First passage leapovers of Lévy flights and the proper formulation of absorbing boundary conditions

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Abstract
An important open problem in the theory of Lévy flights concerns the analytically tractable formulation of absorbing boundary conditions. Although the nonlocal approach, where the dynamical operators are given restrictions on their domain properties, has yielded substantial insights regarding the statistics of first passage, the resultant modifications to the dynamical equations hinder the detailed analysis possible in the absence of these nonlocal restrictions. In this study it is demonstrated that using the first-hit distribution, related to the first passage leapover, as the absorbing sink preserves the tractability of the dynamical equations for a particle undergoing Lévy flight. In particular, knowledge of the first-hit distribution is sufficient to fully determine the first passage time and position density of the particle, without requiring operator domain restrictions or numerical simulations. In addition, we report on the first-hit and leapover properties of first passages and arrivals for Lévy flights of arbitrary skew parameter, and extend these results to Lévy flights in a certain ubiquitous class of potentials satisfying an integral condition.

Keywords: Lévy flights, first passage, leapovers, stochastic analysis

(Some figures may appear in colour only in the online journal)

1. Introduction

A growing body of work has uncovered the ubiquity of anomalous diffusion in natural phenomena [1]. Characterised by long-range correlations in space and time, anomalous diffusive processes distinguish themselves from classical Brownian diffusion by a nonlinear time dependence of the mean squared displacement \( \langle x^2(t) \rangle \propto t^\gamma \), where \( \gamma \neq 1 \), caused by the breakdown of the central limit theorem [2]. In the continuous time random walk description, where
the length of a jump along with the waiting time between two consecutive jumps is jointly described by a probability distribution, the case of a finite mean waiting time and an infinite jump length variance in the continuous long-term limit corresponds to Lévy flight, an example of Markovian (memoryless) superdiffusion ($\gamma > 1$).

Lévy flights in the absence of boundaries are succinctly described in the presence of an arbitrary external potential by the fractional Fokker–Planck equation (FFPE) [3]. Also known as the Kolmogorov forward equation or the Smoluchowski equation, the FFPE is a fractional partial differential equation on the position probability density. However, many physical, chemical, biological, astronomical, geological and meterological situations [1–3] are concerned with the passage of these dynamical processes across a boundary. Unlike classical Brownian motion, the problem of first passage for Lévy flights encounters a number of intricacies due to the discontinuous nature of its sample paths, or trajectories, leading to leapovers over the boundary at the time of first passage. The existence of leapovers leads to the nontriviality of two related distributions: the distribution of first hits, which is the distribution of locations visited by the process immediately upon first passage out of a given region $\Omega$; and the first passage leapover density (FPLD), which is the distribution of distances between the first hit location and the point crossed on the boundary $\partial \Omega$ at the time of first passage. As a result, leapovers are determined from first-hit distributions if and only if it is known which point on the boundary was crossed at the time of first passage; for example, for processes in the half-line, the distribution of first hits is equivalent to the first passage leapover density. Early progress was made in obtaining the distribution of first hits, also known as the harmonic measure, out of both finite and infinite regions in the 1960s [4, 5] and 1970s [6], which was recently revisited in the mathematical [7, 8] and physical [9] context. So far, however, FPLDs of Lévy flights have only been investigated for the symmetric and one-sided cases in the absence of a potential well. Other Markovian pseudoprocesses with discontinuous paths have also been shown to display non-trivial first passage properties, with multipoles occurring in their first hit distributions [10, 11]. That this causes issues for the derivation of first passage times is ultimately unsurprising, as detailed knowledge of the trajectory of the underlying random process is required to uniquely determine the first passage time density (FPTD) [12], explaining the failure of the method of images which had previously succeeded in describing the first passage properties of Brownian motion [13].

In order to determine the properties of first passage for Lévy flights, leaving aside numerical simulations [14–16] and discretisation [17], a number of techniques have hitherto been used in order to analytically account for the required absorbing boundary conditions. It has been known for some time that an analytical formulation of the absorbing boundary condition in the FFPE is consistently phrased in terms of restrictions on domain properties of the fractional Laplacian operator [18], such as the truncation of its integral form [13]. Using the Sonin inversion formula, this technique yielded an analytical form for average properties of symmetric Lévy flights such as the mean first passage time (MFPT) [19, 20]. However, due to the lack of a convenient representation of this nonlocal integral operator in Fourier space, its computational intractability prevented this formulation from being used to determine the position density of the particle [13] or any details about the FPTD on the half-line apart from the universal Sparre–Andersen scaling of its tail [21, 22]. Nevertheless, it is still considered the standard formulation of the absorbing boundary condition in the FFPE [23, 24]. By using a description of the joint FPTD-FPLD in Laplace space [25] the first passage of one-sided Lévy motions on the half-line were able to be analytically described. Due to a theorem by Skorohod [26], the prefactor of the tail of the FPTD for symmetric Lévy flights in the half line was found along with its Sparre–Andersen scaling and the full FPLD [27, 28]. More recently, these FPTD tail prefactor results were extended to Lévy flights of arbitrary skew parameter on
the half-line using the same theorem [29]. Perturbation schemes, inspired by quantum mechanics, have also been investigated around the Brownian solution in order to correctly predict the leading and subleading behaviour of the tail of the position density for symmetric Lévy flights on the half-line [14, 17, 30]. Nevertheless, the correct and analytically tractable formulation of absorbing boundary conditions for general Lévy flights in arbitrary regions, such that the FPTD and position density be analytically determined, remains an open problem.

In this paper, we show that using the first-hit distribution as the absorbing sink term in the FFPE correctly and efficiently formulates the absorbing boundary condition for general Markov processes in arbitrary regions, without modifying the key dynamical operators of the equation. Given an arbitrary source \( f(x, t) \) extended in space and time, we construct a canonical absorbing sink term \( S(x, t) = f(x, t) \) which respects the absorbing boundary condition, and depends only on the FPTD and first-hit distribution of the process along with the source. We obtain an equation for the MFPT of a general Markov process out of an arbitrary region using only the free propagator and the first-hit distribution. In the case of free Lévy flights of arbitrary skew parameter \(-1 \leq \beta \leq 1\), we express the FPTD and position density of the particle in an exact fashion in Laplace space, dependent only on the free propagator and the first-hit distribution. The exact position density expression (29), valid for arbitrary regions and Markov processes with time-independent first-hit distributions, solves the long-standing open problem of the analytical derivation of the position density in Lévy flights with absorbing boundaries [13, 14, 30]. We unify the notions of first passage and first arrival by considering the arrival to a domain as equivalent to the passage out of its complement. This allows us to use the same expressions to find first-arrival densities to arbitrary (not necessarily point-like) regions along with the corresponding position densities. We apply the results to a number of interesting well-known test cases in order to demonstrate the power of this novel framework, and obtain, as an example, the first known analytical expression for the MFPT (45) of Lévy flight with arbitrary skew out of a finite interval. To do this, we review recent mathematical results on first hits for Lévy flights of arbitrary skew and extend them to Lévy flights on the half-line in external potentials termed boundary-centred potentials.

The remainder of the paper is organised as follows. In section 2 we report on the first-hit properties of free Lévy flights of arbitrary skew, focussing on escapes from and arrivals to intervals of half-width \( L \). In section 3 we use these results to investigate the leapover properties of Lévy flights on the half-line of arbitrary skew, in free and boundary-centred potentials, such as the harmonic potential well with the boundary at its local extremum. In section 4 we introduce the general framework outlined above. We discuss the implications of this novel framework in section 5 and compare it in more detail to previous work in the area.

For clarity of expression, Fourier transforms \( \langle e^{ikx} \rangle \) of spatial variables \( x(t) \) are indicated by the explicit use of the variable \( k \), and Laplace transforms of temporal variables \( t \) are indicated by usage of the variable \( s \). We also introduce a rescaled version of the skew parameter \( \beta \),

\[
\beta' := \frac{2}{\pi} \arctan \left( \beta \tan \frac{\pi \alpha}{2} \right)
\]

which is related to the positivity parameter [31, 32] \( \rho \) by the identity \( 2\alpha \rho = \alpha + \beta' \).

2. First arrival and passage of free Lévy flights in finite domains

In this section, we employ a number of results in the mathematical literature to explore spatial properties of one-dimensional Lévy flights of arbitrary skew \(-1 \leq \beta \leq 1\). For easy comparison with previous studies in a physics context, we reproduce the situation set up in [9] for the
symmetric case $\beta = 0$. This situation necessitates the rescaling of those mathematical results, the process of which is described in appendix A. As a result, the results obtained in this section are essentially based on the distributions of first hit obtained in [6–8]; these distributions are used to generalise the results in [9]. We use these results to obtain the first-hit distribution (2) corresponding to the first passage out of $[0, 2L]$, which is equivalent to the first arrival to $\mathbb{R}\setminus [0, 2L] = (-\infty, 0) \cup (2L, \infty)$; along with the first-hit distribution (6) for the first arrival to $[-2L, 0]$, equivalent to the first passage out of $(-\infty, -2L) \cup (0, \infty)$.

The formal setup is as follows [4–8]. Let $X = \{X(t)\mid t \geq 0\}$ be an $\alpha$-stable Lévy process satisfying the stochastic differential equation $dX = dL(\alpha, \beta, \gamma, D)$, where $0 < \alpha \leq 2$ is the Lévy stable index, $-1 \leq \beta \leq 1$ is the skew parameter, $\gamma \in \mathbb{R}$ is the centre, and $D > 0$ is the scale parameter. The classical Gaussian process is recovered when $\alpha = 2$. Since the skew parameter $\beta$ has a slightly different meaning when $\alpha = 1$, we consider the case $\beta = 0$ whenever $\alpha = 1$ for the sake of simplicity, along with $\gamma = 0$ for all $\alpha$. We denote by $\mathbb{P}_{x_0}$ the law of $X$ starting from $x_0$, and the time of first passage out of the region $\Omega$ by $T_\Omega := \inf\{t \geq 0\mid X(t) \notin \Omega\}$. The first-hit distribution $q(x, t) := \mathbb{P}_{x_0}[X(T_\Omega) \in dx \mid T_\Omega \in dt]$, or its marginalised version, the harmonic measure $q(x) := \int_{t=0}^\infty q(x, t)\mathbb{P}_{x_0}[T_\Omega \in dt]$, describes the relative probabilities of points visited at the time of first passage out of $\Omega$.\footnote{Although the results in this section describe the time-marginalised first-hit distribution, the results in section 4.2 for Markov processes with time-independent first-hit distributions remain valid when the relaxation of the first-hit distribution to the time-marginalised case is sufficiently brief (checked empirically for $1 < \alpha < 2$).}

As a result, its support is contained in $\mathbb{R}\setminus \Omega$.

