SECOND TYPE FOLIATIONS OF CODIMENSION ONE

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Abstract. In this article, for holomorphic foliations of codimension one at \((\mathbb{C}^3, 0)\), we define the family of second type foliations. This is formed by foliations having, in the reduction process by blow-up maps, only well oriented singularities, meaning that the reduction divisor does not contain weak separatrices of saddle-node singularities. We prove that the reduction of singularities of a non-dicritical foliation of second type coincides with the desingularization of its set of separatrices.

1. Introduction

An important category of problems in the theory of holomorphic foliations involves the study of equisingularity properties. A question of this type was addressed in \([3]\), where the authors proved that topological equivalent foliation inside the family of generalized curve foliations are equisingular, meaning that their reductions of singularities by blow-up maps are combinatorially equivalent. A germ of foliation at \((\mathbb{C}^2, 0)\) is said to be a generalized curve if there are no saddle-nodes in its reduction of singularities. In this family, separatrices — formal invariant curves — are all analytic and carry an important volume of topological information of the foliation itself. For instance, a generalized curve foliation and its set of separatrices have the same reduction of singularities, meaning that a sequence of blow-ups that desingularizes all separatrices transforms the foliation into one having only simple singularities. Generalized curve foliations are also characterized as those that minimize Milnor numbers, once an equisingular set of separatrices is fixed. For instance, when the set of separatrices of a foliation \(\mathcal{G}\) is finite (the so-called non-dicritical case) having \(g = 0\) as a reduced equation, then \(\mathcal{G}\) is a generalized curve if and only if \(\mu_0(\mathcal{G}) = \mu_0(\mathcal{G})\), where \(\mu\) denotes the Milnor number.

A quite natural development is the extension of the notion of generalized curve for codimension one foliations in higher dimension spaces. In ambient dimension three, the existence of a reduction of singularities \([6, 5]\) has been the starting point in \([9]\) for the definition of generalized surface foliations, comprising non-dicritical foliations (meaning that the divisor in the reduction process is invariant) without saddle-nodes in the reduction of singularities. The main theorem in \([9]\) asserts that a generalized surface foliation and its set of separatrices have the same reduction of singularities. We call the attention that, in the universe of dicritical codimension one foliations, a result of this sort does not make sense, since there are foliations — for instance, the celebrated Jouanolou example \([11]\) — having no separatrix at all. The notion of generalized surface foliations reappears, in arbitrary

\[\mu_0(\mathcal{G}) = \mu_0(\mathcal{G})\]

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ambient dimension, in [7], receiving the designation of complex hyperbolic foliations, a terminology we would rather adopt. The definition now is set in terms of two dimensional sections: a germ of codimension one foliation $\mathcal{F}$ at $(\mathbb{C}^n,0)$ is complex hyperbolic if and only if, for every analytic map $\phi : (\mathbb{C}^2,0) \to (\mathbb{C}^n,0)$ generically transversal to $\mathcal{F}$, the pull-back foliation $\mathcal{G} = \phi^* \mathcal{F}$ is a generalized curve foliation.

Returning to dimension two, the conditions defining generalized curve foliations can be slightly weakened, delimiting the larger family of second type foliations. As introduced in [15], a germ of foliation at $(\mathbb{C}^2,0)$ is of second type or in the second class if all singularities in its reduction of singularities are well oriented (Definition 2.1). This means precisely that saddle-nodes are admitted in the reduction process, provided they lie in the regular part of the divisor, with their weak separatrices transversal to it. As well as generalized curve foliations, second type foliations and their sets of separatrices have equivalent reductions of singularities. However, formal separatrices may exist. The minimization property now works for the algebraic multiplicity. For instance, if $g = 0$ is a reduced equation for the set of separatrices of a non-dicritical foliation $\mathcal{G}$, then $\mathcal{G}$ is second type if and only if $\nu_0(\mathcal{G}) = \nu_0(g)$, where $\nu$ stands for the algebraic multiplicity. These properties are the key ingredients used in [16] in order to prove that second type foliations equivalent by $C^\infty$ diffeomorphisms are equisingular.

The central objective of this article is to propose an extension of the concept of second type foliations to foliations of codimension one. As in [9], we only work in the three dimensional case, since we use a reduction of singularities in our definition. Evidently, for $n > 3$, as soon as the existence of a reduction process for foliations at $(\mathbb{C}^n,0)$ is proved, the notion of second type foliations along with most of the results developed in this article can be properly adapted. Now, a germ of codimension one foliation at $(\mathbb{C}^3,0)$ is of second type if it admits a reduction of singularities in which all singularities are well oriented with respect to the divisor (Definitions 3.1 and 3.4). It turns out that the definition does not depend on the reduction of singularities and that it may also be formulated in terms of two-dimensional sections (Proposition 3.5). Employing arguments similar to those in [9], we can prove the following result:

**Theorem I.** Let $\mathcal{F}$ be a non-dicritical foliation at $(\mathbb{C}^3,0)$. Suppose that $\mathcal{F}$ is second type. Then $\mathcal{F}$ and its set of separatrices have the same reduction of singularities.

This article has the following organization. In Section 2 we briefly recall the main aspects of the reduction of singularities for foliations, both in dimension two and three. Next, in Section 3 we establish the notion of well oriented singularities in dimension three, define second type foliations and prove that the definition does not depend on the reduction of singularities and may also be formulated in terms of two dimensional sections. In Section 4 we prove Theorem I. Finally, in Section 5 we give a characterization of logarithmic foliations in the projective space $\mathbb{P}^3$ in terms of the notion of second type foliation.

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2. SIMPLE SINGULARITIES AND REDUCTION OF SINGULARITIES

We start by establishing a pattern for the notation to be followed. We use $(u,v)$ for analytic or formal local coordinates at $(\mathbb{C}^2,0)$ and $(x,y,z)$ at $(\mathbb{C}^3,0)$. The term foliation
is used as a short for *singular holomorphic foliation of codimension one*. Foliations and normal crossings divisors are denoted, with variations, respectively by $\mathcal{G}$ and $\mathcal{E}$, in dimension two, and by $\mathcal{F}$ and $\mathcal{D}$, in dimension three. The number of local branches of a divisor at a point $p$ is denoted by $e_p(\mathcal{E})$ or $e_p(\mathcal{D})$. Abusing terminology and notation, an empty local normal crossings divisor at a point $p$ is represented by the one point set $\{p\}$. For $n = 2$ or $3$, as usual, $\mathcal{O}_n$ and $\mathcal{\hat{O}}_n$ denote, respectively, the local rings of analytic functions and of formal functions $n$ variables.

