Common fixed point for some generalized contractive mappings in a modular metric space with a graph

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Abstract
In this paper, we investigate the existence and the uniqueness of a common fixed point of a pair of self-mappings satisfying new contractive type conditions on a modular metric space endowed with a reflexive digraph. An application is given to show the use of our main result.

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1 Introduction and preliminaries
More generalized contractive type conditions are considered in the study of the existence and uniqueness of the fixed point. Alber and Guerre-Delabriere in [2] introduced a class of weakly contractive maps on closed convex sets of Hilbert spaces. In [9], Rhoades extended a part of this study to an arbitrary Banach space. The notion of weak contraction has been studied by other authors in the setting of metric spaces (see [8, 12] and the references therein). In [13], Zhang gave some new generalized contractive type conditions for a pair of mappings in a metric space and proved some common fixed point results for these mappings. Let $F : [0, +\infty] \to \mathbb{R}$ be a function satisfying the three conditions:

1. $F(0) = 0$ and $F(t) > 0$ for all $t > 0$;
2. $F$ is nondecreasing on $[0, +\infty]$;
3. $F$ is continuous on $[0, +\infty]$.

Consider the function $\phi : [0, +\infty] \to [0, +\infty]$ such that

1. $\phi(t) < t$ for all $t > 0$;
2. $\phi$ is nondecreasing and right upper semicontinuous on $[0, +\infty]$;
3. $\lim_{t \to +\infty} \phi^n(t) = 0$ for all $t > 0$.

In this paper, motivated by some works as [10], we extend the following theorem to the setting of the modular metric space endowed with a reflexive digraph.
Theorem ([13]) Let $X$ be a complete metric space, and let $T, S : X \to X$ be two self-mappings satisfying

$$F(d(Tx, Sy)) \leq \phi(F(M(x, y))) \quad \text{for each } x, y \in X,$$

where

$$M(x, y) = \max \left\{ d(x, y), d(Tx, x), d(Sy, y), \frac{d(Tx, y) + d(Sy, x)}{2} \right\}.$$  

Then $T$ and $S$ have a unique common fixed point in $X$. Moreover, for each $x_0 \in X$, the iterative sequence $(x_n)$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ converges to the common fixed point of $T$ and $S$.

In the sequel, we recall some basic notions: Let $X$ be a nonempty set. For a function $\omega : [0, +\infty] \times X \times X \to [0, +\infty]$, we will use the notation

$$_-\lambda(x, y) = (\lambda, x, y) \quad \text{for all } \lambda > 0 \text{ and } x, y \in X.$$

Definition 1.1 ([7]) A function $\omega : [0, +\infty] \times X \times X \to [0, +\infty]$ is said to be modular metric on $X$ if it satisfies the following conditions:

(i) Given $x, y \in X$, $x = y$ if and only if $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$;

(ii) For all $x, y \in X$, for all $\lambda > 0$, $\omega_\lambda(x, y) = \omega_\lambda(y, x)$;

(iii) For all $x, y, z \in X$ and for all $\lambda, \mu > 0$, $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$.

In this case, $(X, \omega)$ is called modular metric space.

The modular $\omega$ is said to be regular if condition (i) holds for some $\lambda > 0$.

The modular $\omega$ is said to be convex if, for all $\lambda, \mu > 0$ and $x, y, z \in X$, we have

$$\omega_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda + \mu} \omega_\lambda(x, z) + \frac{\mu}{\lambda + \mu} \omega_\mu(z, y).$$

Let $(X, \omega)$ be a modular metric space. Fix $x_0 \in X$. Set

$$X_\omega = X_\omega(x_0) = \left\{ x \in X : \omega_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty \right\}$$

and

$$X_\omega^* = X_\omega^*(x_0) = \left\{ x \in X : \exists \lambda > 0, \omega_\lambda(x, x_0) < \infty \right\}.$$  

The two linear spaces $X_\omega$ and $X_\omega^*$ are said to be modular spaces (around $x_0$). It is clear that $X_\omega \subseteq X_\omega^*$.

Definition 1.2 ([7]) We say that $\omega$ satisfies the $\Delta_2$-type condition if, for every $\alpha > 0$, there exists a constant $K_\alpha > 0$ such that

$$\omega_\alpha(x, y) \leq K_\alpha \omega_\alpha(x, y)$$

for all $x, y \in X_\omega$ and any $\lambda > 0$.  

\[ \]
Remark 1.3 If \( \omega \) satisfies the \( \Delta_2 \)-type condition, then \( \omega \) is regular and \( X_\omega = X_\omega^* = X \).

A condition weaker than the \( \Delta_2 \)-type condition is often used in the literature:

**Definition 1.4** We say that \( \omega \) satisfies the \( \Delta_2 \)-condition if \( \lim_{n \to +\infty} \omega_\lambda(x_n, x) = 0 \) for some \( \lambda > 0 \) implies that \( \lim_{n \to +\infty} \omega_\lambda(x_n, x) = 0 \) for all \( \lambda > 0 \).

It is clear that if \( \omega \) satisfies the \( \Delta_2 \)-type condition, then \( \omega \) satisfies the \( \Delta_2 \)-condition, and that the converse is not true. Throughout this paper, we consider the modular metrics satisfying the \( \Delta_2 \)-type condition, and we adopt the definitions of some topological notions as stated in [11].

