ON THE QUANTUM POINCARÉ GROUP

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Abstract

The inhomogeneous quantum groups $IGL_q(n)$ are obtained by means of a particular projection of $GL_q(n + 1)$. The bicovariant differential calculus on $GL_q(n)$ is likewise projected into a consistent bicovariant calculus on $IGL_q(n)$. Applying the same method to $GL_q(n, \mathbb{C})$ leads to a bicovariant calculus for the complex inhomogeneous quantum groups $IGL_q(n, \mathbb{C})$. The quantum Poincaré group and its bicovariant geometry are recovered by specializing our results to $ISL_q(2, \mathbb{C})$. 
The study of a continuous deformation of the Poincaré group is worthwhile per se, given the central role of this group in physics. In the context of a quantum group theoretic formulation of gravity theories, it is in fact essential to find a consistent $q$-deformation of the Poincaré group. This we will present in this letter.

Quantum groups [1]-[4] have emerged in the last years as nontrivial deformations of Lie groups, and the differential calculus on them has been developed recently [5, 6, 7, 8, 9]. The general constructive procedure of ref. [7] works for the $q$-groups of the $A, B, C, D$ series, and in ref. [10] we have studied how to extend it to nonhomogeneous quantum groups.

We begin by presenting a general method for constructing inhomogeneous quantum groups and their complexification. In ref. [10] we have found the $R$-matrix and a bicovariant differential calculus for $IGL_q(n)$, using the definition of inhomogeneous $q$-groups as given in [11]. Here, however, we take a different route and obtain the inhomogeneous $q$-groups $IGL_q(n)$ and $IGL_q(n, C)$ (and their bicovariant differential calculi) as projections of $GL_q(n+1)$ and $GL_q(n+1, C)$ (and their bicovariant differential calculi). Both procedures are equivalent and lead to the same $q$-differential calculi; the one we present here has the advantage of giving a new interpretation to the results of ref.s [11, 10].

Finally, our method is specialized to $ISL_q(2, C)$, the quantum Poincaré group, whose bicovariant $q$-Lie algebra is found and given explicitly in the Table.

Other papers concerning the quantum Lorentz group or the quantum Poincaré group are quoted in [12]-[14].

The key observation is that the $R$-matrix of $GL_q(n+1)$ can be written as (A=(0,a)):

$$ R^{AB}_{CD} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1} & 1 & 0 \\ 0 & 0 & 0 & R^{ab}_{cd} \end{pmatrix} $$

(1)

where $R^{ab}_{cd}$ is the $R$-matrix of $GL_q(n)$, and the indices $AB$ are ordered as $00, 0b, a0, ab$. This form of the $R$-matrix allows a consistent projection of $GL_q(n+1)$ into $GL_q(n)$. This projection works also for the corresponding $*$-Hopf algebra structures and bicovariant differential calculi, as was pointed out in ref. [13] in the case of $GL_q(3) \rightarrow GL_q(2)$. The reason why it works will be clarified below.

Let us explain what we mean by “projection”. To be specific, we take again the

1 “bicovariant” meaning that one can define a left and a right action of the $q$-group on the space of quantum one-forms, as in the $q = 1$ case, see ref. [5].
case of $GL_q(3)$, generated by: i) the matrix elements $T^A_B$

$$T^A_B = \begin{pmatrix} T^0_0 & T^0_1 & T^0_2 \\ T^1_0 & T^1_1 & T^1_2 \\ T^2_0 & T^2_1 & T^2_2 \end{pmatrix} \equiv \begin{pmatrix} T_1 & T_2 & T_3 \\ T_4 & T_5 & T_6 \\ T_7 & T_8 & T_9 \end{pmatrix}$$

(2)

ii) the identity $I$, and the inverse $\Xi$ of the $q$-determinant of $T$, defined by:

$$\Xi \det_q T = \det_q T \Xi = I$$

$$\det_q T \equiv \sum_\sigma (-q)^{l(\sigma)} T^1_{\sigma(1)} \cdots T^n_{\sigma(n)}$$

(3)

(4)

where $l(\sigma)$ is the minimum number of transpositions in the permutation $\sigma$. Moreover, the matrix entries in (2) satisfy the “RTT” relations:

$$R^{AB}_{\ EF} T^E_C T^F_D = T^B_F T^A_E R^{EF}_{\ CD}$$

(5)

with $R^{AB}_{\ CD}$ given by:

$$R^{AB}_{\ CD} = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \end{pmatrix}$$

(6)

where $\lambda = q - q^{-1}$. The reader can verify that this $R$-matrix indeed has the form (1). We recall the co-structures of $GL_q(n+1)$, i.e the coproduct $\Delta$, the counit $\varepsilon$ and the coinverse $\kappa$:

$$\Delta(T^A_B) = T^A_B \otimes T^B_C$$

$$\varepsilon(T^A_B) = \delta^A_B$$

$$\kappa(T^A_B) = (T^{-1})^A_B$$

$$\Delta(\det_q T) = \det_q T \otimes \det_q T, \quad \Delta(\Xi) = \Xi \otimes \Xi, \quad \Delta(I) = I \otimes I$$

$$\varepsilon(\det_q T) = 1, \quad \varepsilon(\xi) = 1, \quad \varepsilon(I) = 1$$

$$\kappa(\det_q T) = \xi, \quad \kappa(\xi) = \det_q T, \quad \kappa(I) = I$$

(7)

(8)

(9)

(10)

(11)

(12)

The quantum inverse of $T^A_B$ is given by:

$$(T^{-1})^A_B = \Xi (-q)^{A-B} t^A_B$$

(13)

where $t^A_B$ is the quantum minor, i.e. the quantum determinant of the submatrix of $T$ obtained by removing the $B$-th row and the $A$-th column.
A consistent $\ast$-structure of $GL_q(n)$ is given by [3]:

\[ (T^A_D)^\ast = \kappa(T^B_A), \]  
\[ \Xi^\ast = \det_q T \]  
(14)  
(15)

The unitarity condition (14) restricts $GL_q(n+1)$ to $U_q(n+1)$, while setting $\det_q T = I$ restricts $GL_q(n+1)$ to $SL_q(n+1)$. If both conditions hold we have $SU_q(n+1)$. For sake of generality we consider $GL_q(n+1)$ rather than its restrictions, but our discussion applies to $SL_q(n+1)$, $U_q(n+1)$ and $SU_q(n+1)$ as well. In the following we do not require (14) to hold.

