Radiosity Model and Compensation Theorem

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Abstract. In the preceding conference contribution called “Radiative Heat Transfer in Buildings”, a general model for radiative heat transfer in inner spaces of buildings has been developed. The model is capable of determining radiosities, heat fluxes and heat flows with each surface of the interior. In closed spaces like interiors, the radiative energy should be conserved so that the total heat flow should be zero. This property may be formulated as the compensation theorem and should be mirrored in each reasonable model. Thus, it is desirable to verify whether our developed model satisfies such a property. The mathematical proof has been presented which has confirmed that the developed model obeys the compensation theorem. As a numerical illustration, radiative heat flows in a simple room have been computed and their total sum has been quantified and shown that this summation leads to zero value. The numerical illustrations start with formulating matrix of view factors that describe the ‘mutual visibility’ of interior surfaces and in this way they assist to redistribute radiative heat among the interior surfaces. As soon as the matrix of view factors is established, the system of algebraic equations is formed. These equations serve for computing surface radiosities. The radiosities enable estimating heat fluxes and heat flows occurring in the interior. Some of the heat flows are positive and some are negative. The positive value of heat flow means that the corresponding surface emits energy into the interior whereas the negative value indicates that the corresponding surface absorbs energy from the interior. By summing the positive and negative values of heat flows, the total heat flow can be obtained whose value is zero in agreement with the compensation theorem.

1. Introduction
The history of thermal radiation started in the early 20th Century. The basics of this discipline were laid by Planck, Wien, Stefan, Boltzmann and Kirchhoff. Since that time many different applications have been performed. The properties of radiative heat transfers in enclosures have been the subject of continuous interest of many researchers [1-5] both in the field of mechanical and civil engineering [6-8]. Radiative heat transfer in inner spaces of buildings represents the most effective mechanism of heat exchange.

The radiative heat exchange in enclosures, like inner space of buildings, has an interesting property, namely, the sum of all the radiative heat flows is zero irrespective of temperatures and emissivities of

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the surfaces forming the enclosure. This seems to be a physically understandable property that follows from the principle of conservation energy (first law of thermodynamics) but the question is whether the general model of radiative heat transfer that we derived in the preceding conference contribution called “Radiative Heat Transfer in Buildings” also satisfies that conservation principle. The present contribution is devoted to the mathematical proof showing that our derived model obeys this principle. The proof is actually a methodological complement to our model and together with the conservation property of the model is the manifestation of the conservation theorem following from the mathematical formalism of our model. Prior to start with the proof it would be instructive to recall some important properties of view factors that play core roles in the proof.

2. View factors

The view factors [9-11] are defined by double integrals as follows

\[
F_{ij} = \frac{1}{S_i} \int \int \frac{\cos \varphi_i \cos \varphi_j}{\pi R^2} dS_j dS_i, \quad F_{ji} = \frac{1}{S_j} \int \int \frac{\cos \varphi_j \cos \varphi_i}{\pi R^2} dS_i dS_j
\]

where \(\varphi_i\) and \(\varphi_j\) are the angles between the surface normals and a ray between the two differential areas \(dS_i\) and \(dS_j\) (see Fig.1).

![Figure 1. Differential areas \(dS_i, dS_j\) and their angles \(\varphi_i, \varphi_j\).](image)

The view factors fulfill four basic properties:

(i) \(S_i F_{ij} = S_j F_{ji}\) (Symmetry rule - general property)

(ii) \(S_i = 0\) (Zero rule - surface incapable of irradiating itself)

(iii) \(\sum_{j=1}^{n} F_{ij} = 1\) (Summation rule - applicable only within enclosures)

(iv) \(F_{ij} \in [0,1]\) (General property)