**Definition 1.5** Let \( \omega \) be a modular metric on \( X \).

1. We say that a sequence \( \{x_n\} \subset X_\omega \) is \( \omega \)-convergent to some \( x \in X_\omega \) if

\[
\lim_{n \to +\infty} \omega_\lambda(x_n, x) = 0 \quad \text{for some} \quad \lambda > 0.
\]

We will call \( x \) the \( \omega \)-limit of \( \{x_n\} \).

If \( \omega \) satisfies the \( \Delta_2 \)-type condition, then \( \lim_{n \to +\infty} \omega_\lambda(x_n, x) = 0 \) for all \( \lambda > 0 \).

2. We say that a sequence \( \{x_n\} \subset X_\omega \) is \( \omega \)-Cauchy if, for some \( \lambda > 0 \),

\[
\lim_{n,m \to +\infty} \omega_\lambda(x_n, x_m) = 0.
\]

If \( \omega \) satisfies the \( \Delta_2 \)-type condition, then \( \{x_n\} \) is \( \omega \)-Cauchy if

\[
\lim_{n,m \to +\infty} \omega_\lambda(x_n, x_m) = 0 \quad \text{for all} \quad \lambda > 0.
\]

3. We say that \( M \subset X_\omega \) is \( \omega \)-closed if the \( \omega \)-limit of any \( \omega \)-convergent sequence of \( M \) is in \( M \).

4. We say that \( M \subset X_\omega \) is \( \omega \)-complete if any \( \omega \)-Cauchy sequence in \( M \) is \( \omega \)-convergent and its \( \omega \)-limit belongs to \( M \).

5. We say that \( \omega \) satisfies the Fatou property if, for some \( \lambda > 0 \), we have

\[
\omega_\lambda(x, y) \leq \liminf_{n \to +\infty} \omega_\lambda(x_n, y)
\]

for any sequence \( \{x_n\} \subset X_\omega \) which is \( \omega \)-convergent to \( x \) and for any \( y \in X_\omega \).

Let \( V \) be an arbitrary set. A directed graph, or digraph, is a pair \( G = (V, E) \) where \( E \) is a subset of the Cartesian product \( V \times V \). The elements of \( V \) are called vertices or nodes of \( G \), and the elements of \( E \) are the edges also called oriented edges or arcs of \( G \). An edge of the form \( (v, v) \) is a loop on \( v \). Another way to express that \( E \) is a subset of \( V \times V \) is to say that \( E \) is a binary relation over \( V \). Given a digraph \( G \), the set of vertices (respectively of edges) of \( G \) is denoted by \( V(G) \) (respectively \( E(G) \)). A digraph \( G' = (V', E') \) is said to be an induced subgraph of a digraph \( G = (V, E) \) on \( V \) if \( V' \subseteq V \) and \( E' = E \cap (V' \times V') \). We denote \( G' \) by \( G[V'] \).

The digraph \( G = (V, E) \) is said to be

(i) transitive if whenever \( (x, y) \in E \) and \( (y, z) \in E \), then \( (x, z) \in E \).

(ii) reflexive if \( \Delta := \{(v, v) : v \in V\} \) is a subset of \( E \).

A vertex \( x \) is said to be

(i) a start point of \( G \) if there exists no vertex \( y \) such that \( (y, x) \in E \).

(ii) isolated if, for each vertex \( y \neq x \), we have neither \( (x, y) \in E \) nor \( (y, x) \in E \).
Given two vertices $x, y \in V$. A path in $G$, from (or joining) $x$ to $y$ is a sequence of vertices $p = \{a_i\}_{0 \leq i \leq n}, n \in \mathbb{N}^*$ such that $a_0 = x, a_n = y$ and $(a_i, a_{i+1}) \in E$ for all $i \in \{0, 1, \ldots, n-1\}$. The integer $n$ is the length of the path $p$. If $x = y$ and $n > 1$, the path $p$ is called a directed cycle. An acyclic digraph is a digraph which has no directed cycle.

We denote by $y \in [x]_G$ the fact that there is a directed path in $G$ joining $x$ to $y$.

A sequence $\{x_n\}_{n\in \mathbb{N}}$ is said to be $G$-nondecreasing if $x_{n+1} \in [x_n]_G$ for all $n \in \mathbb{N}$.

A modular metric space $(X, \omega)$ endowed with a digraph $G$ such that $V(G) = X$ is denoted by $(X, \omega, G)$. In recent years, there has been a great interest in the study of the fixed point property in modular metric spaces endowed with a partial order, see [5] and the references therein.

In this work, we investigate the existence and uniqueness of the common fixed point of a pair of mappings satisfying a generalized contractive condition in the setting of a modular metric space with a reflexive digraph. The main result is illustrated by an example and is used to show the existence of a solution of a system of Fredholm integral equations.

As in [6], we use the property (OSC) defined as follows.

**Definition 1.6** Let $(X, \omega, G)$ be a modular metric space endowed with a digraph. We say that $X$ satisfies the property (OSC) if, for any $G$-nondecreasing sequence $\{x_n\} \subseteq X$ which is $\omega$-convergent to $x \in X$, we have $x \in [x_n]_G$ for all $n \in \mathbb{N}$.

**2 Main result**

The following technical lemmas borrowed from [5] are useful in the sequel and highlight the use of the $\Delta_2$-type condition to establish the main result.

