The Riemann-Hilbert approach to the generating function of the higher order Airy point processes

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Abstract

We prove a Tracy-Widom type formula for the generating function of occupancy numbers on several disjoint intervals of the higher order Airy point processes. The formula is related to a new vector-valued Painlevé II hierarchy we define, together with its Lax pair.

1 Introduction

Let us consider the higher order Airy functions

\[ \text{Ai}_n(x) := \frac{1}{\pi} \int_0^\infty \cos \left( \frac{y^{2n+1}}{2n+1} + xy \right) dy, \quad x \in \mathbb{R}, \; n \in \mathbb{N}, \quad (1.1) \]

and the associated kernels

\[ K_n(x, y) := \int_0^\infty \text{Ai}_n(x+z)\text{Ai}_n(y+z)dz. \quad (1.2) \]

It is easy to prove, using standard arguments in the theory of point processes (Theorem 3 in [12]) that the kernels \( K_n \), for any \( n \geq 1 \), define a determinantal point process whose correlation functions are given by the standard formula

\[ \rho_{\ell,n}(x_1, \ldots, x_\ell) := \det \left( K_n(x_i, x_j) \right)_{i,j=1}^\ell, \quad \ell \geq 1, \]

see Appendix A in [4]. The importance of these point processes stems from applications to statistical physics and combinatorics. Indeed, they are associated to new universality classes generalizing the KPZ one (case \( n = 1 \)). These universality classes describe both the limiting behavior of the momenta of non–interacting fermions trapped in an anharmonic potential [11] and the one of multicritical random partitions [3, 9, 8].

Let us denote with

\[ \zeta^{(n)}_1 > \zeta^{(n)}_2 > \zeta^{(n)}_3 > \ldots > \zeta^{(n)}_j > \ldots \]

the (random) points in the process, fix a collection \( \{A_j, j = 1, \ldots, k\} \) of intervals of the form

\[ A_j = (x_j, x_{j-1}), \quad \text{with} \; +\infty := x_0 > x_1 > \ldots > x_k > -\infty \]
and some real constants \(\alpha_1, \ldots, \alpha_k\) such that \(\alpha_j \in [0, 1)\) for all \(j\). We will denote

\[
\#_{A_j}^{(n)} := \# \{ \zeta_{t}^{(n)} \in A_j, t \geq 0 \}
\]

the random variable counting the number of points contained in the interval \(A_j\). We are interested in studying the generating function

\[
F_n(\vec{x}, \vec{\alpha}) = F_n(x_1, \ldots, x_k; \alpha_1, \ldots, \alpha_k) := E \left[ \prod_{j=1}^{k} (1 - \alpha_j)^{\#_{A_j}^{(n)}} \right],
\]

whose derivatives give the joint probability law of \(k\) given particles in the process. More precisely, given \(m_1 < \ldots < m_k\),

\[
\mathbb{P} \left( \bigcap_{j=1}^{k} \left( \zeta_{m_j}^{(n)} < x_j \right) \right) = \sum \frac{(-1)^{j_1 + \ldots + j_k}}{j_1! j_2! \ldots j_k!} \partial^{j_1 + \ldots + j_k} F_n(\vec{x}, \vec{\alpha}) \bigg|_{\vec{\alpha} = (1, \ldots, 1)},
\]

where the sum is taken over all indices \(j_1, \ldots, j_k\) satisfying the conditions

\[
j_1 < m_1, \ j_1 + j_2 < m_2, \ldots, \sum_{\ell=1}^{k} j_\ell < m_k.
\]

(see for instance [1]). The main result of this paper is a Tracy–Widom formula for \(F_n(\vec{x}, \vec{\alpha})\), relating the latter to a vector–valued version of the Painlevé II hierarchy we are going to define. Our formula generalizes both the one obtained by Claeys and Doeraene [5] for the case \(n = 1\) and arbitrary \(k \geq 1\), and the one obtained by one of the authors, Claeys and Girotti for arbitrary \(n \geq 1\) and \(k = 1\) [4].

In order to state precisely our result, we need to introduce some (vector–valued) differential polynomials which will be used to define our hierarchy of equations. We will work with the ring

\[
\mathcal{R} := \mathbb{C}[u_1, \ldots, u_k, Du_1, \ldots, Du_k, D^2 u_1, \ldots, D^2 u_k, \ldots]
\]

generated by \(k\) functions \(u_j : \mathbb{R} \ni t \mapsto u_j(t)\), \(j = 1, \ldots, k\) and its derivatives, and denote with \(D^{-1}\) the left–inverse of the derivation, such that \(D^{-1}Dv = v\) for all \(v \in \text{Im}(D)\). Given \(\vec{v}, \vec{w} \in \mathcal{R}^k\), let us define

\[
\langle \vec{v}, \vec{w} \rangle := \vec{v}^\top \vec{w}, \quad \{ \vec{v}, \vec{w} \} := \vec{v} \vec{w}^\top + \vec{w} \vec{v}^\top \in \text{Mat}(k, \mathcal{R}), \quad [\vec{v}, \vec{w}] := \vec{v} \vec{w}^\top - \vec{w} \vec{v}^\top \in \text{Mat}(k, \mathcal{R}).
\]

We will also denote \(\vec{u} := (u_1, \ldots, u_k)^\top \in \mathcal{R}^k\).

**Definition 1.1.** Suppose that \(\vec{v} \in \mathcal{R}^k\) is such that

\[
\langle \vec{u}, \vec{v} \rangle \in \text{D(}\mathcal{R}) \quad \text{and} \quad \{ \vec{u}, \vec{v} \} \in \text{D(}\text{Mat}(k, \mathcal{R})).
\]

We define

\[
\mathcal{L}_+ \vec{v} := i \vec{D} \vec{v} - i \left( D^{-1} \{ \vec{u}, \vec{v} \} \right) \vec{u} - 2i \left( D^{-1} \langle \vec{u}, \vec{v} \rangle \right) \vec{u}.
\]

Analogously, for any \(\vec{v}\) such that \([\vec{u}, \vec{v}] \in \text{D(}\text{Mat}(k, \mathcal{R}))\), we define

\[
\mathcal{L}_- \vec{v} := i \vec{D} \vec{v} + i \left( D^{-1} [\vec{u}, \vec{v}] \right) \vec{u}.
\]
Theorem 1.2. Let $F_n(\vec{x},\vec{\alpha})$ defined as in (1.3) with $\alpha_j \neq \alpha_{j+1}$ for all $j \leq k - 1$, and $\alpha_{k+1} \equiv 0$. Then
\[
F_n(\vec{x},\vec{\alpha}) = \exp \left(-\int_0^\infty t < \vec{u}(t), \vec{u}(t) > \, dt \right),
\]
where $\vec{u}(t) = \vec{u}(n, \vec{x} + t, \vec{\alpha})$ satisfy the following (vector-valued) ordinary differential equation
\[
\left( L^\alpha_n L_n^\alpha \right)^n \vec{u}(t) = -\text{diag}(x_1 + t, \ldots, x_k + t) \vec{u}(t)
\]
and have the following behavior at $+\infty$
\[
\vec{u}(n, \vec{x} + t, \vec{\alpha}) = \left( \sqrt{\alpha_j - \alpha_{j+1}} \text{Ai}_n(t + x_j) (1 + o(1)) \right)_{j=1,\ldots,k}.
\]
Moreover, if $\alpha_{j+1} < \alpha_j$ then $u_j(n, \vec{x} + t, \vec{\alpha})$ is real-valued for real $t$. If $\alpha_{j+1} > \alpha_j$, then $u_j(n, \vec{x} + t, \vec{\alpha})$ is purely imaginary for real $t$.

