New Static Solutions in $f(T)$ Theory

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Abstract

We consider the equations of motion of an anisotropic space-time in $f(T)$ theory, where $T$ is the torsion. New spherically symmetric solutions of black holes and wormholes are obtained with a constant torsion and the cases for which the radial pressure is proportional to a real constant, to some algebraic functions $f(T)$ and their derivatives $f_T(T)$, or vanish identically.

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1 Introduction

Through a Einstein’s proposal for finding a new version to General Relativity (GR) [1], an alternative gravitation theory, namely Teleparallel Theory (TT) has been embraced and abandoned later for many years. However, from the considerations made by Moller, the proposal for an analogy between TT and GR has been undertaken and developed once again [2]. The RG is a theory that describes the gravitation through the space-time curvature. Since a manifold may possess curvature and torsion, as Cartan spaces, one can separate all the terms coming from the torsion in the geometrical object as Riemann tensor, connection, etc... Hence, we mention that the theory that describes the gravitation as the action of space-time curvature, ie, coming from the Riemann tensor without torsion or without antisymmetric connection, can be similarly viewed as a theory that possesses only torsion and whose Riemann tensor without torsion vanishes identically.

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With the recent observational data about the evolution and the content of the universe, as the accelerated expansion, the existence of dark energy and dark matter, several new proposals for modifying the GR are being tested. Since the theories of unifications in low-energy scales have in their effective actions terms as $R^2$, $R^\mu\nu R_{\mu\nu}$ and $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, a candidate to the modification of the GR that agrees with the cosmological and astrophysical observational data is the $f(R)$ modified gravity [3, 4, 5]. The main problem which appears with the $f(R)$ theory is that the equations of motion are of the order 4, becoming more difficult to be analysed than the GR. Since the GR has an analogous Teleparallel Theory, it has been thought to use a theory called $f(T)$, with $T$ being the torsion scalar, which would be the analogous of the generalization of the GR, namely, the $f(R)$ theory. The $f(T)$ theory is a generalization of the teleparallel one as we shall see later and also does not possess curvature, nor Riemann tensor coming from the terms without torsion.

In cosmology, the $f(T)$ theory was used originally as a source driving the inflation [6]. Thereafter, it has been used as an alternative proposal for the acceleration of the universe, without requiring the introduction of dark energy [20, 7, 9, 11, 12]. In gravitation, the use of $f(T)$ theory started providing solutions for BTZ black holes [16]. Also, using $f(T)$ it is shown that the first thermodynamic law for black hole can be violated due to the lack of local Lorentz invariance [14]. Recently, spherically symmetric static solutions are searched in some $f(T)$ models of gravity theory with a Maxwell term [23], and the existence of relativistic stars in $f(T)$ modified gravity is examined, constructing explicitly several classes of static perfect fluid solutions [22].

In this paper, as well as in others methods of simplification of the equations of motion used in GR, for getting solutions with anisotropic symmetry as in [8], we propose to analyse the possibilities of obtaining new solutions in gravitation with $f(T)$ theory.

The paper is organized as follows. In the section 2, we will present a brief revision of the fundamental concepts of Weitzenbock’s geometry, the action of $f(T)$ theory and the equations of motions. In the section 3, we will fix the symmetries of the geometry and present the equations for the density, the radial and tangential pressures. New solutions of $f(T)$ theory are obtained in the section 4 for the cases in which the torsion is constant and the radial pressure vanishes. Finally, in the section 5, our conclusion and perspectives are presented.

