Approximating \( k \)-Connected \( m \)-Dominating Sets

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Abstract

A subset \( S \) of nodes in a graph \( G \) is a \( k \)-connected \( m \)-dominating set (\((k, m)\)-cds) if the subgraph \( G[S] \) induced by \( S \) is \( k \)-connected and every \( v \in V \setminus S \) has at least \( m \) neighbors in \( S \). In the \( k \)-CONNECTED \( m \)-DOMINATING SET (\((k, m)\)-CDS) problem, the goal is to find a minimum weight \((k, m)\)-cds in a node-weighted graph. For \( m \geq k \) we obtain the following approximation ratios. For unit disk graphs we improve the ratio \( O(k \ln k) \) of Nutov (Inf Process Lett 140:30–33, 2018) to \( \min \{m^2/(m-k+1)^2, k^2/3\} \cdot O(\ln^2 k) \)—this is the first sublinear ratio for the problem, and the first polylogarithmic ratio \( O(\ln^2 k)/\epsilon^2 \) when \( m \geq (1 + \epsilon)k \); furthermore, we obtain ratio \( \min \{m/(m-k+1), \sqrt{k}\} \cdot O(\ln^2 k) \) for uniform weights. For general graphs our ratio \( O(k \ln n) \) improves the previous best ratio \( O(k^2 \ln n) \) of Nutov (2018) and matches the best known ratio for unit weights of Zhang et al. (INFORMS J Comput 30(2):217–224, 2018). These results are obtained by showing the same ratios for the SUBSET \( k \)-CONNECTIVITY problem when the set of terminals is an \( m \)-dominating set.

Keywords \( k \)-connected graph · \( m \)-dominating set · Subset \( k \)-connectivity · Approximation algorithms

1 Introduction

All graphs in this paper are assumed to be simple, unless stated otherwise. A (simple) graph is \( k \)-connected if it has \( k \) pairwise internally node disjoint paths between every pair of its nodes; in this case the graph has at least \( k + 1 \) nodes. A subset \( S \) of nodes in a graph \( G \) is a \( k \)-connected set if the subgraph \( G[S] \) induced by \( S \) is \( k \)-connected; \( S \) is an \( m \)-dominating set if every \( v \in V \setminus S \) has at least \( m \) neighbors in \( S \). If \( S \) is
both $k$-connected and $m$-dominating set then $S$ is a $k$-connected $m$-dominating set, or $(k,m)$-cds for short. A graph is a unit disk graph if its nodes can be located in the Euclidean plane such that there is an edge between $u$ and $v$ iff the Euclidean distance between $u$ and $v$ is at most 1. We consider the following problem for $m \geq k$ both in general graphs and in unit disk graphs.

**$k$-Connected $m$-Dominating Set ($(k,m)$-CDS)**

**Input:** A graph $G = (V, E)$ with node weights $\{w_v : v \in V\}$, and integers $k, m$.

**Output:** A minimum weight $(k,m)$-cds $S \subseteq V$.

The problem generalizes several classic problems including Set-Cover ($k = 0$, $m = 1$), Set-Multicover ($k = 0$), and Connected Dominating Set ($k = m = 1$). The Connected Dominating Set problem is closely related to the Node Weighted Steiner Tree problem, and both problems admit a tight ratio $O(\log n)$ [9,11]. The Connected Dominating Set problem is NP-hard in unit disk graphs [4]. It admits a PTAS for unit weights [3], and ratio $3 + 2.5\rho + \epsilon$ for arbitrary weights [22,24], where $\rho$ is the ratio for the edge-weighted Steiner Tree problem in general graphs; see also [1,25] for additional constant ratio approximation algorithms in unit disk graphs. The $(k,m)$-CDS problem models (fault tolerant) virtual backbones in networks [5,6], and it was studied for $m \geq k$ in general graphs [17,23] and in unit disk graphs [8,17,20,21], and also for $k = 2, m = 1$ [2,19]. For further motivation and literature survey we refer the reader to recent papers of Shi et al. [20] and of Fukunaga [8], where for $m \geq k$ they obtained in unit disk graphs ratios $O(k^3 \ln k)$ and $O(k^2 \ln k)$, respectively. This was improved to $O(k \ln k)$ in [17]. An $O(k^2 \ln n)$-approximation algorithm for general graphs is also given in [17].

Our main results is:

**Theorem 1** $(k,m)$-CDS with $m \geq k$ admits the following approximation ratios:

- $\min \left\{ \frac{m^2}{(m-k+1)^2}, k^{2/3} \right\} \cdot O(\ln^2 k)$ in unit disk graphs.
- $\min \left\{ \frac{m}{m-k+1}, \sqrt{k} \right\} \cdot O(\ln k)$ in unit disk graphs with unit weights.
- $O(k \ln n)$ in general graphs.

For unit disk graphs our ratio $\min \left\{ \frac{k^2}{(m-k+1)^2}, k^{2/3} \right\} \cdot O(\ln^2 k)$ improves the previous best ratio $O(k \ln k)$ of [17]; this is the first sublinear ratio for the problem, and for any constant $\epsilon > 0$ and $m = k(1 + \epsilon)$ the first polylogarithmic ratio $O(\ln^2 k)/\epsilon^2$.

For general graphs our ratio $O(k \ln n)$ improves the previous ratio $O(k^2 \ln n)$ of [17] and matches (while using totally different techniques) the best known ratio for unit weights of Zhang et al. [23].

Let us say that a graph with a set $T$ of terminals and a root $r \in T$ is $k$-$(T,r)$-connected if it has $k$ internally node disjoint $tr$-paths for every $t \in T \setminus \{r\}$. Similarly, a graph is $k$-$T$-connected if it has $k$ internally node disjoint $st$-paths for every $s, t \in T$.

A reason why the case $m \geq k$ is easier than the case $m < k$ is given in the following statement (a proof can be found in many papers, c.f. [8,17,20]).

**Lemma 1** Let $T$ be a $k$-dominating set in a graph $H = (U, F)$. If $H$ is $k$-$(T,r)$-connected then $H$ is $k$-$(U,r)$-connected; if $H$ is $k$-$T$-connected then $H$ is $k$-connected.
Applying Lemma 1 on the graph $H = G[T \cup S]$, we get that $(k, m)$-CDS with $m \geq k$ has the following property: the union $T \cup S$ of a partial solution $T$ that is just $m$-dominating, and a node set $S$ such that $G[T \cup S]$ is $T$-$k$-connected, is a feasible solution; this can be used to construct the solution iteratively. Specifically, most algorithms for the case $m \geq k$ start by computing just an $m$-dominating set $T$; the current best ratios for $m$-DOMINATING SET are $\alpha_{\text{adg}} = O(1)$ in unit disk graphs [8] and $\alpha = \ln(\Delta + m)$ in general graphs [7], where $\Delta$ is the maximum degree in $G$. By invoking just these ratios, Lemma 1 reduces $(k, m)$-CDS with $m \geq k$ to the following problem:

**Subset $k$-Connectivity**

**Input:** A graph $G = (V, E)$ with node-weights $\{w_v : v \in V\}$, a set $T \subseteq V$ of terminals, and an integer $k$.

