Bifurcation calculation in population chaotic dynamics

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Abstract. Chaotic population dynamics present interesting behaviors that are the object of study for biologists, microbiologists, ecologists, epidemiologists and bioanalysts. With the proposal of the mathematician and biologist Pierre Verhulst on the logistic equation as a population discrete mathematical model, studies were initiated on chaotic dynamics regarding the growth factor immersed in the equation. When assigning different values to the factor, orbits are observed that are initially convergent, however, there are values for which there is no convergence due to the bifurcations suffered by the orbit. For values greater than or equal to 3 and less than 4, the forks bend, but not consecutively. Through the logistic equation and the identity function it is possible to find the values of the orbit corresponding to a specific value of the growth factor, however, these data are evidenced graphically, so it is important to know the explicit calculation of these bifurcations. The calculation of the fixed points of the quadratic iterator of the logistic equation is shown by equations.

1. Introduction
Population dynamics is a phenomenon that has been studied from various areas of research, biology, epidemiology, ecology, microbiology, bioanalysis and others related to health sciences. Due to the behavior of this type of dynamics, mathematics through fractal geometry and chaos theory, allow studies to be carried out in order to generate equations that allow the prediction and explanation of the behavior itself. In the basic university courses such as differential and integral calculus and differential equations, several simple models that represent population dynamics are evident, however, in these models it is observed that these are not faithful to behavior when the time variable is greatly increased, by For example, the exponential model presents a difficulty when time increases because at some point there will be a population value for which natural resources will not be sufficient. This problem is common in basic university courses and only in higher courses in some areas is it addressed differently. Because the study of population dynamics permeates not only areas of health, but also those of engineering, it becomes important to include in the basic courses the analysis of this phenomenon based on chaos theory. The development of the population model involves the data of the current population, as the dynamics change from this initial data. In addition to this data, it is important to establish the influence that the population has according to natural resources, environmental and sociodemographic conditions, etc., through a constant population growth. With these data it is possible to analyze the population phenomenon, with special interest in the population growth constant, because with its variation the percentage that we will have the current population at a given time is evidenced. The special interest is due to the fact that for some values of said constant, the percentage of the population may oscillate, but these values are periodic, that is, for some values of the population growth constant, population dynamics have bifurcations. Studying the limit values where the bifurcations are evidenced allows to
predict the behavior of a population with specific conditions. This paper shows the classical logistic equation in chaos theory and using Geogebra software, the dynamics are simulated obtaining the different orbits that represent the percentages of the population with respect to the initial when we vary the population growth constant. At the end it is shown that the forks are of order 2 and the Feigenbaum constant is calculated to show the regularity in the chaotic system.

2. Theoretical framework
Population dynamics is a phenomenon that has been studied from various areas of research, biology, epidemiology, ecology, microbiology, bioanalysis and others related to health sciences. Due to the behavior of this type of dynamics, mathematics through fractal geometry and chaos theory, allow studies to be carried out in order to generate equations that allow the prediction and explanation of the behavior itself. In a large number of countries in the world the population has increased so much that in approximately 40 years we will have twice the population that, at present, is to say that 14600 million human beings would inhabit the earth in 2050 [1]. In 2011 the world population reached 7000 million inhabitants; in 2025, as proposed by the United Nations, we will reach 8000 million. The twentieth century was the period of greatest population growth in history. In a century, from 1900 to 2000, the population of the planet went from approximately 1600 million to 6100 million humans. In a century, the population multiplied by 3.8 times [2]. Generating models that allow the prediction of the amount of population in a given place will depend equally on its initial population, even more so if the behavior is chaotic [3] and a constant that will determine growth or decay. This constant must also be able to determine or relate the sociodemographic context in which the dynamics are presented [4]. Chaos theory makes it possible to demonstrate a certain order in the unpredictable or irregular that a study may present, and also demonstrates the importance of sensitivity to initial conditions, of this the difficulty of extrapolating for prediction purposes [5]. In other words, chaos theory shows us how seemingly small and insignificant things can end if a leading role is assumed in the way they occur [6]. The set of points or situations to which a dynamic system changes over time is defined as attractors [3], some of them show fractal properties due to the self-similarity they present [7-12]. The analysis of population dynamic systems has been carried out with models based on differential equations, however, in some of them the support capacity of the population is not contemplated, which implies an uncontrolled increase of the population to the point of exhausting the possible Natural resources for survival.

