TOMOGRAPHY OF SOLITONS *

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Abstract

We develop the tomographic representation of wavefunctions which are solutions of the generalized nonlinear Schrödinger equation (NLSE) and show its connection with the Weyl–Wigner map. The generalized NLSE is presented in the form of a nonlinear Fokker–Planck–type equation for the standard probability distribution function (marginal distribution). In particular, this theory is applied to the envelope solitons, where tomograms for envelope bright solitons of a wide family of modified NLSE are presented and numerically evaluated.

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http://people.na.infn.it/~fedele/tomosol-figs/ tomosol-fig.htm

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I. INTRODUCTION

In quantum mechanics and quantum optics as well as in signal analysis, the properties of wave functions satisfying the linear Schrödinger equation and the properties of complex analytic signals were studied using the tomographic probability distribution functions called tomograms. In the tomographic representation, the complex wave function is associated with the standard probability distribution by means of invertible map. In signal processing, this method was called noncommutative tomography of analytic signal.

Several tomographic schemes are known, i.e., optical tomography, symplectic tomography, photon-number tomography, and spin tomography. An advantage to use the tomographic representation for quantum states is related to the possibility of measuring the states, described in the standard approach by complex wave functions. The tomographic map of the measured tomogram provides the possibility to reconstruct the wave function (up to the constant phase factor) of the quantum state.

Till now the tomographic representation was used for discussing the solutions of linear Schrödinger equation of quantum mechanics.

On the other hand, there exist nonlinear processes described by well-known nonlinear equations like nonlinear Schrödinger equation with cubic nonlinearity or more complicated nonlinearities. There exist a huge literature to construct solutions of nonlinear equations and different methods to approach the problem. Among solutions of nonlinear equations, there is a set of specific soliton solutions.

Up to our knowledge, till now the tomographic representation was not used to describe the soliton solutions of nonlinear equations. We introduce here the tomographic representation of the soliton solutions of nonlinear equations and concentrate on tomograms of envelope solitons of nonlinear Schrödinger equation.
II. WEYL–VILLE–WIGNER MAP

Now we review the approach called symplectic tomography of quantum state. The real meaning of this scheme is the map of the complex wave function $\psi(x)$ of a real variable $x$ ($-\infty < x < \infty$) onto a family of probability distributions $w(x, \mu, \nu)$ of a random real variable $X$ ($-\infty < X < \infty$) labeled by two real parameters $\mu$ ($-\infty < \mu < \infty$) and $\nu$ ($-\infty < \nu < \infty$).

We use dimensionless variables. The map may be realized by the following steps. First, one constructs the density matrix [1] which is a complex function of two variables $x$ and $x'$$$
\rho_{\psi}(x, x') = \psi(x) \psi^*(x'). \hspace{1cm} (1)$$

Then one uses Wigner–Weyl map of density matrix onto real Wigner function on phase space $W(q, p)$ [2] of two real variables $p$ and $q$$$
W_{\psi}(q, p) = \int \rho_{\psi}(q + u/2, q - u/2) e^{-ipu} du. \hspace{1cm} (2)$$

The Wigner function takes real values. If the wave function is normalized$$$
\int |\psi(x)|^2 dx = 1,$$$

the Wigner function is also normalized$$$
\int W_{\psi}(q, p) dq dp/2\pi = 1.$$$

Ville [3] used this map in the analytic signal theory.

The inverse of the Fourier transform (2) defining the Wigner function in terms of density matrix reads$$$
\psi(x) \psi^*(x') = \frac{1}{2\pi} \int W_{\psi} \left( \frac{x + x'}{2}, p \right) e^{ip(x-x')} dp. \hspace{1cm} (3)$$

One can see that the Wigner function determines the complex wave function up to the constant factor$$$
\psi^*(0) \psi(x) = \frac{1}{2\pi} \int W_{\psi} \left( \frac{x}{2}, p \right) e^{ixp} dp. \hspace{1cm} (4)$$
Modulus of the constant factor $|\psi(0)|$ is determined by the relationship

$$|\psi(0)|^2 = \frac{1}{2\pi} \int W_\psi(0, p) \, dp.$$  \hspace{1cm} (5)

We suppose that $\psi(0)$ is not equal to zero. Thus, given Wigner function one can reconstruct the complex wave function up to constant phase factor. This means that the Wigner function contains the same information which the density matrix does. Also this means that the Wigner function contains the same information as the wave function $\psi(x)$ does (up to the constant phase factor). The Wigner function can be identified with so-called Weyl symbol of density operator describing the quantum state. There exist different kinds of symbols of operators. As it was shown recently [4] tomograms can be also identified with a specific symbol of a density operator.

