Relational thinking in early algebra learning: a systematic literature review

M Andini and S Prabawanto

Abstract. Algebraic thinking in early grades has taken a lot of researchers' attention since a few decades ago until now. A number of students' difficulties in learning early algebra have been linked to deficiencies in learning during students' transition from arithmetic to algebra. This study aims to analyze relational thinking; a way of thinking that is a bridge between arithmetic thinking and algebraic thinking. Through a systematic literature review method, researchers analyzed 26 articles from at least 20 journals. Researchers focus on the characteristics of relational thinking, its relation to early algebra learning, and the problems and teaching solutions that have been raised by previous researchers. As a result and implication, various lesson designs that develop this type of thinking have been presented as solutions to the many problems associated with early algebra learning. This research is expected to be the basis of how to make lesson designs that can anticipate deficiencies in arithmetic learning when students have entered early algebra learning.

1. Introduction

The importance of algebra in mathematics has been revealed by many researchers. Jupri & Drijvers stated that algebra is very important for achievement in other mathematical domains such as analytic geometry, calculus, and statistics [1]. Therefore, early algebra education, which includes students' first steps in this domain, is certainly an important phase in algebra education [2]. Research on algebra is often carried out through the development of algebraic thinking, where this thinking is needed to analyze deeper mathematical structures [3]. Likewise, research on early algebra is often carried out through the development of early algebraic thinking.

Students often meet difficulties in understanding early algebra. The difficulties include difficulties related to the implementation of arithmetic operations such as students' mistakes in using the properties of arithmetic operations and interpreting equal sign [1]; student difficulties in completing algebraic expressions [4]; and student difficulties in solving equations [5]. The difficulties in using arithmetic skills, using equal signs, understanding algebraic expressions, and understanding variable concepts are included in students’ solving equation [5]. These difficulties indicate a link between arithmetic and algebra.

Judging from its characteristics, there are differences in characteristics between arithmetic and algebra. Arithmetic is very close to calculation, whereas algebra focuses on relations [6]. On the other hand, arithmetic and early algebra have close links, especially in the properties of number operations. For some students, learning algebra is a natural development built on their understanding of arithmetic [7], however, for many students, learning algebra is totally different from the experience of learning...
arithmetic. Students do not understand how the basic properties of number operations is applied in calculations, so as a result, students do not realize that arithmetic and algebra are based on the same basic ideas [6] and they feel they have a number of difficulties in transition (from arithmetic to algebra) [8].

Related to the learning process, the shortcomings that result from arithmetic learning during the transition from arithmetic to algebra have an influence on the development of algebraic thinking [3]. Therefore, the transition period of students from (thinking) arithmetic to (thinking) algebra becomes a vital phase in determining students' success in mastering algebra. Thus, facilitating students to make a smooth transition from arithmetic to algebra becomes very important. A number of studies have developed what is called relational thinking as one solution to overcome the transition problem. Relational thinking is said to be a bridge between arithmetic thinking and algebraic thinking [3]. Researchers refer relational thinking to definitions according to Carpenter et al. that relational thinking includes the ability to view mathematical expressions and equations as a whole, not as a process that is carried out step by step [6]. Relational thinking involves the meaning of equal signs, the use of the basic properties of operations and the relation of numbers [6, 7, 9], as well as making strategic decisions [7, 10].

Previous studies related to relational thinking include studies about the meaning of the equal sign. As a result, students have a serious misconception of the meaning of the equal sign [7, 11, 1]; students assume the equal sign as the operator to do calculations, state the results, or as a signal to write answers that come next. Other research related to solving mathematical sentence problems, the majority of students are found doing results-oriented solving [3]. Students do not apply the properties of operations and the relation of numbers but they are still dominant in doing calculations. Learning solutions that have been developed through the use of number sentences, either true or false or open number sentences have brought many successes in developing relational thinking [8, 12, 3, 13]. A number of studies have also been successful in making teaching design based on relational thinking, for instance through pre-service training for teachers [14, 15], as well as several projects related to the development of the teaching profession in this regard also has shown positive implications for the development of students' relational thinking [9, 16].

Many researches on relational thinking since the 2000s along with the developments carried out show the importance of this issue to get attention in early algebra education. Therefore, a study is needed to gather literatures in this field accompanied by critical analysis. In this study, researchers intend to conduct a review related to relational thinking, focusing on the characteristics of relational thinking, its relationship with early algebra learning, and the problems and teaching solutions that have been raised by researchers previously.

