Measures on perfect $e$-free PAC fields

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Abstract

We construct measures on definable sets in $e$-free perfect PAC fields, as well as on perfect PAC fields whose absolute Galois groups are free pro-$p$ of finite rank. We deduce the definable amenability of all groups definable in such fields. As a corollary, we additionally prove the definable amenability of all groups definable in perfect $\omega$-free PAC fields via ultralimit measures.

1 Introduction

In this paper, we construct measures on definable sets in perfect pseudo-algebraically closed fields with free absolute Galois group. A field $K$ is called pseudo-algebraically closed, or PAC, if every absolutely irreducible variety defined over $K$ has a $K$-rational point. The class of PAC fields was isolated by Ax ([1]) in the course of algebraically characterizing pseudo-finite fields; he showed that the class of pseudo-finite fields coincides with the class of perfect PAC fields with free absolute Galois group of rank 1. In the early 80’s, PAC fields were given systematic study by Cherlin, van den Dries, and Macintyre ([3]) and separately by Ershov ([5]). Since then, they have been a central object of study in both model theory and field arithmetic.

Our work was motivated by two results and one question. In [2], van den Dries, Macintyre, and the first author showed how to definably associate a measure to each definable set in a pseudo-finite field. This measure is a non-standard counting measure. Hrushovski shows in [8] that this counting measure is the only finitely additive probability measure on a pseudo-finite field which satisfies Fubini. This result was later extended by Halupczok ([7]) who replaced the Fubini condition by preservation under definable bijections and some invariance condition, and showed, in essence, that in order for a perfect PAC field to have a probability measure on definable sets satisfying these conditions, the absolute Galois group of the field had to be pro-cyclic. So, in a PAC field with non-pro-cyclic absolute Galois group, it is natural to ask if there is still a way to construct a useful measure on definable sets, which necessarily has to satisfy weaker conditions. It is always possible to construct some measures on definable sets

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from types, which determine \(\{0, 1\}\)-valued measures, but these are usually less relevant for the kinds of applications for which measures have been used. A test question for our measures originally came from the problem of determining if all groups in simple theories are definably amenable. Recall that a definable group \(G\) is *definably amenable* if there is a left-translation invariant finitely additive probability measure on definable subsets of \(G\). It is known that all stable groups are definably amenable. Within the class of simple theories it was only recently discovered that there are non-definably amenable groups with simple theories (\([\Pi]\)). Since groups definable in bounded perfect PAC fields form one of the most interesting classes of groups with simple theories, it is natural to ask if they are all definably amenable.

In the particular case of a perfect \(e\)-free PAC field \(k\) (i.e., with absolute Galois group isomorphic to \(\hat{F}_e\), the free profinite group on \(e\) generators), we are able to define a measure on definable subsets of (absolutely irreducible) varieties, which coincides with the non-standard counting measure when \(e = 1\). Our work uses in an essential way the measure introduced by Jarden and Kiehne (\([\Pi]\)). We build on their results by using their measure on sentences to define measures on definable sets.

We also give a description of how the measure changes under definable maps, and show that it is preserved under birational automorphisms of a variety. This latter result applies to show that if \(G\) is an algebraic group defined over \(k\), then \(G(k)\) is definably amenable. Using results of Hrushovski and Pillay (\([9]\)) showing that all groups definable in an \(e\)-free PAC field are virtually isogenous to algebraic groups, we are then able to deduce the definable amenability of all groups definable in perfect \(e\)-free PAC fields.

Our measures are constructed explicitly for definable subsets of perfect PAC fields with free absolute Galois group of \(finite\) rank, but this has useful consequences for perfect \(\omega\)-free PAC fields as well. Recall that a PAC field \(K\) is called \(\omega\)-free if it is elementarily equivalent to a PAC field with absolute Galois group \(\hat{F}_\omega\). Every \(\omega\)-free PAC field is elementarily equivalent to a non-principal ultraproduct of \(e\)-free PAC fields (\(e \in \mathbb{N}\)). Thus, from the definable amenability of all groups definable in \(e\)-free PAC fields, we obtain that groups definable in \(\omega\)-free PAC fields are definably amenable, via the ultralimit measure. On the algebraic group itself, the measure is a \(0-1\) measure, so not very interesting in itself. It raises however interesting questions on the behaviour of the measure on non-algebraic definable groups. As \(\omega\)-free PAC fields are a core example of NSOP\(_1\) theories, for which a satisfactory theory of groups is not yet available, we hope that these results will play a useful role in classifying groups definable in \(\omega\)-free PAC fields.

We are also able to generalize our results to the case of perfect PAC field with free pro-\(p\) absolute Galois group. This involves a similar construction, but working over the fixed field of a fixed \(p\)-Sylow subgroup. As a corollary, we obtain definable amenability for groups definable in these fields as well.

This paper is organized as follows. Section 2 establishes several preliminary facts about definable sets and types in PAC fields, as well as a description of the aforementioned measure of Jarden and Kiehne. The main construction of measures on definable subsets of a perfect \(e\)-free PAC field takes place in Section 3, and we prove there that all groups definable in perfect \(e\)-free PAC fields are definably amenable. In Section 4, we describe extensions to other fields,
namely, the perfect \( \omega \)-free PAC fields and the perfect PAC fields with free pro-\( p \) absolute Galois group. We conclude this section with some questions about the behaviour of these measures and possible extensions.

2 Preliminaries

Definition 2.1. Let \( e \geq 1 \) be an integer. A field \( K \) is \( e \)-free PAC if it is PAC, and has absolute Galois group Gal(\( K \)) isomorphic to \( \hat{F}_e \) (the profinite completion of the free group on \( e \) generators).

Let \( L = \{+,-,\cdot,0,1\} \) be the language of rings, \( E \) a field, and \( L(E) \) the language \( L \) augmented by constant symbols for the elements of \( E \). A test sentence\(^1\) over \( E \) is a Boolean combination of \( L(E) \)-sentences of the form \( \exists y \ f(y) = 0 \), where \( f(y) \in E[y] \) (\( y \) a single variable). Note that in particular any quantifier-free sentence of \( L(E) \) is a test sentence.

Similarly, we say that \( \theta(x) \) is a test formula formula over \( E \) (in the tuple of variables \( x \)) if it is a Boolean combination of \( L(E) \)-formulas of the form \( \exists y \ f(x, y) = 0 \), where \( y \) is a single variable, and \( f \in E[x,y] \).

Notation 2.2. If \( E \) is a field, then \( E^{alg} \) denotes an algebraic closure of \( E \), and \( E^s \) its separable closure. We will view the absolute Galois group Gal(\( E \)) as acting on \( E^{alg} \). \( \text{Var}_k \) denotes the set of (absolutely irreducible) varieties defined over \( k \).