**Lemma 2.1** If $\omega$ satisfies the $\Delta_2$-type condition, then

$$\omega_\lambda(x, y) < \infty \quad \text{for all } \lambda > 0 \text{ and for all } (x, y) \in X^2.$$ 

**Lemma 2.2** Let $s, t \in \mathbb{N}^\ast$. If $\omega$ satisfies the $\Delta_2$-type condition and $\{x_n\}$ is not $\omega$-Cauchy, then there exist $\varepsilon > 0$ and two subsequences of integers $\{n_k\}$ and $\{m_k\}$ such that $n_k > m_k \geq k$, $\omega_\lambda(x_{n_k}, x_{n_k}) \geq \varepsilon$, and $\omega_\frac{1}{2}(x_{n_k-1}, x_{m_k}) < \varepsilon$.

From now on, we mean 1 instead of $\lambda$ for the same reason Abdou and Khamsi used in [1]. One can see that the proof of the main result remains even if we replace 1 with any $\lambda > 0$.

Let $\psi : [0, +\infty[ \longrightarrow [0, +\infty[$ be a function satisfying the two conditions:

(i) $\psi(t) < t$ for all $t > 0$;

(ii) $\psi$ is right upper semicontinuous on $[0, +\infty[$.

Let

$$M(x, y) = \max \left\{ \omega_1(x, y), \omega_1(x, Sx), \omega_1(y, Ty), \frac{\omega_2(x, Ty) + \omega_2(y, Sx)}{2} \right\}$$

and

$$O_{y_0}(S, T) = \{ (TS)^n(x_0), S(TS)^n(x_0) : n \in \mathbb{N} \}.$$
Theorem 2.1 Let \((X, \omega, G)\) be a modular metric space endowed with a reflexive digraph \(G\) where \(\omega\) satisfies the \(\Delta_2\)-type condition and the Fatou property. Let \(C\) be an \(\omega\)-complete nonempty subset of \(X\), and \(T, S : C \to C\) be two self-mappings. If the following conditions are satisfied:

(i) for all \(x, y \in C\),

\[ y \in [x]_G \text{ or } x \in [y]_G \implies F(\omega_1(Sx, Ty)) \leq \psi(F(M(x, y))) ; \quad (1) \]

(ii) there exists an element \(x_0 \in C\) such that the induced subgraph \(G[O_{x_0}(S, T)]\) is a directed path with a unique starting point \(x_0\);

(iii) \(\omega\) satisfies the property \((OSC)\),

then \(S\) and \(T\) have a common fixed point in \(C\).

Proof Let \(x_0\) be an element of \(C\) such that \(G[O_{x_0}(S, T)]\) is a directed path. Consider the sequence \(\{x_n\}\) defined by

\[ x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1} \quad \text{for all } n \in \mathbb{N}. \]

Condition (ii) insures that \(\{x_n\}\) is \(G\)-nondecreasing. If there exists an integer \(n\) such that

\[ x_{2n} = x_{2n+1} = x_{2n+2}, \]

then \(x_{2n}\) is a common fixed point of \(S\) and \(T\). Otherwise, suppose that

\[ x_{2n} \not= x_{2n+1} \quad \text{or} \quad x_{2n} \not= x_{2n+2} \quad \text{for all } n \in \mathbb{N}. \]

Let \(n \in \mathbb{N}\). From \(x_{2n+1} \in [x_{2n}]_G\) and applying (1) for \(x = x_{2n}\) and \(y = x_{2n+1}\), we obtain

\[ F(\omega_1(x_{2n+1}, x_{2n+2})) \leq \psi(F(M(x_{2n}, x_{2n+1}))). \quad (2) \]

From

\[ M(x_{2n}, x_{2n+1}) = \max\left\{ \omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n+1}, x_{2n+2}), \frac{\omega_2(x_{2n}, x_{2n+2})}{2} \right\} \]

and

\[ \frac{\omega_2(x_{2n}, x_{2n+2})}{2} \leq \frac{\omega_1(x_{2n}, x_{2n+1}) + \omega_1(x_{2n+1}, x_{2n+2})}{2}, \]

it follows that

\[ M(x_{2n}, x_{2n+1}) = \max\left\{ \omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n+1}, x_{2n+2}) \right\}. \]

If we suppose that there exists an integer \(n\) such that

\[ \omega_1(x_{2n+1}, x_{2n+2}) \leq \omega_1(x_{2n+1}, x_{2n+2}), \]
then

\[ M(x_{2n}, x_{2n+1}) = \omega_1(x_{2n+1}, x_{2n+2}). \]

Thus

\[ F(\omega_1(x_{2n+1}, x_{2n+2})) \leq \psi(F(\omega_1(x_{2n+1}, x_{2n+2}))), \]

which implies that \( F(\omega_1(x_{2n+1}, x_{2n+2})) = 0 \). Hence, \( x_{2n+1} = x_{2n+2} \) and, from (2), \( x_{2n} = x_{2n+1} \), a contradiction. Hence, for each integer \( n \), we have

\[ \omega_1(x_{2n+1}, x_{2n+2}) \leq \omega_1(x_{2n+1}, x_{2n+1}). \]

