The Soliton-Kähler-Ricci Flow over Fano Manifolds

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Abstract

We introduce a flow of Kähler structures over Fano manifolds with formal limit at infinite time a Kähler-Ricci soliton. This flow correspond to a Perelman’s modified backward Kähler-Ricci type flow that we call Soliton-Kähler-Ricci flow. It can be generated by the Soliton-Ricci flow. We assume that the Soliton-Ricci flow exists for all times and the Bakry-Emery-Ricci tensor preserve a positive uniform lower bound with respect to the evolving metric. In this case we show that the corresponding Soliton-Kähler-Ricci flow converges exponentially fast to a Kähler-Ricci soliton.

1 Introduction

This paper is the continuation of the work [Pal2] in the Kähler setting.

The notion of Kähler-Ricci soliton (in short KRS) is a natural generalization of the notion of Kähler-Einstein metric. A KRS over a Fano manifold $X$ is a Kähler metric in the class $2\pi c_1(X)$ such that the gradient of the default potential of the metric to be Kähler-Einstein is holomorphic. The terminology is justified by the fact that the pull back of the KRS metric via the flow of automorphisms generated by this gradient provides a Kähler-Ricci flow.

We remind that the Kähler-Ricci flow (in short KRF) has been introduced by H. Cao in [Cao]. In the Fano case it exists for all positive times. Its convergence in the classic sense implies the existence of a Kähler-Einstein metric. The fact that not all Fano manifolds admit Kähler-Einstein metrics implies the non convergence in the classic sense of the KRF in general.

Our approach for the construction of Kähler-Ricci solitons is based on the study of a flow of Kähler structures $(X, J_t, g_t)_{t \geq 0}$ associated to any normalized smooth volume form $\Omega > 0$ that we will call $\Omega$-Soliton-Kähler-Ricci flow (in short $\Omega$-SKRF). (See the definition below.) Its formal limit is precisely the KRS equation with corresponding volume form $\Omega$.

It turns out that this flow is generated by the backward KRF via the diffeomorphisms flow corresponding to the gradient of functions satisfying Perelman’s
backward heat equation [Per] for the KRF. In particular our point of view gives a new reason for considering Perelman’s backward heat equation.

Using a result in [Pal1] we can show that the Ω-SKRF can be generated by the Ω-Soliton-Ricci flow (in short Ω-SRF) introduced in [Pal2] via an ODE flow of complex structures of Lax type. (See corollary 2 below.)

Let \( M \) be the space of smooth Riemannian metrics. We have explained in [Pal2] that it make sense to consider the Ω-SRF for a special set \( \mathcal{S}_{\text{SKRF}}^{K, +} \subset M \) of initial data (see [Pal2] for the definition) that we call positive scattering data with center of polarization \( K \).

In this paper we denote by \( K_{\mathcal{J}} \) the set of \( \mathcal{J} \)-invariant Kähler metrics. We define the set of positive Kähler scattering data as the set

\[
\mathcal{S}_{\text{SKRF}}^{K, +} := \mathcal{S}_{\text{SKRF}}^{K} \cap K_{\mathcal{J}}.
\]

With this notations hold the following result which is a consequence of the convergence result for the Ω-SRF obtained in [Pal2].

**Theorem 1.** Let \((X, J_0)\) be a Fano manifold and assume there exist \( g_0 \in \mathcal{S}_{\text{SKRF}}^{K, +} \), for some smooth volume form \( \Omega > 0 \) and some center of polarization \( K \), such that the solution \((g_t)\) of the Ω-SRF with initial data \( g_0 \) exists for all times and satisfies \( \text{Ric}_{g_t}(\Omega) \geq \delta g_t \) for some uniform bound \( \delta \in \mathbb{R}_{>0} \).

Then the corresponding solution \((J_t, g_t)_{t \geq 0}\) of the Ω-SKRF converges exponentially fast with all its space derivatives to a \( J_{\infty} \)-invariant Kähler-Ricci soliton \( g_{\infty} = \text{Ric}_{g_{\infty}}(\Omega) \).

Furthermore assume there exists a positive Kähler scattering data \( g_0 \in \mathcal{S}_{\text{SKRF}}^{K, +} \) with \( g_0 J_0 \in 2\pi c_1(X) \) such that the evolving complex structure \( J_t \) stays constant along a solution \((J_t, g_t)_{t \in [0, T]}\) of the Ω-SKRF with initial data \((J_0, g_0)\). Then \( g_0 \) is a \( J_0 \)-invariant Kähler-Ricci soliton and \( g_t \equiv g_0 \).

### 2 The Soliton-Kähler-Ricci Flow

Let \( \Omega > 0 \) be a smooth volume form over an oriented Riemannian manifold \((X, g)\). We remind that the Ω-Bakry-Emery-Ricci tensor of \( g \) is defined by the formula

\[
\text{Ric}_g(\Omega) := \text{Ric}(g) + \nabla_g d\log \frac{dV_g}{\Omega}.
\]

A Riemannian metric \( g \) is called a Ω-Shrinking Ricci soliton (in short Ω-ShRS) if \( g = \text{Ric}_g(\Omega) \). Let now \((X, J)\) be a complex manifold. A \( J \)-invariant Kähler metric \( g \) is called a \( J \)-Kähler-Ricci soliton (in short \( J \)-KRS) if there exist a smooth volume form \( \Omega > 0 \) such that \( g = \text{Ric}_g(\Omega) \).

