DERIVED CATEGORIES OF COHERENT SHEAVES AND MOTIVES OF K3 SURFACES

A. DEL PADRONE AND C. PEDRINI

Abstract. Let $X$ and $Y$ be smooth complex projective varieties. We will denote by $D^b(X)$ and $D^b(Y)$ their derived categories of bounded complexes of coherent sheaves; $X$ and $Y$ are derived equivalent if there is a $\mathbb{C}$-linear equivalence $F: D^b(X) \xrightarrow{\sim} D^b(Y)$. Orlov conjectured that if $X$ and $Y$ are derived equivalent then their motives $M(X)$ and $M(Y)$ are isomorphic in Voevodsky’s triangulated category of motives $DM_{gm}(\mathbb{C})$ with $\mathbb{Q}$-coefficients. In this paper we prove the conjecture in the case $X$ is a K3 surface admitting an elliptic fibration (a case that always occurs if the Picard rank $\rho(X)$ is at least 5) with finite-dimensional Chow motive. We also relate this result with a conjecture by Huybrechts showing that, for a K3 surface with a symplectic involution $f$, the finite-dimensionality of its motive implies that $f$ acts as the identity on the Chow group of 0-cycles. We give examples of pairs of K3 surfaces with the same finite-dimensional motive but not derived equivalent.

1. Introduction

Let $X$ be a smooth projective variety over $\mathbb{C}$. We will denote by $D^b(X)$ the derived category of bounded complexes of coherent sheaves on $X$. We say that two smooth projective varieties $X$ and $Y$ are derived equivalent if there is a $\mathbb{C}$-linear equivalence $F: D^b(X) \xrightarrow{\sim} D^b(Y)$ ([Ro], [B-B-HR]). It is a fundamental result of Orlov [Or1, Th. 2.19] that every such equivalence is a Fourier-Mukai transform, i.e. there is an object $A \in D^b(X \times Y)$, unique up to isomorphism, called its kernel, such that $F$ is isomorphic to the functor $\Phi_A := p_*(q^*(-) \otimes A)$, where $p_*$, $q^*$ and $\otimes$ are derived functors. Therefore such pairs $X$ and $Y$ are also called Fourier-Mukai partners. Orlov also proved the following Theorem and stated the conjecture below.

Theorem 1. ([Or2, Th. 1]) If $\dim X = \dim Y = n$ and $\Phi_A: D^b(X) \rightarrow D^b(Y)$ is an exact fully faithful functor satisfying the following condition

\[(*) \quad \text{the dimension of the support of } A \in D^b(X \times Y) \text{ is } n,\]

then the motive $M(X)_\mathbb{Q}$ is a direct summand of $M(Y)_\mathbb{Q}$. If in addition the functor $\Phi_A$ is an equivalence then the motives $M(X)_\mathbb{Q}$ and $M(Y)_\mathbb{Q}$ are isomorphic in Voevodsky’s triangulated category of motives $DM_{gm}(\mathbb{C})_\mathbb{Q}$. Moreover the same results hold true at the level of integral motives.

The authors are members of INdAM-GNSAGA.
Conjecture 2. ([Or2, Conj. 1]) Let $X$ and $Y$ be smooth projective varieties and let $F: D^b(X) \to D^b(Y)$ be a fully faithful functor. Then the motive $M(X)_{\mathbb{Q}}$ is a direct summand of $M(Y)_{\mathbb{Q}}$. If $F$ is an equivalence then the motives $M(X)_{\mathbb{Q}}$ and $M(Y)_{\mathbb{Q}}$ are isomorphic.

In [Hu1, 2.7] Huybrechts proved that if $F: D^b(X) \simeq D^b(X)$ is a self equivalence then it acts identically on cohomology if and only if it acts identically on Chow groups (see section 5). This naturally suggests the following conjecture, which appears in [Hu2, Conj. 3.4].

Conjecture 3. Let $X$ be a complex K3 surface and let $f \in Aut(X)$ be a symplectic automorphism, i.e. $f^*$ acts as the identity on $H^{2,0}(X)$. Then $f^* = id$ on $CH^2(X)$.

In section 2 we recall some results on the finite dimensionality of motives and their Chow-K{"u}nneth decompositions. In section 3, after some general remarks on the derived equivalences between two smooth projective varieties $X$ and $Y$, we relate the derived equivalence with ungraded motives and finite-dimensionality (Proposition 15). In section 4 we specialize to the case of K3 surfaces $X$ and $Y$ and prove our main result (Theorem 21): Orlov’s conjecture holds true for K3 surfaces $X$ and $Y$ if the motive of $X$ is finite-dimensional and $X$ admits an elliptic fibration, a case that always occurs if the Picard rank $\rho(X)$ is at least 5. This restriction can possibly be removed, according to a claimed result by Mukai in [Mu2, Th2]. In section 5 we consider the case of a K3 surface with a symplectic involution $\iota$ and prove (Theorem 27) that Huybrechts’ Conjecture 3 holds true for $f = \iota$ if $X$ has a finite-dimensional motive. We also show (Theorem 30 and Examples 31) the existence of K3 surfaces $X$ and $Y$ which are not derived equivalent but with isomorphic motives.

Acknowledgements. We thank Claudio Bartocci for many helpful comments on an early draft of this paper.

2. Categories of motives and finite dimensionality

Let $X$ be a smooth variety over a perfect field $k$ and let $CH^i(X)$ be the Chow group of cycles of codimension $i$ modulo rational equivalence. We will denote by $A^i(X) = CH^i(X)_{\mathbb{Q}}$ the $\mathbb{Q}$-vector space $CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.1. Pure motives. Let $\mathcal{M}^{eff}_{rad}(k)$ be the covariant pseudo-abelian, tensor, additive category of effective Chow motives with $\mathbb{Q}$-coefficients over a perfect field $k$. Its objects are couples $(X, p)$ where $X$ is a smooth projective variety and $p \in CH^i_{dim X}(X \times X)_{\mathbb{Q}}$ is a projector, i.e. $p \circ p = p^2 = p$. Morphisms between $(X, p)$ and $(Y, q)$ in $\mathcal{M}^{eff}_{rad}$ are given by correspondences $\Gamma \in A^i_{dim X}(X \times Y)$. More precisely:

$$\text{Hom}_{\mathcal{M}^{eff}_{rad}(k)}((X, p), (Y, q)) = q \circ CH^i_{dim X}(X \times Y)_{\mathbb{Q}} \circ p.$$
The motive of a smooth projective variety \( X \) is defined as \( h(X) = (X, \Delta_X) \in \mathcal{M}_{\text{rat}}(k) \), thus giving a covariant monoidal functor \( h: \text{SmProj}/k \to \mathcal{M}_{\text{rat}}(k) \) which sends \( f: X \to Y \) to its graph \( h(f) = [\Gamma_f]: h(X) \to h(Y) \). Let \( X = \mathbb{P}^1 \), then the structure map \( X \to \text{Spec}(k) \) together with the inclusion of a closed point \( P \in \mathbb{P}^1 \) (eventually defined over an algebraic extension of \( k \), see [K-M-P, 7.2.8]) induces a splitting

\[
h(\mathbb{P}^1) \simeq 1 \oplus L
\]

where \( 1 = (\text{Spec}(k), \Delta_{\text{Spec}(k)}) \simeq (\mathbb{P}^1, [\mathbb{P}^1 \times P]) \) is the unit of the tensor structure and \( L = (\mathbb{P}^1, [P \times \mathbb{P}^1]) \) is the Lefschetz motive. By \( M_{\text{rat}}(k) \) we will denote the tensor category of covariant Chow motives, obtained from \( M_{\text{rat}}(k) \) by inverting \( L \), as in [K-M-P].