By the same argument, if we take, in inequality (1), \( x = x_{2n-1} \) and \( y = x_{2n} \), we obtain

\[ \omega_1(x_{2n}, x_{2n+1}) < \omega_1(x_{2n-1}, x_{2n}) \quad \text{for all} \quad n \in \mathbb{N}^*. \]

Then \( \omega_1(x_{n+1}, x_{n+2}) < \omega_1(x_n, x_{n+1}) \) for all \( n \in \mathbb{N} \). Thus, the sequence \( \{\omega_1(x_n, x_{n+1})\} \) is decreasing and bounded below. Therefore it is \( \omega \)-convergent to some \( r \geq 0 \). Since

\[ \lim_{n \to +\infty} M(x_{2n}, x_{2n+1}) = \lim_{n \to +\infty} \max \{\omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n+1}, x_{2n+2})\} = r, \]

by letting to limit superior in inequality (2), we obtain

\[ F(r) \leq \lim_{n} \sup \psi(F(M(x_{2n}, x_{2n+1})) \leq \psi(F(r)), \]

which implies that \( r = 0 \). Thus, \( \lim_{n \to +\infty} \omega_1(x_n, x_{n+1}) = 0 \).

Let us prove that the sequence \( \{x_n\} \) is \( \omega \)-Cauchy. For this, it is sufficient to show that the subsequence \( \{x_{2n}\} \) is \( \omega \)-Cauchy. Assume the contrary. Then, according to Lemma 2.2, there exists \( \varepsilon > 0 \) such that we can find two subsequences \( \{m_k\} \) and \( \{n_k\} \) of positive integers satisfying \( n_k > m_k \geq k \) such that the following inequalities hold:

\[ \omega_1(x_{2m_k}, x_{2m_k}) \geq \varepsilon \quad \text{and} \quad \omega_1(x_{2m_k-1}, x_{2m_k}) < \varepsilon. \]

If we take \( x = x_{2m_k} \) and \( y = x_{2m_k-1} \), then \( y \in [x]_G \) and inequality (1) becomes

\[ \psi(F(\omega_1(x_{2m_k+1}, x_{2m_k}))) \leq F(M(x_{2m_k}, x_{2m_k-1})), \]

where

\[ M(x_{2m_k}, x_{2m_k-1}) = \max \left\{ \omega_1(x_{2m_k}, x_{2m_k-1}), \omega_1(x_{2m_k-1}, x_{2m_k}), \omega_1(x_{2m_k-1}, x_{2m_k}), \omega_1(x_{2m_k+1}, x_{2m_k}), \omega_2(x_{2m_k}, x_{2m_k}) + \omega_2(x_{2m_k-1}, x_{2m_k+1}) \right\} / 2 \]

Since

\[ \varepsilon \leq \omega_2(x_{2m_k}, x_{2m_k}) \leq \omega_2(x_{2m_k}, x_{2m_k}) \]
\[ \leq \omega_1(x_{2nk}, x_{2mk}) \]
\[ \leq \frac{1}{2} \omega_1(x_{2nk-1}, x_{2mk}) + \frac{1}{2} \omega_1(x_{2nk-1}, x_{2mk}) \]
\[ \leq \frac{1}{4} \omega_1(x_{2nk-1}, x_{2mk}) + \frac{1}{4} \omega_1(x_{2nk-1}, x_{2mk}) \]
\[ \leq \varepsilon + \frac{1}{4} \omega_1(x_{2nk-1}, x_{2mk}) \]

It follows that \( \lim_{k \to +\infty} \omega_2(x_{2nk}, x_{2mk}) = \lim_{k \to +\infty} \omega_1(x_{2nk}, x_{2mk}) = \varepsilon \).

From
\[ \varepsilon \leq \omega_2(x_{2nk}, x_{2mk}) \leq \omega_1(x_{2nk}, x_{2mk}) + \omega_1(x_{2nk+1}, x_{2mk}), \]
we get
\[ \varepsilon - \omega_1(x_{2nk}, x_{2mk+1}) \leq \omega_1(x_{2nk+1}, x_{2mk}) \]
\[ \leq \frac{1}{4} \omega_1(x_{2nk-1}, x_{2mk}) + \frac{1}{4} \omega_1(x_{2nk-1}, x_{2mk}) \]
\[ \leq \frac{1}{4} \omega_1(x_{2nk-1}, x_{2mk}) + \frac{1}{4} \omega_1(x_{2nk-1}, x_{2mk+1}) \]
\[ \leq \varepsilon + \frac{1}{4} \omega_1(x_{2nk-1}, x_{2mk+1}). \]

Thus
\[ \lim_{k \to +\infty} \omega_1(x_{2nk+1}, x_{2mk}) = \varepsilon. \]

Similarly, using
\[ \varepsilon \leq \omega_2(x_{2nk}, x_{2mk}) \leq \omega_1(x_{2nk}, x_{2mk+1}) + \omega_1(x_{2nk+1}, x_{2mk}), \]
we get
\[ \varepsilon - \omega_1(x_{2nk+1}, x_{2mk}) \leq \omega_1(x_{2nk}, x_{2mk+1}) \]
\[ \leq \frac{1}{4} \omega_1(x_{2nk+1}, x_{2mk}) + \frac{1}{4} \omega_1(x_{2nk+1}, x_{2mk+1}) \]
\[ \leq \frac{1}{4} \omega_1(x_{2nk+1}, x_{2mk}) + \frac{1}{4} \omega_1(x_{2nk+1}, x_{2mk+1}) \]
\[ \leq \varepsilon + \frac{1}{4} \omega_1(x_{2nk+1}, x_{2mk+1}). \]

