Boolean Operations using Generalized Winding Numbers

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Abstract

The generalized winding number function measures insideness for arbitrary oriented triangle meshes. Exploiting this, I similarly generalize binary boolean operations to act on such meshes. The resulting operations for union, intersection, difference, etc. avoid volumetric discretization or pre-processing.

1 Booleans & Classic Winding Numbers

If $A \subset \mathbb{R}^3$ and $B \subset \mathbb{R}^3$ are open subregions of space, then their union contains all points in $A$ or $B$, their intersection all points in $A$ and $B$, and the difference of $B$ from $A$ all points in $A$ but not $B$. Written in set notation, we have respectively:

\[
A \cup B = \{ p \mid p \in A \text{ or } p \in B \}, \\
A \cap B = \{ p \mid p \in A \text{ and } p \in B \}, \\
A \setminus B = \{ p \mid p \in A \text{ and } p \notin B \}.
\]

Meanwhile, the winding number function $w_X: \mathbb{R}^3 \setminus \partial A \to \mathbb{Z}$ determines for every point whether it is inside the set $A$ purely by examining the set’s oriented boundary $\partial A$. The winding number integrates the signed surface area of $\partial A$ projected onto a unit ball around a given point $p$, or in polar coordinates and w.l.o.g. $p = 0$:

\[
w_A(p) = \iint_{\partial A} \sin \varphi \ d\vartheta \ d\varphi = \begin{cases} 1 & \text{if } p \in A, \\ 0 & \text{otherwise } (p \notin A). \end{cases}
\]

If $A$ is an (embedded) solid, its winding number will be exactly one for points inside $A$ and exactly zero for points outside.

We can replace the set inclusions in the definitions of the boolean operations in Equations (1-3) with winding number expressions: replace $p \in X$ with $w_X(p) = 1$ and $p \notin X$ with $w_X(p) = 0$.

Algorithm This immediately reveals an algorithm for conducting boolean operations on solid1 triangle meshes $A$ and $B$ with vertices at general positions in space $A$ and $B$. Refine each mesh at mutual triangle-triangle intersections with the other so that intersections lie only at vertices and along edges (see, e.g., [Jacobson et al. 2013]). For each triangle $t_a$ of $A$, determine its winding number with respect to $B$, e.g., by evaluating $w_B(t_a)$ where $t_a$ is the triangle $t_a$’s barycenter. Likewise determine for each triangle $t_b$ of $B$ its winding number with respect to $A$. Finally—depending on the boolean operation—keep, keep-and-flip, or discard each triangle from $A$ and $B$. For example, for $A \setminus B$ (see Figure 1),

\[
\forall t_a \in A \begin{cases} \text{keep} & \text{if } w_B(t_a) = 0, \\ \text{discard} & \text{otherwise } (w_B(t_a) = 1), \end{cases} \\
\forall t_b \in B \begin{cases} \text{keep-and-flip} & \text{if } w_A(t_b) = 1, \\ \text{discard} & \text{otherwise } (w_A(t_b) = 0). \end{cases}
\]

For non-general position meshes, if the barycenter $t_a$ from a triangle of $A$ lies exactly on $B$ (e.g., due to coplanar overlaps), the winding number $w_B(t_a)$ is undefined and this algorithm will fail. Coplanar overlaps can be handled by identifying perfectly coplanar triangles during triangle-triangle intersection resolution. For each pair of resolved “duplicate” triangles $t_a$ and $t_b$, either keep one or discard both depending on the operation. For example, if conducting the difference of $B$ from $A$, then discard both if $t_a$ and $t_b$ have the same orientation and keep $t_a$ otherwise.

The classic notion of the winding number applies to any closed oriented surface, not just those bounding a solid. The winding number for such surfaces will be a possibly negative integer indicating how many times $\partial A \to \partial B$ determines from a triangle mesh $A$.

2 Generalized Winding Numbers

Jacobson et al. [2013] define the generalized winding number $w_X: \mathbb{R}^3 \setminus X \to \mathbb{R}$ for arbitrary oriented triangle meshes as the sum of the signed solid angles $\Omega_i$ subtended by each triangle $t$:

\[
w_X(p) = \sum_{t \in X} \frac{1}{4\pi} \Omega_i(p). \tag{7}
\]

This smooth function measures for each point in space how much it is inside $X$. For example, a punctured sphere will induce a winding number value close to one inside and far from the hole, and a value close to zero outside and far from the hole. Near the hole, the value will smoothly transition from near one to near zero passing through one half close by the hole.

We may immediately generalize the boolean algorithm in the previous section to arbitrary oriented triangle meshes. Simply replace $w_X(p) = 1$ with $w_X(p) > 1/2$; alternatively, with $\lvert w_X(p) \rvert > 1/2$ if treating negative regions as inside.

This proposed method avoids volumetric discretization of Jacobson et al. [2013] but enjoys their hierarchical winding number evaluation. Computing the generalized winding number as a floating point summation could potentially lead to incorrect assignment due to round-off. If evaluation points (i.e., barycenters from the other mesh) are not very close to open boundaries then this is a non-issue.

Reference

JACOBSON, A., KAVAN, L., AND SORKINE-HORNING, O. 2013. Robust inside-outside segmentation using generalized winding numbers. ACM Trans. Graph.