Some approximation properties of Lupaş $q$-analogue of Bernstein operators

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Abstract

In this paper, we discuss rates of convergence for the Lupaş $q$-analogue of Bernstein polynomials $R_{n,q}$. We prove a quantitative variant of Voronovskaja's theorem for $R_{n,q}$.

Key words: $q$-Bernstein polynomials; Lupaş $q$-analogue; Voronovskaja-type formulas

1 Introduction

Let $q > 0$. For any $n \in N \cup \{0\}$, the $q$-integer $[n] = [n]_q$ is defined by

$$[n] := 1 + q + ... + q^{n-1}, \quad [0] := 0;$$

and the $q$-factorial $[n]! = [n]_q!$ by

$$[n]! := [1] [2] ... [n], \quad [0]! := 1.$$

For integers $0 \leq k \leq n$, the $q$-binomial is defined by

$$\left[ \begin{array}{c} n \\ k \end{array} \right] := \frac{[n]!}{[k]! [n-k]!}.$$

In the last two decades interesting generalizations of Bernstein polynomials were proposed by Lupaş [6]

$$R_{n,q}(f, x) = \sum_{k=0}^{n} f \left( \frac{[k]}{[n]} \right) \left[ \begin{array}{c} n \\ k \end{array} \right] \frac{q^{k(k-1)/2} x^k (1-x)^{n-k}}{(1-x+qx) ... (1-x+q^{n-1}x)}$$

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and by Phillips \[10\]

\[
B_{n,q}(f, x) = \sum_{k=0}^{n} f_{\left(\frac{k}{n}\right)} \binom{n}{k} x^k \prod_{s=0}^{n-k-1} (1 - q^s x).
\]

The Phillips \(q\)-analogue of the Bernstein polynomials \((B_{n,q})\) attracted a lot of interest and was studied widely by a number of authors. A survey of the obtained results and references on the subject can be found in \[8\]. The Lup\u{a}s operators \((R_{n,q})\) are less known. However, they have an advantage of generating positive linear operators for all \(q > 0\), whereas Phillips polynomials generate positive linear operators only if \(q \in (0, 1)\). Lup\u{a}s \[6\] investigated approximating properties of the operators \(R_{n,q}(f, x)\) with respect to the uniform norm of \(C[0,1]\). In particular, he obtained some sufficient conditions for a sequence \(\{R_{n,q}(f, x)\}\) to be approximating for any function \(f \in C[0,1]\) and estimated the rate of convergence in terms of the modulus of continuity. He also investigated behavior of the operators \(R_{n,q}(f, x)\) for convex functions. In \[9\] several results on convergence properties of the sequence \(\{R_{n,q}(f, x)\}\) is presented. In particular, it is proved that the sequence \(\{R_{n,q}(f, x)\}\) converges uniformly to \(f(x)\) on \([0,1]\) if and only if \(q_n \to 1\). On the other hand, for any \(q > 0\) fixed, \(q \neq 1\), the sequence \(\{R_{n,q}(f, x)\}\) converges uniformly to \(f(x)\) if and only if \(f(x) = ax + b\) for some \(a, b \in \mathbb{R}\).

In the paper, we investigate the rate of convergence for the sequence \(\{R_{n,q}(f, x)\}\) by the modulii of continuity. We discuss Voronovskaja-type theorems for Lup\u{a}s operators for arbitrary fixed \(q > 0\). Moreover, for the Voronovskaja’s asymptotic formula we obtain the estimate of the remainder term.

2 Auxiliary results

It will be convenient to use for \(x \in [0,1]\) the following transformations

\[
v = v(q, x) := \frac{qx}{1-x+qx}, \quad v(q^j, v) = \frac{q^j v}{1-v+q^j v}.
\]

Let \(0 < q < 1\). We set

\[
b_{nk}(q; x) := \binom{n}{k} \frac{q^{k(k-1)/2}}{(1-x+qx)/(1-x+q^{n-1}x)} x^k (1-x)^{n-k}, \quad x \in [0,1],
\]

\[
b_{\infty k}(q; x) := \frac{q^{k(k-1)/2}}{(1-q)^k \prod_{j=0}^{\infty} (1 + q^j (x/1-x))} (x/1-x)^k, \quad x \in [0,1].
\]
It was proved in \cite{6} and \cite{9} that for $q \in (0, 1)$ and $x \in [0, 1)$,
\[
\sum_{k=0}^{n} b_{nk}(q; x) = \sum_{k=0}^{\infty} b_{nk}(q; x) = 1.
\]

**Definition 1** Lupaş \cite{6}. The linear operator $R_{n,q} : C[0, 1] \to C[0, 1]$ defined by
\[
R_{n,q}(f, x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) b_{nk}(q; x)
\]
is called the $q$-analogue of the Bernstein operator.

