Abstract
Spin-one color superconductor is a viable candidate phase of dense matter in the interiors of compact stars. Its low-energy excitations will influence the transport properties of such matter and thus have impact on late-stage evolution of neutron stars. It also provides a good example of spontaneous symmetry breaking with rich breaking patterns. In this contribution, we reanalyze the phase diagram of a spin-one color superconductor and point out that a part of it is occupied by noninert states, which have been neglected in literature so far. We classify the collective Nambu–Goldstone modes, which are essential to the transport phenomena.

Key words: Color superconductivity, Spontaneous symmetry breaking, Nambu–Goldstone bosons

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1. Introduction
Nuclear matter is expected to undergo deconfinement at extremely high densities such as can be found in the interiors of compact stars. Due to interactions induced by gluon exchange the resulting quark matter will behave as a color superconductor (see [1] for a recent review). At asymptotically high densities the ground state of quark matter composed of the three light quark flavors will be the color-flavor locked (CFL) phase. However, for densities relevant to the phenomenology of neutron stars where the strange quark mass is large enough, pairing of up and down quarks only (2SC) is expected to take over. Nevertheless, the constraint of electric charge neutrality then imposes stress on the pairing by inducing mismatch between the Fermi surfaces of different flavors. If the mismatch is too large, cross-flavor pairing becomes impossible, and pairing of quarks with the same flavor (spin-one pairing) is a candidate phase. Even in the 2SC phase, the strange quarks left over will still have the possibility to pair with themselves, resulting again in a spin-one phase.

The spin-one pairing affects the low-energy spectrum of the system since it gives a gap, albeit small, to quarks that would otherwise remain ungapped. This gives exponential suppression of transport processes at low temperature. The knowledge of the phase structure of spin-one color superconductors is therefore essential for late-stage evolution of neutron stars [2]. Weak-coupling first-principle calculations [3, 4] show that the single-flavor spin-one color superconductor is in the color-spin locked (CSL) state. However, when the spin-one pairing of a single flavor is considered within three-flavor quark matter, for example as a complement to the primary 2SC pairing, the constraint of color neutrality may lead to other patterns of spin-one pairing [5]. We therefore carry out a phenomenological analysis using the Ginzburg–Landau (GL) theory so as to keep the full spectrum of possible phases.

2. Ground state and phase diagram
The spin-one pair transforms as a color antitriplet and spin triplet. Hence the order parameter may be written as a $3 \times 3$ complex matrix $\Delta$. The transformation rule then reads $\Delta \to U^T \Delta R$, where $U \in SU(3)_c \times U(1)_B \equiv U(3)_L$ and $R \in SU(3)_c \times SU(2)_L \times U(1)_Y$.
$R \in \text{SO(3)}_R$. By exploiting this symmetry, the order parameter can always be cast in the form

$$\Delta = \begin{pmatrix} \Delta_1 & ia_3 & -ia_2 \\ -ia_3 & \Delta_2 & ia_1 \\ ia_2 & -ia_1 & \Delta_3 \end{pmatrix},$$  \quad (1)$$

with real parameters $\Delta_i, a_i$. A detailed analysis shows that there are eight possible inequivalent phases, differing in the way the continuous symmetry is spontaneously broken. In addition there are two states, distinguished only by a discrete symmetry, which occupy a part of the phase diagram. For the full list of phases the reader is referred to [6], see also [2]. We note that the symmetry properties of spin-one color superconductors are similar to those of superfluid Helium 3. Our investigation is to some extent inspired, in both methods and terminology, by this well understood system [8].

Up to fourth order in $\Delta$ and two derivatives, the most general $\text{U(3)}_R \times \text{SO(3)}_R$ invariant GL free energy density functional has the form

$$\mathcal{F}[\Delta] = a_1 \text{Tr}(\partial_i \Delta \partial_i \Delta^\dagger) + a_2 (\partial_i \Delta_{\alpha \beta}) (\partial_j \Delta_{\gamma \delta}^\dagger) + b \text{Tr}(\Delta \Delta^\dagger) + c \varepsilon_{ijk} \Delta_{\alpha \beta} \partial_j \Delta_{\gamma \delta} + d_1 \text{Tr}(\Delta \Delta^\dagger) [\text{Tr}(\Delta \Delta^\dagger)]^2 + d_2 \text{Tr}(\Delta \Delta^\dagger \Delta^\dagger \Delta) + d_3 \text{Tr}[\Delta \Delta^T (\Delta \Delta^T)^\dagger]. \quad (2)$$

The sign of the “mass” term $b$ determines whether the order parameter is zero or nonzero. We will hereafter assume that $b < 0$ so that we are in the ordered phase. Note that the $c$ term breaks parity. In ferromagnets it is responsible for nonuniform, helical ordering in the ground state, as was first explained by Dzyaloshinsky [9] and Moriya [10] fifty years ago. We will therefore refer to it as the DM term.

