Robust Estimation of Reflection Symmetry in Noisy and Partial 3D Point Clouds

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Abstract—Detecting the reflection symmetry plane of an object represented by a 3D point cloud is a fundamental problem in 3D computer vision and geometry processing due to its various applications such as compression, object detection, robotic grasping, 3D surface reconstruction, etc. There exist several efficient approaches for solving this problem for clean 3D point clouds. However, this problem becomes difficult to solve in the presence of outliers and missing parts due to occlusions while scanning the objects through 3D scanners. The existing methods try to overcome these challenges mostly by voting-based techniques but fail in challenging settings. In this work, we propose a statistical estimator for the plane of reflection symmetry that is robust to outliers and missing parts. We pose the problem of finding the optimal estimator as an optimization problem on a 2-sphere that quickly converges to the global solution. We further propose a 3D point descriptor that is invariant to 3D reflection symmetry using the spectral properties of the geodesic distance matrix constructed from the neighbors of a point. This helps us in decoupling the chicken-and-egg problem of finding optimal symmetry plane and correspondences between the reflective symmetric points. We show that the proposed approach achieves the state-of-the-art performance on the benchmarks dataset.

Index Terms—Reflection Symmetry, Point Clouds, Estimation, Optimization.

I. INTRODUCTION

The real-world objects exhibit various kinds of symmetry. For example, architectural sites and houses exhibit reflection and translational symmetry, human-made objects, such as vehicles, household objects, and furniture exhibit rotational and reflectional symmetry. Apart from human-made objects, most living objects such as humans, animals, and insects also exhibit reflection symmetry. The role of symmetry present in objects is to make objects physically balanced for smooth navigation and make objects visually attractive and beautiful. The fundamental problems in computer vision, computer graphics, and geometry processing mainly focus on processing and analyzing objects such as object detection and classification, physics-based photo-realistic rendering of objects, and 3D shape matching and texture transfer. Efficiently solving these problems requires a feature-level understanding of objects. The symmetry of objects plays a critical role in solving these problems [2], [3], [4], [5] and many other application problems such as reconstructing 3D model of face from a single image [6], estimating 3D pose [7], image matching and recognition [8], face expression classification [9], reconstructing 3D models of objects from a single image [10], [11], [12], spectral shape correspondences and texture transfer [13], [14], 3D registration [15], and image re-colorization [16].

The symmetry is classified into three fundamental categories: reflection, rotation, and translation. Also, depending on the amount of occlusion and noise present while acquiring a digital representation of an object, we can classify the symmetry as approximate or exact symmetry, partial or complete symmetry, and single or multiple symmetries. Also, the symmetry can be classified further as extrinsic symmetry (rigid objects such as a building) or intrinsic symmetry (dynamic objects such as animals and humans) based on the distance preserved (Euclidean or Geodesic distance, respectively) under self-isometry transformation. An object can be represented in many formats such as digital images, 3D point clouds, triangle Meshes, implicit surfaces, etc. The problem of detecting various types of symmetries in different forms of object representations has been an active problem of research in the vision and graphics communities [2], [3], [11].

In this work, we propose an algorithm find the global reflection symmetry plane of an object represented as a 3D point cloud acquired through a 3D scanner. We solve this problem in the presence of noisy points and missing parts of the objects due to occlusions or out of the field of view capture while scanning the objects. In literature, there exist many efficient algorithms for detecting exact and complete symmetry of objects represented as 3D point clouds [11]. However, detecting approximate and partial symmetry is an open problem [11] and lacks a generalized formulation for detecting this form of symmetry. Most of the existing methods follow voting-based strategies for handling outliers and missing parts but fail to detect symmetry in the presence of a large number of outliers.

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of outlier points and large missing parts [17], [18], [19]. Also, these methods construct a symmetry affinity matrix between the points that become intractable for large point clouds.

The state-of-the-art approaches [20], [21], [22] pose the problem of detecting the plane of symmetry as an optimization problem. They use the L2-norm as a metric to find the residuals and assume that the set of correspondences between the reflective symmetric points does not contain many outliers correspondences. However, in practice, the set of correspondences between the mirror reflective points may contain many outliers correspondences due to sensor noise, missing parts and non-ideal feature descriptors. Also, the approaches proposed in [21], [20], [22] used iterative closest point (ICP) based approaches where the plane of symmetry and correspondences between symmetric points are found simultaneously using an iterative algorithm. Since the L2-norm is not robust to outliers, the efficiency of these methods degrades in challenging settings. In this work, we propose a statistical estimator for the plane of reflection symmetry that is robust to outliers and missing parts efficiently. The idea is to use $L_2E$ estimator proposed in [23] that has been used (24), [25], [26], [27]) to solve many other problems in computer vision and computer graphics. The penalty curve for the L2 norm is quadratic and assigns high probabilities even for outliers. Hence, its performance is affected by the outliers. Whereas the $L_2E$ estimator assigns very low probabilities for many of the residuals that help it be efficient in the presence of outliers. We further decouple the problem of finding the correspondences and the plane of symmetry to make the proposed approach computationally efficient. We first propose an approach to estimate the set of putative correspondences between mirror symmetric points. We then propose an approach to find reflection symmetry invariant descriptors for points and then match them using an approximate nearest neighbor search approach. We use the spectral properties of the geodesic distance matrix on the local neighborhood of each point to describe a point. Since geodesics are invariant to self-isometry [28], the proposed descriptors are reflection symmetry invariant. Then, we pose the problem of finding an estimator for the plane of reflection symmetry as an optimization problem on a unit 2-Sphere. We show that the proposed approach achieves state-of-the-art performance on the benchmark dataset [1]. In Figure 1 we shown an example result.

