1. Introduction

Artinian modules have been studied by many researchers, but here we discuss new relationships between Artinian module and other concepts. In [6], Wisniewski showed a self projective module is a weakly supplemented module. In [2], Vanuynh and Wisniewski proved that, every f.g continuous right R-module is injective iff all simple right R-modules are injective. In [3], Matlis, developed a structure theory for Artinian modules over a comm Noetherian, 1-dimensional Cohen-Macaulay ring R. In [5], Abed proved that if every sub-module of M is a supplement in M, then M is an Artinian module. In Section 2, some properties of Artinian module are introduced. Also we studied some relations with others concepts in general. In Section 3, we choose two modules namely simple and sem-simple modules in order to explain the strong relation between Artinian module and these concepts. In Theorem 3.2, we proved that if M is projective, then it is Artinian. Finally, in Section 4, we look in general at the Artinian module over division ring. Also, we proved that any module M over a ring Z_n is Artinian module if n = 5 and not Artinian module if n = 6 where n belong to Z and greater than zero.

In this paper, we studied the relationships between Artinian module and two concepts namely semi-simple module and Division ring. Our main results of this paper are Theorem 3.3 and Theorem 4.5.
2. Properties of Artinian Module

In this section, we give some properties of Artinian module and related it with others concepts in general. In order to focus on Art-module we need to define the following basic concepts.

**Definition 2.1. (see [6]).** Let $R$ be a ring. A left $R$ module is an abelian group $M$ and an external law of composition $\mu : R \times M \rightarrow M$, subject to the conditions that for all $r; s \in R$ and $m; n \in M$ we have:

1. $(r; m + n) = (r; m) + (r; n)$;
2. $(r + s; m) = (r; m) + (s; m)$;
3. $(rs; m) = (r; (s; m))$;
4. if $1 \in R$, then $(1; m) = m$.

**Definition 2.2. (see [1]).** A module $M$ has (dcc) if it satisfy $M_1 \supseteq M_2 \supseteq M_n \ldots$ there exists $m+$ such that $M_m = M_n, \forall n \geq m$.

**Definition 2.3. (see [1]).** A module $M$ is called Artinian, if it satisfy all the conditions in Definition 2.2.

From (Definition 2.2 and Definition 2.3), there is a relationship between these definitions and another concept is called minimal element. Therefore we need to introduce some information about this concept. If any sub-modules of a module have minimum condition, then any non empty collection of these sub-modules has minimal element (i.e. any subset $A$ of some partially ordered set that is not greater than any other).

**Example 2.4. ([4]).** Let $M = (a/2^n) : a \in \mathbb{Z}, n \geq 0$ and consider $M$ as a $\mathbb{Z}$-module. Note here that $\mathbb{Z} \subseteq M \subseteq \mathbb{Q}$. Also let $T = M/N$. Then $T$ is Artinian module.

**Lemma 2.5. (see [6]).** IF every non-empty set of sub-modules of $M$ has a minimal element; then every factor module of $M$ is f.co-generated.

**Theorem 2.6.** Every non-zero Artinian $R$-module $M$ has a minimal sub-modules.

**Proof.** Let $M$ be an Artinian $R$-module such that $M \neq 0$. Let us take $A$ equal to the collection of Sub-mods of $M$ such that these Sub-mods are proper. Thus the Sub-mod $(0)$ belong to $A$. Therefore $A \neq \emptyset$. So has minimal element, say $K$. Hence $K$ is a minimal sub-module in $M$.

**Theorem 2.7.** If $\emptyset \neq N$ sub-modules of $M$ has a minimal-element, then $M$ is Artinian module.

**Proof.** From Lemma 2.5, every factor mod of $M$ is finitely co-generated. Let we have (dcc) of $M_i$ of submodule $N = (\cap N)/M$. So $M/N$ is finitely co-generated. Now we must obtain $N=M_k$ for each $k$, i.e. $M_{k+1}=M_k$ for all $i$ in $N$.

**Remark 2.8.** A module $M$ with only finitely many sub-modules is Artinian module and from [5], we found the following three examples.

**Example 2.9.** All finite abelian groups over integer numbers are Artinian module. On the other hand, $\mathbb{Z}_2$ is not Artinian module because $\mathbb{Z} \supseteq 2 \supseteq 4 \supseteq 8 \ldots$ is not (dcc).

**Example 2.10.** As every sub-module is subspace with dimension $\leq \dim(V)$ such that $V$ is vector space, we can say finite-dimensional of $V$ are Artinian module.

