Quantum Mechanics of Lowest Landau Level
Derived from $\mathcal{N}=4$ SYM with Chemical Potential

D. Yamada*
Department of Physics, University of Washington, Seattle, WA. 98195

Abstract

The low energy effective theory of $\mathcal{N}=4$ super-Yang-Mills theory on $S^3$ with an $R$-symmetry chemical potential is shown to be the lowest Landau level system. This theory is a holomorphic complex matrix quantum mechanics. When the value of the chemical potential is not far below the mass of the scalars, the states of the effective theory consist only of the half-BPS states. The theory is solved by the operator method and by utilizing the lowest Landau level projection prescription for the value of the chemical potential less than or equal to the mass of the scalars. When the chemical potential is below the mass, we find that the degeneracy of the lowest Landau level is lifted and the energies of the states are computed. The one-loop correction to the effective potential is computed for the commuting fields and treated as a perturbation to the tree level quantum mechanics. We find that the perturbation term has non-vanishing matrix elements that mix the states with the same $R$-charge.

*dymada@u.washington.edu
1 Introduction

The half-BPS sector of the AdS/CFT correspondence [1, 2, 3] has recently been actively investigated since the works of Berenstein [4] and Lin, Lunin and Maldacena [5] (LLM). The former noted that a certain decoupling limit of $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) on $\mathbb{R} \times S^3$ singles out the half-BPS states of the theory and the theory is simply described by the one dimensional fermionic harmonic oscillators. This motivated the latter authors to construct the geometry of Type IIB supergravity corresponding to the half-BPS states of the field theory. Remarkably, they were able to obtain the generic solution that describes all the half-BPS states. The geometries are determined by solving a function that satisfies a partial differential equation. This procedure requires to specify a boundary condition on a particular two dimensional plane in ten dimensions and LLM conjectured that the boundary condition on the plane is identified to the phase space of the corresponding fermion system in the field theory.

On the field theory side, a half-BPS operator is characterized by its dimension being equal to its $R$-charge. Under the conformal mapping of the theory onto a sphere, the dimension is mapped to the energy of the corresponding state measured in the units of the inverse radius of the sphere. When we introduce a chemical potential corresponding to the $R$-symmetry, the Hamiltonian of the system takes the form,

$$H = h - \mu j,$$

where $h$ is the original Hamiltonian, $\mu$ is the chemical potential and $j$ is the $R$-charge number operator. When the chemical potential is tuned to exactly 1, which
is measured in the units of the inverse radius $R$, any half-BPS states, $|\psi\rangle$, satisfy the condition,

$$H|\psi\rangle = (h - j)|\psi\rangle = 0.$$ 

It was pointed out that such behavior of the half-BPS sector is analogous to the Landau problem. To the knowledge of the author, this was first noted by Berenstein [4].

In the present paper, we consider $\mathcal{N} = 4$ SYM on a sphere with the chemical potential, $\mu$, and show that the states of the low energy effective theory with $\mu \sim 1$ are the half-BPS states. This gives a clear physical picture of the decoupling limit that isolates the half-BPS states from other states of the theory. The Hamiltonian of the low energy effective theory is the one that describes the lowest Landau level system and it is a holomorphic complex matrix quantum mechanics. The attempt to make the precise correspondence between the SYM and the Landau problem has been made by Ghodsi et al. in [6]. However, we would like to emphasize that we are going to directly derive the lowest Landau level system as a low energy effective theory of the SYM with the chemical potential rather than developing the dictionary between the two.

One may start by asking a few questions. First, what happened to the oppositely charged particles? In a relativistic field theory, a particle is necessarily accompanied by the corresponding anti-particle and it has the opposite charge. Therefore, the particle and anti-particle respond differently to the chemical potential applied to the system. In the case of Minkowski space, the dispersion relations appear as $E \pm \mu$, so the above statement, $(h - j)|\psi\rangle = 0$, seems to correspond to either a particle or an anti-particle (in below, we take it as a particle as our convention) and not the both together. They differ in energy by the amount 2 when the chemical potential is 1. This suggests that one should be considering the low energy effective theory of $\mathcal{N} = 4$ SYM on sphere with the scale $1/R$. In such low energy effective theory, the anti-particle excitations are too costly and they are always at the ground state. In the language of the Landau problem mentioned above, one should consider the system projected down to the lowest Landau level. As we shall see, considerations along this line lead to the clarification of the decoupling limit that isolates the half-BPS states and reveal the emergence of the holomorphic complex matrix quantum mechanics.

The second question that one may ask is the response of the system when the chemical potential is slightly off from the value 1. When the chemical potential satisfies $\mu > 1$, it effectively introduces negative mass terms in the scalar fields and the precise determination of the outcome requires higher loop computations. The system may simply become unstable and cease to exist or it may go through a phase transition. In the present work, we limit our analysis to the tree level and to the one-loop with the special case where the fields are assumed to take values in flat directions and do not deal with this question. (However, see [7] for the high temperature analysis of such case where the Higgs phase transition has been found.)

When we have the chemical potential slightly smaller than the value 1, we expect the degeneracy of the lowest Landau level to be lifted, and the half-BPS states respond
to the chemical potential by $H|\psi_n\rangle = (1 - \mu)n|\psi_n\rangle$, where $n$ is the charge of the state. In the free field theory, we do not expect the stress of the chemical potential to mix the half-BPS states and further, the charge conservation requires the states with different charges not to mix. When we turn on the interaction of the theory, we should expect that the states with the same $R$-charge mix to form an energy eigenstate.

In what follows, we deal with the questions addressed above. In Section 2, we construct the tree level effective action of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with chemical potentials corresponding to the $R$-symmetry. In addition, we present the one-loop correction to the effective action for the special case where the fields of the effective theory are taking values in the flat directions. The details of the one-loop computation is given in Appendix A. In Section 3, we solve the non-interacting system first by the operator method. Then, we rearrange the Hamiltonian of the system resulting from the effective action and show explicitly that the Hamiltonian is exactly that of the Landau problem plus a potential that vanishes when $\mu = 1$. We solve the system again by utilizing the lowest Landau level projection prescription of Girvin and Jach [8]. Here, we find that the half-BPS states respond to the chemical potential as expected above and we show that the tree level interaction does not mix the states, while the one-loop correction does mix the states with the same $R$-charges. In Section 4, we discuss the result in the light of the AdS/CFT correspondence mentioned at the beginning of this section, in particular, the non-commutativity of coordinates is discussed from the perspective of the lowest Landau level projection prescription. We conclude by discussing the further directions in Section 5.

2 The Effective Action with Chemical Potential

We consider $U(N) \mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ where the number of the color $N$ is not assumed to be large. Our aim in this section is to obtain the low energy effective action of the theory with chemical potentials where the scale of the effective theory is $1/R$ with $R$ being the radius of the sphere. This theory has scalar mass terms that arise from the conformal scalar-curvature coupling and for the case of $S^3$, the mass of the scalars are the inverse of the radius $R$. The presence of these mass terms motivates us to introduce chemical potentials to the theory where, as we will see, chemical potentials appear as negative mass-squared terms in the Lagrangian.

It is known and also described in [7] that given a non-Abelian global symmetry of a theory, one can introduce chemical potentials corresponding to a Cartan subalgebra of the symmetry. In our case, we have the global $SU(4)_R$ $R$-symmetry, so we can introduce three chemical potentials to the theory corresponding to $U(1)^3$ subgroup of the $R$-symmetry.

In order to write down the Lagrangian, we first would like to determine how the chemical potentials are assigned for each field of the theory. The field content of the theory is six scalars, four (left-handed) Weyl fermions and a vector, all in adjoint representation of the gauge group $U(N)$. The table below summarizes the transformation laws of the fields under the $SU(4)_R$. 

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| Fields | DOF | $SU(4)_R$ |
|--------|-----|----------|
| $\phi$ | $6N^2$ | 6 |
| $\lambda$ | $8N^2$ | 4 |
| $A$ | $2N^2$ | 1 |

Here we have equal number of bosonic and fermionic degrees of freedom as usual in supersymmetric theories. The six scalars $\phi$ are actually complex, but with three complex relations, leaving six real degrees of freedom. To see this, consider the set of six complex scalars, $\phi_{ij}$ with $i, j = 1 \ldots, 4$ and $\phi_{ij} = -\phi_{ji}$, transforming under the anti-symmetric representation $\mathbf{6}$ of $SU(4)_R$. The complex conjugate representation $\bar{\mathbf{6}}$ is given by,

$$\phi^{ij} = \frac{1}{2} \epsilon^{ijkl} \phi_{kl},$$

and we must impose $\phi^{ij} = \phi^{*}_{ij}$. These constraints lead to the relations,

$$\phi^*_{12} = \phi_{34}, \quad \phi^*_{13} = \phi_{42}, \quad \phi^*_{14} = \phi_{23},$$

thus, as claimed, leaving six real degrees of freedom in the scalar sector.

