A regularity criterion of the 3D MHD equations involving one velocity and one current density component in Lorentz space

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Abstract. In this paper, we study the regularity criterion of weak solutions to the three-dimensional (3D) MHD equations. It is proved that the solution \((u, b)\) becomes regular provided that one velocity and one current density component of the solution satisfy

\[
\begin{align*}
u_3 & \in \mathcal{L}^{\frac{30}{7-45\alpha}}(0, T; L^{\alpha, \infty}(\mathbb{R}^3)) \quad \text{with} \quad \frac{45}{7} \leq \alpha \leq \infty, \\
j_3 & \in \mathcal{L}^{\frac{2\beta}{2-3\beta}}(0, T; L^{\beta, \infty}(\mathbb{R}^3)) \quad \text{with} \quad \frac{3}{2} \leq \beta \leq \infty,
\end{align*}
\]

which generalize some known results.

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1. Introduction

This paper deals with the well-known problem of the regularity of the solutions for the 3D magnetohydrodynamical (MHD) system

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - (b \cdot \nabla) b - \Delta u + \nabla \pi &= 0, \\
\partial_t b + u \cdot \nabla b - b \cdot \nabla u - \Delta b &= 0, \\
\nabla \cdot u &= \nabla \cdot b = 0, \\
\end{align*}
\]

where \(u = (u_1, u_2, u_3)\) is the velocity field, \(b = (b_1, b_2, b_3)\) is the magnetic field and \(\pi\) is the scalar pressure, while \(u_0\) and \(b_0\) are the corresponding initial data satisfying \(\nabla \cdot u_0 = \nabla \cdot b_0 = 0\) in the sense of distribution.

Since Duvaut-Lions [9] and Sermange-Temam [25] constructed the so-called well-known weak solution \((u, b)(x, t)\) of the incompressible MHD equation for arbitrary \((u_0, b_0) \in L^2(\mathbb{R}^3)\) with \(\nabla \cdot u_0(x) = \nabla \cdot b_0(x) = 0\) in last century, the problem on the uniqueness and regularity of the weak solutions is one of the most challenging problems of the mathematical community. Hence, many researchers have developed different regularity criteria for the 3D MHD equations under assumption of certain growth conditions on the velocity or on the magnetic field (see, e.g., [3–5, 7, 8, 10, 14–16, 27–30, 37, 38] and the references therein).

Recent years, the problem of so-called regularity criteria via one components was investigated for the MHD equations by some researchers (see [1, 11–13, 17–21, 23, 32, 33, 36] and the references therein). In particular, in [31], Yamazaki established the following regularity criterion by involving one velocity and one current density component, which shows that a weak solution \((u, b)\) is smooth on a time interval \((0, T]\) if

\[
u_3 \in \mathcal{L}^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} \leq \frac{1}{3} + \frac{1}{2q}, \quad \frac{15}{2} < q \leq \infty,
\]

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and
\[ j_3 \in L^{p'}(0, T; L^{p'}(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p'} + \frac{3}{q'} \leq 2, \quad \frac{3}{2} < q' \leq \infty, \]
where \( j_3 \) is the third component of the current density \( j = \nabla \times b = (j_1, j_2, j_3) \). Later, Zhang [34] improved the regularity criterion (1.2) to the following conditions
\[ u_3 \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = \frac{4}{9} - \frac{15}{2q}, \quad 15 \leq q \leq \infty, \] (1.3)
and
\[ j_3 \in L^{p'}(0, T; L^{p'}(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p'} + \frac{3}{q'} \leq 2, \quad \frac{3}{2} < q' \leq \infty. \]
Very recently, this result (1.3) is further refined by Zhang [35] to prove the regularity criterion as long as the following conditions
\[ u_3 \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = \frac{4}{9} + \frac{27}{4} \leq q \leq \infty, \] (1.4)
are satisfied.

Motivated by the papers [31,34,35], the purpose of the present paper is to refine (1.4) and to extend the above regularity criterion to the Lorentz space \( L^{\alpha,\infty} \) which is larger than \( L^\alpha \). More precisely, our main result now read as follows.

**Theorem 1.1.** Suppose \( T > 0 \), \((u_0, b_0) \in L^2(\mathbb{R}^3)\) and \( \nabla \cdot u_0 = \nabla \cdot b_0 = 0 \) in the sense of distributions. Assume that \((u, b)\) is a weak solution of the 3D MHD equations (1.1) on \((0, T)\). If \( u_3 \) and \( j_3 \) satisfy the following growth conditions
\[ \int_0^T \left( \|u_3(\tau)\|^{\frac{30\alpha}{10 - 4\beta}}_{L^{\alpha,\infty}} + \|j_3(\tau)\|^{\frac{2\beta}{2\beta - 3}}_{L^{\beta,\infty}} \right) d\tau < \infty, \] (1.5)
where \( \frac{45}{7} \leq \alpha \leq \infty \) and \( \frac{3}{2} < \beta \leq \infty \), then the weak solution \((u, b)\) is regular on \((0; T)\).

