Abstract

We propose a way to study one-dimensional statistical mechanics models with complex-valued action using transfer operators. The argument consists of two steps. First, the contour of integration is deformed so that the associated transfer operator is a perturbation of a normal one. Then the transfer operator is studied using methods of semi-classical analysis.

In this paper we concentrate on the second step, the main technical result being a semi-classical estimate for powers of an integral operator which is approximately normal.

1 Introduction

Operator-theoretic methods are known to be of great help in one-dimensional statistical mechanics.

Consider the following prototypical example. Let $W > 0$, and let $V : \mathbb{R} \to \mathbb{R}$ be a potential growing sufficiently fast at infinity. It is known that there exists a unique probability measure on $\mathbb{R}^2$ such that, for every $N \in \mathbb{N}$, the
conditional probability density at $\phi \in \mathbb{R}^{\mathbb{Z}}$ given $\phi|_{\{-N,\ldots,N\}^c}$ is proportional to
\[
\exp \left\{ - \sum_{j=-N}^{N} V(\phi_j) - \sum_{j=-N-1}^{N} W^2(\phi_j - \phi_{j+1})^2 \right\}.
\] (1.1)

This model corresponds to the (real-valued) action
\[
S(\phi) = \sum_{j} V(\phi_j) + \sum_{j} W^2(\phi_j - \phi_{j+1})^2.
\] (1.2)

The properties of the model (1.1), such as exponential decay of correlations (between $\phi_j$ and $\phi_k$ as $|j-k| \to \infty$), are encoded in the spectral structure of the self-adjoint operator $K$ (called the transfer operator, or transfer matrix) with kernel given by
\[
K(x, y) = \exp \left\{ -W^2(x - y)^2 - \frac{V'(0)}{4} (x^2 + y^2) \right\}.
\] (1.3)

When $W \gg 1$ is large, semi-classical analysis allows to relate the spectral properties of $K$ to those of a simpler operator $\tilde{K}$ (the harmonic approximation), in which the potential is quadratic and for which computations can be performed explicitly; the small quantity $W^{-1}$ plays the rôle of the semi-classical parameter $\hbar$.

In the case when the unique minimum of $V$ is $V(0) = 0$, $\tilde{K}$ is given by the harmonic oscillator
\[
\tilde{K}(x, y) = \exp \left\{ -W^2(x - y)^2 - \frac{V''(0)}{4} (x^2 + y^2) \right\}.
\] (1.4)

When $V$ has several minima, $K$ is approximated by a direct sum of several harmonic oscillators.

The semi-classical approach to various problems of one-dimensional statistical mechanics is presented in detail in the monograph of Helffer [1].

On the other hand, in many questions in statistical mechanics the potential $V$ is complex-valued. This problem, often referred to as the “sign problem” or “complex action problem”, is inherent, for example, to lattice quantum chromodynamics (see for example Muroya et al. [7] and Splittorff and Verbaarschot [10]), and also arises in supersymmetric models appearing in the study of random operators (see the reviews of Spencer [8, 9]).
A naïve attempt to apply the methods tailored for real-valued action to this situation encounters immediate obstacles. In the context of transfer operators, neither $K$ nor $\tilde{K}$ is self-adjoint; thus perturbation theory is not easily set on a rigorous basis, and on the other hand the spectrum of $K$ is not directly connected to the semigroup $K^n$. We refer to the articles of Davies [1, 2] and further to the monograph of Helffer [5, Chapters 13–15] and to the PhD thesis of Henry [6], where difficulties in the semi-classical analysis of non self-adjoint operators are discussed, along with some positive results.

Our goal in this paper is to suggest a strategy which allows to apply semi-classical analysis to models with complex-valued action, in spite of these difficulties. Here we apply it to a toy model (with one saddle); in a subsequent work, we hope to apply it to a statistical mechanics model arising from the supersymmetric analysis of a class of random band matrices; see [3] for an analysis of a related three-dimensional model, and the review of Spencer [8] for a discussion of supersymmetric models arising from random band matrices, and the possible transfer matrix approach.

The strategy we suggest is as follows. Before setting up the transfer operator, we deform the contour of integration so that the harmonic approximation $\tilde{K}$ is almost normal (in appropriate sense). Then we set up the transfer operator and analyse it using the (semi-) classical tools. In this paper, we restrict ourself to the simplest deformation

$$\phi \leftarrow \zeta \phi, \quad |\zeta| = 1 ; \quad (1.5)$$

in general, a more complicated deformation (similar to (1.5) near each saddle point but different away from the saddles) may be required.

To motivate the idea, let us consider the differential operator

$$L = -\frac{1}{W^2} \frac{d^2}{dx^2} + (a + ib)x^2, \quad a > 0.$$  

One can always find $\zeta, \ |\zeta| = 1$, so that after the change of variables $x \leftarrow \zeta x$ the operator $\hat{L}$ becomes normal, i.e. a scalar multiple of

$$\hat{L} = -\frac{1}{W^2} \frac{d^2}{dx^2} + |a + ib|x^2 \ .$$

In a similar way, for an integral operator $K$, one can rotate the contour so that the harmonic approximation to $K$ near the saddle point (i.e. an operator with quadratic $V$) becomes normal.

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The main result of this paper justifies the approximation $K \approx \tilde{K}$ for a class of operators $K$ of a form similar to (1.3) by the corresponding harmonic approximation $\tilde{K}$, in the case when $\tilde{K}$ is almost normal. The precise statement and conditions are given in Section 2 below. Most of the paper is devoted to the proof of this result. In Section 5 we show an application to a statistical mechanics model with complex-valued potential; namely, we find the sharp exponential decay of correlations for this model.

2 Main result

2.1 Statement of the main technical result

Let $K : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be an operator defined by its kernel

$$K(x, y) = \exp \left\{ -W^2 \zeta^2 (x - y)^2 - \frac{1}{2} U(x) - \frac{1}{2} U(y) \right\}, \quad (2.1)$$

where $\zeta \in \mathbb{C}$ is a complex number with $|\zeta| = 1$ and $\Re \zeta^2 > 0$; $W > 0$ is a large parameter, and the potential $U : \mathbb{R} \to \mathbb{C}$ satisfies the following assumptions:

U1) $U(0) = U'(0) = 0$;

U2) $\zeta^2 U''(0) > 0$; [this condition ensures that the operator is approximately normal near the saddle]

U3) $\Re U(x) \geq \frac{1}{c} \min(1, |x|^2)$ for all $x \in \mathbb{R}$; [in particular, 0 is the unique minimum of $\Re U$]

U4) $U$ has an analytic extension to a strip $|\Im z| \leq c$ about the real axis which satisfies $|U'(z)| \leq C(\max(1, \Re U(z)))^\gamma$ for some $\gamma > 1$ and all $z$ in this strip [we use the analyticity assumption to justify saddle point approximations; the second condition is a mild regularity assumption which rules out wildly oscillating potentials such as $x^2(1 + \exp(\sin e^x))$]

Main proposition. Let $K$ be an operator given by (2.1), where $U$ satisfies the assumptions U1)–U4). Denote

$$\alpha = W \sqrt{\zeta^2 U''(0)/2}, \quad \mu = \sqrt{\frac{\pi}{W^2 \zeta^2 + \alpha}}, \quad (2.2)$$
and
\[ g_\alpha(x) = \left( \frac{2\alpha}{\pi} \right)^{1/4} \exp(-\alpha x^2) . \] (2.3)

Let
\[ K = \sqrt{\frac{\pi}{W^2 \zeta^2 + \alpha}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mu \begin{pmatrix} A & B \\ C & D \end{pmatrix} \] (2.4)
be the block representation of \( K \) corresponding to the decomposition \( L_2(\mathbb{R}) = \mathbb{C}g_\alpha \oplus (\mathbb{C}g_\alpha) \perp \); more formally, if \( \hat{K} = \mu^{-1} K \), and \( P \) is the orthogonal projection to \( g_\alpha \perp \),
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} (1 - P)\hat{K} & (1 - P)\hat{K}P \\ P\hat{K}(1 - P) & P\hat{K}P \end{pmatrix} .
\]

