Deformation of hom-Lie-Rinehart algebras

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ABSTRACT

We study formal deformations of hom-Lie-Rinehart algebras. The associated deformation cohomology that controls deformations is constructed using multiderivations of hom-Lie-Rinehart algebras.

ARTICLE HISTORY

Received 6 August 2018
Accepted 7 November 2019
Communicated by K. Misra

KEYWORDS

Deformation of algebras; deformation complex; differential graded Lie algebras; Hom-Lie-Rinehart algebras

AMS MATHEMATICS SUBJECT CLASSIFICATION (2010)

17A30; 17B55; 17B99

1. Introduction

The aim of this paper is to study formal deformations of hom-Lie-Rinehart algebras. In Ref. [14], hom-Lie-Rinehart algebras are introduced as an algebraic analog of hom-Lie algebroids. This notion generalizes both the notion of a hom-Lie algebra and the notion of a hom-Lie algebroid in Ref. [9]. If we start with a Lie-Rinehart algebra [5–7, 15] and a homomorphism into itself, we obtain a canonical hom-Lie-Rinehart algebra structure (usually referred as “obtained by composition”).

In Ref. [14], we study modules over a hom-Lie-Rinehart algebra and a cohomology with coefficients in a left module. We also characterize this cohomology (in low dimensions) in terms of the group of automorphisms and the equivalence classes of abelian extensions (see Ref. [14] for more details). Later on, a non-abelian tensor product and universal central extensions are introduced in Ref. [13], crossed modules for hom-Lie-Rinehart algebras are studied in Ref. [18], and a relationship between hom-Lie-Rinehart algebras and hom-Gerstenhaber algebras is discussed in Ref. [12].

In the case of classical algebras, the associated cohomologies with coefficients in adjoint representations control deformations. For example, Hochschild (and Chevalley-Eilenberg) cohomology plays the role of deformation cohomology in the case of associative (and Lie) algebras. Likewise, for hom-algebras, in particular for hom-Lie algebras and hom-associative algebras, cohomologies with coefficients in certain adjoint representations are defined (generalizing Chevalley-Eilenberg and Hochschild cohomology, respectively) in Ref. [1], which play the role of deformation cohomologies.
For Lie algebroids, more work is required to find suitable deformation cohomology since the adjoint representation is not defined as representation in the usual sense (see Ref. [3]). In Ref. [2], M. Crainic and I. Moerdijk introduced the notion of multiderivations of a vector bundle. The space of multiderivations of a vector bundle forms a graded Lie algebra. If vector bundle has a Lie algebroid structure, a differential can be associated with this graded Lie algebra to obtain a differential graded Lie algebra (in short DGLA). They also proved that this DGLA controls deformations (in the sense of Ref. [2]) of a Lie algebroid. In the present article, we describe a differential graded Lie algebra for hom-Lie-Rinehart algebra, which gives a deformation complex required to study one-parameter formal deformations.

In Section 2, we recall some basic definitions. Let \( R \) be a commutative ring of characteristic zero and with unity. Also, let \( A \) be an associative commutative \( R \)-algebra, and \( \phi : A \to A \) be an \( R \)-algebra homomorphism.

In Section 3, we first define the notion of \((\phi, \beta)\)-multiderivations of a pair \((M, \beta)\), where \( M \) is an \( A \)-module, and \( \beta : M \to M \) is a \( \phi \)-function linear map. Then we consider a graded Lie algebra structure on the space of \((\phi, \beta)\)-multiderivations. Next, we describe hom-Lie-Rinehart algebra structures on the pair \((M, \beta)\) (over \((A, \phi)\)) in terms of this graded Lie algebra. Consequently, we associate a differential graded Lie algebra (DGLA) to a hom-Lie-Rinehart algebra.

In Section 4, we show that this DGLA controls the deformations of a hom-Lie-Rinehart algebra. We also prove that if \( \phi : A \to A \) is an algebra automorphism, then the space of \( \phi \)-derivations of \( A \) is rigid as a hom-Lie-Rinehart algebra. In the last section, we discuss some particular cases of this deformation complex for hom-Lie-Rinehart algebras.

### 2. Preliminaries

In this section, we recall basic definitions concerning hom-structures from Refs. [1, 4, 10, 11, 14, 16] in order to fix notations and terminology. Let \( R \) be a commutative ring of characteristic zero and with unity. Throughout the paper, we consider all modules, algebras and their tensor products over the ring \( R \) and all linear maps to be \( R \)-linear unless otherwise stated.

**Definition 2.1.** A hom-Lie algebra is a triple \((L, [-, -], \alpha)\), which consists of a \( R \)-module \( L \), a skew-symmetric \( R \)-bilinear map \([-,-] : \mathbf{L} \times \mathbf{L} \to \mathbf{L} \), and a linear map \( \alpha : L \to L \), satisfying

\[
[x(y), [z, x]] + [x(y), [z, x]] + [x(z), [x, y]] = 0 \quad \text{for all } x, y, z \in L.
\]

A hom-Lie algebra \((L, [-, -], \alpha)\) is called multiplicative if the map \( \alpha \) preserves the bracket, i.e. \( \alpha[x, y] = [\alpha(x), \alpha(y)] \). Moreover, if \( \alpha \) is an \( R \)-module automorphism of \( L \), then the hom-Lie algebra \((L, [-, -], \alpha)\) is called a regular hom-Lie algebra.

**Example 2.2.** Given a Lie algebra \((L, [-, -])\) with a Lie algebra homomorphism \( \alpha : L \to L \), we can define a hom-Lie algebra as the triple \((L, \alpha \circ [-, -], \alpha)\).

**Definition 2.3.** A representation of a hom-Lie algebra \((L, [-, -], \alpha)\) on a \( R \)-module \( V \) is a pair \((\theta, \beta)\) consisting of \( R \)-linear maps \( \theta : \mathfrak{g} \to \mathfrak{gl}(V) \) and \( \beta : V \to V \) such that

\[
\theta(\alpha(x)) \circ \beta = \beta \circ \theta(x),
\]

\[
\theta([x, y]) \circ \beta = \theta(\alpha(x)) \circ \theta(y) - \theta(\alpha(y)) \circ \theta(x).
\]

for all \( x, y \in L \).

**Example 2.4.** For \( s \in \mathbb{Z} \), we can define the \( \alpha^s \)-adjoint representation of the regular hom-Lie algebra \((L, [-, -], \alpha)\) on \( L \) by \((ad_s, \alpha)\), where
Definition 2.6. Given an associative commutative algebra $R$ and $A$ such that $q: R \rightarrow A$ is an $R$-module homomorphism.

Let $A$ be an associative commutative $R$-algebra and $\text{Der}(A)$ denote the space of $R$-derivations of the algebra $A$. Then $\text{Der}(A)$ is simultaneously an $A$-module and a Lie algebra with the commutator bracket.

Definition 2.5. A Lie-Rinehart algebra $L$ over (an associative commutative $R$-algebra) $A$ is a Lie algebra over $R$ with an $A$-module structure and a $R$-module homomorphism $\rho : L \rightarrow \text{Der}(A)$, such that $\rho$ is simultaneously an $A$-module homomorphism and a Lie $R$-algebra homomorphism and

$$[x, ay] = a[x, y] + \rho(x)(a)y \quad \text{for all } x, y \in L, \ a \in A.$$ 

Definition 2.6. Given an associative commutative algebra $A$, an $A$-module $M$ and an algebra endomorphism $\phi : A \rightarrow A$, we call an $R$-linear map $\delta : A \rightarrow M$ a $\phi$-derivation of $A$ into $M$ if it satisfies the required identity;

$$\delta(ab) = \phi(a)\delta(b) + \phi(b)\delta(a) \quad \text{for all } a, b \in A.$$ 

Let us denote by $\text{Der}_\phi(A)$ the set of $\phi$-derivations.

Let $M$ be a smooth manifold, and $\psi : M \rightarrow M$ be a smooth map. Then the map $\psi$ induces an algebra homomorphism $\psi^* : C^\infty(M) \rightarrow C^\infty(M)$, and the space of $\psi^*$-derivations of $C^\infty(M)$ into itself can be identified with the space of sections of the pull-back bundle of the tangent bundle $TM$, which is denoted by $\Gamma(\psi^*TM)$.

Definition 2.7. A hom-Lie algebroid is a quintuple $(A, \phi, [-, -], \rho, \varpi)$, where $A$ is a vector bundle over a smooth manifold $M$, the map $\phi : M \rightarrow M$ is a smooth map, the bracket $[-, -] : \Gamma(A) \otimes \Gamma(A) \rightarrow \Gamma(A)$ is a bilinear map, the map $\rho : \phi^!A \rightarrow \phi^!TM$ is called the anchor and $\varpi : \Gamma(A) \rightarrow \Gamma(A)$ is a linear map such that following conditions are satisfied.