### 2.1. Escape from a finite interval and half-line

We begin by considering a particle undergoing free Lévy flight starting at the position $x_0 \in [0, 2L] = \Omega$. The distribution of first hits upon the escape of the particle out of the interval $[0, 2L]$ has (normalised) density (see (A.1))

$$
q(x\mid x_0) = \frac{\sin \left(\frac{\pi \alpha \text{sgn}(x-x_0)}{2}\right)}{\pi} \frac{x-x_0}{|x-x_0|} \begin{cases} 1 & \text{if } x \in (0, 2L) = \text{Int}(\Omega) \\ 0 & \text{otherwise} \end{cases}
$$

when $x \in (-\infty, 0] \cup [2L, \infty) = \mathbb{R}\setminus \Omega$, and $q(x\mid x_0) = 0$ when $x \in (0, 2L) = \text{Int}(\Omega)$. Here, the rescaled skew parameter $\beta'$ is used, defined above in (1). In the Gaussian limiting case $\alpha \to 2$ the density $q(x\mid x_0) = (1-x_0/2L)\delta(x) + (x_0/2L)\delta(2L-x)$ is a sum of delta functions at either end of the allowed region, reflecting the continuous sample paths of the Gaussian process. The weightings of these delta functions represent the probabilities that the particle will hit the respective ends of the boundary first, which in the Gaussian case $\alpha = 2$ is linear in the initial position $x_0$.

This quantity can be used to characterise the first passage leapover length $l \geq 0$, the distance between the crossed boundary and the first hit point, with normalised density $f(l\mid x_0) = q(-l\mid x_0) + q(2L+l\mid x_0)$. This characterisation was rigorously derived in [9, equation (11)] to obtain leapover lengths for the symmetric case $\beta = 0$. Applying this to (2) yields the FPLD

$$
f(l\mid x_0) = \frac{\sin \left(\frac{\pi \alpha \beta}{2}\right)}{\pi} \frac{|x_0|}{l} \begin{cases} 1 & \text{if } \frac{|l-x_0|}{2L} \leq 1 \\ 0 & \text{otherwise} \end{cases}
$$

\[ \begin{align*}
&+ \frac{\sin \left(\frac{\pi (\alpha + \beta)}{2}\right)}{\pi} \frac{x_0}{2L + l} \begin{cases} 1 & \text{if } \frac{|2L - x_0|}{|l+x_0|} \leq 1 \\ 0 & \text{otherwise} \end{cases}
\end{align*}\]
when \( l \geq 0 \), and \( f(l|x_0) = 0 \) when \( l < 0 \). The FPLD decays as \( l^{-(1+\alpha)} \) for all finite half-widths \( L \), independent of the skew \( \beta \). In the half-line limit \( L \to \infty \) (whereby the particle undergoes escape from \( \Omega = [0, \infty) \)), the notion of leapovers becomes equivalent to that of the first hit, and so

\[
f(l|x_0) = q(-l|x_0) = \frac{\sin \left( \frac{\pi (\alpha-x')}{2} \right)}{\pi} \left| \frac{x_0 + x'}{l} \right|^{\alpha-1} \frac{1}{|l + x_0|} \tag{4}
\]

where the FPLD in this limit decays as \( l^{-(1+(-\beta')/2)} \), dependent on \( \beta \). This result is exhibited more generally as an interpolation in the asymptotics of the FPLD with respect to the half-width \( L \):

\[
f(l|x_0) \propto \begin{cases} l^{-(1+(-\beta')/2)} & \text{for } x_0 \ll l \ll L, \\ l^{-(1+\alpha)} & \text{for } x_0 \ll L \ll l. \end{cases} \tag{5}
\]

As a result, the skew parameter \( \beta \) is only important in the intermediate asymptotics of the FPLD, when \( L \) is large compared to \( l \). This can be explained by the observation that when \( L \) is small compared to the leapover length \( l \), jumps of the process in either direction contribute indiscriminately to the FPLD, whereas in the case of large half-width \( L \) relative to \( l \), only those jumps of the process in the negative direction are directly responsible for the FPLD, breaking the directional symmetry and establishing the effect of the rescaled skew \( \beta' \).

### 2.2. Arrival to a finite interval and half-line

We now investigate the first-hit properties of particles arriving at \([-2L, 0]\) outside the interval, which has a number of important differences from the previous case. This corresponds to the first passage out of \( \Omega := \mathbb{R} \setminus [-2L, 0] = (-\infty, -2L) \cup (0, \infty) \). The first-hit distribution for such a particle starting at \( x_0 \in \Omega \) has density (see (A.2))

\[
q(x|x_0) = \frac{\sin \left( \frac{\pi (\alpha-x')}{2} \right)}{\pi (2L + x)^{\alpha-1} |x|^\alpha} \left( \frac{2L + x_0 + x'}{x - x_0} \right) \left( \frac{2L + x_0 + x'}{|x - x_0|} \right)^{\alpha-1} \frac{1}{|x - x_0|}
\]

when \( x \in [-2L, 0] \), and \( q(x|x_0) = 0 \) when \( x \in \Omega \). Because the \( \alpha \)-stable motion \( X(t) \) is transient when \( 0 < \alpha < 1 \), this density is only normalised when \( 1 \leq \alpha \leq 2 \), and so the probability of not hitting the finite first arrival interval \([-2L, 0]\) is nonzero when \( 0 < \alpha < 1 \): (see (A.3))

\[
R(x_0) = \frac{\Gamma \left( 1 - \frac{\alpha + \text{sgn}(x_0 + L)x'}{2} \right)}{\Gamma \left( \frac{\alpha + \text{sgn}(x_0 + L)x'}{2} \right) \Gamma(1 - \alpha)} \int_0^{(x_0 + L)x'} \frac{a^{\alpha - 1}}{1 - a} da.
\tag{7}
\]

When \( L \to \infty \), \( R(x_0) \to 0 \), since in this case we recover the first arrival to the negative half-line, which is equivalent to the first passage out of the positive half-line examined in the previous subsection. On the other hand, when \( L \to 0 \), the classical problem of first arrival to a point is reconstructed and the point is almost certainly not hit by the particle: \( R(x_0) \to 1 \).

The disconnectedness of the allowed region \( \Omega \) has an important consequence for the computation of the finitely supported leapover density. Since the particle can jump over the entire
[−2L, 0] interval without landing inside it, the first-hit distribution does not provide sufficient information on which boundary was crossed between the first hit point and the particle’s location immediately before the first hit point. As a result, the FPLD cannot always be computed from the first-hit distribution alone.

3. Leapover lengths for \( \alpha \)-stable processes with boundary-centred potentials on the half-line

Another scenario of interest in characterising the first-passage properties of Lévy flights (e.g. [27, 28] in the symmetric case \( \beta = 0 \)) consists of free Lévy flight out of the half-line. By using the results reported in the previous section, it is possible to generalise the results on leapover lengths in this scenario to the case of arbitrary skew \( -1 \leq \beta \leq 1 \). Remarkably, using a variable transformation, these results also yield FPLDs for general Lévy flights on the half-line in a wide class of external potentials characterised by the boundary residing at a natural centre of the potential function, which we term boundary-centred potentials. As mentioned in the Introduction, the notion of leapover lengths in the case of the half-line becomes equivalent to that of first hits, and distributions of each in this case can be obtained from the other.

3.1. Leapover lengths for free \( \alpha \)-stable processes

We first reproduce the setup from [27] for the symmetric case \( \beta = 0 \) and extend it to the case of arbitrary skew \( -1 \leq \beta \leq 1 \). Note that the extension of the FPTD in this scenario is covered in section 4 and given by (30). A particle undergoing free Lévy flight starting at the position \( x_0 = 0 \) escapes from the region \( (-\infty, d) \), and we are interested in its FPLD. Reflecting the space axis, we obtain the escape from \( (-d, \infty) \), where \( x_0 = 0 \) (equivalent to the escape from \( (0, \infty) \) where \( x_0 = d \) by translational symmetry of the free Lévy process), allowing us to use the results from the previous section. However, due to the reflection of the space axis, the sign of the skew parameter \( \beta \) must be flipped when adapting the expressions. Adapting (4) we obtain

\[
p_d(l) = \frac{\sin \left( \pi \frac{\alpha + \beta'}{2} \right)}{\pi} \frac{d^{-\frac{\alpha + \beta'}{2}}}{l^{\frac{\alpha + \beta'}{2}} (d + l)} \tag{8}
\]

which applies for all \( 0 < \alpha \leq 2 \) and \( -1 \leq \beta \leq 1 \). This expression is consistent with classical results for the symmetric \( (0 < \alpha \leq 2, \beta = 0) \) [27, equation (12)] and one-sided \( (0 < \alpha < 1, \beta = 1) \) [27, equation (26)] [25, equation (43)] cases, and uncovers a novel one-sided case for \( 1 < \alpha < 2 \), with the exponent \( (\alpha + \beta')/2 = \alpha - 1 \); that is, the first-passage leapovers on the half-line for a one-sided Lévy process with index \( 1 < \alpha < 2 \) are equivalent to those for a one-sided Lévy process with index \( \alpha - 1 \).

This FPLD (8) can be equally obtained from the first-hit distribution (2), translated and reflected, in the limit \( L \to \infty \).

