Frobenius Structures
and Generalized Deformation
of Kodaira Manifolds

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September 20, 2020

Abstract

It is known that generalized deformation in the sense of Hitchin-Gaultieri is a geometric realization of the degree-2 component of Kontsevich-Barannikov’s homological approach to extended deformation. Through extended deformation, one associates a Frobenius structure to the extended moduli space. In this notes, we prove that on primary Kodaira manifolds the restriction of the Frobenius structure on the degree-2 component of the extended moduli space is trivial. It generalizes the author’s past observation on Kodaira surface.

1 Introduction

Inspired by the development of mirror symmetry, especially its homological aspects, Kontsevich and Barannikov developed an extended deformation theory to study mirror symmetry and its higher dimension analogue [2] [3]. In [2], its starting point is with the complex manifold side of a mirror pair. The classical deformation theory of complex structures focuses on the first Dolbeault cohomology $H^1(M, \Theta)$ where $M$ is a complex manifold and $\Theta$ is its holomorphic tangent bundle. Whereas in an

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extended deformation theory, Barannikov and Kontsevich utilize the full cohomology with holomorphic polyvector fields

\[ H^\bullet(M, \wedge \cdot \Theta) = \sum_{p,q} H^q(M, \wedge^p \Theta) \]

to capture the variation of cohomological structures, e.g. Hodge structure. In this development, one of their discoveries is a Frobenius structure on the extended ”moduli” space as a \( \mathbb{Z} \)-graded supermanifold \([3] [12] [13]\). This author had computed the Frobenius structure associated to the primary Kodaira surface as an example \([46]\).

While this development continues to exert its impact on the development of homological aspects of mirror symmetry, see e.g. \([7] [8] [16] [31] [32] [37]\), its degree-2 component

\[ \sum_{p+q=2} H^q(M, \wedge^p \Theta) \subset H^\bullet(M, \wedge \cdot \Theta) \]

of extended deformation could be interpreted geometrically by the deformation theory of generalized geometry \([23] [25] [26]\). In particular, \( H^1(M, \Theta) \) corresponds exactly to the linearized ”virtual moduli” space of classical deformation theory \([33]\). If one finds a de-Rham closed representative in \( H^2(M, \wedge^0 \Theta) = H^2(M, \mathcal{O}) \), it is due to a \( B \)-field transformation within the realm of generalized geometry \([23]\). As we will explain in subsequent computation, closed 2-forms are both \( \bar{\partial} \)-closed and they commute with every element in the Schouten algebra that controls extended deformations, they don’t contribute any non-trivial terms in the homological algebra aspect of deformation theory. \( B \)-field transformation is, therefore, a geometrical realization of this algebraic fact.

However, if an element in \( H^2(M, \mathcal{O}) \) is not de-Rham closed, it represents a deformation in the sense of ”gerbe” as seen in \([24] [27]\). On the other hand, when an element in \( H^0(M, \wedge^2 \Theta) \) represents what is known as holomorphic Poisson structure, it is integrable in the sense of extended deformation theory.

While classical complex deformation has been a century-old subject, holomorphic Poisson structures are also known for some time \([5] [36] [45]\). Its recent role in generalized geometry and related deformation theory arouse further interests in this subject \([17] [20] [21] [28] [50] [51]\). Since generalized geometry encompasses both complex structures and symplectic structures, holomorphic Poisson structure has
been particularly useful as a tool to capture both features for holomorphic symplectic manifolds \([4] [26] [30] [51]\). In addition, it is now known that holomorphic Poisson structures play a fundamental role in generalized geometry \([1]\). Therefore, there has been work towards understanding holomorphic Poisson cohomology \([10] [29] [30] [47]\).

It is not hard to recognize that holomorphic Poisson cohomology is the hypercomplex of a bi-complex. Therefore, theoretically speaking it could be computed by spectral sequence method once a filtration is chosen. The first page of the spectral sequences of one of the two natural filtrations of this bi-complex leads to a holomorphic version of Lichnerowicz-Poisson cohomology \([38] [54]\). The first page of the spectral sequence of another natural filtrations leads to the Dolbeault cohomology of polyvector fields \(H^q(M, \wedge^p \Theta)\). The \(d_1\)-map of the spectral sequence is simply the restriction of the adjoint action of the holomorphic Poisson structure \(\Lambda\), where \(\text{ad}_\Lambda(-) = [\Lambda, -]\). Much of the recent computation of holomorphic Poisson cohomologies takes the second approach. However as seen in \([10] [30] [36] [47]\), this spectral sequence rarely degenerates on the second page on manifolds such as rational surfaces or Hopf manifolds, let alone degeneracy on the first page. Therefore, in general the spectral sequence approach provides a theoretical but not an effective computational framework.

On the other hand, nilmanifolds are also known to be rich source of generalized geometry of various kinds \([6]\). It has long been known that the DeRham cohomology of nilmanifolds is given by algebraic objects \([44]\). Through a series of work by many, e.g. \([10] [11] [12] [52]\), we learn that the Dolbeault cohomology with structure sheaf or tangent sheaf are given by algebraic objects when the holomorphic Poisson structure is also an algebraic object on a finite-dimensional space. The significance of such observation is that the computation of spectral sequence becomes algebraic as well, and much of the computation becomes tractable in an elementary manner.

Based on such foundation, the author and his collaborators recently identify when the spectral sequence degenerates on the first page if the complex structure is abelian and the underlying nilmanifold is 2-steps \([48] [49]\). It enables an effective computation of holomorphic Poisson cohomology. They go on to identify the necessary and sufficient condition for the holomorphic Poisson cohomology, as a Ger-
stenhaver algebra, to be isomorphism to case when the Poisson structure is equal to zero. If one considers $t\Lambda$ to be a deformation with deformation variable $t \in [0, 1]$, it means that the holomorphic Poisson cohomology is invariant under special condition. They also demonstrate that the primary Kodaira surface as seen in [46] is an example.

It takes us back to an observation in [46]. In [46], the author identifies the Frobenius structure of primary Kodaria surfaces. An intriguing observation, after the computation, is that the restriction of the Frobenius structure on the even part of the supermanifold is trivial. The goal of this paper is to prove that the Frobenius structure restricted to the moduli of a generalized deformation of a primary Kodaira manifold of any dimension is trivial. It establishes a conjecture that the same is true for a large class of 2-step nilmanifolds with abelian complex structures. We will provide a rigorous statement of our theorem in the next section after we acquire sufficient preparation to convey the technical details. In the last section, we will present a conjecture to indicate why and potentially how we may extend the results in this paper beyond Kodaira manifolds.

2 Deformation Theory

For any complex manifold $M$, let $\Theta$ be the holomorphic tangent bundle and $\Omega$ the bundle of $(1, 0)$-forms. We use $\Theta$ and $\Omega$ to denote their conjugate bundles. It is well known since Kodaira-Spencer’s deformation theory [34] that a deformation of complex structure is given by $\Gamma$ with $\Gamma \in C^\infty(M, \Theta \otimes \Omega)$ satisfying the Maurer-Cartan equation:

$$\bar{\partial} \Gamma + \frac{1}{2} [\Gamma, \Gamma] = 0,$$

where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket [34] [40]. The Schouten-Nijenhuis bracket, or simply the Schouten bracket, could be extended to the space of sections of the bundle of exterior algebra

$$\wedge^* (\Theta \oplus \Omega) = \oplus_{p,q} \left( \wedge^p \Theta \otimes \wedge^q \Omega \right).$$

We adopt the notations that $\Theta^p = \wedge^p \Theta$ and $\Omega^q = \wedge^q \Omega$. For a section in $\Theta^p \otimes \Omega^q$, we refer $(p, q)$ as its bi-degree, and $p + q$ its (total) degree. We observe that if $U$ and
V are (1, 0)-vector fields, $\alpha$ and $\beta$ are (0, 1)-forms, then $[U, V]$ is simply the usual Lie bracket between vector fields; $[U, \beta] = \mathcal{L}_U \beta = du_U \beta + u_U d\beta$ is the Lie derivative of the 1-form $\beta$ with respect to the vector field $U$; and $[\alpha, \beta] = 0$. It extends to the full exterior algebra through the rules of Schouten algebra.[40][46].