Therefore \( \lim_{k \to +\infty} \omega_1(x_{2nk}, x_{2mk+1}) = \varepsilon. \)

From
\[ \omega_2(x_{2nk}, x_{2mk}) - \omega_4(x_{2nk}, x_{2mk+1}) - \omega_2(x_{2nk+1}, x_{2mk}) \]
\[ \leq \omega_2(x_{2nk+1}, x_{2mk}) \]
\[ \leq \omega_1(x_{2nk+1}, x_{2mk}) + \omega_1(x_{2mk}, x_{2mk+1}), \]
we get \( \lim_{k \to +\infty} \omega_2(x_{2nk+1}, x_{2mk+1}) = \varepsilon. \) Since
\[ \omega_2(x_{2mk}, x_{2mk+1}) \leq \omega_1(x_{2mk}, x_{2mk+1}), \]
we get \( \omega_2(x_{2mk+1}, x_{2mk}) \leq \omega_1(x_{2mk+1}, x_{2mk+1}) \).
and by letting $k \to +\infty$, we obtain $\lim_{k \to +\infty} \omega_1(x_{2m_k-1}, x_{2m_k+1}) = \epsilon$. Therefore

$$\lim_{k \to +\infty} M(x_{2m_k}, x_{2m_k-1}) = \epsilon.$$  

From the continuity of $F$ and the upper semicontinuity of $\psi$, we have

$$F(\epsilon) \leq \psi(F(\epsilon)),$$

a contradiction since $\epsilon > 0$. Therefore the sequence $\{x_n\}$ is $\omega$-Cauchy. Using the $\omega$-completeness of $C$, there exists $x^* \in C$ such that $\lim_{n \to +\infty} \omega_1(x_n, x^*) = 0$. The property (OSC) insures that $x^* \in [x_n]$ for all $n \in \mathbb{N}$. Then

$$F(\omega_1(Sx_{2n}, Tx^*)) \leq \psi(F(M(x_{2n}, x^*))),$$

where

$$M(x_{2n}, x^*) = \max \left\{ \omega_1(x_{2n}, x^*), \omega_1(x_{2n}, x_{2n+1}), \omega_1(x^*, Tx^*), \frac{\omega_2(x_{2n}, Tx^*) + \omega_2(x^*, x_{2n+1})}{2} \right\}.$$  

Since $\omega_2(x_{2n}, Tx^*) \leq \omega_1(x_{2n}, x^*) + \omega_1(x^*, Tx^*)$, $\lim_{n} M(x_{2n}, x^*) = \omega_1(x^*, Tx^*)$.

Using the continuity of $F$ and the upper continuity of $\psi$, we obtain

$$\limsup_{n} \psi(F(M(x_{2n}, x^*))) \leq \psi(F(\omega_1(x^*, Tx^*))).$$

By the Fatou property, we have

$$\omega_1(x^*, Tx^*) \leq \liminf_{n} \omega_1(Sx_{2n}, Tx^*).$$

Since $F$ is continuous and nondecreasing on $[0, +\infty[$, we have

$$F(\omega_1(x^*, Tx^*)) \leq F\left(\liminf_{n} \omega_1(Sx_{2n}, Tx^*)\right) \leq F\left(\liminf_{n} \omega_1(Sx_{2n}, Tx^*)\right) \leq \limsup_{n} F(\omega_1(Sx_{2n}, Tx^*)) \leq \limsup_{n} \psi(F(M(x_{2n}, x^*))) \leq \psi(F(\omega_1(x^*, Tx^*))).$$
which implies that $\omega_1(x^*, Tx^*) = 0$, and according to the regularity of $\omega$, we have $Tx^* = x^*$.

Since $x^* \in [x^*]_G$, $F(\omega_1(Sx^*, Tx^*)) \leq \psi(F(M(x^*, x^*)))$ where

$$M(x^*, x^*) = \max \{\omega_1(x^*, Sx^*), \omega_2(x^*, Sx^*)\} = \omega_1(x^*, Sx^*),$$

which implies that $F(\omega_1(Sx^*, x^*)) \leq \psi(F(\omega_1(Sx^*, x^*)))$. Hence $\omega_1(Sx^*, x^*) = 0$ and the regularity of $\omega$ insures that $Sx^* = x^*$.

□

The next example illustrates Theorem 2.1 and shows that the class of mappings satisfying our main result is a proper nonempty subset of the set of the mappings considered in [13].

