Hamilton-Jacobi formulation for singular systems with second order Lagrangians

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Abstract

Recently the Hamilton-Jacobi formulation for first order constrained systems has been developed. In such formalism the equations of motion are written as total differential equations in many variables. We generalize the Hamilton-Jacobi formulation for singular systems with second order Lagrangians and apply this new formulation to Podolsky electrodynamics, comparing with the results obtained through Dirac’s method.

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1 Introduction

Systems with higher order Lagrangians have been studied with increasing interest because they appear in many relevant physical problems. As examples we have the consistent regularization of ultraviolet divergences in gauge-invariant supersymmetric theories [1] or effective Lagrangians in gauge theories [2]. Besides this, the fact that gauge theories have singular Lagrangians is in itself a motivation to the study of the formalism for second order singular Lagrangians.

The Lagrangian formulation for constrained systems can be found in references [3] and [4] while the Hamiltonian formulation of singular systems is usually made through a formalism developed by Dirac [5, 6, 7]. In this formalism the constraints caused by the Hessian matrix singularity are added to the canonical Hamiltonian and then the consistency conditions are worked out, being possible to eliminate some degrees of freedom of the system. Dirac also showed that the gauge freedom is caused by the presence of first class constraints.

The study of new formalisms for singular systems may provide new tools to investigate these systems. In classical dynamics, different formalisms (Lagrangian, Hamiltonian, Hamilton-Jacobi) provide different approaches to the problems, each formalism having advantages and disadvantages in the study of some features of the systems and being equivalent among themselves. In the same way, different formalisms provide different views of the features of singular systems, which justify the interest in their study.

Here we generalize the Hamilton-Jacobi formalism that was recently developed [8, 9] to include singular second order Lagrangians. We start in Sect. 2 with an overview of the features of singular systems with second order Lagrangians and of
the Dirac’s Hamiltonian formalism for them. In Sect. 3 we develop the Hamilton-Jacobi formulation for a general second order system and apply this formalism to the case of a singular second order system in Sect. 4. An example is solved using both Dirac’s and Hamilton-Jacobi formalism in Sect. 5, while Sect. 6 is devoted to the conclusions.

2 Singular systems with second order Lagrangians

The treatment for theories with higher order derivatives has been first developed by Ostrogradski \[10\] and allows to write the Euler-Lagrange equations, introduce conjugated momenta and develop a Hamiltonian formalism for such systems. Here we center our attention on the case of a system described by a Lagrangian containing time derivatives of the coordinates up to second order (a second order Lagrangian).

In such a theory the configuration space is described by the \( n \) generalized coordinates \( q_i \) and its first and second derivatives with respect to the time parameter \( t \) (we consider a discrete system to simplify the calculations; the generalization to continuous systems is straightforward and will be done in Sect. 5).

The Euler-Lagrange equations, which are obtained from the action integral

\[
S = \int L(q, \dot{q}, \ddot{q}, t) \, dt
\]

(1)

using the Hamilton’s principle, are given by:

\[
\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial q_i} \right) = 0
\]

(2)

We construct the phase space by introducing the generalized momenta

\[
p_i = \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial q_i} \right)
\]

(3)
\( \pi_i = \frac{\partial L}{\partial \dot{q}_i} \) 

(conjugated respectively to \( q_i \) and \( \dot{q}_i \)) and writing the accelerations \( \ddot{q}_i \) as functions of the coordinates \( q \), velocities \( \dot{q} \) and of the momenta \( p \) and \( \pi \) (\( \ddot{q}_i = f_i (q, \dot{q}_i, p_i, \pi_i) \)). The phase space will then be spanned by the canonical variables \((q_i, p_i), (\ddot{q}_i, \pi_i)\)

where \( \ddot{q}_i = \dot{q}_i \).

By introducing the canonical Hamiltonian defined as

\[
H_C = p_i \ddot{q}_i + \pi_i \ddot{q}_i \bigg|_{\ddot{q}_i = f_i} - L \bigg|_{\ddot{q}_i = f_i}
\]

we can write the equations of motion of any function \( g \) of the canonical variables as:

\[
\dot{g} = \{g, H_C\}
\]

But this procedure is only possible if the determinant of the Hessian matrix

\[
H_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}
\]

does not vanish, otherwise it will not be possible to express all the accelerations \( \ddot{q}_i \) as functions of the canonical variables and there will be relations as

\[
\Phi_\alpha (q_i, p_i; \ddot{q}_i, \pi_i) = 0; \ \alpha = 1, ..., m < 2(n - 1)
\]

connecting the momenta variables. As a consequence we will not be able to treat the canonical variables as an independent set and we have to retreat to formalisms specially developed to deal with the dependence among the canonical variables, i.e. a formalism for constrained systems.

The usual treatment of singular systems was developed by Dirac [5, 6, 7] who, in order to deal with this problem, introduced a generalized Hamiltonian formalism (details can be found in references [3], [4] and [11]). Such formalism can also be applied to the case of second order Lagrangians as can be seen in
Dirac’s formalism consist in considering constraints given by equation (8) as *weak* equations, called primary constraints, and represented as:

$$\Phi_\alpha (q_i, p_i; \bar{q}_i, \pi_i) \approx 0$$  \hspace{1cm} (9)

By weak equations we mean those that can’t be used until all Poisson brackets have been calculated.

