ON CURVATURE PROPERTIES OF NARIAI SPACETIMES

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ABSTRACT. The charged Nariai spacetimes are the exact solutions of Einstein-Maxwell field equations with positive cosmological constant and such a spacetime is the direct topological product of a 2-dimensional de-Sitter spacetime with a round 2-sphere of constant radius. The present paper deals with the investigation of curvature restricted geometric structures admitted by the charged Nariai spacetime metric and it is shown that such a spacetime is locally symmetric, 2-quasi-Einstein and Ein(2)-space. Moreover it realizes the pseudosymmetric type conditions such as Ricci generalized projectively pseudosymmetric and pseudosymmetric due to Weyl conformal curvature tensor. It is interesting to note that the energy momentum tensor can be expressed explicitly with the help of some 1-forms. We also evaluate the curvature properties of charged Nariai type topological product metric, and it is found that such metric is semisymmetric but not locally symmetric and under certain restrictions belongs to the family of generalized class of recurrent manifolds. Finally we end up our findings with some concluding remarks.

1. Introduction

In the theory of General Relativity, an important family of analytical solutions of Einstein field equations with positive cosmological constant was introduced by Nariai [47], [48] in 1951. Mathematically these spacetimes can be described as the direct topological product of a 2-dimensional de-Sitter spacetime with a round 2-sphere of constant radius i.e., $dS_2 \times S^2$. Such spacetime solutions became a crucial part of study in both theoretical and mathematical physics when Ginsparg and Perry [32] linked it to the dS-Schwarzschild black hole solution during the study of thermodynamical equilibrium of the black hole. They described that the Nariai solution is generated if the black hole event horizon of the dS-Schwarzschild solution approaches to the cosmological horizon through an appropriate limiting procedure. Nevertheless, the Nariai solution is geodesically complete and its Weyl tensor is of type $D$ in Petrov classification. In

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terms of spherical coordinates \((t, r, \theta, \phi)\), the metric of the Nariai spacetime is given by
\[
ds^2 = R_0^2(-\sin^2 r dt^2 + dr^2 + d\theta^2 + \sin^2 \theta d\phi^2),
\]
(1.1)
where \(R_0^2 = \frac{1}{\Lambda}\), \(\Lambda\) is a positive cosmological constant. The Nariai spacetime has been extended to the charged Nariai spacetime by Bertotti-Robinson [3] (see also [51]) to include Maxwell field. According to Hawking and Ross [36] (see also [39]), the charged Nariai spacetime can also be obtained from the dS-Reissner-Nordström black hole by taking appropriate limit of cosmological horizon going into the outer charged black hole horizon. The AdS part of the charged Nariai solution, called the charged anti-Nariai spacetime, exists and can be generated by considering appropriate AdS black hole solution and limit. The charged Nariai metric with respect to spherical symmetry is given by
\[
ds^2 = \frac{r_0^2}{L_0}(-\sin^2 r dt^2 + dr^2) + r_0^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]
(1.2)
where \(r_0 > 0\) and \(0 < L_0 \leq 1\) are two constants. The Maxwell fields of the solution are
\[
\mathcal{F} = \frac{q_0}{L_0} \sin r dt \wedge dr \quad \text{for the electric case},
\]
(1.3)
\[
\mathcal{F} = q_0 \sin \theta d\theta \wedge d\phi \quad \text{for the magnetic case}
\]
(1.4)
with \(q_0\) being the magnetic or electric charge respectively. The cosmological constant and the charge of the spacetime are related to the constants \(r_0\) and \(L_0\) by
\[
q_0^2 = \frac{1 - L_0}{2} r_0.
\]
(1.5)
\[
\Lambda = \frac{1 + L_0}{2r_0^2}.
\]
(1.6)
We call the metric (1.1) as the neutral Nariai metric. From (1.2)–(1.5) it is obvious that for \(L_0 = 1\) the charged Nariai metric reduces to the neutral Nariai metric (1.1). Also the limit \(\Lambda = 0\) takes the neutral Nariai metric (1.1) into the Minkowski spacetime metric.

The nature and physical properties of the (charged) Nariai and anti-Nariai metrics have been studied by various authors for correct cosmological observation, interpretation and generalization. Bousso [10] extended the Nariai solution to the dilation charged Nariai solution. The rotating Nariai solution was observed by Mellor and Moss [46] (see also [51]). Florian Beyer [6], [7] studied the asymptotics of the generalized Nariai solution and made valuable suggestion that
the Nariai solutions are non-generic among general solutions of Einstein field equations in vacuum with positive cosmological constant. The propagation of non-expanding impulsive waves in the Nariai universe were studied by Ortaggio [49]. However the curvature properties of such a spacetime is yet to known which is the aim of this paper.

The study of the curvature and curvature properties of a certain spacetime metric is important both physically and mathematically. For example, a spacetime represents perfect fluid spacetime if and only if it is quasi-Einstein [62]. On the other hand, the class of pseudosymmetric manifolds are models of various spacetimes. Here by geometric structures we mean the curvature restricted geometric structures obtained by the restriction(s) on several curvature tensors by means of covariant derivatives of first order or higher orders. In the literature of differential geometry there are various geometric structures with their several generalization. For example, the notion of the manifold of constant curvature has been generalized to the notion of locally symmetric manifold by Cartan ([11], [12]). Again this notion has been generalized in more than one direction by various authors. Some of the useful generalizations are the semisymmetric manifolds ([80], [81], [82]), pseudosymmetric manifolds [1], weakly symmetric manifolds [84], recurrent manifolds ([52], [53], [54] and also [86]), quasi generalized recurrent manifolds [74], weakly generalized recurrent manifolds ([57], [75]), hyper generalized recurrent manifolds ([73], [77]), super generalized recurrent manifolds ([64], [76]) etc. Our purpose in this paper is to investigate such type of structures admitted by the charged Nariai spacetime. It is shown that such a spacetime is locally symmetric, 2-quasi-Einstein and $E_{in}(2)$-space and satisfies some pseudosymmetric type conditions. Also the curvature properties of a charged Nariai type metric are investigated here. It is observed that such metric is semisymmetric but not locally symmetric and it admits special type of super generalized recurrent manifold and also has a recurrent type Weyl conformal curvature tensor.

The brief outline of this paper is as follows. Section 2 is concerned with the explanation of notations and definitions of various curvature restricted geometric structures. In Section 3, we investigate the curvature restricted geometric structures admiting by the charged Nariai metric and the charged Nariai type metric. Finally we conclude our findings in Section 3.

2. Preliminaries

In this section we discuss various curvature tensors and some of their related geometric structures which are essential to study the curvature restricted geometric structures of the charged Nariai spacetime. We consider an $n$-dimensional ($n \geq 3$) connected semi-Riemannian smooth
manifold $M$ endowed with a semi-Riemannian metric $g$. Let $\nabla$, $R$, $S$ and $\kappa$ be respectively the Levi-Civita connection, the Reimann-Christoffel curvature tensor, the Ricci tensor and the scalar curvature of $M$. Also let $C^\infty(M)$, $\chi(M)$ and $\chi^*(M)$ be respectively the algebra of all smooth functions, the Lie algebra of all smooth vector fields and the Lie algebra of all smooth 1-forms on $M$.

