EXACT SAMPLING FOR THE ISING MODEL AT ALL TEMPERATURES

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Abstract. The Ising model is often referred to as the most studied model of statistical physics. It describes the behavior of ferromagnetic material at different temperatures. It is an interesting model also for mathematicians, because although the Boltzmann distribution is continuous in the temperature parameter, the behavior of the usual single-spin dynamics to sample from this measure varies extremely. Namely, there is a critical temperature where we get rapid mixing above and slow mixing below this value. Here, we give a survey of the known results on mixing time of Glauber dynamics for the Ising model on the square lattice and present a technique that makes exact sampling of the Ising model at all temperatures possible in polynomial time. At high temperatures this is well-known and although this seems to be known also in the low temperature case since Kramer and Wannier’s paper [KW41] from the 1950s, we did not find any reference that describes exact sampling for the Ising model at low temperatures.

1. Introduction

In this article we summarize the known results about the mixing time of the heat bath dynamics for the Ising model and combine them with some graph theoretic results to an algorithm to sample exactly from the Ising model in polynomial time. By time (or running time) we always mean the number of steps of the underlying Markov chain. The algorithm that will be analyzed (Algorithm 2, given in Section 5) is at high temperatures simply the Coupling from the past algorithm (see Propp and Wilson [PW96]). At low temperatures we have to produce a sample at the dual graph, but this can be traced back to sampling on the initial graph with constant boundary condition.

The main theorem of this article is stated as follows.

Key words and phrases. Ising model, exact sampling, random cluster model.

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Theorem 8. Let $G_L$ be the square lattice with $N = L^2$ vertices. Then, Algorithm 2 outputs an exactly distributed Ising configuration with respect to $\pi_{G_L}^\beta$ in expected time smaller than

- $c_\beta N \log N$ for $\beta \neq \beta_c = \log(1 + \sqrt{2})$ and some $c_\beta > 0$
- $16 N^C \log N$ for $\beta = \beta_c$, where $C$ is given in (1).

As a consequence we get that one can estimate the expectation of arbitrary functions with respect to the Boltzmann distribution in polynomial time. Namely, if we use the simple Monte Carlo method to approximate the expectation of a function $f$ on the Ising model, we need $\epsilon^{-2}\|f\|^2_2$ exact samples from $\pi_\beta$ (i.e. Algorithm 2) to reach a mean square error of at most $\epsilon$. Therefore, if we denote the bounds from Theorem 8 by $T_\beta$, we need on average $T_\beta \epsilon^{-2}\|f - \mathbb{E}_{\pi_\beta} f\|^2_2$ steps of the Markov chain that will be defined in Section 2.

The first polynomial-time algorithm (FPRAS) was shown by Jerrum and Sinclair [JS93]. There they present an algorithm to approximate the partition function $Z_\beta$ and, as a consequence, approximate expectations of functions that are given in terms of the partition function in polynomial time at all temperatures $\beta$.

2. The Ising model

In this section we introduce the two-dimensional Ising model. Let $G = (V, E)$ be a graph with finite vertex set $V \subset \mathbb{Z}^2$ and edge set $E = \left\{ \left\{ u, v \right\} \in \binom{V}{2} : |u - v| = 1 \right\}$, where $\binom{V}{2}$ is the set of all subsets of $V$ with 2 elements. From now, $N := |V|$. We are interested in the square lattice, i.e. $V = \{1, \ldots, L\}^2$ for some $L = \sqrt{N} \in \mathbb{N}$, because it is the most widely used case. We denote the induced graph by $G_L$.

The Ising model on $G_L$ is now defined as the set of possible configurations $\Omega_{\text{IS}} = \{-1, 1\}^V$, where $\sigma \in \Omega_{\text{IS}}$ is an assignment of -1 or 1 to each vertex in $V$, together with the probability measure

$$\pi_\beta(\sigma) := \pi_{G_L}^\beta(\sigma) = \frac{1}{Z_\beta} \exp \left\{ \beta \sum_{u \leftrightarrow v; \sigma(u) = \sigma(v)} \right\},$$

where $u \leftrightarrow v$ means $u$ and $v$ are neighbors in $G_L$, $Z$ is the normalization constant and $\beta \geq 0$ is the called the inverse temperature. This measure is called the Boltzmann (or Gibbs) distribution with free boundary condition.

