Generic partiality for $3^2$-institutions

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Abstract

$3^2$-institutions have been introduced as an extension of institution theory that accommodates implicitly partiality of the signature morphisms together with its syntactic and semantic effects. In this paper we show that ordinary institutions that are equipped with an inclusion system for their categories of signatures generate naturally $3^2$-institutions with explicit partiality for their signature morphisms. This provides a general uniform way to build $3^2$-institutions for the foundations of conceptual blending and software evolution. Moreover our general construction allows for an uniform derivation of some useful technical properties.

1. Introduction

1.1. Institution theory

The broad mathematical context of our work is the theory of institutions [21] which is a three-decades-old category-theoretic abstract model theory that traditionally has been playing a crucial foundational role in formal specification(e.g. [30]). It has been introduced in [20] as an answer to the explosion in the number of population of logical systems there, as a very general mathematical study of formal logical systems, with emphasis on semantics (model theory), that is not committed to any particular logical system. Its role has gradually expanded to other areas of logic-based computer science, most notably to declarative programming and ontologies. In parallel, and often in interdependence to its role in computer science, in the past fifteen years it has made important contributions to model theory through the new area called institution-independent model theory [6] – an abstract approach to model theory that is liberated from any commitment to particular logical systems. Institutions thus allowed for a smooth, systematic, and uniform development of model theories for unconventional logical systems, as well as of logic-by-translation techniques and of heterogeneous multi-logic frameworks.

Mathematically, institution theory is based upon a category-theoretic [25] formalization of the concept of logical system that includes the syntax, the semantics, and the satisfaction relation between them. As a form of abstract model theory, it is the only one that treats all these components of a logical system fully abstractly. In a nutshell, the above-mentioned formalization is a category-theoretic structure $(\text{Sign}, \text{Sen}, \text{Mod}, \models)$, called institution, that consists of (a) a category $\text{Sign}$ of so-called signatures, (b) two functors, $\text{Sen}: \text{Sign} \to \text{SET}$ for the syntax, given by sets of so-called sentences, and $\text{Mod}: \text{Sign}^\Theta \to \text{CAT}$ for the semantics, given by categories of so-called models, and (c) for each signature $\Sigma$, a binary satisfaction relation $\models_\Sigma$ between the $\Sigma$-models, i.e. objects of $\text{Mod}(\Sigma)$, and the $\Sigma$-sentences, i.e. elements of

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Sen(Σ), such that for each morphism \( \varphi : \Sigma \to \Sigma' \) in the category \( \text{Sign} \), each \( \Sigma' \)-model \( M' \), and each \( \Sigma \)-sentence \( \rho \) the following \textit{Satisfaction Condition} holds:

\[
M' \models_{\Sigma} \text{Sen}(\varphi)(\rho) \quad \text{if and only if} \quad \text{Mod}(\varphi)(M') \models_{\Sigma} \rho.
\]

Because of its very high level of abstraction, this definition accommodates not only well established logical systems but also very unconventional ones. Moreover, it has served and it may serve as a template for defining new ones. Institution theory approaches logic and model theory from a relativistic, non-substantialist perspective, quite different from the common reading of formal logic. This does not mean that institution theory is opposed to the established logic tradition, since it rather includes it from a higher abstraction level. In fact, the real difference may occur at the level of the development methodology: top-down in the case of institution theory, versus bottom-up in the case of traditional logic. Consequently, in institution theory, concepts come naturally as presumed features that a logical system might exhibit or not, and are defined at the most appropriate level of abstraction; in developing results, hypotheses are kept as general as possible and introduced on a by-need basis.

1.2. \( \frac{3}{2} \)-institutions

In spite of the broad conceptual coverage provided by institution theory there are specific aspects that require a general treatment but that cannot be addressed by the ordinary concept of institution. This situation has lead to a number of extensions of standard institution theory, such as towards many-valued truth (\( L \)-institutions [9]), implicit Kripke semantics (\( \text{stratified institutions} \) [2, 12]), etc. The most recent such extension are the \( \frac{3}{2} \)-institutions of [11] that accommodate implicitly partiality at the level of the signature morphisms. Signature morphisms that are partial are very difficult to digest from the perspective of logic for several reasons. Firstly, conventional mathematical logic does not usually involve signature morphisms at all, only very rarely in the form extensions of languages (the term “language” often used in conventional mathematical logic corresponds to “signature” in our terminology). It was precisely specification theory that showed the need to consider signature morphisms that are not necessarily inclusions or even injective. Secondly, even within the context of specification theory the idea that translating or mapping between signatures can be a partial has hardly been considered at all. A very notable exception to this is Goguen’s research on algebraic semiotics [17] and conceptual blending [18]. This work, that constitutes a mathematical and computational response to the seminal proposal of Fauconnier and Turner [15] of conceptual blending as a fundamental cognitive operation of language and common-sense, has received much attention within the context of the recent COINVENT project [31]. A serious shortcoming of the Goguen-COINVENT approach to conceptual blending is a lack of an explicit semantic component, and the concept of \( \frac{3}{2} \)-institutions have been proposed precisely as a remedy to this. Moreover in [11] it is argued that partiality of signature morphisms occurs naturally in the mathematical studies of merging of software changes; this can be considered as another application area for \( \frac{3}{2} \)-institutions.

While \( \frac{3}{2} \)-institutions propose an \textit{implicit} approach to partiality of signature morphisms and of their effects on the syntax and of the semantics, many of the examples of \( \frac{3}{2} \)-institutions in [11] display a common pattern in the way they are derived from ordinary institutions on the basis of explicit partiality of signature morphisms. Here we explain this pattern by developing a generic method to construct \( \frac{3}{2} \)-institutions from ordinary institutions that in essence requires only that the category of the signatures of the institution is equipped with an \textit{inclusion system} [14, 6]. Furthermore we exploit this general construction developing general but crucial results on the existence of lax cocones and on model amalgamation properties in \( \frac{3}{2} \)-institutions, obtained on the basis of corresponding properties of the underlying ordinary institutions.
1.3. Contributions and Structure of the Paper

The paper is structured as follows:

1. In a preliminary section we review the theory of \( I \)-institution introduced in [11].

2. A crucial section is dedicated to the development of categories of ‘partial maps’ based upon inclusion systems [14, 6]. This topic is well understood in the category theoretic literature, however the novelty here is that we do this on the basis on inclusion systems rather than factorisation systems (like in the traditional approach), the advantage being that we are able to avoid the quotienting implied in the traditional approach. In this way we get a general concept of partial signature morphism that is simpler and relates more directly to the concrete examples. From this several technical benefits follow in the subsequent developments. In the same section we also study some general properties of partial maps, that are relevant for our aims, such as the inheritance of an inclusion system and pushouts from the original category.

3. We extend the inclusion system based construction of partial signatures morphisms to the other components of the concept of institution, namely the sentence and the model functors. The main result of this section is that the whole construction gets a \( I \)-institution.

4. In the final section we prove some properties of the generic \( I \)-institutions thus constructed that are relevant in the conceptual blending applications, the most important result being the existence of lax cocones with model amalgamation.

2. A review of \( I \)-institutions

2.1. Categories, monads

In general we stick to the established category theoretic terminology and notations, such as in [25]. But unlike there we prefer to use the diagrammatic notation for compositions of arrows in categories, i.e. if \( f : A \to B \) and \( g : B \to C \) are arrows then \( f;g \) denotes their composition. The domain of an arrow/morphism \( f \) is denoted by \( \Box f \) while its codomain is denoted by \( f\Box \). \( \text{SET} \) denotes the category of sets and functions and \( \text{CAT} \) the “quasi-category” of categories and functors.\(^1\)

The dual of a category \( C \) (obtained by formally reversing its arrows) is denoted by \( C^\oplus \).

Given a category \( C \), a triple \((\Delta, \delta, \mu)\) constitutes a monad in \( C \) when \( \Delta : C \to C \), and \( \delta \) and \( \mu \) are natural transformations \( \Delta^2 \Rightarrow \Delta \) and \( 1_C \Rightarrow \Delta \), respectively such that following diagrams commute:

\[
\begin{align*}
\Delta(\Sigma) \xrightarrow{\delta(\Sigma)} & \Delta^2(\Sigma) \xrightarrow{\Delta(\delta)} \Delta(\Sigma) \\
\Downarrow 1_{\Delta(\Sigma)} & \Downarrow 1_{\Delta^2(\Sigma)} \\
\Delta(\Sigma) & \Delta(\Sigma)
\end{align*}
\]

\[
\begin{align*}
\Delta^3(\Sigma) \xrightarrow{\mu(\Sigma)} & \Delta^2(\Sigma) \\
\Downarrow \Delta(\mu) & \Downarrow \mu \\
\Delta^2(\Sigma) & \Delta(\Sigma)
\end{align*}
\]

The Kleisli category \( C_\Delta \) of the monad \((\Delta, \delta, \mu)\) has the objects of \( C \) but an arrow \( \theta_\Delta : A \to B \) in \( C_\Delta \) is an

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\(^1\)This means it is bigger than a category since the hom-sets are classes rather than sets.
The composition in $C\Delta$ is defined as shown below:

$$\begin{array}{ccc}
A & \rightarrow & A \\
\downarrow \theta_\Delta & & \downarrow \theta \\
B & \rightarrow & \Delta(B) \\
\downarrow \theta'_\Delta & & \downarrow \Delta(\theta') \\
C & \rightarrow & \Delta^2(C) \xrightarrow{\mu_C} \Delta(C)
\end{array}$$

The following functor [11] extends the well known power-set functor from sets to categories. The power-set functor on categories $\mathcal{P} : \mathbf{CAT} \rightarrow \mathbf{CAT}$ is defined as follows:

- for any category $C$,
  - $\mathcal{P}C = \{A \mid A \subseteq |C|\}$ and $\mathcal{P}C(A, B) = \{H \subseteq C \mid \exists h \in A, h \subseteq B \text{ for each } h \in H\}$; and
  - composition is defined by $H_1; H_2 = \{h_1; h_2 \mid h_1 \in H_1, h_2 \in H_2, h_1 \subseteq \Delta h_2\};$ then $1_A = \{1_a \mid a \in A\}$ are the identities.

- for any functor $F : C \rightarrow C'$, $\mathcal{P}F(A) = F(A) \subseteq |C'|$ and $\mathcal{P}F(H) = F(H) \subseteq C'$.

Moreover, like in the case of sets, this construction extends to a monad $(\mathcal{P}, \{\_\}, \cup)$ in $\mathbf{CAT}$. Then $\mathbf{CAT}_\mathcal{P}$ denotes its associated Keisli category.

### 2.2. Partial functions

A partial function $f : A \rightarrow B$ is a binary relation $f \subseteq A \times B$ such that $(a, b), (a, b') \in f$ implies $b = b'$. The definition domain of $f$, denoted $\text{dom}(f)$ is the set $\{a \in A \mid \exists b (a, b) \in f\}$. A partial function $\theta : A \rightarrow B$ is called total when $\text{dom}(\theta) = A$. We denote by $\theta^0$ the restriction of $\theta$ to $\text{dom}(\theta) \times B$; this is a total function. Partial functions yield a subcategory of the category of binary relations, denoted $\mathcal{P}\mathbf{fn}$.

### 2.3. $\frac{1}{2}$-categories

A $\frac{1}{2}$-category is just a category such that its hom-sets are partial orders, and the composition preserve these partial orders. In the literature $\frac{1}{2}$-categories are also called ordered categories or locally ordered categories. In terms of enriched category theory [23], $\frac{1}{2}$-category are just categories enriched by the monoidal category of partially ordered sets.

Given a $\frac{1}{2}$-category $C$ by $C^{\square}$ we denote its ‘vertical’ dual which reverses the partial orders, and by $C^{\square\square}$ its double dual $C^{\square\square\square}$. Given $\frac{3}{2}$-categories $C$ and $C'$, a strict $\frac{3}{2}$-functor $F : C \rightarrow C'$ is a functor $C \rightarrow C'$ that preserves the partial orders on the hom-sets. Lax functors relax the functoriality conditions $F(h); F(h') = F(h; h')$ to $F(h); F(h') \leq F(h; h')$ (when $h \subseteq \Delta h'$) and $F(1_A) = 1_{F(A)}$ to $1_{F(A)} \leq F(1_A)$. If these inequalities are reversed then $F$ is an oplax functor. This terminology complies to [3] and to more recent literature, but in earlier literature [24, 22] this is reversed. Note that oplax + lax = strict. In what follows whenever we say “$\frac{3}{2}$-functor” without the qualification “lax” or “oplax” we mean a functor which is either lax or oplax.