**Output:** A minimum weight $k$-$T$-connected subgraph of $G$.

The ratios for this problem are usually expressed in terms of the best known ratio $\beta$ for the following problem (in both problems we assume w.l.o.g. that $w_v = 0$ for all $v \in T$):

**Rooted Subset $k$-Connectivity**

**Input:** A graph $G = (V, E)$ with node-weights $\{w_v : v \in V\}$, a set $T \subseteq V$ of terminals, a root node $r \in T$, and an integer $k$.

**Output:** A minimum weight $k$-$(T, r)$-connected subgraph of $G$.

In the $m$-dominating versions of these problems, $T$ is an $m$-dominating set in $G$ for $m \geq k$. Currently, $\beta = O(k^2 \ln |T|)$ [15], and no better ratio was known for the $k$-dominating versions. From previous work it can be deduced that **Subset $k$-Connectivity** with $|T| \geq k$ admits ratio $\beta + k^2$ (the case $|T| \leq k$ admits a trivial ratio $(\binom{|T|}{2}) = O(k^2)$). Add a new root node $r$ connected to a set $R \subseteq T$ of $k$ nodes by edges of cost zero. Then compute a $\beta$-approximate solution to the obtained **Rooted Subset $k$-Connectivity** instance. Finally, augment this solution by computing for every $u, v \in R$ a min-weight set of $k$ internally disjoint $uv$-paths. For the $(k, m)$-CDS problem with $m \geq k$ this already gives ratio $\beta + k^2 = O(k^2 \ln |T|)$ in general graphs. For the special case when $T$ is a $k$-dominating set, the ratio $\beta + k^2$ was improved in [17] to $\beta + k - 1$, by observing that in the final step it is sufficient to compute a min-weight set of $k$ internally disjoint $uv$-paths only for pairs that form a forest on $R$; this follows from Lemma 1 and the Critical Cycle Theorem of Mader [14]. Summarizing, we have the following.

**Lemma 2** [17] If $m$-dominating **Rooted Subset $k$-Connectivity** admits approximation ratio $\beta$, then $m$-dominating **Subset $k$-Connectivity** (and also $(k, m)$-CDS with $m \geq k$) admits approximation ratio $\beta + k - 1$.

We now consider unit disk graphs. Shi et al. [20] showed that any $k$-connected unit disk graph has a $k$-connected spanning subgraph of maximum degree $\leq 5k$. This implies that the node weighted case is reduced, with a loss of factor $O(k)$, to the case of node induced edge costs $c_{uv} = w_u + w_v$ for every edge $e = uv \in E$. The edge costs version of **Subset $k$-Connectivity** admits ratio $O(k^3 \ln k)$, which gives ratio $O(k^3 \ln k)$ for $(k, m)$-CDS with $m \geq k$. 

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 Independently, Fukunaga [8] obtained ratio $O(k^2 \ln k)$ using a different approach—he considered the Rooted Subset Connectivity Augmentation problem, when $G[T]$ is $\ell$-$(T, r)$-connected and we seek a minimum weight $S \subseteq V \setminus T$ such that $G[T \cup S]$ is $(\ell + 1)$-$(T, r)$-connected. In [15] it is shown that this augmentation problem decomposes into $O(\ell)$ “uncrossable” subproblems, and Fukunaga [8] designed a primal-dual $O(1)$-approximation algorithm for such a subproblem in unit disk graphs. This gives ratio $O(\ell)$ for Rooted Subset Connectivity Augmentation in unit disk graphs. Furthermore, Fukunaga showed that given a Rooted Subset $k$-Connectivity instance with optimal solution value $opt$, his algorithm computes a node set of weight $O\left(\frac{\ell}{k-\ell}\right) \cdot opt$. Hence sequentially increasing the $(T, r)$-connectivity by 1 gives ratio $O(k \ln k)$ for Rooted Subset $k$-Connectivity. He then combined this result with a decomposition of the Subset $k$-Connectivity problem into $k$ Rooted Subset $k$-Connectivity problems, and obtained ratio $O(k^2 \ln k)$ for $(k, m)$-CDS. Using Lemma 2, the ratio improves to $O(k \ln k)$ [17].

However, it seems that previous reductions and methods alone are not sufficient to obtain ratio better than $O(k^2 \ln |T|)$ in general graphs, or a sublinear in $k$ ratio in unit disk graphs. These algorithms rely on the ratios and decompositions for the Rooted/Subset $k$-Connectivity problems from [12,15,16], but these do not consider the specific feature relevant to $(k, m)$-CDS with $m \geq k$—that the set $T$ of terminals is a $k$-dominating set; note that then Subset $k$-Connectivity is equivalent to the problem of finding the lightest $k$-connected subgraph containing $T$, by Lemma 1. Here we change this situation by asking the following question:

*What approximation ratios can be achieved for $m$-dominating Rooted/Subset $k$-Connectivity in unit disk and general graphs?*

We answer this question in the following two theorems, which are of independent interest, and note that they imply Theorem 1, by Lemma 1.

**Theorem 2** In unit disk graphs, the $m$-dominating Subset $k$-Connectivity problem admits approximation ratio $O(\ln^2 k) \cdot \min \left\{ \frac{m^2}{(m-k+1)^2}, k^2/3 \right\}$, and ratio $O(\ln^2 k) \cdot \min \left\{ \frac{m}{m-k+1}, \sqrt{k} \right\}$ for unit weights.

**Theorem 3** The $k$-dominating Rooted Subset $k$-Connectivity problem admits approximation ratio $O(k \ln n)$, and so is the $m$-dominating Subset $k$-Connectivity problem (by Lemma 2).

In the proofs of Theorems 2 and 3 we use several results and ideas from previous works [8,15–17,20]. Both proofs rely on the algorithm from [15], that decomposes the Rooted Subset $k$-Connectivity problem into easier sub-problems. The [15] algorithm has $k$ iterations, where at iteration $\ell = 0, \ldots, k-1$ it considers the Rooted Subset Connectivity Augmentation problem of increasing the $(T, r)$-connectivity from $\ell$ to $\ell + 1$. This is equivalent to “covering” a certain family $\mathcal{F}$ of “deficient sets” (see Sect. 3 for precise definitions). The [15] algorithm decomposes this problem into $O(\ell)$ “uncrossable family covering problems”. For each uncrossable

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1 In fact, Fukunaga [8] considered a slight modification of the Rooted Subset Connectivity Augmentation problem, see Theorem 5. This is ignored in this section, for simplicity of exposition.
family we can find efficiently a cover of weight \( O(\ln n) \cdot \text{opt} \) in general graphs [15], and \( \frac{15}{k-\ell} \cdot \text{opt} \) in unit disk graphs [8]; here \( \text{opt} \) denotes the optimal solution value to the relevant problem in Theorem 2 or 3. For the Theorem 3 problem this gives ratio \( \sum_{\ell=0}^{k-1} O(\ln n) = O(k \ln n) \). For the Theorem 2 problem, applying the rooted version algorithm for \( \ell+1 \) roots gives \( (\ell+1) \)-connected graph while invoking weight \( O(\frac{\ell^2}{k-\ell}) \).

The overall ratio in this case is \( \sum_{\ell=0}^{k-1} O(\frac{\ell^2}{k-\ell}) = O(k^2 \ln k) \) [8].

