Measure synchronization in coupled Duffing Hamiltonian systems

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Abstract. We examine the collective behaviour of coupled Duffing Hamiltonian systems (HS) and show the existence of measure synchronization (MS) in the quasi-periodic (QP) and chaotic states. We show that the dynamics of coupled Duffing Hamiltonians exhibit a transition to coherent invariant measure, their orbits sharing the same phase space as the coupling strength is increased. Transitions from QP measure desynchronous to QP MS state and QP measure desynchronous to chaotic measure synchronous (CMS) state were both identified. Moreover, a transition from partial measure synchronization state to complete CMS state was found for three coupled subsystems.

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1. Introduction

The dynamics of coupled nonlinear oscillators have attracted considerable attention in recent years; because they arise in many branches of science. A collection of coupled oscillators (i.e. arrays) are often used to model many physical, chemical, biological and physiological...
systems explaining reaction diffusion processes. One can visualize such an array as an assembly of a number of subsystems coupled to their neighbours, or as coupled nonlinear networks (CNNs) [1]. They exhibit a great variety of dynamical phenomena such as bifurcation, multistability, spatiotemporal chaos pattern formation, synchronization, etc.

Synchronization, in particular, has a long history going back to Huygens, who observed synchronization of two pendulum clocks in 1673 [2]. This phenomenon is encountered in many different systems in nature and science. In classical terms, synchronization has been defined as the frequency and phase locking of periodic oscillators due to weak interaction. This finding for periodic oscillators has aroused the interest of many researchers in chaotic dynamics during the last decade, such that the study has been extended to chaotic systems [3]–[12]. One of the major important motivations is to understand the coherent dynamical behaviour of coupled systems, which is essential for a wide range of scientific investigations. In this direction, various types of synchronization have been identified. This includes complete synchronization (CS) [3, 4], phase synchronization (PS) [5, 6], lag synchronization (LS) [7, 8], generalized synchronization (GS) [9, 10] and anticipated synchronization (AS) [11, 12].

Most work on synchronization to date has focused mainly on dissipative systems (DS), the reason being that synchronization between two trajectories is normally related to the contraction of phase space volume. Recently, it has been reported that coupled Hamiltonian systems (HS) can exhibit a kind of weak synchronization and this has formed the focus of some researchers [13]–[17]. HS behave quite differently from DS. Unlike DS, HS conserve phase volumes [18] and do not (in the original sense) allow synchronization, i.e. a situation in which two non-identical trajectories approach an identical one asymptotically.

HS are very significant because many of the most important problems in classical and quantum mechanics may be approximated by Hamiltonian flows [18]–[20]. For this reason, exploring the synchronization phenomenon in HS is a very crucial step towards understanding the significance of the concept, mode and possible applications of synchronization in quantum systems.

Hamiltonian equations for autonomous dynamical systems are written as

\[ \dot{q} = \frac{dH}{dp}, \quad \dot{p} = -\frac{dH}{dq}, \]  

(1)

where \( H(q, p) \) is the Hamiltonian, \( q \) the vector of ‘generalized coordinates’ and \( p \) the vector of ‘conjugate momenta’.

In [13], Hampton and Zanette observed certain collective behaviours between two mutually coupled identical HS. The coupled systems were found to exhibit a transition in phase space from a ‘non-synchronous’ state to a ‘synchronous’ state, the synchronous state being referred to as measure synchronization (MS). The main characteristics of MS is that two orbits share the same phase space with the same identical invariant measure, though the two systems are not strictly synchronized. Recently, this phenomenon was reported in a \( \phi^4 \) HS by Wang et al [14]. In addition, Wang et al also studied partial measure synchronization (PMS) [15] in a classical HS of Bambi, Baowen and Hong (BBH) model [21], which describes the heat conduction in one-dimensional non-integrable systems. Explicit characterization of the transition phenomenon associated with MS states has also been undertaken [16]. In a very recent letter [17], we reported the phenomenon of MS in a coupled HS of Njah and Akin-Ojo [22], associated with a nonlinear Schrödinger equation.
While the study of synchronization in coupled HS has received relatively less attention, in the present paper we examine this issue in the context of double–well Duffing Hamiltonian (DDH) systems. The paper is organized as follows. In section 2, we give a brief description of the DDH systems. In section 3, MS in DDH systems is discussed, while section 4 deals with PMS. Conclusions and extension of the present study are finally given in section 4.