**Definition 2** The linear operator defined on $C[0, 1]$ given by
\[
R_{\infty,q}(f, x) = \begin{cases} \sum_{k=0}^{\infty} f \left( 1 - q^k \right) b_{\infty k}(q; x) & \text{if } x \in [0, 1), \\ f(1) & \text{if } x = 1. \end{cases}
\]
is called the limit $q$-Lupaş operator.

It follows directly from the definition that operators $R_{n,q}(f, x)$ possess the end-point interpolation property, that is,
\[
R_{n,q}(f, 0) = f(0), \quad R_{n,q}(f, 1) = f(1)
\]
for all $q > 0$ and all $n = 1, 2, ...$.

**Lemma 3** We have
\[
b_{nk}(q; x) = \binom{n}{k} \prod_{j=0}^{k-1} v(q^j, x) \prod_{j=0}^{n-k-1} \left( 1 - v(q^{k+j}, x) \right), \quad x \in [0, 1],
b_{\infty k}(q; x) = \frac{1}{(1-q)^k \binom{k}{n} \prod_{j=0}^{k-1} v(q^j, x) \prod_{j=0}^{n} \left( 1 - v(q^{k+j}, x) \right)}, \quad x \in [0, 1].
\]

It was proved in \cite{6} and \cite{9} that $R_{n,q}(f, x)$, $R_{\infty,q}(f, x)$ reproduce linear functions and $R_{n,q}(t^2, x)$ and $R_{\infty,q}(t^2, x)$ were explicitly evaluated. Using Lemma 3 we may write formulas for $R_{n,q}(t^2, x)$ and $R_{\infty,q}(t^2, x)$ in the compact form.

**Lemma 4** We have
\[
R_{n,q}(1, x) = 1, R_{n,q}(t, x) = x, R_{\infty,q}(1, x) = 1, R_{\infty,q}(t, x) = x,
\]
\[
R_{n,q}(t^2, x) = xv(q, x) + \frac{x(1 - v(q, x))}{n},
\]
\[
R_{\infty,q}(t^2, x) = xv(q, x) + (1-q)x(1-v(q, x)) = x - qx(1-v(q, x)).
\]
Now define

\[ L_{n,q}(f, x) := R_{n,q}(f, x) - R_{\infty,q}(f, x). \]

**Lemma 5** The following recurrence formulae hold

\[ R_{n,q}(t^{m+1}, x) = R_{n,q}(t^m, x) - (1 - x) \frac{[n - 1]^m}{[n]^m} R_{n-1,q}(t^m, v), \]

\[ R_{\infty,q}(t^{m+1}, x) = R_{\infty,q}(t^m, x) - (1 - x) R_{\infty,q}(t^m, v), \]

\[ L_{n,q}(t^{m+1}, x) = L_{n,q}(t^m, x) + (1 - x) \left( \left( 1 - \frac{[n - 1]^m}{[n]^m} \right) R_{\infty,q}(t^m, v) - \frac{[n - 1]^m}{[n]^m} L_{n-1,q}(t^m, v) \right). \]

**Proof.** First we prove (2.1). We write explicitly

\[ R_{n,q}(t^{m+1}, x) = \sum_{k=0}^n \frac{[k]^{m+1}}{[n]^{m+1}} \left[ \frac{k}{n} \right] \prod_{j=0}^{k-1} v \left( q^j, x \right) \prod_{j=0}^{n-k-1} \left( 1 - v \left( q^{k+j}, x \right) \right) \]

and rewrite the first two factor in the following form:

\[ \frac{[k]^{m+1}}{[n]^{m+1}} \left[ \frac{k}{n} \right] = \frac{[k]^m}{[n]^m} \left( 1 - q^k \frac{n - k}{n} \right) \left[ \frac{n}{k} \right] = \frac{[k]^m}{[n]^m} \left[ \frac{n}{k} \right] - \frac{[n - 1]^m}{[n]^m} \frac{[k]^m}{[n - 1]^m} \left[ \frac{n - 1}{k} \right] q^k. \]

Finally, if we substitute (2.5) in (2.4) we get (2.1):

\[ R_{n,q}(t^{m+1}, x) = \sum_{k=0}^n \frac{[k]^m}{[n]^m} \left[ \frac{n}{k} \right] \prod_{j=0}^{k-1} v \left( q^j, x \right) \prod_{j=0}^{n-k-1} \left( 1 - v \left( q^{k+j}, x \right) \right) \]

\[ - \frac{[n - 1]^m}{[n]^m} (1 - x) \sum_{k=0}^{n-1} \frac{[k]^m}{[n - 1]^m} \left[ \frac{n - 1}{k} \right] \prod_{j=0}^{k-1} v \left( q^j, v(q, x) \right) \prod_{j=0}^{n-k-2} \left( 1 - v \left( q^{k+j}, v(q, x) \right) \right) \]

\[ = R_{n,q}(t^m, x) - \frac{[n - 1]^m}{[n]^m} (1 - x) R_{n-1,q}(t^m, v(q, x)). \]
Formula (2.2) can be obtained from (2.1), by taking the limit as \( n \to \infty \).

