WEIGHTED BANACH SPACES OF HOLOMORPHIC FUNCTIONS IN THE UPPER HALF PLANE

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Abstract. We define a new class of weighted Banach spaces of holomorphic functions in the upper half plane. Three basic theorems on these spaces are proved. Two of them investigate necessary and sufficient conditions on the weight function in order to obtain that the defined spaces are not the trivial ones. The third theorem deals with the behaviour at infinity of some holomorphic functions. These three theorems all are new and do not represent neither generalizations nor particular cases of any previous results.

1. Introduction

Let \( \mathbb{C} \) be the complex plane and let \( \mathbb{R} \) be the real axis. We use the following notations: the real part of \( z \in \mathbb{C} \) is denoted by \( \text{Re} \, z \); the imaginary part of \( z \in \mathbb{C} \) is denoted by \( \text{Im} \, z \); and \( D \) stands for the upper half plane of \( \mathbb{C} \), i.e. \( D = \{ z \in \mathbb{C}, \text{Im} \, z > 0 \} \).

Definition 1.1. A function \( p : (0, \infty) \to (0, \infty) \) is said to be weight function if and only if
\[
\inf_{t, t' \in [\frac{1}{c}, c]} p(t) > 0, \quad \forall c > 1.
\]

Let \( p \) be any weight function. We introduce the notation
\[
\| f \|_p = \sup_{z, z' \in D} p(\text{Im} \, z)|f(z)|,
\]
where \( f \) is any holomorphic function in the upper half plane \( D \). And, we define the spaces \( \Lambda(p) \) and \( \lambda(p) \) as follows:
- \( \Lambda(p) \) is a normed complex function space and \( f \in \Lambda(p) \) if and only if \( f \) is a holomorphic function in the upper half plane \( D \) that \( \| f \|_p < \infty \), and \( \| f \|_p \) is the norm of \( f \in \Lambda(p) \);
- \( \lambda(p) \) is a normed complex function space and \( f \in \lambda(p) \) if and only if \( f \) is a holomorphic function in \( D \) such that satisfies the following two conditions:
  1. \( \| f \|_p < \infty \),
  2. for every positive number \( \varepsilon, \varepsilon > 0 \), there exists a number \( c, c \geq 1 \) such that
\[
\sup_{z, z' \in D \setminus K_c} p(\text{Im} \, z)|f(z)| \leq \varepsilon,
\]
where \( K_c = \{ z \in D, \frac{1}{c} \leq \text{Im} \, z \leq c, |\text{Re} \, z| \leq c \}, c \geq 1 \).

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and \( \| f \|_p \) is the norm of \( f \in \lambda(p) \).

The spaces \( \Lambda(p) \) and \( \lambda(p) \) are called weighted Banach spaces of holomorphic functions in the upper half plane.

Here, in the present Section, the main definition and the main results are stated. Theorems 1.2 and 1.3 solve completely the problem whether the weighted Banach spaces are trivial or not by giving the corresponding necessary and sufficient conditions on the weight function. Theorem 1.4 concerns the behaviour of some functions of \( \lambda(p) \) at infinity.

In Section 2 there are examples of such spaces and it is proved that the weighted Banach spaces of holomorphic functions in the upper half plane all are Banach spaces.

In Section 3 we prove our main results.

In this paper the main results are the following three theorems.

**Theorem 1.2.** Let the function \( p : (0, \infty) \to (0, \infty) \) meet the condition (1.1). Then the space \( \Lambda(p) \neq \{0\} \) if and only if there are two real numbers \( a, b \) such that
\[
- (1) \ln(p(t)) \geq at + b, \quad \forall t > 0.
\]

**Theorem 1.3.** Let the function \( p : (0, \infty) \to (0, \infty) \) meet the condition (1.1). Then the space \( \lambda(p) \neq \{0\} \) if and only if the following two conditions on the weight function \( p \) are fulfilled:

1. there are two real numbers \( a, b \) such that
\[
- (1) \ln(p(t)) \geq at + b, \quad \forall t > 0;
\]
2. \( \lim_{t \to 0^+} p(t) = 0 \).

