INTEGRAL VAN VLECK’S AND KANNAPPAN’S FUNCTIONAL EQUATIONS ON SEMIGROUPS

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Abstract. In this paper we study the solutions of the integral Van Vleck’s functional equation for the sine
\[ \int_S f(x\tau(y)t)d\mu(t) - \int_S f(xy)t)d\mu(t) = 2f(x)f(y), \quad x, y \in S \]
and the integral Kannappan’s functional equation
\[ \int_S f(xy)t)d\mu(t) + \int_S f(x\tau(y)t)d\mu(t) = 2f(x)f(y), \quad x, y \in S, \]
where \( S \) is a semigroup, \( \tau \) is an involution of \( S \) and \( \mu \) is a measure that is linear combinations of point measures \( (\delta_{z_i})_{i \in I} \), such that for all \( i \in I, z_i \) is contained in the center of \( S \).

We express the solutions of the first equation by means of multiplicative functions on \( S \), and we prove that the solutions of the second equation are closely related to the solutions of the classic d’Alembert’s functional equation with involution.

1. Introduction

Throughout this paper \( S \) denotes a semigroup, and \( \tau : S \rightarrow S \) is an involution of \( S \). That is \( \tau(xy) = \tau(y)\tau(x) \) and \( \tau(\tau(x)) = x \) for all \( x, y \in S \). If \( \mu \) denotes a discrete complex measure, we say that \( \mu \) is \( \tau \)-invariant and we write \( \mu = \tau(\mu) \) if \( \int_S f(\tau(t))d\mu(t) = \int_S f(t)d\mu(t) \) for all complex-valued continuous and bounded function \( f \) on a topological semigroup \( S \). A function \( \chi : S \rightarrow \mathbb{C} \) is a multiplicative function if \( \chi(xy) = \chi(x)\chi(y) \) for all \( x, y \in S \).

In 2003, Elqorachi and Akkouchi [4] introduced and studied the bounded and continuous solutions \( f \neq 0 \) of the following generalized d’Alembert’s functional equation

\[ \int_G f(xty)d\mu(t) + \int_G f(xt\tau(y))d\mu(t) = 2f(x)f(y), \quad x, y \in G \]
on a topological group \( G \). They proved that under the conditions that \( \mu = \tau(\mu) \) and \( f \) satisfies the Kannappan’s condition
\[ \int_G \int_G f(xtysz)d\mu(t)d\mu(s) = \int_G \int_G f(ytksz)d\mu(t)d\mu(s), \quad x, y, z \in G, \]
there exists a generalized \( \mu \)-spherical function \( \psi : G \rightarrow \mathbb{C} \):

\[ \int_G \psi(xty)d\mu(t) = \psi(x)\psi(y), \quad x, y \in G \]
such that \( f(x) = \frac{\psi(x)+\psi(\tau(x))}{2} \) for all \( x \in G \).

\( \mu \)-spherical function and related topics are studied in [1, 2].

Key words and phrases. semigroup; d’Alembert’s equation; Van Vleck’s equation; Kannappan’s equation; involution; multiplicative function; complex measure.

2010 Mathematics Subject Classification. 39B32, 39B52.
In the particular case when \( \mu = \delta_{z_0} \) is the Dirac measure, the functional equation (1.1) reduces to Kannappan’s functional equation [5]

\[
f(xz_0y) + f(xz_0\tau(y)) = 2f(x)f(y), \ x, y \in S.
\]

Kannappan proved that any solution \( f: \mathbb{R} \to \mathbb{C} \) of (1.3) with \( \tau(y) = -y \) for all \( y \in \mathbb{R} \) is periodic, if \( z_0 \neq 0 \). Furthermore, the periodic solutions has the form \( f(x) = g(x-z_0) \) where \( g \) is a periodic solution of d’Alembert functional equation (1.4)

\[
g(x + y) + g(x - y) = 2g(x)g(y), \ x, y \in \mathbb{R}.
\]

Perkins and Sahoo [6] studied the following version of Kannappan’s functional equation

\[
f(xyz_0) + f(xy^{-1}z_0) = 2f(x)f(y), \ x, y \in S
\]
on groups. They found the form of any abelian solution \( f \) of (1.5) on groups. They obtained the abelian, complex-valued solutions of Van Vleck’s functional equation

\[
f(xy^{-1}z_0) - f(xyz_0) = 2f(x)f(y), \ x, y \in G,
\]
when \( G \) is a not necessarily abelian group and \( z_0 \) is a fixed element in the center of \( G \). We refer also to [11] and [12].

