Collision integral for multigluon production in a model for scalar quarks and gluons

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Abstract

A model of scalar quarks and scalar gluons is used to derive transport equations for quarks and gluons. In particular, the collision integral is studied. The self-energy diagrams are organized according to the number of loops. A generalized Boltzmann equation is obtained, which involves at the level up to two loops all possible two → two parton scattering processes and corrections to the process q̅q → g.

1 Model of scalar quarks and gluons, and transport theory

The model underpinning our calculations successfully gives a ln s behavior in high energy qq scattering [1]. Quarks and antiquarks are described by complex scalar fields φ, and gluons as the scalar field χ coupled through the Lagrangian

\[ \mathcal{L} = \partial^{\mu} \phi_{i}^{l} \partial_{\mu} \phi_{i}^{l} + \frac{1}{2} \partial^{\mu} \chi_{a,r} \partial_{\mu} \chi_{a,r} - \frac{m^{2}}{2} \chi_{a,r} \chi_{a,r} \\
- gm \phi_{i}^{l} (T^{a})_{j}^{l} (T^{r})_{m}^{r} \phi_{j,m} \chi_{a,r} - \frac{gm}{3!} f_{abc} f_{rst} \chi_{a,r} \chi_{b,s} \chi_{c,t}. \]  

(1)

The quarks are massless, while the gluons are assigned a mass m in order to avoid infra-red divergences. Since in QCD the quartic interaction between gluons leads to

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terms which are sub-leading in \( \ln s \), it is not included here. The two labels on each of the fields refer to the direct product of two color groups which is necessary to make the three gluon vertex symmetric under the exchange of two bosonic gluons.

To describe non-equilibrium phenomena, we use the Green functions of the Schwinger-Keldysh formalism [2]. The equations of motion for the Wigner transforms of the Green functions, the so-called transport and constraint equations, are derived [3, 4] and read

\[
-2ip^\mu \partial_{X^\mu}D^{-+}(X,p) = I_{\text{coll}} + I_A^+ + I_R^+ \quad \text{transport}
\]

and

\[
\left( \frac{1}{2} \partial_{X^\mu} \partial_{P^\mu} - 2p^2 + 2M^2 \right) D^{-+}(X,p) = I_{\text{coll}} + I_A^- + I_R^-, \quad \text{constraint,}
\]

where \( D^{-+} \) is a generic Green function, \( D^{-+} = S^{-+} \) for quarks and \( D^{-+} = G^{-+} \) for gluons. \( M \) is the appropriate mass, and \( I_{\text{coll}} \) is the collision term,

\[
I_{\text{coll}} = \Pi^{-+}(X,p) \hat{\Lambda} D^{-+}(X,p) - \Pi^{+-}(X,p) \hat{\Lambda} D^{-+}(X,p) = I_{\text{coll}}^{\text{gain}} - I_{\text{coll}}^{\text{loss}}.
\]

\( I_R^+ \) and \( I_A^+ \) are terms containing retarded and advanced components:

\[
I_R^+ = -\Pi^{-+}(X,p) \hat{\Lambda} D^R(X,p) \pm D^R(X,p) \hat{\Lambda} \Pi^{-+}(X,p)
\]

\[
I_A^+ = \Pi^A(X,p) \hat{\Lambda} D^{-+}(X,p) \mp D^{-+}(X,p) \hat{\Lambda} \Pi^A(X,p).
\]

In Eqs.(4) to (6), \( \Pi \) is a generic self-energy (\( \Sigma_q \) for quarks and \( \Sigma_g \) for gluons) and the operator \( \hat{\Lambda} \) is given by

\[
\hat{\Lambda} := \exp \left\{ -\frac{i}{2} \left( \overleftarrow{\partial}_X \overrightarrow{p} - \overrightarrow{\partial}_p \overleftarrow{X} \right) \right\}.
\]

