The Large Sieve Inequality for Integer Polynomial Amplitudes

by

Gyan Prakash and D.S. Ramana

Abstract

We obtain a close to the best possible version of the large sieve inequality with amplitudes given by the values of a polynomial with integer coefficients of degree $\geq 2$.

1. Introduction

It is of interest in the context of inequalities of the large sieve type to obtain estimates for the sum $\sum_{x \in \mathcal{X}} \left| \sum_{i \in I} a_i e(x f(i)) \right|^2$, where $e(z)$ denotes $e^{2\pi iz}$ for any complex number $z$, $\mathcal{X}$ is a well-spaced set of real numbers, $I$ is an interval of the integers, $\{a_i\}_{i \in I}$ are complex numbers and $f$ is real valued function on $I$ such that $f(I)$ is sparse, that is, the length of $f(I)$ is much larger than the length of $I$. When $f(I)$ is sparse, the duality argument that is used to establish the classical large sieve inequality generally gives weak bounds thus provoking the search for alternate arguments.

Basic example of sparse sequences are the sequence of values of polynomials of degree $\geq 2$ and Iwaniec and Kowalski, in their book [3] (see page 184, the paragraph following Problem 7.19), pose the problem of determining good large sieve bounds when $f$ is $P(T)$, a given polynomial in $\mathbb{Z}[T]$. Our purpose in this note is to verify the following theorem, which provides a result in this direction.

In the theorem below and thereafter $\omega(n)$ denotes the number of prime divisors of $n$ and $\|a\|^2$ denotes $\sum_{i \in I} |a_i|^2$ for a finite sequence of complex numbers $\{a_i\}_{i \in I}$. Further, for each integer $k \geq 1$, we define

$$\theta(k) = k \left( \frac{k+1}{2} \right).$$

THEOREM 1. — Let $Q$ and $k$ be integers $\geq 1$ and $I$ be an interval in $\mathbb{Z}$ of length $N$. When $\mathcal{F}(Q)$ is the Farey sequence of order $Q$ and $P(T) = c_0 T^k + c_1 T^{k-1} + \ldots + c_k$ is a polynomial of degree $k$ in $\mathbb{Z}[T]$ we have the inequality

$$\sum_{x \in \mathcal{F}(Q)} \left| \sum_{i \in I} a_i e(x P(i)) \right|^2 \ll Q(N + Q)(\log Q)^{\omega(c_0) + \theta(k)} \|a\|^2$$

for every sequence of complex numbers $\{a_i\}_{i \in I}$, where the constant implicit in the $\ll$ depends only on $k$.

While the classical large sieve inequality gives a much better bound than that given by (2) when $P(T)$ is linear, the bound (2) is the best possible up to the term $(\log Q)^{\omega(c_0) + \theta(k)}$ and constant
implicit in the $\ll$ in $\eqref{2}$ when the degree of $P(T)$ is $\geq 2$. We show this by means of an example in Section 3, where we provide a proof of Theorem 1.

A number of authors (see $\cite{5, 6, 1}$) have recently obtained upper bounds for the sum on the left hand side of $\eqref{2}$ from various points of view. These bounds are, however, are comparable to that given by $\eqref{2}$ only when $P(T)$ is of degree 2 and the interval $I$ is of the form $(M, M + N]$ with $M \ll N$. In fact, the only bound for polynomials $P(T)$ of degree $\geq 3$ that we are aware of is due to S. Baier who uses the method of Zhao $\cite{6}$ method to observe that (see Corollary 3 in $\cite{6}$) when $P(T) = T^m$, for any integer $m \geq 3$, and when $I$ is of the form $(0, N]$ then the left hand side of $\eqref{2}$ is $\ll \epsilon (NQ^{2(1-1/m)} + Q^2)N^{1+\epsilon} \sup_{0 \leq i \leq N} \{|a_i|^2$ under Hooley’s hypothesis $K^*$ in the context of Waring’s problem. Baier deduces this from an estimate that is valid even when the Farey series is replaced with an arbitrary well spaced sets as well.

When indeed $P(T)$ is of degree 2 and the interval $I$ is of the form $(0, N]$, Ramaré’s method, based on his theory of local factors, gives the bound $Q(N + Qg(Q))(\log_2 2Q)^2$ for the sum on the left hand side $\eqref{2}$, where $g(Q) = \exp(C \log_2 Q \log_3 Q)$. On the other hand, Zhao $\cite{6}$ gives, for the same sum, upper bounds essentially of the form $(Q(NM)^{1/2} + Q^2)(NM)^\epsilon$, for each $\epsilon > 0$ when $I$ is of the form $(M, M + N]$, via an elegant application of the double large sieve inequality. While Ramaré’s estimate is sharper than that given by Theorem 1 when $N$ is much larger than $Q$, Zhao’s estimate has the advantage of being applicable even when the Farey series is replaced with an arbitrary well spaced set.

