@The Lyapunov stability of the N-soliton solutions in the Lax hierarchy of the Benjamin-Ono equation

Yoshimasa Matsuno¹

Division of Applied Mathematical Science, Graduate School of Science and Engineering, Yamaguchi University, Ube 755-8611, Japan

Abstract

The Lyapunov stability is established for the N-soliton solutions in the Lax hierarchy of the Benjamin-Ono (BO) equation. We characterize the N-soliton profiles as critical points of certain Lyapunov functional. By using several results derived by the inverse scattering transform of the BO equation, we demonstrate the convexity of the Lyapunov functional when evaluated at the N-soliton profiles. From this fact, we deduce that the N-soliton solutions are energetically stable.

¹E-mail address: matsuno@yamaguchi-u.ac.jp
I. INTRODUCTION

The Benjamin-Ono (BO) equation describes the unidirectional propagation of long internal waves in stratified fluids of great depth. It may be written in an appropriate dimensionless form as

\[ u_t + 2uu_x + Hu_{xx} = 0. \]  
\[ (1.1a) \]

Here, \( u = u(x,t) \) represents the amplitude of wave, \( H \) is the Hilbert transform given by

\[ Hu(x,t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(y,t)}{y-x} dy, \]  
\[ (1.1b) \]

and the subscripts \( t \) and \( x \) appended to \( u \) denote partial differentiation. The BO equation can be written as an infinite-dimensional completely integrable Hamiltonian dynamical system.\textsuperscript{1,2} A common feature of integrable evolution equations is the existence of an infinite sequence of conservation laws. The Lax hierarchy of the BO equation is generated by the conservation laws, which we shall denote by \( I_n(n = 2, 3, 4...) \). The first three of \( I_n \) read

\[ I_2 = \frac{1}{2} \int_{-\infty}^{\infty} u^2 dx, \]  
\[ (1.2a) \]

\[ I_3 = - \int_{-\infty}^{\infty} \left( \frac{1}{3} u^3 + \frac{1}{2} uH u_x \right) dx, \]  
\[ (1.2b) \]

\[ I_4 = \int_{-\infty}^{\infty} \left( \frac{1}{4} u^4 + \frac{3}{4} u^2 H u_x + \frac{1}{2} u_x^2 \right) dx. \]  
\[ (1.2c) \]

In (1.2), the mass conservation has been excluded since it is irrelevant in the present analysis. The BO hierarchy is defined by the following nonlinear evolution equations

\[ \frac{\partial u}{\partial t_n} = \frac{\partial}{\partial x} \frac{\delta I_{n+2}}{\delta u}, (n = 0, 1, 2,...), \]  
\[ (1.3) \]

where \( \delta/\delta u \) is the variational derivative defined by

\[ \frac{\partial}{\partial \epsilon} I_{n+2}(u + \epsilon v)\bigg|_{\epsilon=0} = \int_{-\infty}^{\infty} \frac{\delta I_{n+2}}{\delta u(x)} v(x) dx. \]  
\[ (1.4) \]

When \( n = 1 \), (1.3) becomes the BO equation (1.1) with the identification \( t_1 = t \) while when \( n = 2 \), it yields the first higher-order BO equation\textsuperscript{3}

\[ u_{t_2} = \left( u^3 + \frac{3}{2} uH u_x + \frac{3}{2} H(uu_x) - u_{xx} \right)_x. \]  
\[ (1.5) \]
Note that the first member of (1.3) reduces simply to a linear equation \( u_{t_0} = u_x \). As will be shown below, all the members of the BO hierarchy exhibit the \( N \)-soliton solution characterized by the \( 2N \) parameters \( a_j \) and \( x_{j0} (j = 1, 2, \ldots, N) \) where \( N \) is an arbitrary positive integer: \[
\begin{align*}
u &= u_N (x - x_1, x - x_2, \ldots, x - x_N).
\end{align*}
\]
(1.6a)

Here
\[
\begin{align*}
x_j &= \sum_{s=0}^{\infty} (-1)^{s+1} \frac{s+1}{2^s} a_j^s t_s + x_{j0}, (j = 1, 2, \ldots, N),
\end{align*}
\]
(1.6b)
a\_j are amplitude parameters satisfying the conditions \( a_j > 0, a_j \neq a_k \) for \( j \neq k \) (\( j, k = 1, 2, \ldots, N \)) and \( x_{j0} \) are arbitrary phase parameters. Explicitly, \( u_N \) has a simple expression in terms of a tau function \( f \)
\[
\begin{align*}
u_N &= i \frac{\partial}{\partial x} \ln \frac{f}{f^*}, \quad f = \det F,
\end{align*}
\]
(1.7a)

where \( F = (f_{jk})_{1 \leq j,k \leq N} \) is an \( N \times N \) matrix with elements
\[
\begin{align*}
f_{jk} &= \left( x - x_j + \frac{i}{a_j} \right) \delta_{jk} - \frac{2i}{a_j - a_k} (1 - \delta_{jk}).
\end{align*}
\]
(1.7b)

Here, \( f^* \) is a complex conjugate of \( f \) and \( \delta_{jk} \) is Kronecker’s delta. In particular, for \( N = 1 \), (1.7) represents the 1-soliton solution with a Lorenzian profile
\[
\begin{align*}
u_1 = \frac{2a_1}{a_1^2 (x - x_1)^2 + 1}.
\end{align*}
\]
(1.8)

