Non-minimal Wu-Yang monopole

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Abstract

We discuss new exact spherically symmetric static solutions to non-minimally extended Einstein-Yang-Mills equations. The obtained solution to the Yang-Mills subsystem is interpreted as a non-minimal Wu-Yang monopole solution. We focus on the analysis of two classes of the exact solutions to the gravitational field equations. Solutions of the first class belong to the Reissner-Nordström type, i.e., they are characterized by horizons and by the singularity at the point of origin. The solutions of the second class are regular ones. The horizons and singularities of a new type, the non-minimal ones, are indicated.

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1 Introduction

Exact solutions of the monopole type are known to play a significant role in the modern field theory \cite{1,2,3}. The monopole solutions to the self-consistent Einstein-Yang-Mills-Higgs equations (see, e.g., \cite{4,5,6,7,8} and references therein) are of great importance, since they demonstrate explicitly the interplay between the gravitational, gauge and scalar fields in the non-Abelian black hole structure formation. New possibilities for the modeling of the monopole structure appear, when we take into account the so-called non-minimal coupling of the gravitational, gauge and scalar fields. Non-minimal theory has been elaborated in detail for scalar and electromagnetic fields (see, e.g., \cite{9,10} for a review). Müller-Hoissen obtained in \cite{11} the non-minimal Einstein-Yang-Mills (EYM) model from a dimensional reduction of the Gauss-Bonnet action, this model contains one coupling parameter. We follow the alternative derivation of the non-minimal EYM theory, formulated as a non-Abelian generalization of the non-minimal non-linear Einstein-Maxwell theory \cite{12} along the lines proposed by Drummond and Hathrell for the linear electrodynamics \cite{13}. We deal with a non-minimal EYM model linear in curvature, which can be indicated as a three-parameter model, since it contains three coupling constants $q_1$, $q_2$ and $q_3$ with the dimensionality of area. Depending on the type of the model these coupling constants may be associated with three, two specific radii or be reduced to the unique radius, describing the characteristic length of the non-minimal interaction (say, $r_q = \sqrt{2|q_1|}$). Thus, in addition to the standard Schwarzschild radius $r_g$ and Reissner-Nordström radius $r_Q$ we obtain
at least one extra parameter, \( r_q \), for modeling the causal structure of non-minimally extended Einstein-Yang-Mills monopoles.

In this letter we introduce a three-parameter self-consistent Einstein-Yang-Mills model, in which the EYM Lagrangian is gauge invariant, linear in space-time curvature and quadratic in the Yang-Mills field strength tensor \( F_{ik} \). Then we consider exact spherically symmetric static solutions of the obtained model and discuss in detail the non-minimal generalization of the Wu-Yang monopole solution. We distinguish between the solutions of the Reissner-Nordström type, which is irregular in the center of the charged body, and the regular non-minimal Wu-Yang monopole solutions. We discuss also the relation between the values of the parameters \( q_1, q_2, q_3 \) and the radii of the non-minimal horizons and/or singularities, which can be associated with the introduced new (non-minimal) Wu-Yang monopole.

2 Non-minimal Einstein-Yang-Mills field equations

The three parameter non-minimal Einstein-Yang-Mills theory can be formulated in terms of the action functional

\[
S_{\text{NMEYM}} = \int d^4 x \sqrt{-g} \left[ \frac{R}{8\pi\gamma} + \frac{1}{2} F_{ik}^{(a)} F_{ik}^{(a)} + \frac{1}{2} R^{ikmn} F_{ik}^{(a)} F_{mn}^{(a)} \right].
\]

