NONSTANDARD MODEL CATEGORIES AND HOMOTOPY THEORY

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Abstract. In order to apply nonstandard methods to questions of algebraic geometry we continue our investigation from [BS05] and show how important homotopical constructions behave under enlargements.

1. Introduction

Modern algebraic geometry makes heavy use of categorical constructions. Therefore, in order to apply nonstandard methods to algebraic geometry, in [BS05] we started to study how such categorical constructions behave under enlargements. In the papers [BS04], [BS07] and [BS08] we applied the results to constructions in algebraic geometry. In this paper we want to continue the investigation of [BS05]. For reasons of why we consider the use of nonstandard methods in algebraic geometry worthwhile, we refer the reader to the introduction of [BS05].

Homotopical methods have become more and more important in algebraic geometry. For instance, the higher $K$-groups of schemes are defined as homotopy groups of certain simplicial sets, and the construction of Voevodsky’s $A^1$-homotopy category makes heavy use of the concept of Quillen model categories. Here we therefore want to investigate the behaviour of these concept under enlargements.

We start in section 2 with general (strict) $n$-categories and their enlargements. Not surprisingly, it turns out that this notion behaves well under enlargements.

After recalling some definitions, we study the enlargements of model categories in section 3, and we show that the external and internal homotopy category of an internal model category coincide.

In section 4 we consider simplicial sets. In general, the enlargement of a topological space will not again be a topological space. But we will see that at least a *simplicial set can be restricted to become a usual simplicial set. We show that there is a morphism from the external to the internal homotopy group of an internal simplicial set.

In section 5 we study the $K$-theory of exact categories and their enlargements. Then, now formulated in terms of ultraproducts, we construct a nontrivial morphism from the $K$-theory of an ultraproduct to the ultraproduct of the $K$-theory.

2. Internal $n$-Categories

We work in ZFC with the additional assumption that any set is element of a universe (compare [GV72][Exp. I.0]).

Recall that for a natural number $n \in \mathbb{N}_0$, a (small, strict) $n$-category $\mathcal{A}$ is given by the following data (compare [Lei02]):

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A diagram of sets
\[ A_n \xrightarrow{s} A_{n-1} \xrightarrow{s} \cdots \xrightarrow{s} A_1 \xrightarrow{s} A_0, \]
where \( A_0 \) is called the set of objects, \( A_1 \) the set of morphisms and \( A_m \) the set of \( m \)-cells (for \( 0 \leq m \leq n \)), \( s \) is called source and \( t \) is called target,

- maps \( \mathbb{1} : A_m \to A_{m+1}, \alpha \mapsto \mathbb{1}_\alpha \) for \( m = 0, \ldots, n-1 \), called identity, and
- maps \( \circ_p : A_m \times_{A_p} A_m \to A_m, (\alpha', \alpha) \mapsto \alpha' \circ_p \alpha \) for all \( 0 \leq p < m \leq n \) (where the fibred product is taken with respect to \( s^{m-p} \) and \( t^{m-p} \)), called composition,

subject to the following conditions:
- \( ss(\alpha) = st(\alpha) = ts(\alpha) = tt(\alpha) \) for \( \alpha \in A_m, 2 \leq m \leq n \),
- \( s(\alpha' \circ_p \alpha) = \begin{cases} s(\alpha) & , m = p + 1 \\ s(\alpha') \circ_p s(\alpha) & , m \geq p + 2 \end{cases} \) and \( t(\alpha' \circ_p \alpha) = \begin{cases} t(\alpha) & , m = p + 1 \\ t(\alpha') \circ_p t(\alpha) & , m \geq p + 2 \end{cases} \)
  for \( 0 \leq p < m \leq n \) and \( \alpha, \alpha' \in A_m \),
- \( s\mathbb{1}_\alpha = \alpha = t\mathbb{1}_\alpha \) for \( \alpha \in A_m, 0 \leq m < n \),
- \( \mathbb{1}^{m-p} \circ_p (\alpha) \circ_p \alpha = \alpha = \alpha \circ_p \mathbb{1}^{m-p} \circ s^{m-p} (\alpha) \) for \( \alpha \in A_m \) and \( 0 \leq p < m \leq n \),
- \((\alpha'' \circ_p \alpha') \circ_p \alpha = \alpha'' \circ_p (\alpha' \circ_p \alpha) \) for \( \alpha'', \alpha' \) and \( \alpha \) in \( A_m \) with \((\alpha'', \alpha')\) and \((\alpha', \alpha)\) in \( A_m \times_{A_p} A_m \) and \( 0 \leq p < m \leq n \),
- \( \mathbb{1}_{\alpha'} \circ_p \mathbb{1}_\alpha = \mathbb{1}_{\alpha' \circ_p \alpha} \) for \((\alpha', \alpha) \in A_m \times_{A_p} A_m \) and \( 0 \leq p < m < n \) and
- \((\beta' \circ_p \beta) \circ_q (\alpha' \circ_p \alpha) = (\beta' \circ_q \alpha') \circ_p (\beta \circ_q \alpha) \) for \((\beta', \beta)\) and \((\alpha', \alpha)\) in \( A_m \times_{A_p} A_m \) with \((\beta', \alpha')\) and \((\beta, \alpha)\) in \( A_m \times_{A_p} A_m \) and \( 0 \leq q < p < m \leq n \).

Note that a 0-category is just a set and that a 1-category is an ordinary category.

For \( n \in \mathbb{N}_0 \) and \( n \)-categories \( \mathcal{C} = \langle A_m, s, t, \mathbb{1}, \circ_p \rangle \) and \( \mathcal{D} = \langle B_m, s, t, \mathbb{1}, \circ_p \rangle \), a \((\text{covariant}) \) \( n \)-functor \( F : \mathcal{C} \to \mathcal{D} \) is given by maps \( A_m \to B_m, \alpha \mapsto F\alpha \) for \( 0 \leq m \leq n \) that commute with \( s, t, \mathbb{1} \) and \( \circ_p \) in the obvious way.

It is easy to see that we get a category \( n \text{-Cat} \) in this way whose objects are all small \( n \)-categories and whose morphisms are \( n \)-functors.

Note that \( 0 \text{-Cat} = \mathcal{Sets} \) is just the category of sets, and \( 1 \text{-Cat} = \mathcal{Cat} \) is the category of categories.

2.1. Definition. Let \( \mathcal{U} \) be a set, and let \( n \in \mathbb{N}_0 \) be a natural number.

(i) A \( \mathcal{U} \text{-small } n \)-category is an \( n \)-category \( \langle A_m, s, t, \mathbb{1}, \circ_p \rangle \) with \( A_m \in \mathcal{U} \) for all \( 0 \leq m \leq n \).
(ii) \( n \text{-Cat}^{\mathcal{U}} \) is the full subcategory of \( n \text{-Cat} \) with objects the \( \mathcal{U} \text{-small } n \)-categories.
Note that for $\mathcal{U}$ a universe and $n$ as above, $n\text{-Cat}^{\mathcal{U}}$ is not $\mathcal{U}$-small. For example, $0\text{-Cat}^{\mathcal{U}}$ is the category of $\mathcal{U}$-sets and so the set of its objects is $\mathcal{U} \not\in \mathcal{U}$. On the other hand, if we choose a hierarchy $\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2 \in \ldots$ of universes, then $n\text{-Cat}^{\mathcal{U}_n}$ is $\mathcal{U}_{n+1}$-small for all $n \in \mathbb{N}_0$.

