Representation of Crystallographic Subperiodic Groups in Clifford’s Geometric Algebra

Eckhard Hitzer and Daisuke Ichikawa

Abstract. This paper explains how, following the representation of 3D crystallographic space groups in Clifford’s geometric algebra, it is further possible to similarly represent the 162 so called subperiodic groups of crystallography in Clifford’s geometric algebra. A new compact geometric algebra group representation symbol is constructed, which allows to read off the complete set of geometric algebra generators. For clarity moreover the chosen generators are stated explicitly. The group symbols are based on the representation of point groups in geometric algebra by versors (Clifford monomials, Lipschitz elements).

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1. Introduction

Crystals are fundamentally periodic geometric arrangements of atoms, ions or molecules. The directed distance between two such elements is a Euclidean vector in \( \mathbb{R}^3 \). Intuitively all symmetry properties of crystals depend on these vectors. Indeed, the geometric product of vectors \([6]\) combined with the conformal model of 3D Euclidean space \([10,13,25,26,28]\) yields an algebra fully expressing crystal point and space groups \([8,11,14,15]\). Two successive reflections at (non-) parallel planes express (rotations) translations, etc. \([3,4]\). This leads to a 1:1 correspondence of geometric objects and symmetry operators \([21]\) with vectors and their products, ideal for creating a suite of interactive visualizations using CLUCalc \([29]\) and OpenGL \([15,16,18,20]\). Independently it has been shown that the coincidence site lattice transformation groups, important for modeling crystalline interfaces and grain boundaries, can be completely determined by coincidence symmetry reflections through hyperplanes orthogonal to lattice vectors \([32,33,37,38]\).

For material science the subperiodic space groups \([23]\) in 2D and 3D with only one or two degrees of freedom for translations are also of great interest. This
article introduces a new algebraic representation for all subperiodic groups of crys-
tallography, including, for the first time, a complete highly compact multiplicative
presentation of the generators for each group. By presentation we mean an explicit
representation of group elements. We also introduce a compact new system of sub-
periodic group symbols that enables one to write down the generators for each group
directly from the group symbol. Earlier initial work, reported in [19], is thus com-
pleted.

We begin in Section 2 by explaining the representation of point and space
groups in conformal geometric algebra. Then in Section 3 we show how to con-
struct a new compact geometric algebra group representation symbol for subperi-
odic space groups (frieze groups, rod groups and layer groups), which allows to
read off the complete set of geometric algebra generators. For clarity we moreover
state explicitly what generators are chosen.

2. Point groups and space groups in Clifford’s geometric algebra

2.1. Cartan-Dieudonné and Clifford’s geometric algebra

Clifford’s associative geometric product [2, 6] of two vectors simply adds the (sym-
matic) inner product to the (anti-symmetric) outer product of Grassmann

\[ ab = a \cdot b + a \wedge b. \] (2.1)

The mathematical meaning of the left and right side of (2.1) is clear from applying
the geometric product to the \(n\) orthonormal basis vectors \(\{e_1, \ldots, e_n\}\) of the under-
lying vector space \(\mathbb{R}^{p,q}, n = p + q\). We thus have

\[
\begin{align*}
    e_k e_k &= e_k \cdot e_k = +1, & e_k \wedge e_k &= 0, & 1 \leq k \leq p, \\
    e_k e_k &= e_k \cdot e_k = -1, & e_k \wedge e_k &= 0, & p + 1 \leq k \leq n, \\
    e_k e_l &= -e_l e_k = e_k \wedge e_l, & e_k \cdot e_l &= 0, & l \neq k, \ 1 \leq k, l \leq n.
\end{align*}
\] (2.2-2.4)

Under this product parallel vectors commute and perpendicular vectors anti-commute

\[ a x_\parallel = x_\parallel a, \quad a x_\perp = -x_\perp a. \] (2.5)

This allows to write the reflection of a vector \(x\) at a hyperplane through the origin
with normal \(a\) as (see left side of Fig. 1)

\[ x' = -a^{-1} x a, \quad a^{-1} = \frac{a}{a^2}, \quad a^{-1} a = a a^{-1} = 1. \] (2.6)

We can prove (2.6) by beginning with the expression of the reflected vector (see left
side of Fig. 1) for the meaning of \(x_\parallel\) and \(x_\perp\) relative to \(a\):

\[
\begin{align*}
x' &= -x_\parallel + x_\perp = -a^{-1} x_\parallel + a^{-1} x_\perp = -a^{-1} x_\parallel a - a^{-1} x_\perp a \\
    &= -a^{-1} (x_\parallel + x_\perp) a = -a^{-1} x a,
\end{align*}
\] (2.7)

where we have used (2.5) for the third equality.
Subperiodic Groups in Clifford’s Geometric Algebra

The composition of two reflections at hyperplanes, whose normal vectors $a, b$ subtend the angle $\alpha/2$, yields a rotation around the intersection of the two hyperplanes (see center of Fig. 1) by $\alpha$

$$x'' = (ab)^{-1}xab,$$

$$\ (ab)^{-1} = b^{-1}a^{-1}. \quad (2.8)$$

Continuing with a third reflection at a hyperplane with normal $c$ according to the Cartan–Dieudonné theorem $[1, 5, 34, 36]$ yields rotary reflections (equivalent to rotary inversions with angle $\alpha - \pi$)

$$x' = -(abc)^{-1}xabc, \quad (2.9)$$

and inversions

$$x'' = -i^{-1}xi, \quad i \doteq a \wedge b \wedge c, \quad (2.10)$$

where $\doteq$ means equality up to non-zero scalar factors (which cancel out in (2.11)).

In general the geometric product $S = abc \ldots$ of $k$, normal vectors corresponds to the composition of reflections to all symmetry transformations $\ [11, 12]$ of two-dimensional (2D) and 3D point group $[2]$ which describe the symmetry of crystal cells,

$$x' = (-1)^kS^{-1}xS = \hat{S}^{-1}xS = S^{-1}x\hat{S}, \quad (2.11)$$

where $\hat{S} = (-1)^kS$ is the grade involution or main involution in Clifford’s geometric algebra (GA). We call the product of invertible vectors $S$ in $[2, 11]$ versor $[7, 12, 21, 25]$, but the names Clifford monomial of invertible vectors, Clifford group element, or Lipschitz group element are equally in use $[25, 27]$.

1The decomposition of space group transformations in the conformal model $Cl(p + 1, q + 1)$ according to the Cartan–Dieudonné theorem is always possible, though it is clearly not unique, since, e.g., any two Euclidean vectors in the same plane, which enclose the angle $\alpha/2$ will generate the rotation in that plane by the angle $\alpha$. Different choices of lattice vectors allow therefore still to generate the same space group. For example, in the hexagon of Fig. 2 vector $a$ could obviously be replaced by any of the other five side vectors and $b$ then by a vector to a respective neighboring vertex.

