Semi-Discrete and Fully Discrete Mixed Finite Element Methods for Maxwell Viscoelastic Model of Wave Propagation

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Abstract. Semi-discrete and fully discrete mixed finite element methods are considered for Maxwell-model-based problems of wave propagation in linear viscoelastic solid. This mixed finite element framework allows the use of a large class of existing mixed conforming finite elements for elasticity in the spatial discretization. In the fully discrete scheme, a Crank-Nicolson scheme is adopted for the approximation of the temporal derivatives of stress and velocity variables. Error estimates of the semi-discrete and fully discrete schemes, as well as an unconditional stability result for the fully discrete scheme, are derived. Numerical experiments are provided to verify the theoretical results.

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Key words: Maxwell viscoelastic model, mixed finite element, semi-discrete and fully discrete, error estimate, stability.

1 Introduction

Let \( \Omega \subset \mathbb{R}^d (d=2 \text{ or } 3) \) be a bounded open domain with boundary \( \partial \Omega \) and \( T \) be a positive constant. Consider the following Maxwell viscoelastic model of wave propagation:

\[
\begin{aligned}
\rho u_{tt} &= \text{div} \sigma + f, & (x,t) &\in \Omega \times [0,T], \\
\sigma + \sigma_t &= C \varepsilon (u_t), & (x,t) &\in \Omega \times [0,T], \\
u &= 0, & (x,t) &\in \partial \Omega \times [0,T], \\
u(x,0) &= \phi_0(x), & u_t(x,0) &= \phi_1(x), & \sigma(x,0) &= \psi_0(x), & x &\in \Omega.
\end{aligned}
\]

(1.1)

Here \( u = (u_1, \cdots, u_d)^T \) is the displacement field, \( \sigma = (\sigma_{ij})_{d \times d} \) the symmetric stress tensor, \( \varepsilon (u) = (\nabla u + (\nabla u)^T)/2 \) the strain tensor, and \( g_t := \partial g / \partial t \) and \( g_{tt} := \partial^2 g / \partial t^2 \) for any function.

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$g(x,t)$. $\rho(x)$ denotes the mass density, and $\mathbb{C}$ a rank 4 symmetric tensor, with

$$0 < \rho_0 \leq \rho \leq \rho_1 < \infty, \quad \text{a.e. } x \in \Omega,$$

$$0 < M_0 \tau : \tau \leq C^{-1} \tau : \tau \leq M_1 \tau : \tau, \quad \forall \text{ symmetric tensor } \tau = (\tau_{ij})_{d \times d}, \quad \text{a.e. } x \in \Omega,$$

(1.2a)

(1.2b)

where $\rho_0, \rho_1, M_0$ and $M_1$ are four positive constants, and

$$\sigma : \tau := \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_{ij} \tau_{ij}.$$ 

Note that $C\varepsilon(u_t)$ is of the form

$$C\varepsilon(u_t) = 2\mu\varepsilon(u_t) + \lambda \text{div} u_t I$$

(1.3)

for an isotropic elastic medium, where $\mu, \lambda$ are the Lamé parameters, and $I$ the identity matrix. $f = (f_1, \cdots, f_d)^T$ is the body force, and $\phi_0(x), \phi_1(x), \psi_0(x)$ are initial data.

Numerous materials simultaneously display elastic and viscous kinematic behaviours. Such a feature, called viscoelasticity, can be characterized by using springs, which obey the Hooke’s law, and viscous dashpots, which obey the Newton’s law. Different combinations of the springs and dashpots lead to various viscoelastic models, e.g., the three classical models of Zener, Voigt and Maxwell. We note that there is a unified framework to describe the general constitutive law of viscoelasticity by using convolution integrals in time with some kernels [8, 11, 29]; however, the integral forms of constitutive laws, compared with the differential forms, bring more difficulties to the design of algorithms due to the numerical convolution integral. We refer the reader to [5, 9–13, 30–32] for several monographs on the development and applications of viscoelasticity theory.

The numerical simulation of wave propagation in viscoelastic materials was first discussed by Kosloff et al. in [20, 21], where memory variables were introduced to avoid the convolutional integral in the constitutive relation. Later, finite difference methods were developed in [6, 28, 35] for the model with memory variables. There are considerable research efforts on the finite element discretization in this field. In [18] Janovsky et al. studied the continuous/discontinuous Galerkin finite element discretization and used a numerical quadrature formula to approximate the Volterra time integral term. Ha et al. [14] proposed a nonconforming finite element method for a viscoelastic complex model in the space–frequency domain. Bécache et al. [3] applied a family of mixed finite elements with mass lumping, together with a leap-frog scheme in time discretization, to numerically solve the Zener model, and showed that their scheme is stable under certain CFL condition. In [24–26], Rivièrè et al. analyzed discontinuous Galerkin methods with a Crank-Nicolson temporal discrete scheme for quasistatic linear viscoelasticity and linear/nonlinear diffusion viscoelastic models. Rognes and Winther [27] proposed mixed finite element methods for quasistatic Maxwell and Voigt models using weak symmetry, and used a second backward difference scheme in the full discretization. Lee [22] studied mixed finite element methods with weak symmetry for the Zener, Voigt and
Maxwell models and adopted the Crank-Nicolson scheme in temporal discretization. Severino and Guillermo [31] gave an overview of numerical methods for problems in viscoelasticity including finite elements, boundary elements, and finite volume formulations. Kimura et al. [19] studied the gradient flow structure of an extended Maxwell model with a relaxation parameter and proposed a structure-preserving P1/P0 finite element scheme. Recently, Wang and Xie [34] analyzed a hybrid stress finite element method for the Maxwell model and used a second order implicit difference in the fully discrete scheme.

In this paper, we consider semi-discrete and fully discrete mixed finite element discretizations for the Maxwell viscoelasticity model (1.1). Some existing mixed conforming finite elements for elasticity are applied in the spatial discretization to approximate the stress and velocity variables. In the full discretization, the Crank-Nicolson scheme is adopted to discretize the temporal derivatives of stress and velocity. We derive optimal error estimates for both the semi-discrete and fully discrete schemes, and give an unconditional stability result for the fully discrete scheme.

The rest of this paper is arranged as follows. Section 2 introduces notations and weak formulations. Section 3 gives a general mixed conforming finite element framework and carries out the error estimation of the semi-discrete scheme. Section 4 presents the fully discrete scheme and derives stability and error estimates. Finally, numerical examples are provided in Section 5 to verify the theory results when using two low order rectangular elements in the spatial discretization.

2 Weak formulations

We first introduce some notations. For any nonnegative integer \(r\), denote by \(H^r(\Omega)\) and \(H^r_0(\Omega)\) the standard Sobolev spaces with norm \(\|\cdot\|_r\) and semi-norm \(|\cdot|_r\). In particular, \(H^0(\Omega) = L^2(\Omega)\) is the space of square integrable functions. We adopt the convention that an underline (or double underlines) corresponds to a vector-valued (or tensor-valued) space.

For any vector-valued (or tensor-valued) space \(X\), defined on \(\Omega\), with norm \(\|\cdot\|_X\), denote
\[
L^p([0,T];X) := \{ \mathbf{w} : [0,T] \to X ; \|\mathbf{w}\|_{L^p(X)} < \infty \},
\]
where
\[
\|\mathbf{w}\|_{L^p(X)} := \begin{cases} \left( \int_0^T \|\mathbf{w}(t)\|^p_X \right)^{1/p} , & \text{if } 1 \leq p < \infty, \\ \text{esssup}_{0 \leq t \leq T} \|\mathbf{w}(t)\|_X , & \text{if } p = \infty, \end{cases}
\]
and \(\mathbf{w}(t)\) abbreviates \(\mathbf{w}(x,t)\). For integer \(r \geq 0\), the space \(C^r([0,T];X)\) can be defined similarly. In the forthcoming analysis, \(X\) may be taken as \(L^2(\Omega)\), \(H^1(\Omega)\) and
\[
\mathbb{H}(\text{div},\Omega,S) := \{ \mathbf{\tau} = (\tau_{ij})_{d \times d} \in L^2(\Omega) | \tau_{ij} = \tau_{ji} , \text{div}\mathbf{\tau} \in L^2(\Omega) \}.
\]
For convenience, we use the notation \( a \lesssim b \) to denote that there exists a generic positive constant \( C \), independent of the spatial and temporal mesh parameters, \( h \) and \( \Delta t \), such that \( a \leq Cb \).