Additionally we need the notion of boundary conditions, but we restrict ourself here
to the “all plus” and “all minus” case. Let $V^c = \mathbb{Z}^2 \setminus V$. Then we denote the lattice $G_L$ together with the probability measure

$$
\pi^\pm_\beta(\sigma) := \frac{1}{Z_\beta} \pi^{G_L \pm}_\beta(\sigma) \cdot \exp \left\{ \beta \sum_{v \in V, u \in V^c: u \leftrightarrow v} \text{1}_\{\sigma(v) = \pm 1\} \right\}
$$

by the Ising model with plus/minus boundary condition, respectively. One can imagine that this corresponds to the Ising model on $G_L$ with a strip of fixed spins around, so every vertex in $G_L$ has the same number of neighbors.

In 1944 Onsager [Ons44] proved that there is a phase transition at $\beta = \beta_c := \ln(1 + \sqrt{2})$ in the case where $V = \mathbb{Z}^2$ and we will see that this value is also important for finite lattices. Namely, the dynamics that will be defined below is rapidly mixing if and only if $\beta \leq \beta_c$.

We will use the so called heat bath dynamics. These dynamics define a irreducible, aperiodic and reversible Markov chain $X^\beta = (X^\beta_i)_{i \in \mathbb{N}}$ with stationary distribution $\pi_\beta$ by the transition matrix

$$
P(\sigma, \sigma^{v, \xi}) = \frac{1}{N} \left( 1 + \frac{\pi_\beta(\sigma)}{\pi_\beta(\sigma^{v, \xi})} \right)^{-1}, \quad \sigma \in \Omega_{IS}, v \in V,
$$

where $\sigma^{v, \xi}$ with $\xi \in \{-1, 1\}$ is defined by $\sigma^{v, \xi}(v) = \xi$ and $\sigma^{v, \xi}(u) = \sigma(u), u \neq v$. The interpretation of this algorithm is very simple. In each step choose a random $v \in V$ and assign a new value to $v$ according to $\pi_\beta$ conditioned on all the neighbors of $v$.

Note that the results of this article hold in general for all Glauber dynamics as defined in [Gla63] that admit a monotone coupling (see Section 3). For a general introduction to Markov chains see e.g. [LPW09], or [Mar99] in the context of spin systems.

In the sequel we want to estimate how fast such a Markov chain converges to its stationary distribution. Therefore we first introduce the total variation distance to measure the distance between two probability measures $\nu$ and $\pi$, which is defined by

$$
\|\nu - \pi\|_{TV} = \frac{1}{2} \sum_{\sigma \in \Omega_{IS}} |\nu(\sigma) - \pi(\sigma)|.
$$

Now we can define the mixing time of the Markov chain with transition matrix $P$ and stationary distribution $\pi_\beta$ by

$$
\tau_\beta = \min \left\{ n : \max_{\sigma \in \Omega_{IS}} \|P^n(\sigma, \cdot) - \pi_\beta(\cdot)\|_{TV} \leq \frac{1}{2e} \right\}.
$$
This is the expected time the Markov chain needs to get close to its stationary distribution. In fact, one can bound the spectral gap of the transition matrix $P$ in either direction in terms of the mixing time, see e.g. [LPW09, Th. 12.3 & 12.4], so one can bound the error of a MCMC algorithm to integrate functions over $\Omega_{IS}$, as one can read in [Rud09]. Furthermore, if the Markov chain is rapidly mixing (i.e. the mixing time is at most polylogarithmic in the size of the state space $\Omega_{IS}$) we get that the problem of integration (with an unnormalized density) on the Ising model is tractable, see also [NW10]. Unfortunately, there is no Markov chain that is proven to be rapidly mixing at all temperatures.

However, in this article we are interested in sampling exactly from the stationary distribution, but first we present the known mixing time results for the Glauber dynamics for the Ising model. For proofs or further details we refer to the particular articles or the survey of Martinelli [Mar99]. Of course, we can only give a small selection of references, because there are many papers leading to the results given below.

**Theorem 1.** [MO94] Let $\beta < \beta_c$. Then there exists a constant $c_\beta > 0$ such that the mixing time of the Glauber dynamics for the Ising model with arbitrary boundary condition on $G_L$ satisfies

$$\tau_\beta \leq c_\beta N \log N.$$  

**Theorem 2.** [CGMS96] Let $\beta > \beta_c$. Then there exists a constant $c_\beta > 0$ such that the mixing time of the Glauber dynamics for the Ising model on $G_L$ satisfies

$$\tau_\beta \geq e^{c_\beta N}.$$  

The results above can be obtained by the observation that some spatial mixing property of the measure $\pi_\beta$ is equivalent to the mixing in time of the Glauber dynamics. For details for this interesting fact, see [DSVW04].

