs with fewer objective functions. If this is not the case, the results can still give insight into the structure of the Pareto critical set.

Keywords Multiobjective optimization · Many-objective optimization · Pareto set · Pareto critical set

1 Introduction
In many applications the problem arises to optimize several functions at once. In production for example, a typical goal is to maximize the quality of a product and to minimize the production cost at the same time. If the individual goals are in conflict, then there exists no single point that optimizes all objectives at once, and scalar optimization theory cannot be applied. Instead, the goal is to compute the set of optimal compromises, the so-called Pareto set, consisting of all Pareto optimal points. The task of finding the Pareto set is called multiobjective optimization (MOO).

A well-known first-order necessary condition for Pareto optimality is given by Kuhn and Tucker [11]. It states that in a Pareto optimal point of an unconstrained multiobjective optimization problem (MOP) there exists a convex combination of the gradients of the objective functions which is zero. Similar to scalar-valued optimization, the coefficients of this convex combination are called Karush–Kuhn–Tucker (KKT) multipliers. Points for which such a convex combination exists are called Pareto critical and the set of these points is called the Pareto critical set [28]. Roughly speaking, this condition induces a (possibly set-valued) map from the standard simplex to the Pareto critical set. So the question arises whether it is possible to derive properties of the Pareto critical set from the set of KKT multipliers. In particular, we are interested in analyzing properties and computing the boundaries of the
This is particularly useful for MOPs where the number of objectives is large and especially larger than the number of variables, as the Pareto critical sets can be very complicated and expensive to compute in that case [25]. It is therefore of great interest to derive efficient methods to solve these problems, e.g., by exploiting the boundary structure. Until now, MOPs with many objectives are mainly being treated by evolutionary approaches [10] and subdivision and cell mapping techniques [22,30]. 

There already exist some results about the structure of Pareto sets. In [22, Chapter 4] relations between the boundary of the Pareto critical set and subsets of objectives are investigated for a special class of well-behaved objective functions. There the focus lies on the hierarchical structure of Pareto sets, meaning that every neglected objective function results in Pareto critical points that lie on the boundary of the previous problem. In a more theoretical approach, De Melo showed that there is an open and dense subset of the space of smooth vector functions \( C^\infty(\mathbb{R}^n, \mathbb{R}^k) \), where the Pareto critical set is a stratification [2]. This means that the Pareto critical set of a generic smooth vector function is the union of submanifolds of \( \mathbb{R}^n \) called strata, such that the boundary of a stratum and the intersection of strata are composed by strata of lower dimension. Stratifications are also known as “manifolds with boundaries and corners”. In the case where all objective functions are convex (and there are fewer objective functions than there are dimensions in the decision space) the Pareto set is diffeomorphic to a standard simplex and its facets correspond to Pareto sets where a certain number of objectives has been neglected (see [26,28]). Lovison and Pecci extended this result in [16] by showing that for a dense class of smooth (nonconvex) objective vectors, the (local) Pareto set is a Whitney stratification. Besides optimization, the Pareto critical set can also be used in differential topology to generalize Morse theory to vector-valued functions [2,3,7,15,20,31]. All previously mentioned results require some amount of differentiability of the objective functions. For the non-differentiable but convex case, Lowe et al. investigated the relationship between the full problem and subproblems in [17]. It was shown that convex MOPs are Pareto reducible, which means that the Pareto set can be covered by weak Pareto sets of subproblems. These results were later extended to more general cases by Malivert and Boissard [18] and Popovici [23]. Many multiobjective optimization algorithms are focussed on the objective space instead of the decision space. Consequently, it can be of equal interest to investigate the boundary of the Pareto front, i.e., the image of the Pareto set under the objective function. This has been done by Mueller-Gritschneder et al. [21] (see also [5,27]). A related approach is objective reduction in the context of many-objective optimization, where the goal is to eliminate objective functions that are either redundant or have only minor influence on the Pareto front (see, e.g., [1,14,24]).

The goal of this article is to extend the results from [22, Chapter 4] to a much more general setting. We investigate the hierarchical structure of Pareto critical sets and study properties of the boundary. We show that the boundary of the standard simplex is mapped to a covering of the boundary of the Pareto critical set. Since at least one multiplier is zero on the boundary of the simplex, the boundary of the Pareto critical set can be calculated by omitting the objective functions corresponding to the vanishing multipliers. The number of functions that can be omitted can be derived (locally) from the rank of the Jacobian of the objective vector. Although we are mainly interested in the decision space, in all examples we consider where \( k \leq 3 \), we will also show the image of the Pareto critical set under the objective vector. The results will suggest that our observations in the decision space generalize to the image space, thereby extending the results of [21].

The structure of this article is as follows: we start by giving a short introduction to MOO in Sect. 2. In Sect. 3 we first classify Pareto critical points by their respective KKT multipliers by distinguishing between Pareto critical points with positive KKT multipliers (\( P_{\text{int}} \)) and critical
points where at least one KKT multiplier vanishes (P_0). We show some results about the structure and relationships of those sets with varying regularity assumptions on the objective vector. Since we will define the boundary of the Pareto critical set by properties of tangent cones, we then show some results about tangent cones of the Pareto critical set. An important technical result will be that if the MOP is regular enough, then the tangent cone of the Pareto critical set is just the projection of the tangent space of the manifold of Pareto critical points, extended by their KKT multipliers, onto the decision space. This will be used to prove our first main result (Theorem 3), which states that on the boundary of the Pareto critical set, (at least) one KKT multiplier is zero. In Sect. 4 we study how many KKT multipliers are zero, or in other words, how many objective functions have to be considered to compute the boundary of the Pareto critical set. Our second main result (Corollary 3) will be that the number of required components is equal to the maximal rank of the Jacobian on the Pareto critical set. In Sect. 5, we give several examples of how this result can be used to investigate the hierarchical structure of the Pareto critical set. Finally, we draw a conclusion and discuss possible future work in Sect. 6.

We conclude this introduction with a simple example to illustrate the structure we want to investigate in this article. Consider the following convex MOP:

\[
\min_{x \in \mathbb{R}^2} f(x) = \min_{x \in \mathbb{R}^2} \left( \begin{array}{c} f_1(x) \\ f_2(x) \\ f_3(x) \end{array} \right) = \min_{x \in \mathbb{R}^2} \left( \begin{array}{c} (x_1 - 1)^2 + (x_2 + 1)^2 \\ x_1^2 + (x_2 - 1)^2 \\ (x_1 + 1)^2 + (x_2 + 1)^2 \end{array} \right). \tag{1}
\]

The objective functions are spherically symmetric paraboloids, so the Pareto critical (and in this case Pareto optimal) set is given by the triangle with vertices \((-1, -1), (1, -1)\) and \((0, 1)\). If we omit the third objective function, the Pareto critical set of the resulting MOP is the line connecting \((-1, -1)\) and \((1, -1)\), so it is part of the boundary. In the same way we obtain the other parts of the boundary of the original Pareto critical set by omitting different objective functions, see Fig. 1 for an illustration. The Pareto critical set of the subproblem \((f_1, f_2)\) is shown in red, \((f_1, f_3)\) in blue and \((f_2, f_3)\) in green.

This example is simple because we can use the topological definition of “boundary” in \(\mathbb{R}^2\) (endowed with the Euclidean topology) to describe the boundary of the Pareto critical set. Additionally, the simplex of KKT multipliers (as part of a plane in \(\mathbb{R}^3\)) has the same dimension as the decision space, so the map (in this case the diffeomorphism) between the set of KKT multipliers and the Pareto critical set is quite simple. When the number of objectives is smaller than or equal to the number of variables, the situation is more complicated, as the

Fig. 1  a Pareto critical set of the MOP (1).  b Pareto critical sets of all 2-objective subproblems (colored) and 1-objective subproblems (black).  c Image of the Pareto critical set under \(f\) with the images of the subproblems colored as in b. (Color figure online)
Pareto critical set will generically have an empty interior (as a subset of $\mathbb{R}^n$). In that case, the boundary with respect to the topology in $\mathbb{R}^n$ would just be the Pareto critical set itself, which is not helpful. Instead, in what follows we will propose an alternative definition (Definition 6) which is also valid in the more general setting.

2 Multiobjective optimization

Consider the unconstrained multiobjective optimization problem

$$\min_{x \in \mathbb{R}^n} f(x) = \min_{x \in \mathbb{R}^n} \begin{pmatrix} f_1(x) \\ \vdots \\ f_k(x) \end{pmatrix},$$

(MOP)

where $f : \mathbb{R}^n \to \mathbb{R}^k$ is a vector-valued function, called objective vector, with continuously differentiable objective functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, k$. The space of the decision variables $x \in \mathbb{R}^n$ is called the decision space and the function $f$ is a mapping to the $k$-dimensional objective space. In contrast to single objective optimization problems, there exists no natural total order of the image of the objective vector in $\mathbb{R}^k$ for $k \geq 2$ (unless the objective functions are functionally dependent). In particular, the following relation is only a partial order on $\mathbb{R}^k$:

**Definition 1** Let $v, w \in \mathbb{R}^k$. We write $v \leq w$, if $v_i \leq w_i$ for all $i \in \{1, \ldots, k\}$.

A consequence of the lack of a total order is that we cannot expect to find isolated optimal points. Instead, the solution of (MOP) is the set of optimal compromises, the so-called Pareto set named after Vilfredo Pareto:

**Definition 2** (a) A point $x^* \in \mathbb{R}^n$ dominates a point $x \in \mathbb{R}^n$, if $f(x^*) \leq f(x)$ and $f(x^*) \neq f(x)$.

