PRODUCTS OF COMPACT FILTERS AND APPLICATIONS TO CLASSICAL PRODUCT THEOREMS

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Abstract. Two results on product of compact filters are shown to be the common principle behind a surprisingly large number of theorems.

1. Introduction

The terminology and notations are those of the companion paper [25]. In particular, two families $A$ and $B$ of subsets of $X$ are said to mesh, in symbol $A \# B$, if $A \cap B \neq \emptyset$ whenever $A \in A$ and $B \in B$. Given a class $\mathbb{D}$ of filters on $X$ and $A \subset X$, we call a filter $F$ (on $X$) $\mathbb{D}$-compact at $A$ if

$$D \in \mathbb{D}, D \# F \implies \text{adh} D \cap A \neq \emptyset.$$ 

The context of the present paper is that of convergence spaces as defined in [25] and therefore $\text{adh} D$ denotes the union of limit sets of filters finer than $D$. The notion derives from total nets introduced by Pettis [29] and turned out to be very useful in a variety of contexts, for instance in [12], [16], [3], in [7], [5], [6], [17], [2] under the name of compactoid filter, [29], [32], [31] under the name of total filter.

In [25], many classes of maps are characterized as relations preserving $\mathbb{D}$-compactness of filters. The aim of this paper is to establish a pair of theorems on product of $\mathbb{D}$-compact filters and show that, in view of the results of [25], they are the common principle behind a surprising number of results of stability under product of global properties (variants of compactness), local properties (Fréchetness and variants) and maps (variants of quotient and perfect maps).

2. Characterization of $\mathbb{D}$-compact filters in terms of products

The goal of this section is to show that the classical Mrówka-Kuratowski Theorem characterizing compactness of $X$ in terms of closedness of the projections $p_Y : X \times Y \to Y$ for every topological space $Y$ and its variants for other type of compactness (e.g., countable compactness, Lindelöfness), as well as product characterizations of various types of maps are all instances of a simple result on $\mathbb{D}$-compact filters.

If $\mathbb{D}$ and $\mathbb{J}$ are two classes of filters, we say that $\mathbb{J}$ is $\mathbb{D}$-composable if for every $X$ and $Y$, the (possibly degenerate) filter $\mathcal{H}(F) = \{HF : H \in \mathcal{H}, F \in F\}^\uparrow$ belongs to $\mathbb{J}(Y)$ whenever $F \in \mathbb{J}(X)$ and $\mathcal{H} \in \mathbb{D}(X \times Y)$, with the convention that every class of filters contains the degenerate filter. If a class $\mathbb{D}$ is $\mathbb{D}$-composable, we simply say that $\mathbb{D}$ is composable. Notice that

$$\mathcal{H} \# (F \times G) \iff \mathcal{H}(F) \# G \iff \mathcal{H}^-(G) \# F,$$

where $\mathcal{H}^-(G) = \{H^- G = \{x \in X : (x, y) \in H \text{ and } y \in G\} : H \in \mathcal{H}, G \in G\}^\uparrow$.

$HF = \{y \in Y : (x, y) \in H \text{ and } x \in F\}$. 

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Theorem 1. Let \((X, \xi)\) be a convergence space, \(A \subset X\), and let \(F\) be a filter on \(X\). Let \(D\) be a composable class of filters. The following are equivalent:

1. \(F\) is \(D\)-compact at \(A\);
2. For every convergence space \(Y\) and every compact \(D\)-filter \(G\) at \(B \subset Y\), the filter \(F \times G\) is \(D\)-compact at \(A \times B\);
3. For every \(D\)-based atomic \(^2\) topological space \(Y\), every \(y \in Y\) and every \(G\) such that \(y \in \lim Y\), the filter \(F \times G\) is \(F_1\)-compact at \(A \times \{y\}\).

Proof. (1 \(\Rightarrow\) 2).

Let \(D \in D(X \times Y)\) such that \(D \# (F \times G)\). The filter \(D^- (G) \in D(X)\) because \(G \in D(Y)\) and \(D\) is composable. Moreover \(D^- (G) \# F\) so that \(\text{adh}_X D^- (G) \cap A \neq \emptyset\).

Consequently, there exists a filter \(W\) with \(x \in \lim X W \cap A\) such that \(D \cap W \neq \emptyset\). Therefore \(D(W) \# G\) and \(\text{adh}_X D(W) \cap A \neq \emptyset\) by compactness of \(G\). In other words, there is a filter \(U\) with \(y \in \lim Y U \cap B\) such that \(U \# D(W)\). Consequently, \((x, y) \in \text{adh}_X Y D\) because \((W \times U) \# D\).

(2 \(\Rightarrow\) 3) is obvious.

(3 \(\Rightarrow\) 1).

Assume that \(F\) is not \(D\)-compact at \(A\). Then, there exists a \(D\)-filter \(D \# F\) such that \(\text{adh}_X D \cap A = \emptyset\). Chose any point \(x_0 \in X\) and let \(Y\) be a copy of \(X\) endowed with the atomic topology \(\tau\) defined by \(N_{\tau}(x_0) = D \cap \{x_0\}\). Then \(F \times N_{\tau}(x_0)\) is not \(F_1\)-compact at \(A \times \{x_0\}\).

Indeed, \(\{(x, x) : x \neq x_0\} \# F \times N_{\tau}(x_0)\) because \(D \# F\), but \(\text{adh}_{\xi X} \{(x, x) : x \neq x_0\} \cap (A \times \{x_0\}) = \emptyset\). For a filter on \(\{(x, x) : x \neq x_0\}\) is of the form \(G \times G\) and if \(x_0 \in \lim X G\) then \(G \cap D\), so that \(\lim X G\) is empty.

A relation \(R : X \to Y\) is called \(D\)-compact if \(R(F)\) is \(D\)-compact at \(RA\) whenever \(F\) is \(D\)-compact at \(A\). As observed in \(^3[13]\) section 10, preservation of closed sets by a map \(f : (X, \xi) \to (Y, \tau)\) is equivalent to \(F_1\)-compactness of the inverse map \(f^-\) when \((X, \xi)\) is topological, but not if \(\xi\) is a general convergence. More precisely, calling a map \(f : (X, \xi) \to (Y, \tau)\) adherent \(^4\) if

\[ y \in \text{adh}_\tau f(H) \implies \text{adh}_\xi H \cap f^- y \neq \emptyset, \]

we have:

Lemma 2. \(^5\)

1. A map \(f : (X, \xi) \to (Y, \tau)\) is adherent if and only if \(f^- : (Y, \tau) \to (X, \xi)\) is \(F_1\)-compact;
2. If \(f : (X, \xi) \to (Y, \tau)\) is adherent, then it is closed;
3. If \(f : (X, \xi) \to (Y, \tau)\) is closed and if adherence of sets are closed in \(\xi\) (in particular if \(\xi\) is a topology), then \(f\) is adherent.

\(^3[25]\) Theorem 13\(^6\) shows that a map \(f : X \to Y\) is \(D\)-perfect (that is, adherent with \(D\)-compact fibers) if and only if the inverse map \(f^- : Y \to X\) is \(D\)-compact.

Hence, applied for \(F = \{X\} = \{A\}\), Theorem\(^5\) rephrases as:

Corollary 3. Let \(D\) be a composable class of filters and let \(X\) be a convergence space. The following are equivalent:

1. \(X\) is \(D\)-compact;
2. for every \(D\)-based convergence space \(Y\), the projection \(p_Y : X \times Y \to Y\) is \(D\)-perfect;

\(^4\)A topological space with at most one non-isolated point is called atomic. Such spaces have been also called point-spaces and prime topological spaces.

\(^6\)www.encyclopediaofmath.org
Let $D$ be a composable class of filters. Let $X$ be a topological space. The following are equivalent:

1. $X$ is $D$-compact;
2. $X$ is $(\bigvee D)$-compact;
3. $X$ is $(\bigvee D)$-perfect;
4. $X$ is $D$-perfect.

Proof. (1 $\implies$ 2) because the fact that $\{X\} \times D$ is $D$-compact at $X \times \{y\}$ for every $D$-filter $D$ such that $y \in \lim Y \exists \bar{G}$ amounts to $D$-compactness of $p_Y : Y \rightrightarrows X \times Y$, which implies $D$-perfectness of $p_Y : X \times Y \rightarrow Y$.

(2 $\implies$ 3) by definition, and (3 $\implies$ 1) because if $p_Y : X \times Y \rightarrow Y$ is adherent for every $D$-based atomic topological space $Y$, then for every topological space $Y$, every $y \in Y$ and every $D$-filter $D$ that converges to $y$, the filter $\{X\} \times D$ is $F_1$-compact at $X \times \{y\}$. In view of Theorem 3 $\{X\}$ is compact, that is, $X$ is compact.

