THE X-RAY TRANSFORM ON A GENERAL FAMILY OF CURVES ON FINSLER SURFACES

YERNAT M. ASSYLBEKOV AND NURLAN S. DAIRBEKOV

ABSTRACT. We consider a general family of curves Γ on a compact oriented Finsler surface (M, F) with boundary ∂M. Let φ ∈ C∞(M) and ω a smooth 1-form on M. We show that

\[ \int_{\gamma(t)} \{ \varphi(\gamma(t)) + \omega_{\gamma(t)}(\dot{\gamma}(t)) \} \, dt = 0 \]

holds for every γ ∈ Γ whose endpoints belong to ∂M, γ(a) ∈ ∂M, γ(b) ∈ ∂M if and only if φ = 0 and ω is exact.

Similar results were proved when M is closed and some additional conditions on Gaussian curvature are imposed.

We also study the cohomological equations of Anosov generalized thermostats on a closed Finsler surface. Finally, we gave conditions when thermostat is of Anosov type.

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1. **General family of curves on a Finsler surface with boundary**

1.1. **Main definitions and statement of the result.** Given a compact oriented Finsler surface \((M, F)\) with boundary \(\partial M\). \(SM\) is the unit sphere bundle of \((M, F)\).

Let \(\Gamma\) be a family of smooth directed curves \(\gamma\) in \(M\) (constant curves are excluded), which satisfy the following conditions:

1. All curves in \(\Gamma\) are parametrized by arclength with respect to \(F\).
2. For every point \(x \in M\) and every vector \(\xi \in S_xM\), there is at most one curve \(\gamma \in \Gamma\), up to a shift of the parameter, passing through \(x\) in the direction \(\xi\). If such a curve exists, we denote by \(\gamma_{x,\xi}\) this curve with its parameter shifted so that
   \[ \gamma_{x,\xi}(0) = x, \quad \dot{\gamma}_{x,\xi} = \xi. \]
3. For every point \(x \in M^{\text{int}} := M \setminus \partial M\) and every vector \(\xi \in S_xM\) such a curve \(\gamma_{x,\xi}(t)\) exists and hits the boundary for some \(t > 0\) as well as for some \(t < 0\).
4. None of the curves in \(\Gamma\) is defined over the whole real axis.
5. The map \((x, \xi, t) \mapsto \gamma_{x,\xi}(t)\) is smooth, and for every \(x \in M\) the map
   \[ \exp_x^\Gamma (t\xi) := \gamma_{x,\xi}(t) \]
   (which is \(C^1\)-smooth) is a local diffeomorphism. In this case we say that \((M, F, \Gamma)\) has no conjugate points.

If these conditions are satisfied \(\Gamma\) is called *general* family of curves. We will call any subcurve of a curve in \(\Gamma\) a \(\Gamma\)-geodesic. The curves in \(\Gamma\) are then *maximal \(\Gamma\)-geodesics.*

In [19] and [10] \(\Gamma\) is called *path space.* R. G. Mukhometov [12, 13] and V. A. Sharafutdinov [17, Chapter 1] called \(\Gamma\) *regular family of curves.*

Let \(\varphi\) be a smooth function, and \(\omega\) a smooth 1-form on \(M\). Define the X-ray transform of the pair \([\varphi, \omega]\)

\[ I_\Gamma[\varphi, \omega](\gamma) = \int_\gamma \{ \varphi(\gamma(t)) + \omega(\gamma(t)) (\dot{\gamma}(t)) \} \, dt, \quad \gamma \in \Gamma. \]

We say that the pair \([\varphi, \omega]\) *integrates to zero over \(\Gamma\)-geodesics with endpoints on the boundary* if the equality

\[ I_\Gamma[\varphi, \omega](\gamma) = 0 \]

holds for every \(\Gamma\)-geodesic \(\gamma \in \Gamma\).

A natural question of integral geometry arises: when pair \([\varphi, \omega]\) *integrates to zero over \(\Gamma\)-geodesics?* In recent paper we will give answer for this question. Here is one of our main results:

**Theorem A.** If \((M, F)\) is a compact oriented Finsler surface with boundary \(\partial M\) and \(\Gamma\) is a general family of curves, then any pair \([\varphi, \omega]\) integrates to zero over \(\Gamma\)-geodesics with endpoints on the boundary if and only if \(\varphi = 0\) and \(\omega = d\psi\) for some smooth function \(\psi\) on \(M\) which vanishes on \(\partial M\), \(\psi|_{\partial M} = 0\).

In [13] R. G. Mukhometov proved Theorem A in the case when \(M \in \mathbb{R}^2\) is a simply connected domain with boundary, \(\omega = 0\) and \(\Gamma\) is a general family of curves such that
for any two points \( x, y \in M \), there is unique curve in \( \Gamma \) connecting them that depends smoothly on its endpoints. Till now there is no such known result for general family of curves.

On the other hand, if \( \Gamma \) is analytic and \( \omega = 0 \) Theorem A1 was proved in [6] for the case \( \dim(M) \geq 2 \).

1.2. Generalized isokinetic thermostat. We consider a generalized isokinetic thermostat. This consists of a semibasic vector field \( E(x, \xi) \), that is, a smooth map \( TM \ni (x, \xi) \mapsto E(x, \xi) \in TM \) such that \( E(x, \xi) \in T_xM \) for all \( (x, \xi) \in TM \). The equation

\[
\frac{D\dot{\gamma}}{dt} = E(\gamma, \dot{\gamma}) - \frac{\langle E(\gamma, \dot{\gamma}), \dot{\gamma} \rangle}{\dot{\gamma}^2} \dot{\gamma},
\]

defines a flow \( \phi_t \) on the unit sphere bundle \( SM \). Here inner product \( \langle \cdot, \cdot \rangle_\xi \) and norm \( |\cdot|_\xi = F(\xi) \) are all taken with respect to the fundamental tensor in Finsler geometry:

\[
g_{ij}(x, \xi) = \frac{1}{2} [F^2]^{ij}(x, \xi).
\]

These generalized thermostats are no longer reversible unless \( E(x, \xi) = E(x, -\xi) \).

We can write

\[
E(x, \xi) = \kappa(x, \xi)v + \lambda(x, \xi)i\xi
\]

where \( i \) indicates rotation by \( \pi/2 \) according to the orientation of the surface and \( \kappa \) and \( \lambda \) are smooth functions. The evolution of the thermostat on \( SM \) can now be written as

\[
(2) \quad \frac{D\dot{\gamma}}{dt} = \lambda(\gamma, \dot{\gamma})i\dot{\gamma}.
\]

We will refer to a curve \( \gamma \) as a \( \lambda \)-geodesic. Given \( x \in M \), \( \xi \in S_xM \), we denote by \( \gamma_{x,\xi} \) the maximal \( \lambda \)-geodesic with initial data \( \gamma_{x,\xi}(0) = x \), \( \dot{\gamma}_{x,\xi}(0) = \xi \).

Suppose now that \( M \) has boundary. We say that the thermostat is nontrapping if for every \( x \in M^{int} \) and \( \xi \in S_xM \) the \( \lambda \)-geodesic \( \gamma_{x,\xi}(t) \) hits the boundary for some \( t > 0 \) as well as for some \( t < 0 \), and if there is no \( \lambda \)-geodesic defined over the whole real axis. For \( x \in M \), we define the exponential map

\[
\exp^\lambda_x(t\xi) = \gamma_{x,\xi}(t), \quad t \geq 0, \quad \xi \in S_xM,
\]

which is a \( C^1 \)-smooth.

**Theorem 1.1.** Every general family of curves \( \Gamma \) on a compact Finsler surface \( (M, F) \) with boundary can be represented as the family of \( \lambda \)-geodesics of a nontrapping generalized isokinetic thermostat without conjugate points.

**Proof.** Given a general family of curves \( \Gamma \), we define

\[
\lambda(x, \xi) = \left\langle \frac{D\dot{\gamma}_{x,\xi}(t)}{dt}, i\dot{\gamma}_{x,\xi}(t) \right\rangle_{\gamma_{x,\xi}(t)}.
\]

Since there is at most one curve \( \gamma \in \Gamma \), up to a shift of the parameter, passing through \( x \) in the direction \( \xi \) for every point \( x \in M \) and every vector \( \xi \in S_xM \), function \( \lambda \) does
not depend on $t$. Then $\Gamma$ becomes a family of $\lambda$-geodesics of the thermostat $(M,F,\lambda)$. The regularity conditions of $\Gamma$ easily imply that the thermostat has no conjugate points and item (3) of definition of $\Gamma$ implies that the thermostat is nontrapping. □

Then Theorem A can be written in terms of thermostats:

**Theorem A.** Let $(M,F,\lambda)$ be a generalized thermostat on an oriented compact Finsler surface $M$ with boundary $\partial M$. If it is nontrapping and has no conjugate points, then any pair $[\varphi,\omega]$ integrates to zero over $\lambda$-geodesics with endpoints on the boundary if and only if $\varphi = 0$ and $\omega = d\psi$ for a smooth function $\psi$ on $M$ which vanishes on $\partial M$, $\psi|_{\partial M} = 0$.

### 1.3. Canonical coframing

A smooth Finsler structure on $M$ is a smooth hypersurface $SM \subset TM$ for which the canonical projection $\pi : SM \to M$ is a surjective submersion having the property that for each $x \in M$, the $\pi$-fibre $\pi^{-1}(x) = SM \cap T_x M$ is a smooth, closed, strictly convex curve enclosing the origin $0_x \in T_x M$.

