Research Article

Investigating a Class of Generalized Caputo-Type Fractional Integro-Differential Equations

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In this article, we prove some new uniqueness and Ulam-Hyers stability results of a nonlinear generalized fractional integro-differential equation in the frame of Caputo derivative involving a new kernel in terms of another function $\psi$. Our approach is based on Babenko’s technique, Banach’s fixed point theorem, and Banach’s space of absolutely continuous functions. The obtained results are demonstrated by constructing numerical examples.

1. Introduction

It is notable that fractional calculus was and still is a new tool that uses fractional differential and integral equations to construct more modern mathematical models that can precisely describe complex frameworks. There are many definitions of fractional integrals (FIs) and fractional derivatives (FDs) accessible in the literature, for instance, the Riemann-Liouville and Caputo definitions that assumed a significant part in the advancement of the theory of fractional analysis. Referring to all books and papers in this field will be extremely many. In this regard, here, we refer to the most important of main references, e.g., Samko et al. [1] gave a broad comprehensive mathematical handling of fractional derivatives and integrals. Podlubny [2] and Kilbas et al. [3] have been introduced many useful results related to fractional differential equations (FDEs). Several applications have been implemented recently by a wide range of works on this subject, see [4–8].

However, the currently common operator is the generalized FD regarding another function, see [1, 3]. Agrawal [7] studied further various properties for generalized fractional derivatives and integrals. More recently, Almeida [9] inspired an idea of that generalization by projecting this generalization onto the definition of the Caputo fractional derivative with respect to another function $\psi$, so-called $\psi$-Caputo, and introduced many interesting properties, which are more general than the classical Caputo FD. Jarad and Abdeljawad [10] provided interesting properties for generalized FDs and Laplace transform. Specifically, $\psi$-Caputo type FDEs with initial, boundary, and nonlocal conditions have been investigated by many researchers using fixed-point theories, see Almeida et al. [11, 12], Abdo et al. [13], and Wahash et al. [14]. A recent survey on $\psi$-Caputo type FDEs can be found in [15–18]. For more results in this direction, we refer to interesting works provided by Zhang et al. [19], Zhao et al. [20], Baitiche et al. [21], Benchohra et al. [22], Ravichandran et al. [23], Trujillo et al. [24], and Furati et al. [25].
Li in [17, 18] investigated some interesting results of the integral equations and the integro-differential equations involving Hadamard-type. In this work, our goal is to intend to address a general extension of these studies. Precisely, we consider the following $\psi$-Caputo type fractional integro-differential equation (FIDE)

\[
\begin{aligned}
C D^\alpha_a \varphi(x) + a_1 C D^\alpha_a \varphi(x) + a_2 I^\alpha_a \varphi(x) = \int_a^x G\left(\zeta, \varphi'(\zeta)\right) d\zeta, \\
\varphi(a) = 0,
\end{aligned}
\]

where

(i) $0 < \eta_1 < \eta < 1, \eta_2 > 0$, and $a_1, a_2 \in \mathbb{C}$

(ii) The symbol $C D^\alpha_a$ denotes the generalized Caputo FD of order $\sigma \in \{\eta, \eta_1\}$

(iii) The notation $I^\alpha_a$ means the generalized Riemann-Liouville FI of order $\eta_2$

(iv) $G : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and $0 \leq a < b < \infty$

(v) $\varphi \in AC_0[a, b]$ such that $I^\alpha_a$ and $D^\alpha_a$ exist and are both continuous in $[a, b]$

Observe that the considered system (1) covers the previous standard cases of nonlinear FIDEs by defining the kernel, i.e., if $\psi(x) = x$, $\psi(x) = \log(x)$, and $\psi(x) = x^\eta$, then the problem (1) reduces to the Caputo type FIDE, Caputo-Hadamard type FIDE, and Caputo-Katugampola type FIDE, respectively.

The aim of this work is to develop the nonlinear FIDEs. In particular, we investigate the uniqueness and Ulam-Hyers stability of solution for the problem (1) by Banach’s fixed point theorem and Babenko’s technique [26]. Note that the presentation and structuring of the arguments for our problem are new, and our results generalize and cover some of the known results in the literature. In addition, the obtained results here are valid when the left hand side of the considered problem (1) involves many FDs and FIs. For more details, see Remark 22.

The remainder of this paper is organized as follows: in Section 2, we present some important tools related the fractional calculus and the functional spaces, in which we aim to determine our analysis strategies. Section 3 gives the main results and their illustrative examples. Finally, our brief conclusion is included in Section 4.

2. Preliminaries

In this section, we present some properties, lemmas, definitions, and important estimations needed in the proof of our result.

Defining the Banach space as

\[
AC_0[a, b] = \left\{ \varphi : \varphi \in AC[a, b] \text{ with } \varphi(a) = 0 \text{ and } \|\varphi\|_0 = \int_a^b |\varphi'(\zeta)| d\zeta < \infty \right\}.
\]

Next, we present some important definitions and properties of advanced fractional calculus.