On the other hand, $\bar{\partial} \alpha$ is the (0, 2)-part of $d\alpha$, the exterior differential of $\alpha$. $\bar{\partial}U$ is the Cauchy-Riemann differentiation as noted in[18][46]. The $\bar{\partial}$-operator extends to the full exterior algebra $\wedge^\bullet (\Theta \oplus \Omega)$ so that it becomes an exterior differential algebra. Due to the integrability of the complex structure, $\bar{\partial} \circ \bar{\partial} = 0$. In particular, we will consider the cohomology spaces $H^\bullet = \oplus_k H^k(M)$, where the $k$-th cohomology space $H^k(M)$ has a bi-degree decomposition.

$$H^k(M) = \oplus_{p+q=k} H^q(M, \Theta^p). \tag{1}$$

The exterior product $\wedge$ and the Schouten bracket on the space $C^\infty(M, \wedge^\bullet (\Theta \oplus \Omega))$ are compatible in the sense that they form a Gerstenhaber algebra [19]. With the $\bar{\partial}$-operator,

$$(C^\infty(M, \wedge^\bullet (\Theta \oplus \Omega)), \ [\cdot, \cdot], \ \wedge, \ \bar{\partial})$$

form a differential Gerstenhaber algebra DGA(0) [43][46]. In particular, for any sections $K$ and $L$ with degree $k$ and $\ell$ respectively,

$$\bar{\partial}(K \wedge L) = (\bar{\partial}K) \wedge L + (-1)^k K \wedge (\bar{\partial}L), \tag{2}$$

$$\bar{\partial}[K, L] = [\bar{\partial}K, L] + (-1)^{k+1}[K, \bar{\partial}L]. \tag{3}$$

An element $\Gamma \in C^\infty(M, \wedge^\bullet (\Theta \oplus \Omega))$ is an integrable extended deformation if it satisfies the Maurer-Cartan equation

$$\bar{\partial}\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0. \tag{4}$$

For such $\Gamma$, the operator

$$\bar{\partial}_\Gamma = \bar{\partial} + [\Gamma, -] \tag{5}$$

satisfies $\bar{\partial}_\Gamma \circ \bar{\partial}_\Gamma = 0$ and hence determines a cohomology $H^\bullet_\Gamma(M)$ [39]. Due to (2) and (3) above, the cohomology space inherits the structure of a Gerstenhaber algebra.

Roughly speaking, if the collection of integrable extended deformations $\Gamma$ forms a "moduli" space when equivalence is accounted for, the Frobenius structure on the moduli space measures how the cohomology $H^\bullet_\Gamma(M)$ varies when $\Gamma$ varies.
Denote the linear part of $\Gamma$ by $\Gamma_1$. The linear portion of the Maurer-Cartan equation (4) shows that $\overline{\partial}\Gamma_1 = 0$. In particular, it is an element in $H^\bullet(M)$. Conversely, given $\Gamma_1$ in $H^\bullet(M)$, one may attempt the Kuranishi’s recursive method to identify an infinite series of obstruction to the existence of $\Gamma$ [35]. In this notes, we work on primary Kodaira manifolds, which will be defined in the next section. Instead of working with the full extended deformation, we focus on the portion leading to generalized deformations, namely

$$\Gamma_1 \in \bigoplus_{p+q=2} H^q(M, \Theta^p).$$

We consider such $\Gamma_1$ as infinitesimal generalized deformation. Our goal in this paper is to derive the following theorem.

**Theorem 1** Let $M$ be a primary Kodaira manifold. Every infinitesimal generalized deformation sufficiently close to zero is integrable. In other words, for each $\Gamma_1 \in \bigoplus_{p+q=2} H^q(M, \Theta^p)$ near 0, there exists $\Gamma \in C^\infty(M, \wedge^\bullet (\Theta \oplus \Omega))$ with $\Gamma_1$ being its linear part, satisfying the Maurer-Cartan equation. Moreover, there exists a natural isomorphism of Gerstenhaber algebras:

$$(H^\bullet_\Gamma(M), [-,-], \wedge) \cong (H^\bullet(M), [-,-], \wedge).$$

In Theorem 2 below, we will provide further details regarding how close to zero is sufficient. This theorem means that Frobenius structure restricted to the degree-2 portion of the $\mathbb{Z}$-grading of the moduli space associated to a primary Kodaira manifold is trivial. It generalizes a computation on Kodaira surface in [46] from complex 2-dimension to all complex dimensions. It also generalizes an observation in [49] that the holomorphic Poisson cohomology of any holomorphic Poisson structures on primary Kodaira manifolds of all dimensions, as Gerstenhaber algebra, are isomorphic to $H^\bullet(M)$.

## 3 Kodaira Manifolds

We consider primary Kodaira manifolds as nilmanifolds, meaning that they are co-compact quotient of a simply-connected nilpotent Lie group.
Let \( \{X_1, Y_1, \ldots, X_n, Y_n, Z_1, Z_2\} \) be the basis of a real vector space of \( 2n + 2 \)-dimension. Define on it a Lie bracket \([−, −]\) such that the only non-trivial ones are given by
\[
[X_j, Y_j] = −[Y_j, X_j] = Z_1. \tag{6}
\]
The Lie bracket turns the concerned vector space in a Lie algebra, which we denote by \( g \). Its center \( c \) is spanned by \( Z_1, Z_2 \). Let \( G \) be its corresponding simply connected Lie group. Its co-compact quotient is a (primary) Kodaira manifold, which we denote by \( M \). A detailed description of the related lattice and group structure could be found in [22].

Consider the linear map \( J : g \rightarrow g \) such that
\[
JX_j = Y_j, \quad JY_j = −X_j, \quad JZ_1 = Z_2, \quad JZ_2 = −Z_1. \tag{7}
\]
The operator \( J \) becomes an integrable complex structure on \( g \). Through left translation, it becomes an integrable complex structure on \( G \), and descends on the co-compact quotient which is defined by right action. For each \( 1 \leq j \leq n \), define \( T_j = \frac{1}{2}(X_j − iY_j) \) and \( W = \frac{1}{2}(Z_1 − iZ_2) \). The space \( g^{1,0} = \{T_1, \ldots, T_n, W\} \) form the space of \((1, 0)\)-vectors. Let \( t^{1,0} \) and \( c^{1,0} \) respectively be spanned by \( \{T_1, \ldots, T_n\} \) and \( W \). Their conjugate spaces are denoted by \( \overline{g^{1,0}}, \overline{t^{1,0}} \) and \( \overline{c^{1,0}} \) respectively. Given the way the complex structure \( J \) is defined, \( g^{1,0} \) is an abelian complex algebra while on \( g^{1,0} \oplus \overline{g^{1,0}} \) in terms of the basis \( \{T_1, \ldots, T_n, W, \overline{T}_1, \ldots, \overline{T}_n, \overline{W}\} \), the structure equations are given by
\[
[T_j, \overline{T}_j] = \frac{1}{4}[X_j − iY_j, X_j + iY_j] = \frac{i}{2}[X_j, Y_j] = \frac{i}{2}Z_1 = \frac{i}{2}(W + \overline{W}). \tag{8}
\]
Let \( \{ω^1, \ldots, ω^n, ρ\} \) be the dual basis with respective to \( \{T_1, \ldots, T_n, W\} \). Denote the complex linear span of its complex conjugation by \( g^{0,1} \). We will also denote the \( p \)-th exterior product of \( g^{1,0} \) by \( g^{p,0} \), the \( q \)-th exterior product of \( g^{0,1} \) by \( g^{0,q} \), and denote \( g^{p,0} \otimes g^{0,q} \) by \( g^{p,q} \). Then
\[
\bigwedge^k (g^{1,0} \oplus \overline{g^{1,0}}) = \sum_{p+q=k} g^{p,q}. \tag{9}
\]
\( t^{p,q} \) and \( c^{p,q} \) are similarly defined. In this setting the Lie algebra differential \( ∂ : g \rightarrow g \)
\( \mathfrak{g}^{0,1} \to \mathfrak{g}^{0,2} \) is determined by
\[
d\omega^j = 0 \quad \text{for all } j, \quad \text{and} \quad d\bar{\rho} = -\frac{i}{2} \sum_{j=1}^{n} \omega^j \wedge \bar{\omega}^j. \tag{10}\]

Apparently, \( d\rho = d\bar{\rho} \) and
\[
\overline{\partial} \omega^j = 0 \quad \text{for all } j, \quad \text{and} \quad \overline{\partial} \rho = 0. \tag{11}\]