In contrast to the aforementioned results, Theorem 1 is valid for all integer polynomials $P(T)$ and the bound given by this theorem is uniform with regard to the position of the interval $I$.

2. Number of Zeros of $P(T)$ modulo $m$

Let $P(T) = c_0T^k + c_1T^{k-1} + \ldots + c_k$ be a polynomial in $\mathbb{Z}[T]$ and, for any integer $m \geq 1$, let $S(m)$ be the set of congruence classes $l$ modulo $m$ such that $P(l) \equiv 0 \mod m$ and let $\rho(m)$ be Card$(S(m))$. Let $Q$ be a real number $\geq 1$. Proposition 1 below gives for $\sum_{1 \leq m \leq Q} \frac{\rho(m)}{m}$ an upper bound that is independent of the constant term of $P(T)$. The proof of Theorem 1 relies crucially on this feature of this bound.

Let $p$ be a prime number and $m, n$ be integers $\geq 1$. When $m \geq n$ the image of $S(m)$ under the canonical map from $\mathbb{Z}/p^m\mathbb{Z}$ onto $\mathbb{Z}/p^n\mathbb{Z}$ is contained in $S(n)$. Therefore we have $\frac{\rho(p^m)}{p^m} \leq \frac{\rho(p^n)}{p^n}$ whenever $m \geq n \geq 1$.

Suppose now that $p$ is a prime number that does not divide $c_0$. We then have $\rho(p) \leq k$ and hence $\frac{\rho(p^m)}{p^m} \leq \frac{k}{p}$ for all $m \geq 1$. We shall presently improve upon this upper bound for large $m$. To this end we set, for any integer $m \geq 1$, $a(m, k)$ to denote the smallest integer $\geq \frac{m}{(k + 2)}$ and we verify that any interval of the real line of length $p^{a(m, k)}$ contains no more than $k + 2$ integers $x$ such that $P(x)$ is divisible by $p^m$. To verify this it suffices to show that when $x_1, x_2, \ldots, x_{k+1}$ are
distinct integers such that \( P(x_i) \) is divisible by \( p^m \) for each \( i \), we have \( \sup_{i,j} |x_i - x_j| \geq p^{a(m,k)} \).

Indeed, on recalling the well known identity for the vandermonde determinant we have

\[
(1) \quad c_0 \prod_{1 \leq i < j \leq k+1} (x_i - x_j) = \begin{vmatrix}
1 & 1 & \ldots & 1 \\
x_1 & x_2 & \ldots & x_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{k-1} & x_2^{k-1} & \ldots & x_{k+1}^{k-1} \\
P(x_1) & P(x_2) & \ldots & P(x_{k+1})
\end{vmatrix}.
\]

Since the right hand side of (1) is divisible by \( p^m \) and \( p \) does not divide \( c_0 \) we see that \( p^m \) divides \( \prod_{1 \leq i < j \leq k+1} (x_i - x_j) \). Consequently,

\[
(2) \quad \binom{k+1}{2} \sup_{i \neq j} v_p(x_i - x_j) \geq \sum_{1 \leq i < j \leq k+1} v_p(x_i - x_j) \geq m,
\]

from which it follows that \( \sup_{i \neq j} v_p(x_i - x_j) \geq a(m,k) \) and, because the \( x_i \) are distinct, that \( \sup_{i,j} |x_i - x_j| \geq p^{a(m,k)} \).

For each integer \( m \geq 1 \), the set \( S(m) \) is in bijection with the subset of the integers \( x \) in the interval \( [0, p^m) \) such that \( P(x) \) is divisible by \( p^m \). On dividing this interval into subintervals of length \( p^{a(m,k)} \) we then conclude that when \( p \) does not divide \( c_0 \) we have \( \rho(p^m) \leq \frac{(k+2)p^m}{p^{a(m,k)}} \), for all \( m \geq 1 \).

With the aid of the bounds for \( \frac{\rho(p^m)}{p^m} \) given above we then conclude that when \( p \) is a prime number that does not divide \( c_0 \) we have

\[
(3) \quad \sum_{m \geq 0} \frac{\rho(p^m)}{p^m} \leq 1 + \sum_{1 \leq m \leq \binom{k+1}{2}} \frac{k}{p} + \sum_{m > \binom{k+1}{2}} \frac{k+2}{p^{a(m,k)}} = 1 + \frac{\theta(k)}{p} + (k+2) \binom{k+1}{2} \sum_{m \geq \binom{k+1}{2}} \frac{1}{p^m},
\]

where the last identity follows on dividing the sum over \( m \geq \binom{k+1}{2} \) into sums over congruence classes modulo \( \binom{k+1}{2} \) and noting that \( a(l + d\binom{k+1}{2}, k) = d \), when \( d \) is any integer and \( l \) an integer satisfying \( 0 \leq l < \binom{k+1}{2} \).