3. Mathematical method
In basic engineering courses, for example, it is very common to observe in the derivative applications the population model whose solution is given by an exponential equation. In the course of differential equations, algebraic development is carried out to understand the solution of the differential equation, however, we only solve the differential equation and evaluate in the time variable some value to predict the population at that moment. Next, the work carried out in the basic calculation courses regarding population dynamics is shown. The initial population models presented in the basic courses are based on a separable differential equation of the first order. Equation (1) shows this differential equation, however, it presents an inconvenience because as the time increases considerably, the population will grow exponentially with the assumption that such quantity would not be supported by the planet.

\[
\frac{dN(t)}{N(t)} = r \, dt,
\]  

where \( N(t) \) represents the amount of population at time \( t \) and \( r \) the relative growth constant. Equation (2) is a solution to Equation (1), where \( N(0) \) represents the initial population.

\[
N(t) = N(0)e^{rt}.
\]
This model is very basic because immigration and emigration are excluded [2]. Because the population varies with time and is similarly conditioned by the change that the constant r can assume at any time, it is more useful to think of a compound expression where the new calculated population becomes the initial population and resume the process. It is necessary to define the variable x as the percentage of the population with respect to the current population and the constant k as the birth rate to generate the composite function under Equation (3).

\[ x_{n+1} = kx_n \circ x_{n+1} = f^n(x_0) \]  
\[ \text{to } f(x) = kx. \]  

Equation (3) is useful as long as populations have few inhabitants with specific non-ideal conditions [3]. From the above we observe that the proposed models offer conditioned solutions and evidence unreal values which do not allow the generation of confidence in them for decision making. In the didactic transposition is provided by examples that are of interest to those who wish to learn a technique or know different methods of solution to different problems, it is in this sense that it is important to include in these basic courses the development of methods and techniques that they are closer to reality, not only because of the interest of those who learn it, but because there is a practical utility to what they have learned, that is, that knowledge can not only be linked to follow a few steps of solution, but that technique may be applicable once it can be proposed to make significant changes to our environment. By including a new term it was possible that the model described in Equation 3 does not present unreal values once the time variable increases considerably. This new factor \((1 - x_n)\), was introduced to Equation (3) by Pierre Verhulst, which allowed the construction of the logistic equation, shown in Equation (4).

\[ x_{n+1} = kx_n (1 - x_n). \]  

In Equation (4), the terms \(x_n\) and \((1-x_n)\) are rivals in the way they act, since one tries to expand the population while the other tries to decrease it [1]. The constant \(k\), called population growth factor, is introduced to show the population's dependence on environmental conditions, such as the amount of food, predators, birth rate, etc [1]. With the inclusion of this new term given by Verhulst, it is observed that the analysis of this phenomenon is not linear or exponential but quadratic. Under this new consideration, the analysis of the population dynamics becomes important under the values that \(k\) can assume, observing the population change according to the conditions given in the population growth factor. For such an analysis it is necessary to support technological tools that allow the calculation of the results in a faster and more efficient way. Using the Geogebra software based on dynamic geometry, the orbit that shows the different percentage values of the initial population at a certain time is constructed. Interest increases once the dynamic becomes chaotic and several solutions to the problem are observed. Finding the relationship between solution values and chaos theory is the objective of this paper.

4. Results

The logistic equation is based on a deterministic factor such as the population growth described by the variable \(k\). In Figure 1, \(0 \leq k \leq 1\), regardless of the percentage with which the population starts, its behavior is predictable, and the population decreases until its extinction. For this particular case, 33% of the current population and a population growth constant of 0.6 were taken as seed or initial data, with which it is possible to notice that the population tends to extinction.

With the increase in the value of the population growth factor, it is observed that the population does not die out and also remains constant over time. The above occurs if the values of \(k\) range between 1 and 3, see Figure 2.

For the values of \(k\) one and less than three, the convergence in each of these values for the coefficient associated with population growth is noticeable, for example, for \(k = 2.8\) we have that the population value will be 0.642857, that is, approximately 64% of the current population. When the population
growth factor assumes values greater than 3, the phenomenon presents divergence, however, it is possible that, although it does not converge, there are values that oscillate and are assumed as solution values for the orbit of \( k \), for example, for the value of \( k = 3.62 \), the population percentage value ranges between the two values 0.513045 and 0.799455 of its initial value, see Figure 3.