A direct proof of (1.7) using an elementary theory of determinants will be presented in Appendix A.@

The definition of the stability of solitons may be classified according to the following three categories: i) linear (or spectral) stability, ii) energetic stability, iii) nonlinear stability. The energetic stability implies that the second variation of certain Lyapunov functional becomes strictly positive when evaluated at the soliton solutions. It would also lead to the linear stability since the second variation is preserved for the linearized equation. In order to extend the energetic stability to the nonlinear stability which deals with small but finite amplitude perturbations, one must take into account higher-order
nonlinear terms neglected in evaluating the Lyapunov functional and this makes the analysis more difficult. In accordance with the above classification of the stability, we shall briefly review some known results associated with the stability characteristics of the BO solitons. The linear stability of the BO 1-soliton solution has been proved by solving the eigenvalue problem associated with the linearized BO equation. A subsequent nonlinear analysis shows that the soliton is also stable against small but finite perturbations. As for the general \(N\)-soliton solution, its linear stability characteristic has been established by solving explicitly the initial value problem of the linearized BO equation and investigating the large-time asymptotic of the solution. In the process, the completeness relation for the eigenfunctions of the BO equation linearized around the \(N\)-soliton solution has played a central role. The recent study demonstrates the orbital stability of the 2-soliton solution in which the stability problem has been settled based on the Lyapunov method combined with the spectral analysis of the operators associated with the linearized BO equation. The approach used in this paper originates from the stability analysis of the multisoliton solutions of the Korteweg-de Vries (KdV) equation by means of the constrained variational principle. See also an analogous work dealing with the spectral stability of the multisoliton solutions in the KdV hierarchy. All the works mentioned above are concerned with the stability of solitons for the BO equation. The stability characteristics of solitons in the BO hierarchy have not been considered as yet.

The purpose of this paper is to establish the Lyapunov stability of the general \(N\)-soliton solution (1.6). To be more specific, let us consider the following higher-order BO equation which consists of the commuting flows of the BO hierarchy

\[
\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \frac{\delta H_N}{\delta u},
\]

(1.9a)

where the Lyapunov functional \(H_N\) is given by

\[
H_N(u) = I_{N+2} + \sum_{n=1}^{N} \mu_n I_{n+1},
\]

(1.9b)

and \(\mu_n\) are Lagrange multipliers which will be expressed in terms of the elementary symmetric functions of \(a_1, a_2, \ldots, a_N\). See Sec. III for the detail. We define the profile (or shape) of the \(N\)-soliton solution by \(U_N = U_N(x) \equiv u_N|_{t_0=t_1=\ldots=0}\). We observe from (1.7)
that the $N$-soliton profile has the same functional form for all the members of the hierarchy, the only difference being the velocities of the solitons. We show that $U_N$ is a stationary solution of (1.9) decaying at infinity. Namely, $U_N$ is realized as a critical point of the functional $H_N$. Using (1.9b), this condition can be written as the Euler-Lagrange equation

$$\frac{\delta I_{N+2}}{\delta u} + \sum_{n=1}^{N} \mu_n \frac{\delta I_{n+1}}{\delta u} = 0, \text{ at } u = U_N.$$  (1.10)

The Lyapunov stability of $U_N$ may characterize $U_N$ as a minimal point of the functional $I_{N+2}$ subjected to $N$ constraints

$$I_{n+1}(u) = d_n, (n = 1, 2, ..., N),$$  (1.11)

where $d_n$ are real constants and consequently the second variation of $H_N$ is strictly positive at $U_N$. This means that $H_N$ is convex at $U_N$, so that the following inequality holds

$$H_N(U_N + \epsilon v) - H_N(U_N) > 0,$$  (1.12)

where $\epsilon v$ is a perturbation imposed on $U_N$ which belongs to certain function space specified by the $2N$ integral conditions. We assume that the $L^2$ norm of $v$ is finite. The small parameter $\epsilon$ has been introduced to measure the magnitude of the perturbation. The inequality (1.12) shows that the $N$-soliton solutions are energetically stable. We give a direct proof of (1.12) with the aid of the results obtained by the inverse scattering transform (IST) for the BO equation.\textsuperscript{11,12} In Sec. II, we summarize the background results arising from the perturbation theory and the Hamiltonian formulation of the BO equation which provide the necessary machinery in carrying out the stability analysis. In Sec. III, we prove the inequality and hence establish the Lyapunov stability of the $N$-soliton solutions in the Lax hierarchy of the BO equation. In Appendix A, we present a direct proof of the $N$-soliton solution (1.7). In Appendix B, we evaluate the number of positive eigenvalues of the Hessian matrix associated with $H_N$.

II. BACKGROUND RESULTS OF IST

The IST has been applied successfully to solve the initial value problem of the BO equation.\textsuperscript{11,12} Furthermore, for real generic potentials it has been used to prove the com-
plete integrability of the BO equation.\textsuperscript{1,2} Here, we summarize some background results of the IST necessary for the stability analysis.

