Minimizing postulation in a senior undergraduate course in electromagnetism

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An approach to the teaching of electromagnetism to senior undergraduate students, designed for overcoming the fragmentation of the theory is described. As usual it starts from the static case, but it is strictly based on Helmholtz theorem of uniqueness of vector fields, on primary experimental observations upon the sources of electric and magnetic fields, on Lenz’s law and on principles of superposition and reciprocity of interactions. Thereafter, without further postulation, all laws and rules arise in a procedure where electricity and magnetism parallel one another. Maxwell’s equations are built up on a step-by-step basis, showing that electromagnetic field can be considered but a twofold linear vector field obeying the simplest principles of physical theories. It is shown that every system of units fits this formalism, provided suitable values are assigned to some constants arising therein from superposition principle.

I. INTRODUCTION

Contrasting with earlier procedures [1] contemporary physical sciences undergraduate teaching makes frequent use of some powerful mathematical tools pertaining not only to calculus but also to the methods of theoretical physics. Nevertheless, the axiomatic structure hardly takes advantage of such powerful mathematical machinery. Indeed, in spite of their extraordinary contribution in adapting the available mathematical basis to pedagogical needs, most textbooks on electromagnetic theory intended to senior undergraduate students approach the subject by means of the same axioms used in books intended to freshmen: the laws of Coulomb, Ampère, Biot and Savart, Lorentz force equation, etc. [2–5]. On the other hand, outstanding effort has been directed to minimizing the number of axioms of electromagnetic theory, in the form of textbook chapters and papers. Unfortunately, they require a mathematical background that is seldom mastered by students at this course level, thus being labelled “optional matter” and often omitted, although not disregarded [7, 8].

The pedagogical approach presented here stands as a mean term between the conventional one and those highly synthetic works ultimately designed for axiomatic purposes. It has been developed for a course in electromagnetic theory to senior undergraduate students of physics, aiming to encompass the mathematical apparatus and the theoretical concepts into an unitary formalism which provided conciseness and axiomatic economy to the theory, without loss of conceptual or technical knowledge: accordingly, the above mentioned laws, which usually constitute axioms become, here, the consequence of superposition and reciprocity principles applied to a vector field arising from a scalar physical quantity, the electric charge. It provides a better understanding of the theory and prepares the student for the advent of Maxwell’s equations, which are obtained on a step-by-step basis without invoking the usual mosaic of postulates. Throughout the formalism, electricity and magnetism parallel one another, showing their affinity and common origin, even in the stationary case. After obtaining the equation of electromagnetic induction Maxwell’s equations are derived by simply imposing the requirement of self-consistency to the previous set, a procedure that leads to displacement current. Finally, the essentials of time-varying field theory are discussed along with the derivation of the wave equations on potentials and the Poynting theorem. A brief discussion on systems of units for electromagnetic fields in a vacuum and its implications in our general formalism ends its presentation.

The resulting formalism yields great emphasis on concepts, without disregarding the need for training on calculation techniques envisaging the solution of scientific problems. This procedure can better prepare the student for dealing with physical concepts and develop his/her creativeness.

Our approach is founded on a threefold base: (i) a review of mathematical concepts, referring to vector algebra and vector calculus, particularly the theorem of Helmholtz; (ii) on the phenomenological realm, we begin with a set of three elementary thought experiments: first, the pendulum electroscope [9]; second, the interaction between permanent magnets; third, the interaction between an electric discharge and a compass, to which the conservation of electric charge and Lenz’s law are added; (iii) from the conceptual realm, principles of superposition and reciprocity of interactions. Following the general use of courses at this level we assume, critically, the validity of Newtonian mechanics and Galilean transformations. Besides, for reason of conciseness we present here only the approach to electromagnetic systems in a vacuum (microscopic theory). Macroscopic theory will be presented later [10].

In Section II the mathematical basis of our approach is founded and further combined, in Section III, with theoretical postulates and experimental evidence for obtaining the field equations in the static case. Section IV shows how a
reciprocity can be invoked and justified in the formalism and how Lorentz force is derived from it. Section V extends the field equations to the time-dependent case, i.e., we arrive at Maxwell’s equations and its consequences. In Section VI a brief discussion is made about systems of units and in Section VII we present our final remarks.

II. MATHEMATICAL BACKGROUND

In what mathematical support is concerned, we would like to point out the statement of the theorem of Helmholtz \[11,12\]: let \( \mathbf{V}(\mathbf{r}) \) describe a vector field. If we are given the functions \( s(\mathbf{r}) \) and \( c(\mathbf{r}) \), such that,

\[
\nabla \cdot \mathbf{V} = s(\mathbf{r}),
\]

(2.1)

and

\[
\nabla \times \mathbf{V} = c(\mathbf{r}),
\]

(2.2)

then \( \mathbf{V} \) is uniquely determined as:

\[
\mathbf{V} = -\nabla \phi + \nabla \times \mathbf{A},
\]

