Superresolution limits from measurement crosstalk

Manuel Gessner,1 Claude Fabre,1 and Nicolas Treps1

1Laboratoire Kastler Brossel, ENS-Université PSL, CNRS, Sorbonne Université, Collège de France, 24 Rue Lhomond, 75005, Paris, France
(Dated: April 16, 2020)

Superresolution techniques based on intensity measurements after a spatial mode decomposition can overcome the precision of diffraction-limited direct imaging. However, realistic measurement devices always introduce finite crosstalk in any such mode decomposition. Here, we show that any nonzero crosstalk leads to a breakdown of superresolution when the number N of detected photons is large. Combining statistical and analytical tools, we obtain the scaling of the precision limits for weak, generic crosstalk from a device-independent model as a function of the crosstalk probability and N. The scaling of the smallest distance that can be distinguished from noise changes from $N^{-1/2}$ for an ideal measurement to $N^{-1/4}$ in the presence of crosstalk.

The precision of optical imaging devices determines the state of the art in microscopy and astronomy. While diffraction affects our ability to resolve small structures and separations by spatially distributed intensity measurements, it does not pose a fundamental limitation. Historical resolution limits of Abbe, Rayleigh, and others address the effect of diffraction but they become irrelevant if the signal-to-noise ratio is high enough [1]. Moreover, a variety of superresolution techniques have been developed to overcome these limits, e.g., by stimulated-emission microscopy [2], by making use of homodyne measurements [3–5], or by intensity measurements in a transformed basis of modes [6, 7].

A systematic approach to the estimation of the spatial separation between two light sources is provided by the formalism of quantum metrology [8–11]. Tools from quantum information theory allow to optimize over all conceivable measurement techniques, in order to identify the ultimate quantum limits on precision [12]. For example, it was shown in Ref. [13] that spatial demultiplexing in transverse electromagnetic (TEM) Hermite-Gauss modes realizes a quantum-optimal measurement for the estimation of the separation between two incoherent sources. These results have been extended to two-dimensional images [14], thermal states [15, 16], and more general scenarios [17–21]. They have further been implemented experimentally by using interferometric phase-sensitive measurements [22–24], digital holography [25], or by using a local oscillator in an excited TEM01 mode [26, 27]. A decomposition of the detected light in TEMnm modes with $0 \leq n, m \leq 2$ was recently realized [28] using a multi-plane-light-visibility technique [29]. However, any experimental mode decomposition suffers from unavoidable crosstalk between the modes. So far, theoretical treatments of deviations from the ideal scenario have been limited to misaligned centroids [13] and electronic detection noise [30, 31].

In this article, we show that nonzero crosstalk between the detector modes before an intensity measurement leads to a breakdown of superresolution at small source separations. To quantify precision, we introduce the minimal resolvable distance $d_{\text{min}}$ at unit signal-to-noise ratio, as a function of the total number $N$ of photons. For the separation between two incoherent point sources, measurement crosstalk implies a change in the scaling of $d_{\text{min}}$ when $N \gg 1$: While in the ideal, noiseless case $d_{\text{min}}/w = N^{-1/2}$, in the presence of crosstalk, we obtain $d_{\text{min}}/w = \alpha N^{-1/4}$. An analytical model for weak crosstalk predicts that $\alpha \propto \sqrt{\rho}$, where $|\rho|^2$ is the crosstalk probability. These analytical results agree with the statistical predictions of a random-matrix model for generic crosstalk, indicating that this scaling is device independent. At low photon numbers, or for widely separated probes, we find that crosstalk is not a fundamental limitation and the measurement of higher-excited modes becomes increasingly relevant.

**Mode decomposition and measurement.**—We consider the problem of separating two incoherent point sources of equal intensity, located in a two-dimensional plane at positions $\pm r_0$, where

$$r_0 = \frac{d}{2} \left( \frac{\cos \theta}{\sin \theta} \right),$$

see Fig. 1 (a). After passing through a diffraction-limited imaging system, the spatial field distribution is, to a good approximation [1], described by two overlapping Gaussian profiles $u_0(x, y) = \sqrt{2/(\pi w^2)} \exp(-[x^2 + y^2]/w^2)$ and $w$ is the beam width. The function $u_0(r)$ can be extended to a complete Hermite-Gauss basis $(u_k(r))$ with $k = (n, m)$ and $u_0 = u_{00}$. To describe the electromagnetic field in the image plane, it is convenient to introduce two mode bases that are centered at the respective source positions, i.e., $u_{\pm k}(r) = u_k(r \mp r_0)$. Introducing corresponding field operators ($\hat{b}_{\pm k}$), both bases can be used to represent the electric field operator as $\hat{E}(r) = \sum_k u_{\pm k}(r) \hat{b}_{\pm k} = \sum_k u_{-k}(r) \hat{b}_{-k}$. We assume that the emitted photons follow a Poisson distribution and the ele-
detector modes $\hat{d}$ the two non-orthogonal modes $\{\hat{a}_\pm, \hat{b}_{\pm,0}\}$ are coherent states of the two non-orthogonal modes $\hat{b}_{\pm,0}$, respectively. The $P_{\pm}(a)$ are arbitrary probability distributions with $\int d\alpha P_{\pm}(\alpha)|\alpha|^2 = \epsilon$ and $N = M\epsilon$ is the total number of photons. This description applies, e.g., to incoherent superpositions of two weak thermal [13] or coherent light sources.

