\textbf{q-RACAH ENSEMBLE AND $q$-P($E_7^{(1)}/A_1^{(1)}$) DISCRETE PAINLEVÉ EQUATION}

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\textbf{Abstract.} The goal of this paper is to investigate the missing part of the story about the relationship between the orthogonal polynomial ensembles and Painlevé equations. Namely, we consider the $q$-Racah polynomial ensemble and show that the one-interval gap probabilities in this case can be expressed through a solution of the discrete $q$-P($E_7^{(1)}/A_1^{(1)}$) equation. Our approach also gives a new Lax pair for this equation. This Lax pair has an interesting additional involutive symmetry structure.

1. Introduction

The present paper is a continuation of the work on the relationship between the orthogonal polynomial ensembles and Painlevé equations [Kni16], where the $q$-analogue of methods introduced by Arinkin and Borodin in [AB06] was developed. This relationship in the continuous settings was first established in the 90’s [TW94, HS99, WF00, BD02]. First results in the discrete case were obtained in a paper by Borodin and Boyarchenko [BB03] using the formalism of discrete integrable operators and discrete Riemann-Hilbert problems. That paper will be the starting point of our investigation.

Our goal is to establish a certain recurrence procedure for computing the so-called gap probability function for the $q$-Racah orthogonal polynomial ensemble and to show that this function can be expressed through a solution of a $q$-P($E_7^{(1)}/A_1^{(1)}$) discrete Painlevé equation, as written in [KNY17]. For us, the original motivation to study this ensemble comes from its relationship to an interesting tiling model that we describe next.

1.1. The $q$-Racah tiling model. Consider a hexagon, drawn on a regular triangular lattice, whose side lengths are given by integers $a, b, c \geq 1$, see Figure 1. We are interested in random tilings of such a hexagon by rhombi, also called lozenges, that are obtained by gluing two neighboring triangles together. There are three types of rhombi that arise in such a way: , , and , and so, as can be clearly seen in Figure 1,
this model also has a natural interpretation as a random stepped surface formed by a stack of boxes or, equivalently, as a boxed plane partition (that is also called a 3-D Young diagram). In this way we can associate a tiling with a height function \( h \) that assigns to every lattice vertex inside the hexagon its “height” above the “horizontal plane”, as shown on Figure 1.

We are interested in the probability measures on the set of such tilings that were introduced in [BGR10]. These probability measures form a two-parameter family generalization of the uniform distribution. If we denote these parameters by \( q \) and \( \kappa \), the weight of a tiling is defined to be the product of simple factors

\[
w(i,j) = \left( \frac{\kappa q^j - (c+1)/2}{\kappa q^j + (c+1)/2} - \frac{q^{j+1}}{\kappa} \right)
\]

over all horizontal rhombi \( \square \), where \((i,j)\) is the coordinate of the topmost point of the rhombus (the \(i\) and \(j\) axes are shown on Figure 1). The dependence of the factors on the location of the lozenge makes the model inhomogeneous. In order to define a probability measure, the weight of a tiling has to be non-negative. This imposes certain restrictions on the parameters \( q \) and \( \kappa \) that we discuss in Section 2.

An important observation is that each lozenge tiling can be considered as time-dependent configuration of points on the line. To make this connection, we perform a simple affine transformation of the hexagon to get the shifted hexagon and the new coordinates \((x,t)\) as shown in Figure 2. Then each tiling naturally corresponds to a family of \( N \) non-intersecting up-right paths (formed by the midlines of the tiles of the first two types). For each \( 0 \leq t \leq b+c \) we draw a vertical line through the point \((t,0)\) and denote by

\[
x_1^t < x_2^t < \cdots < x_N^t
\]

the points of intersection of the line with the \( N \) up-right paths. In this way, we can view a tiling as an \( N \)-point configuration, which varies in time. Define the gap probability function on a slice \( t \) as

\[
D_t(s) = \text{Prob} \left( x_N^t < s \right);
\]

this function is the main object of our study.

In the same way the Hahn orthogonal polynomial ensemble arises in the analysis of uniform lozenge tilings, our measures are related to the \( q \)-Racah orthogonal polynomials. In this sense, the model goes all the way up to the top of the Askey scheme [KLS10]. The correspondence goes as follows: for a fixed section \( t \), configurations \( x_1^t < x_2^t < \cdots < x_N^t \) form an \( N \)-point process. Under a suitable change of variables this point process has the same distribution as the \( q \)-Racah orthogonal polynomial ensemble for a set of parameters that depend on the location of the vertical slice and the size of the hexagon. We elaborate more on this connection in Section 2.

An interesting aspect of this two-parameter family of probability measures is its various degenerations. For example, the uniform measure on tilings is recovered in the limit \( \kappa \to 0 \) and \( q \to 1 \). Other interesting degenerations include \( \kappa \to 0 \), in which case the weight becomes proportional to \( q^{-V} \), where \( V \) is the number

\[
V = \sum_{j=1}^{b+c+1} \frac{j-1/2}{\kappa q^j + (c+1)/2}.
\]
of boxes in the 3-D interpretation). On one hand, these limits correspond to some arrows in the degeneration cascades in the Askey scheme of hypergeometric and basic hypergeometric orthogonal polynomials. On the other hand, they seem to correspond to the degeneration cascades in Sakai’s classification scheme of discrete Painlevé equations [Sak01], as shown in Figure 3. Specifically, in [BB03] it was shown that gap probabilities of the form (1.1) for many examples of discrete orthogonal polynomial ensembles can be computed using a certain recurrence procedure that is essentially equivalent to the difference and \(q\)-difference discrete Painlevé equations; some cases are labeled on Figure 3. This correspondence has been extended in [Kni16] to the \(q\)-Hahn case that corresponds to the \(q\)-P\(_{\text{VI}}\) discrete Painlevé equation. The \(q\)-Racah case considered in the present paper corresponds to the \(q\)-P\(_{\text{IV}}\) discrete Painlevé equation. Although we do not study these degenerations in detail (we plan to consider this question separately), in Section 4 we show that the weight degeneration from the \(q\)-Racah case to the \(q\)-Hahn case is completely consistent with the degeneration of the \(A^{(1)}_1\) surface (with \(E^{(1)}_7\) symmetry) into the \(A^{(1)}_2\) surface (with \(E^{(1)}_6\) symmetry) in Sakai’s approach.

\[\begin{align*}
(E^{(1)}_4)^q & \quad \rightarrow \quad (E^{(1)}_4)^q \quad \rightarrow \quad (D^{(1)}_5)^q \quad \rightarrow \quad (A^{(1)}_1)^q \quad \rightarrow \quad ((A_1 + A_1)^{(1)})^q \quad \rightarrow \quad (A^{(1)}_1)^q \rightarrow \quad (A^{(1)}_1)^q
\end{align*}\]

| Figure 3. The degeneration cascade for the symmetry-type classification of Painlevé equations |

| Figure 4. A simulation of a tiling for a hexagon with the sides \(a = 60\), \(b = 80\), \(c = 60\) and parameters \(\kappa^2 = 0, 001\), \(q = 0.995\). |

We also want to point out that the \(q\)-Racah tiling model is a source of rich and interesting structures that are worth investigating. In particular, the asymptotic behavior of the height function of the \(q\)-Racah tiling model when the sides of the hexagon become large and simultaneously \(q \rightarrow 1\), \(\kappa \rightarrow \kappa_0\), where \(\kappa_0 \in (0, 1)\) is fixed, was studied in [DK17], (see Figure 4 for a sample tiling in this case), where it was proved that
there exists a deterministic limit shape \( \hat{h} \) and the random height functions \( h \) concentrate near it with high probability as the parameters of the model scale to their critical values. An important feature of that model is that the limit shape develops frozen facets where the height function is linear. In addition, the frozen facets are interpolated by a connected disordered liquid region. In terms of the tiling, a frozen facet corresponds to a region where asymptotically only one type of lozenge is present, and in the liquid region one sees lozenges of all three types, see Figure 4. Similar concentration phenomena for the random height function in the case of the uniform measure and the measure proportional to \( q \) of all three types, see Figure 4. In the liquid region, a region where asymptotically only one type of lozenge is present, and in the liquid region one sees lozenges of all three types, see Figure 4.

The results of the present paper predict the appearance of the Painlevé transcendents in the limit regime for the fluctuations of the height function near the boundary of the limit shape.

### 1.2. Moduli spaces of \( q \)-connections.

Our approach is based on the ideas introduced in [BB03] and [AB06]. First, using Discrete Riemann-Hilbert Problem formalism of [Bor00, Bor03], we express the gap probability function in terms of the matrix entries of a sequence of matrices \( A_s(z) \) of a certain form. We then describe the general moduli space of matrices of this form (equivalently, the moduli space of \( q \)-connections) and show that its smallest compactification is isomorphic to a \( A_1 \)-surface in Sakai’s approach. The evolution \( A_s(z) \to A_{s+1}(z) \) is given by an isomonodromy transformation that can be thought of as an isomorphism between two different surfaces in the \( A_1 \)-family, and so it is not surprising that it is given by a discrete \( q \)-Painlevé equation. We first identify this equation indirectly through the action of the isomonodromic dynamics on the parameters of the moduli space, and then show how to change coordinates to explicitly transform this equation into the standard form.

One new and interesting aspect of the \( q \)-Racah case is a certain involutive symmetry of the problem. Following the ideas of D. Arinkin and A. Borodin, see also [OR17b], we formalize this involutive symmetry structure via the notion of an elliptic connection.

**Definition 1.2.1.** Let \( \mathcal{E} \) be a symmetric bi-quadratic curve in \( \mathbb{P}^1 \times \mathbb{P}^1 \) i.e., \( \mathcal{E} \) a zero locus of a symmetric bi-degree \((2,2)\) polynomial. Note that generically \( \mathcal{E} \) is elliptic. An \( \mathcal{E} \)-connection (or an elliptic connection) is a pair \((\mathcal{L}, \mathcal{A})\), where \( \mathcal{L} \) is a vector bundle on \( \mathbb{P}^1 \), and where for any point \((x, y) \in \mathcal{E}\), we have a map \( \mathcal{A}(x, y) : \mathcal{L}_y \to \mathcal{L}_x \) such that \( \mathcal{A}(x, y) \) is a rational function of \((x, y) \in \mathcal{E}\) satisfying the involutivity condition \( \mathcal{A}(y, x) = \mathcal{A}(x, y)^{-1} \).

For our purposes we need to consider a degenerate case when \( \mathcal{E} \) is a nodal rational curve. Namely, let \( u, q \in (0, 1) \) be two fixed parameters, and let the curve \( \mathcal{E}_u \) be given by the following equation (in the affine chart \( \mathbb{C}^1 \times \mathbb{C}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1 \)):

\[
\mathcal{E}_u : \quad (x - qy)(y - qx) = \frac{u^2}{q^2}(1 - q)^2(1 + q).
\]

This curve has the following rational parameterization in terms of a parameter \( z \in \mathbb{P}^1 \):

\[ x(z) = q^{-1}z + u^2/z, \quad y(z) = z + u^2/(qz) = x(qz). \]

In this way we can identify \( \mathcal{A}(x, y) = \mathcal{A}(z), \mathcal{A}(y, x) = \mathcal{A}(u^2/z), \) and the mapping \( \mathcal{A}(x, y) : \mathcal{L}_y \to \mathcal{L}_x \) induces the mapping \( \mathcal{A}(z) : \mathcal{L}_z \to \mathcal{L}_q \). The latter mapping is the usual definition of a \( q \)-connection, but the above formalism allows us to incorporate into it the symmetry condition.

**Definition 1.2.2.** We say that a point \( z_0 \in \mathbb{P}^1 \) is a pole of \( \mathcal{A} \) if \( \mathcal{A}(z) \) is not regular at \( z = z_0 \). We say that \( z_0 \in \mathbb{P}^1 \) is a zero of \( \mathcal{A} \) if the map \( \mathcal{A}^{-1}(z) : \mathcal{L}_{q^{-1}z} \to \mathcal{L}_z \) is not regular at \( z = z_0 \). Note that \( \mathcal{A} \) can have a zero and a pole at the same point.

**Definition 1.2.3.** Suppose \( \mathcal{R} : \mathcal{L} \to \hat{\mathcal{L}} \) is a rational isomorphism between two vector bundles on \( \mathcal{E}_u \simeq \mathbb{P}^1 \).

We say that \( \hat{\mathcal{L}} \) is a modification of \( \mathcal{L} \) on a finite set \( S \subset \mathcal{E}_u \) if \( \mathcal{R}(z) \) and \( \mathcal{R}^{-1}(z) \) are regular outside \( S \). We call \( \hat{\mathcal{L}} \) an upper modification of \( \mathcal{L} \) if \( \mathcal{R} \) is regular (then \( \mathcal{L} \) is called a lower modification of \( \hat{\mathcal{L}} \)). A \( \mathcal{E}_u \)-connection \((\mathcal{L}, \mathcal{A}(z))\) induces a \( \mathcal{E}_u \)-connection \((\hat{\mathcal{L}}, \hat{\mathcal{A}}(z))\) that we also call a modification of \((\mathcal{L}, \mathcal{A}(z))\).
A class of $E_u$-connections that we consider depends on 8 complex parameters $(z_1, z_2, z_3, z_4, z_5, z_6, d_1, d_2)$. After choosing a trivialization of $\mathcal{L}$ over the affine chart $\mathbb{C} \subset \mathbb{P}^1$, the matrix $A(z)$ of the connection $A$ has the following form (see Section 3):

\[
A(z) = \frac{1}{P(z)} \begin{bmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{bmatrix}, \quad b_{21}(0) = 0,
\]

where $b_{ij}(z)$ are polynomials with $\deg(b_{11}(z)) \leq 6$, $\deg(b_{12}(z)) \leq 8$, $\deg(b_{21}(z)) \leq 5$, $\deg(b_{22}(z)) \leq 6$, and

\[
\det A(z) = \frac{Q(z)}{P(z)},
\]

where

\[
P(z) = (z - z_1)(z - z_3)(z - z_5)(z - u^2/z_4)(z - u^2/z_6),
\]

\[
Q(z) = \frac{z_1 z_3 z_5}{z_2 z_4 z_6} (z - u^2/z_1)(z - z_2)(z - u^2/z_3)(z - z_4)(z - u^2/z_5)(z - z_6).
\]

We also require that $A(z)$ satisfies the asymptotic condition

\[
S \left( \frac{z + u^2}{q} \right) A(z) S^{-1} \left( \frac{z + u^2}{q} \right) \sim \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \quad \text{where } S(z) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix},
\]

and the involution condition

\[
A(u^2/z) = A^{-1}(z).
\]

We consider $A(z)$ modulo gauge transformations of the form

\[
A(z) = R \left( \frac{z + u^2}{q} \right) A(z) R^{-1} \left( \frac{z + u^2}{q} \right), \quad R(z) = \begin{bmatrix} r_{11}(z) & r_{12}(z) \\ 0 & r_{22}(z) \end{bmatrix},
\]

where $r_{ij}(z)$ are polynomials with $\deg(r_{11}(z)) = \deg(r_{22}(z)) = 0$ and $\deg(r_{12}(z)) \leq 1$.

**Lemma 1.2.4.** Under certain non-degeneracy conditions on the parameters $\lambda = (z_1, z_2, \ldots, z_6, d_1, d_2)$ of an $E_u$-connection $A$, there exists its unique modification $\overline{A}$ of type $\overline{\lambda} = (z_1, q_{z_2}, z_3, q_{z_4}, z_5, z_6, q^{-1}d_1, q^{-1}d_2)$.

Let us assume that the parameters $(z_1, \ldots, z_6, d_1, d_2)$ are generic; the precise meaning of this condition is explained in Section 3. We show that the moduli space $M_\lambda$ of $q$-connections of type $\lambda = (z_1, z_2, \ldots, z_6, d_1, d_2)$ modulo $q$-gauge transformations is two-dimensional and its smallest smooth compactification can be identified with $\mathbb{P}^1 \times \mathbb{P}^1$ blown-up at eight points; more precisely, it is a Sakai surface of type $A_1^{(1)}$. We denote the parameters on this surface by $(f, g)$, they are described in (3.9) in terms of the usual spectral coordinates.