In particular, for a topological space $X$, $D$-compactness amounts to $(\bigvee D)$-compactness so that, in view of [25, Lemma 6], we get:

Corollary 4. Let $D$ be a composable class of filters. Let $X$ be a topological space. The following are equivalent:

1. $X$ is $D$-compact;
2. $X$ is $(\bigvee D)$-compact;
3. $X$ is $(\bigvee D)$-perfect;
4. $X$ is $D$-perfect.

A similar result [30, Theorem 1] has been obtained by J. Vaughan for topological spaces. He used nets instead of filters. To a class $\Omega$ of directed set $s$, we can associate a multifilter on $X$ as domain. For each element is a filter on the set of its immediate successors. Every $\Omega$-filter is a filter on the set of its immediate successors. A well-capped tree with least element is called a filter cascade if its every (non maximal) element is a filter on the set of its immediate successors.

A map $\Phi : V \setminus \{\varnothing\} \rightarrow X$, where $V$ is a filter cascade with the least element $\varnothing$, is called a multifilter on $X$. If $D$ is a class of filters, we call $D$-multifilter a multifilter with a cascade of $D$-filters as domain. For each $v \in V$, the subset $V(v)$ of $V$ formed by $v$ and its successors is also a cascade. The contour of $\Phi : V \setminus \{\varnothing\} \rightarrow X$ is defined by induction to the effect that $\int \Phi$ is the filter generated by $\varnothing$ on $\Phi(\max V)$ if $\Phi = \{\varnothing\} \cup \max V$, and

$$\int \Phi = \int \Phi \left(\int \Phi\right)$$

otherwise. With each class $D$ of filters we associate the class $\int D$ of all $D$-contour filters, i.e., the contours of $D$-multifilter. See [3] for details.

A filter $F$ is countably deep if $\bigcap A \in F$ whenever $A$ is a countable subfamily of $F$. \[\square\]
Corollary 5. (Mrówka-Kuratowski \cite{10} Theorem 3.1.16) The following are equivalent for a topological space $X$:

1. $X$ is compact;
2. $p_Y : X \times Y \to Y$ is perfect for every topological space $Y$;
3. $p_Y : X \times Y \to Y$ is closed for every topological space $Y$.

Corollary 6. (Noble \cite{27} Corollary 2.4) The following are equivalent for a topological space $X$:

1. $X$ is countably compact;
2. $p_Y : X \times Y \to Y$ is countably perfect for every subsequential (5) topological space $Y$;
3. $p_Y : X \times Y \to Y$ is closed for every first-countable topological space $Y$.

Corollary 7. (Noble \cite{27} Corollary 2.3) The following are equivalent for a topological space $X$:

1. $X$ is Lindelöf;
2. $p_Y : X \times Y \to Y$ is inversely Lindelöf for every topological $P$-space (6) $Y$;
3. $p_Y : X \times Y \to Y$ is closed for every topological $P$-space $Y$.

To a class $\mathcal{D}$ of filters, S. Dolecki associated in \cite{4} two fundamental concrete functors of the category of convergence spaces: a reflector $A_{\text{adh}}$ where

$$\lim_{\mathcal{D}} A_{\text{adh}} F = \bigcap_{D \in \mathcal{D} \setminus \mathcal{F}} \text{adh} D$$

and a coreflector $\text{Base}_{\mathcal{D}}$ where

$$\lim_{\mathcal{D}} \text{Base}_{\mathcal{D}} F = \bigcup_{D \in \mathcal{D} \leq \mathcal{F}} \lim D.$$

Applied to the case where $A$ is a singleton, Theorem 1 rephrases in convergence theoretic terms as follows.

**Theorem 8.** Let $\mathcal{D}$ be a composable class of filters and let $\xi$ and $\theta$ be two convergences on $X$. The following are equivalent:

1. $\theta \geq A_{\text{adh}} \xi$;
2. $\theta \times \text{Base}_{\mathcal{D}} \tau \geq A_{\text{adh}} (\xi \times \tau)$ for every convergence $\tau$;
3. $\theta \times \tau \geq P (\xi \times \tau)$ for every $\mathcal{D}$-based atomic topology $\tau$.

**Proof.** (1 $\implies$ 2). Let $x \in \lim_{\theta} \mathcal{F}$ and let $y \in \lim_{\tau} \mathcal{G}$ with $\mathcal{G} \in \mathcal{D}$. By assumption, $x \in \lim_{\text{adh}_{\mathcal{D}}} \mathcal{F}$; in other words, $\mathcal{F}$ is $\mathcal{D}$-compact at $\{x\}$ and $\mathcal{G}$ is compact at $\{y\}$. By Theorem 1, $\mathcal{F} \times \mathcal{G}$ is $\mathcal{D}$-compact at $\{(x, y)\}$, that is, $(x, y) \in \lim_{\text{adh}_{\mathcal{D}} (\xi \times \tau)} (\mathcal{F} \times \mathcal{G})$.

(2 $\implies$ 3) is obvious and (3 $\implies$ 1) follows from (3 $\implies$ 1) in Theorem 1. Indeed, if $x \in \lim_{\theta} \mathcal{F}$, then for every atomic topological space $(Y, \tau)$ and every $\mathcal{D}$-filter $\mathcal{G}$ that converges to $y$ in $Y$, $(x, y) \in \lim_{\mathcal{P} (\xi \times \tau)} (\mathcal{F} \times \mathcal{G})$, that is, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathcal{F}_1$-compact at $\{(x, y)\}$, so that $\mathcal{F}$ is $\mathcal{D}$-compact at $\{x\}$. Hence $x \in \lim_{\text{adh}_{\mathcal{D}}} \mathcal{F}$. □

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5 A topological space is sequential if every sequentially closed subset is closed and subsequential if it is homeomorphic to subspace of a sequential space.

6 A topological space is a $P$-space if every countable intersection of open subsets is open; equivalently if it is $\mathcal{F}_{\lambda, \omega}$-based.
The result above was essentially proved in [22, Theorem 7.1] but was not stated explicitly in [22].

Let $\mathcal{D}$ and $\mathcal{J}$ be two classes of filters. A convergence space is called $(\mathcal{J}/\mathcal{D})$-accessible if whenever $x$ is an adherent point of a $\mathcal{J}$-filter $\mathcal{F}$, there exists a $\mathcal{D}$-filter $\mathcal{D}$ which converges to $x$ and meshes with $\mathcal{F}$. S. Dolecki introduced the notion (under a different name) in [4] and noticed that when $\mathcal{D}$ is the class of countably based filters and $\mathcal{J}$ ranges over the classes of all, of countably deep, of countably based, of principal, of principal of closed sets filters, then $(\mathcal{J}/\mathcal{D})$-accessible topological spaces are exactly the bisequential [19], weakly bisequential [18], strongly Fréchet (countably bisequential in [20]), Fréchet and sequential spaces respectively. Additionally, he noticed that a convergence $\xi$ is $(\mathcal{J}/\mathcal{D})$-accessible if and only if $\xi \geq \text{Adh}_{\mathcal{J}}\text{Base}_{\mathcal{D}}\xi$.

In view of [25, Theorem 1], we obtain:

**Corollary 9.** Let $\mathcal{J} \subset \mathcal{D}$ be two classes of filters containing principal filters. Assume that a product of two $\mathcal{D}$-filters is a $\mathcal{D}$-filter. The following are equivalent:

1. $\xi$ is $(\mathcal{J}/\mathcal{D})$-accessible;
2. $\xi \times \tau$ is $(\mathcal{J}/\mathcal{D})$-accessible for every $\mathcal{J}$-based convergence space $(Y, \tau)$;
3. $\xi \times \tau$ is $(\mathcal{F}_1/\mathcal{D})$-accessible for every atomic $\mathcal{J}$-based topological space $(Y, \tau)$.

**Proof.** Notice that $\text{Base}_{\mathcal{F}_1} \geq \text{Base}_{\mathcal{D}}$ because $\mathcal{J} \subset \mathcal{D}$.

(1 $\implies$ 2): If $\xi \geq \text{Adh}_{\mathcal{J}}\text{Base}_{\mathcal{D}}\xi$ and $\tau = \text{Base}_{\mathcal{J}}\tau$, then

$$\xi \times \tau \geq \text{Adh}_{\mathcal{J}}\text{Base}_{\mathcal{D}}\xi \times \tau \geq \text{Adh}_{\mathcal{J}}\text{Base}_{\mathcal{D}}\xi \times \text{Base}_{\mathcal{D}}\tau = \text{Adh}_{\mathcal{J}}\text{Base}_{\mathcal{D}}(\xi \times \tau),$$

so that $\xi \times \tau$ is $(\mathcal{J}/\mathcal{D})$-accessible.