The projection of $GL_q(3)$ onto $GL_q(2)$ is defined by setting

\[ T_2 = T_3 = T_4 = T_7 = 0, \quad T_1 = I \]  
(16)
in (2). The corresponding left-invariant one-forms $\omega$ and the $q$-Lie algebra generators $\chi$ are set to “zero”:

\[ \omega^2 = \omega^3 = \omega^4 = \omega^7 = \omega^1 = 0, \]  
(17)
\[ \chi_2 = \chi_3 = \chi_4 = \chi_7 = \chi_1 = 0. \]  
(18)

(another equivalent projection would be given by $T_3 = T_6 = T_7 = T_8 = 0, \quad T_9 = I$). Using (17) and (18) in the differential calculus of ref. [13], one retrieves the bicovariant differential calculus on $GL_q(2)$. The reason this projection works at the quantum group level is of course the particular form of the $R$-matrix in (2), so that, for example, the “RTT” relations for $GL_q(3)$ correctly reduce to those of $GL_q(2)$. Also, the $\ast$-Hopf algebra structures project into those of $GL_q(2)$, as one can easily deduce by substituting (17) into (2)-(23). As we now discuss, the projection works also for the differential calculi.

A bicovariant differential calculus [3] on $GL_q(n+1)$ can be constructed in terms of the corresponding $R$ matrix [4, 5, 6]. The basic object is the braiding matrix

\[ \Lambda_{A_1 D_1}^A_{C_1} | B_1 C_2 B_2 = d^2 d^{-1} R_{C_2 G_1} R^{-1} C_1 G_1 E_1 A_1 R^{-1} A_2 E_1 G_2 D_1 R_{G_2 D_2 B_2 F_2} \]  
(19)

which is used in the definition of the exterior product of quantum left-invariant one forms $\omega_A^B$:

\[ \omega_{A_1}^{A_2} \wedge \omega_{D_1}^{D_2} \equiv \omega_{A_1}^{A_2} \otimes \omega_{D_1}^{D_2} - \Lambda_{A_1 D_1}^A_{C_1} B_1 C_2 B_2 \omega_{C_1}^{C_2} \otimes \omega_{B_1}^{B_2} \]  
(20)

and in the $q$-commutations of the quantum Lie algebra generators $\chi_A^B$:

\[ \chi_{D_1}^{D_2} \chi_{C_1}^{C_2} - \Lambda_{E_1 F_1}^{E_2 F_2} D_1 C_1 \chi_{E_2}^{E_1} F_2^{F_1} C_2 B_2 \chi_{E_2}^{E_1} F_2^{F_1} C_2 B_2 = C_{D_1}^{D_2} C_1 A_1 A_2 \]  
(21)

where the structure constants are explicitly given by:

\[ C_{A_2 B_2}^{A_1} | B_1 C_1 = \frac{1}{q - q^{-1}} [-\delta_{B_2}^{A_1} \delta_{A_2}^{C_1} + \Lambda_{B_2}^{B_1} C_1 A_2 B_2 | A_2 B_2]. \]  
(22)
and $\chi^{D_1 C_1}_{C_2} \equiv (\chi^{D_1}_{D_2} \otimes \chi^{C_1}_{C_2})\Delta$, cf. ref.s [7, 8, 9].

The $d^A$ vector is defined by

$$\kappa^2(T^A_B) = D^A_C T^C_D (D^{-1})^B_D = d^A d^{-1} B A$$ (23)

For $GL_q(n+1)$ we have $d^A = q^{2A-1}$ (cf. [3]). In the case of $GL_q(3)$, $d^0 = q$, $d^1 = q^3$, $d^2 = q^5$.

The braiding matrix $\Lambda$ and the structure constants $C$ defined in (19) and (22) satisfy the conditions

$$C_{ri}^{\ n} C_{nj}^{\ s} - \Lambda^{kl}_{ij} C_{rk}^{\ n} C_{nl}^{\ s} = C_{ij}^{\ k} C_{rk}^{\ s}$$ (q-Jacobi identities) (24)

$$\Lambda^{nm}_{ij} \Lambda^{rk}_{lp} \Lambda^{ls}_{pq} = \Lambda^{nk}_{ri} \Lambda^{ms}_{kj} \Lambda^{ij}_{pq}$$ (Yang–Baxter) (25)

$$C_{mn}^{\ m} \Lambda^{rs}_{lk} + \Lambda^{il}_{rj} C_{lk}^{\ s} = \Lambda^{pq}_{jk} \Lambda^{il}_{rp} C_{lp}^{\ s} + C_{jk}^{\ m} \Lambda^{is}_{rm}$$ (26)

$$C_{jk}^{\ m} \Lambda^{ns}_{ml} = \Lambda^{ij}_{kl} \Lambda^{nm}_{ri} C_{mj}^{\ s}$$ (27)

where the index pairs $A^B$ and $A^B$ have been replaced by the indices $i$ and $i$ respectively. These are the so-called “bicovariance conditions”, see ref.s [5, 6, 9], necessary in order to have a consistent bicovariant differential calculus.

By using (4) in (19) and (22), one finds that $\Lambda^{A_2 D_2 b_1 c_1}_{A_1 D_1 b_2 c_2} = 0$ unless $A_1 = a_1, A_2 = a_2, D_1 = d_1, D_2 = d_2,$ and $C^{c_1 b_1}_{c_2 b_2 D_1} = 0$ unless $D_1 = d_1, D_2 = d_2$. As a consequence $\Lambda^{a_2 D_2 b_1 c_1}_{a_1 D_1 b_2 c_2}$ and $C^{c_1 b_1}_{c_2 b_2 D_1}$ satisfy by themselves the bicovariance conditions (24)–(27). This explains why the projection from $GL_q(n+1)$ to $GL_q(n)$:

$$T^0_a = T^a_0 = 0, T^0_0 = I,$$ (28)

$$\omega^a_0 = \omega^0_a = \omega^0_0 = 0,$$ (29)

$$\chi^a_0 = \chi_0^a = \chi_0^0 = 0$$ (30)

leads to a consistent bicovariant calculus for $GL_q(n)$.

So far we have seen how to obtain $GL_q(n)$, together with its $*$-Hopf algebra structure and bicovariant differential calculus from the “mother” structures of $GL_q(n+1)$. This is not so exciting, but suggests a way to obtain inhomogeneous quantum groups via another kind of projection.

