General Higher-Order Lipschitz Mappings

Joseph Frank Gordon

Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China

Correspondence should be addressed to Joseph Frank Gordon; jgordon@aims.edu.gh

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In this paper, we introduce a new class of mappings and investigate their fixed point property. In the first direction, we prove a fixed point theorem for general higher-order contraction mappings in a given metric space and finally prove an approximate fixed point property for general higher-order nonexpansive mappings in a Banach space.

1. Introduction

Given a complete metric space \((X, d)\), the most well-studied types of self-maps are referred to as Lipschitz mappings (or Lipchitz maps, for short), which are given by the metric inequality:

\[
d(Tx, Ty) \leq \alpha \, d(x, y),
\]

(1)

for all \(x, y \in X\), where \(\alpha \geq 0\) is a real number, usually referred to as the Lipschitz constant of \(T\). Now, for technical and historical reasons, we classify Lipschitz mappings into three categories, thus contraction mappings for the case where \(0 \leq \alpha < 1\), nonexpansive mappings for the case where \(\alpha = 1\), and expansive mappings for the case where \(\alpha > 1\).

In 2007, Goebel and Japon Pineda [1] introduced the so-called mean nonexpansive mappings which are class of mappings more general than the class of nonexpansive mappings. Thus, given a Banach space \((X, \| \cdot \|)\) and a nonempty subset \(C \subseteq X\), a mapping \(T: C \to C\) is called mean nonexpansive (or \(\alpha\)-nonexpansive) if, for some \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) with \(\sum_{k=1}^{n} \alpha_k = 1\), \(\alpha_k \geq 0\) for all \(k\), and \(\alpha_1, \alpha_n > 0\), we have

\[
\sum_{k=1}^{n} \alpha_k \|T^k x - T^k y\| \leq \|x - y\|, \quad \text{for all } x, y \in C.
\]

(2)

It is obvious that all nonexpansive mappings are mean nonexpansive mappings but the converse may not be true (see for instance [2], examples 2.3 and 2.4). Goebel and Japón Pineda further suggested the class of \((\alpha, p)\)-nonexpansive maps. A self-map \(T: C \to C\) is called \((\alpha, p)\)-nonexpansive if, for some \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) with \(\sum_{k=1}^{n} \alpha_k = 1\), \(\alpha_k \geq 0\) for all \(k\), and \(\alpha_1, \alpha_n > 0\) and for some \(p \in [1, \infty)\),

\[
\sum_{k=1}^{n} \alpha_k \|T^k x - T^k y\|^p \leq \|x - y\|^p,
\]

(3)

for all \(x, y \in C\). It is easy to check that \((\alpha, p)\)-nonexpansive map for \(p > 1\) is also \(\alpha\)-nonexpansive, but the converse does not hold (see [3] for details), whereas nonexpansive mappings are uniformly continuous and that continuous, mean nonexpansive mappings may not generally be continuous as seen in the following example taken from [2], example 2.2.

Example 1. Let \(f: [0, 1] \to [0, 1]\) be given by

\[
f(x) = \begin{cases} 
1, & x = 0, \\
0, & x \neq 0.
\end{cases}
\]

(4)

Clearly, \(f\) is discontinuous but a mean nonexpansive mapping.

In 2015, Ezearn [4] introduced a new class of mappings called higher-order Lipschitz mappings which are seen as a generalization of inequality (1). Thus, a mapping
$T \colon \mathcal{X} \to \mathcal{X}$ on a metric space $(\mathcal{X}, d)$ is an $r$th-order Lipschitz mapping if

$$d(T^r x, T^r y) \leq \sum_{k=0}^{r-1} \alpha_k d(T^k x, T^k y) \quad \forall x, y \in \mathcal{X},$$

(5)

where $r$ is a natural number and $\alpha_k$, for all $0 \leq k \leq r - 1$, are nonnegative real numbers, whereas Lipschitz mappings are (uniformly) continuous mappings; this needs not be the case for higher-order Lipschitz mappings; they may not even be continuous as given by the following example taken from [4].