We also need two Gronwall’s inequalities [33]:

**Continuous Gronwall’s inequality.** Let \( \phi(\cdot) \) be such that

\[
\phi(t) \leq \rho \phi(t) + \eta(t) \quad \text{for} \quad 0 \leq t \leq T,
\]

where \( \rho \geq 0 \) is a constant and \( \eta(\cdot) \geq 0, \eta \in L^1([0,T]) \). Then it holds

\[
\phi(t) \leq e^{\rho T} \left( \phi(0) + \int_0^T \eta(s) ds \right), \quad \forall t \in [0,T].
\]  (2.1)

**Discrete Gronwall’s inequality.** Let \( g_0 \geq 0 \) and two nonnegative sequences \( (k_n)_{n \geq 0}, (p_n)_{n \geq 0} \) be given. Suppose that the sequence \( (\phi_n)_{n \geq 0} \) satisfies

\[
\begin{cases}
\phi_0 \leq g_0, \\
\phi_n \leq g_0 + \sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s \phi_s, \quad \forall n \geq 1.
\end{cases}
\]  (2.2)

Then it holds

\[
\phi_n \leq \left( g_0 + \sum_{s=0}^{n-1} p_s \right) \exp \left( \sum_{s=0}^{n-1} k_s \right), \quad \forall n \geq 1.
\]  (2.3)

We are now in a position to give the weak form of the Maxwell model (1.1). By introducing the velocity variable \( \mathbf{v} = \mathbf{u}_t \), the model changes into a velocity-stress form:

\[
\begin{cases}
\mathbf{\rho} \mathbf{v}_t = \text{div} \mathbf{\sigma} + \mathbf{f}(x,t), & (x,t) \in \Omega \times [0,T], \\
\mathbf{\sigma} + \mathbf{\sigma}_t = \mathbf{C} \epsilon(\mathbf{v}), & (x,t) \in \Omega \times [0,T], \\
\mathbf{v} = 0, & (x,t) \in \partial \Omega \times [0,T], \\
\mathbf{v}(x,0) = \phi_1(x), & \mathbf{\sigma}(x,0) = \psi_0(x), \quad x \in \Omega.
\end{cases}
\]  (2.4)

Based on the Hellinger-Reissner variational principle, the weak problem for (1.1) reads as: Find \( (\mathbf{\sigma}, \mathbf{v}) \in C^1([0,T], H(\text{div}, \Omega, S)) \times C^1([0,T], L^2(\Omega)) \) such that

\[
\begin{cases}
a(\mathbf{\sigma}, \tau) + a(\mathbf{\sigma}, \tau) + b(\mathbf{v}, \tau) = 0, \quad \forall \tau \in \mathbf{H}(\text{div}, \Omega, S), \\
c(\mathbf{v}, w) - b(w, \mathbf{\sigma}) = (\mathbf{f}, w), \quad \forall w \in L^2(\Omega), \\
\mathbf{v}(x,0) = \phi_1, \quad \mathbf{\sigma}(x,0) = \psi_0.
\end{cases}
\]  (2.5)

Here

\[
a(\mathbf{\sigma}, \tau) := \int_{\Omega} \mathbf{C}^{-1} \mathbf{\sigma} : \tau dx, \quad b(\mathbf{v}, \tau) := \int_{\Omega} \mathbf{v} \cdot \text{div} \tau dx, \quad c(\mathbf{v}, w) := \int_{\Omega} \mathbf{\rho} \mathbf{v} \cdot w dx,
\]
where \[ \hat{\tau} : \tau = \sum_{i,j=1}^{d} \hat{\tau}_{ij} \tau_{ij} \quad \text{for} \quad \hat{\tau}, \tau \in H(\text{div}, \Omega, S). \]

For any \( \tau \in H(\text{div}, \Omega, S) \), \( w \in L^2(\Omega) \), define
\[
\| \tau \|^2_a := a(\tau, \tau), \quad \| w \|^2_c := c(w, w).
\]

Then, according to (1.2a) and (1.2b), it holds
\[
\sqrt{M_0} \| \tau \|_0 \leq \| \tau \|_a \leq \sqrt{M_1} \| \tau \|_0, \quad \sqrt{\rho_0} \| w \|_0 \leq \| w \|_c \leq \sqrt{\rho_1} \| w \|_0,
\]
which also give
\[
M_0 \| \tau \|^2_0 \leq a(\tau, \tau), \quad \rho_0 \| w \|^2_0 \leq c(w, w).
\]

Simultaneously, the following stability conditions hold [7]:

(i) Coercivity of \( a(\cdot, \cdot) \) on \( Z := \{ \tau \in H(\text{div}, \Omega, S); b(v, \tau) = 0, \forall v \in L^2(\Omega) \} \):
\[
\| \tau \|^2_{\text{div}} \lesssim a(\tau, \tau), \quad \forall \tau \in Z,
\]
where
\[
\| \tau \|^2_{\text{div}} := \| \tau \|^2_0 + \| \text{div} \tau \|^2_0.
\]

(ii) Inf-sup condition for \( b(\cdot, \cdot) \):
\[
\| w \|_0 \lesssim \sup_{0 \neq \tau \in H(\text{div}, \Omega, S)} \frac{b(w, \tau)}{\| \tau \|_{\text{div}}}, \quad \forall w \in L^2(\Omega).
\]

From [22, Theorem 5.1], the following result of existence and uniqueness holds.

**Lemma 2.1.** Suppose \( \phi_0 \in H^1_0(\Omega) \), \( \phi_1 \in L^2(\Omega) \), \( c_0 \in H(\text{div}, \Omega, S) \) and \( f \in C^0([0, T], L^2(\Omega)) \), then the weak problem (2.5) admits a unique solution \( (\sigma, v) \in C^1([0, T], H(\text{div}, \Omega, S)) \times C^1([0, T], L^2(\Omega)) \).

### 3 Semi-discrete mixed finite element method

In this section, we discuss the semi-discrete finite element discretization of (2.5) and analyze its convergence under a general conforming mixed FEM framework.
3.1 Semi-discrete scheme

Assume that $\Omega$ is a convex polyhedral domain, and let $\mathcal{T}_h = \bigcup \{K\}$ be a shape regular partition of $\Omega$ consisting of triangles/tetrahedrons or rectangles/cuboids. For any $K \in \mathcal{T}_h$, let $h_K$ denote its diameter, and we set $h := \max_{K \in \mathcal{T}_h} h_K$. For any integer $k \geq 0$, let $P_k(K)$ denote the set of all polynomials on $K$ of degree at most $k$, and let $Q_k(K)$ denote the set of all polynomials on $K$ of degree at most $k$ in each variable.

Let $H_h \subset H(\text{div}, \Omega)$ and $V_h \subset L^2(\Omega)$ be two finite-dimensional spaces respectively for stress and velocity approximations on $\mathcal{T}_h$, satisfying the following condition:

(A1) Discrete inf-sup condition:

$$\|w_h\|_0 \lesssim \sup_{0 \neq \tau_h \in H_h} \frac{b(w_h, \tau_h)}{\|\tau_h\|_{\text{div}}}, \quad \forall w_h \in V_h.$$  

From (2.7) we easily obtain the following two inequalities:

\begin{align}
M_0 \|\tau_h\|^2_0 &\leq a(\tau_h, \tau_h), \quad \forall \tau_h \in H_h, \quad (3.1a) \\
\rho_0 \|w_h\|^2_0 &\leq c(w_h, w_h), \quad \forall w_h \in V_h. \quad (3.1b)
\end{align}

Let $\phi_{1,h}, \psi_{0,h}$ be respectively approximations of initial data $\phi_1$ and $\psi_0$, then the generic semi-discrete mixed conforming finite element scheme reads as: Find $(\sigma_h, v_h) \in C^1([0,T], H_h) \times C^1([0,T], V_h)$ such that

$$\begin{cases}
a(\sigma_h, \tau_h) + a(\sigma_h, \tau_h) + b(v_h, \tau_h) = 0, \quad \forall \tau_h \in H_h, \\
c(v_h, w_h) - b(w_h, \sigma_h) = (f, w_h), \quad \forall w_h \in V_h, \\
v_h(x, 0) = \phi_{1,h}, \quad \sigma_h(x, 0) = \psi_{0,h}. 
\end{cases} \quad (3.2)$$

By using standard techniques, we can obtain the following conclusion.

**Lemma 3.1.** The semi-discrete scheme (3.2) admits a unique solution $(\sigma_h, v_h)$.

**Proof.** Let $\{q_i\}_{i=1}^r, \{\kappa_i\}_{i=1}^s$ be bases of $H_h$ and $V_h$ respectively. Let $(i,j)$-th components of matrix $A, B, C$ be

$$(C^{-1} q_i, q_i), \quad (\text{div} q_i, \kappa_i), \quad (\kappa_i, \kappa_i),$$

respectively. We write

$$\sigma_h = \sum_i a_i q_i, \quad v_h = \sum_i \beta_i \kappa_i, \quad \eta_j = (f, \kappa_j),$$

where $a_i, \beta_i, \eta_j$ are coefficients to be determined.
and denote by \( \alpha, \beta, \eta \) the corresponding vectors, respectively. Then we rewrite (3.2) as a matrix equation of the form
\[
\begin{pmatrix}
A & 0 \\
0 & C
\end{pmatrix}
\begin{pmatrix}
\alpha_t \\
\beta_t
\end{pmatrix}
= \begin{pmatrix}
-A & -B \\
0 & B^T
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
+ \begin{pmatrix}
0 \\
\eta
\end{pmatrix}.
\] (3.3)

The coefficient matrix on the left side of the equation is nonsingular because \( A, C \) are symmetric positive definite. Thus, due to the standard theory of ordinary differential equations, the system (3.3), and also (3.2), admits a unique solution.

