Fusion rules
and the
Patera-Sharp generating-function method*

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Abstract

We review some contributions on fusion rules that were inspired by the work of Sharp, in particular, the generating-function fusion method for tensor-product coefficients that he developed with Patera. We also review the Kac-Walton formula, the concepts of threshold level, fusion elementary couplings, fusion generating functions and fusion bases. We try to keep the presentation elementary and exemplify each concept with the simple $\widehat{su}(2)_k$ case.

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1. Introduction

The Patera-Sharp generating function for \( su(2) \) tensor products reads [1]:

\[
G^{su(2)}(L, M, N) = \frac{1}{(1 - LM)(1 - LN)(1 - MN)} \tag{1.1}
\]

where the multiplicity of the representation \( (n) \) in the tensor product \( (\ell) \otimes (m) \) is the coefficient of \( L^\ell M^m N^n \) in the series expansion of (1.1). The derivation of this expression is based on manipulations of the character generating functions [2].

Key concepts on fusion rules have been obtained by looking for the affine-fusion extension of this simple-looking expression and its simplest higher-rank relatives. These are: the threshold level, fusion elementary couplings and fusion bases. Before reviewing these results, we briefly discuss the Kac-Walton formula. This last result can also be linked, albeit loosely, to Bob Sharp. Indeed, it is an affine extension of the Racah-Speiser algorithm for computing tensor-product coefficients, one of Sharp’s favorite techniques. He presented it in his course on group theory, where two of the authors (PM and MW) learned the fundamentals of this subject.

2. Fusion rules: the set up

Fusion rules give the number of independent couplings between three given primary fields in conformal field theories. We are interested in those conformal field theories having a Lie algebra symmetry. These are the Wess-Zumino-Witten models [3], whose spectrum-generating algebra is an affine Lie algebra at integer level. Their primary fields are in 1-1 correspondence with the integrable representations of the appropriate affine Lie algebra at level \( k \). Denote this set by \( P_+^{(k)} \) and a primary field by the corresponding affine weight \( \hat{\lambda} \). Fusion coefficients \( N^{(k)}_{\hat{\lambda}\hat{\mu}\hat{\nu}} \) are defined by the fusion product

\[
\hat{\lambda} \times \hat{\mu} = \sum_{\hat{\nu} \in P_+^{(k)}} N^{(k)}_{\hat{\lambda}\hat{\mu}\hat{\nu}} \hat{\nu} \tag{2.1}
\]

To simplify the presentation, we consider only the algebra \( \hat{su}(N) \).

An affine weight may be written as

\[
\hat{\lambda} = \sum_{i=0}^{N-1} \lambda_i \hat{\omega}_i = [\lambda_0, \lambda_1, ..., \lambda_{N-1}] \tag{2.2}
\]
where $\hat{\omega}_i$ denote the fundamental weights of $\hat{su}(N)$. If the Dynkin labels $\lambda_i$ are nonnegative integers, the weight $\hat{\lambda}$ is the highest weight of an integrable representation of $\hat{su}(N)$ at level $k$, with $k$ defined by $k = \sum_{i=0}^{N-1} \lambda_i$. To the affine weight $\hat{\lambda}$, we associate a finite weight $\lambda$ of the finite algebra $su(N)$:

$$\lambda = \sum_{i=1}^{N-1} \lambda_i \omega_i = (\lambda_1, ..., \lambda_{N-1}) \quad (2.3)$$

where $\omega_i$ are the fundamental weights of $su(N)$. Thus $\hat{\lambda}$ is uniquely fixed by $\lambda$ and $k$.