Remark 1.3. We will call the collection of equations in (1.6) the vector valued PII hierarchy. Their formulation, from an algebraic point of view, is completely analogous to the one we previously introduced, in collaboration with Thomas Bothner [13], for the integro-differential Painlevé II hierarchy, see also [10]. Note, however, that the results contained in this article cannot be deduced from the ones in [13], because of the assumption of smoothness for the weight function $w$, see Section 1.3 in loc.cit.

Remark 1.4. We write down explicitly the first two members of the hierarchy (1.6), using the shorthand notation $\vec{u} = \vec{D}\vec{u}$ to denote the derivative. For $n = 1$, equation (1.6) is
\[
\vec{u} = 2\vec{u}\vec{u}^+ (\vec{x} + t) \vec{u} \quad \text{i. e.}
\begin{align*}
\vec{u}_1 &= 2u_1 \sum_{j=1}^k u_j^2 + (t + x_1)u_1 \\
\vec{u}_2 &= 2u_2 \sum_{j=1}^k u_j^2 + (t + x_2)u_2 \\
&\vdots \\
\vec{u}_k &= 2u_k \sum_{j=1}^k u_j^2 + (t + x_k)u_k
\end{align*}
\]
that coincide indeed with the coupled system of Painlevé II equations introduced in [5]. For $n = 2$, equation (1.6) is
\[
\vec{\vec{u}} = 4\vec{u}\vec{u}^+ \vec{u} + 8\vec{u}\vec{u}^+ \vec{\vec{u}} + 6\vec{u}\vec{\vec{u}}^+ \vec{\vec{u}} + 2\vec{u}\vec{\vec{u}}^+ \vec{\vec{u}} - 6\vec{u}(\vec{\vec{u}}^+ \vec{\vec{u}})^2 - (t + \vec{x})\vec{u}
\]
which is indeed as a vector-valued version of the second member of the Painlevé II hierarchy.

The paper is organised as follows: in Section 2 we prove that the generating function $F_n(\vec{x} + t, \vec{\alpha})$ is equal to the Fredholm determinant of an integrable operator of IIKS type [7]. As a byproduct, we formulate and use the Riemann-Hilbert problem 2.3 to compute the logarithmic derivative of $F_n(\vec{x} + t, \vec{\alpha})$ with respect to $t$, and this concludes Section 2. In Section 3 we associate to the Riemann-Hilbert problem 2.3 a Lax pair for the vector-valued Painlevé II hierarchy (1.6). Section 4 concludes, collecting all the previous results, the proof of Theorem 1.2.

2 $F_n(\vec{x},\vec{\alpha})$ and the associated Riemann-Hilbert problem

It is well known (see for instance [12]) that the generating function $F_n(\vec{x},\vec{\alpha})$ defined in (1.3) can be expressed as a Fredholm determinant. More precisely,
\[
F_n(\vec{x},\vec{\alpha}) = \det \left( 1 - \sum_{j=1}^k \alpha_j \mathcal{K} n|A_j \right),
\]

\[3\]
Figure 1: These are possible choices for the curve $\Gamma_+$ appearing the integral representation of the Airy function $A_{in}$.

where $K_n$ is the integral operator associated to the kernel (1.2) and, for any Borel subset $B \subseteq \mathbb{R}$, $K_n|_B$ indicates the restriction of $K_n$ to $B$. For our purposes, it is convenient to recall a different representation of the kernel $K_n$ as a double contour integral. In what follows, let us denote

$$\psi_n(\lambda; t) := \frac{\lambda^{2n+1}}{2n+1} + \lambda.$$  

(2.2)

It is easy to show (see for instance [13]) that, for any real $t$,

$$A_{in}(t) = \frac{1}{2\pi} \int_{\Gamma_+} \exp(i\psi_n(t; \lambda)) d\lambda = \frac{1}{2\pi} \int_{\Gamma_-} \exp\left(-i\psi_n(t; \lambda)\right) d\lambda,$$  

(2.3)

where $\Gamma_+$ is any smooth contour oriented from $\infty e^{ia}$ to $\infty e^{ib}$ with $a \in \left(\frac{2n\pi}{2n+1}, \pi\right)$ and $b \in (0, \frac{\pi}{2n-1})$ (see Fig. 1), and $\Gamma_-$ its reflection with respect to the real axis. Actually, one can take (2.3) as an alternative definition of $A_{in}$ (or, rather, of its analytical continuation). Then, combining (1.2) with (2.3), one proves [4, 13]

**Lemma 2.1.** The kernel defined in (1.2) admits the double–contour integral representation

$$K_n(x, y) = \frac{i}{(2\pi)^2} \int_{\Gamma_+} d\lambda \int_{\Gamma_-} d\mu \frac{e^{i(\psi_n(\lambda, x) - \psi_n(\mu, y))}}{\lambda - \mu}.$$  

(2.4)

We now define two vector–valued functions$^1$ $f_n, g_n : \Gamma \to \mathbb{R}^{k+1}$, with $\Gamma := \Gamma_+ \cup \Gamma_-:

$$f_n(\lambda) := \frac{1}{2\pi} \left( \begin{array}{c} e^{-\frac{i}{2}\psi_n(\lambda, 0)} \chi_{\Gamma_-}(\lambda) \\ \sqrt{\alpha_1 - \alpha_2 e^{\frac{i}{2}\psi_n(\lambda, 2t_1 + 2x_1)}} \chi_{\Gamma_+}(\lambda) \\ \vdots \\ \sqrt{\alpha_k - \alpha_{k+1} e^{\frac{i}{2}\psi_n(\lambda, 2t_k + 2x_k)}} \chi_{\Gamma_+}(\lambda) \end{array} \right), \quad g_n(\lambda) := \left( \begin{array}{c} e^{\frac{i}{2}\psi_n(\lambda, 0)} \chi_{\Gamma_+}(\lambda) \\ \sqrt{\alpha_1 - \alpha_2 e^{\frac{i}{2}\psi_n(\lambda, 2t_1 + 2x_1)}} \chi_{\Gamma_-}(\lambda) \\ \vdots \\ \sqrt{\alpha_k - \alpha_{k+1} e^{\frac{i}{2}\psi_n(\lambda, 2t_k + 2x_k)}} \chi_{\Gamma_-}(\lambda) \end{array} \right),$$  

(2.5)

and denote with $L_n : L^2(\Gamma) \to L^2(\Gamma)$ the associated integral operator with integrable (in the sense of [7]) kernel defined by the equation

$$(\lambda - \mu)L_n(\lambda, \mu) = f_n^T(\lambda)g_n(\mu).$$  

(2.6)

$^1$The notation we used in (2.5) reflects the fact that it will be convenient, in the sequel, to think about $f_n, g_n$ as functions taking values in $\mathbb{R} \oplus \mathbb{R}^k$
The following proposition is a generalization of Proposition 2.1 in [4] (see also [2] for the case \( n = 1 \) and \( k \) arbitrary).

