Orbifolds with Discrete Torsion and Mirror Symmetry

Maximilian KREUZER

Institut für Theoretische Physik, Technische Universität Wien
Wiedner Hauptstraße 8–10, A-1040 Wien, AUSTRIA

and

Harald SKARKE

Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, D–30167 Hannover, GERMANY

ABSTRACT

For a large class of $N = 2$ SCFTs, which includes minimal models and many $\sigma$ models on Calabi-Yau manifolds, the mirror theory can be obtained as an orbifold. We show that in such a situation the construction of the mirror can be extended to the presence of discrete torsions. In the case of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ torus orbifold, discrete torsion between the two generators directly provides the mirror model. Working at the Gepner point it is, however, possible to understand this mirror pair as a special case of the Berglund–Hübsch construction. This seems to indicate that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ example is a mere coincidence, due to special properties of $\mathbb{Z}_2$ twists, rather than a hint at a new mechanism for mirror symmetry.

* e-mail: kreuzer@tph16.tuwien.ac.at
# e-mail: skarke@itp.uni-hannover.de
1 Introduction

In $N = 2$ superconformal field theories mirror symmetry (MS) is nothing but a change of sign of the left-moving $U(1)$ charge. In some cases, like the minimal models, the $U(1)$ charge conjugation matrix is a simple current modular invariant, i.e. a modular invariant that relates only primary fields on the same orbits of the simple currents under fusion (simple currents, by definition, have a unique fusion product, so that all primary fields are organized into orbits; they can be shown to imply discrete symmetries of the conformal field theory [1]). Via the correspondence between simple current modular invariants and orbifolds [4] this implies that the mirror theory of any orbifold of such a model is also an orbifold of the same conformal field theory. For the case of minimal models this was first shown by Greene and Plesser [3]. If the conformal field theory is related to a $\sigma$ model on a Calabi-Yau manifold, then this simple observation has very non-trivial consequences [4].

More generally, since MS exchanges the $(c,c)$ ring and the $(a,c)$ ring [1], we have a chance to obtain the mirror model of a diagonal SCFT by orbifolding only if the symmetry group is large enough to project out all fields in the chiral ring of the diagonal theory [1]. For Landau–Ginzburg (LG) models [6, 5] this is the case if a non-degenerate superpotential is a sum of $N$ monomials in $N$ superfields. Indeed, Berglund and Hübsch (BH) [7] found a map among orbifolds that produces the mirror theory for Landau–Ginzburg models and for the Calabi–Yau hypersurfaces based on this type of polynomials. More recently, it was checked by two different approaches that this map indeed produces the mirror theory: In [8] the elliptic genera of the dual orbifolds were compared. In [9] the map sending a monomial representation of a chiral ring element to a twist group element was constructed explicitly, and the twist selection rules of the orbifold were shown to be consistent with the OPE selection rule of the original theory.

In the present paper we show how this construction can be extended to arbitrary orbifolds, including discrete torsions [10]. It is well known that the BH construction cannot be extended to more general LG models [11, 12, 13]; in that case a construction of the mirror is available only in the geometrical framework of toric varieties [14]. For these, however, it is not yet clear how the freedom of choosing discrete torsions in the CFT approach can be realized.

Torus orbifolds provide a much smaller class of internal conformal field theories. The corresponding numbers of generations and anti-generations were listed in [15]. This list features two mirror pairs of spectra, each of them requiring discrete torsion (DT). This led to some speculations that DT might be essential for mirror symmetry [16]. More recently, Vafa and Witten [17] reinvestigated the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, whose striking feature is that the inclusion of discrete torsion directly provides the mirror model. The situation is different, however, for the second mirror pair which requires DT at both sides of the correspondence [15, 17]. We

\footnote{1 Note that a general $N = 2$ theory is a product of some neutral CFT with a $U(1)$ current algebra. This product theory, however, does not have a product modular invariant. Hence, although charge conjugation in the $U(1)$ factor is always described by a simple current modular invariant, this is not true in general for the complete $N = 2$ theory.}

\footnote{2 Throughout this paper, we work with the internal SCFT before the generalized GSO projection. In order to make contact with related $\sigma$ models we first need to project to integral charges. This just amounts to a restriction on the allowed twist groups and discrete torsions, so that these cases are covered by our more general discussion.}
will show that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case can be understood as a special case of the BH construction if we reinterpret the DT in a way that is specific to $\mathbb{Z}_2$ twists. In contrast to this, the $\mathbb{Z}_3 \times \mathbb{Z}_3$ and the $\mathbb{Z}_6 \times \mathbb{Z}_6$ orbifolds with $\mathbb{Z}_3$ torsions, despite having the same non-singlet spectrum, have different numbers of singlets and gauge bosons. Consequently, there is no generalized BH map between these two models, although a mirror model is straightforward to construct in each case (the calculation of the singlet number is, in fact, less tedious for the mirror models). This is consistent with the general picture that mirror symmetry, if at all related to orbifolding, operates within the subclass of symmetric orbifolds, but can be extended to accommodate discrete torsions.