Fact 2.3. (Corollary 20.4.2 and Lemma 20.6.4 in [6]) Let \( K \) and \( L \) be perfect \( e \)-free PAC fields, and \( E \) a common subfield. Then

\[ K \equiv_E L \iff E^s \cap K \cong_E E^s \cap L, \]

if and only if they satisfy the same test sentences over \( E \).

An immediate application gives a description of types:

Corollary 2.4. Let \( E \) be a subfield of an \( e \)-free perfect PAC field \( K \), and let \( a, b \) be tuples in \( K \). The following conditions are equivalent:

1. \( \text{tp}(a/E) = \text{tp}(b/E) \);
2. There is an \( E \)-isomorphism \( E(a)^* \cap K \to E(b)^* \cap K \) which sends \( a \) to \( b \);
3. \( a \) and \( b \) satisfy the same test-formulas over \( E \);
4. for every finite Galois extension \( L \) of \( E(a) \), there is a field embedding \( \varphi : L \to E(b)^* \) such that \( \varphi(L \cap K) = \varphi(L) \cap K \).

Remark 2.5. Given a finite Galois extension \( L \) of \( E(a) \), and a subfield \( K \) of \( L \) containing \( E(a) \), we will build a test-formula \( \theta_K(x) \) over \( E \) as follows: for each subfield \( F \) of \( L \) containing

\(^1\)Also called a one-variable statement in [11].
If $L$ is a finite Galois extension of $K$, select a generator $\alpha_F$ over $E(a)$, and let $P_F \in E[x][y]$ be such that $P_F(a, y)$ is a minimal polynomial of $\alpha_F$ over $E(a)$; then the formula $\theta_K(x)$ says the following:

$$\exists y P_K(x, y) = 0 \land \bigwedge_{F \neq K} \forall y P_F(x, y) \neq 0 \land Q(x) \neq 0,$$

where $Q(x)$ is the product of all leading coefficients in $y$ of the polynomials $P_F(x, y)$. This formula has the following property: whenever $K$ contains $E(a)$, then

$$K \models \theta_K(a) \iff K \cap L \simeq K.$$

### 2.6. The measure of Jarden and Kiehne

Let $K$ be a countable Hilbertian field, let $e \geq 1$, and consider the Haar measure on $\text{Gal}(K)^e$. View $\text{Gal}(K)$ as acting on $K^{alg}$. For each $\mathcal{L}(K)$-sentence $\theta$, they define

$$\mu(\theta) = \mu(\{ \bar{\sigma} \in \text{Gal}(K)^e \mid \text{Fix}(\bar{\sigma}) \models \theta \}).$$

Then, for almost all $\bar{\sigma} \in \text{Gal}(K)^e$, $\text{Fix}(\bar{\sigma})$ is $e$-free PAC. Again, by the elementary invariants of $e$-free PAC fields, for every $\mathcal{L}(K)$-sentence $\theta$, there is a finite Galois extension $L$ of $K$, and finitely many subextensions $M_1, \ldots, M_r$ of $L$ such that for all $\bar{\sigma} \in \text{Gal}(K)^e$,

$$\text{Fix}(\bar{\sigma}) \models \theta \iff \bigvee_i \text{Fix}(\bar{\sigma}) \cap L \simeq_K M_i.$$

Hence, assuming that if $i \neq j$, then $M_i \not\simeq K M_j$, we have

$$\mu(\theta) = \sum_i |\{ \bar{\sigma} \in \text{Gal}(L/K)^e \mid \text{Fix}(\bar{\sigma}) \simeq_K M_i \}| [L : K]^{-e}.$$

**Fact 2.7.** (An ingredient in the proof of Gaschütz Lemma - Lemma 17.7.1 of [2].) Let $f : B \to A$ be an epimorphism of finite groups, with $B$ $e$-generated. If $\bar{h} \in A^e$ generates $A$, then $\{ \bar{g} \in B^e \mid f(\bar{g}) = \bar{h}, \langle \bar{g} \rangle = B \}$ is non-empty, and its size does not depend on $\bar{h}$.

**Notation 2.8.** Let $K \leq E \leq F$, with $F$ and $E$ finite Galois over $K$, and $\text{Gal}(F/K)$ $e$-generated. We denote by $\phi(E, K)$, or $\phi(E)$, the number of $\bar{\sigma} \in \text{Gal}(E/K)^e$ which generate $\text{Gal}(E/K)$. This only depends on $e$ and the group $\text{Gal}(E/K)$. We denote by $\phi(F/E, K)$ or $\phi(F/E)$ the number given by Gaschütz Lemma, i.e.: given a set $\bar{\sigma} \in \text{Gal}(E/K)^e$ generating $\text{Gal}(E/K)$, the number of lifts $\bar{\tau} \in \text{Gal}(F/K)^e$ of $\bar{\sigma}$ which generate $\text{Gal}(F/K)$. Note that $\phi(F) = \phi(F/E)\phi(E)$.

### 3 Definition of the measure when $k$ is $e$-free PAC

#### 3.1. Setting and notation

We let $e \in \mathbb{N}$, $e \geq 1$, $k$ be a countable perfect field with $\text{Gal}(k) \simeq \mathbb{F}_e$, and $V \in \text{Var}_k$. We will define a measure $\mu_V$ on definable subsets of $V$. We fix a generic $a$ of $V$ over $k$. If $S \subseteq V$ is definable and not Zariski dense in $V$, then we set $\mu_V(S) = 0$. If $L$ is a finite Galois extension of $k(a)$, we let $k_L = k^{alg} \cap L$. We set $(\sigma_1, \ldots, \sigma_e) \cdot (\tau_1, \ldots, \tau_e) = (\sigma_1 \tau_1, \ldots, \sigma_e \tau_e)$. 

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3.2. Definition of the measure. Using the description of types (2.4), it follows that for every $k$-formula $\varphi(x)$, there is a finite Galois extension $L$ of $k(a)$ such that whether $a$ satisfies $\varphi$ or not in some $e$-free perfect PAC regular extension $K$ of $k$, only depends on the $k(a)$-isomorphism type of $K \cap L$. Let $L$ be finite Galois over $k(a)$, and let $K_1, \ldots, K_r$ be the subfields of $L$ containing $k(a)$, which are regular over $k$ and with $\text{Gal}(L/K_i)$ an image of $G(k)$. We take them up to conjugation over $k(a)$. For each $i$, consider the $L(k)$-formula $\theta_i(x) = \theta_{K_i}(x)$, see 2.5. It defines a subset of $V(k)$.