**Example 2.** Let $T \colon \mathbb{R} \to \mathbb{R}$ given by

$$Tx = \begin{cases} 1 - x, & \text{if } x < 0, \\ x, & \text{if } x \geq 0, \end{cases}$$

(6)

with the metric induced by the usual absolute value on $\mathbb{R}$. Clearly, $T$ is discontinuous at $x = 0$ but a second-order Lipschitz (indeed, a second-order nonexpansive) mapping.

In this paper, we introduce the following new class of mappings which generalizes both inequalities (3) and (5).

**Definition 1.** (general higher-order Lipschitz mappings). A mapping $T \colon \mathcal{X} \to \mathcal{X}$ on a metric space $(\mathcal{X}, d)$ is a $(r, p)$-general $r$th-order Lipschitz mapping if

$$\sum_{k=0}^{r-1} \alpha_k d(T^k x, T^k y)^p \leq \sum_{k=0}^{l} \alpha_k d(T^k x, T^k y)^p \quad \forall x, y \in \mathcal{X},$$

(7)

where $p \in [1, \infty)$, $r$ is a natural number, $\alpha_k \geq 0$ for all $k$, $\alpha_0 \cdot \alpha_r \neq 0$, and $l \in \{0, \ldots, r - 1\}$.

It is clear that inequality (7) reduces to inequality (3) when $l = 0$. Similarly, inequality (7) reduces to inequality (5) when $p = 1$ and $l = r - 1$. Generally, a $(r, p)$-general higher-order Lipschitz mapping may not be continuous as already seen in examples 1 and 2.

Now, to every $(r, p)$-general higher-order Lipschitz mapping, we associate a polynomial defined as

$$h(z) = \sum_{k=1}^{r} \alpha_k z^k - \sum_{k=0}^{l} \alpha_k z^k,$$

(8)

just as Ezearn did in his paper [4]. Now, we classify $(r, p)$-general higher-order Lipschitz mappings into three main categories, thus

(i) $T$ is $(r, p)$-general higher-order contraction mapping if $h(1) > 0$.

(ii) $T$ is $(r, p)$-general higher-order nonexpansive mapping if $h(1) = 0$.

(iii) $T$ is $(r, p)$-general higher-order expansive mapping if $h(1) < 0$.

Now, we provide some fixed point results as it is presently understood and known and their connections to the research focus of this paper.

**Theorem 1.** (Banach contraction mapping theorem). Let $(\mathcal{X}, d)$ be a complete metric space and let $T \colon \mathcal{X} \to \mathcal{X}$ be a contraction mapping. Then, $T$ has a unique fixed point and $\lim_{n \to \infty} T^n x$ converges to this fixed point for any $x \in \mathcal{X}$.

Ezearn [5] provided a version of Theorem 1 in a general Banach space using the concept of the normalized duality mappings and the generalized projection functional. He defined the following set of mappings.

**Definition 2.** (monotone contraction mapping). Let $\mathcal{X}$ be a smooth Banach space and let $\mathcal{C} \subseteq \mathcal{X}$ be a closed subset of $\mathcal{X}$. Then, the mapping $T \colon \mathcal{C} \to \mathcal{C}$ is said to be a monotone contraction mapping if there exists $0 \leq c < 1$ such that for all $x, y \in \mathcal{C}$, the following two conditions are satisfied:

1. $\Re \langle Tx - Ty, JTx - JTy \rangle \leq c \Re \langle x - y, Jx - Jy \rangle$,
2. $\Re \langle T^{m+1} x - T^{m} y, J T^{m+1} x - J T^{m} y \rangle \leq 0$,

where $\Re$ denotes the real part of a complex number and $J$ is the normalized duality mapping for all $m, n \geq 0$ with $m \neq n$.

His result is summarized in the following theorem.

