Abstract

Systems of polynomial equations arise frequently in computer vision, especially in multiview geometry problems. Traditional methods for solving these systems typically aim to eliminate variables to reach a univariate polynomial, e.g., a tenth-order polynomial for 5-point pose estimation, using clever manipulations, or more generally using Grobner basis, resultants, and elimination templates, leading to successful algorithms for multiview geometry and other problems. However, these methods do not work when the problem is complex and when they do, they face efficiency and stability issues. Homotopy Continuation (HC) can solve more complex problems without the stability issues, and with guarantees of a global solution, but they are known to be slow. In this paper we show that HC can be parallelized on a GPU, showing significant speedups up to 26 times on polynomial benchmarks. We also show that GPU-HC can be generically applied to a range of computer vision problems, including 4-view triangulation and trifocal pose estimation with unknown focal length, which cannot be solved with elimination template but they can be efficiently solved with HC. GPU-HC opens the door to easy formulation and solution of a range of computer vision problems.

1 Introduction

Systems of polynomial equations arise frequently in computer vision, especially in multiview geometry problems, because perspective projection is an algebraic model. Examples abound including absolute pose estimation [38, 97, 5], relative pose estimation [79, 91, 56], pose estimation with unknown focal length [16], homography estimation [61, 14], PnP [99, 100], 3-View triangulation [18], pose estimation with unknown principal point [63], rolling shutter camera absolute pose estimation [4], as well as many others. The challenge has been how to solve these polynomial systems efficiently and in a stable way.

The classic 5-point algorithm for relative pose estimation [83, 79] is a case in point. Its formulation begins with 15 equations in 15 unknowns, namely, 10 depths and 5 pose parameters. The traditional approach is to eliminate depths and end up with the epipolar equation which with 5 points results in a 10th-degree univariate polynomial from which pose is determined. A more formal approach to eliminating variables is the Gröbner basis [23, 24] or resultants [23, 24]. Elimination Templates were developed as an automatic solver generator [62] where the Gröbner-based elimination strategy obtained from one input is “remembered” for future inputs. These methods are reviewed in Section 2.

The challenge with the above methods is that they are limited to problems with small number of solutions. They are slow for larger problems whose elimination template can be computed. For even larger problems the computation of elimination template exceeds practical resources, rendering the problem unsolvable. In addition, stability issues might arise in the process of converting a system of polynomials to a single univariate polynomial, e.g., [75, 74].

Homotopy Continuation methods, in contrast, can solve very complex polynomial systems. The basic idea is to find all the solutions of a start system and then to continuously evolve them to the solutions of the target system. They can ensure, with probability 1, to find all solutions [89, 96], provided a “good” starting system. They also avoid the stability issues of symbolic methods as they do not manipulate the input polynomials. Their complexity depends on the number of solutions (tracks) they follow. In this lies the idea to use a GPU to speed up the computation.

GPUs have been used in computer graphics and computer vision to accelerate massively parallel operations. The key
is whether HC can be parallelized to take advantage of many processor in a GPU while avoiding data transfer delays. The HC process consists of prediction and correction steps in the continuation from the start system to the target system. This is done by computing the Jacobian to predict where to go next, and subsequently Newton’s method to correct the solution. We show that by parallelizing the computations in the prediction and correction steps, a track can be implemented on a warp. This is made possible in part by instituting kernel fusion in the MAGMA library for solving batch linear systems. In addition, an indexing system homogenizes the expressions of Jacobian and the two vectors involved so their evaluations can be parallelized. The resulting GPU-HC can be generically applied to systems of up to 32 equations by 32 unknown and speedups of up to 26 times on polynomial benchmarks.

Computer vision problems involving polynomial systems fit these requirements. We have applied GPU-HC to a variety of problems, and found that for moderately complex systems and beyond GPU-HC offers significant savings (with implied stability). We have also explored solutions to two problems, namely, 4-view triangulation and trifocal pose estimation with unknown focal length which have not been explored in the literature. These are introduced as example cases where elimination template fails to produce solutions but GPU-HC solves efficiently. The basic thesis of this paper is that GPU-HC can be applied to all computer vision problems that can be formulated as polynomial systems and produce efficient and stable solutions.

2 Methods for Solving Polynomial Systems

We partition the algorithms for solving systems of polynomial equations in roughly three categories: (i) **Symbolic methods** that rely on algebraic elimination tools, such as Gröbner basis, resultant, etc.; (ii) **Numerical solvers** that are iterative and are generally a variant of Newton’s method, such as homotopy continuation, and/or rely on eigenvalue computations; and, (iii) **Hybrid methods** that combine the benefits of the symbolic and numerical solvers such as elimination templates or subdivision solvers.

**Symbolic solvers** “transform”, using algebraic elimination, the multivariate polynomial system to a univariate polynomial. The roots of this polynomial are computed using dedicated algorithms, like Sturm or Descartes, and are used to recover the system solutions, e.g., [23, 24, 87, 30]. These algorithms mainly rely on exact computations with rational numbers and partially on computations in finite fields. They perform elimination using well-known tools from computational algebraic geometry, such as Gröbner basis and resultants. Gröbner basis manipulate the polynomials “incrementally” (like Gaussian elimination) to deduce the univariate polynomial, while resultants use all the polynomials right from the beginning (similar to Cramer’s rule).

Symbolic methods are used widely in solving minimal problems in computer vision [52, 51, 34, 90, 44]. They always output the exact results with certifications. They deal successfully and rather efficiently with degeneracies such as multiple roots. The efficient implementation of symbolic algorithms is far from a straightforward task; various sub-algorithms must be fine-tuned, implemented, and extended experimentation is needed. However, despite the tremendous recent progress in this direction, systems of more than 5-6 variables of moderate degrees cannot be handled, except if sparsity and the structure is specifically exploited. Even more, we are still very far from having symbolic solvers that solve moderate systems in milliseconds.