Further, for brevity, by \( Mf(y) \) we denote
\[
Mf(y) = \sup_{z, \Im z = y} |f(z)|, \quad \forall y > 0, f \in \Lambda(p),
\]
where \( \Lambda(p) \) is any weighted Banach space of holomorphic functions in the upper half plane.

**Theorem 1.4.** Let the function \( p : (0, \infty) \to (0, \infty) \) be such that
\[
\inf_{t, \in [\frac{c}{c}, c]} p(t) > 0, \quad \forall c > 1,
\]
\( \lambda(p) \neq \{0\} \).

If the holomorphic function \( f \in \lambda(p) \setminus \{0\} \) is such that
\[
\lim_{t \to \infty} \inf \frac{\ln Mf(t)}{t} < \infty,
\]
then for the limit value \( a = \lim_{t \to \infty} \inf \frac{\ln Mf(t)}{t} \) we obtain \( a > -\infty \) and
\[
\lim_{t \to \infty} (\ln Mf(t) - at) = -\infty.
\]
2. Preliminary results

2.1. Examples. We consider some examples of weighted Banach spaces of holomorphic functions in the upper half plane.

Example 2.1. Let $p(t) = 1$, $\forall t > 0$. Then the condition (1.1) is fulfilled. Accordingly to the Definition 1.1 the spaces $\Lambda(p)$ and $\lambda(p)$ are well-defined. In particular,
\[ \| f \|_p = \sup_{z, \text{Im} z > 0} |f(z)|, \]
and $\Lambda(p) = H^\infty$. The assertion $\lambda(p) = \{0\}$ is a consequence of Theorem 1.3.

Example 2.2. Let $p(t) = t$, $\forall t > 0$. Then the condition (1.1) is fulfilled. Accordingly to the Definition 1.1 the spaces $\Lambda(p)$ and $\lambda(p)$ are well-defined. In particular,
\[ \| f \|_p = \sup_{z, \text{Im} z > 0} \text{Im} z |f(z)|. \]

In order to characterize the spaces $\Lambda(p)$, $\lambda(p)$, we first recall the definition of Bloch spaces $B$ and $B_0$ in the open unit disk (for this definition see 1)
\[ f \in B \iff f \text{ is holomorphic function in the open unit disk such that } \sup_{w, |w| < 1} (1 - |w|^2)|f'(w)| < \infty; \]
\[ f \in B_0 \iff f \in B, \lim_{|w| \to 1^-} (1 - |w|^2)|f'(w)| = 0, \]
where $f'$ stands for the derivative of the function $f$, and $\lim_{|w| \to 1^-}$ is uniform with respect to the argument of $w$. We set
\[ g_f(z) = 4 \frac{f'(1+iz)}{(1-iz)^2}, \quad f \in B, \text{Im} z > 0, \]
Then by a simple computation we obtain the following characterization of the spaces $\Lambda(p)$ and $\lambda(p)$:
\[ f \in B \iff g_f \in \Lambda(p), \]
\[ f \in B_0 \iff g_f \in \lambda(p). \]

Example 2.3. Let $p(t) = t + e^{t}$, $\forall t > 0$. Then the condition (1.1) is fulfilled. Accordingly to the Definition 1.1 the spaces $\Lambda(p)$ and $\lambda(p)$ are well-defined. In particular,
\[ \| f \|_p = \sup_{z, \text{Im} z > 0} (\text{Im} z + e^{|\text{Im} z|}) |f(z)|. \]
The space $\Lambda(p)$ does not contain any constant functions, and contains the function $h$,
\[ h(z) = e^{2iz}, \quad \text{Im} z > 0. \]
Thus, $\Lambda(p)$ is a proper subspace of the corresponding spaces of previous two examples. The assertion $\lambda(p) = \{0\}$ is a consequence of Theorem 1.3.
Example 2.4. Let \( p(t) = e^{t^2}, \forall t > 0 \). Then the condition (1.3) is fulfilled. Accordingly to the Definition 1.3, the spaces \( \Lambda(p) \) and \( \lambda(p) \) are well-defined. In particular,

\[
\| f \|_p = \sup_{z, \text{Im} z > 0} e^{(\text{Im} z)^2} |f(z)|.
\]

The assertions \( \Lambda(p) = \{0\} \), \( \lambda(p) = \{0\} \) are consequences respectively of Theorems 1.2 and 1.3.

