Phonon-Coupled Electron Tunneling in Two and Three-Dimensional Tunneling Configurations

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Abstract

We treat a tunneling electron coupled to acoustical phonons through a realistic electron phonon interaction: deformation potential and piezoelectric, in two or three-dimensional tunneling configurations. Making use of slowness of the phonon system compared to electron tunneling, and using a Green function method for imaginary time, we are able to calculate the change in the transition probability due to the coupling to phonons. It is shown using standard renormalization procedure that, contrary to the one-dimensional case, second order perturbation theory is sufficient in order to treat the deformation potential coupling, which leads to a small correction to the transmission coefficient prefactor. In the case of piezoelectric coupling, which is found to be closely related to the piezoelectric polaron problem, vertex corrections need to be considered. Summing leading logarithmic terms, we show that the piezoelectric coupling leads to a significant change of the transmission coefficient.

I. INTRODUCTION

The effect of a bosonic environment on a tunneling particle has been extensively studied. The work of Caldeira and Leggett [1], which originally studied the tunneling of a particle from a metastable state, focused attention on this problem. Today there are many separate fields devoted to related questions. As an example the case of coupling of a tunneling particle to a photonic environment is the basis of three such fields: that of dynamic image potential [2], Coulomb blockade [3] and dissipative macroscopic quantum tunneling (MQT) of Josephson phase [4]. The development of experimental techniques through which different effects now become within the scope of experimental verification [5] is a source of enhanced interest in such problems.

The coupling of a tunneling electron to phonons has received much less attention [6], though this problem is of fundamental interest, being closely related to the polaron problem [7], as well as being of potentially great technological importance. In a previous paper [8] we have shown that in the one-dimensional case, i.e., a quantum wire or narrow constriction, the coupling to the zero-point fluctuations of the acoustic phonon field leads to an exponential enhancement of the tunneling probability. The question of what happens to this enhancement in the the two-dimensional case in which the electron’s wave function is
restricted in one of the the directions perpendicular to the tunneling direction, and the three-dimensional case where the electron is free to move in both the perpendicular directions, is the subject of this paper. We address the problem of an electron tunneling coupled to acoustic phonons via two realistic coupling mechanisms: deformation potential and piezoelectric. For both coupling mechanisms we calculated the change of the tunneling probability and the renormalization of the potential barrier height. In two and three dimensions the change of tunneling probability is smaller than in one dimensional case and for its calculation we developed a perturbation theory employing the Green function method. The problem of calculation of the phonon correction, however, appeared to be nontrivial for the piezoelectric coupling, where the summation of high order terms of the perturbation theory is necessary.

Being interested in the effect of zero-point fluctuations of the phonon field on the tunneling electron we did not consider the coupling to optical phonons. The virtual process of phonon emission and absorption is inversely proportional to the energy difference involved in that process. Since emission and absorption of optical phonons involves a large energy difference the interaction with optical phonons can be neglected.

To calculate the transmission coefficient we make use of an approximation based on the slowness of the phonon field compared to the tunneling electron, which we refer to as the static approximation [8], and obtain a stationary Schrödinger equation for the electron coupled to phonons. We then apply the parabolic (paraxial) approximation which is equivalent to considering the following physical situation (which is typical experimentally) (i) the barrier is so high that the tunneling can be considered semi-classically and (ii) the interaction energy between electrons and phonons is small compared to the height of the barrier. Treating the phonons via a Green function method the self-energy term is then calculated using second order perturbation theory. The resulting self-energy term is divergent at large transferred momenta, for both coupling mechanisms, and these divergences are eliminated by a renormalization procedure. For the deformation potential one has to renormalize the potential barrier height and the electron’s effective mass, this finally leads to a correction to the transition probability. On the other hand for the piezoelectric coupling, after the renormalization of the potential barrier height the second order correction diverges logarithmically. The summation of main logarithmic terms resulted in a factor non-trivially depending on the electron-phonon coupling constant and can lead to a significantly change the transmission coefficient.

The paper is composed in the following way. In Sec.II we formulate the problem, introduce the model, and present the static approximation which helps us facilitate the calculations. In Sec.III the Green function method through which we approach the problem is described, the parabolic approximation is introduced, self-energy diagrams are presented and it is shown how one uses them to derive the transition probability. In Sec.IV the renormalized self-energies for the two and three-dimensional tunneling configurations are given for the deformation potential coupling and the renormalization equations are presented. The resulting change in the transmission coefficient is calculated. In Sec.V the method for including vertex corrections for the piezoelectric case is given and the resulting change in the transmission coefficient is calculated. The calculation of some complicated integrals is given in appendices.
II. FORMULATION OF THE PROBLEM

A. The model

We treat the problem of an electron coupled to acoustic phonons while tunneling through a rectangular barrier, of width L. We will treat two cases that of a two and three-dimensional tunneling configurations. In the three-dimensional, case the electron is free to move in the perpendicular (to the tunneling) directions, whereas in the two-dimensional case the electron’s movement is restricted in one of the perpendicular directions i.e a two dimensional electron gas. In both cases phonons move freely through the bulk and thus are treated as three-dimensional. Being interested in the effect of the zero-point fluctuations of the phonon field on the electron tunneling probability we consider the system at zero temperature.

