The interaction energy of well-separated
Skyrme solitons

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Abstract

We prove that the asymptotic field of a Skyrme soliton of any degree has a non-trivial multipole expansion. It follows that every Skyrme soliton has a well-defined leading multipole moment. We derive an expression for the linear interaction energy of well-separated Skyrme solitons in terms of their leading multipole moments. This expression can always be made negative by suitable rotations of one of the Skyrme solitons in space and iso-space. We show that the linear interaction energy dominates for large separation if the orders of the Skyrme solitons’ multipole moments differ by at most two. In that case there are therefore always attractive forces between the Skyrme solitons.

1 Skyrme solitons

The fundamental field of Skyrme’s theory \([\mathcal{U}]\) is a map

\[
U : \mathbb{R}^3 \rightarrow SU(2).
\]  

We denote points in \(\mathbb{R}^3\) by \(x\) with coordinates \(x_i, i = 1, 2, 3\) and Euclidean length \(r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}\). Sometimes we write \(\hat{x}\) for the unit vector \(x/r\). It is often useful
to parametrise $U$ in terms of the Pauli matrices $\tau_1, \tau_2$ and $\tau_3$ and the triplet of pion fields $\pi_1, \pi_2$ and $\pi_2$ as

$$U(x) = \sigma(x) + i\pi_a(x)\tau_a,$$

where summation over the repeated index $a$ is implied and the field $\sigma$ is determined by the constraint $\sigma^2 + \pi_1^2 + \pi_2^2 + \pi_3^2 = 1$. In this introductory section we do not specify the class of functions to which $U$ belongs. It is assumed to be sufficiently smooth for all the following operations to make sense.

The Skyrme energy functional is best written in terms of the Lie-algebra valued currents

$$L_i = U^\dagger \partial_i U$$

or

$$R_i = U \partial_i U^\dagger,$$

where $\partial_i = \partial/\partial x_i$. It is

$$E[U] = -\int d^3x \left( \frac{1}{2} \text{tr}(L_i L_i) + \frac{1}{16} \text{tr}([L_j, L_i][L_j, L_i]) \right).$$

The Euler-Lagrange equation for stationary points of this functional is conveniently expressed in terms of the modified currents

$$\tilde{L}_i = L_i - \frac{1}{4} [L_j, [L_j, L_i]]$$

and

$$\tilde{R}_i = R_i - \frac{1}{4} [R_j, [R_j, R_i]],$$

where we again use the convention that repeated indices are summed over. It reads

$$\partial_i \tilde{L}_i = 0$$

or, equivalently,

$$\partial_i \tilde{R}_i = 0.$$

Here we are interested in finite-energy solutions of the Euler-Lagrange equation. It is shown in [2] that the finite energy requirement implies that the map $U$ tends to a constant value at infinity in a weak sense. We choose that constant to be the identity element $1 \in SU(2)$ and demand

$$\lim_{r \to \infty} U(x) = 1.$$
have an associated integer degree. The first rigorous proof that for a finite-energy Skyrme
configuration in a very general class of functions the degree
\[ \text{deg}[U] = -\frac{1}{24\pi^2} \int d^3x \, \epsilon_{ijk} \text{tr} (L_i L_j L_k) \] (1.11)
is an integer was given in [3]. This result means that the space
\[ \mathcal{C} = \{ U : \mathbb{R}^3 \to SU(2) \mid E[U] < \infty \} \] (1.12)
of finite-energy configurations is a disjoint union of sectors
\[ \mathcal{C}_k = \{ U : \mathbb{R}^3 \to SU(2) \mid E[U] < \infty \text{ and } \text{deg}[U] = k \} \] (1.13)
labelled by the integers \( k \in \mathbb{Z} \).

The symmetry group of Skyrme’s theory will play an important role in our discussion. The energy functional (1.5), the boundary condition (1.10) and the degree (1.11) are invariant under the action of the Euclidean group of translations and rotations in \( \mathbb{R}^3 \) and under rotations of the pion fields
\[ \pi_a \mapsto G_{ab} \pi_b, \quad G \in SO(3), \] (1.14)
which we call iso-rotations. Reflections in space \( S : x \mapsto -x \) and in iso-space \( \pi_a \mapsto -\pi_a \) both leave the energy invariant but each changes the sign of the degree. The pull-back of Skyrme configurations \( U \) via \( S \) provides a map
\[ \tilde{S} : \mathcal{C}_k \to \mathcal{C}_{-k}, \quad U \mapsto U \circ S \] (1.15)
which preserves the energy.

It was shown in [4] that the energy in each topological sector is bounded below by a multiple of the degree. It follows from the results of [5] that the bound cannot be attained for the standard version of the Skyrme model described here, so that we have the strict inequality
\[ E[U] > 12\pi^2 |k|. \] (1.16)
The bound ensures that the infima
\[ I_k = \inf \{ E[U] \mid U \in \mathcal{C}_k \} \] (1.17)
are well-defined. The question of whether the infima are attained was first addressed by Esteban in the paper [2]. Amongst other things Esteban proved that, for a suitable class of functions,
\[ I_k \leq I_l + I_{k-l} \] (1.18)
for all \( k, l \in \mathbb{Z} \). She also showed that infima are attained provided one assumes the strict inequality
\[ I_k < I_l + I_{k-l} \] (1.19)
for all $k \in \mathbb{Z} - \{0, \pm 1\}$ and $l \in \mathbb{Z} - \{0, k\}$ in the range $|l| + |k - l| < \sqrt{2}|k|$. In [3] it was shown that the result still holds if one widens the class of allowed functions but the inequality (1.19) remains a necessary assumption in the proof. The strict inequality is also of interest in physics. As we shall see it is related to the question of attractive forces in the Skyrme model.

For low values of $k$ the existence and nature of minima of the Skyrme energy functional is understood in more detail. For fields of degree one, the highly symmetric hedgehog ansatz

$$U_H(x) = \exp(\hat{r}_a \tau_a),$$

introduced already by Skyrme, leads to an ordinary differential equation for the profile function $f$. With the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$ the resulting Skyrme configuration has degree 1 and is called the Skyrmion. It was shown in [4] that the Skyrmion minimises the Skyrme energy functional amongst all degree one configurations of the hedgehog form. In [2] the existence of the minimum of the Skyrme energy functional in $\mathcal{C}_1$ was proved, but it has not been established rigorously that minimising configurations have the symmetry of the hedgehog field (1.20). Note that if $U_1$ is a minimal energy configuration in $\mathcal{C}_1$ then the reflected configuration $\tilde{S}(U_1)$ has the same energy and minimises the energy in $\mathcal{C}_{-1}$.

In the following we use the term Skyrme solitons for minimal energy solutions of the Skyrme equation with non-zero degree $k$. There is overwhelming numerical evidence that the minimum in $\mathcal{C}_2$ is attained by a configuration of toroidal symmetry [7]. For higher degree, too, much is known numerically about the minima of Skyrme’s energy functional in the sector $\mathcal{C}_k$. Numerical searches, assisted by analytical ansätze and investigations of the possible symmetries of Skyrme solitons, suggest the existence of Skyrme solitons in all sectors $\mathcal{C}_k$ up to $k = 22$. The energies are sufficiently accurately computed that it appears that the inequality (1.19) is satisfied for all $k$ in the range $2 \leq k \leq 20$ and all $l$ in the range $0 < l < k$, see [3, 4, 10].

The existence of attractive forces between two Skyrmions was already shown by Skyrme, using the product ansatz. In this paper we investigate the existence of attractive forces between general Skyrme solitons. It is perhaps worth stressing that our arguments in fact apply to any finite-energy solution of the Skyrme equation, not just minimal ones. An earlier attempt at proving the existence of attractive forces between Skyrme solitons was made in the unpublished paper [9]. Our approach is partly inspired by ideas in [9] but also fills important gaps left there. Our main tool is an asymptotic expansion of Skyrme solitons. In Section 2 we show that Skyrme fields have an asymptotic expansion in powers of $1/r$ and that for non-trivial Skyrme solitons that expansion always has a non-trivial leading multipole. In Section 3 we study the interaction energy of two Skyrme solitons and show that it is dominated by a certain linear interaction energy provided the orders of the leading multipoles of the Skyrme solitons do not differ by more than 2. In Section 4 we derive an expression for the linear interaction energy of two multipoles, and show that it can always be made negative by suitable relative rotations in space and iso-space. At the end of this paper we briefly comment on the relationship between our results and Esteban’s work, and on the implications for the existence of Skyrme solitons of arbitrary degree.
2 The asymptotics of Skyrme solitons

The aim of this section is to show that if $U$ and the currents $L_j$ are a little better than continuous, then $U$ is smooth in $\mathbb{R}^3$, and has a non-trivial asymptotic expansion in powers of $1/r$ and $\log r$ as $r \to \infty$. By non-trivial we mean here that if $U$ is non-constant, then it cannot happen that all terms in the asymptotic expansion vanish. Put another way, it is not possible for $U$ to approach 1 at infinity faster than every power of $1/r$ unless $U$ is identically equal to 1 in $\mathbb{R}^3$.