**Theorem 1.2.5.** Consider the modification of $\mathcal{L}$ to $\mathcal{L}$ from Lemma 1.2.4 that shifts

\[
z_2 \rightarrow q z_2, \quad z_4 \rightarrow q z_4, \quad d_1 \rightarrow q^{-1}d_1, \quad d_2 \rightarrow q^{-1}d_2.
\]

Then this modification defines a regular morphism between two moduli spaces $M_\lambda$ and $M_{\overline{\lambda}}$. Moreover, the coordinates $(\mathcal{F}, \mathcal{G})$ on the moduli space $M_{\overline{\lambda}}$ are related to $(f, g)$ by the $q$-$P\left( E_7^{(1)}/A_1^{(1)} \right)$ Painlevé equation

\[
\begin{align*}
\left( f g - \frac{\kappa_1}{\kappa_2} \right) (\mathcal{F} g - F) - \left( g - \frac{\nu_1}{\nu_2} \right) \left( g - \frac{\nu_1}{\nu_2} \right) \left( g - \frac{\nu_1}{\nu_2} \right) = 0, \\
\left( f g - \frac{\kappa_1}{\kappa_2} \right) (\mathcal{G} - F) - \left( f - \frac{\nu_1}{\nu_2} \right) \left( f - \frac{\nu_1}{\nu_2} \right) \left( f - \frac{\nu_1}{\nu_2} \right) = 0,
\end{align*}
\]

where we have the following matching of parameters:

\[
\nu_1 = \frac{1}{z_6}, \ \nu_2 = \frac{1}{z_1}, \ \nu_3 = \frac{1}{z_3}, \ \nu_4 = \frac{1}{z_5}, \ \nu_5 = \frac{u z_4}{z_2}, \ \nu_6 = u, \ \nu_7 = \frac{d_1 z_4 z_6}{u}, \ \nu_8 = \frac{d_2 z_4 z_6}{u}, \ \kappa_1 = \frac{u}{z_2}, \ \kappa_2 = \frac{z_4}{u}.
\]

**Remark 1.2.6.** The form (1.2) of the standard $q$-$P\left( E_7^{(1)}/A_1^{(1)} \right)$ equation here follows the recent survey [KNY17] (equation (8.7) in 8.1.3). It is given as two maps $(f, g) \rightarrow (\mathcal{F}, \mathcal{G})$ and $(f, g) \rightarrow (\mathcal{G}, \mathcal{F})$, which reflects the QRT origin of this equation, but it is easy to rewrite it as a mapping $(f, g) \rightarrow (\mathcal{F}, \mathcal{G})$.

We also want to point out that this equation was originally obtained by Grammaticos and Ramani [GR99] (equations (14a) and (14b)), where it is called the the asymmetric $q$-$P_{\nu_1}$ equation.
Correlation kernel $K$ (2.2)

We believe that important new aspects of the present paper are the following. First, it is a good illustration of the power of Sakai’s geometric theory for applications. Here we show how from just the minimal knowledge of the singularity structure of the connection and the evolution of parameters we can identify our dynamics with the standard discrete Painlevé dynamics and produce the required non-trivial change of coordinates that significantly simplifies the further computations. On one hand, our approach is algorithmic and it is adaptable to other applications, but on the other hand it uses the full power of algebro-geometric theory of discrete Painlevé equations. Second, the $q$-Racah weight that we consider is at the top of the degeneration cascade, so other models can be obtained from it through degenerations, as we showed for the $q$-Hahn limit. Further, the $q$-Racah case was not considered in [BB03], so we needed to adopt the computation of the gap probabilities through the discrete Riemann-Hilbert problems from [BB03] to work for this model.

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2. The $q$-Racah Orthogonal Polynomial Ensemble

2.1. Orthogonal Polynomials. Let $X$ be a finite subset of $\mathbb{C}$ such that $\text{card}(X) = M + 1 < \infty$ and $\omega : X \to \mathbb{R}_{>0}$ be any function. Using $\omega$ as a weight function, we can define an inner product on the space $\mathbb{C}[z]$ of complex polynomials via

$$\langle f, g \rangle_\omega := \sum_{x \in X} f(x)g(x)\omega(x), \quad f, g \in \mathbb{C}[z].$$

Given this inner product, a set $\{P_n\}_{n=0}^M$ of complex polynomials is called a collection of orthogonal polynomials associated to the weight function $\omega$ if

- $P_n$ is a polynomial of degree $n$ for all $n = 1, \ldots, M$ and $P_0 \equiv \text{const}$;
- if $m \neq n$ then $(P_m, P_n)_\omega = 0$.

We always take $P_n$ to be monic, i.e. $P_n(z) = z^n + \text{lower order terms}$.

It is clear that a collection of orthogonal polynomials $\{P_n\}_{n=0}^M$ associated to $\omega$ and satisfying the condition $(P_n, P_n)_\omega \neq 0$ for all $n = 0, \ldots, M$ exists if and only if the restriction of $\langle \cdot, \cdot \rangle_\omega$ to the space $\mathbb{C}[z]^{\leq d}$ of polynomials of degree at most $d$ is nondegenerate for all $d = 0, \ldots, M$. If this condition holds we say that the weight function $\omega$ is nondegenerate, and in that case it is clear that the collection $\{P_n\}_{n=0}^M$ (with the monic normalization) is unique.

Definition 2.1.1. Fix $N \in \{1, \ldots, M+1\}$. Under the above assumptions, an $N$-orthogonal discrete polynomial ensemble on $X$ with the weight function $\omega$ is a probability distribution on $N$-tuples $(\chi_1, \ldots, \chi_N)$, $\chi_i \in X$, that is defined by

$$\mathbb{P}(\chi_1, \ldots, \chi_N) = \frac{1}{Z(N, M)} \prod_{1 \leq i<j \leq N} (\chi_i - \chi_j)^2 \prod_{i=1}^N \omega(\chi_i),$$

where $Z(N, M)$ is the usual normalization constant.

It is well known (see, e.g., [Joh06] or [Kô5]) that such an ensemble is a determinantal point process whose correlation kernel $K(x, y)$ can be written in terms of the orthogonal polynomials,

$$K(x, y) = \sqrt{\omega(x)\omega(y)} \sum_{i=0}^{N-1} \frac{P_i(x)P_i(y)}{(P_i, P_i)_\omega} = \begin{cases} \frac{\sqrt{\omega(x)\omega(y)}\phi(x)\psi(y) - \psi(x)\phi(y)}{x - y}, & x \neq y, \\ \omega(x)(\phi'(x)\psi(x) - \phi(x)\psi'(x)), & x = y, \end{cases}$$

where $\phi(z) = P_N(z)$ and $\psi(z) = (P_{N-1}, P_{N-1})_\omega^{-1} \cdot P_{N-1}(z)$. The second equality here follows from the observation that $K(x, y)$ is equal to the product of $\sqrt{\omega(x)\omega(y)}$ with the $N$th Christoffel-Darboux kernel for this system of orthogonal polynomials, see [Sze67].
Let us parametrize the set $X$ as $X = \{ \pi_x \}_{x=0}^{M}$, where $\pi_x < \pi_{x+1}$, $x = 0, \ldots, M$. For any $s \in \mathbb{N}$, $N \leq s \leq M$, let $3_s = \{ \pi_i \}_{i=0}^{s-1}$ and let $R_s = X \setminus 3_s = \{ \pi_i \}_{i=s}^{M}$. It is well-known, see [BB03] or [AGZ10], that the so-called gap probabilities $D_s$ for this ensemble, defined below, can be expressed as a Fredholm determinant of the correlation kernel $K(x, y)$ given by (2.2),

$$D_s = \text{Prob}(\max\{\pi_i\}_{i=1}^N < \pi_s) = \det(1 - K_s), \quad \text{where } K_s = K|_{3_s \times 3_s}. $$

These are the quantities that we are interested in computing.

2.1.1. q-Racah Orthogonal Polynomial Ensemble. In this section we recall some basic properties of the q-Racah orthogonal polynomials, cf. [KS96, Section 3.2].

Definition 2.1.2. Let $q \in (0, 1)$, $M \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\gamma = q^{-M-1}$. For $x = 0, 1, \ldots, M$, the q-Racah weight function $\omega^{qR}(x)$ is defined by

$$\omega^{qR}(x) = \frac{(\alpha q, \beta \delta q, \gamma q, \gamma \delta q; q)_x}{(q, \alpha^{-1} q \delta q, \beta^{-1} q \gamma q, \delta q; q)_x (\alpha \beta q)^x (1 - \gamma \delta q)},$$

where $(y_1, \ldots, y_i; q)_k := (y_1; q)_k \cdots (y_i; q)_k$ and $(y; q)_k := (1 - y) (1 - y q) \cdots (1 - y q^{k-1})$ is the usual q-Pochhammer symbol.

Remark 2.1.3. The condition $\gamma = q^{-M-1}$ can be replaced by $\alpha = q^{-M}$ or $\beta = q^{-M}$, but our choice is due to the fact that under the substitutions $\gamma = q^{-M}$ and $\delta = 0$ the q-Racah weight reduces to the q-Hahn weight $\omega^{qH}(x) = \frac{(\alpha q q^{-M}; q)_x}{(q^{-1} q^{-M}; q)_x (\alpha q)^x}$. Our choice is due to the fact that under the substitutions $\gamma = q^{-M-1}$ and $\delta = 0$ the q-Racah weight reduces to the q-Hahn weight $\omega^{qH}(x) = \frac{(\alpha q q^{-M}; q)_x}{(q^{-1} q^{-M}; q)_x (\alpha q)^x}$.

Definition 2.1.4. Fix $N \in \mathbb{Z}_{\geq 1}$ and let $\alpha, \beta, \gamma, \delta, q$ and $M$ be as in Definition 2.1.2 with $M \geq N - 1$. Denote by $X^N$ a collection of $N$-tuples of non-negative integers,

$$X^N = \{ (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N : 0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_N \leq M \}.$$

The q-Racah ensemble is a probability measure $\mathbb{P}^{qR}$ on the set $X^N$ that is given by

$$\mathbb{P}^{qR}(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z(N, M, \alpha, \beta, \gamma, \delta, q)} \prod_{1 \leq i < j \leq N} (\sigma(q^{-\lambda_i}) - \sigma(q^{-\lambda_j}))^2 \cdot \prod_{i=1}^{N} \omega^{qR}(\lambda_i),$$

where $\sigma(z) = z + \gamma \delta q z^{-1}$ and $Z(N, M, \alpha, \beta, \gamma, \delta, q)$ is the usual probabilistic normalization constant.

For $\mathbb{P}^{qR}$ to be an actual probability measure, expressions in (2.4) have to be non-negative, and this is not necessarily the case for a generic choice of parameters. Thus, some restrictions on the space of parameters have to be imposed and we make one such possible choice in the following assumption.

Assumption 2.1.5. We assume that parameters $\alpha, \beta, \gamma, \delta, q \in \mathbb{R}$ and $M, N \in \mathbb{Z}$ are such that

$$M \geq N - 1 \geq 0, \quad 1 > q > 0, \quad \alpha, \beta > 0, \quad \delta \geq 0, \quad \gamma = q^{-M-1}, \quad 1 > \beta \delta, \quad \beta \geq \gamma, \quad \alpha \geq \gamma.$$

Then expressions in (2.4) are non-negative on all of $X^N$ and indeed define a probability measure $\mathbb{P}^{qR}$.

Remark 2.1.6. Although we chose to consider q-Racah ensemble as a probability measure on $N$-tuples of $(\lambda_1, \ldots, \lambda_N)$, it can also be viewed as a measure on $(\sigma(q^{-\lambda_1}), \ldots, \sigma(q^{-\lambda_N}))$, to agree with Definition 2.1.1.

It is well known that $\omega^{qR}$ is a nondegenerate weight function. Orthogonal polynomials $\{P_n(z)\}_{n=0}^{M}$ associated to it are called the q-Racah Orthogonal Polynomials. They satisfy the following orthogonality relation, written in the argument $\sigma(q^{-x}) := q^{-x} + \gamma \delta q^x$:

$$\sum_{n=0}^{M} \omega^{qR}(x) P_n(\sigma(q^{-x})) P_n(\sigma(q^{-x})) = c_n \cdot \delta_{mn},$$

where $c_n = \frac{(\gamma \delta q^2, \alpha^{-1} \beta^{-1} \gamma q, -\alpha^{-1} \delta q, -\beta^{-1} q; q)_\infty}{(\alpha^{-1} q \delta q, -\beta^{-1} \gamma q, \alpha^{-1} \delta q, \beta^{-1} q^{-1}; q)_\infty (1 - \alpha \beta q)(\gamma \delta q)_n (q, \beta q, \alpha \delta^{-1} q, \alpha \beta^{-1} q; q)_n (1 - \alpha \beta q^{2n+1})(\beta q, q, \alpha \beta q, q; q)_n}$. A connection between q-Racah ensemble and the tiling model described in Section 1.1 is given by the following Theorem, see [BGR10].
Theorem 2.1.7. Consider the tiling of a hexagon with side lengths \(a, b, c\). Let \(N = a, T = b + c, S = c\), and let \(q \in (0, 1)\), \(\kappa \in [0, q^{(T-1)/2}]\). Fix \(t \in \{0, 1, \ldots, T\}\) and let \((x_1^t, \ldots, x_N^t)\) be the corresponding random \(N\)-point configuration, see Figure 2. Then

\[
\mathbb{P}(x_1^t, \ldots, x_N^t) = \mathbb{P}^{QR}(x_1^t, \ldots, x_N^t),
\]

where the parameters of the \(q\)-Racah ensemble are as follows:

1. for \(t < S, t < T - S\), and \(0 \leq x \leq M = t + N - 1\),
   \[
   \alpha = q^{-S - N}, \quad \beta = q^{-S - T + N}, \quad \gamma = q^{-t - N}, \quad \delta = \kappa^2 q^{-S + N};
   \]
2. for \(S - 1 < t < T - S + 1\) and \(x \leq M = S + N - 1\),
   \[
   \alpha = q^{-t - N}, \quad \beta = q^{-T - N}, \quad \gamma = q^{-S - N}, \quad \delta = \kappa^2 q^{-t + N};
   \]
3. for \(T - S + 1 < t < S\) and \(0 \leq x - (t + S - T) \leq M = T - S + N - 1\),
   \[
   \alpha = q^{-T - N + t}, \quad \beta = q^{-T - N}, \quad \gamma = q^{-T - N + S}, \quad \delta = \kappa^2 q^{-T + t + N};
   \]
4. for \(S - 1 < t, T - S - 1 < t, \) and \(0 \leq x - (t + S - T) \leq M = T - t + N - 1\),
   \[
   \alpha = q^{-T - N + S}, \quad \beta = q^{-S - N}, \quad \gamma = q^{-T - N + t}, \quad \delta = \kappa^2 q^{-T + S + N}.
   \]

In particular, we can treat the gap probability function for the tiling model as the gap probability function for the \(q\)-Racah ensemble.

2.2. Discrete Riemann-Hilbert Problems and Gap Probabilities. The connection between Discrete Riemann-Hilbert Problems (DRHP) and gap probabilities goes back to [Bor00, Bor03, BB03]. In this section we review some relevant results from [BB03] and also establishes an easier way (compared to [BB03]) to compute gap probabilities through the solution to the corresponding DRHP.

Let \(X\) and \(\omega\) be as in Section 2.1 and define \(w : X \to \text{Mat}(2, \mathbb{C})\) in terms of the weight function \(\omega\) as

\[
(2.6) \quad w(x) = \begin{bmatrix} 0 & \omega(x) \\ 0 & 0 \end{bmatrix}.
\]

Definition 2.2.1. An analytic function

\[
m : \mathbb{C} \setminus \mathfrak{X} \to \text{Mat}(2, \mathbb{C})
\]

is a solution of the DRHP \((X, w)\) if \(m\) has simple poles at the points of \(X\) and its residues at these points are given by the residue (or jump) condition

\[
(2.7) \quad \text{Res}_{z=x} m(z) = \lim_{z \to x} (m(z)w(x)), \quad x \in X.
\]

Let us introduce the notation

\[
(2.8) \quad c_n := (P_n, P_n)_\omega, \quad H_n(z) := \sum_{x \in \mathfrak{X}} \frac{P_n(x)\omega(x)}{z - x}, \quad n = 0, \ldots, M.
\]

The connection between the collection of orthogonal polynomials \(\{P_n(z)\}_{n=0}^M\) on \(X\) with the weight function \(\omega\) and solutions to DRHP \((X, w)\) was established in [BB03].