In section 2 we will discuss the modding of quantum symmetries, i.e. symmetries that follow from the twist selection rules for orbifolds, and show how this technique can be used to explicitly construct the mirror twists and torsions whenever the mirror theory of some diagonal $N = 2$ SCFT can be obtained as an abelian orbifold of some – possibly different – diagonal theory. In the appendix we demonstrate this mechanism for an asymmetric $(\mathbb{Z}_3)^5$ orbifold in the tensor product $1^9$ of minimal models that has no generations and no anti-generations at all. Here the rank of the twist group for the mirror model is 6, rather than 4, as would be the case without DT. In section 3 we discuss the two candidate mirror pairs of torus orbifolds by applying the methods of section 2 at Gepner points in moduli space.

### 2 Mirror map and discrete torsion

Consider a situation where the mirror model to a diagonal $N = 2$ theory $\mathcal{C}$ is given by an abelian orbifold of a possibly different diagonal $N = 2$ theory $\overline{\mathcal{C}}$. In case of the minimal models we have $\mathcal{C} = \overline{\mathcal{C}} [3]$, but for Landau–Ginzburg models that are described by sums of polynomials of the type

\[
W_{\text{loop}} = X_1^{a_1} X_2 + \ldots + X_n^{a_{n-1}} X_n + X_n^{a_n},
\]

\[
W_{\text{chain}} = X_1^{a_1} X_2 + \ldots + X_n^{a_{n-1}} X_n + X_n^{a_n},
\]

it was first observed in [4] that one has to twist the theories described by the respective ‘transposed’ potentials

\[
\overline{W}_{\text{loop}} = \overline{X}_1^{\overline{a_1}} \overline{X}_2 + \ldots + \overline{X}_n^{\overline{a}_{n-1}} \overline{X}_n + \overline{X}_n^{\overline{a}_n},
\]

\[
\overline{W}_{\text{chain}} = \overline{X}_1^{\overline{a_1}} + \overline{X}_1^{\overline{a}_2} + \ldots + \overline{X}_n^{\overline{a}_n}
\]

in order to obtain the mirror models. (The minimal models with diagonal modular invariant correspond to Fermat potentials $X^a$, i.e. to the case $n = 1$, whereas the $E_7$ invariant and the $D$ series can be described by $W_{\text{chain}}$ with $n = 2$.)

Obviously, the SCFT based on $\overline{W}$ is in general different from the one based on $W$. The orders of the maximal phase symmetry groups of $W$ and $\overline{W}$, however, are equal. Explicitly,

\[
|G_{\text{loop}}| = a_1 a_2 \ldots a_n - (-1)^n, \quad |G_{\text{chain}}| = a_1 a_2 \ldots a_n.
\]

$G$ is generated by

\[
\rho_l X_j = e^{2\pi i \varphi_j^{(l)}} X_j, \quad \varphi_j^{(l)} = \frac{(-1)^{n-j} a_l \ldots a_{l+j-1}}{|G_{\text{loop}}|} \quad \text{for } 0 \leq j < n
\]
for loop potentials and by
\[ \rho_l X_j = e^{2\pi i \varphi_j^{(l)}} X_j, \quad \varphi_j^{(l)} = \frac{(-1)^{l-j+1}}{a_j \ldots a_l} \quad \text{for} \quad 1 \leq j \leq l, \quad \varphi_j^{(l)} = 0 \quad \text{for} \quad j > l \] (7)
for chain potentials.

The transformation of \( X_1 \) determines the complete group action, hence \( \mathcal{G} \) is cyclic. \( \mathcal{G} \) is generated by \( \rho_n \) in both cases; for loop potentials any transformation \( \rho_i \) generates the whole group. It is, nevertheless, useful to consider the complete set of transformations \( \rho_i \), because it can be checked that the phase of the determinant of \( \rho_i \) is equal to \( 2\pi \) times the charge of the field \( X_i \). Furthermore, the phase of \( \det \rho \) essentially determines the left-right asymmetry of the U(1) charges of all states in the sector twisted by \( \rho \) in the orbifold theory [18] (for details see [9]).

This suggests that the mirror map should map the chiral field \( \overline{X}_i \) to a field in the sector twisted by \( \rho_i \), and consequently, because of the chiral ring structure and the twist selection rules, a monomial \( \prod \overline{X}_i^{\alpha_i} \) into the sector twisted by \( \prod \rho_i^{\alpha_i} \). The number of states in such a sector and their charges have been shown to be consistent with the identification of the two theories as a mirror pair [3].

