Properties of two Doubly-Truncated Generalized Distributions

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Abstract. In this paper, some properties of doubly truncated generalized gamma distribution and doubly truncated Generalized Invers Weibull distribution are derived. These properties are the reliability and hazard functions, rth raw moments, stress-strength reliability, Shannon entropy and relative entropy.

1. Introduction

In recent years, many authors have concentrated their consideration on the suggestion of new and more flexible probability distributions, established using different techniques to represent a set of data. Properties of a distribution are very useful to show the ability of that distribution.

The doubly truncated distributions are more realistic to represent phenomena, without losing the generality, since the truncated parameters can take any values.

In this paper, properties of doubly truncated generalized gamma distribution (DTGG) and Doubly truncated Generalized Invers Weibull distribution (DTGIW) are derived. These properties are the rth raw moments, stress-strength reliability, Shannon entropy and relative entropy.

2. The generalized gamma distribution

The generalized Gamma distribution GGD is a continuous probability distribution with three parameters, Presented by Stacy in 1962 [12]. It is contains some of important densities as special cases as Exponential, Gamma, Weibull, half-Normal and lognormal distributions. A lot of literature has been written about GGD, some of which will be mentioned below. Khodabin and Ahmadabadi in 2010 [9] derived some other properties of GGD with Kullback-Leibler discrimination, Akaike and Bayesian information criterion. Cordeiro et al. 2011 [4] derived another generalization of Stacy’s GGD using exponentiated method, and applied it to life time and survival analysis. Cox and Matheson in 2014 [5] compared exponentiated Weibull (EW) and matching GG distributions graphically and using the Kullback-Leibler distance. They found that the survival functions for the EW and matching GG are graphically indistinguishable, and only the hazard functions can sometimes be seen to be slightly different. In 2017, Abid and Abdulrazak [1] presented [0,1] truncated Frechet GGD. They derived the distribution properties such as reliability function, hazard function, the rth raw moment function, Stress-Strength reliability, Shannon and Relative entropies. Barriga in 2018 [3] defined a new extension of the GGD based on the generator pioneered by Marshall and Olkin in 1997. It is shown by Kiche et al. in 2019 [10] that GGD has three sub-families and its application to the analysis of a survival data has also been explored.

The probability density function of the GG variable is,

\[ f(x) = \frac{p}{r(p)} x^{d-1} e^{-\left(\frac{x}{a}\right)^p} \]  (1)

And the cumulative distribution function is,

\[ F(x) = \frac{\gamma\left(d, \left(\frac{x}{a}\right)^p\right)}{r(p)} \]  (2)

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Where \( \Gamma(,.) \) denotes the gamma function which defined as \( \Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, dt \), and \( \gamma\left(\frac{a}{p}, \left(\frac{z}{\varepsilon}\right)^p\right) \) denotes the lower incomplete gamma function, which is generally can written as
\[
\gamma(s, z) = \int_0^z t^{s-1} e^{-t} \, dt = \sum_{s_1=0}^{\infty} \frac{(-1)^{s_1}}{s_1!} \frac{z^{s+s_1}}{s+s_1}
\] (3)

The upper incomplete gamma function also is, \( \Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} \, dt \), where, \( \gamma(s, z) = \Gamma(s) - \Gamma(s, x) \).

2.1. Essential properties of DTGG \((\epsilon, d, p, a, b)\)

We consider here the doubly truncated Generalized Gamma Distribution DTGG \((\epsilon, d, p, a, b)\) of random variable \(X\), where the lower and upper limits are \(a\) and \(b\) respectively, then the probability density function (pdf) and the commutative density function (cdf) are respectively,
\[
g(x) = \frac{\frac{\epsilon}{a} x^{d-1} e^{-\left(\frac{\epsilon}{a}\right)^p}}{\gamma\left(\frac{a}{p}, \left(\frac{z}{\varepsilon}\right)^p\right) - \gamma\left(\frac{d}{p}, \left(\frac{a}{\varepsilon}\right)^p\right)} , \quad a < x < b
\] (4)
\[
G(x) = \frac{\gamma\left(\frac{a}{p}, \left(\frac{x}{\varepsilon}\right)^p\right) - \gamma\left(\frac{d}{p}, \left(\frac{a}{\varepsilon}\right)^p\right)}{\gamma\left(\frac{a}{p}, \left(\frac{z}{\varepsilon}\right)^p\right) - \gamma\left(\frac{d}{p}, \left(\frac{a}{\varepsilon}\right)^p\right)} , \quad a < x < b
\] (5)

