A SIMPLICIAL MODEL FOR INFINITY PROPERADS

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Abstract. We show how the model structure on the category of simplicially-enriched (colored) props induces a model structure on the category of simplicially-enriched (colored) properads.

This short note is an important component in an ongoing project to understand ‘up-to-homotopy’ prop(erad)s. Props and properads are devices like operads, but which are capable of controlling bialgebraic structures. In [8] we construct a combinatorial model for objects like properads, but where the properadic structure only holds up to coherent higher homotopy. There, we present such ‘infinity properads’ as objects of the presheaf category \( \text{Set}^{\Gamma} \) satisfying inner-horn filling conditions, where \( \Gamma \) is a certain category of graphs. The category \( \Gamma \) is an extension of both the simplicial category \( \Delta \) and the Moerdijk-Weiss dendroidal category \( \Omega \) [15], and our definition of infinity properads is analogous to that of quasi-categories [13] (or infinity categories [14]) and dendroidal inner Kan complexes [16]. In a future paper we will prove the existence of a Quillen model structure on the category of graphical sets \( \text{Set}^{\Gamma} \) so that the fibrant objects are precisely the infinity properads; antecedents to this structure are the Joyal model structure on simplicial sets \( \text{Set}^{\Delta} \) [13, 14] and the Cisinski-Moerdijk model structure on dendroidal sets \( \text{Set}^{\Omega} \) [2].

In the present work, we study (small) simplicially-enriched properads, which we expect to be the rigid model for infinity properads, much as simplicially-enriched categories [1] give a model for infinity(-one) categories and simplicially-enriched operads give a model for infinity operads [4]. Namely, in [7] we will present a functor, called the ‘homotopy coherent nerve’

\[ N_{hc}: s\text{Properad} \to \text{Set}^{\Gamma} \]

which we anticipate, in analogy with the corresponding result in the categorical setting [12, 14], will be the right adjoint in a Quillen-equivalence of model categories [3]. For such a theorem to even be stated, we of course require a model structure on \( s\text{Properad} \), the category of small simplicially-enriched properads. In this paper we define [11] the weak equivalences and fibrations in \( s\text{Properad} \) and the main theorem [18] states that they do indeed determine a model structure.

There are three major ingredients used to prove this theorem. The first is the existence of a similar model structure on the category of simplicial props, which was proved by the first two authors in [6]. The second is a precise description of the left adjoint

\[ F: s\text{Properad} \to s\text{Prop} \]

which is a specialization of work of the third author and Mark Johnson [17]. This description allows us to compare solutions to lifting problems in \( s\text{Prop} \) and in

\footnote{This would also provide an alternate proof of the equivalence between the category of simplicial operads and that of dendroidal sets, which appears in [2, 14].}
sProperad (Theorem 8). The third main ingredient is Kan’s recognition theorem for cofibrantly generated model categories (see e.g. [9, 11.3.1] or [10, 2.1.1.9]).

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1. Graph groupoids and pasting schemes

In this section we recall some concepts and examples from [17], though we often use the same terminology for things that are much less general in the present paper.

An $C$-colored graph $G$ consists of
- a directed graph $G$ with half-edges which has no directed cycles,
- a coloring function $\xi : Ed(G) \to C$,
- orderings on the inputs and outputs of the graph
  \[
  \text{ord}_i : \{1, \ldots, n\} \xrightarrow{\xi} \text{in } G
  \]
  \[
  \text{ord}_o : \{1, \ldots, m\} \xrightarrow{\xi} \text{out } G,
  \]
- orderings on the inputs and outputs of each vertex $v \in Vt(G)$
  \[
  \text{ord}_i^v : \{1, \ldots, n_v\} \xrightarrow{\xi} \text{in } v
  \]
  \[
  \text{ord}_o^v : \{1, \ldots, m_v\} \xrightarrow{\xi} \text{out } v.
  \]