Theorem 2.2.2. [BB03, Lemma 2.1 and Theorem 2.4] Let \(X\) be a finite subset of \(\mathbb{C}\), \(\text{card}(X) = M + 1 < \infty\), \(\omega : X \to \mathbb{C}\) a nondegenerate weight function, and \(w\) given by (2.6). Then for any \(N = 1, 2, \ldots, M\) the DRHP \((X, w)\) has a unique solution \(m_X(z)\) satisfying an asymptotic condition

\[
(2.9) \quad m_X(z) \begin{bmatrix} z^{-N} & 0 \\ 0 & z^M \end{bmatrix} = I + O(z^{-1}) \quad \text{as} \quad z \to \infty,
\]

where \(I\) is the identity matrix. This solution is explicitly given by

\[
m_X(z) = \begin{bmatrix} P_N(z) & H_N(z) \\ c_{N-1}^{-1}P_{N-1}(z) & c_{N-1}^{-1}H_{N-1}(z) \end{bmatrix}, \quad \text{where} \quad c_n, H_n \quad \text{are as in (2.8)}.
\]

Since \(w(x)\) is nilpotent, \(\det m_X(z)\) is entire. Moreover, since \(\det m_X(z) \to 1\) as \(z \to \infty\), \(\det m_X(z) \equiv 1\).
Recall that for $N \leq s \leq M$, $\mathcal{Z}_s = \{\pi_j\}_{j=s}^{s-1}$ and $\mathcal{N}_s = \mathfrak{X} \setminus \mathcal{Z}_s = \{\pi_j\}_{j=s}^{M}$. Let

$$m_s(z) = \begin{bmatrix} m^{11}_s(z) & m^{12}_s(z) \\ m^{21}_s(z) & m^{22}_s(z) \end{bmatrix}$$

be the unique solution of DRHP ($\mathcal{Z}_s, \omega|_{\mathcal{Z}_s}$) such that

$$(2.10) \quad m_s(z) \cdot \begin{bmatrix} z^{-N} & 0 \\ 0 & z^N \end{bmatrix} = I + O(z^{-1}) \text{ as } z \to \infty.$$  

Note that $m_s(z)$ is analytic on $\mathcal{N}_s$.

**Lemma 2.2.3** ([BB03, Theorem 3.1(a)]). For each $s \in \mathbb{N}$, $N \leq s \leq M$, there exists a constant nilpotent matrix $T_s$ such that

$$(2.11) \quad m_{s+1}(z) = \left( I + \frac{T_s}{z - \pi_s} \right) m_s(z).$$

**Remark 2.2.4.** Note that any $2 \times 2$ nilpotent matrix can be written in the form

$$(2.12) \quad T_s = \begin{bmatrix} t^{11}_s & t^{12}_s \\ t^{21}_s & t^{22}_s \end{bmatrix}, \quad (t^{11}_s)^2 + t^{12}_s t^{21}_s = 0.$$  

As explained in [BB03, Proposition 5.5], we can assume that $t^{11}_s \neq 0$ (and hence $t^{12}_s, t^{21}_s \neq 0$ as well).

**Proposition 2.2.5.** The following formula holds

$$(2.13) \quad \frac{D_{s+1}}{D_s} = \omega(\pi_s) \cdot \frac{(m^{11}_s(\pi_s))^2}{t^{12}_s}.$$  

**Proof.** From [BB03, Lemma 4.1] it follows that the operator $(1 - K_s)$ is invertible, $D_s = \det(1 - K_s) \neq 0$, and the resolvent $R_s = K_s(1 - K_s)^{-1}$ is well-defined. Moreover, the diagonal values of the resolvent $R_s$ satisfy the important identity (see, for example, [AGZ10, Section 3.4.2])

$$1 + R_s(\pi_s, \pi_s) = \frac{\det(1 - K_{s+1})}{\det(1 - K_s)} = \frac{D_{s+1}}{D_s}.$$  

Finally, in [Bor03, Theorem 2.3 applied in Situation 2.2], it was shown that the diagonal values of the resolvent can be computed explicitly by

$$(2.14) \quad R_s(\pi_s, \pi_s) = -\begin{bmatrix} 0 & \sqrt{\omega(\pi_s)} m^{11}_s(\pi_s) m'_s(\pi_s) \sqrt{\omega(\pi_s)} \\ \sqrt{\omega(\pi_s)} m^{21}_s(\pi_s) m'_s(\pi_s) & 0 \end{bmatrix}^t = -\text{Tr} \left( m^{11}_s(\pi_s) m'_s(\pi_s) \begin{bmatrix} 0 & \omega(\pi_s) \\ 0 & 0 \end{bmatrix} \right).$$

From (2.11) taking residue at $z = \pi_s$ we get

$$(2.15) \quad T_s \begin{bmatrix} m^{11}_s(\pi_s) \\ m^{21}_s(\pi_s) \end{bmatrix} = 0, \quad \text{in particular,} \quad m^{11}_s(\pi_s) m'_s(\pi_s) = \frac{t^{12}_s}{t^{11}_s}.$$  

Second, multiplying by $w(\pi_s)$ we get $T_s m_s(\pi_s) w(\pi_s) = 0$. Note that since $t^{21}_s \neq 0$ we have

$$\ker(T_s) = \text{Span}_\mathbb{C} \left\{ T_s \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} t^{11}_s \\ t^{21}_s \end{bmatrix} \right\}, \quad \begin{bmatrix} m^{11}_s(\pi_s) \\ m^{21}_s(\pi_s) \end{bmatrix} = \lambda \begin{bmatrix} t^{11}_s \\ t^{21}_s \end{bmatrix}.$$  

Now, on the one hand, using the DRHP residue condition (2.7) and (2.11), we get

$$(2.16) \quad \lim_{z \to \pi_s} m_{s+1}(z) w(\pi_s) = \text{Res}_{z=\pi_s} m_{s+1}(z) = T_s m_s(\pi_s).$$  

On the other hand, using (2.11)

$$(2.17) \quad \lim_{z \to \pi_s} m_{s+1}(z) w(\pi_s) = \lim_{z \to \pi_s} \left( \left( I + \frac{T_s}{z - \pi_s} \right) m_s(z) \begin{bmatrix} 0 & \omega(\pi_s) \\ 0 & 0 \end{bmatrix} \right) = m_s(\pi_s) w(\pi_s) + T_s m'_s(\pi_s) w(\pi_s).$$  

Therefore, we get

$$(2.18) \quad T_s m_s(\pi_s) - m_s(\pi_s) w(\pi_s) = T_s m'_s(\pi_s) w(\pi_s).$$

Since $T_s$ is nilpotent, we cannot invert it to find $m'_s(\pi_s) w(\pi_s)$. However, we see that

$$T_s m'_s(\pi_s) \begin{bmatrix} \omega(\pi_s) \\ 0 \end{bmatrix} = T_s \begin{bmatrix} m^{12}_s(\pi_s) \\ m^{22}_s(\pi_s) \end{bmatrix} - \omega(\pi_s) \begin{bmatrix} m^{11}_s(\pi_s) \\ m^{21}_s(\pi_s) \end{bmatrix} = T_s \left( \begin{bmatrix} m^{12}_s(\pi_s) \\ m^{22}_s(\pi_s) \end{bmatrix} - \omega(\pi_s) \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$
Therefore,
\[
m'_s(\pi_s) \begin{bmatrix} \omega(\pi_s) \\ 0 \end{bmatrix} = \begin{bmatrix} m_{12}^s(\pi_s) \\ m_{22}^s(\pi_s) \end{bmatrix} - \omega(\pi_s) \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} t_{11}^s \\ t_{21}^s \end{bmatrix},
\]
where the last vector is some vector in the kernel of \( T_s \). Substituting this in (2.14) and using the fact that \( \det m_s(z) = 1 \) gives, again using (2.12),
\[
R_s(\pi_s, \pi_s) = -\text{Tr} \left( \begin{bmatrix} m_{22}^s(\pi_s) & -m_{12}^s(\pi_s) \\ -m_{21}^s(\pi_s) & m_{11}^s(\pi_s) \end{bmatrix} \right) \begin{bmatrix} 0 & m_{12}^s(\pi_s) - \omega(\pi_s) \lambda + k t_{11}^s \\ 0 & m_{22}^s(\pi_s) + k t_{21}^s \end{bmatrix}
= -\det m_s(\pi_s) - \omega(\pi_s) \lambda m_{21}^s(\pi_s) + k \left( m_{11}^s(\pi_s) t_{21}^s - m_{21}^s(\pi_s) t_{11}^s \right)
= -1 + \omega(\pi_s) \frac{m_{11}^s(\pi_s)}{t_{11}^s} \cdot \frac{m_{11}^s(\pi_s) t_{11}^s}{t_{12}^s} = -1 + \omega(\pi_s) \left( \frac{m_{11}^s(\pi_s)}{t_{12}^s} \right)^2.
\]

\( \Box \)

2.3. Connection matrix for the \( q \)-Racah ensemble. In this section following the steps of [BB03],[AB09] and [Kni16] we introduce a connection matrix for the \( q \)-Racah ensemble which captures all essential information about the gap probability function.

**Assumption 2.3.1.** We assume that \( \gamma \delta q \in (0,1) \). Then \( \sigma(q^{-x}) \) is an increasing function and \( \pi_x = \sigma(q^{-x}) \) is an ordered set for \( x = 0, \ldots, M \). Therefore, the framework of the previous two sections, including formula (2.13) for computing gap probabilities, is applicable.

**Remark 2.3.2.** For the \( q \)-Racah weight, the DHRP condition (2.7) has to be slightly changed, it now takes the form
\[
\text{Res } m(z) = \lim_{z \to \pi_x} (m(z)w(x)), \quad \pi_x = \sigma(q^{-x}) \in \mathfrak{X}.
\]

Let \( u > 0 \) be defined via \( u^2 = \gamma \delta q z^2 \). Then \( \sigma(z) = z + \gamma \delta q z^{-1} = z + u^2/(q z) \), and it is easy to see that
\[
\sigma \left( \frac{u^2}{z} \right) = \sigma \left( \frac{z}{q} \right), \quad \sigma(z) - \sigma(y) = (z - y) \left( 1 - \frac{u^2}{yzq} \right).
\]

**Remark 2.3.3.** Note that
\[
\frac{\omega^{\mathfrak{H}}(x + 1)}{\omega^{\mathfrak{H}}(x)} = \frac{(q^{-x} - \alpha q)(q^{-x} - \beta \delta q)(q^{-x} - \gamma q)(q^{-x} - \gamma \delta q)(q^{-2x} - \gamma \delta q^3)}{(q^{-x} - \alpha q)(q^{-x} - \alpha^{-1} \gamma \delta q)(q^{-x} - \beta^{-1} \gamma \delta q)(q^{-x} - \delta q)(q^{x} - \alpha \beta q)(q^{-2x} - \gamma q)}
\]
\[
= \frac{\Phi^+(q^{-x})(q^{-2x} - q u^2)}{q \Phi^-(q^{-x})(q^{-2x} - u^2/q)}.
\]

where
\[
\Phi^+(z) = (z - \alpha q)(z - \beta \delta q)(z - \gamma q)(z - \gamma \delta q),
\]
\[
\Phi^-(z) = \alpha \beta (z - \alpha^{-1} \gamma \delta q)(z - \beta^{-1} \gamma \delta q)(z - \delta q)(z - q).
\]

The functions \( \Phi^\pm(z) \) appear as coefficients in the Nekrasov’s equation for \( q \)-Racah ensemble, see [DK17] for details.

We are interested in computing the gap probability function for large \( N \). The degrees of the diagonal entries of \( m_s(z) \) grow with \( N \) and that presents a serious computational difficulty. To bypass it, we introduce matrix functions \( A_s(z), \quad N \leq s \leq M \), as follows:
\[
A_s(z) := m_s(\sigma(q^{-1}z)) \, D(z) \, m_s^{-1}(\sigma(z)), \quad \text{where } D(z) = \begin{bmatrix} \Phi^+(z) \\ \Phi^-(z) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

In this definition we used the fact that \( \det m_s(z) = 1 \), see Theorem 2.2.2, and so \( m_s(z) \) is invertible. Matrices \( A_s(z) \) play a central role in the arguments below. In what follows we show that the evolution \( A_s(z) \mapsto A_{s+1}(z) \) can be effectively computed using discrete Painlevé equations and also explain how to extract from this dynamics the relevant information about the recursion on gap probabilities \( D_s \).

**Remark 2.3.4.** In [DK17] the trace of matrix \( A_M(z) \) is linked to the explicit computation of the frozen boundary in the tiling model.
Proposition 2.3.5. Let
\[ z_1(s) = z_2(s) = q^{-s+1}, \quad z_3 = q, \quad z_4 = \alpha q, \quad z_5 = \delta q, \quad z_6 = \beta \delta q. \]
Then \( A_s(z) \) has the following properties:

(i) \( A_s^{-1}(z) = A_s(u^2/z) \) and \( A(u) \) is an identity matrix;

(ii) \( \det A_s(z) = \frac{Q_s(z)}{P_s(z)} \), where
\[
P_s(z) = \left( z - z_1(s) \right) \left( z - \frac{u^2}{z_2(s)} \right) \left( z - z_3 \right) \left( z - \frac{u^2}{z_4} \right) \left( z - z_5 \right) \left( z - \frac{u^2}{z_6} \right),
\]
\[
Q_s(z) = \frac{z_1(s) z_4 z_5}{z_2(s) z_4 z_6} \left( z - \frac{u^2}{z_1(s)} \right) \left( z - z_2(s) \right) \left( z - \frac{u^2}{z_3} \right) \left( z - z_4 \right) \left( z - \frac{u^2}{z_5} \right) \left( z - z_6 \right).
\]

(iii) Matrices \( A_s(z) \) has the form
\[
A_s(z) = \frac{1}{P_s(z)} B_s(z), \quad \text{where}
\]
\[
B_s(z) = \left[ \sum_{i=0}^{3} n_i z_1^{6-i} + n_4 u^2 z_2 + n_5 u^4 z + n_6 u^6 \right] \left( z^2 - u^2 \right) \left( m_0 z^2 + m_1 z + m_0 u^2 \right)
\]
\[
\left( z^2 - u^2 \right) \left( k_0 z^2 + k_1 z + k_0 u^2 \right) \sum_{i=0}^{3} n_{6-i} z_1^{6-i} + n_2 u^2 z^2 + n_4 u^4 z + n_6 u^6.
\]

Proof. Using the fact that \( \det m_s(z) = 1 \), we see that
\[
\det A_s(z) = \det D(z) = \frac{\Phi^+(z)}{\Phi^-(z)} = \frac{(z - \frac{u^2}{z_1})(z - z_4)(z - \frac{u^2}{z_5})(z - z_6)}{z_2(z - z_3)(z - \frac{u^2}{z_4})(z - z_5)(z - \frac{u^2}{z_6})} = Q_s(z),
\]
where in the last equation we note some cancellations since \( z_1(s) = z_2(s) \). Further, since
\[
\left. \frac{z - z_1}{z - \frac{u^2}{z_1}} \right|_{z = \frac{u^2}{z_1}} = \frac{z^2}{u^2 z - z_1} \quad \text{and so} \quad \frac{\Phi^+(u^2/z)}{\Phi^-(u^2/z)} = \frac{\Phi^-(z)}{\Phi^+(z)}, \quad \text{and} \quad \sigma \left( \frac{u^2}{z} \right) = \sigma \left( \frac{z}{q} \right),
\]
we immediately see that \( D^{-1}(z) = D(u^2/z) \) and \( A_s^{-1}(z) = A_s(u^2/z) \).