Here \( g = \det(g_{ik}) \) is the determinant of a metric tensor \( g_{ik} \), \( R \) is the Ricci scalar, \( \gamma \) is the gravitational constant. The Latin indices without parentheses run from 0 to 3, the summation with respect to the repeated group indices \( (a) \) is implied. The tensor \( R^{ikmn} \) is defined as follows (see, e.g., [12, 14]):

\[
R^{ikmn} \equiv \frac{q_1}{2} R (g^{im} g^{kn} - g^{in} g^{km}) + \frac{q_2}{2} (R^{im} g^{kn} - R^{in} g^{km} + R^{kn} g^{im} - R^{km} g^{in}) + q_3 R^{ikmn},
\]

where \( R^{ik} \) and \( R^{ikmn} \) are the Ricci and Riemann tensors, respectively, and \( q_1, q_2, q_3 \) are the phenomenological parameters describing the non-minimal coupling of the Yang-Mills and gravitational fields. Following [1] we assume that the Yang-Mills field, \( F_{mn} \), takes the values in the Lie algebra of the gauge group \( SU(2) \):

\[
A_m = -i \mathcal{G} t_{(a)} A_{m}^{(a)}, \quad F_{mn} = -i \mathcal{G} t_{(a)} F_{mn}^{(a)}.
\]

Here \( t_{(a)} \) are Hermitian traceless generators of \( SU(2) \) group, \( A_{m}^{(a)} \) and \( F_{mn}^{(a)} \) are the Yang-Mills field potential and strength, respectively, the constant \( \mathcal{G} \) is the strength of the gauge coupling, and the group index \( (a) \) runs from 1 to 3. The generators \( t_{(a)} \) satisfy the commutation relations:

\[
[t_{(a)}, t_{(b)}] = i \varepsilon_{(a)(b)(c)} t_{(c)},
\]

where \( \varepsilon_{(a)(b)(c)} \) is the completely antisymmetric symbol with \( \varepsilon_{(1)(2)(3)} = 1 \).

The variation of the action functional with respect to the Yang-Mills potential \( A_{i}^{(a)} \) yields

\[
\hat{D}_k H^{ik} \equiv \nabla_k H^{ik} + [A_k, H^{ik}] = 0, \quad H^{ik} = F^{ik} + R^{ikmn} F_{mn}.
\]

Here the symbol \( \nabla_m \) denotes a covariant space-time derivative. The tensor \( H^{ik} \) is a non-Abelian analogue of the induction tensor well-known in electrodynamics [15]. This analogy shows that
\( R^{ikmn} \) can be considered as a susceptibility tensor \[12\]. In a similar manner, the variation of the action with respect to the metric yields

\[
R_{ik} - \frac{1}{2} R g_{ik} = 8\pi \gamma T^{(\text{eff})}_{ik}.
\]  

(6)

The effective stress-energy tensor \( T^{(\text{eff})}_{ik} \) can be partitioned into four terms:

\[
T^{(\text{eff})}_{ik} = T^{(YM)}_{ik} + q_1 T^{(I)}_{ik} + q_2 T^{(II)}_{ik} + q_3 T^{(III)}_{ik}.
\]  

(7)

The first term \( T^{(YM)}_{ik} \):

\[
T^{(YM)}_{ik} = \frac{1}{4} g_{ik} F_{mn}^{(a)} F_{mn}^{(n(a))} - \frac{1}{2} \left[ \hat{D}_i \hat{D}_k - g_{ik} \hat{D}^l \hat{D}_l \right] \left[ F_{mn}^{(a)} F_{mn}^{(n(a))} \right],
\]  

(8)

is a stress-energy tensor of the pure Yang-Mills field. The definitions of other three tensors are related to the corresponding coupling constants \( q_1, q_2, q_3 \):

\[
T^{(I)}_{ik} = R T^{(YM)}_{ik} - \frac{1}{2} R_{ik} F_{mn}^{(a)} F_{mn}^{(n(a))} + \frac{1}{2} \left[ \hat{D}_i \hat{D}_k - g_{ik} \hat{D}^l \hat{D}_l \right] \left[ F_{mn}^{(a)} F_{mn}^{(n(a))} \right],
\]  