(2 $\implies$ 3) is clear because $\mathcal{F}_1 \subset \mathcal{J}$.

(3 $\implies$ 1) The convergence $\xi$ satisfies

$$\xi \times \tau \geq P\text{Base}_{\mathcal{D}}(\xi \times \tau) = P(\text{Base}_{\mathcal{D}}\xi \times \tau)$$

for every $\mathcal{J}$-based atomic topology $\tau$. By Theorem 8, $\xi \geq \text{Adh}_{\mathcal{J}}\text{Base}_{\mathcal{D}}\xi$. □

In particular, when $\mathcal{J} = \mathcal{D} = \mathcal{F}_\omega$, it shows the following generalization to convergence spaces of [20, Propositions 4.D.4 and 4.D.5]:

**Corollary 10.** A convergence space is strongly Fréchet if and only if its product with every first-countable convergence (equivalently, every atomic first-countable topological space) is strongly Fréchet (equivalently Fréchet).

An $\mathcal{F}_1$-based convergence is called finitely generated. Finitely generated topologies are often called Alexandroff topologies. When $\mathcal{J} = \mathcal{F}_1$ and $\mathcal{D} = \mathcal{F}_\omega$, Corollary 9 particularizes to

**Corollary 11.** [22] A topological (or convergence) space is Fréchet if and only if its product with every finitely generated convergence space (equivalently, Alexandroff topology) is Fréchet.

On the other hand, applying Theorem 1 for the image of a general filter under a relation, we obtain the following corollary for (possibly multi-valued) maps.

**Corollary 12.** Let $\mathcal{D}$ be a composable class of filters and let $R : X \rightrightarrows Z$. The following are equivalent:

1. $R$ is a $\mathcal{D}$-compact relation;
2. $R \times \text{Id}_Y : X \times Y \rightrightarrows Z \times Y$ is a $\mathcal{D}$-compact relation for every $\mathcal{D}$-based convergence space $Y$;
(3) $R \times \text{Id}_Y : X \times Y \Rightarrow Z \times Y$ is an $\mathbb{F}_1$-compact relation for every atomic $\mathcal{D}$-based topological space $Y$.

In view of [24, Theorem 13], the last result leads to:

**Corollary 13.** Let $\mathcal{D}$ be a composable class of filters, let $X$ be a topological space, and let $f : X \to Y$ be a surjective map. The following are equivalent:

1. $f$ is $\mathcal{D}$-perfect;
2. $f \times \text{Id}_W$ is $\mathcal{D}$-perfect for every $\mathcal{D}$-based convergence space $W$;
3. $f \times \text{Id}_W$ is $(\bigwedge \mathcal{D})$-perfect for every $(\bigwedge \mathcal{D})$-based topological space $W$;
4. $f \times \text{Id}_W$ is closed for every $\mathcal{D}$-based topological space $W$.

In particular, [25, Corollary 3.5 (iii), (iv), (v) and (vi)] are special cases:

**Corollary 14.** Let $X$ be a topological space, and let $f : X \to Y$ be a surjective map. The following are equivalent:

1. $f$ is perfect;
2. $f \times \text{Id}_W$ is perfect for every topological space $W$;
3. $f \times \text{Id}_W$ is closed for every topological space $W$.

**Corollary 15.** Let $X$ be a topological space, and let $f : X \to Y$ be a surjective map. The following are equivalent:

1. $f$ is countably perfect;
2. $f \times \text{Id}_W$ is countably perfect for every subsequential topological space $W$;
3. $f \times \text{Id}_W$ is closed for every first-countable topological space $W$.

**Corollary 16.** Let $X$ be a topological space, and let $f : X \to Y$ be a surjective map. The following are equivalent:

1. $f$ is inversely Lindelöf;
2. $f \times \text{Id}_W$ is inversely Lindelöf for every P-space $W$;
3. $f \times \text{Id}_W$ is closed for every P-space $W$.

Similarly, in view of [25, Theorem 14] stating that a map $f : (X, \xi) \to (Y, \tau)$ is $\mathcal{D}$-quotient if and only if $f : (X, f^{-}\tau) \to (Y, f\xi)$ is $\mathcal{D}$-compact [7], we obtain:

**Corollary 17.** Let $\mathcal{D}$ be a composable class of filters and let $f : X \to Y$ be a surjective map. The following are equivalent:

1. $f$ is $\mathcal{D}$-quotient;
2. $f \times \text{Id}_W$ is $\mathcal{D}$-quotient for every $\mathcal{D}$-based convergence space $W$;
3. $f \times \text{Id}_W$ is hereditarily quotient for every $\mathcal{D}$-based topological space $W$.

Notice that even if $X$ and $Y$ are topological, the final convergence $f\xi$ may not be. Therefore, $\mathcal{D}$-quotientness and $(\bigwedge \mathcal{D})$-quotientness are not equivalent. Special instances include the following:

**Corollary 18.** (Michael [19]) The following are equivalent for a surjective map $f : X \to Y$ :

1. $f$ is biquotient;
2. $f \times \text{Id}_W$ is biquotient for every convergence space $W$.

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7 $f\xi$ denotes the final convergence on $Y$ associated to $f : (X, \xi) \to Y$, that is, the finest convergence on $Y$ making $f$ continuous. Dually, $f^{-}\tau$ denotes the initial convergence on $X$ associated to $f : X \to (Y, \tau)$, that is, the coarsest convergence on $X$ making $f$ continuous.
(3) \( f \times \text{Id}_W \) is hereditarily quotient for every topological space \( W \).

**Corollary 19.** (Michael [20], Propositions 4.3 and 4.4) The following are equivalent for a surjective map \( f : X \to Y \):

1. \( f \) is countably biquotient;
2. \( f \times \text{Id}_W \) is countably biquotient for every first-countable convergence space \( W \);
3. \( f \times \text{Id}_W \) is hereditarily quotient for every first-countable topological space \( W \).

Weakly biquotient maps [18] coincide with \( \mathbb{F}_{\omega} \)-quotient maps so that when \( D = \mathbb{F}_{\omega} \), Corollary 14 specializes to:

**Corollary 20.** The following are equivalent for a surjective map \( f : X \to Y \):

1. \( f \) is weakly biquotient;
2. \( f \times \text{Id}_W \) is weakly biquotient for every \( \mathbb{F}_{\omega} \)-based convergence space \( W \);
3. \( f \times \text{Id}_W \) is hereditarily quotient for every topological \( P \)-space \( W \).

Finally, since a multivalued map \( R : X \rightrightarrows Y \) between two topological spaces is upper semicontinuous (usc) if and only if it is an \( F_1 \)-compact relation and compact-valued upper semicontinuous (usco) if and only if it is an \( F \)-compact relation, we have:

**Corollary 21.** Let \( R : X \rightrightarrows Y \) be a multivalued map between two topological space. Then

1. \( R \) is an usco map if and only if \( R \times \text{Id}_W : X \times W \rightrightarrows Y \times W \) is a usc map (equivalently usco map) for every topological space \( W \);
2. \( R \) is an usc map if and only if \( R \times \text{Id}_W : X \times W \rightrightarrows Y \times W \) is a usc map for every Alexandroff topological space \( W \).

3. **Products of \( D \)-compact filters**

In Section 2, \( D \)-compact filters are characterized as those filters whose product with every compact \( D \)-filters is \( D \)-compact. In this section, we consider the following related question: What are the filters whose product with every \( D \)-compact filter (of a given class \( J \)) is \( D \)-compact?

3.1. **Compactly meshable filters.** The question above was answered in [12], where a simplified version of the following notion was introduced:

A filter \( \mathcal{F} \) is \( M \)-compactly \( J \) to \( D \) meshable at \( A \), or \( \mathcal{F} \) is an \( M \)-compactly \( (J/D)_{\#} \)-filter at \( A \), if

\[
\mathcal{J} \in \mathcal{J}, \mathcal{J} \# \mathcal{F} \implies \exists D \in \mathcal{D}, D \# \mathcal{J} \text{ and } D \text{ is } M \text{-compact at } A.
\]

Before proceeding with applications, recall (see [25] for details) that the notion of an \( M \)-compactly \( (J/D)_{\#} \)-filter is instrumental not only in answering the question above but also in characterizing a large number of classical concepts. It generalizes the notions of total countable compactness in the sense of Z. Frolik [11] and more generally of total \( D \)-compactness in the sense of J. Vaughan [22] from sets to filters.