Indeed, consider

$$T^0_a = 0, T^a_0 = a^a, T^0_0 = u$$ (31)

(note that $T^0_0$ is not set to the identity any more), together with

$$\omega^a_0 = \omega^0_a = 0$$ (32)

$$\chi^a_0 = \chi_0^a = 0$$ (33)
The projection \(31\) yields the quantum group \(IGL_q(n)\), generated by \(T^a_b, x^a, u, v\) (the inverse of \(u\), i.e. \(uv = vu = I\)), \(\xi\) (the inverse of \(\det_q T^a_b\)) and the identity \(I\). The commutation relations of these elements can be read off the “RTT” relations \(3\) for \(GL_q(n + 1)\) after using \(1\) and \(31\):

\[
R^{ab}_{\ e \ f} T^e_{\ c} T^f_{\ d} = T^b_{\ f} T^a_{\ c} R^{ef}_{\ cd} \tag{34}
\]

\[
x^a T^b_{\ c} = R^{ba}_{\ e \ f} T^e_{\ c} x^f \tag{35}
\]

\[
A^{ab}_{\ cd} x^c x^d = 0 \tag{36}
\]

\[
T^a_{\ b} u = u T^a_{\ b} \tag{37}
\]

\[
T^a_{\ b} v = v T^a_{\ b} \tag{38}
\]

\[
x^a u = q^{-1} u x^a \tag{39}
\]

\[
x^a v = q v x^a \tag{40}
\]

the \(A\) matrix being the \(q\)-generalization of the antisymmetrizer:

\[
A = \frac{q I - \tilde{R}}{q + q^{-1}} \tag{41}
\]

where \(\tilde{R}^{ab}_{\ cd} \equiv R^{ba}_{\ cd}\).

The “projected” quantum determinant \(\det_q T^A_{\ B} = u \det_q T^a_{\ b}\) and its inverse \(\Xi = v \xi\) are central.

The Hopf algebra co-structures, consistent with the commutation rules, are deduced from those of \(GL_q(n + 1)\) by simply substituting \(31\) into \(7\)-\(14\):

\[
\Delta(T^a_{\ b}) = T^a_{\ c} \otimes T^c_{\ b}, \quad \Delta(I) = I \otimes I, \tag{42}
\]

\[
\Delta(x^a) = T^a_{\ b} \otimes x^b + x^a \otimes u \tag{43}
\]

\[
\Delta(u) = u \otimes u, \quad \Delta(v) = v \otimes v \tag{44}
\]

\[
\Delta(\det_q T) = det_q T \otimes det_q T, \quad \Delta(\xi) = \xi \otimes \xi \tag{45}
\]

\[
\varepsilon(T^a_{\ b}) = \delta^a_b, \quad \varepsilon(I) = 1, \tag{46}
\]

\[
\varepsilon(x^a) = 0 \tag{47}
\]

\[
\varepsilon(u) = \varepsilon(v) = 1 \tag{48}
\]

\[
\varepsilon(\det_q T) = \varepsilon(\xi) = 1 \tag{49}
\]

\[
\kappa(T^a_{\ b}) = (T^{-1})^a_{\ b}, \quad \kappa(I) = I, \tag{50}
\]

\[
\kappa(x^a) = -\kappa(T^a_{\ b}) x^b v \tag{51}
\]

\[
\kappa(u) = v, \quad \kappa(v) = u \tag{52}
\]

\[
\kappa(\det_q T) = \xi, \quad \kappa(\xi) = \det_q T \tag{53}
\]
After using (32) and (33) do we obtain a consistent bicovariant differential calculus for the quantum group $GL_q(n)$? The answer is yes. Indeed consider the $q$-Lie algebra \([21]\) of $GL_q(n+1)$. Using the decomposition \([1]\) for $R^{AB}_{CD}$ we find

\[
\chi^{a_1}_{c_2} \delta^{b_1}_{c_2} \chi^{b_1}_{d_2} - \Lambda^{a_2}_{d_2} \delta^{b_1}_{c_2} \chi^{a_1}_{d_2} = \frac{1}{q-q^{-1}} \left[ -\delta^{b_1}_{d_2} \delta^{c_1}_{d_2} + \Lambda^{a_2}_{d_1} \delta^{b_1}_{c_2} \right] \chi^{d_1}_{d_2} \quad (54)
\]

\[
\chi^{a_1}_{c_2} \delta^{b_1}_{c_2} \chi^{b_1}_{d_2} - \left( R^{1}\right)^{a_2}_{c_1} \epsilon_{c_1}^{b_1} (R^{1})^{a_2}_{d_1} \chi^{d_1}_{d_2} = \frac{1}{q-q^{-1}} \left[ -\delta^{b_1}_{d_2} \delta^{c_1}_{d_2} + \left( R^{1}\right)^{a_1}_{c_2} (R^{1})^{a_2}_{d_1} \right] \chi^{d_1}_{d_2} \quad (55)
\]

\[
\chi^{a_1}_{c_2} \chi^{b_1}_{d_0} - (q-q^{-1}) d^{g_2}_{c_2} d^{i_1}_{c_2} (R^{1})^{c_1}_{g_2i_1} (R^{1})^{a_1}_{i_1} (R^{1})^{a_2}_{d_1} \chi^{a_1}_{d_2} = \quad (56)
\]

\[
\chi^{a_1}_{c_2} \chi^{b_1}_{d_0} - q (R^{1})^{a_1}_{c_2} \chi^{a_1}_{d_0} = 0 \quad (57)
\]

where $\Lambda^{a_2}_{d_2} \delta^{b_1}_{c_2}$ is the braiding matrix of $GL_q(n)$, given in \([58]\), so that the commutations in \([54]\) are those of the $q$-subalgebra $GL_q(n)$. Note that the $q \to 1$ limit on the right hand sides of \([54]\) and \([55]\) is finite, since the terms in square parentheses are a (finite) series in $q-q^{-1}$, and the $0-th$ order part vanishes (see \([9]\), eq. (5.55)). We have written here only a subset $X$ of the commutation relations \([21]\). This subset involves only the $\chi^{a_1}_{b_1}$ and $\chi^{a_1}_{d_0}$ generators, and closes on itself. The $\Lambda$ and $C$ components entering \([54]-[57]\) are

\[
\Lambda^{a_2}_{d_2} \delta^{b_1}_{c_2} = d^{f_2}_{c_2} d^{i_1}_{c_2} (R^{1})^{c_1}_{f_2i_1} (R^{1})^{a_1}_{i_1} (R^{1})^{a_2}_{d_1} R^{g_2d_1}_{d_2} \quad (58)
\]

