Fermion Representation of Quantum Group

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The spinor representation of the quantum group $U_q(su(N))$ is given in terms of a set of fermion creation and annihilation operators. It is shown that the $q$-fermion operators introduced earlier can be identified with the conventional fermion operators. Algebra homomorphisms mapping the fermion operators to their tensor products are discussed. The relation of the coproduct of the quantum group to the above algebra homomorphisms is obtained.
Quantum groups are subject much discussed both by physicists and mathematicians. A quantum group is mathematically defined as a noncommutative and nonco-
commutative Hopf algebra. The representation theory of quantum groups has
also been fully developed and has found many useful applications in physics. It is
interesting that the analogue of the Jordan-Schwinger representation of Lie groups
exists for quantum groups. Biedenharn and Macfarlane introduced q-bosonic
operators satisfying q-deformed commutation relations and constructed the q-boson
representation of the simplest case of quantum groups. A thorough discussion on the
q-boson and q-fermion representations of quantum groups was given by Hayashi. His
discussion covers various quantum groups $U_q(X)$ with $X = A_{N-1}, B_N, C_N, D_N$
and $A^{(1)}_{N-1}$. We, however, concentrate here on the quantum group $U_q(A_{N-1})$ and
assume that the parameter $q$ is real and not equal to ±1. The quantum group
$U_q(A_{N-1})$ is the enveloping algebra generated by $e_i, f_i, k_i, k_i^{-1} (1 \leq i \leq N-1, 2 \leq N)$
satisfying the following relations:

\[
k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \quad (1 \cdot 1a)
\]

\[
k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j, \quad (1 \cdot 1b)
\]

\[
e_i f_j - f_j e_i = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q^2 - q^{-2}}, \quad (1 \cdot 1c)
\]

\[
\sum_{0 \leq n \leq 1-a_{ij}} (-1)^n \left[ \frac{1 - a_{ij}}{n} \right] q^{1-a_{ij} - n} e_j e_i^n = 0, \quad (i \neq j) \quad (1 \cdot 1d)
\]

\[
\sum_{0 \leq n \leq 1-a_{ij}} (-1)^n \left[ \frac{1 - a_{ij}}{n} \right] q^{1-a_{ij} - n} f_j f_i^n = 0, \quad (i \neq j) \quad (1 \cdot 1e)
\]

In (1·1), $a_{ij}$ is the $ij$-element of the $(N-1) \times (N-1)$ generalized Cartan matrix

\[
A_{N-1} = \begin{bmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & \ldots & \cdot \\
0 & -1 & 2 & \ldots & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 2 & -1 & 0 \\
\cdot & \cdot & \cdot & -1 & 2 & -1 \\
0 & \cdot & \cdot & 0 & -1 & 2 \\
\end{bmatrix} \quad (1 \cdot 2)
\]
In (1·1), we have made use of the following notation:

\[
\begin{bmatrix} m \\ n \end{bmatrix}_Q = \frac{[m]_Q! [n]_Q!}{[m-n]_Q! [n]_Q!},
\]

\[
[m]_Q = [m]_Q [m-1]_Q \cdots [2]_Q [1]_Q,
\]

\[
[m]_Q = \frac{Q^m - Q^{-m}}{Q - Q^{-1}}.
\tag{1·3}
\]

Hayashi\(^7\) showed that the relations (1·1) are realized by putting

\[
e_i = \psi_i \psi_{i+1}^\dagger,
\]

\[
f_i = \psi_{i+1} \psi_i^\dagger,
\]

\[
k_i = \omega_i \omega_{i+1}^{-1}, \quad k_i^{-1} = \omega_{i+1} \omega_i^{-1},
\]

for 1 ≤ i ≤ N − 1, where \(\psi_i, \psi_i^\dagger, \omega_i, \omega_i^{-1}\) (1 ≤ i ≤ N) are operators satisfying the following relations:

\[
\omega_i \omega_j = \omega_j \omega_i, \quad \omega_i \omega_i^{-1} = \omega_i^{-1} \omega_i = 1,
\]

\[
\omega_i \psi_j \omega_i^{-1} = \psi_j \quad (i \neq j), \quad \omega_i \psi_i \omega_i^{-1} = q \psi_i,
\]

\[
\omega_i \psi_j^\dagger \omega_i^{-1} = \psi_j^\dagger \quad (i \neq j), \quad \omega_i \psi_i^\dagger \omega_i^{-1} = q^{-1} \psi_i^\dagger,
\]

\[
\psi_i \psi_j + \psi_j \psi_i = \psi_i^\dagger \psi_j^\dagger + \psi_j^\dagger \psi_i^\dagger = 0,
\]

\[
\psi_i \psi_j^\dagger + \psi_j^\dagger \psi_i = 0 \quad (i \neq j),
\]

\[
\psi_i \psi_i^\dagger + q^2 \psi_i^\dagger \psi_i = \omega_i^{-2}, \quad \psi_i \psi_i^\dagger + q^{-2} \psi_i^\dagger \psi_i = \omega_i^2.
\]

Hayashi\(^7\) denoted the algebra generated by \(\psi_i, \psi_i^\dagger, \omega_i, \omega_i^{-1}\) (1 ≤ i ≤ N) by \(A_q(N)\) and called it the \(q\)-Clifford algebra. We should then call \(\psi_i, \psi_i^\dagger\) (1 ≤ i ≤ N) the \(q\)-fermion operators.