(i) $\varpi(f, s) = \phi^*(f)\varpi(s)$ for all $s \in \Gamma(A)$, $f \in C^\infty(M)$.

(ii) The triplet $(\Gamma(A), [-, -], \varpi)$ is a hom-Lie algebra.

(iii) The following hom-Leibniz identity holds:

$$[s, f.s] = \phi^*(f)[s, t] + \rho(s)[f]\varpi(t);$$

for all $s, t \in \Gamma(A)$, $f \in C^\infty(M)$.

(iv) The pair $(\rho, \phi^*)$ is a representation of $(\Gamma(A), [-, -], \varpi)$ on $C^\infty(M)$.

A hom-Lie algebroid $(A, \phi, [-, -], \rho, \varpi)$ is called regular (or invertible) hom-Lie algebroid if the map $\varpi : \Gamma(A) \rightarrow \Gamma(A)$ is an invertible map and the smooth map $\phi : M \rightarrow M$ is a diffeomorphism.

Here, $\rho(s)[f]$ denotes a function on $M$ given by

$$\rho(s)[f](m) = \langle d_{\phi(m)}f, \rho_m(s_{\phi(m)}) \rangle$$

for $m \in M$. The map $\rho_m : (\phi^!A)_m \cong A_{\phi(m)} \rightarrow (\phi^!TM)_m \cong T_{\phi(m)}M$ is the anchor map evaluated at $m \in M$ and $s_{\phi(m)}$ is image of the section $s \in \Gamma(A)$ at $\phi(m) \in M$.

2.1. Hom-Lie-Rinehart algebras

Definition 2.8. A tuple $(A, L, [-, -], \phi, \varpi_L, \rho_L)$ is called a hom-Lie Rinehart algebra over $(A, \phi)$ if $L$ is an $A$-module, $[-, -]_L : L \times L \rightarrow L$ is a skew-symmetric bilinear map, the map $\phi : A \rightarrow A$ is
an algebra homomorphism, \( \alpha_L : L \rightarrow L \) is a linear map preserving the bracket \([\cdot, \cdot]_L\), and \( \rho_L : L \rightarrow \text{Der}_\phi A \) is a \( R \)-linear map satisfying the following conditions:

(i) \( \alpha_L(a \cdot x) = \phi(a) \cdot \alpha_L(x) \) for all \( a \in A, \ x \in L \).

(ii) The triple \((L, [-,-]_L, \alpha_L)\) is a hom-Lie algebra.

(iii) The pair \((\rho_L, \phi)\) is a representation of \((L, [-,-]_L, \alpha_L)\) on \( A \).

(iv) \( \rho_L(a \cdot x) = \phi(a) \cdot \rho_L(x) \) for all \( a \in A, \ x \in L \).

(v) \( [x, a \cdot y]_L = \phi(a)[x, y]_L + \rho_L(x)(a) \alpha_L(y) \) for all \( a \in A, \ x, y \in L \).

Let us denote a hom-Lie Rinehart algebra \((A, L, [-,-]_L, \phi, \alpha_L, \rho_L)\) over \((A, \phi)\) simply by \((\mathcal{L}, \alpha_L)\). In particular,

(a) If \( \alpha_L = \text{Id}_L \) in the above definition, then \( \phi = \text{id}_A \) and the hom-Lie-Rinehart algebra \((A, L, [-,-]_L, \phi, \alpha_L, \rho_L)\) is a Lie-Rinehart algebra \( L \) over \( A \).

(b) Any hom-Lie algebra \((L, [-,-]_L, \alpha_L)\) (over \( R \)) is also a hom-Lie Rinehart algebra \((A, L, [-,-]_L, \phi, \alpha_L, \rho_L)\) with \( A = R \), algebra morphism \( \phi = \text{Id}_R \) and the trivial action of \( L \) on \( R \).

(c) Any hom-Lie algebroid \((A, \phi, [-,-], \rho, \alpha)\) over a smooth manifold \( M \), gives a hom-Lie-Rinehart algebra \((C^\infty(M), \Gamma A, [-,-], \phi^*, \alpha)\) over \((C^\infty(M), \phi^*)\), where \( \Gamma A \) is the space a sections of the underline vector bundle \( A \) over \( M \) and the algebra homomorphism \( \phi^* : C^\infty(M) \rightarrow C^\infty(M) \) is induced by the smooth map \( \phi : M \rightarrow M \).

**Example 2.9.** If we consider a Lie-Rinehart algebra \( L \) over \( A \) along with an endomorphism \((\phi, \alpha) : (A, L) \rightarrow (A, L)\) in the category of Lie-Rinehart algebras then the tuple \((A, L, [-,-], \phi, \alpha, \rho, \alpha, \rho_L)\) is a hom-Lie-Rinehart algebra, called “obtained by composition”, where

(i) \([x, y]_L = \alpha[x, y]\) for \( x, y \in L \);

(ii) \( \rho_L(x)(a) = \phi(\rho(x)(a)) \) for \( x \in L, \ a \in A \).

**Example 2.10.** Let \((A, L, [-,-], \phi, \alpha_L, \rho_L)\) and \((A, M, [-,-], \phi, \alpha_M, \rho_M)\) be hom-Lie-Rinehart algebras over \((A, \phi)\). We consider

\[ L \times_{\text{Der}_\phi A} M = \{(l, m) \in L \times M : \rho_l(l) = \rho_M(m)\} \]

where \( L \times M \) denotes the Cartesian product. Then \((A, L \times_{\text{Der}_\phi A} M, [-,-], \phi, \alpha, \rho)\) is a hom-Lie-Rinehart algebra, where

(i) the bracket is given by

\[ [(l_1, m_1), (l_2, m_2)] = ([l_1, l_2], [m_1, m_2]) \]

(ii) the endomorphism \( \alpha : L \times_{\text{Der}_\phi A} M \rightarrow L \times_{\text{Der}_\phi A} M \) is given by

\[ \alpha(l, m) = (\alpha_L(l), \alpha_M(m)) \]

(iii) and the anchor map \( \rho : L \times_{\text{Der}_\phi A} M \rightarrow \text{Der}_\phi A \) is given by

\[ \rho(l, m)(a) = \rho_L(a) = \rho_M(a) \]

for all \( l, l_1, l_2 \in L, m, m_1, m_2 \in M \), and \( a \in A \). The above structure gives the categorical product in the category \( hLRA_{\phi} \).
**Definition 2.11.** Let $M$ be an $A$-module, and $\beta \in \text{End}_R(M)$. Then the pair $(M, \beta)$ is a left module over a hom-Lie Rinehart algebra $(L, \mathfrak{z}_L)$ if the following holds.

1. There is a map $\theta : L \otimes M \to M$, such that the pair $(\theta, \beta)$ is a representation of the hom-Lie algebra $(L, [-, -], \mathfrak{z}_L)$ on $M$. Let us denote $\theta(x, m)$ by $\{x, m\}$ for $x \in L$, $m \in M$.
2. $\beta(a.m) = \phi(a).\beta(m)$ for all $a \in A$ and $m \in M$.
3. $(\phi(a.X), \beta(m))$ for all $a \in A$, $X \in L$, $m \in M$.
4. $\{X, a.m\} = \phi(a)\{X, m\} + \rho_x(X)(a).\beta(m)$ for all $X \in L$, $a \in A$, $m \in M$.

In particular, for $\mathfrak{z}_L = \text{Id}_L$ and $\beta = \text{Id}_M$, $(L, \mathfrak{z}_L)$ is a Lie-Rinehart algebra and $M$ is a left Lie-Rinehart algebra module over the Lie-Rinehart algebra $L$.

**Example 2.12.** The pair $(A, \phi)$ is a canonical left $(L, \mathfrak{z}_L)$-module, where the left action of $L$ on $A$ is given by the anchor map.

Let $(L, \mathfrak{z}_L)$ be a hom-Lie Rinehart algebra over $(A, \phi)$ and $(M, \beta)$ be a left module over $(L, \mathfrak{z}_L)$. Let us consider the graded $R$-module

$$C^*(L; M) := \bigoplus_{n \geq 1} C^n(L; M)$$

where, $C^n(L; M) \subseteq \text{Hom}_R(\wedge^n L, M)$ consisting of elements $f \in \text{Hom}_R(\wedge^n L, M)$ satisfying the following conditions.

1. $f(\mathfrak{z}_L(x_1), \ldots, \mathfrak{z}_L(x_n)) = \beta(f(x_1, x_2, \ldots, x_n))$ for all $x_i \in L$, $1 \leq i \leq n$
2. $f(x_1, \ldots, a.x_i, \ldots, x_n) = \phi^{n-1}(a)f(x_1, \ldots, x_i, \ldots, x_n)$ for all $x_i \in L$, $1 \leq i \leq n$, and $a \in A$.