Given such a structure it is possible to define a canonical coframing $(\omega_1, \omega_2, \omega_3)$ on $SM$ that satisfies the following structural equations (see [1, Chapter 4]):

\begin{align*}
(3) & \quad d\omega_1 = -\omega_2 \wedge \omega_3, \\
(4) & \quad d\omega_2 = -\omega_3 \wedge (\omega_1 - I\omega_2), \\
(5) & \quad d\omega_3 = -(K\omega_1 - J\omega_3) \wedge \omega_2.
\end{align*}

where $I$, $K$ and $J$ are smooth functions on $SM$. The function $I$ is called the main scalar of the structure. When the Finsler structure is Riemannian, $K$ is the Gaussian curvature.

The form $\omega_1$ is the canonical contact form of $SM$ whose Reeb vector field is the geodesic vector field $X$.

Consider the vector fields $(X,H,V)$ dual to $(\omega_1, \omega_2, \omega_3)$. As a consequence of (3-5) they satisfy the commutation relations

\begin{align*}
(6) & \quad [V,X] = H, \quad [H,V] = X + IH + JV, \quad [X,H] = KV.
\end{align*}

### 1.4. Preparations

We will use the following consequence of [16, Lemma 2.2], which shows that a certain correction can be added to $q(x,\xi) = \varphi(x) + \omega_x(\xi)$ to make it vanish on $\partial M$.

**Lemma 1.2.** Let $g$ be a Riemannian metric on $M$. For every a smooth 1-form $\omega$, there is a function $\psi \in C^\infty_0(M)$ such that

$$d_x \psi(\nu) = \omega_x(\nu)$$

for all $x \in \partial M$ and every vector $\nu \in T_x M$ orthogonal to $\partial M$ with respect to $g$.

Let $\nu(x)$ be an inward unit normal vector field to $\partial M$, i.e. $\nu \in T_x M$, $x \in \partial M$ such that $g_{\nu(x)}(\nu(x), \xi) = 0$ for all $\xi \in T_x \partial M$.

We now construct Riemannian metric in Lemma [12] as follows: restrict fundamental tensor $g_{ij}$ of Finsler metric $F$ to a vector field $\tilde{\nu}(x)$ on $M$ such that $\tilde{\nu}|_{\partial M} = \nu$.
Write
\[ \tilde{q}(x, \xi) := q(x, \xi) - d_x \psi(\xi). \]
Then \( \tilde{q}(x, \nu) = \varphi(x), \) \( x \in \partial M. \)

**Lemma 1.3.** If \( \tilde{q} \in C^\infty(SM) \), a function that was defined above, integrates to zero over \( \lambda \)-geodesics with endpoints on the boundary, i.e.,
\[ \int_a^b \tilde{q}(\gamma(t), \dot{\gamma}(t)) \, dt = 0 \]
for every \( \lambda \)-geodesic \( \gamma \) with endpoints on \( \partial M \), then
\[ \tilde{q}|_{S(\partial M)} = 0. \]

**Proof.** See Appendix A.2. \( \square \)

Further, losing no generality, we assume that \( (M, F) \) is a smooth subset of a compact smooth Finsler surface \( N \) without boundary and extend Finsler metric from \( M \) to \( N \). We preserve the notation \( F \) for the Finsler metric on \( N \). Now, we extend \( q(x, \xi) \) from \( M \) to all of \( N \) by the zero, denoting it again by \( q(x, \xi) \). The boundary condition \( S \) guarantees that the so-obtained \( q(x, \xi) \) is continuous on the whole \( N \) and contains in \( H^1(SN) \).

Henceforth \( X, H \) and \( V \) are the same vector fields on \( SN \) as in 1.3. Let \( \lambda \) be the smooth function on \( SN \) given by (2), and let
\[ F = X + \lambda V \]
be the generating vector field of the generalized thermostat. If \( V(\lambda) = -\lambda I \), then \( \phi \) is magnetic flow.

From (6) we obtain:
\[ [V, F] = H + V(\lambda)V, \]
\[ [H, V] = F + IH + (J - \lambda)V, \]
\[ [F, H] = \{ K - H(\lambda) - \lambda J + \lambda^2 \}V - \lambda F - \lambda IH. \]

Given \( (x, \xi) \in SM \), let \( \gamma_{x,\xi} \) be the complete \( \lambda \)-geodesic in \( N \) issuing from \( (x, \xi) \), \( \dot{\gamma}_{x,\xi}(0) = \xi \). Since \( (M, F) \) is nontrapping, there is no complete \( \lambda \)-geodesic which would be contained entirely in \( M \). Therefore, for any \( (x, \xi) \) there is a number \( l_{x,\xi} \) such that \( \gamma_{x,\xi}(l_{x,\xi}) \notin M \). We define a function \( \chi(x, \xi) : SM \to \mathbb{R} \) to be
\[ \chi(x, \xi) = \int_{l_{x,\xi}}^0 q(\gamma(t), \dot{\gamma}(t)) \, dt. \]

Note that the function \( \chi(x, \xi) \) is independent of the choice of \( l_{x,\xi} \). It follows from \( \Omega \) and that \( q \) vanishes on \( \partial M \) and in the exterior of \( M \).

Call a point \( (x, \xi) \in SM \) **regular** if the \( \lambda \)-geodesic \( \gamma_{x,\xi} \) intersects \( \partial M \) transversally from either side and the open segment of \( \gamma_{x,\xi} \) between the basepoint \( x \) and the point of intersection lies entirely in \( M^{\text{int}} \). We denote by \( RM \subset SM \) the set of all regular points. It is clear that \( RM \) is open in \( SM \) and has full measure in \( SM \).
Lemma 1.4. The function $\chi$ has the following properties:

1. $\chi|_{S(N\setminus M)} = 0$.
2. $\chi \in H^1(SN) \cap C(SN) \cap C^\infty(RM)$.
3. $\chi$ is $C^1$ smooth along the lifts of $\lambda$-geodesic to $SM$ and satisfies
   
   $$F\chi(x, v) = q(x, v)$$

   on $SM$.

Since proof of the lemma repeats the same arguments as in [2, Lemma 2.3] will skip it.

1.5. Pestov identity.

Lemma 1.5 (Pestov identity). For every smooth function $u : SN \to \mathbb{R}$ we have

$$2Hu \cdot VFu = (Fu)^2 + (Hu)^2 - (K - H(\lambda) - \lambda J + \lambda^2)(Vu)^2$$
$$+ F(Hu \cdot Vu) - H(Vu \cdot Fu) + V(Hu \cdot Fu)$$
$$+ Fu \cdot (IHu + JVu) + Hu \cdot Vu \cdot (\lambda I + V(\lambda)).$$

Proof. Using the commutation formulas, we deduce:

$$2Hu \cdot VFu - V(Hu \cdot Fu)$$
$$= Hu \cdot VFu - VHu \cdot Fu$$
$$= Hu \cdot (VFu + [V, F]u) - Fu \cdot (HVu + [V, H]u)$$
$$= Hu \cdot (VFu + Hu + V(\lambda)Vu) - Fu \cdot (HVu - Fu - IHu - (J - \lambda)Vu)$$
$$= (Hu)^2 + (Fu)^2 + Hu \cdot VFu - HVu \cdot Fu + IFu \cdot Hu + (J - \lambda)Fu \cdot Vu$$
$$+ V(\lambda)Hu \cdot Vu$$
$$= (Hu)^2 + (Fu)^2 + F(Hu \cdot Vu) - H(Vu \cdot Fu) - [F, H]u \cdot Vu$$
$$+ IFu \cdot Hu + (J - \lambda)Fu \cdot Vu + V(\lambda)Hu \cdot Vu$$
$$= (Hu)^2 + (Fu)^2 + F(Hu \cdot Vu) - H(Vu \cdot Fu)$$
$$+ (-K - H(\lambda) - \lambda J + \lambda^2)Vu + \lambda Fu + \lambda Hu \cdot Vu$$
$$+ IFu \cdot Hu + (J - \lambda)Fu \cdot Vu + V(\lambda)Hu \cdot Vu$$
$$= (Hu)^2 + (Fu)^2 + F(Hu \cdot Vu) - H(Vu \cdot Fu)$$
$$- (K - H(\lambda) - \lambda J + \lambda^2)(Vu)^2 + IFu \cdot Hu + JFu \cdot Vu$$
$$+ (\lambda I + V(\lambda))Hu \cdot Vu$$

which is equivalent to the Pestov identity. \[\square\]

We will use the following fact. Let $X$ be a vector field and $\Theta$ a volume form. Then

$$L_X \Theta = d(i_X \Theta) + i_X d\Theta.$$  

Now let $\Theta := \omega_1 \wedge \omega_2 \wedge \omega_3$. This volume form gives rise to the Liouville measure $d\mu$ of $SN$. 