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8Let \( \mathcal{D} \) be a class of filters. A topological space is **totally \( \mathcal{D} \)-compact** if every \( \mathcal{D} \)-filter has a finer (relatively) compact \( \mathcal{D} \)-filter. It is easy to see that if \( \mathcal{D} \) is stable under finite supremum, then \( \{X\} \) is \( \mathbb{F} \)-compactly \( \mathcal{D} \) to \( \mathcal{D} \) meshable (at \( X \)) iff \( X \) is totally \( \mathcal{D} \)-compact. The notion of total countable compactness corresponds to \( \mathcal{D} = \mathbb{F}_{\omega} \).
Proposition 22. [25] Proposition 15] Let $\mathcal{D}$, $\mathcal{J}$ and $\mathcal{M}$ be three classes of filters, and let $\xi$ and $\theta$ be two convergences on $X$. The following are equivalent:

1. $\theta \geq \text{Adh}_\mathcal{J}\text{Base}_\mathcal{D}\text{Adh}_\mathcal{M}\xi$;
2. $\mathcal{F}$ is an $\mathcal{M}$-compactly $(\mathcal{J}/\mathcal{D})_\#$ filter at $\{x\}$ in $\xi$ whenever $x \in \lim_{\theta} \mathcal{F}$.

In particular, $\xi = \text{Adh}_\mathcal{M}\xi$ is $(\mathcal{J}/\mathcal{D})$-accessible if and only if $\mathcal{F}$ is an $\mathcal{M}$-compactly $(\mathcal{J}/\mathcal{D})_\#$-filter at $\{x\}$ whenever $x \in \lim \mathcal{F}$.

In view of [4], this means that Fréchet, strongly Fréchet, productively Fréchet, weakly bisequential, bisequential and radial topological spaces among others, can be characterized in terms of $\mathcal{M}$-compactly $(\mathcal{J}/\mathcal{D})_\#$-filters relative to a singleton, for various instances of $\mathcal{J}$, $\mathcal{D}$ and $\mathcal{M}$. Characterizations of Fréchet and strongly Fréchet spaces in terms similar in spirit to those in Proposition 22 were obtained in [9]. We take this opportunity to acknowledge that even though productively Fréchet spaces were not fully characterized in [5], important ideas and tools at work in [13] and [14] were already introduced in [5].

More generally, the notion is instrumental in characterizing a number of classes of maps. A relation $R : (X, \xi) \rightarrow (Y, \tau)$ is $\mathcal{M}$-compactly $(\mathcal{J}/\mathcal{D})$-mesurable if

$$\mathcal{F} \rightarrow x \implies R(\mathcal{F}) \text{ is } \mathcal{M}\text{-compactly } (\mathcal{J}/\mathcal{D})\text{-mesurable at } Rx \text{ in } \tau.$$

Theorem 23. [25] Theorem 16] Let $\mathcal{M} \subset \mathcal{J}$, let $\tau = \text{Adh}_\mathcal{M}\tau$ and let $f : (X, \xi) \rightarrow (Y, \tau)$ be a continuous surjection. The map $f$ is $\mathcal{M}$-quotient with $(\mathcal{J}/\mathcal{D})$-accessible range if and only if $f : (X, f^*\tau) \rightarrow (Y, f\xi)$ is an $\mathcal{M}$-compactly $(\mathcal{J}/\mathcal{D})$-mesurable relation.

A convergence $\xi$ is $P$-diagonal, if $\lim_\xi \mathcal{F} \subset \lim_\xi \int_\mathcal{F} \mathcal{V}_\xi(\cdot)$ for every filter $\mathcal{F}$. The notation $\text{adh}_\mathcal{J}^\omega(\mathcal{M}) \subset \mathcal{M}$ means that the filter generated by $\{\text{adh}_\mathcal{J}\mathcal{M} : \mathcal{M} \in \mathcal{M}\}$ is in the class $\mathcal{M}$ whenever $\mathcal{M}$ is.

Theorem 24. [25] Theorem 17] Let $\mathcal{M} \subset \mathcal{J}$ and $\mathcal{D}$ be three classes of filters, where $\mathcal{J}$ and $\mathcal{D}$ are $F_1$-composable. Let $\tau = \text{Adh}_\mathcal{M}\tau$ and let $\xi$ be a $P$-diagonal convergence such that $\text{adh}_\mathcal{J}^\omega(\mathcal{M}) \subset \mathcal{M}$ . Let $f : (X, \xi) \rightarrow (Y, \tau)$ be a continuous surjection. The map $f$ is $\mathcal{M}$-perfect with $(\mathcal{J}/\mathcal{D})$-accessible range if and only if $f^- : (Y, \tau) \rightarrow (X, \xi)$ is an $\mathcal{M}$-compactly $(\mathcal{J}/\mathcal{D})$-mesurable relation.

The following tables gather instances of these two results, for various classes $\mathcal{M}$, $\mathcal{D}$ and $\mathcal{J}$.

| $\mathcal{M}$ | $\mathcal{J}$ | $\mathcal{D}$ | map $f$ as in Theorem |
|--------------|--------------|--------------|------------------------|
| $F_1$        | $F_1$        | $F_1$        | hereditarily quotient with finitely generated range |
| $F_1$        | $F_1$        | $F_\omega$   | hereditarily quotient with Fréchet range |
| $F_1$        | $F_\omega$   | $F_\omega$   | hereditarily quotient with strongly Fréchet range |
| $F_1$        | $F_\omega$   | $F_1$        | hereditarily quotient with bisequential range |
| $F_\omega$   | $F_\omega$   | $F_1$        | countably biquotient with finitely generated range |
| $F_\omega$   | $F_\omega$   | $F_\omega$   | countably biquotient with strongly Fréchet range |
| $F_\omega$   | $F_\omega$   | $F_\omega$   | countably biquotient with bisequential range |
| $F_\omega$   | $F_\omega$   | $F_\omega$   | countably biquotient |
| $F_\omega$   | $F_\omega$   | $F_\omega$   | biquotient with bisequential range |
| $F_\omega$   | $F_\omega$   | $F_\omega$   | biquotient |
| $F_\omega$   | $F_\omega$   | $F_\omega$   | biquotient |
| $F_\omega$   | $F_\omega$   | $F_\omega$   | biquotient |
| $F_\omega$   | $F_\omega$   | $F_\omega$   | biquotient |
3.2. Main Product Theorem. The purpose of the remaining part of the paper is now to present applications of Theorem 26 below. It is formally more general and also considerably more complicated than [12, Theorem 1] in order to accomodate more applications. It is however based on the same idea. Moreover, similar results were obtained jointly with F. Jordan (Georgia Southern University) and I. Labuda (University of Mississippi) but were not kept in full generality in [12]. I thank them both for their contributions to this subsection.

**Lemma 25.** If $F$ is a compact filter (at $A$) on $X$ such that $MF \in M(Y)$ for every $M \in M(X \times Y)$, and $G$ is $M$-compact (at $B$), then $F \times G$ is $M$-compact (at $A \times B$).

**Proof.** Let $M$ be an $M$-filter such that $M \# F \times G$. Then $M \langle F \rangle \# G$ and $MF$ is an $M$-filter, so that there exists $U \# M \langle F \rangle$ such that $U \rightarrow y$. The filter $M^ \leftarrow (U)$ meshes with the compact filter $F$ and so there exists $W \# M^ \leftarrow (U)$ such that $W \rightarrow x$. Then $(x, y) \in \text{adh} M$. □

In particular, if $F = \{X\}$ and $G = \{Y\}$, it shows that the product of a compact space with a countably compact (resp. Lindelöf, pseudocompact) space is countably compact (resp. Lindelöf, pseudocompact).