To complete the proof, we need to understand the singularity structure of \( A_s(z) \). We see that \( D(z) \) has simple poles at \( z = q = z_3, z = \delta q = z_5, \) \( z = \beta^{-1} \gamma q = u^2/z_6 \), and \( z = \alpha^{-1} \gamma \delta q = u^2/z_4 \). Thus, to show that \( A_s(z) = \frac{1}{P_s(z)} B_s(z) \) where \( B_s(z) \) is regular, we need to show that the only remaining possible poles of \( A_s(z) \) are \( z_1(s) = q^{-s+1} \) and \( u^2/z_2(s) = u^2 q^{s-1} \).

Recall that from the DRHP, \( m_s(z) \) has simple poles at \( \pi_x = \sigma(q^{-x}) \) for \( 0 \leq x \leq s - 1 \),
\[
m_s(z) = A_{-1}(z - \pi_x)^{-1} + A_0 + A_1(z - \pi_x) + \cdots.
\]
Moreover,
\[
\text{Res}_{z = \pi_x} m_s(z) = A_{-1} = \lim_{z \to \pi_x} (m(z) w(x)) = A_0 w(x) \quad \text{and} \quad A_{-1} w(x) = A_0 w(x) = 0,
\]
since \( w(x) \) is nilpotent. Thus,
\[
m_s(z) = A_0 w(x)(z - \pi_x)^{-1} + A_0 + A_1(z - \pi_x) + \cdots = F(z) \left( I + \frac{w(x)}{z - \pi_x} \right),
\]
where \( F(z) \) is regular at \( \pi_x \) and \( F_0 = A_0 \). Since \( \det m_s(z) = 1 \), \( m_s^{-1}(z) \) also has simple poles at \( \pi_x = \sigma(q^{-x}) \) for \( 0 \leq x \leq s - 1 \) and
\[
m_s^{-1}(z) = \left( I - \frac{w(x)}{z - \pi_x} \right) F^{-1}(z), \quad \text{where}
\]
\( F^{-1}(z) \) is regular at \( \pi_x \).

From (2.20) we see that \( \sigma(z) = \sigma(y) \) if \( z = y \) or \( z = u^2/(y q) \). Thus, the first factor \( m_s \left( \sigma(q^{-1} z) \right) \) has simple poles when \( z = q^{-(x-1)} \), then \( \sigma(q^{-1} z) = \pi_x \) and \( \sigma(z) = \pi_{x-1} \), or when \( z = u^2 q^{x-2} \) and then \( \sigma(q^{-1} z) = \pi_x \) and \( \sigma(z) = \pi_{x+1} \). Similarly, the second factor \( m_s^{-1} \left( \sigma(z) \right) \) has simple poles when \( z = q^{-x} \), then \( \sigma(q^{-1} z) = \pi_{x+1} \) and \( \sigma(z) = \pi_x \), or when \( z = u^2 q^{-x-1} \) and then \( \sigma(q^{-1} z) = \pi_{x-1} \) and \( \sigma(z) = \pi_x \).
We need to distinguish between the situation when both factors are singular, which happens when either \( z = q^{-x} \) or \( z = u^2 q^x \), \( 0 \leq x \leq s - 2 \), and the boundary case when only one factor is singular. For \( z \) near \( q^{-x} \), \( 0 \leq x \leq s - 2 \), the matrix \( A_s(z) \) takes the form

\[
A_s(z) = F(\sigma(q^{-1}z)) \left( I + \frac{w(x+1)}{\sigma(q^{-1}z) - \sigma(q^{-x+1})} \right) D(z) \left( I - \frac{w(x)}{\sigma(z) - \sigma(q^{-x})} \right) F^{-1}(\sigma(z))
\]

\[
= F(\sigma(q^{-1}z)) \left[ \frac{\Phi^+(z)}{\Phi^+(z)} \right] \left[ \frac{q^{-x} z^h(z)}{q^{-x} - q^{-x+1}} \right] F^{-1}(\sigma(z)), \quad \text{where}
\]

\[
h(z) = \frac{\omega^{qR}(x+1)}{q^{-1}z - q^{-x}} - \frac{\Phi^+(z)}{\Phi^-(z)} (z - q^{-x+1}).
\]

From (2.21) we see that \( h(q^{-x}) = 0 \), and so \( A_s(z) \) is regular at \( q^{-x} \).

Similarly, for \( z \) near \( u^2 q^{-x} \), \( 0 \leq x \leq s - 2 \), the matrix \( A_s(z) \) takes the form

\[
A_s(z) = F(\sigma(q^{-1}z)) \left( I + \frac{w(x)}{\sigma(q^{-1}z) - \sigma(q^{-x})} \right) D(z) \left( I - \frac{w(x+1)}{\sigma(z) - \sigma(q^{-x+1})} \right) F^{-1}(\sigma(z))
\]

\[
= F(\sigma(q^{-1}z)) \left[ \frac{\Phi^+(z)}{\Phi^+(z)} \right] \left[ \frac{q^{-x} z^h(z)}{q^{-x} - q^{-x+1}} \right] F^{-1}(\sigma(z)), \quad \text{where}
\]

\[
h(z) = \frac{\omega^{qR}(x)}{q^{-1}z - q^{-x}} - \frac{\Phi^+(z)}{\Phi^-(z)} \omega^{qR}(x+1), \quad h \left( \frac{u^2}{z} \right) = \frac{\omega^{qR}(x)}{q^{-1}u^2 - q^{-x}} - \frac{\Phi^-(z)}{\Phi^+(z)} \frac{\omega^{qR}(x+1)}{q u^2 - q^{-x}z}.
\]

Using (2.21) again we see that \( h(u^2 q^{-x}) = h(u^2 q^{-x}) = 0 \), and so \( A_s(z) \) is regular at \( u^2 q^{-x} \).

Consider now the boundary cases. There are four possibilities: when \( z = q \) (resp. \( z = u^2 q^{-s-1} \)), the first factor \( m_s(\sigma(q^{-1}z)) \) has a simple pole at \( \pi_0 \) (resp. \( \pi_{s-1} \)) and the last factor \( m_s^{-1}(\sigma(z)) \) is regular; when \( z = q^{-s+1} \) (resp. \( z = u^2 q^{-1} \)), the last factor has poles at \( \pi_{s-1} \) (resp. \( \pi_0 \)) and the first factor is regular.

Near \( z = q \), we have

\[
A_s(z) = F(\sigma(q^{-1}z)) \left[ \frac{\Phi^+(z)}{\Phi^+(z)} \right] \left[ \frac{q^{-x} z^h(z)}{q^{-x} - q^{-x+1}} \right] m_s^{-1}(\sigma(z)) = F(\sigma(q^{-1}z)) \left[ \frac{\Phi^+(z)}{\Phi^+(z)} \right] \left[ \frac{q^{-x} z^h(z)}{q^{-x} - q^{-x+1}} \right] m_s^{-1}(\sigma(z)),
\]

and so both matrix elements in the top row of the central matrix have a simple pole at \( q = z_3 \) and it is already accounted for. Near \( z = u^2 q^{-1} \) we have

\[
A_s(z) = m_s(\sigma(z)) \left[ \frac{\Phi^+(z)}{\Phi^+(z)} \right] \left[ \frac{q^{-x} z^h(z)}{q^{-x} - q^{-x+1}} \right] F^{-1}(\sigma(z)) = m_s(\sigma(z)) \left[ \frac{\Phi^+(z)}{\Phi^+(z)} \right] \left[ \frac{q^{-x} z^h(z)}{q^{-x} - q^{-x+1}} \right] F^{-1}(\sigma(z)),
\]

and the zero of \( \Phi^+(z) \) at \( u^2/z^3 = u^2 q^{-1} \) cancels the corresponding simple zero of \( (\sigma(z) - \pi_0) \) and so \( A_s(z) \) is regular at that point. Thus, the only new remaining possible poles are at \( u^2/z_2 = u^2 q^{-s-1} \) and \( z_1 = q^{-s+1} \), as we claimed.

Thus, matrix entries of \( B_s(z) \) are polynomials and the condition \( A_s^{-1}(z) = A_s(u^2/z) \) becomes

\[
\begin{bmatrix}
    b^{22}_s(z) & -b^{12}_s(z) \\
    -b^{21}_s(z) & b^{11}_s(z)
\end{bmatrix} = \frac{z^6}{u^6} \begin{bmatrix}
    b^{11}(u^2/z) & b^{12}(u^2/z) \\
    b^{21}(u^2/z) & b^{22}(u^2/z)
\end{bmatrix},
\]

From here it is immediate that \( \deg b^{ij}_s(z) \leq 6 \). Moreover, if we slightly adjust the coefficients and write

\[
b^{11}_s(z) = n_0 z^6 + n_1 z^5 + n_2 z^4 + n_3 z^3 + n_4 u^2 z^2 + n_5 u^4 z + n_6 u^6
\]

\[
b^{22}_s(z) = \frac{z^6}{u^6} b^{11}_s(z) = n_6 z^6 + n_5 z^5 + n_4 z^4 + n_3 z^3 + n_2 u^2 z^2 + n_1 u^4 z + n_0 u^6,
\]

as claimed. In the same way we can see that

\[
b^{12}_s(z) = (z^2 - u^2)(m_4 z^4 + u^2 z^2 + u^4) + m_0 z^2 + m_1 z + m_0 u^2,
\]

but since \( A_s(z) \) is asymptotic to a diagonal matrix when \( z \to \infty \), \( \deg b^{12}_s(z) \leq 5 \), and hence \( m_4 = 0 \) and we get \( b^{12}_s(z) = z (z^2 - u^2) (m_0 z^2 + m_1 z + m_0 u^2) \). The argument for \( b^{21}_s(z) \) is similar, and this completes the proof.

\[
\square
\]
2.4. Initial conditions. Our strategy for computing gap probabilities $D_\ast$ is to use the recursion. In this section we compute the initial conditions for this recursion in the $q$-Racah case. First, we need the following result.

Lemma 2.4.1 ([BB03, Proposition 6.1]). The solution $m_N(z)$ of the DRHP\{$\{\pi_0, \ldots, \pi_{N-1}\}, \omega|_{\pi_0, \ldots, \pi_{N-1}}$\} with the asymptotics $m_N(z) \sim \begin{bmatrix} z^N & 0 \\ 0 & z^{-N} \end{bmatrix}$ as $z \to \infty$ is given by

$$(2.24) \quad m_N(z) = \begin{bmatrix} \Pi(z) \sum_{x=0}^{N-1} \rho_x & 0 \\ \Pi(z) \sum_{x=0}^{N-1} \rho_x & \frac{1}{\Pi(z)} \end{bmatrix}, \quad \text{where} \quad \Pi(z) = \prod_{m=0}^{N-1} (z - \pi_m),$$

and where

$$\rho_x = \omega(x)^{-1}, \quad \prod_{0 \leq m < N-1, m \neq x} (\pi_x - \pi_m)^{-2}, \quad 0 \leq x \leq N - 1.$$ 

Now we are ready to compute the initial conditions for the $q$-Racah case by evaluating the matrix $A_N(z)$. It still has the overall structure described in Proposition 2.3.5, with $z_1(N) = 2q(N) = q^{-N+1}$, but using Lemma 2.4.1, we can now give an explicit formula for $A_N(z)$.

Lemma 2.4.2. The matrix $A_N(z)$ has the following form:

$$A_N(z) = \frac{1}{P_N(z)} B_N(z) = \frac{1}{P_N(z)} \begin{bmatrix} b_{11}^N(z) & b_{12}^N(z) \\ b_{21}^N(z) & b_{22}^N(z) \end{bmatrix}, \quad \text{where}$$

$$b_{11}^N(z) = q^{-N} \frac{z \alpha \beta}{z_3 z_5} \left( z - \frac{u^2}{z_1(N)} \right) \left( z - \frac{u^2}{z_2(N)} \right) \left( z - \frac{u^2}{z_3} \right) \left( z - \frac{u^2}{z_4} \right) \left( z - \frac{u^2}{z_5} \right),$$

$$b_{12}^N(z) = 0,$$

$$b_{21}^N(z) = z(z^2 - u^2)(k_0 z^2 + k_1 z + k_0 u^2), \quad \text{where} \quad k_0 = \sum_{x=0}^{N-1} q \rho_x \left( \frac{q^{-N}}{\alpha \beta} - q^{N-1} \right), \quad \alpha \beta = \frac{z_4 z_6}{z_3 z_5},$$

$$k_1 = \sum_{x=0}^{N-1} q \rho_x \left( \frac{q^{-N}}{\alpha \beta} \left( q \sigma(q^{-x}) - \frac{u^2}{z_1(N)} - \frac{u^2}{z_2(N)} - \frac{u^2}{z_3} - \frac{u^2}{z_4} - \frac{u^2}{z_5} \right) \right),$$

$$b_{22}^N(z) = q^N \left( z - z_1(N) \right) \left( z - z_2(N) \right) \left( z - \frac{u^2}{z_3} \right) \left( z - \frac{u^2}{z_4} \right) \left( z - \frac{u^2}{z_5} \right).$$

Proof. From Lemma 2.4.1 we get

$$A_N(z) = m_N(\sigma(q^{-1}z)) D(z) m_N^{-1}(\sigma(z))$$

$$= \begin{bmatrix} \Phi^+(z) \Pi(\sigma(q^{-1}z)) & 0 \\ \Phi^+(z) \Pi(\sigma(z)) & \Phi^-(z) \Pi(\sigma(z)) \end{bmatrix} \begin{bmatrix} \sum_{x=0}^{N-1} \rho_x \sigma(z) - \frac{u^2}{\sigma^2(z)} & \sum_{x=0}^{N-1} \rho_x \sigma(z) - \frac{u^2}{\sigma^2(z)} \\ \sum_{x=0}^{N-1} \rho_x \sigma(z) - \frac{u^2}{\sigma^2(z)} & \sum_{x=0}^{N-1} \rho_x \sigma(z) - \frac{u^2}{\sigma^2(z)} \end{bmatrix} \begin{bmatrix} \Phi^+(z) \Pi(\sigma(q^{-1}z)) & 0 \\ \Phi^+(z) \Pi(\sigma(z)) & \Phi^-(z) \Pi(\sigma(z)) \end{bmatrix}.$$ 

Direct computation shows that

$$\frac{\Pi(\sigma(q^{-1}z))}{\Pi(\sigma(z))} = \frac{q^{-N} (z - u^2 q^{N-1}) (z - z_3)}{(z - u^2 q^{-1}) (z - q^{-N+1})} = \frac{q^{-N} (z - \frac{u^2}{z_1(N)}) (z - z_3)}{(z - z_1(N)) (z - \frac{u^2}{z_3})},$$

and the expressions for $b_{11}^N(z)$ and $b_{22}^N(z)$ immediately follow. From Proposition 2.3.5(iii) we know that $b_{21}^N(z) = z(z^2 - u^2)(k_0 z^2 + k_1 z + k_0 u^2)$. Moreover, for $b_{21}^N(z)$ we have

$$b_{21}^N(z) = z(z^2 - u^2)(k_0 z^2 + k_1 z + k_0 u^2) = 2q \sum_{x=0}^{N-1} \rho_x \left( \frac{b_{11}^N(z)}{(z - q^x u^2)} - \frac{b_{22}^N(z)}{(z - q^x (q z - q^x u^2))} \right),$$

where
and therefore
\[
k_0 = \frac{q}{\eta_0} \sum_{x=0}^{N-1} \rho_x \left( b_{22}^N(0) - \frac{b_{11}^1(0)}{q} \right) = \sum_{x=0}^{N-1} q \rho_x \left( q^N z_1(N) z_2(N) z_5 - \frac{q^{-N} z_3^2}{z_1(N) z_2(N)} \right)
\]
\[
= \sum_{x=0}^{N-1} q \rho_x \left( \frac{q^{-N}}{\alpha \beta} - q^{N-1} \right),
\]
where \( \alpha \beta = \frac{z_{22} z_5}{z_{32}} \). Similarly,
\[
k_1 = -\frac{1}{u^2} \lim_{z \to 0} \sum_{x=0}^{N-1} q \rho_x \left( \frac{b_{11}^1(z)}{(z - q^{-(x-1)})(z - q^x u^2)} - \frac{b_{22}^N(z)}{(z - q^{-x})(q z - q^x u^2)} - \left( \frac{b_{11}^1(0)}{q} \right) \frac{z^2 - u^2}{u^4} \right)
\]
\[
= \sum_{x=0}^{N-1} q \rho_x \left( \frac{q^{-N}}{\alpha \beta} \left( q \sigma(q^{-x}) - \frac{u^2}{z_1(N)} - \frac{z_3 - z_4 - u^2}{z_5 - z_6} \right) - q^{N-1} \left( \sigma(q^{-x}) - z_1(N) - z_2(N) - \frac{u^2}{z_3 - z_4} - \frac{z_5 - u^2}{z_6} \right) \right).
\]

2.5. The Lax Pair. Equations (2.11) and (2.23) constitute the Lax Pair for solutions of DRHP, c.f. [BB03, Section 3],

\[
\begin{cases}
  m_{s+1}(\sigma(z)) = \left( I + \frac{T_s}{\sigma(z) - \pi_s} \right) m_s(\sigma(z)), \\
  m_s(\sigma(q^{-1} z)) = A_s(z) m_s(\sigma(z)) D^{-1}(z)
\end{cases}
\]

This Lax Pair in turn gives rise to the isomonodromic dynamics for the matrices \( A_s(z) \),

\[
A_{s+1}(z) = \left( I + \frac{T_s}{\sigma(q^{-1} z) - \pi_s} \right) A_s(z) \left( I - \frac{T_s}{\sigma(z) - \pi_s} \right).
\]

To run the recursion computing the gap probability function we will need the values of \( D_k(k), D_k(k+1) \) computed in the next proposition.