A. Eigenvalue problem

The eigenvalue problem associated with the IST of the BO equation may take the form

\[ i\phi_x^+ + \lambda(\phi^+ - \phi^-) = -u\phi^+, \]  

(2.1)

where \( \phi^+(\phi^-) \) is the boundary value of the analytic function in the upper(lower)-half complex \( x \) plane, \( u \) is a real potential rapidly decreasing at infinity and \( \lambda \) is the eigenvalue (or the spectral parameter). We define the two Jost solutions of (2.1) specified by the boundary condition as \( x \to +\infty \)

\[ N(x, \lambda) \to e^{i\lambda x}, \quad \bar{N}(x, \lambda) \to 1, \]  

(2.2)

and the analogous ones as \( x \to -\infty \)

\[ \bar{M}(x, \lambda) \to 1, \quad M(x, \lambda) \to e^{i\lambda x}. \]  

(2.3)

These solutions satisfy the linear integral equations

\[ N_x - i\lambda N = iP_+(uN), \]  

(2.4a)

\[ \bar{N}_x - i\lambda \bar{N} = iP_+(u\bar{N}) - i\lambda, \]  

(2.4b)

\[ M_x - i\lambda M = iP_+(uM) - i\lambda, \]  

(2.4c)

\[ \bar{M}_x - i\lambda \bar{M} = iP_+(u\bar{M}), \]  

(2.4d)

where \( P_+ \) is the projection operator defined by \( P_+ = \frac{1}{2}(1 - iH) \). The solutions of (2.4) subjected to the boundary conditions (2.2) and (2.3) exist for \( \lambda > 0 \). The Jost functions \( M, N \) and \( \bar{N} \) are then related by

\[ M = \bar{N} + \beta N, \]  

(2.5)

where \( \beta \) is a reflection coefficient. For pure soliton potentials, this reflection coefficient vanishes identically.
There exists a set of solutions $\Phi_j(x)$ for negative $\lambda = \lambda_j (j = 1, 2, ..., N)$ which satisfy the equation

$$\Phi_{j,x} - i\lambda_j \Phi_j = iP_+(u\Phi_j), \ (j = 1, 2, ..., N),$$

(2.6a)

with the boundary conditions

$$\Phi_j \to \frac{1}{x}, \ x \to +\infty, \ (j = 1, 2, ..., N).$$

(2.6b)

B. Conservation laws

It follows from (1.1), (2.4b) and the time evolution equation for $\bar{N}$

$$\bar{N}_t - 2\lambda \bar{N}_x - i\bar{N}_{xx} - 2(P_+u_x)\bar{N} = 0,$$

(2.7)

that the quantity $\int_{-\infty}^{\infty} u(x,t)\bar{N}(x,t)dx$ is conserved in time. Expanding $\bar{N}$ in inverse powers of $\lambda$

$$\bar{N} = \sum_{n=0}^{\infty} \frac{(-1)^n \bar{N}_{n+1}}{\lambda^n}, \ \bar{N}_1 = 1,$$

(2.8a)

and substituting (2.8a) into (2.4b), we obtain the following recursion relation that determines $\bar{N}_n$:

$$\bar{N}_{n+1} = i\bar{N}_{n,x} + P_+(u\bar{N}_n), \ n \geq 1.$$  

(2.8b)

The $n$th conservation law may be taken as

$$I_n = (-1)^n \int_{-\infty}^{\infty} u\bar{N}_n dx,$$

(2.9)

where a factor $(-1)^n$ is multiplied for convenience. The first three of $I_n$ except $I_1$ are already given by (1.2). In terms of the scattering data $\beta$ and $\lambda_j$, $I_n$ can be evaluated as

$$I_n = (-1)^n \left\{ 2\pi \sum_{j=1}^{N} (-\lambda_j)^{n-1} + \frac{(-1)^n}{2\pi} \int_{0}^{\infty} \lambda^{n-2} \beta^*(\lambda)\beta(\lambda)d\lambda \right\}, \ (n = 1, 2, ...).$$

(2.10)

The first term on the right-hand side of (2.10) is the contribution from solitons and the second term comes from radiations. It is important that both contributions are additive. A remarkable feature of the conservation laws is that they are in involution, namely $I_n(n = 1, 2, ...)$ commute each other in an appropriate Poisson bracket. In particular

$$\int_{-\infty}^{\infty} \left( \frac{\delta I_n}{\delta u(x)} \right)_{u=U_N} \frac{\partial}{\partial x} \left( \frac{\delta I_m}{\delta u(x)} \right)_{u=U_N} dx = 0, \ (n, m = 1, 2, ...).$$

(2.11)
C. Variational derivatives

The variational derivatives of the scattering data with respect to the potential are calculated explicitly. In developing the Lyapunov stability, we need the formulas of the variational derivatives evaluated for the $N$-soliton potential $u = U_N$. In particular, the following formula plays an important role in our analysis

$$\left( \frac{\delta \lambda_j}{\delta u(x)} \right)_{u=U_N} = \frac{1}{2\pi \lambda_j} \Phi_j^*(x) \Phi_j(x), \quad (j = 1, 2, \ldots, N). \quad (2.12)$$

Here, the eigenfunction $\Phi_j$ corresponding to the discrete spectrum $\lambda_j$ satisfies the system of linear algebraic equations

$$(x - \gamma_j) \Phi_j + i \sum_{k=1}^{N} \frac{1}{\lambda_j - \lambda_k} \Phi_k = 1, \quad (j = 1, 2, \ldots, N), \quad (2.13)$$

where $\gamma_j = x_{j0} + i/(2\lambda_j)$ and $x_{j0}$ are real constants. Recall that $\lambda_j$ are related to the amplitude parameters $a_j$ introduced in (1.7) by the relations $\lambda_j = -a_j/2(j = 1, 2, \ldots, N)$. Taking account of the fact that the reflection coefficient $\beta$ becomes zero for $u = U_N$, we can derive from (2.10) and (2.12) the formula

$$\left( \frac{\delta I_n}{\delta u(x)} \right)_{u=U_N} = (-1)^n(1 - 1)^n \sum_{j=1}^{N} (-\lambda_j)^{n-3} \Phi_j^*(x) \Phi_j(x), \quad (n = 2, 3, \ldots, N). \quad (2.14)$$