We will also consider the \( \mathbb{Q} \)-linear rigid tensor category of ungraded covariant Chow motives \( \mathcal{U}M_{\text{rat}}(k) \) (see for example [Ma, §2, §3, p. 459] and [D-M, 1.3]). It is the pseudo-abelian hull of the \( \mathbb{Q} \)-linear additive category of ungraded correspondences. Hence, its objects are pairs \((X, e)\) with \( X \) a smooth projective variety, \( e \in CH_*(X \times X)_\mathbb{Q} = \bigoplus_{i=0}^{2 \dim X} CH_i(X \times X)_\mathbb{Q} \) a projector, and

\[
\text{Hom}_{\mathcal{U}M_{\text{rat}}(k)}((X, e), (Y, f)) = f \circ CH_*(X \times Y)_\mathbb{Q} \circ e;
\]

the ungraded motive of \( X \) is \( h(X)_{\text{un}} := (X, \Delta_X) \); its endomorphism algebra is the \( \mathbb{Z} \)-graded ring (w.r.t. composition of correspondences, see [Ma, §4 p. 452])

\[
\text{End}_{\mathcal{U}M_{\text{rat}}(k)}(h(X)_{\text{un}}) = CH_*(X \times X)_\mathbb{Q}.
\]

\( \mathcal{U}M_{\text{rat}}(k) \) is a rigid \( \mathbb{Q} \)-linear tensor category in the obvious way.

2.2. Mixed motives. Let \( DM_{\text{eff}}(k) \) be the triangulated category of effective geometrical motives constructed by Voevodsky in [Voev]. We recall that there is a covariant functor \( M: \text{Sm}/k \to DM_{\text{eff}}(k) \) where \( \text{Sm}/k \) is the category of smooth schemes of finite type over \( k \). We shall write \( DM_{\text{eff}}(k, \mathbb{Q}) \) for the pseudo-abelian hull of the category obtained from \( DM_{\text{eff}}(k) \) by tensoring morphisms with \( \mathbb{Q} \), and usually abbreviate it into \( DM_{\text{eff}}(k) \). Then \( M \) induces a covariant functor

\[
\Phi: \mathcal{M}_{\text{rat}}(k) \to DM_{\text{eff}}(k)
\]

which is a full embedding. We will denote by \( DM_{\text{gm}}(k) = DM_{\text{gm}}(k, \mathbb{Q}) \) the category obtained from \( DM_{\text{gm}}(k) \) by inverting the image \( \mathbb{Q}(1) \) of \( L \). Hence, for two smooth projective varieties \( X \) and \( Y \), \( h(X) \simeq h(Y) \) in \( \mathcal{M}_{\text{rat}}(k) \) if and only if the images \( M(X) \) and \( M(Y) \) are isomorphic in \( DM_{\text{gm}}(k) \).
2.3. Finite-dimensional motives. We now recall several notion of "finiteness" on motives (see [Ki, 3.7], [Maz, 1.3], [An1, 12] and [An2, 3]). Let $C$ be a pseudoabelian, $\mathbb{Q}$-linear, symmetric tensor category and let $A$ be an object in $C$. Thanks to the symmetry isomorphism of $C$ the symmetric group on $n$ letters $\Sigma_n$ acts naturally on the $n$-fold tensor product $A^{\otimes n}$ of $A$ by itself for each object $A$: any $\sigma \in \Sigma_n$ defines a map $\sigma_{A^{\otimes n}} : A^{\otimes n} \to A^{\otimes n}$. We recall that there is a one-to-one correspondence between all irreducible representations of the group $\Sigma_n$ (over $\mathbb{Q}$) and all partitions of the integer $n$. Let $V_{\lambda}$ be the irreducible representation corresponding to a partition $\lambda$ of $n$ and let $\chi_{\lambda}$ be the character of the representation $V_{\lambda}$, then

$$d_{\lambda} = \frac{\dim(V_{\lambda})}{n!} \sum_{\sigma \in \Sigma_n} \chi_{\lambda}(\sigma) \cdot \sigma \in \mathbb{Q}\Sigma_n$$

gives, when $\lambda$ varies among the partitions of $n$, a set of pairwise orthogonal central (non primitive) idempotents in the group algebra $\mathbb{Q}\Sigma_n$; the two-sided ideal $(d_{\lambda}) = \text{Im}(A^{\otimes n})$ is the isotypic component of $V_{\lambda}$ inside $\mathbb{Q}\Sigma_n$ hence $(d_{\lambda}) \cong V_{\lambda}$ as $\mathbb{Q}\Sigma_n$-modules. Let

$$d_{\lambda}^A = \frac{\dim(V_{\lambda})}{n!} \sum_{\sigma \in \Sigma_n} \chi_{\lambda}(\sigma) \cdot \sigma_{A^{\otimes n}} \in \text{Hom}_C(A^{\otimes n}, A^{\otimes n})$$

where $\sigma_{A^{\otimes n}}$ is the morphism associated to $\sigma$. Then $\{d_{\lambda}^A\}$ is a set of pairwise orthogonal idempotents in $\text{Hom}_C(A^{\otimes n}, A^{\otimes n})$ such that $\sum d_{\lambda}^A = \text{Id}_{A^{\otimes n}}$. The category $C$ being pseudoabelian, they give a functorial decomposition

$$A^{\otimes n} = \bigoplus_{|\lambda|=n} S_{\lambda}(A) \quad (S_{\lambda}(A) = \text{Im} d_{\lambda}^A),$$

where $S_{\lambda}$ is the isotypic Schur functor associated to $\lambda$ (which is a just "multiple" of the classical one). The $n$-th symmetric product $\text{Sym}^n A$ of $A$ is then defined to be the image $\text{Im}(d_{\lambda}^A)$ when $\lambda$ corresponds to the partition $(n)$, and the $n$-th exterior power $\wedge^n A$ is $\text{Im}(d_{\lambda}^A)$ when $\lambda$ corresponds to the partition $(1, \ldots , 1)$. If $C = \mathcal{M}_{\text{rat}}(k)$ and $A = \mathfrak{h}(X) \in \mathcal{M}_{\text{rat}}(k)$ for a smooth projective variety $X$, then $\wedge^n A$ is the image of $\mathfrak{h}(X^n) = \mathfrak{h}(X)^{\otimes n}$ under the projector $(1/n!)(\sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) \Gamma_{\sigma})$, while $\text{Sym}^n A$ is its image under the projector $(1/n!)(\sum_{\sigma \in \Sigma_n} \Gamma_{\sigma})$.

**Definition 4.** The object $A$ in $C$ is said to be **Schur finite** if $S_{\lambda}(A) = 0$ for some partition $\lambda$ (i.e. $d_{\lambda}^A = 0$ in $\text{End}_C(A^{\otimes n})$); it is said to be **evenly (oddly) finite-dimensional** if $\wedge^n A = 0$ ($\text{Sym}^n A = 0$) for some $n$. An object $A$ is **finite-dimensional** (in the sense of Kimura and O’Sullivan) if it can be decomposed into a direct sum $A_+ \oplus A_-$ where $A_+$ is evenly finite-dimensional and $A_-$ is oddly finite-dimensional.

If $A$ is evenly and oddly finite-dimensional then $A = 0$ (see [Ki, 6.2] and [An2, 6.2]).

**Remark 5.** From the definition it follows that, for a smooth projective variety $X$ over $k$, the motive $\mathfrak{h}(X)$ is finite-dimensional in $\mathcal{M}_{\text{rat}}(k)$ if and only if $M(X)$ is finite-dimensional in $DM_{gm}(k)$. 

Kimura’s nilpotence Theorem in [Ki, 7.5] says that if $M$ is finite-dimensional, any numerically trivial endomorphism universally of trace zero (i.e. given by a correspondence which is numerically trivial as an algebraic cycle) of $M$ is nilpotent; therefore

**Theorem 6.** (Kimura) If $M$ and $N$ are two finite-dimensional Chow motives and $f: M \to N$ is a morphism, then $f$ is an isomorphism if and only if its reduction modulo numerical equivalence is such (see [An2 3.16.2]).

In particular, if $M \in \mathcal{M}_{rat}$ is a finite-dimensional motive such that $H^*(M) = 0$, where $H^*$ is any Weil cohomology, then $M = 0$ ([Ki, 7.3]).