Define the $R$-linear maps $\delta : C^n(L; M) \to C^{n+1}(L; M)$ given by

$$\delta f(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \{\mathfrak{z}_L^{n-i}(x_i), f(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1})\}$$

$$+ \sum_{1 \leq i < j \leq n+1} f([x_i, x_j], \mathfrak{z}_L(x_1), \ldots, \mathfrak{z}_L(x_i), \ldots, \mathfrak{z}_L(x_j), \ldots, \mathfrak{z}_L(x_{n+1}))$$

for all $f \in C^n(L; M)$, $x_i \in L$, and $1 \leq i \leq n + 1$. Here, $(C^*(L; M), \delta)$ forms a cochain complex, see Ref. [14] for more details.

### 3. Deformation complex for hom-Lie-Rinehart algebras

In this section, we construct a deformation complex which encodes all the information about deformation of hom-Lie-Rinehart algebras.

Let $M$ be an $A$-module, $\phi : A \to A$ be an algebra homomorphism, and $\beta : M \to M$ be a $\phi$-function linear map, i.e $\beta(a.x) = \phi(a).\beta(x)$ for $a \in A$, and $x \in M$. A $(\phi, \beta)$-derivation of $M$ is a linear map $D : M \to M$ such that there exists $\sigma_D \in \text{Der}_R(A)$ satisfying the following conditions.

1. $D(fx) = fD(x) + \sigma_D(f)x$, for $f \in A, x \in M$.
2. $D \circ \beta = \beta \circ D$, and $\sigma_D \circ \phi = \phi \circ \sigma_D$.

Let us denote by $\text{Der}_\phi(M, \beta)$, the space of $(\phi, \beta)$-derivations of $M$. It is a Lie algebra with the Lie bracket, given by the commutator bracket.

Next, we consider a graded $R$-module of multiderivations on which we describe a graded Lie algebra structure by extending the canonical Lie algebra structure on the space of $(\phi, \beta)$-derivations of $M$. 
Definition 3.1. Let $M$ be an $A$-module, $\phi : A \to A$ be an algebra homomorphism, and $\beta : M \to M$ be a $\phi$-function linear map. Then a linear map
\[
D : \wedge^{n+1} M \to M
\]
is called a $(\phi, \beta)$-multiderivation of degree $n$ (of the $A$-module $M$) if there exists a linear map $\sigma_D : \wedge^n M \to \text{Der}_A$ such that the following conditions are satisfied.

(i) $D(\beta(x_1), \beta(x_2), \ldots, \beta(x_{n+1})) = \beta(D(x_1, x_2, \ldots, x_{n+1}))$,
(ii) $\sigma_D(\beta(x_1), \beta(x_2), \ldots, \beta(x_n))(\phi(a)) = \phi(\sigma_D(x_1, x_2, \ldots, x_n)(a))$,
(iii) $\sigma_D(x_1, x_2, \ldots, x_n) = \phi^n(a).\sigma_D(x_1, x_2, \ldots, x_n)$, and
(iv) $D(x_0, x_1, \ldots, x_n) = \phi^n(a)D(x_0, \ldots, x_n) + \sigma_D(x_0, \ldots, x_{n-1})(a).\beta^n(x_n)$,

for all $x_0, \ldots, x_n \in M$, and $a \in A$.

The map $\sigma_D$ is called the symbol map of the $(\phi, \beta)$-multiderivation $D$. Let us denote the space of $n$-degree $(\phi, \beta)$-multiderivations of $M$ by $\mathcal{D} \text{er}^n_\phi(M, \beta)$.

Remark 3.2. Let us consider the following cases:

(a) If $n = 0$, then $\mathcal{D} \text{er}^0_\phi(M, \beta) = \text{Der}_\phi(M, \beta)$, the space of $(\phi, \beta)$-derivations of $A$-module $M$.
(b) For $\phi = \text{id}_A$ and $\beta = \text{id}_M$, the above definition describes the notion of $A$-module multiderivations [17] of an $A$-module $M$ (with a change in degree convention).
(c) In particular, for a vector bundle over a smooth manifold $N$, by setting $A = C^\infty(N)$ and $M = \Gamma E$, we get the multiderivations of the vector bundle $E$ (defined in Ref. [2]), i.e. for $n \geq 0$.

$$\mathcal{D} \text{er}^n_\phi(M, \beta) = \text{Der}^n(E).$$

Next, we extend the Lie bracket of $\text{Der}_\phi(M, \beta)$ to a graded Lie bracket on the space of $(\phi, \beta)$-multiderivations of $M$

$$\mathcal{D} \text{er}^*_\phi(M, \beta) := \bigoplus_{n \geq 0} \mathcal{D} \text{er}^n_\phi(M, \beta).$$

Let $D_1 \in \mathcal{D} \text{er}^0_\phi(M, \beta)$, and $D_2 \in \mathcal{D} \text{er}^0_\phi(M, \beta)$, then define a bracket as follows:

$$[D_1, D_2] := (-1)^{pq}D_1 \circ D_2 - D_2 \circ D_1.$$  \hspace{1cm} (1)

In the above expression, the product $D_1 \circ D_2$ is given by the expression below for any $x_0, \ldots, x_p, \ldots, x_{p+q}$.

$$\begin{align*}
(D_1 \circ D_2)(x_0, x_1, \ldots, x_{p+q}) &= \sum_{\tau \in S_{q+1}} (-1)^{|\tau|} D_1(D_2(x_{\tau(0)}, \ldots, x_{\tau(q)}), \beta^\tau(x_{\tau(q+1)}), \ldots, \beta^\tau(x_{\tau(p+q)})),
\end{align*}$$  \hspace{1cm} (2)

Here, by $Sh(q + 1, p)$ we denote the $(q + 1, p)$ shuffles in $S_{q+p+1}$ (the symmetric group on the set $\{1, \ldots, p + q + 1\}$), and for any permutation $\tau \in S_{q+p+1}$, $|\tau|$ denotes the signature of the permutation $\tau$.

It follows that the bracket $[-,-] : \mathcal{D} \text{er}^*_\phi(M, \beta) \times \mathcal{D} \text{er}^*_\phi(M, \beta) \to \mathcal{D} \text{er}^*_\phi(M, \beta)$ is a graded Lie-bracket of degree 0 (see Ref. [1] for details). For any $a \in A$, and $x_0, \ldots, x_{p+q} \in L$, let us consider the following expression

$$[D_1, D_2](x_0, x_1, \ldots, a, x_{p+q}) = \left((-1)^{pq}D_1 \circ D_2 - D_2 \circ D_1\right)(x_0, x_1, \ldots, a, x_{p+q}).$$

Let us denote by $Sh^1(q + 1, p)$, the set of $(q + 1, p)$-shuffles of the set $\{0, 1, \ldots, p + q\}$ fixing $p + q$, and denote by $Sh^2(q + 1, p)$, the set of $(q + 1, p)$-shuffles, which send $q \mapsto p + q$. Clearly, $Sh^1(q + 1, p) \cup Sh^2(q + 1, p) = Sh(q + 1, p)$. Then using equation (2), we get the following equation:
\((D_1 \circ D_2)(x_0, x_1, \ldots, a_{x_{p+q}})\)
\[= \sum_{t \in Sh^2(q+1,p)} (-1)^{|t|}D_1(D_2(x_{t(0)}, \ldots, x_{t(q)}), \beta^g(x_{t(p+q)}), \ldots, \beta^g(x_{t(p+q)})), \beta^g(a_{x_{p+q}})) \]
\[+ \sum_{t \in Sh^2(q+1,p)} (-1)^{|t|}D_1(D_2(x_{t(0)}, \ldots, x_{t(q-1)}, a_{x_{p+q}}), \beta^g(x_{t(p+q)}), \ldots, \beta^g(x_{t(p+q)})) \]
\[= \sum_{t \in Sh^2(q+1,p)} (-1)^{|t|}\phi^{p+q}(\cdot, D_1(D_2(x_{t(0)}, \ldots, x_{t(q)}), \beta^g(x_{t(p+q)}), \ldots, \beta^g(x_{t(p+q)}))) \]
\[+ \sum_{t \in Sh^2(q+1,p)} (-1)^{|t|}(\sigma_{D_1}(D_2(x_{t(0)}, \ldots, x_{t(q)}), \beta^g(x_{t(p+q)}), \ldots, \beta^g(x_{t(p+q)}))) \]
Let us denote the set \(Sh^2(q+1,p)\) simply by \(Sh^2\). The expression in the second summation appeared on the right hand side of equation (3) can be written as \(B_1 + B_2\), where
\[B_1 = \sum_{t \in Sh^2} (-1)^{|t|}\phi^{p+q}(\cdot, D_1(D_2(x_{t(0)}, \ldots, x_{t(p+q)}), \beta^g(x_{t(p+q)}), \ldots, \beta^g(x_{t(p+q)}))),\]
\[B_2 = \sum_{t \in Sh^2} (-1)^{|t|+\rho}\sigma_{D_1}(\beta^g(x_{t(p+q)}), \beta^g(x_{t(p+q)}))) \]
Next, the expression in the third summation appeared on the right hand side of equation (3) can be written as \(C_1 + C_2\), where
\[C_1 = \sum_{t \in Sh^2} (-1)^{|t|}\phi^p(\sigma_{D_1}(x_{t(0)}, \ldots, x_{t(q-1)}, a_{x_{p+q}}).D_1(\beta^g(x_{p+q}), \beta^g(x_{t(p+q)}), \ldots, \beta^g(x_{t(p+q)}))),\]
\[C_2 = \sum_{t \in Sh^2} (-1)^{|t|+\rho}\sigma_{D_1}(\beta^g(x_{t(p+q)}), \beta^g(x_{t(p+q)}))).\]