### III. TOMOGRAPHIC MAP

Let us now construct the tomographic map. To do this, we use the integral Radon transform of the Wigner function

$$w(X, \mu, \nu) = \int W_\psi(q, p) \delta(X - \mu q - \nu p) \frac{dq \, dp}{2\pi}.$$  \hspace{1cm} (6)

In this formula, the Dirac delta-function term $\delta(X - \mu q - \nu p)$ collects values of the Wigner function $W_\psi(q, p)$ from the line in the phase space which is described by the expression obtained by equating the argument of Dirac delta-function to zero. One can prove [5] that for normalized Wigner function the function $w(X, \mu, \nu)$ is normalized probability distribution function of random variable $X$ (called tomogram of the Wigner function), i.e., one has

$$\int w(X, \mu, \nu) \, dX = 1.$$  \hspace{1cm} (7)

This tomogram was called symplectic tomogram [6], since it is related to linear symplectic transform in the phase space.

There is another form of the map [6] given in terms of Fourier transform (we used Fourier transform of delta-term in (6))
\[ w(X, \mu, \nu) = \int W_\psi(q, p) \exp \left[ ik(X - \mu q - \nu p) \right] \frac{dk\,dq\,dp}{(2\pi)^2}. \] (8)

The Fourier integral form of the tomogram (8) gives the possibility to get easily the inverse transform (7)

\[ W(q, p) = \int w(X, \mu, \nu) \exp \left[ i(X - \mu q - \nu p) \right] \frac{dX\,d\mu\,d\nu}{2\pi}. \] (9)

Thus, given the symplectic tomogram \( w(X, \mu, \nu) \) one can find the Wigner function \( W(q, p) \).

Using known relationships (1), (2) and (3) of the Wigner function \( W(q, p) \) and the wave function \( \psi(x) \), one can obtain the expression for tomograms in terms of the wave function (8)

\[ w(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int \psi(y) \exp \left( \frac{i\mu}{2\nu} y^2 - i \frac{Xy}{\nu} \right) dy \right|^2. \] (10)

This formula shows that for a complex function \( \psi(x) \) one can find the tomogram. Formula (10) was found for the quantum wave function and for noncommutative tomography of the analytic signal (8). But it can be used for other arbitrary aims as well. The goal which we are going to reach is to use this formula for the description of solitons.

**IV. TOMOGRAM AS FOURIER TRANSFORM OF A CHIRPED SOLITON**

There are some properties of tomograms to be used. The homogeneity property (7) follows from relations (6) and (10)

\[ w(\lambda X, \lambda \mu, \lambda \nu) = \frac{1}{|\lambda|} w(X, \mu, \nu). \] (11)

This means that, in reality, the tomogram is the function of two real variables. For example, one can take

\[ \mu = \cos \theta, \quad \nu = \sin \theta. \] (12)

In this case, the symplectic tomogram is given by the formula

\[ w(X, \theta) = \int W(q, p) \delta(X - q \cos \theta - p \sin \theta) \frac{dq\,dp}{2\pi}. \] (13)
This relation of the Wigner function $W(q,p)$ and the tomographic probability $w(X,\theta)$ (called tomogram of the optical tomography scheme) was used in the signal theory in [10]. The relation of the optical tomogram $w(X,\theta)$ to the wave function reads

$$w(X,\theta) = \frac{1}{2\pi|\sin\theta|} \left| \int \psi(y) \exp \left( \frac{i}{2} \cot \theta y^2 - \frac{iXy}{\sin \theta} \right) dy \right|^2.$$  \hfill (14)

