Three-loop Euler-Heisenberg Lagrangian and asymptotic analysis in 1+1 QED

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Abstract: In recent years, the Euler-Heisenberg Lagrangian has been shown to be a useful tool for the analysis of the asymptotic growth of the N-photon amplitudes at large N. Moreover, certain results and conjectures on its imaginary part allow one, using Borel analysis, to make predictions for those amplitudes at large loop orders. Extending work by G.V. Dunne and one of the authors to the three-loop level, but in the simpler context of 1+1 dimensional QED, we calculate the corresponding Euler-Heisenberg Lagrangian, analyse its weak field expansion, and study the congruence with predictions obtained from worldline instantons. We discuss the relevance of these issues for Cvitanovic's conjecture.
1. Cvitanovic’s conjecture for g-2 in QED

In their pioneering calculation of the \( g - 2 \) factor of the electron to sixth order in 1974, Cvitanovic and Kinoshita\(^1\) found a coefficient which was much smaller numerically than had been expected by a naive estimate based on the number of Feynman diagrams involved. A detailed analysis revealed extensive cancellations inside gauge invariant classes of diagrams. This led Cvitanovic\(^2\) to conjecture that, at least in the quenched approximation (i.e. excluding diagrams involving virtual fermions) these cancellations would be important enough numerically to render this series convergent for the \( g - 2 \) factor. Although nowadays there exist a multitude of good arguments against convergence of the QED perturbation series (see, e.g., Ref. 3), all of them are based on the presence of an unlimited number of virtual fermions, so that Cvitanovic’s conjecture is still open today. Moreover, should it hold true for the case of the \( g - 2 \) factor, it is natural to assume that it extends to arbitrary QED amplitudes in this quenched approximation. In previous work\(^4,5\) the QED effective Lagrangian in a constant field was used for analyzing the \( N - \) photon amplitudes in the low-energy limit. Based on existing high-order estimates for the imaginary part of this Lagrangian, Borel dispersion relations, and a number of two-loop consistency checks, this very different line of reasoning makes “quenched convergence” appear quite plausible for the case of the \( N - \) photon amplitudes. Its central point is an all-order conjecture for the imaginary part of the constant-field effective Lagrangian for Scalar QED in the weak field limit due to Affleck, Alvarez, and Manton\(^6\) (AAM). Here we present ongoing work towards a first three-loop check of this conjecture.\(^7\)

2. The AAM conjecture

Let us start with recalling the representation obtained by Euler and Heisenberg\(^8\) for the one-loop QED effective Lagrangian in a constant field,

\[
\mathcal{L}^{(1)}_{\text{spin}}(F) = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \left[ \frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{1}{3} (a^2 - b^2) T^2 - 1 \right]
\]

Here \( T \) is the proper-time of the loop particle and \( a, b \) are defined by \( a^2 - b^2 = B^2 - E^2, \quad ab = \mathbf{E} \cdot \mathbf{B} \). The analogous formula for Scalar QED was obtained by Weisskopf\(^9\) but will also be called “Euler-Heisenberg Lagrangian” (EHL) in the following. Except for the magnetic case, these effec-
tive Lagrangians have an imaginary part. Schwinger\textsuperscript{10} found the following representation for the imaginary parts in the purely electric case,

\begin{align*}
\text{Im}\mathcal{L}_{\text{spin}}^{(1)}(E) &= \frac{m^4}{8\pi^3} \beta^2 \sum_{k=1}^{\infty} \frac{1}{k^2} \exp \left[ -\frac{\pi k}{\beta} \right] \\
\text{Im}\mathcal{L}_{\text{scal}}^{(1)}(E) &= -\frac{m^4}{16\pi^3} \beta^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \exp \left[ -\frac{\pi k}{\beta} \right]
\end{align*}

(\beta = eE/m^2). These formulas imply that any constant electric field will lead to a certain probability for electron-positron pair creation from vacuum. The inverse exponential dependence on the field suggests to think of this as a tunneling process in which virtual pairs draw enough energy from the field to turn real. In the following we will be interested only in the weak field limit $\beta \ll 1$, which allows us to truncate the series in (2) to the then dominant first “Schwinger exponential”.

For the Scalar QED case, Affleck et al.\textsuperscript{6} proposed in 1982 the following all-loop generalization of (2),

\begin{align*}
\text{Im}\mathcal{L}_{\text{scal}}^{(\text{all-loop})}(E) \xrightarrow{\beta \to 0} m^4 \beta^2 \frac{1}{16\pi^3} \exp \left[ -\frac{\pi}{\beta} + \alpha\pi \right]
\end{align*}

This formula is highly remarkable for various reasons. Despite of its simplicity it is a true all-loop result; the rhs receives contributions from an infinite set of Feynman diagrams of arbitrary loop order, including also mass renormalization counterdiagrams. Moreover, the derivation given in Ref. 6 is very simple, if formal. Based on a stationary path approximation of Feynman’s worldline path integral representation\textsuperscript{11} of $\mathcal{L}_{\text{scal}}(E)$, it actually uses only a one-loop semiclassical trajectory, and arguments that this trajectory remains valid in the presence of virtual photon insertions.