We may add to the canonical Hamiltonian $H_C$ any linear combination of the primary constraints and define a new Hamiltonian, called total Hamiltonian, given by

$$H_T = H_C + u_\alpha \Phi_\alpha,$$  \hspace{1cm} (10)

where $u_\alpha$ are arbitrary coefficients. Physically $H_C$ and $H_T$ are equivalent and we cannot distinguish between them. The equation of motion for any function $f (q_i, \bar{q}_i, p_i, \pi_i)$ is given in terms of $H_T$ as:

$$\dot{f} \approx \{ f, H_T \} = \{ f, H_C \} + u_\alpha \{ f, \Phi_\alpha \}$$  \hspace{1cm} (11)

The constraints will produce consistency conditions because they must be valid at any time and consequently their time derivative must be weakly zero. The consistency conditions are given by:

$$\dot{\Phi}_\alpha \approx \{ \Phi_\beta, H_T \} = \{ \Phi_\beta, H_C \} + u_\alpha \{ \Phi_\beta, \Phi_\alpha \} \approx 0; \ \alpha, \beta = 1, ..., m$$  \hspace{1cm} (12)

These conditions may be either identically satisfied (when we use the primary constraints), determine some of the arbitrary coefficients $u_\alpha$ or generate new constraints that will be called secondary constraints. The constraints that have null Poisson brackets with all other constraints are called first class constraints otherwise they are called second class ones. This classification is completely independent of the division in primary and secondary constraints. The extended
Hamiltonian is defined as

\[ H_E = H_C + V_\lambda \Psi_\lambda \]  

(13)

were the \( \Psi_\lambda \) include all first class constraints. \( V_\lambda \) are arbitrary coefficients and we use (13) instead of the total Hamiltonian (10) in the equations of motion.

3 Hamilton-Jacobi formalism for second order Lagrangians

Recently a new formalism for singular first order systems was developed by Güler [8, 9] who obtained a set of Hamilton-Jacobi partial differential equations for such systems using Carathéodory’s equivalent Lagrangians method and wrote the equations of motion as total differential equations.

In this section we will use Carathéodory’s method to develop the Hamilton-Jacobi formalism to a general second order Lagrangian. This formalism can be applied to any second order Lagrangian and is not limited to singular ones. The singular case will be considered in the next section.

Carathéodory’s equivalent Lagrangians method to second order Lagrangians says that, given a Lagrangian \( L(q_i, \dot{q}_i, \ddot{q}_i, t) \), we can obtain a completely equivalent one by:

\[ L' = L(q_i, \dot{q}_i, \ddot{q}_i, t) - \frac{dS(q_i, \dot{q}_i, t)}{dt} \]  

(14)

These Lagrangians are equivalent because the action integral given by them have simultaneous extremes. So we can choose the function \( S(q_i, \dot{q}_i, t) \) in such a way that \( L' \) becomes an extreme and then we reduce the variational problem of finding extreme for the Lagrangian \( L \) to a problem of differential calculus. To do this we must find a set of functions \( \varphi_i(q_i, \dot{q}_i, t) \), \( \beta_i(q_i, t) \) and \( S(q_i, \dot{q}_i, t) \) such that

\[ L'(q_i, \beta_i, \varphi_i, t) = 0 \]  

(15)
and for all neighborhood of $\dot{q}_i = \beta_i(q_i, t)$ and $\ddot{q}_i = \varphi_i(q_i, \dot{q}_i, t)$:

(16) \[ L' (q_i, \dot{q}_i, \ddot{q}_i, t) > 0 \]

With these conditions satisfied the Lagrangian $L'$ will have a minimum in $\dot{q}_i = \beta_i(q_i, t)$ and $\ddot{q}_i = \varphi_i(q_i, \dot{q}_i, t)$ and consequently the action integral will have a minimum. So, the solutions of the differential equations will correspond to extremes of the action integral.

From the definition of $L'$ we have:

(17) \[ L' = L (q_i, \dot{q}_i, \ddot{q}_i, t) - \frac{\partial S (q_i, \dot{q}_i, t)}{\partial t} - \frac{\partial S (q_i, \dot{q}_i, t)}{\partial q_i} \frac{dq_i}{dt} - \frac{\partial S (q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \]

Using condition (15) we have:

(18) \[ \left[ L (q_i, \dot{q}_i, \ddot{q}_i, t) - \frac{\partial S (q_i, \dot{q}_i, t)}{\partial t} - \frac{\partial S (q_i, \dot{q}_i, t)}{\partial q_i} \dot{q}_i - \frac{\partial S (q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \ddot{q}_i \right] \bigg|_{q_i = \beta_i} = 0 \]

(19) \[ \frac{\partial S}{\partial t} \bigg|_{q_i = \beta_i} = \left[ L (q_i, \dot{q}_i, \ddot{q}_i, t) - \frac{\partial S (q_i, \dot{q}_i, t)}{\partial q_i} \dot{q}_i - \frac{\partial S (q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \ddot{q}_i \right] \bigg|_{q_i = \beta_i} = 0 \]

Since $\dot{q}_i = \beta_i$ and $\ddot{q}_i = \varphi_i$ are minimum points of $L'$ we must have

(20) \[ \frac{\partial L'}{\partial \ddot{q}_i} \bigg|_{q_i = \beta_i} = 0 \Rightarrow \left[ \frac{\partial L}{\partial \ddot{q}_i} - \frac{\partial S}{\partial \ddot{q}_i} \left( \frac{dS}{dt} \right) \right] \bigg|_{q_i = \beta_i} = 0, \]

(21) \[ \left[ \frac{\partial L}{\partial \ddot{q}_i} - \frac{\partial S}{\partial \ddot{q}_i} \right] \bigg|_{q_i = \beta_i} = 0, \]

or

(22) \[ \frac{\partial S}{\partial q_i} \bigg|_{q_i = \beta_i} = \frac{\partial L}{\partial q_i} \bigg|_{q_i = \beta_i} \]

