Γ-(λ,δ)-Derivation on Semi-Group Ideals in Prime Γ-Near-Ring

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Abstract
The main purpose of this paper is to investigate some results. When \( h \) is \( \Gamma - (\lambda, \delta) \) – Derivation on prime \( \Gamma \)-near-ring \( G \) and \( K \) is a nonzero semi-group ideal of \( G \), then \( G \) is commutative.

Keywords: Prime \( \Gamma \)-near-ring, Semi-group ideal, \( \Gamma - (\lambda, \delta) \) – derivation

1. Introduction
Throughout this paper, \( G \) denotes a zero – symmetric left \( \Gamma \)-near-ring with a multiplicative center \( Z(G) \). For a \( \Gamma \)-near-ring \( G \), the set \( G_0=\{s\in G: \delta ps=0, \ \forall \ p\in\Gamma\} \) is called a zero symmetric part of \( G \). If \( G=G_0 \), then \( G \) is called a zero symmetric \([1,2,3,4]\). An additive mapping \( h: G\rightarrow G \) is called a \( \Gamma - (\lambda, \delta) \)-derivation on a \( \Gamma \)-near-ring \( G \) If there exist two automorphisms mapping \( \lambda, \delta : G\rightarrow G \), such that \( h(spr)=h(s)p\lambda(r)+\delta(s)ph(r), \) for every \( s,r\in G \) and \( p\in \Gamma \) \([4,5]\). A \( \Gamma \)-near-ring \( G \) is said to be a prime \( \Gamma \)-near-ring if \( s\Gamma G\Gamma r=0 \) implies \( s=0 \) or \( r=0 \), for every \( s,r\in G \), and it said to be a semiprime if \( s\Gamma G\Gamma s=0 \) implies \( s=0 \) for every \( s\in G \) \([5,6]\). Further, an element \( s\in G \) is called constant if \( h(s)=0 \) \([4,7]\). A non-empty subset \( K \) of \( G \) is called semi-group ideal if \( K\Gamma G\subset K \) and \( G\Gamma K\subset K \) \([8]\). For \( s,r\in G \) and \( p\in\Gamma \), the symbol \( [s,r]_{\lambda,\delta} \) will denote \( \delta(s)pr - p\lambda(s) \), as previously described \([4,9]\). The other commutators are \([s,r]=spr-rps \) and \( (s,r)=s+r-s-r \) which denote the additive-group commutator \([4,9]\).

The purpose of this paper is to study and generalize some results of previous authors \([4,7,9,10]\) on the commutativity of the prime \( \Gamma \)-near-ring. Some recent results on rings deal with commutativity of prime and semiprime rings admitting suitably constrained derivations. For further details on prime near-ring, we refer to some previous articles \([11-15]\).

As a generalization of near-ring, the \( \Gamma \)-near-ring was discussed by Satyanarayana \([6]\), while Booth and Groenewald \([5,13]\) surveyed various portions in the \( \Gamma \)-near-ring. In this paper, we investigate the condition for a \( \Gamma - (\lambda, \delta) \)-derivation on a prime \( \Gamma \)-near-ring to be commutative.

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2. The Main Results
In this section, we investigate some results of a semi-group ideal of a $\Gamma$-near-ring admitting a $\Gamma$-($\lambda,\delta$)-derivation.

To prove the main theorems, we need the following lemmas.

**Lemma 2.1.** Let $h$ be a $\Gamma$-($\lambda,\delta$)– derivation on a prime $\Gamma$-near-ring and $K$ a semi-group ideal of $G$. If and only if $h(snr) = \delta(s)\eta h(r) + h(s)\eta \lambda(r)$, for all $s,r \in K$ and $\eta \in \Gamma$.

**Proof.** $\forall s, r \in K$ and $\eta \in \Gamma$, we have $s\eta(r + r) = snr + snr$.

By applying $h$ for both sides we obtain

$$h(s\eta(r + r)) = h(s)\eta \lambda(r + r) + \delta(s)\eta h(r + r)$$

and

$$h(snr + snr) = h(snr) + h(snr)$$

Comparing the two relations, we have

$$h(s)\eta \lambda(r) + \delta(s)\eta h(r) = \delta(s)\eta \lambda(r) + h(s)\eta \lambda(r)$$

$$h(snr) = h(s)\eta \lambda(r) + h(s)\eta \lambda(r)$$

$\forall s, r \in K$ and $\eta \in \Gamma$.

