THERE ARE $k$-UNIFORM CUBEFREE BINARY MORPHISMS FOR ALL $k \geq 0$

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Abstract. A word is cubefree if it contains no non-empty subword of the form $xxx$. A morphism $h : \Sigma^* \to \Sigma^*$ is $k$-uniform if $h(a)$ has length $k$ for all $a \in \Sigma$. A morphism is cubefree if it maps cubefree words to cubefree words. We show that for all $k \geq 0$ there exists a $k$-uniform cubefree binary morphism.

1. Introduction

A square is a non-empty word of the form $xx$, and a cube is a non-empty word of the form $xxx$. An overlap is a word of the form $axaxa$, where $a$ is a letter and $x$ is a word (possibly empty). A word is squarefree (resp. cubefree, overlap-free) if none of its factors are squares (resp. cubes, overlaps). The construction of infinite squarefree, cubefree, and overlap-free words is typically done by iterating a suitable morphism. Uniform morphisms have particularly nice properties. In this note we show that for all $k \geq 0$ there exists a $k$-uniform cubefree binary morphism.

Let $\Sigma^*$ denote the set of all finite words over the alphabet $\Sigma$. A morphism $h : \Sigma^* \to \Sigma^*$ is $k$-uniform if $h(a)$ has length $k$ for all $a \in \Sigma$; it is uniform if it is $k$-uniform for some $k$. A morphism $h : \Sigma^* \to \Sigma^*$ is squarefree (resp. cubefree, overlap-free) if $h(w)$ is squarefree (resp. cubefree, overlap-free) whenever $w \in \Sigma^*$ is squarefree (resp. cubefree, overlap-free). Squarefree, cubefree, and overlap-free morphisms have been studied extensively [3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

We denote the Thue–Morse morphism by $\theta$:

\[
\begin{align*}
\theta(0) &= 01 \\
\theta(1) &= 10.
\end{align*}
\]

The Thue–Morse word is the infinite fixed point of $\theta$:

\[
t = \theta^\infty(0) = 01101001100101101\cdots
\]

It is well-known that the Thue–Morse word is overlap-free [14]. Moreover, the Thue–Morse morphism is both overlap-free and cubefree (see [5, 13] for even stronger results). Berstel and Séébold [4] gave a remarkable characterization of overlap-free binary morphisms: namely, that a binary morphism $h$ is overlap-free if and only if $h(01101001)$ is overlap-free. Furthermore, they showed that if $h$ is an overlap-free binary morphism then $h$ is a power of $\theta$ (or its complement). Thus any overlap-free binary morphism is $k$-uniform where $k$ is a power of 2.
It is natural to inquire if cubefree binary morphisms exhibit similar behaviour. In this case the answer is no, as we are able to construct uniform binary morphisms of every length.

For further background material concerning combinatorics on words we refer the reader to [2].

2. Main result

The main result of this note is that for all $k \geq 0$ there exists a $k$-uniform cubefree binary morphism. We begin with some preliminary lemmas.

**Lemma 1.** Let $k \geq 4$ be an integer. Then the Thue–Morse word $t$ contains two distinct words of length $k$ of the form $0y0$ and two distinct words of length $k$ of the form $0z1$.

**Proof.** For $k = 4, 5, 6$ the following table gives the required pairs of subwords.

| $k$  | Words                        | Words                        |
|------|------------------------------|------------------------------|
| 4    | $(0010, 0100)$               | $(0101, 0011)$               |
| 5    | $(00110, 01100)$             | $(01101, 01001)$             |
| 6    | $(001100, 011010)$           | $(001011, 010011)$           |

Suppose then that $k > 6$. If $k$ is even, let $k = 2r$; otherwise, let $k = 2r - 1$. Suppose inductively that $t$ contains two distinct words $0y0$ and $0y'0$ of length $r$ and two distinct words $0z1$ and $0z'1$ of length $r$.

If $k$ is even then the words $01\theta(y)01$, $01\theta(y')01$, and $01\theta(z)10$, $01\theta(z')10$ are the desired words of length $k$. If $k$ is odd then the words $01\theta(y)0$, $01\theta(y')0$, and $01\theta(z)1$, $01\theta(z')1$ are the desired words of length $k$. \[\square\]

The proof of the following lemma essentially follows that of [1, Lemma 4].

**Lemma 2.** Let $k \geq 7$ be an integer. Then $t$ contains two distinct subwords of length $k$ of the form $01x01$ and two distinct subwords of length $k$ of the form $01x10$.