Next we prove (2.3)

\[
L_{n,q}(t^{m+1}, x) = R_{n,q}(t^{m+1}, x) - R_{\infty,q}(t^{m+1}, x)
\]

\[
= R_{n,q}(t^m, x) - (1 - x) \frac{[n-1]_m}{[n]_m} R_{n-1,q}(t^m, v(q, x))
- R_{\infty,q}(t^m, x) + (1 - x) R_{\infty,q}(t^m, v(q, x))
\]

\[
= L_{n,q}(t^m, x) + (1 - x) \times \left( 1 - \frac{[n-1]_m}{[n]_m} \right) R_{\infty,q}(t^m, v(q, x)) - \frac{[n-1]_m}{[n]_m} L_{n-1,q}(t^m, v(q, x)) \right). 
\]

Formula (2.2) can be obtained from (2.1), by taking the limit as \( n \to \infty \). ■

Moments \( R_{n,q}(t^m, x) \), \( R_{\infty,q}(t^m, x) \) are of particular importance in the theory of approximation by positive operators. In what follows we need explicit formulas for moments \( R_{n,q}(t^3, x) \), \( R_{\infty,q}(t^3, x) \).

**Lemma 6** We have

\[
R_{n,q}(t^3, x) = xv(q, x) + x(1-v(q, x)) - \frac{[n-1][n-2]}{[n]^2} q^2 x(1-v(q, x)) v(q^2, x),
\]

\[
R_{\infty,q}(t^3, x) = xv(q, x) + (1-q)^2 x(1-v(q, x)) - q^2 x(1-v(q, x)) v(q^2, x).
\]

**Proof.** Note that explicit formulas for \( R_{n,q}(t^m, x) \), \( R_{\infty,q}(t^m, x) \), \( m = 0, 1, 2 \) were proved in [6], [9]. Now we prove an explicit formula for \( R_{n,q}(t^3, x) \), since formula for \( R_{\infty,q}(t^3, x) \) can be obtained by taking limit as \( n \to \infty \). The proof is based on the recurrence formula (2.1). Indeed,

\[
R_{n,q}(t^3, x) = R_{n,q}(t^2, x) - (1 - x) \frac{[n-1]^2}{[n]^2} R_{n-1,q}(t^2, v)
\]

\[
= xv(q, x) + x(1-v(q, x)) - (1 - x) \frac{[n-1]^2}{[n]^2} v(q, x) v(q^2, x)
- (1 - x) \frac{[n-1]}{[n]^2} v(q, x) + (1 - x) \frac{[n-1]}{[n]^2} v(q, x) v(q^2, x)
\]

\[
= xv(q, x) + \frac{x(1-v(q, x))}{[n]} \left( 1 - \frac{q(n-1)}{[n]} \right)
- \frac{[n-1]}{[n]^2} (q(n-1) - 1) q x(1-v(q, x)) v(q^2, x)
\]

\[
= xv(q, x) + \frac{x(1-v(q, x))}{[n]^2} - \frac{[n-1][n-2]q^2}{[n]^2} x(1-v(q, x)) v(q^2, x).
\]

\[
R_{\infty,q}(t^3, x) = xv(q, x) + (1-q)^2 x(1-v(q, x)) - q^2 x(1-v(q, x)) v(q^2, x).
\]

In order to prove Voronovskaja type theorem for \( R_{n,q}(f, x) \) we also need explicit formulas and inequalities for \( L_{n,q}(t^m, x) \), \( m = 2, 3, 4 \).
Lemma 7 Let $0 < q < 1$. Then

\[
L_{n,q}(t^2, x) = \frac{q^n}{[n]} x (1 - v(q, x)), \quad (2.6)
\]

\[
L_{n,q}(t^3, x) = \frac{q^n}{[n]^2} x (1 - v(q, x))
\times \left[ 2 - q^n + [n - 1] (1 + q) v\left(q^2, x\right) + [n] q v\left(q^2, x\right) \right], \quad (2.7)
\]

\[
L_{n,q}(t^4, x) = \frac{q^n}{[n]^3} x (1 - v(q, x)) M\left(q, v\left(q^2, x\right), v\left(q^3, x\right)\right), \quad (2.8)
\]

where $M$ is a function of $(q, v(q^2, x), v(q^3, x))$.

