Level truncation analysis of regularized identity based solutions

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Abstract

We evaluate the vacuum energy of regularized identity based solutions through level truncation computations in open bosonic string field theory. We show that the level truncated solutions bring a value of the vacuum energy expected for the tachyon vacuum in agreement with Sen’s first conjecture.
1 Introduction

In a previous work [1], we have studied the validity of a proposal for regularizing identity based solutions [2] in open string field theory [3]. The regularization is based on the realization that a one-parameter family of solutions, of the open string field equations of motion, can be constructed by a gauge transformation on an identity based solution [1, 2]

\[ \Psi_\lambda = U_\lambda(Q_B + \Psi_I)U_\lambda^{-1}, \]

(1.1)

where \( \Psi_I \) denotes the identity based solution and \( U_\lambda \) is an element of the gauge transformation defined as

\[ U_\lambda = 1 + \lambda cB K, \quad U_\lambda^{-1} = 1 - \lambda cB \frac{K}{1 + \lambda K}. \]

(1.2)

Explicitly the one-parameter family of solutions \( \Psi_\lambda \), which will be referred as the regularized solution, is obtained by performing the gauge transformation on the identity based solution \( \Psi_I = c(1 - K) \)

\[ \Psi_\lambda = c(1 + \lambda K)Bc \frac{1 + (\lambda - 1)K}{1 + \lambda K}. \]

(1.3)

This solution interpolates between the identity based solution \( \Psi_I = \Psi_{\lambda \to 0} \) and Erler-Schnabl’s solution \( \Psi_{E-S} = \Psi_{\lambda \to 1} \) [4] [5]. The value of the vacuum energy, using the
regularized solution, was computed by evaluating formal CFT correlation functions in the so-called \( KBc \) subalgebra \[6\] and by using \( L_0 \) level truncation computations \[1\].

Nevertheless, it remains the analysis of the regularized solution using the traditional Virasoro \( L_0 \) level truncation scheme \[7, 8, 9, 10, 11, 12\]. This analysis is important since we want to know if the solution behaves as a regular element in the state space constructed out of Fock states. Specifically the analysis of the coefficients appearing in the \( L_0 \) level expansion provides one criterion for the solution being well defined \[13, 14, 15\]. Furthermore the \( L_0 \) level expansion of the solution brings an additional way to numerically test Sen’s first conjecture.

To expand the regularized solution in the Virasoro basis of \( L_0 \) eigenstates, it is convenient to write an integral definition for the rational function

\[
1 + \left( \frac{\lambda - 1}{\lambda} \right) \frac{K}{1 + \lambda K} = \int_0^\infty dt e^{-\frac{t}{\lambda}} \left[ \frac{1}{\lambda} - \frac{\lambda - 1}{\lambda} \partial_t \right] e^{-\lambda t},
\]

where \( \partial_t \equiv \frac{\partial}{\partial t} \) and \( e^{-\lambda t} \equiv \Omega^t \) is the wedge state with \( t \geq 0 \) \[16, 17, 18, 19, 20, 21\]. Let us mention that the wedge state can be expressed in terms of the well known scaling operator \( U_r \) \[17, 22, 23, 24, 25, 26\]

\[
\Omega^t = U_{t+1} W_{t+1} |0\rangle,
\]

where \( U_r \equiv \left( \frac{e^{\pi r}}{e^{\pi i r}} \right) L_0 \).

The operator \( L_0 \) is defined in the sliver frame \[5, 17, 27, 28\], and it is related to the worldsheet energy-momentum tensor as follows

\[
L_0 = \oint \frac{dz}{2\pi i} (1 + z^2)(\arctan z) T(z) = L_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2 - 1} L_{2k}.
\]

Employing the above considerations and noting that under the conformal map defined by \( \tilde{z} = \frac{2}{\pi} \arctan z \) the c ghost transforms as \( \tilde{c}(\tilde{z}) = \frac{2}{\pi} \cos^2(\frac{\pi}{2} \tilde{z}) c \left( \tan \left( \frac{\pi}{2} \tilde{z} \right) \right) \), we will show that the regularized solution can be written like

\[
\Psi_\lambda = \frac{2}{\pi} \int_0^\infty dt e^{-\frac{t}{\lambda}} \left[ \frac{1}{\lambda} - \frac{\lambda - 1}{\lambda} \partial_t \right] \left[ \left( \frac{\lambda - 1}{\lambda} \right) \cos^2 \left( \frac{\pi t}{2t+2} \right) U_{t+1} c \left( \tan \left( \frac{\pi t}{2t+2} \right) \right) \right] |0\rangle
\]

\[
+ Q_B \left\{ \lambda \int_0^\infty dt e^{-\frac{t}{\lambda}} \left[ \frac{1}{\lambda} - \frac{\lambda - 1}{\lambda} \partial_t \right] \left[ \cos^2 \left( \frac{\pi t}{2t+2} \right) U_{t+1} B_0^c \left( \tan \left( \frac{\pi t}{2t+2} \right) \right) \right] |0\rangle \right\}.
\]

This expression allows us to expand the regularized solution using states in the Virasoro basis of \( L_0 \) eigenstates. For example, let us expand the regularized solution up to level two states

\[
\Psi_\lambda = t(\lambda)c_1 |0\rangle + u(\lambda)c_0 |0\rangle + v(\lambda)c_{-1} |0\rangle + w(\lambda)L_{-2} c_1 |0\rangle + \cdots + (Q_B\text{-exact terms}),
\]

\[1\] Remember that a point in the upper half plane \( z \) is mapped to a point in the sliver frame \( \tilde{z} \) via the conformal mapping \( \tilde{z} = \frac{2}{\pi} \arctan z \). Note that we are using the convention of [5] for the conformal mapping.
where the coefficients of the expansion \(t(\lambda), u(\lambda), v(\lambda)\) and \(w(\lambda)\) are given by the following integrals

\[
t(\lambda) = \frac{2}{\pi} \int_0^\infty dt \, e^{-\frac{t}{\lambda}} \left( \frac{1}{\lambda} - \frac{(\lambda - 1)}{\lambda} \partial_t \right) \left[ \frac{1}{4} (t + 1)^2 \cos^2 \left( \frac{\pi t}{2t + 2} \right) \right] , \quad (1.9)
\]

\[
u(\lambda) = \frac{2}{\pi} \int_0^\infty dt \, e^{-\frac{t}{\lambda}} \left( \frac{1}{\lambda} - \frac{(\lambda - 1)}{\lambda} \partial_t \right) \left[ \frac{1}{4} (t + 1) \sin \left( \frac{\pi t}{t + 1} \right) \right] , \quad (1.10)
\]

\[
v(\lambda) = \frac{2}{\pi} \int_0^\infty dt \, e^{-\frac{t}{\lambda}} \left( \frac{1}{\lambda} - \frac{(\lambda - 1)}{\lambda} \partial_t \right) \left[ \sin^2 \left( \frac{\pi t}{2t + 2} \right) \right] , \quad (1.11)
\]

\[
w(\lambda) = \frac{2}{\pi} \int_0^\infty dt \, e^{-\frac{t}{\lambda}} \left( \frac{1}{\lambda} - \frac{(\lambda - 1)}{\lambda} \partial_t \right) \left[ \frac{1}{12} (3 - t^2 - 2t) \cos^2 \left( \frac{\pi t}{2t + 2} \right) \right] . \quad (1.12)
\]

These integrals are convergent provided that the parameter \(\lambda\) belongs to the interval \((0, +\infty)\). The range of validity for this parameter \(\lambda\) was also studied using the \(L_0\) level expansion of the regularized solution [1]. Let us comment that this result, relating to the parameter \(\lambda\), is quite similar to the well known case of Schnabl-Okawa’s one-parameter family of solutions analyzed in [15, 17, 22, 23] where it was shown that the parameter \(\lambda\) belongs to the interval \((0, 1)\) and the limit case \(\lambda \to 1\) precisely corresponds to the original Schnabl’s analytic solution [17].