2.1. In dimension two. A germ of holomorphic foliation $\mathcal{G}$ at $(\mathbb{C}^2, 0)$ is defined, in analytic coordinates $(u, v)$, by an analytic 1-form

$$\eta = A(u, v)du + B(u, v)dv,$$

where $A, B \in \mathcal{O}_2$ are relatively prime. A *separatrix* for $\mathcal{G}$ is an invariant formal irreducible curve, corresponding to an irreducible formal function $f \in \mathcal{\hat{O}}_2$ satisfying

$$\eta \wedge df = (fh)du \wedge dv$$

for some $h \in \mathcal{\hat{O}}_2$. The separatrix is said to be *analytic* if we can take $f, h \in \mathcal{O}_2$. The set of separatrices of $\mathcal{G}$ is denoted by $\text{Sep}_0(\mathcal{G})$. Following the usual notation, its algebraic multiplicity is $\nu_0(\mathcal{G}) = \min\{\nu_0(A), \nu_0(B)\}$ and its Milnor number is $\mu_0(\mathcal{G}) = \dim_{\mathbb{C}} \mathcal{O}_2/(A, B)$.

The foliation $\mathcal{G}$ is *simple* if the linear part of $v = B(u, v)\partial/\partial u - A(u, v)\partial/\partial v$, vector field dual to $\eta$, is non-nilpotent and has eigenvalues with quotient outside $\mathbb{Q}_{>0}$. Simple foliations admit the following formal normal forms:

(a) $$\eta = uv \left( \lambda_1 \frac{du}{u} + \lambda_2 \frac{dv}{v} \right),$$

where $\lambda_1, \lambda_2 \in \mathbb{C}^*$ and $m_1 \lambda_1 + m_2 \lambda_2 \neq 0$ for every $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ with at least one of them non-zero;

(b1) $$\eta = uv \left( \frac{du}{u} + \varphi(u) \frac{dv}{v} \right),$$

where $\varphi \in \mathcal{\hat{O}}_1$ is a non-unity;

(b2) $$\eta = uv \left( p_1 \frac{du}{u} + p_2 \frac{dv}{v} + \varphi(u^{p_1} v^{p_2}) \frac{dv}{v} \right),$$

where $p_1, p_2 \in \mathbb{Z}_{>0}$ and $\varphi \in \mathcal{\hat{O}}_1$ is a non-unity.

Models (a) and (b2) are called *non-degenerate* or *complex hyperbolic* simple singularities. Model (b1), corresponding to the existence of a zero eigenvalue, is a *saddle-node* singularity. For all simple foliations, the separatrix set $\text{Sep}_0(\mathcal{G})$ is formed by two transversal branches, given by $\{u = 0\}$ and $\{v = 0\}$. In the non-degenerate case, both are analytic. For a saddle-node, the separatrix corresponding to $\{u = 0\}$, associated to the eigenspace of the non-zero eigenvalue, is analytic and is called *strong*. On its turn, $\{v = 0\}$ defines a possibly formal separatrix, called *weak*.

Let $\mathcal{E}$ be normal crossings divisor at $(\mathbb{C}^2, 0)$ such that either $e_0(\mathcal{E}) = 1$ or 2. We say that $\mathcal{G}$ is:
the multiplicity also holds in the dicritical case and a formulation for it can be seen in [10].

If $G$ holds if and only if $\nu$ is a simple non-degenerate and thus $\tilde{\nu}$ or $E$ is a fixed local normal crossing divisor $E = \{uv = 0\}$. Then there exists a proper analytic map, formed by a composition of blow-up maps, $\sigma : (\tilde{N}, E) \to (C^2, 0)$, where $E = \sigma^{-1}(0)$ is a normal crossings divisor and $\tilde{N}$ is a germ of complex surface around $E$, such that all points of $E$ are either $E$-regular or $E$-simple for the transformed foliation $\tilde{G} = \sigma^* G$.

The map $\sigma$ above is called reduction of singularities or desingularization for $G$. The concept of simple singularity well oriented with respect to a divisor was established in [17]. Here is the precise definition:

**Definition 2.1.** Let $E$ be a normal crossings divisor and $G$ be an $E$-simple germ of foliation at $(C^2, 0)$. We say that $G$ is $E$-well oriented in one of the following cases:

(i) $G$ is a non-degenerate singularity;

(ii) $G$ is a saddle-node singularity whose weak separatrix is not contained in $E$.

Otherwise, we say that $G$ is an $E$-tangent saddle-node.

The notion of $E$-well oriented simple singularity is invariant by blow-ups in the following sense: if $\sigma$ is a blow-up map at $0 \in C^2$ with exceptional divisor $E = \sigma^{-1}(0)$, then $\tilde{G} = \sigma^* G$ has two simple singularities, corresponding to the two points of intersection, say $p_1$ and $p_2$, between $E$ and the transforms of the branches of $\text{Sep}_0(G)$. If $G$ is $E$-well oriented, then $\tilde{G}$ is well oriented with respect to $\tilde{E} = E \cup \sigma^* E$ at $p_1$ and $p_2$. On the other hand, if $G$ is an $E$-tangent saddle-node whose weak separatrix lies in a component $E_1 \subset E$, then $\tilde{G}$ has a saddle-node singularity at $p_1 = E \cap \sigma^* E_1$ whose weak separatrix is contained in $\sigma^* E_1$, being an $E$-tangent saddle-node, whereas at the other singular point $p_2 \in E$, we have that $\tilde{G}$ is simple non-degenerate and thus $\tilde{E}$-well oriented.

The concept of well oriented simple singularities is the basis for the following definition [15]:

**Definition 2.2.** A germ of foliation $G$ at $(C^2, 0)$ is said to be of second type if, given a reduction process $\sigma : (\tilde{N}, \tilde{E}) \to (C^2, 0)$, all singularities of the transformed foliation $\tilde{G} = \sigma^* G$ are $E$-well-oriented.

Clearly, the definition does not depend on the reduction of singularities. Now, for a fixed local normal crossing divisors $E$ at $(C^2, 0)$, an $E$-reduction of singularities is a map $\sigma : (\tilde{N}, \tilde{E}^\#) \to (C^2, 0)$, composition of a finite number of blow-ups, such that all points in $\tilde{E} = \sigma^{-1}(E) = \tilde{E}^\# \cup \sigma^* E$ are either $\tilde{E}$-simple or $\tilde{E}$-regular for $\tilde{G} = \sigma^* G$. In this case, we say that $G$ is $E$-second type if all singularities of $\tilde{G}$ are $\tilde{E}$-well oriented. For instance, if $G$ is a saddle-node singularity, then it is a second type foliation. However, setting $E = \text{Sep}_0(G)$ or $\tilde{E} = \{\text{weak separatrix}\}$, then $\tilde{G}$ is not $\tilde{E}$-second type.

Let $G$ be a non-dicritical foliation at $(C^2, 0)$ whose set of separatrices has $g = 0$ as a reduced equation, where $g \in \mathcal{O}_2$. Then $\nu_0(G) \geq \nu_0(dg) = \nu_0(\text{Sep}_0(G)) - 1$ and the equality holds if and only if $G$ is second type [15]. This property of minimization of the algebraic multiplicity also holds in the dicritical case and a formulation for it can be seen in [10].
A second type foliation and its set of separatrices have equivalent reductions of singularities. This is a straight consequence of the following lemma, which is a restatement of Lemma 1 in [3]:

**Lemma 2.3.** Let $\mathcal{G}$ be a germ of foliation at $(C^2,0)$. Suppose that

(i) $\text{Sep}_0(\mathcal{G})$ has exactly two transversal branches;

(ii) $\mathcal{G}$ is second type.