2. Methods

This research used a qualitative approach through literature review techniques. In reviewing the literature used as research data sources, researchers used systematic literature review techniques; beginning by searching a number of journal articles from various sources, sorting and selecting the literature gradually based on certain criteria, and finally determining the number of articles to be reviewed. In the searching phase, researchers conducted searches on several sites such as Scopus, Scimago, Google Scholar, Science Direct, and Research Gate. Keywords used in searching process were relational thinking, initial algebra, early algebra, algebraic thinking, and algebraic thinking in early grade. This searching phase was accompanied by a process of sorting and selecting based on abstracts, so articles with inappropriate abstracts were not followed up while articles with the appropriate abstract were searched for the full paper version (if not yet available). From this process, there were 41 articles in full paper version. Furthermore, one by one were read completely and re-sorted according to the criteria of content relevance, recent publication (preferably published in the last 10 years), and journal credibility (preferably indexed international journals). Finally, researchers obtained 26 articles from at least 20 journals for being reviewed and analyzed.
3. Result and Discussion

3.1. early algebra and the problem

Algebra reasoning can take several forms. Kaput in Jacob identifies five interrelated forms of reasoning: algebra as generalized arithmetic, as manipulation of symbols, as the study of structure and systems, as the study of functions, and as modelling [9]. In the context of early algebra, the majority of research refers to algebraic forms as generalized arithmetic. For example, the study of Jacobs et al. refers to the definition of algebra as generalized arithmetic and relations [14]. Carpenter et al. said that is because learning to reason algebraically in elementary schools not only provides a foundation for smoothing the transition to algebra but can also deepen understanding of the basics of arithmetic [14].

The term generalized arithmetic has been used in the context of early algebra in a broad sense to include the properties and relationships that arise in arithmetic operations, without having to use letter symbol notation [13]. However, general arithmetic perspectives on content do not only include numbers / quantities, operations, properties, equality, and related representations and diagrams, but can also include variables, expressions, and equations [17]. This is in line with Kaput & Blanton which uses the term generalized arithmetic to involve the generalization of arithmetic relations which includes the basic properties of numbers and operations and reasoning about the structure of arithmetic expressions rather than computational values [16].

Researches on early algebra are often carried out through the development of early algebraic thinking. Researchers show algebraic thinking students with the support of intervention programs (teaching) that are well designed so as to bring up early algebraic thinking [13]. Although there is no consensus that answers whether the definition of "algebraic thinking in early grades" [8], but Kieran in 2004, referring to previous related studies, has defined what it is to think algebra in the early stages as:

"Algebraic thinking in the early grades involves the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all , such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modelling, justifying, proving, and predicting "[18].

The main problem of early algebra learning is transition problem. The transition period of students from (thinking) arithmetic to (thinking) algebra becomes a vital phase in determining students' success in mastering algebra well. The deficiencies that result from arithmetic learning during the transition from arithmetic to algebra have an influence on the development of algebraic thinking [3]. Actually, arithmetic and early algebra have close links, especially in the properties of number operations. However, students do not understand how the basic properties of number operations are applied in solving mathematical expression, so as a result, students do not realize that arithmetic and algebra are based on the same basic ideas [6]. For some students, learning algebra is a natural development built on their understanding of arithmetic [7], however, for many students, learning algebra is totally different from the experience of learning arithmetic, and they find it difficult to transition (from arithmetic to algebra) [8].

A number of difficulties are met by students during the transition. Some researchers classifies the difficulties had by students in learning early algebra into five categories, including difficulties related to the application of arithmetic operations such as students' mistakes in using the nature of arithmetic operations and the difficulty of students in interpreting equal signs [1]. Difficulties related to the application of arithmetic operations are ranked 3rd (after mathematical difficulties and difficulties in algebraic expression). Specifically, other researchers find that students had difficulty in applying associative and destructive nature when completing algebraic expressions [4]. In addition, difficulties in solving equations was investigated which include difficulties in arithmetic skills, use of equal signs, understanding algebraic expressions, and understanding variable concepts [5].
3.2. relational thinking and its characteristics