Analogously we must have

(23) \[ \frac{\partial L'}{\partial \dot{q}_i} \bigg|_{q_i = \beta_i} = 0 \Rightarrow \left[ \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial S}{\partial \dot{q}_i} \left( \frac{dS}{dt} \right) \right] \bigg|_{q_i = \beta_i} = 0, \]
\[
\begin{align*}
(24) & \quad \left[ \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial}{\partial t} \frac{\partial S}{\partial q_i} - \frac{\partial S}{\partial q_i} \frac{\partial}{\partial \dot{q}_j} \dot{q}_j - \frac{\partial^2 S}{\partial q_i \partial \dot{q}_j} \dot{q}_j \right]_{q_i = \beta_i, \dot{q}_i = \varphi_i} = 0, \\
(25) & \quad \left[ \frac{\partial L}{\partial q_i} - \frac{\partial S}{\partial q_i} - \frac{d}{dt} \frac{\partial S}{\partial \dot{q}_i} \right]_{q_i = \beta_i, \dot{q}_i = \varphi_i} = 0,
\end{align*}
\]

or
\[
(26) \quad \frac{\partial S}{\partial q_i} \bigg|_{q_i = \beta_i} = \left[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial S}{\partial \dot{q}_i} \right]_{q_i = \beta_i, \dot{q}_i = \varphi_i}.
\]

From these results, using the definitions for the conjugated momenta given by equations (3) and (4) and writing \(\dot{q}_i = \tilde{q}_i\), we have from equation (19) that, to obtain an extreme of the action, we must get a function \(S(q_i, \dot{q}_i, t)\) such that:
\[
(27) \quad \frac{\partial S}{\partial t} = -H_0
\]

\[
(28) \quad H_0 = p_i \tilde{q}_i + \pi_i \tilde{q}_i - L
\]

\[
(29) \quad p_i = \frac{\partial S}{\partial q_i}; \quad \pi_i = \frac{\partial S}{\partial \dot{q}_i}
\]

These are the fundamental equations of the equivalent Lagrangian method, equation (27) being called the Hamilton-Jacobi partial differential equation, or simply the HJPDE.

### 4 Formulation for singular second order Lagrangians

We consider now the application of the formalism developed in the previous section to a system with a singular second order Lagrangian. When the Hessian matrix has a rank \(n - R, R < n\), the momenta variables will not be independent variables among themselves. In this case we can choose the order of accelerations \(\tilde{q}_i = \bar{q}_i\) in such a way that the minor of rank \(n - R\) in the bottom right corner has nonvanishing determinant:
\[
(30) \quad \det \left| \frac{\partial^2 L}{\partial \bar{q}_a \partial \bar{q}_b} \right| = \det \left| \frac{\partial \pi_b}{\partial \bar{q}_a} \right| \neq 0; \quad a, b = R + 1, \ldots, n
\]
So we can solve the \( n - R \) accelerations \( \ddot{q}_a \) in terms of the coordinates \((q, \ddot{q})\), the momenta \( \pi_a \) and the unsolved accelerations \( \ddot{q}_\alpha \) (\( \alpha = 1, ..., R \)) as follow:

\[
\ddot{q}_a = f_a \left( q_i, \ddot{q}_i, \pi_b, \ddot{q}_\alpha \right)
\]

(31)

Since the momenta \( \pi \) are functions of the accelerations \( \ddot{q}_i \), we can substitute the expressions (31) and obtain:

\[
\pi_i = g_i \left( q_i, \ddot{q}_i, \ddot{q}_\alpha \right) = g_i \left( q_i, \ddot{q}_i, f_a, \ddot{q}_\alpha \right)
\]

(32)

\[
\pi_i = g_i \left( q_i, \ddot{q}_i, \pi_a, \ddot{q}_\alpha \right)
\]

(33)

Since we have \( \pi_a \equiv g_a \) the other \( n - R \) functions \( g_\alpha \) can’t contain the unsolved accelerations \( \ddot{q}_\alpha \) or we would be able to solve more of the accelerations \( \ddot{q}_i \) as functions of the other variables, which contradicts the fact that the rank of the Hessian matrix is \( n - R \). So, we can write for the momenta \( \pi_\alpha \):

\[
\pi_\alpha = -H^p_\alpha \left( q_i, \ddot{q}_i, p_a, \pi_a \right)
\]

(34)

We can obtain a similar expression for the momenta \( p_\alpha \):

\[
p_\alpha = -H^p_\alpha \left( q_i, \ddot{q}_i, p_a, \pi_a \right)
\]

(35)

Anyway, from a general constraint given by any expression like equation (8) we can always obtain expressions like equations (34) and (35) (see ref. [11]).

The Hamiltonian \( H_0 \), given by equation (28), becomes

\[
H_0 = p_a \ddot{q}_a + \ddot{q}_\alpha \left. p_\alpha \right|_{p_\beta = -H^p_\beta} + \pi_a f_a \\
+ \left. \ddot{q}_\alpha \right|_{\pi_\beta = -H^\pi_\beta} - L \left( q_i, \ddot{q}_i, \ddot{q}_\alpha, \ddot{q}_a = f_a \right)
\]

(36)

where \( \alpha, \beta = 1, ..., R; \ a = R + 1, ..., n \). On the other hand we have

\[
\frac{\partial H_0}{\partial \ddot{q}_\alpha} = \pi_a \frac{\partial f_a}{\partial \ddot{q}_\alpha} + \pi_\alpha - \frac{\partial L}{\partial \ddot{q}_\alpha} - \frac{\partial L}{\partial \ddot{q}_a} \frac{\partial f_a}{\partial \ddot{q}_\alpha} = 0
\]

(37)
so the Hamiltonian $H_0$ does not depend explicitly upon the accelerations $\ddot{q}_\alpha$.