Then $\mathcal{G}$ is simple.

**Proof.** The proof is essentially that of [3]. We take formal coordinates $(u,v)$ such that $\text{Sep}_0(\mathcal{G}) = \{uv = 0\}$, implying that $\mathcal{G}$ is given by a $1-$form of the kind $\eta = u\bar{a} du + v\bar{b} dv$, where $\bar{a}, \bar{b} \in \hat{O}_2$. Since $\mathcal{G}$ is second type, $\nu_0(\mathcal{G}) = \nu_0(\text{Sep}_0(\mathcal{G})) - 1 = 1$. Thus, the linear part of $\eta$ is $\lambda_1 u\bar{a} du + \lambda_3 u\bar{b} dv$, with $\lambda_1, \lambda_3$ not both zero. If both are non-zero, it suffices to see that we cannot have $\lambda_1/\lambda_2 \in \mathbb{Q}_{<0}$. Actually, if this happens, either $\text{Sep}_0(\mathcal{G})$ has a unique branch ($\mathcal{G}$ non-linearizable with either $\lambda_1/\lambda_2 \in \mathbb{Z}_{<0}$ or $\lambda_2/\lambda_1 \in \mathbb{Z}_{<0}$) or $\text{Sep}_0(\mathcal{G})$ has infinitely many branches (all other cases).

2.2. **In dimension three.** A germ of codimension one holomorphic foliation $\mathcal{F}$ at $(C^3,0)$ is defined, in analytic coordinates $(x,y,z)$, by an analytic 1–form

\begin{equation}
\omega = A(x,y,z)dx + B(x,y,z)dy + C(x,y,z)dz,
\end{equation}

where $A, B, C \in O_3$ are without common factors, satisfying the integrability condition $\omega \wedge dz = 0$. A separatrix for $\mathcal{F}$ is an invariant formal irreducible surface, that is, an object given by an irreducible formal function $f \in \hat{O}_3$ such that $\omega \wedge df = (fh)\theta$ for some $h \in \hat{O}_3$ and some formal 2–form $\theta$. We also have the obvious notion of analytic separatrix. The set of separatrices of $\mathcal{F}$ is again denoted by $\text{Sep}_0(\mathcal{F})$.

The dimensional type of a foliation $\mathcal{F}$, denoted by $\tau_0(\mathcal{F})$ or simply by $\tau$, is the smallest number of variables needed to express its defining equation in some system of analytic coordinates. Thus, $\tau_0(\mathcal{F}) = 2$ if and only if there are analytic coordinates $(x,y,z)$ under which $\mathcal{F}$ is an analytic cylinder over a singular foliation in the coordinates $(x,y)$. Note that $\tau_0(\mathcal{F}) = 1$ if and only if $\mathcal{F}$ regular.

Let us list the simple formal models for singularities of a foliation $\mathcal{F}$ at $(C^3,0)$. If $\tau_0(\mathcal{F}) = 2$, they have already been listed in Subsection 2.1. If $\tau_0(\mathcal{F}) = 3$, we say that $\mathcal{F}$ is simple if there are formal coordinates $(x,y,z)$ in which $\mathcal{F}$ is expressed in one of the following models:

\begin{equation}(A)\end{equation}

\[\omega = xyz \left(\lambda_1 \frac{dx}{x} + \lambda_2 \frac{dy}{y} + \lambda_3 \frac{dz}{z}\right),\]

where $\lambda_1, \lambda_2, \lambda_3 \in C^*$ are such that $m_1 \lambda_1 + m_2 \lambda_2 + m_3 \lambda_3 \neq 0$ whenever $m_1, m_2, m_3 \in Z_{\geq 0}$ with at least one of them non-zero;

\begin{equation}(B1)\end{equation}

\[\omega = xyz \left(p_1 \frac{dx}{x} + \varphi(x^{p_1}) \left(\lambda_2 \frac{dy}{y} + \lambda_3 \frac{dz}{z}\right)\right),\]

where $p_1 \in Z_{>0}$, $\varphi \in \hat{O}_1$ is a non-unity and $\lambda_2, \lambda_3 \in C^*$ satisfy $m_2 \lambda_2 + m_3 \lambda_3 \neq 0$ whenever $m_2, m_3 \in Z_{\geq 0}$ with at least one of them non-zero;
where \( p_1, p_2 \in \mathbb{Z}_{>0}, \varphi \in \hat{O}_1 \) is a non-unity and \( \lambda_2, \lambda_3 \in \mathbb{C}^* \) satisfy \( m_2\lambda_2 + m_3\lambda_3 \neq 0 \) whenever \( m_2, m_3 \in \mathbb{Z}_{\geq 0} \) with at least one of them non-zero;}

\[
(B3) \quad \omega = xyz \left( p_1 \frac{dx}{x} + p_2 \frac{dy}{y} + p_3 \frac{dz}{z} + \varphi(x^{p_1}y^{p_2}z^{p_3}) \left( \lambda_2 \frac{dy}{y} + \lambda_3 \frac{dz}{z} \right) \right),
\]

where \( p_1, p_2, p_3 \in \mathbb{Z}_{>0}, \varphi \in \hat{O}_1 \) is a non-unity and \( \lambda_2, \lambda_3 \in \mathbb{C}^* \) satisfy \( m_2\lambda_2 + m_3\lambda_3 \neq 0 \) whenever \( m_2, m_3 \in \mathbb{Z}_{\geq 0} \) with at least one of them non-zero.

Foliations of formal models (A) or (B3) are said to be simple \textit{complex hyperbolic}, whereas those corresponding to models (B1) or (B2) are said to be \textit{saddle-nodes}. In all of them, the singular set is formed by pairwise transversal analytic curves corresponding to the coordinate axis. Outside the origin, the singularities are of dimensional type two. The set of separatrices is precisely the union of the three coordinate planes.

For the complex hyperbolic models, (A) and (B3), the transversal type along all coordinate axes is complex hyperbolic and, furthermore, all separatrices are analytic. Concerning simple saddle-node models, we can say the following:

- In model (B1), the transversal model along the \( x \)-axis is complex hyperbolic. The \( y \)-axis and the \( z \)-axis have transversal models of saddle-node type. For a transversal section \( y = c, c \neq 0 \), the weak separatrix is contained in \( z = 0 \) and the strong separatrix is in \( x = 0 \). On the other hand, for a transversal section \( z = c, c \neq 0 \), the weak separatrix is contained in \( y = 0 \) and the strong separatrix is in \( x = 0 \).

- In model (B2), the \( z \)-axis has complex hyperbolic transversal model, whereas both the \( x \)-axis and the \( y \)-axis have transversal models of saddle-node type. For a transversal section \( x = c, c \neq 0 \), the weak separatrix is contained in \( z = 0 \) and the strong separatrix is in \( y = 0 \). For a transversal section \( y = c, c \neq 0 \), the weak separatrix is contained in \( z = 0 \) and the strong separatrix is in \( x = 0 \).