To describe the statistics of intensity measurements in an arbitrary spatial basis we introduce the basis of detector modes $\{\hat{d}_k\}$ with associated field operators $\{\hat{b}_k\}$. We can express the detector modes $\hat{d}_k = \sum_i f_{\pm,i} \hat{b}_{\pm,i}$ as a function of either one of the two shifted Hermite-Gauss bases with

$$f_{\pm,i} = \int d^2 r v_k^\pm(r) u_i(r \mp r_0). \tag{2}$$

The average photon numbers, i.e., the expectation values of $\hat{N}_k = \langle \hat{N}_k \rangle = \langle \hat{d}_k \rangle^2$ can be easily determined by making use of $\langle \hat{a}_\pm \hat{N}_k |\alpha\rangle = |f_{\pm,k}|^2 |\alpha|^2$ and after measuring all $M$ copies of $\hat{d}(d, \theta)$, we obtain

$$N_k = M\langle \hat{N}_k \rangle_{\hat{d}(d, \theta)} = \frac{N}{2} (|f_{+0}|^2 + |f_{-0}|^2) \tag{3}$$

photons in mode $k$. We assume that the number of photons in each mode is measured with high precision, e.g., by using photon-counting detectors, and the parameter $d$ is estimated from this data. The precision limit of any unbiased estimator is defined by the Cramér-Rao bound [32–34]: $(\Delta d)_{\text{est}} \geq 1/\sqrt{NF(d, \theta)}$, where

$$F(d, \theta) = \sum_{k=0}^{Q} p(k|d, \theta) \left( \frac{\partial}{\partial d} \ln p(k|d, \theta) \right)^2 \tag{4}$$

is the Fisher information of a single photon with detection probabilities

$$p(k|d, \theta) = \frac{N_k}{N}, \tag{5}$$

and $Q$ is the largest index of the measured modes in both spatial dimensions. This bound can be achieved asymptotically, e.g., by a maximum likelihood estimation [32, 33]. Through the dependence of $F(d, \theta)$ on $p(k|d, \theta)$, the achievable precision limit is determined by the measurement basis. Maximizing the Fisher information over all physically implementable measurements $\{\{\hat{d}_i\}\}$ gives rise to the quantum Fisher information $F_{\text{Q}}[\hat{d}(d, \theta)] = \max_{\{\hat{d}_i\}} F(d, \theta)$ [12]. For the estimation of $d$, it can be shown that $F_{\text{Q}}[\hat{d}(d, \theta)] = w^{-2}$ and an ideal intensity measurement in the Hermite-Gauss basis centered at the origin, $v_2(r) = u_0(r)$, achieves this bound in the limit $Q \to \infty$ [13]. For small separations $d \ll 2w$, it suffices to measure the contribution of the first excited mode, $Q = 1$, to saturate the quantum bound. Since the Fisher information stays finite and constant, these results imply that the precision of an estimation of $d$ is independent of the actual value of $d$, and hence, arbitrarily small distances can be resolved equally well as large ones. However, these conclusions only apply to ideal measurements without any noise and imperfections.

**Precision in the presence of mode crosstalk.**—To establish the impact of unavoidable deviations from the ideal mode decomposition, we determine the achievable sensitivity limits in the presence of crosstalk. We model crosstalk between the detector modes by a unitary coupling matrix $c_{kl}$ that maps the actual measurement basis to

$$v_k(r) = \sum_l c_{kl} u_l(r), \tag{6}$$

rather than the ideal Hermite-Gauss modes $u_l(r)$; see Fig. 1 (b). To model weak crosstalk, we consider coupling matrices whose off-diagonal elements are small compared to the diagonal ones. By including couplings into higher-order modes that are not measured, this model can effectively also describe the effect of losses. The relevant mode overlap functions (2) that determine the measurement statistics via (3) are given by $f_{\pm,0} = \sum_l c_{kl} \beta_l(\pm r_0)$, where the $\beta_l(\pm r_0) = \int d^2 r u_l^\pm(r) u_l(r \mp r_0)$ describe the ideal scenario. The precision limits are then obtained by using the corresponding measurement data in Eq. (4) [34].