As we shall discuss later, however, the \(q\)-fermion is nothing but the conventional fermion defined by

\[
\psi_i \psi_j^\dagger + \psi_j^\dagger \psi_i = \delta_{ij},
\]

\[
\psi_i \psi_j + \psi_j \psi_i = \psi_i^\dagger \psi_j^\dagger + \psi_j^\dagger \psi_i^\dagger = 0
\]

(1 ≤ i, j ≤ N)
It turns out that all the relations in (1\cdot5) are obtained from (1 \cdot 6) if we define $\omega_i$ and $\omega_i^{-1}$ by
\[
\omega_i = \psi_i \psi_i^\dagger + q^{-1} \psi_i^\dagger \psi_i, \quad (1 \cdot 7a)
\]
\[
\omega_i^{-1} = \psi_i \psi_i^\dagger + q \psi_i^\dagger \psi_i. \quad (1 \cdot 7b)
\]

After completing this paper, we became aware of the Note added in the proof of Ref.7, in which we found that the above fact was first pointed out by Takeuchi but is yet unpublished. We thus see that the algebra $A_q(N)$ can be identified with the conventional fermion algebra $A(N)$ defined by (1 \cdot 6).

We shall discuss further properties of $A(N)$ and seek the $*$-homomorphism $\delta : A(N) \rightarrow A(N) \otimes A(N)$, i.e., a linear mapping from $A(N)$ to $A(N) \otimes A(N)$ satisfying $\delta(ab) = \delta(a)\delta(b), (\delta(a))^* = \delta(a^*) \ a, b, a^* \in A(N)$ and $\delta(1) = 1 \otimes 1$. Under certain restrictions, we obtain two $*$-homomorphisms, $\delta_1$ and $\delta_2$. Although $\delta_1$ and $\delta_2$ do not satisfy co-associativity and hence cannot be regarded as the coproduct of $A(N)$, they satisfy a relation similar to co-associativity. In contrast to $A(N)$, there exists a coproduct for $U_q(A_{N-1})$. The manner in which the coproduct $\Delta : U_q(A_{N-1}) \rightarrow U_q(A_{N-1}) \otimes U_q(A_{N-1})$ should act on $e_i, f_i, k_i, k_i^{-1}(1 \leq i \leq N-1)$ is well known. It is given by
\[
\Delta(e_i) = k_i \otimes e_i + e_i \otimes k_i^{-1}, \quad (1 \cdot 8a)
\]
\[
\Delta(f_i) = k_i \otimes f_i + f_i \otimes k_i^{-1}, \quad (1 \cdot 8b)
\]
\[
\Delta(k_i) = k_i \otimes k_i. \quad (1 \cdot 8c)
\]

Since elements of $U_q(A_{N-1})$ and $A(N)$ are related by (1 \cdot 4), it is expected that the $\Delta$ can be expressed by $\delta_1$ and/or $\delta_2$. We find that this is the case.

This paper is organized as follows. In §2, we discuss how the $q$-fermionic relations (1\cdot5) are deduced from the conventional fermionic relations (1 \cdot 6) with the $\omega_i$ and $\omega_i^{-1}$ defined by (1 \cdot 7). In §3, we consider a mapping from the fermion algebra to its tensor product and obtain two $*$-homomorphisms $\delta_1$ and $\delta_2$. We investigate the relation of $\delta_1$ and $\delta_2$ to the coproduct $\Delta$ of the quantum group $U_q(su(N))$, i.e., $U_q(A_{N-1})$ supplemented with the $*$-property mentioned below, in §4. In §5, we describe a simple application of $\delta_1$ and $\delta_2$. The final section, §6, is devoted to summary.
§2. $q$-fermion identified with conventional fermion

We consider conventional fermion operators $\psi_i, \psi_i^\dagger (1 \leq i \leq N)$ satisfying the anti-commutation relations $(1 \cdot 6)$. If we define $\omega_i$ by $(1 \cdot 7a)$, we readily see that its inverse is given by $(1 \cdot 7b)$. Moreover, we obtain

$$\omega_i^n = \psi_i \psi_i^\dagger + q^{-n} \psi_i^\dagger \psi_i, \tag{2 \cdot 1}$$

for any integer $n$. Here we have made use of the relations $$(\psi_i \psi_i^\dagger)^2 = \psi_i \psi_i^\dagger, \quad (\psi_i^\dagger \psi_i)^2 = \psi_i^\dagger \psi_i, \quad (\psi_i \psi_i^\dagger)(\psi_i^\dagger \psi_i) = 0 \quad \text{and} \quad (\psi_i^\dagger \psi_i)(\psi_i \psi_i^\dagger) = 0.$$ Putting $n = 2$ or $-2$ in $(2 \cdot 1)$, we are led to $(1 \cdot 5f)$. From the definitions $(1 \cdot 7)$ of $\omega_i$ and $\omega_{-1}^i$, we get

$$(\omega_i - 1)(q\omega_i - 1) = 0, \tag{2 \cdot 2a}$$

$$\omega_i \psi_i = \psi_i, \quad \psi_i \omega_i = q^{-1} \psi_i, \tag{2 \cdot 2b}$$

$$\psi_i^\dagger \omega_i = \psi_i^\dagger, \quad \omega_i \psi_i^\dagger = q^{-1} \psi_i^\dagger. \tag{2 \cdot 2c}$$