Lemma 1.6. We have:

\begin{align}
L_F \Theta &= (\lambda I + V(\lambda)) \Theta, \\
L_H \Theta &= -J \Theta, \\
L_V \Theta &= I \Theta.
\end{align}

Proof. Using (3-5), \( L_X \Theta = (i_X \Theta) = d(\omega_2 \wedge \omega_3) = \omega_2 \wedge \omega_3 - \omega_2 \wedge d\omega_3 = 0 \). Since \( F = X + \lambda V \), we get 
\[ L_F \Theta = L_X \Theta + L_{\lambda V} \Theta = d(\lambda V \Theta) = -d(\lambda \omega_2 \wedge \omega_1) = (\lambda I + V(\lambda)) \Theta. \]

Similarly, \( L_H \Theta = -J \Theta, L_V \Theta = I \Theta. \)

1.6. First integral identity. Below we will use the following consequence of Stokes theorem. Let \( W \) be a compact oriented manifold with boundary and \( \Theta \) a volume form. Let \( X \) be a vector field on \( W \) and \( f : W \to \mathbb{R} \) a smooth function. Then

\[ \int_W X(f) \Theta + \int_W f L_X \Theta = \int_{\partial SW} f i_V \Theta. \]

Let \( D \) be a compact subsurface of \( N \) with boundary \( \partial D \) and \( u : SD \to \mathbb{R} \) be a smooth function such that \( u|_{\partial(SD)} = 0 \). Integrate the Pestov identity, that was derived in Lemma 1.5, over \( SD \) against the Liouville measure \( d\mu \) by making use of (14) and (11)-(13):

\[ \int_{SD} 2Hu \cdot VFu \, d\mu = \int_{SD} (Fu)^2 \, d\mu + \int_{SD} (Hu)^2 \, d\mu \]
\[ - \int_{SD} \{ K - H(\lambda) - \lambda J + \lambda^2 \} (Vu)^2 \, d\mu \]
\[ + \int_{\partial(SD)} \{(Hu \cdot Vu)i_F \Theta + (Fu \cdot Hu)i_V \Theta - (Fu \cdot Vu)i_H \Theta \}. \]

Let \( \Omega_a \) be an area form on \( D \). Note that \( i_V \Theta = \omega_1 \wedge \omega_2 = \pi^* \Omega_a \) vanishes when restricted to \( \partial(SM) \). Hence,

\[ \int_{\partial(SD)} (Fu \cdot Hu)i_V \Theta = 0. \]

So we get

\[ \int_{SD} 2Hu \cdot VFu \, d\mu = \int_{SD} (Fu)^2 \, d\mu + \int_{SD} (Hu)^2 \, d\mu \]
\[ - \int_{SD} \{ K - H(\lambda) - \lambda J + \lambda^2 \} (Vu)^2 \, d\mu \]
\[ + \int_{\partial(SD)} \{(Hu \cdot Vu)i_F \Theta - (Fu \cdot Vu)i_H \Theta \}. \]

By commutation relations, we have

\[ VFu = VFu - Hu - V(\lambda)Vu. \]
Therefore,
\[(FVu)^2 = (VFu)^2 + (Hu)^2 + (V(\lambda)Vu)^2 - 2VFu \cdot Hu + 2Hu \cdot V(\lambda)Vu - 2VFu \cdot V(\lambda)Vu\]
\[= (VFu)^2 + (Hu)^2 + (V(\lambda)Vu)^2 - 2VFu \cdot Hu + 2Hu \cdot V(\lambda)Vu - 2VFu \cdot V(\lambda)Vu\]
\[= (VFu)^2 + (Hu)^2 + (V(\lambda)Vu)^2 - 2VFu \cdot Hu + 2Hu \cdot V(\lambda)Vu - 2V(\lambda)[V,F]Vu \cdot Vu - 2V(\lambda)FVu \cdot Vu\]
\[= (VFu)^2 + (Hu)^2 - (V(\lambda)Vu)^2 - 2VFu \cdot Hu - 2V(\lambda)FVu \cdot Vu.\]

Since
\[-2V(\lambda)FVu \cdot Vu = -F(V(\lambda)(Vu)^2) + (Vu)^2FV(\lambda)\]
we obtain:
\[(FVu)^2 = (VFu)^2 + (Hu)^2 - (V(\lambda)Vu)^2 - 2VFu \cdot Hu - F(V(\lambda)(Vu)^2) + (Vu)^2FV(\lambda).\]

Integrating it over $SD$, we get
\[
\int_{SD} 2Hu \cdot VFu \, d\mu = \int_{SD} (VFu)^2 \, d\mu + \int_{SD} (Hu)^2 \, d\mu - \int_{SD} (FVu)^2 \, d\mu
\]
\[+ \int_{SD} \{K - H(\lambda) - \lambda J + \lambda^2\}(Vu)^2 \, d\mu
\[- \int_{\partial(SD)} V(\lambda)(Vu)^2 i_F \Theta\]

by (14) and (11)
\[-\int_{SD} F(V(\lambda)(Vu)^2) \Theta = \int_{SD} \lambda IV(\lambda)(Vu)^2 (\Theta + \int_{SD} (V(\lambda))^2(Vu)^2 \Theta
\[- \int_{\partial(SD)} V(\lambda)(Vu)^2 i_F \Theta.\]

Combining (15) and (17), we come to first integral identity
\[
\int_{SD} (FVu)^2 \, d\mu - \int_{SD} K(Vu)^2 \, d\mu + \int_{\partial(SD)} \omega(u) = \int_{SD} (VFu)^2 \, d\mu - \int_{SD} (Fu)^2 \, d\mu,
\]
where
\[
\omega(u) := \{(Hu \cdot Vu) + V(\lambda)(Vu)^2\}i_F \Theta - (Fu \cdot Vu)i_H \Theta,
\]
and $K := K - H(\lambda) - \lambda J + \lambda^2 + \lambda IV(\lambda) + FV(\lambda)$.

In view of the boundary condition $u|_{\partial(SD)} = 0$, (18) implies
\[
\int_{SD} (FVu)^2 \, d\mu = \int_{SD} (VFu)^2 \, d\mu - \int_{SD} (Hu)^2 \, d\mu - \int_{SD} (Fu)^2 \, d\mu.
\]
Lemma 1.7. Let $D \subset \mathbb{N}$ be a surface with boundary $\partial D$. Let a function $u : SD \to \mathbb{R}$ be such that $u \in H^1(SD)$, $Fu \in H^1(SD)$, $u$ is smooth in some neighbourhood of $\partial(SD)$ in $SD$, and $u|_{\partial(SD)} = 0$. Then the integral identity (20) is valid for $u$.

Proof. If we construct a sequence of smooth functions $u_k$ coinciding with $u$ in a neighbourhood of $\partial(SD)$ and such that $u_k \to u$ in $H^1(SD)$ and $Fu_k \to Fu$ in $H^1(SD)$ as $k \to \infty$. Applying (18) to each function $u_k$ and passing to the limit as $k \to \infty$, we will come to the desired conclusion.

The construction of such an approximation is quite standard and similar as in [2, Lemma 3.1] so we omit it. □

1.7. Riccati equation. Let $\phi$ denote the flow on $SM$ defined by equation (2),
$$
\phi_t(x, \xi) = (\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t)), \quad (x, \xi) \in SM.
$$

Let $\pi : SM \to M$ be the canonical projection. For $(x, \xi) \in SM$, let
$$
\mathcal{V}(x, \xi) := \ker d\xi \pi,
$$
which a 1-dimensional subspace of $T_{x,\xi}SM$, and
$$
E(x, \xi) := \mathcal{V}(x, \xi) \oplus \mathbb{R}F(x, \xi).
$$

Lemma 1.8. If $\gamma : [0, T] \to M$ is a $\lambda$-geodesic, then
$$
d_{\dot{\gamma}(0)}\phi_t(E) \cap \mathcal{V}(\dot{\gamma}(t)) = 0
$$
for every $t \in (0, T)$.

Proof. Take $(x, \xi) \in SM$ and $t \in (0, T]$. From the definition of $\exp^\lambda$ it is straightforward that
$$
\text{image}(d\xi \exp^\lambda_x) = d_{\dot{\gamma}(t)}(d_{\dot{\gamma}(0)}\phi_t(E)).
$$
By the absence of conjugate point, $d_w \exp^\lambda_x$ is a linear isomorphism for every $w \in T_xM$ at which $\exp^\lambda_x$ is defined, and the lemma follows. □

The lemma implies that, for every $(x, \xi) \in SM$ there exist unique continuous functions $r(t) = r(x, \xi, t)$ on $SM$ such that
$$
H(\phi_t(v)) + r(t) V(\phi_t(v)) \in d_v \phi_t(E).
$$

For $(x, \xi) \in SM$ there is $t_0$ such that $x_0 = \gamma_{x,\xi}(t_0) \in N \setminus M$. If $x_0$ is close enough to $M$, $\gamma_{x_0,\xi_0}$ has no conjugate points either $(\xi_0 = \dot{\gamma}_{x_0,v}(t_0))$. Define $r$ on the orbit of $(x_0, \xi_0)$ by
$$
r(\gamma_{x_0,\xi_0}(t), \dot{\gamma}_{x_0,\xi_0}(t)) = r_{x_0,\xi_0}(t).
$$

So obtained function $r$ is smooth on every orbit and $r \in L^\infty(SM)$. Below we will need to use that the function $r$ satisfies a Riccati type equation along the flow.

Lemma 1.9. (Riccati equation). The function $r$ satisfies
$$
F(r - V(\lambda)) + r(\lambda I - V(\lambda) + r) + K - \lambda IV(\lambda) = 0.
$$
Replacing $H$ we have
$\xi(t) := d\phi_{-t}(H(t) + r(t)V(t))$.

By the definition of $r$, $\xi(t) \in E(x, v)$ for all $t$. Differentiating with respect to $t$ and setting $t = 0$ we obtain:
$$\dot{\xi}(0) = [F, H] + F(r)V + r[F, V].$$

Using that
$$[V, F] = H + V(\lambda)V, \quad [F, H] = (K - H(\lambda) - \lambda J + \lambda^2)V - \lambda F - \lambda IH$$
we have
$$\dot{\xi}(0) = -\lambda F - \lambda I\xi(0) + \{K - H(\lambda) - \lambda J + \lambda^2 + F(r) - rV(\lambda)\}V.$$ 
Replacing $H$ by $\xi(0) - rV$ yields:
$$\dot{\xi}(0) + (r + \lambda I)\xi(0) - \lambda F = \{K - H(\lambda) - \lambda J + \lambda^2 + F(r) + \lambda Ir - rV(\lambda) + r^2\}V.$$
Since $\dot{\xi}(0) + (r + \lambda I)\xi(0) - \lambda F \in E$ we must have
$$K - H(\lambda) - \lambda J + \lambda^2 + F(r) + \lambda Ir - rV(\lambda) + r^2 = 0$$
which is the desired equation since $\mathbb{K} := K - H(\lambda) - \lambda J + \lambda^2 + \lambda IV(\lambda) + FV(\lambda)$.

