Dynamic pull-in for micro–electromechanical device with a current-carrying conductor

Ji-Huan He¹, Daulet Nurakhmetov², Piotr Skrzypacz² and Dongming Wei²

Abstract
The initial value problem for a lumped parameter model arising from design of magneto–electromechanical device with a current-carrying conductor is analyzed. The differential equation is nonlinear because it includes the magnetic force term. The analysis for the dynamic pull-in occurring in the system is presented. The pull-in threshold is given analytically in terms of model parameters. Sufficient conditions for the existence of periodic solutions are proved analytically and verified numerically. The results can be useful for understanding and design of one-degree-of-freedom models of magnetically actuated beams.

Keywords
Micro-electromechanical systems, nonlinear oscillator, pull-in, current-carrying conductor, amplitude–frequency relation

Introduction
Pull-in effect occurs in micro-electromechanical systems (MEMS) at certain thresholds. The pull-in analysis of electrostatically actuated devices is very important for the efficient operation conditions and reliability of these devices. The analysis of the dynamic pull-in for MEMS models under applied voltages has been well established in literature for linear elastic materials, e.g., Younis.¹ It is well known that the static pull-in phenomenon occurs when the electrostatic force balances the linear restoring force at around one-third of the distance between the actuating plate and the base substrate, see Younis;¹ Ganji;²,³ Zhang.⁴ The first mass-spring model for an electrostatically actuated device has been introduced by Nathanson.⁵ For the mass-spring system, Zhang⁴ specifies the dynamic pull-in and describes it as the collapse of the moving structure caused by the combination of kinetic and potential energies. In general, the dynamic pull-in requires a lower voltage to be triggered compared to the static pull-in threshold, see Flores;⁶ Zhang.⁴ Mathematical analysis of the one-degree-of-freedom lumped parameter model for an electrostatically actuated beam has been provided in our previous works, see Skrzypacz;⁷ Omarov.⁸ It is well known in physics and engineering that comparing with electrostatic force the magnetic force stated in Ampère’s law is much larger and therefore in many applications using it as the actuating force has advantages as stated and discussed in Lobato-Dauzier,⁹ Imai and Tsukioka,¹⁰ and Xingdong.¹¹ In particular, in certain devices and applications such as in vibration-based energy harvesting systems, cf. Shishesaz¹² and Shirbani,¹³ the actuation by magnetic force is at an advantage. Authors of these papers have demonstrated the effects of the magnetic force on actuating devices, numerically approximated the corresponding threshold conditions for pull-in, and discussed vibration amplitudes and frequencies with success.

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In some cases, the effects of dispersion forces on the actuation device are also taken into consideration, see Sedighi et al.\textsuperscript{14} and Cheraghbak and Loghman.\textsuperscript{15} In considering the Casimir and van der Waals attractions, the nonlinear lumped parameter becomes more difficult due to inverse cubic and quintic forces.

However, the corresponding mathematical analysis and analytic pull-in conditions appear much more complicated and therefore these forces are omitted here. The models based on the results from Sedighi et al.\textsuperscript{14} can be considered in our forthcoming works.

The purpose of this short note is to present the detailed analysis for the dynamic pull-in that occurs due to magnetic force in a MEMS model with a current-carrying conductor. In contrast to the MEMS models with electrostatically actuated plates where the inverse quadratic term corresponds to the Coulomb force, our model equation contains the nonlinear term which is only the inverse of the linear distance since the acting force is related to the electromagnetic force between two wires with constant currents. We derive analytically the conditions for the dynamic pull-in, see Theorem 1 and its corollary, and show the dichotomy: either the initial value problem has a periodic solution or pull-in occurs. The conditions which separate periodic solutions from pull-in are demonstrated analytically in terms of the operating currents, the linear material parameter, and the associated geometric dimensions.

In the second section, we describe the basic principles in magneto-electromechanical systems, and present the model. In the third section, we demonstrate the analytic pull-in conditions and we illustrate the analytic results numerically. We draw the conclusions in the fourth section.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{MEMS_with_a_current-carrying_wire.png}
\caption{MEMS with a current-carrying wire.}
\end{figure}

\textbf{Model problem}

In the following, we describe the basic principles in systems based on the magnetic actuation. In magnetostatics, the force of attraction or repulsion between two current-carrying wires is often called Ampère's force law. The origin of this force is that each wire generates a magnetic field, following the Biot–Savart law, and the other wire experiences a magnetic force as a consequence, following the Lorentz force law, cf. Assis and AKT.\textsuperscript{16} The force $f$ per unit length between two straight wires can be computed by

$$f = \frac{\mu_0 i_1 i_2}{2\pi r}$$

where $\mu_0 = 4\pi \times 10^{-7}$N/A$^2$ is the magnetic constant, $i_1$, $i_2$ the direct currents through the wires, $r$ the distance between them.