![FIG. 2. Average Fisher information generated by a sample of 500 weak random crosstalk matrices of (a) low (average crosstalk probability $(|c_{ij}|^2 = 0.00017)$, (b) medium ($(|c_{ij}|^2 = 0.0017)$ and (c) high $(|c_{ij}|^2 = 0.0177$) crosstalk for measurements up to $Q = 1$ (blue) and $Q = 2$ (orange) modes in two dimensions. The solid lines and bands represent the average and one standard deviation. Dashed lines correspond to the ideal measurement at $\theta = 0$ and the dotted lines in panel (a) show the ideal measurement at $\theta = \pi/4$. The quantum Fisher information $F_0(d, \theta) = w^{-2}$ is reached by the ideal measurement for $Q \to \infty$ or at very small distances for $Q = 1$. (d) Closeup of the breakdown of the Fisher information $Q = 1$ in the presence of nonzero crosstalk for low (orange), medium (blue), and high (violet) crosstalk. The dashed lines are the analytical predictions (7) of the uniform crosstalk model with $|r|^2 = (|c_{ij}|^2/)$.](image)
In order to assess the impact of generic, weak crosstalk among $D$ modes, we sample randomly generated unitary crosstalk matrices $c_{ij} = C(\alpha)_{ij}$ from SU($D$) as $C(\alpha) = \exp(-i\alpha \sum_{k=1}^{D-1} \lambda_k G_k)$, where $[G_1, \ldots, G_{D-1}]$ are the generalized $D \times D$ Gell-Mann matrices and the real coefficients $\lambda_1, \ldots, \lambda_{D-1}$ are chosen randomly with normalization $\sum_{k=1}^{D-1} \lambda_k^2 = 1$ [34]. For $\alpha \ll 1$, the matrix $C(\alpha) \approx 1 - i\alpha \sum_{k=1}^{D-1} \lambda_k G_k$ is close to the identity matrix $I$ and the measurement basis is described as a small deviation from the ideal decomposition. The average crosstalk probability is determined by the average off-diagonal matrix element $|c_{ij}|^2 = \sum_{k,l=1}^{D-1} |c_{kl}|^2 / D(D-1)$. Figure 2 shows the averaged Fisher information over sets of 500 random crosstalk matrices generated with the same value of $\alpha$ with $D = 9$. The ensemble-averaged crosstalk probability $\langle |c_{ij}|^2 \rangle = 0.0017$ [Fig. 2 (b)] corresponds to the average crosstalk experimentally measured in Ref. [28], and, for comparison, we also show the effect of crosstalk that is ten times weaker [Fig. 2 (a)] or stronger [Fig. 2 (c)]. In the limit $d/2w \to 0$, any nonzero crosstalk causes the Fisher information $F(d,\theta)$ to drop from its ideal value $w^{-2}$ to zero and then to grow approximately quadratically as $d/2w$ increases. As we will explore in detail below, this limits our ability to resolve small distances at large $N$.

To analytically understand the average behavior at small separations, we introduce a uniform crosstalk model that consists in a $D \times D$ unitary matrix with entries $t$ on the diagonal and $r$ on the off-diagonal, satisfying $|r|^2 + (D-1)|t|^2 = 1$. For weak crosstalk probabilities $|r|^2 \ll 1$ and small separations $d \ll 2w$, the Fisher information for any $Q \geq 1$ is given by [34]

$$w^2 F(d,\theta) \approx \left( \frac{d}{2w} \right)^2 \left( \frac{3 + \cos(4\theta)}{4} \right) \frac{1}{|r|^2}. \quad (7)$$

The predictions of this model are shown in Fig. 2 (d) as dashed lines. Note that in contrast to crosstalk, the tilt angle $\theta$ poses no fundamental limitation to resolution since it only affects the proportionality factor $1/2 \leq (3 + \cos(4\theta))/4 \leq 1$.

**Minimal resolvable distance.**—The minimal distance between emitters that can still be resolved is determined by the signal-to-noise ratio (SNR) and requires that $\text{SNR}(d) = d/(\Delta d) \geq 1$. An efficient, unbiased estimator [33] minimizes the noise by saturating the Cramér-Rao bound, leading to $\Delta d = 1 / \sqrt{NF(d,\theta)}$, and we obtain:

$$\text{SNR}(d) = d \sqrt{NF(d,\theta)}. \quad (8)$$

We thus define the minimal resolvable distance as the smallest solution $d_{\text{min}}$ to $\text{SNR}(d_{\text{min}}) = 1$. In Fig. 3, we show $d \sqrt{F(d,\theta)}$ as a function of $d$ for weak, random crosstalk. For a given number $N$ of photons, the minimal resolvable distance is identified as the intersection with $1/\sqrt{N}$. For large photon numbers, $N \gg 1$, $d_{\text{min}}$ is dominated by the behavior of $F(d,\theta)$ in the limit of $d/2w \to 0$. In the case of an ideal measurement (considering either $Q \to \infty$ or $d \ll 2w$ with any $Q \geq 1$), the Fisher information is constant, $F(d,\theta) = w^{-2}$ [13]. Hence, in absence of crosstalk, we obtain the characteristic “shot-noise” scaling

$$d_{\text{min}} = \frac{w}{\sqrt{N}}. \quad (9)$$

In contrast, the quadratic dependence of $F(d,\theta)$ on $d$ observed in Eq. (7) modifies the scaling with the total number of photons $N$ and we obtain in the presence of crosstalk:

$$d_{\text{min}} = \frac{w}{\sqrt{N}} \sqrt{2|r| \left( \frac{4}{3 + \cos(4\theta)} \right)^{\frac{1}{2}}}. \quad (10)$$

The minimal resolvable distance for $N = 10000$ photons is shown in the lower part of Fig. 4. The scaling of the average predictions of the random crosstalk model agrees with that of the analytical, uniform crosstalk model (10) at the same average crosstalk probability.

The crosstalk-induced change of the scaling of $d_{\text{min}}$ with $N$ is due to the behavior of the Fisher information at small
FIG. 4. Minimal resolvable distance \( d_{\text{min}} \) at \( N = 1 \) (top) and \( N = 10,000 \) photons (bottom) as a function of the crosstalk probability \( |r|^2 \), obtained by intersecting \( d \sqrt{F(d, \theta)} \) in Fig. 3 with \( 1/\sqrt{N} \). The dots and errors bars represent the average and standard deviation of the random crosstalk model with average off-diagonal elements \( \langle |c_{ij}|^2 \rangle = |r|^2 \). For \( N = 1 \) (top) we show the results for measurements up to \( Q = 1 \) (blue) and \( Q = 2 \) (orange). The black horizontal lines show \( d_{\text{min}} \) for a measurement without crosstalk for \( N = 1 \) with \( Q = 1 \) (dotted) and \( Q = 2 \) (dot-dashed), and the ideal quantum limit (9), \( Q \to \infty \), yields \( d_{\text{min}}/2w = 0.5 \) in this case. At \( N = 10,000 \) (bottom) there is hardly any improvement by measuring \( Q = 2 \) or higher, and we only show \( Q = 1 \). The thick blue line shows the analytical prediction (10) of the uniform crosstalk model at \( \theta = 0 \). A crosstalk-free measurement with \( N = 10,000 \) and \( Q = 1 \) yields a \( d_{\text{min}} \) that cannot be distinguished from the quantum limit \( d_{\text{min}}/2w = 0.005 \) (dashed) on this scale.