Further evidence for the identification comes from a consideration of discrete symmetries of the SCFT defined by the superpotential \( W \): Since the original LG model is symmetric, all moduli come from the \((c,c)\) ring. But the maximal phase symmetry projects out all \((c,c)\) states (as is necessary for obtaining a SCFT with asymmetric charges for the orbifold). Assume that the orbifold \( W/\mathcal{G} \), which has the correct charge degeneracies of the chiral ring and a set of OPE selection rules that is in one-to-one correspondence with the vanishing relations of the chiral ring, lies in the moduli space of the mirror SCFT of \( \overline{W} \). Then we can conclude that it must indeed be the mirror model since all moduli are fixed by the presence of the quantum symmetry that arises from the modding by \( \mathcal{G} \). Furthermore, the elliptic genera of the two theories have been shown to coincide [8]. That information goes beyond the chiral ring, but it is insensitive to the SUSY preserving moduli of the conformal field theory.

Returning to the general discussion, we consider a ring basis \( \overline{X}_i \) of chiral fields of some diagonal superconformal theory \( \overline{\mathcal{C}} \), which we assume to be the mirror model to an abelian orbifold \( \mathcal{C}/\mathcal{G} \) of another diagonal theory \( \mathcal{C} \). Then the mirror map must induce a map \( \overline{X}_i \rightarrow \rho_i \), where \( \rho_i \in \mathcal{G} \) is the twist of the sector that contains the image of \( \overline{X}_i \) under the mirror map. Since \( \mathcal{C}/\mathcal{G} \) and \( \overline{\mathcal{C}} \) are isomorphic CFTs (up to a redefinition of \( U(1) \) charges), any orbifold of one of the theories must have a counterpart as an orbifold of the other theory. In particular, the mirror model of \( \mathcal{C} \) must be described by the orbifold \( \overline{\mathcal{C}}/\mathcal{G} \) with a certain symmetry group \( \overline{\mathcal{G}} \) of \( \overline{\mathcal{C}} \) that is isomorphic to \( \mathcal{G} \), because we can obtain \( \mathcal{C} \) as an orbifold of \( \overline{\mathcal{C}}/\mathcal{G} \) with respect to the quantum symmetry that comes with the twist group \( \mathcal{G} \) (see below).

Our aim is to give an explicit description of the mirror orbifold in case of a general quotient \( \overline{\mathcal{C}}/\overline{\mathcal{H}} \) with respect to a subgroup \( \overline{\mathcal{H}} \subseteq \overline{\mathcal{G}} \) of the symmetry group \( \overline{\mathcal{G}} \) of \( \overline{\mathcal{C}} \), where we allow arbitrary discrete torsions (as a concrete example, we can keep in mind the case of the BH construction). If some group element \( \overline{\sigma} \in \overline{\mathcal{H}} \) acts on a diagonal basis \( \overline{X}_i \) of the chiral ring of \( \overline{\mathcal{C}} \) as \( \overline{\sigma} \overline{X}_i = \overline{\tau}_i \overline{X}_i \) (with phases \( \overline{\tau}_i \)), then it acts on the chiral primary states (which are all in the untwisted sector) as
\[ \overline{\sigma} \left( \prod \overline{X}_i^{\alpha_i} |0\rangle \right) = \left( \prod \overline{\tau}_i^{\alpha_i} \right) \left( \prod \overline{X}_i^{\alpha_i} |0\rangle \right). \] (8)
Then the mirror twist $q_{\mathcal{T}}$, which is a symmetry transformation of $\mathcal{C}/\mathcal{G}$, has to satisfy
\[ q_{\mathcal{T}}\left(\ldots|\prod\rho_i^{\overline{\tau}_i}\right) = \left(\prod\overline{\tau}_i\right)\left(\ldots|\prod\rho_i^{\overline{\tau}_i}\right), \tag{9} \]
where $|\rho\rangle$ is the ground state of the sector that is twisted by $\rho = \prod\rho_i^{\overline{\tau}_i}$. This can be achieved with the following rule: To any classical symmetry transformation $\tilde{\rho} \in \mathcal{H}$ of $\mathcal{C}$ we assign a quantum symmetry transformation $q_\mathcal{Q}$ of $\mathcal{C}/\mathcal{G}$, which acts trivially on all fields in $\mathcal{C}$ and has discrete torsions with elements $\rho_i$ of $\mathcal{G}$ that are equal to the action of $\tilde{\rho}$ on $X_i$, i.e. $\varepsilon(q_\mathcal{Q}, \rho_i) = \overline{\tau}_i$ (remember that a DT $\varepsilon(g, h)$ between $g$ and $h$ implies that $g|h$ picks up an extra factor of $\varepsilon(g, h)$ compared to the case without DT). Torsions between elements $\tilde{g}, \tilde{h} \in \mathcal{H}$ are translated to torsions between $q_g$ and $q_h$. Modding by a subgroup $\mathcal{H}$ of the full group $\mathcal{G}$ of symmetries of $\mathcal{C}$ hence corresponds to modding $\mathcal{C}$ by $\mathcal{G} \otimes \mathcal{Q}_\mathcal{H}$, where $\mathcal{Q}_\mathcal{H}$ is the group of quantum symmetry transformations $q_\mathcal{Q}$ that correspond to the transformations $\tilde{\rho} \in \mathcal{H}$. The $q_\mathcal{Q}$ act trivially on $\mathcal{C}$ and have the described discrete torsions with $\rho_i \in \mathcal{G}$ and among themselves.