Lax functors can be composed like ordinary functors; we denote by $\mathbf{CAT}^{\frac{3}{2}}$ the category of $\frac{3}{2}$-categories and lax functors.

Most typical examples of a $\frac{3}{2}$-category are $\mathcal{P}\mathbf{fn}$ – the category of partial functions in which the ordering between partial functions $A \rightarrow B$ is given by the inclusion relation on the binary relations $A \rightarrow B$, and
PoSet – the category partial ordered sets (with monotonic mappings as arrows) with orderings between
monotonic functions being defined point-wise ($f \leq g$ if and only if $f(p) \leq g(p)$ for all $p$).

Let us consider the power-set monad on categories as defined above. Given the partial order on each $\mathcal{PC}$
given by category inclusions, the Kleisli category $\mathcal{CAT}_\mathcal{P}$ admits a two-fold refinement to a $\frac{3}{2}$-category:

1. morphisms $C \to \mathcal{PC}'$ are allowed to be lax functors rather than (strict) functors, and
2. we consider the point-wise partial order on the class of the lax functors $C \to \mathcal{PC}'$ that is induced by
the partial order on $\mathcal{PC}'$.

Let us denote the $\frac{3}{2}$-category thus obtained by $\mathcal{3}_2(\mathcal{CAT}_\mathcal{P})$.

Unlike in the case of ordinary categories, colimits in $\mathcal{3}_2$-categories come in several different
flavours according to the role played by the order on the arrows. Here we recall some of these for the particular
emblematic case of pushouts; the extension to other types of colimits being obvious.

Given a span $\varphi_1, \varphi_2$ of arrows in a $\mathcal{3}_2$-category, a lax cocone for the span consists of arrows $\theta_0, \theta_1, \theta_2$ such
that there are inequalities as shown in the following diagram:

When the two inequalities are both equalities, this is a strict cocone. In this case $\theta_0$ is redundant and the
data collapses to the equality $\varphi_1; \theta_1 = \varphi_2; \theta_2$.

A lax cocone like in diagram (1) is:

- **pushout** when it is strict and for any strict cocone $\theta_1', \theta_2'$ there exists and unique arrow $\mu$ that is
  mediating, i.e. $\theta_k; \mu = \theta_k', k = 1, 2$;
- **lax pushout** when for any lax cocone $\theta_0', \theta_1', \theta_2'$ there exists an unique mediating arrow $\mu$, i.e.
  $\theta_k; \mu = \theta_k', k = 0, 1, 2$;
- **weak (lax) pushout** when the uniqueness condition on the mediating arrow is dropped from the
  above properties;
- **near pushout** when for any lax cocone $\theta_0', \theta_1', \theta_2'$ the set of mediating arrows
  $\{\mu \mid \theta_k; \mu \leq \theta_k', k = 0, 1, 2\}$ has a maximal element.

Pushouts are not a proper $\frac{3}{2}$-categorical concept because they do not involve in any way the orders on the
arrows.

Lax pushouts represents the instance of a natural concept of colimit from general enriched category theory
[23] to $\mathcal{3}_2$-categories; however in concrete situations, unlike their cousins from ordinary category theory,
they can be very difficult to grasp and sometimes appearing quite inadequate. For example in $\mathcal{Pfn}$, if
$\text{dom}\varphi_1 \cap \text{dom}\varphi_2 \neq \emptyset$ then the span $(\varphi_1, \varphi_2)$ does not have a lax pushout. This is caused by the discrepancy
between a lot of laxity at the level of diagrams and of the arrows on the one hand (allowing for unbalanced
cocones in which low components may coexist with high components), and the strictness required in the
universal property on the other hand. A remedy for this, that was proposed in [11], is to restrict the
cocones to designated subclasses of arrows as follows.

Given a (1-)subcategory $\mathcal{T} \subseteq \mathcal{C}$ of a $\mathcal{3}_2$-category $\mathcal{C}$, a lax $\mathcal{T}$-cocone for a span $(\varphi_1, \varphi_2)$ is a lax cocone
$(\theta_0, \theta_1, \theta_2)$ for the span such that $\theta_k \in \mathcal{T}, k = 0, 1, 2$. A lax $\mathcal{T}$-pushout is a minimal lax $\mathcal{T}$-cocone, i.e. for any
We may omit the superscripts or subscripts from the notation of the components of institutions when there is no risk of ambiguity. For example, if the considered institution and signature are clear, we may denote $\models_{\Sigma}(\phi)$ just by $\models$. For $M = Mod(\phi)M'$, we say that $M$ is the $\phi$-reduct of $M'$.

**Example 2.1** (Propositional logic $-\mathcal{FL}$). This is defined as follows. $Sign^{\mathcal{FL}} = SET$, and for any set $P$, $Sen(P)$ is generated by the grammar

$$S ::= P \mid S \land S \mid \neg S$$

and $Mod^{\mathcal{FL}}(P) = (2^P, \subseteq)$. For any $M \in Mod^{\mathcal{FL}}(P)$, depending on convenience, we may consider it either as a subset $M \subseteq P$ or equivalently as a function $M : P \to 2 = \{0, 1\}$. For any function $\varphi : P \to P'$, $Sen^{\mathcal{FL}}(\varphi)$ replaces the each element $p \in P$ that occurs in a sentence $\rho$ by $\varphi(p)$, and $Mod^{\mathcal{FL}}(\varphi)(M') = \varphi; M$ for each $M' \in 2^{P'}$. For any $P$-model $M \subseteq P$ and $\rho \in Sen^{\mathcal{FL}}(P)$, $M \models \rho$ is defined by induction on the structure of $\rho$ by $(M \models p) = (p \in M)$, $(M \models \rho_1 \land \rho_2) = (M \models \rho_1) \land (M \models \rho_2)$ and $(M \models \neg \rho) = \neg(M \models \rho)$.

**Example 2.2** (Many-sorted algebra $-\mathcal{MSA}$). The $\mathcal{MSA}$-signatures are pairs $(S, F)$ consisting of a set $S$ of sort symbols and of a family $F = \{F_{w \to s} \mid w \in S^*, s \in S\}$ of sets of function symbols indexed by arities (for
the arguments) and sorts (for the results).\textsuperscript{2} Signature morphisms $\varphi : (S, F) \rightarrow (S', F')$ consist of a function $\varphi^a : S \rightarrow S'$ and a family of functions $\varphi^{op} = \{\varphi^{op}_{w,s} : F_w \rightarrow F'_w | w \in S^*, s \in S\}$.

The $(S, F)$-models $M$, called algebras, interpret each sort symbol $s$ as a set $M_s$ and each function symbol $\sigma \in F_w$ as a function $M_\sigma$ from the product $M_w$ of the interpretations of the argument sorts to the interpretation $M_s$ of the result sort. An $(S, F)$-model homomorphism $h : M \rightarrow M'$ is an indexed family of functions $\{h_s : M_s \rightarrow M'_s | s \in S\}$ such that $h_s(M_\sigma(m)) = M'_s(h_\sigma(m))$ for each $\sigma \in F_w$ and each $m \in M_w$, where $h_{w} : M_{w} \rightarrow M'_{w}$ is the canonical componentwise extension of $h$, i.e.

$$h_{w}(m_{1}, \ldots, m_{n}) = (h_{s_{1}}(m_{1}), \ldots, h_{s_{n}}(m_{n}))$$

for each signature morphism $\varphi : (S, F) \rightarrow (S', F')$, the reduct $\text{Mod}(\varphi)(M')$ of an $(S', F')$-model $M'$ is defined by $\text{Mod}(\varphi)(M') = (h_{w}(M_{w}'), m_{w})$ for each sort or function symbol $x$ from the domain signature of $\varphi$.

For each signature $(S, F)$, $\text{Th}(S, F) = \{(T(S, F))_{s} | s \in S\}$ is the least family of sets such that $\sigma(t) \in (T(S, F))_{\sigma}$ for all $\sigma \in F_w$ and all tuples $t \in (T(S, F))_w$. The elements of $(T(S, F))_{s}$ are called $(S, F)$-terms of sort $s$. For each $(S, F)$-algebra $M$, the evaluation of an $(S, F)$-term $\sigma(t)$ in $M$, denoted $M_{\sigma(t)}$, is defined as $M_{\sigma}(M_t)$, where $M_t$ is the componentwise evaluation of the tuple of $(S, F)$-terms $t$ in $M$.

Sentences are the usual first order sentences built from equational atoms $t = t'$, with $t$ and $t'$ (well-formed) terms of the same sort, by iterative application of Boolean connectives ($\land, \lor, \neg, \forall, \exists$) and quantifiers ($\forall X, \exists X$ – where $X$ is a sorted set of variables). Sentence translations along signature morphisms just rename the sort and function symbols according to the respective signature morphisms. They can be formally defined by recursion on the structure of the sentences. The satisfaction of sentences by models is the usual Tarskian satisfaction defined recursively on the structure of the sentences. (As a special note for the satisfaction of the quantified sentences, defined in this formalisation by means of model reducts, we recall that $M \models_{\Sigma} (\forall X) \rho$ if and only if $M' \models_{\Sigma + X} \rho$ for each expansion $M'$ of $M$ to the signature $\Sigma + X$ that adds the variables $X$ as new constants to $\Sigma$.)

Theories and theory morphisms are one of the crucial concepts in institution theory and its applications to formal specifications. Traditionally theories model logic-based formal specifications, while theory morphisms model relations between specification modules, such as imports, renaming, views, etc. (see [6, 30]). In any institution, a theory is a pair $(\Sigma, E)$ consisting of a signature $\Sigma$ and a set $E$ of $\Sigma$-sentences. A theory morphism $\varphi : (\Sigma, E) \rightarrow (\Sigma', E')$ is a signature morphism $\varphi : \Sigma \rightarrow \Sigma'$ such that $E' \models \text{Sen}(\varphi)E$. It is easy to check that the theory morphisms are closed under the composition given by the composition of the signature morphisms; this gives the category of the theories of $I$ denoted $\text{Th}_I$.

Given a theory $(\Sigma, E)$ its closure is $(\Sigma, E^*)$ where $E^* = \{ e \in \text{Sen}(\Sigma) | E \models_{\Sigma} e \}$.

2.5. $\frac{1}{2}$-institutions

According to [11], a $\frac{1}{2}$-institution $I = (\text{Sign}^I, \text{Sen}^I, \text{Mod}^I, (\models^I)_{\Sigma \in \text{Sign}^I})$ consists of

- a $\frac{1}{2}$-category of signatures $\text{Sign}^I$,
- an $\frac{3}{2}$-functor $\text{Sen}^I : \text{Sign}^I \rightarrow \text{Pfin}$, called the sentence functor,
- an lax $\frac{1}{2}$-functor $\text{Mod}^I : (\text{Sign}^I)^{op} \rightarrow \frac{1}{2}(\text{CAT}_{\text{fin}})$, called the model functor,
- for each signature $\Sigma \in |\text{Sign}^I|$ a satisfaction relation $\models^I_{\Sigma} \subseteq |\text{Mod}^I(\Sigma)| \times \text{Sen}^I(\Sigma)$

such that for each morphism $\varphi \in \text{Sign}^I$, the Satisfaction Condition

$$M' \models^I_{\varphi \sigma} \text{Sen}^I(\varphi) \rho \text{ if and only if } M \models^I_{\sigma} \rho$$

\textsuperscript{2}By $S^*$ we denote the set of strings of sort symbols.
As already mentioned above model homomorphisms do not play yet any role in conceptual blending or in when there is no interpretation. Sen can be either lax or oplax; depending on how is this we may call the respective the model reduct Mod. The divergence from those in other envisaged applications of other 2-institutions and ordinary institutions (also called 1-institutions) is determined by the categorical structure of the signature morphisms which propagates to the sentence and to the model functors. Consequently the Satisfaction Condition (3) takes an appropriate format. Thus, for each signature morphism \( \varphi \) its corresponding sentence translation \( \text{Sen}(\varphi) \) is a partial function \( \text{Sen}(\sqsubseteq \varphi) \rightarrow \text{Sen}(\varphi \sqsubseteq) \) and moreover whenever \( \varphi \leq \theta \) we have that \( \text{Sen}(\varphi) \subseteq \text{Sen}(\theta) \). The sentence functor \( \text{Sen} \) can be either lax or oplax; depending on how is this we may call the respective lax or oplax 2-institution. In many concrete situations it happens that \( \text{Sen} \) is strict while some general results require it to be either lax or oplax or both.