Contributions: Our main contributions are the following.

1) A 3D point descriptor that is invariant to reflection symmetry transformation. For this, we use the spectral properties of the geodesic distance matrix on the local neighborhood of each point.

2) A statistical estimator for the reflection symmetry plane that is robust to significant number of outliers and missing parts due to occlusions.

3) We formulate the problem of finding the estimator as an optimization problem on 2-Sphere and then solve it using an optimization on manifolds technique.

The remaining sections of the papers are organized as follows. In Section II, we report the relevant research available in the literature and their limitations towards solving the symmetry detection problem efficiently. In Section III we describe the proposed algorithm for detecting reflection symmetry. In Section III-B we define a manifold optimization problem for finding an estimator for the plane of reflection symmetry that is robust to outliers and missing parts. In Section III-C we describe each point of the point cloud using a proposed descriptor that is reflection symmetry invariant descriptor. In Section IV we describe the evaluation procedure and the evaluation for measuring the performance of the proposed approach on the benchmark dataset.

II. RELATED WORK

The extrinsic reflection symmetry detection problem in 3D objects represented by triangle meshes and 3D point clouds is a hot topic of research in the computer vision and graphics and 3D geometry processing community due to its various algorithmic advantages in terms of computational complexity, geometric representation, and applications [29], [2]. In computer graphics and geometry processing literature, there exist many automatic algorithms for detecting the plane of reflection symmetry.

The approach proposed in Zabrodsky et al. [5] detect the reflection symmetry plane in a 2D point set. However, it needs a set of correct correspondences between mirror symmetric points computed in advance. The approaches proposed by Lipman et al. [18] and Xu et al. [40] find mirror-symmetric points using surface normals. They follow a voting-based approach and find symmetry orbits but need to tune various hyper-parameter. Also, for the approach proposed in [18], finding the symmetry factored embedding matrix becomes computationally intractable for larger size point clouds. Nagar and Raman used an optimization-based approach to find the plane of reflection symmetry in 3D point clouds with the provable guarantee of convergence of their method with a proper initialization [20]. However, this approach requires solving an integer linear program that becomes computationally intractable for large point clouds and does not detect partial symmetry. The method proposed in [31] uses multiple viewpoints of a 3D model to detect the plane of reflection symmetry. Whereas our approach can detect symmetry using only one viewpoint. The algorithm proposed by Combés et al. [32] detects reflection symmetry in a given 3D point cloud with a mild presence of outliers. This approach requires many hyper-parameters to be tuned, and it is computationally intractable as it requires solving an expectation maximization based problem at multiple scales.

The symmetry detection algorithms proposed by Mitra et al. [17], Speciale et al. [33], and Shi et al. [19] use point features to find reflection symmetry on 3D point clouds. These approaches use a voting-based approach to find the reflection symmetry plane that depends on the proper parameterization of the transformed space and require proper detection of modes in the transformed domain. Also, these approaches require a large number of pairs of points for voting that become intractable for large point clouds. Method proposed by Martinet et al. [34] uses moment functions to detect symmetry and the method proposed by Berner et al. [35] finds reflection symmetry using...
a graph constructed based on slippage features. However, these
require graph connectivity for the input point clouds. The
method proposed by Cohen et al. [36] uses the image features
of pixels to find reflection symmetry in the structure from
motion framework. However, it depends on image features to
find symmetry efficiently. Cicconet et al. [22] proposed a 3D
registration based-approach. They first reflect the input point
cloud about a random symmetry plane and then register both
the point clouds to find the reflection symmetry plane. Ecnis et
al. proposed a symmetric model fitting-based algorithm that is
not robust to missing parts [21]. Also, they require segmentation
of the symmetric object to find the symmetry. These two
algorithms [22] and [21] formulate the problem of symmetry
estimation as a 3D rigid registration problem that increases
the parameters of the reflection symmetry transformation as
rotation matrix has three parameters for 3D registration, but
the reflection symmetry plane has only two parameters to
estimate. This leads an increased computational complexity.
Hruda et al. proposed an efficient method where they proposed
a differential measure for reflection symmetry in 3D point
clouds [47], [48]. However, the performance of this method
does not perform well for non-uniformly sampled point clouds.
Also, it is robust to missing parts but may fail in presence of
many scattered outlier points. Whereas, the proposed approach
is robust to such cases.

There are various exciting approaches that use surface
features to detect symmetry, such as [39], [40], [41]. How-
ever, these approaches can not directly be adapted to work
on volumetric point clouds as the features used in these
approaches assume that the point cloud is sampled from a
2-manifold surface. Ovsjanikov et al. [28], Wang et al. [42],
Nagar and Raman [43], Wang et al. [44], and Sipiran et al.
[45] use spectral properties of the Laplace-Beltrami operator
and Kim et al. [46] use Möbius transformation to detect
intrinsic symmetries of 3D objects represented by triangle
meshes. However, these methods do not generalize to find
extrinsic symmetry in 3D point clouds as they depend on mesh
connectivity between points and assume that the underlying
object is represented by a 2-manifold surface. The problem
of reflection symmetry detection in digital images also is an
active field of research [11], [47]. However, these methods
may not be generalized to detect 3D symmetry in point clouds
due to completely different representations of objects.

III. PROPOSED APPROACH

A. Problem Formulation

Let \( P = \{x_i\}_{i=1}^n \subset \mathbb{R}^3 \) be a set of points sampled from
the surface of a 2-manifold representing a symmetric object.
Let \( v \in \mathbb{R}^3 \) be a unit norm vector that is normal to the
plane of reflection symmetry. We can mathematically model
the reflection symmetry using the Householder transformation
matrix. That is if \( x \) and \( y \) are reflective symmetric points then
\( y = (I - 2vv^T)x \).