**Proposition 2.2.** The generating function \( F_n(\vec{x} + t, \vec{\alpha}) \) coincide with the Fredholm determinant of the integrable operator \( L_n \), i.e.

\[
\det(I - L_n) = F_n(\vec{x} + t, \vec{\alpha}).
\]

**Proof.** From the very definition of \( \mathbb{L}_n \), using the natural polarization of \( L^2(\Gamma) \cong L^2(\Gamma_+) \oplus L^2(\Gamma_-) \), we can write in block form

\[
I - \mathbb{L}_n = \begin{pmatrix}
I & -F_n \\
0 & I \\
\end{pmatrix}
\]

where \( F_n : L^2(\Gamma_+) \to L^2(\Gamma_-) \), \( \mathbb{G}_{n;x_j} : L^2(\Gamma_-) \to L^2(\Gamma_+) \), \( j = 1, \ldots, k \) and \( \mathbb{L}_n, \mathbb{G}_{n;x_j} \) have kernels given by

\[
(\mu - \lambda)F_n(\mu, \lambda) = \frac{1}{2\pi} \int_{\Gamma_+} \frac{e^{i(\psi_n(\lambda;0) - \psi_n(\mu;0)) \chi_{\Gamma_+}(\mu) \chi_{\Gamma_-}(\lambda)}}{|\mu - \lambda|^2} d\mu,
\]

\[
(\xi - \mu)\mathbb{G}_{n;x_j}(\xi, \mu) = \frac{1}{2\pi} \int_{\Gamma_+} \frac{e^{i(\psi_n(\xi;2t+2x_j) - \psi_n(\mu;2t+2x_j))} \chi_{\Gamma_+}(\xi) \chi_{\Gamma_-}(\mu)}}{|\xi - \mu|^2} d\mu.
\]

We consider the two corresponding operators \( F_n, \mathbb{G}_{n;x_j} \) extended to the whole space \( L^2(\Gamma) = L^2(\Gamma_+) \oplus L^2(\Gamma_-) \), acting trivially on the respective orthogonal component. We first notice that both the operators \( F_n, \mathbb{G}_{n;x_j} \) are Hilbert-Schmidt on the whole space. Indeed:

\[
\|F_n\|^2 = \frac{1}{(2\pi)^2} \int_{\Gamma_+} |d\lambda| \int_{\Gamma_-} |d\mu| \frac{e^{-3(\psi_n(\lambda;0) - \psi_n(\mu;0))}}{|\mu - \lambda|^2} < +\infty
\]

and

\[
\|\mathbb{G}_{n;x_j}\|^2 = \frac{1}{(2\pi)^2} \int_{\Gamma_+} |d\mu| \int_{\Gamma_-} |d\xi| \frac{e^{-3(\psi_n(\xi;2t+2x_j) - \psi_n(\mu;2t+2x_j))}}{|\xi - \mu|^2} < +\infty.
\]

Moreover, they are both trace-class, since they both can be obtained as composition of Hilbert-Schmidt operators. To see that, we consider a new contour \( \Gamma_+ := \mathbb{R} + \epsilon \), not intersecting either \( \Gamma_+ \) and \( \Gamma_- \). We start by the case of \( \mathbb{G}_{n;x_j} \). We define the following two operators

\[
B_{n;x_j}^{(1)} : L^2(\Gamma_-) \to L^2(\Gamma_0), \quad \text{with kernel} \quad B_{n;x_j}^{(1)}(\xi, \mu) = \frac{e^{i\psi_n(\mu;2x_j+2t)}}{2\pi i(\xi - \mu)}
\]

\[
B_{n;x_j}^{(1)} : L^2(\Gamma_0) \to L^2(\Gamma_+), \quad \text{with kernel} \quad B_{n;x_j}^{(1)}(\lambda, \xi) = \frac{e^{i\psi_n(\lambda;2x_j+2t)}}{2\pi i(\zeta - \xi)}.
\]

Their composition gives \( \mathbb{G}_{n;x_j} = B_{n;x_j}^{(1)} \circ B_{n;x_j}^{(1)} \), and, since they are both Hilbert-Schmidt, \( \mathbb{G}_{n;x_j} \) is trace-class. The case \( F_n \) is treated similarly using the two operators

\[
C_{n}^{(1)} : L^2(\Gamma_+) \to L^2(\Gamma_0), \quad \text{with kernel} \quad C_{n}^{(1)}(\lambda, \mu) = \frac{e^{i\psi_n(\mu;0)}}{2\pi i(\mu - \lambda)}
\]

\[
C_{n}^{(2)} : L^2(\Gamma_0) \to L^2(\Gamma_-), \quad \text{with kernel} \quad C_{n}^{(2)}(\zeta, \lambda) = \frac{e^{i\psi_n(\zeta;0)}}{2\pi i(\lambda - \zeta)}.
\]
This means, in particular, that the Fredholm determinants det(I − L_n) and det(I − G_{n,x_j}) are well defined, and moreover det(I − G_{n,x_j}) = 1. Hence,

$$\det(I - \mathbb{L}_n) = \det(I - G_{n,x_j}) \det(I - \mathbb{L}_n) = \det \left( \begin{pmatrix} 1 & -F_n \\ 0 & -\sum_{j=1}^{k} (\alpha_j - \alpha_{j+1}) G_{n,x_j} \circ F_n \end{pmatrix} \right) = \det \left( I - \sum_{j=1}^{k} (\alpha_j - \alpha_{j+1}) G_{n,x_j} \circ F_n \right),$$

where the equality between the two lines is easily proven using the block representation of $L_n$ and $G_{n,x_j}$ induced by the polarization $L^2(\Gamma) \simeq L^2(\Gamma_+) \oplus L^2(\Gamma_-)$. Notice that each factor $G_{n,x_j} \circ F_n : L^2(\Gamma_+) \rightarrow L^2(\Gamma_+)$ has kernel equal to

$$(G_{n,x_j} \circ F_n)(\xi, \lambda) = \frac{1}{(2\pi)^2} e^{\frac{i}{\sqrt{2\pi}} \int (\psi_n(\xi; 2t + x_j) + \psi_n(\lambda, 0)) \int_{\Gamma_-} e^{-i\nu_n(\mu, x_j, t)} \lambda - \mu} d\mu.$$

We now conjugate $G_{x_j} \circ F$ by the multiplication operator $\mathbb{P}_n$ with kernel $P_n(\lambda, \mu) = e^{-i\nu_n(\lambda; 0)} \delta(\lambda - \mu)$ so to obtain

$$(P_n \circ G_{x_j} \circ F \circ P_n^{-1})(\xi, \lambda) = \frac{1}{(2\pi)^2} e^{i(t + x_j)\xi} \int \frac{e^{i(\psi_n(\lambda, 0) - \psi_n(\mu, x_j) + t)} \lambda - \mu}{(\xi - \mu)(\mu - \lambda)} d\mu.$$

Next we observe that, in the double-contour integral representation (2.4) of $K_n$, one can deform the contour $\Gamma_+$ into $\mathbb{R}$. Using this property and conjugating once more with the standard Fourier transform (i.e. with the integral operator with kernel $F(x, \xi) = \frac{1}{\sqrt{2\pi}} e^{-ix\xi}$) we finally obtain

$$\left( F \circ P_n \circ G_{n,x_j} \circ F_n \circ P_n^{-1} \right)(x, y) = \frac{1}{(2\pi)^2} \sqrt{2\pi} e^{i(t + x_j - x)\xi} \int_{\Gamma_-} d\mu \int_{\mathbb{R}} e^{i(\psi_n(\lambda; 0) - \psi_n(\mu, x_j) + t)} \frac{\lambda - \mu}{(\xi - \mu)(\mu - \lambda)} e^{-i\lambda} = \begin{cases} \frac{i}{(2\pi)^2} \int_{\Gamma_-} d\mu \int_{\mathbb{R}} e^{i(\psi_n(\lambda; 0) - \psi_n(\mu, x_j))} \frac{\lambda - \mu}{\lambda - \mu} & \text{if } x \geq t + x_j \\ 0 & \text{if } x < t + x_j \end{cases}$$

where the latter equality is obtained deforming the outer contour $\mathbb{R}$ toward $+\infty e^{\pm \pi i}$ (depending on the sign of $t + x_j - x$) and taking a residue. Since the conjugations we performed do not change the value of the Fredholm determinant, we find

$$\det(I - \mathbb{L}_n) = \det \left( I - \sum_{j=1}^{k} (\alpha_j - \alpha_{j+1}) K_{n|[x_j + t, \infty)} \right),$$

and this latter is equal to $F_n(\bar{x} + t, \bar{\alpha})$ because of (2.1).

