A CANTOR-BERNSTEIN THEOREM FOR INFINITE MATROIDS

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ABSTRACT. We give a common matroidal generalisation of ‘A Cantor-Bernstein theorem for paths in graphs’ by Diestel and Thomassen and ‘A Cantor-Bernstein-type theorem for spanning trees in infinite graphs’ by ourselves.

1. INTRODUCTION

Let us reformulate the Cantor-Bernstein theorem in the language of graph theory:

**Theorem 1.1** (Cantor-Bernstein, [4]). If \( G = (V_0, V_1; E) \) is a bipartite graph and matching \( I_i \) covers \( V_i \) for \( i \in \{0, 1\} \), then \( G \) admits a perfect matching.

Ore discovered the following generalisation of the Cantor-Bernstein theorem which is the extension of the Mendelsohn-Dulmage theorem [13, Theorem 1] to infinite graphs:

**Theorem 1.2** (Ore, [14, Theorem 7.4.1]). Let \( G = (V_0, V_1; E) \) be a bipartite graph and let \( I_0, I_1 \subseteq E \) be matchings in \( G \). Then there exists a matching \( I \) such that \( V(I) \cap V_i \supseteq V(I_i) \cap V_i \) for \( i \in \{0, 1\} \).

Diestel and Thomassen examined in their paper ‘A Cantor-Bernstein theorem for paths in graphs’ a more general graph-theoretic setting in which disjoint paths are used to connect two vertex sets. We call a finite path that meets the vertex sets \( V_0 \) and \( V_1 \) and subgraph-minimal with respect to this property a \( V_0V_1 \)-path.

**Theorem 1.3** (Diestel and Thomassen, [5]). Assume that \( G = (V, E) \) is a graph, \( V_0, V_1 \subseteq V \) and \( P_i \) is a system of disjoint \( V_0V_1 \)-paths in \( G \) for \( i \in \{0, 1\} \). Then there exists a system of disjoint \( V_0V_1 \)-paths \( P \) with \( V(P) \cap V_i \supseteq V(P_i) \cap V_i \) for \( i \in \{0, 1\} \).

Note that Theorem 1.2 is the special case of Theorem 1.3 where \( G \) is bipartite and the sets \( V_i \) are its vertex classes.

In our paper entitled ‘A Cantor-Bernstein-type theorem for spanning trees in infinite graphs’ we investigated if the existence of a \( \kappa \)-packing and a \( \kappa \)-covering by spanning trees implies the existence of a \( \kappa \)-family of spanning trees which is both, i.e. a \( \kappa \)-partition:

**Theorem 1.4** (Erde et al. [7, Theorem 1.1]). Let \( G = (V, E) \) be a graph and let \( \kappa \) be a cardinal. If there are \( \kappa \) many pairwise edge-disjoint spanning trees in \( G \) and \( E \) can be covered by \( \kappa \) many spanning trees, then \( E \) can be partitioned into \( \kappa \) many spanning trees.
At first sight the connection between Theorem 1.3 and Theorem 1.4 seems to be only analogical. In this paper, we show that the connection is actually stronger. There is an abstract matroidal “Cantor-Bernstein”-type phenomenon behind these theorems. Let us first state a special case of our main result which is the generalisation of a theorem by Kundu and Lawler (see [12]) to finitary matroids:

**Theorem 1.5.** For \( i \in \{0, 1\} \), let \( M_i \) be a finitary matroid on \( E \) and let \( I_i \in I_{M_0} \cap I_{M_1} \). Then there is an \( I \in I_{M_0} \cap I_{M_1} \) with \( I_i \subseteq \text{span}_{M_i}(I) \) for \( i \in \{0, 1\} \).

The proof for finite matroids by Kundu and Lawler in [12] is quite short: If \( I_0 \) spans \( I_1 \) in \( M_1 \), then \( I := I_0 \) is as desired. Otherwise we add an \( e \in I_1 \setminus \text{span}_{M_1}(I_0) \) to \( I_0 \) and if \( I_0 + e \notin I_{M_0} \), then delete a suitable \( f \in I_0 \setminus I_1 \) in order to restore the \( M_0 \)-independence. This can be done because the fundamental circuit \( C_{M_0}(e, I_0) \) (if exists) cannot be entirely in \( I_1 \). The resulting set \( I_0 + e - f \) (or \( I_0 + e \)) still spans \( I_0 \) in \( M_0 \) and has strictly more edges in \( I_1 \) than \( I_0 \). After finitely many iterations of this step the desired \( I \) is obtained.

