The proximity force approximation for the Casimir energy of plate-sphere and sphere-sphere systems in the presence of one extra compactified universal dimension

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Abstract

The Casimir energies for plate-sphere system and sphere-sphere system under PFA in the presence of one extra compactified universal dimension are analyzed. We find that the Casimir energy between a plate and a sphere in the case of sphere-based PFA is divergent. The Casimir energy of plate-sphere system in the case of plate-based PFA is finite and keeps negative. The extra-dimension corrections to the Casimir energy will be more manifest if the sphere is larger or farther away from the plate. It is shown that the negative Casimir energy for two spheres is also associated with the sizes of spheres and extra space. The larger spheres and the longer distance between them make the influence from the additional dimension stronger.

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I. Introduction

The existence of a very peculiar effect named as Casimir effect was predicted in 1948. This effect shows the attraction between two metallic uncharged parallel plates in vacuum. The existence of such an attraction has been proved experimentally and the measurements of the Casimir force have reached a higher accuracy [2-9]. It should be pointed out that the simple device consisting of parallel plates poses a certain degree of difficulty in the experiment because of the difficulty in achieving parallelism although a plenty of exact results for Casimir effect of parallel plates were obtained theoretically. In fact the experimentalists favour the configurations such as sphere-plate or sphere-sphere system etc. The straightforward computations of geometries involving curved surfaces instead of parallel plates are conceptually complicated. The proximity force approximation (PFA) originated to give rise to a theoretical prediction for the Casimir energy of the setup like sphere-plate or sphere-sphere system as a function of the separation between the two components becomes a standard tool for estimating the curvature effects for nonplanar geometries in all experiments [10, 11]. Some powerful methods have been developed to compute these Casimir interactions further. These approaches include the semiclassical approximation [12, 13], the optical path method [14-16], the world line approach [17-20], the functional determinant or the multiple scattering method [21-28] and the exact mode summation method [29, 30]. The above issues are used to compute the corrections to the PFA. These methods have been applied to derive the Casimir interactions of the geometric configurations such as sphere-plane system [12-18, 20-22, 28], sphere-sphere configurations [13, 22, 24-27] etc. in the four-dimensional spacetime.

In order to unify the interactions in nature, the Kaluza-Klein model was put forward about eight decades ago [31,32]. This theory introduced an additional compactified dimension to unify gravity and classical electrodynamics. The string theory is also developed to unify the quantum mechanics and gravity with the help of introducing seven extra spatial dimensions. The gauge fields may be localized on a four-dimensional brane, our real universe, and only gravitons can propagate in the extra space transverse to the brane [33, 34]. The existence and order of the additional dimensions need to be explored both theoretically and experimentally. The Casimir effect could open a window to observe the extra space. The Casimir effects of the devices such as parallel plates and cavities have been studied in the presence of extra compactified dimensions and the influences from the extra dimensions are shown [35-43].

It is fundamental to investigate the Casimir energies of the sphere-plane and sphere-sphere systems in the world with additional compactified dimension with the help of PFA. The known Casimir energy could lead the further study of Casimir effect. As the first step we choose the Kaluza-Klein model with one extra dimension. We have to discuss the Casimir energy of system such as sphere-plane or sphere-sphere ones in this model according to the easily-established experiments although the theoretical exploration is complicated. As the first step the PFA will be used to estimate the Casimir energy between one sphere and a plate or two spheres in the five-dimensional spacetime and the corrections from the additional compactified dimension must appear. The PFA should be
adequate because the technique gives rise to the leading terms in the derivative expansion of the Casimir energy of nonplanar geometries. The PFA results can afford the acceptable description of the properties. Further the PFA results can be improved. We open a window to probe the extra dimensions in the Casimir effect for nonplanar system except for parallel plates. The purpose of this paper is to reexamine the PFA for the Casimir energy of sphere-plate and sphere-sphere systems in the universe involving the fifth compactified spatial dimension. We wonder how the fifth dimension modifies the Casimir energy. We sum up the Casimir energies for a serious of parallel plates to obtain the Casimir energy of the system under PFA. In particular we can wonder which method of plate-based PFA or sphere-based PFA is adequate for the sphere-plate case in the Kaluza-Klein model. We also compare the Casimir energy in the presence of one additional compactified with that in the four-dimensional spacetime in order to exhibit the extra-dimension corrections theoretically. These energies belong to the kinds of system consisting of plate and sphere or two spheres respectively. The discussions and conclusions are emphasized in the end.

