$Q$-operators for higher spin eight vertex models
with a rational anisotropy parameter
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Workshop: Elliptic Hypergeometric Functions in Combinatorics,
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§0. Introduction

- Solving the lattice models
  \(\iff\) diagonalisation of the transfer matrix
- Most well-known diagonalisation: Bethe Ansatz.
- Alternative method: Baxter’s \(Q\)-operator.

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Today’s topic

Construction of the \(Q\)-operator for the higher spin eight vertex model with a rational anisotropy parameter.
Plan of the talk:

1. The eight vertex model.
2. $Q$-operator.
3. Sklyanin algebra.
4. Higher spin generalisation of the eight vertex model.
§1. Eight vertex model

L-matrix for Baxter’s eight vertex model:

\[ L(u) = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix} = \sum_{a=0}^{3} W_a(u) \sigma_a \otimes \sigma_a, \]

\[ W_a(u) := \frac{\theta_{g_a}(u; \tau)}{\theta_{g_a}(\eta; \tau)}, \quad g_0 = (11), \ g_1 = (10), \ g_2 = (00), \ g_3 = (01). \]

\( u \): spectral parameter, \( \eta \): anisotropy parameter, \( \tau \): elliptic modulus.

\( \sigma_a \): Pauli matrices. (\( \sigma_0 = \text{Id.} \))

Theta functions: (\( \theta_{11} = -\vartheta_1, \ \theta_{10} = \vartheta_2, \ \theta_{00} = \vartheta_3, \ \theta_{01} = \vartheta_4 \))

\[ \theta_{ab}(z; \tau) = \sum_{n \in \mathbb{Z}} \exp \left( \pi i \left( \frac{a}{2} + n \right)^2 \tau + 2\pi i \left( \frac{a}{2} + n \right) \left( \frac{b}{2} + z \right) \right). \]
Definition of the eight vertex model

\[ V_i = \mathbb{C} \uparrow \oplus \mathbb{C} \downarrow (i = 1, \ldots, N), \quad V_0 = \mathbb{C} \leftarrow \oplus \mathbb{C} \rightarrow. \]

\( \mathcal{H} := V_N \otimes \cdots \otimes V_1. \)

- **Monodromy matrix:**

\[ T(u) := L_{N0}(u) \cdots L_{20}(u)L_{10}(u) : \mathcal{H} \otimes V_0 \rightarrow \mathcal{H} \otimes V_0. \]

\( (L_{i0} \curvearrowright V_i \otimes V_0) \)

- **Transfer matrix:**

\[ T = T(u) := \text{tr}_{V_0} T(u) : \mathcal{H} \rightarrow \mathcal{H}. \]

(Recall: we are assuming the periodic boundary condition.)

**Problem**

*Diagonalise* \( T(u) \), or *find* eigenvalues of \( T(u) \).
• Why is the eight vertex model “good”?

$\exists$ 4 $\times$ 4-matrix $R(u)$ (=$L(u + \eta)$) (the $R$-matrix) such that

$$R_{00'}(u - v)L_{i0}(u)L_{i0'}(v) = L_{i0'}(v)L_{i0}(u)R_{00'}(u - v)$$
on $V_i \otimes V_0 \otimes V_{0'}$.

$$\implies R_{00'}(u - v)T_0(u)T_0'(v) = T_0'(v)T_0(u)R_{00'}(u - v)$$
on $\mathcal{H} \otimes V_0 \otimes V_{0'}$.

Taking $\text{tr}_{V_0' \otimes V_0}$ of $R\mathcal{T}_0(u)\mathcal{T}_0'(v)R^{-1} = \mathcal{T}_0'(v)\mathcal{T}_0(u)$,

$$T(u)T(v) = T(v)T(u).$$

$\implies$ Eigenvectors do not depend on $u$. 
§2. $Q$-operator

A tool introduced to find an eigenvalue by Baxter (1972, 1973).

Assume: $\eta \in [-\frac{1}{4}, \frac{1}{4}]$, $\tau = it \ (t \in \mathbb{R}_{>0})$, $N = \text{even as in [Baxter]}$.

