Asymmetric Mach-Zehnder atom interferometers

B. Dubetsky
US Army Research Laboratory, Adelphi, MD 20783 and
Department of Physics, Stanford University, Stanford, California 94305, USA
(Dated: November 26, 2019)

It is shown that using beam splitters with non-equal wave vectors results in a new recoil diagram which is qualitatively different from the well-known diagram associated with the Mach-Zehnder atom interferometer. We predict a new asymmetric Mach-Zehnder atom interferometer (AMZAI) and study it when one uses a Raman beam splitter. The main feature is that the phase of AMZAI contains a quantum part proportional to the recoil frequency. A response sensitive only to the quantum phase was found. A new technique to measure the recoil frequency and fine structure constant is proposed and studied outside of the Raman-Nath approximation.

PACS numbers: 03.75.Dg; 37.25.+k; 04.80.-y

It is well-known that atom interferometry [1] is caused by the quantization of the atomic center-of-mass motion. When the incident atomic momentum state \(| p \rangle\) splits into two states \(| p \rangle\) and \(| p + \hbar k \rangle\) after passing through a beam splitter having effective wave vector \(k\), the coherence between these states evolves as

\[
\rho(p + \hbar k, p, t) \propto \exp(-i\omega p + \hbar k, p, t) \tag{1}
\]

where the frequency of transition between states

\[
\omega_{p + \hbar k, p} = \frac{1}{2M\hbar} [(p + \hbar k)^2 - p^2] = k \cdot \frac{\hbar k^2}{M} + \frac{\hbar k^2}{2M} \tag{2}
\]

contains the quantum term, recoil frequency

\[
\omega_k = \frac{\hbar k^2}{2M} \tag{3}
\]

where \(M\) is an atomic mass. When \(t\) is of the order of interrogation time \(T\) of the given atomic interferometer, one can expect the phase of the interferometer to contain the quantum contribution

\[
\phi_q \sim \omega_k T \tag{4}
\]

which would reveal the quantum nature of the atom interference. This phase leads to the recoil splitting of the optical Ramsey fringes [2], Talbot effect, i.e. quantum beats of the atom interferometer signal, in the standing wave fields [1, 3, 4] and microfabricated structures [5]. Precise measurement of the phase (4) was performed [6, 7] using a Raman analogue of the atom interferometer involving 4 time-separated counterpropagating traveling waves. The value of \(\omega_k\) was proposed for measuring the fine structure constant [6], which resulted [8] in a resolution in this constant of 0.25 ppb.

In spite of this, the phase of the well-known and widely used Mach-Zehnder atom interferometer (MZAI), in which an atom passes through 3 beam splitters separated in time with a delay \(T\), contains no quantum term.

In the uniform gravitational field \(g\) the phase is given by

\[
\phi = k \cdot g T^2 \tag{5}
\]

The reason is that quantum corrections affect the atomic position at the moments of interaction with the 2nd and 3rd beam splitters, but in the uniform gravity field these contributions cancel one another (see for example Appendix in [10]). We would like to underline that derivations of the MZAI phase is purely quantum (see examples of this derivation in [9, 11]), but the result of those derivations (4) is purely classical. The quantum contribution to the MZAI phase arises in the rotating frame [12], or in the non-uniform field, in the presence of the gravity-gradient tensor [13, 14] or in the presence of the gravity curvature tensor [10], or in the strongly non-uniform field of the external test mass [10], the quantum part of the phase in gravity-gradient field of the external test mass has been recently observed [15].

The absence of the quantum phase [4] allows one to doubt that MZAI is caused by matter wave interference. In this article, I propose a modification of the MZAI which contains the term (4). Moreover, we found a response that contains only the quantum phase. Since it is insensitive to gravity, vibration, phase and frequency noise, this response can be used to measure recoil frequency and fine structure constant. An advantage of our approach compared to the approach, using conjugate atom interferometers in the Bragg regime [16], is that our effect is insensitive to phase and frequency noise and has no diffraction phase [17].

In our modification, we first tried to use beam splitters with different effective wave vectors. If the beam splitter is a pulse of standing wave [1], or Raman pulse [11], then usually the field consists of two counterpropagating traveling waves having wave vectors \(q_1\) and \(q_2 \approx -q_1\). The effective wave vector of this beam splitter is

\[
k = q_1 - q_2 \approx 2q_1 \tag{6}
\]

One can obtain a value of \(k\) different from that given by Eq. (6) if the traveling waves are not counterpropagating \((q_1 \neq -q_2)\). In this case, the magnitude of the
The recoil diagram of the MZAI with non-equal wave vectors is shown in Fig. 2. If $|p\rangle$ is an incident momentum state, then the beam splitter produces, in addition to state $|p\rangle$, the scattered momentum state $|p \pm \hbar k_0\rangle$. The scattered momentum states after the 3rd beam splitter are shown in blue.

The recoil diagram of the MZAI is shown in Fig. 1. Calculations show that the change of momentum is accompanied by a change of the internal atomic label and leads to interference. Specifically, the incident state $|g, p\rangle$ splits into states $|g, p\rangle$ and $|e, p + \hbar k\rangle$, while the incident state $|e, p\rangle$ splits into states $|g, p\rangle$ and $|e, p - \hbar k\rangle$. From the recoil diagram it is evident that the choice of the wave vectors and times $T_i$ has to obey 2 constraints. The first constraint is that the blue and black lines in each port have to be parallel, i.e.

$$k_1 = k_2 - k_3,$$

which is a well-known phase matching condition. The second constraint is that $D$ is the point of crossing and therefore $A\overrightarrow{D} = A\overrightarrow{C} + C\overrightarrow{D}$, which means that

$$k_1 (T_3 - T_1) = k_2 (T_3 - T_2).$$

One can get the same equality from the constraint $A\overrightarrow{D} = A\overrightarrow{B} + B\overrightarrow{D}$. Choosing the wave vector $k_0$ as an independent variable, one can get from constraints (7) that other wave vectors have to be equal to

$$k_1 = (1 - s) k_2,$$

$$k_3 = sk_2,$$

where

$$s = (T_2 - T_1) / (T_3 - T_1).$$

If one uses a Raman pulse for beam splitting, then the change of momentum is accompanied by a change of the atomic internal states $|g\rangle$ (ground) and $|e\rangle$ (excited). Specifically, the incident state $|g, p\rangle$ splits into states $|g, p\rangle$ and $|e, p + \hbar k\rangle$, while the incident state $|e, p\rangle$ splits into states $|e, p\rangle$ and $|g, p - \hbar k\rangle$. From the recoil diagram, one finds in each of the output ports different atomic internal states which cannot interfere with each other.

A similar situation occurs for a Stern-Gerlach beam splitter, where the use of an additional microwave pulse changes the internal atomic label and leads to interference [38]. I also propose applying here the microwave
pulse resonant to transition $g \to e$ just before (the case considered here) or after 3rd Raman pulse [26]. It is easier to perform calculations than to draw the recoil diagram corresponding to this set of pulses.