Since the elements of $\mathcal{Q}_\mathcal{H}$ are trivial symmetries of $\mathcal{C}$—they act on $\mathcal{C}$ like the identity—we expect that it should be possible to get rid of them and to replace the modding with twist group $\mathcal{G} \otimes \mathcal{Q}_\mathcal{H}$ by a modding with an equivalent twist group $\mathcal{H} \subseteq \mathcal{G}$ that does not contain any ‘pure’ quantum symmetries. Indeed, this is possible by repeated application of the following lemma.

Reduction of quantum symmetries: If an abelian twist group has a quantum generator $q$ of order $n$, we can rewrite the twists in such a form that $q$ has non-trivial torsion $\varepsilon(q, g) = \zeta_n = e^{2\pi i/n}$ with only a single generator $g$ whose order $N$ is a multiple of $n$. Then the original orbifold is equivalent to the one with $q$ omitted and with $q$ replaced by $q' = q^n$.

In order to prove this statement we have to compare the conformal field theories $\mathcal{C}_0 = \mathcal{C}/(\mathcal{H} \otimes \mathcal{G}')$ and $\mathcal{C}_q = \mathcal{C}/(\mathcal{H} \otimes \mathcal{G} \otimes \mathcal{Q})$, where $g'$, $g$ and $q$ generate the cyclic groups $\mathcal{G}'$, $\mathcal{G}$ and $\mathcal{Q}$, respectively (the discrete torsions are of the form described above; $\mathcal{H}$ could be non-abelian without spoiling the argument). The latter orbifold is a quotient by a quantum symmetry: $\mathcal{C}_q = \mathcal{C}_q/\mathcal{Q}$ with $\mathcal{C}_q = \mathcal{C}/(\mathcal{H} \otimes \mathcal{G})$. It is obvious that this quotient eliminates all sectors with a twist by $g'^i$ whenever $l$ is not a multiple of $n$. More explicitly, the $\mathcal{Q}$ projection keeps exactly the $g$-invariant states in $\mathcal{C}_0$. Since all states $|i\rangle$ in $\mathcal{C}_0$ are invariant under $g'$, the transformation of $|i\rangle$ under $g$ is given by a phase $(\zeta_n)^{\mu_i}$ in a diagonal basis of the Hilbert space. Since $q$ is a pure quantum symmetry, a twist by $q^\mu$ does not change any quantum number of a state except for its transformation property under $g$. Hence there is exactly one copy of each state in $\mathcal{C}_q$, namely the one in the sector twisted by $q^\mu$, which survives the $g-$projection in the orbifold $\mathcal{C}_q = (\mathcal{C}/(\mathcal{H} \otimes \mathcal{G}))/\mathcal{Q}$.

If there are no discrete torsions among the generators of the quantum subgroup $\mathcal{Q}$ of a twist group $\mathcal{G} = \mathcal{Q} \otimes \mathcal{H}$, a strict separation between pure classical and pure quantum symmetries is maintained throughout this elimination process. Each elimination of a quantum generator simply amounts to reducing the group of classical symmetries to the subgroup that has trivial torsion with this generator. So we end up with the group of symmetries that have vanishing torsion with all the elements of $\mathcal{Q}$.

Returning to the case of an orbifold construction of the mirror models we have to twist $\mathcal{C}$ by the group $\mathcal{G} \otimes \mathcal{Q}_\mathcal{H}$, where the modding by $\mathcal{G}$ produces the mirror SCFT of $\mathcal{C}$ and $\mathcal{Q}_\mathcal{H}$ corresponds to the modding $\mathcal{H}$ of $\mathcal{C}$ for which we want to construct the mirror model. If there are no discrete torsions in $\mathcal{H}$, elimination of the quantum twists thus gives us the ‘dual’
subgroup $\mathcal{H}$ satisfying $|\mathcal{H}|/|\mathcal{H}| = |\mathcal{G}| = |\mathcal{G}|$ without any torsions, as in the original work of Berglund and Hübisch\cite{Berglund}. In the case of torsions the situation is more complicated and torsions among the quantum symmetry generators will induce torsions among the classical twists. In particular, the order of the group $\mathcal{H}$ we end up with can be larger than $|\mathcal{G}|/|\mathcal{H}|$.

In the appendix this is demonstrated for an $\mathcal{H} = (\mathbb{Z}_3)^5$ orbifold of the model $1^9$ for which $\mathcal{G} = (\mathbb{Z}_3)^6$. Without torsions, the dual orbifold comes from a twist group of rank 4, but in the example that we discuss we end up with $\mathcal{H} = (\mathbb{Z}_3)^6$.

