GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE DYSTHE EQUATION IN $L^2(\mathbb{R}^2)$

RAZVAN MOSINCAT, DIDIER PILOD, AND JEAN-CLAUDE SAUT

Abstract. This paper focuses on the Dysthe equation which is a higher order approximation of the water waves system in the modulation (Schrödinger) regime and in the infinite depth case. We first review the derivation of the Dysthe and related equations. Then we study the initial-value problem. We prove a small data global well-posedness and scattering result in the critical space $L^2(\mathbb{R}^2)$. This result is sharp in view of the fact that the flow map cannot be $C^3$ continuous below $L^2(\mathbb{R}^2)$. Our analysis relies on linear and bilinear Strichartz estimates in the context of the Fourier restriction norm method. Moreover, since we are at a critical level, we need to work in the framework of the atomic space $U^2_S$ and its dual $V^2_S$ of square bounded variation functions. We also prove that the initial-value problem is locally well-posed in $H^s(\mathbb{R}^2)$, $s > 0$. Our results extend to the finite depth version of the Dysthe equation.

1. Introduction

1.1. Introduction of the model and physical motivation. It is well-known that the full water waves system is too complex to be used to describe rigorously long time dynamics of waves except in physically trivial situations. A well-known procedure (going back to Lagrange!) is instead to derive simpler asymptotic models in various relevant regimes. We refer to the book [27] for a systematic and rigorous approach. The first step is of course to find suitable non dimensional small parameters which will allow to perform approximations (by expansions with respect to the parameters) of the nonlocal Dirichlet to Neumann operator which appears in the Zakharov-Craig-Sulem formulation of the water waves problem.

In this introductory section we motivate and recall the derivation of higher order nonlinear Schrödinger equations in the context of surface water waves.

We are interested here in the so-called modulation regime which aims to describe the weakly nonlinear modulation of a train of surface gravity (or gravity-capillary) waves. More precisely one introduces $h$ a typical depth of the fluid layer, $a$ a typical amplitude of the wave and $\lambda$ a typical wavelength in the horizontal directions (assumed to be isotropic). Then we set $\epsilon = a/h$ and $\mu = h^2/\lambda^2$.

In the modulation regime, the relevant small parameter is the wave steepness, $\varepsilon = a/\lambda = \epsilon \sqrt{\mu}$. We denote $k \in \mathbb{R}^d$, $d = 1, 2$, (respectively $k = |k|$) the wave vector (respectively wave number) of the wave packet. In this regime, one assumes that

$$\frac{|\Delta k|}{k} \ll 1,$$

where $\Delta k$ is the modulated wave vector. The dispersion relation will be in particular Taylor expanded around this wave number. At first order in $\varepsilon$ (that is neglecting the $O(\varepsilon^2)$ terms in the expansion), one obtains nonlinear Schrödinger (NLS) type equations or systems (see [27] Chapter 8 for a rigorous derivation).

Date: July 6, 2020.

2010 Mathematics Subject Classification. Primary: 35A01, 35Q53; Secondary: 35Q60.

Key words and phrases. Dysthe equation, Initial-value problem, Well-posed.
However, as it is recalled in [27] when the steepness \( \varepsilon \) is not very small, the NLS approximation does not match very well with some exact computations, see for instance Longuet-Higgins (32, 33). As recalled in [7], the NLS equation exhibits an unbounded region of Benjamin-Feir instability in the case of two-dimensional sideband perturbations, which extend outside the regime of a narrow-band spectrum. As a result, energy initially contained at low wave numbers can leak out to higher modes, as shown in the numerical simulations of Martin and Yuen [34].

To overcome those shortcomings Dysthe [11] (see also [19]) proposed in the infinite depth case \((h = +\infty)\) to include the \(O(\varepsilon^2)\) terms neglected in the NLS equation, under the bandwidth assumption

\[
\frac{|\Delta k|}{k} = O(\varepsilon).
\]

A consequence of the higher order expansion is the appearance of a mean field \(\phi\) which solves an elliptic boundary-value problem and which can be expressed in function of the amplitude \(\psi\) via a Riesz transform.

In the infinite depth case (thus with the dispersion relation \(\omega(k) = |k|^{1/2}\)), and taking (without loss of generality) the carrier wave number \(k = e_x\), one obtains the following equation in dimensionless form for the slow envelope of the wave (see [11] [19] for a formal derivation and [27] for a rigorous one):

\[
2i(\partial_t \psi + \frac{1}{2} \partial_x \psi) + \varepsilon(-\frac{1}{4} \partial_x^2 + \frac{1}{2} \partial_y^2)\psi - 4\varepsilon|\psi|^2\psi
= \varepsilon^2 \left( \tilde{\mathcal{R}}_x f(\xi, \mu) - \varepsilon \gamma |\psi|^2 \psi \right)
+ 2i\varepsilon^2 \left( \psi \partial_y \bar{\psi} - \bar{\psi} \partial_y \psi \right)
- 10\varepsilon^2 i\psi^2 \partial_x \psi - 4\varepsilon^2 \psi \partial_x \mathcal{R}_x(|\psi|^2),
\]

where \(\mathcal{R}_x\) is the Riesz transform defined by \(\mathcal{R}_x f(\xi, \mu) = -i \frac{\delta}{(\xi, \mu)} f(\xi, \mu)\). If \(\tau\) and \((X, Y)\) are the physical variables, the derivatives \(\partial_t, \partial_x, \partial_y\) are taken with respect to the slow variables \(t = \varepsilon \tau, x = \varepsilon X, y = \varepsilon Y\). Note that the relevant time scale for (1.1) is \(t \sim O(\frac{1}{\varepsilon})\), \(\tau \sim O(\frac{1}{\varepsilon^2})\) for the physical time.

One can obtain similar equations when surface tension is taken into account (gravity-capillary waves), we refer for instance to [18] and to the survey article [9]. They write

\[
2i(\partial_t \psi + c_g \partial_x \psi) + \varepsilon(p \partial_x^2 + q \partial_y^2) - \varepsilon \gamma |\psi|^2 \psi
= \varepsilon^2 \left( -is \partial_x \partial_y^2 \psi - iv \partial_x^3 \psi + iv |\psi|^2 \partial_x \psi + \psi \partial_x \mathcal{R}_x(|\psi|^2) \right),
\]

where \(c_g = \frac{3\kappa^2 + 2\kappa + 1}{2(1 + \kappa)}\) is the group velocity and \(p, q, s, r, u\) and \(v\) are real parameters depending on the surface tension parameter \(\kappa \geq 0\). More precisely

\[
p = \frac{3\kappa^2 + 6\kappa - 1}{4(1 + \kappa)^2}, \quad q = \frac{1 + 3\kappa}{2(1 + \kappa)},
\]

\[
r = -\frac{(1 - \kappa)(1 + 6\kappa + \kappa^2)}{8(1 + \kappa)^3}, \quad s = \frac{3 + 2\kappa + 3\kappa^2}{4(1 + \kappa)^2},
\]

\[
\gamma = \frac{8 + \kappa + 2\kappa^2}{8(1 - 2\kappa)(1 + \kappa)}, \quad u = \frac{(1 - \kappa)(8 + 2\kappa^2)}{16(1 - 2\kappa)(1 + \kappa)^2},
\]

\[
v = \frac{3(4\kappa^4 + 4\kappa^3 - 9\kappa^2 + \kappa - 8)}{8(1 + \kappa)^2(1 - 2\kappa)^2}.
\]

When \(\kappa = 0\), one obtains a variant of the Dysthe equation. Note that \(q\) and \(s\) are strictly positive, while \(p\) can achieve both signs. In particular it is negative for pure gravity waves as in the original Dysthe equation and positive for pure capillary waves. In the latter case (\(\kappa\) infinite) (1.2) becomes a variant of the modified Zakharov-Kuznetsov equation, i.e.

1The order one expansion in \(\varepsilon\) leads to the nonelliptic nonlinear Schrödinger equation, see [11].
\[(1.3) \quad 2i(\partial_t \psi + \frac{3}{2} \partial_x \psi) + \frac{3}{4} \partial_x^2 \psi + \frac{3}{2} \partial_y^2 \psi + \frac{1}{8} |\psi|^2 \psi = - \frac{3}{4} i \partial^3_{xxy} \psi - \frac{i}{8} \partial_x^3 \psi - \frac{1}{16} \psi^2 \partial_x^2 \psi + \frac{3}{8} i |\psi|^2 \partial_x \psi + \psi \partial_x L_h(|\psi|^2) .
\]

The solutions of the Dysthe equation have been compared to those of the NLS equation (see [39 31 17]) showing differences with solutions of the NLS equation, and in particular an increase of the group velocity and asymmetry of the envelope of the surface elevation with respect to the peak of the wave profile. We also refer to [20] for comparisons with experimental data.

**Remark 1.1.** Stiassnie [38] and Stiassnie-Shemer [39] have derived the Dysthe equation as a particular limit of the integral equation introduced by Zakharov in [41].

In the finite depth situation, that is when the domain of the fluid is the strip \([-h < z < 0]\), the complex envelope \(\psi\) of the wave is coupled to the potential \(\phi\) of the induced current (see [39]). One gets (dropping the dependance in \(\varepsilon\)):

\begin{align*}
&\partial_t \psi + \frac{\omega \varepsilon}{2k} \partial_x \psi + i \frac{\omega \varepsilon}{2k} \left( \frac{1}{4} \partial_x^2 \psi - \frac{1}{2} \partial_y^2 \psi \right) + \frac{1}{8} \omega k^2 |\psi|^2 \psi - \frac{1}{16} k \left( \partial_x^3 \psi - 6 \partial^3_{xxy} \psi \right) - \frac{\omega k}{2} \psi^2 \partial_x \psi + \frac{3}{2} \omega k |\psi|^2 \partial_x \psi + i \varepsilon \psi \partial_x \phi = 0, \\
&\partial_x^2 \phi + \partial_x \phi + \partial_x^2 \phi = 0 \quad (-h < z < 0), \\
&\partial_x \phi = \frac{\omega}{2} \partial_x L_h(|\psi|^2) \quad (z = 0), \\
&\partial_x \phi = 0 \quad (z = -h).
\end{align*}

Although this paper is devoted to the two-dimensional case we comment briefly on the interesting equation arising in the one-dimensional case obtained from (1.3) by dropping the terms with a partial derivative in \(y\). The system for \(\phi\) is easily solved by taking the Fourier transform in \(x\) and we find that

\[\partial_x \phi_{z=0} = \frac{\omega}{2} \partial_x L_h(|\psi|^2)\]

where the nonlocal operator \(L_h\) is defined in Fourier variables by

\[\widehat{L_h f}(\xi) = i \coth(h\xi) \hat{f}(\xi).\]

Note that in the infinite depth case \(h = +\infty\), \(L_{+\infty}\) is given by \(\widehat{L_{+\infty} f}(\xi) = i \text{sign}(\xi) \hat{f}(\xi)\) that is \(L_{+\infty} = -\mathcal{H}\) where \(\mathcal{H}\) is the Hilbert transform.