We choose a Cartan subalgebra of the defining representation $\mathbf{4}$ as,

$$Q^4_1 = \frac{1}{2} \text{diag}(1, 1, -1, -1), \quad Q^4_2 = \frac{1}{2} \text{diag}(1, -1, 1, -1),$$

$$Q^4_3 = \frac{1}{2} \text{diag}(1, -1, -1, 1).$$

One can obtain the anti-symmetric representation $\mathbf{6}$, by considering the transformations of an anti-symmetric 2-tensor under $\mathbf{4} \otimes \mathbf{4}$. We choose a basis so that the matrices of $\mathbf{6}$ appear as,

$$Q^6_1 = \text{diag}(1, -1, 0, 0, 0, 0), \quad Q^6_2 = \text{diag}(0, 0, 1, -1, 0, 0),$$

$$Q^6_3 = \text{diag}(0, 0, 0, 0, 1, -1).$$

Then, the introduction of the chemical potentials modifies the Lagrangian by the replacement of the covariant derivatives,

$$D_\nu \rightarrow D_\nu - \mu_j Q_j \delta_{\nu,0},$$

where $\mu_j$ are the chemical potentials and we assumed the Euclidean signature here. One notices that the time component of the covariant derivative is modified by the chemical potentials. One can see this by the spurion argument. Imagine gauging the $U(1)$ global symmetry that you introduce a chemical potential. The chemical potential couples to the conserved $U(1)$ charge, just as the zeroth component of the minimal coupling. So it can be thought as a background gauge field whose components are all zero except the time component. Then, this gauged vector boson must appear in the time component of the covariant derivative as in above.
The modified covariant derivative (2) shows that the chemical potentials are assigned for the four Weyl spinors as,

\[ \tilde{\mu}_1 := \frac{1}{2}(\mu_1 + \mu_2 + \mu_3), \quad \tilde{\mu}_2 := \frac{1}{2}(\mu_1 - \mu_2 - \mu_3), \]
\[ \tilde{\mu}_3 := \frac{1}{2}(-\mu_1 + \mu_2 - \mu_3), \quad \tilde{\mu}_4 := \frac{1}{2}(-\mu_1 - \mu_2 + \mu_3). \]

and the chemical potentials, \( \pm \mu_p \) with \( p = 1, \ldots, 3 \) are given to the scalars \( \phi_{1,p+1} \) and their conjugates as in (1).

We re-express the Weyl fermions in terms of Majorana fermions. In four dimensions, the kinetic terms of the Weyl fermions, \( \lambda_i \), with chemical potentials are related to those of Majorana fermions, \( \psi_i \), by,

\[ \lambda_i \tau_\nu (\bar{\tilde{D}_\nu} - \tilde{\mu}_i \delta_\nu,0) \bar{\lambda}_i = \tilde{\psi}_i (\bar{D} + \bar{M}_i) \psi_i , \]

where

\[ \tau_\nu := (1, i\bar{\sigma}) , \quad \bar{\tau}_\nu := (1, -i\bar{\sigma}) , \]
\[ \gamma_\nu := \begin{pmatrix} 0 & \tau_\nu \\ \bar{\tau}_\nu & 0 \end{pmatrix} , \]
\[ \psi_i := \begin{pmatrix} \lambda_i \\ \bar{\lambda}_i \end{pmatrix} , \quad \bar{\psi}_i := \psi_i \gamma_0 , \]
\[ \bar{M}_i := \begin{pmatrix} 0 & -\tilde{\mu}_i \mathbf{1}_{2\times2} \\ \tilde{\mu}_i \mathbf{1}_{2\times2} & 0 \end{pmatrix} . \]

Finally, the Euclidean Lagrangian density of the theory with those chemical potentials is [9, 10],

\[
\mathcal{L}_E = \frac{1}{4} (F_{\mu\nu}^a)^2 \\
+ \frac{1}{2} (D_\nu X_p^a - i\mu_p \delta_\nu,0 Y_p^a)^2 + \frac{1}{2} (D_\nu Y_p^a + i\mu_p \delta_\nu,0 X_p^a)^2 \\
+ \frac{1}{2} \frac{1}{R^2} (\Phi_A^a)^2 + \frac{1}{4} g^2 (i[\Phi_A, \Phi_B]^a)^2 \\
+ \frac{1}{2} \bar{\psi}_i^a \bar{D} \psi_i^a + \frac{1}{2} \bar{\tilde{\psi}}_i^a \bar{M}_i \psi_i^a + \frac{1}{2} ig \bar{\psi}_i^a \left[ (\alpha^p_{ij} X_p + \beta^q_{ij} \gamma_5 Y_q), \psi_j \right]^a , \tag{3}
\]

where \( X_p \) and \( Y_p \) are the real scalars defined by \( \phi_{1,p+1} = (1/\sqrt{2})(X_p + iY_p) \) (see (11)), the factor of \( i \) in the quartic interaction term accounts for the fact that a commutator of Hermitian fields is anti-Hermitian, and,

\[ F_{\mu\nu}^a = \partial_\mu A_{\nu}^a - \partial_\nu A_{\mu}^a + ig [A_\mu, A_\nu]^a , \]
\[ D_\nu = \partial_\nu + ig [A_\nu, \cdot] , \]
\[ \Phi_A^a = (X_1^a, Y_1^a, X_2^a, Y_2^a, X_3^a, Y_3^a) . \]
with $a = 1, \ldots, N^2$, $A, B = 1, \ldots, 6$, $p, q = 1, \ldots, 3$, and $i, j = 1, \ldots, 4$. The $4 \times 4$ matrices $\alpha^p$ and $\beta^q$ satisfy the relations,

$$\{\alpha^p, \alpha^q\} = -2\delta^{pq}1_{4\times4}, \quad \{\beta^p, \beta^q\} = -2\delta^{pq}1_{4\times4}, \quad [\alpha^p, \beta^q] = 0. \quad (4)$$

We take the convention where the color matrices are Hermitian. The gauge and scalar fields are Hermitian fields and the components of the fermions are complex Grassmann variables that are taking Hermitian color matrix values.

We also have the relations,

$$[T^a, T^b] = if^{abc}T^c, \quad \{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu}, \quad \{\gamma_{\mu}, \gamma_5\} = 0, \quad (\gamma_5)^2 = 1_{4\times4},$$

where $T^a$ are the $N \times N$ Hermitian matrices of the $U(N)$ defining representation. We normalize the basis of the $U(N)$ algebra so that,

$$\text{tr}(T^a T^b) = \delta^{ab}.$$ 

In the present paper, we are interested in the half-BPS states and therefore, we turn on a single chemical potential,

$$\mu_p = \mu \delta_{p,1}.$$ 

In what follows, we set the mass of the scalars, $1/R$, to 1 and assume that all the energies, including the chemical potential, $\mu$, are measured in the units of $1/R$.

To construct the Wilsonian effective action of the theory that is valid below the momentum scale of order $1 = 1/R$, we need to integrate out the fields that are heavier than this scale. Since the theory is on $S^3$, we expand the fields into their spherical harmonics. Note that the topology of the sphere does not allow the zero modes of the spatial vectors and spinors (they have the lowest energies 1 for the spatial vectors and 3/2 for the spinors) and the scalars have masses 1 due to the conformal scalar-curvature coupling. Now, the time component of the gauge field, $A_0$, is a massless scalar and thus the only light mode of the theory is the constant mode of $A_0$. This is the Lagrange multiplier that enforces the Gauss' law constraint.

At first sight, therefore, we do not seem to have any dynamical field whose excitation could be lower than the scale $1 = 1/R$. However, as one can see from the Lagrangian, the chemical potential acts as negative mass terms for $X$ and $Y$ scalars. Thus, those excitation energies could potentially be smaller than the scale of the effective theory, if the chemical potential is sufficiently high, namely, $\mu \sim 1$ and we are going to assume such case. This assumption no longer allows us to integrate out the zero modes of $X, Y$ scalar fields. Therefore, the resulting effective theory is the theory of constant $X$ and $Y$ scalar fields on the time coordinate. This is a matrix quantum mechanics.

The effective Lagrangian at the tree level is therefore given by,

$$L = \frac{1}{2} \text{tr} \left[ (D_0 X - \mu Y)^2 + (D_0 Y + \mu X)^2 - (X^2 + Y^2) - g^2 (i[X, Y])^2 \right]$$

$$= \text{tr} \left[ (D_0 Z^\dagger + i\mu Z^\dagger)(D_0 Z - i\mu Z) - Z^\dagger Z - \frac{1}{2} g^2 ([Z, Z^\dagger])^2 \right], \quad (5)$$
where we Wick rotated to the Minkowski signature and defined the complex fields $Z$ by $Z = (1/\sqrt{2})(X - iY)$.

We now would like to obtain the one-loop effective potential of the theory up to the order of $g^2$. This, in general, is a difficult task because the quartic interaction term mixes the fields in the color space in a complicated way. For this reason, we resort to the special case where the scalar fields of the effective theory are taking values in the flat directions. This special case suffices to show what the loop corrections generally do and might have some relevance to the recent development in the theory with the commuting matrices [11, 12].

The actual one-loop computation is rather involved and presented in Appendix A. The method adopted there is the constant background field method and the fields $(X,Y)$ are shifted to $(x + X, y + Y)$ where $(X,Y)$ are the background fields that are taking values in the flat directions and $(x,y)$ are the arbitrary fluctuations around them. By collecting the quadratic terms in the shifted Lagrangian, the one-loop functional determinant is computed. The result is,

$$V_1 = K g^2 \sum_{m<n} |z_m - z_n|^2,$$

where $z_m$ are the eigenvalues of the diagonal $Z$ matrix and $K$ is defined to be,

$$K := \frac{2}{\pi^2} \left\{ -\frac{3}{4} + \ln 2 + C(\mu) + \frac{\mu^2}{4}(-2 + 2\gamma_E - 2\ln 2 + \ln \Lambda^2) \right\}.$$  

Here, $C(\mu)$ is a monotonically increasing function of $\mu$ and ranges from 0.072 at $\mu = 0.8$ to 0.20 at $\mu = 1.2$. See Figure 2 in Appendix A. The constant $\gamma_E$ is the Euler's constant and $\Lambda$ is an arbitrary renormalization scale measured in the units of $1/R$. There is mass renormalization due to the non-vanishing chemical potential and the quantities $\gamma_E$, $\Lambda$ are associated with the $\overline{MS}$-scheme that we adopted.