Then, the reliability and the hazard functions of \(X\) are respectively,
\[
R(x) = 1 - G(x) = \frac{\gamma\left(\frac{d}{p}, \left(\frac{z}{\varepsilon}\right)^p\right) - \gamma\left(\frac{d}{p}, \left(\frac{a}{\varepsilon}\right)^p\right)}{\gamma\left(\frac{a}{p}, \left(\frac{z}{\varepsilon}\right)^p\right) - \gamma\left(\frac{d}{p}, \left(\frac{a}{\varepsilon}\right)^p\right)}
\] (6)
\[
h(x) = \frac{g(x)}{R(x)} = \frac{\frac{\epsilon}{a} x^{d-1} e^{-\left(\frac{\epsilon}{a}\right)^p}}{\gamma\left(\frac{d}{p}, \left(\frac{z}{\varepsilon}\right)^p\right) - \gamma\left(\frac{d}{p}, \left(\frac{a}{\varepsilon}\right)^p\right)}
\] (7)

Since, \(\int_a^b x^{d-1} e^{-\left(\frac{\epsilon}{a}\right)^p} \, dx = \frac{\epsilon}{p} \left\{ \gamma\left(\frac{a}{p}, \left(\frac{b}{\varepsilon}\right)^p\right) - \gamma\left(\frac{d}{p}, \left(\frac{a}{\varepsilon}\right)^p\right) \right\} \)

(8)

Then, the rth raw moment of DTGG \((\epsilon, d, p, a, b)\) distribution is,
\[
E(X^r) = e^{\frac{r}{p}} \left\{ \gamma\left(\frac{d+r}{p}, \left(\frac{z}{\varepsilon}\right)^p\right) - \gamma\left(\frac{d+r}{p}, \left(\frac{a}{\varepsilon}\right)^p\right) \right\}
\] (9)

Then, the characteristic function can easily get by using the relation,
\[
Q_X(t) = E(e^{itX}) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(X^r) , \quad \text{since} \quad e^{itX} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} X^r
\]

2.2. Stress-Strength Reliability

Inferences about \(R = P[Y < X]\), where \(X\) and \(Y\) are two independent random variables, is very common in the reliability literature. For example, if \(X\) is the strength of a component which is subject to a stress \(Y\), then \(R\) is a measure of system performance and arises in the context of mechanical reliability of a system. The system fails if and only if at any time the applied stress is greater than its strength. Let \(Y\) and \(X\) be the stress and the strength random variables, independent of each other, follow respectively DTGG \((\epsilon_2, d_2, p_2, a, b)\) and DTGG \((\epsilon_1, d_1, p_1, a, b)\), then,
\[
R = P(Y < X) = \int_a^b f_X(x) F_Y(x) \, dx
\]
\[
R = \int_a^b \frac{p_1}{p_2} e^{-\frac{d_1}{p_1} x} - \frac{d_1}{p_2} e^{-\frac{d_1}{p_2} x} \prod \left[ \frac{\gamma \left( \frac{d_2}{p_2} \left( \frac{x}{\epsilon_2} \right)^{p_2} \right) - \gamma \left( \frac{d_2}{p_2} \left( \frac{a}{\epsilon_2} \right)^{p_2} \right)}{\gamma \left( \frac{d_2}{p_2} \left( \frac{b}{\epsilon_2} \right)^{p_2} \right) - \gamma \left( \frac{d_2}{p_2} \left( \frac{a}{\epsilon_2} \right)^{p_2} \right)} \right] dx
\]

Let \( K = \frac{1}{\prod \left[ \frac{\gamma \left( \frac{b}{\epsilon_1} \right)^{p_1} d_1}{\gamma \left( \frac{a}{\epsilon_1} \right)^{p_1} d_1} \right] \prod \left[ \frac{\gamma \left( \frac{b}{\epsilon_2} \right)^{p_2} d_2}{\gamma \left( \frac{a}{\epsilon_2} \right)^{p_2} d_2} \right] \}

\[
R = K \frac{p_1}{\epsilon_1^d_1} \left[ \int_a^b x^{d_1-1} e^{-\frac{d_1}{\epsilon_1} x} \gamma \left( \frac{d_2}{p_2} \left( \frac{x}{\epsilon_2} \right)^{p_2} \right) dx \right] - \int_a^b x^{d_1-1} e^{-\frac{d_1}{\epsilon_1} x} \gamma \left( \frac{d_2}{p_2} \left( \frac{a}{\epsilon_2} \right)^{p_2} \right) dx
\]