B. Static approximation

In the general case of a tunneling particle coupled to the environment two typical time scales arise: the traverse time for the tunneling particle and the typical time in which the environment reacts to the tunneling of the particle. When the tunneling particle is an electron and the environment acoustical phonons, it can be shown that typically the traverse time for the electron is much smaller than the environmental reaction time scale. The small parameter expressing the small ratio between these time-scales is: \( s/v \), where \( s \) is the phonon sound velocity, and \( v \) is the under the barrier electron semi-classical velocity. For a potential barrier of around 10 meV in bulk GaAs, this ratio is of the order of 0.01. This small ratio leads us to what we refer to as the static approximation. In this approximation the phonon dynamics is neglected and the potential created by the barrier has a phonon addition that may have different time independent shapes with the probability equal the wave function squared of the phonon field. [For a more detailed description of the static approximation one is referred to the previous paper [8].] As a result the electron wave function underneath the barrier satisfies the following stationary Schrödinger equation for the electron coupled to phonons

\[
-\frac{\hbar^2}{2m} \nabla^2 \Psi(x, u_q) + (V + f(x, u_q)) \Psi(x, u_q) = E \Psi(x, u_q),
\]

(2.1)

where \( x \) is the electron coordinate, \( m \) is the electrons effective mass, \( V \) is the potential barrier height, \( u_q = u_{-q}^* \) are the Fourier coefficients of the lattice displacement vector \( u(x) \). \( u_q \) is a random variable whose probability is controlled by the ground state phonon wave function. For the two-dimensional case the Laplacian is the two-dimensional Laplacian operator. \( f(x, u_q) \) is the interaction term between the electron and phonons defined in the following way. For the three-dimensional case, in which the electron wave function is not restricted in the perpendicular to tunneling direction which is taken as \( x \), the deformation potential coupling interaction term is given by

\[
f_d(x, u_q) = \frac{i\Lambda}{\sqrt{V_{vol}}} \sum_q |q| u_q \exp(iqx),
\]

(2.2)
where $\Lambda$ is the deformation potential constant and $V_{vol}$ is the normalization volume. For the piezoelectric coupling the interaction term in the Hamiltonian is given by

$$f_p(x, u_q) = \frac{\Xi}{\sqrt{V_{vol}}} \sum_q u_q \exp(iqx), \quad (2.3)$$

where $\Xi = 4\pi e \sqrt{\langle \beta_{ijk}^2 \rangle_{\Omega}/\epsilon}$, $e$ is the electron charge, $\epsilon$ is the dielectric constant, $\beta_{ijk}$ is the piezoelectric modules and $\langle \ldots \rangle_{\Omega}$ denotes the average over angles. Though the piezoelectric interaction is anisotropic the isotropic Hamiltonian approximation is commonly used [9]. Subscripts $d$ and $p$ shall be used throughout this paper, unless otherwise stated to define deformation potential and piezoelectric related terms respectively.

For two dimensional electron gas were the electron wave function is taken as restricted in the $z$ direction the interaction terms are given by

$$f_d(x, u_q) = \frac{i\Lambda}{\sqrt{V_{vol}}} \sum_q |q| u_q M_{qz} \exp[i(q_x x + q_y y)], \quad (2.4)$$

for the deformation potential coupling and

$$f_p(x, u_q) = \frac{\Xi}{\sqrt{V_{vol}}} \sum_q u_q M_{qz} \exp[i(q_x x + q_y y)]. \quad (2.5)$$

for the piezoelectric coupling. Here $M_{qz}$ is the matrix element of the phonon exponent, exp $(iq_z z)$ between the wave functions describing the electron quantization in the $z$ direction.

We neglect screening of the electron-phonon interaction, since underneath the potential barrier there are no free electrons and screening by remote electrons is small.

Although the static approximation resembles the electron tunneling in the presence of static disorder there is a very important difference between the two problems. Whereas a static random field can absorb any momentum from the electron the momentum, absorption in the static approximation is related to phonon emission. Since under the static approximation it is assumed that the phonon configuration does not change the tunneling electron momentum is conserved.

**III. GREEN FUNCTION METHOD**

The commonly used approach for addressing the problem of a tunneling particle coupled to the environment is based on the path-integral approach. The path-integral approach to the problem is extremely convenient in the one-dimensional case where the contribution to the transition probability is essentially from one trajectory. In two and three dimensional cases many different trajectories contribute to the transition probability and we prefer to use Green function method. The application of this method is facilitated by the fact that a reasonable potential barrier (e.g., 10 meV) it is much larger than electron-phonon interaction, so for the calculation of electron Green function we can use perturbation theory. In some points the Green function method that we use differs from commonly used formalism and we present it’s derivation.
A. Parabolic approximation

We consider the case when the coupling between the electron and the acoustic phonons is small compared to the effective potential barrier height $V - E$, which is considered as large. More than this, we assume that $\hbar / \sqrt{2m(V - E)}$ is the smallest length scale in the problem. It is much smaller than the width of the barrier $L$ and in two-dimensional case it is much smaller than the size of the quantization in $z$ direction, $d$. The first of these conditions enables us to use the parabolic (paraxial) approximation [10], and is the basis for our perturbative method.

Under the parabolic approximation we search for a solution to the Schrödinger equation (2.1) of the form: $\Psi(x, u_q) = \exp(-\kappa x)\psi(x, u_q)$ where $x$ is the tunneling direction, the exponential factor characterizes tunneling without interaction with phonons, $\kappa = \sqrt{2m(V - E)}/\hbar$, and $\psi(x, u_q)$ is a slowly varying envelope function. Slow variation of $\psi(x, u_q)$ follows from the weakness of electron-phonon interaction and leads to the possibility of neglect of its second derivative with respect to $x$, $(\partial^2 \psi / \partial x^2) \ll \kappa (\partial \psi / \partial x)$. Then Eq. (2.1) becomes

$$-\frac{1}{2} \nabla^2 \psi + \kappa \frac{\partial \psi}{\partial x} = -\frac{m}{\hbar^2} f(x, u_q) \psi, \quad (3.1)$$

where $\nabla$ designates the gradient with respect to the perpendicular to tunneling direction $x$ (in the two-dimensional case $\nabla = \partial / \partial y$). This Schrödinger equation is just equivalent to a non-stationary Schrödinger equation where $x/\kappa$ plays the role of imaginary time and there are two spatial dimensions instead of the original three.

