ON THE EXISTENCE OF SELF-SIMILAR SOLUTIONS FOR SOME NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We construct solutions of Schrödinger equations which have asymptotic self similar solutions as time goes to infinity. Also included are situations with two-bubbles. These solutions are global, with constant non-zero $L^2$ norm, and are stable. As such they are not of the standard asymptotic decomposition of linear wave and localized waves. Such weakly localized waves were expected in view of previous works on the large time behavior of general dispersive equations. It is shown that one can associate a scattering channel to such solutions, with the Dilation operator as the asymptotic ”hamiltonian”.

1. Introduction

The analysis of dispersive wave equations and systems is of critical importance in the study of evolution equations in Physics and Geometry.

It is well known that the asymptotic solutions of such equations, if they exist, show a dizzying zoo of possible solutions. Besides the “free wave”, which corresponds to a solution of the equation without interaction terms, a multitude or other solutions may appear.

Such solutions are localized around possibly moving center of mass. They include nonlinear bound states, solitons, breathers, hedgehogs, vortices etc... The analysis of such equations is usually done on a case by case basis, due to this complexity. [22]

A natural question then follows: is it true that in general, solutions of dispersive equations converge in appropriate norm ($L^2$ or $H^1$) to a free wave and independently moving localized parts?

In fact this is precisely the statement of Asymptotic Completeness in the case of N-body Scattering[5, 8, 11, 12, 19, 21]. In this case the possible outgoing clusters are clearly identified, as bound states of subsystems.

But when the interaction term includes time dependent potentials (even localized in space) and more general nonlinear terms, we do not have an a-priory knowledge of the possible asymptotic states.

In fact, there are no general scattering results for localized time dependent potentials. The exceptions are charge transfer hamiltonians [34, 9, 33, 15, 17, 4], decaying in time potentials and small potentials [10, 16], time periodic potentials [35, 10] and random (in time) potentials [1]. See also[2, 3]. For potentials with asymptotic energy distribution more could be done [20].

A very recent progress for more general localized potentials without smallness assumptions is obtained in [26]. Some tools from this work will be used in this paper.

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Turning to the nonlinear case, Tao [30, 31, 32] has shown that the asymptotic decomposition holds for NLS with inter-critical nonlinearities in 3 or higher dimensions, in the case of radial initial data.

In particular, in a very high dimension, and with an interaction that is a sum of smooth compactly supported potential and repulsive nonlinearity, Tao was able to show that the localized part is smooth and localized.

In other cases, Tao showed the localized part is only weakly localized and smooth.

Tao’s work uses direct estimates of the incoming and outgoing parts of the solution to control the nonlinear part, via Duhamel representation. In a certain sense, it is in the spirit of Enss’ work. See also [18].

A new approach due to Liu-Soffer [13, 14] is based on proving a-priori estimates on the full dynamics, which hold in suitably localized regions of the extended phase-space. In this way it was possible to show the asymptotic decomposition for general localized interactions, including time and space dependent interactions. Radial initial data is assumed.

More detailed information is obtained on the localized part of the solution. Besides being smooth, its expanding part (if it exists) can grow at most like $|x| \leq \sqrt{t}$, and furthermore, is concentrated in a thin set of the extended phase-space.

The free part of the solution concentrates on the propagation set where $x = vt$, $v = 2P$, and $P$ being the dual to the space variable, the momentum, is given by the operator $-i\nabla_x$.

The weakly localized part is found be localized in the regions where

$$|x|/t^\alpha \sim 1 \quad \text{and} \quad |P| \sim t^{-\alpha}, \quad \forall \quad 0 < \alpha \leq 1/2.$$ 

It therefore shows that the spreading part follows a self similar pattern.

The question is therefore, does there exist solutions which are weakly localized but not localized? For equations which are not dispersive/hyperbolic many such solutions are known. But for hyperbolic/dispersive equations it is harder to see how such global solutions can emerge.

In the energy critical non-linear wave equation, the conformal symmetry implies a specific structure of scaling. Indeed in these models, it was possible to show that the asymptotic states also include self-similar solitons[6, 7]. See also cited references in [6, 7]. These solutions are global and preserve the energy (but have divergent $L^2$ norm).

In [29] it is shown that for small data there is a spreading weakly localized part for the KdV type equation.

In the case of NLS we have both energy and $L^2$ conservation, and no conformal symmetry. The aim of this work is to construct self-similar solutions for NLS type equations which are global and with non-zero mass ($L^2$ norm). We do that first for linear problems with time dependent potentials, which decay like $r^{-2}$ at infinity, and then use it to show that the addition of nonlinear terms does not change much the situation.

We do not know yet if one can get such solutions for a purely nonlinear equation, but it is more plausible now.

Our approach to this problem can be described in the language of scattering theory: We construct a channel of scattering where the asymptotic dynamics is given by a scaling transformation, which is unitary in $L^2$, and a phase transformation. This is to be expected in view of the recent work [13, 14] showing that the asymptotic solutions (in the radial case) concentrates on thin sets of the phase-space, corresponding to self-similar solutions. This is the case for both the free wave and the weakly localized part.
1. Problem and Results. We consider the general class of Nonlinear Schrödinger type equations of the form:

\[
\begin{aligned}
&(i\partial_t \psi - H_0 \psi = V(x, t)\psi + N(|\psi|)\psi \\
&\psi(x, 0) = \psi_0 \in \mathcal{H}_a^s(\mathbb{R}^n), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}
\end{aligned}
\]

with \( H_0 := -\Delta_x \). Here \( \mathcal{H}_a^s \) denotes some suitable Sobolev space. \( (a = 0 \text{ when it comes to linear problem and } a = 1 \text{ when it comes to nonlinear one.}) \)

We start with a linear model

\[
\begin{aligned}
&(i\partial_t \psi + H_0 \psi + g(t)^{-2}V(\frac{\psi}{g(t)})\psi \\
&\psi(x, g(t_0)) = e^{-iD\ln(g(t_0))}\psi_0(x) \in L^2(\mathbb{R}^n), \quad n \geq 3
\end{aligned}
\]

for some \( t_0 > 0 \) \( (t_0 \text{ will be chosen later}), \ g(t) \in C^2(\mathbb{R}) \text{ satisfying that there exist two positive constants } c_g \in (0, 1), c \in (0, 1/2) \text{ such that}

\[
\begin{aligned}
&\inf_{t \in \mathbb{R}} g(t) \geq 1, \\
&g(t) \sim |t|^\epsilon \text{ as } t \to \infty, \\
&g(t) \sim t^\epsilon g''(t) \text{ as } t \to \infty,
\end{aligned}
\]

and \( V(x) \) and \( H := H_0 + V(x) \text{ satisfying that } H \text{ has a unique normalized eigenstate } \psi_0(x) \text{ with an eigenvalue } \lambda < 0 \text{ and}

\[
\begin{aligned}
&\langle x \rangle D\psi_0(x) \in L^2_x, \\
&\langle x \rangle V(x) \in L^\infty_x, V(x) \in L^2_x.
\end{aligned}
\]

where \( P_x := -i\nabla_x, D := \frac{1}{2}(x \cdot P_x + P_x \cdot x) \) and \( P_c \) denotes the projection on the continuous spectrum of \( H \). We refer system (1.2) to mass critical system (MCS). Since \( g(t)^{-2}V(\frac{\psi}{g(t)}) \in L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R}) \) when \( \inf g(t) > 0 \), due to [27], the channel wave operator

\[
\Omega^s_a := s-\lim_{t \to \infty} e^{iH_0} F_c(\frac{|x - 2tP_x|}{t^\alpha} \leq 1) U(t, 0)
\]

exists from \( L^2_x(\mathbb{R}^3) \) to \( L^2_x(\mathbb{R}^3) \) for all \( \alpha \in (0, 1/3) \), where \( F_c \) denotes a smooth characteristic function. When it comes to the case when \( n > 3 \), even though \( V(x, t) \notin L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R}), \) by using a similar propagation estimate introduced in [27, 28], we can get the existence of the free channel wave operator defined by (1.5), see Section 3. In [27, 28], the weakly localized part \( \psi_{w, j}(t) \) is defined by

\[
\psi_{w, j}(t) = F_c(\frac{|x - 2tP_x|}{t^\alpha} \geq 1) \psi(t) + \psi_e(t)
\]

for some \( \psi_e(t) \) satisfying

\[
\lim_{t \to \infty} \|\psi_e(t)\|_{L^2_x} = 0.
\]

Let

\[
\psi_{w, j}(t) = c(t)e^{-iD\ln(g(t))}\psi_0(x) \oplus \psi_{w, j, c}(t)
\]
where
\begin{equation}
(1.9) \quad (e^{-iD \ln(g(t))} \psi_b(x), \psi_{w,b}(t))_{L^2_x} = 0.
\end{equation}

Let \( T : \mathbb{R} \to \mathbb{R} \), \( t \mapsto s = \int_0^t d\mu(u)^{-2} \). Due to monotonicity of \( T \), \( T^{-1} \) exists. Due to assumption (1.3) on \( g(t) \),
\begin{equation}
(1.10) \quad s = T(t) = \int_0^t d\mu(u)^{-2} \geq \int_0^t du(u)^{-2\epsilon} \geq (t)^{1-2\epsilon} \to \infty \text{ as } t \to \infty.
\end{equation}

We prove the weakly localized part \( \psi_{w,b}(t) \) to (1.2) has a non-trivial self-similar part
\begin{equation}
(1.11) \quad \psi_{w,b}(t) := c(t) e^{-iD \ln(g(t))} \psi_b(x)
\end{equation}
in the following sense:

1. Let
\begin{equation}
(1.12) \quad \tilde{a}(t) := (\psi_b(x), e^{-iD \ln(g(t))} \psi(t))_{L^2_x}.
\end{equation}
\begin{equation}
(1.13) \quad \tilde{A}(\infty) := \lim_{t \to \infty} e^{iT(t)} \tilde{a}(t)
\end{equation}
exists.