### 3.2 Error estimation

To carry out the error estimation, we need to introduce, for the solution \((\sigma(t), \nu(t)) \in H(\text{div}, \Omega, S) \times L^2(\Omega)\) to the weak problem (2.5), an “elliptic projection” \((\Pi_1 \sigma, \Pi_2 \nu) \in H_h \times V_h\), which are defined as follows: for \( t \in [0, T]\), let \((\Pi_1 \sigma, \Pi_2 \nu) := (\hat{\sigma}(t), \hat{\nu}(t)) \in H_h \times V_h\) satisfy
\[
\begin{align*}
\begin{aligned}
a(\hat{\sigma}(t), \tau_h) + b(\hat{\nu}(t), \tau_h) &= -a(\partial_t \sigma, \tau_h), & & \forall \tau_h \in H_h, \\
b(\nu_h, \hat{\sigma}(t)) &= b(v_h, \sigma) = c(v_h, w_h) - (f, w_h), & & \forall w_h \in V_h.
\end{aligned}
\end{align*}
\] (3.4)

By (A1) and (3.1) it is easy to see that the “elliptic projection” is well-defined.

To derive convergence rates we also make the following regularity and approximation assumptions:

(A2) Let \((\sigma, \nu)\), the weak solution to (2.5), and its elliptic projection \((\hat{\sigma}(t), \hat{\nu}(t))\) satisfy the regularity conditions
\[
\begin{align*}
\sigma & \in L^\infty([0, T], H^m(\Omega)), & & \sigma_t \in L^2([0, T], H^m(\Omega)), \\
\nu & \in L^\infty([0, T], H^{m'}(\Omega)), & & \nu_t \in L^2([0, T], H^{m'}(\Omega)),
\end{align*}
\] (3.5)

and the approximation conditions
\[
\begin{align*}
\|\hat{\sigma}_t - \sigma_t\|_0 + \|\hat{\nu}_t - \nu_t\|_0 & \lesssim h^l (\|\sigma\|_m + \|\nu\|_{m'}), \\
\|\hat{\sigma}_t - \sigma_t\|_0 + \|\hat{\nu}_t - \nu_t\|_0 & \lesssim h^l (\|\sigma_t\|_m + \|\nu_t\|_{m'}),
\end{align*}
\] (3.6)

where \( l, m, m' \geq 0 \) are integers, and \( g_h \assign \partial g / \partial t \) with \( g = \hat{\sigma}, \hat{\nu} \).

**Remark 3.1.** In the following, we introduce for \( d = 2 \) and 3 several pairs of \( H_h \) and \( V_h \) which satisfy both (3.6) and the discrete inf-sup condition (A1).

- **Arnold-Winther’s triangular elements** (\( d = 2 \)) [2]:
\[
H_h = \left\{ \tau \in H(\text{div}, \Omega, S); \tau_{ij} \mid K \in P_{k+2}(K), \text{ div}(\tau) \in P_k(\Omega)^2, \forall K \in T_h \right\},
\]
\[
V_h = \left\{ w \in L^2(\Omega); w \mid K \in P_k(\Omega), \forall K \in T_h \right\},
\]
where \( k \geq 1 \). The estimates in (3.6) hold with \( l = m = k + 1 \) and \( m' = k + 2 \).
Arnold-Awanou’s rectangular elements \((d=2)\) [1]:

\[
\begin{align*}
\mathbb{H}_h &= \left\{ \tau \in \mathbb{H}(\text{div}, \Omega, S); \quad \tau_{i1}|_K \in P_{k+4,\ell+2}(K), \quad \tau_{i2}|_K \in P_{k+3,\ell+3}(K), \right. \\
&\quad \left. \tau_{22}|_K \in P_{k+2,\ell+4}(K), \quad \forall K \in T_h \right\}, \\
\mathbb{V}_h &= \left\{ w \in L^2(\Omega); \quad w_1|_K \in P_{k+1,\ell}(K), \quad w_2|_K \in P_{k,\ell+1}(K), \quad \forall K \in T_h \right\},
\end{align*}
\]

where \(k \geq 1\), and \(P_{r,s}(K)\) denotes the set of polynomials, defined on \(K\), of degree at most \(r\) in \(x_1\) and of degree at most \(s\) in \(x_2\) for \(x = (x_1,x_2) \in K\). The estimates in (3.6) hold with \(l = m = k+1\) and \(m' = k+2\).

Hu-Zhang’s triangular/tetrahedral elements \((d=2,3)\) [15, 17]:

\[
\begin{align*}
\mathbb{H}_h &= \left\{ \tau \in \mathbb{H}(\text{div}, \Omega, S); \quad \tau_{ij}|_K \in P_{k+d}(K), \quad \forall K \in T_h \right\}, \\
\mathbb{V}_h &= \left\{ w \in L^2(\Omega); \quad w_i|_K \in P_{k+d-1}(K), \quad \forall K \in T_h \right\},
\end{align*}
\]

where \(k \geq 2\). The estimates in (3.6) hold with \(l = m' = k+d-1\) and \(m = k+d\).

Hu-Man-Zhang’s rectangular/cuboid element \((d=2,3)\) [16]:

\[
\begin{align*}
\mathbb{H}_h &= \left\{ \tau \in \mathbb{H}(\text{div}, \Omega, S); \quad \tau_{ij}|_K \in \text{span}\{1,x_i,x_j^2\}, \\
&\quad \quad \text{span}\{1,x_i,x_j,x_{ij}\}, i \neq j, \quad \forall K \in T_h \right\}, \\
\mathbb{V}_h &= \left\{ w \in L^2(\Omega); \quad w_i|_K \in \text{span}\{1,x_i\}, \quad \forall K \in T_h \right\}.
\end{align*}
\]

The estimates in (3.6) hold with \(l = m' = 1\) and \(m = 2\).

Nedelec’s rectangular/cuboid elements \((d=2,3)\) [4, 23]:

\[
\begin{align*}
\mathbb{H}_h &= \left\{ \tau \in \mathbb{H}(\text{div}, \Omega, S); \quad \tau_{ij}|_K \in Q_{k+1}(K), \quad \forall K \in T_h \right\}, \\
\mathbb{V}_h &= \left\{ w \in L^2(\Omega); \quad w_i|_K \in Q_k(K), \quad \forall K \in T_h \right\},
\end{align*}
\]

where \(k \geq 0\). The estimates in (3.6) hold with \(l = k+1\) and \(m = m' = k+2\). We mention that in [4] the degrees of freedom of Nedelec’s rectangular elements \(Q^\text{div}_{k+1} - Q_k\) \((k \geq 0)\) (cf. [23]) are modified so as to allow mass lumping.

In what follows, we choose the initial data in (3.2) as

\[
\psi_{0,h} = I_{\mathbb{H}} \psi_0, \quad \phi_{1,h} = I_{\mathbb{V}} \phi_1,
\]

(3.7)
where \( I_{H_h} : H(\text{div}, \Omega, S) \rightarrow H_h, I_{V_h} : L^2(\Omega) \rightarrow V_h \) be two interpolation operators satisfying

\[
\|\psi_0 - I_{H_h} \psi_0\|_0 \lesssim h^l \|\psi_0\|_l, \quad \|\phi_1 - I_{V_h} \phi_1\|_0 \lesssim h^l \|\phi_1\|_l
\]  

(3.8)

for

\[
\psi_0 \in H^l(\Omega), \quad \phi_1 \in H^l(\Omega).
\]

**Theorem 3.1.** Let \((\sigma, v) \in C^1([0, T], H(\text{div}, \Omega, S)) \times C^1([0, T], L^2(\Omega))\) be the solution of the weak problem (2.5) and \((\sigma_h, v_h) \in C^1([0, T], H_h) \times C^1([0, T], V_h)\) be the solution of the semi-discrete problem (3.2). Then, under the assumptions (A1), (A2) and (3.8) we have

\[
\|\sigma - \sigma_h\|_{L^\infty([0, T], L^2)} + \|v - v_h\|_{L^\infty([0, T], L^2)} \lesssim h^l \left( \|\psi_0\|_l + \|\phi_1\|_l + \|\sigma\|_{L^2(\Omega)} + \|\sigma_t\|_{L^2(\Omega)} + \|\nabla v\|_{L^\infty(\Omega')} + \|v_t\|_{L^2(\Omega')} \right).
\]  

(3.9)

**Proof.** In light of (3.6), (3.4) and the triangle inequality, it suffices to show the estimate

\[
\|\dot{\sigma}_h - \dot{\sigma}_h\|_0 + \|\ddot{\sigma}_h - \ddot{\sigma}_h\|_0 \lesssim h^l \left( \|\psi_0\|_l + \|\phi_1\|_l + \|\sigma\|_{L^2(\Omega)} + \|\sigma_t\|_{L^2(\Omega)} + \|\nabla v\|_{L^\infty(\Omega')} + \|v_t\|_{L^2(\Omega')} \right).
\]  

(3.10)

From (3.4) and (2.5) it follows

\[
a(\partial_t (\sigma - \sigma_h), \tau_h) + a(\dot{\sigma}_h - \sigma_h, \tau_h) + b(\dot{\sigma}_h - v_h, \tau_h) = 0, \quad \forall \tau_h \in H_h,
\]

\[
c(\partial_t (v - v_h), \omega_h) = b(\omega_h, \dot{\sigma}_h - \sigma_h), \quad \forall \omega_h \in V_h.
\]