The discussion below will show that if a compact Kähler manifold admit a Kähler-Ricci soliton \( g \) then this manifold is Fano and the choice of \( \Omega \) corresponding to \( g \) is unique up to a normalizing constant.

We remind first that any smooth volume form \( \Omega > 0 \) over a complex manifold \((X, J)\) of complex dimension \( n \) induces a hermitian metric \( h_\Omega \) over the canonical
bundle $K_{X,J} := \Lambda_{n,0}^{n,0}T_x^*$ given by the formula

$$h_{11}(\alpha, \beta) := \frac{n! \alpha \wedge \bar{\beta}}{\Omega}.$$ 

By abuse of notations we will denote by $\Omega^{-1}$ the metric $h_{11}$. The dual metric $h_{11}^*$ on the anti-canonical bundle $K_{X,J}^{-1} = \Lambda_{n,0}^{n,0}T_x$ is given by the formula

$$h_{11}^*(\xi, \eta) = (-i)^n \Omega(\xi, \bar{\eta}) / n!.$$ 

Abusing notations again, we denote by $\Omega$ the dual metric $h_{11}^*$. We define the $\Omega$-Ricci form

$$\text{Ric}_J(\Omega) := i C_{\Omega}(K_{X,J}^{-1}) = - i C_{\Omega^{-1}}(K_{X,J}),$$

where $C_h(L)$ denotes the Chern curvature of a hermitian line bundle. In particular we observe the identity $\text{Ric}_J(\omega) = \text{Ric}_J(\omega^n)$. We remind also that for any $J$-invariant Kähler metric $g$ the associated symplectic form $\omega := gJ$ satisfies the elementary identity

$$\text{Ric}(g) = - \text{Ric}_J(\omega)J.$$ (2.1)

Moreover for all twice differentiable function $f$ hold the identity

$$\nabla_g df = -(i \partial_J \bar{\partial}_J f)J + g \bar{\partial}_{T_{X,J}} \nabla_g \log \frac{dV_g}{\Omega}.$$ (See the decomposition formula [6.5] in the appendix.) We infer the decomposition identity

$$\text{Ric}_g(\Omega) = - \text{Ric}_J(\Omega)J + g \bar{\partial}_{T_{X,J}} \nabla_g \log \frac{dV_g}{\Omega}.$$ (2.2)

Thus a $J$-invariant Kähler metric $g$ is a $J$-KRS iff there exist a smooth volume form $\Omega > 0$ such that

$$\begin{cases}
g = - \text{Ric}_J(\Omega)J, \\
\bar{\partial}_{T_{X,J}} \nabla_g \log \frac{dV_g}{\Omega} = 0.
\end{cases}$$

The first equation of this system implies that $(X, J)$ must be a Fano variety. We can translate the notion of Kähler-Ricci soliton in symplectic terms. In fact let $(X, J_0)$ be a Fano manifold of complex dimension $n$, let $c_1 := c_1(X, [J_0])$, where $[J_0]$ is the co-boundary class of the complex structure $J_0$ and set

$$\mathcal{J}^{+}_{X,J_0} := \left\{ J \in [J_0] \mid N_J = 0, \exists \omega \in K^{2\pi c_1}_J \right\},$$

where $N_J$ denotes the Nijenhuis tensor and

$$K^{2\pi c_1}_J := \left\{ \omega \in 2\pi c_1 \mid \omega = J^* \omega J, -\omega J > 0 \right\},$$

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is the set of $J$-invariant Kähler forms $\omega \in 2\pi c_1$. It is clear that for any complex structure $J \in \mathcal{J}_{X,J_0}$ and any form $\omega \in \mathcal{K}_j^{2\pi c_1}$ there exist a unique smooth volume form $\Omega > 0$ with $\int_X \Omega = (2\pi c_1)^n$ such that $\omega = \text{Ric}_j(\Omega)$.

This induces an inverse functional $\text{Ric}_j^{-1}$ such that $\Omega = \text{Ric}_j^{-1}(\omega)$. With this notation we infer that a $J$-invariant form $\omega \in 2\pi c_1$ is the symplectic form associated to a $J$-KRS if and only if $0 < g := -\omega J$ and

$$\bar{\partial}_{TX,J} \nabla g \log \frac{\omega^n}{\text{Ric}_j^{-1}(\omega)} = 0.$$ 

In equivalent volume terms we say that a smooth volume form $\Omega > 0$ with $\int_X \Omega = (2\pi c_1)^n$ is a $J$-Soliton-Volume-Form (in short $J$-SVF) if

$$\begin{cases}
0 < g := -\text{Ric}_j(\Omega) J , \\
\bar{\partial}_{TX,J} \nabla g \log \frac{\text{Ric}_j(\Omega)^n}{\Omega} = 0 .
\end{cases}$$

We deduce a natural bijection between the sets $\{ g \ | \ J\text{-KRS}\}$ and $\{ \Omega \ | \ J\text{-SVF}\}$. We define also the set of Soliton-Volume-Forms over $(X, J_0)$ as

$$\mathcal{SV}_{X,J_0} := \left\{ \Omega > 0 \ | \ \int_X \Omega = (2\pi c_1)^n, \ \exists J \in \mathcal{J}_{X,J_0}^+ : \Omega \text{ is a } J\text{-SVF} \right\} .$$

We would like to investigate under which conditions $\mathcal{SV}_{X,J_0} \neq \emptyset$. For this purpose it seem natural to consider the following flow of Kähler structures.