Conversely, assume for every $s, r \in K$ and $\eta \in \Gamma$, that

$$h(snr) = \delta(s)\eta \lambda(r) + h(s)\eta \lambda(r).$$

then,

$$h(s\eta(r + r)) = \delta(s)\eta h(r + r) + h(s)\eta \lambda(r + r)$$

and

$$h(snr + snr) = h(snr) + h(snr)$$

Comparing the two relations provides that:

$$\delta(s)\eta h(r) + h(s)\eta \lambda(r) = h(s)\eta \lambda(r) + \delta(s)\eta h(r)$$

$$h(snr) = h(s)\eta \lambda(r) + \delta(s)\eta h(r)$$

**Lemma 2.2.** If $h$ be a $\Gamma$-($\lambda,\delta$)– derivation on a $\Gamma$-near-ring $G$, $K$ a semi-group ideal of $G$, and $\lambda(K)=K$, then

$$(h(s)\eta \lambda(r) + \delta(s)\eta h(r))\rho v = h(s)\eta \lambda(r)\rho v + \delta(s)\eta h(r)\rho v, \text{ for all } s, r, v \in K \text{ and } \eta, \rho \in \Gamma.$$  

**Proof.** Assume that $\forall s, r, v \in K$ and $\eta, \rho \in \Gamma$. 

$$h((s\eta r)\rho v) = h(s\eta r)\rho \lambda(v) + \delta(s\eta r)\rho h(v)$$

and 

$$h(s\eta(r\rho v)) = h(s)\eta \lambda(r)\rho \lambda(v) + \delta(s)\eta h(r)\rho \lambda(v)$$

Comparing the two relations above of $h(s\eta r\rho v), \forall s, r, v \in K$ and $\eta, \rho \in \Gamma$, and since $\lambda(K)=K$, implies that

$$(h(s)\eta \lambda(r) + \delta(s)\eta h(r))\rho v = h(s)\eta \lambda(r)\rho v + \delta(s)\eta h(r)\rho v.$$  

**Lemma 2.3.** If $h$ be a $\Gamma$-($\lambda,\delta$)– derivation on a $\Gamma$-near-ring $G$ and $K$ a semi-group ideal of $G$ such that $h([s, r]_p) = [s, r]_p$, $\lambda(K)=K$, and $\delta(K)=K$, then 

(i) $h(v) = v$, for every commutator $v$ in $K$.

(ii) $h(k)g[s, r]_p = [s, r]_p \gamma h(k)$, for every $s, k \in K$, $r \in G$ and $\rho, \gamma \in \Gamma$. 

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Proof. (i) Let \( v = [s, r]_\rho \), where \( s \in K \), \( r \in G \) and \( \rho \in \Gamma \).

\[ h([s, r]_\rho) = [s, r]_\rho, \]

for every \( s \in K \), \( r \in G \) and \( \rho \in \Gamma \).

Thus, \( h(v) = v \), for each commutator \( v \) in \( K \).

(ii) By the hypothesis that \( h([s, r]_\rho) = [s, r]_\rho \), we have

\[ -[s, r]_\rho \gamma k + h([s, r]_\rho) \gamma k = -k \gamma (s, r)_\rho + h(k \gamma (s, r)_\rho), \quad \forall s, k \in K, \, r \in G \) and \( \gamma, \rho \in \Gamma \).

By using Lemma 2.1, we have

\[ -[s, r]_\rho \gamma k + h([s, r]_\rho) \gamma k = -k \gamma (s, r)_\rho + h(k) \gamma (s, r)_\rho + \delta(k) \gamma h([s, r]_\rho) \]

By applying \( k \) to both sides, we obtain:

\[ [s, r]_\rho \gamma h(k) = h(k) \gamma (s, r)_\rho, \quad \forall s, k \in K, \, r \in G \) and \( \gamma, \rho \in \Gamma \).

\textbf{Lemma 2.4} \ If \( h \) is a \( \Gamma \)-derivation on a \( \Gamma \)-near-ring \( G \), \( K \) is a nonzero semi-group ideal of \( G \), and \( h([s, r]_\rho) = [s, r]_\rho \), \( \lambda(K) = K, \delta(K) = K \). Then

(i) If \( v \) is a commutator in \( K \) and \( w \gamma v = z \mu v \), where \( w, z \in K \) and \( \mu, \gamma \in \Gamma \), then

\[ v \gamma h(w-z) = 0. \]

(ii) If \( v_1 \) and \( v_2 \) are commutators in \( K \) with \( v_1 \mu v_2 = 0 \), then \( v_1 = 0 \) or \( v_2 = 0 \).

\textbf{Proof.} (i) Let \( v = [s, r]_\rho, \forall s \in K, \, r \in G \) and \( \rho \in \Gamma \).