For the proof of Theorem A we will need the following

**Theorem 1.10.** Let $\psi : SM \to \mathbb{R}$ be a function vanishing on $\partial(SM)$ such that $\psi \in C^\infty(RM)$. Then
$$\int_{SM} (F\psi)^2 \, d\mu - \int_{SM} \mathbb{K}\psi^2 \, d\mu = \int_{SM} [F(\psi) - r\psi + \psi V(\lambda)]^2 \, d\mu \geq 0.$$ 
Moreover,
$$\int_{SM} [F(\psi) - r\psi + \psi V(\lambda)]^2 \, d\mu = 0$$
if and only if $\psi = 0$ on $RM$.

**Proof.** Let us expand $[F(\psi) - r\psi + \psi V(\lambda)]^2$:
$$[F(\psi) - r\psi + \psi V(\lambda)]^2 = [F(\psi)]^2 + \psi^2 r^2 + \psi^2 [V(\lambda)]^2 - 2F(\psi)\psi r + 2F(\psi)\psi V(\lambda) - 2\psi^2 rV(\lambda).$$
Using Lemma (1.9), we obtain:
$$[F(\psi) - r\psi + \psi V(\lambda)]^2 = [F(\psi)]^2 - \mathbb{K}\psi^2 - F((r - V(\lambda))\psi^2) + \psi^2 [V(\lambda)]^2 - \psi^2 r[\lambda I + V(\lambda)] + \lambda IV(\lambda)\psi^2.$$ 
If we integrate the last equality with respect to the measure $\mu$, we obtain as desired:
$$\int_{SM} (F\psi)^2 \, d\mu - \int_{SM} \mathbb{K}\psi^2 \, d\mu = \int_{SM} [F(\psi) - r\psi + \psi V(\lambda)]^2 \, d\mu$$
since by (14) and (11) we have
\[\int_{SM} F((r - V(\lambda))\psi^2) d\mu = \int_{SM} \{\psi^2[V(\lambda)]^2 - \psi^2 r[\lambda I + V(\lambda)] + \lambda IV(\lambda)\psi^2\} d\mu\]
\[+ \int_{\partial(SM)} (r - V(\lambda))\psi^2 iF\Theta\]
and since the last integral in (21) vanishes due to the boundary condition $\psi|_{\partial(SM)} = 0$.

Suppose now
\[\int_{SM} [F(\psi) - r\psi + \psi V(\lambda)]^2 d\mu = 0.\]
which implies
\[F(\psi) - r\psi + \psi V(\lambda) = 0\]
on $RM$. This means that, on almost every orbit, the function $\psi$ satisfies a homogeneous first-order ordinary differential equation with zero boundary data. This surely implies that $\psi \equiv 0$ on such an orbit, which yields $\psi \equiv 0$ on $RM$. \hfill \square

1.8. End of the proof of Theorem A. Let $W \subset N$ be a collar neighbourhood of $\partial M$ in $N$. This means that there is a diffeomorphism $\Psi : \partial M \times (-1, 1) \to W$ such that the restriction $\Psi|_{\partial M \times \{0\}}$ is identity. We also assume that $\Psi(\partial M \times (-1, 0)) \subset M$ and $\Psi(\partial M \times (0, 1)) \subset N \setminus M$.

Put $M_\varepsilon = M \cup \Psi(\partial M \times [0, \varepsilon])$, $0 \leq \varepsilon < 1$, obtaining a nested family of subdomains in $N$, with $M_0 = M$. Write $q_\varepsilon := q|_{SM_\varepsilon}$.

Define
\[u_\varepsilon \equiv \chi.\] (22)

Lemma 1.11. The function $u_\varepsilon$ has the following properties:

1. $u_\varepsilon := u_0 \in H^1(SM_\varepsilon) \cap C(SM) \cap C^\infty(RM)$, $u_\varepsilon|_{\partial(SM_\varepsilon)} = 0$.
2. $u_\varepsilon \in H^1(SM_\varepsilon) \cap C^\infty(S(M_\varepsilon \setminus M))$, $u_\varepsilon|_{\partial(SM_\varepsilon)} = 0$.
3. $Fu \in C^\infty(SM)$.
4. $u_\varepsilon|_{SM} \to u$ in $H^1(SM)$ as $\varepsilon \to 0$.
5. $Fu|_{SM} \to Fu$ in $H^1(SM)$ as $\varepsilon \to 0$.
6. $V u_\varepsilon|_{SM} \to V u$ in $L^2(SM)$ as $\varepsilon \to 0$.
7. $FVu_\varepsilon|_{SM} \to FVu$ in $L^2(SM)$ as $\varepsilon \to 0$.

Proof. Items (1-3) are direct consequences of Lemma 1.4.

Prove item (4). By (22) we have $u_\varepsilon|_{SM} = \chi|_{SM} = \chi$. Then by item (1), for every $0 \leq \varepsilon < 1$, $u_\varepsilon|_{SM} \in H^1(SM)$. So $u_\varepsilon|_{SM} \to u$ in $H^1(SM)$ as $\varepsilon \to 0$.

Item (5) easily follows from item (4) and that $q \in H^1(SM)$.

Item (6) also follows from item (4).

Finally, item (7) is the result of the commutation relation
\[[V, F] = H + V(\lambda)V\]
and items (4-6). \hfill \square
Let us now prove Theorem A. It is clear that $F u_\varepsilon = \omega_\varepsilon + \varphi_\varepsilon$. Each function $u_\varepsilon$ with $\varepsilon > 0$ satisfies the condition of Lemma 1.7 for $D = M_\varepsilon$. By this lemma we have
\[
\int_{SM_\varepsilon} (F V u_\varepsilon)^2 d\mu - \int_{SM_\varepsilon} K(V u_\varepsilon)^2 d\mu = \int_{SM_\varepsilon} (V F u_\varepsilon)^2 d\mu - \int_{SM_\varepsilon} (F u_\varepsilon)^2 d\mu.
\]
It is easy to see that the right-hand side of the last equation is nonpositive. Indeed, since $V F u_\varepsilon = V \omega_\varepsilon(x, v)$ we have
\[
\int_{SM} \varphi_\varepsilon \omega_\varepsilon(x, v) d\mu = 0 \quad \text{and} \quad \int_{SM} (\omega_\varepsilon(x, v))^2 d\mu = \int_{SM} (V \omega_\varepsilon(x, v))^2 d\mu.
\]
This will follow from [4, Lemma 4.4], which holds in any dimension. Thus
\[
\int_{SM_\varepsilon} (F V u_\varepsilon)^2 d\mu - \int_{SM_\varepsilon} K(V u_\varepsilon)^2 d\mu = - \int_{SM} \varphi_\varepsilon^2 d\mu \leq 0.
\]
Items (6) and (7) of Lemma 1.11 allow us to pass to the limit as $\varepsilon \to 0$ in this identity. Setting $\psi = V u$ and using Theorem 1.10 $V u \equiv 0$ on $RM$. This says that $u = f$ almost everywhere, where $f$ is a smooth function on $M$. Since $u \in C(SM)$, then $u = f$ everywhere. But in this case, since $d\pi_{(x, v)}(F) = v$ we have $Fu = df_x(v)$. This clearly implies the claim of the Theorem A.

2. General family of curves on a closed Finsler surface

2.1. Statement of the results. Given a closed connected Finsler surface $(M, F)$. $SM$ is the unit sphere bundle of $(M, F)$, and $\pi : SM \to M$ the canonical projection, $\pi(x, \xi) = x$.

Let $\Gamma$ be a family of smooth directed curves $\gamma$ in $M$ (constant curves are excluded). It is called general family of curves if it satisfies the following conditions:

1. All curves in $\Gamma$ are parametrized by arclength with respect to $F$.
2. For every point $x \in M$ and every vector $\xi \in S_xM$, there is exactly one curve $\gamma \in \Gamma$, up to a shift of the parameter, passing through $x$ in the direction $\xi$. If such a curve exists, we denote by $\gamma_{x, \xi}$ this curve with its parameter shifted so that $\gamma_{x, \xi}(0) = x$, $\dot{\gamma}_{x, \xi}(0) = \xi$.
3. All of the curves in $\Gamma$ is defined over the whole real axis.
4. $(M, F, \Gamma)$ has no conjugate points.

As before $\Gamma$ defines a flow $\phi_t$ on $SM$ which is called generalized thermostat and following holds:

**Theorem 2.1.** Every general family of curves on a closed connected Finsler surface can be represented as the family of $\lambda$-geodesics of a generalized isokinetic thermostat without conjugate points.

So we formulate all results in terms of thermostats.
Theorem B. Let \((M, F, \lambda)\) be a generalized thermostat on a closed Finsler surface \(M\). Let \(h \in C^\infty(M)\) and \(F\) be the vector field generating \(\phi\). Then the cohomological equation \(F(u) = h \circ \pi\) has a solution \(u \in C^\infty(SM)\) if and only if \(h = 0\).

Let \(\pi : SM \to M\) be the canonical projection.

Theorem C. Let \((M, F, \lambda)\) be a generalized thermostat on a closed Finsler surface \(M\). Suppose flow \(\phi_t\) is Anosov and let \(F\) be the vector field generating \(\phi\). Let \(h \in C^\infty(M)\) and let \(\theta\) be a smooth 1-form on \(M\). Then the cohomological equation

\[
F(u) = h \circ \pi + \theta
\]

has a solution \(u \in C^\infty(SM)\) if and only if \(h = 0\) and \(\theta\) is exact.

Note that by the smooth Livsic theorem [11] saying that \(F(u) = h \circ \pi + \theta\) is equivalent to saying that \(h \circ \pi + \theta\) has zero integral over every closed orbit of \(\phi_t\).

Theorem C improves all results previously known:

1. V. Guillemin and D. Kazhdan in [7] proved Theorem C for a negatively curved Riemannian metric. In [15] G. P. Paternain proved for magnetic flows. All these results were based on Fourier analysis.