We pose the problem of estimating the normal vector \( v \) as the below optimization problem.

\[
\begin{align*}
\mathbf{v}^* &= \arg\min_{v \in \mathbb{R}^3, v^Tv = 1} \sum_{i=1}^n \|x_{\pi(i)} - (I - 2vv^T)x_i\|_2^2. \tag{1}
\end{align*}
\]

Here, \( x_{\pi(i)} \) denotes the reflection of the point \( x_i \) about the
symmetry plane defined by the normal vector \( v \). Now, we
rewrite the above formulation by observing that
\( \|x_{\pi(i)} - (I - 2vv^T)x_i\|_2^2 = \|x_{\pi(i)} - x_i\|_2^2 + 4v^T(x_i - x_{\pi(i)})v \). The first term
is a constant with respect to \( v \). Therefore, the optimization
problem defined in Equation (1) is equivalent to the problem
defined in Equation (2).

\[
\mathbf{v}^* = \arg\min_{v \in \mathbb{R}^3, v^Tv = 1} v^T Hv. \tag{2}
\]

Here, the matrix \( H \in \mathbb{R}^{3 \times 3} \) is defined as
\( H = \sum_{i=1}^n x_i x_{\pi(i)}^\top - \frac{1}{n} I \). A
closed form solution to this problem can easily be found by
minimizing the function \( v^T Hv + \lambda(v^Tv - 1) \). The optimal
vector \( v^* \) that minimizes this function is the eigenvector of the
matrix \( H \) corresponding to the smallest eigenvalue. In order
to find the matrix \( H \), we should know the reflection point
\( x_{\pi(i)} \) for each point \( x_i \). However, we can not find the point of
reflection without knowing the plane of reflection symmetry.
Therefore, we have to solve two coupled problems: Find the
normal vector \( v \) to the plane of reflection symmetry using the tuples
\( \{(x_i, x_{\pi(i)})\}_{i=1}^n \) and find the tuples of reflective
symmetry points \( \{(x_i, x_{\pi(i)})\}_{i=1}^n \) using the plane of reflection
symmetry. Here, both \( v \) and \( \{(x_i, x_{\pi(i)})\}_{i=1}^n \) are unknown.
Hence, the problem defined in Equation (2) require both
correspondences as well as the plane of reflection symmetry
defined by \( v \) which depend on each other. We know that given
the correspondences between the mirror symmetric points,
finding the optimal \( v \) is easy as it is the eigenvector corre-
responding to the smallest eigenvalue of the matrix \( H \). However,
finding correspondences \( (x_i, x_{\pi(i)}) \) amounts to solving a linear
assignment problem. This can become intractable for large
point clouds. The In order to solve this problem efficiently,
we make use of surface features. In Section III-C, we propose an
approach for finding occlusion robust symmetry aware feature
descriptors for the input point cloud.

B. Robust Symmetry Plane Estimation

The set of putative correspondences we get through match-
ing the reflection invariant feature descriptors may contains
many outlier correspondences. The reason for this is that there
can be many outlier points present in the given point cloud.
Also, a significant part of the object may be missing, that
leads to wrong matches for points whose mirror reflective
points are missing from the point cloud. Now, the optimal
vector \( v \), found by solving the optimization problem defined
in Equation (2), will not be correct since there are many of
the putative correspondences are wrong and the L2-norm is not
robust to outliers. In order to find the optimal vector \( v \), we
propose a statistical estimation technique based on the \( \ell_2E \)
estimator. Let \( \{(x_i, x_{\pi(i)})\}_{i=1}^n \) be a set of correspondences be-
tween mirror-symmetric points that may contain many outliers
correspondences. Now, we model the estimation problem as
follows:

\[
x_{\pi(i)} = (I - 2vv^T)x_i + \epsilon, \quad i \in \{1, 2, \ldots, n\}. \tag{3}
\]

Here, \( \epsilon \in \mathbb{R}^3 \) is noise and we assume that it follows the
Gaussian distribution with zero mean and \( \sigma^2 \) variance. I.e.,
\( e \sim \mathcal{N}(0, \text{diag}(\sigma^2, \sigma^2, \sigma^2)) \) and all its samples are identically and independently distributed. Let us assume that the true normal vector is \( v_0 \). Now, let \( g(r|v) \) be the parametric density model with respect to the estimation parameter vector \( v \) and \( g(r|v_0) \) be the parametric density model with respect to the true parameter vector \( v_0 \). Here, \( r \) denotes the residual vector. Then, the \( L_2E \) estimator \( \hat{v} \) for \( v_0 \) is found by minimizing the loss \( \int (g(r|v) - g(r|v_0))^2 dr \) with respect to \( v \). As shown in [23], the optimal \( \hat{v} \) can be found by solving the below optimization problem:

\[
\hat{v} = \arg\min_{v\in \mathbb{R}^n} \int g^2(r|v) dr - \frac{2}{n} \sum_{i=1}^{n} g(r_i|v).
\]  

(4)

Now using the fact that the noise \( e \) is the white Gaussian noise, the residual vector \( r_i = x_i - (I - vv^T)x_i \) follows a Gaussian distribution \( g(r|v_i) = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{\|r_i - (I - vv^T)x_i\|^2}{2\sigma^2}} \).