Similarly,
\((D_2 \circ D_1)(x_0, x_1, \ldots, a_{x_{p+q}})\)
\[= \sum_{t \in Sh^2(p+1,q)} (-1)^{|t|}\phi^{p+q}(\cdot, D_2(D_1(x_{t(0)}, \ldots, x_{t(p)}), \beta^g(x_{t(p+q)}), \ldots, \beta^g(x_{t(p+q)}))) \]
\[+ \sum_{t \in Sh^2(p+1,q)} (-1)^{|t|}\phi^p(\sigma_{D_2}(x_{t(0)}, \ldots, x_{t(p-1)}, \beta^g(x_{t(p+q)}), \beta^g(x_{t(p+q)}), \ldots, \beta^g(x_{t(p+q)}))) \]
\[+ \sum_{t \in Sh^2(p+1,q)} (-1)^{|t|}\sigma_{D_1}(\beta^g(x_{t(p+q)}), \beta^g(x_{t(p+q)}))) \]
Let us denote \(Sh^2(p+1,q)\) simply by \(Sh^2\). The expression in the second summation appeared on the right hand side of equation (4) can be written as \(J_1 + J_2\), where
\[J_1 = \sum_{t \in Sh^2} (-1)^{|t|}\phi^{p+q}(\cdot, D_2(D_1(x_{t(0)}, \ldots, x_{t(p+q)}), \beta^g(x_{t(p+q)}), \ldots, \beta^g(x_{t(p+q)}))) \]
\[J_2 = \sum_{t \in Sh^2} (-1)^{|t|+\rho}\phi^p(\sigma_{D_2}(x_{t(p+q)}), \beta^g(x_{t(p+q)}), \beta^g(x_{t(p+q)}), \ldots, \beta^g(x_{t(p+q)}))) \]
The expression in the third summation appeared on the right hand side of equation (4) can be written as
\[K_1 = \sum_{t \in Sh^2} (-1)^{|t|}\phi^p(\sigma_{D_1}(x_{t(0)}, \ldots, x_{t(p-1)}, \beta^g(x_{t(p+q)}), \beta^g(x_{t(p+q)}), \ldots, \beta^g(x_{t(p+q)}))) \]
\[K_2 = \sum_{t \in Sh^2} (-1)^{|t|+\rho}\sigma_{D_2}(\beta^g(x_{t(p+q)}), \beta^g(x_{t(p+q)}))) \sigma_{D_1}(x_{t(0)}, \ldots, x_{t(p-1)}, \beta^g(x_{t(p+q)}))) \]
Next, using the properties of multiderivations \(D_1, D_2\) and there symbols \(\sigma_{D_1}, \sigma_{D_2}\), let us observe the following:
Let $L$ be an $A$-module and $\text{Lie bracket}$ (1) it is clear that

$$\text{Der} \ L = \{ \text{multiderivation \ Der} \ A \}$$

Proof.

Thus, using equations (3) and (4), we get the following identity:

$$[D_1, D_2](x_0, x_1, ..., x_{p+q}, a x_{p+q})$$

Consequently, using equations (3) and (4), we get the following identity:

$$[D_1, D_2](x_0, x_1, ..., x_{p+q}, a x_{p+q}) + \sigma_{[D_1, D_2]}(x_0, x_1, ..., x_{p+q} - 1)(a). \beta^{p+q}(x_{p+q}).$$

In the above identity, the map $\sigma_{[D_1, D_2]} : \wedge^{p+q}M \to \text{Der}_{\phi^{p+q}}(A)$ is given by the following equation

$$\sigma_{[D_1, D_2]} = (-1)^{pq} \sigma_{D_1} \circ D_2 - \sigma_{D_2} \circ D_1 + \{ \sigma_{D_1}, \sigma_{D_2} \}.$$  (5)

The terms appeared in the right hand side of the above equation are given as follows. For any $x_1, ..., x_{p+q} \in M$, and $a \in A$,

(i) $\{ \sigma_{D_1}, \sigma_{D_2} \}$ is defined by

$$\{ \sigma_{D_1}, \sigma_{D_2} \}(x_1, ..., x_{p+q})(a)$$

$$= \sum_{\gamma \in \text{Sh}(p,q)} (-1)^{\gamma} [\sigma_{D_1} (\beta^{q}(x_{r(1)}), \ldots, \beta^{q}(x_{r(p)}))] \sigma_{D_2} (x_{r(p+1)}, \ldots, x_{r(p+q)})$$

$$- \sigma_{D_1} (\beta^{q}(x_{r(p+1)}), \ldots, \beta^{q}(x_{r(p+q)})) \sigma_{D_2} (x_{r(1)}, \ldots, x_{r(p)})(a),$$

(ii) $\sigma_{D_1} \circ D_2$ is defined by

$$(\sigma_{D_1} \circ D_2)(x_1, ..., x_{p+q})(a)$$

$$= \sum_{\gamma \in \text{Sh}(q+1, p-1)} (-1)^{\gamma} \sigma_{D_1} (D_2 (x_{r(1)}), \ldots, x_{r(q+1)}), \beta^{q}(x_{r(q+2)}), \ldots, \beta^{q}(x_{r(p+q)})) (\phi^{q}(a)).$$

Hence, for $D_1 \in \text{Der}_{\phi^{q}}(M, \beta)$, and $D_2 \in \text{Der}_{\phi^{q}}(M, \beta)$, the bracket $[D_1, D_2] \in \text{Der}_{\phi^{q}}(M, \beta)$ with the symbol map $\sigma_{[D_1, D_2]}$. Therefore the space of $(\phi, \beta)$-multiderivation $\text{Der}_{\phi^{q}}(M, \beta)$ is closed under the graded Lie bracket given by equation (1).

Theorem 3.3. Let $M$ be an $A$-module and $\beta : M \to M$ be a $\phi$-function linear map. Then the space of $(\phi, \beta)$-multiderivations of $M$ has a graded Lie algebra structure, where the graded Lie bracket is given by equation (1).

In the next result, we describe a hom-Lie-Rinehart algebra structures in terms of the graded Lie algebra obtained above.

Proposition 3.4. Let $L$ be an $A$-module and $z_L : L \to L$ be a $\phi$-function linear map. Then there is a one-to-one correspondence between hom-Lie-Rinehart algebra structures on the pair $(L, z_L)$ and elements $m \in \text{Der}^1_{\phi}(L, z_L)$ satisfying $[m, m] = 0$.

Proof. Let $(L, z_L)$ be a hom-Lie-Rinehart algebra over $(A, \phi)$. Let us define a bilinear map $m : L \times L \to L$ by $m(x, y) := [x, y]_L$, for $x, y \in L$. By definition for any $x, y \in L$ and $a \in A$, we get

$$m(x, a.y) = \phi(a).m(x, y) + \rho_L(x)(a).z_L(y).$$

By equation (6), it follows that $m$ is a 1-degree $(\phi, z_L)$-derivation of the $A$-module $L$, i.e. $m \in \text{Der}^1_{\phi}(L, z_L)$ with symbol $\sigma_m = \rho_L : L \to \text{Der}_{\phi}(A)$. Moreover, from the definition of the graded Lie bracket (1) it is clear that

$$[m, m](x, y, z) = -2(m \circ m)(x, y, z)$$

$$= 2 \left\{ [z_L(x), [y, z]_L]_L + [z_L(y), [z, x]_L]_L + [z_L(z), [x, y]_L]_L \right\} = 0.$$

Conversely, let us start with an element $m \in \text{Der}^1_{\phi}(L, z_L)$ (with symbol $\sigma_m$) satisfying the identity: $[m, m] = 0$. Let us define a bracket $[-, -]_L : L \times L \to L$ as follows

$$(x, y, z) \mapsto [x, y]_L + m(x, z_L(y)) - m(y, z_L(x)) = 0.$$
Let $[x, y]_L = m(x, y)$ for any $x, y \in L$. Also, define a linear map $\rho_L := \sigma_m : L \to \text{Der}_\phi(A)$. Then it follows that $(A, L, [-, -], \phi, x_L, \rho_L)$ is a hom-Lie-Rinehart algebra over $(A, \phi)$.