FIG. 5. Scaling of the minimal resolvable distance \( d_{\text{min}} \) with the number \( N \) of photons in the presence of random uniform crosstalk with low (violet), medium (blue) and high (orange) crosstalk probability (cf. Fig. 2) for measurements up to \( Q = 2 \). For small \( N \leq 10 \), the minimal resolvable distance \( d_{\text{min}} \) still follows the ideal \( N^{-1/2} \) scaling of Eq. (9) (black line), with a prefactor that increases with the crosstalk. For larger \( N \), the scaling changes and approaches \( N^{-1/4} \), as predicted analytically in Eq. (10) by the uniform crosstalk model and is plotted with \( |r|^2 = \langle |c_{ij}|^2 \rangle \) (dashed lines).

d/2w: In the presence of crosstalk, the leading term in an expansion of the Fisher information \( F(d, \theta) \) around \( d/2w \approx 0 \) scales quadratically in \( d/2w \) [recall Eq. (7)]. This implies a quadratic scaling of \( d \sqrt{F(d, \theta)} \) (see the lower panel in Fig. 3) and leads to the \( N^{-1/4} \)-scaling in Eq. (10) at large \( N \).

In contrast, this is no longer the case at low photon numbers: At \( N \approx 1 \), we find that \( d_{\text{min}} \) is determined by the behavior of \( F(d, \theta) \) at finite values of \( d/2w \) (see the upper panel of Fig. 3). In this case, the leading term in an expansion of \( F(d, \theta) = c + O(d) \) is independent of \( d \), and thus \( d \sqrt{F(d, \theta)} \) scales approximately linearly, which implies \( d_{\text{min}} \approx 1/\sqrt{Nc} \), i.e., the \( N^{-1/2} \)-scaling observed in Eq. (9). The same is true at arbitrary \( N \) for the estimation of small deviations \( d \) from a fixed separation \( d_0 \gg d \). In this case, \( F(d + d_0, \theta) \) is constant to leading order in \( d \) and yields a \( N^{-1/2} \)-scaling for \( d_{\text{min}} \).

As these cases are dominated by the behavior of \( F(d, \theta) \) at values of \( d \approx 2w \), the measurement of higher excited modes becomes increasingly important and can yield significant advantages also in the presence of crosstalk, which no longer poses a fundamental limitation on the precision (see top panel of Fig. 3). The minimal resolvable distance at \( N = 1 \) photon is shown in the upper part of Fig. 4 as a function of the crosstalk. A measurement of \( Q = 2 \) comes close to the ideal quantum resolution limit (9) even in the presence of crosstalk.

The overall scaling of \( d_{\text{min}} \) with \( N \) is displayed in Fig. 5. We note that in the presence of crosstalk, the scaling of \( d_{\text{min}} \) changes from the ideal \( N^{-1/2} \) dependence [Eq. (9)] at small values of \( N \) to a much less favorable \( N^{-1/4} \) scaling [Eq. (10)] in the experimentally relevant regime of large \( N \). This behavior is confirmed by the statistical data of the random crosstalk model. For typical crosstalk probabilities, the transition occurs already at moderate photon numbers of \( N \approx 10^3 \) -- \( 10^4 \).

Conclusions.—We have identified the precision limits for an estimation of the separation of two incoherent point sources by intensity measurements after a realistic spatial mode decomposition. Introducing a general model for measurement crosstalk and losses using basis transformations, we have compared statistical data from random crosstalk matrices to analytical results obtained from a uniform model with tunable crosstalk probability. We observe that within statistical error margins, the scaling of the precision limits with the number of photons is device independent as it only depends on the average crosstalk probability. The uniform crosstalk model further allows us to analytically derive these scaling laws.

To quantify the precision for this estimation, we considered the smallest separation \( d_{\text{min}} \) that can be distinguished from the noise. This definition naturally depends on the number of measured photons. The most relevant information for practical measurements with many photons is contained in the large-\( N \) scaling. Crosstalk, however small, will always be present in realistic experimental mode decompositions and leads to a significant change of scaling from \( N^{-1/2} \) to \( \alpha N^{-1/4} \) with a prefactor \( \alpha \) that depends on the average crosstalk probability. In contrast, measurements at low photon numbers \( N \leq 10 \) are less affected by crosstalk, and in this case \( d_{\text{min}} \) increases by a crosstalk-dependent factor without changing the \( N^{-1/2} \) scaling.
Supplementary Material

In Section I, we derive the Fisher information for the estimation of $d$ from an ideal measurement of up to $Q$ Hermite-Gauss modes for arbitrary separations $d$ and tilt angles $\theta$. In Section II we introduce models for crosstalk and in Section III, we derive the Fisher information including the effect of crosstalk.

