LETTER

On the nonlinear response of a particle interacting with fermions in a 1D lattice

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Abstract. By the Bethe ansatz method we study the energy dispersion of a particle interacting by a local interaction with fermions (or hard core bosons) of equal mass in a one-dimensional lattice. We focus on the period of the Bloch oscillations, which turns out to be related to the Fermi wavevector of the Fermi sea and in particular on how this dispersion emerges as a collective effect in the thermodynamic limit. We also discuss the adiabatic coherent collective response of the system to an applied field.

Keywords: Quantum integrability (Bethe ansatz), connections between chaos and statistical physics, quantum transport in one dimension, optical lattices

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1. Introduction

Prototype integrable quantum many-body models provide valuable insight into the physics of correlated electronic and magnetic compounds. With the development of novel experimental systems, such as quasi-one-dimensional quantum magnets [1] and cold atom systems [2], it became possible to tailor-make and experimentally study them. At the same time because of their non-generic character they exhibit unconventional dynamic and transport properties of academic and potentially technological interest.

One of the simplest models that was studied early on [3], is that of a particle interacting with spinless fermions in a one-dimensional lattice, where the mass of the extra particle is the same as the mass of the fermions. In this model and its lattice version, it was shown that, despite the particle–fermion interaction, the transport of the ‘heavy’ particle remains ballistic at all temperatures, namely its mobility diverges [4,5]. The spinless fermion or hard core boson bath corresponds to the Tonks–Girardeau gas, a one-dimensional boson model with very strong local repulsion.

More recently [6]–[8], motivated by cold atom physics experiments, the dispersion and period of Bloch oscillations of the heavy particle when acted on by a constant field in a continuous system were also studied and shown to be related to the Fermi wavevector $k_F$ of the fermionic sea.

In this work we introduce two main ideas. First, we show that in the competition between the period of oscillations imposed by the lattice in the tight binding version of the model and that imposed by the Fermi wavevector, the period becomes equal to $2k_F$ only in the thermodynamic limit. Second, we show that when the integrable system is driven adiabatically by a constant field, because of level crossing, there is a collective coherent excitation of the system with a macroscopic period.

2. Adiabatic response

The lattice model corresponds to a particular sector of the 1D Hubbard model [5], a prototype integrable model by the Bethe ansatz method [9]. The analytical solution allows us to characterize the low energy spectrum and also study the adiabatic response of the particle to an applied field (in [10] the effect of a flux applied on both spin species was studied).

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Figure 1. Low energy spectrum for $L = 10, u = 0.5, t_h = t$, along with the reconstruction of selected branches by the Bethe ansatz solution.

The system is described by the Hamiltonian,

$$
\hat{H} = -t_h \sum_l (e^{i\phi}d_{l+1}^d + h.c.) - t \sum_l (c_{l+1}^d c_l + h.c.) + U \sum_l d_l^d c_l^d c_l,
$$

where $c_l(c_l^d)$ are annihilation (creation) operators for $N$ spinless fermions and $d_l(d_l^d)$ for the extra particle on an $L$ site chain with periodic boundary conditions. The interaction comes only through the on-site repulsion $U > 0$. If the particle is charged and $\phi$ depends linearly on time, $\phi = -Et$, then a constant electric field $E = -\partial \phi / \partial t$ acts on it (it could also be gravitational [2]). For a vanishing field the system should follow the adiabatic ground state that we will now map.

For illustration of a typical finite size lattice spectrum, we show in figure 1 the evolution of the low lying energy levels for $L = 10$, half-filling and in the $k = 0$ symmetry momentum subspace as a function of $\phi$. It is clear that by scanning $\phi$ we recover, at $\phi = 2\pi n / L$ ($n = 0, L - 1$), the spectra in the successive $k$-subspaces with $\phi = 0$. We will now show how to reconstruct selected branches of the dispersion and study their finite size dependence using the Bethe ansatz solution.