Denote \( \ddot{\sigma}_h := \dot{\sigma}_h - \dot{\sigma}_h, \dddot{\sigma}_h := \ddot{\sigma}_h - v_h, \) and take \( \tau_h = \dot{\sigma}_h, \omega_h = \dddot{\sigma}_h \) in the above two equations, then we get

\[
a(\partial_t (\sigma - \sigma_h), \dot{\sigma}_h) + a(\dot{\sigma}_h, \dddot{\sigma}_h) + c(\partial_t (v - v_h), \dddot{\sigma}_h) = 0,
\]

which yields

\[
a(\partial_t (\sigma - \sigma_h), \dot{\sigma}_h) + c(\partial_t (v - v_h), \dddot{\sigma}_h) \leq 0.
\]

This, together with the relations \( \sigma - \sigma_h = \dot{\sigma}_h + \dddot{\sigma}_h \) and \( v - v_h = \dddot{\sigma}_h + \dddot{\sigma}_h \), implies

\[
a(\partial_t \dot{\sigma}_h, \dddot{\sigma}_h) + c(\partial_t \dddot{\sigma}_h, \dddot{\sigma}_h) \leq a(\partial_t (\dot{\sigma}_h - \sigma), \dot{\sigma}_h) + c(\partial_t (v_h - v), \dddot{\sigma}_h).
\]

Thus, by setting \( E_h := a(\dot{\sigma}_h, \dddot{\sigma}_h) + c(\dddot{\sigma}_h, \dddot{\sigma}_h) \) we can obtain

\[
\frac{dE_h}{dt} \leq 2a(\partial_t (\dot{\sigma}_h - \sigma), \dot{\sigma}_h) + 2c(\partial_t (\dot{\sigma}_h - v), \dddot{\sigma}_h)
\]

\[
\leq E_h + a(\partial_t (\dot{\sigma}_h - \sigma), \partial_t (\dot{\sigma}_h - \sigma)) + c(\partial_t (\dot{\sigma}_h - v), \partial_t (\dot{\sigma}_h - v)),
\]
where we have used the following two inequalities:

\[ 2a(\sigma, \tau) \leq a(\sigma, \sigma) + a(\tau, \tau), \quad 2c(v, w) \leq c(v, v) + c(w, w). \]

By the continuous Gronwall inequality we deduce that

\[
E_h = \| \partial_t \|_a^2 + \| \bar{\sigma}_h \|_2^2 \\
\leq \| \sigma_h(0) \|_a^2 + \| \bar{\sigma}_h(0) \|_2^2 + \int_0^T \| \partial_t (\partial_t (s) - \sigma(s)) \|_2^2 + \| \bar{\sigma}_h (s) - v(s) \|_2^2 ds
\]

As a result, the desired estimate (3.10) follows from the initial data condition (3.8), the assumption (A2) and the equivalence of norms in (2.7).

\[ \square \]

### 4 Fully discrete mixed finite element method

#### 4.1 Fully discrete scheme

Let \( 0 = t_0 < t_1 < \cdots < t_M = T \) be a uniform division of time domain \([0, T]\), with \( t_i = i\Delta t \), \((i = 0, 1, \cdots, M)\), and the time step size \( \Delta t := \frac{T}{M} \). For any function \( \varphi(t) \), we set

\[
q^n := \varphi(t_n), \quad q^{n+\frac{1}{2}} := \frac{q^n + q^{n+1}}{2}, \quad \Delta t q^{n+\frac{1}{2}} := \frac{q^{n+1} - q^n}{\Delta t}.
\]

Based on the semi-discrete scheme (3.2), a Crank-Nicolson full discretization scheme is given as follows: Find \((\sigma_h^{n+1}, v_h^{n+1}) \in \mathbb{H}_h \times \mathbb{V}_h\) for \(0 \leq n \leq M - 1\) such that

\[
\begin{align*}
\quad a(\Delta t \sigma^{n+\frac{1}{2}}_h, \tau_h) + a(\sigma^{n+\frac{1}{2}}_h, \tau_h) + b(v^{n+\frac{1}{2}}_h, \tau_h) &= 0, \quad \forall \tau_h \in \mathbb{H}_h, \\
\quad c(\Delta t \sigma^{n+1}_h, \omega_h) - b(\sigma^{n+1}_h, \sigma^{n+1}_h) &= (f^{n+1}_h, \omega_h), \quad \forall \omega_h \in \mathbb{V}_h,
\end{align*}
\]

with the initial data \(v^0_h = \phi_{1,h}\) and \(a^0_h = \psi_{0,h}\) given by (3.7).

**Lemma 4.1.** The fully discrete scheme (4.1) admits a unique solution \((\sigma_h^{n+1}, v_h^{n+1})\) for \(n = 0, 1, \cdots, M - 1\).

**Proof.** We only need to show that, when given \((\sigma_h^n, v_h^n)\), the linear system (4.1) admits a unique solution \((\sigma_h^{n+1}, v_h^{n+1})\). Since this is a square system, it suffices to show that the homogeneous system

\[
\begin{align*}
\quad a(\sigma^{n+1}_h, \tau_h) + \frac{\Delta t}{2} a(\sigma^{n+1}_h, \tau_h) + \frac{\Delta t}{2} b(v^{n+1}_h, \tau_h) &= 0, \quad \forall \tau_h \in \mathbb{H}_h, \\
\quad c(\sigma^{n+1}_h, \omega_h) - \frac{\Delta t}{2} b(\sigma^{n+1}_h, \sigma^{n+1}_h) &= 0, \quad \forall \omega_h \in \mathbb{V}_h,
\end{align*}
\]
yields the conclusion that
\[ \sigma_h^{n+1} = 0, \quad v_h^{n+1} = 0. \] (4.2)

In fact, taking \( \tau_h = \sigma_h^{n+1} \) and \( \omega_h = v_h \) in the above system leads to
\[ \left( 1 + \frac{\Delta t}{2} \right) a(\sigma_h^{n+1}, \sigma_h^{n+1}) + c(\sigma_h^{n+1}, v_h^{n+1}) = 0, \]
then (4.2) follows. This completes the proof. \( \square \)

4.2 Stability analysis

Lemma 4.2. For \( J = 1, \ldots, M \), it holds
\[ \| \sigma_h^n \|_a^2 + \| v_h^n \|_c^2 \leq 2\Delta t \sum_{n=0}^{J-1} (f^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}}) + \| \sigma_h^n \|_a^2 + \| v_h^n \|_c^2. \] (4.3)

Proof. Take \( \tau_h = \sigma_h^{n+\frac{1}{2}} \) and \( \omega_h = v_h^{n+\frac{1}{2}} \) in (4.1) and add up the two equations, we then get
\[ a(\Delta t, \sigma_h^{n+\frac{1}{2}}, \sigma_h^{n+\frac{1}{2}}) + a(\sigma_h^{n+\frac{1}{2}}, \sigma_h^{n+\frac{1}{2}}) + c(\Delta t, v_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}}) = (f^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}}). \]

By the symmetry of \( C^{-1} \), we deduce that
\[ a(\Delta t, \sigma_h^{n+\frac{1}{2}}, \sigma_h^{n+\frac{1}{2}}) = (C^{-1} \Delta t, \sigma_h^{n+\frac{1}{2}}, \sigma_h^{n+\frac{1}{2}}) \]
\[ = (C^{-1} \Delta t, \sigma_h^{n+\frac{1}{2}} - \sigma_h^n, \sigma_h^{n+\frac{1}{2}}) \]
\[ = \frac{1}{2\Delta t} (\| \sigma_h^{n+1} \|_a^2 - \| \sigma_h^n \|_a^2). \]

Similarly, we have
\[ c(\Delta t, v_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}}) = \frac{1}{2\Delta t} (\| v_h^{n+1} \|_c^2 - \| v_h^n \|_c^2). \]

From the two relations above it follows that
\[ \frac{1}{2\Delta t} (\| \sigma_h^{n+1} \|_a^2 - \| \sigma_h^n \|_a^2 + \| v_h^{n+1} \|_c^2 - \| v_h^n \|_c^2) + \| \sigma_h^{n+\frac{1}{2}} \|_a^2 = (f^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}}). \]

Summing up this equation from \( n = 0, 1, \ldots, J-1 \) gives
\[ \| \sigma_h^n \|_a^2 + \| v_h^n \|_c^2 + 2\Delta t \sum_{n=0}^{J-1} \| \sigma_h^{n+\frac{1}{2}} \|_a^2 = 2\Delta t \sum_{n=0}^{J-1} (f^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}}) + \| \sigma_h^0 \|_a^2 + \| v_h^0 \|_c^2, \]
which indicates the desired result. \( \square \)
\textbf{Theorem 4.1.} Assume $\Delta t < 1$, then the full discretization scheme (4.1) is unconditionally stable in the following sense: For $J = 1, \cdots, M$, it holds
\[
\|\sigma_h^j\|_{\infty}^2 + \|\rho_h^j\|_{\infty}^2 \leq \|\sigma_h^0\|_{\infty}^2 + \|\rho_h^0\|_{\infty}^2 + \|f\|_{L^2(t)}^2.
\] (4.4)