Another major issue with symbolic solvers, especially Gröbner basis, is that they are numerically unstable [53, 74]. This is mainly due to their requirement for a term-ordering that causes instability when the coefficients of the input polynomials are floating point numbers or known up to some precision. Nevertheless, there are efforts to overcome this obstacle using a variant called border basis, e.g., [75]. The same phenomena appear in the resultant computations [80], where there also recent efforts for improvements [15].

**Numerical solvers** are almost exclusively iterative algorithms that exploit a variant of Newton operator and they perform their computations in floating point arithmetic, e.g., [10, 89, 96]. There are also approaches based on numerical linear algebra techniques, mainly on eigenvalue computations e.g., [13, 17]. The most prominent representatives are the Homotopy Continuation (HC) algorithms [7, 9, 21, 45, 96, 42]. They rely on the simple and elegant idea to initially solve a simpler polynomial system (start system) and then deform its roots to the roots of the system we want to solve (target system). Some care is required on choosing an easy-to-solve start system that has at least as many solutions as the target system. They can handle very big problems, especially in the absence of degeneracies, say multiple roots. These solvers are highly efficient in practice and able to handle systems that are out of the reach of symbolic solvers. Nevertheless, they are still comparatively slow, a serious bottleneck to their wide adoption. Their potential for parallelization is a key focus of this paper. HC is used widely in computer vision, especially for minimal problems in multiview geometry [54, 84, 72, 28, 32, 27].

Numerical problems might also occur in HC algorithms, especially if the Jacobian of the system is ill-conditioned and in the many cases we need to use double-precision floating point arithmetic, e.g., [10]. However, HC is an inherit numerical method and does not require an exact input.

Also sometimes it is not easy, if possible at all, to find good, let alone optimal, start systems, the cardinality of the output is not always correct, and extra verification steps are needed. They are in general easier to implement than symbolic methods, even though in all the cases efficient scientific software requires tremendous amount of time, energy, and effort to be efficient and solve real life problems.

**Hybrid solvers** aim to combine the symbolic and numerical approaches e.g. [29, 74, 71], and they have various algorithm-
mic variants. A well-known method in the computer vision community is the “elimination template”, or automatic solver generation [57, 58, 64, 68, 62]. The main idea is to book-keep the steps that an elimination (usually Gröbner basis) algorithm performs for one input and apply these steps to any other input. They generate a “template” of elimination at an offline stage with the random coefficients of a “dummy” system on a finite field. We obtain the solutions by eigenvalue computations or dedicated algorithms. The method is particularly fast for solving systems with low degree and low number of variables [85, 20, 98, 65, 4, 25].

Nevertheless, even though they have turned out to be quite successful in some problems, they cannot always guarantee their result, they might also need to handle very large matrices [62] which are computationally intractable, and, last but not least, it is far from trivial to analyze their stability. The hybrid approaches based on elimination template method try to overcome the instability of the symbolic methods by performing several pre-computations. However, at the end they also must compute with a matrix, similar to Gröbner basis and resultants, which has a dimension at least the number of complex solutions. The condition number of such matrix is not well, if at all, studied, and it is not clear if they can handle problems with \( \geq 300 \) roots.

3 Homotopy Continuation

The idea of Homotopy Continuation (HC) [73, 89] is to evolve the solutions of one polynomial system \( G \), the “start system”, to discover the solutions of another system \( F \). Let \( X = (x_1, x_2, \ldots, x_M) \) represent \( M \) unknowns. Let \( F(X) \) be a system of \( N \) polynomial equations \( F = (f_1, f_2, \ldots, f_N) \); this is the “target system” we want to solve. Let \( G(x) \), \( G = (g_1, g_2, \ldots, g_N) \) be the “start system” whose solutions are all known. The idea of HC is to construct a series of intermediate polynomial systems \( H(X, t), H = (H_1, H_2, \ldots, H_N) \); where \( H(X, 0) = G(X) \) and \( H(X, 1) = F(X) \), e.g., via linear interpolation:

\[
H(x, t) = (1 - t)G(x) + tF(x), \quad t \in [0, 1]. \tag{1}
\]

The basic idea is to find the solution of \( H(X, t + \Delta t) \) from the solution of \( H(X, t) \). Figure 1 illustrates the idea for one solution and one unknown. The black curve is the locus of the solution \( X(t) \) of \( H(X, t) \), the homotopy curve, where \( X_0 \) is the known solution of \( G(X) \) and \( X_1 \) is the desired solution of \( F(X) \). We track solution \( X_1 \) from \( X_0 \) in a number of small steps, each consisting of a prediction and a correction step. Prediction uses a first-order Taylor expansion to estimate \( X \) at \( t + \Delta t \) in the form of

\[
X^*(t + \Delta t) = X(t) + \frac{dX}{dt} \Delta t, \tag{2}
\]

where \( X^* \) is the first order estimation of \( X(t + \Delta t) \). We obtain \( \frac{dX}{dt} \) by differentiating \( H(X(t), t) \), i.e.,

\[
\frac{\partial H}{\partial X} \frac{dX}{dt} + \frac{\partial H}{\partial t} = 0 \rightarrow \frac{dX}{dt} = -\left( \frac{\partial H}{\partial X} \right)^{-1} \frac{\partial H}{\partial t}, \tag{3}
\]

This step, the first-order estimation of \( X^* \) from \( X(t) \), is known as the prediction step (Figure 1). However, we can improve the prediction using a higher-order method like a 4th order Runge-Kutta; alas, we require a correction. Using Newton we update \( X^*(t + \Delta t) \) to \( \hat{X}(t + \Delta t) \), i.e.,

\[
H(X^*, t + \Delta t) + \frac{\partial H}{\partial X}(X^*, t + \Delta t)(\hat{X} - X^*) = 0, \tag{4}
\]

giving the estimate \( \hat{X} \) in the form of

\[
\hat{X} = X^* - \left( \frac{\partial H}{\partial X} \right)^{-1}(X^*, t + \Delta t)H(X^*, t + \Delta t). \tag{5}
\]

This is the correction step. The pairs of prediction and correction steps numerically evolve \( X_0 \) as the solution of \( G(X) \) to \( X_1 \) as the solution of \( F(X) \).