The Bethe ansatz wavefunctions, in the presence of flux $\phi$, are characterized by $M = N + 1$ quantum numbers $k_j$ for $N$ fermions plus the one extra particle, given by the following equations (we take $t_h = t$),

$$
k_j = \frac{2\pi I_j}{L} + \frac{1}{L} \theta(\sin k_j - \Lambda), \quad j = 1, \ldots, M
$$

$$
\theta(p) = -2 \tan^{-1}(p/u), \quad u = U/4t
$$

$$
K = \sum_{j=1}^M k_j = \sum_{j=1}^M \frac{2\pi I_j}{L} + \frac{2\pi J}{L} + \phi = \sum_{j=1}^M k_j^0 + \frac{2\pi J}{L} + \phi.
$$

(1)

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The total energy and momentum are given by the sum of quasi-energies and momenta,
\[ E = \sum_{j=1}^{M} \epsilon(k_j) = -2t \sum_{j=1}^{M} \cos k_j, \quad K = \sum_{j=1}^{M} k_j. \]

Every state is characterized by a set of half-odd integers \( I_j \) and the (half-odd) integer \( J \) for (even) odd number of fermions. We will consider the case of an odd number of fermions, \( M \) even, that is also equivalent to that of hard core bosons in a one-dimensional system. From this Bethe ansatz formulation it is clear that we are dealing with a collective effect as there is no momentum \( k_j \) that is singled out for the extra particle.

We can reconstruct the low energy states by considering the following branches:

(a) place the \( I_j \)s at the successive values \(-M/2 + 1/2 \leq I_j \leq +M/2 - 1/2\) and scan \( \Lambda \) between \( \pm \infty \). At this configuration, for \( u \to 0 \) and \( \Lambda = 0 \), the phase shift term in equation (1) takes the value \(+\pi/L (-\pi/L)\) for \( k_j < 0 (k_j > 0) \). Thus for one particle \( k_j = 0 \) while the rest of fermions uniformly fill a Fermi sea, between \(-(N-1)/2 \leq k_j \leq +(N-1)/2\). The k-occupation corresponds to that of independent particles.

For a finite \( u \), by varying \( \Lambda \), every \( k_j \) shifts by \(-\pi/L (+\pi/L)\) for \( \Lambda \to -\infty (+\infty) \), so that \(-2M\pi/L \leq K \leq +2M\pi/L\). In the thermodynamic limit, \(-k_F \leq K \leq +k_F\) with \( k_F = \pi N/L \). It is clear that the spectrum is scanned by finding the \( \Lambda \) solutions of equations (1) as a function of \( \phi \) for \( J = 0 \), or equivalently for \( \phi = 0 \) as a function of the quantum number \( J \).

(b) Shift each \( I_j \) of branch (a) by +1. Now by varying \(-\infty < \Lambda < +\infty, \pi M/L \leq K \leq 3\pi M/L\), which tends in the thermodynamic limit to \( k_F \leq K \leq 3k_F \).

(c) Shift any \( I_j \) of branch (a) by \( L \) and scan \( \Lambda \).

(d) Shift \( I_M \) of branch (a) by +1 and scan \( \Lambda \).

The adiabatically evolving ground state is obtained by the successive shift of \( I_j \to I_j + 1 \) up to \( L \) times when the \( k_j \)s are shifted by the trivial phase \( 2\pi \). Branch (d) corresponds to a state with an ‘electron–hole’ excitation that is not adiabatically evolving from the ground state.

Next, in figure 2 we show the finite size dependence of the adiabatically evolving ground state for a series of lattices at half-filling \( (k_F = \pi/2) \). It is now clear that in the thermodynamic limit the period becomes \( 2k_F \). We can argue that in this limit the extra particle scatters from a ‘rigid’ background lattice created by the Fermi sea that has only one characteristic length\(^1\), the wavevector \( k_F \).

In order to analyze the finite size effects and obtain the particle dispersion we take to \( O(1/L) \),
\[ k_j \simeq k_j^0 + \frac{1}{L} \theta(\sin k_j^0 - \Lambda), \quad j = 1, \ldots, M. \]

\(^1\) The pair correlation function was studied early on in the pioneering work [3].

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Figure 2. Finite size scaling of the adiabatically evolving ground state for \( t_1 = t, u = 0.5 \) and half-filling.