From this point we will adopt the following notation: the coordinates $t$ and $q_\alpha$ will be called $t_0$ and $t_\alpha$, respectively, and $\ddot{q}_\alpha$ will be called $\ddot{t}_\alpha$. The momenta $p_\alpha$ and $\pi_\alpha$ will be called $P^p_\alpha$ and $P^\pi_\alpha$, respectively, and the momentum $P_0$ will be defined as:

$$P_0 = \frac{\partial S}{\partial t} \quad (38)$$

Then, to obtain an extreme of the action integral, we must find a function $S(q_i, \dot{q}_i, t)$ that satisfies the following set of HJPDE:

$$H'_0 = P_0 + H_0 \left( t_0, t_\alpha, \ddot{t}_\alpha; q_\alpha, \ddot{q}_\alpha; p_\alpha \right) = \frac{\partial S}{\partial q_\alpha} - \pi_\alpha = 0 \quad (39)$$

$$H'^p_\alpha = P^p_\alpha + H^p_\alpha \left( t_0, t_\alpha, \ddot{t}_\alpha; q_\alpha, \ddot{q}_\alpha; p_\alpha \right) = \frac{\partial S}{\partial \dot{q}_\alpha} + \pi_\alpha = 0 \quad (40)$$

$$H'^\pi_\alpha = P^\pi_\alpha + H^\pi_\alpha \left( t_0, t_\alpha, \ddot{t}_\alpha; q_\alpha, \ddot{q}_\alpha; p_\alpha \right) = \frac{\partial S}{\partial q_\alpha} - \pi_\alpha = 0 \quad (41)$$

From the definition above and equation \((36)\) we have

$$\frac{\partial H'_0}{\partial \pi_b} = -\frac{\partial L}{\partial \dot{q}_a} \dot{q}_a \frac{\partial H^\pi_\alpha}{\partial \pi_b} - \ddot{q}_\alpha \frac{\partial H^p_\alpha}{\partial \pi_b} + \pi_\alpha \frac{\partial f_a}{\partial \pi_b}$$

$$\frac{\partial H'_0}{\partial \pi_b} = \ddot{q}_b - \ddot{q}_\alpha \frac{\partial H^\pi_\alpha}{\partial \pi_b} - \ddot{q}_\alpha \frac{\partial H^p_\alpha}{\partial \pi_b}$$

and

$$\frac{\partial H'_0}{\partial p_b} = -\ddot{q}_\alpha \frac{\partial H^\pi_\alpha}{\partial p_b} - \ddot{q}_b + \ddot{q}_\alpha \frac{\partial H^p_\alpha}{\partial p_b} - \ddot{q}_\alpha \frac{\partial H^\pi_\alpha}{\partial p_b}$$

where $\alpha = 1, \ldots, R$; $a, b = R + 1, \ldots, n$.

Remembering that $\dot{q}_i = \ddot{q}_i$ and multiplying by $dt = dt_0$ we have from equations \((43)\) and \((44)\)

$$d \ddot{q}_b = \frac{\partial H'_0}{\partial \pi_b} dt_0 + \frac{\partial H'^p_\alpha}{\partial \pi_b} dt_\alpha + \ddot{q}_\alpha \frac{\partial H'^\pi_\alpha}{\partial \pi_b} d \ddot{t}_\alpha$$

$$dq_b = \frac{\partial H'_0}{\partial p_b} dt_0 + \frac{\partial H'^p_\alpha}{\partial p_b} dt_\alpha + \ddot{q}_\alpha \frac{\partial H'^\pi_\alpha}{\partial p_b} d \ddot{t}_\alpha$$

where $\alpha = 1, \ldots, R$; $a, b = R + 1, \ldots, n$. 
and also
\[ dq_α = \frac{\partial H^0_α}{\partial P^α_α} dt_0 + \frac{\partial H^p_α}{\partial P^α_α} dt_β + \frac{\partial H^p_β}{\partial P^α_α} d \bar{t}_β = \frac{\partial H^p_β}{\partial P^α_α} dt_β = δ_α δ_β dt = dt_α \]
\[ d \bar{t}_α = \frac{\partial H^0_α}{\partial P^π_π} dt_0 + \frac{\partial H^p_β}{\partial P^π_π} dt_β + \frac{\partial H^p_β}{\partial P^π_π} d \bar{t}_β = \frac{\partial H^p_β}{\partial P^π_π} dt_β = δ_α δ_β d \bar{t}_β = d \bar{t}_α \]
\[ dq_0 = dt = \frac{\partial H^0_0}{\partial P^0_0} dt_0 + \frac{\partial H^p_β}{\partial P^0_0} dt_β + \frac{\partial H^p_β}{\partial P^0_0} d \bar{t}_β = \frac{\partial H^p_β}{\partial P^0_0} dt_β = dt_0 \]
for $β = 1, ..., R$; so we can write the equations (44) and (45) as:
\[ dq_i = \frac{\partial H^0_i}{\partial q_i} dt_0 + \frac{\partial H^p_i}{\partial q_i} dt_β + \frac{\partial H^p_β}{\partial q_i} d \bar{t}_β; i = 1, ..., n \]
\[ dq_i = \frac{\partial H^0_0}{\partial p_i} dt_0 + \frac{\partial H^p_0}{\partial p_i} dt_β + \frac{\partial H^p_β}{\partial p_i} d \bar{t}_β; i = 1, ..., n \]

