Evaluation of Differential Equations with Applications to Engineering Problems

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ABSTRACT
Several methods for solving ordinary differential equations (ODE) and partial differential equations (PDE) have been developed over the last century. Though the majority of the methods are only useful for academic purposes, some are critical in the solution of real-world problems arising from science and engineering. Only a subset of the available methods for solving (ODE) and (PDE) are discussed in this paper, as it is impossible to cover all of them in a book. Readers are then encouraged to conduct additional research on this topic if necessary. Afterward, the readers are made known to two major numerical methods commonly used by the engineers for the solution of real-life engineering problems.

Keywords:-Differential Equations, Homogenous, Non-homogenous differential equations

INTRODUCTION
A differential equation is an equation relating an unknown function and one or more of its derivatives. Differential equation can be classified by Type, Order, linearity as shown in Figure 1. Classification by type as an ordinary or partial differential equation which depends on whether only ordinary derivatives are involved or partial derivatives are involved. A differential equation is an ordinary differential equation if the unknown variable or function depends on only one independent variable. If the unknown function depends on two or more independent variables, the equation is called partial differential equation. For instance, equation 1. through 4. are example of ordinary differential equations, since the unknown function \( y \) depends solely on the variable \( x \). equation 5. is a partial differential equation, since \( y \) depends on both the independent variables \( p \) and \( x \). The order of a differential equation (either ODE or PDE) is order of the highest derivative appearing in the equation. For instance, equation 1 is a first-order differential equation; equation 2,4, and 5 are second-order differential equations. Equation 3. is a third-order differential equation.

Fig.1:-Classification of ODE
The following are differential equations involving the unknown function $y$.

$$\frac{dy}{dx} = 6x + 4 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$e^y \frac{d^2y}{dx^2} + 10 \left( \frac{dy}{dx} \right)^2 = 3 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$12 \frac{d^3y}{dx^3} + (\sin x) \frac{d^2y}{dx^2} + 4xy = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\left( \frac{d^2y}{dx^2} \right)^3 + 4y \left( \frac{dy}{dx} \right)^8 + y^3 \left( \frac{dy}{dx} \right)^2 = 4x \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$\frac{\partial^2y}{\partial p^2} - 14 \left( \frac{\partial^2y}{\partial x^2} \right) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$\frac{d^2y}{dp^2} + p^3y(e^y) \left( \frac{dy}{dp} \right)^5 + y = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

The general $n^{th}$ order ODE is expressed in equation 7.

$$\frac{d^ny}{dx^n} = f(x,y,y_1,y_2y_3\ldots y_n) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where $f$ is the real value of the function

when $n = 1$, \quad $\frac{dy}{dx} = f(x,y) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$

when $n = 2$, \quad $\frac{d^2y}{dx^2} = f(x,y,y_1) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$

when $n = 3$, \quad $\frac{d^3y}{dx^3} = f(x,y,y_1,y_2) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$ and so on ....

The differential equation can likewise be classified as (Linearity) linear or non-linear and their method of approach as shown in figure below. A differential equation is termed linear if the independent variables in equation is raised to the power of 1 of $y, y', y''$ or higher order, and all the coefficients depend upon just a single variable $x$ as appeared in equation. (1,2,4,), In Equation. (1), if $f(x) = 0$, at that point we term this equation as homogeneous. The general solution of non-homogeneous ODE or PDE is equivalents to the sum of the solution of the of non-homogeneous ODE or PDE (for example with $f(x) = 0$) plus the particular solution ($y_p$) of the non-homogeneous ODE or PDE. Then again, nonlinear differential equations include nonlinear terms like (trigonometry function, exponential functions and logarithmic functions) in any of $y, y', y''$, or higher order term as appeared in equation (3,6). A nonlinear differential equation is commonly harder to solve than linear equations. Usually, the nonlinear equation is approximated as linear equation for some practical problems, either in an analytical or numerical form. The nonlinear idea of any problem is then approximated as a series of linear differential equation by simple increment from the nonlinear behavior. This methodology is adopted for the solving of numerous non-linear differential equation engineering problems.
A linear differential equation is generally given by an equation (11)
\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = p(x) \ldots \ldots (11) \]

when \( n = 1 \)
\[ a_1(x) \frac{dy}{dx} + a_0(x)y = p(x) \ldots \ldots \ldots \ldots \ldots \ldots (12) \]