**Proof.** We only give the details for $01x01$, the proof for $01x10$ being analogous. If $k$ is even, let $k = 2r$. We have $r = k/2 \geq 4$, so that $t$ contains distinct words $u = 0v0$ and $u' = 0v'0$ of length $r$ by Lemma 1. The words $\theta(u) = 01\theta(v)01$ and $\theta(u') = 01\theta(v')01$ are therefore words of the required form of length $k$.

If $k$ is odd and $k \geq 23$, we can write $k$ as $8r - 9$, $8r - 7$, $8r - 5$ or $8r - 3$ for some $r \geq 4$. Let $u = 0v0$ and $u' = 0v'0$ be distinct words of length $r$ in $t$. The word $\theta^3(u) = 01101001\theta^3(v)01101001$ contains words $01x01$ of lengths $8r - 9$ (including the first and second underlined $01$’s) and $8r - 3$ (including the first and third underlined $01$’s.) Similarly, the word $\theta^3(u') = 01101001\theta^3(v')01101001$ contains words $01x'01$ of lengths $8r - 9$ and $8r - 3$. Moreover, since $v \neq v'$, these words are distinct from the corresponding subwords of $\theta^3(u)$.

Let $z = 0v1$ and $z' = 0v'1$ be distinct words of length $r$ in $t$. The word $\theta^3(z) = 01101001\theta^3(v)10010110$
contains words 01x01 of lengths 8r − 7 (including the first and second underlined 01’s) and 8r − 5 (including the first and third underlined 01’s.) Similarly, the word
\[ \theta^3(z') = 0110100100101101 \]
contains words 01x’01 of lengths 8r − 7 and 8r − 5. Moreover, since v \neq v’, these words are distinct from the corresponding subwords of \( \theta^3(z) \).

For \( k \) odd, \( 7 \leq k \leq 21 \), the following table gives the required pairs of subwords.

| \( k = 7 \) | 0100101 | 0101101 |
| \( k = 9 \) | 010011001 | 011001101 |
| \( k = 11 \) | 01001100101 | 01100101101 |
| \( k = 13 \) | 01001011001 | 0110100101101 |
| \( k = 15 \) | 011001011001101 | 0100110010101101 |
| \( k = 17 \) | 010010110011001 | 01101001011001101 |
| \( k = 19 \) | 010010110011001101 | 010011001011001101 |
| \( k = 21 \) | 011010011001011001101 | 011001101001100101101 |

\[ \square \]

**Lemma 3.** Let \( k \geq 9 \) be an integer. Then there exist two distinct cubefree words of length \( k \) of the form 00x11.

**Proof.** For \( 9 \leq k \leq 14 \), the following table gives the required pairs of subwords.

| \( k = 9 \) | 001001011 | 001010011 |
| \( k = 10 \) | 0010011011 | 0010110011 |
| \( k = 11 \) | 00100110011 | 00101001011 |
| \( k = 12 \) | 001001010011 | 001001011011 |
| \( k = 13 \) | 0010010110011 | 0010011001011 |
| \( k = 14 \) | 00100101001011 | 00100101101011 |

Suppose \( k \geq 15 \). If \( k \) is even, let \( k = 2r − 2 \); otherwise, let \( k = 2r + 1 \). Note that \( r \geq 7 \), so by Lemma 2, there are distinct subwords 01x10 and 01x’10 of \( t \) of length \( r \).

If \( k \) is even, then the complements of the words 0\(^{-1}\)\( \theta(01x10)1\(^{-1}\) = 110\( \theta(x)100 \) and 0\(^{-1}\)\( \theta(01x’10)1\(^{-1}\) = 110\( \theta(x’)100 \) are cubefree words of the desired form of length \( k \).

If \( k \) is odd, then let \( u = 11\( \theta(01x10)1\(^{-1}\) = 110110\( \theta(x)100 \) and \( u’ = 11\( \theta(01x’10)1\(^{-1}\) = 110110\( \theta(x’)100 \). We claim that \( u \) and \( u’ \) are cubefree. Suppose to the contrary that \( u \) contains a cube. Since 0110\( \theta(x)100 \) is overlap-free, any such cube would have to start with either the first or second 1, but in either case, by inspection the period of the cube is at least 3, which forces an overlap in 0110\( \theta(x)100 \), a contradiction. Similarly, \( u’ \) is cubefree. Taking the complements of \( u \) and \( u’ \) gives cubefree words of the desired form of length \( k \). \[ \square \]