**Definition 1** [3, 9]. The $\psi$-Riemann-Liouville FI and $\psi$-Caputo FD are defined by

\[
I^\alpha_a \psi(x) = \frac{1}{\Gamma(\psi)} \int_a^x (x - \zeta)^{\psi-1} \psi(\zeta) \omega(\zeta) d\zeta, \quad \psi > 0, \quad \alpha > 0, \quad a \geq 0,
\]

\[
D^\alpha_a \psi(x) = \frac{1}{\Gamma(n - \psi)} \int_a^x (x - \zeta)^{n-\psi-1} \psi' \omega(\zeta) d\zeta, \quad \psi > 0, \quad \alpha < n, \quad a \geq 0,
\]

respectively, where

\[
n = \lfloor -\psi \rfloor, \quad \omega\psi(x) = \left( \frac{1}{\psi(x)} \right)^n \omega(x).
\]

**Definition 2** [27]. The incomplete gamma function is represented by

\[
\gamma(\zeta, \psi) = \int_0^\infty e^{-u} u^{\psi-1} \omega u^{\zeta-1} du = \zeta e^{-\zeta} \Gamma(\psi) e^{-\zeta} \sum_{n=0}^{\infty} \frac{\zeta^n}{\Gamma(\psi + n + 1)}, \quad \psi > 0, \quad \zeta > 0.
\]

**Property 3** [3, 9]. Let $\eta \geq 0$, and $\kappa > 0$. Then

\[
I^\eta_a \varphi(x) - \psi(a)) = \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa + \eta + 1)} (\psi(x) - \psi(a))^{\kappa+\eta}, x > a,
\]

\[
D^\eta_a \varphi(x) - \psi(a)) = \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa + \eta - 1)} (\psi(x) - \psi(a))^{\kappa-\eta}, x > a.
\]

**Property 4** [3, 9]. Let $\eta, \kappa > 0$, and $\omega \in AC_0[a, b]$. Then

\[
C D^\eta_a I^\eta a \varphi(\zeta) = \varphi(\zeta), \quad \zeta > a,
\]

\[
C D^\eta_a I^\eta a \varphi(\zeta) = I^\eta a \varphi(\zeta), \quad \zeta > a,
\]

\[
C D^\eta_a I^\eta a \varphi(\zeta) = I^\eta a \varphi(\zeta), \quad \zeta > a,
\]

In the following, some very significant lemmas will be given.

**Lemma 5.** Let $\eta, \kappa \in [0, 1]$. If $\omega \in AC_0[a, b]$, then

\[
I^\eta a C D^\eta a \varphi(\zeta) = \varphi(\zeta), \quad \zeta > a,
\]

\[
I^\eta a C D^\eta a \varphi(\zeta) = I^\eta a \varphi(\zeta), \quad \zeta > a.
\]
Proof. Let $\omega \in AC_0[a, b]$. Then by Definition 1 and Property 4, we get

$$
I_{a}^{\psi}e^{\psi}\omega(\zeta) = I_{a}^{\psi}e^{\psi,1-0}I_{a}^{\psi,1-0}(\psi(\zeta) - \omega(a))
= \omega(\zeta), \zeta > a,
$$

(14)

$$
I_{a}^{\psi}e^{\psi}\omega(\zeta) = I_{a}^{\psi}e^{\psi,1-k}I_{a}^{\psi,1-k}(\psi(\zeta) - \omega(a)) = I_{a}^{\psi}e^{\psi}\omega(\zeta), \psi, k, \zeta > a.
$$

(15)

Lemma 7. Let $Q > 0$. Then, $I_{a}^{\psi}$ is bounded from $AC_0[a, b]$ into itself, and

$$
\|I_{a}^{\psi}\omega\|_0 \leq \frac{1}{(Q + 1)^{\alpha}}(\psi(b) - \psi(a))\|\omega\|_0.
$$

(16)

Proof. Let $\omega \in AC_0[a, b]$. Then

$$
\omega(\zeta) = \int_{a}^{\zeta} \omega(s)ds = \int_{a}^{\zeta} z(s)ds, \text{ where } z(\zeta) = \omega(\zeta) \text{ and } \omega(a) = 0.
$$

(17)

By virtue of Definition 1, we get

$$
I_{a}^{\psi}\omega(x) = I_{a}^{\psi}\left(\int_{a}^{x} z(s)ds\right)\left(\frac{1}{(Q + 1)^{\alpha}}(\psi(x) - \psi(\zeta))\psi'(\zeta)d\zeta\right.
$$

(18)

Taking advantage of the Dirichlet's formula, we have

$$
I_{a}^{\psi}\omega(x) = \frac{1}{(Q + 1)^{\alpha}}\int_{a}^{x} z(s)\left(\frac{\psi(x) - \psi(s)}{Q}\right)^{\alpha}ds
$$

(19)

Now, we will provide and prove the next lemma: 

Lemma 9. If $Q \geq 0$, then

$$
I_{a}^{\psi}e^{\psi}\omega(x) = e^{\psi(a)}(\psi(x) - \psi(a)) \sum_{i=0}^{\infty} \left(\frac{\psi(x) - \psi(a)}{Q}\right)^{i} I_{a}^{\psi}e^{\psi}\omega(\zeta).
$$

(20)

Proof. Using Definition 1, we have

$$
I_{a}^{\psi}e^{\psi}\omega(x) = \frac{1}{(Q + 1)^{\alpha}}\int_{a}^{x} (\psi(x) - \psi(\zeta)) \psi'(\zeta)e^{\psi(\zeta)}d\zeta.
$$

(21)

Performing the substitution $s = \psi(x) - \psi(\zeta)$, we get

$$
I_{a}^{\psi}e^{\psi}(\psi(x)) = \frac{1}{(Q + 1)^{\alpha}}\int_{0}^{s} \frac{\psi(x) - \psi(a)}{Q}s^{-1}e^{\psi(s)}d\zeta.
$$

(22)

From Definition 2, we obtain

$$
I_{a}^{\psi}e^{\psi}\omega(\zeta) = \frac{1}{(Q + 1)^{\alpha}}\int_{0}^{s} \left(\frac{\psi(x) - \psi(a)}{Q}\right)^{i} I_{a}^{\psi}e^{\psi}\omega(\zeta).
$$