Provided that we have a good started system the HC algorithms find all the solutions (up to some approximation) with probability one. Even more, there are methods, alas much slower, that can guarantee that we follow accurately the tracks [12] and/or certify the solutions [43].

4 Illustrative Problems

Preliminaries: Let \( \Gamma \) denote a 3D point which projects to an image point \( \gamma = (\xi, \eta, \gamma)^T \) with depth \( \rho \) so that \( \Gamma = \rho \gamma \). The expression of \( \Gamma \) in a camera related by pose \((R,T)\) to another camera where \( R \) is the rotation matrix and \( T \) is translation, is \( \hat{\Gamma} = RT + T \). Due to metric ambiguity the unit direction \( T \) along \( T \) in sought, where \( T = \lambda T' \).

Relative Pose Estimation with Calibrated Cameras is a classic problem most frequently solved by Nister’s 5-point algorithm [78, 79, 83]. Consider five corresponding points \((\gamma_{i}, \bar{\gamma}_{i})\) where \( \gamma_{i} \) in one image is in correspondence with \( \bar{\gamma}_{i} \) in the second image. Since \( \Gamma_{i} = \rho_{i} \gamma_{i}, \hat{\Gamma}_{i} = \rho_{i} \bar{\gamma}_{i} \), and \( \hat{\Gamma}_{i} = \rho_{i} \Gamma_{i} + T \). The relationship between \( \gamma_{i} \) and \( \bar{\gamma}_{i} \) is captured as

\[
\hat{\rho}_{i} \bar{\gamma}_{i} = R \rho_{i} \gamma_{i} + \hat{T}, \quad i = 1, 2, \cdots, N, \tag{7}
\]

where the depths \((\rho_{i}, \rho_{i})\) represent 10 unknowns and \((R, \hat{T}) \) represent 5 unknowns. The above set of five vector equations give 15 constraints in 15 unknowns. Representing \( R \)
with quaternions which involves 4 unknowns with one equation yields 16 polynomial equations in 16 unknowns. Observe that there has been no attempts in the literature to solve these equations, which HC can solve, referred to as relative pose estimation + depth reconstruction in Table 2. Rather, the traditional approach is to reduce the number of unknowns by eliminating the ten depth variables by taking cross product of Equation 7 with \( \tilde{T} \) and then dot product with \( \tilde{\gamma}_i \) giving the classical epipolar relationship

\[
\tilde{\gamma}_i^T E \gamma_i = 0, \quad i = 1, 2, \ldots, 5. \tag{8}
\]

where \( E = [\tilde{T}] \times R \). While this is now 5 equations in 5 unknowns \((R,T)\), these now involve trigonometric equation unless \( R \) is represented with a quaternion giving 6 polynomial equations in 6 unknowns. Again, this can also be solved by HC. Nevertheless the classic approach is to treat \( E \) as nine unknowns and use a Theorem [78] that \( E = [\tilde{T}] \times R \) if and only if

\[
2EE^T E - \text{trace}(EE^T)E = 0. \tag{9}
\]

These are 9 cubic polynomial equations but only four are independent which can be used in conjunction with Eq. 8 to solve for \( E \). Namely, \( E \) is written in vector form as \( \tilde{E} \),

\[
\begin{align*}
\tilde{E}^T &= [E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33}] \\
\tilde{w}^T \tilde{E} &= 0, \quad i = 1, 2, \ldots, 5, \\
\tilde{w}^T &= [\tilde{\xi}_1, \tilde{\eta}_1, \tilde{\zeta}_1, \tilde{\xi}_2, \tilde{\eta}_2, \tilde{\zeta}_2, \tilde{\xi}_3, \tilde{\eta}_3, \tilde{\zeta}_3].
\end{align*} \tag{10}
\]

\( \tilde{E} \) is then an arbitrary linear sum of the four matrices representing the right nullspace, \( \tilde{E} = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + E_4 \), where the last constant \( \alpha_4 \) is set to one due to the scale invariance of \( E \). The only remaining constraint is the set of nine cubic Equations 9, where the unknowns \((\alpha_1, \alpha_2, \alpha_3)\) involve 20 monomials up to order 3 of \((\alpha_1, \alpha_2, \alpha_3)\), so that they can be expressed as a \( 9 \times 20 \) matrix multiplied by a vector of 20 monomials. The idea is to eliminate all monomials except those involving one variable, say \( \alpha_3 \). This can be done by Gauss-Jordan elimination with partial pivoting to make an upper triangular matrix, and after additional manipulations, which are effectively hand-derived Gröbner basis, leads to a single tenth-order polynomial in one variable \( \alpha_3 \) which gives 10 roots. The real roots of \( \alpha_3 \) then can solve for \( \alpha_1, \alpha_2 \) and \( E \) from which \( R \) and \( T \) can be recovered.

Li and Hartley [69] solve Equation 9 with \( \tilde{E} \) as described by Equation 10 using the hidden variable technique, a resultant technique for algebraic elimination [23]. They include \( \text{det}(E) = 0 \) as a tenth equation and solve equating the determinant of the \( 10 \times 10 \) matrix to zero as a function of \( \alpha_3 \), a tenth-order polynomial which can then be solved. The claimed advantage of this technique over Nister’s is its simplicity and ease of implementation.

Observe that both approaches devise ingenious algorithms to turn the basic system of polynomial Equation 7 into a single 10th degree uni-variate polynomial. In contrast, Homotopy Continuation can be used immediately to solve as \( 16 \times 16 \) polynomial system or the reduced \( 6 \times 6 \) of system of Equation 9 avoiding the need for devising such algorithms. Finally, HC can be used to solve \((\alpha_1, \alpha_2, \alpha_3)\) using a \( 3 \times 3 \) system of cubic polynomials. Note that we are not advocating to solve the relative pose using HC (the system is too small to benefit from it). Rather, we are noting that it can be solved by HC as an illustration.