A semi-Reimannian manifold $M$ is said to be Einstein if the relation $S = \frac{\kappa}{n}g$ holds on $M$. The generalization of such notion is given in the following definition:

**Definition 2.1.** ([56], [61], [62], [65]–[68]) A semi-Riemannian manifold is said to be a $k$-quasi-Einstein manifold if $\text{rank } (S - \alpha g) = k$, $0 \leq k \leq (n-1)$, for a scalar $\alpha$. The manifold is called Einstein(resp., quasi-Einstein) if $k = 0$ (resp., $k = 1$). If $\alpha = 0$, then a quasi-Einstein manifold is called Ricci simple.

We define the Ricci tensor of level $k$ by $S^k(X, Y) = S(X, S^{k-1}Y)$ where $S$ is the Ricci operator defined by $g(X, SY) = S(X, Y)$. Again based on the dependency of the $(0,2)$-tensors $S^k(X, Y)$ another generalization of the notion of Einstein manifold is given by the following:

**Definition 2.2.** ([4], [65]) A semi-Reimannian manifold $M$ is said to be $\text{Ein}(2)$, $\text{Ein}(3)$ and $\text{Ein}(4)$ if

$$S^2 + \mu_1 S + \mu_2 g = 0,$$

$$S^3 + \mu_3 S^2 + \mu_4 S + \mu_5 g = 0 \text{ and}$$

$$S^4 + \mu_6 S^3 + \mu_7 S^2 + \mu_8 S + \mu_9 g = 0$$

holds respectively for some $\mu_i \in C^\infty(M)$, $1 \leq i \leq 9$.

Between the class of Ricci symmetric manifold and the manifold of constant scalar curvature Gray [35] introduced a new class of semi-Reimannian manifold called Codazzi type defined as below:

**Definition 2.3.** ([30],[35]) A semi-Riemannian manifold $M$ is said to be of Codazzi type (resp., cyclic parallel) Ricci tensor if

$$\nabla_{X_1} S(X_2, X_3) = \nabla_{X_2} S(X_1, X_3)$$

$$\text{ (resp., } \nabla_{X_1} S(X_2, X_3) + \nabla_{X_2} S(X_3, X_1) + \nabla_{X_3} S(X_1, X_2) = 0 \text{)}$$

holds on $M$. 

The Kulkarni-Nomizu product $E \wedge A$ of two symmetric $(0,2)$ tensors $E$ and $A$ is defined as 
\[(E \wedge A)(X_1, X_2, X_3, X_4) = E(X_1, X_4)A(X_2, X_3) + E(X_2, X_3)A(X_1, X_4) - E(X_1, X_3)A(X_2, X_4) - E(X_2, X_4)A(X_1, X_3).\]

**Definition 2.4.** A semi-Reimannian manifold is said to be a generalized Roter type manifold \([24], [25], [26], [65], [66]\) if

\[R = L_1g \wedge g + L_2g \wedge S + L_3S \wedge S + L_4g \wedge S^2 + L_5S \wedge S^2 + L_6S^2 \wedge S^2\]

holds for some $L_i \in C^\infty(M)$, $1 \leq i \leq 6$. It reduces to Roter type manifold for $L_4 = L_5 = L_6 = 0$ \((19), [20], [22], [27] \text{ and } [31]\).

Now we define the endomorphisms $X \wedge_A Y$, $\overline{\mathcal{R}}(X, Y)$, $\overline{\mathcal{C}}(X, Y)$, $\overline{\mathcal{P}}(X, Y)$, $\overline{\mathcal{W}}(X, Y)$ and $\overline{\mathcal{K}}(X, Y)$ over $\chi(M)$ on $M$ as follows \((17), [21], [30], [65]\)
\[
(X \wedge_A Y) Z = A(Y, Z)X - A(X, Z)Y,
\]
\[
\overline{\mathcal{R}}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},
\]
\[
\overline{\mathcal{C}}(X, Y) = \overline{\mathcal{R}}(X, Y) - \frac{1}{(n - 2)}(X \wedge_g SY + SX \wedge_g Y - \frac{\kappa}{n - 1} X \wedge_g Y),
\]
\[
\overline{\mathcal{P}}(X, Y) = \overline{\mathcal{R}}(X, Y) - \frac{1}{(n - 1)}(X \wedge_S Y),
\]
\[
\overline{\mathcal{W}}(X, Y) = \overline{\mathcal{R}}(X, Y) - \frac{\kappa}{n(n - 1)}(X \wedge_g Y),
\]
\[
\overline{\mathcal{K}}(X, Y) = \overline{\mathcal{R}}(X, Y) - \frac{1}{(n - 2)}(X \wedge_g SY + SX \wedge_g Y).
\]

where $A$ is a symmetric $(0,2)$-tensor. Now we define the following $(0,4)$-curvature tensors namely Gaussian curvature tensor $(G)$, the Riemann-Christoffel curvature tensor $(R)$, the Weyl conformal curvature tensor $(C)$, the projective curvature tensor $(P)$, the concircular curvature tensor $(W)$ and the conharmonic curvature tensor $(K)$ with respect to the above defined endomorphisms
as:

\[ G(X_1, X_2, X_3, X_4) = g((X_1 \wedge g X_2)X_3, X_4), \]
\[ R(X_1, X_2, X_3, X_4) = g(\overline{\mathcal{K}}(X_1, X_2)X_3, X_4), \]
\[ C(X_1, X_2, X_3, X_4) = g(\overline{\mathcal{C}}(X_1, X_2)X_3, X_4), \]
\[ P(X_1, X_2, X_3, X_4) = g(\overline{\mathcal{P}}(X_1, X_2)X_3, X_4), \]
\[ W(X_1, X_2, X_3, X_4) = g(\overline{\mathcal{W}}(X_1, X_2)X_3, X_4), \]
\[ K(X_1, X_2, X_3, X_4) = g(\overline{\mathcal{K}}(X_1, X_2)X_3, X_4). \]