In this paper, by considering the traditional Virasoro \(L_0\) level expansion of the regularized solution \(\Psi_\lambda\), we evaluate the value of the vacuum energy for various values of the parameter \(\lambda > 0\), and show that in the limit case \(\lambda \to 0\), which corresponds to the identity based solution, the regularized solution brings a value of the vacuum energy expected for the tachyon vacuum in agreement with Sen’s first conjecture [29, 30]. The results of our computations provide an additional support to the fact that to extract a finite value for the vacuum energy, the identity based solution can be defined as the limit of a gauge equivalent one-parameter family of solutions. Intuitively this approach works since the vacuum energy is a gauge invariant quantity so that its value is insensitive to the change of the parameter, thus after performing the computations we can safely take the limit. Nevertheless the computation of the relevant gauge invariant quantities using directly the identity based solution remains as an unsolved problem. As noted in [14], in order to solve this issue, it should be interesting to rephrase the problem in terms of distribution theory [31]. We hope that explorations in these directions also might give a lesson to define the space of string fields appropriately.

In addition to the level truncation analysis of the regularized solution \(\Psi_\lambda\), we will study the level truncation of another solution \(\tilde{\Psi}_\lambda\) which satisfies the string field reality condition\(^2\)

\[
\tilde{\Psi}_\lambda = \sqrt{\frac{1 + (\lambda - 1)K}{1 + \lambda K}} (c + \lambda cKBc) \sqrt{\frac{1 + (\lambda - 1)K}{1 + \lambda K}} . \quad (1.13)
\]

\(^2\)In open string field theory, the string field reality condition over a string field \(\Phi\) is given by \(\Phi^\dagger = \Phi\), where the operator \(\dagger\) is defined by the composition of BPZ and Hermitian conjugation [32].
Because of the square root in (1.13), this real solution \( \hat{\Psi} \) seems to be more complicated than the previous one \( \Psi \). But for some purposes the real solution is more convenient. For instance, at some fixed level the number of terms contained in the level expansion of the real solution will be less than the one given in the non-real solution. This difference in the number of terms will be reflected in the evaluation of the vacuum energy, at each level using the real solution we will get less terms to be computed than the ones obtained from the non-real solution. Another advantage of the real solution is related to the convergence property regarding to the evaluation of the vacuum energy. Each time we increase the level of the truncated solution, the value of the vacuum energy gets closer to the expected value predicted from Sen’s first conjecture. For the case of the real solution it turns out that the convergence to this expected value is faster than the one using the non-real solution.

This paper is organized as follows. In section 2, we study the way for regularizing an identity based solution in open bosonic string field theory and by expanding the regularized solution in the Virasoro basis of \( L_0 \) eigenstates, we compute the vacuum energy. In section 3, we introduce another one-parameter family of solutions which satisfies the string field reality condition. By performing a gauge transformation, we show that this real solution can be related to the non-real solution. The level truncation computation of the vacuum energy is analyzed in a similar manner as for the non-real solution. In section 4, a summary and further directions of exploration are given.

2 Regularized identity based solution in open string field theory

In this section, first we are going to review the way for regularizing an identity based solution in open bosonic string field theory and then we will analyze the solution from the level truncation point of view.

2.1 Derivation of the regularized solution

Let us mention that the main results showed in this subsection will be extracted from the references [1, 2, 4]. As discussed in [4] using the methods of [25, 26], an identity based solution in open bosonic string field theory is given by

\[
\Psi_I = c(1 - K),
\]

In what follows, the regularized solution \( \Psi_\lambda \) will be referred as the non-real solution.
where the basic string fields $c$ and $K$ (together with $B$) can be written, using the operator representation \cite{17}, as follows

\begin{align}
K & \to \frac{1}{2} \hat{L} U_1^\dagger U_1 |0\rangle, \\
B & \to \frac{1}{2} \hat{B} U_1^\dagger U_1 |0\rangle, \\
c & \to U_1^\dagger U_1 \tilde{c}(0) |0\rangle. 
\end{align}

The operators $\hat{L}$, $\hat{B}$ and $\tilde{c}(0)$ are defined in the sliver frame \cite{3}, and they are related to the worldsheet energy-momentum tensor, the $b$ and $c$ ghosts fields respectively, for instance

\begin{align}
\hat{L} & \equiv L_0 + L_0^\dagger = \oint \frac{dz}{2\pi i} (1 + z^2)(\arctan z + \text{arccot} z) T(z), \\
\hat{B} & \equiv B_0 + B_0^\dagger = \oint \frac{dz}{2\pi i} (1 + z^2)(\arctan z + \text{arccot} z) b(z),
\end{align}

while the operator $U_1^\dagger U_1$ in general is given by $U_r^\dagger U_r = e^{2z^2z\hat{L}}$, so we have chosen $r = 1$, note that the string field $U_1^\dagger U_1 |0\rangle$ represents the identity string field $1 \to U_1^\dagger U_1 |0\rangle$ \cite{17,22,25,26}.

Using the operator representation \cite{2.2}-\cite{2.4} of the string fields $K$, $B$ and $c$, we can show that these fields satisfy the algebraic relations

\begin{align}
\{ B, c \} &= 1, \\
[B, K] &= 0, \\
B^2 &= c^2 = 0,
\end{align}

and have the following BRST variations

\begin{align}
Q_BK &= 0, \\
Q_BB &= K, \\
Q_Bc &= cKc.
\end{align}

As it is shown in \cite{14}, the direct evaluation of the vacuum energy using the identity based solution \cite{2.1} brings ambiguous result. This phenomenon, as it was noted in \cite{2}, is due to the fact that a naive evaluation of the classical action in terms of CFT methods tends to be indefinite since it corresponds to a correlator on vanishing strip. Recently this problem was overcome and a proposal for regularizing the identity based solution \cite{2.1} has been developed in \cite{2}.

The regularized solution $\Psi_\lambda$ is obtained by considering one-parameter family of classical solutions

\begin{equation}
\Psi_\lambda = \mathcal{U}_\lambda Q_B \mathcal{U}_\lambda^{-1} + \mathcal{U}_\lambda \Psi_1 \mathcal{U}_\lambda^{-1},
\end{equation}

\footnote{Remember that a point in the upper half plane $z$ is mapped to a point in the sliver frame $\tilde{z}$ via the conformal mapping $\tilde{z} = \frac{2}{\pi} \arctan z$. Note that we are using the convention of \cite{5} for the conformal mapping.}
where $\Psi_I$ is the identity based solution (2.1) and

$$U_\lambda = 1 + \lambda cBK, \quad U^-_\lambda = 1 - \lambda cBK \frac{1}{1 + \lambda K} \tag{2.10}$$

is an element of the gauge transformation [2, 4]. Using (2.1), (2.9) and (2.10), it is almost easy to derive the following regularized solution

$$\Psi_\lambda = c(1 + \lambda K)Bc \frac{1 + (\lambda - 1)K}{1 + \lambda K}. \tag{2.11}$$

In order to simplify the evaluation of the vacuum energy, it is convenient to write the regularized solution $\Psi_\lambda$ as an expression containing an exact BRST term

$$\Psi_\lambda = c \left[ 1 + \frac{(\lambda - 1)K}{1 + \lambda K} + Q_B \left\{ \lambda Bc \frac{1 + (\lambda - 1)K}{1 + \lambda K} \right\} \right]. \tag{2.12}$$

Note that this regularized solution interpolates between the identity based solution (2.1) which corresponds to the case $\lambda \to 0$, and the Erler-Schnabl’s solution [5] which corresponds to the case $\lambda \to 1$.