The view factors for \(n\) surfaces may be arranged into a matrix of \(n\)-th order:
\[
\begin{pmatrix}
F_{11} & F_{12} & F_{13} & \ldots & F_{1n} \\
F_{21} & F_{22} & F_{23} & \ldots & F_{2n} \\
F_{31} & F_{32} & F_{33} & \ldots & F_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
F_{n1} & F_{n2} & F_{n3} & \ldots & F_{nn}
\end{pmatrix}
\]

The view factors serve as auxiliary input data for general transfer equations derived in the preceding conference contribution “Radiative Heat Transfer in Buildings”. The equations have assumed the following forms

\[
W_i = \varepsilon_i E_{bi} + \rho_i \sum_{j=1}^{N} F_{ij} W_j \quad \text{(Watt/m}^2\text{)}
\]  
(2)

\[
q_i = W_i - \sum_{j=1}^{N} F_{ij} W_j \quad \text{(Watt/m}^2\text{)}
\]  
(3)

\[
\Phi_i = S_i q_i \quad \text{(Watt)}
\]  
(4)

Formula (2) represents a system of \( n \) linear algebraic equations determining \( n \) values of radiosities \( \{W_i\}_{i=1}^{n} \) that may be used for the calculation of \( n \) heat fluxes \( \{q_i\}_{i=1}^{n} \) according to Eq. (4). The system of Equations (2) - (4) holds for both the closed and open envelopes.

3. Proof of the compensation theorem

According to the compensation theorem, the total radiative heat flow \( \sum_{i=1}^{n} \Phi_i \) in closed envelopes (e.g. in inner spaces of buildings) assumes zero value. The proof of this theorem may be carried out by using the general Equation (2), the symmetry rule (i), and the summation rule (iii).

\[
\Phi = \sum_{i=1}^{n} \Phi_i = \sum_{i=1}^{n} S_i q_i = \sum_{i=1}^{n} S \left( W_i - \sum_{j=1}^{n} F_{ij} W_j \right) = \sum_{i=1}^{n} S W_i - \sum_{i=1}^{n} S \sum_{j=1}^{n} F_{ij} W_j = \sum_{j=1}^{n} S W_j - \sum_{j=1}^{n} S \sum_{i=1}^{n} F_{ij} W_j = \sum_{j=1}^{n} \left( S W_j - \sum_{i=1}^{n} F_{ij} W_j \right) = \sum_{j=1}^{n} \left( S W_j \right) - \sum_{j=1}^{n} \sum_{i=1}^{n} F_{ji} W_j = \sum_{j=1}^{n} \left( S W_j \right) \left( 1 - \sum_{i=1}^{n} F_{ji} \right) = 0 \quad \text{(for closed envelopes only)} \\
\neq 0 \quad \text{(for open envelopes only)}
\]

(5)
The expression \(1 - \sum_{i=1}^{n} F_{ji}\) in Equation (5) is zero only if \(\sum_{i=1}^{n} F_{ji} = 1\) which guarantees the summation rule (iii) holding solely for the closed envelopes and this is the reason why the total radiative heat flow \(\Phi = \sum_{i=1}^{n} \Phi_{i}\) is zero in all closed envelopes in contrast to the open envelopes where the total radiative heat flow \(\Phi\) assumes non-zero value. The validity of the compensation theorem is independent of the temperatures and emissivities of the inner surfaces of enclosures.

4. Compensation theorem in practice

In this section, we illustrate the functionality of the compensation theorem for the case of a simple room. Three basic assumptions are used, namely, the room (Figure 2) contains a perfectly diathermal gas (pure air), the inner surfaces of the room are perfect Lambert's diffuse radiators and are made of perfectly grey materials, i.e. the sums of their emissivities \(\varepsilon_i\) and reflectivities \(\rho_i\) are equal to one, i.e. \((\varepsilon_i + \rho_i = 1)\). The input data for computations of radiative heat flows are summarized in Table 1.