This discussion founds the following definition:
Definition 2.4. Let $\mathcal{F}$ be a simple foliation at $(\mathbb{C}^3,0)$ of saddle-node type. We say that a germ of separatrix of $\mathcal{F}$ is weak if it contains a component $\Lambda$ of $\text{Sing}_0(\mathcal{F})$ with saddle-node transversal type such that, for every $p \in \Lambda \setminus \{0\}$ and every two-dimensional section transversal to $\Lambda$ at $p$, $\Lambda \cap \Sigma$ is the weak separatrix of $\mathcal{F}|_{\Sigma}$.

According to this definition, a saddle-node singularity of model (B1) has two weak separatrices, corresponding to the planes $y = 0$ and $z = 0$. On its turn, a saddle-node of model (B2) has a unique weak separatrix, corresponding to $z = 0$. In dimensional type two, the notion of weak separatrix is clear.

At $(\mathbb{C}^3,0)$, let $\mathcal{F}$ be a foliation defined by an integrable 1-form $\omega$ and $\Gamma$ be a germ of formal curve with Puiseux parametrization $\gamma(t)$. We say that $\Gamma$ is invariant by $\mathcal{F}$ if $\gamma^*\omega = 0$. This notion evidently does not depend on the choices made. For later use, we make explicit the following simple fact:

Lemma 2.5. Let $\mathcal{F}$ be a simple foliation at $(\mathbb{C}^3,0)$. Then all formal curves invariant by $\mathcal{F}$ are contained in $\text{Sep}_0(\mathcal{F})$.

Proof. The result is clear if $\tau_0(\mathcal{F}) = 2$. If $\tau_0(\mathcal{F}) = 3$, it follows from straight calculations with each simple formal model. If the $\mathcal{F}$-invariant curve lies outside the coordinate planes, then it has a parametrization $\gamma(t) = (\gamma_1, \gamma_2, \gamma_3) = (t^{n_1} \tilde{\gamma}_1, t^{n_2} \tilde{\gamma}_2, t^{n_3} \tilde{\gamma}_3)$, where $n_1, n_2, n_3 \in \mathbb{Z}_{>0}$ and $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3 \in \mathcal{O}_1$ are units. For model (B1), for instance,

$$
\gamma^*\omega = \gamma_1 \gamma_2 \gamma_3 \left[ p_1 n_1 \frac{dt}{t} + p_1 \frac{d\tilde{\gamma}_1}{\tilde{\gamma}_1} + \varphi(\gamma_1^{p_1}) \left( (\lambda_2 n_2 + \lambda_3 n_3) \frac{dt}{t} + \lambda_2 \frac{d\tilde{\gamma}_2}{\tilde{\gamma}_2} + \lambda_3 \frac{d\tilde{\gamma}_3}{\tilde{\gamma}_3} \right) \right]
$$

is identically zero, so the residue $p_1 n_1$ of the formal meromorphic 1-form inside brackets is also zero, which is absurd. The argument is the same for the other models. □

Suppose that $\mathcal{F}$ is a foliation having either a simple singularity or a non-singular point at $0 \in \mathbb{C}^3$. Let $\mathcal{D}$ be a normal crossings divisor. We decompose $\mathcal{D} = \mathcal{D}' \cup \mathcal{D}^*$, where $\mathcal{D}'$ assembles the $\mathcal{F}$-invariant components, called non-dicritical, and $\mathcal{D}^*$ the non-invariant ones, called dicritical. We say that $\mathcal{F}$ is adapted to $\mathcal{D}$ if:

- $\tau_0(\mathcal{F}) - 1 \leq e_0(\mathcal{D}') \leq \tau_0(\mathcal{F})$;
- $e_0(\mathcal{D}^*) \leq 3 - \tau_0(\mathcal{F})$.

In analogy with the two dimensional case, a foliation $\mathcal{F}$ adapted to a divisor $\mathcal{D}$ can be:

- $\mathcal{F}$-simple, if either $\tau_0(\mathcal{F}) = 3$ or $\tau_0(\mathcal{F}) = 2$ and there are local coordinates $(x, y, z)$ such that $\mathcal{F}$ is expressed in the variables $(x,y)$ and $\mathcal{D}' \subset \{z = 0\}$;
- $\mathcal{F}$-regular, if there are local coordinates $(x, y, z)$ such that $\tilde{\mathcal{F}}$ is given by $dx = 0$ and $\mathcal{D}^* \subset \{y z = 0\}$.

In dimension three, the existence of a reduction of singularities is stated in the following way:

Theorem (Reduction of singularities, dimension three [6, 5]). Let $\mathcal{F}$ be a holomorphic separatrix foliation of codimension one at $(\mathbb{C}^3,0)$. Then there is an analytic map $\pi : (\bar{\mathcal{M}}, \bar{\mathcal{D}}) \to (\mathbb{C}^3, \text{Sing}(\mathcal{F}))$ formed by the composition of a finite sequence of blow-up maps with non-singular centers such that:

1. at each intermediate step, the blow-up center is invariant for the corresponding transformed foliation and has normal crossings with the corresponding blow-up divisor;
the transformed foliation \( \tilde{F} = \pi^*F \) is such that all points in \( D \) are either \( D \)-simple or \( D \)-regular.

Following the local picture, we decompose that reduction divisor in non-dicritical and dicritical components, getting \( D = D' \cup D^* \). The foliation is said to be non-dicritical if \( D = D' \), that is, if the reduction divisor is \( \tilde{F} \)-invariant. Finally, singularities \( p \in \text{Sing}(\tilde{F}) \) are classified in two groups:

- \textit{simple corners}, if \( e_p(D') = \tau_p(\tilde{F}) \) (local separatrices contained in \( D \));
- \textit{trace singularities}, if \( e_p(D') = \tau_p(\tilde{F}) - 1 \) (there is one local separatrix outside \( D \)).

### 3. Second type foliations in dimension three

The definitions of well oriented singularities and tangent saddle-nodes for foliations in dimension three is analogous to those in dimension two:

**Definition 3.1.** Let \( D \) be a normal crossings divisor and \( F \) be a germ of \( D \)-simple foliation at \((C^3,0)\). The foliation \( F \) is said to be \( D \)-\textit{well oriented} in one of the following cases:

(i) \( F \) is simple complex hyperbolic singularity (types \([A1]\) or \([B3]\));

(ii) \( F \) is a saddle-node singularity having no weak separatrix contained in \( D \).

Otherwise, we say that \( F \) is a \( D \)-\textit{tangent saddle-node}.

Let us identify the tangent saddle-nodes. We have a \( D \)-simple foliation \( F \), for a normal crossings divisor \( D \) that has a decomposition \( D = D' \cup D^* \) in non-dicritical and dicritical components. Suppose that \( \tau_0(F) = 3 \), so that \( \text{Sep}_0(F) \) has three local branches corresponding, in the formal models in Subsection 2.2, to the three coordinate planes. In this case, \( e_0(D) = e_0(D') = 2 \) or \( 3 \). Thus, we have the following possibilities for a \( D \)-tangent saddle-node:

(i) \( e_0(D') = 2 \) and \( F \) is either a saddle-node of model \([B1]\) or a saddle-node of model \([B2]\) having its weak separatrix in \( D \);

(ii) \( e_0(D') = 3 \) and \( F \) is a saddle-node, models \([B1]\) or \([B2]\).