**Definition 1.** (The $\Omega$-Soliton-Kähler-Ricci flow). Let $(X, J_0)$ be a Fano manifold and let $\Omega > 0$ be a smooth volume form with $\int_X \Omega = (2\pi c_1)^n$. A $\Omega$-Soliton-Kähler-Ricci flow (in short $\Omega$-SKRF) is a flow of Kähler structures $(X, J_t, \omega_t)_{t \geq 0}$ which is solution of the evolution system

$$\begin{cases}
\frac{d}{dt} \omega_t = \text{Ric}_{J_t}(\Omega) - \omega_t , \\
\frac{d}{dt} J_t = J_t \bar{\partial}_{TX,J_t} \nabla g_t \log \frac{\omega^n_t}{\Omega} ,
\end{cases} \quad (2.3)$$

where $g_t := -\omega_t J_t$.

The $\Omega$-SKRF equation/system (2.3) can be written in an equivalent way as

$$\begin{cases}
\frac{d}{dt} \omega_t - i \partial_{J_t} \bar{\partial}_{J_t} f_t = \text{Ric}_{J_t}(\omega_t) - \omega_t , \\
\frac{d}{dt} J_t = J_t \bar{\partial}_{TX,J_t} \nabla g_t f_t , \\
e^{-f_t} \omega^n_t = \Omega .
\end{cases} \quad (2.4)$$
We observe also that (2.3) or (2.4) are equivalent to the system
\[
\begin{align*}
\frac{d}{dt} \omega_t &= \text{Ric}_{J_t}(\Omega) - \omega_t, \\
J_t &:= (\Phi_t^{-1})^* J_0 := \left[ (d\Phi_t \cdot J_0) \circ \Phi_t^{-1} \right] \cdot d\Phi_t^{-1}, \\
\frac{d}{dt} \Phi_t &= - \left( \frac{1}{2} \nabla_g \log \frac{\omega_t^n}{\Omega} \right) \circ \Phi_t, \\
\Phi_0 &= \text{Id}_X.
\end{align*}
\]
(2.5)

In fact lemma 4 combined with lemma 5 in the appendix implies
\[
\frac{d}{dt} (\Phi_t^* J_t) = \Phi_t^* \left( \frac{d}{dt} J_t - \frac{1}{2} L\nabla_{g_t,f_t} J_t \right) = \Phi_t^* \left( \frac{d}{dt} J_t - J_t \tilde{\partial}_{TX,J_t} \nabla_{g_t,f_t} \right) = 0.
\]

We define now \(\hat{\omega}_t := \Phi_t^* \omega_t\), \(\hat{g}_t := \Phi_t^* g_t = -\hat{\omega}_t J_0\) and we observe that the evolving family
\((J_0, \hat{\omega}_t) \equiv \Phi_t^* (J_t, \omega_t)\),
represents a backward Kähler-Ricci flow over \(X\). In fact the Kähler condition
\[
\nabla_{\hat{g}_t} J_0 = \Phi_t^* (\nabla_{g_t} J_t) = 0,
\]
hold and
\[
\frac{d}{dt} \hat{\omega}_t = \Phi_t^* \left( \frac{d}{dt} \omega_t - \frac{1}{2} L\nabla_{g_t,f_t} \omega_t \right) = \Phi_t^* \left( \text{Ric}_{J_t}(\omega_t) - \omega_t \right) = \text{Ric}_{J_0}(\hat{\omega}_t) - \hat{\omega}_t,
\]
by the formula (6.4) in the appendix. We observe that the volume form preserving condition \(e^{-f_t} \omega_t^n = \Omega\) in the equation (2.4) is equivalent to the heat equation
\[
2 \frac{d}{dt} f_t = \text{Tr}_{\omega_t} \frac{d}{dt} \omega_t = - \Delta g_t f_t + \text{Scal}(g_t) - 2n,
\]
(2.6)
with initial data \(f_0 := \log \frac{\omega_0^n}{\Omega}\). (In this paper we adopt the sign convention \(\Delta_g := -\text{div}_g \nabla_g\).) In its turn this is equivalent to the heat equation
\[
2 \frac{d}{dt} \hat{f}_t = - \Delta_{\hat{g}_t} \hat{f}_t - |\nabla_{\hat{g}_t} \hat{f}_t|_{\hat{g}_t}^2 + \text{Scal}(\hat{g}_t) - 2n,
\]
(2.7)
with same initial data \( \hat{f}_0 := \log \frac{\omega^n}{\Omega} \). In fact let \( \hat{f}_t := f_t \circ \Phi_t \) and observe that the evolution equation of \( \Phi_t \) in (2.5) implies
\[
\frac{d}{dt} \hat{f}_t = \left( \frac{d}{dt} f_t \right) \circ \Phi_t + \left( \nabla g_t f_t \circ \Phi_t \circ \Phi_t \right) \cdot \phi_t \circ \Phi_t,
\]
\[
= \left( \frac{d}{dt} f_t - \frac{1}{2} |\nabla g_t f_t|_{g_t}^2 \right) \circ \Phi_t.
\]
We observe also that the derivation identity
\[
0 = \frac{d}{dt} \left( \Phi_t^{-1} \circ \Phi_t \right) = \left( \frac{d}{dt} \Phi_t^{-1} \right) \circ \Phi_t + \frac{d}{dt} \Phi_t^{-1} \cdot \frac{d}{dt} \Phi_t,
\]
combined with the evolution equation of \( \Phi_t \) in the system (2.5) implies
\[
2 d \Phi_t \cdot \left( \frac{d}{dt} \Phi_t^{-1} \right) \circ \Phi_t = \nabla g_t f_t \circ \Phi_t = d \Phi_t \cdot \nabla g_t \hat{f}_t.
\]
We infer the evolution formula
\[
\frac{d}{dt} \Phi_t^{-1} = \frac{1}{2} \left( \nabla g_t \hat{f}_t \right) \circ \Phi_t^{-1}.
\] (2.8)
In conclusion we deduce that the \( \Omega \)-SKRF \((J_t, \omega_t)_{t \geq 0}\) is equivalent to the system of independent equations
\[
\begin{aligned}
\frac{d}{dt} \hat{\omega}_t &= \text{Ric}_{g_t} (\hat{\omega}_t) - \hat{\omega}_t, \\
2 \frac{d}{dt} \hat{f}_t &= - \Delta_{\hat{g}_t} \hat{f}_t - |\nabla_{\hat{g}_t} \hat{f}_t|_{\hat{g}_t}^2 + \text{Scal}(\hat{g}_t) - 2n, \\
e^{-\hat{f}_0 \hat{\omega}_0} = \Omega,
\end{aligned}
\]
by means of the gradient flow of diffeomorphisms (2.8).

