ADJOINTS AND MAX NOETHER’S FUNDAMENTALSATZ

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For S. S. Abhyankar on his 70th birthday

Abstract. We give an exposition of the theory of adjoints and conductors for curves on nonsingular surfaces, emphasizing the case of plane curves, for which the presentation is particularly elementary. This is closely related to Max Noether’s “AF + BG” theorem, which is proved for curves with arbitrary multiple components.

Introduction

Our purpose here is to give an elementary exposition of the theory of adjoints of curves in the plane or on a nonsingular algebraic surface. The treatments we have found in the literature are either computationally difficult ([8], [14]), or involve quite a bit of advanced machinery: at least the machinery of sheaves and cohomology ([16]), or even residues and duality ([11]). See Serre [15], Chap. IV, §12 for a discussion, and Kunz [9] for a self-contained treatment of duality in this context. In addition, we have not found Max Noether’s “AF + BG” theorem in its natural generality, which allows the given curves to have irreducible components with arbitrary multiplicities, so we have taken this opportunity to supply a statement and proof.

In honor of Abhyankar, we have made it our goal to be explicit and elementary. We have attempted to make this understandable to one who knows only the basics of plane curves, and is equipped with an undergraduate algebra course, as in [5], for example; indeed, we expect to include a version of this exposition in a revision of [5]. The case of positive characteristic is included. The local theory applies equally to curves on any nonsingular surface, using the less elementary fact that the local ring of a point on a nonsingular surface is a unique factorization domain.

We thank Joe Lipman for stimulating advice.

1. Basic concepts and notation

We work over a fixed algebraically closed ground field $k$. We are concerned with a nonsingular surface $U$, a curve $C$ on $U$, and a point $P$ on $C$. We do not assume $C$ is irreducible or reduced, so it may have several irreducible components, each appearing with arbitrary positive multiplicity. For a local study, we are free to replace $U$ by any smaller neighborhood of $P$, which we often do without changing notation. For example, we may assume $U$ is affine, and $C$ is defined by an element $F$ of the coordinate ring $\Gamma$ of $U$; $F$ is determined up to multiplication by a unit. Instead of working on $U$ and with the coordinate rings $\Gamma$ and $\Gamma/(F)$ of $U$ and $C$, we usually work with the limiting local rings $\Lambda = \mathcal{O}_P U$, the local ring of $U$ at $P$.

Research partially supported by NSF Grant DMS9970435.
and \( A = \Lambda/(F) = \mathcal{O}_P C \), the local ring of \( C \) at \( P \). Let \( \mathfrak{M} \) denote the maximal ideal of \( \Lambda \), and \( \mathfrak{m} \) the maximal ideal of \( A \).

The multiplicity of \( C \) at \( P \) is the largest integer \( r = r_P(C) \) such that \( F \) is in \( \mathfrak{M}^r \). If \( x \) and \( y \) generate \( \mathfrak{M} \), the image \( F_x \) of \( F \) in \( \mathfrak{M}^r/\mathfrak{M}^{r+1} \) can be written \( F_x(x, y) = \sum_{i+j=r} a_{ij} x^i y^j \). The irreducible factors of \( F_x \) give the tangent lines to \( C \) or \( F \) at \( P \). We say that the coordinates \( x \) and \( y \) are suitable for \( C \) or \( F \) if \( F_x(0, 1) \neq 0 \). This can always be achieved by a linear change of coordinates. In this case \( F_x = a \prod (y - \alpha_i x)^{\alpha_i} \), with \( a \neq 0 \), and distinct \( \alpha_i \) in \( k \); the tangent lines are the lines \( y = \alpha_i x \).

For simplicity, we start with (and readers who wish may remain with) the case where \( C \) is planar, by which we mean that, after shrinking \( U \) if necessary, \( U \) is isomorphic to an open subset of the plane \( \mathbb{A}^2 \), and \( C \) is defined by a polynomial \( F = F(x, y) \) in \( k[x, y] \). Applying a translation, we may assume \( P \) corresponds to the origin \((0, 0)\). Then \( F = F_x + F_{r+1} + \ldots + F_n \) where each \( F_i \) is a homogeneous polynomial of degree \( d \) in \( x \) and \( y \), and \( F_r \neq 0 \).

The blow-up of \( U \) at \( P \) can be described as follows. Shrinking \( U \) if necessary, we may assume there are \( x \) and \( y \) in \( \Gamma \) that generate the maximal ideal of \( P \). The blow-up \( U' \) of \( U \) at \( P \) is the subvariety of \( U \times \mathbb{P}^1 \) defined by the equation \( xT = yS \), where \( S \) and \( T \) are the homogeneous coordinates on \( \mathbb{P}^1 \). The exceptional divisor \( E \cong \mathbb{P}^1 \) is defined by \( x = y = 0 \). The blow-up is covered by two affine open subsets \( U_0' \) and \( U_1' \), where \( S \) and \( T \), respectively, are not zero. The first, \( U_0' \), is the subvariety of \( U \times \mathbb{A}^1 \) defined by \( y = tx \), where \( t (= T/S) \) is the coordinate on \( \mathbb{A}^1 \); similarly, \( U_1' \) is the subvariety of \( U \times \mathbb{A}^1 \) defined by \( x = sy \), where \( s (= S/T) \) is the coordinate on \( \mathbb{A}^1 \). Note that if \( U \) is planar, both \( U_0' \) and \( U_1' \) are planar; indeed, when \( U = \mathbb{A}^2 \), \( U_0' \) and \( U_1' \) are both isomorphic to \( \mathbb{A}^2 \), by the maps \((x, y, t) \mapsto (x, t) \) and \((x, y, s) \mapsto (y, s) \).