The integrand in the above expression (14) is similar to the Green function of the quantum harmonic oscillator and the tomogram coincides with modulus squared of fractional Fourier transform of $\psi(y)$ [11]. The tomogram $w(X,\theta)$ is used in quantum optics in the scheme of measuring quantum photon states by means of the so-called optical homodyne tomography [12]. It was also discussed in the context of quantum-optics measurements in [13]. Thus the tomogram $w(X,\mu,\nu)$ determines completely the Wigner function and the complex function $\psi(x)$ (up to the phase factor), if it is known for real parameters satisfying the constraint

$$\mu^2 + \nu^2 = 1.$$ \hfill (15)

But one can use other constraints too. Thus, the homogeneity property (11) implies that the particular values of the symplectic tomogram, e.g., $w(1,\mu,\nu)$, $w(X,1,\nu)$ and $w(X,\mu,1)$ determine the whole tomogram and, consequently, the complex function $\psi(x)$ (up to the phase factor) and the Wigner function $W(q,p)$ completely.

Another property can be obtained by change of variables in (10)

$$\frac{y}{\nu} = z,$$ \hfill (16)

which gives the following expression for the tomogram

$$w(X,\mu,\nu) = \frac{|\nu|}{2\pi} \left| \int \tilde{\psi}(z,\mu,\nu)e^{-iXz}dz \right|^2,$$ \hfill (17)

where the function $\tilde{\psi}(z,\mu,\nu)$ is the function describing “the chirped soliton” (we mean that $\psi(y)$ in (10) is considered as a soliton solution of a nonlinear equation)

$$\tilde{\psi}(z,\mu,\nu) = \psi(z\nu) \exp \left( \frac{i}{2} \mu \nu z^2 \right).$$ \hfill (18)
Expression (17) is convenient for numerical calculations because it gives the tomogram in terms of the standard Fourier transform of the chirped soliton. In terms of optical tomogram, the expression can be rewritten as

\[ w(X, \theta) = \frac{|\sin \theta|}{2\pi} \left| \int \tilde{\psi}(z, \theta) e^{-iXz} dz \right|^2, \]  \hspace{1cm} (19)

where the chirped soliton has the form

\[ \tilde{\psi}(z, \theta) = \psi(z \sin \theta) \exp \left( \frac{iz^2}{4 \sin 2\theta} \right). \]  \hspace{1cm} (20)

The formula obtained is used below to make plots of solitons.

V. NONLINEAR EQUATIONS IN TOMOGRAPHIC AND WEYL–WIGNER–MOYAL REPRESENTATIONS

Let us consider the following generalized nonlinear Schrödinger equation (NLSE):

\[ i\frac{\partial \psi}{\partial s} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + U \left| \psi \right|^2 \psi, \]  \hspace{1cm} (21)

where \( s \) and \( x \) are the time-like and space-like variables and \( \psi = \psi(x, s) \) is a complex wave function describing the system’s evolution in the configuration space; \( U = U \left| \psi \right|^2 \) is an arbitrary real functional of \( |\psi|^2 \).

In this section, we derive the evolution equations for solitons in the phase-space representation. For the case of the cubic NLSE

\[ i\frac{\partial \psi}{\partial s} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + q_0 |\psi|^2 \psi, \]  \hspace{1cm} (22)

the density matrix (11) satisfies the following evolution equation:

\[ i \frac{\partial \rho(x, x', s)}{\partial s} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2} \right) \rho(x, x', s) + \\
+ \left\{ V \left[ \int \delta(x - y) \rho(x, y, s) dy \right] - V \left[ \int \delta(x' - y) \rho(x', y, s) dy \right] \right\} \rho(x, x', s), \]  \hspace{1cm} (23)

where
\[ V \left[ \delta(x - y) \rho(x, y, s) \, dy \right] = q_0 \int \delta(x - y) \rho(x, y, s) \, dy \]  

(24)

is the potential-energy functional of the density matrix for nonlinear Schrödinger equation (22) with cubic nonlinearity.