2. Furthermore, there exists \( t_0 > 0 \) such that with an initial condition
\begin{equation}
(1.14) \quad \psi(t_0) = e^{-iD \ln(g(t_0))} \psi_b(x),
\end{equation}
\begin{equation}
(1.15) \quad |\tilde{A}(\infty)| > 0
\end{equation}
which implies
\begin{equation}
(1.16) \quad \lim_{t \to \infty} \inf |c(t)| \geq 1.
\end{equation}

Indeed,
\begin{equation}
(1.17) \quad \tilde{a}(t) = (e^{-iD \ln(g(t))} \psi_b(x), \psi(t))_{L^2_x}.
\end{equation}

Here we remind you that \( e^{-iD \ln(g(t))} \psi_b(x) \) is a self-similar function
\begin{equation}
(1.18) \quad e^{-iD \ln(g(t))} \psi_b(x) = \frac{1}{g(t)^{\beta/2}} \psi_b(x / g(t))
\end{equation}
satisfying
\begin{equation}
(1.19) \quad \|e^{-iD \ln(g(t))} \psi_b(x)\|_{L^2_x} = \|\psi_b(x)\|_{L^2_x} = 1.
\end{equation}

**Remark 1.1.** Besides the free part of \( \psi(t) \), (1.13) is the unique asymptotic part we can get from \( e^{iD \ln(g(t))} \psi(t) \). Indeed, we find that \( e^{iD \ln(g(t))} \psi(t) \) satisfies (1.71) and after changing variables from \( t \mapsto s = T(t) \), \( \phi(s) := e^{iD \ln(T^{-1}(s))} \psi(T^{-1}(s)) \) enjoys (1.72), see section 1.2. Theorem 1.4 tells us that
\begin{equation}
(1.20) \quad \lim_{s \to \infty} \|F_2 \left( \frac{|x|}{T^{-1}(s)^\beta} \right) \leq 1 \right) e^{isH_0} \phi(s)\|_{L^2_x} = 0
\end{equation}
and
\begin{equation}
(1.21) \quad \lim_{s \to \infty} \|F_2 \left( \frac{|x|}{T^{-1}(s)^\beta} \right) \leq 1 \right) e^{-iD \ln(T^{-1}(s))} e^{isH_0} \phi(s)\|_{L^2_x} = 0
\end{equation}
for some \( \beta > 0 \) when \( n \geq 3 \), which implies
\begin{equation}
(1.22) \quad \text{w- lim}_{s \to \infty} e^{isH_0} \phi(s) = 0 \quad \text{in } L^2_x.
\end{equation}
and
\[ w- \lim_{s \to \infty} e^{-iD\ln(T^{-1}(s))} e^{iH_0} \phi(s) = 0 \quad \text{in } L^2_x. \]

Here \( e^{-iH_0} e^{iD\ln(T^{-1}(s))} \) is the dynamics of the system (See Lemma 2.1)

\[ i\partial_s \phi_f(s) = (H_0 + f(s)D)\phi_f(s). \]

(1.20) implies that
\[ w- \lim_{s \to \infty} P_e e^{iH} e^{iD\ln(T^{-1}(s))} \psi(T^{-1}(s)) = 0 \quad \text{in } L^2_x. \]

**Remark 1.2.** So far the existence of free channel wave operator to (1.1) is missing in 4 or higher dimensions. We prove it in section 3 by using a similar argument introduced by [27].

**Remark 1.3.** The argument in the following context is applicable to the case when \( V(x) \) is of short-range type with more than one bounded state and for such \( V(x) \), 0 regular assumption on \( H \) is used.

**Theorem 1.4.** If \( V(x), H \) satisfy (1.4) and \( g(t) \) satisfies (1.3), then when \( n \geq 3 \) and \( \beta \in (0, 1 - 2/n - 2(n - 2)e/n) \) (for \( n \geq 3, \epsilon < 1/2 \) implies \( 1 - 2/n - 2(n - 2)e/n > 0 \), (1.20) and (1.21) are true.

**Proof.** It is equivalent to show

\[ \lim_{t \to \infty} ||F_x e^{iD\ln(t)} \psi(t)||_{L^2_x} = 0. \]

Now we prove (1.26). Since
\[ e^{-iD\ln(g(t))} H_0 e^{iD\ln(g(t))} = g(t)^2 H_0, \]

which implies
\[ e^{iT(t)H_0} e^{iD\ln(g(t))} \psi(t) = e^{iD\ln(g(t))} e^{i(T(t)g(t)^2 - T_0)H_0} \psi(t), \]

using Duhamel’s formula, one has

\[ e^{iT(t)H_0} e^{iD\ln(g(t))} \psi(t) = e^{iD\ln(g(t))} e^{i(T(t)g(t)^2 - T_0)H_0} \psi(t_0) + \]
\[ (-i) \int_{t_0}^t ds e^{iD\ln(g(t))} e^{i(T(t - s)g(t)^2 - T_0)H_0} \frac{1}{g(s)^2} \psi(s) =: \psi_1(t) + \psi_2(t). \]

Since
\[ \lim_{t \to \infty} \frac{T(t)g(t)^2}{t} = \lim_{t \to \infty} \int_0^t \frac{du}{g(u)^2} g(u)^2 \]
\[ = \lim_{t \to \infty} \frac{g(t)^2}{g(t)^2 - 2tg'(t)/g(t)^3} \quad \text{(L'Hôpital’s rule)} \]
\[ = \lim_{t \to \infty} \frac{g(t)^2}{g(t)^2 - 2tg'/g(t)^3} \]
\[ = 1 + c'_g > 1, \]

one has
\[ T(t)g(t)^2 - t \geq \frac{c'_g}{2} t, \quad t \geq t_M \]
for some sufficiently large $t_M \geq t_0$. Hence, for $t \geq t_M$, $s \in (t_0, t)$,
\begin{equation}
T(t)g(t)^2 - (t - s) \geq T(t)g(t)^2 - t \geq \frac{c'_g t}{2}
\end{equation}
and by using Hölder’s inequality and $L^\infty_x$ decay for a free flow $e^{-itH_0}$,
\begin{equation}
\|F_2(\frac{|x|}{t^\beta} \leq 1)\psi_1(t)\|_{L^2_x(\mathbb{R}^n)} \leq \|F_2(\frac{|x|}{t^\beta} \leq 1)\|_{L^2_x(\mathbb{R}^n)} \|\psi_1(t)\|_{L^\infty_x(\mathbb{R}^n)} \\
\leq t^{n/2} \times g(t)^{n/2} \|e^{i(T(t)g(t)^2-t)H_0}\psi(t_0)\|_{L^2_x(\mathbb{R}^3)} \\
\leq t^{n(\beta+\epsilon)/2} \times \frac{1}{t^{n/2}} \|\psi(t_0)\|_{L^2_x} \\
\leq_0 t^{-n/2(1-\epsilon)} \|\psi_b(x)\|_{L^2_x} \to 0
\end{equation}
as $t \to \infty$ when $\beta < 1 - \epsilon$. Here we use $1 - \frac{n}{2} - 2(n-2)\epsilon/n < 1 - \epsilon$. For $\psi_2(t)$, similarly, by using Hölder’s inequality and $L^\infty_x$ decay for a free flow $e^{-itH_0}$,
\begin{equation}
\|F_2(\frac{|x|}{t^\beta} \leq 1)\psi_2(t)\|_{L^2_x(\mathbb{R}^n)} \leq \|F_2(\frac{|x|}{t^\beta} \leq 1)\|_{L^2_x(\mathbb{R}^n)} \|\psi_2(t)\|_{L^\infty_x(\mathbb{R}^n)} \\
\leq t^{n/2} \times g(t)^{n/2} \int_0^t ds \frac{1}{g(s)^2} \|V(x/g(s))\|_{L^2_x} \|\psi(s)\|_{L^2_x} \\
\leq t^{n(\beta+\epsilon)/2} \times \int_0^t ds \frac{1}{g(s)^{2-n/2}} \|V(x)\|_{L^2_x} \|\psi_b(x)\|_{L^2_x} \\
\leq t^{-(n-2)/2-(n-2)\epsilon/n} \|\psi_b(x)\|_{L^2_x} \to 0
\end{equation}
as $t \to \infty$ when $n \geq 3$, $\epsilon \in (0, 1/2)$ and $\beta \in (0, 1 - 2/n - 2(n-2)\epsilon/n)$. Hence, we get (1.26). Similarly, since
\begin{equation}
\|e^{-iD\ln(g(t))} e^{iT(t)H_0} e^{iD\ln(g(t))} \psi(t)\|_{L^\infty_x(\mathbb{R}^n)} \leq \frac{1}{g(t)^{n/2}} \|e^{iT(t)H_0} e^{iD\ln(g(t))} \psi(t)\|_{L^\infty_x(\mathbb{R}^n)},
\end{equation}
we get
\begin{equation}
\|F_2(\frac{|x|}{T^{-1}(s)^\beta} \leq 1)e^{-iD\ln(g(T^{-1}(s)))} e^{iH_0} e^{iD\ln(g(T^{-1}(s)))} \psi(T^{-1}(s))\|_{L^2_x(\mathbb{R}^n)} \\
\leq_0 \frac{1}{g(t)^{n/2}} \frac{1}{t^{(n-2)/2-(n-2)\epsilon-n/2\beta}} \|V(x)\|_{L^2_x} \|\psi_b(x)\|_{L^2_x} + \frac{1}{g(t)^{n/2}} t^{-n/2(1-\epsilon)} \|\psi_b(x)\|_{L^2_x} \to 0.
\end{equation}
We finish the proof. \hfill \Box