Proposition 2.5.1. ([BB03, Proposition 6.6]) Let \( \mathcal{X} \subset \mathbb{R} \) be a discrete set, let \( \{P_n(z)\} \) be the family of orthogonal polynomials corresponding to a strictly positive weight function \( \omega : \mathcal{X} = \{\pi_0, \ldots, \pi_N\} \to \mathbb{R} \). Then

\[
D_k = \frac{1}{Z} \cdot \prod_{0 \leq i < j \leq k} (\pi_i - \pi_j)^2 \cdot \prod_{l=0}^{k-1} \omega(\pi_l),
\]

(2.27)

\[
D_{k+1} = \omega(\pi_k) \cdot h_k^{-1} \cdot D_k(k) \cdot \prod_{l=0}^{k-1} (\pi_k - \pi_l)^2,
\]

(2.28)

where \( h_k \) is given by

\[
h_k = \rho_k + \sum_{m=0}^{k-1} \frac{\rho_m}{(\pi_k - \pi_m)^2}
\]

and \( \rho_k \) is defined in (2.4.1).

3. Moduli Space of Elliptic Connections and Discrete Painlevé Equations

It is possible to consider \( A_s(z) \) as a matrix representation, with respect to some trivialization, of an \( \mathcal{E}_u \)-connection on the vector bundle \( \mathcal{L} = \mathcal{O} \oplus \mathcal{O}(-1) \).

Remark 3.0.1. Here we follow the approach of [AB06] and twist from the trivial vector bundle to \( \mathcal{L} \), since \( \mathcal{O} \oplus \mathcal{O}(-1) \) has more gauge automorphisms, and this results in significant simplifications in computations.
Thus, we consider the following class of $\mathcal{E}_u$-connections.

\begin{equation}
A(z) = \frac{1}{P(z)} \begin{bmatrix}
  b_{11}(z) & \frac{b_{12}(z)}{z} \\
  b_{21}(z) & b_{22}(z)
\end{bmatrix},
\quad b_{21}(0) = 0,
\end{equation}

where $\deg(b_{11}(z)) \leq 6$, $\deg(b_{12}(z)) \leq 8$, $\deg(b_{21}(z)) \leq 5$, $\deg(b_{22}(z)) \leq 6$ and

$$
\det A(z) = \frac{Q(z)}{P(z)},
\quad P(z) = (z - z_1)(z - u^2/z_2)(z - z_3)(z - u^2/z_4)(z - z_5)(z - u^2/z_6),
\quad Q(z) = \frac{z_1 z_3 z_5}{z_2 z_4 z_6}(z - u^2/z_2)(z - u^2/z_3)(z - z_4)(z - u^2/z_5)(z - z_6).
$$

We also require that $A(z)$ satisfies the asymptotic condition

$$
S \left( \frac{z + u^2}{z} \right) A(z) S^{-1} \left( \frac{z + u^2}{z} \right) \sim \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix},
$$

where $S(z) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}$, and the involution condition

$$
A(u^2/z) = A^{-1}(z) \quad \text{and} \quad A(u) \text{ is an identity matrix.}
$$

Remark 3.0.2. We need to fix that either $A(u)$ is an identity or minus identity to work with a connected component of the moduli space.

Such matrix representation of a connection is not unique, since the choice of the trivialization of $\mathcal{L}$ can be composed with an automorphism of the bundle. Such automorphism can be written as a matrix

\begin{equation}
R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}, \quad r_{11}, r_{22} \in \mathbb{C} - \{0\}, \quad r_{12} \in \Gamma(\mathbb{P}^1, \mathcal{O}(1)).
\end{equation}

As usual, this matrix ansatz is described using the so-called spectral coordinates. To introduce them, we first observe that, using gauge transformations, we can reduce $b_{21}(z)$ to

$$
b_{21}(z) = z(z - u^2)(z - t) \left( z^2 - \frac{u^2}{t} \right),
$$

and so we put $t = t_1/t_2$ to be our first spectral coordinate. The second spectral coordinate $p$ is the $(1, 1)$-entry of $A(z)$ at $t$, $p = b_{11}(t)/P(t)$. Imposing the remaining conditions, such as the asymptotic condition and the determinant condition, allows us to express the remaining entries of $A(z)$ as rational functions of the spectral coordinates. Those rational functions become indeterminate at certain points, and resolving these indeterminacies via blowups and compactifying identifies our moduli space of $q$-connections with (a blowup of) one of the Spaces of Initial Conditions in Sakai’s classification scheme for discrete Painlevé equations, [Sak01].

However, the new feature of this example is that the involution condition above induces the involution on parameters, $t \leftrightarrow u^2/t$ and $p \leftrightarrow 1/p$. As a result, in the spectral coordinates $(t, p)$ we get more than the usual 8 points. Specifically, we get the following six pairs of involution-conjugated points:

\begin{align*}
&\left( \frac{u^2}{z_1}, 0 \right), \quad (z_1, \infty), \quad \left( \frac{u^2}{z_3}, 0 \right), \quad (z_3, \infty), \quad \left( \frac{u^2}{z_5}, 0 \right), \quad (z_5, \infty), \\
&(z_2, 0), \quad \left( \frac{u^2}{z_2}, \infty \right), \quad (z_4, 0), \quad \left( \frac{u^2}{z_4}, \infty \right), \quad (z_6, 0), \quad \left( \frac{u^2}{z_6}, \infty \right),
\end{align*}

as well as points $(u, 1)$ and $(-u, -1)$, and points $(\infty, -\rho_1 = d)$ and $\left( \infty, -\rho_2 = \frac{z_1 z_3 z_5}{z_2 z_4 z_6 q d} \right)$. Note that from the viewpoint of computations of the moduli space we can interchange $d_1$ and $d_2$ and by $d$ we denote one of the choices. In the same way $\rho_1$ and $\rho_2$ are also interchangeable.

To fix this, we need to introduce the involution-invariant coordinates $x = t + \frac{u^2}{t}$ and $y = \frac{t^p - u}{P(t)}$ gluing these pairs of points together. In the involution-invariant $(x, y)$-coordinates we get the point configuration shown on Figure 5. The points

$$
\pi_7(\infty, \rho_1 = -d), \quad \pi_8(\infty, \rho_2 = -\frac{z_1 z_3 z_5}{z_2 z_4 z_6 q d})
$$
lie on the $(1,0)$-curve $C_1 = V(X = 1/x)$, and the points
\[
\pi_i \left( z_i + \frac{u^2}{z_i}, \frac{z_i}{u} \right), \quad i = 1, 3, 5; \quad \pi_i \left( z_i + \frac{u^2}{z_i}, \frac{u}{z_i} \right), \quad i = 2, 4, 6
\]
lie on the $(1,2)$-curve $C_0 = V(u(y^2 + 1) - xy)$; note also that when $x = z_i + \frac{u^2}{z_i}$, the equation $u(y^2 + 1) - xy$ factors as $u(y^2 + 1) - xy = u(y - y(\pi_i))(y - y(\pi_i))$, where the conjugated points $\pi'_i$ are given by
\[
\pi'_i \left( z_i + \frac{u^2}{z_i}, \frac{u}{z_i} \right), \quad i = 1, 3, 5; \quad \pi'_i \left( z_i + \frac{u^2}{z_i}, \frac{z_i}{u} \right), \quad i = 2, 4, 6.
\]
The remaining two points $(t, p) = \pm (u, 1)$ are fixed points of the involution and are also base points of the coordinate $p$. We get one final base point $\pi_9(-2u, -1)$, similar to the $q$-Hahn case. The reason why we are left only with this point in the computation is because due to involution-invariant change of coordinates the singularity at $t = u$ gets resolved (note that for $t = u$ we necessarily have $p = 1$).

3.1. **Reference Example of $q$-$P\left(A_1^{(1)}\right)$**

The goal of this section is to show that the isomonodromic dynamics corresponding to the parameter evolution

\[
\tilde{z}_2 = qz_2, \quad \tilde{z}_4 = qz_4, \quad d = q^{-1}d, \quad \tilde{z}_i = z_i \text{ otherwise.}
\]
is in fact equivalent to the standard $q$-$P(A_1^{(1)})$ discrete Painlevé dynamic and to give the explicit change of variables from the involution-invariant spectral coordinates $(x, y)$ to the Painlevé coordinates $(f, g)$. The approach here is similar to that of [DT18], so we shall be brief and refer the reader to that paper for details. Below we review the geometric setting of Sakai’s theory, as well as introduce some notation. We only consider a generic setup here, see [Sak01] and especially [KHY17] for careful and detailed exposition that also includes special cases.

3.1.1. **The Root Data**

A discrete Painlevé equation describes dynamics on a certain family $\mathbb{X}$ of rational algebraic surfaces obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at eight, possibly infinitely close, points $p_i$ that lie on a, possibly reducible, bi-quadratic curve $\Gamma$. Let $([f_0 : f_1], [g_0 : g_1])$ be homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathbb{P}^1 \times \mathbb{P}^1$ is covered by four affine charts, $(f = f_0/f_1, g = g_0/g_1)$, $(F = 1/f, g)$, $(f, G = 1/g)$, and $(F, G)$.

Parameters $b = \{b_i\}$ of the family are essentially the coordinates of the blowup points. However, since we need to account for various gauge actions, a better choice of parameters is given by the so-called root variables $a = \{a_i\}$, as we explain later. Then a typical surface in the family is $\mathbb{X}_b = Bl_{p_1, \ldots, p_8}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{P}^1 \times \mathbb{P}^1$. The Picard lattices $\text{Pic}(\mathbb{X}_b)$ for all of these surfaces are isomorphic,

$$
\text{Pic}(\mathbb{X}_b) \simeq \text{Pic}(\mathbb{X}) = H^1(\mathcal{X}, \mathcal{O}_\mathbb{X}^*) = \text{Div}(\mathcal{X})/\text{P}(\mathbb{X}) = \text{Span}_\mathbb{Z} \{\mathcal{H}_f, \mathcal{H}_g, \mathcal{F}_1, \ldots, \mathcal{F}_8\},
$$
where $\mathcal{H}_f$ (resp. $\mathcal{H}_g$) are the classes of the total transforms of the vertical (resp. horizontal) lines on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{F}_i$ are the classes of the total transforms of the exceptional divisors of the blowup at $p_i$ under the full blowup map $\eta$. A generic surface $\mathcal{X}_k$ in the family is then a generalized Halphen surface, i.e., it has a unique anti-canonical divisor $-K_\mathcal{X} = \eta^*(\Gamma)$ of canonical type. That is, if

$$-K_\mathcal{X} = 2H_f + 2H_g - F_1 - \cdots - F_8 = \sum_i m_id_i,$$

is the decomposition of the anti-canonical divisor into irreducible components $d_i$ with the multiplicities $m_i$, then $d_i$ is orthogonal to $-K_\mathcal{X}$ w.r.t. the intersection form, $d_i \cdot (-K_\mathcal{X}) = 0$.

We associate with this geometric data two sub-latticed in the Picard lattice Pic($\mathcal{X}$): the surface sub-lattice $\Pi(R) = \text{Span}_\mathbb{Z}\{\delta_i = |d_i|\} \subset \text{Pic}(\mathcal{X})$ that encodes the geometry of the point configuration, and its orthogonal complement $Q$ in Pic($\mathcal{X}$), which is called the symmetry sub-lattice. Both of these sub-lattices are root lattices, i.e., they have bases of simple roots, $R = \{\delta_i \mid \delta_i^2 = -2\}$, and $R^\perp = \{\alpha_j \mid \alpha_j^2 = -2, \alpha_j \cdot \delta_i = 0\}$. Then $Q = \Pi(R^\perp) = \text{Span}_\mathbb{Z}\{\alpha_i\}$. Further, $R$ and $R^\perp$ can be described by affine Dynkin diagrams $D_1$ and $D_2$, whose types are then called the surface (resp. symmetry) type of the corresponding discrete Painlevé equation (and the surface family). To the symmetry root diagram $D_2$ we can associate an affine Weyl group $W(D_2)$ whose action on Pic($\mathcal{X}$) is generated by reflections in the basis symmetry roots, $w_i : \mathcal{C} \mapsto w_i(\mathcal{C}) = \mathcal{C} + (\alpha_i \cdot \mathcal{C})\alpha_i$. Extending this group by the group of automorphism of the Dynkin diagram (same for both $D_i$) we get the full extended affine Weyl group $W(D_2) = \text{Aut}(D_2) \rtimes W(D_2)$, again acting on Pic($\mathcal{X}$) and preserving both sub-lattices (and thus preserving the surface family, that’s why it is called the symmetry group). In the cases we are interested in, this group coincides with the group of Cremona isometries of $\mathcal{X}$, its action on Pic($\mathcal{X}$) can be extended to maps on the family, and a discrete Painlevé equation is a discrete dynamical system on $\mathcal{X}$ that corresponds to a translation element of $\text{Cr}(\mathcal{X})$.

Let us now consider a particular example of the $q$-$P(A_1^{(1)})$-equation, as written in [KNY17]. It is characterized by the following Dynkin diagrams (where the numbers at the nodes are the coefficients of the linear combination describing the class of the anti-canonical divisor $\delta = -K_\mathcal{X}$ in terms of the root classes):

$$\begin{align*}
\text{Dynkin diagram } A_1^{(1)} & : \\
\delta_0 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_3 - \mathcal{F}_4 \\
\delta_1 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_7 - \mathcal{F}_8
\end{align*}$$

The surface data

$$\begin{align*}
\mathcal{F}_1 & : \mathcal{H}_f - \mathcal{H}_g \\
\mathcal{F}_2 & : \mathcal{F}_3 - \mathcal{F}_4 \\
\mathcal{F}_3 & : \mathcal{F}_5 - \mathcal{F}_6 \\
\mathcal{F}_4 & : \mathcal{F}_7 - \mathcal{F}_8
\end{align*}$$

3.1.2. The Point Configuration. To describe a model $A_1^{(1)}$-surface, we start with the following point configuration on $\mathbb{P}^1 \times \mathbb{P}^1$. Take two divisors $d_i \in \delta$ and consider their pushdown $\eta_*(d_i)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. In the affine chart $(f, g)$ these divisors are given by

$$\eta_*(d_i) = V(A_if + B_if + C_ig + D_i), \quad i = 0, 1.$$

Since $\delta_0 \cdot \delta_1 = 2$, generically $|\eta_*(\delta_0) \cap \eta_*(\delta_1)| = 2$ and we can assume that these two points do not lie on the same horizontal or vertical line. Then, using the $\text{PGL}_2 \times \text{PGL}_2$ action, we can arrange that the intersection points are $(0, \infty)$ (thus, $C_i = 0$) and $(\infty, 0)$ (thus, $B_i = 0$). Using rescaling, we can arrange

$$\eta_*(\delta_0) = V(fg - 1) \text{ and } \eta_*(\delta_1) = V(fg - \kappa),$$

where $\kappa = \kappa_1/\kappa_2$ is some parameter.