By using (3), we get

\[
R = K \frac{p_1}{\epsilon_1^d_1} \left[ \sum_{s_1=0}^{\infty} \frac{(-1)^{s_1}}{s_1!} \frac{d_1^{s_1}}{p_1} \prod \left[ \frac{\gamma \left( \frac{d_1 + d_2 + s_1 p_2}{p_1} \left( \frac{b}{\epsilon_1} \right)^{p_1} \right) - \gamma \left( \frac{d_1 + d_2 + s_1 p_2}{p_1} \left( \frac{a}{\epsilon_1} \right)^{p_1} \right)}{\gamma \left( \frac{d_1 + d_2 + s_1 p_2}{p_1} \left( \frac{b}{\epsilon_1} \right)^{p_1} \right) - \gamma \left( \frac{d_1 + d_2 + s_1 p_2}{p_1} \left( \frac{a}{\epsilon_1} \right)^{p_1} \right)} \right] \right]
\]

So the stress- strength reliability of DTGG is,

\[
R = K \frac{p_1}{\epsilon_1^d_1} \left[ \sum_{s_1=0}^{\infty} \frac{(-1)^{s_1}}{s_1!} \frac{d_1^{s_1}}{p_1} \prod \left[ \frac{\gamma \left( \frac{d_1 + d_2 + s_1 p_2}{p_1} \left( \frac{b}{\epsilon_1} \right)^{p_1} \right) - \gamma \left( \frac{d_1 + d_2 + s_1 p_2}{p_1} \left( \frac{a}{\epsilon_1} \right)^{p_1} \right)}{\gamma \left( \frac{d_1 + d_2 + s_1 p_2}{p_1} \left( \frac{b}{\epsilon_1} \right)^{p_1} \right) - \gamma \left( \frac{d_1 + d_2 + s_1 p_2}{p_1} \left( \frac{a}{\epsilon_1} \right)^{p_1} \right)} \right] \right]
\]

2.3. Shannon entropy

The Shannon entropy of a random variable X is a measure of variation of the uncertainty. It is defined for a random variable X with values in a finite set X as \( H = E(-\ln(g(x))) \). So, the Shannon entropy of DTGG random variable is,

\[
H = E \left( -\ln \left( \frac{p}{e^d} \gamma \left( \frac{d}{p} \left( \frac{b}{\epsilon} \right)^{p} \right) - \gamma \left( \frac{d}{p} \left( \frac{a}{\epsilon} \right)^{p} \right) \right) \right)
\]

\[
= E \left( -\ln \left( \frac{p}{e^d} \gamma \left( \frac{d}{p} \left( \frac{b}{\epsilon} \right)^{p} \right) - \gamma \left( \frac{d}{p} \left( \frac{a}{\epsilon} \right)^{p} \right) \right) + \frac{p}{e^d} \gamma \left( \frac{d}{p} \left( \frac{b}{\epsilon} \right)^{p} \right) - \gamma \left( \frac{d}{p} \left( \frac{a}{\epsilon} \right)^{p} \right) \right)
\]

\[
= E \left( -\ln \left( \frac{p}{e^d} \gamma \left( \frac{d}{p} \left( \frac{b}{\epsilon} \right)^{p} \right) - \gamma \left( \frac{d}{p} \left( \frac{a}{\epsilon} \right)^{p} \right) \right) \right)
\]
\[
\begin{aligned}
&= -\ln \left( \frac{p}{\varepsilon \Gamma \left( \frac{d}{p} + \left( \frac{b}{p} \right)^{p} \right)} - \left( \frac{a}{\varepsilon} \right)^{p} \right) - (d - 1)E(\ln(X)) + E \left( \left( \frac{X}{\varepsilon} \right)^{p} \right) \\
\end{aligned}
\]

Since, \(E(\ln(X)) = \int_{a}^{b} \ln(x) \left( \frac{d}{p} + \left( \frac{b}{p} \right)^{p} \right) x^{d-1} e^{-\left( \frac{x}{\varepsilon} \right)^{p}} dx \)