To compute \( \overline{\partial} : \mathfrak{g}^{1,0} \to \mathfrak{g}^{1,1} \), we note that the Cauchy-Riemann operator is given by
\[
\overline{\partial}_BA = \{ [\mathcal{B}, A]\}^{1,0} \]
for any \( A \in \mathfrak{g}^{1,0} \) and \( \mathcal{B} \in \mathfrak{g}^{1,0} \) \([18]\). Therefore for all \( T \in \mathfrak{g}^{1,0} \),
\[
\overline{\partial}T = \sum_{j=1}^{n} [T_j, T]^{1,0} \wedge \bar{\omega}^j. \tag{12}\]

It follows that
\[
\overline{\partial}W = 0, \quad \text{and for all } j, \quad \overline{\partial}T_j = -\frac{i}{2} W \wedge \bar{\omega}^j. \tag{13}\]

Through contraction \( \iota_T d\rho = -\frac{i}{2} \omega^j \), the maps \( d\rho \) and \( d\bar{\rho} \) are treated as linear maps.
\[
d\rho, d\bar{\rho} : t^{1,0} \to t^{0,1}. \tag{14}\]

As explained in \([46]\),
\[
\text{Ger} = (\oplus_{p,q} \mathfrak{g}^{p,q}, \ [-, -], \ \wedge, \ \overline{\partial}) \tag{15}\]
is a finite-dimensional Gerstenhaber sub-algebra contained in
\[
(C^\infty(M, \wedge^\bullet(\Theta \oplus \overline{\Theta})), \ [-, -], \ \wedge, \ \overline{\partial}).
\]

Moreover, the inclusion map is a quasi-isomorphism, meaning that it induces an isomorphism on cohomology level \([46] \text{ Proposition } 6\). Therefore, from now on we focus our computation on the finite-dimensional algebra \( \text{Ger} \).

The vector space \( \oplus_{p,q} \mathfrak{g}^{p,q} \) with the Schouten bracket form a Schouten algebra.
\[
\text{Sch} = (\oplus_{p,q} \mathfrak{g}^{p,q}, \ [-, -]).
\]
The structure of $\text{Sch}$ is generated by the bracket restricted to degree-1 elements

$$\text{Sch}^1 = g^{1,0} \oplus g^{0,1},$$

where $[T, \varpi]$ by definition is equal to $\iota_T d\varpi$. As the algebra $g^{1,0}$ is abelian and all elements in $t^{0,1}$ is $d$-closed, the only non-trivial bracket is due to

$$[T_j, \rho] = \iota T_j d\rho = -i \frac{2}{2} \omega_j.$$

Referencing to the structure of the Schouten algebra $\text{Sch}$, we note the following.

**Lemma 1** The spaces $c^{1,0}$ and $t^{0,1}$ are in the center of the Schouten algebra.

In subsequent analysis, we focus on the even part of the Schouten algebra,

$$\text{Sch}^{\text{even}} = \oplus_{p+q=\text{even}} g^{p,q}.$$  

We start with $\text{Sch}^2 = \oplus_{p+q=2} g^{p,q} = g^{2,0} \oplus g^{1,1} \oplus g^{0,2}$. It contains $t^{1,1}$. We further define the following for all $1 \leq i, j \leq n$.

$$\phi_{ij} = \frac{1}{2} (T_i \wedge \varpi^j + T_j \wedge \varpi^i), \quad \psi_{ij} = \frac{1}{2} (T_i \wedge \varpi^j - T_j \wedge \varpi^i).$$

Apparently, $\phi_{ij} = \phi_{ji}$ and $\psi_{ij} = -\psi_{ji}$ for all $i, j$. We denote the respective linear spans by $\odot^{1,1}$ and $\triangle^{1,1}$ so that

$$t^{1,1} = \odot^{1,1} \oplus \triangle^{1,1}.$$  

So we have

- $g^{2,0} = (c^{1,0} \otimes t^{1,0}) \oplus t^{2,0}$.
- $g^{1,1} = c^{1,1} \oplus (c^{1,0} \otimes t^{0,1}) \oplus (t^{1,0} \otimes c^{0,1}) \oplus \odot^{1,1} \oplus \triangle^{1,1}$.
- $g^{0,2} = (c^{0,1} \otimes t^{0,1}) \oplus t^{0,2}$.

Since $c^{1,0}$ and $t^{0,1}$ are in the center of the Schouten algebra, as far as Schouten bracket is concerned,

- $g^{2,0} \equiv_s (c^{1,0} \otimes t^{1,0}) \oplus t^{2,0}$. 

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\[ \mathfrak{g}^{1,1} \equiv_\mathfrak{s} \mathfrak{c}^{1,1} \oplus (t^{1,0} \otimes \mathfrak{c}^{0,1}) \oplus \odot^{1,1} \oplus \Delta^{1,1}. \]

\[ \mathfrak{g}^{0,2} \equiv_\mathfrak{s} \mathfrak{c}^{0,1} \otimes t^{0,1}. \]

In addition, due to (17) and definitions in (19)

\[ [T_j \wedge T_k, \bar{\rho}] = T_j \wedge [T_k, \bar{\rho}] - T_k \wedge [T_j, \bar{\rho}] = -\frac{i}{2}(T_j \wedge \bar{\omega}^k - T_k \wedge \bar{\omega}^j) = -i\psi_{jk}. \] (21)

Below we compute the Schouten bracket among terms in \( Sch^2 \). Firstly,

\[ [\mathfrak{g}^{2,0}, \mathfrak{g}^{2,0}] = 0 \] (22)

because the complex structure is abelian. Since \( \mathfrak{c}^{1,0} \) is in the center and is one-dimensional,

\[ [\mathfrak{c}^{1,0} \otimes t^{1,0}, \mathfrak{c}^{1,1}] = 0. \] (23)

To compute \( [\mathfrak{c}^{1,0} \otimes t^{1,0}, t^{1,0} \otimes \mathfrak{c}^{0,1}] \), consider \( W \wedge T_j \) in \( \mathfrak{c}^{1,0} \otimes t^{1,0} \) and \( T_k \wedge \bar{\rho} \) in \( t^{1,0} \otimes \mathfrak{c}^{0,1} \).

\[ [W \wedge T_j, T_k \wedge \bar{\rho}] = -W \wedge [T_j, \bar{\rho}] \wedge T_k = -\frac{i}{2}W \wedge T_k \wedge \bar{\omega}^j. \] (24)

In particular,

\[ [\mathfrak{c}^{1,0} \otimes t^{1,0}, t^{1,0} \otimes \mathfrak{c}^{0,1}] \subseteq \mathfrak{c}^{1,0} \otimes t^{1,1}. \] (25)

Given (17),

\[ [\mathfrak{c}^{1,0} \otimes t^{1,0}, \odot^{1,1}] = 0 \quad \text{and} \quad [\mathfrak{c}^{1,0} \otimes t^{1,0}, \Delta^{1,1}] = 0. \] (26)

i.e.

\[ [\mathfrak{c}^{1,0} \otimes t^{1,0}, t^{1,1}] = 0. \] (27)

Regarding \( [\mathfrak{c}^{1,0} \otimes t^{1,0}, \mathfrak{c}^{0,1} \otimes t^{0,1}] \subseteq \mathfrak{c}^{1,0} \otimes t^{0,2} \), we have

\[ [W \wedge T_j, \bar{\rho} \wedge \bar{\omega}^k] = W \wedge [T_j, \bar{\rho}] \wedge \bar{\omega}^k = -\frac{i}{2}W \wedge \bar{\omega}^j \wedge \bar{\omega}^k. \] (28)

In summary,

\[ [\mathfrak{c}^{1,0} \otimes t^{1,0}, \mathfrak{g}^{1,1}] = [\mathfrak{c}^{1,0} \otimes t^{1,0}, \mathfrak{c}^{1,1} \oplus (t^{1,0} \otimes \mathfrak{c}^{0,1})] \subseteq (\mathfrak{c}^{1,0} \otimes t^{1,1}) \oplus (\mathfrak{c}^{1,0} \otimes t^{0,2}). \] (29)