However, a more careful analysis of the [15] algorithm reveals that in fact the number of uncrossable families is \( O(\ell/q) + 1 \), where \( q \) is the minimum number of terminals in a deficient set. In instances with \( q \geq \ell + 1 \), the entire family of deficient sets is uncrossable, hence such instances can be covered by weight \( O(\ln n) \cdot \text{opt} \) in general graphs [15], and \( \frac{15}{k-\ell} \cdot \text{opt} \) in unit disk graphs [8].

The algorithm of [15] has an “inflation phase” that works towards reaching \( q \geq \ell + 1 \), by repeatedly covering \( O(\ell/q) \) uncrossable families to double \( q \). Hence if \( q_0 \) is the initial value of \( q \), the total number of uncrossable families covered is order of \( 1 + \frac{\ell}{q_0} (1 + \frac{1}{2} + \frac{1}{4} + \cdots) = O(\ell/q_0 + 1) \). One of our contributions is designing “lighter” inflation algorithms for increasing the parameter \( q \). These algorithms just aim to cover the inclusion minimal deficient sets, by adding a light set \( S \) of nodes, and then add \( S \) to \( T \); if \( T \) is a \( k \)-dominating set then adding any set \( S \) to \( T \) does not make the problem harder, by Lemma 1.

The problem of covering inclusion minimal deficient sets is essentially a SET COVER problem, that admits ratio \( O(\log n) \). In the Theorem 2 unit disk case, we reduce it to \( O(\log k) \) using a result of Shi et al. [20], that a minimally \( k \)-connected unit disk graph has maximum degree \( \leq 5k \). Furthermore, since any ROOTED SUBSET \( k \)-CONNECTIVITY solution covers every deficient set \( k - \ell \) times, the total weight invoked for increasing \( q \) by 1 is \( O(\frac{\ln k}{k-\ell}) \cdot \text{opt} \).

In the Theorem 3 case of general graphs, the reduction is to a special case considered in [13] of the SUBMODULAR COVERING problem; the ratio invoked by this procedure is \( O(\ln n) \) and applying it \( k \) times we get \( q \geq k \). In fact, we apply this procedure before considering the augmentation problems, but it guarantees that \( q \geq k \) holds in all augmentation iterations.

Our algorithms are combinatorial. We omit the running time analysis, but it is polynomial and dominated by that of applying \( k \) times the algorithms of [8,15] for ROOTED SUBSET \( k \)-CONNECTIVITY.

Theorems 2 and 3 are proved in Sects. 2 and 3, respectively.

## 2 Unit Disk Graphs (Theorem 2)

In this section \( \text{opt} \) denotes the optimal weight of a solution to the SUBSET \( k \)-CONNECTIVITY problem. Later, in Sect. 2.1, we will prove the following.

**Lemma 3** Consider the SUBSET \( k \)-CONNECTIVITY problem on a unit disk graph \( G = (V, E) \) and an \((\ell, m)\)-cds \( T \), \( m \geq k \geq \ell + 1 \). Then for any \( p \leq \ell + 1 \) there exists a polynomial time algorithm that computes \( S \subseteq V \setminus T \) such that \( G[T \cup S] \) is \((\ell + 1)\)-connected and \( \frac{w(S)}{\text{opt}} = O(\frac{\ln k}{k-\ell}) (p + \gamma^2) \) where
\[ \gamma = \gamma(\ell, p) = \frac{m + p}{m - \ell + p} \leq \min \left\{ \frac{m}{m - \ell}, 1 + \frac{\ell}{p} \right\} \]

Furthermore, in the case of unit weights \( \frac{w(S)}{opt} = \frac{O(\ln k)}{k-\ell} (p + \gamma) \).

Let us show that Lemma 3 implies Theorem 2. Note the following.

- For \( p = 1 \) we have \( p + \gamma^2 \leq 1 + \frac{m^2}{(m-\ell)^2} = O\left(\frac{m^2}{(m-\ell)^2}\right) \).
- For \( p = \ell^{2/3} \) we have \( p + \gamma^2 \leq \ell^{2/3} + (1 + \ell^{1/3})^2 = O(\ell^{2/3}) \).

Thus we can compute \( S_\ell \subseteq V \setminus T \) such that \( \frac{w(S_\ell)}{opt} = \frac{O(\ln k)}{k-\ell} \min \left\{ \frac{m^2}{(m-\ell)^2}, \ell^{2/3} \right\} \). Consider an algorithm that at iteration \( \ell = 0, \ldots, k - 1 \) adds such \( S_\ell \) to \( T \). Denoting \( S = S_0 \cup \cdots \cup S_{k-1} \) we get:

\[
\frac{w(S)}{opt} = \sum_{\ell=0}^{k-1} \frac{O(\ln k)}{k-\ell} \min \left\{ \frac{m^2}{(m-\ell)^2}, \ell^{2/3} \right\} = O(\ln^2 k) \cdot \min \left\{ \frac{m^2}{(m-k+1)^2}, k^{2/3} \right\}.
\]

For the case of unit weights, note that \( p + \gamma = O\left(\frac{m}{m-\ell}\right) \) for \( p = 1 \) and \( p + \gamma = O(\sqrt{\ell}) \) for \( p = \sqrt{\ell} \). This gives \( \frac{w(S_k)}{opt} = \frac{O(\ln k)}{k-\ell} \min \left\{ \frac{m}{m-\ell}, \sqrt{\ell} \right\} \), and then a similar analysis gives \( \frac{w(S)}{opt} = O(\ln^2 k) \cdot \min \left\{ \frac{m}{m-k+1}, \sqrt{k} \right\} \).

### 2.1 Proof of Lemma 3

Here we prove Lemma 3, so let \( G, T, \) and \( \ell \) be as in the lemma; namely, \( G[T] \) is an \((\ell, m)\)-cds for \( m \geq k \geq \ell + 1 \). We need some definitions to continue.

**Definition 1** A node subset \( A \) of a graph \( H \) let \( \Gamma_H(A) \) denote the set of neighbors of \( A \) in \( H \) that are not in \( A \). Let \( H = (T, F) \) be an \( \ell \)-connected graph. A proper node subset \( A \) of \( T \) is called a **tight set** of \( H \) if \( |\Gamma_H(A)| = \ell \) and \( T \setminus (A \cup \Gamma_H(A)) \neq \emptyset \). A set \( Q \subset T \) of nodes is an **\( \ell \)-cut** of \( H \) if \( |Q| = \ell \) and \( H \setminus Q \) has at least two connected components.

Note that if \( H \) is \( \ell \)-connected, then \( A \) is tight if and only if \( \Gamma_H(A) \) is an \( \ell \)-cut of \( H \) that disconnects between the non-empty sets \( A \) and \( T \setminus (A \cup \Gamma_H(A)) \); hence if \( A \) is tight, then so is \( T \setminus (A \cup \Gamma_H(A)) \). Conversely, if \( Q \) is an \( \ell \)-cut of \( H \) then the union \( A \) of some but not all of connected components of \( G \setminus Q \) is a tight set with \( \Gamma_H(A) = Q \) (this is so since \( H \) is \( \ell \)-connected). Furthermore, \( H \) is \((\ell + 1)\)-connected if and only if \( H \) has no tight sets/\( \ell \)-cuts.