2Note, that a (geometric) crystal class contains crystals that share the same type of point group. Therefore, the same symbol is used for both point group and crystal class, but there is a fundamental ontological difference between the two concepts.
2.2. Two dimensional point groups

2D point groups [11] are generated by multiplying vectors selected [14, 15] as in Fig. 2. The index $p$ can be used to denote these groups as in Table 1. For example the hexagonal point group is given by multiplying its two generating vectors $\mathbf{a}, \mathbf{b}$

$$6 = \{ \mathbf{a}, \mathbf{b}, \mathbf{R} = \mathbf{ab}, \mathbf{R}^2, \mathbf{R}^3, \mathbf{R}^4, \mathbf{R}^5, \mathbf{R}^6 \} \equiv -1,$$

$$\mathbf{aR}^2, \mathbf{bR}^2, \mathbf{aR}^4, \mathbf{bR}^4 \}.$$

(2.12)

The rotation subgroups are denoted with bars, e.g. $\bar{6}$. The identities $\mathbf{a}^2 \equiv \mathbf{b}^2 \equiv 1$ and $\mathbf{R}^6 \equiv -1$ directly correspond to relations in the group presentation [35] of 6.

2.3. Three dimensional point groups

The selection of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ from each crystal cell [11][16] for generating (cf. Table 2) 3D point groups are indicated in Figs. 3 and 4.

The selection of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, see the right side of Fig. 1 from each crystal cell [11][14][15] for generating all 3D point groups is indicated in Figs. 3 and 4. Using $\angle(\mathbf{a}, \mathbf{b})$ and $\angle(\mathbf{b}, \mathbf{c})$ we can denote all 32 3D point groups (and their associated crystal classes) as in Table 2. For example the monoclinic point groups
\[ \begin{align*}
\angle bc \neq 90^\circ \\
\angle ab \neq 90^\circ
\end{align*} \]

**Figure 3.** From left to right: Triclinic, monoclinic inclined, monoclinic orthogonal, orthorhombic, and tetragonal cell vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) for rod and layer groups.

**Figure 4.** Generating vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) of cell choices for trigonal and hexagonal crystal systems for 3D rod and layer groups. Left: a hexagonal primitive (\( hP \)) cell. Center: an ortho-hexagonal \( C \)-centered cell (\( oC \)). Right: a hexagonal \( H \)-centered cell (\( hH \)) with Bravais symbol in the nomenclature: \( H \) or \( h \).

are then (int. symbols of Hermann-Mauguin: \( 2/m, m \) and \( 2 \), respectively)

\[ 2\overline{2} = \{ \mathbf{c}, R = \mathbf{a} \land \mathbf{b} = i\mathbf{c}, i = \mathbf{c}R, 1 \}, \quad (2.13) \]
\[ 1 = \{ \mathbf{c}, 1 \}, \quad (2.14) \]
\[ \overline{2} = \{ i\mathbf{c}, 1 \}. \quad (2.15) \]

### 2.4. Space groups

The smooth composition with translations is best done in the conformal model \[10, 13, 25, 26, 28\] of Euclidean space (in the GA of \( \mathbb{R}^{4,1} \)), which adds two null-vector dimensions for the origin \( e_0 \) and infinity \( e_\infty \) such that

\[ X = x + \frac{1}{2}x^2e_\infty + e_0, \quad (2.16) \]
\[ e_0^2 = e_\infty^2 = X^2 = 0, \quad (2.17) \]
\[ X \cdot e_\infty = -1. \quad (2.18) \]

The \( +e_0 \) term integrates projective geometry, and the \( +\frac{1}{2}x^2e_\infty \) term ensures \( X^2 = 0 \). The inner product of two conformal points gives their Euclidean distance and therefore a plane \( m \) equidistant from two points \( A, B \) as

\[ X \cdot A = -\frac{1}{2}(x - a)^2 \Rightarrow X \cdot (A - B) = 0, \quad (2.19) \]
\[ m = A - B \propto p - d e_\infty, \quad (2.20) \]
TABLE 3. Computing with reflections and translations. The vectors \( \mathbf{a}, \mathbf{b} \) are pictured in Fig. 2.

| \( \angle(\mathbf{a}, \mathbf{b}) \) | 180° | 90° | 60° | 45° | 30° |
|---------------------------------|------|-----|-----|-----|-----|
| \( T_a \mathbf{b} = \mathbf{b} T_{-a} \) | \( \mathbf{b} T_a \) | \( \mathbf{b} T_{a-b} \) | \( \mathbf{b} T_{a-b} \) |
| \( T_b \mathbf{a} = \mathbf{a} T_{-b} \) | \( \mathbf{a} T_b \) | \( \mathbf{a} T_{b-2a} \) | \( \mathbf{a} T_{b-3a} \) |

where \( \mathbf{p} \) is a unit normal to the plane and \( d \) its signed scalar distance from the origin. Reflecting at two parallel planes \( m, m' \) with distance \( t/2 \) we get the so-called translator (translation operator by \( t \))

\[
X' = m'mXmm' = T_t^{-1}XT_t, \quad T_t = 1 + \frac{1}{2} t \mathbf{e}_\infty. \tag{2.21}
\]

Reflection at two non-parallel planes \( m, m' \) yields the rotation around the \( m, m' \)-intersection line axis by twice the angle subtended by \( m, m' \).

Group theoretically the conformal group \( C(3) \) is isomorphic to \( O(4, 1) \) and the Euclidean group \( E(3) \) is the subgroup of \( O(4, 1) \) leaving infinity \( \mathbf{e}_\infty \) invariant \([11,12,25]\). Now general translations and rotations are represented by geometric products of vectors. To study combinations of versors it is useful to know that (cf. Table 3)

\[
T_t \mathbf{a} = \mathbf{a} T_t', \quad t' = -\mathbf{a}^{-1} t \mathbf{a}. \tag{2.22}
\]

Applying these techniques one can compactly tabulate geometric space group symbols and generators \([12]\). Table 4 implements this for the 13 monoclinic space groups. All this is interactively visualized \([16]\) by the Space Group Visualizer \([31]\).

3. Subperiodic groups represented in Clifford’s geometric algebra

Now we begin to explain the details of the new geometric algebra based representation of so-called subperiodic space groups. These include the seven frieze groups (in 2D space, 1 degree of freedom (DOF) for translation), the 75 rod groups (in 3D space, 1 DOF for translation), and the 80 layer groups (in 3D space, 2 DOF for translations).