The transmission across the barrier is characterized by the solution to Eq.(3.1) with the boundary ("initial") condition $\psi(x = 0, r_{\perp}) = 1$. It can be written as

$$\psi(x, r_{\perp}|u_q) = \int G(x, r_{\perp}; 0, r_{\perp}'|u_q)\psi(0, r_{\perp}')dr_{\perp}'. \quad (3.2)$$

Then the transmission coefficient is given by (multiplied by $\exp(-2\kappa L)$ which is due to the parabolic approximation)

$$\langle |\psi(L, r_{\perp}|u_q)|^2 \rangle = \int \langle G(L, r_{\perp}; 0, r_{\perp}'|u_q)G^*(L, r_{\perp}; 0, r_{\perp}''|u_q)\rangle \psi(0, r_{\perp}')\psi(0, r_{\perp}'')dr_{\perp}'dr_{\perp}'', \quad (3.3)$$

where the angular brackets mean the averaging over the phonon configuration. Due to translational invariance the average of the Green function product depend on the difference $r_{\perp}' - r_{\perp}''$ so that the rhs does not depend of $r_{\perp}$.

In the case of tunneling in the presence of a random field the average of the Green function product mixes both Green functions, i.e., it is not equal to the product of averages. The situation is much simpler in the case of phonons under the static approximation. The neglect of phonon emission means that $\psi(L, r_{\perp}|u_q)$ in the lhs of Eq.(3.3) corresponds to the phonon ground state. In other words, for the calculation of the transmission coefficient we have to keep only the diagonal with respect to phonons part of the Green function. This diagonal part is obtained by the averaging of $G$ over the phonon configuration, $G(x, r_{\perp} - r_{\perp}') = \langle G(x, r_{\perp}; 0, r_{\perp}'|u_q) \rangle$. 

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It is convenient to calculate $G(x, \mathbf{r}_\perp - \mathbf{r}'_\perp)$ in the Fourier representation. Actually we perform Fourier transformation for the perpendicular direction and a Laplace transformation for the $x$ direction. After these transformations Eq. (3.1) becomes

$$
\frac{1}{2} k_\perp^2 \tilde{\psi}_{k_\perp}(p) + \kappa \delta_{k_\perp, 0} + \kappa p \tilde{\psi}_{k_\perp}(p) = \sum_{q_\perp} \sum_{q_x} F_{q_\perp}(q_x) \tilde{\psi}_{k_\perp - q_\perp}(p - iq_x),
$$

(3.4)

where $\tilde{\psi}_{k_\perp}(p)$ are the Fourier and Laplace transforms of $\psi(x, u_q)$ with respect to the perpendicular and $x$ directions respectively and $F_{q_\perp}(q_x) = - (m/\hbar^2) f_{q_\perp}(q_x)$. Throughout the paper we will use the notation $p$ for the Laplace transformation parameter (imaginary $x$-component of the electron wave vector), $k_\perp$ for the electron wave vector in the plane perpendicular to the tunneling direction and $q$ for the phonon wave vector.

The transmission coefficient now is $|t|^2 e^{-2\kappa L}$ where

$$
t = \frac{1}{2\pi i} \int_{-i\infty + \gamma_0}^{i\infty + \gamma_0} dp \ e^{pL} \ G_{k_\perp = 0}(p).
$$

(3.5)

Calculating the transmission coefficient thus amounts to calculating the averaged Green function.

B. Green function calculation

The Green function unperturbed by the phonon potential is $G_{k_\perp}^{(0)}(p) = (\kappa p + k_\perp^2/2)^{-1}$ so

$$
G_{k_\perp}(p) = \left[ \kappa p + k_\perp^2/2 - \Sigma_{k_\perp}(p) \right]^{-1},
$$

(3.6)

where $\Sigma_{k_\perp}(p)$ is the self energy.

We calculate the self-energy with the help of a diagrammatic technique that differs from the regular one in random field by two points. The first is that the energy variable is replaced by $-\kappa p$. The other point, that in the static approximation the two-particle Green function, Eq. (3.3), equals the product of two one-particle Green functions, has already been mentioned. It is also possible to demonstrate this last point diagrammatically, showing that in the static approximation, due to momentum conservation, diagrams with random field lines connecting two Green functions contain the factor $1/V$, where $V$ is the volume of the system.

Due to the weak coupling between electrons and phonons, in most of the cases that we consider a good approximation of the self-energy is given by the first order diagram presented in Fig. 1, i.e.,

$$
\Sigma_{k_\perp}(p) = \sum_{q_\perp, q_x} \langle |F_{k_\perp - q_\perp}(q_x)|^2 \rangle G_{q_\perp}^{(0)}(p - iq_x),
$$

(3.7)

The averaging over phonons is carried out with the help of the phonon wave function squared, $C \exp(-\sum_q \rho|q|s|u_q|^2/\hbar)$, where $\rho$ is the the crystal density $s$ is the sound velocity, and $C$ is the normalization constant. Since $F_{q_\perp}(q_x)$ is proportional to $u_q$ the averaging gives $\langle F_{q_\perp} F_{q'_\perp} \rangle = \alpha |q|^{-1} \delta_{q_\perp, -q'_\perp}$. The +1 corresponding to the deformation potential coupling
and the $-1$ sign corresponding to the piezoelectric coupling. For the deformation potential coupling $\alpha$ is a dimension-less parameter given by

$$\alpha_d = \frac{\hbar/\rho_s}{\hbar^4/m^2} \Lambda^2.$$  

(3.8)

For the piezoelectric coupling mechanism $\alpha$ has units of one over length squared and is defined as

$$\alpha_p = \frac{\hbar/\rho_s}{\hbar^4/m^2} \Xi^2.$$  

(3.9)

For the calculation of the transmission coefficient (3.5) we need the self-energy only for $k_\perp = 0$,

$$\Sigma_{k_\perp=0}(p) = \frac{\alpha_{d,p}}{(2\pi)^3} \int_{-\infty}^{\infty} dq_x \int d^2 q_\perp \frac{1}{\kappa(p - iq_x) + \frac{1}{7}q_\perp^2 \left( \sqrt{q_\perp^2 + q_x^2} \right)^\pm},$$  

(3.10)

the $(d, +1)$ and $(p, -1)$ subscripts and signs correspond to the deformation potential and piezoelectric coupling respectively.