The model reduct \( \text{Mod}(\varphi) \) is a lax functor \( \text{Mod}(\varphi \sqsubseteq) \rightarrow \mathcal{P}\text{Mod}(\varphi \sqsubseteq) \) meaning that for each \( \Sigma' \)-model we have a set of reducts rather than a single reduct. In concrete examples this is a direct consequence of the partiality of \( \varphi \): in the reducts the interpretation of the symbols on which \( \varphi \) requires it to be either lax or oplax or both.

- The fact that \( \text{Mod} \) is a 1-functor implies also that whenever \( \varphi \leq \theta \) we have \( \text{Mod}(\theta) \leq \text{Mod}(\varphi) \), i.e. \( \text{Mod}(\theta) \text{Mod}(\varphi) \text{Mod}(\varphi) \text{Mod}(\varphi) \), etc.
- The lax aspect of \( \text{Mod} \) means that for signature morphisms \( \varphi \) and \( \varphi' \) such that \( \varphi \sqsubseteq = \sqsubseteq \varphi' \) and for any \( \varphi \sqsubseteq \)-model \( M'' \), we have that

\[
\text{Mod}(\varphi)(\text{Mod}(\varphi')(M'')) \subseteq \text{Mod}(\varphi; \varphi')M''
\]

and for each signature \( \Sigma \) and for each \( \Sigma \)-model \( M \) that

\[
M \in \text{Mod}(1_{\Sigma})M.
\]

- The lax aspect of the reduct functors \( \text{Mod}(\varphi) \) means that for model homomorphisms \( h_1, h_2 \) such that \( h_1 \sqsubseteq = \sqsubseteq h_2 \) we have that

\[
\text{Mod}(\varphi)(h_1) \subseteq \text{Mod}(\varphi)(h_2)
\]

and for each \( M' \in \text{Mod}(\varphi \sqsubseteq) \) and each \( M \in \text{Mod}(\varphi)M' \) that

\[
1_M \in \text{Mod}(\varphi)1_{M'}.
\]

As already mentioned above model homomorphisms do not play yet any role in conceptual blending or in other envisaged applications of 2-institutions. Hence the lax aspect of model functors is for the moment a purely theoretical feature which is however supported naturally by all examples.

In [34] there is a 2-categorical generalization of the concept of institution, called 2-institution, that consider \( \text{Sign} \) to be a 2-category, \( \text{Sen} : \text{Sign} \rightarrow \text{CAT} \) and \( \text{Mod} : \text{Sign} \rightarrow \text{CAT} \) to be pseudo-functors, and that takes a (quite sophisticated categorically) many-valued approach to the satisfaction relation. From these we can see immediately that 2-institutions of [34] do not cover the concept of 2-institution through the perspective of 2-categories as special cases of 2-categories, the functors \( \text{Sen} \) and \( \text{Mod} \) in 2-institutions diverging from those in 2-institutions in two ways: they are pseudo-functors (in 2-category theory this means just ordinary functors) and their targets do not match those of 2-institutions. This lack of convergence is due to the two extensions aiming to different application domains.
2.6. $\frac{1}{3}$-institutions: examples

The examples given in this section are imported from [11].

The following expected example shows that the concept of $\frac{1}{3}$-institution constitute a generalisation of the concept of institution.

**Example 2.3** (Institutions). Each 1-institution can be trivially regarded as a $\frac{1}{2}$-institution by regarding its category of signatures as a $\frac{1}{2}$-category with discrete partial orders.

**Example 2.4** (Propositional logic with partial morphisms of signatures – $\frac{1}{3}\mathcal{PL}$). This example extends the ordinary institution $\mathcal{PL}$ to a $\frac{1}{3}$-institution by considering partial functions rather than total functions as signature morphisms; thus $\text{Sign} = \text{Pfn}$.

SENTENCES. While for each set $P$, $\text{Sen}(P)$ is like in $\mathcal{PL}$, for any partial function $\varphi : P \rightarrow P'$ the sentence translation $\text{Sen}(\varphi)$ translates like in $\mathcal{PL}$ but only the sentences containing only propositional variables $P$ that are translated by $\varphi$, i.e. that belong to $\text{dom}(\varphi)$; hence the partiality of $\text{Sen}(\varphi)$. More precisely we have that $\text{dom}(\text{Sen}\varphi) = \text{Sen}(\text{dom}\varphi)$ and for each $\rho \in \text{dom}(\text{Sen}\varphi)$ we have that $\text{Sen}(\varphi)\rho = \text{Sen}(\text{dom}\varphi)\rho$. The sentence functor is a strict $\frac{1}{2}$-functor.

MODELS. The $\frac{1}{3}\mathcal{PL}$ models and model homomorphisms are those of $\mathcal{PL}$, but their reducts differ from those in $\mathcal{PL}$. Given a partial function $\varphi : P \rightarrow P'$ and a $P'$-model $M' : P' \rightarrow 2$,

$$\text{Mod}(\varphi)M' = \{M : P \rightarrow 2 \mid M_p = M'_\varphi(p) \text{ for all } p \in \text{dom}\varphi\}.$$ 

On the model homomorphisms the reduct is defined by

$$\text{Mod}(\varphi)(M' \subseteq N') = \{M \subseteq N \mid M \in \text{Mod}(\varphi)M', N \in \text{Mod}(\varphi)N'\}.$$ 

SATISFACTION. The satisfaction relation of $\frac{1}{3}\mathcal{PL}$ is inherited from $\mathcal{PL}$.

**Example 2.5** (Many sorted algebra with partial morphisms of signatures – $\frac{1}{2}\mathcal{MSA}$). In this example we extend the $\mathcal{MSA}$ institution to its $\frac{1}{2}$ variant in a way that parallels the extension of $\mathcal{PL}$ to $\frac{1}{3}\mathcal{PL}$. For this reason we will give only the definitions and rather skip the arguments.

Given $\mathcal{MSA}$ signatures, a *partial $\mathcal{MSA}$-signatures morphism* $\varphi : (S, F) \rightarrow (S', F')$ consists of:

- a partial function $\varphi^\text{st} : S \rightarrow S'$, and
- for each $w \in (\text{dom}\varphi^\text{st})^*$ and $s \in \text{dom}\varphi^\text{st}$ a partial function $\varphi^\text{op}_{w \rightarrow s} : F_{w \rightarrow s} \rightarrow F'_{\varphi^\text{st}_{w \rightarrow s}}$.

Given $\varphi : (S, F) \rightarrow (S', F')$ and $\varphi' : (S', F') \rightarrow (S'', F'')$ their composition $\varphi; \varphi'$ is defined by:

- $(\varphi; \varphi')^\text{st} = \varphi^\text{st}; \varphi'^\text{st}$, and
- for each $w \in (\text{dom}(\varphi; \varphi')^\text{st})^*$ and $s \in \text{dom}(\varphi; \varphi')^\text{st}$: $(\varphi; \varphi')^\text{op}_{w \rightarrow s} = \varphi'^\text{op}_{\varphi^\text{st}_{w \rightarrow s}} \circ \varphi^\text{op}_{w \rightarrow \varphi'^\text{st}_{w \rightarrow s}}$.

Given $\varphi, \theta : (S, F) \rightarrow (S', F')$, then $\varphi \leq \theta$ if and only if:

- $\varphi^\text{st} \subseteq \theta^\text{st}$, and
- for each $w \in (\text{dom}\varphi^\text{st})^*$ and $s \in \text{dom}\varphi^\text{st}$: $\varphi^\text{op}_{w \rightarrow s} \subseteq \theta^\text{op}_{w \rightarrow s}$.

Under these definitions the partial $\mathcal{MSA}$-signature morphisms form a $\frac{1}{2}$-category, which is the category of the $\frac{1}{2}\mathcal{MSA}$ signatures.

Given a partial $\mathcal{MSA}$-signature morphism $\varphi$ we denote by $\text{dom}\varphi$ the signature $(\text{dom}\varphi^\text{st}, \text{dom}\varphi^\text{op})$ where $(\text{dom}\varphi^\text{op})_{w \rightarrow s} = \text{dom}\varphi^\text{op}_{w \rightarrow s}$ and by $\varphi^\circ : \text{dom}\varphi \rightarrow \varphi^\circ$ the resulting (total) $\mathcal{MSA}$-signature morphism.

For any signature $\Sigma$, $\text{Sen}\frac{1}{2}\mathcal{MSA}(\Sigma) = \text{Sen}\mathcal{MSA}(\Sigma)$ and for any partial $\mathcal{MSA}$-signature morphism $\varphi$, $\text{Sen}\frac{1}{2}\mathcal{MSA}(\varphi)$ is defined by
• dom \( \text{Sen}^{3\text{MSA}}(\varphi) = \text{Sen}^{\text{MSA}}(\text{dom} \varphi) \) and
• for each sentence \( \rho \in \text{dom} \text{Sen}^{3\text{MSA}}(\varphi) \), \( \text{Sen}^{3\text{MSA}}(\varphi)\rho = \text{Sen}^{\text{MSA}}(\varphi^0)\rho \).

Like for \( \frac{1}{2}\mathcal{PL} \) this yields also a strict \( \frac{1}{2} \)-functor. For any signature \( \Sigma \), \( \text{Mod}^{3\text{MSA}}(\Sigma) = \text{Mod}^{\text{MSA}}(\Sigma) \) and for any partial \( \text{MSA} \)-signature morphism \( \varphi \), each \( \varphi \square \)-model \( M' \), \( \text{Mod}^{3\text{MSA}}(\varphi)M' = M \) is defined by

- for each sort symbol \( s \) in \( \text{dom} \varphi \), \( M_s = M'_{\varphi^t,s} \), and
- for each operation symbol \( \sigma \) in \( \text{dom} \varphi \), \( M_{\sigma} = M'_{\varphi^o,\sigma} \).

The definition on model homomorphisms is similar, we skip it here. Under these definitions, \( \text{Mod}^{3\text{MSA}} \) is a lax functor.

The satisfaction relation is inherited from \( \text{MSA} \), and the argument for the Satisfaction Condition in \( \frac{1}{2}\text{MSA} \) is similar to that in \( \frac{1}{2}\mathcal{PL} \).

**Example 2.6.** The \( \frac{1}{2}\text{MSA} \) example can be twisted by considering less partiality in the signature morphisms. This can be done in several ways, in each case a different \( \frac{1}{2} \)-’sub-institution’ of \( \frac{1}{2}\text{MSA} \) emerges.

1. We constrain \( \varphi^t \) to be total functions.
2. We let \( \varphi^t \) to be partial functions but we constrain \( \varphi^o \) to be total.

**Example 2.7.** The pattern of Ex. 2.5 can be applied to the extension of \( \text{MSA} \) that takes the ‘first order views’ of [10] in the role of signature morphisms. Since first order views are more general than the \( \text{MSA} \) signature morphisms, the resulting \( \frac{1}{2} \)-institution based upon partial first order views can thought as an extension of \( \frac{1}{2}\text{MSA} \).

So far the Examples 2.4, 2.5, 2.6 and 2.7 are based upon a pattern that can be described as follows:

1. Consider a concrete 1-institution (that may be quite common).
2. Consider some form of partiality for its signature morphisms; often this can be done in several different ways (see Ex. 2.6).
3. Keep the sentences and the models of the original institution, but based on the partiality of the signature morphisms extend the concepts of sentence translations and of model reducts to \( \frac{1}{2} \)-institutional ones. The partiality of the sentence translations amounts to the fact that only the sentences that only involve symbols from the definition domain of the (partial) signature morphism can be translated. The relation-like aspect of the model reducts amounts to the fact that symbols that are outside the definition domain of the (partial) signature morphisms can be interpreted in several different ways in the models.
4. The satisfaction relation of the resulting \( \frac{1}{2} \)-institution is inherited from the original 1-institution.

