D-instantons and matter hypermultiplet

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Abstract

We calculate the D-instanton corrections (with all D-instanton numbers) to the quantum moduli space metric of a single matter hypermultiplet with toric isometry, in the effective N=2 supergravity arising in type-IIA superstrings compactified on a Calabi-Yau (CY) threefold of Hodge number $h_{2,1} = 1$. The non-perturbative quaternionic hypermultiplet metric is derived by resolution of a complex orbifold singularity, thus generalizing the known (Ooguri-Vafa) solution in flat spacetime to N=2 supergravity.
1 Introduction

D-instanton quantum corrections to the moduli space metric of a single matter hypermultiplet for the CY-compactified type IIA superstrings near a conifold singularity were investigated by Ooguri and Vafa [1]. They found an unique solution consistent with N=2 rigid supersymmetry and toric isometry. The solution [1] was interpreted as the infinite D-instanton sum coming from multiple wrappings of the Euclidean D-branes around the vanishing cycle [1]. The Ooguri-Vafa solution is given by the hyper-Kähler metric in the limit of flat 4d spacetime, i.e. when N=2 supergravity decouples, by taking the Planck mass to infinity. Including N=2 supergravity may lead to new physical phenomena at strong coupling [2], so that it is important to generalize the Ooguri-Vafa solution to the case of local N=2 supersymmetry.

One immediate consequence of the local N=2 supersymmetry is that the hypermultiplet moduli space metric is no longer hyper-Kähler but quaternionic [3]. In particular, the standard (Gibbons-Hawking) hyper-Kähler Ansatz [4] (it was one of the key technical tools in ref. [1]) cannot be applied to quaternionic metrics. Hence, the quaternionic generalization of the Ooguri-Vafa solution is a non-trivial problem.

In our earlier papers [5] we solved a similar problem in the case of the universal hypermultiplet that has a dilaton amongst its bosonic field components. The D-instanton corrections to the universal hypermultiplet moduli space metric were derived in ref. [5] by requiring, in addition to N=2 local supersymmetry and toric isometry, the $SL(2,\mathbb{Z})$ duality invariance that is not the case for a single matter hypermultiplet. The moduli space metric of the matter hypermultiplet is periodic (or T-selfdual) with respect to one of its variables [1] but it is not S-selfdual, so that another solution is needed.

The quaternionic constraints in the case of a single hypermultiplet are given by the (integrable) Einstein-Weyl system of non-linear partial differential equations. The key to the explicit construction of ref. [5] is the remarkable fact that any Einstein-Weyl metric with toric isometry is governed by the real pre-potential that is an eigenfunction of the Laplace-Beltrami operator $\Delta_{\mathcal{H}}$ in the hyperbolic plane $\mathcal{H}$ with the eigenvalue $3/4$ [6]. The four-dimensional hyper-Kähler metrics (including the Ooguri-Vafa metric [1]) with a triholomorphic isometry are known to be governed by harmonic functions in flat three dimensions (except some isolated points) [4]. In this Letter we also use yet another mathematical fact that the eigenfunctions of $\Delta_{\mathcal{H}}$ correspond to the homogeneous harmonic functions [7]. The solution [1] is formally obtained by applying T-duality to the ‘basic’ Green function in three dimensions. By using the Calderbank-Singer correspondence [7] we can lift the Ooguri-Vafa solution [1] to a
curved spacetime of N=2 supergravity by applying T-duality to the corresponding basic eigenfunction of $\Delta_H$. In the next sect. 2 we review the hyper-Kähler solution [1] that is the pre-requisite to our derivation of the corresponding quaternionic solution in sect. 3.

2 Ooguri-Vafa solution

The Ooguri-Vafa (OV) solution [1] describes the D-instanton corrected moduli space metric of a single matter hypermultiplet in N=2 four-dimensional superstrings compactified on a Calabi-Yau threefold of Hodge number $h_{2,1} = 1$, when both N=2 supergravity and the universal hypermultiplet are switched off, and five-brane instantons are suppressed. The corresponding low-energy effective action is given by the four-dimensional N=2 supersymmetric non-linear sigma-model that has the Ooguri-Vafa metric in its target space.

Rigid N=2 supersymmetry of the non-linear sigma-model requires a hyper-Kähler metric [8, 9]. The OV metric also has a (toric) $U(1) \times U(1)$ isometry by construction [1]. There always exist a linear combination of two commuting abelian isometries that is tri-holomorphic, i.e. it commutes with N=2 rigid supersymmetry [10].