In terms of $\Phi_j$, $U_N$ has the following two alternative expressions:

$$U_N = i \sum_{j=1}^{N} (\Phi_j - \Phi_j^*), \quad (2.15)$$

$$U_N = -i \sum_{j=1}^{N} \frac{1}{\lambda_j} \Phi_j^* \Phi_j. \quad (2.16)$$

The positive definiteness of $U_N$ is obvious from (2.16) since all $\lambda_j$ are negative quantities. One can derive (2.16) by using (2.13) and (2.15). The formula (2.16) also follows from (1.2a), (2.10) and (2.14). In Appendix A, we show that $U_N$ can be rewritten in a compact form in terms of a determinant.

The following relation concerning the variational derivative of $\beta$ with respect to $u$ is useful in evaluating the contribution of the continuous part to the functional $H_N$:

$$\frac{\delta \beta(\lambda)}{\delta u(x)} = iM(x, \lambda)N^*(x, \lambda). \quad (2.17)$$
For the $N$-soliton potential $u = U_N$, $M$ reduces to $\bar{N}$ by (2.5) and $\beta \equiv 0$. The function $MN^*$ satisfies the orthogonality conditions

$$\int_{-\infty}^{\infty} M(x, \lambda) N^*(x, \lambda) \frac{\partial}{\partial x} \left( \Phi_j^*(x) \Phi_j(x) \right) dx = 0, \ (j = 1, 2, ..., N). \quad (2.18)$$

Finally, we emphasize that all the results presented here are obtained through the analysis of the spatial part (2.1) of the Lax pair for the BO equation.

III. LYAPUNOV STABILITY

A. Variational characterization of the $N$-soliton profile

We first show that the stationary solution $U_N$ of the higher-order BO equation (1.9) satisfies (1.10) if one prescribes the Lagrange multipliers $\mu_n$ appropriately. This provides a variational characterization of $U_N$. Let $\Psi_j = \Phi_j^* \Phi_j$ and $b_j = -\lambda_j = a_j/2$. With this notation, (1.10) and (2.14) give a linear relation among $\Psi_j$

$$(N + 1) \sum_{j=1}^{N} b_j^{N-1} \Psi_j + \sum_{n=1}^{N} (-1)^N n \mu_n \sum_{j=1}^{N} b_j^{n-2} \Psi_j = 0. \quad (3.1)$$

In view of the fact that $\Psi_j$ are functionally independent squared eigenfunctions, $\mu_n$ must satisfy the following system of linear algebraic equations:

$$\sum_{n=1}^{N} (-1)^{N-n} n b_j^{n-1} \mu_n = (N + 1) b_j^N, \ (j = 1, 2, ..., N). \quad (3.2)$$

To solve (3.2), we introduce an $N \times N$ matrix $V$

$$V = (v_{jk})_{1 \leq j, k \leq N}, \ v_{jk} = b_j^{k-1}, \quad (3.3)$$

and the cofactor of $v_{jk}$ by

$$V_{jk} = \frac{\partial |V|}{\partial v_{jk}}, \ |V| = \det V, \quad (3.4)$$

where $|V|$ is the Vandermonde determinant. Notably, since $|V| = \Pi_{1 \leq j < k \leq N} (b_k - b_j)$ and $b_j \neq b_k$ for $j \neq k$, $|V|$ never vanishes. This fact will be used essentially in the following calculation. It is also convenient to define the polynomials $g(x)$ and $g_k(x)$ by

$$g(x) = \prod_{j=1}^{N} (x - b_j) = \sum_{s=1}^{N} (-1)^s \sigma_s x^{N-s}, \quad (3.5)$$

9
\[ g_k(x) = \prod_{j=1}^{N} (x - b_j) = \sum_{s=1}^{N-1} (-1)^s \sigma_{k,s} x^{N-s}, \]  

(3.6)

where \( \sigma_0 = 1 \) and \( \sigma_s (1 \leq s \leq N) \) are elementary symmetric functions of \( b_1, b_2, \ldots, b_N \):

\[
\sigma_1 = \sum_{j=1}^{N} b_j, \quad \sigma_2 = \sum_{j,k=1}^{N} (j<k) b_j b_k, \ldots, \quad \sigma_N = \prod_{j=1}^{N} b_j,
\]

(3.7a)

and \( \sigma_{k,s} \) are given by the relation

\[
\sigma_{k,s} = \sum_{j=0}^{s} \sigma_j (-b_k)^{s-j}.
\]

(3.7b)

Obviously, all \( \sigma_j \) are positive quantities since \( b_j > 0 (j = 1, 2, \ldots, N) \). Now, applying Cramer’s rule to (3.2) with use of the fact \( |V| \neq 0 \), we find that \( \mu_n \) are determined uniquely as

\[
\mu_n = (-1)^{N-n} \frac{N + 1}{n} \frac{\sum_{k=1}^{N} V_k b_k^N}{|V|}, (n = 1, 2, \ldots, N).
\]

(3.8)