![Figure 2. A simple room with heated floor no.3.](image)

**Table 1.** Input data of the investigated room.

| Parameter | Surface no. 1 | Surface no. 2 | Surface no. 3 |
|-----------|---------------|---------------|---------------|
| S (m²)    | 20            | 54            | 20            |
| T (K)     | 287           | 293           | 303           |
| \(\varepsilon\) | 0.7           | 0.75          | 0.9           |
| \(\rho\)  | 0.3           | 0.25          | 0.1           |
| \(\varepsilon \times E_b\) | 269.28284     | 313.41141     | 430.12638     |
| (Watt / m²) |               |               |               |
Matrix of view factors:

\[
\begin{pmatrix}
0 & 0.6836802 & 0.3163198 \\
0.2532149 & 0.4935702 & 0.2532149 \\
0.3163198 & 0.6836802 & 0
\end{pmatrix}
\]

Radiosities

\[
W_i = \varepsilon_i E_{b_i} + \rho_i \sum_{j=1}^{3} F_{ij} W_j
\]

\[
W_1 = 269.28284 + 0.3 \cdot (0 \cdot W_1 + 0.6836802 \cdot W_2 + 0.3163198 \cdot W_3)
\]

\[
W_2 = 313.41141 + 0.25 \cdot (0.2532149 \cdot W_1 + 0.4935702 \cdot W_2 + 0.2532149 \cdot W_3)
\]

\[
W_3 = 430.12638 + 0.1 \cdot (0.3163198 \cdot W_1 + 0.6836802 \cdot W_2 + 0 \cdot W_3)
\]

\[
1 \cdot W_1 - 0.20510406 \cdot W_2 - 0.09489594 \cdot W_3 = 269.28284
\]

\[
-0.063303725 \cdot W_1 + 0.87660745 \cdot W_2 - 0.063303725 \cdot W_3 = 313.41141
\]

\[
-0.03163198 \cdot W_1 - 0.06836802 \cdot W_2 + 1 \cdot W_3 = 430.12638
\]

\[
W_1 = 400.27278909 \text{ (Watt / m}^2\text{)}
\]

\[
W_2 = 420.48481695 \text{ (Watt / m}^2\text{)}
\]

\[
W_3 = 471.53551523 \text{ (Watt / m}^2\text{)}
\]

\[
\Phi_i = S_i q_i = S_i \left[ W_i - \sum_{j=1}^{N} F_{ij} W_j \right]
\]

\[
\Phi_1 = 20 \cdot [400.27278909 - (0 + 0.6836802 \times 420.48481695 + 0.3163198 \times 471.53551523)]
\]

\[
\Phi_2 = 54 \cdot [420.48481695 - (0.2532149 \times 400.27278909 + 0.4935702 \times 420.48481695 + 0.2532149 \times 471.53551523)]
\]

\[
\Phi_3 = 20 \cdot [471.53551523 - (0.3163198 \times 400.27278909 + 0.6836802 \times 420.48481695)]
\]

\[
\Phi_1 = -727.2075 \text{ (Watt)}, \quad \Phi_2 = -421.6758 \text{ (Watt)}, \quad \Phi_3 = +1148.8833 \text{ (Watt)}
\]

\[
\Phi = \sum_{i=1}^{3} \Phi_i = 0 \text{ (Watt)}
\]

The total radiative heat flow $\sum_{i=1}^{3} \Phi_i$ is zero, as required by the compensation theorem.
5. Conclusions
A mathematical proof of the compensation theorem that is a part of our generalized model for the computation of radiative heat transfer in enclosures (e.g., in inner spaces of buildings) has been presented. The compensation theorem has not only theoretical meaning but also has practical applications. For example, when building technologists estimate radiative heat transfers in interiors between radiant panels and interior walls, they have a straightforward tool for checking their results. Simply, they can sum all the computed heat flows and if the sum considerably differs from zero, they may be sure that the computations contain errors. In such cases the recalculation and careful checking is desirable. However, small differences from zero (less than 1%) may be the result of rounding errors that may cumulate during numerical operations.

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