Given any four-dimensional hyper-Kähler metric with a tri-holomorphic isometry $\partial_t$, it can always be written down in the standard (Gibbons-Hawking) form [4],

$$ds^2_{\text{GH}} = \frac{1}{V}(dt + \hat{\Theta})^2 + V(dx^2 + dy^2 + dz^2),$$

that is governed by linear equations,

$$\Delta V = \nabla^2 V \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)V = 0, \text{ almost everywhere,}$$

and

$$\hat{\nabla}V + \hat{\nabla} \times \hat{\Theta} = 0.$$ 

The one-form $\hat{\Theta} = \Theta_1 dx + \Theta_2 dy + \Theta_3 dz$ is fixed by the ‘monopole equation’ (3) in terms of the real scalar potential $V(x, y, z)$. Equation (2) means that the function $V$ is harmonic, with possible isolated singularities, in three Euclidean dimensions $\mathbb{R}^3$. The singularities are associated with the positions of D-instantons.

Given extra $U(1)$ isometry, after being rewritten in the cylindrical coordinates $(\rho = \sqrt{x^2 + y^2}, \theta = \arctan(y/x), \eta = z)$, the hyper-Kähler potential $V(\rho, \theta, \eta)$ becomes independent upon $\theta$. Equation (1) was used by Ooguri and Vafa [1] in their
analysis of the matter hypermultiplet moduli space near a conifold singularity. The conifold singularity arises in the limit of the vanishing period,
\[ \int_C \Omega \to 0 \ , \]
where the CY holomorphic 3-form $\Omega$ is integrated over a non-trivial 3-cycle $C$ of CY. The powerful singularity theory [11] can then be applied to study the universal behaviour of the hypermultiplet moduli space near the conifold limit.

In the context of the CY compactification of type IIA superstrings, the coordinate $\rho$ represents the ‘size’ of the CY cycle $C$ or, equivalently, the action of the D-instanton originating from the Euclidean D2-brane wrapped about the cycle $C$. The physical interpretation of the $\eta$ coordinate is just the expectation value of another (RR-type) hypermultiplet scalar. The cycle $C$ can be replaced by a sphere $S^3$ for our purposes, since the D2-branes only probe the overall size of $C$, as in ref. [1].

The pre-potential $V$ is periodic in the RR-coordinate $\eta$ since the D-brane charges are quantized [2]. This periodicity should also be valid in curved spacetime. We normalize the corresponding period to be 1, as in ref. [1]. The Euclidean D2-branes wrapped $m$ times around the sphere $S^3$ couple to the RR expectation value on $S^3$ and thus should produce the additive contributions to the pre-potential $V$, with the factor of $\exp(2\pi i m \eta)$ each.

In the classical hyper-Kähler limit, when both N=2 supergravity and all D-instanton contributions are suppressed, the pre-potential $V(\rho, \eta)$ of a single matter hypermultiplet cannot depend upon $\eta$ since there is no perturbative superstring state with a non-vanishing RR charge. Accordingly, the classical pre-potential $V(\rho)$ can only be the Green function of the two-dimensional Laplace operator, i.e.
\[ V_{\text{classical}} = -\frac{1}{2\pi} \log \rho + \text{const.} \ , \]
whose normalization is in agreement with ref. [1].

The calculation of ref. [1] to determine the exact D-instanton contributions to the hyper-Kähler potential $V$ is based on the idea [2] that the D-instantons should resolve the singularity of the classical hypermultiplet moduli space metric at $\rho = 0$. A similar situation arises in the standard (Seiberg-Witten) theory of a quantized N=2 vector multiplet (see, e.g., ref. [9] for a review).

Equation (2) formally defines the electrostatic potential $V$ of electric charges of unit charge in the Euclidean upper half-plane $(\rho, \eta)$, $\rho > 0$, which are distributed along the axis $\rho = 0$ in each point $\eta = n \in \mathbb{Z}$, while there are no two charges at the
same point [1]. A solution to eq. (2) obeying all these conditions is unique,

\[ V_{OV}(\rho, \eta) = \frac{1}{4\pi} \sum_{n=-\infty}^{+\infty} \left( \frac{1}{\sqrt{\rho^2 + (\eta - n)^2}} - \frac{1}{|n|} \right) + \text{const.} \quad (6) \]

After the Poisson resummation eq. (6) takes the singularity resolution form indeed [1],

\[ V_{OV}(\rho, \eta) = \frac{1}{4\pi} \log \left( \frac{\mu^2}{\rho^2} \right) + \sum_{m \neq 0} \frac{1}{2\pi} e^{2\pi \text{i} m \eta} K_0(2\pi |m| \rho) , \quad (7) \]

where the modified Bessel function \( K_0 \) of the 3rd kind has been introduced,

\[ K_s(z) = \frac{1}{2} \int_0^{+\infty} dt t^{s-1} \exp \left[ -\frac{z}{2} (t + \frac{1}{t}) \right] , \quad (8) \]

valid for all \( \text{Re} \, z > 0 \) and \( \text{Re} \, s > 0 \), while \( \mu \) is a constant.

