Abstract

The paper investigates relationship between algebraic expressions and graphs. We consider a digraph called a Fibonacci graph which gives a generic example of non-series-parallel graphs. Our intention in this paper is to simplify the expressions of Fibonacci graphs and eventually find their shortest representations. With that end in view, we describe the optimal decomposition method for generating Fibonacci graph expressions that is conjectured to provide these representations. Proof (or disproof) of this conjecture is presented as an open problem.

Keywords: Fibonacci graph, series-parallel graph, two-terminal directed acyclic graph, decomposition, expression.

1. Introduction

A graph $G = (V, E)$ consists of a vertex set $V$ and an edge set $E$, where each edge corresponds to a pair $(v, w)$ of vertices. If the edges are ordered pairs of vertices (i.e., the pair $(v, w)$ is different from the pair $(w, v)$), then we call the
A graph directed or digraph; otherwise, we call it undirected. If \((v, w)\) is an edge in a digraph, we say that \((v, w)\) leaves vertex \(v\) and enters vertex \(w\). A vertex in a digraph is a source if no edges enter it, and a sink if no edges leave it.

A path from vertex \(v_0\) to vertex \(v_k\) in a graph \(G = (V, E)\) is a sequence of its vertices \([v_0, v_1, v_2, \ldots, v_k]\) such that \((v_{i-1}, v_i) \in E\) for \(1 \leq i \leq k\). \(G\) is an acyclic graph if there is no closed path \([v_0, v_1, v_2, \ldots, v_k, v_0]\) in \(G\). A two-terminal directed acyclic graph (st-dag) has only one source \(s\) and only one sink \(t\). In an st-dag, every vertex lies on some path from \(s\) to \(t\).

A graph \(G' = (V', E')\) is a subgraph of \(G = (V, E)\) if \(V' \subseteq V\) and \(E' \subseteq E\). A graph \(G\) is homeomorphic to a graph \(G'\) (a homeomorph of \(G'\)) if \(G\) can be obtained by subdividing edges of \(G'\) with new vertices.

We consider a labeled graph which has labels attached to its edges. Each path between the source and the sink (a sequential path) in an st-dag can be presented by a product of all edge labels of the path.

**Definition 1.1.** We define the sum of edge label products corresponding to all possible sequential paths of an st-dag \(G\) as the canonical expression of \(G\).

**Definition 1.2.** An algebraic expression is called an st-dag expression (a factoring of an st-dag in [1]) if it is algebraically equivalent to the canonical expression of an st-dag. An st-dag expression consists of terms (edge labels), the operators + (disjoint union) and \(\cdot\) (concatenation, also denoted by juxtaposition when no ambiguity arises), and parentheses.

**Definition 1.3.** We define the complexity of an algebraic expression in two ways. The complexity of an algebraic expression is (i) the total number of terms in the expression including all their appearances (the first complexity characteristic) or (ii) the number of plus operators in the expression (the second complexity characteristic).

We will denote the first and the second complexity characteristic of an st-dag expression by \(T(n)\) and \(P(n)\), respectively, where \(n\) is the number of vertices in the graph.

**Definition 1.4.** An equivalent expression with the minimum complexity is called an optimal representation of the algebraic expression.

**Definition 1.5.** A series-parallel graph is defined recursively so that a single edge is a series-parallel graph and a graph obtained by a parallel or a series composition of series-parallel graphs is series-parallel.
As shown in [11] and [8], a series-parallel graph expression has a representation in which each term appears only once. We proved in [8] that this representation is an optimal representation of the series-parallel graph expression from the perspective of the first complexity characteristic. For example, the canonical expression of the series-parallel graph presented in Figure 1.1 is $abd + abc + acd + ace + f e + fd$. Since it is a series-parallel graph, the expression can be reduced to $(a(b + c) + f)(d + e)$, where each term appears once.

**Definition 1.6.** A Fibonacci graph (FG) [6] has vertices $\{1, 2, 3, \ldots, n\}$ and edges $\{(v, v + 1) \mid v = 1, 2, \ldots, n - 1\} \cup \{(v, v + 2) \mid v = 1, 2, \ldots, n - 2\}$.