Thus, we have a description of hom-Lie-Rinehart algebra structures on the pair $(L, x_L)$ in terms of the graded Lie algebra structure on $(\phi, x_L)$-multiderivations of $L$, obtained by Theorem 3.3.

### 3.1. Deformation complex

Let $(L, x_L)$ be a hom-Lie-Rinehart algebra over $(A, \phi)$. Then by Proposition 3.4, this hom-Lie-Rinehart algebra structure on $(L, x_L)$ corresponds to an element $m \in \text{Der}_\phi^1(L, x_L)$ such that $[m, m] = 0$.

Let us define a cochain complex $(C^*_\text{def}(L, x_L), \delta)$, where

$$C^*_\text{def}(L, x_L) := \bigoplus_{n \geq 1} C^n_{\text{def}}(L, x_L), \quad \text{and} \quad C^n_{\text{def}}(L, x_L) := \text{Der}_\phi^{n-1}(L, x_L).$$

Here, the differential

$$\delta : C^n_{\text{def}}(L, x_L) \to C^{n+1}_{\text{def}}(L, x_L)$$

is given by

$$\delta(D)(x_0, x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i m(x_L^{i-1}(x_i), D(x_0, \ldots, \hat{x}_i, \ldots, x_n)) + \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} D(m(x_i, x_j), x_L(x_0), \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_L(x_n)).$$

Next, let us observe that $m \in \text{Der}_\phi^1(L, x_L)$ satisfies $[m, m] = 0$, therefore graded Jacobi identity for the graded Lie bracket $[\cdot, \cdot, \cdot] : \text{Der}_\phi^0(L, x_L) \times \text{Der}_\phi^0(L, x_L) \to \text{Der}_\phi^0(L, x_L)$ implies that $\delta^2 = 0$.

**Proposition 3.5.** Let $(L, x_L)$ be a hom-Lie-Rinehart algebra over $(A, \phi)$ and $m$ be the corresponding 1-degree $(\phi, x_L)$-derivation satisfying $[m, m] = 0$. Then with the coboundary operator $\delta : C^n_{\text{def}}(L) \to C^{n+1}_{\text{def}}(L)$ defined by $\delta(D) = [m, D]$, for $D \in C^n_{\text{def}}(L, x_L)$, the cochain complex $(C^*_\text{def}(L, x_L), \delta)$ is a differential graded Lie algebra (DGLA with a shift in degree).

Let us denote by $H^*_\text{def}(L, x_L)$, the cohomology of the cochain complex $(C^*_\text{def}(L, x_L), \delta)$. Next, we show that the cohomology $H^*_\text{def}(L, x_L)$ is the deformation cohomology of the hom-Lie-Rinehart algebra $(L, x_L)$.

### 4. Deformation of Hom-Lie-Rinehart algebras

In view of Proposition 3.4, we now consider the hom-Lie-Rinehart algebra structure on $(L, x_L)$ over $(A, \phi)$ as an element $m \in \text{Der}_\phi^1(L, x_L)$ satisfying $[m, m] = 0$. Here, we denote by $R[[t]]$, the space of formal power series ring with parameter $t$.

**Definition 4.1.** A deformation of a hom-Lie-Rinehart algebra structure on $(L, x_L)$ (over $(A, \phi)$) which is given via $m \in \text{Der}_\phi^1(L, x_L)$, is a $R[[t]]$-bilinear map

$$m_t : L[[t]] \otimes L[[t]] \to L[[t]]$$
which is given by

\[ m_t(x, y) = \sum_{i \geq 0} t^i m_i(x, y) \]

for \( m_0 = m \) and \( m_i \in \text{Der}^i_\phi(L, x_L) \) for \( i > 0 \), and satisfy \([ [m_i, m_j] ] = 0\). Here, \([[ -, - ]]\) is a graded Lie algebra bracket on \( \text{Der}^1_\phi(L[[t]], x_L) \) and the map \( \phi_t : A[[t]] \rightarrow A[[t]] \) is an algebra homomorphism extending the algebra homomorphism \( \phi : A \rightarrow A \) such that \( \phi_t(t) = t \).

**Remark 4.2.** Note that \( m_t = \sum_{i \geq 0} t^i m_i \) is a 1-degree \((\phi_t, x_t)\) multiderivation of \( L[[t]] \) with the symbol \( \sigma_{m_t} \) given by

\[ \sigma_{m_t}(x) := \sum_{i \geq 0} t^i \sigma_{m_i}(x) : L[[t]] \rightarrow \text{Der}_\phi(A[[t]]). \]

Since \( m_t \) satisfies \([ [m_t, m_t] ] = 0\), it corresponds to a hom-Lie-Rinehart algebra structure on \((L[[t]], x_t)\) over \((A[[t]], \phi_t)\).

If \( D \in \ker(\delta_1) \) then \( \delta(D) = [m, D] = 0 \) for \( D \in \text{Der}_\phi(L, x_L) \). i.e.,

\[ m(D(x), y) + m(x, D(y)) = D(m(x, y)). \]

Therefore, \( H^1_{\text{def}} \) gives set of all \((\phi, x_L)\)-derivations on hom-Lie-Rinehart algebra \((L, x_L)\).

Let \( m_t \) be a deformation of \( m \). Then

\[ m_t(x_t(a), m_t(b, c)) + m_t(x_L(b), m_t(c, a)) + m_t(x_L(c), m_t(a, b)) = 0. \]

Comparing the coefficients of \( t^n \) for \( n \geq 0 \), we get the following equations:

\[ \sum_{i, j=0}^n m_t(x_L(a), m_t(b, c)) + m_t(x_L(b), m_t(c, a)) + m_t(x_L(c), m_t(a, b)) = 0. \quad (8) \]

For \( n = 1 \), equation (8) implies \([m, m_t] = \delta(m_1) = 0\), i.e. \( m_1 \) is a 2-cocycle.

The 2-cochain \( m_1 \) is called the infinitesimal of the deformation \( m_t \). More generally, if \( m_1 = 0 \) for \( 1 \leq i \leq (n - 1) \) and \( m_n \) is non zero cochain then \( m_n \) is called the \( n \)-infinitesimal of the deformation \( m_t \). By the above discussion the following proposition immediately follows.

**Proposition 4.3.** The infinitesimal of the deformation \( m_t \) is a 2-cocycle in \( C^2_{\text{def}} \). More generally, the \( n \)-infinitesimal is a 2-cocycle.

**Definition 4.4.** Two deformations \( m_t \) and \( \tilde{m}_t \) are said to be equivalent if we have a formal automorphism

\[ \Phi_t : L[[t]] \rightarrow L[[t]] \]

defined as \( \Phi_t = \text{id}_L + \sum_{i \geq 1} t^i \phi_i \)

where for each \( i \geq 1 \), \( \phi_i : L \rightarrow L \) is a \( R \)-linear map such that

\[ \phi_i \circ x_L = x_L \circ \phi_i \quad \text{and} \quad \tilde{m}_t(x, y) = \Phi_t^{-1} m_t(\Phi_t x, \Phi_t y). \]

**Definition 4.5.** An \( y \) deformation equivalent to the deformation \( m_0 = m \) is said to be a trivial deformation.

**Theorem 4.6.** The cohomology class of the infinitesimal of a deformation \( m_t \) is determined by the equivalence class of \( m_t \).

**Proof.** Let \( \Phi_t \) represents an equivalence of deformation given by \( m_t \) and \( \tilde{m}_t \). Then we get,

\[ \tilde{m}_t(x, y) = \Phi_t^{-1} m_t(\Phi_t x, \Phi_t y). \]
Expanding the above identity and comparing the coefficients of $t$ from both sides of the above equation we have

$$m_1 - m_1 = [m, \phi_1].$$

So, cohomology class of infinitesimal of the deformation is determined by the equivalence class of deformation of $m_t$.

**Definition 4.7.** A hom-Lie-Rinehart algebra is said to be rigid if and only if every deformation of it is trivial.

**Theorem 4.8.** A non-trivial deformation of a hom-Lie-Rinehart algebra is equivalent to a deformation whose $n$-infinitesimal is not a coboundary for some $n \geq 1$.

**Proof.** Let $m_t$ be a deformation of hom-Lie-Rinehart algebra with $n$-infinitesimal $m_n$ for some $n \geq 1$. Assume that there exists a 2-cochain $\phi \in C^1_{\text{def}}(\mathcal{L}, \mathcal{z_L})$ with $\delta(\phi) = m_n$. Then set

$$\Phi_t = id_L + \phi t^n$$

and define $\overline{m}_t = \Phi_t \circ m_t \circ \Phi_t^{-1}$.

Then by computing the expression and comparing coefficients of $t^n$, we get

$$\overline{m}_n - m_n = -[m, m_n] = -\delta(\phi).$$

So, $\overline{m}_n = 0$. We can repeat the argument to kill off any infinitesimal, which is a coboundary.