A naive proof-idea for Theorem 1.5 would be to iterate the step above via transfinite recursion. Unfortunately it does not work. To demonstrate this we define a graph \( G = (V, E) \) as a ray (one-way infinite path) \( v_0, v_1, v_2, \ldots \) together with an additional vertex \( w \) connected to each vertex of the ray (see Figure 1). Let \( M_0 \) be the cycle matroid on \( E \) corresponding to \( G \) (i.e. the circuits are the edge sets of the graph-theoretic cycles) and let \( M_1 \) be the free matroid on \( E \) (i.e. every set is independent in \( M_1 \)). We define \( I_0 \) as the set of edges incident with \( w \) and let \( I_1 := E \setminus I_0 \). The naive approach might proceed as:

\[
I_0, \ I_0 + v_0v_1 - wv_0, \ I_0 \cup \{v_0v_1, v_1v_2\} \setminus \{wv_0, w_1\}, \ldots
\]

![Figure 1. The failure of the naive approach for infinite matroids.](image)

It terminates after \( \omega \) steps and transforms \( I_0 \) into \( I_1 \). Since \( I_1 \) does not span \( I_0 \) in \( M_0 \), it fails to provide a desired \( I \). It is easy to see that if we keep \( wv_0 \) and delete only \( wv_1, wv_2, \ldots \) (while the incoming edges are in the same order), then we end up with the same ray together with the edge \( wv_0 \) which is suitable as \( I \). In order to prove Theorem 1.5, we are going to show in Section 3 that it is always possible to choose the leaving edge in each step in such a way that we obtain a solution at the end. The proof of Theorem 1.5 makes it possible to understand quickly the main ideas without dealing with technicalities arising in the general form. Basic knowledge about finite matroids is already sufficient to understand the paper, all the necessary matroidal background is given in Section 2.

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1A matroid is called finitary if all of its circuits are finite. In the older papers of Higgs, Oxley and others it is also called ‘independence space’. For a brief introduction to the concept of infinite matroids see Section 2.
In Section 4 we discuss the general form of our main result. Let us denote the class of finitary matroids by \( \mathfrak{F} \), the class of their duals (i.e. cofinitary matroids) by \( \mathfrak{F}^* \) and let \( \mathfrak{F} \oplus \mathfrak{F}^* \) be the class of matroids that are the direct sums of a finitary and a cofinitary matroid (equivalently the matroids with only finitary and cofinitary components). For a matroid class \( \mathfrak{C} \), let \( \mathfrak{C}(E) \) be the set of matroids on edge set \( E \) that are in class \( \mathfrak{C} \).

Our main result generalises Theorem 1.5 in two ways. On the one hand, we replace the special case of the conjecture where \( E \) is replaced by \( \mathfrak{F} \oplus \mathfrak{F}^*(E) \) by \( \mathfrak{F} \oplus \mathfrak{F}^*(E) \) (see [10, Theorem 1.4]). Then there exists an \( F \subseteq E \) such that \( \text{span}_{M_i}(F) \supseteq F_i \) and \( \text{span}_{M_i}(E \setminus F) \supseteq E \setminus F_{1-i} \) for \( i \in \{0,1\} \).

We are going to prove the following family variant of Theorem 1.6 as well:

**Theorem 1.6.** For \( i \in \{0,1\} \), let \( M_i \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E) \) and \( F_i \subseteq E \). Then there exists an \( F \subseteq E \) such that \( \text{span}_{M_i}(F) \supseteq F_i \) and \( \text{span}_{M_i}(E \setminus F) \supseteq E \setminus F_{1-i} \) for \( i \in \{0,1\} \).

The connection between the Theorems 1.6 and 1.7 is far from obvious. It was shown that it is impossible to extend our results above to arbitrary matroids working in set theory ZFC. Indeed, the analogue of Theorem 1.5 for arbitrary matroids fails under the Continuum Hypothesis even if \( E \) is countable, \( M_i \) is uniform and \( I_i \) is a base of \( M_i \) (take \( U \) and \( U^* \) in [8, Theorem 5.1]).

In the last section (Section 5) we provide an application related to the following conjecture:

**Conjecture 1.8** (Matroid Intersection Conjecture by Nash-Williams, [1, Conjecture 1.2]). For every \( M_0, M_1 \in \mathfrak{F}(E) \), there is an \( I \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1} \) and a partition \( E = E_0 \cup E_1 \) such that \( I \cap E_i \) spans \( E_i \) in \( M_i \) for \( i \in \{0,1\} \).

The special case of the conjecture where \( E \) is assumed to be countable was proved in [11]. This was then generalised to the case where \( E \) is still countable but \( \mathfrak{F}(E) \) is replaced by \( (\mathfrak{F} \oplus \mathfrak{F}^*)(E) \) (see [10, Theorem 1.4]).

A maximal sized common independent set of two finite matroids can always be chosen in such a way that it spans a prescribed common independent set in both matroids. Indeed, if a common independent set is not a largest such a set, then the well-known ‘augmenting path’ method by Edmonds gives a new common independent set which is larger by one and spans the original in both matroids (see in [6]). Iterating such augmenting paths starting with the prescribed common independent set provides a desired largest common independent set.