Then, the hypothesis provides that \( w \mu [s, r]_\rho = z \mu [s, r]_\rho, \forall s, w, z \in K, r \in G \) and \( \rho, \mu \in \Gamma \).

Applying \( h \) for both sides, implies that

\[ h(w \mu [s, r]_\rho) = h(z \mu [s, r]_\rho), \quad \forall s, w, z \in K, \, r \in G \) and \( \rho, \mu \in \Gamma. \]

Thus,

\[ h(w) \mu \lambda([s, r]_\rho) + \delta(w) \mu h([s, r]_\rho) = h(z) \mu \lambda([s, r]_\rho) + \delta(z) \mu h([s, r]_\rho). \]

Using Lemma 2.3 (i, ii) provides that:

\[ h(w) \mu \lambda([s, r]_\rho) = h(z) \mu \lambda([s, r]_\rho), \quad \forall s, w, z \in K, \, r \in G \) and \( \rho, \mu \in \Gamma. \]

So, \( [s, r]_\rho \mu h(w-z) = 0 \). Thus, \( v_1 \gamma h(w-z) = 0 \), for every commutator \( v \) in \( K \), \( w, z \in K \), and \( \mu \in \Gamma \).

(ii) If \( v_1 \mu v_2 = 0 = 0 \mu v_2 \), since \( v_2 \) is a commutator in \( K \), (i) yields

\[ v_2 \mu h(v_1) = 0 \] ........................ (1)

By using Lemma 2.3 (i), since \( v_1 \) is a commutator in \( K \), we obtain

\[ v_2 \mu v_1 = 0 \] ........................ (2)

B substituting \( r \gamma v_1 \) for \( v_1 \), where \( r \in K, \gamma \in \Gamma \) in equation (1), we obtain:

\[ v_2 \mu h(r \gamma v_1) = 0 = v_2 \mu h(r) \gamma \lambda(v_1) + v_2 \mu h(r) \gamma h(v_1) \] ........................ (3)

Using Lemma 2.3 (ii) and equation (2) in equation (3) provides that:

\[ v_2 \mu h(r \gamma v_1) = 0, \quad \text{for every commutator} \ v_1, v_2 \ in \ K, \, r \in K, \) and \( \mu, \gamma \in \Gamma. \]

Hence, \( v_2 \Gamma \mu h(v_1) = 0. \)

By using Lemma 2.3 (i), since \( v_1 \) is commutator, we obtain \( v_2 \Gamma \mu h(v_1) = 0. \)

Since \( K \) is a nonzero semi-group ideal of \( G \) and \( G \) is a prime \( \Gamma \)-near-ring, we obtain \( v_1 = 0 \) or \( v_2 = 0 \).

\textbf{Lemma 2.5} \ If \( G \) be a prime \( \Gamma \)-near-ring and \( K \) is a nonzero semi-group ideal of \( G \), then \( Z(K) \subseteq Z(G) \).

\textbf{Proof.} Suppose that \( t \in Z(K) \), this means that, \( [t, s]_\rho = 0, \quad \forall s \in K \) and \( \rho \in \Gamma. \)

Replacing \( s \) by \( s \mu r, \) so \( r \in G \) in the above equation, we obtain
[t, sμr]_ρ = 0 = sμ[t, r]_ρ + [t, s]_ρμr, ∀ t, s ∈ K, r ∈ G and ρ, μ ∈ Γ.

Thus, Kμ[t, r]_ρ = 0. Since K is a nonzero semi-group ideal of G and G is a prime Γ-near-ring, we get [t, r]_ρ = 0, ∀ t ∈ K, r ∈ G and ρ ∈ Γ. Hence, tε Z(G).

**Lemma 2.6.** If h is a Γ-(λ, δ) - derivation on a prime Γ-near-ring G and K be a semi-group ideal of G.

(i) If u is a nonzero element in Z(G), then u is not a zero divisor.

(ii) If there exists a nonzero element u of Z(G) such that u + u ∈ Z(G), then (K, +) is an abelian.

**Proof.** (i) If u ∈ Z(G)\{0} and uψs = 0, ∀ s ∈ K and ρ ∈ Γ. Then, left multiplication of this equation by ty, where t ∈ G and γ ∈ Γ, provides that 

uψtγs = 0. Since G is a multiplicative with the center Z(G), it implies that 

uψtγs = 0, ∀ t ∈ G and s ∈ K, thus, uψGTG = 0.

Since G is a prime Γ-near-ring and u is a nonzero element, it shows that s = 0.

(ii) Let u ∈ Z(G)\{0} be an element, such that u+ u ∈ Z(G).