By inserting the standard asymptotical expansion of the Bessel function \( K_0 \) near \( \rho = \infty \) into eq. (7) one finds [1]

\[ V_{OV}(\rho, \eta) = \frac{1}{4\pi} \log \left( \frac{\mu^2}{\rho^2} \right) + \sum_{m=1}^{\infty} \exp(-2\pi m \rho) \cos(2\pi m \eta) \times \]

\[ \times \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi n!} \Gamma(-n + \frac{1}{2})} \left( \frac{1}{4\pi m \rho} \right)^{n + \frac{1}{2}} . \quad (9) \]

A dependence upon the string coupling constant \( g_{\text{string}} \) is easily reintroduced into eq. (9) by a substitution \( \rho \to \rho / g_{\text{string}} \). The factors of \( \exp(-2\pi m \rho / g_{\text{string}}) \) in eq. (9) are just the expected multi-D-instanton contributions [1].

The OV pre-potential (6) has the form of the (regularized) T-sum over the T-duality transformations, \( \eta \to \eta + 1 \), applied to the ‘basic’ solution \( V_0 \equiv \frac{1}{4\pi \rho} \equiv \frac{1}{4\pi \sqrt{\rho^2 + \eta^2}} \) of eq. (2),

\[ V_{OV}(\rho, \eta) = A + \sum_T V_0(\rho, \eta) = A + \sum_T \frac{1}{4\pi \sqrt{\rho^2 + \eta^2}} , \quad (10) \]

where \( A \) is a constant. The basic solution \( V_0(\rho, \eta) \) is just the Green function of the three-dimensional Laplace operator \( \Delta \) in eq. (2).

### 3 D-instantons in N=2 supergravity

Any four-dimensional quaternionic manifold has the Einstein-Weyl geometry of negative scalar curvature [3],

\[ W^-_{abcd} = 0 \ , \quad R_{ab} = -\frac{\Lambda}{2} g_{ab} \ , \quad a, b, c, d = 1, 2, 3, 4 \ , \quad (11) \]
where $W_{abcd}$ is the Weyl tensor, $R_{ab}$ is the Ricci tensor of the metric $g_{ab}$, and the constant $\Lambda > 0$ is proportional to the gravitational coupling constant.

It is the theorem [6] that any four-dimensional Einstein-Weyl metric (of non-vanishing scalar curvature) with two linearly independent Killing vectors can be written down in the form

$$\begin{align*}
    ds^2_{\text{CP}} &= \frac{4\rho^2(F_{\rho}^2 + F_{\eta}^2) - F^2}{4F^2} \left( \frac{d\rho^2 + d\eta^2}{\rho^2} \right) \\
    &\quad + \frac{[(F - 2\rho F_{\rho})\hat{\alpha} - 2\rho F_{\eta}\hat{\beta}]^2 + [-2\rho F_{\eta}\hat{\alpha} + (F + 2\rho F_{\rho})\hat{\beta}]^2}{F^2[4\rho^2(F_{\rho}^2 + F_{\eta}^2) - F^2]},
\end{align*}$$

(12)
in some local coordinates $(\rho, \eta, \theta, \psi)$ inside an open region of the half-space $\rho > 0$, where $\partial_\theta$ and $\partial_\psi$ are the two Killing vectors, the one-forms $\hat{\alpha}$ and $\hat{\beta}$ are given by

$$\hat{\alpha} = \sqrt{\rho} d\theta \quad \text{and} \quad \hat{\beta} = \frac{d\psi + \eta d\theta}{\sqrt{\rho}},$$

(13)

while the whole metric (12) is governed by the real function (= hypermultiplet pre-potential) $F(\rho, \eta)$ obeying a linear differential equation,

$$\Delta_{\mathcal{H}} F = \rho^2 \left( \partial_\rho^2 + \partial_{\eta}^2 \right) F = \frac{3}{4} F .$$

(14)

Equation (14) is thus a consequence of 4d, local $\text{N}=2$ supersymmetry and toric isometry. It is highly non-trivial that the linear master equation (14) governs all $U(1) \times U(1)$-symmetric solutions to the highly non-linear Einstein-Weyl system (11).