2 The basic notions and the field equations

The Teleparallel Theory appeared to be equivalent to the GR [17, 18, 19], and so, we will introduce the basic concepts of the $f(T)$ theory. First, to avoid any confusion, let us define the notation of the Latin subscript as those related to the tetrad fields, and the Greek one related to the space-time coordinates.
The line element of the manifold is given by
\[ dS^2 = g_{\mu\nu} dx^\mu dx^\nu. \] (1)
This line element can be converted to a Minkowskian description by the matrix transformation called tetrad, as follows
\[ dS^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ij} \theta^i \theta^j, \] (2)
\[ dx^\mu = e^\mu_i \theta^i, \theta^i = e^i_\mu dx^\mu, \] (3)
where \( \eta_{ij} = \text{diag}[1, -1, -1, -1] \) and \( e^\mu_i e^i_\nu = \delta^\mu_\nu \) or \( e^\mu_i e^j_\mu = \delta^i_j \). The root of the metric determinant is given by \( \sqrt{-g} = \det[e^i_\mu] = e \). For a manifold in which the Riemann tensor part without the torsion terms is null (contributions of the Levi-Civita connection) and only the non zero torsion terms exist, the Weitzenbock’s connection components are defined as
\[ \Gamma^\alpha_{\mu\nu} = e^\alpha_i \partial_\nu e^i_\mu - e^\alpha_\mu \partial_\nu e^i_i. \] (4)
Defining the components of the torsion and the contorsion respectively as
\[ T^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} = e^\alpha_i (\partial_\mu e^i_\nu - \partial_\nu e^i_\mu) , \] (5)
\[ K^\mu^\nu^\alpha = -\frac{1}{2} (T^\mu^\nu^\alpha - T^\nu^\mu^\alpha - T^\mu^\alpha^\nu^\alpha) , \] (6)
and the components of the tensor \( S^\mu^\nu^\alpha \) as
\[ S^\mu^\nu^\alpha = \frac{1}{2} \left( K^\mu^\nu^\alpha + \delta^\nu_\alpha T^\beta^\mu^\beta - \delta^\mu_\alpha T^\beta^\nu^\beta \right) , \] (7)
one can write the torsion scalar as
\[ T = T^\alpha_{\mu\nu} S^\mu^\nu^\alpha. \] (8)
Now, similarly to the \( f(R) \) theory, one defines the action of \( f(T) \) theory being
\[ S[e^i_\mu, \Phi_A] = \int d^4x e \left[ \frac{1}{16\pi} f(T) + \mathcal{L}_{\text{Matter}} (\Phi_A) \right] , \] (9)
where we used the units in which \( G = c = 1 \) and the \( \Phi_A \) are matter fields. Considering the action \( S[e^i_\mu, \Phi_A] \) as a functional of the fields \( e^i_\mu \) and \( \Phi_A \), and vanishing the variation of the functional with respect to the field \( e^i_\mu \), one obtains the following equation of motion \[ \frac{S^\mu^\nu^\rho_{T}}{T_f T} + \left[ e^{-1} e^\mu_\rho \partial_\rho (e e^i_\mu S^\nu^\alpha) + T^\nu^\alpha S^\mu^\nu^\alpha \right] f_T + \frac{1}{4} \delta^\mu_\rho f = 4\pi T^\nu_\mu , \] (10)
where \( T^\nu_\mu \) is the energy momentum tensor. Let us consider that the matter content is an anisotropic fluid, such that the energy-momentum tensor is given by
\[ T^\nu_\mu = (\rho + p_t) u^\mu u^\nu - p_t \delta^\mu_\rho + (p_r - p_t) v^\mu v^\nu , \] (11)
where
\[ u^\mu, v^\mu, \phi \] are the fluid velocity, and \( \phi \) are the scalar fields.
where $u^\mu$ is the four-velocity, $v^\mu$ the unit space-like vector in the radial direction, $\rho$ the energy density, $p_r$ the pressure in the direction of $v^\mu$ (normal pressure) and $p_t$ the pressure orthogonal to $v_\mu$ (transversal pressure). Since we are assuming an anisotropic spherically symmetric matter, one has $p_r \neq p_t$, such that their equality corresponds to an isotropic fluid sphere.

In the next section, we will make some considerations for the manifold symmetries in order to obtain simplifications in the equation of motion and the specific solutions of these symmetries.

3 Specifying the geometry

Assuming that the manifold possesses a stationary and spherical symmetry, the metric can be written as

$$dS^2 = e^{a(r)} dt^2 - e^{b(r)} dr^2 - r^2 \left( d\theta^2 + \sin^2(\theta) d\phi^2 \right).$$  \hfill (12)

In order to re-write the line element (12) into the invariant form under the Lorentz transformations as in (2), we define the tetrad matrix as

$$[e^i_\mu] = \text{diag} \left[ e^{a(r)/2}, e^{b(r)/2}, r, r \sin(\theta) \right].$$ \hfill (13)