\[
\Lambda^{a_0}_{d_2} \delta^{b_1}_{c_2} = d^{f_2}_{c_2} d^{i_1}_{c_2} (R^{1})^{c_1}_{f_2i_1} (R^{1})^{a_1}_{i_1} (R^{1})^{a_2}_{d_1} \quad (59)
\]

\[
\Lambda^{a_1}_{d_2} \delta^{b_1}_{c_2} = (R^{1})^{c_1}_{d_1} (R^{1})^{a_1}_{b_2} \quad (60)
\]

\[
\Lambda^{a_2}_{d_2} \delta^{b_1}_{c_2} = (q-q^{-1}) d^{f_2}_{c_2} d^{i_1}_{c_2} (R^{1})^{c_1}_{f_2i_1} (R^{1})^{a_1}_{i_1} (R^{1})^{a_2}_{d_1} \quad (61)
\]

\[
\Lambda^{a_2}_{d_2} \delta^{b_1}_{c_2} = q (R^{1})^{a_2}_{d_1} \quad (62)
\]

$C^{a_1}_{c_2} \delta^{b_1}_{c_2} \delta^{d_2}_{c_2}$ is the structure constants of $GL_q(n)$. This result was already found in \([1]\) without using the projection discussed in the present Letter. We repeat now the same reasoning as in \([1]\): the components given in \([58]-[59]\) are the only non-vanishing $\Lambda^{a_2}_{d_2} \delta^{b_1}_{c_2}$ components and the only non-vanishing $C^{a_1}_{c_2} \delta^{b_1}_{c_2} \delta^{d_2}_{c_2}$ components with indices $C_1, C_2, B_1, B_2$ corresponding to the subset $X$. Because of this, they satisfy by themselves the bicovariance conditions, as the sums in \([24]-[27]\) do not involve other components. Then \([54]-[57]\) defines a bicovariant quantum Lie algebra, and a consistent differential calculus can be set up, based on a $\Lambda$ tensor whose only nonvanishing components are \([58]-[59]\).
Finally, we consider the complexification \( IGL_q(n, \mathbb{C}) \). This we obtain as the projection of the complex \( q \)-group \( GL_q(n+1, \mathbb{C}) \). Let us recall how to construct \( GL_q(n+1, \mathbb{C}) \) from \( GL_q(n+1) \) [22]. Using the \( \ast \)-structure on \( GL_q(n+1) \), one introduces the conjugated elements

\[
\hat{T}^A_{\bar{B}} \equiv [\kappa(T^B_A)]^* \quad (66)
\]

The complex conjugate version of (5) yields the \( R\hat{T}\hat{T} \) relation:

\[
R^{AB}_{\ EF} \hat{T}^E_C \hat{T}^F_D = \hat{T}^B_F \hat{T}^A_E R^{EF}_{\ CD} \quad (67)
\]

whereas the commutations between \( T^A_B \) and \( \hat{T}^\alpha_{\bar{B}} \) can be defined to be

\[
R^{AB}_{\ EF} \hat{T}^E_C T^F_D = T^B_F \hat{T}^A_E R^{EF}_{\ CD} \quad (68)
\]

An \( RTT \)-formulation for the complexified quantum group \( GL_q(n+1, \mathbb{C}) \) can be found by defining the matrix \( T^J_K \):

\[
T^J_K = \begin{pmatrix} T^A_B & 0 \\ 0 & T^A_{\bar{B}} \end{pmatrix} \quad (69)
\]

where

\[
T^A_{\bar{B}} \equiv \hat{T}^A_B \quad (70)
\]

with the index convention \( J = (A, \bar{A}) \). Then the \( RTT \) relation

\[
\mathcal{R}^{IJ}_{\ MN} T^M_K T^N_L = T^J_N T^I_M \mathcal{R}^{MN}_{\ KL} \quad (71)
\]

with

\[
\mathcal{R}^{IJ}_{\ KL} = \begin{pmatrix} R & 0 & 0 & 0 \\ 0 & (R^+)^{-1} & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{pmatrix} \quad (72)
\]

(indices \( IJ \) ordered as \( AB, \bar{A}B, \bar{A}\bar{B}, \bar{A}\bar{B} \), and \( (R^+)^{AB}_{\ CD} \equiv R^{BA}_{\ DC} \)) reproduces the commutations (5), (67) and (68). A bicovariant calculus on \( GL_q(n+1, \mathbb{C}) \) can be set up in terms of the \( \mathcal{R} \) matrix, via the standard formula given before for the braiding matrix \( \Lambda \).

We define now the projection of \( GL_q(n+1, \mathbb{C}) \) onto the complexified inhomogeneous quantum group \( IGL_q(n, \mathbb{C}) \) by taking the complexified version of (31):

\[
T^0_a = 0, \ T^a_0 \equiv x^a, \ T^0_0 \equiv u, \ T^\alpha_0 = 0, \ T^\alpha_\bar{a} \equiv x^{\bar{a}}, \ T^\bar{0}_0 \equiv \bar{u} \quad (73)
\]

Then \( IGL_q(n+1, \mathbb{C}) \) is defined as the algebra \( \mathcal{A} \) freely generated by the elements \( T^a_b, \ T^a_{\bar{b}}, \ x^a, \ x^{\bar{a}}, \ u \) and its inverse \( v, \bar{u} \) and inverse \( v, \bar{v} \), and the inverses \( \xi, \bar{\xi} \) of the \( q \)-determinants \( \det_q T^a_b, \ det_q T^a_{\bar{b}} \). The “projected” \( q \)-determinants \( u\det_q T^a_b, \ uv \) and their inverses \( v, \bar{v} \) are central. The commutations of these elements
are deduced from (3), (67) and (68) after use of (73); they are given therefore by (44)-(50) for $TT$ and $TT$ commutations, whereas the $\bar{T}T$ commutations are:

\[
R^{ab}_{\, ef} T^{\bar{e}}_{\, c} T^{f}_{\, d} = T^{b}_{\, f} T^{\bar{a}}_{\, \bar{e}} R^{ef}_{\, cd}
\]

(74)

\[
R^{ab}_{\, ef} x^{\bar{e}}_{\, c} x^{f}_{\, d} = x^{b}_{\, f} T^{\bar{a}}_{\, \bar{e}}
\]

