ALGEBRA OF FORMAL VECTOR FIELDS ON THE LINE AND BUCHSTABER’S CONJECTURE

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Abstract. Let $L_1$ denote the Lie algebra of formal vector fields on the line which vanish at the origin together with their first derivatives. $L_1$ is a nilpotent "positive part" of the Witt (Virasoro) algebra. Buchstaber and Shokurov have shown that the universal enveloping algebra $U(L_1)$ is isomorphic to the tensor product $S \otimes \mathbb{R}$, where $S$ is the Landweber-Novikov algebra in complex cobordism theory. Goncharova calculated the cohomology $H^*(L_1) = H^*(U(L_1))$, in particular it follows from her theorem that $H^*(L_1)$ has trivial multiplicative structure. Buchstaber conjectured that $H^*(L_1)$ is generated with respect to non-trivial Massey products by $H^1(L_1)$. Feigin, Fuchs and Retakh found representation of $H^*(L_1)$ by trivial Massey products. Later Artelnykh found non-trivial Massey products for a part of $H^*(L_1)$. In the present article we prove that $H^*(L_1)$ is generated with respect to non-trivial Massey products by two elements from $H^1(L_1)$.

INTRODUCTION

Buchstaber and Shokurov discovered [6] that the Landweber-Novikov algebra in the complex cobordism theory tensored by real numbers $S \otimes \mathbb{R}$ is isomorphic to the universal enveloping algebra $U(L_1)$ of the Lie algebra $L_1$ of polynomial vector fields on the real line $\mathbb{R}^1$ with vanishing non-positive Fourier coefficients. $L_1$ is a maximal (residually) nilpotent subalgebra of the Witt (Virasoro) algebra. In that time the algebra $L_1$ attracted a lot of interest [12] and the computation of $H^*(L_1)$ by Goncharova [15] was one of the most technically complicated results in homology algebra. Her result allowed Buchstaber and Kholodov to obtain some deep results in the complex cobordism theory.

It follows from the Goncharova theorem [15] that the cohomology algebra $H^*(L_1)$ has a trivial multiplication. Buchstaber conjectured

1991 Mathematics Subject Classification. 17B56; 55S30.
Key words and phrases. Massey products, graded Lie algebras, formal connection, Maurer-Cartan equation, representation, cohomology.

The research of the author was partially supported by grants RFBR 05-01-01032 and "Russian Scientific Schools".
that the algebra $H^*(L_1)$ is generated with respect to the non-trivial Massey products by its first cohomology $H^1(L_1)$.

Feigin, Fuchs and Retakh [11] represented the basic homogeneous cohomology classes from $H^*(L_1)$ as Massey products [11]. But all the products considered by them are trivial ones. Twelve years later Artel’nykh [1] represented a part of basic cocycles in $H^*(L_1)$ by means of non-trivial Massey products, but his brief article contains no proof.

In the present article we prove Buchstaber’s conjecture in its original setting. The main result is the Theorem 9.1 stating that the cohomology algebra $H^*(L_1)$ is generated with respect to the non-trivial Massey products by $H^1(L_1)$.

Although we have strengthened the Feigin-Fuchs-Retakh theorem we use some important technical tools from [11]. One of them is the free resolution of the trivial $L_1$-module constructed by means of so-called singular Virasoro vectors (an analogue of Bernshtein-Gelfand-Gelfand resolution). At the time of writing [11] there were no general formula for singular vectors in terms of operators $S_{p,q}(t)$ from $U(L_1)$. The formula for $S_{p,q}(t)$ obtained later by Benoit and Saint-Aubin is another important ingredient of our proof.

The most important part of our construction is the new graded so-called thread $L_1$-module $\tilde{M}$ that we introduce in the Section 21. The structure and important properties of $\tilde{M}$ is the cornerstone of the proof of the main Theorem 9.1.

In the Section 2 we present May’s approach to the definition of Massey products, his notion of formal connection developed by Babenko and Taimanov in [2] for Lie algebras, we introduce also the notion of equivalent Massey products. The analogy with the classical Maurer-Cartan equation is especially transparent in the case of Massey products of 1-dimensional cohomology classes $\langle \omega_1, \ldots, \omega_n \rangle$. The relation of this special case to the representations theory was discovered in [11], [8]. Following [11] we consider Massey products $\langle \omega_1, \ldots, \omega_n, \Omega \rangle$, where $\omega_1, \ldots, \omega_n$ are closed 1-forms and $\Omega$ is a closed $q$-form.

We stress on non-triviality of our Massey products. It was pointed out by May to the author that it follows from some general results [18, 19] that $H^*(L_1)$ is generated by matrix (possibly trivial) Massey products and $H^1(L_1)$. The triple non-trivial Massey product in cohomology $H^*(L_1)$ was used by Babenko and Taimanov in their construction of simply connected non formal symplectic manifolds [2].
1. Cohomology of \( \mathbb{N} \)-graded Lie algebras

Let \( g \) be a Lie algebra over \( K \) and \( \rho : g \to \mathfrak{gl}(V) \) its linear representation (or in other words \( V \) is a \( g \)-module). We denote by \( C^q(g, V) \) the space of \( q \)-linear skew-symmetric mappings of \( g \) into \( V \). Then one can consider an algebraic complex:

\[
V \xrightarrow{d_0} C^1(g, V) \xrightarrow{d_1} C^2(g, V) \xrightarrow{d_2} \ldots \xrightarrow{d_{q-1}} C^q(g, V) \xrightarrow{d_q} \ldots
\]

where the differential \( d_q \) is defined by:

\[
(d_q f)(X_1, \ldots, X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} \rho(X_i)(f(X_1, \ldots, \hat{X}_i, \ldots, X_{q+1})) + \sum_{1 \leq i < j \leq q+1} (-1)^{i+j-1} f([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{q+1}).
\]

The cohomology of the complex \((C^*(g, V), d)\) is called the cohomology of the Lie algebra \( g \) with coefficients in the representation \( \rho : g \to V \).

The cohomology of \((C^*(g, K), d)\) (\( V = K \) and \( \rho : g \to K \) is trivial) is called the cohomology with trivial coefficients of the Lie algebra \( g \) and is denoted by \( H^*(g) \).

One can remark that \( d_1 : C^1(g, K) \to C^2(g, K) \) of the \((C^*(g, K), d)\) is the dual mapping to the Lie bracket \([.,.] : \Lambda^2 g \to g\). Moreover the condition \( d^2 = 0 \) is equivalent to the Jacobi identity for \([.,.]\).

**Definition 1.1.** A Lie algebra \( g \) is called \( \mathbb{N} \)-graded, if it is decomposed to the direct sum of subspaces such that

\[
g = \bigoplus_i g_i, \quad i \in \mathbb{N}, \quad [g_i, g_j] \subset g_{i+j}, \quad \forall i, j \in \mathbb{N}.
\]

**Example 1.2.** Let us recall that the Witt algebra \( W \) is spanned by differential operators on the real line \( \mathbb{R}^1 \) with a fixed coordinate \( x \)

\[
e_i = x^{i+1} \frac{d}{dx}, \quad i \in \mathbb{Z}, \quad [e_i, e_j] = (j - i)e_{i+j}, \quad \forall i, j \in \mathbb{Z}.
\]

We denote by \( L_1 \) a positive part of the Witt algebra, i.e. \( L_1 \) is a subalgebra of \( W \) spanned by all \( e_i \), \( i \geq 1 \).

Obviously \( W \) is a \( \mathbb{Z} \)-graded Lie algebra with one-dimensional homogeneous components:

\[
W = \bigoplus_{i \in \mathbb{Z}} W_i, \quad W_i = \text{Span}(e_i).
\]

Thus \( L_1 \) is a \( \mathbb{N} \)-graded Lie algebra.
Let \( g = \oplus \alpha g_\alpha \) be a \( \mathbb{Z} \)-graded Lie algebra and \( V = \oplus \beta V_\beta \) is a \( \mathbb{Z} \)-graded \( g \)-module, i.e., \( g_\alpha V_\beta \subset V_{\alpha+\beta} \). Then the complex \((C^\ast(g,V),d)\) can be equipped with the \( \mathbb{Z} \)-grading \( C^q(g,V) = \bigoplus_\mu C^q(\mu)(g,V) \), where a \( V \)-valued \( q \)-form \( c \) belongs to \( C^q(\mu)(g,V) \) if and only if for \( X_1 \in g_{\alpha_1}, \ldots, X_q \in g_{\alpha_q} \) we have
\[
c(X_1, \ldots, X_q) \in V_{\alpha_1+\alpha_2+\ldots+\alpha_q+\mu}.
\]

This grading is compatible with the differential \( d \) and hence we have \( \mathbb{Z} \)-grading in the cohomology:
\[
H^q(g,V) = \bigoplus_\mu H^q(\mu)(g,V).
\]

**Remark.** The trivial \( g \)-module \( K \) has only one non-trivial homogeneous component \( K = K_0 \).

The exterior product in \( \Lambda^\ast(g) \) induces a structure of a bigraded algebra in the cohomology \( H^\ast(g) \):
\[
H^p_q(g,V) \wedge H^p_q(g,V) \to H^{p+q}_{k+1}(g).
\]

Let \( g = \oplus_{\alpha>0} g_\alpha \) be a \( \mathbb{N} \)-graded Lie algebra and \( V \) be a \( g \)-module provided with the following invariant flag of linear subspaces \( V_i, i \in \mathbb{Z} \):
\[
V = V_{i_0} \supset V_{i_0+1} \supset \cdots \supset V_i \supset V_{i+1} \supset \cdots,
\]
for some \( i_0 \in \mathbb{Z} \) and such that \( \cap_i V_i = \{0\} \) and
\[
gV_i \subset V_{i+1}, i \in \mathbb{Z}.
\]

One can define a decreasing filtration \( \mathcal{F} \) of \((C^\ast(g,V),d)\):
\[
\mathcal{F}^0 C^\ast(g,V) \supset \cdots \supset \mathcal{F}^q C^\ast(g,V) \supset \mathcal{F}^{q+1} C^\ast(g,V) \supset \cdots
\]
where the subspace \( \mathcal{F}^q C^{p+q}(g,V) \) is spanned by \((p+q)\)-forms \( c \) in \( C^{p+q}(g,V) \) such that
\[
c(X_1, \ldots, X_{p+q}) \in V_q, \forall X_1, \ldots, X_{p+q} \in g.
\]

The filtration \( \mathcal{F} \) is compatible with \( d \).