**Theorem 26.** Let $D$ and $M$ be two composable classes of filters containing principal filters and let $J$ and $K$ be two $D$-composable classes of filters. Let $F \in K(X)$ and $A \subset X$. The following are equivalent:

1. $F$ is a $M$-compactly $(J/D)_#$-filter at $A \subset X$;
2. for every $Y$, every $B \subset Y$ and every $(K/J)_#$-filter $G$ which is a compactly $(D/M)_#$-filter at $B$, the filter $F \times G$ is an $M$-compactly $(D/D \times M)_#$-filter at $A \times B$;
3. for every $(D/M)$-accessible space $Y$, every $B \subset Y$ and every $J$-filter $G$ which is $D$-compact at $B$, the filter $F \times G$ is $(D \cap M)$-compact at $A \times B$;
4. for every $M$-based convergence space $Y$ and $y \in Y$, and for every $J$-filter $G$ which is $D$-compact at $\{y\}$, the filter $F \times G$ is $F_1$-compact at $A \times \{y\}$;
5. for every $M$-based (possibly non-Hausdorff) topological space $Y$ and $B \subset Y$, and for every $J$-filter $G$ which is $D$-compact at $B$, the filter $F \times G$ is $F_1$-compact at $A \times B$.

| $M$ | $J$ | $D$ | Map $f$ as in Theorem 24 |
|-----|-----|-----|--------------------------|
| $F_1$ | $F$ | $F_1$ | closed with finitely generated range |
| $F_1$ | $F_1$ | $F_1$ | closed with Fréchet range |
| $F_1$ | $F_\omega$ | $F_\omega$ | closed with strongly Fréchet range |
| $F_1$ | $F_\omega$ | $F_\omega$ | closed with bisequential range |
| $F_1$ | $F$ | $F$ | closed |
| $F_\omega$ | $F_\omega$ | $F_1$ | countably perfect with finitely generated range |
| $F_\omega$ | $F_\omega$ | $F_\omega$ | countably perfect with strongly Fréchet range |
| $F_\omega$ | $F_\omega$ | $F_\omega$ | countably perfect with bisequential range |
| $F$ | $F$ | $F_1$ | perfect with finitely generated range |
| $F$ | $F$ | $F_\omega$ | perfect with bisequential range |
| $F$ | $F$ | $F$ | perfect |
Proof: (1 $\implies$ 2) Let $D$ be a $D$-filter such that $D \# F \times G$. We can assume without loss of generality that $G \in J$. Indeed, $D(F) \in K$ because $K$ is $D$-composable, and $D(F) \# G$. Therefore, there exists a $J$-filter $G' \# D(F)$ and finer than $G$. Since $G' \geq G$, the filter $G'$ is a compactly $(D/M)\#$-filter at $B$. Moreover, $D \# (F \times G')$.

From now on, assume that $G \in J$. As $J$ is $D$-composable, $D^-(G)$ is a $J$-filter and $D^-(G) \# F$. Since $F$ is an $M$-completely $(J/D)\#$-filter at $A$, there exists $D$-filter $C \# D^-(G)$ which is $M$-compact at $A$. Now $D(C) \# G$ and $D(C)$ is a $D$-filter, so that there exists a filter $M$ in $M$ which is compact at $B$ and meshes with $D(C)$. By Lemma 27, $C \times M$ is an $M$-compact filter at $A \times B$ meshing with $D$ because $M$ is composable. Moreover, $C \times M \in D \times M$. Hence, $F \times G$ is an $M$-completely $(D(D \times M))\#$-filter at $A \times B$.

(2 $\implies$ 3) because a $D$-compact filter in a $(D/M)$-accessible space is compactly $(D/M)$ meshable and a $M$-completely $(D(D \times M))\#$-filter is also $(D \cap M)$-compact.

(3 $\implies$ 4) and (3 $\implies$ 5) are clear, as $F_1 \subset M \cap D$ and every $M$-based convergence space is $(D/M)$-accessible.

(4 $\implies$ 1). If $F$ is not $M$-completely $(J/D)\#$ at $A$, then there exists a $J$-filter $J^\# F$ such that for every $D$-filter $D^\# J$, there exists an $M$-filter $M_D^\# D$ such that $AdhM_D \cap A = \emptyset$. Pick $y_0 \in A$ and denote by $Y$ a copy of $X$ endowed with the atomic $M$-based convergence structure defined by $y_0 \in \lim G$ iff there exists $D^\# J$ such that $G \geq M_D \cap \{y_0\}$. Then $J$ is $D$-compact at $\{y_0\}$ in $Y$, but $F \times J$ is not $F_1$-compact at $A \times \{y_0\}$. Indeed, $\Delta = \{(x, x) : x \in X, x \neq y_0\} \subset X \times Y$ is in $F_1$ and $\Delta^\# (F \times J)$ because $F \# J$. But $Adh\Delta \cap A \times \{y_0\} = \emptyset$. Indeed, a filter on $\Delta$ can be assumed to be of the form $H \times H$. Now if $H$ converges to $\{y_0\}$ in $Y$, then $H \geq M_D$ so that $H$ cannot converge to $y_0 \in A$ in $X$, since $AdhM_D \cap A = \emptyset$.

(5 $\implies$ 1). In the argument (4 $\implies$ 1), consider instead of the convergence space $Y$, the $M$-based topological space $Y = X \uplus \{M_D : D^\# J, D \in D\}$. Then the filter $J$ generated by $J$ on $Y$ is $D$-compact at $B = \{M_D : D^\# J, D \in D\}$ but $F \times J$ is not $F_1$-compact at $A \times B$. Indeed, $\Delta = \{(x, x) : x \in X\} \subset X \times Y$ is in $F_1$ and $\Delta^\# (F \times J)$ because $F \# J$. But $Adh\Delta \cap A \times B = \emptyset$. Indeed, a filter on $\Delta$ is of the form $H \times H$. Now if $H$ converges to some point $\{M_D\}$ in $Y$, then $H \geq M_D$ and $H$ cannot converge to any point of $A$ in $X$, since $AdhM_D \cap A = \emptyset$.

From the viewpoint of convergence, there is no reason to distinguish between a sequence and the filter generated by the family of its tails. Therefore, in this paper, sequences are identified to their associated filter and we will freely treat sequences as filters. For instance, given a filter $M$, we consider the set $E(M) = \{(x_n)_{n \in \mathbb{N}} : (x_n)_{n \in \mathbb{N}} \geq M\}$ of free sequences finer than $M$ by applying this convention.

Lemma 27. Let $M$ be a filter on $X$. The filter $M$ admits a finer free sequence $(E(M) \neq \emptyset)$ if and only if for every family $(G_\alpha)_{\alpha \in I}$ of free filters on $X$ such that $M \geq \bigwedge_{\alpha \in I} G_\alpha$ there exists $\alpha_0 \in I$ and $(x_n)_{n \in \mathbb{N}} \geq G_0$ such that $(x_n)_{n \in \mathbb{N}} \# G_{\alpha_0}$. In particular, $M \# G_{\alpha_0}$.

Proof. Assume that there exists $(x_n)_{n \in \mathbb{N}} \geq M$ and that for every $\alpha \in I$, there exists $G_\alpha \in G_\alpha$ such that $G_\alpha \notin (x_n)_{n \in \mathbb{N}} \#$. Since $G_\alpha$ is free, there exists $G'_\alpha \in G_\alpha$ such that $G'_\alpha \cap \{x_n : n \in \mathbb{N}\} = \emptyset$. Then $\bigcup_{\alpha \in I} G'_\alpha \in \bigwedge_{\alpha \in I} G_\alpha$ but $\bigcup_{\alpha \in I} G'_\alpha \notin (x_n)_{n \in \mathbb{N}}$. Therefore $(x_n)_{n \in \mathbb{N}} \notin \bigwedge_{\alpha \in I} G_\alpha$.

The converse is obvious.

We can now give an alternative version of (2 $\implies$ 1) in Theorem 26.
Proposition 28. Let $\mathcal{M}$ be a class of filters such that $E(M) \neq \emptyset$ whenever $M \in \mathcal{M}$.

Assume that for every $(\mathcal{F}_J/\mathcal{M})$-accessible atomic topological space $Y$ and every $\mathcal{J}$-filter $\mathcal{F}$, which is compactly $D$ to $\mathcal{M}$ meshable at the non-isolated point $\{\infty\}$ of $Y$, the filter $\mathcal{F} \times \mathcal{J}$ is an $\mathcal{F}_1$-compactly $(\mathcal{F}_1/\mathcal{M})_#$-filter at $A \times \{\infty\}$. Then $\mathcal{F}$ is an $\mathcal{M}$-compactly $(\mathcal{J}/\mathcal{D})_#$-filter at $A$.