To make a precise statement, say that the (possibly matrix-valued) function $f$ is in $C^{0,\alpha}_b(\mathbb{R}^3)$, with $0 < \alpha < 1$, if $f$ satisfies

$$
\sup_{x \in \mathbb{R}^3} |f| + \sup_{x,x' \in \mathbb{R}^3, x \neq x'} \frac{(1 + r + r')^\alpha |f(x) - f(x')|}{|r - r'|^\alpha + (1 + r + r')^\alpha d(\omega, \omega')^\alpha} < \infty.
$$

(2.1)

Here $x = r\omega, x' = r'\omega'$, where $\omega$ is regarded as an angular variable (unit vector) living on the unit 2-sphere, and $d(\omega, \omega')$ is the 2-sphere distance between $\omega$ and $\omega'$. If we consider the space $C^{0,\alpha}_b(K)$, where $K$ is a bounded subset of $\mathbb{R}^3$, then this is precisely the same as the usual space of (bounded) Hölder-continuous functions, with Hölder exponent $\alpha$. However, $C^{0,\alpha}_b(\mathbb{R}^3)$ is slightly different from the usual space of bounded Hölder-continuous functions on $\mathbb{R}^3$.

The main result of this section can be summarized as follows:

**Theorem 2.1** Suppose that for some $\delta > 0$,

$$
(1 + r)^\delta (U - 1) \in C^{0,\alpha}_b, \quad (1 + r)^\delta L_j \in C^{0,\alpha}_b.
$$

(2.2)

Suppose further that $U$ satisfies the Skyrme differential equations (1.8) in the sense of distributions.

Then $U$ is smooth in $\mathbb{R}^3$ and has a complete asymptotic expansion in powers of $1/r$ and $\log r$, for large $r$. If $U$ is non-constant, then this expansion has a leading term which is harmonic, hence a multipole.

Note that the hypotheses (2.2) force $U$ to approach 1 and the $L_j$ to approach 0 like $r^{-\delta}$ as $r \to \infty$. As a technical remark, we point out that the assumption of Hölder-continuous currents implies that $U$ will have a Hölder continuous first derivative. The second derivatives $\partial_j \partial_k U$ are then defined only in the sense of distributions, but in the Skyrme equation $\partial_j \partial_k U$ enters linearly, and is multiplied by continuous functions (smooth functions of the currents) see Section 2.1 below. In particular, the left-hand side of the Skyrme equation $\partial_j L_j = 0$ makes sense as a distribution if (2.2) holds.

The assumptions (2.2) of Theorem (2.1) do not follow immediately from the variational analysis used by Esteban in [2]. Her methods only give that the derivatives $\partial_j U$ are locally square-integrable (and that the components of $U$ are locally bounded). On physical grounds, one expects minimizers of the Skyrme energy functional to satisfy (2.2), but it would be desirable to bridge the gap between the analysis given here and what was proved rigorously in [2] and [3]. This issue will not be pursued further here.
The proof of Theorem (2.1) proceeds in four steps, each of which takes up one of the following subsections. In the first subsection we re-write the Skyrme equation in order to make explicit the form of the non-linearities. The equation is *quasilinear*, in the sense that the derivatives of highest order (2) enter linearly. The Skyrme equation can therefore be regarded as a second-order linear elliptic PDE, with Hölder-continuous coefficients. Then standard regularity theorems (Schauder estimates) yield that \( U \) is smooth.

In Section 2.2 we study the behaviour of the Skyrme equation at spatial infinity, by introducing coordinates \((s, \omega)\), where \( s = r^{-1} \), so that the 2-sphere at infinity becomes a genuine boundary at \( s = 0 \). After a rescaling, the Laplacian of \( \mathbb{R}^3 \) is replaced by

\[
\Delta_b = (s \partial_s)^2 - s \partial_s + \Delta_\omega, \tag{2.3}
\]

where \( \Delta_\omega \) is the Laplacian of the unit 2-sphere. The analysis is now guided by the corresponding analysis of a system of ordinary differential equations with regular singular point at \( s = 0 \). In particular, one does not expect the solutions to be smooth near \( s = 0 \), but one does expect a non-trivial expansion in powers of \( s \) (and possibly \( \log s \)). The \( b \)-calculus of [12] enables us to make this kind of argument precise. Thus in Section 2.2 we show that if \( U = 1 + u \), then \( u \) is *conormal* at \( s = 0 \), which is to say that \( u \) and all derivatives of the form \((s \partial_s)^m \nabla_\omega^nu\) are continuous as \( s \to 0 \). (This condition is strictly weaker than \( u \) being smooth near \( s = 0 \).) In Section 2.3, we show that it is not possible for \( U \) to approach 1 faster than \( r^{-\mu} \) for every \( \mu > 0 \) unless \( U = 1 \) in \( \mathbb{R}^3 \). Finally in Section 2.4 we combine this fact with another application of the \( b \)-calculus to show the existence of a non-trivial asymptotic expansion in powers of \( r^{-1} \) and \( \log r \) (or equivalently in powers of \( s \) and \( \log s \)).

### 2.1 Rewriting the Skyrme Equation

To begin with we work near a fixed point of \( \mathbb{R}^3 \), which we may as well take to be the origin 0. By replacing \( U(x) \) by \( U(0)^{-1}U(x) \) we can assume that \( U(0) = 1 \). Write

\[
U(x) = 1 + u(x), \tag{2.4}
\]

so that \( u(0) = 0 \). Because \( U \) is unitary, the \( 2 \times 2 \) complex matrix \( u \) will satisfy the algebraic constraints

\[
u + u^\dagger + uu^\dagger = 0, \quad \text{tr}(u) + \det u = 0. \tag{2.5}
\]

In particular \( u \) is neither exactly skew-hermitian nor trace-free. Then

\[
L_i = \partial_i u + u^\dagger \partial_i u \tag{2.6}
\]

and

\[
\partial_i L_i = (1 + u^\dagger) \Delta u + \partial_i u^\dagger \partial_i u. \tag{2.7}
\]

The cubic term in the currents can be written

\[
\frac{1}{2} L_j [L_j, L_i] = \frac{1}{2} (\partial_j u + u^\dagger \partial_j u) [\partial_j u + u^\dagger \partial_j u, \partial_i u + u^\dagger \partial_i u]. \tag{2.8}
\]

Taking the divergence, and using the notation

\[
v_{ij} = \partial_i u^\dagger \partial_j u, \tag{2.9}
\]
we obtain
\[ \frac{1}{2} L_j[L_j, L_i] = (1 + u^\dagger)(T + F), \] (2.10)
where
\[ T = T(u, \partial u, \partial^2 u) = \frac{1 + u}{2} (L_j[L_j, (1 + u^\dagger)\Delta u] + L_j[(1 + u^\dagger)\partial_i u \partial_j u, L_i]) \] (2.11)
and
\[ F = F(u, \partial u) = \frac{1 + u}{2} (v_{ij}[L_i, L_j] + L_j[v_{ii}, L_i] + L_j[(1 + u^\dagger)v_{ij}, L_i]). \] (2.12)

The nonlinear terms have been divided here so that \( T \) is polynomial in the first derivatives of \( u \) and linear in its second derivatives, while \( F \) only contains \( u \) and its first derivatives.

With this notation, the full Skyrme equation can be written
\[ P(u, \partial u, \partial^2 u) = Q(u, \partial u) + F(u, \partial u) \] (2.13)
where
\[ P(u, \partial u, \partial^2 u) = \Delta u + T(u, \partial u, \partial^2 u) \] and \( Q(u, \partial u) = -(1 + u)v_{ii}. \) (2.14)
Notice that \( Q \) is (approximately) quadratic in \( u \), \( T \) is of degree three and \( F \) is of degree four. Note also that \( T \) is linear in \( \partial_i \partial_j u \) (and quadratic in \( \partial_i u \)).

Because of the assumed Hölder continuity of the currents, the coefficients of the differential operator
\[ f \mapsto P(u, \partial u, \partial^2 f) \] (2.15)
are Hölder continuous, and this operator is linear and elliptic in a small neighbourhood \( K \) of 0. Moreover, the RHS \( Q + F \) of (2.13) is also in \( C^{\beta, \alpha}_b(K) \), so by elliptic regularity, it follows that \( u \in C^{2, \alpha}_b(K) \). Then the currents are in \( C^{1, \alpha}_b(K) \) and the process continues to show that \( u \) is in \( C^{k, \alpha}_b(K) \) for every \( k \). Thus \( u \) is smooth near 0. Since the point 0 was arbitrary, the argument shows that \( u \) is smooth in \( \mathbb{R}^3 \).

### 2.2 Boundary regularity

In order to analyze the asymptotic behaviour of the field \( U \), we shall make a transformation of the problem which involves passing from \( \mathbb{R}^3 \) to a ‘compactification’ \( \overline{\mathbb{R}^3} \) in which the sphere at infinity becomes a genuine boundary. This is easily achieved by introducing the coordinate \( s = 1/r \) along with angular coordinates \( \theta \) and \( \varphi \) in \( \mathbb{R}^3 \). Then in the space \( [0, \infty) \times S^2 \) with coordinates \((s, \theta, \varphi)\), the boundary \( s = 0 \) corresponds to \( r = \infty \) and \( \theta \) and \( \varphi \) give coordinates on the boundary, which is the 2-sphere ‘at infinity’. \( \overline{\mathbb{R}^3} \) is defined to be the union of \( \mathbb{R}^3 \) with the 2-sphere at infinity attached in this way.