Let us consider the corresponding spectral sequence \( E_r^{p,q} \):

**Proposition 1.3.** \( E_1^{p,q} = (V_q/V_{q+1}) \otimes H^{p+q}(g) \).

**Proof.** We have the following natural isomorphisms:

\[
(2) \quad E_0^{p,q} = \mathcal{F}^q C^{p+q}(g,V)/\mathcal{F}^{q+1} C^{p+q}(g,V) = (V_q/V_{q+1}) \otimes \Lambda^{p+q}(g^\ast).
\]

Now the proof follows from the formula for the differential \( d_0^{p,q} : E_0^{p,q} \to E_0^{p+1,q} \):
\[
d_0(v \otimes f) = v \otimes df,
\]
where \( v \in V, f \in \Lambda^{p+q}(\mathfrak{g}^*) \) and \( df \) is the standard differential of the cochain complex of \( \mathfrak{g} \) with trivial coefficients.

**Theorem 1.4** (Goncharova,[15]). The Betti numbers \( \dim H^q(L_1) = 2 \), for every \( q \geq 1 \), more precisely

\[
\dim H^q_k(L_1) = \begin{cases} 
1, & \text{if } k = \frac{3q^2 \pm q}{2} \\
0, & \text{otherwise}
\end{cases}
\]

We will denote in the sequel by \( g^q_{\pm} \) the generators of the spaces \( H^q_k(L_1) \). The numbers \( \frac{3q^2 \pm q}{2} \) that we will denote sometimes in the sequel by \( e_{\pm}(q) \) are so called Euler pentagonal numbers. A sum of two arbitrary pentagonal numbers is not a pentagonal number, hence the algebra \( H^*(L_1) \) has a trivial multiplication.

**Example 1.5.**
1) \( H^1(L_1) \) is generated by \( g^1_1 = [e^1] \) and \( g^1_2 = [e^2] \);

2) the basis of \( H^2(L_1) \) consists of two classes \( g^2_1 = [e^1 \wedge e^4] \) and \( g^2_2 = [e^2 \wedge e^5 - 3e^3 \wedge e^4] \) of weights 5 and 7 respectively.

**Remark.** It is very difficult (almost impossible) to understand all the details of Goncharova’s proof. There is another one proof by Weinstein[24, ?], there is a mistake in[24] corrected later in[25] in the construction of the spectral sequence, but the case \( L_1 \) can be treated apart from the general case as it was shown in[14] (corrected later in[24]).

2. Massey products in cohomology.

In this section we follow [18] and [2] presenting the definitions of Massey products. Let \( \mathcal{A} = \oplus_{l \geq 0} \mathcal{A}^l \) be a differential graded algebra over a field \( \mathbb{K} \). It means that the following operations are defined: an associative multiplication

\[
\wedge : \mathcal{A}^l \times \mathcal{A}^m \rightarrow \mathcal{A}^{l+m}, \quad l, m \geq 0, \quad l, m \in \mathbb{Z},
\]

such that \( a \wedge b = (-1)^{lm} b \wedge a \) for \( a \in \mathcal{A}^l, b \in \mathcal{A}^m \), and a differential \( d, \quad d^2 = 0 \)

\[
d : \mathcal{A}^l \rightarrow \mathcal{A}^{l+1}, \quad l \geq 0,
\]

satisfying the Leibniz rule \( d(a \wedge b) = da \wedge b + (-1)^l a \wedge db \) for \( a \in \mathcal{A}^l \).

**Example 2.1.** \( \mathcal{A} = \Lambda^*(\mathfrak{g}) \) is the cochain complex of a Lie algebra.

For a given differential graded algebra \( (\mathcal{A}, d) \) we denote by \( LT_n(\mathcal{A}) \) a space of all lower triangular \( (n+1) \times (n+1) \)-matrices with entries from \( \mathcal{A} \), vanishing at the main diagonal. \( LT_n(\mathcal{A}) \) has a structure of a differential algebra with a standard matrix multiplication, where matrix...
entries are multiplying as elements of $\mathcal{A}$. A differential $d$ on $LT_n(\mathcal{A})$ is defined by

$$dA = (da_{ij})_{1 \leq i,j \leq n+1}.$$  

An involution $a \rightarrow \bar{a} = (-1)^{k+1}a, a \in A^k$ of $\mathcal{A}$ can be extended to an involution of $LT_n(\mathcal{A})$ as $\bar{A} = (\bar{a}_{ij})_{1 \leq i,j \leq n+1}$. It satisfies the following properties:

$$\bar{A} = A, \quad \bar{AB} = -\bar{A}B, \quad \bar{dA} = -d\bar{A}.$$  

Also we have the generalized Leibniz rule for the differential

$$d(AB) = (dA)B - \bar{A}(dB).$$  

The algebra $LT_n(\mathcal{A})$ has a two-sided center $I_n(\mathcal{A})$ of matrices

$$\begin{pmatrix}
0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & 0 \\
\vdots & & \ddots & \vdots \\
\tau & \ldots & 0 & 0
\end{pmatrix}, \quad \tau \in \mathcal{A}.$$  

**Definition 2.2** ([2]). A matrix $A \in LT_n(\mathcal{A})$ is called the matrix of a formal connection if it satisfies the Maurer-Cartan equation

$$\mu(A) = dA - \bar{A} \cdot A \in I_n(\mathcal{A}).$$  

**Proposition 2.3** ([2]). Let $A$ be the matrix of a formal connection, then the entry $\tau \in \mathcal{A}$ of the matrix $\mu(A) \in I_n(\mathcal{A})$ in the definition is closed.

**Proof.** We have the following generalized Bianci identity for the Maurer-Cartan operator $\mu(A) = dA - \bar{A} \cdot A$ ($A$ is an arbitrary matrix):

$$d\mu(A) = \mu(A) \cdot A + A \cdot \mu(A).$$  

Indeed it’s easy to verify the following equalities:

$$d\mu(A) = -d(\bar{A} \cdot A) = -d\bar{A} \cdot A + A \cdot dA = \bar{dA} \cdot A + A \cdot dA =$$

$$= (\mu(A) + \bar{A} \cdot A) \cdot A + A(\mu(A) + \bar{A} \cdot A) =$$

$$= \mu(A) \cdot A - A \cdot \bar{A} \cdot A + A \cdot \mu(A) + A \cdot \mu(A) =$$

$$= \mu(A) \cdot A + A \cdot \mu(A).$$

□

Now let $A$ be the matrix of a formal connection, then the matrix $\mu(A)$ belongs to the center $I_n(\mathcal{A})$ and hence $d\mu(A) = 0$. One can think of $\mu(A)$ as the curvature matrix of a formal connection $A$. 

Let $A$ be an lower triangular matrix from $LT_n(A)$. One can rewrite it in the following notation:

$$A = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & 0 \\
a(n,n) & 0 & \ldots & 0 & 0 & 0 \\
a(n-1,n) & a(n-1,n-1) & \ldots & 0 & 0 & 0 \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a(2,n) & a(2,n-1) & \ldots & a(2,2) & 0 & 0 \\
a(1,n) & a(1,n-1) & \ldots & a(1,2) & a(1,1) & 0
\end{pmatrix}.$$ 

**Proposition 2.4.** A matrix $A \in LT_n(A)$ is the matrix of a formal connection if and only if the following conditions on its entries hold on

$$a(i,i) = a_i \in A^{p_i}, \quad i = 1, \ldots, n;$$

$$a(i,j) \in A^{p(i,j)+1}, \quad p(i,j) = \sum_{r=i}^{j} (p_r - 1);$$

$$d a(i,j) = \sum_{r=i}^{j-1} \bar{a}(i,r) \cdot a(r+1,j), \quad (i,j) \neq (1,n).$$

**Proof.** The system (5) is just the Maurer-Cartan equation rewritten in terms of the entries of the matrix $A$ and it is a part of the classical definition [17] of the defining system for a Massey product. □

**Definition 2.5 ([17]).** A collection of elements, $A = (a(i,j))$, for $1 \leq i \leq j \leq n$ and $(i,j) \neq (1,n)$ is said to be a defining system for the product $\langle a_1, \ldots, a_n \rangle$ if it satisfies (5).

In this situation the $(p(1,n) + 2)$-dimensional cocycle

$$c(A) = \sum_{r=1}^{n-1} \bar{a}(1,r) a(r+1,n)$$

is called the related cocycle of the defining system $A$.

**Remark.** We saw that the notion of the defining system is equivalent to the notion of the formal connection. However one has to remark that an entry $a(1,n)$ of the matrix $A$ of a formal connection does not belong to the corresponding defining system $A$, it can be taken as an arbitrary element from $A$. In this event the only one (possible) nonzero entry $\tau$ of the Maurer-Cartan matrix $\mu(A)$ is equal to $-c(A) + da(1,n)$.

**Definition 2.6 ([17]).** The $n$-fold product $\langle a_1, \ldots, a_n \rangle$ is defined if there is at least one defining system for it (a formal connection $A$ with entries $a_n, \ldots, a_1$ at the second diagonal). If it is defined, then $\langle a_1, \ldots, a_n \rangle$ consists of all cohomology classes $\alpha \in H^{p(1,n)+2}(A)$ for
which there exists a defining system \( A \) (a formal connection \( A \)) such that \( c(A) \) \((-\tau\) respectively) represents \( \alpha \).

**Theorem 2.7** ([17],[2]). The operation \( \langle a_1, \ldots, a_n \rangle \) depends only on the cohomology classes of the elements \( a_1, \ldots, a_n \).