2. In [5] N.S. Dairbekov and G. P. Paternain proved Theorem C for the case of magnetic flows and \(F\) is Riemannian.

3. Same result was proved in [3] by N.S. Dairbekov and G. P. Paternain for thermostats and \(F\) is Riemannian and in [4] for magnetic flows and \(F\) is Finsler.

The following result was obtained by using a criterion for a flow to be Anosov, which was proved by M. Wojkowski [20].

Theorem D. If \((M, F, \lambda)\) is a generalized thermostat on a closed connected Finsler surface and

\[
K - H(\lambda) - \lambda J + \lambda^2 + \frac{(\lambda I + V(\lambda))^2}{4} < 0
\]

then flow \(\phi_t\) is Anosov.

2.2. An integral identities. Below we will use the following consequence of Stokes theorem. Let \(N\) be a closed oriental manifold and \(\Theta\) a volume form. Let \(X\) be a vector field on \(N\) and \(f : N \to \mathbb{R}\) a smooth function. Then

\[
(23) \quad \int_N X(f)\Theta = -\int_N fL_X\Theta.
\]

Integrating the Pestov identity over \(SM\) against the Liouville measure \(d\mu\), and using (11-13) we obtain:

\[
(24) \quad \int_{SM} 2Hu \cdot VFu d\mu = \int_{SM} (Fu)^2 d\mu + \int_{SM} (Hu)^2 d\mu - \int_{SM} \{K - H(\lambda) - \lambda J + \lambda^2\} (Vu)^2 d\mu.
\]
Integrating equation (16) and using (11), (14) we obtain:

\[
2 \int_{SM} \mathbf{H} \cdot \mathbf{V} \mathbf{F} \mathbf{u} \, d\mu = \int_{SM} (\mathbf{V} \mathbf{F} \mathbf{u})^2 \, d\mu - \int_{SM} (\mathbf{F} \mathbf{V} \mathbf{u})^2 \, d\mu + \int_{SM} (\mathbf{H} \mathbf{u})^2 \, d\mu + \int_{SM} \lambda \mathbf{I} \mathbf{V} (\lambda) (\mathbf{V} \mathbf{u})^2 \, d\mu + \int_{SM} \mathbf{F} \mathbf{V} (\lambda) \cdot (\mathbf{V} \mathbf{u})^2 \, d\mu.
\]

Combining (24) and (25) we arrive at the final integral identity:

**Theorem 2.2.**

\[
\int_{SM} (\mathbf{F} \mathbf{V} \mathbf{u})^2 \, d\mu - \int_{SM} \mathbb{K} (\mathbf{V} \mathbf{u})^2 \, d\mu = \int_{SM} (\mathbf{V} \mathbf{F} \mathbf{u})^2 \, d\mu - \int_{SM} (\mathbf{F} \mathbf{u})^2 \, d\mu,
\]

where \( \mathbb{K} := K - H(\lambda) - \lambda J + \lambda^2 + \lambda \mathbf{I} \mathbf{V} (\lambda) - \mathbf{F} (\lambda \mathbf{I}). \)

2.3. **Proof of Theorem B.** We consider the Jacobi equation (see Appendix A.1)

\[
\ddot{y} - (\lambda \mathbf{I} + \mathbf{V}(\lambda)) \dot{y} + (K - H(\lambda) - \lambda J + \lambda^2 + \lambda \mathbf{I} \mathbf{V}(\lambda) - \mathbf{F}(\lambda \mathbf{I})) y = 0.
\]

The Riccati equation is

\[
\dot{r} + r^2 + (K - H(\lambda) - \lambda J + \lambda^2 + \lambda \mathbf{I} \mathbf{V}(\lambda) - \mathbf{F}(\lambda \mathbf{I})) - (\lambda \mathbf{I} + \mathbf{V}(\lambda)) r = 0,
\]

where \( r = \dot{y}/y. \)

If thermostat has no conjugate points, solution of the Jacobi equation (27) vanish at most once. Set \( K_\lambda = K - H(\lambda) - \lambda J + \lambda^2 + \lambda \mathbf{I} \mathbf{V}(\lambda) - \mathbf{F}(\lambda \mathbf{I}). \)

**Theorem 2.3.** Let \((M, F, \lambda)\) be a thermostat without conjugate points. Then

1. for any \((x, \xi) \in SM\) the solutions \( r^+_R(x, \xi, t) \) and \( r^-_R(x, \xi, t) \) of (28) such that \( r^+_R(x, \xi, -R) = +\infty \) and \( r^-_R(x, \xi, R) = -\infty \) are defined for all \( t > -R \) and all \( t < R \) respectively.

2. the limit solutions \( r^\pm(x, \xi) = \lim_{R \to +\infty} r^\pm_R(x, \xi, 0) \) are well defined for all \((x, \xi) \in SM\).

3. solutions \( r^\pm(x, \xi) \) are bounded.

**Proof.** We prove item (1) for function \( r^+_R(x, \xi, t) \). Similar argument works for \( r^-_R(x, \xi, t) \)

1. Consider solution \( y^+_R(t) \) of (27) satisfying \( y^+_R(-R) = 0, \dot{y}^+_R(-R) = 1 \), where \( R > 0 \). Then \( y^+_R(t) \) does not vanish again for \( t > -R \). So, there is a solution \( r^+_R(x, \xi, t) = y^+_R(t)/y^+_R(t) \) of the Riccati equation (28) well defined for all \( t > -R \).

2. Fix \( t = 0 \) and consider \( R > 0 \). Then \( r^+_R(x, \xi, 0) \) and \( r^-_R(x, \xi, 0) \) are both defined and decreasing and increasing functions of \( R \) respectively. By (11) we have \( r^+_R(x, v, 0) > r^-_R(x, v, 0) \) for all \( R \).

3. From closedness of \( M \) one infers that

\[ |K_\lambda| \leq B^2 \]

and

\[ |\lambda I + V(\lambda)| \leq C \]
on $M$ with suitable constant $B, C \geq 0$. Set $A = \max\{B, C\}$. We will show that

$$|r^\pm| \leq \frac{A}{2}(1 + \sqrt{5}).$$

Suppose that $r^+_R(t_0) > A(1 + \sqrt{5})/2$ for some $t_0 > -R$. Let $w^+(t)$ is the solution of

(29) \( \dot{w}^+ - Aw^+ + [w^+]^2 - A^2 = 0 \)

for which $r^+_R(t_0) = w^+(t_0)$. In fact

$$w^+(t) = \frac{A}{1 - e^{-A\sqrt{5}t + D}} + \frac{A(\sqrt{5} - 1)}{2} \frac{1}{1 - e^{-A\sqrt{5}t + D}}$$

for some $D$.

Subtracting (28) from (29) we get

(30) \( \dot{w}^+ - r^+_R(t) = A^2 + K\lambda - [w^+(t)]^2 + [r^+_R(t)]^2 + Aw^+(t) - (\lambda I + V(\lambda))r^+_R(t). \)

For $t = t_0$ the right-hand side of above equation is positive, so

$$w^+(t) \geq r^+_R(t),$$

at least in a one-sided neighbourhood $(t_0, t_0 + \varepsilon)$ of $t_0$. But now it is clear that above holds for all $t \geq t_0$, since at any point of intersection of the curves $w^+(t), r^+_R(t)$, the right-hand side of (30) would again be positive. Since $w^+(t)$ is bounded above by $A(1 + \sqrt{5})/2$ for $t \geq t_0$, we conclude that $r^+_R(t) \leq A(1 + \sqrt{5})/2$ for $t \geq t_0$.

Suppose now that $r^-_R(t_0) < -A(1 + \sqrt{5})/2$ for some $t_0 < R$. Let $w^-(t)$ is the solution of

(31) \( \dot{w}^- + Aw^- + [w^-]^2 - A^2 = 0 \)

for which $r^-_R(t_0) = w^-(t_0)$. In fact

$$w^-(t) = \frac{-Ae^{A\sqrt{5}t + E}}{e^{A\sqrt{5}t + E} - 1} + \frac{A(1 + \sqrt{5})}{2} \frac{e^{A\sqrt{5}t + E} + 1}{e^{A\sqrt{5}t + E} - 1}$$

for some $E$. Function $w^-(t)$ is defined for $t > E/(A\sqrt{5})$ and

$$\lim_{t \to E/(A\sqrt{5})} \frac{\dot{w}^-}{w^-} = +\infty.$$

As above for $r^+_R$, we show that

$$w^-(t) \geq r^-_R(t),$$

for $t \leq t_0$. This means that the assumption that $r^-_R(t_0) < -A(1 + \sqrt{5})/2$ is incompatible with the hypothesis that $r^-_R$ is defined for all $t < R$. So

$$w^-(t) \leq r^-_R(t).$$
for $t \leq t_0$. Since $w^-(t)$ is bounded below by $-A(1 + \sqrt{5})/2$ for $t \leq t_0$, we conclude that

$$-\frac{A}{2}(1 + \sqrt{5}) \leq r^-_R(t)$$

for $t \leq t_0$.

Set $t = 0$. Since $r^+_R(0) > r^-_R(0)$ for all $R$ we have

$$-\frac{A}{2}(1 + \sqrt{5}) \leq r^-_R(0) < r^+_R(0) \leq \frac{A}{2}(1 + \sqrt{5}).$$

Setting $R \to \infty$ we get

$$-\frac{A}{2}(1 + \sqrt{5}) \leq r^- \leq \frac{A}{2}(1 + \sqrt{5}),$$

i.e. solutions $r^\pm(x, \xi)$ are bounded.

□

Measurability of $r^\pm$ can be proven as in [9]. Together with boundedness it implies that $r^\pm$ are summable.