(b) A point $x^* \in \mathbb{R}^n$ is called (globally) Pareto optimal if there exists no point $x \in \mathbb{R}^n$ dominating $x^*$.

(c) The set of non-dominated points is called the Pareto set, its image the Pareto front.

Consequently, for each solution that is contained in the Pareto set, one can only improve one objective by accepting a trade-off in at least one other objective. A more detailed introduction to multiobjective optimization can be found in [6,19].

Similar to scalar optimization, a necessary condition for optimality is based on the gradients of the objective functions. In the multiobjective situation, the corresponding Karush-Kuhn-Tucker (KKT) condition is as follows [9]:

**Theorem 1** Let $x^*$ be a Pareto optimal point of (MOP). Then there exists some $\alpha \in (\mathbb{R} \geq 0)^k$ such that

$$\sum_{i=1}^k \alpha_i \nabla f_i(x^*) = 0 \quad \text{and} \quad \sum_{i=1}^k \alpha_i = 1. \tag{2}$$

Since this is only a necessary optimality condition, we introduce the following definition of Pareto criticality by Smale [28] as a generalization of critical points in single objective optimization.
Definition 3 Let \( x \in \mathbb{R}^n \). If there is some \( \alpha \in (\mathbb{R}_{\geq 0})^k \) such that Eq. (2) holds, then \( x \) is called Pareto critical and \( \alpha \) a KKT vector of \( x \) containing the KKT multipliers \( \alpha_i, i \in \{1, \ldots, k\} \). The set of Pareto critical points \( P \) of \( \text{(MOP)} \) is called the Pareto critical set.

Remark 1 1. Pareto critical points are sometimes referred to as substationary points [19].
2. In [28], Smale defined a point \( x \in \mathbb{R}^n \) as Pareto critical if
\[
\exists v \in \mathbb{R}^n : Df(x)v > 0.
\]
He later showed in [29] that this condition is equivalent to (2).
3. Second-order conditions for Pareto optimality using the second (intrinsic) derivative of \( f \) were also introduced by Smale [28].

The Pareto critical set contains first-order candidates for Pareto optimal points. It is the main object of interest in this article, and some of its properties will be investigated in the following sections.

3 The structure of the Pareto critical set

In this section we will investigate the structure of the Pareto critical set \( P \) and extend the results from [22]. There it has been assumed that the Jacobian of the objective vector \( f \) has rank \( k - 1 \) everywhere. (In particular, the assumption restricts the results to problems with \( k \leq n + 1 \).) Due to this, \( \text{(MOP)} \) has a unique KKT vector \( \alpha \) for every Pareto critical point \( x \) (cf. Lemma 5) and we obtain a very nice hierarchical structure of the Pareto critical set (see Fig. 2 for a sketch).

Here, we will generalize these results by omitting all assumptions on the rank of the Jacobian of \( f \). In particular, we will also consider the case where \( k > n + 1 \). We begin by classifying Pareto critical points by their respective KKT vectors and then revisit a (slightly modified) result of Hillermeier [9] about the manifold structure of the Pareto critical set extended by the set of KKT vectors. This result will be used to show that under certain conditions, the tangent cone to the Pareto critical set is equal to the projection of the tangent space of the extended Pareto critical set onto the decision space. This can be used to show that the edge of the Pareto critical set—which is defined via the tangent cone (cf. Definition 6)—is a subset of the set of Pareto critical points where (at least) one KKT multiplier is zero (Theorem 3).

Fig. 2  a The Pareto set of a convex MOP with four objectives. b The union of the Pareto sets of the four subproblems with three objectives (shown in different colors) forms the boundary of the original Pareto set. c The Pareto sets of the six subproblems with two objectives are shown in blue. (Color figure online)
For a set $X \subseteq \mathbb{R}^n$, we will denote by $X^\circ$, $\overline{X}$ and $\partial X$ the interior, closure and boundary of $X$, respectively, with respect to the natural topology on $\mathbb{R}^n$. It is important to keep in mind that in general, we are not concerned with the topological boundary of Pareto critical sets.

### 3.1 Classifying $P$ via KKT multipliers

We begin by interpreting the KKT conditions (2) as a nonlinear system of equations: Define $F : \mathbb{R}^n \times (\mathbb{R}^\geq 0)^k \to \mathbb{R}^{n+1}$,

$$F(x, \alpha) := \left( \sum_{i=1}^k \alpha_i \nabla f_i(x) \right) - \left( \sum_{i=1}^k \alpha_i - 1 \right) \nabla f_k(x).$$

Then $x \in \mathbb{R}^n$ is Pareto critical if there exists a KKT vector $\alpha \in (\mathbb{R}^\geq 0)^k$ with $F(x, \alpha) = 0$.

Let $pr_x$ be the projection onto the first $n$ components. The set of Pareto critical points $P$ is then given by $P = pr_x(F^{-1}(0))$.

In order to investigate the structure of $P$, we take a closer look at the structure of the set of KKT vectors. We distinguish between points $x \in P$ for which the corresponding KKT vectors have at least one zero component and points that have a strictly positive KKT vector:

**Definition 4** For $k > 1$ let

$$P_0 := \{x \in P : \forall \alpha \in (\mathbb{R}^\geq 0)^k \text{ with } F(x, \alpha) = 0 \text{ there is at least one } i \in \{1, \ldots, k\} \text{ with } \alpha_i = 0\}$$

and $P_{\text{int}} := P \setminus P_0$.

**Remark 2** The index int (for interior) has been chosen since we will show that $P_{\text{int}}$ can in fact be interpreted as the “interior” of the Pareto critical set. Until now, $P_{\text{int}}$ is only defined as the set of Pareto critical points that have a strictly positive KKT vector.

The remainder of this section is dedicated to the question how $P_0$ and $P_{\text{int}}$ are related to differential geometrical and topological properties of $P$. We start by simplifying $F$. Obviously, if $F(x, \alpha) = 0$ we must have $\alpha_i = 1 - \sum_{j \neq i} \alpha_j$ for all $i \in \{1, \ldots, k\}$, i.e., it suffices to consider $k-1$ entries of $\alpha$. Define

$$\Delta^{k-1} := \left\{ \alpha \in (\mathbb{R}^\geq 0)^{k-1} : \sum_{i=1}^{k-1} \alpha_i < 1 \right\}$$

with the closure

$$\overline{\Delta^{k-1}} = \left\{ \alpha \in (\mathbb{R}^\geq 0)^{k-1} : \sum_{i=1}^{k-1} \alpha_i \leq 1 \right\}$$

and $\tilde{F} : \mathbb{R}^n \times \overline{\Delta^{k-1}} \to \mathbb{R}^n$ via

$$\tilde{F}(x, \alpha) := \sum_{i=1}^{k-1} \alpha_i \nabla f_i(x) + \left( 1 - \sum_{i=1}^{k-1} \alpha_i \right) \nabla f_k(x)$$

$$= \sum_{i=1}^{k-1} \alpha_i (\nabla f_i(x) - \nabla f_k(x)) + \nabla f_k(x).$$
Then $P_{\text{int}} = pr_x((\tilde{F}|_{\mathbb{R}^n \times \Delta^{k-1}})^{-1}(0))$ by Definition 4. Since we require differentiability of $\tilde{F}$ (e.g., in order to ensure applicability of the Implicit Function Theorem), we assume from now on that $f$ is twice continuously differentiable. Then the derivative of $\tilde{F}$ is

$$D \tilde{F}(x, \alpha) = (D_x \tilde{F}(x, \alpha), D_{\alpha} \tilde{F}(x, \alpha)) \in \mathbb{R}^{n \times (n+k-1)}$$

with

$$D_x \tilde{F}(x, \alpha) = \sum_{i=1}^{k-1} \alpha_i \nabla^2 f_i(x) + \left(1 - \sum_{i=1}^{k-1} \alpha_i\right) \nabla^2 f_k(x) \in \mathbb{R}^{n \times n}$$

and

$$D_{\alpha} \tilde{F}(x, \alpha) = (\nabla f_1(x) - \nabla f_k(x), \ldots, \nabla f_{k-1}(x) - \nabla f_k(x)) \in \mathbb{R}^{n \times (k-1)}.$$
The important case for the investigation in this article is the special situation where
\( r_k(D_x \tilde{F}(x, \alpha)) = n \) for all \((x, \alpha) \in \mathcal{M} \). This means that we can apply the Implicit Function Theorem to locally obtain a differentiable mapping between the set of KKT vectors and the Pareto critical set. If the assumption of Theorem 2 holds but \( r_k(D_x \tilde{F}(x, \alpha)) < n \) for some \((x, \alpha) \in \mathcal{M} \), then \( \mathcal{M} \) is still a manifold except in these points, which were called dent border points in [32]. There, they were characterized as the preimage of points on the (local) Pareto front where the curvature of the front changes.

To illustrate the consequence of the assumption of Theorem 2 not being valid, we consider the following example.