**Corollary 4.9.** If $H^2_{\text{def}}(\mathcal{L}, \mathcal{z_L}) = 0$, then hom-Lie-Rinehart algebra is rigid.

### 4.1. An example of a rigid Hom-Lie-Rinehart algebra

Here, we deduce that the space of $\phi$-derivations of an associative commutative algebra is a rigid Hom-Lie-Rinehart algebra.

Let $A$ be an associative commutative $R$-algebra and $\phi : A \to A$ be an algebra automorphism, then the space of all $\phi$-derivations of $R$-algebra $A$, denoted by $\text{Der}_\phi(A)$ has a hom-Lie-Rinehart algebra structure. Let us recall that

$$(\text{Der}_\phi(A), Ad_\phi) := (A, \text{Der}_\phi(A), [-, -]_\phi, \phi, Ad_\phi, Ad_\phi)$$

is a hom-Lie-Rinehart algebra over $(A, \phi)$.

Let $D \in \mathcal{D}_{\text{Der}}(\text{Der}_\phi(A), Ad_\phi)$ with symbol $\sigma_D$. Let $D$ be a 2-cocycle, i.e.

$$\delta(D) = [m, D] = 0,$$

where $m = [-, -]_\phi$, and therefore it follows that $\sigma_{[m, D]} = 0$. Now from equation (5), we have the following relation

$$0 = \sigma_{[m, D]} = -\sigma_m \circ D - \sigma_D \circ m + \{\sigma_m, \sigma_D\}.$$
Therefore, we get the following equation
\[
\phi(D(x, y)(a)) = -\sigma_D([x, y]_\phi)(\phi(a)) + Ad_\phi^2(x)\sigma_D(y)(a) \\
- \sigma_D(Ad_\phi(y)Ad_\phi(x)(a)) - Ad_\phi^2(y)\sigma_D(x)(a) \\
+ \sigma_D(Ad_\phi(x))Ad_\phi(y)(a).
\]

Let us consider, a map $Ad^{-1}_\phi \circ \sigma_D : \text{Der}_\phi(A) \to \text{Der}_\phi(A)$, then clearly it is an $A$-linear map and it commutes with the map $Ad_\phi : \text{Der}_\phi(A) \to \text{Der}_\phi(A)$. Therefore, $Ad^{-1}_\phi \circ \sigma_D$ is a zero-degree $(\phi, Ad_\phi)$-derivation of the pair $\text{Der}_\phi(A)$.

\[
\delta(Ad^{-1}_\phi \circ \sigma_D)(x, y)(a) = [m, Ad^{-1}_\phi \circ \sigma_D](x, y)(a) \\
= [Ad^{-1}_\phi \circ \sigma_D(x), y]_\phi(a) - [Ad^{-1}_\phi \circ \sigma_D(y), x]_\phi(a) \\
- Ad^{-1}_\phi \circ \sigma_D([x, y]_\phi)(a).
\]

The first two terms on the right-hand side of the previous expression can be written as follows:

\[
\begin{align*}
\left([Ad^{-1}_\phi \circ \sigma_D(x), y]_\phi - [Ad^{-1}_\phi \circ \sigma_D(y), x]_\phi\right)(a) &= \sigma_D(x)(y(\phi^{-1}(a)) - \phi(y(\phi^{-1}(\sigma_D(x)(a)))) \\
&\quad - \sigma_D(y)(x(\phi^{-1}(a)) + \phi(x(\phi^{-1}(\sigma_D(y)(a))))) \\
&= \phi^{-1}(\sigma_D(Ad_\phi(x))Ad_\phi(y)(a) - Ad_\phi^2(y)\sigma_D(x)(a) \\
&\quad - Ad_\phi^2(x)\sigma_D(y)(a)) + Ad_\phi^2(x)\sigma_D(y)(a).
\end{align*}
\]

By using equations (9–11), we deduce that

\[
D(x, y)(a) = [m, Ad^{-1}_\phi \circ \sigma_D](x, y)(a) = \delta(Ad^{-1}_\phi \circ \sigma_D)(x, y)(a).
\]

Therefore, any 2-cocycle is a 2-coboundary, i.e. $H^2_{\text{def}}(\text{Der}_\phi(A), Ad_\phi) = 0$. Consequently, we have the following proposition

**Proposition 4.10.** The hom-Lie-Rinehart algebra $(\text{Der}_\phi(A), Ad_\phi)$ is rigid.

### 4.2. Obstructions to the extension of deformations

Let $(\mathcal{L}, \mathcal{z}_L)$ be a hom-Lie-Rinehart algebra and the hom-Lie-Rinehart structure on the pair $(L, \mathcal{z}_L)$ corresponds to $\mathfrak{m} \in C^2_{\text{def}}(\mathcal{L}, \mathcal{z}_L)$. Now, we consider the problem of extending a deformation of $\mathfrak{m}$ of order $N$ to a deformation of $\mathfrak{m}$ of order $(N + 1)$. Let $m_i$ be a deformation of $\mathfrak{m}$ of order $N$. Then

\[
m_i = \sum_{i=0}^{N} m_it^i
\]

where $m_i \in C^2_{\text{def}}(\mathcal{L}, \mathcal{z}_L)$ for each $1 \leq i \leq N$ such that $[[m_i, m_i]] = 0$. If there exists a 2-cochain $m_{N+1} \in C^2_{\text{def}}(\mathcal{L}, \mathcal{z}_L)$ such that $m_i = m_i + m_{N+1}t^{N+1}$ is a deformation of $\mathfrak{m}$ of order $N + 1$. Then we say that $m_i$ extends to a deformation of order $(N + 1)$.

**Definition 4.11.** A deformation of $\mathfrak{m}$ of order $N$ is given by $m_i = \sum_{i=0}^{N} m_it^i$ modulo $t^{N+1}$ such that $[[m_i, m_i]] = 0$. If there exists a 2-cochain $m_{N+1} \in C^2_{\text{def}}(\mathcal{L}, \mathcal{z}_L)$ such that
is a deformation of order \(N + 1\). Then we say that \(\tilde{m}_t\) extends to a deformation of order \((N + 1)\).

**Definition 4.12.** Let \(m_t\) be a deformation of \(m\) of order \(N\). Consider the cochain

\[
\Theta(L, x_L) \in C_{def}^3(L, x_L),
\]

where

\[
\Theta(L, x_L)(a, b, c) = \sum_{i+j=N+1; \ i, j \geq 0} (m_i(x_L(a), m_j(b, c)) + m_i(x_L(b), m_j(c, a)) + m_i(x_L(c), m_j(a, b)))
\]

for \(a, b, c \in L\). The 3-cochain \(\Theta(L, x_L)\) is called the obstruction cochain for extending the deformation of \(m\) of order \(N\) to a deformation of order \(N + 1\). We can write \(\Theta(L, x_L)\) as follows

\[
\Theta(L, x_L) = -\frac{1}{2} \sum_{i+j=N+1; \ i, j \geq 0} [m_i, m_j]
\]

By equation (12), and using graded Jacobi identity it follows that \(\Theta(L, x_L)\) is a 3-cocycle.

**Theorem 4.13.** Let \(m_t\) be a deformation of \(m\) of order \(N\). Then \(m_t\) extends to a deformation of order \(N + 1\) if and only if the cohomology class of \(\Theta(L, x_L)\) vanishes.

**Proof.** Suppose that a deformation \(m_t\), of order \(N\), extends to a deformation of order \(N + 1\). Then

\[
\sum_{i+j=N+1; \ i, j \geq 0} m_i(x_L(a), m_j(b, c)) + m_i(x_L(b), m_j(c, a)) + m_i(x_L(c), m_j(a, b)) = 0.
\]

As a result, we get \(\Theta(L, x_L) = \delta(m_{N+1})\). So, cohomology class of \(\Theta(L, x_L)\) vanishes.

Conversely, let \(\Theta(L, x_L)\) be a coboundary. Suppose that

\[
\Theta(L, x_L) = \delta(m_{N+1})
\]

for some 2-cochain \(m_{N+1}\). Define a map \(\tilde{m}_t : L[[t]] \times L[[t]] \rightarrow L[[t]]\) as follows

\[
\tilde{m}_t = m_t + m_{N+1} t^{N+1}.
\]

Then for any \(x, y, z \in L\), the map \(\tilde{m}_t\) satisfies the following identity

\[
\sum_{i+j=N+1; \ i, j \geq 0} m_i(x_L(x), m_j(y, z)) + m_i(x_L(y), m_j(z, x)) + m_i(x_L(z), m_j(x, y)) = 0.
\]

This, in turn, implies that \(\tilde{m}_t\) is a deformation of \(m\) extending \(m_t\). \(\square\)

**Corollary 4.14.** If \(H^3_{def}(L, x_L) = 0\), then every 2-cocycle in \(C_{def}^2(L, x_L)\) is an infinitesimal of some deformation of \(m\).