Recently, Stetkær [8] took \( z_0 \) in the center and expressed the complex-valued solutions of Kannappan’s functional equation (1.3) on semigroups in terms of solutions of d’Alembert’s functional equation (1.6)

\[
g(xy) + g(x\tau(y)) = 2g(x)g(y), \ x, y \in S,
\]
The complex-valued solutions of (1.6) are formulated by Davison [3] on monoids that need not be commutative.

Stetkær [9, Exercise 9.18] found the complex-valued solution of Van Vleck’s functional equation

\[
f(xy^{-1}z_0) - f(xyz_0) = 2f(x)f(y), \ x, y \in G,
\]
when \( G \) is a not necessarily abelian group and \( z_0 \) is a fixed element in the center of \( G \). We refer also to [11] and [12].

Perkins and Sahoo [6] replaced the group inversion by an involution \( \tau: G \to G \) and they obtained the abelian, complex-valued solutions of equation

\[
f(x\tau(y)z_0) - f(xyz_0) = 2f(x)f(y), \ x, y \in G.
\]
Stetkær [7] extends the results of Perkins and Sahoo [6] about equation (1.7) to the case where \( G \) is a semigroup and the solutions are not assumed to be abelian.

The main purpose of this paper is to extend Stetkær’s results [7, 8] to the following generalizations of Van Vleck’s functional equation for the sine

\[
\int_{S} f(xy)\mu(t) - \int_{S} f(xy)\mu(t) = 2f(x)f(y), \ x, y \in S,
\]
and of Kannappan’s functional equation

\[
\int_{S} f(xy)\mu(t) + \int_{S} f(x\tau(y)t)\mu(t) = 2f(x)f(y), \ x, y \in S,
\]
where \( S \) is a semigroup, \( \tau \) is an involution of \( S \) and \( \mu \) is a measure that is linear combinations of point measures \( (\delta_{z_i})_{i \in I} \), with \( z_i \) contained in the center of \( S \), for all \( i \in I \).

We express the solutions of (1.9) in terms of multiplicative functions on \( S \) and we prove that the solutions of (1.10) are closely related to the solutions of the classic d’Alembert’s functional equation (1.6).
2. The Solutions of the integral Van Vleck’s functional equation on semigroups

In this section we obtain the complex-valued solutions of the integral Van Vleck’s functional equation (1.9) on semigroups. The following lemmas will be used later. They are generalizations of Stetkær’ lemmas obtained in [7] for \( \mu = \delta_{z_0} \), where \( z_0 \) is a fixed element in the center of the semigroup \( S \).

**Lemma 2.1.** Let \( S \) be a semigroup with an involution \( \tau: S \rightarrow S \). Let \( \mu \) be a complex measure with support contained in the center of \( S \). Let \( f \) be a non-zero solution of equation (1.9). Then for all \( x \in S \) we have

\[
(2.1) \quad f(x) = -f(\tau(x)),
\]

\[
(2.2) \quad \int_S f(t)d\mu(t) \neq 0,
\]

\[
(2.3) \quad \int_S \int_S f(ts)d\mu(t)d\mu(s) = \int_S \int_S f(\tau(t)s)d\mu(t)d\mu(s) = 0,
\]

\[
(2.4) \quad \int_S \int_S f(x\tau(t)s)d\mu(t)d\mu(s) = f(x) \int_S f(t)d\mu(t),
\]

\[
(2.5) \quad \int_S \int_S f(xts)d\mu(t)d\mu(s) = -f(x) \int_S f(t)d\mu(t),
\]

\[
(2.6) \quad \int_S f(\tau(x)t)d\mu(t) = \int_S f(xt)d\mu(t).
\]

**Proof.** Replacing \( y \) by \( \tau(y) \) in the functional equation (1.9) and using \( \tau(\tau(y)) = y \) we get

\[
\int_S f(xyt)d\mu(t) - \int_S f(x\tau(y)t)d\mu(t) = 2f(x)f(\tau(y))
\]

\[
= -\left[ \int_S f(x\tau(y)t)d\mu(t) - \int_S f(xyt)d\mu(t) \right] = -2f(x)f(y),
\]

which implies the formula (2.1).