**Proof.** First we find a formula for $L_{n,q}(t^3, x)$. To do this we use the recurrence formula (2.3):

\[
L_{n,q}(t^3, x)
= L_{n,q}(t^2, x) + (1 - x)
\]

\[
\times \left[ \left(1 - \frac{[n-1]^2}{[n]^2}\right) R_{\infty,q}(t^2, v(q, x)) - \frac{[n-1]^2}{[n]^2} L_{n-1,q}(t^2, v(q, x)) \right]
\]

\[
= \frac{q^n}{[n]} x (1 - v(q, x)) + (1 - x) \left(1 - \frac{[n-1]^2}{[n]^2}\right) \left[ (1 - q) v(q, x) + q v(q, x) v(q^2, x) \right]
\]

\[
- (1 - x) \frac{[n-1]^2}{[n]^2} \frac{q^{n-1}}{[n-1]} v(q, x) (1 - v(q^2, x))
\]

\[
= \frac{q^n}{[n]^2} x (1 - v(q, x))
\]

\[
\times \left[ [n] + \left(\frac{[n]^2 - [n-1]^2}{q^{n-1}}\right) (1 - q + q v(q^2, x)) - [n-1] \left(1 - v(q^2, x)\right) \right]
\]

\[
= \frac{q^n}{[n]^2} x (1 - v(q, x))
\]

\[
\times \left[ [n] + ([n-1] + [n]) (1 - q + q v(q^2, x)) - [n-1] \left(1 - v(q^2, x)\right) \right]
\]

\[
= \frac{q^n}{[n]^2} x (1 - v(q, x))
\]

\[
\times \left[ [n] + 1 - q^{n-1} + 1 - q^n + [n-1] (1 + q) v(q^2, x) + [n] q v(q^2, x) - [n-1] \right]
\]

\[
= \frac{q^n}{[n]^2} x (1 - v(q, x)) \left[ 2 - q^n + [n - 1] (1 + q) v(q^2, x) + [n] q v(q^2, x) \right].
\]

The proof of the equation (2.8) is also elementary, but tedious and complicated. Just notice that we use recurrence formula for $L_{n,q}(t^4, x)$ and clearly each term of the formula contains $\frac{q^n}{[n]^2} x (1 - v(q, x))$. \hfill \blacksquare
Lemma 8 We have

\[ L_{n,q}(t-x^2, x) = \frac{q^n}{[n]} x (1 - v(q,x)), \tag{2.9} \]
\[ L_{n,q}(t-x^3, x) = \frac{q^n}{[n]^2} x (1 - v(q,x)) \tag{2.10} \]
\[ \times \left[ 2 - q^n + [n-1] (1+q) v(q^2,x) + [n]qv(q^2,x) - 3[n]x \right], \]
\[ L_{n,q}(t-x^4, x) \leq K_1 \frac{q^n}{[n]^2} x (1 - v(q,x)), \tag{2.11} \]

where \( K_1 \) is a positive constant.

Proof. Proofs of (2.10) and (2.11) are based on (2.7), (2.8) and on the following identities.

\[ L_{n,q}(t-x^3, x) = L_{n,q}(t^3, x) - 3xL_{n,q}(t-x^2, x), \]
\[ L_{n,q}(t-x^4, x) = L_{n,q}(t^4, x) - 4xL_{n,q}(t-x^3, x) - 6x^2L_{n,q}(t-x^2, x). \]

\[ \square \]

3 Convergence properties

For \( f \in C[0,1], t > 0 \), the modulus of continuity \( \omega(f,t) \) and the second modulus of smoothness \( \omega_2(f,t) \) of \( f \) are defined by

\[ \omega(f,t) = \sup_{|x-y| \leq t} |f(x) - f(y)|, \]
\[ \omega_2(f,t) = \sup_{0 \leq h \leq t} \sup_{0 \leq x \leq 1-2h} |f(x+2h) - 2f(x+h) + f(x)|. \]

In [9], it is proved that \( b_{nk}(q;x) \to b_{\infty k}(q;x) \) uniformly in \( x \in [0,1] \) as \( n \to \infty \). In the next lemma we show that this convergence is uniform on \( (0,q_0] \times [0,1) \) and give some estimates for \( |b_{nk}(q;x) - b_{\infty k}(q;x)| \).

Lemma 9 Let \( 0 < q \leq q_0 < 1, k \geq 0, n \geq 1 \).

(i) For any \( \varepsilon > 0 \) there exists \( M > 0 \) such that

\[ |b_{nk}(q;x) - b_{\infty k}(q;x)| \leq b_{nk}(q;x) M(\varepsilon) \frac{(q_0 + \varepsilon)^n}{1 - (q_0 + \varepsilon)} + b_{\infty k}(q;x) \frac{q_0^{n-k+1}}{1-q_0} \]

for all \( (q,x) \in (0,q_0] \times [0,1) \). In particular, \( b_{nk}(q;x) \) converges to \( b_{\infty k}(q;x) \) uniformly in \( (q,x) \in (0,q_0] \times [0,1) \).
(ii) For any \( x \in [0, 1) \) we have
\[
|b_{nk}(q; x) - b_{\infty k}(q; x)| \leq b_{nk}(q; x) \frac{x}{1 - x} \frac{q^n}{1 - q} + b_{\infty k}(q; x) \frac{q^{n-k+1}}{1 - q}.
\]

In particular, \( b_{nk}(q; x) \) converges to \( b_{\infty k}(q; x) \) uniformly in \( (q, x) \in (0, q_0] \times [0, a] \), \( 0 < a < 1 \).