It can be cumbersome to work with explicit coordinates on the 2-sphere, so we again use \( \omega \) for points on \( S^2 \) and represent any point other than the origin of \( \overline{\mathbb{R}^3} \) in the form \((s, \omega)\).

Next, introduce rescaled derivatives
\[ D_i = r \partial_i = \frac{1}{s} \partial_i. \] (2.16)
These vector fields have the property that they are linear combinations, with coefficients that are smooth, down to \( s = 0 \), of the basic vector fields \( s \partial_s \), \( \partial_\theta \) and \( \partial_\phi \). The Euclidean Laplacian takes the form

\[
\Delta = s^2 \Delta_b
\]

with \( \Delta_b \) defined in (2.3).

Now we write \( U = 1 + u \) for large \( r \) (that is, for small positive \( s \)) and make the rescalings of (2.13) suggested by (2.16) and (2.17). The result is a ‘b’ version of the Skyrme equation,

\[
P_b(u, Du, D^2u) = Q_b(u, Du) + s^2 F_b(u, Du),
\]

where

\[
P_b(u, Du, D^2u) = \Delta_b u + s^2 T_b(u, D)u,
\]

\[
Q_b(u, Du) = -(1 + u) D_i u^\dagger D_i u,
\]

these terms being got by replacing \( \partial_i \) by the rescaled derivative \( D_i \) wherever they occur.

The reason for reformulating the equation in this way is that there is a well-established theory, called the \( b \)-calculus, which can be used to analyze equations of this kind \([12]\). The \( b \)-calculus is concerned with \( b \)-differential operators. In the present situation, a \( b \)-differential operator is just a differential operator of the form

\[
P = \sum_{a+b+c \leq m} C_{abc} (s \partial_s)^a \partial_\theta^b \partial_\phi^c,
\]

where the coefficients \( C_{abc} \) are smooth up to the boundary \( s = 0 \). From (2.3) it is clear that the rescaled Laplacian \( \Delta_b \) is an example of such an operator. The set of all such operators will be denoted by \( \text{Diff}_b \), those of order \( k \) by \( \text{Diff}_b^k \).

The first aspect of this theory that is needed is the counterpart of the elliptic regularity for Hölder spaces that we used in the previous subsection. For this we need \( b \)-Hölder spaces, already introduced in (2.1). With the new variable \( s \), we have \( f(s, \omega) \in C_{b}^{0, \alpha} \) if

\[
\sup_{x \in \mathbb{R}^3} |f| + \sup_{(s, \omega) \neq (s', \omega')} \frac{(s + s')^\alpha |f(s, \omega) - f(s', \omega')|}{|s - s'|^\alpha + (s + s')^\alpha d(\omega, \omega')^\alpha} < \infty.
\]

Now put

\[
C_{b}^{k, \alpha} = \{ f : Lf \in C_{b}^{0, \alpha} \text{ for all } b \text{-differential operators } L \in \text{Diff}_b^k \}
\]

and

\[
A = \{ u : Lu \in C_{b}^{0, \alpha} \text{ for all } L \in \text{Diff}_b \}.
\]

In order to force functions to decay as \( s \to 0 \), we introduce \textit{weighted} versions of these spaces,

\[
s^\delta C_{b}^{k, \alpha} = \{ u = s^\delta v : v \in C_{b}^{k, \alpha} \}, \quad s^\delta A = \{ u = s^\delta v : v \in A \}.
\]

The following is a very special case of elliptic regularity for \( b \)-differential operators \([13]\).
Theorem 2.2 Consider the differential operator $P = \Delta_b + s^2 E$, where $E$ is a second-order differential operator with coefficients smooth up to the boundary $s = 0$. Suppose that $Pu = f$ near $s = 0$, with $u$ and $f$ in $s^\delta C_b^{0,\alpha}$. Then if $\delta$ is not an integer, it follows that $u \in s^\delta C_b^{2,\alpha}$.

We want to apply this to the Skyrme equation, written in the form (2.19). After the last section, we know that the coefficients of the perturbing term $E$ are smooth for $s > 0$, but all we know at $s = 0$ is the original assumption that the currents are in $s^\delta C_b^{0,\alpha}$. The elliptic regularity statement still holds in this case (though this statement does not seem to be available in the literature). Applying this result, for $\delta$ a small positive number, we obtain first that $u \in s^\delta C_b^{2,\alpha}$, which gives that the coefficients are in $s^\delta C_b^{1,\alpha}$, so $u \in s^\delta C_b^{3,\alpha}$ and so on. Iterating, we find $u \in s^\delta A$.

This is a major step forward: in particular, the angular variation of $u$ is now very well controlled. However $u$ may still be very far from being smooth up to the boundary or having an asymptotic expansion there. Indeed, horrors like $s^\delta \sin \log s$ lie in $s^\delta A$.

2.3 Power-law decay of $u$

In this section and the next we establish that a topologically non-trivial solution of the Skyrme equation must have a non-trivial asymptotic expansion in powers of $1/r$. We show first that it is not possible for a topologically non-trivial solution to approach 1 faster than every power of $1/r$. This result is then fed into an iterative analysis of the equation in the next section. These two sections can, however, be read in either order. The main result of this section is as follows.

Theorem 2.3 If the topological charge of the Skyrme field $U$ is non-zero and if it satisfies the hypotheses of Theorem 2.1, then there exists some $\mu \in \mathbb{R}^+$ such that $r^\mu u(r, \omega)$ does not tend to zero as $r \to \infty$.

To set this result in a more general context, recall that a partial differential equation $Lu = 0$ is said to have the unique continuation property at a point $0$, say, if the following is true:

If all derivatives of $u$ vanish at $0$, then $u = 0$ in some neighbourhood of $0$. \hfill (2.23)

The methods used to establish that a second-order PDE has the unique continuation property establish analogous statements with somewhat weaker hypotheses. For example, a simplified version of Theorem 17.2.6 of \cite{16} is as follows.

Theorem 2.4 Let $a_{jk}(x)$ be smooth and positive-definite in a neighbourhood $X$ of $0$, and suppose that $a_{jk}(0) = \delta_{jk}$. Suppose that for $x \in X$, the smooth function $u$ satisfies

$$|a_{jk}(x) \partial_j \partial_k u| \leq A (|u(x)| + |\nabla u(x)|)$$ \hfill (2.24)

for some constant $A$, and

$$|x|^{-\mu} |u(x)| \to 0 \text{ as } |x| \to 0 \text{ for every } \mu \in \mathbb{R}^+. \hfill (2.25)$$

Then $u = 0$ identically in a neighbourhood of $0$.\hfill (9)
Using this result we can prove the following:

**Theorem 2.5** Let $U$ satisfy the hypotheses of Theorem 2.4. Then if the topological charge of $U$ is non-zero, $U$ cannot be constant in any open subset of $\mathbb{R}^3$.

**Proof** Define

$$W = \{ x \in \mathbb{R}^3 : U(y) = U(x) \text{ for all } y \text{ in some neighbourhood of } x \}. \quad (2.26)$$

Then $W$ is open by definition. By Theorem 2.4, $W$ is also closed. To see this, suppose that $x_n \in W$, $x_n \to x_0 \in \mathbb{R}^3$. Then all derivatives of $U$ are zero at $x_0$, so (2.25) holds (with 0 replaced by $x_0$). The Skyrme equation, in the form (2.13), implies a differential inequality of the form (2.24) in a neighbourhood of $x_0$. It follows that $U$ is identically constant in a neighbourhood of $x_0$, so that $x_0 \in W$. Since $\mathbb{R}^3$ is connected, $W = \emptyset$ or $W = \mathbb{R}^3$.

We will use this theorem to give an indirect proof of Theorem 2.3. Suppose $\mu u(x) \to 0$ as $r \to \infty$ for every $\mu$. Adapting Theorem 2.4 we shall show that then $u = 0$ for all sufficiently large $r$. Thus $U$ is constant in an open set, hence by the previous result constant everywhere.

So consider the $b$-differential operator

$$P_b = \Delta_b + s^2 T_b \quad (2.27)$$

on $\mathbb{R}^3$, where the coefficients of $T_b$ are in $C^{0,\alpha}_b$. By rescaling Theorem 17.2.6 in [16], we obtain

**Theorem 2.6** Suppose that

$$|P_b u(s, \omega)| \leq A s^8 (|u(s, \omega)| + |Du(s, \omega)|) \text{ for all } 0 < s < s_0, \omega \in S^2 \quad (2.28)$$

and that

$$s^{-\mu} |u(s, \omega)| \to 0 \text{ as } s \to 0 \text{ for all } \mu. \quad (2.29)$$

Then $u(s, \omega) = 0$ for $0 \leq s < s_1$, where $s_1$ is some small positive number.

The ‘$b$’ version (2.18) of the Skyrme equation implies a differential inequality of the form (2.28), just as before. It follows that if $u$ decays faster than any power of $r$, then $u = 0$ for all sufficiently large $r$. By the remarks before the statement of Theorem 2.6, the proof of Theorem 2.3 is now complete.