**Theorem 2.** (monotone contraction mapping theorem). Let $\mathcal{C}$ be a closed subset of a uniformly convex smooth Banach space $\mathcal{X}$ and let $T \colon \mathcal{C} \to \mathcal{C}$ be a monotone contraction mapping. Then, $T$ has a unique fixed point, that is, $F(T) = \{p\}$ and that the Picard iteration associated to $T$, that is, the sequence defined by $x_n = T(x_{n-1}) = T^n(x_0)$ for all $n \geq 1$ converges to $p$ for any initial guess $x_0 \in \mathcal{C}$.

Ezearn [4] extended the conclusion of the Banach contraction mapping theorem (Theorem 1) to higher-order contraction mappings as summarized in the following theorem.

**Theorem 3.** (higher-order contraction mapping theorem). Let $(\mathcal{X}, d)$ be a complete metric space and let $T \colon \mathcal{X} \to \mathcal{X}$ be an $r$th-order contraction mapping. Then, $T$ has a unique fixed point and $\lim_{n \to \infty} T^n x$ converges to this fixed point for arbitrary $x \in \mathcal{X}$.

Ezearn later provided a remetisation argument that relates his higher-order Lipschitz mappings to (first-order) Lipschitz mappings. His remetisation of the original metric space does not necessarily result in a complete metric space, and as a result, he provided a completion of the remetised space and an extension of the higher-order Lipschitz mapping into a complete remetised space. Before stating this theorem, let us introduce the following: a new metric on the space $\mathcal{X}$ as already defined by Ezearn in his paper,
\[ D(y, x) = \sum_{k=0}^{r-1} b_k d(T^k y, T^k x), \quad \text{where } b_k = \sum_{j=0}^{k} \lambda^{j-k-1}. \]  

(9)

His theorem (already stated as Theorem 3.5 in his paper) is stated as follows.

**Theorem 4.** Define the mapping,

\[ T: \mathcal{X} \to \mathcal{X}, \quad [x_n] \to [Tx_n]. \]  

(10)

Then, we have

\[ D(T[y_n], T[x_n]) \leq \lambda D([y_n], [x_n]), \]  

(11)

where \( \{y_n\}_{n \geq 1}, \{x_n\}_{n \geq 1} \) are Cauchy sequences in \( (\mathcal{X}, D) \) and \([x_n]\) denotes the equivalence class of \( x_n \) in \( (\mathcal{X}, D) \). In particular, if \( (\mathcal{X}, d) \) is complete, then \( T \) has a fixed point in \( (\mathcal{X}, D) \) if and only if \( T \) has a fixed point in \( (\mathcal{X}, \mathcal{D}) \) where \( (\mathcal{X}, \mathcal{D}) \) is the canonical completion of the metric space \( (\mathcal{X}, D) \); that is, \( \mathcal{D}([y_n], [x_n]) = \lim_{n \to \infty} D(y_n, x_n) \).

In light of Theorem 4, this paper carefully uses ideas of the auxiliary results leading to the proof of Theorem 4 to establish entirely similar proof of Theorem 4 as far as \((r, p)\)-general higher-order contraction mappings are concerned.

These references [6–9] can be consulted for fixed point problems in metric spaces.

An approximate fixed point sequence of a nonexpansive self-map \( T \) on a closed convex subset \( \mathcal{C} \) of Banach space \( \mathcal{X} \) is any sequence \( \{x_n\}_{n \geq 1} \) such that

\[ \lim_{n \to \infty} \|x_n - Tx_n\| = 0. \]  

(12)

When \( \mathcal{C} \) is bounded, then such a sequence always exists. One of the results in this paper is to show that given any Banach space \( \mathcal{X} \) and a closed bounded convex subset \( \mathcal{C} \) containing the origin, an approximate fixed point sequence always exists for positively homogeneous \((r, p)\)-general higher-order nonexpansive mappings.

### 2. Preliminaries

**Theorem 5.** (Bolzano’s intermediate value [10].) If a continuous real function defined on an interval is sometimes positive and sometimes negative, then it must be 0 at some point in the interval.

**Proposition 1.** Let \( h(z) = \sum_{k=0}^{r} a_k z^k \) be an \( r \)th degree polynomial over the real numbers \( a_k \). Then, the number of positive real roots of \( f \) is bounded above by the number of sign changes of the coefficients \( a_k \) as one proceeds from \( k = 0 \) to \( k = r \) (ignoring zero coefficients).