\textbf{Proof.} From (4.3) and Cauchy-Schwarz inequality we get
\[
\|\sigma_h^j\|_{\infty}^2 + \|\rho_h^j\|_{\infty}^2 \leq \Delta t \sum_{n=0}^{J} \rho^{-1} \|f^{n+\frac{1}{2}}\|_0^2 + \Delta t \sum_{n=0}^{J} \|\sigma_h^{n+\frac{1}{2}}\|_{\infty}^2 + \|\sigma_h^0\|_{\infty}^2 + \|\rho_h^0\|_{\infty}^2.
\]

On the other hand,
\[
\sum_{n=0}^{J-1} \|\sigma_h^{n+\frac{1}{2}}\|_{\infty}^2 = \sum_{n=0}^{J-1} \left| \frac{\sigma_h^{n+1} - \sigma_h^n}{\Delta t} \right|_{\infty}^2 \\
\leq \frac{1}{2} \left( \sum_{n=0}^{J-1} \|\sigma_h^{n+1}\|_{\infty}^2 + \sum_{n=0}^{J-1} \|\sigma_h^n\|_{\infty}^2 \right) \\
\leq \frac{1}{2} \sum_{n=0}^{J-1} \|\sigma_h^n\|_{\infty}^2 + \frac{1}{2} \|\sigma_h^0\|_{\infty}^2.
\]

Since $\Delta t < 1$, the two inequalities above indicate
\[
\frac{1}{2} \|\sigma_h^j\|_{\infty}^2 + \|\rho_h^j\|_{\infty}^2 \leq \Delta t \sum_{n=0}^{J} \rho^{-1} \|f^{n+\frac{1}{2}}\|_0^2 + \Delta t \sum_{n=0}^{J} \|\sigma_h^{n+\frac{1}{2}}\|_{\infty}^2 + \|\sigma_h^0\|_{\infty}^2 + \|\rho_h^0\|_{\infty}^2,
\]

which, together with the discrete Gronwall’s inequality, yields
\[
\|\sigma_h^j\|_{\infty}^2 + \|\rho_h^j\|_{\infty}^2 \leq \left( \|\sigma_h^0\|_{\infty}^2 + \|\rho_h^0\|_{\infty}^2 + T \rho^{-1} \|f\|_{L^2(t)}^2 \right) \times \exp(2T),
\]

i.e., (4.4) holds true. \hfill \Box

\subsection{4.3 Error estimation}

\textbf{Lemma 4.3.} Let $(\sigma_h^j, \rho_h^j)$ $(j = 1, \cdots, M)$ and $(\hat{\sigma}_h, \hat{\rho}_h)$ be respectively the solutions to the fully discrete scheme (4.1) and the semi-discrete "elliptic projection" problem (3.4), then it holds
\[
\max_{1 \leq j \leq M} \|\sigma_h^j - \sigma_h(t_j)\|_a + \max_{1 \leq j \leq M} \|\rho_h^j - \rho_h(t_j)\|_c \\
\leq \|\sigma_h^0 - \sigma_h(0)\|_0 + \|\rho_h^0 - \rho_h(0)\|_0 \\
+ \Delta t \left( \sum_{j=0}^{M-1} \|\partial_t \sigma_h^{j+\frac{1}{2}} - \Delta_t \sigma_h^{j+\frac{1}{2}}\|_0 + \sum_{j=0}^{M-1} \|\partial_t \rho_h^{j+\frac{1}{2}} - \Delta_t \rho_h^{j+\frac{1}{2}}\|_0 \right).
\] (4.5)
Proof. Setting \( \psi_h^n := \sigma_h^a - \partial_h(t_a) \), \( r_h^n := v^a - \hat{v}_h(t_a) \) for any index \( a \) and taking \( t = t_j, t_{j+1} \) in (3.4) respectively, from (4.1) we have, for \( \forall \tau_h \in V_h \) and \( w_h \in \mathbb{V}_h \),

\[
\begin{align*}
\begin{cases}
 a(\psi_h^{j+\frac{1}{2}}, \tau_h) + b(r_h^{j+\frac{1}{2}}, \tau_h) &= -a(\Delta_t \psi_h^{j+\frac{1}{2}}, \tau_h) + a(\partial_t \sigma^{j+\frac{1}{2}} - \Delta_t \partial_h^{j+\frac{1}{2}}, \tau_h), \\
b(w_h, \psi_h^{j+\frac{1}{2}}) &= c(\Delta_t r_h^{j+\frac{1}{2}}, w_h) + c(\Delta_t \hat{v}_h^{j+\frac{1}{2}} - \partial_t v^{j+\frac{1}{2}}, w_h).
\end{cases}
\end{align*}
\]

Take \( \tau_h = \psi_h^{j+\frac{1}{2}} \) and \( w_h = r_h^{j+\frac{1}{2}} \) in these two equations, respectively, then we obtain

\[
\begin{align*}
a(\psi_h^{j+\frac{1}{2}}, \psi_h^{j+\frac{1}{2}}) + b(r_h^{j+\frac{1}{2}}, \psi_h^{j+\frac{1}{2}}) &= -a(\Delta_t \psi_h^{j+\frac{1}{2}}, \psi_h^{j+\frac{1}{2}}) + a(\partial_t \sigma^{j+\frac{1}{2}} - \Delta_t \partial_h^{j+\frac{1}{2}}, \psi_h^{j+\frac{1}{2}}), \\
b(r_h^{j+\frac{1}{2}}, \psi_h^{j+\frac{1}{2}}) &= c(\Delta_t r_h^{j+\frac{1}{2}}, r_h^{j+\frac{1}{2}}) + c(\Delta_t \hat{v}_h^{j+\frac{1}{2}} - \partial_t v^{j+\frac{1}{2}}, r_h^{j+\frac{1}{2}}).
\end{align*}
\]

Subtracting the second one of the above two equations from the first one, we arrive at

\[
\begin{align*}
a(\psi_h^{j+\frac{1}{2}}, \psi_h^{j+\frac{1}{2}}) + c(\Delta_t r_h^{j+\frac{1}{2}}, r_h^{j+\frac{1}{2}}) + c(\Delta_t \hat{v}_h^{j+\frac{1}{2}} - \partial_t v^{j+\frac{1}{2}}, r_h^{j+\frac{1}{2}}) = a(\partial_t \sigma^{j+\frac{1}{2}} - \Delta_t \partial_h^{j+\frac{1}{2}}, \psi_h^{j+\frac{1}{2}}) + c(\partial_t v^{j+\frac{1}{2}} - \Delta_t \hat{v}_h^{j+\frac{1}{2}}, r_h^{j+\frac{1}{2}}),
\end{align*}
\]

which implies

\[
\frac{1}{2\Delta t} \left( \| \psi_h^{j+\frac{1}{2}} \|_a^2 - \| \psi_h^j \|_a^2 + \| r_h^{j+\frac{1}{2}} \|_c^2 - \| r_h^j \|_c^2 \right) 
\leq a(\partial_t \sigma^{j+\frac{1}{2}} - \Delta_t \partial_h^{j+\frac{1}{2}}, \psi_h^{j+\frac{1}{2}}) + c(\partial_t v^{j+\frac{1}{2}} - \Delta_t \hat{v}_h^{j+\frac{1}{2}}, r_h^{j+\frac{1}{2}}) + c(\Delta_t \hat{v}_h^{j+\frac{1}{2}} - \partial_t v^{j+\frac{1}{2}}, r_h^{j+\frac{1}{2}}).
\]

(4.7)

Multiplying this inequality by \( 2\Delta t \), and summing these equations for \( j = 0, 1, \ldots, n - 1 \) \( (1 \leq n \leq M) \), we get

\[
\| \psi_h^n \|_a^2 + \| r_h^n \|_c^2 \leq \| \psi_h^0 \|_a^2 + \| r_h^0 \|_c^2 + A_1 + A_2,
\]

with

\[
A_1 := \Delta t \sum_{j=0}^{n-1} a(\partial_t \sigma^{j+\frac{1}{2}} - \Delta_t \partial_h^{j+\frac{1}{2}}, \psi_h^{j+\frac{1}{2}}),
\]

\[
A_2 := \Delta t \sum_{j=0}^{n-1} c(\partial_t v^{j+\frac{1}{2}} - \Delta_t \hat{v}_h^{j+\frac{1}{2}}, r_h^{j+\frac{1}{2}}).
\]

For the term \( A_1 \), it holds

\[
A_1 \leq \tilde{C} \Delta t \sum_{j=0}^{n-1} \| \partial_t \sigma^{j+\frac{1}{2}} - \Delta_t \partial_h^{j+\frac{1}{2}} \|_0 \| \psi_h^{j+\frac{1}{2}} + \psi_h^j \|_0
\leq 2\tilde{C} \Delta t \max_{0 \leq j \leq n} \| \psi_h^j \|_0 \left( \sum_{j=0}^{n-1} \| \partial_t \sigma^{j+\frac{1}{2}} - \Delta_t \partial_h^{j+\frac{1}{2}} \|_0 \right)
\leq \frac{1}{2} \max_{0 \leq j \leq n} \| \psi_h^j \|_0^2 + 2(\tilde{C} \Delta t)^2 \left( \sum_{j=0}^{n-1} \| \partial_t \sigma^{j+\frac{1}{2}} - \Delta_t \partial_h^{j+\frac{1}{2}} \|_0 \right)^2,
\]