**Theorem 1.5.** Let $\tilde{a}(t)$ be as in (1.12). If $V(x), H$ satisfy (1.4) and $g(t)$ satisfies (1.3), then when $n \geq 3$,
\begin{equation}
\tilde{A}(\infty) := \lim_{t \to \infty} e^{iT(t)} \tilde{a}(t)
\end{equation}
exists and
\begin{equation}
\psi_{w,s}(x, t) = c(t)e^{-iD\ln(g(t))} \psi_b(x) \oplus \psi_c(x, t)
\end{equation}
where
\begin{equation}
c(t) := (e^{-iD\ln(g(t))} \psi_b(x), \psi_{w,s}(x, t))_{L^2_x},
\end{equation}
We show that the weakly localized part defined in (1.43) exists and

\[(1.44) \quad \tilde{\Omega}_g \psi(0) := \lim_{s \to 0^+} e^{i\Theta s} e^{iD \ln(g(T^{-1}(s)))} \psi(T^{-1}(s))\]

exists in $L_x^2$ and

\[(1.45) \quad \lim_{r \to \infty} \sup_{x \in \mathbb{R}} \|\psi(t, x)\|_{H^1_x} = 1\]

Moreover, based on (1.25), the $g(t)$-self-similar channel wave operator

\[(1.46) \quad \Omega_g^* \psi(0) := \rho \lim_{s \to 0^+} e^{i\Theta s} e^{iD \ln(g(T^{-1}(s)))} \psi(T^{-1}(s))\]

exists in $L_x^2$ and

\[(1.47) \quad \lim_{r \to \infty} \sup_{x \in \mathbb{R}} \|\psi(t, x)\|_{H^1_x} = 1\]

Based on (1.2), we also consider a class of mixture models

\[(1.48) \quad \left\{ \begin{array}{l}
\psi(x, t) = \psi_d(x) + e^{-iD \ln(g(t))} \psi_d(x) \\
H_0 + W(x) \text{ has a normalized eigenvector } \psi_d(x) \text{ with an eigenvalue } \lambda_0 < 0 \\
W(x) \in L_x^2(\mathbb{R}^n) \end{array} \right.\]

We show that the weakly localized part defined in (1.6) asymptotically has at least two bubbles: a non-trivial self-similar part and a non-trivial localized part near the origin.

**Theorem 1.6.** Let $\tilde{a}(t)$ be as in (1.12). If $W(x), V(x), H$ satisfy (1.42) and $g(t)$ satisfies (1.3), then when $n \geq 5$, $\epsilon \in (2/n, 1/2)$,

\[(1.49) \quad \tilde{\Omega}(\infty) := \lim_{t \to \infty} e^{i\Theta t} \tilde{a}(t)\]

exists and

\[(1.50) \quad \tilde{\psi}(x, t) = \tilde{c}(t) e^{-iD \ln(g(t))} \psi_b(x) \oplus \psi_c(x, t)\]

where $c(t)$ satisfies (1.16) and there exists $M > 1$ such that

\[(1.51) \quad \lim_{t \to \infty} \inf_{x \in \mathbb{R}} \|\psi_c(x, t)\|_{L_x^2} \geq c.\]

Moreover, based on (1.25), the $g(t)$-self-similar channel wave operator

\[(1.52) \quad \Omega_g^* \psi(0) := \rho \lim_{s \to 0^+} e^{i\Theta s} e^{iD \ln(g(T^{-1}(s)))} \psi(T^{-1}(s))\]

exists in $L_x^2$ and

\[(1.53) \quad \tilde{\Omega}_g^* \psi(0) = \tilde{\Omega}(\infty) \psi_b(x).\]
As an application of Theorem 4.2, consider a focusing nonlinear Schrödinger equation
\begin{align}
\begin{cases}
    i\partial_t \psi = H_0 \psi + g(t)^{-2} V \left( \frac{\psi}{g(t)} \right) \psi + \mathcal{N}(|\psi|) \psi \\
    \psi(x, t_0) = \psi_s(x) + e^{-iD\ln(g(t_0))} \psi_b(x) \in \mathcal{H}_x^1(\mathbb{R}^n)
\end{cases}
\end{align}
(1.51)

There is a global $H_x^1$ solution $\psi(t)$ such that
\[
\sup_{t \in \mathbb{R}} \|\psi(t)\|_{H_x^1} \leq \psi(t_0), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R},
\]
Both $\psi(t_0)$ and $V(x)$ are radial in $x$

$\mathcal{N} \leq 0$

when $n \geq 5$. Assume that $V(x), \psi_b(x)$ satisfy (1.4)($H := H_0 + V(x)$) and
\begin{align}
\left\{ \begin{array}{c}
    \|\mathcal{D}\psi_0(x)\|_{L_x^\infty} \leq 1 \\
    \| \frac{1}{x-H} P_{\mathcal{E}} c_{L_x^\infty \rightarrow L_x^\infty} \|_1 \leq 1
\end{array} \right.
\end{align}
(1.52)

and $\psi_s(x)$ is a soliton of
\begin{align}
i\partial_t \phi = H_0 \phi + \mathcal{N}_{F,0}(|\phi|) \phi,
\end{align}
(1.53)

where
\begin{align}
\mathcal{N}_{F,0}(k) := \int_0^k dq q \mathcal{N}(q)/k^2 < 0,
\end{align}
(1.54)

that is,
\begin{align}
(H_0 + 2\mathcal{N}_{F,0}(|\psi_s(x)|))\psi_s(x) = E\psi_s(x), \text{ for some } E < 0.
\end{align}
(1.55)

If the nonlinearity satisfies that
\begin{align}
(1) \text{ there exists } T \geq 1 \text{ such that for all } t_0 \geq T,
\end{align}
(1.56)

\[
(\psi(t_0), (H_0 + \frac{1}{g(t_0)^2} V \left( \frac{x}{g(t_0)} \right) + 2\mathcal{N}_{F,0}(|\psi(t_0)|))\psi(t_0))_{L_x^2} \leq \frac{E}{2} \|\psi_s(x)\|_{L_x^2}^2
\]

with $\psi(t_0) = \psi_s(x) + e^{-iD\ln(g(t_0))} \psi_b(x),$

(2) $\mathcal{N}$ satisfies that
\begin{align}
|\mathcal{N}_{F,0}(k)| \leq |k|^{\beta}, \text{ for some } \beta > 0
\end{align}
(1.57)

and for $f \in H_x^1$,
\begin{align}
\begin{cases}
    \|\mathcal{N}(|f(x)|)|L_x^2 \leq C\|f(x)\|_{H_x^1} \\
    \|\mathcal{N}(|f(x)|)f(x)|L_x^2 \leq C\|f(x)\|_{H_x^1}
\end{cases}
\end{align}
(1.58)

then there are at least two bubbles in $\psi(t)$, a solution to system (1.51).

**Remark 1.7.** The radial assumption implies that
\begin{align}
|\psi(t)| \leq \frac{1}{|x|^{\frac{d-1}{2}}}, \quad |x| \geq 1.
\end{align}
(1.59)

Based on (1.57) and (1.59), one has that
\begin{align}
\left| (\psi(t), \mathcal{N}_{F,0}(|\psi(t)|)\psi(t))_{L_x^2} \right| \leq \frac{1}{M^{\frac{d-1}{2}}} C\|\psi(t)\|_{H_x^1} + \|\chi(|x| \leq M)\psi(t)|_{L_x^2} C\|\psi(t)\|_{H_x^1}
\end{align}
(1.60)

for all $t \geq t_0$ and $t_0$, sufficiently large.
Remark 1.8. Here

\[(1.61) \quad \left\| \frac{1}{\lambda - H} P_c \right\|_{L^\infty_v \to L^\infty_x} \leq 1\]

is true when the wave operator, associated with a pair $H_0, H$, from high frequency cut-off $L^p$ space to $L^p$, is bounded for $p = \infty$. See [26] for more details about such potentials.

We show that the weakly localized part of (1.51) has at least two bubbles: a non-trivial self-similar part and a non-trivial localized part near the origin.

**Theorem 1.9.** Let $\tilde{u}(t)$ be as in (1.12). If $N, V(x), H$ satisfy (1.51) and $g(t)$ satisfies (1.3), then when $n \geq 5, \epsilon > (2/n, 1/2)$,

\[(1.62) \quad \tilde{A}(\infty) := \lim_{t \to \infty} e^{i\lambda(t)} \tilde{u}(t) \]

exists and

\[(1.63) \quad \psi_{w,t}(x, t) = c(t) e^{-iD \ln(g(t))} \psi_b(x) \oplus \psi_c(x, t) \]

\[(1.64) \quad c(t) := (e^{-iD \ln(g(t))} \psi_b(x), \psi_{w,t}(x, t))_{L^2_v}, \]

\[(1.65) \quad (e^{-iD \ln(g(t))} \psi_b(x), \psi_c(x, t))_{L^2_v} = 0, \]

with $c(t)$ satisfying (1.16). Furthermore, there exists some large number $M \geq 1$ such that

\[(1.66) \quad \lim_{t \to \infty} \inf \|\chi(|x| \leq M)\psi(t)\|_{L^2_v} \geq c' \]

for some $c' > 0$. Moreover, based on (1.25), the $g(t)$-self-similar channel wave operator

\[(1.67) \quad \Omega_g^* \psi(0) := \text{w- lim}_{s \to \infty} e^{iH} e^{iD \ln(g(T^{-1}(s)))} \psi(T^{-1}(s)) \]

exists in $L^2_v$ and

\[(1.68) \quad \Omega_g^* \psi(0) = \tilde{A}(\infty) \psi_b(x). \]

**Typical example** of Theorem 1.9 is

\[(1.69) \quad \chi(|\psi(t)|) = -\lambda \frac{|\psi(t)|}{1 + |\psi(t)|^2}, \quad g(t) = (t)^\epsilon \]

for some $\epsilon \in (2/5, 1/2)$ and some sufficiently large $\lambda > 0$ in 5 space dimensions. In this case, by taking $\lambda > 0$ large enough, there is a soliton to (1.53). By using standard iteration scheme, there is a global $L^2$ solution to (1.51) for any initial $H_1^2$ data. The $H_1^2$ norm of the solution is uniformly bounded in $t$ since this system has an asymptotic energy. See Lemma 5.1 in Section 5 for more details.