We thus can write (1.3) in the one-dimensional case as a single equation:

\[(1.5) \quad \partial_t \psi + \frac{\omega}{2k} \partial_x \psi + i \frac{\omega}{8k^2} \partial_x^2 \psi + \frac{i}{2} \omega k^2 |\psi|^2 \psi - \frac{1}{16} k^3 \partial_x^2 \psi - \frac{\omega k}{4} \psi^2 \partial_x \psi + \frac{3}{2} \omega k |\psi|^2 \partial_x \psi + i \frac{\omega k}{2} \psi \partial_x L_h(|\psi|^2) = 0 .\]

This equation is reminiscent of the complex modified KdV equation

\[(1.6) \quad \partial_x u + u^2 \partial_x u + \partial_x^3 u = 0 .\]

In fact by eliminating the transport term in (1.5) by the change of variable \(X = x - \frac{\omega \varepsilon}{2k} t\) (we will keep the notation \(x\) for the spatial variable), then writing, with \(\alpha = -\frac{1}{16} k^3\) and \(\beta = \frac{\omega k}{2}\),

\[\left( \xi + \frac{\beta}{3\alpha} \right)^3 = \xi^3 + \frac{\beta}{\alpha} \xi^2 + \frac{\beta^2}{3\alpha^2} \xi + \frac{\beta^3}{27\alpha^3},\]

an easy computation shows that the fundamental solution of the linearization of (1.5) can be expressed as

\[\mathcal{R}(x,t) = \frac{1}{(t\alpha)^{1/3}} \exp \left( \frac{2it\beta^3}{27\alpha^2} \right) \exp \left( -\frac{i\beta x}{2\alpha^2} \right) \mathcal{A} \left( \frac{1}{t^{1/3} \alpha^{1/3}} \left( x - \frac{\beta^2}{3\alpha} t \right) \right),\]

where we have used here the Airy function

\[\mathcal{A}(z) = \int_{-\infty}^{\infty} e^{i(\xi^3 + iz)} d\xi .\]
It follows that the dispersive estimates for $\mathfrak{A}$ are essentially the same as those of the linearized KdV equation and one obtains for (1.6) the same results as for the complex modified KdV equation (see [22]), that is the local well-posedness of the initial value problem in $H^s(\mathbb{R})$, $s > \frac{1}{4}$.

In the two-dimensional case, the mean flow potential is solution of the system (see [39])

\begin{align}
\frac{\partial_t^2 \phi + \partial_x^2 \phi + \partial_z^2 \phi = 0}{\partial_z \phi = \frac{\omega}{2} \partial_x |\psi|^2, \quad z = 0,} \\
\partial_x \phi = 0, \quad z = -h.
\end{align}

This system is easily solved after Fourier transform in the $(x, y)$ variables and one finds

\begin{align}
\partial_t \psi + \frac{\omega}{2k} \partial_x \psi + i \frac{\omega}{2k} \left( \frac{1}{4} \partial_x^2 \psi - \frac{1}{2} \partial_y^2 \psi \right) + i \frac{\omega k}{3} |\psi|^2 \psi \\
- \frac{1}{16} \frac{w}{k^3} \left( \partial_x^2 \psi - 6 \partial_{xy} \psi \right) - \frac{\omega k}{4} \psi^2 \partial_x \psi + \frac{3}{2} \frac{\omega k}{2} |\psi|^2 \partial_x \psi + i \frac{\omega k}{2} \psi \partial_z \mathcal{L}_h(|\psi|^2) = 0.
\end{align}

Observe that formally $\mathcal{L}_h \rightarrow \mathcal{R}_x$ as $h \rightarrow +\infty$. Moreover, the symbol of the operator $\partial_z \mathcal{L}_h$ is bounded close to the origin since

\begin{align}
-i \frac{\xi^2}{\sqrt{\xi^2 + \mu^2}} \coth[(\xi^2 + \mu^2)^{1/2} h] \sim -i \frac{\xi^2}{h \xi^2 + \mu^2} \lesssim 1 \quad \text{whenever } |(\xi, \mu)| \lesssim 1.
\end{align}

**Remark 1.2.** Trulsen and Dysthe [12] have extended the Dysthe equation by relaxing the narrow bandwidth condition to

\[ \frac{|\Delta k|}{k} = O(\varepsilon^{1/2}). \]

More precisely, in the finite depth case, Trulsen and Dysthe obtained the system posed on the strip $\mathbb{R}^2 \times (-h, 0)_z$ (dropping the dependence on $\varepsilon$), where the complex envelope $\psi$ of the wave is coupled to the potential $\phi$ of the induced current:

\begin{align}
\partial_t \psi + \frac{1}{4} \partial_x^2 \psi + \frac{1}{8} \partial_y^2 \psi - \frac{1}{16} \partial_x^3 \psi + \frac{3}{8} \partial_{xy} \psi \\
- \frac{5}{16} \partial_x^2 \psi + \frac{1}{16} \partial_{xy} \psi - \frac{3}{16} \partial_y^2 \psi + \frac{7}{32} \partial_x^3 \psi + \frac{3}{32} \partial_{xy} \psi \\
+ \frac{1}{32} \partial_{xy} \psi + \frac{3}{32} \partial_x^2 \psi - \frac{1}{32} \psi \partial_x \psi + \frac{\omega}{2} \psi \partial_z \phi = 0 \quad \text{at } z = 0,
\end{align}

\begin{align}
&\partial_x \phi = 0 \quad (z = -h), \\
&\partial_z \phi = 0 \quad (z = 0),
\end{align}

This has the effect of adding to the Dysthe equation fourth- and fifth-order linear dispersive terms, which do not add any further complication to the well-posedness analysis.

In the rest of this paper we fix $\varepsilon = 1$ and study the initial-value problem for the Dysthe equation written in the form

\begin{align}
2i(\partial_t v + \frac{1}{2} \partial_x v) + (-\frac{1}{4} \partial_x^2 v + \frac{1}{2} \partial_y^2 v) - i \frac{\omega}{8} \partial_x^2 v - 6 \partial_{xy} v \\
= 4 |v|^2 v + 2ivv \partial_x v - \bar{v} \partial_x v - 10i |v|^2 \partial_x v - 4v \partial_x \mathcal{R}_x(|v|^2),
\end{align}

where $v = v(t, x, y)$ is a complex-valued function, $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$ and $\mathcal{R}_x$ denotes the Riesz transform in the $x$ variable.

Concerning the conservation laws of Dysthe type systems, one checks readily that they conserve formally the $L^2$-norm, namely that

\begin{align}
M[v](t) = \int |v(t, x, y)|^2 dx dy = M[v](0).
\end{align}
On the other hand and contrary to the nonlinear Schrödinger equation, they do not seem to possess a conserved energy. Note however that Hamiltonian versions of the Dysthe equation were derived in [13, 7, 6, 8] by expanding the Hamiltonian in the Zakharov-Craig-Sulem formulation of the water waves system.

1.2. Statement of main results and comments. We now review known results on the initial-value problem (IVP) for Dysthe type systems. Local well-posedness in analytic classes for a family of systems comprising the Dysthe ones has been established by A. de Bouard [1]. The local well-posedness for initial data in $H^3(\mathbb{R}^2)$ has been proved by Chihara [5], who applied techniques developed for derivative nonlinear Schrödinger equations to these third-order equations. Koch and Saut [24] obtained the local well-posedness for data in $H^2(\mathbb{R}^2)$ by using solely the local smoothing effects of the underlying group. The result in [24] was improved very recently to $s > 1$ by Grande, Kurianski and Staffilani [14]. The proof is based on a nontrivial adaptation of the Kenig, Ponce and Vega techniques [22] to this two dimensional problem. In particular a new maximal function estimate associated to the linear part of Dysthe is proved. The authors also proved that the IVP associated to (1.12) is ill-posed in $H^s(\mathbb{R}^2)$ for $s < 0$ in the sense that the flow map data-solution cannot be $C^3$ in $H^s(\mathbb{R}^2)$, $s < 0$.

Finally we recall that other papers have been devoted to the initial-value problem for relevant third order nonlinear Schrödinger equations involving spatial derivatives in the nonlinearities, see for instance [27].

The aim of this article is to improve these results by using the Fourier restriction norm method. Our first theorem proves global well-posedness and scattering for small initial data down to the critical space $L^2(\mathbb{R}^2)$.

**Theorem 1.1.** There exists $\delta > 0$ such that for any $v_0 \in L^2(\mathbb{R}^2)$ with $\|v_0\|_{L^2} < \delta$, there exists a unique $v \in \mathcal{Y} \subset C([0,T] : L^2(\mathbb{R}^2))$ solution to (1.12) with $v|_{t=0} = v_0$, the flow map $v_0 \mapsto v$ from $\{v_0 \in L^2(\mathbb{R}^2) : \|v_0\|_{L^2} < \delta\}$ to $\mathcal{Y}$ is smooth, and there exist $v_0^\pm \in L^2(\mathbb{R}^2)$ such that $v(t) - v_{0\text{lim}}(t) \to 0$ in $L^2(\mathbb{R}^2)$, as $t \to \pm \infty$,

where $v_{0\text{lim}}^\pm$ is the solution to the linear Dysthe equation starting from $v_0^\pm$.

**Remark 1.3.** The resolution space $\mathcal{Y}$ is defined in Section 4 – see (4.11) and (2.11).

**Remark 1.4.** This result is sharp due to the ill-posedness result proved in [14].

We also obtain a local well-posedness result in $H^s(\mathbb{R}^2)$, $s > 0$.

**Theorem 1.2.** Let $s > 0$. For any $v_0 \in H^s(\mathbb{R}^2)$ there exist $T = T(\|v_0\|_{H^s}) > 0$ and a unique solution $v \in \mathcal{X}_{-T^+} \subset C([0,T] : H^s(\mathbb{R}^2))$ to (1.12) with $v|_{t=0} = v_0$. Moreover, for any $T' \in (0,T)$, there exists a neighborhood $\mathcal{U}$ of $v_0 \in H^s(\mathbb{R}^2)$ such that the flow map data solution $w_0 \in \mathcal{U} \mapsto w \in C([0,T] : H^s(\mathbb{R}^2))$ is smooth.

**Remark 1.5.** The well-posedness results obtained in Theorems 1.1 and 1.2 for the Dysthe equation in dimension $d = 2$ hold in lower regularity than the ones for the corresponding one-dimensional equation (1.2), for which well-posedness only holds in $H^{s}(\mathbb{R})$, $s \geq \frac{1}{2}$ by using the Kenig, Ponce and Vega techniques of [22]. This is due to the fact that the two-dimensional equation is more dispersive than its one-dimensional counterpart which allows to recover 1/4 of a derivative instead of only 1/8 of a derivative in the $L^4$-Strichartz estimates. As explained below, these estimates are crucial to handling the resonant high×high×high frequency interactions region in the nonlinear term.

**Remark 1.6.** It is clear from the proofs in Section 4 and the bound (1.10) that the results of Theorems 1.1 and 1.2 also hold for the finite depth Dysthe equation (1.9).

We now discuss the main ingredients for the proofs of Theorems 1.1 and 1.2. First observe that by a change of variables of the form

$$ (t,x,y,v) \mapsto (t,x + a_1 t, y, e^{ia_2 x} e^{ia_3 t} v) $$

(1.14)
where \(a_1, a_2, a_3\) are real numbers (see Lemma 2.1 below) we can remove the first- and second-order derivative linear terms on the left-hand side of (1.12). Thus, for the sake of simplicity, we work on the following simplified version of the Dysthe equation

\[
\partial_t u + \partial_x \left( \partial_x^2 u - 3 \partial_y^2 u \right) = \sum_{j=1}^{4} c_j N_j(u, u, u),
\]

where \(c_1, c_2, c_3 \in \mathbb{C}\) are constants and the nonlinearities \(N_j(u, u, u)\) are given by

\[
\begin{align*}
N_1(u, u, u) &= |u|^2 u, \\
N_2(u, u, u) &= |u|^2 \partial_x u, \\
N_3(u, u, u) &= u^2 \partial_x \bar{u}, \\
N_4(u, u, u) &= u \partial_x \mathcal{R}_x(|u|^2).
\end{align*}
\]

We note that for \(c_1 = 0\), the equation is invariant under the scaling transformation

\[
u(t, x, y) \mapsto \nu(t, \lambda x, \lambda y) := \lambda \nu(t, \lambda x, \lambda y)
\]

and since \(\|\nu_\lambda(0)\|_{L^2(\mathbb{R}^2)} = \|\nu(0)\|_{L^2(\mathbb{R}^2)}\) we regard \(L^2(\mathbb{R}^2)\) as the scaling critical Sobolev space of (1.15) (and of (1.12) due to the invariance of the Sobolev norms under (1.14)). We will prove the equivalent results of Theorems 1.1 and 1.2 for the IVP associated to (1.15) which has the advantage of having a homogeneous linear part. Theorems 1.1 and 1.2 will then be concluded by inverting the change of variables (1.14).