Let’s assume that atoms are launched at $t = 0$ and interact with 3 Raman pulses. Pulse $i$ is applied at time $T_i$ and has the area, effective wave vector, phase and Raman detuning $\theta_i$, $k_i$, $\phi_i$ and $\delta_i$. In addition the microwave pulse, having the area, effective wave vector, phase and detuning $\theta_m, k_m = 0, \phi_m$ and $\delta_m$, is applied at time $T_m$.

I assume, for simplicity, that all pulses have the same duration $\tau$. The calculations were performed using Wigner representation for the atomic density matrix. To get the AMZAI phase, I used equations for the density matrix evolution in times between pulses and equations for the density matrix change immediately after the interaction with a given pulse, i.e. Eqs. (21, 48) in [14]. Regarding the microwave pulse time $T_m$, I assume that

$$0 < T_3 - T_m \ll T_3$$

so that one can ignore the density matrix evolution at $T_m + \tau < t < T_3$. If initially all atoms are in the ground state,

$$\rho_{gg} (x, p, 0) = f (x, p), \rho_{ee} (x, p, 0) = \rho_{eg} (x, p, 0) = 0,$$

then I calculate the probability of atoms’ excitation immediately at time $T_3 + \tau$

$$w = \int dp dx \rho_{ee} (x, p, T_3 + \tau). \quad (13)$$

The straightforward, but extended, calculations of response [13] will be omitted here. The interaction which I consider here is a particular case of the interaction with pulses considered previously in Sec. VII of the article [13], where the 3rd pulse is a microwave pulse, the 4th pulse becomes the 3rd one, and the processes shown in Fig. 2 correspond to processes called in [13] "stimulated echo". Saving only these processes and background terms one finds

\[
\begin{align}
  w &= \bar{w} + w_{1+} + w_{1-} + w_{11+} + w_{11-}, \\
  \bar{w} &= \frac{1}{2} \left( 1 - \cos \theta_1 \cos \theta_2 \cos \theta_3 \right), \\
  \begin{cases}
  w_{1\pm} \\
  w_{11\pm}
\end{cases} &= -\frac{1}{8} \sin \theta_3 \sin \theta_m \sin \theta_2 \cos \theta_1 \int dxdp f (x, p) \begin{cases}
  \cos (\phi_{1\pm} + \phi_1) \\
  \cos (\phi_{11\pm} + \phi_{11})
\end{cases}, \\
  \phi_{11} &= -\delta_3 T_3 + \delta_m T_m \pm \delta_2 T_2 \mp \delta_1 T_1 - \phi_3 + \phi_m \mp \phi_2 + \phi_1 - \arg (\Omega_3 \Omega_m) \mp \arg (\Omega_2 \Omega_1^*), \\
  \phi_{1\pm} &= \begin{cases}
  k_3 \cdot x_{3\pm} - k_2 \cdot x_2 + k_1 \cdot x_1 \\
  k_3 \cdot x_{3\pm} + k_2 \cdot x_2 - k_1 \cdot x_1
\end{cases}, \\
  \begin{cases}
  x_1, p_1 \\
  x_{3\pm}
\end{cases} &= \begin{cases}
  X (x, p, T_1), P (x, p, T_1) + \frac{\hbar k_1}{2} \\
  X (x_{1, p_1, T_2 - T_1}), P (x_{1, p_1, T_2 - T_1}) \mp \frac{\hbar k_2}{2}
\end{cases}, \\
  x_{2, p_{2\pm}} &= X (x_{2, p_{2\pm}, T_3 - T_2), \\
  x_{3\pm} &= \begin{cases}
  \begin{cases}
  X (x_{1, p_1, T_2 - T_1}), P (x_{1, p_1, T_2 - T_1}) \mp \frac{\hbar k_2}{2}
\end{cases}
\end{cases},
\end{align}
\]

where $\Omega_n$ is a two-photon Raman Rabi frequency of the Raman pulse $n$, $\Omega_m$ is a Rabi frequency of the microwave pulse, \{X (x, p, t), P (x, p, t)\} is a point of the atom trajectory in the phase space, and \{x, p\} is the initial point of this trajectory. I consider here for simplicity only atom motion in the uniform gravity field $g$, when

\[
\begin{align}
  X (x, p, t) &= x + p M t + g t^2, \\
  P (x, p, t) &= p + M g t.
\end{align}
\]
Using these dependencies one obtains
\[
\phi_{I\pm} = (k_3 - k_2 + k_1) \cdot x + (k_3 T_3 - k_2 T_2 + k_1 T_1) \cdot \frac{\mathbf{P}}{M} + \frac{1}{2} (k_3 T_3^2 - k_2 T_2^2 + k_1 T_1^2) \cdot \mathbf{g} + \frac{\hbar}{2M} \{k_3 \cdot [k_1 (T_2 - T_1) + (k_1 \mp k_2) (T_3 - T_2)] - k_2 \cdot k_1 (T_2 - T_1)\}.
\] (16)

One sees that, owing to the first two terms in Eq. (16), the integrand for \( w_{I\pm} \) in Eq. (14c) rapidly oscillates in the phase space, which can wash out the interference signal. However, owing to the constraints (7, 8) the first two terms are exactly equal to 0. Substituting Eq. (9) into Eq. (10), one finds
\[
\phi_{I\pm} = \phi_c \mp \phi_q,
\] (17a)
\[
\phi_c = \frac{1}{2} k_2 \cdot \mathbf{g} (T_3 - T_1)^2 s (1 - s),
\] (17b)
\[
\phi_q = \omega_k (T_3 - T_1) s (1 - s).
\] (17c)

One can find that for wave vectors (9), terms \( w_{I\pm} \) are washed out and the sum of the interference terms \( w_{I+} + w_{I-} \) contains separately a factor depending on the quantum phase (17c) and a factor depending on the all other phases, \( \phi_c, \phi_1, \delta I T_j \), i.e.
\[
w = \bar{w} - \frac{1}{4} \sin \theta_3 \sin \theta_m \sin \theta_2 \sin \theta_1 \cos (\phi_q) \cos (\phi_c + \phi_I).
\] (18)

If one monitors the signal (18) as a function of any phase or detuning of the Raman or microwave field, then the difference of the maximum and minimum of the signal is given by
\[
A = w_{\text{max}} - w_{\text{min}} = \frac{1}{2} |\sin \theta_3 \sin \theta_m \sin \theta_2 \sin \theta_1 \cos (\phi_q)|.
\] (19)

The oscillating dependence in (19) on the interrogation time \( T_3 - T_1 \) is caused only by the quantization of the atomic center-of-mass motion and reveals the quantum nature of the atom interference. Evidently, the signal (19) is insensitive to the gravity field, vibration noise, and phase noise of the laser fields.