For the Green functions, we use the quasiparticle approximation:

\[
i D^{-+}(X,p) = \frac{\pi}{E_p} \delta(E_p - p^0) f_a(X,p) + \delta(E_p + p^0) \bar{f}_a(X,-p)
\]

\[
i D^{++}(X,p) = \frac{\pm i}{p^2 - M^2 \pm i\epsilon} + \Theta(-p^0) i D^{-+}(X,p) + \Theta(p^0) i D^{-+}(X,p),
\]

\( D^{++} \) is obtained from the expression for \( D^{-+} \) by replacing \( f \) with \( \bar{f} \) and vice versa. \( f_a \) is the quark (\( a = q \)) or the gluon (\( a = g \)) distribution function, \( \bar{f}_a \) is an abbreviation for \( 1+f_a \). Then an evolution equation for the parton distribution function is obtained by integrating Eq.(2) over an interval \( \Delta^+ \) that contains the energy \( E_p \). To lowest order in an expansion that sets \( \hat{\Lambda} = 1 \), the integral for the collision term is performed in the next section.
2 The collision integral

For the collision integral, one has

\[ J_{\text{coll}} = \int_{\Delta^+} dp_0 \Pi^{++}(X, p) D^{+-}(X, p) - \int_{\Delta^+} dp_0 \Pi^{+-}(X, p) D^{+-}(X, p) \]
\[ = -i \frac{\pi}{E_p} \Pi^{++}(X, p_0 = E_p, \vec{p}) \bar{f}_a(X, \vec{p}) + i \frac{\pi}{E_p} \Pi^{+-}(X, p_0 = E_p, \vec{p}) f_a(X, \vec{p}), \]

(10)

i.e. the off-diagonal quasiparticle self-energies are required to be calculated on shell.

In the following, only the results for the quarks are presented, the generalization to the gluons is then obvious. The self-energies are organized now according to their number of loops.

2.1 Hartree and Fock self-energies

The Hartree self-energies (self-energy with a quark or a gluon loop) vanish due to their color factors as it is the case for real QCD.

The Fock self-energy gives a contribution to the loss term of the collision integral in Eq.(10) given by

\[ J_{\text{loss}}^{\text{coll}} = -\frac{\pi}{E_p} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \delta^{(4)}(p + p_1 - p_2) \times |\mathcal{M}_{q\bar{q} \rightarrow g}|^2 f_q(X, \vec{p}) f_{\bar{q}}(X, \vec{p}_1) f_g(X, \vec{p}_2). \]

(11)

Here, \( \mathcal{M}_{q\bar{q} \rightarrow g} \) means the scattering amplitude of the process \( q\bar{q} \rightarrow g \) of order \( \alpha^g m^3 \) which is pictured in Fig.1 a). Processes as \( qg \rightarrow q, q \rightarrow qg \) or \( q\bar{q}g \rightarrow \phi \) are kinematically forbidden and therefore do not occur in this collision integral. The gain term can be obtained by exchanging \( f \) with \( \bar{f} \) in Eq.(11).

![Figure 1: The process \( q\bar{q} \rightarrow g \) up to order \( \alpha^g m^3 \)](image-url)
### 2.2 Two loop self-energies

The two loop self-energy graphs are pictured in Fig.2. According to their topology, they are called rainbow (R), ladder (L), cloud (C), exchange (E) and quark-loop (QL) graphs.

![Two loop self-energy diagrams](image)

Let us first consider the contribution of the seven diagrams Ra), La), Ca), Cb), Ea), Eb) and QLa) to the collision integral. The first five of these diagrams lead to the (lowest order) scattering amplitudes of the processes $q\bar{q} \rightarrow gg$ and $qg \rightarrow qg$, while the latter two lead to the (lowest order) scattering amplitudes of the processes $qq \rightarrow qq$ and $q\bar{q} \rightarrow q\bar{q}$ [5]. Thus, to obtain all possible $2 \rightarrow 2$ processes it is necessary to consider all types of diagrams. But an analysis of the color factors shows that the quark-loop diagrams are subleading as it is the case in real QCD. Therefore, in an additional expansion in the inverse number of colors, $1/N_c$, one can neglect the processes $qq \rightarrow qq$ and $q\bar{q} \rightarrow q\bar{q}$. That means that gluon production is favored.