**Perspective-n-Point problem (PnP)** estimates the pose of a calibrated camera \((R,T)\) using \( n \) correspondences between 3D world coordinate points \( \Gamma \) and their 2D projections in the image \( \gamma \) (known as space resection in photogrammetry). The P3P problem where 3D points \((\Gamma_1, \Gamma_2, \Gamma_3)\) correspond to 2D image points \((\gamma_1, \gamma_2, \gamma_3)\), respectively, has a long history [36, 35, 39, 86] and it has 4 solutions requiring a 4th correspondence to disambiguate.

The basic formulation can be posed using \( \Gamma_i = \rho_i \gamma_i \) as

\[
\begin{align*}
\Gamma_1 &= \rho_1 R \gamma_1 + T \\
\Gamma_2 &= \rho_2 R \gamma_2 + T \\
\Gamma_3 &= \rho_3 R \gamma_3 + T
\end{align*} \tag{11}
\]

a set of nine equations in the nine unknowns. At this point where the formulation is completed, HC can be used to solve for \((R,T)\), as well as depths! Using a quaternion representation of \( R \) which involves 4 unknowns and one equation, this becomes a set of \( 10 \times 10 \) polynomials with 10 unknowns. The traditional approach eliminates \( R \) and \( T \) to solve depth from

\[
\begin{align*}
(\Gamma_2 - \Gamma_1)^T (\Gamma_2 - \Gamma_1) &= (\rho_2 \gamma_2 - \rho_1 \gamma_1)(T) (\rho_2 \gamma_2 - \rho_1 \gamma_1) \\
(\Gamma_3 - \Gamma_1)^T (\Gamma_3 - \Gamma_1) &= (\rho_3 \gamma_3 - \rho_1 \gamma_1)(T) (\rho_3 \gamma_3 - \rho_1 \gamma_1), \\
(\Gamma_2 - \Gamma_1)^T (\Gamma_3 - \Gamma_1) &= (\rho_2 \gamma_2 - \rho_1 \gamma_1)(T) (\rho_3 \gamma_3 - \rho_1 \gamma_1) - (\rho_2 \gamma_2 - \rho_1 \gamma_1)(T) (\rho_3 \gamma_3 - \rho_1 \gamma_1)
\end{align*} \tag{12}
\]

a set of three quadratic in three unknowns \((\rho_1, \rho_2, \rho_3)\). Again, this reduced form can be easily solved by HC, but the traditional approach is to apply Silvester’s resultant to get an 8th degree polynomial, containing even terms so that it is effectively a quartic [86].

The general PnP problem relies on \( n \) correspondences between 3D points \( \Gamma \) and 2D image points \( \gamma_i \), \( i = 1, 2, \ldots, n \). A direct minimization of the algebraic reconstruction error [99] uses a non-unit quaternion representing of \( R \) and explicitly optimize for \( R \). This gives four polynomials of degree three in four variables, which are solved by Gröbner bases, from which an elimination template is constructed using the automatic generator in [57]. This gives at most 81 solutions with an 575x656 elimination template and 81x81 action matrix. Alternatively, these equations can be solved using HC without any further processing with about a factor of 5 times speedup on a GPU, Table 2. In this larger case, HC features both simplicity and efficiency.

**N-view Triangulation** aims to find the 3D world point \( \Gamma \) that is most consistent with a set of projection, \( \gamma_1, \ldots, \gamma_N \) from \( N \) views, given relative pose of all cameras in the form of the pairwise essential matrix \( E_{ij} \) between views \( i \) and \( j \). Due to noise, the projection rays from corresponding points do not necessarily meet in space. For two cameras, the mid-point between the closest points on the projection rays is used [11]. But this can have a large error, especially with large calibration error. Rather then minimizing the latent 3D error, the reprojection error can be minimized [40, 41, 49]. Let
\[ \gamma_i = \hat{\gamma}_i + \Delta \gamma_i \quad \text{where} \quad \hat{\gamma}_i \text{ is the true 2D observation and} \ \Delta \gamma_i \text{ is the error introduced by noise, i.e.,} \]
\[ \hat{\gamma}_j^T E_{ij} \hat{\gamma}_i = 0, \quad (\gamma_j - \Delta \gamma_j)^T E_{ij} (\gamma_i - \Delta \gamma_i) = 0. \quad (13) \]

Minimizing reprojection errors \( \Delta \gamma_i \) and \( \Delta \gamma_j \) subject to this constraint solves the optimal estimate

\[
(\Delta \gamma_1^*, \Delta \gamma_2^*) = \arg \min_{(\gamma_j - \Delta \gamma_j)^T E_{ij} (\gamma_i - \Delta \gamma_i)=0} (||\Delta \gamma_1||^2 + ||\Delta \gamma_2||^2).
\]

Using Lagrange multipliers and notation \( \Delta \gamma_i^* = (u_i, v_i, 0) \) the problem becomes

\[
(u_i^*, v_i^*, u_j^*, v_j^*, \lambda^*) = \arg \min_{u_i, v_i, u_j, v_j, \lambda} (u_i^2 + v_i^2 + u_j^2 + v_j^2 + \lambda (\gamma_j^T - [u_j v_j 0]) E_{ij} (\gamma_i - [u_i v_i 0])^2).
\]

This can be solved by differentiating with respect to the five variables and setting to zero. Specifically,

\[
\begin{align*}
2u_i - \lambda (\gamma_j^T - [u_j v_j 0]) E_{ij} [1 0 0]^T = 0 \\
2v_i - \lambda (\gamma_j^T - [u_j v_j 0]) E_{ij} [0 1 0]^T = 0 \\
2u_j - \lambda [1 0 0]^T E_{ij} (\gamma_j^T - [u_j v_j 0])^T = 0 . \\
2v_j - \lambda [1 0 0]^T E_{ij} (\gamma_j^T - [u_j v_j 0])^T = 0 \\
(\gamma_j^T - [u_j v_j 0]) E_{ij} (\gamma_i - [u_i v_i 0])^T = 0
\end{align*}
\]

(14)

This is a set of five multi-linear polynomial equations in five unknowns. Setting the first derivative with respect to the five variables gives a \( 5 \times 5 \) polynomial system. This system can be solved using HC without any further effort. Traditionally, however, the system is solved by eliminating four of five variables, gives a single 6-th order polynomial [41]. This gives excellent results but it is slow prompting [49, 70] to use an iterative method which is faster but is prone to being stuck in local minima.