3. Fusion rules and tensor products: the Kac-Walton formula

The fusion coefficient $N^{(k)}_{\hat{\lambda}\hat{\mu}} \hat{\nu}$ is fixed to a large extent by the tensor-product coefficient pertaining to the product of the corresponding finite representations. We denote by $N^{(k)}_{\lambda\mu} \nu$ the multiplicity of the representation $\nu$ in the tensor product $\lambda \otimes \mu$:

$$\lambda \otimes \mu = \sum_{\nu \in P_+} N^{(k)}_{\lambda\mu} \nu \quad (3.1)$$

By abuse of notation, we use the same symbol for the highest weight and the highest-weight representation. $P_+$ represents the set of integrable finite weights. The precise relation between tensor-product and fusion-rule coefficients is given by the Kac-Walton formula [4,5]:

$$N^{(k)}_{\hat{\lambda}\hat{\mu}} \hat{\nu} = \sum_{w \in \hat{W}, \xi = \tilde{\nu} \in P_+^{(k)}} N_{\lambda\mu}^{(k)} \xi \epsilon(w) \quad (3.2)$$

$w$ is an element of the affine Weyl group $\hat{W}$, of sign $\epsilon(w)$, and the dot indicates the shifted action: $w \cdot \hat{\lambda} = w(\hat{\lambda} + \hat{\rho}) - \hat{\rho}$, where $\hat{\rho}$ stands for the affine Weyl vector $\hat{\rho} = \sum_{i=0}^{N-1} \hat{\omega}_i$.

The Kac-Walton formula can be transcribed into a simple algorithm: one first calculates the tensor product of the corresponding finite weights and then extends every weight to its affine version at the appropriate level $k$. Weights with negative zeroth Dynkin label are then shift-reflected to the integrable affine sector. Weights that cannot be shift-reflected to the integrable sector are ignored (this is the case, for example, for those with zeroth Dynkin label equal to $-1$).
Here is a simple example: consider the $su(2)$ tensor-product

$$(2) \otimes (4) = (2) \oplus (4) \oplus (6) \quad (3.3)$$

and its affine extension at level 4:

$$[2, 2] \times [0, 4] = [2, 2] + [0, 4] + [-2, 6] \quad (3.4)$$

The last weight must be reflected since it is not integrable: the shifted action of $s_0$, the reflection with respect to the zeroth affine root, is

$$s_0 \cdot [-2, 6] = s_0([-2, 6] + [1, 1]) - [1, 1] = [0, 4] \quad (3.5)$$

and this contributes with a minus sign ($\epsilon(s_0) = -1$), cancelling then the other $[0, 4]$ representation; we thus find:

$$[2, 2] \times [0, 4] = [2, 2] \quad (3.6)$$

The relation between tensor products and fusion was further explored in [6], published in a special volume of the Canadian Journal of Physics dedicated to Prof. R.T. Sharp.

4. The idea of threshold level

The result of the above computation is manifestly level dependent. Let us reconsider the same product, but at level 5. The affine extension of the product becomes

$$[3, 2] \times [1, 4] = [3, 2] + [1, 4] + [-1, 6] \quad (4.1)$$

The last weight is thus ignored and the final result is $[3, 2] \times [1, 4] = [3, 2] + [1, 4]$. For $k > 5$ it is clear that there are no truncations, hence no difference between the fusion coefficients and the tensor products. Moreover, we see that the representation (4) occurs at level 5 and higher. We then say that its threshold level, denoted by $k_0$, is 5. The threshold level is thus the smallest value of $k$ such that the fusion coefficient $\mathcal{N}_{\lambda\hat{\mu}}^{(k)} \hat{\nu}$ is non-zero, when $\mathcal{N}_{\lambda\hat{\mu}}^{(k)} \hat{\nu} \in \{0, 1\}$, for all levels $k$.\[\Box\] If we indicate the threshold level by a subscript, we can write

$$(2) \otimes (4) = (2)_4 \oplus (4)_5 \oplus (6)_6 \quad (4.2)$$

\[1\] More generally, we say there are $\mathcal{N}_{\lambda\mu}^{\nu}$ couplings, each having its own threshold level $k_0$. For fixed $\{\lambda, \mu; \nu\}$ then, one gets a multi-set of threshold levels. A simple example: $su(3)$ with $\lambda = \mu = \nu$ labelling the adjoint representation has threshold levels $\{2, 3\}$. 
To read off a fusion at fixed level $k$, we only keep terms with index not greater than $k$. This implies directly the inequality

$$\mathcal{N}_{\lambda \mu}^{(k)} \hat{\nu} \leq \mathcal{N}_{\lambda \mu}^{(k+1)} \hat{\nu}$$