PROPOSITION 1. — Let \( P(T) = c_0 T^k + c_1 T^{k-1} + \ldots + c_k \) be a polynomial of degree \( k \) in \( \mathbb{Z}[T] \) and let \( \rho(m) \), for each integer \( m \geq 1 \), be the number of residue classes \( l \) modulo \( m \) such that \( P(l) \equiv 0 \mod m \). For any real number \( Q \geq 1 \) we then have \( \sum_{1 \leq m \leq Q} \frac{\rho(m)}{m} \ll (\log Q)^{\omega(c_0) + \theta(k)} \), where the implied constant in the \( \ll \) is depends only on \( k \).

PROOF. — Since \( \rho(m) \) is multiplicative, the sum \( \sum_{1 \leq m \leq Q} \frac{\rho(m)}{m} \) is evidently majorised by

\[
(4) \quad \prod_{1 \leq p \leq Q} \left( \sum_{\substack{m \geq 0, \\
p^m \leq Q}} \frac{\rho(p^m)}{p^m} \right) \leq (\log Q)^{\omega(c_0)} \prod_{1 \leq p \leq Q} \left( 1 + \frac{\theta(k)}{p} + (k+2) \binom{k+1}{2} \sum_{m \geq \binom{k+1}{2}} \frac{1}{p^m} \right),
\]

where we have used (3) when \( p \) does not divide \( c_0 \) and \( \rho(p^m) \leq p^m \) otherwise. The proposition now follows on dropping the condition \( (c_0,p) = 1 \) in the product on the right hand side of (4) and noting that \( \prod_{1 \leq p \leq Q} \left( 1 + \frac{a}{p} + \frac{b}{p^2} \right) \ll_{b} (\log Q)^{a} \), for any real numbers \( a \) and \( b \).
3. Proof of the Large Sieve Inequality

For each \((i, j) \in I \times I\) let us set

\[
K(i, j) = \sum_{x \in \mathcal{F}(Q)} e(x(P(i) - P(j))
\]

so that \(|K(i, j)| = |K(j, i)|\), for each \((i, j)\). On squaring out the sum over \(i \in I\) in the left hand side of the inequality given by Theorem 1, interchanging the summations and applying the triangle inequality together with \(|a_i\bar{a}_j| \leq \frac{1}{2}|a_i|^2 + |a_j|^2\) for each \((i, j)\) we obtain

\[
\sum_{x \in \mathcal{F}(Q)} \left| \sum_{i \in I} a_i e(xP(i)) \right|^2 = \sum_{(i, j) \in I \times I} a_i \bar{a}_j K(i, j) \leq \left( \sup_{j \in I} \sum_{i \in I} |K(i, j)| \right) \|a\|^2.
\]

Let \(c(i, j)\) denote \(P(i) - P(j)\) for each \((i, j)\). The classical estimate \(|\sum_{0 \leq p \leq q-1} e\left(\frac{ap}{q}\right)\| \leq (a, q)\), valid for any integer \(a\) with the convention that \((0, q) = q\), then implies

\[
|K(i, j)| \leq \sum_{1 \leq q \leq Q} \left| \sum_{0 \leq p \leq q-1, (p, q) = 1} e\left(\frac{pc(i, j)}{q}\right)\right| \leq \sum_{1 \leq q \leq Q} (c(i, j), q).
\]

Since for any integer \(k\) with \(1 \leq k \leq Q\), the number of multiples \(q\) of \(k\) with \(1 \leq q \leq Q\) does not exceed \(\frac{Q}{k} + 1 \leq \frac{2Q}{k}\), we obtain

\[
\sum_{1 \leq q \leq Q} (c(i, j), q) \leq \sum_{1 \leq k \leq Q} k \sum_{1 \leq q \leq Q, k \mid c(i, j)} 1 \leq 2Q \sum_{1 \leq k \leq Q} 1.
\]

For any \(j \in I\), let us set \(\rho_j(k)\) to denote the number of congruence classes \(l\) modulo \(k\) for which \(P(l) \equiv P(j) \mod k\). On combining (4) with (3) and recalling that \(I\) is an interval of length \(N\), we then conclude that for each \(j \in I\)

\[
\sum_{i \in I} |K(i, j)| \leq 2Q \sum_{i \in I, k \mid c(i, j)} 1 = 2Q \sum_{1 \leq k \leq Q} \sum_{j \in I, e(i, j) \equiv 0 \mod k} 1 \leq 2Q \sum_{1 \leq k \leq Q} \rho_j(k) \left( \frac{N}{k} + 1 \right).
\]