Let $S$ be a set of “individuals” with cardinality $|S| \geq \bigcup_{n \in \mathbb{N}_0} |\mathcal{U}_n|$, and let $*: \hat{S} \to \hat{\hat{S}}$ be an enlargement.

Our discussion shows that the categories $0\text{-Cat}^{\mathcal{U}_0}$, $1\text{-Cat}^{\mathcal{U}_1}$, $2\text{-Cat}^{\mathcal{U}_2}$, ... are all $\hat{S}$-small (in the sense of [BS05] if we consider the universes $\mathcal{U}_i$ as subsets of $S$ and hence as elements of $S \setminus \mathcal{S}$), so we can consider their enlargements $*[n\text{-Cat}^{\mathcal{U}_n}]$ in $\hat{\hat{S}}$ which are $\hat{\hat{S}}$-small categories.

### 2.2. Definition

For $n \in \mathbb{N}_0$, an $n$-*category* is an object of $n$-*Cat* := $*[n\text{-Cat}^{\mathcal{U}_n}]$, and a (covariant) $n$-*functor* $F: \mathcal{C} \to \mathcal{D}$ between $n$-*categories* $\mathcal{C}$ and $\mathcal{D}$ is a morphism $\mathcal{C} \to \mathcal{D}$ in $n$-*Cat*.

### 2.3. Proposition

$n$-*Cat* is the category whose objects are $*\mathcal{U}_n$-small internal $n$-categories and whose morphisms are internal $n$-functors.

*Proof.* The objects of $n\text{-Cat}^{\mathcal{U}_n}$ are all tuples $\langle A_m, s, t, 1, \circ_p \rangle$ with $A_m \in \mathcal{U}_n$ that satisfy the conditions stated above. Since these conditions are obviously first order, transfer proves the object part of the proposition. The morphism part is analogous but even simpler.

q.e.d.

In view of 2.3, we will from now on use the terms “internal $n$-categories” and “internal $n$-functors” for $n$-*categories* and $n$-*functors* respectively (and drop “$*\mathcal{U}_n$-small” from the notation).

Note that 2.3 shows in particular that if $\mathcal{C}$ is a $\mathcal{U}_n$-small $n$-category, then $*\mathcal{C}$ is also an $n$-category. Also note that — as $n\text{-Cat}^{\mathcal{U}_n}$ clearly has infinitely many objects — not all internal $n$-categories are of the form $*\mathcal{C}$ for an $n$-category $\mathcal{C}$.

For example, since $1\text{-Cat}^{\mathcal{U}_1}$ obviously contains the categories of finite dimensional vector spaces over $\mathbb{F}_p$ for all finite primes $p$, $1$-*Cat* contains the internal categories of *finite* dimensional $\mathbb{F}_p$-vector spaces for all infinite primes $P \in *\mathbb{N}_0$, and these categories are clearly not of the form $*\mathcal{C}$ for any category $\mathcal{C}$.

### 3. Internal Model Categories

Recall the notion of model category from [Qui69]:

A (small) model category is a quadruple $\langle \mathcal{M}, W, F, C \rangle$, with a small category $\mathcal{M} = \langle M_0, M_1, s, t, 1, \circ \rangle$ and sets of morphisms $W, F, C \subseteq M_1$, subject to the following conditions:

- All finite limits and colimits exist in $\mathcal{M}$,
- $\forall X \in M_0 : 1_\mathcal{M} X \in W \cap F \cap C$,
- $\forall X \in \{W, F, C\} : \forall (f, g) \in X \times_{M_0} X : fg \in X$, 

Recall the notion of model category from [Qui69]:

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- $\forall X \in \{W, F, C\} : \forall (f, g) \in X \times_{M_0} X : fg \in X$, 

\[ \forall (f, g) \in M_1 \times M_0 \mathcal{M}_1 : [fg \in W \land (f \in W \lor g \in W)] \implies \{f, g\} \subseteq W, \]

\[ \forall X \in \{W, F, C\} : \forall (r, i), (r', i') \in M_1 \times M_0 \mathcal{M}_1 : \forall f \in M_1 : \forall g \in X : [si = sf \lor tr \lor si' = tf \lor tr' \lor sg = ti \lor tg = ti' \lor ri = 1_{si} \lor r' i' = 1_{si'}] \implies f \in X, \]

\[ \forall (p, f) \in F \times M_0 \mathcal{M}_1 : \forall (g, i) \in M_1 \times M_0 \mathcal{C}_1 : [tp = tg \lor si = sf \lor (i \in W \lor p \in W)] \implies \exists h \in M_1 : sh = ti \lor th = sp \lor hi = f \lor ph = g, \]

\[ \forall f \in M_1 : [\exists (p, i) \in (F \cap W) \times M_0 \mathcal{C}_1 : f = pi] \lor [\exists (p, i) \in F \times M_0 (C \cap W) : f = pi]. \]

If \( \mathcal{M} \) is a category, then a triple \( \langle W, F, C \rangle \) of classes of morphisms of \( \mathcal{M} \) is called a model structure on \( \mathcal{M} \) if \( \langle \mathcal{M}, W, F, C \rangle \) is a model category.

By definition, a Quillen functor between model categories \( \langle \mathcal{M}, W, F, C \rangle \) and \( \langle \mathcal{M}', W', F', C' \rangle \) is an adjunction \( \langle L, R, \varphi \rangle \) between \( \mathcal{M} \) and \( \mathcal{M}' \) satisfying

\[ \forall c \in C : Lc \in C' \] and

\[ \forall f \in F' : Rf \in F. \]

A natural transformation between Quillen functors \( \langle L, R, \varphi \rangle, \langle L', R', \varphi' \rangle : \mathcal{M} \rightarrow \mathcal{M}' \) is just a natural transformation from \( L \) to \( L' \).

We chose to put the definition in this — admittedly not very readable — way in order to show that everything (except maybe the first condition) is first order. That also this first condition on the existence of finite limits and colimits is first order was explained in detail in [BS05].

3.1. Definition. Call a model category \( \langle \mathcal{M}, W, F, C \rangle \) \( \mathcal{U}_1 \)-small if \( \mathcal{M} \), the underlying category, is \( \mathcal{U}_1 \)-small, and let \( \mathcal{M}_{\mathcal{U}_1} \) be the 2-category with \( \mathcal{U}_1 \)-small model categories as objects, Quillen functors as morphisms and natural transformations between Quillen functors as 2-cells.