The constant $c_\beta$ of Theorem 1 is widely believed to be of order $\frac{1}{\beta - \beta_c}$. To determine the mixing time in the case $\beta = \beta_c$ was a challenging problem for a long time. It was solved by Lubetzky and Sly in their recent paper [LS10].

**Theorem 3.** [LS10] There exists a constant $C > 0$ such that the mixing time of the Glauber dynamics for the Ising model on $G_L$ at the critical temperature satisfies

$$\tau_\beta \leq 4 N^C.$$
Remark 4. We give here only a brief description of the constant $C$, which can be given explicitly. For more details see [LS10, p.19]. However, numerical experiments on the “true” exponent suggest that $C \approx 3.08$ (see e.g. [WHS95], [NB96] and note the explanation below).

The constant $C$ in Theorem 3 is given by

$$C = 2 + \log_{3/2} \left( \frac{2}{1 - p^+} \right).$$

Here, $p^+$ is the limiting vertical crossing probability in the random cluster model on a fully-wired rectangle, where the width of the lattice is 3 times its height. The $C$, as given here, differs from the one given in [LS10] by eliminating a factor of 2 in front of the log term and by the additional 2. The reason is that we state their result in terms of $N$ and not in the side-length $L$ of the lattice (therefore without factor 2) and that we are interested in the discrete time single-spin algorithms. Therefore we get an additional factor $N$ in their spectral gap result ([LS10 Th. 1]) and a factor $N$ by (see e.g. [LPW09])

$$\tau_{\beta} \leq \log \left( \frac{e}{\min_\sigma \pi_{\beta_c}(\sigma)} \right) \text{gap}(X^{\beta})^{-1} \leq 4N \text{gap}(X^{\beta})^{-1},$$

because $\min_\sigma \pi_{\beta_c}(\sigma) \geq \exp(-3N)$.

The results of this section show that the Glauber dynamics is rapidly mixing for $\beta \leq \beta_c$, but very slowly mixing for larger $\beta$. In Section 4 we will see how to avoid this problem.

3. Exact Sampling

In this section we briefly describe the so called Coupling from the past algorithm (CFTP) to sample exactly from the stationary distribution of a Markov chain. This algorithm works under weak assumptions on the Markov chain for every finite state space and every distribution, but to guarantee that the algorithm is efficient we need some monotonicity property of the model and that the chain is rapidly mixing. For a detailed description of CFTP and the proof of correctness see [PW96].

We restrict ourself to the heat bath dynamics for the Ising model. First note that the heat bath dynamics, as defined above, admits a monotone coupling, that is, given two realizations of the heat bath chain $X = (X_t)_{t \in \mathbb{N}}$ and $Y = (Y_t)_{t \in \mathbb{N}}$, there exists a coupling $(X, Y)$ (i.e. using the same random numbers) such that

$$X_t \leq Y_t \implies X_{t+1} \leq Y_{t+1} \quad \text{for all } t \in \mathbb{N},$$

because $\min_\sigma \pi_{\beta_c}(\sigma) \geq \exp(-3N)$. 

The Glauber dynamics is rapidly mixing for $\beta \leq \beta_c$, but very slowly mixing for larger $\beta$. In Section 4 we will see how to avoid this problem.
where \( \leq \) means smaller or equal at each vertex.

Additionally we know that \(-1 \leq \sigma \leq 1\) for all \(\sigma \in \Omega_{\text{IS}}\), where \(-1 = (-1)^V\) and \(1 = (1)^V\). Therefore if we set \(X_0 = -1\) and \(Y_0 = 1\) we know that \(X_0 \leq \sigma \leq Y_0\) for all \(\sigma\) and so \(X_t \leq Z_t \leq Y_t\) for the realization \(Z = (Z_t)_{t \in \mathbb{N}}\) with \(Z_0 = \sigma\). Since this holds for all \(\sigma\), one can choose \(Z_0 \sim \pi_\beta\) and we get that whenever \(X_t\) and \(Y_t\) coalesce, they also coalesce with \(Z_t\) which has the right distribution.