The definition of relational thinking is generally quite diverse, mostly related to mathematical connections, but researchers find definitions that are quite uniform in perspective of arithmetic and algebra. Relational thinking according to Carpenter et al. include the ability to view expressions and mathematical equations as a whole, not as a process that is carried out step by step [6]. Kiziltoprak & Kose suggested that relational thinking mostly focuses on examining relations between given quantities rather than finding the results of operations [3]. Thus, relational thinking is not just a procedural calculation, but involves the use of fundamental properties of numbers and operations to transform mathematical expressions [6, 3]. This is in line with the statement of Jacobs et al. that relational thinking involves the application of the properties of numbers, operations, and equations in solving mathematical problems. Slightly different, but with the same meaning, the definition expressed by Stephens that relational thinking is the ability in which a person can express an understanding of the objects involved without using literal symbols that solving the problem used can be general, the understanding is about equivalence and compensation [19].

For example, the sentence number \(1986 + 8 + \Box = 1986 + 12\) [12]. Students who use the relational mind to the equal sign will be able to determine answer 4 by looking at the relationship between the numbers. The equal signs allow students to ignore 1986 numbers on each side and only focus on the \(8 + \Box = 12\) which is possible for students to reduce the level of computing [12]. The example also implied one of the goals of relational thinking that is to help students realize the fact that both sides of the equation represent the same number without doing any calculations [3]. Therefore, in order to think relational, students must first understand the relational meaning of the equal signs “=” [6,11, 3, 13].

Byrd et al. focus on how specific misconceptions of the equal signs can inhibit students' early algebra learning [18]. The researchers distinguished the interpretation of the sign in three ways, namely: (a) arithmetic-specific (for example, "when students add something, they get the total"); (b) non-relational (for example, "end of question", "symbol to tell students the next answer") and (c) relational (for example, "something is equivalent to something else"). Students in the arithmetic-specific category show lower performance in solving initial algebra problems than those who define the equal sign in a non-relational way but without using specific arithmetic words [13].

Related to solving mathematical problem, the curriculum encourages students to apply various strategies, but more than just implementing strategies, relational thinking involves making strategic decisions [10]. Students integrate relational thinking strategies with computational skills so that students prepare to succeed in advanced mathematics [7]. Thus, the role of relational thinking is very important, especially in relation to problem solving.

Developing students' relational thinking in solving mathematical problems also contributes to efforts to realize learning with understanding (meaningful learning). With relational thinking, students have to use understanding of the relationship between quantities on both sides of the same sign to focus and think about the properties of operations as reasoning for the equation as a whole. Students are not only skilled in doing calculations, but are able to make mathematical connections, in this case specific to relationships that refer to experts' definitions of relational thinking. Thus, relational thinking can be considered as one of the productive ways in meaningful learning [6].

Based on the review of the results of the studies, the researcher concludes that relational thinking includes the following aspects: looking at mathematical sentences as a whole; interpreting the equal signs as an equality or balance; applying the properties of operations and the relation of numbers in problem solving; not really doing calculations; and involving making strategic decisions.

In a study conducted by Stephens & Wang, relational thinking was divided into three categories namely Established Relational Thinking, Consolidating Relational Thinking, and Emerging Relational Thinking [20]. The other researchers grouped students based on the operation process carried out in solving number sentences into three themes, namely: (a) the process of operations based on relational thinking, consisting of subthemes using the basic properties of arithmetic operations and using relations between numbers; (b) the operation process of introducing relational thinking, consisting of sub-themes
explaining relations after discovering unknown and pre-relational thinking; and (c) result-oriented operation process [3].

Based on the analysis of relational thinking and early algebra, it can be concluded that there is a clear connection between the two, that algebra focuses on relationships rather than calculating answers (solutions) where students should view an equation as 'equality' and solve problems by thinking relatively [6]. Thus, relational thinking is the initial stage of thinking algebra [21].

3.3. relational thinking in early algebra teaching: influenced factors and how to develop

Based on previous studies, the dominant factor influencing the development of students' relational thinking is the teaching process conducted by teachers [6, 9, 3]. Carpenter et al. used the term scaffolding to refer to the way teachers develop students' relational thinking frameworks [6], while Kiziltoprak & Kose mentions the term teaching interrogative approach to refer to questions that lead students to develop their relational thinking [3]. Other factor is that students are not used to mathematical expressions such as number sentences that involve operations on both sides of the equation due to the limited number of student books that provide such sentence numbers [3]. In fact, solving number sentences can improve relational thinking [6]. Furthermore, Banerjee concludes that arithmetic with number sentences is very useful for delivering students to algebra [22].