**Example 1** Consider the MOP \( \min_{x \in \mathbb{R}^2} f(x) \) with
\[
f(x) := \left( -2x_1x_2 - 2x_1^2 - 2x_2^2 + x_2^2 \right) / (x_1x_2 + x_1^2 + x_2).
\]
For this problem \( \mathcal{M} \) can be calculated analytically:
\[
\mathcal{M} = (\mathbb{R} \times \{0\} \times \{1/3\}) \cup \left\{ \left( -\frac{1 - 3\alpha}{7\alpha - 1}, -\frac{2 - 3\alpha}{7\alpha - 1}, \alpha \right) : \alpha \in (0, 1), \alpha \neq 1/7 \right\}.
\]
We see in Fig. 3a that \( \mathcal{M} \) is the union of three smooth curves and that there is an intersection at \((0, 0, 1/3)\). Therefore, \( \mathcal{M} \) is not a manifold. With the same argument it follows that
\[
P_{\text{int}} = \text{pr}_x(\mathcal{M}) = (\mathbb{R} \times \{0\}) \cup \{(t, -2t) : t \in \mathbb{R}\setminus[-1, -1/3]\}
\]
is not a manifold (due to the intersection at \((0, 0)\)), cf. Fig. 3b.

Observe that in Example 1, \( \mathcal{M} \) would again be a manifold if we removed the singular points (in this case \((0, 0, 1/3)\)) from \( \mathcal{M} \). This is made precise in the following lemma.

**Lemma 2** Let \( N := \{(x, \alpha) \in \mathcal{M} : r_k(D_x \tilde{F}(x, \alpha)) \leq n - 1\} \). Then \( \mathcal{M} \setminus N \) is an \((k - 1)\)-dimensional \( C^2 \) submanifold of \( \mathbb{R}^{n+k-1} \) and \( T_{(x, \alpha)}(\mathcal{M} \setminus N) = \ker(D \tilde{F}(x, \alpha)) \).

**Proof** \( N \) is closed since \( r_k \circ D_x \tilde{F} \) is lower semicontinuous as the composition of the lower semicontinuous function \( r_k : \mathbb{R}^{n \times n} \to \mathbb{N} \) (see, e.g., [12]) and the continuous function \( D_x \tilde{F} \). This implies that \( U := (\mathbb{R}^n \times \Delta^{k-1}) \setminus N \) is open and we can apply the Submersion Theorem to \( \tilde{F}|_U \) as in the proof of Theorem 2. \( \square \)

**Remark 3** 1. It is sufficient to remove the set \( N' := \{(x, \alpha) \in \mathcal{M} : r_k(D \tilde{F}(x, \alpha)) \leq n - 1\} \subseteq N \) from \( \mathcal{M} \) (i.e., to relax the condition to the entire Jacobian instead of the
columns w.r.t. $x$) if one is only interested in the manifold structure on $\mathcal{M}$. This follows from the proof of Lemma 2 by replacing $N$ by $N'$. But since we want to apply the Implicit Function Theorem as described above, we remove $N$ instead of only $N'$.

2. Lemma 2 implies that $\mathcal{M}$ is locally a manifold in all points that satisfy the rank condition. This has also been observed by Hillermeier [9] in a similar way.

Note that the previous results on manifolds only hold in the augmented $(x, \alpha)$ space and have no direct implication for the decision space. To illustrate this, the following example shows an MOP where $rk(D_x \tilde{F}(x, \alpha)) = n$ for all $(x, \alpha) \in \mathbb{R}^n \times \Delta^{k-1}$, i.e. $\mathcal{M}$ is a manifold by Theorem 2, but $\mathcal{P}_{\text{int}}$ is not a manifold.

Example 2 Consider the MOP $\min_{x \in \mathbb{R}^2} f(x)$ with

$$f(x) := \begin{pmatrix} x_1^2 + x_2^2 \\ (x_1 - 1)^2 + (x_2 - 1)^4 \\ (x_1 - 2)^2 + (x_2 - 2)^2 \end{pmatrix}.$$ 

The Pareto critical set can be calculated analytically and is shown in Fig. 4. $\mathcal{P}_{\text{int}}$ is given by the gray area united with the points $(1, 1)^\top$, $(1 + \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}})^\top \approx (1.7071, 1.7071)^\top$ and $(1 - \frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}})^\top \approx (0.2929, 0.2929)^\top$. Due to these additional points, $\mathcal{P}_{\text{int}}$ is not a manifold even though $\mathcal{M}$ is.

3.2 Tangent cones and the uniqueness of KKT vectors

Motivated by Theorem 2 and the discussion thereafter, we will from now on assume that

$$rk(D_x \tilde{F}(x, \alpha)) = n \text{ for all } (x, \alpha) \in \mathcal{M},$$

such that that $\mathcal{M}$ is a manifold with the properties stated in Theorem 2.

We will now look at tangent vectors of the Pareto critical set which will later be used to define the edge of the set (Definition 6). Tangent vectors are elements of the tangent cone, see [8] for the following and equivalent definitions:

![Fig. 4](image-url)  
(a) Pareto critical set for Example 2, (b) Image of the Pareto critical set with $f((1, 1)^\top)$, $f((1 + 1/\sqrt{2}, 1 + 1/\sqrt{2})^\top)$ and $f((1 - 1/\sqrt{2}, 1 - 1/\sqrt{2})^\top)$ marked by dots.
**Definition 5** Let $Y \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then

$$Tan(Y, x) := \left\{ v \in \mathbb{R}^n : \exists (v_i)_i \subseteq \mathbb{R}^n \setminus \{0\} : v_i \to 0, x + v_i \in Y, \frac{v_i}{\|v_i\|} \to \frac{v}{\|v\}} \right\} \cup \{0\}$$

is the tangent cone of $Y$ at $x$.

The tangent cone $Tan(Y, x)$ contains the directions in $x$ that point inside or alongside $Y$. Note that in contrast to the tangent space of a manifold, we do not require any additional structure to define the tangent cone of a set. The following lemma shows that if $D_x \bar{F}(x, \alpha)$ is also regular for $x \in P$ and $\alpha \in A(x)$ (and not only for $(x, \alpha) \in \mathcal{M}$), then there are no isolated directions in $Tan(P, x)$.

**Lemma 3** If $D_x \bar{F}(x, \alpha)$ is regular for all $x \in P$ and $\alpha \in A(x)$ then

$$Tan(P_{\text{int}}, x_0) = Tan(P, x_0) \quad \forall x_0 \in P_{\text{int}}.$$  

**Proof** As shown in [8], for all $Y \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we have

$$Tan(Y, x) = \left\{ v \in \mathbb{R}^n : \liminf_{h \to 0} \frac{d_Y(x + hv)}{h} = 0 \right\}$$

with $d_Y(x) := \inf_{y \in Y} \|y - x\|$. It is easy to see that $d_Y(x) = d_{\bar{P}}(x) \forall x \in \mathbb{R}^n$ and consequently, $Tan(Y, x) = Tan(\bar{Y}, x)$. By Lemma 1 we have $\bar{P}_{\text{int}} = P$ such that

$$Tan(P_{\text{int}}, x_0) = Tan(\bar{P}_{\text{int}}, x_0) = Tan(P, x_0).$$

In order to show the irregularities that may occur in MOPs where (6) holds but $D_x \bar{F}(x, \alpha)$ is not regular for some $x \in P$ and $\alpha \in A(x)$ with $(x, \alpha) \notin \mathcal{M}$, we will now give an example where Lemma 3 (and Lemma 1) cannot be applied.

**Example 3** Consider the MOP $\min_{x \in \mathbb{R}^2} f(x)$ with

$$f(x) := \begin{pmatrix} \frac{x_1^2}{2} + (x_2 - 1)^2 \\ \frac{x_1^2}{2} + (x_2 + 1)^2 \\ x_2^2 \end{pmatrix}, \quad Df(x) = \begin{pmatrix} 2x_1 & 2(x_2 - 1) & 0 \\ 2x_1 & 2(x_2 + 1) & 0 \\ 0 & 2x_2 & 0 \end{pmatrix}.$$  

Let $F$ be as in (3), i.e.,

$$F(x, \alpha) = \begin{pmatrix} 2x_1\alpha_1 + 2x_1\alpha_2 \\ 2(x_2 - 1)\alpha_1 + 2(x_2 + 1)\alpha_2 + 2x_2\alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1(\alpha_1 + \alpha_2) \\ 2x_2(\alpha_1 + \alpha_2 + \alpha_3) + 2(\alpha_2 - \alpha_1) \\ \alpha_1 + \alpha_2 + \alpha_3 - 1 \end{pmatrix}.$$  

It is easy to verify that $P = (\mathbb{R} \times \{0\}) \cup (\{0\} \times [-1, 1])$. We will now determine $P_{\text{int}}$ and $P_0$ for this MOP:

- $x_1 \neq 0$: $F(x, \alpha) = 0$ iff $x \in \mathbb{R} \times \{0\}$ and $\alpha = (0, 0, 1)^\top$.
- $x_1 = 0$: $F(x, \alpha) = 0$ iff $x \in \{0\} \times [-1, 1]$ and $\alpha \in \{(\lambda_1, \lambda_1 - x_2, 1 - 2\lambda_1 + x_2)^\top : \lambda_1 \in [0, 1] \cap [x_2, 1 + x_2] \cap [\frac{x_2}{2}, \frac{1 + x_2}{2}]\}$.  