The transition to the evolution equation for the Wigner function can be done using the standard algebra, which provides the following recepie. One has to make in (23) the replacement

\[ \rho \rightarrow W \] along with the following replacement:

\[
\begin{align*}
\frac{\partial}{\partial x} \rho(x, x') &\quad \rightarrow \quad \left( \frac{1}{2} \frac{\partial}{\partial q} + i p \right) W(q, p); \\
\frac{\partial}{\partial x'} \rho(x, x') &\quad \rightarrow \quad \left( \frac{1}{2} \frac{\partial}{\partial q} - i p \right) W(q, p); \\
x \rho(x, x') &\quad \rightarrow \quad \left( q + \frac{i}{2} \frac{\partial}{\partial p} \right) W(q, p); \\
x' \rho(x, x') &\quad \rightarrow \quad \left( q - \frac{i}{2} \frac{\partial}{\partial p} \right) W(q, p).
\end{align*}
\]  

(25)

It provides the following Moyal-like form of the nonlinear equation (23) for the Wigner function

\[
\frac{\partial W(q, p, s)}{\partial s} = -p \frac{\partial W(q, p, s)}{\partial q} + \frac{1}{i} \left\{ V \left[ \rho \left( q + \frac{i}{2} \frac{\partial}{\partial p}, q + \frac{i}{2} \frac{\partial}{\partial p}, s \right) \right] - \text{c.c.} \right\} W(q, p, s).  
\]  

(26)

Here the arguments of the potential energy are replaced by the operators which act onto the Wigner function. For the cubic NLSE

\[ V(z) = q_0 z, \]  

(27)

one has a simple form of the equation

\[
\frac{\partial W(q, p, s)}{\partial s} + p \frac{\partial W(q, p, s)}{\partial q} - 2q_0 \text{Im} \rho \left( q + \frac{i}{2} \frac{\partial}{\partial p}, q + \frac{i}{2} \frac{\partial}{\partial p}, s \right) W(q, p, s) = 0. \tag{28}
\]

Using the relation

\[ \rho(x, x) = \int W(x, p) \frac{dp}{2\pi}, \]  

(29)

one has

\[
\frac{\partial W(q, p, s)}{\partial s} + p \frac{\partial W(q, p, s)}{\partial q} - 2q_0 \text{Im} \int W \left( q + \frac{i}{2} \frac{\partial}{\partial p}, P, s \right) \frac{dP}{2\pi} W(q, p, s) = 0. \tag{30}
\]
In (26), (28), and (30) the arguments of the density matrix and Wigner function are replaced by operators and the operators act on the Wigner function itself. Equations (26) and (28) can be presented in the form of Moyal-like series [14]. Equation (26) for an arbitrary nonlinear potential \( V(z) \) can be written in terms of the functional partial differential equation for the Wigner function only

\[
\frac{\partial W(q, p, s)}{\partial s} + p \frac{\partial W(q, p, s)}{\partial q} - 2 \text{Im} \left\{ V \left( \int W \left( q + \frac{i}{2} \frac{\partial}{\partial p}, P, s \right) \frac{dP}{2\pi} \right) \right\} W(q, p, s) = 0. \tag{31}
\]

Now we consider the relation of the density matrix, Wigner function and tomogram

\[
W(q, p) = \frac{1}{2\pi} \int w(X, \mu, \nu) \exp \left[ -i(\mu q + \nu p - X) \right] d\mu d\nu dX; \tag{32}
\]

\[
\rho(X, X') = \frac{1}{2\pi} \int w(Y, \mu, X - X') \exp \left[ i \left( Y - \mu X + \frac{X'}{2} \right) \right] d\mu dY. \tag{33}
\]

In view of this relation, the following rule for substitution in the evolution equation (23)

\[
\rho(x, x', s) \rightarrow w(X, \mu, \nu, s);
\]

\[
x\rho \rightarrow \left[ - \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} + i \frac{\nu}{2} \frac{\partial}{\partial X} \right] w;
\]

\[
\frac{\partial}{\partial x} \rho \rightarrow \left[ \mu \frac{\partial}{\partial X} - i \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \nu} \right] w; \tag{34}
\]

\[
x' \rho \rightarrow \left[ - \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} - i \frac{\nu}{2} \frac{\partial}{\partial X} \right] w;
\]

\[
\frac{\partial}{\partial x'} \rho \rightarrow \left[ \mu \frac{\partial}{\partial X} + i \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \nu} \right] w
\]