Let s, r ∈ K and ρ ∈ Γ so,

(s + r)ρ(u + u) = (u + u)ρ(s + r)

sρu + sρu + rρu + rρu = uρ + uρ + uρ + uρ

Since u ∈ Z(G), we get 

uψs + uψr = uψr + uψs

Thus, uψ(s+r-r-s) = 0, ∀ s, r ∈ K and ρ ∈ Γ.

Left multiplication this equation by ay, where aG, γ ∈ Γ, provides that:

aψuρ(s, r) = 0, ∀ s, r ∈ K, a ∈ G and γ, ρ ∈ Γ.

Because G is a multiplicative with the center Z(G), this provides that:

uψa dissolution (s, r) = 0. Hence, uψGTG = 0.

Because G is a prime Γ-near-ring and u is a nonzero element, it implies that 

(s, r) = 0, ∀ s, r ∈ K. Thus, (K, +) is an abelian. □

**Lemma 2.7.** If h be a nonzero Γ-(λ, δ)-derivation on a prime Γ-near-ring G and K be a nonzero semi-group ideal of G. Then sψh(K) = 0, which implies that s = 0 and h(K)Γs = 0, which means that s = 0, where s ∈ G.

**Proof.** Assume that sψh(K) = 0, ∀ r ∈ G, t ∈ K and β ∈ Γ.

Then, sψh(tβr) = 0, showing that:

sψh(tβr) = 0, tψh(tβr) + sψh(tβr) = 0

Therefore, ∀ s, t ∈ G, t ∈ K and η, β ∈ Γ, we have sψh(tβr) = 0.

Since δ(K) = K, then sψGTG = 0.

Since K is a nonzero semi-group ideal and G is a prime Γ-near-ring, h ≠ 0, it implies that s = 0.

Similarly, we can show that if h(K)Γs = 0, ∀ s ∈ G, it implies that s = 0. □

**Lemma 2.8.** If G is a 2-torsion free prime Γ-near-ring, h be a nonzero Γ-(λ, δ)-derivation of G, and K be a nonzero semi-group ideal of G. If hγ(K) = 0 and λ, δ commute with h, then h(K) = 0.

**Proof.** ∀ s, r ∈ K and ρ ∈ Γ.

0 = h2(sψr) = h(h(sψr)) = h(h(s)ρλ(r) + δ(s)ρφh(r))

= h(h(s)ρλ(r)) + h(δ(s)ρφh(r))

= h2(s)ρλ2(r) + δ(h(s))ρλ(h(r)) + δ2(s)ρφ2(r)

By the hypothesis, we obtain that 2h(δ(s))ρφh(λ(r)) = 0, ∀ s, r ∈ K and ρ ∈ Γ.

Because G is a 2-torsion free and λ(Γ) = K, this provides h(δ(s))ρφ(K) = 0

By using Lemma 2.7, we obtain that h = 0. □

**Lemma 2.9.** Let G be a prime Γ-near-ring and K be a nonzero semi-group ideal of G. If K is a commutative then G is a commutative ring.

**Proof.** ∀ s, r ∈ K, [s, r]_ρ = 0.

By taking sψa instead of s and rψb instead of r, where a, b ∈ G and γ ∈ Γ, we obtain that [sψa, rψb]_ρ = 0. Since K is a commutative and semi-group ideal of G, this provides
If \( s\gamma t = r\gamma b \) then \( s\gamma t = s\gamma r b \).

\[ a, b \in G, s, r \in K \text{ and } t, \gamma, \delta \in \Gamma, \text{ this implies that } s \Gamma K[a, b] = 0. \]

Because \( K \) is a nonzero semi-group ideal of \( G \), \( G \) is a prime \( \Gamma \)-near-ring, thus \([a, b]_\Gamma = 0, \forall a, b \in G. \text{ Thus, } G \) is a commutative ring.

**Lemma 2.10.** If \( G \) is a prime \( \Gamma \)-near-ring and \( K \) is a nonzero semi-group ideal of \( G \). If \( (K, +) \) is an abelian, then \( (G, +) \) is an abelian.

**Proof.** Since \( (K, +) \) is an abelian, we obtain that \( z + c = c + z, \forall z, c \in K \).

By substituting \( s\eta z \) for \( z \) and \( r\eta z \) for \( c \), for \( s, r \in G \) and \( \eta \in \Gamma \), we have

\[ s\eta z + r\eta z = r\eta z + s\eta z, \forall z \in K, s, r \in G \text{ and } \eta \in \Gamma. \]

Which gives \( (s + r - r - s)\eta z = 0. \)

Thus, \( (s, r) \Gamma K = 0. \) Since \( K \neq 0 \) is a semi-group ideal and \( G \) is a prime, then

\( (s, r) = 0, \forall s, r \in G \). \( (G, +) \) is abelian.