### 3 Quantum Mechanics of the Theory

In this section, we solve the resulting quantum mechanics without the interaction in two ways; the operator method and the Schrödinger equation. Each method gives different perspective on the decoupling limit that isolates the half-BPS states from the other states of the full theory. The former method shows that it is the low energy effective theory of the scale $1 = 1/R$ and anti-particles of the theory are restricted to the ground state. The latter shows that the system is the same low energy effective theory but that is equivalent to the Landau problem and the states are restricted to the lowest Landau level.

We treat the quartic interaction term in (5) and the one-loop effect (6) as perturbations to the free theory. Thus, we first solve the free theory in Subsection 3.1 and in Subsection 3.5 we show that the tree level quartic interaction has no non-vanishing matrix elements. Finally in Subsection 3.6 we discuss the matrix elements of the one-loop potential (6).
3.1 Operator Method

We first solve the theory by using the creation and annihilation operators. Although the physical setup and the motivations are somewhat different, we would like to acknowledge the papers by Corley et al. [13], Caldarelli and Silva [14] and especially by Takayama and Tsuchiya [15] that have carried out similar analyses to the one presented in this subsection.

We start the analysis from the complex field form of the Lagrangian (5) without the interaction term. The conjugate momenta are,

\[ \Pi^a := \frac{\partial L}{\partial \dot{Z}^a} = \dot{Z}^a + i\mu Z^a - ig[A_0, Z]^a \]
\[ \Pi^{a*} := \frac{\partial L}{\partial \dot{Z}^{a*}} = \dot{Z}^{a*} - i\mu Z^a - ig[A_0, Z]^a , \]

where \( a = 1, \ldots, N^2 \) is the color index. The Hamiltonian is obtained through Legendre transformation. Taking into account of the Hermiticity of the Hamiltonian, we have,

\[ H = \Pi^{a*} \dot{Z}^{a*} + \dot{Z}^a \Pi^a - L \]
\[ = \Pi^{a*} \Pi^a - i\mu \Pi^{a*} Z^a + i\mu Z^a \Pi^a + Z^a Z^{a*} \]
\[ - ig \left( \Pi^{a*} [A_0, Z]^a + [A_0, Z]^a \Pi^a \right) \]
\[ = \text{tr} \left[ (\Pi^\dagger, Z) \begin{pmatrix} 1 & -i\mu \\ i\mu & 1 \end{pmatrix} \begin{pmatrix} \Pi \\ Z^\dagger \end{pmatrix} \right] \]
\[ - ig \left( \Pi^\dagger [A_0, Z]^\dagger + [A_0, Z] \Pi \right) , \]

where the trace is taken over the color space.

The cyclic generalized coordinates \( A_0 \) are the constants of motion and their equations of motion \( \partial H/\partial A_0 = 0 \) impose the Gauss’ law to the states. In Subsection 3.3, we construct the global \( U(N) \) invariant states by hand and we are going to ignore the \( A_0 \) part from now on.

Now define a unitary matrix,

\[ U := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} , \]

which satisfies,

\[ U \begin{pmatrix} 1 & -i\mu \\ i\mu & 1 \end{pmatrix} U^\dagger = \begin{pmatrix} 1 + \mu & 0 \\ 0 & 1 - \mu \end{pmatrix} . \]

Then, the Hamiltonian (without the \( A_0 \) factors) takes the form,

\[ H = \text{tr} \left[ (\Pi^\dagger, Z) U^\dagger U \begin{pmatrix} 1 & -i\mu \\ i\mu & 1 \end{pmatrix} U^\dagger U \begin{pmatrix} \Pi \\ Z^\dagger \end{pmatrix} \right] \]
\[ = \text{tr} \left[ (A^\dagger, B) \begin{pmatrix} 1 + \mu & 0 \\ 0 & 1 - \mu \end{pmatrix} \begin{pmatrix} A \\ B^\dagger \end{pmatrix} \right] \]
\[ = (1 + \mu) \text{tr}(A^\dagger A) + (1 - \mu) \text{tr}(BB^\dagger) , \]
where we have defined,
\[
\begin{pmatrix}
A \\
B^\dagger
\end{pmatrix} := U \begin{pmatrix}
\Pi \\
Z^\dagger
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\Pi - iZ^\dagger \\
\Pi + iZ^\dagger
\end{pmatrix}.
\tag{9}
\]

It is clear from (8) that we must have the condition \(|\mu| \leq 1\), otherwise the Hamiltonian is not bounded from below. This condition is always true for free theories with chemical potentials.

To quantize the system, we impose the canonical quantization conditions,
\[
[A^a, A^{b*}] = \delta^{ab} = [B^a, B^{b*}] ,
\]
otherwise commute .

These are equivalent to the conditions,
\[
[Z^a, \Pi^b] = i\delta^{ab} \iff [Z^{a*}, \Pi^{b*}] = i\delta^{ab} ,
\]
otherwise commute .

The commutation relations introduce zero point energy in the Hamiltonian (8) but we are going to drop this by declaring the energy to be measured from the ground state of the \(A, B\) operators.

The authors of [13, 14], who have carried out similar analyses, diagonalize the matrix corresponding to the \(B\) operator at this point. However, this is not valid. A matrix \(M\) can be diagonalized by a unitary matrix if and only if \(M\) is normal, \(i.e., M\) satisfies \(MM^\dagger = M^\dagger M\). The operator \(B\) is not normal because of the commutation relations and hence one may not diagonalize it. The justification for their procedure is more subtle and will be mentioned in Subsection 3.5.

Assuming \(\mu > 0\), we see from (8) that the operators \(A^{a*}\) create the quanta of unit energy \(1 + \mu\), \(i.e.,\) the ones that we are calling anti-particles in the original relativistic theory and \(B^{a*}\) create particles of energy quanta \(1 - \mu\). However, recall that we have constructed the effective action that is valid below the scale \(1 = 1/R\). Therefore, our system here cannot afford to excite the anti-particles and the states of our theory are always at the ground state of the \(A\) operators. Thus, our effective theory is a low energy effective theory of the scale \(1 = 1/R\) that constrains the anti-particles to be at the ground state.

Obtaining the wavefunction is straightforward. From the definitions of \(A, B\) operators (9), we see that the ground state is,
\[
f_{0,0} = \prod_a \frac{1}{\sqrt{\pi}} \exp\left(-|Z^a|^2\right)
\]

Then by applying \(B^{a*}\) operators repeatedly, we obtain,
\[
f_{0,[n]}[Z^*] = \prod_a \left(\frac{(Z^{a*})^{n_a}}{\sqrt{\pi n_a!}}\right) \exp\left(-|Z^a|^2\right),
\]

where \([n]\) and \([Z]\) are the shorthands for \((n_1, \ldots, n_{N^2})\) and \((Z^1, \ldots, Z^{N^2})\), respectively. We note that the \(n_i\) are non-negative integers. We thus have,
\[
Hf_{0,[n]}[Z^*] = (1 - \mu) \left(\sum_a n_a\right) f_{0,[n]}[Z^*],
\]
and therefore this state evolves in time with the phase factor \((1 - \mu) \sum_a \eta_a\).

Note that the set, \(\{(Z^a)^{\eta_a}\}\), forms a basis of holomorphic functions with respect to \(Z^*\), so in general, any wavefunction of the theory can be written as,

\[
\psi[Z^*] = f[Z^*] \exp \left( -\sum_a |Z^a|^2 \right),
\]

where \(f[Z^*]\) is a holomorphic function. We will take full advantage of this fact in the formalism presented in the next subsection.

There is a problem with these solutions. Not all of them are satisfying the Gauss’ law and they contain unphysical solutions. We must project the Hilbert space to the physical subspace and we will deal with this issue in Subsection 3.3.

Before we move on to the next method of solving the theory, we would like to make comments about the special case with \(\mu = 1\). From the equation (7), the Hamiltonian can be written as,

\[
H = \text{tr}(\Pi^\dagger \Pi + Z^\dagger Z) - \mu i \text{tr}(\Pi^\dagger Z^\dagger - Z \Pi)
\]

\[
= h - \mu j,
\]

where we have defined,

\[
h := \text{tr}(\Pi^\dagger \Pi + Z^\dagger Z), \quad j := i \text{tr}(\Pi^\dagger Z^\dagger - Z \Pi).
\]

One notices that \(h\) is the Hamiltonian of harmonic oscillators and \(j\) is the charge number operator.

When we have \(\mu = 1\), as one can see from (8), any particle states (as opposed to the anti-particle states) have the same energy. By dropping this constant from the Hamiltonian, we have,

\[
H|\psi\rangle = (h - j)|\psi\rangle = 0,
\]

for any state of the effective theory, \(|\psi\rangle\), and thus the physical states of our effective theory with \(\mu \sim 1\) purely consist of the half-BPS states. Moreover, since \(j\) is a conserved charge, \(h\) and \(j\), or equivalently, \(H\) and \(j\) commute. Therefore, there are infinitely many degenerate states at the zero energy level and those states are labeled by the quantum number of \(j\). This is analogous to the system of the lowest Landau level and in the next subsection, we show that it is such a system.