**Proof.** A changing of an arbitrary entry \( a_{ij}, j > i \) of the matrix \( A \) of a formal connection to \( a_{ij} + db \) leads to a replacement of \( A \) by

\[
A' = A + db \cdot E_{ij} + A \cdot b \cdot E_{ij} - \bar{b} \cdot E_{ij} \cdot A,
\]

where \( E_{ij} \) is a scalar matrix which has 1 on \((i, j)\)-th place and zeroes on all others. For the corresponding Maurer-Cartan matrix we will have

\[
\mu(A') = \mu(A) + d((A \cdot b \cdot E_{ij} - \bar{b} \cdot E_{ij} \cdot A) \cap I_n).
\]

\[\square\]

**Definition 2.8** ([17]). A set of closed elements \( a_i, i = 1, \ldots, n \) from \( A \) representing some cohomology classes \( \alpha_i \in H^p(A), i = 1, \ldots, n \) is said to be a defining system for the Massey \( n \)-fold product \( \langle \alpha_1, \ldots, \alpha_n \rangle \) if it is one for \( \langle a_1, \ldots, a_n \rangle \). The Massey \( n \)-fold product \( \langle \alpha_1, \ldots, \alpha_n \rangle \) is defined if \( \langle a_1, \ldots, a_n \rangle \) is defined, in which case \( \langle \alpha_1, \ldots, \alpha_n \rangle = \langle a_1, \ldots, a_n \rangle \) as subsets in \( H^{p(1,n)+2}(A) \).

**Example 2.9.** For \( n = 2 \) the matrix \( A \) of a formal connection has a form \( A = \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ c & a & 0 \end{pmatrix} \) and the matrix Maurer-Cartan equation is equivalent to two equations \( db = 0 \) and \( da = 0 \). Evidently \( \langle \alpha, \beta \rangle = \langle a, b \rangle = \bar{\alpha} \cdot \beta \).

**Example 2.10** (Triple Massey products). Let \( \alpha, \beta, \) and \( \gamma \) be the cohomology classes of closed elements \( a \in A^p, b \in A^q, \) and \( c \in A^r \). The Maurer-Cartan equation for

\[
A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ g & b & 0 & 0 \\ h & f & a & 0 \end{pmatrix}
\]

is equivalent to

\[
df = (-1)^{p+1} a \wedge b, \quad dg = (-1)^{q+1} b \wedge c.
\]

Hence the triple Massey product \( \langle \alpha, \beta, \gamma \rangle \) is defined if and only if

\[
\alpha \cdot \beta = \beta \cdot \gamma = 0 \quad \text{in} \quad H^*(A).
\]
If these conditions are satisfied then the Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined as a subset in $H_{p+q+r-1}(A)$ of the following form

$$\langle \alpha, \beta, \gamma \rangle = \left\{ [(−1)^{p+1}a \land g + (−1)^{p+q}f \land c] \right\}.$$

Since $f$ and $g$ are defined by (6) up to closed elements from $A_{p+q-1}$ and $A_{q+r-1}$ respectively, the triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is an affine subspace of $H_{p+q+r-1}(A)$ parallel to $\alpha \cdot H_{q+r-1}(A) + H_{p+q-1}(A) \cdot \gamma$.

Remark. We defined Massey products as the multi-valued operations in general. More often in the literature the triple Massey product is defined as a quotient-space $\langle \alpha, \beta, \gamma \rangle / (\alpha \cdot H_{q+r-1}(A) + H_{p+q-1}(A) \cdot \gamma)$ and it is single-valued in this case (see [12]).

Definition 2.11. Let an $n$-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ be defined. It is called trivial if it contains the trivial cohomology class: $0 \in \langle \alpha_1, \ldots, \alpha_n \rangle$.

Proposition 2.12. Let a Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined. Then all Massey products $\langle \alpha_l, \ldots, \alpha_q \rangle$, $1 \leq l < q \leq n$, $q-l < n-1$ are defined and trivial.

Proof. It follows from (5). □

Remark. The triviality of all Massey products $\langle \alpha_l, \ldots, \alpha_q \rangle$, $1 \leq l < q \leq n$, $q-l < n-1$ is only a necessary condition for a Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ to be defined. It is sufficient only in the case $n = 3$.

Let us denote by $GLT_n(\mathbb{K})$ a group of non-degenerate lower triangular $(n+1, n+1)$-matrices of the form:

$$C = \begin{pmatrix}
a_{1,1} & 0 & \ldots & 0 & 0 \\
a_{1,2} & a_{2,2} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{1,n} & a_{2,n} & \ldots & a_{n,n} & 0 \\
a_{1,n+1} & a_{2,n+1} & \ldots & a_{n,n+1} & a_{n+1,n+1}
\end{pmatrix}.$$

Proposition 2.13. Let $A \in LT_n(A)$ be the matrix of a formal connection and $C$ an arbitrary matrix from $GLT_n(\mathbb{K})$. Then the matrix $C^{-1}AC \in LT_n(A)$ and satisfies the Maurer-Cartan equation, i.e. is again the matrix of a formal connection.

Proof.

$$d(C^{-1}AC) - C^{-1}\bar{A}C \land C^{-1}AC = C^{-1} (dA - \bar{A} \land A) C = 0.$$ □
Example 2.14. Let $A \in LT_n(A)$ be the matrix of a formal connection (defining system) for a Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$. Then a matrix $C^{-1}AC$ with

$$C = \begin{pmatrix}
  x_1 \ldots x_{n-1} x_n & 0 & \ldots & 0 & 0 \\
  0 & x_1 \ldots x_{n-1} & \ldots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \ldots & x_1 & 0 \\
  0 & 0 & \ldots & 0 & 1
\end{pmatrix}$$

is a defining system for $\langle x_1 \alpha_1, \ldots, x_n \alpha_n \rangle = x_1 \ldots x_n \langle \alpha_1, \ldots, \alpha_n \rangle$.

Definition 2.15. Two matrices $A$ and $A'$ of formal connections from $LT_n(A)$ are equivalent if there exists a matrix $C \in GL(n+1, \mathbb{K})$ such that

$$A' = C^{-1}AC.$$

Example 2.16. Triple products $\langle \alpha, \beta, \gamma \rangle$ and $\langle x\alpha, y\beta, z\gamma \rangle$, where $x, y, z \neq 0$, are equivalent with

$$C = \begin{pmatrix}
  xyz & 0 & 0 & 0 \\
  0 & xy & 0 & 0 \\
  0 & 0 & x & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}$$

and

$$\langle x\alpha, y\beta, z\gamma \rangle = xyz \langle \alpha, \beta, \gamma \rangle, \quad x, y, z \in \mathbb{K}.$$

Remark. Following the original Massey work [20] some higher order cohomological operations that we call now Massey products were introduced in the 60s in [17] and [18]. The relation between Massey products and the Maurer-Cartan equation was first noticed by May [18] and this analogy was not developed until [2].

In the present article we deal only with Massey products of non-trivial cohomology classes. It is possible to take some of them trivial, but in this situation is more natural to work with so-called matrix Massey products that were first introduced by May [18]. This approach was also developed in [2]. We will not treat this case in the sequel.

3. Formal connections and representations

Let $LT_n(\mathbb{K})$ be a Lie algebra of lower triangular $(n+1, n+1)$-matrices over a field $\mathbb{K}$ of zero characteristic and $\rho : \mathfrak{g} \rightarrow LT_n(\mathbb{K})$ be a representation of a Lie algebra $\mathfrak{g}$. 
Example 3.1. We take $n = 1$ and consider a linear map
\[ \rho : x \in g \rightarrow \begin{pmatrix} 0 & 0 \\ a(x) & 0 \end{pmatrix}. \]
It is evident that $\rho$ is a Lie algebra homomorphism if and only if the linear form $a \in g^\ast$ is closed
\[ da(x, y) = a([x, y]) = a(x)a(y) - a(y)a(x) = 0, \forall x, y \in g. \]
In other words the matrix $A = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ satisfies the "strong" Maurer-Cartan equation $dA - \bar{A} \wedge A = 0$.

Remark. We recall that we defined in the Section 2 the involution of a graded $\mathcal{A}$ as $\bar{a} = (-1)^{k+1}a, a \in \mathcal{A}^k$. Thus for a matrix $A$ with entries from $g^\ast$ we have $\bar{A} = A$. One has to remark that $\bar{a}$ differs by the sign from the definition of $\bar{a}$ in [17], however in [19] one meets our sign rule.

Proposition 3.2. A matrix $A$ with entries from $g^\ast$ defines a representation $\rho : g \to T_n(\mathbb{K})$ if and only if $A$ satisfies the strong Maurer-Cartan equation
\[ dA - \bar{A} \wedge A = 0. \]

Proof.
\[ (dA - \bar{A} \wedge A)(x, y) = A([x, y]) - [A(x), A(y)], \forall x, y \in g. \]
\[ \square \]

Example 3.3. For $n = 2$ the matrix $A$ of a representation $\rho$ has a form
\[ A = \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ c & a & 0 \end{pmatrix}, \text{ where } a, b, c \in g^\ast \text{ and the strong } \text{Maurer-Cartan equation is equivalent to the following equations on entries } a, b, c: \]
\[ da = 0, \quad db = 0, \quad dc = a \wedge b. \]

The Lie algebra $LT_n(\mathbb{K})$ has a one-dimensional center $I_n(\mathbb{K})$ spanned by the matrix
\[ \begin{pmatrix} 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 0 \\ & \ldots \\ 1 & \ldots & 0 & 0 \end{pmatrix}. \]
One can consider an one-dimensional central extension
\[ 0 \longrightarrow \mathbb{K} \cong I_n(\mathbb{K}) \longrightarrow LT_n(\mathbb{K}) \overset{\pi}{\longrightarrow} \tilde{LT}_n(\mathbb{K}) \longrightarrow 0. \]
Proposition 3.4 ([11], [8]). Fixing a Lie algebra homomorphism \( \tilde{\varphi} : g \to \mathcal{L}T_n(K) \) is equivalent to fixing a defining system \( A \) with elements from \( g^* = \Lambda^1(g) \). The related cocycle \( c(A) \) is cohomologous to zero if and only if \( \tilde{\varphi} \) can be lifted to a homomorphism \( \varphi : g \to \mathcal{L}T_n(K) \), \( \tilde{\varphi} = \pi \varphi \).