**5. Some particular cases**

In this section, we discuss some particular cases of hom-Lie-Rinehart algebras and the associated deformation complex governed by suitable differential graded Lie algebras.

**5.1. Lie-Rinehart algebras**

Let \((L, x_L)\) be a hom-Lie-Rinehart algebra over \((A, \phi)\). If \(x_L = \text{Id}_L\), and \(\phi = \text{id}_A\), then the hom-Lie-Rinehart algebra \((L, x_L)\) is a Lie-Rinehart algebra \(L\) over \(A\).
Next, let \( L \) be an \( A \)-module and \( \beta : L \to L \) be a \( \phi \)-function linear map, then in the case when \( \beta = id_L \), and \( \phi = id_A \), the definition of degree \( n \) \( (n \geq 0) \) \( (\phi, \beta) \)-multiderivation of the pair \( (L, \beta) \) gives the notion of an \( A \)-module multiderivation [17] of degree \( n + 1 \). However, here we consider the following definition of \( A \)-module multiderivations (with a change in the convention of degree from Ref. [17]).

**Definition 5.1.** An \( A \)-module multiderivation (of degree \( n \geq 0 \)) of \( L \) is a skew-symmetric \( R \)-multilinear map:

\[
D : \wedge^{n+1}L \to L.
\]

for which we have an \( A \)-multilinear map \( \sigma_D \in Hom_A(\wedge^n L, Der(A)) \) such that

\[
D(x_0, \ldots, f . x_i, \ldots, x_n) = f.D(x_0, \ldots, x_i, \ldots, x_n) + (-1)^{n-i} \sigma_D(x_0, \ldots, \tilde{x}_i, \ldots, x_{n-1})(f).x_i.
\]

Denote the space of all \( n \)-degree multiderivations as \( Der^n(L) \). Set \( Der^{-1}(L) = L \). This gives a graded \( A \)-module \( Der^i(L) = \bigoplus_{i \in \mathbb{Z}} Der^i(L) \), where \( Der^i(L) = 0 \) for \( i \leq -2 \). Note that for any \( n \geq 0 \), we have

\[
Der^n(L) = \bigoplus_{i \leq n} Der^n_i(L, id_L).
\]

**Remark 5.2.** Let \( E \) be a faithful \( A \)-module. A Koszul connection on \( E \) is an \( A \)-linear mapping \( \nabla : Der(A) \to Hom_R(E, E) \), sending \( X \mapsto \nabla_X \) such that

\[
\nabla_X(a.m) = X(a).m + a.\nabla_X(m)
\]

for \( m \in E \), \( X \in Der(A) \), and \( a \in A \). In general, an \( A \)-module may not admit a Koszul connection. But for any Projective \( A \)-module, there exists a Koszul connection.

Following lemma generalizes its geometric counterpart for Lie algebroid case, given in Ref. [2]:

**Lemma 5.3.** Space of \( A \)-module multiderivations of degree \( n \) on \( L \), i.e. \( Der^n(L) \) fits into an exact sequence of \( A \)-modules:

\[
0 \to Hom_A(\wedge^{n+1} L, L) \to Der^n(L) \to Hom_A(\wedge^n L, Der(A)) \to 0,
\]

for \( n \geq 0 \).

**Proof.** Define, \( F_D \in Hom_R(\otimes^{n+1} L, L) \) by

\[
F_D(x_0, \ldots, x_n) = D(x_0, \ldots, x_n) + (-1)^n \sum_{i=0}^{n} (-1)^{i+1} \nabla_{\sigma_D(x_0, \ldots, \tilde{x}_i, \ldots, x_n)}(x_i).
\]

It follows that \( F_D \) is skew-symmetric and \( A \)-multilinear. Also, one can check that a connection \( \nabla \) on \( L \) determines an isomorphism of \( A \)-modules

\[
Der^n(L) \cong Hom_A(\wedge^{n+1} L, L) \oplus Hom_A(\wedge^n L, Der(A))
\]

assigning \( D \to (F_D, \sigma_D) \). \( \square \)

### 5.1.1. Graded lie algebra structure on \( Der^1(L) \)

From Theorem 3.3, it is immediate that there is a graded Lie algebra structure on the graded \( A \)-module \( Der^1(L) \). If \( D_1 \in Der^0(L) \), and \( D_2 \in Der^1(L) \), then the graded Lie bracket is defined as follows:

\[
[D_1, D_2] = (-1)^{pq}D_1 \circ D_2 - D_2 \circ D_1
\]

where,
be a projective Rinehart algebra structures on $L$ and elements $m$

If $L$ is an $A$-module, then there exists a one-one correspondence between Lie-Rinehart algebra structures on $L$ and elements $m \in \text{Der}^1(L)$ satisfying $[m, m] = 0$.

\[ D_1 \circ D_2(x_0, \ldots, x_{p+q}) = \sum_{\tau \in S(p+q)} \text{sgn}(\tau) D_1(D_2(x_{\tau(0)}, \ldots, x_{\tau(q+1)})), \ldots, x_{\tau(p+q)}) \text{.} \]

It follows that $[D_1, D_2] \in \text{Der}^{p+q}(L)$ and

\[ \sigma_{[D_1, D_2]} = (-1)^{pq}\sigma_{D_1} \circ D_2 - \sigma_{D_2} \circ D_1 + [\sigma_{D_1}, \sigma_{D_2}] \]

where,

\[ [\sigma_{D_1}, \sigma_{D_2}](x_1, \ldots, x_{p+q}) = \sum_{\tau \in S[p, q]} \text{sgn}(\tau) [\sigma_{D_1}(x_{\tau(1)}, \ldots, x_{\tau(p)}), \sigma_{D_2}(x_{\tau(p+1)}, \ldots, x_{\tau(p+q)})] \]

Next result is a particular case of Proposition 3.4, which describes Lie-Rinehart algebra structures on $L$ in term of the above-mentioned graded Lie algebra.

**Proposition 5.4.** If $L$ is an $A$-module, then there exists a one-one correspondence between Lie-Rinehart algebra structures on $L$ and elements $m \in \text{Der}^1(L)$ satisfying $[m, m] = 0$.

### 5.1.2. Deformation complex for Lie-Rinehart algebra

Let $L$ be a Lie-Rinehart algebra over $A$, then it corresponds to an element $m \in \text{Der}^1(L)$ satisfying $[m, m] = 0$. We define

\[ C_{\text{def}}^n(L) := C_{\text{def}}^n(\mathcal{L}, \text{id}_L) \]

for $n \geq 1$, and $C_{\text{def}}^0(L) = \text{Der}^{-1}(L) = L$. The coboundary $\delta : C_{\text{def}}^n(L) \to C_{\text{def}}^{n+1}(L)$ is given by $\delta(D) = [m, D]$. Thus, $(C_{\text{def}}^0(L), \delta)$ is a differential graded Lie algebra and it forms a cochain complex. We denote the cohomology of this cochain complex by $H_{\text{def}}^n(L)$.

**Example 5.5.** In particular, for Lie algebra $\mathfrak{g}$ the deformation complex $C_{\text{def}}^0(\mathfrak{g})$ is the usual Chevalley-Eilenberg complex $C^n(\mathfrak{g}, \mathfrak{g})$ with coefficients in the adjoint representation.

**Example 5.6.** For any Lie algebroid, the above deformation complex will be the deformation complex, defined in Ref. [2].

**Example 5.7.** For $L = \text{Der}(A)$, the commutator bracket $[-, -]_C$ gives a Lie algebra bracket and by considering the anchor map $\rho = \text{id}_L([-, -, \rho])$ is a Lie-Rinehart algebra. Consider $\text{Der}(A)$ to be a projective $A$-module. then

\[ Z^k(C_{\text{def}}^n(L)) = \text{Hom}_A(\wedge^{k-1} L, L). \]

$\delta(D) = 0$. Now, as we know

\[ \sigma_{\delta(D)} = \delta(\sigma_D) + (-1)^{k-1}\rho \circ D \]

i.e., $D = (-1)^k \delta(\sigma_D)$. Therefore,

\[ H_{\text{def}}^n(\text{Der}(A)) = 0. \]

**Definition 5.8.** A deformation of a Lie-Rinehart algebra structure on $L$, which is given via $m \in \text{Der}^1(L)$, is defined as a $R[[t]]$-bilinear map $m_i : L[[t]] \otimes L[[t]] \to L[[t]]$, given by

\[ m_i[x, y] = \sum_{i \geq 0} t^i m_i[x, y], \text{ where } m_0 = m \text{ and } m_i \in \text{Der}^i(L) \text{ for } i \geq 0 \]

satisfying $[[m_i, m_i]] = 0$. Here, $[[-, -]]$ is the graded Lie algebra bracket on $\text{Der}^*(L[[t]])$. 
Remark 5.9. If \( D \in \ker(\delta_1) \), then \( \delta(D) = 0 \) for \( D \in \text{Der}(L) \). i.e.,
\[
m(D(x),y) + m(x,D(y)) = D(m(x,y)).
\]
if \( D \in \text{Im}(\delta_0) \), then \( D(y) = m(x,y) \). So, \( H^1_{\text{def}} \) gives set of all outer derivation on \( L \).