• $Q$-operator: $Q(u) : \mathcal{H} \rightarrow \mathcal{H}$: a linear operator satisfying:
  • $TQ$-relation: $(h_{\pm}(u) = (2[u \mp \eta])^N, [u] = \theta_{11}(u).)$
    \[ T(u)Q(u) = Q(u)T(u) = h_-(u)Q(u - 2\eta) + h_+(u)Q(u + 2\eta), \]
  • Commutativity: $[Q(u), Q(v)] = 0$.
  • Holomorphicity and quasi-periodicity in $u$:
    \[ \sigma_1^{\otimes N} Q(u) = Q(u)\sigma_1^{\otimes N} = e^{-N\pi i/2}Q(u + 1), \]
    \[ \sigma_3^{\otimes N} Q(u) = Q(u)\sigma_3^{\otimes N} = e^{N\pi i(\tau - 1)/2 + N\pi iu}Q(u + \tau), \]
\( T(u), Q(u), \sigma_1^\otimes N, \sigma_3^\otimes N \): all commute with each other for any \( u \).

\[ \iff \] eigenvectors are common for all \( u \) and independent of \( u \).

If \( \Psi \) is a common eigenvector:

\[
T(u)\Psi = t(u)\Psi, \\
Q(u)\Psi = q(u)\Psi, \\
\sigma_1^\otimes N \Psi = (-1)^{\nu_1} \Psi, \\
\sigma_3^\otimes N \Psi = (-1)^{\nu_3} \Psi. \quad (\nu_1, \nu_3 \in \{0, 1\}).
\]

Holomorphicity & quasi-periodicity \( \implies \exists n_1 \in \mathbb{Z}, u_j \in \mathbb{C} \ (j = 1, \ldots, N/2) \):

\[
q(u) = Ce^{(\nu_1 + 2n_1)\pi i u} \prod_{j=1}^{N/2} [u - u_j].
\]

\( u \mapsto u_j \) in the \( TQ \)-relation: \( 0 = h_-(u_j)q(u_j - 2\eta) + h_+(u_j)q(u_j + 2\eta) \).


\[ \left( \frac{[u_j + \eta]}{[u_j - \eta]} \right)^N = e^{4(\nu_1 + 2n_1)\pi i \eta} \prod_{k=1, k \neq j}^{N/2} \frac{[u_j - u_k + 2\eta]}{[u_j - u_k - 2\eta]} . \]

(Bethe equations; same as those of the Bethe Ansatz.)

\( TQ \)-relation gives the eigenvalue of \( T(u) \):

\[ t(u) = h_-(u) \frac{q(u - 2\eta)}{q(u)} + h_+(u) \frac{q(u + 2\eta)}{q(u)} . \]

Quasi-periodicity says more: the sum rule of Bethe roots.

\[ \sum_{j=1}^{N/2} u_j \equiv -\frac{\nu_1 \tau}{2} + \frac{\nu_3}{2} \pmod{\mathbb{Z} + \tau\mathbb{Z}).} \]

(The Bethe Ansatz does not give the sum rule.)
§3. Sklyanin algebra

\[ [T(u), T(v)] = 0 \iff RTT = TTR \iff RLL = LLR. \]

Generalise \( L(u) = \sum_{a=0}^{3} W_a(u) \sigma_a \otimes \sigma_a \)

\[ \longrightarrow L(u) = \sum_{a=0}^{3} W_a(u) S^a \otimes \sigma_a, \text{ so that } RLL = LLR \text{ still holds.} \]

[Sklyanin (1982)]

\[ RLL = LLR \text{ gives the commutation relation of } S^a \text{'s:} \]

\[ [S^\alpha, S^0]_- = -iJ_{\alpha, \beta}[S^\beta, S^\gamma]_+ \quad \quad [S^\alpha, S^\beta]_- = i[S^0, S^\gamma]_+, \]

\(([A, B]_\pm = AB \pm BA, (\alpha, \beta, \gamma) = \text{cyclic permutation of } (1, 2, 3).)\)

The structure constants \( J_{\alpha, \beta} = \frac{(W_\alpha)^2 - (W_\beta)^2}{(W_\gamma)^2 - (W_0)^2} \) do not depend on \( u. \)