By replacing \( x \) by \( \tau(s) \) in (1.9) and using (2.1) we have

\[
(2.7) \quad \int_S f(\tau(s)\tau(y)t)d\mu(t) - \int_S f(\tau(s)yt)d\mu(t) = -2f(s)f(y)
\]

for all \( x, s \in S \). By integrating the two members of equation (2.7) with respect to \( s \) we obtain

\[
(2.8) \quad \int_S \int_S f(\tau(s)\tau(y)t)d\mu(s)d\mu(t) - \int_S \int_S f(\tau(s)yt)d\mu(s)d\mu(t)
\]

\[
= -2f(x) \int_S f(s)d\mu(s).
\]

By using (2.1) we find

\[
\int_S \int_S f(\tau(s)\tau(y)t)d\mu(s)d\mu(t) = -\int_S \int_S f(\tau(t)ys)d\mu(s)d\mu(t)
\]
From (2.5) and (2.1), we have

By setting

the assumption that

If

so, we obtain

which proves (2.4).

Setting \( y = s \) in (1.9) and integrating the result obtained with respect to \( s \) we get by using (2.4) that

So, we deduce formula (2.5).

By replacing \( x \) by \( xs \) in the functional equation (1.9) and integrating the result obtained with respect to \( s \) we get by using (2.5) and the support of \( \mu \) contained in the center of \( S \) that

If \( \int_S f(s)\,d\mu(s) = 0 \), then \( f(y) \int_S f(xs)\,d\mu(s) = 0 \) for all \( x, y \in S \). Since \( f \neq 0 \) then \( \int_S f(xs)\,d\mu(s) = 0 \) for all \( x \in S \), so we have

for all \( x, y \in S \) from which we deduce that \( f(x) = 0 \) for all \( x \in S \). This contradicts the assumption that \( f \neq 0 \) and it follows that \( \int_S f(s)\,d\mu(s) \neq 0 \), so, we have (2.2).

From (2.5) and (2.1), we have

By setting \( x = \tau(t)s' \), \( y = s \) in (1.9) and integrating the result obtained with respect to \( t \) and \( s' \) we get by a computation that

which can be written as follows

\[ \int_S \int_S f(st)\,d\mu(s)\,d\mu(t) \int_S f(s)\,d\mu(s) = \int_S \int_S f(st)\,d\mu(s)\,d\mu(t) \int_S f(s)\,d\mu(s). \]
+ \int_S \int_S f(\tau(t)s)d\mu(s)d\mu(t) \int_S f(s)d\mu(s)
= 2 \int_S \int_S f(\tau(t)s')d\mu(t)d\mu(s') \int_S f(s)d\mu(s).

This implies that
\int_S \int_S f(st)d\mu(s)d\mu(t) = \int_S \int_S f(\tau(t)s)d\mu(s)d\mu(t).

On the other hand, since \( f(\sigma(x)) = -f(x) \) for all \( x \in S \), we get
\int_S \int_S f(\tau(t)s)d\mu(t)d\mu(s) = -\int_S \int_S f(\tau(t)s)d\mu(t)d\mu(s),
and then we obtain
\int_S \int_S f(\tau(t)s)d\mu(t)d\mu(s) = 0 = \int_S \int_S f(ts)d\mu(t)d\mu(s),
which proves (2.3).

By replacing \( x \) by \( st \) in (1.9) and integrating the result obtained with respect to \( s \) and \( t \) we get
\[
\int_S \int_S f(st\tau(y)t')d\mu(s)d\mu(t')
- \int_S \int_S f(styt')d\mu(s)d\mu(t')
= 2f(y)\int_S \int_S f(st)d\mu(t)d\mu(s) = 0.
\]