As $\mathbb{L}_n$ is of integrable type, in the sense of [7], it is naturally associated to a Riemann-Hilbert problem we are going to define, and whose jumps are given by the $k + 1$ dimensional matrix $J_Y(\lambda) := 1_{k+1} - 2\pi if_n(\lambda)g_n(\lambda) \delta(\lambda)$ for $\lambda \in \Gamma$.

**Riemann-Hilbert Problem 2.3.** Find a sectionally-analytic function $Y(\bullet; n, \bar{x} + t, \bar{\alpha}) : \mathbb{C}/\Gamma \rightarrow \text{GL}(k + 1, \mathbb{C})$ such that
\( a) \) \( Y \) has continuous boundary values \( Y_\pm \) as \( \lambda \in \Gamma \) is approached from the left \((+)) \) or right \((-)) \) side, and they are related by

\[
Y_+ (\lambda) = Y_- (\lambda) \begin{pmatrix} 1 & -i\Theta_n^\top (-\lambda; \vec{x} + t, \vec{\alpha}) \chi_{\Gamma_-} (\lambda) \\ -i\Theta_n (\lambda; \vec{x} + t, \vec{\alpha}) \chi_{\Gamma_+} (\lambda) & \mathbf{1}_k \end{pmatrix},
\]

where \( \Theta_n (\lambda; \vec{x} + t, \vec{\alpha}) \) is the \((column)\) vector

\[
\Theta_n (\lambda; \vec{x} + t, \vec{\alpha}) := \left( \sqrt{\alpha_j - \alpha_{j+1}} \right)_{j=1}^k.
\]

\( b) \) There exists a matrix \( Y_1 = Y_1 (n, \vec{x} + t, \vec{\alpha}) \), independent of \( \lambda \), such that \( Y \) satisfies

\[
Y (\lambda) = \mathbf{1}_{k+1} + Y_1 \lambda^{-1} + \mathcal{O}(\lambda^{-2}), \quad \lambda \to \infty.
\]

We record in the following remark some symmetries that will be useful in the sequel.

**Remark 2.4.** The jump matrix \( J_Y (\lambda; n, \vec{x} + t, \vec{\alpha}) \equiv J_Y (\lambda) \) of the Riemann-Hilbert problem 2.3 satisfies the following two symmetries:

\[
J_Y^\top (-\lambda) = D_1 J_Y (\lambda) D_1, \quad \overline{J_Y (\lambda)} = D_2 J_Y (-\lambda) D_2,
\]

where \( D_1 := \text{diag}(1, -1, \ldots, -1) \) and \( D_2 := \text{diag}(1, c_1, \ldots, c_k) \) with \( c_j := -\text{sgn}(\alpha_j - \alpha_{j+1}) \). Consequently, the unique solution \( Y \) to the Riemann-Hilbert problem satisfy the symmetry relations

\[
D_1 Y^\top (-\lambda) D_1 = Y^{-1} (\lambda), \quad D_2 Y (\lambda) D_2 = Y (\lambda).
\]

In particular, using the expansion of the function \( Y \) at \( \lambda \to \infty \) together with the symmetries above, we obtain that

\[
Y_1 = \begin{pmatrix} -\delta & \vec{u}^\top \\ \vec{u} & \Delta \end{pmatrix},
\]

where the entry \( u_j, j = 1, \ldots, k \) of \( \vec{u} = \vec{u} (n, \vec{x} + t, \vec{\alpha}) \) is real if \( \alpha_j - \alpha_{j+1} > 0 \) and purely imaginary if \( \alpha_j - \alpha_{j+1} < 0 \). Moreover, since \( \det Y \equiv 1 \), we also have that \( \delta = \text{Tr}(\Delta) \).

**Remark 2.5.** The jump matrix \( J_Y (\lambda; n, \vec{x} + t, \vec{\alpha}) \equiv J_Y (\lambda) \) can be factorized as

\[
J_Y (\lambda) = \exp(M) J_Y (0) \exp(-M)
\]

with \( M \equiv M (\lambda; n, \vec{x} + t, \vec{\alpha}) := \text{diag}(M_0, M_1, \ldots, M_k) \).

\[
M_0 := -\frac{i}{k+1} \sum_{j=1}^k \psi_n (\lambda; t + x_j), \quad M_\ell := M_0 + i\psi_n (\lambda; x_\ell + t), \quad \ell = 1, \ldots, k.
\]

Note that the matrix \( J_Y (0) \) does not depend on \( \vec{x}, t \) or \( n \).
Proposition 2.6. The unique solution $Y$ of the Riemann–Hilbert problem 2.3 is related to the Fredholm determinant $F_n(\vec{x}, \vec{a})$ via the formula

$$\frac{\partial}{\partial t} F_n(\vec{x} + t, \vec{a}) = i(Y_1)_{1,1}. \quad (2.22)$$

Proof. As we proved that $F_n(\vec{x} + t, \vec{a})$ is the Fredholm determinant associated to the integrable kernel $L_n$ defined in (2.6), the following general formula, proven in [2], Theorem 3.2, relates $F_n(\vec{x} + t, \vec{a})$ to the (unique) solution of the Riemann–Hilbert problem 2.3:

$$\frac{\partial}{\partial t} F(\vec{x} + t, \vec{a}) = \int_{\Gamma} \text{Tr} \left( Y^{-1}_-(\lambda) Y'_-(\lambda) \frac{\partial}{\partial \lambda} J_Y(\lambda) J^{-1}_Y(\lambda) \right) \frac{d\lambda}{2\pi i}, \quad (2.23)$$

where the symbol $'$ denotes the derivative w.r.t. the complex parameter $\lambda$. Thanks to the factorization of the jump matrix $J_Y(\lambda)$ given in (2.20) and the nature of the contour $\Gamma = \Gamma_+ \cup \Gamma_-$, the integral in the right hand side is computed as a formal residue at infinity. Indeed, we start noticing that

$$\int_{\Gamma_+ \cup \Gamma_-} \text{Tr} \left( Y^{-1}_-(\lambda) Y'_-(\lambda) \frac{\partial}{\partial \lambda} J_Y(\lambda) J^{-1}_Y(\lambda) \right) \frac{d\lambda}{2\pi i} =$$

$$\int_{\Gamma_+ \cup \Gamma_-} \text{Tr} \left( Y^{-1}_-(\lambda) Y'_-(\lambda) \left( \frac{\partial}{\partial \lambda} M(\lambda) - J_Y(\lambda) \frac{\partial}{\partial \lambda} M(\lambda) J^{-1}_Y(\lambda) \right) \right) \frac{d\lambda}{2\pi i} =$$

$$\int_{\Gamma_+ \cup \Gamma_-} \text{Tr} \left( Y^{-1}_-(\lambda) Y'_-(\lambda) \left( \frac{\partial}{\partial \lambda} M(\lambda) \right) \right) \frac{d\lambda}{2\pi i} - \int_{\Gamma_+ \cup \Gamma_-} \text{Tr} \left( Y^{-1}_-(\lambda) Y'_-(\lambda) \left( \frac{\partial}{\partial \lambda} M(\lambda) \right) \right) \frac{d\lambda}{2\pi i}$$

$$+ \int_{\Gamma_+ \cup \Gamma_-} \text{Tr} \left( J_Y(\lambda) J'_Y(\lambda) \left( \frac{\partial}{\partial \lambda} M(\lambda) \right) \right) \frac{d\lambda}{2\pi i} =$$

$$- \lim_{R \to +\infty} \int_{C_R} \text{Tr} \left( Y^{-1}_-(\lambda) Y'_-(\lambda) \left( \frac{\partial}{\partial \lambda} M(\lambda) \right) \right) \frac{d\lambda}{2\pi i}.$$