After we presented the idea of the algorithm, we state the algorithm in detail. Note that the algorithm is called Coupling from the past, because we run the chains from the past to the present. The algorithm CFTP\((G, \beta)\) to sample from the distribution \(\pi_\beta^G\) works as described in Algorithm 1.

**Algorithm 1** Coupling from the past

**Input:** The graph \(G = (V, E)\) and the value of \(\beta\)

**Output:** An Ising configuration \(\sigma \sim \pi_\beta\)

1: procedure CFTP\((G, \beta)\)
2: \hspace{1em} Set \(t = 0\)
3: \hspace{1em} Set \(X_0 = -1\) and \(Y_0 = 1\)
4: \hspace{1em} while \(X_0 \neq Y_0\) do
5: \hspace{2em} \(t = t + 1\)
6: \hspace{2em} Generate random numbers \(U_{-2^t+1}, \ldots, U_{-2^t-1}\) that are sufficient to run the Markov chain.
   \hspace{2em} (e.g. \(U_t \sim \text{Uniform}\ \{V \times [0, 1]\}\))
7: \hspace{2em} Set \(X_{-2^t+1} = 0\) and \(Y_{-2^t+1} = 1\) and run the chains until time 0 by using only the random numbers \(U_{-2^t+1}, \ldots, U_{-1}\)
8: \hspace{1em} end while
9: \hspace{1em} return \(\sigma = X_0\)
10: end procedure

We denote the algorithm by CFTP\(^\pm\)(\(G, \beta\)) if we sample with respect to \(\pi_\beta^\pm\), i.e. with plus/minus boundary condition.

See [Häg02] for examples that show that it is necessary to go from the past in the future and that we have to reuse the random numbers.

Now we state the connection between the expected running time of the CFTP algorithm and the mixing time of the Markov chain.
Proposition 5. [PW96] Let $T_\beta$ be the expected running time of CFTP($G, \beta$) from Algorithm 1 with $G = (V, E)$ and $|V| = N$. Then

$$T_\beta \leq 4 \tau_\beta \log N,$$

where $\tau_\beta$ is the mixing time of the underlying Markov chain.

We see that exact sampling from the Boltzmann distribution is efficient whenever the Markov chain is rapidly mixing. By the results of Section 2 we know that this is the case for $\beta \leq \beta_c$. In the case $\beta > \beta_c$ we need a different technique to generate exact samples. Therefore we need essentially the so called random cluster model, as we will see in the next section.

4. THE RANDOM CLUSTER MODEL

The random cluster model (also known as the FK-model) was introduced by Fortuin and Kasteleyn in [FK72] to study lattice spin systems with a graph structure. It is defined on a graph $G = (V, E)$ by its state space $\Omega_{RC} = \{\omega : \omega \subseteq E\}$ and the RC measure

$$\mu_p(\omega) = \frac{1}{Z} p^{\vert \omega \vert} (1 - p)^{|E| - |\omega|} 2^{C(\omega)},$$

where $p \in (0, 1)$, $Z$ is the normalization constant and $C(\omega)$ is the number of connected components in the graph $(V, \omega)$. For a detailed introduction and related topics see the book [Gri06].

There is a tight connection between the Ising model and the random cluster model. Namely, if we set $p = 1 - e^{-\beta}$, we can translate an Ising configuration $\sigma \sim \pi_\beta$ to a random cluster state $\omega \sim \mu_p$ and vice versa. To get an Ising configuration $\sigma \in \Omega_{IS}$ from $\omega \in \Omega_{RC}$ assign independent and uniformly random spins to each connected component of $\omega$. For the reverse way include all edges $e = \{e_1, e_2\} \in E$ with $\sigma(e_1) = \sigma(e_2)$ to $\omega$ with probability $p$. For details see [ES88].

Therefore sampling an Ising configuration according to $\pi_\beta$ is equivalent to sampling a RC state from $\mu_p$ whenever both models are defined on the same graph $G$ and $p = 1 - e^{-\beta}$.