The quantum phase is maximal when \( k_2 = k \) and \( s = 1/2 \), or \( k_1 = k_3 = k/2, T_2 = (T_1 + T_3)/2 \). This means that the 2nd pulse still consists of counterpropagating fields, but the 1st and 3rd pulses have to consist of traveling waves having \( \pi/3 \) angle between their wave vectors. In this case
\[
\phi_q = \frac{1}{4} \omega_k (T_3 - T_1).
\] (20)

The contrast of the signal (19),
\[
c = \frac{\left( w_{\text{max}} - w_{\text{min}} \right)_{\phi_x = 0}}{\bar{w}}
\] achieves 100% for \( \pi/2 - \pi/2 - \pi/2 - \pi/2 \) sequence of the pulses.

Phases \( \phi_{I\pm} \) can be considered in the same manner. To avoid washing out the \( w_{I\pm} \) terms, one has to choose the same value \( \text{im} \) of the wave vector \( k_1 \) and opposite value of the wave vector \( k_3, k_3 = -sk_2. \) With this choice of wave vectors, the terms \( w_{I\pm} \) are washed out, while the phases change their sign, \( \phi_{I\pm} = -\phi_{I\pm}. \)

The equations for the density matrix in the Wigner representation, derived in (13) have a limited region of validity. One can use them only if the Raman detunings \( \delta_n \) compensate for the recoil frequency \( \omega_{k_n}. \) One can easily achieve the compensation for MZAI, since all beam splitters have the same effective wave vector \( k. \) The situation changes for AMZAI: since one could not achieve compensation for all wave vectors simultaneously, the approach (13) becomes valid only in the Raman-Nath approximation, when the pulse duration \( \tau \) is significantly smaller than an inverse recoil frequency
\[
\omega_{k_n} \tau \ll 1.
\] (21)

For the 2-quantum transition in Rb\(^{87} \omega_{k_1}^{-1} \approx 10.6 \mu s. The minimal duration of the pulses used in the atom interferometer is 50ns (27), which is sufficient for the Raman-Nath approximation. But in experiments exploring Raman beam splitters, pulse durations are in the range \((4 \div 35) \mu s (21, 28, 38). To describe AMZAI outside of the Raman-Nath approximation, I used the equation for an atomic wave function in momentum space. The solution of this equation in the uniform field (23) can be applied for both the wave function evolution between pulses and inside a given Raman pulse if one chirps the pulse frequency with a rate \( \alpha_n \) and
\[
|\alpha_n - k_n \cdot \mathbf{g}| \tau^2 \ll 1,
\] (22)
(see appendices \( \text{E} \) and \( \text{C}. \) The background of the probability of excitation \( \bar{w} \) is given by Eq. (C31), while for the interferometric part of this probability one obtains from Eqs. (C29a, C30a)
\[
\bar{w} = -2 |\mathbf{x}| \cos (\phi_q) \sqrt{|x_+|^2 + |x_-|^2 + 2|x_+||x_-| \cos \left\{ 2r + \arctan \frac{\text{Im} (x_-)}{\text{Re} (x_-)} - \arctan \frac{\text{Im} (x_+)}{\text{Re} (x_+)}\right\}},
\] (23a)
\[
\phi_q = \phi_I + \phi_c + \left( \delta_m - \delta_1 \mathbf{p}_0 + \omega_k (1 - s)^2 \right) \tau
- \arctan \frac{\text{Im} (x)}{\text{Re} (x)} - \arctan \frac{\cos r \text{Im} (x_+ + x_-) - \sin r \text{Re} (x_+ - x_-)}{\cos r \text{Re} (x_+ + x_-) + \sin r \text{Im} (x_+ - x_-)}.
\] (23b)
where $P_0$ is an initial atomic momentum and $\delta_n, P$ is a detuning associated with pulse $n$. Expression (23) has been derived for a rectangular shape of the Raman and microwave pulses. One can accept that the pulses are rectangular if the durations of the forward and backward fronts are significantly smaller than the inverse recoil frequency.

The quantum phase $\phi_q$ is included now in parameter $r$ [see Eq. (C30b)]. It cannot now be separated from other phase factors. In spite of this, after monitoring the signal (23a), as a function of any Raman field phase, one finds that the response (19) still depends only on the quantum phase,

$$A = 4|x| \sqrt{|x_+|^2 + |x_-|^2 + 2|x_+| x_- \cos \left[2r + \arctan \frac{\text{Im} (x_-)}{\text{Re} (x_-)} - \arctan \frac{\text{Im} (x_+)}{\text{Re} (x_+)} \right]},$$  \hspace{1cm} (24)

One can define the magnitude of the response (24) as the difference $R = A_{\text{max}} - A_{\text{min}}$, which is equal to

$$R = 8|x| \text{Min} \{|x_+|, |x_-|\}. \hspace{1cm} (25)$$

To maximize this magnitude one evidently has to choose $\delta_1 P_n = (1-s)^2 \Omega_k \omega_k$, $\delta_m = 0$, $\theta_1 = \theta_m = \pi/2$, where we defined pulses’ areas as

$$\theta_n = \Omega_n \tau. \hspace{1cm} (26)$$

For this choice one finds that $|x| = 1/4$. I assumed that 2nd and 3rd pulses still have areas $\theta_2 = \theta_3 = \pi/2$ and found numerically optimum values of the Raman detunings for $s = 1/2$. These values are shown in Fig. 4 while the maximal magnitude $R$, background $\bar{w}$ and contrast $c = R/w$ are shown in Fig. 5. One period of the dependence (24) for the different values of recoil frequency is shown in Fig. 6.

One sees that outside of the Raman-Nath approximation, one can achieve almost the same magnitude of the AMZAI with the proper choice of the Raman detunings, while the interference picture shifts as a whole. The AMZAI becomes sensitive to the fields’ frequencies, but requirements for the frequencies’ stabilization are less stringent compared to the conjugate interferometers technique \[10\] in a parameter

$$\tau/T. \hspace{1cm} (27)$$

In this article I consider only the Raman pulse beam splitter. I expect that AMZAI should also be built using a standing wave beam splitter \[1\] and Raman standing wave beam splitter \[39\]. I plan to examine these cases elsewhere.

AMZAI can be used to measure recoil frequency $\omega_k$. In contrast to other interferometers used for these measurements \[3,4,6–8\], AMZAI does not use counterpropagating effective wave vectors or standing waves. Another useful feature is that after monitoring the response as a function of the Raman field phase and measuring the difference between maximal and minimal values of the re-
sponse, one gets the signal\(^\text{[19]}\) insensitive to the gravity, vibration and phase noise. Calculations showed that the similar signal occurs in the interferometers\(^\text{[8]}\), but to our knowledge monitoring of the response for given time delay between pulses and measuring the difference between response’s maximum and minimum has never been used in those experiments. Here I propose this technique to get the signal sensitive only to the recoil frequency.