The self energy for the two-dimensional tunneling configuration is given by

$$\Sigma_2(p) = \frac{\alpha_{d,p}}{(2\pi)^3} \int_{-\infty}^{\infty} dq_z \left| M_{q_z} \right|^2 \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y \frac{1}{\kappa(p - iq_x) + \frac{1}{7}q_y^2 \left( \sqrt{q_x^2 + q_y^2 + q_z^2} \right)^\pm},$$  

(3.11)

where the subscript 2 will be used to define the two-dimensional case throughout the paper.

**IV. DEFORMATION POTENTIAL COUPLING**

The self-energy integrals for the three-dimensional tunneling configuration (3.10) have an ultra-violet divergence for both coupling mechanisms. Actually these divergences are cut off by the inverse lattice constant $a$, but it goes against physical intuition when for a phenomena involving momenta much smaller than the inverse lattice constant such cut-off is necessary. The divergences can be eliminated with the help of the standard renormalization procedure (see, e. g., Ref. [11]). For the deformation potential coupling the large constant term $\Sigma(0)$ (which has a form $\alpha_d(c_0/a^2)$ where $c_0$ is a numerical constant) can be viewed as a renormalization of the potential barrier height, $V$ and, hence, the renormalization of the propagation constant $\kappa$,

$$\kappa_{\text{ren}}^2 = 2m(V - E) + \Sigma_d(0).$$  

(4.1)

The abbreviation ren is used to denote renormalized quantities, also it is assumed that $\kappa$ is $\kappa_{\text{ren}}$ (unless otherwise stated), through the rest of the paper. Actually the renormalization (4.1) has to be made before one passes to the parabolic approximation.

Canceling the non-$p$ dependent diverging part of $\Sigma(p)$ for the case of deformation potential coupling does not cancel all the divergences. One needs further to cancel a term $\Sigma_d(0)p$, ($\Sigma_d(0)$ is logarithmically divergent, it can be estimated as $\alpha_d \kappa c_1 \ln(\kappa a)$ where $c_1$ is a numerical constant) which can be viewed as a further renormalization due to a renormalization of the electron’s effective mass,
\[
\frac{\kappa}{m_{\text{ren}}} = \frac{\kappa - \tilde{\Sigma}_d(0)}{m}. \tag{4.2}
\]

Subtracting the two diverging terms from the self-energy integral one obtains
\[
\Sigma_{\text{ren}}^d(p) = \Sigma_d(p) - \Sigma_d(0) - \tilde{\Sigma}_d(0)p = -\alpha_d \frac{\kappa^2 \lambda^3}{6\pi^2} \ln 2\lambda, \tag{4.3}
\]
where \(\lambda = p/\kappa\).

For the two-dimensional tunneling in the deformation potential coupling the self-energy also has an ultra-violet divergence. After subtracting the divergent term from the self-energy integral, we obtain a finite expression, for the self-energy
\[
\Sigma_{\text{ren}}^{d,2}(p) = \Sigma_{d,2}(p) - \Sigma_{d,2}(0) = -\alpha_d \frac{\sqrt{3}/2 - 1}{2\pi^2} \int_{-\infty}^{\infty} dq_z |M_{q_z}|^2, \tag{4.4}
\]
For details of this calculation one is referred to the Appendix A.

In the tunneling problem, contrary to the propagation one, in the case of weak coupling the Green function can be expanded in the self-energy. Then after the substitution of Eqs. (4.3) and (4.4) into Eq. (3.5) the inverse Laplace transformation for the three-dimensional case one gets that \(t_d = 1 - (\alpha_d/6\pi^2)(\kappa L)^{-2}\). Thus the transition probability for the electron coupled via the deformation potential coupling mechanism is given by
\[
T = T^0 \left[ 1 - \frac{\alpha_d}{3\pi^2} \frac{1}{(\kappa L)^2} \right], \tag{4.5}
\]
where \(T^0\) is the transition probability for the tunneling problem without phonons.

In the two-dimensional tunneling case the transition probability is given by
\[
T_2 = T_2^0 \left[ 1 - \alpha_d \frac{\sqrt{3}/2 - 1}{\pi^2 \kappa} \int_{-\infty}^{\infty} dq_z |M_{q_z}|^2 \right]. \tag{4.6}
\]

V. PIEZOELECTRIC COUPLING

In the two-dimensional case the integral for the piezoelectric self-energy is finite. The finite independent of \(p\) part of the integral, \(\Sigma_p(0) \sim (\alpha_p/d\kappa)|M_0|^2 \ln(d\kappa)\), can be viewed as a finite renormalization of \(\kappa\), where \(M_0\) is the matrix element of the phonon exponent, for \(q_z = 0\), and \(d\) is the length of the potential barrier in the \(z\) direction. \(1/d\) is the scale that cuts off the \(q_z\) integration, and it is assumed that in the physical situation we are studying, \(1 \ll d\kappa\). Renormalizing \(\Sigma\) we have
\[
\Sigma_{\text{ren}}^{p,2}(p) = \Sigma_{p,2}(p) - \Sigma_{p,2}(0) = \frac{\alpha_p}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{P}{\kappa} |M_0|^2 \ln (pd) \tag{5.1}
\]

In the three-dimensional case the renormalization correction diverges in the parabolic approximation and the renormalized self-energy is
\[ \Sigma_p^{ren}(p) = \Sigma_p(p) - \Sigma_p(0) = \frac{\alpha_p}{2\pi^2} \lambda \left( \ln \lambda + \ln 2 - 1 \right). \] (5.2)

Details of the calculation of the integrals can be found Appendix A.