Compared to the geometric 2D and 3D space group symbols in \([12]\) we have introduced dots: If one or two dots occur between the Bravais symbol (\( \varphi, p, c \)) and index 1, the vector \( \mathbf{b} \) or \( \mathbf{c} \), respectively, is present in the generator list. If one or two dots appear between the Bravais symbol and the index 2 (without or with bar), then the vectors \( \mathbf{b}, \mathbf{c} \) or \( \mathbf{a}, \mathbf{c} \), respectively, are present in the generator list.

In agreement \([12]\) the indexes \( a, b, c, n \) (and \( g \) for frieze groups) in first, second or third position after the Bravais symbol indicate that the reflections \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) (in this order) become glide reflections. An index \( n \) indicates diagonal glides. The dots also serve as symbolic \( a, b, c \) position indicators. For example rod group 5: \( \varphi, 1 \) has glide reflection \( \mathbf{a} T_c^{1/2} \), rod group 19: \( \varphi, c \) 2 has \( \mathbf{b} T_c^{1/2} \), and layer group 39: \( p_b 2a 2n \) has \( \mathbf{a} T_b^{1/2}, \mathbf{b} T_a^{1/2} \) and \( \mathbf{c} T_{a+b}^{1/2} \).
TABLE 4. Monoclinic space group versor generators, $T_A = T_{b+c}$, col. 1: international space group number [9], col. 2: international space group symbol [9], col. 3: geometric space group symbol [12], col. 4: geometric space group versor generators [12], cols. 5 and 6: alternative geometric space group versor generators [17]. The pure translators $T_a, T_b, T_c$ are omitted.

|   | 1. | 2. | 3. | 4. | 5. | 6. |
|---|----|----|----|----|----|----|
| 3 | $P2$ | $P_2^2$ | $i c = a \wedge b$ |    |    |    |
| 4 | $P2_1$ | $P_2^1$ | $i c T_c^{1/2}$ |    |    |    |
| 5 | $C2$ | $A_2^2$ | $i c, T_A$ |    |    |    |
| 6 | $Pm$ | $P1$ | $c$ |    |    |    |
| 7 | $Pc$ | $P_{a1}$ | $c T_a^{1/2}$ |    |    |    |
| 8 | $Cm$ | $A_1$ | $c, T_A$ |    |    |    |
| 9 | $Cc$ | $A_{a1}$ | $c T_a^{1/2}, T_A$ |    |    |    |
| 10 | $P2/m$ | $P_2 \bar{2}$ | $c, i c$ | $i, i c$ | $i, c$ |    |
| 11 | $P2_1/m$ | $P_{2\bar{2}}_1$ | $c, i c T_c^{1/2}$ | $i, i c T_c^{1/2}$ | $i, c T_c^{1/2}$ |    |
| 12 | $C2/m$ | $A_{2\bar{2}}$ | $c, i c, T_A$ | $i T_A, i c T_A, T_A$ | $i, c, T_A$ |    |
| 13 | $P2/c$ | $P_{a2\bar{2}}$ | $c T_a^{1/2}, i c$ | $i, i c T_a^{1/2}$ | $i, c T_a^{1/2}$ |    |
| 14 | $P2_1/c$ | $P_{a\bar{2}2\bar{1}}$ | $c T_a^{1/2}, i c T_c^{1/2}$ | $i, i c T_a^{1/2}$ | $i, c T_a^{1/2}$ |    |
| 15 | $C2/c$ | $A_{a2\bar{2}}$ | $c T_a^{1/2}, i c, T_A$ | $i, i c T_a^{1/2}, T_A$ | $i, c T_a^{1/2}, T_A$ |    |

The notation $\bar{n}_p$ indicates a right handed screw rotation of $2\pi/n$ around the $\bar{n}$-axis, with pitch $T^{p/n}_t$ where $t$ is the shortest lattice translation vector parallel to the axis, in the screw direction. For example the layer group 21: $p\bar{2}\bar{2}_1\bar{2}_1$ has the screw generators $bc T_a^{1/2}$ and $ac T_b^{1/2}$.

In the following we discuss specific issues for frieze groups, rod groups and layer groups.

3.1. Frieze groups

Figure 5 shows the generating vectors $a, b$ of oblique and rectangular cells for 2D frieze groups. The only translation direction is $a$, frieze groups are thus subgroups of plane space groups, which can also be fully visualized with the interactive Space Group Visualizer software [20]. Table 5 lists the seven frieze groups with new geometric symbols and generators.

3.2. Rod groups

Figure 3 shows the generating vectors $a, b, c$ of triclinic, monoclinic, orthorhombic and tetragonal cells for 3D rod and layer groups. Figure 4 shows the same for trigonal and hexagonal cells. For rod groups the only translation direction is $c$. There is a total of 75 rod groups in all 3D crystal systems. Table 6 lists the triclinic, monoclinic and orthorhombic rod groups with new geometric symbols and generators:
FIGURE 5. Generating vectors \( \mathbf{a}, \mathbf{b} \) of oblique and rectangular cells for 2D frieze groups.

Rod group number (col. 1), international rod group notation \([23]\) (col. 2), international 3D space super group numbers \([9]\) (col. 3), and notation \([9]\) (col. 4), geometric 3D space super group notation \([12]\) (col. 5), geometric rod group notation (col. 6), geometric algebra generators (col. 7). The tetragonal and trigonal rod groups are listed in Table 7, and the hexagonal rod groups in Table 8.

Note that in the last two rows of Table 7 we give in col. 5 the geometric 3D space super group notation exactly as found in \([12]\). But for full consistency with the choice of vectors in Table 2 and Fig. 4 we have decided to modify the rod group notation (and their versor generators) specified in col. 6 (and col. 7) of the last two rows of Table 7.

3.3. Layer groups

For layer groups the two translation directions are \( \mathbf{a}, \mathbf{b} \). There is a total of 80 layer groups. Table 9 lists the triclinic and monoclinic 3D layer groups with new geometric symbols and generators: Layer group number (col. 1), international layer group notation \([3]^{23}\) (col. 2), international 3D space super group numbers \([9]\) (col. 3), and notation \([9]\) (col. 4), geometric 3D space super group notation \([12]\) (col. 5), geometric layer group notation (col. 6), geometric algebra generators (col. 7). Table 10 lists the orthorhombic/rectangular layer groups, and Table 11 the tetragonal/square, trigonal/hexagonal and hexagonal/hexagonal layer groups. The layer groups are classified according to their 3D crystal system/2D Bravais system\(^4\). The monoclinic/oblique (rectangular) system corresponds to the monoclinic/orthogonal (inclined) system of Fig. 3. Figure 4 shows the hexagonally centered cell with Bravais symbols \( \text{H} \) (space group) and \( h \) (layer group).