The proof of the small data global well-posedness and scattering result in \(L^2(\mathbb{R}^2)\) for (1.15) relies on linear and bilinear Strichartz estimates in the context of the Fourier restriction norm method. We refer for instance to the work of Hadac, Herr and Koch [10] and Molinet, Saut and Tzvetkov [30] for the the KP-II equation, Grünrock and Herr [13] and Molinet, Pilod [35] for the Zakharov-Kuznetsov equation, Kinoshita [23] for the modified Zakharov-Kuznetsov equation and Angelopoulos [1] and Kazeykina, Muñoz [21] for the Novikov-Veselov equation. The bilinear Strichartz estimates are used to deal with the low \(\times\) low \(\times\) high and low \(\times\) high \(\times\) high frequency interactions in the nonlinearity. To handle the resonant case corresponding to the high \(\times\) high \(\times\) high frequency interactions, we use a sharp linear Strichartz estimate, whose proof is a corollary of a result of Carbery, Kenig, and Ziesler [11] for homogeneous dispersive operators (see Subsection 3.1). These estimates could also be obtained from the dispersion estimates of Ben-Artzi, Koch and Saut in [2]. Finally, since we are at a critical level, we need to work in the framework of the atomic space \(U^2_3\) and its dual \(V^2_3\) of square bounded variation functions introduced by Koch and Tataru in [25] (see also [10]) in order to derive the main trilinear estimate.

The proof of the arbitrarily large data local well-posedness in \(H^s(\mathbb{R}^2)\), \(s > 0\), for (1.15) follows the same strategy as above. Note however that the use of \(U^2 - V^2\) space is not needed anymore since the problem is subcritical. It is thus enough to work with the usual \(X^{s,b}\) spaces.

It is worth noting that this method of proof only relies on the fact that the nonlinearities \(N_j(u, u, u), j = 1, \ldots, 4\) in (1.15) are cubic with at most one derivative, but not on their specific structures. Indeed, it is clear from Plancherel’s identity that the linear Strichartz estimate (3.6) and the bilinear Strichartz estimates (3.9)-(3.10) also hold if one replace \(\varphi, u_1, u_2\) by \(\mathcal{F}^{-1}(|\varphi|), \mathcal{F}^{-1}(|\tilde{u}_1|)\) and \(\mathcal{F}^{-1}(|\tilde{u}_2|)\).

We observe that contrary to the linear Zakharov-Kuznetsov symbol, the Hessian of the dispersion relation of the Dysthe equation (1.15) is sign-definite, which allows us to recover 1/4 of a derivative in the \(L^4\)-Strichartz estimate (see (3.7)). A similar estimate only holds outside of cones centered at the origin for the linear Zakharov-Kuznetsov equation or with a weaker gain of 1/8 derivatives (see the discussion in Remark 3.1. of [35]). This is the reason why we can reach \(L^2(\mathbb{R}^2)\) in Theorem 1.1 while a similar result only holds for \(H^s(\mathbb{R}^2), s \geq \frac{1}{4}\) for the modified Zakharov-Kuznetsov equation (see [28, 37, 23]). In particular, Kinoshita constructed in [23] a

\[\text{We prefer not to normalize both constants in } \partial_x (\partial_x^2 - 3 \partial_y^2) \text{ and thus work with a simpler Hessian expression for its Fourier symbol (see 3.8).}\]
counterexample localized in the resonant high×high×high frequency interaction region proving
the sharpness of this result.

The argument of Kinoshita [23] could be applied to the limit version of the Dysthe equation
with surface tension (1.2) providing a local well-posedness result in $H^s\hat{\mathbb{R}^2}$ for the associated
Cauchy problem. Note that a change of variable similar to the one in Lemma 2.1 could also be
applied in this case to obtain a homogeneous linear part, but this time of the Zakharov-Kuznetsov
type.

The rest of the paper is organized as follows: in the next section we introduce the notations, the
crucial change of variable which allows to work with a homogenous linear part, define the function
spaces and recall some of their important properties. In Section 3 we recall the linear Strichartz
estimates and derive the bilinear estimates. Those estimates are used in Section 4 to prove the
trilinear estimates in $\mathbb{R}^2$. Finally, we conclude the paper in Section 5 with some interesting open
questions.

2. Preliminaries and linear estimates

2.1. Notation. For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a
positive constant $c$ such that $a \leq cb$, we use $a \lesssim b$ when we find it necessary to make it explicit
that the constant $c$ depends on the parameter $\theta$, and $a \ll b$ when $c$ is small (e.g. $c \leq 10^{-2}$).
We also write $a \sim b$ when $a \leq b$ and $b \leq a$. We denote by $|S|$ the Lebesgue measure of a measurable set
$S$ of $\mathbb{R}^d$, whereas $\#F$ denotes the number of elements of a finite set $F$. For $u = u(t, x, y) \in \mathcal{S}(\mathbb{R}^3)$,
we use the notation $\mathcal{F}(u)$, or $\hat{u}$ to denote the space-time Fourier transform of $u$, whereas $\mathcal{F}_{xy}(u)$,
or $(u)^{\hat{\mathbb{R}^2}}$, respectively $\mathcal{F}_{t}(u) = (u)^{\hat{\mathbb{R}}} \in \mathcal{S}(\mathbb{R}^3)$, will denote its Fourier transform in space, respectively in
time.

For $s \in \mathbb{R}$, we define $J^s$ and $D^s$, the Bessel and Riesz potentials of order $-s$, by

$$J^s u = \mathcal{F}_{xy}^{-1}((1 + |(\xi, \mu)|^2)^{s/2} \mathcal{F}_{xy}(u)) \quad \text{and} \quad D^s u = \mathcal{F}_{xy}^{-1}((|(\xi, \mu)|^s)^{s} \mathcal{F}_{xy}(u)).$$

We also recall here that the Riesz transform $\mathcal{R}_x$ that appears in the nonlinearity of the Dysthe
equation is given by

$$\mathcal{R}_x u = -i\mathcal{F}_{xy}^{-1}\frac{\xi}{\sqrt{\xi^2 + \mu^2}} \mathcal{F}_{xy}(u)(\xi, \mu).$$

Let $S(t) := e^{-it\partial_\xi(\partial^2_{xy} - 3\partial_y^3)}$ denote the unitary group associated with the linear part of equation
(1.14), which is to say,

$$\mathcal{F}_{xy}(S(t)\varphi)(\xi, \mu) = e^{iw(\xi, \mu) \mathcal{F}_{xy}(\varphi)(\xi, \mu)},$$

where

$$w(\xi, \mu) = \xi^3 - 3\xi\mu^2.$$

We also define the resonance function $R$ by

$$R(\xi_1, \mu_1, \xi_2, \mu_2) := w(\xi_1 + \xi_2, \mu_1 + \mu_2) - w(\xi_1, \mu_1) - w(\xi_2, \mu_2)$$

$$= 3\xi_1\xi_2(\xi_1 + \xi_2) - 3\xi_2\mu_1^2 - 3\xi_1\mu_2^2 - 6(\xi_1 + \xi_2)\mu_1\mu_2.$$

Throughout the paper, we fix a smooth cutoff function $\eta$ such that

$$\eta \in C^\infty_0(\mathbb{R}), \quad 0 \leq \eta \leq 1, \quad \eta_{[-5/4, 5/4]} = 1 \quad \text{and} \quad \text{supp}(\eta) \subset [-8/5, 8/5].$$

For $k \in \mathbb{N}^* = \mathbb{N} \cap [1, +\infty)$, we define

$$\phi(\xi) = \eta(\xi) - \eta(2\xi), \quad \phi_2(\xi, \mu) := \phi(2^{-k}|(\xi, \mu)|).$$

and

$$\psi_{2k}(\xi, \mu, \tau) = \phi(2^{-k}(\tau - w(\xi, \mu))).$$

By convention, we also denote

$$\phi_1(\xi, \mu) = \eta(|(\xi, \mu)|), \quad \text{and} \quad \psi_1(\xi, \mu, \tau) = \eta(\tau - w(\xi, \mu)).$$
Let us define the Littlewood-Paley multipliers by
\begin{equation}
\alpha(1.16)
\end{equation}
and
\begin{equation}
\partial(2.4)
\end{equation}
Next we proceed arguing as in Section 2 in [29]. If Lemma 2.1. The function \( v = v(t, x, y) \) is a solution to
\begin{equation}
\partial_t v + \alpha_1 \partial_x (\partial_x^2 - 3 \partial_y^2)v + \alpha_2 i (\partial_x^2 - \partial_y^2)v + \alpha_3 \partial_x v = \sum_{j=1}^{4} c_j N_j(v, v, v),
\end{equation}
where \( \alpha_1 \in \mathbb{R} \setminus \{0\}, \alpha_2, \alpha_3 \in \mathbb{R}, \) and \( N_j(v, v, v), j = 1, \ldots, 4 \) are the nonlinearities defined in [1,10], if and only if \( u = \mathcal{T}(v) \) is a solution to
\begin{equation}
\partial_t u + \partial_x (\partial_x^2 - 3 \partial_y^2)u = \sum_{j=1}^{4} c_j N_j(u, u, u),
\end{equation}
where
\begin{equation}
\mathcal{T}(v)(t, x, y) := e^{ia_2x}e^{-i\omega t}v(t, x + a_1t, y)
\end{equation}
and
\begin{equation}
a_1 = \frac{\alpha_2^2}{3\alpha_1^2} + \frac{\alpha_3}{\alpha_1}, \quad a_2 = \frac{\alpha_2}{3\alpha_1}, \quad a_3 = \frac{\alpha_2\alpha_3}{3\alpha_1} + \frac{2}{27} \frac{\alpha_3^2}{\alpha_1^2}.
\end{equation}
Remark 2.1. Before applying the transformation in Lemma 2.1 to equation (1.12) with \( \epsilon = 1 \), we first need to rescale the \( y \) variable by a factor \( \sqrt{2} \) to be exactly in the configuration (2.4).
Proof. First, we observe that the linear symbol \( \omega(\xi, \mu) \) associated to (2.4) can be factorized as
\begin{equation}
\omega(\xi, \mu) = \alpha_1 (\xi^3 - 3\xi \mu^2) + \alpha_2 (\xi^2 - \mu^2) - \alpha_3 \xi
\end{equation}
Next we proceed arguing as in Section 2 in [29]. If \( v \) is a solution of the linear part of (2.4), we first define \( f(x, y, t) = e^{i\frac{\alpha_2}{3\alpha_1} x}v(x, y, t) \) and verify that
\begin{equation}
\partial_t f + \alpha_1 \partial_x (\partial_x^2 f - 3 \partial_y^2 f) + \left( \alpha_3 + \frac{\alpha_2^2}{3\alpha_1} \right) \partial_x f - i \left( \frac{\alpha_2\alpha_3}{3\alpha_1} + \frac{2}{27} \frac{\alpha_3^2}{\alpha_1^2} \right) f = 0.
\end{equation}
Now define \( g(x, y, t) = f(x + (\alpha_3 + \frac{\alpha_2^2}{3\alpha_1})t, y, t) \), so that
\begin{equation}
\partial_t g + \alpha_1 \partial_x (\partial_x^2 g - 3 \partial_y^2 g) - i \left( \frac{\alpha_2\alpha_3}{3\alpha_1} + \frac{2}{27} \frac{\alpha_3^2}{\alpha_1^2} \right) g = 0.
\end{equation}
Finally, by defining \( u(t, x, y) = e^{i \left( \frac{\alpha_2\alpha_3}{3\alpha_1} + \frac{2}{27} \frac{\alpha_3^2}{\alpha_1^2} \right) }g(x, y, \frac{t}{\alpha_1}) = \mathcal{T}(v)(t, x, y) \), we conclude that
\begin{equation}
\partial_t u + \partial_x (\partial_x^2 u - 3 \partial_y^2 u) = 0.
\end{equation}
Under this change of variables, one easily checks that the nonlinear terms of (1.12) transform as follows:
\[
\begin{align*}
|v|^2 v &\mapsto e^{-ia_2 x} e^{i\alpha t} |u|^2 u, \\
|v|^2 \partial_x v &\mapsto e^{-ia_2 x} e^{i\alpha t} \left( -i a_2 |u|^2 u + |u|^2 \partial_x u \right), \\
v^2 \partial_x \overline{v} &\mapsto e^{-ia_2 x} e^{i\alpha t} \left( i a_2 |u|^2 u + u^2 \partial_x \overline{v} \right), \\
\partial_x \mathcal{R}_x (|v|^2) &\mapsto e^{-ia_2 x} e^{i\alpha t} u \partial_x \mathcal{R}_x (|u|^2),
\end{align*}
\]
which concludes the proof of the lemma. \(\square\)