### 3 Torus orbifolds

The mirror pair considered in ref. \cite{Berglund} consists of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the product of three tori without and with DT between the two $\mathbb{Z}_2$ factors. The twist group is generated by $\sigma_1 \sigma_2$ and $\sigma_1 \sigma_3$, where $\sigma_i : z_i \rightarrow -z_i$ changes the sign of the complex coordinate of the $i^{th}$ torus. In the Weierstraß normal form of the embedding of an elliptic curve in $\mathbb{P}^2$,

$$x_0 x_2^2 = 4x_1^3 - g_2 x_1 x_0^2 - g_3 x_0^3$$ \hspace{1cm} (10)

this symmetry corresponds to $x_2 \leftrightarrow -x_2$. At $g_2 = 0$, which corresponds to a torus with complex structure modulus $\tau = \exp(2\pi i/3)$, we have an additional $\mathbb{Z}_3$ symmetry which will allow us to apply the BH construction. To this end consider the potential

$$W_D = X^2 Y + Y^3 + Z^3,$$ \hspace{1cm} (11)

obtained from (10) with $g_3 = g_2 = 0$ by a change of variables, and its phase symmetries

$$\rho_X(X, Y, Z) = (-X, Y, Z), \quad \rho_Y(X, Y, Z) = (\zeta_6 X, \zeta^2 Y, Z), \quad \rho_Z(X, Y, Z) = (X, Y, \zeta^2 Z).$$ \hspace{1cm} (12)

Since $\rho_X = \rho_Y^3$, the group of phase symmetries is generated by $\rho_Y$ and $\rho_Z$; a torus is obtained by modding the $\mathbb{Z}_3$ generated by $j = \rho_Y^3 \rho_Z^2$. The first two terms in $W_D$ are just the LG representation of the $D$ invariant of the minimal model at level 4 \cite{Berglund}, which is the $\mathbb{Z}_2$ orbifold of the $A$ invariant.

The classical $\mathbb{Z}_2$ symmetry $\rho_X$ of $W_D$ corresponds to the quantum $\mathbb{Z}_2$ symmetry of

$$W_A = X^2 + Y^6 + Z^3$$ \hspace{1cm} (13)

modded by its canonical $\mathbb{Z}_6$ symmetry

$$j(X, Y, Z) = (\zeta_2 X, \zeta_6 Y, \zeta_3 Z).$$ \hspace{1cm} (14)

Therefore the product of three tori, modded by $\mathbb{Z}_2 \times \mathbb{Z}_2$, can be represented by

$$W = \sum_{i=1}^3 \left(X_i^2 + Y_i^6 + Z_i^3\right),$$ \hspace{1cm} (15)

modded by $(\mathbb{Z}_6)^3 \times (\mathbb{Q}\mathbb{Z}_2)^2$. Let us choose generators $j_i^{(2)} = j_i^3$ (of order 2) and $j_i^{(3)} = j_i^2$ (of order 3) for each of the $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$ factors and $q_{12}$ and $q_{13}$ for $\mathbb{Q}\mathbb{Z}_2$, such that $q_{12}$ has torsion $-1$ with $j_1^{(2)}$ and $j_2^{(2)}$, $q_{13}$ has torsion $-1$ with $j_1^{(2)}$ and $j_3^{(2)}$ and all other torsions
between quantum and classical symmetries are trivial. The case without torsion between the two $\mathbb{Z}_2$ factors acting on the torus corresponds to trivial torsion between $q_{12}$ and $q_{13}$. In this case the rules given above can easily be used to show that the modding of $W$ by $(\mathbb{Z}_6)^3 \times (\mathbb{QZ}_2)^2$ can be reduced to a modding by $(\mathbb{Z}_3)^3 \times \mathbb{Z}_2$ generated by $j_i^{(3)}$, $i = 1, 2, 3$ and $j^{(2)} = j_1^{(2)} j_2^{(2)} j_3^{(2)}$.

The mirror of this model is $W$ modded by the group $(\mathbb{Z}_6)^6 \times (\mathbb{QZ}_3)^3 \times \mathbb{QZ}_2$ generated by

$$\rho_{x_i}, \rho_{y_j}, \rho_{z_k}, q_j^{(3)}, q_j^{(2)}.$$ (16)

The cancellations of the quantum generators take place independently among generators of order 2 and generators of order 3: We can replace

$$\rho_{x_i}, \rho_{y_j}^{(2)}, q_j^{(2)}$$ (17)

by

$$j_i^{(2)}, \rho_{x_1} \rho_{x_2}, \rho_{x_1} \rho_{x_3},$$ (18)

and we can replace each of the sets

$$\rho_{y_i}^{(3)}, \rho_{z_i}, q_j^{(3)}$$ (19)

(with $i = 1, 2, 3$) by $j_i^{(3)}$. So the mirror model is given by $(\mathbb{Z}_6)^3 \times (\mathbb{Z}_2)^2$, where the $\mathbb{Z}_6$ factors again correspond to the canonical $\mathbb{Z}_6$ symmetries of the three parts and the two $\mathbb{Z}_2$'s are generated by $\rho_{x_1} \rho_{x_2}$ and $\rho_{x_1} \rho_{x_1}$.