Note that we use in Table 11 for the symmorphic space group No. 81 the rotary reflection generator \( \mathbf{abc} \) and not \( \mathbf{bac} \) of Table 5 of \([12]\). But the point groups and symmorphic space groups generated by \( \mathbf{abc} \) and \( \mathbf{bac} \) are the same, because for \( p = 4 \) we have the following equalities (up to non-zero scalar factors, which cancel out in (2.11))

\[
\mathbf{ba} \equiv (\mathbf{ab})^3, \quad (\mathbf{ba})^2 \equiv (\mathbf{ab})^2, \quad (\mathbf{ba})^3 \equiv \mathbf{ab},
\]

and hence with \( q = 2 \), \( \mathbf{bac} = \mathbf{cba}, \mathbf{abc} = \mathbf{cab} \), and \( c^2 \equiv 1 \) we also have

\[
\mathbf{bac} \equiv (\mathbf{abc})^3, \quad (\mathbf{bac})^2 \equiv (\mathbf{abc})^2, \quad (\mathbf{bac})^3 \equiv \mathbf{abc}.
\]

\(^3\)Further well known and used layer group symbols are due to Wood, and to Bohm and Dornberger-Schiff \([24]\).

\(^4\)Note that Bravais systems have officially been renamed lattice systems since 2002.
That is, the two sets of point transformations generated by the integer powers of the generators $abc$ and $bac$ are identical for $p = 4$, $q = 2$. We have therefore decided for consistency with Table 2 to use for the space group No. 81 the geometric symbol $P42$ and the generator $abc$. A similar argument is valid for our use of the generator $abc$ for space group No. 147 in our Table 11 instead of $bac$ in Table 5 of [12].

4. Conclusion

We have devised a new representation for the 162 subperiodic space groups in Clifford’s geometric algebra using versors (Clifford monomials, Lipschitz elements). In the future this may be extended to magnetic subperiodic space groups. We expect that the present work forms a suitable foundation for interactive visualization software of subperiodic space groups [16]. Fig. 6 shows how the rod groups 13: $\overline{2}2\overline{2}$ and 14: $\overline{2}1\overline{2}2$, and the layer group 11: $p1$, might be visualized in the future, based on [16].

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Table 5. Table of frieze groups. Group number (col. 1), international frieze group notation \([23]\) (col. 2), international 3D space super group numbers \([9]\) (col. 3), and notation \([9]\) (col. 4), geometric 3D space super group notation \([12]\) (col. 5), international 2D space super group numbers \([9]\) (col. 6), and notation \([9]\) (col. 7), geometric 2D space super group notation \([12]\) (col. 8), geometric frieze group notation (col. 9), geometric algebra frieze group versor generators (col. 10). The pure translator \(T_a\) is omitted.

| 1. | 2. | 3. | 4. | 5. | 6. | 7. | 8. | 9. | 10. |
|----|----|----|----|----|----|----|----|----|----|
| Oblique | | | | | | | | | |
| 1 | ∞1 | 1 | \(P1\) | \(P\overline{1}\) | 1 | \(p1\) | \(p\overline{1}\) | ∞ | \(\mathbf{a} \wedge \mathbf{b}\) |
| 2 | ∞211 | 3 | \(P2\) | \(P\overline{2}\) | 2 | \(p2\) | \(p\overline{2}\) | \(\sqrt{2}\) | |
| Rectangular | | | | | | | | | |
| 3 | ∞1m1 | 6 | \(Pm\) | \(P1\) | 3 | \(pm\) | \(p1\) | ∞ | \(\mathbf{a}\) |
| 4 | ∞11m | 6 | \(Pm\) | \(P1\) | 3 | \(pm\) | \(p1\) | ∞ | \(\mathbf{b}\) |
| 5 | ∞11g | 7 | \(Pc\) | \(P_g1\) | 4 | \(p_g\) | \(p_g1\) | ∞ | \(\mathbf{b}T_a^{1/2}\) |
| 6 | ∞2mm | 25 | \(Pmm2\) | \(P2\) | 6 | \(p2mm\) | \(p2\) | ∞ | \(\mathbf{a, b}\) |
| 7 | ∞2mg | 28 | \(Pma2\) | \(P2_a\) | 7 | \(p2mg\) | \(p2_g\) | ∞ | \(\mathbf{a, b}T_a^{1/2}\) |

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Table 6. Table of triclinic, monoclinic and orthorhombic rod groups. The pure translator $T_c$ is omitted.

|   | 1. | 2. | 3. | 4. | 5. | 6. | 7. |
|---|----|----|----|----|----|----|----|
| Triclinic |    |    |    |    |    |    |    |
| 1 | $\alpha_1$ | 1 | $P1$ | $P\bar{1}$ | $P\alpha_1$ | $a \land b \land c$ |
| 2 | $\alpha \bar{1}$ | 2 | $P\bar{1}$ | $P22$ | $\bar{2}2$ | $b \land c$ |
| Monoclinic/inclined |    |    |    |    |    |    |    |
| 3 | $\bar{2}11$ | 3 | $P112$ | $P\bar{2}$ | $\bar{2}$ | $a \land b \land c$ |
| 4 | $m11$ | 6 | $Pm$ | $P1$ | $\bar{1}$ | $a$ |
| 5 | $c11$ | 7 | $Pc$ | $Pc1$ | $\alpha 1$ | $a T_c^{1/2}$ |
| 6 | $2/m11$ | 10 | $P2/m$ | $P22$ | $22$ | $a, b \land c$ |
| 7 | $2/c11$ | 13 | $P2/c$ | $P\bar{2}2$ | $\bar{2}2$ | $a T_c^{1/2}, b \land c$ |
| Monoclinic/orthogonal |    |    |    |    |    |    |    |
| 8 | $\bar{1}12$ | 3 | $P112$ | $P\bar{2}$ | $\bar{2}$ | $a \land b$ |
| 9 | $112_1$ | 4 | $P2_1$ | $P\bar{2}_1$ | $\bar{2}_1$ | $(a \land b) T_c^{1/2}$ |
| 10 | $11m$ | 6 | $Pm$ | $P1$ | $\bar{1}$ | $c$ |
| 11 | $112/m$ | 10 | $P2/m$ | $P22$ | $22$ | $a \land b, c$ |
| 12 | $\bar{1}12_1/m$ | 11 | $P2_1/m$ | $P\bar{2}_12$ | $\bar{2}_12$ | $(a \land b) T_c^{1/2}, c$ |
| Orthorhombic |    |    |    |    |    |    |    |
| 13 | $\bar{2}22$ | 16 | $P22$ | $P\bar{2}22$ | $\bar{2}22$ | $a, b, c$ |
| 14 | $222_1$ | 17 | $P22_1$ | $P\bar{2}_22$ | $\bar{2}_22$ | $ab T_c^{1/2}, b c$ |
| 15 | $\alpha mm2$ | 25 | $P\alpha mm2$ | $P2$ | $\bar{2}$ | $a, b$ |
| 16 | $cc2$ | 27 | $Pcc2$ | $Pc_2c$ | $\bar{2}c_2c$ | $a T_c^{1/2}, b T_c^{1/2}$ |
| 17 | $\alpha mc2_1$ | 26 | $Pmc2_1$ | $Pc_2c$ | $\bar{2}c_2c$ | $a, b T_c^{1/2}$ |
| 18 | $\alpha mm2$ | 25 | $P\alpha mm2$ | $P2$ | $\alpha 2$ | $a, b, c$ |
| 19 | $\alpha 2cm$ | 28 | $Pma2$ | $P\bar{2}a$ | $\bar{2}a$ | $b T_c^{1/2}, c$ |
| 20 | $\alpha mmm$ | 47 | $P\alpha mmm$ | $P22$ | $\bar{2}2$ | $a, b, c$ |
| 21 | $cccm$ | 49 | $Pcccm$ | $Pc_2c_2$ | $\bar{2}c_2c_2$ | $a T_c^{1/2}, b T_c^{1/2}, c$ |
| 22 | $mcm$ | 51 | $Pmm$ | $P22$ | $\bar{2}2$ | $a, b T_c^{1/2}, c$ |