The exceptional divisor \( E \) is defined on \( U_0' \) by \( x \), and on \( U_1' \) by \( y \). The coordinate functions \( x \) and \( y \) determine a basis for \( \mathfrak{M}/\mathfrak{M}^2 \), where \( \mathfrak{M} \) is the maximal ideal of \( \Gamma \); this determines an identification of \( E \) with the projective tangent space to \( U \) at \( P \). (This can also be seen by the intrinsic construction of the blow-up as \( \text{Proj}((\oplus \mathfrak{M}^i)) \), with \( E = \text{Proj}((\oplus \mathfrak{M}^i/\mathfrak{M}^{i+1})) \), but we do not need this description.)

The proper transform \( C' \) of \( C \) is the curve on \( U' \) defined by the equation \( \pi^*(C) = C' + rE \), where \( r \) is the multiplicity of \( C \) at \( P \). Explicitly, \( C' \) is defined on \( U_0' \) by the \( F' \) such that \( F = x^r F' \), and \( C' \) is defined on \( U_1' \) by an \( F'' \) with \( F = y^r F'' \). When \( U \) is planar, on \( U_0' \) we have \( F(x, y) = F(x, tx) = x^r F' \), where

\[
F' = F_r(1, t) + xF_{r+1}(1, t) + \ldots + x^{n-r}F_n(1, t),
\]

and similarly on \( U_1' \).

The coordinates \( x \) and \( y \) are suitable for \( C \) if the line \( x = 0 \) is not tangent to \( C \) at \( P \). This means that \( C' \) does not contain the point \([0 : 1]\) in \( \mathbb{P}^1 = E \), i.e., \( C' \) is contained in the affine piece \( U_0' \). In this case we set \( \Lambda' = \Lambda[t]/(y - tx) \), and \( A' = \Lambda'/F' = \Lambda[t]/(y - tx, F') \). Note that \( \mathfrak{M} \Lambda' = (x) \). The projection from \( C' \) to \( C \) corresponds to the natural homomorphism from \( A = \Lambda/(F) \) to \( A' \). We often abuse notation by writing \( x, y, \) and \( t \) for their images in \( A \) or \( A' \). (In fancier and more intrinsic language, the morphism \( \pi: C' \to C \) is a finite morphism, and \( A' \) is the localization of \( \pi_* C_{C'} \) at \( P \).) We will always assume that coordinates are suitable for any finite number of given curves passing through \( P \).

We use the fact that \( \Lambda \) is a unique factorization domain, which is a general fact about regular local rings (cf. [11], §19 or [3], §19.4); in the planar case it follows
from the fact that the polynomial ring $k[x, y]$ is a unique factorization domain. If $G$ is an element of $\Lambda$ that has no irreducible factors in common with $F$, it follows that the image of $G$ in $A = \Lambda/(F)$ is a non-zero-divider. The same holds for local rings on the blow-up, and for $A'$. In particular, if two elements $G$ and $H$ of $A'$ have only a finite number of common zeros, then each is a non-zero divisor in the ring modulo the ideal generated by the other. For example, $x$ is a non-zero-divisor in $A$ and in $A'$; in the latter case this follows from the fact that the common zeros of $x$ and $F'$ correspond to the finite number of tangent lines to $C$ at $P$.

2. Conductors

Recall that for any subring $A$ of a ring $A'$, the conductor $I$ of $A$ in $A'$ is the ideal of elements $a$ in $A$ such that $aA' \subset A$; it is the largest ideal of $A$ that is also an ideal in $A'$; and any element $a$ in $I$ satisfies $aA' \subset I$.

Our results depend on the following elementary computation.

**Proposition 2.1.** Suppose $A \to A'$ arises as in Section 1 from the blowup of a curve at a point.

(i) The homomorphism $A \to A'$ is injective; $A'$ is a finitely generated $A$-module, generated by the elements $t^j$, $0 \leq j \leq r - 1$.

(ii) The images of the elements $x^i y^j$, for $0 \leq i < j \leq r - 1$, form a basis for $A'/A$ over $k$.

(iii) The conductor $I$ of $A$ in $A'$ is $m^{r-1} = x^{r-1}A'$; the images of the elements $x^i y^j$, for $0 \leq i + j \leq r - 2$, form a basis for $A/I$ over $k$.

**Proof.** For any $H$ in $A'$, there is a positive $N$ such that $x^N H$ is in $A$. It follows that if $G$ in $A$ is not divisible by $F$, then the image of $G$ in $A'$ is not divisible by $F'$; for if $G = F' H$ in $A'$, then $x^N G = F J$ in $A$ for some integer $N$ and some element $J$ in $A$; since $x$ is not a zero-divisor in $A$, this is a contradiction. This shows that $A$ is a subring of $A'$. To show that $A'$ is generated over $A$ by the $r$ elements $1$, $t$, $t^2$, $\ldots$, $t^{r-1}$, by Nakayama’s Lemma (cf. [4], §4.1 or [13], §4.4), it suffices to show that $A'/xA'$ is generated over $A/xa$ by the images of these elements. But $A'/xA' = A[t]/(x, y, F'(1, t)) = k[t]/(F'(1, t))$, and $F'(1, t)$ is a polynomial in $t$ of degree $r$. This proves (i).