**Notation.** Let \((X, g, J)\) be a Kähler manifold with symplectic form \(\omega := gJ\) and consider \(v \in S^2 g T_X, \alpha \in \Lambda^2 g T_X\). We define the endomorphisms \(v^* g := g^{-1} v\) and \(\alpha^* g := \omega^{-1} \alpha\). For example we will define the endomorphisms
\[
\text{Ric}^*_g (\Omega) := g^{-1} \text{Ric}_g (\Omega),
\]
and
\[
\text{Ric}^*_J (\Omega)_g := \omega^{-1} \text{Ric}_J (\Omega).
\]
With this notations formula (2.2) implies the decomposition identity
\[
\text{Ric}_g^* (\Omega) = \text{Ric}_J^* (\Omega)_g + \bar{\partial}_{\bar{x}, J} \nabla_g \log \frac{dV_g}{\Omega}.
\] (2.9)
3 The Riemannian nature of the Soliton-Kähler-Ricci Flow

The goal of this section is to show that the Kähler structure along the SKRF comes for free from the SRF introduced in [Pal2] by means of a Lax type ODE for the complex structures which preserves the Kähler condition. For this purpose let \((J_t, g_t)_{t \geq 0}\) be a \(\Omega\)-SKRF. Time deriving the identity \(g_t = -\omega_t J_t\) we obtain

\[
\frac{d}{dt} g_t = - \frac{d}{dt} \omega_t J_t - \omega_t \frac{d}{dt} J_t
\]

\[
= - \text{Ric}_{\omega_t}(\Omega) J_t + \omega_t \frac{d}{dt} J_t - \omega_t J_t \bar{\partial}_{X, J_t} \nabla_{g_t} f_t
\]

\[
= - \text{Ric}_{\omega_t}(\Omega) J_t + g_t \bar{\partial}_{X, J_t} \nabla_{g_t} f_t - g_t
\]

\[
= \text{Ric}_{g_t}(\Omega) - g_t,
\]

thanks to the complex decomposition (2.2). We have obtained the evolving system of Kähler structures \((J_t, g_t)_{t \geq 0}\),

\[
\left\{ \begin{array}{l}
\frac{d}{dt} g_t = \text{Ric}_{g_t}(\Omega) - g_t, \\
2 \frac{d}{dt} J_t = J_t \nabla^2_{g_t} \log \frac{d\Omega}{g_t} - \nabla^2_{g_t} \log \frac{d\Omega}{g_t} J_t,
\end{array} \right.
\]

which is equivalent to (2.3). (The second equation in the system follows from the fact that in the Kähler case the Chern connection coincides with the Levi-Civita connection.) We observe that the identity (2.1) implies that the Ricci endomorphism

\[
\text{Ric}^*(g) = \text{Ric}^*_\omega(\omega)_{g_t},
\]

is \(J\)-linear. Thus the system (3.1) is equivalent to the evolution of the couple \((J_t, g_t)\) under the system

\[
\left\{ \begin{array}{l}
\dot{g}_t = \text{Ric}_{g_t}(\Omega) - g_t, \\
2 \dot{J}_t = J_t \dot{g}_t - \dot{g}_t J_t, \\
J_t^2 = - \mathcal{L}_{T_X}, \quad (J_t)_{g_t}^T = - J_t, \quad \nabla_{g_t} J_t = 0,
\end{array} \right.
\]

where for notation simplicity we set \(\dot{g}_t := \frac{d}{dt} g_t\) and \(\dot{J}_t := \frac{d}{dt} J_t\). Moreover \((J_t)_{g_t}^T\) denotes the transpose of \(J_t\) with respect to \(g_t\). We remind now an elementary fact (see lemma 4 in [Pal1]).