Let us consider a Massey product of the form
\[
\langle \omega_1, \ldots, \omega_n, \Omega \rangle, \omega_i \in H^1(g), i = 1, \ldots, n, \Omega \in H^p(g).
\]
If it is defined then \( n \)-fold product \( \langle \omega_1, \ldots, \omega_n \rangle \) is trivial and the existence of the homomorphism \( \varphi : g \to LT_n(K) \) means that there is a \((n + 1)\)-dimensional \( g \)-module \( V \) with a basis \( f_1, \ldots, f_{n+1} \), such that
\[
gf_j \in \text{Span}(f_{j+1}, \ldots, f_{n+1}), j = 1, \ldots, n; \ gf_{n+1} = 0.
\]
One can consider the spectral sequence \( E_{r}^{p,q} \) from the Proposition 1.3 converging to the cohomology \( H^*(g,V) \) of \( g \) with coefficients in \( V \).

Theorem 3.5. Let \( g \) be a Lie algebra and a \((n+1)\)-fold Massey product
\[
\langle \omega_1, \ldots, \omega_n, \Omega \rangle, \omega_i \in H^1(g), i = 1, \ldots, n, \Omega \in H^p(g),
\]
be defined. Let also \( A \) be a corresponding formal connection. Then it exists the \((n + 1)\)-dimensional \( g \)-module \( V \) and the spectral sequence \( E_r^{p,q} \) converging to the cohomology \( H^*(g,V) \) such that
\[
f_1 \otimes \Omega \in E_1^{p-1,1},
\]
(7)
\[
d_1(f_1 \otimes \Omega) = \cdots = d_{n-1}(f_1 \otimes \Omega) = 0,
\]
\[
d_n(f_1 \otimes \Omega) = f_{n+1} \otimes [c(A)],
\]
where \( d_i : E_r^{i-1,i} \to E_r^{i+2,i+1} \) is the \( i \)-th differential of the spectral sequence \( E_r^{p,q} \) and \( c(A) \) is the cocycle of the formal connection \( A \).

Proof. The proof is almost evident, one has to follow only the definitions. Namely we denote the entries of the matrix \( A \) of the formal connection by
\[
A = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\Omega & 0 & 0 & \cdots & 0 & 0 & 0 \\
\Omega_1 & \omega_n & 0 & \cdots & 0 & 0 & 0 \\
\Omega_2 & a(n-1,n) & \omega_{n-1} & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\Omega_{n-1} & a(2,n) & a(2,n-1) & \cdots & \omega_2 & 0 & 0 \\
\Omega_n & a(1,n) & a(1,n-1) & \cdots & a(1,2) & \omega_1 & 0
\end{pmatrix}.
\]
The Maurer-Cartan equation gives us the following system of equations on the elements $\Omega_i$ standing at the first column of $A$:

\begin{equation}
\begin{align*}
    d\Omega_1 &= \omega_n \wedge \Omega = \omega_n \wedge \Omega, \\
    d\Omega_2 &= \omega_{n-1} \wedge \Omega_1 + a(n-1, n) \wedge \Omega, \\
    &\vdots \\
    d\Omega_{n-1} &= \omega_{2} \wedge \Omega_{n-1} + \cdots + a(2, n-2) \wedge \Omega_2 + a(2, n-1) \wedge \Omega_1 + a(2, n) \wedge \Omega.
\end{align*}
\end{equation}

We recall that $\Omega_n$ can be an arbitrary form. The corresponding cocycle $c(A)$ will be equal to

\begin{equation}
c(A) = \sum_{i=0}^{n-1} a(1, n-i) \wedge \Omega_i, \quad \Omega_0 = \Omega, \ a(1, 1) = \omega_1.
\end{equation}

On the other hand in the cochain complex $C^*(g, V)$ we have

\begin{equation}
\begin{align*}
    d(f(X)) &= [X, f], \ f \in C^0(g, V) = V, \ X \in V; \\
    d(v \otimes \Omega) &= dv \wedge \Omega - v \otimes d\omega, \ v \in V, \ \Omega \in \Lambda^*(g),
\end{align*}
\end{equation}

where $d$ denotes the differential in the complex $C^*(g, V)$ and the standard $d$ is the differential of the complex $C^*(g)$ with trivial coefficients. Hence $df_{n+1} = 0$ and for $i = 1, \ldots, n$, we have

\begin{equation}
    df_i = f_{i+1} \otimes \omega_{n-i+1} + f_{i+2} \otimes a(n-i, n-i+1) + \cdots + f_{n+1} \otimes a(1, n-i+1).
\end{equation}

And

\begin{equation}
\begin{align*}
    d(f_1 \otimes \Omega) &= f_2 \otimes \omega_n \wedge \Omega + \ldots, \\
    d(f_1 \otimes \Omega + f_2 \otimes \Omega_1) &= f_3 \otimes (\omega_{n-1} \wedge \Omega_1 + a(n-1, n) \wedge \Omega) + \ldots, \\
    &\vdots \\
    d \left( \sum_{i=1}^{n} f_i \otimes \Omega_{i-1} \right) &= f_{n+1} \otimes \left( \sum_{i=0}^{n-1} a(1, n-i) \wedge \Omega_i \right) = f_{n+1} \otimes c(A).
\end{align*}
\end{equation}

The end of the proof. \qed

4. The Feigin-Fuchs-Retakh theorem

We recall that the algebra $H^*(L_1)$ has a trivial multiplication. Buchstaber conjectured that the algebra $H^*(L_1)$ is generated with respect to the Massey products by $H^1(L_1)$, moreover all corresponding Massey products can be chosen non trivial. The weak version of Buchstaber’s conjecture was proved by Feigin, Fuchs and Retakh \[\ref{11}\].
Theorem 4.1 (Feigin, Fuchs, Retakh [11]). Let $g^k_\pm$ be a non trivial cocycle in $H^k_{\frac{1}{2}(3q^2\pm q)}(L_1)$. For any $k \geq 2$ we have

$$g^-_k \in \langle e^1, \ldots, e^1, g^{k-1}_- \rangle_{2k-1}, \quad g^+_k \in \langle e^1, \ldots, e^1, g^{k-1}_+ \rangle_{3k-1}.$$

Feigin, Fuchs and Retakh proposed the following formal connection:

$$A = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
g^{k-1}_+ & 0 & 0 & \ddots & \ddots & \ddots & 0 \\
\Omega_1 & e^1 & 0 & \ddots & \ddots & \ddots & 0 \\
\Omega_2 & \alpha e^2 & e^1 & \ddots & \ddots & \ddots & 0 \\
\vdots & 0 & \alpha e^2 & \ddots & \ddots & \ddots & 0 \\
\Omega_{n-1} & \ddots & \ddots & e^1 & 0 & 0 & 0 \\
* & 0 & \ldots & 0 & \alpha e^2 & e^1 & 0
\end{pmatrix},$$

with homogeneous forms $\Omega_i \in \Lambda^{k+i-1}_{\frac{1}{2}(3(k-1)^2+(k-1))}(L_1)$ and parameter $\alpha \in \mathbb{K}$. The system (8) on $\Omega_i$, $i = 1, \ldots, n-1$, is solvable for all values of parameter $\alpha$ if $n \leq 2k$. The corresponding cocycle $c(A) \in \Lambda^{k+1}_{\frac{1}{2}(3(k-1)^2+(k-1))}(L_1)$ can be non-trivial only if $n = 2k$ or $n = 3k$ that corresponds to $H^k_{\frac{1}{2}(3k^2\pm k)}(L_1)$. According to the Theorem 3.5 the triviality of the cocycle $c(A)$ is equivalent to the triviality of the differential $d_n$ of the spectral sequence $E^p_{r,q}$ converging to $H^*(L_1, V)$.

By methods that we are going to discuss in next sections Feigin, Fuchs and Retakh established that

1) $d_{2k}$ is trivial if and only if $\alpha \in \{\frac{1}{6}, \frac{1}{24}, \ldots, \frac{1}{6(k-1)^2}\}$;

2) $d_{3k}$ is defined and trivial if and only if:

a) $\alpha \in \{\frac{1}{6}, \frac{1}{54}, \ldots, \frac{1}{6(k-3)^2}, \frac{1}{6(k-1)^2}\}$ in the case of even $k$;

b) $\alpha \in \{\frac{1}{27}, \frac{1}{90}, \ldots, \frac{1}{6(k-3)^2}, \frac{1}{6(k-1)^2}\}$ if $k$ is odd.

Corollary 4.2. All Massey products from the Theorem 4.1 are trivial.

A few words about the proof of the Theorem 4.1. The main technical problem is obvious: one has to write out explicit formulas for the forms $\Omega_i$ from the formal connection $A$. It easy to see that cocycles $g^2_- = e^2 \wedge e^3$ and $g^2_+ = e^2 \wedge e^5 - e^3 \wedge e^4$ span the homogeneous subspaces $H^2_2(L_1)$ and $H^2_2(L_1)$ respectively. But it is still an open question how to write out explicit formulas for all Goncharova’s cocycles $g^k_\pm$ in terms of exterior forms from $\Lambda^*(L_1)$. Fuchs, Feigin and Retakh proposed an elegant and effective way how to present an alternative proof of non-triviality of differentials (for the special choice of the parameter $\alpha$) in the spectral sequence $E^p_{r,q}$ without providing explicit formulas for $\Omega_i$. 
In 2000 Buchstaber’s PhD-student Artel’nykh attacked Buchstaber’s conjecture in its original setting. In particular he claimed the following

**Theorem 4.3** (Artel’nykh [1]). There are non-trivial Massey products

\[ g_k \in \langle e^2, \ldots, e^2, g^{k-1}_+, e^1 \rangle, \quad k \geq 2; \quad g^{2l+1}_+ \in \langle e^2, \ldots, e^2, g^{2l}_+, e^1 \rangle, \quad l \geq 1. \]

One can see that Artel’nykh have not found non-trivial Massey products for cohomology classes \( g^{2l+1}_+ \). On the another hand Artel’nykh’s brief article [1] does not contain any proof.