1.2. **Outline of the proof.** For the linear problem, the proof scheme of Theorem 1.5 is first to set

\[(1.70) \quad \tilde{\phi}(t) := e^{iD \ln(g(t))} \psi(x, t), \]

with $\tilde{\phi}(t)$ satisfying

\[(1.71) \quad \begin{cases} 
    i\partial_t \tilde{\phi} = g(t)^{-2} H \tilde{\phi} - (\partial_t g(t)) g(t)^{-1}) D \tilde{\phi} \\
    \tilde{\phi}(t_0) = \psi_b(x)
  \end{cases} \]
see Lemma 2.1. Secondly, using change of variables from \( t \) to \( s = T(t) \), (1.71) can be rewritten as

\[
\begin{cases}
i\partial_s \phi = H\phi + f(s)D\phi \\
\phi(s_0) = \psi_b(x)
\end{cases}
\]

by setting

\[
\phi(s) := \tilde{\phi}(T^{-1}(s)), \quad t_0 := T^{-1}(s_0)
\]

where

\[
f(s) := -(\partial_t [g(t)]g(t))|_{t=T^{-1}(s)}.
\]

Based on (1.3), we have

\[
f(s) \sim \frac{1}{\langle s \rangle} \quad \text{and} \quad f'(s) \sim \frac{1}{\langle s \rangle^2},
\]

see Lemma 2.2. So up to here, the problem is reduced to study the ionization problem (for ionization problem, see [23]). To be precise, it is reduced to study the asymptotic behavior of \( a(s) \) with

\[
a(s) := (\psi_b, \phi(s))_{L_2^2}.
\]

Indeed

\[
\tilde{a}(t) = a(T(t)).
\]

Let

\[
A(s) := e^{i\lambda s}a(s).
\]

In the end, we show that the limit

\[
A(\infty) := \lim_{s \to \infty} e^{i\lambda s}a(s)
\]

exists which implies

\[
\tilde{A}(\infty) = A(\infty)
\]

exists since

\[
\tilde{A}(T^{-1}(s)) = A(s).
\]

And if we choose \( s_0 \) wisely (large enough),

\[
|A(\infty)| \geq \frac{1}{2} > 0
\]

and finish the proof.

For the nonlinear problem or the mixture problem, it is like the linear one except for the nonlinear term and \( W(x) \) term. For these terms, we use

\[
|g(t)^2 e^{-iD\ln(g(t))} \psi_b(x), N(|\psi|)\psi|_{L_2^2} | \leq g(t)^{-(n/2-2)}||\psi_b(x)||_{L_2^2}||N(|\psi|)|\psi||_{L_4^2},
\]

\[
|g(t)^2 e^{-iD\ln(g(t))} \psi_b(x), W(x)\psi|_{L_2^2} | \leq g(t)^{-(n/2-2)}||\psi_b(x)||_{L_2^2}||W(x)\psi||_{L_4^2},
\]

and

\[
g(T^{-1}(u))^{-(n/2-2)} \sim \langle u \rangle^{-(n/2-2)e/(1-2\epsilon)} \in L_4^1[1, \infty)
\]

when \( n \geq 5 \) and \( \epsilon \in (2/n, 1/2) \).
This follows stability analysis of coherent structures [24, 25].

In order to prove the existence of another bubble near the origin, we use the fact that for these systems, there is an asymptotic energy which is negative. Since the self-similar part carries no energy, there must be a part of the solution localized on the support of the potential $W$.

2. Proof of Theorem 1.5: Self-similar Time-dependent Linear Problem

2.1. Tool box.

Lemma 2.1. Let $\tilde{\phi}$ be as in (1.70) and $\psi$ denote the solution to (1.1). Then $\tilde{\phi}$ satisfies (1.71).

Proof. Let $U(t, 0)$ denote the solution operator to (1.1). Then $\psi(x, t)$ can be rewritten as

\[ \psi(x, t) = [U(t, 0)\psi_0](x, t). \]

Thus, based on the definition of $\tilde{\phi}$, it can be rewritten as

\[ \tilde{\phi}(x, t) = e^{iD\ln g(t)}\psi(t) = g(t)^{n/2} [U(t, 0)\psi_0](g(t)x, t). \]

Using Chain rule, compute $i\partial_t[\tilde{\phi}(x, t)]$

\[ i\partial_t[\tilde{\phi}(x, t)] = e^{iD\ln g(t)}(H_0 + g(t)^{-2}V(x/g(t)))\psi(t) - (\partial_t[g(t)]g(t)^{-1})D\tilde{\phi} \]

\[ = g(t)^{-2}H\tilde{\phi} - (\partial_t[g(t)]g(t)^{-1})D\tilde{\phi}. \]

We finish the proof.

Lemma 2.2. If $g(t)$ satisfies (1.3), (1.75) is true.

Proof. Based on the definition of $T$, we have

\[ s = \int_0^t du g^{-2} \sim \langle t \rangle^{1-2\varepsilon}. \]

Since as $t \to \infty$,

\[ |f(s)| = |g'(t)g(t)| \sim \frac{1}{t}g(t)^2 \sim \frac{1}{\langle t \rangle^{1-2\varepsilon}}, \]

we get

\[ |f(s)| \leq \frac{1}{\langle s \rangle}. \]

Now we show

\[ f'(s) \sim \frac{1}{\langle s \rangle^2}. \]

Since

\[ f'(s) = -(g'(t)^2 + g''(t)g(t)) \times \frac{dt}{ds} = -(g'(t)^2 + g''(t)g(t))^2 = -f(s)^2 - g''(t)g(t)^3, \]

and since due to (1.3), (2.4),

\[ g''(t)g(t)^3 \sim \frac{1}{\langle t \rangle^2} \times \langle t \rangle^{4\varepsilon} \sim \frac{1}{\langle s \rangle^2}, \]
according to (2.6), we get

\begin{equation}
 f'(s) \sim f(s)^2 + g''(t)g(t)^3 \sim \frac{1}{\langle s \rangle^2}.
\end{equation}

We finish the proof. \hfill \Box

**Lemma 2.3.** Let \( \lambda, H, V, g \) be as in (1.4).

\begin{equation}
 \| D_\lambda \frac{1}{\lambda - H} P_c \langle x \rangle^{-1} \|_{L^2_x} \lesssim 1.
\end{equation}

**Proof.** Using second resolvent identity, we have

\begin{equation}
 D_\lambda \frac{1}{\lambda - H} P_c \langle x \rangle^{-1} = D_\lambda \frac{1}{\lambda - H_0} P_c \langle x \rangle^{-1} - D_\lambda \frac{1}{\lambda - H} V(x) \frac{1}{\lambda - H} P_c \langle x \rangle^{-1}.
\end{equation}

(2.11) follows from

\begin{equation}
 \| D_\lambda \frac{1}{\lambda - H_0} \langle x \rangle^{-1} \|_{L^2_x} \lesssim 1,
\end{equation}

\begin{equation}
 \| \langle x \rangle P_c \langle x \rangle^{-1} \|_{L^2_x} \lesssim 1,
\end{equation}

and

\begin{equation}
 \| \langle x \rangle V(x) \frac{1}{\lambda - H} P_c \|_{L^2_x} \lesssim \| \langle x \rangle V \|_{L^\infty_x}.
\end{equation}

\hfill \Box

### 2.2. Ionization Problem.

Based on (1.76) and (1.72), we write out an equation of \( a \)

\begin{equation}
 i \partial_s [a(s)] = \lambda a(s) + f(s)(\psi_b, D\psi_b)_{L^2_x} a(s) + f(s)(\psi_b, DP_c \phi(s))_{L^2_x}
\end{equation}

where \( P_c \) denotes the projection on the continuous spectrum of \( H \). Without loss of generality, we can choose \( \psi_b(x) \) real. Then

\begin{equation}
 (\psi_b, D\psi_b)_{L^2_x} = 1/2 \int d^3 x P_x \cdot (x|\psi_b|^2) = 0.
\end{equation}

Let

\begin{equation}
 A(s) := e^{i \lambda s} a(s).
\end{equation}

\( A(s) \) satisfies

\begin{equation}
 i \partial_s [A(s)] = e^{i \lambda s} f(s)(\psi_b, DP_c \phi(s))_{L^2_x}.
\end{equation}

Thus,

\begin{equation}
 A(s) = A(s_0) + (-i) \int_{s_0}^s du e^{i \lambda u} f(u)(\psi_b, DP_c \phi(u))_{L^2_x}.
\end{equation}