In the last passage we used that the terms denoted with (”) and (”) are zero (as we will see in a moment) and we rewrote the remaining term by deforming the contour $\Gamma_+ \cup \Gamma_-$ into a circle $C_R$ centered in zero and of increasing radius $R$. Indeed, (”) is zero because of the form of the jump matrix $J_Y$ (we are computing the trace of a strictly lower/upper triangular matrix), while (”) is zero because the integrations along $\Gamma_+$ and $\Gamma_-$ cancel out. Finally, using the asymptotic expansion of $Y$ at infinity combined with

$$\frac{\partial M}{\partial t} = i\frac{\lambda}{k+1} \text{diag}(-k, 1, \ldots, 1),$$

we explicit compute

$$- \lim_{R \to +\infty} \int_{C_R} \text{Tr} \left( Y^{-1}_-(\lambda) Y'_-(\lambda) \left( \frac{\partial}{\partial \lambda} M(\lambda) \right) \right) \frac{d\lambda}{2\pi i} = i(Y_1)_{1,1}.$$

This result will be used in the last section to relate $F_n(\vec{x} + t, \vec{a})$ to a distinguished solution of the vector–valued PII hierarchy, whose Lax pair is given in the following section.
3 A Lax pair associated to a vector valued PII hierarchy

We now introduce a new matrix $\Psi(\lambda) = \Psi(\lambda; n, \vec{x} + t, \vec{\alpha})$ solving a Riemann-Hilbert problem with constant jumps. More specifically, let

$$T^{(1)}(\lambda; n, \vec{x} + t) := \text{diag}\left(1, e^{i\psi_n(\lambda; x_1 + t)}, \ldots, e^{i\psi_n(\lambda; x_k + t)}\right),$$

and $T(\lambda) \equiv T(\lambda; n, \vec{x} + t) := T^{(1)}(\lambda; n, \vec{x} + t)T^{(2)}(\lambda; n, \vec{x} + t)$. We define

$$\Psi(\lambda; n, \vec{x} + t, \vec{\alpha}) := Y(\lambda; n, \vec{x} + t, \vec{\alpha})T(\lambda; n, \vec{x} + t).$$

It is easy to prove that this latter satisfies the following Riemann–Hilbert problem.

**Riemann-Hilbert Problem 3.1.** Find a sectionally-analytic function $\Psi: \mathbb{C}/\Gamma \rightarrow GL(k + 1, \mathbb{C})$ such that

a) $\Psi$ has continuous boundary values $\Psi_{\pm}$ as $\lambda \in \Gamma$ is approached from the left (+) or right (−) side, and they are related by

$$\Psi_{+}(\lambda) = \begin{pmatrix} 1 & -i\hat{\Theta}^\top \chi_{\Gamma_-}(\lambda) \\ -i\hat{\Theta} \chi_{\Gamma_+}(\lambda) & 1_k \end{pmatrix},$$

where $\hat{\Theta} := \Theta(\lambda; n, \vec{x} + t, \vec{\alpha})|_{\lambda=0}$ is the (column) vector

$$\hat{\Theta} := \left(\sqrt{\alpha_j - \alpha_{j+1}}\right)_{j=1}^k.$$

b) $\Psi$ has the following asymptotic behavior

$$\Psi(\lambda) = \left(1 + Y_1 \lambda^{-1} + \mathcal{O}(\lambda^{-2})\right)T(\lambda), \quad \text{as } \lambda \rightarrow \infty.$$

The properties of the function $\Psi(\lambda)$ listed above are then used to prove the following proposition.

**Proposition 3.2.** There exists two matrices $A(\lambda) \equiv A(\lambda; n, \vec{x} + t, \vec{\alpha})$ and $B(\lambda) \equiv B(\lambda; n, \vec{x} + t, \vec{\alpha})$, polynomial in $\lambda$, such that

$$\frac{\partial}{\partial \lambda} \Psi(\lambda) = A(\lambda)\Psi(\lambda),$$

$$\frac{\partial}{\partial t} \Psi(\lambda) = B(\lambda)\Psi(\lambda).$$

Moreover,

$$B(\lambda) = \begin{pmatrix} \frac{-i\lambda}{k + 1} & -i\vec{u}^\top \\
\frac{i\vec{u}}{k + 1} & 1_k \end{pmatrix}$$
and

\[ A(\lambda) = \sum_{j=0}^{2n} A_j \lambda^{2n-j} + \hat{A}_{2n}, \]  

(3.9)

with

\[ A_0 = \frac{i}{k + 1} \text{diag} (-k, 1, \ldots, 1), \]  

(3.10)

\[ \hat{A}_{2n} = \frac{i}{k + 1} \text{diag} \left( -kt - \sum_{j=1}^{k} x_j, t + kx_1 - \sum_{j \neq 1} x_j, \ldots, t + kx_k - \sum_{j \neq k} x_j \right). \]  

(3.11)

and \( \bar{u} \) as in (2.19).

**Proof.** For both equations the proof follows applying Liouville theorem and using straightforward computations. For the first equation, we define

\[ A(\lambda) := \frac{\partial}{\partial \lambda} \Psi(\lambda) \left( \Psi(\lambda) \right)^{-1}. \]

The matrix-valued function \( A(\lambda) \) is analytic for every \( \lambda \in \mathbb{C} \setminus \Gamma \). Moreover, for \( \lambda \in \Gamma \) we have that \( A_\lambda(\lambda) = A_\lambda(\lambda) \), thanks to the fact that \( \Psi(\lambda) \) has constant jump condition along \( \Gamma \). Thus \( A(\lambda) \) is entire and behaves like a polynomial in \( \lambda \) of degree \( 2n \) at \( \infty \). By Liouville theorem, we conclude that \( A(\lambda) \) is a polynomial of degree \( 2n \) in \( \lambda \). Using the asymptotic condition written in equation (2.17), we can then compute explicitly the leading coefficient and the constant coefficient of \( A(\lambda) \) as in (3.10), (3.11). For the second equation, in an analogue way we define

\[ B(\lambda) := \frac{\partial}{\partial t} \Psi(\lambda) \left( \Psi(\lambda) \right)^{-1}. \]

Using the same reasoning, we conclude that \( B(\lambda) \) is a polynomial in \( \lambda \) of degree 1 and we compute its coefficients as in (3.8).