The N-view triangulation is not as well-explored despite the formulation of minimizing reprojection error is identical

\[
(\Delta \gamma_1^*, \Delta \gamma_2^*, \ldots, \Delta \gamma_N^*) = \arg \min_{\Delta \gamma_1, \Delta \gamma_2, \ldots, \Delta \gamma_N \text{ such that} \forall i, j (\gamma_j - \Delta \gamma_j)^T E_{ij} (\gamma_i - \Delta \gamma_i) = 0}
\]

or

\[
(u_1^*, v_1^*, u_2^*, v_2^*, \ldots, u_N^*, v_N^*, \lambda^*) = \arg \min_{u_1, v_1, u_2, v_2, \ldots, u_N, v_N, \lambda} \sum_{k=1}^{N} (u_k^2 + v_k^2) + \sum_{i=1}^{N} \sum_{j=i+1}^{N} \lambda_k (\gamma_j^T - [u_j v_j 0]) E_{ij} (\gamma_i - [u_i v_i 0]),
\]

where \( \Lambda = \{\lambda_k | i = 1, 2, \ldots, N, j = i + 1, \ldots, N\} \). Note that there are \( 2N + \frac{N(N-1)}{2} = \frac{N^2 + 3N}{2} \) unknowns and setting first derivatives to zero gives \( 5 \times 5, 9 \times 9 \) and \( 14 \times 14 \), for two, three, and four views, respectively, becoming exponentially more difficult to use with Gröbner basis and other traditional methods. [60] restricts the consideration of all sequential pairwise essential matrices to these with the previous view, i.e., \( E_{12}, E_{23}, \text{etc.} \) and uses the elimination template method with a \( 274 \times 305 \) template to reduce the size of the elimination template to \( 239 \times 290 \). Note that the full problem gives an elimination template of \( 1866 \times 1975 \) which is not practical to solve. Similarly, the 4-view triangulation gives improvederror but it leads to large polynomial systems which are simply impractical to solve. Homotopy Continuation, however, can solve these problems and with improved efficiency, Table 2.

**Trifocal relative pose estimation with unknown focal length** aims to estimate the relative pose between three views as well as the focal length. Trifocal pose estimation has drawn more attention recently [67, 19, 48, 50, 33]. These approaches often assume that the intrinsic camera calibration is available. Recently, [66] estimates trifocal tensor with radial distortion, a minimal problem of one pinhole camera and two radial cameras. We now consider another minimal problem with three pinhole cameras with common unknown focal length. This minimal problem needs only 4 points correspondences across three views. Let the calibration matrix be \( K = \text{diag}(f,f,1) \), where \( f \) is the focal length. Consider three corresponding points \( (\gamma_1, \gamma_2, \gamma_3) \) in image \( (1,2,3) \), respectively, with unknown depth \( (\rho_1, \rho_2, \rho_3) \) respectively. Then, denoting \( (R_{12}, T_{12}) \) and \( (R_{13}, T_{13}) \) the relative pose of the second and third cameras with respect to the first, respectively, we have

\[
\begin{align*}
\rho_2 K^{-1} \gamma_2 &= \rho_1 K^{-1} R_{12} \gamma_1 + \hat{T}_{12} \\
\rho_3 K^{-1} \gamma_3 &= \rho_1 K^{-1} R_{13} \gamma_1 + \hat{T}_{13},
\end{align*}
\]

(17)

where \( \hat{T}_{12} \) is taken to have unit length. Thus, there are 11 poses and 3 depth unknowns. Since there are four sets of correspondences, there are in total one focal length, 11 pose and 12 depth unknowns, for a total of 24 unknowns. There are also four sets of vector Equation 17 which each gives 6 equations. If \( R_{12} \) and \( R_{13} \) are represented by quaternions there are a total of 26 equation in 26 unknowns which can be solve by HC.

Alternatively, \( \rho_2 \) and \( \rho_3 \) can be written in terms of \( \rho_1 \) as

\[
\begin{align*}
\rho_2 e_{\frac{\pi}{2}}^T K^{-1} \gamma_2 &= e_{\frac{\pi}{2}}^T \rho_1 K^{-1} R_{12} \gamma_1 + e_{\frac{\pi}{2}}^T K^{-1} T_{12} \\
\rho_3 e_{\frac{\pi}{2}}^T K^{-1} \gamma_3 &= e_{\frac{\pi}{2}}^T \rho_1 K^{-1} R_{13} \gamma_1 + e_{\frac{\pi}{2}}^T K^{-1} T_{13}.
\end{align*}
\]

(18)

