Mean Curvature Flows and Isotopy of Maps Between Spheres

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Abstract

Let $f$ be a smooth map between unit spheres of possibly different dimensions. We prove the global existence and convergence of the mean curvature flow of the graph of $f$ under various conditions. A corollary is that any area-decreasing map between unit spheres (of possibly different dimensions) is isotopic to a constant map.

1 Introduction

Let $\Sigma_1$ and $\Sigma_2$ be two compact Riemannian manifolds and $M = \Sigma_1 \times \Sigma_2$ be the product manifold. We consider a smooth map $f : \Sigma_1 \to \Sigma_2$ and denote the graph of $f$ by $\Sigma$; $\Sigma$ is a submanifold of $M$ by the embedding $id \times f$. In [17], [18], and [19], the second author studies the deformation of $f$ by the mean curvature flow (see also the work of Chen-Li-Tian [2]). The idea is to deform $\Sigma$ along the direction of its mean curvature vector in $M$ with the hope that $\Sigma$ will remain a graph. This is the negative gradient flow of the volume functional and a stationary point is a “minimal map” introduced by Schoen in [12]. In [19], the second author proves various long-time existence and convergence results of graphical mean curvature flows in arbitrary codimensions under assumptions on the Jacobian of the projection from $\Sigma$ to $\Sigma_1$. This quantity is denoted by $*\Omega$ in [19] and $*\Omega > 0$ if and only if $\Sigma$ is a graph over $\Sigma_1$ by the implicit function theorem. A crucial observation
in [19] is that $\star \Omega$ is a monotone quantity under the mean curvature flow when $\star \Omega > \frac{1}{\sqrt{2}}$.

In this paper, we discover new positive geometric quantities preserved by the graphical mean curvature flow. To describe these results, we recall the differential of $f$, $df$, at each point of $\Sigma$ is a linear map between the tangent spaces. The Riemannian structures enables us to define the adjoint of $df$. Let $\{\lambda_i\}$ denote the eigenvalues of $\sqrt{(df)^T df}$, or the singular values of $df$, where $(df)^T$ is the adjoint of $df$. Note that $\lambda_i$ is always nonnegative. We say $f$ is an area decreasing map if $\lambda_i \lambda_j < 1$ for any $i \neq j$ at each point. In particular, $f$ is area-decreasing if the $df$ has rank one everywhere. Under this condition, the second author proves the Bernstein type theorem [21] and interior gradient estimates [22] for solutions of the minimal surface system. It is also proved in [23] that the set of graphs of area-decreasing linear transformations forms a convex subset of the Grassmannian. We prove that this condition is preserved along the mean curvature flow and the following global existence and convergence theorem.

**Theorem A.** Let $\Sigma_1$ and $\Sigma_2$ be compact Riemannian manifolds of constant curvature $k_1$ and $k_2$ respectively. Suppose $k_1 \geq |k_2|$, $k_1 + k_2 > 0$ and $\dim(\Sigma_1) \geq 2$. If $f$ is a smooth area decreasing map from $\Sigma_1$ to $\Sigma_2$, the mean curvature flow of the graph of $f$ remains the graph of an area decreasing map, exists for all time, and converges smoothly to the graph of a constant map.

We remark that the condition $k_1 \geq |k_2|$ is enough to prove the long time existence of the flow. The following is an application to determine when a map between spheres is homotopically trivial.

**Corollary A.** Any area-decreasing map from $S^n$ to $S^m$ with $n \geq 2$ is homotopically trivial.

When $m = 1$, the area-decreasing condition always holds and the above statement follows from the fact that $\pi_n(S^1)$ is trivial for $n \geq 2$. We remark that the result when $m = 2$ is proved by the second author in [20] using a somewhat different method. The higher homotopy groups $\pi_n(S^m)$ has been computed in many cases and it is known that homotopically nontrivial maps do exist when $n \geq m$. Since an area-decreasing map may still be surjective when $n > m$, we do not know any topological method that would imply such a conclusion.

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2 Preliminaries

In this section, we recall notations and formulae for mean curvature flows. Let \( f : \Sigma_1 \to \Sigma_2 \) be a smooth map between Riemannian manifolds. The graph of \( f \) is an embedded submanifold \( \Sigma \) in \( M = \Sigma_1 \times \Sigma_2 \). At any point of \( \Sigma \), the tangent space of \( M \), \( T_M \) splits into the direct sum of the tangent space of \( \Sigma \), \( T_\Sigma \) and the normal space \( N_\Sigma \), the orthogonal complement of the tangent space \( T_\Sigma \) in \( T_M \). There are isomorphisms \( T_\Sigma \to T_\Sigma \) by \( X \mapsto X + (df)(X) \) and \( T_\Sigma \to N_\Sigma \) by \( Y \mapsto Y - (df)^T(Y) \) where \((df)^T : T_\Sigma \to T_\Sigma \) is the adjoint of \( df \).

We assume the mean curvature flow of \( \Sigma \) can be written as a graph of \( f_t \) for \( t \in [0, \epsilon) \) and derive the equation satisfied by \( f_t \). The mean curvature flow is given by a smooth family of immersions \( F_t \) of \( \Sigma \) into \( M \) which satisfies

\[
\left( \frac{\partial F}{\partial t} \right)^+ = H
\]

where \( H \) is the mean curvature vector in \( M \) and \((\cdot)^+\) denotes the projection onto the normal direction since the difference is only a tangential diffeomorphism (see for example White [24] for the issue of parametrization). By the definition of the mean curvature vector, this equation is equivalent to

\[
\left( \frac{\partial F}{\partial t} \right)^+ = \left( \Lambda^{ij} \nabla^M_{\frac{\partial}{\partial x^i}} \frac{\partial F}{\partial y^A} \right)^+
\]

where \( \Lambda^{ij} \) is the inverse to the induced metric \( \Lambda_{ij} = \langle \frac{\partial F}{\partial y^i}, \frac{\partial F}{\partial y^j} \rangle \) on \( \Sigma \).