The three diagrams Rd), Ld) and QLd) vanish due to the momentum structure of their propagators [6].

To see the purpose of the remaining ten diagrams [6], i.e. Rb), Rc), Lb), Lc), Cc), Cd), Ec), Ed), QLb) and QLc), we investigate first the process $q\bar{q} \rightarrow g$ which is shown up to order $g^3 m^3$ in Fig.1. $|\mathcal{M}_{q\bar{q} \rightarrow g}|^2$ was given in lowest order by the Fock term. To construct it up to order $g^4 m^4$, one has to take the hermitian conjugate of the scattering amplitude of Fig.1a) and multiply it with each of the scattering amplitudes of Fig.1b)-g). It is also necessary to take the hermitian conjugate of each of these products, too. Using a symbolical notation, we write these products as $a^\dagger b, ab^\dagger, a^\dagger c, \ldots$. Then a detailed analysis shows that each of these products, with the
exception of $a^\dagger g$ and $ag^\dagger$, is provided by one of the above mentioned ten self-energy diagrams. The diagram $g$ of Fig.1 does not enter into the collision integral, as it is a renormalization diagram for the incoming quark, for which the momentum $p$ is fixed externally.

So far, we have constructed the collision integral out of self-energy diagrams with graphs up to two loops. Let us now return to the transport equation (9). If we integrate the left hand side over the interval $\Delta^+$ and neglect the contributions of $I_A$ and $I_R$ on the right hand side, then we get for the transport equation up to order $g^4 m^4$

$$2p^\mu \partial_{X^\mu} f_q(X, \vec{p}) = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(p + p_1 - p_2)$$

$$\times |M_{qq\rightarrow g}|^2 \left\{ \tilde{f}_q(X, \vec{p}) \tilde{f}_g(X, \vec{p}_1) f_q(X, \vec{p}_2) - f_q(X, \vec{p}) f_q(X, \vec{p}_1) \tilde{f}_g(X, \vec{p}_2) \right\}$$

$$+ \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3} (2\pi)^4 \delta^{(4)}(p + p_1 - p_2 - p_3)$$

$$\times \left[ \frac{1}{2} |M_{qg\rightarrow gg}|^2 \left\{ \tilde{f}_q(X, \vec{p}) \tilde{f}_q(X, \vec{p}_1) f_g(X, \vec{p}_2) f_g(X, \vec{p}_3) \right. \right.$$  

$$- f_q(X, \vec{p}) f_q(X, \vec{p}_1) \tilde{f}_g(X, \vec{p}_2) \tilde{f}_g(X, \vec{p}_3) \right\}$$

$$+ |M_{qg\rightarrow qg}|^2 \left\{ \tilde{f}_q(X, \vec{p}) \tilde{f}_q(X, \vec{p}_1) f_q(X, \vec{p}_2) f_q(X, \vec{p}_3) \right.$$

$$- f_q(X, \vec{p}) f_q(X, \vec{p}_1) \tilde{f}_q(X, \vec{p}_2) \tilde{f}_q(X, \vec{p}_3) \right\} \right]. \quad (12)$$

The generalization to higher orders is then obvious. Including, e.g., the three loop self-energy into the collision term leads to cross sections of all kinematically allowed processes with two (three) partons in the initial state and three (two) partons in the final state. Furthermore, they give all additional corrections of order $g^6 m^6$ to the process $q\bar{q} \rightarrow g$, e.g. the product of scattering amplitudes of the diagrams b)-f) in Fig.2, and also corrections to the scattering processes of two partons into two partons.

### 3 Closing comments

One may speculate that pinch singularities could enter into the collision integral up to the two loop level. In fact this is not so, and has been explicitly demonstrated [3].

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