(4.3)

which in turn yields

$$\lim_{k \to \infty} \mathcal{N}_{\lambda \mu}^{(k)} \hat{\nu} = \mathcal{N}_{\lambda \mu} \hat{\nu}$$

(4.4)

The concept of threshold level was first introduced in [7]. Its origin is directly rooted in the generating-function method applied to fusion rules. This is reviewed in the next section, focusing again on the simple $\hat{su}(2)$ case.

5. Fusion generating functions

The result (1.1) on the $su(2)$ tensor-product generating function can be understood as follows: all couplings can be described by appropriate products of three tensor-product elementary couplings:

$$E_1 = LM \quad E_2 = LN \quad E_3 = MN$$

(5.1)

$G_{su}(2)$ can thus be written compactly as

$$G_{su}(2)(L, M, N) = \prod_{i=1}^{3} \frac{1}{1 - E_i}$$

(5.2)

How can we construct the affine extension of this generating function? One certainly needs to introduce a further dummy variable, say $d$, in order to keep track of the extra variable $k$. Then one could try to introduce factors of $d$ appropriately. A natural guess is

$$G_{\hat{su}(2)} = \frac{1}{(1 - d)(1 - dLM)(1 - dLN)(1 - dMN)}$$

(5.3)

That turns out to be the right answer: this reproduces the $\hat{su}(2)_k$ fusion rules. The prefactor is justified as follows: the fusion of the ‘vacuum’ with itself, $[k, 0] \times [k, 0] = [k, 0]$, exists at every level and this is precisely taken into account by the factor $1/(1 - d)$. With the threshold level insight, we can also naturally justify the factors of $d$ multiplying the three elementary couplings: their power yields their threshold level. This expression
was first proved in [7]. From this example and that of the $\widehat{su}(3)$ generating function, we conjectured that any fusion coupling is characterized by a threshold level. This was subsequently checked with the $\widehat{so}(5)$ case [8]. This is now understood to be a consequence of the Gepner-Witten depth rule [3], as shown in [9].

From the above expression, we also infer the existence of fusion elementary couplings. For $\widehat{su}(2)$, there are four of them

$$
\begin{align*}
\hat{E}_0 : d : (0) \otimes (0) & \supset (0)_1, \\
\hat{E}_1 : dLM : (1) \otimes (1) & \supset (0)_1, \\
\hat{E}_2 : dLN : (1) \otimes (0) & \supset (1)_1, \\
\hat{E}_3 : dMN : (0) \otimes (1) & \supset (1)_1.
\end{align*}
$$

As explained above, subscripts indicate the threshold level.

A re-derivation of (5.3) was presented in [10]. The method used there was amenable to generalization, unlike the original proof in [7]. Consequently, [10] displays further examples of fusion generating functions.

6. Tensor products, linear inequalities and elementary couplings

There are simple combinatorial methods that can be used for calculating $su(N)$ tensor products, for instance, the Littlewood-Richardson (LR) rule. It is thus natural to ask whether we can read off the threshold level of a coupling from its LR tableau.