In terms of coordinates \( \{y^A\}_{A=1 \ldots n+m} \) on \( M \), we have

\[
\Lambda^{ij} \nabla^M_{\frac{\partial}{\partial x^i}} \frac{\partial F}{\partial y^A} = \Lambda^{ij} \left( \frac{\partial^2 F^A}{\partial x^i \partial x^j} + \frac{\partial F^B}{\partial x^i} \frac{\partial F^C}{\partial x^j} \Gamma^A_{BC} \right) \frac{\partial}{\partial y^A}
\]

where \( \Gamma^A_{BC} \) is the Christoffel symbol of \( M \) and thus

\[
\left( \Lambda^{ij} \nabla^M_{\frac{\partial}{\partial x^i}} \frac{\partial F}{\partial x^j} \right)^+ = \Lambda^{ij} \left( \frac{\partial^2 F^A}{\partial x^i \partial x^j} + \frac{\partial F^B}{\partial x^i} \frac{\partial F^C}{\partial x^j} \Gamma^A_{BC} - \tilde{\Gamma}^k_{ij} \frac{\partial F^A}{\partial x^k} \right) \frac{\partial}{\partial y^A}
\]

where \( \tilde{\Gamma}^k_{ij} \) is the Christoffel symbol of the induced metric on \( \Sigma \).
By assumption, the embedding is given by the graph of $f_t$. We fix a coordinate system $\{x^i\}$ on $\Sigma_1$ and consider $F: \Sigma_1 \times [0, T) \to M$ given by

$$F(x^1, \ldots, x^n, t) = (x^1, \ldots, x^n, f^{n+1}, \ldots, f^{n+m}).$$

We shall use $i, j, k, l = 1 \cdots n$ and $\alpha, \beta, \gamma = n+1 \cdots n+m$ for the indices.

Therefore $\frac{\partial F}{\partial t} = \frac{\partial f}{\partial t} \frac{\partial}{\partial y}$ and

$$\Lambda^{ij}(\frac{\partial^2 F^A}{\partial x^i \partial x^j} + \frac{\partial F^B}{\partial x^i} \frac{\partial F^C}{\partial x^j} \Gamma^{A}_{BC}) \frac{\partial}{\partial y^A} = \Lambda^{ij}(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \Gamma^l_{ij} \frac{\partial}{\partial y^l} + \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j} \Gamma^\alpha_{\beta\gamma} \frac{\partial}{\partial y^\alpha}).$$

Thus the mean curvature flow equation is equivalent to the normal part of

$$\frac{\partial f^\alpha}{\partial t} - \Lambda^{ij}(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j} \Gamma^\alpha_{\beta\gamma}) \frac{\partial}{\partial y^\alpha} - \Lambda^{ij} \Gamma^l_{ij} \frac{\partial}{\partial y^l} = 0$$

is zero.

Now given any vector $a^i \frac{\partial}{\partial y^i} + b^\alpha \frac{\partial}{\partial y^\alpha}$, the equation that the normal part being zero is equivalent to

$$b^\alpha - a^i \frac{\partial f^\alpha}{\partial x^i} = 0 \tag{2.1}$$

for each $\alpha$. Therefore we obtain the evolution equation for $f$

$$\frac{\partial f^\alpha}{\partial t} - \Lambda^{ij}(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j} \Gamma^\alpha_{\beta\gamma} + \Gamma^k_{ij} \frac{\partial f^\alpha}{\partial x^k}) = 0. \tag{2.2}$$

where $\Lambda^{ij}$ is the inverse to $g_{ij} + h_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}$ and $g_{ij} = \langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \rangle$ and $h_{\alpha\beta} = \langle \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \rangle$ are the Riemannian metrics on $\Sigma_1$ and $\Sigma_2$, respectively. $\Gamma^k_{ij}$ and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of $g_{ij}$ and $h_{\alpha\beta}$ respectively.

(2.2) is a nonlinear parabolic system and the usual derivative estimates do not apply to this equations. However, the second author in [19] identifies a geometric quantity in terms of the derivatives of $f^\alpha$ that satisfies the maximum principle; this quantity and its evolution equation are recalled in the next section.
3 Two evolution equations

In this section, we recall two evolution equations along the mean curvature flow. The basic set-up is a mean curvature flow $F : \Sigma \times [0, T) \to M$ of an $n$ dimensional submanifold $\Sigma$ inside an $n + m$ dimensional Riemannian manifold $M$. Given any parallel tensor on $M$, we may consider the pullback tensor by $F_t$ and consider the evolution equation with respect to the time-dependent induced metric on $F_t(\Sigma) = \Sigma_t$. For the purpose of applying maximum principle, it suffices to derive the equation at a space-time point. We write all geometric quantities in terms of orthonormal frames keeping in mind all quantities are defined independent of choices of frames. At any point $p \in \Sigma_t$, we choose any orthonormal frames $\{e_i\}_{i=1 \cdots n}$ for $T_p \Sigma_t$ and $\{e_\alpha\}_{\alpha = n+1 \cdots n+m}$ for $N_p \Sigma_t$. The second fundamental form $h_{\alpha ij}$ is denoted by $h_{\alpha ij} = \langle \nabla^M e_i e_j, e_\alpha \rangle$ and the mean curvature vector is denoted by $H_\alpha = \sum_i h_{\alpha ii}$. For any $j, k$, we pretend $h_{n+i, jk} = 0$ if $i > m$.

When $M = \Sigma_1 \times \Sigma_2$ is the product of $\Sigma_1$ and $\Sigma_2$, we denote the projections by $\pi_1 : M \to \Sigma_1$ and $\pi_2 : M \to \Sigma_2$. By abusing notations, we also denote the differentials by $\pi_1 : T_p M \to T_{\pi_1(p)} \Sigma_1$ and $\pi_1 : T_p M \to T_{\pi_2(p)} \Sigma_2$ at any point $p \in M$. The volume form $\Omega$ of $\Sigma_1$ can be extended to a parallel $n$-form on $M$. For an oriented orthonormal basis $e_1, \ldots, e_n$ of $T_p \Sigma$, $\Omega(e_1, \ldots, e_n) = \Omega(\pi_1(e_1), \ldots, \pi_1(e_n))$ is the Jacobian of the projection from $T_p \Sigma$ to $T_{\pi_1(p)} \Sigma_1$. This can also be considered as the pairing between the $n$-form $\Omega$ and the $n$-vector $e_1 \wedge \cdots \wedge e_n$ representing $T_p \Sigma$. We use $*\Omega$ to denote this function as $p$ varies along $\Sigma$. By the implicit function theorem, $*\Omega > 0$ at $p$ if and only if $\Sigma$ is locally a graph over $\Sigma_1$ at $p$. The evolution equation of $*\Omega$ is calculated in Proposition 3.2 of [10].