**Remark.** The inclusion \( g^{2l+1}_+ \in \langle e^2, \ldots, e^2, g^{2l}_+, e^1 \rangle, \quad l \geq 1 \), can be found in [11]. The question is the corresponding Massey product non-trivial or not have not been discussed.

5. **Verma modules and Virasoro singular vectors**

**Definition 5.1.** The Virasoro algebra \( \text{Vir} \) is an infinite dimensional Lie algebra defined by its basis \( \{ z, e_i, i \in \mathbb{Z} \} \) and the structure relations:

\[ [e_i, z] = 0, \quad \forall i \in \mathbb{Z}, \quad [e_i, e_j] = (j - i)e_{i+j} + \frac{j^3 - j}{12}\delta_{-i,j}z. \]

\( \text{Vir} \) is \( \mathbb{Z} \)-graded Lie algebra where \( z, e_0 \) have gradings equal to zero, and a generator \( e_i \) has grading equal to \( i \). \( \text{Vir} \) is the one-dimensional central extension of the Witt algebra (the one-dimensional centre of \( \text{Vir} \) is spanned by \( z \)).

**Definition 5.2.** A Virasoro Verma module \( V(h, c) \) is a free module over the subalgebra \( L_1 \subset \text{Vir} \) generated by some vector \( v \) such that

\[ zv = cv, \quad e_0v = hv, \quad e_iv = 0, \quad \text{if } i < 0, \]

where \( c, h \in \mathbb{C} \).

A Verma module \( V(h, c) \) is \( \mathbb{N} \)-graded module:

\[ V(h, c) = \bigoplus_{n=0}^{+\infty} V_n(h, c), \quad V_n(h, c) = \langle e_{i_1} \ldots e_{i_s}v, \quad i_1 + \cdots + i_s = n \rangle. \]

\( V_n(h, c) \) is an eigenspace of \( e_0 \) with the eigenvalue \( (h + n) \):

\[ e_0(e_{i_1} \ldots e_{i_s}v) = (h + i_1 + \cdots + i_s)e_{i_1} \ldots e_{i_s}v. \]

Besides of this \( zw = cw \) for any \( w \in V(h, c) \).

**Definition 5.3.** A vector \( w \in V(h, c) \) is called singular, if \( e_iw = 0 \) for \( i < 0 \).
A homogeneous singular vector \( w \in V_n(h, c) \) of degree \( n \) generates in \( V(h, c) \) a submodule isomorphic to \( V(h+n, c) \).

It is well-known \([9]\) that there is a singular vector \( w \in V_n(h, c) \) of degree \( n \) if and only if, for some positive integers \( p \) and \( q \) and complex number \( t \), we have \( n = pq \) and

\[
c = c(t) = 13 + 6t + 6t^{-1},
\]

\[
h = h_{p,q}(t) = -\frac{p^2-1}{4}t - \frac{pq-1}{2} - \frac{q^2-1}{4}t^{-1}.
\]

The singular vector \( w_{p,q}(t) \) of degree \( n = pq \) is unique up to scalar multiplication. One can write \( w_{p,q}(t) \) as a continuous function of \( t \) \([13]\):

\[
w_{p,q}(t) = S_{p,q}(t) = \sum_{|I|=pq} a_I^{p,q}(t) e_I v = \sum_{i_1,\ldots,i_s=pq} a_{i_1,\ldots,i_s}^{p,q}(t) e_{i_1} \cdots e_{i_s} v.
\]

The coefficients \( a_I^{p,q}(t) \) depend polynomially on \( t \) and \( t^{-1} \). We assume that the coefficient \( a_{1,1}^{p,q}(t) \) is equal to 1. It is natural to consider the last sum over ordered partitions \( i_1 \geq i_2 \geq \cdots \geq i_s \geq 1 \), for instance

\[
S_{1,1}(t) = e_1, \quad S_{2,1}(t) = e_1^2 + te_2, \quad S_{3,1}(t) = e_1^3 + 4te_2e_1 + (4t^2 + 2t)e_3, \\
S_{4,1}(t) = e_1^4 + 10te_2e_1^2 + 6t^2e_2^2 + (24t^2 + 10t)e_3e_1 + (36t^3 + 24t^2 + 6t)e_4.
\]

It is still unclear how to write out the general formula for all \( S_{p,q}(t) \) with the ordering \( i_1 \geq i_2 \geq \cdots \geq i_s \geq 1 \).

6. \( L_1 \)-resolution and cohomology of thread modules

Let us consider the Verma module \( V(0,0) \). We fix the value \( t = -\frac{3}{2} \) and denote \( S_{p,q}(-\frac{3}{2}) \) by \( S_{p,q} \) for simplicity.

**Proposition 6.1** (Kac \([16]\), Feigin-Fuchs \([9, 10]\)). The module \( V(0,0) \) has a singular vector \( w_n \) of degree \( n \) (at the level \( n \)) if and only if \( n \) is a pentagonal number \( n = \frac{3k^2 \pm k}{2} \).

Denote by \( V \left( \frac{3k^2 \pm k}{2} \right) \) the submodule of \( V(0,0) \) generated by the singular vector of degree \( \frac{3k^2 \pm k}{2} \).

**Proposition 6.2.** We have the following properties:

1) the sum \( V(1)+V(2) \) is the subspace of codimension one in \( V(0,0) \);
2) \( V \left( \frac{3k^2-k}{2} \right) \cap V \left( \frac{3k^2+k}{2} \right) = V \left( \frac{3(k+1)^2-(k+1)}{2} \right) + V \left( \frac{3(k+1)^2+(k+1)}{2} \right) \), \( k \geq 1 \).
Example 6.3. One can verify the following equality in the universal enveloping algebra $U(L_1)$ that illustrates the property considered above:

\[
(e_1^3 - 6e_2e_1 + 6e_3) \left( e_1^2 - \frac{2}{3} e_2 \right) = \left( e_1^4 - \frac{20}{3} e_2e_1^2 + 4e_2^2 + 4e_3e_1 - 4e_1 \right) e_1.
\]

It means that

\[ S_{3,1}S_{1,2} = S_{1,4}S_{1,1}. \]

The inclusions of submodules $V \left( \frac{3k^2+k}{2} \right)$ considered above provides us with the exact sequence of Vir-modules [23, 9, 10]

(13)

\[
\ldots \rightarrow V(\frac{3(3k+1)^2-(k+1)}{2}, 0) \oplus V(\frac{3(k+1)^2+(k+1)}{2}, 0) \xrightarrow{\delta_{k+1}} V(\frac{3k^2-k}{2}, 0) \oplus V(\frac{3k^2+k}{2}, 0) \rightarrow \ldots
\]

\[
\ldots \xrightarrow{\delta_3} V(5, 0) \oplus V(7, 0) \xrightarrow{\delta_2} V(1, 0) \oplus V(2, 0) \xrightarrow{\delta_1} V(0, 0) \rightarrow \mathbb{C} \rightarrow 0,
\]

where $\delta_k$ are defined with the help of operators $S_{p,q} \in U(L_1)$:

\[
\delta_{k+1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_{1,3k+1} & S_{2k+1,2} \\ -S_{2k+1,1} & -S_{1,3k+2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad k \geq 1;
\]

\[
\delta_{1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_{1,1} & S_{1,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\]

$\varepsilon$ is the canonical surjective homomorphism.

Theorem 6.4 (Rocha-Carridi-Wallach [23], Feigin-Fuchs [9]). The exact sequence (13) regarded as the sequence of $L_1$-modules is a free resolution of the one-dimensional trivial $L_1$-module $\mathbb{C}$.

Corollary 6.5. Let $M$ be a $L_1$-module. Then the cohomology $H^*(L_1, M)$ is isomorphic to the cohomology of the following complex

(15)

\[
\ldots \xleftarrow{d_{k+1}} M \oplus M \xleftarrow{d_k} M \oplus M \xleftarrow{d_{k-1}} \ldots \xleftarrow{d_1} M \oplus M \xleftarrow{d_0} M,
\]

with differentials

\[
d_k \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} S_{1,3k+1} & -S_{2k+1,1} \\ S_{2k+1,2} & -S_{1,3k+2} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad k \geq 1;
\]

\[
d_0(m) = \begin{pmatrix} S_{1,1m} \\ S_{1,2m} \end{pmatrix}, \quad m, m_1, m_2 \in M.
\]

Definition 6.6 ([11]). An infinite dimensional thread $L_1$-module is a $\mathbb{Z}$-graded $L_1$-module $V = \oplus_{j \in \mathbb{Z}} V_j$ such as

\[
\dim V_j = 1, \quad e_iV_j \subset V_{i+j}, \quad \forall e_i \in L_1, j \in \mathbb{Z}.
\]

One can fixe an infinite basis $f_j, j \in \mathbb{Z}$, such that $f_j$ spans $V_j$. 

Example 6.7. Let us define a thread $L_1$-module $A_\alpha$ by its basis $f_i$, $i \in \mathbb{Z}$, and some parameter $\alpha \in \mathbb{K}$:

$$e_1 f_j = f_{j+1}, \quad e_2 f_j = \alpha f_{j+2}, \quad \forall j \in \mathbb{Z}.$$ 

It is evident that $e_1 f_j = 0, i \geq 3, \forall j$.

Example 6.8. Another thread $L_1$-module $F_{\lambda,\mu}$ came from the well-known infinite dimensional representation of the Witt algebra in the tensor densities \[12\]:

$$e_1 f_j = (j + \mu - \lambda(i + 1)) f_{i+j}, \forall i \in \mathbb{N}, j \in \mathbb{Z},$$

where $\lambda, \mu \in \mathbb{K}$ are two parameters.