Integrable weights in $su(N)$ can be represented by tableaux: the weight $(\lambda_1, \lambda_2, \ldots, \lambda_{N-1})$ is associated to a left justified tableau of $N-1$ rows with $\lambda_1 + \lambda_2 + \ldots + \lambda_{N-1}$ boxes in the first row, $\lambda_2 + \ldots + \lambda_{N-1}$ boxes in the second row, etc. Equivalently, the tableau has $\lambda_1$ columns of 1 box, $\lambda_2$ columns of 2 boxes, etc. The scalar representation has no boxes, or equivalently, any number of columns of $N$ boxes. For instance:

$$
\begin{align*}
su(3) : (1, 1) & \leftrightarrow \boxed{\phantom{\boxed{}}} \\
su(4) : (2, 3, 0) & \leftrightarrow \boxed{\phantom{\boxed{}}} \\
su(4) : (2, 3, 0) & \leftrightarrow \boxed{\phantom{\boxed{}}}
\end{align*}
$$

The Littlewood-Richardson rule is a simple combinatorial algorithm that calculates the decomposition of the tensor product of two $su(N)$ representations $\lambda \otimes \mu$. The second tableau ($\mu$) is filled with numbers as follows: the first row with 1’s, the second row with 2’s, etc. All the boxes with a 1 are then added to the first tableau according to following restrictions: (1) the resulting tableau must be regular: the number of boxes in a given row
must be smaller or equal to the number of boxes in the row just above; (2) the resulting tableau must not contain two boxes marked by 1 in the same column. All the boxes marked by a 2 are then added to the resulting tableaux according to the above two rules (with 1 replaced by 2) and the further restriction: (3) in counting from right to left and top to bottom, the number of 1’s must always be greater or equal to the number of 2’s. The process is repeated with the boxes marked by a 3, 4, . . . , N − 1, with the additional rule that the number of i’s must always be greater or equal to the number of i + 1’s when counted from right to left and top to bottom. The resulting Littlewood-Richardson (LR) tableaux are the Young tableaux of the irreducible representations occurring in the decomposition.

Here is a simple su(3) example: (1, 1) ⊗ (1, 1) ⊃ 2(1, 1) since we can draw two LR tableaux with shape (1, 1) and an extra column of three boxes (the total number of boxes being preserved, the resulting LR tableau must have 6 boxes):

```
  1 2 1
  2 1 2
```

These rules can be rephrased in an algebraic way as follows [11]. Define \( n_{ij} \) to be the number of boxes \( i \) that appear in the LR tableau in the row \( j \). The LR conditions read:

\[
\lambda_{j - 1} + \sum_{i=1}^{k-1} n_{i,j-1} - \sum_{i=1}^{k} n_{ij} \geq 0 \quad 1 \leq k < j \leq N \quad j \neq 1 \tag{6.3}
\]

and

\[
\sum_{j=i}^{k} n_{i-1,j-1} - \sum_{j=i}^{k} n_{ij} \geq 0 \quad 2 \leq i \leq k \leq N \quad \text{and} \quad i \leq N - 1. \tag{6.4}
\]

The weight \( \mu \) of the second tableau and the weight \( \nu \) of the resulting LR tableau are easily recovered from these data.

The combined equations (6.3) and (6.4) constitute a set of linear and homogeneous inequalities. We call this the LR (or tensor-product) basis. As described in [12], the Hilbert basis theorem guarantees that every solution can be expanded in terms of the elementary solutions of these inequalities. This is a key concept for the following (see [13] for an extensive discussion of these methods). A sum of two solutions translates into the product of the corresponding couplings, more precisely, to the stretched product (denoted by \( \cdot \)) of the corresponding two LR tableaux. This is defined as follows. Denote the void boxes of
a LR tableau by a 0, so that \( n_{0j} = \sum_{i=j}^{N-1} \lambda_i \). A tableau is thus completely characterized by the data \( \{ n_{ij} \} \) where now \( i \geq 0 \). Then, the tableau obtained by the stretched product of the tableaux \( \{ n_{ij} \} \) and \( \{ n'_{ij} \} \) is simply described by the numbers \( \{ n_{ij} + n'_{ij} \} \), e.g.,

\[
\begin{array}{ccc}
1 & 2 & 2 \\
2 & 3 & 2 \\
4 & &
\end{array}
\cdot
\begin{array}{ccc}
1 & 1 & 2 \\
1 & 2 & 2 \\
2 & 2 & 3 \\
4 & &
\end{array}
= 
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 2 \\
2 & 2 & 2 \\
4 & &
\end{array}
\]

