Random disturbance in cavity boundaries

Kangjian Wang, Guang Rong*, Qing Mu, Hongqiao Yin
National Key Laboratory of Transient Physics, Nanjing University of science and technology, Nanjing, 210094, P.R. China
E-mail: rongg@njust.edu.cn

Abstract. The water condition usually is complicated due to many uncertain factors such as internal wave, density differences and so on. So the real cavity boundaries is hard to predict for a supercavition projectile. In this work, a random disturbance model is established based on Poison process of random mathematics. This model is employed to the Logvinovich cavitation expansion equation for simulating a cavity in real ocean environment. A detail of cavity outline is obtained by this calculation. The results indicate that the cavitation boundary of higher velocity case is smoother than lowers. The parameter $\lambda$ of Poison model is a key factor of disturbance frequency, which largely affects the length of cavity. The $\lambda$ is proportion to the cavity length. This provides a new method for water environment cavity analysis.

1. Introduction
Supercavitation is a form of liquid cavitation [1]. When the pressure is reduced to the saturation vapor pressure of water temperature, the cavitation phenomenon will occur in liquid. Generally speaking, the cavity firstly produce at the head of the projectile because of the thinning of boundary layer. Then the cavity would spread over entire surface of the projectile. As the density of water is far greater than the vapors, the cavity could greatly reduce the navigation resistance of the projectile under water [2]. For supercavitation projectile, the projectile is always equipped with a sharp-edged, flat-headed and cylinder-shaped cavitator. Its beneficial to produce a well-shaped cavity out of the projectile. In order to obtain the cavity shape by analytical method, Logvinovich established a partial differential equation based on energy conservation, which could get the precise cavity geometry in undisturbed environment.

However, cavity shape of supercavitation projectile is greatly influenced by fluid disturbance [3] which includes the pressure fluctuation and the instability of projectile motion underwater. It is inevitably that a supercavitation projectile would be affected by ocean wave during launching or navigating. Waves are one of the most important external forces acting on ocean moving objects. The wave pressure results in an additional pressure at projectile external environment, which impacts the cavity shape. Due to the compication of ocean environment, the ocean waves show randomness and uncertainty. Therefore, the use of continuity function has some limitations on the description of this problem. That is useful to use random mathematical thought to research this issue.

In this paper, a random disturbance Poison process model is established to simulate ocean random disturbance process [4]. That can significantly cause the cavity formation, and that is a major reason of cavity instability. The result show that the cavity length and boundaries stability is influenced by fluctuation of pressure. The higher velocity of supercavitation projectile
is beneficial to the stability of cavitation. And the cavity near the projectile head has a stable cavity boundaries. However, the parameter $\lambda$ is related to the length of cavity. The parameter $\lambda$ is larger, the length cavitation is longer.

2. Random process model

2.1. Random disturbance Poison process model

It is assumed that $N(t)$ is the random disturbance times from some uncertain factors during the projectile motion phase $[0, t]$. So $N(t)$, $t \geq 0$ has the characters which include the continuous parameter of time and the discrete state space. There are a few conditions about this process as follow [5]:

1. The property of zero initial value: $N(0) = 0$, which means that the disturbance is zero at initial time.
2. Independent increment: $\forall n, \forall t_{i+1} > t_i > 0$, $N(t_{i+1}) - N(t_i), i = 0, 1, \ldots, n - 1$ are mutual independent. That means that the disturbances are mutual independent at the no overlapping time interval.
3. Stationary increment: $\forall s, t \geq 0$, $n \in N$, $N(s + t) - N(s)$ and $N(t)$ has same distribution regular, that is, the probability of the disturbances are the same regular at the same time interval.
4. The property of general: for $\forall t > 0$ and small enough $\Delta t \geq 0$, there is

$$P(t) = \begin{cases} 
\frac{P[N(t + \Delta t) - N(t) = 1]}{\lambda \Delta t + o(\Delta t)} \\
\frac{P[N(t + \Delta t) - N(t) \geq 2]}{o(\Delta t)}
\end{cases}$$

Where $o(\Delta t)$ is the a high order infinitesimal of $\Delta t$, and $\lambda = \text{const} > 0$. At the short time interval $[t, t + \Delta t]$, there is a conclusion that the probability of disturbance occurring is proportional to $\lambda \Delta t$.