An independent derivation of (3), as well as extension to the spinor QED case, was given by Lebedev and Ritus\textsuperscript{12} through the consideration of higher-order corrections to the pair creation energy in the vacuum tunneling picture. At the two-loop level, (3) has also been verified by a direct calculation of the EHL\textsuperscript{13} (for the spinor QED case), as well as been extended to the case of a self-dual field.\textsuperscript{4}
3. Connection between the AAM and Cvitanovic conjectures

Writing the AAM formula (3) as

\[ \text{Im} \mathcal{L}^{(\text{all-loop})}_{\text{scal}}(E) = \sum_{l=1}^{\infty} \text{Im} \mathcal{L}^{(l)}_{\text{scal}}(E) \sim \text{Im} \mathcal{L}^{(1)}_{\text{scal}}(E) e^{\alpha \pi} \]  

it states that an all-loop summation has produced the convergent factor \( e^{\alpha \pi} \), clearly an observation similar in vein to Cvitanovic’s. Moreover, at least at a formal level it is not difficult to transfer this loop summation factor from \( \text{Im} \Gamma(E) \) to the QED photon amplitudes\(^4,5\). Consider the weak field expansion of the \( l \)-loop contribution to the electric EHL:

\[ \mathcal{L}^{(l)}(E) = \sum_{n=2}^{\infty} c^{(l)}(n) \left( \frac{eE}{m^2} \right)^{2n} \]  

Using Borel dispersion relations, (3) can be shown\(^4,13\) to imply that, at any fixed loop order \( l \), the weak field expansion coefficients have the same asymptotic growth,

\[ c^{(l)}(n) \xrightarrow{n \to \infty} c^{(l)}_\infty \pi^{-2n} \Gamma[2n - 2] \]  

where the constant \( c^{(l)}_\infty \) relates directly to the prefactor of the corresponding leading Schwinger exponential in the weak field limit:

\[ \text{Im} \mathcal{L}^{(l)}(E) \xrightarrow{\beta \to 0} c^{(l)}_\infty e^{-\pi} \]  

As is well-known, the \( n \)-th term in the weak field expansion of the \( l \)-loop EHL carries information on the corresponding \( N = 2n \)-photon amplitudes in the low energy limit. Let us assume that the asymptotic behaviour should not depend on the choice of photon polarizations \( \varepsilon \) (this is plausible and supported by two-loop results\(^3\)). Since the kinematical structure of the \( N \)-photon amplitudes in this limit reduces to a prefactor which is the same at any loop order,\(^14\) one can eliminate it by dividing the \( l \)-loop amplitude by the one-loop one. Expanding (3) in \( \alpha \) and combining it with (7) and (6) one then arrives at a formula for the ratio of amplitudes in the limit of large photon number,

\[ \lim_{N \to \infty} \frac{\Gamma^{(l)}[k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N]}{\Gamma^{(1)}[k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N]} = \frac{(\alpha \pi)^{l-1}}{(l-1)!} \]
If we could now sum both sides over \( l \) and interchange the sum and limit, we could reconstruct the \( e^{\alpha \pi} \) factor, and conclude that the perturbation series for the \( N \) - photon amplitudes, at least in this low energy limit, is perfectly convergent! But this is too good to be true, since so far we have nowhere made a distinction between quenched and unquenched contributions to the photon amplitudes, and convergence of the whole perturbation series can certainly be excluded. However, as was noted in Ref. 5 this distinction comes in naturally if one takes into account that in the path integral derivation of (3) in Ref. 6 the rhs comes entirely from the quenched sector; all non-quenched contributions are suppressed in the weak field limit. And since (switching back to the usual Feynman diagram picture) the importance of non-quenched diagrams is growing with increasing loop order, it is natural to assume that their inclusion will slow down the convergence towards the asymptotic limit with increasing \( l \), sufficiently to invalidate the above naive interchange of limits. On the other hand, there is no obvious reason to expect such a slowing down of convergence inside the quenched sector, which led to the prediction\(^5\) that Cvitanovic’s “quenched convergence” will indeed be found to hold true for the photon amplitudes.

As a further step in this line of reasoning, one should now check that the convergence of (6) does not show a slowing down when going from two to three loops if one keeps only quenched diagrams. However, a calculation of any three-loop EHL, be it in Scalar or Spinor QED, for an electric or self-dual field, poses an enormous computational challenge.