Springer
Thus, \( P_{\text{int}} = \{0\} \times (-1, 1) \) and \( P_0 = (\mathbb{R} \times \{0\}) \setminus \{(0, 0)^\top, (0, 1)^\top\} \), as shown in Fig. 5. Consequently, \( \overline{P_{\text{int}}} = \{0\} \times [-1, 1] \neq P \). In \((0, 0)^\top\) we have \( \text{Tan}(P_{\text{int}}, (0, 0)^\top) = \{0\} \times \mathbb{R} \) and \( \text{Tan}(P, (0, 0)^\top) = \{(0) \times \mathbb{R}\} \cup (\mathbb{R} \times \{0\}) \), hence
\[
\text{Tan}(P_{\text{int}}, (0, 0)^\top) \neq \text{Tan}(P, (0, 0)^\top).
\]
The reason for this is that
\[
\nabla^2 f_3(x) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad \forall x \in \mathbb{R}^n,
\]
\( i.e., D_x F((0, 0)^\top, (0, 0, 1)^\top) = \nabla^2 f_3(x) \) is not regular.

We will now take a look at the relationship between \( \text{Tan}(P_{\text{int}}, x) = \text{Tan}(pr_x(\mathcal{M}), x) \) and \( pr_x(T_p\mathcal{M}) \) for \( p = (x, \alpha) \in \mathcal{M} \). Note that for a general submanifold \( \mathcal{M} \) of \( \mathbb{R}^n \), we have \( \text{Tan}(pr_x(\mathcal{M}), x) \neq pr_x(T_p\mathcal{M}) \). For example, if \( \mathcal{M} = S^1 \) is the unit circle in \( \mathbb{R}^2 \) and \( pr_x \) is the projection onto the first coordinate (i.e., \( pr_x(\mathcal{M}) = [-1, 1] \)), then for \( \tilde{x} = (1, 0)^\top \) the (one-dimensional) vector \(-1\) is a tangent vector in \( pr_x(\tilde{x}) = 1 \), but \( v_1 = 0 \) for all tangent vectors \( v = (v_1, v_2)^\top \in T_{\tilde{x}}\mathcal{M} \), so \( pr_x(v) = 0 \) for all \( v \in T_{\tilde{x}}\mathcal{M} \).

However, for the manifolds that arise in our context, we will show that, under some conditions, \( \text{Tan}(P_{\text{int}}, x) = pr_x(T_p\mathcal{M}) \). The following lemma will show the first half of this statement, which is that the projection of a tangent vector of \( \mathcal{M} \) is always in the tangent cone of \( P_{\text{int}} \) (without requiring any additional regularity assumptions).

**Lemma 4** \( pr_x(T(x_0, \alpha_0), \mathcal{M}) \subseteq \text{Tan}(P_{\text{int}}, x_0) \) \( \forall (x_0, \alpha_0) \in \mathcal{M} \).

**Proof** Let \( v \in T(x_0, \alpha_0), \mathcal{M} \) and let \( \gamma : (-1, 1) \to \mathcal{M} \) be the \( C^1 \)-curve corresponding to \( v \). It is easy to see that \( v \in \text{Tan}(P_{\text{int}}, x_0) \) by considering the sequence \( (v_i) \subseteq \mathbb{R}^n \) with
\[
 v_i := pr_x(\gamma(1/i)) - x_0.
\]
In order to prove the opposite inclusion, i.e., \( pr_x(T(x_0, \alpha_0), \mathcal{M}) \supseteq \text{Tan}(P_{\text{int}}, x_0) \), we first derive additional results about the uniqueness of KKT vectors, i.e., about the set \( A(x) \subseteq \Delta^{k-1} \) [see (4) and (5)].

**Lemma 5** Let \( x_0 \in P \).

1. If \( rk(Df(x_0)) = k - 1 \) then \( |A(x_0)| = 1 \).
2. If \( x_0 \in P_{\text{int}} \) and \( |A(x_0)| = 1 \) then \( rk(Df(x_0)) = k - 1 \).
3. If \( x_0 \in P_{\text{int}} \) and \( rk(Df(x_0)) < k - 1 \) then \( |A(x_0)| > 1 \) and \( A(x_0) \cap \partial \Delta^{k-1} \neq \emptyset \).

**Proof** 1. Since \( x_0 \in P \) we have \( |A(x_0)| > 0 \). Assume \( |A(x_0)| > 1 \) and let \( \alpha^1, \alpha^2 \in A(x_0) \) with \( \alpha^1 \neq \alpha^2 \). Let \( \tilde{\alpha}^1 := (\alpha^1, 1 - \sum_{i=1}^{k-1} \alpha^1_i)^T \in \mathbb{R}^k \) and \( \tilde{\alpha}^2 := (\alpha^2, 1 - \sum_{i=1}^{k-1} \alpha^2_i)^T \in \mathbb{R}^k \).

By definition we have \( \tilde{\alpha}^1, \tilde{\alpha}^2 \in ker(Df(x_0)^T) \) and \( \sum_{i=1}^{k} \tilde{\alpha}^1_i = \sum_{i=1}^{k} \tilde{\alpha}^2_i = 1 \). If there would be some \( s \in \mathbb{R} \) with \( \tilde{\alpha}^1 = s \tilde{\alpha}^2 \) then this would result in

\[
1 = (1, \ldots, 1)^T \tilde{\alpha}^1 = s(1, \ldots, 1)^T \tilde{\alpha}^2 = s,
\]

so \( \tilde{\alpha}^1 \) and \( \tilde{\alpha}^2 \) have to be linearly independent. Thus \( \dim(ker(Df(x_0)^T)) > 1 \) which—by the Rank-nullity theorem—implies \( rk(Df(x_0)) = rk(Df(x_0)^T) = k - dim(ker(Df(x_0)^T)) < k - 1 \).

2. Assume \( rk(Df(x_0)) < k - 1 \). Let \( \alpha \in A(x_0) \). Then there must be some \( w \in \mathbb{R}^k \) with \( Df(x_0)^T w = 0 \) such that \( (\alpha, 1 - \sum_{i=1}^{k-1} \alpha_i)^T \in \mathbb{R}^k \) and \( w \) are linearly independent.

**Case 1** \( \sum_{i=1}^{k} w_i \neq 0 \). Assume w.l.o.g. \( \sum_{i=1}^{k} w_i = 1 \). It follows that \( \tilde{F}(x_0, (w_1, \ldots, w_{k-1})^T) = 0 \). For \( \lambda \in \mathbb{R} \) define \( \beta(\lambda) := \alpha + \lambda((w_1, \ldots, w_{k-1})^T - \alpha) \). Then we have \( \tilde{F}(x_0, \beta(\lambda)) = 0 \) for all \( \lambda \in \mathbb{R} \). Since \( \alpha \in \Delta^{k-1} \) and \( \Delta^{k-1} \) is open there has to be some \( \lambda \neq 0 \) such that \( \beta(\lambda) \in \Delta^{k-1} \), i.e., \( \alpha \) is not unique.

**Case 2** \( \sum_{i=1}^{k} w_i = 0 \). This implies

\[
\sum_{i=1}^{k} w_i \nabla f_i(x_0) = 0 \iff \sum_{i=1}^{k-1} w_i \nabla f_i(x_0) + \left( \sum_{i=1}^{k-1} w_i \right) \nabla f_k(x_0) = 0 \iff \sum_{i=1}^{k-1} w_i (\nabla f_i(x_0) - \nabla f_k(x_0)) = 0.
\]

Define \( \beta(\lambda) := \alpha + \lambda(w_1, \ldots, w_{k-1})^T \). Then \( \tilde{F}(x_0, \beta(\lambda)) = 0 \) for all \( \lambda \in \mathbb{R} \). The contradiction follows as in Case 1.

3. Let \( \alpha \in A(x_0) \cap \Delta^{k-1} \). From the proof of 2. we know that there is some \( \gamma \in \mathbb{R}^{k-1} \) with \( \tilde{F}(x_0, \alpha + \lambda \gamma) = 0 \) for all \( \lambda \in \mathbb{R} \). Since \( \alpha \in \Delta^{k-1} \) and \( \Delta^{k-1} \) is bounded, there has to be some \( \lambda \in \mathbb{R} \) such that \( \alpha + \lambda \gamma \in \partial \Delta^{k-1} \). In particular \( |A(x_0)| > 1 \).

Lemma 5 has the following obvious implications:

**Corollary 1** 1. Let \( x_0 \in P \). If \( rk(Df(x_0)) < k - 1 \) then \( A(x_0) \cap \partial \Delta^{k-1} \neq \emptyset \).

2. Let \( x_0 \in P_{\text{int}} \). Then \( rk(Df(x_0)) = k - 1 \iff |A(x_0)| = 1 \).

We now use the results about the uniqueness of the KKT vectors to show that if the rank of \( Df \) is large enough, the converse of Lemma 4 holds as well. For this we require \( f \) to be three times continuously differentiable (by which \( \tilde{F} \) is \( C^2 \)) which we will assume from now on.

**Lemma 6** Let \( x_0 \in P_{\text{int}} \) with \( rk(Df(x_0)) = k - 1 \). Then there exists \( \alpha_0 \in A(x_0) \cap \Delta^{k-1} \) with

\[
\Tan(P_{\text{int}}, x_0) \subseteq pr_x(T(x_0, \alpha_0, \mathcal{M})).
\]

**Proof** Let \( v \in \Tan(P_{\text{int}}, x_0) \) and \((v_i)_i\) be a sequence as in Definition 5. Since \((x_0 + v_i)_i \subseteq P_{\text{int}}\), this induces a sequence \((\alpha_i)_i \subseteq \Delta^{k-1} \) with \( \tilde{F}(x_0 + v_i, \alpha_i) = 0 \forall i \in \mathbb{N} \). Since \( \Delta^{k-1} \) is bounded, we can w.l.o.g. assume that \( \alpha_i \to \alpha_0 \in \Delta^{k-1} \) and by continuity of \( \tilde{F} \) we have \( \tilde{F}(x_0, \alpha_0) = 0 \). By Corollary 1 we have \( \alpha_0 \in \Delta^{k-1} \).