However, this proposition is equivalent to that mutual independence $T_k, k = 1, 2, \ldots$ are subject to exponential distribution according to the theory of random mathematics. The $T_k$ is the time interval between two disturbances.

The probability density function of the exponential distribution is that [5]:

$$f(x) = \begin{cases} 
\lambda e^{-\lambda x} & x > 0 \\
0 & x \leq 0
\end{cases}$$

We can obtain the distribution function that

$$F(x) = \begin{cases} 
1 - e^{-\lambda x} & x > 0 \\
0 & x \leq 0
\end{cases}$$

For $x \geq 0$, we can obtain the inverse function that:

$$x = -\frac{1}{\lambda} \ln[1 - F(x)]$$

However, the disturbance can be obtained by the uniform distribution on time. The probability density function of the uniform distribution is as follows:

$$f(x) = \begin{cases} 
\frac{1}{b-a} & a < x < b \\
0 & \text{others}
\end{cases}$$

Where the $a$ and $b$ are the uniform distributions parameters. That is related to the ocean environment condition.

So the uniform distribution $U[0,1]$ could be used for the $F(x)$. Because $U$ is uniform distributed in $[0, t]$, so $1 - U$ is uniform distribution in $[0, t]$ either. The equation could be expressed as

$$x = -\frac{1}{\lambda} \ln u$$
2.2. Logvinovich cavity expansion equation

The normal projectile has large resistance when it moves in water. However, supercavitation projectile could largely decrease the resistance underwater because it could produce a cavity wrapped around the body. This cavity can separate the projectile from the water. It has significant prospect for future. But the motion of supercavitation projectile is influenced by the cavity’s outline. The principle of independence of the cavity sections expansion proposed by G. V. Logvinovich, which provides an easier way for investigating the non-stationary cavities and the cavities formation with the variable external pressure. It is assumed that the fluid is an incompressible, homogeneous and potential fluid. The pressure on the free surface of the fluid is equal to the pressure of the atmospheric pressure [6–8].

\[ \frac{\partial^2 S(x,t)}{\partial t^2} = -\frac{k(p - p_v)}{\rho} \]

Where \( k \) is coefficient which depends on cavitation number function \( \sigma = \Delta p/(0.5\rho v^2) \), usually \( k = 4\pi/A^2 \), \( A \approx 2 \) its an empirical constant. \( x(t) - l(t) \leq x \leq x(t) \)

\[ S_0 = \frac{\pi D_n^2}{4} \]

\[ \dot{S}_0 = \frac{kA}{4} D_n^2 v_\infty \sqrt{C_{x0}} \]

In these variables, \( S_0 \) is the cavitator area, and \( \dot{S}_0 \) is the initial expansion velocity of the cavity in the maximum section, the \( D_n \) is the cavitator diameter, and the \( C_{x0} \) is the resistance coefficient, which is 0.82, when \( \sigma = 0 \).

The Logvinovich equation can be rewritten as:

\[ S(\tau, t) = S_0 + \dot{S}_0(t - \tau) - \frac{k}{\rho} \int_{\tau}^{t} \int_{\tau}^{u} (p - p_v) dvdu \]  

3. Numerical method and algorithm

3.1. Discrete method of Logvinovich equation

Trapezoidal formula [9] is used for discrete the equations double integration. First, divide [\( u, \tau \)] into \( m \) equal parts and assume that \( h_x = (\tau - t)/m, x_i = a + ih(i = 0, 1, 2, \ldots, m) \), so

\[ I = h_x \left[ \frac{G(t) + G(\tau)}{2} + \sum_{i=1}^{m-1} G(x_i) \right] \]

Where

\[ G(x_i) = \int_{\tau(x_i)}^{u(x_i)} f(x_i, y)dy \]

Then we divide [\( u(x_i), \tau(x_i) \)] into \( n \) equal parts and assume that \( h_y = (\tau(x_i) - u(x_i))/m, \) so

\[ y_{ij} = \tau_1(x_i) + jh_y(i) \]

Where \( i = 0, 1, \ldots m, j = 0, 1, \ldots n \).