1Unique continuation theorems do not extend wholesale to systems. However, in our case, the leading term is the Laplacian and the other second-order terms $C(u, \partial u, \partial^2 u)$ are non-scalar but small near $x_0$. The proof of Theorem 2.4 goes through in this case.
2.4 Refined regularity, asymptotic expansions

The main result of this section can be stated as follows:

**Theorem 2.7** Let \( U = 1 + u \) satisfy the hypotheses of Theorem 2.1. Then there is some integer \( M \geq 1 \) and an asymptotic expansion

\[
u \sim \sum_{j=M}^{2M} Y_j(\omega)r^{-(j+1)} + \sum_{j=2M+1}^\infty r^{-(j+1)}w_j(\omega, \log r) \text{ for large } r. \tag{2.30}
\]

Here the \( Y_j \) are Lie-algebra valued spherical harmonics,

\[
\Delta_\omega Y_j = -j(j+1)Y_j, \tag{2.31}
\]

and \( Y_M \neq 0 \), so that the piece \( \sum_{j=M}^{2M} Y_j(\omega)r^{-(j+1)} \) is a non-zero harmonic function. The functions \( w_j \) are smooth in \( \omega \) and polynomial in \( \log r \).

It will follow from the proof that the terms \( w_{2M+1} \) to \( w_{3M+1} \) are of at most degree 1 in \( \log r \), the terms \( w_{3M+2} \) to \( w_{4M+2} \) are of at most degree 2 in \( \log r \) and so on. We remark also that the asymptotic expansion can safely be differentiated term by term to give asymptotic expansions of all derivatives of \( u \).

In order to motivate the proof, consider the equation

\[
\Delta_b u = f \tag{2.32}
\]

where \( u \) and \( f \) are defined for small \( s \). If \( f \) has the form \( f = s^{\lambda}g(\omega) \), then a solution can be found as follows. Expand \( g \) as a sum of spherical harmonics,

\[
g = \sum_{j=0}^\infty g_j, \text{ where } \Delta_\omega g_j = -j(j+1)g_j
\]

and seek a solution \( u = \sum u_j(s)g_j \). Then \( u_j \) must satisfy

\[
[(s\partial_s)^2 - (s\partial_s) - j(j+1)]u_j(s) = s^\lambda.
\]

This is solved by

\[
u_j(s) = \frac{s^\lambda}{\lambda(\lambda - 1) - j(j+1)}
\]

provided there is no resonance, that is to say

\[
\lambda \neq -j, j + 1.
\]

In the resonant case, \( u_j(s) \) has the form \( s^\lambda(A + B \log s) \). The general solution is obtained by combining this with an arbitrary solution of the homogeneous equation \( \Delta_b v = 0 \). If we require \( u \to 0 \) as \( s \to 0 \), then \( v \) must itself go to zero and hence will be a sum of multipoles,

\[
v = \sum_{j=0}^\infty Y_j(\omega)s^{j+1}, \text{ where } Y_j \text{ satisfies (2.31).}
\]
The $b$-calculus extends results of this kind to functions in $s^\delta A$ (which, as we have seen, can be far from having expansions in powers of $s$). In order to summarize the needed results, write
\[ f = \mathcal{O}(s^\delta) \] instead of $f \in s^\delta A$. (2.33)
This conforms to the use of the ‘$\mathcal{O}$’-notation in the rest of the paper, but has the additional property
\[ \text{If } f = \mathcal{O}(s^\delta) \text{ then } Lf = \mathcal{O}(s^\delta) \text{ for all } L \in \text{Diff}_b. \] (2.34)

**Lemma 2.8** Suppose that $u$ and $f$ defined near $s = 0$ satisfy (2.32). If $u = \mathcal{O}(s^\delta)$ and $f = \mathcal{O}(s^{n+\delta})$, where $\delta > 0$ and $n$ is a positive integer, then
\[ u = \sum_{j=0}^{n-1} Y_j(\omega)s^{j+1} + \mathcal{O}(s^{n+\delta}) \] (2.35)
where $Y_j$ satisfies (2.31). In particular the sum on the RHS is harmonic
\[ \Delta_b \sum_{j=0}^{n-1} Y_j(\omega)s^{j+1} = 0. \] (2.36)

**Lemma 2.9** Suppose that $u$ and $f$ defined near $s = 0$ satisfy (2.32). If $u = \mathcal{O}(s^\delta)$ and
\[ f = \sum_{j=0}^{n-1} f_j(\omega, \log s)s^{j+1} + \mathcal{O}(s^{n+\delta}), \] (2.37)
where $\delta > 0$, $n$ is a positive integer and $f_j$ is polynomial of degree $m_j$ in $\log s$, then
\[ u = \sum_{j=0}^{n-1} w_j(\omega, \log s)s^{j+1} + \mathcal{O}(s^{n+\delta}) \] (2.38)
where $w_j$ is a polynomial of degree $m_j + 1$ in $\log s$.

These results will be applied to the Skyrme equation, now rewritten as
\[ \Delta_b u = Z_b(u) := Q_b(u) - s^2T_b(u) + s^2F_b(u). \] (2.39)

**Lemma 2.10** Suppose that $U = 1 + u$ satisfies the Skyrme equation and $u = \mathcal{O}(s^\delta)$. Then $u = \mathcal{O}(s^2)$.

**Proof:** On the RHS of (2.39), $Q_b$ is quadratic in $Du$ and the other terms in $Z_b$ are of even higher degree. Hence $Z_b = \mathcal{O}(s^{2\delta})$. Applying Lemma 2.8 we obtain that $u = \mathcal{O}(s^{2\delta}) + \mathcal{O}(s)$. If $2\delta < 1$, we can iterate this argument to obtain eventually $u \in \mathcal{O}(s)$. In [13], it is shown that a solution of the Skyrme equation cannot have leading term $1/r = s$ in its
asymptotic expansion. It follows that \( u = \mathcal{O}(s^{1+\delta}) \) (for a possibly smaller \( \delta > 0 \)). Hence \( Z_0(u) = \mathcal{O}(s^{2+2\delta}) \) and so, applying Lemma 2.8 again, \( u = Y_1s^2 + \mathcal{O}(s^{2+\delta}) \).

Combining Theorem 2.3 with Lemma 2.8, we see that there exists an integer \( M \geq 1 \) with the property that

\[
u = Y_M(\omega)s^{M+1} + \mathcal{O}(s^{M+1+\delta}),
\]

where \( Y_M \) is a non-vanishing spherical harmonic. We can now complete the proof of Theorem 2.7 in the following iterative fashion. From the structure of \( Z_b(u) \) it follows from (2.40) that

\[
Z_b(u) = f_2s^{2M+2} + \mathcal{O}(s^{2M+2+\delta}).
\]

Applying Lemma 2.9,

\[
u = 2\sum_{j=M}^{M+1} Y_j(\omega)s^{j+1} + w_2s^{2M+2} + \mathcal{O}(s^{2M+2+\delta}),
\]

where \( w_2(s, \log s) \) is of degree at most 1 in \( \log s \). Now this expression for \( u \) is substituted into \( Z_b \), to give

\[
Z_b = 3(M+1)\sum_{j=2M+1}^{3M+1} f_j(\omega)s^{j+1} + f_3s^{3M+3} + \mathcal{O}(s^{3M+3+\delta}),
\]

where \( f_3 \) is of degree at most 1 in \( \log s \). Now apply Lemma 2.8 to get

\[
u = \sum_{j=M}^{2M} Y_j(\omega)s^{j+1} + \sum_{j=2M+1}^{3M+1} w_j(\omega, \log s)s^{j+1} + w_3s^{3M+3} + \mathcal{O}(s^{3M+3+\delta}).
\]

The upshot of this section is that every Skyrme soliton has a leading Lie-algebra valued multipole field (called a \( 2M \)-pole)

\[
u_M(x) = i\tau_a \sum_{m=-M}^{M} \frac{4\pi}{2M+1} Q_{Mm}^a Y_{Mm}(\theta, \varphi),
\]

where \( Y_{Mm} \) are the usual spherical harmonics on \( S^2 \), see appendix A. The leading multipole moment \( Q_{Mm}^a \) is independent of the location of the Skyrme soliton, and is acted on naturally by rotations and iso-rotations. It is a key ingredient in the calculations of the following sections. As already mentioned one can show that Skyrme solitons never have asymptotic monopole fields [14]. The leading multipole field of the \( B = 1 \) hedgehog (1.20) is a triplet of dipoles, and dipoles are known to occur as leading multipoles in a number of Skyrme solitons. The highest leading multipole known from numerical work is an octupole, which occurs in a \( B = 7 \) configuration with icosahedral symmetry [3].
3 The interaction energy of two Skyrme solitons

Suppose we have Skyrme solitons $U^{(1)}$ and $U^{(2)}$ of degrees $k$ and $l$ which minimise the energy in the sectors $\mathcal{C}_k$ and $\mathcal{C}_l$. Since the total energies of $U^{(1)}$ and $U^{(2)}$ are finite there must be balls $B_1$ and $B_2$ in $\mathbb{R}^3$ so that most of the energy of $U^{(1)}$ and $U^{(2)}$ is concentrated in, respectively, $B_1$ and $B_2$. Outside the balls $B_1$ and $B_2$ the asymptotic analysis of the previous section applies. Suppose that the leading multipole of $U^{(1)}$ is a $2M$-pole and the leading multipole of $U^{(2)}$ is a $2N$-pole. Denoting the radii of $B_1$ and $B_2$ by $D_1$ and $D_2$, and with the abbreviation (2.45) for a generic Lie-algebra valued multipole field we have

\[
U^{(1)}(x) \sim 1 + u_M(x) \quad \text{for} \quad x \notin B_1
\]  

(3.1)

and

\[
U^{(2)}(x) \sim 1 + u_N(x) \quad \text{for} \quad x \notin B_2.
\]  

(3.2)

Using the translational invariance of the Skyrme energy functional we can assume without loss of generality that $B_1$ is centred at $X_+ = (0,0,R/2)$ and that $B_2$ is centred at $X_- = (0,0,-R/2)$, where $R$ is so large that $B_1$ and $B_2$ do not overlap, i.e. $R > D_1 + D_2$. The parameter $R$ will be interpreted as the separation of the Skyrme solitons. There is clearly an ambiguity in the definition of such a separation parameter, but this does not affect our calculation of leading terms in the limit where $R$ becomes large. Then we define the following product configuration

\[
U_R(x) = U^{(1)}(x)U^{(2)}(x).
\]  

(3.3)

This configuration has degree $k + l$ and we shall see shortly that its energy is finite, so that $U_R \in \mathcal{C}_{k+l}$.