Substituting the formulas into (3.12)

\[
V_{kn} = \frac{(-1)^{N-n} \sigma_{k,N-n} |V|}{g_k(b_k)}, (k, n = 1, 2, \ldots, N),
\]

(3.9)

\[
\sum_{k=1}^{N} \frac{\sigma_{k,N-n} b_k^N}{g_k(b_k)} = \sigma_{N-n+1}, (n = 1, 2, \ldots, N),
\]

(3.10)

into (3.8), we arrive at a simple expression of \( \mu_n \)

\[
\mu_n = \frac{N + 1}{n} \sigma_{N-n+1}, (n = 1, 2, \ldots, N).
\]

(3.11)

If we use the relations \( b_j = a_j/2 (j = 1, 2, \ldots, N) \), we can see that \( \mu_n \) are expressed in terms of elementary symmetric functions of \( a_1, a_2, \ldots, a_N \).

B. Stability

Let us now prove the inequality (1.12) which assures that the functional \( H_N \) is convex at the \( N \)-soliton profile \( U_N \). The method used here is based on the ideas developed in a recent work on the spectral stability of the \( N \)-soliton solution of the KdV hierarchy as well as an earlier work on the algebraic structure of the BO \( N \)-soliton solution. We first rewrite (2.10) as

\[
I_{n+1}(u) = (-1)^{n+1} \left\{ 2\pi \sum_{j=1}^{N} b_j^n + (-1)^{n+1} r_n \right\},
\]

(3.12a)
where we have put $b_j = -\lambda_j$ and

$$r_n = \frac{1}{2\pi} \int_0^\infty \lambda^{n-1}\beta^*(\lambda)\beta(\lambda)d\lambda. \quad (3.12b)$$

Let $\Delta Q$ be the increment of any functional $Q(u)$ around $u = U_N$, i.e.,

$$\Delta Q = Q(U_N + \epsilon v) - Q(U_N). \quad (3.13)$$

It then follows from the constraints (1.11) that

$$\Delta I_{n+1} = 0, (n = 1, 2, ..., N). \quad (3.14)$$

We then use (3.12) to rewrite (3.14) in the form

$$2\pi n \sum_{j=1}^N b_j^{n-1}\Delta b_j + (-1)^{n+1}\Delta r_n = 0, (n = 1, 2, ..., N), \quad (3.15)$$

where we have neglected the higher-order terms $(\Delta b_j)^s (s = 2, 3, ..., N)$. These relations indicate that the increments of soliton amplitudes are balanced with the increments of radiations. We recall that $\beta \equiv 0$ for $u = U_N$ and consequently $\Delta(\beta^*\beta) = \Delta\beta^*\Delta\beta$. This leads to the estimates $\Delta r_n \sim O(\epsilon^2)$ and $\Delta r_n > 0$ for all $n$. The case $\Delta r_n = 0$ calls a special attention and it will considered in detail later. Hence, (3.15) can be solved consistently in $\Delta b_j$ only if $\Delta b_j \sim O(\epsilon^2)$. Since by the definition (1.4)

$$\Delta b_j = \epsilon \int_{-\infty}^{\infty} \left( \frac{\delta b_j}{\delta u} \right)_{u=U_N} v(x)dx + O(\epsilon^2), \quad (3.16)$$

one must impose the integral conditions on the perturbation $v(x)$

$$\int_{-\infty}^{\infty} \left( \frac{\delta b_j}{\delta u} \right)_{u=U_N} v(x)dx = 0, \ (j = 1, 2, ..., N), \quad (3.17)$$

in accordance with the above estimate for $\Delta b_j$. We can see from (2.12), (2.14) together with the relations $b_j = -\lambda_j (j = 1, 2, ..., N)$ and $|V| \neq 0$ that (3.17) are equivalent to

$$\int_{-\infty}^{\infty} \left( \frac{\delta I_{n+1}}{\delta u} \right)_{u=U_N} v(x)dx = 0, \ (n = 1, 2, ..., N). \quad (3.18)$$
Owing to (3.14), however, these conditions are satisfied automatically. The above observations allow us to solve (3.15). Indeed, the solutions are written, with use of Cramer’s rule, as

$$\Delta b_j = \frac{1}{2\pi} \sum_{n=1}^{N} \frac{(-1)^n V_n \Delta r_n}{|V|}. \quad (j = 1, 2, \ldots, N).$$  \hspace{1cm} (3.19)

It now follows from (1.9b), (3.12) and (3.14) that

$$\Delta H_N = (-1)^N \left\{ 2\pi (N + 1) \sum_{j=1}^{N} b_j^N \Delta b_j + (-1)^{N+2} \Delta r_{N+1} \right\}. \quad (3.20)$$

If we substitute (3.19) into (3.20) and use the formulas (3.9) and (3.10), $\Delta H_N$ simplifies to

$$\Delta H_N = (N + 1) \sum_{n=1}^{N} \frac{\sigma_{N-n+1}}{n} \Delta r_n + \Delta r_{N+1}. \quad (3.21)$$

Since $\sigma_{N-n+1} > 0$ for $n = 1, 2, \ldots, N$ by the definition (3.7), we find that if at least one of $\Delta r_n$ is not zero, then $\Delta H_N > 0$. On the other hand, if all $\Delta r_n$ become zero, then $\Delta H_N = 0$. In the latter case, we see from (3.15) that $\Delta b_j = 0$ for all $j$. This situation will happen when the perturbation $\epsilon v$ represents the small variation of $U_N$ with respect to the phase parameters $x_{j0}$. Specifically

$$\epsilon v(x) = \sum_{j=1}^{N} \frac{\partial U_N}{\partial x_{j0}} \delta x_{j0}, \quad (3.22)$$

where $\delta x_{j0}$ are small perturbations of order $\epsilon$. If we impose the following $N$ integral conditions on $v(x)$ in addition to (3.28)