This pattern pervades a lot of useful \( \frac{1}{2} \)-institutions and can be captured as a generic mathematical construction that derives \( \frac{1}{2} \)-institutions from 1-institutions. The main topic of this paper is precisely to explain mathematically this pattern, and then on such basis to derive general properties that are useful in the envisaged applications of \( \frac{3}{2} \)-institution theory. However in [11] there are interesting examples of \( \frac{3}{2} \)-institutions that fall short off this pattern.

### 3. Generic partial signature morphisms

In this section we present a generic method for constructing \( \frac{1}{2} \)-institutions on top of 1-institutions that is based on extending the category of the signatures by considering partiality for the signature morphisms.
Instances of this generic construction include \( \mathfrak{P L} \), \( \mathfrak{MSA} \) but also the \( \mathfrak{I} \)-subinstitutions of \( \mathfrak{MSA} \) from Ex. 2.6.

The structure of the section is as follows:

1. We recall from the literature the concept of inclusion system that we employ for building generic partiality for the signature morphisms.
2. Given a 1-category of \( \text{Sign} \) endowed with an inclusion system we build a \( \mathfrak{I} \)-category \( \mathfrak{pSign} \), that extends \( \text{Sign} \), and whose arrows are ‘partial maps’ in \( \text{Sign} \). The categorical literature has an established approach to those via spans (e.g. [27, 22], etc.), and in principle we follow that. However the distinctive feature of our approach is the use of inclusion systems, which leads to somehow simpler constructions and proofs as it avoids the quotentioning inherent in the standard span-based approaches to partial maps. We show how inclusion systems and colimits in \( \mathfrak{pSign} \) are inherited from \( \text{Sign} \).
3. Then the sentence and the model structures of the constructed \( \mathfrak{I} \)-institution are developed from those of the base 1-institution on the basis of the partial maps in \( \text{Sign} \). The satisfaction relation of the \( \mathfrak{I} \)-institution is inherited from the base 1-institution.
4. We show how lax cocones of signature morphisms admitting model amalgamation in the constructed \( \mathfrak{I} \)-institution can be obtained from cocones of signature morphisms admitting model amalgamation in the base 1-institution.
5. We provide a taxonomy of theory morphisms in the constructed \( \mathfrak{I} \)-institutions, that reflects various ways to achieve partiality for theory morphisms.

3.1. Inclusion systems

Inclusion systems were introduced in [14] as a categorical device supporting an abstract general study of structuring of specification and programming modules that is independent of any underlying logic. They have been used in a series of general module algebra studies such as [14, 16, 6] but also for developing axiomatisability [28, 5, 6] and definability [1] results within the framework of the so-called institution-independent model theory [6]. Inclusion systems capture categorically the concept of set-theoretic inclusion in a way reminiscent of how the rather notorious concept of factorization system [3] captures categorically the set-theoretic injections; however, in many applications the former are more convenient than the latter. Here we recall from the literature the basics of the theory of inclusion systems. The definition below can be found in the recent literature on inclusion systems (e.g. [6]) and differs slightly from the original one of [14].

A pair of categories \( \langle I, E \rangle \) is an inclusion system for a category \( C \) if \( I \) and \( E \) are two broad subcategories of \( C \) such that

1. \( I \) is a partial order (with the order relation denoted by \( \subseteq \)), and
2. every arrow \( f \) in \( C \) can be factored uniquely as \( f = e_f; i_f \) with \( e_f \in E \) and \( i_f \in I \).

The arrows of \( I \) are called abstract inclusions, and the arrows of \( E \) are called abstract surjections. The domain of the inclusion \( i_f \) in the factorization of \( f \) is called the image of \( f \) and is denoted as \( \text{Im}(f) \) or \( f(A) \) when \( A \) is the domain of \( f \). An inclusion \( i : A \to B \) may also be denoted simply by \( A \subseteq B \).

In [4] it is shown that the class \( I \) of abstract inclusions determines the class \( E \) of abstract surjections. In this sense, [4] gives an explicit equivalent definition of inclusion systems that is based only on the class \( I \) of abstract inclusions.

Given categories \( C \) and \( C' \), each endowed with an inclusion system, a functor \( C \to C' \) is called inclusive when it maps abstract inclusions to abstract inclusions. This is the established structure-preserving
mapping between inclusion systems (see [14, 6, 7], etc.).
The literature contains many other examples of inclusion systems for the categories of signatures and for the categories of models of various institutions from logic or from specification theory. We recall here only a couple of them.

**Example 3.1** (Inclusion system for $\mathcal{P}C$ signatures). The standard example of inclusion system is that from $\text{SET}$, with set theoretic inclusions in the role of the abstract inclusions and surjective functions in the role of the abstract surjections.

**Example 3.2** (Inclusion systems for $\text{MSA}$-signatures). Besides the trivial inclusion system that can be defined in any category (i.e. identities as abstract inclusions and all arrows as abstract surjections) the category of $\text{MSA}$-signatures admits also the following three non-trivial inclusion systems:

- **Closed**
  \[ \phi^c: (\Sigma, E) \to (\Sigma', E') \quad \Sigma \subseteq \Sigma' \quad F_{\Sigma \to \Sigma'} = F'_{\Sigma \to \Sigma'} \text{ for } w \in S^*, s \in S \]

- **Strong**
  \[ \varnothing : (\Sigma, E) \to (\Sigma', E') \quad \Sigma \subseteq \Sigma' \quad F'_{\Sigma \to \Sigma'} = \bigcup_{w \in S^*, s \in S} \varnothing^o(F_{w \to s}) \quad F_{\Sigma \to \Sigma'} = F'_{\Sigma \to \Sigma'} \text{ for } w \in S^*, s \in S \]

- **Nearly Strong**
  \[ \varnothing : (\Sigma, E) \to (\Sigma', E') \quad \Sigma \subseteq \Sigma' \quad F'_{\Sigma \to \Sigma'} = \bigcup_{w \in S^*, s \in S} \varnothing^o(F_{w \to s}) \quad F_{\Sigma \to \Sigma'} = F'_{\Sigma \to \Sigma'} \text{ for } w \in S^*, s \in S \]

**Example 3.3** (Inclusion systems for theory morphisms). In any institution such that its category $\text{Sign}$ of signatures is endowed with an inclusion system such that $\text{Sen}$ is inclusive, its category of closed theories (which is the corresponding full subcategory of $\text{Th}^I$) may inherit this inclusion system in two different ways. This is well known in the literature (e.g. [6]3) and goes as shown in the following table:

| inclusion system | abstract surjections | abstract inclusions |
|------------------|----------------------|---------------------|
| $\varnothing$    | $\varnothing: \Sigma \to \Sigma'$ surjective | $\Sigma \subseteq \Sigma'$ and $E = \text{Sen}(\Sigma) \cap E'$ |

**Definition 3.1.** In any category endowed with an inclusion system, a cospan of arrows $f_1: A_1 \to A_2, f_2: A_2 \to A$ is called semi-inclusive when one of $f_1$ or $f_2$ is an abstract inclusion.

The following property of inclusion systems, which can be found in [6], has a special relevance in what follows.

**Lemma 3.1.** In a category endowed with an inclusion system and which has pullbacks of semi-inclusive cospans, for any $f: A \to B$ and any inclusion $B' \subseteq B$ there exists an unique pullback such that $A' \subseteq A$:

\[
\begin{array}{c}
A \\
\downarrow_f \\
A' \\
\downarrow_f \\
B' \\
\end{array}
\]

\[ (4) \]
3.2. Partial signature morphisms

Partial maps in abstract categories are well known in the literature, one of the earliest references being [27]. There are only slight differences between different approaches, all of them defining partial maps as equivalences classes of spans of arrows. Here we come up with an inclusion systems-based variant that avoids quotients.

Definition 3.2. Given a category $\text{Sign}$ endowed with an inclusion system and which has pullbacks of semi-inclusive cospans, for any $\Sigma, \Sigma' \in |\text{Sign}|$, a partial $\text{Sign}$-morphism $\varphi : \Sigma \rightarrow \Sigma'$ consists of a $\text{Sign}$-morphism $\varphi^0 : \Sigma_0 \rightarrow \Sigma'$ such that $\Sigma_0 \subseteq \Sigma$. We may denote $\Sigma_0$ by $\text{dom}\varphi$.

Given $\varphi : \Sigma \rightarrow \Sigma'$ and $\varphi' : \Sigma' \rightarrow \Sigma''$ their composition $\varphi;\varphi'$ is defined by the following diagram:

\[
\begin{array}{c}
\Sigma \xrightarrow{\varphi} \Sigma' \\
\downarrow^{\text{dom}\varphi} \downarrow^{\varphi^0} \downarrow^{\varphi} \\
\text{dom}\varphi' \downarrow^{\varphi'} \\
\Sigma'' \xleftarrow{(\varphi;\varphi')^0} \\
\end{array}
\]

where the square $(\circ)$ is the unique pullback of $\varphi^0$ and $\text{dom}\varphi' \subseteq \Sigma'$ given by Lemma 3.1.

Given $\varphi, \theta : \Sigma \rightarrow \Sigma'$, then $\varphi \leq \theta$ if and only if $\text{dom}\varphi \subseteq \text{dom}\theta$ and $\varphi^0 = (\text{dom}\varphi \subseteq \text{dom}\theta); \theta^0$.

\[
\begin{array}{c}
\varnothing \theta = \varnothing \varphi \\
\downarrow^{\text{dom}\theta} \downarrow^{\theta^0} \downarrow^{\varnothing} \\
\varnothing \varphi = \theta \varnothing \\
\end{array}
\]

Note the overloading of notations $\rightarrow$ and $\text{dom}\varphi$ here with the corresponding ones from partial functions. In the abstract context they are meant to suggest abstract partiality rather than concrete partiality. However in the example of $\mathcal{P}\mathcal{L}$ and $\frac{1}{2}\mathcal{P}\mathcal{L}$ their meanings do coincide. Also giving the pair $\varphi^0$ such that $\varnothing \varphi^0 \subseteq \varnothing \varphi$ is the same with giving the span $\text{dom}\varphi \subseteq \varnothing \varphi, \varphi^0$ in $\text{Sign}$ (the first arrow being an abstract inclusion).

Proposition 3.1. Let $\text{pSign}$ have the same objects as $\text{Sign}$ and the partial $\text{Sign}$-morphisms as arrows. Under the definitions given in Dfn. 3.2, $\text{pSign}$ is a $\frac{1}{2}$-category.

Proof. The associativity of the composition in $\text{pSign}$ can be determined by chasing the following diagram
and by resorting to Lemma 3.1.

The squares (1), (2), (3) are unique pullbacks as determined by Lemma 3.1. Then the square (2)+(3) corresponds to the square (⊔) in the diagram of Dfn. 3.2 for the composition \((\varphi_1; \varphi_2); \varphi_3\) while the square (1)+(3) corresponds to the square (⊔) for the composition \(\varphi_1; (\varphi_2; \varphi_3)\).

The identities of \(p\text{Sign}\) are the identities of \(\text{Sign}\) (we skip here the straightforward proof that these are identities in \(p\text{Sign}\) indeed).

Now we prove the preservation of the partial orders on the hom-sets by the composition; let us do here only one side of that, the argument for the other side being similar. We consider \(\varphi_1 \leq \varphi_2 : \Sigma \to \Sigma'\) and \(\theta : \Sigma' \to \Sigma''\). The argument for \(\varphi_1; \theta \leq \varphi_2; \theta\) is apparent by analysing the following diagram:

**Fact 3.1.** There is a canonical faithful functor \([\cdot] : \text{Sign} \to p\text{Sign}\) which is the identity on the objects and such that \([\chi]^0 = \chi\) for each arrow \(\chi \in \text{Sign}\).