**Lemma 1.** Let \((g_t)_{t \geq 0}\) be a smooth family of Riemannian metrics and let \((J_t)_{t \geq 0}\) be a family of endomorphisms of \(T_X\) solution of the ODE

\[
2 \dot{J}_t = J_t \dot{g}_t - \dot{g}_t J_t,
\]
with initial conditions $J_0^2 = -1_{T_X}$ and $(J_0)^T_{g_0} = -J_0$. Then this conditions are preserved in time i.e. $J_t^2 = -1_{T_X}$ and $(J_t)^T_{g_t} = -J_t$ for all $t \geq 0$.

We deduce that the system (3.2) is equivalent to the system

\[
\begin{cases}
\dot{g}_t = \Ric_{g_t}(\Omega) - g_t, \\
2 \dot{J}_t = J_t \dot{g}^*_t - \dot{g}_t^* J_t, \\
\nabla_{g_t} J_t = 0,
\end{cases}
\tag{3.3}
\]

with Kähler initial data $(J_0, g_0)$. We show now how we can get rid of the last equation. We define the vector space

$$
\mathbb{F}_g := \left\{ v \in C^\infty(X, S^2 T_X^*) \mid \nabla_{T_X, g} v = 0 \right\},
$$

where $\nabla_{T_X, g}$ denotes the covariant exterior derivative acting on $T_X$-valued differential forms and we remind the following key result obtained in [Pal1].

**Proposition 1.** Let $(g_t)_{t \geq 0}$ be a smooth family of Riemannian metrics such that $\dot{g}_t \in \mathbb{F}_g$, and let $(J_t)_{t \geq 0}$ be a family of endomorphisms of $T_X$ solution of the ODE

\[
\dot{J}_t = J_t \dot{g}^*_t - \dot{g}_t^* J_t,
\]

with Kähler initial data $(J_0, g_0)$. Then $(J_t, g_t)_{t \geq 0}$ is a smooth family of Kähler structures.

In particular using lemma 1 in [Pal2] we infer the following corollary which provides a simple way to generate Kähler structures.

**Corollary 1.** Let $(J_t, g_t)_{t \geq 0} \subset C^\infty(X, \text{End}_{g}(T_X)) \times \mathcal{M}$ be the solution of the ODE

\[
\begin{cases}
\frac{d}{dt} \dot{g}_t^* = 0, \\
2 \frac{d}{dt} J_t = J_t \dot{g}_t^* - \dot{g}_t^* J_t,
\end{cases}
\tag{3.4}
\]

with $(J_0, g_0)$ Kähler data and with $\nabla_{T_X, g_0} (\dot{g}_0^*)^p = 0$ for all $p \in \mathbb{Z}_{>0}$. Then $(J_t, g_t)_{t \geq 0}$ is a smooth family of Kähler structures.

We remind also the definitions introduced in [Pal2]. We define the set of pre-scattering data

$$
\mathcal{S}_\Omega := \left\{ g \in \mathcal{M} \mid \nabla_{T_X, g} \Ric_g(\Omega) = 0 \right\}.
$$

**Definition 2.** (The $\Omega$-Soliton-Ricci flow). Let $\Omega > 0$ be a smooth volume form over an oriented Riemannian manifold $X$. A $\Omega$-Soliton-Ricci Flow (in short $\Omega$-SRF) is a Flow of Riemannian metrics $(g_t)_{t \geq 0} \subset \mathcal{S}_\Omega$ solution of the evolution equation $\dot{g}_t = \Ric_g(\Omega) - g_t$. 

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From Proposition 1 we deduce the following fact which shows the Riemannian nature of the $\Omega$-SKRF. Namely that the $\Omega$-SKRF can be generated by the $\Omega$-SRF.

**Corollary 2.** Let $\Omega > 0$ be a smooth volume form over a Kähler manifold $(X, J_0)$ and let $(g_t)_{t \geq 0}$ be a solution of the $\Omega$-SRF with Kähler initial data $(J_0, g_0)$. Then the family $(J_t, g_t)_{t \geq 0}$ with $(J_t)_{t \geq 0}$ the solution of the ODE

$$2 J_t = J_0 g_t^* - \dot{g}_t^* J_t ,$$

is a solution of the $\Omega$-SKRF equation.

## 4 The set of Kähler pre-scattering data

We define the set of Kähler pre-scattering data as $S_{\Omega, J} := S_\omega \cap K_J$. Using the complex decomposition formula (2.9) we infer the equality

$$S_{\Omega, J} = \{ g \in K_J \mid \partial g_{TX,J} \bar{\partial} g_{TX,J} \nabla g \log \frac{dV_g}{\Omega} = - \bar{\partial} g_{TX,J} \text{Ric}_J^* (\Omega) g \} .$$

In fact the identity $d \text{Ric}_J (\Omega) = 0$ is equivalent to the identity $\partial g_{TX,J} \text{Ric}_J^* (\Omega) g = 0$, which in its turn is equivalent to the identity

$$\partial g_{TX,J} \text{Ric}_J^* (\Omega) g = 0 .$$

We observe now the following quite elementary facts.

**Lemma 2.** Let $(X, J)$ be a Fano manifold and let $g \in S_{\Omega, J}$ such that $\omega := gJ \in 2\pi c_1(X)$. Then $g$ is a $J$-invariant KRS iff $\text{Ric}_J (\Omega)$ is $J$-invariant.