Also, since \( g(r|v) \) is a Gaussian distribution the term \( \int g^2(r|v) dr = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \) which is constant with respect to \( v \). Hence, the \( L_2E \) estimator of \( v \) can be defined as below:

\[
\hat{v} = \arg\max_{v\in \mathbb{R}^n} -\frac{2}{n} \sum_{i=1}^{n} \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{\|r_i - (I - vv^T)x_i\|^2}{2\sigma^2}}
\]  

(5)

\[
= \arg\max_{v\in \mathbb{R}^n} -\frac{2}{n} \sum_{i=1}^{n} \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{vv^T C_i v}{2\sigma^2}}
\]  

(6)

\[
= \arg\max_{v\in \mathbb{R}^n} f(v)
\]  

(7)

Here, \( C_i = x_i x_i^T \in \mathbb{R}^{3\times3} \). Therefore, in order to find the optimal estimator \( \hat{v} \) for \( v \), we have to find the maximum of the function \( f \) with respect to \( v \). We observe that the domain of the function \( f \) is the 2-sphere \( S^2 = \{v \in \mathbb{R}^3|v^Tv = 1\}\), which is a smooth 2-manifold [48]. Therefore, in order to solve the problem defined in Equation (7), we use the optimization on manifold technique [48]. We find the Riemannian gradient of the function \( f \) on the manifold \( S^2 \) by first finding the Euclidean gradient on the tangent plane and then projecting back to the sphere as follows. Let \( \nabla f(v) \) be the Euclidean gradients and \( \text{grad} f(v) \) be the Riemannian gradient of the function \( f \). Then, the Riemannian gradient and the Euclidean gradient are related as \( \text{grad} f(v) = \mathbb{P}_v(\nabla f(v)) \). Here, \( \mathbb{P}_v \) is the projection operator for \( S^2 \) that project an ambient space vector onto the tangent space at \( v \) and defined as \( \mathbb{P}_v(x) = x - (x^Tv)v \). The Euclidean gradient of the function \( f(v) = \frac{1}{n(2\pi\sigma^2)^{\frac{n}{2}}} \sum_{i=1}^{n} e^{-\frac{vv^T C_i v}{2\sigma^2}} \) can be easily found by some algebraic manipulation and is defined as below:

\[
\nabla f(v) = -\frac{4}{n} \sum_{i=1}^{n} \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{vv^T C_i v}{2\sigma^2}} (C_i + C_i^T)v \sigma^2
\]  

(8)

Here, the matrix \( A \) is equal to \( \text{diag}(e^{-\frac{4v^T C_i v}{2\sigma^2}}, \ldots, e^{-\frac{4v^T C_n v}{2\sigma^2}}) \), the matrix \( B \) is equal to \( \text{diag}(C_i + C_i^T, \ldots, C_n + C_n^T) \), \( I \) is the identity matrix of size \( 3n \times 3n \), \( 1 \) is the vector of all ones and size equal to \( 3n \times 1 \), \( \otimes \) is the Kronecker matrix product operator, and \( \circ \) is the element-wise matrix multiplication operator. We use the Manifold-BFGS optimization algorithm, proposed in [48], for finding the optimal solution. We use the manifold toolbox for optimization on manifolds [49]. We further observe that the function \( f \) is not a convex function and locally convex around the global maximum. Therefore, the final solution and the convergence rate completely depend on the initialization of vector \( v \). In order to converge to the final solution quickly, we propose the following initialization strategy for \( v \).

**Initialization of \( v \):** First, we sample at most 1000 points uniformly at random from the given point cloud. This random sampling is inspired from the analysis proposed in [50] and shown to preserve symmetry in the sampled point cloud. Then, we quantize the surface of 2-sphere \( S^2 \) into 162 patches by binning the parametrization of the sphere. The spherical parametrization of \( S^2 \) is given by two parameters \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \) and a point \( v \) on \( S^2 \) is defined as \( v = [\sin \theta \cos \phi \sin \theta \sin \phi \cos \theta]^T \). We quantize the parameter space as \( (\theta, \phi) \in \{0, 20^\circ, \ldots, 180^\circ\} \times \{0, 20^\circ, \ldots, 360^\circ\} \).

Then, we find the error \( -f(v) \) for each vector defined by the center of these bins. We choose the vector for which the error is the lowest. Let this vector be \( v_1 \). We observe that \( v_1 \) is an approximation and the angle between \( v_1 \) and the ground-truth vector \( v_g \) could be more than \( 20^\circ \). Therefore, it requires further improvement to find the actual solution.

**Symmetry Plane Center:** Ideally, the center of reflection lies on the mid-point of the line segment, joining two mirror symmetric points. In order to find the center, we use the method proposed in Section III.C to find putative correspondences. We use the RANSAC algorithm for finding the inlier correspondences. We then take the center of reflection to be the median of the mid-points of the inlier correspondences. We translate the input point cloud so that the plane of reflection symmetry passes through the origin.

**A Remark on MLE:** We would like to clarify that the cost function, defined in Equation (7), of the \( L_2E \) estimator is different than the maximum likelihood estimator (MLE). We can easily determine the MLE estimator \( v_{mle} \) as by solving the below optimization problem:

\[
v_{mle} = \arg\min_{v\in \mathbb{R}^n, v^Tv=1} \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \sum_{i=1}^{n} v^T C_i v
\]  

(9)

\[
v_{mle} = \arg\min_{v\in \mathbb{R}^n} \sum_{i=1}^{n} v^T C_i v.
\]  

(10)

We can observe that the cost functions of \( L_2E \) estimator and MLE estimators defined in Equations (6) and (9), respectively, are different. In Section III.D, we analyze the robustness of the \( L_2E \) and the MLE estimators in presence of outliers.