Then
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 + O(W^{-1-\delta}) \\ O(W^{-1-\delta}) \end{pmatrix}
\]
of norm \( \leq 1 - \sqrt{\frac{|U''(0)|}{2W} \Re \zeta^2} + O(W^{-1-\delta}) \),
meaning that
\[
A = (1 + O(W^{-1-\delta})) , \quad \quad (2.5)
\]
\[ \|B\|, \|C\| = O(W^{-1-\delta}) , \quad \quad (2.6) \]
\[ \|D\| \leq 1 - \sqrt{\frac{|U''(0)|}{2W} \Re \zeta^2} + O(W^{-1-\delta}) ; \quad \quad (2.7) \]
the exponent \( \delta > 0 \) depends only on \( \gamma \), and the implicit constants in the \( O \)-notation depend on \( \gamma \) and the implicit constants in the assumptions.

Here and forth, we slightly abuse notation and identify scalar multiples of \( 1 - P \) with complex numbers.

### 2.2 The main corollary

Let \( \lambda_0 \) be the largest eigenvalue (in absolute value) of \( K \), and let \( u_0 \) be the corresponding eigenfunction with normalisation conditions
\[ \|u_0\|^2 = \langle u_0, u_0 \rangle = 1, \quad \text{and} \quad \langle u_0, g_\alpha \rangle \geq 0. \] (2.8)

The Main Proposition yields the following corollary:
Corollary 2.1. In the setting of the main proposition, the largest eigenvalue (in absolute value) of $K$ satisfies

$$\lambda_0 = \sqrt{\frac{\pi}{W^2 \zeta^2 + \alpha}} (1 + O(W^{-1-\delta})) = \mu(1 + O(W^{-1-\delta})),$$

and the corresponding eigenfunction $u_0$ (with the normalisation conditions (2.8)) satisfies $\|u_0 - g_\alpha\| \leq CW^{-\delta}$. For any natural $n$ and any $u$ in the invariant subspace $\tilde{u}_0^\perp$ of $K$,

$$\|K^n u\| \leq |\lambda_0|^n \left(1 - \sqrt{\frac{|U''(0)|^2}{2} \frac{3R\zeta^2}{W} + O(W^{-1-\delta})}\right)^n \|u\|.$$

We remark that (2.9) can be restated as

$$\lambda_0 = \tilde{\lambda}_0(1 + O(W^{-1-\delta})),$$

where $\tilde{\lambda}_0$ is the largest eigenvalue of the harmonic approximation

$$\tilde{K}(x, y) = \exp \left\{-W^2 \zeta^2 (x - y)^2 - \frac{U''(0)}{4} (x^2 + y^2)\right\}.$$

This remark is justified by the formulæ of Section 3 below.

Proof of Corollary 2.1. According to the Main Proposition, $\hat{K} = \mu^{-1} K$ (with the normalising factor $\mu$ given by (2.2)) has the block structure

$$\hat{K} = \begin{pmatrix}
1 + O(W^{-1-\delta}) & O(W^{-1-\delta}) \\
O(W^{-1-\delta}) & D
\end{pmatrix}$$

with respect to the decomposition $L_2(\mathbb{R}) = \mathbb{C}g_\alpha \oplus (\mathbb{C}g_\alpha)^\perp$. Set

$$\hat{K}_t = \begin{pmatrix}
A_t & B_t \\
C_t & D
\end{pmatrix} = t\hat{K} + (1 - t) \begin{pmatrix}
1 & 0 \\
0 & D
\end{pmatrix}, \quad 0 \leq t \leq 1.$$

From (2.7), the largest eigenvalue of $\hat{K}_0$ is close to 1, whereas the rest of the spectrum lies in a disc of radius $1 - \text{const}/W$. Let us show that, as $t$ varies from 0 to 1, the resolvent $R_z[\hat{K}_t] = (\hat{K}_t - z)^{-1}$ of $\hat{K}_t$ remains bounded on a circle $\mathcal{C}$ of radius $O(W^{-1-\delta})$ about 1. This will imply that the spectral projection

$$P_t = \frac{1}{2\pi i} \oint_{\mathcal{C}} R_z[\hat{K}_t] dz$$

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is a continuous function of $t \in [0, 1]$, whence the rank of $P_t$ is identically equal to 1, its value at $t = 0$.

According to the Schur–Banachiewicz formula for the inverse of a block operator, the resolvent $R_z[\hat{K}_t]$ is given by

$$R_z[\hat{K}_t] = \begin{pmatrix} S_t^{-1} & -S_t^{-1}B_t(D - z)^{-1} \\ -(D - z)^{-1}C_tS_t^{-1} & (D - z)^{-1} + (D - z)^{-1}C_tS_t^{-1}B_t(D - z)^{-1} \end{pmatrix},$$

where $S_t = (A_t - z) - B_t(D - z)^{-1}C_t$ is the Schur complement. If $|z - 1| = RW^{-1-\delta}$, where $R > 0$ is sufficiently large,

$$|S_t| \geq \frac{R}{2} W^{-1-\delta} - O(W^{-2-2\delta} \times W) \geq \frac{R}{4} W^{-1-\delta}.$$

Therefore

$$|S_t^{-1}| \leq \frac{4}{R} W^{1+\delta}.$$

Similarly, the norms of the other three blocks of $R_z[\hat{K}_t]$ are bounded, thus the norm of the resolvent is bounded for these $z$, uniformly in $0 \leq t \leq 1$. Hence $\hat{K}_0$ and $\hat{K}_1$ have the same number of eigenvalues in the circle $|z - 1| \leq RW^{-1-\delta}$, i.e. $\hat{K}$ has a unique eigenvalue $\lambda_0$ satisfying (2.9). The eigenvalue $\lambda_0$ is the largest one in absolute value due to the estimate (2.10) which we shall prove shortly.

Next, let $u_0$ be an eigenfunction corresponding to $\lambda_0$, with normalisation conditions (2.8). Then

$$u_0 = pg_\alpha + u_0^\perp, \quad \langle u_0^\perp, g_\alpha \rangle = 0;$$

the Main Proposition implies

$$u_1 = \hat{K}u_0 = \left(1 + O(W^{-1-\delta})\right)p + O(W^{-1-\delta})\sqrt{1 - |p|^2} g_\alpha + u_1^\perp,$$

where

$$||u_1^\perp|| \leq O(W^{-1-\delta})|p| + \left(1 - \frac{1}{CW}\right) \sqrt{1 - |p|^2}.$$

On the other hand,

$$u_1 = \hat{\lambda}_0 u_0 = p\hat{\lambda}_0 g_\alpha + \hat{\lambda}_0 u_0^\perp,$$

where $\hat{\lambda}_0 = \mu^{-1}\lambda_0$; comparing the norms of $u_1^\perp$ and $\hat{\lambda}_0 u_0^\perp$, using (2.9) and $p = \langle u_0, g_\alpha \rangle \geq 0$, we obtain:

$$\sqrt{1 - |p|^2} \leq CW^{-\delta}, \quad \text{whence} \quad ||u_0 - g_\alpha|| \leq ||u_0^\perp|| + ||g_\alpha(p - 1)|| \leq C'W^{-\delta}.$$
Finally, let \( u \) be a unit vector lying in the invariant subspace \( \bar{u}_0 \), of \( \hat{K} \). Then

\[
|\langle u, g_\alpha \rangle| \leq |\langle u, \bar{u}_0 \rangle| + |\langle u, \bar{u}_0 - g_\alpha \rangle| \leq CW^{-\delta}.
\]