### 2.2 Level expansion of the regularized solution

To expand the previous regularized solution in the Virasoro basis of $L_0$ eigenstates, it is convenient to write an integral definition for the rational function

$$\frac{1 + (\lambda - 1)K}{1 + \lambda K} = \int_0^\infty dt \ e^{-t} \left\{ \frac{1}{\lambda - 1} \partial_t \right\} e^{-Kt}, \tag{2.13}$$

where $\partial_t \equiv \frac{\partial}{\partial t}$ and $e^{-Kt} \equiv \Omega^t$ is the wedge state with $t \geq 0$. Let us mention that the wedge state can be expressed in terms of the well known scaling operator $U_r$ [17, 22, 23, 24, 25, 26]

$$\Omega^t = U^t_{t+1} U_{t+1} |0\rangle, \quad \text{where} \quad U_r \equiv \left( \frac{2}{r} \right)^{L_0} e^{u_{2,r}L_2} e^{u_{4,r}L_4} e^{u_{6,r}L_6} e^{u_{8,r}L_8} e^{u_{10,r}L_{10}} \cdots, \tag{2.14}$$

For level truncation computations, it is useful to write the scaling operator as the following canonical ordered form

$$U_r = \left( \frac{2}{r} \right)^{L_0} e^{u_{2,r}L_2} e^{u_{4,r}L_4} e^{u_{6,r}L_6} e^{u_{8,r}L_8} e^{u_{10,r}L_{10}} \cdots, \tag{2.15}$$

where the operators $L_n$ are the usual Virasoro generators. To find the coefficients $u_{n,r}$ appearing in the exponentials, we use

$$\frac{r}{2} \tan \left( \frac{r}{2} \arctan z \right) = \lim_{N \to \infty} \left[ f_{2,u_{2,r}} \circ f_{4,u_{4,r}} \circ f_{6,u_{6,r}} \circ f_{8,u_{8,r}} \circ f_{10,u_{10,r}} \circ \cdots \circ f_{N,u_{N,r}}(z) \right] = \lim_{N \to \infty} \left[ f_{2,u_{2,r}}(f_{4,u_{4,r}}(f_{6,u_{6,r}}(f_{8,u_{8,r}}(f_{10,u_{10,r}}(\cdots (f_{N,u_{N,r}}(z)) \cdots)))) \right], \tag{2.16}$$
where
\[ f_{n,u_n,r}(z) = \frac{z}{(1 - u_n r^n z^n)^{1/n}}. \] (2.17)

Using the above expressions, we get
\[ u_{2,r} = \frac{4 - r^2}{3r^2}, \] (2.18)
\[ u_{4,r} = \frac{r^4 - 16}{30r^4}, \] (2.19)
\[ u_{6,r} = \frac{-16 (r^6 - 21r^2 + 20)}{945r^6}, \] (2.20)
\[ u_{8,r} = \frac{109r^8 - 2688r^4 + 5120r^2 - 5376}{11340r^8}, \] (2.21)
\[ u_{10,r} = \frac{-16 (9r^{10} - 253r^6 + 660r^4 - 1056r^2 + 640)}{22275r^{10}}. \] (2.22)

Plugging the definition of the rational function (2.13) into the expression for the regularized solution (2.12), we obtain
\[ \Psi_\lambda = \int_0^\infty dt e^{-\frac{t}{\lambda}} \left[ \frac{1}{\lambda} - \frac{(\lambda - 1)}{\lambda} \partial_t \right] c \Omega^t + Q_B \left\{ \lambda \int_0^\infty dt e^{-\frac{t}{\lambda}} \left[ \frac{1}{\lambda} - \frac{(\lambda - 1)}{\lambda} \partial_t \right] B c \Omega^t \right\}. \] (2.23)

Using the operator representation for the c ghost (2.4), the expression for the wedge state (2.14), and noting that under the conformal map \( \tilde{z} = \frac{2}{\pi} \arctan z \), the c ghost transforms as \( \tilde{c}(\tilde{z}) = \frac{2}{\pi} \cos^2 \left( \frac{\pi}{2} \tilde{z} \right) c \left( \tan \left( \frac{\pi}{2} \tilde{z} \right) \right) \), from equation (2.23) we derive
\[ \Psi_\lambda = \frac{2}{\pi} \int_0^\infty dt e^{-\frac{t}{\lambda}} \left[ \frac{1}{\lambda} - \frac{(\lambda - 1)}{\lambda} \partial_t \right] \left[ \cos^2 \left( \frac{\pi t}{2t + 2} \right) U^t_{t+1} c \left( \tan \left( \frac{\pi t}{2t + 2} \right) \right) \right] |0\rangle + Q_B \left\{ \frac{\lambda}{\pi} \int_0^\infty dt e^{-\frac{t}{\lambda}} \left[ \frac{1}{\lambda} - \frac{(\lambda - 1)}{\lambda} \partial_t \right] \left[ \cos^2 \left( \frac{\pi t}{2t + 2} \right) U^t_{t+1} B^t_0 c \left( \tan \left( \frac{\pi t}{2t + 2} \right) \right) \right] |0\rangle \right\}. \] (2.24)

Since in the next subsection it will be argued that to evaluate the vacuum energy, we only need to consider the first term of the regularized solution (2.24), let us study in some detail this first term
\[ \Psi_\lambda^{(1)} = \frac{2}{\pi} \int_0^\infty dt e^{-\frac{t}{\lambda}} \left[ \frac{1}{\lambda} - \frac{(\lambda - 1)}{\lambda} \partial_t \right] \left[ \cos^2 \left( \frac{\pi t}{2t + 2} \right) U^t_{t+1} c \left( \tan \left( \frac{\pi t}{2t + 2} \right) \right) \right] |0\rangle. \] (2.25)

Using the expression (2.15) for the operator \( U_r \), from equation (2.25) we obtain an expression for \( \Psi_\lambda^{(1)} \) which is ready to be used in level truncation computations
\[ \Psi_\lambda^{(1)} = \frac{2}{\pi} \int_0^\infty dt e^{-\frac{t}{\lambda}} \left[ \frac{1}{\lambda} - \frac{(\lambda - 1)}{\lambda} \partial_t \right] \left[ \cos^2 \left( \frac{\pi t}{2t + 2} \right) U^t_{t+1} c \left( \frac{2}{t + 1} \tan \left( \frac{\pi t}{2t + 2} \right) \right) \right] |0\rangle, \] (2.26)
where
\[ \tilde{U}_{t+1} \equiv \cdots e^{u_{10,t+1}L^{-10}}e^{u_{8,t+1}L^{-8}}e^{u_{6,t+1}L^{-6}}e^{u_{4,t+1}L^{-4}}e^{u_{2,t+1}L^{-2}}. \] (2.27)

The coefficients \( u_{n,t+1} \) appearing in the exponentials can be computed by performing the substitution \( r \rightarrow t + 1 \) in the set of equations (2.18)-(2.22).