**C. Reflection Symmetry Point Descriptors**

Our goal is to find a set of candidate matchings between mirror symmetric points to find the plane of reflective symmetry plane. Since the proposed algorithm for finding the normal vector to the plane of reflection symmetry is robust to a significant number of wrong matches, our aim is to find a set with a
few correct correspondences where other correspondences can be outliers. In order to find a set \( \{(x_i, x_{\pi(i)})\}_{i=1}^n \) of putative correspondences between the mirror-symmetric points, we need to find the mapping \( \pi : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\} \).

In order to find the mapping \( \pi \), we first find missing parts aware feature descriptors for each point of the point cloud and then match them using an approximate nearest neighbor search approach \([51]\). We propose the following approach to find missing part-aware feature descriptors that are invariant to reflection symmetry. Consider a point \( x_i \in \forall \) on the input point cloud. To make the feature invariant to missing parts and outliers, we only consider the local neighborhood of the point. If we consider all the points, then the outliers and missing parts will influence the feature descriptors. Now, let \( G_{r, x_i} \) be a sphere of radius \( r \) centered at the point \( x_i \). Let \( N_i = (G_{r, x_i} \cap \forall) \cup \{x_i\} \) be the set of points from the set \( \forall \) that lie within the sphere of radius \( r \) and centered at the point \( x_i \). To make the descriptors invariant to reflection symmetry transformation, we have to work in the intrinsic coordinates system. The reason is that in the extrinsic coordinate system, the coordinates of the neighbors of two symmetric points would be different and therefore, their descriptors may be different. In order to find the differential coordinate, we propose a very simple yet very effective approach. We translate all points in \( N_i \) such that the point \( x_i \) is at the origin of the local coordinate system, i.e., shift each point \( x \) of the set \( N_i \) to \( x - x_i \). Now, to make the features invariant to reflection, we use the pairwise geodesic distance between the neighboring points of \( x_i \) since the pairwise distances are invariant to reflection. This can be seen easily as follows. Let \( x_b \) be the mirror reflection of the point \( x_i \) and \( x_d \) be the mirror reflection of the point \( x_i \). Then, the distance \( ||x_b - x_d||_2 \) between the points \( x_b \) and \( d \) is equal to \( ||x_b - x_c||_2 \) as we show below.

\[
\frac{|x_b - x_d|_2}{2} = \frac{|| (I - 2vv^T)x_a - (I - 2vv^T)x_c ||_2}{2} = (x_a - x_c)^T (I - 2vv^T)^2 (x_a - x_c) = \frac{||x_a - x_c||_2}{2}. \tag{12}
\]

Here, we have used the result \((I - 2vv^T)^T(I - 2vv^T) = (I - 2vv^T)^2 = I\). To find an efficient descriptor for the point \( x_i \) by using the pairwise distances between its neighboring points, we construct a matrix \( D_i \in \mathbb{R}^{||N_i| | \times |N_i|} \) such that its \((p, q)\)-th entry is equal to the distance between the points \( x_p \in N_i \) and \( x_q \in N_i \), i.e., \( D_i(p, q) = ||x_p - x_q||_2 \). Let \( \lambda_1, \lambda_2, \ldots, \lambda_k \) be the first \( k \) smallest sorted eigenvalues of \( D_i \). Then, we define a descriptor \( f_i \in \mathbb{R}^k \) for the point \( x_i \) as

\[
f_i = \frac{1}{\sum_{i=1}^k \lambda_i} \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \end{bmatrix}^T. \tag{12}
\]

If two points \( x_i \) and \( x_j \) are mirror images of each other, the neighborhoods will also remain the same. Therefore, the descriptors \( f_i \) and \( f_j \) will also be the same. Although, the ordering of the neighboring points may change for two reflective symmetric points \( x_i \) and \( x_{\pi(i)} \). Therefore, the matrices \( D_i \) and \( D_{\pi(i)} \) will not be the same. However, the neighboring points for \( x_i \) and \( x_{\pi(i)} \) remains the same with permuted ordering of the indices. Therefore, the matrix \( D_{\pi(i)} \) is the permutation of the matrix \( D_i \). Let \( P \) be the permutation matrix then \( D_{\pi(i)} = PD_i^TP \). Now, we can easily show that the eigenvalue spectrum of both the matrices are the same. Let \( (\lambda, \beta) \) be an eigenvalue-eigenvector pair of the matrix \( D_{\pi(i)} \). Then, we show that \( (\lambda, \beta^T) \) will be an eigenvalue-eigenvector pair of the matrix \( D_i \), as follows:

\[
D_{\pi(i)}\beta = \lambda\beta \Rightarrow PD_i^TP\beta = \lambda\beta \\
P^TPD_i^TP\beta = \lambda\beta^T \\
\Rightarrow D_i(\beta^T) = \lambda(\beta^T). \tag{13}
\]

Therefore, eigenvalue spectrum of both the matrices are the same. Here, we have used the fact that a permutation matrix is an orthonormal matrix, i.e., \( P^TP = I \). After, obtaining feature descriptors for each point, we use the approximate nearest neighbor method proposed in \([51]\) to find the set \( \{(x_i, x_{\pi(i)})\}_{i=1}^n \) of correspondences between the mirror-symmetric points as defined in Section III-C.