In this article a technique to obtain a signal insensitive to gravity is proposed. It was justified above only for uniform gravity. I verified that this signal can also be obtained in the presence of the gravity-gradient and Coriolis forces, if these forces are small. Only if these forces are sufficiently large, or the time delay between pulses is sufficiently long, then one can use exact expressions for the atomic trajectories\(^\text{[14][14][14]}\) derived in\(^\text{[41]}\) to study corresponding systematic errors in the measurement of the recoil frequency using AMZAI or technique\(^\text{[8]}\).

Acknowledgments

I appreciate fruitful discussions and suggestions from Paul R. Berman and Mark A. Kasevich. This research was sponsored by Stanford University, VPFF program, and by the Army Research Laboratory and was accomplished under Cooperative Agreement Number W911NF-16-2-0146.

[1] B. Dubetsky, A. P. Kazantsev, V. P. Chebotayev, V. P. Yakovlev, Pis’ma Zh. Eksp. Teor. Fiz. 39, 531 (1984) [JETP Lett. 39, 649 (1984)].
[2] Ye. V. Baklanov, B. Dubetsky, V. M. Semibalamut, Zh. Eksp. Teor. Fiz. 76, 482 (1979) [JETP 49, 244 (1979)].
[3] V. P. Chebotayev, B. Dubetsky, A. P. Kazantsev, V. P. Yakovlev, J. Opt. Soc. Am. B 2, 1791 (1985).
[4] S. B. Cahn, A. Kumarakrishnan, U. Shim, T. Sleator, P. R. Berman, B. Dubetsky, Phys. Rev. Lett. 79, 784 (1997).
[5] M. S. Chapman, C. R. Ekstrom, T. D. Hammond, J. Schmiedmayer, B. E. Tannian, S. Wehinger, D. E. Pritchard, Phys. Rev. A 51, R14 (1995).
[6] D. S. Weiss, B. C. Young, S. Chu, Phys. Rev. Lett., 70, 2706 (1993).
[7] Ch. J. Borde, Phys. Lett. 140, 10 (1989).
[8] B. Œstey, C. Yu, and H. Müller, Phys. Rev. Lett. 115, 083002 (2015).
[9] Expression\(^\text{[5]}\) for the phase has been first derived for neutron interferometer, see Eq. (4.8) in D. M. Greenberger, A. W. Overhauser, Rev. Mod. Phys. 51, 43 (1979).
[10] B. Dubetsky, S. B. Libby and P. Berman, Atoms 4, 14 (2016).
[11] M. Kasevich, S. Chu, Phys. Rev. Lett., 67, 181 (1991).
[12] B. Dubetsky and P. R. Berman, Phys. Rev. A 56, R1091 (1997).
[13] K. Bongs, R. Launay, and M. A. Kasevich, Appl. Phys. B 84, 599 (2006).
[14] B. Dubetsky, M. A. Kasevich, Phys. Rev. A 74, 023615 (2006).
[15] P. Asenbaum, C. Overstreet, T. Kovachy, D. D. Brown, J. M. Hogan, and M. A. Kasevich, Phys. Rev. Lett. 118, 183602 (2017).
[16] S.-w. Chiow, S. Herrmann, S. Chu, and H. Müller, Phys. Rev. Lett. 103, 050402 (2009).
[17] M. A. Kasevich, Private communication (2017)
[18] T. Sleator, T. Pfau, V. Balykin, J. Mlynek, Appl. Phys. B 54, 375 (1992).
[19] B. Dubetsky and P. R. Berman, Phys. Rev. A 58, 2413 (1998).
[20] J. L. Cohen, B. Dubetsky, and P. R. Berman, Phys. Rev. A 60, 3982 (1999).
[21] T. Kovachy, P. Asenbaum, C. Overstreet, C. A. Donnelly, S. M. Dickerson, A. Sugarbaker, J. M. Hogan & M. A. Kasevich, Nature 528, 530 (2015).
[22] A. Roura, Phys. Rev. Lett. 118,160401 (2017).
[23] C. Overstreet, P. Asenbaum, T. Kovachy, R. Notermans, J. M. Hogan, M. A. Kasevich, https://arxiv.org/abs/1711.09986.
[24] G. D’Amico, G. Rosi, S. Zhan, L. Cacciapuoti, M. Fattori, and G. M. Tino, Phys. Rev. Lett. 119, 253201 (2017).
[25] B. Dubetsky, G. Raithel, http://arxiv.org/abs/physics/0206029.
[26] One can expect that a necessity of the microwave pulse disappears if one uses the standing wave beam splitter\(^\text{[1]}\) in the Raman-Nath or Bragg regime, which does not change the internal state of the atom.
[27] C. Mok, B. Barrett, A. Carew, R. Berthiaume, S. Beattie, and A. Kumarakrishnan, Phys. Rev. A 88, 023614 (2013).
[28] B. Barrett, L. Antoni-Micollier, L. Chichet, B. Battevier, P.-A. Gominet, A. Bertoldi, P. Bouyer, A. Landragin, New J. Phys. 17, 085010 (2015).
[29] X. Zhang, R. P. del Aguila, T. Mazzoni, N. Poli, and G. M. Tino, Phys. Rev. A 94, 043608 (2016).
[30] B. Barrett, L. Antoni-Micollier, L. Chichet, B. Battevier, T. Lévéque, A. Landragin & P. Bouyer, Nat. Commun. 7, 13786 (2016).
[31] J. M. Hogan, D. M. S. Johnson, M. A. Kasevich, https://arxiv.org/abs/0806.3261.
[32] T. Farah, P. Gillot, B. Cheng, A. Landragin, S. Merlet, and F. Pereira Dos Santos, Phys. Rev. A 90, 023606 (2014).
[33] G. D’Amico, F. Borselli, L. Cacciapuoti, M. Prevedelli, G. Rosi, F. Sorrentino, and G. M. Tino, Phys. Rev. A 93, 063628 (2016).
[34] J. K. Stockton, T. Takase, and M. A. Kasevich, Phys. Rev. Lett. 107, 130401 (2011).
[35] M. Meunier, I. Dutta, R. Geiger, C. Guerlin, C. L. Garrido Alzar, and A. Landragin, Phys. Rev. A 90, 063633 (2014).
[36] G. Rosi, G. D’Amico, L. Cacciapuoti, F. Sorrentino, M. Prevedelli, M. Zych, C. Brukner, G. M. Tino, Nat. Commun. 8, 15529 (2017).
[37] J. M. McGuirk, M. J. Snadden, and M. A. Kasevich, Phys. Rev. Lett. 85, 4498 (2000).

[38] S. M. Dickerson, J. M. Hogan, A. Sugarbaker, D. M. S. Johnson, and M. A. Kasevich, Phys. Rev. Lett. 111, 083001 (2013).

[39] B. Dubetsky and P. R. Berman, Laser Physics, 12, 1161 (2002), the Raman standing wave scheme is also known as double-diffraction technique [40].

