Even Values of Ramanujan’s Tau-Function

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Abstract
In the spirit of Lehmer’s speculation that Ramanujan’s tau-function never vanishes, it is natural to ask whether any given integer $\alpha$ is a value of $\tau(n)$. For odd $\alpha$, Murty, Murty, and Shorey proved that $\tau(n) \neq \alpha$ for sufficiently large $n$. Several recent papers have identified explicit examples of odd $\alpha$ which are not tau-values. Here we apply these results (most notably the recent work of Bennett, Gherga, Patel, and Siksek) to offer the first examples of even integers that are not tau-values. Namely, for primes $\ell$ we find that

$$\tau(n) \notin \{\pm 2\ell : 3 \leq \ell < 100\} \cup \{\pm 2\ell^2 : 3 \leq \ell < 100\}$$
$$\cup \{\pm 2\ell^3 : 3 \leq \ell < 100 \text{ with } \ell \neq 59\}.$$ 

Moreover, we obtain such results for infinitely many powers of each prime $3 \leq \ell < 100$. As an example, for $\ell = 97$ we prove that

$$\tau(n) \notin \{2 \cdot 97^j : 1 \leq j \neq 0 \pmod{44}\} \cup \{-2 \cdot 97^j : j \geq 1\}.$$ 

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The method of proof applies mutatis mutandis to newforms with residually reducible mod 2 Galois representation and is easily adapted to generic newforms with integer coefficients.

**Keywords** Lehmer’s conjecture · Ramanujan’s tau-function · Newforms · Modular forms

## 1 Introduction and Statement of Results

Ramanujan’s tau-function [7,15], the coefficients of the unique normalized weight 12 cusp form for \( \text{SL}_2(\mathbb{Z}) \) (note: \( q := e^{2\pi i z} \) throughout)

\[
\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - \cdots,
\]

(1.1)

has been a remarkable prototype in the theory of modular forms. Despite many advances that reveal its deep properties, Lehmer’s conjecture [13] that \( \tau(n) \) never vanishes remains open.

In the spirit of this conjecture, it is natural to ask whether any given integer \( \alpha \) is a value of \( \tau(n) \). Much is known for odd \( \alpha \) thanks to the convenient fact that

\[
\Delta(z) = \sum_{n=0}^{\infty} q^{(2n+1)^2} \pmod{2}.
\]

(1.2)

Murty et al. [14] proved that \( \tau(n) \neq \alpha \) for sufficiently large \( n \). Craig and the authors [4,5] proved some effective results concerning potential odd values of \( \tau(n) \) and, more generally, coefficients of newforms with residually reducible mod 2 Galois representation. Their methods have been carried further in subsequent work by Amir and Hong [2], Dembner and Jain [11], and Hanada and Madhukara [12]. For example, for \( n > 1 \), these papers prove that

\[
\tau(n) \notin \{ \pm 1, \pm 691 \} \cup \{ \pm \ell : 3 \leq \ell < 100 \text{ prime} \}.
\]

(1.3)

Recently, Bennett et al. [6] proved a number of spectacular results regarding odd values of \( \tau(n) \). For example, they prove (see Theorem 6 of [6]) that \( |\tau(n)| \neq \ell^b \), where \( 3 \leq \ell < 100 \) is prime and \( b \) is a positive integer.

Much less is known for even \( \alpha \). To this end, we make use of lower bounds for the number of prime divisors of tau-values. Craig and the authors proved (see\footnote{Theorem 2.5 of [5] concerns the case of generic newforms with integer coefficients.} 1 Theorem