The terms \( \lambda \ln \lambda \) in Eq. (5.2) and \( p \ln(pd) \) in Eq. (5.1) become large compared to \( \kappa p \) as \( p \) goes to zero. Thus the perturbation theory in this case is not sufficient.

A similar problem was encountered by Édel’shtein when he studied the piezoelectric polaron [12]. He considered high order vertex corrections and showed that in each order of the perturbation theory the highest power of the logarithm comes from diagrams with non-crossing phonon lines. Neglecting all other diagrams he succeeded in obtaining an equation for a vertex correction. Because the origin of the logarithmic divergence in Édel’shtein’s case and ours is the same we can use the same approximation. In the following calculation we follow closely the derivation presented in Ref. [12].

In the following discussion and calculation we will treat the three dimensional case the treatment of the two dimensional case is very similar therefore only the results shall be presented for it.

We start by considering vertex corrections. The first vertex correction, \( \Gamma_1 \), presented in Fig. 2:

\[ \Gamma_1(p_1, k_{1\perp}, p_2, k_{2\perp}) = \alpha_p \int \frac{d^3q}{(2\pi)^3} \frac{1}{q} G^{(0)}_{k_{1\perp}-q_{\perp}}(p_1 - iq_x) G^{(0)}_{k_{2\perp}-q_{\perp}}(p_2 - iq_x). \] (5.3)

The integral is calculated in Appendix B giving

\[ \Gamma_1 \approx -\left( \frac{\alpha_p}{2\pi^2\kappa^2} \right) |M_0|^2 \ln \left( \frac{\eta}{\kappa^2} \right). \] (5.4)

where \( \eta = \kappa p + k_{\perp}^2/2 \) \( (p \) is either \( p_1 \) or \( p_2 \), since under logarithmic accuracy the difference between them can be ignored, for the same reason \( k_{\perp} \) is either \( k_{1\perp} \) or \( k_{2\perp} \)). For \( k_{\perp} = 0 \) the same result can be obtained from the self-energy (5.2) with the help of the Ward identity.

For the two dimensional case the first vertex correction is

\[ \Gamma_1 \approx -\left( \frac{\alpha_p}{2\pi^2\kappa^2} \right) |M_0|^2 \ln \left( \frac{\tilde{\eta} d}{\kappa} \right), \] (5.5)

where \( \tilde{\eta} = \kappa p + k_{\perp}^2/2 \). The calculation of this integral can also be found in Appendix B.

The second order diagrams for the vertex corrections are presented in Fig. 3. The diagram (3,a) is reduced to the renormalization of the Green function. It is possible to show that the leading order contribution to the diagram (3,d) is of the order of \( (\alpha_p/2\pi^2\kappa^2)^2 \ln(\eta/k^2) \) containing lower power of the logarithm than the power of \( \alpha_p \). The leading order contribution to the other two diagram is of the order of \( (\alpha_p/2\pi^2\kappa^2)^2 \ln^2(\eta/k^2) \), i.e., the power of the logarithm equals the power of the coupling constant. The same relation holds true for more complicated diagrams. That is, diagrams containing intersections of phonon lines contain lower power of the logarithm than the power of the coupling constant and diagrams without such an intersection get another power of the logarithm in every next order of the perturbation theory.

The last statement can be proven in the following way. The increase of the order of the perturbation theory by unity in a vertex diagram without intersection of phonon lines
corresponds to the replacement of one of the bare vertices in this diagram by \( \Gamma_1 \). Similar to the calculation in Appendix B, the values of the momentum in the contour of this new \( \Gamma_1 \) that give the main contribution are much larger than the momentum in external contours but much smaller than \( \kappa \). So this \( \Gamma_1 \) again gives a logarithmic contribution.

Another result of such a calculation is that in diagrams without phonon line crossings the variables \( p \) and \( k_\perp \) enter only in the combination \( \eta = \kappa p + k_\perp^2 / 2 \).

Thus when \( \eta / \kappa^2 \) is small enough, e.g. the effective potential barrier is high enough, one can neglect diagrams containing phonon intersection lines. The summation of all other diagrams leads to the equation for the vertex \( \Gamma \) presented in Fig.4.

Since the Green function and the vertex correction are functions of \( \eta \), the Ward identity is expressed in the following form:

\[
\frac{\partial G^{-1}}{\partial \eta} = \Gamma, \tag{5.6}
\]

Integrating Eq. (5.6) and making use of the fact \( \Gamma \) is a slow function we get

\[
G(\eta) = G^{(0)}(\eta) \Gamma^{-1}(\eta). \tag{5.7}
\]

The substitution of this expression in the equation presented in Fig.4 leads to the cancellation of two vertices and resulting in a linear equation for the vertex function. The main contribution to the integral term in this equation comes from the region where the momentum in the contour in Fig.4 is much larger than the external momenta. Similar to the calculation of \( \Gamma_1 \), the difference between the external momenta is not important and they have to be kept only to limit the value of otherwise logarithmically divergent integral. So the equation can be written in the following form

\[
\Gamma(\eta) = 1 + \alpha_p \frac{\int_0^\infty dq}{(2\pi^2)^2} \int_{-1}^1 dt \frac{\Gamma(-ikqt + q^2/2 + \eta)}{(-ikqt + q^2/2(1 - t^2) + \eta)^2}. \tag{5.8}
\]