[40] T. Leveque, A. Gauguet, F. Michaud, F. Pereira Dos Santos, A. Landragin, Phys. Rev. Lett. 103, 080405 (2009).

[41] M. A. Kasevich and B. Dubetsky, U.S. Patent 7,317,184, 8 January 2008.

[42] V. S. Letokhov, B. D. Pavlik, Optika i Spektosk. 32, 856 (1972) [Opt. Spectrosc. 32, 456 (1972)].

[43] N. F. Ramsey, Phys. Rev. 76, 996 (1949).

[44] A. N. Oraevsky, IEEE Trans. IM-17, 346 (1968)
Supplemental Material

Appendix A: MZAI with nonequal effective wave vectors

In this section, we calculate the phase of the MZAI shown in the Fig. 1. Let us consider an interaction of the atom with 3 Raman pulses having effective wave vectors \( k_1 \neq k_2 \neq k_3 \). The choice of the wave vectors \( k_i \) and times \( T_i \) has to obey 2 constraints again. From the requirement that the blue and black lines in each port in Fig. 1 have to be parallel, i.e.

\[
k_1 - 2k_2 + k_3 = 0,
\]  

(A1)

The second constraint is that \( D \) is the point of crossing and therefore \( \vec{AB} + \vec{BD} = \vec{AC} + \vec{CD} \), which means that

\[
k_1 (T_2 - T_1) + (k_1 - k_2) (T_3 - T_2) = k_2 (T_3 - T_2).
\]  

(A2)

Resolving Eqs. (A1, A2) in respect to the wave vectors \( k_1 \) and \( k_3 \), one gets

\[
k_1 = 2(1 - s)k_2,
\]  

(A3a)
\[
k_3 = 2sk_2,
\]  

(A3b)

where \( s \) is still given by Eq. (10). If initially an atomic density matrix is given by Eq. (12), then applying consequently Eq. (21) from the article for the density matrix evolution before and between the pulses and Eqs. (48) from the same article for the density matrix jump inside given pulse, one finds that after the 3rd pulse action, the upper state
distribution is given by

\[ \rho_{ee}(x, p, T_{3+}) = \rho_b(x, p) + \rho_R(x, p) + \rho_i(x, p), \tag{A4a} \]

\[ \rho_b(x, p) = \cos^2(\theta_3/2) \left\{ \cos(\theta_2/2) \sin^2(\theta_1/2) \times f \left[ X(x_1, p_1 - \hbar k_1, -T_1), P(x_1, p_1 - \hbar k_1, -T_1) \right] \right. \]

\[ + \left. \sin^2(\theta_2/2) \cos^2(\theta_1/2) f \left[ X(x_2, p_2 - \hbar k_2, -T_2), P(x_2, p_2 - \hbar k_2, -T_2) \right] \right\} \left\{ x_1, p_1 \right\} \]

\[ \times \left\{ x_2, p_2 \right\} = \left\{ X(x, p, T_{3+}), P(x, p, T_{3+}) \right\} \tag{A4b} \]

\[ \rho_R(x, p) = \frac{1}{2} \sin \theta_2 \sin \theta_1 \cos \left( k_1 \cdot x_1 - \delta_1 T_1 - \phi_1 - (k_2 \cdot x_2 - \delta_2 T_2 - \phi_2) \right) \]

\[ \times f \left[ \left\{ X(x_1, p_1 - \hbar k_1/2, -T_1), P(x_1, p_1 - \hbar k_1/2, -T_1) \right\} \right. \left\{ x_1, p_1 \right\} \]

\[ \times \left\{ x_2, p_2 \right\} = \left\{ X(x, p, T_{3+}), P(x, p, T_{3+}) \right\} \tag{A4c} \]

\[ \rho_i(x, p) = \frac{1}{2} \sin \theta_3 \sin^2(\theta_2/2) \sin \theta_1 \cos \left[ 2 \left( k_2 \cdot x_2 - \delta_2 T_2 - \phi_2 \right) - (k_1 \cdot x_1 - \delta_1 T_1 - \phi_1) - \left( k_3 \cdot x - \delta_3 T_3 - \phi_3 \right) \right] \]

\[ \times f \left[ \left\{ X(x_1, p_1 - \hbar k_1/2, -T_1), P(x_1, p_1 - \hbar k_1/2, -T_1) \right\} \right. \left\{ x_1, p_1 \right\} \]

\[ \times \left\{ x_2, p_2 \right\} = \left\{ X(x, p, T_{3+}), P(x, p, T_{3+}) \right\} \tag{A4d} \]

where \( \theta_i \) is the area of the pulse \( i \), \( \{ X(x, p, t), P(x, p, t) \} \) are atomic classical position and momentum subject to initial value \( \{ x, p \} \). The expression \( A4b \) can be used to calculate any response associated with atoms on the upper level. The term \( A4b \) is caused by transfering of the atomic populations between Raman pulses first considered in \[42\]. This term is responsible for the background of the atomic excitation to the state \( |e⟩ \).

The term \( A4c \) is caused by transfering of the atomic coherence between two Raman pulses, saturated in the field of third Raman pulse. This term is responsible for the set of Ramsey fringes \[43\]. In the Doppler limiting case

\[ k_3 \delta p \left( T_2 - T_1, T_3 - T_2 \right) \gg 1, \tag{A5} \]

where \( \delta p \) is the width of the atomic momentum distribution. Ramsey fringes are washed out \[44\].

The term \( A4d \) is caused by the transfering coherence \( \rho_{ee} \) between first and second pulse, mirroring it to the coherence \( \rho_{ee} \) in the field of the second Raman pulse and further transfering between second and third Raman pulses. This process is responsible for the atom interference.

We use Eqs. \( A4 \) to get the probability of the excitation

\[ w = \int dx \rho_{ee}(x, p, T_{3+}). \tag{A6} \]
which consists of the background and interference terms,

\[ w = \bar{w} + \bar{w}. \]  

(A7)

For the background term one finds from the Eq. (A4b)

\[ \bar{w} = 1 - \cos \theta_3 \cos \theta_2 \cos \theta_1. \]  

(A8)

To get interference term from Eq. (A4d) it is convenient to use initial atomic position and momentum,

\[ \{x', p'\} = \{X(x_1, p_1 - \hbar k_1/2, -T_1), P(x_1, p_1 - \hbar k_1/2, -T_1)\}, \]  

(A9)

as the integration variables. Expressing other phase points in Eq. (A4d) through the point (A9) after replacement \(\{x', p'\} \rightarrow \{x, p\}\) one gets

\[ \bar{w} = -\frac{1}{2} \sin \theta_3 \sin^2(\theta_2/2) \sin \theta_1 \int dx dp f(x, p) \cos [\phi(x, p) - \delta_1 T_1 + 2\delta_2 T_2 - \delta_3 T_3 - \phi_1 + 2\phi_2 - \phi_3], \]  