Since these events are mutually exclusive on the generics of $V$, it suffices to assign a measure to each $\theta_i$, and compute $\mu_V(\varphi)$ as the sum of the appropriate $\mu_V(\theta_i)$. Indeed, they will be mutually exclusive on a Zariski open subset of $V(k)$, the complement of which will have measure 0. See also Proposition 20.6.6 in [6]. Letting $\mu$ denote the measure of Jarden and Kiehne (see 2.6), we define

$$\mu_V(\theta_i) = \frac{\mu(\theta_i(a))}{\sum_j \mu(\theta_j(a))}.$$ 

Note that

$$\sum_j \mu(\theta_j(a)) = \mu(\{ \bar{\sigma} \in \text{Gal}(k(a))^e \mid \text{Fix}(\bar{\sigma}) \cap k^s = k \})$$

$$= |\{ \bar{\sigma} \in \text{Gal}(L/k(a))^e \mid \text{Fix}(\bar{\sigma}) \cap k_L = k \}| [L : k(a)]^{-e},$$

i.e., those $\bar{\sigma}$ whose fixed field (within $L$) is regular over $k$. We do this for every finite Galois extension of $k(a)$. Another way of expressing $\mu_V(\theta_i)$ as defined above is simply as

$$\mu_V(\theta_i) = \frac{|\{ \bar{\sigma} \in \text{Gal}(L/k(a))^e \mid \text{Fix}(\bar{\sigma}) \simeq_{k(a)} K_i \}|}{|\{ \bar{\sigma} \in \text{Gal}(L/k(a))^e \mid \text{Fix}(\bar{\sigma}) \cap k_L = k \}|}.$$ 

Proposition 3.3. The measure $\mu_V$ is well-defined.

Proof. We need to show that the measure does not depend on the choice of the Galois extension $L$. I.e., that if $M$ contains $L$, and we do the counting in $\text{Gal}(M/k(a))$ to compute $\mu_V(\theta_i)$, we obtain the same number. This will follow from the following Claim, which we will prove below:

Claim. Let $\bar{\sigma} \in \text{Gal}(L/k(a))^e$, and suppose that $\text{Fix}(\bar{\sigma})$ is regular over $k$. Then the number of $\bar{\tau} \in \text{Gal}(M/k(a))$ which restrict to $\bar{\sigma}$ and with $\text{Fix}(\bar{\tau})$ regular over $k$, equals $\phi(k_M/k_L, k)[M : k_M L]^e$, and therefore does not depend on $\bar{\sigma}$.

Indeed, let $M'_1, \ldots, M'_s$ be a list (up to conjugation over $k(a)$) of all regular extensions of $k$ which are contained in $M$, and contain $k(a)$, and $\text{Gal}(M/M'_j)$ $e$-generated. Then given $K_i$, if $\theta'_j$ denotes the formula expressing that the relative algebraic closure of $k(a)$ inside $M$ is isomorphic to $M'_j$, then one has, for $\bar{\tau} \in \text{Gal}(M/k(a))^e$ and $\bar{\sigma}$ its restriction to $L$: $\text{Fix}(\bar{\tau}) \cap L \simeq_{k(a)} \text{Fix}(\bar{\sigma})$ if and only if $\text{Fix}(\bar{\tau}) \simeq_{k(a)} M'_j$ and $M'_j \cap L \simeq \text{Fix}(\bar{\sigma})$. I.e., computed in $M$, we will have

$$\mu_V(\theta_i) = \frac{\sum_{j \in J_i} |\{ \bar{\tau} \in \text{Gal}(M/k(a))^e \mid \text{Fix}(\bar{\tau}) \simeq_{k(a)} M'_j \}|}{\sum_{1 \leq j \leq r} |\{ \bar{\tau} \in \text{Gal}(M/k(a))^e \mid \text{Fix}(\bar{\tau}) \simeq_{k(a)} M'_j \}|},$$

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where $J_i = \{ j \mid M'_j \cap L \simeq K_i \}$. But by the claim, the number on the right hand side equals
\[
\frac{|\{ \sigma \in \text{Gal}(L/k(a))e \mid \text{Fix}(\sigma) \cap L \simeq k_i(a) K_i \}| \cdot \phi(k_M/k_L,k)[M : k_M]_e}{|\{ \sigma \in \text{Gal}(L/k(a))e \mid \text{Fix}(\sigma) \cap k_L = k \}| \cdot \phi(k_M/k_L,k)[M : k_M]_e} = \frac{\mu(\theta_i(a))}{\sum_j \mu(\theta_j(a))},
\]
which equals $\mu_V(\theta_i)$ computed in $L$. It therefore suffices to prove the claim.

**Proof of the claim.** Consider $\bar{\sigma} \in \text{Gal}(L/k(a))e$, which projects onto a set of generators of $\text{Gal}(k_L/k)$. Then the following two sets have the same cardinality $\phi(k_M/k_L,k)$:
\[
\begin{align*}
\{ \bar{\tau} \in \text{Gal}(k_M/k) & \mid \bar{\tau}|_{k_L} = \bar{\sigma}|_{k_L}, \langle \bar{\tau} \rangle = \text{Gal}(k_M/k) \} \\
\{ \bar{\tau} \in \text{Gal}(k_M L/k(a)) & \mid \bar{\tau}|_{L} = \bar{\sigma}, \langle \bar{\tau}|_{k_M} \rangle = \text{Gal}(k_M/k) \}.
\end{align*}
\]
Hence
\[
\frac{|\{ \bar{\tau} \in \text{Gal}(M/k(a))e \mid \bar{\tau}|_{L} = \bar{\sigma}, \langle \bar{\tau}|_{k_M} \rangle = \text{Gal}(k_M/k) \}| \cdot \phi(k_M/k_L,k)[M : k_M]_e}{|\{ \bar{\sigma} \in \text{Gal}(L/k(a))e \mid \text{Fix}(\bar{\sigma}) \cap k^s = k, \, \bar{\sigma} \supset \bar{\sigma}_0|_{k^s} \} |} = \frac{\mu(\theta_i(a))}{\sum_j \mu(\theta_j(a))}.
\]
\]

\[\Box\]

### 3.4. Another way of counting

Let $\bar{\sigma}_0 \in \text{Gal}(k(a))e$ be a lift of an $e$-tuple generating $\text{Gal}(k)$. Then the set $\{ \bar{\sigma}_0 \cdot \bar{\tau} \mid \bar{\tau} \in \text{Gal}(k(a))e \}$ coincides with the set $\{ \bar{\sigma} \in \text{Gal}(k(a))e \mid \text{Fix}(\bar{\sigma}) \cap k^s = k, \, \bar{\sigma} \supset \bar{\sigma}_0|_{k^s} \}$. We fix a finite Galois extension $L$ of $k(a)$ such that $\text{Gal}(L/k(a))$ splits, i.e.,
\[
\text{Gal}(L/k(a)) \simeq \text{Gal}(L/k_L(a)) \times \text{Gal}(k_L/k)
\]
and without loss of generality, the restriction of $\bar{\sigma}_0$ to $L$ belongs to $\text{Gal}(k_L/k) \leq \text{Gal}(L/k(a))$. (The fact that $\text{Gal}(k(a)) \simeq \text{Gal}(k^s(a)) \times \text{Gal}(k)$ is well known: since $\text{Gal}(k)$ is projective, the restriction map $\text{Gal}(k(a)) \to \text{Gal}(k)$ splits.)