Now, we show that the system (3.7) is the Lax pair for the vector-valued Painlevé II hierarchy (1.6). For any \( j = 0, \ldots, 2n \), we will denote with \( a_{j1}^{11}, a_{j1}^{12}, a_{j1}^{21}, a_{j1}^{22} \) the block-entries of \( A_j \) (and the same for \( \hat{A}_{2n} \)). Hence, \( a_{j1}^{11} \) will be a scalar, \( a_{j1}^{12}, a_{j1}^{22} \) will be, respectively, a row and a column vector, and \( a_{j1}^{22} \) a square matrix of size \( k \). We will now study the compatibility condition

\[ A(\lambda)B(\lambda) - B(\lambda)A(\lambda) = \frac{\partial B}{\partial \lambda}(\lambda) - \frac{\partial A}{\partial t}(\lambda). \]  

(3.12)

The following two Lemmas are the analogue, for the vector-valued case, of Lemma 5.4 and 5.5 in [13] and the technique used in their proofs is inspired by the one used in [14]. The dependence of the variable \( \bar{u} \) and \( \{a_k^{(j)}\} \) on \( n, t, \bar{x}, \bar{a} \) will not be made explicit in the following formulas. We will denote with a dot the derivative with respect to \( t \), as this will be the only dynamical variable (the other variables will be considered as parameters).

**Lemma 3.3.** The compatibility condition of the Lax pair (3.7) is equivalent to the system of equations

\[ a_1^{12} = -i\bar{u}^T, \quad a_1^{11} = i\bar{u}, \]  

(3.13)

\[ \begin{cases} \dot{a}_{j1}^{11} = -i(\bar{u}^T a_{j1}^{21} + a_{j1}^{12} \bar{u}), & \dot{a}_{j1}^{12} = -i(a_{j1}^{12} + \bar{u}^T a_{j1}^{22} - a_{j1}^{11} \bar{u}^T), \\ \dot{a}_j^{22} = i(a_{j1}^{21} + a_{j1}^{11} \bar{u} - a_{j1}^{22} \bar{u}), & \dot{a}_j^{21} = i(a_{j1}^{21} + a_{j1}^{11} \bar{u} - a_{j1}^{22} \bar{u}) \end{cases}, \quad j = 1, \ldots, 2n - 1 \]  

(3.14)
\[
\begin{cases}
\hat{a}_{2n}^{11} = -i(\bar{u}^T a_{2n}^{11} + a_{2n}^{21} \bar{u}), \\
\hat{a}_{2n}^{12} = -i(\bar{u}^T a_{2n}^{12} - a_{2n}^{11} \bar{u} + i\bar{u}^T m_{t,\bar{x}}) \\
\hat{a}_{2n}^{22} = i(\bar{u}a_{2n}^{12} + a_{2n}^{21} \bar{u}^T), \\
\hat{a}_{2n}^{21} = i(a_{2n}^{11} \bar{u} - a_{2n}^{22} \bar{u} - im_{t,\bar{x}} \bar{u}),
\end{cases}
\] (3.15)

where \( m_{t,\bar{x}} := \text{diag}(x + t_1, \ldots, x + t_k) \).

**Proof.** The proof follows by direct computation, exploiting the polynomiality of the matrices \( A(\lambda), B(\lambda) \) in \( \lambda \). In particular, equation (3.13) corresponds to the term associated to the monomial \( \lambda^{2n} \) in the compatibility condition (3.12). Then system (3.14) corresponds, for every \( j = 1, \ldots, 2n - 1 \), to the monomial \( \lambda^{2n-j} \) and, finally, (3.15) corresponds to \( \lambda^0 \).

From equations (3.13), (3.14), and the first two equations of (3.15) we can derive some symmetries for the entries of \( A_j \), for \( j = 1, \ldots, 2n \). In particular,

\[
\frac{\partial}{\partial t} \text{Tr}(a_j^{22}) = \text{Tr}(\dot{a}_j^{22}) = i \text{Tr}(\bar{u}a_j^{12} + a_j^{21} \bar{u}^T) = i \text{Tr}(\bar{u}a_j^{12}) + i \text{Tr}(a_j^{21} \bar{u}^T) = ia_j^{12} \bar{u} + i\bar{u}^T a_j^{21} = -\dot{a}_j^{11}.
\] (3.16)

Thus we conclude that

\[
- \text{Tr}(a_j^{22}) = a_j^{11}, \quad j = 1, \ldots, 2n,
\] (3.16)

up to constant of integration. This constant is actually zero. This can be proven observing that \( \lim_{t \to \infty} a_j^{22} = \lim_{t \to \infty} a_j^{11} = 0 \). Indeed, both of them are polynomials in the entries of the matrix elements of \( Y_i, i \geq 1 \), and \( Y_i \to 0 \) for \( t \to +\infty \), which is easily proven using the small norm theorem (see Section 4 below and, in particular, the proof of Proposition 4.1). Moreover, we can also deduce the following symmetries:

\[
a_j^{21} = (-1)^j(a_j^{12})^T, \quad a_j^{22} = (-1)^j(a_j^{21})^T \quad \forall j = 1, \ldots, 2n.
\] (3.17)

This is proved by induction over \( j \) using the equations appearing in the Lemma 3.3.

**Lemma 3.4.** For every \( \ell = 1, 2, \ldots, 2n \)

\[
a^{11}_\ell = -\frac{1}{\ell} \sum_{j=1}^{\ell-1} (a_j^{11} a_{\ell-j}^{11} + a_j^{12} a_{\ell-j}^{21}) \quad \text{and} \quad a^{22}_\ell = \frac{1}{\ell} \sum_{j=1}^{\ell-1} (a_j^{22} a_{\ell-j}^{22} + a_j^{21} a_{\ell-j}^{12}).
\] (3.18)

**Proof.** Thanks to equation (3.16), we only need to prove the formula for \( a^{22}_\ell \). Consider the matrix

\[
C := A^2 = 
\begin{pmatrix}
-\frac{\lambda^2}{(k+1)^2} & 0 \\
0^T & -\frac{1}{(k+1)^2}
\end{pmatrix} \lambda^{4n} + \sum_{\ell=1}^{4n} \lambda^{4n-\ell} C_\ell,
\]

where \( C_\ell = \sum_{j=0}^{\ell} A_j A_{\ell-j} \) for \( \ell = 1, \ldots, 2n - 1 \) and \( C_{2n} = \sum_{j=0}^{2n} A_j A_{2n-j} + A_0 \hat{A}_{2n} + \hat{A}_{2n} A_0 \) (these are the only coefficients we are going to use in the proof). We can write \( C_\ell \), for every value of \( \ell \), in the usual block-form

\[
C_\ell := 
\begin{pmatrix}
\begin{pmatrix}
\ell^1 & \ell^2 \\
\ell^2 & \ell^2
\end{pmatrix} & \\
\begin{pmatrix}
\ell^1 & \ell^2 \\
\ell^2 & \ell^2
\end{pmatrix}
\end{pmatrix},
\]

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and, in particular, each block-entry can be written in terms of the block entries of the matrix $A$

$$
\begin{align*}
\ell_1^{11} &= \sum_{j=0}^\ell (a_{j,1}^{11}a_{1-j}^{11} + a_{j,2}^{12}a_{2-j}^{21}) + 2a_{1}^{1}a_{2n}^{11}\delta_{\ell,2n} \\
\ell_1^{22} &= \sum_{j=0}^\ell (a_{j}^{11}a_{1-j}^{12} + a_{j}^{22}a_{2-j}^{22}) + (a_{0}^{22}a_{2n}^{22} + a_{2n}^{22}a_{0}^{22})\delta_{\ell,2n} \\
\ell_1^{22} &= \sum_{j=0}^\ell (a_{j}^{11}a_{1-j}^{12} + a_{j}^{22}a_{2-j}^{22}) \\
\ell_1^{11} &= \sum_{j=0}^\ell (a_{j}^{21}a_{1-j}^{11} + a_{j}^{21}a_{2-j}^{21})
\end{align*}
$$

(3.19)