If we consider that we have a solution $S(q_1, q_2, t)$ of the set of HJPDE given by equations (39), (40) and (41) then, differentiating those equations with respect to $q_1$, we obtain
\[ \frac{\partial H^0_0}{\partial q_i} + \frac{\partial H^0_0}{\partial P^0_0} \partial q_i + \frac{\partial H^p_0}{\partial q_i} \partial q_i + \frac{\partial H^p_0}{\partial q_i} \partial q_i = 0 \]
\[ \frac{\partial H^0_0}{\partial P^π_π} \partial q_i + \frac{\partial H^p_0}{\partial P^π_π} \partial q_i + \frac{\partial H^p_0}{\partial P^π_π} \partial q_i + \frac{\partial H^p_0}{\partial P^π_π} \partial q_i = 0 \]
\[ \frac{\partial H^0_0}{\partial P^π_π} \partial q_i + \frac{\partial H^p_0}{\partial P^π_π} \partial q_i + \frac{\partial H^p_0}{\partial P^π_π} \partial q_i + \frac{\partial H^p_0}{\partial P^π_π} \partial q_i = 0 \]
whereas for $q_i$ we get
\[ \frac{\partial H^0_0}{\partial q_i} + \frac{\partial H^0_0}{\partial P^0_0} \partial q_i + \frac{\partial H^p_0}{\partial q_i} \partial q_i + \frac{\partial H^p_0}{\partial q_i} \partial q_i = 0 \]
\[ \frac{\partial H^0_0}{\partial P^π_π} \partial q_i + \frac{\partial H^p_0}{\partial P^π_π} \partial q_i + \frac{\partial H^p_0}{\partial P^π_π} \partial q_i + \frac{\partial H^p_0}{\partial P^π_π} \partial q_i = 0 \]
\[ \frac{\partial H^0_0}{\partial P^π_π} \partial q_i + \frac{\partial H^p_0}{\partial P^π_π} \partial q_i + \frac{\partial H^p_0}{\partial P^π_π} \partial q_i + \frac{\partial H^p_0}{\partial P^π_π} \partial q_i = 0 \]
and for $t_0$ we have:

$$\frac{\partial H_0'}{\partial t_0} + \frac{\partial H_0'}{\partial P_0} \frac{\partial^2 S}{\partial t_0^2} + \frac{\partial H_0'}{\partial p_a} \frac{\partial^2 S}{\partial q_a \partial t_0} + \frac{\partial H_0'}{\partial \pi_a} \frac{\partial^2 S}{\partial q_a \partial t_0} = 0$$

(58)

$$\frac{\partial H_0'}{\partial t_0} + \frac{\partial H_0'}{\partial P_0} \frac{\partial^2 S}{\partial t_0^2} + \frac{\partial H_0'}{\partial p_a} \frac{\partial^2 S}{\partial q_a \partial t_0} + \frac{\partial H_0'}{\partial \pi_a} \frac{\partial^2 S}{\partial q_a \partial t_0} = 0$$

(59)

$$\frac{\partial H_0'}{\partial t_0} + \frac{\partial H_0'}{\partial P_0} \frac{\partial^2 S}{\partial t_0^2} + \frac{\partial H_0'}{\partial p_a} \frac{\partial^2 S}{\partial q_a \partial t_0} + \frac{\partial H_0'}{\partial \pi_a} \frac{\partial^2 S}{\partial q_a \partial t_0} = 0$$

(60)

Making $Z = S(q_t, \dot{q}_t, t)$ and using the momenta definitions together with equations (50) and (51) we have

$$dZ = \frac{\partial S}{\partial t} dt_0 + \frac{\partial S}{\partial t} dt_0 + \frac{\partial S}{\partial q_a} dq_a + \frac{\partial S}{\partial \pi_a} d\pi_a,$$

(61)

$$dZ = -H_0 dt_0 - H_0^p dt_0 - H_0^\pi d\pi_a$$

(62)

$$dZ = \left( \left( -H_0 + p_a \frac{\partial H_0'}{\partial p_a} + \pi_a \frac{\partial H_0'}{\partial \pi_a} \right) dt_0 \right.$$

$$+ \left. \left( -H_0 + p_a \frac{\partial H_0'}{\partial p_a} + \pi_a \frac{\partial H_0'}{\partial \pi_a} \right) dt_0 \right.$$

$$+ \left. \left( -H_0 + p_a \frac{\partial H_0'}{\partial p_a} + \pi_a \frac{\partial H_0'}{\partial \pi_a} \right) d\pi_a \right).$$

and, from momenta definitions:

$$dP_0 = \frac{\partial^2 S}{\partial t^2} dt_0 + \frac{\partial^2 S}{\partial t \partial t_\alpha} dt_0 + \frac{\partial^2 S}{\partial t \partial q_a} dq_a + \frac{\partial^2 S}{\partial t \partial \pi_a} d\pi_a + \frac{\partial^2 S}{\partial t \partial q_a} d\pi_a.$$

(64)
\( dp_i = \frac{\partial^2 S}{\partial q_i \partial t} dt_0 + \frac{\partial^2 S}{\partial q_i \partial t_\alpha} dt_\alpha + \frac{\partial^2 S}{\partial q_i \partial \bar{q}_a} dq_a + \frac{\partial^2 S}{\partial q_i \partial \bar{\bar{q}}_a} d \bar{q}_a \) \( (65) \)

\( d\pi_i = \frac{\partial^2 S}{\partial \bar{q}_i \partial \bar{t}_\alpha} dt_\alpha + \frac{\partial^2 S}{\partial \bar{q}_i \partial \bar{q}_a} d \bar{q}_a + \frac{\partial^2 S}{\partial \bar{q}_i \partial \bar{\bar{q}}_a} d \bar{\bar{q}}_a \) \( (66) \)