(A10a)

\[ \phi(x, p) = k_1 \cdot x_1 - 2k_2 \cdot x_2 + k_3 \cdot x_3, \]  

(A10b)

\[ \{x_1, p_1\} = \{X(x, p, T_1), P(x, p, T_1) + \hbar k_1/2\}, \]  

(A10c)

\[ \{x_2, p_2\} = \{X(x_1, p_1, T_2 - T_1), P(x_1, p_1, T_2 - T_1)\}, \]  

(A10d)

\[ \{x_3, p_3\} = \{X(x_2, p_2, T_3 - T_2), P(x_2, p_2, T_3 - T_2) + \hbar k_3/2\}, \]  

(A10e)

where the phase of the MZAI is given by Eq. (A10a). In the case of the uniform gravity field \(g\)

\[ \{X(x, p, t), P(x, p, t)\} = \left\{x + \frac{p}{M}t + \frac{gt^2}{2}, p + Mg t\right\}, \]  

(A11)

for the wave vectors (A3) one arrives for the phase

\[ \phi(x, p) = s(1 - s) k_2 \cdot g(T_3 - T_1)^2, \]  

(A12)

which still contains no quantum term.

**Appendix B: Atomic amplitudes in the momentum state**

In this appendix, we use the expressions for the atomic amplitudes’ evolution in the uniform field [25]. For the purpose of integrity we consider both the amplitudes’ evolution between Raman pulses [25] and during the interaction with a given pulse. Consider an interaction of the 3-level atom with a rectangular pulse of the 2 traveling waves

\[ E(x, t) = \left\{E_1 \exp \left[i \left(q_1 \cdot x - \omega_1 t - \phi(1)(t)\right)\right] + E_2 \exp \left[i \left(q_2 \cdot x - \omega_2 t - \phi(2)(t)\right)\right]\right\} f(t) + c.c. \]  

(B1)

where \(E_i, q_i, \omega_i\) and \(\phi(i)(t)\) are amplitude, wave vector, frequency and phase of the waves,

\[ f(t) = \left\{\begin{array}{ll}
1, & \text{for } T < t < T + \tau \\
0, & \text{for } t < T \text{ and } t > T + \tau
\end{array}\right. \]  

(B2)

is the shape of the rectangular pulse acting at the moment \(T\) and having a duration \(\tau\). We assume that the 2 atomic states, \(g\) and \(e\), are the sublevels of the atomic ground state, while the third state 0 is an excited state, fields \(E_1\) and \(E_2\) are resonant to the transitions \(g \rightarrow 0\) and \(e \rightarrow 0\) correspondingly. The Hamiltonian of the atom is given by

\[ H = \frac{P^2}{2M} - Mg \cdot x + \hbar \left\{\Omega^{(1)} \exp \left[i \left(q_1 \cdot x - \Delta_1 t - \phi(1)(t)\right)\right] \right\} |0\rangle \langle g| + \Omega^{(2)} \exp \left[i \left(q_2 \cdot x - \Delta_2 t - \phi(2)(t)\right)\right] |0\rangle \langle e| + H.c., \]  

(B3)

where \(\Omega^{(1)} = -d_{0g} \cdot E_1 / \hbar\) and \(\Omega^{(2)} = -d_{0e} \cdot E_2 / \hbar\) are Rabi frequencies, \(d_{0g}\) and \(d_{0e}\) are the matrix elements of the dipole moment operator, \(\Delta_1 = \omega_1 - \omega_{0g}\) and \(\Delta_2 = \omega_2 - \omega_{0e}\) are the detunings of the frequencies. Let us eliminate an amplitude of the excited state, which evolves as

\[ i \dot{a}(0, p, t) = \left(\frac{P^2}{2M \hbar} - iMg \cdot \partial_p\right)a(0, p, t) + \left\{\Omega^{(1)} e^{-i[\Delta_1 t + \phi(1)(t)]} a(g, p - \hbar q_1, t) + \Omega^{(2)} e^{-i[\Delta_2 t + \phi(2)(t)]} a(e, p - \hbar q_2, t)\right\} f(t). \]  

(B4)
When the detunings are closed

$$\Delta_1 \approx \Delta_2 \approx \Delta \quad (B5)$$

and sufficiently large,

$$|\Delta| \gg \text{Max} \left\{ |\tilde{\delta}|, \tau^{-1}, \left| \phi^{(i)} \right|, q_2 g T, q_0 \frac{P}{M}, \omega_q \right\}, \quad (B6)$$

where

$$\tilde{\delta} \equiv \Delta_1 - \Delta_2 \quad (B7)$$

is Raman detuning, one can neglect the first term in the right-hand-side of Eq. (B4) to get

$$a(0, p, t) = \left\{ \Omega^{(1)} \exp \left[ -i \left( \Delta_1 t + \phi^{(1)}(t) \right) \right] a(g, p - \hbar q_1, t) + \Omega^{(2)} \exp \left[ -i \left( \Delta_2 t + \phi^{(2)}(t) \right) \right] a(e, p - \hbar q_2, t) \right\} / \Delta, \quad (B8)$$

where the usual assumption was accepted that the atomic ground state amplitudes vary slowly at the time $|\Delta|^{-1}$. Substituting this expression in the equations for the atomic ground state amplitudes, one gets

$$i \left( \partial_t + M g \cdot \partial_p + i \frac{p^2}{2M\hbar} \right) \tilde{a}(e, p, t) = \frac{\Omega}{2} f(t) \exp \left[ -i\tilde{\delta}t - i\phi(t) \right] \tilde{a}(g, p - \hbar k, t), \quad (B9a)$$

$$i \left( \partial_t + M g \cdot \partial_p + i \frac{p^2}{2M\hbar} \right) \tilde{a}(g, p, t) = \frac{\Omega^*}{2} f(t) \exp \left[ i\tilde{\delta}t + i\phi(t) \right] \tilde{a}(e, p + \hbar k, t), \quad (B9b)$$

where $\tilde{a}(g, p, t) = \exp \left( i \frac{\Omega^{(1)^2} t}{2\Delta} \right) a(g, p, t); \tilde{a}(e, p, t) = \exp \left( i \frac{\Omega^{(2)^2} t}{2\Delta} \right) a(g, p, t)$, $k = q_1 - q_2$ is an effective wave vector, $\delta = \tilde{\delta} - \left( \frac{\Omega^{(2)^2}}{2} - \frac{\Omega^{(1)^2}}{2} \right) / \Delta$, $\Omega = 2\Omega^{(1)}\Omega^{(2)*} / \Delta$ is Rabi Raman frequency, $\phi(t) = \phi^{(1)}(t) - \phi^{(2)}(t)$.