Our goal is to study the energy of the product configuration $U_R$ perturbatively in the limit of large $R$ and to compute the leading terms in powers of $1/R$. A similar calculation for moving and spinning Skyrmsions was performed in [15], where some further details are given. Let $L_i^{(1)}$ and $\tilde{L}_i^{(1)}$ be the currents (1.3) and (1.6) constructed out of $U^{(1)}$, and $R_i^{(2)}$ and $\tilde{R}_i^{(2)}$ be the currents (1.4) and (1.7) constructed out of $U^{(2)}$. Then one computes

\[
E[U_R] = E[U^{(1)}] + E[U^{(2)}] + W_2 + W_4.
\]  

(3.4)

The energies $E[U^{(1)}]$ and $E[U^{(2)}]$ are simply the energies of the Skyrme solitons $U^{(1)}$ and $U^{(2)}$ and therefore independent of $R$. The interaction terms $W_2$ and $W_4$ are given by integrals over $\mathbb{R}^3$

\[
W_2 = \int d^3x w_2 \quad \text{and} \quad W_4 = \int d^3x w_4,
\]  

(3.5)

with integrands

\[
w_2 = \text{tr} \left( L_i^{(1)} \tilde{R}_i^{(2)} + \tilde{L}_i^{(1)} R_i^{(2)} - L_i^{(1)} R_i^{(2)} \right)
\]  

(3.6)
\[ w_4 = -\frac{1}{8} \text{tr} \left( [L_i^{(1)}, R_j^{(2)}][L_i^{(1)}, R_j^{(2)}] + [L_i^{(1)}, R_j^{(2)}][R_j^{(2)}, L_i^{(1)}] + [L_i^{(1)}, L_j^{(1)}][R_j^{(2)}, R_j^{(2)}] \right). \] (3.7)

We shall see shortly that the term \( W_2 \) contains the leading contribution to the interaction energy. However, for the presentation of our method of computation it is more convenient to begin with the quartic term \( W_4 \). We split the integration region \( \mathbb{R}^3 \) into the balls \( B_1 \) and \( B_2 \) and the complement \( C = \mathbb{R}^3 - (B_1 \cup B_2) \). To illustrate our method, consider the integral

\[ I = -\frac{1}{8} \int_{B_1} d^3x \text{tr} \left( [L_i^{(1)}, R_j^{(2)}][L_i^{(1)}, R_j^{(2)}] \right) \leq \frac{1}{4} \int_{B_1} d^3x \text{tr} \left( L_i^{(1)} L_i^{(1)} \right) \text{tr} \left( R_j^{(2)} R_j^{(2)} \right). \] (3.8)

The currents \( L_i^{(1)} \) are smooth functions and hence bounded on the compact domain \( B_1 \). Therefore

\[ |I| \leq -K \int_{B_1} d^3x \text{tr} \left( R_j^{(2)} R_j^{(2)} \right). \] (3.9)

for some positive constant \( K \). Since \( B_1 \) is far away from the centre of soliton \( U^{(2)} \), the leading contribution to the integral (3.9) is obtained by replacing \( R_j^{(2)} \) by the leading multipole component \(-i\partial_j v_N\). Using

\[ |\partial_j v_N|(x) \leq \frac{K'}{|x - X|^{N+2}} \] (3.10)

for a further positive constant \( K' \) we conclude that

\[ I = O \left( \frac{1}{R^{2N+4}} \right). \] (3.11)

A similar calculation for the other terms in \( W_4 \) shows that

\[ \int_{B_1} d^3x w_4 = O \left( \frac{1}{R^{2N+4}} \right). \] (3.12)

Considering the contribution from \( B_2 \) we find by the same argument

\[ \int_{B_2} d^3x w_4 = O \left( \frac{1}{R^{2M+4}} \right). \] (3.13)

Finally the remaining integral over \( C \) can be estimated as follows. Define

\[ F(R) = \int_C d^3x \frac{1}{|x - X_+|^{2M+4}} \frac{1}{|x - X_-|^{2N+4}}, \] (3.14)

noting that the integral converges for all values of \( R > D_1 + D_2 \). Then there is a positive constant \( K'' \) such that

\[ \int_C d^3x w_4 < K'' F(R). \] (3.15)
The large $R$ behaviour of $F$ can be estimated by a scaling analysis. Fix $R_0 > D_1 + D_2$ and consider $R > R_0$. Changing integration variables $x \mapsto (R_0/R)x$ one computes

$$F(R) = \left(\frac{R_0}{R}\right)^{2M+2N+5} \left[ F(R_0) + \int_{S_1(R) \cup S_2(R)} d^3x \frac{1}{|x - X_+|^{2M+4}} \frac{1}{|x - X_-|^{2N+4}} \right], \quad (3.16)$$

where $S_1(R)$ and $S_2(R)$ are the thick shells

$$S_1(R) = \{ x \in \mathbb{R}^3 | (R_0/R)D_1 \leq |(x_1, x_2, x_3 - (R_0/2)| < D_1 \}$$

$$S_2(R) = \{ x \in \mathbb{R}^3 | (R_0/R)D_2 \leq |(x_1, x_2, x_3 + (R_0/2)| < D_2 \} \quad (3.17)$$

which converge to punctured balls

$$B_1^0 = \{ x \in \mathbb{R}^3 - \{(0,0,R_0/2)\} ||(x_1, x_2, x_3 - (R_0/2)| < D_1 \}$$

$$B_2^0 = \{ x \in \mathbb{R}^3 - \{(0,0,-R_0/2)\} ||(x_1, x_2, x_3 + (R_0/2)| < D_2 \} \quad (3.18)$$

in the limit $R \to \infty$. In that limit the integral over $S_1(R)$ diverges like $R^{2N+1}$ and that over $S_2(R)$ like $R^{2M+1}$. Combining this with with the factor $R^{-(2M+2N+5)}$ we deduce that $F(R)$ decays for large $R$ like $R^{-(2N+4)}$ and $R^{-(2M+4)}$, just like the contributions (3.12) and (3.13). Combining all the terms, we conclude that the leading terms in $W_4$ decay for large $R$ according to $R^{-(2N+4)}$ and $R^{-(2M+4)}$.

In order to evaluate $W_2$ we first divide the region of integration into the half-spaces

$$H^+ = \{ x \in \mathbb{R}^3 | x_3 > 0 \} \quad \text{and} \quad H^- = \{ x \in \mathbb{R}^3 | x_3 < 0 \}. \quad (3.19)$$

In $H^+$ we replace $U^{(2)}$ by the leading term $1 + v_N$ and in $H^-$ we replace $U^{(1)}$ by the leading contribution $1 + u_M$. The result is

$$W_2 \approx \int_{H^+} d^3x \left[ \frac{1}{4} \right] \left( \partial^t_i \tilde{L}_i^{(1)} \right) - \left( \partial^t_i v_N \tilde{L}_i^{(1)} \right) - \int_{H^-} d^3x \left[ \frac{1}{4} \right] \left( \partial^t_i \tilde{R}_i^{(2)} \right) + \left( \partial^t_i u_M \tilde{R}_i^{(2)} \right). \quad (3.20)$$

Now integrating by parts and using the Euler-Lagrange equations $\partial^t_i \tilde{L}_i^{(1)} = \partial^t_i \tilde{R}_i^{(2)} = 0$ for the individual Skyrme solitons we convert two of the terms into an area integral

$$- \int_{H^+} d^3x \left( \partial^t_i v_N \tilde{L}_i^{(1)} \right) + \int_{H^-} d^3x \left( \partial^t_i u_M \tilde{R}_i^{(2)} \right) = \int_{x_3=0} dx_1 dx_2 \left( v_N \tilde{L}_3^{(1)} + u_M \tilde{R}_3^{(2)} \right). \quad (3.21)$$

Since the $x_1x_2$-plane is far away from both Skyrme solitons the leading contribution to this area integral can be expressed entirely in terms of the asymptotic fields:

$$\Delta E = 2 \sum_{a=1}^3 \int_{x_3=0} dx_1 dx_2 \left( u_M^0 \partial_3 v_N^a - v_N^a \partial_3 u_M^a \right). \quad (3.22)$$
A simple scaling analysis shows that $\Delta E$ falls off like $R^{-(N+M+1)}$ for large $R$. We skip the details here because we shall show how to evaluate $\Delta E$ exactly in the next section. The remaining terms in (3.20) can be estimated with the techniques used in estimating $W_2$. The result is

$$W_2 = \Delta E + O \left( \frac{1}{R^{3N+6}} \right) + O \left( \frac{1}{R^{3M+6}} \right)$$

(3.23)

Combining all terms in (3.4) we conclude that

$$E[U_R] = E[U^{(1)}] + E[U^{(2)}] + \Delta E + O \left( \frac{1}{R^{2N+4}} \right) + O \left( \frac{1}{R^{2M+4}} \right).$$

(3.24)

Note that $\Delta E$ is the leading contribution to the interaction energy if $|N - M| \leq 2$ i.e. if the orders of the leading multipoles of the two Skyrme solitons differ by at most two. We will comment on the validity of this assumption at the end of this paper.