In this integral \( t \sim 1 \) and the terms \( q^2/2 \) and \( \eta \) are small compared to \( \kappa q \). So these terms can be neglected in the argument of \( \Gamma \) (it is a slow function) and \( t^2 \) can be neglected in the coefficient of \( q^2/2 \) in the denominator (the exact value of this term is not important). Now it is convenient to consider \( \Gamma(\eta) \) as a function of the large logarithmic variables \( x = \ln(\eta/\kappa^2) \). Then keeping just the large logarithm \( x' = \ln(q/\kappa) \) in the argument of \( \Gamma \) in the integrand we can easily integrate with respect to \( t \). As a result, with the logarithmic accuracy,

\[
\Gamma(x) = 1 - \frac{\alpha_p}{2\pi^2\kappa^2} \int_0^x dx' \Gamma(x'), \tag{5.9}
\]

The solution of this equation and using Eq. (5.9) gives

\[
G(\eta) = \frac{1}{\eta} \left( \frac{\eta}{\kappa^2} \right)^{-\frac{\alpha_p}{2\pi^2\kappa^2}}. \tag{5.10}
\]

In the two dimensional case the Green function is given by:

\[
G(\tilde{\eta}) = \frac{1}{\tilde{\eta}} \left( \frac{\tilde{\eta}}{\kappa} \right)^{-\frac{\alpha_p}{2\pi^2\kappa^2}|M_0|^2}. \tag{5.11}
\]
Inserting these results in the equation for the transition amplitude (3.5) and performing the inverse Laplace transform we obtain for the three-dimensional tunneling configuration

\[ T = T^0 \left( \kappa L \right)^{-\frac{\alpha}{\pi^2 \kappa^2}}. \] (5.12)

The two-dimensional case gives the following transition probability

\[ T = T^0 \frac{1}{(\kappa L)} \left( \frac{d}{L} \right)^{-\frac{\alpha}{\pi^2 \kappa^2}|M_0|^2}. \] (5.13)

VI. SUMMARY AND CONCLUSIONS

In this paper we have presented a detailed study of the effect of coupling of an electron tunneling across a rectangular barrier, to the zero-point fluctuations of the acoustic phonon field. Two coupling mechanisms, the piezoelectric and deformation potential, were studied for two and three dimensional tunneling configurations. We considered a small coupling and used two main approximations: the static approximation and the parabolic approximation.

Contrary to the one-dimensional case, the main effect of the deformation potential coupling is the barrier and the electron mass renormalization. On the top of this the second order perturbation theory is sufficient. As a result only a prefactor in the tunneling probability is changes.

For the piezoelectric coupling mechanism, the situation is quite different. The perturbation theory is not sufficient and an accurate calculation of the vertex is necessary. As a result of the coupling, in both two-dimensional and three-dimensional cases the tunneling probability acquires a prefactor proportional to a power of the barrier width. The small coupling constants enters in the power. This result resembles piezoelectric polaron. When the barrier is wide enough the piezoelectric coupling can lead to a substantial reduction of the tunneling probability.

It is interesting also to compare the sign of the coupling effect in one- and higher dimensions. In one-dimensional case the coupling led to an enhancement of the tunneling probability. The physics behind this is that an electron chooses for tunneling the most favorable phonon configuration. In high-dimensional case, due to ultra-violet divergences, such favorable phonon configurations led to overall renormalization of the barrier which cannot be separated from the bare barrier. The measurable effect depending on the length of the barrier is a reduction of the tunneling probability.

In both cases the effects mentioned are enhanced when tunneling is near the top of the barrier, though as the effective potential barrier height, \( V - E \), is lowered two effects become important: dynamical corrections to our static approximation and an effect which can be described as channeling, the change of the electron wave function due to the interaction with phonons. These are subjects for future study.

VII. ACKNOWLEDGMENTS

The work was supported by The Israel Science Foundation, grant No. 174/98.
APPENDIX A: CALCULATION OF THE SELF-ENERGIES

In this appendix we present the main stages in the calculation of the self-energy integrals appearing in the text, except the self-energy integral for the three-dimensional deformation potential coupling which is straightforwardly evaluated. We start with the calculation of the self-energy integral, for the deformation potential coupling in the two-dimensional case, after the subtraction of the divergent term.

$$\Sigma_2^{\text{ren}}(p) = \frac{\alpha_d}{(2\pi)^3} \int_{-\infty}^{\infty} dq_z |M_{q_z}|^2 \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y \left( \frac{1}{\kappa p - \kappa q_x + \frac{1}{2} q_y^2} - \frac{1}{-\kappa q_x + \frac{1}{2} q_y^2} \right) \times \sqrt{q_x^2 + q_y^2 + q_z^2}. \quad (A1)$$

Noticing that for the $q_x$ integration poles are located in the lower half of the complex $q_x$ plane we shift the $q_x$ integration contour to the upper half plane. The shifted contour goes from $i\infty$ along the imaginary axis, rounds the point $x_0 = i\sqrt{q_y^2 + q_z^2}$ in the positive direction and comes back to $i\infty$. Thus we obtain

$$\Sigma_2^{\text{ren}}(p) = \frac{\alpha_d}{4\pi^3} \int_{-\infty}^{\infty} dq_z |M_{q_z}|^2 \int_{-\infty}^{\infty} dq_y \int_0^{\kappa} dx \left( \frac{1}{\kappa p + \kappa x + \frac{1}{2} q_y^2} - \frac{1}{\kappa x + \frac{1}{2} q_y^2} \right) \sqrt{x^2 - q_y^2 - q_z^2}. \quad (A2)$$