Assigning four blowup points to each of the curves,

$$p_i \left( \nu_i, \frac{1}{\nu_i} \right), \quad i = 1, \ldots, 4; \quad p_i \left( \frac{\kappa_1}{\nu_i}, \frac{\nu_i}{\kappa_2} \right), \quad i = 5, \ldots, 8.$$
we get the \( A_1^{(1)} \) point configuration as in [KNY17], see Figure 6.

Figure 6. The standard point configuration for the \( A_1^{(1)} \)-surface

This point configuration has 10 parameters, \( \nu_i, \ i = 1, \ldots, 8 \) and \( \kappa_1, \kappa_2 \), however, there are two rescaling actions, one is internal on the parameters \( (\kappa_1, \kappa_2) \sim (\mu \kappa_1, \mu \kappa_2) \), and the other is the scaling of the axes preserving the curve \( fg = 1 \),

\[
\begin{pmatrix}
\nu_1 & \nu_2 & \nu_3 & \nu_4 & \kappa_1 & f \\
\nu_5 & \nu_6 & \nu_7 & \nu_8 & \kappa_2 & g
\end{pmatrix} \sim \left( \frac{\lambda \nu_1}{\chi \nu_5}, \frac{\lambda \nu_2}{\chi \nu_6}, \frac{\lambda \nu_3}{\chi \nu_7}, \frac{\lambda \nu_4}{\chi \nu_8}, \frac{\mu \kappa_1}{\chi}, \frac{\lambda f}{\chi g} \right),
\]

so the actual number of parameters is 8. As usual, the invariant parameterization is given by the root variables \( a_i \) that can be obtained using the period map.

3.1.3. The Period Map.

Proposition 3.1.1. For our model of the \( A_1^{(1)} \)-surface, the period map and the root variables \( a_i = \chi(a_i) \) are given by

\[
a_0 = \frac{\kappa_1}{\kappa_2}, \quad a_1 = \frac{\nu_3}{\nu_4}, \quad a_2 = \frac{\nu_2}{\nu_3}, \quad a_3 = \frac{\nu_1}{\nu_2}, \quad a_4 = \frac{\kappa_2}{\nu_1 \nu_5}, \quad a_5 = \frac{\nu_5}{\nu_6}, \quad a_6 = \frac{\nu_6}{\nu_7}, \quad a_7 = \frac{\nu_7}{\nu_8}.
\]

This gives us the following parameterization by the root variables \( a_i \)

\[
\begin{pmatrix}
\nu_1 & \nu_2 & \nu_3 & \nu_4 & \kappa_1 \\
\nu_5 & \nu_6 & \nu_7 & \nu_8 & \kappa_2
\end{pmatrix} = \begin{pmatrix}
\nu_1 & \frac{\nu_1}{a_1} & \frac{\nu_1}{a_2} & \frac{\nu_3}{a_1 a_2} & \frac{\nu_3}{a_1 a_2 a_3} & \frac{\nu_4}{a_1 a_4} & \frac{a_4}{a_0 a_4} & \frac{a_4}{a_0 a_4 a_5} & \frac{a_4}{a_0 a_4 a_5 a_7} & \frac{f}{g}
\end{pmatrix}.
\]

Proof. To compute the period map, we first need to define a symplectic form \( \omega \) whose pole divisor is \( \eta_*(d_0) + \eta_*(d_1) \). Let us put \( s = fg \). Then, up to a normalization constant \( C \), we can take \( \omega \) (in the affine \((f, g)\)-chart) to be

\[
\omega = C \frac{df \wedge dg}{(f - 1)(fg - \kappa)} = C \frac{df \wedge ds}{f(s - 1)(s - \kappa)} = C \frac{ds \wedge dg}{g(s - 1)(s - \kappa)},
\]

\[
\text{res}_{\eta_*(d_0)} \omega = \frac{C}{\kappa - 1} \frac{df}{f} = -\frac{C}{\kappa - 1} \frac{dg}{g}, \quad \text{res}_{\eta_*(d_1)} \omega = -\frac{C}{\kappa - 1} \frac{df}{f} = \frac{C}{\kappa - 1} \frac{dg}{g}.
\]
Then

\[
\chi(\alpha_0) = \chi(\mathcal{H}_f \setminus \mathcal{H}_g) = \chi([H_f - F_1] - [H_g - F_1]) = \frac{\langle H_f - F_1 \rangle \cap \eta_*(d_1)}{(H_g - F_1) \cap \eta_*(d_1)} = \frac{C}{\kappa - 1} \int_{\kappa \nu_1}^{\nu_1} df \int f
\]

\[
\chi(\alpha_1) = \chi(\mathcal{F}_3 - \mathcal{F}_4) = \chi([F_3] - [F_4]) = \int_{p_3} \text{res}_{\eta_*(d_0)} \omega = \frac{C}{\kappa - 1} \log \left( \frac{\nu_3}{\nu_4} \right)
\]

\[
\chi(\alpha_2) = \chi(\mathcal{F}_2 - \mathcal{F}_3) = \frac{C}{\kappa - 1} \log \left( \frac{\nu_2}{\nu_3} \right), \quad \chi(\alpha_3) = \chi(\mathcal{F}_1 - \mathcal{F}_2) = \frac{C}{\kappa - 1} \log \left( \frac{\nu_1}{\nu_2} \right)
\]

\[
\chi(\alpha_4) = \chi(\mathcal{H}_g - \mathcal{F}_1 - \mathcal{F}_5) = \chi([H_g - F_1] - [F_5]) = \int_{\mathcal{H}_g - F_1 \cap \eta_*(d_1)} \text{res}_{\eta_*(d_1)} \omega = \frac{C}{\kappa - 1} \int_{\frac{\nu_5}{\nu_7}}^{\nu_7} dg \frac{g}{g}
\]

\[
\chi(\alpha_5) = \chi(\mathcal{F}_5 - \mathcal{F}_6) = \chi([F_5] - [F_6]) = \int_{p_5} \text{res}_{\eta_*(d_1)} \omega = \frac{C}{\kappa - 1} \log \left( \frac{\nu_6}{\nu_7} \right), \quad \chi(\alpha_6) = \chi(\mathcal{F}_7 - \mathcal{F}_8) = \frac{C}{\kappa - 1} \log \left( \frac{\nu_7}{\nu_8} \right)
\]

From these computations we see that it is convenient to choose the normalization constant \(C = \kappa - 1\), which gives \((3.4)\). Also, note that, using the decomposition of the anti-canonical divisor class, \(\delta = \delta_0 + \delta_1 = 2\alpha_0 + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\), we get the following expression for the step \(q\) of the dynamics:

\[
q = \exp(\chi(\delta)) = a_0^2 a_1 a_2 a_3 a_4 a_5 a_6 a_7 = \frac{\kappa_1^2 \kappa_2^2}{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5 \nu_6 \nu_7 \nu_8}
\]

\[
\square
\]

3.1.4. The Symmetry Group. The symmetry group of the \(A^{(1)}\)-surface family is the extended affine Weyl group \(\tilde{W} \left( E^{(1)}_7 \right) = \text{Aut} \left( E^{(1)}_7 \right) \times W \left( E^{(1)}_7 \right)\), where \(\text{Aut} \left( E^{(1)}_7 \right) \simeq \mathbb{Z}_2\) and the affine Weyl group \(W \left( E^{(1)}_7 \right)\) is defined in terms of generators \(w_i = w_{\alpha_i}\) and relations that are encoded by the affine Dynkin diagram \(E^{(1)}_7\),

\[
W \left( E^{(1)}_7 \right) = W \left( \begin{array}{cccccccc}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\end{array} \right) = \left\{ \begin{array}{c}
w_0, \ldots, w_6 \\
\end{array} \right\}
\]

\[
w_i \circ w_j = w_j \circ w_i \quad \text{when} \quad \begin{array}{c}
\circ \circ \circ \\
\circ \circ \circ
\end{array}
\]

3.1.5. The Standard Dynamic. The evolution of the parameters considered in [KNY17] is \(\frac{\kappa_1}{q}, \kappa_2 = q\kappa_2\), and \(\nu_i = \nu_i\) for all \(i\) which, in the \(q\)-\(P \left( A^{(1)}_1 \right)\)-case gives us equations (8.7) in Section 8.1.3 of [KNY17]:

\[
\begin{align*}
\left( f g - \frac{\kappa_2}{\kappa_2} \right) \left( f g - \frac{\nu_1}{\nu_2} \right) &= \left( g - \frac{\kappa_2}{\kappa_2} \right) \left( g - \frac{\nu_2}{\nu_2} \right) \left( g - \frac{\nu_1}{\nu_2} \right), \\
\left( f g - \frac{\kappa_1}{\kappa_1} \right) \left( f g - \frac{\nu_1}{\nu_2} \right) &= \left( f - \frac{\kappa_1}{\kappa_1} \right) \left( f - \frac{\nu_1}{\nu_2} \right)
\end{align*}
\]

Further, we get the following action on the root variables: \(\pi_0 = q^{-2} a_0, \pi_4 = q a_4\) and \(\pi_i = a_i\) otherwise.
Remark 3.1.2. In view of the birational representation, the following observation is very helpful. Let $w \in \tilde{W}(E^{(1)}_7)$, and let $\eta : X_b \to X_b$ be the corresponding mapping, i.e., $w = \eta$, and $w^{-1} = \eta^*$, where $\eta$ and $\eta^*$ are the induced push-forward and pull-back actions on the divisors (and hence on Pic($X$)) that are inverses of each other. Since $\eta$ is just a change of the blowdown structure that the period map $\chi$ does not depend on, $\chi_{X}(\alpha_i) = \chi_{\eta(X)}(\eta_*(\alpha_i))$. Thus, we can compute the evolution of the root variables directly from the action on Pic($X$) via the formula

$$\tilde{a}_i = \chi_{\eta(X)}(\tilde{a}_i) = \chi_X(w^{-1}(\tilde{a}_i)).$$

Thus the action of $\eta$ on the root variables is inverse to the action of $w$ on the roots. This is not essential for the generating reflections, that are involutions, but it is important for composed maps.

In view of Remark 3.1.2, we can now identify the translation element in $\tilde{W}(E^{(1)}_7)$ (w.r.t. the given choice of root vectors) through its action on the symmetry roots:

$$\varphi_* : \alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) \mapsto \tilde{\alpha} = \alpha + (2, 0, 0, 0, -1, 0, 0, 0)\delta.$$  

Proposition 3.1.3. The $q$-$P(A_1^{(1)})$ discrete Painlevé dynamics, given our choices, corresponds to the following element of the $\tilde{W}(E^{(1)}_7)$, written in terms of the generators

$$\varphi_* : w = (w_9 w_6 w_5 w_4 w_3 w_2 w_1 w_0) \mapsto (w_9 w_6 w_5 w_4 w_3 w_2 w_1 w_0)\delta.$$  

Proof. The proof of this statement is a standard computation, see [DT18] for an example.

Remark 3.1.4. Note that this representation of the dynamic allows us to recover the action of the mapping on Pic($X$) and, provided that we know the birational representation of the symmetry group $\tilde{W}(E^{(1)}_7)$, the equation itself (although there are better ways of obtaining the equation).

3.2. Matching the Isomonodromic and the Standard Dynamic. We are now ready to prove the following theorem.

Theorem 3.2.1. The isomonodromic dynamics corresponding to the parameter evolution (3.3) is equivalent to the standard dynamics through the following change of coordinates from isomonodromic to Painlevé:

$$f(x, y) = \frac{\sigma_3(xy + u(y - 1)) - u^2(x^2 - \sigma_1 x + \sigma_2(y + 1)) + u^3(1 - y)(\sigma_1 - x) + u^4(1 + y)}{\sigma_2(x^2 + u(y - 1)) - u^2(\sigma_2 xy + \sigma_3(y + 1)) + u^3\sigma_2(1 - y) + u^4(\sigma_1(1 + y) - x) + u^5(y - 1)},$$  

$$g(x, y) = \frac{xy z_{6} + u z_{6}(y - 1) - u^2(1 + y)}{z_{6}(1 + y) - x - u(1 + y)},$$

where $\sigma_i$ are the standard symmetric functions, $\sigma_1 = z_2 + z_4 + z_6$, $\sigma_2 = z_2 z_4 + z_4 z_6 + z_6 z_2$, and $\sigma_3 = z_2 z_4 z_6$. The inverse change of coordinates is given by

$$x(f, g) = \frac{(\kappa_1 - \kappa_2)g + \nu_6(1 + \kappa_1 \kappa_2)(1 - f g) + \nu_6^2(\kappa_1 - \kappa_2)g}{\kappa_1 - \kappa_2 fg},$$  

$$y(f, g) = \frac{\nu_1 \nu_6(1 - f g)(\nu_6 \kappa_1 - (1 + \kappa_1 \kappa_2)g) + \kappa_2 fg ((\nu_1 \nu_6 - 1)g - \nu_6) + \nu_1 \kappa_2 g^2 + \kappa_1 (1 - \nu_1 g)(g + \nu_6)}{(1 - f g)(\nu_6 - \kappa_2 (g - \nu_6 \kappa_1)) - \nu_6 ((g + \nu_6)(\kappa_1 \nu_1 + \kappa_2 f(1 - g \nu_1)) - \kappa_1 (1 + \nu_6 f))}.$$

Corresponding to these changes of coordinates we have the following matching of parameters:

$$\nu_1 = \frac{1}{z_6}, \nu_2 = \frac{1}{z_1}, \nu_3 = \frac{1}{z_3}, \nu_4 = \frac{1}{z_5}, \nu_5 = \frac{u z_4}{z_2}, \nu_6 = u, \nu_7 = \frac{-\rho_1 z_4 z_6}{u}, \nu_8 = \frac{-\rho_2 z_4 z_6}{u}, \kappa_1 = \frac{u}{z_2}, \kappa_2 = \frac{z_4}{u}.$$  

Proof. Looking at the point configuration for the moduli space of the $q$-connections in the $q$-Racah case, we see that it is not minimal (the divisor $\Delta_0$ has self-intersection degree $-3$), so we need to blow down one of the $-1$-curves. Instead, it is easier to blow up a point on one of the curves $\eta_*(d_i)$ for the $A_1^{(1)}$-surface model. Without loss of generality, we can let this point $p_0$ be on the curve $\eta_*(d_0)$, see Figure 7.
Next, we need to find a map (change of basis) from Pic(\(\mathcal{X}^R\)) to Pic(\(\mathcal{X}'\)) that will transform the components of the anti-canonical divisor class \(\Delta_i\) to \(d_i\) and then extend this map to the isomorphism between the surfaces, which, when written as a birational map \((x, y) \rightarrow (f, g)\), will give us the required change of variables. However, finding such an identification between the surfaces does not guarantee that the dynamics will also match. First, it may turn out that the dynamics are non-equivalent. Second, even if they are equivalent, our preliminary change of variables may result in a conjugated translation vector. Below we explain that there is a systematic procedure that resolves this issue.