Substituting these back into Equation 17 gives 4 equations for each triplet of corresponding for a total of 16 equations. The unknowns are 1 focal length, 11 pose and 4 depth unknowns for a total of 16 unknowns. If \( R \) is represented as a quaternion one additional unknown and one additional equations is introduced per rotation matrix giving a total of 18 polynomial equations in 18 unknowns. This minimal problem cannot be solved using elimination template since the required memory is unavailable even on a high performance computing machine. However, our HC implementation can solve this system with 1376ms in CPU and 154.19ms in GPU, Table 2.
and computationally intensive, and therefore a good fit for
independent computations (algorithms, like the one mentioned in Section 6, involve many
MAGMA (LAPACK for GPUs), that can be expressed using
GPU [77]. Subsequently, many dense linear algebra al-
bound operations, like general matrix-matrix multiplica-
share memory, which enables GPUs to outperform multi-
many images into batched GEMMs [22]. Many-to-many
computational patterns can be mapped efficiently to GPUs as well [82]. Operations that can not be mapped efficiently to
GPU have been left in general for the CPUs. This usually in-
volves irregular computations on small data sets where there
is not enough parallelism, and computations with a lot of
data dependencies (like solving a small system of equations).
Still, techniques like batching computations to increase parallel-
and developments in numerical linear algebra libraries
for GPUs [1, 37], have laid the groundwork for many more
algorithms to be easily ported and benefit GPU use. Often
algorithms that have been avoided before due to their compu-
tational cost are becoming preferred for GPUs when current
advances make their GPU mapping very efficient. This is the
case for the HC that we target to develop and illustrate in this
paper.

5 GPUs and Computer Vision

GPUs are often preferred over CPUs because of their superior
computational power, memory bandwidth, and energy effi-
ciency. For example, a V100 GPU provides an FP64 compute
peak of 7 TFlop/s and memory bandwidth of 900 GB/s at 250
Watts. While one CPU core is faster and provides wider in-
struction sets, GPUs have many more cores, e.g., 5,120 in the
V100. The key to unlocking the computational power of the
GPU is to design algorithms that are highly parallel and use
efficiently all the cores.

Figure 2 shows the GPU architecture. The CUDA cores
are organized into Streaming Multiprocessors (SMs) where
each SM has a number of CUDA cores. The GPU work is
organized into kernels that have two levels of nested parallel-
ism - a coarse level that is data parallel and is spread across
the SMs, and a fine level within each SM. The parallelism is
organized in terms of thread blocks (TBs). A TB is sched-
uled for execution on one of the SMs and is data parallel with
respect to the other TBs. Each TB is composed of multiple
threads running in groups of 32 called warps. Threads in a
TB can share data through a shared memory module. Private
variables that have the scope of one thread are usually stored
in the register file. Algorithms must be designed to support
this type of parallelism.

The multi-level memory hierarchy enables compute-
bound operations, like general matrix-matrix multiplica-
tion (GEMM), perform close to the compute peak of the
GPU [77]. Subsequently, many dense linear algebra al-
gorithms such as the ones in LAPACK, and subsequently
MAGMA (LAPACK for GPUs), that can be expressed using
GEMMs and BLAS in general [93]. Some numerical algo-
rithms, like the one mentioned in Section 6, involve many
independent computations (e.g. dense factorizations) on rel-
atively small matrices. These algorithms, are limited by the
memory bandwidth, but have a high degree of parallelism,
which is suitable for GPUs. Maximizing data reuse is pos-
sible by caching each matrix entirely in the register file or
shared memory, which enables GPUs to outperform multi-
core CPUs in these types of algorithms.

Algorithms in computer vision are naturally data-parallel
and computationally intensive, and therefore a good fit for
modern GPUs. Computational patterns involving one-to-one
mappings like an image being modified by different filters,
can benefit from the data parallelism and the memory hier-
archy to chain the applications of filters. Many-to-one mapp-
ings that involve summations of certain buffers also can ben-
efit the memory hierarchy and do it fast in MPs, vs. cre-
ating a sequence of fragment programs to simulate summa-
tions in older GPUs without memory hierarchy. CNNs for
example map many convolutions (involving summation) on
many images into batched GEMMs [22]. Many-to-many
computational patterns can be mapped efficiently to GPUs as
well [82].

6 GPU Implementation of HC

The homotopy continuation process can be parallelized in two
ways: First, observe that since HC follows many independent
tracks to convergence, a straightforward approach would be
to assign each track to a thread. However, the efficiency of
GPU processing depends on (i) number of threads processing
many tracks in parallel, and (ii) avoiding costly data transfer
rates by using the fast register files, or at least L1 caches v.s.
the slower L2 cache or even slower global memory, Figure 2.
In our application, each track requires a few Kbytes while the
available memory is 125, 46, 37, and 97K bytes for register
file, L1 cache, L2 cache, and global memory, respectively, for
one thread per track. Thus any process requiring more than
125 + 46 + 37 = 208 bytes of memory is forward to use the
very slow global memory. As a result, each track must make
use of many threads, and not only the processing must be
parallelized, but so must the use of memory with the aim of
keeping everything in register file, shared memory, or at least
L1 cache.

Observe that the other extreme of spreading a trade over
numerously many threads starts becoming counterproductive
because the synchronization of threads employs the slower
shared memory (2 clock cycles per 32 threads) so that if 2048
threads are employed per track. 128 clock cycles (~104 ns)
times the tens of thousands of their synchronization is needed
which becomes an unnecessary overhead.

The optimal balance for the target applications is to assign
a track to a warp (32 threads) using one GPU core. This gives
the application access to 256K/64 = 4K very fast register
file memory and $96K/64 = 1.5K$ of fast L1 cache (if no shared memory is used), well satisfying the memory requirement of the target application. On the other hand, the cost of thread synchronization is only 2 clock cycles.

The second intuition aims for parallelizing HC within each warp by (i) solving a system of linear equations in both the prediction step, Eq. 3 and the correction step, Eq. 5, and (ii) evaluating the Jacobian matrix $\partial H/\partial x$, $\partial H/\partial t$, and the homotopy $H$, Eq. 4 and 6.