**Proof.** In the case $\text{Ric}_J (\Omega) = \mathbb{H}_{TX}$ the condition $g \in S_{\Omega, J}$ is equivalent to the condition

$$\partial g_{TX,J} \nabla g \log \frac{dV_g}{\Omega} = 0 . \quad (4.1)$$

(i.e the $J$-invariance of $\text{Ric}_g (\Omega)$ and thus that $g$ is a $J$-invariant KRS.) In fact in this case

$$\partial g_{TX,J} \nabla g \log \frac{dV_g}{\Omega} = 0 ,$$

which by a standard Kähler identity implies

$$\partial^* g_{TX,J} \nabla g \log \frac{dV_g}{\Omega} = 0 .$$

Thus an integration by parts yields the required identity (4.1).

On the other hand if we assume that $\text{Ric}_J (\Omega)$ is $J$-invariant i.e we assume (4.1) then the condition $g \in S_{\Omega, J}$ is equivalent to the condition

$$\partial g_{TX,J} \text{Ric}_J^* (\Omega) g = 0 .$$
For cohomology reasons hold the identity
\[ \omega = \text{Ric}_J(\Omega) + i \partial_J \bar{\partial}_J u , \]
for some \( u \in C^\infty(X, \mathbb{R}) \). We deduce the equalities
\[ 0 = \bar{\partial}_{\mathcal{T}_X,J} \left( i \partial_J \bar{\partial}_J u \right)^* = \bar{\partial}_{\mathcal{T}_X,J} \partial^\mathcal{T}_X,J \nabla g u . \]
Using again a standard Kähler identity we infer
\[ \partial^* \nabla g u = 0 . \]
An integration by parts yields the conclusion \( i \partial_J \bar{\partial}_J u \equiv 0 \), i.e. \( u \equiv 0 \), which implies the required KRS equation.

**Lemma 3.** Let \((X, J)\) be a Kähler manifold and let \((g_t)_{t \in [0, T)}\) be a smooth family of \(J\)-invariant Kähler metrics solution of the equation
\[ \dot{g}_t = \text{Ric}_{g_t}(\Omega) - g_t . \]
Then this family is given by the formula
\[ g_t = - \text{Ric}_J(\Omega) J + \left( g_0 + \text{Ric}_J(\Omega) J \right) e^{-t} , \]
with \(J\)-invariant Kähler initial data \( g_0 \) solution of the equation
\[ \bar{\partial}_{\mathcal{T}_X,J} \nabla g_0 \log \frac{dV_{g_0}}{\Omega} = 0 . \]
**Proof.** The fact that \( g_t \) is \(J\)-invariant implies that \( \dot{g}_t \) is also \(J\)-invariant. Then the decomposition formula \([2.2]\) combined with the evolution equation of \( g_t \) provides
\[ \dot{g}_t = - \text{Ric}_J(\Omega) J - g_t , \]
which implies the required conclusion.

From the previous lemmas we deduce directly the following corollary.

**Corollary 3.** Let \( g_0 \in \mathcal{S}_{\Omega,J} \) with \( gJ \in 2\pi c_1(X) \) be an initial data for the \(\Omega\)-SKRF such that the complex structure stays constant along the flow. Then \( g_0 \) is a \(J\)-invariant KRS and \( g_t \equiv g_0 = - \text{Ric}_J(\Omega) J \).

The last statement in the theorem follows directly from this corollary.
5 On the smooth convergence of the Soliton-Kähler-Ricci flow

We show now the convergence statement in theorem 1. According to the convergence result for the Ω-SRF obtained in [Pal2] we just need to show the smooth convergence of the complex structures. We consider the differential system

\[
\begin{align*}
2 \dot{J}_t &= [J_t, \dot{g}_t^*], \\
2 \dot{g}_t &= -\Delta_{g_t} g_t - 2 \dot{g}_t,
\end{align*}
\]

along the Ω-SRF (see [Pal2]) and we remind the uniform estimates

\[
|\dot{g}_t|_{g_t} \leq |\dot{g}_0|_{C^0(X),g_0} e^{-\delta t/2},
\]

\[
e^{-C} g_0 \leq g_t \leq e^C g_0,
\]

proved in [Pal2]. We consider also the estimate of the norm

\[
|\dot{J}_t|_{g_t} \leq \sqrt{2n} |J_t|_{g_t} |\dot{g}_t|_{g_t},
\]

where the constant $\sqrt{2n}$ comes from the equivalence between the Riemannian norm and the operator norm on the space of endomorphisms of $T_{X,x}$. We observe now the trivial identities

\[
|J_t|_{g_t}^2 = \text{Tr}_n \left[ J_t (J_t)^T_{g_t} \right] = - \text{Tr}_n J_t^2 = \text{Tr}_n \|_{T_{X,x}} = 2n.
\]

We deduce the exponential estimate of the variation of the complex structure

\[
|\dot{J}_t|_{g_t} \leq 2n |\dot{g}_0|_{C^0(X),g_0} e^{-\delta t/2},
\]

and thus the convergence of the integral

\[
\int_0^{+\infty} |\dot{J}_t|_{g_0} dt < + \infty.
\]

In its turn this shows the existence of the integral

\[
J_\infty := J_0 + \int_0^{+\infty} \dot{J}_t dt,
\]

thanks to Bochner’s theorem. Moreover hold the exponential estimate

\[
|J_\infty - J_t|_{g_0} \leq \int_t^{+\infty} |\dot{J}_s|_{g_0} ds \leq C' e^{-\delta t/2}.
\]