From (2.5) we have
\[
\int_S \int_S f(st\tau(y)t')d\mu(s)d\mu(t')
= \int_S \left[ \int_S f(\tau(y)t's)d\mu(s)d\mu(t') \right]d\mu(t')
= -\int_S f(s)d\mu(s) \int_S \int_S f(\tau(y)t')d\mu(t')
\]
and
\[
\int_S \int_S f(styt')d\mu(s)d\mu(t)d\mu(s)
= \int_S \left[ \int_S f(y't's)d\mu(s)d\mu(t') \right]d\mu(t')
= -\int_S f(s)d\mu(s) \int_S \int_S f(y't)d\mu(t').
\]
Since \( \int_S f(s)d\mu(s) \neq 0 \) we deduce that \( \int_S f(y't)d\mu(t') = \int_S f(\tau(y)t')d\mu(t') \) for all \( y \in S \). This completes the proof. \( \square \)

**Lemma 2.2.** Let \( f \) be a non-zero solution of equation (1.4). Then (1) the function defined by
\[
g(x) := \frac{\int_S f(xt)d\mu(t)}{\int_S f(s)d\mu(s)} \text{ for } x \in S
\]
is a non-zero abelian solution of d’Alembert’s functional equation (1.7).

(2) \[ \int_S \int_S g(ts) d\mu(t)d\mu(s) \neq 0; \quad \int_S g(s)d\mu(s) = 0 \]

(3) The function \( g \) from (1) has the form \( g = \frac{x^2 + y^2}{2} \), where \( \chi : S \rightarrow \mathbb{C}, \chi \neq 0 \), is a multiplicative function on \( S \).

Proof. (1). From [2.4], (2.3) and the definition of \( g \) we get by a computation that

\[
(\int_S f(s)d\mu(s))^2[g(xy) + g(x\tau(y))] = \\
\int_S f(s)d\mu(s) \int_S f(xyt)d\mu(t) + \int_S f(s)d\mu(s) \int_S f(x\tau(y)t)d\mu(t) \\
- \int_S \int_S \int_S f(xytss') d\mu(t)d\mu(s)d\mu(s') \\
+ \int_S \int_S \int_S f(x\tau(y)t\tau(s)s') d\mu(t)d\mu(s)d\mu(s') \\
= \int_S \int_S \int_S f(xs'\tau(y)s't)d\mu(t)d\mu(s)d\mu(s') - \int_S \int_S \int_S f(xs'yts)d\mu(t)d\mu(s)d\mu(s') \\
= 2 \int_S f(xs')d\mu(s') \int_S f(y)s)d\mu(s)
\]

which implies the desired result.

(2). From (2.3) and the definition of \( g \) we get

\[
\int_S \int_S g(ts)d\mu(t)d\mu(s) = \frac{\int_S \int_S f(s'ts)d\mu(t)d\mu(s)d\mu(s')}{\int_S f(s)d\mu(s)} \\
= \frac{-\int_S f(s)d\mu(s) \int_S f(s')d\mu(s)}{\int_S f(s)d\mu(s)} = -\int_S f(s)d\mu(s) \neq 0.
\]

From (2.3) and the definition of \( g \) we get

\[
\int_S g(s)d\mu(s) = \frac{\int_S f(s)d\mu(s)d\mu(t)}{\int_S f(s)d\mu(s)} = \frac{0}{\int_S f(s)d\mu(s)} = 0
\]

Furthermore, \( \int_S \int_S g(st) d\mu(t)d\mu(s) \neq 0 \), so \( g \neq 0 \).

As \( g \) is a solution of equation (1.7) then by [9, Proposition 9.17(c)] \( g \) is a solution of pre-d’Alembert function. Now, according to [9, Proposition 8.14(a)] we discuss two cases:

**Case 1.** If there is a \( t \) in the center of \( S \) such that \( g(t)^2 \neq d(t) \), then \( g \) is abelian.

**Case 2** If for all \( t \) in the center of \( S \) satisfies \( g(t)^2 = d(t) \), then we get \( g(xt) = g(x)g(t) \) for all \( x \in S \) and \( t \) in the center of \( S \). By integrating the expression with respect to \( t \) we get \( \int_S g(xt)d\mu(t) = g(x) \int_S g(t)d\mu(t) = 0 \) for all \( x \in S \) and then \( \int_S \int_S g(st)d\mu(t)d\mu(s) = 0 \). This contradicts the first assertion of Lemma 2.2 (2). Finally, we conclude that \( g \) is abelian and for the rest of the proof we use [9, Theorem 9.12].