2.1. Phase diagram without parity violation

Let us first neglect the DM term. The free energy can be minimized by a uniform order parameter. While its size depends on the balance between the $b$ and $d_{1,2,3}$ parameters, its orientation in color and spin space is governed solely by the ratio of $d_2$ and $d_3$. The phase diagram in the $(d_2, d_3)$ plane is plotted in the left panel of Fig. [1]. The four phases that appear in the phase diagram are defined by the following characteristic shapes of the order parameter,

$$\Delta_{\text{CSL}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Delta_{\text{polar}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Delta_{\Lambda} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & i & 0 \end{pmatrix}, \quad \Delta_{c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta \\ \alpha & i \alpha & 0 \end{pmatrix}, \quad (3)$$

the latter two being obtained from the general form [1] by a left unitary transformation. Note that the same GL analysis for the spin-one color superconductor was performed more than twenty years ago [11]. However, the authors just compared the energies of the four inert states (that is, CSL, polar, planar, and $\Lambda$). A proper minimization of the free energy [2] reveals that a part of the phase diagram is occupied by the noninert $c$-state.

The finding of a noninert state is one of our main messages. Previous literature on spin-one color superconductors has rather focused exclusively on the inert states. It was argued [5] that while the ground state of isolated one-flavor quark matter is most likely CSL, color imbalance induced by the presence of other quark flavors (such as in the “2SC+ss” scheme) may favor the polar phase. Our result opens up the possibility for the ground state to belong to a whole continuous family of states interpolating between the CSL and polar ones, the noninert axial phase, which will be discussed in some detail below. This will lead to a further energy gain and thus a slight expansion of the region that the spin-one color superconductor occupies in the phase diagram of dense neutral quark matter.

2.2. Phase diagram with parity violation

When the parity-violating DM term is included, the ground state changes qualitatively. This may be understood by noting that the DM term contains a single derivative, and leads to a spatially varying ground state. A detailed analysis [6] reveals that the absolute minimum of the free energy [2] is either a complex single plane wave, or a real double (standing) one, depending on the specific phase. Its wavelength is equal to $4\pi a_1/c$. All plane waves are transverse so...
that the $a_2$ term does not play any role. The modified phase diagram is shown in the right panel of Fig. 1. The shapes of the five states appearing in the phase diagram are now, for plane waves propagating in the $z$ direction, given by

\[
\Delta_{\text{DM}}^\text{axial} = \begin{pmatrix} \alpha \cos kz & \alpha \sin kz & 0 \\ -\alpha \sin kz & \alpha \cos kz & 0 \\ 0 & 0 & \beta \end{pmatrix}, \quad \Delta_{\text{DM}}^\text{planar} = \begin{pmatrix} \cos kz & \sin kz & 0 \\ -\sin kz & \cos kz & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Delta_{\text{DM}}^\text{polar} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cos kz & \sin kz & 0 \end{pmatrix},
\]

\[
\Delta_{\lambda}^\text{DM} = e^{ikz} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -i & 0 \end{pmatrix}, \quad \Delta_{\varphi}^\text{DM} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha e^{ikz} & -i \alpha e^{ikz} & \beta \end{pmatrix}.
\]

Note that the change in the topology of the phase diagram: a narrow window of the planar phase appears, and the CSL phase is replaced by the axial one. The latter is easily understood by observing that the axial state reduces to CSL for $\alpha = \beta$. Since only two colors participate in the transverse nonuniform structure while the third color is uniform and aligned with the $z$ axis, it is natural that a small color imbalance appears. In fact, the two nonuniformly distributed colors gain energy from the DM term and hence are favored. As a consequence, in presence of parity violation the ground state of a spin-one color superconductor is never truly color neutral.

The second and necessary part of our argument is that in dense quark matter parity violation actually does appear. Quantum chromodynamics itself is parity conserving, but the DM term can be induced by weak interactions. Another example of parity violation in a spin-one color superconductor can be found in the anisotropy of neutrino emission [12]. Since the electroweak scale is far above the strong one, we expect the DM parameter $c$ to be tiny. Consequently, the ground state of a spin-one color superconductor will exhibit nonuniform ordering with a very long length scale. A concrete order-of-magnitude estimate for a typical spin-one color superconductor [6] yields the wavelength of about a millimeter. Of course, once the $c$ coefficient is small, the energy gain from the formation of the nonuniform ordering will be tiny. The helical structure will be destroyed by thermal phonon fluctuations at temperatures higher than about 0.1 keV. We therefore conclude that the DM ordering will only play some role in the very late stage of the neutron star life. On the other hand it should be stressed that the phenomenon is general, the only prerequisites are a vector order parameter and parity violation.
3. Collective modes