4 Harmonic functions and their interaction energy

In order to compute the interaction energy $\Delta E$ we need to derive some general results about harmonic functions. We define the regions

$$H^- = \{ x \in \mathbb{R}^3 \mid x_3 < \delta \} \quad \text{and} \quad H^+ = \{ x \in \mathbb{R}^3 \mid x_3 > -\delta \},$$

(4.1)

where the positive parameter $\delta$ is introduced for technical reasons. Then we introduce the spaces

$$\mathcal{H}^- = \{ f : H^- \to \mathbb{R} \mid \Delta f = 0, \lim_{r \to \infty} f(x) = 0 \}$$

(4.2)

and

$$\mathcal{H}^+ = \{ g : H^+ \to \mathbb{R} \mid \Delta g = 0, \lim_{r \to \infty} g(x) = 0 \}.$$ 

(4.3)

Elements of $\mathcal{H}^-$ tend to zero at the boundary “at infinity” of $H^-\delta$, elements of $\mathcal{H}^+$ tend to zero at the boundary “at infinity” of $H^+\delta$. No additional restriction is placed on behaviour at the boundaries $x_3 = \pm \delta$.

For the calculations in this section it is convenient to split $\mathbb{R}^3$ into $\mathbb{R}^2 \times \mathbb{R}$ and denote vectors in $\mathbb{R}^2$ by bold letters, e.g. $\vec{x} = (x_1, x_2)$. We then write $x = (\vec{x}, x_3)$. The most general element of $\mathcal{H}^-$ can be written as

$$f(x) = \int \frac{d^2k}{(2\pi)^22k} p(k) \exp(ik \cdot \vec{x} + kx_3),$$

(4.4)

where $k = \sqrt{k^2}$ and the volume element $d^2k/(2\pi)^22k$ arises from the combination of $d^3k$ with the delta-function $\delta(k^2 - k^2)$ which ensures that $f$ satisfies the Laplace equation. Since $f$ is real the Fourier transform $p$ satisfies

$$\bar{p}(k) = p(-k).$$

(4.5)
Similarly, the most general element of $\mathcal{H}^+$ can be written as

$$g(x) = \int \frac{d^2l}{(2\pi)^2} q(l) \exp(i l \cdot x - lx_3), \quad (4.6)$$

with $l = \sqrt{l^2}$ and

$$\bar{q}(k) = q(-k). \quad (4.7)$$

There is a natural pairing between elements of the $\mathcal{H}^-$ and those of $\mathcal{H}^+$

$$\langle f, g \rangle = \int_{x_3=0} dx_1 dx_2 (g \partial_3 f - f \partial_3 g) \quad \text{for} \quad f \in \mathcal{H}^-, g \in \mathcal{H}^+, \quad (4.8)$$

Using the expansion (4.4) and (4.6) we find, in terms of the Fourier modes,

$$\langle f, g \rangle = \int \frac{d^2k}{(2\pi)^2} p(k)q(-k). \quad (4.9)$$

This pairing is of interest to us since the interaction energy (3.22) is proportional to the sum over the pairings $\langle u_M^a, v^a_N \rangle$. Therefore, we also refer to the expressions (4.8) and (4.9) as the interaction energy of the harmonic functions $f$ and $g$.

It is clear that the pairing (4.8) may vanish for some pairs of harmonic functions $f \in \mathcal{H}^-$, $g \in \mathcal{H}^+$. This happens for example if the support of the Fourier transform $p$ is complementary to that of the Fourier transform $q$. However, we shall now show that the interaction energy of multipoles can always be made non-zero by rotating one of the functions.

### 4.1 Multipole fields

For $f \in \mathcal{H}^-$ we have an alternative expansion in spherical harmonics $Y_{Mm}$

$$f(x) = \sum_{M \geq 0} \sum_{m=-M}^{M} \frac{4\pi}{(2M + 1)} Q_{Mm} Y_{Mm}(\theta_+, \varphi_+) \frac{1}{r_+^{M+1}}, \quad (4.10)$$

where $r_+ = |x - X_+|$ and $(\theta_+, \varphi_+)$ are spherical coordinates centred at $X_+ = (0, 0, R/2)$. In this section we only need to assume $R > 0$, but in our applications we will be interested in the large $R$ limit. The coefficients $Q_{Mm}$ are called the multipole moments of the function $f$. Assume that $f$ has non-vanishing multipole moments and suppose $M$ is the smallest integer such that $Q_{Mm} \neq 0$ for some $m = -M, ..., M$. The function

$$f_M(x) = \frac{4\pi}{(2M + 1)} \sum_{m=-M}^{M} Q_{Mm} Y_{Mm}(\theta_+, \varphi_+) \frac{1}{r_+^{M+1}}, \quad (4.11)$$

is a $2^M$-pole field and $Q_{Mm}$ are the leading multipole moments of $f$. 

It is often convenient to write multipole fields in terms of partial derivatives of the Coulomb potential centred at \( X_+ \):

\[
\phi_+(x) = \frac{1}{r_+} = \frac{1}{\sqrt{\rho^2 + (x_3 - R/2)^2}}, \tag{4.12}
\]

where \( \rho^2 = x^2 \). The function \( \partial_3^{m_3} \partial_2^{m_2} \partial_1^{m_1} \phi_+(x) \) is a \( 2^M \)-pole field if \( m_1 + m_2 + m_3 = M \). However, not all of the fields obtained in this way are independent. We introduce the complex derivatives

\[
\partial = \frac{1}{2} (\partial_1 - i \partial_2) \quad \text{and} \quad \bar{\partial} = \frac{1}{2} (\partial_1 + i \partial_2) \quad (4.13)
\]

and note that \( \Delta = \partial_3^2 + 4 \partial \bar{\partial} \). Then, since \( \Delta \phi_+(x) = 0 \) we have

\[
\partial \bar{\partial} \phi_+(x) = -\frac{1}{4} \partial_3^2 \phi_+(x). \tag{4.14}
\]

Thus a basis for \( 2^M \)-pole fields is given by \( \partial_3^{m_3} \partial_2^{m_2} \partial_1^{m_1} \phi_+(x) \) where \( M = m_3 + n + \bar{n} \) and either \( n \) or \( \bar{n} \) can be taken to be zero. In appendix A we derive the exact relation between the functions \( \partial_3^{M+m} \partial^{-m} \phi_+(x) \), \(-M \leq m < 0\), and \( \partial_3^{M-m} \partial^m \phi_+(x) \), \( 0 \leq m \leq M \), on the one hand and the spherical harmonics \( Y_{Mm} \) centred at \( X_+ \) on the other. The result is that we have the alternative expansion of an \( 2^M \)-pole field

\[
f_M(x) = \sum_{-M \leq m \leq 0} A_{Mm} \partial_3^{M+m} \partial^{-m} \phi_+(x) + \sum_{1 \leq m \leq M} A_{Mm} \partial_3^{M-m} \partial^m \phi_+(x), \tag{4.15}
\]

where the coefficients \( A_{Mm}, m = -M, ..., M \) are directly proportional to the multipole moments \( Q_{Mm} \). It follows from the results in appendix A that

\[
A_{Mm} = \sqrt{\frac{4\pi}{2M + 1}} \frac{(-1)^{M+m}2^m}{\sqrt{(M-m)!(M+m)!}} Q_{Mm} \tag{4.16}
\]

for \( m \geq 0 \) and

\[
A_{Mm} = \sqrt{\frac{4\pi}{2M + 1}} \frac{(-1)^{M-|m|}2^{|m|}}{\sqrt{(M-m)!(M+m)!}} Q_{Mm} \tag{4.17}
\]

for \( m < 0 \). Note in particular that the reality of \( f_M \) is equivalent to

\[
A_{M(-m)} = \overline{A_{Mm}}. \tag{4.18}
\]

Multipole fields have a remarkably simple Fourier transform, which will be important for us. We use the representation

\[
\phi_+(x) = \int \frac{d^2k}{2\pi k} e^{-k|x_3 - R/2|} \exp(i k \cdot x), \tag{4.19}
\]