3.2. Leapover lengths for Lévy flights in a boundary-centred harmonic potential well

Before proceeding to the general variable transformation technique, we demonstrate it for the case of the harmonic potential well in which the restoring force is linear, which has been of particular interest in Lévy flights [3, 33]. We find that when the boundary lies at the extremum of the well, the FPLD (12) can be obtained by transformation of (8). As before we are interested
Figure 1. Lévy flights in a boundary-centred harmonic potential well. (a) Schematic describing the setup. A particle undergoes Lévy flight in the presence of a harmonic potential $V(x) \propto (x - d)^2$ starting at $x = 0$, and crosses the boundary point $x = d$ at the first passage time $t_{FP}$. The probability that the particle overshoots the boundary by a distance $l$ is given by the leapover density $f_d(l, t_{FP})$. (b) The FPLD $fd(l, t_{FP})$ with $(\alpha, \beta, \gamma, D) = (3/2, 0, 0, 1)$ and $(d, \mu_1, x_0) = (1, -1, 0)$ varies as a function of the first passage time $t$, plotted here for $t = 1, 2, 3, 4$ (blue, yellow, green, red lines). The density approaches a delta function as $t$ increases, in agreement with numerical simulations (squares) of $10^5$ flights with $dt = 10^{-3}$, and constant scale terms $\sigma$ randomly chosen for each flight according to a power law between $10^{-3}$ and $10^0$.

in the escape from $(-\infty, d)$ for a process $X$ starting at $X_0 = 0$, but with stochastic differential equation

$$dX = (\mu_1 X + \mu_2)dt + \sigma(X, t)dL(\alpha, \beta, 0, D)$$

(9)

describing a form of Lévy Ornstein–Uhlenbeck process [33] (see figure 1(a)). For simplicity we assume that $\sigma(X, t)$ is positive everywhere. Denote the first passage time $t_{FP} := T_{(-\infty, d)} = \inf\{t|X(t) \geq d\}$. We consider the transformation defined by the function

$$Y(X, t) := \left(X + \frac{\mu_2}{\mu_1}\right)e^{-\mu_1 t}.$$  

(10)

This function has the property $dY = e^{-\mu_1 t}\sigma(X, t)dL$, so $Y$ is a free Lévy process with the time-dependent scale parameter remaining positive everywhere, with initial condition $Y(X_0, 0) = \frac{\mu_2}{\mu_1}$ and boundary $Y(d, t_{FP}) = \left(d + \frac{\mu_2}{\mu_1}\right)e^{-\mu_1 t_{FP}}$. This boundary depends on the first passage time $t_{FP}$, which is itself a random variable, and so the boundary is in general time-dependent. Time-dependent boundaries have been considered within the framework of stopping problems [34], but determining their leapovers remains an open problem. We note that this problem is equivalent to having a constant boundary and time-dependent drift term; this can be seen by using the transformation $Z(X, t) = (X - d)e^{-\mu_1 t}$ instead.

However, if $d = -\frac{\mu_2}{\mu_1}$, that is, the boundary of $X$ is at the centre of the harmonic potential, the boundary of $Y(X, t)$ is no longer time-dependent and resides at zero. Given that the leapover in the original spatial variable $X$ is $l_X = X(t_{FP}) - d$, the leapover in the transformed variable $Y$ is $l_Y = Y(X, t_{FP}) - Y(d, t_{FP}) = (X - d)e^{-\mu_1 t_{FP}} = l_X e^{-\mu_1 t_{FP}}$. As a result the FPLD for $Y$ is given by (8):
\[
p_{0}(l_{y})dl_{y} = \frac{\sin \left( \frac{\alpha + \beta}{2} \right)}{\pi} \frac{d\frac{\alpha + \beta}{2}}{\left\{ \frac{\alpha + \beta}{2} \right\}^{2} (d + l_{y})} \\
= \frac{\sin \left( \frac{\alpha + \beta}{2} \right)}{\pi} \frac{d\frac{\alpha + \beta}{2}}{(e^{-\mu l_{y}}l_{y})^{\frac{\alpha + \beta}{2}} (d + e^{-\mu l_{y}}l_{y})}.
\] (11)

Thus the FPLD of \( X \) is

\[
f_{0}(l, t) = \frac{\sin \left( \frac{\alpha + \beta}{2} \right)}{\pi} \frac{d\frac{\alpha + \beta}{2}}{l^{\frac{\alpha + \beta}{2}} (d + e^{-\mu t})} e^{-\mu t} \left( 1 - \frac{\alpha + \beta}{2} \right)
\] (12)

where \( t \) is the first passage time. The above measure-based transformation ensures that the FPLD remains normalised: \( \int_{0}^{\infty} f_{0}(l, t)dl = 1 \), regardless of the time at which first passage occurs. Importantly, this FPLD (12) is distinguished from the FPLD (8) in the absence of a potential by the dependence on the time of first passage, despite being normalised in space for any given time. The time-dependent form of (12) is confirmed by numerical simulations of (9) using the Euler–Maruyama method [9, 15], as shown in figure 1(b). This is a development over current understandings of the leapover density, which have assumed it to be always independent of time; we further explore the meaning of this in section 4.1.2 and the Discussion.

The physical behaviour of the FPLD (12) compared with the classical FPLD in the free case (8) can be understood by considering the physical dynamics of the steepness \( \mu \) of the potential well, which appears as a scaling factor on time \( t \), and vice versa. At the beginning of the flight \( t = 0 \) (or equivalently, as the steepness \( \mu \rightarrow 0 \)), the Lévy flight behaves approximately as a free flight in the absence of the potential well. As time passes, large leapovers are suppressed in favour of smaller leapovers; the larger the steepness \( \mu \) of the well, the faster the FPLD (12) approaches its end state \( f_{0}(l, t) \sim \delta(t) \) as \( t \rightarrow \infty \). In this end state, the process resides close to the boundary at \( x = d \), with infinitesimally small leapovers dominating the presence of any larger leapovers. This tendency is reminiscent of ageing effects arising in modified continuous time random walks [35].

This constitutes the first known leapover result for Lévy flights in nontrivial potential wells.

3.3. Leapover lengths for a general class of boundary-centred potentials

The above technique can be generalised to potentials which satisfy a certain integral condition. Here we consider a Lévy-driven process \( X \) satisfying

\[
dX = \mu(X) dt + \sigma(X, t)dL(\alpha, \beta, 0, D)
\] (13)

which begins at \( X_{0} = 0 \) and has boundary \( d \). Furthermore, we consider potentials \( V(x) \) where the force \( -V(x) = \mu(x) \) satisfies \( \int_{0}^{d} \frac{dx}{\mu(x)} = \pm \infty \) only when \( X = d \), and where the function \( X \mapsto \int_{0}^{X} \frac{dx}{\mu(x)} \) is (independently) injective on either side of the boundary. As a result, there exists a constant \( r \) such that

\[
r \int_{0}^{d} \frac{dx}{\mu(x)} = -\infty.
\] (14)

Observe that the boundary-centring condition (14) is a stronger version of the stationary point condition \( -V(d) = \mu(d) = 0 \) of the potential function: the first implies the second,
but the reverse is not necessarily true. Let us consider the transformation \(Y(X, t) = b(X) e^{-rt}\) for some function \(b\) to be determined. The transformed starting position is time-independent: \(Y(x_0, 0) = b(0)\). The corresponding stochastic differential equation for the transformed variable is
\[
dY = e^{-rt}(b'(X)\mu(X) - rb(X))dt + \sigma(X, t)b'(X)e^{-rt}dL.
\]
To remove the drift term we impose the condition \(b'(X)\mu(X) - rb(X) = 0\), yielding
\[
|b(X)| = |b(0)|e^{\int_0^X \frac{dY}{|b|}}.
\]
The transformed boundary is at \(Y(d, t_{FP}) = b(d)e^{-r t_{FP}}\), so to ensure the boundary is zero (and thus non-moving) we require \(b(d) = 0\), which is satisfied by the boundary-centring condition (14).

To guarantee that the boundary is properly retained in the transformed variable \(Y\), the function \(b\) needs to have differing sign on either side of the boundary point. This is possible if and only if (14) is satisfied for only the boundary point \(X = d\), for if this equation were satisfied for a second point \(X = d' \neq d\), then \(b(d) = b(d') = 0\) and so \(Y(d, t) = Y(d', t) = 0\), making the boundary indistinguishable from \(d'\) in the transformed space. The injectivity condition ensures that each point in the original space is uniquely represented in the transformed space, eliminating the possibility of artificial unwanted ‘teleports’ between points in nonsingleton preimages of \(Y\) with respect to \(X\).

To obtain the FPLD, we note that the leapover distance for \(Y\) is \(l_Y = |Y(X, t_{FP})| = |b(0)|e^{\int_0^{d+y} \frac{dX}{|b|}}\) and so \(dl_Y = \frac{|b(0)|}{\mu + \mu_2} dl_X\). Hence from (8)
\[
p_{b|b(0)}(l_Y)dl_Y = \frac{\sin \left( \frac{\pi \alpha + \beta'}{2} \right)}{\pi} \frac{|b(0)|^{\alpha + \beta'} dl_Y}{l_Y^\alpha (|b(0)| + l_Y)}
\]
\[
= \frac{\sin \left( \frac{\pi \alpha + \beta'}{2} \right)}{\pi} \frac{|b(0)|^{\alpha + \beta'} r}{(|b(0)| + l_Y) \mu (d + l_x)}\frac{1}{l_Y^{\alpha + \beta'}} dl_X
\]
\[
= \frac{\sin \left( \frac{\pi \alpha + \beta'}{2} \right)}{\pi} \frac{r}{\mu (d + l_x)} e^{\int_0^{d+y} \frac{dX}{|b|}} \left( \frac{1}{1 + e^{\int_0^{d+y} \frac{dX}{|b|}}} \right) dl_X.
\]

Therefore the FPLD of the original process \(X\) is
\[
f_{d}(l, t) = \frac{\sin \left( \frac{\pi \alpha + \beta'}{2} \right)}{\pi} \frac{r}{\mu (d + l)} e^{\int_0^{d+y} \frac{dX}{|b|}} \left( \frac{1}{1 + e^{\int_0^{d+y} \frac{dX}{|b|}}} \right).
\]

In all potentials from this class the dependence on the first passage time \(t\) is exponential, with growth or decay of the exponential terms depending on the shape of the potential, via the sign of \(r\). The dependence on the first passage leapover \(l\) is more ambiguous, and depends on the particular form of the force \(\mu(x)\); we present the FPLDs \(f_d(l, t)\) for a number of example forces \(\mu(x) = -V(x)\) in appendix B. The boundary-centred harmonic case (section 3.2) is recovered when setting \(\mu(x) = \mu_1 x + \mu_2\).
4. Fractional Fokker–Planck equation with absorbing boundaries

In this section we outline the general framework for constructing the absorbing boundary condition consistently and efficiently in the FFPE, without modifying the dynamical operators of the equation. Consider a Markov process \( \mathcal{M} \) with master equation \( \frac{\partial P}{\partial t} = \mathcal{A} P \), where \( \mathcal{A} \), which we call the dynamical operator, is the adjoint of the infinitesimal generator of \( \mathcal{M} \) [32]. The term \( P(x, t) \) can be understood as a position concentration density of a mass of particles driven by \( \mathcal{M} \); when \( P(x, t) \) is normalised over position \( x \) at a certain time \( t \), it can equivalently be understood as the position probability density of the process \( \mathcal{M} \). In the following, we will refer to \( P(x, t) \) in both cases as the position density.