II. The PFA for the Casimir energy of plate-sphere system in the presence of one extra compactified universal dimension

Within the frame of Kaluza-Klein approach the scalar field satisfying the Dirichlet boundary conditions on the parallel plates was studied in the spacetime with only one extra dimension, the Casimir energy density is \[\varepsilon_C(R) = -\frac{\pi^2}{720} \frac{1}{R^3} + \frac{1}{16\pi^5} \Gamma(2)\zeta(4) \frac{1}{L^3} - \frac{1}{16\pi^5} \frac{1}{\Gamma(\frac{5}{2})\zeta(5)} R \frac{1}{L^4} - \frac{1}{4\pi^2} \frac{1}{RL^2} \sum_{n_1, n_2=1}^{\infty} \frac{(n_2)}{n_1}^2 K_2(\frac{2R}{L} n_1 n_2)\] (1)

where \(R\) is the separation of the two parallel plates and \(L\) is the radius of the extra dimension. \(K_\nu(z)\) is the modified Bessel functions of the second kind and falls exponentially with \(z\). This Casimir energy density can be inserted into the surface integral to obtain the Casimir energy of two arbitrary smooth surfaces under PFA like [9, 22],

\[\varepsilon_{PFA} = \int \int_A \varepsilon_C[z(\sigma)] d\sigma\] (2)

where the scalar field also obey the Dirichlet boundary conditions on the smooth surfaces. \(A\) stands for the area of one of the opposing surfaces which are locally separated by the distance \(z(\sigma)\). \(\varepsilon_C[z(\sigma)]\) represents the corresponding Casimir energy density and is also surface-dependent. It should be pointed out that the local distance vector \(\vec{z}(\sigma)\) is perpendicular only to the plate segment \(d\sigma\) which is tangential to only one of the surfaces.

In the case of two surfaces with different shapes, the Casimir energy \(\varepsilon_{PFA}\) will not be uniquely defined. For example the surfaces in the sphere-plate system are different, so there will be two kinds of expressions of Casimir energy for the sphere-based PFA and plate-based PFA respectively. In the case of sphere-based PFA, the Casimir energy expression is [22].
\[ E_{S0} = -\frac{\pi^3}{720} \frac{a}{d^2} \left\{ 1 - \frac{3d}{a} - 6 \left( \frac{d}{a} \right)^2 [1 - (1 + \frac{d}{a}) \ln(1 + \frac{a}{d})] \right\} \]

On the other hand, the Casimir energy for the plate-based PFA was expressed as [22],
\[ E_{P0} = -\frac{\pi^3}{720} \frac{a}{d^2} \frac{1}{1 + \frac{d}{a}} \]

where \( a \) is the radius of sphere and \( d \) is the shortest distance between the sphere and the plate.

We start to discuss the extra-dimension corrections to the Casimir energy of plate-sphere system with PFA. We let the Casimir energy density in Eq. (1) replace the integrand in Eq. (2). In the case of sphere-based PFA, the Casimir energy is,
\[ E_S = \int \int_{\text{half-sphere}} \varepsilon C(\vec{z}) a^2 d\Omega \]
\[ = -\frac{\pi^3}{720} \frac{\xi}{\mu^2} \left\{ 1 - \frac{3\mu}{\xi} - 6 \left( \frac{\mu}{\xi} \right)^2 [1 - (1 + \frac{\mu}{\xi}) \ln(1 + \frac{\xi}{\mu})] \right\} \frac{1}{L} \]
\[ + \frac{1}{8\pi^4} \Gamma(2) \zeta(4) \xi^2 \frac{1}{L} \]
\[ - \frac{1}{2\pi} \frac{\xi^2}{L} \sum_{n_1,n_2=1}^{\infty} \left( \frac{n_2}{n_1} \right)^2 \int_{0}^{1} \frac{x}{\mu + \xi - \xi x} K_2(2n_1n_2\mu + \xi - \xi x)x dx \]
\[ + \frac{1}{8\pi^4} \Gamma(5) \zeta(5) \xi^2 \left[ \int_{0}^{\frac{\pi}{2}} (\xi - \frac{\mu + \xi}{\cos \theta}) \sin \theta d\theta \right] \frac{1}{L} \]

with the choice of \( |\vec{z}| = \frac{d+a}{\cos \theta} - a \) and here
\[ \xi = \frac{a}{L} \]

and
\[ \mu = \frac{d}{L} \]