For a given $L_1$-module $M$ the elements $S_{p,q} \in U(L_1)$ are well-defined operators acting on $M$. For instance if $M = F_{\lambda,\mu}$ then

$$S_{2,1} f_j = \left((j+\mu-2\lambda)(j+1+\mu-2\lambda)-\frac{3}{2}(j+\mu-3\lambda)\right) f_{j+2}.$$ 

Let us introduce the numbers $S_{p,q} f_j = \sigma_{p,q}(j) f_{j+pq}$.

Corollary 6.9 (Feigin, Fuchs, \[9\]). Let $M = \oplus_i M_i$ be a thread $L_1$-module over a field $\mathbb{K}$. Then the homogeneous cohomology $H^*_s(L_1, M)$ is isomorphic to the cohomology of the following complex:

\[\begin{align*}
\cdots & \xleftarrow{D_{k+1}} \mathbb{K} \oplus \mathbb{K} \xleftarrow{D_k} \mathbb{K} \oplus \mathbb{K} \xrightarrow{D_{k-1}} \cdots \xleftarrow{D_1} \mathbb{K} \oplus \mathbb{K} \xleftarrow{D_0} \mathbb{K}, \\
\end{align*}\]

were differentials $D_k$ are defined by the matrices

\[\begin{pmatrix}
\sigma_{1,3k+1}(s+e(-k)) & -\sigma_{2k+1,1}(s+e(-k)) \\
\sigma_{2k+1,2}(s+e(k)) & -\sigma_{1,3k+2}(s+e(k))
\end{pmatrix}, k \geq 1; \\
\begin{pmatrix}
\sigma_{1,1}(s) \\
\sigma_{1,2}(s)
\end{pmatrix}.
\]

Proof. The homogeneous cohomology $H^*_s(L_1, M)$ is isomorphic to the cohomology of the following subcomplex of \[15\].

\[\begin{align*}
\cdots & \xleftarrow{d_{k+1}} M_{s+e(-k-1)} \oplus M_{s+e(k+1)} \xleftarrow{d_k} M_{s+e(-k)} \oplus M_{s+e(k)} \xrightarrow{d_{k-1}} \cdots \\
\cdots & \xrightarrow{d_2} M_{s+5} \oplus M_{s+7} \xrightarrow{d_1} M_{s+1} \oplus M_{s+2} \xleftarrow{d_0} M_s,
\end{align*}\]

with differentials \[15\] $d_k$ restricted on $M_{s+e(-k)} \oplus M_{s+e(k)}$. In its turn each subspace $M_{s+e(\pm k)}$ is isomorphic to $\mathbb{K}$. \hfill $\square$

At the time of writing the article \[9\] Feigin and Fuchs were not able to write the general formula for all operators $S_{p,q} \in U(L_1)$. However they managed to find explicit expressions for all entries $\sigma_{p,q}(j) \in \mathbb{K}$ for both examples $F_{\lambda,\mu}$ and $A_\alpha$ of thread modules considered above. In
particular for the $L_1$-module $A_{1,\alpha}$ the matrix $D_k$ will have the following form:

$$D_k = \begin{pmatrix} \sigma_{1,3k+1} & -\sigma_{2k+1,1} \\ \sigma_{2k+1,2} & -\sigma_{1,3k+2} \end{pmatrix},$$

where in particular

$$\sigma_{2k+1,1} = \prod_{i=1}^{k} (1 - 6\alpha^2).$$

The corresponding formulas for the module $F_{\lambda,\mu}$ were also obtained by Feigin and Fuchs [9].

It is useful to consider also finite-dimensional thread $L_1$-modules of the following type (short thread modules in [11])

$$V^{m,n} = \bigoplus_i V_i, \quad V_i = 0, \text{ if } i < m \text{ or } i > n, \quad \dim V_i = 1, \quad m \leq i \leq n,$

for some integers $m, n$.

For a given infinite dimensional thread module $V = \bigoplus_i V_i$ one can consider its so-called subquotient $V^{m,n}$:

$$V^{m,n} = (\bigoplus_{i \geq m} V_i) / (\bigoplus_{i \geq n} V_i)$$

that is a finite-dimensional thread module: $\dim V^{m,n} = n-m+1$.

**Example 6.10.** The subquotient $A_{1,\alpha}^{m,n}$ can be defined by its basis $f_m, \ldots, f_n$ and non-trivial structure relations

$$e_1 f_j = f_{j+1}, \quad j = m, \ldots, n-1, \quad e_2 f_j = \alpha f_{j+2}, \quad j = m, \ldots, n-2.$$  

7. **Special thread $L_1$-module $\tilde{M}$**

Let us consider a new thread $L_1$-module $\tilde{M}$ defined by its infinite basis $\{f_j, j \in \mathbb{Z}\}$.

$$e_i f_j = \begin{cases} 
    j f_{i+j}, & j \geq 0; \\
    (i+j) f_{i+j}, & i+j \leq 0, \quad j < 0; \\
    f_{i+j}, & i+j > 0, \quad j < 0.
\end{cases}$$  

(21)

The verification that the formulas (21) define a $L_1$-module is straightforward. One can think of $\tilde{M}$ as the result of "gluing together" of two modules: the quotient of $F_{-1,1}$ with submodule of $F_{0,0}$.

**Remark.** The module $\tilde{M} = \bigoplus_i \tilde{M}_i$ is decomposable as the direct sum of two $L_1$-modules

$$\tilde{M} = \tilde{M}_0 \oplus \left( \bigoplus_{i \neq 0} \tilde{M}_i \right),$$  

(22)
where $\tilde{M}_0$ is trivial one-dimensional $L_1$-module and $\tilde{M}_{\neq 0} = \oplus_{i \neq 0} \tilde{M}_i$ is infinite dimensional cyclic, i.e.

$$e_1 \tilde{M}_i = \tilde{M}_{i+1}, i \leq -2, \ e_2 \tilde{M}_{-1} = \tilde{M}_1, \ e_1 \tilde{M}_i = \tilde{M}_{i+1}, i \geq 1.$$ 

Let us consider a subquotient $\tilde{M}_{m,n}^{m,n} = \text{Span} \ (f_m, \ldots, f_{-1}, f_1, \ldots, f_n)$ that can be defined by

$$e_i f_j = 
\begin{cases} 
  j f_{i+j}, & j > 0, \ i+j \leq n; \\
  (i+j)f_{i+j}, & n \geq i+j > 0, \ m \leq j < 0; \\
  f_{i+j}, & n \geq i+j > 0, \ m \leq j < 0.
\end{cases}
$$

The module $\tilde{M}_{m,n}^{m,n}$ defines a representation $L_1 \to LT_{n-m+1}$ and hence a formal connection $A_{\tilde{M}}$. We present below an example of formal connection $A_{\tilde{M}}$ with $m = -2, n = 3$.

$$A_{\tilde{M}} = 
\begin{pmatrix}
  0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2e^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-e^2 & -e^1 & 0 & 0 & 0 & 0 & 0 \\
e^4 & e^3 & e^2 & 0 & 0 & 0 & 0 \\
e^5 & e^4 & e^3 & e^1 & 0 & 0 & 0 \\
e^6 & e^5 & e^4 & e^2 & 2e^1 & 0 & 0 \\
e^7 & e^6 & e^5 & e^3 & 2e^2 & 3e^1 & 0
\end{pmatrix},
$$

Now we want to establish the uniqueness in some sense of the module $\tilde{M}$: up to an isomorphism $\tilde{M}$ is a unique thread $L_1$-module with the decomposition (22), where the first summand is one-dimensional trivial module and the second one is cyclic in the sense (23).

It will be convenient for us to consider a new basis of $L_1$: $\tilde{e}_1 = e_1, \ \tilde{e}_i = 6(i-2)!e_i$.

Now we have in particular that

$$[\tilde{e}_1, \tilde{e}_i] = \tilde{e}_{i+1}, \ i \geq 2.$$ 

It was proved by Benoist [3] that $L_1$ is generated by two elements $\tilde{e}_1, \tilde{e}_2$ with the following two relations on them

$$[\tilde{e}_2, \tilde{e}_3] = \tilde{e}_5, \ [\tilde{e}_2, \tilde{e}_5] = \frac{9}{10} \tilde{e}_7,$$

where $\tilde{e}_3, \tilde{e}_5, \tilde{e}_7$ are defined by (7).

Hence the defining relations (25) will give us the following set of equations on a $L_1$-module $V = \text{Span} \ (f_m, \ldots, f_n)$:

$$R_i^5 : \ ([\tilde{e}_2, \tilde{e}_3] - \tilde{e}_5) f_i = 0, \ i = m, \ldots, n-5,$$
$$R_j^7 : \ ([\tilde{e}_2, \tilde{e}_5] - \frac{9}{10} \tilde{e}_7) f_i = 0, \ i = m, \ldots, n-7.$$
Obviously if $e_1 f_i \neq 0$ in a thread module $M$ one can consider the vector $f'_{i+1} = e_1 f_i$ instead of $f_{i+1}$.

**Theorem 7.1.** Let $M^{m,n}$ be a $(n-m+1)$-dimensional thread $L_1$-module defined by its basis $f_i, i = m, \ldots, -1, 0, 1, \ldots, n$, with $n-m+1 \geq 11$ such that:

$$\tilde{e}_1 f_i = f_{i+1}, \quad i = m, \ldots, -2, 1, \ldots, n-1;$$
$$\tilde{e}_1 f_{-1} = \tilde{e}_1 f_0 = 0, \quad \tilde{e}_2 f_j = b_j f_{j+2}, \quad j = m, \ldots, n-2, \quad b_{-2} = 0, \quad b_{-1} \neq 0.$$

Then $M^{m,n}$ is isomorphic to the module $\tilde{M}^{m,n}$.

**Proof.** 2) $R^5_{-2}, R^5_{-1}, R^5_0, R^7_{-2}$ will have the following form:

$$-b_{-1} b_1 - b_{-2} b_0 + 3b_{-1} = 0,$$
$$2b_{-1} b_2 - b_{-1} b_1 - b_{-1} = 0,$$
$$2b_0 b_3 - b_0 b_2 - b_0 = 0,$$
$$-3b_{-1} b_3 - b_{-2} b_0 + \frac{9}{2} b_{-1} = 0.$$  \hfill (26)

Now after rescaling the basic vectors if necessary one can assume that $b_{-2} = 0, \quad b_{-1} = b_0 = 1$.