By writing the c ghost in terms of its modes \( c(z) = \sum_m c_m/z^m \) and employing equations (2.26) and (2.27), we can expand \( \Psi^{(1)}_\lambda \) in terms of elements contained in the Virasoro basis of \( L_0 \) eigenstates. For example, let us expand \( \Psi^{(1)}_\lambda \) up to level two states
\[ \Psi^{(1)}_\lambda = t(\lambda)c_1|0\rangle + u(\lambda)c_0|0\rangle + v(\lambda)c_{-1}|0\rangle + w(\lambda)L_{-2}c_1|0\rangle + \cdots, \] (2.28)
where the coefficients of the expansion \( t(\lambda), u(\lambda), v(\lambda) \) and \( w(\lambda) \) are given by the following integrals
\[ t(\lambda) = \frac{2}{\pi} \int_0^\infty dt \ e^{-\frac{\pi}{4}(t + 1)^2 \cos^2 \left( \frac{\pi t}{2t + 2} \right)}, \] (2.29)
\[ u(\lambda) = \frac{2}{\pi} \int_0^\infty dt \ e^{-\frac{\pi}{4}(t + 1)^2 \sin \left( \frac{\pi t}{t + 1} \right)}, \] (2.30)
\[ v(\lambda) = \frac{2}{\pi} \int_0^\infty dt \ e^{-\frac{\pi}{4}(t + 1)^2 \sin^2 \left( \frac{\pi t}{2t + 2} \right)}, \] (2.31)
\[ w(\lambda) = \frac{2}{\pi} \int_0^\infty dt \ e^{-\frac{\pi}{4}(t + 1)^2 \cos^2 \left( \frac{\pi t}{2t + 2} \right)}. \] (2.32)

These integrals are convergent provided that the parameter \( \lambda \) belongs to the interval \((0, +\infty)\). The range of validity for this parameter \( \lambda \) was also studied using the \( L_0 \) level expansion of the regularized solution \([1]\). Let us comment that this result, relating to the parameter \( \lambda \), is quite similar to the well known case of Schnabl-Okawa’s one-parameter family of solutions analyzed in \([15, 17, 22, 23]\) where it was shown that the parameter \( \lambda \) belongs to the interval \((0, 1)\) and the limit case \( \lambda \rightarrow 1 \) precisely corresponds to the original Schnabl’s analytic solution \([17]\).

### 2.3 Evaluation of the vacuum energy

Assuming the validity of the string field equation of motion when contracted with the solution itself \([1]\), we can show that the normalized value of the vacuum energy is given by
\[ E = \frac{\pi^2}{3} \langle \Psi_\lambda, Q_B \Psi_\lambda \rangle. \] (2.33)

Since the regularized solution \( \Psi_\lambda \) can be written as an expression containing an exact BRST term, to compute the normalized value of the vacuum energy (2.33), we only need
to consider the first term of (2.24)
\[ E = \frac{\pi^2}{3} \langle \Psi_\lambda^{(1)}, Q_B \Psi_\lambda^{(1)} \rangle. \] (2.34)

As described in [5, 15, 17, 24], it is convenient to replace the string field \( \Psi_\lambda^{(1)} \) with \( z^{L_0} \Psi_\lambda^{(1)} \) in the \( L_0 \) level truncation scheme, so that states in the \( L_0 \) level expansion of the solution acquire different integer powers of \( z \) at different levels. As we are going to see, the parameter \( z \) is needed because we will need to express the normalized value of the vacuum energy (2.34) as a formal power series expansion if we want to use Padé approximants [24, 33]. After doing our calculations, we will simply set \( z = 1 \).

Using (2.26) and (2.27) the string field \( \Psi_\lambda^{(1)} \) can be readily expanded and the individual coefficients can be numerically integrated. For instance, employing some numerical values for the parameter \( \lambda \), we obtain

\[ \Psi_{\lambda=1/10}^{(1)} = 0.43430476 \, c_1 |0\rangle + 0.47429914 \, c_0 |0\rangle + 0.18754065 \, c_{-1} |0\rangle + 0.00492478 \, L_{-2} c_1 |0\rangle + 0.09189138 \, c_{-2} |0\rangle + 0.30188401 \, L_{-2} c_0 |0\rangle + 0.05141509 \, c_{-3} |0\rangle + 0.07875502 \, L_{-4} c_1 |0\rangle + 0.10209574 \, L_{-2} c_{-1} |0\rangle - 0.13289997 \, L_{-2} L_{-2} c_1 |0\rangle + \cdots, \] (2.35)

\[ \Psi_{\lambda=2/10}^{(1)} = 0.40070791 \, c_1 |0\rangle + 0.43550897 \, c_0 |0\rangle + 0.25391982 \, c_{-1} |0\rangle - 0.00600265 \, L_{-2} c_1 |0\rangle + 0.17221906 \, c_{-2} |0\rangle + 0.19467134 \, L_{-2} c_0 |0\rangle + 0.12798546 \, c_{-3} |0\rangle + 0.01431572 \, L_{-4} c_1 |0\rangle + 0.08883011 \, L_{-2} c_{-1} |0\rangle - 0.09200582 \, L_{-2} L_{-2} c_1 |0\rangle + \cdots, \] (2.36)

\[ \Psi_{\lambda=3/10}^{(1)} = 0.37467203 \, c_1 |0\rangle + 0.39938623 \, c_0 |0\rangle + 0.27811748 \, c_{-1} |0\rangle - 0.00538991 \, L_{-2} c_1 |0\rangle + 0.21805409 \, c_{-2} |0\rangle + 0.13091926 \, L_{-2} c_0 |0\rangle + 0.18356534 \, c_{-3} |0\rangle + 0.03887291 \, L_{-4} c_1 |0\rangle + 0.06563034 \, L_{-2} c_{-1} |0\rangle - 0.06299155 \, L_{-2} L_{-2} c_1 |0\rangle + \cdots, \] (2.37)

\[ \Psi_{\lambda=4/10}^{(1)} = 0.35396897 \, c_1 |0\rangle + 0.36790312 \, c_0 |0\rangle + 0.28461930 \, c_{-1} |0\rangle - 0.00065616 \, L_{-2} c_1 |0\rangle + 0.24293724 \, c_{-2} |0\rangle + 0.08996485 \, L_{-2} c_0 |0\rangle + 0.21992140 \, c_{-3} |0\rangle + 0.02519440 \, L_{-4} c_1 |0\rangle + 0.04537003 \, L_{-2} c_{-1} |0\rangle - 0.04177194 \, L_{-2} L_{-2} c_1 |0\rangle + \cdots, \] (2.38)

\[ \Psi_{\lambda=5/10}^{(1)} = 0.33709802 \, c_1 |0\rangle + 0.34076058 \, c_0 |0\rangle + 0.28293698 \, c_{-1} |0\rangle + 0.00552825 \, L_{-2} c_1 |0\rangle + 0.25561048 \, c_{-2} |0\rangle + 0.06216305 \, L_{-2} c_0 |0\rangle + 0.24287102 \, c_{-3} |0\rangle + 0.01431572 \, L_{-4} c_1 |0\rangle + 0.02931597 \, L_{-2} c_{-1} |0\rangle - 0.02570229 \, L_{-2} L_{-2} c_1 |0\rangle + \cdots. \] (2.39)

Once we have the level expansion of the string field \( \Psi_\lambda^{(1)} \), we can compute the normalized value of the vacuum energy. By performing the replacement \( \Psi_\lambda^{(1)} \rightarrow z^{L_0} \Psi_\lambda^{(1)} \) from
equation (2.34), we define
\[ E_\lambda(z) = \frac{\pi^2}{3} \langle z^{L_0} \Psi^{(1)}_\lambda, Q_B z^{L_0} \Psi^{(1)}_\lambda \rangle. \]  
(2.40)

For example, plugging the level expansions (2.35)-(2.39) into the definition (2.40), we obtain
\[ E_\lambda=1/10(z) = -\frac{0.620536}{z^2} - 1.480175 - 0.097159 z^2 + 0.730101 z^4 - 0.680451 z^6 + \cdots, \]  
(2.41)
\[ E_\lambda=2/10(z) = -\frac{0.528243}{z^2} - 1.247965 - 0.241727 z^2 + 0.882372 z^4 + 0.064902 z^6 + \cdots, \]  
(2.42)
\[ E_\lambda=3/10(z) = -\frac{0.461828}{z^2} - 1.049529 - 0.283676 z^2 + 0.751339 z^4 + 0.313303 z^6 + \cdots, \]  
(2.43)
\[ E_\lambda=4/10(z) = -\frac{0.412200}{z^2} - 0.890585 - 0.270186 z^2 + 0.575221 z^4 + 0.327130 z^6 + \cdots, \]  
(2.44)
\[ E_\lambda=5/10(z) = -\frac{0.373844}{z^2} - 0.764024 - 0.232088 z^2 + 0.418196 z^4 + 0.244691 z^6 + \cdots. \]  
(2.45)