When \( n = 2 \)
\[ a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = p(x) \ldots \ldots \ldots \ldots \ldots \ldots (13) \]

METHODS AND TECHNIQUES
Methods and techniques involved in determining the DE are as follows:

Variable Separation Techniques
In the following Variable separation techniques, Examples shows the basic procedures in solving separable ODEs and rounding up the solution. A separability first-order differential equation of the form,

\[ y' \equiv \frac{dy}{dx} = F(x, y) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (14) \]

the right-hand side can then be simplified as “a formula of just \( x \)” times “a formula of just \( y \)

\[ y' = \frac{dy}{dx} = F(x, y) = f(x)g(y) \ldots \ldots \ldots \ldots \ldots \ldots \ldots (15) \]

If this simplification is not possible, the equation is not separable.
To summary it, a first-order differential equation is separable if and only if it can be written as

\[ \frac{dy}{dx} = f(x)g(y) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (16) \]

where \( f \) and \( g \) are known functions of DE.

Example 1
Solve the differential equation \( \frac{dy}{dx} = y(1 + x^5) \)

Solution:
The equation is in form, \[ \frac{dy}{dx} = f(x)g(y) \text{ with } f(x) = (1 + x^5) \text{ and } g(y) = y \]

Separate the variables, we have

\[ \frac{dy}{y} = (1 + x^5)dx \]

Integrate both sides, gives

\[ \int \left( \frac{1}{y} \right) dy = \int (1 + x^5) dx \]

\[ \ln(y) = x + \frac{x^6}{6} + c \]

Rounding up, we have

\[ y = e^{x + \frac{x^6}{6}} \]

\[ y = e^{x + \frac{x^6}{6}} e^c \]

\[ y = Ae^{x + \frac{x^6}{6}} \]

**Example 2**

Solve the ordinary differential equation \[ \frac{dy}{dx} = (1 + x^8) \]

**Solution:**

Separate the variables, gives

\[ dy = (1 + x^8)dx \]

Integrate both sides, we have

\[ \int dy = \int (1 + x^8) dx \]

Rounding up

\[ y = x + \frac{x^9}{9} + c \]

**Example 3**

Solve ODE \[ \frac{dy}{dx} = \frac{x^2}{y^2} \]

**Solution**

Separate the variables, gives

\[ y^2 dy = x^2 dx \]

Integrate both sides, we have

\[ \int y^2 dy = \int x^2 dx \]

Rounding up

\[ \frac{y^3}{3} = \frac{x^3}{3} + c \]

\[ y = \sqrt[3]{\frac{x^3}{3} + c} \]

**Example 4**

Solve ODE \[ \frac{dy}{dx} + y = 0 \]

**Solution:**

Re-arrange the variables

\[ \frac{dy}{dx} = -y \]

Separate the variables, and integrate both-side gives

\[ (-1) \int \frac{1}{y} dy = \int x dx \]

Rounding-up

\[ -\ln y = x^2 + c \]
\[ y = e^{-(x^2+c)} = e^{-x^2}e^{-c} \text{ where } A = e^{-c} \]
\[ y = \frac{A}{e^{x^2}} \]

**Example 5**
Considering an example having boundary/initial condition like this \( \frac{dy}{dx} + y = 0 \) with \( y(2) = 1 \)
From example (4):
\[ -\ln y = x^2 + c \]
Applying the boundary conditions, we have \( y = 2, x = 1 \)
\[ \ln(1) = (-1)^2 - c \]
\[ 0 = 1 - c \]
\[ c = 1 \]
Substitute value of \( c \)
\[ -\ln y = -(x^2 + 1) \]
Rounding up
\[ y = e^{-x^2+1} = \frac{1}{e^{x^2+1}} \]