2.2. The completeness of \( \Lambda(p) \) and \( \lambda(p) \).

Proposition 2.1. Let the function \( p : (0, \infty) \to (0, \infty) \) meet the condition (1.3). Then \( \Lambda(p) \) and \( \lambda(p) \), both are Banach spaces.

Remark 2.1. The main idea of the proof of Proposition 2.1 is demonstrated in [1].

Proof of proposition 2.1. It is sufficient to prove proposition 2.1 with additional assumption that \( \Lambda(p) \neq \{0\} \) and resp. \( \lambda(p) \neq \{0\} \). Thus let us assume the corresponding one.

For brevity, we set \( K_c = \{ z | z \in \mathcal{D}, \frac{1}{c} \leq \text{Im} z \leq c, |\text{Re} z| \leq c \}, c \geq 1 \).

First we shall prove that \( \Lambda(p) \) is complete.

Suppose \( \{f_n\}_{n=1}^\infty \) be any Cauchy’s sequence in \( \Lambda(p) \). Then for every positive number \( \varepsilon \) and for every \( c \geq 1 \) there is a number \( n_0 \) such that \( \forall m, n > n_0 \)

\[
\sup_{z, z \in K_c} p(\text{Im} z)|f_n(z) - f_m(z)| \leq \| f_n - f_m \|_p \leq \varepsilon.
\]

Hence, in particular,

\[
\sup_{z, z \in K_c} |f_n(z) - f_m(z)| \leq \frac{\varepsilon}{\inf_{t \in [\frac{1}{c}, c]} p(t)}.
\]

Thus we obtain that the sequence \( \{f_n\}_{n=1}^\infty \) is uniformly convergent on any compact in the upper half plain \( \mathcal{D} \). Then there is a holomorphic function \( f \) in \( \mathcal{D} \) which is the uniform limit of \( \{f_n\}_{n=1}^\infty \) on every compact in \( \mathcal{D} \). Therefore, by the inequality (2.1) it follows

\[
\sup_{z, z \in K_c} p(\text{Im} z)|f_n(z) - f(z)| \leq \varepsilon.
\]

Hence, accordingly to the choice of \( c \) we obtain

\[
\| f_n - f \|_p \leq \varepsilon, \quad n > n_0.
\]

In particular, \( f \in \Lambda(p) \) and \( \lim_{n \to \infty} \| f_n - f \|_p = 0 \).

Thus we obtain that \( \Lambda(p) \) is complete and hence it is a Banach space.

Second, we shall prove that \( \lambda(p) \) is complete.

Note \( \lambda(p) \) is a linear subspace of the Banach space \( \Lambda(p) \). So, to obtain the completeness of \( \lambda(p) \) it is enough to prove that \( \lambda(p) \) is a closed subspace of \( \Lambda(p) \).

We claim that \( \lambda(p) \) is a closed subspace of \( \Lambda(p) \).

Suppose \( \{f_n\}_{n=1}^\infty \) is such a sequence of elements of \( \lambda(p) \) that is convergent in the space \( \Lambda(p) \). Let \( f \in \Lambda(p) \) be such that \( \lim_{n \to \infty} \| f_n - f \|_p = 0 \).