We will apply these definitions on the \( \ell \)-connected graph \( H = G[T] \). Let \( q = q(T) \) be the minimum size of a tight set of \( G[T] \). Since the problem admits ratio \( O\left(\frac{\ell + q(T)}{q(T)}\right) \) (see Theorem 5 to follow), our goal will be to find a cheap set \( S \subseteq V \setminus T \) such that \( q(T \cup S) \) is relatively large. Specifically, the algorithm relies on the following key statement, to be proved later in Sect. 2.2.
Lemma 4 (Inflation Lemma for unit disk graphs) Consider the \textit{Subset $k$-Connectivity} problem on a unit disk graph $G = (V, E)$ and an $(\ell, m)$-cds $T$, $m \geq k \geq \ell + 1$. There exists a polynomial time algorithm that finds $S \subseteq V \setminus T$ of weight $w(S) = O(\text{opt}) \cdot \frac{k\ln k}{k-\ell}$ such that either $q(T \cup S) \geq m - \ell + 1, q(T) + 1$ or $G[T \cup S]$ is $(\ell + 1)$-connected.

Note that sequentially adding $p$ sets $S_1, \ldots, S_p$ to an $m$-dominating set $T$, where each set is as Lemma 4, we get the following.

Corollary 1 There exists a polynomial time algorithm that for any $p \leq \ell + 1$ computes a set $S_p \subseteq V \setminus T$ of weight $w(S_p) = O(\text{opt}) \cdot \frac{p\ln k}{k-\ell}$ such that either $q(T \cup S_p) \geq m + p - \ell$ and $|T \cup S_p| \geq m + p$, or $G[T \cup S_p]$ is $(\ell + 1)$-connected.

We also need the following two results from previous works.

Theorem 4 [12,16] Any $\ell$-connected graph $H = (T, F)$ has a set $R \subseteq T$ of $|R| = O \left( \frac{|T|}{|T| - \ell} \ln \ell \right)$ nodes, such that for any $\ell$-cut $Q$ of $H$ there exists $r \in R$ with $r \notin Q$. Furthermore, such $R$ can be found in polynomial time.

Theorem 5 (Fukunaga [8]) Consider the $k$-dominating \textit{Subset $k$-Connectivity} problem on a unit disk graph $G = (V, E)$ with an $\ell$-connected set $T \subseteq V$ and $k \geq \ell + 1$. Given $r \in T$ there exists a polynomial time algorithm that computes $S_r \subseteq V \setminus T$ of weight $w(S_r) = O \left( \frac{\ell + q(T)}{q(T)} \right) \cdot \frac{\text{opt}}{k-\ell}$ such that $r$ belongs to every $\ell$-cut of $G[T \cup S]$.

Theorem 5 is Corollary 3 in Fukunaga’s paper [8]. We note that in [8, Corollary 3] is stated explicitly only the bound $w(S_r) = O(\ell) \cdot \frac{\text{opt}}{k-\ell}$. The proof of Fukunaga of [8, Corollary 3] relies on two ingredients.

1. An algorithm of Fukunaga [8] for a certain subproblem (called in [8] “covering a $T$-intersecting uncrossable family of demand cuts”) of the problem of computing $S_r$ as in Theorem 5; this algorithm computes a solution to the subproblem of weight at most $\frac{15}{\ell} \cdot \text{opt}$.

2. A decomposition of the Theorem 5 problem of computing $S_r$ into $O(\ell)$ such subproblems [15].

However the bound in [15] (see also [18, Lemmas 13.8 and 13.9]) on the number of the subproblems in the [15] decomposition is $O \left( \frac{\ell + q(T)}{q(T)} \right)$, which is better than $O(\ell)$ if $q(T)$ is large. Now we state the Lemma 3 algorithm.

\begin{algorithm}
\textbf{Algorithm 1:} $(G, w, T, k, \ell, m, p)$ $T$ is an $(\ell, m)$-cds in a unit disk graph $G$, $m \geq k \geq \ell + 1 \geq p$
\begin{enumerate}
\item $T \leftarrow T \cup S_p$, where $S_p$ is as in Corollary 1
\item if $G[T]$ is $(\ell + 1)$ connected then $S_R \leftarrow \emptyset$
\item else
\begin{enumerate}
\item find a set $R \subseteq T$ of $O \left( \frac{|T|}{|T| - \ell} \ln \ell \right)$ nodes as in Theorem 4
\item for each $r \in R$ compute $S_r$ as in Theorem 5
\item $S_R \leftarrow \cup_{r \in R} S_r$
\end{enumerate}
\item $\text{return } S = S_p \cup S_R$
\end{enumerate}
\end{algorithm}
To prove the correctness of the algorithm we need the following lemma.

**Lemma 5** Let $T$ be an $(\ell, \ell + 1)$-cds in a graph $G = (V, E)$, let $S \subseteq V \backslash T$, and let $Q$ be an $\ell$-cut of $G[T \cup S]$. Then $Q \subseteq T$ and $Q$ is also an $\ell$-cut of $G[T]$.

**Proof** We will show that if $A$ is a connected component of $G[T \cup S] \backslash Q$, then $A \cap T$ is a tight set of $G[T]$, and $\Gamma_{G[T]}(A \cap T) = Q$.

In $G[T \cup S]$, a node in $S \cap A$ has at least $\ell + 1$ and at most $|A \cap T| + |Q|$ neighbors in $T$, thus $\ell + 1 \leq |A \cap T| + |Q| = |A \cap T| + \ell$. This implies that $A \cap T \neq \emptyset$. Consequently, $G[T] \backslash (Q \cap T)$ is disconnected, hence $|Q \cap T| \geq \ell$. Since $|Q| = \ell$, we get $Q \subseteq T$. Since $\Gamma_{G[T]}(A \cap T) \subseteq \Gamma_{G[T \cup S]}(A) = Q$ and $|\Gamma_{G[T]}(A \cap T)| \geq \ell$, we must have $\Gamma_{G[T]}(A \cap T) = Q$. \qed

**Proof of Lemma 3** Let $S$ be the node set returned by Algorithm 1. We will show that $G[T \cup S]$ is $(\ell + 1)$-connected and has weight as in Lemma 3.

If $G[T \cup S]$ is not $(\ell + 1)$-connected then it has an $\ell$-cut $Q$. Let $r \in R$ be as in Theorem 4, so $r \notin Q$. The graph $G[T \cup S_r]$, and thus also $G[T \cup S]$ (by Lemma 5), may have only $\ell$-cuts that contain $r$, giving a contradiction.