When $\Sigma$ is the graph of $f : \Sigma_1 \to \Sigma_2$, the equation at each point can be written in terms of singular values of $df$ and special bases adapted to $df$. Denote the singular values of $df$, or eigenvalues of $(df)^T df$, by $\{\lambda_i\}_{i=1 \cdots n}$. Let $r$ denote the rank of $df$. We can rearrange them so that $\lambda_i = 0$ when $i$ is greater than $r$. By singular value decomposition, there exist orthonormal bases $\{a_i\}_{i=1 \cdots n}$ for $T_{\pi_1(p)} \Sigma_1$ and $\{a_\alpha\}_{\alpha = n+1 \cdots n+m}$ for $T_{\pi_2(p)} \Sigma_2$ such that

$$df(a_i) = \lambda_i a_{n+i}$$
for \( i \) less than or equal to \( r \) and \( df(a_i) = 0 \) for \( i \) greater than \( r \). Moreover,

\[
e_i = \begin{cases} 
\frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i a_{n+i}) & \text{if } 1 \leq i \leq r \\
a_i & \text{if } r + 1 \leq i \leq n
\end{cases}
\]  

(3.1)

becomes an orthonormal basis for \( T_p\Sigma \) and

\[
e_{n+p} = \begin{cases} 
\frac{1}{\sqrt{1+\lambda_{n+p}^2}}(a_{n+p} - \lambda_{n+p} a_p) & \text{if } 1 \leq p \leq r \\
a_{n+p} & \text{if } r + 1 \leq p \leq m
\end{cases}
\]  

(3.2)

becomes an orthonormal basis for \( N_p\Sigma \).

In terms of the singular values \( \lambda_i \),

\[
\ast \Omega = \frac{1}{\sqrt{\prod_{i=1}^n (1 + \lambda_i^2)}}
\]  

(3.3)

With all the notations understood, the following result is essentially derived in Proposition 3.2 of [19] by noting that \( (\ln \ast \Omega)_k = -(\sum_i \lambda_i h_{n+i,ik}) \).

**Proposition 3.1** Suppose \( M = \Sigma_1 \times \Sigma_2 \) and \( \Sigma_1 \) and \( \Sigma_2 \) are compact Riemannian manifolds of constant curvature \( k_1 \) and \( k_2 \) respectively. With respect to the particular bases given by the singular value decomposition of \( df \), \( \ln \ast \Omega \) satisfies the following equation.

\[
\left( \frac{d}{dt} - \Delta \right) \ln \ast \Omega = \sum_{\alpha,i,k} h_{\alpha ik}^2 + \sum_{k,i} \lambda_i^2 h_{n+i,ik}^2 + 2 \sum_{k,i<j} \lambda_i \lambda_j h_{n+j,ik} h_{n+i,jk} + \sum_i \frac{\lambda_i^2}{1 + \lambda_i^2} \left[ (k_1 + k_2) \left( \sum_{j \neq i} \frac{1}{1 + \lambda_j^2} \right) + k_2 (1 - n) \right]
\]  

(3.4)

Next we recall the evolution equation of parallel two tensors from [15]. The calculation indeed already appears in [17]. The equation will be used later to obtain more refined information. Given a parallel two-tensor \( S \) on \( M \), we consider the evolution of \( S \) restricted to \( \Sigma_t \). This is a family of time-dependent symmetric two tensors on \( \Sigma_t \).
Proposition 3.2 Let $S$ be a parallel two-tensor on $M$. Then the pull-back of $S$ to $\Sigma_t$ satisfies the following equation.

\[
\left(\frac{d}{dt} - \Delta\right) S_{ij} = -h_{ai}H_{\alpha}S_{lj} - h_{aj}H_{\alpha}S_{li} \\
+ R_{kika}\ S_{aj} + R_{kjka}\ S_{ai} \\
+ h_{aki}h_{aki}S_{lj} + h_{aki}h_{akj}S_{li} - 2h_{aki}h_{bjk}\ S_{\alpha\beta}
\]

where $\Delta$ is the rough Laplacian on two-tensors over $\Sigma_t$ and $S_{ai} = S(e_\alpha, e_i)$, $S_{\alpha\beta} = S(e_\alpha, e_\beta)$, and $R_{kika} = R(e_k, e_i, e_k, e_\alpha)$ is the curvature of $M$.

The evolution equations (3.5) of $S$ can be written in terms of evolving orthonormal frames as in Hamilton [8]. If the orthonormal frames

\[
F = \{F_1, \cdots, F_a, \cdots, F_n\}
\]

are given in local coordinates by

\[
F_a = F^i_a \frac{\partial}{\partial x^i} .
\]

To keep them orthonormal, i.e. $g_{ij}F^i_a F^j_a = \delta_{ab}$, we evolve $F$ by the formula

\[
\frac{\partial}{\partial t} F^i_a = g^{ij}g^{\alpha\beta}h_{\alpha jl}H_{\beta}F^l_a .
\]

Let $S_{ab} = S_{ij} F^i_a F^j_b$ be the components of $S$ in $F$. Then $S_{ab}$ satisfies the following equation

\[
\left(\frac{d}{dt} - \Delta\right) S_{ab} = \ R{caca} S_{ab} + R{cbca} S_{oa} \\
+ h_{acd}h_{aaco}S_{db} + h_{acd}h_{acbo}S_{da} \\
- 2h_{aca}h_{bcb}\ S_{\alpha\beta} .
\]

4 Preserving the distance-decreasing condition

In this section, we show the condition $|df| < 1$, or each singular value $\lambda_i < 1$, is preserved by the mean curvature flow. This result will not be used
in proof of the Theorem A. But the proof of Theorem A depends on the 
computation in this section. The tangent space of $M = \Sigma_1 \times \Sigma_2$ is identified 
with $T\Sigma_1 \oplus T\Sigma_2$. Let $\pi_1$ and $\pi_2$ denote the projection onto the first and second 
summand in the splitting. We define the parallel symmetric two-tensor $S$ by 

$$S(X, Y) = \langle \pi_1(X), \pi_1(Y) \rangle - \langle \pi_2(X), \pi_2(Y) \rangle$$

(4.1)

for any $X, Y \in TM$.

Let $\Sigma$ be the graph of $f : \Sigma_1 \to \Sigma_1 \times \Sigma_2$. $S$ restricts to a symmetric 
two-tensor on $\Sigma$ and we can represent $S$ in terms of the orthonormal basis 
(3.1).

