DIFFERENTIAL FORMS CANONICALLY ASSOCIATED TO
EVEN-DIMENSIONAL COMPACT CONFORMAL MANIFOLDS

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Abstract. On a 6-dimensional, conformal, oriented, compact manifold
without boundary, we compute a whole family of differential forms \( \Omega_6(f, h) \) of
order 6, with \( f, h \in C^\infty(M) \). Each of these forms will be symmetric on \( f \) and
\( h \), conformally invariant, and such that \( \int_M f_0 \Omega_6(f_1, f_2) \) defines a Hochschild
2-cocycle over the algebra \( C^\infty(M) \). In the particular 6-dimensional confor-
mally flat case, we compute the unique one satisfying \( \text{Wres}(f_0[F, f][F, h]) = \int_M f_0 \Omega_6(f, h) \) for \( (H, F) \) the Fredholm module associated by A. Connes \[6\] to
the manifold \( M \), and \( \text{Wres} \) the Wodzicki residue.

Keywords: Conformal geometry, Wodzicki residue, Fredholm module.

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1. Introduction

For a compact, oriented manifold \( M \) of even dimension \( n = 2l \), endowed with a
conformal structure, there is a canonically associated Fredholm module \((H, F)\) \[6\].
\( H \) is the Hilbert space of square integrable forms of middle dimension
\[
H = L^2(M, \Lambda^{n/2}_0 T^* M),
\]
in which functions on \( M \) act as multiplication operators. \( F \) is the pseudodifferential
operator of order 0 acting in \( H \), obtained from the orthogonal projection \( P \) on the
image of \( d \), by the relation \( F = 2P - 1 \). From the Hodge decomposition theorem \[10\]
it is easy to see that \( F \) preserves the finite dimensional space of harmonic forms
\( H^l \), and that \( F \) restricted to the \( H \种种 \) is given by
\[
F = \frac{d\delta - \delta d}{d\delta + \delta d}.
\]

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in terms of a Riemannian metric compatible with the conformal structure of \( M \). Both \( \mathcal{H} \) and \( F \) are independent of the metric in the conformal class \( [\mathcal{M}] \). Section IV.A.γ.

By considering a Riemann surface \( \Sigma \), and by considering instead of \( dX \) its quantized version \([F,X] \), Connes (Chapter 4 [17]) quantized the Polyakov action as the Dixmier trace of the operator \( \eta_{ij}dX^idX^j \)

\[
\frac{1}{2\pi} \int_{\Sigma} \eta_{ij} dX^i \wedge *dX^j = -\frac{1}{2} \text{trace}_{\omega}(\eta_{ij}[F,X^i][F,X^j]).
\]

Because of the fact that the Wodzicki residue extends uniquely the Dixmier trace as a trace on the algebra of pseudodifferential operators [17], this quantized Polyakov action has sense in the general even-dimensional case. Because of the Connes’ trace theorem (see Theorem 7.18 [9]), the Dixmier trace and the Wodzicki residue of an elliptic pseudodifferential operator of order \( -n \) in an \( n \)-dimensional manifold are proportional by a factor of \( n(2\pi)^n \). In the 2-dimensional case the factor is \( 8\pi^2 \) and so the quantized Polyakov action [2] can be written as,

\[
-16\pi^2 \mathcal{L} = \text{Wres}(\eta_{ij}[F,X^i][F,X^j])
\]

which determines, by using the general formula for the total symbol of the product of two pseudodifferential operators, an \( n \)-dimensional differential form \( \Omega_n \). Note that we decided to write the constant on the left of \( \Omega_n \) to simplify the typing. This differential form is symmetric, conformally invariant and uniquely determined, for every \( f_0, f, h \in C^\infty(M) \), by the relation:

\[
\text{Wres}(f_0[F,f][F,h]) = \int_M f_0 \Omega_n(f,h).
\]

\( \Omega_n(f,h) \) is given as the Wodzicki 1-density (which we denote \( \text{wres} \) following [9]) of the product of commutators \([F,f][F,h]\):

\[
\Omega_n(f,h) = \text{wres}([F,f][F,h]) = \left\{ \sum_{\alpha' \alpha'' \beta \delta} A_{\alpha',\alpha'',\beta,\delta}(D^\beta_x(f)D^\alpha''_x(h)) \right\} d^n x,
\]

with the sum taken over \( |\alpha'| + |\alpha''| + |\beta| + |\delta| + j + k = n, |\beta| \geq 1, |\delta| \geq 1, \) and

\[
A_{\alpha',\alpha'',\beta,\delta} = \int_{|\xi|=1} \text{trace}\left( \partial^{\alpha'+\alpha''+\beta}_\xi (\sigma_{-j}(F)) \partial^{\gamma}_\xi \left( D^{\alpha''}_x \left( \sigma_{-k}(F) \right) \right) \right) d^{n-1}\xi
\]

where \( \sigma_{-j}(F) = \sigma_{-j}(F)(x,\xi) \) is the component of order \(-j\) in the total symbol of \( F \), \( |\xi| = 1 \) means the Euclidean norm of the coordinate vector \((\xi_1, \cdots, \xi_n)\) in \( \mathbb{R}^n \), and \( d^{n-1}\xi \) is the normalized volume on \( \{ |\xi| = 1 \} \).