Note that $x^{r-1} y^j = x^{r-1-j} y^j$ for $j \leq r - 1$, so $x^{r-1}$ is in the conductor $I$. And since $m = (x, y) \subset xA'$, we have $m^{r-1} \subset x^{r-1} A' \subset I$. The other assertions come from looking at the exact sequence

$$A/m^{r-1} \to A'/x^{r-1} A' \to A'/A \to 0,$$

where the second map is the canonical surjection, arising from the fact that $x^{r-1} A' \subset A$; and the first is induced from the inclusion of $A$ in $A'$, noting that $m^{r-1} \subset x^{r-1} A'$. Since $F$ is in $m^r$, we know that the images of the elements $x^i y^j$, for $i + j < r - 1$, form a basis for $A/m^{r-1} \cong \Lambda/m^{r-1}$. We claim that the images of the elements $x^i y^j$, for $i < r - 1$ and $j < r$, form a basis for $A'/x^{r-1} A'$. Since $x$ is a non-zero-divisor in $A'$, looking at the filtration $A' \supset xA' \supset x^2 A' \supset \ldots \supset x^{r-1} A'$, it suffices to show that the elements $t^j$, $j < r$, form a basis for $A'/xA'$. And this is clear since, as we have seen, $A'/xA' = k[t]/(F'(1, t))$.

The mapping from $A/m^{r-1}$ to $A'/x^{r-1} A'$ takes $x^i y^j$ to $x^{i+j} t^j$. It follows that the images of the remaining $x^i y^j$, with $i < j < r$, form a basis for $A'/A$, which proves (ii). (We also see that the displayed sequence is exact on the left.)
To finish the proof of (iii), we must show that \( I \subseteq \mathfrak{m}^{r-1} \). Since we know that \( \mathfrak{m}^{r-1} \subseteq I \), if this were not true there would be an element \( z \) in \( I \) of the form \( \sum_{i+j<r-1} a_{ij} x^i y^j \), with each \( a_{ij} \) in \( k \), and not all \( a_{ij} = 0 \). Let \( \ell \) be minimal such that some \( a_{ij} \neq 0 \). Then
\[
z \cdot t^{\ell+1} = \sum_j a_{ij} x^j t^{\ell+1} + \sum_{i+j>\ell} a_{ij} x^i y^j t^{\ell+1} = \sum_j a_{ij} x^j t^{\ell+1} + \sum_{i+j>\ell} a_{ij} x^{i-\ell} y^j t^{\ell+1}.
\]
Since \( z \cdot t^{\ell+1} \) is in \( A \), and the second term on the right is in \( A \), the first term must also be in \( A \). But since each \( \ell + j + 1 < r \), it follows from (ii) that no such linear combination can be in \( A \). \( \Box \)

**Corollary 2.2.** The dimensions (over \( k \)) of \( A'/A \) and \( A/I \) are both equal to \( r(r-1)/2 \).

**Corollary 2.3.** The image of any non-zero-divisor in \( A \) is a non-zero-divisor in \( A' \).

**Proof.** Suppose \( a \) is in \( A \), and \( a \cdot a' = 0 \) for some \( a' \) in \( A' \). Then \( a \cdot x^{r-1} \cdot a' = 0 \), and \( x^{r-1} \cdot a' \) is in \( A \). If \( a \) is a non-zero-divisor, then \( x^{r-1} \cdot a' = 0 \). But we have seen that \( x \) is a non-zero-divisor in \( A' \), so \( a' = 0 \). \( \Box \)

These results extend readily to the case of the blow-up of several points. This case is not quite local, so one needs a slightly more general blowing up. If \( V \) is a nonsingular surface, embedded as a locally closed subvariety in some projective space \( \mathbb{P}^n \), and \( P \) is a point in \( V \), one may choose homogeneous linear polynomials \( L_0, \ldots, L_n \) whose restrictions to \( V \) vanish only at \( P \). Then \([L_0 : \ldots : L_n]\) determines a morphism from \( V \setminus P \) to \( \mathbb{P}^n \). The blowup \( V' \) of \( V \) at \( P \) can be taken to be the closure of the graph of this morphism in \( V \times \mathbb{P}^n \). (This shows that \( V' \) can also be embedded as a locally closed subvariety of a projective space, since by Segre \( \mathbb{P}^n \times \mathbb{P}^n \) is a closed subvariety of a larger projective space.) The projection from \( V' \) to \( V \) is an isomorphism over \( V \setminus P \). To see that it is isomorphic to the blowup considered before, over some affine neighborhood \( U \) of \( P \), take such a neighborhood, with functions \( x \) and \( y \) generating the maximal ideal of \( P \). Let \( L \) be a linear form that does not vanish at \( P \), and write \( L_i/L = a_{i1}x + a_{i2}y \) for some functions \( a_{ij} \). Shrink \( U \) if necessary so that the matrix \((a_{ij})\) has rank 2 everywhere on \( U \). Then this matrix \((a_{ij})\) determines a closed embedding of \( U \times \mathbb{P}^1 \) in \( U \times \mathbb{P}^n \), and one verifies easily that the blowup we defined earlier in \( U \times \mathbb{P}^1 \) is mapped to the closure of the graph just defined.