**Theorem 2.4.** Let \( A(s) \) be as defined above. \( A(\infty) \) exists and if \( s_0 > 0 \) large enough,

\begin{equation}
 |A(\infty)| \geq \frac{1}{2} > 0.
\end{equation}
Proof. Writing $e^{i\lambda u}P_c\phi(u)$ as
\begin{equation}
(2.22) \quad e^{i\lambda u}P_c\phi(u) = e^{i\lambda u}e^{-iuH}P_c e^{iuH}\phi(u)
\end{equation}
and taking integration by parts in $u$ variable, we obtain that for $s > s_1 \geq s_0$
\begin{equation}
(2.23) \quad A(s) = A(s_1) + (-1)f(u)(\psi_b, D\frac{P_c}{\lambda - H}e^{i\lambda u}P_c\phi(u))_{L^2_T}\big|_{u=s_1} +
\end{equation}
\begin{equation}
(2.24) \quad \int_{s_1}^{\infty} du f'(u)(\psi_b, D\frac{P_c}{\lambda - H}e^{i\lambda u}P_c\phi(u))_{L^2_T} +
\end{equation}
\begin{equation}
(2.25) \quad \int_{s_1}^{\infty} du f(u)^2(\psi_b, D\frac{P_c}{\lambda - H}e^{i\lambda u}P_cD\phi(u))_{L^2_T}
\end{equation}
\begin{equation}
(2.26) \quad =: A(s_1) + \sum_{j=1}^{3} A_j(s).
\end{equation}
Here
\begin{equation}
(2.27) \quad (\psi_b, D\frac{1}{\lambda - H}e^{i\lambda u}P_cD\phi(u))_{L^2_T}
\end{equation}
is understood in weak sense, that is,
\begin{equation}
(2.28) \quad (\psi_b, D\frac{1}{\lambda - H}e^{i\lambda u}P_cD\phi(u))_{L^2_T} = (P_c\frac{1}{\lambda - H}D\psi_b, De^{i\lambda u}\phi(u))_{L^2_T}.
\end{equation}
For $s \geq s_1 \geq s_0$, using $L^2_T$ conservation law, Hölder’s inequality and Lemma 2.2,
\begin{equation}
(2.29) \quad |A_1(s)| \leq |f(s_1)||D\psi_b(x)||_{L^2_T} \times \|\frac{1}{\lambda - H}P_c\|_{L^2_T} \|\phi(s_0)||_{L^2_T} \leq |f(s_1)| \leq \frac{1}{\langle s_1 \rangle},
\end{equation}
\begin{equation}
(2.30) \quad |A_2(s)| \leq \int_{s_1}^{\infty} du |f'(u)||D\psi_b(x)||_{L^2_T} \times \|\frac{1}{\lambda - H}P_c\|_{L^2_T} \|\phi(s_1)||_{L^2_T} \leq \frac{1}{\langle s_1 \rangle},
\end{equation}
and due to Lemma 2.2 and Lemma 2.3,
\begin{equation}
(2.31) \quad |A_3(s)| \leq \int_{s_1}^{\infty} du f(u)^2||D\psi_b(x)||_{L^2_T} \times \|\frac{1}{\lambda - H}P_c\|_{L^2_T} \|\phi(s_1)||_{L^2_T} \leq \frac{1}{\langle s_1 \rangle}.
\end{equation}
So $\{A(s)\}_{s \geq s_0}$ is Cauchy and therefore $A(\infty)$ exists with
\begin{equation}
(2.32) \quad |A(s)| \geq |A(s_0)| - C \times \frac{1}{\langle s_0 \rangle} = 1 - C \times \frac{1}{\langle s_0 \rangle} \geq \frac{1}{2}, \text{ for all } s \in [s_0, \infty]
\end{equation}
if we choose $s_0$ large enough. We finish the proof. □

Lemma 2.5. Let $V, H, g$ be as in 1.4. For $t \geq t_0$ and $t_0$ large enough,
\begin{equation}
(2.33) \quad |(\psi_b(x), \tilde{\phi}(x, t))_{L^2_T}| \geq 1/2.
\end{equation}

Proof. Taking $t_0 = T^{-1}(s_0)$ for $s_0$ satisfying (2.33), we have
\begin{equation}
(2.34) \quad |(\psi_b(x), \tilde{\phi}(x, t))_{L^2_T}| = |(\psi_b(x), \phi(x, s))_{L^2_T}| = |A(s)| \geq 1/2
\end{equation}
with $s = T(t)$. □
2.3. **Linear Problem.** Now we prove Theorem 1.5.

*Proof.* Express \( \psi(x, t) \) in terms of \( \tilde{\phi}(x, t) \)

\[
\psi(x, t) = e^{-iD\ln g(t)} \tilde{\phi}(x, t) = c(t)e^{-iD\ln g(t)} \psi_b(x) + \psi_c(x, t)
\]

where

\[
(\psi_c(x, t), e^{-iD\ln g(t)} \psi_b(x))_{L^2_t} = 0.
\]

Then due to Lemma 2.5,

\[
|\tilde{a}(t)| \geq \frac{1}{2}, \quad \text{for all } t \geq t_0
\]

for some sufficiently large \( t_0 > 0 \). Let

\[
\psi_{g,b}(x, t) := e^{-iD\ln g(t)} \psi_b(x).
\]

Using \( L^p \) decay estimates for the free flow, we obtain

\[
\|F_c(\frac{|x - 2tP_x|}{t^\alpha}) \leq 1)\psi_{g,b}(x, t)||_{L^2_t} \leq \|F_c(\frac{|x|}{t^\alpha}) \leq 1\) e^{it\tilde{H}_0} \psi_{g,b}(x, t)||_{L^2_t}
\]

\[
\leq \|F_c(\frac{|x|}{t^\alpha}) \leq 1\)||_{L^2_t} \times \|e^{it\tilde{H}_0} \psi_{g,b}(x, t)||_{L^2_t}
\]

\[
\leq t^{-(n(1-\alpha)/2)} \times ||\psi_{g,b}||_{L^2} \leq \frac{g(t)^{n/2}}{p^{n(1-\alpha)/2}} \leq \frac{\langle t \rangle^{n/2}}{p^{n(1-\alpha)/2}} \to 0
\]

as \( t \to \infty \) when \( 1 - \alpha - \epsilon > 0 \). Given \( t_0 \) satisfying (2.38),

\[
|c(t)| = |(\psi_{g,b}(x, t), \psi_{w,\tilde{a}}(t))_{L^2_t}| = |a(t) - (\psi_{g,b}(x, t), F_c(\frac{|x - 2tP_x|}{t^\alpha}) \leq 1)\psi(t) - \psi_c(x, t))_{L^2_t}|
\]

\[
\geq \frac{1}{2} - \frac{1}{8} - \frac{1}{8} = \frac{1}{4}
\]

when \( t \geq t_M \) for some sufficiently large \( t_M \geq t_0 \), which implies (1.16). We finish the proof.

\( \square \)

3. **Existence of Free Channel Wave Operator for a MCS in 4 or Higher Space Dimensions**

In this section, we prove the existence of free channel wave operator in system

\[
\begin{cases}
  i\partial_t \psi(t) = (H_0 + V(x, t))\psi(t) \\
  \psi(0) = \psi_0 \in L^2_t(R^n)
\end{cases} \quad n \geq 4
\]

provided that \( \langle t \rangle^{-n/2+2}V(x, t) \in L^\infty_t L^2_x(R^n \times R) \).

**Theorem 3.1.** If \( \langle t \rangle^{-n/2+2}V(x, t) \in L^\infty_t L^2_x(R^n \times R) \), then when \( n \geq 4 \), the free channel wave operator of (3.1)

\[
\Omega^*_\alpha \psi(0) := s-\lim_{t \to 0^+} e^{itH_0} F_c(\frac{|x - 2tP|}{t^\alpha} \leq 1)\psi(t),
\]

exists in \( L^2_x(R^n) \) for \( \alpha \in (0, 2/n) \).
Proof. Let
\begin{equation}
\Omega_\alpha^*(t)\psi(0) := e^{it\hat{H}_0}F_c(\frac{|x - 2tP|}{t^\alpha} \leq 1)\psi(t).
\end{equation}
Since
\begin{equation}
e^{it\hat{H}_0}F_c(\frac{|x - 2tP|}{t^\alpha} \leq 1)\psi(t) = F_c(\frac{|x|}{t^\alpha} \leq 1)e^{it\hat{H}_0}\psi(t),
\end{equation}
then by using Fundamental Theorem in Calculus,
\begin{equation}
\Omega_\alpha^*(t)\psi(0) = \Omega_\alpha^*(1)\psi(0) + \int_1^n ds\partial_s[e^{is\hat{H}_0}F_c(\frac{|x - 2sP|}{s^\alpha} \leq 1)\psi(s)]
= \Omega_\alpha^*(1)\psi(0) + \int_1^n ds\partial_s[F_c(\frac{|x|}{s^\alpha} \leq 1)]e^{is\hat{H}_0}\psi(s) + \int_1^n dsF_c(\frac{|x|}{s^\alpha} \leq 1)e^{is\hat{H}_0}V(x, s)\psi(s)
=: \Omega_\alpha^*(1)\psi(0) + \psi_1(t) + \psi_2(t).
\end{equation}
\(\psi_2(\infty) \in L^2_\alpha(\mathbb{R}^n)\) since for \(T \geq t \geq 1\),
\begin{equation}
\int_1^T ds\|F_c(\frac{|x|}{s^\alpha} \leq 1)e^{is\hat{H}_0}V(x, s)\psi(s)\|_{L^2_\alpha(\mathbb{R}^n)}
\leq \int_1^T ds\|F_c(\frac{|x|}{s^\alpha} \leq 1)\|_{L^2_\alpha}e^{is\hat{H}_0}V(x, s)\|_{L^\infty_\alpha(\mathbb{R}^n)}\psi(s)\|_{L^2_\alpha}
\leq \int_1^T \frac{1}{s^{2-n/2\alpha}}\|t^{-n/2+2}\|V(x, s)\|_{L^\infty_\alpha(\mathbb{R}^n)}\|\psi(s)\|_{L^2_\alpha}
\leq \frac{1}{t^{1-n/2\alpha}}\|t^{-n/2+2}\|V(x, s)\|_{L^\infty_\alpha(\mathbb{R}^n)}\|\psi(s)\|_{L^2_\alpha} \to 0
\end{equation}
as \(t \to \infty\) when \(\alpha \in (0, 2/n)\).
For \(\psi_1(t)\), we use Propagation estimates, see [27]. To be precise, choose
\begin{equation}
B(t) = F_c(\frac{|x|}{t^\alpha} \leq 1)
\end{equation}
as our observable. Observe
\begin{equation}
\langle B(t) \rangle := (e^{it\hat{H}_0}\psi(t), B(t)e^{it\hat{H}_0}\psi(t))_{L^2_\alpha}.
\end{equation}
Compute \(\partial_t\langle B(t) \rangle\)
\begin{equation}
\partial_t\langle B(t) \rangle = \langle \partial_t[B(t)] \rangle + (-i)(e^{it\hat{H}_0}\psi(t), B(t)e^{it\hat{H}_0}V(x, t)\psi(t))_{L^2_\alpha} +
\langle i(e^{it\hat{H}_0}V(x, t)\psi(t), B(t)e^{it\hat{H}_0}\psi(t))_{L^2_\alpha} =: A_1(t) + A_2(t) + A_3(t).
\end{equation}
\(A_2(t), A_3(\infty) \in L^1_t[1, \infty)\) due to (3.6). Since \(A_1(t) \geq 0\) and
\begin{equation}
\langle B(t) \rangle \leq ||\psi(0)||^2_{L^2_\alpha},
\end{equation}
one has that for \(T \geq t \geq 1\),
\begin{equation}
\int_t^T dsA_1(s) \leq 2||\psi(0)||^2_{L^2_\alpha} + ||A_2(s)||_{L^1_t[1, \infty)} + ||A_3(s)||_{L^1_t[1, \infty)}
\end{equation}
which implies $A_1(t) \in L^1_t[1, \infty)$. Then by using Hölder’s inequality in $s$ variable, for $T \geq t \geq 1$,