In the 4-dimensional case, Connes [18] showed that the Paneitz operator \( P_4 \), analogue of the scalar Laplacian in 4-dimensional conformal geometry, can be derived from the Wodzicki residue by dropping some information. That is to say, by setting \( f_0 = 1 \) and integrating by parts he has obtained

\[
\int_M \Omega_4(f,h) = \int_M f P_4(h) dv
\]

producing \( P_4 \) by the arbitrariness of \( f \) and \( h \).

The relation (6) is of importance in the study of conformally invariant differential operators generalizing the Yamabe operator. There are (see [11]) invariant operators (\( GJMS \) operators) on scalar densities with principal parts \( \Delta^k \), unless the
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dimension is even and $2k > n$. The $n$-th order operator is called critical GJMS operator. When one calculates the Polyakov action for the quotient of functional determinants of a conformally covariant operator $D$ at conformally related metrics, the operator $P_n$ shows, see for example [3] and [5]. That is why the study of such a differential form $\Omega_n$, is of considerable importance in the case $n \geq 4$. We proposed an approach using automated symbolic computation to (5).

In this paper, we present partial results from [14], see also [15]. In section 2 we introduce a general construction which associates to any pseudodifferential operator $S$ of order 0 acting on sections of a bundle $B$ on a compact manifold without boundary $M$, a differential form of order $n$ acting on $C^\infty(M) \times C^\infty(M)$, $\Omega_{n,S}(f,h)$. This $\Omega_{n,S}(f,h)$ is uniquely given by the relation

$$Wres(f_0[S,f_1][S,f_2]) = \int_M f_0 \Omega_{n,S}(f_1,f_2) \, dvol$$

for every $f_i \in C^\infty(M)$, with Wres the Wodicki residue. The first result is given in Lemma 2 where we give an explicit expression for $\Omega_{n,S}$ in terms of the total symbol of $S$. In the particular case of a compact, conformal, oriented, even-dimensional manifold, with $S = F$ the operator given by (4), the differential form $\Omega_{n,F}$ is furthermore, symmetric on $f$ and $h$ and conformal invariant (Theorem 2.5). The rest of the paper focuses on computing $\Omega_{n,F} = \Omega_n$ in this case, in particular for $n = 6$. In section 3, we give an explicit expression for $\Omega_n(f_1,f_2)$ in the flat case. This expression (see Proposition 3.2) is given in terms of the Taylor expansion of the function $\text{trace}(\sigma(\xi)\sigma(\eta))$ (see (6)). In section 4, we present $\Omega_6(f_1,f_2)$, given by Theorem 4.5, in the 6-dimensional conformally flat case. The last result is presented as Theorem 6.2 in section 6, where we compute a whole family $\Omega_6(f,h)$ of differential forms of order 6 associated to a conformal, oriented, compact manifold without boundary. Each of these forms will be symmetric on $f$, and $h$, conformally invariant and such that $\int_M f_0 \Omega_6(f_1,f_2)$ defines a Hochschild 2-cocycle over the algebra $C^\infty(M)$. To compute the unique form $\Omega_6$ satisfying the relation $Wres(f_0[F,f_1][F,f_2]) = \int_M f_0 \Omega_6(f_1,f_2)$ more information is needed in the 6-dimensional case.

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Notations and conventions

A conformal manifold is an equivalence class of Riemannian manifolds where two metrics $g$ and $\hat{g}$ are said to be equivalent if one is a positive scalar multiple of the other, for this work, it is convenient to write $\hat{g} = e^{\eta}g$ for some $\eta \in C^\infty(M)$.

The Laplace-Beltrami operator on $k$-forms is defined as $\Delta = \delta\delta + \delta d$ where we assume the sign convention $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ on $\mathbb{R}^2$. The contraction of a $k$-form $\eta$ with a vector field $X$ is defined by

$$\iota_\eta(X)(X_1,\cdots,X_{k-1}) := \eta(X,X_1,\cdots,X_{k-1}).$$

The contraction of $\eta$ with a 1-form $\xi_X$ which is determined by the vector field $X$ is given by $\iota_\eta(\xi_X) = \iota_\eta(X)$. The exterior multiplication by a $k$-form $\eta$ will be denoted by $\iota_\eta : \xi \mapsto \eta \wedge \xi$.

In this work, the Riemann curvature tensor will be represented with the letter $R$, the Ricci tensor will be represented by $R_{ij} = R^{ik}_{\quad kij}$, and the scalar curvature
by $\text{Sc} = \text{Re}^i_{\cdot i}$. The conformal change equation for the Ricci tensor:

$$\eta;ij = -V_{ij} - \eta;\cdot j + \frac{1}{2} \eta;\cdot k \eta;\cdot l g_{ij},$$

allows to replace, in the case $g = e^{2\eta}g_{\text{flat}}$, the second derivatives on $\eta$ with terms with the Ricci tensor. $V$ represents a normalized translation of the Ricci tensor, useful in conformal geometry, given in terms of the normalized scalar curvature $J$ by

$$V = \frac{\text{Re} - Jg}{n-2} \quad \text{with} \quad J = \frac{\text{Sc}}{2(n-1)}.$$ 

In (7), the indices after the semicolon represents covariant derivatives, $\eta;ij = \nabla_j \nabla_i \eta$. In terms of $V$, the relation between the Weyl tensor and the Riemann tensor is given by

$$W^i_{\cdot jkl} = R^i_{\cdot jkl} + V_{jk} \delta^i_l - V_{jl} \delta^i_k + V^i_{\cdot l} g_{jk} - V^i_{\cdot k} g_{jl},$$

where $\delta$ represents the Kronecker’s delta tensor.