**Lemma 2.11.** If \( h \) be a \( \Gamma \)-\( (\lambda, \delta) \)-derivation on a prime \( \Gamma \)-near-ring \( G \) and \( K \) is a semi-group ideal of \( G \). Suppose that \( \delta \) is not a left zero divisor. If \( [t, h(t)]^{(\lambda, \delta)} = 0, \) then \( (s, t) \) is a constant for every \( s \in K \) and \( \beta \in \Gamma. \)

**Proof.** From \( \beta(s + t) = \beta s + \beta t, \forall t \in K \text{ and } \beta \in \Gamma. \)

By applying \( h \) for both sides, we have

\[ h(t \beta(s + t)) = h(t) \beta \lambda(s + t) + \delta(t) \beta h(s + t) \]

and

\[ h(t \beta s + t \beta t) = h(t \beta s) + h(t \beta t) \]

Which gives that \( h(t) \beta \lambda(t) + \delta(t) \beta h(s) = \delta(t) \beta h(s) + h(t) \beta \lambda(t), \forall t, s \in K \) and \( \beta \in \Gamma. \)

Therefore, \( \delta(t) \Gamma K \beta h((s, t)) = 0, \forall t, s \in K. \)

Because \( t \) is not a left zero divisor and \( \delta(K) = K \), \( K \) is a semi-group ideal and \( G \) is a prime \( \Gamma \)-near-ring, we obtain that

\( h((s, t)) = 0. \) Thus, \( (s, t) \) is a constant for every \( s \in K. \)

Now we can prove the main theorems.

**Theorem 2.12.** Let \( h \) be a \( \Gamma \)-\( (\lambda, \delta) \)-derivation of a prime \( \Gamma \)-near-ring \( G \) and \( K \) is a semi-group ideal of \( G \) which has no nonzero divisors of zero, where \( h \) is commuting on \( K, \lambda(K) = K, \) then \( (G, +) \) is an abelian.

**Proof.** Let \( v \) be any additive commutator in \( K. \)

So, the application of Lemma 2.11 yields that \( v \) is a constant.

For any \( s \in K, svv \) is also an additive commutator in \( K. \) Then, \( svv \) is also a constant.

Therefore, \( 0 = h(svv) = h(s)\gamma \lambda(v) + \delta(s)\gamma h(v) = h(s)\gamma \lambda(v), \forall s \in K \text{ and } \gamma \in \Gamma. \)

Because \( h(s) \neq 0, \) for some \( s \in K, \) and \( K \) has no nonzero divisors of zero.

Which gives \( \lambda(v) = 0, \) thus \( v = 0 \) for every additive commutator \( v \) in \( K. \)

Hence, \( (K, +) \) is an abelian. By using Lemma 2.10, we obtain that \( (G, +) \) is an abelian. \( \square \)
We need the following lemma to prove the main theorem.

**Lemma 2.13.** Let $h$ be a nonzero $\Gamma$-$\delta$-derivation on a prime $\Gamma$-near-ring $G$, and $K$ is a semi-group ideal of $G$, so $\lambda \gamma h = h \lambda \gamma$, $\delta \gamma h = h \delta \gamma$ for every $\gamma \in \Gamma$, $\lambda(K) = K$, where $h(K) \subseteq Z(G)$, then $(K, +)$ is an abelian. If $G$ is a 2-torsion free and $h(K) \subseteq K$, then $K$ is a central ideal.

**Proof.** Since $h(K) \subseteq Z(G)$ and $h$ is a nonzero $\Gamma$-$\delta$-derivation. There exists a nonzero element $t$ in $K$, such that $t = h(t) + Z(G)$.

And, $u + u = h(t) + h(t) = h(t + t) \in Z(G)$.

Therefore, $(K, +)$ is an abelian by Lemma 2.6 (ii).

Using the hypothesis, $\forall s, r \in K$, $\forall \beta \in \Gamma$ and $\gamma \in \Gamma$ gives $\lambda(c)\gamma h(s) \beta r = h(s) \beta r \gamma \lambda(c)$.

Uusing Lemma 2.2, it provides

$$\lambda(c)\gamma h(s)\beta \gamma(r) + \lambda(c)\gamma \delta(s)\beta h(r) = h(s)\beta \lambda(r)\gamma \lambda(c) + \delta(s)\beta h(r)\gamma \lambda(c).$$

Now, by using $h(K) \subseteq Z(G)$ and since $(K, +)$ is an abelian, $\lambda \gamma h = h \lambda \gamma$, and $\delta \gamma h = h \delta \gamma$, it shows that

$$h(s)\beta \lambda(c)\gamma(r) - h(s)\beta \lambda(r)\gamma \lambda(c) = h(r)\beta \delta(s)\gamma \lambda(c) - h(r)\beta \lambda(c)\gamma \delta(s)$$

Then, $h(s)\beta \lambda(c)\gamma(r) = h(r)\beta \delta(s)\gamma \lambda(c) - h(r)\beta \lambda(c)\gamma \delta(s)$

Suppose that $K$ is not a central ideal.