Then for every positive number \( \varepsilon \) there is a number \( n_0 \) such that

\[
\sup_{z, \text{Im} z > 0} p(\text{Im} z)|f_n(z) - f(z)| \leq \frac{\varepsilon}{2}, \quad \forall n > n_0.
\]
Fix an \( n, \ n > n_0 \). Then it follows by \( f_n \in \lambda(p) \) that there is \( c, \ c \geq 1 \), such that
\[
\sup_{z, z \in D \setminus K_c} p(\text{Im } z) |f_n(z)| \leq \frac{\varepsilon}{2}.
\]
Hence
\[
\sup_{z, z \in D \setminus K_c} p(\text{Im } z) |f(z)| \leq \varepsilon.
\]
and therefore \( f \in \lambda(p) \). Thus the assertion which we claim above is proved. \( \square \)

### 2.3. A preliminary lemma.

**Lemma 2.2.** Let the function \( p : (0, \infty) \to (0, \infty) \) be such that the following two conditions are fulfilled:
\[
\inf_{t, t \in \left[ \frac{1}{c}, c \right]} p(t) > 0, \quad \forall c \geq 1,
\]
\[
\Lambda(p) \neq \{0\}.
\]
If \( f \in \Lambda(p) \setminus \{0\} \) then
1. \( \ln Mf \) is convex in \((0, \infty)\);
2. there are two numbers \( a, b \) such that
\[
\ln Mf(t) \geq at + b, \quad \forall t > 0.
\]

**Proof of Lemma 2.2.** Let \( f, \ f \in \Lambda(p) \setminus \{0\} \), be an arbitrary non zero element of \( \Lambda(p) \). Then
\[
\sup_{t, t > 0} p(t) Mf(t) = \| f \|_p < \infty.
\]
Hence
\[
\sup_{t, t \in \left[ \frac{1}{c}, c \right]} Mf(t) \leq \frac{\| f \|_p}{\inf_{t \in \left[ \frac{1}{c}, c \right]} p(t)} < \infty, \quad \forall c \geq 1,
\]
and in particular the holomorphic in the upper half plane \( D \) function \( f \) is bounded in the band
\[
\{ z \mid z \in D, \frac{1}{c} \leq \text{Im } z \leq c \}, \quad \forall c \geq 1.
\]
Then by the Phragmén-Lindelöf principle we obtain that \( \ln Mf \) is convex in \( \left[ \frac{1}{c}, c \right] \), \( \forall c \geq 1 \). Thus \( \ln Mf \) is convex in \((0, \infty)\).

So, the first assertion of Lemma 2.2 is proved. In order to prove the second assertion of this Lemma we set
\[
\mathcal{E} = \{ (u, v) \mid u > 0, v > \ln Mf(u) \}.
\]
From the first assertion of Lemma 2.2 it follows that \( \mathcal{E} \) is an open convex subset of Euclidean plane \( \{ (u, v) \mid u \in \mathbb{R}, v \in \mathbb{R} \} \). Further, by \( f \neq 0 \) it follows that there is \( t, \ t > 0 \), such that \( \ln Mf(t) \neq -\infty \) (in fact this inequality holds for every \( t > 0 \), but we do not use it). Hence there exists a point \( (u_0, v_0) \), \( u_0 > 0, v_0 \in \mathbb{R} \), which does not belong to \( \mathcal{E} \). Then there exists a line that distinguishes the set \( \mathcal{E} \) and the point \( (u_0, v_0) \).

Hence, there are two numbers \( a, b \) such that
\[
\ln Mf(t) \geq at + b, \quad \forall t > 0.
\]
So, Lemma 2.2 is proved. \( \square \)
3. Proof of main results

Proof of Theorem 1.2 First we prove that if \( \Lambda(p) \neq \{0\} \) then there are two numbers \( a, b \) such that

\[
(-1) \ln p(t) \geq at + b, \quad \forall t > 0.
\]

Suppose \( \Lambda(p) \neq \{0\} \) and let \( f, f \in \Lambda(p) \setminus \{0\} \) be an arbitrary non zero function that belongs to \( \Lambda(p) \). Then by Lemma 2.2 there exist two numbers \( a_0, b_0 \) such that

\[
\ln Mf(t) \geq a_0 t + b_0, \quad \forall t > 0.
\]

Further, by \( f \in \Lambda(p) \) and \( p(t)Mf(t) \leq \| f \|_p, \forall t > 0 \), it follows

\[
(-1) \ln p(t) \geq (-1) \ln \| f \|_p + \ln Mf(t) \geq (-1) \ln \| f \|_p + a_0 t + b_0, \quad \forall t > 0.
\]

We set \( a = a_0, b = b_0 - \ln \| f \|_p \) and obtain

\[
(-1) \ln p(t) \geq at + b, \quad \forall t > 0.
\]

Thus we prove that if \( \Lambda(p) \neq \{0\} \) then the condition on the function \( p \) which is stated in Theorem 1.2 is fulfilled.