We now impose an explicit modification of position density, independently of the action of the Markov process \( \mathcal{M} \), using an auxiliary term \( g(x, t) \), yielding the continuity equation

\[
\frac{\partial P(x, t)}{\partial t} = \mathcal{A} P(x, t) + g(x, t).
\]

The key to our framework arises from a reinterpretation of the role of the term \( g(x, t) \) when it is independent of \( P(x, t) \). This term represents an explicit effect on how the density \( P(x, t) \) changes at a specific point in time, and is commonly referred to as a source (sink) if its magnitude is positive (negative). For this reason, we refer to the term \( g(x, t) \) as the combined source–sink. Through the source–sink, one can encode the behaviour of a wide variety of physical effects on particles, including boundary conditions.

The most basic type of source–sinks encountered in such equations have the spatial form of a delta function, representing the injection, or removal, of position density at a single point, happening at a rate determined by its coefficient, which may be time-dependent. Another commonly seen example is a spatially extended (e.g. Gaussian) source representing the injection of a particle at a randomly selected point determined by the form of the source function, at a rate determined by the coefficient. Applying this principle in reverse, one finds that the effect of a spatially extended sink term is to randomly remove the particle with the relative probability of removal at a given point determined by the spatial form of the sink term, at a rate determined by its (possibly time-varying) coefficient. As a result, the effect of a sink with the spatial form of the first-hit distribution out of a region \( \Omega \), along with the FPTD as its coefficient, is to remove the particle as soon as it first exits the region \( \Omega \), consistently phrasing the first passage problem for Markov processes \( \mathcal{M} \). This is the central result of this study: the absorbing boundary condition is consistently formulated by the usage of the first-hit distribution multiplied by the FPTD as the source–sink term, without modifying the dynamical operator \( \mathcal{A} \). This allows the usage of standard methods to analytically solve these dynamical equations, which in the case of Lévy flights involves moving into Fourier–Laplace space, where the operator \( \mathcal{A} \) has a simple formulation. This is not possible when imposing domain restrictions, such as truncating the integral representation, on the operator \( \mathcal{A} \), as was classically required for the consistent phrasing of the absorbing boundary condition [13, 18].

This section is organised as follows. In section 4.1 we solve the FFPE (19) for general Markov processes \( \mathcal{M} \) and obtain the position density (22) given only the source–sink term and the free propagator. When the dynamical operator \( \mathcal{A} \) is a Fourier multiplier, the position density (25) can be expressed directly in Fourier–Laplace space with knowledge of only the source–sink, and when the source–sink term is separable (e.g. when the first-hit distribution is time-independent) the temporal distribution of the source–sink (28) (e.g. the first passage or first arrival time densities) can be expressed in Laplace space, along with the position density (29), using only the propagator and the spatial source–sink distribution (e.g. the corresponding first-hit distribution). In section 4.2 we use this formulation to consider the
absorbing boundary condition for a free Lévy flight with arbitrary skew $\beta$ out of an arbitrary region $\Omega$, and obtain the exact FPTD (30) and position density (32) in Laplace space using only the first-hit distribution and the propagator. Using this result, the classical test scenario of free symmetric Lévy flight out of the half-line [13, 14, 27, 30] is revisited, and the position density is analytically plotted for the first time. Returning to general Markov processes, in section 4.3 a canonical absorbing sink term (36) is constructed for arbitrary sources extended in both space and time, which respects the absorbing boundary condition, and modifies the source–sink in the FFPE (37) while leaving the dynamical operator $\mathcal{A}$ unchanged. In section 4.4 an equation for the MFPT (38) is obtained for general Markov processes using only the propagator and the first-hit distribution, even when the first-hit density is time-dependent.

This technique is used to analytically determine for the first time the MFPT (45) of free Lévy processes of arbitrary skew $\beta$ out of a finite interval. An implicit equation for the MFPT is also constructed in the case of Lévy flights in the boundary-centred harmonic potential wells considered in section 3; this, along with additional details on the MFPT derivation, is presented in appendix C. It should be noted that the source–sink framework is capable of implementing more general conditions on stochastic processes; a discussion of how this framework might be used to analytically construct reflecting boundary conditions in the FFPE is presented in appendix D.

4.1. Framework for general Markov processes with dynamical operator $\mathcal{A}$

We begin from (19), the equation describing the evolution of a Markov process with dynamical operator $\mathcal{A}$ and source–sink $g(x,t)$. Its Laplace transform is

$$(s - \mathcal{A}_s)P(x, s) = P(x, t = 0^-) + g(x, s)$$

where $P(x, t = 0^-)$ is the initial condition, assumed to be known a priori, yielding

$$P(x, s) = G_{s - \mathcal{A}_s}(x) * s (P(x, t = 0^-) + g(x, s))$$

where $G_{s - \mathcal{A}_s}(x)$ is the Green’s function of the operator $s - \mathcal{A}_s$ (note $L G_{s}(x) = \delta(x)$), and $f(x) * s g(x)$ denotes the convolution $\int f(x - y)g(y)dy$. The Green’s function can be reinterpreted as the free propagator $W(x, s)$, so that the position density in real space is expressed as

$$P(x, t) = W(x, t) * s P(x, t = 0^-) + W(x, t) * s_{x,t} g(x, t)$$

where $*_{x,t}$ denotes combined space-time convolution. For clarity of expression, the expressions in this section are presented in the case where the dynamical operator $\mathcal{A}_s$ is translation invariant. When it is not, the expressions continue to apply with the following modifications: replace all instances of the Green’s function $G_{s - \mathcal{A}_s}(x)$ with $G_{s - \mathcal{A}_s}(x, y)$ (where $L_s G_{s}(x, y) = \delta(x - y)$); redefine the spatial convolution to be the noncommutative operator $G(x) *_{s} f(x) := \int G(x, y)f(y)dy$; and replace instances of the propagator $W(x_{(b)} = x_{(0)}, \cdots)$ with $W(x_{(b)}, x_{(0)}, \cdots)$.

When the source–sink $g(x, t)$ is not fully known a priori, conditions on the position density $P(x, t)$ (such as boundary conditions) impose restrictions on the source–sink term, allowing the
operators, in general, are no longer Fourier multipliers. On dynamical operators \([18]\), such as the truncated integral in \([13]\), because those restricted obtainable using the classical formulation of the boundary conditions as domain restrictions.

This concise expression (25) of the position density.

4.1.2. Source–sink term separable.

4.1.1. Dynamical operator as Fourier multiplier. When the dynamical operator \(A\) can be expressed as a Fourier multiplier \(\tilde{A}\) (such as in free Lévy flights), the position density can be expressed directly in Fourier–Laplace space. Note that all multiplier operators are translation invariant. Taking the Fourier transform of (20) and rearranging gives

\[
P(k, s) = \frac{P(k, t = 0^-) + g(k, s)}{s - \tilde{A}_k} = \frac{W(k, s)}{s - \tilde{A}_k}P(k, t = 0^-) + g(k, s),
\]

where the free propagator \(W(x, t)\) is the position density in the case of no source–sink and initial condition at zero,

\[
W(k, t) = \frac{1}{s - \tilde{A}_k}.
\]

This concise expression (25) of the position density \(P(k, s)\) in Fourier–Laplace space is not obtainable using the classical formulation of the boundary conditions as domain restrictions on dynamical operators \([18]\), such as the truncated integral in \([13]\), because those restricted operators are, in general, no longer Fourier multipliers.

4.1.2. Source–sink term separable. Any source–sink \(g(x, t)\) can be decomposed into a purely temporal part \(p(t) = \int g(x, t)dx\) and a normalised spatial part \(q(x, t) := g(x, t)/p(t)\) such that \(g(x, t) = p(t)q(x, t)\) and \(\int q(x, t)dx = 1\) for all times \(t \geq 0\).

In many instances, such as free Lévy flights, the source–sink \(g(x, t) = p(t)q(x)\) is separable, where \(q(x)\) represents the spatial distribution of injection, or removal, of position density over time with temporal distribution \(p(t)\). For example, when \(q(x)\) is the first-hit distribution out of an arbitrary region \(\Omega\), and \(P(x_0, t) = 0\) for all \(t > 0\) and \(x_0 \notin \Omega\), then \(-p(t)\) becomes the FPTD out of \(\Omega\). As shown in [13], when \(q(x)\) is the delta sink instead then \(-p(t)\) becomes the first arrival time density, in the classical sense of arrival to a point (e.g. the limit of zero half-width \(L \to 0\) in section 2 for free Lévy flights).

The equation for the boundary condition (23) is rendered in the separable source–sink case as

\[
W(x_b, s)\ast_{x_b} P(x_b, t = 0^-) = P(x_b, s) - W(x_b, s)\ast_{x_b} q(x_b)p(s)
\]

which, by rearranging, fully determines the temporal distribution of the source–sink in Laplace space

\[
p(s) = \frac{P(x_b, s) - W(x_b, s)\ast_{x_b} P(x_b, t = 0^-)}{W(x_b, s)\ast_{x_b} q(x_b)}
\]

regardless whether the distribution represents first passage, first arrival, or a more exotic case. Substituting this distribution into (21) yields the exact expression of the position density...
using only the first-hit distribution \( q(x) \) and propagator \( W(x, s) \).

### 4.2. Free Lévy flight with absorbing boundaries

We now apply the above general results to the absorbing boundary condition problem for free Lévy flights undergoing first passage out of an arbitrary region \( \Omega \), given the first-hit distribution \( q_{\Omega}(x) \) out of \( \Omega \). Suppose that the particle starts at \( x_0 \in \Omega \), so \( P(x, t = 0^-) = \delta(x - x_0) \).

Here, the dynamical operator is a Fourier multiplier \( \hat{A} = -D|\alpha|^{\alpha}(1 - i\beta \tan \frac{\pi \alpha}{2} \text{sgn}(k)) \), and the source–sink is assumed separable. Since the boundary at \( x_b \in \partial \Omega \) is absorbing, \( P(x_b, t) = 0 \) for all \( t \geq 0 \).