\( U_{\tau, \eta} := \langle S^0, S^1, S^2, S^3 \rangle: \text{ Sklyanin algebra. (the “first” quantum algebra) } \)
spin \( l \) representation: \( \rho^l : U_{\tau,\eta} \to \text{End}_\mathbb{C}(\Theta^{4l+}_0) \),

\[
\Theta^{4l+}_0 := \{ f(z) \mid f(z+1) = f(-z) = f(z), f(z+\tau) = e^{-4l\pi i(2z+\tau)} f(z) \},
\]

\[
(\rho^l(S^a)f)(z) = \frac{s_a(z - l\eta)f(z + \eta) - s_a(-z - l\eta)f(z - \eta)}{\theta_{11}(2z, \tau)},
\]

\[
s_0(z) = \theta_{11}(\eta, \tau)\theta_{11}(2z, \tau), \quad s_1(z) = \theta_{10}(\eta, \tau)\theta_{10}(2z, \tau),
\]

\[
s_2(z) = i\theta_{00}(\eta, \tau)\theta_{00}(2z, \tau), \quad s_3(z) = \theta_{01}(\eta, \tau)\theta_{01}(2z, \tau).
\]

- Deformation of spin \( l \) representation of \( sl_2(\mathbb{C}) \).

- \( l \in \frac{1}{2}\mathbb{Z}_{\geq 0} \implies \text{dim} \Theta^{4l+}_0 = 2l + 1. \)

- \( l = \frac{1}{2} : \rho^{1/2}(S^a) \propto \sigma_a \implies \text{eight vertex model}. \)
Assume: $\eta \in \left[-\frac{1}{2(2l+1)}, \frac{1}{2(2l+1)}\right]$, $l \in \frac{1}{2}\mathbb{Z}_{>0}$. Recall $\tau = it$, $t > 0$.

### Sklyanin form

$\langle \cdot, \cdot \rangle : \Theta_{00}^{4l+} \times \Theta_{00}^{4l+} \to \mathbb{C}$: sesquilinear form, such that

$$\langle \rho^l(S^a)f, g \rangle = \langle f, \rho^l(S^a)g \rangle.$$ 

### Explicit definition:

$$\langle f(z), g(z) \rangle := \int_0^1 dx \int_0^t dy \overline{f(x+iy)} g(x+iy) \mu(x+iy, x-iy),$$

$$\mu(z, w) := \frac{\theta_{11}(2z, \tau)\theta_{11}(2w, \tau)}{\prod_{j=0}^{2l+1} \theta_{00}(z+w+(2j-2l-1)\eta, \tau) \theta_{00}(z-w+(2j-2l-1)\eta, \tau)}. $$
§4. Higher spin generalisation of the eight vertex model

Idea: Use the spin \( l \) representation to define the \( L \)-matrix!

\[
L(u) := \sum_{a=0}^{3} W_a(u) \rho^l(S^a) \otimes \sigma_a : \Theta_{00}^{4l+} \otimes V_0 \to \Theta_{00}^{4l+} \otimes V_0.
\]

\( V_0 \cong \mathbb{C}^2 \) as before. Hereafter \( l \in \frac{1}{2} \mathbb{Z}_{>0} \).

- **Monodromy matrix:**

\[
\mathcal{T}(u) := L_{N0}(u) \cdots L_{20}(u)L_{10}(u) : \mathcal{H} \otimes V_0 \to \mathcal{H} \otimes V_0,
\]

where \( \mathcal{H} := V_N \otimes \cdots \otimes V_1, V_i \cong \Theta_{00}^{4l+} \).

- **Transfer matrix:** \( T(u) = \text{tr}_{V_0} \mathcal{T}(u) \).
Goal: Construct the $Q$-operator for this $T(u)$.

Main strategy: ([Baxter (1972, 1973)], [Baxter’s book])

1. Find $Q_R(u) : (\mathbb{C}^\dim H) \rightarrow H$ such that

$$T(u)Q_R(u) = h_-(u)Q_R(u-2\eta) + h_+(u)Q_R(u+2\eta), \quad h_\pm(u) = (2[u\mp2l\eta])^N.$$ 

2. $Q_L(u) := (Q_R(-\bar{u}))^\ast$ (hermitian conjugate with respect to $\langle , \rangle$):

$$Q_L(u)T(u) = h_-(u)Q_L(u - 2\eta) + h_+(u)Q_L(u + 2\eta).$$

3. Commutation relation $Q_L(u)Q_R(u') = Q_L(u')Q_R(u)$.

4. $Q(u) = Q_R(u)Q_R(u_0)^{-1} = Q_L(u_0)^{-1}Q_L(u)$ is the $Q$-operator, where $u_0$ is a suitably fixed parameter.