As a pedagogical illustration of the numerical method based on Padé approximants, let us compute in detail the normalized value of the vacuum energy using the expansion (2.45) which corresponds to the case \( \lambda = 1/2 \). First, we express \( E_{\lambda=1/2}(z) \) as the rational function \( P_{2+4}^4(\lambda, z) \)
\[ E_{\lambda=1/2}(z) = P_{2+4}^4(1/2, z) = \frac{1}{z^2} \left[ \frac{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4}{1 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4} \right]. \]  
(2.46)

Expanding the right hand side of (2.46) around \( z = 0 \) up to sixth order in \( z \) and equating the coefficients of \( z^{-2}, z^{-1}, z^0, z^1, z^2, z^3, z^4, z^5, z^6 \) with the expansion (2.45), we get a system of seven algebraic equations for the unknown coefficients \( a_0, a_1, a_2, a_3, a_4, b_1, b_2, b_3 \) and \( b_4 \). Solving those equations we get
\[ a_0 = -0.37384441, \quad a_1 = 0, \quad a_2 = -0.67402070, \quad a_3 = 0, \quad a_4 = -0.28011517, \]  
(2.47)
\[ b_1 = 0, \quad b_2 = -0.24075156, \quad b_3 = 0, \quad b_4 = 0.62049213. \]  
(2.48)

Replacing the value of these coefficients into the definition of \( P_{2+4}^4(1/2, z) \) (2.46), and evaluating this at \( z = 1 \), we get the following normalized value of the vacuum energy
\[ P_{2+4}^4(1/2, z = 1) = -0.96248549. \]  
(2.49)

To evaluate higher Padé approximants \( P_{2+n}^n(\lambda, z) \), we need to know the expansion of the normalized value of the vacuum energy \( E_\lambda(z) \) up to the order \( z^{2n-2} \). We have computed this \( z \)-dependent energy using the components of the string field \( \Psi^{(1)}_\lambda \) up to level 12. The results of our calculations are summarized in table 2.1. As we can see, the normalized value of the vacuum energy computed numerically using Padé approximants confirm Sen’s
For instance \( \Psi^0 \) two-parameter family of solutions is given by transformation (2.9) and (2.10) with \( \Psi^1 = 1 \) corresponds to the Erler-Schnabl’s solution which was already studied in [5].

The particular case \( \lambda = 1 \) corresponds to the Erler-Schnabl’s solution which was already studied in [5].

It is interesting to observe that we can use another identity-based solution for instance \( \Psi^0 = \alpha c - \lambda K \), with \( \alpha \) being an arbitrary parameter. Employing the gauge transformation (2.9) and (2.10) with \( \Psi_I = c(\alpha - K) \), we can show that the resulting two-parameter family of solutions is given by

\[
\Psi_{\lambda,\alpha} = c(1 + \lambda K)Bc \frac{\alpha + (\alpha \lambda - 1)K}{1 + \lambda K}.
\]

To simplify the evaluation of the vacuum energy for this solution, it is convenient to write \( \Psi_{\lambda,\alpha} \) as an expression containing an exact BRST term

\[
\Psi_{\lambda,\alpha} = c \frac{\alpha + (\alpha \lambda - 1)K}{1 + \lambda K} + Q_B \left\{ \lambda Bc \frac{\alpha + (\alpha \lambda - 1)K}{1 + \lambda K} \right\}.
\]

Following the same procedures we used for the previous asymmetric solution, we obtain an expression for the first term of the right-hand side of (2.51) \( \Psi_{\lambda,\alpha}^{(1)} \) which is ready for level truncation computations,

\[
\Psi_{\lambda,\alpha}^{(1)} = \frac{2}{\pi} \int_0^\infty dt \ e^{-i \frac{\alpha}{\lambda} \ t} \left( \frac{\alpha \lambda - 1}{\lambda} \right)^2 \left( \frac{\pi t}{2t + 2} \right)^2 \cos^2 \left( \frac{\pi t}{2t + 2} \right) U_{t+1} c \left( \frac{2}{t+1} \tan \left( \frac{\pi t}{2t + 2} \right) \right) |0\rangle.
\]

For example, let us expand \( \Psi_{\lambda,\alpha}^{(1)} \) up to level two states,

\[
\Psi_{\lambda,\alpha}^{(1)} = t(\lambda, \alpha)c_1|0\rangle + u(\lambda, \alpha)c_0|0\rangle + v(\lambda, \alpha)c_1|0\rangle + w(\lambda, \alpha)L_{-2}c_1|0\rangle + \cdots,
\]
where the coefficients of the expansion, \( t(\lambda, \alpha) \), \( u(\lambda, \alpha) \), \( v(\lambda, \alpha) \) and \( w(\lambda, \alpha) \), are given by the following integrals

\[
t(\lambda, \alpha) = \frac{2}{\pi} \int_{0}^{\infty} dt \, e^{-t} \left( \frac{\alpha}{\lambda} - \frac{(\alpha - 1)}{\lambda} \partial_t \right) \left[ \frac{1}{4} (t + 1)^2 \cos \left( \frac{\pi t}{2t + 2} \right) \right], \tag{2.54}
\]

\[
u(\lambda, \alpha) = \frac{2}{\pi} \int_{0}^{\infty} dt \, e^{-t} \left( \frac{\alpha}{\lambda} - \frac{(\alpha - 1)}{\lambda} \partial_t \right) \left[ \frac{1}{4} (t + 1) \sin \left( \frac{\pi t}{t + 1} \right) \right], \tag{2.55}
\]

\[
v(\lambda, \alpha) = \frac{2}{\pi} \int_{0}^{\infty} dt \, e^{-t} \left( \frac{\alpha}{\lambda} - \frac{(\alpha - 1)}{\lambda} \partial_t \right) \left[ \sin \left( \frac{\pi t}{t + 1} \right) \right], \tag{2.56}
\]

\[
w(\lambda, \alpha) = \frac{2}{\pi} \int_{0}^{\infty} dt \, e^{-t} \left( \frac{\alpha}{\lambda} - \frac{(\alpha - 1)}{\lambda} \partial_t \right) \left[ \frac{1}{12} \left( 3 - t^2 - 2t \right) \cos \left( \frac{\pi t}{2t + 2} \right) \right]. \tag{2.57}
\]

These integrals are convergent provided that the parameter \( \lambda \) belongs to the interval \((0, +\infty)\), with \( \alpha \) being an arbitrary parameter.

The numerical evaluation of the vacuum energy follows the same steps developed in this subsection; in other words, we first define the normalized value of the vacuum energy

\[
E_{\lambda, \alpha}(z) \equiv \frac{\pi^2}{3} \langle z L_0 \Psi_{\lambda, \alpha}^{(1)} , Q_{BZ} z L_0 \Psi_{\lambda, \alpha}^{(1)} \rangle, \tag{2.58}
\]

we then plug (2.53) into (2.58), and truncate the series to order \( z^{2n-2} \), and, finally, we employ Padé approximants of order \( P_{n+2}^{n}(\lambda, \alpha, z) \). We have performed the numerical computations at level \( n = 8 \) with \( \lambda = 1/2 \), \( \lambda = 1 \) and for various values of the parameter \( \alpha \). The results are shown in Table 2.2. We see that the normalized value of the vacuum energy computed numerically using Padé approximants agrees with the answer expected from Sen’s first conjecture.