**The dual of (Example 2.10) is not true. (see Example.2.11).**

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**Example 2.11.** All the finite dimensional vector spaces are not Artinian module.
Theorem 2.12. Let $N_i$ be a sub-modules of an Artinian module $M$. Let be a quotient-module. Then $N_i$ and the quotient $M/N$ are Artinian modules.

Proof. We take $M$ is an Artinian module and $N$ sub-module of $M$. We have any sub-mod of $N$ is also a sub-module of $M$. Hence any (dcc) of $N$ is chain sub-mods in $M$. So the result follows. Now any sub-module of $M/N$ as $L/M$ such that $L$ is a sub-mod of $M$ and contain $N$. Let us say $L_1/N\supseteq L_2/N \supseteq \ldots$ be (dcc) of sub-modules in $M/N$. Thus $L_1 \supseteq L_2 \supseteq \ldots$ is a chain in $M$. Thus $L_m=L_n\ n\geq m$ for some $m$.

The next theorem explain that the sum of any Artinian modules is also Artinian module.

Theorem 2.13. Let $A=\sum A_i; i=1,\ldots, n$. If $A_1, A_2,\ldots, A_n$ are Artinian modules, then $A$ is also Artinian module.

Proof. Let $A=\sum A_i; i=1,\ldots, n$. Assume that $A$ is Artinian module. By induction on $n; if n=1$, there is nothing to prove. We claim that the case is true if $m<n$. Now $A=\sum A_i+An; i=1,\ldots, n-1$. Let $N=\sum A_i; i=1,\ldots, n-1$. Hence by induction hypothesis $N$ is Artinian. We have

$$(A/N)=(N+(A_n/N))=A_n/(N\cap A_n)$$

is also Artinian. Since $(A/N)$ and $N$ are Artinian, then $A$ is Artinian module.

3. Artinian Module and Semi-simple Property

In order to describe the relationship between Artinian module and semi-simple property, we need to introduce some information about semi-simple-mod. Any module $M$ is called semisimple module if every submodule of $M$ is simple. One of the most important idea say any sem-simple module is Artinian module.

Lemma 3.1. Any semi-simple module $M$ is Artinian module.

Proof. The proof is very easy, because for any sequence $M_1\supseteq M_2\supseteq \ldots M_n\supseteq \ldots$ of sub-module have decreasing property. This imply to the length is also decreasing (i.e $M$ is Artinian module).

In the next theorem, we explain the relationship between projective module and Artinian module.

Theorem 3.2. Let $M$ be an $R$-module. If $M$ is projective module, then it is Artinian module.

Proof. Let $M$ be an arbitrary sub-module of $R$. Then we have the following sequence

$$0\rightarrow M\rightarrow R\rightarrow R\rightarrow 0\ldots(*)$$

But all the modules over $R$ are projective. Then $N/M$ Then $N/M$ is projective. Hence $(*)$ is splits. such that $N\leq R$ and $N\cong (R/M)$ . Thus $R$ is semi-simple. So, every module $M$ of $R$ is semi-simple, Then $M$ is Artinian module (see Lemma 3.1).

The following theorem shows a sub-module of semi-simple module is Artinian module.

Theorem 3.3. (see [1]). Let $M$ be a semi-simple $R$-module. If $N$ is a non-zero sub-modules of $M$, then $N$ is Artinian as a module.

In order to understand Theorem 3.3, we must prove the next theorem.