We can easily extentd the operation of the endomorphism \( \mathcal{H}(X, Y) \) on a \((0, k)\)-tensor \( T \), \( k \geq 1 \), and obtain the tensor \( H \cdot T \) given by

\[
(H \cdot T)(X_1, X_2, \cdots, X_k; X, Y) = (\mathcal{H}(X, Y)T)(X_1, X_2, \cdots, X_k) = -T(\mathcal{H}(X, Y)X_1, X_2, \cdots, X_k) - \cdots - T(X_1, X_2, \cdots, \mathcal{H}(X, Y)X_k),
\]

where \( H \) is the associated \((0, 4)\)-tensor corresponding to the endomorphism \( \mathcal{H}(X, Y) \). Again if \( A \) is a symmetric \((0, 2)\) tensor then for \( \mathcal{H}(X, Y) = X \wedge_A Y \) we define a \((0, k + 2)\) Tachibana tensor \( Q(A, T)([21], [23], [29], [83]) \) by

\[
Q(A, T)(X_1, X_2, \cdots, X_k; X, Y) = ((X \wedge_A Y).T)(X_1, X_2, \cdots, X_k) = -T((X \wedge_A Y)X_1, X_2, \cdots, X_k) - \cdots - T(X_1, X_2, \cdots, (X \wedge_A Y)X_k) = A(X, X_1)T(Y, X_2, \cdots, X_k) + \cdots + A(X, X_k)T(X_1, X_2, \cdots, Y) - A(Y, X_1)T(X, X_2, \cdots, X_k) - \cdots - A(Y, X_k)T(X_1, X_2, \cdots, X).
\]

**Definition 2.5.** ([12], [18], [23], [25], [80]) A semi-Riemannian manifold \( M \) is said to be \( T \)-semisymmetric due to \( H \) if \( H \cdot T = 0 \) and it is said to be \( T \)-pseudosymmetric type due to \( H \) if \( H \cdot T = L_T Q(g, H) \), where \( L_T \) is some scalar function on \( \{ x \in M : Q(g, T)_x \neq 0 \} \).

In particular, for \( H = R \) and \( T = R \) (resp., \( S, C, W \) and \( K \)) a \( T \)-semisymmetric manifold is said to be semisymmetric (resp., Ricci, conformally, concircularly and conharmonically semisymmetric). Also for \( H = R \) and \( T = R \) (resp., \( S, C, W \) and \( K \)) a \( T \)-pseudosymmetric manifold is said to be Deza pseudosymmetric (resp., Ricci, conformally, concircularly and conharmonically pseudosymmetric).
Definition 2.6. A semi-Riemannian manifold $M$ is said to be weakly symmetric \cite{84} (see \cite{60}, \cite{63} and also references therein) if

$$\nabla_X R(X_1, X_2, X_3, X_4) = \Pi(X) R(X_1, X_2, X_3, X_4) + \Phi(X_1) R(X_1, X_2, X_3, X_4)$$

$$+ \overline{\Phi}(X_2) R(X_1, X, X_3, X_4) + \Psi(X_3) R(X_1, X_2, X, X_4) + \overline{\Psi}(X_4) R(X_1, X_2, X_3, X)$$

holds for $X, X_i \in \chi(M)$ ($i = 1, 2, 3, 4$) and some 1-forms $\Pi, \Phi, \overline{\Phi}, \Psi$ and $\overline{\Psi}$ on $\{x \in M : R_x \neq 0\}$. In particular, if $\frac{1}{2} \Pi = \Phi = \overline{\Phi} = \Psi = \overline{\Psi}$ (resp., $\Phi = \overline{\Phi} = \Psi = \overline{\Psi} = 0$), then the manifold is called Chaki pseudosymmetric manifold \cite{13} (resp., recurrent manifold \cite{52}).

It is also noted that the notion of Chaki pseudosymmetry is different from Deszcz pseudosymmetry. For details about the weak symmetry and its interrelation with Deszcz pseudosymmetry, we refer the reader to see \cite{60} and also references therein.

Definition 2.7. \cite{70, 73, 74, 75} and also \cite{77}) For a $(0, 4)$-tensor $T$, $M$ is said to be $T$-super generalized recurrent manifold if the condition

$$\nabla T = \Pi \otimes T + \Phi \otimes S \wedge S + \Psi \otimes g \wedge S + \Theta \otimes g \wedge g$$

holds on $\{x \in M : \nabla T \neq \xi \otimes T \text{ at } x \}$ for some $\Pi, \Phi, \Psi, \Theta \in \chi^*(M)$ called the associated 1-forms. In particular for $\Phi = 0$ (resp., $\Psi = 0$, and $\Phi = \Psi = \Theta = 0$) it is called $T$-hyper generalized recurrent manifold (resp., $T$-weakly generalized recurrent and $T$-recurrent manifold).

A $T$-recurrent (resp., $T$-weakly generalized recurrent, $T$-hyper generalized recurrent and $T$-super generalized recurrent) manifold is briefly denoted as $T$-$K_n$ (resp., $T$-$WGR_n$, $T$-$HGK_n$ and $T$-$SGK_n$). In particular for $T = R$, $T$-$K_n$ (resp., $T$-$WGR_n$, $T$-$HGK_n$ and $T$-$SGK_n$) is simply denoted as $K_n$ (resp., $WGR_n$, $HGK_n$ and $SGK_n$) and called recurrent (resp., weakly generalized recurrent, hyper generalized recurrent and super generalized recurrent ) manifold.

Definition 2.8. \cite{40, 41} A symmetric $(0, 2)$ tensor $E$ on a semi-Riemannian $M$ manifold is said to be Riemann compatible if

$$R(\mathcal{E}X_1, X, X_2, X_3) + R(\mathcal{E}X_2, X, X_3, X_1) + R(\mathcal{E}X_3, X, X_1, X_2) = 0$$

holds, where $\mathcal{E}$ is the endomorphism corresponding to $E$ defined as $g(\mathcal{E}X_1, X_2) = E(X_1, X_2)$. Again an 1-form $\Phi$ is said to be Riemann compatible if $\Phi \otimes \Phi$ is Riemann compatible.
In the similar manner, we can define conformal compatibility, concircular compatibility and conharmonic compatibility.

**Definition 2.9.** Let $B$ be a $(0, 4)$ tensor and $A$ be a $(0, 2)$ tensor on $M$. Then the corresponding curvature $2$-forms $\Omega^m_{(B)\ell}$ ($43, 38$) are recurrent if

$$(\nabla_{X_1} B)(X_2, X_3, X_4, X) + (\nabla_{X_2} B)(X_3, X_1, X_4, X) + (\nabla_{X_3} B)(X_2, X_1, X_4, X) = \Pi(X_1)B(X_2, X_3, X_4, X) + \Pi(X_2)B(X_3, X_1, X_4, X) + \Pi(X_3)B(X_2, X_1, X_4, X)$$

and the $1$-forms $\Lambda_{(A)\ell}$ ($79$) are recurrent if

$$(\nabla_{X_1} A)(X_2, X) - (\nabla_{X_2} A)(X_1, X) = \Pi(X_1)A(X_2, X) - \Pi(X_2)A(X_1, X)$$

for some $1$-form $\Pi$.

**Definition 2.10.** ($50, 85$) Let $\mathcal{L}(M)$ be the vector space formed by all $1$-forms $\Theta$ on $M$ satisfying

$$\Theta(X_1)B(X_2, X_3, X_4, X_5) + \Theta(X_2)B(X_3, X_1, X_4, X_5) + \Theta(X_3)B(X_1, X_2, X_4, X_5) = 0,$$

where $B$ is a $(0, 4)$ tensor. Then $M$ is said to be a $B$–space by Venzi if $\dim \mathcal{L}(M) \geq 1$.