Using (13), one can obtain $e = \det [e^i_\mu] = e^{(a+b)/2} r^2 \sin(\theta)$, and with (4)-(8), we determine the torsion scalar and its derivatives in terms of $r$ as

$$T(r) = \frac{2e^{-b}}{r} \left( a' + \frac{1}{r} \right),$$ \hfill (14)

$$T'(r) = \frac{2e^{-b}}{r} \left( a'' - \frac{1}{r^2} \right) - T \left( b' + \frac{1}{r} \right),$$ \hfill (15)

where the prime (') denotes the derivative with respect to the radial coordinate $r$. One can now re-write the equations of motion (10) for an anisotropic fluid as

$$4\pi \rho = \frac{f}{4} \left( T - \frac{1}{r^2} - \frac{e^{-b}}{r} (a' + b') \right) \frac{f_T}{2},$$ \hfill (16)

$$4\pi p_r = \left( T - \frac{1}{r^2} \right) \frac{f_T}{2} - \frac{f}{4},$$ \hfill (17)

$$4\pi p_t = \left[ T + e^{-b} \left( \frac{a''}{2} + \left( \frac{a'}{4} + \frac{1}{2r} \right) (a' - b') \right) \right] \frac{f_T}{2} - \frac{f}{4},$$ \hfill (18)

$$\cot\theta = \frac{2}{2r^2} T' \frac{f_T}{2} = 0,$$ \hfill (19)

where $p_r$ and $p_t$ are the radial and tangential pressures respectively. In the Eqs. (16)-(18), we used the imposition (19), which arises from the non-diagonal components $\theta - r (2 - 1)$ of the equation of motion (11). This imposition does not appear in the static case of the GR, but making its use in (19), we get...
only the following possible solutions

\[ T' = 0 \Rightarrow T = T_0, \tag{20} \]

\[ f_{TT} = 0 \Rightarrow f(T) = a_0 + a_1 T, \tag{21} \]

\[ T' = 0, f_{TT} = 0 \Rightarrow T = T_0, f(T) = f(T_0), \tag{22} \]

which always relapse into the particular case of Teleparallel theory, with \( f(T) \) a constant or a linear function.

In the next section, we will determine new solutions for the \( f(T) \) theory making some consideration about the matter components \( \rho(r), p_r(r) \) and \( p_t(r) \).

4 Obtaining new solutions

Several works have been done in cosmology, modeling and solving some problems, using the \( f(T) \) theory as basis. Actually, in local and astrophysical phenomena, their is still slowly moving to obtain new solutions. Recently, Deliduman and Yapidzan [21] shown that it could not exist relativistic stars, such as that of neutrons and others, in \( f(T) \) theory, except in the linear trivial case, the usual Teleparallel Theory. However, Boehmer et al [22] showed that for the cases where \( T = 0 \) and \( T' = 0 \), there exists solutions of relativistic stars. In the same way, we would like to show some classes of spherically symmetric static solutions of the theory coming from \( f(T) \).

1. Let us start with the simple case, in which the torsion is constant \( T = T_0 \), which satisfies the imposition (20). In this case, considering the condition

\[ a''(r) = \frac{1}{r^2}, \tag{23} \]

which generalizes the case of Boehmer et al [22], where \( T = 0 \), for

\[ a'(r) = -\frac{1}{r} + c_0, \tag{24} \]

one gets

\[ e^{a(r)} = \left(\frac{r_0}{r}\right) e^{c_0 r}, \tag{25} \]

where \( c_0 \) is a real integration constant. From (25) and (14), one obtains

\[ e^{b(r)} = \left(\frac{2c_0}{T_0}\right) \frac{1}{r}. \tag{26} \]

It is important to note that from the equation (14) we can distinguish two interesting situations about \( T_0 \). Hence, \( c_0 > 0 \) implies that \( T_0 > 0 \), while \( c_0 < 0 \) leads to \( T_0 < 0 \). So, \( e^b \) is always positive in (20).
The line element (12) is given by
\[ dS^2 = \left( \frac{r_0}{r} \right) e^{c_0 r} dt^2 - \left( \frac{2c_0}{T_0} \right) \frac{dr^2}{r} - r^2 d\Omega^2 . \] (27)

Here, the condition \( r_0 > 0 \) determines the signature of the metric as diag\(+ - - -\). The energy density, the radial and tangential pressures can be obtained from the expressions (16)-(18) as
\[
\rho(r) = \frac{f(T_0) - f_T(T_0)T_0}{16\pi} - \frac{f_T(T_0)T_0}{8\pi c_0 r} + \frac{f_T(T_0)}{8\pi r^2} , \tag{28}
\]
\[
p_r(r) = \frac{f_T(T_0)T_0}{8\pi} - \frac{f(T_0)}{16\pi} - \frac{f_T(T_0)}{8\pi r^2} , \tag{29}
\]
\[
p_t(r) = \frac{5c_0 f_T(T_0)T_0}{64\pi} - \frac{f(T_0)}{16\pi} + \frac{f_T(T_0)T_0}{32\pi c_0 r} + \frac{c_0 f_T(T_0)T_0 r}{64\pi} . \tag{30}
\]

It is easy to observe through (28)-(30) that \( \rho(r) \), \( p_r(r) \) and \( p_t(r) \) diverge as \( r \) goes to zero, which means that there is a singularity at the origin. It appears here that the imposition (20) makes free the algebraic function \( f(T) \). Here, we confirm our position against that of Deliduman and Yapiskan [21], where there exists solutions of stars for \( f(T) \) theory, and (27) is one of them. This solution can only be analyzed in parts, for example, the exponential term in \( g_{00} \) term of type Yukawa potential) was obtained in [29] and related to the \( f(R) \) theory in [30].