**Linearity**
An ODE is said to be non-linear if it encompasses powers and/or products of the dependent variable or its derivatives, and linear if otherwise. For example, equation (29) is said to be linear whereas equation 30 is said to be non-linear.

\[ a_1(x) \frac{dy}{dx} + a_0(x)y = p(x) \] \hspace{1cm} (29)
\[ a_1(x) \frac{dy}{dx} + a_0(x)y^3 = p(x) \] \hspace{1cm} (30)

The remaining types of ODEs that we will consider are on the whole linear and we will use the special properties of linear equations in evaluating their solutions. Additionally, we will change to variables \( x \) also, \( t \) since numerous uses of these kinds of ODEs are time-dependent. If we have two functions, \( x_0(t) \) and \( x_1(t) \), then a linear combination of these functions takes the form.

\[ \frac{dx}{dt} = a_0x_0(t) + a_1x_1(t) \] \hspace{1cm} (31)

Where \( a_0 \) and \( a_1 \) are constants. A non-linear differential equation would contain powers and/or products of \( y \)'s.

**First Order, Linear Ordinary Differential Equations**

**Example 6**
\[ \frac{dx}{dt} + Rx = 0 \]
Where \( R \) is constant
We can re-solve this equation using separability of variables
\[ \frac{dx}{dt} = -Rx \] \hspace{1cm} (32)
\[ \frac{1}{x} dx = -R dt \]
Integrate both sides, gives
\[ f \left( \frac{1}{x} dx \right) = \int (-R dt) \] \hspace{1cm} (33)
\[ \ln x = -Rt + c \] \hspace{1cm} (34)
\[ x = e^{-Rt+c} \]
\[ e^{-Rt}.e^c \text{ where } A = e^c \]
\[ x = A e^{-Rt} \]

**Second Order, Linear Ordinary Differential Equation with Constant Coefficients**

This second order linear ordinary differential equation is of two forms:

a. Homogeneous ordinary differential equation

b. Non-homogeneous ordinary differential equation.

\[ a_0 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + cx = 0 \text{ (homogenous) } \tag{35} \]

\[ a_0 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + cx = f(x) \text{ (non-homogenous) } \tag{36} \]

Where \( a_0 \) and \( a_1 \) and \( c \) are constant

The form of the Equations (35 and 36) above are extremely useful for modelling a simple suspension system linking a mass to some fixture through a spring and dash-pot.

a. Homogeneous ordinary differential equation

Given:

\[ a_0 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + cx = 0 \text{ (homogenous) } \tag{37} \]

To solve this homogeneous ODE, let us consider a function \( x \) of the form

\[ x = e^{kt} \text{ (homogenous) } \tag{38} \]

where \( k \) is unspecified, if we differentiate equation (38) to the respect of \( t \) we get,

\[ \frac{dx}{dt} = ke^{kt} \text{ (homogenous) } \tag{39} \]

Further differential of equation (39) gives,

\[ \frac{d^2x}{dt^2} = k^2 e^{kt} \]

If substitute \( x \) and its derivatives \( \frac{dx}{dt} \text{ and } \frac{d^2x}{dt^2} \) in to equation (37) we get,

\[ (a_0 k^2 + a_1 k + c)e^{kt} = 0 \text{ (homogenous) } \tag{40} \]

Or

\[ (a_0 k^2 + a_1 k + c) = 0 \text{ (homogenous) } \tag{41} \]

Equation [41] is known as the auxiliary equation or characteristic polynomial of the ODE. Solving it permits us to build particular solutions and, henceforth, the general solution of the ODE. Since we have to solve a quadratic equation, we have to deal separately with the diverse types of solution that can occur. Recall that

\[ k = \frac{-a_1 \pm \sqrt{(a_1^2 - 4a_0c)}}{2a_0} \text{ (homogenous) } \tag{42} \]