Let $r$ denote the rank of $df$. By (3.1), it is not hard to check

$$\pi_1(e_i) = \frac{a_i}{\sqrt{1 + \lambda_i^2}}, \pi_2(e_i) = \frac{\lambda_i a_{n+i}}{\sqrt{1 + \lambda_i^2}} \quad \text{for } 1 \leq i \leq r ,$$

and $\pi_1(e_i) = a_i, \pi_2(e_i) = 0$ for $r + 1 \leq i \leq n$. (4.2)

Similarly, by (3.2) we have

$$\pi_1(e_{n+p}) = \frac{-\lambda_p a_p}{\sqrt{1 + \lambda_p^2}}, \pi_2(e_{n+p}) = \frac{a_{n+p}}{\sqrt{1 + \lambda_p^2}} \quad \text{for } 1 \leq p \leq r ,$$

and $\pi_1(e_{n+p}) = 0, \pi_2(e_{n+p}) = a_{n+p}$ for $r + 1 \leq p \leq m$. (4.3)

From the definition of $S$, we have

$$S(e_i, e_j) = \frac{1 - \lambda_i^2}{1 + \lambda_i^2} \delta_{ij} .$$

(4.4)

In particular, the eigenvalues of $S$ are

$$\frac{1 - \lambda_i^2}{1 + \lambda_i^2}, \quad i = 1 \ldots n .$$

(4.5)

Notice that $S$ is positive-definite if and only if

$$|\lambda_i| < 1$$

for any singular value $\lambda_i$ of $df$. 8
Now, at each point we express $S$ in terms of the orthonormal basis \( \{e_i\}_{i=1}^n \) and \( \{e_\alpha\}_{\alpha=n+1}^{n+m} \). Let \( I_{k\times k} \) denote a \( k \) by \( k \) identity matrix. Then \( S \) can be written in the block form

\[
S = \begin{pmatrix}
B & 0 & D & 0 \\
0 & I_{n-r\times n-r} & 0 & 0 \\
D & 0 & -B & 0 \\
0 & 0 & 0 & -I_{m-r\times m-r}
\end{pmatrix}
\]

(4.6)

where \( B \) and \( D \) are \( r \) by \( r \) matrices with \( B_{ij} = \frac{1-\lambda^2}{1+\lambda^2} \delta_{ij} \) and \( D_{ij} = S(e_i, e_{n+j}) = \frac{-2\lambda}{1+\lambda^2} \delta_{ij} \) for \( 1 \leq i, j \leq r \). We show that the positivity of \( S \) is preserved by the mean curvature flow. We remark that a similar positive definite tensor has been considered for the Lagrangian mean curvature flow in Smoczyk [14] and Smoczyk-Wang [15]. The following lemma shows that the distance decreasing condition is preserved by the mean curvature flow if \( k_1 \geq |k_2| \).

**Lemma 4.1** The condition

\[
T_{ij} = S_{ij} - \epsilon g_{ij} > 0 \quad \text{for some } \epsilon \geq 0
\]

(4.7)

is preserved by the mean curvature flow if \( k_1 \geq |k_2| \).

**Proof.** We compute the evolution equation for \( T_{ij} \). From Proposition (3.2) and

\[
\frac{\partial}{\partial t} q_{ij} = -2h_{\alpha ij} H_{\alpha},
\]

we have

\[
(\frac{d}{dt} - \Delta)T_{ij} = -h_{\alpha il} H_{\alpha} T_{lj} - h_{\alpha j} H_{\alpha} T_{li} + R_{kika} S_{\alpha j} + R_{kka} S_{\alpha i} + h_{\alpha kl} h_{\alpha k} T_{lj} + h_{\alpha kl} h_{\alpha k} T_{li} + 2\epsilon h_{\alpha i} h_{\alpha k} T_{k} + 2\epsilon h_{\alpha i} h_{\alpha k} T_{k} - 2h_{\alpha i} h_{\beta kj} S_{\alpha \beta}.
\]

(4.8)

To apply Hamilton’s maximum principle, it suffices to prove that \( N_{ij} V^i V^j \geq 0 \) for any null eigenvector \( V \) of \( T_{ij} \), where \( N_{ij} \) is the right hand side of (4.8). Since \( V \) is a null eigenvector of \( T_{ij} \), it satisfies \( \sum_j T_{ij} V^j = 0 \) for any \( i \), and thus \( N_{ij} V^i V^j \) is equal to

\[
2\epsilon h_{\alpha i} h_{\alpha k} V^i V^j + 2R_{kika} S_{\alpha j} V^i V^j - 2h_{\alpha i} h_{\beta kj} S_{\alpha \beta} V^i V^j.
\]

(4.9)
Obviously, the first term of (4.9) is nonnegative. Applying the relation in (4.6) to the last term of (4.9) we obtain

\[-2h_{\alpha ki}h_{\beta kj}S_{\alpha \beta}V^iV^j = \sum_{1 \leq p, q \leq r} 2h_{n + pk_i h_{n + qk_j} S_{pq} V^iV^j} + \sum_{r + 1 \leq p, q \leq m} 2h_{n + pk_i h_{n + qk_j} V^iV^j}.\]

Since \(T_{pq} \geq 0\) implies that \(S_{pq} \geq \epsilon g_{pq}\), we obtain \(-2h_{\alpha ki}h_{\beta kj}S_{\alpha \beta}V^iV^j \geq 0\). In the next lemma we show that \(R_{kik\alpha}S_{\alpha j}\) is nonnegative definite whenever \(S_{ij}\) is under the curvature assumption \(k_1 \geq |k_2|\).

\[\square\]

**Lemma 4.2**

\[R_{kik\alpha}S_{\alpha j} = \lambda_1^2 \left( k_1 - k_2 \right) (n - 1) + (k_1 + k_2) \sum_{k \neq i} \frac{1 - \lambda_2^2}{1 + \lambda_2^2} \delta_{ij}.\] (4.10)

**Proof.** We follow the calculation of the curvature terms in \[19\].

\[
\sum_k R(e_\alpha, e_k, e_k, e_i)
= \sum_k R_1(\pi_1(e_\alpha), \pi_1(e_k), \pi_1(e_k), \pi_1(e_i)) + R_2(\pi_2(e_\alpha), \pi_2(e_k), \pi_2(e_k), \pi_2(e_i))
= \sum_k k_1 \left[ \langle \pi_1(e_\alpha), \pi_1(e_k) \rangle \langle \pi_1(e_k), \pi_1(e_i) \rangle - \langle \pi_1(e_\alpha), \pi_1(e_i) \rangle \langle \pi_1(e_k), \pi_1(e_k) \rangle \right]
+ k_2 \left[ \langle \pi_2(e_\alpha), \pi_2(e_k) \rangle \langle \pi_2(e_k), \pi_2(e_i) \rangle - \langle \pi_2(e_\alpha), \pi_2(e_i) \rangle \langle \pi_2(e_k), \pi_2(e_k) \rangle \right].
\]