Where the upper cut-off, $\kappa$, for the $x$ integration is due to the parabolic approximation. The integration with respect to $q_z$ is cut off by the matrix element at $q_z \sim 1/d$ where $d$ is the size of the quantization in the $z$ direction. We assume that $d\kappa \gg 1$ and hence the main contribution to the integral \((A2)\) comes from the region $q_y \gg q_z$. Then we can neglect the $q_z^2$ appearing in the square roots, and

$$\Sigma_2^{\text{ren}}(p) = -\frac{\alpha_d\kappa p}{4\pi^3} \int_{-\infty}^{\infty} dq_z |M_{q_z}|^2 \int_{-\infty}^{\infty} dq_y \int_{|y|}^{\kappa} dx \frac{\sqrt{x^2 - q_y^2}}{(\kappa p + \kappa x + \frac{1}{2} q_y^2)(\kappa x + \frac{1}{2} q_y^2)}. \quad (A3)$$

Since we are interested in the limit $p \sim 1/L$ which is much smaller than all other scales, we neglect $\kappa p$ denominator, scaling out $\kappa$ by changing variables $x$ to $\kappa x$ and $q_y$ to $\kappa y$ we arrive at

$$\Sigma_2^{\text{ren}}(p) = -\frac{\alpha_d p}{2\pi^3} \int_{-\infty}^{\infty} dq_z |M_{q_z}|^2 \int_{0}^{\infty} dy \int_{|y|}^{1} dx \frac{\sqrt{x^2 - y^2}}{(x + \frac{1}{2} y^2)^2}. \quad (A4)$$

Changing variables to: $x = ty$ and changing the order of integration with respect to $t$ and $y$ we get

$$\Sigma_2^{\text{ren}}(p) = -\frac{\alpha_d p}{2\pi^3} \int_{-\infty}^{\infty} dq_z |M_{q_z}|^2 \int_{1}^{\infty} dt \sqrt{t^2 - 1} \int_{0}^{1/t} \frac{dy}{(t + \frac{1}{2} y^2)^2}. \quad (A5)$$

Were this integral is already easily evaluated resulting in Eq. \((4.5)\).

We now evaluate the self energy term for the piezoelectric coupling in the two-dimensional case,
\[
\Sigma^\text{ren}_2(p) = \frac{\alpha_p}{(2\pi)^3} \int_{-\infty}^{\infty} dq_x |M_{q_x}|^2 \int_{-\infty}^{\infty} dq_y \int_{-\infty}^{\infty} dq_y \left( \frac{1}{\kappa p - i\kappa q_x + \frac{1}{2}q_y^2} - \frac{1}{-i\kappa q_x + \frac{1}{2}q_y^2} \right) \\
\times \frac{1}{\sqrt{q_x^2 + q_y^2 + q_z^2}}.
\] 

(A6)

Using the same contour integration we derive

\[
\Sigma^\text{ren}_2(p) = -\frac{\alpha_p\kappa p}{2\pi^3\kappa} \int_{-\infty}^{\infty} dq_x |M_{q_x}|^2 \int_{-\infty}^{\infty} dq_y \int_{-\infty}^{\infty} dq_y \frac{dx}{\sqrt{q_y^2 + q_z^2}} \frac{1}{(\kappa p + \kappa x + \frac{1}{2}q_z^2)(\kappa x + \frac{1}{2}q_z^2)} \\
\times \frac{1}{\sqrt{x^2 - q_y^2 - q_z^2}}.
\] 

(A7)

\(q_y\) in the integrand is smaller than \(x\) and as we will see \(x \ll \kappa\). Due to this we neglect \(q_y^2\) compared to \(\kappa x\) and we obtain

\[
\Sigma^\text{ren}_2(p) = -\frac{\alpha_p p}{2\pi^3\kappa} \int_{-\infty}^{\infty} dq_x |M_{q_x}|^2 \int_{0}^{\infty} dy \int_{1}^{\infty} dy \int_{1}^{\infty} dt \frac{1}{(p + x)t^2 - 1}.
\] 

(A8)

Changing variables to: \(x = t\sqrt{q_y^2 + q_z^2}\); \(q_y = yq_z\), we get

\[
\Sigma^\text{ren}_2(p) = -\frac{\alpha_p p}{2\pi^3\kappa} \int_{-\infty}^{\infty} dq_x |M_{q_x}|^2 \int_{0}^{\infty} dy \int_{y^2 + 1}^{\infty} dy \int_{1}^{\infty} dt \frac{1}{t^2\sqrt{t^2 - 1}}.
\] 

(A9)

When \(p \ll 1/d\) the integral with respect to \(q_z\) contains a large logarithm. In the logarithmic approximation

\[
\Sigma^\text{ren}_2(p) = -\frac{\alpha_p p}{2\pi^3\kappa} |M_0|^2 \int_{0}^{1/d} dq_z \int_{0}^{\infty} dy \int_{y^2 + 1}^{\infty} dy \int_{1}^{\infty} dt \frac{1}{t^2\sqrt{t^2 - 1}},
\] 

(A10)

which results in Eq. (5.1).