Equations $(2 \cdot 2b)$ and $(2 \cdot 2c)$ should be regarded as the detailed versions of $(1 \cdot 5b)$ and $(1 \cdot 5c)$ and the former reproduces the latter. Thus we see that the conventional fermions reproduce the $q$-fermions. Although there exists a freedom to define a $q$-fermion as a constant multiple of a conventional fermion, we hereafter fix $\psi_i, \psi_i^\dagger, \omega_i$ and $\omega_{-1}^i$ by $(1 \cdot 6)$ and $(1 \cdot 7)$ and regard the $q$-fermions equivalent to the conventional fermions. Guided by the above discussion, we consider the algebra $A(N)$ solely generated by $\psi_i, \psi_i^\dagger (1 \leq i \leq N)$ satisfying $(1 \cdot 6)$. It should be noted that the relation $(2 \cdot 2a)$ was neither used nor noted by Hayashi. It turns out, however, to be important to discuss the algebra homomorphism of $A(N)$.

In $A(N)$, we define the $*$-operation by

$$(\psi_i)^* = \psi_i^\dagger, \quad (\psi_i^\dagger)^* = \psi_i. \tag{2 \cdot 3}$$

$$(c\phi\phi')^* = c^* (\phi')^* (\phi)^*, \quad (\phi, \phi' \in A(N)) \tag{2 \cdot 4}$$

where $c$ and $c^*$ are a complex number and its complex conjugate, respectively. Since we are assuming that $q$ is a real number, we have

$$(\omega_i)^* = \omega_i, \quad (\omega_{-1}^i)^* = \omega_{-1}^i. \tag{2 \cdot 5}$$
The ∗-operations for $U_q(A_{N-1})$ are naturally defined with the help of (2 · 3), ∼ (2cd15) and (1 · 4). They are given by

\[ (e_i)^* = f_i, \quad (f_i)^* = e_i, \]  

\[ (k_i)^* = k_i, \quad (k_i^{-1})^* = (k_i)^{-1} \]  

(2 · 6a, 6b)

The group $U_q(A_{N-1})$ with the ∗-property (2 · 6) is denoted by $U_q(su(N))$ and has some interesting applications in physics. In other words, to consider $U_q(su(N))$, the ∗-property (2 · 6) is indispensable.

The representation (1 · 4) is called the spinor representation of $U_q(su(N))$. Its representation space is the fermion Fock space $V$ spanned by the vectors

\[ |m\rangle \equiv (\psi_1^\dagger)^{m_1}(\psi_2^\dagger)^{m_2} \cdots (\psi_N^\dagger)^{m_N}|0\rangle, \]  

(2 · 7)

where $m$ is given by

\[ m = (m_1, m_2, \cdots, m_n) \in \{0, 1\}^N \]  

(2 · 8)

and $|0\rangle$ is the vacuum satisfying

\[ \psi_i|0\rangle = 0. \quad (1 \leq i \leq N) \]  

(2 · 9)

If we denote the vector $(0, 0, \cdots, 0, 1, 0, \cdots, 0)$ with the $i$th component equal to 1 and all other components 0 by $e_i$, we have

\[ \psi_i|m\rangle = (-1)^{m_1 + m_2 + \cdots + m_{i-1} + 1} \delta_{m,i,1} |m - e_i\rangle, \]  

(2 · 10a)

\[ \psi_i^\dagger|m\rangle = (-1)^{m_1 + m_2 + \cdots + m_{i-1}} \delta_{m,i,0} |m + e_i\rangle, \]  

(2 · 10b)

and hence

\[ \omega_i|m\rangle = q^{-m_i}|m\rangle. \]  

(2 · 10c)

The space $V$ is decomposed as

\[ V = \bigoplus_{r=0}^{N} V_r, \]  

(2 · 11)

where $V_r$ is a subspace of $V$ defined by

\[ V_r = \bigoplus_{|m|=r} \mathbb{C}|m\rangle, \quad |m| \equiv \sum_{i=1}^{N} m_i. \]  

(2 · 12)
It is known\(^7\) that the space \(V\) is irreducible under the actions of \(A_q(N)\), while each \(V_r(0 \leq r \leq N)\) is irreducible under the action of \(U_q(A_{N-1})\) with \(e_i, f_i, k_i, k_i^{-1}\) given by (1·4).

We note that, in the representations mentioned above, the generators \(e_i, f_i, k_i, k_i^{-1}\) defined by (1·4) satisfy relations in addition to those in (1·1):

\[
e_i^2 = f_i^2 = 0,
\]

\[
(k_i - 1)(k_i - q)(k_i - q^{-1}) = 0,
\]

\[
e_i f_i e_i = e_i, \quad f_i e_i f_i = f_i,
\]

\[
k_i = 1 + \frac{1 - q}{q}(f_i e_i - q e_i f_i),
\]

\[
k_i^{-1} = 1 + \frac{1 - q}{q}(e_i f_i - q f_i e_i),
\]

\[
e_i k_i = q^{-1} e_i, \quad f_i k_i = q f_i,
\]

\[
k_i e_i = q e_i, \quad k_i f_i = q^{-1} f_i.
\]

(2·13)

For example, we have \(e_i^2 v = 0\) for any vector \(v\) belonging to \(V\). The equality \(\Delta(e_i^2) = \{\Delta(e_i)\}^2\), however, is in general a non-vanishing operator on \(V \otimes V\). To obtain general representations, we are to consider tensor products of spinor representations.