If needed, we will “raise” and “lower” indices without explicit mention following for example, $g_{mn} R^i_{\cdot jkl} = R_{m jkl}$.

When working with the total symbol of a pseudodifferential operator $P$, we will denote its leading symbol by $\sigma^P_k$, or $\sigma_k(P)$ in case $P$ has a long expression. If the operator $P$ is of order $k$ then its total symbol (in some given local coordinates) will be represented as

$$\sigma(P) = \sigma_k^P + \sigma_{k-1}^P + \sigma_{k-2}^P + \cdots,$$

where $\sigma_{j}^P = \sigma_{j}^P$. It is important to note that the different $\sigma_{j}^P$ for $j < k$ are defined only in local charts and are not diffeomorphism invariant [12]. However, Wodzicki [17] has shown that the term $\sigma_{-n}^P$ enjoys a very special significance. For a pseudodifferential operator $P$, acting on sections of a bundle $B$ over a manifold $M$, there is a 1-density on $M$ expressed in local coordinates by

$$\text{wres}(P) = \int_{|\xi|=1} \{\text{trace}(\sigma_{-n}^P(x, \xi)) \, d^{n-1} \xi\} \, d^n x.$$ 

This Wodzicki residue density is independent of the local representation. Here we are using the same notations as in [9], where an elementary proof of this matter can be found. The Wodzicki residue, $\text{Wres}(P)$, is then computed [6] by choosing any local coordinates $x^j$ on $M$ and any local basis of sections $s_k$ for $B$. $P$ is represented in terms of the chosen basis $s_k$ as a matrix $P^i_k$ of scalar pseudodifferential operators:

$$P(f^k \alpha_k) = (P^i_k f^k) \alpha_i.$$ 

The residue $\text{Wres}(P)$ is given by

$$\text{Wres}(P^i_k) = \int_M \left\{ \int_{|\xi|=1} \text{trace}(\sigma_{-n}(x, \xi)) \, d^{n-1} \xi \right\} \, d^n x$$

where $\sigma_{-n}(x, \xi)$ is the component of order $-n$ in the total symbol of $P$, $|\xi| = 1$ means the Euclidean norm of the coordinate vector $(\xi_1, \ldots, \xi_n)$ in $\mathbb{R}^n$, and $d^{n-1} \xi$ is the normalized volume on $\{ |\xi| = 1 \}$. $\text{Wres}(P)$ is independent of the choice of the local coordinates on $M$, the local basis $(s_k)$ of $B$, and defines a trace (see [17]).

To study the conformal invariance, there is no need to study the whole conformal deformation. It is enough to study the conformal deformation up to order one in $\eta$ as follows. If we set a metric $g$ in $M$ and consider another metric $\tilde{g}$ conformally related to $g$ by the relation $\tilde{g} = e^{2z \eta} g$ where $\eta \in C^\infty(M)$ and $z$ a constant, then the conformal variation of each expression is a polynomial in $z$ whose coefficients
are expressions in the metric and the conformal factor \( \eta \) (actually, this is an abuse of the language since the conformal factor is \( e^{2z\eta} \)). In this way, the conformal deformation up to order one in \( \eta \) is given by \( \frac{1}{2}\alpha_0|_{z=0} \). As appointed in [10], if a natural tensor or a differential operator is invariant up to order one in \( \eta \), i.e. if its conformal deformation up to order one is equal to zero, then by integration it follows that it is fully invariant, for details see [2].

2. Existence of \( \Omega_n \)

In this section, we associate to any pseudodifferential operator \( S \) of order 0 acting on sections of a bundle \( B \) on an arbitrary compact manifold without boundary \( M \), a bilinear differential form \( \Omega_{n,S} \) acting on \( C^\infty(M) \times C^\infty(M) \). Most of the properties of \( \Omega_{n,S} \) are related with the properties of the Wodzicki residue. The total symbol up to order \( -n \) of the pseudodifferential operator of order \( -2 \) given by the product \( P = f_0[S, f_1][S, f_2] \) with each \( f_i \in C^\infty(M) \), is represented as a sum of \( r \times r \) matrices of the form \( \sigma_{-2}^P + \sigma_{-3}^P + \cdots + \sigma_{-n}^P \), with \( r \) the rank of \( B \). We aim to study

\[
\text{Wres}(P) = \int_M \left\{ \int_{|\xi|=1} \text{trace}(\sigma_{-n}^P(x, \xi)) d^{n-1}\xi \right\} d^n x.
\]

In general, the total symbol of the product of two pseudodifferential operators \( P_1 \) and \( P_2 \) is given by

\[
\sigma(P_1P_2) = \sum_1^\alpha \frac{1}{\alpha!} \partial_\xi^\alpha (\sigma^{P_1}) D_\xi^\alpha (\sigma^{P_2})
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a multi-index, \( \alpha! = \alpha_1! \alpha_2! \cdots \alpha_n! \) and \( D_\xi^\alpha = (-i)^{\alpha!} \partial_\xi^\alpha \). Using this formula it is possible to deduce an expression for \( \sigma_{-n}([S, f_1][S, f_2]) \) finding first \( \sigma([S, f]) \).