Example 1. Given a biprofile $(c, d) = (c_1, \ldots, c_n; d_1, \ldots, d_m)$ with $c_i, d_j \in C$, the standard corolla $C(c, d)$ is the graph with one vertex $v$, half-edges $1, \ldots, n + m$ with
\[
\text{ord}_i = \text{ord}_i^v : \{1, \ldots, n\} \xrightarrow{\xi} \{1, \ldots, n\} = \text{in } G = \text{in } v
\]
\[
\text{ord}_o = \text{ord}_o^v : \{1, \ldots, m\} \xrightarrow{\xi} \{n + 1, \ldots, n + m\} = \text{out } G = \text{out } v
\]
and
\[
\xi(i) = \begin{cases} 
  c_i & 1 \leq i \leq n \\
  d_{i-n} & n + 1 \leq i \leq n + m.
\end{cases}
\]

A strict isomorphism between $C$-colored graphs preserves all structure, while a weak isomorphism does not necessarily preserve the orderings. The category of graphs (as $C$-varies) along with weak isomorphism gives us our first example of a graph groupoid, which we denote by $Gr^\uparrow$. It contains a (full) subgroupoid $Gr^\uparrow_C$ whose objects are the connected graphs. If we fix a set of colors $C$, then we will write $G^C \subset Gr^\uparrow$ (resp. $G^C_C \subset Gr^\uparrow_C$) for the full subgroups of $C$-colored graphs. For a fixed biprofile $(c, d) = (c_1, \ldots, c_n; d_1, \ldots, d_m)$ with $c_i, d_j \in C$, there is a (non-full) subgroupoid $G^C(c, d) \subset G^C \subset Gr^\uparrow$ with
\[
\begin{align*}
\text{objects} & \quad \text{those graphs with } \xi(\text{ord}_i(s)) = c_s \in C \text{ and } \xi(\text{ord}_o(t)) = d_t \in C, \\
\text{morphisms} & \quad \text{the strict isomorphisms.}
\end{align*}
\]

The use of strict isomorphism guarantees preservation of the colors of the inputs and outputs. There is an analogously defined subgroupoid $G^C_C(c, d) \subset G^C_C \subset Gr^\uparrow_C$.

Both $Gr^\uparrow$ and $Gr^\uparrow_C$ are $C$-colored pasting schemes [17, 8.2] for any color set $C$, which essentially means that they are closed under the operation of graph substitution.
For a fixed color set $\mathcal{C}$, the so-called Kontsevich groupoid $[17] 9.1.4$

$$K = K^\mathcal{C} = \text{Kont}^\mathcal{C}(\mathfrak{Gr}^+_c, \mathfrak{Gr}^+)$$

is defined to be the maximal subgroupoid of $\mathfrak{G}^\mathcal{C}$ which is both orthogonal and prime to $\mathfrak{G}^\mathcal{C}$. Orthogonality means that $G \cap K$ consists only of (permuted) corollas. A graph $G$ is prime to $\mathfrak{G}^\mathcal{C}$ if any graph substitution decomposition $G = K(H_v)$ with $K \in \mathfrak{G}^\mathcal{C}$ and $\uparrow \neq H_v \in \mathfrak{G}^\mathcal{C}$ necessarily has that each $H_v$ is a corolla. Further, $K$ being prime to $\mathfrak{G}^\mathcal{C}$ means exactly that every $G \in K$ is prime to $\mathfrak{G}^\mathcal{C}$.

We have, by [17, Example 9.12(4)], that

**Proposition 2.** The Kontsevich groupoid $K$ consists of relabeled unions of corollas.

2. THE LEFT ADJOINT AND LIFTING

Let $\mathcal{C}$ be a set of colors. A $\mathcal{C}$-colored simplicial properad (resp. $\mathcal{C}$-colored simplicial prop) consists of the data of

- a family of simplicial sets $P(d c, \ell) \in \mathbf{sSet}$, one for each biprofile $(d c, \ell)$ in $\mathcal{C}$;
- unit elements $\text{id}_c \in P(c c, 0)$; and
- composition maps $\gamma_G : P[G] \to P(\xi \text{ out } G, \xi \text{ in } G)$ for each $\mathcal{C}$-colored graph $G \in \mathfrak{G}^\mathcal{C}$ (resp. $G \in \mathfrak{G}^\mathcal{C}$), where
  $$P[G] = \prod_{v \in Vt(G)} P(\xi \text{ out } (v), \xi \text{ in } (v))$$
  is the graph $G$ with each vertex decorated by some element in the family.