(75)

\[
T^{\bar{a}}_{\, \bar{e}} u = u T^{\bar{a}}_{\, \bar{e}}, \quad T^{\bar{a}}_{\, \bar{e}} v = v T^{\bar{a}}_{\, \bar{e}}
\]

(76)

\[
R^{ab}_{\, ef} x^{\bar{e}}_{\, c} x^{f}_{\, d} = (q - q^{-1}) x^{b}_{\, f} T^{\bar{a}}_{\, \bar{e}} + T^{b}_{\, d} x^{\bar{a}}
\]

(77)

\[
R^{ab}_{\, ef} x^{\bar{e}}_{\, c} x^{f}_{\, d} = q x^{b}_{\, f} x^{\bar{a}}
\]

(78)

\[
x^{\bar{a}} u + (q - q^{-1}) \bar{u} x^{a} = q u x^{\bar{a}}
\]

(79)

\[
v x^{\bar{a}} + (q - q^{-1}) v \bar{u} x^{a} v = q x^{\bar{a}} v
\]

(80)

\[
\bar{u} T^{a}_{\, b} = T^{\bar{a}}_{\, \bar{b}} \bar{u}, \quad \bar{v} T^{a}_{\, b} = T^{\bar{a}}_{\, \bar{b}} \bar{v}
\]

(81)

\[
\bar{u} x^{a} = q x^{\bar{a}} \bar{u}, \quad \bar{v} x^{a} = q^{-1} x^{\bar{a}} \bar{v}
\]

(82)

\[
\bar{u} u = u \bar{u}, \quad \bar{v} v = v \bar{v}, \quad \bar{v} u = u \bar{v}, \quad \bar{v} v = v \bar{v}
\]

(83)

The co-structures of the conjugated elements $T^{A}_{\, B\bar{}}$ are given by the same formulas (with barred indices) as in (39)-(42).

The bicovariant differential calculus on $GL_{q}(n + 1, \C)$ is found by the usual procedure, described in formulas (19)-(22), after replacing $A, B...$ indices by $I, J...$ indices. Here again we find a subset $X$ of the $q$-Lie algebra of $GL_{q}(n + 1, \C)$ that closes on itself, and allows therefore a consistent projection onto a bicovariant differential calculus for $IGL_{q}(n, \C)$. This subset is given by (54)-(57) and:

\[
\chi^{\bar{c}}_{\bar{b}} x_{a_{2}} = \Lambda_{a_{2}}^{\bar{a}} d_{1} | c_{2} b_{2} \chi^{\bar{a}}_{a_{2}} x_{a_{2}} = \frac{1}{q - q^{-1}} \left( -\delta^{\bar{b}_{1}}_{b_{1}} c_{1} d_{2}^{1} + \Lambda_{a_{2}}^{\bar{a}} c_{1} b_{1} \right) \chi^{\bar{a}}_{a_{2}} x_{a_{2}} = 0 (84)
\]

\[
\chi^{\bar{c}_{2}}_{\bar{b}} x_{a_{2}} = d^{2}_{2} d^{2}_{2} R^{b_{2}}_{c_{2} a_{2}} R^{a_{2}}_{b_{2} f_{2}} \chi^{\bar{a}_{2}}_{a_{2}} x_{a_{2}} = \frac{1}{q - q^{-1}} \left( -\delta^{\bar{b}_{2}}_{b_{2}} c_{2} d_{2}^{2} + \delta^{\bar{b}_{2}}_{b_{2}} c_{2} d_{2}^{2} \right) \chi^{\bar{a}_{2}}_{a_{2}} x_{a_{2}} = 0 (85)
\]

\[
\chi^{\bar{c}_{2}}_{\bar{b}} x_{a_{2}} = (q - q^{-1}) \delta^{\bar{a}_{2}}_{a_{2}} R^{a_{2}}_{b_{2} c_{2}} \chi^{\bar{a}_{2}}_{a_{2}} x_{a_{2}} = \frac{1}{\lambda_{a_{2}} x_{a_{2}}} = 0 (86)
\]

\[
\chi^{\bar{c}_{2}}_{\bar{b}} x_{a_{2}} = q^{-1} R^{a_{2}}_{b_{2} c_{2}} \chi^{\bar{a}_{2}}_{a_{2}} x_{a_{2}} = 0 (87)
\]

\[
\chi^{\bar{b}_{1}}_{\bar{b}_{2}} = (q - q^{-1}) d^{0}_{2} d^{0}_{2} c_{1} c_{2} (R^{1})^{a_{2}}_{a_{2} b_{1} d_{1}} \chi^{\bar{a}_{2}}_{a_{2}} x_{a_{2}} = \frac{1}{q - q^{-1}} \left( -\delta^{\bar{b}_{1}}_{b_{1}} c_{1} d_{1}^{1} + \Lambda_{a_{1}}^{\bar{a}_{1}} c_{1} b_{1} \right) \chi^{\bar{a}_{2}}_{a_{2}} x_{a_{2}} = 0 (88)
\]

\[
\chi^{\bar{b}_{1}}_{\bar{b}_{2}} = (q - q^{-1}) d^{0}_{2} d^{0}_{2} c_{1} c_{2} (R^{1})^{a_{2}}_{a_{2} b_{1} d_{1}} \chi^{\bar{a}_{2}}_{a_{2}} x_{a_{2}} = \frac{1}{q - q^{-1}} \left( -\delta^{\bar{b}_{1}}_{b_{1}} c_{1} d_{1}^{1} + \Lambda_{a_{1}}^{\bar{a}_{1}} c_{1} b_{1} \right) \chi^{\bar{a}_{2}}_{a_{2}} x_{a_{2}} = 0 (88)
\]

\[
\chi^{\bar{b}_{1}}_{\bar{b}_{2}} = (q - q^{-1}) d^{0}_{2} d^{0}_{2} c_{1} c_{2} (R^{1})^{a_{2}}_{a_{2} b_{1} d_{1}} \chi^{\bar{a}_{2}}_{a_{2}} x_{a_{2}} = \frac{1}{q - q^{-1}} \left( -\delta^{\bar{b}_{1}}_{b_{1}} c_{1} d_{1}^{1} + \Lambda_{a_{1}}^{\bar{a}_{1}} c_{1} b_{1} \right) \chi^{\bar{a}_{2}}_{a_{2}} x_{a_{2}} = 0 (88)
\]

\[
\chi^{\bar{b}_{1}}_{\bar{b}_{2}} = (q - q^{-1}) d^{0}_{2} d^{0}_{2} c_{1} c_{2} (R^{1})^{a_{2}}_{a_{2} b_{1} d_{1}} \chi^{\bar{a}_{2}}_{a_{2}} x_{a_{2}} = \frac{1}{q - q^{-1}} \left( -\delta^{\bar{b}_{1}}_{b_{1}} c_{1} d_{1}^{1} + \Lambda_{a_{1}}^{\bar{a}_{1}} c_{1} b_{1} \right) \chi^{\bar{a}_{2}}_{a_{2}} x_{a_{2}} = 0 (88)
\]
\( \chi_{c_2}^0 x_{b_2} - d^2 d_{c_2}^{-1} R f_{c_2 a_1} (R^{-1})^{a_2 e_1} f_{b_2} \chi_{a_2}^0 x_{d_2} = 0 \) \hfill (91)