Hereafter $\eta = \frac{r'}{2lr} \in \left[ -\frac{1}{2(2l+1)}, \frac{1}{2(2l+1)} \right]$, $Nl \in \mathbb{Z}$. 
Construction of $Q_R$

Find an auxiliary operator $S_n(u) : (C^{2l+1}) \otimes N \otimes C^r \to \mathcal{H} \otimes C^r$ such that

$$M^{-1}L_n(u) \otimes S_n(u)M = \begin{pmatrix} A_n(u) & 0 \\ \ast & D_n(u) \end{pmatrix}$$

for some $M : C^2 \otimes C^r \to C^2 \otimes C^r$. (Recall: $\eta = \frac{r'}{2lr}$.)

Defining

$$Q_R(u) := \text{tr}_{C^r} S_N(u)S_{N-1}(u) \cdots S_1(u),$$

we obtain

$$T(u)Q_R(u) = \text{tr}_{C^r} A_N(u) \cdots A_1(u) + \text{tr}_{C^r} D_N(u) \cdots D_1(u),$$

which is “close” to the $TQ$-relation.
As the matrix $M$, we take $M := \tilde{M}_1 \otimes E_1^1 + \tilde{M}_2 \otimes E_2^2 + \cdots + \tilde{M}_r \otimes E_r^r$, where

$$\tilde{M}_i = \tilde{M}_i(u) = \begin{pmatrix} 1 & p_{i''} \\ 0 & 1 \end{pmatrix}, \quad p_{i''} := -\frac{\theta_{00}((2i'' - 1)l\eta, \tau)}{\theta_{01}((2i'' - 1)l\eta, \frac{\tau}{2})}.$$  

Then

$$\tilde{M}_i^{-1}L(u)\tilde{M}_{i \pm 1} = \begin{pmatrix} \alpha_{i,i \pm 1}(u) & \beta_{i,i \pm 1}(u) \\ \gamma_{i,i \pm 1}(u) & \delta_{i,i \pm 1}(u) \end{pmatrix}$$

has a degenerate $(2,1)$-component $\beta_{i,i \pm 1}(u)$:

$$\exists f_{\pm}(u) \in V_i \cong \Theta_{00}^{4l+}, \quad \begin{cases} 
\beta_{i,i \pm 1}f_{\pm}(u) = 0, \\
\alpha_{i,i \pm 1}f_{\pm}(u) = 2[u - 2l\eta]c_{i}c_{i \pm 1}^{-1}f_{\pm}(u + 2\eta), \\
\gamma_{i,i \pm 1}f_{\pm}(u) = 2[u + 2l\eta]c_{i}^{-1}c_{i \pm 1}f_{\pm}(u - 2\eta). \end{cases}$$
Define $S_n(u)$ using these vectors:

$$
S_n(u) = \begin{pmatrix}
S_1^1(u)_n & S_2^1(u)_n \\
S_2^2(u)_n & 0 & S_3^2(u)_n \\
& & & & & \ddots \\
& & & & & & & & 0 & S_{r-1}^r(u)_n \\
& & & & & & & & & S_r^r(u)_n \\
\end{pmatrix}
$$

$$
S_{j''}^{j''\pm 1}(u) : \mathbb{C}^r \ni e_k \mapsto \tau_{k,j''} f_{\pm}(2(2j'' - 1)l\eta, u, z) \in \Theta_{00}^{4l++},
$$

$$
S_1^1(u) : \mathbb{C}^r \ni e_k \mapsto \tau_{k1} f_{-}(2l\eta, u, z) \in \Theta_{00}^{4l++},
$$

$$
S_r^r(u) : \mathbb{C}^r \ni e_k \mapsto \tau_{kr} f_{+}(2(2r - 1)l\eta, u, z) \in \Theta_{00}^{4l++},
$$

$$(\{e_k\}_{k=1, \ldots, r} : \text{the standard basis of } \mathbb{C}^r, \tau_{k,j''} : \text{generic parameters})$$

$$
\Rightarrow T(u) Q_R(u) = (2[u - 2l\eta])^N Q_R(u + 2\eta) + (2[u + 2l\eta])^N Q_R(u - 2\eta).
$$
• Hermitian conjugate of $T(u)$ and definition of $Q_L(u)$