Therefore

\[
u = pg_\alpha + u^\perp, \quad u^\perp \perp g_\alpha, \quad |p| \leq CW^{-\delta}.
\]

Denote \( q = \|u^\perp\| = \sqrt{1 - |p|^2} \). Then

\[
\|\hat{K}u\|^2 \leq \left\{ (1 + O(W^{-1-\delta}))|p| + O(W^{-1-\delta})|q| \right\}^2 + \left\{ O(W^{-1-\delta})|p| + \left(1 - \frac{c_0}{W} + O(W^{-1-\delta}) \right) |q| \right\}^2,
\]

where \( c_0 = \sqrt{\frac{U''(0)}{2}} \Re \zeta^2 \). Therefore

\[
\|\hat{K}u\|^2 \leq \left[ 1 + O(W^{-1-\delta}) \right] |p|^2 + \left[ (1 - \frac{c_0}{W})^2 + O(W^{-1-\delta}) \right] |q|^2 + O(W^{-1-\delta})
\]

\[
\leq (1 - \frac{c_0}{W})^2(|p|^2 + |q|^2) + O(W^{-1-\delta})
\]

\[
\leq \left(1 - \frac{c_0}{W} + O(W^{-1-\delta}) \right)^2.
\]

Recalling that

\[
\langle \hat{K}u, \bar{u}_0 \rangle = \langle u, \hat{K}^* \bar{u}_0 \rangle = \langle u, \bar{\lambda}_0 \bar{u}_0 \rangle = 0,
\]

we can iterate this estimate, thus obtaining

\[
\|\hat{K}^n u\| \leq \left(1 - \frac{c_0}{W} + O(W^{-1-\delta}) \right)^n
\]

for any \( n \), as claimed.

\[\square\]

### 3 Preliminaries: harmonic oscillator

In this section, we collect the properties of the harmonic oscillator, defined by

\[
K_{hr}(x, y) = \exp \left\{ -W^2 \zeta^2 (x - y)^2 - \frac{a + ib}{2} (x^2 + y^2) \right\}
\]

for

\[
a > 0, \quad |\zeta| = 1, \quad \Re \zeta^2 > 0.
\]
The operator $K_{hr}$ defined by (3.1) is compact under the conditions (3.2), hence it has pure point spectrum. The eigenvalues of $K_{hr}$ are given by the formula:

$$\lambda_{hr}^{j} = \sqrt{\frac{\pi}{W^2 \zeta^2 + \alpha_{hr} + \frac{a+ib}{2}}} \left( \frac{W^2 \zeta^2}{W^2 \zeta^2 + \alpha_{hr} + \frac{a+ib}{2}} \right)^j,$$  \hspace{1cm} (3.3)

where $\alpha_{hr}$ is the solution of

$$\alpha_{hr}^2 = W^2 \zeta^2 (a + ib) + \frac{(a + ib)^2}{4} = \alpha^2 + \frac{(a + ib)^2}{4} = \alpha^2 \left[ 1 + O(W^{-2}) \right], \hspace{1cm} (3.4)$$

with $\Re\alpha_{hr} > 0$, and $\alpha^2 = W^2 \zeta^2 (a + ib)$. This definition is consistent with (2.2). Note that the constraints $a > 0$, $|\zeta|^2 = 1$, $\Re \zeta^2 > 0$ and $\zeta^2 (a + ib) \in \mathbb{R}$ ensure that $\zeta^2 (a + ib) = \Re[\zeta^2 (a + ib)] = a/\Re \zeta^2 > 0$, hence the solution $\Re\alpha_{hr} > 0$ always exists. The eigenfunction corresponding to $\lambda_{hr}^0$ is exactly the function $g_{\alpha_{hr}}$ given by (2.3) (with $\alpha$ replaced by $\alpha_{hr}$); if $\alpha_{hr}$ is real, the $L^2$ norm of $g_{\alpha_{hr}}$ is equal to one.

For real $\zeta$ and $a + ib$, (3.3) is well-known (see [4, Section 5.2]). Since both sides of the identity

$$\det(1 - zK_{hr}) = \prod_{j=0}^{\infty} (1 - z\lambda_{hr}^j)$$

(expressing the Fredholm determinant $\det(1 - zK_{hr})$ in terms of the eigenvalues) are analytic functions of $\zeta$ and $a + ib$, the formula (3.3) remains valid in the full range of parameters (3.2).

Now we turn to $K_{hr}^* K_{hr}$. Set $A = 2W^2 \Re \zeta^2 + a$. One may check that

$$(K_{hr}^* K_{hr})(x, y)$$

\begin{align*}
&= \sqrt{\frac{\pi}{A}} \exp \left\{ - \frac{W^4 + W^2 (a \zeta^2 + (a - ib) \Re \zeta^2) + \frac{\alpha_{hr}}{2} (a - ib)}{A} x^2 
- \frac{W^4 + W^2 (a \zeta^2 + (a + ib) \Re \zeta^2) + \frac{\alpha_{hr}}{2} (a + ib)}{A} y^2 + \frac{2W^4}{A} xy \right\} \\
&= \sqrt{\frac{\pi}{A}} \exp \left\{ - \frac{\alpha_{hr}^2 + \Re[W^2 \zeta^2 (a - ib)] + \frac{1}{4} (a^2 + b^2)}{A} x^2 
- \frac{\alpha_{hr}^2 + \Re[W^2 \zeta^2 (a - ib)] + \frac{1}{4} (a^2 + b^2)}{A} y^2 - \frac{W^4}{A} (x - y)^2 \right\} .
\end{align*} \hspace{1cm} (3.5)
In particular, $K_{hr}$ is normal ($K_{hr}^*K_{hr} = K_{hr}K_{hr}^*$) if and only if $\alpha_{hr}^2$ is real (which happens if and only if $\alpha_{hr} > 0$). More generally, two operators of the form (3.1) commute if and only if they share the same $\alpha_{hr}$.

From (3.5), $K_{hr}^*K_{hr}$ is similar (conjugate) to the operator

$$T_{hr}(x, y) = e^{-i\frac{3\alpha_{hr}^2}{2}x^2}(K_{hr}^*K_{hr})(x, y)e^{i\frac{3\alpha_{hr}^2}{2}y^2} = \sqrt{\frac{\pi}{A}} \exp \left\{ -\frac{W^4}{A} (x - y)^2 - 2a \left( 1 - \frac{a}{2A} \right) \frac{x^2 + y^2}{2} \right\}$$

of the form (3.1). This allows to compute the singular values

$$s_{0}^{hr} \geq s_{1}^{hr} \geq \cdots$$

of $K_{hr}$:

$$(s_{j}^{hr})^2 = \frac{\pi^2}{W^4 + 2aA - \sqrt{[2W^4 + aA]aA}} \left( \frac{W^4}{W^4 + 2aA + \sqrt{[2W^4 + aA]aA}} \right)^j.$$

If $\alpha_{hr} > 0$, we have $s_{j}^{hr} = |\lambda_{j}^{hr}|$ for any $j$. If instead we require $\alpha > 0$, then $\alpha_{hr}$ is real up to an error term of order $O(W^{-2})$. The corresponding operator is almost normal. More precisely, we have the following result.

**Lemma 3.1.** If $K_{hr}$ is an operator of the form (3.1) with real (positive) $\zeta^2(a + ib)$, then for any fixed $j$

$$\frac{s_{j}^{hr}}{|\lambda_{j}^{hr}|} = 1 + O(W^{-2}),$$

where the implicit constant may depend on $j$. Moreover, for any $0 < \epsilon < 1$,

$$\|K_{hr}g_\alpha - \mu g_\alpha\| = |\mu|O(W^{-2+2\epsilon}) \quad \text{and} \quad \|\tilde{g} - g_\alpha\| \leq O(W^{-2}),$$

where $\tilde{g}$ is the top normalized eigenfunction for $K_{hr}^*K_{hr}$, and $g_\alpha$ and $\mu$ are given by (2.3) and (2.2) respectively.