**Remark 1.1.** Theorem 1.1 extends the previous results on Navier–Stokes equations due to the fact that the MHD equations with \( b(x, t) = 0 \) reduce the Navier–Stokes equations. According to the embedding relation \( L^\alpha \subseteq L^{\alpha,\infty} \), it is easy to see that our result of Theorem 1.1 is an improvement in the recent works by Yamazaki [31] and Zhang [34,35].

## 2. Preliminaries

Throughout this paper, we use the following usual notations. \( L^p(\mathbb{R}^3) \) denotes the Lebesgue space associated with norm
\[ \|f\|_{L^p} = \begin{cases} \left( \int_{\mathbb{R}^3} |f(x)|^p \, dx \right)^{\frac{1}{p}}, & \text{for } 1 \leq p < \infty, \\ \text{ess sup} \|f(x)\|_{L^2}, & \text{for } p = \infty. \end{cases} \]
\( H^k(\mathbb{R}^3) \) denotes the Hilbert space \( \{ u \in L^2(\mathbb{R}^3) : \|\nabla^k u\|_{L^2} < \infty \} \). Let \((X, \mathcal{M}, \mu)\) be a non-atomic measurable space. For a complex- or real-valued \( \mu \)-measurable function \( f(x) \) defined on \( X \), its distributional function is defined by
\[ f_*(\sigma) = \mu \{ x \in X : f(x) > \sigma \}, \quad \text{for } \sigma > 0, \]
which is non-increasing and continuous from the right. Furthermore, its non-increasing rearrangement \( f^* \) is defined by
\[ f^*(t) = \inf \{ s > 0 : f_*(s) \leq t \}, \quad \text{for } t > 0, \]
which is also non-increasing and continuous from the right and has the same distributional function as $f(x)$.

The Lorentz space $L^{p,q}$ on $(X,\mathcal{M},\mu)$ is the collection of all real- or complex-valued $\mu$–measurable functions $f(x)$ defined on $X$ such that $\|f\|_{L^{p,q}} < \infty$, where

$$
\|f\|_{L^{p,q}} = \begin{cases}
\left( \frac{1}{q} \int_0^\infty \left( \frac{1}{p} \int_0^x (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t} \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}, & \text{if } 1 \leq p < \infty, \ 1 < q < \infty \\
\sup_{t>0} (t^{\frac{1}{p}} f^*(t)), & \text{if } 1 \leq p < \infty, \ q = \infty.
\end{cases}
$$

Moreover,

$$
\|f\|_{L^{p,\infty}} = \sup_{t>0} (t^{\frac{1}{p}} f^*(t)) = \sup_{\sigma>0} (f_*^*(\sigma))^{\frac{1}{p}}
$$

for any $f \in L^{p,\infty}$. For details, we refer to [2] and [26].

The space definition implies the following continuous embeddings:

$$
L^p(\mathbb{R}^3) = L^{p,p}(\mathbb{R}^3) \hookrightarrow L^{p,q}(\mathbb{R}^3) \hookrightarrow L^{p,\infty}(\mathbb{R}^3), \quad 1 \leq p \leq q < \infty.
$$

In order to prove Theorem 1.1, we need the following Hölder inequality in Lorentz spaces (see, e.g., O’Neil [24] and [22]).

**Lemma 2.1.** ([24], Theorems 3.4 and 3.5) Let $f \in L^{p_2,q_2}(\mathbb{R}^3)$ and $g \in L^{p_3,q_3}(\mathbb{R}^3)$ with $1 \leq p_2, p_3 \leq \infty, 1 \leq q_2, q_3 \leq \infty$. Then, $fg \in L^{p_1,q_1}(\mathbb{R}^3)$ with

$$
\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3}, \quad \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}
$$

and the Hölder inequality of Lorentz spaces

$$
\|fg\|_{L^{p_1,q_1}} \leq C \|f\|_{L^{p_2,q_2}} \|g\|_{L^{p_3,q_3}},
$$

holds true for a positive constant $C$.

We also recall Gagliardo–Nirenberg inequality in Lorentz spaces which plays an important role in the proofs of Theorem 1.1.