(2.3)

where \( \phi \) and \( \mathbf{A} \) are the scalar and vector potentials, respectively, given by:

\[
\phi(\mathbf{r}) = \left( \frac{1}{4\pi} \right) \int \frac{[s(\mathbf{r}')] / ||\mathbf{r} - \mathbf{r}'||} {dV'}
\]

(2.4)

and

\[
\mathbf{A}(\mathbf{r}) = \left( \frac{1}{4\pi} \right) \int \frac{[c(\mathbf{r}')] / ||\mathbf{r} - \mathbf{r}'||} {dV'}.
\]

(2.5)

According to Eqs. (2.3)-(2.3), the resulting field has the form

\[
\mathbf{V}(\mathbf{r}) = \left( \frac{1}{4\pi} \right) \int [s(\mathbf{r}') (\mathbf{r} - \mathbf{r}') / ||\mathbf{r} - \mathbf{r}'||^3]dV' + \left( \frac{1}{4\pi} \right) \int [c(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}') / ||\mathbf{r} - \mathbf{r}'||^3]dV'.
\]

(2.6)

All integrals above are taken over the whole space. This theorem is a powerful tool for developing the theory, because establishing \( s \) and \( c \) in Eqs. (2.1) and (2.2), leads to the unique determination of \( \mathbf{V} \). Thus, the search for those two functions constitutes an advantageous starting point for developing a theory of the field \( \mathbf{V} \). Furthermore, Eqs. (2.4) and (2.5) show that, from the mathematical standpoint, \( s(\mathbf{r}')dV' \) and \( c(\mathbf{r}')dV' \) play the role of field sources located at point \( \mathbf{r}' \) and having scalar and vector nature, respectively. When applied to the field vectors \( \mathbf{E} \) and \( \mathbf{B} \), this theorem provides not only a precise correlation between these fields and their sources, but also the proper definition of other physical quantities as, for instance, the magnetic vector potential. In addition, it has a number of corollaries \[13\] which provide better understanding of the theory clarifying, particularly, the origin of the magnetic scalar potential \[14\]. On the other hand, it has to be pointed out that, for practical purposes, the theorem use is almost entirely restricted to the stationary case, for in the general time-dependent situation the finite character of the velocity of interactions impedes an \textit{a priori} determination of the right-hand side term in Eqs. (2.1) and (2.2). In other words, despite being useful for theoretical inspection of time-dependent fields, Helmholtz theorem cannot afford the complete solution for them. Furthermore, in electromagnetism we deal with a twofold vector field, a field described by vectors \( \mathbf{E} \) and \( \mathbf{B} \) which can be depicted by independent field equations in the static case but not in the general (time-dependent) case, because of their common origin in the dynamics of electric charges. Therefore our starting point must be the field originated from stationary source configurations, i.e., the case when time is ignorable and the non-instantaneous nature of interactions is irrelevant. It is underlined that the solution for the time-dependent case - as well as its analysis - will be worked out after establishing the general equations, Maxwell’s equations.

A second point to be stressed on the mathematical foundations of the formalism is the uniqueness of products involving vectors, as long as only linear operations are taken into account. Accordingly, it can be shown by simple means that:

1. The multiplication of a vector \( \mathbf{v} \) by a scalar \( c \) is the only linear operation leading from the pair \( \{c, \mathbf{v}\} \) to a vector quantity.
2. The usual scalar multiplication of two vectors \( \mathbf{u} \) and \( \mathbf{v} \) is the only linear operation leading from them to a scalar quantity.

3. Analogously, the vector multiplication of an ordered pair of vectors \( \{\mathbf{u}, \mathbf{v}\} \) is the only linear operation leading from this pair to a vector quantity.

III. EQUATIONS OF THE STATIC FIELD

Assuming that electric and magnetic fields have vector nature, our formalism starts from the determination of the analogues of Eqs. (2.1) and (2.2) for both these fields. There are powerful theoretical arguments which can be used for attaining that aim but, in order to weaken our set of axioms, we prefer to approach the subject by means of some trivial experimental concepts and facts. It wouldn’t be important describing here a detailed experiment. For our purpose a simple enumeration of the experimental goals and the possibility of reaching them is enough. These goals are proving:

1. that the electric field is irrotational;
2. that the magnetic field is divergenceless;
3. the equivalence between permanent magnets and electric discharges as sources of magnetic field.

Electricity: Electric interactions are here defined by means of well known elementary experiments consisting of rubbing some bodies against wool, fur or certain fabrics and observing that they attract pieces of paper and other light objects. Electrometers and pendulum electrosopes are among the useful measuring instruments. At this stage of development we also introduce the concept of insulators and conductors, as enlightened by the concept of conduction, which can be easily checked from the experimental standpoint. Then a Van de Graaff generator is presented, its operation being described in terms of the preceding concepts and phenomena. Electric field, represented by the vector \( \mathbf{E} \), is defined as the responsible for the electric interaction and its sources are shown to have scalar nature. This fact can be demonstrated by charging a spherical metal ball - using a Van de Graaff generator, for instance - and observing that a rotation around an arbitrary axis passing across its center doesn’t change its field, i.e., the attraction/repulsion it causes on, say, the pendulum electroscope. Thus we prove that the electric field is irrotational, since we must have

\[
\nabla \times \mathbf{E} = 0,
\]

in order to cope with the non-vector nature of its sources. On the other hand, taking into account that, in Eq. (2.1), \( s(\mathbf{r}) \) is the scalar source density, we can write

\[
\nabla \cdot \mathbf{E} = k_1 \rho(\mathbf{r}),
\]

being \( \rho(\mathbf{r}) \) the electric charge density and \( k_1 \) a constant to be established according to a system of units.