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## Table 7. Table of tetragonal and trigonal rod groups. The pure translator \( T_e \) is omitted.

|   | 1. | 2. | 3. | 4. | 5. | 6. | 7. |
|---|----|----|----|----|----|----|----|
| **Tetragonal** | | | | | | | |
| 23 | \( \alpha 4 \) | 75 | \( P4 \) | \( \alpha 4 \) | \( P4 \) | \( \alpha 4 \) | \( ab \) |
| 24 | \( \alpha 4_1 \) | 76 | \( P4_1 \) | \( \alpha 4_1 \) | \( P4_1 \) | \( \alpha 4_1 \) | \( abT_e^{\frac{1}{4}} \) |
| 25 | \( \alpha 4_2 \) | 77 | \( P4_2 \) | \( \alpha 4_2 \) | \( P4_2 \) | \( \alpha 4_2 \) | \( abT_e^{\frac{1}{2}} \) |
| 26 | \( \alpha 4_3 \) | 78 | \( P4_3 \) | \( \alpha 4_3 \) | \( P4_3 \) | \( \alpha 4_3 \) | \( abT_e^{\frac{3}{2}} \) |
| 27 | \( \alpha^4 \) | 81 | \( P4^4 \) | \( \alpha^4 \) | \( P4^2 \) | \( \alpha^2 \) | \( abc \) |
| 28 | \( \alpha 4/m \) | 83 | \( P4/m \) | \( \alpha 4 \) | \( P4 \) | \( \alpha 2 \) | \( ab, c \) |
| 29 | \( \alpha 4_2/m \) | 84 | \( P4_2/m \) | \( \alpha 4_2 \) | \( P4 \) | \( \alpha 4_2 \) | \( abT_e^{\frac{1}{4}}, c \) |
| 30 | \( \alpha 4/m \) | 89 | \( P4_4 \) | \( \alpha 4_1 \) | \( P4 \) | \( \alpha 4_1 \) | \( ab, bc \) |
| 31 | \( \alpha 4_1 \) | 91 | \( P4_1 \) | \( \alpha 4_2 \) | \( P4 \) | \( \alpha 4_2 \) | \( abT_e^{\frac{1}{2}}, bc \) |
| 32 | \( \alpha 4_2 \) | 93 | \( P4_2 \) | \( \alpha 4_2 \) | \( P4 \) | \( \alpha 4_2 \) | \( abT_e^{\frac{3}{2}}, bc \) |
| 33 | \( \alpha 4_3 \) | 95 | \( P4_3 \) | \( \alpha 4_3 \) | \( P4 \) | \( \alpha 4_3 \) | \( abT_e^{\frac{5}{2}}, bc \) |
| 34 | \( \alpha 4/4 \) | 99 | \( P4/4 \) | \( \alpha 4 \) | \( P4 \) | \( \alpha 4 \) | \( a, b \) |
| 35 | \( \alpha 4_2 \) | 101 | \( P4_2 \) | \( \alpha 4_2 \) | \( P4 \) | \( \alpha 4_2 \) | \( abT_e^{\frac{1}{4}}, b \) |
| 36 | \( \alpha 4/2 \) | 103 | \( P4_2 \) | \( \alpha 4_2 \) | \( P4 \) | \( \alpha 4_2 \) | \( abT_e^{\frac{1}{2}}, bT_e^{\frac{1}{2}} \) |
| 37 | \( \alpha 4/m \) | 115 | \( P4/m \) | \( \alpha 4_4 \) | \( P4 \) | \( \alpha 4_4 \) | \( a, bc \) |
| 38 | \( \alpha 4_1 \) | 116 | \( P4_4 \) | \( \alpha 4_1 \) | \( P4 \) | \( \alpha 4_1 \) | \( aT_e^{\frac{1}{2}}, bc \) |
| 39 | \( \alpha 4/4 \) | 123 | \( P4/4 \) | \( \alpha 4_1 \) | \( P4 \) | \( \alpha 4_1 \) | \( a, b, c \) |
| 40 | \( \alpha 4_2 \) | 124 | \( P4_2 \) | \( \alpha 4_2 \) | \( P4 \) | \( \alpha 4_2 \) | \( aT_e^{\frac{1}{2}}, bT_e^{\frac{1}{2}}, c \) |
| 41 | \( \alpha 4_2 \) | 131 | \( P4_2 \) | \( \alpha 4_2 \) | \( P4 \) | \( \alpha 4_2 \) | \( a, bT_e^{\frac{1}{2}}, c \) |
| **Trigonal** | | | | | | | |
| 42 | \( \alpha 3 \) | 143 | \( P3 \) | \( \alpha 3 \) | \( P3 \) | \( \alpha 3 \) | \( ab \) |
| 43 | \( \alpha 3_1 \) | 144 | \( P3_1 \) | \( \alpha 3_1 \) | \( P3_1 \) | \( \alpha 3_1 \) | \( abT_e^{\frac{1}{3}} \) |
| 44 | \( \alpha 3_2 \) | 145 | \( P3_2 \) | \( \alpha 3_2 \) | \( P3_2 \) | \( \alpha 3_2 \) | \( abT_e^{\frac{3}{2}} \) |
| 45 | \( \beta 3 \) | 147 | \( P3 \) | \( \beta 3 \) | \( P3 \) | \( \beta 3 \) | \( abc \) |
| 46 | \( \beta 3_1 \) | 149 | \( P3_1 \) | \( \beta 3 \) | \( P3_1 \) | \( \beta 3 \) | \( ab, bc \) |
| 47 | \( \beta 3_2 \) | 151 | \( P3_2 \) | \( \beta 3 \) | \( P3_2 \) | \( \beta 3 \) | \( aT_e^{\frac{1}{3}}, bc \) |
| 48 | \( \beta 3_1 \) | 153 | \( P3_1 \) | \( \beta 3 \) | \( P3_1 \) | \( \beta 3 \) | \( abT_e^{\frac{1}{2}}, bc \) |
| 49 | \( \beta 3 \) | 156 | \( P3_1 \) | \( \beta 3 \) | \( P3_1 \) | \( \beta 3 \) | \( a, b \) |
| 50 | \( \beta 3_1 \) | 158 | \( P3_1 \) | \( \beta 3 \) | \( P3_1 \) | \( \beta 3 \) | \( aT_e^{\frac{1}{2}}, bT_e^{\frac{1}{2}} \) |
| 51 | \( \beta 3_2 \) | 162 | \( P3_2 \) | \( \beta 3 \) | \( P3_2 \) | \( \beta 3 \) | \( a, bc \) |
| 52 | \( \beta 3_1 \) | 163 | \( P3_1 \) | \( \beta 3 \) | \( P3_1 \) | \( \beta 3 \) | \( abT_e^{\frac{1}{3}}, bc \) |