As the generators $\rho_{x_i}$ act only on the trivial fields $X_i$ which do not contribute to the chiral ring, their action depends only on the twist of chiral state in the orbifold. But a symmetry acting only on the twisted vacua, and not on the chiral fields of the untwisted theory, may as well be regarded as a quantum symmetry with properly chosen discrete torsion.\(^3\) To see how this works in detail we use the result of Intriligator and Vafa\(^4\) for the action of a group element $g$ on the ground state $|h\rangle$ of a twisted sector\(^5\):

$$g|h\rangle = \varepsilon(g, h) \det_h(g) |h\rangle,$$ (20)

where $\det_h(g)$ is the determinant of $g$ restricted to the subspace of superfields of the LG theory that are twisted by $h$ (we work in a basis where both $g$ and $h$ are diagonal). If $g$ acts only on trivial fields, then $\det_h(g) = \det_g(h) = (-1)^{n_{g,h}}$, where $n_{g,h}$ is the number of trivial fields on which the actions of $g$ and $h$ are $-1$. Hence we can replace $g$ by a symmetry that acts trivially on all fields (i.e. a quantum symmetry) and, at the same time, $\varepsilon(g, h)$ by $(-1)^{n_{g,h}} \varepsilon(g, h)$. It is easily checked that this substitution produces the correct transformations also in all twisted sectors. In our above representation of the mirror of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold we can thus identify $\rho_{x_1} \rho_{x_2}$ with $q_{12}$ and $\rho_{x_1} \rho_{x_3}$ with $q_{13}$, with the same torsions between classical and quantum generators as above, but with nontrivial DT of $-1$ between $q_{12}$ and $q_{13}$. Thus we have indeed related the $\mathbb{Z}_2 \times \mathbb{Z}_2$ torus orbifold without torsion to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ torus orbifold with torsion.

In the list of torus orbifolds in ref.\(^3\) there is another candidate for a mirror pair: The $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold with the actions of the generators $g_i$ on the complex coordinates $z_i$ given

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\(^3\) The fact that $\mathbb{Z}_2$ symmetries acting on trivial fields can mimic DT has already been exploited in\(^2\).

\(^4\) Our conventions differ from\(^1\) by a factor $(-1)^{K_g K_h}$ in the definition of the discrete torsion $\varepsilon(g, h)$, where $(-1)^{K_g}$ is the discrete torsion between a group element $g$ and the $\mathbb{Z}_2$ twist that generates the Ramond sector.
by multiplication with \((\zeta_3, 1, \zeta_3^{-1})\) and \((1, \zeta_3, \zeta_3^{-1})\), and with a discrete torsion of \(\varepsilon(g_1, g_2) = \zeta_3\)
among the generators, has the spectrum \(n_{27} = 27\) and \(n_{27} = 3\). The \(\mathbb{Z}_6 \times \mathbb{Z}_6\) orbifold with
the actions of the generators \(g_i\) given by \((\zeta_6, 1, \zeta_6^{-1})\) and \((1, \zeta_6, \zeta_6^{-1})\) and with the same discrete torsion \(\varepsilon(g'_1, g'_2) = \zeta_3\) has \(n_{27} = 3\) and \(n_{27} = 27\). We will show now, however, that this mirror
pairing of spectra cannot be understood by the BH construction. Moreover, if we compute
the complete massless spectrum, it turns out that the numbers of gauge bosons disagree.

In order to implement the \(\mathbb{Z}_3 \times \mathbb{Z}_3\) symmetry in a LG model we have to represent each
of the three tori by a potential \(W_i, i = 1, 2, 3\) like, for example, \(W_A\) or \(W_D\) (cf. eqs. (13,11)). The \(\mathbb{Z}_3 \times \mathbb{Z}_3\) orbifold is then represented by the LG potential \(W = W_1 + W_2 + W_3\) modded
by the group \(\mathcal{H}\) generated by \(j_1, j_2, j_3, g_1^{(3)}, g_2^{(3)}\), with nontrivial DT only between \(g_1^{(3)}\) and \(g_2^{(3)}\). The latter generators (of orders 3) correspond to the generators acting on the complex
coordinates \(z_i\) by multiplication with \((\zeta_3, 1, \zeta_3^{-1})\) and \((1, \zeta_3, \zeta_3^{-1})\). The BH mirror of this model is described by \(\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3\) modded by some group \(\mathcal{H}\). Here we note again that the construction of the mirror twist group takes place independently for each prime number. In particular going from \(\mathcal{H}\) to a different twist group \(\mathcal{H}\) by dropping \(g_1^{(3)}\) and \(g_2^{(3)}\) does not affect the mirror construction in the rank 2 sector. But \(W/\mathcal{H}\) is just the product of three tori, whose
mirror \(W/\mathcal{H}\) is well known to be the product of three tori, again. If we could identify \(W/\mathcal{H}\) with a \(\mathbb{Z}_6 \times \mathbb{Z}_6\) torus orbifold, we could identify \(W/\mathcal{H}\) with a \(\mathbb{Z}_2 \times \mathbb{Z}_2\) torus orbifold, which is clearly a contradiction.