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which can be verified as follows. Exploiting the invariance of \( \phi_+ \) under rotations in the \( x_1x_2 \)-plane we may assume that \( \mathbf{x} = (\rho, 0) \). Using polar coordinates \((k, \psi)\) for \( k \) we first carry out the \( dk \) integration and then the angular integration:

\[
\int_0^{2\pi} \frac{d\psi}{2\pi} \int_0^\infty dk \, e^{-k|x_3-\frac{R}{2}|} \exp(i k \cdot \mathbf{x}) = \int_0^{2\pi} \frac{d\psi}{2\pi} \frac{-1}{(i \rho \cos \psi - |x_3 - R/2|)} = \oint_{S^1} \frac{dw}{2\pi i} \frac{-2}{(i \rho w^2 - 2|x_3 - R/2|w + i \rho)},
\]

where we changed variables to \( w = e^{i\psi} \) in the last line. Expanding the integrand in partial fractions and using the residue theorem then yields the expression (4.12). We compute the Fourier transform of the multipole field (4.15) by differentiating (4.19) under the integral sign. Note that

\[
\partial \exp(i k \cdot \mathbf{x}) = i \frac{2}{k} e^{-i \psi} \exp(i k \cdot \mathbf{x}) \quad \text{and} \quad \bar{\partial} \exp(i k \cdot \mathbf{x}) = i \frac{2}{k} e^{i \psi} \exp(i k \cdot \mathbf{x}),
\]

so we find

\[
f_M(x) = \frac{1}{2\pi} \int d^2 k \, e^{-k(x_3 - \frac{R}{2})} k^{M-1} \sum_{-M \leq m \leq M} \left( \frac{i}{2} \right)^{|m|} A_M e^{im \psi} \exp(i k \cdot \mathbf{x}).
\]

Here we have used that \( x_3 < \delta \) so that in particular \( x_3 < \frac{R}{2} \). Thus with the normalisation (4.14) we arrive at the following simple expression for the Fourier transform

\[
p_M(k) = 4\pi e^{-k \frac{R}{2}} k^{M} \sum_{-M \leq m \leq M} \left( \frac{i}{2} \right)^{|m|} A_M e^{im \psi}.
\]

Remarkably, this function factorises into a \( k \)-dependent part and the function

\[
\Theta(\psi) = \sum_{-M \leq m \leq M} \left( \frac{i}{2} \right)^{|m|} A_M e^{im \psi}
\]

of the angle \( \psi \). The \( k \)-dependent part \( e^{-k \frac{R}{2}} k^{M} \) is non-zero for \( k \neq 0 \) and the function \( \Theta \) only vanishes identically if \( A_M = 0 \) for all \( m = -M, ..., M \), i.e. if the \( 2M \)-pole field is trivial.

### 4.2 The interaction energy of two scalar multipoles

The interaction energy of two multipoles can be expressed in a remarkably compact way. Let

\[
\phi_-(x) = \frac{1}{|x - X_-|}
\]

be the Coulomb potential centred at \( X_- = (0, 0, -R/2) \) and consider the multipole field

\[
g_N(x) = \sum_{-N \leq n \leq 0} B_{Nn} \partial_3^{N+n} \partial^{-n} \phi_-(x) + \sum_{1 \leq n \leq N} B_{Nn} \partial_3^{N-n} \bar{\partial}^n \phi_-(x)
\]
with $B_{N(-n)} = B_{Nn}$. By the same calculation as for $f_M$ above we find the Fourier transform of $g_N$ in the $x_1x_2$-plane:

$$q_N(k) = 4\pi e^{-kR^{N+M}} \sum_{-N \leq n \leq N} \left(i \frac{n}{2}\right)^{|n|} B_{Nn} e^{i\psi}.$$  \hspace{1cm} (4.28)

The interaction energy of the two multipole fields $f_M$ and $g_N$

$$V_{MN} = \langle f_M, g_N \rangle$$  \hspace{1cm} (4.29)

can now be computed using the formula (4.9). Using the factorisation property of the Fourier transforms $p_N$ and $q_N$, it is easy to perform the integration over $k$. Assuming without loss of generality that $M \leq N$ we first carry out the integration over the angle $\psi$ to find

$$V_{MN} = 4\pi \int_0^\infty dk e^{-kR^{N+M}} \sum_{m=-M}^M 2^{-2|m|} \bar{A}_{Mm} B_{Nm},$$  \hspace{1cm} (4.30)

where we have used the reality condition for the coefficients $A_{Mm}$ and $B_{Nm}$. Computing the remaining integral we obtain the final result

$$V_{MN} = 4\pi \frac{(M+N)!}{R^{M+N+1}} \sum_{m=-M}^M 2^{-2|m|} \bar{A}_{Mm} B_{Nm}.$$

This formula has a number of interesting features. The interaction energy depends only on the separation of the multipoles and on the combination $\sum_{m=-M}^M 2^{-2|m|} \bar{A}_{Mm} B_{Nm}$ of the multipole components. As explained in appendix A, the multipole moments $Q_{Nn}$ of a $2^N$-pole can be thought of as elements of the $(2N+1)$-dimensional unitary irreducible representation $W_N$ of $SO(3)$. The vector $B$ with $2N+1$ components $B_{Nn}$, $-N \leq n \leq N$ is naturally an element of $W_N$. Rotations $G \in SO(3)$ about the centre $X$ of the multipole field $g_N$ act on the multipole components via $B_{Nn} \mapsto \sum_{n'=\{-N,\ldots,N\}} U_N^{Nn'}(G) B_{Nn'}$, where $U_N$ is a $(2N+1)$-dimensional irreducible representation of $SO(3)$ (because of the rescaling (4.16) and (4.17) this is not the standard unitary representation). With our assumption that $M \leq N$ we can use the multipole components $A_{Mm}$, $-M \leq m \leq M$, to define the linear form

$$F_A : W_N \rightarrow \mathbb{R}, \quad B \mapsto \sum_{m=-M}^M 2^{-2|m|} \bar{A}_{Mm} B_{Nm}.$$  \hspace{1cm} (4.32)

By assumption, the components $A_{Mm}$ are not all zero, and therefore the map $F_A$ is non-degenerate. Writing the formula (4.31) in terms of this map as

$$V_{MN} = 4\pi \frac{(M+N)!}{R^{M+N+1}} F_A(B),$$  \hspace{1cm} (4.33)

we immediately deduce the following result.
**Theorem 4.1** The interaction energy of an $2^M$-pole and a $2^N$-pole separated by a distance $R$ is always non-vanishing for some relative orientation of the two multipoles. When such an orientation is chosen, the modulus of the interaction energy decreases with the separation as $R^{-(M+N+1)}$.

**Proof:** Assuming without loss of generality that the multipoles are separated along the $x_3$-axis and that $M \leq N$ we have the formula (4.33) for the interaction energy. Since the map (4.32) defined in terms of the (non-vanishing) multipole components $A_{Mm}$ of the $2^M$ pole at $X_+$ is non-degenerate it has a $2N$-dimensional kernel. It then follows from the irreducibility of the $(2N+1)$-dimensional representation $W_N$ that there exists a rotation $G \in SO(3)$ such that $U^N(G)B$ is not in the kernel of $F_A$ for some $G$. For that $G$ we thus have $F_A(U^N(G)B) = \kappa \neq 0$ and $V_{MN} = 4\pi \kappa(M + N)!R^{-(M+N+1)}$. \hfill $\Box$

**5 Attractive forces and existence of minima**

The arguments of the previous section apply to the asymptotic pion fields of the Skyrme solitons $U^{(1)}$ and $U^{(2)}$ discussed in Section 3. In particular we note that the interaction energy $\Delta E$ (3.22) for the leading multipole fields $u_M$ and $v_N$ is just the sum over iso-components of pairings of the form (4.29)

$$\Delta E_a = -2\langle u^a_M, v^a_N \rangle = 2 \int_{x_3=0} dx_1 dx_2 \left( u^a_M \partial_3 v^a_N - v^a_N \partial_3 u^a_M \right). \quad (5.1)$$

Now pick one of the iso-indices, say $a = 1$, and use iso-rotations of the Skyrme solitons to make sure that the first iso-components $u^1_M$ and $v^1_N$ are non-vanishing. It then follows from theorem (4.1) that we can make the multipole interaction energy $\Delta E_1$ non-zero by spatial rotations of one of the Skyrme solitons. This fact is the main upshot of the calculations in the previous section and a crucial input for the following argument which was missing in [11].

Now consider the sum

$$\Delta E = \Delta E_1 + \Delta E_2 + \Delta E_3. \quad (5.2)$$

We would like to show that we can always arrange for $\Delta E$ to be negative by a suitable iso-rotation of one of the Skyrme solitons. We may assume that, possibly after re-labelling the pion fields,

$$\Delta E_1 \geq \Delta E_2 \geq \Delta E_3, \quad (5.3)$$

If $\Delta E < 0$ we are done, so suppose that $\Delta E \geq 0$. Since we know that not all $\Delta E_a$ vanish we can conclude that $\Delta E_1 > 0$. Now perform an iso-rotation of Skyrme soliton 1 by 180 degrees around the third iso-spin axis. This reverses the sign of $\pi_1^{(1)}$ and $\pi_2^{(1)}$ and hence also the sign of $\Delta E_1$ and $\Delta E_2$. The new value of $\Delta E$ is

$$\Delta E = -\Delta E_1 - \Delta E_2 + \Delta E_3$$

$$= -\Delta E_1 - (\Delta E_2 - \Delta E_3) < 0 \quad (5.4)$$

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since \(-\Delta E_1 < 0\) and, with our ordering, \(- (\Delta E_2 - \Delta E_3) \leq 0\).