From (28) we obtain the exact FPTD \( p_{\text{FP}}(t) = -p(t) \) in Laplace space\(^2\):

\[
p_{\text{FP}}(s) = \frac{W(x_b - x_0, s)}{W(x_b, s)_{x_b} q_{\Omega}(x_b)} = \frac{F^{-1}_k \left[ \frac{1}{s - \hat{A}_x} \right] (x_b - x_0)}{F^{-1}_k \left[ \frac{1}{s - \hat{A}_x} \right] (x_b)} \tag{30}
\]

where \( F^{-1} \) denotes the inverse Fourier transform, and the free propagator is given by (26):

\[
W(k, s) = \frac{1}{s - \hat{A}_x} = \frac{1}{s + D|\alpha|^{\alpha}(1 - i\beta \tan \frac{\pi \alpha}{2} \text{sgn}(k))}. \tag{31}
\]

Using (29) the position density in Laplace space is given by

\[
P(x, s) = W(x - x_0, s) - W(x_b - x_0, s)\frac{W(x, s)_{x} q_{\Omega}(x)}{W(x_b, s)_{x_b} q_{\Omega}(x_b)} \tag{32}
\]

or alternatively in Fourier–Laplace space, due to (25)

\[
P(k, s) = \frac{e^{ikx_0} - q_{\Omega}(k)p_{\text{FP}}(s)}{s - \hat{A}_x}. \tag{33}
\]

This resolves the problem of first passage for free Lévy flights of arbitrary skew parameter \( \beta \) in arbitrary regions \( \Omega \). One requires only the first-hit distribution \( q_{\Omega}(x) \), along with the free propagator \( W(x, s) \) in Laplace space, to obtain both the FPTD and position density in Laplace space. In section 2, first-hit distributions \( q_{\Omega}(x) \) for connected intervals and their complements are given. We note that the classical problem of first arrival to a point \( x_b \) is recovered by setting the region \( \Omega = \mathbb{R} \setminus \{x_b\} \), whereby the equations (30) and (32) reduce to the well-known forms for first arrival \([13, 36]\), thus unifying the notions of first passage and arrival.

\(^2\)To see how (30) reproduces the result for Brownian motion \( L(2,0,0,D) \) in the half-line, note that \( q_{\Omega,2L}(s) = \delta(s) \) from (2) in the limit \( L \to \infty, \alpha \to 2 \), and \( W(x, s) = e^{-(|x|D)^{1/\alpha}}/(2\sqrt{D\pi}) \) is the well-known propagator for Brownian motion in Laplace space \([1, 2]\). Hence (30) yields \( p_{\text{FP}}(s) = e^{-s\sqrt{D/\alpha}} \), the classical result \([29, \text{equation (D.15)}]\) in Laplace space.
Figure 2. The analytical position density of free symmetric Lévy flight on the positive half-line with \((\alpha, \beta, \gamma, D) = (3/2, 0, 0, 1)\) starting at \(x_0 = 1\). (a) The position density \(P(x, s)\) in Laplace space (32) for \(s = 1, 2, 3, 4\) (blue, yellow, green, red) flattens as the Laplace variable \(s\) increases; the zero density on the negative half-line demonstrates that the absorbing boundary condition has correctly been set up. (b) The inverse Laplace transform of (32) yields the position density \(P(x, t)\) in real space, plotted here for \(t = 1, 2, 3, 4\) (blue, yellow, green, red lines). The density flattens out as \(t\) increases, in agreement with numerical simulations (squares) of \(10^5\) flights with \(dt = 10^{-3}\). Compare with [14, figure 3], [30, figure 3].

4.2.1. Example: free symmetric Lévy flight, \(1 < \alpha < 2, \beta = 0\) [13, 14, 27, 30]. We now use the section 2 results on the first-hit distribution to investigate the escape from the positive half-line of a particle undergoing free symmetric Lévy flight \((x_b = 0)\). The dynamical operator has a particularly simple form in this case, \(\mathcal{A}_x = -D|\kappa|^{\alpha}\), and the first-hit distribution is, up to a reflection, equivalent to the first passage leapover density (4).

We evaluate (32) in order to plot the position density of the particle in real and Laplace space (Figure 2), using an implementation of the inverse Laplace transform algorithm [37, 38]. Previously, due to the computational difficulty of solving the FFPE using the classical formulation of the absorbing boundary condition as a domain-restricted dynamical operator, the position density had hitherto only been obtained using numerical simulations of Lévy flights. This efficiency problem was noted in [13] when establishing the truncated dynamical operator as a consistent phrasing of the FPTD problem for Lévy flights. Here, in order to analytically obtain the position density plots, we require only the first-hit distribution

\[
g_{(0,\infty)}(x) = \frac{\sin(\pi\alpha/2)}{\pi} \frac{|x_0|^\alpha/2}{|x|^\alpha/2|x - x_0|} \tag{34}
\]

supported on \(x \leq 0\), given by (2) with \(L \rightarrow \infty\) and \(\beta = 0\) (see also [27, equation (12)]); along with the propagator

\[
W(x, s) = F_k^{-1}[1/(s + D|\kappa|^{\alpha})](x). \tag{35}
\]

The power of this novel framework is therefore demonstrated by the fact that this hitherto unknown position density (32) can be expressed and plotted using known, established quantities.

4.3. Absorbing sink for arbitrary extended sources

Returning to general Markov processes, the above interpretation of a source–sink satisfying the absorbing boundary condition as the product of the FPTD and the first-hit distribution is
only applicable for an FFPE describing a single particle, where the only source is of a delta form happening at time \( t = 0 \). Note that the initial condition \( a(x) \equiv P(x, t = 0) \) can be reinterpreted as a source term \( \delta(t)a(x) \) with zero initial condition everywhere. However, in more general cases where the FFPE describes concentration densities, the source may be nontrivial and extended in both space and time, and it becomes important to determine what the corresponding extended sink term would be, such that the absorbing boundary condition remain respected.

Suppose firstly that we add \( n \) particles at \((x_i, t_i)\) for \( i \in \{1, \ldots, n\} \). The source term has magnitude \( \sum_{i=1}^{n} \delta(t - t_i) \delta(x - x_i) \) and the corresponding sink term has magnitude \( \sum_{i=1}^{n} p_{x_i}(t - t_i) q_{x_i}(x, t - t_i) \), where \( p_{x_i}(t) \) and \( q_{x_i}(x, t) \) are the FPTD and first-hit distribution for a particle starting at \( x_i \). Thus, for an extended source \( \nu(t) \mu(x, t) = \int dt_1 \nu(t_1) \delta(t - t_1) \int dx_1 \mu(x_1, t_1) \delta(x - x_1) \) the canonical absorbing sink is

\[
S_{x,\nu}[\nu(t)\mu(x, t)](x, t) := \int dt_1 \int dx_1 \nu(t_1) p_{x_1}(t - t_1) \mu(x_1, t_1) q_{x_1}(x, t - t_1)
= \int dx_1 \nu(t) \mu(x_1, t) s_{x_1} p_{x_1}(t) q_{x_1}(x, t).
\] (36)

Therefore an FFPE with arbitrary source respecting the absorbing boundary condition also contains its corresponding canonical absorbing sink as an additional (negative) contribution to the source–sink

\[
\frac{\partial P(x, t)}{\partial t} = \mathcal{A} P(x, t) - p_{x_0}(t) q_{x_0}(x, t) + \nu(t) \mu(x, t) - S_{x,\nu}[\nu(t)\mu(x, t)](x, t)
\] (37)

where \( x_0 \) is the starting position of the particle. Note that the term \( p_{x_0}(t) q_{x_0}(x, t) = S_{x,\nu}[\delta(t) \delta(x - x_0)] \) in (37) is the canonical sink term in the case of a delta source in both space and time, and can be modified as appropriate for more exotic initial conditions.

### 4.4. MFPT for general Markov processes

Next, we compute the MFPT for general Markov processes \( \mathcal{M} \) out of arbitrary regions \( \Omega \) starting at \( x_0 \in \Omega \), even when their first-hit distributions \( q(x, t) \) are time-dependent. The method is inspired by a technique used in theoretical neuroscience for the analysis of networks of neurons [39–42] in the Gaussian case; the details of the method can be found in appendix C. Using the method, we find that the MFPT \( \langle \tau_{\text{FP}} \rangle \) for any Markov process \( \mathcal{M} \) with infinitesimal generator \( \mathcal{A} \) starting from \( x_0 \) out of any region \( \Omega \) is given by the solution to

\[
0 = \mathcal{A} P_0(x) + \frac{1}{\langle \tau_{\text{FP}} \rangle} (\delta(x - x_0) - q_{x_0}(x, t = \langle \tau_{\text{FP}} \rangle)), \quad P_0(k = 0) = 1,
\] (38)

and so only knowledge of the dynamical operator \( \mathcal{A} \) and first-hit distribution is required for the evaluation of the MFPT. The term \( P_0(x) \) represents the relative proportion of time spent by the process in the neighbourhood of \( x \) before being absorbed,

\[
P_0(x) = \frac{1}{\langle \tau_{\text{FP}} \rangle} G_{\mathcal{A}}(x, s_0) \delta(x - x_0) - q_{x_0}(x, t = \langle \tau_{\text{FP}} \rangle))
= \frac{W(x - x_0, s = 0) - W(x, s = 0) q_{x_0}(x, t = \langle \tau_{\text{FP}} \rangle)}{\langle \tau_{\text{FP}} \rangle}.
\] (39)

The normalisation condition \( P_0(k = 0) = 1 \) implicitly determines the MFPT \( \langle \tau_{\text{FP}} \rangle \). When the dynamical operator \( \mathcal{A} \) of the process \( \mathcal{M} \) is translation invariant, this leads to an expression
for the MFPT using these quantities in Fourier space without the need for convolution
operators,
\[ \langle \tau_{FP} \rangle = W(k = 0, s = 0)(1 - q_{x_0}(k = 0, t = \langle \tau_{FP} \rangle)). \] (41)

It is necessary to take the value \( k = 0 \) directly, instead of the limit \( \lim_{k \to 0} P_0(k) \), as we demonstrate in the following example. Whenever the first-hit distribution is time-independent, (40) and (41) yield explicit expressions for the MFPT, dependent only on the propagator and the first-hit distribution.

4.4.1. Example: free Lévy process, escape from a finite interval [15–17, 19, 20]. To verify the validity of this method, we find the MFPT of a particle undergoing free Lévy flight from a finite interval \([0, 2L]\), where the method yields an explicit expression. The MFPT has previously been known for this situation in the symmetric case \( \beta = 0 \) [5, 9, 16]; using the above method we reproduce those results and extend them to arbitrary skew parameter \(-1 \leq \beta \leq 1\), which are subsequently verified using numerical simulations.

To do this, we make use of the following result: when \( q(x) \sim A_\pm /|x|^{1+\alpha} \) as \( x \to \pm \infty \) in real space, then \( 1 - q(k) \sim \frac{\pm A_\pm + k}{\alpha} \left( \frac{\pi}{\sin(2\pi \alpha)} \right) |k|^{\alpha} \) for asymptotically small \( k \) (positive and negative) in Fourier space. This can be shown, for instance, by arguments similar to those in [43, p. 10–11].