### 3. Curvature Restricted Geometric Structures

In terms of spherical coordinates $(t, r, \theta, \phi)$ the non-zero components of the metric tensor of the charged Nariai metric (1.2) are given by

$$g_{11} = -\frac{r_0^2}{L_0} \sin^2 r, \quad g_{22} = \frac{r_0^2}{L_0}, \quad g_{33} = r_0^2, \quad g_{44} = r_0^2 \sin^2 \theta.$$

In view of Section 2 we can calculate the non-zero components (upto symmetry) of its Riemann curvature tensor $R$, Ricci tensor $S$ and scalar curvature $\kappa$ as follows

$$R_{1212} = \frac{r_0^2}{L_0} \sin^2 r, \quad R_{3434} = r_0^2 \sin^2 \theta;$$

$$S_{11} = \sin^2 r, \quad S_{22} = S_{33} = -1, \quad S_{44} = -\sin^2 \theta;$$

$$\kappa = -\frac{2(1+L_0)}{r_0^2}.$$

Also from (3.1) and (3.2) one can easily obtain the components of $g \wedge S$, $S \wedge S$ and $S^2$. We see that the covariant derivative of the local components of $R$ vanishes and hence we get the following:
Proposition 3.1. The charged Nariai metric (1.2) is not a manifold of constant curvature but it is (i) locally symmetric i.e., $\nabla R = 0$ and (ii) Roter type i.e., satisfies $R = a_1 S \wedge S + a_2 g \wedge S + a_3 g \wedge g$ for $a_1 = -\frac{(1+L_0)r_0^2}{2(L_0-1)^2}$, $a_2 = -\frac{2L_0}{(L_0-1)^2}$, $a_3 = -\frac{(1+L_0)L_0^2}{2r_0(L_0-1)^2}$.

Corollary 3.1. From the proposition (3.1) it follows that the charged Nariai metric (1.2) satisfies $\nabla C = \nabla P = \nabla W = \nabla K = \nabla S = 0$.

With the help of local components if we study the Ricci tensor of the metric (1.2) and the operation of its corresponding endomorphism on other curvature tensors, we get the following:

Proposition 3.2. The charged Nariai metric possesses the following curvature properties:

(i) rank$(S - \alpha g) = 2$ for $\alpha = -\frac{1}{r_0}$,

(ii) $\mu_1 S^2 + \mu_2 S + \mu_3 g = 0$ for $\mu_1 = 1$, $\mu_2 = \frac{(1+L_0)}{r_0^2}$, $\mu_3 = \frac{L_0}{r_0^2}$,

(iii) $S = \alpha g + \beta \Pi \otimes \Pi + \gamma(\Pi \otimes \Phi + \Phi \otimes \Pi)$ for $\alpha = -\frac{1}{r_0}$, $\beta = -1$, $\gamma = 1$, $\Pi = \{\sin \theta, 1, 0, 0\}$ and $\Phi = \left\{\frac{(1-2L_0)}{2L_0 \sin \theta}, \frac{1}{2r_0}, 0, 0\right\}$

(iv) $S = \alpha g + \beta \Pi \otimes \Pi + \gamma \Phi \otimes \Phi$, for $\alpha = -\frac{1}{r_0}$, $\beta = 1$, $\gamma = -1$, $\Pi = \left\{1, \frac{\sqrt{(1-L_0)\sin^2 \theta + L_0}}{\sin \theta}, 0, 0\right\}$ and $\Phi = \left\{-\frac{\sqrt{(1-L_0)\sin^2 \theta + L_0}}{\sqrt{L_0}}, -\frac{1}{\sin \theta}, 0, 0\right\}$

(v) $R(SX_1, X, X_2, X_3) + R(SX_2, X, X_3, X_1) + R(SX_3, X, X_1, X_2) = 0$,

$C(SX_1, X, X_2, X_3) + C(SX_2, X, X_3, X_1) + C(SX_3, X, X_1, X_2) = 0$,

$P(SX_1, X, X_2, X_3) + P(SX_2, X, X_3, X_1) + P(SX_3, X, X_1, X_2) = 0$.

From (3.2), we can get the non-zero components (upto symmetry) of the conformal curvature tensor $C$, the projective curvature tensor $P$, the concircular curvature tensor $W$ and the conharmonic curvature tensor $K$ for the metric (1.2) as follows

\begin{equation}
\begin{aligned}
C_{1212} &= -\frac{(1+L_0)r_0^2\sin^2 \theta}{3L_0^2}, & C_{1313} &= \frac{(1+L_0)r_0^2\sin^2 \theta}{6L_0}, & C_{1414} &= \frac{(1+L_0)r_0^2\sin^2 \theta}{6L_0}, \\
C_{2323} &= -\frac{(1+L_0)r_0^2}{6L_0^2}, & C_{2424} &= -\frac{(1+L_0)r_0^2\sin^2 \theta}{6L_0}, & C_{3434} &= \frac{(1+L_0)r_0^2\sin^2 \theta}{3L_0}.
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
P_{1212} &= P_{1221} = 2P_{1313} = -2P_{1331} = -\frac{2r_0^2\sin^2 \theta}{3L_0}, \\
P_{1414} &= \frac{r_0^2\sin^2 \theta}{3}, & P_{1441} &= -\frac{r_0^2\sin^2 \theta}{3L_0}, & P_{2323} &= -\frac{r_0^2}{3}, \\
P_{1331} &= \frac{r_0^2}{3L_0}, & P_{2442} &= \frac{r_0^2\sin^2 \theta}{3L_0}, & P_{2424} &= -\frac{1}{2}, & P_{3434} &= -\frac{1}{2} P_{3443} = -\frac{r_0^2\sin^2 \theta}{3}.
\end{aligned}
\end{equation}
\begin{align}
W_{1212} &= \frac{(1-5L_0)r_0^2 \sin^2 r}{6 L_0}, \quad W_{1313} = \frac{(1+L_0)r_0^2 \sin^2 r}{6 L_0}, \quad W_{1414} = \frac{(1+L_0)r_0^2 \sin^2 r \sin^2 \theta}{6 L_0}, \\
W_{2323} &= -\frac{(1+L_0)r_0^2}{6 L_0}, \quad W_{2424} = -\frac{(1+L_0)r_0^2 \sin^2 \theta}{6 L_0}, \quad W_{3434} = -\frac{(-5+L_0)r_0^2 \sin^2 \theta}{6}.
\end{align}

Again in view of Section 2 and by using (3.2), (3.3) and (3.4) we can calculate the non-zero components (upto symmetry) of the \((0,6)\)-tensors \(C \cdot R, \ C \cdot C, \ P \cdot R, \ Q(g,R), \ Q(g,C)\) and \(Q(S,R)\). Studying the linear dependency among them we get the following:

**Proposition 3.3.** The charged Nariai metric (1.2) satisfies

(i) \( C \cdot Z = L_1 Q(g,Z) \) for \( L_1 = \frac{(1+L_0)}{6 L_0} \) and

(ii) \( P \cdot Z = L_2 Q(S,Z) \) for \( L_2 = -\frac{1}{3} \)

where \( Z = R \) or \( C \) or \( W \) or \( K \).