The question can be phrased with respect to Conjecture 1.8 by replacing ‘maximal sized’ by ‘strongly maximal’ which we define as satisfying the property described in Conjecture 1.8. The same argument for the positive answer does not work because finitely many
iteration of augmenting paths does not lead to a strongly maximal one in general. Even so, we can answer the question affirmatively based on our main results. Let us denote the set of strongly maximal common independent sets by $\text{SM}(M_0, M_1)$. For $I, J \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$, let $J \preceq_{M_0, M_1} I$ iff $J \subseteq \text{span}_{M_0}(I) \cap \text{span}_{M_1}(I)$.

**Theorem 1.9.** Let $E$ be countable and let $M_i \in (\mathcal{F}_i \oplus \mathcal{F}_i^*)(E)$ for $i \in \{0, 1\}$. Then $\text{SM}(M_0, M_1)$ is cofinal but not necessarily upward closed in $(\mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}, \preceq_{M_0, M_1})$.

2. Preliminaries

Rado asked in 1966 if there is an infinite generalisation of matroids preserving the key concepts (like duality and minors) of the finite theory. Based on some early results of Higgs [9] and Oxley [15], Bruhn, Diestel, Kriesell, Pendavingh and Wollan answered the question affirmatively and gave a set of cryptomorphic axioms for infinite matroids, generalising the usual independent set-, bases-, circuit-, closure- and rank-axioms of finite matroids (see [3]). They showed that several fundamental facts of the theory of finite matroids are preserved in the infinite case. It opened the door for a more systematic investigation of infinite matroids. An $M = (E, \mathcal{I})$ is a matroid (also called B-matroid) if $\mathcal{I} \subseteq \mathcal{P}(E)$ with

(I) $\emptyset \in \mathcal{I}$;

(II) $\mathcal{I}$ is downward closed;

(III) For every $I, J \in \mathcal{I}$ where $J$ is $\subseteq$-maximal in $\mathcal{I}$ and $I$ is not, there exists an $e \in J \setminus I$ such that $I + e \in \mathcal{I}$;

(IV) For every $X \subseteq E$, any $I \in \mathcal{I} \cap \mathcal{P}(X)$ can be extended to a $\subseteq$-maximal element of $\mathcal{I} \cap \mathcal{P}(X)$.

For a finite $E$, axioms (I)-(III) are equivalent to the usual axiomatization of finite matroids in terms of independent sets (while (IV) is automatically true).

The terminology and the basic facts we will use are well-known for finite matroids. The elements of $\mathcal{I}$ are called independent sets while the sets in $\mathcal{P}(E) \setminus \mathcal{I}$ are dependent. The maximal independent sets are the bases and the minimal dependent sets are the circuits of the matroid. Every dependent set contains a circuit (which fact is not obvious if $E$ is infinite). A singleton circuit is called a loop. The components of a matroid are the connected components of the hypergraph of its circuits on $E$. The dual of matroid $M$ is the matroid $M^*$ on the same edge set whose bases are the complements of the bases of $M$. By the deletion of an $X \subseteq E$ we obtain the matroid $M - X := (E \setminus X, \{Y \in \mathcal{I} : Y \subseteq E \setminus X\})$ and the contraction of $X$ gives $M/X := (M^* - X)^*$. If $I$ is independent in $M$ but $I + e$ is dependent for some $e \in E \setminus I$ then there is a unique circuit $C_M(e, I)$ of $M$ through $e$ contained in $I + e$ which is called the fundamental circuit of $e$ on $I$ in $M$. We say $X \subseteq E$ spans $e \in E$ in matroid $M$ if either $e \in X$ or there exists a circuit $C \ni e$ with $C - e \subseteq X$. We denote the set of edges spanned by $X$ in $M$ by $\text{span}_M(X)$. A matroid is called finitary if all of its circuits are finite. A matroid is cofinitary if its dual is finitary. If $C_1$ and $C_2$ are circuits with $e \in C_1 \setminus C_2$ and $f \in C_1 \cap C_2$, then there is a circuit $C_3$ with $e \in C_3 \subseteq C_1 \cup C_2 - f$. This fact is called (strong) circuit elimination. For more information about infinite matroids we refer to [2].
3. The infinite generalisation of the Kundu-Lawler theorem

**Theorem 1.5.** For \( i \in \{0,1\} \), let \( M_i \) be a finitary matroid on \( E \) and let \( I_i \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1} \). Then there is an \( I \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1} \) with \( I_i \subseteq \text{span}_{M_i}(I) \) for \( i \in \{0,1\} \).

**Proof.** We may assume without loss of generality that \( E \) is the disjoint union of \( I_0 \) and \( I_1 \) since otherwise we can simply contract \( I_0 \cap I_1 \) and delete \( E \setminus (I_0 \cap I_1) \) in both matroids.