provides the tomographic form of the nonlinear equation under consideration

\[
\frac{\partial w(X, \mu, \nu, s)}{\partial s} + \mu \frac{\partial w(X, \mu, \nu, s)}{\partial \nu} +
\]

\[
- 2 \text{Im} V \left\{ \int w(y, \mu', 0, s) \exp \left[ i \left( y + \mu' \left[ \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} - i \frac{\nu}{2} \frac{\partial}{\partial X} \right] \right) \right] \right\} \frac{dy d\mu'}{2\pi} w(X, \mu, \nu, s) = 0. \tag{35}
\]

In the above equation (35), one has the integro-differential operator in the exponent, which acts on the tomographic probability function. The integral operator \( (\partial/\partial X)^{-1} \) is defined by the following action on Fourier component of the function \( f(X) \):

\[
\left( \frac{\partial}{\partial X} \right)^{-1} f(X) = \left( \frac{\partial}{\partial X} \right)^{-1} \int \tilde{f}(k)e^{ikx} dk = \int \frac{\tilde{f}(k)}{ik} e^{ikx} dk.
\]
For the case of the cubic NLSE, one has

\[
\frac{\partial w(X, \mu, \nu, s)}{\partial s} + \mu \frac{\partial w(X, \mu, \nu, s)}{\partial \nu} + 2q_0 \text{Im} \int w(y, \mu', 0, s) \exp \left[ i \left( y + \mu' \left( \frac{\partial}{\partial X} - \frac{i}{2} \nu \frac{\partial}{\partial X} \right) \right) \right] \frac{dy d\mu'}{2\pi} w(X, \mu, \nu, s) = 0. \tag{36}
\]

It should be pointed out that Eq. (36) has the solutions which in the course of the evolution process preserve the positivity and normalization. Thus the soliton solutions of the nonlinear equations can be mapped onto probability distribution functions.

The meaning of the probability distributions is the following. If \( x \) is a coordinate and \( p \) is the momentum, the value \( X = \mu x + \nu p \) is the position in the reference frame in the phase space \( (x, p) \), the reference frame being scaled and rotated. The parameters of the scaling \( \lambda \) and rotation \( \theta \) are determined by the real parameters \( \mu \) and \( \nu \), namely, \( \mu = e^{\lambda} \cos \theta \) and \( \nu = e^{-\lambda} \sin \theta \).

The probability distributions \( w(X, \mu, \nu) \) determine soliton solutions \( \psi(x) \) in the corresponding representation. Consequently, in the tomographic representation soliton solutions of nonlinear dynamic systems are solutions of generalized Fokker–Planck-type equations for the standard probability distributions. Such representation can be useful from mathematical point of view since analysis of probabilities and their asymptotics can be additionally incorporated (using existing theorems) on the behaviour of the probability distribution functions.

**VI. EXAMPLES**

In this section, we study bright soliton in both the tomographic and Weyl–Wigner representations. The envelope bright soliton of the cubic NLSE, i.e.

\[
\Psi(x, s) = \left( \frac{2|E|}{|q_0|} \right)^{1/2} \text{sech} \left[ \sqrt{2|E|} \xi \right] \exp \left[ i \left( V_0 x - \frac{E + V_0^2}{2} \right) s \right],
\tag{37}
\]

where \( E \) is a negative real constant, \( V_0 \) is an arbitrary real constant and \( \xi = x - V_0 s \) (see f.i., Ref. [15]). Thus, the corresponding optical tomogram is given by the formula

\[
w_b(X, \theta, s) = \frac{|E \sin \theta|}{|q_0| \pi} \int \text{sech} \left[ \sqrt{2|E|}(y \sin \theta - V_0 s) \right] \times
\]
\begin{equation}
\times \exp \left\{ i \left[ V_0 y \sin \theta - \left( E + \frac{V_0^2}{2} \right) s \right] + \frac{i \sin 2\theta}{4} y^2 - iX y \right\} dy \right|^2.
\end{equation}