**Theorem 6.** (Descartes’ rule of signs, see for instance, [11]). Let \( f(z) = \sum_{k=0}^{r} a_k z^k \) be an \( r \)th degree polynomial over the real numbers \( a_k \). Then, the number of positive real roots of \( f \) is bounded above by the number of sign changes of the coefficients \( a_k \) as one proceeds from \( k = 0 \) to \( k = r \) (ignoring zero coefficients).

3. **Main Result**

We give the proof of the main result of this paper, which is accomplished in Theorems 7 and 8. The following lemma, corollary, and proposition shall aid us in arriving at the conclusion of the main result.

3.1. **Fixed Point Theorem for General Higher-Order Contraction Mappings.** Let \( T \) be a \((r, p)\)-general higher-order Lipschitz mapping on a complete metric space \( (\mathcal{X}, d) \) as given in inequality (13) and let \( \lambda \) be the unique root of the polynomial \( h(z) = \sum_{k=0}^{r} a_k z^k \) as guaranteed by Proposition 1. Now, let us define the following on the space \( \mathcal{X} \):

\[ D_p(x, y) = \left( \sum_{k=0}^{r-1} b_k d(T^k x, T^k y)^p \right)^{(1/p)}, \quad \text{where } b_k = \sum_{j=0}^{k} \lambda^{j-k-1} \text{sgn}(j). \]  

(13)
Corollary 1. Let \( b_k \) be defined in equation (13). Then, \( b_k \) is nonnegative.

Proof. To see this, if \( k \leq l \), then \( \lambda^{k+1} b_k = \sum_{j=0}^{k} \alpha_j \lambda^j \geq 0 \). On the other hand, if \( k \geq l + 1 \), then
\[
\lambda^{k+1} b_k = \sum_{j=0}^{l} \alpha_j \lambda^j - \sum_{j=l+1}^{k} \alpha_j \lambda^j,
\]
\[
= \sum_{j=0}^{l} \alpha_j \lambda^j - \sum_{j=l+1}^{k} \alpha_j \lambda^j, = -h(\lambda) + \sum_{j=k+1}^{\lambda} \alpha_j \lambda^j,
\]
\[
= 0 + \sum_{j=k+1}^{\lambda} \alpha_j \lambda^j \geq 0,
\]
and that completes the proof. \( \square \)

Lemma 1. Let \( D_p(x, y) \) be defined in equation (13). Then, \( D_p \) is a metric on the space \( \mathcal{X} \).

Proof. By Corollary 1, \( b_k \) is nonnegative. Now, for \( p = 1 \) is straightforward. For \( p > 1 \), it is obvious that \( D_p \) is nonnegative since \( d \) and \( b_k \) are nonnegative and it is subadditive. We have also that \( D_p(x, y) = 0 \) if and only if \( x = y \) since \( b_0 \neq 0 \) for all \( x, y \). Also, \( D_p(x, y) = D_p(y, x) \). Finally, without loss of generality, suppose \( x \neq y \). Then, we have the following evaluation:

Using the Hölder’s inequality on each of the terms on the right hand side, we have

\[
D_p(x, z)^p \leq \sum_{k=0}^{r-1} b_k d(T^k x, T^k z)^p,
\]
\[
= \sum_{k=0}^{r-1} b_k d(T^k x, T^k z)^p d(T^k x, T^k z)^{p-1},
\]
\[
\leq \sum_{k=0}^{r-1} b_k^{(1/p)} d(T^k x, T^k y) b_k^{1-(1/p)} d(T^k x, T^k z)^{p-1} + \sum_{k=0}^{r-1} b_k^{(1/p)} d(T^k y, T^k z) b_k^{1-(1/p)} d(T^k x, T^k z)^{p-1}.
\]

From \( (1/p) + (1/q) = 1, ((p - 1)/p) = q \Rightarrow (p - 1)q = p \), we obtain
\[
D_p(x, z) \leq D_p(x, y) + D_p(y, z),
\]
and that completes the proof. \( \square \)