**Lemma 2.2.** Let $f \in L^{p,q}(\mathbb{R}^3)$ with $1 \leq p, q, p_4, q_4, p_5, q_5 \leq \infty$. Then, the Gagliardo–Nirenberg inequality of Lorentz spaces

$$
\|f\|_{L^{p,q}} \leq C \|f\|_{L^{p_4,q_4}} \|f\|_{L^{p_5,q_5}}^{1-\theta}
$$

holds for a positive constant $C$ and

$$
\frac{1}{p} = \frac{\theta}{p_4} + \frac{1-\theta}{p_5}, \quad \frac{1}{q} = \frac{\theta}{q_4} + \frac{1-\theta}{q_5}, \quad \theta \in (0,1).
$$

3. Proof of main result

In this section, under the assumptions of Theorem 1.1, we prove our main result. Before proving our result, we recall the following multiplicative Sobolev imbedding inequality in the whole space $\mathbb{R}^3$ (see, for example, [6]) :

$$
\|f\|_{L^6} \leq C \|\nabla_h f\|_{L^2}^2 \|\partial_3 f\|_{L^2} \frac{1}{2}, \quad (3.1)
$$

where $\nabla_h = (\partial_{x_1}, \partial_{x_2})$ is the horizontal gradient operator. We are now given the proof of our main theorem.
Proof. To prove our result, it suffices to show that for any fixed \( T > T^* \), there holds
\[
\sup_{0 \leq t \leq T^*} \| \nabla u(t) \|^2_{L^2} + \| \nabla b(t) \|^2_{L^2} \leq C_T,
\]
where \( T^* \), which denotes the maximal existence time of a strong solution and \( C_T \) is an absolute constant which only depends on \( T, u_0 \) and \( b_0 \).

The method of our proof is the standard energy estimates as in [31]. We will base on two major parts. The first one establishes the bounds of \((\| \nabla u \|^2_{L^2} + \| \nabla b \|^2_{L^2})\), while the second gives the bounds of the \( H^1 \)-norm of velocity \( u \) and magnetic field \( b \) in terms of the results of part one.

For this purpose, we multiply the first and second equations of (1.1) by \(-\Delta_h u\) and \(-\Delta_h b\), respectively, and integrate them over \( \mathbb{R}^3 \) with respect to the spatial variable. Then, integration by parts gives the following identity:
\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla u \|^2_{L^2} + \| \nabla b \|^2_{L^2} \right) + \| \nabla \nabla u \|^2_{L^2} + \| \nabla \nabla b \|^2_{L^2} = \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta_h u dx - \int_{\mathbb{R}^3} (b \cdot \nabla) b \cdot \Delta_h u dx + \int_{\mathbb{R}^3} (u \cdot \nabla) b \cdot \Delta_h b dx - \int_{\mathbb{R}^3} (b \cdot \nabla) u \cdot \Delta_h b dx = \text{RHS},
\]
where \( \Delta_h = \partial^2_{x_1} + \partial^2_{x_2} \) is the horizontal Laplacian. For simplicity of exposition, we denote
\[
\mathcal{L}^2(t) = \sup_{\tau \in [\Gamma, t]} (\| \nabla u(\tau) \|^2_{L^2} + \| \nabla b(\tau) \|^2_{L^2}) + \int_{\Gamma}^t (\| \nabla \nabla u(\tau) \|^2_{L^2} + \| \nabla \nabla b(\tau) \|^2_{L^2}) d\tau,
\]
\[
\mathcal{J}^2(t) = \sup_{\tau \in [\Gamma, t]} (\| \nabla u(\tau) \|^2_{L^2} + \| \nabla b(\tau) \|^2_{L^2}) + \int_{\Gamma}^t (\| \Delta u(\tau) \|^2_{L^2} + \| \Delta b(\tau) \|^2_{L^2}) d\tau,
\]
for \( t \in [\Gamma, T^*] \). We choose \( \epsilon, \eta > 0 \) to be precisely determined subsequently and then select \( \Gamma < T^* \) sufficiently close to \( T^* \) such that for all \( \Gamma \leq t < T^* \),
\[
\int_{\Gamma}^t (\| \nabla u(\tau) \|^2_{L^2} + \| \nabla b(\tau) \|^2_{L^2}) d\tau \leq \epsilon \ll 1 \quad \text{and} \quad \int_{\Gamma}^t \| j_3(\tau) \|^2_{L^2} d\tau \leq \eta \ll 1. \tag{3.2}
\]
Applying the divergence-free condition, \( \nabla \cdot u = \nabla \cdot b = 0 \), we find that RHS can be estimated as
\[
\text{RHS} \leq \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla \nabla u| dx + \int_{\mathbb{R}^3} |u_3| |\nabla b| |\nabla \nabla b| dx + \int_{\mathbb{R}^3} |b_3| |\nabla u| |\nabla \nabla b| dx + \int_{\mathbb{R}^3} |b_3| |\nabla b| |\nabla \nabla u| dx + \int_{\mathbb{R}^3} |\nabla u| |\nabla b| j_3 | dx
\]
\[
= L_1 + L_2 + L_3 + L_4 + L_5, \tag{3.3}
\]
where the last inequality was proved in [31] (see, Proposition 3.1 in [31] for details).