With view to show \( \Lambda(p) \neq \{0\} \), suppose that the function \( p \) satisfies condition (1.1) and in addition there exist two numbers \( a \) and \( b \) such that

\[
(-1) \ln p(t) \geq at + b, \quad \forall t > 0.
\]

Then for such numbers \( a, b \) we set \( f(z) = e^{-iaz+b}, \) where \( z \in \mathcal{D}. \) Then a direct computation shows \( \| f \|_p \leq 1. \) So, \( f \in \Lambda(p) \setminus \{0\}. \)

Thus the proof of Theorem 1.2 is completed.

Proof of Theorem 1.3. Suppose \( \lambda(p) \neq \{0\} \) and let \( f, f \in \lambda(p) \setminus \{0\} \) be an arbitrary non zero function that belongs to \( \lambda(p) \). In particular, from the inclusion \( \lambda(p) \subseteq \Lambda(p) \) it follows \( f \in \Lambda(p) \setminus \{0\}. \) Therefore, by Theorem 1.2 the first condition on the function \( p \) is fulfilled. Furthermore, according to Definition 1.1

\[
\lim_{t \to 0^+} p(t)Mf(t) = 0.
\]

Moreover, by Lemma 2.2, there exist two numbers \( a_0, b_0 \) such that

\[
\ln Mf(t) \geq a_0 t + b_0, \quad \forall t > 0.
\]

Hence,

\[
0 \leq \liminf_{t \to 0^+} p(t) \leq \limsup_{t \to 0^+} p(t) \leq \limsup_{t \to 0^+} p(t)e^{at+b}e^{-(at+b)} \leq \limsup_{t \to 0^+} p(t)Mf(t) \lim_{t \to 0^+} e^{-(at+b)} = 0
\]

which yields \( \lim_{t \to 0^+} p(t) = 0. \)

Thus, as we assert in Theorem 1.3, both conditions on function \( p \) are fulfilled.

Further, suppose that the function \( p \) meets the condition (1.3) and in addition the following two conditions are fulfilled, too:

1. there exist two numbers \( a, b \) such that

\[
(-1) \ln p(t) \geq at + b, \quad \forall t > 0.
\]

2. \( \lim_{t \to 0^+} p(t) = 0. \)
We claim that \( \lambda(p) \neq \{0\} \).

Indeed, the function \( f \) defined by
\[
f(z) = \frac{e^{i(a+1)z}}{z+i}, \quad \forall z \in \mathcal{D},
\]
belongs to \( \lambda(p) \setminus \{0\} \). One can verify this by a simple direct computation and we omit the details.

Thus Theorem 1.3 is proved. \( \square \)

**Remark 3.1.** In this remark, we explain the main idea of the proof of theorem 1.4.

In general, its feature is demonstrably application of scheme of Phragmén-Lindelöf theorems generalizing the maximum modulus principle. Thus, for a given function belonging to \( \lambda(p) \), some auxiliary holomorphic function is constructed and explored with the help of the usual maximum modulus principle.

**Proof of Theorem 1.4.** Let \( f \in \lambda(p) \setminus \{0\} \) be such that
\[
\liminf_{t \to \infty} \frac{\log M_f(t)}{t} < \infty.
\]

In particular, from the inclusion \( \lambda(p) \subset \Lambda(p) \), it follows \( f \in \Lambda(p) \setminus \{0\} \). Then by Lemma 2.2, there exist two numbers \( a_0, b_0 \) such that
\[
\log M_f(t) \geq a_0 t + b_0, \quad \forall t > 0.
\]

Hence \( \liminf_{t \to \infty} \frac{\log M_f(t)}{t} \geq a_0 \). We set
\[
(3.1) \quad a = \liminf_{t \to \infty} \frac{\log M_f(t)}{t}.
\]