### 3.2 The Lowest Landau Level Projected Schrödinger Equation

We solve the theory again, this time, by utilizing the lowest Landau level projection of Girvin and Jach [8]. The first key observation is that the Hamiltonian of the theory is the kinetic term that describes the Landau problem and a potential. This can
be seen by a simple rearrangement of the Hamiltonian resulting from the tree level Lagrangian (3).

From the real scalar expression of the Lagrangian, we have the canonical momenta,

\[ P_a^X = \frac{\partial L}{\partial \dot{X}^a} = \dot{X}^a - \mu Y^a \]
\[ P_a^Y = \frac{\partial L}{\partial \dot{Y}^a} = \dot{Y}^a + \mu X^a , \]

where we discarded the \( A_0 \) fields as in the previous subsection. The Legendre transformation leads to the Hamiltonian of the the form,

\[ H = P_a^X \dot{X}^a + P_a^Y \dot{Y}^a - L \]
\[ = \frac{1}{2} \text{tr} \left[ (P_X)^2 + (P_Y)^2 + 2\mu(P_X Y - P_Y X) + (X^2 + Y^2) \right] \]
\[ = \frac{1}{2} \text{tr} \left[ (P_X + \mu Y)^2 + (P_Y - \mu X)^2 + (1 - \mu^2)(X^2 + Y^2) \right] . \]

The field redefinition in the Lagrangian, \((X, Y) \rightarrow (X, Y)/\sqrt{c_1}\) with \(c_1 = 2\mu/(1 + \mu)^2\) and a trivial canonical transformation \(H \rightarrow c_2 H\) with \(c_2 = (1 + \mu)/2\mu\) can bring the Hamiltonian into the form,

\[ H = \frac{c_2}{2c_1} \text{tr} \left[ (P_X + c_1 \mu Y)^2 + (P_Y - c_1 \mu X)^2 \right] + \frac{1}{2}(1 - \mu) \text{tr}(X^2 + Y^2) \]
\[ = H_0 + V_0 , \]

where we have defined,

\[ H_0 := \frac{c_2}{2c_1} \text{tr} \left[ (P_X + c_1 \mu Y)^2 + (P_Y - c_1 \mu X)^2 \right] , \]
\[ V_0 := (1 - \mu) \text{tr}(Z^\dagger Z) . \] (10)

In above, we redefined \(Z^a := (1/\sqrt{2})(X^a + iY^a)\). The kinetic term of the Hamiltonian, \(H_0\), exactly is the Hamiltonian of the Landau problem\(^1\) and the Hamiltonian \(H_0\) gives

\(^1\)Recall that a non-relativistic electrically charged particle in a uniform magnetic field is described by the Hamiltonian,

\[ \tilde{H} = \frac{1}{2m} \left( \vec{p} + \frac{e}{c} \vec{A} \right)^2 . \]

When one chooses the symmetric gauge,

\[ \vec{A} = (yB/2, -xB/2, 0) , \]

which is related to the more common Landau gauge \(\vec{A} = (0, -xB, 0)\) by a gauge transformation, the Hamiltonian becomes,

\[ \hat{H} = \frac{1}{2m} \left[ (p_x + \frac{eB}{2c} y)^2 + (p_y - \frac{eB}{2c} x)^2 \right] . \]

The energy of this system is quantized in the units of the cyclotron frequency \(eB/mc\).
rise to the Landau level spacing of $1 + \mu$. As argued similarly in the previous subsection, any nonzero excitation of the Landau level goes beyond the scale of our effective theory. Therefore, our effective theory is described by the Hamiltonian $H$ projected down to the lowest Landau level. It is interesting to note that the projection to the ground state of the anti-particles corresponds to the lowest Landau level projection.

The solutions to the Schrödinger equation for $H_0$ is well-known (e.g., see [17]) and the eigenfunctions of the lowest Landau level are given by,

$$f_{0,[n]}[Z] := \prod_a \frac{(Z^a)^{n_a}}{\sqrt{\pi n_a!}} \exp \left( -|Z^a|^2 \right),$$

(11)

where $n_a$ are non-negative integers. This is the same solution as the one obtained in the previous subsection. However, this is the solution for the kinetic part, $H_0$, of the full Hamiltonian and the eigenvalues of $H_0$ for the lowest Landau level is zero. (More precisely, the kinetic energy in this case is a constant of motion and we set the constant to zero.) We must take into account of the potential $V_0$.

As mentioned in the previous subsection, the set $\{(Z^a)^{n_a}\}$ forms a basis of the space of holomorphic functions. Thus, any solutions of the Hamiltonian $H$ projected down to the lowest Landau level must have the form,

$$\psi[Z] = f[Z] \exp \left( -\sum_a |Z^a|^2 \right),$$

(12)

where $f[Z]$ is a holomorphic function of $[Z]$. Thus, all the wavefunctions of the lowest Landau level are holomorphic functions of $[Z]$ times the exponential factor. This is a powerful constraint on the form of wavefunctions and the systematic projection onto the lowest Landau level had been developed by Girvin and Jach [8].

We briefly summarize the prescription here. The important idea involved is to absorb the non-holomorphic exponential factor into the definition of the Hilbert space inner product. More specifically, if $f[Z]$ and $g[Z]$ are the holomorphic parts of wavefunctions, $\psi'$ and $\psi$, respectively, then the inner product of the Hilbert space is defined to be,

$$(f,g) := \int d\mu[Z] f[Z]^* g[Z],$$

(13)

where the integration measure $d\mu[Z]$ is given by,

$$d\mu[Z] := \prod_a \frac{1}{\pi} e^{-2|Z^a|^2} dX^a dY^a,$$

(14)

such that the inner product of the wavefunctions is,

$$\langle \psi' | \psi \rangle = (f,g).$$

In this way, our Hilbert space now is the space of holomorphic functions with the inner product defined as above.
The Hamiltonian of the system needs to be projected. Consider a Hamiltonian of the form,
\[ H[Z^*, Z] = H_0 + V[Z^*, Z], \]
where \( H_0 \) is the Landau problem kinetic term and \( V \) is a potential. In general, the factors of \( Z^{a*} \) in the potential bring the states out of our Hilbert space and this is what requires the projection. Girvin and Jach’s prescription is to “normal order” the \( Z^a \) in the potential to the left of \( Z^a \) and replace them as,
\[ Z^a \to \frac{\partial}{\partial Z^a}. \] (15)
We have the factor of 2 difference here from the Girvin and Jach and this is due to the factor of \( 1/\sqrt{2} \) in our definition of \( Z \). The projected potential, \( \hat{V} \), satisfies the Schrödinger equation for the states in the lowest Landau level,
\[ \hat{V} f[Z] = E f[Z]. \] (16)
Thus, in our case with (10), the projected potential is,
\[ \hat{V}_0 = (1 - \mu) \sum_a \frac{\partial}{\partial Z^a} Z^a = (1 - \mu) \left( N^2 + \sum_a Z^a \frac{\partial}{\partial Z^a} \right). \]
The first term contributes to the zero point energy of the system and as usual, we set that to zero. Hence the differential equation that we must solve is,
\[ (1 - \mu) \sum_a Z^a \frac{\partial}{\partial Z^a} f[Z] = E f[Z], \]
and one can readily see that the solution is,
\[ f_{[n]}[Z] = \prod_a \frac{(Z^a)^{n_a}}{\sqrt{n_a!}}, \]
with the eigenvalue \( E_{[n]} = (1 - \mu) \sum a n_a \). Recalling the fact that we absorbed the non-holomorphic exponential factor (and \( \pi \)) into the Hilbert space inner product, we in fact see that the eigenfunctions of \( H_0 \) are also the eigenfunctions of \( \hat{V}_0 \) with the eigenvalues \( E_{[n]} \) as above. This result is in agreement with the previous subsection. But again, the solutions contain the ones that do not satisfy the Gauss’ law. In the next subsection, we project the Hilbert space to the physical subspace.

3.3 Orthogonal Basis of \( U(N) \) Invariant States: Schur Polynomials

As described before, the solutions to the lowest Landau level projected Schrödinger equation are holomorphic functions of \( Z \) and therefore, the Hilbert space of the theory
is the space of holomorphic functions with the inner product defined as (13). However, this Hilbert space is too large and contains unphysical states. Recall that we discarded the equations of motion for $A_0$ in the previous subsections by promising that we choose the states that satisfy the Gauss’ law as our physical states. Therefore, the general holomorphic function is not the physical solution to the theory at hand. The physical wavefunctions are the holomorphic functions of $Z$ that is invariant under $U(N)$ conjugation. A convenient basis for such holomorphic functions is the Schur polynomials and the use of this basis was suggested by Corley et al. [13]. We are going to adopt this basis in our analysis and will see the advantage of this.