On the other hand, we can choose another condition for the line element (12). Then, embracing the condition of quasi-global coordinate
\[ a'(r) = -b'(r) , \tag{31} \]
from (14), we get the differential equation
\[
(e^{-b})' + \left( \frac{1}{r} \right) (e^{-b}) - \frac{T_0}{2r} r = 0 , \tag{32} \]
whose the solution yields the line element
\[ dS^2 = \left( \frac{c_0}{r} + \frac{T_0}{6} r^2 \right) dt^2 - \left( \frac{c_0}{r} + \frac{T_0}{6} r^2 \right)^{-1} dr^2 - r^2 d\Omega^2 . \tag{33} \]

This is similar to Schwarzschild-(Anti)de Sitter’s solution when we fix \( c_0 = -2M \) and \( T_0 = -2\Lambda \). Wang [23] obtained similar solution, in which the unique difference is the fixation of the function \( f(T) = T = -2\Lambda \). The energy density, the radial and tangential pressures can be obtained from the expressions (16)-(18) as
\[
\rho(r) = \frac{f(T_0)}{16\pi} + \frac{f_T(T_0)}{8\pi r^2} - \frac{f_T(T_0)T_0}{8\pi} , \tag{34} \]
\[
p_r(r) = -\rho(r) , \tag{35} \]
\[
p_t(r) = \frac{f_T(T_0)T_0}{8\pi} - \frac{f(T_0)}{16\pi} . \tag{36} \]
In fact, we can show that this is the first solution for a wormhole in the $f(T)$ theory. Wormholes solutions also appear in the theory $f(R)$ [24].

Through (34) and (35), we see that the matter content solution is singular in $r = 0$. This solution seems to obey to a condition of dark energy, $p_r(r) = -\rho(r)$, but which is quite different, with $p_t$ constant and different from $p_r$, thus being anisotropic. In fact, we can show that this is the first solution of a wormhole for the $f(T)$ theory.

The conditions of existence of a wormhole are the following. First we define the metric (12) in terms of the proper length $l$ as

$$dS^2 = e^{a(r)} dl^2 - d\beta(r)\, d\Omega^2,$$  \hspace{1cm} (37)

where $a(r)$ is denoted redshift function, and through the redefinition $\beta(r) = r [1 - e^{-b(r)}]$, with $b(r)$ being the metric function given in (12), $\beta(r)$ is called shape function. Therefore, the conditions of existence of a traversable wormhole are [31]: a) the function $r(l)$ must possess a minimum value $r_0$ for $r$, which imposes $d^2 r(l)/dl^2 > 0$; b) $\beta(r_0) = r_0$; c) $a(r_0)$ has a finite value; and finally d) $\beta(r)/dr|_{r=r_0} \leq 1$.

For our solution (33), making $dr/dl = \sqrt{e^{-b(r)}} = 0$, we obtain the following minimum value $r_0 = \sqrt{-6c_0/T_0}$. The redshift function $a(r)$ has a finite value in $r_0$, and $\beta(r_0) = r_0$. Imposing the condition $d\beta(r)/dr|_{r=r_0} \leq 1$, one gets $T_0 \geq 0$, which implies that $c_0 < 0$ ($T_0 > 0$), for getting $r_0 > 0$. Therefore, the solution (33) is a traversable wormhole. This wormhole connects two non-asymptotically flat regions, which is asymptotically anti-de Sitter (AdS) for $T_0 > 0$. The wormholes solutions asymptotically (A)dS had been developed in [32]. The most interesting aspect in this solution is that the energy conditions are satisfied here. In General Relativity, wormholes solutions violate some energy conditions, but here, we get a solution that satisfies all energy conditions.