(6.5)

Let us now turn to the \( su(2) \) case. The complete set of inequalities for \( su(2) \) variables \( \{ \lambda_1, n_{11}, n_{12} \} \) is simply

\[
\lambda_1 \geq n_{12} \quad n_{11} \geq 0 \quad n_{12} \geq 0
\]

(6.6)

The first one expresses the fact that two boxes marked by a 1 cannot be in the same column while the other two are obvious. The other weights are fixed by the relation \( \mu_1 = n_{11} + n_{12} \) and \( \nu_1 = \lambda_1 + n_{11} - n_{12} \). Any solution of these inequalities describes a coupling. By inspection, the elementary solutions of this set of inequalities are

\[
(\lambda_1, n_{11}, n_{12}) = (1, 0, 1), \quad (1, 0, 0), \quad (0, 1, 0)
\]

(6.7)

(For more complicated cases, we point out that powerful methods to find the elementary solutions are described in [10].) These correspond to the following LR tableaux, denoted respectively \( E_1, E_2, E_3 \):

\[
E_1 : \begin{array}{c}
\boxed{1}
\end{array}, \quad E_2 : \begin{array}{c}
\phantom{1}
\end{array}, \quad E_3 : \begin{array}{c}
\boxed{1}
\end{array}
\]

(6.8)

It is also manifest that there are no linear relations between these couplings. Any stretched product of these elementary tableaux is an allowed \( su(2) \) coupling. Because there are no relations between the elementary couplings, this decomposition is unique. We thus see that the description of the elementary couplings captures, in a rather economical way, the whole set of solutions of (6.6), that is, the whole set of \( su(2) \) couplings.

7. Reformulating the fusion rules in terms of linear inequalities

Consider now the affine-fusion extension of the reformulation of the \( su(2) \) tensor products in terms of linear inequalities. The elementary couplings have a natural affine extension, denoted by a hat, and their threshold levels are easily computed from the Kac-Walton
The result is: $k_0(\hat{E}_i) = 1$ for $i = 1, 2, 3$. We observe that these values of $k_0$ are the same as the number of columns. Since the product of fusion elementary couplings is also a fusion and because this decomposition is unique, we can read off the threshold level of any coupling, hence of any LR tableau, simply from the number of its columns:

$$k_0 = \#\text{columns} = \lambda_1 + n_{11} \quad (7.1)$$

And since $k$ is necessarily greater than $k_0$, we have obtained the extra inequality:

$$k \geq \lambda_1 + n_{11} \quad (7.2)$$

This together with (6.6) yield a set of inequalities describing completely the fusion rules. This is what we call a fusion basis, here the fusion basis of $\hat{su}(2)$. As in the finite case, the fusion couplings can be described in terms of elementary fusions. These correspond to the elementary solutions of the four inequalities, which are easily found to be

$$(k, \lambda_1, n_{11}, n_{12}) = (1, 0, 0, 0), \ (1, 1, 0, 1), \ (1, 1, 0, 0), \ (1, 0, 1, 0) \quad (7.3)$$

They correspond respectively to the coupling

$$\hat{E}_0 : [1, 0] \times [1, 0] \supset [1, 0] \quad \hat{E}_2 : [0, 1] \times [1, 0] \supset [0, 1],$$

$$\hat{E}_1 : [0, 1] \times [0, 1] \supset [1, 0] \quad \hat{E}_3 : [1, 0] \times [0, 1] \supset [0, 1]. \quad (7.4)$$

Any fusion has an unique decomposition in terms of these elementary couplings. For instance

$$[3, 2] \times [1, 4] \supset [1, 4] \leftrightarrow \begin{array}{|c|c|c|c|} \hline & & & \hline & 1 & 1 & 1 \hline \end{array} \leftrightarrow \hat{E}_1 \hat{E}_2 \hat{E}_3 : \ k_0 = 5 \quad (7.5)$$