(23)

3. Main Results

Theorem 11. Let $a_i \in C(i = 1, 2), 0 \leq q_1 < q < 1, \text{ and } Q_2 > 0$. If $h \in AC_0[a, b]$, then, the following linear problem

$$
\left\{ C_{D_{a}}^{q_1}q_iq_iq_iq_ih(x) + a_{i}C_{D_{a}}^{q_1}q_iq_iq_iq_ih(x) + a_{i}D_{a}^{q_1}q_iq_iq_iq_iq_ih(x) = h(x),\right.
$$

(24)

has a solution in the space $AC_0[a, b]$, that is

$$
q_i(x) = \sum_{i=0}^{\infty} (-1)^{q} \sum_{\ell_i, \ell_2 = 0}^{Q} \left(\frac{\psi(x) - \psi(\zeta)}{Q}\right)^{\ell} a_{i}a_{i} \frac{\psi(x) - \psi(\zeta)}{Q}\left\{ a_{i}D_{a}^{q_1}q_iq_iq_iq_iq_ih(x).
$$

(25)

Proof. Applying the operator $I_{a}^{\psi}$ to both sides of Eq. (24), we obtain

$$
I_{a}^{\psi}C_{D_{a}}^{q_1}q_iq_iq_iq_iq_ih(x) + a_{i}I_{a}^{\psi}C_{D_{a}}^{q_1}q_iq_iq_iq_iq_ih(x) + a_{i}I_{a}^{\psi}D_{a}^{q_1}q_iq_iq_iq_iq_ih(x) = I_{a}^{\psi}h(x).
$$

(26)

According to Lemma 5, we find that

$$
q_i(x) + a_{i}I_{a}^{\psi}q_iq_iq_iq_iq_iq_ih(x) + a_{i}I_{a}^{\psi}q_iq_iq_iq_iq_iq_ih(x) = I_{a}^{\psi}h(x).
$$

(27)

Observe that $q_i(a) = 0$ and $0 < q_1 < q < 1$. It follows that

$$
(1 + a_{i}I_{a}^{\psi}q_iq_iq_iq_iq_iq_i + a_{i}I_{a}^{\psi}q_iq_iq_iq_iq_iq_i)q_i(x) = I_{a}^{\psi}h(x).
$$

(28)

In view of Babenko approach, we have

$$
q_i(x) = (1 + a_{i}I_{a}^{\psi}q_iq_iq_iq_iq_iq_i + a_{i}I_{a}^{\psi}q_iq_iq_iq_iq_iq_i)^{-1}I_{a}^{\psi}h(x).
$$

(29)

By using the multinomial theorem and Property 4, we
obtain

\[
\varphi(x) = \sum_{\ell=0}^{\infty} (-1)^{\ell} \int_{\ell+1}^{\infty} (a_{\ell} t^{-\alpha+\nu_1}) t^{\rho \varphi}(x) dt
\]

\[
= \sum_{\ell=0}^{\infty} (-1)^{\ell} \sum_{\ell_1, \ell_2+\ell} \left( a_{\ell} t^{-\alpha+\nu_1} \right) \int_{\ell_1}^{\infty} (a_{\ell_2} t^{-\alpha+\nu_2}) t^{\rho \varphi}(x) dt
\]

\[
= \sum_{\ell=0}^{\infty} (-1)^{\ell} \sum_{\ell_1, \ell_2+\ell} \left( a_{\ell} a_{\ell_2} t^{-\alpha+\nu_1+\nu_2} \right) (a_{\ell_2} t^{-\alpha+\nu_2}) t^{\rho \varphi}(x).
\]  

(30)

As \( x \to a \), we get \( \varphi(a) = 0 \). Now, we need to prove the series is absolutely continuous on \([a, b]\) and converges in the space \( AC^0[a, b] \). Indeed, by Lemma 7, we have

\[
\left\| \int_{a}^{x} (a_{\ell}(t-q)) t^{\rho \varphi}(x) \right\|_{0} \leq \eta \|h\|_{0}.
\]  

(31)

where

\[
\eta = \frac{(\psi(b) - \psi(a)) t^{\rho \varphi}(x) + \epsilon_2 (t^{\rho \varphi}(x) + \epsilon) + 1}{1 + \epsilon_1 (t^{\rho \varphi}(x) + \epsilon_2 (t^{\rho \varphi}(x) + \epsilon) + 1)}.
\]  

(32)

It follows that

\[
\|\varphi\| \leq \eta \sum_{\ell=0}^{\infty} \sum_{\ell_1, \ell_2+\ell} \left( a_{\ell} \right) \left| (a_{\ell}) t^{-\alpha+\nu_1} \right| \left( (a_{\ell_2}) t^{-\alpha+\nu_2} \right) \left( (t^{\rho \varphi}(x) + \epsilon_2 (t^{\rho \varphi}(x) + \epsilon) + 1) \right)^{1}
\]

\[
= \eta \sum_{\ell=0}^{\infty} \sum_{\ell_1, \ell_2+\ell} \left( a_{\ell} \right) \left| (a_{\ell}) t^{-\alpha+\nu_1} \right| \left( (a_{\ell_2}) t^{-\alpha+\nu_2} \right) \left( (t^{\rho \varphi}(x) + \epsilon_2 (t^{\rho \varphi}(x) + \epsilon) + 1) \right)^{1}
\]

\[
= \eta \frac{E_{(\rho \varphi, \rho \varphi, \epsilon \varphi_2)}}{1 + \epsilon_1 (t^{\rho \varphi}(x) + \epsilon_2 (t^{\rho \varphi}(x) + \epsilon) + 1)}
\]