For operators \(a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n \in A(N)\), we define the multiplication rule of tensor products by

\[
(a_1 \otimes a_2 \otimes \cdots \otimes a_n)(b_1 \otimes b_2 \otimes \cdots \otimes b_n) = (-1)^M a_1 b_1 \otimes a_2 b_2 \otimes \cdots \otimes a_n b_n,
\]

(2·14)

\[
M = \sum_{i=2}^{n} \sum_{j=1}^{i-1} d(a_i)d(b_j),
\]

(2·15)

where \(d(a) = 0\) (respectively 1) if \(a \in A(N)\) consists of an even (respectively, odd) number of fermions operators. The *-operation on \(a_1 \otimes a_2 \otimes \cdots \otimes a_n\) is defined by

\[
(a_1 \otimes a_2 \otimes \cdots \otimes a_n)^* = (-1)^L (a_1)^* \otimes (a_2)^* \otimes \cdots \otimes (a_n)^*,
\]

(2·16)

\[
L = \sum_{i=2}^{n} \sum_{j=1}^{i} d(a_i)d(a_j),
\]

(2·17)
The key tool to discuss the tensor product representations of $A(N)$ preserving the $\ast$-property is the $\ast$-homomorphism $\delta$ of $A(N)$, i.e. a linear mapping from $A(N)$ to $A(N) \otimes A(N)$ satisfying

$$\delta(ab) = \delta(a)\delta(b), \quad \delta(1) = 1 \otimes 1, \quad (\delta(a))^\ast = \delta(a^\ast), \quad (2 \cdot 18)$$

$$(a, b, a^\ast \in A(N). \quad \delta(a), \delta(b), \delta(a^\ast) \in A(N) \otimes A(N))$$

which we seek in the next section.

§ 3. Homomorphic mapping of $A(N)$ to $A(N) \otimes A(N)$

We now seek a $\ast$-homomorphism $\delta$ of $A(N)$. We assume that $\delta(\psi_i)$ takes the form

$$\delta(\psi_i) = \alpha_i \otimes \psi_i + \psi_i \otimes \beta_i, \quad (3 \cdot 1)$$

where $\alpha_i$ and $\beta_i$ consist only of $\psi_i$ and $\psi_i^\dagger$ and satisfy $d(\alpha_i) = d(\beta_i) = 0$. Because of the relations (1 · 6) and (2 · 1), the most general form of $\alpha_i$ and $\beta_i$ are given by

$$\alpha_i = a_i \psi_i \psi_i^\dagger + b_i \psi_i^\dagger \psi_i, \quad \beta_i = c_i \psi_i \psi_i^\dagger + d_i \psi_i^\dagger \psi_i, \quad a_i, b_i, c_i, d_i \in C.$$  

Assuming that $a_i, b_i, c_i$ and $d_i$ are independent of $i$, we have

$$\alpha_i = a \psi_i \psi_i^\dagger + b \psi_i^\dagger \psi_i,$$

$$\beta_i = c \psi_i \psi_i^\dagger + d \psi_i^\dagger \psi_i,$$

$$a, b, c, d \in C. \quad (3 \cdot 2)$$

We are thus led to consider a $\ast$-homomorphism of the form

$$\delta(\psi_i) = (a \psi_i \psi_i^\dagger + b \psi_i^\dagger \psi_i) \otimes \psi_i + \psi_i \otimes (c \psi_i \psi_i^\dagger + d \psi_i^\dagger \psi_i), \quad (3 \cdot 3a)$$

and

$$\delta(\psi_i^\dagger) = (a^\ast \psi_i \psi_i^\dagger + b^\ast \psi_i^\dagger \psi_i) \otimes \psi_i^\dagger + \psi_i^\dagger \otimes (c^\ast \psi_i \psi_i^\dagger + d^\ast \psi_i^\dagger \psi_i). \quad (3 \cdot 3b)$$
The requirements $\delta(\psi_i)\delta(\psi_j) + \delta(\psi_j)\delta(\psi_i) = 0$, $\delta(\psi_i)\delta(\psi_j^\dagger) + \delta(\psi_j^\dagger)\delta(\psi_i^\dagger) = 0$ and
$\delta(\psi_i^\dagger)\delta(\psi_j^\dagger) + \delta(\psi_j)\delta(\psi_i) = 0 \ (i \neq j)$ are obtained from (1.6) and (2.14) irrespectively of the values of $a, b, c, d$. Another requirement $\delta(\psi_i)\delta(\psi_i) = 1 \otimes 1$ is satisfied if $a, b, c$ and $d$ are related by

$$|a|^2 + |c|^2 = 1$$

$$|a|^2 = |b|^2, \ |c|^2 = |d|^2,$$

$$a^*c - b^*d = 0.$$  \hspace{1cm} (3.4)

If we further require that $\delta$ satisfies the co-associativity $(\delta \otimes \text{id}) \circ \delta(\phi) = (\text{id} \otimes \delta) \circ \delta(\phi)$, $\phi \in A(N)$, $\text{id} =$identity, we are left only with the trivial homomorphisms $\delta(\phi) = 1 \otimes \phi$ and $\delta(\phi) = \phi \otimes 1$. In other words, under the restrictions (3cdot1) and (3cdot2), there exists no nontrivial coassociative $*$-homomorphism for $A(N)$. A somewhat looser requirement

$$m((\text{id} \otimes \delta) \circ \delta(\phi)) = m((\delta \otimes \text{id}) \circ \delta(\phi)), \quad (\phi \in A(N)) \hspace{1cm} (3.5)$$

allows essentially four solutions, where $m$ is the multiplication

$$m(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = a_1a_2 \cdots a_n. \quad (a_1, a_2, \cdots, a_n \in A(N)) \hspace{1cm} (3.6)$$