The propositions (3.1), (3.2) and (3.3) lead us to the following theorem on curvature properties of the charged Nariai metric (1.2).

**Theorem 3.1.** The charged Nariai metric (1.2) possesses the following curvature restricted geometric structures:

(i) locally symmetric,

(ii) 2-quasi Einstein manifold,

(iii) generalized quasi-Einstein both by Chaki [14] and by De and Ghosh [15],

(iv) Ein(2)-space,

(v) Roter type manifold,

(vi) satisfies pseudosymmetric type condition \( C \cdot R = \frac{(1+L_0)}{6 L_0} Q(g,R) \) and \( P \cdot R = -\frac{1}{3} Q(S,R) \)

i.e., pseudosymmetric due to Weyl conformal curvature tensor and Ricci generalized projectively pseudosymmetric respectively,

(vii) Ricci tensor is Reimann compatible, conformal compatible, projectively compatible, concircularly compatible and conharmonically compatible.
(vii) the general form of compatible tensors for \( R, C, P, W, \) and \( K \) are given by

\[
\begin{pmatrix}
a_{11} & a_{12} & 0 & 0 \\
a_{12} & a_{22} & 0 & 0 \\
0 & 0 & a_{33} & a_{34} \\
0 & 0 & a_{34} & a_{44}
\end{pmatrix}
\]

where \( a_{ij} \) are arbitrary scalars.

The energy momentum tensor \( T \) of the charged Nariai spacetime metric (1.2) can be expressed in view of the Einstein field equations as

\[
T = \frac{c^4}{8\pi G} [S - \left( \frac{\kappa}{2} - \Lambda \right) g]
\]

where \( c = \) speed of light in vacuum, \( G = \) gravitational constant and \( \Lambda = \) cosmological constant.

The non-zero components of the energy momentum tensor \( T \) are given by

\[
T_{11} = -\frac{c^4}{16\pi GL_0} (3 + L_0) \sin^2 r, \quad T_{22} = \frac{c^4}{16\pi GL_0} (3 + L_0),
\]

\[
T_{33} = \frac{c^4}{16\pi G} (1 + 3L_0), \quad T_{44} = \frac{c^4}{16\pi G} (1 + 3L_0) \sin^2 \theta.
\]

Since \( T \) is a linear combination of \( S \) and \( g \) and \( \nabla S = 0 \), we have \( \nabla T = 0 \) and hence we can state the following:

**Theorem 3.2.** The energy momentum tensor \( T \) of the charged Nariai metric (1.2) fulfills the following:

(i) covariant derivative of \( T \) vanishes,

(ii) \( T = \alpha e_1 \otimes e_1 + \beta e_2 \otimes e_2 + \gamma e_3 \otimes e_3 + \mu e_4 \otimes e_4 \) for \( \alpha = -1, \beta = \frac{1}{L_0}, \gamma = \frac{1}{4}, \mu = 1, e_1 = \left\{ \frac{\sqrt{3+L_0}}{4L_0} \sin r, 0, 0, 0 \right\}, e_2 = \left\{ 0, \frac{\sqrt{3+L_0}}{4}, 0, 0 \right\}, e_3 = \left\{ 0, 0, \sqrt{1+3L_0}, 0 \right\} \) and \( e_4 = \left\{ 0, 0, 0, \frac{\sqrt{1+3L_0}}{4} \sin \theta \right\} \).

The neutral Nariai spacetime metric is a special case of the charged Nariai metric. Thus for the metric (1.1) we have the following

**Corollary 3.2.** The neutral Nariai metric (1.1) satisfies

(i) \( S = -\frac{1}{r^6} g \) i.e., Einstein manifold and hence \( C = P = W \),

(ii) \( \nabla R = 0 \) and hence \( \nabla C = \nabla W = 0 \) and

(iii) \( C \cdot R = \frac{1}{3\pi^6} Q(g, R) \).
Remark 3.1. The metric (1.2) does not fulfill the following structures:
(a) $R$–space or $C$–space or $P$–space or $W$–space or $K$–space by Venzi,
(b) $C \cdot Z \neq 0$ for $Z = R, C, P, W, K, S$,
(c) $P \cdot Z \neq 0$ for $Z = R, C, P, W, K$,
(d) $W \cdot Z \neq 0$ for $Z = R, C, P, W, K, S$,
(e) $K \cdot Z \neq 0$ for $Z = R, C, P, W, K, S$,
(f) Einstein or quasi-Einstein.

Remark 3.2. It is interesting to note that the charged Nariai metric (1.2) is conformally (resp., projectively, concircularly and conharmonically) symmetric but does not satisfy the semisymmetric type condition due to conformal (resp., projective, concircular and conharmonic) curvature tensor.

Remark 3.3. The charged anti-Nariai metric with respect to the coordinates $(\tau, \rho, \omega, \psi)$ is given by

$$ds^2 = \frac{a_0^2}{L_0} (-\sinh^2 \rho d\tau + d\rho^2) + a_0^2 (d\omega^2 + \sinh^2 \omega d\psi^2)$$

where $a_0 > 0$ and $1 \leq K_0 < 2$ are two constants. The above metric reduces to the form of charged Nariai metric (1.2) by the coordinate transformation $\tau = t$, $\rho = i r$, $\omega = i \theta$, $\psi = \phi$ where $i = \sqrt{-1}$. Hence both the metrics possess the same curvature restricted geometric structures.

During the study of asymptotic behaviour of the Nariai spacetime Beyer in his two sequential papers [6] and [7] considered a family of Nariai type solutions called them generalized Nariai solutions. Batista [5] also studied the generalized form of charged Nariai solutions in arbitrary even dimensions. Motivated from that we consider a charged Nariai type metric which is also a direct topological product of two 2-dimensional manifolds given by

(3.7) $$ds^2 = \frac{r_0^2}{L_0} (-f_1^2(r) dt^2 + dr^2) + r_0^2 (d\theta^2 + f_2^2(\theta) d\phi^2)$$

where $r_0$, $L_0$ are the constants defined in (1.2) and $f_1(r)$, $f_2(\theta)$ are everywhere non-vanishing smooth functions. From (3.7) it is obvious that for $f_1(r) = \sin r$ and $f_2(\theta) = \sin \theta$ the above metric reduces to the metric (1.2).