Now we bound the weight of $S = S_p \cup S_R$. Note that after step 2 we have $|T| \geq m + p$ and $q(T) \geq m + p - \ell$. This implies that $\frac{|T|}{|T| - \ell} \leq \frac{m + p}{m + p - \ell}$ and $\frac{\ell + q(T)}{q(T)} \leq \frac{m + p}{m + p - \ell}$. We have $|R| = O\left(\frac{|T|}{|T| - \ell} \ln \ell\right)$ and $\frac{w(S_p)}{\text{opt}} = O\left(\frac{\ell + q(T)}{q(T)} \cdot \frac{1}{k - \ell}\right)$. Combining we get

$$\frac{w(S_R)}{\text{opt}} = O\left(|R| \cdot \frac{\ell + q(T)}{q(T)} \cdot \frac{1}{k - \ell}\right) = O\left(\frac{\ln k}{k - \ell}\right) \cdot \frac{(m + p)^2}{(m + p - \ell)^2}.$$  

Also, $w(S_p) = p \frac{O(\ln k)}{k - \ell} \cdot \text{opt}$. Thus

$$w(S) = w(S_p) + w(S_R) = O\left(\frac{\ln k}{k - \ell}\right) \left(\frac{p + \frac{(m + p)^2}{(m + p - \ell)^2}}{\frac{k - \ell}{m}}\right) \cdot \text{opt}.$$  

In the case of unit weights, we add arbitrary $\ell$ nodes to $T$; this step invokes an additive term of $O(1)$ to the ratio, and $|T| \geq m + \ell$ holds after this step. Then we will have $\frac{|T|}{|T| - \ell} \leq \frac{m + \ell}{m} \leq 2$, and the overall weight of the node sets computed will be $O(\text{opt}) \cdot \frac{\ln k}{k - \ell} \left(\frac{p + \frac{(m + p)^2}{(m + p - \ell)^2}}{\frac{k - \ell}{m}}\right)$. \qed

### 2.2 Proof of Lemma 4

Assume that we are given a SUBSET $k$-CONNECTIVITY instance on a unit disk graph $G = (V, E)$ with an $(\ell, m)$-cds $T$, where $m \geq k \geq \ell + 1$. We will give a polynomial time algorithm that computes $S \subseteq V \backslash T$ of weight $w(S) = O(\text{opt}) \cdot \frac{\ln k}{k - \ell}$ such that either $q(T \cup S) \geq \max\{m - \ell + 1, q(T) + 1\}$ or $G[T \cup S]$ is $(\ell + 1)$-connected.

We need some definitions. Inclusion minimal tight sets of $H = G[T]$ are called **cores**. Let $\mathcal{C}$ denote the family of cores of $G[T]$. We say that a node $v \in V \backslash T$ **covers a core** $C \in \mathcal{C}$ if $v \in \Gamma_G(C)$, namely, if in $G$ there is an edge between $v$ and some node.
in $C$. A set $S \subseteq V \setminus T$ covers a subfamily $C' \subseteq C$ of cores if every $C \in C'$ is covered by some $v \in S$. From Lemma 5 we have the following.

**Corollary 2** If $S \subseteq V \setminus T$ covers all cores of $G[T]$, and if $G[T \cup S]$ is not $(\ell + 1)$-connected, then $q(T \cup S) \geq \max\{m - \ell + 1, q(T) + 1\}$.

**Proof** By Lemma 5, if $A$ is a core of $G[T \cup S]$, then $A$ contains some core $C$ of $G[T]$ and any node $v \in S$ that covers $C$; we cannot have $v \in \Gamma_{G[T \cup S]}(A)$ since $\Gamma_{G[T \cup S]}(A) \subseteq T$, by Lemma 5. Furthermore, $v$ has at most $\ell$ neighbors not in $A$, and thus has at least $m - \ell$ neighbors in $A$. $\square$

Consequently, to finish the proof of Lemma 4, it is sufficient to prove that we can find in polynomial time a cover $S$ of $C$ of weight $w(S) = O(\text{opt}) \cdot \frac{\ln k}{k - \ell}$. The problem of covering $C$ is essentially a (weighted) SET COVER problem where for each $v \in V \setminus T$ the corresponding set has weight $w_v$ and consists of the cores covered by $v$. Let $\Delta$ be the maximum number of cores that can be covered by a node. Then the greedy algorithm for SET COVER computes a solution of weight $O(\ln \Delta)$ times the value of the standard SET COVER LP

$$\tau(C) = \min \sum_{v \in V \setminus T} w_v x_v$$

s.t. $\sum_{v \in \Gamma(C) \setminus T} x_v \geq 1 \quad \forall C \in C$

$$x_v \geq 0 \quad \forall v \in V \setminus T$$

For any $S' \subseteq V \setminus T$ such that $G[T \cup S']$ is $k$-connected, any tight set $A$ of $G[T]$ has in $G$ at least $k - \ell$ neighbors in $S'$, since $A$ has exactly $\ell$ neighbors in $G[T]$ and at least $k$ neighbors in $G[T \cup S]$. Hence if $\lambda'$ is a characteristic vector of $S'$ then $\frac{x'}{k - \ell}$ is a feasible solution to the above LP. Consequently, $\tau(C) \leq \frac{\text{opt}}{k - \ell}$, and the greedy algorithm computes a cover $S \subseteq V \setminus T$ of $C$ of weight $w(S) = O(\ln \Delta) \cdot \frac{\text{opt}}{k - \ell}$. However, we may have $\Delta = \Omega(n)$. We need some results from previous work to obtain a bound that depends on $k$ only. The following property of tight sets is a folklore, c.f. [10, Lemma 1.2].

**Lemma 6** Let $A, B$ be tight sets with $A \cap B \neq \emptyset$ of an $\ell$-connected graph $H = (T, F)$. Then $|\Gamma_H(A \cap B)| \geq \ell$ and the following holds:

(i) If $|T \setminus (A \cup B)| \geq \ell$ then $A \cap B$ is tight.

(ii) If $A \cup B \cup \Gamma_H(A) \cup \Gamma_H(B) \neq T$ then $A \cap B, A \cup B$ are both tight.

The following property of cores was proved implicitly in [[10], Lemma 3.5]; we provide a proof for completeness of exposition.

**Lemma 7** (Jordán [10]) If an $\ell$-connected graph $H = (T, F)$ has two distinct cores $A, B$ with $A \cap B \neq \emptyset$, then $H$ has at most $\ell(\ell + 1)$ distinct cores.

**Proof** Let $A, B$ be as in the lemma with $|A \cup B|$ minimum. Let $a \in A \setminus B$ and $b \in B \setminus A$. Let $P = (T \setminus (A \cup B)) \cup \{a, b\}$. Note that $|T \setminus (A \cup B)| \leq \ell - 1$, as otherwise $A \cap B$ is tight (by Lemma 6(i)), contradicting the minimality of $A, B$. Thus $|P| = |T \setminus (A \cup B)| + 2 \leq \ell + 1$. We claim that $P \cap C \neq \emptyset$ for every core $C$. Suppose to the contrary that $P \cap C = \emptyset$
for some core \( C \). Then \( C \) intersects \( A \) or \( B \), say \( C \cap A \neq \emptyset \). Then \( C \cup A \subseteq (A \cup B)\setminus \{b\} \), and thus \(|A \cup C| < |A \cup B|\), contradicting the choice of \( A \), \( B \).

There are at most \(|P|(|P| - 1) \leq \ell(\ell + 1)\) distinct \( \ell \)-cores, since:

- For every core \( C \) there is \( s \in P \cap C \), and there is \( t \in P \cap (V \setminus (C \cup \Gamma_H(C))) \), since the set \( T \setminus (C \cup \Gamma_H(C)) \) is tight.
- For each \((s, t) \in P \times P\) there is at most one such \( C \), by Lemma 6(ii).

This concludes the proof of the lemma.

We also need the following result.