Magnetism: In a similar manner, magnetism is defined as the interaction between permanent magnets at rest. A compass may be used as measuring instrument. The tendency to alignment of the compass in a preferred direction is observed and assigned to an external field, due to the Earth. It is neglected as compared to the magnetic field created by the magnet. Alternatively, a double magnetic needle, as that used by Faraday for neutralizing the Earth’s magnetic field [16], can be used in place of the common compass. Magnetic field, represented by the induction vector \( \mathbf{B} \), is the responsible for magnetic interaction. Its sources are proven to have vector nature, as their orientation affects the field they produce, according to the orientation and reorientation of the compass under such field. That these sources don’t have a scalar component can be observed by superposing two identical permanent magnets in opposite directions and noticing that their fields vanish [17] by opposition of the vector sources. If there were a scalar component of the field source it would, instead, double its contribution to the field as the magnets were superposed, regardless of the common axis of their directions. Therefore, we must have:

\[
\nabla \cdot \mathbf{B} = 0.
\]

In the third and last part of the thought experiment, a Van de Graaff generator supplies charge to a rough capacitor [18] which is then conveniently discharged by means of a conducting wire. The resulting magnetic field is detected by a compass. Thus we conclude that the motion of electric charges is equivalent to permanent magnets, in the sense that it also gives rise to a magnetic field. Although remarking that not all magnetic fields are originated by moving electric charges, we claim, in view of the referred experimental observation and our definition of magnetic interaction
that, from the mathematical standpoint, magnetic fields can always be seen as being so originated. In other words, it is always possible to find distributions of electric currents leading to a prescribed field, whatever its true sources [19].

Before deriving the last equation on the vector fields, namely, the analogue of Eq. (2.2) for vector B, we shall work out the hydrodynamic picture of electric charge flows. We recall those concepts referring to mass flows, as density, flux, etc., and establish, as usual, the charge density

\[ \rho = Nq \]  

and the current density

\[ \mathbf{J} = Nqv, \]  

for a system of moving charged particles, being N the number of particles per unit volume, q the charge of each particle and v its drift velocity. Conservation of electric charge is then imposed, leading to continuity equation,

\[ \nabla \cdot \mathbf{J} + \left( \frac{\partial \rho}{\partial t} \right) = 0. \]  

An important remark is that the hydrodynamic picture is enough for macroscopically describing the steady flow of charges. Therefore, \( \rho \) and \( \mathbf{J} \), given by Eqs. (3.4) and (3.5), are the only quantities needed for describing this flow and, consequently, all its effects. Whereas \( \rho \) describes the charge configuration, \( \mathbf{J} \) describes its motion; when combined they describe a sort of electrodynamic state of the system, similarly to what occurs in strictly dynamic flows. As long as the vorticity of \( \mathbf{B} \) is considered as deriving from charge flows, it must be a function of \( \mathbf{J} \). Furthermore, for obeying superposition principle it must be a linear function of \( \mathbf{J} \). Thus we have

\[ \nabla \times \mathbf{B} = k_2 \mathbf{J}(\mathbf{r}), \]  

where, as for \( k_1 \), \( k_2 \) will be evaluated in accordance with the chosen system of units. Equation (3.7) completes the aimed set of equations on \( \mathbf{E} \) and \( \mathbf{B} \). Obtaining \( \mathbf{E} \) and \( \mathbf{B} \) is now straightforward. According to Eqs. (2.6), (3.1), (3.2), (3.3) and (3.7) we have:

\[ \mathbf{E}(\mathbf{r}) = \left( \frac{k_1}{4\pi} \right) \int \left[ \frac{\rho(\mathbf{r'}) (\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3} \right] dV' \]  

and

\[ \mathbf{B}(\mathbf{r}) = \left( \frac{k_2}{4\pi} \right) \int \left[ \frac{\mathbf{J}(\mathbf{r'}) \times (\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3} \right] dV'. \]  

Finally we must underline a special feature of the field equations, namely, the asymmetry shown by Eq. (3.3) as compared to Eq. (3.2), and by Eq. (3.1) as compared to Eq. (3.7), which amounts to the magnetic field not having scalar sources, i.e., for it arising from the motion of the electric field sources, the electric charges. It is tempting to admit the converse, i.e., the existence of magnetic poles directly originating a magnetic field as well as - by its motion - an electric field and, consequently, the possibility of reversing and eliminating that asymmetry. In practice, although magnets - our original source of magnetic field - are endowed with north and south “poles”, attempts to divide them into their “polar” parts are unsuccessful: the resulting parts are again magnets, with north and south “poles”. However, since long the existence of magnetic poles has been assumed as theoretically possible [20, 21] and, more recently, it became an imposition to certain theories. Besides, a great number of experimental attempts to detect magnetic poles have been performed, leading in a few cases to positive events [22], being the most outstanding of these the one reported by Cabrera [23]. In view of the only positive experiments being cosmic rays or matter searches, of unpredictable repeatability, as well as in most cases having divergent interpretations, the existence of magnetic poles is yet an open theme in scientific investigation. Due to the scope of this approach we won’t consider it therein.