The main content of this section is the following theorem.
Theorem 2.3. The non-zero solutions \( f : S \to \mathbb{C} \) of the functional equation (1.9) are the functions of the form

\[
(2.9) \quad f = \left[ \chi - \frac{\chi \circ \tau}{2} \right] \int_S \chi(\tau(t))d\mu(t),
\]
where \( \chi : S \to \mathbb{C} \) is a multiplicative function such that \( \int_S \chi(t)d\mu(t) \neq 0 \) and \( \int_S \chi(\tau(t))d\mu(t) = -\int_S \chi(t)d\mu(t) \).

If \( S \) is a topological semigroup and that \( \tau : S \to S \), is continuous, then the non-zero solution \( f \) of equation (1.9) is continuous, if and only if \( \chi \) is continuous.

Proof. Let \( f : S \to \mathbb{C} \), \( f \neq 0 \), be a solution of equation (1.9). Then by replacing \( y \) by \( s \) and integrating the result obtained with respect to \( s \) we get

\[
f(x) = \frac{\int_S \int_S f(x\tau(s)t)d\mu(s)d\mu(t) - \int_S \int_S f(xst)d\mu(s)d\mu(t)}{2 \int_S f(s)d\mu(s)} = \frac{\int_S g(x\tau(s))d\mu(s) - \int_S g(xs)d\mu(s)}{2}
\]

for all \( x \in S \) and where \( g \) is the function given by Lemma 2.2(1). According to \( g = \frac{\chi + \chi \circ \tau}{2} \) we get the following formula

\[
(2.10) \quad f = \left[ \frac{\int_S \chi(s)d\mu(s) - \int_S \chi(\tau(s))d\mu(s)}{2} \right] \left[ \chi - \frac{\chi \circ \tau}{2} \right].
\]

In view of Lemma 2.1 we have \( \int_S f(\tau(x)t)d\mu(t) = \int_S f(xt)d\mu(t) \) for all \( x \in S \). Substituting (2.10) into (2.6) we find after simple computations that

\[
\left[ \int_S \chi(\tau(s))d\mu(s) + \int_S \chi(s)d\mu(s) \right] [\chi - \chi \circ \tau] = 0.
\]

The rest of the proof is similar to Stetkær’s proof [7]. \( \square \)

Corollary 2.4. [7] Let \( S \) be a semigroup with an involution \( \tau : S \to S \). If \( \mu = \delta_{z_0} \), where \( z_0 \) is a fixed element in the center of \( S \). The non-zero solutions \( f : S \to \mathbb{C} \) of the functional equation (1.9) are the functions of the form

\[
(2.11) \quad f = \chi(\tau(z_0))\left[ \frac{\chi - \chi \circ \tau}{2} \right],
\]
where \( \chi : S \to \mathbb{C} \) is a multiplicative function such that \( \chi(z_0) \neq 0 \) and \( \chi(\tau(z_0)) = -\chi(z_0) \).

3. Integral Kannappan’s functional equation on semigroups

In this section we study the complex-valued solutions of the functional equation (1.10). The support of the discrete complex measure \( \mu \) is assumed to be contained in the center of the semigroup \( S \).

The following useful lemma will be used later. It’s a natural generalization of Lemma 1 and Lemma 2 obtained by Stetkær [8] for \( \mu = \delta_{z_0} \).

Lemma 3.1. (1) If \( f : S \to \mathbb{C} \) is a solution of (1.10), then for all \( x \in S \) we have

\[
(3.1) \quad f(x) = f(\tau(x)),
\]

\[
(3.2) \quad \int_S f(t)d\mu(t) \neq 0 \iff f \neq 0.
\]
We find that

Assume that

which implies the formula (3.4).

\[ \int f(x(t)s) dx(t) dx(s) = f(x) \int f(t) dx(t), \]

(3.4)

\[ \int \int f(xt) dx(t) dx(s) = f(x) \int f(t) dx(t), \]

2.1.

Proof. (1). The formula (3.1) is proved like the corresponding statement in Lemma 2.1.

By putting \( x = \tau(s) \) in (1.10) and integrating the result obtained with respect to \( s \) to get

\[
\int_S \int_S f(\tau(s)t) d\mu(t) d\mu(s) + \int_S \int_S f(\tau(s)\tau(y)t) d\mu(t) d\mu(s) = 2f(y) \int_S f(\tau(t)) d\mu(t) = 2f(y) \int_S f(t) d\mu(t),
\]

where the last equality holds, because \( f \) satisfies (3.1).