It is clear that the last term in the equation above,
\[ \frac{1}{8\pi^4} \Gamma(5) \zeta(5) \xi^2 \left[ \int_{0}^{\frac{\pi}{2}} (\xi - \frac{\mu + \xi}{\cos \theta}) \sin \theta d\theta \right] \frac{1}{L} = \infty \]

is divergent. The Casimir energy of plate-sphere system under the sphere-based PFA certainly approaches the infinity because of the divergent term. We introduce the plate-based PFA to this system, then the Casimir energy becomes,
\[ E_P = \int \int_{x^2+y^2 \leq a^2} \varepsilon C(\vec{z}) dx dy \]
\[ = -\frac{\pi^3}{720} \frac{\xi}{\mu^2} \frac{1}{1 + \frac{\xi}{\mu} \frac{1}{L}} \]
\[ + \frac{1}{16\pi^4} \Gamma(2) \zeta(4) \frac{\xi^2}{L} - \frac{1}{8\pi^2} \Gamma\left(\frac{5}{2}\right) \zeta(5) \left(\frac{1}{2} \xi^2 \mu + \frac{1}{6} \xi^3\right) \frac{1}{L} \]

\[ - \frac{1}{2\pi} \frac{1}{L} \sum_{n_1, n_2 = 1}^{\infty} \left(\frac{n_2}{n_1}\right)^2 \int_{\mu}^{\mu + \xi} \int_{\mu + \xi - t}^{\mu + \xi - t} K_2(2n_1n_2t) \, dt \]

while we choose \(|\vec{z}| = d + a - \sqrt{a^2 - x^2 - y^2}\). This Casimir energy is finite. If the plate and the sphere are sufficiently far away from each other, the asymptotic behaviour of the casimir energy is,

\[ \lim_{\mu \to -\infty} E_P = -\frac{1}{16\pi^2} \Gamma\left(\frac{5}{2}\right) \zeta(5) \xi^2 \mu \frac{1}{L} \]

\[ < 0 \] (10)

The Casimir energy of plate-sphere device with plate-based PFA in the spacetime with one extra dimension is depicted in Fig. 1. The curves of energy are similar with different size of sphere. The larger spheres make the Casimir energy higher. The nature of the Casimir energy keeps negative no matter how far the plate and the sphere localize each other. When the two components are moved farther, the value of the Casimir energy is linearly decreasing like in Eq. (10). In order to show the influence from the additional dimension evidently, we compare the two kinds of energies with or without extra space. We demonstrate the ratio \( h = \frac{E_P}{E_{P0}} \) in Fig. 2. It is interesting that the larger sphere and the larger distance between the plate and sphere both make the ratio larger, and the larger ratio exhibits the extra-dimension correction more manifest.

**III. The PFA for the Casimir energy of sphere-sphere system in the presence of one extra compactified universal dimension**

Now we pay our attention to the Casimir energy between two spheres under PFA in the world involving one additional compactified dimension. At first we write the Casimir energy of sphere-sphere system under PFA in the four-dimensional spacetime as follow according to Ref. [22],

\[ E_{SS0} = -\frac{\pi^3}{720} \frac{a^2}{d^2(2a + d)} \] (11)

It is necessary that we take the Casimir energy density for parallel plates subject to one extra dimension in Eq. (1) the place of the integrand in Eq. (2). We derive the Casimir energy of the sphere-sphere system as,

\[ E_{SS} = \int \int_{x^2 + y^2 \leq a^2} \varepsilon_C[2z(\sigma)] \, d\sigma \]

\[ = -\frac{\pi^3}{720} \frac{\xi^2}{\mu^2(2\xi + \mu)} \frac{1}{L} \]

\[ + \frac{1}{16\pi^4} \Gamma(2) \zeta(4) \frac{\xi^2}{L} - \frac{1}{8\pi^2} \Gamma\left(\frac{5}{2}\right) \zeta(5) \left(\frac{1}{2} \xi^2 \mu + \frac{1}{3} \xi^3\right) \frac{1}{L} \]

\[ - \frac{1}{2\pi} \frac{1}{L} \sum_{n_1, n_2 = 1}^{\infty} \left(\frac{n_2}{n_1}\right)^2 \int_{\mu}^{\mu + \xi} \int_{\mu + \xi - t}^{\mu + \xi - t} K_2(4n_1n_2t) \, dt \]