These data should satisfy appropriate identity, associativity, and equivariance properties; we refer the reader to [8] or [17, 10.39] for precise definitions. A morphism $P \to Q$ from a $\mathcal{C}$-colored prop(erad) to a $\mathcal{D}$-colored prop(erad) consists of a set map $f : \mathcal{C} \to \mathcal{D}$ and a family of morphisms

$$\left\{ P\left(\frac{d}{\ell}\right) \to Q\left(\frac{f d}{f \ell}\right) \right\}$$

which are compatible with the composition maps and unit elements.

**Theorem 3.** The left adjoint $F : \mathbf{sProperad} \to \mathbf{sProp}$ is given by

$$FP\left(\frac{d}{\ell}\right) = \prod_{[G], w \in \mathcal{C}(d)} P[G]$$

where $[G]_w$ is the orbit of $G$ under weak isomorphisms which preserve the functions $\xi \circ \text{ord}_i$ and $\xi \circ \text{ord}_o$.

*Proof.* The pair $\mathfrak{Gr}^+_c \subset \mathfrak{Gr}^+$ is well-matched [17, Example 9.20], so [17, Lemma 12.3 & Proposition 12.4] applies. □
Remark 4. Suppose that we have a map $\phi: P \to Q$, which, on color sets, is a function $f: \mathcal{C} \to \mathcal{D}$. Then on the component $[G]_w$, the map $F(\phi)$ is given by

$$F(\phi) = \prod_{v \in V(G)} P(\xi_{\text{out}}(v)) \to \prod_{v \in V(fG)} Q(f\xi_{\text{out}}(v)) = Q(fG) \leftrightarrow \prod_{[H]_w \in K(fd_f \sigma)} Q[H] = FQ(fd_f \sigma),$$

where $fG$ is the graph $G$ tweaked to have color function $f \circ \xi$ instead of $\xi$.

One could ask if the operation $G \mapsto fG$ can take graphs from different wheeled isomorphism classes in $K(fd_f \sigma)$ to the same wheeled isomorphism class in $K(fd_f \tau)$. It turns out this is not possible, as we shall see in Proposition 6.

Lemma 5. Fix a color set $\mathcal{C}$. Then there is a bijection between the set

$$\{ [G]_w : G \in K \text{ is connected} \}$$

and the set of pairs of unordered lists $[c_1, \ldots, c_n], [d_1, \ldots, d_m]$ of colors of $\mathcal{C}$. Consequently, for any profile $(c \cdot d)$, the category $K(c \cdot d)$ has only a single weak isomorphism class whose constituent graphs are connected.

Proof. The connected graphs in $K$ are precisely the relabeled corollas. By [17, 4.18(1,2)], for a pair of ordered lists $\underline{c} = c_1, \ldots, c_n$ and $\underline{d} = d_1, \ldots, d_m$ we have weak isomorphisms

$$\gamma_{\underline{c}, \underline{d}} \delta \cong C_{\underline{c}, \underline{d}} \xrightarrow{\gamma_{\underline{c}, \underline{d}}} C_{\underline{c}, \underline{d}} \xrightarrow{\delta} C_{\underline{c}, \underline{d}}$$

for all $\sigma, \gamma \in \Sigma_m$ and $\tau, \delta \in \Sigma_n$. \hfill \Box

Proposition 6. Suppose that $f: \mathcal{C} \to \mathcal{D}$ is a map of colors, $(c \cdot d)$ is a $\mathcal{C}$-biprofile, and consider the map

$$\psi: K(c \cdot d) \to K(d_f \cdot d_f) \quad G \mapsto fG.$$ 

Then $\psi$ is a bijection which descends to a bijection

$$[\psi]_w : \left\{ [G]_w : G \in K(c \cdot d) \right\} \cong \left\{ [H]_w : H \in K(d_f \cdot d_f) \right\}$$

between weak isomorphism (preserving profiles) classes.

Proof. We first construct an inverse to $\psi$. Given $H \in K(d_f \cdot d_f)$, let $\gamma(H)$ be the $\mathcal{C}$-colored graph with the same underlying graph, the same ordering functions
ord, ord, ord, ord, and with the coloring function \( \xi \) defined to be the composite

\[
\xi \circ \ord^{-1} : \{1, \ldots, n\} \coprod \{1, \ldots, m\} \to C.
\]

We know that \( \text{Ed}(G) = \text{in}(G) \coprod \text{out}(G) \) since \( G \) consists of unions of corollas. But then we have that \( \psi_\gamma(H) = H \) and \( \gamma \psi(G) = G \).