Table 2.2: The \( P_{2+8}^{8} \) Padé approximation for the normalized value of the vacuum energy \( \frac{\pi^2}{3} \langle z L_0 \Psi_{\lambda, \alpha}^{(1)} , Q_{BZ} z L_0 \Psi_{\lambda, \alpha}^{(1)} \rangle \) evaluated at various values of the parameter \( \alpha \), with \( \lambda = 1/2 \) and \( \lambda = 1 \).

| \( \alpha \) | \( P_{2+8}^{8}(\lambda = 1/2, \alpha, z = 1) \) | \( P_{2+8}^{8}(\lambda = 1, \alpha, z = 1) \) |
|---|---|---|
| \(-1\) | -1.003161547 | -0.728764899 |
| \(0\) | -0.994968610 | -0.795944823 |
| \(1\) | -0.991643419 | -0.93565316 |
| \(1.5\) | -0.996141185 | -0.948563810 |
| \(2\) | -0.972530090 | -0.940860097 |

3 Real one-parameter family of solutions

In this section, we will study another one-parameter family of solutions which satisfies the string field reality condition. By performing a gauge transformation, this real solution
will be related to the previous non-real solution. The level truncation computation of the vacuum energy will be analyzed in a similar manner as for the non-real solution, namely by means of Padé approximants.

3.1 Generating the real solution

A solution $\hat{\Psi}_\lambda$ which satisfies the string field reality condition can be generated by performing a gauge transformation on the non-real solution (2.11)

$$\hat{\Psi}_\lambda = \sqrt{1 + (\lambda - 1)K} \frac{(\Psi_\lambda + QB)}{1 + \lambda K} \sqrt{1 + \lambda K}.$$

Because of the square root in (3.1), we should be careful about whether the solution derived from this equation is a well-behaved element in the space of string fields introduced in [14]. A standard criterion for determining whether or not a string field is a well behaved element, consists in studying the convergence properties of its level expansion using the Virasoro basis of $L_0$ eigenstates.

It turns out that the real solution derived from (3.1)

$$\hat{\Psi}_\lambda = \sqrt{1 + (\lambda - 1)K} \frac{(c + \lambda cKBc)}{1 + \lambda K} \sqrt{1 + \lambda K},$$

as in the case of the non-real solution, can be written as an expression containing an exact BRST term

$$\hat{\Psi}_\lambda = \sqrt{1 + (\lambda - 1)K} \frac{c}{1 + \lambda K} + QB \left\{ \sqrt{1 + (\lambda - 1)K} \frac{\lambda Bc}{1 + \lambda K} \right\}.$$

Since we assume the validity of the string field equation of motion when contracted with the solution itself, the exact BRST term will not contribute to the evaluation of the vacuum energy. So in the next subsection, we will only concentrate in the level expansion of the first term of the real solution (3.3)

$$\hat{\Psi}_\lambda^{(1)} = \sqrt{1 + (\lambda - 1)K} \frac{c}{1 + \lambda K} \sqrt{1 + (\lambda - 1)K}.$$

3.2 Level expansion of the real solution

To expand the string field $\hat{\Psi}_\lambda^{(1)}$ in the Virasoro basis of $L_0$ eigenstates, it is convenient to write an integral representation for the square root function

$$\sqrt{1 + (\lambda - 1)K} \frac{1}{1 + \lambda K} = \int_0^\infty dt \frac{e^{\frac{(1-2\lambda)}{2(\lambda-1)}} I_0 \left( \frac{1}{2(\lambda-1)} t \right)}{\sqrt{\lambda - 1} t} \left[ 1 - (\lambda - 1) \partial_t \right] e^{-Kt},$$

14
where \( I_0(x) \) is the modified Bessel function of the first kind. This integral is well defined if the parameter \( \lambda > 1 \). Since the limit case \( \lambda \to 1 \) corresponds to the function \( 1/\sqrt{1 + \lambda} \), we should have \[5\]

\[
\lim_{\lambda \to 1} \left[ \frac{e^{(1-2\lambda)} I_0 \left( \frac{1}{2(\lambda-1)^2} \right)}{\sqrt{(\lambda - 1)\lambda}} \right] = \frac{e^{-t}}{\sqrt{\pi t}}.
\]

(3.6)

Plugging the definition of the square root function (3.5) into the expression for the string field (3.4), we get

\[
\hat{\Psi}^{(1)}_\lambda = \int_0^\infty ds dt \frac{e^{-\frac{(s+t)(\lambda-1)}{2(\lambda-1)^2}} I_0 \left( \frac{1}{2(\lambda-1)^2} s \right) I_0 \left( \frac{1}{2(\lambda-1)^2} t \right)}{(\lambda - 1)\lambda [1 - (\lambda - 1)\partial_s] [1 - (\lambda - 1)\partial_t]} \Omega^s c \Omega^t.
\]

(3.7)

Using equation (2.4), the representation for the wedge state (2.14), and noting that under the conformal map \( \tilde{z} = \frac{2}{\pi} \arctan z \) the c ghost transforms as \( \tilde{c}(\tilde{z}) = \frac{2}{\pi} \cos^2 \left( \frac{\pi}{4} \tilde{z} \right) c(\tan(\frac{\pi}{4} \tilde{z})) \), from equation (3.7), we obtain

\[
\hat{\Psi}^{(1)}_\lambda = \frac{2}{\pi} \int_0^\infty ds dt \mathcal{J}(s, t, \lambda) \hat{\mathcal{O}}_t \mathcal{J}(s, t) \tilde{U}_{s+t+1} e^{\left( \frac{2}{s + t + 1} \tan \left( \frac{\pi}{2} \frac{s - t}{s + t + 1} \right) \right)} \langle 0 |,
\]

(3.8)

where the functions \( \mathcal{I}(s, t, \lambda) \), \( \mathcal{J}(s, t) \) and the operators \( \hat{\mathcal{O}}_t \), \( \tilde{U}_{s+t+1} \) are defined as follows

\[
\mathcal{I}(s, t, \lambda) = \frac{e^{-\frac{(s+t)(\lambda-1)}{2(\lambda-1)^2}} I_0 \left( \frac{1}{2(\lambda-1)^2} s \right) I_0 \left( \frac{1}{2(\lambda-1)^2} t \right)}{(\lambda - 1)\lambda},
\]

(3.9)

\[
\mathcal{J}(s, t) = \left( \frac{s + t + 1}{2} \right)^2 \cos^2 \left( \frac{\pi}{2} \frac{s - t}{s + t + 1} \right),
\]

(3.10)

\[
\tilde{U}_{s+t+1} = \cdots e^{u_{10,s+t+1} L_{-10} e^{u_{9,s+t+1} L_{-9} e^{u_{8,s+t+1} L_{-8} e^{u_{7,s+t+1} L_{-7} e^{u_{6,s+t+1} L_{-6} e^{u_{5,s+t+1} L_{-5} e^{u_{4,s+t+1} L_{-4} e^{u_{3,s+t+1} L_{-3} e^{u_{2,s+t+1} L_{-2} e^{u_{1,s+t+1} L_{-1} e^{u_0,s+t+1} L_{-0}}}}}}}}}}
\]

(3.11)

\[
\hat{\mathcal{O}}_t = [1 - (\lambda - 1)\partial_t].
\]

(3.12)

By writing the c ghost in terms of its modes \( c(z) = \sum_m c_m / z^{m-1} \) and employing equations (3.8)-(3.12), we can expand \( \hat{\Psi}^{(1)}_\lambda \) in terms of elements contained in the Virasoro basis of \( L_0 \) eigenstates. For instance, let us expand \( \hat{\Psi}^{(1)}_\lambda \) up to level two states

\[
\hat{\Psi}^{(1)}_\lambda = t(\lambda) c_1 |0\rangle + v(\lambda) c_{-1} |0\rangle + w(\lambda) L_{-2} c_1 |0\rangle + \cdots,
\]