(33)

where

\[
E_{(\rho \varphi, \rho \varphi, \epsilon \varphi_2)} \left( (a_{\ell_1}) t^{\rho \varphi}(x) - \psi(a) \right) < \infty,
\]

(34)

which is the value at \( v_1 = \psi(b) - \psi(a) \) and \( v_2 = \psi(b) - \psi(a) \) of the multivariate Mittag-Leffler function \( E_{(\rho \varphi, \rho \varphi, \epsilon \varphi_2)}(v_1, v_2) \) given in [3]. So, we conclude that the series to the right of Eq. (25) is convergent. Obviously, \( \varphi(x) \in AC[a, b] \) due to \( h \in AC[a, b] \). To affirm that the obtained series could be a solution, we must see that

\[
\text{it fulfills Eq. (24), i.e.,}
\]

\[
C D_{\alpha}^{\rho \varphi} \sum_{\ell=0}^{\infty} (-1)^{\ell} \sum_{\ell_1, \ell_2+\ell} \left( a_{\ell} a_{\ell_2} t^{\rho \varphi}(x) \right) + c_{D_{\alpha}^{\rho \varphi}} \left( a_{\ell} a_{\ell_2} t^{\rho \varphi}(x) \right) + a_{\ell} a_{\ell_2} t^{\rho \varphi}(x) = 0,
\]

(35)

by the cancellation. Notice that each series is absolutely convergent and also the term arrangements are possibly cancelled. In fact,

\[
- \sum_{\ell_1, \ell_2+\ell} \left( a_{\ell_1} a_{\ell_2} t^{\rho \varphi}(x) \right) + \sum_{\ell_1, \ell_2+\ell} \left( a_{\ell_1} a_{\ell_2} t^{\rho \varphi}(x) \right) + \sum_{\ell_1, \ell_2+\ell} \left( a_{\ell_1} a_{\ell_2} t^{\rho \varphi}(x) \right) = 0.
\]

(36)

The remainder terms cancel each other similarly. Plainly, the uniqueness follows promptly from the fact that the FIDE

\[
C D_{\alpha}^{\rho \varphi} \varphi(x) + a_{\ell} C D_{\alpha}^{\rho \varphi} \varphi(x) + a_{\ell} a_{\ell_2} t^{\rho \varphi}(x) = 0,
\]

(37)

only has solution zero due to the Babenko approach. \( \square \)

Remark 13. Notice that the solution of Eq. (24) in \( AC[a, b] \) is stable, if for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( \|\varphi\| < \epsilon \)
with \( \|h\|_0 < \delta \). Taking advantage of the following inequality
\[
\|\varphi\|_0 \leq \eta E_{[0,0,0,0,0]}(\|a_1(\psi(b) - \psi(a))^{e^{\varphi_1}}, a_2(\psi(b) - \psi(a))^{e^{\varphi_2}}\|)\|h\|_0,
\]
we conclude that \( \varphi \) is stable.

**Example 1.** The following \( \psi \)-Caputo type FIDE
\[
C G\psi^{0.9} \varphi(x) + 2 C G\psi^{0.7} \varphi(x) - E\psi\varphi(x) = (\psi(x) - \psi(a))^k,
\]
has the solution in \( AC_0[a, b] \), that is
\[
\varphi(x) = \sum_{\ell=0}^{\infty} (-1)^\ell \sum_{\ell_1 + \ell_2 = \ell} \left( \frac{\ell}{\ell_1, \ell_2} \right) (2)^{\ell_1}(-1)^{\ell_2} \\
\times \Gamma(k + 1) \Gamma(k + 0.2\ell_1 + 1.3\ell_2 + 1.9) (\psi(x) - \psi(a))^k.
\]
(39)

So, according to Theorem 11, we have
\[
\varphi(x) = \sum_{\ell=0}^{\infty} (-1)^\ell \sum_{\ell_1 + \ell_2 = \ell} \left( \frac{\ell}{\ell_1, \ell_2} \right) (2)^{\ell_1}(-1)^{\ell_2} \\
\times \Gamma(k + 1) \Gamma(k + 0.2\ell_1 + 1.3\ell_2 + 1.9) (\psi(x) - \psi(a))^k.
\]
(40)

**Example 2.** The following \( \psi \)-Caputo type FIDE
\[
C G\psi^{0.8} \varphi(x) + C G\psi^{0.7} \varphi(x) - 3 I\psi^{0.2} \varphi(x) = \varphi(x),
\]
has the solution in \( AC_0[a, b] \) described as
\[
\varphi(x) = \psi(a) \sum_{\ell=0}^{\infty} (-1)^\ell \sum_{\ell_1 + \ell_2 = \ell} \left( \frac{\ell}{\ell_1, \ell_2} \right) (3)^{\ell_1} \psi(x) \\
- \psi(a))^{0.1\ell_1 + \ell_2 + 0.8} \times \sum_{\ell=0}^{\infty} \Gamma(0.1\ell_1 + \ell_2 + 1.8 + i).
\]
(43)
\( (44) \)

So, as stated by Theorem 11, we obtain
\[
\varphi(x) = \sum_{\ell=0}^{\infty} (-1)^\ell \sum_{\ell_1 + \ell_2 = \ell} \left( \frac{\ell}{\ell_1, \ell_2} \right) (1)^{\ell_1}(3)^{\ell_2} \psi^{0.1\ell_1 + \ell_2 + 0.8} \psi(x).
\]
(45)

By virtue of Lemma 9, we obtain
\[
\varphi(x) = \sum_{\ell=0}^{\infty} (-1)^\ell \sum_{\ell_1 + \ell_2 = \ell} \left( \frac{\ell}{\ell_1, \ell_2} \right) (3)^{\ell_2} \psi^{0.1\ell_1 + \ell_2 + 0.8} \times \sum_{\ell=0}^{\infty} \Gamma(0.1\ell_1 + \ell_2 + 1.8 + i).
\]
(46)

The uniqueness result of Eq. (1) will be proved through the following theorem.