With the use of Lemma 2.1, (3.1), and the Young inequality, we derive the estimate of the first term \( L_1 \) of (3.3) as follows:
\[
L_1 \leq C \| u_3 \|_{L^{\infty}} \| \nabla u \|_{L^{\frac{2n}{n-2}}_{x,2}} \| \nabla \nabla u \|_{L^2} \]
\[
\leq C \| u_3 \|_{L^{\infty}} \| \nabla u \|_{L^4_x}^{1-\frac{n}{2}} \| \nabla u \|_{L^6}^{\frac{n}{2}} \| \nabla \nabla u \|_{L^2}.
\]
\[ \leq C \| u_3 \|_{L^{p, \infty}} \| \nabla u \|_{L^2}^{1 - \frac{2}{\alpha}} \| \Delta u \|_{L^2}^{\frac{1}{2}} \| \nabla \nabla_h u \|_{L^2}^{1 + \frac{2}{\alpha}} \]
\[ \leq C \| u_3 \|_{L^{p, \infty}} \| \nabla u \|_{L^2}^{2 - \frac{2}{\alpha} - \frac{2}{\beta}} \| \Delta u \|_{L^2}^{\frac{2}{\alpha}} + \frac{1}{8} \| \nabla \nabla_h u \|_{L^2}^2, \]

where we have used the following Gagliardo–Nirenberg inequality in Lorentz spaces:
\[ \| \nabla u \|_{L^\frac{2}{\alpha - 2}} \leq C \| \nabla u \|_{L^2}^{1 - \frac{2}{\alpha}} \| \nabla u \|_{L^\infty}. \]

Similarly, employing the Hölder inequality and the Gagliardo–Nirenberg inequality gives that for \( L_1 \),
\[ L_2 \leq C \| u_3 \|_{L^{p, \infty}} \| \nabla b \|_{L^2}^{2 - \frac{2}{\alpha} - \frac{2}{\beta}} \| \Delta b \|_{L^2}^{\frac{2}{\alpha}} + \frac{1}{8} \| \nabla \nabla_h b \|_{L^2}^2. \]

We now estimate \( L_3 \),
\[ L_3 \leq \| b_3 \|_{L^{p, \infty}} \| \nabla u \|_{L^2}^{\frac{1}{2}} \| \nabla \nabla_h b \|_{L^2} \leq C \| b_3 \|_{L^{p, \infty}} \| \nabla u \|_{L^2}^{\frac{2}{7}} \| \nabla \nabla_h b \|_{L^2} \]
\[ \leq C \| b_3 \|_{L^{p, \infty}} \| \nabla \nabla_h b \|_{L^2} \| \Delta u \|_{L^2}^{\frac{1}{2}} \| \nabla \nabla_h b \|_{L^2} \]
\[ \leq C \| b_3 \|_{L^{p, \infty}} \| \nabla \nabla_h b \|_{L^2} \| \Delta u \|_{L^2}^{\frac{1}{2}} + \frac{1}{8} (\| \nabla \nabla_h b \|_{L^2}^2 + \| \nabla \nabla_h b \|_{L^2}^2), \]

where we have used the fact \( \| \nabla u \|_{L^2}^{\frac{1}{2}} \leq C \| \nabla u \|_{L^2}^{\frac{2}{7}} \| \nabla u \|_{L^\infty}. \)

Likewise,
\[ L_4 \leq C \| b_3 \|_{L^{p, \infty}} \| \nabla b \|_{L^2}^{\frac{1}{2}} \| \Delta b \|_{L^2}^{\frac{1}{2}} + \frac{1}{8} (\| \nabla \nabla_h b \|_{L^2}^2 + \| \nabla \nabla_h b \|_{L^2}^2). \]

For \( L_5 \), applying the Hölder inequality, the Gagliardo–Nirenberg inequality and the Young inequality, one shows that
\[ L_5 = \int_{\mathbb{R}^3} |\nabla_h u| |\nabla_h b| |j_3| \, dx \leq \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla_h u|^2 + |\nabla_h b|^2 \right) |j_3| \, dx \]
\[ \leq C \| j_3 \|_{L^{p, \infty}} \left( \| \nabla_h u \|_{L^2}^{2 - \frac{2}{\alpha} - \frac{2}{\beta}} \| \nabla \nabla_h b \|_{L^2} + \| \nabla \nabla_h u \|_{L^2}^{2 - \frac{2}{\alpha} - \frac{2}{\beta}} \| \nabla h b \|_{L^2} \right) \]
\[ \leq C \| j_3 \|_{L^{p, \infty}} \left( \| \nabla_h u \|_{L^2}^{2 - \frac{2}{\alpha} - \frac{2}{\beta}} + \| \nabla \nabla_h u \|_{L^2}^{2 - \frac{2}{\alpha} - \frac{2}{\beta}} \| \nabla h b \|_{L^2}^{\frac{1}{2}} \right) \]
\[ \leq C \| j_3 \|_{L^{p, \infty}} \left( |\nabla_h u|_{L^2}^2 + |\nabla_h b|_{L^2}^2 \right) + \frac{1}{8} (\| \nabla \nabla_h b \|_{L^2}^2 + \| \nabla \nabla_h u \|_{L^2}^2). \]