Proof. If $\mathcal{F}$ is not $\mathcal{M}$-compactly $(\mathcal{J}/\mathcal{D})_#$ at $A$, then there exists a $\mathcal{J}$-filter $\mathcal{J} \# \mathcal{F}$ such that for every $\mathcal{D}$-filter $\mathcal{D} \# \mathcal{J}$, there exists an $\mathcal{M}$-filter $\mathcal{M}_D \# \mathcal{D}$ such that $\text{adh}M_D \cap A = \emptyset$. Let $Y = X \oplus \{M_D : D \in \mathcal{D}, \mathcal{D} \# \mathcal{J}\}$ and denote by $\infty$ the point $\bigwedge \{M_D : D \in \mathcal{D}, \mathcal{D} \# \mathcal{J}\}$ of $Y$. Since infima of $\mathcal{M}$-filters are exactly $(\mathcal{F}_J/\mathcal{M})_#$-filters, $Y$ is an $(\mathcal{F}_J/\mathcal{M})$-accessible topological space $Y$. By definition, $\mathcal{J}$ is a compactly $(\mathcal{D}/\mathcal{M})_#$-filter at $\{\infty\}$, but $\mathcal{F} \times \mathcal{J}$ is not an $\mathcal{F}_1$-compactly $(\mathcal{F}_1/\mathcal{M})_#$-filter at $A \times \{\infty\}$. Indeed, $\Delta = \{(x, x) : x \in X\}$ meshes with $\mathcal{F} \times \mathcal{J}$ because $\mathcal{J} \# \mathcal{F}$ in $X$. An $\mathcal{M}$-filter on $\Delta$ is of the form $\mathcal{M} \times \mathcal{M}$ where $\mathcal{M} \in \mathcal{M}(X)$. Assume that $\mathcal{M} \times \mathcal{M} \rightarrow (x, \infty)$ in $X \times Y$. Then $\mathcal{M} \geq \bigwedge_{D \in \mathcal{D}} M_D$ because $\mathcal{M} \rightarrow \infty$. By Lemma 27 there exists a $\mathcal{D}$-filter $\mathcal{D} \# \mathcal{J}$ such that $\mathcal{M} \# \mathcal{M}_D$. Consequently, $x \notin A$ because $\text{adh}X M_D \cap A = \emptyset$. \hfill \Box

4. Further applications

4.1. Global properties. As observed in [12], the part $(1 \implies 3)$ of Theorem 26 applied to principal filters $\mathcal{F} = \{X\}$ and $\mathcal{G} = \{Y\}$, for various instances of $\mathcal{D} = \mathcal{J}$ and of $\mathcal{M}$ allows to recover results of J. Vaughan [32], and also to provide new variants. For instance:

Theorem 29. [12]

(1) The product of a countably compact space and a compactly $(\mathcal{F}_\omega/\mathcal{F}_\omega)$-meshable space is countably compact.

(2) The product of a strongly Fréchet countably compact space and a $\mathcal{F}_\omega$-compactly $(\mathcal{F}_\omega/\mathcal{F}_\omega)$-meshable space is countably compact.

For example, compact, sequentially compact, countably compact $k$-spaces are all examples of compactly $(\mathcal{F}_\omega/\mathcal{F}_\omega)$-meshable space and every countably compact space is a $\mathcal{F}_\omega$-compactly $(\mathcal{F}_\omega/\mathcal{F}_\omega)$-meshable space.

If $\mathcal{A} = \{A \subseteq X : (X, \mathcal{O}) \text{ is a convergence space}, A \text{ is open}, \exists O \in \mathcal{O}, A \subseteq O\}$. Accordingly, $\mathcal{O}(\mathcal{O}(\mathcal{D}))$ will denote the class of $\mathcal{D}$-filters $\mathcal{D}$ such that $\mathcal{D} = \{O(\mathcal{O}(\mathcal{D}))\}^\uparrow$. A topological space $X$ is feebly compact if and only if $\{X\}$ is $\mathcal{O}(\mathcal{F}_\omega)$-compact [1].

Theorem 30. [12]

(1) The product of a feebly compact space and a compactly $(\mathcal{O}(\mathcal{F}_\omega)/\mathcal{O}(\mathcal{F}_\omega))$-meshable space is feebly compact.

(2) The product of a $(\mathcal{O}(\mathcal{F}_\omega)/\mathcal{F}_\omega)$-accessible (in particular strongly Fréchet) feebly compact space and a $\mathcal{F}_\omega$-compactly $(\mathcal{O}(\mathcal{F}_\omega)/\mathcal{O}(\mathcal{F}_\omega))$-meshable space is feebly compact.

Theorem 31. [12]

(1) The product of a Lindelöf space and a compactly $(\mathcal{F}_\lambda/\mathcal{F}_\lambda)$-meshable space is Lindelöf.

A Tychonoff space is feebly compact if and only if it is pseudocompact.
The product of a weakly bisequential Lindelöf space and a $\mathbb{F}_\omega$-compactly $(\mathbb{F}/\mathbb{F}_\omega)$-meshable space is Lindelöf.

4.2. Local properties. Theorem 26 and Proposition 28 applied in the case of compactness at a singleton leads to the following.

**Theorem 32.** Let $\mathcal{D} \subset \mathcal{M}$ be two composable classes of filters containing principal filters and assume that there exists a sequence $(x_n)_{n \in \mathbb{N}} \geq \mathcal{M}$ whenever $\mathcal{M} \in \mathcal{M}$. The following are equivalent for a topological space $X$:

1. $X$ is $((\mathcal{D}/\mathcal{M})\# \geq \mathcal{D})$-accessible;
2. $X \times Y$ is $((\mathcal{D}/\mathcal{M})\# \geq \mathcal{D})$-accessible for every $(\mathcal{D}/\mathcal{M})\# \geq \mathcal{M}$-accessible topological space $Y$;
3. $X \times Y$ is $(\mathbb{F}_1/\mathcal{M})\#$-accessible for every $(\mathcal{D}/\mathcal{M})\# \geq \mathcal{M}$-accessible atomic topological space $Y$.

**Proof.** (1 $\implies$ 2). Let $x \in \lim_X \mathcal{F}$ and $y \in \lim_Y \mathcal{G}$. In view of Proposition 22, $\mathcal{F}$ is an $\mathcal{M}$-compactly $((\mathcal{D}/\mathcal{M})\# \geq \mathcal{D})\#$-filter at $\{x\}$ and $\mathcal{G}$ is a compactly $(\mathcal{D}/\mathcal{M})\# \geq \mathcal{D}$-filter at $\{y\}$ because $X$ is $((\mathcal{D}/\mathcal{M})\# \geq \mathcal{D})$-accessible and $Y$ is $(\mathcal{D}/\mathcal{M})$-accessible. Moreover, $\mathcal{G}$ can be assumed to be in $(\mathcal{D}/\mathcal{M})\# \geq \mathcal{D}$, which is a $\mathcal{D}$-composable class of filters.

By (1 $\implies$ 2) of Theorem 26 with $J = (\mathcal{D}/\mathcal{M})\# \geq \mathcal{D}$, $\mathcal{F} \times \mathcal{G}$ is a $\mathcal{M}$-compactly $(\mathcal{D}/\mathcal{M})\# \geq \mathcal{D}$-filter at $(\{x,y\})$. Hence, $X \times Y$ is $(\mathcal{D}/\mathcal{M})$-accessible.

(2 $\implies$ 3) is trivial.

(3 $\implies$ 1). Let $x \in \lim_X \mathcal{F}$. We use Proposition 28 (with $J = (\mathcal{D}/\mathcal{M})\# \geq \mathcal{D}$) to show that $\mathcal{F}$ is an $\mathcal{M}$-compactly $((\mathcal{D}/\mathcal{M})\# \geq \mathcal{D})\#$-filter at $\{x\}$, which will show that $X$ is $((\mathcal{D}/\mathcal{M})\# \geq \mathcal{D})$-accessible by Proposition 22. To this end, consider a $(\mathcal{D}/\mathcal{M})\# \geq \mathcal{D}$-filter $\mathcal{J}$ which is a compactly $(\mathcal{D}/\mathcal{M})\# \geq \mathcal{D}$-filter at $\{\infty\}$ where $\infty$ is the non-isolated point of an $(\mathbb{F}_1/\mathcal{M})\# \geq \mathcal{M}$-accessible atomic topological space $Y$. Notice that $\mathcal{J}$ is $\mathbb{F}_1$-compact at $\{\infty\}$, hence converges to $\infty$ in $Y$. Let $Y'$ be the (finer) atomic topological space obtained from $Y$ by letting $\mathcal{N}_{\mathcal{J}}(\infty) = \mathcal{J} \wedge \{\infty\}$.