Now, multiplying equations \( (58) \) by \( dt_0 \), contracting equations \( (59) \) and \( (60) \) with \( dt_\alpha \) and \( d \bar{t}_\alpha \) (respectively) and adding them all to equation \( (64) \) we get:

\( dP_0 + \frac{\partial H'_0}{\partial t_0} dt_0 + \frac{\partial H'^p_\alpha}{\partial t_0} dt_\alpha + \frac{\partial H'^\pi_\alpha}{\partial t_0} d \bar{t}_\alpha = \)

\( = \frac{\partial^2 S}{\partial t_0 \partial q_a} \left( dq_a - \frac{\partial H'_0}{\partial p_a} dt_0 - \frac{\partial H'^p_\alpha}{\partial p_a} dt_\alpha - \frac{\partial H'^\pi_\alpha}{\partial p_a} d \bar{t}_\alpha \right) \)

\( + \frac{\partial^2 S}{\partial t_0 \partial \bar{q}_a} \left( d \bar{q}_a - \frac{\partial H'_0}{\partial \bar{\pi}_a} dt_0 - \frac{\partial H'^p_\alpha}{\partial \bar{\pi}_a} dt_\alpha - \frac{\partial H'^\pi_\alpha}{\partial \bar{\pi}_a} d \bar{t}_\alpha \right) \) \( (67) \)

In a similar way, using the same steps with equations \( (53-57) \) and \( (65) \) we obtain:

\( dp_i + \frac{\partial H'_0}{\partial \bar{q}_i} dt_0 + \frac{\partial H'^p_\alpha}{\partial \bar{q}_i} dt_\alpha + \frac{\partial H'^\pi_\alpha}{\partial \bar{q}_i} d \bar{t}_\alpha = \)

\( = \frac{\partial^2 S}{\partial \bar{q}_i \partial q_a} \left( dq_a - \frac{\partial H'_0}{\partial p_a} dt_0 - \frac{\partial H'^p_\alpha}{\partial p_a} dt_\alpha - \frac{\partial H'^\pi_\alpha}{\partial p_a} d \bar{t}_\alpha \right) \)

\( + \frac{\partial^2 S}{\partial \bar{q}_i \partial \bar{q}_a} \left( d \bar{q}_a - \frac{\partial H'_0}{\partial \bar{\pi}_a} dt_0 - \frac{\partial H'^p_\alpha}{\partial \bar{\pi}_a} dt_\alpha - \frac{\partial H'^\pi_\alpha}{\partial \bar{\pi}_a} d \bar{t}_\alpha \right) \) \( (68) \)

And, finally, using equations \( (52-54) \) and \( (66) \) we have:

\( d\pi_i + \frac{\partial H'_0}{\partial \bar{\bar{q}}_i} d \bar{t}_\alpha + \frac{\partial H'^p_\alpha}{\partial \bar{\bar{q}}_i} d \bar{t}_\alpha + \frac{\partial H'^\pi_\alpha}{\partial \bar{\bar{q}}_i} d \bar{\bar{t}}_\alpha = \)

\( (69) \)
\[ \frac{\partial^2 S}{\partial \bar{q}_i \partial q_a} \left( dq_a - \frac{\partial H'_0}{\partial p_a} dt_0 - \frac{\partial H'_{\alpha p}}{\partial p_a} dt_\alpha - \frac{\partial H'_{\alpha \pi}}{\partial p_a} d \bar{t}_\alpha \right) \]

\[ + \frac{\partial^2 S}{\partial \bar{q}_i \partial q_a} \left( d \bar{q}_a - \frac{\partial H'_0}{\partial \pi_a} dt_0 - \frac{\partial H'_{\alpha p}}{\partial \pi_a} dt_\alpha - \frac{\partial H'_{\alpha \pi}}{\partial \pi_a} d \bar{t}_\alpha \right) \]

If the total differential equations given by (45) and (46) are valid, the equations (67), (68) and (69) become:

\[ dP_0 = \frac{\partial H'_0}{\partial t_0} dt_0 - \frac{\partial H'_{\alpha p}}{\partial t_0} dt_\alpha - \frac{\partial H'_{\alpha \pi}}{\partial t_0} d \bar{t}_\alpha \] (70)

\[ dp_i = -\frac{\partial H'_0}{\partial q_i} dt_0 - \frac{\partial H'_{\alpha p}}{\partial q_i} dt_\alpha - \frac{\partial H'_{\alpha \pi}}{\partial q_i} d \bar{t}_\alpha \] (71)

\[ d\pi_i - \frac{\partial H'_0}{\partial \bar{q}_i} dt_0 - \frac{\partial H'_{\alpha p}}{\partial \bar{q}_i} dt_\alpha - \frac{\partial H'_{\alpha \pi}}{\partial \bar{q}_i} d \bar{t}_\alpha \] (72)

These equations together with equations (49), (50), (51) and (63) are the total differential equations for the characteristics curves and, if they form a completely integrable set, their simultaneous solutions determine \( S(q_i, \bar{q}_i, t_0) \) uniquely by the initial conditions.