Working in $\text{Gal}(k^s(a))$, we denote the Jarden-Kiehn measure by $\bar{\mu}$. Then the fields $K_i$ are contained in $\text{Fix}(\bar{\sigma}_0)$. If $\theta'_i$ is the $L(k^s(a))$-sentence expressing that the relative algebraic closure of $k^s(a)$ in $k^s L$ is isomorphic over $k^s(a)$ to $k^s K_i$, then $\theta'_i$ is a sentence of $L(k_L(a))$, and we define
\[
\bar{\mu}_V(\theta'_i) = \bar{\mu}(\theta'_i) = |\{ \bar{\sigma} \in \text{Gal}(L/k_L(a))e \mid \text{Fix}(\bar{\sigma}) \cap L \simeq k_L K_i \}| |L : k_L(a)|^{-e}.
\]

**Proposition 3.5.** Assumptions and notation as above. Then $\bar{\mu}_V = \mu_V$.

**Proof.** We have
\[
|\{ \bar{\sigma} \in \text{Gal}(L/k(a))e \mid \text{Fix}(\bar{\sigma}) \cap k_L = k \}| = \phi(k_L)[L : k_L(a)]^{-e}.
\]
Note that the fields $K_i$ are linearly disjoint from $k^s(a)$ over $k(a)$. Hence, if $K'$ is a regular extension of $k$ containing $k(a)$ and contained in $L$, then $K' \simeq_{k(a)} K_i$ if and only if $k_L K' \simeq_{k_L(a)} k_L K_i$. It follows that if $\bar{\tau} \in \text{Gal}(k^s(a))e$, then
\[
\text{Fix}(\bar{\sigma}_0 \cdot \bar{\tau}) \cap L \simeq_{k(a)} K_i \iff \text{Fix}(\bar{\tau}) \cap L \simeq_{k_L(a)} k_L K_i,
\]
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We then have
\[ \{(\bar{\sigma} \in \text{Gal}(L/k(a))^e \mid \bar{\sigma}|_{k_L} = \bar{\sigma}_0|_{k_L}, \text{Fix}(\bar{\sigma}) \cong k(a) K_i)\} = [L : k_L(a)^e \bar{\mu}(\theta'_i)]. \]

Hence, \( \mu(\theta_i) = \phi(k_L)\bar{\mu}(\theta'_i)[L : k_L(a)]^e \), which gives \( \mu_V(\theta_i) = \bar{\mu}_V(\theta_i) \).

### 3.6 Change of measure under definable maps

**Lemma 3.7.** If \( e > 1 \), then the measure \( \mu_V(V \in \text{Var}_k) \) defined above is not preserved under definable bijection.

**Proof.** Let \( a \) be a generic of \( V \). If \( L \) is a finite Galois extension of \( k(a) \), say with \( \text{Gal}(L/k(a)) = \text{Gal}(L/k(a)) \) \( k_L/k \), and \( \bar{\sigma} \in \text{Gal}(L/k(a))^e \) lifts a set of generators of \( \text{Gal}(k_L/k) \), we let \( K := \text{Fix}(\bar{\sigma}) \). Then, in any perfect \( e \)-free PAC field \( K \) containing \( k(a) \) and intersecting \( L \) in \( K \), we will have that \( \text{dcl}(ka) \cap L = \text{Fix}(N) \), where \( N \) is the normaliser in \( \text{Gal}(L/k(a)) \) of \( H := \langle \bar{\sigma} \rangle \). Indeed, it is the subfield \( E \) of \( K \) fixed by the elements of \( \text{Aut}(K/k(a)) \), and if \( \rho \in \text{Gal}(L/k(a)) \) restricts to an automorphism of \( K \), this means that it normalizes \( H \), i.e., belongs to \( N \).

Assume that \( E \neq k(a) \), let \( b \in E \) be such that \( E = k(a,b) \), and let \( \pi : W \to V \) be the natural projection, where \( W \) is the algebraic locus of \( (a,b) \) over \( k \). Then \( \pi \) defines a bijection between the generics \( a' \) of \( V \) which satisfy \( \theta_K \) and the generics \( (a',b') \) of \( W \) which satisfy \( \theta'_K \) (where \( \theta'_K \) is the formula expressing that the relative algebraic closure of \( k(a,b) \) in \( L \) is isomorphic to \( K \)). By compactness, it defines a bijection between some definable subset \( S \) of \( V(k) \) and some definable subset \( S' \) of \( W(k) \). Let us now count.

By definition (using the second way of counting), \( \mu_V(S) = \bar{\mu}(\theta_K) \), computed over \( k^e(a) \), and \( \mu_W(S') = \bar{\mu}_1(\theta'_K) \), where \( \bar{\mu} \) and \( \bar{\mu}_1 \) are the Haar measures on \( \text{Gal}(k^e(a))^e \) and \( \text{Gal}(k^e(a),b)^e \) respectively, and \( \theta'_K \) is the analogue of \( \theta'_K \) over \( W \). We then have
\[ \mu_V(S) = \phi(k_L k_L(a))\text{[N : H][L : k_L(a)]}^{-e} \]
Proof there is a definable bijection $f : S \to k$ and $k$ an\textit{ }.

The result follows from $(\ast)$. Indeed, the number of field extensions within $L$ which are $k_L(a)$-isomorphic to $K$ is $[N : H]$, and $K$ is Galois over $k(a, b)$. From $[k_L(a, b) : k_L(a)] = [N : H]$ we obtain the result, i.e.

$(\ast) \quad \mu_V(S)[N : H]^{e-1} = \mu_W(S').$

It therefore remains to find such an $(L, K, b)$. Let $L$ be a Galois extension of $k(a)$, which is regular over $k$, and with Galois group $S_n$, with $n \geq 5$. (Such extensions exist, see e.g. [\textit{R}, 16.2.5(a) and 16.2.6]). The only non-trivial normal subgroup of $S_n$ is $A_n$, so take any $\sigma$ which generates a proper subgroup of $S_n$ not contained in $A_n$ (e.g., all $\sigma_i = (1, 2)$). Then the normalizer of $\langle \sigma \rangle$ is a proper non-trivial subgroup of $\text{Gal}(L/k(a)) = S_n$, and because $e > 1$, $(\ast)$ gives the desired inequality.