The space $Y'$ is an atomic $(\mathcal{D}/\mathcal{M})\# \geq \mathcal{D}$-accessible topological space, so that $X \times Y'$ is an $(\mathbb{F}_1/\mathcal{M})\# \geq \mathcal{M}$-accessible topological space. Therefore $\mathcal{F} \times \mathcal{J}$ is an $\mathbb{F}_1$-compactly $(\mathbb{F}_1/\mathcal{M})\# \geq \mathcal{D}$-filter at $(\{x,\infty\})$. By Proposition 28, $\mathcal{F}$ is an $\mathcal{M}$-compactly $((\mathcal{D}/\mathcal{M})\# \geq \mathcal{D})\#$-filter at $\{x\}$. □

In particular if $\mathcal{D} = \mathcal{M} = \mathbb{F}_\omega$, we obtain:

**Corollary 33.** 13 The following are equivalent:

1. $X$ is productively Fréchet;
2. $X \times Y$ is strongly Fréchet for every strongly Fréchet topological space $Y$;
3. $X \times Y$ is Fréchet for every atomic strongly Fréchet topological space $Y$.

For $\mathcal{D} = \mathbb{F}_1$ and $\mathcal{M} = \mathbb{F}_\omega$, we obtain:

**Corollary 34.** 22 The following are equivalent:

1. $X$ is finitely generated (i.e., every point has a minimal neighborhood);
2. $X \times Y$ is Fréchet for every Fréchet topological space $Y$;
3. $X \times Y$ is Fréchet for every atomic Fréchet topological space $Y$.

4.3. Products of Maps. In view of Theorems 24 and 23 and of 25 Theorems 13 and 14, Theorem 26 has important consequences in terms of product of maps. More specifically:
Theorem 35. Let \( \mathbb{D} \) and \( \mathbb{M} \) be two composable classes of filters containing principal filters and let \( \mathbb{J} \) be a \( \mathbb{D} \)-composable class of filters. The following are equivalent for a relation \( R : X \to Y \):

1. \( R \) is an \( \mathbb{M} \)-compactly \( (\mathbb{J}/\mathbb{D}) \)-meshable relation;
2. for every compactly \( (\mathbb{D}/\mathbb{M}) \)-meshable relation \( G : Z \to W \) where the convergence space \( Z \) is \( \mathbb{J} \)-based, the relation \( R \times G : X \times Z \to Y \times W \) is an \( \mathbb{M} \)-compactly \( (\mathbb{D}/\mathbb{D} \times \mathbb{M}) \)-meshable relation;
3. for every \( \mathbb{D} \)-compact relation \( G : Z \to W \) where the convergence spaces \( W \) and \( Z \) are respectively \( (\mathbb{D}/\mathbb{M}) \)-accessible and \( \mathbb{J} \)-based, the relation \( R \times G : X \times Z \to Y \times W \) is \( \mathbb{D} \)-compact;
4. for every \( \mathbb{M} \)-based convergence space \( W \), the relation \( R \times Id : X \times \text{Base}_2 \text{Adh}_5 W \to Y \times W \) is \( \mathbb{F}_1 \)-compact;
5. for every map \( g : W \to Z \), where \( W \) is an \( \mathbb{M} \)-based topological space and \( Z \) is a \( \mathbb{J} \)-based atomic topological space, whose inverse relation \( g^- : Z \to W \) is \( \mathbb{D} \)-compact, the relation \( R \times g^- : X \times Z \to Y \times W \) is \( \mathbb{F}_1 \)-compact.

Proof. (1 \( \implies \) 2) Let \( x \in \lim_x \mathcal{F} \) and \( z \in \lim_z \mathcal{G} \). We can assume \( \mathcal{G} \) and hence \( G(\mathcal{G}) \) to be \( \mathbb{J} \)-filters. By assumption, \( G(\mathcal{G}) \) is a \( \mathbb{J} \)-filter that is compactly \( (\mathbb{D}/\mathbb{M}) \)-meshable at \( Gy \), and \( R(\mathcal{F}) \) is \( \mathbb{M} \)-compactly \( (\mathbb{J}/\mathbb{D}) \)-meshable at \( Rx \). By Theorem 26 (1 \( \implies \) 2), \( R(\mathcal{F}) \times G(\mathcal{G}) \) is \( \mathbb{M} \)-compactly \( (\mathbb{D}/\mathbb{D} \times \mathbb{M}) \)-meshable at \( Rx \times Gy \) in \( Y \times W \).

(2 \( \implies \) 3), (3 \( \implies \) 4) and (3 \( \implies \) 5) are obvious.

(4 \( \implies \) 1). In view of Theorem 26 it is sufficient to show that \( x \in \lim_x \mathcal{F} \) implies that \( R(\mathcal{F}) \times G \) is \( \mathbb{F}_1 \)-compact at \( Rx \times \{ w \} \) in \( Y \times W \) whenever \( G \) is a \( \mathbb{D} \)-compact at \( \{ w \} \) \( \mathbb{J} \)-filter, where \( W \) is an \( \mathbb{M} \)-based convergence space. Notice that \( w \in \lim_{\text{Base}_2 \text{Adh}_5} W G \). Therefore \( (R \times Id)(\mathcal{F} \times G) = R(\mathcal{F}) \times G \) is \( \mathbb{F}_1 \)-compact at \( Rx \times \{ w \} \) in \( Y \times W \) and the conclusion follows.

(5 \( \implies \) 1). In view of Theorem 26 it is sufficient to show that \( x \in \lim_x \mathcal{F} \) implies that \( R(\mathcal{F}) \times G \) is \( \mathbb{F}_1 \)-compact at \( Rx \times B \) whenever \( G \) is \( \mathbb{D} \)-compact at \( B \subset W \), where \( W \) is an \( \mathbb{M} \)-based topological space. For each such \( G \), consider the relation \( G_G : Z \to W \), where \( Z \) is the \( \mathbb{J} \)-based atomic topological space \( W \oplus \{ \} \), defined by \( G_G(w) = \{ w \} \) for every \( w \in W \) and \( G_G(\{ \} ) = B \). The filter \( G_G(\mathcal{G}) = G \) is \( \mathbb{D} \)-compact at \( B \subset W \) by construction, so that, by hypothesis, \( R(\mathcal{F}) \times G \) is \( \mathbb{F}_1 \)-compact at \( Rx \times B \) and the conclusion follows. Notice that the inverse relation is a map \( g_G \).

Theorem 35 (restricted to \( \mathbb{J} = \mathbb{F} \)) can be combined with Theorem 24 to the effect that:

Corollary 36. Let \( \mathbb{D} \) and \( \mathbb{M} \) be two composable classes of filters containing principal filters. Let \( f : X \to Y \) be a continuous surjection between two topological spaces. The following are equivalent:

1. \( f \) is \( \mathbb{M} \)-perfect with \( (\mathbb{F}/\mathbb{D}) \)-accessible range;
2. \( f \times g \) is \( (\mathbb{D} \cap \mathbb{M}) \)-perfect, for every \( \mathbb{D} \)-perfect map \( g \) with \( (\mathbb{D}/\mathbb{M}) \)-accessible domain;
3. \( f \times g \) is closed, for every \( \mathbb{D} \)-perfect map \( g \) with \( \mathbb{M} \)-based domain.

Notice that the statement corresponding to Theorem 35 (2) is omitted in Corollary 36. The reason is that the hypothesis \( \mathbb{M} \subset \mathbb{J} \) of Theorem 24 is in general not fulfilled so that this statement cannot be interpreted in terms of \( \mathbb{D} \)-perfect maps via Theorem 24. However, when \( \mathbb{D} = \mathbb{F} \), Theorem 35 (2) and (4) can be interpreted properly, leading to the following generalization of Corollary 14.
Corollary 37. Let \( \mathcal{M} \) be a composable classes of filters containing principal filters. Let \( f : X \to Y \) be a continuous surjection between two topological spaces. The following are equivalent:

1. \( f \) is \( \mathcal{M} \)-perfect;
2. \( f \times g \) is \( \mathcal{M} \)-perfect, for every perfect map \( g \) with \( (\mathcal{F}/\mathcal{M}) \)-accessible domain;
3. \( f \times g \) is closed, for every perfect map \( \mathcal{M} \)-based domain;
4. \( f \times \text{Id}_Y \) is closed for every \( \mathcal{M} \)-based topological space \( Y \).