**Lemma 2.1.** \([S, f]\) is a pseudodifferential operator of order \(-1\) with total symbol \( \sigma([S, f]) = \sum_{k\geq 1} \sigma_{-k}([S, f]) \) where

\[
\sigma_{-k}([S, f]) = \sum_{|\beta| = 1}^k \frac{1}{\beta!} D_\xi^\beta (f) \partial_\xi^\beta (\sigma_{-k}^S) (\sigma_{-k}^S).
\]

**Lemma 2.2.** With the sum taken over \( |\alpha'| + |\alpha''| + |\beta| + |\delta| + j + k = n \), \( |\beta| \geq 1 \), and \( |\delta| \geq 1 \),

\[
\sigma_{-n}([S, f_1][S, f_2]) = \sum \frac{1}{\alpha!\alpha''!\beta!\delta!} D_\xi^\beta (f_1) D_\xi^{\alpha''+\delta} (f_2) \partial_\xi^{\alpha'+\alpha''+\beta} (\sigma_{-j}^S) \partial_\xi^\delta (D_\xi^\alpha (\sigma_{-k}^S)).
\]

As a consequence

\[
\text{wres}([S, f_1][S, f_2]) = \int_{|\xi|=1} \text{trace}\left\{ \sum \frac{1}{\alpha''!\beta!\delta!} D_\xi^\beta (f_1) D_\xi^{\alpha''+\delta} (f_2) \times \partial_\xi^{\alpha'+\alpha''+\beta} (\sigma_{-j}^S) \partial_\xi^\delta (D_\xi^\alpha (\sigma_{-k}^S)) \right\} d^{n-1}\xi \right\} d^n x.
\]

**Definition 2.3.** For every \( f_1 \) and \( f_2 \) in \( C^\infty(M) \) we define

\[
\Omega_{n,S}(f_1, f_2) := \text{wres}([S, f_1][S, f_2]).
\]
Theorem 2.4. For any pseudodifferential operator $S$ of order 0 acting on sections of a bundle $B$ on a manifold $M$, there is a unique $n$-differential form $\Omega_{n,S}$ such that

$$\text{Wres}(f_0[S,f_1][S,f_2]) = \int_M f_0 \Omega_{n,S}(f_1,f_2)$$

for all $f_i \in C^\infty(M)$.

We restrict ourselves to an even dimensional, compact, oriented, conformal manifold without boundary $M$, and $(B,S)$ given by the canonical Fredholm module $(H,F)$ associated to $M$ by A. Connes [6].

In this particular case, $F = (d\delta - \delta d)(d\delta + \delta d)^{-1}$ is the pseudodifferential operator of order 0 acting on the component $\mathfrak{H}(d+\delta)$ of $\mathcal{H} = L^2(M,\Lambda^\infty T^*M)$. We can conclude that the differential form $\Omega_{n,F}$ is symmetric and conformally invariant.

Theorem 2.5. In the particular case in which $M$ is a even dimensional compact conformal manifold without boundary and $(H,F)$ is the Fredholm module associated to $M$ by A. Connes [6], there is a unique, symmetric, and conformally invariant $n$-differential form $\Omega_n = \Omega_{n,F}$ such that

$$\text{Wres}(f_0[F,f_1][F,f_2]) = \int_M f_0 \Omega_n(f_1,f_2)$$

for all $f_i \in C^\infty(M)$.

All the proofs for the lemmas and theorems in this paper, as well as the detail for the computations in here presented can be read in [14].

3. $\Omega_n$ in the flat case

Proposition 3.1. The leading symbol of $F$ is given by

$$\sigma^F_{\ell}(x,\xi) = |\xi|^{-2}(\epsilon\xi\xi - \iota\xi\epsilon\xi)$$

for all $(x,\xi) \in T^*M$, $\xi \neq 0$. In the particular case of a flat metric, we also have $\sigma^F_{-k} = 0$ for all $k \geq 1$.

Using this result, and the explicit expression for $\Omega_n$ (Definition 12), it is possible to give a formula for $\Omega_n$ in the flat case using the Taylor expansion of the function

$$\text{trace}(\sigma^F_{\ell}(\xi)\sigma^F_{\ell}(\eta)) = a_{n,m} \frac{(|\xi|^2|\eta|^2 + b_{n,m}}{\xi^2|\eta|^2}.$$

Here $\xi, \eta \in T^*_x M \setminus \{0\}$, $\sigma^F_{\ell}(\xi)\sigma^F_{\ell}(\eta)$ is acting on $n/2$-forms, and the values of the constants are given by

$$\begin{pmatrix} n \\ m \end{pmatrix} - a_{n,m} = b_{n,m} = \begin{pmatrix} n - 2 \\ m - 2 \end{pmatrix} + \begin{pmatrix} n - 2 \\ m \end{pmatrix} - 2 \begin{pmatrix} n - 2 \\ m - 1 \end{pmatrix}.$$