Two of the above four solutions satisfy a slightly modified coassociativity:

$$Y \circ (\delta_k \otimes \text{id}) \circ \delta_k \circ Y(a) = (\text{id} \otimes \delta_k) \circ \delta_k(a). \quad (a \in A(N), \ k = 1, 2) \hspace{1cm} (3.7)$$

Here the linear mapping $Y$ is defined by

$$Y(a_1a_2) = Y(a_2)Y(a_1), \hspace{1cm} (3.8a)$$

$$Y(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = Y(a_n) \otimes \cdots \otimes Y(a_2) \otimes Y(a_1), \hspace{1cm} (3.8b)$$

$$(a_1, a_2, \cdots, a_n \in A(N))$$

and by

$$Y(\psi_i) = \psi_i, \quad Y(\psi_i^\dagger) = \psi_i^\dagger. \quad (1 \leq i \leq N) \hspace{1cm} (3.9)$$
It can be readily seen that $Y$ satisfies

$$Y^2 = \text{id.} \quad (3 \cdot 10)$$

When an algebra homomorphism satisfies the co-associativity, the coalgebra can be defined as an associative algebra. In the case that we have only $(3 \cdot 7)$, the coalgebra $B(N)$ dual to $A(N)$ is not associative but turns out to be a flexible algebra. The two homomorphisms satisfying conditions $(3 \cdot 5)$ as well as $(3 \cdot 7)$ are given by

$$\delta_1(\psi_i) = \frac{i}{\sqrt{2}}(\zeta_i \otimes \psi_i - \psi_i \otimes \zeta_i),$$

$$\delta_1(\psi_i^\dagger) = -\frac{i}{\sqrt{2}}(\zeta_i \otimes \psi_i^\dagger - \psi_i^\dagger \otimes \zeta_i), \quad (1 \leq i \leq N) \quad (3 \cdot 11)$$

and

$$\delta_2(\psi_i) = \frac{1}{\sqrt{2}}(1 \otimes \psi_i + \psi_i \otimes 1),$$

$$\delta_2(\psi_i^\dagger) = \frac{1}{\sqrt{2}}(1 \otimes \psi_i^\dagger + \psi_i^\dagger \otimes 1), \quad (1 \leq i \leq N) \quad (3 \cdot 12)$$

where $\zeta_i$ is defined by

$$\zeta_i = \psi_i \psi_i^\dagger - \psi_i^\dagger \psi_i, \quad (1 \leq i \leq N) \quad (3 \cdot 13)$$

The equality $(3 \cdot 7)$ is derived in the following way. By direct calculation, we see that Equation $(3 \cdot 7)$ holds for $a = \psi_i$ and $a = \psi_i^\dagger$. For example, we have

$$Y \circ (id \otimes \delta_1) \circ \delta_1 \circ Y(\psi_i)$$

$$= -\frac{1}{2}(\psi_i \otimes \zeta_i \otimes \zeta_i - \zeta_i \otimes \psi_i \otimes \zeta_i) + \frac{i}{2\sqrt{2}}(\zeta_i \otimes 1 \otimes \psi_i + 1 \otimes \zeta_i \otimes \psi_i)$$

$$+ \frac{i}{\sqrt{2}}(\psi_i^\dagger \otimes \psi_i \otimes \psi_i - \psi_i \otimes \psi_i^\dagger \otimes \psi_i)$$

$$= (\delta_1 \otimes id) \circ \delta_1 (\psi_i). \quad (3 \cdot 14)$$

From the definition of $\delta_1$ and $\delta_2$, we observe the relation

$$d(a) = d(a'_{k,l}) + d(a''_{k,l}), \quad (\text{mod} 2) \quad (3 \cdot 15)$$
where $a^t_{k,l}$ and $a^r_{k,l}$ are defined by $\delta_k(a) = \sum_l a^t_{k,l} \otimes a^r_{k,l}$. With the aid of (3cdot15), it can be shown that the mappings $U_k, W_k : A(N) \rightarrow A(N) \otimes A(N) \otimes A(N)$ defined by

$$U_k = Y \circ (\delta_k \otimes id) \circ \delta_k \circ Y, \quad W_k = (id \otimes \delta_k) \circ \delta_k$$

(3·16)

satisfy

$$U_k(ab) = U_k(a)U_k(b), \quad (3·17a)$$

$$W_k(ab) = W_k(a)W_k(b), \quad (3·17b)$$

for $a, b \in A(N), \ k = 1, 2$. This discussion yields (3·7). We have thus obtained two fermionic homomorphisms $\delta_1$ and $\delta_2$ by requiring (3cdot3), (3·5) and (3·7).