We already know that \( \psi \) is functorial with respect to weak isomorphisms. Since the definition of the coloring function for \( \gamma(H) \) does not utilize the ord, ord, we see that \( \gamma \) is also functorial with respect to weak isomorphisms. \( \square \)

Notice that the unit of the adjunction is levelwise an inclusion

\[
P \left( \frac{d}{L} \right) = P[C(\underline{w})] \to \coprod_{[G] \in K(\underline{w})} P[G] = \text{FP} \left( \frac{d}{L} \right);
\]

in particular, \( F : \text{sProperad} \to \text{sProp} \) is faithful.

**Lemma 7.** If \( f : P \to Q \) is a morphism in \( \text{sProperad} \) and \( \left( \frac{d}{L} \right) \) is a biprofile in \( \text{Col} P \), then

\[
P \left( \frac{d}{L} \right) \xrightarrow{i_P} \text{FP} \left( \frac{d}{L} \right) \xrightarrow{F(f)} \text{FP} \left( \frac{f}{L} \right)
\]

is a pullback.

**Proof.** Let \( z \in Q \left( \frac{f}{L} \right) \). Suppose that \( i_Q z = F(f)y \) for some \( y \in FP \left( \frac{d}{L} \right) \). Since \( i_Q z = F(f)y \) is in the \( [C(\underline{w})]_w \) component of \( FQ \left( \frac{d}{L} \right) \), we know that \( y \) must be in the \( [C(\underline{w})]_w \) component of \( FQ \left( \frac{d}{L} \right) \) by remark \( [4] \) But then \( y = iy' \) for a unique element \( y' \in P \left( \frac{d}{L} \right) \). Then \( i_Q f(y') = F(f)i_P y' = F(f)y = i_Q x \), so injectivity of \( i_Q \) gives that \( f(y') = x \). \( \square \)

The following theorem is key. The functor \( F : \text{sProperad} \to \text{sProp} \) is not full, but it is ‘full enough’ to do all of our liftings in \( \text{sProperad} \).

**Theorem 8.** Suppose that

\[
A \xrightarrow{a} P \xleftarrow{g} B \xrightarrow{b} Q
\]
is a commutative diagram in $sProperad$ and that there exists $q \in sProp$ making the diagram

\[
\begin{array}{ccc}
FA & \xrightarrow{Fa} & FP \\
\downarrow{Fq} & & \downarrow{Ff} \\
FB & \xrightarrow{Fb} & FQ
\end{array}
\]

commute. Then there exists $\tilde{q} \in sProperad$ with $F(\tilde{q}) = q$, $\tilde{q}g = a$, and $f\tilde{q} = b$.

**Proof.** We must show that the adjoint of $q$, $\tilde{q} : B \to UF\mathcal{P}$ factors through $\mathcal{P}$:

\[
\tilde{q} : B \xrightarrow{\tilde{q}} \mathcal{P} \to UF\mathcal{P}.
\]

But this is immediate since at each biprofile $(\tilde{b} \tilde{a})$ we have a diagram

\[
\begin{array}{ccc}
B((\tilde{b})) & \xrightarrow{\tilde{q}} & \mathcal{P}((\tilde{b})) \\
\downarrow{\tilde{q}} & & \downarrow{i_{\mathcal{P}}} \\
\mathcal{P}((\tilde{b})) & \xrightarrow{i_{\mathcal{P}}} & FP((\tilde{b})_{\mathcal{P}}) \\
\downarrow{f} & & \downarrow{F(\tilde{f})} \\
Q((d)) & \xrightarrow{i_{\mathcal{Q}}} & FQ((d)_{\mathcal{Q}})
\end{array}
\]

(where $\tilde{c} = qa$ and $\tilde{d} = qb$) which admits $\tilde{q}$ by the preceding lemma. That $\tilde{q}$ is a map of properads follows from the fact that $q = F\tilde{q} : FB \to FP$ is a map of props. $\square$

### 3. Cofibrantly generated model categories

Suppose that $K$ is a class of maps in some bicomplete category $\mathcal{C}$. Recall from [10, 2.1.7, 2.1.9], the following classes of maps in $\mathcal{C}$:

- A map $f$ is $K$-injective, that is $f \in K$-inj, if $f$ has the right lifting property with respect to every map in $K$. In other words, given any solid arrow diagram

\[
\begin{array}{ccc}
A & \xrightarrow{k} & X \\
\downarrow{f} & & \downarrow{f} \\
B & \xrightarrow{} & Y
\end{array}
\]

with $k \in K$, then the dotted arrow exists and makes the diagram commute.