\[
\|\psi_1(T) - \psi_1(t)\|_{L^2_x} \lesssim \left( \int d^3x \int_t^T ds|\partial_s[F_c(\frac{|x|}{s^\alpha} \leq 1)]|e^{i\mu H_0} \psi(s)\right)^{1/2} \lesssim \left( \int_t^T dsA_1(s) \right)^{1/2} \to 0
\]

as $t \to \infty$, which implies $\psi_1(\infty)$ exists in $L^2_x$. Thus, that $\psi_1(\infty), \psi_2(\infty) \in L^2_x$ implies that $\Omega^*_a \psi(0)$ exists in $L^2_x$ and we finish the proof.

\[\square\]

Since

\[
\sup_{t \in \mathbb{R}} \|\langle t \rangle^{-n/2+2} g(t)^{-2} V(x/g(t))\|_{L^2_x} = \sup_{t \in \mathbb{R}} \left( \frac{g(t)}{t} \right)^{n/2-2} \|V(x)\|_{L^2_x}
\]

\[
\lesssim \sup_{t \in \mathbb{R}} \left( \frac{1}{t^{1-\epsilon}} \right)^{n/2-2} \|V(x)\|_{L^2_x}
\]

for $\epsilon \in (0, 1]$, in (1.2), (1.42) and (1.51) with space dimension $n \geq 4$, the free channel wave operator

\[
\Omega^*_a := s^{-1} \lim_{t \to \infty} e^{iH_0} F_c(\frac{|x - 2tP|}{t^n} \leq 1) U(t, 0)
\]

exists from $L^2_x$ to $L^2_x$.

### 4. Proof of Theorem 4.2: The Linear Mixture Problem

In this section, we keep using

\[
s = T(t) := \int_0^t du g(u)^{-2}.
\]

#### 4.1. Tool box.

**Lemma 4.1.** Let $\psi$ be the global solution to (1.42) and $f(s)$ be as in (1.75). Let

\[
\tilde{\phi}(x, t) := e^{iD \ln g(t)} \psi(x, t),
\]

and

\[
\phi(x, s) := \tilde{\phi}(x, T^{-1}(s)).
\]

Then with $s_0 = T(t_0)$, $s = T(t)$,

\[
\begin{cases}
    i\partial_s \phi = H\phi + g(T^{-1}(s))^2 W(g(T^{-1}(s)) x) \phi + f(s) D\phi \\
    \phi(s_0) = g(t)^n/2 \psi_a(g(t) x) + \psi_b(x)
\end{cases}
\]

where

\[
\psi_g(t) := \psi(g(t) x, t).
\]
Proof. It follows from the same proof for Lemma 2.1 by replacing \( V(x, t) = g(t)^{-2}V \left( \frac{x}{g(t)} \right) \) with \( V(x, t) = g(t)^{-2}V \left( \frac{x}{g(t)} \right) + W(x) \) and then change variable from \( t \) to \( s = T(t) \).

\( \square \)

4.2. Ionization Problem of a mixture. Let

\[
(a(s) := (\psi_b(x), \phi(s)))_{L^2_x}.
\]

As what we did for linear problem, we derive an equation for \( a(t) \) first. Compute \( i\partial_s[a(s)](s = T(t)) \)

\[
i\partial_s[a(s)] = \lambda a(s) + f(s)(\psi_b(x), DP_c\phi(s))_{L^2_x} +
\]

\[
g(T^{-1}(s))^2(\psi_b(x), W(g(T^{-1}(s))\phi(s))_{L^2_x}
\]

where we use (2.17). Set

\[
A(s) := e^{is\lambda}a(s).
\]

Then \( A(s) \) satisfies

\[
i\partial_s[A(s)] = e^{is\lambda}f(s)(D\psi_b(x), P_c\phi(s))_{L^2_x} +
\]

\[
e^{is\lambda}g(T^{-1}(s))^2(\psi_b(x), W(g(T^{-1}(s))\phi(s))_{L^2_x}.
\]

Theorem 4.2. Let \( A(s) \) be as defined above. If \( g(t) \sim \langle t \rangle^\varepsilon \) with \( \varepsilon \in \left( \frac{2}{n}, \frac{1}{2} \right) \), \( W(x) \in L^2_x \), when \( n \geq 5, A(\infty) \) exists and if \( s_0 > 0 \) large enough,

\[
|A(s)| \geq \frac{1}{2} > 0 \quad \text{for all} \quad s \geq s_0.
\]

Proof. Set

\[
N_g(x, u) := W(g(T^{-1}(u))x).
\]

\[
A(s) = A(s_0) + (-i) \int_{s_0}^{s} d\varepsilon u e^{i\varepsilon \lambda}f(u)e^{i\varepsilon \lambda}(D\psi_b(x), P_c\phi(u))_{L^2_x} +
\]

\[
(-i) \int_{s_0}^{s} e^{i\varepsilon \lambda}g(T^{-1}(u))^2(\psi_b(x), N_g(x, u)e^{i\varepsilon \lambda}\phi(u))_{L^2_x}
\]

\[
= A(s_0) + A_1(s) + A_2(s).
\]

For \( A_1(s) \), take integration by parts in \( s \) variable by setting

\[
e^{is\lambda}P_c\phi(s) = \frac{1}{i(\lambda - H)}P_c\partial_s[e^{is\lambda}e^{-i\varepsilon H}]e^{isH}\phi(s)
\]

and we obtain

\[
A_1(s) = f(u)(D\psi_b(x)), \frac{1}{i(\lambda - H)}P_c\partial_s[e^{is\lambda}e^{-i\varepsilon H}]e^{isH}\phi(s)
\]

\[
- \int_{s_0}^{s} d\varepsilon u f(u)(D\psi_b(x)), \frac{1}{i(\lambda - H)}P_c\partial_s[e^{is\lambda}e^{-i\varepsilon H}]e^{isH}\phi(s)
\]

\[
- \int_{s_0}^{s} d\varepsilon u f(u)^2(D\psi_b(x)), \frac{1}{i(\lambda - H)}P_c\partial_s[e^{is\lambda}e^{-i\varepsilon H}]e^{isH}\phi(s)
\]

\[
- \int_{s_0}^{s} d\varepsilon u g(T^{-1}(u))^2 f(u)(D\psi_b(x)), \frac{1}{i(\lambda - H)}P_c\partial_s[e^{is\lambda}e^{-i\varepsilon H}]e^{isH}\phi(s)
\]
(4.21) \[ = \sum_{j=1}^{4} \tilde{A}_{1j}(s). \]

As what we did in linear case, for \( s_2 \geq s_1 \geq s_0, \)
(4.22) \[ |\tilde{A}_{1j}(s_2) - \tilde{A}_{1j}(s_1)| \leq \frac{1}{\langle s_1 \rangle}, \quad j = 1, 2, 3. \]

For \( \tilde{A}_{14}, \) let
(4.23) \[ \tilde{\psi}_b(x) := P_c \frac{1}{-i(\lambda - H)} D\psi_b(x). \]

Then
(4.24) \[ \|\tilde{\psi}_b(x)\|_{L^\infty} = \|P_c \frac{1}{-i(\lambda - H)} D\psi_b(x)\|_{L^\infty} \leq \|D\psi_b(x)\|_{L^\infty}. \]

Using change of variable from \( x \) to \( y = \langle T^{-1}(u) \rangle^x x, \) we have
(4.25) \[ (\tilde{\psi}_b(x), N_q(x, u)e^{iu\lambda}\phi(u))_{L^2} = \]
\[ \left( \frac{1}{g(T^{-1}(u))^{y/2}} \tilde{\psi}_b(g(T^{-1}(u))^{-1}y), W(y)e^{iu\lambda}\psi(y, T^{-1}(u)) \right)_{L^2}. \]