The teacher’s questions can be as followed: "Can you decide if the given equality is right or wrong without calculation?"; "Can you find the missing value in the equation without calculation?"; "What is the unknown in the equation and how you obtain these answers?"; and "Do you think the properties of number operations is valid for all numbers?" [13].

Specifically, interviews conducted by researchers with children involved in the classroom as well as the type of questioning, scaffolding, and problem order support the development of relational thinking. Carpenter et al. shows how grade 3 elementary school students use distributive properties to learn the multiplication of numbers and the types of scaffolding and the order of problems that support learning [6]. The students’ answers have shown that she uses the relational mind, not doing calculations such as by calculating the results from each side (see [6] p.55).

Kiziltoprak & Koses’s study lead to significant changes in students' thought processes between process before the intervention (teaching experiment) and after the intervention. It shows that during preliminary interviews, the majority of students respond to questions with a result-oriented process. Significant change was seen in the results of the last-interview which showed that almost all students respond to the operation process by involving relational thinking.

Because arithmetic has been considered as the main context for early algebraic thinking [23], many studies have been conducted to investigate children's understanding of equal signs, expressions, and similarities [13, 24]. In connection with misconceptions of equal signs as explained before, researchers improve relational understanding of the equal sign in various ways, such as students’ intervention through the use of various representations and exposure equation (in standard and nonstandard formats) [7]. While Jones & Pratt taught meaning substitution of equal signs by offering children access to the way of thinking in investigative way and exploration as well as working with a formal notation [11].

Generally, one of teaching designs in this context was developed by Blanton et al., that is a teaching design for early algebra by adapting the Curriculum Research Framework (CRF) which consists of three categories; a principle foundation, learning model, and evaluation. The content of algebra material for teaching consists of four essential practices, namely algebraic thinking; generalizing, representing, justifying, and reasoning with mathematical structures and relationships [25]. Positive impacts from the implementation of this design are shown that the early algebraic innovations carried out in this study provide an effective curriculum concept map and can strengthen the implementation of standards and practices regarding algebraic thinking.

Thus, due to the importance of the teaching design created by the teacher, training for teachers to carry out teaching based on relational thinking becomes very necessary. This can be in the form of web-based professional development programs and in-service training [3], or pre-service training for teachers to develop teaching based on relational thinking [14, 15]. Several projects related to the development of
the teaching profession in this regard were also carried out [9, 16]. It has been shown that teachers who participate in training produce many variations of student learning strategies, including strategies that reflect the use of relational thinking, compared to teachers who do not participate. Students taught by participating teachers showed significantly better understanding of equal signs and used relational thinking during interviews than students taught by teachers who did not participate [9].

4. Conclusion
Learning deficiency during the transition from arithmetic to algebra seems to be very influential on the success of early algebra learning. With many cases of students’ difficulties in understanding early algebra, it becomes a challenge for both elementary and secondary school teachers to present teaching and learning that can develop students’ relational thinking. Research-related research shows that relational thinking is very important to make learning arithmetic richer, to think flexibly about the operation, to help students recognize the expression in mathematics without calculation, to help students make sense of the equal sign as an equality, and most importantly to provide a basis to extend the move to algebra. As a solution, the teaching through the development of number sentences are claimed by many researchers as an effective tool in developing relational thinking, but the majority of studies applying in pre-algebra (mostly in primary schools). Through this research, it is hoped that it can be used as a basis for how to make lesson designs that can anticipate deficiencies in arithmetic learning when students have entered into early algebra learning (high school students).