The observation that there is no BH construction of a mirror map relating the two torus orbifolds under consideration suggests that, after all, they may not be a real mirror pair. We should thus perform a more stringent test of this possibility. Since we have a representation of the orbifolds as exactly solvable conformal field theories, the natural next step is to compute
the complete massless spectrum. There exists an extensive list of such spectra [20], which
should contain the information we need. Indeed, we find the numbers \(\{27, 3\}\) for \(\{n_{27}, n_9\}\)
among the \((2,2)\) vacua that were constructed in [20] from the tensor products \(1^9\) and \(1^34^3\)
of minimal models. For \(1^9\), which corresponds to \(W_i = X^3 + Y^3 + Z^3\), the only result that is consistent with the tables supplement of [20] is \(n_S = 252\) and \(n_V = 8\), where \(n_S\) is the number of singlets and \(n_V\) is the number of extra gauge bosons. For the product \(1^34^3\), which corresponds to \(W_i = X^3 + Y^6 + Z^2\), there is also a second possibility, which is \(n_S = 213\) and \(n_V = 5\).

The \(\mathbb{Z}_3 \times \mathbb{Z}_3\) orbifold can be represented as a phase orbifold of \(1^9\), hence we know that
its spectrum must be given by \(n_S = 252\) and \(n_V = 8\). Unfortunately, this is not quite enough
for our purpose since we still need to compute at least one of the number \(n_S\) and \(n_V\) for the
\(\mathbb{Z}_6 \times \mathbb{Z}_6\) orbifold. Once more, the BH construction can be used to simplify the calculation:
Starting with the representation of the \(\mathbb{Z}_6 \times \mathbb{Z}_6\) torus orbifold as a \((\mathbb{Z}_6)^5\) quotient of the LG model with the potential \(W = W_1 + W_2 + W_3\) with \(W_i = X_i^3 + Y_i^6 + Z_i^2\) it is straightforward to construct the BH mirror. We find a \(\mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_3\) quotient of \(W\) with generators \(g_1 = j_1j_2j_3\),
\(g_2 = (j_1)^2\), \(g_3 = (j_2)^2\), where \(j_i\) generates the canonical \(\mathbb{Z}_6\) twist of \(W_i\), and the only torsion is \(\varepsilon(g_2, g_3) = \zeta_3\). For that orbifold the number of extra \(U(1)\)'s can easily be computed to be \(n_V = 5\) (all of them come from descendents due to the \(U(1)\)'s of the factors of the tensor
products). Hence the two torus orbifolds indeed have different singlet spectra and do not form
a mirror pair\footnote{As a check, the same calculation can be repeated for the \(\mathbb{Z}_3 \times \mathbb{Z}_3\) orbifold, represented as a \((\mathbb{Z}_6)^3 \times (\mathbb{Z}_3)^2\)
Appendix

As an example for the general construction of the mirror orbifold consider the model (1)
which can be described by the LG–potential
\[ W = X_1^3 + X_2^3 + X_3^3 + Y_1^3 + Y_2^3 + Y_3^3 + Z_1^3 + Z_2^3 + Z_3^3. \]  
(21)
The symmetry group \( G = (\mathbb{Z}_3)^9 \) is generated by
\[ \rho_{X_i}, \quad \rho_{Y_i}, \quad \rho_{Z_i}, \quad i = 1, 2, 3, \]  
(22)
where \( \rho_{X_i} \) acts on \( X_i \) with a phase \( 1/3 \) and leaves all other fields invariant, and similarly for \( Y \) and \( Z \). We choose the twist group \( \mathcal{H} \) that is generated by
\[ \rho_X = \rho_{X_1}\rho_{X_2}\rho_{X_3}, \quad \rho_Y = \rho_{Y_1}\rho_{Y_2}\rho_{Y_3}, \quad \rho_Z = \rho_{Z_1}\rho_{Z_2}\rho_{Z_3}, \quad \rho_1 = \rho_{X_1}\rho_{Y_1}\rho_{Z_1}, \quad \rho_2 = \rho_{X_2}\rho_{Y_2}\rho_{Z_2} \]  
(23)
and the discrete torsions
\[ \varepsilon(\rho_X, \rho_Y) = \varepsilon(\rho_X, \rho_Z) = \varepsilon(\rho_Y, \rho_Z) = 1, \]
\[ \varepsilon(\rho_X, \rho_1) = \varepsilon(\rho_Y, \rho_1) = \varepsilon(\rho_Z, \rho_1) = \varepsilon(\rho_1, \rho_2) = \exp(4\pi i/3). \]  
(24)
In a matrix representation
\[ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \]
(25)
where the phases of the twists and discrete torsions are all in units of \( 1/3 \). The ‘Hodge diamond’ of this model can be calculated with the methods of refs. \[18\]. In practice we used a computer program that we developed for the investigation of ref. \[21\] to obtain
\[ \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \]
\[ \begin{pmatrix} 3 & 0 \\ 3 & 0 \\ 1 \end{pmatrix}, \]
(26)
i.e. the charge degeneracy is 3 for chiral ring elements with left and right \( U(1) \) charges \( (q_L, q_R) \in \{(1, 0), (2, 0), (1, 3), (2, 3)\} \), whereas there are no chiral states with \( q_R = 1 \) or \( q_R = 2 \).