Because $\sigma^F_{-k}(x,\xi) = 0$ for all $k > 0$ in the flat case, (12) reduces to

$$\Omega_{n,\text{flat}}(f,h) = \left\{ \int_{|\xi|=1} \frac{1}{\alpha!\beta!\delta!} (D^\beta_x f)(D^\alpha_x h)\partial^{\alpha+\beta}_\xi D^{\delta}_\xi (\sigma^F_{\ell}(\xi)\sigma^F_{\ell}(\eta)) d^{n-1}\xi \right\} d^n x,$$

with the sum taken over $|\alpha| + |\beta| + |\delta| = n$, $1 \leq |\beta|$, $1 \leq |\delta|$.
We denote by $T^n_\psi(\xi,\eta,u,v)$ the term of order $n$ in the Taylor expansion of $\psi(\xi,\eta,u,v)$ minus the terms with only powers of $u$ or only powers of $v$. That is to say,

$$T^n_\psi(\xi,\eta,u,v) = \sum_{|\beta|+|\delta|=n,|\beta|\geq 1,|\delta|\geq 1} \frac{u^\beta v^\delta}{\beta! \delta!} \text{trace}(\partial^{\beta}_\xi(\sigma^F_L(\xi))\partial^{\delta}_\eta(\sigma^F_L(\eta))).$$

**Proposition 3.2.**

$$\Omega^n_{\text{flat}}(f,h) = \left(\sum A_{a,b}(D^a_x f)(D^b_x h)\right) dx^n,$$

where

$$\sum A_{a,b} u^a v^b = \int_{[\xi]} (T^n_\psi(\xi,\xi,u+v,v) - T^n_\psi(\xi,\xi,v,v)) dx^{n-1}\xi.$$

4. The 6-dimensional conformally flat case

In the 6-dimensional case, symmetry and conformal invariance are not enough to fully describe $\Omega_6$ as we will find terms like

$$\left\{ A f_{;ij} h^{ij} W_{jklm} W^{jklm} + B f_{;ij} h^{ij} W_{jklm} W^{jklm} \right\} d^6 x.$$

which are symmetric on $f$ and $h$, and conformally invariant.

Other important property that we will exploit, is the fact that, by definition, $\text{Wres}(f_0[F,f_1][F,f_2])$ is a Hochschild 2-cocycle over the algebra $C^\infty(M)$. If we define $\tau(f_0,f,h)$ as

$$\tau(f_0,f,h) = \int_M f_0 \left\{ A f_{;ij} h^{ij} W_{jklm} W^{jklm} + B f_{;ij} h^{ij} W_{jklm} W^{jklm} \right\} d^6 x,$$

then we obtain a Hochschild 2-cocycle for any value of $A$ and $B$, that is to say [8]

$$0 = (tr)(f_0,f_1,f_2)$$

$$= \tau(f_0,f_1,f_2) - \tau(f_0,f_1,f_3) + \tau(f_0,f_1,f_2,f_3) - \tau(f_3,f_0,f_1,f_2).$$

In the 4-dimensional case, this property was used merely to make sure the constants found had the right values. In the 6-dimensional case, as we shall see, this property will play a more important role in the non-conformally flat case, even so, the fully description of $\Omega_6$ escapes these properties, requiring some more information to be used.

In the 6-dimensional flat case, using proposition 3.2 to compute $\Omega_6$ we have found

$$\Omega_6_{\text{flat}}(f,h) = Q_6(df,dh)$$

$$= \left\{ 12(f_{;ij} h^{ij} k + f_{;ij} k^{ij} h) + 24(f_{;ij} h^{ij} k + f_{;ij} k^{ij} h) + 6(f_{;ij} h^{ij} k + f_{;ij} k^{ij} h) + 24(f_{;ij} h^{ij} k + 16 f_{;ij} h^{ij} k) \right\} d^6 x$$

$$= \left\{ 12 \Delta^2(\langle df,dh \rangle) - 6 \Delta(\Delta f \Delta h) - 12 \langle \nabla^2 f, \nabla^2 h \rangle \right\} d^6 x$$

where each summand in the last expression is explicitly symmetric on $f$ and $h$.

Studying $\Omega_n$ in the flat metric gives information about the conformally flat case, in particular, by growing the flat expression for $\Omega_n$ we can find its expression in
the conformally flat case. To do that, we consider a metric \( \hat{g} \) conformally related to the flat metric \( g \) by the relation \( \hat{g} = e^{2\eta}g \) with \( \eta \) an smooth function on \( M \). Each time we express a component of \( \Omega_6 \) in the conformally related metric, terms containing derivatives on \( \eta \) will show. Using the conformal change equation for the Ricci tensor \( \Omega_6 \) until we reduce all the higher derivatives on \( \eta \) to derivatives of order one, the Ricci tensor, and the scalar curvature related to \( g \), we obtain the expression for \( \Omega_6 \) in the conformally flat case. The computations could be a little tedious, so we did them using Ricci.m to obtain:

\[
\begin{align*}
\Omega_{6\text{conf-flat}}(f, h) &= \Omega_{6\text{flat}}(f, h) \\
&+ \left\{ -72 f_{;ij} h_{;i}^j + f_{;ij} h_{;i j} - 24 f_{;i} h_{;i j} - 96 f_{;ij} h_{;i j} \right\} J \\
&+ 96 f_{;i} h_{;i} J^2 \\
&+ 24 f_{;i} h_{;i} J^2 + 64 f_{;ij} h_{;ij} J^2 - 24 (f_{;ij} h_{;ij} J^2 + f_{;i} h_{;i j}) \\
&- 24 f_{;i} h_{;i} J^2 + 64 f_{;ij} h_{;ij} J^2 - 32 (f_{;ij} h_{;ij} J^2 + f_{;i} h_{;i j}) + 64 (f_{;ij} h_{;ij} J^2 + f_{;i} h_{;i j}) \\
&- 192 f_{;ij} h_{;ij} (J^2 - 64 f_{;ij} h_{;ij} V_{jk} + 128 f_{;ij} h_{;ij} V_{jk} + V_{ij}^k V_{jk}) d^6 x.
\end{align*}
\]

An interesting introduction to automated symbolic computations can be found in [1].