Let \( \prec \) be a well-order on \( E \) in which \( I_1 \) is an initial segment, i.e. \( e \prec f \) for every \( e \in I_1 \) and \( f \in I_0 \). From now on, the maximum of a finite subset of \( E \) is interpreted corresponding to \( \prec \). We define a well-order \( \prec \) on the set \( E^{<\aleph_0} \) of finite subsets of \( E \). For \( X \neq Y \in E^{<\aleph_0} \) let \( X \prec Y \) iff one of the following holds:

- \( X = \emptyset \),
- \( \max X < \max Y \),
- \( \max X = \max Y =: z \) and \( X - z \prec Y - z \).

It is not too hard to check that \( \prec \) is indeed a well-order.

**Observation 3.1.** If \( X \prec Y \) then \( X + z \prec Y + z \) for every \( z \in I_0 \cup I_1 \).

Let \( \langle E_\beta : \beta < \alpha \rangle \) be a sequence of subsets of \( E \) where \( \alpha \) is a limit ordinal. If

\[
\bigcup_{\gamma < \alpha} \bigcap_{\beta > \gamma} E_\beta = \bigcap_{\gamma < \alpha} \bigcup_{\beta > \gamma} E_\beta,
\]

then we call this set the limit of the sequence and denote it by \( \lim \langle E_\beta : \beta < \alpha \rangle \). We apply transfinite recursion starting with \( J_0 := I_0 \). Suppose that \( J_\alpha \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1} \) is defined and spans \( I_0 \) in \( M_0 \). If \( J_\alpha \) spans \( I_1 \) in \( M_1 \) as well, then \( I := J_\alpha \) as desired. Otherwise let \( e \in I_1 \setminus \text{span}_{M_1}(J_\alpha) \) be arbitrary and let

\[
J_{\alpha+1} := \begin{cases} 
J_\alpha + e & \text{if it is independent in } M_0 \\
J_\alpha + e - \max C_{M_0}(e,J_\alpha) & \text{otherwise.}
\end{cases}
\]

Note that \( e \in I_1 \setminus I_0 \) and \( \max C_{M_0}(e,J_\alpha) \in I_0 \setminus I_1 \). In limits steps we take the limit of the earlier members (which is well-defined). Clearly, \( J_\alpha \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1} \) remains true for limit ordinals because a finite circuit cannot show up first in a limit step. It is enough to show that \( J_\beta \subseteq \text{span}_{M_0}(J_\alpha) \) for \( \beta < \alpha \). Let \( \beta \) and \( g \in I_\beta \) be fixed and suppose for a contradiction that there is a (smallest) \( \alpha \) with \( g \notin \text{span}_{M_0}(J_\alpha) \). It is obvious from the definition of successor steps that \( \alpha \) must be a limit ordinal. For \( \gamma \in [\beta, \alpha) \), let \( S_\gamma \) be the unique minimal subset of \( J_\gamma \) that spans \( g \) in \( M_0 \). It is enough to show that \( S_{\gamma+1} \preceq S_\gamma \) for \( \gamma \in [\beta, \alpha) \). Indeed, since there is no infinite \( \prec \)-decreasing sequence, \( S_\gamma \) is the same set \( S \) for every large enough \( \gamma \). But then \( S \subseteq J_\alpha \) and it spans \( g \) in \( M_0 \), a contradiction.

Let \( \gamma \in [\beta, \alpha) \) be fixed. We may assume that \( S_{\gamma+1} \neq S_\gamma \) since otherwise we are done. Suppose first that \( S_\gamma = \{g\} \). Then \( g \notin S_{\gamma+1} \) because otherwise \( S_\gamma = S_{\gamma+1} = \{g\} \). But then there is an edge \( e \) such that \( g = \max C_{M_0}(e,J_\alpha) \) and \( J_{\gamma+1} = J_\gamma + e - g \). Therefore

\[
S_{\gamma+1} = C_{M_0}(e,J_\gamma) - g \prec \{g\} = S_\gamma.
\]

If \( S_\gamma \neq \{g\} \), then \( S_\gamma = C_{M_0}(g,J_\gamma) - g \) and there is an edge \( e \) such that \( J_{\gamma+1} = J_\gamma + e - \max C_{M_0}(e,J_\gamma) \) with \( \max C_{M_0}(e,J_\gamma) \in C_{M_0}(g,J_\gamma) - g \). By strong circuit elimination we
know that
\[ C_{M_0}(g, J_{\gamma+1}) \subseteq C_{M_0}(g, J_\gamma) \cup C_{M_0}(e, J_\gamma) - \max C_{M_0}(e, J_\gamma) \]
and therefore
\[ S_{\gamma+1} \subseteq S_\gamma \cup C_{M_0}(e, J_\gamma) - \max C_{M_0}(e, J_\gamma). \]

It follows that \( S_{\gamma+1} \setminus S_\gamma \prec S_\gamma \setminus S_{\gamma+1} \) because \( \max C_{M_0}(e, J_\gamma) \in S_{\gamma+1} \setminus S_\gamma \) is \( \prec \)-larger than any element of \( S_\gamma \setminus S_{\gamma+1} \). Finally, this implies \( S_{\gamma+1} \prec S_\gamma \) by applying Observation 3.1 repeatedly with the edges in \( S_\gamma \cap S_{\gamma+1} \).