**Example 3.4.** The \(\frac{1}{2}\)-category of the signatures of \(\frac{1}{2}\mathcal{P}_{\mathcal{L}}\) is the category of partial \(\text{Sign}^{\mathcal{P}_{\mathcal{L}}}\)-morphisms when considering the standard inclusion system in \(\text{SET}\). The \(\frac{1}{2}\)-category of the signatures of \(\frac{1}{2}\mathcal{MSA}\) is the category of partial \(\text{Sign}^{\mathcal{MSA}}\)-morphisms when considering the strong inclusion system in \(\text{Sign}^{\mathcal{MSA}}\). The \(\frac{1}{2}\)-categories of the signatures of the \(\frac{1}{2}\)-subinstitutions of \(\frac{1}{2}\mathcal{MSA}\) from Ex. 2.6 arise as categories of partial \(\text{Sign}^{\mathcal{MSA}}\)-morphisms when considering the closed and nearly strong inclusion systems in \(\text{Sign}^{\mathcal{MSA}}\).

3.3. **Inclusion systems for partial signature morphisms**

The functor of Fact 3.1 transfers the inclusion system of \(\text{Sign}\) to \(p\text{Sign}\); however this is not completely trivial as the additional following technical property is needed:
**Definition 3.3.** In any category endowed with an inclusion system and with pullbacks of semi-inclusive cospans, we say that abstract surjections are stable under semi-inclusive pullbacks when for each pullback square like in diagram (4) if f is an abstract surjection then f′ is an abstract surjection too.

**Example 3.5.** The stability property under inclusive pullbacks holds widely in examples. It is not difficult to check that all four inclusion systems of Examples 3.1 and 3.2 have this property. Let us do it here only for the strong inclusion system for the \( \mathcal{MSA} \) signatures. Consider an inclusive pullback square with respect to the strong inclusion system of \( \mathcal{MSA} \) signatures like in the diagram (4):

\[
\begin{array}{ccc}
(S, F) & \xrightarrow{\varphi} & (S_1, F_1) \\
\subseteq & & \subseteq \\
(S′, F′) & \xrightarrow{\varphi′} & (S′_1, F′_1)
\end{array}
\]

such that \( \varphi \) is an abstract surjection. We have to show that \( \varphi′ \) is an abstract surjection too.

Since \( \varphi^{st} \) is a surjective function it follows that for each \( s′ \in S′_1 \subseteq S_1 \) there exists \( s \in S \) such that \( \varphi^{st}(s) = s′ \). The pullback square (6) implies that the following is a pullback square in \( \text{SET} \) (see [6] for a detailed general construction of pullbacks off signature morphisms in \( \mathcal{MSA} \)):

\[
\begin{array}{ccc}
S & \xrightarrow{\varphi^{st}} & S_1 \\
\subseteq & & \subseteq \\
S′ & \xrightarrow{\varphi′^{st}} & S′_1
\end{array}
\]

which means that \( S′ = \{ x \in S \mid \varphi^{st}(x) \in S′_1 \} \). Consequently \( s \in S′ \) and \( \varphi^{st}(s) = s′ \). Thus shows that \( \varphi^{st} \) is a surjective function too.

The remaining part of the argument is slightly more intricate. Let \( w_1, s_1 \) and \( \sigma_1 \in (F_1′)_{w_1 \to s_1} \). Since \( (S_1′, F_1′) \subseteq (S_1, F_1) \) we have that \( \sigma_1 \in (F_1)_{w_1 \to s_1} \). Since \( \varphi \) is abstract surjection there exists \( w, s \) and \( \sigma \in F_{w \to s} \) such that \( \varphi^{op}(\sigma) = \sigma_1 \). By the construction of pullbacks in \( \mathcal{MSA} \) (see diagram (7)) we know that \( w \in S′′ \) and \( s \in S′ \) and that \( \varphi^{st}(w) = w_1 \) and \( \varphi^{st}(s) = s_1 \). But the construction of pullbacks of \( \mathcal{MSA} \) signatures also gives us that

\[
F′_{w \to s} = \{ x \in F_{w \to s} \mid \varphi^{op}(x) \in (F_1′)_{w_1 \to s_1} \}.
\]

Consequently \( \sigma \in F′_{w \to s} \) and \( \varphi^{op}(\sigma) \in (F_1′)_{w_1 \to s_1} \), which completes the proof that \( \varphi′ \) is an abstract surjection of the strong inclusion system of the \( \mathcal{MSA} \) signature morphisms.

**Proposition 3.2.** Assuming that in \( \text{Sign} \) the abstract surjections are stable under inclusive pullbacks, the following gives an inclusion system in \( p\text{Sign} \):

- **abstract inclusions:** \( [i] \), where \( i \) is an abstract inclusion in \( \text{Sign} \); and
- **abstract surjections:** \( \varphi \), such that \( \varphi^0 \) is an abstract surjection in \( \text{Sign} \).

**Proof.** That the abstract inclusions of \( p\text{Sign} \) form a partial order follows from the functoriality and the faithfulness of the embedding \([\_]\). That the abstract surjections in \( p\text{Sign} \) form a subcategory follows by inspecting the diagram (S) and by applying the stability property under inclusive pullbacks to the square (\( \bigcirc \)) and to \( \varphi^0 \). Then \( (\varphi^0)^′ \) is abstract surjection (in \( \text{Sign} \)) and consequently \( (\varphi;\varphi^0)^0 = (\varphi^0)^′;\varphi^0 \) is abstract surjection (in \( \text{Sign} \)) too.
Any $\varphi \in p\text{Sign}$ can be factored as shown in the following figure (with $e_\varphi$ and $i_\varphi$ being abstract surjection and inclusion, respectively):

For showing the uniqueness of the factoring in $p\text{Sign}$ let us assume $\varphi = e; i$ where $e$ and $i$ are abstract surjections and inclusions, respectively. There exists an abstract inclusion $i'$ in $\text{Sign}$ such that $i = [i']$. It follows that $\varphi^0 = e^0; i'^0$. By the uniqueness of the factoring in the inclusion system of $\text{Sign}$ it follows that $e^0 = (e_\varphi)^0$ and that $i'^0 = i_\varphi$, hence $e = e_\varphi$ and $i = i_\varphi$.

**Corollary 3.1.** The categories $p\text{Sign}^{\mathcal{PE}}$ and $p\text{Sign}^{\mathcal{MSA}}$ have inclusion systems that inherit the respective inclusion systems of $\text{Sign}^{\mathcal{PE}}$ (Example 3.1) and of $\text{Sign}^{\mathcal{MSA}}$ (Example 3.2).

3.4. Pushouts in the category of partial signature morphisms

The following result shows that a relevant class of lax pushouts in $p\text{Sign}$ is determined on the basis of pushouts in $\text{Sign}$. It can also be extended easily to other colimits.

**Proposition 3.3.** If $\text{Sign}$ has (weak) pushouts then $p\text{Sign}$ has (weak) lax $\text{Sign}$-pushouts.\(^4\)

*Proof.* The proof for the weak case is obtained from the proof of the non-weak case by discarding the uniqueness properties. We will therefore consider here only the non-weak case. We consider a span $\varphi_k : \Sigma_0 \to \Sigma_k$, $k = 1, 2$ of partial $\text{Sign}$-morphisms. Then

1. (in $\text{Sign}$) we consider pushout cocones $(\alpha_k, \chi_k)$ for the two spans $(\varphi_k^0, \text{dom}\varphi_k \subseteq \Sigma_0)$, $k = 1, 2$ (see diagram (8) below);
2. (in $\text{Sign}$) we consider a pushout cocone $(\beta_1, \beta_2)$ for the span $(\alpha_1, \alpha_2)$;
3. for $k = 1, 2$ we define $\theta_k^0 = \chi_k; \beta_k$ and we also define $\theta_0^0 = \alpha_k; \beta_k$.

\(^4\)Where $\text{Sign}$ is considered as a subcategory of $p\text{Sign}$ via the embedding of Fact 3.1.
It follows that, in \( pSign \), \( (\theta_0 = [\theta_0^0], \theta_1 = [\theta_1^0], \theta_2 = [\theta_2^0]) \) constitutes a lax cocone for the span \( (\varphi_1, \varphi_2) \) (see diagram (8)).

Now we consider a lax \( Sign \)-cocone \( (\gamma_0 = [\gamma_0^0], \gamma_1 = [\gamma_1^0], \gamma_2 = [\gamma_2^0]) \) for the same span. It follows that for \( k = 1, 2 \), \( (\gamma_k^0, \gamma_k^0) \) is a cocone for the span \( (\varphi_k^0, \dom \varphi_k \subseteq \Sigma_0) \). By the pushout property in \( Sign \), for \( k = 1, 2 \) there exists an unique \( \delta_k : \Sigma_k^+ \rightarrow \Sigma' \) such that \( \chi_k \cdot \delta_k = \gamma_k^0 \) and \( \alpha_k \cdot \delta_k = \gamma_k^0 \). This yields \( (\delta_1, \delta_2) \) a cocone in \( Sign \) for the span \( (\alpha_1, \alpha_2) \). By the pushout property in \( Sign \) for the span \( (\alpha_1, \alpha_2) \) there exists an unique \( \mu^0 : \Sigma \rightarrow \Sigma' \) such that for \( k = 1, 2 \), \( \beta_k \cdot \mu^0 = \delta_k \).

By chasing diagram (8) we have that, for \( k = 1, 2 \)
\[
(9) \quad \theta_0^0, \mu^0 = \alpha_k; \beta_k; \mu^0 = \alpha_k; \delta_k = \gamma_0^0
\]
and
\[
(10) \quad \theta_k^0, \mu^0 = \chi_k; \beta_k; \mu^0 = \chi_k; \delta_k = \gamma_k^0.
\]

Let \( \mu = [\mu^0] \). From (9) and (10) we obtain that \( (pSign) \theta_k ; \mu = \gamma_k, k = 0, 1, 2 \). The uniqueness of \( \mu \) follows from the uniqueness side of the pushout properties involved. \( \square \)

4. Sentences, models and satisfaction with partial signature morphisms

In this section we complete the development of \( \frac{3}{2} \)-institutions on the basis of the results of the previous section. Therefore here the underlying technical assumption is that the category of signatures \( Sign \) is endowed with an inclusion system such that it has pullbacks of semi-inclusive cospans.

4.1. The sentence functor \( pSen \)

The following construction represents and extension of the sentence functor \( Sen \) of a base institution to a \( \frac{3}{2} \)-institution theoretic sentence functor \( pSen \).

**Definition 4.1.** Given an inclusive functor \( Sen : Sign \rightarrow SET \), for each partial signature \( Sign \)-morphism \( \varphi \in pSign \) we define a partial function \( pSen(\varphi) : Sen(\Box \varphi) \rightarrow Sen(\varphi \Box) \) by letting

- \( \dom pSen(\varphi) = Sen(\dom \varphi) \) and
- for each \( \rho \in \dom pSen(\varphi) \), \( pSen(\varphi) \rho = Sen(\varphi^0 \rho) \).

**Proposition 4.1.** Dfn. 4.1 gives a oplax \( \frac{3}{2} \)-functor \( pSen : pSign \rightarrow Pfn \).

**Proof.** Note that \( Sen \) and \( pSen \) are the same on the signatures, they differ only on the signature morphisms. The oplax property of \( pSen \) on the identities is rather immediate; in fact it holds in the strict form \( pSign(1_\Sigma) = 1_{Sen(\Sigma)} \).

Let us now focus on proving that
\[
(11) \quad pSen(\varphi; \varphi') \subseteq pSen(\varphi); pSen(\varphi').
\]

We consider \( \rho \in \dom pSen(\varphi; \varphi') \) which by Dfn. 4.1 means \( \rho \in Sen(\dom \varphi; \varphi') \).

- Since by Dfn. 3.2 we have that \( \dom \varphi; \varphi' \subseteq \dom \varphi \) and because \( I \) is inclusive, we get that \( \rho \in Sen(\dom \varphi) = \dom pSen(\varphi) \).
- By the commutativity of the square \( (\Box) \) of Dfn. 3.2 we have that \( Sen(\varphi^0 \rho) = Sen(\varphi'^0 \rho) \in Sen(\dom \varphi') = \dom pSen(\varphi') \).
This means \( \rho \in \text{dom } p\text{Sen}(\varphi); p\text{Sen}(\varphi') \). Hence \( \text{dom } p\text{Sen}(\varphi; \varphi') \subseteq \text{dom } p\text{Sen}(\varphi); p\text{Sen}(\varphi') \).