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1.5 of [5]) that

$$\Omega(\tau(n)) \geq \sum_{\substack{p|n \text{ prime} \}} \left( \sigma_0(\text{ord}_p(n) + 1) - 1 \right) \geq \omega(n), \quad (1.4)$$

where $\omega(n)$ (resp. $\Omega(\tau(n))$) is the number of distinct prime factors of $n$ (resp. $\tau(n)$ with multiplicity), and $\sigma_0(N)$ is the number of positive divisors of $N$. Therefore, if $\tau(n) = \pm 2$, then $n = p^m$, where $p$ and $m + 1$ are both prime.\(^2\) Similarly, if $\tau(n) = \pm 2\ell$, where $\ell$ is an odd prime, then this inequality implies that $n$ has at most two distinct prime factors. Moreover, if $n = p_1^{m_1} p_2^{m_2}$, where $p_1 \neq p_2$ are prime and $m_1, m_2 \geq 1$, then $m_1 + 1$ and $m_2 + 1$ are both prime.

Combining these results with the recent work of Bennett et al. [6], we show that certain even numbers never arise as tau-values. To make this precise, we require sets of triples $(\ell, r, t)$, where $3 \leq \ell < 100$ is prime and $r \pmod{t}$ is an arithmetic progression with modulus $t \mid 44$:

$$S^+ := \{(3, 0, 44), (5, 0, 22), (7, 0, 44), (7, 19, 44), (11, 0, 22), (13, 0, 44), (17, 0, 44), (19, 0, 22), (23, 0, 4), (29, 0, 22), (31, 0, 22), (37, 0, 44), (37, 35, 44), (41, 0, 22), (43, 0, 44), (43, 37, 44), (47, 0, 4), (53, 0, 44), (59, 0, 22), (61, 0, 22), (67, 0, 44), (67, 43, 44), (71, 0, 22), (73, 0, 44), (79, 0, 22), (83, 0, 44), (89, 0, 22), (97, 0, 44)\} \quad (1.5)$$

$$S^- := \{(3, 15, 44), (5, 11, 22), (17, 33, 44), (59, 3, 22), (83, 11, 44), (89, 11, 22)\}. \quad (1.6)$$

Then we define the set of pairs

$$N^\pm := \{ (\ell, j) : 1 \leq j \not\equiv r \pmod{t} \text{ for all } (\ell, r, t) \in S^\pm \}. \quad (1.7)$$

These sets determine values of the form $\pm 2 \cdot \ell^j$ that we rule out as possible even tau-values.

**Theorem 1.1** If $j \geq 1$ and $3 \leq \ell < 100$ is prime, then for every $n$ we have

$$\tau(n) \notin \{ 2\ell^j : (\ell, j) \in N^+ \} \cup \{-2\ell^j : (\ell, j) \in N^- \}.$$ 

Moreover, we have that $\tau(n) \notin \{ \pm 2 \cdot 691 \}$.

**Example** The triples $(7, r, t) \in S^+$ are $(7, 0, 44)$ and $(7, 19, 44)$. Therefore, Theorem 1.1 gives

$$\tau(n) \notin \{ 2 \cdot 7^j : j \not\equiv 0, 19 \pmod{44} \}.$$ 

\(^2\) In Sect. 2 we shall show that $\tau(n) = \pm 2$ requires that $n$ is prime.
Example Let $\Omega := \{7, 11, 13, 19, 23, 29, 31, 37, 41, 43, 47, 53, 61, 67, 71, 73, 79, 97\}$ be the set of primes $3 \leq \ell < 100$ for which there are no triples of the form $(\ell, r, t) \in S^-$. For these primes, $N^-$ contains $(\ell, j)$ for every $j \geq 1$, and so Theorem 1.1 gives

$$\tau(n) \notin \{-2\ell^j : \ell \in \Omega \text{ and } j \geq 1\}.$$ 

As an immediate corollary, we obtain the following conclusion for primes $3 \leq \ell < 100$.

Corollary 1.2 For every $n$, we have

$$\tau(n) \notin \{\pm2\ell : 3 \leq \ell < 100\} \cup \{\pm2\ell^2 : 3 \leq \ell < 100\} \cup \{\pm2\ell^3 : 3 \leq \ell < 100 \text{ with } \ell \neq 59\}.$$

Remark The first examples of $\tau(n) = \pm2\ell$, where $\ell$ is prime, are

$$\tau(277) = -2 \cdot 8209466002937 \quad \text{and} \quad \tau(1297) = 2 \cdot 58734858143062873.$$

We note that 277 and 1297 are both prime. Every such value with $n \leq 200,000$ has prime $n$.