We show this as follows: From the equation (35), one gets $\rho(r) + p_r(r) = 0$, which satisfies the condition $\rho(r) + p_r(r) \geq 0$. From the equations (34) and (35), we obtain $\rho(r) + p_t(r) = f_T(T_0)/8\pi r^2$, which, for $f_T(T_0) \geq 0$, satisfies the condition $\rho(r) + p_t(r) \geq 0$. Through the equation (34), for $f(T_0) \geq 2T_0 f_T(T_0)$, always $\rho(r) \geq 0$, and for $f(T_0) < 2T_0 f_T(T_0)$, if $r_0 \leq r \leq r_1$, with $r_1 = \sqrt{2f_T(T_0)/[2T_0 f_T(T_0) - f(T_0)]}$, we obtain $\rho(r) \geq 0$. Thus, the weak energy condition (WEC) and the null energy condition (NEC) are satisfied.

In both solutions (27) and (33) presented here, for the constant torsion scalar, the value of $T_0$ and the functions $f(T_0)$ and $f_T(T_0)$ cannot be arbitrary as suggested by Boehmer [22].

2. The second case we present here is that for which the radial pressure (17) is taken to be identically null. This has been done originally in GR by Florides [25] and used later by Boehmer et al [15]. In
this case the equation (17) reads
\[ f(T) = 2f_T(T) \left( T - \frac{1}{r^2} \right). \] (38)
Assuming that the function \( f(T) \) is given for the imposition (21), we obtain
\[ T(r) = \frac{a_0}{a_1} + \frac{2}{r^2}. \] (39)

We distinguish the following two interesting sub-cases:

(a) For the case in which \( a(r) \) obeys \( \beta \), we calculate \( b(r) \) from the expression of the torsion scalar (14), yielding the line element
\[ dS^2 = \frac{r_0}{e^{\omega a}} dl^2 - \left( \frac{a_0 r}{2a_1 c_0} + \frac{1}{c_0 r} \right)^{-1} dr^2 - r^2 d\Omega^2. \] (40)
The energy density, radial and tangential pressures are given by
\[ \rho(r) = \frac{a_0}{16\pi} + \frac{a_1}{8\pi r^2} - \frac{a_0}{8\pi c_0 r}, \] (41)
\[ p_r(r) = 0, \] (42)
\[ p_t(r) = -\frac{3a_0}{64\pi} + \frac{a_0 c_0 r}{64\pi} + \frac{1}{32\pi} \left( \frac{a_0}{c_0} + a_1 c_0 \right) - \frac{a_1}{32\pi r^2}. \] (43)
This solution is clearly singular in \( r = 0 \). This solution is a traversable wormhole that connects two non asymptotically flat regions. Comparing (12), (37) and (40), we observe that for \( dr/dl = 0 \), the radial coordinate \( r \) possesses a minimum value \( r_1 = \sqrt{2a_1/a_0} \), which implies that \( \text{sign}(a_1/a_0) = -1 \). The redshift function \( a(r) \) has a finite value in \( r_1 \), and the shape function obeys \( \beta(r_1) = r_1 \). Imposing the condition \( d\beta(r)/dr|_{r=r_1} \leq 1 \), and as \( r_1 > 0 \), one gets \( c_0 < 0 \). As \( p_r(r) = 0 \), the energy conditions \( \rho(r) \geq 0 \) and \( \rho(r) + p_r(r) \geq 0 \) are the same, ie, are equivalent to \( \rho(r) \geq 0 \). Taking into account the both conditions \( \rho(r) \geq 0 \) and \( \rho(r) + p_t(r) \geq 0 \), we get two possibilities to satisfy these energy conditions: for \( a_0 > 0 \) \( (a_1 < 0) \), we must get \( 3|c_0| \leq r_1 \leq r_2 \), with \( r_2 = (1/|c_0|) \left( \sqrt{1 + 2a_1/a_0 c_0} - 1 \right) \); for \( a_0 < 0 \) \( (a_1 > 0) \), we must obtain \( 3|c_0| \leq r_1 \leq r < +\infty \).

(b) Considering now the condition (31), we obtain from (14),
\[ (e^{-b})^\prime + \left( \frac{1}{r} \right) (e^{-b}) - \frac{T}{2} (T(r) = 0, \] (44)
which, using (39), leads to the line element
\[ dS^2 = \left( 1 + \frac{c_0}{r} + \frac{a_0}{6a_1} r^2 \right) dl^2 - \left( 1 + \frac{c_0}{r} + \frac{a_0}{6a_1} r^2 \right)^{-1} dr^2 - r^2 d\Omega^2. \] (45)
This is similar to S-(Anti)dS solution when we fix \( c_0 = -2M \) and \( \frac{a_0}{2a_1} = -\Lambda \). Note that, once again, Wang [23] found a similar solution in which the difference is about the fixation of the
function $f(T) = T = -2\Lambda$, which is not the case here, because $T(r)$ in (39) depends on the radial coordinate and is not constant. The density, the radial and tangential pressures are identically zero

$$\rho(r) = p_r(r) = p_t(r) = 0 .$$  \hspace{1cm} (46)

This interior and regular solution is identified as the vacuum of matter, ie, $T^\nu_{\mu} = 0$ in (10). But as mentioned in [27], the frame in (12) represents a material vacuum but with a non-vanishing torsion scalar, given in (39). This is an aspect of the $f(T)$ theory, which is no longer invariant under local Lorentz transformations.