- A map $f$ is an $K$-cofibration, that is, $f \in K$-cof, if it has the left lifting property with respect to every $K$-injective map.

- A map $f$ is an $K$-cell complex, that is, $f \in K$-cell, if it is a transfinite composition of pushouts of elements of $K$.

Let $\mathcal{C}$ be a cocomplete category and $A \in \mathcal{C}$ an object. We say that $A$ is finite if for every sequence $X_0 \to X_1 \to \cdots \to X_n \to \cdots$ indexed by the natural numbers $\mathbb{N}$, the map

\[
\colim C(A, X_i) \to C(A, \colim X_i)
\]

is an isomorphism. There is a more general version of this, where one can speak of an object $A$ being small relative to a class of maps $D$ in $\mathcal{C}$ (see [10, 2.1.3]), but we only deal with finite objects, which are small relative to any class of maps in $\mathcal{C}$.
Definition 9. A model category $C$ is cofibrantly generated if there are sets $I$ and $J$ of maps such that

- The domains of $I$ are small relative to $I$-cell;
- The domains of $J$ are small relative to $J$-cell;
- The class of fibrations is $J$-inj; and
- The class of acyclic fibrations is $I$-inj.

Recall the following recognition theorem [10, 2.1.19] for cofibrantly generated model categories.

Theorem 10. Let $C$ be a bicomplete category, $W$ a subcategory of $C$, and $I$, $J$ sets of maps of $C$. Then there is a cofibrantly generated model category structure on $C$ with $I$ as the set of generating cofibrations, $J$ as the set of generating acyclic cofibrations, and $W$ as the subcategory of weak equivalences if the following are satisfied:

1. The subcategory $W$ has the two out of three property and is closed under retracts
2. The domains of $I$ are small relative to $I$-cell.
3. The domains of $J$ are small relative to $J$-cell.
4. $J$-cell $\subset W \cap I$-cof.
5. $I$-inj $= W \cap J$-inj.

4. THE MODEL STRUCTURE ON SIMPLICIAL PROPERADS

We begin by making some definitions. Given a simplicial prop or properad $P$, we can look at its underlying simplicial category by discarding all $P(c,d)$ with $|c| \neq 1 \neq |d|$. Further, given a simplicial category $C$, we can get a discrete category of components $\pi_0 C$ by setting $\text{Ob} \pi_0 C = \text{Ob} C$ and $(\pi_0 C)(a,b) = \pi_0(C(a,b))$. For concision, we will just write $\pi_0$ for either of the composites

$$s\text{Prop} \to s\text{Properad} \to s\text{Cat} \xrightarrow{\pi_0} \text{Cat}$$

or

$$s\text{Properad} \to s\text{Cat} \xrightarrow{\pi_0} \text{Cat}.$$

A functor $f : C \to D$ in $\text{Cat}$ is called an isofibration if for each isomorphism $h : f(c) \to d$ in $D$, there exists an isomorphism $g : c \to c'$ in $C$ with $f(g) = h$.

Definition 11. Let $f : P \to Q$ be a morphism in $s\text{Prop}$ or $s\text{Properad}$. We say that $f$ is a weak equivalence if

W1: for each input-output profile $(\frac{b}{a})$ in $\text{Col}(P)$ the morphism

$$f : P\left(\frac{b}{a}\right) \to Q\left(\frac{fb}{fa}\right)$$

is a weak homotopy equivalence of simplicial sets; and

W2: the functor $\pi_0 f : \pi_0 P \to \pi_0 Q$ is an equivalence of categories.

We say that the morphism $f$ is a fibration if

F1: for each input-output profile $(\frac{b}{a})$ in $\text{Col}(P)$ the morphism

$$f : P\left(\frac{b}{a}\right) \to Q\left(\frac{fb}{fa}\right)$$

is a Kan fibration of simplicial sets; and
\textbf{F2}: the functor \( \pi_0 f : \pi_0 \mathcal{P} \to \pi_0 \mathcal{Q} \) is an isofibration.