**IV. SUPERPOSITION AND RECIPROCITY LEADING TO LORENTZ FORCE**

Comparing Eqs. (2.1) and (2.3) with Eqs. (3.1), (3.2), (3.3) and (3.7) and taking into account our discussion about Helmholtz theorem in Section II, we conclude that \( \rho dV \) and \( \mathbf{J}dV \) are elementary sources of \( \mathbf{E} \) and \( \mathbf{B} \), respectively. Nevertheless we must consider the different meanings of the word source. We may call elementary source of a certain field a minute part of the material system which originates such field, or we may so entitle the physical quantity, carried by that material element, which measures its intensity as field source. In other words, by source we mean the
physical medium or else its measurable attribute as physical agent. The latter is the choice when we talk about, say, “a point charge \( q \)” and give its position instead of referring to “a particle bearing an amount \( q \) of charge” located somewhere. It is a shorthand inherited from mathematics, i.e., we use the mathematical concept of field source (see Section II) as a shorthand description of the material source. On the other hand, one may ask how such a particle or an elementary material system would respond to an external field. One shall expect that, as long as it is a source of a given field, it is also sensitive to the presence of an external field of the same kind and, even more, that the physical quantity which measures its intensity as field source will, reciprocally, measure its sensitivity to that external field. Particularly, this implies that a material medium only suffers the action of fields of the kind it originates. The above argumentation may be summarized as the following qualitative reciprocity principle: an elementary material system suffers the action of an applied field in the same measure as it originates a field of that kind. In association with superposition principle it brings immediately on Lorentz force law. Indeed, let us initially calculate the force exerted by an external electric field on the elementary source of another. According to superposition principle stated, e.g., as “summing up causes leads to summing up their effects”, the resulting force has to be linear in both physical quantities which measure its immediate causes: the intensity of the external electric field, given by the field vector \( E_{\text{ext}} \), and the elementary source sensitivity given by \( \rho dV \), according to reciprocity principle. In view of the discussion in Section II about products involving vectors, the only way of combining those two quantities for obtaining a linear result is the multiplication of the vector \( E_{\text{ext}} \) by the scalar \( \rho dV \). Thus we must have, for that force:

\[
dF_e = \alpha (\rho dV) E_{\text{ext}},
\]

being \( \alpha \) another constant referring to the system of units. Analogously, being the vector product the only possibility of linearly combining two vectors (see, again, Section II), the magnetic force resulting from the action of an external magnetic field, \( B_{\text{ext}} \), on an elementary magnetic field source, \( JdV \), must be given by

\[
dF_m = \beta (JdV) \times B_{\text{ext}},
\]

being \( \beta \) the last constant attached to a system of units. Now, in view of Eq. (3.4), Eq. (4.1) takes the form

\[
dF_e = \alpha (dn) qE_{\text{ext}},
\]

being

\[
dn = NdV
\]

the number of particles in volume \( dV \). It is immediately noticed from Eq. (4.3) that the force exerted by an electric field \( E \) on a point charge \( q \) is

\[
F_e = \alpha qE.
\]

Analogously, introducing the expression of \( J \), given by Eq. (3.5), into Eq. (4.2) we get:

\[
dF_m = \beta (dn) qv \times B_{\text{ext}},
\]

being \( dn \) as defined by Eq. (4.4). Then, in the discrete limit, when a charge \( q \) moves with velocity \( v \) under a field \( B \), we have

\[
F_m = \beta qv \times B.
\]

Summing up Eqs. (4.5) and (4.7) we get Lorentz force equation:

\[
F_{em} = q (\alpha E + \beta v \times B).
\]

The above equation has been derived under the assumption that the particle which suffers the force \( F_{em} \) belongs to a steady flow. Nevertheless it must be remarked that such condition doesn’t restrict the type of motion of an individual particle in the flow. Indeed, it can be shown that the particle may have any instantaneous acceleration - not to mention its velocity - even in this case. Therefore, we arrive to the conclusion that Eq. (4.8) is valid whatever the particle motion.

Introducing \( \rho' = q' \delta (r') \) into Eq. (3.8) and applying the result in Eq. (4.3) we obtain Coulomb’s force between charges \( q \) and \( q' \):

\[
F_c = (\alpha k_1/4\pi)(qq'r/r^3),
\]

being

\[
k_1 = \epsilon_0 c^2
\]

the electrostatic constant.