On the other hand the Kähler identity $\nabla_{g_t} J_t \equiv 0$ implies the equality

\[
2 \nabla_{g_t}^p \dot{J}_t = [J_t, \nabla_{g_t}^p \dot{g}_t^*],
\]

where $p$ denotes the complex conjugate of $p$. This completes the proof of the smooth convergence statement.
for all $p \in \mathbb{N}$. We deduce the estimates
\[
|\nabla_{g_t} J_t|_{g_t} \leq \sqrt{2n} |J_t|_{g_t} |\nabla_{g_t} g_t|_{g_t} \leq 2n C_p e^{-\varepsilon_p t},
\]
thanks to the exponential decay of the evolving Riemannian metrics proved in [Pal2]. The fact that the flow of Riemannian metrics $(g_t)_{t \geq 0}$ is uniformly bounded in time for any $C^p(X)$-norm implies the uniform estimate
\[
|\nabla_{g_0} J_t|_{g_0} \leq C'_p e^{-\varepsilon_p t}.
\]
We infer the convergence of the integral
\[
\int_0^{+\infty} |\nabla_{g_0} J_t|_{g_0} dt < +\infty,
\]
and thus the existence of the integral
\[
I_p := \nabla_{g_0} J_0 + \int_0^{+\infty} \nabla_{g_0} J_t dt.
\]
We deduce the exponential estimate
\[
|I_p - \nabla_{g_0} J_t|_{g_0} \leq \int_t^{+\infty} |\nabla_{g_0} J_s|_{g_0} ds \leq C'_p e^{-\varepsilon_p t}.
\]
A basic calculus fact combined with an induction on $p$ implies $I_p = \nabla_{g_0} J_\infty$. We deduce that $(J_\infty, g_\infty)$ is a Kähler structure. Then the convergence result in [Pal2] implies that $g_\infty$ is a $J_\infty$-invariant KRS.

6 Appendix. Basic differential identities

The results explained in this appendix are well known. We include them here for readers convenience.

**Lemma 4.** Let $M$ be a differentiable manifold and let
\[
(\xi_t)_{t \geq 0} \subset C^\infty(M, T^*_M), \quad (\alpha_t)_{t \geq 0} \subset C^\infty(M, (T^*_M)^{\otimes p} \otimes T^*_M),
\]
be smooth families and let $(\Phi_t)_{t \geq 0}$ be the flow of diffeomorphisms induced by the family $(\xi_t)_{t \geq 0}$, i.e.
\[
\frac{d}{dt} \Phi_t = \xi_t \circ \Phi_t, \quad \Phi_0 = \text{Id}_M.
\]
Then hold the derivation formula
\[
\frac{d}{dt} (\Phi_t^* \alpha_t) = \Phi_t^* \left( \frac{d}{dt} \alpha_t + L_{\xi_t} \alpha_t \right),
\]

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Proof. We prove first the particular case
\[
\frac{d}{dt}_{|t=0} (\Phi_t^* \alpha) = L_{\xi_0} \alpha , \tag{6.1}
\]
where \( \alpha \) is \( t \)-independent. For this purpose we consider the 1-parameter sub-

group of diffeomorphisms \((\Psi_t)_{t \geq 0}\) induced by \( \xi_0 \), i.e
\[
\frac{d}{dt} \Psi_t = \xi_0 \circ \Psi_t, \quad \Psi_0 = \text{Id}_M.
\]
Let \( \hat{\Psi} : \mathbb{R}_{\geq 0} \times M \to M \) given by \( \hat{\Psi}(t, x) = \Psi_t^{-1}(x) \) and observe the equalities
\[
\frac{d}{dt}_{|t=0} \Psi_t^{-1} = -\xi_0 = \frac{d}{dt}_{|t=0} \Phi_t^{-1} , \tag{6.2}
\]
We will note by \( \partial \) the partial derivatives of the coefficients of the tensors with
respect to a trivialization of the tangent bundle over an open set \( U \subset M \). Let
\( v \in T_{M,x}^{O} \). Then
\[
(L_{\xi_0} \alpha) \cdot v = \left( \frac{d}{dt}_{|t=0} (\Psi_t^*) \alpha \right) \cdot v \\
= \frac{d}{dt}_{|t=0} \left[ (d\Psi_t^{-1})^r \cdot (\alpha \circ \Psi_t) \cdot (d\Psi_t)^p \cdot v \right] \\
= \left[ \frac{d}{dt}_{|t=0} (\partial_x \hat{\Psi})^r(t, \Psi_t(x)) \right] \cdot \alpha \cdot v \\
+ \frac{d}{dt}_{|t=0} \left[ (\alpha \circ \Psi_t) \cdot (d\Psi_t)^p \cdot v \right] \\
= (\partial_x \hat{\Psi})^r(0, x) \cdot \alpha \cdot v + \left( (\partial^2_x \hat{\Psi})^r(0, x) \cdot \xi_0^p(x) \right) \cdot \alpha \cdot v \\
+ \left( \frac{d}{dt}_{|t=0} \alpha(\Psi_t(x)) \right) \cdot v + \alpha(x) \cdot (\partial_t \partial_x \Psi)^p(0, x) \cdot v \\
= - (\partial_x \xi_0)^r(x) \cdot \alpha \cdot v \\
+ (\partial_x \alpha(x) \cdot v) \cdot \xi_0(x) + \alpha(x) \cdot (\partial_x \xi_0)^p(x) \cdot v ,
\]