The homogeneous ordinary differential equation is divided into three cases

**Case (1)** \( a_1^2 - 4a_0c > 0 \) \text{ (homogenous) } \tag{43} \]

When we have \( a_1^2 - 4a_0c > 0 \), it gives two real values of auxiliary equation \( k = k_1 \text{ and } k = k_2 \)

Then the particular solution of ODE shall be recall from

\[ x = Ae^{k_1t} + Be^{k_2t} \text{ (homogenous) } \tag{44} \]

**Case (2)** \( a_1^2 - 4a_0c < 0 \) \text{ (homogenous) } \tag{45} \]

When we have \( a_1^2 - 4a_0c < 0 \), it gives two complex values of the auxiliary equation \( k_1 = \alpha + i\beta \text{ and } k_2 = \alpha - i\beta \)

Then the particular solution of ODE shall be recall from

\[ x = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t} \text{ (homogenous) } \tag{46} \]

Or
\[ x = A e^{at} e^{i \beta t} + B e^{at} e^{-i \beta t} \] ............................. (47)

Or
\[ x = e^{at} (A e^{i \beta t} + B e^{-i \beta t}) \] ............................. (48)

Using a complex number theory, we can write
\[ e^{i \beta t} = \cos(\beta t) + i \sin(\beta t) \] ............................. (49)
\[ e^{-i \beta t} = \cos(\beta t) - i \sin(\beta t) \] ............................. (50)

Therefore, the general solution of the ODE is given as
\[ e^{at}(A \cos(\beta t) + B \sin(\beta t)) \] ............................. (51)

**Case (3) \[ a_1^2 - 4a_0 c = 0 \] ............................. (52)**

When we have \[ a_1^2 - 4a_0 c = 0 \], it gives only one root of the auxiliary equation \( k = k_0 \)

Then, general solution will be
\[ x = e^{k_0 t}(A + Bt) \] ............................. (53)

Based on the explanation made on the homogeneous and non-homogeneous, the general procedure is as follows:

To determine ordinary differential equation

Given:
\[ \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + cx = 0 \] ............................. (55)

i. generate the auxiliary equation: \( (k^2 + k + c) = 0 \)

ii. Solve the auxiliary equation;

iii. write down the general solution of the ODE, depending on the nature of the solution of the quadratic:

- Two real roots: \( x = A e^{k_1 t} + B e^{k_2 t} \)
- Two complex roots: \( e^{at}(A \cos(\beta t) + B \sin(\beta t)) \)
- One real root: \( x = e^{k_0 t}(A + Bt) \)

**Examples 6**

Solve the ordinary differential equation
\[ \frac{d^2 y}{dt^2} - 6 \frac{dy}{dt} + 5x = 0 \]

Write out the auxiliary equation as shown:
\[ (k^2 - 6k + 5) = 0 \] ............................. (56)

\[ k(\beta - 5) - 1(\beta - 5) = 0 \]
\[ (\beta - 5)(\beta - 1) = 0 \]
\[ k = 5 \text{ and } k = 1 \]

It forms two distinct roots
\[ \therefore x = A e^{5t} + B e^{t} \]

**Examples 7**

Solve the ordinary differential equation
\[ \frac{d^2 y}{dx^2} - \frac{dy}{dx} + 4 = 0 \]

Write out the auxiliary equation as shown:
\[ k^2 - k + 4 = 0 \]
\[ k = \frac{-a_1 \pm \sqrt{(a_1^2 - 4a_0 c)}}{2a_0} \] ............................. (57)

Where \( a_0 = 1, a_1 = -1, c = -4 \)
\[ = \frac{(1 \pm \sqrt{-15})}{2} \]
\[ = \frac{1}{2} \pm i \left( \frac{\sqrt{15}}{2} \right) \]