\( \chi_{b_2}^c x_{b_2} - R^{b_1 c_{1 e_1}} (R^{-1})^{a_{2 e_1}} b_{2 d_1} \chi_{a_2}^0 x_{d_2} = 0 \) \hfill (92)

\( \chi_{c_2}^0 x_{b_2} - d^2 d_{c_2}^{-1} R f_{c_2 a_1} (R^{-1})^{a_2 e_1} b_{2 a_1} \chi_{a_2}^0 x_{d_2} = 0 \) \hfill (93)

\( \chi_{b_2}^c x_{b_2} - q^{-1} (R^{-1})^{a_{2 c_1}} b_{2 d_1} \chi_{a_2}^0 x_{d_2} = 0 \) \hfill (94)

\( \chi_{c_2}^0 x_{b_2} - q d^2 d_{c_2}^{-1} R f_{c_2 a_1} (R^{-1})^{a_{2 e_1}} b_{2 a_1} \chi_{a_2}^0 x_{d_2} = 0 \) \hfill (95)

The commutations in (54)-(57) and (58)-(61) are those of the two \( q \)-commuting subalgebras \( GL_q(n) \). Again we call \( X \) this subset of the \( q \)-Lie algebra commutation relations of \( GL_q(n+1,C) \). It closes on the generators \( \chi^a_b, \chi^a_0, \chi^b_0, \chi^0_b \). The \( \Lambda \) and \( C \) components entering the subset \( X \) are given by (58)-(63) and by

\( \Lambda_{a_2 d_2}^{c_1 b_1} e_{c_1 b_1} = \Lambda_{a_2 d_2}^{c_1 b_1} e_{c_2 b_1} = \Lambda_{a_1 d_1}^{c_1 b_1} e_{c_2 b_2} = \Lambda_{a_2 d_2}^{c_1 b_0} e_{c_2 b_1} = (R^{-1})^{a_{2 e_1}} b_{1 a_1} \chi_{a_2 e_1} (R^{-1})^{a_2 e_1} b_{2 d_1} \) \hfill (96)

\( \Lambda_{a_2 d_2}^{c_1 b_0} e_{c_2 b_1} = (R^{-1})^{a_{2 e_1}} b_{1 a_1} \chi_{a_2 e_1} (R^{-1})^{a_2 e_1} b_{2 d_1} \) \hfill (97)

\( \Lambda_{a_1 d_1}^{c_1 b_0} e_{c_2 b_1} = (R^{-1})^{a_{2 e_1}} b_{1 a_1} \chi_{a_2 e_1} (R^{-1})^{a_2 e_1} b_{2 d_1} \) \hfill (98)

\( \Lambda_{a_2 d_2}^{c_1 b_0} e_{c_2 b_1} = (R^{-1})^{a_{2 e_1}} b_{1 a_1} \chi_{a_2 e_1} (R^{-1})^{a_2 e_1} b_{2 d_1} \) \hfill (99)

\( \Lambda_{a_2 d_2}^{c_1 b_0} e_{c_2 b_1} = (R^{-1})^{a_{2 e_1}} b_{1 a_1} \chi_{a_2 e_1} (R^{-1})^{a_2 e_1} b_{2 d_1} \) \hfill (100)

Again we find that these are the only non-vanishing \( \Lambda_{A_2 D_2}^{C_1 B_1} \) components and the only non-vanishing \( C_{C_2 B_2}^{D_2} \) components with indices \( C_1, C_2, B_1, B_2 \) corresponding to the subset \( X \). By the same reasoning used in the case of \( IGL_q(n) \), we
conclude that \( (34)-(37), (68)-(72) \) define a bicovariant quantum Lie algebra, and a consistent differential calculus can be set up, based on a \( \Lambda \) tensor whose only non-vanishing components are \( (58)-(62), (96)-(110) \). This differential calculus is obtained from the one of \( GL_q(n+1, C) \) by setting:

\[
\begin{align*}
\omega_0^a &= \omega_0^a = \omega_0^0 = 0 \\
\chi^0_a &= \chi^0_a = \chi^0_0 = 0
\end{align*}
\] (114)

In the Table we present the \( q \)-Lie algebra for \( IGL_q(2, C) \), i.e. the quantum Poincaré group with the addition of two dilatations, generated by \( \chi^1_1 + \chi^2_2 \). The generators \( \chi^a_0 \) and \( \chi_0^a \) close on the \( q \)-Lie algebra of \( GL_q(2, C) \), while \( \chi_0^0 \) and \( \chi^0_0 \) are the four \( q \)-momentum generators. To obtain \( ISL_q(2, C) \) we must require \( \text{udet} q T^a_b - \bar{\text{udet}} q T^\bar{a}_b = I \). This implies the relation

\[
\chi^1_1 + \chi^2_2 - (q - q^{-1})\chi^1_1 \chi^2_2 + q^2(q - q^{-1})\chi^1_1 \chi^2_1 = 0
\] (116)

(cf. ref. [8]) and a similar one for barred generators, which reduce the number of independent generators from 12 to 10.

The co-structures of \( \chi^I_J \) are given by:

\[
\begin{align*}
\Delta'(\chi^I_J) &= I' \otimes \chi^I_J + \chi^K_L \otimes f_{KL}^{LI} J \\
\varepsilon'(\chi^I_J) &= 0 \\
\kappa'(\chi^I_J) &= -\chi^K_L f_{KL}^{LI} J
\end{align*}
\] (117) (118) (119)

where

\[
f_{KL}^{LI} J \equiv \kappa'((L^+)^I_K)(L^-)^L_J
\] (120)

and the functionals \( (L^\pm)^I_J \) are defined below. A detailed account of the bicovariant differential calculus on the quantum Poincaré group is given in ref. [16].