Spontaneous symmetry breaking by the nonzero vacuum expectation value of the order parameter gives rise to collective Nambu–Goldstone (NG) modes. Being gapless, they dominate low-energy physics of the system, in particular transport phenomena. (It is nevertheless important to keep in mind that in most of the spin-one phases, some of the quarks also remain ungapped, or at least their gap function has nodes on the Fermi surface.) In the following, we will classify the NG modes in the phases that appear in the phase diagram. We will for simplicity neglect the DM term which gives rise to parity violation. The formation of the nonuniform ground state in presence of parity violation brings additional anisotropy in the low-energy spectrum, leading to non-Fermi liquid behavior of the system [13].

Spontaneous symmetry breaking in many-body systems can be intricate provided some of the conserved charges develop nonzero density [14, 15]. The conventional one-to-one correspondence of NG bosons and broken generators in general does not hold. One recognizes two qualitatively different classes of NG modes: type-I whose energy is proportional to an odd (typically first) power of momentum in the low-momentum limit, and type-II whose energy is proportional to an even (typically second) power of momentum. One may use the following rule of thumb: if the commutator of two broken generators develops nonzero vacuum expectation value, then they give rise to one type-II NG boson, instead of two usual (type-I) NG bosons. The same phenomenon already appears in the spin-zero 2SC phase in the Nambu–Jona-Lasinio (NJL) type model with a global color symmetry, where one of the color charges acquires nonzero density, giving rise to two type-II NG modes. However, this is just an artifact of the NJL model. Color neutrality is automatically satisfied in the full gauge theory of strong interactions and such modes are absent. On the contrary, we will see that some spin-one color superconducting phases possess physical type-II NG modes coming from the spontaneously broken rotational symmetry.

Below we list the symmetry-breaking patterns, the unbroken and broken generators (conveniently defined to be orthogonal), and classify the NG modes into multiplets of unbroken symmetry. In the general transformation, \( \Lambda \rightarrow U R \Lambda \), we parameterize the left unitary and right orthogonal matrices as \( U(\pi_\alpha) = \exp(i\pi_\alpha \lambda^\alpha) \) and \( R(v_j) = \exp(i v_j M_j) \). Here \( \lambda^\alpha \) are the Gell-Mann matrices extended by \( \lambda^3 = \frac{1}{\sqrt{3}} 1 \). The specific expressions for the (un)broken generators correspond to the forms of the order parameters in Eq. (3).

3.1. CSL phase

The symmetry breaking pattern is \( U(3)_L \times SO(3)_R \rightarrow SO(3)_V \). The unbroken subgroup \( SO(3)_V \) is generated by the diagonal generators, \( (M_j \otimes 1 + 1 \otimes M_j)/\sqrt{2} \). The nine broken generators fall into multiplets as follows:

- \( (M_j \otimes 1 + 1 \otimes M_j)/\sqrt{2} \), \( j = 1 \) to \( 3 \). Correspond to a triplet of type-I NG modes.
- \( \lambda^\alpha \otimes 1 \), \( \alpha = 1 \) to \( 8 \). Correspond to a 5-plet of type-I NG modes.
- \( \lambda^0 \otimes 1 \). Corresponds to a type-I NG singlet.

In this phase all charge densities but the \( U(1) \) are zero, and therefore each broken generator gives rise to one type-I NG boson. After gauging the color group, eight of the NG bosons are absorbed by gluons and only the singlet survives in the spectrum, similar to the CFL phase.

3.2. Polar phase

The symmetry-breaking pattern is \( U(3)_L \times SO(3)_R \rightarrow U(2)_L \times SO(2)_R \). The unbroken symmetry is not locked so that the left unitary and right rotational groups are broken separately. The unbroken generators are: \( \lambda^1 \), \( \lambda^2 \), \( \lambda^3 \otimes 1 \) \( [SU(2)_L] \), \( \Pi_{12} \otimes 1 \) \( [U(1)_L] \), and \( 1 \otimes M_3 \) \( [SO(2)_R] \). We have denoted \( \Pi_{12} = (\sqrt{2} \lambda_0 + \lambda_3)/\sqrt{2} \) as the projector on the first two colors. The generator \( \lambda^3 \otimes 1 \) acquires nonzero density, giving rise to type-II NG bosons in the \( \lambda^4 \), \( \lambda^6 \), \( \lambda^7 \) sector. The complete list of broken generators and the associated NG modes are:

- \( \lambda^\alpha \otimes 1 \), \( \alpha = 4 \) to \( 7 \). Correspond to an \( SU(2)_L \) doublet of type-II NG bosons.
- \( 1 \otimes M_j \), \( j = 1 \) to \( 2 \). Correspond to an \( SO(2)_R \) doublet of type-I NG modes.
- \( \sqrt{2} \Pi_3 \otimes 1 \). Corresponds to a type-I NG singlet.