Similarly,

\[ [t^{2,0}, \odot^{1,1}] = 0, \quad [t^{2,0}, \Delta^{1,1}] = 0, \quad [t^{2,0}, \mathfrak{c}^{0,1} \otimes t^{0,1}] = t^{0,1} \otimes \Delta^{1,1}, \] (30)
because by (21),
\[
[T_j \wedge T_k, W \wedge \overline{p}] = iW \wedge \psi_{jk},
\]
\[
[T_j \wedge T_k, T_\ell \wedge \overline{p}] = iT_\ell \wedge \psi_{jk},
\]
\[
[T_j \wedge T_k, \overline{p} \wedge \overline{w}^\ell] = -i\psi_{jk} \wedge \overline{w}^\ell.
\]

Next \([c^{1,1}, c^{1,1}] = 0\). Since
\[
[W \wedge \overline{p}, T_j \wedge \overline{p}] = W \wedge [\overline{p}, T_j] \wedge \overline{p} = -\frac{i}{2} W \wedge \overline{p} \wedge \overline{w}^j,
\]
\[
[\overline{p}, \phi_{jk}] = \frac{1}{2} ( [\overline{p}, T_j] \wedge \overline{w}^k + [\overline{p}, T_k] \wedge \overline{w}^j ) = 0,
\]
\[
[\overline{p}, \psi_{jk}] = \frac{i}{4} ( \overline{w}^j \wedge \overline{w}^k - \overline{w}^k \wedge \overline{w}^j ) = \frac{i}{2} \overline{w}^j \wedge \overline{w}^k
\]
for all \(i, j\). We observe that
\[
[c^{1,1}, t^{1,0} \otimes c^{0,1}] = c^{1,1} \otimes t^{0,1}, \quad [c^{1,1}, \odot^{1,1}] = 0, \quad [c^{1,1}, \triangle^{1,1}] = c^{1,0} \otimes t^{0,2}.
\]

As a consequence, we note the following

**Lemma 2** The spaces \(\odot^{1,1}, c^{1,0} \otimes t^{0,1}\) and \(t^{0,2}\) in \(\text{Sch}^2\) are in the center of the Schouten algebra \(\text{Sch}\), and hence up to equivalence in Schouten algebra,

- \(g^{2,0} \equiv_s (c^{1,0} \otimes t^{1,0}) \oplus t^{2,0}\).
- \(g^{1,1} \equiv_s c^{1,1} \oplus (t^{1,0} \otimes c^{0,1}) \oplus \triangle^{1,1}\).
- \(g^{0,2} \equiv_s c^{0,1} \otimes t^{0,1}\).

It follows \((36)\) that
\[
[W \wedge \overline{p}, \psi_{jk}] = \frac{i}{2} W \wedge \overline{w}^j \wedge \overline{w}^k
\]
for all \(i, j\). Also, \([c^{1,1}, c^{0,1} \otimes t^{0,1}] = 0\). By \((17)\),
\[
[T_j \wedge \overline{p}, T_k \wedge \overline{p}] = T_j \wedge [\overline{p}, T_k] \wedge \overline{p} + [T_j, \overline{p}] \wedge T_k
\]
\[
= \frac{i}{2} ( T_j \wedge \overline{w}^k \wedge \overline{p} - \overline{p} \wedge \overline{w}^j \wedge T_k ) = i\psi_{jk} \wedge \overline{p}.
\]

By \((36)\),
\[
[T_\ell \wedge \overline{p}, \psi_{jk}] = \frac{i}{2} T_\ell \wedge \overline{w}^j \wedge \overline{w}^k.
\]
It follows that
\[
[t^{1,0} \otimes c^{0,1}, t^{1,0} \otimes c^{0,1}] = \Delta^{1,1} \otimes c^{0,1},
\]
\[
[t^{1,0} \otimes c^{0,1}, \Delta^{1,1}] = t^{1,0} \otimes \Delta^{1,1},
\]
\[
[t^{1,0} \otimes c^{0,1}, c^{0,1} \otimes t^{0,1}] = c^{0,1} \otimes t^{0,2}.
\]

Next, it is clear that \([\Delta^{1,1}, \Delta^{1,1}] = 0\). We also note that \([\Delta^{1,1}, c^{0,1} \otimes t^{0,1}] = t^{0,3}\) because
\[
[\psi_{jk}, \bar{\rho} \wedge \omega^i] = \frac{i}{2} \omega^j \wedge \omega^k \wedge \omega^i. \quad (40)
\]

Finally, \([c^{0,1} \otimes t^{0,1}, c^{0,1} \otimes t^{0,1}] = 0\). Therefore, we could now summarize the Schouten bracket on \(\text{Sch}^2\) in a 5 \(\times\) 5-symmetric matrix.

**Proposition 1** All non-zero terms of Schouten bracket among elements in \(\text{Sch}^2\) are given below.

|     | \(c^{1,0} \otimes t^{1,0}\) | \(t^{2,0}\) | \(c^{1,1}\) | \(t^{1,0} \otimes c^{0,1}\) | \(\Delta^{1,1}\) | \(c^{0,1} \otimes t^{0,1}\) |
|-----|----------------------------|-----------|------------|----------------------------|----------------|-----------------------------|
| \(c^{1,0} \otimes t^{1,0}\) | 0           | 0         | 0          | \(c^{1,0} \otimes t^{1,1}\) | 0              | \(c^{1,0} \otimes t^{0,2}\) |
| \(t^{2,0}\)                  | 0           | 0         | \(c^{1,0} \otimes \Delta^{1,1}\) | \(t^{1,0} \otimes \Delta^{1,1}\) | 0              | \(t^{0,1} \otimes \Delta^{1,1}\) |
| \(c^{1,1}\)                  | 0           | \(c^{1,0} \otimes \Delta^{1,1}\) | 0          | \(c^{1,1} \otimes t^{0,1}\) | \(c^{1,0} \otimes t^{0,2}\) | 0              |
| \(t^{1,0} \otimes c^{0,1}\) | \(t^{1,0} \otimes t^{1,1}\) | \(t^{1,0} \otimes \Delta^{1,1}\) | \(c^{1,1} \otimes t^{0,1}\) | \(\Delta^{1,1} \otimes c^{0,1}\) | \(t^{1,2}\) | \(c^{0,1} \otimes t^{0,2}\) |
| \(\Delta^{1,1}\)            | 0           | 0         | \(c^{1,0} \otimes t^{0,2}\) | \(t^{1,2}\) | 0              | \(t^{0,3}\) |
| \(c^{0,1} \otimes t^{0,1}\) | \(c^{1,0} \otimes t^{0,2}\) | \(t^{0,1} \otimes \Delta^{1,1}\) | 0          | \(c^{0,1} \otimes t^{0,2}\) | \(t^{0,3}\) | 0              |

*Table 1: Schouten bracket among degree-2 elements*

### 4 Exterior differential algebra

In this section, we consider the exterior differential algebra structure within the Gerstenhaber algebra: \((\oplus_{p,q} g^{p,q}, \wedge, \bar{\partial})\). All are dictated by the equations
\[
\bar{\partial}W = 0, \quad \bar{\partial}T_j = -\frac{i}{2} W \wedge \omega^j, \quad \bar{\partial} \bar{\rho} = 0, \quad \bar{\partial} \omega^j = 0 \quad (41)
\]
for all \(1 \leq j \leq n\). We now begin to compute the cohomology.

\[
H^{p,q} = \frac{\ker \bar{\partial} : g^{p,q} \rightarrow g^{p,q+1}}{\text{image } \bar{\partial} : g^{p,q-1} \rightarrow g^{p,q}} \quad (42)
\]
To facilitate further computation in deformation theory, we consider a vector space decomposition \( g_{p,q} \) in three components. Its so-called harmonic part is isomorphism to \( H_{p,q} \). Its \( \bar{\partial} \)-exact part is \( D_{p,q} = \text{image } \bar{\partial} : g_{p,q-1} \to g_{p,q} \). Taking advantage of the finite-dimensional situation, we will choose a base so that the remaining part is denoted by \( G_{p,q} \). In respect to classical deformation theory, we address \( G_{p,q} \) the Green’s part. We consider

\[
g_{p,q} = H_{p,q} \oplus D_{p,q} \oplus G_{p,q}
\]
a Hodge decomposition of the type-(\( p, q \)) space \( g_{p,q} \). From observations above (41), we conclude the following.