Finally we calculate the self-energy for piezoelectric coupling in the three-dimensional case,

\[
\Sigma^\text{ren}_p(p) = \frac{\alpha_p}{(2\pi)^3} \int_{-\infty}^{\infty} dq_x \int d^2q_\perp \left( \frac{1}{\kappa p - i\kappa q_x + \frac{1}{2}q_\perp^2} - \frac{1}{-i\kappa q_x + \frac{1}{2}q_\perp^2} \right) \frac{1}{\sqrt{q_x^2 + q_\perp^2}}.
\] 

(A11)

Performing the angle integration and scaling out \(\kappa\) and changing variables to: \(q_\perp^2 = 2\kappa p z\); \(q_x = px\), we get

\[
\Sigma^\text{ren}_p(p) = -\frac{\alpha_p}{(2\pi)^2} \int_{0}^{\infty} dz \int_{-\infty}^{\infty} dx \frac{1}{(1 - ix + z)(-ix + z)} \frac{1}{\sqrt{x^2 + 2z/\lambda}},
\] 

(A12)

where \(\lambda = p/\kappa\). Noticing that for the \(x\) integration poles are located in the lower half of the complex \(x\) plane we shift the contour as we did before. Now the shifted contour rounds the point \(x = i\sqrt{2z/\lambda}\) and the result becomes
\[
\Sigma_p^{ren}(p) = -\frac{\alpha_p}{2\pi^2} \int_0^\infty dz \int_0^\infty dy \frac{1}{\left(1 + \sqrt{\frac{2\pi}{\lambda} + z + y}\right) \left(\sqrt{\frac{2\pi}{\lambda} + z + y}\right) \sqrt{2y\sqrt{\frac{2\pi}{\lambda} + y^2}}}
\]

\[
= -\frac{\alpha_p\lambda}{2\pi^2} \int_0^\infty \frac{dt}{\sqrt{(2 + t)t}} \int_0^\infty \frac{du}{(1 + u + \frac{\lambda u^2}{2} + ut) \left(1 + \frac{\lambda u}{2} + t\right)}.
\]

In the integral with respect to \(u\) we neglect terms of the order of \(\lambda\) so that

\[
\Sigma_p^{ren}(p) = -\frac{\alpha_p\lambda}{2\pi^2} \int_0^\infty \frac{dt}{\sqrt{(2 + t)t}} \frac{1}{(1 + t)^2} \left[\ln \frac{2}{\lambda} - 1 + 2\ln(1 + t)\right]
\]

\[
= -\frac{\alpha_p\lambda}{2\pi^2} \left[\ln \frac{2}{\lambda} - 1 + 2(1 - \ln 2)\right].
\]  

**APPENDIX B: LEADING ORDER CALCULATION OF FIRST VERTEX CORRECTIONS**

Under logarithmic accuracy the main contribution to the integral (5.3) comes from the region \(\kappa \gg q \gg p, k_\perp\). For this reason difference between \(p_1\) and \(p_2\) as well as between \(k_{1\perp}\) and \(k_{2\perp}\) is not important. So the integral (5.3) can be written as

\[
\Gamma_1(\eta) = \alpha_p \int_0^\infty \frac{qdq}{(2\pi)^2} \int_0^\pi \frac{d\theta \sin \theta}{(-iq\kappa \cos \theta + \frac{q^2}{2} \sin^2 \theta + \eta)^2},
\]

where \(\eta = \kappa p + k_\perp^2/2\). The term \(q^2/2\) cuts off the integration with respect to \(q\) from above and the term \(\eta\) does the same from below. In the region between these limits the integral is logarithmic and its value is accumulated from the whole interval of the integration. So that the exact value of these limits is not important and we can simplify the integration by the replacement of \(\sin^2 \theta\) with unity. As a result

\[
\Gamma_1(\eta) = \frac{2\alpha_p}{(2\pi)^2\kappa^2} \int_0^\infty \frac{qdq}{q^2 + \left(\frac{q^2}{2\kappa} + \frac{\eta}{\kappa}\right)^2} = -\frac{\alpha_p}{2\pi^2\kappa^2} \ln \left(\frac{\eta}{\kappa^2}\right).
\]  

Where a logarithmic approximation has been used in calculating the last integral.

Considering the first vertex correction in the two dimensional case again under logarithmic accuracy the difference between \(p_1\) and \(p_2\) as well as between \(k_{1y}\) and \(k_{2y}\) is not important. Thus the first vertex correction is given by

\[
\Gamma_1(\bar{\eta}) = \frac{\alpha_p}{(2\pi)^2} \int_{-\infty}^\infty dq_x |M_{q_x}|^2 \int_{-\infty}^{\infty} dq_y \int_{-\infty}^{\infty} \frac{dq_y}{\sqrt{q_x^2 + q_y^2 + q_z^2}} \frac{1}{\left(-i\kappa x + \frac{q^2}{2} + \bar{\eta}\right)^2},
\]

where \(\bar{\eta} = \kappa p + k_{2y}^2/2\). Using the contour integration described in Appendix (A) one gets an equation which is very similar to Eq. (A7)

\[
\Gamma_1(\bar{\eta}) = \frac{\alpha_p}{2\pi^3} \int_{-\infty}^\infty dq_x |M_{q_x}|^2 \int_0^\infty dq_y \int_0^\infty \frac{dq_y}{\sqrt{q_y^2 + q_z^2}} \frac{dx}{\left(\kappa x + \frac{1}{2} q_y^2 + \bar{\eta}\right)^2 \sqrt{x^2 - q_y^2 - q_z^2}}.
\]
Treating this integral in the same way Eq. (A7) has been evaluated results in Eq. (5.5). The main difference between this $\Gamma_1$ and that for three dimensional case is that now the logarithmic integral is cut off from above by the upper limit of $q_z$ which is of the order of $1/d \ll \kappa$. 
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FIGURES

FIG. 1. The self energy term. The solid line represents the bare electron propagator whereas the dashed line represents interaction with phonons.

FIG. 2. The first correction to the vertex $\Gamma$.

FIG. 3. Second order vertex corrections.

FIG. 4. The equation for the renormalized vertex $\Gamma$ resulting from the summation of leading logarithmic corrections. Double lines represent the full electron Green function.
Figure 1:
Figure 2:
Figure 3:
\[ \Gamma = 1 + \quad \text{Figure 4:} \]