**Algorithm 1 Robust Reflection Symmetry Plane Estimation**

**Input:** A 3D Partial Point cloud \( \forall = \{x_i\}_{i=1}^n \)

1. Normalize the point cloud \( \forall \) to the unit cube.
2. Initialize \( r \) and \( k \). We choose \( r = 0.1 \) and \( k = 20 \).
3. Find a set \( \{(x_i, x_{\pi(i)})\}_{i=1}^n \) of correspondences between mirror symmetric points as defined in Section III-C.
4. Find the matrices \( C_i = x_{\pi(i)}x_i^T, \forall \{1, 2, \ldots, n\} \).
5. Initialize the vector \( v(0) = v_i \) and \( t = 0 \).
6. \( B \leftarrow \text{diag}(C_1 + C_1^T \cdots C_n + C_n^T) \)
7. while not converged do
8. \( A \leftarrow \text{diag}\left(\begin{bmatrix} e^{-4v(0)^T C_1 v} & \cdots & e^{-4v(0)^T C_n v} \end{bmatrix}\right) \)
9. \( \beta \leftarrow \frac{4v(t)^T (A \circ B)(1 - 4v(t)^Tv(t))}{\beta(t)^2} \)
10. \( \text{grad} f(v(t)) \leftarrow -\beta - \beta^T v(t)v(t)^T \)
11. \( v(t+1) \leftarrow \text{Manifold-BFGS} (\text{grad} f(v(t)), f, v(t)) \)
12. \( t \leftarrow t + 1 \)
13. end while

**Output:** \( v(t) \).
be the eigenvalue-eigenvector pairs of the matrices \( \mathbf{PBP}^\top \) and \( \mathbf{P}(\mathbf{B} + \delta \mathbf{B})\mathbf{P}^\top \), respectively. Then, using the eigenvalue-perturbation theory, we have

\[
P(\mathbf{B} + \delta \mathbf{B})\mathbf{P}^\top (\mathbf{v}_i + \delta \mathbf{v}_i) = (\lambda_i + \delta \lambda_i)(\mathbf{v}_i + \delta \mathbf{v}_i).
\] (14)

Now, using the result \( \mathbf{PBP}^\top \mathbf{v}_i = \lambda_i \mathbf{v}_i \) and neglecting the second order small error matrix \( \delta \mathbf{B} \) and vector \( \delta \mathbf{v}_i \), we have

\[
\mathbf{PBP}^\top \delta \mathbf{v}_i + \mathbf{P}\delta \mathbf{BP}^\top \mathbf{v}_i = \lambda_i \delta \mathbf{v}_i + \delta \lambda_i \mathbf{v}_i.
\] (15)

We further observe that the set \( \{\mathbf{v}_i\}_{i=1}^n \) of eigenvectors of the matrix \( \mathbf{PBP}^\top \) forms an orthonormal basis for \( \mathbb{R}^n \) as the matrix \( \mathbf{PBP}^\top \) symmetric. Therefore, we can write \( \delta \mathbf{v}_i = \sum_{j=1}^n c_{ij} \mathbf{v}_j \). Hence, we have

\[
\mathbf{PBP}^\top \sum_{j=1}^n c_{ij} \mathbf{v}_j + \mathbf{P}\delta \mathbf{BP}^\top \mathbf{v}_i = \lambda_i \sum_{j=1}^n c_{ij} \mathbf{v}_j + \delta \lambda_i \mathbf{v}_i
\]

\[
\sum_{j=1}^n c_{ij} \mathbf{PBP}^\top \mathbf{v}_j + \mathbf{P}\delta \mathbf{BP}^\top \mathbf{v}_i = \lambda_i \sum_{j=1}^n c_{ij} \mathbf{v}_j + \delta \lambda_i \mathbf{v}_i
\]

\[
\sum_{j=1}^n c_{ij} \lambda_j \mathbf{v}_j + \mathbf{P}\delta \mathbf{BP}^\top \mathbf{v}_i = \lambda_i \sum_{j=1}^n c_{ij} \mathbf{v}_j + \delta \lambda_i \mathbf{v}_i.
\]

Now, we take the inner product on both the sides by \( \mathbf{v}_i^\top \) and observing that \( \mathbf{v}_i^\top \mathbf{v}_j = 0 \) and \( \mathbf{v}_i^\top \mathbf{v}_i = 1 \), we have

\[
\delta \lambda_i = \mathbf{v}_i^\top \mathbf{P}\delta \mathbf{BP}^\top \mathbf{v}_i
\]

\[
= \mathbf{u}_i^\top \delta \mathbf{Bu}_i.
\] (16)

Here, \( (\lambda_i, \mathbf{u}_i) \) is the eigenvalue-eigenvector pair \( \mathbf{B} \). We can further assume that \( \sum_{i=1}^k (\lambda_i + \delta \lambda_i) \approx \sum_{i=1}^k \lambda_i \) since \( \lambda_i > \delta \lambda_i \). Therefore, for the mirror symmetric points, the distance is equal to

\[
\|\mathbf{f}_i - \mathbf{f}_{\pi(i)}\|^2 = \frac{1}{k} \sum_{i=1}^k (\lambda_i - (\lambda_i + \delta \lambda_i))^2
\]

\[
= \frac{1}{k} \sum_{i=1}^k (\delta \lambda_i)^2
\]

\[
= \frac{1}{k} \sum_{i=1}^k (\mathbf{u}_i^\top \delta \mathbf{Bu}_i)^2
\] (18)

\[
\approx 0.
\] (21)