(4.9)
where $\hat{C} > 0$ is a constant depending on $C^{-1}$ and $\rho$. Similarly, we have

$$A_2 \leq \frac{1}{2} \max_{0 \leq i \leq n} \| r_h^i \|_0^2 + 2(\rho \Delta t)^2 \left( \sum_{j=0}^{n-1} \| \partial_t v^{j+\frac{1}{2}} - \Delta_t \hat{v}^{j+\frac{1}{2}} \|_0 \right)^2. \quad (4.10)$$

Putting (4.10) and (4.9) into (4.8) and noticing that $1 \leq n \leq M$, we finally get

$$\max_{1 \leq j \leq M} \| \psi_h^j \|_a^2 + \max_{1 \leq j \leq M} \| r_h^j \|_c^2 \lesssim \| \psi_h^0 \|_0^2 + \| r_h^0 \|_0^2 + (\Delta t)^2 \left( \sum_{j=0}^{M-1} \| \partial_t v^{j+\frac{1}{2}} - \Delta_t \hat{v}^{j+\frac{1}{2}} \|_0 \right)^2$$

$$+ (\Delta t)^2 \left( \sum_{j=0}^{M-1} \| \partial_t v^{j+\frac{1}{2}} - \Delta_t \hat{v}^{j+\frac{1}{2}} \|_0 \right)^2, \quad (4.11)$$

i.e., (4.5) holds true.

**Lemma 4.4.** Under the assumption (A2) and the condition that

$$\sigma_{ht} \in L^\infty([0,T], L^2(\Omega)), \quad v_{ht} \in L^\infty([0,T], L^2(\Omega)), \quad (4.12)$$

it holds, for $0 \leq j \leq M - 1$,

$$\Delta t \left( \| \partial_t \sigma^{j+\frac{3}{2}} - \Delta_t \hat{\sigma}^{j+\frac{3}{2}} \|_0 + \| \partial_t v^{j+\frac{1}{2}} - \Delta_t \hat{v}^{j+\frac{1}{2}} \|_0 \right)$$

$$\lesssim h^l \left( \| \sigma \|_{L^\infty(H^0)} + \| v \|_{L^\infty(H^0')} \right) + (\Delta t)^2 \left( \| v_{ht} \|_{L^\infty(L^2)} + \| \sigma_{ht} \|_{L^\infty(L^2)} \right).$$

**Proof.** On one hand, using the Taylor expansion, we have

$$\partial_t \sigma^{j+\frac{3}{2}} - \Delta_t \hat{\sigma}^{j+\frac{3}{2}} = \frac{1}{2} \int_{t_j}^{t_{j+1}} \sigma_{ht}(s)ds - \frac{1}{\Delta t} \int_{t_j}^{t_{j+1}} (t_{j+1} - s) \sigma_{ht}(s)ds,$$

which gives

$$\| \partial_t \sigma^{j+\frac{3}{2}} - \Delta_t \hat{\sigma}^{j+\frac{3}{2}} \|_0 \lesssim \Delta t \| \sigma_{ht} \|_{L^\infty(L^2)}.$$ 

On the other hand, from (3.6) it follows

$$\Delta t \| \Delta_t \sigma^{j+\frac{1}{2}} - \Delta_t \hat{\sigma}^{j+\frac{1}{2}} \|_0 = \| \sigma(t_{j+1}) - \sigma_h(t_{j+1}) - (\sigma(t_j) - \sigma(t_j)) \|_0$$

$$\lesssim h^l \left( \| \sigma \|_{L^\infty(H^0)} + \| v \|_{L^\infty(H^0')} \right).$$

As a result, by the triangle inequality we get

$$\Delta t \| \partial_t \sigma^{j+\frac{3}{2}} - \Delta_t \hat{\sigma}^{j+\frac{3}{2}} \|_0 \lesssim h^l \left( \| \sigma \|_{L^\infty(H^0)} + \| v \|_{L^\infty(H^0')} \right) + (\Delta t)^2 \| \sigma_{ht} \|_{L^\infty(L^2)}.$$
In the same way, we can obtain
\[
\Delta t \left| \partial_t v^j + \frac{1}{2} - \Delta t \frac{1}{\delta_h} \hat{v}^{j+1} \right|_{0} \lesssim h \left( \| \sigma \|_{L^\infty(H^m)} + \| \psi_0 \|_1 + \| \phi_1 \|_1 \right) + (\Delta t)^2 \| \psi_h \|_{L^\infty(L^2)}.
\]
This finishes the proof.

Based on (3.6), (3.8), and Lemmas 4.3, 4.4, it is easy to give the following error estimate for the fully discrete finite element scheme.

**Theorem 4.2.** Let \((\sigma(t), v(t))\) be the solution to the weak problem (2.5) and \((\sigma_h^n, v_h^n)\) \((n = 1, \cdots, M)\) be the solution to the fully discrete scheme (4.1) such that the assumptions (A1), (A2), (3.8), (4.12) hold. Then it holds the error estimate
\[
\max_{1 \leq n \leq M} \| \sigma(t_n) - \sigma_h(t_n) \|_a + \max_{1 \leq n \leq M} \| v(t_n) - v_h(t_n) \|_c \lesssim C_1 h^l + C_2 (\Delta t)^2,
\]
where
\[
C_1 := \| \sigma \|_{L^\infty(H^m)} + \| \psi \|_{L^\infty(H^{m'})} + \| \psi_0 \|_1 + \| \phi_1 \|_1,
\]
\[
C_2 := \| \sigma_{tt} \|_{L^\infty(L^2)} + \| \psi_{tt} \|_{L^\infty(L^2)}.
\]

**Remark 4.1.** From Remark 3.1, Theorem 4.2, we easily see that the error estimate (4.13) holds for

- Arnold-Winther’s triangular elements \((d=2)\) [2] and Arnold-Awanou’s rectangular elements \((d=2)\) [1] with \(l = m = k + 1\) \((k \geq 1)\) and \(m' = k + 2\);
- Hu-Zhang’s triangular/cuboid elements \((d=2,3)\) [15,17] with \(l = m' = k + d - 1\) \((k \geq 2)\) and \(m = k + d\);
- Hu-Man-Zhang’s rectangular/cuboid element \((d=2,3)\) [16] with \(l = m' = 1\) and \(m = 2\);
- Nedelec’s rectangular/cuboid elements \((d=2,3)\) [4, 23] with \(l = k + 1\) \((k \geq 0)\) and \(m = m' = k + 2\).

5 Numerical results

As shown in Remark 4.1, there are many existing mixed conforming finite elements that can be used in the discretization of the two- or three-dimensional Maxwell viscoelastic model (1.1). In this section, we only consider two-dimensional numerical examples (Examples 5.1-5.3) and apply the following two low order rectangular elements in the full discretization scheme (4.1):
• The lowest order modified Nedelec’s rectangular element \([k = 0]\) with mass lumping: \(Q_1^{\text{div}} - Q_0\) element. The corresponding finite-dimensional spaces are

\[
\begin{align*}
\mathbb{H}_h &= \left\{ \tau \in \mathcal{H}(\text{div}, \Omega, S); \tau_{ij}|_K \in Q_1(K), \forall K \in \mathcal{T}_h \right\}, \\
\mathbb{V}_h &= \left\{ w \in L^2(\Omega); w_i|_K \in Q_0(K), \forall K \in \mathcal{T}_h \right\},
\end{align*}
\]

and the local degrees of freedom for the stress tensor \(\sigma_h \in \mathbb{H}_h\) are \(\sigma_h(T_i) (i = 1, 2, 3, 4)\), i.e., the values of \(\sigma_h\) at the four vertices of rectangular element \(K\). In the computation of \(a(\cdot, \cdot)\), the following quadrature formula on \(K\) is used for mass lumping [4]:

\[
\int_K g dx \approx \frac{h_x h_y}{4} \sum_{i=1}^{4} g(T_i), \quad \forall g \in C^0(K),
\]

where \(h_x\) and \(h_y\) are the side lengths of \(K\).

• Hu-Man-Zhang’s (abbr. HMZ) rectangular element [16]. We recall in this case that

\[
\begin{align*}
\mathbb{H}_h &= \left\{ \tau \in \mathcal{H}(\text{div}, \Omega, S); \tau_{ij}|_K \in \text{span}\{1, x_i, x^2_i\}, \tau_{12}|_K \in Q_1(K), \forall K \in \mathcal{T}_h \right\}, \\
\mathbb{V}_h &= \left\{ w \in L^2(\Omega); w_i|_K \in \text{span}\{1, x_i\}, \forall K \in \mathcal{T}_h \right\}.
\end{align*}
\]

The local nodal degrees of freedom for the stress tensor are shown in Fig. 1.

For the numerical quadrature on each element \(K\), we divide \(K\) into two triangles and use the seven-points Gauss quadrature formula on each triangle.