**Note 1:** we have chosen \( R^{\bar{A}\bar{B}}_{\bar{C}\bar{D}} = R^{AB}_{CD} \) in (72). In fact, another choice is possible, i.e. \( R^{AB}_{CD} = [(R^+)^{-1}]^{AB}_{CD} \), since it reproduces the same commutations (67). This last choice is favoured in ref. [12]. However a consistent projection on \( IGL_q(n, C) \) does not seem to exist in this case. Note that our choice (72) is still consistent with a \( \ast \)-structure on the space of regular functionals. Indeed a conjugation on the functionals \( (L^\pm)^A_B \) can be defined as:

\[
[(L^\pm)^A_B]^\ast(a) \equiv [(L^\pm)^B_A(a^\ast)]
\] (121)

where \( a \in GL_q(n+1, C) \) and the bar indicates the usual conjugation on \( C \). We recall that these functionals are defined by their action on the group elements:

\[
(L^\pm)^I_J(T^K_L) = (R^\pm)^{IK}_{JL}
\] (122)
Then

\[ (L^\pm)^A_B = [(L^\pm)^A_B]^\dagger \]  \hspace{1cm} (123)

**Note 2:** the right-hand sides of eqs (88) and (89) vanish because of the identities:

\[ d^f_2 d^{-1}(R^{-1})^{b_1 f_2}_{c_2} R^{g d_2}_{b_2 f_2} = \delta^{d_2}_{c_2} \delta^{b_1}_{b_2} \]  \hspace{1cm} (124)

\[ d^f_2 d^{-1}(R^{-1})^{f_2 b_1}_{c_2 g} R^{d_2 g}_{f_2 b_2} = \delta^{d_2}_{c_2} \delta^{b_1}_{b_2} \]  \hspace{1cm} (125)

valid for any \( GL_q(n) \) \( R \)-matrix.

**Note 3:** the quantum Lorentz group \( SL_q(2, \mathbb{C}) \) is obviously contained in the \( q \)-Poincaré group \( ISL_q(2, \mathbb{C}) \). This inclusion holds also for the corresponding \( q \)-Lie algebras, since \( \chi^a\bar{b} \) and \( \chi^a\bar{b} \) close on the quantum Lorentz \( q \)-Lie algebra, cf. the Table. This fact is of relevance for the construction of a \( q \)-Minkowski spacetime as the quantum coset space \( q \)-Poincaré / \( q \)-Lorentz.
Table
The bicovariant $q$-Lie algebra of the quantum Poincaré group

\[
\begin{align*}
\chi_1^1\chi_2^1 - \chi_1^2\chi_1^1 + (1 - q^2)\chi_1^2\chi_2^2 &= q\chi_2^1, \\
\chi_1^2\chi_2^1 - \chi_1^1\chi_1^2 - (1 - q^2)\chi_2^2\chi_1^1 &= -q\chi_1^2, \\
\chi_1^1\chi_2^2 - \chi_2^1\chi_1^1 &= 0, \\
\chi_1^1\chi_2^2 - \chi_1^1\chi_2^1 + (1 - q^2)\chi_2^2\chi_1^1 - (1 - q^2)\chi_2^2\chi_2^1 &= q(\chi_1^1 - \chi_2^2), \\
\chi_1^2\chi_2^2 - q^2\chi_2^2\chi_2^1 &= q\chi_1^1, \\
\chi_1^2\chi_2^2 - q^{-2}\chi_2^2\chi_1^1 &= -q^{-1}\chi_2^1, \\
\text{and same with barred indices}
\end{align*}
\]

\[
\begin{align*}
\chi_0^1\chi_1^1 - q^{-2}\chi_1^1\chi_0^1 - (q^{-2} - 1)\chi_1^2\chi_0^2 &= -q^{-1}\chi_0^1, \\
\chi_0^1\chi_1^2 - q^{-1}\chi_2^1\chi_0^1 &= 0, \\
\chi_0^1\chi_2^1 - q^{-1}\chi_1^2\chi_0^1 - (q^{-1} - q)\chi_2^2\chi_0^2 &= -\chi_0^1, \\
\chi_0^1\chi_2^2 - \chi_2^1\chi_0^1 &= 0, \\
\chi_0^1\chi_1^2 - q^{-2}\chi_1^2\chi_0^1 - (q^{-2} - 1)\chi_1^1\chi_0^1 - (-2 + q^{-2} + q^2)\chi_2^2\chi_0^2 &= (q - q^{-1})\chi_0^1, \\
\chi_0^1\chi_2^2 - q^{-1}\chi_2^2\chi_0^1 - (q^{-1} - q)\chi_2^1\chi_0^1 &= -\chi_0^1, \\
\chi_0^1\chi_2^2 - q^{-1}\chi_2^1\chi_0^2 &= 0, \\
\chi_0^1\chi_2^2 - q^{-2}\chi_2^2\chi_0^2 &= -q^{-1}\chi_0^2
\end{align*}
\]

\[
\begin{align*}
\chi_0^1\chi_1^1 - q^{-2}\chi_1^1\chi_0^1 - (1 - 2q^2 + q^4)\chi_2^1\chi_0^2 - (-q + q^3)\chi_2^2\chi_0^2 &= q^3\chi_1^0, \\
\chi_0^1\chi_1^2 - q\chi_1^2\chi_0^0 - (q^2 + q^4)\chi_2^2\chi_0^2 &= q^3\chi_1^2, \\
\chi_0^0\chi_1^1 - \chi_1^1\chi_0^2 - (q - q^{-1})\chi_2^1\chi_0^1 &= 0, \\
\chi_0^0\chi_1^2 - q\chi_1^2\chi_0^0 &= 0, \\
\chi_0^1\chi_2^1 - q\chi_2^1\chi_0^1 &= 0, \\
\chi_0^1\chi_2^2 - q\chi_1^2\chi_0^1 &= 0, \\
\chi_0^2\chi_1^1 - q\chi_2^2\chi_1^0 - (q^2 - 1)\chi_2^2\chi_1^0 &= q\chi_1^0, \\
\chi_0^2\chi_1^2 - q^2\chi_2^2\chi_1^0 &= q\chi_1^2
\end{align*}
\]