\textbf{Corollary 3.8.} Assume that $e = 1$, and let $\mu_V$ be defined as above for $V \in \text{Var}_k$. Then $\mu_V$ is preserved under definable bijections between sets of positive measure; more precisely, let $S \subset V$ and $S' \subset W$ be definable subsets of the varieties $V$ and $W$, and which are Zariski dense. If there is a definable bijection $f$ between $S$ and $S'$ then $\mu_V(S) = \mu_W(S')$.

\textit{Proof.} Let $a$ be a generic of $V$, and let $L$ be a finite Galois extension of $k(a)$, such that whenever an $e$-free PAC field $K$ contains $a$ and is regular over $k$, then whether $a$ belongs to $S$ or not is determined by $L \cap K$. We may assume that $\text{Gal}(L/k(a)) = \text{Gal}(L/k_L(a)) \rtimes \text{Gal}(k_L/k)$, and that for some subfield $K$ of $L$, $a \in S(K)$ if and only if $K \cap L \cong k(a)$. Let $e = f(a)$, and let $b \in K \cap L$ generate $\text{dcl}(ka) \cap L$ over $k(a)$. Note that because $f$ is a bijection, $\text{dcl}(kc) = \text{dcl}(ka)$. The result follows from $(\ast)$ in Lemma 3.7.\hfill$\Box$

\textbf{Example 3.9.} (Another example). We will build an example of a definable bijection $f$ between two definable subsets of some $V \in \text{Var}_k$ in a perfect $e$-free PAC field $k$ with $e > 1$ which does not preserve the measure.

\textit{Proof.} We assume $k$ contains a primitive 5th root of 1. Let $u, v \in k^\times$, a transcendental over $k$, and consider $u + av, u - av$, and let $L = k(a, \sqrt[5]{u + av}, \sqrt[5]{u - av})$. Then $L$ is Galois over $k(a^2)$, with Galois group $\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. Let $K = k(a, \sqrt[5]{u + av})$, and $K$ any $e$-free perfect PAC with $e > 1$, which is regular over $k$ and intersects $L$ in $K$. Then, inside $K$, we have $\text{dcl}(ka^2) = \text{dcl}(ka) = k(a)$. Indeed, $a$ satisfies the following formula

$$x^2 = a^2 \land \exists y \ y^5 = u + xv,$$

while $-a$ satisfies its negation.\hfill$\Box$

\textbf{Example 3.10.} (And a third example). Assume now that $k$ does not contain a primitive 5th root $\zeta$ of 1, and consider the map $f : x \mapsto x^5$. It is an injective map on any regular extension of $k$. If $L = k(a, \zeta, \sqrt[5]{a})$, then $\text{Gal}(L/k(a)) \cong \mathbb{Z}/5\mathbb{Z} \rtimes \text{Gal}(k(\zeta)/k)$. If $\sigma \in \text{Gal}(L/k(a))$ restricts to a generator of $\text{Gal}(k(\zeta)/k)$, then its order equals $[k(\zeta) : k]$. So, when $e = 1$, every $e$-free field which is a regular extension of $k$ is closed under 5th roots. Suppose $e \geq 2$, and let $S$ be the image of $f$ in an $e$-free PAC $K$ which is regular over $k$. Then $\mu_{\text{gm}}(S) = 5^{1-e}$.\hfill$\Box$
Corollary 2.4, there is a finite extension \( L \) of \( k \). If \( R \) is made in Lemma 3.7 then gives, for \( \mu \) such that the Zariski closure of the graph \( \Gamma \) of \( V \) dimension \( \dim(V) \), and these in turn induce a partition of \( S_1 \) into finitely many definable subsets. We will therefore assume that the Zariski closure of the graph \( \Gamma \) of \( f \) is irreducible. Let \( (a, b) \) be a generic of \( \Gamma \) over \( k \), in some elementary extension of \( k \), and with \( f(a) = b \). By Corollary 2.4 there is a finite extension \( L \) of \( k(a, b) \) such that the isomorphism type over \( k(a, b) \) of \( k(a, b)_{\text{alg}} \cap L \) implies \( "f(a) = b" \). Let \( c \in L \) be such that \( \text{dcl}(ka) \cap L = k(c) \). The computation made in Lemma 3.7 then gives, for \( R_1 \) a definable subset of \( S_1 \):

\[
\mu_V(R_1)[k(c) : k(a)]^{\epsilon-1} = \mu_W(f(R_1))[k(c) : k(b)]^{\epsilon-1},
\]

i.e.,

\[
\mu_V(R_1) = \left( \frac{[k(a, b) : k(b)]}{[k(a, b) : k(a)]} \right)^{\epsilon-1} \mu_W(f(R_1)).
\]
Proposition 3.14. If $V, W \in \text{Var}_k$ are birationally isomorphic by a map $f$, then $f$ preserves the measure: if $S \subseteq V$ is definable, then $\mu_V(S) = \mu_W(f(S))$.

Proof. The rational map $f : V \to W$ map defines a $k$-automorphism of $k(V) = k(W)$. As $k(V)$ is regular over $k$, $f$ extends to a $k^{\text{alg}}$-automorphism $\tilde{f}$ of $k(V)^{\text{alg}}$; then $\tilde{f}$ induces a continuous automorphism of $\text{Gal}(k(V))$, which induces the identity on $\text{Gal}(k)$. I.e., if $L$ is a finite Galois extension of $k(V)$, then $\tilde{f}(L)$ is a finite Galois extension of $k(W)$, which intersects $k^{\text{alg}}$ in $k_L = k^{\text{alg}} \cap L$, and $\text{Gal}(\tilde{f}(L)/k(W))$ and $\text{Gal}(L/k(V))$ are isomorphic, by an isomorphism which is the identity on $\text{Gal}(k_L/k)$. This implies $\mu_V(S) = \mu_W(f(S))$. \qed

The following lemma records some closure properties of the family of definably amenable groups in an arbitrary theory.

Lemma 3.15. Suppose $G$ and $H$ are definable groups.

(1) If $H \leq G$ is a definably amenable subgroup of $G$ of finite index and $H$ is definably amenable, then $G$ is definably amenable.

(2) If $\pi : G \to H$ is a definable surjective homomorphism with finite kernel and $H$ is definably amenable, then $G$ is definably amenable.

Proof. (1) Let $n = [G : H]$ and let $g_1H, \ldots, g_nH$ list the left cosets of $H$ in $G$ with $g_1 = e$. Let $\mu_H$ denote the measure on $H$. For a definable subset $X \subseteq G$, define $\mu_G$ by

$$\mu_G(X) = \frac{1}{n} \sum_{i=1}^{n} \mu_H(g_i^{-1}(X \cap g_iH)).$$

It is clear that $\mu_G(G) = 1$, $\mu_G$ is $G$-invariant, and the finite additivity of $\mu_G$ follows from that of $\mu_H$.