We consider the motion of a current-carrying wire of length $l$ and mass $m$ in the field of an infinite current-carrying conductor and restrained by linear elastic springs, see Figure 1. The mathematical model and the bifurcation analysis for the static pull-in have been presented in the textbook of Nayfeh and Mook.\textsuperscript{17} Here, we consider the corresponding analysis for the dynamic pull-in.
The dynamic lumped parameter differential equation describing the motion of the wire as a point mass can be derived based on the equation (1) and the theory of lumped parameter modeling of elastic Euler beam as follows

\[ m \ddot{x} + k\dot{x} - \frac{\mu_0 i_1 i_2 l}{2\pi(b - \bar{x})} = 0 \]  

(2)

where \(-k\ddot{x}\) is the restoring force due to the springs and \(\frac{\mu_0 i_1 i_2 l}{2\pi(b - \bar{x})}\) is the attraction force between the conductors due to the magnetic fields produced by the currents \(i_1, i_2\). The differential equation (2) can be rewritten as

\[ \ddot{x} + \frac{1}{1 - x} \dot{x} = 0 \]  

(3)

where \(x = \bar{x}/b\) and \(t = \bar{t} \omega_0\), \(\omega_0^2 = k/m\), and \(K = \mu_0 i_1 i_2 l/(2\pi kb^2)\). We prescribe zero initial conditions \(x(0) = 0\), \(\dot{x}(0) = 0\), and assume that the currents in both wires are unidirectional, i.e., \(K \geq 0\). The initial value problem for the nonlinear differential equation (3) can be rewritten as a system of first order differential equations

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x + \frac{K}{1 - x}
\end{align*}
\]  

(4)

with initial values \(x(0) = 0, y(0) = 0\).

**Analysis of dynamic pull-in**

Multiplying equation (3) by \(\dot{x}\) and integrating with respect to time, we get the conservation of energy

\[ E(t) = \frac{1}{2}(\dot{x}(t))^2 + \frac{1}{2}x^2(t) + K\ln|1 - x(t)| \]

i.e.,

\[ \frac{1}{2}(\dot{x}(t))^2 + \frac{1}{2}x^2(t) + K\ln|1 - x(t)| = C \]

where \(C = 0\) due to zero initial conditions. Thus

\[ (\dot{x}(t))^2 = -x^2(t) - 2K\ln|1 - x(t)| \]  

(5)

The phase diagrams for several values of \(K\) are presented in Figure 2. The closed orbits in phase diagrams represent periodic solutions. We see in Figure 2 that the periodic solutions are expected for small parameter values of \(K > 0\). The following theorem determines the range of the positive parameter \(K\) for which the dynamic pull-in occurs.

**Theorem 1.** Let \(K > 0\). The initial value problem (equation (3)) with zero initial values has a periodic solution if \(K \leq K_c\), whereas the pull-in occurs if \(K > K_c\), where

\[ K_c = 0.203632188 \ldots \]  

(6)

is a positive root of the transcendental equation

\[
\left( \frac{1 + \sqrt{1 - 4K}}{2} \right)^2 + 2K\ln \left| 1 - \frac{1 + \sqrt{1 - 4K}}{2} \right| = 0
\]
The solution $x(t)$ is periodic if and only if the phase diagram, $x$ vs. $x'$, produces a closed curve. In the case of zero initial conditions, this means that it is necessary and sufficient for energy equation (5) to have a closed curve. This is the case when

$$f_K(s) = -s^3 - 2K \ln|1 - s|$$

has a root in $(0, 1)$. Note that $f_K > 0$ when $s \in (0, 1)$ approaches to $0^+$ and $1^-$. Hence, due to the Mean Value theorem, the existence of a root in $(0, 1)$ is equivalent to existence of a local minimum of $f_K(s)$ in $(0, 1)$ that is at most 0. Therefore, $f_K(s)$ must have a non-positive minimum at some $s \in (0, 1)$. To find the critical points, we compute $f_K'(s) = -2s + 2K/(1 - s)$. Then, considering the second derivative, we see that $f_K(s)$ attains its minimum at the greatest critical point

$$s_1 = \frac{1 + \sqrt{1 - 4K}}{2}$$

which is in $(0, 1)$ and we must have $f_K(s_1) \leq 0$. This results in

$$-\left(\frac{1 + \sqrt{1 - 4K}}{2}\right)^2 - 2K \ln\left|1 - \frac{1 + \sqrt{1 - 4K}}{2}\right| \leq 0$$

The assertion follows from the numerical solution of the above inequality.

Consequently, it holds true.

**Corollary 2.** The dynamic pull-in for zero initial value problem (equation (2)) occurs if

$$i_1i_2 > K_s \frac{2\pi kh^2}{\mu_0 l}$$

where $K_s$ is given by equation (6).