E. Complete Algorithm and Analysis

Now, we present the complete method for finding the normal vector to the plane of reflection symmetry given a 3D point cloud in Algorithm 1. In Figure 2, we demonstrate the robustness of our approach for outliers. In the first row, we add a different number of outlier correspondences (blue color lines) and perturb the ground-truth correspondences (orange color lines) between the symmetric mirror points. In the second row, we show the actual global minimum (green color point) of the cost function defined in Equation (1).
We observe that if we directly optimize this cost function, the estimated solution (red color point) quickly moves away from the global solution as we increase the number of outlier correspondences. This is the main limitation of the state-of-the-art methods (\cite{21,22,20,52}) as they minimize the L2-norm based cost function. Whereas, we minimize the $L_2E$ cost function $\min_{E}$ that can assign low values of the threshold $t$ on the distance criterion in the range $[0, \frac{\pi}{2}]$ and the value of the threshold $t$ on the distance criterion in the range $[0, 2\pi]$. Here, the constant $s$ is defined to be equal to $\min\{||p_a - p_b||_2, ||p_a - p_c||_2, ||q_a - q_b||_2, ||q_a - q_c||_2\}$. Since this dataset contains perfectly symmetric objects with no outliers, we define the center of the plane of symmetry as the mean center of the input point cloud. Hence, the distance threshold criterion is true for all the approaches.

IV. RESULTS AND EVALUATION

A. Evaluation of Reflection Symmetry Plane

Benchmark Dataset and Evaluation Metric: In order to test the performance of the proposed approach, we find the accuracy of detecting the symmetry plane. We use the benchmark dataset and the standard evaluation metric proposed by Funk et al. \cite{1}. Funk et al. published a dataset of 1354 objects represented as 3D point clouds exhibiting single reflection symmetry. We compare the accuracy of detection of plane of symmetry with the accuracies of the state-of-the-art methods proposed by Cicconet et al. \cite{22}, Ecins et al. \cite{21}, Speciale et al. \cite{33}, Hruda \cite{38}, and Nagar and Raman \cite{52}. We use the F-score as a performance measure metric proposed by Funk et al. \cite{1}. We also compare the accuracy of symmetry detection with the accuracy of the methods proposed by Cicconet et al. \cite{22} and Nagar and Raman \cite{52} for the detection of partial and approximate symmetry detection. We first find the precision and recall rates and then use them to find the F-score. The precision rate of detecting symmetry plane is defined as $p_r = \frac{t_p}{t_p + f_p}$, and the value of the F-Score is defined as $f_s = \frac{2tp}{tp + fp}$. Here, we define the quantities $t_p$ (true positives), $f_p$ (false positives), and $f_n$ (false negatives) as follows: $t_p$ is the number of correctly detected planes of symmetry, $f_p$ is the number of incorrectly detected planes of symmetry, and $f_n$ is the number of undetected ground-truth planes of symmetry. In the benchmark dataset, the ground symmetry plane defined as $p_c = \frac{p_a + p_b}{2}$, and the center of the detected plane of reflection symmetry $q_c = \frac{q_a + q_b}{2}$ should be smaller than a given threshold $t$ on the distance. We change the value of the threshold $t$ on angle criterion in the range $[0, \frac{\pi}{2}]$ and the value of the threshold $t$ on the distance criterion in the range $[0, 2\pi]$. Here, the constant $s$ is defined to be equal to $\min\{||p_a - p_b||_2, ||p_a - p_c||_2, ||q_a - q_b||_2, ||q_a - q_c||_2\}$. Since this dataset contains perfectly symmetric objects with no outliers, we define the center of the plane of symmetry as the mean center of the input point cloud. Hence, the distance threshold criterion is true for all the approaches.

Comparison Results: In Figure 4 we plot the recall vs precision curves for the algorithms proposed by Cicconet et al. \cite{22}, Ecins et al. \cite{21}, Speciale et al. \cite{33}, Hruda \cite{38}, and Nagar and Raman \cite{52}. We observe that the value of the F-score (0.93) for the proposed approach is the highest among all methods. The benchmark dataset \cite{1} contains complete objects. Therefore the performance of the method proposed in \cite{52} is comparable to the performance of the proposed approach. In Figure 5 we plot the estimated normal vectors to
symmetry planes, a few sampled points (red color points) on the detected plane, and correspondences between mirror symmetric points (blue color lines) using the proposed approach for a few 3D point clouds from the dataset [53]. We observe that our method can detect symmetry of objects using their partial scans in the presence of outlier points.

![Image](image1.png)

Fig. 5. Results of the proposed approach on the four models from the dataset [53]. The first point cloud is the complete one and the second point cloud is the incomplete one. We observe that we are able to robustly detect symmetries even for partial scans.

![Image](image2.png)

Fig. 6. Results of the proposed approach on the two models from the partial and real scan dataset [1].

B. Evaluation of Partial Symmetry Detection

In order to measure the performance of our approach and compare it with that of the state-of-the-art methods for partial symmetry, we remove a set $P_m$ of connected points from the input 3D point cloud $P$ such that $|P_m| = \gamma |P|$, $\gamma \in \{0, 0.15, 0.20, 0.28\}$ and find the value of F-score for the methods proposed in [22], [52], and the proposed algorithm. We use the dataset [1] for evaluation. In Table I we present the obtained F-score values for different values of $\gamma$. We note that even for $\gamma = 0.28$, F-score for the proposed approach remains 0.89. However, for the algorithm proposed in [22], it drops to 0.43. We present a qualitative result compared with [52] in Figure 7. The main reason for the failure of the method [52] is that the symmetry plane detection algorithm fails to converge to the final solution if the angle between the initialized normal vector and the ground-truth normal vector is more than around 20°. We observe that there are many outlier matches to a part (neck and chest) for which the mirror part does not exist. In the presence of many outliers and missing parts, this initialize algorithm fails to properly initialize the normal vector. Whereas, the proposed estimator is significantly robust (angle error < 7%) to many outlier correspondences (65%) and produces good matches at the convergence.