**Example 2.3** Consider the modular metric space $(X, \omega)$ where

$$X = [0, 1] \quad \text{and} \quad \omega_\lambda(x, y) = \frac{|x - y|^2}{2\lambda} \quad \text{for all} \quad \lambda \in [0, +\infty) \quad \text{and} \quad x, y \in X.$$

Consider the reflexive digraph $G = (X, E)$ represented in Fig. 1, where

$$E = \Delta \cup \left\{ \left( \frac{1}{3^n}, 0 \right), \left( \frac{1}{3^n}, \frac{1}{3^n + 1} \right) : n \in \mathbb{N} \right\}.$$

Consider the two self-mapping $S$ and $T$ defined on $X$ by

$$Tx = \frac{x}{3} \quad \text{and} \quad Sx = \frac{x}{9} \quad \text{for all} \quad x \in X,$$

and the two functions $F$ and $\psi$ defined on $[0, +\infty[$ by

$$F(t) = \sqrt{t} \quad \text{and} \quad \psi(t) = \frac{t}{\sqrt{2}} \quad \text{for all} \quad t \in [0, +\infty[.$$

We can see that

1. $X$ is $\omega$-complete;

![Figure 1](image_url)
Figure 2 The digraph $G(O_1(S, T))$ (the loops are not represented)

2. $\omega$ satisfies the $\Delta_2$-type condition and the Fatou property;
3. $G[O_1(S, T)]$ is a directed path with a unique starting point $x_0$ (see Figure 2).

Let us show that, for all $x, y \in C$,

$$\left( y \in [x]_G \text{ or } x \in [y]_G \right) \implies F(\omega_1(Sx, Ty)) \leq \psi(F(M(x, y))).$$

For this, we proceed by disjunction of the cases:

- The case where $x = y = 0$ is avoided.
- If $x = \frac{1}{3^m}$ for $n \in \mathbb{N}$ and $y = 0$, then
  $$F(\omega_1(Sx, Ty)) = \frac{1}{\sqrt[3]{2} \cdot 3^{n+2}} \leq \frac{1}{2 \cdot 3^n} \leq \psi(F(M(x, y))).$$
- If $x = 0$ and $y = \frac{1}{3^n}$ for $n \in \mathbb{N}$, then
  $$F(\omega_1(Sx, Ty)) = \frac{1}{\sqrt[3]{2} \cdot 3^{n+1}} \leq \frac{1}{2 \cdot 3^n} \leq \psi(F(M(x, y))).$$
- If $x = y = \frac{1}{3^n}$ for $n \in \mathbb{N}$, then
  $$F(\omega_1(Sx, Ty)) = \frac{\sqrt[3]{2}}{3^{n+2}} \leq \frac{4}{3^n} \leq \psi(F(M(x, y))).$$
- If $x = \frac{1}{3^m}$ and $y = \frac{1}{3^n}$ for $m, n \in \mathbb{N}$ such that $m > n$, then
  $$F(\omega_1(Sx, Ty)) = \frac{1}{\sqrt[3]{2}} \left( \frac{1}{3^{m+2}} - \frac{1}{3^{n+1}} \right) \leq \frac{4}{3^{n+2}} \leq \psi(F(M(x, y))).$$
- If $x = \frac{1}{3^m}$ and $y = \frac{1}{3^n}$ for $m, n \in \mathbb{N}$ such that $m > n$, then
  $$F(\omega_1(Sx, Ty)) = \frac{1}{\sqrt[3]{2}} \left( \frac{1}{3^{m+2}} - \frac{1}{3^{n+1}} \right) \leq \frac{\sqrt[3]{2}}{3^{n+1}} \leq \psi(F(M(x, y))).$$

All assumptions of Theorem 2.1 are satisfied and $S$ and $T$ have a fixed point $x^* = 0$.

Remark 2.4 In Example 2.3, if we consider the function $\psi(t) = 0.8 \times \ln(1 + t)$ for all $t \in [0, +\infty[$, we get

$$F(d(Sx, Ty)) = \frac{1}{2} \times 0.8 \ln \left( 1 + \frac{\sqrt{3}}{2} \right) = \psi(F(M'(x, y))) \quad \text{for } x = 0 \text{ and } y = \frac{3}{4}.$$
where \( d(x, y) = |x - y| \) and

\[
M'(x, y) = \max \left\{ \frac{d(x, y) + d(Tx, y) + d(Sy, x)}{2}, d(x, y), d(Tx, x), d(Sy, y) \right\}.
\]

Theorem on page 2 is not applicable, but by Theorem 2.1, we obtain the existence of a common fixed point of \( S \) and \( T \). Indeed, we have, for all \( x, y \in X \),

\[
(y \in [x]_G \text{ or } x \in [y]_G) \implies F(\omega_1(Sx, Ty)) \leq \psi(F(M(x, y))).
\]

**Corollary 2.2** Let \((X, \omega, G)\) be a modular metric space endowed with a reflexive digraph \( G \) where \( \omega \) satisfies the \( \Delta_2 \)-type condition and the Fatou property. Let \( C \) be an \( \omega \)-complete nonempty subset of \( X_\omega \) and \( T, S : C \to C \) be two self-mappings. If the following conditions are satisfied:

(i) there exists \( k \in [0,1[ \) such that, for all \( x, y \in C \),

\[
(y \in [x]_G \text{ or } x \in [y]_G) \implies \omega_1(Sx, Ty) \leq (1 + \omega_1(x, y))k - 1;
\]

(ii) there exists an element \( x_0 \in C \) such that \( G[C_\omega(S, T)] \) is a directed path with a unique starting point \( x_0 \),

(iii) \( \omega \) satisfies the property (OSC),

then \( S \) and \( T \) have a common fixed point in \( C \).