\(\square\) Springer
Since $D_x \tilde{F}(x_0, \alpha_0)$ is regular and $\tilde{F}$ is $C^2$, we can apply the Implicit Function Theorem at $(x_0, \alpha_0)$ to obtain neighborhoods $U \subseteq \Delta^k$, $V \subseteq \mathbb{R}^n$ of $x_0$ and a (unique) $C^2$ function $\phi : U \to V$ such that $\tilde{F}(x, \alpha) = 0 \iff \phi(\alpha) = x$ and $D\phi(\alpha) = -(D_x \tilde{F}(x, \alpha))^{-1} D_\alpha \tilde{F}(x, \alpha)$ for all $(x, \alpha) \in V \times U$. Since $\alpha_i \to \alpha_0$ and $v_i \to 0$ we can assume w.l.o.g. that $\alpha_i \in U$ and $x_0 + v_i \in V$ for all $i \in \mathbb{N}$. Thus $\phi(\alpha_i) = x_0 + v_i$ and we obtain

$$\frac{v}{\|v\|} = \lim_{i \to \infty} \frac{v_i}{\|v_i\|} = \lim_{i \to \infty} \frac{\phi(\alpha_i) - \phi(\alpha_0)}{\|v_i\|} = \lim_{i \to \infty} \frac{\phi(\alpha_0 + \|v_i\| \frac{\alpha_i - \alpha_0}{\|v_i\|}) - \phi(\alpha_0)}{\|v_i\|} = \phi(\alpha_0) - \phi(\alpha_0) = 0.$$

(\text{Taylor series})

Assume that $\left(\frac{\alpha_i - \alpha_0}{\|v_i\|}\right)_i$ is unbounded. Then by the above equation we have

$$D\phi(\alpha_0) \frac{\alpha_i - \alpha_0}{\|v_i\|} \to 0.$$

W.l.o.g. we assume that $\frac{\alpha_i - \alpha_0}{\|v_i\|} \to w$, which implies $D\phi(\alpha_0)w = 0$, and therefore

$$D_\alpha \tilde{F}(x_0, \alpha_0)w = 0$$

with $\|w\| = 1$. Thus,

$$\begin{align*}
(\nabla f_1(x_0) - \nabla f_k(x_0))w_1 + \cdots + (\nabla f_{k-1}(x_0) - \nabla f_k(x_0))w_{k-1} &= 0 \\
\iff w_1\nabla f_1(x_0) + \cdots + w_{k-1}\nabla f_{k-1}(x_0) + \left(\sum_{i=1}^{k-1} (-w_i)\right)\nabla f_k(x_0) &= 0 \\
\iff Df(x_0)^\top \left(w_1, \ldots, w_{k-1}, \sum_{i=1}^{k-1} (-w_i)\right)^\top &= 0 \\
\iff \tilde{w} &\in \ker(Df(x_0)^\top)
\end{align*}$$

for $\tilde{w} := \left(w, \sum_{i=1}^{k-1} (-w_i)\right)^\top \in \mathbb{R}^k$. Let $\tilde{\alpha} := (\alpha, 1 - \sum_{i=1}^{k-1} \alpha_i)^\top \in \mathbb{R}^k$. Then $\tilde{\alpha}$ and $\tilde{w}$ are linearly independent since $\sum_{i=1}^k \tilde{w}_i = 0$ and $\sum_{i=1}^k \tilde{\alpha}_i = 1$. As they are both in $\ker(Df(x_0)^\top)$ we must have $rk(Df(x_0)) < k - 1$, which is a contradiction.

As a consequence, $\left(\frac{\alpha_i - \alpha_0}{\|v_i\|}\right)_i$ has to be bounded and we can assume w.l.o.g. that $\frac{\alpha_i - \alpha_0}{\|v_i\|} \to v^\alpha$. Thus

$$\frac{v}{\|v\|} = \lim_{i \to \infty} \frac{v_i}{\|v_i\|} = \lim_{i \to \infty} \frac{\phi(\alpha_0 + \|v_i\| \frac{\alpha_i - \alpha_0}{\|v_i\|}) - \phi(\alpha_0)}{\|v_i\|} = D\phi(\alpha_0)v^\alpha.$$

From this we obtain

$$D\tilde{F}(x_0, \alpha_0)z = 0$$

with $z := (v, \|v\|v^\alpha)^\top \in \mathbb{R}^{n+k-1}$. By Theorem 2 we have $T_{x_0,\alpha_0}.\mathcal{M} = \ker(D\tilde{F}(x_0, \alpha_0))$ which completes the proof.
3.3 The edge of the Pareto critical set

We are now in the position to extend the results in [22] to a more general situation and show that if \( x \) lies on the edge of the Pareto critical set, then there has to be a corresponding KKT vector \( \alpha \) with at least one zero component. The topological boundary \( \partial P \) is in general not suitable for what we want to describe with the term edge (see, e.g., Example 3 where \( \partial P = P \)). Instead we define it in the following way:

**Definition 6** We call 
\[ P_E := \{ x \in P : \text{Tan}(P_{\text{int}}, x) \neq -\text{Tan}(P_{\text{int}}, x) \} \]
the edge of the Pareto critical set \( P \). In other words, \( x_0 \in P_E \) iff there exists \( v \in \text{Tan}(P_{\text{int}}, x_0) \) with \( -v \notin \text{Tan}(P_{\text{int}}, x_0) \).

On the basis of Lemmas 4 and 6, we can now prove our first main result, namely that points on the edge of the Pareto critical set also satisfy the optimality conditions for a subproblem with at least one neglected objective.

**Theorem 3** If \( x_0 \in P_E \) then \( A(x_0) \cap \partial \Delta^{k-1} \neq \emptyset \).

**Proof** Assume that the assertion does not hold, so \( x_0 \in P_E \) and \( \exists \alpha \in \Delta^{k-1} \setminus \Delta^{k-1} \) with \( \bar{F}(x_0, \alpha) = 0 \). In particular, we obtain \( x_0 \in P_{\text{int}} \). Then according to Lemma 5—\( \alpha \) has to be unique, \( \alpha \in \Delta^{k-1} \) and \( rk(Df(x_0)) = k - 1 \). We can thus apply Lemmas 4 and 6 to see that \( \text{Tan}(P_{\text{int}}, x_0) = \text{pr}_x(T(x_0, \alpha_0), \mathcal{M}) \). Since \( \text{pr}_x(T(x_0, \alpha_0), \mathcal{M}) \) is a vector space, we obviously have \( \text{Tan}(P_{\text{int}}, x_0) = -\text{Tan}(P_{\text{int}}, x_0) \) which contradicts our assumption. \( \square \)

If we additionally assume that the rank of \( Df \) is large enough, then we can use Lemma 5 to obtain the following corollary which states that on the edge, there is no Pareto critical point with a strictly positive KKT vector.

**Corollary 2** Let \( rk(Df(x)) = k - 1 \) for all \( x \in P_{\text{int}} \). Then \( P_E \subseteq P_0 \).

**Remark 4** Theorem 3 shows that some objective functions may be discarded if we are only interested in calculating the edge of the Pareto critical set. Note that at this point, it is unclear how many objective functions we are allowed to discard to still obtain a covering of the edge. This will be investigated in the following section.

**Remark 5** Although \( \partial P \) and \( P_0 \) do in general not coincide, there is a special situation where \( \partial P = P_0 \). This is the case when \( rk(Df(x)) = k - 1 = n \) for all \( x \in P \).

4 The decomposition of an MOP into lower-dimensional subproblems

In Sect. 3 we have shown that points on the edge \( P_E \) of the Pareto critical set have a KKT vector where one component is zero. We want to exploit this and consider the \( k \) subproblems of (MOP) where one objective function is neglected. By the results from Sect. 3, \( P_E \) is a subset of the union of the Pareto critical sets of these \( k \) subproblems. (As we will see in Example 4, there are situations where this union is a non-trivial superset of \( P_E \)). Furthermore, we are going to study problems where more than one KKT multiplier is zero.

The subproblems mentioned above arise by omitting certain objective functions or, in other words, by only taking a subset of the set of objective functions into account: For
Lemma 7 Let $P_I$ be the set of Pareto critical points of $(MOP_I)$. Then

$$\min_{x \in \mathbb{R}^n} f^I(x) \quad (MOP^I)$$

the MOP where $f^I(x) := (f_i(x))_{i \in I}$, i.e., $f^I$ contains only those components of the objective vector $f$ with indices in $I$. Let $P^I$ be the corresponding set of Pareto critical points and let $F^I, \tilde{F}^I$ and $A^I$ be defined analogously to $F, \tilde{F}$ and $A$ in Sect. 3. Since in our setup $P_0$ and $P_{\text{int}}$ are not defined for scalar-valued MOPs, we set $P^0_I := P^I$ and $P^0_{\text{int}} := 0$ if $|I| = 1$ for ease of notation. For $I = \emptyset$ let $P^I := \emptyset$ and $P_{\text{int}} := \emptyset$.