**Proof.** We only prove part (i), since the proof of (ii) is similar to that of (i). Standard computations show that
\[
|b_{nk}(q; x) - b_{\infty k}(q; x)| = \left| \left( \begin{array}{c} n \\ k \end{array} \right) \prod_{j=0}^{k-1} v(q^j, x) \prod_{j=0}^{n-k-1} (1 - v(q^{k+j}, x)) \right|
\]
\[
- \frac{1}{(1-q)^k} \prod_{j=0}^{k-1} v(q^j, x) \prod_{j=0}^{\infty} (1 - v(q^{k+j}, x))
\]
\[
= \left| \left( \begin{array}{c} n \\ k \end{array} \right) \prod_{j=0}^{k-1} v(q^j, x) \left( \prod_{j=0}^{n-k-1} (1 - v(q^{k+j}, x)) - \prod_{j=0}^{\infty} (1 - v(q^{k+j}, x)) \right) \right|
\]
\[
+ \prod_{j=0}^{k-1} v(q^j, x) \prod_{j=0}^{\infty} (1 - v(q^{k+j}, x)) \left( \left( \begin{array}{c} n \\ k \end{array} \right) - \frac{1}{(1-q)^k} \left( \begin{array}{c} k \\ j \end{array} \right) \right) \right|
\]
\[
\leq b_{nk}(q; x) \left| 1 - \prod_{j=n}^{\infty} (1 - v(q^j, x)) \right| + b_{\infty k}(q; x) \prod_{j=n-k+1}^{n} (1 - q^j) - 1. \quad (3.1)
\]

Now using the inequality
\[
1 - \prod_{j=1}^{k} (1 - a_j) \leq \sum_{j=1}^{k} a_j, \quad (a_1, a_2, ..., a_k \in (0, 1), \ k = 1, 2, ..., \infty),
\]
we get from (3.1) that
\[
|b_{nk}(q; x) - b_{\infty k}(q; x)| \leq b_{nk}(q; x) \sum_{j=n}^{\infty} v(q^j, x) + b_{\infty k}(q; x) \sum_{j=n-k+1}^{n} q^j. \quad (3.2)
\]

On the other hand, \( \lim_{j \to \infty} \frac{v(q^{j+1}, x)}{v(q^j, x)} = q < 1 \) and observe for any \( \varepsilon > 0 \) such that \( q_0 + \varepsilon < 1 \) there exists \( n^* \in \mathbb{N} \) such that
\[
\frac{v(q^{j+1}, x)}{v(q^j, x)} < q_0 + \varepsilon = \frac{(q_0 + \varepsilon)^{j+1}}{(q_0 + \varepsilon)^j}
\]
for all \( j > n^* \). Hence, the sequence \( v(q^j, x) / (q_0 + \varepsilon)^j \) is decreasing for large \( j \).
and thus uniformly bounded in \((q, x) \in (0, q_0] \times [0, 1]\) by

\[
M(\varepsilon) = \max \left\{ \frac{v(q^{n+1}, x)}{(q_0 + \varepsilon)^{n+1}}, \frac{v(q^n, x)}{(q_0 + \varepsilon)^n}, \ldots, \frac{v(q, x)}{q_0 + \varepsilon} \right\}.
\]

So, for such \(M(\varepsilon) > 0\) we have \(|v(q^j, x)| \leq M(\varepsilon)(q_0 + \varepsilon)^j\) for all \(j = 1, 2, \ldots\) and \((q, x) \in (0, q_0] \times [0, 1)\).

Now from (3.2) we get the desired inequality

\[
|b_{nk}(q; x) - b_{\infty k}(q; x)| \leq b_{nk}(q; x)M(\varepsilon) \frac{(q_0 + \varepsilon)^n}{1 - (q_0 + \varepsilon)} + b_{\infty k}(q; x) \frac{q_0^{n-k+1}}{1 - q_0}.
\]

Before proving the main results notice that the following theorem proved in [9] will allow us to reduce the case \(q \in (1, \infty)\) to the case \(q \in (0, 1)\).

**Theorem 10** Let \(f \in C[0, 1], g(x) := f(1 - x).\) Then for any \(q > 1\),

\[
R_{n,q}(f, x) = R_{n,\frac{1}{q}}(g, 1 - x) \quad \text{and} \quad R_{\infty,q}(f, x) = R_{\infty,\frac{1}{q}}(g, 1 - x).
\]

Using Lemma 9 we prove the following quantitative result for the rate of local convergence of \(R_{n,q}(f, x)\) in terms of the first modulus of continuity.