So, \( a_0 \leq a < \infty \). In particular,
\[
F(z) = e^{iaz} f(z), \quad \forall z \in \mathcal{D},
\]
is a well-defined holomorphic function in the upper half plane \( \mathcal{D} \). We prove that the function \( F \) has the following properties:

1. \( \log M_F(t) = \log M_f(t) - at, \forall t > 0 \) — which is obvious and we omit the details;
2. \( \log M_F \) is convex in \( (0, \infty) \) — indeed, \( \log M_f \) is convex in \( (0, \infty) \) by Lemma 2.2 and hence, according to the previous item, \( \log M_F \) is convex, too;
3. \( \liminf_{t \to \infty} \frac{\log M_F(t)}{t} = 0 \) — this follows immediately from both the equation (3.1) and the first item;
4. \( \log M_F \) is a decreasing function in \( (0, \infty) \) — we deduce this from both the second and the third items: for arbitrary \( t_1 \) and \( t_2, 0 < t_1 < t_2 < \infty \), and for \( t > t_2 \) by the second item it follows
\[
\log M_F(t_2) \leq \frac{t - t_2}{t - t_1} \log M_F(t_1) + \frac{t_2 - t_1}{t - t_1} \log M_F(t)
\]
and we obtain by the third item \( \log M_F(t_2) \leq \log M_F(t_1) \), i.e. \( \log M_F \) decreases in \( (0, \infty) \);
5. \( \log M_F \) is bounded from above on \( \left[\frac{1}{2}, \infty\right) \) — it follows immediately from the previous item that \( \log M_F(t) \leq \log M_F(\frac{1}{2}), \forall t \geq \frac{1}{2} \). Further from \( f \in \lambda(p) \) it follows that \( \log M_f(\frac{1}{2}) \) is finite and hence by the first item \( \log M_F(\frac{1}{2}) \) is finite, too, which yields the boundedness of \( \log M_F \) on \( \left[\frac{1}{2}, \infty\right) \).
6. $F(x + iy)$ tends uniformly to 0 as $|x| \to \infty$, with respect to $y$, $y \in [\frac{1}{c}, c]$, where $c \geq 1$ is any. We obtain this by the corresponding property of the function $f \in \Lambda(p)$, namely according to the Definition 1.1, $f(x + iy)$ has the same property.

We set $A = \sup_{z, \text{Im} z \geq 1} |F(z)|$. According to the fifth item $A < \infty$. Moreover, $A \neq 0$ because $F$ is a non zero element of $\Lambda(p)$.

Note, according to the first item, the assertion of Theorem 1.4 can be stated in terms of the function $F$ as follows

$$\lim_{t \to \infty} M F(t) = 0. \quad (3.2)$$

In order to prove the equation (3.2), we have to obtain that for every positive number $\varepsilon$ there exists a number $y_\varepsilon > 0$ such that $M F(t) \leq \varepsilon$, $\forall t > y_\varepsilon$.

Let $\varepsilon$ be an arbitrary number such that $\varepsilon \in (0, A)$. It follows by the sixth item of the above listed properties of the function $F$ that there exists a number $x_1$, $x_1 > 0$, such that

$$\sup_{x, |x| \geq x_1} |F(x + i)| \leq \frac{\varepsilon}{2}.$$ 

Further, we fix $y_\varepsilon$, $y_\varepsilon > 1$, in such a way that

$$\sup_{y, y \geq y_\varepsilon} \sqrt{\frac{x_1^2 + 1}{x_1^2 + (y + 1)^2}} \leq \frac{\varepsilon}{2A}.$$ 

We intend to show that $M F(t) \leq \varepsilon$, $\forall t > y_\varepsilon$.

Let $z_0 \in D$ be such that $\text{Im} z_0 > y_\varepsilon$.