By choosing $r \in K$ and $\forall s \in G$, such that $[c, r] \neq 0$.

And since $h(K) \subseteq K$, let $s = h(x) \in Z(G)$, where $x \in K$, which gives

$$h^\gamma(x)\beta \lambda(c, r) = h(r)\beta \delta(s)\gamma \lambda(c), \forall s, r \in K, c \in G \text{ and } \gamma, \beta \in \Gamma.$$}

Then, $h^\gamma(x)\beta \lambda(c, r) = 0$

By Lemma 2.6 (i), the central element $h^\gamma(x)$ cannot be a nonzero divisor of zero, then we conclude that

$$h^\gamma(x) = 0, \forall x \in K.$$}

By using Lemma 2.8, we obtain that $h(x) = 0$

This contradicts that $h$ is a nonzero $\Gamma$-$\delta$-derivation on $G$.

So, we obtain that $\lambda(c, r) = 0, \forall s \in K, c \in G$.

Because $\lambda(K) = K$, this gives a contradiction with the assumption. Then $K$ is a central ideal. □

**Theorem 2.14.** Let $h$ be a nonzero $\Gamma$-$\delta$-derivation on a prime $\Gamma$-near-ring $G$ and $K$ a semi-group ideal of $G$, so $\lambda \gamma h = h \lambda \gamma$, $\delta \gamma h = h \delta \gamma$ for every $\gamma \in \Gamma$, $\lambda(K) = K$, where $h(K) \subseteq Z(G)$, then $(K, +)$ is an abelian. If $G$ is a 2-torsion free and $h(K) \subseteq K$, then $G$ is a commutative ring.

**Proof.** By using Lemma 2.13, it gives that $(K, +)$ is an abelian.

By using Lemma 2.10, it gives that $(G, +)$ is an abelian.

Now, assume that $G$ is a 2-torsion free. The application of Lemma 2.13. shows that $K$ is a central ideal.

Thus, $K$ is a commutative. By Lemma 2.9, it implies that $G$ is a commutative ring. □

**Theorem 2.15.** Let $h$ be a nonzero $\Gamma$-$\delta$-derivation on a prime $\Gamma$-near-ring $G$, and $K$ is a nonzero semi-group ideal of $G$, so $h([s, r]) = [s, r]_\alpha$, where $\gamma \delta \alpha u = \delta \gamma \beta u$ for every $\alpha, \beta, \gamma \in \Gamma$ and $\gamma, \beta \in \Gamma$, then $(G, +)$ is an abelian. If $G$ is a 2-torsion free and $h(K) \subseteq K$, then $G$ is a commutative ring.

**Proof.** Since $[s, s \gamma]_{\alpha} = s \gamma[s, r]_{\alpha}, \forall s \in K, r \in G$ and $\gamma, \beta \in \Gamma$.

By using Lemma 2.3 (ii), we have

$$h(v)\beta \gamma [s, r]_\alpha = s \gamma [s, r]_\alpha \beta \gamma h(v) = s \gamma h(v) \beta [s, r]_\alpha, \forall s, v \in K, r \in G \text{ and } \gamma, \beta \in \Gamma.$$}

By using Lemma 2.4 (i), we have

$$[s, r]_\alpha \beta h(v)\gamma s - s \gamma h(v) = 0, \forall s, v \in K \text{ and } \gamma, \beta \in \Gamma.$$}

Hence, $[s, r]_\alpha \beta h(v\gamma s - s \gamma h(v)) = 0, \forall s, v \in K \text{ and } \gamma, \beta \in \Gamma$.

By Lemma 2.3 (i), we obtain that $[s, r]_\alpha \beta [(h(v), s)_{\alpha}] = 0$

The application of Lemma 2.4 (ii) gives

either $[s, r]_\alpha = 0$ or $h(v, s) = 0, \forall s, v \in K \text{ and } \gamma, \beta \in \Gamma$.

If $[h(v), s]_\alpha = 0, \forall s, v \in K \text{ and } \gamma \in \Gamma$.

Hence, $h(K) \subseteq Z(K)$. By Lemma 2.5, we obtain that $h(K) \subseteq Z(G)$. Then, by Theorem 2.14, we complete this theorem.