**Proof.** By the formulæ for $\lambda_{j}^{hr}$ and $s_{j}^{hr}$ given above and using $\zeta^2(a + ib) > 0$

$$\frac{|\lambda_{0}^{hr}|^2}{(s_{0}^{hr})^2} = \frac{W^4 + 2W^3\sqrt{aR\zeta^2} + O(W^2)}{W^4 + 2W^3\sqrt{\zeta^2(a + ib)} + O(W^2)}. $$
Using the constraints $\Re \zeta^2 > 0$, $|\zeta|^2 = 1$ and $\zeta^2(a + ib) > 0$ we see that $\zeta^2(a + ib)\Re \zeta^2 = a$, whence $|\lambda_{hr}^0|^2/(\zeta_{hr}^0)^2 = (1 + O(W^{-2}))$. The same proof applies to the case $j > 0$. To prove the second part, we see that

$$(K_{hr}g_\alpha)(x) = \mu g_\alpha(x) c(\alpha)e^{-x^2d(\alpha)},$$

where

$$c(\alpha) = \frac{1}{\sqrt{1 + \frac{a+ib}{2(a+W^2\zeta^2)}}} = 1 + O(W^{-2}), \quad d(\alpha) = \frac{a+ib}{1 + \frac{2(a+W^2\zeta^2)}{a+ib}} = O(W^{-2}).$$

Then, using the exponential decay of $g_\alpha$,

$$\|K_{hr}g_\alpha - \mu g_\alpha\| \leq |\mu||c(\alpha) - 1| + |\mu c(\alpha)| \left\|g_\alpha[e^{-x^2d(\alpha)} - 1]1_{|x| \leq W^2}\right\|
+ |\mu c(\alpha)| \left\|g_\alpha[e^{-x^2d(\alpha)} - 1]1_{|x| > W^2}\right\| = |\mu| O(W^{-2+2\epsilon}).$$

Finally to prove the last inequality, we remark that

$$\tilde{g}(x) = (\frac{2\alpha x}{\pi})^{1/4} e^{\frac{3\alpha^2}{4}x^2} e^{-\alpha_T x^2}$$

where $e^{[-\alpha_T x^2]}$ is a top eigenfunction for $T_h$ and $\alpha_T$ is the real positive solution of

$$\alpha_T^2 = \frac{W^4}{A} - 2a \left(1 - \frac{a}{2A}\right) + a^2 \left(1 - \frac{a}{2A}\right)^2 = \alpha^2[1 + O(W^{-2})].$$

By assumption, $\Im \alpha_{hr}^2 = O(1)$ and $\alpha_T = \alpha(1 + O(1/W^2)) = \alpha + O(1/W)$, therefore

$$\tilde{g}(x) = (1 + O(W^{-2}))e^{O(W^{-1})x^2} g_\alpha(x).$$

Hence

$$\|	ilde{g} - g_\alpha\|^2 \leq \int g_\alpha(x)^2 |1 - e^{O(W^{-1})x^2}|^2 dx + O(W^{-4})$$

$$\leq O(W^{-2}) \int g_\alpha(x)^2 x^4 e^{O(W^{-1})x^2} dx + O(W^{-4}) = O(W^{-4}).$$

where in the last line we applied $|1 - e^x| \leq |x|e^x$. \qed
4 Proof of the main proposition

Similarly to the semi-classical arguments in the self-adjoint case (see § (5.6.1)), we separate the contribution of the vicinity of the saddle point and the rest of the real line as follows. Let \( T(x, y) \) be a kernel, and suppose \( \chi_1^2 + \chi_2^2 = 1 \) is a partition of unity. Then

\[
T(x, y) = \sum_{j=1}^{2} \chi_j(x) T(x, y) \chi_j(y) + \sum_{j=1}^{2} R_j(x, y) ,
\]

(4.1)

where

\[
R_j(x, y) = \frac{1}{2}(\chi_j(x) - \chi_j(y))^2 T(x, y) .
\]

In operator notation,

\[
T = \sum_{j=1}^{2} \chi_j T \chi_j + \sum_{j=1}^{2} R_j .
\]

Another ingredient is Schur’s bound (see [4, Lemma 4.4.1] for a proof)

\[
\|T\| \leq \sqrt{\sup_x \int dy |T(x, y)|} \sqrt{\sup_x \int dy |T(y, x)|} ,
\]

(4.2)

which, in the case when \( |T(x, y)| = |T(y, x)| \), assumes the form

\[
\|T\| \leq \sup_x \int dy |T(x, y)| .
\]

The difference from the usual setting stems from the fact that \( K \) is not self-adjoint. This is why we work with the self-adjoint operator \( K^* K \), and our main effort will be invested in decent bounds on the kernel.

The main proposition will follow from the next three lemmata, which are applied to estimate the four blocks \( A, B, C, D \) of (2.4). We shall compare our operator \( K \) with its harmonic approximation \( \tilde{K} \) introduced in (2.11), which is approximately normal due to assumption U2) of Section 2.1 and Lemma 3.1.
Lemma 4.1. Let $K$ be an operator given by (2.1), so that $U$ satisfies the assumptions $U1)$ and $U3)$. If $\alpha > 0$ is such that
\[
\left| \alpha^2 - W^2 \zeta^2 \frac{U''(0)}{2} \right| \leq CW^{3/2} ,
\] (4.3)
then the asymptotics of the integral
\[
I(\alpha) = \iint dx dy \exp \left\{ -W^2 \zeta^2 (x - y)^2 - \frac{1}{2} U(x) - \alpha x^2 - \frac{1}{2} U(y) - \alpha y^2 \right\}
\]
is given by
\[
I(\alpha) = (1 + O(W^{-3/2+\epsilon})) \sqrt{\frac{\pi}{W^2 \zeta^2 + \alpha}} \sqrt{\frac{\pi}{2\alpha}} ,
\]
for any $\epsilon > 0$.

Remark 4.2. Although the bound is valid for any $\alpha > 0$ satisfying (4.3), we shall only apply it to $\alpha = W \sqrt{\zeta^2 U''(0)/2}$ of (2.2).

Lemma 4.3. In the setting of the Main Proposition,
\[
\| (K - \tilde{K}) g_\alpha \| = O(W^{-3/2+\epsilon} |\mu|)
\]
for any $\epsilon > 0$.

Lemma 4.4. In the setting of the Main Proposition, let $u \in L^2$ be a function of unit norm. Then there exists $\delta > 0$ so that
\[
\| Ku \| \leq |\mu| \left( 1 + O(W^{-1-\delta}) \right) .
\]
Moreover, if $u \perp g_\alpha$, then
\[
\| Ku \| \leq |\mu| \left( 1 - \sqrt{\frac{|U''(0)|}{2}} \frac{\Re \zeta^2}{W} + O(W^{-1-\delta}) \right) .
\]
The same estimates hold for $\| K^* u \|$.

Proof of Main Proposition. The estimate on $A$ follows from Lemma 4.1
\[
\mu A = \langle Kg_\alpha, g_\alpha \rangle = \sqrt{\frac{2\alpha}{\pi}} I(\alpha) = (1 + O(W^{-3/2+\epsilon})) \mu .
\] (4.4)
The estimate on $B$ and $C$ follows from Lemma 4.3 and 3.1

$$\|\mu C\| = \|PKg_\alpha\| = \inf_{w \in C} \|(K - w)g_\alpha\|$$

$$\leq \|(K - \tilde{K})g_\alpha\| + \|\tilde{K}g_\alpha - \mu g_\alpha\| = O(W^{-3/2 + \epsilon} |\mu|).$$

In a similar way $\|\mu B\| = \|PK^*g_\alpha\| \leq \|(K^* - \tilde{\mu})g_\alpha\| = \|(K - \mu)g_\alpha\|$ since $g_\alpha$ is real. Therefore the arguments for $C$ apply. Finally, the bound on $\|D\|$ follows from the second statement of Lemma 4.4 since

$$\|PKP\| \leq \sup_{u \perp g_\alpha, \|u\|=1} \|Ku\|.$$

Now we turn to the proofs of the lemmata.