Notice that \(\langle \pi_2(X), \pi_2(Y) \rangle = \langle X, Y \rangle - \langle \pi_1(X), \pi_1(Y) \rangle\) since \(T\Sigma_1 \perp T\Sigma_2\). Therefore

\[
\sum_k R(e_\alpha, e_k, e_k, e_i)
= \sum_k \left( k_1 + k_2 \right) \left[ \langle \pi_1(e_\alpha), \pi_1(e_k) \rangle \langle \pi_1(e_k), \pi_1(e_i) \rangle - \langle \pi_1(e_\alpha), \pi_1(e_i) \rangle \langle \pi_1(e_k) \rangle^2 \right]
+ k_2 (n - 1) \langle \pi_1(e_\alpha), \pi_1(e_i) \rangle
\]
Now use $\pi_1(e_α) = -\lambda_p \pi_1(e_p) \delta_{α,n+p}$ and $S(e_j, e_{n+p}) = -\frac{2\lambda_j \delta_{jp}}{1 + \lambda_j^2}$ in (4.6), we have

$$\sum_{α,k} R_{kikα} S_{αj} = - \sum_{p,k} R_{n+p,kki} S_{n+p,j}$$

$$= \sum_{p,k} \left\{ \lambda_p (k_1 + k_2) \left[ \langle \pi_1(e_p), \pi_1(e_k) \rangle \langle \pi_1(e_k), \pi_1(e_i) \rangle - \langle \pi_1(e_p), \pi_1(e_i) \rangle \langle \pi_1(e_k) \rangle^2 \right] + \lambda_p k_2 (n - 1) \langle \pi_1(e_p), \pi_1(e_i) \rangle \right\} S_{n+p,j}$$

$$= - \frac{2\lambda^2}{1 + \lambda^2} (k_1 + k_2) \left[ \frac{\delta_{ij}}{(1 + \lambda^2)^2} - \frac{\delta_{ij}}{1 + \lambda^2} \sum_k |\pi_1(e_k)|^2 \right]$$

$$+ k_2 (n - 1) \frac{\delta_{ij}}{1 + \lambda^2} \}.$$ 

Recall that $|\pi_1(e_k)|^2 = \frac{1}{1 + \lambda_k^2}$ and we obtain

$$R_{kikα} S_{αj} = \frac{2\lambda^2 \delta_{ij}}{(1 + \lambda^2)^2} \left[ (k_1 + k_2) \left( \sum_{k \neq i} 1 \frac{1}{1 + \lambda_k^2} \right) + k_2 (1 - n) \right].$$

This can be further simplified by noting

$$(k_1 + k_2) \left( \sum_{k \neq i} 1 \frac{1}{1 + \lambda_k^2} \right) + k_2 (1 - n) = \frac{(k_1 - k_2) (n - 1)}{2} + (k_1 + k_2) \sum_{k \neq i} 1 \frac{1 - \lambda_k^2}{2(1 + \lambda_k^2)} \quad (4.11)$$

where we use the following identity for each $i$

$$\left( \sum_{k \neq i} \frac{1}{1 + \lambda_k^2} \right) - \frac{n - 1}{2} = \sum_{k \neq i} \left( \frac{1}{1 + \lambda_k^2} - \frac{1}{2} \right) = \sum_{k \neq i} \frac{1 - \lambda_k^2}{2(1 + \lambda_k^2)}.$$

\[\square\]

5 Preserving the area-decreasing condition

In this section, we show that the area decreasing condition is preserved along the mean curvature flow. In the following, we require that $n = \text{dim}(\Sigma_1) \geq 2$. By (4.5), the sum of any two eigenvalues of $S$ is

$$\frac{1 - \lambda_i^2}{1 + \lambda_i^2} + \frac{1 - \lambda_j^2}{1 + \lambda_j^2} = \frac{2(1 - \lambda_i^2 \lambda_j^2)}{(1 + \lambda_i^2)(1 + \lambda_j^2)}. \quad (5.1)$$
Therefore the area decreasing condition $|\lambda_i \lambda_j| < 1$ for $i \neq j$ is equivalent to the two-positivity of $S$, i.e. the sum of any two eigenvalues is positive. We remark that curvature operator being two-positive is preserved by the Ricci flow, see Chen [1] or Hamilton [8] for detail.

The two-positivity of a symmetric two tensor $P$ can be related to the convexity of another tensor $P^{[2]}$ associated with $P$. The following notation is adopted from Caffarelli-Nirenberg-Spruck [3]. Let $P$ be a self-adjoint operator on an $n$-dimensional inner product space. From $P$ we can construct a new self-adjoint operator

$$P^{[k]} = \sum_{i=1}^{k} 1 \otimes \cdots \otimes P_i \otimes \cdots \otimes 1$$

acting on the exterior powers $\Lambda^k$ by

$$P^{[k]}(\omega_1 \wedge \cdots \wedge \omega_k) = \sum_{i=1}^{k} \omega_1 \wedge \cdots \wedge P(\omega_i) \wedge \cdots \wedge \omega_k.$$

With the definition of $P^{[k]}$, we have the following lemma.

**Lemma 5.1** Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be the eigenvalues of $P$ with corresponding eigenvectors $v_1 \cdots v_n$. Then $P^{[k]}$ has eigenvalues $\mu_{i_1} + \cdots + \mu_{i_k}$ and eigenvectors $v_{i_1} \wedge \cdots \wedge v_{i_k}$, $i_1 < i_2 \cdots < i_k$.

Recall that the Riemannian metric $g$ and $S$ are both in $T\Sigma \otimes T\Sigma$, the space of symmetric two tensor on $\Sigma$. We can identify $S$ with a self-adjoint operator on the tangent bundle through the metric $g$. Therefore $S^{[2]}$ and $g^{[2]}$ are both sections of $(\Lambda^2(T\Sigma))^* \otimes \Lambda^2(T\Sigma)$ associated to $S$ and $g$ respectively. We shall use orthonormal frames in the following calculation; this has the advantage that $g$ is the identity matrix and we will not distinguish lower index and upper index. With the above interpretation and (5.1), we have the following lemma.

**Lemma 5.2** The area decreasing condition is equivalent to the convexity of $S^{[2]}$.

To show that the area decreasing condition is preserved, it suffices to prove that the convexity of $S^{[2]}$ is preserved. In fact, we prove the stronger result that the convexity of $S^{[2]} - \epsilon g^{[2]}$ for $\epsilon > 0$ is preserved.
We compute the evolution equation of $S^{[2]} - \epsilon g^{[2]}$ in terms of the evolving orthonormal frames $\{F_a\}_{a=1\ldots n}$ introduced earlier in (3.6). We will use indices $a, b, \ldots$ to denote components in the evolving frames. Denote $S_{ab} = S(F_a, F_b)$ and $g_{ab} = g(F_a, F_b) = \delta_{ab}$. Since $\{F_a \wedge F_b\}_{a,b}$ form a basis for $\Lambda^2 T\Sigma$, we have

\[
S^{[2]}(F_a \wedge F_b) = S(F_a) \wedge F_b + F_a \wedge S(F_b) = S_{ac}F_c \wedge F_b + F_a \wedge S_{ac}F_c
\]

\[
= \sum_{c<d} (S_{ac}\delta_{bd} + S_{bd}\delta_{ac} - S_{ad}\delta_{bc} - S_{bc}\delta_{ad}) F_c \wedge F_d\quad \text{and}
\]

\[
g^{[2]}(F_a \wedge F_b) = \sum_{c<d} (2\delta_{ad}\delta_{bd} - 2\delta_{ad}\delta_{bc}) F_c \wedge F_d.
\]