Now, in 2006 M. Krasnansky\(^{15}\) calculated the two-loop EHL in 1+1 dimensional Scalar QED and found it, surprisingly, to have a structure almost identical to the one of the corresponding self-dual EHL in the four-dimensional case:

\[
\begin{align*}
L_{\text{scal}}^{(2)}(4D)(\kappa) &= \alpha \frac{m^4}{(4\pi)^3} \kappa^2 \left[ \frac{3}{2} \xi^2 - \xi' \right], \quad \xi(\kappa) := -\kappa \left( \psi(\kappa) - \ln(\kappa) + \frac{1}{2\kappa} \right) \\
L_{\text{scal}}^{(2)}(2D)(\kappa) &= -\frac{e^2}{32\pi^2} \left[ \xi_{2D}^2 - 4\kappa \xi'_{2D} \right], \quad \xi_{2D} := -\left( \psi(\kappa + \frac{1}{2}) - \ln(\kappa) \right)
\end{align*}
\]

(9)

\[ (\psi(x) = \Gamma'(x)/\Gamma(x), \kappa := m^2/(2\epsilon f), f^2 = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}). \] This led us to consider 2D QED as a toy model for studying the above asymptotic predictions.
4. Extension of the AAM conjecture to 1+1 QED

Of course, this will make sense only if the AAM formula (3) can be extended to the 2D case. The worldline instanton approach of\(^{6}\) can be extended to the 2D case straightforwardly,\(^{7}\) yielding the following analogue of (3):

\[
\text{Im} \mathcal{L}(E) \sim e^{-\frac{\omega^2}{\pi \kappa} + \hat{\alpha} \pi^2 \kappa^2}
\]  

(\(\hat{\alpha} := \frac{2e^2}{\pi m^2}\)). Note that, contrary to the 4D case, the second term in the exponent also involves the external field. This leads also to a somewhat more complicated form of the corresponding asymptotic limit statement:

\[
\lim_{n \to \infty} \frac{c^{(1)}(n)}{c^{(1)}(n + \ell - 1)} = \frac{(\hat{\alpha} \pi^2)^{l-1}}{(l-1)!}
\]

5. Three loop Euler-Heisenberg Lagrangian in 1+1 QED

At the one and two-loop level, we have obtained the EHL in 2D Spinor QED explicitly in terms of the gamma and digamma functions:\(^{7}\)

\[
\mathcal{L}^{(1)}(\kappa) = -\frac{m^2}{4\pi \kappa} \left[ \ln \Gamma(\kappa) - \kappa (\ln \kappa - 1) + \frac{1}{2} \ln \left( \frac{\kappa}{2\pi} \right) \right]
\]

\[
\mathcal{L}^{(2)}(\kappa) = \frac{m^2 \hat{\alpha}}{4\pi} \left[ \tilde{\psi}(\kappa) + \kappa \tilde{\psi}'(\kappa) + \ln(\lambda_0 m^2) + \gamma + 2 \right]
\]

Here \(\tilde{\psi}(\kappa) := -\xi(\kappa)/\kappa\), and \(\lambda_0\) is an IR cutoff for the photon propagator which becomes necessary at two loops in 2D. Curiously, in the 2D case the two-loop Spinor QED result (13) is simpler (just linear in the digamma function) than the corresponding Scalar QED one (9). Using the well-known large - \(x\) expansion of \(\ln \Gamma(x)\) in terms of the Bernoulli numbers \(B_n\) one can then easily verify that (11) does indeed hold true for \(l = 2\).

At three loops our results are rather preliminary. There are three diagrams contributing to the EHL, depicted in fig. 1 (the solid line represents the electron propagator in a constant field). For all three we have obtained representation in terms of fourfold proper-time integrals. The first six coefficients \(c^{(3)}(n)\) for the quenched part (diagrams A and B) were then obtained in part analytically, in part by numerical integration. As it turns out, at three loops all coefficients of the weak field expansion except the first one depend on the IR cutoff \(\lambda_0\). Introducing the modified cutoff \(\Lambda := \ln(\lambda_0 m^2) + \gamma\) the coefficients can be written in the form
where the coefficients \( c_{1,2}^{(3)}(n) \) are rational numbers, while \( c_0^{(3)}(n) \) contains a \( \zeta(3) \) already for \( n = 0 \). Since the prediction (11) is cutoff-independent, it can involve only the \( c_0^{(3)}(n) \)'s, so that the \( c_{1,2}^{(3)}(n) \)'s must be subdominant. For the series \( c_2^{(3)}(n) \) we have been able to compute a sufficient number of coefficients to verify that this is the case. Showing that the series \( c_0^{(3)}(n) \) indeed satisfies (11) is, however, not possible with the coefficients obtained.

6. Summary

Extending the worldline instanton method of \(^6\) to 2D QED we have obtained a prediction for the asymptotic growth of the weak field expansion coefficients of the 2D EHL at any loop order. At two loops we have verified this prediction by an analytic calculation of the EHL. At three loops we have obtained an integral representation of the EHL suitable for a numerical calculation of the expansion coefficients, and we expect to be able shortly to verify (or refute) the three main facts relevant for the AAM conjecture, namely that (11) holds at the \( \tilde{\alpha}^2 \) level, independence of spin, and asymptotic suppression of the non-quenched diagram C. On the slowing down issue, relevant for Cvitanovic’s conjecture, it unfortunately seems not to be possible to get information from the 2D QED case, due to the dependence of the three-loop expansion coefficients on the IR cutoff \( \Lambda \); although its numerical value does not affect the asymptotic limit, it does have an influence on the rate of convergence towards it, which thus remains ambiguous. Thus
further progress in this line of attack on Cvitanovic’s conjecture presumably has to await the calculation of the three-loop EHL in 4D.

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