Let \( \mathcal{W}_c \) (resp. \( \mathcal{F}_c \)) be the weak equivalences (resp. fibrations) in \sProp, and \( \mathcal{W} \) (resp. \( \mathcal{F} \)) be the weak equivalences (resp. fibrations) in \sProp.

Notice that with this definition, a map \( f : \mathcal{P} \to \mathcal{Q} \) of simplicial props is a weak equivalence (resp., fibration) if and only if \( U f : U \mathcal{P} \to U \mathcal{Q} \) is a weak equivalence (resp., fibration) of simplicial properads. Also notice that if \( f \) satisfies \( \text{W1} \), then \( \pi_0 f \) is fully-faithful, so to check \( \text{W2} \) it is enough to check that \( \pi_0 f \) is essentially surjective.

\textbf{Lemma 12.} Let \( f : \mathcal{P} \to \mathcal{Q} \) be a morphism in \sProp. Then \( f \) satisfies \( \text{F2} \) if and only if \( F(f) : \mathcal{P} \to \mathcal{Q} \) in \sProp satisfies \( \text{F2} \).

\textbf{Proof.} A key fact is the fact if \( x \in \mathcal{P}[G] \subset \mathcal{F} \mathcal{P}(\bar{e}) \) and \( y \in \mathcal{P}[H] \subset \mathcal{F} \mathcal{P}(\bar{e}) \) where \( G \) has \( n \) connected components and \( H \) has \( m \) connected components, then \( y \circ x \) is in the summand \( \mathcal{P}[K] \subset \mathcal{F} \mathcal{P}(\bar{e}) \) of a graph \( K \) with \( n+m-1 \) connected components. It follows (since \( \text{id}_a \in \mathcal{P}[C(a,a)] \)) that when \( c = e \), if \( y \circ x \) is in the same connected component as \( e \), or \( x \circ y \) is in the same connected component as \( e \), then \( G = C^{(c,d)} \) and \( H = C^{(d,c)} \).

Suppose that \( \pi_0 f \) is an isofibration, and there is an \( h \in \mathcal{Q}(\mathcal{P} \mathcal{F}^p_0) \) so that \( [h] \in \pi_0 \mathcal{Q} \) is an isomorphism. By the previous paragraph, we actually have that \( [h] \) is an isomorphism in \( \pi_0 \mathcal{Q} \), hence there exists \( g \in \mathcal{P}(\mathcal{P} \mathcal{F}^p_0) \) in \( \mathcal{P} \) so that \( (\pi_0 f)(g) = h \).

Thus \( \pi_0 F(f) \) is an isofibration as well.

For the reverse implication, suppose that we have \( h \in \mathcal{Q}(\mathcal{P} \mathcal{F}^p_0) \), which is an isomorphism in \( \pi_0 \mathcal{Q} \). Then it is also an isomorphism in \( \pi_0 \mathcal{F} \), hence there exists a vertex \( g \in \mathcal{P}(\mathcal{P} \mathcal{F}^p_0) \) with \( \pi_0(g) \) an isomorphism and \( \pi_0 F(f)(g) = h \). Since \( \pi_0(g) \) is an isomorphism, we actually have \( g \in \mathcal{P}(\mathcal{P} \mathcal{F}^p_0) \), and \( \pi_0 f(g) = h \). Thus \( \pi_0 f \) is an isofibration as well.

\textbf{Proposition 13.} We have an equality of classes of maps

\[ F(\mathcal{W}_c) = \mathcal{W} \cap \text{Im } F \]

in \sProp; since \( F \) is injective on objects and faithful, this establishes that \( \mathcal{W}_c \cong \mathcal{W} \cap \text{Im } F \). Similarly, we have \( \mathcal{F}_c \cong F(\mathcal{F}_c) = \mathcal{F} \cap \text{Im } F \).

\textbf{Proof.} First observe that \( f : \mathcal{P} \to \mathcal{Q} \) satisfies \( \text{W1} \) if and only if \( F(f) : F \mathcal{P} \to F \mathcal{Q} \) satisfies \( \text{W1} \): this follows from Remark 4, Theorem 3, and Proposition 6. Analogously \( f \) satisfies \( \text{F1} \) if and only if \( F(f) \) satisfies \( \text{F1} \). We also have that \( f \) satisfies \( \text{F2} \) if and only if \( F(f) \) satisfies \( \text{F2} \) by Lemma 12, so the second statement is proved.