We denote $S_{(ab)(cd)}^{[2]} = (S_{ac}\delta_{bd} + S_{bd}\delta_{ac} - S_{ad}\delta_{bc} - S_{bc}\delta_{ad})$ and $g^{[2]}_{(ab)(cd)} = 2\delta_{ad}\delta_{bd} - 2\delta_{ad}\delta_{bc}$. Thus the evolution equation of $S^{[2]} - \epsilon g^{[2]}$ in terms of the evolving orthonormal frames is

\[
\left(\frac{d}{dt} - \Delta\right) (S_{ac}\delta_{bd} + S_{bd}\delta_{ac} - S_{ad}\delta_{bc} - S_{bc}\delta_{ad} - 2\epsilon\delta_{ad}\delta_{bd} + 2\epsilon\delta_{ad}\delta_{bc})
\]

\[
= Re_{aea}S_{ac}\delta_{bd} + Re_{eae}S_{ac}\delta_{bd} + Re_{eaS_{ac}\delta_{ad}} - Re_{eae}S_{ac}\delta_{ad} - Re_{eae}S_{ac}\delta_{ad}
\]

\[
+ h_{ae}h_{ae}S_{ac}\delta_{bd} + h_{ae}h_{ae}S_{ac}\delta_{bd} + h_{ae}h_{ae}S_{ac}\delta_{bd} + h_{ae}h_{ae}S_{ac}\delta_{bd} + h_{ae}h_{ae}S_{ac}\delta_{bd}
\]

\[
- h_{ae}h_{ae}S_{ac}\delta_{bd} + h_{ae}h_{ae}S_{ac}\delta_{bd} + h_{ae}h_{ae}S_{ac}\delta_{bd} + h_{ae}h_{ae}S_{ac}\delta_{bd} + h_{ae}h_{ae}S_{ac}\delta_{bd}
\]

\[
- 2h_{ae}S_{ac}\delta_{bd} + h_{ae}h_{ae}S_{ac}\delta_{bd} + h_{ae}h_{ae}S_{ac}\delta_{bd} + h_{ae}h_{ae}S_{ac}\delta_{bd} + h_{ae}h_{ae}S_{ac}\delta_{bd}.
\]

Now, we are ready to prove that the area decreasing condition is preserved along the mean curvature flow.

**Lemma 5.3** Under the assumption of Theorem A, with $S$ defined in (4.1) and $S^{[2]}$ defined in (5.2), suppose there exists an $\epsilon > 0$ such that

\[
S^{[2]} - \epsilon g^{[2]} \geq 0
\]

holds on the initial graph. Then this is preserved along the mean curvature flow.

**Proof.** Set

\[
M_\eta = S^{[2]} - \epsilon g^{[2]} + \eta g^{[2]}.
\]

Suppose the mean curvature flow exists on $[0, T)$. Consider any $T_1 < T$, it suffices to prove that $M_\eta > 0$ on $[0, T_1]$ for all $\eta < \frac{1}{2T_1}$. If not, there will
be a first time $0 < t_0 \leq T_1$ where $M_\eta = S[2] - \epsilon g[2] + \eta t g[2]$ is nonnegative definite and has a null eigenvector $V = V^{ab} F_a \wedge F_b$ at some point $x_0 \in \Sigma_{t_0}$. We extend $V^{ab}$ to a parallel tensor in a neighborhood of $x_0$ along geodesic emanating out of $x_0$, and defined $V^{ab}$ on $[0, T]$ independent of $t$. Define

$$f = \sum_{a < b, c < d} V^{ab} M_\eta V^{cd},$$

then by (5.2), $f$ equals

$$\sum_{a < b, c < d} (S_{ac} g_{bd} + S_{bd} g_{ac} - S_{bc} g_{ad} + 2(\eta t - \epsilon)(g_{ac} g_{bd} - g_{ad} g_{bc})) V^{ab} V^{cd}.$$

At $(x_0, t_0)$, we have $f = 0$, $\nabla f = 0$ and $(\frac{d}{dt} - \Delta) f \leq 0$ where $\nabla$ denotes the covariant derivative and $\Delta$ denotes the Laplacian on $\Sigma_{t_0}$. We may assume that at $(x_0, t_0)$ the orthonormal frames $\{F_a\}$ is given by $\{e_i\}$ in (3.1). In the following, we use the orthonormal basis $\{e_i\}$ to write down the condition $f = 0$ and $\nabla f = 0$ at $(x_0, t_0)$. The basis $\{e_i\}$ diagonalizes $S$ with eigenvalues $\{\lambda_i\}$ and we order $\{\lambda_i\}$ such that

$$\lambda_1^2 \geq \lambda_2^2 \geq \cdots \geq \lambda_n^2$$

and

$$S_{nn} = \frac{1 - \lambda_n^2}{1 + \lambda_n^2} \geq \cdots \geq S_{22} = \frac{1 - \lambda_2^2}{1 + \lambda_2^2} \geq S_{11} = \frac{1 - \lambda_1^2}{1 + \lambda_1^2}.$$  (5.5)

It follows from Lemma (5.1) that $\{e_i \wedge e_j\}_{i < j}$ are the eigenvectors of $M_\eta$. Thus we may assume that

$$V = e_1 \wedge e_2.$$  (5.6)

At $(x_0, t_0)$, the condition $f = 0$ is the same as

$$S_{11} + S_{22} = 2\epsilon - 2\eta t_0 > 0.$$  (5.7)

This is equivalent to

$$\frac{2(1 - \lambda_1^2 \lambda_2^2)}{(1 + \lambda_1^2)(1 + \lambda_2^2)} = 2(\epsilon - \eta t_0) > 0.$$

Thus

$$\lambda_1 \lambda_2 < 1$$

and

$$\lambda_i < 1$$

for $i \geq 3$.  (5.8)
Next, we compute the covariant derivative of the restriction of $S$ on $\Sigma$.

$$(\nabla_{e_i}S)(e_j, e_k) = e_i(S(e_j, e_k)) - S(\nabla_{e_i}e_j, e_k) - S(e_j, \nabla_{e_i}e_k) = S(\nabla_{e_i}^M e_j - \nabla_{e_i}e_j, e_k) + S(e_j, \nabla_{e_i}^M e_k - \nabla_{e_i}e_k) = h_{aij}S_{ak} + h_{\beta ik}S_{\beta j}. $$