So using Cauchy Schwarz inequality, due to (1.42), (1.43) and (1.3)(\( g(t) \sim \langle t \rangle^\epsilon \)), we obtain
(4.26) \[ |\tilde{A}_{14}(s_2) - \tilde{A}_{14}(s_1)| \leq \int_{s_1}^{s_2} du |f(u)| \frac{1}{\langle T^{-1}(u) \rangle^{y/2-2\epsilon}} \|\tilde{\psi}_b(y)\|_{L^\infty} \|W(y)\psi(y, T^{-1}(u))\|_{L^1} \]
\[ \leq \frac{\|W(x)\|_{L^2} \|\psi(0)\|_{L^2}}{\langle T^{-1}(s_1) \rangle^{x/2-2\epsilon}} \leq \frac{\|W(x)\|_{L^2} \|\psi(0)\|_{L^2}}{\langle s_1 \rangle^{(n/2-2\epsilon)/(1-2\epsilon)}} \to 0 \]
as \( s_1 \to \infty \) when \( \epsilon \in (0, 1/2) \) and \( n \geq 5. \) Thus,
(4.28) \[ \tilde{A}_{1j}(\infty) \text{ exists for all } j = 1, 2, 3, 4 \]
and therefore \( A_1(\infty) \) exists. And for \( s_0 \) large enough,
(4.29) \[ |A_1(s)| \leq C \frac{\|W(x)\|_{L^2} + 1}{\langle s_0 \rangle^{\min(1, (n/2-2\epsilon)/(1-2\epsilon))}} \leq \frac{1}{4}. \]

For \( A_2(s), \) using Cauchy Schwarz inequality, due to (1.42) and (1.3), we obtain that for \( s_2 \geq s_1 \geq s_0, \)
(4.30) \[ |A_2(s_2) - A_2(s_1)| \leq \int_{s_1}^{s_2} du \frac{1}{\langle T^{-1}(u) \rangle^{y/2-2\epsilon}} \|\tilde{\psi}_b(y)\|_{L^\infty} \|W(y)\psi(T^{-1}(u))\|_{L^1} \]
\[ \leq \int_{s_1}^{s_2} \frac{\|W(x)\|_{L^2} \|\psi(0)\|_{L^2}}{\langle u \rangle^{(n/2-2\epsilon)/(1-2\epsilon)}} \leq \epsilon \frac{\|W(x)\|_{L^2} \|\psi(0)\|_{L^2}}{\langle s_1 \rangle^{(n/2-2\epsilon)/(1-2\epsilon)-1}} \to 0 \]
since when \( \epsilon \in \left( \frac{2}{n}, \frac{1}{2} \right), n \geq 5, \)
(4.32) \[ [(n/2-2\epsilon)/(1-2\epsilon)] - 1 > 0. \]

Therefore \( A_2(\infty) \) exists. And for \( s_0 \) large enough,
(4.33) \[ |A_2(s)| \leq C \frac{\|W(x)\|_{L^2} \|\psi(0)\|_{L^2}}{\langle s_0 \rangle^{\min(1, (n/2-2\epsilon)/(1-2\epsilon))}} \leq \frac{1}{4}. \]
According to (4.29) and (4.33), we have that \( A(\infty) \) exists and for \( s_0 \) large enough,
\[
|A(s)| \geq |A(s_0)| - |A_1(s)| - |A_2(s)| \geq \frac{9}{10} - |A_1(s)| - |A_2(s)| \geq \frac{1}{2}
\]
where we use that for large \( t_0 > 0 \),
\[
|\langle g(t_0)^{\alpha/2}\psi(x)g(t_0), \psi(x) \rangle_{L^2_x}| \leq \frac{1}{10}.
\]
We finish the proof.

**Remark 4.3.** Based on the proof of Theorem 4.2, if we have \( W(x, t) \) instead of \( W(x) \) in (1.42), in order to get the conclusion of Theorem 4.2, we only need that \( W(x, t) \in L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R}) \) for \( n \geq 5, \epsilon \in \left( \frac{2}{n}, \frac{1}{2} \right) \).

**Lemma 4.4.** Let \( W, V, H, g \) be as in 1.42. For \( t \geq t_0 \) and \( t_0 \) large enough,
\[
|\langle \psi(x), \phi(x, t) \rangle| \geq 1/2.
\]

**Proof.** Taking \( t_0 = T^{-1}(s_0) \) for \( s_0 \) satisfying (4.34), we have
\[
|\langle \psi_b(x), \phi(x, t) \rangle| = |\langle \psi_b(x), \phi(x, s) \rangle| = |A(s)| \geq 1/2
\]
with \( s = T(t) \).

**Corollary 4.5.** Let \( N, V, H, g \) be as in (1.51). For \( t \geq t_0 \) and \( t_0 \) large enough,
\[
|\langle \psi_b(x), \phi(x, t) \rangle| \geq 1/2.
\]

**Proof.** Based on the proof of Theorem 4.2, all we need is that \( N(\psi(t)) \in L^\infty_t L^2_x \). Also, see Remark 4.3. So based on (1.51), we have (4.38).

**Proof of Theorem 1.6.** Based on Lemma 4.4, we have \( c(t) \) satisfies (1.16). (1.48) follows from that this system has an asymptotic energy
\[
\partial_t(\psi(t)), (H_0 + W(x) + g(t)^{-2}V(\frac{x}{g(t)})\psi(t)) \in L^1_t \text{ with }
\]
\[
\int_{t_0}^T dt \left| \partial_t(\psi(t)), (H_0 + W(x) + g(t)^{-2}V(\frac{x}{g(t)})\psi(t)) \right|_{L^2_x} \leq C \int_{t_0}^T dt \left| \langle \psi(t), (H_0 + W(x) + g(t)^{-2}V(\frac{x}{g(t)})\psi(t) \rangle \right|_{L^2_x} \leq \frac{|A_0|}{2} \left| \psi(x) \right|_{L^2_x}
\]
for all \( T \geq t_0 \) if we choose \( t_0 \) large enough and recall that \( g(t) \sim \langle t \rangle^\epsilon \). Since
\[
(\psi(t_0), (H_0 + W(x) + g(t_0)^{-2}V(\frac{x}{g(t_0)})\psi(t_0))_{L^2_x} = \lambda_0 \left| \psi_b(x) \right|_{L^2_x}^2 + (g(t_0)^{-2}A\left| \psi_b(x) \right|_{L^2_x}^2 + \langle \psi_b(x), g(t_0)^{-2}V(\frac{x}{g(t_0)})\psi_b(x) \rangle_{L^2_x} + \langle W(x)e^{-iD\ln g(t_0)}\psi_b(x), W(x)e^{-iD\ln g(t_0)}\psi_b(x) \rangle_{L^2_x}
\]
if we choose \( t_0 \) large enough, then for all \( T \geq t_0 \),
\[
(\psi(T), (H_0 + W(x) + g(T)^{-2}V(\frac{x}{g(T)})\psi(T))_{L^2_x} \leq \frac{3}{4} \lambda_0 \left| \psi_b(x) \right|_{L^2_x}^2.
\]
If $t_0$ is large enough, for all $T \geq t_0$,
\begin{equation}
(\psi(T), (H_0 + W(x))\psi(T))_{L^2_x} \leq \frac{1}{8} \lambda_0 \|\psi_d(x)\|^2_{L^2_x}
\end{equation}
since
\begin{equation}
(\psi(t), g(t)^{-2}V\left(\frac{x}{g(t)}\right)\psi(t))_{L^2_x} \to 0
\end{equation}
as $t \to \infty$. Hence for $t \geq t_0$,
\begin{equation}
\frac{1}{\|\psi_d(x)\|_{L^2_x}} |(\psi(t), \psi_d(x))_{L^2_x}| \geq \frac{1}{2 \sqrt{2}},
\end{equation}
which implies (1.48) since
\begin{equation}
|g(t)^{-n/2} \psi_b(\frac{x}{g(t)}), \psi_d(x))_{L^2_x}| \leq \frac{1}{g(t)^{n/2}}
\end{equation}
and
\begin{equation}
|\left(F_x(\frac{|x - 2tP|}{P^2} \leq 1)\psi(t), \psi_d(x)\right)_{L^2_x}| \leq \frac{1}{P^{n/2(1 - a)}} \|\psi_d(x)\|_{L^2_x} \|\psi(t_0)\|_{L^2_x}.
\end{equation}
We finish the proof.