Proposition 2. Let \( b_k \) be defined in equation (13). Then, the following recurrence relations hold:

\[
\begin{align*}
\alpha_0 &= \lambda b_0, \\
b_{r-1} &= 0, \\
\lambda b_k &= b_{k-1} + \alpha_k \text{sgn}(k), \quad 1 \leq k \leq r - 1.
\end{align*}
\]

Proof. \( b_0 = \sum_{j=0}^{0} \alpha_j \lambda^{j-1} \text{sgn}(j) = \alpha_0 \lambda^{-1} \Rightarrow \alpha_0 = \lambda b_0. \)

Also,
\[ b_{k-1} + a_k \text{sgn}(k) = k \sum_{j=0}^{k} \alpha_i \lambda^{-k} \text{sgn}(j) + a_k \text{sgn}(k) = \sum_{j=0}^{k} \alpha_i \lambda^{-k} \text{sgn}(j), \]

\[ \lambda^{-1} \left( b_{k-1} + a_k \text{sgn}(k) \right) = \lambda^{-1} \sum_{j=0}^{k} \alpha_i \lambda^{-k} \text{sgn}(j) = \sum_{j=0}^{k} \alpha_i \lambda^{1-k} \text{sgn}(j) = b_k, \]

and multiplying through again by \( \lambda \) completes the proof. Finally, for \( k = r - 1 \), we have

\[
\lambda^r b_{r-1} = \sum_{j=0}^{r} \alpha_i \lambda^j - \sum_{j=1}^{r} \alpha_i \lambda^j
\]

\[
= \sum_{j=0}^{r} \alpha_i \lambda^j - \sum_{j=1}^{r} \alpha_i \lambda^j - \alpha_r \lambda^r + \alpha_r \lambda^r,
\]

\[
= \sum_{j=0}^{r} \alpha_i \lambda^j - \alpha_r \lambda^r, = h(\lambda) + \alpha_r \lambda^r.
\]

Hence, \( \lambda^r b_{r-1} = \alpha_r \lambda^r \Rightarrow b_{r-1} = \alpha_r. \]

Lemma 2. Let \( (\mathcal{X}, d) \) be a (not necessarily complete) metric space and let \( T : \mathcal{X} \rightarrow \mathcal{X} \) be a \((r, p)\)-general higher-order Lipschitz mapping. Let \( D_p \) be the new metric defined in equation (13). Then,

\[
D_p(Tx, Ty) \leq \lambda^{(1/p)} D_p(x, y).
\]

Moreover, a sequence \( \{x_n\}_{n \geq 1} \subset (\mathcal{X}, D_p) \) is Cauchy in \( (\mathcal{X}, D_p) \) if and only if the sequence \( \{T^k x_n\}_{n \geq 1} \subset (\mathcal{X}, d) \) is Cauchy in \( (\mathcal{X}, d) \) for all \( 0 \leq k \leq r - 1 \).

Proof

\[
D_p(Tx, Ty)^p = \sum_{k=0}^{r} b_k d(T^{k+1}x, T^{k+1}y)^p
\]

\[
= \sum_{k=0}^{r} b_k d(T^k x, T^k y)^p + \sum_{k=1}^{r} (b_{k-1} - a_k) d(T^k x, T^k y)^p + \sum_{k=1}^{r} b_{k-1} d(T^k x, T^k y)^p,
\]

\[
\leq \sum_{k=0}^{r} a_k d(T^k x, T^k y)^p + \sum_{k=1}^{r} (b_{k-1} - a_k) d(T^k x, T^k y)^p + \sum_{k=1}^{r} b_{k-1} d(T^k x, T^k y)^p,
\]

\[
= a_0 d(x, y)^p + \sum_{k=1}^{r} a_k d(T^k x, T^k y)^p + \sum_{k=1}^{r} (b_{k-1} - a_k) d(T^k x, T^k y)^p + \sum_{k=1}^{r} b_{k-1} d(T^k x, T^k y)^p,
\]

\[
= a_0 d(x, y)^p + \sum_{k=1}^{r} a_k d(T^k x, T^k y)^p + \sum_{k=1}^{r} (b_{k-1} + a_k) d(T^k x, T^k y)^p + \sum_{k=1}^{r} b_{k-1} d(T^k x, T^k y)^p,
\]

\[
= \lambda b_0 d(x, y)^p + \lambda \sum_{k=1}^{r} b_k d(T^k x, T^k y)^p,
\]