The proof of Theorem 1.1 is a modification of the method employed in [4,5]. These tools are based on the observation that integer sequences of the form \{1, $\tau(p)$, $\tau(p^2)$, $\tau(p^3)$, \ldots\}, where $p$ is prime, are Lucas sequences. Important work of Bilu et al. [8] on primitive prime divisors of Lucas sequences applies to $\alpha$-variants of Lehmer’s conjecture. Loosely speaking, their work implies that each $\tau(p^m)$ is divisible by at least one prime $\ell$ for which $\ell \nmid \tau(p)\tau(p^2)\cdots\tau(p^{m-1})$. In [4,5], this property is combined with the theory of newforms to obtain variants of Lehmer’s conjecture. Namely, certain odd integers $\alpha$ are ruled out as tau-values, as well as coefficients of newforms with residually reducible mod 2 Galois representation. Such conclusions follow from the absence of special integer points $(X, Y)$ on specific curves, including hyperelliptic curves and curves defined by Thue equations. These special points (if any) have the property that $X = p$ or $p^{2k-1}$, where $p$ is prime and $2k$ is the weight of the newform.

In Sect. 2, we recall the main tools from [5] and essential facts about newform coefficients, such as Ramanujan’s tau-function. In Sect. 3 we combine these facts with (1.3), the work of Bennett et al. (i.e. Theorem 6 of [6]), and Ramanujan’s famous tau-congruences to prove Theorem 1.1.

Remark The proof of Theorem 1.1 applies mutatis mutandis to integer weight newforms with integer coefficients and residually reducible mod 2 Galois representation. A minor modification holds for arbitrary integer weight newforms $f(z)$ with integer coefficients, regardless of its 2-adic properties. Indeed, suppose that $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n$, and let $\alpha$ be any non-zero integer. We consider the “equation” $a_f(n) = \alpha$. Theorem 2.5 of [5] offers the generalization of (1.4) which constrains the
possible prime factorizations of $n$; the number of distinct prime factors of $n$ generally
does not exceed $\omega(\alpha)$. By the multiplicativity of newform coefficients, for $d \mid \alpha$, we
must solve the equation $a_f(p^m) = d$, where $m \geq 1$, and $p$ is prime. To this end, one
applies Theorem 3.2 of [5] which identifies the finitely many $m$ that must be consid-
ered. As explained in [5], a solution for $p$, when $m \geq 2$, requires special integer points
on specific curves. In many cases, there are no such points, which leads to restrictions
such as those in Theorem 1.1 using the methods employed in [4–6].

2 Nuts and Bolts

Here we recall essential facts about Lucas sequences and properties of newform coef-
ficients.

2.1 Properties of Newforms

We recall basic facts about even integer weight newforms (see [3]), along with the
deep theorem of Deligne [9,10] that bounds their Fourier coefficients.

**Theorem 2.1** Suppose that $f(z) = q + \sum_{n=2}^{\infty} a_f(n)q^n \in S_{2k}(\Gamma_0(N))$ is a newform
with integer coefficients. Then the following are true:

1. If $\gcd(n_1, n_2) = 1$, then $a_f(n_1 n_2) = a_f(n_1) a_f(n_2)$.
2. If $p \nmid N$ is prime and $m \geq 2$, then

$$a_f(p^m) = a_f(p) a_f(p^{m-1}) - p^{2k-1} a_f(p^{m-2}).$$

3. If $p \nmid N$ is prime and $\alpha_p$ and $\beta_p$ are roots of $F_p(x) := x^2 - a_f(p)x + p^{2k-1}$,
then

$$a_f(p^m) = \frac{\alpha_p^{m+1} - \beta_p^{m+1}}{\alpha_p - \beta_p}.$$

Moreover, we have $|a_f(p)| \leq 2p^{\frac{2k-1}{2}},$ and $\alpha_p$ and $\beta_p$ are complex conjugates.