**Lemma 3** The Hodge decomposition of degree-1 cohomology is given below.

\[
\begin{align*}
H^{1,0} &= c^{1,0}, & D^{1,0} &= 0, & G^{1,0} &= t^{1,0}, \\
H^{0,1} &= c^{0,1} \oplus t^{0,1}, & D^{0,1} &= 0, & G^{0,1} &= 0.
\end{align*}
\]

Moreover, the map \( \bar{\partial} : t^{1,0} \to c^{1,0} \otimes t^{0,1} \) where

\[
\bar{\partial} T_j = -\frac{i}{2} W \wedge \bar{\omega}^j,
\]

is an isomorphism.

Due to the computation above, it is now apparent that for all \( T \), \( \bar{\partial}(W \wedge T) = 0 \). However,

\[
\begin{align*}
\bar{\partial}(T_j \wedge T_k) &= -\frac{i}{2} \left( W \wedge \bar{\omega}^j \wedge T_k - T_j \wedge W \wedge \bar{\omega}^k \right) \\
&= -\frac{i}{2} W \wedge (T_j \wedge \bar{\omega}^k - T_k \wedge \bar{\omega}^j) = -i W \wedge \psi_{jk}.
\end{align*}
\]

**Lemma 4** The Hodge decomposition of Type-(2,0) space is given below.

\[
\begin{align*}
H^{2,0} &= c^{1,0} \otimes t^{1,0}, & D^{2,0} &= 0, & G^{2,0} &= t^{2,0}.
\end{align*}
\]

Moreover, the map \( \bar{\partial} : t^{2,0} \to c^{1,0} \otimes \Delta^{1,1} \) where

\[
\bar{\partial}(T_j \wedge T_k) = -i W \wedge \psi_{jk}
\]

is an isomorphism.
As vector spaces,
\[ g^{1,1} = c^{1,1} \oplus (c^{1,0} \otimes t^{0,1}) \oplus (t^{1,0} \otimes c^{0,1}) \oplus \circ^{1,1} \oplus \Delta^{1,1}. \] (48)

It is apparent that \( W \wedge \bar{\rho} \) is \( \overline{\partial} \)-closed but not \( \partial \)-exact. Due to Lemma 3, the second summand above is \( \overline{\partial} \)-exact. The same implies that for all \( T_j \) in \( t^{1,0} \),
\[ \overline{\partial}(T_j \wedge \bar{\rho}) = -\frac{i}{2} W \wedge \bar{\omega} \wedge \bar{\rho}. \] (49)

Regarding the remaining summands, we observe that
\[ \partial(T_j \wedge \omega_k) = -\frac{i}{4} W \wedge (\bar{\omega} \wedge \bar{\omega}^k + \bar{\omega}^k \wedge \bar{\omega}^j). \] (50)
\[ \partial \phi_{jk} = -\frac{i}{4} W \wedge (\bar{\omega} \wedge \bar{\omega}^k - \bar{\omega}^k \wedge \bar{\omega}^j) = -\frac{i}{2} W \wedge \bar{\omega} \wedge \bar{\omega}^j. \] (51)

**Lemma 5**  The Hodge decomposition of type-(1,1) space is given below.
\[ H^{1,1} = c^{1,1} \oplus \circ^{1,1}, \quad D^{1,1} = c^{1,0} \otimes t^{0,1} = \overline{\partial} t^{1,0}, \quad G^{1,1} = (t^{1,0} \otimes c^{0,1}) \oplus \Delta^{1,1}. \] (53)

Moreover, the following maps are isomorphisms:
\[ \overline{\partial} : t^{1,0} \otimes c^{0,1} \to c^{1,1} \otimes t^{0,1}, \quad \overline{\partial} : \Delta^{1,1} \to c^{1,0} \otimes t^{0,2} \] (54)

On the other hand, the following is clear from the structure equations.

**Lemma 6**  The Hodge decomposition of type-(0,2) space is given below.
\[ H^{0,2} = g^{0,2} = (c^{0,1} \otimes t^{0,1}) \oplus t^{0,2}, \quad D^{0,2} = 0, \quad G^{0,2} = 0. \] (55)

Define \( H^k = \oplus_{p+q=k} H^{p,q} \), \( D^k = \oplus_{p+q=k} D^{p,q} \), and \( G^k = \oplus_{p+q=k} G^{p,q} \) respectively, we summarize our observations in the last few lemmas as below.

**Proposition 2**  The Hodge decomposition of degree-2 space is given below.
\[ H^2 = (c^{1,0} \otimes t^{1,0}) \oplus c^{1,1} \oplus \circ^{1,1} \oplus (c^{0,1} \otimes t^{0,1}) \oplus t^{0,2} \]
\[ D^2 = c^{1,0} \otimes t^{0,1} = \overline{\partial} t^{1,0} \]
\[ G^2 = t^{2,0} \oplus (t^{1,0} \otimes c^{0,1}) \oplus \Delta^{1,1}. \]

Moreover, \( \overline{\partial} t^{2,0} = c^{1,0} \otimes \Delta^{1,1}, \quad \overline{\partial}(t^{1,0} \otimes c^{0,1}) = c^{1,1} \otimes t^{0,1}, \) and \( \overline{\partial} \Delta^{1,1} = c^{1,0} \otimes t^{0,2} \).
5 Solving the Extended Maurer-Cartan Equation

In view of Proposition 1, up to non-trivial Schouten brackets

\[ H^2 \equiv \mathbb{c}^{1,0} \otimes t^{1,0} \oplus \mathbb{c}^{1,1} \oplus (\mathbb{c}^{0,1} \otimes t^{0,1}), \]

and the only non-trivial Schouten bracket among elements in \( H^2 \) is due to a single type of brackets, namely

\[ \left[ \mathbb{c}^{1,0} \otimes t^{1,0}, \mathbb{c}^{0,1} \otimes t^{0,1} \right] \subseteq \mathbb{c}^{1,0} \otimes t^{0,2} = \overline{\partial} \Delta^{1,1}. \]

To be concrete, by (28) and (56),

\[ \left[ W \wedge T_j, \bar{\rho} \wedge \bar{\omega}^k \right] = -\frac{i}{2} W \wedge \bar{\omega}^j \wedge \bar{\omega}^k = \overline{\partial} \psi_{jk}. \]  

As elements in \( \Delta^{1,1} \) appear as potentials of Schouten bracket between harmonic elements in \( \text{Sch}^2 \), it is necessary to compute the Schouten bracket between elements in \( H^2 \) and \( \Delta^{1,1} \) as well as Schouten bracket between two elements in \( \Delta^{1,1} \). However, we have seen that the latter is equal to zero.

Computation in (27) leads to \( \left[ \Delta^{1,1}, \mathbb{c}^{1,0} \otimes t^{1,0} \right] = 0 \). Identities (37) and (56) lead to \( \left[ \Delta^{1,1}, \mathbb{c}^{1,1} \right] = \mathbb{c}^{1,0} \otimes t^{0,2} = \overline{\partial} \Delta^{1,1} \). In fact,

\[ \left[ W \wedge \bar{\rho}, \psi_{ij} \right] = \frac{i}{2} W \wedge \bar{\omega}^j \wedge \bar{\omega}^k = -\overline{\partial} \psi_{jk}. \]  

Similarly, (40) leads to \( \left[ \Delta^{1,1}, \mathbb{c}^{0,1} \otimes t^{0,1} \right] = t^{0,3} \), i.e.

\[ \left[ \psi_{jk}, \bar{\rho} \wedge \bar{\omega}^l \right] = \frac{i}{2} \bar{\omega}^j \wedge \bar{\omega}^k \wedge \bar{\omega}^l. \]  

On the other hand, elements in \( t^{0,3} \) are in the center of the Schouten algebra, and they are always in the harmonic part. Therefore, we have the following observation.