2. The DDH system

The Duffing oscillator is a well-known model of nonlinear oscillator, which find several important applications in many science and engineering modelling and is usually considered as a paradigm for nonlinear dynamics of systems. For instance, if the oscillation of a driven torsion pendulum is confined to moderate angles (smaller than say $30^\circ$), the motion may be accurately modelled by the Duffing equation [23]. Another application arises in the modelling of magnetoelastic mechanical systems, which consists of a beam placed vertically between two magnets with the top end fixed and the bottom end free to swing [24]. There are also known electric circuits that behave according to the Duffing equations [25, 26].

Duffing oscillators comprise one of the canonical examples of HS [23, 24]. For the present study, we adopt the nearest diffusive coupling and periodic boundary condition. The Hamiltonian describing the DDH system of interest is given by

$$H = \frac{p_i^2}{2} - \frac{q_i^2}{2} + q_i^4 + K(q_{i+1} - q_i)^2, \quad (2)$$

where the middle term in $q_i$ describes the double-well potential

$$V(q_i) = -\frac{q_i^2}{2} + \frac{q_i^4}{4}. \quad (3)$$

The last term is the coupling, with $K$ being a parameter that determines the strength of the coupling. From equation (2), we obtain the following canonical equations governing the motion of each oscillator as:

$$\dot{q}_i = p_i, \quad \dot{p}_i = q_i - q_i^3 + K(q_{i+1} + q_{i-1} - 2q_i), \quad i = 1, 2, \ldots, N, \quad (4)$$

where $q_i$ and $p_i$ represent the spatial position and momentum of site $i$; and $N$ is the system size. The dynamics of system (4) depend on $K$ as well as on its initial conditions. By varying the coupling strength, $K$, the total energy can be regarded as an irrelevant parameter. We fix $q_i(t = 0) = 0$, a configuration which ensures that the interaction energy,

$$E_i = K(q_{i+1} + q_{i-1} - 2q_i), \quad (5)$$

is given zero initial value, so that any slight adjustment of $K$ does not change the total energy, $E$, for any initial choice of $p_i(0)$. Thus in our model, there are two adjustable parameters—the initial conditions $p_i$ and the coupling strength $K$. 

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3. Measure synchronization

We simulated the coupled DDH systems (4) and studied its dynamical behaviour using the initial conditions and the coupling parameter $K$ as the control parameters. We found different kinds of MS states including chaotic MS.

Let us start by considering a case with $N = 2$. Equation (4) reduces to

$$\dot{q}_1 = p_1, \quad \dot{p}_1 = q_1 - q_1^3 + K(q_2 - q_1),$$
$$\dot{q}_2 = p_2, \quad \dot{p}_2 = q_2 - q_2^3 + K(q_1 - q_2).$$

(6)

The initial condition were set as follows: $q_1(0) = q_2(0) = 0$, $p_1(0) = 0.2$ and $p_2(\sqrt{2E - p_1^2})$. $E = 0.2$ was fixed throughout the study. In figure 1(a), we plot the orbits in $(q_i, p_i)(i = 1, 2)$, phase plane of the two identical oscillators at zero coupling. Both oscillators have different periodic orbits in different energy surfaces determined by its initial conditions. When a small
nonzero coupling (e.g. $K = 5.0 \times 10^{-3}$) is switched on, the orbits of the two oscillators becomes quasi-periodic (QP) (figure 1(b)).

With a further increase in $K$ to $9.15 \times 10^{-3}$, we find that the external border of the inner tori (oscillator 2) approaches the internal border of the outer tori (oscillator 1) and vice versa until a critical $K$ value (about $K_c = 1.1 \times 10^{-2}$) above which MS of the two oscillators can be reached. Thus, in figures 1(c) and (d), an MS state is shown for $K = 0.05$ where the two regions completely merge to an identical enlarged torus, and both oscillators share the same phase domain with their QP orbits indicating the phenomenon of MS in QP states.

Next, we examined the system behaviour for a different set of initial conditions given by: $q_1(0) = q_2(0) = 0$, $p_1(0) = 1.42 \times 10^{-2}$ and $p_2(0) = 0.632$. The results are shown in figure 2. For $K = 0$ (figure 2(a)) the orbits are periodic, similar to the previous result. For very small nonzero coupling (of the order of $10^{-3}$), the orbits are QP (see figure 2(b)). However, with sufficiently large coupling ($K > 5.2 \times 10^{-2}$), the orbits are chaotic and measure synchronized.
The motion is periodic and the oscillators are not synchronized for zero coupling.