**Proof** If we consider the two functions \( F \) and \( \psi \) defined on \([0, +\infty[\) by

\[
F(t) = \ln(1 + t) \quad \text{and} \quad \psi(t) = kt,
\]

then we can verify that the second part of implication (4) is equivalent to

\[
F(\omega_1(Sx, Ty)) \leq \psi(F(\omega_1(x, y))),
\]

which implies that \( F(\omega_1(Sx, Ty)) \leq \psi(F(M(x, y))) \), since \( F \) and \( \psi \) are nondecreasing on \([0, +\infty[\). By applying Theorem 2.1, we terminate the demonstration. \( \square \)

In the sequel, we use the following lemma.

**Lemma 2.5** ([5]) Let \((X, \omega)\) be a modular space such that \( \omega \) is convex and satisfies the \( \Delta_2 \)-condition. If \( \{x_n\} \) is a sequence in \( X_\omega \) such that \( \lim_{n \to +\infty} \omega_1(x_n, x_{n+1}) = 0 \), then \( \{x_n\} \) is \( \omega \)-Cauchy.

**Theorem 2.3** Let \((X, \omega, G)\) be a modular metric space endowed with a reflexive digraph \( G \) where \( \omega \) is convex and satisfies the \( \Delta_2 \)-type condition and the Fatou property. Let \( C \) be an \( \omega \)-complete nonempty subset of \( X_\omega \) and \( T, S : C \to C \) be two self-mappings. If the following conditions are satisfied:

(i) for all \( x, y \in C \),

\[
(y \in [x]_G \text{ or } x \in [y]_G) \implies F(\omega_1(Sx, Ty)) \leq \psi(F(M(x, y)));
\]

(ii) \( \omega \) satisfies the property (OSC),

then \( S \) and \( T \) have a common fixed point in \( C \).

Proof If we consider the two functions \( F \) and \( \psi \) defined on \([0, +\infty[\) by

\[
F(t) = \ln(1 + t) \quad \text{and} \quad \psi(t) = kt,
\]

then we can verify that the second part of implication (4) is equivalent to

\[
F(\omega_1(Sx, Ty)) \leq \psi(F(\omega_1(x, y))),
\]

which implies that \( F(\omega_1(Sx, Ty)) \leq \psi(F(M(x, y))) \), since \( F \) and \( \psi \) are nondecreasing on \([0, +\infty[\). By applying Theorem 2.1, we terminate the demonstration. \( \square \)
where
\[ M(x, y) = \max \{ \omega_1(x, y), \omega_1(x, Sx), \omega_1(y, Ty), \omega_2(x, Ty) + \omega_2(y, Sx) \}; \]

(ii) there exists an element \( x_0 \in C \) such that \( G[O_{x_0}(S, T)] \) is a directed path with a unique starting point \( x_0 \);

(iii) \( \omega \) satisfies the property (OSC),

then \( S \) and \( T \) have a common fixed point in \( C \) and \( \mathcal{S}(S, T) = \mathcal{S}(S) = \mathcal{S}(T) \), where \( \mathcal{S}(T) \) is the set of fixed points of \( T \).

Proof Let \( x_0 \) an element of \( C \) such that \( G[O_{x_0}(S, T)] \) is a directed path. Consider the sequence \( \{x_n\} \) defined by

\[ x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1} \quad \text{for all} \quad n \in \mathbb{N}. \]

Condition (ii) insures that \( \{x_n\} \) is \( G \)-nondecreasing. If there exists an integer \( n \) such that \( x_{2n} = x_{2n+1} = x_{2n+2} \), then \( x_{2n} \) is a common fixed point of \( S \) and \( T \). Otherwise, suppose that

\[ x_{2n} \neq x_{2n+1} \quad \text{or} \quad x_{2n} \neq x_{2n+2} \quad \text{for all} \quad n \in \mathbb{N}. \]

Let \( n \in \mathbb{N} \). From \( x_{2n+1} \in [x_{2n}]_G \) and applying (5) for \( x = x_{2n} \) and \( y = x_{2n+1} \), we obtain

\[ F(\omega_1(x_{2n+1}, x_{2n+2})) \leq \psi(F(M(x_{2n}, x_{2n+1}))). \]  (6)

From

\[ M(x_{2n}, x_{2n+1}) = \max \{ \omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n+1}, x_{2n+2}), \omega_2(x_{2n}, x_{2n+2}) \}, \]

since \( \omega \) is convex,

\[ \omega_2(x_{2n}, x_{2n+2}) \leq \frac{\omega_1(x_{2n}, x_{2n+1}) + \omega_1(x_{2n+1}, x_{2n+2})}{2}, \]

from which it follows that

\[ M(x_{2n}, x_{2n+1}) = \max \{ \omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n+1}, x_{2n+2}) \}. \]

By the same arguments as in the proof of Theorem 2.1, we prove that

\[ \lim_{n \to +\infty} \omega_1(x_n, x_{n+1}) = 0. \]

According to Lemma 2.5, the sequence \( \{x_n\} \) is \( \omega \)-Cauchy, and since \( C \) is \( \omega \)-complete, then \( \{x_n\} \) is \( \omega \)-convergent to an element \( x^* \in C \). Again similar to the proof of Theorem 2.1, we prove that \( x^* \) is a common fixed point of \( S \) and \( T \). □
3 Application

Consider the space $X = C^1([0,1], \mathbb{R})$. Let $G = (X, E)$ be the digraph such that, for all $x, y \in X$,

$$(x, y) \in E \iff x(t) \leq y(t) \quad \text{for each } t \in [0,1].$$

Consider the function $\omega : [0, +\infty[ \times X \times X \rightarrow [0, +\infty]$ defined, for each $\lambda \in ]0, +\infty[$ and $x, y \in X$, by

$$\omega(\lambda, x, y) = \frac{1}{\lambda} \| x - y \|_{\infty}^2 = \frac{1}{\lambda} \left( \sup_{t \in [0,1]} |x(t) - y(t)| \right)^2.$$

It is easy to check the following result.