\( \mathcal{M}_{\mathcal{U}_1} \) is obviously \( \mathcal{U}_2 \)-small and hence an object of \( \mathcal{2}_{\mathcal{U}_2} \). Put \( *\mathcal{M}_{\mathcal{U}_1} := *[[\mathcal{M}_{\mathcal{U}_1}]] \). This is an object of \( *\mathcal{2} \) and hence by definition an internal 2-category — call its objects *model categories, its morphisms *Quillen functors and its 2-cells *natural transformations.

3.2. Proposition. *\( \mathcal{M}_{\mathcal{U}_1} \) is the 2-category having *\( \mathcal{U}_2 \)-small internal model categories as objects, internal Quillen functors as morphisms and internal natural transformations as 2-cells.

Proof. By transfer, an object of *\( \mathcal{M}_{\mathcal{U}_1} \) is a quadruple \( \langle \mathcal{M}, W, F, C \rangle \) with \( \mathcal{M} \) a *\( \mathcal{U}_1 \)-small category, subject to the transferred version of the conditions defining a model category. Since these conditions are first order, the object part of the proposition follows. The rest follows from [BS05], where we show that adjunctions and natural transformations are first order.

q.e.d.

3.3. Remark. Examples of 2-categories whose structure is defined by first order conditions abound and can be treated in a similar way: In [BS05] we have seen (although in a
slightly different setup) that the 2-categories of additive, abelian, triangulated categories and posets as well as those of (additive, abelian, triangulated) fibrations all belong to that category.

Another example is the 2-category of *exact categories* that we will need for our treatment of algebraic $K$-theory below.

In all these cases, the analogue of [3.2] holds: *Additive categories are precisely the (small) internal additive categories, *abelian categories are precisely the (small) internal abelian categories, and so on.

Because of [3.2], we will refer to *model categories, *Quillen functors and *natural transformations as internal model categories (again dropping “$\ast U_1$-small” from the notation), internal Quillen functors and internal natural transformations respectively.

Before we can formulate the next result, we have to recall some more definitions and facts from the theory of model categories. Let $\langle \mathcal{M}, W, F, C \rangle$ be a model category.

- Since $\mathcal{M}$ has all finite limits and colimits, it in particular has a terminal and an initial object, which we denote by $\ast$ and $\emptyset$ respectively.

- An object $X$ of $\mathcal{M}$ is called *cofibrant* if the unique morphism $\emptyset \to X$ is in $C$, and *fibrant* if the unique morphism $X \to \ast$ is in $F$.

- For all objects $X$ in $\mathcal{M}$, there is a morphism $QX \to X$ in $F \cap W$ with $QX$ cofibrant and a morphism $X \to RX$ in $C \cap W$ with $RX$ fibrant. (It follows from the axioms that $RQX$ is both fibrant and cofibrant.)

- Let $A$ and $X$ be cofibrant respectively fibrant objects of $\mathcal{M}$, and let $f, g \in \text{Mor}_\mathcal{M}(A, X)$ be morphisms. Then $f$ and $g$ are called *homotopic*, $f \sim g$, if there exists a commutative diagram

\[
\begin{array}{ccc}
A & \xleftarrow{i_0} & I & \xrightarrow{H} & X \\
\downarrow{p} & & & \downarrow{g} \\
A & \xleftarrow{i_1} & I & \xrightarrow{H} & X \\
\end{array}
\]

in $\mathcal{M}$ with $p \in W$. Homotopy is an equivalence relation, and the set of equivalence (or homotopy) classes $\text{Mor}_\mathcal{M}(A, X)/\sim$ is denoted by $\pi(A, X)$.

By transfer, we get corresponding notions of *(co-)fibrant objects and *homotopic morphisms for internal model categories.

3.4. Proposition. Let $\mathcal{M} = \langle \mathcal{M}, W, F, C \rangle$ be an internal model category, let $A$ and $X$ be objects of $\mathcal{M}$ with $A$ *cofibrant and $X$ *fibrant, and let $f, g : A \to X$ be morphisms in $\mathcal{M}$. 
Then $A$ is cofibrant, and $X$ is fibrant, and $f$ and $g$ are homotopic if and only if they are homotopic. In particular, the set of homotopy classes equals $\pi(A,X)$, the set of homotopy classes.

**Proof.** This is easy, because the terminal object of $\mathcal{M}$ is the terminal object, the initial object is the initial object, and the condition defining homotopy is obviously first order.

q.e.d.

By definition, the homotopy category $\text{Ho}(\mathcal{M})$ of a model category $\langle \mathcal{M}, W, F, C \rangle$ is the localization $\mathcal{M}[W^{-1}]$ of $\mathcal{M}$ by the set $W$. By transfer, we get the notion of the homotopy category $\ast \text{Ho}(\mathcal{M})$ of an internal category $\langle \mathcal{M}, F, C, W \rangle$, which is the localization of $\mathcal{M}$ by $W$.

Note that for an arbitrary internal category $\mathcal{C}$, it will not be true that its localization by an (internal) set $W$ of morphisms agrees with its localization by $W$, because in general the morphisms in the localization are finite words built from morphisms in $\mathcal{C}$ and formal inverses of morphisms in $W$, whereas words in the localization can be of finite length and do not have to be finite.

Nevertheless, the situation is much better with model categories, as the next result shows:

3.5. **Theorem.** Let $\langle \mathcal{M}, F, C, W \rangle$ be an internal model category. Then $\text{Ho}(\mathcal{M})$ and $\ast \text{Ho}(\mathcal{M})$ are canonically isomorphic.

**Proof.** By the universal property of localization, the canonical functor $\mathcal{M} \rightarrow \ast \text{Ho}(\mathcal{M})$ factors uniquely as $\mathcal{M} \rightarrow \text{Ho}(\mathcal{M}) \xrightarrow{F} \ast \text{Ho}(\mathcal{M})$, where $F$ is the identity on objects. We claim that $F$ is also fully faithful. For this, let $A$ and $X$ be arbitrary objects of $\mathcal{M}$. The diagrams $RQA \leftarrow QA \rightarrow A$ and $RQX \leftarrow QX \rightarrow X$ give isomorphisms in $\text{Ho}(\mathcal{M})$ and $\ast \text{Ho}(\mathcal{M})$, so by composing with them we can assume without loss of generality that $A$ and $X$ are both fibrant and cofibrant. It is a well known fact that in this case $\text{Mor}_{\text{Ho}(\mathcal{M})}(A, X) = \pi(A, X)$, and the theorem follows from this fact, its transferred version and 3.4.

q.e.d.

4. **Internal Simplicial Sets**

Let $\Delta$ be the simplicial category, i.e. the category whose objects are the finite ordinals $[n] = \{0, 1, \ldots, n\}$ for $n \in \mathbb{N}_0$ and whose morphisms are monotonic maps; this is of course a $\mathcal{U}_1$-small category, and it follows immediately by transfer that $\ast \Delta$ has objects $[n]$ for $n \in \ast \mathbb{N}_0$ and again monotonic maps as morphisms.