It forms two complex roots:

\[ \alpha = \frac{1}{2}, \text{ and } \beta = \frac{\sqrt{15}}{2} \]

\[ \therefore x = e^{\frac{t}{2}} \left( A \cos \left( \frac{\sqrt{15}}{2} t \right) + B \sin \left( \frac{\sqrt{15}}{2} t \right) \right) \] ... ... ... ... ... ... ... ... \( (58) \)

**Example 8**

Solve the equation \( \frac{d^2y}{dx^2} - 16 = 0 \)

Write out the auxiliary equation as shown:

\[ k^2 - 16 = 0 \]
\[ k = \sqrt{16} \]
\[ k = 4 \]

It forms a single root:

\[ x = (A + Bt)e^{4t} \] ... ... ... ... ... ... ... ... \( (59) \)

**Example 9**

Solve the ordinary differential equation:

\[ \frac{d^2x}{dt^2} + 16x = 0 \text{ subject to} \]

(a) \( x(0) = 1 \quad x \left( \frac{\pi}{4} \right) = -\sqrt{2} \)

(b) \( x(0) = 1 \quad \frac{dx}{dt}(0) = 4 \)

a) Write out the auxiliary equation as shown:

\[ k^2 + 16 = 0 \]
\[ k^2 = -16 \]
\[ k = 0 \pm 4i \]
\[ \alpha = 0, \beta = 4 \]

General solution:

\[ x = e^{\alpha t} \left( A \cos(\beta t) + B \sin(\beta t) \right) \]
\[ x = e^{\alpha t} \left( A \cos(4t) + B \sin(4t) \right) \]
\[ x = e^{\frac{t}{2}} \left( A \cos(4t) + B \sin(4t) \right) \]
\[ x = (A \cos(4t) + B \sin(4t)) \]

Apply \( x = x(0) = 1 \):

\[ 1 = (A \cos(4 \times 0) + B \sin(4 \times 0)) \]
\[ A = 1 \]

Apply \( x \left( \frac{\pi}{8} \right) = -\sqrt{2} \):

\[ -\sqrt{2} = \left( A \cos \left( \frac{\pi}{2} \right) + B \sin \left( \frac{\pi}{2} \right) \right) \]
\[ -\sqrt{2} = (A(0) + B(1)) \]
\[ -\sqrt{2} = B(1) \]
\[ B = -\sqrt{2} \]

\[ \therefore x = \cos(4t) - \sqrt{2} \sin(4t) \] ... ... ... ... ... ... ... ... \( (60) \)

b) Everything starts exactly as in parts a)

The general solution earlier obtained as \( x = (A \cos(4t) + B \sin(4t)) \)

Apply \( x = x(0) = 1 \):

\[ 1 = (A \cos(4 \times 0) + B \sin(4 \times 0)) \]
Before applying the second condition, we must differentiate the general solution:

\[ x = (A\cos(4t) + B\sin(4t)) \]

\[ \frac{dy}{dt} = (-4A\cos(4t) + 4B\sin(4t)) \]

now apply \( \frac{dy}{dt}(0) = 4: \)

\[ \frac{dy}{dx} = (-4A\sin(4t) + 4B\cos(4t)) \]

\[ 4 = (-4A\sin(4x0) + 4B\cos(4x0)) \]

\[ 4 = (0 + 4B) \]

\[ B = 1 \]

Putting the two values of A and B into the general solution gives the particular solution

\[ x = \cos(4t) + \sin(4t) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (61) \]

**Non-homogeneous ordinary differential equation**

The solution of a second order non-homogeneous linear differential equation of the form

\[ a \frac{d^2x}{dy^2} + b \frac{dy}{dx} + cy = f(x) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (62) \]

The general solution of non-homogeneous ODE of the form equation directly above is found as follows:

1. Obtain the general solution of the corresponding homogeneous ODE and this is called the complementary function and denote it by \( y_c(x) \)
2. Obtain one particular solution of the entire non-homogeneous ODE using method of undetermined coefficients and it is denoted as \( y_p(x) \)
3. Then the general solution of the full, non-homogeneous ODE is given by \( y = y_c(x) + y_p(x) \)

**The Method of Undermined Coefficients**

This technique is best illustrated by examples. Let’s consider the following ODE

\[ \frac{d^2x}{dy^2} - \frac{dy}{dx} + 2y = x^2 + 5 \]

Determine the complementary function as follows:

Auxiliary equation:

\[ k^2 - k + 2 = 0 \]

\[ (k - 2)(k + 1) = 0 \]

\[ k = 2 \text{ and } k = -1 \]

Since it is two real roots, so

\[ y_c(x) = Ae^{2t} + Be^{-t} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ld...

Where a, b and c are undetermined coefficients. The aim now is to obtain those values of coefficients that make solution of ODE. We solve this by putting the general form into the LHS of ODE and equate it to given RHS. To do this we form the 1st and 2nd derivatives of the above equation

\[ \frac{dy_p}{dx} = 2ax + b \]
\[
\frac{d^2y_p}{dx^2} = 2a
\]
Then we have:
\[
(2a) - (2ax + b) - 2(ax^2 + bx + c) = x^2 + 5
\]
\[
2a - 2ax - b - 2ax^2 - 2bx - 2c = x^2 + 5
\]
Collect like terms:
\[
(2a - b - 2c) - (2ax + 2bx) - 2ax^2 = x^2 + 5
\]
Equating the powers of \(x\) and contacts:
\[
x^2: - 2ax^2 = x^2
\]
\[
-2a = 1
\]
\[
a = -\frac{1}{2}
\]
\[
x: (-2a - 2b)x = 0x
\]
\[
-2a - 2b = 0
\]
\[
-2 \left(-\frac{1}{2}\right) - 2b = 0
\]
\[
1 - 2b = 0
\]
\[
b = \frac{1}{2}
\]

constant: \(2a - b - 2c = 5\)
\[
2 \left(-\frac{1}{2}\right) - \frac{1}{2} - 2c = 5
\]
\[
-\frac{3}{2} - 2c = 5
\]
\[
-2c = 5 + \frac{3}{2}
\]
\[
-2c = \frac{13}{2}
\]
\[
= -\frac{13}{4}
\]
The particular solution is found as:
\[
y_p(x) = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{13}{4}
\]
Then the general solution of the full, non-homogeneous ODE is given by
\[
y = yc(x) + y_p(x)
\]
\[
y = Ae^{2t} + Be^{-t} + -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{13}{4} 
\]
...
(64)

**Example 2:** Solve the differential equation: \(y'' + 3y' + 2y = 0\)
Determine the complementary function as follows:
Auxiliary equation:
\[
k^2 + 3k + 2 = 0 
\]
...(65)
Roots: \((k + 1)(k + 2) = 0\)
Distinct real roots: \(k_1 = -1\) and \(k_2 = -2\)
Since it is two real roots, so
\[
y_c(x) = Ae^{-t} + Be^{-2t}
\]
We now need a particular solution:
\[
y_p(x) = ax^2 + bx + c
\]
\[
\frac{dy_p}{dx} = 2ax + b
\]
\[
\frac{d^2y_p}{dx^2} = 2a
\]
Then we have:
\[
(2a) - 3(2ax + b) - 2(ax^2 + bx + c) = x^2
\]
We plug \( a \) into the middle equation, we get
\[
x: (6a + 2b) = 0
\]
\[
b = -\frac{3}{2}
\]
Using values for \( a \) and \( b \) in the third equation, we get
\[
constant: 1 - \frac{3}{2} + 2c = 0
\]
\[
c = -\frac{1}{4}
\]
The particular solution is found as: \( y_p(x) = \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4} \)
Then the general solution of the full, non-homogeneous ODE is given by
\[
y = y_c(x) + y_p(x)
\]
\[
y = Ae^{-t} + Be^{-2t} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4} \quad \text{............... .......... .......... (66)}
\]