When \( \tau_0(F) = 2 \), the picture is essentially two-dimensional: \( \text{Sep}_0(F) \) has two local branches crossing normally, and \( e_0(D') = 1 \) or \( 2 \). A \( D \)-tangent saddle-node corresponds to the following cases:

(i) \( e_0(D') = 1 \) and \( F \) is a saddle-node having its weak separatrix in \( D \);

(ii) \( e_0(D') = 2 \) and \( F \) is a saddle-node.

We need the following definition:

**Definition 3.2.** Let \( F \) be a germ of foliation at \((C^3,0)\) defined by \( \omega = 0 \) and let \( E \) be a normal crossings divisor \((C^2,0)\). An analytic map \( \phi : (C^2,0) \to (C^3,0) \) is generically transversal to \( F \) outside \( E \) if \( \text{Sing}(\phi^*\omega) \subset E \). When \( E = \{0\} \) we simply say that \( \phi \) is generically transversal to \( F \).

If \( F \) is a foliation at \((C^3,0)\) then there are embeddings \( \phi : (C^2,0) \to (C^3,0) \) generically transversal to \( F \) satisfying \( \nu_0(\mathcal{G}) = \nu_0(F) \), where \( \mathcal{G} = \phi^*F \). This fact is needed in the proof of Lemma 4.1 below, a three-dimensional analogue of Lemma 2.3. For our purposes, a relevant property of generically transversal analytic maps is that they preserve well orientation of simple singularities:
Lemma 3.3. Let $D$ be a normal crossings divisor and $F$ be a $D$-well oriented germ of $D$-simple foliation at $(\mathbb{C}^3,0)$. Let $E$ be a normal crossing divisor at $(\mathbb{C}^2,0)$ and suppose that $\phi : (\mathbb{C}^2,E) \to (\mathbb{C}^3,D)$ is an analytic map generically transversal to $F$ outside $E$. If $G = \phi^* F$ is $E$-simple, then $G$ is $E$-well oriented.

Proof. If $F$ is a simple complex hyperbolic, then $G = \phi^* F$ is simple non-degenerate by Lemma 4.3. Thus, we can restrict ourselves to the case where $F$ is a $D$-well oriented saddle-node. We fix $(u,v)$ normalizing formal coordinates for $G$, so that $E \subset \text{Sep}(G) = \{uv = 0\}$. We also fix normalizing formal coordinates $(x,y,z)$ for $F$. Thus, all our objects become formal, including the map $\phi : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$, that can be written as $\phi(u,v) = (\phi_1,\phi_2,\phi_3) = (x,y,z)$. We factorize the maximal powers of $u$ and $v$ from each $\phi_i$, getting

$$\phi_i(u,v) = u^{r_i} v^{s_i} \tilde{\phi}_i(u,v), \quad \text{for } i = 1,2,3.$$ 

Since $E$ is $G$-invariant, by Lemma 2.5 it is mapped into the invariant part of $D$. Thus, the eventual dicritical components do not intervene in our analysis and we make a simplification by supposing that that $D$ is $F$-invariant. We have two situations, depending on the dimensional type of $F$:

Case I: $\tau_0(F) = 2$. We take for $F$, in coordinates $(x,y,x)$, the formal normal form \[b1\]:

$$\omega = xy \left( \frac{dx}{x} + \varphi(x) \frac{dy}{y} \right) = xgy.$$ 

In this way, $D = \{x = 0\}$ and the weak separatrix is $\{y = 0\}$. No curve outside $\{uv = 0\}$ can be $G$-invariant. Hence both $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are unities. We have

$$\phi^* \tilde{\omega} = \frac{du}{u} + s_1 \frac{dv}{v} + \frac{d\tilde{\phi}_1}{\tilde{\phi}_1} + \varphi(u^{r_1} v^{s_1} \tilde{\phi}_1) \left( \frac{du}{u} + s_2 \frac{dv}{v} + \frac{d\tilde{\phi}_2}{\tilde{\phi}_2} \right).$$

Our analysis splits into two possibilities:

Subcase 1.1: $E$ has two branches, $E = \{uv = 0\}$. Since $\phi$ takes $E$ into $D$, we must have $r_1 > 0$ and $s_1 > 0$. By cancelling poles in \[b1\], we find

$$uv\phi^* \tilde{\omega} = r_1 vdu + s_1 udv + uv \theta,$$

for some formal 1-form $\theta$. Thus, $G$ is simple non-degenerate (of formal type \[b2\]).

Subcase 1.2: $E$ has a single branch. Let us suppose $E = \{v = 0\}$. Since $\phi$ takes $E$ into $D$, we have that $s_1 > 0$. By Lemma 2.5, the image of $\Gamma = \{u = 0\}$ must be contained in some $F$-invariant plane — either in $D = \{x = 0\}$ or in $\{y = 0\}$ — not in both, for $\phi$ is generically transversal outside $E$. According to this, we have:

- If $\phi(\Gamma) \subset D$, then $r_1 > 0$ and $r_2 = 0$. We recover the situation of the previous subcase: $G$ is induced by a 1-form of the type \[b3\], having a simple non-degenerate singularity at the origin.
- If $\phi(\Gamma) \subset \{y = 0\}$, then $r_1 = 0$ and $r_2 > 0$. By cancelling poles in \[b1\], we get

$$uv\phi^* \tilde{\omega} = s_1 udv + r_2 v\varphi(u^{s_1} \tilde{\phi}_1) du + uv \theta,$$

for some formal 1-form $\theta$. Thus, $G$ has a saddle-node singularity at the origin whose weak separatrix is $\{u = 0\}$. It is therefore $E$-well oriented.
Case II: \( \tau_0(\mathcal{F}) = 3 \). We can suppose that \( \mathcal{F} \) is of type \([\mathcal{B}2]\) and that \( \mathcal{D} = \{xy = 0\} \). The weak separatrix is \( \{z = 0\} \). This time, \( \tilde{\phi}_1, \tilde{\phi}_2 \) and \( \tilde{\phi}_3 \) in \([3]\) are unities. Writing \( \omega = xyz\bar{\omega} \), we find

\[
\phi^*\tilde{\omega} = p_1 \left( \frac{du}{r_1 u} + s_1 \frac{dv}{v} + \frac{d\tilde{\phi}_1}{\tilde{\phi}_1} \right) + p_2 \left( \frac{du}{r_2 u} + s_2 \frac{dv}{v} + \frac{d\tilde{\phi}_2}{\tilde{\phi}_2} \right) + \\
\varphi(\phi_1^{p_1} \phi_2^{p_2}) \left( \lambda_2 \left( \frac{du}{r_2 u} + s_2 \frac{dv}{v} + \frac{d\tilde{\phi}_2}{\tilde{\phi}_2} \right) + \lambda_3 \left( \frac{du}{r_3 u} + s_3 \frac{dv}{v} + \frac{d\tilde{\phi}_3}{\tilde{\phi}_3} \right) \right)
\]

\[
= \frac{du}{u} + s \frac{dv}{v} + p_1 \frac{d\tilde{\phi}_1}{\phi_1} + p_2 \frac{d\tilde{\phi}_2}{\phi_2} + \\
\varphi(u^r v^s \phi_1^{p_1} \phi_2^{p_2}) \left( (\lambda_2 r_2 + \lambda_3 r_3) \frac{du}{u} + (\lambda_2 s_2 + \lambda_3 s_3) \frac{dv}{v} + \lambda_2 \frac{d\tilde{\phi}_2}{\phi_2} + \lambda_3 \frac{d\tilde{\phi}_3}{\phi_3} \right),
\]

where \( r = p_1 r_1 + p_2 r_2 \) and \( s = p_1 s_1 + p_2 s_2 \). Once more we have two subcases:

Subcase II.1 \( \mathcal{E} \) has two branches, \( \mathcal{E} = \{uv = 0\} \). Since \( \phi \) takes \( \mathcal{E} \) into \( \mathcal{D} \), we have that either \( r_1 > 0 \) or \( r_2 > 0 \) and, by the same token, either \( s_1 > 0 \) or \( s_2 > 0 \). Thus, both \( r > 0 \) and \( s > 0 \).