For any \( \rho \in \text{Sen}(\text{dom } \varphi; \varphi') = \text{dom } p\text{Sen}(\varphi; \varphi') \) we have the following:

\[
\begin{align*}
p\text{Sen}(\varphi; \varphi')\rho &= \text{Sen}(\varphi; \varphi')^0 \rho & \text{by the definition of } p\text{Sen}(\varphi; \varphi') \\
&= \text{Sen}(\varphi^0; \varphi^0) \rho & \text{by the definition of } (\varphi; \varphi')^0 \text{ cf. diagram (5)} \\
&= \text{Sen}(\varphi^0)\text{Sen}(\varphi^0) \rho & \text{by the functoriality of } \text{Sen} \\
&= \text{Sen}(\varphi^0) \rho & \text{by applying } \text{Sen} \text{ to the square } (\bigcirc) \text{ of Dfn. } 3.2 \\
&= p\text{Sen}(\varphi')(p\text{Sen}(\varphi)\rho) & \text{by the definition of } p\text{Sen}(\varphi), p\text{Sen}(\varphi') \\
&= (p\text{Sen}(\varphi); p\text{Sen}(\varphi'))\rho.
\end{align*}
\]

This concludes the proof of (11) and of the proposition. \( \square \)

In many concrete situations of interest in fact the sentence \( \frac{1}{2} \)-functor \( p\text{Sen} \) is strict. The following result gives a widely applicable general condition for that.

**Corollary 4.1.** If \( \text{Sen} \) maps each pullback square of semi-inclusive cospanst to a weak pullback square, then \( p\text{Sign} \) is a strict \( \frac{1}{2} \)-functor.

**Proof.** By Prop. 4.1 it is enough to prove that \( p\text{Sen} \) is lax. Since \( p\text{Sign} \) is strict on the identities anyway we prove only that

\[(12) \quad p\text{Sen}(\varphi); p\text{Sen}(\varphi') \subseteq p\text{Sen}(\varphi; \varphi').\]

For this we consider \( \rho \in \text{dom } p\text{Sen}(\varphi); p\text{Sen}(\varphi') \). This means \( \rho \in \text{dom } p\text{Sen}(\varphi) = \text{Sen}(\text{dom } \varphi) \) and furthermore that \( \text{Sen}(\varphi^0) \rho \in \text{dom } p\text{Sen}(\varphi') = \text{Sen}(\text{dom } \varphi') \). By the hypothesis when we apply \( \text{Sen} \) to the square \( (\bigcirc) \) of Dfn. 3.2 we still get a pullback square, which means that there exists a sentence in \( \text{Sen}(\text{dom } \varphi; \varphi') \) that gets mapped to \( \rho \) by the inclusion \( \text{Sen}(\text{dom } \varphi; \varphi') \subseteq \text{Sen}(\text{dom } \varphi) \) and to \( \text{Sen}(\varphi^0) \rho \) by \( \text{Sen}(\varphi^0); \text{Sen}(\varphi^0) \rho \). Of course because of the inclusion \( \text{Sen}(\text{dom } \varphi; \varphi') \subseteq \text{Sen}(\text{dom } \varphi) \) this sentence must be \( \rho \). Hence we have just proved that \( \text{dom } p\text{Sen}(\varphi); p\text{Sen}(\varphi') \subseteq \text{dom } p\text{Sen}(\varphi; \varphi') \). The rest of the argument is similar to the corresponding part from the proof of Prop. 4.1. \( \square \)

**Example 4.1.** The sentence functors of both \( \frac{1}{2} \text{PL} \) and \( \frac{1}{2} \text{MS} / \text{A} \) arise as instances of Dfn. 4.1. While in both cases the existence of pullbacks in \( \text{Sign} \) is easy or well known (see [6, 33] for the \( \text{MS} / \text{A} \)), the assumption on \( \text{Sen} \) being inclusive deserves here a bit of attention.

While the sentence functor \( \text{Sen}^{\text{PL}} \) of \( \text{PL} \) (propositional logic) is obviously inclusive and while for the definitions of \( \text{MS} / \text{A} \) that take a global approach to quantification variables this is true as well (such as in [21, 6, 30], etc.), when considering a local approach to quantifiers (like in some more recent publications; see [8] for an ample discussion on the issue) the sentence functor is not inclusive anymore, signature inclusions being mapped to designated injections that are subject to some coherence properties. This is basically due to the fact that in the local approach to quantification variables these are rather heavily qualified, very much like in the implementations of specification languages (e.g. CafeOBJ [13]), and for example the qualifications by the signatures are not preserved by sentence translations along inclusions. In such cases the solution has been already formulated above, namely to weaken the concept of inclusion system to a system of designated “abstract injections”. However for the sake of simplicity of presentation we stick here with the concept of inclusive functor, but keeping in the mind that for the situations when this does not really work there is exists technical remedy.

In both the \( \frac{1}{2} \text{PL} \) and the \( \frac{1}{2} \text{MS} / \text{A} \) cases the sentence functors are strict, which means they are are also lax \( \frac{3}{2} \)-functors. This is due to the fact the condition of preservation of pullbacks of Cor. 4.1 holds both in \( \text{PL} \)
Definition 4.2. Given any functor $\text{Mod} : \text{Sign}^{\otimes} \to \text{CAT}$ we define

- for each $\Sigma \in |\text{Sign}| (= |p\text{Sign}|)$, $p\text{Mod}(\Sigma) = \text{Mod}(\Sigma)$,
- for each partial $\text{Sign}$-morphism $\varphi : \Sigma \to \Sigma'$,
  \[ p\text{Mod}(\varphi)M' = \{ M \in |\text{Mod}(\Sigma)| \mid \text{Mod}(\text{dom}\varphi \subseteq \Sigma)M = \text{Mod}(\varphi^0)M' \} . \]

$p\text{Mod}(\varphi)$ is defined on the arrows like on the objects.

Proposition 4.2. $p\text{Mod}$ is a lax functor $(p\text{Sign})^{\otimes} \to \frac{3}{2}(\text{CAT}_p)$.

Proof. First we show that for each partial $\text{Sign}$-morphism $\varphi : \Sigma \to \Sigma'$, the model reduct $\frac{3}{2}$-functor $p\text{Mod}(\varphi) : \text{Mod}(\Sigma') \to p\text{Mod}(\Sigma)$ is lax. We denote the inclusion $\text{dom}\varphi \subseteq \Sigma$ by $d_\varphi$.

For any model homomorphisms $f', g' \in p\text{Mod}(\Sigma')$ with $f' \square = \square g'$ the homomorphisms in $p\text{Mod}(\varphi)^0f'; p\text{Mod}(\varphi)^0g'$ are $f; g$ with $f, g \in \text{Mod}(\Sigma)$ such that $f \square = \square g, \text{Mod}(d_\varphi)f = \text{Mod}(\varphi^0)f'$, and $\text{Mod}(d_\varphi)g = \text{Mod}(\varphi^0)g'$. The calculation

\[
\text{Mod}(d_\varphi)(f; g) = \text{Mod}(d_\varphi)(f); \text{Mod}(d_\varphi)(g)
\]

by the functoriality of $\text{Mod}(d_\varphi)$

\[
\text{Mod}(\varphi^0)f'; \text{Mod}(\varphi^0)g'
\]

since $f \in p\text{Mod}(\varphi)^0f'$ and $g \in p\text{Mod}(\varphi)^0g'$

\[
\text{Mod}(\varphi^0)(f'; g')
\]

by the functoriality of $\text{Mod}(\varphi^0)$

shows that $f; g \in p\text{Mod}(\varphi)^0f'; p\text{Mod}(\varphi)^0g'$. Hence $p\text{Mod}(\varphi)^0f'; p\text{Mod}(\varphi)^0g' \subseteq p\text{Mod}(\varphi)(f'; g')$.

Now we show that $p\text{Mod}$ is a lax $\frac{3}{2}$-functor.
Let \( \varphi \leq \theta \) be partial \( \text{Sign} \)-morphisms, which means the diagram below commutes:

\[
\begin{array}{c}
\varphi^0 \ar[r]^\subseteq \ar[d] & \varphi \ar[d]
\end{array}
\]
\[
\theta^0 \ar[r]_{\subseteq} \ar[d] & \theta \ar[d]
\]
\[
\begin{array}{c}
\varphi \ar[r] & \varphi^0
\end{array}
\]
\[
\theta \ar[r]_{\subseteq} & \theta^0
\]

For any \( M \in \text{pMod}(\theta)M' \) we have that
\[
\text{Mod}(\varphi^0)M' = \text{Mod}(\text{dom}\varphi \subseteq \text{dom}\theta)(\text{Mod}(\theta)M')
\]
\[
= \text{Mod}(\text{dom}\varphi \subseteq \text{dom}\theta)(\text{Mod}(\text{dom}\theta \subseteq \varnothing)M)
\]
\[
= \text{Mod}(\text{dom}\varphi \subseteq \varnothing)M.
\]

Hence \( M \in \text{pMod}(\varphi)M' \) which shows that \( \text{pMod}(\theta)M' \subseteq \text{pMod}(\varphi)M' \).

- We consider partial \( \text{Sign} \)-morphisms \( \varphi, \varphi' \) such that \( \varnothing \varphi = \varnothing \varphi' \) and consider \( M \in \text{pMod}(\varphi)(\text{pMod}(\varphi')M'') \). This means that there exists \( M' \) such that \( M \in \text{pMod}(\varphi)M' \) and \( M' \in \text{pMod}(\varphi')M'' \). We have that
\[
\text{Mod}((\varphi; \varphi')^0)M'' = \text{Mod}((\varphi^0); \varphi^0)M''
\]
\[
= \text{Mod}(\varphi^0)(\text{Mod}(\varphi^0)M'')
\]
\[
= \text{Mod}(\varphi^0)(\text{Mod}(\varphi \subseteq \varnothing)M'')
\]
\[
= \text{Mod}(\varphi^0; (\text{dom}\varphi \subseteq \varnothing)M'')
\]
\[
= \text{Mod}(\varphi^0; \theta)M'\text{ from diagram (5)}
\]
\[
= \text{Mod}(\varphi^0; \theta)M'\text{ by the functoriality of Mod}
\]
\[
= \text{Mod}(\varphi^0; \theta)M'\text{ since } M' \in \text{pMod}(\varphi')M'
\]
\[
= \text{Mod}(\varphi^0; \theta)M'\text{ by the functoriality of Mod}
\]
\[
= \text{Mod}(\varphi^0; \theta)M'\text{ from diagram (5)}
\]
\[
= \text{Mod}(\varphi^0; \theta)M'\text{ by the functoriality of Mod}
\]
\[
= \text{Mod}(\varphi^0; \theta)M'\text{ since } M \in \text{Mod}(\varphi)M'
\]
\[
= \text{Mod}(\varphi^0; \theta)M'\text{ by the functoriality of Mod}
\]
\[
= \text{Mod}(\varphi^0; \theta)M.\]

This shows that \( M \in \text{Mod}(\varphi; \varphi')M'' \), hence \( \text{pMod}(\varphi'); \text{pMod}(\varphi) \leq \text{pMod}(\varphi; \varphi') \).

- We know that the identities in \( \text{pSign} \) are \([1_\Sigma]\) where \( 1_\Sigma \) is an identity in \( \text{Sign} \). From the definition, it follows that \( \text{pMod}([1_\Sigma]M) = [M] \), hence \( M \in \text{pMod}([1_\Sigma]M) \).

\( \Box \)

**Example 4.2.** Both the model functors of \( \mathfrak{P}_{\Sigma} \) and of \( \frac{1}{2}\text{MSR} \) arise as instances of Dfn. 4.2 (as \( \text{pMod} \), where \( \text{Mod} \) is the model functor of \( \mathfrak{P}_{\Sigma} \) and \( \text{MSR} \), respectively). The same holds for the model functors of the \( \frac{1}{2} \)-substitutions of \( \text{MSR} \) discussed in Ex. 2.6.