I. FISHER INFORMATION: IDEAL MEASUREMENT

A. Fisher information for Poissonian distributed intensity measurements in a finite set of modes

We begin by briefly recalling the derivation of the Fisher information for a Poissonian photodetection model, see, e.g., Ref. [1]. Assume that intensity measurements of a finite set of detector modes $1, \ldots, K$ are performed in order to estimate the parameter of interest. Photons in modes of higher order $k > K$ are not detected. This gives rise to events of the kind $x = (n_1, \ldots, n_K)$, where $n_1, \ldots, n_K$ denote the number of registered clicks in each of the respective modes. We further assume no correlations or bunching effects between the recorded photons, such that the statistics in each mode $k$ is given by a Poissonian distribution with average $N_k$, respectively. This yields the following probability for the event $x$:

$$p(n_1, \ldots, n_K|d, \theta) = \frac{1}{n_1! \cdots n_K!} N_k^{n_1} \cdots N_k^{n_K} e^{-N_k}, \quad (S1)$$

where $N_D = \sum_{k=1}^{K} N_k$ is the total number of detected photons. Each of the average values $N_k$ depends on the parameters $d$ and $\theta$. With ideal detectors we recover the total number of photons as $N = \lim_{K \to \infty} N_D$.

The Fisher information for estimations of the parameter $d$ with this measurement is given by

$$\mathcal{F}(d, \theta) = \left(\frac{\partial}{\partial d} \log p(n_1, \ldots, n_K|d, \theta)\right)^2 \quad (S2)$$

where the average of an arbitrary function $f(n_1, \ldots, n_K)$ is obtained from (S1) as

$$\langle f(n_1, \ldots, n_K) \rangle = \sum_{n_1=0}^{\infty} \cdots \sum_{n_K=0}^{\infty} p(n_1, \ldots, n_K|d, \theta)f(n_1, \ldots, n_K).$$

To determine (S2), we use (S1) to write

$$\log p(n_1, \ldots, n_K|d, \theta) = \sum_{k=1}^{K} (\log(-n_k!) + n_k \log N_k - N_k),$$

and we obtain

$$\frac{\partial}{\partial d} \log p(n_1, \ldots, n_K|d, \theta) = \sum_{k=1}^{K} \left( n_k \frac{\partial}{\partial d} \log N_k - \frac{\partial}{\partial d} N_k \right)$$

$$= \sum_{k=1}^{K} \left( \frac{n_k}{N_k} - 1 \right) \frac{\partial}{\partial d} N_k. \quad (S3)$$

Inserting (S3) into Eq. (S2), we find

$$\mathcal{F}(d, \theta) = \sum_{k=1}^{K} \left( \frac{\langle n_k \rangle}{N_k} - \frac{\langle n_k \rangle}{N_k} + 1 \right) \left( \frac{\partial}{\partial d} N_k \right) \left( \frac{\partial}{\partial d} N_k \right).$$

Making use of the Poissonian average and variance,

$$\langle n_k \rangle = N_k, \quad \langle n_k n_l \rangle = N_k N_l + \delta_{kl} N_k, \quad (S4)$$

we obtain

$$\mathcal{F}(d, \theta) = \sum_{k=1}^{K} \frac{1}{N_k} \left( \frac{\partial}{\partial d} N_k \right)^2. \quad (S5)$$

We define

$$p(k|d, \theta) = \frac{N_k}{N} \quad (S6)$$

as the probability for a single photon to be detected in mode $k$, such that (S6) reads

$$\mathcal{F}(d, \theta) = N \mathcal{F}(d, \theta), \quad (S7)$$

where

$$\mathcal{F}(d, \theta) = \sum_{k=1}^{K} \frac{1}{p(k|d, \theta)} \left( \frac{\partial}{\partial d} p(k|d, \theta) \right)^2 \quad (S8)$$

is the Fisher information associated with a single photon in the form of Eq. (4) in the main text. Notice that the probability obeys the normalization

$$\sum_{k=1}^{\infty} p(k|d, \theta) = 1. \quad (S9)$$

but only the first $K$ terms contribute to Eq. (S9). Since each term in the sum (S9) is non-negative, we see that the Fisher information increases as more modes are measured.

In practical situations, rather than the total number $N$ of emitted photons, only the number $N_D$ of detected photons may be known. In contrast to $N$, this number may in principle depend on the parameter $d$. Defining $p_D(k|d, \theta) = N_k/N_D$ and following the same approach as above yields from Eq. (S6):

$$\mathcal{F}(d, \theta) = N_D \mathcal{F}(d, \theta) + (2N_D + 1) \frac{1}{N_D} \left( \frac{\partial}{\partial d} N_D \right)^2. \quad (S10)$$

where $\mathcal{F}(d, \theta) = \sum_{k=1}^{K} \frac{1}{p_D(k|d, \theta)} \left( \frac{\partial}{\partial d} p_D(k|d, \theta) \right)^2$. Since the second term is non-negative, the bound $\mathcal{F}(d, \theta) \geq N_D \mathcal{F}(d, \theta)$ holds. This implies that we may effectively replace $N$ by $N_D$ in Eqs. (S7) and (S9): With this replacement, Eq. (S8) generally becomes a lower bound on the actual Fisher information, and this bound is tight when $N_D$ is independent of $d$. For measurements in the Hermite-Gauss basis and small values of $d$ this is a good approximation since only few photons go by undetected in highly excited modes with $k > K$. 
B. Mode overlap and statistics