Inserting all the estimates into (3.3), Gronwall’s type argument using
\[ 1 \leq \sup_{\lambda \in [\Gamma, T]} \exp \left( c \int_{\lambda}^{T} \| j_3 (\varphi) \|_{L^{p, \infty}}^{2 - \frac{2}{\alpha} - \frac{2}{\beta}} \, d\varphi \right) \leq \exp \left( c \int_{0}^{T} \| j_3 (\varphi) \|_{L^{p, \infty}}^{2 - \frac{2}{\alpha} - \frac{2}{\beta}} \, d\varphi \right) \leq 1, \]
due to (1.5) leads to, for every \( \tau \in [\Gamma, t] \)
\[ \mathcal{L}^2 (t) \leq C + C \int_{\Gamma}^{t} \| u_3 \|_{L^{p, \infty}} \left( |\nabla u|_{L^2}^{2 - \frac{2}{\alpha} - \frac{2}{\beta}} + |\nabla \nabla_h u|_{L^2}^{\frac{2}{\alpha}} \| \Delta b \|_{L^2}^{\frac{2}{\alpha}} \| \nabla \nabla_h b \|_{L^2}^{\frac{2}{\alpha}} \right) \, d\tau \]
\[ + C \int_{\Gamma}^{t} \| b_3 \|_{L^{p, \infty}} \left( |\nabla u|_{L^2}^{\frac{1}{2}} \| \Delta u \|_{L^2}^{\frac{1}{2}} + |\nabla b|_{L^2}^{\frac{1}{2}} \| \Delta b \|_{L^2}^{\frac{1}{2}} \right) \, d\tau \]
\[ + C \int_{\Gamma}^{t} \| j_3 \|_{L^{p, \infty}} \left( |\nabla u|_{L^2}^{2} + |\nabla b|_{L^2}^{2} \right) \, d\tau \]
\[ = C + I_1 (t) + I_2 (t) + I_3 (t). \]  

(3.4)
Next, we analyze the right-hand side of (3.4) one by one. First, due to (3.2) and the definition of $J^2$, we have

$$I_1(t) \leq C \left( \sup_{\tau \in [\Gamma, t]} \| \nabla u(\tau) \|_{L^2}^{\frac{3}{2} - \frac{3}{4}} \right)^{\frac{3}{2} - \frac{1}{4}} \int_{\Gamma} \| u_3(\tau) \|_{\frac{20}{3} L^{\frac{20}{3}, \infty}} \| \nabla u(\tau) \|_{L^2} \| \Delta u(\tau) \|_{L^2}^{\frac{1}{2}} d\tau$$

$$+ C \left( \sup_{\tau \in [\Gamma, t]} \| \nabla b(\tau) \|_{L^2}^{\frac{3}{2} - \frac{3}{4}} \right)^{\frac{3}{2} - \frac{1}{4}} \int_{\Gamma} \| u_3(\tau) \|_{\frac{20}{3} L^{\frac{20}{3}, \infty}} \| \nabla b(\tau) \|_{L^2} \| \Delta b(\tau) \|_{L^2}^{\frac{1}{2}} d\tau$$

$$\leq C J^{\frac{3}{2} - \frac{3}{4}}(t) \left( \int_{\Gamma} \| u_3(\tau) \|_{\frac{20}{3} L^{\frac{20}{3}, \infty}} \| \nabla u(\tau) \|_{L^2} \| \Delta u(\tau) \|_{L^2} d\tau \right)^{\frac{3}{4} - \frac{1}{4}} \left( \int_{\Gamma} \| \nabla u(\tau) \|_{L^2}^{\frac{1}{2}} d\tau \right)^{\frac{3}{4} - \frac{1}{4}} \left( \int_{\Gamma} \| \Delta u(\tau) \|_{L^2}^{\frac{1}{2}} d\tau \right)^{\frac{3}{4} - \frac{1}{4}}$$

$$+ C J^{\frac{3}{2} - \frac{3}{4}}(t) \left( \int_{\Gamma} \| u_3(\tau) \|_{\frac{20}{3} L^{\frac{20}{3}, \infty}} \| \nabla b(\tau) \|_{L^2} \| \Delta b(\tau) \|_{L^2} d\tau \right)^{\frac{3}{4} - \frac{1}{4}} \left( \int_{\Gamma} \| \nabla b(\tau) \|_{L^2}^{\frac{1}{2}} d\tau \right)^{\frac{3}{4} - \frac{1}{4}} \left( \int_{\Gamma} \| \Delta b(\tau) \|_{L^2}^{\frac{1}{2}} d\tau \right)^{\frac{3}{4} - \frac{1}{4}}$$