### 2.3. Function spaces.

For \(1 \leq p \leq \infty\), \(L^p(\mathbb{R}^d)\) is the usual Lebesgue space with the usual norm \(\| \cdot \|_{L^p}\), and for \(s \in \mathbb{R}\), the Sobolev space \(H^s(\mathbb{R}^d)\) denotes the space of all complex-valued functions with the norm \(\|u\|_{H^s} = \|J^s u\|_{L^2}\). If \(u = u(x, y, t)\) is a function defined for \((x, y) \in \mathbb{R}^2\) and \(t \in [0, T]\) with \(T > 0\), and if \(B\) is one of the spaces defined above, we define the mixed space-time spaces \(L^p_T B_{xy}, L^p_T B_{xy}, L^q_T B^p_{xy}\) by the norms

\[
\|u\|_{L^p_T B_{xy}} = \left( \int_0^T \|u(\cdot, \cdot, t)\|_{B_{xy}}^p dt \right)^{\frac{1}{p}}, \\
\|u\|_{L^p_T B_{xy}} = \left( \int_0^T \|u(\cdot, \cdot, t)\|_{B_{xy}}^p dt \right)^{\frac{1}{p}},
\]
for \(1 \leq p, q < \infty\) with the obvious modifications in the case \(p = +\infty\) or \(q = +\infty\).

Next, we introduce the functional framework introduced by Koch and Tataru in [25] that allows us to reach the critical regularity for the Dysthe equation. We mainly work with the function spaces \(U^p, V^p, W^p\), however the machinery also requires the use of \(U^p, V^p, W^p\) spaces with \(p \neq 2\) (for example in the proof of (3.17) in Corollary [3.7] below).

**Definition 2.2.** Let \(1 \leq p < \infty\) and let \(Z\) be the set of finite partitions \(-\infty = t_0 < t_1 < \cdots < t_K = +\infty\). For \(\{t_k\}_{k=0}^K \in Z\) and \(\{\phi_k\}_{k=0}^{K-1} \subset L^2(\mathbb{R}^2)\) with \(\sum_{k=0}^{K-1} \|\phi_k\|_{L^2} = 1\) and \(\phi_0 = 0\) we call the function \(a : \mathbb{R} \rightarrow L^2(\mathbb{R}^2)\) given by

\[
a = \sum_{k=1}^K \chi_{(t_{k-1}, t_k)} \phi_{k-1}
\]
a \(U^p\)-atom and we define the atomic space

\[
U^p := \left\{ u = \sum_{j=1}^\infty \lambda_j a_j : a_j U^p\text{-atom and } \lambda_j \in \mathbb{R} \text{ with } \sum_{j=1}^\infty |\lambda_j| < \infty \right\}
\]
with norm

\[
\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| : u = \sum_{j=1}^\infty \lambda_j a_j \text{ with } \lambda_j \in \mathbb{R} \text{ and } a_j U^p\text{-atom} \right\}.
\]

The function space \(V^p\) is defined as the normed space of all functions \(v : \mathbb{R} \rightarrow L^2(\mathbb{R}^2)\) such that the limits \(\lim_{t \to -\infty} v(t), \lim_{t \to +\infty} v(t)\) exist, and for which the norm

\[
\|v\|_{V^p} := \sup_{(t_k)_{k=0}^K \in Z} \left( \sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{1/p}
\]
is finite, where we use the convention that \(v(-\infty) := \lim_{t \to -\infty} v(t)\) and \(v(+\infty) := 0\).

For basic properties of these spaces we refer the reader to [16, 25]. Here we recall without proof the following results.
Lemma 2.3. Let $1 \leq p < \infty$ and let $V^p_\mathbb{R}$ denote the space of all functions $v : \mathbb{R} \to L^2(\mathbb{R}^2)$ such that $\lim_{t \to -\infty} v(t) = 0$ and $\lim_{t \to +\infty} v(t)$ exists, endowed with the norm $\| \cdot \|_{V^p_\mathbb{R}}$ given by (2.9). Also, let $V^p_{\mathbb{R}, c}$ denote the closed subspace of all right-continuous $V^p_\mathbb{R}$-functions. Then, the following embeddings are continuous:

$$U^p \subset V^p_{\mathbb{R}, c} \quad \text{and} \quad V^p_{\mathbb{R}, c} \subset U^q, \quad p < q.$$ 

Also, $U^p \subset U^q \subset L^\infty(\mathbb{R}; L^2(\mathbb{R}^2))$, $V^p \subset V^q$, $V^p_\mathbb{R} \subset V^q_\mathbb{R}$, and $V^p_{\mathbb{R}, c} \subset V^q_{\mathbb{R}, c}$, provided $1 \leq p < q < \infty$.

Lemma 2.4 (Duality lemma). Let $1 < p < \infty$ and $p'$ denote the Hölder conjugate, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. There exists a unique bilinear form $B : U^p \times V^{p'} \to \mathbb{C}$ such that $V^{p'} \ni v \mapsto B(\cdot, v) \in (U^p)^*$ is an isometric isomorphism. In particular,

$$\|u\|_{U^p} = \sup_{v \in V^{p'}, \|v\|_{V^{p'}} = 1} |B(u, v)|, \quad \|v\|_{V^{p'}} = \sup_{u \in U^p, \text{atom}} |B(u, v)|.$$ 

Moreover, if $u \in V^1_\mathbb{R}$ is absolutely continuous on compact intervals, then

$$B(u, v) = -\int_\mathbb{R} \langle u'(t), v(t) \rangle_{L^2(\mathbb{R}^2)} dt.$$ 

Next, by recalling the notation $S(t) := e^{-i\alpha_x(\beta^2 - 3\sigma_0)}$ of the linear group associated to the linear part of the Dysthe equation ([1,5]), we define the spaces

$$U^p_S = S(\cdot)U^p \quad \text{and} \quad V^p_S = S(\cdot)V^p,$$

$$\text{and similarly } V^p_{\mathbb{R}, S} \text{ and } V^p_{\mathbb{R}, c, S}. \quad \text{In particular, by Lemma 2.3, any } u \in U^2_S \text{ also lies in } V^2_{\mathbb{R}, c, S} \text{ and we have}$$

$$\|u\|_{V^2_S} \lesssim \|u\|_{U^2_S}.$$ 

The above spaces behave well with respect to sharp cut-off functions since $\|\mathbb{1}_{(a,b)} f\|_{U^2} \lesssim \|f\|_{U^2}$ and $\|f\|_{V^2} \lesssim \|\mathbb{1}_{(a,b)} f\|_{V^2} \lesssim 2 \|f\|_{V^2}$, for any $-\infty < a < b \leq +\infty$. A simple application of the duality lemma 2.4 provides the following linear non-homogeneous estimate.

Lemma 2.5. Let $1 < p, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, and $-\infty < a < b \leq +\infty$. If $f \in L^1((a,b); L^2(\mathbb{R}^2))$, then

$$\bigg\| \mathbb{1}_{(a,b)}(t) \int_a^t S(t-t')f(t') dt' \bigg\|_{U^p_S} \lesssim \sup_{\|g\|_{V^{p'}_{\mathbb{R}}}} \bigg| \int_a^b \int_{\mathbb{R}^2} f(t') g(t') dt' \bigg|.$$ 

We are now ready to define our resolution space for constructing solutions to ([1,5]). We denote by $Y^s$ the closure of all functions $u \in C(\mathbb{R} : H^s(\mathbb{R}^2))$ with respect to the norm

$$\|u\|_{Y^s} := \left( \sum_N N^{2s} \|P_N u\|_{L^2_S}^2 \right)^{1/2} \leq \infty.$$ 

The following proposition is a transference principle in the $U^p_S$ setting.

Proposition 2.6 ([16, Proposition 2.19.(i)]). Let $L_0 : L^2(\mathbb{R}^3) \times \cdots \times L^2(\mathbb{R}^3) \to L^1_{\text{loc}}(\mathbb{R}^3)$ be a $n$-linear operator. Assume that for some $1 \leq p, q \leq \infty$, we have

$$\|L_0(S(\cdot)\phi_1, \ldots, S(\cdot)\phi_n)\|_{L^p(\mathbb{R}; L^q_{\mathbb{R}, c}(\mathbb{R}^2))} \lesssim \prod_{i=1}^n \|\phi_i\|_{L^p}.$$ 

Then, there exists $\mathcal{L} : U^p_S \times \cdots \times U^p_S \to L^p(\mathbb{R}; L^q_{\mathbb{R}, c}(\mathbb{R}^2))$ satisfying

$$\|\mathcal{L}(u_1, \ldots, u_n)\|_{L^p(\mathbb{R}; L^q_{\mathbb{R}, c}(\mathbb{R}^2))} \lesssim \prod_{i=1}^n \|u_i\|_{U^p_S}.$$
such that \( \mathcal{L}(u_1, \cdots, u_n)(t)(x, y) = \mathcal{L}_0(u_1(t), \cdots, u_n(t))(x, y) \) almost everywhere.

For the proof of Theorem 1.2 we use Bourgain spaces related to the linear part of \( e^{it\partial_x^2} \). Thus, for any \( s, b \in \mathbb{R} \), let \( X^{s,b} \) denote the completion of the Schwartz space \( S(\mathbb{R}^3) \) under the norm

\[
\|u\|_{X^{s,b}} = \left( \sum_{N,L} N^{2s} L^{2b} \|P_N Q_L u\|_{L^2(\mathbb{R}^3)}^2 \right)^{\frac{1}{2}},
\]

and we define the localized-in-time version of these spaces by the norm

\[
\|u\|_{X^{s,b}_T} = \inf \left\{ \|\tilde{u}\|_{X^{s,b}} : \tilde{u} \in S(\mathbb{R}^3), \tilde{u}|_{[0,T]} = u \right\}.
\]

The basic properties of these spaces that we need here are stated in the following proposition (see e.g. \[40\]).

**Proposition 2.7.** Let \( s \in \mathbb{R} \) and \( \chi \in C_0^\infty(\mathbb{R}) \) be a time cutoff function.

(i) For any \( b > \frac{1}{2} \),

\[
\|\chi(t)S(t)\varphi\|_{X^{s,b}} \lesssim \|\varphi\|_{H^s(\mathbb{R}^2)}.
\]

(ii) For any \( 0 < \delta < \frac{1}{2} \),

\[
\left\| \chi(t) \int_0^t S(t-t')f(t') \right\|_{X^{s-b+\delta}} \lesssim \|f\|_{X^{s,b}}.
\]

(iii) For any \( T > 0 \) and \(-\frac{1}{2} < b' \leq b < 1\),

\[
\|\chi(t/T)u\|_{X^{s,b'}} \lesssim T^{b-b'} \|f\|_{X^{s,b}}.
\]

(iv) For any \( T > 0 \) and \( b > \frac{1}{2} \), we have \( X^{s,b}_T \subset C([0,T] : H^s(\mathbb{R}^2)) \).

3. Linear and bilinear Strichartz estimates

3.1. Linear Strichartz estimates on \( \mathbb{R}^2 \).

In \[10\], Carbery, Kenig and Ziesler proved an optimal \( L^4 \)-restriction theorem for homogeneous polynomial hypersurfaces in \( \mathbb{R}^3 \).