(c) Another way to regain these results of the previous sub-item is differentiating the equation (35) with respect to $T$, and considering (21), which results into

$$\frac{dr^2}{dT} = -\frac{1}{2} .$$  \hspace{1cm} (47)

Integrating (47) in the limits of $r_0$ to $r$ and $T_0$ to $T$, we get

$$T(r) = T_0 - \frac{2}{r_0^2} + \frac{2}{r^2} ,$$  \hspace{1cm} (48)

in which, choosing $T_0 = (2/r_0^2) + (a_0/a_1)$, the two previous cases are recovered.

3. Taking the case in which the radial pressure is a constant,

$$p_r(r) = p_r ,$$  \hspace{1cm} (49)

where $p_r \in \mathcal{R}$, with the imposition (21), we obtain the torsion scalar on the form

$$T(r) = \frac{16\pi p_r + a_0}{a_1} + \frac{2}{r^2} .$$  \hspace{1cm} (50)

Then, we have two possibilities

(a) When we use the condition (23), the line element is given by

$$dS^2 = \frac{r_0}{r}c_0^2 c_0 r^2 - \left[\frac{(16\pi p_r + a_0)}{2a_1 c_0} + \frac{1}{c_0 r^2}\right]^{-1} dr^2 - r^2 d\Omega^2 .$$  \hspace{1cm} (51)

The energy density, the radial and tangential pressures are given by

$$\rho(r) = \frac{a_0}{16\pi} + \frac{a_1}{8\pi r^2} - \frac{a_0}{8\pi c_0 r} - \frac{2p_r}{c_0 r} ,$$  \hspace{1cm} (52)

$$p_r(r) = p_r ,$$  \hspace{1cm} (53)

$$p_t(r) = -\frac{3a_0}{64\pi} + \frac{p_r}{4} + \frac{a_0 c_0 r}{64\pi} + \frac{1}{32\pi r} \left(\frac{a_0}{c_0} + a_1 c_0\right) - \frac{a_1}{32\pi r^2} + \frac{p_r}{2c_0} + \frac{c_0 p_r}{4r} .$$  \hspace{1cm} (54)

This solution is singular in $r = 0$, and generalizes the solution (40). When $p_r = 0$ in (51)-(54), we regain (40)-(43). This solution is a traversable wormhole that connects two non-asymptotically flat regions. The demonstration follows the same steps as in (40), substituting $a_0$ by $a_0 \rightarrow a_0 + 16\pi p_r$. 

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(b) For the condition (31), we have the equation (44), which integrated, yields the line element

\[ dS^2 = \left[ 1 + \frac{c_0}{r} + \frac{(16\pi p_r + a_0)^2}{6a_1} r^2 \right] dt^2 - \left[ 1 + \frac{c_0}{r} + \frac{(16\pi p_r + a_0)^2}{6a_1} r^2 \right]^{-1} dr^2 - r^2 d\Omega^2. \]  (55)

The density, the radial and tangential pressures are given by

\[ p_r(r) = -\rho(r) = p_t(r) = p_r. \]  (56)

This interior solution is regular for all values of the radial coordinate \( r \). It is similar to the S-(A)dS’s solution when \( c_0 = -2M \) and \( (16\pi p_r + a_0)/2a_1 = -\Lambda \). It also obeys some dark energy condition between the density, the radial and tangential pressures in (56). Our model starts considering an anisotropic spacetime with a constant radial pressure. Due to the choice of the quasi-global coordinate condition (31), the equations lead to the isotropization of the spacetime, where \( p_r = p_t \). This solution yields (45) - (46) for \( p_r = 0 \) in (55) - (56). For the case in which the solution looks like S-dS (or S-AdS), we will obtain, according to equation (50), a possibility of a change in the sign of the torsion scalar. The radial pressure (equal to the tangential one) should always be a negative constant for obtaining the positivity of the energy density \( \rho \) in (56).