$$\leq C J^{\frac{3}{2} - \frac{3}{4}}(t) \left( \int_{\Gamma} \| u_3(\tau) \|_{\frac{20}{3} L^{\frac{20}{3}, \infty}} \| \nabla u(\tau) \|_{L^2} \| \Delta u(\tau) \|_{L^2} d\tau \right)^{\frac{3}{4} - \frac{1}{4}} \left( \int_{\Gamma} \| \nabla u(\tau) \|_{L^2}^{\frac{1}{2}} d\tau \right)^{\frac{3}{4} - \frac{1}{4}} \left( \int_{\Gamma} \| \Delta u(\tau) \|_{L^2}^{\frac{1}{2}} d\tau \right)^{\frac{3}{4} - \frac{1}{4}}$$

Now, we estimate the term $I_2(t)$ as

$$I_2(t) \leq C \left( \sup_{\tau \in [\Gamma, t]} \| b_3(\tau) \|_{L^1}^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \int_{\Gamma} \| \nabla u(\tau) \|_{L^2} \| \Delta u(\tau) \|_{L^2} d\tau \right)^{\frac{1}{8}}$$

$$+ \left( \sup_{\tau \in [\Gamma, t]} \| b_3(\tau) \|_{L^1}^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \int_{\Gamma} \| \nabla b(\tau) \|_{L^2} \| \Delta b(\tau) \|_{L^2} d\tau \right)^{\frac{1}{8}}$$

$$\leq \left( \sup_{\tau \in [\Gamma, t]} \| b_3(\tau) \|_{L^1}^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \int_{\Gamma} \| \nabla u(\tau) \|_{L^2} \| \Delta u(\tau) \|_{L^2} d\tau \right)^{\frac{1}{8}} \left( \int_{\Gamma} \| \Delta u(\tau) \|_{L^2} d\tau \right)^{\frac{1}{8}}$$

$$+ \left( \sup_{\tau \in [\Gamma, t]} \| b_3(\tau) \|_{L^1}^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \int_{\Gamma} \| \nabla b(\tau) \|_{L^2} \| \Delta b(\tau) \|_{L^2} d\tau \right)^{\frac{1}{8}} \left( \int_{\Gamma} \| \Delta b(\tau) \|_{L^2} d\tau \right)^{\frac{1}{8}}$$

$$\leq C \left( \sup_{\tau \in [\Gamma, t]} \| b_3(\tau) \|_{L^1}^{\frac{1}{2}} \right)^{\frac{1}{2}} \epsilon^{\frac{1}{8}} J^{\frac{1}{8}}(t).$$

For $I_3(t)$, applying Hölder’s and Young’s inequalities, we get

$$I_3(t) \leq C \sup_{\tau \in [\Gamma, t]} \left( \| \nabla u(\tau) \|_{L^2}^2 + \| \nabla b(\tau) \|_{L^2}^2 \right) \int_{\Gamma} \| \Delta u(\tau) \|_{L^{\frac{20}{3}, \infty}} d\tau$$

$$\leq C \eta L^2(t).$$
Therefore, combining the estimates of $I_1(t), I_2(t)$ and $I_3(t)$ together with (3.4) and taking $\eta$ small enough, it is easy to see that for all $\Gamma \leq t < T^*$:

\[
L^2(t) \leq C + C\epsilon^{\frac{1}{2}} J^\frac{3}{2}(t) \left( \int_{\Gamma}^{t} \|u_3(\tau)\|_{ \frac{8\alpha}{3(4\alpha-5)} } \, d\tau \right)^{\frac{3\alpha-10}{4(\alpha-2)}} + C \left( \sup_{\tau \in [\Gamma,t]} \|b_3(\tau)\|_{ \frac{5}{L_{10}^2} } \right) \epsilon \frac{7}{2} J^\frac{3}{2}(t) \tag{3.5}
\]

Now, we will establish the bounds of $L^{10}$-norm of the magnetic field $b_3$. In order to do it, we recall the third equation of the magnetic field:

\[ \partial_t b_3 - \Delta b_3 + (u \cdot \nabla) b_3 = (b \cdot \nabla) u_3, \]

and multiply this equation by $|b_3|^8$, integrating by parts, using incompressibility conditions to obtain

\[
\frac{1}{10} \frac{d}{dt} \int_{\mathbb{R}^3} |b_3|^{10} \, dx + \frac{9}{25} \int_{\mathbb{R}^3} |\nabla(b_5^2)|^2 \, dx = \int_{\mathbb{R}^3} (b \cdot \nabla u_3)(|b_3|^8 b_3) \, dx
\]

\[
= -9 \int_{\mathbb{R}^3} b \cdot |b_3|^4 (|b_3|^4 \nabla b_3) u_3 \, dx
\]

\[
\leq \frac{9}{5} \int_{\mathbb{R}^3} |b| (|b_3|^5)^\frac{4}{5} |u_3| |\nabla(b_3^5)| \, dx = I. \tag{3.6}
\]