We begin by showing that points that are Pareto critical with respect to a subset of the set of objective functions are also Pareto critical for the full problem:

**Lemma 7** Let $P^I$ be the set of Pareto critical points of $(MOP_I)$. Then

$$P^I \subseteq P \quad \forall I \subseteq \{1, \ldots, k\}.$$  

**Proof** Let $I = \{i_1, \ldots, i_{|I|}\} \subseteq \{1, \ldots, k\}$ and $x_0 \in P^I$. Then

$$\exists \alpha^I \in (\mathbb{R}^{\geq 0})^{|I|} : \nabla f_{i_1}(x_0)\alpha^I_1 + \cdots + \nabla f_{i_{|I|}}(x_0)\alpha^I_{|I|} = 0 \quad \text{and} \quad \sum_{i=1}^{|I|} \alpha^I_i = 1.$$  

Define $\alpha \in \mathbb{R}^k$ via

$$\alpha_i := \begin{cases} \alpha^I_i & i \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_{i=1}^k \alpha_i = \sum_{j=1}^{|I|} \alpha^I_j = 1$ and

$$\nabla f_1(x_0)\alpha_1 + \cdots + \nabla f_k(x_0)\alpha_k = 0.$$  

In particular, $x_0 \in P$ which yields the desired result. \hfill $\square$

**Remark 6**

1. It immediately follows from Lemma 7 that

$$P_{I_1} \subseteq P_{I_2} \text{ if } I_1 \subseteq I_2.$$  

2. Lemma 7 still holds if we replace Pareto criticality by weak Pareto optimality [19]: If $x$ is weakly Pareto optimal for the subset $I \subseteq \{1, \ldots, k\}$, i.e., $\exists y \in \mathbb{R}^n : f_i(y) < f_i(x) \forall i \in I$, then in particular $\exists y \in \mathbb{R}^n : f_i(y) < f_i(x) \forall i \in \{1, \ldots, k\}$, i.e., $x$ is weakly Pareto optimal for the full problem.

In Corollary 2 we had to assume that $rk(Df(x)) = k - 1$ for all $x \in P_{\text{int}}$ to see that $P_E \subseteq P_0$. The following lemma shows that if this rank condition is violated, then we can still find a subproblem $(MOP^I)$ such that $x$ is Pareto critical with respect to $(MOP^I)$ and

$$rk(Df_I(x)) = |I| - 1.$$  

In particular, Corollary 2 can be applied to that subproblem. This way, we obtain a decomposition of $(MOP)$ into subproblems that satisfy the rank condition.

**Lemma 8** Let $x_0 \in P$. Then there exists some $I \subseteq \{1, \ldots, k\}$ with $|I| = rk(Df(x_0)) + 1$ such that $rk(Df_I(x_0)) = rk(Df(x_0))$ and $x_0 \in P^I$. Moreover, if $x_0 \in P_0$ then $x_0 \in P^I_0$. In particular, there is a subset $\mathcal{I}$ of the power set $\mathcal{P}(\{1, \ldots, k\})$ with

$$P = \bigcup_{I \in \mathcal{I}} P^I \quad \text{and} \quad P_0 \subseteq \bigcup_{I \in \mathcal{I}} P^I_0$$

and $\forall x \in P \exists I \in \mathcal{I}$ with $rk(Df_I(x)) = rk(Df(x)) = |I| - 1$.  

$\square$
Proof If \(rk(Df(x_0)) = k - 1\) then we can simply choose \(I = \{1, \ldots, k\}\). We therefore now assume \(rk(Df(x_0)) < k - 1\). Let

\[
J := \{j \in \{1, \ldots, k\} : rk(Df^{I\setminus\{j\}}(x_0)) < rk(Df(x_0))\}
\]

be the set of linearly independent objectives and \(K := \{1, \ldots, k\}\setminus J\). Then we have \(\alpha_j = 0\) for all \(j \in J\) and all \(\alpha \in (\mathbb{R}^\geq 0)^k\) with \(F(x_0, \alpha) = 0\) (since the \(j\)th gradient is not in the span of the other gradients). By construction, we have \(rk(Df^J(x_0)) = |J|\) and

\[
k - 1 > rk(Df(x_0)) = rk(Df^K(x_0)) + |J|
\]

\[
\Leftrightarrow rk(Df^K(x_0)) < k - |J| - 1 = |K| - 1.
\]

Consequently, \(x_0 \in P^K\) and we can apply Corollary 1 to \(f^K\). This yields the existence of some \(\alpha' \in (\mathbb{R}^\geq 0)^{|K|}\) with \(F^K(x_0, \alpha') = 0\) and some \(l \in K\) with \(\alpha'_l = 0\) and \(rk(Df^K^{\{l\}}(x_0)) = rk(Df^K(x_0))\) (since \(K := \{1, \ldots, k\}\setminus J\)). Thus, if we set \(I = \{1, \ldots, k\}\setminus\{l\}\) then we have \(rk(Df^I(x_0)) = rk(Df(x_0))\) and \(x_0 \in P^I\).

Now assume that \(x_0 \in P_0\). Let \(A'(x_0) := \{\alpha \in (\mathbb{R}^\geq 0)^k : F(x_0, \alpha) = 0\}\). We will show that after neglecting the \(l\)th objective, there still exists some \(j \in I = \{1, \ldots, k\}\setminus\{l\}\) with \(\alpha_j = 0\) for all \(\alpha \in A'(x_0)\) such that \(x_0 \in P^I_0\). For all \(\alpha \in A'(x_0)\) there exists some \(j \in \{1, \ldots, k\}\) with \(\alpha_j = 0\). By the structure of \(A'(x_0)\), there has to be \(j \in \{1, \ldots, k\}\) with \(\alpha_j = 0\) for all \(\alpha \in A'(x_0)\). If there are two such indices, then we are done since only one element of \(I\) is removed. Hence, we assume there is only one such. If \(j \in I\) we are done, since \(l \notin J\). Consequently, assume \(j \notin J\). Then \(rk(Df(x_0)) = rk(Df^{I\setminus\{j\}}(x_0))\), and there is some \(\beta \in \mathbb{R}^k\) with \(\beta_j = 0\) and \(Df(x_0)^\top \beta = \nabla f_j(x_0)\). If we set \(\beta' := -\beta + e_j\), where \(e_j\) is the \(j\)th unit vector (in Cartesian coordinates), then \(Df(x_0)^\top \beta' = 0\). By the assumption of the uniqueness of \(j\), there has to be some \(\alpha' \in A'(x_0)\) with \(\alpha'_j > 0\) for all \(i \neq j\). It follows that there has to be some \(s > 0\) such that \(\gamma \in A'(x_0)\) where

\[
\gamma := \frac{\alpha' + s \beta'}{1 - s \sum_{i=1}^k \beta'_i}.
\]

But we have \(\gamma_j = \frac{s}{1 - s \sum_{i=1}^k \beta'_i} \neq 0\) which is a contradiction to \(\alpha_j = 0\) for all \(\alpha \in A'(x_0)\), hence \(j \in J\). As a consequence, in the above step \(j\) cannot be removed since \(l \notin J\).

Each time we apply the above procedure, \(|I|\) is decreased by 1 and \(rk(Df^{I\setminus\{l\}}(x_0))\) does not change. Therefore, there has to be some \(I\) with \(|I| - 1 = rk(Df(x_0))\) and \(x_0 \in P^I\). Moreover, \(x_0 \in P_0^I\) if \(x_0 \in P_0\). \(\Box\)

Lemma 8 also shows that it suffices to solve a number of subproblems with fewer objective functions instead of the full MOP to obtain the complete Pareto critical set (and not only the edge) if the rank of the Jacobian of \(f\) is small (relative to \(k\)). (For instance, such a situation always occurs when \(k > n + 1\)). This is particularly useful since the complexity for solving MOPs in general increases significantly with the number of objectives. In practice, we are obviously interested in solving as few subproblems as possible, i.e., in finding the smallest \(I\) in Lemma 8. Unfortunately, while it may be possible to find a \(I\) as in Lemma 8 that is sufficiently large to cover the complete Pareto critical set, if we want to find the smallest \(I\), we have to consider all possible subproblems. We illustrate Lemma 8 with the following example.
Example 4  (a) Consider the MOP $\min_{x \in \mathbb{R}^2} f(x)$ with

$$f(x) := \begin{pmatrix}
(x_1 + 1)^2 + (x_2 + 1)^2 \\
(x_1 - 1)^2 + (x_2 + 1)^2 \\
x_1^2 + (x_2 - 1)^2 \\
x_1^4 + x_2^4
\end{pmatrix}.$$ 

The Pareto critical set is the triangle with vertices $(-1, -1), (1, -1)$ and $(0, 1)$. Since $k = 4$ and $n = 2$, we must have $\text{rk}(Df(x)) < k - 1 = 3$. By Lemma 8 we can write $P$ as the union of several $P^I$ with $I \subseteq \{1, 2, 3, 4\}$ and $|I| = 3$. We can choose for example to solve (MOP$^I$) with $I \in \{(1, 2, 4), (2, 3, 4), (1, 3, 4)\}$ or with $I = \{1, 2, 3\}$. The situation is shown in Fig. 6. (Note that in general, it will obviously not be sufficient to only solve one (MOP$^I$) as will be seen in the next example).