The idea of the basis starts with the observation that general half-BPS operators of a fixed $R$-charge, $n$, have the form,

$$\{\text{tr}(Z^{l_1})\}^{k_1}\cdots\{\text{tr}(Z^{l_m})\}^{k_m},$$

where the integers $l_i$ and $k_i$ satisfy,

$$n = \sum_{i=1}^{m} l_i k_i.$$

Schur polynomials for the complex matrix are linear combinations of such polynomials and they are given by,

$$\chi_{n,R}(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \text{tr}(\sigma Z),$$

where $S_n$ is the symmetric group, $R$ is its representation and $\chi_R(\sigma)$ is the character of $\sigma$ in $S_n$. The last factor can be explicitly written as,

$$\text{tr}(\sigma Z) = \sum_{i_1, \ldots, i_n} Z_{i_1\sigma(1)}^{i_1} \cdots Z_{i_n\sigma(n)}^{i_n},$$

where we packaged the $N^2$ complex degrees of freedom $Z^a$ into a $N \times N$ complex matrix, that is, each degree of freedom now is given by $Z_{ij}$ which is defined by $Z_{ij} = \sum_a Z^a(T^a)_{ij}$. For example, when $n = 2$, there are symmetric and antisymmetric representations for the symmetric group and the Schur polynomials in this case are,

$$\chi_{2,S}(Z) = \frac{1}{2}\{(\text{tr } Z)^2 + \text{tr}(Z^2)\}, \quad \chi_{2,A}(Z) = \frac{1}{2}\{(\text{tr } Z)^2 - \text{tr}(Z^2)\}. \quad (17)$$

The virtue of this basis is that they are orthogonal with respect to the inner product defined in (13),

$$(\chi_{n_1,R_1}(Z), \chi_{n_2,R_2}(Z)) \sim \delta_{n_1,n_2} \delta_{R_1,R_2},$$

and spans the space of $U(N)$ invariant functions of the complex matrix $Z$. The proof of Corley et al. [13] for the orthogonality is valid here with the only modification
being the expectation value of the operators is replaced by the inner product \((13)\). The modification does not change the proof by noting that we have,
\[
((Z_{ij})^m, (Z_{kl})^n) = m! \delta_{i,k} \delta_{j,l} \delta^{m,n}.
\]

It is important to note that the basic building block of the Schur polynomials \(\{\text{tr}(Z^m)\}^n\) can be reduced to a polynomial of the eigenvalues. In fact, given a complex matrix \(Z\), there exists a unitary matrix \(U\) such that,
\[
Z = UWU^\dagger, \tag{18}
\]
where \(W\) is a triangular complex matrix, i.e., we have \(W_{ij} = 0\) for all \(i > j\) and the diagonal elements \(z_i := W_{ii}\) are the eigenvalues. Thus we have,
\[
\{\text{tr}(Z^m)\}^n = \left\{\text{tr}(W^m)\right\}^n = \left(\sum_{i=1}^N z_i^m\right)^n,
\]
as claimed. This shows that the physical wavefunctions are functions only of \([z] := (z_1, \ldots, z_N)\) and do not involve the off-diagonal elements of the matrix \(W\).

Let us consider the eigenvalues of the Schur polynomials with respect to the potential \(V_0\) as in (10). Since we have,
\[
\text{tr}(Z^\dagger Z) = \sum_i |z_i|^2 + \sum_{l < m} |W_{lm}|^2,
\]
the projected potential is,
\[
\hat{V}_0 = (1 - \mu) \left(\sum_i z_i \frac{\partial}{\partial z_i} + \sum_{l < m} W_{lm} \frac{\partial}{\partial W_{lm}}\right),
\]
where we again dropped the constant. For a polynomial of the form,
\[
\left(\sum_i z_i^{l_1}\right)^{k_1} \cdots \left(\sum_i z_i^{l_m}\right)^{k_m},
\]
where the integers \(l_i\) and \(k_i\) satisfy,
\[
n = \sum_{i=1}^m l_i k_i,
\]
it is not hard to see that this is an eigenfunction of the operator \(\sum_i z_i (\partial/\partial z_i)\) with the eigenvalue \(n\). Thus, by construction, the Schur polynomial \(\chi_{n,R}[z]\) also is an eigenfunction of the operator with the same eigenvalue. Hence we have,
\[
\hat{V}_0 \chi_{n,R}[z] = (1 - \mu) n \chi_{n,R}[z]. \tag{19}
\]
This is the physical solution to the lowest Landau level projected Schrödinger equation. But the story is not complete. The triangulation of the matrix \(Z\) in (18) causes the change in the inner product measure \(d\mu[Z]\) of (13) and we discuss the change of the measure in the next subsection.
3.4 Change of the Measure and the Solution

In this subsection, we reduce the complex matrix quantum mechanics into the quantum mechanics of the eigenvalues. This procedure is described in Chapter 15 and Appendix 33 of the book by Mehta [16] and also reproduced in the paper by Takayama and Tsuchiya [15].

We first rewrite the inner product measure of the Hilbert space (14) as,

\[ d\mu[Z] = \prod_{i,j} d\text{Re} \, Z_{ij} d\text{Im} \, Z_{ij} \exp \left( -2 \text{tr}(Z^\dagger Z) \right) . \]

We dropped the irrelevant overall constant that comes from the Jacobian associated with the change of the variables from \( X^a \) and \( Y^a \) to \( \text{Re} \, Z_{ij} \) and \( \text{Im} \, Z_{ij} \).

Now, under the change of the variables (18), a straightforward computation for the variation of \( Z \) shows that the measure takes the form,

\[ d\mu[Z] = \prod_{i<j} |z_i - z_j|^2 \prod_k dz_k d\text{z}_k^* \prod_{l<m} dW_{lm} dW_{lm}^* dM_{lm} dM_{lm}^* \]

\[ \times \exp \left( -2 \sum_i |z_i|^2 - 2 \sum_{j<k} |W_{jk}|^2 \right) , \quad (20) \]

where \( dM \) is the Hermitian matrix defined by \( dM = -iU^\dagger dU \) with the constraints \( dM_{ii} = 0 \). The latter constraints are possible due to the fact that the unitary matrix \( U \) that triangulates \( Z \) is not unique.

Since our potential and wavefunctions do not involve \( M \), these degrees of freedom integrate to a constant. Also in the previous subsection, we saw that \( W \) sector of the potential had vanishing matrix element due to the fact that the wavefunctions are independent of \( W \). Therefore, the factors of \( M \) and \( W \) may be dropped from the measure. We will further elaborate on this point in the next subsection.

Note that the first piece in the measure is the conjugate squared Van der Monde determinant \( \Delta := \prod_{i<j} (z_i - z_j) \). It is customary to absorb this factor into a wavefunction of the theory by,

\[ \psi_F[z] := \Delta \psi[z] , \quad (21) \]

and at the same time, modify the Hamiltonian by,

\[ H_F := \Delta H \frac{1}{\Delta} . \quad (22) \]

Under these redefinitions, the Van der Monde determinant factor in the measure (20) must be omitted. By dropping the factors of \( M \) and \( W \), we now have the reduced measure,

\[ d\hat{\mu}[z] = \prod_i \frac{1}{\pi} dz_i d\text{z}_i^* \exp \left( -2 |z_i|^2 \right) , \quad (23) \]

where we reinserted the normalization factor of \( \pi \). Since \( \Delta \) is anti-symmetric under the exchange of any \( z_i \), the wavefunction \( \psi_F \) is fermionic, hence our theory boils down
to the system of identical fermions and this is analogous to the system of electrons under a strong uniform magnetic field.

Given the redefinition (21) and the solution of the previous subsection (19), we now have the final time-dependent physical solution,

$$\psi_F[z,t] = \sum_{n,R} C_{n,R} e^{-i(1-\mu)nt} \Delta \chi_{n,R}[z],$$

where $C_{n,R}$ are constants and the inner product of the Hilbert space is given by the measure $d\hat{\mu}[z]$ of (23).

We see from the solution that the degenerate states of the lowest Landau level in $\mu = 1$ case is lifted for $\mu \neq 1$. The ground state is the lowest filling lowest Landau level where the $N$ states are filled from the state with the lowest angular momentum to the higher without any gap and the excited states are the state filling with gaps. The energies of the states are proportional to the $R$-charges, $n$, and the states are responding to the chemical potential, which is lower than the mass of the scalars, according to the size of the $R$-charges that they have.

However, the degeneracy is not completely lifted. At the energy level $n$, the number of degenerate states is the number of $S_n$ irreducible representations, that is, the number of Young diagrams for $n$ boxes. This is also the number of the ways to partition $n$ and as $n$ becomes large, this number grows exponentially. This is the Hagedorn behavior. We note that $n$ may not be too large that the energy goes beyond the scale of the effective theory, namely, we must have the condition,

$$n < \frac{1}{1-\mu}. \quad (24)$$

Thus, in this effective theory, such Hagedorn behavior occurs only when $N$ is large and the chemical potential is slightly below the mass of the scalars.

We also note that when $\mu \neq 1$, the states acquire relative phase differences along with the time evolution. This is not the case for $\mu = 1$ where all the states evolve with the same phase factor and they do not have relative phases.

### 3.5 The Tree Level Interaction Term

Recall that the measure (20) contains the factors of $M$ and $W$, and we argued that these factors may be dropped. In this subsection, we elaborate on this point and show that the quartic interaction in (5) does not play a role at the tree level.

First, observe that any potential that is invariant under $U(N)$ conjugation does not contain the factor of $M$ because the triangulation matrix $U$ in (18) must drop out. Moreover, the physical wavefunctions do not contain the factor of $M$ either, so in general, this factor integrates to a constant.

Second, again, since the wavefunctions do not contain the factors of $W_{lm}$, non-vanishing matrix elements of the potential that contain $W_{lm}$ must have the form $|W_{lm}|^{2k}$ where $k$ is an integer. However, this form of the potential does not occur under
the lowest Landau level projection. The prescription for the projection instructs us to replace \( W_{lm}^* \) to the derivatives \( \partial/\partial W_{lm} \), and therefore, apart from the zero point energy, the matrix elements of the potential that contain only \( W \) are always zero with respect to the basis discussed in Subsection 3.3. This means that we may begin with the potential whose \( W \) factors, that are not mixing with \( z_i \), are all set to zero. In this very subtle sense, the incorrect diagonalization of \( B \) mentioned in Subsection 3.1 is “valid.”