First, comparing the expressions for the irreducible components of the anti-canonical divisor class in Pic(\(\mathcal{X}^R\)) and Pic(\(\mathcal{X}'\)),

\[
\begin{align*}
\delta_0 &= H_f + H_g - T_1 - T_2 - T_3 - T_4 - T_9 = H_x + 2H_y - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_9, \\
\delta_1 &= H_f + H_g - T_5 - T_6 - T_7 - T_8 = H_x - E_7 - E_8,
\end{align*}
\]

we see that we can preliminary do the following change of bases:

\[
\begin{align*}
H_f &= H_x + H_y - E_2 - E_9, & H_x &= H_f + H_g - T_7 - T_8, \\
H_g &= H_x + H_y - E_4 - E_9, & H_y &= H_f + H_g - T_7 - T_9, \\
T_1 &= E_1, & E_1 &= T_2, \\
T_2 &= E_6, & E_2 &= H_g - T_7, \\
T_3 &= E_3, & E_3 &= T_3, \\
T_4 &= E_5, & E_4 &= H_f - T_7, \\
T_5 &= E_7, & E_5 &= T_4, \\
T_6 &= E_8, & E_6 &= T_1, \\
T_7 &= H_x + H_y - E_2 - E_4 - E_9, & E_7 &= T_5, \\
T_8 &= H_y - E_9, & E_8 &= T_6, \\
T_9 &= H_x - E_9, & E_9 &= H_f + H_g - T_7 - T_8 - T_9.
\end{align*}
\]

From this correspondence we see that the \(f\) is a coordinate on a pencil of \((1, 1)\)-curves in the \((x, y)\)-plane passing through the points \(\pi_2\) and \(\pi_9\). Taking \(u^2(y + 1) - z_2(xy + u(y - 1))\) and \(z_2(y + 1) - x + u(y - 1)\) to be the basis of this pencil, we get

\[
f = \frac{f_0}{f_1} = \frac{A(u^2(y + 1) - z_2(xy + u(y - 1))) + B(z_2(y + 1) - x + u(y - 1))}{C(u^2(y + 1) - z_2(xy + u(y - 1))) + D(z_2(y + 1) - x + u(y - 1))},
\]

Similarly,

\[
g = \frac{g_0}{g_1} = \frac{K(u^2(y + 1) - z_4(xy + u(y - 1))) + L(z_4(y + 1) - x + u(y - 1))}{M(u^2(y + 1) - z_4(xy + u(y - 1))) + N(z_4(y + 1) - x + u(y - 1))}.
\]
Adjusting the coefficients $A$, $B$, $C$, $D$, $K$, $L$, $M$, and $N$ of the Möbius transformations using the mapping between exceptional divisors, we get the following change of coordinates:
\[
f = \frac{x - u(y - 1) - z_2(y + 1)}{u^2(y + 1) - z_2(xy + u(y - 1))}, \quad g = \frac{u^2(y + 1) - z_4(xy + u(y - 1))}{x - u(y - 1) - z_4(y + 1)}.
\]

This change of variables results in the following identification between two sets of parameters:
\[
\nu_i = \frac{z_i}{u^2}, \quad \nu_2 = \frac{1}{z_1}, \quad \nu_3 = \frac{1}{z_3}, \quad \nu_4 = \frac{1}{z_5}, \quad \nu_5 = -\rho_1 z_2 z_4, \quad \nu_6 = -\rho_2 z_2 z_4, \quad \nu_7 = \frac{u^2}{z_2 z_4}, \quad \nu_8 = z_2 z_4, \quad \kappa_1 = z_4, \quad \kappa_2 = z_2.
\]

From here, using (3.4), we can recompute the root variables in terms of the parameters of $q$-Racah setting,
\[
a_0 = \frac{z_4}{z_2}, \quad a_1 = \frac{z_5}{z_3}, \quad a_2 = \frac{z_3}{z_1}, \quad a_3 = \frac{z_1 z_6}{u^2}, \quad a_4 = -\frac{u^2}{\rho_1 z_4 z_6}, \quad a_5 = \frac{\rho_1}{\rho_2}, \quad a_6 = -\frac{\rho_2 z_2 z_4}{u^2}, \quad a_7 = \frac{u^2}{z_2 z_4},
\]
we see, using Remark 3.1.2, that the corresponding translation vector is
\[
\psi : \alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) \mapsto \bar{\alpha} = \alpha + (0, 0, 0, 0, 0, -1, 2) \delta,
\]
and that it turns out to be different from the standard translation vector
\[
\varphi : \alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) \mapsto \bar{\alpha} = \alpha + (2, 0, 0, 0, -1, 0, 0) \delta.
\]

However, these elements are conjugated. This can be observed, for example, by looking at the corresponding words in the affine Weyl symmetry group:
\[
\psi : w_7 w_6 w_5 w_4 w_3 w_2 w_1 w_8 w_7 w_6 w_5 w_4 w_3 w_2 w_1 w_8 w_7 w_6 w_5 w_4, \quad \varphi : w_8 w_7 w_6 w_5 w_4 w_3 w_2 w_1 w_8 w_7 w_6 w_5 w_4 w_3 w_2 w_1 w_8 w_7 w_6 w_5 w_4,
\]
and then using the far commutativity and the braid relations in $W(E_7^{(1)})$ to write
\[
\psi = (w_8 w_7 w_6 w_5 w_4) \varphi (w_6 w_5 w_4 w_3 w_2 w_1 w_8 w_7 w_6 w_5 w_4)^{-1}.
\]

Conjugating by the element $w_8 w_7 w_6 w_5 w_4 w_3 w_2 w_1 w_8 w_7 w_6 w_5 w_4$ adjusts the divisor matching as
\[
\mathcal{H}_f = 2 \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_9, \quad \mathcal{H}_x = \mathcal{H}_f + \mathcal{H}_y - \mathcal{F}_5 - \mathcal{F}_6, \quad \mathcal{H}_y = \mathcal{H}_f + 2 \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_9, \quad \mathcal{E}_1 = \mathcal{F}_2, \quad \mathcal{E}_2 = \mathcal{H}_y - \mathcal{F}_5, \quad \mathcal{E}_3 = \mathcal{F}_3, \quad \mathcal{E}_4 = \mathcal{H}_y - \mathcal{F}_6, \quad \mathcal{E}_5 = \mathcal{F}_4, \quad \mathcal{E}_6 = \mathcal{H}_y + \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_5 - \mathcal{F}_6, \quad \mathcal{E}_7 = \mathcal{F}_7, \quad \mathcal{E}_8 = \mathcal{F}_8, \quad \mathcal{E}_9 = \mathcal{H}_f + \mathcal{H}_y - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_9.
\]

Proceeding as before, we get the final change of variables (3.9), as well as the matching of parameters (3.12). The inverse change of variables (3.10) can be computed in a similar way.

Finally, it is now easy to verify that the parameter dynamic $\bar{z}_2 = qz_2$, $\bar{z}_4 = qz_4$, $\bar{d} = q^{-1}d$ (and so $\bar{\rho}_i = q^{-1} \rho_i$) gives us the correct translation element:
\[
\psi : \alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) \mapsto \bar{\alpha} = \alpha + (2, 0, 0, 0, -1, 0, 0) \delta.
\]

□
4. Degeneration from the \(q\)-Racah to the \(q\)-Hahn case

Note that, as shown in Figure 3, there exists a degeneration cascade for the \(q\)-Racah weight that matches (a part of) the degeneration scheme of discrete Painlevé equations. In this section we show that our choice of coordinates is compatible with the weight degeneration from the \(q\)-Racah to the \(q\)-Hahn case. We plan to consider the degenerations to Racah and Hahn cases in a separate publication. The \(q\)-Hahn case was considered in detail in [Kni16], however, to match the \(q\)-P\((A^{(1)}_2)\)-equation as written in [KNY17], we need to make a slightly different change of coordinates. Below we briefly summarize the relevant data.

4.1. Reference Example of \(q\)-P\((A^{(1)}_2)\).

4.1.1. The Root Data. As before, we take the standard example of the \(q\)-P\((A^{(1)}_2)\)-equation from [KNY17]. It is characterized by the following Dynkin diagrams:

\[
\begin{align*}
\delta_0 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_3 - \mathcal{F}_4 \\
\delta_1 &= \mathcal{H}_f - \mathcal{F}_5 - \mathcal{F}_6 \\
\delta_2 &= \mathcal{H}_g - \mathcal{F}_7 - \mathcal{F}_8
\end{align*}
\]

The surface data

4.1.2. The Point Configuration. Our model \(A^{(1)}_2\)-surface is obtained from the \(A^{(1)}_1\)-surface on Figure 6 via the following degeneration. We rescale parameters \(\kappa_1 \sim \varepsilon \kappa_1, \nu_7 \sim \varepsilon \nu_7, \text{ and } \nu_8 \sim \varepsilon \nu_8\) and then let \(\varepsilon \to 0\). Under this degeneration, \(\eta_*(d_1) = V(fg - \kappa_1/\kappa_2) = V(f) + V(g)\) decomposes into \(\eta_*(d_1) = V(f) = H_f - F_5 - F_6\) and \(\eta_*(d_2) = V(g) = H_g - F_7 - F_8, \eta_*(d_0) = H_f + H_g - F_1 - F_2 - F_3 - F_4\) remains unchanged, and the new point configuration becomes

\[
p_i \left( \frac{\nu_i}{\kappa_i} \right), \quad i = 1, \ldots, 4; \quad p_i \left( 0, \frac{\nu_i}{\kappa_i} \right), \quad i = 5, 6; \quad p_i \left( \frac{\kappa_i}{\nu_i}, 0 \right), \quad i = 7, 8.
\]

see Figure 8.

**Figure 8.** The standard point configuration for the \(A^{(1)}_2\)-surface
4.1.3. The Period Map. The $A_1^{(1)}$ symplectic form $\omega$ degenerates (in the affine $(f,g)$-chart) to

$$\omega = (\kappa - 1) \frac{df \wedge dg}{(fg - 1)(fg - \kappa)} \sim \omega = - \frac{df \wedge dg}{fg(fg - 1)}$$

$$\text{res}_{\kappa(d_0)} \omega = \frac{df}{f} = - \frac{dg}{g}, \quad \text{res}_{\kappa(d_1)} \omega = \frac{dg}{g}, \quad \text{res}_{\kappa(d_1)} \omega = - \frac{dg}{g}.$$ 

The same computation as before gives us the following root variables $a_i = \exp(\chi(\alpha_i))$:

$$a_0 = \frac{\nu_7}{\nu_8}, \quad a_1 = \frac{\nu_6}{\nu_5}, \quad a_2 = \frac{\nu_2}{\nu_1 \nu_6}, \quad a_3 = \frac{\nu_1}{\nu_2}, \quad a_4 = \frac{\nu_2}{\nu_3}, \quad a_5 = \frac{\nu_3}{\nu_4}, \quad a_6 = \frac{\nu_1}{\nu_1 \nu_7}.$$ 

To reduce the number of parameters, it is convenient to introduce the variables $b_i$ for the coordinates of the blowup points as follows:

$$p_i \left( b_i = \nu_i, \frac{1}{b_i} \right), \quad i = 1, \ldots, 4; \quad p_i \left( 0, \frac{1}{b_i} = \nu_i \right), \quad i = 5, 6; \quad p_i \left( b_i = \frac{\nu_1}{\nu_i}, 0 \right), \quad i = 7, 8.$$ 

This gives us 8 parameters, however, there is still a rescaling action of the axes preserving the curve $fg = 1$,

$$\left( b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, f, g \right) \sim \left( \lambda b_1, \lambda b_2, \lambda b_3, \lambda b_4, \lambda f, \frac{1}{\chi} g \right),$$

so the true number of parameters is 7 and they are given by the root variables $a_i$. We use the parameter $b_4$ as a free parameter and normalize birational maps to keep it fixed. We then have the following relationship between $b_i$ and the root variables $a_i$:

$$a_0 = \frac{b_8}{b_7}, \quad a_1 = \frac{b_5}{b_6}, \quad a_2 = \frac{b_6}{b_1}, \quad a_3 = \frac{b_1}{b_2}, \quad a_4 = \frac{b_2}{b_3}, \quad a_5 = \frac{b_3}{b_4}, \quad a_6 = \frac{b_7}{b_1},$$

and the root variable parameterization

$$\left( b_1, b_2, b_3, b_4, \frac{b_5}{b_6}, \frac{b_6}{b_7}, \frac{b_7}{b_8}, f, g \right) = \left( a_3 a_4 a_5 b_4, a_4 a_5 b_4, a_3 a_4 a_5 b_4, b_4, \frac{a_3 a_4 a_5 a_6 b_4}{b_4} \right).$$

Using the decomposition of the anti-canonical divisor class, $\delta = \delta_0 + \delta_1 + \delta_2 = a_0 + a_1 + 2a_2 + 3a_3 + 2a_4 + a_5 + 2a_6$, we get the following expression for the step $q$ of the dynamics:

$$q = \exp(\chi(\delta)) = a_0 a_1 a_2 a_3 a_4 a_5 a_6 = \frac{\kappa_1^2 \kappa_2^2}{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5 \nu_6 \nu_7 \nu_8} = \frac{b_5 b_6 b_7 b_8}{b_1 b_2 b_3 b_4}.$$ 

4.1.4. The Standard Dynamic. Using the same parameter evolution as in the $A_1^{(1)}$-case, $\kappa_1 = \frac{\kappa_1}{q}, \kappa_2 = q \kappa_2$, and $\nu_i = \nu_i$ for all $i$, gives us the evolution $\tilde{a}_2 = qa_2, \tilde{a}_6 = q^{-1}a_6, \tilde{a}_i = a_i$ otherwise. This, in view of Remark 3.1.2, gives us the translation

$$\phi_* : \alpha = \langle a_0, a_1, a_2, a_3, a_4, a_5, a_6 \rangle \mapsto \varpi = \alpha + (0, 0, -1, 0, 0, 0, 0)\delta,$$

on the symmetry sub-lattice which, when written in terms of generators of $\hat{W} \left( E_6^{(1)} \right)$, becomes

$$\phi_* = r w_2 w_3 w_1 w_2 w_6 w_3 w_4 w_0 w_6 w_3 w_5 w_4 w_2 w_3 w_1 w_2,$$

where $r = (\delta_0 \delta_1 \delta_2) = (a_0 a_3 a_1) (a_2 a_6 a_4)$ is a Dynkin diagram automorphism corresponding to the counterclockwise rotation of the diagram. The resulting dynamics, written in the affine chart $(f, g)$, is given by equations (8.8) in Section 8.1.3 of [KNY17]:

$$\begin{cases}
\frac{(fg - 1)(fg - 1)}{ff} = \left( g - \frac{1}{\nu_1} \right) \left( g - \frac{1}{\nu_2} \right) \left( g - \frac{1}{\nu_3} \right) \left( g - \frac{1}{\nu_4} \right) \\
\frac{(fg - 1)(fg - 1)}{gg} = \left( f - \frac{1}{\nu_1} \right) \left( f - \frac{1}{\nu_2} \right) \left( f - \frac{1}{\nu_3} \right) \left( f - \frac{1}{\nu_4} \right)
\end{cases}.$$
4.2. Moduli space for the $q$-Hahn connections. As shown in [Kni16], the $q$-Hahn case corresponds to the moduli space of $q$-connections of type $\lambda = (z_1, \ldots, z_6; u, qv, w, w; 3)$ on the $\mathcal{O} \oplus \mathcal{O}(-1)$ bundle over $\mathbb{P}^1$. (We write here $u$ in bold to distinguish it from the parameter $u$ in the context of the present paper). After a trivialization, a generic connection of this type is represented by a matrix $A(z)$ that has the following form:

$$A(z) = \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix}, \quad A(0) = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix},$$

where $\deg(a_{11}) \leq 3$, $\deg(a_{12}) \leq 2$, $\deg(a_{21}) \leq 2$, $\deg(a_{22}) \leq 3$ and

$$\det A(z) = uw(z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6).$$

We also impose the following asymptotic conditions:

$$\det \det(S^{-1}(qz)A(z)S(z)) = qwz^6 + \mathcal{O}(z^5) \quad \text{tr}(S^{-1}(qz)A(z)S(z)) = (u + qv)z^3 + \mathcal{O}(z^2),$$

where $S(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}$ gives the trivialization of the bundle in the neighborhood of $z = \infty$. We consider these matrices modulo gauge transformations of the form $\hat{A}(z) = R(qz)A(z)R^{-1}(z)$, where the gauge matrix $R(z)$ has the form

$$R(z) = \begin{bmatrix} r_{11}(z) & r_{12}(z) \\ 0 & r_{22}(z) \end{bmatrix}, \quad \deg(r_{11}) \leq 1, \deg(r_{12}) \leq 2, \deg(r_{22}) \leq 1.$$ 

The isomonodromic dynamic $A(z) \to \hat{A}(z)$ that we consider corresponds to the following parameter evolution:

$$(z_1, z_2, \ldots, z_6, u, v, w, \rho) \to (\tau_1, \tau_2, \ldots, \tau_6, u, v, w) = (z_1, qz_2, z_3, qz_4, z_5, z_6, u, v, qw).$$

Let us now explicitly describe the moduli space of $q$-Hahn connections of type $\lambda = (z_1, \ldots, z_6; u, qv, w, w; 3)$. After gauging we can put $a_{21}(z) = z(z - t)$, where $t = t_0/t_1$ is our first spectral coordinate. The second spectral coordinate we adjust slightly and put

$$p = \frac{p_0}{p_1} = \frac{z_1z_3z_5a_{11}(t)}{(t - z_1)(t - z_3)(t - z_5)}.$$ 

If we just use $p = a_{11}(t)$, we get singular points $(z_i, 0)$ that results in a $-6$ curve that appears after we resolve the singularities of the parameterization using blowup, the above change of variables results in two $-3$-curves that are easier to handle. In the coordinates $(t, p)$ we get the following base points:

$$\pi_i(z_i, 0), \quad i = 1, 2, 3; \quad \pi_i(z_i, \infty), \quad i = 4, 5, 6; \quad \pi_7(\infty, \rho_1 = \frac{w^2}{vz_1z_2z_3}), \quad \pi_8(\infty, \rho_2 = \frac{vz_4z_5z_6}{q}); \quad \pi_9(0, w).$$

This gives us the point configuration shown on Figure 9 on the right. Note that the resulting surface is $q$-Hahn surface is again not minimal and requires blowing down the $-1$-curve $t = 0$. This follows from the properties of the matrix and the nature of the parametrization: for $t = 0$ we will always have $p = w$. To match it with the standard $A_2^{(1)}$-surface, is easier to first blow up the point $\pi_9(\infty, 0)$ in the standard $(f, g)$-coordinates and establishing the identification on the level of Picard lattices, and then extending it to the birational change of coordinates.