In the accelerated frame

$$p = P + Mg t, \quad (B10)$$

using the initial atomic momentum $P$ as an independent variable, one obtains

$$i \left( \partial_t + i \frac{(P + Mg t)^2}{2M\hbar} \right) \tilde{a}(e, P, t) = \frac{\Omega}{2} f(t) \exp \left[ -i\tilde{\delta}t - i\phi(t) \right] \tilde{a}(g, P - \hbar k, t), \quad (B11a)$$

$$i \left( \partial_t + i \frac{(P + Mg t)^2}{2M\hbar} \right) \tilde{a}(g, P, t) = \frac{\Omega^*}{2} f(t) \exp \left[ i\tilde{\delta}t + i\phi(t) \right] \tilde{a}(e, P + \hbar k, t). \quad (B11b)$$

Then one finds that in the interaction representation

$$\tilde{a}(n, P, t) = \exp \left[ -i \int_0^t dt' \frac{(P + Mg t')^2}{2M\hbar} \right] c(n, P, t) \quad (B12)$$

the vector state

$$c(t) = \begin{pmatrix} c\left(e, P + \frac{\hbar k}{2}, t\right) \\ c\left(g, P - \frac{\hbar k}{2}, t\right) \end{pmatrix} \quad (B13)$$

evolves as

$$i\dot{c} = \frac{f(t)}{2} \begin{pmatrix} 0 & \Omega \exp \left\{ -i \left[ \delta t + \phi(t) - \frac{k\cdot P}{M} t - \frac{1}{2} k \cdot gt^2 \right] \right\} \\ \Omega^* \exp \left\{ i \left[ \delta t + \phi(t) - \frac{k\cdot P}{M} t - \frac{1}{2} k \cdot gt^2 \right] \right\} & 0 \end{pmatrix} c. \quad (B14)$$
One sees that in the accelerated frame the atomic amplitude in the interaction representation stays unchanged at the pulses’ free time,

\[ c(t) = \text{constant, at } t < T \text{ or } t > T + \tau. \]  

(B15)

If one chirps linearly the fields’ frequencies, i.e. if

\[ \phi(t) = \phi + \alpha t^2 / 2 \]  

(B16)

and if the chirping rate \( \alpha \) is close to \( k \cdot g \),

\[ |\alpha - k \cdot g| \tau^2 \ll 1, \]  

(B17)

then one can neglect the quadratic over terms in the phase factors in Eq. (B14) and arrives at the well-known equation for the amplitudes of a two-level atom in the rectangular resonant pulse with permanent detuning and phase,

\[ i \frac{dc}{dt'} = \frac{f(T + t')}{2} \left( \begin{array}{cc} 0 & \Omega \exp \{ -i [\delta_P t' + \phi(P)] \} \\ \Omega^* \exp \{ i [\delta_P t' + \phi(P)] \} & 0 \end{array} \right) c, \]  

(B19)

where

\[ \phi(P) = \phi + \left( \frac{\delta - k \cdot P}{\Delta T} \right) T + \frac{1}{2} (\alpha - k \cdot g) T^2, \]  

(B20a)

\[ \delta_P = \partial \phi(P) / \partial T. \]  

(B20b)

Using the solution of the Eq. (B19), one finds that during an interaction with Raman pulse the atomic amplitudes jump as

\[ c(e, P, T + \tau) = F_{ee}(P - h\kappa/2) c(e, P, T) + F_{eg}(P - h\kappa/2) c(g, P - h\kappa, T), \]  

(B21a)

\[ c(g, P, T + \tau) = F_{ge}(P + h\kappa/2) c(e, P + h\kappa, T) + F_{gg}(P + h\kappa/2) c(g, P, T), \]  

(B21b)

where \( F_{m,n}(P) \) is an element of the matrix

\[ F(P) = \left( \begin{array}{cc} f_{ee}(P) & f_{eg}(P) \exp \{ -i\phi(P) \} \\ f_{ge}(P) \exp \{ i\phi(P) \} & f_{gg}(P) \end{array} \right), \]  

(B22)

where

\[ f(\Omega, \delta) = \cos \frac{\Omega r(\Omega, \delta) \tau}{2} + i \frac{\delta}{\Omega r(\Omega, \delta)} \sin \frac{\Omega r(\Omega, \delta) \tau}{2}, \]  

(B23b)

\[ f_a(\Omega, \delta) = \frac{\Omega}{\Omega r(\Omega, \delta)} \sin \frac{\Omega r(\Omega, \delta) \tau}{2}, \]  

(B23c)

\[ \Omega r(\Omega, \delta) = \sqrt{\Omega^2 + \delta^2}. \]  

(B23d)

Appendix C: AMZAI response

Let us now apply Eqs. (B15, B21) to consider AMZAI, i.e. to calculate the total probability of the atoms’ excitation

\[ w = \int dP |c(e, P, T_3 + \tau)|^2 \]  

(C24)
with 3 Raman pulses and one microwave pulse having Rabi frequencies \{\Omega_1, \Omega_2, \Omega_m, \Omega_3\}, detunings \{\delta_1, \delta_2, \delta_m, \delta_3\}, phases \{\phi_1, \phi_2, \phi_m, \phi_3\}, wave vectors and chirping rates

\[
\{k_1, k_2, k_m, k_3\} = k_2 \{1 - s, 1, 0, s\}, \quad (C25a)
\]
\[
\{\alpha_1, \alpha_2, \alpha_m, \alpha_3\} = \alpha_2 \{1 - s, 1, 0, s\}, \quad (C25b)
\]

and acting at the moments \{T_1, T_2, T_m, T_3\}. Assuming that before the pulses action atoms are in the ground state only \[c(e, P, T_1) = 0\], and using consequently the Eqs. \[1221\] for each pulse one arrives to the following result

\[
c(e, P, T_3 + \tau) = c_{1+} + c_{2+} + c_{1-} + c_{2-}
\]
\[
+ F_{3ec}(P - shk_2/2) [F_{mem}F_{2ec}(P - \hbar k_2/2) F_{1eg}(P - (1 - s) \hbar k_2, T_1)
\]
\[
+ F_{meg}F_{2ge}(P + \hbar k_2/2) F_{1eg}(P + (1 + s) \hbar k_2/2) c(g, P + shk_2, T_1)]
\]
\[
+ F_{3eg}(P - shk_2/2) [F_{meg}F_{2eg}(P - (2s + 1) \hbar k_2/2) F_{1ggg}(P - (3s + 1) \hbar k_2/2) c(g, P - (s + 1) \hbar k_2, T_1)
\]
\[
+ F_{megg}F_{2ggg}(P - (2s - 1) \hbar k_2/2) F_{1ggg}(P - (3s - 1) \hbar k_2/2) c(g, P - shk_2, T_1)] ,
\]

