AUTOMORPHISMS OF PERFECT POWER SERIES RINGS

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Abstract. Let $R$ be a perfect ring of characteristic $p$. We show that the group of continuous $R$-linear automorphisms of the perfect power series ring over $R$ is generated by the automorphisms of the ordinary power series ring together with Frobenius; this answers a question of Jared Weinstein.

1. Introduction

Let $R$ be a ring. Let $S = R[[t]]$ be the ring of formal power series with coefficients in $R$, equipped with the $t$-adic topology, and let $m$ be the ideal of $S$ consisting of series with zero constant term. For each $y \in m$, the formula

$$
\sum_i x_i t^i \mapsto \sum_i x_i y^i
$$

defines an element $\text{Sub}(y)$ of the algebra $\text{End}_{ct}^{cts}(S)$ of continuous $R$-linear endomorphisms of $S$. The resulting map $\text{Sub} : m \to \text{End}_{ct}^{cts}(S)$ is inverse to the map $\text{End}_{ct}^{cts}(S) \to m$ sending $f : S \to S$ to $f(t)$. It is well-known (and easy to check) that $\text{Sub}(y)$ is invertible if and only if the coefficient of $t$ in $y$ is a unit in $R$, i.e., $y \in R^* t + m^2$; that is, $\text{Sub}$ identifies $R^* t + m^2$ with the group $\text{Aut}_{ct}^{cts}(S)$ of continuous $R$-linear automorphisms of $S$.

From now on, assume that $R$ is of characteristic $p > 0$ and is perfect, i.e., the Frobenius endomorphism $x \mapsto x^p$ on $k$ is bijective. The analogue of the power series construction in the category of perfect rings is the $t$-adic completion of $R[[t^{1/p}, t^{1/p^2}, \ldots]]$, which we call $S'$. The elements of $S'$ may be viewed as formal sums $\sum_{i \in \mathbb{Z}_{[p^{-1}]} \geq 0} x_i t^i$ with $x_i \in R$ whose support (i.e., the set of $i$ for which $x_i \neq 0$) is either finite or an unbounded increasing sequence.

Let $m'$ be the ideal of $S'$ consisting of series with zero constant coefficients. Then the formula (1) again defines a substitution homomorphism $\text{Sub}' : m' \to \text{End}_{ct}^{cts}(S')$ which is inverse to the map $\text{End}_{ct}^{cts}(S') \to m'$ given by evaluation at $t$. In particular, we may construct a commutative diagram

$$
\begin{array}{ccc}
m & \xrightarrow{\text{Sub}} & \text{End}_{ct}^{cts}(S) \\
\downarrow & & \downarrow \\
m' & \xrightarrow{\text{Sub}'} & \text{End}_{ct}^{cts}(S')
\end{array}
$$

in which the left vertical arrow is the obvious inclusion. In particular, we get an injective homomorphism $\text{Aut}_{ct}^{cts}(S) \to \text{Aut}_{ct}^{cts}(S')$ of groups of continuous $R$-linear automorphisms.

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We show that while \( \text{End}_{R}^{\text{cts}}(S') \) is much bigger than \( \text{End}_{R}^{\text{cts}}(S') \), the map of automorphism groups is quite close to being an isomorphism.

**Theorem 1.** The map \( \text{Aut}_{R}^{\text{cts}}(S) \times \mathbb{Z} \to \text{Aut}_{R}^{\text{cts}}(S') \) taking \( n \in \mathbb{Z} \) to the map \( x \mapsto x^{p^{n}} \) is an isomorphism of groups.

This answers a question posed to us by Jared Weinstein, motivated by the following considerations. Let \( S'' \) be the \((t_1, t_2)\)-adic completion of \( R[t_1^{1/p^\infty}, t_2^{1/p^\infty}] \). By a (one-dimensional commutative) perfect formal group law over \( R \), we will mean an element \( f \in S'' \) satisfying the following conditions.

(a) We have \( f(t_2, t_1) = f(t_1, t_2) \).
(b) We have \( f(t_1) \equiv t_1 \, (\text{mod } t_2^{1/p^\infty}) \).
(c) In the \((t_1, t_2, t_3)\)-adic completion of \( R[t_1^{1/p^\infty}, t_2^{1/p^\infty}, t_3^{1/p^\infty}] \), we have \( f(f(t_1, t_2), t_3) = f(t_1, f(t_2, t_3)) \).