Let \(A_{1} = \frac{p}{\varepsilon \Gamma \left( \frac{d}{p} + \left( \frac{b}{p} \right)^{p} \right)} - \left( \frac{a}{\varepsilon} \right)^{p} \), then,

\[
E(\ln(X)) = A_{1} \int_{a}^{b} \ln(x) x^{d-1} e^{-\left( \frac{x}{\varepsilon} \right)^{p}} dx = A_{1} \int_{a}^{b} \ln(x) x^{d-1} \sum_{s_{1}=0}^{\infty} \left( \frac{X}{\varepsilon} \right)^{p} \frac{s_{1}}{s_{1}!} dx
\]

Since, \(f_{a}^{b} x^{m} \ln(x) dx = x^{m+1} \int_{m+1}^{b} \ln(x) \left( \frac{1}{x} \right) dx \), then,

\[
E(\ln(X)) = A_{1} \sum_{s_{1}=0}^{\infty} \left[ \frac{b^{d+p} s_{1}}{s_{1}! (d+p s_{1})^{2}} - \frac{1}{(d+p s_{1})^{2}} \right] - a^{d+p s_{1}} \left[ \ln(a) \frac{1}{d+p s_{1}} - \frac{1}{(d+p s_{1})^{2}} \right]
\]

And by using (9), we get,

\[
E \left( \left( \frac{X}{\varepsilon} \right)^{p} \right) = \left( \frac{1}{\varepsilon} \right)^{p} E(X)^{p} = \frac{\left( \frac{d}{p} + \left( \frac{b}{p} \right)^{p} \right) - \left( \frac{a}{p} \right)^{p}}{\left( \frac{d}{p} + \left( \frac{b}{p} \right)^{p} \right) - \left( \frac{a}{p} \right)^{p}}
\]

Then the entropy of DTGG random variable is:-

\[
H = -\ln \left( \frac{p}{\varepsilon \Gamma \left( \frac{d}{p} + \left( \frac{b}{p} \right)^{p} \right)} - \left( \frac{a}{\varepsilon} \right)^{p} \right) - (d - 1)A_{1} \sum_{s_{1}=0}^{\infty} \frac{(-1)^{s_{1}}}{s_{1}! (d+p s_{1})^{2}} - \frac{1}{(d+p s_{1})^{2}}
\]

2.4. The relative entropy

The relative entropy (or the Kullback–Leibler divergence) is a measure of the difference between two probability distributions \(G\) and \(G^{*}\). In applications \(G\) typically represents the "true" distribution of data, observations, or a precisely calculated theoretical distribution, while \(G^{*}\) typically represents a theory, model, description, or approximation of \(G\). Specifically, the Kullback–Leibler divergence of \(G^{*}\) from \(G\), denoted \(D_{KL}(G||G^{*})\), is a measure of the information gained when one revises ones beliefs from the prior probability distribution \(G^{*}\) to the posterior probability distribution \(G\). More exactly, it is the amount of information that is lost when \(G^{*}\) is used to approximate \(G\), defined operationally as the expected extra number of bits required to code samples from \(G\) using a code optimized for \(G^{*}\) rather than the code optimized for \(G\).

The relative entropy \(D_{KL} = (G||G^{*})\) for DTGG random variable is,

\[
D_{KL} = (G||G^{*}) = \int_{a}^{b} g(x) \ln \left( \frac{g(x)}{g^{*}(x)} \right) dx
\]

= \int_{a}^{b} g(x) \ln(g(x)) dx - \int_{a}^{b} g(x) \ln(g^{*}(x)) dx
\[ = -H - \int_a^b g(x) \ln(g^*(x)) \, dx \]

So, \[\int_a^b g(x) \ln(g^*(x)) \, dx = \int_a^b \frac{\frac{a}{\varepsilon_1}}{\gamma(\frac{d_1}{p_1}, \varepsilon_1)}^\frac{b}{\varepsilon_1} \cdot \gamma(\frac{d_1}{p_1}, \varepsilon_1) x^{\frac{d_1}{p_1} - 1} e^{-\left(\frac{x}{\varepsilon_2}\right)^{p_2}} \ln\left(\frac{\frac{d_2}{p_2}}{\varepsilon_2} \cdot \gamma(\frac{d_2}{p_2}, \varepsilon_2)ight) x^{\frac{d_2}{p_2} - 1} e^{-\left(\frac{x}{\varepsilon_2}\right)^{p_2}} \, dx \]

\[= A_2 + (d_2 - 1)E[\ln(X)] - E\left(\frac{X}{\varepsilon_2}\right)^{p_2} \]

