ON THE UNIVERSALITY OF THE ENTROPY-AREA RELATION

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Abstract. We present an argument that, for a large class of possible dynamics, a canonical quantization of gravity will satisfy the Bekenstein-Hawking entropy-area relation. This result holds for temperatures low compared to the Planck temperature and for boundaries with areas large compared to Planck area. We also relate our description, in terms of a grand canonical ensemble, to previous geometric entropy calculations using area ensembles.

1. Introduction

When formulating a new physical theory, especially one as speculative as quantum gravity, it is essential to have limiting behavior to test possible theories. It is also useful to understand how tight this limiting behavior constrains the theory. For quantum gravity, the semiclassical Bekenstein-Hawking relation provides one such limit. This paper is an exploration of the nature of the constraint due to the entropy-area relation. We find, in fact, that the restriction is extremely mild.

To make this argument, we use a model motivated by the formulation of geometry in loop quantum gravity. In this formulation, the states of the gravitational field are given by intersecting one-dimensional excitations of geometry called spin networks. States of a surface, such as an horizon, are determined by the quantum states at the intersections of the spin network and the surface. These intersections or “geometric particles” in turn determine the resulting statistical mechanics.

The current status of the theoretical development of loop quantum gravity (see Ref. 1 for a review) is that while the kinematics is relatively established, the dynamics remains controversial. One intriguing possibility is that the study a four dimensional formulations called spin foams may resolve the form of the Hamiltonian constraint. Another possibility is that the correct formulation of the Hamiltonian constraint could be settled with acceptable limiting behavior such as the Bekenstein-Hawking relation. The present work suggest that the entropy-area proportionality is largely insensitive to the microdynamics of the boundary theory. Instead, the relation is seen to be a consequence of the scaling property of a class of very simple statistical systems.

A number of previous papers have studied the Bekenstein-Hawking relation in loop quantum gravity. Early work 2, 3, 4 concentrated on computing the geometrical entropy associated with fixed total area. Later work 5, 6 provided a more careful analysis of the boundary conditions and the resulting intrinsic geometry on the horizon. In all these cases, the proportionality of entropy and area was recovered. The Bekenstein-Hawking relation could then be used to fix the undetermined parameter in the theory. In addition, it was found that the black hole entropy could be accounted for by the quantum states of the horizon geometry 6. More recent
work by the present authors [7] uses a definition of the quasilocal energy to explore the statistical mechanics of a region of space bounded by a surface. A precise form of the Hamiltonian on the boundary was used in that derivation. Unlike that work, in this paper we find that a wide class of possible Hamiltonians also recover the Bekenstein-Hawking relation.

Many properties of extremal and near extremal black holes have been derived in string theory. (See Ref. [8] for a review.) It would be interesting to explore the universality of the entropy-area relation in that context as well. However, we focus on the specific question, how tight does the Bekenstein-Hawking relation constrain the form of the boundary Hamiltonian? This question is more relevant to loop quantum gravity than string theory.

2. Geometric particles

We begin with a brief discussion of the statistical mechanics of a non-interacting ensemble of distinguishable particles. The assumption of distinguishability makes this system a bit unusual in that garden variety systems in statistical mechanics satisfy either Bose-Einstein or Fermi-Dirac statistics. This subsection also serves to fix notation. The results of this analysis will then be applied to our geometric particles in Section 2.2.

2.1. Distinguishable particles and low temperatures. Consider a system of distinguishable particles. With each state of definite energy, \( \nu \), we associate a probability proportional to the Boltzmann factor

\[
P_\nu \sim e^{-\beta (E_\nu - \mu N_\nu)},
\]

where \( E_\nu \) is the total energy and \( N_\nu \) is the particle number of the state. This assumption amounts to modeling the system as a grand canonical ensemble. We assume two characteristics of the Hamiltonian. First, the particle interactions are negligible. Second, the Hamiltonian has a discrete set of energy levels with increasing energies \( \{\epsilon_1, \epsilon_2, \ldots\} \). These levels may be degenerate, so we denote the respective (finite) degeneracies \( \{g_1, g_2, \ldots\} \). The total Hamiltonian for the system cleanly splits into a sum over the individual energies of the particles. Notice that we do not assume a precise form for the energy of the particles.

A state of the system \( \nu \) is uniquely specified (up to energy-level degeneracy) by the number of particles \( N_\nu \) and an ordered \( N_\nu \)-tuple of integers \( \{n_i\} \), where \( n_i \) is the state label of the \( i \)th particle (for \( i = 1, 2, \ldots, N_\nu \)). Thus, \( E_\nu = \sum_{i=1}^{N_\nu} \epsilon_{n_i} \). It is a simple matter to compute the partition function

\[
Z = \sum_{\{\nu\}} e^{-\beta (E_\nu - \mu N_\nu)} = \sum_{N=1}^{\infty} e^{\beta \mu N} \sum_{\{n_i\}} \prod_{i=1}^{N} g_{n_i} e^{-\beta \epsilon_{n_i}},
\]

\[
= \frac{f_z}{1 - f z},
\]

where

\[
f = \sum_{n=1}^{\infty} g_n e^{-\beta \epsilon_n}
\]

is the single particle partition function and \( z = e^{\beta \mu} \) is the fugacity. With \( Z \), we can compute expectation values for the total particle number, \( N \), and the occupation
number for the $k$th energy level, $N_k$

$$\langle N \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial \mu} = \frac{1}{1 - f z}$$

$$\langle N_k \rangle = -\frac{1}{\beta} \frac{\partial \ln Z}{\partial \epsilon_k} = \frac{g_k e^{-\beta \epsilon_k}}{f} \left( \frac{1}{1 - f z} \right).$$