**Theorem 11** Let \(0 < q < 1\) and \(f \in C[0, 1]\). Then for all \(0 \leq x < 1\) we have

\[
|R_{n,q}(f, x) - R_{\infty,q}(f, x)| \leq \frac{2}{1 - q} \frac{1}{1 - x} \omega(f, q^n).
\]

**Proof.** Consider

\[
\Delta(x) := R_{n,q}(f, x) - R_{\infty,q}(f, x) = \sum_{k=0}^{n} f \left( \frac{[k]}{[n]} \right) b_{nk}(q; x) - \sum_{k=0}^{\infty} f \left( 1 - q^k \right) b_{\infty k}(q; x).
\]

Since \(R_{n,q}(f, x)\) and \(R_{\infty,q}(f, x)\) possess the end point interpolation property \(\Delta(0) = \Delta(1) = 0\). For all \(x \in (0, 1)\) we rewrite \(\Delta\) in the following form

\[
\Delta(x) = \sum_{k=0}^{n} \left[ f \left( \frac{[k]}{[n]} \right) - f \left( 1 - q^k \right) \right] b_{nk}(q; x)
+ \sum_{k=0}^{n} \left[ f \left( 1 - q^k \right) - f(1) \right] (b_{nk}(q; x) - b_{\infty k}(q; x))
- \sum_{k=n+1}^{\infty} \left[ f \left( 1 - q^k \right) - f(1) \right] b_{\infty k}(q; x) =: I_1 + I_2 + I_3.
\]
We start with estimation of $I_1$ and $I_3$. Since

$$0 \leq \frac{[k]}{n} - (1 - q^k) = \frac{1 - q^k}{1 - q^n} - (1 - q^k) = q^n \frac{1 - q^k}{1 - q^n} \leq q^n,$$

$$0 \leq 1 - (1 - q^k) = q^k \leq q^n, \quad k > n,$$

we get

$$|I_1| \leq \omega (f, q^n) \sum_{k=0}^{n} b_{nk}(q; x) = \omega (f, q^n), \quad (3.3)$$

$$|I_3| \leq \omega (f, q^n) \sum_{k=n+1}^{\infty} b_{\infty k}(q; x) \leq \omega (f, q^n). \quad (3.4)$$

Finally we estimate $I_2$. Using the property of the modulus of continuity

$$\omega (f, \lambda t) \leq (1 + \lambda) \omega (f, t), \quad \lambda > 0$$

and Lemma[9] we get

$$|I_2| \leq \sum_{k=0}^{n} \omega (f, q^k) |b_{nk}(q; x) - b_{\infty k}(q; x)|$$

$$\leq \omega (f, q^n) \sum_{k=0}^{n} \left(1 + q^{k-n}\right) |b_{nk}(q; x) - b_{\infty k}(q; x)|$$

$$\leq 2\omega (f, q^n) \frac{1}{q^n} \sum_{k=0}^{n} q^k |b_{nk}(q; x) - b_{\infty k}(q; x)|$$

$$\leq 2\omega (f, q^n) \frac{1}{q^n} \sum_{k=0}^{n} q^k \left(b_{nk}(q; x) \frac{x}{1 - x} \frac{q^n}{1 - q} + b_{\infty k}(q; x) \frac{q^{n-k+1}}{1 - q}\right)$$

$$\leq \frac{2}{1 - q} \left(\frac{x}{1 - x} + 1\right) \omega (f, q^n) = \frac{2}{1 - q} \frac{1}{1 - x} \omega (f, q^n). \quad (3.5)$$

From (3.3), (3.4), and (3.5), we conclude the desired estimation. ■

**Corollary 12** Let $q > 1$ and $f \in C[0, 1]$. Then for all $0 < x \leq 1$ we have

$$|R_{n,q} (f, x) - R_{\infty, q} (f, x)| \leq 2q \frac{1}{q - 1} \omega (g, q^{-n}).$$

**Proof.** Proof follows from Theorems[11] and[10] ■

Next corollary gives quantitative result for the rate of uniform convergence of $R_{n,q} (f, x)$ in $C[0, a]$ and $C[a, 1]$, $0 < a < 1$.

**Corollary 13** Let $f \in C[0, 1], 0 < a < 1$.

1. If $0 < q < 1$, then

$$\|R_{n,q} (f) - R_{\infty, q} (f)\|_{C[0,a]} \leq 2q \frac{1}{1 - q} \frac{1}{1 - a} \omega (f, q^n).$$
(2) If $q > 1$, then
\[
\|R_{n,q}(f) - R_{\infty,q}(f)\|_{C[a,1]} \leq \frac{2q}{q-1} \frac{1}{a} \omega(g,q^{-n}).
\]

In order to prove the estimation in terms of the second modulus of continuity we need the following theorem proved in [16].

**Theorem 14** [16] Let \( \{T_n\} \) be a sequence of positive linear operators on \( C[0,1] \) satisfying the following conditions:

(A) the sequence \( \{T_n(t^2)(x)\} \) converges uniformly on \([0,1]\);
(B) the sequence \( \{T_n(f)(x)\} \) is nonincreasing in \( n \) for any convex function \( f \) and any \( x \in [0,1] \).