**Example 3:** solve \( y'' + 9y' = e^{-4x} \)
We first find the solution of the corresponding homogeneous equation,
\( y'' + 9y' = 0 \)
Determine the complementary function as follows:
Auxiliary equation: \( k^2 + 9 = 0 \)
Roots: \( k^2 = -9; k = \alpha + i\beta = 3j - 3j \)
Complex roots: \( \alpha = 3j \) and \( \beta = -3j \)
Since it is Complex roots, so
\[
y_c(x) = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}
\]
\[
= e^{\alpha t}(Acos(\beta t) + Bsin(\beta t))
\]
\[
= e^{3t}(Acos(-3t) + Bsin(3t))
\]
Next we find a particular solution. Our trial solution is of the form
\( y_p(x) = Ae^{-4x} \)
\( y'_p(x) = -4Ae^{-4x} \)
\( y''_p(x) = 16Ae^{-4x} \)
We plug \( y_p(x), y'_p(x) \) and \( y''_p(x) \) into the equation to get:
\[
16Ae^{-4x} + 9Ae^{-4x} = e^{-4x}
\]
\[
e^{-4x}(16A + 9A) = e^{-4x}
\]
Since \( e^{-4x} > 0 \), we can cancel to get: \( 25A = 1; \quad A = \frac{1}{25} \)
Hence a particular solution is given by \( y_p(x) = \frac{1}{25}e^{-4x} \)
Then the general solution of the full, non-homogeneous ODE is given by
\[
y = y_c(x) + y_p(x)
\]
\[
y = e^{3t}(Acos(-3t) + Bsin(3t)) + \frac{1}{25}e^{-4x} \quad \text{............... .......... .......... (67)}
\]

**Problems**
1. Check whether each of the following differential equations is or is not separable, and, if it is separable, rewrite the equation in the form.
\[
\frac{dy}{dx} = f(x)g(y)
\]

\begin{enumerate}
\item \[\frac{dy}{dx} = 3y^2 - y^2 \sin(x)\]
\item \[\frac{dy}{dx} = \sin(x + y)\]
\item \[\frac{dy}{dx} = \sqrt{2 + x^2}\]
\item \[2\frac{dy}{dx} + 8y = 16\]
\item \[4\frac{dy}{dx} + 16y = 4x^2\]
\item \[\sqrt{\frac{dy}{dx}} = \sqrt{\sin(x + y)}\]
\item \[\frac{dy}{dx} + xy = 6x\]
\item \[\frac{dy}{dx} = xy - 6x - 4y + 12\]
\item \[xy\frac{dy}{dx} = e^{x-6y^2}\]
\end{enumerate}

2. Solve the following homogenous differential linear equations
\begin{enumerate}
\item \[y'' + 4y' = 0\]
\item \[y'' + 2y' = 0\]
\item \[y'' + 9y' + 4y = 0\]
\item \[y'' + y' - 3y = 0\]
\end{enumerate}

3. Solve the following non-homogenous differential linear equations
\begin{enumerate}
\item \[y'' + 4y' = 5x\]
\item \[y'' + 2y' = 24x + 4\]
\item \[y'' + 9y' + 4y = x - 4\]
\item \[y'' + y' - 3y = 2x\]
\end{enumerate}

CONCLUSION
The works of Lee and Schiesser [5] and Jovanoic and Suli [4]; Veiga et al. [11]; Sewell [10]; Morton and Mayers [7]; Logg et al. [6] are also provided in various publications about the numerical solutions of differential equations (2012). Therefore, the author has chosen certain methods, which actually are used for teachers and research, that will not cover any available analytical or numerical methods. The numerous resources available in several books and publications will be strongly encouraged by readers. New developments for solutions to special differential equations are still available for large-scale problems and this is also the current trend in developing differential equation solutions.

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