The non zero local components (upto symmetry) of Reiman curvature tensor $R$, the Ricci tensor $S$ and the scalar curvature $\kappa$ are given by

$$R_{1212} = \frac{r_0^2}{L_0} f_1 f_1', \quad R_{3434} = -\frac{r_0^2}{L_0} f_2 f_2'; \quad S_{11} = -f_1 f_1'', \quad S_{22} = \frac{f_1''}{f_1}, \quad S_{33} = \frac{f_2''}{f_2}, \quad S_{44} = f_2 f_2'';$$
\[ \kappa = \frac{2}{r_0^2 f_1 f_2} (L_0 f_2 f''_1 - f_1 f''_1). \]

Also the non-zero local components of \( \nabla R \) and \( \nabla S \) are given by

\[ R_{1212,1} = -\frac{r_0^2}{L_0} (f'_1 f''_1 - f_1 f''_1), \quad R_{3434,1} = \frac{r_0^2}{L_0} (f'_2 f''_2 - f_2 f''_2); \]

\[ S_{11,2} = (f'_1 f''_1 - f_1 f''_1), \quad S_{22,2} = -\frac{1}{f_1^2} (f'_1 f''_1 - f_1 f''_1), \quad S_{33,3} = -\frac{1}{f_2^2} (f'_2 f''_2 - f_2 f''_2), \quad S_{44,4} = -(f'_2 f''_2 - f_2 f''_2) \]

where \( f'_1 = \frac{df}{dr}, \quad f''_1 = \frac{d^2 f}{dr^2}, \quad f'_2 = \frac{df}{dr}, \quad f''_2 = \frac{d^2 f}{dr^2} \) and \( f'''_2 = \frac{d^3 f}{dr^3} \).

We easily observe that the local components of \( \nabla R \) vanishes if \( f'_1 f''_1 - f_1 f''_1 = 0 \) and \( f''_2 f''_2 - f_2 f''_2 = 0 \) hold simultaneously. Also we see that \( f_1(r) = \sin r \) and \( f_2(\theta) = \sin \theta \) satisfy the above equations.

The non zero components of \( g \wedge g, \ g \wedge S \) and \( S \wedge S \) (upto symmetry) are given below:

\[ L_0 (g \wedge g)_{1212} = (g \wedge g)_{1313} = \frac{2r_0^4 f_1^2}{L_0} , \quad (g \wedge g)_{1414} = \frac{2r_0^4 f_1 f_2}{L_0} , \]

\[ (g \wedge S)_{1212} = \frac{2r_0^2 f_1 f''_1}{L_0} , \quad (g \wedge S)_{3434} = -2r_0^2 f_2 f''_2 ; \]

\[ f_2^2 (g \wedge S)_{1313} = (g \wedge S)_{1414} = -f_2^2 f_2^2 (g \wedge S)_{2233} = -f_1^2 (g \wedge S)_{2424} = \frac{r_0^2 f_1 f_2}{L_0} (L_0 f_2 f''_1 + f_1 f''_2) ; \]

\[ (S \wedge S)_{1212} = 2f_1^{n2} , \quad (S \wedge S)_{3434} = 2f_2^{n2} , \]

\[ f_2^2 (S \wedge S)_{1313} = (S \wedge S)_{1414} = -f_2^2 f_2^2 (S \wedge S)_{2233} = -f_1^2 (S \wedge S)_{2424} = \frac{1}{2} f_1 f_2 f''_1 f''_2 . \]

From the above local components of tensors we can state the following:

**Proposition 3.4.** The metric \((3.7)\) admits the following curvature properties:

(i) 2-quasi Einstein (i.e., rank \((S - \alpha g) = 0\)) for \( \alpha = \frac{f''_1}{r_0^2 f_2} \),

(ii) generalized quasi-Einstein in the sense of Chaki \((S = \alpha g + \beta \eta \otimes \eta + \gamma (\eta \otimes \delta + \delta \otimes \eta))\) for \( \alpha = \frac{f''_1^2}{r_0^2 f_2^2} , \quad \beta = -1 , \quad \gamma = 1 , \quad \eta = \{ f_1 , 1 , 0 , 0 \} , \quad \delta = \left\{ \frac{-L_0 f_2 f''_1 f'_1 f''_1 - L_0 f_1 f''_2}{2L_0 f_2 f_1} , \frac{L_0 f_1 f''_1 f''_1 + L_0 f_1 f_2}{2L_0 f_2 f_1} , 0 , 0 \right\} \),

(iii) generalized quasi Einstein in the sense of De and Ghosh \((S = \alpha g + \beta \eta \otimes \eta + \gamma \delta \otimes \delta)\) for \( \alpha = \frac{f''_1^2}{r_0^2 f_2^2} , \quad \beta = -1 , \quad \gamma = 1 , \quad \eta = \left\{ \sqrt{f_1 (L_0 f_2 f''_1 f''_1 - f_1 f''_1)} , 0 , 0 , 0 \right\} , \quad \delta = \left\{ 0 , \frac{L_0 f_1 f''_1 f''_1 - f_1 f''_1}{L_0 f_2 f_1} , 0 , 0 \right\} \),

(iv) if \( L_0 f_2 f''_1 = f_1 f''_1 \neq 0 \), then it is Roter type manifold \((R = L_1 S \otimes S + S \otimes S + L_2 g \otimes g)\) for \( L_1 = \frac{r_0^2 f_1^2 (L_0 f_2 f''_1 f''_1 + f_1 f''_1)}{2(L_0 f_2 f''_1 - f_1 f''_1)^2} \), \( L_2 = \frac{2L_0 f_2 f''_1 f''_1 f''_1}{(L_0 f_2 f''_1 - f_1 f''_1)^2} \), \( L_3 = \frac{L_0 f_2 f''_1 f''_1 f''_1 f''_1 + f_1 f''_1 f''_1}{2r_0^2 (L_0 f_2 f''_1 - f_1 f''_1)^2} \),

(v) if \( L_0 f_2 f''_1 = f_1 f''_1 \neq 0 \) then it is a special type of super generalized recurrent manifold \((\nabla R = \Pi \otimes R + \Phi \otimes S \otimes S + \Psi \otimes g \otimes g)\) with \( \Pi = \left\{ 0 , \frac{f''_1 f''_1 - f''_1 f''_1}{2f_1 f''_1} , \frac{-f''_1 f''_1 - f''_1 f''_1}{2f_1 f''_1} , 0 \right\} , \quad \Phi = \left\{ \right\).
\[
\left\{ 0, -\frac{r_0^2 f_2(f_1'' f'' - f_2 f''')}{4 f_1^2 (L_0 f_2 f_1 f_2 f_3 - f_1 f_2^2)}, \frac{r_0^2 f_2(f_1'' f'' - f_2 f''')}{4 f_2^2 (L_0 f_2 f_1 f_2 f_3 - f_1 f_2^2)}, 0 \right\}, \quad \Psi = \left\{ 0, \frac{L_0 f_2^2 f_1'' f_2''}{4 f_1 f_2 (L_0 f_2 f_1 f_2 f_3 - f_1 f_2^2)}, -\frac{L_0 f_2^2 f_1'' f_2''}{4 f_2 f_2 (L_0 f_2 f_1 f_2 f_3 - f_1 f_2^2)}, 0 \right\}.
\]

**Corollary 3.3.** From the proposition (3.4) we see that the metric (3.7) is

(i) Roter type and hence it is Ein(2)-space \((\lambda_1 S^2 + \lambda_2 S + \lambda_3 g = 0)\) for \(\lambda_1 = -\frac{L_0 f_2 f_1'' + f_1 f_2''}{r_0^2 f_1 f_2},\)

(ii) Roter type and (special type) super generalized recurrent manifold and hence by [76] it is Ricci generalized recurrent manifold.