**Proposition 3** Suppose that \( \Gamma_1 \in H^2 \). There exists \( \Psi \in \Delta^{1,1} \) and \( \bar{\partial} \in t^{0,3} \) such that

\[ \bar{\partial}(\Gamma_1 + \Psi) + \frac{1}{2}[\Gamma_1 + \Psi, \Gamma_1 + \Psi] + \bar{\partial} = 0. \]  

After Merkulov [43], the element \( \bar{\partial} \) above is called the Chen vector and it constitutes the obstruction for \( \Gamma_1 \) to be integrable in such a way that \( \Gamma = \Gamma_1 + \Psi \) solves the Maurer-Cartan equation.
We use Kuranishi recursive formula to solve the Maurer-Cartan equation. Let
\[ \Gamma = \sum_{m=1}^{\infty} t^m \Gamma_m. \]
Substitute it into the Maurer-Cartan equation, we have
\[
\bar{\partial} (\sum_{m=1}^{\infty} t^m \Gamma_m) + \frac{1}{2} \sum_k t^k [\Gamma_k, \sum_{\ell} t^\ell \Gamma_{\ell}] \\
= t^m (\bar{\partial} \Gamma_m + \frac{1}{2} \sum_{k+\ell=m} [\Gamma_k, \Gamma_{\ell}] ).
\]
Our goal is that for all \( m \geq 1 \)
\[ \bar{\partial} \Gamma_m + \frac{1}{2} \sum_{k+\ell=m} [\Gamma_k, \Gamma_{\ell}] = 0. \]
Whenever \( \sum_{k+\ell=m} [\Gamma_k, \Gamma_{\ell}] \) has a component failing to be \( \bar{\partial} \)-exact, we compensate it by \( \bar{\partial} \). We address it as the \( m \)-th order term of the Chen vector. Therefore, we always have
\[ \bar{\partial} \Gamma_m + \frac{1}{2} \sum_{k+\ell=m} [\Gamma_k, \Gamma_{\ell}] + \bar{\partial} = 0. \]
Define \( \bar{\partial} = \sum_{m=2}^{\infty} t^m \bar{\partial}_m \), then
\[ \bar{\partial} \Gamma + \frac{1}{2} [\Gamma, \Gamma] + \bar{\partial} = 0. \]
The Kuranishi space of solution of the Maurer-Cartan equation is given by \( \ker \bar{\partial} \).
In particular,
\[ \bar{\partial} \Gamma_1 = 0, \quad \frac{1}{2} \sum_{k+\ell=2} [\Gamma_k, \Gamma_{\ell}] = -\bar{\partial} \Gamma_2 - \bar{\partial}_2 \]
where \( -\bar{\partial}_2 \) is equal to the harmonic part of \( \frac{1}{2} \sum_{k+\ell=2} [\Gamma_k, \Gamma_{\ell}] \). Now, set \( T = \sum_i \lambda_i T_i, \omega = \sum_j \alpha_j \omega^j \). Then
\[ \Gamma_1 = \lambda_j W \wedge T_j + \gamma W \wedge \bar{\partial} + \alpha_j \bar{\partial} \wedge \omega^j + \sum_{j \leq k} \gamma_{jk} \phi_{jk} + \sum_{j<k} \beta_{jk} \omega^j \wedge \omega^k \]
\[ = W \wedge T + \gamma W \wedge \bar{\partial} + \bar{\partial} \wedge \omega + \sum_{j \leq k} \gamma_{jk} \phi_{jk} + \sum_{j<k} \beta_{jk} \omega^j \wedge \omega^k. \]
Since both $\phi_{jk}$ and $\omega_j \wedge \omega_k$ are $\bar{\partial}$-closed and in the center of the Schouten algebra, in subsequent calculation of Kuranishi recursive formula, $\Gamma_1$ is equivalent to

$$\Gamma_1 \equiv s \ W \wedge \ T + \gamma W \wedge \bar{\rho} + \bar{\rho} \wedge \bar{\omega}. \quad (61)$$

By Proposition 1 and identity (56),

$$\frac{1}{2} \mathcal{[}[\Gamma_1, \Gamma_1] = [W \wedge T, \bar{\rho} \wedge \bar{\omega}]$$

$$= \sum_{j,k} \lambda_j \alpha_k W \wedge [T_j, \bar{\rho}] \wedge \bar{\omega}^k = -i \sum_{j,k} \lambda_j \alpha_k W \wedge \bar{\omega}^j \wedge \bar{\omega}^k$$

$$= \bar{\partial} \left( \sum_{j,k} \lambda_j \alpha_k \psi_{jk} \right).$$

Therefore,

$$\Gamma_2 = - \sum_{j,k=1}^n \lambda_j \alpha_k \psi_{jk} \quad \text{and} \quad \bar{\partial}_2 = 0.$$

Next,

$$\frac{1}{2} \mathcal{[}[\Gamma_k, \Gamma_2] = \frac{1}{2} (\mathcal{[}[\Gamma_1, \Gamma_2] + [\Gamma_2, \Gamma_1]) = [\Gamma_1, \Gamma_2]$$

$$= - \sum_{j,k=1}^n \lambda_j \alpha_k \left( \mathcal{[}[W \wedge T, \psi_{jk}] + \gamma \mathcal{[}[W \wedge \bar{\rho}, \psi_{jk}] + [\bar{\rho} \wedge \bar{\omega}, \psi_{jk}] \right)$$

$$= - \sum_{j,k=1}^n \lambda_j \alpha_k \left( \gamma \mathcal{[}[W \wedge \bar{\rho}, \psi_{jk}] + \sum_{\ell} \alpha_\ell \mathcal{[}[\bar{\rho} \wedge \bar{\omega}^\ell, \psi_{jk}] \right).$$

By (57) and (58), it is equal to

$$= \gamma \sum_{j,k=1}^n \lambda_j \alpha_k \bar{\partial} \psi_{jk} + \frac{i}{2} \sum_{j,k=1}^n \lambda_j \alpha_k \bar{\omega}^j \wedge \bar{\omega}^k \wedge \bar{\omega}$$

$$= \gamma \sum_{j,k=1}^n \lambda_j \alpha_k \bar{\partial} \psi_{jk} + \frac{i}{2} \left( \sum_{j,k=1}^n \lambda_j \bar{\omega}^j \right) \wedge \bar{\omega} \wedge \bar{\omega}$$

$$= -\gamma \bar{\partial} \Gamma_2.$$

It means that

$$\Gamma_3 = \gamma \Gamma_2 \quad \text{and} \quad \bar{\partial}_3 = 0. \quad (62)$$
Moreover, we obtain

\[ \left[ \Gamma_1, \Gamma_2 \right] = -\gamma \partial \Gamma_2. \] (63)

**Lemma 7** For all \( k \geq 3 \), \( \Gamma_k = \gamma^{k-2} \Gamma_2 \), \( \bar{\partial}_k = 0 \).

**Proof:** We have seen that the above statement is true when \( k = 3 \). Fix any \( j \) and suppose that this statement holds for all \( n \) such that \( j \geq n \geq 3 \). To compute \( \Gamma_{j+1} \) and \( \bar{\partial}_{j+1} \), we consider

\[
\frac{1}{2} \sum_{k+\ell = j+1} \left[ \Gamma_k, \Gamma_\ell \right]
= \frac{1}{2} \left[ \Gamma_j, \Gamma_1 \right] + \frac{1}{2} \sum_{k+\ell = j+1, k, \ell \geq 2} \left[ \Gamma_k, \Gamma_\ell \right] + \frac{1}{2} \left[ \Gamma_1, \Gamma_j \right]
= \left[ \Gamma_j, \Gamma_1 \right] + \frac{1}{2} \sum_{k+\ell = j+1, k, \ell \geq 2} \left[ \gamma^{k-2} \Gamma_2, \gamma^{\ell-2} \Gamma_2 \right]
= \left[ \Gamma_j, \Gamma_1 \right] = \gamma^{j-2} \left[ \Gamma_2, \Gamma_1 \right]
\]

because \( \left[ \triangle^{1,1}, \triangle^{1,1} \right] = 0 \). In addition, by (63) it is equal to \( -\gamma^{j-1} \partial \Gamma_2 \) as claimed. ■

Since \( \bar{\partial}_k = 0 \) for all \( k \), \( \bar{\partial} = 0 \). Moreover,

\[
\Gamma(t) = \sum_{m=2}^{\infty} t^m \Gamma_m = t \Gamma_1 + \sum_{m=2}^{\infty} t^m \gamma^{m-2} \Gamma_2 = t \Gamma_1 + \sum_{m=2}^{\infty} t^m \gamma^{m-2} \Gamma_2
= t \Gamma_1 + t^2 \left( \sum_{m=2}^{\infty} t^{m-2} \gamma^{m-2} \right) \Gamma_2 = t \Gamma_1 + \frac{t^2}{1-t\gamma} \Gamma_2
\] (64)

when \( |t\gamma| < 1 \).