When the elements of \( W \) are mixing with \( z \), the situation is different. For example, the term of the form \( |W|^2 |z|^{2n} \) would give the contribution \( (N(N-1)/2)^2 |z|^{2n} \). The quartic interaction in (5) indeed contains the \( W-z \) mixing components. However, we are going to show that they do not contribute to the matrix elements of the Hamiltonian. The quartic term can be written as,

\[
\frac{1}{2} \text{tr}([Z, Z^\dagger]^2) = \frac{1}{2} \text{tr}([W, W^\dagger]^2) = W_{ji}^* W_{lk}^* W_{jk} W_{li} - W_{ji}^* W_{kj}^* W_{kl} W_{li},
\]

where we “normal ordered” the \( W^* \) according to the lowest Landau level projection prescription. After replacing the \( W^* \) to the derivatives \( \partial/\partial W \), one may try to bring the derivatives toward right. In this process, each term gives a constant, the terms of the form \( W(\partial/\partial W) \) and the terms with the form \( WW(\partial^2/\partial W^2) \) where the indices are assumed to be contracted properly. The constants from the two terms cancel. Now, because the wavefunctions are functions only of \( W_{ii} = z_i \), the terms of the form \( W(\partial/\partial W) \) in the both terms only have non-vanishing elements with \( W_{ii}(\partial/\partial W_{ii}) \) and hence they cancel. Similar argument applies to the terms of the form \( WW(\partial^2/\partial W^2) \). Thus, at the tree level, the interaction does not play a role and the absence of it implies that the states of the theory do not mix.

One might have expected this result from the usual behavior of the half-BPS states. However, it is nice to see that this works in terms of the lowest Landau level projection. And we point out that the nonrenormalization theorem is not valid for the effective theory with the chemical potential. In fact, we have the one-loop corrections to the effective potential as in (6). In the next subsection, we are going to discuss the perturbation theory of this one-loop potential.

### 3.6 The One-Loop Perturbation

We are going to treat the one-loop potential obtained in (6) as a perturbation. We remind the reader that the one-loop correction to the effective action was obtained by restricting the fields \( X, Y \) to take the values only in the flat directions. This is rather a special case, however, we will see the general behavior of the loop corrections from this case.

The lowest Landau level projected form of the potential is,

\[
\hat{V}_1 = K g^2 \sum_{m<n} \left( \frac{\partial}{\partial z_m} - \frac{\partial}{\partial z_n} \right) (z_m - z_n).
\]
First, note that the perturbation does not change the order of polynomials on which it acts, implying that the potential does not have non-vanishing matrix elements that connect states with different charges. This is a general statement for any corrections to the effective action because of the gauge invariance. This is consistent with the fact that the $R$-charge is a conserved quantity.

However, we expect the perturbation to mix the states with the same $R$-charge and lifts the degeneracy that we observed in the end of Subsection 3.4. To illustrate this, we consider a concrete system with the number of the color $N = 3$, and the states with $R$-charge $n = 2$. The states are given in (17) for $n = 2$ and they can be explicitly written as,

$$\chi_{2,S} = \frac{1}{24} \left\{ (\sum_{i=1}^{3} z_i)^2 + \sum_{i=1}^{3} z_i^2 \right\}, \quad \chi_{2,A} = \frac{1}{12\sqrt{2}} \left\{ (\sum_{i=1}^{3} z_i)^2 - \sum_{i=1}^{3} z_i^2 \right\},$$

where we normalized the states with respect to $|\Delta|^2 d\hat{\mu}[z]$ which was discussed in Subsection 3.4. Then, we have,

$$\hat{V}_1 \chi_{2,S} = Kg^2 (8\chi_{2,S} - 2\sqrt{2}\chi_{2,A}) ,$$
$$\hat{V}_1 \chi_{2,A} = Kg^2 (-2\sqrt{2}\chi_{2,S} + 10\chi_{2,A}) ,$$

and in the basis of $(\chi_{2,S}, \chi_{2,A})^T$, we have the matrix expression,

$$(\chi_{2,R'}, \hat{V}_1 \chi_{2,R}) = Kg^2 \begin{pmatrix} 8 & -2\sqrt{2} \\ -2\sqrt{2} & 10 \end{pmatrix}.$$ 

Given this, one can solve the degenerate perturbation theory up to $g^2$. The energy eigenstates and their energies are,

$$\frac{1}{\sqrt{3}}(\sqrt{2}\chi_{2,S} + \chi_{2,A}) , \quad E = 2(1 - \mu) + 6Kg^2 ,$$
$$\frac{1}{\sqrt{3}}(-\chi_{2,S} + \sqrt{2}\chi_{2,A}) , \quad E = 2(1 - \mu) + 12Kg^2 .$$

We see here the mixing of the states and the splitting in the energy level.

Even though we worked out a simple example, we should expect the mixing and the energy level splitting for any corrections to the effective action. We would like to emphasize that we do not know how the theory behaves at the strong coupling region.

4 AdS/CFT Interpretation/Speculation

As described in Subsection 3.1 at the value of the chemical potential $\mu \sim 1$, the states of the effective theory are the half-BPS states, $|\psi\rangle$, satisfying $(h - j)|\psi\rangle = 0$. In AdS/CFT correspondence, those states are conjectured to be the various gravitons.
in the dual gravity theory. When the $R$-charge is small compared to $N$, those states correspond to the Kaluza-Klein gravity modes propagating in the bulk. When the $R$-charge is comparable to the large number of the color $N$, the Schur polynomial of totally symmetric representation corresponds to the giant graviton in AdS bulk while totally antisymmetric case corresponds to the giant graviton in $S^5$. (See the papers by Balasubramanian et al. [18] and Corley et al. [13].) And as mentioned in Introduction, all the geometries that correspond to the half-BPS states of $\mathcal{N} = 4$ SYM are constructed by LLM in [5]. In the effective theory that we have considered, the conjectured dual of those gravity systems are all degenerate at $\mu = 1$ and they evolve in time without phase differences.

Now let us discuss the case with $\mu \sim 1$ but $\mu < 1$. Even though our analysis has been carried out for general $N$, in order for the gravity calculation to be valid, we must take the large $N$ limit. Since the giant graviton states correspond to the particular representations of $S_n$ when $n$ is comparable to $N$, the condition (24) tells us that this picture is valid only when the chemical potential is slightly lower than the mass of the scalars. More specifically, we must have $N \lesssim 1/(1 - \mu)$. Put it another way, the excitations that correspond to the giant gravitons cost too much in the low energy effective theory and they are not excited. The gravitons with $R$-charges that satisfy (24) are allowed to appear and they respond to the chemical potential proportional to their charges as expected.

Note that the ground state is the lowest filling lowest Landau level, and according to LLM [5], this state is conjectured to be the geometry, AdS$_5 \times S^5$. Thus, when the chemical potential is lower than the mass of the scalars, AdS$_5 \times S^5$ geometry acquires a special status as the ground state of quantum mechanically superposed other geometries. However, because of the $R$-charge conservation, the theory does not have non-vanishing matrix elements that connect the states with different $R$-charges and in this sense, the ground state is not very special.

So far in this section, we have been discussing the correspondence between the field theory and the gravity pictures as if the identification is proved. The author, however, feels that this is too naive especially for the theory with the chemical potential. In our effective theory, the half-BPS states are not protected against the change in the coupling constant, and as we mentioned at the end of Subsection 3.6, we do not know how the theory behaves at the strong coupling region where the gravity picture is supposed to be valid. The discussions above should be treated as speculations rather than interpretations.

In the work of LLM [5], the geometry is obtained by specifying the boundary condition of a particular function on a two dimensional plane in ten dimensions. It is conjectured that this two dimensional plane corresponds to the phase space of the free fermions that arises from the reduction of the SYM on $\mathbb{R} \times S^3$ onto $\mathbb{R}$ (c.f. Berenstein's work [4]). Notice that the phase space coordinates do not commute quantum mechanically and the question was raised; in what sense the two coordinates become non-commutative? As already noted by Berenstein in [4], this non-commutativity seems to arise from the familiar feature of the lowest Landau level. In our analysis above, the variable $z^*$ is replaced by the derivative $\partial/\partial z$ under the lowest Landau level.
projection \[\text{[15]}\] and hence the commuting variables \(z, z^*\) become non-commutative. In fact, we have \(\left[ z, -i\partial/\partial z \right] = i\). Let us rewrite this in terms of the real variables defined by \(x_1 = (1/i\sqrt{2})(z - z^*)\) and \(x_2 = (1/\sqrt{2})(z + z^*)\). Upon the lowest Landau level projection, these variables become \(\hat{x}_1 = (1/i\sqrt{2}) (z - \partial/\partial z)\) and \(\hat{x}_2 = (1/\sqrt{2}) (z + \partial/\partial z)\), and they satisfy the commutation relation \(\left[ \hat{x}_1, \hat{x}_2 \right] = i\). This appears to be the relation of conjugate variables that form two dimensional phase space.

In the decoupling limit of Maldacena \[\Pi\], the low energy field theory living on a stack of \(N\) D3-branes in Type IIB string theory is \(\mathcal{N} = 4\) SYM and the transverse coordinates of the D3-branes are the scalar fields of the SYM. If we literally carry over this identification to our SYM on \(\mathbb{R} \times S^3\), then the eigenvalues \(z_i\) are the coordinates of the D3-branes. Thus, projecting the (low energy effective) theory down to the lowest Landau level, the positions of the D3-branes no longer commute, again, if we boldly carry over the analogy. Also this seems to suggest that the phase space distribution (known as the droplet configuration) of LLM is the distribution of the D3-branes. In fact, the total area of the phase space configuration is calculated to be \(N\) by LLM. However, the author of the present paper is not aware of such identification and the proof.