since the map
\[
(\partial^2_x \hat{\Psi})^r(0, x) : S^2 T^r_U \to T^r_U,
\]
is zero. Observe in fact the identity
\[
\partial_x \Psi_t^{-1} = \text{Id}_{T^r_U}.
\]
Moreover the same computation and conclusion work for $\Phi_t$ thanks to (6.2). We infer the identity (6.1). We prove now the general case. We expand the time derivative

$$
\frac{d}{dt} (\Phi_t^* \alpha_t) = \frac{d}{ds} \big|_{s=0} \Phi_{t+s}^* \alpha_{t+s}
$$

$$
= \Phi_t^* \left( \frac{d}{dt} \alpha_t \right) + \frac{d}{ds} \big|_{s=0} \Phi_{t+s}^* \alpha_t
$$

$$
= \Phi_t^* \left( \frac{d}{dt} \alpha_t \right) + \frac{d}{ds} \big|_{s=0} (\Phi_t^{-1} \Phi_{t+s})^* \Phi_t^* \alpha_t.
$$

We set $\Phi_s^t := \Phi_t^{-1} \Phi_{t+s}$ and we observe the equalities

$$
\frac{d}{ds} \big|_{s=0} \Phi_s^t = \frac{d}{ds} \big|_{s=0} \Phi_{t+s}
$$

$$
= \frac{d}{ds} \big|_{s=0} (\Phi_t^{-1} \Phi_s)
$$

$$
= \Phi_t^* \xi_t.
$$

Then the identity (6.1) applied to the family $(\Phi_s^t)_s$ implies

$$
\frac{d}{dt} (\Phi_t^* \alpha_t) = \Phi_t^* \left( \frac{d}{dt} \alpha_t \right) + L_{\xi_t} \Phi_t^* \alpha_t
$$

$$
= \Phi_t^* \left( \frac{d}{dt} \alpha_t + L_{\xi_t} \alpha_t \right).
$$

Lemma 5. Let $(X, J)$ be an almost complex manifold and let $N_j$ be the Nijenhuis tensor. Then for any $\xi \in C^\infty(X, T_X)$ hold the identity

$$
L_{\xi} J = 2 J \left( \bar{\partial}_{\tau X,j} \xi - \xi \circ N_j \right).
$$

Proof. Let $\eta \in C^\infty(X, T_X)$. Then

$$
\bar{\partial}_{\tau X,j} \xi(\eta) = [\eta^{0,1}, \xi^{1,0}]^{1,0} + [\eta^{1,0}, \xi^{0,1}]^{0,1},
$$

$$
N_j(\xi, \eta) = [\xi^{1,0}, \eta^{0,1}]^{0,1} + [\xi^{0,1}, \eta^{1,0}]^{1,0},
$$

and the conclusion follows by decomposing in type $(1,0)$ and $(0,1)$ the identity

$$
(L_{\xi} J) \eta = [\xi, J \eta] - J[\xi, \eta].
$$
We observe now that if \((X, J, \omega)\) is a Kähler manifold and \(u \in C^\infty(X, \mathbb{R})\), then hold the identities
\[
\nabla_\omega u \cdot \omega = - (du) \cdot J = - i \partial J u + i \bar{\partial} J u ,
\]
and
\[
L_{\nabla_\omega u} \omega = d (\nabla_\omega u \cdot \omega) = 2 i \partial J \bar{\partial} J u . \quad (6.4)
\]

**Lemma 6.** Let \((X, J, g)\) be a Kähler manifold and let \(u \in C^\infty(X, \mathbb{R})\). Then hold the decomposition formula
\[
\nabla_g d u = i \partial J \bar{\partial} J u \left( \cdot, J \cdot \right) + g \left( \cdot, \bar{\partial}_{TX,J} \nabla_g u \cdot \right) . \quad (6.5)
\]

**Proof.** Let \(\xi, \eta, \mu \in C^\infty(X, T_X)\). By definition of Lie derivative hold the identity
\[
\xi \cdot g(\eta, \mu) = (L_\xi g)(\eta, \mu) + g(L_\xi \eta, \mu) + g(\eta, L_\xi \mu) .
\]

Let \(\omega := g(J, \cdot)\) be the induced Kähler form. Then by using again the definition of Lie derivative we infer the equalities
\[
\xi \cdot g(\eta, \mu) = \xi \cdot \omega(\eta, J \mu)
\]
\[
= (L_\xi \omega)(\eta, J \mu) + \omega(L_\xi \eta, J \mu) + \omega(\eta, L_\xi (J \mu))
\]
\[
= (L_\xi \omega)(\eta, J \mu) + g(L_\xi \eta, \mu) + \omega(\eta, (L_\xi J) \mu) + g(\eta, L_\xi \mu) .
\]

We deduce the identity
\[
L_\xi g = L_\xi \omega \left( \cdot, J \cdot \right) + \omega \left( \cdot, L_\xi J \cdot \right) .
\]

We apply this identity to the vector field \(\xi := \nabla_g u\). Then the conclusion follows from the identity
\[
L_{\nabla_g u} g = 2 \nabla_g d u ,
\]
combined with (6.4), and (6.3).

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