Where, \(A_2 = \ln\left(\frac{\frac{d_2}{p_2}}{\varepsilon_2} - \gamma(\frac{d_2}{p_2}, \varepsilon_2)\right)\). By using (11) and (12), we get,

\[\int_a^b g(x) \ln(g^*(x)) \, dx = A_2 + (d_2 - 1) \left[ A_1 \sum_{s_1=0}^{\infty} \frac{(-1)^{s_1}}{s_1! (\varepsilon_1)^{p_1} s_1} \left[ b^{d_1+p_1 s_1} \left\{ \ln(b) - \frac{1}{(d_1 + p_1 s_1)^2} \right\} \right] - A_2 \right] \]

Then, the relative entropy is,

\[D_{KL} = (G||G^*) = \ln(A_1) + (d_1 - 1) \left[ A_1 \sum_{s_1=0}^{\infty} \frac{(-1)^{s_1}}{s_1! (\varepsilon_1)^{p_1} s_1} \left[ b^{d_1+p_1 s_1} \left\{ \ln(b) - \frac{1}{(d_1 + p_1 s_1)^2} \right\} \right] - A_2 \right] \]

3. The generalized inverse Weibull distribution

A three parameter generalized inverse Weibull distribution (GIWD) with decreasing and unimodal failure rate is introduced and studied by de Gusmao et al in 2011 [6]. They provided a comprehensive treatment of the mathematical properties of GIWD. The mixture model of two generalized inverse Weibull distributions is investigated. They also proposed a location-scale regression model based on the log- GIWD for modeling lifetime data. In addition, some diagnostic tools for sensitivity analysis is developed. Khan and King in 2014 [8], introduced five parameter transmuted GIWD. They derived moments, moment generating function, entropy, mean deviation, Bonferroni and Lorenz curves. Maximum likelihood for estimating the model parameters is used and based on the observed information matrix is obtained. Elbatal and Muhammed in 2014 [7], presented the Exponentiated GIWD. They derived the moment generating function and the rth moment. Expressions for the
density, moment generating function and rth moment of the order statistics also are obtained. They discussed the parameters estimation by maximum likelihood and provide the information matrix. GIW- GIW distribution is proposed by Abid et al in 2019 [2] as new distribution. The probability density function, cumulative distribution function, reliability and hazard rate functions are introduced. Furthermore, they derived most important statistical properties of GIW- GIW distribution such as Shannon entropy, relative entropy, stress-strength model. Salem in 2019 [11] studied the Marshall–Olkin GIWD. The new distribution is flexible and contains sub-models such as inverse exponential, inverse Rayleigh, Weibull, inverse Weibull, Marshall–Olkin inverse Weibull and Fréchet distributions. Some properties are obtained. Maximum likelihood, least square estimators, interval estimators, estimators, fisher information matrix and asymptotic confidence intervals are described.

The probability density and cumulative functions of GIWD random variable with three parameters \( \alpha > 0, \beta > 0 \) and \( \sigma > 0 \) are respectively given by,

\[
\begin{align*}
f(x) &= \sigma \beta \alpha^\beta x^{-(\beta+1)} e^{-\sigma(x) / \alpha}, \quad x > 0 \\
F(x) &= e^{-\sigma(x) / \alpha}, \quad x > 0
\end{align*}
\]

(15) (16)

3.1. Essential properties of DTGIW(\( \alpha, \sigma, \beta, a, b \))

The pdf and cdf of DTGIW(\( \alpha, \sigma, \beta, a, b \)) random variable are respectively,

\[
\begin{align*}
g(x) &= \frac{\sigma \beta \alpha^\beta x^{-(\beta+1)} e^{-\sigma(x) / \alpha}}{e^{-\sigma(x) / \alpha} - e^{-\sigma(x) / \alpha}}, \quad a < x < b \\
G(x) &= \frac{\sigma \beta \alpha^\beta}{e^{-\sigma(x) / \alpha} - e^{-\sigma(x) / \alpha}}, \quad a < x < b
\end{align*}
\]

(17) (18)

Also the reliability and hazard functions of DTGIW(\( \alpha, \sigma, \beta, a, b \)) distribution are respectively,

\[
\begin{align*}
R(x) &= \frac{e^{-\sigma(x) / \alpha} - e^{-\sigma(x) / \alpha}}{e^{-\sigma(x) / \alpha} - e^{-\sigma(x) / \alpha}}, \\
H(x) &= \frac{\sigma \beta \alpha^\beta x^{-(\beta+1)} e^{-\sigma(x) / \alpha}}{e^{-\sigma(x) / \alpha} - e^{-\sigma(x) / \alpha}}
\end{align*}
\]