Using the Hölder, Young inequalities and interpolation, the estimates of $I$ are given by

\[
I \leq \frac{9}{5} \|b\|_{L^\infty} \left\| u_3 \left( |b_3|^5 \right)^\frac{4}{5} \right\|_{L^3} \left\| \nabla(b_3^5) \right\|_{L^2}
\]

\[
\leq C \left\| \nabla h b \right\|_{L^2}^\frac{2}{5} \left\| \nabla b \right\|_{L^2}^\frac{1}{5} \left\| u_3 \right\|_{L^{\alpha \infty}} \left( \|b_3\|_{L^\frac{6(\alpha-5)}{4\alpha-15}}^5 \right)^{\frac{4}{5}} \left\| \nabla(b_3^5) \right\|_{L^2}
\]

\[
\leq C \left\| \nabla h b \right\|_{L^\frac{20\alpha}{4\alpha-15}} \left\| \nabla b \right\|_{L^\frac{10\alpha}{4\alpha-15}} \left\| u_3 \right\|_{L^{\alpha \infty}} \left\| b_3 \right\|_{L^\frac{30(\alpha-5)}{4\alpha-15}}.
\]

Putting $I$ in (3.6), we get

\[
\frac{d}{dt} \int_{\mathbb{R}^3} |b_3|^{10} \, dx \leq C \left\| \nabla h b \right\|_{L^\frac{20\alpha}{4\alpha-15}} \left\| \nabla b \right\|_{L^\frac{10\alpha}{4\alpha-15}} \left\| u_3 \right\|_{L^{\alpha \infty}} \left\| b_3 \right\|_{L^\frac{30(\alpha-5)}{4\alpha-15}}.
\]

Dividing by $\|b_3\|_{L_{10}^{10}}$, we arrive at

\[
\frac{d}{dt} \|b_3\|_{L_{10}^{10}}^{\frac{10\alpha}{4\alpha-15}} \leq C \left\| \nabla h b \right\|_{L^\frac{20\alpha}{4\alpha-15}} \left\| \nabla b \right\|_{L^\frac{10\alpha}{4\alpha-15}} \left\| u_3 \right\|_{L^{\alpha \infty}} \left\| b_3 \right\|_{L^\frac{30(\alpha-5)}{4\alpha-15}}.
\]

Integrating over interval $[\Gamma, \tau]$, it follows that

\[
\|b_3(\tau)\|_{L_{10}^{10}} \leq \left[ \|b_3(\Gamma)\|_{L_{10}^{10}}^{\frac{10\alpha}{4\alpha-15}} + C \int_{\Gamma}^{\tau} \left( \left\| \nabla h(b(\lambda)) \right\|_{L^\frac{20\alpha}{4\alpha-15}} \left\| \nabla(b(\lambda)) \right\|_{L^\frac{10\alpha}{4\alpha-15}} \left\| u_3(\lambda) \right\|_{L^{\alpha \infty}} \right) d\lambda \right]^{\frac{4\alpha-15}{10\alpha}} , \tag{3.7}
\]
for all $\tau \in [\Gamma, t)$. It follows from (3.7) and (3.5) that

$$
\mathcal{L}^2(t) \leq C + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t) \left( \int_\Gamma \|u_3(\tau)\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} d\tau \right) + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t) \left( \int_\Gamma \left[ \frac{4\alpha-15}{4\alpha} \right] d\tau \right) + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t) \left( \int_\Gamma \|u_3(\tau)\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} d\tau \right) + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t) \left( \int_\Gamma \|u_3(\tau)\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} d\tau \right)
$$

$$
\leq C + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t) \left( \int_\Gamma \|u_3(\tau)\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} d\tau \right) + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t) \left( \int_\Gamma \|u_3(\tau)\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} d\tau \right)
$$

$$
\leq C + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t) \left( \int_\Gamma \|u_3(\tau)\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} d\tau \right) + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t) \left( \int_\Gamma \|u_3(\tau)\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} d\tau \right)
$$

$$
\leq C + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t) \left( \int_\Gamma \|u_3(\tau)\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} d\tau \right) + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t) \left( \int_\Gamma \|u_3(\tau)\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} d\tau \right)
$$

which leads to

$$
\mathcal{L}^2(t) \leq C + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t) + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t) + C \epsilon^\frac{7}{2} \mathcal{J}^\frac{2}{5}(t).
$$

(3.8)