From the compatibility condition (3.12), we deduce the following equation for the matrix $C$

$$
\frac{\partial C}{\partial \lambda}(\lambda) = B(\lambda)C(\lambda) - C(\lambda)B(\lambda) + \frac{\partial B}{\partial \lambda}(\lambda)A(\lambda) + A(\lambda)\frac{\partial B}{\partial \lambda}(\lambda).
$$

(3.20)

In particular, by looking at the coefficients of the powers $\lambda^m$ with $m = 4n, \ldots, 2n$ in equation (3.20), we obtain the following system of difference and differential equations for the block entries of each coefficient $C_\ell$

$$
\begin{align*}
\ell_1^{11} &= -i(\vec{u}^T\mu_1^2 + c_1^{12}\vec{u}) + 2(a_0^{11})^2\delta_{\ell,2n}, \\
\ell_1^{12} &= -i(c_1^{12} + \vec{u}^T\mu_2^2 - c_1^{11}\vec{u}^T) \\
\ell_2^{11} &= i(\vec{u}_2^{11} + c_1^{21}\vec{u}^T) + 2(a_{0}^{22})^2\delta_{\ell,2n}, \\
\ell_2^{21} &= i(c_1^{21} + \vec{u}_1^{11} - c_1^{22}\vec{u})
\end{align*}
$$

(3.21)

Note that this is almost the same system satisfied by the matrix elements of $A$, see (3.14). In particular, the equations giving the entries (1, 2), (2, 1) in (3.14) and (3.21) are exactly the same, while for the entries (2, 2), (1, 1) the two sets of equations differ just for $\ell = 2n$.

Now, by using the equations above, we first prove that the entries of $C_\ell$, for $\ell = 1, \ldots, 2n$ are multiple of the ones of $A_\ell$, by induction over $\ell$. Keeping in mind that the coefficient $C_0$ is explicitly written in the definition of the matrix $C$, we start by computing $C_1$:

$$
\begin{align*}
c_1^{12} &= \frac{1-k}{k+1}a_1^{22}, \\
c_1^{21} &= \frac{1-k}{k+1}a_1^{21}, \\
c_1^{11} &= 0 = \frac{1-k}{k+1}a_1^{21}, \\
c_1^{22} &= 0 = \frac{1-k}{k+1}a_1^{22}.
\end{align*}
$$

(3.22)

We suppose now that the equation above holds for $\ell$ and we prove that it holds for $\ell + 1$ too, by using the equations (3.21) together with equations (3.13), (3.14). In particular, from the third equation in the system (3.21) we recover $c_{\ell+1}^{22}$ as

$$
c_{\ell+1}^{22} = i(\vec{u}_2^{12} - \vec{u}_1^{22} + c_1^{21}\vec{u}^T) = i(\frac{1-k}{k+1}(\vec{u}_2^{12} - \vec{u}_1^{22} + a_{0}^{22}\vec{u}^T),
$$

(3.23)

using the induction hypothesis and the equation for $a_{0}^{22}$ in (3.14). The same procedure can be applied for the block entry $c_{\ell+1}^{21}$, to get the analogue result. Now, for the diagonal entries, we use instead the first two equations in the system (3.21) and obtain

$$
c_{\ell+1}^{22} = i(\vec{u}_2^{12} + c_1^{21}\vec{u}^T) = i(\frac{1-k}{k+1}(\vec{u}_2^{12} + a_{0}^{22}\vec{u}^T).
$$

(3.24)

Thus we conclude that $c_{\ell+1}^{22} = i(\frac{1-k}{k+1}a_{0}^{22})$ (from the equation above, this is true up to a constant of integration, that can be fixed to zero thanks to equations (3.19)). The same can be done for $c_{\ell+1}^{11}$. Thus the proportionality relation between the block entries of type (1, 2) or (2, 1) of $A_\ell$ and of $C_\ell$ is
proved for \( \ell = 1 \ldots 2n - 1 \), while for the diagonal block entries it holds for \( \ell = 1 \ldots 2n - 2 \). For \( \ell = 2n - 1 \) we have for the diagonal block entries

\[
c_{2n} = \frac{1 - k}{k + 1} a_{2n}^{22} + 2(a_{0}^{22})^2,
\]

and an analogue relation for \( a_{1n}^{12} \). Now, by using the first two equations in the system (3.19) for \( c_{\ell}^{22} \), we deduce the following chain

\[
\frac{1 - k}{k + 1} a_{\ell}^{22} = c_{\ell}^{22} = \sum_{j=1}^{\ell-1} \left( a_{j}^{21} a_{\ell-j}^{12} + a_{j}^{22} a_{\ell-j}^{22} + a_{0}^{21} a_{\ell-j}^{12} + a_{0}^{22} a_{\ell-j}^{22} \right) + 2(a_{0}^{22})^2
\]

for \( \ell = 1 \ldots 2n - 1 \), from which the statement for \( a_{\ell}^{22} \) is directly obtained. For the case \( \ell = 2n \) the chain obtained by replacing the equation (3.19) for \( c_{2n}^{22} \) is the following one

\[
\frac{1 - k}{k + 1} a_{2n}^{22} + 2(a_{0}^{22})^2 = c_{2n}^{22} = \frac{d}{d\ell} \left( \sum_{j=0}^{\ell} \left( a_{j}^{21} a_{\ell-j}^{12} + a_{j}^{22} a_{\ell-j}^{22} \right) \right) + 2(a_{0}^{22})^2.
\]

Thus, by simplifying both sides the term \( 2(a_{0}^{22})^2 \) and then integrating, the formula for \( a_{2n}^{22} \) is obtained as for the previous values of \( \ell \).

Combining the two lemmas 3.3 and 3.4, we are able to express all the coefficients of the Lax matrix \( A(\lambda) \) in function of the vector \( u \) and its derivatives.

**Proposition 3.5.** The entries of the matrix \( A(\lambda) \) are differential polynomials on \( u \), given recursively by the formulas

\[
a_{1}^{21} = iu, \quad a_{1}^{22} = 0,
\]

\[
a_{j+1}^{21} = -i a_{j}^{21} - a_{j}^{11} u + a_{j}^{22} \ddot{u}, \quad a_{j+1}^{22} = i \sum_{\ell=1}^{j} \left( a_{\ell}^{21} a_{j+1-\ell}^{12} + a_{\ell}^{22} a_{j+1-\ell}^{22} \right), \quad j = 1, \ldots, 2n
\]

together with (3.16) and (3.17).

Moreover, using (3.29) together with (3.15) and the definition of the operators \( L_{\pm}^{\ddot{u}} \), we prove that the vectors \( \dot{a}_{j}^{21}, \quad j = 1, \ldots, 2n \) satisfy the recursion

\[
a_{2j+1}^{21} = -L_{+}^{\ddot{u}} a_{2j}^{21}, \quad a_{2j}^{21} = -L_{-}^{\ddot{u}} a_{2j-1}^{21}, \quad j = 1, \ldots, n - 1,
\]

while \( a_{2n}^{21} \) satisfies the differential equation

\[
-L_{+}^{\ddot{u}} a_{2n}^{21} = -im_{t, \tilde{x}} \ddot{u}.
\]

Hence, we proved recursively that \( \ddot{u} \), as defined in (2.19), satisfies the equation (1.6). Moreover, the entries of \( \ddot{u} \) are real/purely imaginary depending on the sign of \( (\alpha_{j+1} - \alpha_{j}) \), as already observed in Remark 2.4. We are now left with the proofs of the equations (1.5) and (1.7).