Remark 5.10. Let \( m_t \) be a deformation of \( m \). Then we have:
\[
m_t(a, m_t(b, c)) + m_t(b, m_t(c, a)) + m_t(c, m_t(a, b)) = 0.
\]
From equation (1), \( \delta(m_1) = 0 \), i.e. \( m_1 \) is a 2-cocycle.

By Theorem 4.6, it follows that \( H^2_{\text{def}}(L) \) characterizes the non-trivial infinitesimal deformations of the Lie-Rinehart algebra \( L \). Thus, if \( H^2_{\text{def}}(L) = 0 \), then Lie-Rinehart algebra is rigid. Moreover, Theorem 4.13 implies that obstructions to extend a deformation of order \( n \) of Lie-Rinehart algebra \( L \) to a deformation of order \( n + 1 \) are contained in the 3-rd cohomology group \( H^3_{\text{def}}(L) \). Therefore, \( H^3_{\text{def}}(L) \) is deformation cohomology for the Lie-Rinehart algebra \( L \), obtained as a particular case of hom-Lie-Rinehart algebra.

Remark 5.11. In Ref. [8], the authors have discussed deformations of Lie-Rinehart algebras as an application of the deformation theory of Courant algebroids. In particular, they identified the deformation complex of a Lie-Rinehart algebra in terms of a specific Rothstein algebra.

5.2. Hom-Lie algebras

Let \((L, [-,-], x_L)\) be a hom-Lie algebra (over \( R \)), then it is a hom-Lie module over itself by adjoint action. Recall that \((L, [-,-], x_L)\) is also a hom-Lie-Rinehart algebra \((L, x_L)\) over \((R, id_R)\). Then a \((id_R, x_L)\)-multiderivation \( \varphi \) of degree \( n \) is simply a \( n + 1 \)-linear alternating map
\[
\varphi : \wedge^{n+1}L \to L,
\]
satisfying \( x_L \circ \varphi = \varphi \circ x_L^{\otimes (n+1)} \). Therefore,
\[
\mathcal{D} \text{er}^n_{\text{dl}}(L, x_L) = C^n_{\text{HL}}(L, L),
\]
where, \( C^n_{\text{HL}}(L, L) \) is the space of \( (n+1) \)-linear alternating cochains defined in Section 2 of Ref. [1].

Next, it is easy to see that the differential graded Lie algebra structure on \( \mathcal{D} \text{er}^n_{\text{dl}}(L, x_L) \) is also the same as the one discussed in Ref. [1]. Consequently, in the case of hom-Lie algebras, the cohomology \( H^n_{\text{def}}(L, x_L) \) is the same as the deformation cohomology \( H^n_{\text{HL}}(L, L) \) defined in Ref. [1].

5.3. Lie algebroids

A Deformation complex for Lie algebroids is defined by considering the associated differential graded Lie algebra structure on the space of multiderivations in Ref. [2]. On the other hand, one can deduce a deformation complex for Lie algebroids by considering them as a hom-Lie-Rinehart algebra.

Let \( E \) be a Lie algebroid over a smooth manifold \( M \) with the anchor map \( \rho \) and the Lie-bracket \([-,-]\) on the space of sections \( \Gamma E \). Then the Lie algebroid structure on \( E \) yields a hom-Lie-Rinehart algebra \((L, x_L)\) over \((\mathcal{A}, \phi)\) where \( A = C^\infty(M) \), \( \phi = id_A \), \( L = \Gamma E \), \([-,-]\) = \([-,-]\), \( x_L = id_L \), and \( \rho_L = \rho \). By considering \( \phi = id_A \) and \( \beta = id_L \), the space of \((\phi, \beta)\)-multiderivations turns out to be the space of multiderivations of the vector bundle \( E \). Moreover, the differential graded Lie algebra structure on the space of \((\phi, \beta)\)-multiderivations is the one (except in degree \(-1\)) studied for Lie algebroids.
**Conclusion**

Here, we described a differential graded Lie algebra (DGLA) for hom-Lie-Rinehart algebras, which controls the formal one-parameter deformations. This study goes in line with particular cases such as hom-Lie algebra, Lie-Rinehart algebras. One can expect to associate such a differential graded Lie algebra to a hom-Lie algebroid since any hom-Lie algebroid is also a particular type of hom-Lie-Rinehart algebras. Next, it is natural to ask: can we interpret this differential graded Lie algebra in terms of deformations of a hom-Lie algebroid? This type of questions we plan to address in a separate note by introducing deformations of a hom-Lie algebroid as a smooth family of hom-Lie algebroids over an interval $I \subset \mathbb{R}$.

**Funding**

This article was funded by Council of Scientific and Industrial Research, India.

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**References**

[1] Ammar, F., Ejbehi, Z., Makhlouf, A. (2011). Cohomology and deformations of Hom-algebras. *J. Lie Theory* 21(4):813–836.

[2] Crainic, M., Moerdijk, I. (2008). Deformations of Lie brackets: cohomological aspects. *J. Eur. Math. Soc. (JEMS)*. 10(4):1037–1059.

[3] Evens, S., Lu, J.-H., Weinstein, A. (1999). Transverse measures, the modular class and a cohomology pairing for Lie algebroids. *Quart. J. Math. Oxford Ser. (2)*. 50(200):417–436. DOI: 10.1093/qmath/50.200.417.

[4] Hartwig, J. T., Larsson, D., Silvestrov, S. D. (2006). Deformations of lie algebras using $\sigma$-derivations. *J. Algebra* 295(2):314–361. DOI: 10.1016/j.jalgebra.2005.07.036.

[5] Huebschmann, J. (1998). Lie-Rinehart algebras, Gerstenhaber algebras and Batalin-Vilkovisky algebras. *Ann. Inst. Fourier (Grenoble)*. 48(2):425–440. DOI: 10.5802/aif.1624.

[6] Huebschmann, J. (1999). Duality for Lie-Rinehart algebras and the modular class. *J. Reine Angew. Math*. 510:103–159. DOI: 10.1515/crll.1999.043.

[7] Huebschmann, J. (2004). Lie-Rinehart algebras, descent, and quantization. In *Galois Theory, Hopf Algebras, and Semiabelian Categories, Volume 43 of Fields Inst. Commun*. Providence, RI: Amer. Math. Soc., pp. 295–316.

[8] Keller, F., Waldmann, S. (2015). Deformation theory of Courant algebroids via the rothstein algebra. *J. Pure Appl. Algebra* 219(8):3391–3426. DOI: 10.1016/j.jpaa.2014.12.002.

[9] Laurent-Gengoux, C., Teles, J. (2013). Hom-Lie algebroids. *J. Geom. Phys*. 68:69–75. DOI: 10.1016/j.geomphys.2013.02.003.

[10] Makhlouf, A., Silvestrov, S. (2010). Notes on 1-parameter formal deformations of hom-associative and Hom-Lie algebras. *Forum Math.* 22(4):715–739.

[11] Makhlouf, A., Silvestrov, S. D. (2008). Hom-algebra structures. *J. Gen. Lie Theory Appl.* 2(2):51–64. DOI: 10.4303/jgta/S070206.

[12] Mandal, A., Kumar Mishra, S. (2018). On Hom-Gerstenhaber algebras and Hom-Lie algebroids. *J. Geom. Phys.* 133:287–302. DOI: 10.1016/j.geomphys.2018.07.018.

[13] Mandal, A., Kumar Mishra, S. Universal Central extensions and non-abelian tensor product of Hom-Lie-Rinehart algebras. arXiv:1803.00936v2 [math.KT].

[14] Mandal, A., Kumar Mishra, S. (2018). Hom-Lie-Rinehart algebras. *Commun. Algebra* 46(9):3722–3744. DOI: 10.1080/00927872.2018.1424865.

[15] Rinehart, G. S. (1963). Differential forms on general commutative algebras. *Trans. Am. Math. Soc.* 108(2):195–222. DOI: 10.2307/1993603.

[16] Sheng, Y. (2012). Representations of hom-Lie algebras. *Algebra Represent. Theory* 15(6):1081–1098. DOI: 10.1007/s10468-011-9280-8.
[17] Vitagliano, L. (2015). Representations of homotopy lie rinehart algebras. Math. Proc. Camb. Phil. Soc. 158(1):155–191. DOI: 10.1017/S0305004114000541.

[18] Zhang, T., Han, F., Bi, Y. (2018). Crossed modules for Hom-Lie-Rinehart algebras. Colloq. Math. 152(1): 1–14. DOI: 10.4064/cm7170-3-2017.