By cancelling poles, we find

\[
uvw\phi^*\tilde{\omega} = rvdu + sudv + uv\theta
\]

for some formal 1–form \( \theta \). This shows that \( \mathcal{G} \) has a simple non-degenerate singularity, which is \( \mathcal{E} \)-well oriented.

Subcase II.2 \( \mathcal{E} \) has a single branch, which we suppose to be \( \mathcal{E} = \{v = 0\} \). Since \( \phi \) takes \( \mathcal{E} \) into \( \mathcal{D} \), and then either \( s_1 > 0 \) or \( s_2 > 0 \), which gives at once that \( s > 0 \). The curve \( \Gamma = \{u = 0\} \) is mapped into one of the \( \mathcal{F} \)-invariant planes. We have two possibilities:

- If \( \phi(\Gamma) \subset \mathcal{D} \), then either \( r_1 > 0 \) or \( r_2 > 0 \) (not both), implying that also \( r > 0 \).
- \( \phi(\Gamma) \nsubseteq \mathcal{D} \), then \( \phi(\Gamma) \subset \{z = 0\} \). This implies that \( r_3 > 0 \) and \( r_1 = r_2 = 0 \), giving \( r = 0 \).

By cancelling poles, we get

\[
uvw\phi^*\tilde{\omega} = sudv + \lambda_3 r_3 v \varphi(v^s \phi_1^{p_1} \phi_2^{p_2}) du + uv\theta,
\]

for some formal 1–form \( \theta \). Thus, \( \mathcal{G} \) has a saddle-node singularity at the origin whose weak separatrix is \( \Gamma = \{u = 0\} \). It is therefore \( \mathcal{E} \)-well oriented.

\[\square\]

**Remark.** We can replace, in Lemma \([x, x]\) the hypothesis “\( \mathcal{G} = \phi^* \mathcal{F} \) is \( \mathcal{E} \)-simple” for “\( \mathcal{G} \) has exactly two transversal separatrices”. The fact of \( \mathcal{G} \) being \( \mathcal{E} \)-simple is a consequence of the proof.

We can now extend to the three dimensional case the definition of *second type foliation*. As in dimension two, we take into account the final models in the process of reduction of singularities:

**Definition 3.4.** Let \( \mathcal{F} \) be a germ of codimension one holomorphic foliation at \((\mathbb{C}^3, 0)\). We say that \( \mathcal{F} \) is a *second type* foliation if there exists a reduction process \( \pi : (\tilde{M}, \mathcal{D}) \to (\mathbb{C}^3, \text{Sing}(\mathcal{F})) \) such that all singularities of \( \tilde{\mathcal{F}} = \pi^* \mathcal{F} \) are \( \mathcal{D} \)-well oriented.
The definition of second type foliations is independent of the reduction of singularities. Besides, it may be expressed in terms of two dimensional sections. This is the content of the following result:

**Proposition 3.5.** For a germ of holomorphic foliation $F$ at $(\mathbb{C}^3, 0)$, the following facts are equivalent:

1. $F$ is a second type foliation;
2. for every normal crossings divisor $E$ at $(\mathbb{C}^2, 0)$ and every analytic map $\phi : (\mathbb{C}^2, E) \to (\mathbb{C}^3, \text{Sing}(F))$ generically transversal to $F$ outside $E$ we have that $\mathcal{G} = \phi^* F$ is $E$-second type.
3. for every reduction of singularities $\pi : (\tilde{M}, D) \to (\mathbb{C}^3, \text{Sing}(F))$ for $F$, all singularities of $\tilde{F} = \pi^* F$ are $D$-well oriented.

**Proof.** (1) $\Rightarrow$ (2). Let $\pi : (\tilde{M}, D) \to (\mathbb{C}^3, \text{Sing}(F))$ be a reduction of singularities for $F$ such that all singularities of $\tilde{F} = \pi^* F$ are $D$-well oriented and $\phi : (\mathbb{C}^2, E) \to (\mathbb{C}^3, \text{Sing}(F))$ be an analytic map generically transversal to $\tilde{F}$ outside a normal crossings divisor $E$. Take $\sigma : (\check{N}, \check{E}^\#) \to (\mathbb{C}^2, 0)$ an $E$-reduction of singularities and denote $\tilde{E} = \sigma^{-1}(E) = E^\# \cup \sigma^* E$. Then, by the universal property of blow-up maps, there exists an analytic map $\psi : (\check{N}, \check{E}) \to (\tilde{M}, D)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
(\check{N}, \check{E}) & \xrightarrow{\psi} & (\tilde{M}, D) \\
\sigma \downarrow & & \downarrow \pi \\
(\mathbb{C}^2, E) & \xrightarrow{\phi} & (\mathbb{C}^3, \text{Sing}(F))
\end{array}
$$

Note that

$$
\tilde{G} = \sigma^* G = \sigma^* \phi^* F = \psi^* \pi^* F = \psi^* \tilde{F}.
$$

The result then follows by applying Lemma 3.3 to the local map defined by $\psi$ at each simple singularity of $\tilde{G}$ in order to conclude that they are all $\check{E}$-well oriented and, as a consequence, that $\mathcal{G}$ is $\check{E}$-second type.

(2) $\Rightarrow$ (3). Let $\pi : (\tilde{M}, D) \to (\mathbb{C}^3, \text{Sing}(F))$ be a reduction of singularities for $F$. Suppose that $\tilde{F} = \pi^* F$ has a tangent saddle-node, say at $p \in D$. We can choose $p$ such that $\tau_p(\tilde{F}) = 2$. Take $\rho : (\mathbb{C}^2, 0) \to (\tilde{M}, p)$ an analytic embedding transversal to $\tilde{F}$. Setting $E = \rho^{-1}(D)$, we have that $\mathcal{G} = \rho^* \tilde{F}$ is an $E$-tangent saddle-node. Define $\phi = \pi \circ \rho : (\mathbb{C}^2, E) \to (\mathbb{C}^3, \text{Sing}(F))$, making the diagram commute. We have that $\mathcal{G} = \phi^* F$ is an $E$-tangent saddle-node singularity, which is in contradiction with (2).