Several solution profiles of $x(t)$ obtained by the standard Maple Ordinary Differential Equation (ODE) solver Hunt et al. for different sets of parameter $K > 0$ are presented in Figures 3 and 4. Notice that the time axis corresponds to the normalized time. Clearly, the periods and pull-in times depend on the positive parameter $K$. The critical value $K_s$ for the dynamic pull-in is less than 1/4, which represents the threshold for the static pull-in. Moreover, we notice that the value of $K_s$ is bigger than 1/8, which corresponds to the threshold for the dynamic pull-in in the MEMS model based on the inverse square electrostatic force, see Skrzypacz.
Remark 3. By analogy, we can show that the zero initial value problem (equation (3)) has always periodic solution for \( K < 0 \). Several solution profiles of \( x(t) \) for different sets of parameter \( K < 0 \) are presented in Figure 5.

Remark 4. The exact pull-in time \( t_{\text{pull-in}} \) and the period \( T \) can be computed as follows

\[
t_{\text{pull-in}} = \int_0^1 \frac{ds}{\sqrt{-s^2 - 2K \ln|1-s|}}, \quad K > K_s
\]

and

\[
T = \int_0^{x_{\text{min}}} \frac{2ds}{\sqrt{-s^2 - 2K \ln|1-s|}}, \quad 0 < K < K_s
\]

where \( x_{\text{min}} \in (0, 1) \) is the smallest positive root of \( f_K(s) \) and \( K_s \) is given by equation (6). We notice that \( x_{\text{min}} \) also represents the amplitude.
In Figure 6, we numerically demonstrated how the pull-in time decreases when the parameter $K_{2.5}^{0.2}$ increases. Notice that the scaled pull-in time for our model is bigger than the pull-in time for the MEMS model with the inverse square force term, see Skrzypacz,7 due to the fact that $K/(1 - x) < K/(1 - x)^2$ for $K > 0$ and $x \in (0, 1)$.

The formula

$$x_{ap}(t) = \frac{1 - \sqrt{1 - 4K}}{2} \left( 1 - \cos \left( t \sqrt{\frac{2K}{1 - \sqrt{1 - 4K}}} \right) \right)$$

based on the truncated Fourier series can be used in order to approximate the periodic solutions of (equation (3)) with zero initial values in the case of $0 < K \leq 0.1$, see Figure 7. Notice that the approximated amplitude $1 - \sqrt{1 - 4K}$ in

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**Figure 5.** Solution profiles for different values of $K < 0$.

**Figure 6.** Scaled pull-in time for $K \in [0.205, 1]$. 

In Figure 6, we numerically demonstrated how the pull-in time decreases when the parameter $K \in [0.205, 1]$ increases. Notice that the scaled pull-in time for our model is bigger than the pull-in time for the MEMS model with the inverse square force term, see Skrzypacz,7 due to the fact that $K/(1 - x) < K/(1 - x)^2$ for $K > 0$ and $x \in (0, 1)$.

The formula

$$x_{ap}(t) = \frac{1 - \sqrt{1 - 4K}}{2} \left( 1 - \cos \left( t \sqrt{\frac{2K}{1 - \sqrt{1 - 4K}}} \right) \right)$$

based on the truncated Fourier series can be used in order to approximate the periodic solutions of (equation (3)) with zero initial values in the case of $0 < K \leq 0.1$, see Figure 7. Notice that the approximated amplitude $1 - \sqrt{1 - 4K}$ in
our model is less than $(1 - \sqrt{1 - 8K})/2$ which represents the amplitude of the periodic solutions to the MEMS model equation with the inverse square force, cf. Skrzypacz.\(^7\) Our simple numerical approach and the obtained results can be useful for design of magneto-electromechanical devices.

**Conclusions**

Dynamic pull-in conditions for a MEMS switch with current carrying conductor subject to the electromagnetic force with the linear restoring force are obtained. Specific conditions for dynamic pull-in phenomenon to occur in the model are presented in the case of zero initial conditions in terms of the operating currents, the spring parameter, and the geometric dimensions. Numerical illustrations and validations of the analytic solutions are also presented. The results obtained in this short note are novel and can be useful for design of some MEMS with current-carrying conductor.

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Appendix

Notation

- \( b \) gap between the moving wire with current \( i_1 \) and fixed wire with current \( i_2 \)
- \( i_1 \) current in the moving wire
- \( i_2 \) current in the fixed wire
- \( K \) force parameter in the dimensionless model equation
- \( k \) elastic constant of spring
- \( m, l \) mass and length of the moving wire
- \( T \) period
- \( t \) normalized time
- \( \tilde{t} \) time
- \( t_{\text{pull-in}} \) pull-in time
- \( \tilde{x}(\tilde{t}) \) displacement of the moving wire at time \( \tilde{t} \)
- \( x(t) \) normalized displacement of the moving wire at normalized time \( t \)
- \( x_{\text{ap}}(t) \) approximated normalized displacement of the moving wire at normalized time \( t \)
- \( \mu \) permeability parameter
- \( \mathcal{E}(t) \) energy at time \( t \)