4. A condition taken directly from the equations of motion (17), is when the radial pressure is proportional to the algebraic function \( f(T) \). For the particular case of constant radial pressure, we regain the imposition (20) or (22), treated in the previous subsection. Then, we have now to treat the case where \( f(T) \) obeys the imposition (21), for which, assuming the dependence of the radial pressure as

\[ p_r(r) = \frac{f(T)}{c_0}, \]  (57)

and using the equation (17), we get

\[ T(r) = \frac{a_0}{a_1} \left( \frac{c_0 + 16\pi}{c_0 - 16\pi} \right) + \frac{2c_0}{(c_0 - 16\pi)r^2}. \]  (58)

By considering the imposition (21) and integrating within the limits of \( r_0 \) to \( r \) and \( T_0 \) to \( T \), with \( T_0 = (a_0/a_1)[(c_0 + 16\pi)/(c_0 - 16\pi)] + [2c_0/(c_0 - 16\pi)r_0^2] \), the result (58) can be obtained using the condition (57) in the equation of motion (17), and differentiating with respect to the torsion scalar \( T \). We then have two possibilities
(a) For the condition (23), the solution is given by

\[ dS^2 = \frac{r_0}{r} e^{c_0 t} dt^2 - \left[ \left( \frac{c_0 + 16\pi}{c_0 - 16\pi} \right) \frac{a_0}{2a_1c_1} r + \frac{c_0}{c_1(c_0 - 16\pi)r} \right]^{-1} dr^2 - r^2 d\Omega^2 , \]  
\[ \rho(r) = \frac{a_0}{(16\pi - c_0)} \left( 1 - \frac{c_0}{16\pi} + \frac{1}{2c_1r} \left[ 2 + \frac{c_0}{8\pi c_1} \right] + \frac{a_1}{(16\pi - c_0)r^2} \left( 2 - \frac{c_0}{8\pi} \right) \right), \]  
\[ p_r(r) = \frac{2}{(c_0 - 16\pi)} \left( a_0 + \frac{a_1}{r^2} \right), \]  
\[ p_t(r) = \frac{a_0}{2(16\pi - c_0)} \left[ \frac{3c_0}{32\pi} - \frac{5}{2} - \frac{1}{c_1r} - \frac{c_0}{4\pi c_1 r} - \frac{c_1r}{2} \left( 1 + \frac{c_0}{16\pi} \right) \right] + \frac{a_1c_0}{32\pi(c_0 - 16\pi)r} \left( \frac{1}{r} - c_1 \right). \]

This solution is singular in \( r = 0 \). With the limit \( c_0 \to +\infty \) in (59)-(62), we regain the solution (40)-(43), where \( p_r = 0 \). This is a traversable wormhole that connects two non asymptotically flat regions. The demonstration follows the same steps as in (40).

(b) For the condition (31), the solution is given by

\[ dS^2 = \left[ \frac{c_0}{c_0 - 16\pi} + \frac{c_1}{r} + \frac{a_0}{6a_1(c_0 - 16\pi)} \right] dt^2 - \left[ \frac{c_0}{c_0 - 16\pi} + \frac{c_1}{r} + \frac{a_0}{6a_1(c_0 - 16\pi)} \right]^{-1} dr^2 - r^2 d\Omega^2 , \]  
\[ \rho(r) = -\frac{2a_0}{c_0 - 16\pi} - \frac{2a_1}{(c_0 - 16\pi)r^2}, \]  
\[ p_r(r) = -\rho(r), \]  
\[ p_t(r) = \frac{2a_0}{c_0 - 16\pi}. \]

This is a solution similar to that of S-(A)dS. The constants \( c_0, a_0 \) and \( a_1 \) should take values such that the positivity of the energy density is guaranteed in (64). This is another singular solution at \( r = 0 \). Again, for the limit \( c_0 \to +\infty \) in (63)-(66), we regain the solution (55)-(56), where \( p_r = 0 \).

5. Making use of the condition

\[ p_r(r) = \frac{\eta}{8\pi r^2} f_T(T), \]

where \( \eta \in \mathbb{R} \), the equation (17) leads to

\[ T(r) = \frac{a_0}{a_1} + \frac{2(1+\eta)}{r^2}. \]