**Theorem 14.** Let \( G : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R} \) be a continuous function, and assume that there exists a constant \( C \) such that
\[
|G(x, \varphi_1) - G(x, \varphi_2)| \leq C|\varphi_1 - \varphi_2|, x \in [a, b], \varphi_1, \varphi_2 \in \mathbb{R}.
\]
(47)

Then, the problem (1) has a unique solution in \( AC_0[a, b] \).

**Proof.** Consider the operator \( \mathfrak{S} \) on \( AC_0[a, b] \) defined by
\[
\mathfrak{S} (\varphi) = \sum_{\ell=0}^{\infty} (-1)^\ell \sum_{\ell_1 + \ell_2 = \ell} \left( \frac{\ell}{\ell_1, \ell_2} \right) a_1^{\ell_1} a_2^{\ell_2} \\
\times \Gamma(\ell_1 + 0.2\ell_1 + 1.3\ell_2 + 1.9) (\psi(x) - \psi(a))^k.
\]
(49)

For \( \varphi \in AC_0[a, b] \), we have \( \int_a^b G(\zeta, \varphi'(\zeta))d\zeta \in AC_0[a, b] \), since \( \varphi'(\zeta) \in L(a, b) \) and \( G(\zeta, \varphi'(\zeta)) \in L(a, b) \). Hence,
\[
\left\| \int_a^b G(\zeta, \varphi'(\zeta))d\zeta \right\| = \int_a^b \left| G(x, \varphi'(\zeta)) \right| dx \leq \int_a^b \left| G(x, \varphi'(\zeta)) \right| dx + \int_a^b |G(x, \varphi'(\zeta))| dx + \int_a^b |G(x, \varphi'(\zeta))| dx < \infty.
\]
(50)

Using the inequality (39), we obtain
\[
\left\| \mathfrak{S} (\varphi) \right\|_0 < \infty \text{ and } \mathfrak{S} (\varphi)(a) = 0.
\]
(51)

Besides, \( \mathfrak{S} (\varphi) \) is absolutely continuous on \([a, b]\) via Theorem 11. So, \( \mathfrak{S} : AC_0[a, b] \longrightarrow AC_0[a, b] \). Now, we just have
to show that \( \mathfrak{F} \) is a contraction mapping. Let \( \varphi, \varphi^* \in AC_0[a, b] \). Then

\[
\| \mathfrak{F}(\varphi) - \mathfrak{F}(\varphi^*) \|_0 \leq \eta \mathcal{E}_{[\varphi - \varphi^*]} + \varepsilon
\]

\[
\cdot \left( |a_1|(\psi(b) - \psi(a))^{(\varphi - \varphi^*)}, |a_2|(\psi(b) - \psi(a))^{(\varphi - \varphi^*)} \right)
\]

\[
\times \left| \int_a^\kappa \mathcal{G}(\zeta, \varphi'((\zeta))) d\zeta - \int_a^\kappa \mathcal{G}(\zeta, \varphi'((\zeta))) d\zeta \right|
\]

(52)

Since

\[
\left| \int_a^\kappa \mathcal{G}(\zeta, \varphi'((\zeta))) d\zeta - \int_a^\kappa \mathcal{G}(\zeta, \varphi'((\zeta))) d\zeta \right|
\]

\[
= \int_a^\kappa \mathcal{G}(x, \varphi'(x)) - \mathcal{G}(x, \varphi'(x)) dx
\]

\[
\leq C \int_a^\kappa |\varphi' - \varphi^*| dx = C\|\varphi - \varphi^*\|_0
\]

we obtain

\[
\| \mathfrak{F}(\varphi) - \mathfrak{F}(\varphi^*) \|_0 
\leq C\eta \mathcal{E}_{[\varphi - \varphi^*]} + \varepsilon \times \|\varphi - \varphi^*\|_0.
\]

(54)

Inequality (48) leads us to that \( \mathfrak{F} \) is contraction mapping. \( \square \)

3.1. Ulam-Hyers Stability (UHS). The first results about this type of stability emerged in 1940 by Ulam [28, 29]. From that point forward, the UHS is studied via several researchers. With the vast development of fractional calculus, the studying of stability for FDEs also attracted the numerous authors, see [30–32].

In this regard, we investigate some recent results on the UHS and generalized UHS of (1). For \( \varepsilon > 0, x \in [a, b] \), and \( \varphi_1 \in AC_0[a, b] \), the following inequality

\[
\bar{C}D_\alpha^{q_0} \varphi_1(x) + a_1 \bar{C}D_\alpha^{p_0} \varphi_1(x) + a_2 \bar{C}D_\alpha^{q_0} \varphi_1(x) - \int_a^\kappa \mathcal{G}(\zeta, \varphi'((\zeta))) d\zeta \leq \varepsilon.
\]

(55)

is satisfied.