Now, given that the sequence \( \{x_n\}_{n \geq 1} \subset (\mathcal{X}, D_p) \) is Cauchy, that is, \( \lim_{n \rightarrow \infty} D_p(x_n, x_m) = 0 \), then it is obvious that \( \lim_{n \rightarrow \infty} d(T^k x_n, T^k x_m) = 0 \) for all \( 0 \leq k \leq r - 1 \). Similarly, if \( \{T^k x_n\}_{n \geq 1} \subset (\mathcal{X}, d) \) is Cauchy, that is, \( \lim_{n \rightarrow \infty} d(T^k x_n, T^k x_m) = 0 \) for all \( 0 \leq k \leq r - 1 \), then we have \( \lim_{n \rightarrow \infty} D_p(x_n, x_m) = 0 \) which completes the proof. \( \square \)

As noted by Ezearn in his paper, Lemma 2 does not imply that \( T \) is uniformly continuous or even continuous in \( (\mathcal{X}, d) \) as seen in the introduction; rather, \( T \) is Lipschitz continuous and therefore uniformly continuous in \( (\mathcal{X}, D_p) \). We must also note that when \( \lambda < 1 \), Theorem 1 cannot be used to conclude that \( T \) has a fixed point in \( (\mathcal{X}, D_p) \) unless \( T \) is continuous in \( (\mathcal{X}, d) \). However, the following theorem
remedies the case when $T$ is discontinuous on $(X,d)$. Here, our proof follows exactly as in Ezearn’s Theorem 3.5, but we give the proof for the sake of completeness. To begin, let $(X, D_p)$ be the canonical completion of the metric space $(X, D_p)$; that is,
\[
D_p([y_n], [x_n]) = \lim_{n \to \infty} D_p(y_n, x_n),
\]
where $\{y_n\}_{n \geq 1}$ and $\{x_n\}_{n \geq 1}$ are Cauchy sequences in $(X, D_p)$. Define the mapping,
\[
T: X \to X,
\]
\[
[x_n] \mapsto [Tx_n].
\]
Then, we have
\[
D_p(T[y_n], T[x_n]) \leq \lambda^{(1/p)} D_p([y_n], [x_n]).
\]
In particular, if $(X, d)$ is complete, then $T$ has a fixed point in $(X, d)$ if and only if $T$ has a fixed point in $(X, D_p)$.

Proof. Since $\{x_n\}_{n \geq 1}$ is Cauchy in $(X, D_p)$, then, by Lemma 2,
\[
D_p(Tx_n, Tx_m) \leq \lambda^{(1/p)} D_p(x_n, x_m),
\]
and so $\{Tx_n\}_{n \geq 1}$ is Cauchy in $(X, D_p)$; thus, $\overline{T}$ is well-defined. Now, given Cauchy sequences $\{x_n\}_{n \geq 1}, \{y_n\}_{n \geq 1}$ in $(X, D_p)$, then we have
\[
D_{\overline{T}}([T[y_n]], [T[x_n]]) = D_p([Ty_n], [Tx_n]),
\]
\[
= \lim_{n \to \infty} D_p(Ty_n, Tx_n),
\]
\[
\leq \lambda^{(1/p)} \lim_{n \to \infty} D_p(y_n, x_n),
\]
\[
= \lambda^{(1/p)} D_p([y_n], [x_n]).
\]
Finally, if $x = Tx$ in $(X, d)$, then let $[x]$ be the equivalence class of the constant sequence $\{x, x, x, \ldots\} \in (X, D_p)$. Then, we have
\[
T[x] = [Tx] = [x].
\]
On the other hand, if $[x_n] = T[x_n] = [Tx_n]$ in $(X, D_p)$, then, by Lemma 2, $T$ is continuous at $x$: $\lim_{n \to \infty} x_n$ in $(X, d)$, hence
\[
Tx = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_n = x,
\]
which completes the proof.