Now we are able to define a \( \frac{1}{2} \)-institution on the basis of any institution (with its category of signatures satisfying the required technical assumptions) by putting together the results of the Propositions 4.1 and 4.2. Since both the sentences and the models of the \( \frac{1}{2} \)-institution are exactly those of the base institution, the satisfaction relation of the \( \frac{1}{2} \)-institution is inherited from the base institution.

**Corollary 4.2.** For any institution \( I = (\text{Sign}, \text{Sen}, \text{Mod}, \models) \) such that
- \( \text{Sign} \) is endowed with an inclusion system,
- \( \text{Sign} \) has pullbacks of semi-inclusive cospans,
Sen is an inclusive functor, 
\(\frac{3}{2}I = (p\text{Sign}, p\text{Sen}, p\text{Mod}, \models)\) is an oplax \(\frac{3}{2}\)-institution.

Proof. By the virtue of the Propositions 4.1 and 4.2 it only remains to show the Satisfaction Condition for \(\frac{3}{2}I\). We consider \(\varphi \in p\text{Sign}, \rho \in \text{dom } p\text{Sen}(\varphi)\), and models \(M, M'\) such that \(M \in p\text{Mod}(\varphi)M'\). Then

\[
M' \models_{\varphi \Box} p\text{Sen}(\varphi)\rho \quad \text{if and only if} \quad M' \models_{\varphi \Box} \text{Sen}(\varphi^0)\rho \quad \text{by definition of } p\text{Sen}
\]
\[
\text{if and only if} \quad \text{Mod}(\varphi^0)M' \models_{\text{dom}_\varphi} \rho \quad \text{by the Satisfaction Condition in } I
\]
\[
\text{if and only if} \quad \text{Mod}(\text{dom}_\varphi \subseteq \Box \varphi)M \models_{\text{dom}_\varphi} \rho \quad \text{by definition of } p\text{Mod}(\varphi)M'
\]
\[
\text{if and only if} \quad M \models_{\Box \varphi} \text{Sen}(\text{dom}_\varphi \subseteq \Box \varphi)\rho \quad \text{by the Satisfaction Condition in } I
\]
\[
\text{if and only if} \quad M \models_{\Box \varphi} \rho \quad \text{since } \text{Sen} \text{ is inclusive.}
\]

5. Properties of \(\frac{3}{2}I\)

In this section we determine some properties of \(\frac{3}{2}I\) that are significant within the context of the general theory of \(\frac{3}{2}\)-institutions and its envisaged applications as developed in [11].

5.1. Total signature morphisms

In [11] a couple of complementary concepts, one of syntactic and the other of semantic nature, have been introduced in order to reflect abstractly the situation when a signature morphism is total. These concepts have been applied in [11] for deriving crucial properties on colimits and model amalgamation.

A signature morphism \(\varphi\) in a \(\frac{3}{2}\)-institution \((\text{Sign}, \text{Sen}, \text{Mod}, \models)\) is

- **Sen-maximal** when \(\text{Sen}(\varphi)\) is total;
- **Mod-maximal** when for each \(\varphi \Box\)-model \(M', \text{Mod}(\varphi)M'\) is a singleton; and
- **total** when it is both Sen-maximal and Mod-maximal.

The following is an expected straightforward property.

**Fact 5.1.** For any signature morphism \(\chi\) in \(I\), \([\chi]\) is total in \(\frac{3}{2}I\).

The following is yet another semantic technical expression of the totalness of the signature morphisms that has been used in [11] in connection to model amalgamation properties.

In a \(\frac{3}{2}\)-institution a signature morphism \(\varphi\) is **Mod-strict** when for each signature morphism \(\theta\) such that \(\theta \Box = \Box \varphi\) we have that 

\[
\text{Mod}(\varphi); \text{Mod}(\theta) = \text{Mod}(\theta; \varphi).
\]

In general, in many concrete situations of interest – \(\frac{3}{2}\text{PL}\) and \(\frac{3}{2}\text{MSA}\) included – a signature morphism is Mod-strict whenever it is total.

**Proposition 5.1.** For any signature morphism \(\chi\) in \(I\), \([\chi]\) is pMod-strict (in \(\frac{3}{2}I\)).

**Proof.** Let \(\chi \in \text{Sign}\) (the category of the signatures of \(I\)). Let \(\varphi \in p\text{Sign}\) such that \(\varphi \Box = \Box \chi\). We have to show that

\[
p\text{Mod}([\chi]); p\text{Mod}(\varphi) = p\text{Mod}(\varphi; [\chi]).
\]
Since $p\text{Mod}$ is lax we have to show only that the right hand side is less than the left hand side. Therefore let $M'' \in \text{Mod}(\chi \Box)$ and let $M \in p\text{Mod}(\varphi; [\chi])M''$. We have to show that $M \in p\text{Mod}(\varphi)(p\text{Mod}([\chi])M'')$. Note that

(13) $p\text{Mod}([\chi])M'' = \{\text{Mod}(\chi)M''\}$

and that the collapse of the square ($\Diamond$) in diagram (5) to $\varphi^0$ means also that

(14) $\text{dom}(\varphi; [\chi]) = \text{dom} \varphi$ and $(\varphi; [\chi])^0 = \varphi^0; \chi$.

By (14) and by the definition of $p\text{Mod}$, $M \in p\text{Mod}(\varphi; [\chi])M''$ translates to

$$\text{Mod}(\varphi^0)(\text{Mod}(\chi)M'') = \text{Mod}(\text{dom} \varphi \subseteq \Box \varphi)M.$$

which means that $M \in p\text{Mod}(\varphi)(\text{Mod}(\chi)M'')$. By (13) it follows that $M \in p\text{Mod}(\varphi)(p\text{Mod}([\chi])M'')$. 

5.2. Lax cocones and model amalgamation

The main proposal of [11] regarding the $\frac{1}{3}$-institution theoretic foundations of conceptual blending is based upon two concepts: lax cocones and model amalgamation. Both of them constitute $\frac{1}{3}$-institution theoretic extension of corresponding ordinary institution theoretic concepts. In this section we develop a result on the existence of lax cocones and model amalgamation in $\frac{1}{3}I$ based upon existence of cocones and model amalgamation in the base institution $I$. By considering the mere fact that these properties are very common in concrete institutions, this result is applicable to a wide range of concrete situations.

We start by recalling the well established notion of model amalgamation in ordinary institution theory, then we move to recalling its $\frac{1}{3}$-institution theoretic extension from [11] and finally we develop the above mentioned result.

Model amalgamation properties for institutions formalize the possibility of amalgamating models of different signatures when they are consistent on some kind of generalized ‘intersection’ of signatures. It is one of the most pervasive properties of concrete institutions and it is used in a crucial way in many institution theoretic studies. A few early examples are [29, 32, 26, 14]. For the role played by this property in specification theory and in institutional model theory see [30] and [6], respectively.

A model of a diagram of signature morphisms in an institution consists of a model $M_k$ for each signature $\Sigma_k$ in the diagram such that for each signature morphism $\varphi : \Sigma_i \rightarrow \Sigma_j$ in the diagram we have that $M_i = \text{Mod}(\varphi)M_j$.

A commutative square of signature morphisms

$$\begin{array}{ccc}
\Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\
\varphi_2 \downarrow & & \downarrow \varphi_1 \\
\Sigma_2 & \xrightarrow{\varphi_2} & \Sigma_j
\end{array}$$

is an amalgamation square if and only if each model of the span $(\varphi_1, \varphi_2)$ admits an unique completion to a model of the square. When we drop off the uniqueness requirement we call this a weak model amalgamation square.

In most of the institutions formalizing conventional or non-conventional logics, pushout squares of signature morphisms are model amalgamation squares [6].

In the literature there are several more general concepts of model amalgamation. One of them is model amalgamation for cocones of arbitrary diagrams (rather than just for spans), another one is model amalgamation for model homomorphisms. Both are very easy to define by mimicking the definitions
presented above. While the former generalisation is quite relevant for the intended applications of our work, the latter is less so since at this moment model homomorphisms do not seem to play any role in conceptual blending or in merging of software changes. Moreover amalgamation of model homomorphisms is known to play a role only in some developments in institution-independent model theory [6], but even there most involvements of model amalgamation refers only to amalgamation of models.

In [11] this notion is extended to \(\frac{3}{2}\)-institutions. For the sake of simplicity of presentation, this is presented for lax cocones of spans, the general concept for lax cocones over arbitrary diagrams of signature morphisms being an obvious generalisation.

A model for a diagram of signature morphisms in a \(\frac{3}{2}\)-institution consists of a model \(M_k\) for each signature \(\Sigma_k\) in the diagram such that for each signature morphism \(\varphi : \Sigma_i \rightarrow \Sigma_j\) in the diagram we have that \(M_i \in \text{Mod}(\varphi)M_j\).

The diagram is consistent when it has at least one model.

In any \(\frac{3}{2}\)-institution, a lax cocone for a span in the \(\frac{3}{2}\)-category of the signature morphisms

\[
\begin{array}{ccc}
\Sigma_1 & \leq & \Sigma_2 \\
\varphi_1 & \leq & \varphi_2 \\
\Sigma_0 & \geq & \Sigma_0
\end{array}
\]

has model amalgamation when each model of the span admits an unique completion to a model (called the amalgamation) of the lax cocone.

When dropping the uniqueness condition, the property is called weak model amalgamation.

**Proposition 5.2.** Let \(I\) be an inclusive institution with pullbacks of semi-inclusive cospans.

1. If each span of signature morphisms in \(I\) admits a cocone

   then each span of signature morphisms \((\varphi_1, \varphi_2)\) in \(\frac{3}{2}I\) admits a lax cocone.

2. If each span of signature morphisms in \(I\) admits a cocone that has (weak) model amalgamation, then each span of signature morphisms in \(\frac{3}{2}I\) admits a lax cocone that has (weak) model amalgamation.

**Proof.**

1. We define the a lax cocone for \((\theta_0, \theta_1, \theta_2)\) for the span \((\varphi_1, \varphi_2)\) as follows. We consider any \(\text{dom}\theta_0 \subseteq \Sigma_0\) such that \(\text{dom}\theta_0 \subseteq \text{dom}\varphi_k, k = 1, 2\). Note that in general there may be several choices for \(\text{dom}\theta_0\), and definitely at least one exists by letting \(\text{dom}\theta_0 = \Sigma_0\). In \(I\) we consider successively:

   - for \(k = 1, 2\), \((u_k, v_k)\) a cocone for the span \((\varphi_k^0, \text{dom}\varphi_k \subseteq \text{dom}\theta_0)\), and
   - \((w_1, w_2)\) a cocone for the span \((v_1, v_2)\).
For $k = 1, 2$ let $\theta_k^0 = u_k; v_k$.

(15)

Then $\theta_k^0, k = 0, 1, 2$ define the $\Sigma^3$ signature morphisms $\theta_0 : \Sigma_0 \rightarrow \Sigma, \theta_k = [\theta_k^0] : \Sigma_k \rightarrow \Sigma, k = 1, 2$. So $\theta_1$ and $\theta_2$ are total, which implies that for $k = 1, 2$,

$$\text{dom}(\varphi_k; \theta_k) = \text{dom}\varphi_k$$

and $(\varphi_k; \theta_k)^0 = \varphi_k^0; \theta_k^0$.

By chasing diagram (15) we establish that for $k = 1, 2$, $\varphi_k^0; \theta_k^0 = (\text{dom}\varphi_k \subseteq \text{dom}\theta_0); \theta_0^0$. It thus follows that $\varphi_k; \theta_k \leq \theta_0$.

2. For the second part of the proposition we consider models $M_0, M_1, M_2$ such that for $k = 1, 2, M_0 \in p\text{Mod}(\varphi_k)M_k$. When considering the three cocones for the spans of signature morphisms in $I$ as above, we may also consider them to admit weak model amalgamation.

For $k = 1, 2$ let $N_k = \text{Mod}(\text{dom}\varphi_k \subseteq \Sigma_0)M_0$ and let $N_0 = \text{Mod}(\text{dom}\theta_0 \subseteq \Sigma_0)M_0$. For $k = 1, 2$ we have that $\text{Mod}(\varphi_k^0)M_k = N_k = \text{Mod}(\text{dom}\varphi_k \subseteq \text{dom}\theta_0)N_0$.