Now suppose \( U \) is an affine nonsingular surface, and \( P_1, \ldots, P_s \) are distinct points of \( U \). In this case we take \( \Lambda \) to be the semi-local ring that is the localization of the coordinate ring \( \Gamma \) of \( U \) at the multiplicative set of elements not vanishing at any \( P_i \). If \( C \) is a curve on \( U \) (with irreducible components of arbitrary multiplicities), set \( A = \Lambda/I(C) \), where \( I(C) \) is the ideal of elements which are divisible by a local equation for \( C \) at each \( P_i \). Let \( U' \to U \) be the simultaneous blow-up of \( U \) at each of the points \( P_i \) (i.e., the result of successively blowing up each \( P_i \), the result being independent of the order of blow-up). We again have the proper transform \( C' \) of \( C \), with its finite morphism \( C' \to C \), which corresponds to a monomorphism \( A \to A' \) of \( k \)-algebras. If \( I \) is the conductor, then \( A/I \) and \( A'/A \)
both have dimension $\sum r_i(r_i - 1)/2$, where $r_i$ is the multiplicity of $C$ at $P_i$. The point is that, since the $A$-modules and $A'/A$ have support at these points $P_i$, we have canonical decompositions $A/I = \oplus_i A_i/I A_i$, and $A'/A = \oplus_i A_i'/A_i$, where $A_i = S_i^{-1}A$, with $S_i$ the multiplicative set of elements in $A$ not vanishing at $P_i$, and $A_i' = S_i^{-1}A'$ (cf. §2.9, §2.4). Each of $A_i \to A_i'$ is an extension as studied above, so we know the dimensions of each summand in these decompositions.

Because of this we may repeat the blowing up process. Starting from the blow-up $U^{(1)} = U' \to U^{(0)} = U$ of $U$ at $P$, one can construct the blow-up $U^{(2)} \to U^{(1)}$ of $U^{(1)}$ at a finite number of points mapping to $P$ (lying on the exceptional divisor). Repeating, at each stage blowing up points in the exceptional divisors from the preceding stage, we get a sequence

$$U^{(n)} \to U^{(n-1)} \to \ldots \to U^{(2)} \to U^{(1)} = U' \to U^{(0)} = U.$$ Points in any $U^{(n)}$ mapping to $P$ are called infinitely near points to $P$, in the $n^{th}$ neighborhood, for $n \geq 0$. If at each stage $C^{(i)}$ is the proper transform of $C^{(i-1)}$, we have a sequence

$$\bar{C} = C^{(n)} \to C^{(n-1)} \to \ldots \to C^{(2)} \to C^{(1)} = C' \to C^{(0)} = C,$$

and corresponding finite extensions of $k$-algebras:

$$A = A^{(0)} \subset A' = A^{(1)} \subset A^{(2)} \subset \ldots \subset A^{(n-1)} \subset A^{(n)} = \bar{A}.$$ If $Q$ is an infinitely near point, in the neighborhood $U^{(i)}$, we let

$$r_Q = r_Q(C) = r_Q(F)$$

be the multiplicity of the proper transform $C^{(i)}$ at $Q$.

**Proposition 2.4.** Let $I$ be the conductor of $A$ in $A'$, $J$ the conductor of $A'$ in $\bar{A}$, and $K$ the conductor of $A$ in $\bar{A}$. Then $K = I \cdot J$.

**Proof.** From the definition of conductors we have $I \cdot J \subset K$; we must show that $K \subset I \cdot J$. Since forming conductors commutes with localization, we may assume $A$ is the local ring of one point $P$. Choosing coordinates as above, we have seen that $I = m^{-1} = x^{-1}A'$. If $u$ is an element of $K$, since $K \subset I$ from the definition, we may write $u = x^{-v} \cdot v$, for some $v \in A'$. It suffices to show that $v$ is in $J$, i.e., that $v \cdot b$ is in $A'$ for any $b$ in $\bar{A}$. Since $u$ is in $K$, $u \cdot b$ is in $K \subset I = x^{-1}A'$, so we can write $u \cdot b = x^{-1} \cdot a'$ for some $a'$ in $A'$. But then $x^{-1} \cdot v \cdot b = x^{-1} \cdot a'$. By Corollary 2.3 $x$ is a non-zero-divisor in $\bar{A}$, and it follows that $v \cdot b = a'$, as desired. $\square$

**Corollary 2.5.** $\dim(\bar{A}/A) = \dim(A/K) = \sum r_Q(r_Q-1)$, the sum over all infinitely near points $Q$ in some $U^{(i)}$, $0 \leq i \leq n - 1$.

**Proof.** As in the proposition, we may assume $A$ is local. We know that $\dim(A'/A) = \dim(A'/I) = r_P(r_P - 1)/2$. By induction on the length of the chain, we have $\dim(\bar{A}/A') = \dim(A'/J) = \sum r_Q(r_Q - 1)$, the sum over infinitely near $Q$ in some $U^{(i)}$, $1 \leq i \leq n - 1$. From the inclusions $A \subset A' \subset \bar{A}$ we have $\dim(\bar{A}/A') = \dim(\bar{A}/A') + \dim(\bar{A}/A)$. It therefore suffices to show that $\dim(A/K) = \dim(A'/J) + \dim(A'/I)$; adding $\dim(A'/A)$ to both sides, we are reduced to proving that $\dim(A'/K) = \dim(A'/J) + \dim(A'/I)$. Since $K \subset J$, so $\dim(A'/K) = \dim(A'/J) + \dim(J/K)$, this is equivalent to proving that $\dim(J/K) = \dim(A'/I)$. Since $K = x^{-1}J$ and
$I = x^r - 1 A'$, we must show that $\dim(J/x^r - 1 J) = \dim(A'/x^r - 1 A')$. From the inclusions

$$x^r - 1 J \subset J \subset A' \quad \text{and} \quad x^r - 1 J \subset x^r - 1 A' \subset A',$$

we have $\dim(J/x^r - 1 J) + \dim(A'/J) = \dim(x^r - 1 A'/x^r - 1 J) + \dim(A'/x^r - 1 A')$, so we are reduced to showing that $\dim(A'/J) = \dim(x^r - 1 A'/x^r - 1 J)$. But multiplication by $x^r - 1$ gives an isomorphism of $A'/J$ with $x^r - 1 A'/x^r - 1 J$, since $x^r - 1$ is a non-zero divisor in $A'$.