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showing that Coulomb’s law is but a measurable result of the basic conditions satisfied by electrostatic interaction: (i) superposition principle, (ii) reciprocity principle and (iii) scalar nature of the electrostatic field source. Particularly, the conservative nature of electrostatic field is due to the scalar nature of the field source, as can be noted in the derivation of Eq. (3.3). Analogously, for obtaining the laws of magnetism it is only required to add to those principles the assumption that the static magnetic field arises from the steady flow of electric field sources. We may also obtain the derivation of Eq. (3.1). Analogously, for obtaining the laws of magnetism it is only required to add to those principles the conservative nature of electrostatic field is due to the scalar nature of the field source, as can be noted in the superposition principle, (ii) reciprocity principle and (iii) scalar nature of the electrostatic field source. Particularly, showing that Coulomb’s law is but a measurable result of the basic conditions satisfied by electrostatic interaction: (i) superposition principle, (ii) reciprocity principle and (iii) scalar nature of the electrostatic field source. Particularly, the conservative nature of electrostatic field is due to the scalar nature of the field source, as can be noted in the derivation of Eq. (3.3). Analogously, for obtaining the laws of magnetism it is only required to add to those principles the assumption that the static magnetic field arises from the steady flow of electric field sources. We may also obtain the interaction force of a pair of parallel wires carrying currents I and I’ by combining, for instance, Eq. (3.7) in integral form (Ampère’s circuital law) with Eq. (4.2), and using the correspondence between a volume distribution of charges and a current circuit: \( \int \mathbf{J} dV \leftrightarrow \mathbf{I} \oint \mathbf{D} - \) where the asterisk stands for an arbitrary factor (an scalar or a vector) and the multiplication sign, if needed. There results, for the magnetic force per unit length of wire:

\[
\mathbf{F}_{m1} = -\left(\beta k_2 II'/2\pi d\right)t,
\]

where \( t \) represents the unit vector in a direction on the plane containing the wires which is perpendicular to them and points outward.

The set of fundamental equations of static electric and magnetic interactions is now accomplished. The choice of constants \( k_1, k_2, \alpha \) and \( \beta \) in Eqs. (3.2), (3.7) and (4.8) implies defining a system of units. For performing applications and solving problems at this stage of development one may ascribe these constants the values pertaining to the preferred system of units as well as, correctly, the positive sign to all them, and postpone the detailed analysis of different possibilities of signs and values until obtaining Maxwell’s equations and analyzing its implications. Then it will be possible to restrict their signs to be positive, find the only relation among them: \( \alpha k_1/\beta k_2 = c^2 \), and determine the rules governing the systems of units used in electromagnetism, as we do in Sections V and VI.

All laws and useful relations valid for static electric and magnetic interactions follow from the equations formerly derived. Accordingly, Eqs. (3.2) and (3.7) are, respectively, Gauss’ law and Ampère’s circuital law in differential form. As previously pointed out, the Coulombian field and the laws governing magnetic phenomena follow from Eqs. (3.9) and (4.7). The meaning of electric potential and the proof that it obeys Poisson’s or Laplace’s equation, the derivation of relations which establish the electric and magnetic energy and energy density, etc., also follow straightforwardly from the previous relations. In brief, no concept or relation is omitted in our approach, although they may appear in a different order.

V. TIME-DEPENDENT FIELDS

In this Section we extend the analysis of electromagnetic phenomenology to the time-dependent case, starting from electromagnetic induction. It can be rigorously shown that it isn’t an independent phenomenon. On the contrary, the law of electromagnetic induction - Faraday’s law - and, in general, Maxwell’s equations can be derived from the phenomenology of static fields by suitably applying a Lorentz transformation to the corresponding quantities \( \mathbf{A}, \phi \). However, as we already noticed (Section I), such derivation requires the knowledge of concepts and techniques seldom mastered by students at this course level, but we can avoid further postulation by adopting an approach \( \mathbf{A} \) based on a Galileian transformation imposed to the Lorentz force that a purely magnetic field exerts on charges at rest on a circuit attached to a moving frame, as well as on the invariance of physical laws, particularly Lorentz force equation. It is important to note that, although Lorentz transformation would be the only rigorous choice, the Galileian one leads to the same result, namely:

\[
\nabla \times \mathbf{E} = -\left(\beta/\alpha\right)(\partial\mathbf{B}/\partial t),
\]

as the general form of the electromagnetic induction law, Faraday’s law. Constants \( \alpha \) and \( \beta \) arise here in accordance to our general formalism.

Imposition of self-consistency to the set of Eqs. (3.2), (3.3), (3.7), (5.1) and the equation of continuity, Eq. (3.6), leads to displacement current and to substitution of Ampère’s law, Eq. (3.7), by

\[
\nabla \times \mathbf{B} = k_2 \mathbf{J} + \left(k_2/k_1\right)(\partial\mathbf{E}/\partial t).
\]

The unchanged equations plus the latter constitute the set of microscopic Maxwell’s equations.