(3.13)

\[5\]In fact this limit can be derived by using the asymptotic expansion \( I_0(x) \approx e^x / \sqrt{2\pi x} [1 + O(1/x)] \) which is valid for \( x \gg 1 \).
where the coefficients of the expansion \( t(\lambda), v(\lambda) \) and \( w(\lambda) \) are given by the following integrals

\[
\begin{align*}
t(\lambda) &= \frac{2}{\pi} \int_0^\infty ds dt I(s,t,\lambda) \hat{\mathcal{O}}_s \hat{\mathcal{O}}_t \left[ \mathcal{J}(s,t) \right], \\
v(\lambda) &= \frac{2}{\pi} \int_0^\infty ds dt I(s,t,\lambda) \hat{\mathcal{O}}_s \hat{\mathcal{O}}_t \left[ \frac{4 \mathcal{J}(s,t) \tan^2 \left( \frac{\pi}{2} s + t \right)}{(s + t + 1)^2} \right], \\
w(\lambda) &= \frac{2}{\pi} \int_0^\infty ds dt I(s,t,\lambda) \hat{\mathcal{O}}_s \hat{\mathcal{O}}_t \left[ \frac{4 - (s + t + 1)^2}{3(s + t + 1)^2} \mathcal{J}(s,t) \right].
\end{align*}
\]

These integrals are convergent provided that the parameter \( \lambda \) belongs to the interval \((1, +\infty)\). Employing numerical values for the parameter \( \lambda \), the integrals can be evaluated numerically. The limit case \( \lambda \to 1 \) precisely corresponds to the Erler-Schnabl’s twist even (real) solution which was analyzed in [5].

At this point we would like to explain why the symmetric solution does not exist for \( \lambda < 1 \). The answer is related to the holomorphicity property of the square root function \( F(K) \), which is defined on the left-hand side of (3.5). Recall that a function \( f \) defined on a non-empty open set \( \mathcal{O} \) is holomorphic if its derivative \( f' \) is well defined at every point in \( \mathcal{O} \). For our case, the derivative

\[
F'(K) = -\frac{1}{2(1 + (\lambda - 1)K)^{1/2}(1 + \lambda K)^{3/2}}
\]

is well defined in the region where \( \lambda > 1 \) and \( \text{Re} K > 0 \), whereas in the region where \( \lambda < 1 \) and \( \text{Re} K > 0 \) it is not well defined. Therefore the left-hand side of (3.5) is not holomorphic in the region where \( \lambda < 1 \) and \( \text{Re} K > 0 \), and consequently using the criterion for safe string fields derived in reference [14], the symmetric solution does not exist.

### 3.3 Evaluation of the vacuum energy

In this subsection, in order to evaluate the normalized value of the vacuum energy for the real solution

\[
E = \frac{\pi^2}{3} \langle \hat{\Psi}_\lambda^{(1)}, Q_B \hat{\Psi}_\lambda^{(1)} \rangle,
\]

we will perform the replacement \( \hat{\Psi}_\lambda^{(1)} \to z^{L_0} \hat{\Psi}_\lambda^{(1)} \) in the \( L_0 \) level truncation scheme, so that states in the \( L_0 \) level expansion of the solution will acquire different integer powers of \( z \) at different levels. The parameter \( z \) is needed because we will need to express the normalized value of the vacuum energy (3.18) as a formal power series expansion if we want to use Padé approximants [24, 33]. After doing our calculations, we will simply set \( z = 1 \).
Using the equations (3.18)-(3.22), the string field $\hat{\Psi}^{(1)}_\lambda$ can be readily expanded and the individual coefficients can be numerically integrated. For instance, employing some numerical values for the parameter $\lambda$, we obtain

\[
\hat{\Psi}^{(1)}_{\lambda=6/5} = 0.48079059 c_1|0\rangle + 0.15487181 c_{-1}|0\rangle + 0.00031916 L_{-2}c_1|0\rangle + 0.13495200 c_{-3}|0\rangle - 0.01975863 L_{-4}c_1|0\rangle - 0.00920921 L_{-2}c_{-1}|0\rangle + 0.03282467 L_{-2}L_{-2}c_1|0\rangle + \cdots, \\
\hat{\Psi}^{(1)}_{\lambda=7/5} = 0.45869460 c_1|0\rangle + 0.16142345 c_{-1}|0\rangle + 0.00550055 L_{-2}c_1|0\rangle + 0.15824508 c_{-3}|0\rangle - 0.02526528 L_{-4}c_1|0\rangle - 0.01365061 L_{-2}c_{-1}|0\rangle + 0.04027528 L_{-2}L_{-2}c_1|0\rangle + \cdots, \\
\hat{\Psi}^{(1)}_{\lambda=8/5} = 0.44104558 c_1|0\rangle + 0.16132987 c_{-1}|0\rangle + 0.01141476 L_{-2}c_1|0\rangle + 0.17029934 c_{-3}|0\rangle - 0.02990375 L_{-4}c_1|0\rangle - 0.01729818 L_{-2}c_{-1}|0\rangle + 0.04603466 L_{-2}L_{-2}c_1|0\rangle + \cdots, \\
\hat{\Psi}^{(1)}_{\lambda=9/5} = 0.42662772 c_1|0\rangle + 0.15821733 c_{-1}|0\rangle + 0.01725819 L_{-2}c_1|0\rangle + 0.17597281 c_{-3}|0\rangle - 0.03380422 L_{-4}c_1|0\rangle - 0.01997450 L_{-2}c_{-1}|0\rangle + 0.05058763 L_{-2}L_{-2}c_1|0\rangle + \cdots, \\
\hat{\Psi}^{(1)}_{\lambda=2} = 0.41461941 c_1|0\rangle + 0.15372018 c_{-1}|0\rangle + 0.02275988 L_{-2}c_1|0\rangle + 0.17784600 c_{-3}|0\rangle - 0.03711080 L_{-4}c_1|0\rangle - 0.02185865 L_{-2}c_{-1}|0\rangle + 0.05426470 L_{-2}L_{-2}c_1|0\rangle + \cdots.
\]

Once we have the level expansion of the string field $\hat{\Psi}^{(1)}_\lambda$, we can compute the normalized value of the vacuum energy. By performing the replacement $\hat{\Psi}^{(1)}_\lambda \rightarrow z^{L_0}\hat{\Psi}^{(1)}_\lambda$ from equation (3.18), we define

\[
E_\lambda(z) \equiv \frac{\pi^2}{3} \langle z^{L_0}\hat{\Psi}^{(1)}_\lambda, Q_B z^{L_0}\hat{\Psi}^{(1)}_\lambda \rangle.
\]

For example, plugging the level expansions (3.19)-(3.23) into the definition (3.24), we obtain

\[
E_{\lambda=6/5}(z) = -\frac{0.76048459}{z^2} - 0.07793135z^2 - 0.25756015z^6 + \cdots, \\
E_{\lambda=7/5}(z) = -\frac{0.69219068}{z^2} - 0.06780088z^2 - 0.38256079z^6 + \cdots, \\
E_{\lambda=8/5}(z) = -\frac{0.63994910}{z^2} - 0.04756125z^2 - 0.48705471z^6 + \cdots, \\
E_{\lambda=9/5}(z) = -\frac{0.59879290}{z^2} - 0.02453608z^2 - 0.57161143z^6 + \cdots, \\
E_{\lambda=2}(z) = -\frac{0.56555878}{z^2} - 0.00186180z^2 - 0.63933036z^6 + \cdots.
\]
Using these kind of series in $z$ for $E_{\lambda}(z)$, by the standard procedure based on Padé approximants of order $P_{2+n}^n(\lambda, z)$, we numerically compute the normalized value of the vacuum energy. The label $n$ corresponds to the power of $z$ in the series expansion of $E_{\lambda}(z)$ truncated up to the order $z^{2n-2}$. The results of our calculations are summarized in table 3.1. As we can see, the normalized value of the vacuum energy evaluated using Padé approximants nicely confirm Sen’s first conjecture [29, 30].