**Theorem 3.1.** Let \( \Gamma(\xi, \mu) = (\xi, \mu, \Omega(\xi, \mu)) \), where \( \Omega(\xi, \mu) \) is a polynomial, homogeneous of degree \( d \geq 2 \). Then there exists a positive constant \( C \) (depending on \( \phi \)) such that

\[
\left( \int_{\mathbb{R}^2} \left| \hat{f}(\Gamma(\xi, \mu)) \right|^2 |K_\Omega(\xi, \mu)|^\frac{1}{\delta} d\xi d\mu \right)^\frac{\delta}{2} \leq C \|f\|_{L^4/3},
\]

for all \( f \in L^{4/3}(\mathbb{R}^3) \) and where

\[
|K_\Omega(\xi, \mu)| = |\det \text{Hess} \Omega(\xi, \mu)|.
\]

As a consequence, we have the following corollary (see for example the proof of Corollary 3.4 in \[35\]).

**Corollary 3.2.** Let \( |K_\Omega(D_x, D_y)|^\frac{1}{\delta} \) and \( e^{it\Omega(D_x,D_y)} \) be the Fourier multipliers associated to \( |K_\Omega(\xi, \mu)|^\frac{1}{\delta} \) and \( e^{it\Omega(\xi, \mu)} \), i.e.

\[
\mathcal{F}_{xy}\left(|K_\Omega(D_x, D_y)|^\frac{1}{\delta} \varphi\right)(\xi, \mu) = |K_\Omega(\xi, \mu)|^\frac{1}{\delta} \mathcal{F}_{xy}(\varphi)(\xi, \mu)
\]

where \( |K_\Omega(\xi, \mu)| \) is defined in \( \[5.2\] \), and

\[
\mathcal{F}_{xy}(e^{it\Omega(D_x,D_y)} \varphi)(\xi, \mu) = e^{it\Omega(\xi, \mu)} \mathcal{F}_{xy}(\varphi)(\xi, \mu).
\]

Then,

\[
\| |K_\Omega(D_x, D_y)|^\frac{1}{\delta} e^{it\Omega(\xi, \mu)} \varphi \|_{L^4_{x,y,t}} \lesssim \|\varphi\|_{L^2},
\]

for all \( \varphi \in L^2(\mathbb{R}^2) \).

Now, we apply Corollary \[3.2\] in the case of the unitary group \( e^{-it\partial_x^2 - 3t\partial_y^2} \).
Proposition 3.3. (i) For all \( \varphi \in L^2(\mathbb{R}^2) \), we have that
\[
\|D_x^+ e^{-t\partial_x (\partial_x^2 - 3\partial_y^2)} \varphi\|_{L^2_{x,y}(\mathbb{R}^3)} \lesssim \|\varphi\|_{L^2(\mathbb{R}^2)}.
\]
(ii) Let \( N \) be a dyadic number in \( \{2^k : k \in \mathbb{N}^*\} \). For all \( u \in V^2_{x,y} \) we have
\[
\|P_N u\|_{L^4(\mathbb{R})} \lesssim N^{-\frac{1}{4}} \|P_N u\|_{V^2}.
\]

Proof. The symbol associated to \( e^{-t\partial_x (\partial_x^2 - 3\partial_y^2)} \) is given by \( w(\xi, \mu) = \xi^3 - 3\xi \mu^2 \). An easy computation shows that
\[
det \text{Hess} w(\xi, \mu) = -36(\xi^2 + \mu^2).
\]
Estimate (3.6) follows then as a direct application of Corollary 3.2 and estimate (3.7) follows from (3.6) \( \square \).

3.2. Bilinear Strichartz estimates. In this subsection, we state crucial bilinear estimates related to the Dysthe dispersion relation for functions defined on \( \mathbb{R}^3 \).

Proposition 3.4. Let \( N_1, N_2, L_1, L_2 \) be dyadic numbers in \( \{2^k : k \in \mathbb{N}\} \) and assume that \( u_1, u_2 \) are two functions in \( L^2(\mathbb{R}^3) \).

(i) We have the basic estimate
\[
\|(P_{N_1}Q_{L_1}u_1)(P_{N_2}Q_{L_2}u_2)\|_{L^2(\mathbb{R}^3)} \lesssim \min\{N_1, N_2\} \min\{L_1, L_2\} \|P_{N_1}Q_{L_1}u_1\|_{L^2(\mathbb{R}^3)} \|P_{N_2}Q_{L_2}u_2\|_{L^2(\mathbb{R}^3)}.
\]

(ii) If \( N_1 \geq 4N_2 \), then
\[
\|(P_{N_1}Q_{L_1}u_1)(P_{N_2}Q_{L_2}u_2)\|_{L^2(\mathbb{R}^3)} \lesssim N_1^{\frac{1}{4}} N_2^{-1} L_1^{\frac{1}{4}} L_2^{\frac{1}{4}} \|P_{N_1}Q_{L_1}u_1\|_{L^2(\mathbb{R}^3)} \|P_{N_2}Q_{L_2}u_2\|_{L^2(\mathbb{R}^3)}.
\]

The proof of Proposition 3.4 is given in Lemma 9 in [1] (see also Proposition 3.6 in [36]). For the sake of completeness we give the argument here. The proof relies on some basic Lemmas stated for example in [36].

Lemma 3.5. Consider a set \( \Lambda \subset \mathbb{R} \times \mathbb{R} \). Let the projection on the \( \mu \) axis be contained in a set \( I \subset \mathbb{R} \). Assume in addition that there exists \( C > 0 \) such that for any fixed \( \mu_0 \in I \), \( \{|\xi \in \mathbb{R} : (\xi, \mu_0) \in \Lambda| \} \leq C \). Then, we get that \(|\Lambda| \leq C|I|\).

The second one is a direct consequence of the mean value theorem.

Lemma 3.6. Let \( I \) and \( J \) be two intervals on the real line and \( f : J \rightarrow \mathbb{R} \) be a smooth function. Then,
\[
\{|x \in J : f(x) \in I| \} \leq \frac{|J|}{\inf_{\xi \in J} |f'(\xi)|}.
\]

Proof of Proposition 3.4. The Cauchy-Schwarz inequality and Plancherel’s identity yield
\[
\|(P_{N_1}Q_{L_1}u_1)(P_{N_2}Q_{L_2}u_2)\|_{L^2(\mathbb{R}^3)} = \|(P_{N_1}Q_{L_1}u_1)^\wedge \ast (P_{N_2}Q_{L_2}u_2)^\wedge\|_{L^2(\mathbb{R}^3)} \lesssim \sup_{(\xi, \mu, \tau) \in \mathbb{R}^3} |A_{\xi, \mu, \tau}|^{\frac{1}{4}} \|P_{N_1}Q_{L_1}u_1\|_{L^2(\mathbb{R}^3)} \|P_{N_2}Q_{L_2}u_2\|_{L^2(\mathbb{R}^3)},
\]
where
\[
A_{\xi, \mu, \tau} := \left\{(\xi_1, \mu_1, \tau_1) \in \mathbb{R}^3 : |(\xi_1, \mu_1)| \in I_{N_1}, |(\xi - \xi_1, \mu - \mu_1)| \in I_{N_2}, |	au_1 - \omega(\xi_1, \mu_1)| \in I_{L_1}, |	au - \tau_1 - \omega(\xi_1, \mu_1)| \in I_{L_2} \right\}.
\]
It remains then to estimate the measure of the set \( A_{\xi, \mu, \tau} \) uniformly in \( (\xi, \mu, \tau) \in \mathbb{R}^3 \).

To obtain (3.5), we use the trivial estimate
\[
|A_{\xi, \mu, \tau}| \lesssim \min\{L_1, L_2\} \min\{N_1, N_2\}^2,
\]
for all \((\xi, \mu, \tau) \in \mathbb{R}^3\).

Now we turn to the proof of estimate \(3.10\). First, we get easily from the triangle inequality we get that
\[
|A_{\xi, \mu, \tau}| \lesssim \min\{L_1, L_2\}|B_{\xi, \mu, \tau}|,
\]
where
\[
B_{\xi, \mu, \tau} := \left\{ (\xi_1, \mu_1) \in \mathbb{R}^2 : \left| (\xi_1, \mu_1) \right| \in I_{N_1}, \left| (\xi - \xi_1, \mu - \mu_1) \right| \in I_{N_2} \right\}
\]
and
\[
|\tau - w(\xi, \mu) - R(\xi_1, \mu_1, \xi - \xi_1, \mu - \mu_1)| \lesssim \max\{L_1, L_2\}
\]
and where \(R(\xi_1, \mu_1, \xi_2, \mu_2)\) is the resonance function defined in \(2.2\).

In the case where \(|\xi_1| \gg |\mu_1|\) or \(|\mu_1| \gg |\xi_1|\), we observe from the hypothesis \(N_1 \geq 4N_2\) that
\[
|\frac{\partial R}{\partial \xi_1}(\xi_1, \xi - \xi_1, \mu_1 - \mu_1)| = |3(\xi_1^2 - \mu_1^2) - 3((\xi - \xi_1)^2 - (\mu - \mu_1)^2)| \gtrsim N_2^2.
\]
Then, if we define \(B_{\xi, \mu, \tau}(\mu_1) = \{\xi_1 \in \mathbb{R} : (\xi_1, \mu_1) \in B_{\xi, \mu, \tau}\}\), we deduce applying estimate \(6.11\) that
\[
|B_{\xi, \mu, \tau}(\mu_1)| \lesssim \frac{\max\{L_1, L_2\}}{N_1^2},
\]
for all \(\mu_1 \in \mathbb{R}\). Thus, it follows from Lemma \(6.3\) that
\[
|B_{\xi, \mu, \tau}| \lesssim \frac{N_2}{N_1^2} \max\{L_1, L_2\}.
\]
In the case where \(|\xi_1| \sim |\mu_1|\), then we use that
\[
|\frac{\partial R}{\partial \mu_1}(\xi_1, \xi - \xi_1, \mu_1 - \mu_1)| = |6\xi_1\mu_1 - 6(\xi - \xi_1)(\mu - \mu_1)| \gtrsim N_2^2.
\]
Then, if we define \(B_{\xi, \mu, \tau}(\xi_1) = \{\mu_1 \in \mathbb{R} : (\xi_1, \mu_1) \in B_{\xi, \mu, \tau}\}\), we deduce applying estimate \(3.11\) that
\[
|B_{\xi, \mu, \tau}(\xi_1)| \lesssim \frac{\max\{L_1, L_2\}}{N_1^2},
\]
for all \(\xi_1 \in \mathbb{R}\), so that estimate \(3.15\) also follows in this case.

Finally, we conclude the proof of the estimate \(3.10\) by gathering estimates \(3.12\)–\(3.15\).

\[
\square
\]

\textbf{Corollary 3.7.} Let \(N_1, N_2\) be dyadic numbers in \(\{2^k : k \in \mathbb{N}\}\).