2.2 Implications of Properties of Lucas Sequences for Newforms

Suppose that $\alpha$ and $\beta$ are algebraic integers for which $\alpha + \beta$ and $\alpha \beta$ are relatively prime
non-zero integers, where $\alpha/\beta$ is not a root of unity. Their Lucas numbers $\{u_n(\alpha, \beta)\} =
\{u_1 = 1, u_2 = \alpha + \beta, \ldots \}$ are the integers

$$u_n(\alpha, \beta) := \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$   (2.1)
In particular, in the notation of Theorem 2.1, for primes $p \nmid N$ and $m \geq 1$, we have

$$a_f(p^m) = u_{m+1}(\alpha_p, \beta_p) = \frac{\alpha_p^{m+1} - \beta_p^{m+1}}{\alpha_p - \beta_p}. \quad (2.2)$$

The following well-known relative divisibility property is important for the proof of Theorem 1.1.

**Proposition 2.2** (Prop. 2.1 (ii) of [8]) If $d \mid n$, then $u_d(\alpha, \beta) \mid u_n(\alpha, \beta)$.

To prove Theorem 1.1, we employ bounds on the first occurrence of a multiple of a prime $\ell$ in a Lucas sequence. We let $m_\ell(\alpha, \beta)$ be the smallest $n \geq 2$ for which $\ell \mid u_n(\alpha, \beta)$. We note that $m_\ell(\alpha, \beta) = 2$ if and only if $\alpha + \beta \equiv 0 \pmod{\ell}$. The following proposition is well known.

**Proposition 2.3** (Corollary 3.2.2 of [8]) If $\ell \nmid \alpha\beta$ is an odd prime with $m_\ell(\alpha, \beta) > 2$, then the following are true.

1. If $\ell \mid (\alpha - \beta)^2$, then $m_\ell(\alpha, \beta) = \ell$.
2. If $\ell \nmid (\alpha - \beta)^2$, then $m_\ell(\alpha, \beta) \mid (\ell - 1)$ or $m_\ell(\alpha, \beta) \mid (\ell + 1)$.

**Remark** If $\ell \mid \alpha\beta$, then either $\ell \mid u_n(\alpha, \beta)$ for all $n$, or $\ell \nmid u_n(\alpha, \beta)$ for all $n$.

A prime $\ell \mid u_n(\alpha, \beta)$ is a *primitive prime divisor* of $u_n(\alpha, \beta)$ if $\ell \nmid (\alpha - \beta)^2 u_1(\alpha, \beta) \cdots u_{n-1}(\alpha, \beta)$. Bilu, Hanrot, and Voutier [8] proved that every Lucas number $u_n(\alpha, \beta)$, with $n > 30$, has a primitive prime divisor. Their work is comprehensive; they have classified defective terms, the integers $u_n(\alpha, \beta)$, with $n > 2$, that do not have a primitive prime divisor. Their work, combined with a subsequent paper by Abouzaid [1], gives the complete classification of defective Lucas numbers. In [4,5], these results were applied to even weight newforms, including $\Delta(z)$. Arguing as in these papers, we obtain the following lemma.

**Lemma 2.1** Suppose $2k \geq 4$ is even, and $\alpha$ and $\beta$ are roots of the integral polynomial

$$F(X) = X^2 - AX + p^{2k-1} = (X - \alpha)(X - \beta), \quad (2.3)$$

where $p$ is prime, $|A| = |\alpha + \beta| \leq 2p^{\frac{2k-1}{2}}$, and $\gcd(\alpha + \beta, p) = 1$. Then there are no defective Lucas numbers $u_n(\alpha, \beta) \in \{\pm 2\ell^i\}$, where $i \geq 0$ and $\ell$ is an odd prime. Also, if $u_n(\alpha, \beta) = \pm \ell$ is a defective Lucas number, then one of the following is true.