Theorem 3.4. Let $M$ be an $R$-module. If $N\leq M$ and $N= M$, then is the sum of a collection of simple sub-modules.
**Proof.** First we must show $N=\bigoplus M$. Let $N\leq M$ and $K\subseteq N$. We have $M=K\bigoplus L$ for submodule $L$ of $M$. Hence $N=(K\bigoplus L)\cap N=(K\cap N)\bigoplus (L\cap N)=K\bigoplus (L\cap N)$. Thus $N=\bigoplus N$. We claim that every sub-module $N\neq 0$ contains a simple module. Let $0 \neq x \in N$. Let $x=L$ be a submodule of $N$ such that $x$ not in $L$. Now $Rx$ is a sub-modules of $N$ and hence $N=Rx+U$ for some $U$. Thus $x$ not in $U$. $\xi$ is non-empty. In order to benefit from zorn’s Lemma, we obtain a max-element $k$ in $J$. Then $N=K\bigoplus S$ for some sub-modules $S$ of $N$. We need to show $S$ is simple module. Suppose $S_0$ is a proper sub-modules of $S$. Then $S=S_0\bigoplus T$, $T\neq (0)$. Hence $N=K\bigoplus S_0\bigoplus T$. Now $K\bigoplus S_0$ and $K\bigoplus T$ both do not belong to $I$. Thus $x\in (K\bigoplus S_0)\cap (K\bigoplus T)$. Therefore $x=k_1+s=k_2+t$ for some $k_1$, $k_2\in K$, $s\in S_0$, $t\in T$. So $k_1-k_2+s-t=0$ and $k_1=k_2$, $s=t=0$. Thus $x=k_1=k_2\in K$ a contradiction. Hence $S$ is simple. Let $M_0=\sum \{S: S$ is simple of $M\}$ there exists a sub-module $U$ of $M$ such that $M=M_0\bigoplus U$. If $U=(0)$, we are done. If $U\neq (0)$, then $U$ contains a simple Sub-mod. Thus $M=M_0$ is a sum of collection of simple sub-modules and so $M$ is simple (see [1]). Hence $N$ semi-simple and so is Artinian sub-module.

**Corollary 3.5.** Sum of the semi-simple modules are Artinian module.  
**Proof.** Let $S=S_1+S_2+\ldots+S_n$ such that $S_1$, $S_2$, $\ldots$, $S_n$ are semi-simple mods. We have every sub-module of each $S_n$ is simple. Hence each $S_n$ are semi-simple and so $S$ is a semi-simple Thus $S$ Artinian module. (see Lemma 3.1).

**Remark 3.6.** We can rewrite a proof of (Theorem 2.12) by using semi-simple property in order to obtain $M/N$ is Artinian module. See the following. To prove quotient modules are Art-mod , we need to say if $M$ is a semi-simple module and $N$ subset of $M$ so $N$ Sub-mod of $M/N$. Then $N$ has complement of $N \in M$. i.e. $M=N\bigoplus NN^\perp$. Then $Q$ equal to complement of $N$ and so $Q$ is equivalent to a Sub-module $N$ of $M$. Thus $Q$ is semi-simple and hence is Artinian module.

### 4. Artinian Module Over Division Ring

In this section, we study Artinian module over important ring namely Division ring. Some new results has been obtained in this section.

**Definition 4.1.** (See [4]). Let $R$ be a ring. Then $R$ is called Division ring if every $0\neq a$ is an element of $R$ is a unit.

**Remark 4.2.** We know that $Q$, $R$, and $C$ are Division rings. But $Z$ and $M_n(R)$ are not Division rings. We say $R$ is commutative field if $R$ is a commutative Division ring. Also $Q$, $R$ and $C$ are fields but $Z$ is not a field (not a Division ring).

**Lemma 4.3.** Every Division ring has no zero divisors.  
**Proof.** Suppose that $R$ is a Division ring. We must prove that $R$ is integral domain. By contradiction, if $a,b=0$, so $a\neq 0$ and $b\neq 0$. But $0\neq a$ in $R$. So the inverse of $a$ in $R$. Thus $a^{-1}(ab)=a^{-1}(0)\neq 0=(a^{-1}a)b=1$. Contradiction.

**Theorem 4.4.** Let $D$ be a Division ring. Then every Division module is Artinian module.  
**Proof.** Let $V$ be a left Division module, and let $W$ be a sub-module of $V$. Let $B$ be a basis for $W$. Then there exists a basis, $B_1$, of $V$ which contains $B$. Let $B_2=\bigcap B$; and let $W_1$ be the span of $B_2$. Then $W_1\bigoplus W=V$. Thus $V$ is semi-simple module and so is Artinian module. Any module $M$ over integral domain is not Artinian module, because the converse of (Lemma 4.3) is not true in general. (i.e. there are integral domains which are not Division rings.

**Remark 4.5.** Let $(Z_n^*, +, \cdot)$ be the ring of integers (where $n^* \in Z$). This is a commutative ring.
**Theorem 4.6.** Let $D$ be a Division ring and let $n^+$ be an integer number. If $\text{End}_{D}(D_n)$ is a simple ring that is semi-simple as a left $\text{End}_{D}(D_n)$-module, then it is Artinian module.