In view of (3.1), we have

\[
\int_S \int_S f(\tau(s)\tau(y)t) d\mu(t) d\mu(s) = \int_S \int_S f(\tau(t)s) d\mu(t) d\mu(s).
\]

So, we obtain

\[
2 \int_S \int_S f(\tau(s)t) d\mu(t) d\mu(s) = 2f(y) \int_S f(t) d\mu(t),
\]

which proves (3.3).

By setting \( y = s \) in (1.10) and integrating the result obtained with respect to \( s \) we get

\[
\int_S \int_S f(xs) dx(s) dx(t) d\mu(t) d\mu(s) + \int_S \int_S f(xs) dx(s) dx(t) d\mu(t) d\mu(s) = \int_S \int_S f(xs) dx(s) d\mu(s) + f(x) \int_S f(s) d\mu(s),
\]

which implies the formula (3.4).

Assume that \( f \) is a solution of equation (1.10) and that \( \int_S f(t) d\mu(t) = 0 \). Replacing \( x \) by \( xs \), \( y \) by \( yt \) in (1.10) and integrating the result obtained with respect to \( s \) and \( t \) we find

\[
\int_S \int_S \int_S f(xsyt) dx(t) dx(s) d\mu(s) d\mu(k) + \int_S \int_S \int_S f(xs\tau(t) \tau(y)k) dx(t) dx(s) d\mu(s) d\mu(k) = 2 \int_S f(xs) d\mu(s) \int_S f(yt) d\mu(s).
\]
Since, from (3.3) we have
\[
\int_S \int_S \int_S f(xs\tau(t)\tau(y)k)d\mu(t)d\mu(s)d\mu(k) = \int_S \int_S \int_S f(xs\tau(y)\tau(t)k)d\mu(t)d\mu(k)d\mu(s)
\]
\[
= \int_S \int_S f(t)d\mu(t)f(xs\tau(y))d\mu(s) = \int_S 0d\mu(s) = 0.
\]
In view of (3.4) we have
\[
\int_S \int_S \int_S f(xsytk)d\mu(t)d\mu(s)d\mu(k) = \int_S \int_S \int_S f(xystk)d\mu(t)d\mu(k)d\mu(s)
\]
\[
= \int_S \int_S f(t)d\mu(t)f(xys)d\mu(s) = \int_S 0d\mu(s) = 0,
\]
and it follows that \(\int_S f(xs)d\mu(s)\int_S f(yt)d\mu(s) = 0\) for all \(x, y \in S\). So, we obtain
\[
\int_S f(xyt)d\mu(t) + \int_S f(x\tau(y)t)d\mu(t) = 2f(x)f(y) = 0
\]
for all \(x, y \in S\). Consequently, \(f(x) = 0\) for all \(x \in S\) and this proves (3.2).

(2) Let \(g\) be a solution of (1.6). Assume that \(\int_S g(xt)d\mu(t) = \int_S g(x\tau(t))d\mu(t)\) holds for all \(x \in S\). Since \(g(xt) + g(x\tau(t)) = 2g(x)g(t)\) for all \(x, t \in S\), then by integrating the statement with respect to \(t\), we get
\[
\int_S g(xt)d\mu(t) + \int_S g(x\tau(t))d\mu(t) = 2g(x) \int_S g(t)d\mu(t) = 2 \int_S g(xt)d\mu(t).
\]
Conversely,
\[
2 \int_S g(xt)d\mu(t) = 2g(x) \int_S g(t)d\mu(t) = \int_S [g(xt) + g(x\tau(t))]d\mu(t)
\]
\[
= \int_S g(xt)d\mu(t) + \int_S g(x\tau(t))d\mu(t),
\]
which implies that \(\int_S g(xt)d\mu(t) = \int_S g(x\tau(t))d\mu(t)\) for all \(x \in S\) and that (3.5) and (3.6) are equivalent.