So

$$S_{jk,i} = h_{aij}S_{ak} + h_{\beta ik}S_{\beta j}.$$ 

Recall that $V_{ab}$ is parallel at $(x_0, t_0)$, $V_{12} = 1$ and all other components of $V^{ab}$ is zero. At $(x_0, t_0)$, $\nabla f = 0$ is equivalent to

$$0 = \sum_{i<j,k<l} \nabla_{e_p}((S_{ik}\delta_{jl} + S_{jl}\delta_{ik} - S_{il}\delta_{jk} - S_{jk}\delta_{il} + 2(\eta t - \epsilon)(\delta_{ik}\delta_{jl} - \epsilon\delta_{il}\delta_{jk}))V^{ij}V^{kl}) = \nabla_{e_p}S_{11} + \nabla_{e_p}S_{22} = 2h_{\alpha p1}S_{\alpha 1} + 2h_{\beta p2}S_{\beta 2}. $$

Since $S_{n+q,l} = -\frac{2\lambda_{n+q,l}}{1+\lambda^2}$, we have

$$\frac{\lambda_1}{1+\lambda^2_1}h_{n+1,p1} + \frac{\lambda_2}{1+\lambda^2_2}h_{n+2,p2} = 0 \quad (5.9)$$

for any $p$.

By (5.3), at $(x_0, t_0)$, we have

$$\left(\frac{d}{dt} - \Delta\right)f = 2\eta + 2R_{k1\alpha}S_{\alpha 1} + 2R_{k2\alpha}S_{\alpha 2} + 2h_{akj}h_{ak1}S_{j1} + 2h_{akj}h_{ak2}S_{j2} - 2h_{ak1}h_{\beta k1}S_{\alpha \beta} - 2h_{ak2}h_{\beta k2}S_{\alpha \beta}. \quad (5.10)$$

The ambient curvature term can be calculated using Lemma 4.2 and we derive

$$\sum_{k,\alpha} R_{k1\alpha}S_{\alpha 1} + R_{k2\alpha}S_{\alpha 2}. $$

$$= (k_1 - k_2)(n - 1)\sum_{i=1}^{2} \frac{\lambda^2_i}{(1 + \lambda^2_i)^2} + (k_1 + k_2)\sum_{i=1}^{2} \frac{\lambda^2_i}{(1 + \lambda^2_i)^2} \left[\sum_{j \neq i} \frac{1 - \lambda^2_j}{(1 + \lambda^2_j)^2}\right]. \quad (5.11)$$
This can be simplified as

\[
(k_1 - k_2)(n - 1) \sum_{i=1}^{2} \frac{\lambda_i^2}{(1 + \lambda_i^2)^2} + (k_1 + k_2) \sum_{i=1}^{2} \frac{\lambda_i^2}{(1 + \lambda_i^2)^2} \left[ \sum_{j>3} \frac{1 - \lambda_j^2}{(1 + \lambda_j^2)} \right]
\]

\[
+ (k_1 + k_2) \left[ \frac{\lambda_1^2}{(1 + \lambda_1^2)^2} \frac{1 - \lambda_2^2}{(1 + \lambda_2^2)^2} + \frac{\lambda_2^2}{(1 + \lambda_2^2)^2} \frac{1 - \lambda_1^2}{(1 + \lambda_1^2)^2} \right]
\]

\[
= (k_1 - k_2)(n - 1) \sum_{i=1}^{2} \frac{\lambda_i^2}{(1 + \lambda_i^2)^2} + (k_1 + k_2) \sum_{i=1}^{2} \frac{\lambda_i^2}{(1 + \lambda_i^2)^2} \left[ \sum_{j>3} \frac{1 - \lambda_j^2}{(1 + \lambda_j^2)} \right]
\]

\[
+ (k_1 + k_2) \left[ \frac{(\lambda_1^2 + \lambda_2^2)(1 - \lambda_1^2 \lambda_2^2)}{(1 + \lambda_1^2)(1 + \lambda_2^2)^2} \right].
\]

\[(5.12)\]

This is nonnegative by equation (5.8).

Using the relations in (4.6) again, the last four terms on the right hand side of (5.10) can be rewritten as

\[
\sum_{p,k} 2h_{n+p,k1}^2 S_{11} + 2h_{n+p,k2}^2 S_{22} + 2h_{n+p,k1}^2 S_{pp} + 2h_{n+p,k2}^2 S_{pp}
\]

\[
= \sum_{k} (2h_{n+k1}^2 S_{11} + 2h_{n+k2}^2 S_{22} + 2h_{n+k1}^2 S_{pp} + 2h_{n+k2}^2 S_{pp})
\]

\[
+ 2h_{n+k1}^2 S_{11} + 2h_{n+k2}^2 S_{22} + 2h_{n+k1}^2 S_{11} + 2h_{n+k2}^2 S_{22}
\]

\[
+ \sum_{q \geq 3,k} 2h_{n+q,k1}^2 S_{11} + 2h_{n+q,k2}^2 S_{22} + 2h_{n+q,k1}^2 S_{pp} + 2h_{n+q,k2}^2 S_{pp}.
\]

\[(5.13)\]

Since \(S_{ii} \geq S_{11}\) for \(i \geq 2\), it is clear that (5.13) is nonnegative if \(S_{11} \geq 0\). Otherwise, from (5.7), we may assume that

\[
S_{11} < 0, \quad S_{22} > 0 \text{ and } S_{11} + S_{22} > 0.
\]

\[(5.14)\]

In particular, we have \(\lambda_2^2 < \lambda_1^2\) and \(\lambda_1^2 \lambda_2^2 < 1\). From (5.9), we have

\[
h_{n+1,p1}^2 = \frac{\lambda_1^2(1 + \lambda_2^2)^2}{\lambda_1^2(1 + \lambda_2^2)^2} h_{n+2,p2}^2.
\]

Since \(\lambda_2^2 < \lambda_1^2\) and \(\lambda_1^2 \lambda_2^2 < 1\), we have \(\frac{\lambda_2^2(1 + \lambda_2^2)^2}{\lambda_1^2(1 + \lambda_2^2)^2} < 1\). Thus

\[
h_{n+1,p1}^2 \leq h_{n+2,p2}^2 \text{ for all } p \geq 1.
\]

\[(5.15)\]
Recall that $S_{qq} \geq S_{22}$ for $q \geq 3$. The right hand side of (5.13) can be regrouped as

$$\sum_k \left[ (4h_{n+1,k1}^2S_{11} + 4h_{n+2,k2}^2S_{22}) + 2h_{n+2,k1}^2(S_{11} + S_{22}) + 2h_{n+1,k2}^2(S_{11} + S_{22}) \right] + \sum_{q \geq 3,k} \left[ 2h_{n+q,k1}^2(S_{11} + S_{qq}) + 2h_{n+q,k2}^2(S_{22} + S_{qq}) \right].$$

This is nonnegative by (5.5), (5.14), and (5.15). Thus, we have $$(d/dt - \Delta)f \geq 2\eta > 0$$ at $(x_0, t_0)$ and this is a contradiction. \hfill \Box

Remark: The condition $S_{[2]} - \epsilon g^{[2]} \geq 0$ is equivalent to 

$$\left(1 - \lambda_i^2 \lambda_j \right) \left(1 + \lambda_i^2 \right) \left(1 + \lambda_j^2 \right) \geq \epsilon$$

for all $i \neq j$. In particular, we have $\lambda_i^2 \leq \frac{1-\epsilon}{\epsilon}$. This implies that the Lipschitz norm of $f$ is preserved along the mean curvature flow.