The category $\Delta\text{-Sets}$ of simplicial sets is by definition the category of functors $X_\bullet : \Delta^{\text{op}} \rightarrow \text{Sets}$, $[n] \mapsto X_n$, and for a set $\mathcal{U}$, we denote by $\Delta\text{-Sets}^\mathcal{U}$ the full subcategory of $\Delta\text{-Sets}$ of functors $X_\bullet$ with $X_n \in \mathcal{U}$ for all $n \in \mathbb{N}_0$.

$\Delta\text{-Sets}^\mathcal{U}_0$ is obviously $\mathcal{U}_1$-small, and we put $\ast \Delta\text{-Sets} := \ast \Delta\text{-Sets}^\mathcal{U}_0$ and call the resulting category the category of simplicial sets.

It is clear that $\ast \Delta\text{-Sets}$ is the category of internal functors $X_\bullet$ from $\ast \Delta^{\text{op}}$ to the category of $\mathcal{U}_0$-sets with internal maps as morphisms, and we use the term internal simplicial set as a synonym for “*simplicial set”.

4.1. **Definition.** The functor $\ast : \Delta \rightarrow \ast \Delta$ (which is the identity on objects and morphisms and obviously a full embedding) induces by composition a canonical “restriction”
functor \( \Delta \text{-Sets} \to \Delta \text{-Sets} \), which “cuts off” the infinite part of an internal simplicial set. We denote this functor by \( \text{res} \).

Recall that there is an adjunction \( \langle |.|, \text{Sing}, \varphi \rangle \) from \( \Delta \text{-Sets} \) to \( \text{Top} \), the category of topological spaces, in terms of which \( \Delta \text{-Sets} \) can be given a model structure by setting

- \( W := \{ w : X \to Y \mid |w| : |X| \to |Y| \text{ is a weak homotopy equivalence} \} \),
- \( F := \{ f : X \to Y \mid \forall n \in \mathbb{N}^+ : \forall 0 \leq k \leq n : \forall x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in X_{n-1} : \forall y \in Y_n : \left( \forall 0 \leq i < j \leq n : i \neq j \Rightarrow \delta^{-1}x_i = \delta^i x_j \right) \wedge \left( \forall i \in [n] \setminus \{k\} : \delta_i y = f x_i \right) \} \)
- \( C := \{ c : X \to Y \mid c_n : X_n \to Y_n \text{ is injective for all } n \in \mathbb{N}_0 \} \).

Furthermore, there is a model structure on \( \text{Top} \) that turns \( \langle |.|, \text{Sing}, \varphi \rangle \) into a Quillen functor (which induces an equivalence on the associated homotopy categories).

By restricting to \( \Delta \text{-Sets}^{\mathbb{N}_0} \) and \( \text{Top}^{\mathbb{N}_0} \) (the category of topological spaces whose underlying sets are \( \mathbb{N}_0 \)-small), we get a model structure on \( \Delta \text{-Sets}^{\mathbb{N}_0} \) and a Quillen functor \( \Delta \text{-Sets}^{\mathbb{N}_0} \to \text{Top}^{\mathbb{N}_0} \) and hence an internal model structure on \( \ast \Delta \text{-Sets} \) and an internal Quillen functor \( \ast \Delta \text{-Sets} \to \ast \text{Top}^{\mathbb{N}_0} \).

For \( n \in \mathbb{N}_0 \) and \( i \in [n+1] \), \( \delta^i : [n] \to [n+1] \) denotes the unique injective monotonic map whose image is \([n+1] \setminus \{i\}\), and for \( n \in \mathbb{N}_0 \) and \( i \in [n] \), \( s^i : [n+1] \to [n] \) denotes the unique surjective monotonic map with \( s^i(i) = s^i(i+1) \), for a simplicial set \( X_\bullet \), \( \delta^i : X_{n+1} \to X_n \) denotes \( X(d^i) \), and \( \sigma_i : X_n \to X_{n+1} \) denotes \( X(s^i) \).

**4.2. Proposition.** The functor \( \text{res} : \ast \Delta \text{-Sets} \to \Delta \text{-Sets} \) respects \( F \), \( C \) and the terminal, the initial and fibrant and cofibrant objects.

**Proof.** Since \( F \) and \( C \) for \( \Delta \text{-Sets} \) are defined by first order formulas, the same formulas define \( F \) and \( C \) for \( \ast \Delta \text{-Sets} \) (except that \( \mathbb{N}_+ \) and \( \mathbb{N}_0 \) have to be replaced by \( \ast \mathbb{N}_+ \) respectively \( \ast \mathbb{N}_0 \)). This shows that \( \text{res} \) respects \( F \) and \( C \).

The terminal object of \( \Delta \text{-Sets} \) is the simplicial set \( \ast \) with \( \ast_n = \{\ast\} \) for all \( n \in \mathbb{N}_0 \), so the terminal object of \( \ast \Delta \text{-Sets} \) is the internal simplicial set \( \ast \) with \( \ast_n = \{\ast\} \) for all \( n \in \ast \mathbb{N}_0 \), which is clearly mapped to \( \ast \) under \( \text{res} \). Similarly, the initial object of \( \Delta \text{-Sets} \) is the constant simplicial set \( \emptyset \) with \( \emptyset_n = \emptyset \) for all \( n \in \mathbb{N}_0 \), the initial internal simplicial set is \( \ast \emptyset \) with \( \ast \emptyset_n = \emptyset \) for all \( n \in \ast \mathbb{N}_0 \), and again \( \text{res} \) \( \emptyset \) = \( \emptyset \) holds trivially.

Thus by definition of “fibrant” and “cofibrant”, the rest of the proposition follows as well.

q.e.d.

Recall the following definitions and results from [Kan58] for simplicial sets:
• If $X_\bullet$ is fibrant and $n \in \mathbb{N}_0$, then there is an equivalence relation $\sim$ on $X_n$ defined by putting $y \sim z$ if and only if

$$\exists x \in X_1 : \delta^0 x = y \land \delta^1 x = z$$

$$\left\{ \begin{array}{ll}
[0 \leq i \leq n : \delta^i y = \delta^i z] \land \exists x \in X_{n+1} : \\
\delta^i x = \begin{cases}
\sigma^0 \delta^i y , & 0 \leq i \leq n \\
y, & i = n \\
z, & i = n + 1
\end{cases}
\right\}, \quad n \geq 1$$

• A simplicial set with base point is a pair $\langle X_\bullet, x \rangle$, where $X_\bullet$ is a simplicial set and $x \in X_0$. A morphism $\langle X_\bullet, x \rangle \rightarrow \langle Y_\bullet, y \rangle$ of simplicial sets with base points is a morphism $F : X_\bullet \rightarrow Y_\bullet$ of simplicial sets with $f_0(x) = y$. We denote the resulting category by $\Delta\text{-Sets}^+$. 