**Remark 7.** For Schur-finite objects such a nilpotency result holds only under some extra assumptions as shown in [DP-M1] and [DP-M2], but not in general. In fact let $C$ be the $\mathbb{Q}$-linear rigid tensor category of bounded chain complexes of finitely generated $\mathbb{Q}$-vector spaces with the usual tensor structure and the “Koszul” commutativity constraint. Then $\text{Id}_\mathbb{Q}: \mathbb{Q} \to \mathbb{Q}$ can be thought of as an object $A$ of $C$, concentrated in homological degrees 1 and 0. It is indecomposable as $\text{End}_C(A) \cong \mathbb{Q}$, and it is not finite-dimensional for $\wedge^n(A) \neq 0$ and $\text{Sym}^n(A) \neq 0$ (as complexes) for each $n \in \mathbb{N}$. On the other hand $S_{(2,2)}(A) = 0$, i.e. $A$ is Schur-finite, for it is so under the obvious faithful (but not full) $\mathbb{Q}$-linear tensor functor towards $\mathbb{Z}/2$-graded $\mathbb{Q}$-vector spaces. Moreover, due to the Koszul rule, $\text{Id}_A$ is universally of trace zero but not nilpotent.

**Examples 8.**

(1) Finite-dimensionality and Schur-finiteness are stable under direct sums, tensor products, and direct summand. More precisely: $S_\lambda(B) = 0$ whenever $B$ is a direct summand of $A$ with $S_\lambda(A) = 0$. It is also true that a direct summand of a finite-dimensional object is such ([An2, 3.7]). Finite-dimensionality implies Schur-finiteness, but the converse does not hold not even in $DM_{gm}(k)$. In fact Peter O’Sullivan showed that there exist smooth surfaces $S$ whose motives in $DM_{gm}(k)$ is Schur-finite but not finite dimensional, see [Maz, 5.11].

(2) Clearly we have $\wedge^2 1 = 0$ in any symmetric tensor category. It is also straightforward that $\wedge^2 \mathbb{L} = 0$ for the Lefschetz motive, and $\wedge^3 h_1(\mathbb{P}^1) = 0$. Kimura showed $\text{Sym}^{2g+1}(h^1(C)) = 0$ for any smooth projective curve $C$ of genus $g$ [Ki, 4.2].

We also have Kimura’s conjecture:

**Conjecture 9.** Any motive in $\mathcal{M}_{rat}$ is finite-dimensional.

**Remark 10.** The status of the conjecture is the following.

(1) The conjecture is true for curves, abelian varieties, Kummer surfaces, complex surfaces not of general type with $p_g = 0$ (e.g. Enriques surfaces), Fano 3-folds [G-G]. For a complex surface $X$ of general type with $p_g(X) = 0$ the finite-dimensionality of the motive $h(X)$ is equivalent to Bloch’s conjecture, i.e. to the vanishing of the Albanese Kernel of $X$ (see [G-P, Th. 7]). If the conjecture holds for $h(X)$ then it holds
true for $h(Y)$ with $Y$ a smooth projective variety dominated by $X$. The full subcategory of $\mathcal{M}_{rat}$ on finite-dimensional objects is a $\mathbb{Q}$-linear rigid tensor subcategory closed under direct summand.

(2) Let $X$ be a K3 surface; then $h(X)$ is finite-dimensional in the following cases, see [Pe3]

- $\rho(X) = 19$ or $\rho(X) = 20$. In these cases $X$ has a Nikulin involution which gives a Shioda-Inose structure, in the sense of [Mo, 6.1], and the transcendental motive $t_2(X)$ of $X$ (see [2.4]) is isomorphic to the transcendental motive of a Kummer surface [Pe3, Th. 4].
- $X$ has a non-symplectic group $G$ acting trivially on the algebraic cycles and the order of the kernel (a finite group) of the map $\text{Aut}(X) \to \mathcal{O}(\text{NS}(X))$ is different from 3, where $\mathcal{O}(\text{NS}(X))$ is the group of isometries of $\text{NS}(X)$. Then, by a result in [L-S-Y, Th. 5], $X$ is dominated by a Fermat surface $F_n$, whose motive is of abelian type (hence finite-dimensional) by the Shioda-Katsura inductive structure [S-K, Th. 1]. K3 surfaces satisfying these conditions have $\rho(X) = 2, 4, 6, 10, 12, 16, 18, 20$.

By a result of Deligne ([De, 6.4]), for every complex polarized K3 surface there exists a smooth family of polarized K3 surfaces $\{X_t\}_{t \in \Delta}$, with $\Delta$ the unit disk, such that the central fibre $X_0$ is isomorphic to $X$. Therefore the finite-dimensionality of the motive of a general K3 surface, i.e. with $\rho(X) = 1$, implies the finite-dimensionality of the motive of any K3 surface, see [Pe1, 4.3].

(3) In all the known cases where the motive $h(X)$ is finite-dimensional, it lies in the tensor subcategory of $\mathcal{M}_{rat}(k)$ generated by the motives of abelian varieties (see [An, 2.5]).

The following result will appear in [DP].

**Proposition 11.** Let $M = (X, p)$ be an effective Chow motive. Then

(a) The (graded) motive $M$ is Schur-finite if and only if the ungraded motive $M_{an}$ is such. More precisely for any partition $\lambda$ we have $S^{\mathcal{M}_{rat}(k)}(M) = 0$ if and only if $S^{\mathcal{M}_{rat}(k)}(M_{an}) = 0$. In particular, being $M$ even or odd depends only on the ungraded isomorphism class of the ungraded motive $M_{an}$.

(b) If $M$ is finite-dimensional in $\mathcal{M}_{rat}(k)$ then $M_{an}$ is so in $\mathcal{U}\mathcal{M}_{rat}(k)$. Moreover, if $M = h(X)$ with $X$ a variety such that the projections on the even and the odd part of the cohomology (w.r.t. a given Weil cohomology theory) are algebraic then $h(X)$ is finite-dimensional if and only if $h(X)_{an}$ is.

**Remark 12.** The hypothesis in (b) of Proposition 11 is Jannsen’s homological sign conjecture $C^+(X)$ [An2, 5.1.3], called $S(X)$ in [Ja, 13.3].
2.4. The refined Chow-K"unneth decomposition. Let for simplicity $k = \mathbb{C}$ in what follows. We recall from [K-M-P, 2.1] that the covariant Chow motive $h(S) \in M_{rat}(\mathbb{C})_Q$ of any smooth projective surface $S$ has a refined Chow-K"unneth decomposition
\[ \sum_{0 \leq i \leq 4} h_i(S) \]
corresponding to a splitting $\Delta_S = \sum_{0 \leq i \leq 4} \pi_i$ of the diagonal in $H^*(S \times S)$. Here $h_0(S) = (S, [S \times P]) \simeq (\text{Spec}(\mathbb{C}), \text{Id}) = 1$ and $h_4(S) = (S, [P \times S]) \simeq \mathbb{L}^2$, where $P$ is a rational point on $S$. Also
\[ h_2(S) = h_2^{\text{alg}}(S) \oplus t_2(S) \]
with $h_2^{\text{alg}}(S) = (S, \pi_2^{\text{alg}})$ the effective Chow motive defined by the idempotent
\[ \pi_2^{\text{alg}}(S) = \sum_{1 \leq h \leq \rho} \frac{[D_h \times D_h]}{D_h^2} \in A_2(S \times S) \]
where $\rho = \rho(S)$ is the rank of the Neron-Severi $\text{NS}(S)$ and $\{D_h\}$ is an orthogonal bases of $\text{NS}(S)_\mathbb{Q}$. It follows that $h_2^{\text{alg}}(S) \simeq \mathbb{L}^{\rho}$.

Definition 13. The Chow motive $t_2(S) = (S, \pi_2^{tr}, 0)$, with $\pi_2^{tr} = \pi_2 - \pi_2^{\text{alg}}$, is the transcendental part of the motive $h(S)$. Then $H^j(t_2(S)) = 0$ if $i \neq 2$ and $H^2(t_2(S)) = H^2_{\text{tr}}(S) = \pi_2^{tr}H^2(S, \mathbb{Q}) = H^2_{\text{tr}}(S, \mathbb{Q})$.

The Chow motive $t_2(S)$ does not depend on the choices made to define the refined Chow-K"unneth decomposition, it is functorial on $S$ for the action of correspondences, and it is a birational invariant of $S$ (see [K-M-P]).

Remark 14. For any smooth projective surface $S$, all the motives $h_i(S)$ appearing in a refined Chow-K"unneth decomposition, except possibly for $t_2(S)$ are finite dimensional. Therefore the motive $h(S)$ of a surface $S$ is finite dimensional if and only if the motive $t_2(S)$ is evenly finite dimensional, i.e. $\wedge^n t_2(S) = 0$ for some $n$. If $S$ has no irregularity (i.e. $q(S) := \dim H^1(S, \mathcal{O}_S) = 0$) then $h_1(S) = h_3(S) = 0$.