Now, we will establish the bounds of $H^1$-norm of the velocity and magnetic field. In order to do it, we multiply the first and second equations of (1.1) by $-\Delta u$ and $-\Delta b$, respectively, and integrate them over $\mathbb{R}^3$ with respect to the spatial variable. Then, integration by parts gives the following identity:

$$
\frac{1}{2} \frac{d}{dt} \left( \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) + \|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2
$$
Applying the divergence-free condition, \( \nabla \cdot u = \nabla \cdot b = 0 \), by using the Hölder inequality, the interpolation inequality and (3.1), it follows that

\[
\frac{1}{2} \frac{d}{dt} (\|u\|^2_{L^2} + \|b\|^2_{L^2}) + \|\Delta u\|^2_{L^2} + \|\Delta b\|^2_{L^2} \\
\leq C \int_{\mathbb{R}^3} (|\nabla x u|^2 + |\nabla x b|^2) \, dx \\
\leq C (\|\nabla x u\|^2_{L^2} + \|\nabla x b\|^2_{L^2}) (\|u\|^2_{L^4} + \|b\|^2_{L^8}) \\
\leq C (\|\nabla x u\|^2_{L^2} + \|\nabla x b\|^2_{L^2}) (\|u\|^2_{L^2} \|\nabla u\|^2_{L^2} + \|b\|^2_{L^2} \|\nabla b\|^2_{L^2}) \\
\leq C (\|\nabla x u\|^2_{L^2} + \|\nabla x b\|^2_{L^2}) (\|u\|^2_{L^2} \|\nabla u\|^2_{L^2} \|\Delta u\|^2_{L^2} + \|b\|^2_{L^2} \|\nabla b\|^2_{L^2} \|\Delta b\|^2_{L^2}).
\]

Integrating this last inequality in time, we deduce that for all \( \tau \in [\Gamma, t] \)

\[
\mathcal{J}^2(t) \leq \|\nabla u(\Gamma)\|^2_{L^2} + \|\nabla b(\Gamma)\|^2_{L^2} + C \sup_{\tau \in [\Gamma, t]} (\|\nabla x u(\tau)\|_{L^2} + \|\nabla x b(\tau)\|_{L^2}) \\
\times \left( \int_{\Gamma}^{t} \|\nabla u(\tau)\|^2_{L^2} \, d\tau \right) \left( \int_{\Gamma}^{t} \|\nabla x u(\tau)\|^2_{L^2} \, d\tau \right) \left( \int_{\Gamma}^{t} \|\Delta u(\tau)\|^2_{L^2} \, d\tau \right) \left( \int_{\Gamma}^{t} \|\Delta b(\tau)\|^2_{L^2} \, d\tau \right) \\
+ C \sup_{\tau \in [\Gamma, t]} (\|\nabla x u(\tau)\|_{L^2} + \|\nabla x b(\tau)\|_{L^2}) \\
\times \left( \int_{\Gamma}^{t} \|\nabla b(\tau)\|^2_{L^2} \, d\tau \right) \left( \int_{\Gamma}^{t} \|\nabla x b(\tau)\|^2_{L^2} \, d\tau \right) \left( \int_{\Gamma}^{t} \|\Delta b(\tau)\|^2_{L^2} \, d\tau \right) \\
\leq \|\nabla u(\Gamma)\|^2_{L^2} + \|\nabla b(\Gamma)\|^2_{L^2} + 2C \mathcal{L}(t) \epsilon^2 \mathcal{L}(t) \mathcal{J}^{1/2} (t) \\
= \|\nabla u(\Gamma)\|^2_{L^2} + \|\nabla b(\Gamma)\|^2_{L^2} + C \epsilon^{1/2} \mathcal{L}^2(t) \mathcal{J}^{1/2} (t). \tag{3.10}
\]

Inserting (3.8) into (3.10) and taking \( \epsilon \) small enough, then it is easy to see that for all \( \Gamma \leq t < T^* \), there holds

\[
\mathcal{J}^2(t) \leq \|\nabla u(\Gamma)\|^2_{L^2} + \|\nabla b(\Gamma)\|^2_{L^2} + C \epsilon^{1/2} \mathcal{J}^{1/2} (t) + C \epsilon^{1/2} \mathcal{J}^2 (t) \\
+ C \|b_3(\Gamma)\|_{L^{10}}^{5} \|\nabla b(\Gamma)\|_{L^2}^{2} \mathcal{J}^{3/2} (t) + C \epsilon^{1/2} \mathcal{J}^{3/2} (t) \\
< \infty,
\]

which proves

\[
\sup_{\Gamma \leq t < T^*} \|\nabla u(t)\|^2_{L^2} + \|\nabla b(t)\|^2_{L^2} < \infty.
\]

This completes the proof of Theorem 1.1.  \( \square \)
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