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left( \frac{\delta I_{n+1}}{\delta u} \right)_{u=U_N} v(x) dx = 0, \quad (n = 1, 2, \ldots, N), \quad (3.23)$$

then the perturbation of the form (3.22) ceases to be admissible, as we shall now demonstrate. We first notice that the right-hand side of (3.22) can be expressed in terms of the $x$ derivative of $(\delta I_{n+1}/\delta u)_{u=U_N}$. Indeed, we take $u = u_N$ in (1.3) and then put $t_0 = t_1 = \ldots = 0$ to obtain

$$(-1)^n \sum_{j=1}^{N} b_j^{n-1} \frac{\partial U_N}{\partial x_{j0}} = \frac{\partial}{\partial x} \left( \frac{\delta I_{n+1}}{\delta u} \right)_{u=U_N}, \quad (n = 1, 2, \ldots, N), \quad (3.24)$$
where we have used (1.6), the definition of $U_N$ and $b_j = a_j/2$. Thanks to the fact $|V| \neq 0$, the relations (3.24) can be inverted to give

$$\frac{\partial U_N}{\partial x_j} = \sum_{n=1}^{N} \frac{(-1)^n}{n} \frac{V_{jn}}{|V|} \frac{\partial}{\partial x} \left( \frac{\delta I_{n+1}}{\delta u} \right)_{u=U_N}, \quad (j = 1, 2, \ldots, N). \quad (3.25)$$

An alternative expression of (3.22) follows immediately upon introducing (3.25) into (3.22), which reads

$$\epsilon v(x) = \sum_{j=1}^{N} \sum_{n=1}^{N} \frac{(-1)^n}{n} \frac{V_{jn}}{|V|} \frac{\partial}{\partial x} \left( \frac{\delta I_{n+1}}{\delta u} \right)_{u=U_N} \delta x_j. \quad (3.26)$$

We observe that this perturbation satisfies the conditions (3.18) by virtue of (2.11). It is important that the $N \times N$ matrix $C = (c_{jk})_{1 \leq j, k \leq N}$ with elements

$$c_{jk} = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left( \frac{\delta I_{j+1}}{\delta u} \right)_{u=U_N} \frac{\partial}{\partial x} \left( \frac{\delta I_{k+1}}{\delta u} \right)_{u=U_N} dx, \quad \text{(3.27)}$$

is positive definite and hence $|C| \neq 0$. In view of this fact, we deduce from (3.23) and (3.26) that $\delta x_j = 0 (j = 1, 2, \ldots, N)$ and consequently $v = 0$, which implies the assertion mentioned above. An additional relation which deserves remark is

$$\int_{-\infty}^{\infty} \left( \frac{\delta \beta}{\delta u} \right)_{u=U_N} v(x) dx = 0, \quad \text{(3.28)}$$

which follows from (2.14), (2.17), (2.18) and (3.26). This leads to the estimates $\Delta \beta \sim O(\epsilon^2)$ and $\Delta r_n \sim O(\epsilon^4)$. As a result, the perturbation (3.26) gives rise to higher-order contributions to (3.21) which also means that the second variation of $H_N$ turns out to be zero. In conclusion, the inequality $\Delta H_N > 0$ holds under the simultaneous conditions (3.18) and (3.25) imposed on $v(x)$, which completes the proof of (1.12). The convexity of $H_N$ implies that the second variation of $H_N$ is strictly positive and consequently the $N$-soliton solutions are energetically stable.

C. Remark

In this paper, the convexity of $H_N$ has been proved by invoking some results obtained by the IST of the BO equation. There exists, however another method to establish the same convex property without recourse to the IST. To illustrate this, we put $u(x,t) =$


\( U_N(x) + \epsilon v(x)e^{\lambda t} \) and linearize (1.9) around \( U_N \). The resulting eigenvalue equation can be written as

\[
\frac{\partial}{\partial x} \mathcal{L}_N v = \lambda v, \tag{3.29}
\]

where \( \mathcal{L}_N \) is a self-adjoint operator. This operator may be defined through the relation

\[
\delta^2 \mathcal{H}_N = \frac{\epsilon^2}{2} \int_{-\infty}^{\infty} v(x) \mathcal{L}_N v(x) dx, \tag{3.30}
\]

where \( \delta^2 \mathcal{H}_N \) denotes the second variation of \( \mathcal{H}_N \). Let \( n(\mathcal{L}_N) \) be the number of negative eigenvalues of \( \mathcal{L}_N \) and \( p(\mathcal{H}_N) \) be the number of positive eigenvalues of the Hessian matrix defined by

\[
\mathcal{H}_N = (h_{jk})_{1 \leq j,k \leq N}, \quad h_{jk} = \frac{\partial^2 H_N}{\partial \mu_j \partial \mu_k}. \tag{3.31}
\]