• For a fibrant simplicial set with base point $\langle X_\bullet, x \rangle$ and $n \in \mathbb{N}_0$, define the $n$-th simplicial homotopy set of $\langle X_\bullet, x \rangle$ as $\pi_n(X_\bullet, x) := I_n/\sim$, where

$$X_n \supseteq I_n := \left\{ \begin{array}{ll}
X_0 & , n = 0 \\
\left\{ y \in X_1 \mid \delta^0 y = \delta^1 y = x \right\} & , n = 1 \\
\left\{ y \in X_{n+1} \mid \forall 0 \leq i \leq n : \delta^i y = \sigma^{n-2} \ldots \sigma^0 x \right\} & , n \geq 2
\end{array} \right.$$

The set $\pi_0(X_\bullet, x)$ is pointed by $\bar{x}$, and for $n \geq 1$, $\pi_n(X_\bullet, x)$ is a group, the $n$-th simplicial homotopy group of $\langle X_\bullet, x \rangle$, whose multiplication is given as follows: Let $y, y' \in I_n$. Since $X_\bullet$ is fibrant, there exists $z \in X_{n+1}$ satisfying

$$\delta^{n-1} z = y \land \delta^{n+1} z = y' \land \forall 0 \leq i \leq n - 2 : \delta^i z = \sigma^{n-1} \ldots \sigma^0 x.$$

Put $\overline{y} \cdot \overline{y'} := \overline{\delta^n z} \in \pi_n(X, x)$. 

• If $f_\bullet : \langle X_\bullet, x \rangle \rightarrow \langle Y_\bullet, y \rangle$ is a morphism of fibrant simplicial sets with base points, then $f_n : X_n \rightarrow Y_n$ induces a map $\pi_n(f) : \pi_n(X_\bullet, x) \rightarrow \pi_n(Y_\bullet, y)$ for all $n \in \mathbb{N}_0$ which is a morphism of pointed sets for $n = 0$ and a group homomorphism for $n \geq 1$. In this way, we get functors $\pi_n$ from the full subcategory of $\Delta\text{-Sets}^+$ of fibrant objects to the category of pointed sets (for $n = 0$) respectively the category of groups (for $n \geq 1$).

• If $\langle X_\bullet, x \rangle$ is an arbitrary object of $\Delta\text{-Sets}^+$, the $n$-th homotopy set respectively group is defined as $\pi_n(X_\bullet, x) := \pi_n(|X_\bullet|, |x|)$ for $n \in \mathbb{N}_0$, so $\pi_n$ defines a functor from $\Delta\text{-Sets}^+$ to the category of pointed sets respectively groups. This functor — when restricted to the full subcategory of fibrant objects — is canonically isomorphic to the functor $\pi_n$ for all $n \in \mathbb{N}_0$.

• A sequence of morphisms in $\Delta\text{-Sets}^+$

$$\langle F_\bullet, f \rangle \xrightarrow{q_\bullet} \langle E_\bullet, e \rangle \xrightarrow{\nu_\bullet} \langle B_\bullet, b \rangle$$

is called a fibre sequence if

- $p \in F \land \forall n \in \mathbb{N}_0 : p_n$ is surjective,
- $q \in C$ and
- $\forall n \in \mathbb{N}_0 : q_n(F_n) = \begin{cases} p_{n-1}^{-1}(b) & , n = 0 \\
p_n^{-1}(\sigma^{n-1} \ldots \sigma^0 b) & , n \geq 1
\end{cases}$

A fibre sequence as above functorially defines a long exact sequence

$$\ldots \overset{\pi_{n+1}(p_\bullet)}{\xrightarrow{}} \pi_{n+1}(B_\bullet, b) \overset{\delta}{\rightarrow} \pi_n(F_\bullet, f) \overset{\pi_n(p_\bullet)}{\xrightarrow{}} \pi_n(E_\bullet, e) \overset{\pi_n(\nu_\bullet)}{\xrightarrow{}} \pi_n(B_\bullet, b) \overset{\delta}{\rightarrow} \ldots$$
of groups respectively pointed sets.

- There is an endofunctor $\text{Ex}^\infty$ of $\Delta$-$\text{Sets}$ and a natural transformation $\text{ex}^\infty : \mathbb{1}_{\Delta$-$\text{Sets}} \to \text{Ex}^\infty$ with the following properties:
  - $\text{Ex}^\infty X_\bullet$ is fibrant for all simplicial sets $X_\bullet$.
  - $\text{ex}^\infty : X_\bullet \to \text{Ex}^\infty X_\bullet$ is in $W$ for all simplicial sets $X_\bullet$.
- If $\langle F_\bullet, f \rangle \xrightarrow{q} \langle E_\bullet, e \rangle \xrightarrow{\nu} \langle B_\bullet, b \rangle$ is a fibre sequence, then
  $\langle \text{Ex}^\infty F_\bullet, \text{ex}^\infty(f) \rangle \xrightarrow{\text{Ex}^\infty q} \langle \text{Ex}^\infty E_\bullet, \text{ex}^\infty(e) \rangle \xrightarrow{\text{Ex}^\infty \nu} \langle \text{Ex}^\infty B_\bullet, \text{ex}^\infty(b) \rangle$
  is also a fibre sequence.

These notions can be restricted to $\Delta$-$\text{Sets}^{q_0}$ and then enlarged to $^\ast\Delta$-$\text{Sets}$, i.e. we have the category $^\ast\Delta$-$\text{Sets}^+$ of internal simplicial sets with base points, internal (simplicial) homotopy functors $^\ast\pi_n$ and $^\ast \tilde{\pi}_n$ (for $n \in \mathbb{N}_0$) and internal fibred sequences as well as $^\ast\text{Ex}^\infty$ and $^\ast\text{ex}^\infty$ with analogous properties.

Note that $\text{res} : ^\ast\Delta$-$\text{Sets} \to \Delta$-$\text{Sets}$ induces a functor $\text{res} : ^\ast\Delta$-$\text{Sets}^+ \to ^\ast\Delta$-$\text{Sets}^+$ by sending the internal base point to itself.

4.3. Theorem. There is a canonical natural transformation $\varphi_n : \pi_n \circ \text{res} \Rightarrow ^\ast\pi_n$ of functors from $^\ast\Delta$-$\text{Sets}^+$ to groups respectively sets for all $n \in \mathbb{N}_0$. Moreover, if

$$\langle F_\bullet, f \rangle \xrightarrow{q} \langle E_\bullet, e \rangle \xrightarrow{\nu} \langle B_\bullet, b \rangle$$

is an internal fibre sequence, then

$$\langle \text{res} F_\bullet, f \rangle \xrightarrow{\text{res} \, q} \langle \text{res} E_\bullet, e \rangle \xrightarrow{\text{res} \, \nu} \langle \text{res} B_\bullet, b \rangle$$

is a fibre sequence and

$$\pi_{n+1}(\text{res} B_\bullet, b) \xrightarrow{\delta} \pi_n(\text{res} F_\bullet, f)$$

$$\varphi_{n+1} \downarrow \quad \quad \quad \varphi_n \downarrow$$

$$^\ast\pi_n(B_\bullet, b) \xrightarrow{\delta} ^\ast\pi_n(F_\bullet, f)$$

commutes for all $n \in \mathbb{N}_0$.