1. We have $(A, \ell, n) = (\pm m, 3, 3)$, where $3 \mid m$ and $(p, \pm m)$ satisfies $Y^2 = X^{2k-1} \pm 3$.
2. We have $(A, \ell, n) = (\pm \ell, \ell, 4)$, where $(p, \pm \ell)$ satisfies $Y^2 = 2X^{2k-1} - 1$.

**Remark** Thanks to Lemma 2.1 and (1.4), if $\tau(n) = \pm 2$, then $n$ must be prime.

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3 This corollary is stated for Lehmer numbers. The conclusions hold for Lucas numbers because $\ell \nmid (\alpha + \beta)$.
4 This paper included a few cases that were omitted in [8].
Proof As mentioned above, [1, 8] classify defective Lucas numbers. This classification includes a finite list of sporadic examples and a list of parameterized infinite families. Theorem 2.2 of [5] uses these results to describe the defective Lucas numbers that can arise as newform coefficients, i.e. sequences defined by (2.3). Tables 1 and 2 of [5] list the possible defective cases.

An inspection of Table 1 of [5], which concerns the sporadic examples, reveals that the only possible defective numbers with \(2k \geq 4\) have \(2k = 4\). Moreover, they are the odd numbers \(u_3(\alpha, \beta) = 1\) or \(u_4(\alpha, \beta) = \pm 85\).

To complete the proof, we consider the parametrized infinite families in Table 2 of [5]. If \(u_n(\alpha, \beta)\) is even, then we only have to consider rows four, five, six, and seven of the table. A simple inspection reveals that \(\{\pm 2 \ell^i\}\) for \(i \geq 0\) never arises. This then leaves \(u_n(\alpha, \beta) = \pm \ell\) as the only cases to consider. However, Lemma 2.1 of [5] includes these cases, giving (1) and (2) above. \(\square\)

3 Proof of Theorem 1.1

Here we use the previous section to prove Theorem 1.1.

3.1 Ramanujan’s Congruences

Ramanujan’s classical congruences for the tau-function imply the following convenient fact involving the sets \(N^\varepsilon\) defined in (1.7).

Lemma 3.1 If \(3 \leq \ell < 100\) is prime and \((\ell, j) \in N^\varepsilon\), then for every prime \(p\) we have that

\[
\tau(p) \neq \varepsilon 2\ell^j.
\]

Proof We recall the famous Ramanujan congruences (see [7, 15]):

\[
\tau(n) \equiv \begin{cases} n^3\sigma_1(n) \pmod{4}, \\ n^2\sigma_1(n) \pmod{3}, \\ n\sigma_1(n) \pmod{5}, \\ n\sigma_3(n) \pmod{7}, \end{cases}
\]

where \(\sigma_v(n) := \sum_{1 \leq d \mid n} d^v\). Furthermore, if \(p \neq 23\) is prime, Ramanujan proved that

\[
\tau(p) \equiv \begin{cases} 0 \pmod{23} & \text{if } (\frac{p}{23}) = -1, \\ \sigma_{11}(p) \pmod{23^2} & \text{if } p = a^2 + 23b^2 \text{ with } a, b \in \mathbb{Z}, \\ -1 \pmod{23} & \text{otherwise}. \end{cases}
\]
If $p \neq 23$ is prime, then the collection of these congruences imply
\[
\tau(p) \equiv 0 \pmod{2}, \quad \tau(p) \equiv 0, 2 \pmod{3}, \quad \tau(p) \equiv 0, 1, 2 \pmod{5}, \\
\tau(p) \equiv 0, 1, 2, 4 \pmod{7}, \quad \text{and} \quad \tau(p) \equiv 0, -1, 2 \pmod{23}.
\]

These congruences are easily reformulated in terms of $N^\varepsilon$. This completes the proof for $p \neq 23$. Finally, we note that $\tau(23) = 18643272 = 2^3 \cdot 3 \cdot 617 \cdot 1259$. \qed

### 3.2 Proof of Theorem 1.1

Theorem 1.1 consists of two different types of $\alpha$.