**Theorem 2** Let \( \Gamma_1 \) in \( H^2 \) be given by

\[
\Gamma_1 = \sum_j \lambda_j (W \wedge T_j) + \gamma (W \wedge \overline{p}) + \sum_k \alpha_k (\overline{p} \wedge \overline{\omega}^k) + \sum_{j \leq k} \gamma_{jk} \phi_{jk} + \sum_{j < k} \beta_{jk} (\overline{\omega}^j \wedge \overline{\omega}^k).
\]

If \( |\gamma| < 1 \) and \( \Gamma_2 = -\sum_{j,k=1}^{n} \lambda_j \alpha_k \psi_{jk} \),

\[
\Gamma = \Gamma_1 + \frac{1}{1-\gamma} \Gamma_2
\]

is a solution of the Maurer-Cartan equation.
It recovers an observation in [16, Theorem 12], and constitutes the foundation for our main observation next.

6 Frobenius Structure

We begin to examine the Frobenius structure associated to the primary Kodaira manifolds. Once again, since $\otimes^{1,1}$ and $\mathfrak{j}^{0,2}$ are in the center of the Schouten algebra and the kernel of the $\overline{\partial}$-operator, elements in these spaces do not affect the Frobenius structure. Therefore, up to equivalence in the Gerstenhaber algebra, 

$$\Gamma_1 \equiv_{\text{Ger}} \sum_j \lambda_j (W \wedge T_j) + \gamma (W \wedge \mathfrak{p}) + \sum_k \alpha_k (\mathfrak{p} \wedge \mathfrak{w}^k)$$

and hence 

$$\Gamma \equiv_{\text{Ger}} \sum_j \lambda_j (W \wedge T_j) + \gamma (W \wedge \mathfrak{p}) + \sum_k \alpha_k (\mathfrak{p} \wedge \mathfrak{w}^k) - \frac{1}{1 - \gamma} \sum_{j,k=1}^{n} \lambda_j \alpha_k \psi_{jk} \quad (65)$$

Let us adopt the following notations. 

$$T = \sum_j \lambda_j T_j, \quad \mathfrak{w} = \sum_j \alpha_j \mathfrak{w}^j, \quad X = \sum_j \alpha_j T_j, \quad \overline{\Omega} = \sum_j \lambda_j \mathfrak{w}^j. \quad (66)$$

In such case, 

$$\Gamma_2 = -\frac{1}{2} \lambda_j \alpha_k (T_j \wedge \mathfrak{w}^k - T_k \wedge \mathfrak{w}^j) = -\frac{1}{2} (\lambda_j T_j \wedge \alpha_k \mathfrak{w}^k - \alpha_k T_k \wedge \lambda_j \mathfrak{w}^j) = -\frac{1}{2} (T \wedge \mathfrak{w} - X \wedge \overline{\Omega}).$$

Therefore, 

$$\Gamma \equiv_{\text{Ger}} W \wedge T + \gamma W \wedge \mathfrak{p} + \mathfrak{p} \wedge \mathfrak{w} + \frac{1}{1 - \gamma} \Gamma_2 \quad (67)$$

We are going to compare the generators for $\text{DGA}(\overline{\partial})$ and $\text{DGA}(\overline{\partial}_\Gamma)$, where 

$$\text{DGA}(\overline{\partial}) = (\oplus_{p,q} \mathfrak{g}^{p,q}, \quad [-,-], \wedge, \overline{\partial}), \quad \text{DGA}(\overline{\partial}_\Gamma) = (\oplus_{p,q} \mathfrak{g}^{p,q}, \quad [-,-], \wedge, \overline{\partial}_\Gamma).$$
Recall that the structure equations for \( DGA(\bar{\partial}) \) are generated by
\[
\bar{\partial} T_k = -\frac{i}{2} W \wedge \bar{w}^k, \quad [T_k, \bar{\rho}] = -\frac{i}{2} \bar{w}^k. \tag{68}
\]
Otherwise, \( c^{1,0} \) and \( t^{0,1} \) are in the center of the Schouten algebra and are in the kernel of the exterior \( \bar{\partial} \)-operator.

In order to compute \( \bar{\partial}_\Gamma = \bar{\partial} + \text{ad}_\Gamma \), we take advantage of linearity:
\[
\text{ad}_\Gamma = \text{ad}_W \wedge \bar{T} + \gamma \text{ad}_W \wedge \bar{\rho} + \text{ad}_{\bar{\rho}} \wedge \omega + \frac{1}{1 - \gamma} \text{ad}_{\Gamma_2}.
\]

We test each summand above on \( T_k \) and \( \bar{\rho} \) respectively.
\[
[W \wedge T, T_k] = 0, \\
[W \wedge T, \bar{\rho}] = \lambda_j W \wedge [T_j, \bar{\rho}] = -\frac{i}{2} \lambda_j W \wedge \bar{w}^j = \bar{\partial} T. \tag{69}
\]
\[
[W \wedge \bar{\rho}, T_k] = -W \wedge [T_k, \bar{\rho}] = i \frac{i}{2} W \wedge \bar{w}^k = -\bar{\partial} T_k, \tag{70}
\]
\[
[W \wedge \bar{\rho}, \bar{\rho}] = 0, \tag{71}
\]
\[
[\bar{\rho} \wedge \bar{w}, T_k] = -\bar{w} \wedge [\bar{\rho}, T_k] = \bar{w} \wedge [T_k, \bar{\rho}] = -\frac{i}{2} \bar{w} \wedge \bar{w}^k. \tag{72}
\]
\[
[\bar{\rho} \wedge \bar{w}, \bar{\rho}] = 0. \tag{73}
\]
\[
[\Gamma_2, T_k] = 0.
\]

By (36),
\[
[\Gamma_2, \bar{\rho}] = -\frac{1}{2} \lambda_j \alpha_k ([T_j \wedge \bar{w}^k - T_k \wedge \bar{w}^j, \bar{\rho}]) + \frac{1}{2} \lambda_j \alpha_k (\bar{w}^k \wedge [T_j, \bar{\rho}] - \bar{w}^j \wedge [T_k, \bar{\rho}])
= -\frac{i}{4} \lambda_j \alpha_k (\bar{w}^k \wedge \bar{w}^j - \bar{w}^j \wedge \bar{w}^k) = -\frac{i}{2} \bar{w} \wedge \bar{\Omega}. \tag{73}
\]

Now we are ready to compute \( \bar{\partial}_\Gamma T_k \) and \( \bar{\partial}_\Gamma \bar{\rho} \). By (70) and (72),
\[
\bar{\partial}_\Gamma T_k = \bar{\partial} T_k + \gamma [W \wedge \bar{\rho}, T_k] + [\bar{\rho} \wedge \bar{w}, T_k]
= (1 - \gamma) \bar{\partial} T_k - \frac{i}{2} \bar{w} \wedge \bar{w}^k. \tag{74}
\]