Proof. Let $\langle X_\bullet, x \rangle$ be an object of $^\ast\Delta$-$\text{Sets}^+$. Then on the one hand, we have canonical isomorphisms

$$\alpha_n : ^\ast\pi_n(X_\bullet, x) \xrightarrow{^\ast\pi_n(\text{ex}^\infty)} \sim^\ast\pi_n(^\ast\text{Ex}^\infty X_\bullet, ^\ast\text{ex}^\infty x) \xrightarrow{\sim} ^\ast \tilde{\pi}_n(^\ast\text{Ex}^\infty X_\bullet, ^\ast\text{ex}^\infty x)$$

for all $n \in \mathbb{N}_0$, on the other hand $\text{res} \, \text{Ex}^\infty X_\bullet$ is fibrant by \cite{12}, so we get morphisms

$$\beta_n : \pi_n(\text{res} X_\bullet, x) \xrightarrow{\pi_n(\text{res} \, \text{ex}^\infty)} \pi_n(\text{res} ^\ast\text{Ex}^\infty X_\bullet, ^\ast\text{ex}^\infty x) \xrightarrow{\sim} \tilde{\pi}_n(\text{res} ^\ast\text{Ex}^\infty X_\bullet, ^\ast\text{ex}^\infty x)$$

for all $n \in \mathbb{N}_0$. But the definition of the $n$-th simplicial homotopy set respectively group of a simplicial set with base point $\langle Y_\bullet, y \rangle$ is obviously first order and only depends on the restriction of $Y$ to the full subcategory of $\Delta$ with objects $\{[0], [1], \ldots, [n+1]\}$, so that

$$\tilde{\pi}_n(\text{res} ^\ast\text{Ex}^\infty X_\bullet, ^\ast\text{ex}^\infty x) = ^\ast \tilde{\pi}_n(^\ast\text{Ex}^\infty X_\bullet, ^\ast\text{ex}^\infty x)$$

for $n \in \mathbb{N}_0$. Putting everything together, we can define $\varphi_n := \alpha_n^{-1} \beta_n$, and this is clearly functorial and hence defines a natural transformation.
That res maps internal fibre sequences to fibre sequences is clear from 4.2 and the
definition of “fibre sequence”. So the last part of the theorem follows from the fact that
\(\text{Ex}^\infty\) respects fibre sequences and from the construction of \(\varphi_n\).
q.e.d.

5. Nonstandard \(K\)-Theory

Recall the notion of an exact category from [Qui73]:

Let \(\mathcal{M}\) be a small additive category, and let \(\mathcal{E}\) be a set of composable pairs \(\langle M' \xrightarrow{i} M, M \xrightarrow{j} M'' \rangle\) of morphisms in \(\mathcal{M}\). A morphism \(i\) in \(\mathcal{M}\) is called an admissible monomorphism if there is a morphism \(j\) in \(\mathcal{M}\) with \(\langle i, j \rangle \in \mathcal{E}\), and \(j \in \text{Mor}_{\mathcal{M}}\) is called an admissible epimorphism if there exists \(i \in \text{Mor}_{\mathcal{M}}\) with \(\langle i, j \rangle \in \mathcal{E}\).

The pair \(\langle \mathcal{M}, \mathcal{E} \rangle\) is a (small) exact category if the following conditions are satisfied:

- If

\[
\begin{array}{c}
M' \xrightarrow{i} M \xrightarrow{j} M'' \\
\downarrow \alpha \quad \downarrow \beta \quad \downarrow \gamma \quad \downarrow i \\
N' \xrightarrow{i} N \xrightarrow{j} N''
\end{array}
\]

is a commutative diagram in \(\mathcal{M}\) with \(\langle i, j \rangle \in \mathcal{E}\), then \(\langle \tilde{i}, \tilde{j} \rangle \in \mathcal{E}\),

- for all objects \(M', M''\) of \(\mathcal{M}\),

\[
\left\langle M' \xrightarrow{(1_{M'}, 0)} M' \oplus M'', M' \oplus M'' \xrightarrow{pr_2} M'' \right\rangle \in \mathcal{E},
\]

- for all \(\langle i, j \rangle \in \mathcal{E}\), \(i\) is a kernel of \(j\), and \(j\) is a cokernel of \(i\),

- the set of admissible epimorphisms is closed under composition,

- the set of admissible monomorphisms is closed under composition,

- pullbacks of admissible epimorphisms along arbitrary morphisms exist in \(\mathcal{M}\) and are again admissible epimorphisms,

- pushouts of admissible monomorphisms along arbitrary morphisms exist in \(\mathcal{M}\) and are again admissible monomorphisms,

- Let \(M \xrightarrow{j} M''\) be a morphism in \(\mathcal{M}\) which has a kernel. If there is a morphism \(N \xrightarrow{\beta} M\) in \(\mathcal{M}\) such that \(j\beta\) is an admissible epimorphism, then \(j\) is an admissible epimorphism,

- Let \(M' \xrightarrow{i} M\) be a morphism in \(\mathcal{M}\) which has a cokernel. If there is a morphism \(M \xrightarrow{\beta} N\) in \(\mathcal{M}\) such that \(\beta i\) is an admissible monomorphism, then \(i\) is an admissible monomorphism.
Recall further that an exact functor between exact categories \( \langle M, E \rangle \) and \( \langle M', E' \rangle \) is an additive functor \( F : M \longrightarrow M' \) with
\[
\forall (i, j) \in E : \langle Fi, Fj \rangle \in E',
\]
and a natural transformation between exact functors is just a natural transformation in the ordinary sense.

Note that if \( A \) is an abelian category, then \( \langle A, E \rangle \) is exact with
\[
E := \{ (i, j) \mid 0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} B'' \longrightarrow 0 \text{ exact} \},
\]
and if not stated otherwise, an abelian category is always considered as an exact category in this way.

5.1. **Definition.** Call an exact category \( \langle M, E \rangle \) \( U_1 \)-small if \( M \), the underlying additive category, is \( U_1 \)-small, and let \( \mathcal{E}x\text{Cat}^{U_1} \) be the 2-category with \( U_1 \)-small exact categories as objects, exact functors as morphisms and natural transformations as 2-cells.

\( \mathcal{E}x\text{Cat}^{U_1} \) is obviously \( U_2 \)-small and hence an object of \( 2\text{-Cat}^{U_2} \). Put \( *\mathcal{E}x\text{Cat} := *[\mathcal{E}x\text{Cat}^{U_1}] \). This is an object of \( 2\text{-Cat}^* \) and hence by definition an internal 2-category — call its objects *exact categories, its morphisms *exact functors and its 2-cells *natural transformations.

5.2. **Proposition.** \( *\mathcal{E}x\text{Cat} \) is the 2-category having \( U_2 \)-small internal exact categories as objects, internal exact functors as morphisms and internal natural transformations as 2-cells.

**Proof.** Since the conditions defining an exact category are all first order, this is completely analogous to the proof of 3.2 q.e.d.