**Lemma 3.1** The function $\omega$ is a modular metric satisfying the following:

(i) $\omega$ satisfies the $\Delta_{2}$-type condition and the Fatou property;
(ii) $X_{\omega} = X$ is $\omega$-complete;
(iii) $\omega$ satisfies the (OSC) property.

Let us consider the following integral equations system:

$$(IES): \quad \begin{cases} x(t) = \int_{0}^{1} f(t, y(s)) \, ds + a(t) & \forall t \in [0,1], \\ y(t) = \int_{0}^{1} g(t, x(s)) \, ds + a(t) & \forall t \in [0,1], \end{cases}$$

where $a \in X$ and $f, g : [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ are two mappings such that $f$ and $g$ are of the class $C^1$ on $[0,1] \times \mathbb{R}$.

Let us consider the two mappings $T$ and $S$ defined in $X$ as follows:

$$\begin{cases} Tx(t) = \int_{0}^{1} f(t, x(s)) \, ds + a(t), \\ Sx(t) = \int_{0}^{1} g(t, x(s)) \, ds + a(t), \end{cases} \quad t \in [0,1].$$

One can see that $Tx$ and $Sx$ are in $X$ for all $x \in X$.

**Theorem 3.2** If the following two conditions are satisfied:

(i) for every $s, t \in [0,1]$ and for all comparable elements $x, y \in X$,

$$|f(t, x(s)) - g(t, y(s))| \leq -1 + \sqrt{1 + |x(s) - y(s)|},$$

(ii) there exists $x_0 \in X$ such that, for all $t \in [0,1]$, we have

$$x_0(t) \leq Sx_0(t) \leq TSx_0(t) \leq STSx_0(t) \leq (TS)^2 x_0(t) \leq S(TS)^2 x_0(t) \leq \cdots,$$

then the system (IES) admits at least a solution which belongs to the diagonal of $X^2$.

**Proof** Let $x$ and $y$ be two comparable elements in $X$, that is, $x \in [y]_G$ or $y \in [x]_G$. Since, for each $t, s \in [0,1]$,

$$|f(t, x(s)) - g(t, y(s))| \leq -1 + \sqrt{1 + |x(s) - y(s)|} \leq -1 + \sqrt{1 + \|x(s) - y(s)\|_{\infty}}$$
and
\[
\|Tx - Sy\|_\infty = \sup_{t \in [0,1]} |x(t) - y(t)| = \sup_{t \in [0,1]} \int_0^1 |f(t, x(s)) - g(t, y(s))| \, ds,
\]
we have
\[
\|Tx - Sy\|_\infty \leq -1 + \sqrt{1 + \|x - y\|_\infty^2}.
\]
Since
\[
(-1 + \sqrt{1 + \|x - y\|_\infty})^2 \leq -1 + \sqrt{1 + \|x - y\|_\infty^2},
\]
we have
\[
\omega_1(Tx, Sy) \leq -1 + (1 + \omega_1(x, y))^\frac{1}{2}.
\]
Since, for all \( t \in [0,1] \),
\[
x_0(t) \leq Sx_0(t) \leq TSx_0(t) \leq STSx_0(t) \leq (TS)^2x_0(t) \leq S(TS)^2x_0(t) \leq \cdots,
\]
the induced subgraph \( G(O_{x_0}(S, T)) \) is a directed path with the unique starting point \( x_0 \).

According to Corollary 2.2, \( T \) and \( S \) have a common fixed point in \( X \), i.e., there exists an element \( x^* \in X \) such that \( (x^*, x^*) \) verifies the system (IES). Then the system (IES) admits at least a solution in \( X^2 \) which belongs to \( \Delta(X \times X) = \{(u, u) / u \in X\} \) the diagonal of \( X^2 \).

**Conclusion** Our results improve, extend, and generalize some classical results:

(i) *In Theorem 2.3, if we take \( \omega_\lambda(x, y) = \frac{d(x, y)}{\lambda} \) for all \( \lambda \in [0, +\infty[ \), we get an improved version of the main result of Zhang [13, Theorem 1] by removing condition (iii) verified by the function \( \phi \) and the monotony of \( \phi \).*

(ii) *In Theorem 2.1, if the function \( F \) is the identity and the function \( \psi \) is nondecreasing, we obtain an analogue of [4, Theorem 2] but for a common fixed point in the setting of modular metric spaces with graph.*

(iii) *Theorem 2.3 generalizes and extends [3, Theorem 2.1] in the setting of a modular metric space with graph.*

(iv) *Corollary 2.2 generalizes and extends [1, Theorem 3.1] in the setting of modular metric spaces with graph, since*
\[
\omega_1(Sx, Ty) \leq k\omega_1(x, y) \implies \omega_1(Sx, Ty) \leq (1 + \omega_1(x, y))^k - 1.
\]

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