Then, under the conditions (3.18) and (3.23) the positivity of \( \delta^2 H_N \) is satisfied if and only if \( n(\mathcal{L}_N) = p(\mathcal{H}_N) \). The above criterion of the positivity property has been proved in Ref. 15 and has been applied to the Lyapunov stability of the \( N \)-soliton solution of the KdV equation. In particular, the spectral property of the \( 2N \)th-order differential operator associated with the linearized KdV equation has been investigated by extending the classical Sturmian theory. See also a related work dealing with the stability of the \( N \)-soliton solutions in the KdV hierarchy. In the case of the BO equation, however, the eigenvalue equation (3.29) is not purely differential equation but actually integrodifferential equation since it includes the Hilbert transform. This makes the spectral analysis more difficult. Quite recently, a new method was developed to characterize the spectral property of \( \mathcal{L}_N \) for \( N = 2 \). The extension to the general \( N \)-soliton solutions of the BO equation and its hierarchy is still to be resolved. It is noteworthy that \( p(\mathcal{H}_N) \) can be evaluated explicitly for the \( N \)-soliton profile \( U_N \). This calculation is presented in Appendix B. The stability analysis developed in this paper would suggest that \( n(\mathcal{L}_N) \) is equal to \( p(\mathcal{H}_N) \). This interesting issue will be pursued in a future study.

**APPENDIX A: PROOF OF THE \( N \)-SOLITON SOLUTION**

In this Appendix, we provide a direct proof of the \( N \)-soliton solution (1.7) of the \( n \)th higher-order BO equation (1.3) by means of an elementary theory of determinants. For convenience, we write down some basic formulas for determinants upon which our proof
relies. Let $F$ be an $N \times N$ matrix with elements $f_{jk}$ given by (1.7b) and $F_{jk}$ be the cofactor of $f_{jk}$. The expansion of $|F|$ by elements and their cofactors is given by the two ways:

$$\sum_{k=1}^{N} f_{jk} F_{lk} = \delta_{jl} |F|, \quad (A1a)$$

$$\sum_{j=1}^{N} f_{jk} F_{jl} = \delta_{kl} |F|. \quad (A1b)$$

The following formula is a consequence of (A1)

$$\sum_{j,k=1}^{N} (f_j + g_k) f_{jk} F_{jk} = \sum_{j=1}^{N} (f_j + g_j) |F|. \quad (A2)$$

The differential rule applied to the determinant $|F|$ gives

$$|F|_x = \sum_{j=1}^{N} F_{jj}, \quad (A3a)$$

$$|F|_{t_n} = (-1)^n \sum_{j=1}^{N} c_j F_{jj}, \quad (A3b)$$

where $c_j = (n + 1)a_j^n / 2^n$. To carry out the proof, it is necessary to assign the time dependence of the eigenfunction $\Phi_j$ for the discrete spectrum $\lambda_j$. This can be accomplished simply by replacing the phase factor $\gamma_j$ introduced in (2.13) by $x_j$ which is defined in (1.6).

We first show that (2.15) can be rewritten in an alternative determinantal form (1.7). The solution to (2.13) is found by using Cramer’s rule as

$$\Phi_j = i \sum_{k=1}^{N} \frac{F_{kj}}{|F|}. \quad (A4)$$

We put $f_j = a_j$ and $g_j = -a_j$ in (A2) to derive the relation

$$\sum_{j,k=1}^{N} F_{jk} = \sum_{j=1}^{N} F_{jj}. \quad (A5)$$

It follows from (A3)-(A5) that

$$\sum_{j=1}^{N} \Phi_j = i (\ln |F|)_x. \quad (A6)$$
Substituting (A6) and its complex conjugate expression into (2.15), we find that (2.15) coincides with (1.7).

Let us now proceed to the proof of the $N$-soliton solution. We substitute (1.7) and (2.14) into (1.3) and integrate it once with respect to $x$ to recast (1.3) into the form

\[ i(|F|^*|F|_{t_n} - |F||F|^*_{t_n})/|F|^*|F| = (-1)^n(n + 1) \sum_{j=1}^{N} \left( \frac{a_j}{2} \right)^{n-1} \Phi_j^* \Phi_j, \quad (A7) \]

where we have used the relation $\lambda_j = -a_j/2$. The following identity has been established by using Jacobi’s formula for determinants:

\[ i \left( \frac{|F|^* F_{j,k}}{a_k} - \frac{|F|^* k_{j,i}}{a_j} \right) = 2 \left( \frac{|F|^* \Phi_j|^*}{a_j a_k} \right) \left( \frac{|F|^* \Phi_j}{a_j a_k} \right), \quad (j, k = 1, 2, ..., n). \quad (A8) \]

Indeed, (A8) coincides with (A20) in Ref. 7 with the identification $f = |F|^*$, $\Delta_{jk} = F_{jk}^*$, $\psi_j = \Phi_j^*$. If we multiply (A8) with $j = k$ by $c_j$ and sum up with respect to $j$, we obtain

\[ \frac{i}{2} \sum_{j=1}^{N} c_j (|F|^* F_{jj} - |F|^* F_{jj}^*) = \frac{n+1}{2^n} \sum_{j=1}^{N} a_j^n \Phi_j^* \Phi_j |F|^* |F|. \quad (A9) \]

The left-hand side of (A9) is modified further by introducing the formula (A3b) and its complex conjugate expression. It leads, after dividing the resultant expression by $|F|^* |F|$, to (A7) and thus completing the proof.