(9)

\[
T^{(II)}_{ik} = \frac{1}{2} g_{ik} \left[ \hat{D}_m \hat{D}_l \left( F_{mn}^{(a)} F_{nl}^{(n(a))} \right) - R_{lm} F_{mn}^{(a)} F_{nl}^{(n(a))} \right] - F^{ln(a)} \left( R_{kl} F_{kn}^{(a)} + R_{kl} F_{kn}^{(n(a))} \right) - R_{mn} F_{ln}^{(a)} F_{kl}^{(n(a))} - \frac{1}{2} \hat{D}_m \hat{D}_m \left( F_{kn}^{(a)} F_{mn}^{(n(a))} \right) +
\]

\[
+ \frac{1}{2} \hat{D}_l \left[ \hat{D}_i \left( F_{kn}^{(a)} F^{ln(a)} \right) + \hat{D}_k \left( F_{ln}^{(a)} F^{ln(a)} \right) \right],
\]  

(10)

\[
T^{(III)}_{ik} = \frac{1}{4} g_{ik} R_{mnls}^{(a)} F_{mn}^{(a)} F_{ls}^{(a)} - \frac{3}{4} F_{ls}^{(a)} \left( F_{kn}^{(a)} R_{knls} + F_{k}^{(n(a))} R_{knls} \right) -
\]

\[
- \frac{1}{2} \hat{D}_m \hat{D}_n \left[ F_{k}^{(n(a))} F_{k}^{(m(a))} + F_{k}^{(n(a))} F_{l}^{(m(a))} \right].
\]  

(11)

One can check directly that the tensor \( T^{(\text{eff})}_{ik} \) satisfies the equation \( \nabla^k T^{(\text{eff})}_{ik} = 0 \), as in the case of non-minimal electrodynamics \[12\]. The self-consistent system of equations (5) and (6) with (7) - (11) is a direct non-Abelian generalization of the three-parameter non-minimal Einstein-Maxwell model discussed in \[12\]. This system can also be considered as one of the variants of a non-minimal generalization of the Einstein-Yang-Mills model.

### 3 Wu-Yang monopole

Let us consider a static spherically symmetric non-minimal Einstein-Yang-Mills model with the space-time metric

\[
ds^2 = \sigma^2 N dt^2 - \frac{dr^2}{N} - r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right).
\]  

(12)
The Einstein-Maxwell model for such metric with a central electric charge was studied in [16] for the special case \( q_1 + q_2 + q_3 = 0, \) \( 2q_1 + q_2 = 0. \) We focus on the gauge field characterized by the special ansatz (see, [17]):

\[
A_0 = A_r = 0,
A_\theta = -i \left( \frac{w}{\nu} - 1 \right) t_\varphi, \quad A_\varphi = i \left( w - \nu \right) \sin \theta t_\theta.
\]

Here \( \sigma \) and \( N \) are functions depending on the radius \( r \) only and satisfying the asymptotic conditions

\[
\sigma (\infty) = 1, \quad N (\infty) = 1.
\]

Generally, one can consider \( w \) as a function of the radius, however, in this paper we focus on the model with constant \( w, \) keeping in mind the well-known Wu-Yang solution. The parameter \( \nu \) is a non-vanishing integer. The generators \( t_r, t_\theta, \) and \( t_\varphi \) are the position-dependent ones and are connected with the standard generators of the \( SU(2) \) group as follows:

\[
t_r = \cos \nu \varphi \sin \theta t_{(1)} + \sin \nu \varphi \sin \theta t_{(2)} + \cos \theta t_{(3)},
\]

\[
t_\theta = \partial_\theta t_r, \quad t_\varphi = \frac{1}{\nu \sin \theta} \partial_\varphi t_r.
\]

The field strength tensor

\[
F_{ik} = \partial_i A_k - \partial_k A_i + [A_i, A_k]
\]

has only one non-vanishing component:

\[
F_{\theta\varphi} = -i \frac{(w^2 - \nu^2)}{\nu} \sin \theta t_r,
\]

which does not depend on the variable \( r. \) Due to the discussed ansatz the system of Yang-Mills equations (15) reduces to the single equation

\[
\frac{w(w^2 - \nu^2)}{r^4} \left( 1 + 2\mathcal{R}^{\theta\varphi}_{\theta\varphi} \right) = 0,
\]

which is a non-minimal generalization of the well-known key equation resulting in the Wu-Yang monopole solution [18].