(3) $\Rightarrow$ (1). This is evident, using the fact that a foliation in ambient dimension three has a reduction of singularities.

□
Corollary 3.6. Let $\mathcal{F}$ be a non-dicritical second type foliation at $(\mathbb{C}^3,0)$ and $f = 0$, where $f \in \hat{\mathcal{O}}_3$, be a reduced equation of separatrices. Then $\nu_0(\mathcal{F}) = \nu_0(df)$.

Proof. Let $\phi : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ be an embedding generically transversal to $\mathcal{F}$. We can take $\phi$ satisfying $\nu_0(\mathcal{G}) = \nu_0(\mathcal{F})$, where $\mathcal{G} = \phi^* \mathcal{F}$, and $\nu_0(df) = \nu_0(\mathcal{G})$, where $g = \phi^* f = f \circ \phi$. The foliation $\mathcal{G}$ is non-dicritical, second type by Proposition 3.5, having $g = 0$ as a reduced equation of separatrices. Thus, the minimization property of the algebraic multiplicity gives $\nu_0(\mathcal{G}) = \nu_0(df)$, proving the corollary.

4. Desingularization of separatrices

In this section, we prove that, in the non-dicritical case, a second type foliation and its set of separatrices have the same reduction of singularities. The arguments are similar to those in [9].

We begin by a brief comment on the method of Cano-Cerveau for the construction of separatrices for a non-dicritical foliation $\mathcal{F}$ at $(\mathbb{C}^3,0)$ [6, Part IV]. Let $\hat{\mathcal{F}} = \pi^* \mathcal{F}$, where $\pi : (\hat{\mathcal{M}}, \hat{\mathcal{D}}) \to (\mathbb{C}^3,\text{Sing}(\mathcal{F}))$ is a reduction of singularities for $\mathcal{F}$. Let $\mathcal{U} \subset \hat{\mathcal{D}}$ be the analytic set formed by all trace singularities, that is, points $p \in \text{Sing}(\mathcal{F})$ such that $e_p(\mathcal{D}) = \tau_p(\mathcal{F}) - 1$. Then, the blow-up map $\pi$ defines canonically a bijection between $\text{Sep}_0(\mathcal{F})$ and the connected components of $\mathcal{U}$. This, in particular, has the following consequence: if $\mathcal{F}$ is a non-dicritical foliation at $(\mathbb{C}^3,0)$ and $\Gamma$ is an $\mathcal{F}$-invariant formal curve outside $\text{Sing}(\mathcal{F})$, then $\Gamma \subset \text{Sep}_0(\mathcal{F})$. Indeed, since $\mathcal{D}$ is $\hat{\mathcal{F}}$-invariant, $\tilde{\Gamma} = \pi^* \Gamma$ touches $\mathcal{D}$ at a point $p$ that is singular for $\mathcal{F}$. By Lemma 2.3, $\Gamma$ must be contained in a component of $\text{Sep}_p(\mathcal{F})$ which lies outside $\mathcal{D}$. Thus, $\Gamma$ lies in the correspondent irreducible component of $\text{Sep}_0(\mathcal{F})$.

We have the following analogue of Lemma 2.3 for foliations in dimension three:

Lemma 4.1. Let $\mathcal{F}$ be a germ of non-dicritical foliation at $(\mathbb{C}^3,0)$ and $\mathcal{D}$ be an $\mathcal{F}$-invariant normal crossings divisor. Suppose that $\text{Sep}_0(\mathcal{F})$ is formed by $s_0(\mathcal{F}) = 2$ or $3$ smooth surfaces with normal crossings and that

(i) $s_0(\mathcal{F}) - 1 \leq e_0(\mathcal{D}) \leq s_0(\mathcal{F})$;

(ii) $\mathcal{F}$ is second type.

Then $\mathcal{F}$ is $\mathcal{D}$-simple.

Proof. An important point here is that $\text{Sing}(\mathcal{F})$ is an analytic set, formed by $s_0(\mathcal{F})!/2$ pairwise transversal smooth curves corresponding to the intersections of components of $\text{Sep}_0(\mathcal{F})$.

Let us first suppose $s_0(\mathcal{F}) = 2$. Let $\phi : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ be a germ of analytic embedding generically transversal to $\mathcal{F}$. It follows from Cano-Cerveau’s method that $\mathcal{G} = \phi^* \mathcal{F}$ has exactly two invariant curves, which are smooth and transversal. Besides, $\mathcal{G}$ is second type, by Proposition 3.5, and simple, by Lemma 2.3. To see that $\mathcal{F}$ is also simple, it suffices to see it — after a coordinate change — as the unfolding of the simple foliation $\mathcal{G}$ at $(\mathbb{C}^2,0)$. This unfolding must be trivial by [13, Lemma 1.1.5]. Thus, $\mathcal{F}$ has dimensional type $\tau = 2$ and is $\mathcal{D}$-simple.

Suppose now $s_0(\mathcal{F}) = 3$ and, once more, take a germ $\phi : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ of analytic embedding generically transversal to $\mathcal{F}$, also satisfying $\nu_0(\mathcal{F}) = \nu_0(\mathcal{G})$, where $\mathcal{G} = \phi^* \mathcal{F}$. We have that $\mathcal{G}$ is a second type foliation with exactly three invariant curves, smooth and pairwise transversal — corresponding to the pre-images of the components of $\text{Sep}_0(\mathcal{F})$ —
which implies \( \nu_0(\mathcal{G}) = 2 \). Choosing formal normalizing coordinates \((x, y, z)\) for \( \text{Sep}_0(\mathcal{F})\), we find that \( \mathcal{F} \) is induced by a 1-form of the kind
\[
\omega = xyz \left( \frac{dx}{a} + \frac{dy}{b} + \frac{dz}{c} \right),
\]
where \( a, b, c \in \hat{\mathcal{O}}_3 \) and, since \( \nu_0(\mathcal{F}) = 2 \), at least one of them must be a unity. Then, it is a pre-simple singularity (see [6, Def. 2.2]) adapted to \( \mathcal{D} \). On the other hand, \( \mathcal{F} \) has exactly two smooth transversal separatrices at every point in \( \text{Sing}(\mathcal{F}) \setminus \{0\} \) — this too results from Cano-Cerveau’s method. Thus, by the case \( s_0(\mathcal{F}) = 2 \), all these points are \( \mathcal{D} \)-simple singularities. We conclude from [6, Prop. 4.7 and Def. 4.8] that \( \mathcal{F} \) is also \( \mathcal{D} \)-simple at \( 0 \in \mathbb{C}^3 \). 