**Linear System Solver:** The vast majority of work on solving linear systems on GPU is centered around large matrices, motivating a hybrid CPU+GPU approach [95, 94]. For smaller matrices like ours, cuBLAS or MAGMA [3, 2, 47] can be used. A linear system is generally solved by an LU factorization with partial pivoting followed by two triangular solves. The LU factorization in MAGMA is fast, typically 15% to 80% faster than cuBLAS for small matrices. However, we found out that cuBLAS is faster than MAGMA for the combined (factorization + solve) operation. This is mainly due to a slow triangular solver kernel in MAGMA, which does not take advantage of small matrices.

Our contribution to improving these standard libraries for our purposes is twofold. First, solving the linear system as two separate GPU kernels causes redundant global memory traffic. The two kernels can be fused into one if the matrices are small, thus maximizing data reuse in the register file. The proposed kernel fusion significantly speeds up the solution. Second, in solving a linear system $Ax = b$, the decomposition can act on the augmented matrix $[A \ b]$, which implicitly carries out the triangular solve with respect to the $L$ factor of $A$. The second triangular solve uses the cached $U$ factor after the factorization is complete. The proposed fused kernel is now integrated into the MAGMA library.

**Parallel Evaluations of the Jacobian and Vectors:**

The main bottleneck to parallel evaluations of the elements of the Jacobian matrix $\partial H/\partial x$ and the vectors $\partial H/\partial t$ and $H$ is the heterogeneity of its elements which prevents evaluation by many threads requiring a uniform format. This heterogeneity can be illustrated by a simple example of a system with two variables $X = (x_1, x_2)$ where the Jacobian elements are spanned by monomials, for example, $A = 2a_1 x_1 + 4a_2 x_1 x_2 + 8a_3 x_3^2$ or $B = 5a_4 x_1 x_2 + 7a_5 x_2^3$, where the coefficients $a_i$ are linear interpolation of corresponding elements in the start and target systems. The straightforward approach to homogenize these expressions is to write each as a sum over all possible monomials and associate a scalar zero with those absent from the Jacobian elements in parallel. However, due to the extreme sparsity, the process is highly inefficient.

Alternatively, consider $K$ the maximum number of terms in the Jacobian matrix elements; in the above examples, $A$ has three terms and $B$ has two terms, so that $K = 3$ if these were the only elements of the Jacobian matrix. Furthermore, consider that each term consists of a scalar multiplied with a coefficient and a number of variables, e.g., the third term of $A$ is a product of $(8, a_3, x_3, x_3)$ while the first term of $B$ is $(5, a_4, x_1, x_2)$. Note that the first term of $A$ is a product of $(2, a_1, x_1)$. Thus, to homogenize the expression, it is written as $(2, a_1, x_1, x_3)$ where the auxiliary variable $x_3 = 1$. Now all terms of both $A$ and $B$ can be written as $U = \sum_{k=1}^{K} s_k a_{k,j} x_{k,m_1} x_{k,m_2} \cdots x_{k,m_M}$, where $s_k$ is a scalar, $a_{k,j}$ identifies a coefficient, $x_{k,m_i}$ identifies one of the variables, including $x_3 = 1$, and $M$ is the maximal number of variables in a term. With this in mind, the only data to be communicated for the parallel computation of $U$ is $(s_{k}, a_{k,j}, x_{k,m_1}, x_{k,m_2}, \ldots, x_{k,m_M})$ where $a_{k,j}$, $x_{k,m_i}$ are pointers to data stored in shared memory and accessed by an index, i.e., $A$ is represented by $((2, 1, 1, 3), (4, 2, 1, 2), (8, 3, 2, 2))$ and $B$ is represented by $((5, 4, 1, 2), (7, 5, 2, 2), (0, 1, 1, 1))$. Note that $\partial H/\partial t$ and $H$ are evaluated in the same way although the coefficients $a_{k,j}$ are different. This homogeneous form allows for parallel computation of all elements of the Jacobian matrix $\partial H/\partial x$ and the vectors $\partial H/\partial t$ and $H$.

Finally, there is an issue on how to allocate the parallel computations per thread. Recall that each track is assigned to a warp which has 32 threads. Since the matrices are generally less than $32 \times 32$, and since the subsequent operation of LU decomposition is row-by-row with one thread per row, it makes sense to assign one row per thread.

**Experiments**

The experiments aim at testing kernel fusion for batch linear systems, and measuring performances on polynomial system benchmarks as well as computer vision problems. We use an 8-core 2.6GHz Intel Xeon CPU and an nVidia Quadro RTX
Kernel-Fused Batch Linear Systems: The performance of the batched linear systems with kernel fusion and augmented matrix, Section 6, is compared with cuBLAS and MAGMA in Figure 3 on a Tesla V100-PCIe GPU for 1000 matrices with sizes ranging from $4 \times 4$ to $20 \times 20$. Evidently, kernel-fused MAGMA outperforms cuBLAS with speedup of $2.23 \times$ to $3.65 \times$ and MAGMA with speedups ranging from $3.11 \times$ to $4.91 \times$.

Polynomial System Benchmarks: We selected four representative benchmark systems [81, 8, 46, 73] to evaluate the performance of our GPU-HC. Table 1 shows GPU-HC speedup ranging from $10 \times$ to $26 \times$. Figure 4 (a) shows that the speedup is not at the cost of lower accuracy, i.e., the GPU-HC computed solutions satisfy the polynomial system with high accuracy.

| Problems | # of Unkn. | # of Sol. | CPU (ms) | GPU (ms) | CPU GPU |
|----------|------------|----------|---------|---------|--------|
| Alea-6 [81] | 6 | 387 | 156.71 | 5.94 | 26.39× |
| Cyclic-7 [8] | 7 | 924 | 219.02 | 8.35 | 26.23× |
| D-1 [46] | 12 | 192 | 100.57 | 7.09 | 14.18× |
| Eco-12 [73] | 12 | 1024 | 279.12 | 27.43 | 10.18× |

Table 1: Performance of GPU-HC on benchmark problems.