The system (26) has the unique solution

$$b_1 = 3, \quad b_2 = 2, \quad b_3 = \frac{3}{2}.$$

In order to find $b_{-3}$ we have to consider the following equations:

$$b_{-2} b_0 - b_{-3} b_{-1} - 3b_{-1} = 0,$$
$$3b_{-1} b_2 - b_{-3} b_{-1} - 9b_{-1} = 0.$$  \hfill (27)

Evidently we have the answer $b_{-3} = -3$.

For $b_{-4}$ we have two new additional equations:

$$2b_{-4} b_{-1} - b_{-3} b_{-1} + b_{-1} = 0,$$
$$-b_{-1} b_1 + 3b_{-4} b_{-1} + 9b_{-1} = 0.$$  \hfill (28)

It follows that $b_{-4} = -2$.

Again one can remark that there are two equations on $b_{-5}$:

$$2b_{-5} b_{-2} - b_{-4} b_{-2} + b_{-2} = 0,$$
$$-b_0 b_{-2} - 3b_{-5} b_{-1} - \frac{9}{2} b_{-1} = 0.$$  \hfill (29)

And we have $b_{-5} = -\frac{3}{2}$.  


Proceeding in the same way we came to the thread $L_1$-module:

| $b_m$ | $b_{-4}$ | $b_{-3}$ | $b_{-2}$ | $b_{-1}$ | $b_0$ | $b_1$ | $b_2$ | $b_{n-2}$ |
|-------|----------|----------|----------|----------|------|------|------|----------|
| $-6/(m+1)$ | $-2$ | $-3$ | $0$ | $1$ | $1$ | $3$ | $2$ | $6/(n-1)$ |

One can verify that this module is isomorphic to $\tilde{M}^{m,n}$.

**Corollary 7.2.** Let $A'$ be an arbitrary formal connection corresponding to the trivial Massey product $\langle e^1, \ldots, e^1, e^2, e^1, \ldots, e^1, e^1 \rangle$, $m+n = 2k-1$.

Then for an arbitrary entry $a'(i, j), i < j$ of $A'$ we have the following property

$$a'(i, j) = a_{\tilde{M}^{m,n}_0}(i, j) + \delta(i, j),$$

where $\delta(i, j)$ is a linear combination of $e^r$ with $r < \deg a_{\tilde{M}^{m,n}_0}(i, j)$ and $a_{\tilde{M}^{m,n}_0}(i, j)$ is an entry of the formal connection $A_{\tilde{M}^{m,n}_0}$ that corresponds to the module $\tilde{M}^{m,n}_0$.

**Remark.** We will consider in the sequel both modules $\tilde{M}^{m,n}$ and $\tilde{M}^{m,n}_{\neq 0}$. The explanation is very simple: we need the module $\tilde{M}^{m,n}_{\neq 0}$ for the construction of Massey products but in the same time it is more convenient to compute the cohomology $H^*(L_1, \tilde{M}^{m,n})$ because of the very simple graded thread structure of $\tilde{M}^{m,n}$. The relation between $H^*(L_1, \tilde{M}^{m,n})$ and $H^*(L_1, \tilde{M}^{m,n}_{\neq 0})$ is very simple:

$$H^*(L_1, \tilde{M}^{m,n}) = H^*(L_1, \tilde{M}^{m,n}_{\neq 0}) \oplus \tilde{M}_0 \otimes H^*(L_1).$$

**8. Benoit-Saint-Aubin formula and thread module $\tilde{M}$.**

Now it is clear that in order to have the general combinatorial formula for singular Virasoro vectors in terms of operators $S_{p,q}$ it is useful to consider the sums over all unordered sequences $pq = \sum i_k$ by positive integers $i_1, \ldots, i_s$.

The first important step in this direction was made by Benoit and Saint-Aubin.
Theorem 8.1 (Benoit, Saint-Aubin [4]).

\[ S_{p,1}(t) = \sum_{i_1, \ldots, i_s} c_p(i_1, \ldots, i_s) t^{p-s} e_{i_1} \cdots e_{i_s}, \]
\[ S_{1,q}(t) = \sum_{i_1, \ldots, i_s} c_q(i_1, \ldots, i_s) t^{-q+s} e_{i_1} \cdots e_{i_s}, \]

where the sums are all over all partitions of \( p \) and \( q \) by positive numbers without any ordering restriction, and the coefficients are defined by

\[ c_r(i_1, \ldots, i_s) = \prod_{1 \leq k < r, k \neq i_1 + \cdots + i_l, \ l = 1, \ldots, s-1} k(r-k). \]

Example 8.2.

\[ S_{1,1}(t) = e_1, \quad S_{2,1}(t) = e_1^2 + te_2, \quad S_{3,1}(t) = e_1^3 + t(2e_1e_2 + 2e_2e_1) + 4t^2 e_3, \]
\[ S_{4,1}(t) = e_1^4 + t(3e_1^2e_2 + 4e_1e_2e_1 + 3e_2^2e_1) + t^2(12e_1e_3 + 9e_2^2 + 12e_3e_1) + 36t^3 e_4. \]

Later an important results on the structure of singular Virasoro vectors were obtained by Bauer, Di Francesco, Itzykson and Zuber [5] and others, especially interesting is interpretation of singular vectors in terms of Jack symmetric polynomials [22].

Now we are going to study the action of operators \( S_{p,1}(t) \) on the infinite dimensional thread module \( \tilde{M} \).

Example 8.3.

\[ S_{3,1}(t) f_{-2} = (e_1^3 + t(2e_1e_2 + 2e_2e_1) + 4t^2 e_3) f_{-2}. \]

Obviously \( e_1^3 f_{-2} = e_2 f_{-2} = 0 \). Hence

\[ S_{3,1}(t) f_{-2} = (-2t + 4t^2) f_1 = 4t \left( t - \frac{1}{2} \right) f_1 = (2!)^2 \prod_{i=1}^2 \left( t + \frac{i-2}{i(3-i)} \right) f_1. \]

Lemma 8.4. Let \( j < 0 \) and \( p + j > 0 \) then

\[ S_{p,1}(t) f_j = (p-1)^2 \prod_{i=1}^{p-1} \left( t + \frac{i+j}{i(p-i)} \right) f_{j+p}. \]

Proof.
Proposition 8.5. Let \( j < 0 \) and \( i_1 + \cdots + i_s + j > 0, i_t > 0 \). Then we have the following formula

\[
e_{i_1} \cdots e_{i_{s-1}} e_{i_s} f_j = \prod_{t=2}^{s} (i_1 + \cdots + i_s + j) f_{j+i_1+\cdots+i_s}
\]

**Proof.** There are three possibilities:

1) \( i_s + j < 0 \) and it follows that \( \exists m, s \geq m \geq 2 \) such that

\[
i_s + \cdots + i_m + j < 0, \quad \text{and} \quad i_s + \cdots + i_{m-1} + j \geq 0.
\]

Hence one can see that

\[
e_{i_m} \cdots e_{i_s} f_j = (i_s + j) \cdots (i_s + \cdots + i_m + j) f_{j+i_s+\cdots+i_m},
\]

\[
e_{i_m-1} f_{j+i_s+\cdots+i_m} = f_{j+i_s+\cdots+i_{m-1}},
\]

\[
e_{i_1} \cdots e_{i_{m-1}} f_{j+i_1+\cdots+i_m} = (i_s + \cdots + i_2 + j) \cdots (i_s + \cdots + i_{m-1} + j) f_{j+i_s+\cdots+i_1}.
\]

2) \( i_s + j > 0 \) then we have

\[
e_{i_s} f_j = f_{j+i_s},
\]

3) \( i_s + j = 0 \) then we have \( e_{i_s} f_j = (i_s + j) f_{j+i_s} = 0. \) \( \square \)

Let us introduce new positive integers

\[
k_1 = i_s, \quad k_2 = i_s + i_{s-1} > k_1, \ldots, \quad k_{s-1} = i_s + \cdots + i_2 > k_{s-2} > \cdots > k_1.
\]

Also one can remark that

\[
c_p(i_1, \ldots, i_s) = \frac{(p-1)!^2}{i_1(i_1+i_2) \cdots (i_1+\cdots+i_{s-1}) (p-i_1) (p-i_1-i_2) \cdots (p-i_1-\cdots-i_{s-1})}.
\]

Now one can consider the Benoit-Saint-Aubin formula (30). It follows that we have the formula for \( S_{p,1}(t)f_j \):

\[
= \sum_{(i_1, \ldots, i_s) \in \mathbb{N}^s, \ i_1 + \cdots + i_s = p} \frac{t^{p-s}(p-1)!^2(i_s+j) \cdots (i_s+i_2+j)}{i_1 \cdots (i_1+\cdots+i_s-1)(p-i_1) \cdots (p-i_1-\cdots-i_{s-1})} f_{j+i_1+\cdots+i_s} =
\]

\[
= (p-1)!^2 \sum_{1 \leq k_1 < \cdots < k_{s-1} \leq p-1} \frac{t^{p-s}}{k_1(p-k_1)} \cdots \frac{(k_{s-1}+j)}{k_{s-1}(p-k_{s-1})} f_{j+p} =
\]

\[
= (p-1)!^2 \prod_{i=1}^{p-1} \left( t + \frac{i+j}{i(p-i)} \right) f_{j+p}.
\]
Let us introduce a polynomial $F_{j,p}(t)$:

$$F_{j,p}(t) = (p-1)!^2 \prod_{i=1}^{p-1} \left( t + \frac{i+j}{i(p-i)} \right).$$

**Proposition 8.6.** Let $j < 0$ and $p \geq 2$, then

$$F_{j,p} \left( -\frac{3}{2} \right) \neq 0; \quad F_{j,p} \left( -\frac{2}{3} \right) = 0 \iff p + 3j = 1.$$

**Proof.** Let us suppose that $F_{j,p} \left( -\frac{3}{2} \right) = 0$. It means that

$$3i^2 + (2-3p)i + 2j = 0$$

for some positive integer $i, 1 \leq i \leq p-1$. For the roots $i_1, i_2$ of this square equation we have $3i_1 i_2 = 2j < 0$. On the other hand $i_1 + i_2 = \frac{3p-2}{3}$ and it implies that for the positive root $i_2$ we have $i_2 > \frac{3p-2}{3}$ that is impossible in the virtue of $i_2 \leq p - 1$.