(2) Let $\mu_H$ denote the measure witnessing the definable amenability of $H$ and set $n = |\text{Ker}(\pi)|$. Given a definable set $X \subseteq G$, define, for each $1 \leq i \leq n$,

$$(X)_i = \{x \in X : |X \cap \pi^{-1}(\pi(x))| = i\}.$$  

The $(X)_i$'s form a definable partition of $X$. We define $\mu_G$ by setting

$$\mu_G(X) = \frac{1}{n} \sum_{i=1}^{n} i \mu_H(\pi((X)_i)).$$

This is finitely additive because $\mu_H$ is, and since $(G)_n = G$, we have $\mu_G(G) = \frac{1}{n}(n\mu_H(H)) = 1$. Invariance follows from the left transitivity of $\mu_H$ and the fact that $(gX)_i = g(X)_i$ for all $1 \leq i \leq n$. \qed

Theorem 3.16. Let $e \geq 1$, let $k$ be a perfect $e$-free PAC field, and let $G$ be a group definable in $k$. Then $G$ is definably amenable.

Proof. By Theorem C in [9], there is a definable subgroup $G_0$ of finite index in $G$, and a definable homomorphism $\pi : G_0 \to H(k)$, where $H$ is a (connected) algebraic group, with Ker $(\pi)$ finite, and $\pi(G_0)$ Zariski dense in $H$. By Remark 3.13, the measure $\mu_H$ is stable under translation, so $H(k)$ is definably amenable (via $\mu_H$). As $\pi(G)$ is definable and has finite index in $H(k)$, so is $\pi(G)$. The conclusion then follows by Lemma 3.15. \qed

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4 Extension of the measure to other fields

4.1 $\omega$-free perfect PAC fields

4.1. Definition and useful facts. Recall that a field is $\omega$-free if it has an elementary substructure with Galois group isomorphic to $\hat{F}_\omega$, the free profinite group on countably many generators. Equivalently, if every finite embedding problem of its absolute Galois group has a solution, see §3 in [11], or Chapters 25, 27 in [6].

The theory of perfect $\omega$-free PAC fields can be viewed as the limit of the theories of perfect $e$-free PAC fields, see also Lemma 4.2 below. Fact 2.3 and its Corollary [2.4] both generalise to $\omega$-free PAC fields, see §4 in [11].

Lemma 4.2. Suppose $K$ is an $\omega$-free perfect PAC field. Then there is a collection of PAC fields $(K_e)_{e \in \mathbb{N}}$ and a non-principal ultrafilter $U$ on $\mathbb{N}$ such that each $K_e$ is a PAC field with free absolute Galois group of rank $e$ and $K \equiv \prod_{e \in \mathbb{N}} K_e/U$. If $K$ is countable, then $K$ embeds elementarily in $\prod_{e \in \mathbb{N}} K_e/U$.

Proof. We know that $Th(K)$ is axiomatised within the class of perfect $\omega$-free PAC fields by a description of the isomorphism type of its absolute numbers (see Thm 4.2, [11]). If the characteristic is positive, then the absolute Galois group of the field of absolute numbers of $K$ is procyclic, and therefore there is an $e$-free perfect PAC field $K_e$ with the same absolute numbers as $K$ for every $e \geq 1$.

If the characteristic is 0, however, it may be that the absolute Galois group of the field of absolute numbers of $K$ is not finitely generated. Let $\sigma_j, j \in \mathbb{N}$, be a set of topological generators of $Gal(K \cap \overline{Q}_{alg})$, and for each $e \geq 1$, choose an $e$-free perfect PAC field $K_e$, satisfying $K_e \cap \overline{Q}_{alg} = \text{Fix}(\sigma_0, \ldots, \sigma_e)$.

If $K^*$ is a non-principal ultraproduct of the $K_e$, then $K^*$ is perfect PAC, with field of absolute numbers equal to $K \cap \overline{Q}_{alg}$, and absolute Galois group free on infinitely many generators ([6], Thm 25.2.3), hence is $\omega$-free.

The last assertion follows from the fact that $\prod_{e \in \mathbb{N}} K_e/U$ is $\aleph_1$-saturated when $U$ is non-principal.

4.3. In [11], Jarden introduces a measure on $G = \bigcup_{e \geq 1} Gal(K)^e$, for $K$ a field, as follows. First, one defines a topology on $G$ by stating that a subset $A$ of $G$ is open if $A_e = A \cap Gal(K)^e$ is open for all $e$. Then $G$ is Hausdorff, locally compact, and totally disconnected. A closed subset $C$ is compact if and only if it is bounded, i.e., $C_e = \emptyset$ for $e \gg 0$, and $Gal(K)$ acts continuously on $G$ from the left and the right. A subset $A$ is measurable if $A_e$ is measurable for all $e$ (for the Haar measure $\mu_e$ on $Gal(K)^e$). If $A$ is measurable, then one defines

$$\mu(A) = \sum_{e=1}^{\infty} \mu_e(A_e).$$

Then $\mu$ is a complete regular Borel measure of $G$ and is invariant under the action of $Gal(K)$. Note that many sets have infinite measure, but some do not. One result we will use is the following:
Fact 4.4. (Lemma 7.2 in [11]). Let $K$ be a countable Hilbertian field, and $\theta$ an $L(K)$-elementary statement of one variable (i.e., a Boolean combinations of $L(K)$-sentences of the form $[\exists x f(x) = 0]$, where $f \in K[X]$).

(a) There are integers $n, r > 0$ and $1 \leq n_1, \ldots, n_r \leq n$, $\varepsilon_1, \ldots, \varepsilon_r \in \{\pm 1\}$ such that

$$\mu_e(\theta) = \sum_{i=1}^{r} \varepsilon_i \left(\frac{n_i}{n}\right)^e$$

for every $e > 0$.

(b) If $K \not\models \theta$, then $n_i < n$ for $i = 1, \ldots, r$. Hence $\lim_{e \to +\infty} \mu_e(\theta) = 0$, and $\mu(\theta) \in \mathbb{Q}$.

(c) If $K \models \theta$, then $\lim_{e \to +\infty} \mu_e(\theta) = 1$, and $\mu(\theta) = \infty$.

Much of what follows can be found in Jarden’s paper [11] or in [6], at least implicitly, but we chose to give proofs.

Lemma 4.5. (See also §20.7 in [6]) Suppose $k_0$ is a perfect field and $V \in \text{Var}_{k_0}$. Let $a$ be a generic of $V$ over $k_0$.

(1) Let $L$ be a finite Galois extension of $k_0(a)$. Let $K$ be a subfield of $L$ containing $k_0(a)$, which is regular over $k_0$, and consider the formula $\theta_K$. There are integers $n, r > 0$ and $1 \leq n_1, \ldots, n_r \leq n$, $\varepsilon_1, \ldots, \varepsilon_r \in \{\pm 1\}$ such that if $k$ is an $e$-free perfect PAC which is regular over $k_0$, then

$$\mu_V(\theta_K) = \sum_{i=1}^{r} \varepsilon_i \left(\frac{n_i}{n}\right)^e$$

Here $V$ is identified with $V_{k_0}$, and $\mu_V$ is the measure defined in Paragraph 3.2.