Using (31), the implicit equation (41) for the MFPT becomes explicit due to the first-hit distribution (2) being time-independent:

\[ \langle \tau_{FP} \rangle = \frac{1 - q(k)}{D|k|^{\alpha}(1 - i\beta \tan \frac{\pi \alpha}{2} \text{sgn}(k))} \bigg|_{k=0}. \] (42)

For large \( x \), the first-hit distribution (2) asymptotically behaves as

\[ q(x) \sim \frac{\sin \left( \frac{\pi}{2}(\alpha + \text{sgn}(x)\beta') \right)}{\pi} |x_0|^{-\frac{\alpha}{2}} |2L - x_0|^{\frac{\alpha}{2}} |x|^{-1-\alpha} \] (43)

so that for small \( k \)

\[ 1 - q(k) \sim \frac{\cos \frac{\pi \alpha'}{\Gamma(1 + \alpha)} |x_0|^{-\frac{\alpha}{2}} |2L - x_0|^{\frac{\alpha}{2}} |k|^\alpha}{1 + \beta\tan(\pi \alpha/2)}. \] (44)

Therefore the MFPT of free Lévy flight from \([0, 2L]\) is

\[ \langle \tau_{FP} \rangle = \frac{\cos \frac{\pi \alpha'}{\Gamma(1 + \alpha)} |x_0|^{-\frac{\alpha}{2}} |2L - x_0|^{\frac{\alpha}{2}}}{1 + \beta\tan(\pi \alpha/2)}. \] (45)

When \( \beta = 0 \), the classical MFPT for the symmetric case [9, equation (1)] is recovered. For the term \( 1 - i\beta \tan \frac{\pi \alpha}{2} \text{sgn}(k) \) in the denominator of (42), it is necessary to take the value \( k = 0 \) directly since the right and left limits are not equal. Numerical simulations (figure 3) corroborate this finding; taking the average of the left and right limits leads to an incorrect result, which would amount to dividing (45) by \( 1 + \beta^2 \tan^2(\pi \alpha/2) \).

In appendix C.3, we demonstrate how the method is used when the dynamical operator \( A \) is not translation invariant, and the first-hit distribution \( q(x,t) \) depends on time. In this case, an implicit equation for the MFPT \( \langle \tau_{FP} \rangle \) is obtained.
5. Discussion

In this study we investigated the problem of Markov processes in nontrivial geometries. An analytically tractable method was devised to construct absorbing boundary conditions in their deterministic dynamical equations. A fundamental link was established between the absorbing boundary condition and the first-hit distribution in the consideration of the first passage time problem. Using this, an equation for the mean first passage time of Markov processes out of arbitrary regions was constructed. For free Lévy flights of arbitrary skew \( \beta \), the first passage time density out of connected regions and their complements, along with the position density, was analytically expressed in Laplace space, using only the free propagator and the corresponding first-hit distribution. With the additional importance now assigned to the first-hit distribution, the distributions of first hits and leapovers were investigated for free Lévy flights of arbitrary skew \( \beta \) in finite intervals, and Lévy flights on the half-line in an ubiquitous class of external potentials.

The first-hit results for free Lévy flights in section 2 generalise those presented recently for the symmetric case in [9]. However, these escape results have been known in principle for almost half a century [6] using coupled integral equations [8], and those in the form presented here were obtained by a variable transformation on these classical results (see appendix A). This is indicative of a more general mathematician-physicist disconnect in the first passage leapover problem, where the same quantity is referred to variously as the first-hit distribution or place [4, 6, 7, 9–11], the harmonic measure [4, 8], and, in the case of the half-line, the FPLD [9, 25, 27, 28]. This disconnect is evident in statements such as the claim that the prefactor of the tail of the FPTD for free symmetric Lévy flights on the half-line was first derived in [27]; in fact, as mentioned in [29], the prefactor has also been known in the mathematical literature since at least 1973 [31, 44] for both symmetric and asymmetric free Lévy flights.

The FPLD for free Lévy flights on the half-line in section 3 generalise the FPLD for the symmetric case in [27]. In that study, both the FPLD and the tail of the FPTD were obtained
using a theorem due to Skorokhod [26, p 303], which gives a formula for the joint FPTD–FPLD $p(s, u)$ in Laplace space for homogeneous processes with independent increments out of the half-line. The FPTD and FPLD, it is said, then follow from Laplace inversion of $p(s, u = 0)$ and $p(s = 0, u)$ respectively [27]. However, as we see from the existence of time-dependent FPLDs in section 3, the Laplace inversion of $p(s = 0, u)$ actually gives the time-averaged FPLD, which is equal to the FPLD only when it is independent of time. To obtain the FPLD when it is time-dependent, it is necessary to divide the joint distribution in real space by the FPTD, which continues to be expressed as $p(s, u = 0)$ in Laplace space. In any case, the theorem due to Skorohod is formidable in practice: the tail of the FPTD was obtained using the theorem for asymmetric Lévy flights only recently [29], and the FPLD for arbitrary skew (8) has not yet, to the best of our knowledge, been deduced using the theorem (cf the speculative fit in [28, figure 13]). More work needs to be done to ascertain whether the theorem can be generalised to more general classes of Markov processes, such as Lévy flights in finite regions or those modulated by external potentials.

The first passage leapover problem for general Lévy flights in the presence of external potential wells has not yet been explored, despite its relevance to a wide plethora of physical problems [3]. One reason for this is the sheer difficulty of the problem: the variable transformation technique in section 3 reveals that the problem is equivalent to that for free Lévy processes with time-dependent fluctuating boundaries [34]. The fluctuating boundary problem has not yet been fully explored for Brownian processes [45, 46], and the difficulty associated with the addition of only a linear drift has been noted in the case of heat-type Markov pseudoprocesses [10, 11]. The validity of the variable transformation technique demonstrated here arises from the observation that the first passage leapover for a free Lévy process is independent of the scale parameter, and thus these results also apply when the scale parameter is time-dependent. The existence of time-dependent FPLDs in the case of nontrivial potential wells adds another layer of difficulty to the problem, especially when the first passage times are not known a priori. More generally, the variable transformation technique utilised here may be used for any stochastic process whose FPLD is scale-independent and known in the free case.

The usage of the first-hit distribution as the spatially extended absorbing sink properly phrases the FPTD problem for general Markov processes in a local, pointwise manner. The previous global formulation of the absorbing boundary condition [13, 18, 23], while consistent, involved domain restrictions on the dynamical operator, such as the truncation of its integral representation to the allowed region $\Omega$. The resultant operator, in general, then lost those properties which allowed for techniques amenable to the process to be used to solve for the position density of the particle. This issue was recognised by previous authors [15, 17, 30], and is arguably the principal reason the first passage problem for Lévy flights remains relatively untouched compared to Brownian motion. Physically, domain restrictions (such as integral truncation) corresponds to acting on the particle until it first exits the allowed region $\Omega$, at which point the particle remains stationary for all future times. The probability density outside the region $\Omega$ then, by definition, builds up at a rate determined temporally by the FPTD, and spatially by the first-hit distribution. Thus, subtracting this from the FFPE removes the necessity to perform domain restrictions, such as integral truncation, on the equation, since the inclusion of the complementary domain in the operator has no effect on the particle after the particle itself is removed. Our formulation thus recovers the Fourier representation and translational invariance of the dynamical operators for free Lévy flights in the presence of boundaries. The derivation of the canonical absorbing sink term takes this into account in order to construct the absorbing boundary condition for cases where concentration densities are considered instead of discrete particles. The first passage problem in Markov processes
is thus reduced to the determination of first-hit distributions, motivating a greater focus on
the problem of first hit for Lévy flights in external potentials, and Markov processes more
generally. Whether similar techniques can be used to formulate the absorbing boundary con-
dition problem for non-Markovian processes (e.g. time-fractional diffusion) remains to be
elucidated.

The source–sink framework in section 4 reconciles the notions of first passage and first
arrival. Classically, first arrival has been understood as arrival to a point, which is equivalent
to first passage in the Gaussian case of Brownian motion due to its continuous sample paths.
Consequently, the method by which to distinguish between the two cases in Lévy motion,
where the discontinuous sample paths create an important distinction between first passage
and arrival, has not always been clear. First passage, on the other hand, has classically referred
to the passage out of finite or semi-infinite regions, which in both cases has an infinite region
as its complement. The approach taken in this study recognises that a particle passing out
of a region arrives at its complement, and hence each of the first passage and arrival prob-
lems may be phrased in terms of the other. For example, the classical first arrival problem
can be phrased as the first passage out of the entire ambient space with the arrival point
removed. The phenomenon that the first arrival to a point (i.e. first passage out of \( \mathbb{R} \setminus \{0\} \))
is equivalent to the first passage out of the half-line in Brownian motion can then be under-
stood in the light of the continuous sample paths of the process: when the allowed region
\( \Omega \) is disconnected, the Brownian motion resides in only one connected component for the
entirety of the process’s lifetime. This is, of course, no longer the case in general for Lévy
processes.

The formulation of analytically efficient absorbing boundary conditions in this study has
been one of the longstanding open problems of Lévy flights. As a result, it is hoped that this
framework will contribute to the future study of Lévy flights in nontrivial geometries, along
with Markov processes more generally. Some immediate directions for future study in terms of
raw data include the usage of the exact Laplace expressions for the FPTD and position density
to obtain the remaining unknown real space FPTD and position density expressions for special
cases of the parameters \( \alpha, \beta \) for Lévy flights; obtaining FPLDs for more instances of boundary-
centred potentials for Lévy flights on the half-line; and deriving equations for the MFPT for
more instances of Markov processes. More generally, with the absorbing boundary condition
problem for Lévy flights now placed on a similar level of tractability with that for classical
Brownian motion, the process of generalising the extensive literature on Brownian motion in
nontrivial domains to the case of Lévy flights can now be realistically envisioned.