We here mention that, as for a type of $q$-bosonic algebra, there has been invented a coproduct.\textsuperscript{10} This coproduct, however, does not possess, the $*$-property necessary for the discussion of $U_q(su(N))$. In our case of fermion algebra, we respect the $*$-property and modify the co-associativity to (3·7). We thus see that the fermion algebra $A(N)$ is a bialgebra with the homomorphism $\delta_1$ or $\delta_2$. The higher dimensional representations of $U_q(su(N))$ can be obtained by making use of (1·4) and $\delta_1$ or $\delta_2$. For example, $\delta_k(\psi_i\psi_{i+1}^\dagger), \, \delta_k(\psi_{i+1}\psi_i^\dagger), \, \delta_k(((\psi_i^\dagger \psi_i + q^{-1}\psi_i^\dagger \psi_i)(\psi_{i+1}^\dagger \psi_{i+1}^\dagger + q\psi_{i+1}^\dagger \psi_{i+1}))$ with $1 \leq i \leq N - 1, \ k = 1$ or $2$, constitute the representation of $e_i, \ f_i, \ k_i \in U_q(su(N))$, respectively, on $V \otimes V$. The mappings $U_k$ and $W_k$ defined by (3·16) can be used to construct representations of $U_q(su(N))$ on $V \otimes V \otimes V$.

We note that $\delta_1$ together with $\delta_2$ generates a $2N$-fermion algebra on $A(N) \otimes A(N)$. This result is obtained by observing

$$\delta_1(\psi_i)\delta_2(\psi_j^\dagger) + \delta_2(\psi_j^\dagger)\delta_1(\psi_i) = 0, \ (i \neq j)$$

(3·18)

$$\delta_2(\psi_i)\delta_1(\psi_j^\dagger) + \delta_1(\psi_j^\dagger)\delta_2(\psi_i) = 0, \ (i \neq j)$$

and

$$\delta_1(\psi_i)\delta_2(\psi_j) + \delta_2(\psi_j)\delta_1(\psi_i) = \delta_1(\psi_i^\dagger)\delta_2(\psi_j^\dagger) + \delta_2(\psi_j^\dagger)\delta_1(\psi_i^\dagger) = 0$$

(3·19)

and recalling that both $\delta_1$ and $\delta_2$ are homomorphisms of the $N$-fermion algebra. If we define $\Psi_I, \ 1 \leq I \leq 2N$ by

$$\Psi_i = \delta_1(\psi_i), \quad \Psi_i^\dagger = \delta_1(\psi_i^\dagger), \quad (1 \leq i \leq N)$$

$$\Psi_{i+N} = \delta_2(\psi_i), \quad \Psi_{i+N}^\dagger = \delta_2(\psi_i^\dagger), \quad (1 \leq i \leq N)$$

(3·20)
the properties mentioned above yield the $2N$-fermion algebra on $A(N) \otimes A(N)$:

$$
\Psi_I \Psi_J^\dagger + \Psi_J^\dagger \Psi_I = \delta_{IJ} 1 \otimes 1,
$$

$$
\Psi_I \Psi_J + \Psi_J \Psi_I = \Psi_J^\dagger \Psi_I^\dagger + \Psi_I^\dagger \Psi_J = 0.
$$

(3 \cdot 21) 

$$(1 \leq I, J \leq 2N)$$

§4. $\Delta$ in term of $\delta_1$ and $\delta_2$

As was discussed in the previous sections, the quantum group $U_q(A_{N-1})$ is related to the algebra $A(N)$ generated by conventional fermion operators. The coproduct $\Delta$ of $U_q(A_{N-1})$ maps $U_q(A_{N-1})$ into $U_q(A_{N-1}) \otimes U_q(A_{N-1})$ homomorphically. On the other hand, under certain restrictions we have found two homomorphic mappings $\delta_1, \delta_2 : A(N) \rightarrow A(N) \otimes A(N)$. It would be very interesting if there existed some relations between $\Delta$ and $\delta_1$ and/or $\delta_2$.

To explore this possibility, it is sufficient to express $\Delta(e_i), \Delta(f_i), \Delta(k_i), \Delta(k_i^{-1}), 1 \leq i \leq N - 1$, in terms of $\psi_i, \psi_i^\dagger, 1 \leq i \leq N$ and $\delta_k, k = 1, 2$. We first discuss $\Delta(k_i)$. From $(1 \cdot 7a), (3 \cdot 11)$ and $(3 \cdot 12)$, we obtain

$$2\delta_1(\omega_i) = 1 \otimes \omega_i + \omega_i \otimes 1 + \frac{q-1}{q} (\psi_i \otimes \psi_i^\dagger - \psi_i^\dagger \otimes \psi_i),$$

(4 \cdot 1)

$$2\delta_2(\omega_i) = 1 \otimes \omega_i + \omega_i \otimes 1 - \frac{q-1}{q} (\psi_i \otimes \psi_i^\dagger - \psi_i^\dagger \otimes \psi).$$

Their product is given by

$$\delta_1(\omega_i)\delta_2(\omega_i) = \delta_2(\omega_i)\delta_1(\omega_i) = \omega_i \otimes \omega_i,$$

(4 \cdot 2)

where we have made use of $(2 \cdot 2a)$. We also have

$$\delta_1(\omega_i^{-1})\delta_2(\omega_i^{-1}) = \delta_2(\omega_i^{-1})\delta_1(\omega_i^{-1}) = \omega_i^{-1} \otimes \omega_i^{-1}.$$  