### 4 The logarithmic derivative of \( F_{n}(\tilde{x}, \tilde{\alpha}) \)

In this last section we finish the proof of Theorem 1.2 giving the relation between \( F_{n}(\tilde{x}, \tilde{\alpha}) \) and a particular solution \( \ddot{u} \) of the vector-valued Painlevé II hierarchy. The main ingredient is the first logarithmic derivative of the Fredholm determinant \( F_{n}(\tilde{x} + t, \tilde{\alpha}) \), computed at the end of Section 2.
Proposition 4.1. The distribution $F_n$ defined in (1.3) satisfies the equation

$$
\frac{\partial^2}{\partial t^2} F_n(x + t, \alpha) = - < \bar{u}(t), \bar{u}(t) >
$$

(4.1)

and its integrated version

$$
F_n(x, \alpha) = \exp \left( - \int_0^\infty t < \bar{u}(t), \bar{u}(t) > \, dt \right),
$$

(4.2)

where $\bar{u}(t) = \bar{u}(n, x + t, \alpha)$ satisfy the following (vector-valued) ordinary differential equation

$$(L_+^n L_-^n) \bar{u}(t) = - \text{diag}(x_1 + t, \ldots, x_k + t) \bar{u}(t)
$$

(4.3)

and have the following behavior at $+\infty$

$$
\bar{u}(n, x + t, \alpha) = \begin{pmatrix}
\sqrt{\alpha_j - \alpha_{j+1}} A_{in}(t + x_j)(1 + o(1))
\end{pmatrix}_{j=1, \ldots, k}.
$$

(4.4)

Proof. We start computing the $\lambda^{-1}$-term in the asymptotic expansion of $(\partial_t \Psi) \Psi^{-1}$ for $\lambda \to \infty$, where $\Psi$ is the solution of the Riemann-Hilbert problem 3.1 with constant jump condition. The $(1, 1)$-entry of this term, which is equal to zero because $B(\lambda)$ is polynomial in $\lambda$, leads to

$$
\frac{\partial}{\partial t} (Y_1)_{11} - i \bar{u}^\top \bar{u} = 0
$$

and this equation, together with (2.22), gives (4.1).

We describe now the boundary behavior of $\bar{u}$ for $t \to +\infty$. We start by proving that the jump matrix $J_Y(\lambda)$ of the Riemann–Hilbert problem 2.3, for $t \to +\infty$, behaves like the identity matrix, so that the small norm theorem can be applied. We consider a rescaled complex variable $w$ defined through the equation $\lambda = wt^2$, so that the entries $(1, j+1)$ and $(j+1, 1)$ of $J_Y(\lambda)$ for $j = 1, \ldots, k$ are rewritten respectively as

$$
- i (\Theta(\bar{x} + t, \bar{\alpha}, -\lambda))_j \chi_\cdot(\lambda) = - \sqrt{\alpha_j - \alpha_{j+1}} w^{2n+1} \left( \frac{1}{\alpha_j - \alpha_{j+1}} - 1 \right) \chi_{\Gamma_-}(wt^{1/2}),
$$

(4.5)

We describe now the boundary behavior of $\bar{u}$ for $t \to +\infty$. We start by proving that the jump matrix $J_Y(\lambda)$ of the Riemann–Hilbert problem 2.3, for $t \to +\infty$, behaves like the identity matrix, so that the small norm theorem can be applied. We consider a rescaled complex variable $w$ defined through the equation $\lambda = wt^2$, so that the entries $(1, j+1)$ and $(j+1, 1)$ of $J_Y(\lambda)$ for $j = 1, \ldots, k$ are rewritten respectively as

$$
- i (\Theta(\bar{x} + t, \bar{\alpha}, \lambda))_j \chi_\cdot(\lambda) = - \sqrt{\alpha_j - \alpha_{j+1}} w^{2n+1} \left( \frac{1}{\alpha_j - \alpha_{j+1}} - 1 \right) \chi_{\Gamma_-}(wt^{1/2}),
$$

(4.5)

Note that the quantity $d_j := 1 + \frac{\bar{w}}{\alpha_{j+1}}$ is bounded in the regime $t \to \infty$ for any fixed $x_j$, as it converges to 1. Thus we can modify the curves $\Gamma_-$ and $\Gamma_+$ into $\tilde{\Gamma}_-$ and $\tilde{\Gamma}_+$ (as done in [4], Section 3) so that

$$
\mathcal{J} \left( \frac{w^{2n+1}}{2n + 1} + d_j w \right) < 0, \quad \text{for} \quad w \in \tilde{\Gamma}_- \quad \text{and} \quad \mathcal{J} \left( \frac{w^{2n+1}}{2n + 1} + d_j w \right) > 0, \quad \text{for} \quad w \in \tilde{\Gamma}_+.
$$

(4.6)

In this way we obtain

$$
\| J_Y(wt^{1/2}) - 1_{(1, k+1)} \|_\infty = \sqrt{\kappa_j - \kappa_{j+1}} \sup_{w \in \tilde{\Gamma}_+} e^{\frac{1}{2n+1} \mathcal{J} \left( \frac{w^{2n+1}}{2n + 1} + d_j w \right)} \to 0
$$

(4.7)

for $t \to +\infty$ and any fixed $x_j \in \mathbb{R}$. Thus, the rescaled function $X(w) := Y(wt^{1/2})$, thanks to the Riemann-Hilbert problem 2.3, satisfies the following conditions:

- it is analytic on $\mathbb{C} \setminus \tilde{\Gamma}_- \cup \tilde{\Gamma}_+$;
it admits continuous boundary values $X_\pm$ while approaching from the left or from the right of the curves $\tilde{\Gamma}_- \cup \tilde{\Gamma}_+$ and they are related through $X_+(w) = X_-(w) J_Y(wt)$ for all $w$ along the curves;

- for $|w| \to \infty$ it behaves like the identity matrix, i.e. $X(w) \sim 1 + \sum_{j \geq 1} \frac{X_j}{w^j}$.

Note that, in particular, $X_1 = t^{-\frac{1}{2}} Y_1$ (we will use it in a moment). Moreover, by applying the small norm theorem (see for instance Theorem 5.1.5 in [6]) we can conclude that $X(w) \sim 1 + \sum_{j \geq 1} \frac{X_j}{w^j}$ for all $w$ along the curves;

On the other hand, because of its properties described above, $X$ satisfies the integral equation

$$X(w) = 1 + \int_{\tilde{\Gamma}_- \cup \tilde{\Gamma}_+} X_-(u) f_n(vt) g_n^\top(vt) du/dv. \quad (4.8)$$

Expanding the right hand side for $w \to \infty$ we find

$$X_1 = \int_{\tilde{\Gamma}_- \cup \tilde{\Gamma}_+} X_-(v) f_n(vt) g_n^\top(vt) dv, \quad (4.9)$$

and thus we can conclude that, for $t \to +\infty$ and fixed $x_j$, $u_j(t; \vec{x}, \vec{a}) = (Y_1)_{1,j+1} = t^{\frac{1}{2}} (X_1)_{1,j+1} \sim \sqrt{\alpha_j - \alpha_{j+1}} e^{i (\frac{2n+1}{2n^2} + (t+x_j)z)} dz = \sqrt{\alpha_j - \alpha_{j+1}} \text{Ai}_n(t+x_j)$$

for any $j = 1, \ldots, k$, where in the last two equalities we used the small norm theorem and the integral representation of the $n$-th Airy function. Finally, the equation (4.2) is obtained integrating twice (4.1), and checking that one can set the integration constants equal to zero because of the asymptotic behavior of $\vec{u}$ as $t \to +\infty$.

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