3.2. Approximate Fixed Point Property for Positively Homogeneous General Higher-Order Nonexpansive Mappings. In this subsection, we prove the approximate fixed point property for $(r, p)$-general higher-order nonexpansive mapping in a Banach space, which is accomplished in the following theorem.

**Theorem 8.** Let $\mathcal{C}$ be a closed bounded convex subset containing the origin of a Banach space $X$ and let $T: \mathcal{C} \to \mathcal{C}$ be a positively homogeneous general higher-order nonexpansive mapping. Then, $T$ has an approximate fixed point sequence.

Proof. Now, for all $y \in (0, 1)$, define $T_y: \mathcal{C} \to \mathcal{C}$ by
\[
T_y x = y Tx, \quad \forall x \in \mathcal{C}.
\]
Then, $T^2_y x = y^2 Tx, T^3_y x = y^3 Tx, \ldots, T^n_y x = y^n Tx$ since $T$ is a homogeneous general higher-order nonexpansive mapping. By inequality (7), we have the following evaluation:
\[
\sum_{k=1}^{r} \alpha_k y^{-k} \|T_k x - T_k y\|^p = \sum_{k=1}^{r} \alpha_k \|T_k x - T_k y\|^p,
\]
\[
\leq \sum_{k=0}^{r} \alpha_k \|T_k x - T_k y\|^p,
\]
\[
\sum_{k=0}^{r} \alpha_k \|T_k x - T_k y\|^p.
\]
Hence, we have
\[
\sum_{k=1}^{r} \alpha_k y^{-k} \|T_k x - T_k y\|^p = \sum_{k=0}^{r} \alpha_k \|T_k x - T_k y\|^p.
\]
Now, we have that
\[
\sum_{k=0}^{r} \alpha_k (y^{-1})^k - \sum_{k=0}^{r} \alpha_k (y^{-1})^{k+1} \geq \sum_{k=0}^{r} \alpha_k - \sum_{k=0}^{r} \alpha_k (y^{-1})^{k+1},
\]
\[
\sum_{k=0}^{r} \alpha_k (y^{-1})^k - \sum_{k=0}^{r} \alpha_k (y^{-1})^{k+1} \geq \sum_{k=0}^{r} \alpha_k - \sum_{k=0}^{r} \alpha_k (y^{-1})^{k+1}.
\]
Since $T$ is a general higher-order nonexpansive mapping, then, by Proposition 1, we have that $\sum_{k=1}^{r} \alpha_k = \sum_{k=0}^{r} \alpha_k$. Now, let $C = \sum_{k=1}^{r} \alpha_k = \sum_{k=0}^{r} \alpha_k$. Then, inequality (33) becomes
\[
\sum_{k=0}^{r} \alpha_k (y^{-1})^k - \sum_{k=0}^{r} \alpha_k (y^{-1})^{k+1} \geq \sum_{k=0}^{r} \alpha_k - \sum_{k=0}^{r} \alpha_k (y^{-1})^{k+1}.
\]
Since $\alpha_k \cdot \alpha_k \neq 0$. Hence, equation (32) is a general higher-order contraction mapping and thus, by Theorem 7, $T_y$ has a unique fixed point $x_\gamma$ in $\mathcal{C}$. That is, $T_\gamma x_\gamma = x_\gamma$. Let $\gamma = 1 - (1/n)$ for all $n \geq 1$. Then,
\[
\|x_n - Tx_n\| = \frac{1}{n} \|T_n x_n\| \geq \frac{1}{n} \text{diam}(\mathcal{C}).
\]
In the limit as $n \to \infty$, it follows that $x_n - Tx_n \to 0$ and that completes the proof.

**Data Availability**

No data were used for this research.

**Conflicts of Interest**

The author declares that he has no conflicts of interest.
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