Therefore the fraction of particles in the $k$th state is strictly a function of the temperature of the system (independent of $\mu$)

$$\frac{\langle N_k \rangle}{\langle N \rangle} = \frac{g_k e^{-\beta \epsilon_k}}{f}.$$ (6)

Since

$$\frac{\langle N_2 \rangle}{\langle N_1 \rangle} = \frac{g_2}{g_1} e^{-\beta (\epsilon_2 - \epsilon_1)},$$

it is easy to see that for sufficiently low temperatures (as determined by the natural scale of the theory), the system condenses into the ground state. This is the expected result; turning down the temperature causes particles to fall into their lowest energy states. What may be surprising is that there exists a theory for which all observed temperatures are in this regime. That theory is quantum gravity and we will see in the next section how this drives the proportionality between area and entropy.

The reader might wonder why we see any interesting behavior at all for such low temperatures, since the particles are in their ground states. However, since our system is a grand canonical ensemble, rather than merely canonical ensemble, the system may still exhibit interesting behavior in the form of particle number fluctuations (fluctuations in $N$). This may be derived with the relative dispersion relation

$$\frac{(\Delta N^2)}{\langle N \rangle^2} = 1 - \frac{1}{\langle N \rangle}.$$ (8)

Although the system condenses into the ground state fluctuations in particle number remain significant. At sufficiently low temperatures, the relative dispersion goes as $1 - 1/\langle N \rangle$ so that particle fluctuation is due to fluctuations in the number of particles in the ground state.

2.2. Entropy and Area. We now show how the statistical mechanics of geometric particles leads to the proportionality of entropy and area. To accomplish this we must characterize states of the system as observed from the exterior. Following \[2\], given a foliation of spacetime the horizon is a closed two dimensional surface. On account of the horizon, different states of the interior are regarded as physically indistinguishable from the outside. Thus, the quantum state of the system is specified entirely by the state of the boundary \[3, 4\]. Further, we consider distinguishable particles since local geometric fluctuations on the horizon relative to the exterior geometry are diffeomorphism invariant \[4\].

The natural scale of the theory is set by the Planck energy, $E_P \sim 10^{27} eV$, so that the natural temperature scale of the theory is the Planck temperature $T_P \sim 10^{32} K$. We previously found that condensation occurs when temperatures are “sufficiently low.” For quantum gravity this is $T \ll T_P$, a condition that is certainly satisfied.
for all observed temperatures. Since the spacing between the ground and excited state energy levels is on the order of the Planck energy, Eq. (7) tells us that

$$\frac{\langle N_2 \rangle}{\langle N_1 \rangle} \approx e^{-T_P/T}.$$  (9)

Temperatures commonly found in the universe are minuscule in comparison to the Planck temperature, so the ratio in Eq. (9) is, for all intents and purposes, zero. Thus, we come to the key result that all of the particles condense into the ground state; at the quantum gravity scale, the universe is such a chilly place that the geometry of space “freezes” into the lowest energy level.

It is straightforward to compute the quantities needed to check the entropy-area relation. For low temperatures, each particle rests in the ground state, so the average energy simply scales with

$$\langle N \rangle \langle E \rangle = \epsilon_1 \langle N \rangle,$$  (10)

for $T \ll T_P$.

Now let’s take a careful look at the area of the surface. Since these are non-interacting, geometric particles, each contributes independently to the area. Thus, for a state $\nu$ of the system $A_\nu = \sum_{i=1}^{N_\nu} a_i$, where $a_i$ is the area contribution of the $i$th particle. Upon condensation, each particle will be found with equal likelihood in one of the $g_1$ degenerate ground states. Note that each of these states could possibly yield a different area contribution (although, they certainly do not have to), so there could be anywhere from 1 to $g_1$ “area levels” within the ground state energy level. Let $\langle a \rangle_1$ be the mean area contribution among these $g_1$ states. Clearly, $\langle a \rangle_1$ will be a constant of the system, independent of temperature or particle number. We find that

$$\langle A \rangle = \langle a \rangle_1 \langle N \rangle,$$  (11)

so the area, too, scales with $\langle N \rangle$ at low temperatures. Our regime is quantum gravitational, so the constant of proportionality, $\langle a \rangle_1$, is approximately equal to the Planck area $l_P^2 \sim 10^{-70}$ m$^2$. The immediate consequence is that all systems with bounding area large compared with the Planck area – what we call “macroscopic” – satisfy $\langle N \rangle \gg 1$.