2.5. Refined C-K decomposition of a $K3$ surface. Let now $S$ be a smooth (irreducible) projective K3 surface over $\mathbb{C}$. As $S$ is a regular surface (i.e. $q(S) = 0$), its refined Chow-K"unneth decomposition has the following shape
\[ h(S) = 1 \oplus h_2^{\text{alg}}(S) \oplus t_2(S) \oplus \mathbb{L}^{\otimes 2} \simeq 1 \oplus \mathbb{L}^{\rho} \oplus t_2(S) \oplus \mathbb{L}^{\otimes 2} \]
with $1 \leq \rho \leq 20$. Moreover
\[ A_i(t_2(S)) = \pi_2^{tr} A_i(S) = 0 \text{ for } i \neq 0; \quad A_0(t_2(S)) = A_0(S)_0, \]
where the last $\mathbb{Q}$-vector space is the group of 0-cycles of degree 0 tensored with $\mathbb{Q}$. We also have
\[ \dim H^2(S) = b_2(S) = 22; \quad \dim H^2_{\text{tr}}(S) = b_2(S) - \rho(S) = 22 - \rho. \]
By \( T_{S,Q} = H^2_t(S,\mathbb{Q}) \) we will denote the lattice of transcendental cycles, tensored with \( \mathbb{Q} \), it coincides with the orthogonal complement to the Neron-Severi \( \text{NS}(S) \otimes \mathbb{Q} \) in \( H^2(S,\mathbb{Q}) \).

3. Derived equivalence and motives

Let \( X \) and \( Y \) be smooth projective varieties over \( \mathbb{C} \). If \( X \) and \( Y \) are derived equivalent then (see e.g. [Ro], [Hu], [B-B-HR]) \( \dim X = \dim Y, \kappa(X) = \kappa(Y) \) (where \( \kappa \) is the Kodaira dimension), and \( H^*(X,\mathbb{Q}) \cong H^*(Y,\mathbb{Q}) \) (isomorphism of \( \mathbb{Z}/2 \)-graded vector spaces). If \( \dim X = 2 \) the surfaces \( X \) and \( Y \) have the same Picard number and the same topological Euler number; and \( X \) is a K3 surface, respectively an abelian surface, if and only if \( Y \) is.

Kawamata conjectured that, up to isomorphism, \( X \) has only a finite number of Fourier-Mukai partners \( Z \) [Ka]. This conjecture is true for curves (and in this case \( Z \cong X, [B-B-HR, 7.16] \)), surfaces ([B-M]), abelian varieties (see [Ro, 3] and [H-NW, 0.4]), and varieties with ample or antample canonical bundle, in which case \( Z \cong X \) (due to Bondal-Orlov, see [B-B-HR, 2.51]).

The following result is somewhat in the same spirit, with respect to the relation between derived equivalence of smooth projective varieties and their associated Chow motives.

**Proposition 15.** Let \( \Phi_A : D^b(X) \to D^b(Y) \) an exact equivalence, then

(a) The ungraded Chow motives \( h(X)_{un} \) and \( h(Y)_{un} \) are isomorphic. If the condition \( (\ast) \) in Theorem 2 is satisfied then the isomorphism is given by a correspondence of degree zero, hence \( h(X) \) and \( h(Y) \) are isomorphic as Chow motives.

(b) The (graded) motive \( h(X) \) is Schur-finite if and only if \( h(Y) \) is such.

(c) If \( X \) is curve, a surface, an abelian variety, or a finite product of them (or any variety if \( k \) is algebraic over a finite field), then \( h(X) \) is finite-dimensional if and only if \( h(Y) \) is such.

**Proof.** (a) The argument in [Or 1, p. 1243], which we briefly recall can be used to prove that \( h(X)_{un} \cong h(Y)_{un} \) in \( \mathcal{U} \mathcal{M}_{rat}(k) \). Let \( B \in D^b(X \times Y) \) be the kernel of the quasi-inverse of \( \Phi_A \). Using Huybrechts’ notation ([Hu1, p. 1534] and [Hu2, 4.1]), we then have (non homogeneus, \( \mathbb{Q} \)-linear) algebraic cycles

\[
a = v^{CH}(A) := \text{ch}(A) \cdot \sqrt{\text{td}_{X \times Y}} = p_1^* \left( \sqrt{\text{td}_X} \right) \cdot \text{ch}(A) \cdot p_2^* \left( \sqrt{\text{td}_Y} \right) \in CH_*(X \times Y)_\mathbb{Q},
\]

and

\[
b = v^{CH}(B) = p_1^* \left( \sqrt{\text{td}_Y} \right) \cdot \text{ch}(B) \cdot p_2^* \left( \sqrt{\text{td}_X} \right) \in CH_*(Y \times X)_\mathbb{Q},
\]

where \( \text{td} \) is the Todd class and \( \text{ch} : D^b(Z) \to CH_*(Z)_\mathbb{Q} \) is the composition of the Chern character with the Euler characteristic \( \chi(\mathcal{E}) = \sum (-1)^i [H^i(\mathcal{E})] \in K_0(Z) \) of the complex of sheaves \( \mathcal{E} \). Orlov proved, by Grothendieck-Riemann-Roch, that

\[
b \circ a = [\Delta_X] = \text{Id}_{h(X)_{un}}, \quad \text{and} \quad a \circ b = [\Delta_Y] = \text{Id}_{h(Y)_{un}}
\]
as (ungraded) correspondences.

In case the kernel \( A \) satisfies the hypothesis (\( \ast \)) of Theorem 1, that is \( \dim \text{supp}(A) = \dim X \), it turns out that the “middle components” \( a_d \in CH_d(X \times Y)_Q \) and \( b_d \in CH_d(Y \times X)_Q \) of the above cycles \( a \) and \( b \) (which are correspondences of degree zero) give an isomorphism at the level of usual Chow motives.

(b) As already observed in Proposition 11, being Schur-finite for a graded motive \( M \) can be tested on \( M_{\text{un}} \).

(c) In all these cases \( C^+(X) \) holds true, hence Proposition 11(b) applies.

\[ \square \]

Example 16. Let \( X = A \) be an abelian variety, \( Y = \hat{A} \) its dual and let \( A = P_A \in \text{Pic}(A \times \hat{A}) \) be the sheaf complex given by the Poincaré bundle. The corresponding isomorphism of ungraded Chow motives is given by

\[ \text{ch}(P_A) : h(A)_{\text{un}} \to h(\hat{A})_{\text{un}} \]

because the Todd classes are 1 for abelian varieties. It can be shown (see [B-L 16.3]) that it coincides with the \textit{motivic Fourier-Mukai transform} of Deninger and Murre ([D-M, 2.9]). We note that in this case the dimension of the support of \( A \) is equal to \( \dim(A \times \hat{A}) = 2 \cdot \dim A \). As \( A \) and \( \hat{A} \) are isogenous it follows that their Chow motives (with \( \mathbb{Q} \)-coefficients) are isomorphic (see for example [An1, 4.3.3]).

Remarks 17. Let us make two comments on Orlov’s hypothesis (\( \ast \)), that is “the dimension of the support of the kernel \( A \) of the equivalence \( D^b(X) \simeq D^b(Y) \) equals \( \dim X \”).

(1) If \( \Phi_A \) is an equivalence then the natural projections

\[ \text{supp}(A) \to X, \quad \text{supp}(A) \to Y \]

are surjective [Hu, 6.4]. Therefore, in general, \( \dim \text{supp}(A) \geq \dim X \) whenever \( \Phi_A \) is an equivalence.

(2) If \( \Phi_A \) is an equivalence and Orlov’s hypothesis (\( \ast \)) holds true then \( X \) and \( Y \) are \textbf{K-equivalent}, a notion due to Kawamata [Ka] (see [B-B-HR, 2.48]). In case \( X \) and \( Y \) are smooth projective complex \textit{surfaces}, they are K-equivalent if and only if they are isomorphic [B-B-HR, 7.19]. This is, in general, not the case for K3 surfaces, see for example [So].