**Theorem 2.4.** Let $\psi: SM \to \mathbb{R}$ be a smooth function. Then

$$\int_{SM} (F\psi)^2 d\mu - \int_{SM} K\psi^2 d\mu = \int_{SM} [F(\psi) - r\psi + \psi(\lambda I + V(\xi))]^2 d\mu \geq 0.$$

Proof is similar as in Theorem 1.10.

Let $\theta$ be a smooth 1-form on $M$. If $Fu = h \circ \pi + \theta$, then it is easy to see that the right-hand side of (26) is nonpositive. Indeed, since $V F(u) = V\theta_x(v)$ we have

$$\int_{SM} (h \circ \pi)\theta_x(v) d\mu = 0 \quad \text{and} \quad \int_{SM} (\theta_x(v))^2 d\mu = \int_{SM} (V\theta_x(v))^2 d\mu.$$

This will follow from [4, Lemma 4.4], which holds in any dimension. Thus

$$\int_{SM} (F u)^2 d\mu - \int_{SM} (F u)^2 d\mu = - \int_{SM} (h \circ \pi)^2 d\mu \leq 0.$$

Theorem 2.4 implies that $h = 0$. Assuming that $\theta = 0$ we conclude Theorem B.

2.4. **Anosov property.** The Finsler metric $F$ induces a Riemannian metric of Sasaki type on $TM \setminus \{0\}$,

$$\hat{g} := g_{ij}(y)dx^i \otimes dx^j + g_{ij}(y)dy^i \otimes dy^j,$$

where $\delta y^i = dy^i + N^i_j(y)dx^j$.

Let $\hat{g}$ denote induced Riemannian metric on $SM$.

Recall that the Anosov property means that $T(SM)$ splits as $T(SM) = \mathbb{R}F \oplus E^u \oplus E^s$ in such a way that there are constants $C > 0$ and $0 < \rho < 1 < \eta$ such that for all $t > 0$ we have

$$\|d\phi_{-t}|E^u\| \leq C \eta^{-t} \quad \text{and} \quad \|d\phi_t|E^s\| \leq C \rho^t,$$

where norms are taken with respect to $\hat{g}$.
The subbundles are then invariant and Hölder continuous and have smooth integral manifolds, the stable and unstable manifolds, which define a continuous foliation with smooth leaves.

Let us introduce the weak stable and unstable bundles:

\[ E^+ = \mathbb{R}F \oplus E^s, \]
\[ E^- = \mathbb{R}F \oplus E^u. \]

**Lemma 2.5.** For any \((x, v) \in SM, V(x, v) \notin E^\pm(x, v).\)

**Proof.** Let \(\Lambda(SM)\) be the bundle over \(SM\) such that at each point \((x, v) \in SM\) consists of all 2-dimensional subspaces \(W\) of \(T_{(x, v)}SM\) with \(F(x, v) \in W.\)

The map \((x, v) \mapsto \mathbf{V} := \mathbb{R}F(x, v) \oplus \mathbb{R}V(x, v)\) is a section of \(\Lambda(SM)\) and its image is a codimension one submanifold that we denote by \(\Lambda^V.\) Similarly the map \((x, v) \mapsto \mathbb{R}F(x, v) \oplus \mathbb{R}H(x, v)\) is a section of \(\Lambda(SM)\) and its image is a codimension one submanifold that we denote by \(\Lambda^H.\)

The flow \(\phi\) naturally lifts to a flow \(\phi^*\) acting on \(\Lambda(SM)\) via its differential. Let \(F^*\) be the infinitesimal generator of \(\phi^*\).

**Claim.** \(F^*\) is transversal to \(\Lambda^V.\)

To prove the claim we define a function \(m : \Lambda(SM) \setminus \Lambda_H \to \mathbb{R}\) as follows. If \(W \in \Lambda(SM) \setminus \Lambda_H,\) then \(H \notin W.\) Thus there exists a unique \(m = m(W)\) such that \(mH + V \in W.\) Clearly \(m\) is smooth and \(\Lambda_V = m^{-1}(0) \subset \Lambda(SM) \setminus \Lambda_H.\) Fix \((x, v) \in SM\) and set \(m(t) := m(\phi_t^*(\mathbf{V}(x, v))).\) By the definition of \(m,\) there exist functions \(x(t)\) and \(y(t)\) such that

\[ m(t)H(t) + V(t) = x(t)F(t) + y(t)d\phi_t(V). \]

Equivalently

\[ m(t)d\phi_{-t}(H(t)) + d\phi_{-t}(V(t)) = x(t)F + y(t)V. \]

Differentiating with respect to \(t\) and setting \(t = 0\) (recall that \(m(0) = 0\)) we obtain:

\[ \dot{m}(0)H + [F, V] = \dot{x}(0)F + \dot{y}(0)V. \]

But \([V, F] = H + V(\lambda)V.\) Thus \(\dot{m}(0) = 1\) which proves the Claim.

From the Claim it follows that \(\Lambda_V\) determines an oriented codimension one cycle in \(\Lambda(SM)\) and by duality it defines a cohomology class \(m \in H^1(\Lambda(SM), \mathbb{Z}).\) Set \(E = E^\pm.\)

Given a continuous closed curve \(\alpha : S^1 \to SM,\) the index of \(\alpha\) is \(\nu(\alpha) := \langle m, [E \circ \alpha] \rangle\) (i.e. \(\nu = E^*m \in H^1(SM, \mathbb{Z}).\)) The index of \(\alpha\) only depends on the homology class of \(\alpha.\) Since \(E\) is \(\phi\)-invariant, the Claim also ensures that if \(\gamma\) is any closed orbit of \(\phi,\) then \(\nu(\gamma) \geq 0.\)

Recall that according to Ghys [8] we know that \(\phi\) is topologically conjugate to the geodesic flow of a metric of constant negative curvature. In particular, every homology class in \(H_1(SM, \mathbb{Z})\) contains a closed orbit of \(\phi.\) Thus \(\nu\) must vanish.

If there exists \((x, v) \in SM\) for which \(V(x, v) \in E(x, v),\) then using that every point of \(\phi\) is non-wandering, we can produce exactly as in [14, Lemma 2.49] a closed curve \(\alpha : S^1 \to SM\) with \(\nu(\alpha) > 0.\) This contradiction shows the lemma. \(\square\)
The lemma implies that there exist unique continuous functions \( r^\pm(x, v) \) on \( SM \) such that
\[
H(x, v) + r^+(x, v)V(x, v) \in E^+,
\]
\[
H(x, v) + r^-(x, v)V(x, v) \in E^-.
\]
Note that the Anosov property implies that \( r^+ \neq r^- \) everywhere. Below we will need to use that the functions \( r^\pm \) satisfy a Riccati type equation along the flow. Note that \( r^\pm \) are smooth along \( \phi \) because \( E^\pm \) are \( \phi \)-invariant.

**Lemma 2.6. (Riccati equation).** The function \( r = r^\pm \) satisfies
\[
F(r - V(\lambda)) + r(\lambda I - V(\lambda) + r) + K - \lambda IV(\lambda) = 0.
\]

**Proof.** Let \( E = E^\pm \). Fix \( (x, v) \in SM \), flow along \( \phi \) and set
\[
\xi(t) := d\phi_{-t}(H(t) + r(t)V(t)).
\]
By the definition of \( r \), \( \xi(t) \in E(x, v) \) for all \( t \). Differentiating with respect to \( t \) and setting \( t = 0 \) we obtain:
\[
\dot{\xi}(0) = [F, H] + F(r)V + r[F, V].
\]
Using that
\[
[V, F] = H + V(\lambda)V, \quad [F, H] = (K - H(\lambda) - \lambda J + \lambda^2)V - \lambda F - \lambda IH
\]
we have
\[
\dot{\xi}(0) = -\lambda F - \lambda I\xi(0) + \{K - H(\lambda) - \lambda J + \lambda^2 + F(r) - rV(\lambda)\}V.
\]
Replacing \( H \) by \( \dot{\xi}(0) - rV \) yields:
\[
\dot{\xi}(0) + (r + \lambda I)\xi(0) - \lambda F = \{K - H(\lambda) - \lambda J + \lambda^2 + F(r) + \lambda Ir - rV(\lambda) + r^2\}V.
\]
Since \( \dot{\xi}(0) + (r + \lambda I)\xi(0) - \lambda F \in E \) we must have
\[
K - H(\lambda) - \lambda J + \lambda^2 + F(r) + \lambda Ir - rV(\lambda) + r^2 = 0
\]
which is the desired equation since \( K := K - H(\lambda) - \lambda J + \lambda^2 + \lambda IV(\lambda) + FV(\lambda) \). \( \square \)

Here is the main result of this subsection:

**Theorem 2.7.** Let \( \psi : SM \to \mathbb{R} \) be a smooth function and suppose \( \phi \) is Anosov. Then for \( r = r^\pm \)
\[
\int_{SM} (F\psi)^2 d\mu - \int_{SM} K\psi^2 d\mu = \int_{SM} [F(\psi) - r\psi + \psi V(\lambda)]^2 d\mu \geq 0.
\]
Moreover,
\[
\int_{SM} [F(\psi) - r\psi + \psi V(\lambda)]^2 d\mu = 0
\]
if and only if \( \psi = 0 \).
Proof. We omit the proof of first part. Suppose now
\[ \int_{SM} [F(\psi) - r\psi + \psi V(\lambda)]^2 \, d\mu = 0 \]
which implies
\[ F(\psi) - r\psi + \psi V(\lambda) = 0 \]
everywhere. Since this holds for \( r = r^\pm \) we deduce:
\[ (r^+ - r^-)\psi = 0 \]
But for an Anosov flow \( r^+ - r^- \neq 0 \). This surely implies that \( \psi \equiv 0 \) on \( SM \).
□

2.5. End of the proof of Theorem C. If \( Fu = h \circ \pi + \theta \), then it is easy to see that (as in section 2.3)
\[ \int_{SM} (Fu)^2 \, d\mu - \int_{SM} K(Vu)^2 \, d\mu \leq 0. \]
By Theorem 2.7 this happens if and only if \( Vu = 0 \), which says that \( u = f \circ \pi \) where \( f \) is a smooth function on \( M \). But in this case, since \( d\pi(x,v)(F) = v \) we have \( Fu = df_x(v) \). This means that Theorem C is proved.