Using the same techniques as before we get the following change of basis on the level of the Picard lattice,

$$\mathcal{H}_f = \mathcal{H}_t, \quad \mathcal{F}_1 = \mathcal{E}_1, \quad \mathcal{F}_3 = \mathcal{E}_3, \quad \mathcal{F}_5 = \mathcal{E}_7, \quad \mathcal{F}_7 = \mathcal{E}_2, \quad \mathcal{F}_9 = \mathcal{H}_t - \mathcal{E}_9, \quad \mathcal{H}_g = \mathcal{H}_t + \mathcal{H}_t - \mathcal{E}_6 - \mathcal{E}_9, \quad \mathcal{F}_3 = \mathcal{E}_4, \quad \mathcal{F}_6 = \mathcal{E}_8, \quad \mathcal{F}_8 = \mathcal{E}_4,$$

as well as the corresponding change of variables

$$(4.4) \quad f = \frac{1}{t}, \quad g = \frac{twz_6}{z_6(p - w) + tw}.$$ 

The resulting parameter matching is

$$k_1 = \frac{1}{w}, \quad \nu_1 = \frac{1}{z_1}, \quad \nu_3 = \frac{1}{z_3}, \quad \nu_5 = \rho_1 z_6, \quad \nu_7 = \frac{z_2}{w}, \quad k_2 = w, \quad \nu_2 = \frac{1}{z_6}, \quad \nu_4 = \frac{1}{z_5}, \quad \nu_6 = \rho_2 z_6, \quad \nu_8 = \frac{z_4}{w},$$

as well as the corresponding change of variables

$$(4.4) \quad f = \frac{1}{t}, \quad g = \frac{twz_6}{z_6(p - w) + tw}.$$ 

The resulting parameter matching is

$$k_1 = \frac{1}{w}, \quad \nu_1 = \frac{1}{z_1}, \quad \nu_3 = \frac{1}{z_3}, \quad \nu_5 = \rho_1 z_6, \quad \nu_7 = \frac{z_2}{w}, \quad k_2 = w, \quad \nu_2 = \frac{1}{z_6}, \quad \nu_4 = \frac{1}{z_5}, \quad \nu_6 = \rho_2 z_6, \quad \nu_8 = \frac{z_4}{w},$$

as well as the corresponding change of variables
Proposition 5.0.1.

\( A_{s+1}(z) = R_s(q^{-1}z)A_s(z)R_s^{-1}(z), \) where \( R_s(z) = I + \frac{T_s}{\sigma(z) - \sigma(q^{-1})}. \)

First, recall that our nilpotent matrix \( T_s \) has the form (2.12),

\[
T_s = \begin{bmatrix}
\frac{t^{11}}{t_s} & \frac{t^{12}}{t_s} \\
-t^{21}_s & t^{11}_s
\end{bmatrix} = \lambda v v_1^t, \quad \text{where} \quad v \in \text{Ker}(T_s) = \text{Span}\left\{ \begin{bmatrix} t^{11} \\ t^{21} \end{bmatrix} \right\}, \quad v_1 \in \text{Ker}(T_s^t) = \text{Span}\left\{ \begin{bmatrix} t^{11}_s \\ t^{12}_s \end{bmatrix} \right\}.
\]

Fix vectors \( v \) and \( v_1 \). Then take \( v_2 \notin \text{Span}\{v\} \), note that \( v_1^tv_2 \neq 0 \). Then \( T_s v_2 \in \text{Span}\{v\} \), so after rescaling we can pick a vector \( v_2 \) so that \( T_s v_2 = v \); such \( v_2 \) is unique up to adding a multiple of \( v \in \text{Ker}(T_s) \). Then

\[
\lambda = (v_1^tv_2)^{-1} \quad \text{and} \quad T_s = \frac{vv_1^t}{v_1^tv_2}.
\]

Proposition 5.0.1. In this parameterization of \( T_s \), if we take

\[
v = \begin{bmatrix} m^{11}_s(\pi_s) \\ m^{21}_s(\pi_s) \end{bmatrix},
\]

expression (2.12) for gap probabilities takes the form

\[
\frac{D_{s+1}}{D_s} = \omega(s) \det[v,v_2].
\]

Proof. We can take \( v_2 = \mu \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). We know from (2.16) that

\[
T_s v = \begin{bmatrix} t^{11}_s m^{11}_s(\pi_s) + t^{12}_s m^{21}_s(\pi_s) \\ t^{21}_s m^{11}_s(\pi_s) - t^{11}_s m^{21}_s(\pi_s) \end{bmatrix} = 0, \quad T_s v_2 = \mu \begin{bmatrix} t^{11}_s \\ t^{21}_s \end{bmatrix} = \begin{bmatrix} m^{11}_s(\pi_s) \\ m^{21}_s(\pi_s) \end{bmatrix}, \quad \text{and so} \quad \mu = \frac{m^{11}_s(\pi_s)}{t^{11}_s}.
\]
Thus,

$$\det[v, v_2] = -\mu m_s^{21}(\pi_s) = \frac{m_{s1}^{11}(\pi_s)}{t_s^{11}} \cdot \frac{t_{s2}^{11}(\pi_s)}{t_s^{12}} = \frac{1}{\omega(s)} \cdot \frac{D_{s+1}}{D_s}. $$

The computation is the same as in 2.2.5

Consider now the singularity structure of matrices in the Lax Pair (5.1). Since

$$R_s(z) = I + \frac{T_s}{\sigma(z) - \sigma(q^{-s})} = I + T_s(z - q^{-s})(z - u^2q^{-s-1}) = I + \frac{T_s}{q^{-s} - u^2q^{-s-1}} \left( \frac{q^{-s}}{z - q^{-s}} - \frac{u^2q^{-s-1}}{z - u^2q^{-s-1}} \right),$$

we see that $R_s(z)$ and $R^{-1}(z)$ have simple poles at $z = q^{-s}$ and $z = u^2q^{-s-1}$, $R_s(q^{-1}z)$ and $R^{-1}(q^{-1}z)$ have simple poles at $z = q^{-s+1}$ and $z = u^2q^s$, $A_s(z)$ and $A_{s+1}(z)$ share simple poles at $z_3$, $u^2/z_3$, $z_5$, $u^2/z_5$, $A_s^{-1}(z)$ and $A_{s+1}^{-1}(z)$ share simple poles at $u^2/z_3$, $z_4$, $u^2/z_5$, $z_6$. Finally, $A_s(z)$ and $A_{s+1}^{-1}(z)$ have simple poles at $z_1(s) = q^{-s+1}$ and $u^2/z_2(s) = u^2q^{s-1}$ and $A_{s+1}(z)$ and $A_{s+1}^{-1}(z)$ have simple poles at $z_1(s + 1) = q^{-s}$ and $u^2/z_2(s + 1) = u^2q^s$. Thus,

$$A_s(z) = R_s(q^{-1}z)H_s(z), \quad A_{s+1}^{-1}(z) = H_s^{-1}(z)R_s^{-1}(q^{-1}z),$$

where $H_s(z) = A_{s+1}(z)R_s(z)$ and both $H_s(z)$ and $H_s^{-1}(z)$ are regular at $q^{-s+1}$. Thus,

$$\text{Res}_{z=q^{-s+1}} A_s(z) = \frac{vv_1H_s(q^{-s+1})q^{-s+1}}{(v_1v)v(q^{-s} - u^2q^{s-1})}, \quad \text{Im} \left( \text{Res}_{z=q^{-s+1}} A_s(z) \right) = \text{Span}\{v\},$$

and we can take any $v \in \text{Im} \left( \text{Res}_{z=q^{-s+1}} A_s(z) \right)$. Similarly, $v_1 \in \text{Im} \left( \text{Res}_{z=q^{-s+1}} (A_s^{-1}(z))^t \right)$. To find $v_2$, observe that if we impose the condition $T_sv_2 = v$ then $v_2$ is characterized by

$$R_s(q^{-1}z) \left( \frac{(z - q^{-s+1})(q^{-s} - u^2q^{s-1})}{q^{-s+1}} v_2 \right) = v + O(z - q^{-s+1}).$$

Thus,

$$\lim_{z \to q^{-s+1}} R_s(q^{-1}z) \left( v - \frac{(z - q^{-s+1})(q^{-s} - u^2q^{s-1})}{q^{-s+1}} v_2 \right) = 0,$$

and since $H_s^{-1}(z)$ is regular at $z = q^{-s+1}$,

$$\lim_{z \to q^{-s+1}} A_s^{-1}(z) \left( v - \frac{(z - q^{-s+1})(q^{-s} - u^2q^{s-1})}{q^{-s+1}} v_2 \right) = 0.$$

We now show how, given the triple $(v, v_1, v_2)$ for $A_s(z)$, to compute the triple $(\hat{v}, \hat{v}_1, \hat{v}_2)$ for $A_{s+1}(z)$. Since

$$\hat{v} \in \text{Im} \left( \text{Res}_{z=q^{-s}} A_{s+1}(z) \right) = \text{Span}\{R_s(q^{-s-1})A_s(q^{-s})v\},$$

we can take $\hat{v} = R_s(q^{-s-1})A_s(q^{-s})v$. Similarly, $\hat{v}_1 = v_1A_s^{-1}(q^{-s})R_s^{-1}(q^{-s-1})$.

Finally, to find $v_2$ we will solve

$$\lim_{z \to q^{-s}} A_{s+1}^{-1}(z) \left( \hat{v} - \frac{(z - q^{-s})(q^{-s-1} - u^2q^s)}{q^{-s}} \hat{v}_2 \right) = 0.$$

Since $A_{s+1}^{-1}(z)R_s^{-1}(q^{-1}z)$ is regular at $z = q^{-s}$ we can replace it in the expression for $A_{s+1}^{-1}(z)$ with the series expansion near $z = q^{-s}$ to get

$$A_{s+1}^{-1}(z) = \left( I + \frac{T_s}{q^{-s} - u^2q^{-s-1}} \left( \frac{q^{-s}}{z - q^{-s}} - \frac{u^2q^{-1}}{z - u^2q^{s-1}} \right) \right) \cdot \left( A_{s+1}^{-1}(q^{-s})R_{s+1}^{-1}(q^{-s-1}) + \frac{d(A_{s+1}(z)R_{s+1}^{-1}(q^{-1}z))}{dz} \bigg|_{z=q^{-s}} \cdot (z - q^{-s}) + O(z - q^{-s})^2 \right).$$
Thus,
\[
\lim_{z \to q^{-s}} A_{s+1}^{-1}(z) \left( \hat{v} - \frac{(z - q^{-s})(q^{-s-1} - u^2 q^s)}{q^{-s}} \right) = v^+
\]
\[
\frac{q^{-s} T_s}{q^{-s} - u^2 q^s - 1} \left( \frac{d(A_{s}^{-1}(z) R_s^{-1}(q^{-1}z))}{dz} \right)_{z=q^{-s}} \hat{v} - \frac{q^{-s-1} - u^2 q^s}{q^{-s}} A_{s}^{-1}(q^{-s}) R_s^{-1}(q^{-s-1}) \hat{v}_2 = 0.
\]
Since \( v = T_s v_2 \), we can now solve for \( \hat{v}_2 \) (again, modulo adding a vector in the kernel of \( T_s \)):
\[
\frac{q^{-s}}{q^{-s} - u^2 q^s - 1} \left( \frac{d(A_{s}^{-1}(z) R_s^{-1}(q^{-1}z))}{dz} \right)_{z=q^{-s}} \hat{v} - \frac{q^{-s-1} - u^2 q^s}{q^{-s}} A_{s}^{-1}(q^{-s}) R_s^{-1}(q^{-s-1}) \hat{v}_2 = -v_2,
\]
or
\[
\hat{v}_2 = R_s \left( q^{-s-1} \right) A_s(\sigma(q^{-s})) \left( q^{-s} \frac{d(A_{s}^{-1}(z) R_s^{-1}(q^{-1}z))}{dz} \right)_{z=q^{-s}} \hat{v} + (q^{-s} - u^2 q^s - 1)v_2.
\]
We are now ready to prove the following result.

**Proposition 5.0.2.** Gap probabilities can be computed with the help of the following recursion:

\[
D_{s+2} D_s \frac{D_{s+1}}{D_{s+1}^2} = \omega(s + 1) = \omega(s) + \left( \Phi^-(q^{-s}) \right)^2 \det[\hat{v}, \hat{v}_2] \det[v, v_2].
\]

**Proof.** Let \( v = \begin{bmatrix} m_{s+1}^{11}(\pi_s) \\ m_{s+1}^{12}(\pi_s) \end{bmatrix} \). From (2.12) and (2.23) we see that

\[
m_{s+1}(\sigma(q^{-1}q^{-s})) = R_s(\sigma(q^{-1}q^{-s})) m_s(\sigma(q^{-1}q^{-s})) = R(q^{-s-1}) A_s(\sigma(q^{-s})) m_s(\sigma(q^{-s})) D^{-1}(q^{-s}).
\]

Multiplying on the right by \( D(q^{-s}) \) and looking at the first column, we get

\[
\begin{bmatrix} m_{s+1}^{11}(\pi_{s+1}) \\ m_{s+1}^{12}(\pi_{s+1}) \end{bmatrix} = R_s(q^{-s-1}) A_s(q^{-s}) \begin{bmatrix} m_{s+1}^{11}(\pi_s) \\ m_{s+1}^{12}(\pi_s) \end{bmatrix} = R_s(q^{-s-1}) A_s(q^{-s}) v = \hat{v}.
\]

In view of linearity, \( \hat{v}_2 \) scales in the same way as \( \hat{v} \), and so by Proposition 5.0.1, we get

\[
\det[\hat{v}, \hat{v}_2] = \left( \frac{\Phi^+(q^{-s})}{\Phi^-(q^{-s})} \right)^2 \frac{1}{\omega(s + 1)} \frac{D_{s+2}}{D_{s+1}^2}.
\]

Thus,
\[
\frac{D_{s+2} D_s}{D_{s+1}^2} = \omega(s + 1) = \omega(s) + \left( \frac{\Phi^+(q^{-s})}{\Phi^-(q^{-s})} \right)^2 \det[\hat{v}, \hat{v}_2] \det[v, v_2],
\]
where the final equation is now independent of the choice of the initial scaling for \( v \). \( \square \)

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