Another important concept in connection with the RC model is the duality of graphs (see e.g. [Gri10]). Let $G = (V, E)$ be a finite, planar graph, i.e. without intersecting edges if we draw it in the plane (like our $G_L$). The dual graph $G^* = (V^*, E^*)$ of $G$ is constructed as follows. Put a vertex in each face (including the infinite outer one) of the graph and connect 2 vertices by a edge if and only if the corresponding faces of $G$ share a boundary edge. It is clear, that the number of vertices can differ in the
dual graph, but we have the same number of edges. Additionally we define a dual configuration \( \omega^* \subseteq E^* \) in \( G^* \) to a RC state \( \omega \subseteq E \) in \( G \) by

\[
e \in \omega \iff e^* \notin \omega^*,
\]

where \( e^* \) is the edge in \( E^* \) that “crosses” \( e \). (By the construction, this edge is unique.) See Figure 1 for the graph \( G_L \) with \( L = 3 \) and its dual graph \( G^*_L \) together with 2 corresponding RC states.

Now we can state the following theorem about the relation of the distribution of a RC state and its dual, see [Gri10].

**Proposition 6.** [Gri10, p. 164] Let \( G = (V,E) \) be a finite, planar graph and \( \mu_p \) be the random cluster measure on \( G \). Furthermore let \( G^* = (V^*,E^*) \) be the dual graph of \( G \) and \( \mu^*_p \) be the random cluster measure on \( G^* \). Then

\[
\omega \sim \mu_p \iff \omega^* \sim \mu^*_p,
\]

where

\[
p^* = 1 - \frac{p}{2-p}.
\]

Obviously, \((p^*)^* = p\). By Proposition 6 one can see that sampling from \( \mu_p \) and sampling from \( \mu^*_p \) is equivalent. It is straightforward to get the following Proposition.

**Proposition 7.** Sampling from the Boltzmann distribution \( \pi^G_\beta \) is equivalent to sampling from the Boltzmann distribution \( \pi^{G^*_*}_\beta \), where

\[
\beta^* = \log \left( \frac{\coth \beta}{2} \right).
\]
Additionally,
\[ \beta > \beta_c \iff \beta^* < \beta_c. \]

**Proof.** The equivalence was shown by the above procedure, i.e. if we want to sample from \( \pi^G_\beta \), we can sample from \( \pi^G_{\beta^*} \) generate a RC state with respect to \( \mu_{p^*} \), go to the dual lattice with measure \( \mu_p \) and finally generate a state according to \( \pi^G_\beta \). Since \( p^{(\ast)} = 1 - e^{-\beta^{(\ast)}} \), the formula for \( \beta^* \) comes from
\[ \beta^* = -\log(1 - p^*) \equiv \log\left( \frac{2 - p}{p} \right) = \log\left( \coth \frac{\beta}{2} \right). \]

This proves the statement. \( \square \)

## 5. Efficient sampling for the Ising model

In this section we show an efficient algorithm to sample exactly from the Boltzmann distribution. But, before we prove that it is efficient, we state our algorithm.

Therefore we first have to explain how the graph \( G^*_L \) looks like. It is easy to obtain (see Figure 1) that \( G^*_L = (V^*_L, E^*_L) \) is also a square lattice with \((L - 1)^2\) vertices and an additional auxiliary vertex \( v^* \), which is connected to every vertex on the boundary of it. We denote the operation of adding a vertex to a graph and connect it to all boundary vertices by \( \cup_b \). So \( G^*_L = G_{L-1} \cup_b v^* \).

**Algorithm 2**  
Sampling from the Ising model on the square lattice

**Input:** An integer \( L \) and the value of \( \beta \)

**Output:** An Ising configuration \( \sigma \sim \pi^G_{\beta} \)

1: **if** \( \beta \leq \beta_c \) **then**
2: \( \sigma = \text{CFTP}(G_L, \beta) \)
3: **else**
4: \( \tilde{\sigma} = \text{CFTP}^+(G_{L-1}, \beta^*), \) where \( \beta^* \) is given in (3)
5: Define a Ising configuration \( \sigma^* \) on \( G^*_L = G_{L-1} \cup_b v^* \) by \( \sigma^*(v) = \tilde{\sigma}(v) \) on \( V(G_{L-1}) \) and \( \sigma^*(v^*) = 1. \)
6: Generate a RC state \( \omega^* \) from \( \sigma^* \)
7: Take the dual RC state \( \omega = (\omega^*)^* \)
8: Generate an Ising configuration \( \sigma \) from \( \omega \)
9: **end if**
10: **return** \( \sigma \)
Theorem 8. Let $G_L$ be the square lattice with $N = L^2$ vertices. Then, the algorithm from above outputs an exactly distributed Ising configuration with respect to $\pi^G_{\beta_c}$ in expected time smaller than

- $c_\beta N (\log N)^2$ for $\beta \neq \beta_c = \log(1 + \sqrt{2})$ and some $c_\beta > 0$
- $16 N^C \log N$ for $\beta = \beta_c$, where $C$ is given in (1).