According to the construction we gave in section 2, the mirror to this model is given by the quotient of \( W \) by the group \( \mathcal{G} \otimes \mathcal{Q_H} \) that is generated by
\[ \rho_{X_i}, \quad \rho_{Y_i}, \quad \rho_{Z_i}, \quad q_X, \quad q_Y, \quad q_Z, \quad q_1, \quad q_2. \]  
(27)
quote
The $q$-generators act trivially on all fields, but have torsions given by

$$\varepsilon(\rho_X, q_X) = \varepsilon(\rho_Y, q_Y) = \varepsilon(\rho_Z, q_Z) =$$
$$\varepsilon(\rho_{X_1}, q_1) = \varepsilon(\rho_{Y_1}, q_1) = \varepsilon(\rho_{Z_1}, q_1) = \varepsilon(\rho_{X_2}, q_2) = \varepsilon(\rho_{Y_2}, q_2) = \varepsilon(\rho_{Z_2}, q_2) =$$
$$\varepsilon(q_X, q_1) = \varepsilon(q_X, q_2) = \varepsilon(q_Y, q_1) = \varepsilon(q_Y, q_2) = \varepsilon(q_Z, q_1) = \varepsilon(q_Z, q_2) =$$
$$\varepsilon(q_1, q_2) = \exp(4\pi i/3),$$

with all other torsions being trivial. Changing our set of generators to

$$\rho_X, \rho_{X_1}/\rho_{X_2}, \rho_Y, \rho_{Y_1}/\rho_{Y_2}, \rho_Z, \rho_{Z_1}/\rho_{Z_2},$$
$$\rho_{X_1}/q_1, \rho_{X_1}\rho_{Y_1}/q_1, \rho_{X_1}\rho_{Y_1}\rho_{Z_1}/q_1, q_X, q_Y, q_Z, q_1/q_2, q_1,$$

we see that in this set $q_X$ has nontrivial DT only with $q_1$, so we can eliminate these two generators. But then $q_Y$ has nontrivial DT only with $\rho_{X_1}/q_1$, so we also cancel these two generators. In a third step we notice that now $q_Z$ has nontrivial DT only with $\rho_{X_1}\rho_{Y_1}/q_1$, so after eliminating these two generators we have reduced our set to

$$\rho_X, \rho_{X_1}/\rho_{X_2}, \rho_Y, \rho_{Y_1}/\rho_{Y_2}, \rho_Z, \rho_{Z_1}/\rho_{Z_2}, \rho_{X_1}\rho_{Y_1}\rho_{Z_1}/q_1, q_1/q_2,$$

which is equivalent to a new basis given by

$$\rho_X, \rho_{X_3}\rho_{Y_1}\rho_{Z_1}/q_1, \rho_Y, \rho_{Y_3}\rho_{X_1}\rho_{Z_1}/q_1, \rho_Z, \rho_{Z_3}\rho_{X_1}\rho_{Y_1}/q_1, \rho_{X_1}\rho_{Y_1}\rho_{Z_1}/q_1, q_1/q_2,$$

where we can cancel the last two generators. Now all torsions come from the $q_1$'s, so this is equivalent to an orbifold with a twist group $\mathcal{H}$ generated by

$$\rho_X, \rho_Y, \rho_Z, \rho_{X_3}\rho_{Y_1}\rho_{Z_1}, \rho_{Y_3}\rho_{X_1}\rho_{Z_1}, \rho_{Z_3}\rho_{X_1}\rho_{Y_1},$$

with trivial torsions among the first and last three generators and torsions of $\exp(4\pi i/3)$ between any of the first and any of the last three generators. In a matrix notation we thus find

$$\mathcal{H} \sim \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad \varepsilon \sim \begin{pmatrix}
0 & 0 & 0 & 2 & 2 & 2 \\
0 & 0 & 0 & 2 & 2 & 2 \\
0 & 0 & 0 & 2 & 2 & 2 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}$$

for the mirror of the model of (26). The spectrum of this model is indeed again described by (26), which is its own mirror spectrum.

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