\[ \square \]

4. The proof of the main results

We are going to derive Theorems 1.6 and 1.7 from the following statement:

**Proposition 4.1.** For \( i \in \Theta \), let \( M_i \in (\mathfrak{F} \oplus \mathfrak{F}^*)^*(E) \) and \( P_i, R_i \subseteq E \) such that the sets \( P_i \) form a packing and the sets \( R_i \) form a covering, i.e. \( P_i \cap P_j = \emptyset \) for \( i \neq j \) and \( \bigcup_{i \in \Theta} R_i = E \). Then there are \( T_i \subseteq P_i \cup R_i \) for \( i \in \Theta \) forming a partition of \( E \) such that \( \text{span}_{M_i}(T_i) \supseteq P_i \) and \( \text{span}_{M_i}(E \setminus T_i) \supseteq E \setminus R_i \).

**Proof.** We may assume without loss of generality by “trimming” that the sets \( R_i \) form a partition of \( E \). We can also assume that \( P_i \in \mathcal{I}_{M_i} \) since otherwise we replace \( P_i \) with a maximal \( M_i \)-independent subset of it. It is enough to consider the case where \( P_i \cap R_i = \emptyset \) for \( i \in \Theta \). Indeed, if it is not the case, then we contract \( P_i \cap R_i \) and delete \( P_j \cap R_j \) for \( j \neq i \) in \( M_i \). Finally, by decomposing each \( M_i \) into a finitary and a cofinitary matroid (which we extend to \( E \) by loops) and partition the sets \( R_i \) and \( P_i \) accordingly, it is enough to deal with matroid families where each \( M_i \) is either finitary or cofinitary.

Let \( \prec_i \) be a well-order on \( P_i \cup R_i \) where \( R_i \) is an initial segment. Then \( \prec_i \) induces a well-order \( \prec_i \) on the set \([P_i \cup R_i]^{<\aleph_0}\) the same way as in Section 3.

**Observation 4.2.** Suppose that \( E_\alpha \) is the limit of \( \langle E_\beta : \beta < \alpha \rangle \).

(i) If \( E_\alpha \) contains an \( M_i \)-circuit \( C \not\subseteq R_i \) where \( M_i \) is finitary, then so does \( E_\beta \) for every large enough \( \beta < \alpha \);

(ii) If \( g \in \text{span}_{M_i}(E_\beta) \) for \( \beta < \alpha \) where \( M_i \) is cofinitary, then \( g \in \text{span}_{M_i}(E_\alpha) \).

To construct the desired partition \( \langle T_i : i \in \Theta \rangle \), we apply transfinite recursion. Let \( T_i^0 := P_i \) for \( i \in \Theta \). Suppose that \( T_i^\beta \) is defined for \( \beta < \alpha \) and \( i \in \Theta \) satisfying the following properties:

(1) \( T_i^\beta \cap T_j^\beta = \emptyset \) for \( i \neq j \in \Theta \);

(2) \( T_i^\beta \subseteq P_i \cup R_i \);

(3) \( T_i^\beta \cap P_i \) is \( \subseteq \)-decreasing and \( T_i^\beta \cap R_i \) is \( \subseteq \)-increasing in \( \beta \);

(4) \( T_i^\beta = \lim \langle T_i^{\delta} : \delta < \beta \rangle \) if \( \beta \) is a limit ordinal;

(5) \( \text{span}_{M_i}(T_i^\beta) \supseteq P_i \);

(6) For every finitary \( M_i \), each \( M_i \)-circuit \( C \subseteq T_i^\beta \) is a subset of \( R_i \);

(7) For every finitary \( M_i \) and \( g \in P_i \), the \( \prec_i \)-smallest finite \( S_g^\beta \subseteq T_i^\beta \) that is witnessing \( g \in \text{span}_{M_i}(T_i^\beta) \) is a \( \subseteq_i \)-decreasing function of \( \beta \);

(8) \( \langle T_i^\delta : i \in \Theta \rangle \neq \langle T_i^{\delta+1} : i \in \Theta \rangle \) for \( \delta + 1 < \alpha \).
Note that condition (6) is a rephrasing of \( \text{span}_{M_i}(E \setminus T_i^\beta) \supseteq E \setminus R_i \) for finitary \( M_i \). Assume first that \( \alpha \) is a limit ordinal. Then conditions (2) and (3) guarantee that \( T_i^\alpha := \lim \langle T_i^\beta : \beta < \alpha \rangle \) is well-defined. Preservation of conditions (1)-(4) and (8) is straightforward. The restriction of condition (5) to cofinitary matroids and condition (6) are kept by Observation 4.2. To check condition (5) for a finitary \( M_i \), let \( g \in P_i \) be arbitrary. Since \( \preceq_i \) is a well-order, it follows from condition (7) that there is an \( S_g \) such that \( S_g^\beta = S_g \) for all large enough \( \beta < \alpha \). But then \( S_g \subseteq T_i^\alpha \) from which \( g \in \text{span}_{M_i}(T_i^\alpha) \) follows. Furthermore, clearly \( S_g^\alpha = S_g \) since a finite set which is \( <_i \)-smaller than \( S_g \) and \( M_i \)-spans \( g \) would have appeared already before the limit.