The fact that $\dim(\tilde{A}/A) = \dim(A/K) = \frac{1}{2} \dim(\tilde{A}/A)$ is known as Gorenstein’s theorem. It depends on the fact that $C$ is a curve on a nonsingular surface. For example, if $C$ is the curve in affine 3-space which is the image of the map $t \mapsto (t^3, t^4, t^5)$ from the affine line, the conductor $K$ of $A = k[t^3, t^4, t^5]$ in $A = k[t]$ is generated by the maximal ideal at the origin, but the images of $t$ and $t^2$ give a basis for $\tilde{A}/A$. The same is true after localizing at the origin, so one has an example where $\dim(A/K) = 1$ but $\dim(\tilde{A}/A) = 2$. For another proof that $\dim(\tilde{A}/A) = \sum r_Q(r_Q - 1)$, see [3].

**Corollary 2.6.** Let $G$ and $H$ be elements of $\Lambda$, with images $g$ and $h$ in $A$. Let $D$ be the curve defined by $G$, and assume that the proper transforms $D^{(n)}$ and $C^{(n)}$ in $U^{(n)}$ have no points in common. If $r_Q(H) \geq r_Q(D) + r_Q(C) - 1$ for all infinitely near points $Q$ to $P$ in $C$, then $h$ is in $g \cdot K$, with $K$ the conductor of $A$ in $\tilde{A}$. In particular, $h$ is divisible by $g$ in $A$.

**Proof.** If $n = 0$, there is nothing to prove. Let $a$ and $b$ be the multiplicities of $G$ and $H$ at $P$. For the first blowup, choosing coordinates that are suitable for $G$ and $H$ as well as $F$, we have $g = x_a - 1 g'$, $h = x_b - 1 h'$, with $g', h' \in A'$. By induction on the length of the chain, we know that $h' = g' \cdot z$, with $z$ in $J$. Since $b - a - r + 1 \geq 0$, $x_b - a - r + 1 z$ is in $J$, so $x_b - a - r + 1 z = x_r - 1 (x_b - a - r + 1 z)$ is in $x_r - 1 J = K$. Therefore $h = x_b - 1 h' = g' x_b - 1 z = g \cdot x_b - a z$ is in $g \cdot K$, as required.

Now suppose $C$ is an irreducible curve at $P$, so its local ring $A$ is an integral domain. Since $\tilde{A}$ is the localization of a finitely generated algebra $\Gamma$ over the field $k$, it is a general theorem of E. Noether (see [8], §13.3 or [33], §36) that the integral closure $\tilde{A}$ of $A$ in its quotient field is a finitely generated $A$-module. If at each stage of blowing up, one blows up at all the singular points in exceptional divisor of each proper transform $C^{(i)}$, one arrives at a chain $A = A^{(0)} \subset A^{(1)} \subset \ldots \subset A^{(n)} \subset \tilde{A}$. It follows that this process must terminate, so $\tilde{A} = A^{(n)}$ for some $n$. Indeed the dimension of $\tilde{A}/A$ puts a bound on the number of steps required.

In the planar case, one can see that this process terminates directly, without using Noether’s theorem. We include a proof in the appendix.

Suppose $C$ is irreducible, and one performs the sequence of blowups to resolve the singularities of $C$, so $C^{(n)} = \tilde{C}$ is nonsingular. In this case the conductor $K$ of $A$ in $\tilde{A}$ is called the conductor of $C$ at $P$. An element $G$ in $A$ is adjoint to $C$ at $P$ if the image of $G$ in $A$ is in the ideal $K$. Define an effective divisor $\Delta_P = \sum d_Q Q$ on $\tilde{C}$ by defining $d_Q$ to be the order of the ideal $K$ at $Q$, i.e., $K \cdot Q \tilde{C} = m_Q \tilde{C}^{d_Q}$. The degree of $\Delta_P$ is the dimension of $\tilde{A}/K$, which, by Corollary 2.5, is $2 \cdot \delta_P$, with $\delta_P = \dim(\tilde{A}/A) = \dim(A/K)$. An element $h$ of the function field $R(C)$, i.e., the quotient field of $A$, is in the conductor $K$ if and only if $\operatorname{ord}_Q(h) \geq d_Q$ for all $Q$ in $\tilde{C}$, where $\operatorname{ord}_Q$ is the order function on $R(C)$ defined by the discrete valuation ring
\( \mathcal{O}_Q(\tilde{C}) \). If \( g \) and \( h \) are in \( R(C) \), and \( \text{ord}_Q(h) \geq \text{ord}_Q(g) + d_Q \) for all \( Q \) in \( \tilde{C} \), then \( h \) is in \( g \cdot K \subset g \cdot \mathcal{O}_P(C) \).

**Remark 2.7.** For any \( g \) in \( A \) which is a non-zero-divisor in \( A' \), we have
\[
\dim(A'/gA') = \dim(A/gA).
\]
. As in Corollary 2.4, this follows by comparing \( gA \subset gA' \subset A' \) and \( gA \subset A \subset A' \), and noting that \( A'/A \) is isomorphic to \( gA'/gA \). In particular, we see that \( \dim(A'/xA') = \dim(A/xA) \).