So

\[ G(x_i) = h_y(i) \left[ \frac{f(x_i, \tau(x_i)) + f(x_i, \tau(x_i))}{2} + \sum_{j=1}^{n-1} f(x_i, y_{ij}) \right] \]

The equation can be expressed as

\[ S(\tau, t) = S_0 + (\dot{S}_0(t - \tau) - I \]
3.2. Algorithm

(1) Initial value is required in the program such as initial velocity $V_0$, water density $\rho$, atmosphere pressure $P_0$, projectile motion depth $h$, average arrival intensity of random disturbance $\lambda$, calculation stop condition and time interval $\Delta t$.

(2) The disturbance pressure is chosen by uniformed distribution $U[0, 101325]$, because the supercavitation projectile motion depth is about 2-3 m. This depth limitation pressure usually has small fluctuation range. And most pressure change is no more than the atmosphere pressure in standard status. So the 101325 Pa is used for the pressure upper limit.

(3) Then calculate the value $I$.

(4) The cavity cross-section can be calculated by the equation that:

$$S(\tau, t) = S_0 + \dot{S}_0(t - \tau) - I$$  \hspace{1cm} (15)

Where the $\tau$ is a certain time of projectile in space $x$, but it is variable parameter for different position $x$.

(5) The next step calculation method is same with above. The next position time is calculated by $t_{i+1} = t_i + \Delta t$. Then continue the step from (1) to (5) until satisfying the calculation end condition. The condition can be set as the projectile motion stop time.

4. Calculation results and discussion

4.1. Projectiles velocity influences

Because the water density is much greater than the air, the projectile has totally different motion status in the two different mediums. So the velocity is a significant parameter during the projectile motion analyzing underwater. Figure 1 show the cavity boundaries at two different velocity. That illustrates the cavity of the supercavitation projectile, which means that the initial infinitesimal disturbance will be amplified later. This disturbance and stationary pressure become the reason of cavity collapsing.

![Figure 1. The cavity boundaries at different velocities ($\lambda=1000$).](image)

We can also obtain that the random disturbance greatly influences the cavity boundaries, and the velocity 300 m/s cavity boundary is more stable than the 200 m/s. Because the cavity is phase boundary, so it is a free surface. The surface properties is like the ocean surface.
The ocean is always unstable caused by some factors such as wind, density gap between cold and warm area. While the cavity boundary is underwater, the only difference between ocean surface and cavity boundary is that the cavity inner pressure is negative pressure. So the cavity boundary can be disturbed by some factors from ocean wave, which could cause the boundaries instable and fluctuation.

When the velocity of the projectile is high, the relative velocity of the stationary fluid in the environment will increase. So the fluid will produce higher inertia force. And the stationary fluid will smooth the surface of the cavity boundary, meanwhile the surface will become smoother (figure 2). This is beneficial to the stability of the cavity. Therefore, we can improve initial velocity to reduce the cavitation boundaries fluctuation. The cavity latter part has large fluctuation, which proves that the disturbance on the cavity will be amplification. With these fluctuations amplifying, the cavity outline will impact and fused together. This is the nature of cavity collapsing.