On applying Proposition 1 of Section 2 to the polynomial \(P(T) - P(j)\), we see, for each \(j \in I\), that

\[
\sum_{1 \leq k \leq Q} \rho_j(k) \left( \frac{N}{k} + 1 \right) \leq \sum_{1 \leq k \leq Q} (N + Q) \frac{\rho_j(k)}{k} \ll (N + Q)(\log Q)^{\omega(\epsilon_0) + \theta(k)},
\]

combining which with (2) and (5) we obtain Theorem 1.

Let us verify that upto the term \((\log Q)^{\omega(\epsilon_0) + \theta(k)}\) and the constant implicit in \(\ll\) the bound given by Theorem 1 is the best possible. To this end, let us take \(P(T) = T^n\), where \(n \geq 2\). We then learn on page 24 of [1] that when \(q\) is a prime number we have
Moreover, we have the bound \( |\sum_{1 \leq i \leq q} e^{(pP(i)/q)}| \leq (n-1)q^{1/2} \) from the estimate of Weil for exponential sums, for all prime numbers \( q > n \) and all integers \( p, k \) with \( (p, q) = 1 \). On using Theorem 2, page 12 of [4] we then deduce that

\[
\left( \sum_{1 \leq i \leq m} e^{(pP(i)/q)} \right) \leq 2(n-1)q^{1/2} \log q,
\]

for all prime numbers \( q > n \) and all integers \( p, m \) with \( (p, q) = 1 \) and \( 1 \leq m \leq q \). Let us now take \( N \) and \( Q \) integers \( \geq 1 \) with \( N \geq 8(n-1)Q \log Q \) and \( Q \geq n^2 \). On dividing the interval \( (0, N] \) into subintervals of length \( q \) and applying the triangle inequality together with (7) we then see that

\[
\sum_{1 \leq p \leq q-1} \left| \sum_{1 \leq i \leq q} e^{(pP(i)/q)} \right|^2 \geq \sum_{1 \leq p \leq q-1} \left( \frac{N}{q} \right) \left| \sum_{1 \leq i \leq q} e^{(pP(i)/q)} \right| - 2(n-1)q^{1/2} \log q \geq \frac{(n-1)N^2}{8},
\]

for every prime number \( q \) satisfying \( n < q \leq Q \). Consequently, we obtain the minorisation

\[
\sum_{x \in \mathbb{Z}(Q)} \left| \sum_{0 < i \leq N} e(xP(i)) \right|^2 \geq \sum_{1 \leq q \leq Q \, q \text{ prime}, \, q > n} \sum_{1 \leq p \leq q-1} \sum_{0 < i \leq N} e^{(pP(i)/q)} \geq \frac{(n-1)N^2Q}{16 \log Q},
\]

which may be compared with the upper bound \( \ll_n N^2Q(\log Q)^{\theta(n)} \) for the first term on left hand side of (10) supplied by Theorem 1 when applied with \( I \) taken to be the interval of integers \( (0, N] \), the polynomial \( P(T) = T^n \) and all the \( a_i = 1 \). When \( N < 8(n-1)Q \log Q \), this theorem gives the upper bound \( \ll_n Q^2(\log Q)^{\theta(n)+1} \|a\|^2 \) for the left hand side of (2), which may be compared with the lower bound \( Q^2 \|a\| \) obtained when \( I \) taken to be the interval \((0, 1]\) in this expression.

The method of proof of Gallagher’s inequality (see page 144 of [2]) immediately implies the following corollary to Theorem 1.

**Corollary.** — Let \( D \) be an integer \( \geq 1 \). When \( I \) is an interval of the integers of length \( N \) and \( P(T) = c_0T^k + c_1T^{k-1} + \ldots + c_k \) is a polynomial of degree \( k \) in \( \mathbb{Z}[T] \) we have the inequality

\[
\sum_{1 \leq d \leq D} \frac{\phi(d)}{d} \sum_{\chi \bmod d} \left| \sum_{i \in I} a_i \chi(P(i)) \right|^2 \ll D(N + D)(\log D)^{\omega(c_0)+\theta(k)} \|a\|^2
\]

for every sequence of complex numbers \( \{a_i\}_{i \in I} \), where the constant implicit in the \( \ll \) depends only on \( k \).
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Harish-Chandra Research Institute,
Chhatnag Road, Jhunsi,
Allahabad - 211 019, India.
email: gyan@mri.ernet.in, suri@mri.ernet.in