(b) Consider the MOP $\min_{x \in \mathbb{R}^2} f(x)$ with

$$f(x) := \begin{pmatrix}
(x_1 + 1)^2 + (x_2 + 1)^2 \\
(x_1 - 1)^2 + (x_2 + 1)^2 \\
x_1^2 + (x_2 - 1)^2 \\
x_1^4 + (x_2 - 1)^2
\end{pmatrix}.$$ 

The Pareto critical set is the square with vertices $(-1, -1), (1, -1), (1, 1)$ and $(-1, 1)$. In analogy to (a) we can write $P$ as the union of several $P^I$ with $I \subseteq \{1, 2, 3, 4\}$ and $|I| = 3$. Here we can choose for example $I \in \{(1, 2, 4), (2, 3, 4)\}$ (the bottom-left triangle and the upper-right triangle), which is shown in Fig. 7. Note that Lemma 8 does not state anything about $P_{\text{int}}^I$. In fact, if we apply this lemma to some $x_0 \in P$ we generally do not know whether there is some $I$ (that satisfies the rank condition) with $x_0 \in P_{\text{int}}^I$. In this example there is no $I$ with $|I| = 3$ so that $(0, 0)^\top \in P_{\text{int}}^I$, although $(0, 0)^\top \in P_{\text{int}}$.

As described above, Lemma 8 states how the complete Pareto critical set can be obtained by solving subproblems with less objective functions if the rank of the Jacobian of $f$ is small in relation to $k$ (e.g., when $k > n + 1$). Roughly speaking, these subproblems are identified in such a way that Corollary 2 is applicable. This is done in the following lemma.

Lemma 9  Let $x_0 \in P_E$ and $m := \max_{x \in P} \text{rk}(Df(x))$. There exists $I \in \mathcal{P}(\{1, \ldots, k\})$ with $|I| \leq m + 1$ such that either $I = \{i\}$ and $\nabla f_i(x_0) = 0$ or $A^I(x_0) \cap \partial A^{|I|-1} \neq \emptyset.$

\[\text{Springer}\]
Proof Let \( \mathcal{I} \) be as in Lemma 8 and \( x_0 \in P_E \). By construction we have \(|I| \leq m + 1\) for all \( I \in \mathcal{I} \). If there is some \( I \in \mathcal{I} \) with \( x_0 \in P^I \) and \(|I| = 1\) we have \( \nabla f_i(x_0) = 0 \) for \( I = \{i\} \) and we are done. Therefore we now assume that \(|I| > 1\) for all \( I \in \mathcal{I} \) with \( x_0 \in P^I \).

Case 1 \( \text{Tan}(P_{\text{int}}, x_0) \neq \text{Tan}(P, x_0) \). We have \( v \in \text{Tan}(P_{\text{int}}, x_0) \) with \( v \notin \text{Tan}(P_{\text{int}}, x_0) \). Since \( P_0 = P \setminus P_{\text{int}} \), this implies \( U \cap P_0 \neq \emptyset \) for all neighborhoods \( U \) of \( x_0 \) and thus, \( x_0 \in \overline{P_0} \). Let \( (y_j)_j \in P_0 \) with \( \lim_{j \to \infty} y_j = x_0 \). By Lemma 8 and since \(|\mathcal{I}| \) is finite, there has to be some \( K \in \mathcal{I} \) with \( y_j \in P^K_0 \) infinitely many times such that \( x_0 \in \overline{P^K_0} \). In particular, \( \partial \Delta^{|K| - 1} \cap A^K(x_0) \neq \emptyset \).

Case 2 \( \text{Tan}(P_{\text{int}}, x_0) = \text{Tan}(P, x_0) \). This implies \( \text{Tan}(P_{\text{int}}, x_0) = \text{Tan}(P, x_0) = \text{Tan} \left( \bigcup_{I \in \mathcal{I}} P^I, x_0 \right) = \bigcup_{I \in \mathcal{I}} \text{Tan}(P^I, x_0) \).

For the last equality note that \( \supseteq \) is obvious and \( \subseteq \) follows from the fact that \( P^I \) is closed for all \( I \). Since \( x_0 \in P_E \) there has to be \( I \in \mathcal{I} \) with \( \text{Tan}(P^I, x_0) \neq -\text{Tan}(P^I, x_0) \). If \( \text{Tan}(P^I_{\text{int}}, x_0) \neq \text{Tan}(P^I, x_0) \), we have \( x_0 \in \overline{P^I_0} \) and we are done as in Case 1. Otherwise we obtain \( x_0 \in P^I_0 \) if \( \text{rk}(Df^I(x_0)) = |I| - 1 \) then we can apply Corollary 2 to obtain \( x_0 \in P^I_0 \) (so \( A^I(x_0) \subseteq \partial \Delta^{|I| - 1} \)). If \( \text{rk}(Df^I(x_0)) < |I| - 1 \) then we can apply Corollary 1 to \((\text{MOP})^I\) which yields \( \partial \Delta^{|I| - 1} \cap A^I(x_0) \neq \emptyset \).

By considering the union of all possible subsets of \( \{1, \ldots, k\} \) of appropriate size, the last lemma can be used to state our second main result:

**Corollary 3** Assume \( m := \max_{x \in P} \text{rk}(Df(x)) > 0 \). Then

\[
P_E \subseteq \bigcup_{I \in \mathcal{I} \setminus \{(1, \ldots, k)\}, |I| = m} P^I.
\]

Observe that Corollary 3 basically implies that one only needs to consider \(|I| = \text{rk}(Df)\) objective functions in order to compute the edge of the Pareto critical set. We will demonstrate this by several examples in the following section.
5 Examples

In this section, we will demonstrate how our results can be used to investigate the hierarchical structure of Pareto critical sets. We start with the special case where $rk(Df(x)) = k - 1 = n$ everywhere before revisiting the simple MOPs in Example 4 and then moving on to more complicated MOPs.

**Example 5** Consider the MOP $\min_{x \in \mathbb{R}^3} f(x)$ with

\[
 f(x) := \begin{pmatrix}
 (x_1 - 1)^4 + (x_2 - 1)^2 + (x_3 - 1)^2 \\
 (x_1 + 1)^2 + (x_2 + 1)^4 + (x_3 - 1)^2 \\
 (x_1 - 1)^2 + (x_2 + 1)^2 + (x_3 + 1)^4 \\
 (x_1 + 1)^2 + (x_2 - 1)^4 + (x_3 + 1)^2
\end{pmatrix}.
\]

In this simple example, the rank of $Df$ is $k - 1$ everywhere on the Pareto critical set (which, in this case, coincides with the Pareto set). Consequently, all KKT vectors in $P_{\text{int}}$ are strictly positive and $P_0 = \partial P$, cf. Remark 5. In this situation, we can clearly identify the hierarchical structure of the Pareto set. The entire set as well as the 3-objective and 2-objective solutions are shown in Fig. 2.

**Example 6** We again consider the MOPs from Example 4.

(a) We have $rk(Df(x)) = 2$ for all $x \in \mathbb{R}^2$, so by Corollary 3 it suffices to consider only pairs of 2 objective functions. The corresponding Pareto critical sets are shown in Fig. 8a. To compute $P_E$ it would suffice to consider the three subproblems in

\[
 \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}.
\]

(b) We have $rk(Df(x)) = 2$ for all $x \in \mathbb{R}^2$, so again it suffices to consider only pairs of 2 objective functions. In this case we need to solve the four subproblems in

\[
 \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}
\]

to obtain $P_E$ as shown in Fig. 8b.

The MOPs in the previous example are obviously very simple such that the relation between $P$, $P_E$, $P_{\text{int}}$ and $P_0$ is relatively easy to see. We will now consider more complicated examples:
Example 7 Consider the MOP min \( x \in \mathbb{R}^2 \) f(x) with

\[
 f(x) := \begin{pmatrix}
 x_1^4 + x_2^4 \\
 (x_1 - 1/3)^6 + (x_2 - 1/3)^2 \\
 (x_1 - 2/3)^2 + (x_2 - 2/3)^4 \\
 0.25(x_1 - 1)^2 + (x_2 - 1)^4
\end{pmatrix}.
\]

By construction the hessian matrices of the individual objectives are diagonal and it is easy to see that \( D_x \tilde{F}(x, \alpha) \) is regular for all \( (x, \alpha) \in \mathbb{R}^2 \times \Delta^2 \), so the assumption (6) is satisfied. The Pareto critical set is shown in Fig. 9a. Figure 9b shows the solutions to all possible 2-objective subproblems, colored according to which objectives were considered. The black dots mark the critical points of each objective individually.

In contrast to Example 6, we see that it is possible for the Pareto critical set of a subproblem to be partly on \( P_E \) and partly inside \( P \), for example in the case of objective 1 and 3 (green). Additionally, there are intersections of critical sets aside from the solutions of the 1-objective subproblems, for example the intersection of the red and the blue line. This indicates that these points have two KKT vectors with different zero components.

The previous example indicates that if we have a “kink” in \( P_E \) then it is either a critical point of a \((m - 1)\)-objective subproblem (with \( m \) as in Corollary 3) or a critical point with multiple KKT vectors on the boundary of the standard simplex. The classification of those non-differentiabilities in the boundary of the Pareto critical set highlights an additional advantage of the approach presented in this paper.

As shown in Lemma 2, we can remove points that do not satisfy the assumption (6) from the extended Pareto critical set \( \mathcal{A} \) to preserve the manifold structure. Since the techniques we used in Sects. 3 and 4 are essentially of local nature, this encourages that our results can also be applied to MOPs that do not satisfy assumption (6). This will be done in the following examples.