We have all elements to prove the main result of this paper:

**Proof.** (of Theorem I) Evidently, a reduction of singularities for \( \mathcal{F} \) desingularizes \( S = \text{Sep}_0(\mathcal{F}) \) and we only have to work the inverse assertion. Let \( \pi : (M, \mathcal{D}) \to (\mathbb{C}^3, \text{Sing}(\mathcal{F})) \) be a composition of blow-ups with non-singular centers that desingularizes \( S \), that is, such that \( \tilde{S} = \pi^* S \) is smooth and has normal crossings with \( \mathcal{D} \). All such centers are permissible for the reduction process of \( \mathcal{F} \), since the foliation is non-dicritical and \( S \) is \( \mathcal{F} \)-invariant. Now, the crucial fact is that a point in the regular part of \( \mathcal{D} \) not lying in \( \tilde{S} \) is regular for \( \tilde{\mathcal{F}} = \pi^* \mathcal{F} \). Indeed, this follows from Corollary 3.6, considering that, at such a point, the unique local separatrix is contained in \( \tilde{\mathcal{D}} \). Therefore, all singularities of \( \tilde{\mathcal{F}} \) satisfy the conditions of Lemma 4.1, leading to the conclusion that they are all \( \mathcal{D} \)-simple. 

\[ \square \]

5. **Logarithmic foliations**

A singular holomorphic foliation \( \mathcal{F} \) of degree \( d \geq 0 \) in \( \mathbb{P}^3 = \mathbb{P}^3_{\mathbb{C}} \) is given, in homogeneous coordinates \([X : Y : Z : W] \in \mathbb{C}^4\), by a polynomial 1-form
\[
\omega = \frac{A}{X} dX + \frac{B}{Y} dY + \frac{C}{Z} dZ + \frac{D}{W} dW,
\]
where \( A, B, C, D \in \mathbb{C}[X, Y, Z, W] \) are homogeneous of degree \( d + 1 \) satisfying the Euler condition, \( AX + BY + CZ + DW = 0 \), and the integrability condition, \( \omega \wedge d\omega = 0 \). Let \( S \subset \mathbb{P}^3 \) be a surface with reduced equation \( F_1 \cdots F_\ell = 0 \), where \( F_1, \ldots, F_\ell \in \mathbb{C}[X, Y, Z, W] \) are irreducible homogeneous polynomials. The foliation \( \mathcal{F} \) is said to be *logarithmic with poles on \( S \)* if it is defined by a 1-form of the kind
\[
\omega = F_1 \cdots F_\ell \left( \lambda_1 \frac{dF_1}{F_1} + \cdots + \lambda_\ell \frac{dF_\ell}{F_\ell} \right) = F_1 \cdots F_\ell \tilde{\omega},
\]
where \( \lambda_1, \ldots, \lambda_\ell \in \mathbb{C}^* \) comply with
\[
\lambda_1 \deg F_1 + \cdots + \lambda_\ell \deg F_\ell = 0,
\]
which is imposed by the Residue Theorem. The set of poles of \( \tilde{\omega} \) is precisely \( S \), which is invariant by \( \mathcal{F} \). Note that
\[
\deg \mathcal{F} = \deg F_1 + \cdots + \deg F_\ell - 2 = \deg S - 2.
\]

The foliation \( \mathcal{F} \) may have some isolated singularities, where it admits local holomorphic first integrals by Malgrange’s Theorem [12]. Let us then denote by \( \text{Sing}_2(\mathcal{F}) \) the union of all components of codimension two in \( \text{Sing}(\mathcal{F}) \). Some of these components are contained in \( S \): the pairwise intersections of pole components \( \{F_i = F_j = 0\} \), as well as codimension
two components of the singular set of each surface \( \{ F_i = 0 \} \). Outside \( S \), the 1–form \( \tilde{\omega} \) is holomorphic and closed, thus each singularity has a local holomorphic first integral. Besides, if \( \mathcal{F} \) is non-dicritical, at each point of \( \text{Sing}_2(\mathcal{F}) \cap S \), all local separatrices of \( \mathcal{F} \) are contained in \( S \). Taking into account this description, we propose the following characterization of logarithmic foliations:

**Proposition 5.1.** Let \( \mathcal{F} \) be a non-dicritical second type foliation in \( \mathbb{P}^3 \). Suppose that there exists an algebraic surface \( S \subset \mathbb{P}^3 \) invariant by \( \mathcal{F} \) such that:

(i) for every \( p \in \text{Sing}_2(\mathcal{F}) \cap S \), the local set of separatrices \( \text{Sep}_p(\mathcal{F}) \) is contained in \( S \);

(ii) at every point outside \( S \), the foliation \( \mathcal{F} \) admits a holomorphic first integral.

Then \( \mathcal{F} \) is a logarithmic foliation with poles on \( S \).

**Proof.** We apply the two-dimensional version of this result proved in [8]. Let \( i : \mathbb{P}^2 \to \mathbb{P}^3 \) be a linear embedding generically transversal to \( \mathcal{F} \), set \( \mathcal{G} = i^*\mathcal{F} \) and identify \( \mathbb{P}^2 \) and the plane \( H = i(\mathbb{P}^2) \). Such a map exists by [4]. Denoting by \( d_0 = \deg(S) \) and \( d = \deg(\mathcal{F}) \), we also have \( d_0 = \deg(S \cap H) \) and \( d = \deg(\mathcal{G}) \). Denote by \( p_1, \ldots, p_r \in \text{Sing}(\mathcal{G}) \) the points of intersection between \( H \) and the components of \( \text{Sing}_2(\mathcal{F}) \) in \( S \). At the other points in \( \text{Sing}(\mathcal{G}) \), say \( q_1, \ldots, q_s \), the foliation \( \mathcal{G} \) has local holomorphic first integrals. Note that \( S \cap H \) contains all local separatrices of \( \mathcal{G} \) at the points \( p_1, \ldots, p_r \). Our calculations is based on the following fact [8, Theorem I]: if \( \mathcal{G} \) is a non-dicritical second type foliation, then \( \text{BB}_p(\mathcal{G}) = \text{CS}_p(\mathcal{G}) + 2\text{GSV}_p(\mathcal{G}) \), where \( \text{BB} \) is the Baum-Bott index, \( \text{CS} \) and \( \text{GSV} \) are, respectively, the total — that is, with respect to the complete set of separatrices — Camacho-Sad and Gómez-Mont-Seade-Verjovski indices (see [2] for definitions and properties of indices). Using known formulas for the sum of the CS and GSV-indices along an invariant algebraic curve, we find

\[
\sum_{i=1}^{r} \text{BB}_{p_i}(\mathcal{G}) = \sum_{i=1}^{r} \text{CS}_{p_i}(\mathcal{G}) + 2\text{GSV}_{p_i}(\mathcal{G}) \\
= d_0^2 + 2((d+2)d_0 - d_0^2) \\
= 2(d+2)d_0 - d_0^2.
\]

On the other hand, \( \text{BB}_{q_j}(\mathcal{G}) \leq 0 \) for all \( j = 1, \ldots, s \), since local holomorphic first integrals exist [1]. Thus

\[
\sum_{i=1}^{r} \text{BB}_{p_i}(\mathcal{G}) \geq \sum_{p \in \text{Sing}(\mathcal{G})} \text{BB}_p(\mathcal{G}) = (d+2)^2.
\]

Comparing these two expressions, we find \((d_0 - (d+2))^2 \leq 0\), which is possible if and only if \( d_0 = (d+2) \). This implies, by [2], that \( \mathcal{G} \) is a logarithmic foliation, giving that also \( \mathcal{F} \) is logarithmic.

\[ \square \]

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