Suppose now that \( \alpha = \beta + 1 \). If \( \bigcup_{i \in \Theta} T_i^\beta \supseteq E \) and the analogue of condition (6) for the cofinitary \( M_i \), holds, then \( (T_i^\beta : i \in \Theta) \) is a desired partition of \( E \) and we are done. Suppose it is not the case. If there is some \( T_j^\beta \) that contains an \( M_j \)-circuit \( C \) with \( C \not\subseteq R_j \), then we take an \( e \in P_j \cap C \) (see property (2)) and define \( T_j^{\beta+1} := T_j^\beta - e \) and \( T_i^{\beta+1} := T_i^\beta \) for \( i \neq j \). The preservation of the conditions (1)-(8) is trivial. If there is no such a \( T_j^\beta \), then there must be some \( e \in E \) which is not covered by the sets \( T_i^\beta \). Then there is a unique \( k \in \Theta \) with \( e \in R_k \). If \( M_k \) is cofinitary then let \( T_k^{\beta+1} := T_k^\beta + e \) and \( T_i^{\beta+1} := T_i^\beta \) for \( i \neq k \). We proceed the same way if \( M_k \) is finitary and \( T_k^\beta + e \) does not contain any \( M_k \)-circuit \( C \) with \( C \not\subseteq R_k \). The preservation of the conditions is again straightforward in both cases.

Finally assume that \( M_k \) is finitary and \( T_k^\beta + e \) contains an \( M_k \)-circuit \( C \) with \( C \subseteq R_k \). Let \( f \) be the \( <_k \)-maximal element of such a \( C \) and we define \( T_k^{\beta+1} := T_k^\beta + e - f \) and \( T_i^{\beta+1} := T_i^\beta \) for \( i \neq k \). Since \( C \cap P_k \neq \emptyset \) (because \( C \subseteq R_k \)) and the elements of \( P_k \) are \( <_k \)-larger than the elements of \( R_k \), we have \( f \in P_k \). Conditions (1)-(5) remain true for obvious reasons. Suppose for a contradiction that condition (6) fails and \( C' \) is an \( M_k \)-circuit in \( T_k^{\beta+1} \) with \( C' \not\subseteq R_k \). Then \( f \notin C' \) and we must have \( e \in C' \) since otherwise \( C' \subseteq T_k^\beta \) and therefore this condition would have been already violated with respect to \( T_k^\beta \).

By applying strong circuit elimination with the \( M_k \)-circuits \( C \) and \( C' \), we obtain a circuit \( C'' \subseteq C \cup C' - e \) through \( f \). But then \( C'' \subseteq T_k^\beta \) is an \( M_k \)-circuit and \( f \) witnesses \( C'' \not\subseteq R_k \) in violation of condition (6) for \( \beta \) which is a contradiction. To check (7), we may assume that \( f \in S_g^\beta \) since otherwise \( S_g^\beta \subseteq T_k^{\beta+1} \) and thus \( S_g^{\beta+1} \not\subseteq_k S_g^\beta \). If \( S_g^\beta = \{ g \} \), then \( f = g \) by \( f \in S_g^\beta \) and by the choice of \( f \) we have \( S_f^{\beta+1} \not\subseteq_k C - f \). Otherwise there is an \( M_k \)-circuit \( C' \) such that \( S_g^\beta = C' - g \subseteq T_k^\beta \). By applying strong circuit elimination with \( C \) and \( C' \), we obtain a circuit \( C'' \subseteq C \cup C' - f \) through \( g \). Since \( f \in C'' \setminus C'' \) and each element of \( C'' \setminus C' \) is \( <_k \)-smaller than \( f \) (because \( f = \max_{<_k} C \) we may conclude that \( C'' \setminus C' \not\subseteq_k C'' \). Thus by applying Observation 3.1 iteratively we get \( C'' - g \not\subseteq_k C'' - g \). Therefore

\[
S_g^{\beta+1} \not\subseteq_k C'' - g \not\subseteq_k C'' - g = S_g^\beta.
\]

The recursion is done and it terminates at some ordinal since the constructed set families \( (T_i^\beta : i \in \Theta) \) are pairwise distinct by conditions (2), (3) and (8).