5 Example: Podolsky generalized electrodynamics

In this section we will consider a continuous system with Lagrangian density dependent on the dynamical field variables and its derivatives upon second order: \( \mathcal{L} = \mathcal{L}(\psi, \partial \psi, \partial^2 \psi) \). We adopt the metric \( \eta_{\mu \nu} = diag(+1, -1, -1, -1) \) with Greek indices running from 0 to 3 while Latin indices run from 1 to 3. As stated previously, the generalization of the formalism presented in Sections 4 and 5 is straightforward, being necessary only to consider that the Euler-Lagrange equa-
tions of motion are now given by
\[
\frac{\partial L}{\partial \dot{\psi}^a} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \psi^a)} \right) + \partial_\mu \partial_\nu \left( \frac{\partial L}{\partial (\partial_\mu \partial_\nu \psi^a)} \right) = 0
\]
and that the momenta, conjugated respectively to \( \dot{\psi}^a \) and \( \ddot{\psi}^a \), are:
\[
p_a = \frac{\partial L}{\partial \dot{\psi}^a} - 2\partial_k \left( \frac{\partial L}{\partial (\partial_k \dot{\psi}^a)} \right) - \partial_0 \left( \frac{\partial L}{\partial \dot{\psi}^a} \right)
\]
\[
\pi_a = \frac{\partial L}{\partial \ddot{\psi}^a}
\]

The Hessian matrix is now:
\[
H_{ab} = \frac{\partial^2 L}{\partial \ddot{\psi}^a \partial \ddot{\psi}^b}
\]

With these modifications we consider the case of Podolsky electrodynamics which is based on the following Lagrangian
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + a^2 \partial_\lambda F^{\alpha\lambda} \partial_\rho F_{\alpha\rho}
\]
were \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \).

An analysis of the Hamiltonian formalism for this theory was carried out in ref.\[12\] and we compare some of the results presented there with the formalism developed here. The Euler-Lagrange equations are
\[
\left( 1 + 2a^2 \Box \right) \partial_\mu F_{\alpha\rho} = 0
\]
with our dynamical variables chosen as \( A^\mu \) and \( A_\mu^\nu = \dot{A}^\mu \). The conjugated momenta given by definitions (74) and (75) are:
\[
p_\mu = -F_{0\mu} - 2a^2 \left( \partial_\kappa \partial_\lambda F^{\alpha\lambda\delta_\kappa\mu} - \partial_0 \partial_\lambda F_{\mu}^{\alpha\lambda} \right)
\]
\[
\pi_\mu = 2a^2 \left( \partial_\lambda F^{\alpha\lambda\delta_\mu} - \partial_\lambda F_{\mu}^{\alpha\lambda} \right)
\]
The primary constraints are:

(81) \[ \Phi_1 = \pi_0 \approx 0 \]

(82) \[ \Phi_2 = p_0 - \partial^k \pi_k \approx 0 \]

Using the definition of \( \pi \) we can write the accelerations \( \ddot{A} \) as:

(83) \[ \ddot{A}^i = \frac{1}{2a^2} \pi^i + \partial_k F^{ik} + \partial^i \dot{A}_0 \]

The canonical Hamiltonian is given by:

(84) \[ H_c = \int d^3 x \left[ p_\mu \ddot{A}^\mu + \pi_\mu \dddot{A}^\mu - \mathcal{L} \right] \]

Using equation (83) we get:

(85) \[ H_c = \int d^3 x \left[ \ddot{A}^0 \partial^i \pi_i + p_i \ddot{A}^i + \frac{1}{4a^2} \pi_i \pi^i + \pi_i \partial_k F^{ik} + \pi_i \partial^i \ddot{A}_0 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right. \]

\[ + \frac{1}{2} \left( \ddot{A}_i - \partial_i \dot{A}_0 \right) \left( \dddot{A}^i - \partial^i \ddot{A}_0 \right) - a^2 \left( \partial_k \dddot{A}^k - \partial_k \partial^k A_0 \right) \left( \partial_i \dddot{A}^i - \partial_i \partial^i A_0 \right) \]

According to Dirac’s formalism the total Hamiltonian is:

(86) \[ H_T = H_C + \int d^3 x \left( C_1(x) \Phi_1 + C_2(x) \Phi_2 \right) \]

The consistency conditions result in:

(87) \[ \dot{\Phi}_1 = \{ \Phi_1, H_T \} \approx 0 \]

(88) \[ \dot{\Phi}_2 = \{ \Phi_2, H_T \} = \partial^k p_k \approx 0 \]

So we have a secondary constraint given by

(89) \[ \Phi_3 = \partial^k p_k \approx 0 \]
and the consistency condition results in $\dot{\Phi}_3 = \{\Phi_3, H_T\} \approx 0$. All constraints are first class so the extended Hamiltonian is:

\begin{equation}
H_E = H_C + \int d^3x \left( C_1(x) \Phi_1 + C_2(x) \Phi_2 + C_3(x) \Phi_3 \right)
\end{equation}

The equations of motion for the dynamical variables, given by $\dot{A}^\alpha = \{A^\alpha, H_E\}$, are:

\begin{equation}
\dot{A}^0 = \bar{A}^0 + C_2; \quad \dot{A}^i = \bar{A}^i - \partial^i C_3
\end{equation}

This simply means that $\bar{A}^\alpha$ is defined as $\dot{A}^\alpha$ plus additive arbitrary functions. Besides, $\dot{\bar{A}} = \{\bar{A}, H_E\}$ gives

\begin{equation}
\dot{\bar{A}} = C_1; \quad \dot{\bar{A}} = \frac{1}{2a^2} \pi^i + \partial_k F^i_k + \partial^i \bar{A}_0,
\end{equation}

which mean that both $\bar{A}^0$ and $A^0$ are arbitrary while we obtained again equation (83).