| $n$ | $P_{2+n}^n[\lambda = \frac{11}{10}]$ | $P_{2+n}^n[\lambda = \frac{12}{10}]$ | $P_{2+n}^n[\lambda = \frac{13}{10}]$ | $P_{2+n}^n[\lambda = \frac{14}{10}]$ | $P_{2+n}^n[\lambda = \frac{15}{10}]$ |
|-----|----------------|----------------|----------------|----------------|----------------|
| 0   | -0.80293844   | -0.76048459   | -0.69219068   | -0.66442720   |               |
| 4   | -0.75437392   | -0.72667434   | -0.67758602   | -0.6554662    |               |
| 8   | -0.98896263   | -0.98497033   | -0.97427446   | -0.96760968   |               |
| 12  | -0.99001621   | -0.98660724   | -0.97749570   | -0.97151457   |               |
|     | $P_{2+n}^n[\lambda = \frac{16}{10}]$ | $P_{2+n}^n[\lambda = \frac{17}{10}]$ | $P_{2+n}^n[\lambda = \frac{18}{10}]$ | $P_{2+n}^n[\lambda = 2]$ |
| 0   | -0.63994910   | -0.61821502   | -0.59879290   | -0.5655878    |               |
| 4   | -0.63480210   | -0.61557936   | -0.59769248   | -0.5655334    |               |
| 8   | -0.96019541   | -0.95214843   | -0.94358184   | -0.9252928    |               |
| 12  | -0.96462560   | -0.95692661   | -0.94853355   | -0.93013994   |               |

### 3.4 Closed string tadpole

There is another gauge invariant quantity which is known in the literature as the closed string tadpole [34,35]. This quantity was already evaluated in [2] for the case of the non-real solution (2.12), and it was shown that its value coincides with the expected answer of closed string tadpole on the disk [34].

In this subsection, we would like to perform a similar computation for the case of the real solution

$$\hat{\Psi}_{\lambda} = \sqrt{\frac{1 + (\lambda - 1)K}{1 + \lambda K}} c \sqrt{\frac{1 + (\lambda - 1)K}{1 + \lambda K}} + Q_B \left\{ \sqrt{\frac{1 + (\lambda - 1)K}{1 + \lambda K}} \lambda B c \sqrt{\frac{1 + (\lambda - 1)K}{1 + \lambda K}} \right\},$$

(3.30)

The second term on the right hand side of (3.30) does not contribute to the tadpole due to the BRST invariance of the tadpole. Then we are going to compute

$$\text{Tr}[V \hat{\Psi}_{\lambda}] = \text{Tr} \left[ V \sqrt{\frac{1 + (\lambda - 1)K}{1 + \lambda K}} c \sqrt{\frac{1 + (\lambda - 1)K}{1 + \lambda K}} \right],$$

(3.31)
where $V = c\tilde{c}\mathcal{V}_{\text{matter}}$ is a closed string vertex operator insertion at the open string midpoint. We can write an integral representation for the square root function

$$\sqrt{1 + \frac{(\lambda - 1)K}{1 + \lambda K}} = \int_0^\infty dt \frac{e^{\frac{(1-2\lambda)}{2(\lambda-1)\lambda}} I_0\left(\frac{1}{2(\lambda-1)\lambda} t\right)}{\sqrt{\lambda - 1}} \left[1 - (\lambda - 1)\partial_t\right] e^{-Kt}, \quad (3.32)$$

where $I_0(x)$ is the modified Bessel function of the first kind. Using (3.32), from equation (3.31) we get

$$\text{Tr}[V\hat{\Psi}_{\lambda}] = \int_0^\infty ds dt \mathcal{I}(s, t, \lambda) \left[1 - (\lambda - 1)\partial_s\right] \left[1 - (\lambda - 1)\partial_t\right] \text{Tr}[V\Omega^*c\Omega'], \quad (3.33)$$

where the function $\mathcal{I}(s, t, \lambda)$ was defined in (3.9).

The inner product $\text{Tr}[V\Omega^*c\Omega']$ is a correlator on a cylinder of circumference $s + t$; by a scale transformation we can reduce it to a cylinder of unit circumference, producing a factor of $s + t$ for the $c$ ghost from the conformal transformation. Thus $\text{Tr}[V\hat{\Psi}_{\lambda}]$ can be written as

$$\text{Tr}[V\hat{\Psi}_{\lambda}] = \int_0^\infty ds dt \mathcal{I}(s, t, \lambda) \left[1 - (\lambda - 1)\partial_s\right] \left[1 - (\lambda - 1)\partial_t\right] ((s + t)\text{Tr}[Vc\Omega]). \quad (3.34)$$

From this equation (3.34), we obtain

$$\text{Tr}[V\hat{\Psi}_{\lambda}] = \text{Tr}[Vc\Omega] \times \int_0^\infty ds dt \frac{e^{-\frac{(s+t)(2\lambda-1)}{2(\lambda-1)\lambda}} (s + t - 2\lambda + 2) I_0\left(\frac{s}{2(\lambda-1)\lambda}\right) I_0\left(\frac{t}{2(\lambda-1)\lambda}\right)}{\lambda - 1}. \quad (3.35)$$

This double integral is well defined for $\lambda > 1$ and it can be analytically computed giving as a result a value which does not depend on $\lambda$

$$\int_0^\infty ds dt \frac{e^{-\frac{(s+t)(2\lambda-1)}{2(\lambda-1)\lambda}} (s + t - 2\lambda + 2) I_0\left(\frac{s}{2(\lambda-1)\lambda}\right) I_0\left(\frac{t}{2(\lambda-1)\lambda}\right)}{\lambda - 1} = 1. \quad (3.36)$$

Plugging the answer of this integral (3.36) into the equation (3.35), we get the desired result

$$\text{Tr}[V\hat{\Psi}_{\lambda}] = \text{Tr}[Vc\Omega] = \langle \mathcal{V}(i\infty)c(0)\rangle_{C_1}. \quad (3.37)$$

This result, as similar to the case of the non-real solution [2], coincides with the expected answer of closed string tadpole on the disk [3].
4 Summary and discussion

Using states in the Virasoro basis of $L_0$ eigenstates, we have analyzed the level expansion of regularized identity based solutions in open bosonic string field theory. We have evaluated the vacuum energy for these solutions by means of the numerical method based on Padé approximants. Our results confirm the expected answer for the tachyon vacuum in agreement with Sen’s first conjecture.

In order to simplify the computation of the vacuum energy, we have assumed the validity of the string field equations of motion when contracted with the regularized solution itself. This assumption has been proven correct by explicit evaluation of correlation functions in the so-called $KBC$ subalgebra and by using $L_0$ level truncation computations [1]. Nevertheless, it should be nice to confirm the validity of this assumption by using a third option, namely by employing the usual Virasoro $L_0$ level truncation scheme.

The procedures developed in this work, related to the level truncation analysis of regularized identity based solutions, can be extended for solutions in the case of the modified cubic superstring field theory [36], as well as in Berkovits non-polynomial open superstring field theory [37, 38]. Since in the case of the modified cubic superfield theory, we have already proposed a way for regularizing identity based solutions [1], the level truncation analysis should be straightforward. However, constructing a similar regularization for Berkovits superstring field theory remains an unsolved problem.

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