**Theorem 6** (Shi et al. [20]) Any \( k \)-connected unit disk graph has a \( k \)-connected spanning subgraph of maximum degree \( \leq 5k \).

We now finish the proof of Lemma 4. In the case \(|\mathcal{C}| \leq \ell(\ell + 1)\) we get a solution of weight \( O(\ln |\mathcal{C}|) \cdot \tau(C) = O(\ln \ell) \cdot \frac{\text{opt}}{k - \ell} \).

In the case \(|\mathcal{C}| > \ell(\ell + 1)\), \( C \cap C' = \emptyset \) for any \( C, C' \in \mathcal{C} \), by Lemma 7. Then relying on Theorem 6 we modify this reduction such that every \( v \in V \setminus T \) can cover at most \( 5k \) cores; this is essentially the \( \text{SET COVERAGE} \) with (soft) capacities problem. Specifically, for each pair \((v, J)\) where \( v \in V \setminus S \) and \( J \) is a set of at most \( 5k \) edges incident to \( v \), we add a new node \( v_J \) of weight \( w_v \) with corresponding copies of the edges in \( J \). In the obtained \( \text{SET COVERAGE} \) instance the maximum size of a set is at most \( 5k \), since the cores are pairwise disjoint. Note that we do not need to construct this \( \text{SET COVERAGE} \) instance explicitly to run the greedy algorithm—we just need to determine for each \( v \in V \) the maximum number of at most \( 5k \) not yet covered cores that can be covered by \( v \). Since the cores are pairwise disjoint, this can be done in polynomial time. Note that during the greedy algorithm we may pick pairs \((v, J)\) and \((v, J')\) with distinct \( J, J' \) but with the same node \( v \), but this only makes the solution lighter. Since in the \( \text{SET COVERAGE} \) instance the maximum set size is \( 5k \), the computed solution has weight \( O(\ln k) \cdot \tau \), where here \( \tau \) is an optimal LP-value of the modified instance. Now we argue in the same way as before that \( \tau \leq \frac{\text{opt}}{k - \ell} \). Consider a feasible solution \( S' \subseteq V \setminus T \) and an edge \( J' \) such that \( G[T] \cup S' \cup J' \) is a spanning \( k \)-connected subgraph of \( G[T \cup S'] \) and \( \deg_{J'}(v) \leq 5k \) for all \( v \in S' \); such \( J' \) exists by Theorem 6. Let \( x' \) be the characteristic vector of the pairs \((v, J'_v)\) where \( v \in S' \) and \( J'_v \) is the set of edges in \( J' \) incident to \( v \). Any tight set \( A \) of \( G[T] \) has at least \( k - \ell \) neighbors in \( S' \) in the graph \( G[T] \cup S' \cup J' \), hence \( x' \frac{1}{k - \ell} \) is a feasible solution to the LP. Consequently, \( \tau \leq \frac{\text{opt}}{k - \ell} \).

This concludes the proof of Lemma 4, and thus also of Lemma 3. Consequently, the proof of Theorem 2 is also complete.

**3 General Graphs (Theorem 3)**

Here we consider the \( k \)-dominating \( \text{ROOTED SUBSET} \ \ k \text{- CONNECTIVITY} \) problem, and prove Theorem 3. In the previous section, we obtained a feasible solution to the \( \text{SUBSET} \ \ k \text{- CONNECTIVITY} \) problem by repeatedly adding to \( T \) an augmenting set \( S_\ell \) that increased the connectivity from \( \ell \) to \( \ell + 1 \), \( \ell = 0, \ldots, k - 1 \). At the beginning of each iteration \( \ell \), we computed a cheap set \( S'_\ell \) such that \( q(T \cup S'_\ell) \) (the minimum
size of a tight set in \( G(T \cup S') \) was large enough. Here we will use a similar approach, except that we will compute such \( S' \) only once—before the first iteration. This \( S' \) will guarantee that \( q(T) \geq k \) will hold in all augmentation iterations. In the augmentation problems in the previous section, the objects we needed to cover were the tight sets, that all had “deficiency” 1. Here we need to cover higher deficiency objects, as follows.

**Definition 2** An ordered pair \( \mathbb{A} = (A, A^+) \) of subsets of a groundset \( V \) with \( A \subseteq A^+ \) is called a **biset**; the set \( \partial \mathbb{A} = A^+ \setminus A \) is called the **cut** of \( \mathbb{A} \). For an edge set/graph \( H \), let \( d_H(\mathbb{A}) \) denote the number of edges in \( H \) that go from \( A \) to \( V \setminus A^+ \). For \( T \subseteq V \) we use the notation \( \mathbb{A} \cap T = (A \cap T, A^+ \cap T) \), and for \( r \in T \) we say that \( \mathbb{A} \) is a \((T, r)\)-**biset** if \( A \cap T \neq \emptyset \) and \( r \in V \setminus A^+ \).

Let \( S \) be a partial solution (possibly \( S = \emptyset \)) to a \( k \)-dominating ROOTED SUBSET \( k \)-**CONNECTIVITY** instance \((G = (V, E), T, r, k)\). From Menger’s Theorem it follows that \( G[T \cup S] \) is \( k-(T, r) \)-connected iff

\[
|\partial \mathbb{A}| + d_{G[T \cup S]}(A) \geq k \text{ for every } (T, r)\text{-biset } \mathbb{A} \text{ on } T \cup S.
\]

Since the instance we consider is \( k \)-dominating, it is equivalent to require that the above inequality will hold for every \((T \cup S, r)\)-biset \( \mathbb{A} \) on \( T \cup S \), by Lemma 1. The \((T \cup S)\)-**deficiency** of a biset \( \mathbb{A} \) on \( T \cup S \) is defined by

\[
f_{T \cup S}(\mathbb{A}) = \max\{k - |\partial \mathbb{A}| - d_{G[T \cup S]}(A), 0\} \text{ if } \mathbb{A} \text{ is a } (T \cup S, r)\text{-biset},
\]

and \( f_{T \cup S}(\mathbb{A}) = 0 \) otherwise. We say that \( \mathbb{A} \) is a \((T \cup S)\)-**deficient biset** if \( f_{T \cup S}(\mathbb{A}) \geq 1 \).

Similarly to the previous section we let

\[
q(T \cup S) = \min\{|A| : \mathbb{A} \text{ is } (T \cup S)\text{-deficient}\}
\]

We have the following from previous work [15, Theorem 2.3].

**Lemma 8** [15] Consider the ROOTED SUBSET CONNECTIVITY AUGMENTATION problem when \( G[T] \) is \((k - 1)-(T, r)\)-connected and we seek a min-weight \( S \subseteq V \setminus T \) such that \( G[T \cup S] \) is \( k-(T, r)\)-connected. If \( q(T) \geq k \) then the problem admits ratio \( O(\log n) \).

We can obtain a feasible solution to a ROOTED SUBSET \( k \)-**CONNECTIVITY** instance by repeatedly adding to \( T \) an augmenting set \( S_\ell \) that increases the \((T, r)\)-connectivity from \( \ell \) to \( \ell + 1 \), \( \ell = 0, \ldots, k - 1 \). The following lemma implies that if \( q(T) \geq k \) for the original ROOTED SUBSET \( k \)-**CONNECTIVITY** instance, then \( q(T) \geq k \) also holds for every augmentation instance.