So, if $[s, r]_\alpha = 0, \forall s \in K, r \in G$ and $\gamma \in \Gamma$.  

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We substitute \( s \gamma h(v) \) for \( s \), where \( v \in K \), which gives

\[
[s \gamma h(v), r]_p = 0 = s h[h(v), r], \quad \forall s,v \in K, r \in G \text{ and } \rho, \gamma \in \Gamma
\]

Which gives, \( K \gamma h(v), r]_p = 0, \quad \forall v \in K, r \in G \text{ and } \rho, \gamma \in \Gamma \)

Because \( K \) is a nonzero semi-group ideal of \( G \) and \( G \) is a prime \( \Gamma \)-near–ring, this implies that

\[
[h(v), r]_p = 0, \quad \forall v \in K, r \in G \text{ and } \rho \in \Gamma.
\]

So, \( h(K) \subseteq Z(G) \). By Theorem 2.14, the proof will be complete. \( \Box \)

**Theorem 2.16.** Let \( h \) be a nonzero \( \Gamma \)-(\( \lambda, \delta \))-derivation on a prime \( \Gamma \)-near–ring \( G \), and \( K \) is a nonzero semi-group ideal of \( G \), so \( h([s,r])=[s,r] \), and \( tyk\beta u = t\beta k\gamma u \) for every \( t,k,u \in K \) and \( \gamma,\beta \in \Gamma \), then \( G \) is a commutative ring.

**Proof.** Since \( h([s,r])_p = -spr + rps, \quad \forall s \in K, r \in G \) and \( \rho, \gamma \in \Gamma \).

By replacing \( r \) by \( rys \) in this equation, we get

\[
h([s, rys]) = -sprys + rpsys = (-spr + rps)ys, \quad \forall s, r \in K, r \in G \text{ and } \rho, \gamma \in \Gamma
\]

And,

\[
h([s, rys])_p = h([s, r])_p \gamma h(s) = h([s, r])_p \gamma h(s) = (-spr + rps)ys = (-spr + rps)ys
\]

It follows from the two expressions for \( h([s, rys])_p \) that

\[
\delta(s) \rho \delta(v) h(s) = \delta(v) \rho \delta(s) h(s), \quad \forall s, v \in K, r \in G \text{ and } \rho, \gamma \in \Gamma
\]

Replacing \( r \) by \( vfr \) in equation (4), where \( v \in K \) and \( \beta \in \Gamma \), we obtain that

\[
\delta(s) \rho \delta(v) h(s) = \delta(v) \rho \delta(s) h(s), \quad \forall s, v \in K, r \in G \text{ and } \rho, \gamma \in \Gamma
\]

Left multiplicative equation (4) by \( \delta(v) \beta \), gives

\[
\delta(v) \beta \delta(s) \rho \delta(r) h(s) = \delta(v) \beta \rho \delta(s) h(s), \quad \forall s, v \in K, r \in G \text{ and } \rho, \gamma \in \Gamma
\]

The combining of equation (5) and equation (6) gives

\[
\delta(s) \rho \delta(v) h(s) = 0, \quad \forall s, v \in K, r \in G \text{ and } \rho, \gamma \in \Gamma
\]

Since \( \delta(K) = K, \) and \( \delta \) is automorphism on \( G \).

Hence, \( [s, v]_p \Gamma GIh(s) = 0 \).

Because \( G \) is a prime \( \Gamma \)-near–ring, this implies that \( h(s) = 0 \text{ or } [s, v]_p = 0 \).

Since \( h \neq 0 \), therefore \( [s, v]_p = 0, \quad \forall s, v \in K \) and \( \beta \in \Gamma \).

Thus, \( K \) is a commutative. By Lemma 2.9, we obtain that \( G \) is a commutative ring. \( \Box \)

**Lemma 2.17:** Let \( h \) be a nonzero \( \Gamma \)-(\( \lambda, \delta \))-derivation on a prime \( \Gamma \)-near–ring \( G \), and \( K \) a nonzero semi-group ideal of \( G \), if \( [s, r]_p = [h(s), h(r)]_p \), then the constant in \( K \) is in \( Z(G) \).

**Proof:** Let \( s \) be a constant in \( K \), i.e., \( h(s) = 0 \). then

\[
[s, r]_p = [h(s), h(r)]_p = [0, h(r)]_p = 0, \quad \forall r \in K \text{ and } \rho \in \Gamma.
\]

Then, \( s \in Z(K) \). By Lemma 2.5, we obtain that \( s \in Z(G) \).