In connection with the result in [Or2, Th. 1] Orlov made the following more precise conjecture [Or2, Conj. 2]:

\[ \textbf{Conjecture 18.} \text{Let } A \text{ be an object on } X \times Y \text{ for which } \Phi_A : D^b(X) \to D^b(Y) \text{ is an equivalence. Then there are line bundles } L \text{ and } M \text{ on } X \text{ and } Y, \text{ respectively, such that the } \dim X \text{ component of the cycle associated to } A' := p_1^*L \otimes A \otimes p_2^*M \text{ determines an isomorphism between the motives } M(X)_\mathbb{Q} \text{ and } M(Y)_\mathbb{Q} \text{ in } DM_{gm}(\mathbb{C})_\mathbb{Q}. \]
4. Derived equivalence and complex K3 surfaces

Let us now consider Orlov’s Conjecture 2 in low dimension; a case of particular interest is that of K3 surfaces. We recall that if \( Y \) is a Fourier-Mukai partner of a K3 surface \( X \) (respectively abelian surface), then also \( Y \) is a K3 surface (respectively abelian surface).

We fix some notation. For a K3, or abelian, smooth projective complex surface \( X \) we have the Mukai lattice, also called extended Hodge lattice in [B-M, 5], that is the cohomology ring
\[
\tilde{H}(X, \mathbb{Z}) := H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}),
\]
endowed with the symmetric bilinear form
\[
\langle (r_1, D_1, s_1), (r_1, D_1, s_1) \rangle := D_1 \cdot D_2 - r_1s_2 - r_2s_1,
\]
and the following Hodge decomposition
\[
\tilde{H}^{(0,2)}(X, \mathbb{C}) = H^{0,2}(X, \mathbb{C}), \quad \tilde{H}^{(2,0)}(X, \mathbb{C}) = H^{2,0}(X, \mathbb{C}),
\]
\[
\tilde{H}^{(1,1)}(X, \mathbb{C}) = H^0(X, \mathbb{C}) \oplus H^{1,1}(X, \mathbb{C}) \oplus H^4(X, \mathbb{C}).
\]

Inside \( H^2(X, \mathbb{Z}) \) we have two sublattices, the Neron-Severi lattice \( \text{NS}(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C}) \), and its orthogonal complement \( T_X \), the transcendental lattice of \( X \). The transcendental lattice inherits a Hodge structure from \( H^2(X, \mathbb{Z}) \).

Definition 19. Let \( X \) and \( Y \) be two complex K3 surfaces. A map \( T_X \to T_Y \) (resp. \( T_X, \mathbb{Q} \to T_Y, \mathbb{Q} \)) is a Hodge homorphism of (resp. rational) Hodge structures if it preserves the Hodge structures of \( H^2_{tr}(X) \otimes \mathbb{C} \) and of \( H^2_{tr}(Y) \otimes \mathbb{C} \), i.e. if the one dimensional subspace \( H^2_{tr}(X) \subset T_X \otimes \mathbb{C} \) goes to \( H^2_{tr}(Y) \subset T_Y \otimes \mathbb{C} \). A Hodge isomorphism \( T_X \to T_Y \) is an Hodge isometry if it is an isometry with respect to the quadratic form induced by the usual intersection form. A rational Hodge isometry \( \phi: T_X, \mathbb{Q} \to T_Y, \mathbb{Q} \) is induced by an algebraic cycle \( \Gamma \in CH^2(X \times Y)_\mathbb{Q} \) if \( \phi = \Gamma_*: T_X, \mathbb{Q} \to T_Y, \mathbb{Q} \) (cf. [Mu, pp. 346-347]).

Due to work of Mukai and Orlov ([Mu], [Or1, 3.3 and 3.13], [B-M, 5.1]) we have the following result:

Theorem 20. Let \( X \) and \( Y \) be a pair of K3 (resp. abelian) surfaces. The following statements are equivalent.

(a) \( X \) and \( Y \) are derived equivalent,
(b) the transcendental lattices \( T_X \) and \( T_Y \) are Hodge isometric,
(c) the extended Hodge lattices \( \tilde{H}(X, \mathbb{Z}) \) and \( \tilde{H}(Y, \mathbb{Z}) \) are Hodge isometric,
(d) \( Y \) is isomorphic to a fine, two-dimensional moduli space of stable sheaves on \( X \).
The next result relates the finite-dimensionality of the motive of a K3 surface with Orlov’s conjecture.

**Theorem 21.** Let $X, Y$ be smooth projective K3 surfaces over $\mathbb{C}$ such that $X$ has an elliptic fibration and the Chow motive $h(X)$ is finite dimensional. If $D^b(X) \simeq D^b(Y)$ then the motives $M(X)$ and $M(Y)$ are isomorphic in $DM_{gm}(\mathbb{C})$.

**Proof.** By point (b) of Proposition 15 we know that $h(Y)$ is finite-dimensional. Theorem 20 ensures the existence of a Hodge isometry $\phi : TX,\mathbb{Q} \overset{\sim}{\rightarrow} TY,\mathbb{Q}$ which, by [Ni, Th. 3], is induced by an algebraic cycle, i.e. there exists an algebraic correspondence $\Gamma \in CH_2(X \times Y)_{\mathbb{Q}}$ such that $\Gamma_* = \phi$. Then $\pi^Y_2 \circ \Gamma \circ \pi^X_2$ induces an isomorphism between the transcendental motives as homological motives, hence numerical ones; thus, thanks to Theorem 6 it is an isomorphism at the level of Chow motives by finite-dimensionality. Then $h(X)$ and $h(Y)$ are isomorphic in $\mathcal{M}_{rat}(\mathbb{C})$, hence $M(X)$ and $M(Y)$ are isomorphic in $DM_{gm}(\mathbb{C})$. □

**Remark 22.** Besides the properties of finite-dimensional objects, the other key point in the previous argument is the algebraicity of $\phi$. This question goes back to a Săfarevîc’s conjecture stated at the ICM 1970 in Nice [Sh, B4 p. 416]. Shioda and Inose verified the conjecture in [S-I] for singular K3 surfaces (those having the maximum possible Picard number, i.e. $\rho(X) = 20$). Then Mukai proved it in [Mu1, 1.10] for K3 surfaces with $\rho(X) \geq 11$, and Nikulin showed its validity in [Ni, proof of Th.3] whenever $\text{NS}(X)$ contains a (nonzero) square zero element; this is is certainly the case if $\rho \geq 5$ and, according to Pjatetskiĭ-Sâpiro and Săfarevîc [PS-S], it is equivalent to the existence of an elliptic fibration on $X$. Eventually Mukai claimed to have completely solved the problem at ICM 2002 in Beijing [Mu2, Th. 2], hence the hypothesis on the elliptic fibration could be removed.

5. **Nikulin involutions**

Let $X$ be a smooth projective K3 surface over $\mathbb{C}$ and let $\Phi_A : D^b(X) \overset{\sim}{\rightarrow} D^b(X)$ be an autoequivalence. To $\Phi_A$ we can associate an Hodge isometry

$$\Phi^H_A : \tilde{H}(X,\mathbb{Z}) \simeq \tilde{H}(X,\mathbb{Z}),$$

as well as an automorphism of the Chow group

$$\Phi^{CH}_A : CH^*(X) \simeq CH^*(X)$$
induced by the correspondence \( v^{CH}(A) \in CH^*(X \times X) \) defined in [Hu2, 4.1]. We therefore get the two representations

\[
\begin{array}{c}
\text{Aut}(CH^*(X)) \\
\rho^{CH} \\
\text{Aut}(D^b(X)) \\
\rho^H \end{array}
\]

\( O(\tilde{H}(X, \mathbb{Z})) \)

Here \( O(\tilde{H}(X, \mathbb{Z})) \) is the group of all integral Hodge isometries of the weight two Hodge structure defined on the Mukai lattice \( \tilde{H}(X, \mathbb{Z}) \) and \( \text{Aut}(CH^*(X)) \) denotes the group of all automorphisms of the additive group \( CH^*(X) \). The following Theorem has been proved by D. Huybrechts in [Hu1, 2.7].