2.6. Proof of Theorem D. We consider the quotient bundle \( \hat{T}(SM) \) defined by
\[ \hat{T}(x,v)(SM) := T(x,v)(SM)/RF(x,v). \]
Since \( d\phi_t F(x,v) = F(\phi_t(x,v)) \), \( d\phi_t \) descends to the quotient to define a map \( A_t : \hat{T}(x,v)(SM) \to \hat{T}(\phi_t(x,v)(SM)) \) satisfying
\[ A_{s+t} = A_s \circ A_t. \]
As we said before following result was proved in [20]

Theorem 2.8. Let \( M \) be a closed manifold and \( \phi_t : SM \to SM \) a flow on \( SM \) with generating vector field \( F \). If there exists a quadratic form \( Q : T(SM) \to \mathbb{R} \) satisfying the following properties:

(a) For each \( (x,v) \in SM \) the form \( Q_{(x,v)} := Q|_{T(x,v)(SM)} \) depends continuously on \( (x,v) \).

(b) For all \( (x,v) \in SM \), \( \xi \in T(x,v)(SM) \) and \( \mu \in \mathbb{R} \), we have
\[ Q_{(x,v)}(\xi + \mu F(x,v)) = Q_{(x,v)}(\xi). \]

(c) Quadratic form \( Q \) can be projected onto the quotient bundle \( \hat{T}(SM) \) to define
\[ \hat{Q} : \hat{T}(SM) \to \mathbb{R} \]
which is non-degenerate.

(d) The Lie derivative \( L_\xi Q \) must be continuous and
\[ \hat{L}(\xi) := \left. \frac{d}{dt} \right|_{t=0} \hat{Q}(A_t(\xi)), \]
must be positive definite on \( \hat{T}(SM) \). Then the flow \( \phi_t \) is Anosov.
Given $\xi \in T_{(x,v)}(SM)$ we may say
\[ \xi = aF(x,v) + yH(x,v) + zV(x,v). \]

Define quadratic form $Q$ by
\[ Q(\xi) = yz. \]

Checking the first three conditions of Theorem 2.8 is trivial, but the fourth requires the Jacobi equations.

\[ \hat{L}(\xi) = \left. \frac{d}{dt} \right|_{t=0} Q(A_t(\xi)), \]
\[ = \left. \frac{d}{dt} \right|_{t=0} y(t)z(t), \]
\[ = -\{K - H(\lambda) - \lambda J + \lambda^2\}y^2 + (V(\lambda) + \lambda I)yz + z^2. \]

By Silvestor’s criterion it is positively definite if and only if
\[ K - H(\lambda) - \lambda J + \lambda^2 + \frac{(\lambda I + V(\lambda))^2}{4} < 0. \]

**Appendix A.**

**A.1. Jacobi equation for thermostats.** For $\xi \in T(SM)$ write
\[ d\phi_t(\xi) = a(t)F + y(t)H + z(t)V, \]
equivalently,
\[ \xi = a(t)d\phi_{-t}(F) + y(t)d\phi_{-t}(H) + z(t)d\phi_{-t}(V). \]

If we differentiate the last equality with respect to $t$ we obtain:
\[ 0 = \dot{a}F + \dot{y}H + y[F, H] + \dot{z}V + z[F, V]. \]

Using the bracket relations and regrouping we have
\[ \dot{a} = \lambda y, \]
\[ \dot{y} = \lambda I y + z, \]
\[ \dot{z} = -\{K - H(\lambda) - \lambda J + \lambda^2\}y + V(\lambda)z. \]

From these equations we get the Jacobi equation:
\[ \ddot{y} - (\lambda I + V(\lambda))\dot{y} + (K - H(\lambda) - \lambda J + \lambda^2 + \lambda IV(\lambda) - F(\lambda I))y = 0. \]
A.2. Proof of Lemma 1.5. Without loss of generality we may denote function \( q \) by \( q \), because \( q \) also satisfies (7). So, we have to show that

\[
q(x, \xi) = 0
\]

for any boundary points \( x \in \partial M \) and any vector \( \xi \in T_x(\partial M) \).

Take \( x \in \partial M \) and \( \xi \in S_x(\partial M) \). Let \( \nu_x \in T_x M \) be the inward unit normal vector to \( \partial M \) at \( x \). For \( \varepsilon > 0 \), put \( \xi_\varepsilon = \xi + \varepsilon \nu_x \). Consider the \( \lambda \)-geodesic \( \gamma_\varepsilon = \gamma_{x, \xi_\varepsilon} \). Clearly, \( \gamma_\varepsilon(t) \in M^{\text{int}} \) for \( t > 0 \) small enough.

Let \( l_\varepsilon \) be the first time instance at which \( \gamma_\varepsilon \) meets the boundary, \( \gamma_\varepsilon(l_\varepsilon) \in \partial M \). So \( \gamma_\varepsilon : [0, l_\varepsilon] \to M, \gamma_\varepsilon(0) \in \partial M, y_\varepsilon := \gamma_\varepsilon(l_\varepsilon) \in \partial M, \) and \( \gamma_\varepsilon(t) \in M^{\text{int}} \) for \( 0 < t < l_\varepsilon \). We shift time parameter of \( \gamma_\varepsilon(t) \) so that \( \gamma_\varepsilon(-l_\varepsilon) = x, \dot{\gamma}_\varepsilon(-l_\varepsilon) = \xi_\varepsilon \) and \( \gamma_\varepsilon(0) = y_\varepsilon \) and denote this \( \lambda \)-geodesic again by \( \gamma_\varepsilon(t) \).

We separately consider two possible cases: (1) there is a sequence \( 0 < \varepsilon_k \to 0 \) such that \( l_{\varepsilon_k} \to 0 \) as \( k \to \infty \); and (2) \( l_\varepsilon \geq l_0 > 0 \) for all \( 0 < \varepsilon \leq \varepsilon_0 \).

In the first case, assume (37) fails, for definiteness

\[
q(x, \xi) > 0.
\]

For \( k \) large enough, the points \((\gamma_{\varepsilon_k}(t), \dot{\gamma}_{\varepsilon_k}(t))\) belong to any prescribed neighborhood of \((x, \xi)\) for all \( t \in [0, l_{\varepsilon_k}] \). Therefore, (38) implies that the integrand in

\[
\int_{-l_{\varepsilon_k}}^{0} q(\gamma_{\varepsilon_k}(t), \dot{\gamma}_{\varepsilon_k}(t)) \, dt
\]

is strictly positive on \([0, l_{\varepsilon_k}]\). This contradicts to condition of the lemma.

Now, we consider the second case. Fix \( \varepsilon \in (0, \varepsilon_0) \) and simplify the notations to: \( \gamma = \gamma_\varepsilon, l = l_\varepsilon, y = y_\varepsilon \). Let \( \tau \to x_\tau \in \partial M (0 \leq \tau < \delta) \) be a parametrization of \( \partial M \) near \( x \) such that \( x_0 = x \) and \( \frac{dx_\tau}{d\tau} \bigg|_{\tau=0} = \xi \).

We first assume that \( \gamma \) meets \( \partial M \) at \( y = \gamma(0) \) transversally. Then for \( \tau \) small enough there is a unique \( \lambda \)-geodesic \( \gamma_\tau \) joining \( x_\tau \) and \( y \) runs completely in \( M \) for a sufficiently small \( \tau \). We parametrize \( \lambda \)-geodesic \( \gamma_\tau \) by \([-l_\tau, 0] \). So \( \gamma_\tau : [-l_\tau, 0] \to M, \gamma_\tau(-l_\tau) = x_\tau, \gamma_\tau(0) = y \). Moreover, \( l_\tau \) depends smoothly on \( \tau \) and \( \gamma_\tau(t) \) depends smoothly jointly on \((\tau, t)\) in \([0, \delta] \times [-l_\tau, 0] \).

Since \( q \) satisfies (7), we have

\[
\int_{-l_\tau}^{0} q(\gamma_\tau(t), \dot{\gamma}_\tau(t)) \, dt = 0.
\]

Taking the derivative with respect to \( \tau \) at \( \tau = 0 \), we get

\[
q(\gamma(-l), \dot{\gamma}(-l)) \frac{dl_\tau}{d\tau} \bigg|_{\tau=0} + \int_{-l}^{0} \frac{\partial q(\gamma_\tau(t), \dot{\gamma}_\tau(t))}{\partial x^k} \frac{\partial \gamma^k_\tau(t)}{\partial \tau} \bigg|_{\tau=0} \, dt.
\]

Vector field

\[
J(t) = \frac{\partial \gamma^k(t)}{\partial \tau} \bigg|_{\tau=0}
\]
is Jacobi vector field along $\gamma$ – because it is a field of $\lambda$-geodesic variation $\gamma_\tau$. This field satisfies to initial conditions:

$$J(-l) = \xi, \quad J(0) = 0.$$  

To calculate $dl_\tau/d\tau|_{\tau=0}$, put $c(\tau, t) = \gamma_\tau((l_\tau/l)t)$. Then $l_\tau$ is the length of the curve $c(\tau, \cdot) : [-l, 0] \to M$, and first variation formula gives

$$\left.\frac{dl_\tau}{d\tau}\right|_{\tau=0} = -\langle \dot{\gamma}(-l), \xi \rangle - \int_{-l}^{0} \langle \frac{D\dot{\gamma}}{dt}, J \rangle \dot{\gamma} dt.$$  