There are three formal possibilities to satisfy the equation (19): first, \( w = 0, \) second, \( w = \pm \nu, \) third, \( \left( 1 + 2\mathcal{R}^{\theta\varphi}_{\theta\varphi} \right) = 0. \) When the space-time is asymptotically flat ( \( R_{ikmn}(r \to \infty) = 0) \) the last term in the key equation (19) cannot vanish identically. When \( w = \pm \nu, \) we obtain from (18) that \( F_{ik} \) vanishes, and this exact solution describes the so-called pure gauge. Finally, when \( w = 0 \) we deal with the Wu-Yang monopole solution. The strength of the Yang-Mills field now gets the form \( F_{\theta\varphi} = i \nu \sin \theta t_r, \) as in the case of minimal Wu-Yang monopole in the Minkowski space-time [18]. This solution is known to be effectively Abelian, i.e., by the suitable gauge transformation \( U = \exp(-i \theta t_\varphi) \) it can be converted into the product of the Dirac type potential and the gauge group generator \( t_{(3)}. \)
4 Exact solutions to the gravitational field equations

4.1 Key equations

For the metric (12) only four components of the Einstein tensor $G^k_i = R^k_i - \frac{1}{2} \delta^k_i R$ are non-vanishing:

$$G^0_0 = \frac{1 - N}{r^2} - \frac{N'}{r}, \quad G^r_i = \frac{1 - N}{r^2} - \frac{N'}{r} - \frac{2 N \sigma'}{r^2},$$

$$G^0_\sigma = G^\sigma_r = -\frac{1}{2r^2} (2 \sigma N' + 2 N \sigma' + 3 r \sigma' N' + 2 r N \sigma'' + r \sigma N'').$$

The prime denotes the derivative with respect to the radius $r$. The corresponding four non-vanishing components of the effective stress-energy tensor (see (8)-(11)) take the form

$$T^{0(\text{eff})}_0 = \nu^2 \frac{G^2}{G} \left[ \frac{N}{r^6} - \frac{q_1 N'}{r^5} + \frac{1}{r^6} \left( 13 q_1 + 4 q_2 + q_3 \right) - (q_1 + q_2 + q_3) \frac{1}{r^6} \right],$$

$$T^{\sigma(\text{eff})}_r = \frac{\nu^2}{G^2} \left[ \frac{1}{2r^4} - q_1 \frac{N'}{r^5} - 2 q_1 \frac{N \sigma'}{r^5} - \frac{2}{2r^4} + (q_1 + q_2 + q_3) \frac{1}{r^6} \right],$$

$$T^{\theta(\text{eff})}_\theta = T^{\varphi(\text{eff})}_\varphi = -\nu^2 \frac{G^2}{G} \left[ \frac{1}{2r^4} - \frac{3 q_1 \sigma' N'}{2 \sigma r^4} - q_1 N \sigma'' \frac{2 r^4}{2r^4} \right] - (q_1 + q_2 + q_3) \left[ \frac{(\sigma N')'}{\sigma r^5} - \frac{2 N}{r^6} \right] + (q_1 + q_2 + q_3) \frac{2}{r^6}.$$

Analogously to the case of minimal electrodynamics the equation $G^\theta_\theta = 8 \pi \gamma T^{\theta(\text{eff})}_\theta$ is a differential consequence of two first Einstein equations. Thus, in order to find two quantities, $N(r)$ and $\sigma(r)$, we have two independent equations. Moreover, the difference of the first and second equations, $G^0_0 - G^r_r = 8 \pi \gamma \left( T^{0(\text{eff})}_0 - T^{r(\text{eff})}_r \right)$, gives the equation for the function $\sigma(r)$ only:

$$r \frac{\sigma'}{\sigma} \left( 1 - \frac{\kappa q_1}{r^4} \right) = \frac{\kappa (10 q_1 + 4 q_2 + q_3)}{r^4}. \quad (25)$$