**Theorem 2.18.** Let \( h \) be a nonzero \( \Gamma \)-(\( \lambda, \delta \))-derivation on a prime \( \Gamma \)-near–ring \( G \), and \( K \) is a nonzero semi-group ideal of \( G \) that has no nonzero divisors of zero. If \( K \) has a right cancellation and \( [h(s), h(r)]_p = [s, r]_p \), \( t\beta k\gamma u = t\beta k\gamma u \) for every \( t,k,u \in K \) and \( \gamma, \beta \in \Gamma \), and \( h(K) \subseteq K \), then \( h \) is commuting and \( (G,+) \) is an abelian.

**Proof.**

\[
\forall s \in K, \quad [s, s \gamma h(s)]_p = [h(s), s \gamma h(s)]_p \quad \text{ (7)}
\]

By using Lemma 2.1 and 2.2, the right–hand side of equation (7) equals

\[
[h(s), s \gamma h(s)]_p = h(s) \gamma h(s) + h(s) \gamma h(s) h^2(s) - h(s) \gamma h(s) \rho h(s) - h(s) \gamma h(s) \rho h(s) = h(s) \gamma h(s) h^2(s) - h(s) \gamma h(s) \rho h(s)
\]

The left–hand side of equation (7) equals

\[
s \gamma h(s) \gamma h(s) = s \gamma h(s) h^2(s) - s \gamma h^2(s) \rho h(s), \quad \forall s \in K \text{ and } \rho, \gamma \in \Gamma.
\]

It follows from equation (7), since \( \delta(K) = K \), it implies that

\[
s \gamma h(s) h^2(s) = h(s) s \gamma h^2(s), \quad \forall s \in K \text{ and } \rho, \gamma \in \Gamma.
\]

Hence, by using the hypotheses, we obtain that \( [s, h(s)]_p = 0, \quad \forall s \in K \text{ and } \rho, \gamma \in \Gamma. \)

If \( h^2(s) = 0, \quad \forall s \in K. \)

Then, \( h(s) \) is constant in \( K \), by using Lemma 2.17, we obtain that \( h \) is central. Thus, \( h \) is commuting in \( K \).
By Theorem 2.12, we obtain that \((G, +)\) is an abelian. Otherwise, \(h^2(x)\) can be cancelled on the right in equation (8).

In either event, \([s, h(s)]_p = 0 \ \forall \ s \in K \ \text{and} \ \gamma \in \Gamma\).

Then, by using Theorem 2.12, we obtain that \((G, +)\) is an abelian. □

**Theorem 2.19.** Let \(h\) be a nonzero \((\lambda,\delta)\) - derivation on a prime \(\Gamma\)-near-ring \(G\), and \(K\) be a nonzero semi-group ideal of \(G\) that has no nonzero divisors of zero. If \(h\) is commuting on \(K\) and \([s, r]_p = [h(s), h(r)]_p\), then \(G\) is a commutative ring.

**Proof.** For every \(s, r \in K\), we have

\([s, s\beta r]_p = [h(s), h(s\beta r)]_p = [h(s), h(s)\beta(\lambda) + \delta(s)\beta(h(r))]_p\)

By using Lemma 2.2, we have

\([h(s), h(s\beta r)]_p = h(s)\beta[h(s), \lambda(r)]_p + \delta(s)\beta[h(s), h(r)]_p\).

Since \(h\) is commuting, by using Theorem 2.12, \((G, +)\) is an abelian, and \(\delta(K) = K\), we obtain that

\([s, r]_p = s\beta[h(s), h(r)]_p = h(s)\beta[h(s), \lambda(r)]_p + \delta(s)\beta[h(s), h(r)]_p, \forall s, r \in K \ \text{and} \ \rho, \ \beta \in \Gamma\).

Hence, \(h(s)\beta[h(s), h(r)]_p = 0, \forall s, r \in K \ \text{and} \ \rho, \ \beta \in \Gamma\).

In particular, \(\forall s, r, t \in K\), we have

\([h(s), t\beta(h(r))]_p = 0 = t\beta[h(s), h(r)]_p\).

Hence,

\(K\beta[h(s), h(r)]_p = 0, \forall s, r \in K \ \text{and} \ \rho, \ \beta \in \Gamma\).

Since \(K\) is a nonzero semi-group ideal of \(G\) and \(G\) is a prime \(\Gamma\)-near-ring, we obtain that

\([h(s), h(r)]_p = 0, \forall s, r \in K \ \text{and} \ \rho \in \Gamma\).

Then, we conclude that \([s, r]_p = 0, \forall s, r \in K \ \text{and} \ \rho \in \Gamma\).

Hence, \(K\) is commutative. By Lemma 2.9, we have \(G\) is a commutative ring.

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