We now have the following exact result for the entropy

$$S = -k \sum_\nu P_\nu \ln P_\nu$$

$$= k [\beta \langle E \rangle - \mu \langle N \rangle + \ln(\langle N \rangle - 1)].$$  (12)

For temperatures much lower than the Planck temperature, this becomes

$$S \simeq k \ln g_1 \langle N \rangle + k \ln \langle N \rangle - k \ln \left(1 - \frac{1}{\langle N \rangle}\right)(\langle N \rangle - 1).$$  (13)

Or, rewriting this in terms of the expectation value of the area we have

$$S \simeq \frac{k \ln g_1}{\langle a \rangle_1} \langle A \rangle + k \ln \frac{\langle A \rangle}{\langle a \rangle_1} - k \ln \left(1 - \frac{\langle a \rangle_1}{\langle A \rangle}\right) \left(\frac{\langle A \rangle}{\langle a \rangle_1} - 1\right).$$

It is clear what happens for macroscopic bounding surfaces ($\langle A \rangle \gg \langle a \rangle_1$)

$$S \simeq \frac{k \ln g_1}{\langle a \rangle_1} \langle A \rangle + k \ln \frac{\langle A \rangle}{\langle a \rangle_1}.$$  (14)

1Since $\epsilon_k > \epsilon_2$ for $k > 2$, it follows that $\langle N_k \rangle/\langle N_1 \rangle = 0$ as well.
The proportionality between entropy and area is recovered! The first correction term is logarithmic. Note that the relation holds, irrespective of the form of the microscopic dynamics.

3. Conclusions

In summary, we have shown that for temperatures \( T \ll T_P \) and for macroscopic black holes \( A \gg l_P^2 \), a gas of geometric particles will exhibit the entropy-area proportionality (with logarithmic correction) irregardless of the details of the dynamics. This occurs when (i) the particles are distinguishable (ii) the particles are non-interacting.

We reach this result from the following steps:

- The quantum state of the system is specified entirely by the state of the boundary theory.
- With each state of definite energy, \( \nu \), we associate the “grand canonical” probability

\[
P_{\nu} \sim e^{-\beta(E_{\nu} - \mu N_{\nu})}.
\]

- Each particle is noninteracting and may occupy a number of discrete, possibly degenerate, energy levels with increasing energies.
- The difference in energy between the ground state and the first excited state is on the order of the Planck energy; that is, \( \epsilon_2 - \epsilon_1 \approx E_P \).
- The average area of the surface, \( \langle A \rangle \), is proportional to the average particle number, \( \langle N \rangle \). Since our regime is quantum gravitational, the constant of proportionality is approximately equal to the Planck area. Thus, any macroscopic system requires \( \langle N \rangle \gg 1 \).

The entropy-area result follows. Consider, now, a few things we are not assuming:

- We do not assume any particular form of the area and energy operators. The result is insensitive to the microstructure of the theory.
- Nor do we assume any relationship between the energy and area contributions of a single particle. Indeed, a degenerate energy level could even correspond to many different areas.

In regards to previous work, we note that Eq. (14) does not explicitly mention the Immirzi parameter \( \gamma \). In fact, it has merely been absorbed into the constant \( \langle a \rangle_1 \). If we use the appropriate values for the degeneracy and area coming from loop quantum gravity, we find that (leaving out the logarithmic correction term)

\[
S \simeq k \ln \frac{2}{4\pi \sqrt{3\gamma l_P^2}} \langle A \rangle,
\]

which is precisely what was found in the isolated horizons black hole calculation \( \mathcal{B} \). (We use \( l_P = \sqrt{\hbar G} \).) This is despite the fact that the calculations are based on an entirely different ensembles and statistical weights (although see below).

An interesting consequence of the condensation of particles into the lowest energy level is that our grand canonical probability factor becomes

\[
e^{-\beta(E - \mu N)} \simeq e^{-\beta(\epsilon_1 - \mu)N} = e^{-\alpha A}
\]

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\(^2\)This is unlike Ref. \( \mathcal{B} \) in which a precise relation was assumed.
where we used the low temperature condition in the first equality. So, in the limit \( T \ll T_P \) we recover the statistical weight used in the “area canonical ensemble” of Ref. [3]. Further,

\[
\alpha = \frac{\beta(\epsilon_1 - \mu)}{\langle a \rangle_1} \simeq \ln g_1/\langle a \rangle_1.
\]  

(18)

Thus, for macroscopic configurations the weight reduces to \( g_1 \exp(-A/\langle a \rangle_1) \).

The argument presented in this paper indicates that, although full quantum results must match semiclassical ones, using the entropy-area relation offers little guidance to the form of the microdynamics. There is a huge universality class of theories which reproduce the proportionality of entropy and area. Thus, there are many Hamiltonians which meet the very mild test of matching the entropy-area proportionality. Second, turning the argument around, since the \( S \propto A \) is true for a wide class of theories in the low temperature and large area (compared to the Planck scale) regime, a theory which merely matches this relation may not offer a glimpse into Planck scale physics.

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