Let $\eta$, $\eta > 0$, be any. We set

$$y_\eta = 1 + \max \left\{ \text{Im} z_0, \frac{1}{\eta} \ln \frac{2A}{\varepsilon} \right\}.$$ 

Then it follows from the fifth item that there exists $x_\eta$, $x_\eta > \max\{|\text{Re} z_0|, x_1\}$ such that

$$\sup_{z, |\text{Re} z| \geq x_\eta, \text{Im} z \in [1, y_\eta]} |F(z)| \leq \frac{\varepsilon}{2}.$$ 

We construct the rectangle $R$

$$R = \{ z | |\text{Re} z| \leq x_\eta, \text{Im} z \in [1, y_\eta] \},$$

and define the holomorphic function $G$ in the upper half plane $D$ by

$$G(z) = e^{i\eta y_\eta} \frac{z}{z + iy_\eta} F(z), \quad \forall z \in D.$$ 

By the choice of $x_\eta$, $y_\eta$, $y_\varepsilon$ it is easy to verify that the modulus $|G|$ of the function $G$ is less or equal to $\frac{\varepsilon}{2}$ on the boundary of the rectangle $R$. Indeed:

- for $z$ such that $\text{Re} z \in [-x_\eta, x_\eta]$, $\text{Im} z = y_\eta$, we obtain

$$|G(z)| = e^{-\eta y_\eta} \left| \frac{z}{z + iy_\eta} \right||F(z)| \leq e^{-\eta y_\eta} A \leq \frac{\varepsilon}{2}.$$
• for $z$ such that $\text{Re} \ z \in [-\eta, \eta]$, $\text{Im} \ z = 1$, we obtain

$$|G(z)| = e^{-\eta} \left| \frac{z}{z + iy_\epsilon} \right| |F(z)| \leq \left| \frac{z}{z + iy_\epsilon} \right| |F(z)|$$

and further there are two cases

1. if $|\text{Re} \ z| \leq x_1$ then we have to continue with the following inequalities in which we use the notation $x = \text{Re} \ z$

   $$|G(z)| \leq \sqrt{\frac{x^2 + 1}{x^2 + (y_\epsilon + 1)^2}} |F(z)|$$

   $$\leq \sqrt{\frac{x^2 + 1}{x^2 + (y_\epsilon + 1)^2}} |F(z)|$$

   $$\leq \frac{\epsilon}{2A} A = \frac{\epsilon}{2},$$

2. if $|\text{Re} \ z| \geq x_1$ then it is obvious that $|G(z)| \leq \frac{\epsilon}{2}$, i.e. in both cases $|G(z)| \leq \frac{\epsilon}{2}$;

• for $z$ such that $\text{Im} \ z \in [1, y_\eta]$, $|\text{Re} \ z| = \eta$ we obtain

$$|G(z)| = e^{-\eta \text{Im} \ z} \left| \frac{z}{z + iy_\epsilon} \right| |F(z)| \leq |F(z)| \leq \frac{\epsilon}{2}.$$

Thus, the modulus $|G|$ of the function $G$ is less or equal to $\frac{\epsilon}{2}$ on the boundary of the rectangle $R$.

Then, by the maximum modulus principle, it follows

$$\sup_{z \in R} |G(z)| \leq \frac{\epsilon}{2},$$

and hence, in particular, $|G(z_0)| \leq \frac{\epsilon}{2}$.

Therefore,

$$|F(z_0)| \leq e^{\eta \text{Im} \ z_0} \left| \frac{z_0 + iy_\epsilon}{z_0} \right| \frac{\epsilon}{2}$$

$$\leq e^{\eta \text{Im} \ z_0} \left( 1 + \frac{y_\epsilon}{|z_0|} \right) \frac{\epsilon}{2}$$

$$\leq e^{\eta \text{Im} \ z_0} \left( 1 + \frac{y_\epsilon}{\text{Im} \ z_0} \right) \frac{\epsilon}{2}$$

$$\leq \epsilon e^{\eta \text{Im} \ z_0}.$$

Then $|F(z_0)| \leq \epsilon$ because of the choise of $\eta$.

So, $MF(\text{Im} \ z_0) \leq \epsilon$.

Thus the claim $MF(t) \leq \epsilon$, $\forall t > y_\epsilon$, is proved.

The proof of Theorem 1.4 is completed.

References

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