(4 \cdot 3)
We here note the interesting formula
\[ \delta_1(\omega_i^n) + \delta_2(\omega_i^n) = 1 \otimes \omega_i^n + \omega_i^n \otimes 1, \quad (4 \cdot 4) \]
where \( n \) is an arbitrary integer. Recalling \((1 \cdot 4c)\) and \((1 \cdot 8c)\), we readily obtain
\[ \Delta(k_i) = \delta_1(k_i) \delta_2(k_i) = \delta_2(k_i) \delta_1(k_i), \quad (4 \cdot 5a) \]
\[ \Delta(k_i^{-1}) = \delta_1(k_i^{-1}) \delta_2(k_i^{-1}) = \delta_2(k_i^{-1}) \delta_1(k_i^{-1}) \quad (4 \cdot 5b) \]
from \((4 \cdot 2)\) and \((4 \cdot 3)\). Turning to the discussion of \( \Delta(e_i) \) and \( \Delta(f_i) \), we define \( \tilde{\delta} \) and a mapping \( Z : A(N) \otimes A(N) \to A(N) \otimes A(N) \) by
\[ \tilde{\delta}(a) = \frac{1}{2\sqrt{q}} \left\{ (q + 1) \delta_2(a) - i(q - 1) \delta_1(a) \right\}, \quad (a \in A(N)) \quad (4 \cdot 6) \]
\[ Z(a \otimes b) = Y^{d(a)d(b)}(a \otimes b) = \begin{cases} a \otimes b & \text{if } d(a)d(b) = 0, \\ Y(a \otimes b) & \text{if } d(a)d(b) = 1. \end{cases} \quad (4 \cdot 7) \]
Then, with the aid of the relations
\[ \omega_i = \frac{1}{2} \{ (q^{-1} + 1) - (q^{-1} - 1) \zeta_i \}, \quad (4 \cdot 8) \]
\[ \omega_i^{-1} = \frac{1}{2} \{ (q + 1) - (q - 1) \zeta_i \}, \quad (4 \cdot 9) \]
we obtain
\[ \tilde{\delta}(\psi_i) = \frac{1}{\sqrt{2}} (\sqrt{q} \omega_i \otimes \psi_i + \frac{1}{\sqrt{q}} \psi_i \otimes \omega_i^{-1}), \quad (4 \cdot 10) \]
\[ \tilde{\delta}(\psi_{i+1}^\dagger) = \frac{1}{\sqrt{2}} (\sqrt{q} \psi_{i+1}^\dagger \otimes \omega_{i+1} + \frac{1}{\sqrt{q}} \psi_{i+1} \otimes \omega_{i+1}^{-1} \psi_{i+1}^\dagger) \]
and hence
\[ \tilde{\delta}(\psi_i) \tilde{\delta}(\psi_{i+1}^\dagger) \]
\[ = \frac{1}{2} \left\{ \omega_i \omega_{i+1}^{-1} \otimes \psi_i \psi_{i+1}^\dagger + \psi_i \psi_{i+1}^\dagger \otimes \omega_i^{-1} \omega_{i+1} \\ -q \omega_i \psi_{i+1}^\dagger \otimes \psi_i \omega_{i+1} + q^{-1} \psi_i \omega_{i+1}^{-1} \otimes \omega_i^{-1} \psi_{i+1}^\dagger \right\} \]
\[ = \frac{1}{2} \left\{ k_i \otimes e_i + e_i \otimes k_i^{-1} - q(1 - Y)(\omega_i \psi_{i+1}^\dagger \otimes \psi_i \omega_{i+1}) \right\}. \quad (4 \cdot 11) \]
Noting the relations
\[ Y(\omega_i) = q^{-1} \omega_i^{-1}, \quad Y(\omega_i^{-1}) = q \omega_i, \]
\[ Y(k_{i-1}) = k_i, \quad Y(k_i) = k_{i-1}, \quad Y(e_i) = -e_i, \]

we obtain
\[ \Delta(e_i) = (id + Z) \circ \{ \tilde{\delta}(\psi_i)\tilde{\delta}(\psi_{i+1}^\dagger) \}. \quad (4 \cdot 13) \]

Similarly, we have
\[ \Delta(f_i) = (id + Z) \circ \{ \tilde{\delta}(\psi_{i+1})\tilde{\delta}(\psi_i^\dagger) \}. \quad (4 \cdot 14) \]

It can be seen that the relations (3 \cdot 18) and (3 \cdot 19) ensure \( \Delta(f_i) = \{\Delta(e_i)\}^* \). We note that equations (4 \cdot 13) and (4 \cdot 14) are also written as
\[
\Delta(e_i) = \{ 2\tilde{\delta}(\psi_i)\tilde{\delta}(\psi_{i+1}^\dagger) \} \cap (S \otimes S), \quad (4 \cdot 15a) \\
\Delta(f_i) = \{ 2\tilde{\delta}(\psi_{i+1})\tilde{\delta}(\psi_i^\dagger) \} \cap (S \otimes S), \quad (4 \cdot 15b) 
\]

where \( S \) is the linear algebra spanned by \( e_i, f_i, k_i, k_{i-1} \) of (1 \cdot 4), or
\[
\Delta(e_i) = \tilde{\delta}(\psi_i)\tilde{\delta}(\psi_{i+1}^\dagger) + \tau(\{ \tilde{\delta}(\psi_{i+1}^\dagger) \}^* \{ \tilde{\delta}(\psi_i) \}^*), \quad (4 \cdot 16a) \\
\Delta(f_i) = \tilde{\delta}(\psi_{i+1})\tilde{\delta}(\psi_i^\dagger) + \tau(\{ \tilde{\delta}(\psi_i) \}^* \{ \tilde{\delta}(\psi_{i+1}^\dagger) \}^*), \quad (4 \cdot 16b) 
\]

where \( \tau \) is defined by
\[
\tau(a \otimes b) = b \otimes a. \quad (a, b \in A(N)) \quad (4 \cdot 17) 
\]