8. Constructing the fusion basis: Farkas’ lemma

For algebras other than $\hat{su}(2)$, the threshold level is not simply the number of columns. So the question is: how can we derive the fusion basis? The following strategy was developed in [10]:

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2 Let us mention here that for the classical simple Lie algebras, the tableau methods for tensor products have been modified to implement the Kac-Walton formula for fusions – see [14] in the Sharp volume of the Canadian Journal of Physics.
1) Write the LR inequalities;
2) from these, find the tensor-product elementary couplings;
3) from these, find fusion elementary couplings;
4) from these, reconstruct the fusion basis.

To go from step 2 to step 3, we need some tools; we describe below a method based on the outer automorphism group. Similarly to go from 3 to 4, we need a further ingredient: this is the Farkas’ lemma. We discuss these techniques in turn.

Let us start from the set of tensor-product elementary couplings \( \{ E_i, i \in I \} \) for some set \( I \) fixed by the particular \( su(N) \) algebra under study. For each \( E_i \), we calculate the threshold level \( k'_0(E_i) \) and this datum specifies the affine extension of \( E_i \), denoted \( \hat{E}_i \). We have then a partial set of fusion elementary couplings with the set \( \{ \hat{E}_i, i \in I \} \). Our conjecture is that the missing fusion elementary couplings can all be generated by the action of the outer-automorphism group. For \( \hat{su}(N) \), this group is simply \( \{ a^n | n = 0, \ldots, N - 1 \} \), with

\[
a[\lambda_0, \lambda_1, \ldots, \lambda_{N-1}] = [\lambda_{N-1}, \lambda_0, \ldots, \lambda_{N-2}]
\]

The conjecture is based on the invariance relation

\[
\mathcal{N}_{a^n \lambda, a^m \mu}^{(k)} a^{n+m} \nu = \mathcal{N}_{\lambda, \mu}^{(k)} \nu
\]

It amounts to supposing that the full set is contained in \( \{ \mathcal{A} \hat{E}_i, i \in I, \forall \mathcal{A} \} \). Here \( \mathcal{A} \hat{E}_i \) indicates a coupling of weights

\[
\mathcal{A}\{\hat{\lambda}, \hat{\mu}; \hat{\nu}\} = \{a^n \hat{\lambda}, a^m \hat{\mu}; a^{n+m} \hat{\nu}\},
\]

\( n, m \) being arbitrary integers defined modulo \( N \), if \( \hat{E}_i \) has weights \( \{\hat{\lambda}, \hat{\mu}; \hat{\nu}\} \). The conjectured completeness requires the consideration of all possible pairs \( (n, m) \).

Let us illustrate this with the \( \hat{su}(2) \) case. Start with the elementary coupling \( E_1 : (1) \otimes (1) \supset (0) \), which, as already indicated, arises at level 1: \( k_0(E_1) = 1 \). The corresponding

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3 Note that we do not suppose that the action of \( \mathcal{A} \) on an elementary coupling will necessarily produce another elementary coupling. Indeed, the resulting coupling could be a product of elementary couplings. What is conjectured here is that all fusion elementary couplings can be generated in this way.
fusion is thus \([0, 1] \times [0, 1] \supset [1, 0]\), denoted as \(\hat{E}_1\). We now consider all possible actions of the outer-automorphims group on it. Since this group is of order 2, there are 4 possible choices for the doublet \((n, m)\):

\[(a^n, a^m) \in \{(a, a), (1, 1), (1, a), (a, 1)\}\]  

(8.4)

with \(a[\lambda_0, \lambda_1] = [\lambda_1, \lambda_0]\). This generates the set of four elementary couplings found previously, in the respective order \(\hat{E}_0, \hat{E}_1, \hat{E}_2, \hat{E}_3\). Thus, from one tensor-product elementary coupling, all four fusion elementary couplings are deduced.