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Table 8. Table of hexagonal rod groups. The pure translator $T_c$ is omitted.

| 1. | 2. | 3. | 4. | 5. | 6. | 7. |
|----|----|----|----|----|----|----|
| 53 | $\bar{6}$ | 168 | $P6$ | $P\bar{6}$ | $\bar{6}$ | ab |
| 54 | $\bar{6}1$ | 169 | $P6_1$ | $P\bar{6}_1$ | $\bar{6}1$ | ab$T_c^{\frac{1}{6}}$ |
| 55 | $\bar{6}2$ | 171 | $P6_2$ | $P\bar{6}_2$ | $\bar{6}2$ | ab$T_c^{\frac{1}{3}}$ |
| 56 | $\bar{6}3$ | 173 | $P6_3$ | $P\bar{6}_3$ | $\bar{6}3$ | ab$T_c^{\frac{1}{2}}$ |
| 57 | $\bar{6}4$ | 172 | $P6_4$ | $P\bar{6}_4$ | $\bar{6}4$ | ab$T_c^{\frac{5}{3}}$ |
| 58 | $\bar{6}5$ | 170 | $P6_5$ | $P\bar{6}_5$ | $\bar{6}5$ | ab$T_c^{\frac{5}{6}}$ |
| 59 | $\bar{6}$ | 174 | $P6$ | $P32$ | $\bar{3}2$ | ab, c |
| 60 | $\bar{6}/m$ | 175 | $P6/m$ | $P\bar{6}2$ | $\bar{6}2$ | ab, c |
| 61 | $\bar{6}3/m$ | 176 | $P63/m$ | $P\bar{6}32$ | $\bar{6}32$ | ab$T_c^{\frac{1}{2}}$, c |
| 62 | $\bar{6}22$ | 177 | $P622$ | $P\bar{6}22$ | $\bar{6}22$ | ab, bc |
| 63 | $\bar{6}122$ | 178 | $P6122$ | $P\bar{6}122$ | $\bar{6}122$ | ab$T_c^{\frac{1}{6}}, bc$ |
| 64 | $\bar{6}222$ | 180 | $P6222$ | $P\bar{6}222$ | $\bar{6}222$ | ab$T_c^{\frac{1}{3}}, bc$ |
| 65 | $\bar{6}322$ | 182 | $P6322$ | $P\bar{6}322$ | $\bar{6}322$ | ab$T_c^{\frac{2}{3}}, bc$ |
| 66 | $\bar{6}422$ | 181 | $P6422$ | $P\bar{6}422$ | $\bar{6}422$ | ab$T_c^{\frac{5}{3}}, bc$ |
| 67 | $\bar{6}522$ | 179 | $P6522$ | $P\bar{6}522$ | $\bar{6}522$ | ab$T_c^{\frac{5}{6}}, bc$ |
| 68 | $\bar{6}mmm$ | 183 | $P6mm$ | $P6$ | $\bar{6}6$ | a, b |
| 69 | $\bar{6}cc$ | 184 | $P6cc$ | $P6_c$ | $\bar{6}c6_c$ | a$T_c^{\frac{1}{2}}, bT_c^{\frac{1}{2}}$ |
| 70 | $\bar{6}3cm$ | 185 | $P63cm$ | $P6_c$ | $\bar{6}c6$ | a$T_c^{\frac{1}{2}}, b$ |
| 71 | $\bar{6}m2$ | 187 | $P6m2$ | $P32$ | $\bar{3}2$ | a, b, c |
| 72 | $\bar{6}c2$ | 188 | $P6c2$ | $P3c2$ | $\bar{6}c3c2$ | a$T_c^{\frac{1}{2}}, bT_c^{\frac{1}{2}}, c$ |
| 73 | $\bar{6}mmm$ | 191 | $P6/mmm$ | $P62$ | $\bar{6}2$ | a, b, c |
| 74 | $\bar{6}mcc$ | 192 | $P6/mcc$ | $P6_c2$ | $\bar{6}c6_c2$ | a$T_c^{\frac{1}{2}}, bT_c^{\frac{1}{2}}, c$ |
| 75 | $\bar{6}3/mcm$ | 193 | $P63/mcm$ | $P62$ | $\bar{6}c6$ | a$T_c^{\frac{1}{2}}, b, c$ |

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Table of triclinic and monoclinic layer groups. The pure translators $T_a, T_b$ are omitted.