5. References
[1] Jupri A & Drijvers P 2014 Difficulties in initial algebra learning in Indonesia Mathematics Education Research Journal 26 p.683-710
[2] Van Amerom B A 2002 Reinvention of early algebra: developmental research on the transition from arithmetic to algebra Dissertation (Utrecht) CD-B Press.
[3] Kiziltoprak A & Kose N Y 2017 Relational thinking: The bridge between arithmetic and algebra International Electronic Journal of Elementary Education 10 1 p.131-145
[4] Muchoko C, Jupri A & Prabawanto S 2019 Algebraic visualization difficulties of students in junior high school Journal of Physics: Conference Series 1157
[5] Jupri A, Drijvers P & Heuvel-panhuizen M. Van Den 2014 Student Difficulties in Solving Equations from an Operational and a Structural Perspective International Society of Educational Research 9 1 p.39-55
[6] Carpenter T P, Levi L, Franke M L & Zeringue J K 2005 Algebra in Elementary School: Developing Relational Thinking ZDM - International Journal on Mathematics Education 37 p.53-59
[7] Harbour K E, Karp K S & Lingo A S 2017 Inquiry to Action Diagnosing and Addressing Students’ Relational Thinking About the Equal Sign Teaching Exceptional Children 49 2 p.126-133
[8] Cai J & Moyer P 2008 Developing algebraic thinking in earlier grades: Some insights from international comparative studies. In C. E. Greene (Ed.) Algebra and Algebraic thinking in School Mathematics (Reston, VA: NCTM) p.169-179
[9] Jacobs V R, Franke M L, Carpenter T P, Levi L & Battey D 2007 Professional Development Focused on Children’s Algebraic Reasoning in Elementary School of Journal for Research in Mathematics Education 38 3 p.258-288
[10] Whitacre I, Schoen R C, Champagne Z and Goddard A 2016 Relational Thinking: What ’ s the Difference ? Teaching Children Mathematics 23 5 p.302-308
[11] Jones I & Pratt D 2012 A Substituting Meaning for the Equals Sign in Arithmetic Notating Tasks Journal for Research in Mathematics Education, 43 1 p.2-33
[12] Kindrat A N & Osana H P 2018 The relationship between mental computation and relational thinking in the seventh grade Fields Mathematics Education Journal, 3 6
[13] Pang J & Kim J 2018 Characteristics of Korean Students’ Early Algebraic Thinking: A Generalized Arithmetic Perspective. In Kieran (ed.) Teaching and Learning Algebraic Thinking with 5- to 12-Year-Olds (Switzerland: Springer)

[14] Fisher M H, Thomas J, Jong C, Schack E O & Dueber D 2019 Comparing preservice teachers’ professional noticing skills in elementary mathematics classrooms School Science and Mathematics 119 3 p.142-149

[15] Osana H P & Proulx J 2018 A tale of two researchers: Commonalities, complementarities, and contrasts in an examination of mental computation and relational thinking Journal of Numerical Cognition 4 1 p.59-83

[16] Kaput J J & Blanton M L 2005 Characterizing a classroom practice that promotes algebraic reasoning Journal for Research in Mathematics Education 36 5 p.412-446

[17] Kieran C, Pang J, Schifter D & Fong S Ng 2016 Early algebra: Research into its nature, its learning, its teaching (ICME 13 Topical surveys) (Switzerland: Springer)

[18] Kieran C 2004 Algebraic thinking in the early grades: What is it. The Mathematics Educator 8 1 p.139-151

[19] Stephens A C 2006 Equivalence and relational thinking: Preservice elementary teachers’ awareness of opportunities and misconceptions Journal of Mathematics Teacher Education 9 p. 249–278

[20] Stephens M & Wang X 2008 Investigating some junctures in relational thinking: A study of year 6 and year 7 students from Australia and China Journal of Mathematics Education 1 1 p.28-39.

[21] Brendefur J, Johnson E S, Thiede K W, Smith E V, Strother S, Severson H H & Beaulieu J 2015 Developing a comprehensive mathematical assessment tool to improve mathematics intervention for at-risk students International Journal for Research in Learning Disabilities, 2 2 p.65-90.

[22] Banerjee R 2011 Is Arithmetic Useful for the Teaching and Learning of Algebra? Contemporary Education Dialogue, 8 2 p.137-159

[23] NCTM 2014 What Does Research Tell Us about Fostering Algebraic (703) 20191.

[24] Stephens A C, Knuth E J, Blanton M L, Isler I, Gardiner A M & Marum T 2013 Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students’ understandings Journal of Mathematical Behavior 32 2 p.173-182

[25] Blanton M, Stroud R, Stephens A, Gardiner A M, Stylianou D A, Knuth E, Baykal I I and Strachota S 2019 Does Early Algebra Matter? The Effectiveness of an Early Algebra Intervention in Grades 3 to 5 American Educational Research Journal 56 p.1930-1972

Acknowledgments
The author would like to acknowledge Lembaga Pengelola Dana Pendidikan (LPDP) for giving fully funded scholarship to the first author and also the Postgraduate School of Universitas Pendidikan Indonesia for supporting this research.