As shown in [3], an st-dag is series-parallel if and only if it does not contain a subgraph which is a homeomorphic of the forbidden subgraph positioned between vertices 1 and 4 of the Fibonacci graph illustrated in Figure 1.2. Thus, Fibonacci graphs are of interest as "through" non-series-parallel st-dags.

**Figure 1.2:** A Fibonacci graph.

Mutual relations between graphs and algebraic expressions are discussed in [1], [4], [5], [8], [9], [10], [11], [12], [13], [14], and other works. Specifically, [11], [12], and [14] consider the correspondence between series-parallel graphs and read-one functions. A Boolean function is defined as read-one if it may be computed by some formula in which no variable occurs more than once (read-one formula).
On the other hand, a series-parallel graph expression can be reduced to the representation in which each term appears only once. Hence, such a representation of a series-parallel graph expression can be considered as a read-once formula (Boolean operations are replaced by arithmetic ones).

An expression of a homeomorph of the forbidden subgraph belonging to any non-series-parallel st-dag has no representation in which each term appears once. For example, consider the subgraph positioned between vertices 1 and 4 of the Fibonacci graph shown in Figure 1. Possible optimal representations of its expression are \( a_1 (a_2 a_3 + b_2) + b_1 a_3 \) or \((a_1 a_2 + b_1) a_3 + a_1 b_2 \). For this reason, an expression of a non-series-parallel st-dag can not be represented as a read-once formula. However, for arbitrary functions, which are not read-once, generating the optimum factored form is NP-complete [15].

Our intention is to simplify the expressions of Fibonacci graphs (we denote them by \( Ex(FG) \)) and eventually find their optimal representations. The last goal is an open problem. In this paper we survey a method which is conjectured to provide an optimal representation for \( Ex(FG) \).

2. Preliminary Results

The number of methods for generating Fibonacci graph expressions is described in [7]. Most of them derive representations with complexities which increase exponentially as the number of the graph’s vertices increases.

Specifically, the sequential-paths method is based directly on the definition of an st-dag expression as the canonical expression of the st-dag. For example, for a 9-vertex Fibonacci graph, the corresponding algebraic expression is

\[
\begin{align*}
& a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 + a_1 a_2 a_3 a_4 a_5 a_6 b_7 + a_1 a_2 a_3 a_4 a_5 b_6 a_8 + a_1 a_2 a_3 a_4 b_5 a_7 a_8 + \\
& a_1 a_2 a_3 a_4 b_5 b_7 + a_1 a_2 a_3 b_4 a_6 a_7 a_8 + a_1 a_2 a_3 b_4 a_6 b_7 + a_1 a_2 a_3 b_5 b_6 a_8 + \\
& a_1 a_2 b_3 a_5 a_6 a_7 a_8 + a_1 a_2 b_3 a_5 a_6 b_7 + a_1 a_2 b_3 a_5 b_6 a_8 + a_1 a_2 b_3 b_5 a_7 a_8 + \\
& a_1 a_2 b_3 b_5 b_7 + a_1 b_2 a_3 a_4 a_5 a_6 a_7 a_8 + a_1 b_2 a_3 a_4 a_5 a_6 b_7 + a_1 b_2 a_4 a_5 b_6 a_8 + \\
& a_1 b_2 a_4 b_5 a_7 a_8 + a_1 b_2 a_4 b_5 b_7 + a_1 b_2 b_4 a_6 a_7 a_8 + a_1 b_2 b_4 a_6 b_7 + \\
& a_1 b_2 b_4 b_6 a_8 + b_1 a_3 a_4 a_5 a_6 a_7 a_8 + b_1 a_3 a_4 a_5 a_6 b_7 + b_1 a_3 a_4 a_5 b_6 a_8 + \\
& b_1 a_3 a_4 b_5 a_7 a_8 + b_1 a_3 a_4 b_5 b_7 + b_1 a_3 b_4 a_6 a_7 a_8 + b_1 a_3 b_4 a_6 b_7 + \\
& b_1 a_3 b_4 b_6 a_8 + b_1 b_2 a_3 a_4 a_5 a_6 a_7 a_8 + b_1 b_2 a_3 a_5 a_6 b_7 + b_1 b_2 a_3 b_6 a_8 + \\
& b_1 b_3 b_5 a_7 a_8 + b_1 b_3 b_5 b_7.
\end{align*}
\]
It contains 201 terms and 33 plus operators.