**Proof of Lemma 4.1.** Changing variables

$y \leftarrow \frac{y + x}{\sqrt{2}}, \quad x \leftarrow \frac{y - x}{\sqrt{2}},$

we obtain:

$$I(\alpha) = \iint dx dy \exp \left\{ -(2W^2 \zeta^2 + \alpha)x^2 - \alpha y^2 - \frac{1}{2} U\left(\frac{y + x}{\sqrt{2}}\right) - \frac{1}{2} U\left(\frac{y - x}{\sqrt{2}}\right) \right\}.$$

The integration over the complement of the rectangle defined by the inequalities $|x| \leq W^{-1+\epsilon/3}, |y| \leq W^{-1/2+\epsilon/3}$ is exponentially suppressed according to U3), where we took into account that $\alpha$ is of order $W$. Inside the rectangle we expand about $y/\sqrt{2}$ (since $x$ is typically smaller in absolute value):

$$U\left(\frac{y \pm x}{\sqrt{2}}\right) = U\left(\frac{y}{\sqrt{2}}\right) \pm U'\left(\frac{y}{\sqrt{2}}\right) \frac{x}{\sqrt{2}} + O(x^2),$$

whence by U1)

$$\frac{1}{2} \left( U\left(\frac{y + x}{\sqrt{2}}\right) + U\left(\frac{y - x}{\sqrt{2}}\right) \right) = U\left(\frac{y}{\sqrt{2}}\right) + O(x^2) = \frac{U''(0)}{4} y^2 + O(x^2 + |y|^3).$$

Therefore

$$I(\alpha) = (1 + O(W^{-3/2 + \epsilon})) \sqrt{\frac{\pi}{2W^2 \zeta^2 + \alpha}} \sqrt{\frac{\pi}{\alpha + \frac{U''(0)}{4}}}. $$
We have:
\[
(2W^2\zeta^2 + \alpha)(\alpha + \frac{U''(0)}{4}) = 2W^2\zeta^2\alpha + \alpha^2 + W^2\zeta^2\frac{U''(0)}{2} + O(W),
\]
whereas
\[
2\alpha(W^2\zeta^2 + \alpha) = 2W^2\zeta^2\alpha + 2\alpha^2.
\]
Under the assumption (4.3) on \(\alpha\), the two expressions differ by \(O(W^{3/2})\).

**Proof of Lemma 4.3.** We start with the identity
\[
\|(K - \tilde{K})g_\alpha\|^2 = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy (K^* - \tilde{K}^*)(x, r)(K - \tilde{K})(r, y)g_\alpha(x)g_\alpha(y).
\]
This time the integral over the complement of the polytope
\[
|x - r|, |r - y| \leq W^{-1+\epsilon/3}, |x|, |r| \leq W^{-1/2+\epsilon/3}
\]
is exponentially suppressed, whereas inside the polytope
\[
|(K^* - \tilde{K}^*)(x, r)| \leq CW^{-3/2+\epsilon}|\tilde{K}^*(x, r)|
\]
and
\[
|(K - \tilde{K})(r, y)| \leq CW^{-3/2+\epsilon}|\tilde{K}(r, y)|;
\]
the statement follows from these inequalities.

To prove Lemma 4.4 we need several estimates on the kernel of \(K^*K\), which are collected in the next lemma. We shall apply the first estimate when \(|x|, |y| \leq W^{-1/2+\delta}\), the second one when either \(\Re U \geq W^{\eta}\) or \(|x - y| \geq W^{-1+\eta}\) (for a small \(\eta > 0\) to be chosen later), and the third one in the remaining range of parameters.

**Lemma 4.5.** The kernel \(K^*K(x, y)\) satisfies the following estimates.

1. For \(|x|, |y| \leq c_0\) (where \(c_0 > 0\) may depend on \(\zeta\) and on the width of the strip in which \(U\) is analytic),

\[
(K^*K)(x, y) = \left[1 + O(|x|^3 + |y|^3 + W^{-3+\epsilon})\right](\tilde{K}^*\tilde{K})(x, y).
\]

where \(\epsilon > 0\) is an arbitrary positive number.
2. For any \(x, y\),
\[
| (K^* K)(x, y) | \leq \sqrt{\frac{\pi}{2W^2\Re^2}} \exp \left\{ -\frac{\Re U(x) + \Re U(y)}{2} - \frac{W^2}{2\Re^2}(x - y)^2 \right\}
\]

3. If \(|x - y| \leq W^{-1+\eta}\), \(\Re U(x) \leq W^\eta\) and \(\Re U(y) \leq W^\eta\), where \(\eta > 0\) is sufficiently small and \(\gamma\) is the parameter appearing in \(U^4\), then we have
\[
(K^* K)(x, y) = (1 + O(W^{-1+5\eta})) \sqrt{\frac{\pi}{2W^2\Re^2}} \exp \left\{ -\frac{\bar{U}(x) + U(y)}{2} - \Re U(x + y) - \frac{W^2}{2\Re^2}(x - y)^2 \right\}.
\]

We postpone the proof of Lemma 4.5 and start with Proof of Lemma 4.4.

Let \(\delta > 0\) be a small number. Construct a partition of unity \(\chi_1^2 + \chi_2^2 = 1\). We choose \(\chi_1, \chi_2 : \mathbb{R} \to \mathbb{R}^+_0\) such that

1. \(\chi_1\) is supported on \([-W^{-1/2+\delta}, W^{-1/2+\delta}]\) and is identically equal to one in \([-\frac{1}{2}W^{-1/2+\delta}, \frac{1}{2}W^{-1/2+\delta}]\);
2. \(\chi_2\) is supported outside \([-\frac{1}{2}W^{-1/2+\delta}, \frac{1}{2}W^{-1/2+\delta}]\) and is identically equal to one outside \([-W^{-1/2+\delta}, W^{-1/2+\delta}]\);
3. the two functions are differentiable, and \(|\chi_1'|, |\chi_2'| \leq CW^{1/2-\delta}\).

Also denote \(\mathbb{I}_1 = 1_{\chi_1 > 0}\), \(\mathbb{I}_2 = 1_{\chi_2 > 0}\); then \(\chi_1 \mathbb{I}_1 = \chi_1\) and \(\chi_2 \mathbb{I}_2 = \chi_2\). According to the decomposition (4.1),
\[
\| Ku \|^2 = \langle K^* Ku, u \rangle = \sum_{j=1}^2 \langle \chi_j K^* K \chi_j u, u \rangle + \sum_{j=1}^2 \langle R_j u, u \rangle. \tag{4.5}
\]

Let \(\tilde{s}_0 \geq \tilde{s}_1 \geq \cdots\) be the singular values of \(\widetilde{K}\), and let \(\tilde{g}\) be the top eigenfunction of \(\widetilde{K}^* \widetilde{K}\). From the properties of the harmonic oscillator collected in Lemma 3.1
\[
\tilde{s}_0 = |\mu|(1 + O(W^{-2})) \, , \, \frac{\tilde{s}_1}{\tilde{s}_0} = 1 - \sqrt{\frac{|U''(0)|}{2W}} \Re^2 + O(W^{-2}). \tag{4.6}
\]