\section*{§5. An application of \( \delta_1 \) and \( \delta_2 \)}

As an application of the homomorphisms \( \delta_1 \) and \( \delta_2 \), we consider an eigenvalue problem on the product space \( V \otimes V \), where \( V \) in the fermion Fock space discussed in §2. We define the operator \( H \) by
\[
H = \sum_{i,j=1}^{N} (a_{ij} \psi_i \otimes \psi_j^\dagger + b_{ij} \psi_i^\dagger \otimes \psi_j), \quad (a_{ij} \in \mathbb{C}) \quad (5 \cdot 1) 
\]
Reguiring that $H$ satisfies $H^* = H$ with the $*$-operation defined by $(2 \cdot 16)$, we have $b_{ij} = -a_{ij}^*$. The further requirement (a) $\tau(H) = -H$ yields $a_{ij} = a_{ij}^*$, while the requirement (b) $\tau(H) = H$ yeilds $a_{ij} = -a_{ij}^*$. Through an appropriate unitary transformation, the matrix $a = (a_{ij})$ is diagonalized, and we are left with

$$H_a = \sum_{i,j=1}^{N} a_{ij}(\psi_i \otimes \psi_j^\dagger - \psi_i^\dagger \otimes \psi_j) = \sum_{l=1}^{N} \lambda_l (\phi_l \otimes \phi_l^\dagger - \phi_l^\dagger \otimes \phi_l), \quad (\lambda_l, a_{ij} \in \mathbb{R}) \quad (5 \cdot 2a)$$

$$H_b = \sum_{i,j=1}^{N} a_{ij}(\psi_i \otimes \psi_j^\dagger + \psi_i^\dagger \otimes \psi_j) = \sum_{l=0}^{N} \sigma_l (\phi_l \otimes \phi_l^\dagger + \phi_l^\dagger \otimes \phi_l)). \quad (\sigma_l, ia_{ij} \in \mathbb{R}) \quad (5 \cdot 2b)$$

Here $\phi_l$ is a unitary transformation of $\psi_m$:

$$\phi_l = \sum_{M=1}^{N} u_{lm} \psi_m, \quad (u_{lm}) = u : \text{unitary}, \quad (5 \cdot 3)$$

It t satisfies the same $N$ fermion algebra as $(1 \cdot 6)$. The eigenvalue problem of $H_a$ is solved in the following way. As we saw at the end of §3, the set of operators $\{ \Phi_I : 1 \leq I \leq 2N \}$ defined by

$$\Phi_i = \delta_1(\phi_i), \quad \Phi_{i+N} = \delta_2(\phi_i), \quad (1 \leq i \leq N) \quad (5 \cdot 4)$$

constitutes a $2N$ fermion algebra. It is easy to rewrite $H_a$ as

$$H_a = \sum_{i=1}^{n} \lambda_i (\Phi_i^\dagger \Phi_i - \Phi_{i+N}^\dagger \Phi_{i+N}). \quad (5 \cdot 5)$$

We now see that the vector

$$|\mathbf{M}\rangle = (\Phi_1^\dagger)^{M_1} (\Phi_2^\dagger)^{M_2} \cdots (\Phi_{2N}^\dagger)^{M_{2N}} (|0\rangle \otimes |0\rangle) \quad (5 \cdot 6)$$

$$(M_I \in \{0, 1\}, \quad 1 \leq I \leq 2N)$$

is the eigenvector of $H_a$ belonging to the eigenvalue

$$E_{\mathbf{M}} = \sum_{i=1}^{n} \lambda (M_i - M_{i+N}). \quad (5 \cdot 7)$$

The orthonormality and the completeness of $\{|\mathbf{M}\rangle, \mathbf{M} \in \{0, 1\}^{2N}\}$ in $V \otimes V$ is assured by the $2N$-fermion algebraic property of $\{ \Phi_I : 1 \leq I \leq 2N \}$.

We note that the eigenvalue problem of $H_a$ can be solved with the aid of the two other homomorphisms mentioned between $(3 \cdot 5)$ and $(3 \cdot 6)$. 

§6. Summary

In this paper, we have discussed how Hayashi’s $q$-fermion algebra $A_q(N)$ is realized in terms of the conventional fermion algebra $A(N)$. A simple but important observation is that the quantity $\omega_i$ defined by (1·7a) satisfies (2·1). We have found the algebra homomorphisms $\delta_1, \delta_2 : A(N) \to A(N) \otimes A(N)$ satisfying the pseudo-co-associativity, (3·7). We have seen that $\delta_1$ and $\delta_2$ can be used to generate a $2N$-fermion algebra on $A(N) \otimes A(N)$. In terms of $\delta_1$ and $\delta_2$, the $U_q(su(N))$ coproducts $\Delta(k_i)$ and $\Delta(k_i^{-1})$ are given by (4·5a) and (4·5b), respectively. The combination $\tilde{\delta}$ is convenient for the discussion of $\Delta(e_i)$ and $\Delta(f_i)$ as is seen in (4·13) $\sim$ (4·16). The expression for $\Delta$ is rather simple but is not simple. Some deeper meaning of them should be sought.

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