Remark 16. Let \( \varepsilon > 0 \). Then, \( \varphi_1 \in AC_0[a, b] \) satisfies (55) iff there exists \( \xi(x) \in AC_0[a, b] \) with \( \xi(0) = 0 \) such that

(i) \( \|\xi\|_0 = \int_a^x \xi'((\zeta)) d\zeta \leq \varepsilon \), for \( x \in [a, b] \)

(ii) for \( x \in [a, b] \)

\[
\bar{C}D_\alpha^{q_0} \varphi_1(x) + a_1 \bar{C}D_\alpha^{p_0} \varphi_1(x) + a_2 \bar{C}D_\alpha^{q_0} \varphi_1(x) = \int_a^\kappa \mathcal{G}(\zeta, \varphi'((\zeta))) d\zeta
\]

\[
+ \int_a^\kappa \xi'((\zeta)) d\zeta.
\]

(56)

Lemma 17. The solution of the problem (56) with \( \varphi_1(0) = 0 \) satisfies the following inequality

\[
\|\varphi_1 - Z_G\|_0 \leq \eta \mathcal{E}_{\varphi_1} + \varepsilon\left( |a_1|(\psi(b) - \psi(a))^{(\varphi - \varphi^*)}, |a_2|(\psi(b) - \psi(a))^{(\varphi - \varphi^*)} \right),
\]

(57)

where

\[
Z_G(x) := \sum_{t, \xi, \xi_1, \xi_2, \xi_3 = 0}^{\infty} (-1)^t \sum_{\xi_1, \xi_2, \xi_3 = \ell \xi, \ell_2, \ell_3} \left( \begin{array}{l} \ell \xi \ell_2 \ell_3 \\ \xi_1 \xi_2 \xi_3 \end{array} \right) a_1^\ell \ell_2 \ell_3 \times \bar{C}D_\alpha^{p_0} \xi_1^{(\varphi - \varphi^*)} + \int_a^\kappa \mathcal{G}(\zeta, \varphi'((\zeta))) d\zeta
\]

(58)

and \( \eta \) is defined by (32).

Proof. By virtue of Lemma 8, the solution of Eq. (56) is described as

\[
\varphi_1(x) = \sum_{t, \xi, \xi_1, \xi_2, \xi_3 = 0}^{\infty} (-1)^t \sum_{\xi_1, \xi_2, \xi_3 = \ell \xi, \ell_2, \ell_3} \left( \begin{array}{l} \ell \xi \ell_2 \ell_3 \\ \xi_1 \xi_2 \xi_3 \end{array} \right) a_1^\ell \ell_2 \ell_3 \times \bar{C}D_\alpha^{p_0} \xi_1^{(\varphi - \varphi^*)} + \int_a^\kappa \mathcal{G}(\zeta, \varphi'((\zeta))) d\zeta
\]

(59)

It follows from Eq. (59), Remark 16, and Eq. (38) that

\[
\|\varphi_1 - Z_G\|_0 \leq \eta \mathcal{E}_{\varphi_1} + \varepsilon\left( |a_1|(\psi(b) - \psi(a))^{(\varphi - \varphi^*)}, |a_2|(\psi(b) - \psi(a))^{(\varphi - \varphi^*)} \right)
\]

\[
\times \int_a^\kappa \mathcal{G}(\zeta, \varphi'((\zeta))) d\zeta - \int_a^\kappa \mathcal{G}(\zeta, \varphi'((\zeta))) d\zeta \leq \eta \mathcal{E}_{\varphi_1} + \varepsilon\left( |a_1|(\psi(b) - \psi(a))^{(\varphi - \varphi^*)}, |a_2|(\psi(b) - \psi(a))^{(\varphi - \varphi^*)} \right)
\]

(60)

\[
\square
\]

Theorem 19 (UHS). Suppose that the hypotheses of Theorem 14 with Eq. (55) are satisfied. Then, Eq. (1) is UH stable.

Proof. Assume that \( \varepsilon > 0 \) and \( \varphi_1 \in AC_0[a, b] \) satisfy Eq. (55), and let \( \varphi \in AC_0[a, b] \) be a unique solution of

\[
\bar{C}D_\alpha^{q_0} \varphi_1(x) + a_1 \bar{C}D_\alpha^{p_0} \varphi_1(x) + a_2 \bar{C}D_\alpha^{q_0} \varphi_1(x) = \int_a^\kappa \mathcal{G}(\zeta, \varphi'((\zeta))) d\zeta,
\]

\[
\varphi(a) = \varphi_1(a) = 0.
\]

(61)
that is
\[
\varphi(x) = \varphi(a) + \sum_{\ell=0}^{\infty} (-1)^\ell \left( \frac{\ell}{\ell_1, \ell_2} \right) \times d_1^\ell d_2^\ell \varphi^0_{a^\ell}(1+\ell) + \int_0^x G(a, \varphi'(a)) \, d\zeta.
\]
(62)

Since \( \varphi(a) = \varphi_1(a) = 0 \), we obtain
\[
\varphi(x) = \sum_{\ell=0}^{\infty} (-1)^\ell \left( \frac{\ell}{\ell_1, \ell_2} \right) \times d_1^\ell d_2^\ell \varphi^0_{a^\ell}(1+\ell) + \int_0^x G(a, \varphi'(a)) \, d\zeta.
\]
(63)

According to Lemma 17 and (38), we get
\[
\|\varphi_1 - \varphi\|_0 \leq \|\varphi_1 - \varphi_0\|_0 + \|\varphi_0 - \varphi\|_0 \leq \eta E_{(t,0,0,0,0,0)} \left[ \begin{array}{c} \eta_1 \left| \psi(b) - \psi(a) \right|^{(\alpha_0, 1)} \end{array} \right] \epsilon
\]
(64)