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We shall prove the following estimates:

\[ \langle \chi_1 K^* K \chi_1 u, u \rangle \leq \tilde{s}_1^2 \| \chi_1 u \|^2 + C \tilde{s}_0^2 W^{-3/2 + 3\delta} \| u \|^2 \quad (u \perp g_\alpha), \]

\[ \langle \chi_1 K^* K \chi_1 u, u \rangle \leq \tilde{s}_0^2 (1 + CW^{-3/2 + 3\delta}) \| \chi_1 u \|^2, \]

\[ |\langle \chi_2 K^* K \chi_2 u, u \rangle| \leq \tilde{s}_0^2 (1 - \frac{1}{C} W^{-1+2\delta}) \| \chi_2 u \|^2, \]

\[ |\langle R_j u, u \rangle| \leq CW^{-1-2\delta} \tilde{s}_0^2 \| u \|^2. \]

Once these bounds are established, the proof of the lemma is concluded as follows. For \( u \perp g_\alpha \), we use (4.7), (4.9), (4.10) to estimate the addends in (4.5); then from the inequality

\[ \tilde{s}_0^2 (1 - \frac{1}{C} W^{-1+2\delta}) \leq \tilde{s}_1^2 \]

and the identity

\[ \| \chi_1 u \|^2 + \| \chi_2 u \|^2 = \| u \|^2 \]

we obtain:

\[ |\langle K^* K u, u \rangle| \leq (1 + O(W^{-1-2\delta})) \tilde{s}_0^2 \| u \|^2. \]

For arbitrary \( u \) we apply (4.8) in place of (4.7), and obtain:

\[ |\langle K^* K u, u \rangle| \leq (1 + O(W^{-1-2\delta})) \tilde{s}_0^2 \| u \|^2. \]

This is what we claimed.

**Proof of (4.7)** starts from

\[ \langle \chi_1 K^* K \chi_1 u, u \rangle = \langle \tilde{K}^* \tilde{K} \chi_1 u, \chi_1 u \rangle + \langle \chi_1^* K^* K \chi_1 - \chi_1^* \tilde{K}^* \tilde{K} \chi_1 \rangle \chi_1 u, \chi_1 u \rangle. \]

From the decomposition

\[ \chi_1 u = \langle \chi_1 u, \tilde{g} \rangle \tilde{g} + \{ \chi_1 u - \langle \chi_1 u, \tilde{g} \rangle \tilde{g} \} \]

we obtain

\[ \langle \tilde{K}^* \tilde{K} \chi_1 u, \chi_1 u \rangle \leq \tilde{s}_0^2 |\langle \chi_1 u, \tilde{g} \rangle|^2 + \tilde{s}_1^2 \| \chi_1 u \|^2. \]

If \( u \perp g_\alpha \),

\[ |\langle \chi_1 u, \tilde{g} \rangle| \leq \| \chi_1 u \| \| \tilde{g} - g_\alpha \| + |\langle u - \chi_1 u, g_\alpha \rangle| \leq \| u \| \| \tilde{g} - g_\alpha \| + \| u \| \| (1 - \chi_1) g_\alpha \| \leq C_1 W^{-2} \| u \|. \]
where in the first term of the sum we applied Lemma 3.1. Finally
\[ \| \mathbb{1}_1 K^* K \mathbb{1}_1 - \bar{\mathbb{1}}_1 \bar{K}^* \bar{K} \bar{\mathbb{1}}_1 \| \leq C_2 W^{-1/2+3\delta} \tilde{s}_0^2 \]  
(4.12)
according to item 1. of Lemma 4.5 and Schur’s bound (4.2). Hence
\[
\langle \chi_1 K^* K \chi_1 u, u \rangle \leq C_2 W^{-4} \| u \|^2 + \tilde{s}_1^2 \| \chi_1 u \|^2 + C_2 \tilde{s}_0 W^{-3/2+3\delta} \| u \|^2 
\] 
(4.13)
\[
\leq \tilde{s}_1^2 \| \chi_1 u \|^2 + C_2 \tilde{s}_0 W^{-3/2+3\delta} \| u \|^2 .
\] 
(4.14)

**Proof of (4.8)** also starts from (4.11). From
\[
\langle \chi_1 \bar{K}^* \bar{K} \chi_1 u, u \rangle \leq \tilde{s}_0^2 \| \chi_1 u \|^2 
\] 
and (4.12), we obtain the bound
\[
\langle \chi_1 K^* K \chi_1 u, u \rangle \leq \tilde{s}_0^2 (1 + CW^{-3/2+3\delta}) \| \chi_1 u \|^2 .
\] 
(4.15)

**Proof of (4.9)** We plug into Schur’s bound (4.2) the estimates on the kernel of \((K^* K)(x, y)\) obtained in Lemma 4.5, as follows. Set \(\eta = \delta/(5\gamma)\). We use the estimate given in item 2 when either \(|x - y| > W^{-1+\eta}\), or \(\Re U(x) > W^\eta\), or \(\Re U(y) > W^\eta\). In the complementary region
\[
|x - y| \leq W^{-1+\eta} \land \Re U(x) \leq W^\eta \land \Re U(y) \leq W^\eta
\]
we use the estimate given in item 3. Then
\[
\| \mathbb{1}_2 K^* K \mathbb{1}_2 \| \leq \left( \frac{\pi}{W^2} \right) W^{-1+2\delta} e^{-C_1 W^\eta} \leq \tilde{s}_0^2 (1 - \frac{1}{C} W^{-1+2\delta}) ,
\]
where in the first term we used \(\Re U(x) \geq \frac{|x|^2}{C} \geq \frac{W^{-1+2\delta}}{C}\) (from U3) and the definition of \(\chi_2\) and the relations between \(|\mu| = \frac{\pi}{W} (1 + O(W^{-2}))\) and \(\tilde{s}_0\) given in (4.6). The second term comes from the estimate in item 2 of Lemma 4.5. Hence
\[
\| \langle \chi_2 K^* K \chi_2 u, u \rangle \| \leq \tilde{s}_0^2 (1 - \frac{1}{C} W^{-1+2\delta}) \| u \|^2 .
\] 
(4.16)

**Proof of (4.10)** is similar: Schur’s bound, item 2 of Lemma 4.5 and the bounds
\[
|\chi_j(x) - \chi_j(y)| \leq CW^{1/2-\delta} |x - y|
\]
are used to show that
\[
\| R_j \| \leq CW^{-1-2\delta} \tilde{s}_0^2 .
\] 
\[\square\]
Proof of Lemma 4.5. First,

$$(K^*K)(x, y) = E(x, y)I(x, y),$$

where

$$E(x, y) = \exp \left\{ -\frac{\bar{U}(x) + U(y)}{2} - W^2\zeta^2 x^2 - W^2\zeta^2 y^2 \right\},$$

and

$$I(x, y) = \int dr \exp \left\{ -2W^2 \left[ \Re\zeta^2 r^2 - (\bar{\zeta}^2 x + \zeta^2 y)r - \Re U(r) \right] \right\}.$$

On the real line, $\Re U(z)$ coincides with the analytic function

$$U_{Schw}(z) = \frac{(U(z) + \bar{U}(\bar{z}))}{2}, \quad (4.17)$$

therefore we replace $\Re U$ with $U_{Schw}$.