From

$$\left.\frac{D\dot{\gamma}}{dt}, J \right|_{\dot{\gamma}} = \frac{d}{dt} \langle \dot{\gamma}, J \rangle_{\dot{\gamma}} - \langle \dot{\gamma}, \frac{DJ}{dt} \rangle_{\dot{\gamma}}$$  

we then find

$$\left.\frac{dl_\tau}{d\tau}\right|_{\tau=0} = \int_{-l}^{0} \langle \dot{\gamma}, \frac{DJ}{dt} \rangle_{\dot{\gamma}} dt.$$  

So we have

$$q(\gamma(-l), \dot{\gamma}(-l)) \int_{-l}^{0} \langle \dot{\gamma}, \frac{DJ}{dt} \rangle_{\dot{\gamma}} dt + \int_{-l}^{0} \langle \partial q(\gamma_\tau(t), \dot{\gamma}_\tau(t)), \frac{\partial}{\partial x^k} J^k(t) \rangle dt = 0.$$  

Now we estimate Jacobi field $J$. By (40), $J$ depends linearly on $\xi$. Since

$$\dot{\gamma}(0) = \xi = \xi + \varepsilon \nu_x, \quad F(\xi) = F(\nu_x) = 1, \quad \langle \xi, \nu_x \rangle_{\nu_x} = 0$$  

vector $\xi$ can be written in the following form:

$$\xi = \frac{\dot{\gamma}(-l)}{[F(\xi + \varepsilon \nu_x)]^2} + \varepsilon \eta,$$

where

$$\langle \eta, \dot{\gamma}(-l) \rangle_{\eta} = 0.$$

So

$$J(t) = \frac{a(t)\dot{\gamma}(t)}{[F(\xi + \varepsilon \nu_x)]^2} + \varepsilon J_\eta(t),$$

where $a(t)$ is a smooth function such that $a(-l) = 1$, $a(0) = 0$ and $J_\eta$ is a Jacobi field along $\gamma$, defining by

$$J_\eta(-l) = \eta, \quad \langle \eta, \dot{\gamma}(-l) \rangle_{\eta} = 0; \quad J_\eta(0) = 0.$$  

Using this relations and vectorial Jacobi equation (we do not derive vectorial Jacobi equation, because it is same as in [4]) we get following estimates:

$$\| J_\eta(t) \| \leq C, \quad \| D_{\dot{\gamma}} J_\eta(t) \| \leq C,$$

which is uniform over all $\lambda$-geodesics $\gamma : [0, l] \to M$ with $l \geq l_0 > 0$, $1 \leq \| \dot{\gamma} \| \leq 2$ and over all vectors $\eta$ satisfying $\langle \eta, \dot{\gamma}(-l) \rangle_{\eta} = 0$. $C$ depends only on $l_0$ and $C^1$-norms of the curvature tensor and other summands of right-hand side of Jacobi equation [4, Section 4.4].
Substituting (43) for \( J(t) \) in (41), we obtain
\[
q(\gamma(-l), \gamma(-l)) \left( \int_{-l}^{0} \dot{a}(t) \| \dot{\gamma} \|^2 dt - 1 + \varepsilon [F(\xi + \varepsilon \nu_x)]^2 \int_{-l}^{0} \left\langle \dot{\gamma}, \frac{\partial J_{\eta}}{\partial t} \right\rangle dt \right)
\]
(46)
\[
= \int_{-l}^{0} \dot{a}(t) q(\gamma(t), \gamma(t)) dt - \varepsilon [F(\xi + \varepsilon \nu_x)]^2 \int_{-l}^{0} \frac{\partial q(\gamma(t), \gamma(t))}{\partial x^k} J^k_\eta(t) dt.
\]
Note that
\[
\int_{-l}^{0} \left\langle \dot{\gamma}, \frac{\partial J_{\eta}}{\partial t} \right\rangle dt \leq C' \| J_{\eta} \|_{C^1}
\]
(47)
By smoothness of \( a(t) \) its derivative is bounded \( |\dot{a}(t)| \leq c < \infty \). Write
\[
q(\gamma(-l), \gamma(-l)) = q(x, \xi)
\]
by (42). So we have
\[
q(x, \xi) = \frac{\int_{-l}^{0} \dot{a}(t) q(\gamma(t), \gamma(t)) dt - \varepsilon [F(\xi + \varepsilon \nu_x)]^2 \int_{-l}^{0} \frac{\partial q(\gamma(t), \gamma(t))}{\partial x^k} J^k_\eta(t) dt}{\int_{-l}^{0} \dot{a}(t) \| \dot{\gamma} \|^2 dt - 1 + \varepsilon [F(\xi + \varepsilon \nu_x)]^2 \int_{-l}^{0} \left\langle \dot{\gamma}, \frac{\partial J_{\eta}}{\partial t} \right\rangle dt}
\]
(48)
If, in particular
\[
\frac{1}{2C' \| J_{\eta} \|_{C^1}} \geq \varepsilon, \quad C' = \text{const},
\]
by using that \( \dot{a}(t) \) is bounded we get
\[
|q(x, \xi)| \leq \frac{\left| \int_{-l}^{0} \dot{a}(t) q(\gamma(t), \gamma(t)) dt \right| + \varepsilon [F(\xi + \varepsilon \nu_x)]^2 \int_{-l}^{0} \left| \frac{\partial q(\gamma(t), \gamma(t))}{\partial x^k} J^k_\eta(t) dt \right|}{2 - \varepsilon [F(\xi + \varepsilon \nu_x)]^2 \int_{-l}^{0} \left| \dot{\gamma}, \frac{\partial J_{\eta}}{\partial t} \right| dt}
\]
(49)
\[
\leq \varepsilon \frac{\left| \int_{-l}^{0} \frac{\partial q(\gamma(t), \gamma(t))}{\partial x^k} J^k_\eta(t) dt \right|}{1 - 2\varepsilon}
\]
\[
\leq 4 \frac{\varepsilon}{1 - 2\varepsilon},
\]
since (7). Estimating the last integral by means of (15), we arrive at the final estimate
\[
|q(x, \xi)| \leq \frac{\varepsilon}{1 - 2\varepsilon} C \| q \|_{C^1}
\]
(50)
with some new constant \( C \) independent of \( \varepsilon \) and \( q \).

Recall that the above arguments were carried out under the assumption that the \( \lambda \)-geodesic \( \gamma(t) \) intersects \( \partial M \) transversally at the point \( y = y(0) \). If this is not the case, the \( \lambda \)-geodesic \( \gamma_\tau \) in (7) can go partly outside \( M \). Of course, we can extend \( q \) by zero outside \( M \) without violating (7). However, the integrand of (7) may then become discontinuous, making the differentiation of (7) problematic. Fortunately, there is another way allowing us to avoid discontinuous integrands.

So, we consider the case in which the vector \( \dot{\gamma}(0) \) belongs to \( T_y(\partial M), y = \gamma(0) \). Introduce the following definition: a point \( x' \in \partial M \) is said to be visible from \( y \) if there is a \( \lambda \)-geodesic \( \gamma' : [0, a] \to M \), such that \( \gamma'(0) = y \), \( \gamma'(a) = x' \), and \( \gamma'(t) \in \partial M \).

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for $t \in (0, a)$. For example, $x$ is visible from $y$ (recall that $x$ is a point fixed during the proof of Lemma). The following statement describes the local structure near $x$ of the set of points visible from $y$.

**Claim 1.** There exists a curve $H \subset \partial M$ through $x$ which separates each sufficiently small neighborhood $U \subset \partial M$ of $x$ into the disjoint union $U = U^+ \cap (H \cup U) \cap U$ such that one of the two open half-neighborhoods $U^+$ and $U$ consists of points visible from $y$.

The proof of Claim 1 is similar as in [17], despite metric is Finsler, so we omit it. Let

$$T_x(\partial M) = T_x^+(\partial M) \cup T_x H \cup T_x^-(\partial M)$$

be the decomposition of the vector space $T_x(\partial M)$ into the disjoint union of two open rays and $\{0\}_x$. At least one of the two vectors $\xi$ and $-\xi$ belongs to the closed half-rays $T_x^+(\partial M) \cup T_x H$. This implies the existence of a unit vector $\xi' \in T_x^+(\partial M)$ such that

$$\|\xi' - \xi\| < \varepsilon_1, \quad \|\xi' + \xi\| < \varepsilon_1$$

holds with an arbitrary $\varepsilon_1 > 0$. In first case $\xi' = \xi$ and in second case $\xi' = -\xi$. In both cases

$$|\langle \xi, \xi' \rangle_{\xi}| = 1.$$  

We represent $\xi'$ as

$$\xi' = \frac{\langle \xi, \xi' \rangle_{\xi}}{\|F(\xi + \varepsilon \nu_x)\|^2} \dot{\gamma}(-l) + \varepsilon \eta.$$

According to this, the Jacobi field $J$ can be represented as

$$J(t) = \frac{a(t) \langle \xi, \xi' \rangle_{\xi}}{\|F(\xi + \varepsilon \nu_x)\|^2} \dot{\gamma}(t) + \varepsilon J_\eta(t)$$

with some Jacobi field satisfying (41). Inserting (51) in (41) and repeating the arguments in the derivation of (50), we obtain

$$|q(x, \xi)| \leq \frac{\varepsilon}{1 - 2 \varepsilon} C \|q\| C^1.$$  

This proves (50) in the case of nontransversal intersection.

Finally, (50) implies (37), because $\varepsilon$ is arbitrary.

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Laboratory of Geometry, Institute of Mathematics, Informatics and Mechanics, Pushkina 125, Almaty, 050010, Kazakhstan

E-mail address: y.assylbekov@yahoo.com

Kazakh British Technical University, Tole bi 59, 050000 Almaty, Kazakhstan, & Laboratory of Geometry, Institute of Mathematics, Informatics and Mechanics, 125 Pushkin st., Almaty, 050010, Kazakhstan

E-mail address: Nurlan.Dairbekov@gmail.com