For the momenta variables $\pi_i = \{\pi_i, H_E\}$ and $\dot{p}_\alpha = \{p_\alpha, H_E\}$ give:

\begin{align}
\dot{\pi}_i &= -F_{0i} - 2a^2 \partial_i \partial_k F^i_k - p_i \\
\dot{p}_0 &= -\partial_i F^0_i - 2a^2 \partial^j \partial_i \partial_k F^i_k \\
\dot{p}_i &= -\partial_i \partial^k \pi_k + \partial_k \partial^k \pi_i - \partial_k F^i_k
\end{align}

Equation (93) is the definition of $p_i$ given by equation (79) and together with (94) it gives constraint $\Phi_3$.

Now, using Hamilton-Jacobi formalism we have:

\begin{align}
H'_0 &= H_C + P_0; \quad P_0 = \frac{\partial S}{\partial t} \\
H'_1 &= \pi_0; \quad H'_2 = p_0 - \partial^k \pi_k
\end{align}

The total differential equation for $A^i$ is

\begin{equation}
dA^i = \frac{\partial H'_0}{\partial p_i} dt + \frac{\partial H'_1}{\partial p_i} d\bar{A}_0 + \frac{\partial H'_2}{\partial p_i} dA_0.
\end{equation}
\[ dA^i = \frac{\partial H_0}{\partial p_i} dt = \frac{\partial H_C}{\partial p_i} dt \Rightarrow dA^i = \bar{A}^i dt \]

which is completely equivalent to equation (91) since \( C_3 \) is arbitrary. For \( \bar{A}^i \) we have:

\[ d\bar{A}^i = \frac{\partial H'_0}{\partial \pi_i} dt + \frac{\partial H'_1}{\partial \pi_i} d\bar{A}_0 + \frac{\partial H'_2}{\partial \pi_i} dA_0 = \frac{\partial H'_0}{\partial \pi_i} dt + \frac{\partial H_C}{\partial \pi_i} dt \]

\[ d\bar{A}^i = \left( \frac{1}{2a^2} \pi^i + \partial_k F^{ik} + \partial^i \bar{A}_0 \right) dt \]

Again we have a result in agreement with Dirac’s method result given in (92). For the momenta \( p_i \) and \( p_0 \) we have

\[ dp^i = -\frac{\partial H'_0}{\partial A^i} dt - \frac{\partial H'_1}{\partial A^i} d\bar{A}_0 - \frac{\partial H'_2}{\partial A^i} dA_0 = -\frac{\partial H'_0}{\partial A^i} dt - \frac{\partial H_C}{\partial A^i} dt \]

\[ dp^i = -\int d^3x \left[ \pi_j \partial_k \left( \frac{\partial F^{jk}}{\partial A^i} \right) - \frac{1}{2} F^{jn} \partial^j F_{jn} \right] dt \]

\[ dp^i = \left[ -\partial^i \partial^k \pi_k + \partial_k \partial^k \pi^i - \partial_k F^{ki} \right] dt \]

and

\[ dp^0 = -\frac{\partial H'_0}{\partial A_0} dt - \frac{\partial H'_1}{\partial A_0} d\bar{A}_0 - \frac{\partial H'_2}{\partial A_0} dA_0 = -\frac{\partial H_C}{\partial A_0} dt \]

\[ dp^0 = -\int d^3x \left[ \left( \bar{A}^i - \partial^i A_0 \right) \frac{\partial \left( \bar{A}^i - \partial^i A_0 \right)}{\partial A_0} \right. \]

\[ \left. -2a^2 \left( \partial_i \bar{A}^i - \partial_i \partial^i A_0 \right) \frac{\partial \left( \bar{A}_k - \partial_k \partial^i A_0 \right)}{\partial A_0} \right] dt \]

\[ dp^0 = \left[ -\partial_i \left( \bar{A}^i - \partial^i A_0 \right) - 2a^2 \partial^k \partial^i \left( \partial_i \bar{A}^i - \partial_i \partial^i A_0 \right) \right] dt \]

Finally for \( \pi^i \) we have:

\[ d\pi^i = -\frac{\partial H'_0}{\partial \bar{A}^i} dt - \frac{\partial H'_1}{\partial \bar{A}^i} d\bar{A}_0 - \frac{\partial H'_2}{\partial \bar{A}^i} dA_0 = -\frac{\partial H'_0}{\partial \bar{A}^i} dt - \frac{\partial H_C}{\partial \bar{A}^i} dt \]
\(d\pi^i = -\int d^3x \left[ p^i \frac{\partial \tilde{A}_j}{\partial A_i} + (\tilde{A}_j - \partial^j A_0) \frac{\partial (\tilde{\lambda}_j - \partial_j A_0)}{\partial A_0} \right. \\
-2a^2 (\partial_j \tilde{A}_j - \partial_j \partial^j A_0) \frac{\partial (\partial_k \tilde{A}_k - \partial_k \partial^k A_0)}{\partial A_0} \left. \right] dt \)

\(d\pi^i = \left[ -p^i - F^{0i} - 2a^2 \partial^i \partial_k F^{0k} \right] dt \)

Equations (104), (107) and (110) are completely equivalent to (93), (94) and (95); consequently equations (107) and (110) give us the secondary constraint that isn’t present in the total differential equations.

6 Conclusions

We obtained a generalization of Hamilton-Jacobi formalism whose results agree with those obtained using Dirac’s formalism. In this formalism those coordinates whose correspondent accelerations can’t be solved in function of the momenta are arbitrary variables of the theory. We obtained a set of Hamilton-Jacobi partial differential equations in terms of these variables and from this set we obtained the equations of motion of the system as the total differential equations for the characteristics. These total differential equations so obtained must satisfy integrability conditions and for these conditions to be satisfied the nature of the constraints (first class or second class) will play an essential role. The study of these integrability conditions is in progress as well as the generalization of the present formalism for Lagrangians of order higher than two.

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