The analysis carried out in the present work has another aspect in AdS/CFT correspondence. In \[7\], the effective theory of the finite temperature free SYM on \(S^1 \times S^3\) with the \(R\)-symmetry chemical potentials has been considered. There, all the massive modes, including the scalars with chemical potentials, are integrated out and the effective theory becomes the matrix model of the constant \(A_0\) mode that implements the Gauss’ law constraint. The partition function is derived for the theory and a phase structure of the theory in the grand canonical ensemble has been found. The schematic phase diagram of the theory is shown in Figure \[\Pi\]. As indicated in the diagram, the phases are named as “confining and deconfining” phases. This is because the Polyakov loop and the large \(N\) scaling behaviors are the same as the ones observed in the flat space gauge theories. (It still is too naive to directly identify those phases to the confining/deconfining phases. For example, the real QCD does not confine at zero coupling.)
The AdS/CFT correspondence conjectures that the confining and deconfining phases are dual to the AdS space with string gas and AdS space with a black hole, respectively [3, 19]. When one turns on the chemical potential, the conjecture reads that the confining phase as the dual of the AdS space with charged gas and the deconfining phase as the Reissner-Nordström-AdS black hole. Now recall that we have worked out zero temperature effective field theory with the chemical potential less than or equal to 1. If the states that we obtained in the field theory side correspond to the geometries of LLM, then they are smooth and do not have horizons. When the back reactions of the gravitons can be ignored, which we think is plausible when the $R$-charge is small compared to $N$, those geometries are gravitons in AdS space. This seems to resemble the conjectured gravity picture of the confining phase in the field theory side and fits to the zero temperature axis of Figure 1 below the chemical potential 1.

5 Outlook

We have carried out the analysis for the effective action at the tree level and the one-loop level with the commuting fields. We also turned on one out of three possible chemical potentials. It is natural to extend the analysis to two or three nonzero chemical potentials and also construct the general higher loop corrections to the effective action.

For the case with more than one nonzero chemical potentials, we have more than one complex fields in the effective theory and one may not be able to diagonalize or triangulate those matrices simultaneously. We expect the theory to become very complicated to analyze. If we assume that the half-BPS states do not receive corrections from the interactions as in the flat space case, it is plausible to consider the theory of the commuting matrices so that the quartic interactions do not contribute. There are arguments that at the classical level, the matrices must be commuting and the works on such commuting matrices have been carried out by Berenstein et al. [11, 12]. Some investigations need to be done to see how this scenario works in the effective theory with chemical potentials.

The higher loop calculations are worth doing because the loop corrections to the effective action is crucial to answer the question; what happens to the theory when $\mu > 1$? We could not investigate this issue in the present work, because the one-loop computation was limited to the commuting fields. The theory may go through a phase transition as found at the high temperature end [7]. We should expect that the phase structure of a theory to be robust against the change in the coupling, so an analysis at the weak coupling has more hope of capturing the dynamics at the strong coupling region. This is currently under investigation.

We have considered the theory on $\mathbb{R} \times S^3$ and this is a zero temperature field theory on sphere. As mentioned at the end of the previous section, there is a finite temperature version of the theory on $S^1 \times S^3$ where the circumference of $S^1$ is the inverse of the temperature. Witten pointed out in [19] that the conformal symmetry
requires the parameter of the theory be the ratio of the two radii, $TR$, where $T$ is the temperature and $R$ is the radius of $S^3$. Thus, the low and high temperature limits are equivalent to the theory on a small sphere and on a nearly flat three space, respectively. The high temperature end of the theory with chemical potentials has been carried out in [7]. Thus the natural next step is to consider the low temperature limit of the theory. This is also under investigation.

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A The One-Loop Effective Potential

In this appendix, we derive the one-loop effective potential (6) of the theory. We restrict ourselves to the the special case where the matrix valued fields of interest are taking values in the flat directions and they are diagonal. We adopt the background field method. The $X,Y$ fields of the theory in the Lagrangian (3) are shifted to $x + X$, $y + Y$ where $X,Y$ are now the diagonal constant backgrounds and $x,y$ are the arbitrary fluctuations around them. We denote the diagonal components of $X,Y$ by $X_n,Y_n$. The shift causes the quadratic terms that mix the gauge fields and the scalars. To get rid of the mixed terms as much as possible, we employ the $R_\xi$-gauge,

$$\frac{1}{2\xi} \{\partial_\mu A_\mu - i\xi g ([x,X] + [y,Y])\}^2 ,$$

with $\xi = 1$.

The quadratic terms in the shifted (Euclidean) Lagrangian with the above choice of the gauge are,

$$\sum_{m<n} \left[ (A_\iota)_{mn}^* (-\partial^2 + g^2 M_{mn}^2) (A_\iota)_{mn} 
+ (\Phi_A)_{mn}^* (-\partial^2 + 1 + g^2 M_{mn}^2) (\Phi_A)_{mn} 
+ ((A_0)_{mn}^*, x_{mn}^*, y_{mn}^*) \mathcal{M} \begin{pmatrix} (A_0)_{mn} \\ x_{mn} \\ y_{mn} \end{pmatrix} 
+ (\bar{\psi}_i)_{mn}^* \left\{ \delta_{ij} (\bar{\theta} + \bar{M}_i) + ig (\alpha_{ij} X_{mn} + i\beta_{ij} \gamma_5 Y_{mn}) \right\} (\psi_j)_{mn} \right] , \tag{25}$$

where $m,n$ are the color indices, the index $\iota$ runs from 1 to 3 for the spatial components of the gauge fields, the index $A$ on the scalar fields $\Phi$ runs from 3 to 6 excluding the $x,y$
scalars, and we have defined $X_{mn} := X_m - X_n$, $Y_{mn} := Y_m - Y_n$, $M^2_{mn} := X^2_{mn} + Y^2_{mn}$ and a matrix,

$$
\mathcal{M} := \begin{pmatrix}
-\partial^2 + g^2 M^2_{mn} & -2\mu g Y_{mn} & 2\mu g X_{mn} \\
2\mu g Y_{mn} & -\partial^2 + 1 - \mu^2 + g^2 M^2_{mn} & 2i\mu \partial_0 \\
-2\mu g X_{mn} & -2i\mu \partial_0 & -\partial^2 + 1 - \mu^2 + g^2 M^2_{mn}
\end{pmatrix}.
$$

For the explicit computation of the fermion sector, one may use the matrices, $\mathcal{M}$.

The zeroth component tors are resulting from the functional determinants of $x, y$ taken over the momentum and the color space. The primed trace operator implies that the zero spatial momentum is denoted by $0$ and $\mu$ is non-negative integers. The eigenvalue $\Delta^2$ is shown in the table and they are assumed in the trace operators.

$$
\alpha^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix},
\beta^1 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \beta^2 = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix},
$$

that satisfy the relations (4).

Now the logarithms of the functional determinants that result from (25) are,

$$
-3 \text{tr} \ln(\omega_0^2 + \Delta^2_g + g^2 M^2_{mn}) - 4 \text{tr} \ln(\omega_0^2 + \Delta^2_s + g^2 M^2_{mn})
- \text{tr} \ln(\omega_0^2 + \Delta^2_{A_0} + g^2 M^2_{mn})
- \text{tr}' \ln \left\{ \omega_0^2 + (\sqrt{\Delta^2_g + g^2 M^2_{mn}} - \mu)^2 \right\}
+ 4 \text{tr} \ln \left\{ \omega_0^2 + (\sqrt{\Delta^2_f + g^2 M^2_{mn}} - \mu/2)^2 \right\}
- \text{tr} \ln \left[ 1 + \frac{4\mu^2 g^2 M^2_{mn}(\omega_0^2 + \Delta^2_s - \mu^2 + g^2 M^2_{mn})}{(\omega_0^2 + \Delta^2_{A_0} + g^2 M^2_{mn})\{\omega_0^2 + (\sqrt{\Delta^2_s + g^2 M^2_{mn}} - \mu)^2\}\{\omega_0^2 + (\sqrt{\Delta^2_s + g^2 M^2_{mn}} + \mu)^2\}} \right].
$$

Let us define all the quantities appearing in the above expression. The traces are taken over the momentum and the color space. The primed trace operator implies that the zero spatial momentum is excluded in the trace. This is because the operators are resulting from the functional determinants of $x, y$ fields and we are specifically excluding the zero modes of those fields from the integration. The zeroth component of the momentum is denoted by $\omega_0$ and $\Delta^2$’s denote the eigenvalues of the Laplacians for the $S^3$ spherical harmonics. The specific values of $\Delta^2$’s are shown in the table below.

| Fields | Symbol | Eigenvalue | Degeneracy |
|--------|--------|------------|------------|
| $A_0$  | $\Delta^2_g$ | $(h + 1)^2$ | $h(h + 2)$ |
| $\Phi_A$ | $\Delta^2_s$ | $(h + 1)^2$ | $h + 1)$ |
| $A_0$  | $\Delta^2_{A_0}$ | $h(h + 2)$ | $h + 1)$ |
| $x, y$ | $\Delta^2_s$ | $(h + 1)^2$ | $h + 1)$ |
| $\psi_i$ | $\Delta^2_f$ | $(h + 1/2)^2$ | $h(h + 1)$ |

The $h$ are non-negative integers. The eigenvalue $\Delta^2_s$ includes the mass factor, i.e., $\Delta^2_s = \omega_0^2 + \Delta^2_s = (h + 1)^2$. Since the $A_0$-scalar does not have mass, the eigenvalue is $h(h + 2)$. At each value of $h$, the spherical harmonics are degenerate and the degeneracy factors are shown in the table and they are assumed in the trace operators.
Before we proceed to compute the momentum integrals, we comment on the ghost contributions. The spatial components of the gauge fields can be decomposed into the gradient of a scalar and a divergenceless vector. The ghosts, which are the scalars, cancel the former piece in the decomposition and the scalar $A_0$ contributions.