This analysis gives a quick proof of the following formula of Max Noether for the intersection multiplicity of two curves \( C \) and \( D \) at \( P \). Here we assume \( C \) and \( D \) have no irreducible components in common through \( P \). The intersection multiplicity \( I(P, C \cdot D) \) is defined to be the dimension over \( k \) of \( \Lambda/(F, G) \), where \( F \) and \( G \) are local equations for \( C \) and \( D \).

**Proposition 2.8.** The intersection multiplicity is given by the formula

\[
I(P, C \cdot D) = \sum r_Q(C) \cdot r_Q(D),
\]

where the sum is over \( Q = P \) and all infinitely near points \( Q \) of \( P \) that lie in proper transforms of both \( C \) and \( D \).

**Proof.** By induction, we need only show that \( I(P, C \cdot D) = r_P(C) \cdot r_P(D) + \sum I(P', C' \cdot D') \), where \( P' \) varies over the points lying over \( P \) in both proper transforms \( C' \) and \( D' \), in the blowup \( U' \) of \( U \) at \( P \). Let \( g \) be the image in \( A = \mathcal{O}_P(C) \) of a local equation for \( D \) at \( P \). By Remark 2.7,

\[
I(P, C \cdot D) = \dim(A/gA) = \dim(A'/gA').
\]

In \( A' \), \( g = x^s g' \), where \( s = r_P(D) \). There is an exact sequence

\[
0 \to A'/g'A' \to A'/x^s g'A' \to A'/x^s A' \to 0,
\]

where the first map is multiplication by \( x^s \), and the second is the natural projection; the exactness follows from the fact that \( x \) is a non-zero-divisor in \( A' \) (cf. [3], §3.3). Therefore

\[
\dim(A'/gA') = \dim(A'/x^s A') + \dim(A'/gA').
\]

Since \( \dim(A'/x^s A') = \dim(A/xA) = s \cdot \dim(A/xA) = s \cdot r_P(C) \), and \( \dim(A'/gA') = \sum I(P', C' \cdot D') \), the conclusion follows.

**Corollary 2.9.** For any \( C \) and \( D \) with no irreducible components in common at \( P \), there is a sequence of blowups so that \( C^{(n)} \) and \( D^{(n)} \) are disjoint, and each is a disjoint union of curves, each consisting of a nonsingular curve with some positive multiplicity.

**Proof.** We know that a sufficient number of blowups will make each irreducible component of \( C \) and of \( D \) nonsingular. By the proposition, a sufficient number of blowups will then make the proper transform of pairs of these components disjoint.

**Remark 2.10.** The same reasoning shows that one can make the total transforms, including all the exceptional divisors and their proper transforms, a union of nonsingular curves, with some multiplicities, with each pair of irreducible components meeting transversally, and no three components passing through any point.
3. Adjoints and Differentials

For any $k$-algebra $R$, we have the $R$-module $\Omega_{R/k}$ of differentials over $k$. It can be defined to be the free $R$-module on symbols $df$, for $f$ in $R$, modulo the submodule generated by all: 1) $df, f$ in $k$; 2) $d(f + g) - df - dg, f, g$ in $R$; 3) $d(fg) - f \cdot dg - g \cdot df, f, g$ in $R$. It is constructed so that for any $R$-module $M$, the $k$-linear derivations from $R$ to $M$ correspond to $R$-linear homomorphisms from $\Omega_{R/k}$ to $M$. See [3], §8.4 or [1], §16 for basic facts about differentials.

Let $K$ be the field of rational functions on $U$, i.e., the quotient field of $\Lambda$. The differentials $\Omega_{K/k}$ form a vector space over $K$ of dimension 2. If $x$ and $y$ generate the maximal ideal of $\Lambda$, then $dx$ and $dy$ give a basis for $\Omega_{K/k}$ over $K$. If $F$ is a local equation for an irreducible curve $C$, then $dF = F_x dx + F_y dy$ is not zero on $U$, although it vanishes on $C$. If $F_y \neq 0$, then its image in $A$ is not zero. This can be seen by induction on the length of steps needed to resolve the singularity, it being clear when $P$ is a nonsingular point of $C$. Starting with the equation $F = x^r F'$, with $y = xt$, differentiating both sides with respect to $t$ gives $xF_y = x^r F'_t$, i.e.,

$$F_y = x^{-r} F'_t.$$ 

By induction, we know that the image $F'_t$ is not zero in $A'$, and since $x$ is a non-zero-divisor, $F_y$ is not zero in $A$.

The differentials $\Omega_{R(C)/k}$ of the function field of $C$ over $k$ form a 1-dimensional vector space over $R(C)$. It is generated by $dz$, where $z$ is any element of $R(C)$ such that $R(C)$ is a finite separable extension of $k(z)$. We consider the differential

$$\omega = dx/F_y \text{ on } C.$$ 

(If $F_y = 0$ on $C$, then we use $\omega = -dy/F_x$.) This differential on $C$ is independent of choice of coordinates, up to multiplication by a function not vanishing at $P$.

Explicitly, if $x = x(u, v)$, $y = y(u, v)$, for $u$ and $v$ other coordinates, and we set $\tilde{F}(u, v) = F(x(u, v), y(u, v))$, then, if $\tilde{F}_v \neq 0$, a calculation as in calculus shows that

$$dx/F_y = J \cdot du/\tilde{F}_v,$$

where $J = xu_y - xv_y$ is the Jacobian determinant; if $\tilde{F}_u \neq 0$, then $dx/F_y = -J \cdot dv/\tilde{F}_u$.