Proceeding in the analogous way in the case $F_{j,p} \left( -\frac{2}{3} \right) = 0$ one can easily see that for the positive root $i_2$ of the corresponding square equation $2i^2 + (3-2p)i + 3j = 0$ we have $i_2 > \frac{2p-3}{2}$. It follows that $i_2 = p-1$ and $p-1+3j = 0$. □

9. Main theorem

**Theorem 9.1.** The cohomology $H^*(L_1)$ is generated by two elements $e^1, e^2 \in H^1(L_1)$ by means of two series of non-trivial Massey products. More precisely the recurrent procedure is organized as follows:

1) we take $e^1$ and $e^2$ as a basis of $H^1(L_1)$;

2) the triple Massey product $\langle e^1, e^2, e^2 \rangle$ is single-valued and determines non-trivial cohomology class $g^2_+ = \langle e^1, e^2, e^2 \rangle \in H^2_+(L_1)$;

3) the 5-fold product $\langle e^1, e^2, e^1, e^1, e^2 \rangle$ is non-trivial and it is an affine line $\{ g^2_+ + tg^1_-, t \in \mathbb{K} \}$ on the plane $H^2(L_1)$, where $g^2_+$ denotes some generator from $H^2_+(L_1)$. One can take an arbitrary element $\tilde{g}^2_+ \in \langle e^1, e^2, e^1, e^1, e^2 \rangle$ as the second basic element of $H^2(L_1)$.

Let us suppose that we have already constructed some basis $g^k_-, \tilde{g}^k_+$ of $H^k(L_1), k \geq 2$, such that the cohomology class $g^k_-$ spans the homogeneous subspace $H^k_{\text{hom},-}(L_1)$. Then

4) the $(2k+1)$-fold Massey product

$$\langle e^1, \ldots, e^1, e^2, e^1, \ldots, e^1, \tilde{g}^k_+ \rangle, \ m + n = 2k - 1,$$
is single-valued and determines non-trivial cohomology class \( g_{k+1} \) from the subspace \( H^{k+1}_{\frac{1}{2}(3(k+1)^2-(k+1))}(L_1) \).

5) the \((3k+2)\)-fold product

\[
\langle e^1, \ldots, e^1; e^2, \ldots, e^1, \tilde{g}^k \rangle
\]

is non-trivial and it is an affine line on the two-dimensional plane \( H^{k+1}(L_1) \) parallel to the one-dimensional subspace \( H^{k+1}_{\frac{1}{2}(3(k+1)^2-(k+1))}(L_1) \).

One can take an arbitrary element \( \tilde{g}^{k+1}_+ \) in \( \langle e^1, \ldots, e^1; e^2, \ldots, e^1, \tilde{g}^k \rangle \) as the second basic element of \( H^{k+1}(L_1) \).

**Proof.** 2) for an arbitrary choice of formal connection \( A \) for the product \( \langle e^2, e^2, e^1 \rangle \) the corresponding cocycle will be equal to \( c(A) = -e^2 \wedge e^3 + \alpha d(e^3) \) for some constant \( \alpha \). Hence \( \langle e^1, e^2, e^2 \rangle = -[e^2 \wedge e^3] \neq 0 \) in \( H^2(L_1) \).

3) in the second case it is more convenient to consider the equivalent product \( \langle e^1, e^2, -e^1, -2e^1, -e^2 \rangle \) instead of \( \langle e^1, e^2, e^1, e^1, e^2 \rangle \). One can take the following formal connection \( A \):

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
-e^2 & 0 & 0 & 0 & 0 & 0 \\
2e^3 & -2e^1 & 0 & 0 & 0 & 0 \\
-e^4 - te^2 & -e^2 & -e^1 & 0 & 0 & 0 \\
0 & e^4 & e^3 & e^2 & 0 & 0 \\
* & e^5 & e^4 & e^3 & e^1 & 0
\end{pmatrix}.
\]

The corresponding cocycle will be \( c(A) = (e^2 \wedge e^5 - 3e^3 \wedge e^4) + te^2 \wedge e^3 \). On the other hand for an arbitrary defining system \( A' \) the corresponding cocycle will have the form \( c(A') = (e^2 \wedge e^5 - 3e^3 \wedge e^4) + \ldots \), where dots stand for the summands with the second grading strictly less than 7.

4) Let us consider a finite dimensional thread \( L_1 \)-module \( M_{-m-1,n} = \langle f_{-m-1}, f_m, \ldots, f_n \rangle \) where \( m + n = 2k - 1 \). In the second grading \( -m - 1 - \frac{3k^2+1}{2} \) in the spectral sequence we have only one differential that can be non trivial:

\[
g^k_+ \otimes f_{-m-1} \xrightarrow{d_{2k+1}} g^{k+1}_- \otimes f_n.
\]

We recall that

\[
d_{2k+1} (g^k_+ \otimes f_{-m-1}) = c(A_{M_{-m-1,n}}) \otimes f_n,
\]
where $c(A_{\tilde{M} - m-1,n})$ is the cocycle that corresponds to the formal connection $A_{\tilde{M} - m-1,n}$ of the $(2k+1)$-fold Massey product

$\langle e^1, \ldots, e^1, e^2, e^1, \ldots, e^1, g^k_+ \rangle$, $m + n = 2k - 1$,

defined by the module $\tilde{M}_{m-1,n}^{-m-1}$.

**Proposition 9.2.** The differential $d_{2k+1}$ (and hence the cocycle $c(A_{\tilde{M} - m-1,n})$) is non trivial.

**Proof.** We will compute the cohomology by means of the free resolution. Here we have also only one possibly non trivial differential:

$$(0, f_{-m-1}) \xrightarrow{D_k} (f_n, 0), \quad D_k = \begin{pmatrix} 0 & 0 \\ -S_{2k+1,1} & 0 \end{pmatrix}.$$

It follows that

$$S_{2k+1,1}(f_{-m-1}) = F_{-m-1,2k+1} \begin{pmatrix} -3 \\ 2 \end{pmatrix} f_n \neq 0.$$ 

□

**Corollary 9.3.** It exists a formal connection $A$ for the Massey product

$\langle e^1, \ldots, e^1, e^2, e^1, \ldots, e^1, g^k_+ \rangle$, $m + n = 2k - 1$,

such that $c(A) = g_{-}^{k+1}$.

**Proposition 9.4.** The product $\langle e^1, \ldots, e^1, e^2, e^1, \ldots, e^1, g^k_+ \rangle$, $m + n = 2k - 1$, is non trivial and single-valued, moreover one can replace it the element $g^k_+$ by $\tilde{g}^k_+ = g^k_+ + tg^k_-$ with arbitrary $t$.

**Proof.** It directly follows from the Corollary 7.4 for an arbitrary formal connection $A'$ that corresponds to our Massey product we will have

$$c(A') = c(A_{\tilde{M}}) + \ldots,$$

where dots stand for summands with the gradings strictly less than $\deg c(A_{\tilde{M}})$ and hence $[c(A')] = [c(A_{\tilde{M}})]$ because the grading $\deg c(A_{\tilde{M}}) = e_-(k + 1)$ is the minimal where there is non trivial cohomology class from $H^{k+1}(L_1)$.

□

5) Let us consider a finite dimensional graded $L_1$-module $\tilde{M}^{-2k-1,k}$. In the second grading $-2k-1 - \frac{3k^2+1}{2}$, in the spectral sequence we have only one possibly non trivial differential

$\langle g^k_- \otimes f_{-2k-1} \rangle \xrightarrow{d_{2k+1}} \langle g^k_- \otimes f_k \rangle$. 

We recall that
\[ d_{2k+1} \left( g^k_+ \otimes f_{-2k-1} \right) = c(A_M) \otimes f_k, \]
where \( c(A_M) \) is the cocycle that corresponds to the formal connection \( A_M \) of the Massey product
\[ \langle e^1, \ldots, e^1_k, e^2, e^1, \ldots, e^1_{2k}, g^k_+ \rangle \]
defined by the module \( M_{-2k-1,k} \).

**Proposition 9.5.** The differential \( d_{3k+2} \) (and hence the cocycle \( c(A_M) \)) is non trivial.

**Proof.** The computation by means of the free resolution gives us that in this complex we have also the only one possibly non trivial differential:
\[ \langle (0, f_{-2k-1}) \rangle \xrightarrow{D_k} \langle (f_k, 0) \rangle, \quad D_k = \begin{pmatrix} 0 & 0 \\ -S_{2k+1,1} & -S_{1,3k+2} \end{pmatrix}. \]
It is obvious that
\[ S_{2k+1,1}(t)f_{-2k-1} = 0. \]
On the another hand
\[ S_{1,3k+2} \left( -\frac{3}{2} \right) f_{-2k-1} = S_{3k+2,1} \left( -\frac{2}{3} \right) f_{-2k-1} \neq 0. \]

**Corollary 9.6.** It exists a formal connection \( A \) for the Massey product
\[ \langle e^1, \ldots, e^1_k, e^2, e^1, \ldots, e^1_{2k}, g^k_+ \rangle \]
such that \( c(A) = g^{k+1}_+ \).

**Proposition 9.7.** The product \( \langle e^1, \ldots, e^1_k, e^2, e^1, \ldots, e^1_{2k}, g^k_+ \rangle \) is non trivial, but not single-valued. It value will not change if we replace in it the element \( g^k_+ \) by \( \tilde{g}^k_+ = g^k_+ + tg^k_- \) with arbitrary constant \( t \).

**Proof.** It follows from the Corollary \( \square \)

\( \square \)
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