In the diagram

\[
\begin{array}{ccc}
\pi_0 \mathcal{P} & \longrightarrow & \pi_0 F \mathcal{P} \\
\downarrow \pi_0 f & & \downarrow \pi_0 F(f) \\
\pi_0 \mathcal{Q} & \longrightarrow & \pi_0 F \mathcal{Q}
\end{array}
\]

the horizontal maps are the identity on object sets. Thus, if \( \pi_0 f \) is essentially surjective then so too is \( \pi_0 F(f) \). To complete the proof, we will show that if \( \pi_0 F(f) \) is essentially surjective, then so too is \( \pi_0 f \).
Suppose that \( c \in \text{Ob} \pi_0 Q = \text{Ob} \pi_0 FQ \). Since \( F(f) \) is essentially surjective, there is an object \( d \in \text{Ob} \pi_0 FP = \text{Ob} \pi_0 P \) and morphisms
\[
x \in FQ \left( \frac{f(d)}{c} \right) \quad \& \quad y \in FQ \left( \frac{c}{f(d)} \right)
\]
with
\[
x \circ y \sim \text{id}_c \text{ in } FQ \left( \frac{c}{c} \right) \quad \text{and} \quad y \circ x \sim \text{id}_{f(d)} \text{ in } FQ \left( \frac{f(d)}{f(d)} \right).
\]
But using the description in Theorem 3, we know that
\[
\text{id}_a \in Q \left( \frac{a}{a} \right) = Q[C(a;a)] \subset FQ \left( \frac{a}{a} \right).
\]
By the first paragraph of the proof of Lemma 12, this implies that
\[
x \in Q[C(c;f(d))] = Q \left( \frac{f(d)}{c} \right) \quad \& \quad y \in Q \left( \frac{c}{f(d)} \right).
\]
Since \( Q(a) \rightarrow FQ(a) \) is injective on components, we know that
\[
x \circ y \sim \text{id}_c \text{ in } Q \left( \frac{c}{c} \right) \quad \text{and} \quad y \circ x \sim \text{id}_{f(d)} \text{ in } Q \left( \frac{f(d)}{f(d)} \right)
\]
as well. Thus \( c \) is isomorphic to \( f(d) \) in \( \pi_0 Q \). Since \( c \) was arbitrary, \( \pi_0 f \) is essentially surjective. \( \square \)

Notice it is not true that \( f \) satisfies (W2) if and only if \( F(f) \) satisfies (W2).

The following is the main theorem of [6].

**Theorem 14.** The category of simplicial props \( \text{sProp} \) admits a cofibrantly generated model category structure with the classes of weak equivalences and fibrations from definition 11. The set of generating cofibrations is \( FI \) and the set of generating acyclic cofibrations is \( FJ \). Cofibrations are those morphisms which have the left lifting property (LLP) with respect to the acyclic fibrations.

**Definition 15.** For \( n, m \geq 0 \), let \( \mathcal{G}_{n,m} : \text{sSet} \rightarrow \text{sProperad} \) be the functor characterized by the property that
\[
\text{Hom}_{\text{sProperad}}(\mathcal{G}_{n,m}[X], P) = \left\{ (c,d), f : |c| = n, |d| = m, f \in \text{Hom}_{\text{sSet}} \left( X, P \left( \frac{d}{c} \right) \right) \right\}.
\]

Let \( \mathcal{I} \) be the category with one object \( x \) and no non-identity morphisms. We consider the class of simplicial categories \( \mathcal{H} \) with two objects \( x \) and \( y \), weakly contractible function complexes, and only countably many simplices in each function complex. Furthermore, we require that each such \( \mathcal{H} \) is cofibrant in the Dwyer-Kan model category structure on \( \text{sCat}_{[x,y]} \) [5]. Let \( \mathcal{H} \) denote a set of representatives of isomorphism classes of such categories.