Let us point out that the special case of Proposition 4.1 in which \( P_i \) and \( R_i \) are bases of \( M_i \) is exactly [8, Theorem 1.2]. Now we derive Theorems 1.6 and 1.7 from Proposition 4.1:

**Theorem 1.6.** For \( i \in \{ 0, 1 \} \), let \( M_i \in (\mathfrak{F} \oplus \mathfrak{F}')(E) \) and \( F_i \subseteq E \). Then there exists an \( F \subseteq E \) such that \( \text{span}_{M_i}(F) \supseteq F_i \) and \( \text{span}_{M_{1-i}}(E \setminus F) \supseteq E \setminus F_{1-i} \) for \( i \in \{ 0, 1 \} \).
Proof. We can assume by contracting $F_0 \cap F_1$ and deleting $E \setminus (F_0 \cup F_1)$ in both matroids that the sets $F_i$ form a bipartition of $E$. We apply Proposition 4.1 with $\Theta = \{0,1\}$, matroids $M_0$ and $M_1^*$ and sets $P_0 := R_1 := F_0$ and $P_1 := R_0 := F_1$. From the resulting bipartition $E = T_0 \cup T_1$ we take $F := T_0$. Then

\begin{enumerate}
  \item $\text{span}_{M_0}(F) \supseteq F_0$,
  \item $\text{span}_{M_1^*}(E \setminus F) \supseteq F_1$,
  \item $\text{span}_{M_0}(E \setminus F) \supseteq F_0$,
  \item $\text{span}_{M_1}(F) \supseteq F_1$.
\end{enumerate}

\hfill $\square$

Theorem 1.7. For $i \in \Theta$, let $M_i \in (\mathfrak{S} \oplus \mathfrak{S}^*)(E)$, $P_i, R_i \subseteq E$ and for $e \in E$, let $N_e \in (\mathfrak{S} \oplus \mathfrak{S}^*)(\Theta)$. Then there are $T_i \subseteq P_i \cup R_i$ for $i \in \Theta$ such that

\begin{enumerate}
  \item $\text{span}_{M_i}(T_i) \supseteq P_i$;
  \item $\text{span}_{M_i^*}(E \setminus T_i) \supseteq E \setminus R_i$;
  \item For every $e \in E$, the set \{ $i \in \Theta : e \in T_i$ \} spans \{ $i \in \Theta : e \in R_i$ \} in $N_e$;
  \item For every $e \in E$, the set \{ $i \in \Theta : e \notin T_i$ \} spans \{ $i \in \Theta : e \notin P_i$ \} in $N_e^*$.
\end{enumerate}

Proof. We may assume that $\Theta \cap E = \emptyset$. For $i \in \Theta$, we construct a matroid $M_i'$ by “copying” $M_i$ to $\{i\} \times E$ and then extending to $\Theta \times E$ by loops. For $e \in E$, we construct a matroid $N_e'$ by copying $N_e^*$ to $\Theta \times \{e\}$ and then extending to $\Theta \times E$ by loops. The sets $R_i' := \{i\} \times R_i$ for $i \in \Theta$ together with the sets $R_e' := \{i \in \Theta : e \notin R_i\} \times \{e\}$ for $e \in E$ cover $\Theta \times E$. Furthermore, the elements of the family consisting of $P_i' := \{i\} \times P_i$ for $i \in \Theta$ and $\{i \in \Theta : e \notin P_i\} \times \{e\}$ for $e \in E$ are pairwise disjoint. Let \{ $T_i', T_e' : i \in \Theta, e \in E$ \} be a partition of $\Theta \times E$ obtained by applying Proposition 4.1 with the matroids $M_i', N_e'$, covering $R_i', R_e'$ and packing $P_i', P_e'$ ($i \in \Theta, e \in E$). It is easy to check that the family consisting of the projections $T_i'$ of $T_e'$ to $E$ for $i \in \Theta$ is as desired. \hfill $\square$

5. Applications

5.1. Cantor-Bernstein for path-systems. We derive Theorem 1.3 from Theorem 1.5.

Theorem 1.3 (Diestel and Thomassen, [5]). Assume that $G = (V,E)$ is a graph, $V_0, V_1 \subseteq V$ and $P_i$ is a system of disjoint $V_0V_1$-paths in $G$ for $i \in \{0,1\}$. Then there exists a system of disjoint $V_0V_1$-paths $\mathcal{P}$ with $V(\mathcal{P}) \cap V_i \supseteq V(P_i) \cap V_i$ for $i \in \{0,1\}$.