(19) (20)

So, the rth raw moment is,

\[
E(X^r) = \int_a^b x^r \frac{\sigma \beta \alpha^\beta x^{-(\beta+1)} e^{-\sigma(x) / \alpha}}{e^{-\sigma(x) / \alpha} - e^{-\sigma(x) / \alpha}} \, dx
\]

\[
= \frac{\sigma \beta \alpha^\beta}{e^{-\sigma(x) / \alpha} - e^{-\sigma(x) / \alpha}} \int_a^b x^r e^{-\sigma(x) / \alpha} \, dx
\]

\[
= \frac{\sigma \beta \alpha^\beta}{e^{-\sigma(x) / \alpha} - e^{-\sigma(x) / \alpha}} \sum_{s_1=0}^{\infty} \frac{(-\sigma(x))^s_1}{s_1!} \int_a^b x^{r-s_1-1} \, dx
\]

\[
= \frac{\sigma \beta \alpha^\beta}{e^{-\sigma(x) / \alpha} - e^{-\sigma(x) / \alpha}} \sum_{s_1=0}^{\infty} \frac{(-\sigma(x))^s_1}{s_1!} \left[ \frac{b^{r-s_1} - a^{r-s_1}}{r - \beta(s_1 + 1)} \right]
\]

Let \( \frac{\sigma \beta \alpha^\beta}{e^{-\sigma(x) / \alpha} - e^{-\sigma(x) / \alpha}} \), then,
Then, the characteristic function can easily get by using the relation,

\[ Q_x(t) = E(e^{itx}) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(x^r) \], since \( e^{itx} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} x^r \)

### 3.2. Stress-Strength Reliability

Suppose \( X \sim DTGIW(\alpha_1, \sigma_1, \beta_1, a, b) \) and \( Y \sim DTGIW(\alpha_2, \sigma_2, \beta_2, a, b) \) with unknown parameters \( \alpha_1, \sigma_1, \beta_1, \alpha_2, \sigma_2, \beta_2, a, b \), where \( X \) and \( Y \) are independently distributed, then the stress-strength reliability function is,

\[ R = \int_a^b \frac{\sigma_1 \beta_1 \alpha_1 \beta_1 x^{-(\beta_1+1)} e^{-\sigma_1(\frac{\alpha_1}{x})^{\beta_1}} e^{-\sigma_2(\frac{\alpha_2}{x})^{\beta_2}}}{\left( e^{-\sigma_1(x)}^{\beta_1} - e^{-\sigma_1(\frac{\alpha_1}{x})^{\beta_1}} \right) \left( e^{-\sigma_2(x)}^{\beta_2} - e^{-\sigma_2(\frac{\alpha_2}{x})^{\beta_2}} \right)} \, dx \]

\[ - \int_a^b \frac{\sigma_1 \beta_1 \alpha_1 \beta_1 x^{-(\beta_1+1)} e^{-\sigma_1(\frac{\alpha_1}{x})^{\beta_1}} e^{-\sigma_2(\frac{\alpha_2}{x})^{\beta_2}}}{\left( e^{-\sigma_1(x)}^{\beta_1} - e^{-\sigma_1(\frac{\alpha_1}{x})^{\beta_1}} \right) \left( e^{-\sigma_2(x)}^{\beta_2} - e^{-\sigma_2(\frac{\alpha_2}{x})^{\beta_2}} \right)} \, dx \]

\[ = \int_a^b \frac{\sigma_1 \beta_1 \alpha_1 \beta_1 x^{-(\beta_1+1)} e^{-\sigma_1(\frac{\alpha_1}{x})^{\beta_1}} e^{-\sigma_2(\frac{\alpha_2}{x})^{\beta_2}}}{\left( e^{-\sigma_1(x)}^{\beta_1} - e^{-\sigma_1(\frac{\alpha_1}{x})^{\beta_1}} \right) \left( e^{-\sigma_2(x)}^{\beta_2} - e^{-\sigma_2(\frac{\alpha_2}{x})^{\beta_2}} \right)} \, dx \]