**Definition 16.** The set \( I \) consists of the following morphisms of simplicial properads:

**C1:** For \( n, m, p \geq 0 \), the maps \( \mathcal{G}_{n,m}[\partial \Delta[p]] \rightarrow \mathcal{G}_{n,m}[\Delta[p]] \).
C2: The \( \mathbf{sSet} \)-functor \( \emptyset \xrightarrow{} \mathcal{I} \) viewed as a morphism of simplicial properads.

The set \( J \) consists of the following morphisms of simplicial properads:

A1: For \( n, m, p \geq 0 \) and \( 0 \leq k \leq p \), the maps \( G_{n,m}[\Lambda[k,p]] \to G_{n,m}[\Delta[p]] \).

A2: The \( \mathbf{sSet} \)-functors \( \mathcal{I} \xleftarrow{} \mathcal{H} \) for \( \mathcal{H} \in \mathcal{H} \) which take \( x \) to \( x \), viewed as morphisms of simplicial properads.

Theorem 14 was proved by applying the recognition Theorem 10 to \( W \) and the sets \( FI, FJ \subset \mathbf{sProp} \).

The following lemma follows from the key theorem 8.

**Lemma 17.** If \( K \) is a class of morphisms of \( \mathbf{sProperad} \), then

\[
\begin{align*}
(1) & \quad F(K-\text{inj}) = (FK)-\text{inj} \cap \text{Im } F \\
(2) & \quad F(K-\text{cell}) \subset (FK)-\text{cell} \cap \text{Im } F \\
(3) & \quad (FK)-\text{cof} \cap \text{Im } F \subset F(K-\text{cof}).
\end{align*}
\]

**Proof.** For (2), note that applying \( F \) to a pushout of a map in \( K \) gives a pushout of a map in \( FK \), since \( F \) is a left adjoint hence preserves colimits. Cocontinuity also guarantees that \( F \) preserves transfinite composition. The other two follow from Theorem 8. \( \square \)

**Theorem 18.** There is a cofibrantly-generated model structure on \( \mathbf{sProperad} \) with fibrations \( \mathcal{F}_c \) and weak equivalences \( \mathcal{W}_c \).

**Proof.** Take \( I \) and \( J \) to be the sets from definition 16. Now that we have Proposition 13 and (1–3) of Lemma 17, we apply show why the conditions of Theorem 11 hold. First note that \( \mathbf{sProperad} \) is complete and cocomplete.

(1): It is clear that \( \mathcal{W}_c \) satisfies 2-of-3 and is closed under retracts.

(2-3): Clearly \( \emptyset \) and \( \mathcal{I} \) are finite. By the characterization of definition 15 and a variation on [10, 3.1.2], we also have that \( G_{n,m}[\partial \Delta[p]] \) and \( G_{n,m}[\Lambda[k,p]] \) are finite. This implies that all of these objects are small relative to both \( I \)-cell and \( J \)-cell.

(4): We have

\[
\begin{align*}
F(J-\text{cell}) & \subset F(\text{J-cell} \cap \text{Im } F) \\
& \subset (\mathcal{W} \cap FI-\text{cof}) \cap \text{Im } F \\
& \overset{13}{=} F(\mathcal{W}_c) \cap (FI-\text{cof} \cap \text{Im } F) \\
& \subset F(\mathcal{W}_c) \cap F(I-\text{cof}).
\end{align*}
\]

Since \( F \) is injective on objects and faithful, we have

\( I \)-cell \( \subset \mathcal{W}_c \cap I \)-cof.

(5): Using Lemma 17 and Proposition 13, we have

\[
\begin{align*}
F(I-\text{inj}) & \overset{4}{=} FI-\text{inj} \cap \text{Im } F = FJ-\text{inj} \cap \mathcal{W} \cap \text{Im } F \\
& \overset{11}{=} F(J-\text{inj}) \cap F(\mathcal{W}_c).
\end{align*}
\]

Since \( F \) is injective on objects and faithful, we have

\( I \)-inj \( = J \)-inj \( \cap \mathcal{W}_c. \)
Thus $s\text{Properad}$ is a cofibrantly generated model category with $W_c$ as the subcategory of weak equivalences. The characterization of fibrations is given by

$$F_c \cong F(F_c) = F \cap \text{Im } F = (FJ\text{-inj}) \cap \text{Im } F \cong F(J\text{-inj}) \cong J\text{-inj}.$$ 

□

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