By definition, the Quillen category \( Q\langle M, E \rangle \) (also simply denoted by \( QM \) if the set \( E \) is understood) associated to an exact category \( \langle M, E \rangle \) is the categories whose objects are the objects of \( M \) and whose morphisms from \( M \) to \( M' \) are isomorphism classes of diagrams
\[
(j, i) : M \xrightarrow{j} N \xrightarrow{i} M'
\]
in \( M \) with an admissible epimorphism \( j \) and an admissible monomorphism \( i \), where two diagrams \((j, i)\) and \((\tilde{j}, \tilde{i})\) are isomorphic if there is a commutative diagram
\[
\begin{array}{ccc}
M & \xrightarrow{j} & \tilde{N} & \xrightarrow{i} & M' \\
\downarrow^{\tilde{j}} & \downarrow^\beta & \downarrow_{\tilde{i}} & \downarrow\beta & \downarrow^i \\
N & \xrightarrow{i} & M' \\
\end{array}
\]
in \( M \) with an isomorphism \( \beta \). Composition of morphisms \( M \xrightarrow{j} N \xrightarrow{i} M' \) and \( M' \xrightarrow{j'} N' \xrightarrow{i'} M'' \) is defined as \((jj'', i'i'') : M \longrightarrow M''\), where \( j'' : M'' \longrightarrow N \) is a pullback of \( j' \) along \( i \) and \( i'' : M'' \longrightarrow N' \) is the corresponding projection.

\[
\begin{array}{ccc}
M'' & \xrightarrow{j''} & N' & \xrightarrow{i'} & M'' \\
\downarrow^{j''} & \downarrow^{j'} & \downarrow^i & \downarrow^{j'} & \downarrow^{j''} \\
N & \xrightarrow{i} & M' & \xrightarrow{i'} & M'' \\
\downarrow^j & \downarrow^i & \downarrow^{j'} & \downarrow^i & \downarrow^j \\
M & & & & \\
\end{array}
\]
(it follows from the axioms for an exact category that $i''$ and $j''$ are an admissible monomorphism respectively epimorphism).

In this way, we get the Quillen 2-functor $Q : \mathcal{ExCat} \to 1\text{-Cat}$, $\langle \mathcal{M}, \mathcal{E} \rangle \mapsto Q\langle \mathcal{M}, \mathcal{E} \rangle$, and by transfer the *Quillen 2-functor $^*Q : *\mathcal{ExCat} \to 1^*\text{-Cat}$ from internal exact categories to internal categories.

5.3. Lemma. Let $\langle \mathcal{M}, \mathcal{E} \rangle$ be an internal exact category. Then $Q\langle \mathcal{M} \rangle$ and $^*Q\langle \mathcal{M} \rangle$ are canonically isomorphic.

Proof. This follows immediately from the fact that the definition of morphisms and their composition in $Q\langle \mathcal{M} \rangle$ is obviously first order. \textbf{q.e.d.}

Recall that the nerve of a small category $\mathcal{C}$ is the simplicial set $N\mathcal{C}_*$, where $N\mathcal{C}_n$ is the set of sequences

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n$$

in $\mathcal{C}$ and where $\delta_i$ (respectively $\sigma_i$) is obtained by deleting the object $X_i$ (respectively replacing $X_i$ by $X_i \xrightarrow{1_{X_i}} X_i$) in the evident way. If $F : \mathcal{C} \to \mathcal{C}'$ is a functor, we obviously get an induced morphism of simplicial sets $NF : N\mathcal{C} \to N\mathcal{C}'$ between the nerves.

By transfer, we define the notion of the *nerve $^*N\mathcal{C}$ of an internal category $\mathcal{C}$, which is a *simplicial set.

5.4. Lemma. If $\mathcal{C}$ is an internal category, then $\text{res}(^*N\mathcal{C}) = N\mathcal{C}$, and if $F : \mathcal{C} \to \mathcal{D}$ is an internal functor between internal categories, then $\text{res}(^*NF) = NF$.

Proof. This is obvious from the definition of the nerve. \textbf{q.e.d.}

Let $\langle \mathcal{M}, \mathcal{E} \rangle$ be a small exact category, and let $n \in \mathbb{N}_0$. Then the $n$-th K-group of $\mathcal{M}$ is defined as

$$K_n(\mathcal{M}) := \pi_{n+1}(NQ\mathcal{M}, 0)$$

(this is abelian even for $n = 0$), and by transfer, for $n \in *\mathbb{N}_0$, the $n$-th *K-group of an internal exact category $\langle \mathcal{M}, \mathcal{E} \rangle$ is

$$^*K_n(\mathcal{M}) := ^*\pi_{n+1}(^*N^*Q\mathcal{M}, 0).$$

Let $\mathcal{A}$ be a (small) abelian category, and let $\mathcal{B} \subseteq \mathcal{A}$ be a Serre subcategory, i.e. a full abelian subcategory such that for every short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, $A$ is in $\mathcal{B}$ if and only if $A'$ and $A''$ are both in $\mathcal{B}$. Recall that in this situation, $\mathcal{A}/\mathcal{B}$ is the abelian category whose objects are the objects of $\mathcal{A}$ and whose morphisms from $A$ to $A'$ are represented by diagrams in $\mathcal{A}$ of the form

$$(\sigma, \varphi) : A \xleftarrow{\sigma'} C \xrightarrow{\varphi'} A'$$

with kernel and cokernel of $\sigma$ in $\mathcal{B}$, where another diagram

$$(\sigma', \varphi') : A \xleftarrow{\sigma''} C' \xrightarrow{\varphi''} A'$$

represents the same morphism if and only if there is a commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\varphi} & A' \\
\sigma^\prime & \searrow & \\
\downarrow \tau & & \\
A & \xleftarrow{\sigma''} & C'' & \xrightarrow{\varphi''} & A'
\end{array}
\]
in \( \mathcal{A} \) where \( \tau, \tau' \) and \( \sigma'' \) have kernel and cokernel in \( \mathcal{B} \). The composition of two morphisms \([A \xrightarrow{\sigma} C \xleftarrow{\varphi} A']\) and \([A' \xrightarrow{\sigma'} C' \xleftarrow{\varphi'} A'']\) is represented by \((\sigma'' \varphi \sigma', \varphi' \sigma'')\), where \( \sigma'' \) is the pullback of \( \sigma' \) along \( \varphi \) and \( \varphi'' \) is the corresponding projection:

\[
\begin{array}{ccc}
A & \xleftarrow{\sigma} & C \\
\varphi & \cong & \varphi'' \\
A' & \xleftarrow{\sigma'} & C' \\
\varphi' & \cong & \\
& & A''
\end{array}
\]

(One can easily check that this is well defined.)

The canonical exact functor \( F : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{B} \) has the property that an object \( B \) of \( \mathcal{A} \) is in \( \mathcal{B} \) if and only if \( FB \) is isomorphic to 0 in \( \mathcal{A}/\mathcal{B} \), and if \( F' : \mathcal{A} \rightarrow \mathcal{A}' \) is an exact functor between abelian categories satisfying \( F' B \cong 0 \) for all objects \( B \) of \( \mathcal{B} \), then \( F' \) factors uniquely over \( F \).