(C26)

where

\[
c_{1+} = F_{3ec}(P - shk_2/2) F_{mem}F_{2ec}(P - \hbar k_2/2) F_{1ggg}(P - (1 + s) \hbar k_2/2) c(g, P - \hbar k_2, T_1) ,
\]
\[
c_{2+} = F_{3eg}(P - shk_2/2) F_{meg}F_{2ec}(P - (2s + 1) \hbar k_2/2) F_{1eg}(P - (1 + s) \hbar k_2/2) c(g, P - \hbar k_2, T_1) ,
\]
\[
c_{1-} = F_{3ec}(P - shk_2/2) F_{meg}F_{2ggg}(P + \hbar k_2/2) F_{1ggg}(P + (1 - s) \hbar k_2/2) c(g, P, T_1) ,
\]
\[
c_{2-} = F_{3eg}(P - shk_2/2) F_{megg}F_{2ge}(P - (2s - 1) \hbar k_2/2) F_{1eg}(P + (1 - s) \hbar k_2/2) c(g, P, T_1) ,
\]

(C27a–d)

\[F_{N\alpha\beta}\] is \((\alpha\beta)\) element of matrix \[\{1222\}\] associated with the pulse \(N\). The sum of the 8 integrals of the absolute squares of the each term in \[\{C26\}\] is a background \(\bar{w}\) of the response \[\{C24\}\], while the cross terms are responsible for the interference. If the initial atomic distribution \[|c(g, P, T_1)|^2\] is centered near the momentum \(P_0\) and, after using the filtering technique \[\{13, 21\}\], it is narrowed to the width

\[
\delta P \ll \frac{M}{kT}.
\]

(C28)

and if the pulse duration is comparable to the inverse recoil frequency, then different terms in Eq. \[\{C26\}\] are centered in the momentum space which are far distance from one another and cannot interfere. There are only 2 exclusions, 2 pairs of the terms \(\{c_{1+}, c_{2+}\}\), located at the point \(P = P_0 + \hbar k_2\), and \(\{c_{1-}, c_{2-}\}\), located at the point \(P = P_0\), leading to the interferometric part of the excitation probability

\[
\bar{w} = w_+ + w_- .
\]

(C29a)
\[
w_\pm = 2 \Re \int dP c_\pm c_{\pm}^* .
\]

(C29b)

For the parameter \(s\) given by Eq. \[\{10\}\] the rapid oscillations of the integrands in Eq. \[\{C29b\}\] as functions of the momentum with period \(\sim M/kT\) disappear. Other factors originated from the matrices \(f_N(P)\), given by Eq. \[\{1223\}\], are smooth functions of the momentum with size \(\sim M/kT\), so that one can put in Eqs. \[\{C27\}\] \(c(g, P, T_1) = \sqrt{\delta(P - P_0)}\) to find

\[
w_\pm = -2 \Re \exp \left\{ i \left[ -\phi_f - \phi_c + (-\delta_m + \delta_{1-P_0} - \omega k_2 (1 - s)^2) \tau \right] \mp r \right\} xx_\pm ,
\]
\[
r = \phi_q - \alpha_2 \rho k_2 T;
\]
\[
\phi_c = \frac{1}{2} (\hbar k_2 \cdot g - \alpha_2) s (1 - s) (T_3 - T_1)^2 ,
\]
\[
x = f_d [\Omega_1, \delta_{1-P_0} - (1 - s)^2 \omega k_2] f_a [\Omega_1, \omega_0, \omega, \omega_1] f_a [\Omega_m, \delta_m] f_d [\Omega_m, \delta_m] .
\]
\[
x_+ = f_d [\Omega_3, \delta_{3-P_0} - s (2s - 2) \omega k_2] f_a [\Omega_3, \delta_{3-P_0} - s (2s - 2) \omega k_2]
\]
\[
\times f_a [\Omega_3, \delta_{3-P_0} - s (2s - 2) \omega k_2] f_d [\Omega_2, \delta_{2-P_0} - s (2s - 2) \omega k_2] .
\]
\[
x_- = f_d [\Omega_3, \delta_{3-P_0} + s^2 \omega k_2] f_a [\Omega_3, \delta_{3-P_0} + s^2 \omega k_2]
\]
\[
\times f_a [\Omega_3, \delta_{3-P_0} + s^2 \omega k_2] f_d [\Omega_2, \delta_{2-P_0} + s (2s - 2) \omega k_2] ,
\]

(C30a–f)

where the quantum phase \(\phi_q\) and phase \(\phi_f\) are given by Eqs. \[\{17\}, \{14b\}\], and Eq. \[\{C30c\}\] now defines the classical phase \(\phi_c\).
Calculating the absolute squares of the terms in Eq. (C26), one arrives at the following results for the background term

\[ \bar{w} = \left| f_a \left[ \Omega_1, \delta_1, P_0 - (1 - s)^2 \omega_{k_2} \right] \right|^2 \left\{ \left| f_d \left[ \Omega_m, \delta_m \right] \right|^2 \left( \left| f_d \left( \Omega_3, \delta_3, P_0 - s (2 - s) \omega_{k_2} \right) f_d \left( \Omega_2, \delta_2, P_0 - (1 - 2s) \omega_{k_2} \right) \right|^2 \\
+ \left| f_a \left( \Omega_3, \delta_3, P_0 + s^2 \omega_{k_2} \right) f_a \left( \Omega_2, \delta_2, P_0 - (1 - 2s) \omega_{k_2} \right) \right|^2 \\
+ \left| f_a \left[ \Omega_3, \delta_3, P_0 - s (2 - s) \omega_{k_2} \right] \right|^2 \left( \left| f_a \left( \Omega_3, \delta_3, P_0 - s (2 - s) \omega_{k_2} \right) f_a \left( \Omega_2, \delta_2, P_0 - (1 - 2s) \omega_{k_2} \right) \right|^2 \\
+ \left| f_d \left[ \Omega_1, \delta_1, P_0 - (1 - s)^2 \omega_{k_2} \right] \right|^2 \left( \left| f_d \left[ \Omega_m, \delta_m \right] \right|^2 \left( \left| f_d \left( \Omega_3, \delta_3, P_0 - s (2 - s) \omega_{k_2} \right) f_d \left( \Omega_2, \delta_2, P_0 - (1 - 2s) \omega_{k_2} \right) \right|^2 \\
+ \left| f_a \left( \Omega_3, \delta_3, P_0 - s^2 \omega_{k_2} \right) f_d \left( \Omega_2, \delta_2, P_0 - (1 - 2s) \omega_{k_2} \right) \right|^2 \\
+ \left| f_a \left[ \Omega_m, \delta_m \right] \right|^2 \left( \left| f_d \left( \Omega_3, \delta_3, P_0 + s^2 \omega_{k_2} \right) f_d \left( \Omega_2, \delta_2, P_0 - (1 - 2s) \omega_{k_2} \right) \right|^2 \\
+ \left| f_a \left( \Omega_3, \delta_3, P_0 - s (2 + s) \omega_{k_2} \right) f_a \left( \Omega_2, \delta_2, P_0 - (1 - 2s) \omega_{k_2} \right) \right|^2 \right\} \right\} . \]  

(C31)