To prove the first item of the lemma, let $|x|, |y| < c_0$ for a small constant $c_0$, and let $\tilde{E}$ and $\tilde{I}$ be expressions analogous to $E$ and $I$ which correspond to $\tilde{K}$;

$$\tilde{I}(x, y) = \sqrt{\frac{\pi}{2W^2\Re\zeta^2 + \frac{\Re U''(0)}{2}}} \exp \left\{ \frac{W^4(\bar{\zeta}^2 x + \zeta^2 y)^2}{2W^2\Re\zeta^2 + \frac{\Re U''(0)}{2}} \right\}.$$

Let $\xi \geq 2|x| + 2|y|$ be a small number to be fixed later (1.15), and set

$$r_0 = \frac{W^2(\bar{\zeta}^2 x + \zeta^2 y)}{2W^2\Re\zeta^2 + \frac{\Re U''(0)}{2}}.$$

Then $|r_0| \leq \xi/4$. Deform the contour of integration to

$$(-\infty, \Re r_0 - \xi) \cup (\Re r_0 - \xi, r_0 - \xi) \cup (r_0 - \xi, r_0 + \xi) \cup (r_0 + \xi, \Re r_0 + \xi) \cup (\Re r_0 + \xi, \infty).$$

Let $I = I_1 + I_2$, where $I_1$ is the integral over $(r_0 - \xi, r_0 + \xi)$, and $I_2$ is the integral over the remaining part of the contour. Let $\tilde{I} = \tilde{I}_1 + \tilde{I}_2$ be the analogous decomposition of $\tilde{I}$. (Observe that, for sufficiently small $c_0$, the deformed contour is within the domain of analyticity of $U$.) Then

$$|I_2|, |\tilde{I}_2| \leq \exp \left\{ -C^{-1}W^2\zeta^2 \right\}.$$
To estimate the difference between the dominant parts $I_1, \tilde{I}_1$, we write

$$I_1 - \tilde{I}_1 = \exp \left\{ \frac{W^4 (\zeta^2 x + \zeta y)^2}{2W^2 \Re \zeta^2 + \frac{\Re U''(0)}{2}} \right\} \int_{r_0 - \xi}^{r_0 + \xi} dr \exp \left\{ -2W^2 \Re \zeta^2 + \frac{\Re U''(0)}{2} (r-r_0)^2 \right\} \left[ e^{R(r)} - 1 \right]$$

where

$$R(r) = \frac{U''\text{Schw}(0)}{2} r^2 - U\text{Schw}(r).$$

We obtain:

$$|I_1 - \tilde{I}_1| = O((|r_0| + |\xi|)^3)|\tilde{I}|,$$

To conclude the proof of the first item, set

$$\xi = 2(|x| + |y|) + W^{-1+\epsilon/3}, \quad (4.18)$$

and observe that

$$E(x, y) = (1 + O(|x|^3 + |y|^3)) \tilde{E}(x, y).$$

To prove the second item, we insert absolute values:

$$|E(x, y)| \leq \exp \left\{ -\frac{\Re U(x) + \Re U(y)}{2} - W^2 \Re \zeta^2 (x^2 + y^2) \right\},$$

and

$$|I(x, y)| \leq \exp \left\{ \frac{W^2}{2} \Re \zeta^2 (x + y)^2 \right\} \times \int dr \exp \left\{ -2W^2 \Re \zeta^2 (r - \frac{x + y}{2})^2 - \Re U(r) \right\} \leq \sqrt{\frac{\pi}{2W^2 \Re \zeta^2}} \exp \left\{ \frac{W^2}{2} \Re \zeta^2 (x + y)^2 \right\}.$$

To prove the third item, let us rewrite $I(x, y)$ as

$$\exp \left\{ \frac{W^2 (\zeta^2 x + \zeta y)^2}{2 \Re \zeta^2} \right\} \int dr \exp \left\{ -2W^2 \Re \zeta^2 (r - r_0)^2 - \Re U(r) \right\},$$

where

$$r_0 = \frac{y + x}{2} + \frac{\Im \zeta^2 y - \Re \zeta^2 x}{\Re \zeta^2} i.$$
Then performing a contour deformation similar to the one in the proof of the first item we have
\[ \int dr \exp \{-2W^2\Re c^2(r-r_0)^2 - \Re U(r)\} = I_1 + I_2 , \]
where
\[ I_1 = e^{-U_{\text{Schw}}(r_0)} \int_{-\xi}^{\xi} dr e^{-2W^2\Re c^2} e^{U_{\text{Schw}}(r_0) - U_{\text{Schw}}(r+r_0)} \]
(4.19)
\[ |I_2| \leq e^{-W^2(\xi^2 - |\Im r_0|^2)/C} \leq e^{-W^2\xi^2/C'} , \]
(4.20)
if we choose \( \xi > |\Im r_0|/4 \). The imaginary part \( |\Im r_0| \) may be as large as \( W^{-1+\eta} \), therefore we have to take \( \xi = W^{-1+\eta+\epsilon} \) for some \( \epsilon > 0 \). We later set \( \epsilon = 2\eta\gamma \).

Let us show that for any \( \eta_1 \in (\eta, 2\eta) \) the following estimate holds for \( r \) in a complex neighbourhood of \( x \):
\[ |r - x| \leq 2\xi = 2W^{-1+\eta+\epsilon} \implies |U'(r)| < 2CW^{\eta \gamma} . \]
Indeed, by the inequalities \( \Re[U(x)] \leq W^\eta \) and U4), \( U'(x) \) satisfies \( |U'(r)| < 2CW^{\eta \gamma} \), and the smoothness of \( U \) guarantees there exists some constant \( c_x > 0 \) such that (4.21) holds inside the ball of radius \( c_x \) centred at \( x \). Let \( r_1 \in \mathbb{C} \) be a point such that \( |r_1 - x| > c_x \), (4.21) holds for all \( |r - x| < |r_1 - x| \) and fails at \( r_1 \). Then by U4)
\[ \Re[U(r_1)] > W^\eta . \]
(4.22)
Performing a Taylor expansion with first order integral remainder we have
\[ \frac{U(r_1) - U(x)}{r_1 - x} = \int_0^1 U'(x + t(r_1 - x))dt. \]
Inserting absolute values,
\[ \frac{|U(r_1) - U(x)|}{|r_1 - x|} \leq \int_0^1 |U'(x + t(r_1 - x))|dt \leq 2CW^{\eta \gamma} \]
since \( |x + t(r_1 - x)| < |r_1 - x| \) for all \( 0 \leq t < 1 \). From (4.22) and the assumptions \( \Re[U(x)] \leq W^\eta \ll W^m \) and \( |r_1 - x| \leq 2W^{-1+\eta+\epsilon} \) we get
\[ \frac{1}{4} W^{m+1-\eta-\epsilon} \leq \frac{|\Re[U(r_1)] - \Re[U(x)]|}{|r_1 - x|} \leq \frac{|U(r_1) - U(x)|}{|r_1 - x|} \leq 2CW^{\eta \gamma} . \]
hence

\[ W^{1-\eta-\epsilon} \leq 8C W^{n(\gamma-1)} \]  \tag{4.23}

as long as \( \eta_1(\gamma-1) < 1-\eta-\epsilon \). When \( \eta, \epsilon > 0 \) are sufficiently small we have \( \eta_1(\gamma-1) < 1-\eta-\epsilon \) for all \( \eta_1 \in (\eta, 2\eta) \), in contradiction with (4.23). Thus (1.21) is established.