The non zero components of conformal curvature tensor \(C\), projective curvature tensor \(P\), concircular curvature tensor \(W\) and conharmonic curvature tensor \(K\) are given by

\[-\frac{L_0}{2} f_2^2 C_{1212} = f_2^2 C_{1313} = C_{1414} = -f_1^2 f_2^2 C_{2323} = -f_1^2 C_{2424} = \frac{1}{L_0} f_1^2 C_{3434} = -\frac{r_0^2 f_1 f_2}{6 L_0} (L_0 f_2 f_1'' + f_1 f_2'');
\]

\[f_2^2 P_{1212} = -f_2^2 P_{1221} = -\frac{2 f_2^2}{L_0} P_{1313} = -\frac{2}{L_0} P_{1414} = \frac{2}{L_0} f_2^2 P_{2323} = \frac{2}{L_0} f_1^2 P_{2424} = \frac{2 r_0^2}{3 L_0} f_1 f_2 f_1'';
\]

\[2 f_2^2 P_{1331} = P_{1441} = -2 f_1^2 f_2^2 P_{2323} = -2 f_1^2 P_{2442} = -\frac{f_1}{L_0} P_{3434} = \frac{f_1}{L_0} P_{3443} = \frac{2 r_0^2}{3 L_0} f_1^2 f_2 f_2'';
\]

\[W_{1212} = \frac{r_0^2 f_2}{6 L_0^2 f_2} (5 L_0 f_2 f_1'' - f_1 f_2''), \quad W_{3434} = \frac{r_0^2 f_2}{6 f_1} (L_0 f_2 f_1'' - 5 f_1 f_2'),
\]

\[f_2^2 W_{1313} = W_{1414} = -f_1^2 f_2^2 W_{2323} = -f_1^2 W_{2424} = -\frac{r_0^2 f_1 f_2}{6 L_0} (L_0 f_2 f_1'' + f_1 f_2'');
\]

\[f_2^2 K_{1313} = K_{1414} = -f_1 f_2 K_{2323} = -f_1^2 K_{2424} = \frac{r_0^2 f_1 f_2}{2 L_0} (L_0 f_2 f_1'' + f_1 f_2'').
\]

Also the non zero local components (upto symmetry) of \(\nabla C\), \(\nabla P\), \(\nabla W\) and \(\nabla K\) are given by

\[-2 C_{1212,2} = -\frac{2}{L_0} C_{1313,2} = \frac{2}{L_0 f_2^2} C_{1414,2} = -\frac{f_1^2}{L_0^2} C_{2323,2} = \frac{f_1^2 f_2}{L_0 f_2^2} C_{2424,2} = \frac{f_1^2}{L_0^2 f_2^2} C_{3434,2} = \frac{r_0^2}{3 L_0} (f_1' f_2'' - f_1 f_2'''),
\]

\[-\frac{L_0 f_2^2}{f_1^2} C_{1212,3} = \frac{2 f_2^2}{f_1^2} C_{1313,3} = \frac{2}{f_1^2} C_{1414,3} = -2 f_2^2 C_{2323,3} = -2 C_{2424,3} = \frac{1}{L_0} C_{3434,3} = \frac{r_0^2}{3 L_0} (f_2 f_2'' - f_2 f_2''');
\]

\[-\frac{1}{2} P_{1212,2} = \frac{1}{2} P_{1221,2} = P_{1313,2} = \frac{1}{f_1^2} P_{1414,2} = -\frac{f_2^2}{L_0} P_{2323,2} = -\frac{f_2^2}{L_0 f_2} P_{2424,2} = \frac{r_0^2}{3 L_0} (f_1' f_1'' - f_1 f_1'''),
\]

\[-\frac{f_2^2}{f_1^2} P_{1331,3} = -\frac{1}{f_1^2} P_{1441,3} = f_2^2 P_{2323,3} = f_2^2 P_{2442,3} = \frac{L_0}{2} P_{3434,3} = -\frac{L_0}{2} P_{3443,3} = \frac{r_0^2}{3 L_0} (f_2 f_2'' - f_2 f_2''');
\]

\[-\frac{1}{5} W_{1212,2} = \frac{1}{L_0} W_{1313,2} = \frac{1}{L_0 f_2^2} W_{1414,2} = -\frac{f_2^2}{L_0} W_{2323,2} = -\frac{f_2^2}{L_0 f_2} W_{2424,2} = -\frac{f_2^2}{L_0^2 f_2^2} W_{3434,2} = \frac{r_0^2}{6 L_0} (f_1' f_1'' - f_1 f_1'''),
\]
\[
\frac{L_0f_2^2}{f_1^2}W_{1212,3} = \frac{f_2^2}{f_1^2}W_{1313,3} = \frac{1}{f_1^2}W_{1414,3} = -f_2^2W_{2323,3} = -W_{2424,3} = \frac{1}{5L_0}W_{3434,3} = \frac{r_0^2}{6L_0}(f_2''f_2'' - f_2f_2''');
\]
\[
\frac{1}{L_0}K_{1313,2} = \frac{1}{L_0f_2^2}K_{1414,2} = -f_2^2K_{2323,2} = \frac{f_2^2}{L_0f_2^2}K_{2424,2} = \frac{r_0^2}{2L_0}(f_1f_1'' - f_1f_1'''),
\]
\[
\frac{f_2^2}{f_1^2}K_{1313,3} = \frac{1}{f_1^2}K_{1414,3} = -f_2^2K_{2323,3} = -K_{2424,3} = \frac{r_0^2}{2L_0}(f_2f_2'' - f_2f_2''').
\]

From above we can state the following:

**Proposition 3.5.** The metric (3.7) admits the following curvature properties:

1. If \(L_0f_2f_2'' + f_1f_2'' \neq 0\), it is conformally recurrent manifold \((\nabla C = \Pi \otimes C)\) for 

\[
\Pi = \left\{ 0, -\frac{f_2''(f_2'' - f_2f_2''')}{(f_2f_2'' - f_2f_2'')}f_1(f_0f_2f_2'' + f_1f_2'''), f_2(f_0f_2f_2'' + f_1f_2'''), 0 \right\},
\]