For example, any ordinary (one-dimensional commutative) formal group law over \( R \), as an element of \( \mathbb{R}[t_1, t_2] \), is a perfect formal group law.

Recall that for every formal group law over a ring of characteristic \( p \), the formal multiplication by integers interpolates continuously to a formal action of \( \mathbb{Z}_{p} \). The same holds for perfect formal group laws, and Theorem 1 implies that for \( m \in \mathbb{Z}_{p}^\times \), the formal multiplication map, which \emph{a priori} is a perfect power series in one variable, is in fact always an ordinary power series. This suggests a possible affirmative answer to the following question.

**Question 2.** Is every perfect formal group law an ordinary formal group law? That is, is any perfect formal group law contained in \( \mathbb{R}[t_1, t_2] \)?

It may be possible to gain additional insight into Question 2 by classifying continuous \( R \)-linear automorphisms of \( S'' \); however, this approach is complicated by the fact that the map

\[
\text{Aut}_{R}^{\text{cts}}(\mathbb{R}[t_1, t_2]) \times \mathbb{Z}^{2} \to \text{Aut}_{R}^{\text{cts}}(S''),
\]

in which \((n_1, n_2) \in \mathbb{Z}^{2}\) maps to the substitution \( t_1 \mapsto t_1^{p^{n_1}}, t_2^{p^{n_2}} \), is far from being surjective. For example, for any \( f \in S' \), the substitution

\[
t_1 \mapsto t_1, t_2 \mapsto t_2 + f(t_1)
\]

is an automorphism of \( S'' \) with inverse

\[
t_1 \mapsto t_1, t_2 \mapsto t_2 - f(t_1).
\]

2. Proof of Theorem 1

The remainder of this document consists of the proof of Theorem 1. The argument is loosely inspired by an analogous calculation of automorphism groups of certain rings of Hahn-Mal’cev-Neumann generalized power series [1, §3], although the details turn out to be somewhat different.

By evaluating maps at \( t \), we see that the map \( \text{Aut}_{R}^{\text{cts}}(S) \times \mathbb{Z} \to \text{Aut}_{R}^{\text{cts}}(S') \) is injective. To check surjectivity, let \( v_{t} : S' \to \mathbb{Z}[p^{-1}]_{\geq 0} \) denote the \( t \)-adic valuation, and note that

\[
v_{t}(\text{Sub}(y)(z)) = v_{t}(y)v_{t}(z) \quad (y, z \in m').
\]

Consequently, any \( y \in m' \) for which \( \text{Sub}(y) \) is an automorphism must satisfy \( v_{t}(y) \in p\mathbb{Z} \); there is thus no harm in assuming that \( v_{t}(y) = 1 \).
It suffices to derive a contradiction assuming that there exist \( y, z \in m' \setminus m \) such that \( v(y) = v(z) = 1 \) and that \( \text{Sub}(y) \circ \text{Sub}(z) = \text{id}_S \). (Note that this immediately implies that \( \text{Sub}(z) \circ \text{Sub}(y) = \text{id}_S \) because \( \text{Sub}(y) \) is injective whenever \( y \neq 0 \).) Write \( y = \sum_i y_it^i, z = \sum_j z_jt^j \). Let \( v_p \) denote the \( p \)-adic valuation on \( \mathbb{Z}[p^{-1}] \). For \( n \in \mathbb{Z} \), put
\[
\begin{align*}
a_n &= \min \{ i - 1 : v_p(i) \leq -n, y_i \neq 0 \}, \\
b_n &= \min \{ i - 1 : v_p(i) \leq -n, z_i \neq 0 \},
\end{align*}
\]
ine each case interpreting the minimum over an empty set as \(+\infty\). From our hypotheses,
\[
a_n, b_n = 0 \ (n \leq 0); \ 0 < a_1, b_1 < +\infty; \ \lim_{n \to -\infty} a_n, \lim_{n \to -\infty} b_n = +\infty.
\]
Consequently,
\[
c = \min \left\{ \frac{p^n - 1}{p^n - 1}a_n, \frac{p^n - 1}{p^n - 1}b_n : n = 1, 2, \ldots \right\}
\]
eexists, is finite and positive, and is achieved by only finitely many indices. Put
\[
c_n = \frac{p^n - 1}{p^n}c \quad (n \in \mathbb{Z}).
\]
Then
\[
a_l \geq c_l, \quad b_m \geq c_m \quad (l, m \in \mathbb{Z});
\]
there exist maximal indices \( l, m \) for which equalities occur; and these maximal indices are nonnegative and not both zero. Moreover, if \( l > 0 \), then we must have \( v_p(a_l) = -l \), as otherwise we would have the contradiction
\[
c_l = a_l = a_{l+1} \geq \frac{p^{l+1} - 1}{p^{l+1}}c > \frac{p^l - 1}{p^l}c = c_l;
\]
similarly, if \( m > 0 \), then \( v_p(b_m) = -l \). Since we either have \( a_l = c_l \) for some \( l > 0 \) or \( b_m = c_m \) for some \( m > 0 \), we may deduce that for all \( n > 0 \), \( c_n > 0 \) and \( v_p(c_n) = -n \).