Now, we will show that (3.7) and (3.6) are equivalent. If \(\int_S g(st)d\mu(t) = g(s) \int_S g(t)d\mu(t)\) for all \(s \in S\), then by integrating this expression with respect to \(s\), we get \(\int_S \int_S g(st)d\mu(s)d\mu(t) = (\int_S g(t)d\mu(t))^2\). Conversely, suppose that \(\int_S \int_S g(st)d\mu(s)d\mu(t) = (\int_S g(t)d\mu(t))^2\). Since \(g\) is a solution of d’Alembert’s functional equation (1.6), then \(g\) is a solution of the pre-d’Alembert functional equation [9, Proposition 9.17]. So, from [9, Proposition 8.14(a)] we will discuss the following two cases.

**Case 1.** If for all \(s\) in the center of \(S\) satisfies \(g(s)^2 = d(s)\), then \(g(xs) = g(x)g(s)\) for all \(x \in S\). So, by integrating this expression with respect to \(s\) we get \(\int_S g(xs)d\mu(s) = g(x) \int_S g(s)d\mu(s)\) for all \(x \in S\).

**Case 2.** If there is \(s\) in the center of \(S\) such that \(g(s)^2 \neq d(s)\), then \(g\) is abelian and there exists a multiplicative function \(\chi: S \rightarrow \mathbb{C}\) such that \(g = \frac{\chi + \chi \circ \tau}{2}\). Substituting this into \(\int_S \int_S g(st)d\mu(s)d\mu(t) = (\int_S g(t)d\mu(t))^2\), gives after an elementary computations that
\[
\int_S \chi(t)d\mu(t) - \int_S \chi(\tau(t))d\mu(t) = 0.
\]
Thus, we get
\[
\int_S g(xt)d\mu(t) = \int_S \frac{\chi + \chi \circ \tau}{2}(xt)d\mu(t)
\]
Furthermore, let
\[
\int S \chi(t) d\mu(t) + \chi(\tau(x)) \int S \chi(t) d\mu(t) = \int S \chi(t) d\mu(t) \frac{\chi(x) + \chi \circ \tau(x)}{2}
\]
\[= g(x) \int S g(t) d\mu(t). \]
This completes the proof. \(\square\)

Now, we are ready to prove the second main result of this paper. We use the following notations [8]:
- \(\mathcal{A}\) consists of the solution of \(g : S \to \mathbb{C}\) of d’Alembert’s functional equation (1.6) with \(\int_S g(t) d\mu(t) \neq 0\) and satisfying the conditions of Lemma 3.1(2)(ii).
- To any \(g \in \mathcal{A}\) we associate the function \(Tg = \int_S g(t) d\mu(t) g : S \to \mathbb{C}\).
- \(\mathcal{K}\) consists of the non-zero solutions \(f : S \to \mathbb{C}\) of integral Kannappan’s functional equation (1.10).

**Theorem 3.2.** (1) \(T\) is a bijection of \(\mathcal{A}\) onto \(\mathcal{K}\). The inverse \(T^{-1} : \mathcal{K} \to \mathcal{A}\) is defined by
\[
(T^{-1} f)(x) = \frac{\int_S f(xt) d\mu(t)}{\int_S f(t) d\mu(t)}
\]
for all \(f \in \mathcal{K}\) and \(x \in S\).

(2) Any non-zero solution \(f : S \to \mathbb{C}\) of integral Kannappan’s functional equation (1.10) is of the form \(f = \int_S g(t) d\mu(t) g\), where \(g \in \mathcal{A}\). Furthermore, \(f(x) = \int_S g(xt) d\mu(t) = \int_S g(x\tau(t)) d\mu(t) = \int_S g(t) d\mu(t) g(x)\) for all \(x \in S\).

(3) \(f\) is abelian [9] if and only if \(g\) is abelian.

(4) If \(S\) is equipped with a topology then \(f\) is continuous if and only if \(g\) is continuous.

**Proof.** If \(g \in \mathcal{A}\), then
\[
\int_S Tg(xyt) d\mu(t) + \int_S Tg(x\tau(y)t) d\mu(t) = \int_S g(s) d\mu(s) \left[ \int_S g(xyt) d\mu(t) + \int_S g(x\tau(y)t) d\mu(t) \right]
\]
\[= \int_S g(s) d\mu(s) \left[ g(xy) \int_S g(t) d\mu(t) + g(x\tau(y)) \int_S g(t) d\mu(t) \right]
\]
\[= \left( \int_S g(s) d\mu(s) \right)^2 [2g(x)g(y)] = 2Tg(x)Tg(y).
\]
Furthermore, \(\int_S Tg(s) d\mu(s) = (\int_S g(s) d\mu(s))^2 \neq 0\). So, we get \(Tg \in \mathcal{K}\).