We now turn to Farkas’ lemma. For its presentation, it is convenient to use an exponential description of the couplings, that is,

\[(k, \lambda_i, n_{ij}) \rightarrow d^k L_i^k N_{ij}^{n_{ij}}\]  

(8.5)

d, \(L_i\), \(N_{ij}\) being dummy variables. For instance \(\hat{E}_1\) is represented by \(dL_1 N_{12}\). If we collectively describe a coupling by the complete set of variables \(\{x_i\}\), we have

\[\{x_i\} \rightarrow \{X_i^{x_i}\}\]  

(8.6)

A particular coupling is thus described by a given product \(\prod_i X_i^{x_i}\) with fixed \(x_i\). Its decomposition in terms of elementary couplings take the form \(\prod_i \hat{E}_i^{a_i}\). Now, since \(\hat{E}_i\) can be decomposed in terms of the \(X_j\) as

\[\hat{E}_i = \prod_j X_j^{e_{ij}}\]  

(8.7)

it means that reading off a particular coupling means that we are interested in a specific choice set of positive integers \(\{x_i\}\) fixed by

\[\sum_i a_i e_{ij} = x_j\]  

(8.8)

in terms of some positive integers \(a_i\). We are thus looking for the existence conditions for such a coupling, i.e., the underlying set of linear and homogeneous inequalities. This is exactly what the Farkas’ lemma [15] gives us: given the knowledge of the elementary couplings, it allows us to recover the underlying set of inequalities. For tensor products, this is of no interest since we know the corresponding set of inequalities and our elementary
couplings have been extracted from them. But the situation is quite different in the fusion case, where the fusion basis is unknown.

For our application we need the following modification of the lemma, proved in [10]:

Lemma: Let $A$ be an $r \times m$ integer matrix and let $\epsilon_j$ ($j = 1, \ldots, n$) be a set of fundamental solutions to

$$Ax \geq 0, \quad x \in \mathbb{N}^m. \quad (8.9)$$

Let $V$ be the $m \times n$ matrix with entries $V_{ij} = (\epsilon_j)_i$ (for $i = 1, \ldots, m, \ j = 1, \ldots, n$), i.e., the columns of $V$ are a set of fundamental solutions to (8.9). Let $e_w$ ($w = 1, \ldots, \ell$) be a fundamental system of solutions of $u^TV \geq 0$, (not necessarily positive) $u \in \mathbb{Z}^m$, and let $E$ be the $\ell \times m$ matrix with entries $E_{wi} = (e_w)_i$, i.e., the rows of $E$ are the fundamental solutions $e_w$ ($w = 1, \ldots, \ell$). Then the solution set of the system

$$Ex \geq 0, \quad x \in \mathbb{N}^m \quad (8.10)$$

is the same as the solution set of (8.9).

To link the lemma to the situation presented above, we note that the entries $V_{ij}$ of the matrix $V$ are given here by the numbers $\epsilon_{ji}$ appearing in (8.7). The relation $V a = x$ describes a generic coupling $\prod \hat{E}_i^{a_i}$, and our goal is to find the defining system of inequalities for $x$ that underly the existence of this coupling.

Take a simple example, the $\widehat{su}(2)$ case. The elementary couplings and the corresponding vectors $\epsilon_i$ are

$$\hat{E}_0 : d : \quad \epsilon_0 = (1, 0, 0, 0), \quad \hat{E}_2 : dL_1 : \quad \epsilon_2 = (1, 1, 0, 0) \quad (8.11)$$

$$\hat{E}_1 : dL_1N_{12} : \quad \epsilon_1 = (1, 1, 0, 1), \quad \hat{E}_3 : dN_{11} : \quad \epsilon_3 = (1, 0, 1, 0)$$

From the vectors $\epsilon_i$ with components $\epsilon_{ij}$, we form the matrix $V$ with entries $V_{ij} = \epsilon_{ji}$:

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (8.12)$$

With $a$ and $x$ denoting the column matrices of entries $a_i$ and $x_i$ respectively, we have the matrix equation

$$Va = x \quad (8.13)$$
Again, this equation describes a general fusion coupling $\prod_i \hat{E}_i^{a_i}$. We now want to unravel the underlying system of inequalities. For this, we consider the fundamental solutions of

$$u^\top V \geq 0$$

where $u$ is the vector of entries $u_i$. These inequalities read

$$u_0 \geq 0, \quad u_0 + u_1 + u_3 \geq 0, \quad u_0 + u_1 \geq 0, \quad u_0 + u_2 \geq 0$$

In this simple case, the elementary couplings can be found by inspection and these are:

$$e_0 = (1, -1, -1, 0), \quad e_1 = (0, 0, 0, 1), \quad e_2 = (0, 1, 0, -1), \quad e_3 = (0, 0, 1, 0)$$

Finally, we consider the conditions $e_i x \geq 0$, with $(x_0, x_1, x_2, x_3) = (k, \lambda_1, n_{11}, n_{12})$. They read, in order,

$$k \geq \lambda_1 + n_{11}, \quad n_{12} \geq 0, \quad \lambda_1 \geq n_{12}, \quad n_{11} \geq 0$$

The last three conditions define the LR basis. The first one is the additional fusion constraint. Together, they form the $\hat{su}(2)$ fusion basis.

9. Constructing the fusion basis: polytope techniques

In the previous section, the Farkas’ lemma has been used to construct the fusion basis out of the set of fusion elementary couplings. There are alternative approaches, however. Another one is based on the reinterpretation of the fusion-rule computations in terms of counting points inside a polytope. A polytope can be described by its vertices or its facets. In our context, the vertices are represented by the fusion elementary couplings and the facets are the inequalities for which the elementary couplings are the elementary solutions. The reconstruction of the facets of a polytope from its vertices is thus another way to generate the fusion basis. This method is described in [16].

In the special case of $su(N)$, Berenstein-Zelevinsky triangles can also be used to derive the polytope description of a fusion basis [17], by considering so-called virtual triangles. Multiple sum formulas can then be written for fusion coefficients of various types. However, as are all methods to date, this one is difficult to extend to higher rank. Assigning a threshold level to a Berenstein-Zelevinsky triangle becomes very rapidly more difficult with increasing rank (see [18]).
The fusion bases have been constructed for $\hat{su}(3)$, $\hat{su}(4)$ and $\hat{sp}(4)$. Note that for algebras other than $\hat{su}(N)$, we replace the LR basis by the Berenstein and Zelevinsky basis [19]. This leads to explicit expressions for the threshold levels, hence for the fusion coefficients.

We stress that the reformulation of the problem of computing fusion rules in terms of a fusion basis solves, in principle, the quest for a combinatorial method, since it reduces a fusion computation to solving inequalities. But it is probably not the optimal solution to the quest for an efficient combinatorial description.

The main open problem concerning fusion bases is to find a fundamental and Lie algebraic way of deriving the basis, analogous in spirit to the Berenstein-Zelevinsky conjectures for generic Lie algebras in [19].

The methods described in section 8, involving Farkas’ lemma, are general and powerful, but they may not be Lie algebraic enough. Perhaps one should step back and look at a first principles description of the tensor product couplings, and its adaptation to fusion. This was done in [20]. Three-point functions were calculated that can be regarded as generating functions for tensor product couplings, and a very simple method was found for adapting the results to fusion couplings. In principle, the procedure works for any semi-simple Lie algebra. Unfortunately, these more Lie algebraic methods are inevitably more involved. It gives more information (such as operator product coefficients instead of just fusion coefficients), but it is not clear that it can be implemented effectively on higher rank algebras.

\[\text{Related methods for obtaining the threshold level are presented in [18].}\]
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