$(\cdot)^*$: Hermitian conjugate with respect to the Sklyanin form.

$$(S^a)^* = S^a \implies (T(u))^* = (-1)^N T(-\bar{u})$$

$$\implies Q_L(u) := (Q_R(-\bar{u}))^* : \mathcal{H} \to \mathbb{C}^{\text{dim} \mathcal{H}} \text{ satisfies}$$

$$Q_L(u)T(u) = h_-(u)Q_L(u - 2\eta) + h_+(u)Q_L(u + 2\eta).$$

**Theorem**

$$Q_L(u)Q_R(u') = Q_L(u')Q_R(u).$$
Idea of the proof:

Want:

\[(i, j)\text{-element of } Q_L(u) Q_R(u') \equiv (i, j)\text{-element of } Q_L(u') Q_R(u)\]

\[\implies \text{reduces to symmetry of the Sklyanin form}\]

\[\Phi(u, u') := \langle f_\pm(-\bar{u}; \text{parameters}), f_\pm'(u'; \text{parameters'}) \rangle\]

with respect to \( u \leftrightarrow u' \) for any choice of parameters.

This can be computed explicitly thanks to the results of [Rosengren (2004, 2007)].

(Computation is extremely tedious!)
Assumption: $Q_R(u_0)$ is generically invertible.

$$Q(u) := Q_R(u)Q_R(u_0)^{-1} = Q_L(u_0)^{-1}Q_L(u) : \mathcal{H} \to \mathcal{H}.$$ 

From the commutation relation $Q_L(u)Q_R(u') = Q_L(u')Q_R(u)$ follows:

**Theorem**

- $T(u)Q(u) = Q(u)T(u) = h_-(u)Q(u - 2\eta) + h_+(u)Q(u + 2\eta)$.
- $[Q(u), Q(u')] = 0$.

Holomorphicity and quasi-periodicity can be also proved.
Problems:

- Restrictions:
  - $\eta = \frac{r'}{2lr}$ ($r, r' \in \mathbb{Z}_{>0}$). $r = \dim(\text{auxiliary space})$.
  - $Nl \in \mathbb{Z}$. ($l$ even $\Rightarrow$ $\forall N$, $l$ odd $\Rightarrow$ for even $N$.)
    
    This is weaker than the assumption $N \in 2\mathbb{Z}$ in [Baxter (1973)] (higher spin version [Takebe (2016)]).

- [Fabricius-McCoy (2003)] (numerical study for the eight vertex model):
  
  $\exists N, r$ such that $Q_R(u)$ degenerates, i.e., $\nexists Q_R(u)^{-1}$
  
  $\implies$ cannot construct $Q(u) = Q_R(u) Q_R(u_0)^{-1}$.

[Fabricius (2007)] proposes a modified version, which conjecturally gives non-degenerate $Q_R$. (Higher spin version: [T] arXiv:1810.12969)
Comparison of various constructions

|                  | Baxter72(&T18) | Fabricius07 | Baxter73(&T16) | alg. BA |
|------------------|----------------|-------------|----------------|---------|
| $N$              | $Nl \in \mathbb{Z}$ | $N/2 \in \mathbb{Z}$ | $N/2 \in \mathbb{Z}$ | $Nl \in \mathbb{Z}$ |
| $\exists Q_R^{-1}$? | not always(?) | $+(?)$ | $+(?)$ | No problem |
| sum rule         | $+$           | $+$         | $+$           | $-$     |
Conclusion

- Baxter’s construction: *seemingly* too technical.
  (heavily dependent on the explicit structure of the transfer matrix)

*However*, it can be generalised to higher spin cases.

imbus existence of mathematical background?

Tack så mycket!