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Appendix A. Classical results on first arrival and passage for general
\( \alpha \)-stable processes

In this section, we collate a number of results on first-hit distributions for \( \alpha \)-stable Lévy pro-
cesses of arbitrary skew parameter \( \beta \). The variables corresponding to original formulae are
to obtain leapover length distributions for a number of example forces. In this section we substitute the FPLD formula (18) and boundary-centring condition (14) to obtain leapover length distributions for a number of example forces. The case $\beta = 0$, $\beta' = 0$ for all $\alpha$. The case $\beta = \pm 1$ demonstrates a key difference between the two $\alpha$ regimes: when $0 < \alpha < 1$, $\beta' = \pm \alpha$, whereas when $1 < \alpha < 2$, $\beta' = \pm (\alpha - 2) = \mp (2 - \alpha)$. For the sake of simplicity, the only 1-stable process considered here is the symmetric Cauchy process: $\beta = 0$ when $\alpha = 1$.

$$[8, \text{theorem A}(a)][6, \text{theorem 1}]$$

To get the first hit density on $(-\infty, 0] \cup [2L, \infty)$ ($x_0 \in [0, 2L]$), make the transformation $x_{00} \rightarrow x_{00}/L - 1$ (where $x_{00} \in \{x, x_0\}$) and multiply by the Jacobian $1/L$.

$$[8, \text{theorem A}(b)][7, \text{theorem 1}]$$

To get the first hit density on $[-2L, 0]$ ($x_0 \notin [-2L, 0]$), make the transformation $x_{00} \rightarrow x_{00}/L + 1$ (where $x_{00} \in \{x, x_0\}$) and multiply by the Jacobian $1/L$.

$$[7, \text{corollary 1.2}]$$

$\alpha < 1$, $|x_0| > 1$, probability $R_o(x_{00})$ of not hitting $[-1, 1]$ is

$$R_o(x_{00}) = \frac{\Gamma \left(1 - \frac{\alpha + \text{sgn}(x_{00})\beta'}{2}\right)}{\Gamma \left(\frac{\alpha + \text{sgn}(x_{00})\beta'}{2}\right) \Gamma(1 - \alpha)} \int_0^{\left|\frac{x_{00}}{L}\right|} t^{-\alpha / 2} (1 - t)^{\alpha / 2 - 1} \, dt \, dr. \quad (A.3)$$

To get the probability of not hitting $[-2L, 0]$ ($x_0 \notin [-2L, 0]$), make the transformation $x_0 \rightarrow x_0/L + 1$.

Cases on the half-line for extreme $\beta$ hold by Skorokhod continuity [8].

**Appendix B. Leapover lengths for example boundary-centred potentials**

In this section we substitute the FPLD formula (18) and boundary-centring condition (14) to obtain leapover length distributions for a number of example forces $\mu(x) = -V(x)$ which correspond to boundary-centred potentials $V(x)$. In particular, applying (14) yields relations for the constant $t$, the boundary point $d$, and the FPLD $f_d(l, t)$ for Lévy flights in several boundary-centred potentials $V(x)$.
The examples presented below use formulae from the Wikipedia page ‘List of integrals of rational functions’. The injectivity condition on the integral discussed in section 3.3 is satisfied by functions of terms such as log|x| and 1/x^n with n even.

(a) Regularised logarithmic potential:

Set \( \mu(x) = a_1 + \frac{2}{x} \) where \( a_1 \neq 0 \), so the Lévy Bessel process is not included. This corresponds to a potential well \( V(x) = -(a_1 x + a_2 \ln|x|) \). We find that \( d = -a_2/a_1 > 0 \) (where the potential is flat), \( r = -a_1/a_2 \), and the FPLD is

\[
\begin{align*}
  f_d(l, t) &= \frac{\sin \left( \frac{\pi (a_1 + \beta f)}{2} \right)}{\pi} \frac{(a_1 l)^{2}}{d(d + l)} \frac{d^{a_1 f + \beta f} d}{d + e^{1 + \frac{d^{2} + \beta f}{2}}} e^{\left(1 + \frac{d^{2} + \beta f}{2}\right)} \left(1 - \frac{d^{2} + \beta f}{2}\right). 
\end{align*}
\]

(B.1)

This cannot be applied to the Bessel case as when \( a_1 \to 0, d \to \infty \).

(b) Even-degree polynomial potential:

Suppose we have the potential well \( V(x) = (a_1 x + a_2)^n \), where \( n \geq 4 \) is even. Then \( \mu(x) = -V(x) = -na_1(a_1 x + a_2)^{n-1}, d = -a_2/a_1 \) (the centre of the potential well where it is flat), \( r = -(2 - n)a_1^2 \), and the FPLD is

\[
\begin{align*}
  f_d(l, t) &= \frac{\sin \left( \frac{\pi (a_1 + \beta f)}{2} \right)}{\pi} \frac{(2 - n)a_1^n}{na_1(la_1)^{n-1}} e^{\left(\frac{n(a_1 l)^{2} - (a_1 d)^{2} - (2 - n)a_1 l t}{1 + e^{a_1 l d^{2} - (a_1 d)^{2} + (2 - n)a_1 l t}}\right)} f'((a_1 d)^{2} + (2 - n)a_1 l t). 
\end{align*}
\]

(B.2)

(c) Forces of the form \( f(x)/f(x) \):

Suppose we have a potential well \( V(x) \) such that the force \( -V'(x) = \mu(x) = \frac{f(x)}{f(x)} \), and suppose that \( f \) has only one zero. We find that the boundary is at that zero, i.e. \( f(d) = 0 \), \( r = 1 \), and

\[
\begin{align*}
  f_d(l, t) &= \frac{\sin \left( \frac{\pi (a_1 + \beta f)}{2} \right)}{\pi} \frac{|f(0)|^{a_1 f + \beta f}}{|f(d + l)e^{-t}|^{a_1 f + \beta f}} |f'(d + l)e^{-t}|. 
\end{align*}
\]

(B.3)

**Appendix C. Derivation of the MFPT**

In this section, we provide details of the derivation of the MFPT for general Markov processes. We begin by recalling a classical method for deriving the first passage time in Gaussian Ornstein–Uhlenbeck processes, which was used to obtain spike rates for leaky integrate-and-fire neurons with Gaussian noise. We then generalise this method to arbitrary Markov processes \( \mathcal{M} \), by properly accounting for the first-hit properties of \( \mathcal{M} \). Afterwards, we use this method to obtain an implicit equation for the MFPT of a Markov process whose dynamical operator is not translation invariant, out of a region where the corresponding first-hit distribution is time-dependent. We finish with a brief discussion of the usefulness of this MFPT procedure in light of the FPTD expressions obtained in the main text for processes where the first-hit distribution is independent of time.

**C.1. Classical MFPT technique for Gaussian Ornstein–Uhlenbeck processes**

We consider a quantity \( X(t) \) satisfying the stochastic differential equation \([42, \text{equation (9.203)}]\) [39, equations (1) and (3)] [41, equation (7)]
where $W = L(2, 0, 0, 1/2)$ is a unit Gaussian process. In the classical theoretical neuroscience literature, this quantity models the membrane potential of a leaky integrate-and-fire neuron in a balanced, homogeneous randomly connected network of such neurons. The neuron fires when the membrane potential $X(t)$ reaches a threshold $\theta$, at which point it is reset to the reset potential $x_r < \theta$ after a refractory period $\tau_{rp}$ during which the membrane potential is unresponsive to input.

By applying the theory of Gaussian processes, we obtain the membrane potential probability density $P(x, t)$ corresponding to (C.1), which satisfies a Fokker-Planck equation

$$
\tau \frac{\partial P(x, t)}{\partial t} = \frac{\sigma^2(t)}{2} \frac{\partial^2 P(x, t)}{\partial x^2} + \frac{\partial}{\partial x} [(x - \mu(t))P(x, t)].
$$

(C.2)

In order to map the spiking behaviour of the neuron onto the Fokker-Planck equation (C.2), the boundary conditions need to be specified at the reset potential $x_r$ and threshold $\theta$, along with a normalisation condition. In the conventional Fokker-Planck framework, neural spiking has been defined as first arrival to the threshold $\theta$ (which equals first passage through $\theta$ due to the continuous trajectories of the Gaussian noise). The firing of the neuron thus imposes a point sink term $\nu(t)\delta(x - \theta)$ at the threshold $\theta$, where probability density is actively removed with rate $\nu(t)$; the term $\nu(t)$ is then interpreted as the firing rate of the neuron at time $t$. Neurons exiting the refractory period are injected back into the probability density using a point source term $\nu(t - \tau_{rp})\delta(x - x_r)$ at the reset potential $x_r$. This injection happens at a rate equal to the firing rate when the neuron had just fired, $\nu(t - \tau_{rp})$. These point sources and sinks can also be expressed in terms of discontinuity boundary conditions of the probability current [39], which is (up to a sign) the space derivative of the right hand side of (C.2).

Next, by definition of the integrate-and-fire neuron, $P(x, t) = 0$ for $x > \theta$, and thus by continuity $P(\theta, t) = 0$. Since $P(x, t)$ is a probability distribution, it is integrable (so its spatial Fourier transform $P(k, t)$ is smooth): \(\lim_{x \to -\infty} P(x, t) = \lim_{x \to +\infty} P(x, t) = 0\); and it satisfies the normalisation condition $P(k = 0, t) + p(t) = 1$, where $p(t) := \int_{-\infty}^{\theta} \nu(u)du$ is the probability that the neuron is refractory at time $t$. Incorporating these boundary conditions into (C.2) yields a Fokker-Planck equation for characterising motion with resetting in the presence of an absorbing threshold,

$$
\tau \frac{\partial P}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial x^2} + \frac{\partial}{\partial x} [(x - \mu)P] + \nu(t - \tau_{rp})\delta(x - x_r) - \nu(t)\delta(x - \theta).
$$

(C.3)

This equation is then solved under the condition $\partial P/\partial t = 0$, in order to obtain the stationary state of the system; in a stationary solution, $P(x, t) = P_0(x)$ and $\nu(t) = \nu_0$. This yields an expression for the stationary membrane potential distribution, $P_0(x)$ (see, e.g. [39, equation (19)]) with $V$ in place of $x$ as the membrane potential) in terms of the stationary firing rate $\nu_0$. The normalisation condition then provides the self-consistency condition, which determines $\nu_0$ (see [39, equation (21)]).

The classical technique [39, 41, 42] is to interpret the MFPT $\langle t_{fp} \rangle$ as the inter-spike interval in the stationary state. The inter-spike interval is the inverse $1/\nu(t)$ of the resetting rate, so that $\langle t_{fp} \rangle = 1/\nu_0$. We will use this interpretation to derive MFPTs for general Markov processes $\mathcal{M}$. 