For completing the picture given by this formalism we will analyze a few outstanding consequences of Maxwell’s equations: the wave equations on electromagnetic potentials and the energy balance described by Poynting’s theorem.

Adopting a procedure analogous to the usual one \( \mathbf{A} \) we derive, from Eqs. (3.3) and (5.1), the electromagnetic vector and scalar potentials, \( \mathbf{A}_{em}(\mathbf{r}) \) and \( \phi_{em}(\mathbf{r}) \), respectively, related to the time-dependent field vectors according to:
\[ \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}_{em} \]  

(5.3)

and

\[ \mathbf{E}(\mathbf{r}) = -\nabla \phi_{em} - (\beta/\alpha)(\partial \mathbf{A}_{em}/\partial t). \]  

(5.4)

Applying to them the remaining Maxwell’s equations, Eqs. (3.2) and (5.2), and imposing to the result the condition:

\[ \nabla \cdot \mathbf{A}_{em} + (k_2/k_1)(\partial \phi_{em}/\partial t) = 0, \]  

(5.5)

to be identified as the Lorenz condition in its general form, we are lead to the wave equations

\[ \nabla^2 \phi_{em} - (\beta k_2/\alpha k_1)(\partial^2 \phi_{em}/\partial t^2) = -k_1 \rho \]  

(5.6)

and

\[ \nabla^2 \mathbf{A}_{em} - (\beta k_2/\alpha k_1)(\partial^2 \mathbf{A}_{em}/\partial t^2) = -k_2 \mathbf{J}. \]  

(5.7)

In the absence of charges and currents the above equations reduce to d’Alembert’s equations in \( \phi_{em} \) and \( \mathbf{A}_{em} \), provided that \( \beta k_2/\alpha k_1 > 0 \). Otherwise those equations lead to decaying or building-up solutions, without physical meaning. Moreover, in the meaningful case \( \alpha k_1/\beta k_2 \) represents the square of the electromagnetic wave velocity, which is experimentally known to equal the velocity of light, \( c \), in view of light waves belonging to the electromagnetic spectrum. Thus we have,

\[ \alpha k_1/\beta k_2 = c^2 > 0. \]  

(5.8)

The physically meaningful solutions of Eqs. (5.6) and (5.7) are the well-known retarded potentials.

On the other hand, applying to Eqs. (5.1) and (5.2) a treatment analogous to the one adopted by Landau \[27\] we are lead to the energy relation:

\[
\partial \left\{ \int_V \left( \frac{1}{2} \left[ (\alpha/k_1)E^2 + (\beta/k_2)B^2 \right] dV + \sum K \right) \right\} / \partial t = - \oint_S \mathbf{S} \cdot n dS, 
\]  

(5.9)

being \( V \) the volume enclosed by surface \( S \). The summation \( \sum K \) represent the total kinetic energy of the enclosed charged particles or, alternatively, their total energy, including their rest masses (these are constants, thus their inclusion don’t alter the final value of the derivative). The vector \( \mathbf{S} \) in the right-hand side is Poynting vector, defined here as:

\[ \mathbf{S} = (\alpha/k_2)(\mathbf{E} \times \mathbf{B}). \]  

(5.10)

If \( V \) is extended to comprise the whole space and the charges and currents form a bounded system, Eq. (5.9) reduces to

\[
\partial \left\{ \int_V \left( \frac{1}{2} \left[ (\alpha/k_1)E^2 + (\beta/k_2)B^2 \right] dV + \sum K \right) \right\} / \partial t = 0. 
\]  

(5.11)

This equation tells us that the quantity into the outer brackets is conserved. Being \( \sum K \) the kinetic (or the total) energy of the particles in the system, we conclude that the quantity into outer brackets represents the mechanical energy of the system of particles, being the integral - now taken over the whole space - the energy stored in the electromagnetic field. Accordingly, the integrand

\[ \omega = (1/2) \left[ (\alpha/k_1)E^2 + (\beta/k_2)B^2 \right] \]  

(5.12)

is associated with the energy density of the field. Thus, Eq. (5.3) represents the energy balance of the system - Poynting’s theorem - being \( \mathbf{S} \) a measure of the energy flow per unit area and per unit time. Yet all this familiar digression has been performed up to the determination of individual signs on the constants of the set \( \{ k_1, k_2, \alpha, \beta \} \), which we are now in a position to accomplish, thus confirming the positive value we ascribed them \( a \ priori \) (see Section IV).