Proof. The running time of the algorithm follows directly from Theorems 1 and 3 and Prop. 5. Therefore we only have to prove that the output $\sigma$ of the algorithm has the right distribution. In the case of $\beta \leq \beta_c$ this is obvious. For $\beta > \beta_c$ we know from Proposition 7 that $\sigma \sim \pi^G_{\beta_c}$, if the dual configuration $\sigma^* \in G^*_L$ (line 5 of Algorithm 2) is distributed according to $\pi^G_{\beta_c}$. But by the construction of lines 4 and 5 of Algorithm 2, this is true. For this, note that $\pi^G_{\beta}(\eta) = \pi^G_{-\beta}(\eta)$ for all $\eta \in \Omega_{IS}$. We get that for each vertex $v \in V$ (especially for $v^*$)

$$
\pi^G_{\beta}(\eta) = \pi^G_{\beta}(\eta \cap \{\sigma: \sigma(v) = 1\}) + \pi^G_{\beta}(\eta \cap \{\sigma: \sigma(v) = -1\})
$$

$$
= \pi^G_{\beta}(\{\sigma: \sigma(v) = 1\}) \pi^G_{\beta}(\eta \mid \{\sigma: \sigma(v) = 1\})
+ \pi^G_{\beta}(\{\sigma: \sigma(v) = -1\}) \pi^G_{\beta}(\eta \mid \{\sigma: \sigma(v) = -1\})
$$

$$
= \frac{1}{2} \left[ \pi^G_{\beta}(\eta \mid \{\sigma: \sigma(v) = 1\}) + \pi^G_{\beta}(\eta \mid \{\sigma: \sigma(v) = -1\}) \right]
$$

$$
= \frac{1}{2} \pi^G_{\beta}(\{\eta, -\eta\} \mid \{\sigma: \sigma(v) = 1\}).
$$

The last equality comes from the fact that

$$
\pi^G_{\beta}(\eta \mid \{\sigma: \sigma(v) = -1\}) = \pi^G_{\beta}(-\eta \mid \{\sigma: \sigma(v) = 1\}).
$$

Therefore we can sample from $\pi^G_{\beta}$ on $G^*_L$ by sampling $\eta$ from the conditional measure $\pi^G_{\beta}(\cdot \mid \{\sigma: \sigma(v^*) = 1\})$ and then choose with probability $\frac{1}{2}$ either $\eta$ or $-\eta$. If we now use that $G^*_L = G_{L-1} \cup_b v^*$ one can see that sampling on $G^*_L$ with respect to $\pi^G_{\beta}(\cdot \mid \{\sigma: \sigma(v^*) = 1\})$ is the same as sampling $\tilde{\sigma}$ from $\pi^{G_{L-1} +}_{\beta_c}$ and setting

$$
\sigma(v) = \begin{cases} 
\tilde{\sigma}(v), & v \in V(G_{L-1}) \\
1, & v = v^*.
\end{cases}
$$

Note that we omit the step of choosing $\sigma$ or $-\sigma$ with probability $\frac{1}{2}$, because the RC state that will be generated would be the same.

This completes the proof. \qed
Remark 9. Note that the same technique works also for the $q$-state Potts model. This model consists of the state space $\Omega_p = \{1, \ldots, q\}^V$ and the same measure $\pi_\beta$. In this case we consider the random cluster measure

$$\mu_{p,q}(\omega) = \frac{1}{Z} p^{|\omega|} (1 - p)^{|\partial\omega|} q^{C(\omega)}$$

and the connection of the models is again given by $p = 1 - e^{-\beta}$.

A recent result of Beffara and Duminil-Copin [BD10] shows that the self-dual point of the RC model corresponds to the critical temperature of the Potts model $\beta_c(q) = \ln(1 + \sqrt{q})$ in the same way as in the case $q = 2$ (i.e. the Ising case). Therefore, a sampling algorithm for the Potts model above (and at) the critical temperature is enough to sample at all temperatures.

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