**Lemma 9** Let \( T \) be a \( k \)-dominating set and let \( \mathbb{A} \) be a \((T \cup S)\)-deficient biset. Then \( A \cap T \neq \emptyset \) and \( f_T(\mathbb{A} \cap T) \geq f_{T \cup S}(\mathbb{A}) \). Consequently, \( q(T \cup S) \geq q(T) \).

**Proof** We cannot have \( A \subseteq S \) as then \( |\partial \mathbb{A}| + d_{G[T \cup S]}(A) \geq k \), since every node in \( S \) has at least \( k \) neighbors in \( T \); this contradicts that \( \mathbb{A} \) is \((T \cup S)\)-deficient. To see that \( f_T(\mathbb{A} \cap T) \geq f_{T \cup S}(\mathbb{A}) \), note that \( |\partial \mathbb{A} \cap T| \leq |\partial \mathbb{A}| \) and \( d_{G[T]}(A \cap T) \leq d_{G[T \cup S]}(A) \).
To see that $q(T \cup S) \geq q(T)$, note that if $A$ is a $(T \cup S)$-deficient biset with $|A|$ minimum, then $A \cap T$ is $T$-deficient (since $f_T(A \cap T) \geq f_{T \cup S}(A) \geq 1$), and that $|A \cap T| \leq |A|$.

From Lemmas 8 and 9 we get the following.

**Corollary 3** $k$-dominating ROOTED SUBSET $k$-CONNECTIVITY instances with $q(T) \geq k$ admit ratio $O(k \ln n)$.

We will show how to find an $O(k \ln n)$-approximate set $S \subseteq V \setminus T$ such that $q(T \cup S) \geq k$. Later, we will prove the following.

**Lemma 10** (Inflation Lemma for general graphs) There exists a polynomial time algorithm that, given a $k$-dominating $I$-deficient biset $G\subseteq V \cup E$, finds $S \subseteq V \setminus T$ of weight $w(S) = O(\ln n) \cdot \text{opt}$ such that $|A \cap S| \geq f_{T \cup S}(A)$ holds for any $(T \cup S)$-deficient biset $A$.

**Lemma 11** Consider the algorithm that given an $m$-dominating ROOTED SUBSET $k$-CONNECTIVITY instance starts with $T_0 = T$ and at iteration $i = 1, \ldots, p$ obtains $T_i$ by adding to $T_{i-1}$ a set $S_i$ as in Lemma 10. Let $A$ be a $T_p$-deficient biset (so $f_{T_p}(A) \geq 1$). Then $|A \cap T_p| \geq 1 + p \cdot f_{T_p}(A)$. 

**Proof** Note that $f_{T_1}(A \cap T_1) \geq f_{T_2}(A \cap T_2) \geq \cdots \geq f_{T_p}(A)$, by Lemma 9. Thus $f_{T_i}(A \cap T_i) \geq f_{T_p}(A)$ for all $i$. Consequently, applying Lemma 10 on $T_{i-1}$, $S_i$, and $A \cap T_i$ for $i = 1, \ldots, p$, we get

$$(A \cap T_1) \cap S_1 \geq f_{T_{i-1} \cup S_i}(A \cap T_i) = f_{T_i}(A \cap T_i) \geq f_{T_p}(A).$$

By Lemma 9, $A \cap T_0 \neq \emptyset$. Since $T_0, S_1, \ldots, S_p$ are pairwise disjoint we get $|A \cap T_p| \geq |A \cap T_0| + \sum_{i=1}^p |A \cap S_i| \geq 1 + p \cdot f_{T_p}(A)$, as required.

Applying Lemma 11 with $p = k - 1$ implies that we can find $S$ of weight $w(S) = O(k \log n) \cdot \text{opt}$ such that $q(T \cup S) \geq k$. By Lemma 1, substituting $T \leftarrow T \cup S$ gives an equivalent instance, that by Corollary 3 admits ratio $O(k \log n)$. Thus to finish the proof of Theorem 3 it remains only to prove Lemma 10.

**Remark** Note that in the proof of Lemma 11 we have $|A \cap T_0| \geq m - k + 1$. This is so since any $v \in A \cap S_1$ has $m$ neighbors in $T_0$, and at most $\psi_{T \cup S}(A) \leq k - 1$ of them are not in $A$. Thus in Lemma 11 we have a better bound $|A \cap T_p| \geq m - k + 1 + p$, and for $m \geq 2k - 2$ can reach a $T$-independent free instance for $p = 1$. However, this does not improve the overall ratio $O(k \log n)$ in Theorem 3, since the best ratio known for solving the problem on a $T$-independent free instance is $O(k \log n)$ [15], see Corollary 3.

In the rest of this section we prove Lemma 10. Given a ROOTED SUBSET $k$-CONNECTIVITY instance $I = (G = (V, E), w, r, T, k)$, the Lemma 10 algorithm returns a solution $S$ to a new directed instance $I' = (G' = (V, E'), w, T, r, k)$, constructed as follows (see Fig. 1a,b):
1. Remove from \( E \) all edges with both ends in \( V \setminus T \).
2. Bidirect edges with both ends in \( T \setminus \{r\} \).
3. For every \( v \in V \setminus T \), direct into \( v \) the edges between \( v \) and \( T \setminus \{r\} \).
4. Direct into \( r \) all edges incident to \( r \).
5. For every \( v \in V \setminus (T \cup \{r\}) \), add the edge \( vr \), if such does not exist.

In this new directed instance \( \mathcal{I}' \), the goal is to find min-weight \( S \subseteq V \setminus T \) such that \( G'[T \cup S] \) is \( k \)-(\( T \), \( r \))-connected, meaning that it contains \( k \) internally disjoint directed \( tr \)-paths (paths from \( t \) to \( r \)) for every \( t \in T \). The proof of the performance of the algorithm easily follows from the following three points.

1. The new instance \( \mathcal{I}' \) admits approximation ratio \( O(\log n) \).
2. \( \text{opt}(\mathcal{I}') \leq \text{opt}(\mathcal{I}) \).
3. Any feasible solution \( S \) for \( \mathcal{I}' \) obeys the requirement of Lemma 10, that \( |A \cap S| \geq f_{T \cup S}(A) \) holds for any \((T \cup S)\)-deficient biset \( A \) of the instance \( \mathcal{I} \).

We prove these three points in the following three lemmas.

**Lemma 12** The new instance \( \mathcal{I}' \) admits approximation ratio \( O(\log n) \).

**Proof** Note that the directed instance has the following special property—every directed edge incident to a non-terminal \( v \) either enters \( v \) or enters \( r \). This enables us further to reduce the problem to the case of edge costs, by assigning cost \( c(vr) = w(v) \) to the edge \( vr \) for every \( v \in V \setminus T \); other edges in \( G' \) have cost zero. In particular, every edge of positive cost is incident to the root \( r \). In [13, Theorem 1.5], this directed version of Rooted Subset \( k \)-Connectivity (such that every edge of positive cost is incident to the root), was shown to admit ratio \( O(\log n) \), by a reduction to the Submodular Cover problem. \( \square \)

**Lemma 13** If \( S \) be a feasible solution for \( \mathcal{I} \) then \( S \) is a feasible solution for \( \mathcal{I}' \).