It has been recently shown that the following modified NLSE \((U[|\psi|^2] = q_0|\psi|^{2\beta})\)
\begin{equation}
\frac{i \partial \Psi}{\partial s} = -\frac{\partial^2 \Psi}{2 \partial x^2} + q_0|\Psi|^{2\beta} \Psi,
\end{equation}
for \(q_0 < 0\) and any real positive value of \(\beta\), has the following envelope soliton-like solutions \([16]\):
\begin{equation}
\Psi(x, s) = \left[ \frac{|E| (1 + \beta)}{|q_0|} \right]^{1/2\beta} \sech^{1/\beta} \left[ \beta \sqrt{2|E|} \xi \right] \exp \left[ i \left[ V_0 x - \left( E + \frac{V_0^2}{2} \right) s \right] \right],
\end{equation}
where the real numbers \(V_0\) and \(E\) are arbitrary and negative, respectively, and still \(\xi = x - V_0 s\) (note that \(V_0\) is the soliton velocity). It should be noted that the case \(\beta = 1\) (ordinary envelope bright soliton of the cubic NLSE \([17]\)) can be very easily recovered \([16]\).

The Wigner function of bright soliton for \(V_0 = 0\) is given by the formula
\begin{equation}
W(x, p) = \frac{|E|}{|q_0|} \int \sech \left[ \sqrt{2|E|} \left( x + \frac{u}{2} \right) \right] \sech \left[ \sqrt{2|E|} \left( x - \frac{u}{2} \right) \right] \exp^{-ipu} du.
\end{equation}
The tomogram of the soliton solution of generalized nonlinear Schrödinger equation is given by the formula \((V_0 = 0)\)
\begin{equation}
w(X, \mu, \nu) = \frac{1}{2\pi |\nu|} \left| \left[ \frac{|E|(1 + \beta)}{|q_0|} \right]^{1/\beta} \int \sech^{1/\beta} \left[ \beta \sqrt{2|E|} y \right] \exp \left( \frac{i\mu}{2\nu} y^2 - \frac{iX y}{\nu} \right) dy \right|^2.
\end{equation}

The Wigner function of the soliton solution of generalized nonlinear Schrödinger equation is given by the formula \((V_0 = 0)\)
\begin{equation}
W(x, p) = \left[ \frac{|E|(1 + \beta)}{|q_0|} \right]^{1/\beta} \int \sech^{1/\beta} \left[ \beta \sqrt{2|E|} \left( x + \frac{u}{2} \right) \right] \sech^{1/\beta} \left[ \beta \sqrt{2|E|} \left( x - \frac{u}{2} \right) \right] \exp^{-ipu} du.
\end{equation}

\textbf{VII. SOME NUMERICAL COMPUTATIONS}

3D Plots and density plots of both tomograms and Wigner functions for bright solitons with \(\beta = 0.5, 1.0, 2.0\) and \(2.5\) are given in Fig.1 - Fig.5. We have fixed the free parameters as follows:
$V_0 = 0, E = -1,$ and $q_0 = -1$. Fig.1 represents the 3D plots of the tomogram of the solitons for the different values of $\beta$. The corresponding density plots are displayed in Fig.2. Fig.3 displays the 3D plots of the Wigner quasidistribution of the solitons for the diverse values of $\beta$. For such solitons, the phase-space regions where the Wigner function is negative are not easily visible in Fig.3, whilst the corresponding density plots displayed by Fig.4 show clearly this behaviour. They correspond to the black and grey regions. The ”deepness” of the negativity is represented in scale of grey. Black regions correspond to the deepest negative parts. The size of these negative parts of the Wigner function are, for instance, clearly represented by the cross section shown in Fig.5 at $p = 2$ for the diverse values of $\beta$.

**VIII. REMARKS AND PERSPECTIVES**

We discussed the tomography of solitons considering the tomograms as additional characteristics of the soliton solutions of nonlinear dynamic equations. But there exists another experimental aspect of tomograms because the tomograms can be directly measured in different situations. Thus the problem of tomography of some phenomena which is described by a complex function $\psi(x)$ is equivalent to the problem of measuring the amplitude $|\psi(x)|$ and the phase $\varphi(x)$ of the complex function $\psi(x) = |\psi(x)| \exp i\varphi(x)$. The function $\psi(x)$ can describe a soliton but also this function can describe some signal connected with different processes, e.g., in optical fibers, in plasma, etc. In all processes where one needs to measure the amplitude and phase by measuring experimentally only intensities, the tomography can be used as an instrument for achieving this aim. We consider two different possibilities which are based on the symplectic tomogram [10]. The density matrix can be reconstructed either by measuring the tomogram of the optical tomography scheme $w(X, \theta)$ or tomogram $w(X, 1, \nu)$.