Using the assumption of Theorem 14, we have
\[
\left\| \int_a^x G(\zeta, \varphi'_{a^\ell}(\zeta)) \, d\zeta - \int_a^x G(\zeta, \varphi'(\zeta)) \, d\zeta \right\|_0 \leq C \|\varphi_1 - \varphi\|_0.
\]
(65)

So,
\[
\|\varphi_1 - \varphi\|_0 \leq \eta E_{(t,0,0,0,0,0)} \left[ \begin{array}{c} \eta_1 \left| \psi(b) - \psi(a) \right|^{(\alpha_0, 1)} \end{array} \right] \epsilon
\]
(66)

From the inequality (48), we find that
\[
\|\varphi_1 - \varphi\|_0 \leq C_G \epsilon,
\]
(67)

where \( C_G = \mathcal{R}/1 - \mathcal{R} \mathcal{C} \) and
\[
\mathcal{R} = \eta E_{(t,0,0,0,0,0)} \left[ \begin{array}{c} \eta_1 \left| \psi(b) - \psi(a) \right|^{(\alpha_0, 1)} \end{array} \right] \epsilon.
\]
(68)

Corollary 21. Under assumptions of Theorem 19, if we put \( \Phi(\epsilon) = C_G \epsilon \) along with \( \Phi(0) = 0 \), then Eq. (1) is a generalized UH stable.
Let us choose a positive $C$ such that

$$C < \frac{1}{0.04 + \varepsilon}.$$  \hspace{1cm} (76)

there exists a unique solution $\varphi(x) \in AC_0[a, b]$ of Eq. (69) such that

$$\|\varphi_1 - \varphi\|_0 \leq C_G \varepsilon,$$  \hspace{1cm} (78)

where $C_G = 8/1 - 8R > 0$, $R = \eta/0.04 + \varepsilon$ and $\eta = 1/\ell(i_0 + 0.9 + 1.5\varepsilon_i + 1.9)$. Consequently, Eq. (69) is UH stable.

\[
\begin{align*}
C_{\alpha,a}D^{\psi,a}_{\alpha} \varphi(x) + a_1 C_{\alpha,a}D^{\psi,a}_{\alpha} \varphi(x) + \ldots + a_n C_{\alpha,a}D^{\psi,a}_{\alpha} \varphi(x) + b_{n+1} I^{\psi,k_{n+1}}_{\alpha,a} \varphi(x) + b_{n+2} I^{\psi,k_{n+2}}_{\alpha,a} \varphi(x) + \ldots + b_m I^{\psi,k_m}_{\alpha,a} \varphi(x) &= h(x),
\end{align*}
\]

has a solution

$$\varphi(x) = \sum_{\ell=0}^{\infty} (-1)^{\ell} \sum_{\ell_1+\ell_2+\cdots+\ell_m=\ell} a_{\ell,\ell_1,\ell_2,\ldots,\ell_m} \psi(x) \times \prod_{i=1}^{m} I^{\psi,k_i}_{\alpha,a}(\alpha_a) + \cdots + \prod_{i=\ell+1}^{n+m} \psi(x) = h(x).$$  \hspace{1cm} (80)

where $C_{\alpha,a}D^{\psi,a}_{\alpha} \varphi(x)$ is the $\psi$-Caputo FD of order $\delta(>0) \in \{a, \alpha_i; i = 1, \ldots, n\}$ and $I^{\psi,k}_{\alpha,a} \varphi(x)$ is generalized FI of order $\sigma(>0) \in \{k_j; j = n+1, \ldots, m\}$.

4. Conclusions

$\psi$-Caputo FD, a general fractional operator, is of great use because of its wide freedom to cover many classical fractional operators. In this work, we have studied the uniqueness of solution for the nonlinear $\psi$-Caputo type FIDE (1) by using the Banach space $AC_0[a, b]$, Banach’s fixed point theorem, and Babenko’s method. Moreover, the UH stability results to the proposed problem have been discussed. Also, some pertinent examples have been provided to justify the main results. The obtained results in this study extended and developed the current results introduced by [17, 18]. We have already concluded that our results are valid when the left-hand side of the considered problem (1) involves many FDs and IDs as shown in Remark 22. Furthermore, problem (1) covers previous standard cases of nonlinear FDEs and FIDEs by selecting the suitable standard kernel in the studied problem. More specifically, our results generalize some known results in literature like those that include Hammad and Katugampola FDs.

For future research, we will consider a class of nonlinear FIDEs with the fuzzy initial conditions in a fractional case. It would also be interesting to study the same results for our current problem under the $\psi$-Hilfer operator [33] or Atangana-Baleanu operator [8].

Data Availability

The data of this study were used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors made equal contributions and read and supported the last manuscript.
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References

[1] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Longhorne, PA, 1993.

[2] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, “Theory and applications of fractional differential equations,” in North-Holland Mathematics Studies 204, Elsevier, Amsterdam, 2006.

[4] V. Kiryakova, Generalized Fractional Calculus and Applications, Longman & J. Wiley, Harlow, New York, 1994.

[5] Y. Luchko, “Operational rules for a mixed operator of the Erdélyi-Kober type,” Fractional Calculus and Applied Analysis, vol. 7, no. 3, pp. 339–364, 2004.

[6] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[7] O. P. Agrawal, “Some generalized fractional calculus operators and their applications in integral equations,” Fractional Calculus and Applied Analysis, vol. 15, no. 4, pp. 700–711, 2012.