(i) If \(N_1 \geq 4N_2\), then
\[
\|P_N u_1P_N u_2\|_{L^2(\mathbb{R}^3)} \lesssim N_2^\frac{1}{2} N_1^{-1}\|P_N u_1\|_{L^2_3} \|P_N u_2\|_{L^2_3}, \quad \text{for any } u_1, u_2 \in U_{3}^z,
\]
and
\[
\|P_N u_1P_N u_2\|_{L^2(\mathbb{R}^3)} \lesssim N_2^\frac{1}{2} N_1^{-1}\|P_N u_1\|_{L^2_3} \|P_N u_2\|_{L^2_3}, \quad \text{for any } u_1, u_2 \in V_{2}^{2}.
\]

(ii) If \(N_1, N_2 \leq 1\), we have
\[
\|P_N u_1P_N u_2\|_{L^2(\mathbb{R}^3)} \lesssim \|P_N u_1\|_{L^2_{3}} \|P_N u_2\|_{L^2_{3}}, \quad \text{for any } u_1, u_2 \in V_{2}^{2}.
\]

\textbf{Proof.} Let \(\chi \in C_0^\infty(\mathbb{R})\) be a cut-off function satisfying \(0 \leq \chi \leq 1\), \(\chi|_{[-1,1]} = 1\) and supp \(\chi \in [-2, 2]\). We deduce by using Plancherel’s identity that \(\varphi \in L^2(\mathbb{R}^2)\) and any dyadic number \(L \geq 1\)
\[
\mathcal{F}(Q_L \chi S(\cdot) \varphi)(\xi, \mu, \tau) = \phi_L(\tau - w(\xi, \mu))\tilde{\chi}(\tau - w(\xi, \mu))\mathcal{F}_{xy}(\varphi)(\xi, \mu).
\]
Then by using again Plancherel’s identity and taking into account that \(\tilde{\chi}\) is rapidly decreasing, it holds that
\[
\|Q_L \chi S(\cdot) \varphi\|_{L^2_{x,y,\mu,\tau}} \leq \|\phi_L \tilde{\chi}\|_{L^2(\mathbb{R})} \|\varphi\|_{L^2(\mathbb{R})} \lesssim L^{-1} \|\varphi\|_{L^2(\mathbb{R})}.
\]
This ensures by using \(3.10\) that
\[
\|P_N S(\cdot) \varphi_1)(P_N S(\cdot) \varphi_2)\|_{L^2([-1,1] \times \mathbb{R}^2)} \leq \|\chi \frac{3}{2} P_N S(\cdot) \varphi_1)\|_{L^2(\mathbb{R}^3)} \|\chi \frac{3}{2} P_N S(\cdot) \varphi_2)\|_{L^2(\mathbb{R}^3)} \lesssim N_2^\frac{1}{2} N_1^{-1}\|P_N \varphi_1\|_{L^2(\mathbb{R}^3)} \|P_N \varphi_2\|_{L^2(\mathbb{R}^3)}.
\]
We can remove the time cut off \( \chi \) by using the scaling invariance (1.17). Indeed let \( u_1 = S(t)\varphi_1 \) and \( u_2 = S(t)\varphi_2 \). Then it follows by using the notation in (1.17) that for any \( \lambda > 0 \)
\[
\| P_{N_1}u_1P_{N_2}u_2 \|_{L^2([-\lambda^3,\lambda^3] \times \mathbb{R}^2)} = \lambda^\frac{3}{2} \| (P_{N_1}u_1)\lambda(P_{N_2}u_2) \|_{L^2([-1,1] \times \mathbb{R}^2)}.
\]

Since \( (P_N)_{\lambda} = P_{\lambda N}(u) \), we conclude from (3.19) that
\[
\| P_{N_1}S(\cdot)\varphi_1(P_{N_2}S(\cdot)\varphi_2) \|_{L^2([-\lambda^3,\lambda^3] \times \mathbb{R}^2)} \lesssim N_1^\frac{3}{2}N_2^{-1} \| P_{N_1}\varphi_1 \|_{L^2(\mathbb{R}^2)} \| P_{N_2}\varphi_2 \|_{L^2(\mathbb{R}^2)},
\]
which implies by letting \( \lambda \to +\infty \),
\[
\| P_{N_1}S(\cdot)\varphi_1(P_{N_2}S(\cdot)\varphi_2) \|_{L^2(\mathbb{R}^2)} \lesssim N_1^\frac{3}{2} \| P_{N_1}\varphi_1 \|_{L^2(\mathbb{R}^2)} \| P_{N_2}\varphi_2 \|_{L^2(\mathbb{R}^2)}.
\]

Estimate (3.16) follows then by applying Proposition 2.6, while (3.17) follows as in [16, Corollary 2.21]. We obtain (3.18) analogously.

\[\square\]

Corollary 3.8. For \( 0 < \theta \leq \frac{1}{4} \), we have
\[
\| P_N u \|_{L^4(\mathbb{R}^2)} \lesssim_{\theta} N^{-\frac{1}{4} + \frac{\theta}{1 + 8\theta}} \| P_N u \|_{X^0,\frac{1}{4} + \theta}.
\]

Proof. By using the transference principle for \( X^{s,b} \)-spaces (see e.g. [40, Lemma 2.9]), from (3.20) we have
\[
\| P_N u \|_{L^4(\mathbb{R}^2)} \lesssim_{\theta} N^{-\frac{1}{4}} \| P_N u \|_{X^0,\frac{1}{4} + \theta}.
\]
The estimate (3.21) follows by interpolating (3.22) with the trivial \( L^4 \)-estimate that follows from (3.19), namely with
\[
\| P_N u \|_{L^4(\mathbb{R}^2)} \lesssim N^{\frac{1}{2}} \| P_N u \|_{X^0,\frac{1}{4}}.
\]
\[\square\]

4. Proof of the main results

For \( T \in (0, +\infty) \), we consider
\[
I_{T,j}(u_1, u_2, u_3)(t) := \mathbb{1}_{[0,T]}(t) \int_0^t S(t - t')N_j(u_1, u_2, u_3)(t') \, dt'.
\]

Proposition 4.1. Let \( s \geq 0 \) and \( u_1, u_2, u_3 \in Y^s \).
(i) There exists \( C > 0 \) such that for all \( 0 < T < +\infty \) and \( 1 \leq j \leq 4 \), we have
\[
\| I_{T,j}(u_1, u_2, u_3) \|_{Y^s} \leq C \| u_1 \|_{Y^s} \| u_2 \|_{Y^s} \| u_3 \|_{Y^s}.
\]
(ii) The estimate (4.2) also holds for \( T = +\infty \) and for any \( 1 \leq j \leq 4 \), we have
\[
\| I_{T,j}(u_1, u_2, u_3) - I_{+\infty,j}(u_1, u_2, u_3) \|_{Y^s} \to 0 \text{ as } T \to +\infty
\]
and the limit
\[
\lim_{t \to +\infty} S(-t)I_{+\infty,j}(u_1, u_2, u_3)(t)
\]
exists in \( L^2(\mathbb{R}^2) \).

Proof. (i) Note that it suffices to prove the estimates for \( s = 0 \) and by Littlewood-Paley decomposition it suffices to prove the following
\[
\left( \sum_{N_1} \sum_{N_1:N_2:N_3} P_{N_1}D_j(P_{N_1}u_1, P_{N_2}u_2, P_{N_3}u_3) \right)^2 \lesssim \| u_1 \|_{Y^0} \| u_2 \|_{Y^0} \| u_3 \|_{Y^0},
\]
where
\[
D_j(P_{N_1}u_1, P_{N_2}u_2, P_{N_3}u_3) := \int_0^t S(t - t')N_j(P_{N_1}u_1, P_{N_2}u_2, P_{N_3}u_3)(t') \, dt',
\]
with \( u_i := \mathbb{1}_{[0,T]} u_i, 1 \leq i \leq 4 \). We treat the contribution of the worst parts of the nonlinearities \( N_j \)'s, namely when there is a derivative and it falls on the largest frequency factor; the nonlinearity involving the Riesz transform \( \mathcal{R}_x \) (which is bounded on \( L^2(\mathbb{R}^2) \)) does not pose a further difficulty.
Thus, in what follows we focus on the nonlinear term
\[(4.7)\] 
\[N(P_{N_1}u_1, P_{N_2}u_2, P_{N_3}u_3) := (P_{N_1}u_1)(P_{N_2}u_2)\partial_x(P_{N_3}u_3)\]
with \(N_1 \leq N_2 \leq N_3\). Thus, by Lemma 2.5\(^4\) it remains to estimate the following term
\[(4.8)\]
\[\left(\sum_{N_4} \sup_{u_4 \parallel u_4}_2 1^{\frac{1}{2}} \right),\]
where
\[I_{N_1,\ldots,N_4} := \sum_{N_1 \leq N_2 \leq N_3} N_3 \int_{\mathbb{R}^3} (P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3)(P_{N_4}u_4) \, dx dy dt,\]
and where we assume without loss of generality that the Fourier transforms \(F_{xy}(u_i)\), \(1 \leq i \leq 4\), are non-negative.

We recall that due to the frequency hyperplane, the largest two dyadic numbers are comparable and thus we distinguish the following cases.

\textbf{Case 1:} \(N_1 \leq N_2 \ll N_3 \sim N_4\). We apply the bilinear estimates (3.10) and (3.17) together with (2.10). For fixed \(N_4\), we have
\[I_{N_1,\ldots,N_4} \lesssim N_3 \parallel P_{N_1}u_1P_{N_2}u_2 \parallel_{L^2_{x,y,t}} \parallel P_{N_2}u_2P_{N_4}u_4 \parallel_{L^2_{x,y,t}} \]
\[\lesssim \parallel u_1 \parallel_{L^{\infty}} \parallel u_2 \parallel_{L^{\infty}} \sum_{N_1 \leq N_2 \ll N_3 \sim N_4} N_4^\frac{1}{2}N_{N_2}^{\frac{\gamma}{N_2}} \parallel P_{N_3}u_3 \parallel_{U^3_{y}} \parallel P_{N_4}u_4 \parallel_{V^3_{y}},\]
where in the last step we used with \(\parallel P_{N_4}u_4 \parallel_{V^3_{y}} \leq 1\). Then, by the Cauchy-Schwarz inequality we have
\[I_{N_1,\ldots,N_4} \lesssim \parallel u_1 \parallel_{L^{\infty}} \parallel u_2 \parallel_{L^{\infty}} \left(\sum_{N_4} \left(\sum_{N_3 \sim N_4} \parallel P_{N_3}u_3 \parallel_{U^3_{y}}\right)^2\right)^\frac{1}{2} \lesssim \parallel u_1 \parallel_{L^{\infty}} \parallel u_2 \parallel_{L^{\infty}} \parallel u_3 \parallel_{L^{\infty}} \parallel u_4 \parallel_{L^{\infty}}.\]

\textbf{Case 2:} \(N_1, N_4 \ll N_2 \sim N_3\). By arguing similarly to Case 1, for fixed \(N_4\) we now have
\[I_{N_1,\ldots,N_4} \lesssim \parallel u_1 \parallel_{L^{\infty}} \sum_{N_1 \ll N_2 \sim N_3} \left(\sum_{N_4} \left(\frac{N_1}{N_2} \right)^{1-2\varepsilon} \right) \parallel P_{N_2}u_2 \parallel_{U^3_{y}} \parallel P_{N_3}u_3 \parallel_{U^3_{y}} \parallel P_{N_4}u_4 \parallel_{V^3_{y}} \]
\[\lesssim \parallel u_1 \parallel_{L^{\infty}} \sum_{N_2 \sim N_3} \left(\frac{N_1}{N_2} \right)^{1-\varepsilon} \parallel P_{N_2}u_2 \parallel_{U^3_{y}} \parallel P_{N_3}u_3 \parallel_{U^3_{y}} \parallel P_{N_4}u_4 \parallel_{U^3_{y}}.\]

By Minkowski and Cauchy-Schwarz inequalities we then get
\[(4.8) \lesssim \parallel u_1 \parallel_{L^{\infty}} \sum_{N_2 \sim N_3} \left(\sum_{N_4} \left(\frac{N_1}{N_2} \right)^{1-2\varepsilon} \right) \parallel P_{N_2}u_2 \parallel_{U^3_{y}} \parallel P_{N_3}u_3 \parallel_{U^3_{y}} \parallel P_{N_4}u_4 \parallel_{U^3_{y}} \lesssim \parallel u_1 \parallel_{L^{\infty}} \parallel u_2 \parallel_{L^{\infty}} \parallel u_3 \parallel_{L^{\infty}} \parallel u_4 \parallel_{L^{\infty}}.\]

\textbf{Case 3:} \(N_1 \ll N_2 \sim N_3 \sim N_4\). We apply the bilinear estimate of Corollary 3.7 and the \(L^4\)-Strichartz estimate (3.17) twice. Thus, arguing similarly to Case 1, we obtain
\[I_{N_1,\ldots,N_4} \lesssim \left(\sum_{N_4} \left(\sum_{N_1 \ll N_2 \sim N_3 \sim N_4} \right) N_3 \parallel P_{N_1}u_1P_{N_2}u_2 \parallel_{L^2_{x,y,t}} \parallel P_{N_2}u_2P_{N_4}u_4 \parallel_{L^4_{x,y,t}} \parallel P_{N_3}u_3 \parallel_{L^1_{x,y,t}}\right)^2 \]
\[\lesssim \parallel u_1 \parallel_{L^{\infty}} \parallel u_2 \parallel_{L^{\infty}} \parallel u_3 \parallel_{L^{\infty}} \parallel u_4 \parallel_{L^{\infty}}.\]

\textbf{Case 4:} \(N_4 \ll N_1 \sim N_2 \sim N_3\). In this case the desired estimate follows as in Case 3 via an application of Minkowski’s inequality (similarly to the last step of Case 2).