Recall that a differential $\omega$ on a nonsingular curve $\tilde{C}$ is regular at a point $Q$ if, for a uniformizing parameter $t$ for $\tilde{C}$ at $Q$, $\omega = h dt$, with $h$ regular at $Q$, i.e., $h$ is in $O_Q(\tilde{C})$. More generally, the order $\text{ord}_Q(\omega)$ is the order of such a function $h$.

**Proposition 3.1.** An element $g$ in $A = O_P(C)$ is in the conductor $K$ of $A$ over $A$ if and only if the differential $g \cdot dx/F_y$ is regular at each point of $\tilde{C}$ that maps to $P$.

**Proof.** Take coordinates as before at $P$. If $P$ is a nonsingular point on $C$, then $F_y$ is a unit at $P$, and the assertion is clear. Otherwise perform a blowup, and write $F = x^r F'$. As we have just seen, $F_y = x^{-r} F'_t$, where $x$ and $t$ are coordinates on $U'$. By induction on the number of blowups needed to resolve the singularity, a function $g$ is in the conductor $J$ for $A'$ in $A$ exactly when $g \cdot dx/F'_t$ is regular at all points of $\tilde{C}$ over $P$. For $g$ to be in the conductor $K = x^{-r} J$, $g/x^{-r}$ must be in $J$, i.e., $g \cdot dx/F_y = (g/x^{-r}) dx/F'_t$ must be regular at all points of $\tilde{C}$ over $P$, as required. \qed
4. Plane Curves

The results on adjoints lead to a sharp form of Noether’s theorem, allowing curves with arbitrary multiple components.

**Theorem 4.1** (Max Noether’s Fundamentalsatz). Let \( C \) and \( D \) be plane curves with no common components, defined by homogeneous polynomials \( F \) and \( G \) of degrees \( c \) and \( d \). Suppose \( H \) is a homogeneous polynomial of degree \( e \), and suppose that \( r_P(H) \geq r_P(C) + r_P(D) - 1 \) for all points \( P \) in and infinitely near to \( C \) and \( D \). Then there is an equation

\[
H = A \cdot F + B \cdot G,
\]

where \( A \) and \( B \) are homogeneous polynomials of degrees \( e - c \) and \( e - d \).

**Proof.** By Corollary 2.6, at every point \( P \) in the plane, a local equation for \( H \) in \( \mathcal{O}_P(\mathbb{P}^2) \) is in the ideal generated by local equations for \( F \) and \( G \). The fact that this is true locally if and only if it is true globally, so that there is an identity as shown, is proved in [5], §5.5.

It follows from what we have done here that most of the results proved in [5] for plane curves with only ordinary singularities extend without essential change to curves with arbitrary singularities. For example, suppose \( C \) is an irreducible plane curve, defined by a homogeneous polynomial \( F(X, Y, Z) \), and \( X' = C \to C \) is its nonsingular model (constructed by a succession of blowups over singular points of \( C \)), the adjoint divisor \( \Delta \) is the sum \( \sum d_Q Q \), where \( d_Q \) is the order of vanishing at \( Q \) in \( C \) of the conductor ideal at the image of \( Q \) in \( C \). The genus \( g_X \) of \( X' \) can be defined to be \( (n-1)(n-2)/2 - \delta \), where \( \delta = \sum \delta_P = 1/2 \deg(\Delta) \). We choose coordinates so that the line \( Z = 0 \) intersects \( C \) only at nonsingular points.

The divisor \( \text{div}(\omega) \) of a differential \( \omega \) of \( R = R(X') = R(C) \) over \( k \) is \( \sum \text{ord}_Q(\omega) Q \), the sum over the (finite) set of \( Q \) in \( X' \) at which the order of \( \omega \) is not zero. If \( f = F(x, y, 1) \), it follows from Proposition 3.3 that the order of \( dx/f_y \) at each point of the affine plane \( \mathbb{A}^2 \) is \( -d_Q \). One can calculate the order at the points of \( Z = 0 \) by changing coordinates from the given copy of \( \mathbb{A}^2 \) to the other two copies.

One finds that

\[
\text{div}(dx/f_y) = -\Delta + (n-3) \text{div}(Z).
\]

A homogeneous polynomial \( G(X, Y, Z) \) is adjoint to \( C \) if the divisor \( \text{div}(G) \) cut out on \( X' \) by \( G \) contains \( \Delta \), i.e., \( \text{div}(G) = \Delta + A \), for some effective divisor \( A \). If \( G \) is an adjoint to \( C \) of degree \( n-3 \), it follows that \( \text{div}(G) = \Delta + A \), where \( A = \text{div}(\omega) \) for some everywhere regular differential \( \omega \) on \( X' \), namely \( \omega = (G/Z^{n-3}) dx/f_y \). Such adjoints exist whenever the genus is positive, since the condition for \( G \) to be in the adjoint ideal at \( P \) is defined by \( \delta_P \) linear equations, and the projective space of such forms has dimension \( (n-1)(n-2)/2 \).