|    | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|----|-----|-----|-----|-----|-----|-----|-----|
| 1. Triclinic/oblique | $p1$ | $P1$ | $P1$ | $P1$ | $P1$ | $P1$ | $a \land b \land c$
| 2. | $p1$ | $P1$ | $P2$ | $P2$ | $p2$ | $a \land b$
| 3. Monoclinic/oblique | $P2$ | $P1$ | $P2$ | $P2$ | $p2$ | $a \land b, c$
| 4. | $P1$ | $P1$ | $cT_a^{1/2}$ | $cT_a^{1/2}$ | $cT_a^{1/2}$ | $cT_a^{1/2}$ | $cT_a^{1/2}$
| 5. | $Pm$ | $Pm$ | $P1$ | $p1$ | $a$
| 6. | $Pc$ | $P_{a1}$ | $p_{b1}$ | $aT_b^{1/2}$ | $aT_b^{1/2}$ | $aT_b^{1/2}$ | $aT_b^{1/2}$
| 7. | $C2$ | $A2$ | $c2$ | $b \land c, T_{a+b}^{1/2}$ | $b \land c, T_{a+b}^{1/2}$ | $b \land c, T_{a+b}^{1/2}$ | $b \land c, T_{a+b}^{1/2}$
| 8. Monoclinic/rectangular | $Pm$ | $P1$ | $P2$ | $P2$ | $p2$ | $b \land c$
| 9. | $P2$ | $P2$ | $p2$ | $p2$ | $b \land c$
| 10. | $P2$ | $P2$ | $p2$ | $p2$ | $b \land c, T_{a+b}^{1/2}$ | $b \land c, T_{a+b}^{1/2}$ | $b \land c, T_{a+b}^{1/2}$
| 11. | $P1$ | $p1$ | $a$
| 12. | $Pc$ | $P_{a1}$ | $p_{b1}$ | $aT_b^{1/2}$ | $aT_b^{1/2}$ | $aT_b^{1/2}$ | $aT_b^{1/2}$
| 13. | $Cm$ | $A1$ | $c1$ | $a, T_{a+b}^{1/2}$ | $a, T_{a+b}^{1/2}$ | $a, T_{a+b}^{1/2}$ | $a, T_{a+b}^{1/2}$
| 14. | $P2/m$ | $P2/1$ | $P2$ | $P2$ | $p2$ | $a, b \land c$
| 15. | $P2$ | $P2$ | $p2$ | $p2$ | $a, b \land c$
| 16. | $P2/m$ | $P2$ | $P2$ | $P2$ | $p2$ | $a, b \land c$
| 17. | $P2/c$ | $P_2$ | $P_2$ | $P_2$ | $aT_b^{1/2}$ | $aT_b^{1/2}$ | $aT_b^{1/2}$ | $aT_b^{1/2}$
| 18. | $C2/m$ | $C2$ | $A2$ | $A2$ | $a, b \land c, T_{a+b}^{1/2}$ | $T_{a+b}^{1/2}$ | $T_{a+b}^{1/2}$ | $T_{a+b}^{1/2}$

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Table 10. Table of orthorhombic/rectangular layer groups. The
pure translators \(T_a, T_b\) are omitted.

| 1. | 2. | 3. | 4. | 5. | 6. | 7. |
|----|----|----|----|----|----|----|
| 19 | \(p22\) | 16 | \(P22\) | \(P2\) | \(22\) | \(p22\) | \(ab, bc\) |
| 20 | \(p2_1\) | 17 | \(P22_1\) | \(P2\_1\) \(\bar{2}\) | \(p\_2\_1\) \(\bar{2}\) | \(ab, bc T_a \frac{1}{2}\) |
| 21 | \(p2_1\) | 18 | \(P2_1 2_1\) | \(P2\_1\) \(2\) \(\bar{1}\) | \(p\_2\_1\) \(2\) \(\bar{1}\) | \(bc T_a \frac{1}{2}, ac T_b \frac{1}{2}\) |
| 22 | \(c22\) | 21 | \(C22\) | \(\bar{C}2\) \(2\) | \(c2\) \(\bar{2}\) | \(ab, bc, T_{a+b} \frac{1}{2}\) |
| 23 | \(pmm\) | 25 | \(Pmm\) | \(P2\) | \(p2\) | \(a, b\) |
| 24 | \(pma\) | 28 | \(Pma\) | \(P2\_a\) | \(p2\_a\) | \(a, b T_a \frac{1}{2}\) |
| 25 | \(pba\) | 32 | \(Pba\) | \(Pb2\_a\) | \(pb2\_a\) | \(aT_b \frac{1}{2}, b T_a \frac{1}{2}\) |
| 26 | \(cmm\) | 35 | \(Cmm\) | \(C2\) | \(c2\) | \(a, b, T_{a+b}^{1/2}\) |
| 27 | \(pm2\) | 25 | \(Pmm\) | \(P2\) | \(p2\) | \(a, c\) |
| 28 | \(pm2\_b\) | 26 | \(Pmc\_2\) | \(P2\_c\) | \(p\_b\_2\) | \(a, c T_b \frac{1}{2}\) |
| 29 | \(pb2\_m\) | 26 | \(Pmc\_2\) | \(P2\_c\) | \(p\_b\_2\) | \(aT_b \frac{1}{2}, c\) |
| 30 | \(pb2\) | 27 | \(Pcc\) | \(P2\_c\) | \(p\_b\_2\) | \(aT_b \frac{1}{2}, c T_b \frac{1}{2}\) |
| 31 | \(pm2\_a\) | 28 | \(Pma\) | \(P2\_a\) | \(p\_a\_2\) | \(a, c T_a \frac{1}{2}\) |
| 32 | \(pm2\_m\) | 31 | \(Pmm\) | \(P2\_a\) | \(p\_a\_2\) | \(a, c T_{a+b}^{1/2}\) |
| 33 | \(pb2\_a\) | 29 | \(Pca\_2\) | \(Pc2\_a\) | \(p\_b\_a\_2\) | \(aT_b \frac{1}{2}, c T_a \frac{1}{2}\) |
| 34 | \(pb2\_n\) | 30 | \(Pnc\) | \(Pn2\_c\) | \(p\_b\_n\_2\) | \(aT_b \frac{1}{2}, c T_{a+b}^{1/2}\) |
| 35 | \(cm2\) | 35 | \(Cmm\) | \(C2\) | \(c2\) | \(a, c, T_{a+b}^{1/2}\) |
| 36 | \(cm2\_e\) | 39 | \(Aem\) | \(A2\) | \(c\_2\) | \(a, c T_{a+b}^{1/2}, T_{a+b}^{1/2}\) |
| 37 | \(pmmm\) | 47 | \(Pmmm\) | \(P22\) | \(p22\) | \(a, b, c\) |
| 38 | \(pmaa\) | 49 | \(Pcm\) | \(P2\_c\_2\) | \(p2\_a\_2\) | \(a, b T_{a+b}^{1/2}, c T_a \frac{1}{2}\) |
| 39 | \(pbam\) | 50 | \(Pbam\) | \(P\_b\_2\_a\_2\) | \(p\_b\_2\_a\_2\) | \(aT_b \frac{1}{2}, b T_a \frac{1}{2}, c T_{a+b}^{1/2}\) |
| 40 | \(pmam\) | 51 | \(Pmma\) | \(P22\_a\) | \(p2\_a\_2\) | \(a, b T_a \frac{1}{2}, c\) |
| 41 | \(pmm\_m\) | 51 | \(Pmma\) | \(P22\) | \(p22\) | \(a, b, c T_a \frac{1}{2}\) |
| 42 | \(pma\_n\) | 53 | \(Pma\) | \(P2\_n\_2\) | \(p2\_a\_2\) | \(a, b T_{a+b}^{1/2}, c T_a \frac{1}{2}\) |
| 43 | \(pbam\) | 54 | \(Pcma\) | \(P\_2\_c\_2\) | \(p\_2\_a\_2\) | \(aT_b \frac{1}{2}, b T_a \frac{1}{2}, c T_a \frac{1}{2}\) |
| 44 | \(pb\_m\) | 55 | \(Pbam\) | \(P\_b\_2\_a\_2\) | \(p\_b\_2\_a\_2\) | \(aT_b \frac{1}{2}, b T_a \frac{1}{2}, c\) |
| 45 | \(pbm\_a\) | 57 | \(Pbcm\) | \(P\_b\_2\_c\_2\) | \(p\_b\_2\_a\) | \(aT_b \frac{1}{2}, b, c T_{a+b}^{1/2}\) |
| 46 | \(pmm\_n\) | 59 | \(Pmmm\) | \(P22\_n\) | \(p22\_a\) | \(a, b, c T_{a+b}^{1/2}\) |
| 47 | \(cm\_m\) | 65 | \(Cmmm\) | \(C22\) | \(c2\) | \(a, b, c, T_{a+b}^{1/2}\) |
| 48 | \(cm\_m\_e\) | 67 | \(Cmme\) | \(C22\_a\) | \(c2\_a\) | \(a, b, c T_a \frac{1}{2}, T_{a+b}^{1/2}\) |