2.1. Decomposition method

In [8] we consider the decomposition method which provides an algorithm for constructing \( Ex(FG) \) with polynomial complexity.

This method is based on revealing subgraphs in the initial graph. The resulting expression is produced by a special composition of subexpressions describing these subgraphs.

Consider the \( n \)-vertex \( FG \) presented in Figure 2.1. Denote by \( E(p, q) \) a subexpression related to its subgraph (which is an \( FG \) as well) having a source \( p \) (1 \( \leq \) \( p \) \( \leq \) \( n \)) and a sink \( q \) (1 \( \leq \) \( q \) \( \leq \) \( n \), \( q \geq p \)). If \( q - p \geq 2 \), then we choose any decomposition vertex \( i \) (1 \( \leq \) \( i \) \( \leq \) \( q - 1 \)) in a subgraph, and, in effect, split it at this vertex (Figure 2.1). Otherwise, we assign final values to \( E(p, q) \). As follows from the structure of a Fibonacci graph, any path from vertex \( p \) to vertex \( q \) passes through vertex \( i \) or avoids it via edge \( b_{i-1} \). Therefore, \( E(p, q) \) can be generated by the following recursive procedure (decomposition procedure):

1. **case** \( q = p : E(p, q) \leftarrow 1 \)
2. **case** \( q = p + 1 : E(p, q) \leftarrow a_p \)
3. **case** \( q \geq p + 2 : \text{choice}(p, q, i) \)
4. \[ E(p, q) \leftarrow E(p, i)E(i, q) + E(p, i - 1)b_{i-1}E(i + 1, q) \]

Lines [1] and [2] contain conditions of exit from the recursion. The special case when a subgraph consists of a single vertex is considered in line [1]. It is clear that such a subgraph can be connected to other subgraphs only serially. For this reason, it is accepted that its subexpression is 1, so that when it is multiplied by another subexpression, the final result is not influenced. Line [2] describes a
subgraph consisting of a single edge. The corresponding subexpression consists of a single term equal to the edge label. The general case is processed in lines 3 and 4. The procedure, choice\((p, q, i)\), in line 3 chooses an arbitrary decomposition vertex\(i\) on the interval\((p, q)\) so that\(p < i < q\). A current subgraph is decomposed into four new subgraphs in line 4. Subgraphs described by subexpressions \(E(p, i)\) and \(E(i, q)\) include all paths from vertex\(p\) to vertex\(q\) passing through vertex\(i\). Subgraphs described by subexpressions \(E(p, i-1)\) and \(E(i+1, q)\) include all paths from vertex\(p\) to vertex\(q\) passing through edge\(b_{i-1}\).

\(E(1, n)\) is the expression of the initial \(n\)-vertex\( \mathit{FG} (Ex (\mathit{FG}))\). Hence, the decomposition procedure is initially invoked by substituting parameters \(1\) and \(n\) instead of \(p\) and \(q\), respectively.

In [8] we proved the following theorem that determines an optimal location of the decomposition vertex\(i\) in an arbitrary interval\((p, q)\) of a Fibonacci graph from the perspective of the first complexity characteristic.

**Theorem 2.1.** The representation with a minimum total number of terms among all possible representations of\(Ex (\mathit{FG})\) derived by the decomposition method is achieved if and only if in each recursive step\(i\) is equal to \(\frac{q+p}{2}\) for odd\(q-p+1\) and to \(\frac{q+p-1}{2}\) or \(\frac{q+p+1}{2}\) for even\(q-p+1\), i.e., when \(i\) is a middle vertex of the interval\((p, q)\). Such a decomposition method is called optimal.

The following theorem for the second complexity characteristic is proven in [7].

**Theorem 2.2.** The representation with a minimum number of plus operators among all possible representations of\(Ex (\mathit{FG})\) derived by the decomposition method can be achieved by the optimal decomposition method.