2. If \(L_0f_2f_2'' - f_1f_2'' \neq 0\), it is projectively supergeneralized recurrent manifold \((\nabla P = \Pi \otimes P + \Phi \otimes S \wedge + \Phi \otimes g \wedge S + \Psi g \wedge g)\) for

\[
\Pi = \left\{ 0, \frac{-f_2''(f_2'' - f_2f_2''')}{(f_2f_2'' - f_2f_2'')}f_1(f_0f_2f_2'' + f_1f_2'''), f_2(f_0f_2f_2'' + f_1f_2'''), 0 \right\},
\]

\[
\Phi = \left\{ 0, \frac{L_0f_2f_2f_2''(f_2'' - f_2f_2'')}{(f_2f_2'' - f_2f_2'')}, f_2L_0f_2f_2''(f_2'' - f_2f_2'''), 0 \right\},
\]

\[
\Theta = \left\{ 0, \frac{L_0f_2f_2f_2''(f_2'' - f_2f_2'')(f_0f_2f_2'' + f_1f_2''')}{(f_2f_2'' - f_2f_2'')(f_0f_2f_2'' + f_1f_2''')), f_2L_0f_2f_2f_2''(f_0f_2f_2'' + f_1f_2'''), 0 \right\},
\]

3. If \(L_0f_2f_2' - f_1f_2'' \neq 0\), it is a special type of conicircularly super generalized recurrent manifold \((\nabla W = \Pi \otimes W + \Phi \otimes S \wedge + \Psi g \wedge g)\) for

\[
\Pi = \left\{ 0, -\frac{f_2''(f_2'' - f_2f_2'')}{2f_1f_2}, -\frac{f_2f_2'' - f_2f_2'''}{2f_2f_2''}, 0 \right\},
\]

\[
\Phi = \left\{ 0, -\frac{f_2''(f_2'' - f_2f_2'')}{4f_1f_2}, \frac{f_2f_2'' - f_2f_2'''}{4f_2f_2''}, 0 \right\},
\]

\[
\Psi = \left\{ 0, \frac{(f_2'' - f_2f_2')f_2''(f_0f_2f_2'' + f_1f_2''')}{2L_0f_2f_2f_2''(f_0f_2f_2'' + f_1f_2''')}, \frac{(f_2'' - f_2f_2')(f_0f_2f_2'' + f_1f_2''')}{2L_0f_2f_2f_2''(f_0f_2f_2'' + f_1f_2''')}, 0 \right\},
\]

4. If \(L_0f_2f_2' + f_1f_2'' \neq 0\), it is conharmonically recurrent manifold \((\nabla K = \Pi \otimes K)\) for

\[
\Pi = \left\{ 0, -\frac{L_0f_2f_2f_2''(f_2'' - f_2f_2'')}{f_1(f_0f_2f_2'' + f_1f_2''')}, -\frac{f_2f_2' - f_2f_2'''}{f_1(f_0f_2f_2'' + f_1f_2''')} \right\}.
\]

In view of section 2 we can calculate the non zero components of the \((0, 6)\)-tensors \(C \cdot R, P \cdot R, Q(g, R)\) and \(Q(S, R)\). On observation of their linear dependency we have the following:

**Proposition 3.6.** The metric (3.7) admits the following structures:

(i) \(R \cdot R = 0\) i.e., semisymmetric, (ii) \(C \cdot R = L_{11}Q(g, R)\) for \(L_{11} = -\frac{L_0f_2f_2'' + f_1f_2''}{6r_0^2f_1f_2}\) i.e., pseudosymmetric due to conformal curvature tensor, (ii) \(P \cdot R = L_{22}Q(S, R)\) for \(L_{22} = -\frac{1}{3}\) i.e., Ricci generalized projectively pseudosymmetric.

**Corollary 3.4.** (i) \(R \cdot R = 0\) but \(\nabla R = 0\) if the conditions \(f_1''f_1'' - f_1f_1''' = 0\) and \(f_2''f_2'' - f_2f_2''' = 0\) hold,
(ii) $R \cdot R = 0$ and hence $R \cdot C = R \cdot P = R \cdot W = R \cdot K = R \cdot S = 0$,

(iii) $C \cdot R = L_{11} Q(g, R)$ for $L_{11} = -\frac{L_0 f_2 f_1'' + f_2 f_1 f_1'}{6r_0^2 f_1 f_2}$ and hence $C \cdot C = L_{11} Q(g, C)$, $C \cdot P = L_{11} Q(g, P)$, $C \cdot W = L_{11} Q(g, W)$, $C \cdot K = L_{11} Q(g, K)$, $C \cdot S = L_{11} Q(g, S)$.

From the propositions (3.4), (3.5) and (3.6) we can state the following theorem:

**Theorem 3.3.** The charged Nariai type metric (3.7) is (i) semisymmetric, Ricci generalized projectively pseudosymmetric and pseudosymmetric due to conformal curvature tensor (ii) 2-quasi Einstein and Roter type manifold, (iii) generalized quasi Einstein both in the sense of Chaki [14] and De and Ghosh [15], (iv) special type of super generalized recurrent and projectively super generalized recurrent manifold, (v) conformally recurrent and special type of concircularly super generalized recurrent manifold, (vi) Reimann compatible as well as projective compatible.

**Remark 3.4.** The metric (3.7) does not satisfy the following geometric structures:

1. the metric is not R–space or C–space or P–space or W–space or K–space by Venzi,
2. it is not weakly symmetric and hence not Chaki pseudosymmetric,
3. it is neither recurrent nor generalized recurrent or hypergeneralized recurrent or weakly generalized recurrent,
4. its curvature 2-forms for Reimann curvature or projective curvature or concircular curvature is not recurrent,
5. $\text{div} R \neq 0$, $\text{div} C \neq 0$, $\text{div} P \neq 0$, $\text{div} W \neq 0$, $\text{div} K \neq 0$,
6. the Ricci tensor is neither conformally nor concircularly or cnharmonicaly compatible.

4. **Conclusion**

The charged Nariai spacetime is an exact solution of Einstein-Maxwell field equations and mathematically a topological product spacetime metric. In this present paper, we have investigated the curvature restricted geometric structures of the charged Nariai spacetime and it is found that such spacetime is locally symmetric and satisfies the pseudosymmetric type conditions $C \cdot R = \frac{1+L_0}{6r_0} Q(g, R)$ and $P \cdot R = -\frac{1}{3} Q(S, R)$. Also it is 2–quasi Einstein, generalized quasi-Einstein and Roter type manifold. We have also investigated the curvature restricted geometric structures of the charged Nariai type metric and showed that such metric is not locally symmetric but semisymmetric and its Ricci tensor is neither Codazzi nor cyclic parallel or recurrent but generalized recurrent. Also under certain restrictions, it admits recurrent type structures on several curvature tensors such as special type of $SGK_n$, special type of $W-SGK_n$, $P-SGK_n$, $C-K_n$ and $K-K_n$. Moreover it is also 2–quasi Einstein, generalized quasi Einstein, Roter type.
The Ricci tensor is Riemann compatible as well as projective compatible. As a consequence of the investigation of curvature restricted geometric structures of the charged Nariai type metric we obtain a new class of semisymmetric and generalized recurrent type semi Riemannian manifolds and the charged Nariai type metric belongs to that class.

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