\(^4\)The estimates for \(T = +\infty\) need to be derived separately as in this case we cannot apply Lemma 2.5.
Case 5: \( 1 \ll N_1 \sim N_2 \sim N_3 \sim N_4 \). In this case we use the \( L^4 \)-Strichartz estimate (5.7) together with (2.10) in order to get
\[
\begin{align*}
&4.8 \lesssim \left( \sum_{N_1 \ll N_2 \sim N_3 \sim N_4} \| P_{N_1} u_1 \|_{L^4_{t,x,y}} \| P_{N_2} u_2 \|_{L^4_{t,x,y}} \| P_{N_3} u_3 \|_{L^4_{t,x,y}} \| P_{N_4} u_4 \|_{L^4_{t,x,y}} \right)^2 \lesssim \| u_1 \|_{Y^0} \| u_2 \|_{Y^0} \| u_3 \|_{Y^0} \| u_4 \|_{Y^0}.
\end{align*}
\]

Case 6: \( N_1 \sim N_2 \sim N_3 \sim N_4 \lesssim 1 \). We simply use the bilinear estimate (3.18) twice together with (2.10) and we get (4.5).

Part (ii) follows analogously to the proof of [16, Corollary 3.4].

\[ \square \]

Proof of Theorem 1.1. It uses a standard argument via contraction mapping principle applied to the simplified equation (1.15) after which we undo the the transformation (2.10).

Let \( v_0 \in L^2(\mathbb{R}^2) \) such that \( \| v_0 \|_{L^2(\mathbb{R}^2)} < \delta \), with \( \delta \) to be chosen later. First, we construct a solution \( u(t) \) to (1.15) for \( t \in (0, \infty) \) with initial data \( u_0(x,y) = e^{ia2x}v_0(x,\sqrt{2}y) \), for some \( a \in \mathbb{R} \) (see Lemma 2.1 and Remark 2.1). Thus, we consider \( \Phi(u) \) given by
\[
\Phi(u)(t) = \mathbb{I}_{[0,\infty)}(t) S(t) u_0 + \sum_{j=1}^{4} c_j \mathbb{I}_{[0,\infty)}(t) \int_{0}^{t} S(t-t')N_j(u,u,u)(t')dt'.
\]

Since \( \mathbb{I}_{[0,\infty)}(t)u_0 \) is a \( U^2 \)-atom, one easily checks that \( \| \mathbb{I}_{[0,\infty)}(t) S(t) u_0 \|_{Y^0} \sim \| u_0 \|_{L^2} \) and thus by Proposition 4.1 we get
\[
\| \Phi(u) \|_{Y^0} \lesssim C\delta + C\| u_0 \|_{Y^0}^3,
\]
for some \( C > 0 \). By setting \( r := 2C\delta \), we have that \( \Phi \) maps \( B_r := \{ u \in Y^0 : \| u \|_{Y^0} \leq r \} \) into itself, provided that \( 8C^3\delta^2 \leq 1 \). Similarly, by using telescoping sums, there exists \( \bar{C} > 0 \) such that
\[
\| \Phi(u_1) - \Phi(u_2) \|_{Y^0} \lesssim \bar{C} \| u_1 - u_2 \|_{Y^0} \left( \| u_1 \|_{Y^0}^2 + \| u_2 \|_{Y^0}^2 \right).
\]

By ensuring that \( 8C^2\bar{C}\delta^2 < 1 \), we have that \( \Phi \) is a contraction on \( B_r \), and thus it has a unique fixed point in \( B_r \) which solves (1.15) for \( t \in [0, +\infty) \).

For the scattering claim, we use Proposition 4.1 (ii) and take
\[
u^+_0 := u_0 + \sum_{j=1}^{4} c_j \lim_{t \to +\infty} S(-t) I_{+\infty,j}(u,u,u)(t) \in L^2(\mathbb{R}^2).
\]

Since the embedding \( Y^0 \subset L^\infty(\mathbb{R} : L^2(\mathbb{R}^2)) \) is continuous, we get that for any \( T > t \),
\[
\| u(t) - S(t) u^+_0 \|_{L^2(\mathbb{R}^2)} \leq \sum_{j=1}^{4} |c_j| \left( \| I_{T,j}(u,u,u) - I_{+\infty,j}(u,u,u) \|_{Y^0} + \| S(-\tau) I_{+\infty,j}(u,u,u)(\tau) \|_{L^2(\mathbb{R}^2)} \right)
\]
and by (1.13) and (1.14), we then get \( \| u(t) - S(t) u^+_0 \|_{L^2(\mathbb{R}^2)} \to 0 \) as \( t \to +\infty \).

Second, we construct the solution \( u_{neg}(t) \) to (1.15) for \( t \in (-\infty, 0) \) and the scattering data \( u^-_0 \), by running the analogous argument for the equation obtained from (1.15) after the change of variable
\[
u(t,x,y) \mapsto v(-t,-x,-y) =: \mathcal{J}(u),
\]
with initial data \( u_0(-x,-y) \), and then undoing the transformation (1.10). Since the transformations (2.6) and (4.10) commute, the solution \( v \) to the original Dysthe equation (1.12) is given by
\[
(4.11) \quad v = \mathcal{J}^{-1}(u + u_{neg}) \in \bigcup \left\{ \mathbb{I}_{[0,\infty)} v_1 + \mathbb{I}_{(-\infty,0)} v_2 : \mathcal{T}(v_1) \in Y^0 \text{ and } \mathcal{T}(v_2) \in Y^0 \right\},
\]
and the corresponding scattering data are \( v^+_0(x,y) := e^{-ia2x}u^+_0(x,\sqrt{2}y) \).

\[ \square \]
In order to obtain the local well-posedness result for arbitrarily large initial data (i.e. Theorem 1.2), we prove the trilinear estimates in $X^{s,b}$ spaces.

**Proposition 4.2.** Let $s > 0$ and $\nu > 0$ sufficiently small. For all $1 \leq j \leq 4$, we have

\[
\|N_j(u_1, u_2, u_3)\|_{X^{s,\frac{1}{2}+\nu}} \lesssim \|u_1\|_{X^{s,\frac{1}{2}+\nu}} \|u_2\|_{X^{s,\frac{1}{2}+\nu}} \|u_3\|_{X^{s,\frac{1}{2}+\nu}}.
\]

**Proof.** As in the proof of Proposition 4.1 we focus on the contribution of the worst parts of the nonlinearities and thus we prove the estimate for the nonlinear term

\[
N(P_{N_1}u_1, P_{N_2}u_2, P_{N_3}u_3) := (P_{N_1}u_1)(P_{N_2}u_2)\partial_x(P_{N_3}u_3)
\]

with $N_1 \leq N_2 \leq N_3$. By duality and Littlewood-Paley decomposition it suffices to prove the following

\[
\sum_{N_1 \leq N_2 \leq N_3} \Gamma^2_{N_1,\ldots,N_4} \Lambda_{L_1,\ldots,L_4} \left| I_{N_1,\ldots,N_4}(v_1, v_2, v_3, v_4) \right|^2 \lesssim \prod_{i=1}^4 \|v_i\|_{L^2(\mathbb{R}^3)},
\]

where

\[
\Gamma_{N_1,\ldots,N_4} := N_1^{-s} N_2^{-s} N_3^{-s} N_4^{-s},
\]

\[
\Lambda_{L_1,\ldots,L_4} := L_1^{-\frac{\nu}{2}} L_2^{-\frac{\nu}{2}} L_3^{-\frac{\nu}{2}} L_4^{-\frac{\nu}{2} + 2\nu},
\]

\[
I_{N_1,\ldots,N_4}(v_1, v_2, v_3, v_4) := \int_{\mathbb{R}^3} (P_{N_1}Q_{L_1}v_1)(P_{N_2}Q_{L_2}v_2)(P_{N_3}Q_{L_3}v_3)(P_{N_4}Q_{L_4}v_4) \, dx dy dt,
\]

and where we assume without loss of generality that the Fourier transforms $\mathcal{F}v_i$, $1 \leq i \leq 4$, are non-negative.

We discuss the same cases as in the proof of Proposition 4.1.

**Case 1:** $N_1 \leq N_2 \ll N_3 \sim N_4$. As in the proof of Proposition 4.1, we interpolate between the two bilinear Strichartz estimates (3.9) and (3.10), and so we have

\[
\|/(P_{N_2}Q_{L_2}u_2)(P_{N_3}Q_{L_3}u_4)\|_{L^2(\mathbb{R}^3)} \lesssim \frac{N_1^{s} N_2^{s} N_3^{s} N_4^{s}}{N_4^{-1}} \min\{L_2, L_4\}^\frac{\nu}{2} \max\{L_2, L_4\}^\frac{1-s}{2} \|P_{N_2}Q_{L_2}u_2\|_{L^2(\mathbb{R}^3)} \|P_{N_3}Q_{L_3}u_4\|_{L^2(\mathbb{R}^3)}.
\]

We now assume that $L_2 \leq L_4$ as the case $L_4 < L_2$ is easier. Then, by the Cauchy-Schwarz inequality, (4.17), and (3.10) for $(P_{N_3}Q_{L_3}u_3)$, we get

\[
\Gamma_{N_1,\ldots,N_4} \Lambda_{L_1,\ldots,L_4} \left| I_{N_1,\ldots,N_4}(v_1, v_2, v_3, v_4) \right| \lesssim \frac{N_1^{s} N_2^{s} N_3^{s} N_4^{s}}{N_4^{1-s}} L_1^{-\frac{\nu}{2}} L_2^{-\nu} L_3^{-\nu} L_4^{-\nu + 2\nu} \prod_{i=1}^4 \|v_i\|_{L^2(\mathbb{R}^3)}.
\]

By choosing $\theta$ and $\nu$ such that $\frac{1+\theta}{2} - s + \frac{1}{2} - s < -1 - \theta$ and $2\nu + \frac{\nu}{2} < 0$ (e.g. $\theta = 6\nu$ and $0 < \nu < \frac{3}{5}s$), the estimate (4.13) follows immediately.

**Case 2:** $N_1, N_4 \ll N_2 \sim N_3$. The estimate (4.13) follows as in Case 1 above (by interchanging $N_2$ and $N_4$ in (4.18)) via Minkowski’s inequality for the summation in $N_4$ (see also Case 2 in the proof of Proposition 4.1).

For the remaining cases, we follow essentially the same arguments as in the proof of Proposition 4.1.

**Case 3:** $N_1 \ll N_2 \sim N_3 \sim N_4$. We apply the bilinear Strichartz estimate (3.10) and the $L^4$-Strichartz estimate (3.21) twice (with $\theta = \frac{s}{2}$ and $\theta = 3\nu$) and we get

\[
\Gamma_{N_1,\ldots,N_4} \Lambda_{L_1,\ldots,L_4} \left| I_{N_1,\ldots,N_4}(v_1, v_2, v_3, v_4) \right| \lesssim N_1^{-s} N_2^{-s} N_3^{-s} N_4^{-s} \prod_{i=1}^4 L_i^{-\nu} \|v_i\|_{L^2(\mathbb{R}^3)}.
\]

By choosing $\nu > 0$ such that $\frac{3\nu}{2(1+\nu)} + \frac{9\nu}{1+6\nu} < s$ we ensure summability over $N_2 \sim N_3 \sim N_4$ and thus we get (4.13).
Case 4: $N_4 \ll N_1 \sim N_2 \sim N_3$. We use the interpolated bilinear Strichartz estimate as in Case 1 (namely (4.17) with $N_2^{-\theta}/N_4^{1-\theta}$ replaced by $N_4^{-\theta}/N_2^{1-\theta}$) and the $L^4$-Strichartz estimate (5.21). We have

$$\Gamma_{N_1, \ldots, N_4} A_{L_1, \ldots, L_4} \left| I_{N_1, \ldots, N_4}^{L_1, \ldots, L_4}(v_1, v_2, v_3, v_4) \right| \lesssim N_1^{0-3\theta} N_4^{-\theta} L_1^{-\nu} L_2^{-\nu} L_3^{-\nu} L_4^{2\nu-2} \prod_{i=1}^{4} \|v_i\|_{L^2(\mathbb{R}^3)}$$

and we choose $\theta$ and $\nu$ such that $0 < 4\nu < \theta < 2s$.