Then there exists an operator \( T_\infty \) on \( C[0,1] \) such that
\[
T_n(f)(x) \rightarrow T_\infty(f)(x)
\]
as \( n \rightarrow \infty \) uniformly on \([0,1]\). In addition, the following estimation holds:
\[
|T_n(f)(x) - T_\infty(f)(x)| \leq C \omega_2\left(f; \sqrt{\lambda_n(x)}\right),
\]
where \( \omega_2 \) is the second modulus of smoothness, \( \lambda_n(x) = |T_n(t^2)(x) - T_\infty(t^2)(x)| \), and \( C \) is a constant depending only on \( T_1(1) \).

**Theorem 15** Let \( 0 < q < 1 \). Then
\[
\|R_{n,q}(f) - R_{\infty,q}(f)\| \leq C \omega_2\left(f, \sqrt{q^n}\right).
\]
Moreover,
\[
\sup_{0 < q \leq 1} \|R_{n,q}(f) - R_{\infty,q}(f)\| \leq c \omega_2\left(f, n^{-1/2}\right),
\]
where \( c \) is a constant.

**Proof.** From [6], we know that the \( q \)-Bernstein operators satisfy condition (B) of Theorem [14]. On the other hand
\[
0 \leq R_{n,q}(t^2,x) - R_{\infty,q}(t^2,x) = \frac{q^n}{n} x(1-v(q,x)) \leq q^n \frac{x(1-x)}{1-x+qx} \leq q^n
\]
and
\[
\sup_{0 < q \leq 1} \left|R_{n,q}(t^2,x) - R_{\infty,q}(t^2,x)\right| = \sup_{0 < q \leq 1} \frac{q^n}{n} \frac{x(1-x)}{1-x+qx} = \frac{x(1-x)}{n}.
\]
Since
\[
|R_{n,1}(t^2,x) - x^2| = \frac{x(1-x)}{n},
\]
we conclude that
\[
\sup_{0 < q \leq 1} \left|R_{n,q}(t^2,x) - R_{\infty,q}(t^2,x)\right| \leq \frac{x(1-x)}{n} \leq \frac{1}{n},
\]
Theorem follows from (3.9), (3.8) and Theorem 14.

**Theorem 16** Let \( q > 1 \). Then
\[
\| R_{n,q}(f) - R_{\infty,q}(f) \| \leq c \omega_2 \left( g, \sqrt{q^{-n}} \right).
\]
Moreover,
\[
\sup_{1 \leq q < \infty} \| R_{n,q}(f) - R_{\infty,q}(f) \| \leq c \omega_2 \left( g, n^{-1/2} \right), \tag{3.10}
\]
where \( c \) is a constant.

**Proof.** The proof is similar to that of Theorem 15.

**Remark 17** From (3.7) and (3.10), we conclude that the rate of convergence \( \| R_{n,q}(f) - R_{\infty,q}(f) \| \) can be dominated by \( c \omega_2 \left( f, n^{-1/2} \right) \) uniformly with respect to \( q \neq 1 \).

**Remark 18** We may observe here that for \( f(x) = x^2 \), we have
\[
\| R_{n,q}(f) - R_{\infty,q}(f) \| \asymp q^n \asymp \omega_2 \left( f, \sqrt{q^n} \right), \quad 0 < q < 1,
\]
where \( A(n) \asymp B(n) \) means that \( A(n) \ll B(n) \) and \( A(n) \gg B(n) \) means that there exists a positive constant \( c \) independent of \( n \) such that \( A(n) \leq cB(n) \). Hence the estimate (3.6) is sharp in the following sense: the sequence \( q^n \) in (3.6) cannot be replaced by any other sequence decreasing to zero more rapidly as \( n \to \infty \).

4 Voronovskaja type results

**Theorem 19** Let \( 0 < q < 1 \), \( f \in C^2[0,1] \). Then there exists a positive absolute constant \( K \) such that
\[
\left| \frac{[n]}{q^n} (R_{n,q}(f,x) - R_{\infty,q}(f,x)) - \frac{f''(x)}{2} x (1 - v(q,x)) \right| \leq K x (1 - v(q,x)) \omega(f'', [n]^{-\frac{1}{2}}). \tag{4.1}
\]

**Proof.** Let \( x \in (0,1) \) be fixed. We set
\[
g(t) = f(t) - \left( f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 \right).
\]
It is known that (see [6]) if the function \( h \) is convex on \([0,1]\), then
\[
R_{n,q}(h,x) \geq R_{n+1,q}(h,x) \geq ... \geq R_{\infty,q}(h,x),
\]
and therefore,
\[
L_{n,q}(h,x) := R_{n,q}(h,x) - R_{\infty,q}(h,x) \geq 0.
\]
Thus $L_{n,q}$ is positive on the set of convex functions on $[0, 1]$. But in general $L_{n,q}$ is not positive on $C[0, 1]$.