From Eq. (5.8) we conclude that the products \( \alpha k_1 \) and \( \beta k_2 \) have the same sign. Consequently, the same holds for \( \alpha/k_1 \) and \( \beta/k_2 \). Thus the field energy, given by the integral in Eq. (5.11), will have this common sign. In other words, it will be positive or negative according to these products (ratios) having the positive or the negative sign.
However the field energy can’t be negative. In fact, in deriving Eq. (5.11) no relation has been assumed between the charge of particle-sources and their masses or velocities. Therefore, under the hypothesis of negative field energy we could imagine a system of particles with sufficiently great values of charges and sufficiently small values of masses and velocities, so that its total relativistic energy, i.e., its rest mass, given by the quantity into the outer brackets in Eq. (5.11), would be negative, which is obviously forbidden [28]. Thus we must rule out negative signs from the products (ratios) $\alpha k_1 (\alpha/k_1)$ and $\beta k_2 (\beta/k_2)$. It results that the ordered set \{ $k_1$, $k_2$, $\alpha$, $\beta$ \} can only bear signs according to the schemes: \{(1) \{+ + + +\}; (2) \{+ − + −\}; (3) \{− + + −\}, or (4) \{− − − −\}, each sign belonging to the constant which occupies the same site into the brackets. Now we proceed to show that among the four cases, those which differ by an overall sign inversion - case 1 as compared to case 4 and case 2 as compared to case 3 - are equivalent. Indeed, Maxwell’s equations in absence of charges and currents, i.e., for $\rho = 0$ and $\mathbf{J} = 0$, don’t change by an overall inversion of the signs on these constants (cf. Eqs. (3.2), (3.3), (3.4) and (3.7)). Neither is Coulomb’s law, Eq. (1.3), and the force between current circuits (see, e.g., Eq. (4.10)) changed under such an operation. This means that both the field-field relation and the particle-particle interaction remain unchanged under a general inversion of signs on those constants. The only change is noted in the particle-field relation, but it isn’t significant here. It represents an inversion on charge sign: we would name positive what is normally considered negative and vice-versa. In view of this analysis the number of cases are reduced, even more, to a half. Accordingly, we choose $k_1 > 0$ - which amounts to establishing that a positive source gives rise to divergent lines of force and conversely - so eliminating cases 4 and 3 in view of cases 1 and 2, respectively. Now, from Eq. (5.1) we notice that only case 1 obeys Lenz’s law, case 2, for which $\alpha$ and $\beta$ have different signs, doesn’t follow that law and must also be discarded. Thus we conclude that, as long as electromagnetism is described by our postulates all constants of the set \{ $k_1$, $k_2$, $\alpha$, $\beta$ \} are positive [29]. We have thus confirmed the signs we anticipated for those constants in Section IV, as well as the relation given by Eq. (5.8), among them.

As for the results we obtained in the static case, the above results represent the core of electromagnetic theory for time-dependent fields. Indeed, Maxwell’s equations and Lorentz force equation are enough for describing all phenomena related to electromagnetic waves, electrodynamics of moving particles, etc. As a whole, the previous analyses of static and time-dependent cases form an axis from which many specific subjects and problems branch, whose formulation entails a corresponding number of interesting approaches and solutions yet not fitting into the space here available. However, in view of its importance and general applicability, that of systems of units represents a subject worth while analyzing here. Particularly, when suitably employed the results of such analysis unveil the generality of this formalism, under which problems and applications can be worked out regardless of the system of units used in their formulation.

VI. SYSTEMS OF UNITS

Theoretical and historical reasons converged to transform the systems of units in electromagnetism into one of the most entangled theme of the undergraduate physics course. Theoretical freedom to choose, represented by the four constants previously defined - having only one required relation linking them - enhanced by other undetermined constants related to fields in material media, was the background for early definition of practical units and their dictating the establishment and renewal of several systems, leading eventually, in 1960, to the SI (Système International d’Unités) [30]. Although ending in a scientific system this process betrays its origin through the complex interrelations of the resulting units and between them and those of systems created directly for scientific purposes - particularly the gaussian system, the most used among the latter. Determination of Maxwell’s equations governing fields in material media requires an extended discussion about the properties of those materials as well as about coherence between the corresponding electric and magnetic constitutive equations. We shall avoid it here and only discuss the subject of system of units envisaging applications to fields in a vacuum.

Units of electromagnetic quantities are linked to the mechanical ones by Eq. (5.8), or by Eqs. (4.9) and (4.10), or by other equivalent relations. In other words, those units aren’t independent from mechanical units but, instead, shall be defined in connection to them. Thus, systems of electromagnetic units are based on mechanical systems of units, CGS and MKS, and accomplish them. In conclusion, they must obey two requirements:

1. being attached to the CGS or the MKS system of units;

2. the corresponding constants being interrelated according to Eq. (5.8).

As we pointed out, there are two commonly used systems: the SI, linked to the MKS system and having constants given by \{1/$\varepsilon_0$, $\mu_0$, 1, 1\} - in the order previously defined - and the gaussian system, linked to the CGS and having the constants \{4$\pi$, 4$\pi$/c, 1, 1/c\}. Note that both obey Eq. (5.8) [31]. Whenever needed, our equations can be readily
adapted to one system or another by substituting the values of these constants in them. Nevertheless we believe that
this shall be done only in applications, leaving the constants and the whole theory in their general form insofar as
theoretical developments are aimed.