Figure 3. The same as in figure 1, but with $N = 3$ and the following initial conditions: $q_1(0) = q_2(0) = q_3(0) = 0$, $p_1(0) = 0.2$, $p_2(0) = 0.6$ and $p_3(0) = 0.8$. The motion is periodic and the oscillators are not synchronized for zero coupling.

as shown in figures 2(c) and (d). Chaotic MS (CMS) state persists for much higher coupling with different kinds of structure in the phase space.

4. Partial measure synchronization

The case $N = 2$ is the trivial case for which the oscillators are either measure synchronized or non-synchronized. Let us consider the DDH system with a larger system size (i.e. $N > 2$). For $N = 3$, equation (4) becomes:

$$\dot{q}_1 = p_1, \quad \dot{p}_1 = q_1 - q_1^3 + K(q_2 - q_1),$$
$$\dot{q}_2 = p_2, \quad \dot{p}_2 = q_2 - q_2^3 + K(q_1 - 2q_2 + q_3),$$
$$\dot{q}_3 = p_3, \quad \dot{p}_3 = q_3 - q_3^3 + K(q_2 - q_3).$$

Each oscillator is represented by different letters $a, b$ and $c$, indicating different orbit measures. In this case, the motion of the oscillators in a set can be non-synchronous and is denoted by the $abc$ structure. They can also be partially synchronized (i.e. $aab, abb, aba$ structure) or completely synchronized ($aaa$ structure). The $abc$ structure implies different orbit measure. On the other
Figure 4. Same as in figure 3; (a) oscillator 1 (b) oscillator 2 and (c) oscillator 3 in PMS state with \(abb\) structure for \(K = 0.003\). (d) Oscillator 1 (e) oscillator 2 and (f) oscillator 3 in complete CMS state for \(K = 0.15\).

Hand, the \(aab\) structure means that two oscillators (\(i = 1\) and 2) have the same measure, \(a\), while the remaining oscillator (\(i = 3\)) have orbit with different measure, \(b\). Similarly, the \(abb\) and \(aba\) structures imply that the oscillators 2 and 3 have identical orbit measure, \(b\), while the 1 and 3 also have identical orbit measure, \(a\). In the \(aaa\) structure, the three oscillators (\(i = 1, 2\) and 3) share the same orbit measure. The generic feature of MS in HS is that all oscillators having the same orbit measured practically have different orbits.

In figure 3, we plot the orbits for three oscillators for zero coupling using the following set of initial conditions: \(q_1 = q_2 = q_3 = 0, p_1 = 0.2, p_2 = 0.6\) and \(p_3 = 0.8\). Clearly, the oscillators are in desynchronized state, that is \(abc\) structure. As an illustration of PMS phenomenon in our model, the coupling is switched on and the orbits are shown in figure 4. Here, we observe in figures 4(a)–(c), a PMS state (for \(K = 0.003\)) built up by the system. Here, the structure is basically \(abb\); in which case oscillator 2 synchronizes with oscillator 3, both being in QP states. A further increase in the coupling strength forces the oscillators to achieve a complete CMS state as shown in figures 4(d)–(f) for \(K = 0.15\). Thus, we find that a transition from QP PMS to CMS state exists for \(N = 2\) coupled oscillators described by the Hamiltonian (7).

For \(N = 4\), possible PMS structures are \(abab, aabb, abc\) and \(abcc\). In this study, we have limited our investigation to a maximum system size of \(N = 3\) coupled oscillator. However,
the case for $N > 3$ has been studied by Wang et al [15] using the BBH [21] model of heat conduction.

5. Conclusion

In summary, we have shown the existence of MS in a Duffing HS, whose classical equation of motion has been widely studied in the field of nonlinear dynamics. Our numerical findings reveal that transitions from QP measure desynchronization to QP MS and QP measure desynchronization to CMS exist among coupled subsystems. In particular, we found the phenomenon of PMS for larger subsystems ($N = 3$). The characterization of this phenomenon, using for example, the long-term average energy of the subsystems [15], remains an open issue, which we hope to address in a future paper which is in progress.

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