Here $\kappa = \frac{8 \pi \nu^2}{G}$ is a new convenient constant with the dimensionality of area. The function $N(r)$ satisfies the linear differential equation

$$r \frac{N'}{N} \left( 1 - \frac{\kappa q_1}{r^4} \right) + N \left[ 1 + \frac{\kappa}{r^4} (13 q_1 + 4 q_2 + q_3) \right] = 1 - \frac{\kappa}{2} \frac{r}{r^2} + \frac{\kappa}{r^4} (q_1 + q_2 + q_3). \quad (26)$$

It is worth mentioning that all the non-minimal contributions to the equations (25) and (26) have the similar form: they contain products of $\kappa$ and linear combinations of the coupling constants divided by $r^4$. 
4.2 Minimal limit $q_1 = q_2 = q_3 = 0$

When $q_1$, $q_2$, $q_3$ vanish, the equation (25) and the asymptotic conditions (14) give $\sigma(r) = 1$. The equation (26) yields

$$N = 1 - \frac{2M}{r} + \frac{\kappa}{2r^2},$$

(27)

where $M$ is a constant of integration describing the asymptotic mass of the monopole (in the geometrical units $2M$ is equal to the Schwarzschild radius $r_s$). This solution is of the Reissner-Nordström type.

4.3 Non-minimal models with $q_1 \neq 0$

For generic $q_1$, $q_2$, $q_3$ the equations (25) and (26) with the conditions (14) yield

$$\sigma = \left(1 - \frac{\kappa q_1}{r^4}\right)\beta, \quad \beta \equiv \frac{10q_1 + 4q_2 + q_3}{4q_1},$$

(28)

$$N = 1 - \frac{1}{r} \cdot \left(1 - \frac{\kappa q_1}{r^4}\right)^{-\beta+1}\left\{2M - \frac{\kappa}{2} \int_0^\infty \frac{dx}{x^2} \left[1 + \frac{6}{x^2}(4q_1 + q_2)\right] \left(1 - \frac{\kappa q_1}{x^4}\right)^\beta\right\}.$$  

(29)

When $r \to \infty$, these solutions asymptotically behave as

$$\sigma = 1 - \frac{\kappa q_1}{r^4} \beta + \ldots, \quad N = 1 - \frac{2M}{r} + \frac{\kappa}{2r^2} + \frac{\kappa}{r^4} (4q_1 + q_2) + \ldots,$$

(30)

thus the leading order terms recover the Reissner-Nordström solution, and the non-minimal contributions contain the terms $\frac{1}{r^7}$, $\frac{1}{r^9}$, etc.

When $r \to 0$, the function $\sigma(r)$ can tend to infinity, if $\beta$ is positive, can tend to zero, if $\beta$ is negative, and remains equal to one, if $\beta = 0$. In other words, the metric coefficient $g_{00} = \sigma^2 N$ can be irregular at the point of origin $r = 0$, when $10q_1 + 4q_2 + q_3 \neq 0$. Moreover, when the parameter $q_1$ is positive, the metric coefficient $\sigma(r)$ takes zero value at the point $r_s = (\kappa q_1)^{\frac{1}{7}}$, if $\beta > 0$, and becomes infinite, if $\beta < 0$, providing the curvature invariants to be infinite at $r_s$.

To illustrate this remark, let us assume that $\beta = 1$, or, equivalently, $6q_1 + 4q_2 + q_3 = 0$. Then one obtains the exact solution

$$\sigma = 1 - \left(\frac{r_s}{r}\right)^4,$$

(31)

$$N = 1 - \left[1 - \left(\frac{r_s}{r}\right)^4\right]^{-2}\left\{\frac{2M}{r} - \frac{\kappa}{2r^2}\left[1 - \frac{1}{5} \left(\frac{r_s}{r}\right)^4 + \frac{2(4q_1 + q_2)}{r^2} \left[1 - \frac{3}{7} \left(\frac{r_s}{r}\right)^4\right]\right]\right\},$$