VII. FINAL REMARKS

Starting from the theorem of Helmholtz, which establishes the conditions for uniqueness of vector fields, and
supported by elementary experimental concepts on the field source, a theoretical basis is readily accomplished as
differential equations for the stationary state field vectors, \( \mathbf{E} \) and \( \mathbf{B} \), representing the basic laws regarding electro-
static and magnetostatic fields. Then, Lorentz force arises from the principles of superposition and reciprocity of
electromagnetic interactions through a very simple reasoning, and the law of electromagnetic induction - Faraday’s
law - is also obtained from them by imposing a Galilean transformation of reference frame on forces due to stationary
magnetic fields. As a consequence of self-consistency being imposed to the previous set of equations, Maxwell’s equa-
tions are obtained up to the determination of four constants whose signs are uniquely defined with the help of Lenz’s
law and whose values are linked by a single relation, derived from the experimental fact of light waves belonging to
the electromagnetic spectrum. In short, this approach has the following essential features:

1. The field equations are proposed, from the beginning, in Maxwell’s format, i.e., in terms of divergence and curl
differential operators, being continuously adjusted until transforming into Maxwell’s equations.

2. Beginning with the theorem of Helmholtz, it clarifies some concepts as, for instance, those of field source, both
in its mathematical and physical meaning, and vector magnetic potential.

3. The whole theory follows from trivial principles and from elementary experimental observations.

4. Along its development, electric and magnetic phenomenology are always presented side by side, and their affinity
is clearly shown even for static fields. The presentation thus requires less expenditure of time in class.

5. Uniqueness of Maxwell’s equations and their consequences are unambiguously derived from those primary prin-
ciples.

6. Alternatively, the formalism would apply to other kinds of vector fields obeying different primary principles,
restricting their possible number and properties, according to the signs and values of constants \( \{k_1, k_2, \alpha, \beta\} \).

7. Every system of units fits the formalism, provided that suitable values are assigned to those constants.

In essence, the formalism was based on primary experimental concepts, through which we have shown that an scalar
quantity - the electric charge - gives rise to both the electric and the magnetic fields, being the former a consequence
of its presence and the latter one of its motion. Thus, electromagnetic laws and rules represent the simplest picture
Nature can stand for those vector field. However we must remark that it was the available mathematical apparatus
which provided these conclusions, thus it is an underutilization of technical resources not reaching them in a senior
undergraduate course.

The emphasis on calculation techniques - brought to undergraduate teaching in physics some forty years ago - seems
to have looked at the urgent demand for development of new technologies, but it can do much more. It provides the
means for presenting to students a sequential theory of electromagnetism, with closely intertwined concepts, thus
enlightening its overall picture and overcoming its usual fragmentation. In other words, the final result may be a clear
and unitary course.

ACKNOWLEDGMENTS

I am greatly indebted to Olival Freire Jr. and Saulo Carneiro for their careful reading of a previous version of the
manuscript and their invaluable suggestions, and to Ademir Santana and Marcelo Moret for their encouraging help
on solving specific problems. I also address my thanks to the students whose comments and even doubts were very
worthy in the process of transforming the initial ideas into a defined pedagogical formalism.
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[7] See Ref. 4, pp. 232-243.
[8] D. H. Kobe, “Generalization of Coulomb’s law to Maxwell’s equations using special relativity,” Am. J. Phys. 54 (7), 631-636 (1986).
[9] **Pendulum electroscope**: a device consisting of a tiny ball whose core is made out of a light material as, for example, cork, covered by a thin conducting layer, the whole being suspended from a support by a fine thread of silk or another light insulating fiber. Under influence of the nonuniform electric field of a bounded system of charges, the ball suffers electric induction and moves toward that source. This movement can be reversed by neutralizing the opposite induced charge through contact with the inducing source.
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[13] See Ref. 11, pp. 5 and 6.
[14] See Ref. 10.
[15] Among “vector quantities” we also count the pseudo-vectors.
[16] E. M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1963), 2nd. ed., pp. 226-228.
[17] Unless for a short-range residual field due to quadrupoles and higher-order multipoles, which could at this moment be justified by the non-interpenetration of the magnets.
[18] We mean a kind of capacitor whose functioning can be easily explained and understood - in terms of the concepts previously established here - as for example a Leyden bottle. Thus we avoid introducing a “black-box” into the experiment.
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[28] Although being Galilean in general, our approach can easily bear such a relativistic digression, based on a matter of fundamental scientific knowledge.
[29] On the other hand it can be shown, by interchanging the roles of electric and magnetic fields in the theory and reinterpretting Lenz’s law, that the theory of an electromagnetic field arising analogously from magnetic poles would obey case 2. Thus, the generalized theory comprises both physically meaningful cases!
[30] E. C. Jordan and K. G. Balmain, *Electromagnetic Waves and Radiating Systems*, 2nd ed. (Prentice-Hall, Inc., New Jersey, 1968), pp. 22-23.
[31] We recall that $\varepsilon_0\mu_0 = 1/c^2$.
[32] Its apparent intricacy is due to a sort of logical inversion: the usual choice of following historical chronology when developing a course in electromagnetic theory leads to reversing the roles of suitable premises and suitable consequences. So are, for example, the statements of fields and potentials in the conventional formalism.