It can be easily shown that for an \(n\)-vertex\( \mathit{FG}\):

1. The total number of terms\(T(n)\) in the expression\(Ex (\mathit{FG})\) derived by the optimal decomposition method is defined recursively as follows:

\[
T(1) = 0 \\
T(2) = 1 \\
T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + T\left(\left\lceil \frac{n}{2} \right\rceil - 1 \right) + T\left(\left\lceil \frac{n}{2} \right\rceil \right) + 1 \quad (n > 2).
\]
2. The number of plus operators \( P(n) \) in the expression \( Ex(FG) \) derived by the optimal decomposition method is defined recursively as follows:

\[
P(1) = 0 \\
P(2) = 0 \\
P(n) = P\left(\left\lceil \frac{n}{2} \right\rceil \right) + P\left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + P\left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + P\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + 1 \quad (n > 2).
\]

For large \( n \)

\[
T(n) \approx 4T\left(\left\lceil \frac{n}{2} \right\rceil \right) + 1
\]

and, by the master theorem \[2\], \( T(n) \) and \( P(n) \) are \( \Theta(n^2) \).

For \( n = 9 \), the possible algebraic expression derived by the optimal decomposition method is

\[
((a_1a_2 + b_1)(a_3a_4 + b_3) + a_1b_2a_4)((a_5a_6 + b_5)(a_7a_8 + b_7) + a_5b_6a_8) + \]
\[
(a_1(a_2a_3 + b_2) + b_1a_3)b_4(a_6(a_7a_8 + b_7) + b_6a_8).
\]

It contains 31 terms and 11 plus operators.

As shown in \[7\], the optimal decomposition method is not always the only one that provides an expression for a Fibonacci graph with a minimum number of plus operators. There exist special values of \( n \) when an \( n \)-vertex Fibonacci graph has several expressions with the same minimum number of plus operators (among expressions derived by the decomposition method). These special values are grouped as follows:

\[
7, 13 \div 15, 25 \div 31, 49 \div 63, 97 \div 127, 193 \div 255, \ldots
\]

In the general view, they can be presented in the following way:

\[
\begin{align*}
n_{\text{first}_\nu} &\leq n_{\text{sp}_\nu} \leq n_{\text{last}_\nu}; \\
n_{\text{first}_1} &= n_{\text{last}_1} = 7, \\
n_{\text{first}_\nu} &= 2n_{\text{first}_{\nu-1}} - 1, \\
n_{\text{last}_\nu} &= 2n_{\text{last}_{\nu-1}} + 1.
\end{align*}
\]

Here \( \nu \) is a number of a group of special numbers; \( n_{\text{sp}_\nu} \) is a special number of the \( \nu \)-th group; \( n_{\text{first}_\nu} \) and \( n_{\text{last}_\nu} \) are the first value and the last value, respectively, in the \( \nu \)-th group. For all these values of \( n \), not only the values of \( i \) which are mentioned in Theorem \[2.1\] provide a minimum number of plus operators in \( Ex(FG) \).
For example, for $n = 7$, the possible algebraic expression derived by the optimal decomposition method ($i$ is equal to 4 in the first recursive step) is

\[
(a_1a_2a_3 + b_4) + b_1a_3)(a_4(a_5a_6 + b_5) + b_4a_7) + \]
\[
(a_1a_2 + b_1)b_3(a_5a_7 + b_5).
\]

It contains 19 terms and 7 plus operators. For $i$ chosen equal to 3 in the first recursive step, the possible expression is

\[
(a_1a_2 + b_1)((a_3a_4 + b_3)(a_5a_6 + b_5) + a_3b_4a_6) + \]
\[
a_1b_2(a_4(a_5a_6 + b_5) + b_4a_6).
\]

This expression contains 20 terms but the number of its plus operators is also equal to 7.

### 2.2. Generalized decomposition (GD) method

As follows from the previous section, the decomposition method is based on splitting a Fibonacci graph in each recursive step into two parts via decomposition vertex $i$ and edge $b_{i-1}$. The GD method entails splitting a Fibonacci graph in each recursive step into an arbitrary number of parts (we will denote this number by $m$) via decomposition vertices $i_1, i_2, \ldots, i_{m-1}$ and edges $b_{i_1-1}, b_{i_2-1}, \ldots, b_{i_{m-1}-1}$, respectively. An example for $m = 3$ is illustrated in Figure 2.2.