Thus, the contribution \(\Delta E\) to the interaction energy of two Skyrme solitons \(U^{(1)}\) and \(U^{(2)}\) can always be made less than zero by suitable rotations and iso-rotations of Skyrme soliton 1. It follows from the discussion at the end of Section 3 that for \(|N - M| \leq 2\) and sufficiently large separation parameter \(R\),

\[
E[U_R] < E[U^{(1)}] + E[U^{(1)}]. \tag{5.5}
\]

We conclude with a few comments on the implications of our result for the question of existence of general Skyrme solitons. As explained in Section 1, Esteban proved the existence of Skyrme solitons of arbitrary degree provided the strict inequality (1.19) holds. Our result (5.5) implies the inequality in those cases where minima exist in the sectors \(l\) and \(k - l\), and where the associated multipoles have orders which do not differ by more than two. Since monopole fields do not arise in Skyrme solitons, the interaction energy \(\Delta E\) dominates at large separation if the leading multipole moments in Skyrme solitons are at most octupoles. As explained at the end of Section 2, the \(B = 7\) Skyrme soliton is believed to have octupoles as leading multipoles, but there is no numerical evidence for leading multipoles of higher order. Unfortunately, it seems very difficult to rule out this possibility in general.

Even if one could prove (or circumvent) the assumption concerning multipoles, the existence of attractive forces between Skyrme solitons is not sufficient to establish the inequality (1.19) for infima. Physically it seems reasonable that the existence of attractive forces should imply the existence of minima in every sector. However, we have not been able to develop this observation into a mathematical proof. Further thoughts and speculations in this direction can be found in [17].

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A Spherical Harmonics

In this appendix we derive the relation between the standard spherical harmonics and the following functions on \(\mathbb{R}^3 - \{0\}\) used in the multipole expansion in Section 4.1

\[
F_{Nn}(x) = \begin{cases} 
\bar{\partial}^n \partial^N_n \left( \frac{1}{r} \right) & \text{if } n \geq 0 \\
\bar{\partial}^{-n} \partial^{N+n}_n \left( \frac{1}{r} \right) & \text{if } n < 0
\end{cases} \tag{A.1}
\]

Here \(N \geq 0\) and \(-N \leq n \leq N\) and \(\bar{\partial} = \frac{1}{2}(\partial_1 - i\partial_2)\). These functions are harmonic in their domain:

\[
\Delta F_{Nn} = 0. \tag{A.2}
\]
They are also homogeneous of degree \(-(N + 1)\) so that they can be written as

\[
F_{Nn} = \frac{1}{r^{N+1}} \Phi_{Nn}(\theta, \varphi),
\]

(A.3)

where \((\theta, \varphi)\) are the usual spherical coordinates on the two-sphere centred at the origin. Since the Laplace operator takes the following form in spherical coordinates

\[
\Delta = \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{1}{r^2} \Delta_\omega,
\]

(A.4)

where \(\Delta_\omega\) is the Laplace operator on the 2-sphere of unit radius, it follows from (A.2) that

\[
\Delta_\omega \Phi_{Nn} = -N(N + 1)\Phi_{Nn}.
\]

(A.5)

Define the generator of rotations about the 3-axis

\[
J_3 = -i \frac{\partial}{\partial \varphi}
\]

(A.6)

and express it in terms of complex coordinates \(z = x_1 + ix_2\) and complex derivatives in the \(x_1x_2\) plane:

\[
J_3 = z\partial - \bar{z}\bar{\partial}.
\]

(A.7)

The operator \(\partial\) acts as a raising operator and the operator \(\bar{\partial}\) acts as a lowering operator for \(J_3\):

\[
[J_3, \bar{\partial}] = \bar{\partial} \quad \text{and} \quad [J_3, \partial] = -\partial.
\]

(A.8)

Thus if \(\phi_n\) is a function on \(\mathbb{R}^3\) which is an eigenfunction of \(J_3\) with eigenvalue \(n\) then \(\bar{\partial}\phi_n\) is an eigenfunction with eigenvalue \(n + 1\) provided it is not zero. Similarly \(\partial\phi_n\) is an eigenfunction of \(J_3\) with eigenvalue \(n - 1\) provided it is not zero.

It follows from (A.1) that

\[
F_{Nn} = \begin{cases} 
\bar{\partial}^n F_{N-n,0} & \text{if } n \geq 0 \\
\partial^{-n} F_{N+n,0} & \text{if } n < 0.
\end{cases}
\]

(A.9)

Noting that, by rotational symmetry about the 3-axis,

\[
J_3 \Phi_{N0} = 0 \quad \text{for all } N,
\]

(A.10)

we conclude that

\[
J_3 F_{Nn} = n F_{Nn}.
\]

(A.11)

It follows that

\[
\Phi_{Nn} = r^{N+1} F_{Nn}
\]

(A.12)
also satisfies

$$J_3 \Phi_{Nn} = n \Phi_{Nn}. \quad (A.13)$$

Thus, to sum up, the $\Phi_{Nn}$ are functions on $S^2$, which are eigenfunctions of both the Laplace operator (A.3) and the operator $J_3$ with eigenvalues respectively $-N(N+1)$ and $n$. It follows from standard harmonic analysis on $S^2$ that they must be proportional to the spherical harmonics $Y_{Nn}$.

In the case $n = 0$ we can determine the proportionality constant by evaluating both $\Phi_{N0}$ and $Y_{N0}$ on the positive 3-axis. With the usual normalisation [18] we find

$$Y_{N0} = \sqrt{\frac{2N+1}{4\pi}} (-1)^N N! \Phi_{N0}. \quad (A.14)$$

The relation between $Y_{Nn}$ and $\Phi_{Nn}$ for $n \neq 0$ is harder to compute. Let us assume initially that $n > 0$. Starting with the standard expression for the associated Legendre function in terms of Legendre polynomials

$$P_n^m(\cos \theta) = (-1)^n \sin^n \theta \left( \frac{d}{d \cos \theta} \right)^n P_n(\cos \theta) \quad (A.15)$$

and the expression of the spherical harmonic in terms of associated Legendre function (see e.g. [18] p. 99) we have the relation

$$Y_{Nn}(\theta, \varphi) = (-1)^n \sqrt{\frac{(N-n)!}{(N+n)!}} \left( \sin \theta e^{i\varphi} \right)^n \left( \frac{\partial}{\partial \cos \theta} \right)^n Y_{N0}(\theta, \varphi). \quad (A.16)$$

Then using (A.14) and the definition of $\Phi_{N0}$ we deduce

$$Y_{Nn}(\theta, \varphi) = r^{N+1} \sqrt{\frac{2N+1}{4\pi}} (-1)^{N+n} \frac{(N-n)!}{(N+n)!} z^n D_\theta \partial_3 \left( \frac{1}{r} \right), \quad (A.17)$$

where

$$D_\theta = \frac{1}{r} \frac{\partial}{\partial \cos \theta} = \frac{3}{r} \frac{\partial}{\partial \rho} + \partial_3 \quad (A.18)$$

and $z = \rho e^{i\varphi}$ with $\rho = r \sin \theta$ as in the main text of the paper. Then we use the commutation relation

$$[D_\theta, \partial_3] = \frac{1}{\rho} \frac{\partial}{\partial \rho} \quad (A.19)$$

to move $D_\theta$ past $\partial_3$ in (A.17). Noting that $D_\theta r = 0$ we find

$$Y_{Nn}(\theta, \varphi) = r^{N+1} \sqrt{\frac{2N+1}{4\pi}} (-1)^{N+n} \sqrt{(N-n)!(N+n)!} z^n \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \right)^n \partial_3^{N-n} \left( \frac{1}{r} \right). \quad (A.20)$$
Now exploit that on any function $f$ which only depends on $\rho$ and $x_3$

$$\frac{1}{\rho} \frac{\partial f}{\partial \rho}(\rho, x_3) = \frac{2}{z} \bar{\partial} f(\rho, x_3)$$  \hspace{1cm} (A.21)

to conclude

$$Y_{Nn}(\theta, \varphi) = r^{N+1} \sqrt{\frac{2N+1}{4\pi}} \frac{(-1)^{N+n} 2^n}{\sqrt{(N-n)!(N+n)!}} \bar{\partial}^n \partial_3^{N-n} \left( \frac{1}{r} \right).$$ \hspace{1cm} (A.22)

Thus we finally arrive at promised relationship between $Y_{Nn}$ and $\Phi_{Nn}$, valid for $n \geq 0$:

$$Y_{Nn} = \sqrt{\frac{2N+1}{4\pi}} \frac{(-1)^{N+n} 2^n}{\sqrt{(N-n)!(N+n)!}} \Phi_{Nn}.$$ \hspace{1cm} (A.23)

To deduce the corresponding result for $n < 0$ we note that $\Phi_{Nn} = \bar{\Phi}_{N(-n)}$ and $Y_{Nn} = (-1)^n \bar{Y}_{N(-n)}$ for $n < 0$. Thus for $n < 0$:

$$Y_{Nn} = \sqrt{\frac{2N+1}{4\pi}} \frac{(-1)^N 2^{\lvert n \rvert}}{\sqrt{(N-n)!(N+n)!}} \Phi_{Nn}.$$ \hspace{1cm} (A.24)

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