\[
\begin{align*}
\chi_1^0\chi_0^1 - q\chi_1^0\chi_0^1 &= 0, \\
\chi_0^1\chi_0^2 - q^{-1}\chi_0^2\chi_0^1 &= 0, \\
\chi_1^0\chi_0^2 - q^{-2}\chi_1^0\chi_0^1 + (1 - q^{-2})\chi_0^2\chi_0^2 &= 0, \\
\chi_0^1\chi_0^2 - q^{-1}\chi_2^1\chi_0^2 &= 0, \\
\chi_0^2\chi_0^2 - q^{-1}\chi_1^1\chi_0^2 &= 0, \\
\chi_0^2\chi_0^2 - q^{-2}\chi_2^2\chi_0^2 &= 0
\end{align*}
\]
\[ \chi^1_1\chi^1_2 - \chi^1_1\chi^1_1 + (-q^{-4} + q^{-2})\chi^0_1\chi^0_0 + (-1 - q^{-4} + 2q^{-2})\chi^0_2\chi^0_2 - 
\vspace{-0.5cm}
- (1 - q^{-2})\chi^1_2\chi^2_1 = 0
\]
\[ \chi^1_1\chi^1_2 - \chi^1_2\chi^1_1 - (-q^{-3} + q^{-1})\chi^0_2\chi^0_0 = 0
\]
\[ \chi^1_1\chi^1_2 - \chi^1_1\chi^1_1 - (q^2 - 1)\chi^1_1\chi^1_1 - (1 - q^2)\chi^2_2\chi^2_1 = 0
\]
\[ \chi^1_1\chi^1_2 - \chi^1_2\chi^1_1 - (1 - q^{-2})\chi^1_2\chi^1_1 = 0
\]
\[ \chi^2_1\chi^1_2 - (q^2 - 1)\chi^2_2\chi^2_1 - \chi^2_1\chi^2_1 - (1 - q^{-2})\chi^2_1\chi^2_2 = 0
\]
\[ \chi^1_2\chi^2_1 - q^2\chi^1_2\chi^2_1 = 0
\]
\[ \chi^2_1\chi^1_2 - q^{-2}\chi^2_1\chi^1_2 + (-q^{-6} + q^{-4})\chi^0_1\chi^0_0 - q^{-6}(1 - q^{-2})\chi^0_2\chi^0_2 - 
\vspace{-0.5cm}
- (1 - q^{-2})\chi^1_1\chi^1_1 - (1 - q^{-2})\chi^2_2\chi^2_1 - (q^{-4} - q^{-2} + q^2 - 1)\chi^1_2\chi^2_1 - 
\vspace{-0.5cm}
- (1 - q^{-2})\chi^1_2\chi^2_2 = 0
\]
\[ \chi^1_2\chi^2_1 - q^2\chi^1_2\chi^2_1 = 0
\]
\[ \chi^1_2\chi^1_2 - \chi^1_1\chi^1_1 + (-q^{-3} + q^{-1})\chi^0_1\chi^0_0 = 0
\]
\[ \chi^1_2\chi^2_1 - q^{-2}\chi^1_2\chi^2_1 + (-q^{-4} + q^{-2})\chi^2_2\chi^2_1 = 0
\]
\[ \chi^1_2\chi^1_2 - q^2\chi^1_2\chi^1_1 = 0
\]
\[ \chi^1_2\chi^1_2 - \chi^1_2\chi^1_2 = 0
\]
\[ \chi^1_2\chi^1_2 - \chi^1_1\chi^1_2 - (1 - q^{-2})\chi^2_2\chi^1_1 = 0
\]
\[ \chi^2_2\chi^1_2 - \chi^2_2\chi^2_2 = 0
\]
\[ \chi^2_2\chi^1_2 - \chi^1_2\chi^2_2 + (-q^{-5} + q^{-3})\chi^0_2\chi^0_0 - (-1 + q^{-2})\chi^1_2\chi^1_1 - 
\vspace{-0.5cm}
- (1 - q^{-2})\chi^2_2\chi^2_2 = 0
\]
\[ \chi^1_2\chi^0_1 - \chi^0_1\chi^1_1 - (1 - q^2)\chi^0_2\chi^1_1 = 0
\]
\[ \chi^0_1\chi^0_1 - \chi^0_1\chi^0_1 = 0
\]
\[ \chi^1_2\chi^0_1 - q^{-1}\chi^0_2\chi^1_2 + (q - q^{-1})\chi^0_2\chi^1_1 - (q^{-1} - q)\chi^0_2\chi^2_2 = 0
\]
\[ \chi^1_2\chi^2_2 - q^2\chi^0_2\chi^1_1 = 0
\]
\[ \chi^1_2\chi^0_1 - q\chi^0_1\chi^1_1 = 0
\]
\[ \chi^2_2\chi^0_2 - q^{-1}\chi^0_2\chi^2_2 = 0
\]
\[ \chi^0_2\chi^0_1 - \chi^0_1\chi^2_2 - (1 - q^{-2})\chi^0_2\chi^1_1 = 0
\]
\[ \chi^2_2\chi^0_2 - \chi^0_2\chi^2_2 = 0
\]
\[ \chi^1_0\chi^1_1 - \chi^1_1\chi^0_1 - (1 - q^2)\chi^1_2\chi^0_2 = 0
\]
\[ \chi^0_1\chi^0_1 - q\chi^0_2\chi^1_1 = 0
\]
\[ \chi^0_2\chi^0_1 - q^{-1}\chi^0_2\chi^1_1 - (q - q^{-1})\chi^0_2\chi^1_1 - (q^{-1} - q)\chi^0_2\chi^2_2 = 0
\]
\[ \chi^0_2\chi^2_2 - \chi^0_2\chi^1_1 = 0
\]
\[ \chi^2_0\chi^1_1 - \chi^1_1\chi^0_1 = 0
\]
\[ \chi^1_0\chi^0_1 - q^{-1}\chi^1_1\chi^0_1 - (1 - q^{-2})\chi^1_1\chi^0_2 = 0
\]
\[ \chi^0_0\chi^1_1 - \chi^1_1\chi^0_1 = 0
\]
\[ \chi^0_1\chi^0_1 - q\chi^0_2\chi^1_1 = 0
\]
\[ \chi^0_2\chi^0_1 - \chi^0_2\chi^0_1 = 0
\]
\[ \chi^2_0\chi^1_1 - q\chi^1_2\chi^0_1 = 0
\]
\[ \chi^0_2\chi^1_1 - q^{-1}\chi^1_2\chi^0_1 = 0
\]
\[ \chi^0_2\chi^2_2 - \chi^2_2\chi^2_2 = 0
\]
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