We consider the ideal measurement in a basis of Hermite-Gauss modes, $u_k(r) = u_{nm}(r)$ with $k = (n, m)$. In this case, we can write the overlap integrals as $f_{x, y} = \beta_{nm}(\pm x)\beta_{nm}(\pm y)$, where

$$\beta_{nm}(a):= \int d^2r u^*_m(r) u_0(r-a). \tag{S12}$$

The Hermite-Gauss function are defined for $r = (x, y)$ as

$$u_{nm}(x, y) = \frac{1}{\sqrt{(\pi/2)w^2n!m!}} H_n\left(\sqrt{2x/w}\right) H_m\left(\sqrt{2y/w}\right) e^{-(x^2+y^2)/w^2}, \tag{S13}$$

where $H_n(x)$ are the Hermite polynomials and $w$ is the point spread function that determines the width of the 00-mode as

$$u_{00}(x, y) = \frac{\sqrt{2}}{w} e^{-x^2/w^2}. \tag{S14}$$

The integral (S12) corresponds to the overlap of a coherent state of a two-dimensional quantum harmonic oscillator (displaced to phase space coordinates $a$) with the excited state that contains $n$ and $m$ excitations in the respective directions. We use polar coordinates to express $x$ and $y$ in function of their separation $d$ and tilt angle $\theta$. Using polar coordinates has the advantage that $d$ represents the distance between emitters for arbitrary values of $\theta$, whereas in cartesian coordinates, a nonzero tilt requires the estimation of the nonlinear function $\sqrt{x^2+y^2}$ of the parameters $x$ and $y$. In polar coordinates, we obtain

$$\beta_{nm}(r_0) = \frac{1}{\sqrt{n!m!}} \left(\frac{d}{2w}\right)^{n+m} (\cos \theta)^n (\sin \theta)^m e^{-d^2/(4w^2)}. \tag{S15}$$

We obtain the average photon numbers in each mode as

$$N_{nm} = \frac{N}{2} (|\beta_{nm}(r_0)|^2 + |\beta_{nm}(-r_0)|^2) = N|\beta_{nm}(r_0)|^2. \tag{S16}$$

Moreover, the probability to find a detector click in the mode $nm$ is given by

$$p(nm|d, \theta) = \frac{N_{nm}}{N} = |\beta_{nm}(r_0)|^2. \tag{S17}$$

This probability is conditioned on the true value of the displacement being $r_0$. In this scenario, the problem of resolving two incoherent sources is equivalent to the estimation of the position of a single emitter.

C. Fisher information for the estimation of the emitter distance

We first make use of Eq. (S15) to obtain

$$\frac{\partial}{\partial \theta} \beta_{nm}(\pm x) = \frac{d}{\sqrt{n!m!}} \left(\frac{d}{2w}\right)^{n+m} (\cos \theta)^n (\sin \theta)^m e^{-d^2/(4w^2)}. \tag{S18}$$

Since $\beta_{nm}(r) \in \mathbb{R}$, this implies that

$$\frac{\partial}{\partial \theta} p(nm|d, \theta) = \frac{2}{\sqrt{n!m!}} \left(\frac{d}{2w}\right)^{n+m} (\cos \theta)^n (\sin \theta)^m e^{-d^2/(4w^2)}. \tag{S19}$$

Assuming that all modes $nm$ with $0 \leq n \leq Q$ and $0 \leq m \leq Q$ are measured, we obtain the Fisher information

$$F(d, \theta) = \frac{\partial}{\partial \theta} p(nm|d, \theta) \left(\frac{\partial}{\partial \theta} p(nm|d, \theta)\right)^2 \tag{S20}$$

$$= \frac{4}{\sqrt{n!m!}} \left(\frac{d}{2w}\right)^{n+m} (\cos \theta)^n (\sin \theta)^m e^{-d^2/(4w^2)} \tag{S21}$$

Substituting

$$x = \frac{d}{2w} \tag{S22}$$

and inserting Eq. (S15) leads to

$$F(d, \theta) = \frac{Q}{w^2} \sum_{n=m=0}^{Q} \left(\frac{d}{2w}\right)^{n+m} (\cos \theta)^n (\sin \theta)^m e^{-d^2/(4w^2)}. \tag{S23}$$

We obtain the limits

$$\lim_{Q \to \infty} F(d, \theta) = \frac{1}{w^2} \tag{S24}$$

and for any $Q > 0$:

$$\lim_{d \to 0} F(d, \theta) = \frac{1}{w^2} \tag{S25}$$

which are independent of $\theta$ and correspond to the quantum Fisher information $F_Q[d, \theta] = \frac{w^2}{2} \Gamma(Q+1, x^2) \tag{13}$.

Moreover, we find the following explicit expressions for specific values of the orientation angle $\theta$:

$$F(d, 0) := \lim_{\theta \to 0} F(d, \theta) = \lim_{\theta \to \frac{\pi}{2}} F(d, \theta) \tag{S26}$$

$$= \frac{1}{w^2} \Gamma(Q+1, x^2) \tag{S27}$$

As we can see from Fig. S1, in these limits we achieve the lowest sensitivity over all angles $\theta$, while the maximal sensitivity is achieved at $\theta = \pi/4$. 