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Table 11. Table of tetragonal, trigonal and hexagonal layer groups. Layer groups 57, 58 and 71 use special vector notation. The pure translators $T_a$, $T_b$ are omitted.

| 1. | 2. | 3. | 4. | 5. | 6. | 7. |
|----|----|----|----|----|----|----|
| **Tetragonal/square** | | | | | | |
| 49 | $p4$ | 75 | $P4$ | $P4$ | $p4$ | ab |
| 50 | $p4$ | 81 | $P4$ | $P42$ | $p42$ | abc |
| 51 | $p4/m$ | 83 | $P4/m$ | $P42$ | $p42$ | ab, c |
| 52 | $p4/n$ | 85 | $P4/n$ | $P42_1$ | $p42_1$ | ab, $cT_b^{1/2}$ |
| 53 | $p422$ | 89 | $P422$ | $P422$ | $p422$ | ab, bc |
| 54 | $p42_12$ | 90 | $P42_12$ | $P42_12$ | $p42_12$ | ab, $bcT_{2a-b}$ |
| 55 | $p4mm$ | 99 | $P4mm$ | $P4$ | $p4$ | a, b |
| 56 | $p4bm$ | 100 | $P4bm$ | $P_{b4}$ | $p_{b4}$ | $aT_{a-b}$, b |
| 57 | $p42m$ | 111 | $P42m$ | $P24$ | $p24$ | ac, b |
| 58 | $p42_1m$ | 113 | $P42_1m$ | $P2_14$ | $p2_14$ | $aT_{a-b}$, b |
| 59 | $p4m2$ | 115 | $P4m2$ | $P42$ | $p42$ | a, bc |
| 60 | $p4b2$ | 117 | $P4b2$ | $P_{b4}$ | $p_{b4}$ | $aT_{a-b}$, bc |
| 61 | $p4/nmm$ | 123 | $P4/nmm$ | $P42$ | $p42$ | a, b, c |
| 62 | $p4/nbm$ | 125 | $P4/nbm$ | $P_{b4}$ | $p_{b4}$ | $aT_{a-b}$, b, $cT_b^{1/2}$ |
| 63 | $p4/mbm$ | 127 | $P4/mbm$ | $P_{b4}$ | $p_{b4}$ | $aT_{a-b}$, b, c |
| 64 | $p4/nmm$ | 129 | $P4/nmm$ | $P42_n$ | $p42_n$ | a, b, $cT_b^{1/2}$ |
| **Trigonal/hexagonal** | | | | | | |
| 65 | $p3$ | 143 | $P3$ | $P3$ | $p3$ | ab |
| 66 | $p3$ | 147 | $P3$ | $P62$ | $p62$ | abc |
| 67 | $p312$ | 149 | $P312$ | $P32$ | $p32$ | ab, bc |
| 68 | $p321$ | 150 | $P321$ | $H32$ | $h32$ | ab, bc |
| 69 | $p3m1$ | 156 | $P3m1$ | $P3$ | $p3$ | a, b |
| 70 | $p31m$ | 157 | $P31m$ | $H3$ | $h3$ | a, b |
| 71 | $p31m$ | 162 | $P31m$ | $P26$ | $p26$ | ac, b |
| 72 | $p3m1$ | 164 | $P3m1$ | $P62$ | $p62$ | a, bc |
| **Hexagonal/hexagonal** | | | | | | |
| 73 | $p6$ | 168 | $P6$ | $P6$ | $p6$ | ab |
| 74 | $p6$ | 174 | $P6$ | $P32$ | $p32$ | ab, c |
| 75 | $p6/m$ | 175 | $P6/m$ | $P62$ | $p62$ | ab, c |
| 76 | $p622$ | 177 | $P622$ | $P62$ | $p62$ | ab, bc |
| 77 | $p6mm$ | 183 | $P6mm$ | $P6$ | $p6$ | a, b |
| 78 | $p6m2$ | 187 | $P6m2$ | $P32$ | $p32$ | a, b, c |
| 79 | $p62m$ | 189 | $P62m$ | $H32$ | $h32$ | a, b, c |
| 80 | $p6/mmm$ | 191 | $P6/mmm$ | $P62$ | $p62$ | a, b, c |
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Eckhard Hitzer  
College of Liberal Arts, Department of Material Science,  
International Christian University,  
181-8585 Tokyo, Japan  
e-mail: hitzer@icu.ac.jp

Daisuke Ichikawa  
Department of Applied Physics,  
University of Fukui,  
910-8507 Fukui, Japan  
e-mail: seiuunnedved3032@yahoo.co.jp