Since \( l + m > 0 \), we have \( c_{l+m} > 0 \), so the coefficient of \( t^{l+c_{l+m}} \) in \( t = (\text{Sub}(y) \circ \text{Sub}(z))(t) = \sum_{i \geq 0} y_iz^i \) must be zero. To obtain the desired contradiction, it will thus suffice to verify that the coefficient of \( t^{l+c_{l+m}} \) in \( y_iz^i \) is nonzero for exactly one value of \( i \); we check this by distinguishing options for \( d = -v_p(i) \).

- For \( d \geq l + m \), we have
\[
v_i(y_iz^i) = v_i(y_it^i) \geq 1 + a_d \geq 1 + c_d \geq 1 + c_{l+m}.
\]
For the coefficient of \( t^{l+c_{l+m}} \) in \( y_iz^i \) to be nonzero, this chain of inequalities must become a chain of equalities, yielding
\[
i = 1 + a_d, \quad d \leq l, \quad d = l + m.
\]
Since \( m \geq 0 \), this is only possible if \( d = l, m = 0, i = 1 + a_l \); in this case, the coefficient of \( t^{l+c_{l+m}} \) in \( y_iz^i \) is the nonzero value \( y_1z^1z_1^{1+a_l} \).

- For \( d < l + m \), we have
\[
y_iz^i = y_i(z^{p^{-d}})^i p^d = y_i z^i \left(1 + \sum_{j > 1} (z_j/z_1) p^{p^{-d}(j-1)} \right)^{ip^d} t^i.
\]
By the definition of $b_n$, the sum over $j$ can be rewritten as
\[(\text{element of } (S')^{p-l-m+1}) + (\text{nonzero element of } R) \cdot t^{p-db_{l+m-d}} + (\text{higher order terms}).\]

Since $v_p(ip^d) = 0$ and $t^i \in (S')^{p-l-m+1}$, the binomial expansion yields
\[y_i z^i = (\text{element of } (S')^{p-l-m+1}) + (\text{nonzero element of } R) \cdot t^{i+p-db_{l+m-d}} + (\text{higher order terms}).\]

Since $v_p(c_{l+m}) = -l - m$, the coefficient of $t^{1+c_{l+m}}$ in any element of $(S')^{p-l-m+1}$ is zero. On the other hand, we have
\[i + p^{-d}b_{l+m-d} \geq 1 + a_d + p^{-d}b_{l+m-d} \geq 1 + c_{l+m} = 1 + c_{l+m}.\]

For the coefficient of $t^{1+c_{l+m}}$ in $y_i z^i$ to be nonzero, this chain of inequalities must become a chain of equalities, yielding
\[i = 1 + a_d, \quad d \leq l, \quad l + m - d \leq m.\]

This is only possible if $d = l, \quad i = 1 + a_l, \quad m > 0$; in this case, the coefficient of $t^{1+c_{l+m}}$ in $y_i z^i$ is the nonzero value $y_{1+a_l} z^{1+a_l}$. Since exactly one of the two boundary cases can occur (depending on whether $m = 0$ or $m > 0$), this yields the desired contradiction.

References

[1] K.S. Kedlaya and B. Poonen, Orbits of automorphism groups of fields, *J. Alg.* 293 (2005), 167–184.