FIG. S1. Fisher information $w^2 F(d, \theta)$ for the estimation of the separation $d$ in a two-dimensional setup with a tilt angle $\theta$ between the separation axis of the two sources and the measurement apparatus as a function of $d/2w$. We show the sensitivity $F(d, \theta)$ at its lowest ($\theta = 0$ or $\theta = \pi/2$, continuous lines) and highest ($\theta = \pi/4$, dashed lines) values, with $Q = 1$ (black) and $Q = 3$ (red). The inset shows the difference $F(d, \theta) - F(d, 0)$ as a function of $d/2w$ and $\theta$.

II. MODELS FOR CROSSTALK

A. Random crosstalk

We introduce a random-matrix model for unitary crosstalk matrices from a basis $\{G_1, \ldots, G_{D^2-1}\}$ of the Lie algebra $\mathfrak{su}(D)$, the generalized $D \times D$ Gell-Mann matrices [3]. We generate a random unitary $D \times D$ matrix by sampling the random, real coefficients $\lambda_1, \ldots, \lambda_{D^2-1}$ with $\sum_{k=1}^{D^2-1} \lambda_k^2 = 1$ producing a unitary matrix

$$ C(\alpha) = \exp(-i\alpha \sum_{k=1}^{D^2-1} \lambda_k G_k), \quad \text{(S28)} $$

with fixed $\alpha > 0$.

We characterize these random matrices, $C(\alpha)_{ij} = c_{ij}$, by analyzing the average absolute square of diagonal and off-diagonal elements, respectively: For each random matrix, we define

$$ |c_{ii}|^2 = \frac{1}{D} \sum_{k=1}^{D^2} |c_{kk}|^2, \quad \text{(S29)} $$

$$ |c_{ij}|^2 = \frac{1}{D(D-1)} \sum_{k,l=1, k\neq l}^{D^2} |c_{kl}|^2. \quad \text{(S30)} $$

Unitary ensures that

$$ 1 - |c_{ii}|^2 = (D - 1)|c_{ij}|^2. \quad \text{(S31)} $$

Figure S2 (a) and (b) shows $|c_{ii}|^2$ and $|c_{ij}|^2$ for 500 random matrices with fixed value $\alpha$. These quantities have relatively low fluctuations around a well-defined average value which is determined by $\alpha$. Figure S2 (c) and (d) displays the dependence of the ensemble averaged values $\langle |c_{ii}|^2 \rangle$ and $\langle |c_{ij}|^2 \rangle$ over 500 random matrices on the parameter $\alpha$. The quantity $\langle |c_{ij}|^2 \rangle$ represents the average probability for crosstalk into a specific mode.

B. Uniform crosstalk

A simple analytical crosstalk model is given by the uniform $D \times D$ uniform coupling matrix

$$ C = \begin{pmatrix} t & r & \cdots & r \\ r & t & & r \\ \vdots & \ddots & \ddots & \vdots \\ r & \cdots & r & t \end{pmatrix}, \quad \text{(S32)} $$

with $|t|^2 + (D-1)|r|^2 = 1$. 

III. FISHER INFORMATION: PRESENCE OF CROSSTALK

Crosstalk is modelled by a unitary coupling matrix that describes the actual measurement basis as a linear combination of the ideal basis. Expressed with two-dimensional indices, \( k = (n, m) \), we obtain

\[
\nu_{nm}(r) = \sum_{kl} c_{nm,kl} u_{kl}(r), \quad (S33)
\]

with ideal Hermite-Gauss modes \( u_{kl}(r) \). We obtain the average photon numbers

\[
N_{nm} = \frac{N}{2} \left( |\gamma_{nm}(r_0)|^2 + |\gamma_{nm}(-r_0)|^2 \right), \quad (S34)
\]

where \( \gamma_{nm}(r) = \sum_{kl} c_{nm,kl} \beta_{kl}(r) \) and \( \beta_{kl}(r) \) are the ideal overlap functions defined in Eq. (S12). Using again \( p(nm|d, \theta) = N_{nm}/N \), we find the probability distribution

\[
p(nm|d, \theta) = \frac{1}{2} \sum_{klpq} c_{nm,kl}^* c_{nm,pq} \left( \beta_{kl}(r_0) \beta_{pq}(r_0) + \beta_{kl}(-r_0) \beta_{pq}(-r_0) \right)
\]

\[
= \sum_{klpq} c_{nm,kl}^* c_{nm,pq} \frac{1 + (-1)^{k+l+p+q}}{2} \beta_{kl}(r_0) \beta_{pq}(r_0)
\]

\[
= \sum_{klpq} c_{nm,kl}^* c_{nm,pq} \beta_{kl}(r_0) \beta_{pq}(r_0), \quad (S35)
\]

where we used that \( \beta_{nm}(r) \in \mathbb{R} \). Notice that due to

\[
\beta_{kl}(-r_0) \beta_{pq}(-r_0) = (-1)^{k+l+p+q} \beta_{kl}(r_0) \beta_{pq}(r_0), \quad (S36)
\]

only terms where \( k + l + p + q \in 2\mathbb{N} \) is an even number contribute. We obtain the derivative

\[
\frac{\partial}{\partial d} p(nm|d, \theta) = \sum_{klpq} c_{nm,kl}^* c_{nm,pq} \frac{1}{d} \left( k + l + p + q - 2 \left( \frac{d}{2w} \right)^2 \right) \beta_{kl}(r_0) \beta_{pq}(r_0), \quad (S37)
\]

which permits us to calculate the Fisher information for arbitrary crosstalk matrices using Eq. (S20).