Now we are going to integrate the momenta. Consider the generic expression,

$$ I = \int_{-\infty}^{\infty} \frac{d\omega_0}{2\pi} \ln(\omega_0^2 + M^2), $$

where we assume $M > 0$. Then,

$$ \frac{\partial I}{\partial M^2} = \int_{-\infty}^{\infty} \frac{d\omega_0}{2\pi} \frac{1}{\omega_0^2 + M^2} = \frac{1}{2\sqrt{M^2}}. $$

Integrating back $M^2$, we obtain $I = M$. This result tells us that the chemical potentials appearing in the third and fourth lines of (26) cancel and this is interpreted as the cancellation between the particle and anti-particle pairs. Thus, except for the last line of (26), the computations are similar and we demonstrate the explicit computation for the first term in (26).

We have,

$$ - \text{tr} \ln(\omega_0^2 + \Delta_g^2 + g^2M_{mn}^2) = - \text{tr} \sqrt{\Delta_g^2 + g^2M_{mn}^2} $$

$$ = - \text{tr} \left( \Delta_g + \frac{g^2M_{mn}^2}{2\Delta_g} + \cdots \right). $$

In the first equality, the $\omega_0$ integration has been done and in the second equality, the square root has been expanded with respect to $g^2M_{mn}^2$. The first term in the last line gives the Casimir energy of the $S^3$ and we are going to neglect this. We also concentrate on the term of $g^2M_{mn}^2$ and drop the higher order terms. The second term in the last line is divergent and we adopt the dimensional regularization which boils down to the usual $\zeta$-function regularization. However, as we will see, the dimensional regularization has the advantage that it nicely picks up the divergent $1/\epsilon$ pole. The idea is to extend the spatial $S^3$ manifold to $S^3 \times \mathbb{R}^\epsilon$ and use the following replacements,

$$ \sum_h \rightarrow \sum_h \left( \frac{e^{\gamma_E}\Lambda^2}{4\pi} \right)^\epsilon \int \frac{d^{-2\epsilon}p}{(2\pi)^\epsilon}, $$

$$ \Delta_g \rightarrow \sqrt{p^2 + (h+1)^2}, $$

where we adopted the $\overline{MS}$-scheme and $\gamma_E$ is the Euler’s constant and $\Lambda$ is an arbitrary renormalization scale measured in the units of $1/R$. 

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The regularization gives,
\[
\text{tr} \frac{1}{\Delta_g} = \frac{1}{2\pi^2} \sum_{h=0}^{\infty} h(h+2) \left( e^{\gamma_E} \Lambda^2 \right)^\epsilon \int \frac{d^{-2\epsilon}p}{(2\pi)^\epsilon} \frac{1}{\sqrt{p^2 + (h+1)^2}}
\]
\[
= \frac{1}{2\pi^2} \sum_{h=0}^{\infty} h(h+2) \left( e^{\gamma_E} \Lambda^2 \right)^\epsilon \frac{\Gamma(\epsilon + 1/2)}{\Gamma(1/2)} \frac{1}{(h+1)^{1+2\epsilon}}
\]
\[
= \frac{1}{2\pi^2} \left( e^{\gamma_E} \Lambda^2 \right)^\epsilon \frac{\Gamma(\epsilon + 1/2)}{\Gamma(1/2)} \{ \zeta(-1 + 2\epsilon, 1) - \zeta(1 + 2\epsilon, 1) \}
\]
\[
= \frac{1}{2\pi^2} \left\{ -\frac{1}{2\epsilon} - \frac{1}{12} - \gamma_E + \ln 2 - \frac{1}{2} \ln \Lambda^2 + O(\epsilon) \right\},
\]
where we have the generalized \( \zeta \)-function defined by,
\[
\zeta(x, y) := \sum_{h=0}^{\infty} \frac{1}{(h + y)^x},
\]
with \( y \neq 0 \). Other terms in (26) except the last line can be evaluated similarly and they give the contribution,
\[
-\frac{1}{\pi^2} g^2 \left( \sum_{m<n} M_{mn}^2 \right) \left( -\frac{3}{4} + \ln 2 \right).
\]  
(27)

We remind the reader that one of the spatial gauge field contributions and \( A_0 \) contribution have been canceled by the ghosts and the primed trace appearing in (26) is evaluated from \( h = 1 \), excluding the light mode of the \( x, y \) fields. In the expression (27), the \( 1/\epsilon \) poles and the renormalization scheme factors (such as \( \gamma_E \) and \( \Lambda \)) canceled among the bosonic and fermionic contributions.

We are now going to evaluate the last line in (26). We expand the logarithm with respect to \( g^2 M_{mn}^2 \) and calculate the first term in the expansion. The \( \omega_0 \) integral can be done and yields,
\[
-\text{tr} 4\mu^2 g^2 M_{mn}^2 \left( \frac{\Delta_{A_0} + \Delta_s}{2\Delta_{A_0}\Delta_s (\Delta_{A_0} + \Delta_s - \mu)(\Delta_{A_0} + \Delta_s + \mu)} \right).
\]

The argument of the parenthesis is logarithmically divergent and asymptotically approaches to \( 1/4\Delta_s \) as \( h \to \infty \). What we are going to do is to subtract \( 1/4\Delta_s \) from the above expression and add the same factor \( 1/4\Delta_s \). The above expression subtracted by \( 1/4\Delta_s \) gives a finite sum and we dimensionally regulate the added \( 1/4\Delta_s \). So,
\[
-4\mu^2 g^2 \left( \sum_{m<n} M_{mn}^2 \right) \frac{1}{2\pi^2} \sum_{h=1}^{\infty} \left\{ (h+1)^2 \frac{\Delta_{A_0} + \Delta_s}{2\Delta_{A_0}\Delta_s (\Delta_{A_0} + \Delta_s - \mu)(\Delta_{A_0} + \Delta_s + \mu)} - \frac{1}{4\Delta_s} \right\}
\]
\[
+ \left( e^{\gamma_E} \Lambda^2 \right)^\epsilon \int \frac{d^{-2\epsilon}p}{(2\pi)^\epsilon} \frac{1}{4\sqrt{p^2 + (h+1)^2}}
\]
\[
= -\frac{1}{\pi^2} g^2 \left( \sum_{m<n} M_{mn}^2 \right) \left\{ C(\mu) + \frac{\mu^2}{4\epsilon} + \frac{\mu^2}{4} \left( -2 + 2\gamma_E - 2 \ln 2 + \ln \Lambda^2 \right) \right\},
\]  
(28)
where we have defined a monotonically increasing function $C(\mu)$ as,

$$C(\mu) := 2\mu^2 \sum_{h=1}^{\infty} \left\{ (h+1)^2 \frac{\Delta_{A_0} + \Delta_s}{2\Delta_{A_0}\Delta_s(\Delta_{A_0} + \Delta_s - \mu)(\Delta_{A_0} + \Delta_s + \mu)} - \frac{1}{4\Delta_s} \right\}.$$  

The behavior of this function around the value $\mu = 1$ is plotted in Figure 2.

The $1/\epsilon$ pole present in (28) is subtracted by the mass renormalization counterterm associated with the chemical potential. To see this, consider the diagram below:

The vertices arise from the factor in the Lagrangian (3),

$$2\mu g \text{tr}(A_0[A, Y]) = 2i\mu gf^{abc} A^a_0 X^b Y^c.$$  

The diagram is logarithmically divergent and in the zero external momentum limit, it yields the $1/\epsilon$ pole,

$$-\frac{\lambda \mu^2}{4\pi^2\epsilon},$$  

where $\lambda$ is the ’t Hooft coupling $\lambda := C_A g^2$ with the second Casimir $C_A$ defined as $f^{abc}f^{dbc} = C_A \delta^{ad}$. Now observe that,

$$\sum_{m<n} \mu^2 g^2 M^2_{mn} = \frac{1}{2} \lambda \mu^2 \sum_{a=1}^{N^2-1} \left\{ (X^a)^2 + (Y^a)^2 \right\},$$  

where we set the generator of the $U(1)$ part of the $U(N)$ gauge group, $T^{N^2}_N = 1_{N\times N}/\sqrt{N}$, to the last $N^2$th one. We are considering the fields that take values
in the flat directions, so we have $X^a, Y^a \neq 0$ only in the Cartan subalgebra valued fields and the $U(1)$ part is excluded since the corresponding $U(1)$ field is free and does not contribute to the one-loop correction. Thus, we see that the subtraction of the $1/\epsilon$ pole in fact comes from the mass renormalization counterterm associated with the chemical potential. We note that this is not the field strength renormalization of $X, Y$ fields because the divergent diagram considered above is not proportional to the external momentum and it is proportional to $\mu^2$.

The one-loop contribution to the effective potential is the negative of $27$ and $28$ with $1/\epsilon$ pole subtracted. The result is $6$.

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