**Theorem 23.** Ker(\( \rho^H \)) = Ker(\( \rho^{CH} \)).

From Theorem 23 if \( \rho^H(\Phi_A) = \Phi_A^H \) is the identity in \( O(\tilde{H}(X, \mathbb{Z})) \), then the correspondence \( v^{CH}(A) \) acts as the identity on \( CH^*(X) \). In particular \( \phi_A^H \) acts as the identity on \( H^{2,0}(X) \simeq H^0(X, \Omega^2_X) \subset H^2_\tau(X, \mathbb{C}) \). The above Theorem suggested Huybrechts’ conjecture \( 3 \), that is that any symplectic automorphism \( f \in \text{Aut}(X) \) acting trivially on \( H^{2,0}(X) \) acts trivially also on \( CH^2(X) \).

In this section we deal with the case of a symplectic involution.

**Definition 24.** A Nikulin involution \( \iota \) on a K3 surface \( X \) is a symplectic involution, i.e. \( \iota^*(\omega) = \omega \) for all \( \omega \in H^{2,0}(X) \).

A Nikulin involution \( \iota \) on a complex projective K3 \( X \) has the following special properties, as proved by Nikulin (see e.g. [Mo, 5.2]):

- the fixed locus of \( \iota \) consists of precisely eight distinct points and
- the minimal resolution \( Y \) of the quotient \( X/\iota = X/<\iota> \) is a K3 surface.

The surface \( Y \) can also be obtained as the quotient of the blow up \( \tilde{X} \) of \( X \) in the 8 fixed points by the extension \( \tilde{i} \) of \( i \) to \( \tilde{X} \) ([Mo, 3], [VG-S, 1.4]). In other words we get the commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{b} & X \\
\downarrow{g} & & \downarrow \\
Y & \longrightarrow & X/\iota
\end{array}
\]

where \( Y \) is a desingularization of the quotient surface \( X/\iota \) and \( Y \simeq \tilde{X}/\tilde{i} \), with \( \tilde{i} \) the involution induced by \( i \) on \( \tilde{X} \).
As explained in [VG-S, 2.1] a K3 surface with a Nikulin involution has $\rho(X) \geq 9$. Moreover ([VG-S, 2.4]) $\iota$ induces an isomorphism $\phi_\iota: T_{X,Q} \sim T_{Y,Q}$ of rational Hodge structures.

Let $X$, $\tilde{X}$, and $Y$ be as in the diagram above and let $t_2(X)$ be the transcendental part of the motive of $X$. By [Ma, §3 Example 1] the degree 2 map $g$ induces a splitting in $\mathcal{M}_{rat}(\mathbb{C})$

$$h(\tilde{X}) = (X,p) \oplus (X, \Delta_X - p) \simeq h(Y) \oplus (X, \Delta_X - p)$$

where $p = 1/2(\Gamma'_g \circ \Gamma_g) \in A_2(X \times X)$. Since $t_2(\iota)$ is a birational invariant we have $t_2(X) = t_2(\tilde{X})$. From the above splitting it follows that $t_2(Y)$ is a direct summand of $t_2(X)$, i.e. $t_2(X) = t_2(Y) \oplus N$.

**Proposition 25.** Let $X$, $\tilde{X}$, and $Y$ be as in the diagram above. Then $t_2(X) \simeq t_2(Y) \iff A_0(X)^{\iota} = A_0(X)$

i.e. if and only if the involution $\iota$ acts as the identity on $A_0(X)$. If $t_2(X) \simeq t_2(Y)$, then the rational map $X \to Y$ induces an isomorphism between the motives $h(X)$ and $h(Y)$ and therefore also between $M(X)$ and $M(Y)$ in $DM_{gm}(\mathbb{C})$.

**Proof.** Let $k(X)$ be the field of rational functions of $X$; then the Chow group of 0-cycles on $X_{k(X)}$ may be identified with

$$\lim_{U \subset X} A^2(U \times X) \simeq A_0(X_{k(X)})$$

where $U$ runs among the open sets of $X$ (see [Bl, Lecture 1. Appendix]). Since $\text{Alb}(X) = 0$, the Albanese kernel $T(X_{k(X)})$ coincides with $A_0(X_{k(X)})_0$. By [K-M-P, 5.10] there is an isomorphism

$$\text{End}_{\mathcal{M}_{rat}}(t_2(X)) \simeq \frac{A_0(X_{k(X)})}{A_0(X)}$$

where the identity map of $t_2(X)$ corresponds to the class of $[\xi]$ in $\frac{A_0(X_{k(X)})}{A_0(X)}$. Here $\xi$ denotes the generic point of $X$ and $[\xi]$ its class as a cycle in $A_0(X_{k(X)})$. The involution $\iota$ induces an involution $\tilde{\iota}$ on $A_0(X_{k(X)})$. The splitting

$$[\xi] = 1/2([\xi] + \tilde{\iota}([\xi])) + 1/2([\xi] - \tilde{\iota}([\xi]))$$

in $A_0(X_{k(X)})$ corresponds to the splitting of the identity map of $t_2(X)$ in $t_2(X) = t_2(Y) \oplus N$. Therefore $N = 0$ if and only if $\tilde{\iota}([\xi]) = [\xi]$. From the equalities $A_0(t_2(X)) = A_0(X)_0$, $A_0(t_2(Y)) = A_0(Y)_0$ and $A_0(X)^\iota = A_0(Y)$ we get

$$t_2(X) \simeq t_2(Y) \iff N = 0 \iff \tilde{\iota}([\xi]) = [\xi] \iff A_0(X)^\iota = A_0(X).$$

The rest follows from [2.5] because $X$ and $Y$ are K3 surfaces, with $\rho(X) = \rho(Y)$. $\Box$

Next we show that for every K3 surface with a Nikulin involution $\iota$ the finite dimensionality of $h(X)$ implies that $\iota$ acts as the identity on $A_0(X)$. Therefore for such $X$ Conjecture [3] holds true.
Lemma 26. Let $X$ be a K3 surface with a Nikulin involution $\iota$. Then $\rho(X) = \rho(Y)$ and $t = 6$, where $t$ denotes the trace of the involution $\iota$ on $H^2(X, \mathbb{C})$.

Proof. Let $X$ be a smooth projective surface over $\mathbb{C}$ with $q(X) = 0$ and an involution $\sigma$ and let $Y$ be a desingularization of $X/\sigma$. Let $e(-)$ be the topological Euler characteristic. Then we have (see [D-ML-P, 4.2])

$$e(X) + t + 2 = 2e(Y) - 2k$$

where $t$ is the trace of the involution $\sigma$ on $H^2(X, \mathbb{C})$ and $k$ is the number of the isolated fixed points of $\sigma$. If $X$ and $Y$ are K3 surfaces and $\sigma = \iota$ is a Nikulin involution, then $e(X) = e(Y) = 24$ and $k = 8$. Therefore we get $t = 6$. Since $\dim H^2_{tr}(X) = \dim H^2_{tr}(Y)$ and $b_2(X) = b_2(Y) = 22$, we have $\rho(X) = \rho(Y)$. \hfill $\square$

Theorem 27. Let $X$ be a K3 surface with a Nikulin involution $\iota$. If $h(X)$ is finite dimensional then $h(X) \simeq h(Y)$, therefore $\iota$ acts as the identity on $A_0(X)$.

Proof. Let $Y$ be the desingularization of $X/\iota$. Then $Y$ is a K3 surface and we have $\tau_2(\tilde{X}) \simeq \tau_2(X)$ because $\tau_2(-)$ is a birational invariant for surfaces, see [K-M-P]. Also

$$H^2_{tr}(X) \simeq H^2_{tr}(\tilde{X}) \simeq H^2_{tr}(Y)$$

because the Nikulin involution acts trivially on $H^2_{tr}(X)$. Let $E_{r1}, 1 \leq i \leq 8$ be the exceptional divisors of the blow-up $\tilde{X} \to X$ and let $g_*(E_i) = C_i$ be the corresponding $(-2)$-curves on $Y$. We have $\rho = \text{rank}(\text{NS}(X)) \geq 9$, $b_2(X) = b_2(Y) = 22$ and $e(X) = e(Y) = 24$, where $e(X)$ is the topological Euler characteristic. Let $t$ be the trace of the action of the involution $\iota$ on the vector space $H^2(X, \mathbb{C})$. By Lemma 26, we have $t = 6$. The involution $\iota$ acts trivially on $H^2_{tr}(X)$ which is a subvector space of $H^2(X, \mathbb{C})$ of dimension $22 - \rho$; therefore the trace of the action of $\iota$ on $\text{NS}(X) \otimes \mathbb{C}$ equals $\rho - 16$. Since the only eigenvalues of an involution are $+1$ and $-1$ we can find an orthogonal basis for $\text{NS}(X) \otimes \mathbb{C}$ of the form $H_1, \ldots, H_r; D_1, \ldots, D_8$, with $r = \rho - 8 \geq 1$ such that $\iota_*(H_j) = H_j$ and $\iota_*(D_i) = -D_i$. Then $\text{NS}(\tilde{X}) \otimes \mathbb{C}$ has a basis of the form $E_1, \ldots, E_8; H_1, \ldots, H_r; D_1, \ldots, D_8$. Since $X$ and $Y$ are K3 surfaces we have $q(X) = q(Y) = q(\tilde{X}) = 0$. Therefore we can find Chow-Künneth decompositions for the motives $h(X), h(\tilde{X})$ such that $h_1 = h_3 = 0$ and