**Figure 1.** Evolution of the initial population with 33% with 60 iterations for \( k = 0.6 \).

**Figure 2.** Evolution of the initial population with 33% with 60 iterations for \( k = 2.8 \).

**Figure 3.** Evolution of the initial population with 33% with 80 iterations for \( k = 3.62 \).
Due to the fractal nature of the diagram, such as that found in Figure 3, from values of $k$ greater than 3 there are periods $2, 4, 8, 16, \ldots, 2^n$, which indicates that periods are doubled. In 1975, the mathematician Feigenbaum calculated a constant that shows regularity in this chaotic system. The Feigenbaum constant $\delta$ (Equation (5)) is the number that satisfies:

$$\delta = \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 4.6692016091 \ldots,$$  \hspace{1cm} (5)

where $a_n = |c_{n-1} - c_n|$ and $c_n$ the points where the fork is born, then $\delta$ is the ratio of the scale [1]. From Figure 3, one might think that population dynamics presents bifurcations of order 2, that is, every time the value of $k$ increases, the amount of possible results of the percentage of the population increases as double the previous value of the factor. To analyze the behavior of the existing bifurcations, the fixed points of Equation (4) are observed and performing an iterative process converting this equation into a discrete function, we obtain an equation of order 4, Equation (6) [13].

$$-a^3x^4 + 2a^3x^2 - (a^2 + a^3)x + (a^2 - 1) = 0. \hspace{1cm} (6)$$

Equation (6) then becomes an equation of order 3 if we take as a solution $\frac{(a-1)}{a}$, which dividing it by the factor $-a^2$, it allows us to find the bifurcations of the function, Equation (7).

$$-a^3x^2 + (a^2 + a^3)x + (a^2 + a) = 0. \hspace{1cm} (7)$$

The solutions of Equation (7) are $x_h(a) = \frac{a+1+\sqrt{a^2-2a-3}}{2a}$ and $x_l(a) = \frac{a+1-\sqrt{a^2-2a-3}}{2a}$ since that equation is grade 2.

5. Conclusions

Learning through an appropriate didactic transposition allows the generation of proposals for the purpose of improving our environment, since knowledge must be permeated and evidenced in the cultural and social transformation of individuals, therefore, exemplify with models that present serious differences with reality, will leave the impression of only learning a technique to solve problems created for it and not to innovate.

Modeling population dynamics based on the quadratic model exposed by Verhulst, demands further study in the growth constant, since it contemplates the incidence of the environment, that is, the importance of natural resources, climatic changes, social behavior, birth rate. The above allows that, when analyzing this type of behavior, we can predict what happens to the population according to the resources it possesses. On the other hand, in the bifurcation we can observe that the population will undergo changes of which there is a type of behavior established under the oscillation of the solution values.

The dynamics of a population is observed from models that present chaotic behaviors, so that when the constant $k$ that represents the growth factor assumes values between 0 and 1, we have that the population tends to disappear, while when it is greater than 1 but less than 3, we observe convergence and it is possible to estimate the percentage of the population. For values greater than 3, bifurcations are observed, however, since this type of phenomena shows periodic behavior, it was possible to find a constant that can predict when chaos will occur in a system, the Feigenbaum constant.

The calculation of the bifurcations allows us to establish the reason why the value of the coefficient $k$ cannot exceed the upper boundary 4, because initially it starts from a function of order 4 and decomposes until its behavior can be found. Through these types of activities, it is possible to observe that there are more refined models that allow us to approach reality in order to carry out actions for continuous improvement, not only in the academic environment but also in formative research.
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References
[1] Rubiano G 2009 Iteración y Fractales (Bogotá: Universidad Nacional de Colombia)
[2] Ordorica M 2014 Recordando a Thomas Malthus y a Alfred Lotka Miscelánea Matemática de la Sociedad Matemática Mexicana (58) 53
[3] Braun E 2011 Caos, Fractales y Cosas Raras (México: Fondo de Cultura Económica) p 44
[4] Arboleda A 2016 La escuela bajo los preceptos de la teoría del caos: Incertidumbre, caos, complejidad, lógica difusa y bioaprendizajes Biocien (11) 91
[5] Rivero R, Ramírez M 1992 Caos: Definición, detección y ejemplos Rev. Desa. y Soc. (30) 189
[6] Briggs J, Peat D 1999 Las Siete Leyes del Caos (New York: Harper Collins Publisher)
[7] Almarza F 2002 La teoría del caos. Modelo de interpretación epistémica e instrumento de solución: Reconciliación entre ciencias y humanidades Rev. Univ. Arte. y Cult. Esc. Arte. Univ. Cent. Venez. (3) 107
[8] Mandelbrot B 2006 Los Objetos Fractales; Forma, Azar y Dimensión 6 (España: Metatemas)
[9] Barnsley M 2012 Fractals Everywhere (United States: Dover Publications)
[10] Mandelbrot B 2009 La Geometría Fractal de la Naturaleza 2 (España: Metatemas)
[11] Arenas G, Sabogal S 2011 Una Introducción a la Geometría Fractal (Bucaramanga: Ediciones UIS)
[12] Falconer K 2004 Fractal Geometry Mathematical Foundations and Applications (London: John Wiley & Sons)
[13] Peitgen H, Harmut J, Dietmar S 2004 Chaos and Fractals: New Frontiers of Science (New York: Springer)