Let \( K_1 = \frac{\sigma_1 \beta_1 \alpha_1 \beta_1}{e^{-\sigma_1(x)}^{\beta_1} - e^{-\sigma_1(\frac{\alpha_1}{x})^{\beta_1}}} \) and \( K_2 = \frac{e^{-\sigma_2(x)}^{\beta_2} - e^{-\sigma_2(\frac{\alpha_2}{x})^{\beta_2}}}{\left( e^{-\sigma_2(x)}^{\beta_2} - e^{-\sigma_2(\frac{\alpha_2}{x})^{\beta_2}} \right)} \), then,

\[ R = K_1 \int_a^b x^{-(\beta_1+1)} \sum_{s_1=0}^{\infty} \left( -\sigma_1(\frac{\alpha_1}{x})^{\beta_1} \right)^{s_1} \sum_{s_2=0}^{\infty} \left( -\sigma_2(\frac{\alpha_2}{x})^{\beta_2} \right)^{s_2} \, dx - K_2 \]

\[ = K_1 \sum_{s_1=0}^{\infty} \left( -\sigma_1(\frac{\alpha_1}{x})^{\beta_1} \right)^{s_1} \sum_{s_2=0}^{\infty} \left( -\sigma_2(\frac{\alpha_2}{x})^{\beta_2} \right)^{s_2} \int_a^b x^{-(\beta_1 s_1 + \beta_2 s_2 - 1)} \, dx - K_2 \]

Then the stress-strength reliability function will be,

\[ R = K_1 \sum_{s_1=0}^{\infty} \left( -\sigma_1(\frac{\alpha_1}{x})^{\beta_1} \right)^{s_1} \sum_{s_2=0}^{\infty} \left( -\sigma_2(\frac{\alpha_2}{x})^{\beta_2} \right)^{s_2} \left[ \frac{-(\beta_1 s_1 + \beta_2 s_2 - \alpha^{-\alpha(\beta_1 s_1 + \beta_2 s_2)})}{(\beta_1 s_1 + \beta_2 s_2)} \right] - K_2 \]
Let $A_1 = \frac{\sigma \beta \alpha^b}{e^{-\sigma(\beta)} - e^{-\sigma(\frac{a}{\beta})}}$, then

$$E(\ln(X)) = \int_a^b \ln(x) \frac{\sigma \beta \alpha^b x^{-(\beta+1)} e^{-\sigma(\frac{a}{x})}}{e^{-\sigma(\beta)} - e^{-\sigma(\frac{a}{\beta})}} \, dx$$

$$= A_1 \int_a^b \ln(x) x^{-(\beta+1)} \sum_{s_1=0}^{\infty} \left( -\sigma \left( \frac{a}{x} \right) \right)^{s_1} \, dx$$

$$= A_1 \sum_{s_1=0}^{\infty} \frac{(-\sigma(a))^b}{s_1!} \int_a^b \ln(x) x^{-(\beta+1)+s_1} \, dx$$

Since $\int_a^b x^{-m} \ln(x) \, dx = -\frac{1}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2 x^{m-1}}$, then,

$$E(\ln(X)) = A_1 \sum_{s_1=0}^{\infty} \frac{(-\sigma(a))^b}{s_1!} \left\{ \frac{1}{b(\beta+1+s_1)} \left[ \ln(b) + \frac{1}{\beta(1+s_1)} \right] + \frac{1}{a(\beta+1+s_1)} \left[ \ln(a) + \frac{1}{\beta(1+s_1)} \right] \right\}$$

(23)

With $m = (\beta(1+s_1) + 1)$. Let $A_2 = \sigma \beta a^b / \left[ \exp - \sigma \left( \frac{a}{b} \right)^b - \exp \left( -\sigma \left( \frac{a}{a} \right)^b \right) \right]$, then,

$$E(\alpha^b X^{-\beta}) = A_2 \int_a^b (x)^{-2\beta-1} e^{-\sigma(\frac{a}{x})} \, dx = A_2 \int_a^b (x)^{-2\beta-1} \sum_{s_1=0}^{\infty} \frac{(-\sigma(a))^b}{s_1!} \, dx$$

$$= A_2 \sum_{s_1=0}^{\infty} \frac{(-\sigma(a))^b}{s_1!} \int_a^b (x)^{-(\beta(2+s_1)+1)} \, dx$$

$$= A_2 \sum_{s_1=0}^{\infty} \frac{(-\sigma(a))^b}{s_1!} \left[ \frac{1}{b(\beta(2+s_1)+1) - (\beta(2+s_1))} \right]$$

(24)