(2) Let $S \subset V$ be definable with parameters in $k_0$ by a formula $\varphi(x)$. There are integers $n, r, e_1 > 0$ and $1 \leq n_1, \ldots, n_r \leq n$, $\varepsilon_1, \ldots, \varepsilon_r \in \{\pm 1\}$ such that whenever $k$ is an $e$-free perfect PAC field which is regular over $k_0$ and $e \geq e_1$, then

$$\mu_V(\varphi) = \sum_{i=1}^{r} \varepsilon_i \left(\frac{n_i}{n}\right)^e$$

If $\varphi$ is a Boolean combination of test formulas (cf[2,5]), then we may take $e_1 = 1$.

Proof. Let us first show how (2) follows from (1). The last assertion is clear, since such a formula can be written as a disjunction of mutually incompatible test formulas. (Note however that the statement may be vacuous if $\text{Gal}(k_0)$ is not $e$-generated.) The general case then follows: modulo the theory of perfect $\omega$-free PAC fields which are regular extensions of $k_0$, we know that the formula $\varphi$ is equivalent to a disjunction of mutually incompatible test-formulas. By ultraproduct, the same holds modulo the theory of perfect $e$-free PAC fields which are regular extensions of $k_0$ for $e$ sufficiently large. We take $e_1$ be such that the equivalence holds for all $e \geq e_1$.
(1) Let $K_1, \ldots, K_s$ enumerate all regular extensions of $k_0$ which are between $k_0(a)$ and $L$. Let $k$ be an $e$-free perfect PAC field regular over $k_0$ and linearly disjoint from $k_0(a)$ over $k_0$. As we saw in Proposition 3.5 if $K = K_1$,

$$\mu_V(\theta_K) = \bar{\mu}_e(\theta_K) = \mu'_e(\theta_K),$$

where $\bar{\mu}_e$ is the Jarden-Kiehne measure on $\text{Gal}(k^s(a))^e$ and $\mu'_e$ the Jarden-Kiehne measure on $\text{Gal}(k_0^s(a))^e$. The second equality is because $k$ and $k_0(a)$ are linearly disjoint over $k_0$, so that $k^s$ and $k_0^sL$ are linearly disjoint (because free) over $k_0^s$, and therefore $\text{Gal}(k^sL/k^s(a)) \cong \text{Gal}(k_0^sL/k_0^s(a))$. The result then follows by Lemma 4.4.

**4.6. A first definition of the measure.** Let $k$ be a perfect countable $\omega$-free PAC field, $V \in \text{Var}_k$, and consider, for each $e \geq 1$, the measure $\mu_{e,V}$ defined above. If $S$ is a definable subset of $V$, defined by the formula $\varphi(x)$, we then set $\mu'_V(\varphi) = \sum_e \mu_{e,V}(\varphi)$.

As in Fact 4.5 one of the consequences of Lemma 4.5(2) is that if $k_0(a) \not\models \theta$, then $\mu'(\theta) \in \mathbb{Q}$, as we will see below. The proof goes as follows: if $\theta$ is determined by the finite Galois extension $L$ of $k_0(a)$, where $a$ is a generic of $V$ and $k_0$ is the relative algebraic closure in $k$ of the field of definition of $V$, then $\mu_{e,V}(\theta) = \bar{\mu}_e(\theta)$, by 3.5. Let $n = [k_0^{alg} L : k_0^{alg}(a)]$, $n_i, \varepsilon_i, 1 \leq i \leq r$ and $e_1$ be given by Lemma 4.5. Then

$$\mu'_V(\theta) = \sum_{e=1}^{e_1-1} \bar{\mu}_e(\theta) + \sum_{i=1}^r \sum_{j \geq e_1} \varepsilon_i \left(\frac{n_j}{n}\right)^e.$$ 

We may assume that the sum is reduced, i.e., that for no $i \neq j$ we have $n_i = n_j$ and $\varepsilon_i \varepsilon_j = -1$. Note that $\sum_{e=1}^{e_1-1} \bar{\mu}_e(\theta) \in \mathbb{Q}$, and that if $n_i < n$, then $\sum_{j \geq e_1} \varepsilon_i \left(\frac{n_i}{n}\right)^e \in \mathbb{Q}$. However, any $i$ with $n_i = n$ will contribute to $+\infty$ (The measure being positive, and the sum reduced, if $n_i = n$, then $\varepsilon_i = 1$).

If $k_0(a) \models \theta(a)$, then all $\bar{\mu}_e(\theta)$ are equal to 1. If not, then all $n_i$ are $< n$, so that we get

$$\mu'_V(\theta) = \left(\sum_{e=1}^{e_1-1} \bar{\mu}_e(\theta)\right) + \sum_{i=1}^r \varepsilon_i \left(\frac{n_i}{n}\right)^{\varepsilon_i} \left(\frac{n_i}{n-n_i}\right).$$

**Corollary 4.7.** Let $k$ be a perfect $\omega$-free PAC field, $V \in \text{Var}_k$, and $\mu'_V$ the measure defined above. Then $\mu'_V(V) = \infty$. Let $S \subset V$ be definable by a formula $\varphi(x)$ over $k$. If $k(a) \not\models \varphi(a)$, then $\mu'_V(\varphi) \in \mathbb{Q}$.

**Proof.** The result follows by Lemma 4.5.

**Example 4.8.** Consider $V = \mathbb{G}_m$, and $S$ the set of squares. Then $\mu_{e,V}(S) = 2^{-e}$, so that $\mu_V(S) = 1/2$. We do have that $k(a) \not\models \exists y \ y^2 = a$; and therefore $\mu_V(V \setminus S) = \infty$, one computes that it equals $\sum 1^e - 2^{-e} = \infty$.

**4.9. Another definition of a measure.** Let $k$ be a perfect countable $\omega$-free PAC field, $V \in \text{Var}_k$, let $K_e$ be a family of perfect $e$-free PAC fields and $\mathcal{U}$ an ultrafilter such that $k \prec \prod_e K_e/\mathcal{U}$ (cf Lemma 4.2), and let $\mu_V$ be defined as the limit of the measures $\mu_{V,e}$ along the ultrafilter $\mathcal{U}$.
Lemma 4.10. Then the limit measure $\mu_V$ only takes the values 0 and 1, and does not depend on $U$.