**Proof.** If $R$ is semi-simple, then by Wedderburn’s Theorem (Let $R$ be a semi-simple ring, then there is a finitely collection of integers $n_1,\ldots, n_k$, and Division rings $D_1,\ldots, D_k$ such that $R \oplus \text{End}_{D_i}(D_{n_i}), i=1$ to $k$). Then when $k=1$ is true. Thus $\text{End}_{D}(D_n)$ is Artinian module.

**Example 4.7.** If $M$ is an $R$-module over the ring $(Z_n,+)$, then $M$ is Artinian module. Let $r$ any element in $Z_n$. We must prove $r$ has unit element (i.e. $r^{-1}$ exists). It is clear that the elements of $Z_n$ are $1,\ldots, n-1$, for example, if $n=5$, then the elements are $1, 2, 3, 4$ and so $1^{-1} = 1, 2^{-1} = 3, 3^{-1} = 2$ and $4^{-1} = 4$. So $Z_n$ is a division ring. Thus $M$ is Artinian module.

**Example 4.8.** Any $R$-module $M$ over $(Z_n,+)$ is not Art-mod such that $n=6$. We have $1^{-1} = 1$, but $2^{-1}, 3^{-1}, 4^{-1}$ have no inverse and $5^{-1} = 5$. So unit of $(Z_n) = 1, 5= <5 >$. Thus $Z_n$ is not a division ring such that $n=6$, and hence $M$ is not Artinian module.

**Example 4.9.** The Quaternions: Let $D$ be the set of all symbols $a_0 + a_1i + a_2j + a_3k$ where $a_i \in \mathbb{R}$. by term using the relations $i^2 = j^2 = k^2 = -1$ and $ij = -jk = k$; $jk = -ki = i$; $ki = -ik = j$. Then $D$ is a non-commutative ring with zero and identity. Let $a_0 + a_1i + a_2j + a_3k$ be a non-zero element of $D$. Then not all the ai are zero. We have $(a_0+ a_1i + a_2j + a_3k)(a_0+ a_1i + a_2j + a_3k) = (a_0)^2 + (a_1)^2+ (a_2)^2+ (a_3)^2 \neq 0$. So letting $n = (a_0)^2 + (a_1)^2+ (a_2)^2+ (a_3)^2$, the element $(a_0/n)+(a_1/n)i+(a_2/n)j+(a_3/n)k$ is the inverse of $a_0 + a_1i + a_2j + a_3k$. Thus $D$ is a D-ring. So p any module over $D$ is Artinian.

**Theorem 4.10.** Let $M$ be an $R$-module. If $R$ has identity element, then not necessary $R$ is comm, $R$ and $(0)$ are only left ideals. Then any module over $R$ is Artinian module.

**Proof.** We have $R$ not equal to $\{0\}$ as $1 \in R$. Assume $a \in R$ not equal zero. Let $aR=\{ar: r \in R\}$. So $aR$ is a right-ideal in $R$. Also $\{0\}$ and $R$ are right ideals in $R$. Hence $aR$ equal to $\{0\}$ or $aR$ equal to $R$. But $aR$ not equal to $\{0\}$ in case 1 in $R$, so $ar = a \in ar$. Therefore $aR= R$. We have 1 in $R$, so $a$ in $R$, $a=1$. So $a^3$ in $R$. Hence $a$ is an element such that only stipulation that $a$ not equal to zero, so any $0 \neq x \in R$, the $x^{-1}$ in $R$. Thus $R$ is a Division ring and hence $M$ is Artinian module.

**Corollary 4.11.** Let $R$ be a ring with $p^4$ elements. If $R$ have only $(0)$ and $R$ satisfy the conditions of ideals, then $R$ has not Artinian module.

**Proof.** Let $M$ be an $R$-module. Since $p\neq 2$, then $Q_p$ is a non-commutative f.ring. But Wedderburn’s Theorem gives a finite Division ring is comm. So $Q_p$ not be a Division ring, Thus $M$ is not Art-mod.

**Corollary 4.12.** Let $M$ be an $R$-module such that $R$ have only two right ideals $(0)$ and $R$. Then $R$ is a Division ring and then $M$ has Artinian module or $R$ is a ring has many prime number such that $ab=0$; $a$ and $b$ in $R$.

5. Conclusion

The relationship between Artinian module and other concepts was discussed in this paper. The main results are appeared clearly to achieve all objectives of this research, for example the first aim result is the sum of semi-simple-mod are Artinian module. The other result is, if $M$ satisfy all conditions of projective module, then it is Artinian. Also, any module $M$ over Division ring $(Z_n,+)$ is Artinian module.
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