Applying the definition (4.17) of \( U_{\text{Schw}} \) we have \( U'_{\text{Schw}}(r) = \frac{U'(r) + U'(\bar{r})}{2} \), and from (4.21)

\[ |r-x| \leq 2\xi = 2W^{-1+\eta+\epsilon} \Rightarrow |U'_{\text{Schw}}(r)| < 2CW^{n\gamma}, \]

where \( |\bar{r} - x| = |r - x| \) since \( x \in \mathbb{R} \). Now set \( \epsilon = 2\eta\gamma \). Then for any \( \eta < \eta_1 < 2\eta \) we have \( \epsilon > \eta_1\gamma \) and

\[ |U_{\text{Schw}}(r_0) - U_{\text{Schw}}(r)| \leq |r - r_0| \int_0^1 |U'_{\text{Schw}}(r + t(r_0 - r))| dt \]

\[ \leq O(W^{-1+\eta+\epsilon+\gamma\eta_1}) = O(W^{-1+5\gamma\eta}) , \]

hence \( I_1 \) of (4.19) satisfies

\[ I_1 = e^{-\Re U(\frac{\epsilon}{W^{n\gamma}})} \sqrt{\frac{\pi}{2W^2\Re^2}} \left[ 1 + O(W^{-1+5\gamma\eta}) \right] . \]

This concludes the proof of the third item of Lemma 4.5.

\[ \square \]

5 Application to a complex statistical mechanics model

Let \( V(x) = a \log(1+bx^2) \), where \( a > 0 \) and \( \Re b > 0 \). We are interested in the statistical mechanics model corresponding to the action

\[ W^2 \nabla \phi^2 + V(\phi) ; \]

for simplicity of notation, we set the inverse temperature to one. Without going into the details of the construction of infinite-volume measures, let us define the “mean” of a local observable \( F : \mathbb{R}^{n+1} \rightarrow \mathbb{C} \) as follows:

\[ \langle F(\phi_0, \ldots, \phi_n) \rangle = \lim_{M,N \rightarrow \infty} \frac{\int \prod_{j=-M}^{N} d\phi_j e^{-\sum_{j=-M}^{N} V(\phi_j) - W^2 \sum_{j=-M}^{N} (\phi_j - \phi_{j+1})^2} F(\phi_0, \ldots, \phi_n)}{\int \prod_{j=-M}^{N} d\phi_j e^{-\sum_{j=-M}^{N} V(\phi_j) - W^2 \sum_{j=-M}^{N} (\phi_j - \phi_{j+1})^2}} \]  \tag{5.1}
We are interested in the long-distance correlations, e.g.
\[
\langle (F(\phi_0) - \langle F(\phi_0) \rangle)(G(\phi_n) - \langle G(\phi_n) \rangle) \rangle.
\]
with \(F, G : \mathbb{R} \to \mathbb{C}\). Define \(\zeta \in \mathbb{C}\) by
\[
|\zeta| = 1, \quad |\arg \zeta| < \pi/4, \quad \zeta^4 V''(0) = 2 \zeta^4 ab > 0; \quad (5.2)
\]
let \(\Sigma = \text{conv } (\mathbb{R} \cup \mathbb{R} \zeta)\), and set
\[
U(x) = V(\zeta x). \quad (5.3)
\]
Our transfer operator method can be applied to study observables \(F : \mathbb{R}^{n+1} \to \mathbb{C}\) which have an analytic extension to \(\Sigma^{n+1}\) and do not grow too fast in this sector. For simplicity, let us focus on \(n = 0\): assume that

F1) \(F : \Sigma \to \mathbb{C}\) is analytic;

F2) \(|F(z)| \leq C(1 + |z|)^{2a-1-\epsilon}\) for some \(C > 0\) and \(\epsilon > 0\), and all \(z \in \Sigma\).

**Theorem 1.** Suppose that \(V(x) = a \log(1 + bx^2)\) for some \(a > 0\) and \(\Re b > 0\), and that \(F, G\) are observables which satisfy F1)–F2). Then
\[
|\langle (F(\phi_0) - \langle F(\phi_0) \rangle)(G(\phi_n) - \langle G(\phi_n) \rangle) \rangle| \leq C_F C_G \left( 1 - \sqrt{\frac{|V''(0)|}{2} \frac{\Re \zeta^2}{W - O(W^{-1-\delta})} n} \right),
\]
where \(\zeta\) is defined in (5.2).

**Proof.** Let \(F, G\) be observables satisfying F1)–F2). According to Cauchy’s theorem, we can rotate the contour of integration in (5.1):
\[
\int \prod_{j=M}^N d\phi_j e^{-\sum_{j=-M}^N V(\phi_j) - W^2 \sum_{j=-M}^{N-1} (\phi_j - \phi_{j+1})^2} F(\phi_0) G(\phi_n)
\]
\[
\int \prod_{j=M}^N d\phi_j e^{\sum_{j=-M}^N V(\phi_j) - W^2 \sum_{j=-M}^{N-1} (\phi_j - \phi_{j+1})^2}
\]
\[
= \int \prod_{j=-M}^N d\phi_j e^{-\sum_{j=-M}^N U(\phi_j) - W^2 \zeta^2 \sum_{j=-M}^{N-1} (\phi_j - \phi_{j+1})^2} F(\zeta_0) G(\zeta_n) \quad (5.4)
\]
\[
\int \prod_{j=-M}^N d\phi_j e^{-\sum_{j=-M}^N U(\phi_j) - W^2 \zeta^2 \sum_{j=-M}^{N-1} (\phi_j - \phi_{j+1})^2}
\]
Now we set up an integral operator \(K\), the kernel of which is given by (2.1); the Main Proposition and Corollary 2.1 are applicable, so the largest
eigenvalue of $K$ (in absolute value) $\lambda_0$ is given by (2.9). Let $u_0$ be a corresponding eigenfunction satisfying the normalisation conditions (2.8). Set $B(x) = \exp(-U(x)/2)$. Then the denominator of (5.4) is equal to

$$\int K^{M+N}(x_1, x_2)B(x_1)B(x_2)dx_1dx_2 .$$

From Corollary 2.1 we have

$$\langle u_0, \bar{u}_0 \rangle = \langle u_0, g_\alpha \rangle + \langle u_0, \bar{u}_0 - g_\alpha \rangle = 1 + O(W^{-\delta}) \neq 0.$$

Decomposing $B = \langle B, \bar{u}_0 \rangle \langle u_0, \bar{u}_0 \rangle u_0 + B_1$ (this is well defined since $\langle u_0, \bar{u}_0 \rangle = 1 + O(W^{-\delta}) \neq 0$) and setting $\hat{K} = \lambda_0^{-1}K$, we obtain from Corollary 2.1:

$$\lim_{M,N \to \infty} \int \hat{K}^{M+N}(x_1, x_2)B(x_1)B(x_2)dx_1dx_2 = \frac{\langle B, u_0 \rangle^2}{\langle u_0, \bar{u}_0 \rangle^2}.$$

Similarly, for $F : \mathbb{R} \to \mathbb{C}$ satisfying F1)–F2),

$$\int \prod_{j=1}^{3} dx_j B(x_1)\hat{K}^M(x_1, x_2)F(\zeta x_2)\hat{K}^N(x_2, x_3)B(x_3) \to \frac{\langle B, u_0 \rangle^2}{\langle u_0, \bar{u}_0 \rangle^2} \langle F^\zeta u_0, \bar{u}_0 \rangle ,$$

where $F^\zeta(x) = F(\zeta x)$. Hence we obtain:

$$\langle F(\phi_0) \rangle = \langle F^\zeta u_0, \bar{u}_0 \rangle . \quad (5.5)$$

Similarly,

$$\langle F(\phi_0)G(\phi_n) \rangle = \langle \hat{K}^n F^\zeta u_0, \hat{G}^\zeta \bar{u}_0 \rangle . \quad (5.6)$$

If $\langle F(\phi_0) \rangle = 0$ (i.e. $F^\zeta u_0 \in \bar{u}_0$), Corollary 2.1 yields:

$$\|\hat{K}^n F^\zeta u_0\| \leq \left( 1 - \sqrt{\frac{|V''(0)|}{2W} \Re \zeta^2 + O(W^{-1-\delta})} \right) \|F^\zeta u_0\| ,$$

thus for any $F,G$ satisfying F1)–F2)

$$|\langle (F(\phi_0) - \langle F(\phi_0) \rangle)(G(\phi_n) - \langle G(\phi_n) \rangle) \rangle| \leq C_F C_G \left( 1 - \sqrt{\frac{|V''(0)|}{2W} \Re \zeta^2 + O(W^{-1-\delta})} \right) ^n .$$

\[\square\]
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