From [8, Lemma 3] the map \(T\) is injective. Now, we will show that \(T\) is surjective.

Let \(f \in \mathcal{K}\) and define the function
\[
g(x) = \frac{\int_S f(xt) d\mu(t)}{\int_S f(t) d\mu(t)}.
\]
In view of (3.3) and (3.4) we have
\[
\left( \int_S f(s) d\mu(s) \right)^2 [g(xy) + g(x\tau(y))]
\]
\[= \int_S f(s) d\mu(s) \int_s f(xyt) d\mu(t) + \int_s f(s) d\mu(s) \int_s f(x\tau(y)t) d\mu(t)
\]
\[= \int_S \int_S f(xyt) d\mu(t) d\mu(s) + \int_S \int_S f(x\tau(y)t) d\mu(t) d\mu(s)
\]
\[= \int_S \int_S f(xys) d\mu(k) + \int_S f(x\tau(y)s) d\mu(k) d\mu(s) = 2 \int_S f(xt) d\mu(t) \int_S f(y) d\mu(s)
\]
It follow that \( g \) is a solution of d’Alembert’s functional equation (1.6). On the other hand, in view of (3.3) and (3.4) we have

\[
(\int_S g(s)d\mu(s))^2 = \int_S g(s)d\mu(s) \int_S g(t)d\mu(t) = \frac{1}{2} \int_S \int_S [g(st) + g(s\tau(t))]d\mu(s)d\mu(t)
\]

\[
= \frac{1}{2} \int_S \int_S \left[ \int_S f(stk)d\mu(k) \int_S f(s)d\mu(s) \right]d\mu(t) \\
= \frac{1}{2} \left[ \int_S \int_S f(stk)d\mu(s)d\mu(t) \int_S f(s)d\mu(s) \right] \\
+ \left[ \int_S \int_S f(s\tau(t)k)d\mu(s)d\mu(t) \int_S f(s)d\mu(s) \right] \\
= \frac{1}{2} \left[ \int_S f(s)d\mu(s) \int_S f(s)d\mu(s) \right] \int_S f(s)d\mu(s),
\]

and

\[
\int_S g(st)d\mu(s)d\mu(t) = \frac{\int_S \int_S f(ts)k d\mu(t) d\mu(k)}{\int_S f(s)d\mu(s)}
\]

\[
= \frac{\int_S f(t)d\mu(t) \int_S f(s)d\mu(s)}{\int_S f(s)d\mu(s)} = \int_S f(s)d\mu(s),
\]

which proves that \( g \) satisfies the conditions of Lemma 3.1(2)(ii).

Finally, \((\int_S g(s)d\mu(s))^2 = \int_S f(s)d\mu(s) \neq 0, \) then \( \int_S g(s)d\mu(s) \neq 0. \) This completes the proof. \( \square \)

**Corollary 3.3.** If \( \mu = \delta_{z_0}, \) where \( z_0 \) is a fixed element in the center of a semi group \( S. \) Then, any non-zero solution \( f: S \rightarrow \mathbb{C} \) of Kannappan’s functional equation (1.3) is of the form \( f = g(z_0)g, \) where \( g \) is a solution of d’Alemmbert’s functional equation (1.6) with \( g(z_0) \neq 0 \) and satisfying the conditions of Lemma 3.1(ii).

**Proposition 3.4.** The non-zero abelian solutions of integral Kannappan’s functional equation (1.10) are the functions of the form

\[
f(x) = \frac{1}{2} \left[ \chi(x) + \chi(\tau(x)) \right] \int_S \chi(t)d\mu(t), \ x \in S,
\]

where \( \chi : S \rightarrow \mathbb{C} \) is a multiplicative function such that \( \int_S \chi(t)d\mu(t) \neq 0 \) and \( \int_S \chi(\tau(t))d\mu(t) = \int_S \chi(t)d\mu(t). \)
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