Consider the MOP min \( x \in \mathbb{R}^2 \) f(x) with

\[
 f(x) := \begin{pmatrix}
 0.5(x_1 - 1)^2 + x_2^2 \\
 2x_1^2 + 2(x_2 - 1)^2 \\
 2(x_1 + 1)^2 + x_2^5 \\
 -2x_1^3 + 2(x_2 + 1)^2
\end{pmatrix}.
\]
Since each objective function is polynomial, it is still (relatively) easy to calculate the Pareto critical set analytically. The part of interest is shown in Fig. 10a. Figure 10b shows the corresponding Pareto critical sets of all 2-objective subproblems, colored according to which objectives were considered. The black dots mark the critical points of each objective individually.

One can see that for this example, it is not necessary to consider the subproblem \( \{1, 4\} \) since its Pareto critical set is in the interior—i.e., it does not lie on the edge—of the actual Pareto critical set. (Strictly speaking \( P_E \cap P^{\{1,4\}} = \{(0, -1)\} \), but this point is also in \( P^{\{2,4\}} \) and \( P^{\{3,4\}} \), so it is already covered). Other than that, all subproblems have to be solved to obtain \( P_E \).

The following example from [22, Example 4.1.5] shows how the Pareto critical set can be derived from \( P_E \) if additional properties of the MOP are known, like in this case boundedness of \( P \).

**Example 9** Consider the MOP \( \min_{x \in \mathbb{R}^2} f(x) \) with

\[
 f(x) := \begin{pmatrix}
 -6x_1^2 + x_1^4 + 3x_2^2 \\
 (x_1 - 0.5)^2 + 2(x_2 - 1)^2 \\
 (x_1 - 1)^2 + 2(x_2 - 0.5)^2
\end{pmatrix}.
\]

Figure 11 shows the Pareto critical sets of the three 2-objective subproblems. It is possible to show that \( rk(Df(x)) = 2 \) for all \( x \in P \), so \( \partial P = P_0 \). The Pareto critical set of this problem is bounded so we know that it is given by the interior (and boundary) of the two disconnected sets depicted in Fig. 11.

### 6 Conclusion and outlook

We have presented results about the structure of the set of Pareto critical points and the relation to the corresponding KKT vectors. Our first main result is that the boundary of the Pareto critical set can be covered by Pareto critical sets of subproblems where only subsets of
the set of objective functions are considered. Our second main result shows that the number of objective functions required for the subproblems (locally) depends on the rank of the Jacobian of the objective vector. To prove these results, we have investigated the relationship between tangent cones of the Pareto critical set and the tangent spaces of the manifold of Pareto critical points extended by their KKT vectors. The boundary of the Pareto critical set can give useful insight into the global Pareto set or—if it coincides with the topological boundary—even describe it completely. This allows us to design very efficient algorithms for solving MOPs with many objective functions by exploiting this structure.

For future work, there are some theoretical aspects that should be investigated further, for example the relationship between $\partial P$ and $P_E$ and what the requirements are such that $P_0 = P_E$. Furthermore, the examples in Sect. 5 have shown that our theoretical results still hold even if the regularity assumption (6) is violated. This indicates that there is a way to weaken this assumption. Additionally, since we have only considered a first-order necessary condition for Pareto optimality, it may be possible to extend our results by using nondominance tests (see, e.g., [4]) or information about higher-order derivatives (as in [15], where the second-order conditions from [29] were used). Moreover, we have only considered the unconstrained case, so it will be interesting to see how the results generalize to equality and inequality constrained MOPs. Finally, our results may be used to build new methods for solving MOPs via computation of the boundary of the Pareto critical set. A first approach in this direction has been discussed in [22], where the well-known $\epsilon$-constraint method was generalized to considering subproblems with fewer objective functions instead of scalar problems.

Acknowledgements This research was funded by the DFG Priority Programme 1962 "Non-smooth and Complementarity-based Distributed Parameter Systems".

References

1. Brockhoff, D., Zitzler, E.: Objective reduction in evolutionary multiobjective optimization: theory and applications. Evolut. Comput. 17(2), 135–166 (2009)
2. De Melo, W.: On the structure of the Pareto set of generic mappings. Bull. Braz. Math. Soc. 7(2), 121–126 (1976)
3. Degiovanni, M., Lucchetti, R., Ribarska, N.: Critical point theory for vector valued functions. J. Convex Anal. 9, 415–428 (2002)
4. Dellnitz, M., Schütze, O., Hestermeyer, T.: Covering pareto sets by multilevel subdivision techniques. J. Optim. Theory Appl. 124(1), 113–136 (2005)
5. di Pierro, F., Khu, S.T., Savic, D.A.: An investigation on preference order ranking scheme for multiobjective evolutionary optimization. IEEE Trans. Evolut. Comput. 11(1), 17–45 (2007)
6. Ehrgott, M.: Multicriteria Optimization. Springer, Berlin (2005)
7. Enrico, M., Elena, M., Matteo, R.: A Morse-type index for critical points of vector functions. Economics and quantitative methods, Department of Economics, University of Insubria (2007). URL https://EconPapers.repec.org/RePEc:ins:quaeco:qf0702. Accessed 9 Jan 2019
8. Giorgi, G., Guerraggio, A.: On the notion of tangent cone in mathematical programming. Optimization 25(1), 11–23 (1992)
9. Hillermeier, C.: Nonlinear Multiobjective Optimization: A Generalized Homotopy Approach. Springer, Birkhäuser, Basel (2001)
10. Ishibuchi, H., Tsukamoto, N., Nojima, Y.: Evolutionary many-objective optimization: a short review. In: Proceedings of the 2008 IEEE Congress on Evolutionary Computation, pp. 2419–2426 (2008)
11. Kuhn, H.W., Tucker, A.W.: Nonlinear programming. In: Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, pp. 481–492. University of California Press, Berkeley (1951)
12. Lange, K.: Optimization, 2nd edn. Springer, New York (2013)
13. Lee, J.: Introduction to Smooth Manifolds. Springer, New York (2012)
14. López Jaimes, A., Coello Coello, C.A., Chakraborty, D.: Objective reduction using a feature selection technique. In: Proceedings of the 10th Annual Conference on Genetic and Evolutionary Computation, GECCO ’08, pp. 673–680. ACM (2008)
15. Lovison, A.: Singular continuation: generating piecewise linear approximations to pareto sets via global analysis. SIAM J. Optim. 21(2), 463–490 (2011)
16. Lovison, A., Pecci, F.: Hierarchical Stratification of Pareto Sets. arXiv:1407.1755 (2014)
17. Lowe, T.J., Thisse, J.F., Ward, J.E., Wendell, R.E.: On efficient solutions to multiple objective mathematical programs. Manag. Sci. 30, 1346–1349 (1984)
18. Malivert, C., Boissard, N.: Structure of efficient sets for strictly quasi convex objectives. J. Convex Anal. 1(2), 143–150 (1994)
19. Miettinen, K.: Nonlinear Multiobjective Optimization. Springer, New York (1998)
20. Miglierina, E., Molho, E., Rocca, M.: Critical points index for vector functions and vector optimization. J. Optim. Theory Appl. 138(3), 479–496 (2008)
21. Mueller-Gritschneder, D., Graeb, H., Schlichtmann, U.: A successive approach to compute the bounded pareto front of practical multiobjective optimization problems. SIAM J. Optim. 20(2), 915–934 (2009)
22. Peitz, S.: Exploiting structure in multiobjective optimization and optimal control. In: Ph.D. Thesis, Paderborn University, Paderborn (2017)
23. Popovici, N.: Pareto reducible multicriteria optimization problems. Optimization 54(3), 253–263 (2005)
24. Saxena, D.K., Duro, J.A., Tiwari, A., Deb, K., Zhang, Q.: Objective reduction in many-objective optimization: linear and nonlinear algorithms. IEEE Trans. Evolut. Comput. 17(1), 77–99 (2013)
25. Schütze, O., Lara, A., Coello Coello, C.A.: On the influence of the number of objectives on the hardness of a multiobjective optimization problem. IEEE Trans. Evolut. Comput. 15(4), 444–455 (2011)
26. Shoval, O., Sheftel, H., Shinar, G., Hart, Y., Ramote, O., Mayo, A., Dekel, E., Kavanagh, K., Alon, U.: Evolutionary trade-offs, pareto optimality, and the geometry of phenotype space. Science 336(6085), 1157–1160 (2012)
27. Singh, H.K., Isaacs, A., Ray, T.: A pareto corner search evolutionary algorithm and dimensionality reduction in many-objective optimization problems. IEEE Trans. Evolut. Comput. 15(4), 539–556 (2011)
28. Smale, S.: Global analysis and economics I: Pareto optimum and a generalization of Morse theory. In: Dynamical Systems, pp. 531–544. Academic Press, Cambridge (1973)
29. Smale, S.: Optimizing several functions. In: Proceedings of International Conference on Manifolds and Related Topics in Topology, pp. 69–75. University of Tokyo, Tokyo (1975)
30. Sun, J.Q., Xiong, F.R., Schütze, O., Hernández, C.: Cell Mapping Methods. Springer, New York (2018)
31. Wan, Y.: Morse theory for two functions. Topology 14(3), 217–228 (1975)
32. Witting, K.: Numerical algorithms for the treatment of parametric multiobjective optimization problems and applications. In: Ph.D. Thesis, Paderborn University, Paderborn (2012)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.