### 6 Long time existence and convergence

In this section, we prove Theorem A using the evolution equation (3.4) of $\ln \ast \Omega$.

**Proof of Theorem A.** Since $|\lambda_i \lambda_j| < 1$ for $i \neq j$ and $\Sigma_1$ is compact, we can find an $\epsilon > 0$ such that $\frac{(1-\lambda_i^2 \lambda_j^2)}{(1+\lambda_i^2)(1+\lambda_j^2)} \geq \epsilon$ for all $i \neq j$. By Lemma (5.3), the condition $\frac{(1-\lambda_i^2 \lambda_j^2)}{(1+\lambda_i^2)(1+\lambda_j^2)} \geq \epsilon$ for all $i \neq j$ is preserved along the mean curvature flow. In particular, we have $|\lambda_i \lambda_j| \leq \sqrt{1-\epsilon}$ and $\lambda_i^2 \leq \frac{1-\epsilon}{\epsilon}$. This implies $\Sigma_t$ remains the graph of a map $f_t : \Sigma_1 \to \Sigma_2$ whenever the flow exists. Each $f_t$ has uniformly bounded $|df_t|$.

We look at the evolution equation (3.4) of $\ln \ast \Omega$. The quadratic terms of the second fundamental form in equation (3.4) is

$$\sum h_{\alpha \gamma k}^2 + \sum \lambda_i^2 h_{n+i,ik}^2 + 2 \sum \lambda_i \lambda_j h_{n+j,ijk} h_{n+i,jk} = \delta |A|^2 + \sum \lambda_i^2 h_{n+i,ik}^2 + (1 - \delta)|A|^2 + 2 \sum \lambda_i \lambda_j h_{n+j,ijk} h_{n+i,jk}.$$

Let $1 - \delta = \sqrt{1-\epsilon}$. Using $|\lambda_i \lambda_j| \leq 1 - \delta$, we conclude that this term is bounded below by $\delta |A|^2$. 

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By equation (4.11), the curvature term in (3.4) equals

\[
\frac{(k_1 - k_2)(n - 1)}{2} \sum_{i=1}^{n} \frac{\lambda_i^2}{1 + \lambda_i^2} + (k_1 + k_2) \sum_{i=1}^{n} \frac{\lambda_i^2}{1 + \lambda_i^2} \left[ \sum_{j \neq i}^{n} \frac{1 - \lambda_j^2}{2(1 + \lambda_j^2)} \right]. \tag{6.1}
\]

The second term on the right hand side of (6.1) can be simplified as

\[
\sum_{i=1}^{n} \frac{\lambda_i^2}{1 + \lambda_i^2} \left[ \sum_{j \neq i}^{n} \frac{1 - \lambda_j^2}{2(1 + \lambda_j^2)} \right] = \sum_{i=1}^{n} \sum_{j \neq i}^{n} \frac{\lambda_i^2 - \lambda_i^2 \lambda_j^2}{2(1 + \lambda_i^2)(1 + \lambda_j^2)} \tag{6.2}
\]

This is non-negative because \(|\lambda_i \lambda_j| \leq 1 - \delta\). Thus \(\ln \ast \Omega\) satisfies the following differential inequality with \(k_1 \geq |k_2|\):

\[
\frac{d}{dt} \ln \ast \Omega \geq \Delta \ln \ast \Omega + \delta|A|^2. \tag{6.3}
\]

According to the maximum principle for parabolic equations, \(\min_{\Sigma^t} \ln \ast \Omega\) is nondecreasing in time. In particular, \(\ast \Omega \geq \min_{\Sigma^t} \ast \Omega = \Omega_0\) is preserved and \(\ast \Omega\) has a positive lower bound. Let \(u = \frac{\ln \ast \Omega - \ln \Omega_0 + c}{-\ln \Omega_0 + c}\) where \(c\) is a positive number such that \(-\ln \Omega_0 + c > 0\). Recall that \(0 < \ast \Omega \leq 1\). This implies that \(0 < u \leq 1\) and \(u\) satisfies the following differential inequality

\[
\frac{d}{dt} u \geq \Delta u + \frac{\delta}{-\ln \Omega_0 + c}|A|^2.
\]

Because \(u\) is also invariant under parabolic dilation, it follows from the blow-up analysis in the proof of Theorem A [19] that the mean curvature flow of the graph of \(f\) remains a graph and exists for all time under the assumption that \(k_1 \geq |k_2|\).

Using \(\lambda_i^2 \leq \frac{1-\epsilon}{\epsilon}\) and \(\lambda_i \lambda_j \leq \sqrt{1 - \epsilon}\), it is not hard to show

\[
(k_1 + k_2) \sum_{i<j} \frac{\lambda_i^2 + \lambda_j^2 - 2\lambda_i^2 \lambda_j^2}{2(1 + \lambda_i^2)(1 + \lambda_j^2)} \geq c_1 \sum_{i=1}^{n} \lambda_i^2 \geq c_1 \ln \prod_{i=1}^{n} (1 + \lambda_i^2) \tag{6.4}
\]

where \(c_1\) is a constant that depends on \(\epsilon, k_1\) and \(k_2\).
Recall equation (3.3) and we obtain

$$\frac{d}{dt} \ln *\Omega \geq \Delta \ln *\Omega - c_3 \ln *\Omega.$$ 

By the comparison theorem for parabolic equations, $\min_{\Sigma_t} \ln *\Omega$ is non-decreasing in $t$ and $\min_{\Sigma_t} \ln *\Omega \to 0$ as $t \to \infty$. This implies that $\min_{\Sigma_t} *\Omega \to 1$ and $\max |\lambda_i| \to 0$ as $t \to \infty$. We can then apply Theorem B in [19] to conclude smooth convergence to a constant map at infinity.

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