Here \( \Pi_3 = (\lambda^0 - \sqrt{2} \lambda_3)/\sqrt{6} \) is the projector on the third color. Altogether there are 7 broken generators, but only 5 NG bosons. After gauging the color symmetry, only the \( SO(2)_R \) vector of type-I NG bosons survives. They represent two linearly polarized spin waves.
3.3. A phase

The symmetry-breaking pattern is $\text{U}(3)_L \times \text{SO}(3)_R \rightarrow \text{U}(2)_L \times \text{SO}(2)_V$. The unbroken $\text{U}(2)_L$ is the same as in the polar phase, given by the $2 \times 2$ unitary matrices in the upper left corner of $U$. The unbroken $\text{SO}(2)_V$ is a diagonal one, generated by $\sqrt{2}(P_3 \otimes 1 - 1 \otimes M_3)$. Now the generators $\lambda_8 \otimes 1$ and $1 \otimes M_3$ acquire nonzero density so that there is both color and spin density in the ground state. As a consequence there are type-II NG bosons in both the color and the spin sector:

- $\lambda_\alpha \otimes 1$, $\alpha = 4, 5, 6, 7$. Correspond to an $\text{SU}(2)_L$ doublet of type-II NG bosons.
- $1 \otimes (M_1 \pm iM_2)/\sqrt{2}$. Correspond to one type-II NG boson of $\text{SO}(2)_V$.
- $\sqrt{2}(P_3 \otimes 1 + 1 \otimes M_3)$. Corresponds to a type-I NG singlet.

There are 7 broken generators, but only 4 NG bosons. Only the type-II NG singlet survives after gauging the color group. It represents a circularly polarized spin wave.

3.4. $\epsilon$ phase

The symmetry-breaking pattern is $\text{U}(3)_L \times \text{SO}(3)_R \rightarrow \text{U}(1)_L \times \text{SO}(2)_V$, with the unbroken generators given by $\sqrt{2}P_1 \otimes 1$ [$\text{U}(1)_L$], and $\sqrt{2}(P_3 \otimes 1 - 1 \otimes M_3)$ [$\text{SO}(2)_V$]. Here $P_1$ is obviously the projector on the first color. The generators $\lambda_3 \otimes 1$ and $1 \otimes M_3$ acquire nonzero density. Thanks to low unbroken symmetry the collective excitation spectrum has a rich structure, in particular there are no nontrivial multiplets of states:

- $1 \otimes 1/\sqrt{2}(M_1 \pm iM_2)$. Correspond to one type-II NG boson of $\text{SO}(2)_V$.
- $1/\sqrt{2}(\lambda_{1,4,6} \pm i\lambda_{2,5,7}) \otimes 1$. Correspond to three type-II NG singlets.
- $\sqrt{2}P_2, \sqrt{2}(P_3 \otimes 1 + 1 \otimes M_3)$. Correspond to two type-I NG singlets.

Altogether there are 10 broken generators, but only 6 NG bosons. All gluons acquire mass by the Higgs mechanism, and the only NG boson that remains after gauging the color symmetry is again the circularly polarized spin wave.

3.5. Mode mixing at nonzero momentum

It is interesting to note that in absence of the $\alpha_2$ (and $c$) term, the free energy (\ref{eq:2}) is invariant under separate space-time rotations and orthogonal $\text{SO}(3)_R$ rotations of the order parameter. It is only the $\alpha_2$ term which ties these two symmetries together and makes the order parameter a space three-vector. Some of the NG bosons listed above therefore stem from spontaneous breaking of a spacetime symmetry. This has an intriguing consequence. Our classification of the NG modes was based on the symmetry breaking by a uniform order parameter. However, when fluctuations of the order parameter, that is, excitations above the ground state, are considered, their nonzero momentum leads to further breaking of the rotation symmetry. Consequently, some of the modes belonging to different multiplets as defined above may mix. Strictly speaking our classification is only accurate in the long-wavelength limit. A full treatment of the general case will be reported in a forthcoming publication.

4. Conclusions

In analogy to the coupling of orbital angular momentum and spin in atomic physics, the color–spin coupling is a new kind of interaction involving subatomic degrees of freedom and appears in the spin-one color superconductivity. It provides a perfect example of spontaneously symmetry breaking in a condensed matter system with its rich symmetry breaking patterns. We analyzed the phase diagram of a spin-one color superconductor and pointed out that a part of it is occupied by noninert states, which have been neglected in literature so far. We classified the collective excitations (the NG modes) in a systematic way and obtained their dispersion relations. The details of our calculations will be published elsewhere.
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