[8] A. Atangana and D. Baleanu, “New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model,” Thermal Science, vol. 20, no. 2, pp. 763–769, 2016.

[9] R. Almeida, “A Caputo fractional derivative of a function with respect to another function,” Communications in Nonlinear Science and Numerical Simulation, vol. 44, pp. 460–481, 2017.

[10] F. Jarad and T. Abdeljawad, “Generalized fractional derivatives and Laplace transform,” Discrete & Continuous Dynamical Systems-S, vol. 13, no. 3, pp. 709–722, 2020.

[11] R. Almeida, A. B. Malinowska, and M. T. T. Monteiro, “Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications,” Mathematical Methods in the Applied Sciences, vol. 41, no. 1, pp. 336–352, 2018.

[12] R. Almeida, “Fractional differential equations with mixed boundary conditions,” Bulletin of the Malaysian Mathematical Sciences Society, vol. 42, no. 4, pp. 1687–1697, 2019.

[13] M. D. Kassim, T. Abdeljawad, W. Shatanawi, S. M. Ali, and M. S. Abdo, “A qualitative study on generalized Caputo fractional integro-differential equations,” Advances in Difference Equations, vol. 2021, no. 1, 2021.

[14] H. A. Wahash, M. S. Abdo, S. K. Panchal, and S. P. Bhairat, “Existence of solution for Hilfer fractional differential problem with nonlocal boundary condition in Banach spaces,” Studia Universitatis Babeş-Bolyai Mathematica, vol. 66, no. 3, pp. 521–536, 2021.

[15] H. A. Wahash, M. S. Abdo, and S. K. Panchal, “Existence and Ulam-Hyers stability of the implicit fractional boundary value problem with \( \psi \)-Caputo fractional derivative,” Journal of Applied Mathematics and Computational Mechanics, vol. 19, no. 1, pp. 89–101, 2020.

[16] A. Boutiara, M. S. Abdo, M. A. Alqudah, and T. Abdeljawad, “On a class of Langevin equations in the frame of Caputo function-dependent-kernel fractional derivatives with antiperiodic boundary conditions,” AIMS Mathematics, vol. 6, no. 6, pp. 5518–5534, 2021.

[17] C. Li, “Uniqueness of the Hadamard-type integral equations,” Advances in Difference Equations, vol. 2021, no. 1, 2021.

[18] C. Li, “On the nonlinear Hadamard-type integro-differential equation,” Fixed Point Theory Algorithms Sciences and Engineering, vol. 2021, no. 1, pp. 1–15, 2021.

[19] J. Zhang, B. Ahmad, G. Wang, and R. P. Agarwal, “Nonlinear fractional integro-differential equations on unbounded domains in a Banach space,” Journal of Computational and Applied Mathematics, vol. 249, pp. 51–56, 2013.

[20] K. Zhao, L. Suo, and Y. Liao, “Boundary value problem for a class of fractional integro-differential coupled systems with Hadamard fractional calculus and impulses,” Boundary Value Problems, vol. 2019, no. 1, 2019.

[21] Z. Baitiche, C. Derbazi, and M. Benchohra, “\( \psi \)-Caputo fractional differential equations with multi-point boundary conditions by topological degree theory,” Nonlinear Analysis, vol. 3, no. 4, pp. 167–178, 2020.

[22] M. Benchohra, S. Bouriah, and J. J. Nieto Roig, “Existence and Ulam stability for nonlinear implicit differential equations with Riemann-Liouville fractional derivative,” Demonstration Mathematica, vol. 52, no. 1, pp. 437–450, 2019.

[23] C. Ravichandran, K. Logeswari, and F. Jarad, “New results on existence of solutions of fractional integro-differential equations,” Fractional Calculus and Applied Analysis, vol. 15, no. 1, pp. 44–69, 2012.

[24] K. M. Furati, M. D. Kassim, and N.-E. Tatar, “Existence and uniqueness for a problem involving Hilfer fractional derivative,” Computers & Mathematics with Applications, vol. 64, no. 6, pp. 1616–1626, 2012.

[25] Y. I. Babenko, Heat and Mass Transfer, Khimiya, Leningrad (in Russian), 1986.

[26] C. M. Aslam and S. M. Zuhair, “Extended gamma and digamma functions,” Fractional Calculus and Applied Analysis, vol. 4, pp. 303–326, 2001.

[27] S. M. Ulam, A Collection of the Mathematical Problem, Interscience, New York, 1960.

[28] D. H. Hyers, “On the stability of the linear functional equation,” Proceedings of the National Academy of Sciences of the United States of America, vol. 27, pp. 222–224, 1941.

[29] J. V. C. Sousa and C. E. Oliveira, “Ulam-Hyers stability of a nonlinear fractional Volterra integro-differential equation,” Applied Mathematics Letters, vol. 81, pp. 50–56, 2018.

[30] A. Khan, K. Shah, Y. Li, and T. S. Khan, “Ulam type stability for a coupled system of boundary value problems of nonlinear fractional differential equations,” Journal of Function Spaces, vol. 2017, Article ID 3046013, 8 pages, 2017.

[31] M. S. Abdo, K. Shah, S. K. Panchal, and H. A. Wahash, “Existence and Ulam stability results of a coupled system for terminal value problems involving \( \psi \)-Hilfer fractional operator,” Advances in Difference Equations, vol. 2020, no. 1, 2020.

[32] J. V. C. Sousa and C. E. Oliveira, “On the \( \psi \) -Hilfer fractional derivative,” Communications in Nonlinear Science and Numerical Simulation, vol. 60, pp. 72–91, 2018.