![Image](image3.png)

Fig. 7. Results of the proposed approach on the four models from the dataset [53]. We observe that we are able to robustly detect symmetries even for partial scans.

**Table I**

| $\gamma \rightarrow$ | 0     | 0.15  | 0.20  | 0.28  |
|-----------------------|-------|-------|-------|-------|
| Cicconet et al.       | 0.67  | 0.60  | 0.54  | 0.52  |
| Nagar and Raman       | 0.90  | 0.86  | 0.86  | 0.85  |
| Proposed              | 0.93  | 0.92  | 0.90  | 0.89  |

C. Effect of Outliers and Missing Parts

In order to measure the performance in presence of a significant amount of outliers, we use the strategy proposed in [52]. We again consider the 3D objects provided in dataset [1] and introduce random points into the input 3D point cloud to get a modified noisy point cloud $P_n = P \cup P_r$. Here, the point cloud $P_r$ is a set of outliers. We choose different number of noise points such that $|P_r| = \alpha |P|$, $\alpha \in \{0, 20, \ldots, 120\}$. We find the value of the F-score for the methods in [22], [52], and the proposed algorithm for every value of the new point cloud $P_n$. We use the dataset [1] for evaluation. In Table II we present the obtained F-score values for different values of $\alpha$. We observe that even for the case where half of the points are outliers, the F-score for the proposed approach remains around 0.88. Whereas, the F-score for the method proposed in [22] decreases to 0.47. To measure the performance of our method and compare it with that of the state-of-the-art methods for partial symmetry, we remove a set $P_m$ of connected points from the input 3D point cloud $P$ such that $|P_m| = \gamma |P|$, $\gamma \in \{0, 0.15, 0.20, 0.28\}$ and find the value of F-score for the methods proposed in Table II.

**Table II**

| $\gamma \rightarrow$ | 0     | 0.15  | 0.20  | 0.28  |
|-----------------------|-------|-------|-------|-------|
| Cicconet et al.       | 0.67  | 0.61  | 0.52  | 0.43  |
| Nagar and Raman       | 0.90  | 0.82  | 0.75  | 0.67  |
| Proposed              | 0.93  | 0.91  | 0.89  | 0.89  |

**Time Complexity:** The BFGS method takes $O(n)$ [48]. The descriptor finding step takes $O(nh^2)$, $O(h^2)$ for finding eigenvalues of $D$. Here, $h$ is the average number of neigh-

![Image](image4.png)

Fig. 8. Results of the proposed approach on the four models from the dataset [53]. We observe that we are able to robustly detect symmetries even for partial scans.
neighboring points. For our experiments $h \approx 100$. Therefore, the overall approximate time complexity of our approach is $O(n) + O(nh^2)$. The time complexity of the method proposed in [20] is $O(n^{3/2})$. For a model with 5k points, $k = 20$ and $r = 0.1$, our algorithm takes around 1.58 seconds on a Linux OS with an Intel-i7 processor. We report the time comparison in Table III. We observe that the computation time for the method proposed in [52] increases drastically ($O(n \log n)$) for large point clouds as it solves a nearest-neighbor search problem for updating mirror correspondences in each iteration. Our algorithm finds a robust estimator for the symmetry plane even from noisy mirror correspondences and does not require solving the nearest-neighbor problem. We need to run Manifold-BFGS solver to find $v$ which has only two variables. Hence, our computational complexity is almost linear in point cloud size. In Table IV we report the accuracy and average time for feature computation in terms of number of neighboring points.

![Fig. 7. Left: Result of the approach proposed in [52] on a partial model. Right: Result of the proposed approach on the same partial model. We observe that our approach is robust to significant missing parts and finds good symmetry plane (blue points) correspondences between mirror symmetric points (red lines).](image)

### Table III

| #points | 5k     | 10k    | 50k    | 100k   | 300k   |
|---------|--------|--------|--------|--------|--------|
| [35]    | 0.38s  | 0.48s  | 6.12s  | 16.03s | 61.01s |
| Proposed| 1.58s  | 1.98s  | 2.11s  | 2.83s  | 6.92s  |

### Table IV

| $r \to$ | 0.01 | 0.05 | 0.1  | 0.15 | 0.20 |
|---------|------|------|------|------|------|
| Time    | 0.29s | 0.31s | 1.35s | 1.46s | 1.54s |
| F1-Score| 0.89  | 0.91  | 0.93  | 0.93  | 0.93  |

V. Conclusion and Future Work

In this work, we have proposed a fast and robust algorithm for detecting symmetry of 3D objects exhibiting single reflection symmetry and represented by partial and noisy points clouds. We used a statistical estimation technique for finding the plane of reflection symmetry. For this purpose, we first found a 3D point descriptor for each point that is invariant to reflection symmetry transformation. We used the spectral properties of the geodesic distance matrix computed on the local neighborhood of each point to describe it. Then, we used an approximate nearest neighbor matching technique for finding a set of candidate correspondences between mirror reflective points. We used this noisy set of correspondences and a statistical estimator to estimate the reflection symmetry plane that is robust to a significant number of outliers and missing parts. We first formulated the problem of finding the estimator as an optimization problem on 2-Sphere. We showed that our approach achieves the best performance on the benchmark dataset.

A limitation of our approach is that it detects the plane of reflection symmetry for an object having single symmetry. In future work, we would like to extend our framework for detecting symmetries objects exhibiting multiple reflection symmetries. We hope that the proposed algorithm would be helpful for solving various problems in the field of computer vision and graphics and geometry processing.

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