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C.2. Method for computing the MFPT of Markov processes

We proceed by the addition of a resetting mechanism to the process $M$ whenever it exits a given region $\Omega$, which in our case leads to the inclusion of its corresponding canonical absorbing sink term as in (37),

$$\frac{\partial P(x, t)}{\partial t} = A_c P(x, t) - p_{x_0}(t) q_{x_0}(x, t) + \nu(t) \delta(x - x_0) - \nu(t) \delta(x - x_0) q_{x_0}(x, t).$$  \hfill (C.4)

When the resetting rate $\nu(t)$ is equal to the rate at which the particle is absorbed (so that $P(x, t)$ maintains the same normalisation over time), we obtain the integral condition $\nu(t) = p_{x_0}(t) + \nu(t) q_{x_0}(x, t)$, whose solution in Laplace space is given by $\nu(s) = p_{x_0}(s)/(1 - p_{x_0}(s))$. Physically, this means that the particle is placed at the starting position $x_0$ as soon as it is absorbed outside $\Omega$.

As time passes, a nonequilibrium stationary state (NESS) [47] is established in an expanding region around the resetting centre $x_0$. Inside this region, the FPTD in the convolution term in (C.4) behaves as a delta function $p_{x_0}(t) \approx \delta (t - \langle t_{FP} \rangle)$ centred at the MFPT. Using the classical MFPT interpretation in the above section (MFPT as inverse of resetting rate) yields

$$\frac{\partial P(x, t)}{\partial t} = A_c P(x, t) + \nu(t) \delta(x - x_0) - \nu(t) q_{x_0}(x, t = 1/\nu(t))$$  \hfill (C.5)

in the NESS region. Once the system, after a large amount of time, has relaxed to the NESS, all the parameters of the system become time-independent, and the dominant contribution to the dynamics of the system arises from the MFPT $\langle t_{FP} \rangle = 1/\nu_0$, yielding from the NESS-FFPE (C.5) the relation $0 = A_c P_0(x) + \nu_0 \delta(x - x_0) - \nu_0 q_{x_0}(x, t = 1/\nu_0)$ over the entire space. $P_0(x)$ is the non-equilibrium stationary position density at $x$; it can be interpreted as the average proportion of time spent in the region immediately around $x$ by a random walker with absorbing boundaries outside $\Omega$.

C.3. Example: boundary-centred harmonic potential case [3,33]

To demonstrate the usage of this method for more difficult cases where the first-hit distribution is time-dependent, we obtain an equation for the MFPT of a particle undergoing Lévy flight out of the half-line in the boundary-centred harmonic potential well explored in section 3.2. Here, the dynamical operator $A_c$ in the presence of a harmonic potential well is no longer a simple multiplier, but contains a differential operator when acting in Fourier space,

$$A_c f(x) = -\sigma(-\Delta)_\beta^{\alpha/2} f(x) - \frac{\partial}{\partial x}((\mu_1 x + \mu_2) f(x)), \hfill (C.6)$$

$$A_c = -\sigma |k|^\alpha + \mu_1 k \frac{\partial}{\partial k} + i \mu_2 k, \hfill (C.7)$$

where we assume that $\sigma$ is constant and hence, without loss of generality, $D = 1$. The operator $(-\Delta)_\beta^{\alpha/2} := (-\Delta)^{\alpha/2} + \beta \tan \frac{\alpha}{2} \frac{\partial}{\partial k}(-\Delta)^{(\alpha-1)/2}$ is the skew-adjusted fractional Laplacian with Fourier multiplier $|k|^\alpha := |k|^{\alpha(1 - i \beta \tan \frac{\alpha}{2})}$ (where we restrict ourselves to the case $1 < \alpha \leq 2$; when $\alpha = 1$ it suffices to replace $\tan \pi \alpha / 2$ in the above Fourier multiplier expression for $|k|^{\alpha}$ with $-2/(2\pi) \ln |k|$).

For processes such as these with dynamical operators which are not translation invariant, it is simplest to begin with (38). We thus proceed from the equation $A_c P_0(x) = h(x)$ where $h(x) := -\nu_0 \delta(x - x_0) + \nu_0 q(x, t = 1/\nu_0)$, and the first-hit distribution $q(x, t)$ is obtained in the boundary-centred case from the FPLD (12). Note that the propagator
tion determines the MFPT explicitly only when the appearance of the MFPT

The equation for the MFPT for general Markov processes is useful since the simple expression (30) for the FPTD in Laplace space only applies when the first-hit distribution is independent of time (i.e. when the source–sink is separable). More generally, the FPTD satisfies only an integral condition akin to (23). An important question that thus arises is whether this method for the MFPT can be adapted in order to compute higher-order moments of the FPTD, or the FPTD directly, possibly using techniques inspired by [42, equations (9.135-7)]. If so, the position density would then follow immediately from (22) given the first-hit distribution and propagator.

In this equation, *σ* refers to the regular spatial convolution, while *pL(α, β, γ, D)* is the probability density of the free Lévy process with index α, skew β, centre γ and scale D.

The boundary point is at *x₀ = d* and hence *C₁ = *e⁻*⁻*αμ|k|*²/µ₁. The normalisation condition *P(k = 0) = 1* yields *C₁ = 1* which fully and implicitly determines *ν₀* and hence the MFPT 1/ν₀. A situation where this case is applied will be found in [48]. The normalisation condition determines the MFPT explicitly only when α = 2, since the implicit part arises from the appearance of the MFPT 1/ν₀ in the first-hit distribution term when α ≠ 2.

**C.4. Discussion**

The equation for the MFPT for general Markov processes is useful since the simple expression (30) for the FPTD in Laplace space only applies when the first-hit distribution is independent of time (i.e. when the source–sink is separable). More generally, the FPTD satisfies only an integral condition akin to (23). An important question that thus arises is whether this method for the MFPT can be adapted in order to compute higher-order moments of the FPTD, or the FPTD directly, possibly using techniques inspired by [42, equations (9.135-7)]. If so, the position density would then follow immediately from (22) given the first-hit distribution and propagator.

As for the method of the MFPT, the application of the NESS ansatz can be understood by considering the convolution between ν(t) and *pL(α, β, γ, D)* in (C.4). In the NESS, where the resetting rate limₜ→∞ ν(t) > 0 is nonzero in the limit of large time, it has infinite mass compared to the finite mass of the normalised FPTD, and hence the FPTD *pL(α, β, γ, D)* behaves as a delta function centred at the mean of the FPTD, inside the convolution.

**Appendix D. Implementing other boundary conditions**

The techniques introduced in this framework can also be used to implement more exotic effects on the particle. In this final appendix, we explore how the multiple definitions of the reflecting boundary condition [16] may be implemented in the FFPPE using extended sources and sinks without the modification of the dynamical operator.

Here, we consider two different formulations of the reflecting boundary condition defined on the behaviour of the trajectory [16]: motion reversal or wrapping, which for wrapping-absorbing intervals is equivalent to an absorbing-absorbing interval of double the width; and motion stopping, which is equivalent to the imposition of an infinite potential well without boundary conditions. A pertinent question then is how these reflecting boundary conditions...
might be implemented in the FFPE without modifying the dynamical operator $A$, as was classically required [18]:

(a) Motion reversal: absorb the particle outside the allowed region (an absorbing sink of magnitude $p_{\infty}(t)q_{\infty}(x, t) = S_{\infty}[\delta(t)\delta(x - x_0)](x, t)$), and place it back equidistantly on the other side of the boundary (a source term of magnitude $S_{\infty}[\delta(t)\delta(x - x_0)](x_b - x, t)$).

However, we now face the problem of managing particles which have been absorbed and replaced once and exit the region a second time, and so on. Accounting for this, we arrive at a source–sink term comprised of an infinite sum

$$\frac{\partial P}{\partial t} = AP + \sum_{n=0}^{\infty} (-S_n(x, t) + S_n(x_b - x, t))$$ (D.1)

where we use the recursively defined function

$$S_n(x, t) := S_{\infty}[S_{n-1}(x_b - x, t)](x, t),$$ (D.2)

$$S_0(s, t) := p_{\infty}(t)q_{\infty}(x, t) = S_{\infty}[\delta(t)\delta(x - x_0)](x, t),$$ (D.3)

where $S_n$ is the absorbing sink for the particle which had previously exited the region $n$ times before.

(b) Motion stopping: absorb the particle outside the allowed region (as before) and place it on the other side of the boundary at a distance of $\epsilon$ (a source term of magnitude $p_{\infty}(t)\delta(x - (x_0 + \epsilon))$). Managing the same problem as before leads to the infinite-sum source–sink

$$\frac{\partial P}{\partial t} = AP + \sum_{n=0}^{\infty} \left(-S_n(x, t) + \left(\int S_n(x_1, t)dx_1\right)\delta(x - (x_b + \epsilon))\right)$$ (D.4)

where

$$S_n(x, t) := S_{\infty}\left[\left(\int S_{n-1}(x_1, t)dx_1\right)\delta(x - (x_b + \epsilon))\right](x, t),$$ (D.5)

$$S_0(s, t) := p_{\infty}(t)q_{\infty}(x, t) = S_{\infty}[\delta(t)\delta(x - x_0)](x, t).$$ (D.6)

The important advantage of this approach, compared to a possible domain restriction implementation of the reflecting condition in the FFPE [18, 23], is that the dynamical operator $A$ remains untouched and the sinks never depend explicitly on the position density $P(x, t)$, simplifying the solution of the FFPE.

The technique here reminds one of the scattering matrix approach used to compute scattering amplitudes in quantum field theory (pictorially represented as Feynman diagrams).

**D.1. Review of approaches to other boundary problems for Lévy flights**

The difficulty of the problem of first passage for Lévy flights has led to some groups turning to other variants of boundary problems for Lévy flights. The usage of a delta-function sink in the FFPE is known to describe the first arrival of the particle to a single point [49], regardless of the underlying trajectory of the random process; which differs from the first passage in all non-Gaussian Lévy flights. This case has been investigated in the context of search [36], and both the first arrival time density and corresponding position density have been obtained in Laplace space for Lévy flights in free, linear and harmonic potential wells.
for arbitrary point sink strengths representing various probabilities of absorption [50]. Since first arrival is equivalent to first passage in the Gaussian case of Brownian motion, the same technique is used to formulate absorbing boundary conditions in classical Brownian motion. A reverse kind of first arrival, stochastic resetting, has also been investigated [47], where the particle is stochastically reset to a given region. The sources and sinks employed in stochastic resetting generally explicitly depend on the position density, which can lead to difficulties in solving the corresponding dynamical equations. Finally, reflecting boundary conditions have had some discussion with regard to the implementation of its different realisations in Lévy flights [16]. In particular, the motion stopping formulation was found to be equivalent to an infinite potential well with no other restrictions on the flight, leading to an easily solvable FFPE [51]. Approaches which directly modify or redefine the dynamical operator (e.g. the fractional Laplacian in free, symmetric Lévy flight) have also been studied [18] for the reflecting boundary condition.

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