Now we show that both tomograms can be obtained in two different and realizable processes. The optical tomogram can be rewritten in terms of fractional Fourier transform (which is reduced to the Green function of the harmonic oscillator) [11]. In fact, one has

$$w(X, \Theta) = \left| \frac{1}{\sqrt{2\pi i \sin \Theta}} \int \psi(y) \exp \left[ \frac{i}{2} \cot \Theta \left( y^2 + X^2 \right) - \frac{iXy}{\sin \Theta} \right] dy \right|^2.$$  

(44)
In this formula, we take $\nu = \sin \Theta$ and $\mu = \cos \Theta$. The phase factor $\exp iX^2/2 \cot \Theta$ does not change the value of the tomogram.

The tomogram presented in such form coincides with the value of the wave function at the point $x$ at the time moment $t$ if the initial value of the wave function at the time moment $t = 0$ is equal to $\psi(y)$. This means that to reconstruct the initial value of the wave function $\psi(x)$ including both the amplitude $|\psi(x)|$ and the phase $\varphi(x)$

$$\psi(x) = |\psi(x)| \exp i\varphi(x),$$

one can measure the tomogram, i.e., amplitude squared of the wave function which evolves in the quadratic potential well. This situation can be perfectly done for optical fibers with a parabolic profile of the refractive index called “selfoc” (linear propagation). In fact, the light beams in optical fibers obey to the Schrödinger-like equation which follows from the Helmholtz equation in the Fock–Leontovich approximation [18]. But time $t$ in the Schrödinger equation is replaced by the longitudinal coordinate $z$ and the Planck’s constant is replaced by the wavelength. Thus, to measure the input field amplitude and phase, it is sufficient to measure the tomogram which is the field intensity in each cross-section of the fiber given by longitudinal coordinate $0 < z \leq 2\pi$.

Another possibility is related to the formula

$$w(X, 1, \nu) = \frac{1}{\sqrt{2\pi i\nu}} \int \exp \frac{i(X - y)^2}{2\nu} \psi(y) dy \left| \right|^2. \quad (45)$$

This formula is equivalent to formula (10) in which we took $\mu = 1$ and added unessential phase factor $\exp \frac{iX^2}{2\nu}$. Thus the tomogram for “time moment” $\nu$ is equivalent to the intensity of the free propagating signal. In fact, the kernel in (45) is the Green function of a free particle. Since due to homogeneity the tomogram $w(X, 1, \nu)$ is equivalent to the tomogram $w(X, \mu, \nu)$, while measuring the intensity of free propagating signal one measures both the phase and amplitude of the input signal $\psi(y)$. If one measures the field in optical fiber, the structure of the output field can be evaluated by measuring the free propagation of the beam. There is a peculiarity in using formula (45). For complete reconstructing the amplitude, one needs to know the intensity for
arbitrary large values of time (or longitudinal coordinate $z$). Practically the length or duration can be chosen to fit appropriate accuracy of the measurement. As an example, Fig.6 shows the tomographic map and the corresponding density plot for the case of free propagation of a pulse whose initial profile is solitonlike with $\beta = 1.0$ (also here we have fixed the free parameters as follows: $V_0 = 0$, $E = -1$, and $q_0 = -1$).

**IX. CONCLUSIONS**

We introduced the tomographic probability distribution associated with soliton solutions of nonlinear equations.

Nonlinear dynamical equations like nonlinear Schrödinger equation were presented in the form of equation (a nonlinear generalization of the Fokker–Planck equation) for the standard probability distribution function.

Specific cases of solitons for a wide family of modified NLSE were studied in the tomographic representation explicitly.

The possibility to use tomograms to reconstruct the phase of linear or nonlinear signals by measuring the signals’ intensities was finally discussed.

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