This result can be obtained by replacing the condition (67) in the equation of motion (17), differentiating with respect to the torsion scalar \( T \), imposing (24) and integrating within the limits \( r_0 \) to \( r \) and \( T_0 \) to \( T \), with \( T_0 = (a_0/a_1) + [2(1+\eta)/r_0^2] \). Then, we have the following possibilities
(a) For the condition \((23)\), we get the following solution

\[
dS^2 = \frac{r_0}{r} e^{c_0 r} dt^2 - \left[ a_0 \left( \frac{1}{2} \frac{1}{c_0 r} \right) + \frac{a_1}{8 \pi r^2} \right] dr^2 - r^2 d\Omega^2,
\]
\[
\rho(r) = \frac{a_0}{8 \pi} \left( \frac{1}{2} \frac{1}{c_0 r} \right) + \frac{a_1}{8 \pi r^2},
\]
\[
p_r(r) = \frac{\eta a_1}{8 \pi r^2},
\]
\[
p_t(r) = \frac{a_0}{32 \pi} \left( - \frac{3}{2} \frac{1}{c_0 r} + \frac{c_0 r}{2} \right) + \frac{a_1(1 + \eta)}{32 \pi r} \left( c_0 - \frac{1}{r} \right).
\]

This is a singular solution in \(r = 0\). With \(\eta = 0\) in \((69)-(72)\), we re-obtain \((40)-(43)\), where \(p_r = 0\). This is a new traversable wormhole solution that connects two non asymptotically flat regions. This can be easily observed following the same steps as in \((40)\).

(b) For the condition \((31)\), we have the following solution

\[
dS^2 = \left[ (1 + \eta) + \frac{c_0}{r} + \frac{a_0}{6 a_1} r^2 \right] dt^2 - \left[ (1 + \eta) + \frac{c_0}{r} + \frac{a_0}{6 a_1} r^2 \right]^{-1} dr^2 - r^2 d\Omega^2,
\]
\[
\rho(r) = -p_r(r) = -\frac{\eta a_1}{8 \pi r^2}, \quad p_t(r) = 0.
\]

This solution resembles to that of S-(A)dS. Once again, we have a singular solution at \(r = 0\). When we make use of \(\eta = 0\) in \((73)-(74)\), we regain the solution \((45)-(46)\) for \(p_r = 0\). The energy density \((74)\) obeys the restriction \(\eta a_1 < 0\), for ensuring its positivity. But the relation \(p_r(r) = -\rho(r)\) in this case does not provide us a model like dark energy, due to \(p_t \neq p_r\), then being anisotropic.

5 Conclusion

The equations of motion for the \(f(T)\) theory are presented in section 2, showing a surprising result. The fact is that there is an off-diagonal equation for a static case with spherical symmetry. This was first shown by Boehmer et al. [22], and now confirming the conclusion made in [21] for a diagonal static metric. But Boehmer et al still make another statement that, in the case of non diagonal tetrad \(e^a_{\mu}\), the constraint equation \((19)\) no longer holds. Thus, the algebraic function \(f(T)\) and its derivative are arbitrary, only with the constraint of positivity on the energy density in \((16)\).

Through the consideration that the torsion of the Weitzenbock’s geometry is constant, we obtained new spherically symmetric static solutions for the \(f(T)\) theory, fixing a generalization of the case of Boehmer et al. [22], in [23], and a new assumption for which \(a'(r) = -b'(r)\). The solution which generalizes the case of Boehmer et al presents a signature such that the coordinates \(t\) and \(\tau\) are time-like and space-like respectively. There is other solution similar to the S-(Anti)dS’s one, obtained by Wang [23], for which the unique difference is fixing the function \(f(T) = T = -2\Lambda\). Note that here, this assumption does not
make free $f(T_0)$. In fact, this solution is a traversable wormhole which fixes the positivity of the torsion scalar and the algebraic function $f_T(T_0)$.

We also constructed four conditions that fix the matter content as a function of $r$. The first is when the radial pressure is identically zero, the second when it is a real constant, the third is when it is proportional to the algebraic function $f(T)$, and the last is when it is proportional to $f_T(T)$ multiplied by $\eta/8\pi r^2$. Through the coordinates conditions (23) and (31), we obtain new classes of static black holes solutions and wormholes to the $f(T)$ theory.

Using the simplification methods for the equations of motion in GR [25, 10, 13], we shown that it is possible to obtain new spherically symmetric static anisotropic solutions for $f(T)$ theory. With this, we hope that others conditions and symmetries reveal new solutions, even analogous to GR, as we see here, or also shown in [28].

An important point to be observed in this work is that, according to the equation (16), the torsion $T$, the functions $f(T)$ and $f_T$ cannot be arbitrary functions because conditions are required on these functions following the energy conditions in $f(T)$ gravity. We presented here a brief discussion about some particular wormholes solutions. However, this discussion could be made in more detail. We propose to present this aspect in a future work.

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