By transfer, the *quotient \( \mathcal{A} */\mathcal{B} \) of a *abelian category \( \mathcal{A} \) by a *Serre subcategory \( \mathcal{B} \) is also defined.

5.5. **Lemma.** Let \( \mathcal{B} \) be a *Serre subcategory of an internal abelian category \( \mathcal{A} \). Then \( \mathcal{B} \) is a Serre subcategory of \( \mathcal{A} \), and the abelian categories \( \mathcal{A}/\mathcal{B} \) and \( \mathcal{A} */\mathcal{B} \) are canonically isomorphic.

**Proof.** This is again obvious, because both the definitions of "Serre subcategory" and of "quotient by a Serre subcategory" are obviously first order. q.e.d.

The *localization theorem* in (higher) algebraic K-theory states that if \( \mathcal{B} \) is a Serre subcategory of an abelian category \( \mathcal{A} \), then

\[
\xymatrix{ \text{NQ} \mathcal{B} \ar[r] & \text{NQ} \mathcal{A} \ar[r] & \text{NQ}(\mathcal{A}/\mathcal{B}) }
\]

is a fibre sequence and hence induces a long exact sequence of K-groups

\[
\ldots \rightarrow K_{n+1}(\mathcal{A}/\mathcal{B}) \xrightarrow{\delta} K_n(\mathcal{B}) \rightarrow K_n(\mathcal{A}) \rightarrow K_n(\mathcal{A}/\mathcal{B}) \rightarrow \ldots .
\]

Furthermore, if \( \mathcal{B}' \) is a Serre subcategory of \( \mathcal{A}' \) and if \( F : \mathcal{A} \rightarrow \mathcal{A}' \) is an exact functor that maps \( \mathcal{B} \) to \( \mathcal{B}' \), then \( F \) induces a morphism of fibre sequences

\[
\xymatrix{ \text{NQ} \mathcal{B} \ar[r] & \text{NQ} \mathcal{A} \ar[r] & \text{NQ}(\mathcal{A}/\mathcal{B}) \ar[d]_{\text{NQF}} \ar[r] & \\
\text{NQ} \mathcal{B}' \ar[r] & \text{NQ} \mathcal{A}' \ar[r] & \text{NQ}(\mathcal{A}'/\mathcal{B}') \ar[d]_{\text{NQF}} \ar[r] & 
}
\]

5.6. **Theorem.** For an internal exact category \( \mathcal{M} \) and \( i \in \mathbb{N}_0 \), there are canonical group homomorphisms \( \tilde{N} : K_i(\mathcal{M}) \rightarrow {}^*K_i(\mathcal{M}) \) which are functorial in \( \mathcal{M} \) in the following sense:

- If \( F : \mathcal{M} \rightarrow \mathcal{M}' \) is an internal exact functor between internal exact categories, then the diagram

\[
\xymatrix{ {}^*K_n(\mathcal{M}) \ar[r]^{\tilde{K}_n(F)} & {}^*K_n(\mathcal{M}') \\
K_n(\mathcal{M}) \ar[u]^\tilde{N} \ar[r]_{K_n(F)} & K_n(\mathcal{M}') \ar[u]_\tilde{N} }
\]

commutes for every \( n \in \mathbb{N}_0 \).
• If \( \mathcal{B} \) is a *Serre subcategory of an internal abelian category \( \mathcal{A} \), then the diagram

\[
\begin{array}{ccc}
*K_{n+1}(\mathcal{A}/\mathcal{B}) & \xrightarrow{\delta} & *K_n(\mathcal{B}) \\
\mathcal{N} & \downarrow & \mathcal{N} \\
K_{n+1}(\mathcal{A}/\mathcal{B}) & \xrightarrow{\delta} & K_n(\mathcal{B})
\end{array}
\]

commutes for all \( i \in \mathbb{N}_0 \).

**Proof.** By [5,3] and [5,4] the simplicial set \( \text{NQ}\mathcal{M} \) is functorially isomorphic to \( \text{res}^* \text{NQ}\mathcal{M} \), so we can apply [4,3] and define \( \mathcal{N} \) by

\( \mathcal{N} : K_n(\mathcal{M}) = \pi_{n+1}(\text{NQ}\mathcal{M}, 0) = \pi_{n+1}(\text{res}^* \text{NQ}\mathcal{M}, 0) \xrightarrow{\varphi_{n+1}} \pi_{n+1}(\text{NQ}\mathcal{M}, 0) = K_n(\mathcal{M}) \)

for \( n \in \mathbb{N}_0 \). The functoriality of \( \mathcal{N} \) follows from the functoriality of \( \varphi_n \) proven in [4,3] q.e.d.

5.7. **Corollary.** Let \( k \) be an internal field. Then there is a canonical morphism of functors

\[ N : K_n() \longrightarrow *K_n() \circ N : (\text{Sch}_k^{fp})^{\text{op}} \longrightarrow \mathcal{A} \mathcal{B} \]

for all \( n \in \mathbb{N}_0 \), i.e. for every \( k \)-scheme of finite type \( X \) and every \( n \in \mathbb{N}_0 \), there is a canonical group homomorphism

\[ N : K_n(X) \longrightarrow *K_n(N X) \]

which is functorial in \( X \).

**Proof.** For a \( k \)-scheme of finite type \( X \), denote the category of vector bundles of finite rank on \( X \) by \( \mathcal{P}_X \). Then \( (\mathcal{P}_X, \mathcal{E}) \) is an exact category with

\[ \mathcal{E} := \left\{ (i, j) \mid 0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} B'' \longrightarrow 0 \text{ exact in } \text{Coh}_X \right\}, \]

and by definition \( K_n(X) := K_n(\mathcal{P}_X) \). Let \( \mathcal{P}_X \) be the internal exact category of *vector bundles of *finite rank on \( N X \). We know from [BS07, 5.5] that \( N : \text{Coh}_X \longrightarrow *\text{Coh}_{N X} \) restricts to an exact functor \( N : \mathcal{P}_X \longrightarrow *\mathcal{P}_{N X} \), and if \( f : X' \longrightarrow X \) is a morphism of \( k \)-schemes of finite type, then the functors \( N \circ f^* \) and \( [N f]^* \circ N \) from \( \mathcal{P}_X \) to \( *\mathcal{P}_{N X} \) are obviously canonically isomorphic.

This implies that we get functorial group homomorphisms \( K_n(X) \longrightarrow K_n(*\mathcal{P}_{N X}) \) for all \( n \in \mathbb{N}_0 \), so we can define \( N \) as the composition

\[ N : K_n(X) \longrightarrow K_n(*\mathcal{P}_{N X}) \xrightarrow{\mathcal{N}} *K_n(*\mathcal{P}_{N X}) = *K_n(N X). \]

The functoriality of \( \mathcal{N} \) proven in [5,6] shows that \( N \) thus defined is functorial. q.e.d.

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