Proof. For $i \in \{0,1\}$, we define $M_i$ to be the cycle matroid of the graph we obtain from $G$ by contracting $V_i$ to a single vertex. Then $E(P_i) \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ for $i \in \{0,1\}$. By applying Theorem 1.5 with $I_i := E(P_i)$ and $M_{1-i}$, we can find an $I \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ with $E(P_{1-i}) \subseteq \text{span}_{M_i}(I)$ for $i \in \{0,1\}$. Then $G[I]$ is a forest in which every tree meets each $V_i$ at most once. Each connected component of $G[I]$ which meets both $V_i$ contains a unique $V_0V_1$-path. We define $\mathcal{P}$ to be the set of these paths. It remains to show that $\mathcal{P}$ satisfies the requirements. Let $v_0 \in V(P_i) \cap V_i$. It is enough to show that $v_0$ is reachable from $V_{1-i}$ in $G[I]$ because then the (unique) path witnessing this is in $\mathcal{P}$. Consider the path $P \in \mathcal{P}_i$ through $v_0$. Let the vertices of $P$ be $v_0, \ldots, v_n$ enumerated in the path-order starting from $V_i$. It follows from $E(P) \subseteq \text{span}_{M_{1-i}}(I)$ that for every $k < n$ either $G[I]$ contains a path between $v_k$ and $v_{k+1}$ or both of them are reachable from $V_{1-i}$ in $G[I]$. Vertex $v_n$
is obviously reachable from $V_{1-i}$ because it is an element of it. If we already know that $v_{k+1}$ is reachable from $V_{1-i}$ in $G[I]$, then it follows that $v_k$ is reachable as well. Thus by induction $v_0$ is reachable from $V_{1-i}$ in $G[I]$ which completes the proof. \qed

5.2. Matroid Intersection.

**Theorem 1.9.** Let $E$ be countable and let $M_i \in (\mathfrak{F} \oplus \mathfrak{F}^+)(E)$ for $i \in \{0,1\}$. Then $\text{SM}(M_0, M_1)$ is cofinal but not necessarily upward closed in $(\mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}, \subseteq_{M_0,M_1})$.

**Proof.** We start with the ‘cofinal’ part of the statement. Let $J \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ be given. We take an $I' \in \text{SM}(M_0, M_1)$ and fix a partition $E = E_0 \cup E_1$ such that $I_i := I' \cap E_i$ spans $E_i$ in $M_i$ for $i \in \{0,1\}$. By applying Theorem 1.6 with the matroids $M_i \restriction E_i$ and $M_{1-i} \restriction E_i$ and sets $I_i := I \cap E_i$, we obtain a base $I_i$ of $M_i \restriction E_i$ which is independent in $M_{1-i} \restriction E_i$ and spans $J_i$ in $M_{1-i} \restriction E_i$. We claim that $I := I_0 \cup I_1$ is as desired. Indeed, $I \in \text{SM}(M_0, M_1)$ because $I_i$ is an $M_{1-i} \restriction E_i$-independent base of $M_i \restriction E_i$. Finally, $I_{1-i}$ spans $J_{1-i}$ in $M_{1-i} \restriction E_i = M_i \restriction E_i$ and $I_i \subseteq E_i$ spans $E_i$ (which contains $J_i$) in $M_i$ by construction thus $J \subseteq \text{span}_{M_i}(I)$. Therefore $J \subseteq \text{span}_{M_0}(I) \cap \text{span}_{M_1}(I)$ which means $J \subseteq_{M_0,M_1} I$.

In order to show the ‘not necessarily upward closed’ part we shall construct first a bipartite graph $G = (V_0, V_1; E)$. We start with a double ray $\ldots, v_{-1}, v_0, v_1, \ldots$ and add a new vertex $w_i$ and new edge $v_iw_i$ for $i \in \{0,1\}$ (see Figure 2). The bipartite graph $G$ induces two partition matroids $M_0$ and $M_1$ on $E$ in the way that $I \subseteq E$ is defined to be independent in $M_i$ if no two edges in $I$ have a common end-vertex in $V_i$. Then the elements of $\mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ are exactly the matchings, moreover, matching $I$ is in $\text{SM}(M_0, M_1)$ iff one can choose exactly one vertex from each $e \in I$ such that the resulting set is a vertex cover. Let

$$I_i := \{v_{2k+i}v_{2k+1+i} : k < \omega\} \text{ for } i \in \{0,1\}.$$ 

On the one hand, the matchings $I_i$ cover the same vertices thus

$$I_0 \subseteq_{M_0,M_1} I_1 \subseteq_{M_0,M_1} I_0.$$ 

On the other hand, we claim that $I_1$ is strongly maximal but $I_0$ is not. Indeed, $\{v_{-2k}, v_{2k+1} : k < \omega\}$ is a vertex cover (upper-left and lower-right corners on Figure 2) that consists of choosing exactly one end-vertex of each edge in $I_1$ and therefore witnessing $I_1 \in \text{SM}(M_0, M_1)$. But there is no such a vertex cover for $I_0$ because if we pick $v_i$ from the edge $v_0v_1$, then we cannot choose any end-vertex of $v_{1-i}w_{1-i}$. Thus $\text{SM}(M_0, M_1)$ is not upward closed in $(\mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}, \subseteq_{M_0,M_1})$.

**Figure 2.** Matching $I_0$ consists of the dashed and $I_1$ consists of the normal edges.

\[\framebox{\begin{array}{c}
V_0 \\
\vdots \\
V_1
\end{array}}\]
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