$$h(X) = 1 \oplus h_2^{alg}(X) \oplus \tau_2(X) \oplus \mathbb{L}^2 \simeq 1 \oplus \mathbb{L}^{2\rho} \oplus \tau_2(X) \oplus \mathbb{L}^2$$

$$h(\tilde{X}) = 1 \oplus h_2^{alg}(\tilde{X}) \oplus \tau_2(X) \oplus \mathbb{L}^2 \simeq h(X) \oplus \mathbb{L}^{alg}$$

where $h_2^{alg}(\tilde{X}) = (\tilde{X}, \pi_2^{alg}(\tilde{X}))$ with $\pi_2^{alg}(\tilde{X}) = \Gamma + I$ and

$$\Gamma = \sum_{1 \leq k \leq 8} \frac{[E_k \times E_k]}{E_k^2} + \sum_{1 \leq j \leq r} \frac{[H_j \times H_j]}{H_j^2}, \quad I = \sum_{1 \leq h \leq r} \frac{[D_h \times D_h]}{D_h^2}.$$

Also

$$\mathbb{L}^{alg} \simeq \left(\tilde{X}, \sum_{1 \leq k \leq 8} \frac{[E_k \times E_k]}{E_k^2}\right).$$
Let $g: \tilde{X} \to Y$ and let $p = 1/2(\Gamma_\rho' \circ \Gamma_\rho) \in A^2(\tilde{X} \times \tilde{X})$: then $p$ is a projector and

$$h(\tilde{X}) = (\tilde{X}, p) \oplus (\tilde{X}, \Delta_{\tilde{X}} - p) \simeq h(Y) \oplus (\tilde{X}, \Delta_{\tilde{X}} - p)$$

because $(\tilde{X}, p) \simeq h(Y)$ by [Ma, §3 Example 1]. The set of $r + 8 = \rho$ divisors $g_*(E_k) = C_k$, for $1 \leq k \leq 8$ and $g_*(H_j) \simeq H_j$, for $1 \leq j \leq r$ gives an orthogonal basis for $\text{NS}(Y) \otimes \mathbb{Q}$. Therefore we can find a Chow-K¨unneth decomposition of $h(Y)$ such that

$$h_2^\text{alg}(Y) \simeq (\tilde{X}, \Gamma) \simeq L^{\oplus 8}$$

and we get

$$h_2(\tilde{X}) = h_2^\text{alg}(\tilde{X}) \oplus t_2(\tilde{X}) \simeq h_2^\text{alg}(Y) \oplus L^{\oplus 8} \oplus t_2(Y) \oplus M$$

where $H^*(M) = 0$ because $H^2_\text{tr}(\tilde{X}) = H^2_\text{tr}(X) = H^2_\text{tr}(Y)$. From Theorem 6 it follows that $M = 0$ and we get an isomorphism

$$h_2(\tilde{X}) \simeq h_2(Y) \oplus L^{\oplus 8} \simeq h_2(X) \oplus L^{\oplus 8}$$

which implies $h(X) \simeq h(Y)$. The rest follows from Proposition 25.

The following result gives examples of K3 surfaces with a Nikulin involution $\iota$ such that $\iota$ acts as the identity on $A_0(X)$.

**Theorem 28.** Let $X$ be a smooth projective K3 surface over $\mathbb{C}$ with $\rho(X) = 19, 20$. Then $X$ has a Nikulin involution $\iota$, $h(X)$ is finite dimensional and $\iota$ acts as the identity on $A_0(X)$.

**Proof.** By [Mo, 6.4] $X$ admits a Shioda-Inose structure, i.e. there is a Nikulin involution $\iota$ on $X$ such that the desingularization $Y$ of the quotient surface $X/\iota$ is a Kummer surface, associated to an abelian surface $A$; hence $h(Y)$ is finite dimensional by [Pe1, 5.8]. The rational map $f: X \to Y$ induces a splitting $t_2(X) \simeq t_2(Y) \oplus N$. Since $t_2(Y)$ is finite dimensional we are left to show that $N = 0$. By the same argument as in the proof of Proposition 25 the vanishing of $N$ is equivalent to $A_0(X)^{\iota} = A_0(Y)$. By [Mo, 6.3 (iv)] the Neron Severi group of $X$ contains the sublattice $E_b(-1)^2$. Hence by the results in [Hu2, 6.3, 6.4] the symplectic automorphism $\iota$ acts as the identity on $A_0(X)$. As, by [K-M-P, 6.13], we have $t_2(Y) = t_2(A)$, the motive $h(X)$ is finite dimensional and it lies in the subcategory of $\mathcal{M}_{rat}(\mathbb{C})$ generated by the motives of abelian varieties.

The next theorem gives examples of surfaces $X$ and $Y$ such that $M(X) \simeq M(Y)$ but the derived categories $D^b(X)$ and $D^b(Y)$ are not equivalent. We will use the following result by Van Geemen and Sarti

**Proposition 29.** ([VG-S 2.5]) Let $X$ be a complex K3 surface with a Nikulin involution $\iota$ and let $Y$ be a desingularization of the quotient surface $X/\iota$. The involution induces an isomorphism of Hodge structures between $T_{X, \mathbb{Q}}$ and $T_{Y, \mathbb{Q}}$. If the dimension of the $\mathbb{Q}$-vector space $T_{X, \mathbb{Q}}$ is odd there is no isometry between $T_{X, \mathbb{Q}}$ and $T_{Y, \mathbb{Q}}$. 
Theorem 30. Let $X$ be a complex K3 surface with a Nikulin involution $\iota$ such that $\rho(X) = 9$ and let $Y$ be the desingularization of $X/\iota$. Assume that the map $f: X \to Y$ induces an isomorphism between $t_2(X)$ and $t_2(Y)$. Then $\iota$ acts as the identity on $A_0(X)$, the rational map $f: X \to Y$ induces an isomorphism $M(X) \to M(Y)$ in $DM_{gm}(\mathbb{C})$, but the isomorphism of Hodge structures $\phi_\iota: T_{X,Q} \to T_{Y,Q}$ is not an isometry.

Proof. The Nikulin involution $\iota$ induces an isomorphism of Hodge structures $\phi_\iota: T_{X,Q} \to T_{Y,Q}$ which by Proposition 29 is not an isometry because $\dim T_{X,Q} = 22 - 9$ is odd. Since $X$ and $Y$ are both K3 surfaces the isomorphism $t_2(X) \simeq t_2(Y)$ implies $h(X) \simeq h(Y)$ in $M_{\text{rat}}(\mathbb{C})$, hence also $M(X) \simeq M(Y)$. □

Examples 31. The following are examples of K3 surfaces $X$ with a Nikulin involution $\iota$ and $\rho(X) = 9$ such that $t_2(X) \simeq t_2(Y)$ hence $h(X) \simeq h(Y)$. Therefore $X$ satisfies Huybrechts’ conjecture 3, i.e. $\iota$ acts as the identity on $A_0(X)$. On the other hand, $X$ and $Y$ are not Fourier-Mukai partner because, as in Theorem 30, there is no Hodge isometry between their transcendental lattices. The proof of the isomorphism $t_2(X) \simeq t_2(Y)$ in these cases follows directly from the geometric description of $X$ and $Y$ given by Van Geemen and Sarti in [VG-S], see [Pe2].