In the model problem (1.1), we take \(\Omega = [0,1] \times [0,1], \ T = 1\), and assume that the elastic medium is isotropic with \(\rho = 1, \ \mu = 1, \ \lambda = 1\). We use \(N \times N\) square meshes and \(M\) uniform grids for the spatial region \(\Omega\) and the time region \([0, T]\). To test the accuracy, we compute
the following errors for the stress and velocity approximations:

\[ E_{\sigma}^a = \max_{1 \leq n \leq M} \| \sigma(t_n) - \sigma_h(t_n) \|_{\sigma}, \]
\[ E_{\nu}^a = \max_{1 \leq n \leq M} \| \nu(t_n) - \nu_h(t_n) \|_{\nu}. \]

According to Theorem 4.2, and Remark 4.1, the theoretical accuracy of the full discretiza-
tion is

\[ E_{\sigma}^a + E_{\nu}^a \lesssim h + (\Delta t)^2 \approx N^{-1} + M^{-2}. \]

We consider the following three examples.

**Example 5.1.** The exact displacement field \( u(x,y,t) \) and symmetric stress tensor \( \sigma(x,y,t) = (\sigma_{ij})_{2 \times 2} \) are respectively given by

\[
\begin{pmatrix}
-2 e^{-t} (x^4 - 2 x^3 + x^2) + 4 y^3 - 6 y^2 + 2 y \\
-2 e^{-t} (y^4 - 2 y^3 + y^2) + 4 x^3 - 6 x^2 + 2 x
\end{pmatrix},
\]
\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{12} \\
\sigma_{22}
\end{pmatrix} = \begin{pmatrix}
16 t e^{-t} (2 x^3 - 3 x^2 + x) (2 y^3 - 3 y^2 + y) \\
2 t e^{-t} [(x^4 - 2 x^3 + x^2) (6 y^3 - 6 y + 1) + (y^4 - 2 y^3 + y^2) (6 x^2 - 6 x + 1)] \\
16 t e^{-t} (2 x^3 - 3 x^2 + x) (2 y^3 - 3 y^2 + y)
\end{pmatrix}.
\]

Notice that the velocity field \( v = u_t \). Numerical results of \( E_{\sigma}^a \) and \( E_{\nu}^a \) are shown in Tables 1, 4.

**Example 5.2.** The exact displacement field \( u \) and symmetric stress tensor \( \sigma \) are respectively given by

\[
\begin{pmatrix}
-2 e^{-t} \sin(\pi x) \sin(\pi y) \\
-2 e^{-t} \sin(\pi x) \sin(\pi y)
\end{pmatrix},
\]
\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{12} \\
\sigma_{22}
\end{pmatrix} = \begin{pmatrix}
\pi t e^{-t} (3 \cos(\pi x) \sin(\pi y) + \sin(\pi x) \cos(\pi y)) \\
\pi t e^{-t} (\sin(\pi x) \cos(\pi y) + \cos(\pi x) \sin(\pi y)) \\
\pi t e^{-t} (3 \sin(\pi x) \cos(\pi y) + \cos(\pi x) \sin(\pi y))
\end{pmatrix}.
\]

Numerical results are shown in Tables 2, 5.

**Example 5.3.** The exact displacement field \( u \) and symmetric stress tensor \( \sigma \) are respectively given by

\[
\begin{pmatrix}
e^t \sin(\pi x) (y^{3/2} - y^{5/2}) \\
e^t \sin(\pi y) (x^{3/2} - x^{5/2})
\end{pmatrix},
\]
\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{12} \\
\sigma_{22}
\end{pmatrix} = \begin{pmatrix}
\pi e^t \left( \frac{3}{2} \cos(\pi x) (y^3 - y^5) + \frac{1}{2} \cos(\pi y) (x^3 - x^5) \right) \\
\frac{1}{2} e^t \left( \sin(\pi x) \left( \frac{3}{2} y^3 - \frac{5}{2} y^5 \right) + \sin(\pi y) \left( \frac{3}{2} x^3 - \frac{5}{2} x^5 \right) \right) \\
\pi e^t \left( \frac{3}{2} \cos(\pi y) (x^3 - x^5) + \frac{1}{2} \cos(\pi x) (y^3 - y^5) \right)
\end{pmatrix}.
\]

Numerical results are shown in Tables 3, 6.
Table 1: Convergence history: Example 5.1 with $\Delta t = 0.005$.

| $N \times N$ | $E^a_{\sigma}$ error | $E^a_{\sigma}$ order | $E^a_E$ error | $E^a_E$ order |
|--------------|-----------------------|----------------------|----------------|----------------|
| $Q^1_{\text{div}} - Q_0$ | $4 \times 4$ | 0.0207 | - | 0.0066 | - |
| | $8 \times 8$ | 0.0111 | 0.89 | 0.0033 | 0.98 |
| | $16 \times 16$ | 0.0053 | 1.08 | 0.0016 | 1.08 |
| | $32 \times 32$ | 0.0019 | 1.49 | 0.0007 | 1.11 |
| | $64 \times 64$ | 0.0004 | 1.97 | 0.0003 | 1.22 |
| HMZ | $4 \times 4$ | 0.0097 | - | 0.0032 | - |
| | $8 \times 8$ | 0.0054 | 0.83 | 0.0018 | 0.86 |
| | $16 \times 16$ | 0.0028 | 0.96 | 0.0008 | 0.97 |
| | $32 \times 32$ | 0.0014 | 0.99 | 0.0004 | 0.99 |
| | $64 \times 64$ | 0.0007 | 1.00 | 0.0002 | 1.00 |

Table 2: Convergence history: Example 5.2 with $\Delta t = 0.005$.

| $N \times N$ | $E^a_{\sigma}$ error | $E^a_{\sigma}$ order | $E^a_E$ error | $E^a_E$ order |
|--------------|-----------------------|----------------------|----------------|----------------|
| $Q^1_{\text{div}} - Q_0$ | $4 \times 4$ | 0.9423 | - | 0.4120 | - |
| | $8 \times 8$ | 0.5323 | 0.82 | 0.1925 | 1.10 |
| | $16 \times 16$ | 0.2245 | 1.25 | 0.0862 | 1.16 |
| | $32 \times 32$ | 0.0663 | 1.76 | 0.0355 | 1.28 |
| | $64 \times 64$ | 0.0157 | 2.08 | 0.0156 | 1.18 |
| HMZ | $4 \times 4$ | 0.3524 | - | 0.1587 | - |
| | $8 \times 8$ | 0.1784 | 0.98 | 0.0797 | 0.99 |
| | $16 \times 16$ | 0.0896 | 0.99 | 0.0399 | 1.00 |
| | $32 \times 32$ | 0.0448 | 1.00 | 0.0199 | 1.00 |
| | $64 \times 64$ | 0.0224 | 1.00 | 0.0100 | 1.00 |

Table 3: Convergence history: Example 5.3 with $\Delta t = 0.005$.

| $N \times N$ | $E^a_{\sigma}$ error | $E^a_{\sigma}$ order | $E^a_E$ error | $E^a_E$ order |
|--------------|-----------------------|----------------------|----------------|----------------|
| $Q^1_{\text{div}} - Q_0$ | $4 \times 4$ | 0.6230 | - | 0.2136 | - |
| | $8 \times 8$ | 0.3480 | 0.84 | 0.1000 | 1.10 |
| | $16 \times 16$ | 0.1522 | 1.19 | 0.0422 | 1.25 |
| | $32 \times 32$ | 0.0460 | 1.73 | 0.0179 | 1.24 |
| | $64 \times 64$ | 0.0111 | 2.05 | 0.0081 | 1.14 |
| HMZ | $4 \times 4$ | 0.2531 | - | 0.0833 | - |
| | $8 \times 8$ | 0.1307 | 0.95 | 0.0425 | 0.97 |
| | $16 \times 16$ | 0.0661 | 0.98 | 0.0214 | 0.99 |
| | $32 \times 32$ | 0.0332 | 0.99 | 0.0107 | 1.00 |
| | $64 \times 64$ | 0.0167 | 1.00 | 0.0053 | 1.00 |

Tables 1, 2 and 3 give some numerical results with a fixed time step $\Delta t = 0.005$ to verify the theoretical first order spatial-accuracy of the schemes. Tables 4, 5 and 6 give numer-
Table 4: Convergence history: Example 5.1 with $N = M^2/4$.

| $M$ | $E_a^\sigma$ error | $E_a^\sigma$ order | $E_v^\sigma$ error | $E_v^\sigma$ order |
|-----|---------------------|---------------------|---------------------|---------------------|
| 4   | 0.0096              | -                   | 0.0054              | -                   |
| 8   | 0.0014              | 2.76                | 0.0013              | 2.02                |
| 12  | 0.0004              | 2.91                | 0.0005              | 2.27                |
| 16  | 0.0001              | 2.97                | 0.0002              | 2.22                |

HMZ

| $M$ | $E_a^\sigma$ error | $E_a^\sigma$ order | $E_v^\sigma$ error | $E_v^\sigma$ order |
|-----|---------------------|---------------------|---------------------|---------------------|
| 4   | 0.0097              | -                   | 0.0025              | -                   |
| 8   | 0.0028              | 1.79                | 0.0007              | 1.66                |
| 12  | 0.0013              | 1.98                | 0.0003              | 1.88                |
| 16  | 0.0007              | 2.00                | 0.0002              | 1.93                |