1. The case where $\alpha = \pm 2\ell$, where $3 \leq \ell \leq 100$ is prime or $\ell = 691$.
2. The case where $\alpha = \pm 2\ell^j$, where $3 \leq \ell \leq 100$ is prime and $j \geq 2$.

By Lemma 2.1 with $2k = 12$, the numbers $\{\pm 2\ell^i\}$ for $i \geq 0$ (if they arise) are never defective Lucas numbers in $\{\tau(p), \tau(p^2), \tau(p^3), \ldots\}$, where $p$ is prime. Lemma 2.1 (1) and (2) covers the cases apart from Case (1), where the conclusion is that
\[
\tau(p) = \pm 2\ell.
\]

Thanks to (1.4), if $\tau(n) = \pm 2\ell$, where $\ell$ is an odd prime, then either $n = p_1^{m_1} \cdot p_2^{m_2}$, where the $p_i$ are prime and the $m_i \geq 1$. Using Theorem 2.1 (1) and (1.3), the latter case requires $|\tau(p_1^{m_1})| = 2$ and $|\tau(p_2^{m_2})| = \ell$. Thanks to (1.3) again, this is impossible for $\ell = 691$ and primes $3 \leq \ell < 100$.

Therefore, we may assume that $\tau(p_1^{m_1}) = \pm 2\ell$. Thanks to Theorem 2.1, we have that $p_1 \neq 2$, as $4 \mid \tau(2^m)$ for every positive integer $m$. Therefore, (1.2) implies that $m_1$ is odd. Moreover, since $\tau(p_1)$ is even, it must be that $\tau(p_1^{m_1})$ is the first term in the Lucas sequence that is divisible by $\ell$. Otherwise, $\pm 2\ell$ would be defective, contradicting Lemma 2.1. If $m_1 + 1$ has a non-trivial divisor other than 2, then by the relative divisibility of Lucas numbers given in Proposition 2.2, and the nondefectivity of $\pm 2$ in Lemma 2.1, we obtain a contradiction. Hence, it boils down to considering the case when $m_1 + 1 = 4$, $\tau(p) = \pm 2\ell$, and $\tau(p^3) = \pm 2\ell$ for some prime $p$. However, using the Hecke relation ((2) in Theorem 2.1), we have that $\pm 2\ell = \pm 4(p^{11} - 2)$, and there is no such $p$, as the left hand side is 2 (mod 4) while the right hand side is 0 (mod 4). Hence, it follows that $m_1 + 1$ is prime. Therefore, we have $m_1 = 1$, which in turn leads to $\tau(p_1) = \pm 2\ell$. The proof in this case is complete as Lemma 3.1 shows that $\tau(p) \neq \pm 2\ell$.

#### Case (2). Since $3 \leq \ell < 100$ is prime, (1.3) and Theorem 6 of [6] implies that $|\tau(n)| \neq \ell^b$ for all $n$ and $b \geq 1$. Therefore, we may assume that $\tau(p^m) = \pm 2\ell^j$, where $p$ is an odd prime. Here we again use the fact that $4 \mid \tau(2^m)$ for every positive integer $m$, and consider the degenerate case $m_1 + 1 = 4$, $\tau(p) = \pm 2\ell$, and $\tau(p^3) = \pm 2\ell$ for some prime $p$, which gives the corresponding equation $\pm 2\ell^j = \pm 4(p^{11} - 2)$. The argument in Case (1), where the conclusion is that $m = 1$, applies mutatis mutandis. Therefore, the proof is complete as Lemma 3.1 shows that $\tau(p) \neq \pm 2\ell^j$.

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