Equation (14) means that the quaternionic pre-potential $F$ is a local eigenfunction (of the eigenvalue $3/4$) of the two-dimensional $\text{SL}(2, \mathbb{R})$ Laplace-Beltrami operator

$$\Delta_{\mathcal{H}} = \rho^2 (\partial_\rho^2 + \partial_{\eta}^2)$$

on the hyperbolic plane $\mathcal{H}$ equipped with the metric

$$ds_{\mathcal{H}}^2 = \frac{1}{\rho^2}(d\rho^2 + d\eta^2) , \quad \rho > 0 .$$

(16)

The ‘basic’ $\eta$-independent solutions to eq. (14) are given by

$$\rho^{-1/2} \quad \text{and} \quad \rho^{3/2} .$$

(17a)

Their linear combination gives some perturbative contributions to the hypermultiplet pre-potential [5]. The only $\eta$-dependent ‘basic’ solution to eq. (14) is given by [6, 7]

$$F_0(\rho, \eta) = \sqrt{\rho + \frac{\eta^2}{\rho}} .$$

(17b)
A linear combination of those solutions with T-shifts \( \eta \to \eta + n \) is known to describe a generic multi-instanton quaternionic metric via eqs. (12) and (14) [6, 5].

The basic eigenfunction (17b) is simply related to the harmonic (outside the origin) function \( V_0 \) (see the end of sect. 2),

\[
4\pi V_0(\rho, \eta) = \frac{\partial(\sqrt{\rho} F_0)}{\rho \partial \rho} .
\]

(18)

This is an example of the general correspondence between the harmonic functions \( V \) of homogeneity \( \alpha \) in \( \mathbb{R}^3 \) and the \( \Delta_{\mathcal{H}} \)-eigenfunctions \( F \) in \( \mathcal{H} \) of eigenvalue \( \alpha(\alpha - 1) \) [7].

Applying T-duality (i.e. summing up over the orbit with respect to the T-transformation \( \eta \to \eta + 1 \)) to the linear relation (18) yields

\[
V_{OV}(\rho, \eta) = A + \sum_T V_0(\rho, \eta) = \frac{\partial(\sqrt{\rho} \sum_T F_0)}{4\pi \rho \partial \rho} = \frac{\partial(\sqrt{\rho} F)}{4\pi \rho \partial \rho} .
\]

(19)

The quaternionic hypermultiplet potential \( F \), corresponding to the Ooguri-Vafa solution \( V_{OV} \), is thus given by eq. (19). A general solution to eq. (19) reads

\[
\sqrt{\rho} F(\rho, \eta) = 4\pi \int_0^\rho d\xi \xi [V_{OV}(\xi, \eta) + B] + f(\eta)
\]

(20)

with some function \( f(\eta) \), and a constant \( B \). Substituting eq. (7) into eq. (20) and using the identities

\[
\frac{d}{dx} [xK_1(x)] = -xK_0(x) , \quad \lim_{x \to 0} xK_1(x) = 1 , \quad \sum_{k=1}^\infty \frac{\cos(kx)}{k^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} ,
\]

(21)

we find

\[
\sqrt{\rho} F(\rho, \eta) = \tilde{f}(\eta) + \frac{\rho^2}{2} \ln \frac{\tilde{\mu}^2}{\rho^2} - \frac{2\rho}{\pi} \sum_{k=1}^\infty \frac{\cos(2\pi k\eta)}{k} K_1(2\pi k\rho) ,
\]

(22)

where we have redefined the integration function \( f(\eta) \) and the renormalization parameter \( \mu \) as

\[
\tilde{f}(\eta) = f(\eta) + \frac{1}{6} [1 + 6\eta(\eta - 1)] , \quad 1 + 4\pi B + \ln \mu^2 = \ln \tilde{\mu}^2 .
\]

(23)

In the perturbative region at large \( \rho \) the sum in eq. (22) represents the non-perturbative contributions with all D-instanton numbers, when using the asymptotical expansion of the \( K_1 \)-function at \( x \to \infty \),

\[
K_1(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \left[1 + \mathcal{O}(x^{-1})\right] ,
\]

(24)

in full similarity to ref. [1]. The first two terms in eq. (22) are perturbative contributions that are to be \( \eta \)-independent. Hence, the function \( \tilde{f}(\eta) \) is actually a constant,
\( \tilde{f} = C \). Our final result for the D-instanton-corrected pre-potential of the matter hypermultiplet moduli space metric is thus given by

\[
F(\rho, \eta) = \frac{C}{\sqrt{\rho}} + \frac{\rho^{3/2}}{2} \ln \frac{\mu^2}{\rho^2} - \frac{2\sqrt{\rho}}{\pi} \sum_{k=1}^{\infty} \cos(2\pi k \eta) \frac{K_1(2\pi k \rho)}{k}.
\]

The logarithmic factor in the second term of eq. (25) is apparently to be interpreted as the N=2 superstring (one-loop) renormalization effect.

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