(i) $X$ a double cover of $\mathbb{P}^2$ branched over a sextic curve and $Y$ a double cover of a quadric cone in $\mathbb{P}^3$;
(ii) $X$ is a double cover of a quadric in $\mathbb{P}^3$ and $Y$ is the double cover of $\mathbb{P}^2$ branched over a reducible sextic;
(iii) $X$ is the intersection of 3 quadrics in $\mathbb{P}^5$ and $Y$ is a quartic surface in $\mathbb{P}^3$.

References

[An1] Y. André Une introduction aux motifs Panoramas et Synthéses, 17. Société Mathématique de France, Paris, 2004.
[An2] Y. André Motifs de dimension finie (d’après S.-I. Kimura, P. O’Sullivan…) Séminaire Bourbaki. Vol. 2003/2004. Astérisque No. 299 (2005), Exp. No. 929, viii, 115-145.
[B-B-HR] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez Fourier-Mukai and Nahm transforms in geometry and mathematical physics Progress in Mathematics, 276. Birkhäuser Boston, Inc., Boston, MA, 2009.
[B-L] C. Birkenhake, H. Lange Complex abelian varieties Second edition. Grundlehren der Mathematischen Wissenschaften, 302. Springer-Verlag, Berlin, 2004.
[Bl] S. Bloch Lectures on algebraic cycles Duke University Mathematics Series IV, Duke University Press, Durham U.S.A., (1980).
[Br] T. Bridgeland Fourier-Mukai transforms for elliptic surfaces J. Reine Angew. Math. 498 (1998), 115-133.
[B-M] T. Bridgeland, A. Maciocia Complex surfaces with equivalent derived categories Math. Z. 236 (2001), no. 4, 677-697.
[De] P. Deligne La conjecture de Weil pour les surfaces K3 Invent. Math. 15 (1972), 206-226.
[DP-M1] A. Del Padrone, C. Mazza Schur finiteness and nilpotency C. R. Math. Acad. Sci. Paris 341 (2005), no. 5, 283-286.
DERIVED EQUIVALENCE AND MOTIVES

[DP-M2] A. Del Padrone, C. Mazza Schur-finiteness and endomorphisms universally of trace zero via certain trace relations Comm. Algebra 37 (2009), no. 1, 32-39.

[DP] A. Del Padrone A note on derived equivalence and finite dimensional motives in preparation.

[D-M] C. Deninger, J. Murre Motivic decomposition of abelian schemes and the Fourier transform J. Reine Angew. Math. 422 (1991), 201-219.

[D-ML-P] I. Dolgachev, M. Mendes Lopez and R. Pardini Rational surfaces with many nodes Compositio Math. 132 (2002), no. 3, 349-363.

[VG-S] B. van Geemen and A. Sarti Nikulin Involutions on K3 Surfaces Math. Z. (2007), 731-753.

[G-G] S. Gorchinskiy and V. Guletskiĭ Motives and representability of algebraic cycles on threefolds over a field arXiv:0806.0173v2 [math.AG].

[G-P] V. Guletskiĭ and C. Pedrini Finite-dimensional Motives and the Conjectures of Beilinson and Murre K-Theory 30 (2003), 243-263.

[Hu] D. Huybrechts, Fourier-Mukai transforms in algebraic geometry Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006.

[Hu1] D. Huybrechts Chow groups of K3 surfaces and spherical objects J. Eur. Math. Soc. 12 (2010), no. 6, 1533-1551.

[Hu2] D. Huybrechts Chow groups and derived categories of K3 surfaces to appear in “Proc. Classical Algebraic Geometry today”, MSRI January 2009, arXiv:0912.5299v1 [Math.AG].

[H-NW] D. Huybrechts, M. Nieper-Wisskirchen Remarks on derived equivalences of Ricci-flat manifolds Math. Z. (2011) 267:939-963

[In] H. Inose Defining equations of singular K3 surfaces and a notion of isogeny in “Proceedings of the International Symposium on Algebraic Geometry” (Kyoto Univ., Kyoto, 1977), pp. 495-502, Kinokuniya Book Store, Tokyo, 1978.

[Ja] U. Jannsen On finite-dimensional motives and Murre’s conjecture Algebraic cycles and motives. Vol. 2, 112-142, London Math. Soc. Lecture Note Ser., 344, Cambridge Univ. Press, Cambridge, 2007.

[Ka] Y. Kawamata D-equivalence and K-equivalence J. Differential Geom. 61 (2002), no. 1, 147-171.

[Ki] S. I. Kimura Chow groups can be finite-dimensional, in some sense Math. Ann. 331 (2005), 173-201.

[K-M-P] B. Kahn, J. Murre and C. Pedrini On the transcendental part of the motive of a surface in “Algebraic cycles and Motives” Vol. II, London Math. Soc. LNS 344 (2008), Cambridge University Press, 1-58.

[L-S-Y] R. Livné, M. Schütt, N. Yui The modularity of K3 surfaces with non-symplectic group actions Math. Ann. 348 (2010), no. 2, 333-355.

[Ma] Yu. I. Manin Correspondences, motives and monoidal transformations Math. USSR Sb. 6 (1968), 439-470.

[Maz] C. Mazza Schur functors and motives K-Theory 33 (2004), no. 2, 89-106.

[Mo] D. R. Morrison On K3 surfaces with large Picard number Inv. Math. 75 (1984), 105-121.

[Mu1] S. Mukai On the moduli space of bundles on a K3 surface I in “Vector bundles on algebraic varieties”, Tata Inst. of Fund. Research Stud. Math. 11 (1987), 34-413.

[Mu2] S. Mukai Vector bundles on a K3 surface in “Proceedings of the International Congress of Mathematicians Vol. II (Beijing, 2002)”, 495-502, Higher Ed. Press, Beijing, 2002.

[Ni] V. Nikulin On correspondences between surface of K3 type Math. USSR Izvestia 30 (1988), no. 2, 375-383.

[Or1] D. Orlov Equivalence of derived categories and K3 surfaces in “Algebraic geometry, 7”, J. Math. Sci. (New York) 84 (1997), no. 5, 1361-1381.

[Or2] D. Orlov Derived categories of coherent sheaves and motives Usp. Mat. Nauk 60 (6) (2005) [Russ. Math. Surv. 60, 1242-1244 (2005)]; arXiv: math/0512620.
C. Pedrini On the motive of a $K^3$ surface in “The geometry of algebraic cycles”, Clay Math. Proc., 9, (2010), 53-74.

C. Pedrini The Chow Motive of a $K^3$ surface Milan J. Math. 77 (2009), 151-170.

C. Pedrini On the finite-dimensionality of a $K^3$ surface submitted.

I. I. Pjatetskii-Shapiro, I. R. Safarevich A Torelli theorem for algebraic surfaces of type $K^3$ Izv. AN SSSR. Ser. mat., 35 (1971), no. 3, 530-572; English transl.: Math. USSR Izv. 5 (1971), no. 3, 547-588.

R. Rouquier Catégories dérivées et géométrie birationnelle (d’après Bondal, Orlov, Bridgeland, Kawamata et al.) Séminaire Bourbaki. Vol. 2004/2005. Astérisque No. 307 (2006), Exp. No. 946, viii, 283-307.

I. R. Safarevich Le théorème de Torelli pour les surfaces algébriques de type $K^3$ in “Actes du Congrès International des Mathématiciens” (Nice, 1970), Tome 1, pp. 413-417. Gauthier-Villars, Paris, 1971.

T. Shioda, H. Inose On singular $K^3$ surfaces in “Complex analysis and algebraic geometry”, pp. 119-136. Iwanami Shoten, Tokyo, 1977.

T. Shioda, T. Katsura On Fermat varieties Tôhoku Math. J. (2) 31 (1979), no. 1, 97-115.

P. Sosna, Derived equivalent conjugate $K^3$ surfaces Bull. Lond. Math. Soc. 42 (2010), no. 6, 1065-1072.

V. Voevodsky Triangulated categories of motives over a field in “Cycles, transfers and motivic homology theories”, Ann. of Math. Stud. 143 (2000), 188-238.

Dipartimento di Matematica, Università degli Studi di Genova, Via Dodecaneso 35, 16146 Genova, Italy
E-mail address, A. Del Padrone: delpadro@dima.unige.it

Dipartimento di Matematica, Università degli Studi di Genova, Via Dodecaneso 35, 16146 Genova, Italy
E-mail address, C. Pedrini: pedrini@dima.unige.it