\section{Nonlinear Problem}

\textit{Proof of Theorem 1.9.} The first part of Theorem 1.9 follows by using Corollary 4.5. For the second part of Theorem 1.9, prove by contradiction. Assume that (1.66) is not true. Then for any $M \geq 1$, given $t_0 \geq 1$, there exists $t_M \geq t_0$ such that
\begin{equation}
\|\chi(|x| \leq M)\psi(t_M)\|_{L^2_x} \leq \frac{1}{M}.
\end{equation}
We will get contradiction from the fact that this system has an asymptotic energy
\begin{equation}
\partial_t(\psi(t), (H_0 + g(t)^{-2}V(\frac{x}{g(t)})) + N(|\psi(t)|))_{L^2_x} =
(\psi(t), \partial_t[g(t)^{-2}V(\frac{x}{g(t)})]\psi(t))_{L^2_x} + (N'(\psi(t)), \partial_t[|\psi(t)|])_{L^2_x}
\end{equation}
where
\begin{equation}
N_F(k) := \int_0^k dq q^2 N'(q).
\end{equation}
Then
\begin{equation}
(\psi(T), (H_0 + g(T)^{-2}V(\frac{x}{g(T)})) + N(|\psi(T)|))_{L^2_x} =
(\psi(t_0), (H_0 + g(t_0)^{-2}V(\frac{x}{g(t_0)})) + N(|\psi(t_0)|))_{L^2_x} + \int_{t_0}^T ds g_0(s) + (G(T) - G(t_0))
\end{equation}
where
\begin{equation}
g_0(t) := (\psi(t), \partial_t[g(t)^{-2}V(\frac{x}{g(t)})]\psi(t))_{L^2_x} \in L^1_t[1, \infty),
\end{equation}
\( G(t) := \int d^n x N_F(|\psi(t)|) \)

\( = (\psi(t), N(|\psi(t)|)\psi(t))_{L^2} - 2(\psi(t), N_{F,0}(|\psi(t)|)\psi(t))_{L^2} \)

with

\( N_{F,0}(k) = \int_0^k dq N(q)/k^2 < 0. \)

Then (5.4) can be rewritten as

\( (\psi(T), (H_0 + g(T))^{-2}V(\frac{x}{g(T)}) + N(|\psi(T)|))\psi(T))_{L^2} = \)

\( (\psi(t_0), (H_0 + g(t_0))^{-2}V(\frac{x}{g(t_0)}) + 2N_{F,0}(|\psi(t_0)|)\psi(t_0))_{L^2} + \int_{t_0}^T ds g_0(s) + G(T), \)

which is equivalent to

\( (\psi(t_0), (H_0 + g(t_0))^{-2}V(\frac{x}{g(t_0)}) + 2N_{F,0}(|\psi(t_0)|)\psi(t_0))_{L^2} + \int_{t_0}^T ds g_0(s). \)

On the one hand, due to (1.56) and that \( g(s) \in L_1^1 \), we have that there exists \( \tilde{t}_0 \geq 1 \) such that for all \( T \geq t_0 \geq \tilde{t}_0 \),

\( (\psi(t_0), (H_0 + g(t_0))^{-2}V(\frac{x}{g(t_0)}) + 2N_{F,0}(|\psi(t_0)|)\psi(t_0))_{L^2} + \int_{t_0}^T ds g_0(s) \leq \frac{E}{4} \|\psi_s(x)\|_{L^2}^2. \)

On the other hand, based on assumption (5.1) and Remark 1.7, there exists \( \bar{M} \geq 1 \) such that for all \( M \geq \bar{M}, t_m \geq t_0, t_0 \) sufficiently large,

\( 2|\psi(t_M), N_{F,0}(|\psi(t_M)|)\psi(t_M)|_{L^2} \leq (\frac{1}{M^{\alpha - \beta w}} + \frac{1}{M}) C(\sup_T \|\psi(t)\|_{H^1}) \leq -\frac{E}{100} \|\psi_s(x)\|_{L^2}^2 \)

and

\( |(\psi(t_M), g(t_M)^{-2}V(x/g(t_M))\psi(t_M)| \leq t_M^{-2\alpha} \|V(x)\|_{L^2} \|\psi(t_M)\|_{L^2} \leq -\frac{E}{100} \|\psi_s(x)\|_{L^2}^2 \)

which implies that for all \( M \geq \bar{M}, t_m \geq t_0, t_0 \) sufficiently large,

\( (\psi(t_M), (H_0 + g(t_M))^{-2}V(\frac{x}{g(t_M)}) + 2N_{F,0}(|\psi(t_M)|)\psi(t_M))_{L^2} \geq \frac{E}{50} \|\psi_s(x)\|_{L^2}^2 \)

since

\( (\psi(t_M), H_0\psi(t_M))_{L^2} \geq 0. \)

Based on (5.10), (5.11) and (5.14), contradiction and we finish the proof. \( \square \)

A typical example in this case is

\( N(k) = -\lambda \frac{k}{1 + k^2}. \)
Lemma 5.1. When

\[ N(k) = -\lambda \frac{k}{1 + k^2} \tag{5.17} \]

for some sufficiently large \( \lambda > 0 \), the assumption of Theorem 1.9 is satisfies and we get at least two bubbles as \( t \) goes to infinity.

Proof. By taking \( \lambda > 0 \) large enough, there is a soliton \( \psi_s(x) \) to (1.53). By using standard iteration scheme, there is a global \( L^2 \) solution to (1.51) for any initial \( H^1 \) data for any \( V \) satisfying (1.4) and (1.52). Now we show that the \( H^1 \) norm of the solution is uniformly bounded in \( t \). Compute

\[ \partial_t \left[ (\psi(t), (-\Delta_x + \langle t \rangle)^{-2}\nu V(x/\langle t \rangle^\epsilon) - \lambda \frac{|\psi(t)|}{1 + |\psi(t)|^2} \right] \psi(t)_{L^2} \tag{5.18} \]

\[ = (\psi(t), \partial_t \left[ (\langle t \rangle)^{-2}\nu V(x/\langle t \rangle^\epsilon) \right] \psi(t))_{L^2} - \lambda (\psi(t), \partial_t \left[ \frac{|\psi(t)|}{1 + |\psi(t)|^2} \right] \psi(t))_{L^2} \]

\[ =: g_0(t) - \lambda (\psi(t), \partial_t \left[ \frac{|\psi(t)|}{1 + |\psi(t)|^2} \right] \psi(t))_{L^2}. \tag{5.19} \]

\[ \int_{0}^{T} dt |g_0(t)| \lesssim \frac{1}{\langle t \rangle^{1+2\epsilon}} \left( \|V(x)\|_{L^infty} + \|x \cdot \nabla V(x)\|_{L^infty} \right) \|\psi(t_0)\|_{L^2}^2 \]

\[ \lesssim (\|V(x)\|_{L^infty} + \|x \cdot \nabla V(x)\|_{L^infty}) \|\psi(t_0)\|_{L^2}^2. \]

Let

\[ \mathcal{N}_F(k) = \int_0^k dq q^2 \mathcal{N}'(q) = \int_0^k dq q^2 \partial_q \left[ \frac{2}{1 + q^2} \right]. \tag{5.20} \]

Then

\[ |\mathcal{N}_F(k)| \lesssim k^2 \tag{5.21} \]

and

\[ (\psi(t), \partial_t \left[ \frac{|\psi(t)|}{1 + |\psi(t)|^2} \right] \psi(t))_{L^2} = \int d^n x \partial_t \left[ \mathcal{N}_F(|\psi(t)|) \right]. \tag{5.22} \]

Thus, by using (5.21),

\[ \int_{0}^{T} dt \left( \psi(t), \partial_t \left[ \frac{|\psi(t)|}{1 + |\psi(t)|^2} \right] \psi(t) \right)_{L^2} \]}

\[ \leq \|\psi(T)\|_{L^2}^2 + \|\psi(t_0)\|_{L^2}^2 \lesssim \|\psi(t_0)\|_{L^2}^2. \tag{5.23} \]

Based on (5.19) and (5.23), we have

\[ (\psi(t), (-\Delta_x)\psi(t))_{L^2} \lesssim \|\psi(t_0)\|_{L^2}^2 + (\psi(t), \langle t \rangle^{-2\epsilon}|V(x/\langle t \rangle^\epsilon)\psi(t))_{L^2} + \lambda (\psi(t), \frac{|\psi(t)|}{1 + |\psi(t)|^2} \psi(t))_{L^2} + \lambda (\psi(t_0), \frac{|\psi(t_0)|}{1 + |\psi(t_0)|^2} \psi(t_0))_{L^2} + (\psi(t_0), \langle t_0 \rangle^{-2\epsilon}|V(x/\langle t_0 \rangle^\epsilon)\psi(t_0))_{L^2} + (\psi(t_0), (-\Delta_x)\psi(t_0))_{L^2}. \tag{5.24} \]
\[ \leq \|\psi(t_0)\|_{L_x^2}^2 (1 + \|V(x)\|_{L_x^\infty}) + \|\psi(t_0)\|_{H_x^1}^2, \]

where we also use
\[ (5.25) \quad \frac{|\psi(t)|}{1 + |\psi(t)|^2} \leq 1. \]

Hence,
\[ (5.26) \quad \sup_{t \in \mathbb{R}} \|\psi(t)\|_{H_x^1} \leq \|\psi(t_0)\|_{L_x^2} \sqrt{(1 + \|V(x)\|_{L_x^\infty}) + \|\psi(t_0)\|_{H_x^1}}, \]

In addition, taking \( \beta = 1 > 0 \),
\[ (5.27) \quad |N_{F,0}(k)| = \left| \int_0^k dq q N(q)/k^2 \right| = \lambda \int_0^k dq \frac{q^2}{1 + q^2} \leq \lambda k^\beta \]
and for any \( f \in H_x^1 \),
\[ (5.28) \quad \begin{cases} \left\| \frac{f(x)}{1 + |f(x)|} \right\|_{L_x^2} \leq \|f(x)\|_{L_x^2} \leq \|f(x)\|_{H_x^1} \\ \left\| \frac{f(x)}{1 + |f(x)|} \right\|_{L_x^2} \leq \frac{1}{2} \|f(x)\|_{L_x^2} \leq \frac{1}{2} \|f(x)\|_{H_x^1} \end{cases} \]
and (1.56) follows from that \( g(t) \to \infty \) as \( t \to \infty \) and \( N_{F,0}(|\psi(t)|) \) is localized in \( x \) since we have radial symmetry. Thus, the assumption of Theorem 1.9 is satisfied and we get that the solution \( \psi(t) \) to system
\[ (5.29) \quad \begin{cases} i\partial_t \psi = (\mathcal{H}_0 + \langle t \rangle^{-2e} V(x/\langle t \rangle^e) - \lambda \frac{|\psi(t)|}{1 + |\psi(t)|^2}) \psi \\ \psi(x, t_0) = \psi_x(x) + e^{-iD \ln(g(t_0))} \psi_b(x) \end{cases} \]
has at least two bubbles with different patterns. We finish the proof. \( \square \)

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