The classical proof of the Riemann-Roch theorem, given in [5], Chap. 8, for curves with ordinary singularities, then applies without change for curves with arbitrary singularities. This proof is based on Max Noether’s Fundamentalsatz. In particular, one sees that the adjoints of degree \( n-3 \) cut out, besides the fixed component \( \Delta \), the complete linear series of canonical divisors. See [5], Chap. I, App. A, for a modern discussion of adjoints and differentials for complex curves; there Gorenstein’s theorem is deduced from the Riemann-Roch theorem. Zariski ([17], §15) discusses adjoints in higher dimensions. One can find a comparison with
other notions of adjoints in \[3\] and \[7\], and more about adjoints and conductors in \[1\].

**APPENDIX. Resolution of Singularities for Planar Curves**

We keep the notation of Sections 1 and 2. We show that, for planar curves, the blowing up process must stop, by induction on the multiplicity. Note that for one blowup,

\[
\sum rp_i \leq \dim(A'/xA') = \dim(A/xA) = rP = r,
\]

by Remark [2, 3]. It therefore suffices to show that it is impossible for there to be, at every stage, just one point on the proper transform of the curve over \(P\), with the same multiplicity \(r > 1\).

We rule this out by a power series calculation. We may assume that the leading term of \(F\) is \(Y^r\). Set \(F^{(1)} = F\), and construct inductively a sequence of polynomials \(F^{(n)} = F^{(n)}(X, Y)\), each of whose leading terms is \(Y^r\), and a sequence of elements \(a_n\) in the ground field \(k\), such that \(F^{(n-1)}(X, XY) = X^r F^{(n)}(X, Y - a_n X)\). It follows by induction that for all \(n \geq 2\), with \(\varphi_n(X) = \sum_{i=2}^{n} a_i X^i\),

\[
F(X, X^{n-1} Y + \varphi_n(X)) = X^{r(n-1)} F^{(n)}(X, Y).
\]

Setting \(\varphi(X) = \sum_{i=2}^{\infty} a_i X^i\) in \(k[[X]]\), we see that \(F(X, \varphi(X)) = 0\), and so \(Y - \varphi(X)\) divides \(F(X, Y)\). If \(\varphi(X)\) is a polynomial, this contradicts the irreducibility of \(F\), so an infinite number of its coefficients must be nonzero. We claim that \(F(X, Y) = (Y - \varphi(X))^r\). If not, write \(F(X, Y) = (Y - \varphi(X))^s \cdot G(X, Y)\), for some \(G(X, Y)\) in \(k[[X]][Y]\) with \(G(X, \varphi(X)) \neq 0\), and \(s < r\). From the displayed equation we see that

\[
(X^{n-1} Y + \varphi_n(X) - \varphi(X))^s G(X, X^{n-1} Y + \varphi_n(X)) =
X^{r(n-1)} F^{(n)}(X, Y).
\]

If \(a_{n+1} \neq 0\), setting \(Y = 0\) in this equation and computing the order of vanishing with respect to \(X\), one sees that \(G(X, \varphi_n(X))\) is divisible by \(X^{n(r-s)-s}\). Since there are arbitrarily large \(n\) with \(a_{n+1} \neq 0\), this shows that \(G(X, \varphi(X)) = 0\), a contradiction.

To complete the proof it remains to verify that if \((Y - \varphi(X))^r\) is a polynomial, then \(\varphi(X)\) must be a polynomial. This is clear in characteristic zero, so assume the characteristic is \(p\), and write \(r = q \cdot u\), with \(q\) a power of \(p\) and \(u\) relatively prime to \(p\). Since the binomial coefficient \(\binom{r}{q}\) is not zero modulo \(p\), \(\varphi(X)^q\) must be in \(k[X]\), and this implies that \(\varphi(X)\) is in \(k[[X]]\).

This calculation shows that if \(\Lambda\) is the localization of \(k[X, Y]\) at its maximal ideal \((X, Y)\), and \(F\) is any irreducible element in \(\Lambda\), then \(F\) cannot be a power of an irreducible element in the completion \(k[[X, Y]]\) of \(\Lambda\); in other words, the completion of the ring \(\Lambda = \Lambda/(F)\) cannot have nilpotent elements. This illustrates the general fact that the integral closure of a one-dimensional Noetherian domain \(A\) is a finitely generated \(A\)-module if and only if its completion has no nilpotents; see \([3], \S 33, \S 31\), or \([2], \S 33\).

An example from Nagata \([3]\), Appendix, shows that this is not true for all two dimensional regular local rings \(\Lambda\). To see such an example, let \(\{a_{ij} \mid i, j \geq 0\}\) be a collection of indeterminates over \(\mathbb{F}_p\). Let \(K = \mathbb{F}_p(a_{ij})\) be the field generated

\[\text{For example, if } F(X, Y) = Y^2 + 2X^2 Y + X^4 + X^7, \text{ then } F(X, XY) = X^2((Y + X)^2 + X^5), \text{ so } F^{(2)}(X, Y) = Y^2 + X^7, \text{ and } a_2 = -1; \text{ then } F^{(3)} = Y^2 + X^3, \text{ with } a_3 = 0.\]
over $\mathbb{F}_p$ by these indeterminates, and let $\Lambda$ be the subring of $K[[X,Y]]$ consisting of power series whose coefficients lie in some finite extension of $K^p = \mathbb{F}_p(a_{ij}^p)$. This $\Lambda$ is a regular local ring, with maximal ideal generated by $X$ and $Y$, and $F = \sum a_{ij}^p X^{i} Y^{j}$ is an element of $\Lambda$ which is a $p^\text{th}$ power in the completion of $\Lambda$, but $F$ is not a $p^\text{th}$ power in $\Lambda$. One can verify directly that the blowing up process on this $F$ continues indefinitely.

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