Finally, the Shannon entropy will be,

$$H = (\beta + 1) \left[ A_1 \sum_{s_1=0}^{\infty} \frac{(-\sigma(a))^b}{s_1!} \left\{ \frac{-1}{b(\beta+1+s_1)} \left[ \ln(b) + \frac{1}{\beta(1+s_1)} \right] + \frac{1}{a(\beta+1+s_1)} \left[ \ln(a) + \frac{1}{\beta(1+s_1)} \right] \right\} \right]$$

$$+ \sigma A_2 \sum_{s_1=0}^{\infty} \frac{(-\sigma(a))^b}{s_1!} \left[ \frac{1}{b(\beta(2+s_1)+1) - (\beta(2+s_1))} \right] - \ln(A_1)$$

(25)

3.4. The relative entropy
The relative entropy $D_{KL} = \langle G \| G^* \rangle$ for DTGIW random variable is,

$$D_{KL} = \langle G \| G^* \rangle = -H - \int_a^b g(x) \ln(g^*(x)) \, dx$$

then,

$$\int_a^b g(x) \ln(g^*(x)) \, dx = \int_a^b \left( \frac{\sigma_2 \beta_2 a_2^b x^{-(\beta_2+1)} e^{-\sigma_2(a_2/x)}}{e^{-\sigma_2(\beta_2/x)} - e^{-\sigma_2(\frac{a_2}{\beta_2})}} \right) \ln \left( \frac{\sigma_2 \beta_2 a_2^b x^{-(\beta_2+1)} e^{-\sigma_2(a_2/x)}}{e^{-\sigma_2(\beta_2/x)} - e^{-\sigma_2(\frac{a_2}{\beta_2})}} \right) \, dx$$
By using the same steps for equations (23) and (24), we get,

\[
D_{KL} = -\left( \beta_1 + 1 \right) \left[ A_{11} \sum_{s_1=0}^{\infty} \frac{\left( -\sigma_1 (\alpha_1) \beta_1 \right)^{s_1}}{s_1! \beta_1} \left( \frac{-1}{b(\beta_1(1+s_1))} \left( \frac{\ln(b)}{1 + s_1} + \frac{1}{\beta_1(1 + s_1)^2} \right) \right) + \frac{1}{a(\beta_1(1+s_1))} \left( \frac{\ln(a)}{1 + s_1} + \frac{1}{\beta_1(1 + s_1)^2} \right) \right] + \ln(A_{11})
\]

\[
+ A_3 \sum_{s_1=0}^{\infty} \frac{\left( -\sigma_1 (\alpha_1) \beta_1 \right)^{s_1}}{s_1! \beta_1} \left( \frac{-1}{b(\beta_1(1+s_1))} \left( \frac{\ln(b)}{1 + s_1} + \frac{1}{\beta_1(1 + s_1)^2} \right) \right) + \frac{1}{a(\beta_1(1+s_1))} \left( \frac{\ln(a)}{1 + s_1} + \frac{1}{\beta_1(1 + s_1)^2} \right) \right] + \ln(A_{14})
\]

Where

\[
A_{11} = \frac{\sigma_1 \beta_1 \alpha_1 \beta_1}{e^{-\sigma_1(\alpha_1)^2/\beta_1^2} - e^{-\sigma_1(\alpha_1)^2/\beta_1^1}} \quad A_{21} = \frac{-\beta_1(1+s_1) + \beta_2}{\sigma_1 \beta_1 \alpha_1 \beta_1} \quad A_3 = \frac{\sigma_1 \beta_1 \alpha_1 \beta_1}{e^{-\sigma_1(\alpha_1)^2/\beta_1^2} - e^{-\sigma_1(\alpha_1)^2/\beta_1^1}}
\]

and

\[
A_4 = \frac{\sigma_1 \beta_1 \alpha_1 \beta_1}{e^{-\sigma_1(\alpha_1)^2/\beta_1^2} - e^{-\sigma_1(\alpha_1)^2/\beta_1^1}}
\]

4. Summary and Conclusion

Distributions are used to represent set(s) of data in statistical analysis. The composing of some distributions with each other’s in some way to generate new distributions more flexible than the others to model real data. In this paper, we derived Properties of DTGG and DTGIW distributions, since doubly truncated distributions are more realistic to represent phenomena. We provided forms of rth raw moment, reliability function, hazard rate function, Shannon entropy function and Relative entropy function. This paper deals also with the determination of stress-strength R=p[y<x] when x (strength) and y (stress) are two independent DTGG (DTGIW) distribution with different parameters.

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