![Figure 2.2: Decomposition of a Fibonacci subgraph at vertices $i_1$ and $i_2$.](image)

In all cases when $m > 2$, the decomposition procedure used in the previous section is transformed to the more complex form. Specifically, for $m = 3$, the general line of the new decomposition procedure, corresponding to line 4 of the decomposition procedure with $m = 2$ is presented as:

\[
E(p, q) \leftarrow E(p, i_1)E(i_1, i_2)E(i_2, q) + \]
\[
E(p, i_1 - 1)b_{i_1-1}E(i_1 + 1, i_2)E(i_2, q) + \]
\[
E(p, i_1)E(i_1, i_2 - 1)b_{i_2-1}E(i_2 + 1, q) + \]
\[
E(p, i_1 - 1)b_{i_1-1}E(i_1 + 1, i_2 - 1)b_{i_2-1}E(i_2 + 1, q).
\]
The sum above consists of four parts, with each part including three subexpressions corresponding to the three parts of a split subgraph. Hence, a current subgraph is decomposed into twelve new subgraphs.

Suppose that a Fibonacci graph is split into approximately equal parts in each recursive step (distances between decomposition vertices are equal or approximately equal). It will be the uniform GD method.

The following theorem is proven in [9].

**Theorem 2.3.** For an \( n \)-vertex \( FG \), both the total number of terms \( T(n) \) and the number of plus operators \( P(n) \) in the expression \( Ex(FG) \) derived by the uniform GD method (the \( FG \) is split into \( m \) parts) are \( O\left(n^{1+\log_m2^{n-1}}\right) \).

As follows from Theorem 2.3, \( T(n) \) and \( P(n) \) reach the minimum complexity among \( 2 \leq m \leq n-1 \) when \( m = 2 \). Substituting 2 for \( m \) gives \( O\left(n^2\right) \) (we have the optimal decomposition method in this case). Further, the complexity increases with the increase in \( m \). For example, we have \( O\left(n^{1+\log_34}\right) \) for \( m = 3 \), \( O\left(n^{2.5}\right) \) for \( m = 4 \), etc. In the extreme case, when \( m = n - 1 \), all inner vertices (from 2 to \( n - 1 \)) of an \( n \)-vertex \( FG \) are decomposition vertices. The single recursive step is executed in this case, and all revealed subgraphs are individual edges (labeled \( a \) with an index) connected by additional edges (labeled \( b \) with an index). That is, in this instance, the uniform GD method is reduced to the sequential-paths method. Substituting \( n - 1 \) for \( m \) gives

\[
O\left(n^{1+\log_{n-1}2^{n-2}}\right) > O\left(n^{1+\log_n2^{n-2}}\right) = O\left(2^{n-2}n\right).
\]

### 3. Open Problems

We conjecture that the optimal decomposition method provides an optimal representation (for both our complexity characteristics) of an algebraic expression related to a Fibonacci graph. The results obtained in section 2.2 do not contradict this conjecture. At least, the optimal decomposition method is the best one among uniform GD methods (asymptotically).

However, we did not prove that splitting a Fibonacci graph into approximately equal \( m \) parts gives the optimal result for arbitrary \( m \) (as in Theorems 2.1 and 2.2 for \( m = 2 \)). Besides, the GD method entails splitting a Fibonacci graph into the same number of parts in each recursive step. One further generalization of the method assigns to any subgraph its own number of decomposition vertices.
Finally, there exist representations that are obtained through algorithms which are not appropriate to any generalized decomposition method. Thus, we have the following open problems.

**Problem 3.1.** Prove (or disprove) that the optimal decomposition method is the only one that provides an optimal representation of an algebraic expression related to a Fibonacci graph from the perspective of the first complexity characteristic.

**Problem 3.2.** Prove (or disprove) that the optimal decomposition method provides an optimal representation of an algebraic expression related to a Fibonacci graph from the perspective of the second complexity characteristic.

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