Proof. Indeed, we know that $\mu_{V,e}(\theta) = \sum_{i=1}^{r} \varepsilon_i \left( \frac{n_i}{[L:k_L(a)]} \right)^e$ for some integers $r, n_1, \ldots, n_r, \varepsilon_1, \ldots, \varepsilon_r$ and $e \gg 0$, and that $\mu_{V,e}(\theta) \in [0,1]$. Hence,

$$\mu_V(\theta) = \sum_{i=1}^{r} \lim_{e \to +\infty} \varepsilon_i \left( \frac{n_i}{[L:k_L(a)]} \right)^e.$$ 

Note that if $n_i < [L:k_L(a)]$, then it will contribute 0 to the sum, and if $n_i = n$, it will contribute 1. Hence $\mu_V(\theta) \in [0,1] \cap \mathbb{Z} = \{0,1\}$. \hfill \qed

Remark 4.11. This means that there is a unique generic type $p_V$ on the variety $V$: If $a$ realises $p_V$ over $k$, then $a$ is a generic of $V$ over $k$, and its relative algebraic closure in a model is $k(a)$.

Proposition 4.12. Let $k$ be perfect $\omega$-free PAC, $H$ a connected algebraic group defined over $k$, and $G \leq H(k)$ a definable subgroup. Then either $G = H(k)$, or $\mu_H(G) = 0$, and $[H(k) : G] = \infty$.

Proof. If $G \leq H(k)$ is proper, then $\mu_H(G) = [H(k) : G]^{-1} < 1$, so must equal 0.

Proposition 4.13. Let $G$ be a group definable in a $\omega$-free perfect PAC field $k$. Then $G$ is definably amenable.

Proof. By Lemma 4.12 $G$ embeds elementarily in an ultraproduct $\prod_{e \in \mathbb{N}} G_e/\mathcal{U}$, where $G_e$ is definable in the $e$-free perfect PAC field $K_e$. Each $G_e$ is definably amenable, hence so is $G$. \hfill \qed

4.14. Some questions.

- What are the possible sets of values of $\mu_G$?
- Can this set be infinite?
- Can it contain irrational numbers?
- Does it depend on the ultrafilter $\mathcal{U}$?

4.2 Perfect PAC fields with free pro-$p$ absolute Galois group

4.15. Measure on pro-$p$-$e$-free PAC fields. We consider the theory of perfect PAC fields, with absolute Galois group free pro-$p$ on $e$ generators —i.e., the absolute Galois group is the pro-$p$-completion of $F_e$. Let $k$ be a perfect field with absolute Galois group free pro-$p$ on $e$ generators, let $V \in \text{Var}_k$, and $a$ a generic of $V$ over $k$. Choose a $p$-Sylow subgroup $P$ of $\text{Gal}(k(a))$, and let $F = \text{Fix}(P)$.

Let $L$ be a finite Galois extension of $k(a)$, and $K$ a subfield of $L$ containing $k(a)$, regular over $k$, and with $\text{Gal}(L/K)$ an $e$-generated $p$-group. We wish to define $\mu(\theta_K)$. Since all $p$-Sylow
of Gal(k(a)) are conjugate by an element of Gal(k(a)), and Gal(k^aL) is a normal subgroup of Gal(k(a)). P projects onto Gal(k^aL/K). We now set

$$\mu_V(\theta_K) = \frac{|\{\bar{\sigma} \in \text{Gal}(FL/F)^e \mid \text{Fix}(\bar{\sigma}) \simeq F K \}|}{|\{\bar{\sigma} \in \text{Gal}(FL/F)^e \mid \text{Fix}(\bar{\sigma}) \cap k^a = k \}|}.$$

Note that the choice of P only depends on the field k(a) (i.e., on the variety V), and not on the fields L, K. We need to show that the definition of $\mu_V$ does not depend on L, i.e., that if we compute it in some FM, with M Galois over k(a) and containing L, we will get the same number. But this follows from the following remarks:

Let $\bar{\sigma} \in \text{Gal}(FL/F)^e$ project onto a set of generators of Gal(kL/k); then $\bar{\sigma}|_{k_L}$ has $\phi(k_M/k_L, k)$ many extensions to $k_M F$ which project onto a set of generators of Gal(k_M/k), and therefore $\bar{\sigma}$ has $\phi(k_M/k_L, k)$ many extensions to $k_M FL$ with the same property; since Gal(FM/FL) is a $p$-group, $\bar{\sigma}$ has exactly $\phi(k_M/k_L, k)[FM : k_MFL]^e$ many extensions to FM with fixed field a regular extension of k. The result follows.

**Corollary 4.16.** Suppose k is a perfect PAC field whose absolute Galois group is free pro-p. Then every group definable in k is definably amenable.

**Proof.** In the case that the absolute Galois group of k is free pro-p of finite rank e, then, as k is a bounded perfect PAC field, Theorem C of [9] implies that G is virtually isogenous to the k points of an algebraic group H and we may argue as in Theorem 3.16, using that the measure $\mu_H$ witnesses the definable amenability of $H(k)$. If the absolute Galois group of k is free pro-p of infinite rank, then k is elementary equivalent to a non-principal ultraproduct of fields $F_e$ with $F_e$ perfect PAC and pro-p-e. We then obtain a translation invariant ultralimit measure as in Proposition 4.13. \qed

**References**

[1] J. Ax, The elementary theory of finite fields, Annals of Math. 88 (1968), 239 – 271.

[2] Z. Chatzidakis, L. van den Dries, A. Macintyre, Definable sets over finite fields, J. reine u. ang. Math. 427 (1992), 107 – 135.

[3] G. Cherlin, L. van den Dries, A. Macintyre, Decidability and Undecidability Theorems for PAC-Fields, Bull.AMS 4 (1981), 101 – 104.

[4] A. Chernikov, E. Hrushovski, A. Kruckman, K. Krupinski, S. Moconja, A. Pillay, N. Ramsey, Invariant measures in simple and in small theories, arXiv 2105.07281

[5] Ju. L. Ershov, Regularly closed fields, Soviet Math. Doklady 21 (1980), 510 – 512.

[6] M. Fried, M. Jarden, *Field arithmetic*, Ergebnisse 11, Springer Berlin-Heidelberg 2008.
[7] Immanuel Halupczok, A measure for perfect PAC fields with pro-cyclic Galois group, J. of Algebra 310 (2007), 371 – 395.

[8] E. Hrushovski, Pseudo-finite fields and related structures, in: Model Theory and Applications, Bélair et al. ed., Quaderni di Matematica Vol. 11, Aracne, Rome 2005, 151 – 212.

[9] E. Hrushovski, A. Pillay, Groups definable in local fields and pseudo-finite fields, Israel J. of Math. 85 (1994), 203 – 262.

[10] M. Jarden, U. Kiehne, The elementary theory of algebraic fields of finite corank, Inv. Math. 30 (1975), 275 – 294.

[11] M. Jarden, The elementary theory of $\omega$-free Ax fields, Inv. Math. 38 (1976), 187 – 206.

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