Simple calculation gives

$$L_{n,q} (g, x) = (R_{n,q}(f, x) - R_{\infty,q}(f, x)) - \frac{q^n}{[n]^2} f''(x) x (1 - v(q, x)).$$

In order to prove the theorem, we need to estimate $L_{n,q}(g, x)$. To do this, it is enough to choose a function $S(t)$ such that the functions $S(t) \pm g(t)$ are convex on $[0, 1]$. Then $L_{n,q}(S \pm g, x) \geq 0$, and therefore,

$$|L_{n,q}(g(t), x)| \leq L_{n,q}(S(t), x).$$

So the first thing to do is to find such function $S(t)$. Using the well-known inequality $\omega(f, \lambda \delta) \leq (1 + \lambda^2) \omega(f, \delta)$ ($\lambda, \delta > 0$), we get

$$|g''(t)| = |f''(t) - f''(x)| \leq \omega(f'', |t - x|)
= \omega \left( f'', \frac{1}{[n]^2} [n]^{\frac{1}{2}} |t - x| \right) \leq \omega \left( f'', \frac{1}{[n]^\frac{1}{2}} \right) \left( (1 + [n] (t - x)^2) \right).$$

Define $S(t) = \omega \left( f'', [n]^{-\frac{1}{2}} \right) \left[ \frac{1}{2} (t - x)^2 + \frac{1}{12} [n] (t - x)^4 \right]$. Then

$$|g''(t)| \leq \frac{1}{6} \omega \left( f'', [n]^{-\frac{1}{2}} \right) \left( 3 (t - x)^2 + \frac{1}{2} [n] (t - x)^4 \right)'' = S''(t)$$

Hence the functions $S(t) \pm g(t)$ are convex on $[0, 1]$, and therefore,

$$|L_{n,q}(g(t), x)| \leq L_{n,q}(S(t), x),$$

and

$$L_{n,q}(S(t), x) = \frac{1}{6} \omega \left( f'', [n]^{-\frac{1}{2}} \right) \left( 3 q^n [n^2] x (1 - v(q, x)) + \frac{1}{2} [n] L_{n,q} ((t - x)^4, x) \right).$$

Since by the formula (2.11)

$$L_{n,q} ((t - x)^4, x) \leq K_1 \frac{q^n}{[n]^2} x (1 - v(q, x))$$

we have

$$L_{n,q}(S(t), x) \leq \frac{1}{6} \omega \left( f'', [n]^{-\frac{1}{2}} \right) \left( 3 q^n [n^2] x (1 - v(q, x)) + \frac{1}{2} [n] K_1 \frac{q^n}{[n]^2} x (1 - v(q, x)) \right).$$

By (4.2) and (4.3), we obtain (4.1). Theorem is proved. ■

**Corollary 20** Let $q > 1$, $f \in C^2[0, 1]$. Then there exists a positive absolute
constant $K$ such that

$$
\left| q^n \left[ n \right]_{\frac{1}{q}} (R_{n,q}(f, x) - R_{\infty,q}(f, x)) - \frac{f''(1 - x)}{2} v(q, x) (1 - x) \right|
\leq K v(q, x) (1 - x) \omega(g'', [n]_{\frac{1}{q}}^{\frac{1}{2}}).
$$

**Corollary 21** If $f \in C^2[0, 1]$ and $q_n \to 1$ as $n \to \infty$, then

$$
\lim_{q_n \uparrow 1} [n]_{q_n} (R_{n,q_n}(f, x) - f(x)) = \frac{f''(x)}{2} x (1 - x),
$$

$$
\lim_{q_n \downarrow 1} [n]_{q_n} (R_{n,q_n}(f, x) - f(x)) = \frac{f''(1 - x)}{2} x (1 - x)
$$

uniformly on $[0, 1]$.

**Remark 22** When $q_n \equiv 1$, (4.4) reduces to the classical Voronovskaja’s formula. For the function $f(t) = t^2$, the exact equality

$$
\frac{[n]_{q}}{q^n} \left( R_{n,q}(t^2, x) - R_{\infty,q}(t^2, x) \right) = x (1 - v(q, x)), \quad 0 < q < 1,
$$

$$
q^n \left[ n \right]_{\frac{1}{q}} \left( R_{n,q}(t^2, x) - R_{\infty,q}(t^2, x) \right) = v(q, x) (1 - x), \quad q > 1,
$$

takes place without passing to the limit, but in contrast to the Phillips $q$-analogue of the Bernstein polynomials the right hand side depends on $q$. In contrast to the classical Bernstein polynomials and Phillips $q$-analogue of the Bernstein polynomials the exact equality

$$
[n] (B_{n,q}(t^2, x) - x^2) = (x^2)'' x (1 - x) / 2
$$

does not hold for the Lupà s $q$-analogue of the Bernstein polynomials.

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