- for $k = 1, 2$ let $M_k'$ be the amalgamation of $N_0$ and $M_k$ (in $I$, for the cocone $(u_k, v_k)$ of the span $(\varphi_k^0, \text{dom}\varphi_k \subseteq \text{dom}\theta_0)$), and
- let $M$ be the amalgamation of $M_1'$ and $M_2'$ (in $I$, for the cocone $(v_1, w_2)$ of the span $(v_1, v_2)$).

Then $M$ is the amalgamation of $M_0, M_1, M_2$ for the lax cocone $(\theta_0, \theta_1, \theta_2)$ of the span $(\varphi_1, \varphi_2)$.

For the strict (non-weak) case, it is enough to note that

- $N_0$ is uniquely determined by the equation $\text{Mod}(\text{dom}\theta_0 \subseteq \Sigma_0)M_0 = N_0$ (which is a consequence of $M_0 \in p\text{Mod}(\theta_0)M$),
- for $k = 1, 2, M_k'$ are uniquely determined by $M_k$ (that are given) and $N_0$, and
- $M$ is uniquely determined by $M_1'$ and $M_2'$.

$\square$

The condition that each span of signature morphisms admits a cocone is usually easy in concrete situations, for example the stronger version of existence of pushout cocones holds in most situations of interest. Very often pushout cocones have model amalgamation property at least in its weak form (see for example [6, 30]).

Note that in principle there are several choices for $\text{dom}\theta_0$. Each of them determines a different lax cocone of signature morphisms in $\Sigma^3 I$. Another parameter that may lead to different constructed variants of the lax
cocone is the choice of the cocones in the category of the signatures of \( I \); for example in the weak model amalgamation case, in general there can be multiple choices.

5.3. A taxonomy of “partial” theory morphisms

In [11] there is a study of how the concept of theory can be extended to the \( \frac{3}{2} \)-institutional framework, which is motivated by the applications to conceptual blending. In Goguen’s approach to conceptual blending [17, 19] concept translation is modelled as translation of logical theories (but this has not been achieved there in a proper way due to the lack of an institution theoretic framework). While theories in \( \frac{3}{2} \)-institutions are the same as theories in 1-institutions, the \( \frac{3}{2} \)-institution theoretic concept of theory morphism is much more subtle because of the partiality of the sentence translations. In [11] it is shown how mathematically there are four possible extensions of the institution theoretic concept of theory morphism to \( \frac{3}{2} \)-institutions but also that only the following two of these make real sense technically and in the applications.

In a \( \frac{3}{2} \)-institution given two theories \((\Sigma, E)\) and \((\Sigma', E')\), a signature morphism \( \varphi : \Sigma \to \Sigma' \) is

- a weak \( \frac{3}{2} \)-theory morphism \((\Sigma, E) \to (\Sigma', E')\) when \( \text{Sent}(\varphi)E^* \subseteq E'^* \).  
- a strong \( \frac{3}{2} \)-theory morphism \((\Sigma, E) \to (\Sigma', E')\) when for each \( \Sigma' \)-model \( M' \) such that \( M' \models E' \) there exists \( M \in \text{Mod}(\varphi)M' \) such that \( M \models E \).

The relationship between these two concepts is an inclusive one:

Fact 5.2. [11] Each strong \( \frac{3}{2} \)-theory morphism is weak.

In the light of the above considerations, given an institution \( I \) endowed with an inclusion system and with pullbacks of semi-inclusive cospans, there are four ways to think about partiality for theory morphisms. These four can be grouped as follows:

1. Consider the closed and the strong inclusion systems for theories in \( I \); each of them determines a concept of partial theory morphisms by considering the category of closed theories in the role of \( \text{Sign} \) in Dfn. 3.2. For this it is necessary to recall (from [6]) that in any institution the pullbacks of (closed) theories are inherited from the underlying category of the signatures. Let us call the former concept closed-partial theory morphism and the latter one strong-partial theory morphism.

2. Consider \( \frac{3}{2} I \), the \( \frac{3}{2} \)-institution built on top of \( I \). Then we may consider the two significant concepts of \( \frac{3}{2} \)-theory morphisms discussed above, the weak and the strong one.

In the following we analyse and establish the relationships between these four concepts of partiality for theory morphisms, a first step being given by Fact 5.2. The following is a crucial result in this direction:

Proposition 5.3. Given an inclusive institution \( I = (\text{Sign}, \text{Sen}, \text{Mod}, \models) \) with pullbacks of semi-inclusive cospans, then the category of weak \( \frac{3}{2} \)-theory morphisms in \( \frac{3}{2} I = (\text{pSign}, \text{pSen}, \text{pMod}, \models) \) is equivalent to the category of closed-partial theory morphisms.

Proof. Any weak \( \frac{3}{2} \)-theory morphisms in \( \frac{3}{2} I \), \( \varphi : (\Sigma, E) \to (\Sigma', E') \), gets mapped to the closed-partial theory morphism \( \overline{\varphi} : (\Sigma, E^*) \to (\Sigma', E'^*) \)

where
• $\text{dom} \varphi = (\text{dom} \varphi, E^* \cap \text{Sen}(\text{dom} \varphi))$,
• $\varphi^0 = \varphi^0$.

That $\varphi^0$ is a theory morphism in $\mathcal{J}$ follows from

$\text{Sen}(\varphi^0)(E^* \cap \text{Sen}(\text{dom} \varphi)) \subseteq E^{**}$

which can be expressed as

$p\text{Sen}(\varphi)E^* \subseteq E^{**}$

which is exactly the condition that $\varphi$ is a weak $\frac{3}{2}$-theory morphism.

It is rather straightforward to check that this mapping $\varphi \mapsto \varphi^0$ is functorial and that it is an equivalence of categories.

**Corollary 5.1.** The following figure shows the relationships between the four concepts of “partial” theory morphisms:

| strong-partial theory morphism |
|--------------------------------|
| weak $\frac{3}{2}$-theory morphism = closed-partial theory morphism |
| strong $\frac{3}{2}$-theory morphism |

**References**

[1] Marc Aiguier and Fabrice Barbier. An institution-independent proof of the Beth definability theorem. *Studia Logica*, 85(3):333–359, 2007.
[2] Marc Aiguier and Răzvan Diaconescu. Stratified institutions and elementary homomorphisms. *Information Processing Letters*, 103(1):5–13, 2007.
[3] Francis Borceux. *Handbook of Categorical Algebra*. Cambridge University Press, 1994.
[4] Virgil Emil Căzănescu and Grigore Roșu. Weak inclusion systems. *Mathematical Structures in Computer Science*, 7(2):195–206, 1997.
[5] Răzvan Diaconescu. Elementary diagrams in institutions. *Journal of Logic and Computation*, 14(5):651–674, 2004.
[6] Răzvan Diaconescu. *Institution-independent Model Theory*. Birkhäuser, 2008.
[7] Răzvan Diaconescu. Grothendieck inclusion systems. *Applied Categorical Structures*, 19(5):783–802, 2011.
[8] Răzvan Diaconescu. From universal logic to computer science, and back. In G. Ciobanu and D. Méry, editors, *Theoretical Aspects of Computing – ICTAC 2014*, volume 8687 of *Lecture Notes in Computer Science*. Springer, 2014.
[9] Răzvan Diaconescu. Graded consequence: an institution theoretic study. *Soft Computing*, 18(7):1247–1267, 2014.
[10] Răzvan Diaconescu. Functorial semantics of first-order views. *Theoretical Computer Science*, 656:46–59, 2016.
[11] Răzvan Diaconescu. $\frac{3}{2}$-institutions: an institution theory for conceptual blending. *arXiv:1708.09675 [math.LO]*, 2017.
[12] Răzvan Diaconescu. Implicit Kripke semantics and ultraproducts in stratified institutions. *Journal of Logic and Computation*, 27(5):1577–1606, 2017.
[13] Răzvan Diaconescu and Kokichi Futatsugi. *CafeOBJ Report: The Language, Proof Techniques, and Methodologies for Object-Oriented Algebraic Specification*, volume 6 of *AMAST Series in Computing*. World Scientific, 1998.
[14] Răzvan Diaconescu, Joseph Goguen, and Petros Stefanac. Logical support for modularisation. In Gerard Huet and Gordon Plotkin, editors, *Logical Environments*, pages 83–130. Cambridge, 1993. Proceedings of a Workshop held in Edinburgh, Scotland, May 1991.
[15] Gilles Fauconnier and Mark Turner. Conceptual integration networks. *Cognitive Science*, 22(2):133–187, 1998.
[16] Joseph Goguen and Grigore Roșu. Composing hidden information modules over inclusive institutions. In Olaf Owe, Stein Krogdahl, and Tom Lyche, editors, *From Object-Orientaion to Formal Methods*, volume 2635 of *Lecture Notes in Computer Science*, pages 96–123. Springer, 2004.
[17] Joseph A. Goguen. *An Introduction to Algebraic Semiotics, with Application to User Interface Design*, pages 242–291. Springer Berlin Heidelberg, 1999.

[18] Joseph A. Goguen. What is a concept? In Frithjof Dau, Marie-Laure Mugnier, and Gerd Stumme, editors, *Conceptual Structures: Common Semantics for Sharing Knowledge*, volume 3596 of *Lecture Notes in Computer Science*, pages 52–77. Springer, 2005.

[19] Joseph A. Goguen. Mathematical models of cognitive space and time. (A preliminary version was published in *Reasoning and Cognition*, edited by Daniel Andler and Mitsu Okada.), 2006.

[20] Joseph A. Goguen and Rod M. Burstall. Introducing institutions. In Edmund M. Clarke and Dexter Kozen, editors, *Logic of Programs*, volume 164 of *Lecture Notes in Computer Science*, pages 221–256. Springer, 1983.

[21] Joseph A. Goguen and Rod M. Burstall. Institutions: abstract model theory for specification and programming. *Journal of the ACM*, 39(1):95–146, 1992.

[22] C. Barry Jay. Partial functions, ordered categories, limits and cartesian closure. In G. Birtwistle, editor, *IV Higher Order Workshop, Banff 1990: Proceedings of the IV Higher Order Workshop, Banff, Alberta, Canada 10–14 September 1990*, pages 151–161. Springer London, 1991.

[23] Max Kelly. *Basic Concepts of Enriched Category Theory*. Cambridge University Press, 1982.

[24] Max Kelly and Ross Street. Review of elements of 2-categories. In *Category Seminar Sydney 1972/1973*, volume Lecture Notes in Mathematics, pages 75–103. Springer, 1974.

[25] Saunders Mac Lane. *Categories for the Working Mathematician*. Graduate texts in mathematics. Springer, 1998.

[26] José Meseguer. General logics. In H.-D. Ebbinghaus et al., editors, *Proceedings, Logic Colloquium, 1987*, pages 275–329. North-Holland, 1989.

[27] Edmund Robinson and Giuseppe Rosolini. Categories of partial maps. *Information and Computation*, 79(2):95–130, 1988.

[28] Grigore Rosu. Axiomatizability in inclusive equational logic. *Mathematical Structures in Computer Science*, 12(5):541–563, 2002.

[29] Donald Sannella and Andrzej Tarlecki. Specifications in an arbitrary institution. *Information and Control*, 76:165–210, 1988.

[30] Donald Sannella and Andrzej Tarlecki. *Foundations of Algebraic Specifications and Formal Software Development*. Springer, 2012.

[31] Marco Schorlemmer, Alan Smaill, Kai-Uwe Kühnberger, Oliver Kutz, Simon Colton, Emilios Cambouropoulos, and Alison Pease. COINVENT: towards a computational concept invention theory. In Simon Colton, Dan Ventura, Nada Lavrac, and Michael Cook, editors, *International Conference on Computational Creativity*, pages 288–296. computationalcreativity.net, 2014.

[32] Andrzej Tarlecki. On the existence of free models in abstract algebraic institutions. *Theoretical Computer Science*, 37:269–304, 1986.

[33] Andrzej Tarlecki, Rod Burstall, and Joseph Goguen. Some fundamental algebraic tools for the semantics of computation, part 3: Indexed categories. *Theoretical Computer Science*, 91:239–264, 1991.

[34] J. Climent Vidal and J. Soliveres Tur. A 2-categorial generalization of the concept of institution. *Studia Logica*, 95(3):301–344, 2010.