**Theorem 4.1.** In the 6-dimensional conformally flat case, the expression for \( \Omega_6 \) given by Theorem 2.5 as a sum of explicitly symmetric components on \( f \) and \( h \), is given by

\[
\begin{align*}
\Omega_{6\text{conf-flat}}(f, h) &= \left\{ 12 \Delta^2 \langle df, dh \rangle - 6 \Delta (\Delta f \Delta h) - 12 \langle \nabla \Delta f, \nabla \Delta h \rangle \\
&+ 24 \Delta (\nabla df, \nabla dh) + 16 \langle \nabla^2 df, \nabla^2 dh \rangle + 72 \Delta \langle df, dh \rangle J \\
&- 24 \Delta (\Delta(h) J) + 48 \langle \nabla df, \nabla dh \rangle J + 96 \langle df, dh \rangle J^2 + \\
&+ 24 \langle df, dh \rangle \Delta(J) - 24 (\Delta(f) dh + \Delta(h) df, dJ) \\
&- 24 \langle d(df, dh), dJ \rangle - 96 \langle \Delta(h) df + \Delta(f) \nabla dh, V \rangle \\
&+ 32 \langle \nabla (\Delta(f) \otimes dh) + \nabla (\Delta(h) \otimes df), V \rangle \\
&+ 64 \langle \nabla^2 (df, dh), V \rangle - 64 \langle df, dh \rangle \langle V, V \rangle \\
&- 128 \text{trace}(df \otimes dh) V^2 + 64 \text{trace}((\text{Hess} f)(\text{Hess} h) V) \right\} d^6 x.
\end{align*}
\]

In the last two terms, both factors are considered as \((1,1)\) tensors (one contravariant and one covariant).

Actually, the difference in between the two expressions \((15)\) and \((16)\) is given by the term

\[
\begin{align*}
\text{15} - \text{16} &= 96 f_{;ij} h_{;kl} W^{ijlk} - 32 (f_{;ij} h_{;kl} W^{ijlk} + f_{;ij} h_{;kl} W^{ijlk}) d^6 x,
\end{align*}
\]

which vanishes in the conformally flat case.

Leaving for an instant the conformally flat case, in the general conformally curved case, the conformal variation of \( \Omega_6(f, h) \), up to order one in \( \eta \) is given by

\[
\begin{align*}
-32 \left\{ \eta_{;ij} f_{;i} h_{;kl} W^{ijlk} + \eta_{;ij} f_{;ij} h_{;kl} W^{ijlk} - \eta_{;ij} f_{;ij} h_{;kl} W^{ijlk} \right\} d^6 x,
\end{align*}
\]

which vanishes in the conformally flat case, meaning that our expression is conformally invariant inside the conformally flat class of metrics on \( M \). In the general conformally curved case, this variation will be useful in finding the extra terms we are missing, that is to say, those terms that vanish in the conformally flat case.
If we define using $\{18\}$ the trilinear form on $C^\infty(M)$
\[ \tau(f_0, f_1, f_2) := \int_M f_0 \Omega_{6,\text{conf, flat}}(f_1, f_2) \]
then
\[
(b \tau)(f_0, f_1, f_2, f_3) = \int_M f_0 \left( -96 (f_{1; j} f_{2; i} f_{3; k} W^{ijk}; l_1) + f_{1; j} f_{2; i} f_{3; k} W^{ijk}; l_1) \\
+ 128 (f_{1; jk} f_{2; i} f_{3; k} W^{ijkl} + f_{1; j} f_{2; i} f_{3; kl} W^{ijkl}) \right) d^6 x
\]
which vanishes in the conformally flat case meaning that $\tau$ is a Hochschild 2-cocycle, in the conformally flat case.

5. A filtration by degree

To simplify the notation, and because of the factor $d^6 x$ in the definition of $\Omega_n$ we will write $\Omega_n$ in the conformally flat case.

For any occurrence of $W, R, f, V, S,$ or $J$ is counted as an occurrence of $R.$ By closing under addition, we denote by $P_n$ the space of these polynomials.