![Figure 2. The process of stationary fluid smooth.](image)

Figure 3 shows the ideal smooth cavity boundaries which means that there is no any disturbance. So the random disturb model reveals actual situation of cavity boundary in details. For the cavity boundary, the nearer part of projectile head is smoother than the latter’s. Because the cavity near the projectile head has large curvature, which result in the surface tension effect is more obviously. There is a strong ability to resist external disturbance. Another phenomenon is that the disturbed cavity length is shorter than undisturbed. The disturbance produced additionally pressure on the fluid environment, which curb the cavity expansion on space. Therefore, the cavity length will greatly reduce.
4.2. Disturbance intensify influences

The parameter $\lambda$ has a specific meaning in the field of probability theory. Generally speaking, the parameter $\lambda$ is the function of time. That can be defined as follow:

$$\lambda(t) = \lim_{h \to 0} \frac{P[N(t + h) - N(t) > 0]}{h}$$

(16)

$\lambda(t)$ named strength function. The strength function can well describe the nature of the pressure disturbance, which reflects the infinitesimal characteristic of $N(t)$, that is, the rate of change in the unit time in the sense of probability. But for the Poison process $\lambda = const$.

Figure 3. The cavity boundaries at different velocities (Undisturbed).

Figure 4. The cavitation boundaries for different $\lambda$.

According to the random mathematics theory that the disturbance time interval is subject to the exponential distribution. The parameter $\lambda$ has specified physical mean in this case. The
disturbance time interval expectation $E(x)$ is $1/\lambda$. Because the $E(x)$ describe the average of disturbance time interval, so when the higher $\lambda$ the lower $E(x)$. So the figure 4 show the cavity boundaries changing with the parameter $\lambda$. As the parameter increases to the 100 times that of origin 10, the length of the supercavity is increasing to 1.17 times than the original. It is indicated that the decrease in the time interval of the disturbance, which increases the length of the cavity (Table 1).

| Table 1. Parameter $\lambda$ and cavity length |
|-----------------------------------------------|
| Number | $\lambda$ | Length of cavitation(m) |
|--------|-----------|-------------------------|
| 1      | 10        | 1.52                    |
| 2      | 100       | 1.72                    |
| 3      | 1000      | 1.78                    |

5. Conclusion

This paper reports on the calculation of the supercavitating projectile cavity outline in ocean environment which has uncertain factors. A random pressure disturbance model is established for researching the stability of cavitation boundaries. The following conclusions have been drawn through theoretical studies:

1. The higher velocity of projectile can greatly smooth the cavity boundaries. That is benefit for the stability of projectile. To some extent, the cavity boundaries are similar to the ocean surface.

2. The cavity collapses rising from the small disturbance and stationary pressure of water. And these disturbance will be amplified at every cavity cross-section with the projectile motion.

3. With changing of the parameter $\lambda$, we found that the cavity length is related to it. And the larger of $\lambda$ the larger of cavity length, which indicate that the frequency of disturbance can increase the cavity length.

Acknowledgments

This work was supported by the Postgraduate Research &Practice Innovation Program of Jiangsu Province (Grant No. KYCX19_0338).

References

[1] Kulkarni S S and Pratap R 2000 Applied Mathematical Modelling 24 113–129
[2] Serebryakov V V, Arndt R E A and Dzielski J E Journal of Physics Conference 656 012169
[3] Kubenko V D and Gavrilenko O Journal of Fluids and Structures 25 794–814
[4] Ross S M 2014 Introduction to Probability and Statistics for Engineers and Scientists (Fifth Edition) (Boston: Academic Press) pp 89 – 140
[5] Ross S 2014 Introduction to Probability Models (Eleventh Edition) (Boston: Academic Press) pp 277 – 356
[6] Pellone C, FRANC J and PERRIN M Comptes Rendus Mécanique 332 827–833
[7] Hongde, Qin, Linyue, Zhao, Jing and Shen Journal of Harbin Engineering University 184–189
[8] Li K B, Wang A W, Shi L H and Deng L Dandao Xuebao/journal of Ballistics 25 103–106
[9] Todd J 1979 Numerical Analysis (Academic Press) pp 97 – 117