(12)
where $\sigma = \sigma(x, y)$ and $d\sigma = dxdy$. Here the local distance is denoted as $\vec{z}(x, y) = \frac{d}{2} + a - \sqrt{a^2 - x^2 - y^2}$. $a$ is a common radius of the two spheres and $d$ is the shortest separation between the sphere surfaces. We can move the two spheres farther away from each other to investigate the asymptotic value of the Casimir energy as follow,

$$\lim_{\mu \to \infty} E_{SS} = -\frac{1}{16\pi^{\frac{5}{2}}} \Gamma\left(\frac{5}{2}\right) \zeta(5) \xi^2 \mu \frac{1}{L}$$

which is the same as the results of plate-sphere system under the plate-based PFA in Eq. (10). We show the Casimir energy between two spheres controlled by PFA in the presence of one additional dimension graphically in Fig. 3. The shapes of the energy associated with the size of the spheres are similar. The Casimir energy becomes higher as the two spheres enlarge. The sign of the Casimir energy of the two spheres remains negative. The Casimir energy will also be linearly decreasing as in Eq. (13) if the distance between the spheres is longer. We can also make use of the sphere-sphere experiment to explore the extra space. We plot the ratio $h = \frac{E_{SS}}{E_{SS0}}$ in Fig. 4 to compare the two-sphere Casimir energy including or excluding the fifth dimension. It is found that the larger ration is due to the larger spheres which localize farther each other. The larger ratio originated by the extra dimensions could be measured in the experiment.

IV. Discussion and conclusion

In this work it is the first time to investigate the Casimir energies of plate-sphere and sphere-sphere under PFA in the presence of one extra compactified universal dimension. Having studied the Casimir energy of plate-sphere device in the cases of sphere-based PFA and plate-based PFA respectively, we discover that the Casimir energy belonging to the sphere-based PFA is divergent, but the energy for plate-based PFA is finite and keeps negative. It is shown that the Casimir energy of plate-sphere system governed by plate-based PFA depends on the experimental structure and the additional dimension. Further we find that the larger sphere and longer distance between plate and sphere make the influence from extra dimension more obvious. We also scrutinize the Casimir energy between two identical spheres limited by PFA in the spacetime with one extra compactified dimension. We demonstrate that the negative Casimir energy has something to do with the radii of the spheres and the gap between them. It is similar to the results of the plate-sphere system in the case of plate-based PFA that the extra-dimension corrections will become more visible in the experiment including larger spheres which are farther away from each other. Our predictions from PFA are leading terms of Casimir energies and certainly can become a window to explore the extra compactified space. The related topics need further research and in progress.

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Figure 1: The solid, dot and dashed curves of the Casimir energy of plate-sphere system in unit of $\frac{1}{L}$ as functions of ratio of the shortest distance between the plate and sphere and extra-dimension radius $\mu = \frac{d}{L}$ under the plate-based PFA for $\xi = 200, 210, 220$ respectively.
Figure 2: The solid, dot and dashed curves of the ratio of Casimir energies of plate-sphere system under the plate-based PFA with and without the additional dimension as functions of $\mu = \frac{d}{L}$ for $\xi = 200, 210, 220$ respectively.
Figure 3: The solid, dot and dashed curves of the Casimir energy of two spheres system limited by PFA in unit of $\frac{1}{L}$ as functions of ratio of the shortest distance between two spheres and extra-dimension radius $\mu = \frac{d}{L}$ under the plate-based PFA for $\xi = 200, 210, 220$ respectively.
Figure 4: The solid, dot and dashed curves of the ratio of Casimir energies of two spheres system limited by PFA with and without the additional dimension as functions of $\mu = \frac{d}{L}$ for $\xi = 200, 210, 220$ respectively.