Case 5: $1 \ll N_1 \sim N_2 \sim N_3 \sim N_4$. We use the $L^4$-Strichartz estimate (5.21) to obtain

$$\Gamma_{N_1, \ldots, N_4} A_{L_1, \ldots, L_4} \left| I_{N_1, \ldots, N_4}^{L_1, \ldots, L_4}(v_1, v_2, v_3, v_4) \right| \lesssim N_1^{-2s+\frac{9\nu}{2(1+\nu)} + \frac{9\nu}{1+6\nu}} \prod_{i=1}^{4} L_i^{-\nu} \|v_i\|_{L^2(\mathbb{R}^3)}$$

and we choose $\nu > 0$ such that $\frac{9\nu}{2(1+\nu)} + \frac{9\nu}{1+6\nu} < 2s$.

Case 6: $N_1 \sim N_2 \sim N_3 \sim N_4 \lesssim 1$. We simply use the bilinear estimate (3.9) twice to obtain

$$\Gamma_{N_1, \ldots, N_4} A_{L_1, \ldots, L_4} \left| I_{N_1, \ldots, N_4}^{L_1, \ldots, L_4}(v_1, v_2, v_3, v_4) \right| \lesssim L_1^{-\nu} L_2^{-\nu} L_3^{-\nu} L_4^{2\nu+2\nu} \prod_{i=1}^{4} \|v_i\|_{L^2(\mathbb{R}^3)}$$

and we choose $0 < \nu < \frac{\theta}{4}$.

\[ \square \]

Proof of Theorem 1.2. It uses a standard argument via contraction mapping principle and Proposition 2.4 applied to

$$\Phi(u)(t) = \chi(t) S(t) u_0 + \sum_{j=1}^{4} c_j \eta_j(t) \int_0^t \chi(t'/T) S(t-t') N_j(u, u, u)(t') \, dt'. \tag{4.19}$$

in a ball of $X_T^{s,b+\nu}$, by choosing $T = T(\|u_0\|_{L^2(\mathbb{R}^2)})$, $\nu > 0$ sufficiently small, and some $\chi \in C_0^\infty(\mathbb{R})$ with $\chi \equiv 1$ on $[0, 1]$. Lastly, we note that the solution $v$ corresponding to the original equation lies in the space $X_T^{s,b}$ defined analogous to $X_T^{s,b}$, using the non-homogeneous symbol $\omega$ (defined by (2.41)) instead of the homogeneous version $w$.

\[ \square \]

5. Final comments

Some interesting issues remain open for the Cauchy problem of Dysthe type equations, for instance the possible finite time blow-up of large local solutions.

Other questions concern the Dysthe equations as a water waves model and are summarized in [27] Section 8.5.4, for instance the comparison of the Dysthe equation with other models with improved dispersion.

More precisely, as recalled in Lannes [27], Section 8.5.4, the Dysthe equation contains both higher nonlinear and dispersive terms neglected in the standard cubic NLS equation, while the full dispersion equation derived in [27] Section 8.5.3 contains only higher order dispersive terms (at infinite order actually). A comparison between these models (and between a possible full-dispersion Dysthe equation) would bring some insight to the relative importance of dispersive and nonlinear effects in situations where the standard NLS approximation compares poorly with the experiments (see for instance [34]). We plan to come back to those issues in a subsequent paper.

Acknowledgements. D.P. and R.M. were supported by a Trond Mohn Foundation grant. J.-C. S. was partially supported by the ANR project ANuI (ANR-17-CE40-0035-02).


References

[1] Y. Angelopoulos, Well-posedness and ill-posedness results for the Novikov-Veselov equation, Comm. Pure Appl. Anal., 15 (2016), 727–760.
[2] M. Ben-Artzi, H. Koch and J.-C. Saut, Dispersion estimates for third-order equations in two dimensions, Comm. Part. Diff. Eq., 28 (2003), 1943–1974.
[3] L. Bergé, S. Skupin, R. Nutter, J. Kasparian and J.-P; Wolf, Ultrashort filaments of light in weakly ionized optically transparent media, Rep. Prog. Phys., 70 (2007), 1633–1713.
[4] A. de Bouard, Analytic solutions to nonelliptic nonlinear Schrödinger equations, J. Diff. Equ, 104 (1) (1993), 196–213.
[5] H. Chihara, Third order dispersive equations related to deep water waves, preprint (2004), arXiv:math/0404005
[6] W. Craig, P. Guyenne, D. Nichols and C. Sulem, A Hamiltonian approach to nonlinear modulation of surface water waves, Wave Motion, 47 (2010), 552–563.
[7] W. Craig, P. Guyenne and C. Sulem, Hamiltonian higher-order nonlinear Schrödinger equations for broader-banded waves on deep water, European J. Mechanics B/Fluids, 32 (2010), 22–31.
[8] W. Craig, P. Guyenne and C. Sulem, Normal form transformations and Dyson’s equation for the nonlinear modulation of deep water waves, Water Waves (2020), in press.
[9] F. Dias and C. Kharif, Nonlinear gravity and capillary -gravity waves, Annu. Rev. Fluid Mech., 31 (1999), 301–346.
[10] A. Carbery, C. E. Kenig and S. Ziesler, Restriction for homogeneous polynomial surfaces in $\mathbb{R}^3$, Trans. Amer. Math. Soc., 365 (2013), 2367–2407.
[11] K. B. Dysthe, Note on a modification to the nonlinear Schrödinger equation for application to deep water waves, Proc. R. Soc. Lond. A., 369 (1979), 105–114.
[12] K. B. Dysthe and K. Trulsen, A modified nonlinear Schrödinger equation for broader bandwidth gravity waves on deep water, Wave Motion 24 (1996), 281–289.
[13] F. Fedele and D. Dutykh, Hamiltonian form and solitary waves of the spatial Dysthe equation, JETP Letters, 94 (2011), 840–844.
[14] R. Grande, K. Kurianski and G. Staffilani, On the nonlinear Dysthe equation, preprint (2020), arXiv:2006.13392.
[15] A. Grünrock and S. Herr, The Fourier restriction norm method for the Zakharov-Kuznetsov equation, Disc. Contin. Dyn. Syst. Ser. A, 34 (2014), 2061–2068.
[16] M. Hadac, S. Herr and H. Koch, Well-posedness and scattering for the KP-II equation in a critical space, Ann. Inst. H. Poincaré Anal. Non Lin., 26 (2009), 917–941. ERRATUM Ann. Inst. H. Poincaré Anal. Non Lin., 27 (2010), 971–972.
[17] D. Henderson, D.H. Peregrine and J.W. Dold, Unsteady water wave modulations: Fully nonlinear solutions and comparison with the nonlinear Schrödinger equation, Wave Motion 20 (1999), 341–361.
[18] S. J. Hogan, The fourth-order evolution equation for deep-water gravity-capillary waves., Proc. R. Soc. Lond. A, 402 (1985), 359–372.
[19] P.A.E.M. Janssen, Fourth-order envelope equation for deep-water waves, J. Fluid Mech. 126 (1983), 1-11.
[20] H.-H. Hwang, W.-S.Chiang and S.-C.Hsiao, Observations on the evolution of wave modulation, Proc. Royal Society A., 463 (2007), 85–112.
[21] A. Kazezina and C. Muñoz, Dispersive estimates for rational symbols and local well-posedness of the nonzero energy NV equation, J. Funct. Anal., 270 (2016), no. 5, 1744–1791.
[22] C.E. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Comm. Pure Appl. Math., 46 (1993), 527–620.
[23] S. Kinoshita, Well-posedness for the Cauchy problem of the modified Zakharov-Kuznetsov equation, preprint (2019), arXiv:1911.13295
[24] H. Koch and J.-C. Saut, Dispersive smoothing and local solvability for third order dispersive equations, SIAM J. Math. Anal., 38 (2007), 1528–1541.
[25] H. Koch and D. Tataru, Dispersive estimates for principally normal pseudodifferential operators, Comm. Pure Appl. Math., 58 (2005), 217–284.
[26] H. Koch and D. Tataru, A priori bounds for the 1D cubic NLS in negative Sobolev spaces, Int. Math. Res. Not., (2007), article ID nm053, 36 pages.
[27] D. Lannes, Water waves : mathematical theory and asymptotics, Mathematical Surveys and Monographs, vol 188 (2013), AMS, Providence.
[28] F. Linares and A. Pastor, Well-posedness for the two-dimensional modified Zakharov-Kuznetsov equation, SIAM J. Math. Anal., 41 (2009), 1323–1339.
[29] F. Linares, D. Pilod and G. Ponce, Well-posedness for a higher-order Benjamin-Ono equation, J. Diff. Eq., 250 (2011) 450–475.
[30] E. Lo and Chiang C. Mei, A numerical study of wave-wave modulation based on a higher-order nonlinear Schrödinger equation, J. Fluid Mech. 150 (1985), 395–416.
[31] E. Lo and Chiang C. Mei, Slow evolution of nonlinear deep water water waves in two horizontal dimensions: A numerical approach, Wave Motion 9 (1987), 245–259.
M.S. Longuet-Higgins, The instabilities of gravity waves of finite amplitude in deep water. I. Superharmonics, Proc. Royal Society A, 360 (1978), 47–488.

M.S. Longuet-Higgins, The instabilities of gravity waves of finite amplitude in deep water. II. Subharmonics, Proc. Royal Society A, 360 (1978), 489–505.

D.U. Martin and H.C. Yuen, Quasi-recurring energy leakage in the two-space dimensional nonlinear Schrödinger equation, Phys. Fluids, 13 (1980), 881–883.

L. Molinet and D. Pilod, Bilinear Strichartz estimates for the Zakharov-Kuznetsov equation and applications, Ann. Inst. H. Poincaré An. Non Lin., 32 (2015), 347–371.

L. Molinet, J.-C. Saut and N. Tzvetkov, Global well-posedness for the KP-II equation on the background of non localized solution, Ann. Inst. H. Poincaré Anal. Non Lin., 28 (2011), no. 5, 653–676.

F. Ribaud and S. Vento, A note on the Cauchy problem for the 2D generalized Zakharov-Kuznetsov equations, C. R. Math. Acad. Sci. Paris, 350 (2012), 499–503.

M. Stiassnie, Note on the modified nonlinear equation for deep water waves, Wave Motion, 6 (1984), 43–433.

M. Stiassnie and L. Shemer, On modifications of the Zakharov equation for surface gravity waves, J. Fluid Mech., 143 (1984), 47–67.

T. Tao, Nonlinear Dispersive Equations: Local and Global Analysis, no. 106 in CBMS, Amer. Math. Soc., 2007.

V.E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, J. Appl. Mech. Tech. Phys., 9 (1968), 190–194.

Department of Mathematics, University of Bergen, Postbox 7800, 5020 Bergen, Norway
E-mail address: Razvan.Mosincat@uib.no

Department of Mathematics, University of Bergen, Postbox 7800, 5020 Bergen, Norway
E-mail address: Didier.Pilod@uib.no

Laboratoire de Mathématiques, UMR 8628, Université Paris-Saclay and CNRS, 91405 Orsay, France
E-mail address: jean-claude.saut@universite-paris-saclay.fr