We can now see how the dispersion becomes symmetric around \( k_F \) in the thermodynamic limit by evaluating the difference in energy between the state \( K = 0 \) and \( 2k_F \). By shifting \( k_1 = (2\pi/L)(-M/2 + 1/2) \) to \( k'_1 = (2\pi/L)(M/2 + 1/2) \) and taking \( \Lambda = 0 \), we obtain,

\[
K_{2k_F} - K_0 = \frac{2\pi}{L} M + O(1/L) \simeq 2k_F
\]

\[
E_{2k_F} - E_0 = +2t\cos(k_1) - 2t\cos(k'_1) \simeq \frac{4\pi}{L} \sin k_F.
\]

So the finite size effects decrease as \( 1/L \).

Next we can obtain an analytical solution of the dispersion of branch (a) and by symmetry (b) in the thermodynamic limit, including corrections \( O(1/L) \),

\[
K = \sum_{j=1}^{M} k_j \simeq K_0 + \delta K = \sum_{j=1}^{M} k^0_j + \frac{1}{L} \sum_{j=1}^{M} \theta(\sin k^0_j - \Lambda),
\]

\[
E = -2t \sum_{j=1}^{M} \cos k_j \simeq E_0 + \delta E = -2t \sum_{j=1}^{M} \cos k^0_j + \frac{2t}{L} \sum_{j=1}^{M} \sin k^0_j \theta(\sin k^0_j - \Lambda).
\]

Replacing sums by integrals for the branch (a) we find for the momentum and energy dependent terms on \( \Lambda \) and thus \( \phi \),

\[
\delta K = \frac{1}{2\pi} \int_{-k_F}^{+k_F} dk^0 \theta(\sin k^0 - \Lambda), \quad \delta E = \frac{2t}{2\pi} \int_{-k_F}^{+k_F} dk^0 \sin k^0 \theta(\sin k^0 - \Lambda).
\]

Scanning \( -\infty < \Lambda < +\infty \), we obtain the dispersion shown in figure 2 for \( L \to +\infty \).

If we interpret \( \delta E, \delta K \) as the dispersion of the correlation energy and momentum we can deduce their temperature dependence by inserting a Fermi–Dirac thermal distribution \( f^0_k \) for the momenta \( k^0_j \),

\[
\delta K = \frac{1}{2\pi} \int_{-\pi}^{+\pi} dk^0 f^0_k \theta(\sin k^0 - \Lambda), \quad \delta E = \frac{2t}{2\pi} \int_{-\pi}^{+\pi} dk f^0_k \sin k^0 \theta(\sin k^0 - \Lambda).
\]

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Figure 3. Time evolution for $L = 10$, half-filling and $u = 0.5$. $E = 0, t_h = t$ (blue), $E = 0, t_h = 0.5t$ (black), $E = 0.1, t_h = t$ (red), $E = 0.1, t_h = 0.5t$ (green curve).

Now, by electron–hole symmetry we see that at half-filling, $\mu = 0$, the thermal distribution factors cancel, so that the dispersion becomes temperature independent. This is due to the particular form of scattering phase shifts in this integrable model.

3. Non-adiabatic response

The next issue we want to discuss is the response of the particle to a constant external field $E$ created by a time dependent flux $\phi = -Et$.

The calculation proceeds numerically, first, by exact diagonalization of the Hamiltonian on a $L = 10$ site lattice, in a given ($k = 0$) momentum subsector corresponding to a Hilbert space of 252 states, in order to obtain the eigenvalues and eigenfunctions. The time evolution is subsequently carried out using a Runge–Kutta method. Slightly larger lattices are possible to study, e.g. $L = 12$, although the computing time requirements and precision rapidly deteriorate. Approximate Lanczos type methods can be used in the study of larger lattices, but this is not in the scope of this work, which is the demonstration of prototype effects.

As is evident from figure 1 the collective ground state evolves through a maze of excited ‘electron–hole’ states by level crossing. The coherent evolution through the energy spectrum is protected by the macroscopic number of conservation laws characterizing an integrable system that suppresses level repulsion and also results in Poisson statistics. There is a coherent drag of the Fermi sea by the ‘heavy’ particle. We should note that a similar level crossing trajectory also characterizes the time evolution of every excited state. We can plausibly argue that this is a generic behavior of integrable quantum many-body systems.