**Proof** Let \( t \in T \) and let \( P_t \) be a set of \( k \) internally disjoint \( tr \)-paths in \( G[T \cup S] \). Let \( S_t \) be obtained by picking for each path in \( P_t \) the node in \( S \) that is closest to \( t \) on this path, if such a node exists. The graph \( G'[T \cup S] \) has \( |S_t| \) internally disjoint directed \( tr \)-paths of length 2 each that go through \( S_t \), and \( k - |S_t| \) paths that have all nodes in \( T \). \( \square \)
Lemma 14 Let $S$ be a feasible solution for the instance $I'$ and let $A$ be a $(T \cup S)$-deficient biset. Then $|A \cap S| \geq f_{T \cup S}(A) = k - |\partial A| - d_{G[T \cup S]}(A)$.

Proof Let $d_{G[T \cup S]}(A)$ be the number of directed edges from $A$ to $(T \cup S) \setminus A^+$. By Lemma 9, $A \cap T \neq \emptyset$. Let $S' = \{v \in S : vr \notin E\}$; see Fig. 1c. We claim that

$$|A \cap S'| + d_{G[T \cup S]}(A) \geq d_{G'[T \cup S]}(A).$$

To see this consider a directed edge $e'$ of $G'$ that contributes 1 to the term $d_{G'[T \cup S]}(A)$, so $e$ goes from $A$ to $(T \cup S) \setminus A^+$. If $e'$ was obtained by directing the undirected edge $e$ of $G$ in steps 2, 3, 4 of the construction, then $e$ also contributes 1 to the term $d_{G[T \cup S]}(A)$. If $e'$ was added at step 5, then $e$ contributes 1 to the term $|A \cap S'|$. Now note that since $S$ is a feasible solution to the instance $I'$, then by Menger’s Theorem $d_{G'[T \cup S]}(A) \geq k - |\partial A|$. Combining we get

$$|A \cap S| \geq |A \cap S'| \geq d_{G'[T \cup S]}(A) - d_{G[T \cup S]}(A) \geq k - |\partial A| - d_{G[T \cup S]}(A),$$

as claimed. □

Remark Note that in Lemma 14 a strict inequality may hold. This may happen if $|A \cap S| > |A \cap S'|$ (see the biset $B$ in see Fig. 1). Another possibility to consider is when an undirected edge $e \in E$ that contributes 1 to $d_{G[T \cup S]}(A)$ was removed at step 1, or was directed into $A$ at step 3; such $e$ has no contribution to $d_{G'[T \cup S]}(A)$.

This concludes the proof of Lemma 10, and thus Theorem 3 is also proved.

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References

1. Ambühl, C., Erlebach, T., Mihalákov, M., Nunkesser, M.: Constant-factor approximation for minimum weight (connected) dominating sets in unit disk graphs. In: APPROX-RANDOM, pp. 3–14 (2006)
2. Belgi, A., Nutov, Z.: An $O(\log^2 n)$-approximation algorithm for 2-edge-connected dominating set. Inf. Process. Lett. (2021) (to appear)
3. Cheng, X., Huang, X., Li, D., Weili, W., Du, D.Z.: A polynomial-time approximation scheme for the minimum-connected dominating set in ad hoc wireless networks. Networks 42(4), 202–208 (2003)
4. Clark, B.N., Colbourn, C.J., Johnson, D.S.: Unit disk graphs. Discrete Math. 86(1–3), 165–177 (1990)
5. Das, B., Bharghavan, V.: Routing in ad-hoc networks using minimum connected dominating sets. In: Proc. IEEE International Conference on Communications, pp. 376-380 (1997)
6. Ephremides, A., Wieselthier, J.E., Baker, D.I.: A design concept for reliable mobile radio networks with frequency hopping signaling. Proc. IEEE 75(1), 56–73 (1987)
7. Förster, K.T.: Approximating fault-tolerant domination in general graphs. In: ANALCO, pp. 25–32 (2013)
8. Fukunaga, T.: Approximation algorithms for highly connected multi-dominating sets in unit disk graphs. Algorithmica 80(11), 3270–3292 (2018)
9. Guha, S., Khuller, S.: Approximation algorithms for connected dominating sets. Algorithmica 20(4), 374–387 (1998)
10. Jordán, T.: On the optimal vertex-connectivity augmentation. J. Combin. Theory Ser. B 63, 8–20 (1995)
11. Klein, P., Ravi, R.: A nearly best-possible approximation algorithm for node-weighted Steiner trees. J. Algorithms 19(1), 104–115 (1995)
12. Kortsarz, G., Nutov, Z.: Approximating k-node connected subgraphs via critical graphs. SIAM J. Comput. 35(1), 247–257 (2005)
13. Kortsarz, G., Nutov, Z.: Approximating source location and star survivable network problems. Theor. Comput. Sci. 674, 32–42 (2017)
14. Mader, W.: Ecken vom grad n in minimalen n-fach zusammenhängenden graphen. Arch. Math. 23, 219–224 (1972)
15. Nutov, Z.: Approximating minimum cost connectivity problems via uncrossable bifamilies. ACM Trans. Algorithms 9(1), 1:1-1:16 (2012)
16. Nutov, Z.: Approximating subset k-connectivity problems. J. Discrete Algorithms 17, 51–59 (2012)
17. Nutov, Z.: Improved approximation algorithms for k-connected m-dominating set problems. Inf. Process. Lett. 140, 30–33 (2018)
18. Nutov, Z.: Chapter 13: Node-connectivity survivable network problems. In: Gonzalez, T.F. (ed.) Handbook on Approximation Algorithms and Metaheuristics, vol. 2, 2nd edn. Chapman & Hall/CRC, New York (2018)
19. Nutov, Z.: 2-Node-connectivity network design. In: WAOA, pp. 220–235 (2020)
20. Shi, Y., Zhang, Z., Mo, Y., Du, D.Z.: Approximation algorithm for minimum weight fault-tolerant virtual backbone in unit disk graphs. IEEE/ACM Trans. Netw. 25(2), 925–933 (2017)
21. Thai, M., Zhang, N., Tiwari, R., Xu, X.: On approximation algorithms of k-connected m-dominating sets in disk graphs. Theor. Comput. Sci. 385, 49–59 (2007)
22. Willson, J., Zhang, Z., Wu, W., Du, D.Z.: Fault-tolerant coverage with maximum lifetime in wireless sensor networks. In: INFOCOM, pp. 1364–1372 (2015)
23. Zhang, Z., Zhou, J., Tang, S., Huang, X., Du, D.Z.: Computing minimum k-connected m-fold dominating set in general graphs. INFORMS J. Comput. 30(2), 217–224 (2018)
24. Zou, F., Li, X., Gao, S., Weili, W.: Node-weighted Steiner tree approximation in unit disk graphs. J. Comb. Optim. 18(4), 342–349 (2009)
25. Zou, F., Wang, Y., XiaoHua, X., Li, X., Hongwei, D., Wan, P.J., Weili, W.: New approximations for minimum-weighted dominating sets and minimum-weighted connected dominating sets on unit disk graphs. Theor. Comput. Sci. 412(3), 198–208 (2011)

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