**APPENDIX B: POSITIVE EIGENVALUES OF THE HESSIAN MATRIX $\mathcal{H}_N$**

The Hessian matrix $\mathcal{H}_N$ is defined by (3.31). It is a real symmetric matrix whose elements are calculated explicitly for the $N$-soliton solution. Indeed, by taking $\beta = 0$ in (2.10), the $n$th conservation law corresponding to $u = u_N$ reduces to

\[ I_n = 2\pi (-1)^n \sum_{l=1}^{N} b_l^{n-1}, \quad (b_l = -\lambda_l). \quad (B1) \]

If we regard $H_N$ as a function of $\mu_j (j = 1, 2, ..., N)$, we obtain from (1.9b) and (1.10)

\[ \frac{\partial H_N}{\partial \mu_j} = I_{j+1}, \quad (j = 1, 2, ..., N). \quad (B2) \]

Hence

\[ h_{jk} = \frac{\partial I_{j+1}}{\partial \mu_k} = 2\pi (-1)^{j+1} \sum_{l=1}^{N} b_l^{j-1} \frac{\partial b_l}{\partial \mu_k}. \quad (B3) \]
Let $P = (p_{jk})_{1 \leq j, k \leq N}$ and $Q = (q_{jk})_{1 \leq j, k \leq N}$ be $N \times N$ matrices with elements
\[ p_{jk} = 2\pi(-1)^{j+1}jb_k^{j-1}, \quad (B4a) \]
\[ q_{jk} = \frac{\partial \mu_j}{\partial b_k} = \frac{N + 1}{j} \frac{\partial \sigma_{N-j+1}}{\partial b_k}, \quad (B4b) \]
respectively. Note that the right-hand side of (B4b) follows from (3.11). Using the above definition, we can rewrite (B3) in the form
\[ H_N = PQ^{-1}, \quad (B5) \]
if $Q^{-1}$ exists. To show the nonsingular nature of $Q$, we use the definition (3.7a) of $\sigma_{N-j+1}$ and (B4b) to evaluate the determinant of $Q$. A simple calculation immediately leads to
\[ |Q| = \frac{(N + 1)^N}{N!} \prod_{1 \leq j < k \leq N} (b_k - b_j). \quad (B6) \]
Since $b_j \neq b_k$ for $j \neq k$, we confirm that $|Q| \neq 0$, implying that $Q$ is invertible.

It now follows from (B5) that
\[ Q^T H_N Q = Q^T P. \quad (B7) \]
In accordance with Sylvester's law of inertia, one can see from (B7) that the number of positive eigenvalues of $H_N$ coincides with that of $Q^T P$. The latter can be counted easily, as we shall now demonstrate. Using (B4), the $(j, k)$ element of $Q^T P$ becomes
\[ (Q^T P)_{jk} = 2\pi(N + 1) \sum_{l=1}^{N} (-1)^{l+1} \frac{\partial \sigma_{N-l+1}}{\partial b_j} b_k^{l-1}. \quad (B8) \]
We differentiate (3.5) by $b_j$ and then put $x = b_k$ to derive the relation
\[ \sum_{l=1}^{N} (-1)^{l+1} \frac{\partial \sigma_{N-l+1}}{\partial b_j} b_k^{l-1} = \delta_{jk} \prod_{l=1}^{N} (b_l - b_k). \quad (B9) \]
Introducing (B9) into (B8), we find that $Q^T P$ is a diagonal matrix. We can order the magnitude of $b_j$ as $b_1 > b_2 > ... > b_N > 0$ without loss of generality. Then, (B8) and (B9) indicate that the number of positive eigenvalues of $Q^T P$ is equal to $\left[ \frac{N+1}{2} \right]$ where $[x]$ denotes the integer part of $x$. If we take account of (B7) and Sylvester's law of inertia, we conclude that $\rho[H_N] = \left[ \frac{N+1}{2} \right]$. 

17
REFERENCES

1. D.J. Kaup, T.I. Lakoba and Y. Matsuno, Phys. Lett. A 238, 123 (1998).
2. D.J. Kaup, T.I. Lakoba and Y. Matsuno, Inverse Problems 15, 215 (1999).
3. Y. Matsuno, J. Phys. Soc. Jpn. 47, 1745 (1979).
4. H.H. Chen and D.J. Kaup, Phys. Fluids 23, 235 (1980).
5. D.P. Bennett, R.W. Brown, S.E. Stansfield, J.D. Stroughair and J.L. Bona, Math. Proc. Camb. Phil. Soc. 94, 351 (1983).
6. Y. Matsuno and D. J. Kaup, Phys. Lett. A 228, 176 (1997).
7. Y. Matsuno and D. J. Kaup, J. Math. Phys. 38, 5198 (1997).
8. A. Neves and O. Lopes, Comm. Math. Phys. 262, 757 (2006).
9. J. Maddocks and R. Sachs, Comm. Pure Appl. Math. 46, 867 (1993).
10. Y. Kodama and D. Pelinovsky, J. Phys. A: Math. Gen. 38, 6129 (2005).
11. A.S. Fokas and M.J. Ablowitz, Stud. Appl. Math. 68, 1 (1983).
12. D.J. Kaup and Y. Matsuno, Stud. Appl. Math. 101, 73 (1998).
13. R. Vein and P. Dale, Determinants and Their Applications in Mathematical Physics (Springer, New York, 1999).
14. Y. Matsuno, J. Phys. Soc. Jpn. 51, 3375 (1982).
15. M. Grillakis, J. Shatah and W. Strauss, J. Func. Anal. 94, 308 (1990).