**Theorem 4.** Let \( k \geq 5 \) be an integer. Let \( w_0, w_1 \in 00\{0,1\}^k11 \) be distinct cube-free words. The morphism \( \phi : \{0,1\}^* \to \{0,1\}^* \) given by
\[ \phi(i) = \theta(w_i)(010)^{-1} \]
is cubefree.
Proof. The existence of \( w_0 \) and \( w_1 \) is guaranteed by Lemma 3. Suppose that \( v \in \{0,1\}^* \) is cube-free, but \( \phi(v) \) contains a cube \( xxx \). Let \( p = |x| \). For \( i = 0,1 \), since neither 000 nor 111 is a factor of \( w_i \), word \( \phi(i) \) cannot have 10101 as a factor; for the same reason, word \( \phi(i) \) has prefix 01011 and suffix 11. Thus 10101 occurs as a factor in \( \phi(v) \) exactly at the boundaries between images of letters of \( v \). It follows that the indices of any occurrences of 10101 in \( \phi(v) \) differ by multiples of \( |\phi(0)| \). Again, since 10101 always occurs in \( \phi(v) \) in the context 1101011, no proper extension of 10101 in \( v \) has period 1, 2, 3 or 4.

Since \( w_i \) and \( \theta \) are cube-free, for \( i \in \{0,1\} \), the word \( \phi(i)010 = \theta(w_i) \) is cube-free. It follows that \( xxx \) spans the border between \( \phi(i) \) and \( \phi(j) \) for some \( i, j \in \{0,1\} \) and in fact \( xxx \) contains factor 10101. Since 10101 is cube-free, \( xxx \) is a proper extension of 10101, and thus has period at least 5. Note that any factor \( u \) of \( xxx \) with \( |u| \leq p \) occurs twice in \( xxx \) with indices differing by \( p \). In particular, since \( |10101| = 5 \leq p \), two occurrences of 10101 in \( xxx \) have indices differing by \( p \). We conclude that \( p \) is a multiple of \( |\phi(0)| \).

Write \( x = a\phi(u)b \) where \( u \in \{0,1\}^* \), \( |ab| = |\phi(0)| \). We have \( xxx = a\phi(u)b\phi(u)b\phi(u)b \), and \( ba = \phi(i_0) \) for some \( i_0 \in \{0,1\} \). However, since \( w_1 \neq w_2 \), we also have \( \phi(0) \neq \phi(1) \) so that either

- at most one of \( \phi(0), \phi(1) \) has \( b \) as a prefix OR
- at most one of \( \phi(0), \phi(1) \) has \( a \) as a suffix.

Suppose that at most one of \( \phi(0), \phi(1) \) has \( b \) as a prefix. (The other case is similar.) Without loss of generality, say that \( \phi(0) \) has \( b \) as a prefix. It follows that \( \phi(v) \) contains \( \phi(u0u0u0) \), and \( v \) contains \( u0u0u0 \) as a factor. This is a contradiction. \( \square \)

Corollary 5. For every integer \( k \geq 0 \), there exists a \( k \)-uniform cube-free binary morphism.

Proof. If \( k \) is odd and \( k \geq 15 \), then Theorem 4 gives a cube-free morphism of length \( k \). For \( k \in \{3, 5, 7, 11, 13\} \), the morphisms given in the table below are cube-free.

| \( \phi \) | \( 0 \to \) | \( 1 \to \) |
|---|---|---|
| \( \phi_3 \) | 001 | 011 |
| \( \phi_5 \) | 01001 | 10110 |
| \( \phi_7 \) | 0010011 | 10011011 |
| \( \phi_{11} \) | 001010011011 | 1001010011011 |
| \( \phi_{13} \) | 0010010110011 | 10010011100111 |

The cube-freeness of these morphisms can be established by a criterion of Keränen, which states that to confirm that a uniform binary morphism is cube-free, it suffices to check that the images of all words of length at most 4 are cube-free.

For \( k = 1 \), the identity morphism is certainly cube-free, and for \( k = 9 \), clearly we may take \( \phi_3^2 \). This establishes the result for all odd \( k \).

If \( k = 0 \), the morphism that maps every word to the empty word is trivially cube-free. If \( k \) is positive, even and not a power of 2, then \( k = 2^a(2r + 1) \) for some positive \( a, r \). If \( \phi \) is a \((2r + 1)\)-uniform cubefree morphism, then the morphism \( \theta^a \circ \phi \) is a \( k \)-uniform cubefree morphism. Similarly, if \( k = 2^a \), then \( \theta^a \) is a \( k \)-uniform cubefree morphism. This completes the proof. \( \square \)

Brandenbug [5] gave an example of an 11-uniform squarefree ternary morphism and stated further that there are no smaller uniform squarefree ternary morphisms (excluding 0-uniform and 1-uniform morphisms). We therefore conclude by asking:
Do there exist $k$-uniform squarefree ternary morphisms for all $k \geq 11$?

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