On the other hand by (69) (71) and (73),
\[
\bar{\partial}_\Gamma \bar{\rho} = [W \wedge T, \bar{\rho}] + \gamma [W \wedge \bar{\rho}, \bar{\rho}] + \frac{1}{1 - \gamma} [\Gamma_2, \bar{\rho}]
= \bar{\partial} T - \frac{i}{2(1 - \gamma)} \bar{w} \wedge \bar{\Omega}.
\]
Let \( \sum_k \mu_k T_k + \mu \bar{\rho} \) be a generic element in \( t^{1,0} \oplus \mathfrak{c}^{0,1} \). Then
\[
\bar{\partial}_T (\sum_k \mu_k T_k + \mu \bar{\rho}) = (1 - \gamma) \mu_k \partial T_k - i \frac{\mu}{2} \omega \wedge (\mu \omega^k) + \mu \bar{\partial} T - i \frac{\mu}{2} \frac{1 - \gamma}{1 - \gamma} \omega \wedge \bar{\Omega}
\]
\[
= \bar{\partial} ((1 - \gamma) \mu_k T_k + \mu \lambda_k T_k) - i \frac{\mu}{2} \omega \wedge \left( \mu_k \omega^k + \frac{\mu}{1 - \gamma} \bar{\Omega} \right)
\]
\[
= ((1 - \gamma) \mu_k + \mu \lambda_k) \bar{\partial} T_k - i \frac{\mu}{2} \frac{1 - \gamma}{1 - \gamma} \omega \wedge ((1 - \gamma) \mu_k + \mu \lambda_k) \omega^k .
\]
It could be equal to zero only if \((1 - \gamma) \mu_k + \mu \lambda_k = 0\) for each \(k\), i.e.
\[
\mu_k = -\frac{\mu \lambda_k}{1 - \gamma}.
\]
Therefore, the only \( \bar{\partial}_T \)-closed element in \( t^{1,0} \oplus \mathfrak{c}^{0,1} \) is
\[
\sum_k \mu_k T_k + \mu \bar{\rho} = \mu (\bar{\rho} - \sum_k \frac{\lambda_k}{1 - \gamma} T_k) = \mu (\bar{\rho} - \frac{1}{1 - \gamma} T) .
\]  \(\tag{75}\)
Consider a basis for \( g^1 = g^{1,0} \oplus g^{0,1} \) as below.
\[
\frac{1}{1 - \gamma} T_k, \quad (1 - \gamma) W + \omega, \quad \frac{1}{1 - \gamma} \omega^k, \quad \bar{\rho} - \frac{1}{1 - \gamma} T .
\]  \(\tag{76}\)
The above computation shows that
\[
\bar{\partial}_T ((1 - \gamma) W + \omega) = 0, \quad \bar{\partial}_T \left( \frac{1}{1 - \gamma} \omega^k \right) = 0, \quad \bar{\partial}_T \left( \bar{\rho} - \frac{1}{1 - \gamma} T \right) = 0 .
\]
Moreover, regarding the non-trivial terms by \(\tag{74}\),
\[
\bar{\partial}_T \left( \frac{1}{1 - \gamma} T_k \right) = \bar{\partial} T_k - i \frac{1}{2} \omega \wedge \left( \frac{1}{1 - \gamma} \omega^k \right) = -i \frac{1}{2} W \wedge \omega^k - i \frac{1}{2} \omega \wedge \left( \frac{1}{1 - \gamma} \omega^k \right)
\]
\[
= -i \frac{1}{2} ((1 - \gamma) W + \omega) \wedge \left( \frac{1}{1 - \gamma} \omega^k \right) ;
\]
\[
\left[ \frac{1}{1 - \gamma} T_k, \bar{\rho} - \frac{1}{1 - \gamma} T \right] = \frac{1}{1 - \gamma} [T_k, \bar{\rho}]
\]
\[
= \frac{1}{1 - \gamma} \left( -i \frac{1}{2} \omega^k \right) = -i \frac{1}{2} \left( \frac{1}{1 - \gamma} \omega^k \right) .
\]  \(\tag{77}\)
\(\tag{78}\)

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Theorem 3  For any generalized deformation $\Gamma$ of the complex structure of the primary Kodaira manifold given in Theorem 2, there exists a natural isomorphism of Gerstenhaber algebras:

$$(H^\bullet_\Gamma(M), [-,-], \wedge) \cong (H^\bullet(M), [-,-], \wedge).$$

Proof: Choose a map $\Phi$ so that $\Phi(\bar{\partial}) = \bar{\partial}_\Gamma$.

$$\Phi(T_k) = \frac{1}{1-\gamma} T_k, \quad \Phi(W) = (1-\gamma)W + \overline{\omega}, \quad (79)$$

$$\Phi(\overline{\omega}^k) = \frac{1}{1-\gamma} \overline{\omega}^k, \quad \Phi(\overline{\rho}) = \overline{\rho} - \frac{1}{1-\gamma} T. \quad (80)$$

Comparing the structure equations for $H^\bullet(M)$ as given in (68) with those for $H^\bullet_\Gamma(M)$ as given in (77) and (78), we complete the proof of our theorem and the main result of this paper.  

7  Further Development

As noted in the Introduction, the result in Theorem 1 is inspired by an investigation on the Frobenius structures on primary Kodaira surfaces. Although the observation in Theorem 1 focuses on a generalization to primary Kodaira manifolds in all dimensions, we anticipate more because several key elements in this paper work in a general context.

When working with primary Kodaira manifold, we consider it as a nilmanifold with an abelian complex structure. In this context, we make use of the fact [41] [46, Proposition 6] that the inclusion map is a quasi-isomorphism of Gerstenhaber algebras:

$$\left( \bigoplus_{p,q} B^{p,q}, \quad [-,-], \quad \wedge, \quad \overline{\partial} \right) \hookrightarrow (C^\infty(M, \wedge^\bullet(\Theta \oplus \overline{\Omega})), \quad [-,-], \quad \wedge, \quad \overline{\partial}).$$

Given the work of others [11] [12] [41] [44] [52], the above quasi-isomorphism is established for all nilmanifolds with abelian complex structures [3, Theorem 1]. Therefore, computation of Frobenius structures could be reduced to a finite-dimensional setting on all these manifolds.
To make a computation effective, one may consider an *ascending basis* for the algebra $\mathfrak{g}^{1,0}$ and the conjugation of its dual for $\mathfrak{g}^{0,1}$ as in [49]. Precisely, given the work of [13] [14], Salamon finds [53] that there exists a basis $\{\omega^1, \ldots, \omega^n, \omega^{n+1}\}$ for $\mathfrak{g}^{0,1}$ such that for all $j$,

$$d\omega^{j+1} \in I(\omega^1, \ldots, \omega^j) \wedge I(\overline{\omega}^1, \ldots, \overline{\omega}^j),$$

where $I(\omega^1, \ldots, \omega^j)$ denotes the ideal generated by $\{\omega^1, \ldots, \omega^j\}$. Let $\{T_1, \ldots, T_{n+1}\}$ be the dual basis for $\mathfrak{g}^{1,0}$. We call both $\{T_1, \ldots, T_{n+1}\}$ and $\{\omega^1, \ldots, \omega^{n+1}\}$ ascending basis for the complex structure. Let $\mathfrak{t}^{1,0}$ and $\mathfrak{t}^{0,1}$ be the complex linear spans of $\{T_1, \ldots, T_n\}$ and $\{\overline{\omega}^1, \ldots, \overline{\omega}^n\}$ respectively. It is not hard to find that the constraints in (81) above imply that $\Lambda = T_n \wedge T_{n+1}$ is an invariant holomorphic Poisson structure [48]. In [49], the author and his collaborator further identify conditions to secure an isomorphism between Gerstenbaher algebras:

$$(H^\bullet(M), [-,-], \wedge) \cong (H^\bullet(M), [-,-], \wedge).$$

On 2-step nilmanifolds with abelian complex structures, including primary Kodaira manifolds, the necessary condition could be easily stated. The constraint in (81) above implies that the contraction of $d\omega^{n+1}$ is a linear map:

$$d\omega^{n+1} : \mathfrak{t}^{1,0} \rightarrow \mathfrak{t}^{0,1}.$$  

**Theorem 4** [49 Theorem 6] Suppose that $M$ is a 2-step complex $(n+1)$-dimensional nilmanifold $M$ with abelian complex structure. Given an ascending basis with $d\omega^{n+1}$ being non-degenerate, $\Lambda = V_n \wedge V_{n+1}$ is a holomorphic Poisson structure. Moreover, there exists a natural isomorphism between Gerstenhaber algebras:

$$(H^\bullet_M(M), [-,-], \wedge) \cong (H^\bullet(M), [-,-], \wedge).$$

Since the holomorphic Poisson structure represents a portion of the infinitesimal generalized deformation:

$$H^2(M) = H^0(M, \Theta^2) \oplus H^1(M, \Theta) \oplus H^2(M, \mathcal{O}),$$

the last theorem and the work in this paper lead to the following conjecture.
Conjecture 1 On a 2-step nilmanifold $M$ with abelian complex structure. Suppose that there exists an ascending basis $\{\omega^1, \ldots, \omega^{n+1}\}$ such that $d\omega^{n+1}$ is non-degenerate. When an element $\Gamma_1 \in H^2(M)$ is integrable to a generalized deformation $\Gamma$, the Gerstenbaher algebra $(H^*_{\Gamma}(M), [-, -], \wedge)$ is naturally isomorphic to $(H^*(M), [-, -], \wedge)$.

We remark that there are examples to demonstrate that the non-degeneracy condition for $d\omega^{n+1}$ is necessary [48] [49]. On the other hand, there are many examples satisfying the conditions of this conjecture [15] [41] [48].

Acknowledgments. Y.S. Poon is grateful for hospitality of the Institute of Mathematical Sciences at the Chinese University of Hong Kong during his visits in 2019.

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