To give evidence of this evolution we show, in figure 3, the long time adiabatic energy evolution of an $L = 10$ system as a function of phase $\phi = -Et$ in units of $\pi$ for $E \to 0$.

When the momentum of each particle is displaced by $2\pi$, so that $\phi = 2\pi M$, we have an identical state (in the example at $\phi/\pi = 2M, M = 6$). Thus the periodicity of the
coherent collective motion becomes macroscopic. In the adiabatic limit there is reversible pumping of a macroscopic amount of energy into the system that could experimentally be observed in a mesoscopic system. In contrast, in the nonintegrable case, e.g. \( t_h = 0.5t \), the periodicity is simply \( 2\pi \) for a finite size system, and it becomes \( 2k_F \) in the thermodynamic limit. This behavior is generic for any interaction \( U \) and hopping difference \( t_h \neq t \).

Note also that in the continuous system, hard core bosons in 1D, the driving leads to an indefinite increase of energy, as there is no \( \phi = 2\pi M \) periodicity.

Furthermore, as shown in figure 3, in a non-adiabatic evolution obtained by a finite field, e.g. \( E = 0.1 \), in the integrable case the energy deviates from the adiabatic—nonlinear in energy-trajectory, in sharp contrast to the non-adiabatic case where it spreads diffusively upwards from the ground state. It is clear that in the integrable system the departure from the adiabatic evolution is by mixing with highly excited states, while in the nonintegrable case it is by tunneling between level repelled nearest neighbor states. For long times, the integrable system rapidly evolves through the whole energy spectrum to thermalize at the infinite temperature energy limit (mean value of energy spectrum) in a non-monotonic way, while in the nonintegrable system it diffuses monotonically to the same mean energy (not shown). Of course this unconventional thermalization is irrelevant in the thermodynamic limit but it might be observable in a finite size system.

To quantify the spreading of the wavefunction during the evolution, we show, in figure 4, \( \Delta(\phi) \) defined by [11],

\[
\Delta(\phi) = \langle (H(\phi) - \langle H(\phi) \rangle)^2 \rangle, \quad \langle \cdots \rangle = \langle \Psi(\phi) | \cdots | \Psi(\phi) \rangle.
\]

Here we also observe that the spreading is highly irregular in the integrable case while it is almost linear in time, and diffusive, in the nonintegrable one.

The main issues emerging from these results are; first, whether it is possible to observe experimentally the adiabatic dispersion of the driven particle in a macroscopic system, that is in fact qualitatively similar in the integrable and nonintegrable case. In this context we should mention that recent works argue on the absence of an adiabatic limit in low-dimensional gapless systems [12] in macroscopic systems. Still, despite the spreading of
the wavefunction, it might still be possible to observe the characteristic energy–momentum
dependence for at least a few Bloch oscillations and the remnants of the singular integrable
behavior.

Second, in a finite size—mesoscopic—system, there is a striking difference between
the nonlinear evolution of an integrable and a nonintegrable system. It is worth studying
to what extent this coherent behavior is affected by non-adiabatic effects, for at least some
modes of driving. Experimentally, it is interesting to tune the boson–boson interaction
from the weak (nonintegrable) to the strong Tonks–Girardeau (integrable) limit and
observe the resulting dynamics. Or alternatively vary the ‘heavy’ particle mass.

Of course in experiments on optical lattices we are dealing with open end systems
and/or an harmonic well potential. However studies show, at least in the linear response
regime \[13,14\], a robustness of the integrable regime that is reflected in nearly diverging
conductivities over a finite region of parameter space near an integrable point. Thus open
boundaries, a \(1/L\) effect or a weak potential might not affect the proposed behavior, which
is due to bulk ‘heavy particle’–fermion bath interactions.

We conclude that this prototype integrable model exhibits an unconventional
collective nonlinear evolution when driven by a constant field, which motivates the
exploration of other integrable quantum many-body models.

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