(32)

for which $\sigma(r_s) = 0$. As for $N(r_s)$, it can be infinite, equal to zero or take a finite value depending on relationships between $q_1$, $q_2$, $M$ and $\kappa$. For instance, when $q_1 = \frac{16}{165}$, $q_2 = -\frac{6}{8}$ and $M = \frac{3t}{2\sqrt{3}}$ one obtains that $r_s = \frac{\sqrt{3}}{2}$ and $N(r_s) = 0$, however, the Ricci scalar $R$ and quadratic curvature invariants $R_{ik}R^{ik}$, $R_{ikmn}R^{ikmn}$ are regular at the point $r_s$. In other cases these invariants become infinite, and the point $r = r_s$ can be indicated as a specific non-minimal singularity. When $q_1$ is negative, such a singularity does not appear.
4.4 Non-minimal models with \( q_1 = 0 \)

Since \( q_1 \) appears in the denominator of the expression (28) for \( \beta \), let us consider the case \( q_1 = 0 \) as a special one. Now the metric functions are

\[
\sigma = \exp \left\{ -\frac{\kappa}{4r^4} (4q_2 + q_3) \right\},
\]

\[
N = 1 - \frac{1}{r} \cdot \exp \left\{ \frac{\kappa(4q_2 + q_3)}{4r^4} \right\} \cdot \left( 2M - \frac{\kappa}{2r} \int_0^\infty \frac{dx}{x^2} \left( 1 + \frac{6q_2}{x^2} \right) \exp \left\{ -\frac{\kappa(4q_2 + q_3)}{4x^4} \right\} \right). \tag{34}
\]

Clearly, the analytical progress is possible, when \( q_3 = -4q_2 \). Indeed, for this model \( \sigma(r) = 1 \), and the explicit exact solution for the function \( N(r) \) is

\[
N = 1 - \frac{2M}{r} + \frac{\kappa}{2r^2} + \frac{\kappa q_2}{r^4}. \tag{35}
\]

We deal with the non-minimal generalization of the Reissner-Nordström star with \( N(0) = \infty \). Such a star possesses horizons, when the algebraic equation of the fourth order

\[
r^4 - 2Mr^3 + \frac{\kappa}{2}r^2 + \kappa q_2 = 0 \tag{36}
\]

has real positive roots. There are two explicit cases admitting specific non-minimal horizons.

(i) \( M = 0 \) and \( q_2 < 0 \)

Then the positive real root of (36) is

\[
r = r_{(H)} = \frac{1}{2} \sqrt{\kappa} \sqrt[4]{1 + \frac{16|q_2|}{\kappa} - 1}. \tag{37}
\]

In the minimal limit \( r_{(H)} \) coincides with \( r = 0 \) and tends to \( \sqrt{2|q_2|} \) when \( |q_2| \ll \kappa \).

(ii) \( \kappa = 2M^2 \) and \( q_2 < 0 \)

The equation (36) possesses the positive real root

\[
r_{(H1)} = \frac{M}{2} \left( 1 + \sqrt{1 + \frac{4\sqrt{2}|q_2|}{M}} \right). \tag{38}
\]

When \( M > 4\sqrt{2|q_2|} \), there are two additional roots

\[
r_{(H2,3)} = \frac{M}{2} \left( 1 \pm \sqrt{1 - \frac{4\sqrt{2}|q_2|}{M}} \right). \tag{39}
\]

In the minimal limit the condition \( \kappa = 2M^2 \) leads to the so-called extremal Reissner-Nordström black hole, for which two horizons coincide. When \( q_2 < 0 \), the specific radii \( r_{(H1)} \), \( r_{(H2)} \) and \( r_{(H3)} \) play the roles of the non-minimal horizons radii. When \( q_2 \) tends to zero, \( r_{(H1)} \rightarrow r_{(H2)} \rightarrow M \) and \( r_{(H3)} \rightarrow 0 \).
4.5 Regular one-parameter model

When $10q_1 + 4q_2 + q_3 = 0$, $4q_1 + q_2 = 0$, i.e., $q_1 = -q$, $q_2 = 4q$, $q_3 = -6q$, and $q$ is positive, we obtain a new explicit exact solution

$$\sigma(r) = 1, \quad N = 1 + \frac{r^2(k - 4Mr)}{2(r^4 + \kappa q)}. \quad (40)$$