Computer Vision Problems: We consider a sample of minimal problem in computer vision ranging from the classic pose estimation P3P to the more complex 3-view triangulation as well as two problems that have not been explored previously: (i) 4-view triangulation is an extension of 3-view triangulation [18]. As far as we know, this is the first attempt to explore this problem. (ii) trifocal relative pose estimation without focal length is an extension of existing trifocal relative pose estimation problems [32, 66] to the uncalibrated scenario; as far as we know, this problem has not been explored previously. The most popular technique for solving polynomial system is the elimination template approach [62] and is used to gauge the performances of GPU-HC. Table 2 shows a comparison of the elimination template performances with that of HC on CPU and on GPU. Each problem is instantiated 20 times with random parameters and its performance is averaged. The start systems for HC are generated with monodromy module in Macaulay2 [26]. Numerous factors affect the speedup of GPU-HC over CPU-HC, including the number of solutions, number of unknowns, and the number of terms in the polynomial evaluations of the Jacobian matrix. The performances of elimination template is dependent on the size of the linear system it solves which itself is related to the number of solutions of the polynomial system. Note that for the top three problems the elimination template cannot compute the basis of the quotient ring of the system even with ample memory. A review of Table 2 which is ordered by elimination template time, shows that with the exception simpler problem such as P3P (the bottom four rows), the GPU-HC outperforms the elimination template. GPU-HC opens the door to more complex problems that the elimination template cannot handle.

Conclusion: We presented GPU-HC, a GPU implementation of HC that is generic and can be easily applied to any computer vision problem formulated as a system of polynomial equations. The significant speedup of GPU-HC is an enabler in that HC can now be efficiently used for moderately complex problems in place of completing approaches. GPU-HC also enables the exploration of problems whose complexity has thus far evaded a practical solution.

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### Table 2: Performance of GPU-HC and CPU-HC vs. elimination template in application to several minimal problems.

| Problems | # of Unkns. | # of Sols. | Elim. Temp. (ms) | CPU (ms) | GPU (ms) | Elim. Temp. (ms) |
|----------|-------------|------------|-----------------|---------|---------|-----------------|
| trifocal rel. pose, unknown focal length | 18 | 1784 | X | 1456.22 | **154.19** | 9.44× | N.A. |
| 4-view triangulation | 14 | 296 | X | 156.28 | **18.60** | 8.32× | N.A. |
| 5 pt rel. pose & depth recon. | 16 | 160 | X | 150.94 | **26.89** | 5.61× | N.A. |
| 6 pt rolling shutter abs. pose w. 1-lin. [6] | 18 | 160 | X | 158.48 | **27.11** | 5.85× | N.A. |
| 3-view triangulation [18] | 9 | 94 | 612.432 | 101.86 | **8.17** | 24.19× | 38.24× |
| optimal PnP with quaternion [76] | 4 | 128 | 36.329 | 80.26 | **7.18** | 11.18× | 5.06× |
| P4P, unknown focal length & radial distortion [15] | 5 | 192 | 9.03 | 130.79 | **7.51** | 17.42× | 1.2× |
| 2-view triangulation with radial distortion [59] | 5 | 28 | 5.92 | 66.22 | **3.06** | 21.64× | 1.93× |
| optimal P4P abs. pose [92] | 5 | 32 | 1.864 | 53.13 | **1.57** | 33.84× | 1.19× |
| 3 pt rel. pose w. homography constraint [88] | 8 | 8 | 1.472 | 51.25 | **0.95** | 53.95× | 1.55× |
| PnP, unknown principal point [63] | 10 | 12 | **1.466** | 58.31 | 3.87 | 15.07× | 0.38× |
| rel. pose w. quiver, unknown focal length [55] | 4 | 28 | **1.082** | 56.01 | 1.23 | 45.54× | 0.88× |
| P3P abs. pose [57] | 3 | 8 | **0.063** | 39.64 | 0.22 | 180.18× | 0.29× |
| 5 pt rel. pose w.o. depth recon. [79] | 3 | 27 | **0.035** | 55.48 | 0.96 | 57.79× | 0.036× |

X: it is impossible for elimination template to solve because of an out of memory issue.
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In this supplementary material, we aim to show the effectiveness of the proposed 4-view triangulation problem using a synthetic multiview dataset and a real dataset. Furthermore, we also release full polynomial formulations of all the tested problems in: https://anonymous.4open.science/r/Minimal-Problems-in-Computer-Vision-201D.

1 Effectiveness of the Four-View Triangulation Problem

To show the usefulness of 4-view triangulation method, synthetic multiview dataset from [33] is used for evaluation. We randomly select 200 points from random 4 pairs of views. To simulate the noise on the observed points and the calibration, the poses are perturbed with $\mathcal{N}(0, 2.0)$ and the observed points are perturbed with $\mathcal{N}(0, 1.0)$. We compare our 4-view triangulate result with 3-view triangulation [60], Figure 5. It is clear to see that the error between the optimized 2D points and the ground truths of 4-view triangulation is smaller than 3-view triangulation. Quantitatively, the mean error of 4-view triangulation is 2.9712 pixels, while in 3-view triangulation, the error is 4.2074 pixels. Therefore, adding one more view to do triangulation would drop the projection error significantly.

Apart from using the synthetic dataset, a real dataset [101] is also employed to challenge the effectiveness of the 4-view triangulation. We use the dinosaur sequence from [101] which contains 36 images with 4,983 corresponding points. To deploy our 4-view triangulation method, only 1,516 points that are co-visible by more than 3 views are used. The triangulation result is shown in Figure 6. Note that after GPU-HC computes the optimal positions of the image points, the projection lines from all four images always intersect in the 3D space. In such a case, we are free to select any two views to find the position of the 3D points given the camera extrinsic matrix of these two views. The processing time of this whole sequence using the proposed GPU-HC is around 10.6 seconds.
Figure 6: (a) Sample image of dinosaur sequence from VGG Multiview Dataset [101]. (b) Triangulation result using GPU-HC 4-view triangulation method.