This same idea is used in $\{4\}$ to study leading terms in the heat invariants produced by the Laplacian of de Rham and other complexes. We borrow from there the idea of filtration by degree. For a homogeneous polynomial $Q$ in $P_n,$ we denote by $k_R$ its degree in $R$ and by $k_V$ its degree in $V.$ In this way, $2k_R + k_V = n.$ Because $|\beta| \geq 1,$ and $|\delta| \geq 1$ in Lemma $\{22\}$ we have $k_V \geq 2$ and hence $2k_R \leq n - 2.$

We say that $Q$ is in $P_{n,l}$ if $Q$ can be written as a sum of monomials with $k_R \geq l,$ or equivalently, $k_V \leq n - 2l.$ We have
\[ P_n = P_{n,0} \supseteq P_{n,1} \supseteq P_{n,2} \supseteq \cdots \supseteq P_{n, \frac{n-2}{2}}, \]
and $P_{n,l} = 0$ for $l > (n - 2)/2.$ There is an important observation to make. An expression which a priori appears to be in, say $P_{6,1},$ may actually be in a subspace of it, like $P_{6,2}.$ For example,
\[
\sum_{\in P_{6,1}} f_{1; i j k} h_{i j k} W^{ijkl} = f_{1; i j k} V_{ijkl} W^{ijkl} + f_{1; i j k} W^{ijkl} \kappa^{lm} W_{ijkl},
\]
by reordering covariant derivatives and making use of the symmetries of the Weyl tensor. Because of this filtration, we use a fix convention on how the indices should be placed when representing each expression in its index notation. For example, $f_{1; i j k} h_{i j k}$ will be preferred over $f_{1; ij k} h_{i j k}.$ Also $f_{1; i j} k$ will be preferred over $f_{1; j i} k.$
or any other variation. Once we have defined the filtration on $P_n$, and accepted our notational convention, we can state the following proposition.

**Proposition 5.1.** There exists a universal bilinear form $Q_n(df, dh)$ in $P_{n,0} \sim P_{n,1}$ and a form $Q_{R,n}(df, dh)$ in $P_{n,1}$ such that

$$\Omega_n(f, h) = Q_n(df, dh) + Q_{R,n}(df, dh).$$

In the particular case of the flat metric $\Omega_n(f, h) = Q_n(df, dh)$ since the curvature vanishes.

In the particular case $n = 4$, $k_R$ can be 0 or 1, hence $\Omega_4$ can be written as

$$\Omega_4(f, h) = Q_4(df, dh) + Q_{R,4}(df, dh).$$

where $Q_{R,4}(df, dh)$ is a trilinear form on $R$, $df$, and $dh$.

In the 6-dimensional case, $k_R \in \{0, 1, 2\}$ thus

$$\Omega_6(f, h) = Q_6(df, dh) + Q_{R,6}^{(1,0)}(df, dh) + Q_{R,6}^{(1,1)}(df, dh) + Q_{R,6}^{(1,2)}(df, dh) + Q_{R,6}^{(2,0)}(df, dh),$$

(19)

where

- $Q_{R,6}^{(1,0)}(df, dh) \in P_{6,1} \setminus P_{6,2}$, without covariant derivatives on $R$,
- $Q_{R,6}^{(1,1)}(df, dh) \in P_{6,1} \setminus P_{6,2}$, with a single covariant derivative on $R$,
- $Q_{R,6}^{(1,2)}(df, dh) \in P_{6,1} \setminus P_{6,2}$, with two covariant derivatives on $R$, and
- $Q_{R,6}^{(2,0)}(df, dh) \in P_{6,2}$, without covariant derivatives on $R$.

From the previous expressions, it is evident that there exists a sub-filtration inside each $P_{n,l}$ for $l \geq 1$. Such a filtration is a lot more complicated to describe in higher dimension because of the presence of terms like $\nabla^a R \nabla^c R \cdots$.

6. The 6-dimensional non-conformally flat case

We do not restrict ourselves anymore to the conformally flat case. Now we are going to find those terms we need to add to $\Omega_6, \text{conf. flat}$ in order to get the expression for $\Omega_6$ in the general conformally curved case.

The first set of terms to be added will complete the expression for $Q_{R,6}^{(1,0)}$. In this case, there is just one possibility to be consider, that is

$$f_{;ij}h_{;kl}W^{ijkl}.$$

Any other possibility is ruled out by the relation

$$f_{;i}h_{;jkl}W^{ijkl} = f_{;i}h_{;jkl}W^{ijkl} + f_{;i}h_{;jkl}W^{iklm}W^{jklm}$$

which express the right hand side, an element of $P_{6,1}$, as the sum of two elements of $P_{6,2}$.

The second set of terms completes the expression for $Q_{R,6}^{(1,1)}$, it is given by the following symmetric term on $f$ and $h$:

$$f_{;ij}h_{;kl}W^{ijkl} + f_{;ij}h_{;kl}W^{ijkl}.$$

which express the right hand side, an element of $P_{6,1}$, as the sum of two elements of $P_{6,2}$. 
The only term to complete the expression for $Q^{(1,2)}_{R,6}$ is

$$f; i h; j W^{i}_{k}^{j} l^{k;l},$$

symmetric on $f$ and $h$.

For $Q^{(2,0)}_{R,6}$ we consider at this time, just one term

$$f; i h; j V_{kl} W^{ikjl}.$$

It happens that the other possible terms

$$f; i h; j W_{jklm} W^{jklm}_{klm} d^{6} x$$

and $f; i h; j W^{i}_{klm} W^{iklm}_{km} d^{6} x$.

are conformally invariant.