A. Approximation for small displacements in the presence of crosstalk

To study the limitations imposed by crosstalk on the Fisher information at small source separation, we perform a perturbative expansion of \( F(d, \theta) \), assuming

\[
x = \frac{d}{2w} \ll 1. \quad (S38)
\]

We obtain from Eq. (S15)

\[
\beta_{00}(r_0) = 1 - \frac{1}{2} x^2 + O(x^4)
\]

\[
\beta_{10}(r_0) = x \cos \theta + O(x^3)
\]

\[
\beta_{01}(r_0) = x \sin \theta + O(x^3)
\]

\[
\beta_{11}(r_0) = x^2 (\cos \theta \sin \theta) + O(x^4)
\]

\[
\beta_{nm}(r_0) = O(x^3) \quad \forall n + m \geq 3. \quad (S39)
\]

According to Eq. (S35) we find

\[
p(nm|d, \theta) = |c_{nm,00}|^2 + O(x^2). \quad (S40)
\]

Moreover, from Eq. (S37) follows

\[
 w \frac{\partial}{\partial d} p(nm|d, \theta) = \sum_{klpq} c_{nm,kl}^* c_{nm,pq} \frac{1}{2w} \left( k + l + p + q - 2 \left( \frac{d}{2w} \right)^2 \right) \beta_{kl}(r_0) \beta_{pq}(r_0),
\]

\[
= x \left[ (\cos \theta)^2 |c_{nm,10}|^2 + (\sin \theta)^2 |c_{nm,01}|^2 - |c_{nm,00}|^2 \right]
\]

\[
+ (2\theta) \text{Re}(c_{nm,10}^* c_{nm,01} + c_{nm,11}^* c_{nm,00})) + O(x^2). \quad (S41)
\]

B. Uniform crosstalk model

For the uniform crosstalk matrix given in Eq. (S32), we obtain for \( Q \geq 1 \):

\[
 w \frac{\partial}{\partial d} p(00|d, \theta) = x \left[ |r|^2 - |r|^2 + g(r, t, \theta) \right] + O(x^2),
\]

\[
 w \frac{\partial}{\partial d} p(10|d, \theta) = x \left[ (\cos \theta)^2 (|r|^2 - |r|^2) + g(r, t, \theta) \right] + O(x^2),
\]

\[
 w \frac{\partial}{\partial d} p(01|d, \theta) = x \left[ (\sin \theta)^2 (|r|^2 - |r|^2) + g(r, t, \theta) \right] + O(x^2),
\]

\[
 w \frac{\partial}{\partial d} p(11|d, \theta) = x g(r, t, \theta) + O(x^2),
\]

\[
 w \frac{\partial}{\partial d} p(nm|d, \theta) = 2x \sin(2\theta) |r|^2 + O(x^2) \quad \forall n + m \geq 3,
\]

with \( g(r, t, \theta) = \sin(2\theta)(|r|^2 + \text{Re}(\star r)) \). The Fisher information now reads

\[
w^2 F(d, \theta) = \frac{\sum_{n,m=0}^{Q} p(nm|d, \theta) \left( w \frac{\partial}{\partial d} p(nm|d, \theta) \right)^2}{(n^2 + m^2)} \quad (S42)
\]

\[
= x^2 \frac{|r|^2 - |r|^2 + g(r, t, \theta)|^2}{|r|^2}
\]

\[
+ x^2 \frac{(\cos \theta)^2 (|r|^2 - |r|^2) + g(r, t, \theta)}{|r|^2}
\]

\[
+ x^2 \frac{(\sin \theta)^2 (|r|^2 - |r|^2) + g(r, t, \theta)}{|r|^2}
\]

\[
+ x^2 \frac{g(r, t, \theta)}{|r|^2}
\]

\[
+ 4x^2 (Q-1)^2 \sin(2\theta)^2 |r|^2 + O(x^4).
\]
Simple expressions can be obtained, e.g., in the limit $\theta \to 0$ or $\theta \to \frac{\pi}{2}$, i.e., when the separation axis between the two emitters is aligned with the measurement basis. In this case, we obtain $F(d, 0) = \lim_{\theta \to 0} F(d, \theta) = \lim_{\theta \to \frac{\pi}{2}} F(d, \theta)$ with

$$w^2 F(d, 0) = x^2 \left( \frac{|r|^4}{|r|^2} + \frac{|t|^4}{|t|^2} - |r|^2 - |t|^2 \right) + O(x^3). \quad (S43)$$

In the limit of weak uniform crosstalk, $|r|^2 \ll 1$, we obtain

$$w^2 F(d, \theta) = x^2 \left( \frac{3 + \cos(4\theta)}{4} \frac{1}{|r|^2} + O(|r|^{-1}) \right) + O(x^3), \quad (S44)$$

where $\frac{1}{2} \leq \frac{3 + \cos(4\theta)}{4} \leq 1$.

[1] J. Chao, E. S. Ward, and R. J. Ober, Fisher information theory for parameter estimation in single molecule microscopy: tutorial, J. Opt. Soc. Am. A 33, B36 (2016).

[2] G. B. Arfken, H.-J. Weber, and F. E. Harris, Mathematical Methods for Physicists (Academic Press, 2012).

[3] R. A. Bertlmann and P. Krammer, Bloch vectors for qudits, J. Phys. A 41, 235303 (2008).