The obtained function $N(r)$ takes the value $N = 1$ at three points: $N(0) = 1$, $N\left(\frac{\kappa}{4M}\right) = 1$, $N(\infty) = 1$ (asymptotically). When $M = 0$ the second and the third points coincide, $N(r) \geq 1$ and $N(r)$ has only one extremum (maximum) at the point $r_{(\text{max})} = (\kappa q)^{\frac{1}{2}}$. For small $M$ one has a minimum at some point $r_{(\text{min})}$ ($r_{(\text{min})} > \frac{\kappa}{4M}$), for which $0 < N_{(\text{min})} < 1$. When the mass $M$ increases, this minimum reaches the value $N_{(\text{min})} = 0$ with the mass taking a critical value $M_{(\text{crit})}$ of the following form

$$M_{(\text{crit})} = \frac{r_*}{6} \left(4 + \frac{\kappa}{r_*^2}\right). \quad (41)$$

Here

$$r_* = \frac{\sqrt{\kappa}}{2} \sqrt{\left(1 + \frac{48q}{\kappa} + 1\right)}. \quad (42)$$

Thus, when $M < M_{(\text{crit})}$ the metric (40) has no horizons; when $M > M_{(\text{crit})}$ there are two horizons, $r_-$ and $r_+$. When $M = M_{(\text{crit})}$ the function $N(r)$ takes zero value only at $r = r_*$, i.e., in this case the metric (40) is a non-minimal analogue of the extremal Reissner-Nordström solution. When $q = 0$, the parameter $r_*$ coincides with the Reissner-Nordström radius, $r_Q = \sqrt{\frac{\kappa}{\kappa}}$.

The solution (40) is regular at the point $r = 0$, since the denominator cannot reach zero value. In addition, direct calculations show that the curvature invariants $R$, $R_{ik}R^{ik}$, $R_{ikmn}R^{ikmn}$ take finite values at $r = 0$.

5 Conclusions

We have shown that the three-parameter non-minimally extended Einstein-Yang-Mills theory admits the exactly solvable generalization of the Wu-Yang monopole model. Indeed, the non-minimal Yang-Mills subsystem admits the exact solution of the standard explicit form (13), (18). The solutions to the gravitational field equations are also presented in the explicit form: in the quadratures for generic $q_1$, $q_2$ and $q_3$ (see, (29), (34)), and in the elementary functions for the special choices of the coupling parameters (see, (32), (35)). The analysis of these exact solutions permits the following three features to be emphasized.

(i) On the inheritance of the structure of the Wu-Yang monopole solution.
Non-minimal interaction of the Yang-Mills and gravitational fields results in essentially complicated master equations (see, (5), (2)). Nevertheless, the well-known Wu-Yang solution with the ansatz (13) keeps its form in the non-minimally extended theory, the coupling parameters $q_1$, $q_2$ and $q_3$ do not enter the expression for $A_i$.

(ii) On the regularity of the Wu-Yang monopole.
The analytical solution (40) to the gravitational field equations is regular at $r = 0$ ($\sigma(0) = 1$, $N(0) = 1$) and has no horizons, when $M < M_{(\text{crit})}$. The curvature invariants, $R$, $R_{ik}R^{ik}$, $R_{ikmn}R^{ikmn}$, for such a gravity field are finite for arbitrary $r$. In contrast to the curvature invariants the invariant of the gauge field, $I_{(1)} = \frac{1}{2} F_{ik}^{(a)} F^{(a)ik} = \frac{\mu^2}{Q^2 r^4}$, is singular at $r = 0$. Thus, we give an example, which demonstrates that the non-minimal interaction can eliminate the singularity of the gravitational field.

(iii) On the non-minimal horizons and singularities.

The formulas in Sec.4 show that the space-time metric, describing the gravitational field of the Wu-Yang monopole, can contain a number of horizons and singularities depending on the relationships between $q_1$, $q_2$, $q_3$, as well as on their signs and values. When the coupling constants vanish, all these horizons and singularities convert into inner, outer horizons and point of origin for the Reissner-Nordström metric, respectively. In other words, the non-minimal coupling splits the characteristic surfaces, and makes the causal structure of the object much more sophisticated. This problem requires a special discussion.

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