Up to this point, what we must add to $\Omega_{6,\text{conf, flat}}$ is a linear combination of the form

$$\left\{ A f; i j h; k W_{i j k l}^{i j k l} + B (f; i j h; k W^{i j k l}_{i j k l} + h; i j f; k W^{i j k l}_{i j k l}) + C f; i j h; k W^{i j k l}_{i j k l} + D f; i j h; k V_{kl} W^{ikjl}_{ikjl} \right\} d^{6} x.$$

Its conformal variation up to order one in $\eta$ is given by

$$\left\{ (B + 2C) (\eta; i j h; k W^{i j k l}_{i j k l} + h; i j f; k W^{i j k l}_{i j k l}) + (3B - 2A) (\eta; i j f; k h; i W^{ijkl}_{ijkl} + \eta; i f; j h; k W^{ijkl}_{ijkl}) + (D - 3C) (\eta; i j f; k h; i W^{ijkl}_{ijkl}) \right\} d^{6} x.$$

By comparing it with the conformal variation of $\Omega_{6,\text{conf, flat}}(f, h)$ \[17\], we deduce the conditions $B + 2C = -32 = 3B - 2A$, and $D - 3C = 0$, which means, conformally invariant and symmetry is not enough to find the right values for all the constants. So far, the term to be added to $\Omega_{6}(f, h)$ is given by

$$\left\{ -(32 + 3C) f; i j h; k W^{ikjl}_{ikjl} - (32 + 2C) f; i j h; k W^{ijkl}_{ijkl} + h; i j f; k W^{ijkl}_{ijkl} + C f; i j h; k W^{ijkl}_{ijkl} + 3C f; i j h; k V_{kl} W^{ikjl}_{ikjl} + E f; i h; j W_{jklm} W^{jklm}_{jklm} + G f; i h; j W^{ijkl}_{ijkl} W^{jklm}_{jklm} \right\} d^{6} x$$

where the last two terms come from \[20\].

Using the Hochschild 2-cocycle property

**Proposition 6.1.** If we define the trilinear form on $C^{\infty}(M)$

$$\tau'(f_{0}, f_{1}, f_{2}) :=$$

$$\int_{M} f_{0} \left\{ C f_{1} ; i f_{2} ; j W^{i}_{k}^{j} l^{i k} + D f_{1} ; i f_{2} ; j V_{kl} W^{ikjl} + E f_{1} ; i f_{2} ; j W_{jklm} W^{jklm} + G f_{1} ; i f_{2} ; j W^{ijkl}_{ijkl} W^{jklm}_{jklm} \right\} d^{6} x$$

then

$$(b \tau')(f_{0}, f_{1}, f_{2}, f_{3}) = 0$$

for any $f_{i} \in C^{\infty}(M)$ meaning that we obtain a Hochschild 2-cocycle on the algebra $C^{\infty}(M)$ for any value of the constants $C, D, E, G$. 
On the other hand, if we define
\[
\tau''(f_0, f_1, f_2) := \int_M f_0 \left\{ A f_1 ; ij f_2 ; kl W^{ijkl} \\
+ B \left( f_1 ; ij f_2 ; kl W^{ijkl} ; l \right) + f_2 ; ij f_1 ; kl W^{ijkl} ; l \right\} d^6 x
\]
then
\[
(b \tau'')(f_0, f_1, f_2, f_3)
= \int_M f_0 \left( 3 B \left( f_1 ; ij f_2 ; kl W^{ijkl} ; l \right) + f_1 ; ij f_2 ; kl W^{ijkl} ; l \right)
- 2 A \left( f_1 ; ij f_2 ; kl W^{ijkl} + f_1 ; ij f_2 ; kl W^{ijkl} \right) d^6 x.
\]
(22)

To have that \( \int_M f_0 \Omega_6(f_1, f_2) \) is a Hochschild 2-cocycle on \( C^\infty(M) \) we need \( 13 \) + \( 22 \) = 0, for any \( f_i \in C^\infty(M) \). Thus \( 3B = 96 \) and \( 2A = 128 \). Because \( B + 2C = -32 \), \( 3B - 2A = 96 \) we must have \( C = -32 \) and hence using \( 13 \), \( 20 \), and \( 21 \) we conclude
\[
\Omega_6(f, h) = \Omega_{6 \text{ conf. flat}}(f, h)
+ \left\{ 64 f ; ij h ; kl W^{ijkl} + 32 \left( f ; ij h ; kl W^{ijkl} ; l \right) + f ; ij k ; kl W^{ijkl} ; l \right\}
- 32 f ; ij k ; kl W^{ijkl} ; l - 96 f ; ij h ; kl V^{ijkl}
+ E f ; ij h ; kl W^{ijkl} ; l + G f ; ij k ; kl W^{ijkl} \}
\]
(23)
where the last two terms are the needed ones to fully complete the expression for \( Q^{[2,0]}_{R,6} \) as in \( 20 \).

**Theorem 6.2.** The expression (23) gives a family of 6-dimensional differential forms associated to \( M \), each of these differential forms is symmetric on \( f \) and \( h \), conformally invariant, and defines a Hochschild 2-cocycle on the algebra \( C^\infty(M) \) by the relation \( \tau(f_0, f_1, f_2) = \int_M f_0 \Omega_6(f_1, f_2) \).

In the 6-dimensional conformally curved case, more information is needed to find the unique one satisfying the relation
\[
\text{Wres}(f_0[F, f_1][F, f_2]) = \int_M f_0 \Omega_6(f_1, f_2).
\]

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