The following table gathers the corresponding results. Conditions in parenthesis are equivalent to the condition given in the same cell.

| \( \mathcal{D} \) | \( \mathcal{M} \) | \( f \times g \) is | for every \( g \) | iff \( f \) is |
|---|---|---|---|---|
| \( \mathcal{F}_1 \) | \( \mathcal{F}_1 \) | closed | closed with finitely generated range | closed with finitely generated range |
| \( \mathcal{F}_\omega \) | \( \mathcal{F}_1 \) | closed | countably perfect with finitely generated range | closed with bisequential range |
| \( \mathcal{F}_1 \) | \( \mathcal{F}_\omega \) | closed with Fréchet range (first-countable domain) | countably perfect with finitely generated range |
| \( \mathcal{F}_\omega \) | \( \mathcal{F}_\omega \) | countably perfect with strongly Fréchet range (first-countable domain) | countably perfect with bisequential range |
| \( \mathcal{F}_1 \) | \( \mathcal{F} \) | closed | hereditarily quotient |
| \( \mathcal{F}_\omega \) | \( \mathcal{F} \) | countably perfect with finitely generated range (identity of finitely generated) | perfect with bisequential range |
| \( \mathcal{F}_1 \) | \( \mathcal{F}_1 \) | closed | perfect with finitely generated range (identity of first-countable) |
| \( \mathcal{F} \) | \( \mathcal{F}_1 \) | closed | perfect (identity map) |

Similarly, Theorem 35 (restricted to \( J = \mathcal{F} \)) can also be combined with Theorem 23 to the effect that (taking again into account the restrictions applying to Theorem 23):

Corollary 38. Let \( \mathcal{D} \) and \( \mathcal{M} \) be two composable classes of filters containing principal filters. Let \( f : X \to Y \) be a continuous surjection between two topological spaces. The following are equivalent:

1. \( f \) is \( \mathcal{D} \)-quotient with \( (\mathcal{F}/\mathcal{D}) \)-accessible range;
2. \( f \times g \) is \( (\mathcal{D} \cap \mathcal{M}) \)-quotient, for every \( \mathcal{D} \)-perfect map \( g \) with \( (\mathcal{D}/\mathcal{M}) \)-accessible domain;
3. \( f \times g \) is hereditarily quotient, for every \( \mathcal{D} \)-quotient map \( g \) with \( \mathcal{M} \)-based domain.

Corollary 39. Let \( \mathcal{M} \) be a composable class of filters containing principal filters. Let \( f : X \to Y \) be a continuous surjection between two topological spaces. The following are equivalent:

...
(1) \( f \) is \( \mathcal{M} \)-quotient;
(2) \( f \times g \) is \( \mathcal{M} \)-quotient, for every biquotient map \( g \) with \( (\mathcal{F}/\mathcal{M}) \)-accessible domain;
(3) \( f \times g \) is hereditarily quotient, for every biquotient map \( \mathcal{M} \)-based domain;  
(4) \( f \times \text{Id}_Y \) is hereditarily quotient for every \( \mathcal{M} \)-based topological space \( Y \).

| \( \mathcal{D} \) | \( \mathcal{M} \) | \( f \times g \) is | for every \( g \) | iff \( f \) is |
|-------------|-------------|------------------|-----------------|-----------------|
| \( F_1 \)   | \( F_1 \)   | hereditarily quotient | hereditarily quotient with finitely generated range | hereditarily quotient with finitely generated range |
| \( F_\omega \) | \( F_1 \) | hereditarily quotient | countably biquotient with finitely generated range | hereditarily quotient with biquotient range |
| \( F_1 \)   | \( F_\omega \) | hereditarily quotient | hereditarily quotient with Fréchet range (first-countable domain) | countably biquotient with finitely generated range |
| \( F_\omega \) | \( F_\omega \) | countably biquotient (hereditarily quotient) | countably biquotient with strongly Fréchet range (first-countable domain) | countably biquotient with biquotient range |
| \( F_1 \)   | \( F \)     | hereditarily quotient | hereditarily quotient | biquotient with finitely generated range |
| \( F \)     | \( F_1 \)   | hereditarily quotient | biquotient with finitely generated range (identity of finitely generated) | hereditarily quotient |
| \( F \)     | \( F \)     | countably biquotient (hereditarily quotient) | countably biquotient (hereditarily quotient) | biquotient with biquotient |
| \( F \)     | \( F_\omega \) | countably biquotient (hereditarily quotient) | countably biquotient with biquotient range (identity of first-countable) | countably biquotient |
| \( F \)     | \( F \)     | biquotient (hereditarily quotient) | biquotient (identity map) | biquotient |

4.4. **Coreflectively modified duality.** In a series of papers [9], [22], [24], [26] the author developed a categorical method to deal with topological product theorems, which relates product problems with properties of function spaces and commutation of functors with products. Applications of this method range from a unified treatment of a wide number of classical results [22], [24] to solutions of an old topological problem [21] on one hand, and of a problem of convergence theory (pertaining to Lindelöf and countably compact convergence spaces) [23] on the other hand. The key to concretely apply the abstract results of [22], [24], [26] is to internally characterize couples of convergences \((\xi, \theta)\) (on the same underlying set) satisfying

\[
\theta \times F\tau \geq G(\xi \times \tau),
\]

for every \( \tau \geq H\tau \) for specific instances of concrete endofunctors \( F, G \) and \( H \) of the category of convergence spaces and continuous maps.

In view of Proposition [22] Theorem [26] rephrases as follows when \( A \) is a singleton.

**Theorem 40.** Let \( \mathcal{D} \) and \( \mathcal{M} \) be two composable classes of filters containing principal filters and let \( \mathcal{J} \) be a \( \mathcal{D} \)-composable class of filters. The following are equivalent:

(1) \( \theta \geq \text{Adh}_\mathcal{J}\text{Base}_\mathcal{D}\text{Adh}_\mathcal{M}\xi \);
(2) \( \theta \times \text{Base}_D \text{Adh}_D \text{Base}_M \sigma \geq \text{Adh}_D \text{Base}_M \sigma (\xi \times \tau) \);

(3) for every \( \tau \geq \text{Adh}_D \text{Base}_M \tau \),
\[
\theta \times \text{Base}_D \text{Adh}_D \tau \geq \text{Adh}_D \text{Base}_M \sigma (\xi \times \tau) ;
\]

(4) for every \( \tau = \text{Base}_M \tau \),
\[
\theta \times \text{Base}_D \text{Adh}_D \tau \geq P (\xi \times \tau) .
\]

This generalizes \[22\], Corollary 7.2 and Proposition 7.3 (corresponding to the case \( J = F \) and \( D \subset M \)) whose important consequences are exposed in \[22\] and \[24\]. In particular, relationships between a topological (or convergence) space and the function spaces over it endowed with the continuous convergence \((10)\) can be deduced from Theorem 26. Beattie and Butzmann \[1\] call a pseudotopological space a Choquet space and call a space countably Choquet if a countably based filter converges to a point whenever all of its ultrafilter do. In other words, a convergence \( \xi \) is countably Choquet, or in our terminology countably pseudotopological, if \( \xi \leq \text{FirstS}_{\xi} \). More generally, we call \( J \)-pseudotopological a convergence satisfying \( \xi \leq \text{Base}_J \text{S}_{\xi} \) and \( J \)-paratopological a convergence satisfying \( \xi \leq \text{Base}_J \mathcal{P}_\omega \xi \).

Combining Theorem \[10\] and \[22\], Theorem 3.1, we get (for \( \theta = \xi \)) the following new characterizations of bisequentiality, strong and productive Fréchetness in terms of function spaces:

**Corollary 41.** Let \( D \) and \( M \) be two composable classes of filters containing principal filters and let \( J \) be a \( D \)-composable class of filters. A convergence \( \xi = \text{Adh}_M \xi \) is \((J/D)\)-accessible if and only if \( \text{Base}_D \text{Adh}_D \text{Base}_M [\xi, \sigma] \geq [\xi, \sigma] \) for every \( \sigma = \text{Adh}_D \sigma \) (equivalently for every pretopology \( \sigma \)).

In particular, when \( D = F_\omega \) and \( M = F \):

1. A pseudotopology \( \xi \) is bisequential if and only if the continuous convergence \([\xi, \sigma]\) is a paratopology for every paratopology (equivalently every pretopology) \( \sigma \);
2. A pseudotopology \( \xi \) is productively Fréchet if and only if \([\xi, \sigma]\) is \((F_\omega/F_\omega)\)\#\(-\)paratopological for every paratopology (equivalently every pretopology) \( \sigma \);
3. A pseudotopology \( \xi \) is strongly Fréchet if and only if \([\xi, \sigma]\) is countably paratopological for every paratopology (equivalently every pretopology) \( \sigma \).

This is a sample example. Many others can be found in \[22\] and \[24\].

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\(^{10}\)The continuous convergence \([\xi, \sigma]\) on the set of continuous functions from \((X, \xi)\) to \((Y, \sigma)\) is the coarsest convergence making the evaluation map jointly continuous. See \[1\] for details.
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