Table 5: Convergence history: Example 5.2 with $N = M^2/4$.

| $M$ | $E_a^\sigma$ error | $E_a^\sigma$ order | $E_v^\sigma$ error | $E_v^\sigma$ order |
|-----|---------------------|---------------------|---------------------|---------------------|
| 4   | 0.3531              | -                   | 0.3231              | -                   |
| 8   | 0.0404              | 3.13                | 0.0667              | 2.28                |
| 12  | 0.0118              | 3.03                | 0.0268              | 2.24                |
| 16  | 0.0049              | 3.05                | 0.0145              | 2.14                |

HMZ

| $M$ | $E_a^\sigma$ error | $E_a^\sigma$ order | $E_v^\sigma$ error | $E_v^\sigma$ order |
|-----|---------------------|---------------------|---------------------|---------------------|
| 4   | 0.3536              | -                   | 0.1253              | -                   |
| 8   | 0.0896              | 1.98                | 0.0354              | 1.82                |
| 12  | 0.0399              | 2.00                | 0.0164              | 1.90                |
| 16  | 0.0224              | 2.00                | 0.0094              | 1.93                |

Table 6: Convergence history: Example 5.3 with $N = M^2/4$.

| $M$ | $E_a^\sigma$ error | $E_a^\sigma$ order | $E_v^\sigma$ error | $E_v^\sigma$ order |
|-----|---------------------|---------------------|---------------------|---------------------|
| 4   | 0.2601              | -                   | 0.2126              | -                   |
| 8   | 0.0362              | 2.85                | 0.0396              | 2.42                |
| 12  | 0.0117              | 2.78                | 0.0153              | 2.34                |
| 16  | 0.0055              | 2.60                | 0.0081              | 2.21                |

HMZ

| $M$ | $E_a^\sigma$ error | $E_a^\sigma$ order | $E_v^\sigma$ error | $E_v^\sigma$ order |
|-----|---------------------|---------------------|---------------------|---------------------|
| 4   | 0.2528              | -                   | 0.0837              | -                   |
| 8   | 0.0661              | 1.94                | 0.0215              | 1.96                |
| 12  | 0.0295              | 1.99                | 0.0096              | 2.00                |
| 16  | 0.0166              | 1.99                | 0.0054              | 2.00                |

Numerical results with synchronous refinement of temporal and spatial meshes, $h = 4(\Delta t)^2$ or equivalently $N = M^2/4$, to verify the theoretical second order temporal-accuracy. From all the numerical results we have the following observations:

- As shown in Tables 1, 2 and 3, the HMZ element is of first order spatial accuracy, and the Nedelec’s $Q_1^\text{div} - Q_0$ element gives better convergence rates than the first order for both $E_a^\sigma$ and $E_v^\sigma$. 
• As shown in Tables 4, 5 and 6, the HMQ element is of second order temporal-
accuracy, and the $Q_1^\text{div} - Q_0$ element yields higher than 2nd order convergence rates.

• For the $Q_1^\text{div} - Q_0$ element, the better convergence behaviours than the theoretical
prediction may be due to some superconvergence of the element on square meshes.

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References

[1] D. N. ARNOLD AND G. AWANOU, Rectangular mixed finite elements for elasticity, Math. Models Methods Appl. Sci., 15(9) (2005), pp. 1417–1429.
[2] D. N. ARNOLD AND R. WINther, Mixed finite elements for elasticity, Numerische Mathematik, 92(3) (2002), pp. 401–419.
[3] E. BÉCACHE, A. EZZIANI, AND P. JOLY, A mixed finite element approach for viscoelastic wave propagation, Comput. Geo., 8(3) (2005), pp. 255–299.
[4] E. BÉCACHE, P. JOLY, AND C. TSOGKA, A new family of mixed finite elements for the linear elastodynamic problem, SIAM J. Numer. Anal., 39(6) (2001), pp. 2109–2132.
[5] D. R. BLAND, The Theory of Linear Viscoelasticity, Pergamon Press, 1960.
[6] T. BOHLEN, Parallel 3-d viscoelastic finite difference seismic modelling, Comput. Geosci., 28(8) (2002), pp. 887–899.
[7] F. BREZZI AND M. FORTIN, Mixed and Hybrid Finite Element Methods, Springer-Verlag, 1991.
[8] R. M. CHRISTENSEN, Theory of Viscoelasticity, An Introduction, Academic Press, 1982.
[9] E. H. DILL, Continuum Mechanics : Elasticity, Plasticity, Viscoelasticity, CRC Press, 2007.
[10] A. D. DROZDOV, Mechanics of Viscoelastic Solids, Wiley, 1998.
[11] Y. C. FUNG, International series on dynamics, (book reviews: foundations of solid mechanics), Science, 152 (1966).
[12] J. M. GOLDEN AND G. A. C. GRAHAM, Boundary Value Problems in Linear Viscoelasticity, Springer, 1988.
[13] M. E. GURTIN AND E. STERNBERG, On the linear theory of viscoelasticity, Arch. Rational Mech. Anal., 11(1) (1962), pp. 291–356.
[14] T. HA, J.E. SANTOS, AND D. SHEEN, Nonconforming finite element methods for the simulation of waves in viscoelastic solids, Comput. Methods Appl. Mech. Eng., 191 (2002), pp. 5647–5670.
[15] J. HU, Finite element approximations of symmetric tensors on simplicial grids in $\mathbb{R}^n$: the high order case, J. Comput. Math., 33(3) (2015), pp. 283–296.
[16] J. HU, H. Y. MAN, AND S. ZHANG, A simple conforming mixed finite element for linear elasticity on rectangular grids in any space dimension, J. Sci. Comput., 58 (2014), pp. 367–379.
[17] J. HU AND S. Y. ZHANG, A family of conforming mixed finite elements for linear elasticity on triangular grids, arXiv:1406.7457 (2014).
[18] V. Janovsky, S. Shaw, M. K. Warby, and J. R. Whiteman, Numerical methods for treating problems of viscoelastic isotropic solid deformation, J. Comput. Appl. Math., 63(1-3) (1995), pp. 91–107.

[19] M. Kimura, Notsu H., Y. Tanaka, and H. Yamamoto, The gradient flow structure of an extended maxwell viscoelastic model and a structure-preserving finite element scheme, J. Sci. Comput., (2018).

[20] D. Kosloff, J. M. Carcione, and R. Kosloff, Wave propagation simulation in a linear viscoelastic medium, Geophys. J., 93 (1988), pp. 393–407.

[21] D. Kosloff, J. M. Carcione, and R. Kosloff, Wave propagation simulation in a visco-elastic medium, Geophys. J., 95 (1988), pp. 597–611.

[22] J. Lee, Mixed Methods with Weak Symmetry for Time Dependent Problems of Elasticity and Viscoelasticity, PhD thesis, University of Minnesota, (2012).

[23] J. C. Nedelec, A new family of mixed finite elements in $\mathbb{R}^3$, Numerische Mathematik, 50(1) (1986), pp. 57–81.

[24] B. Rivièere and S. Shaw, Discontinuous galerkin finite element approximation of nonlinear non-fickian diffusion in viscoelastic polymers, SIAM J. Numer. Anal., 44(6) (2006), pp. 2650–2670.

[25] B. Rivièere, S. Shaw, M. F. Wheeler, and J. R. Whiteman, Discontinuous galerkin finite element methods for linear elasticity and quasistatic linear viscoelasticity, Numerische Mathematik, 95(2) (2003), pp. 347–376.

[26] B. Rivièere, S. Shaw, and J. R. Whiteman, Discontinuous galerkin finite element methods for dynamic linear solid viscoelasticity problems, Numer. Methods for Partial Differential Equations, 23(5) (2007), pp. 1149–1166.

[27] M. E. Rognes and R. Winther, Mixed finite element methods for linear viscoelasticity using weak symmetry, Math. Models Methods Appl. Sci., 20(06) (2010), pp. 955–985.

[28] V. Sabinin, T. Chichinina, and G. R. Jarillo, Numerical Model of Seismic Wave Propagation in Viscoelastic Media, Springer Berlin Heidelberg, 2003.

[29] J. Salencon, Viscoelasticité pour le Calcul des Structures, 2016.

[30] R. A. Schapery, Nonlinear viscoelastic solids, Int. J. Solids Structures, 37(1–2) (2000), pp. 359–366.

[31] P. C. M. Severino and J. C. Guillermo, Computational Viscoelasticity, Springer New York, 2012.

[32] Fei Teng and Zhendong Luo, A highly efficient reduced-order extrapolating model for the 2d viscoelastic wave equation, Adv. Appl. Math. Mech., 13(2) (2021), pp. 355–377.

[33] V. Thomee, Galerkin finite element methods for parabolic problems, Math. Comput., 17(2) (2006), pp. 186–187.

[34] S. Wang and X. Xie, Semi-discrete and fully discrete hybrid stress finite element methods for maxwell viscoelastic model of wave propagation, Numer. Math. J. Chinese Universities (in Chinese), 42(3) (2020).

[35] T. Xu and G. A. McMechan, Efficient 3-d viscoelastic modeling with application to near-surface land seismic data, Geophys., 63(2) (1998), pp. 601–612.