Asymmetric $\lambda$–deformed cosets

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Abstract

We study the integrable asymmetric $\lambda$-deformations of the $SO(n+1)/SO(n)$ coset models, following the prescription proposed in [1]. We construct all corresponding deformed geometries in an inductive way. Remarkably we find a $Z_2$ transformation which maps the asymmetric $\lambda$–deformed models to the symmetric $\lambda$–deformed models.

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1 Introduction

Integrability plays a key role in obtaining exact results in field theories and string theory. After witnessing a remarkable progress in understanding the integrable string theories it is clearly interesting and important to construct new integrable models. In the last ten years, based on sigma models on group or coset manifolds powerful tools such as $\eta$–[2, 3, 4] and $\lambda$–deformations [5] [6] [7] have been developed to investigate this issue.

The original $\lambda$–deformation [5] gives an interpolation between the (gauged) Wess-Zumino-Witten (WZW) model [11] and the non-Abelian T-dual of the Principle Chiral model (PCM) [12]. The deformations of $AdS_p \times S^q$ have been successfully constructed in [6] [7]. The novel idea behind the construction of $\lambda$–deformation is to combine different integrable models through a gauging procedure. Following the same idea, various of generalizations of $\lambda$–deformation have been proposed [8] [9] [10]. Recently, the authors in [1] introduced a new generalization named asymmetric $\lambda$–deformation by modifying the gauging procedure. The key observation in [1] is that different choices of anomaly free gauges can be made in the deformation. It is somehow similar to the gauged WZW model with $U(1)$ symmetry, in which case one can choose either the vector or axial gauge and the two resulted gauged theories are T-dual to each other [13]. Now the deformation breaks the axial-vector duality since the deformation destroys the isometries of the background. To have a non-trivial asymmetric $\lambda$–deformation, the starting model has to possess a Lie algebra with non-trivial outer automorphism group.

In this article, we will study the asymmetric $\lambda$–deformation of the $SO(n+1)/SO(n)$ coset models, and pay special attention to the case that $n$ is odd and the corresponding Lie algebra admits a $Z_2$ outer automorphism group. A physical motivation to study the deformation of this class of coset models is that the coset $SO(n+1)/SO(n)$ is isomorphic to $S^n$ and by an analytic continuation they can be transformed into $AdS_n$. Then it would be possible to embed the deformed models into supergravity.

The paper has the following organization: In section 2, we review the construction of asymmetric $\lambda$–deformation introduced in [1]. To apply the construction, we need to find a suitable coset representative and explicit forms of the outer automorphism in this given representative. So in section 3, we first find the outer automorphism of the $SO(n)$ group following [16] and then develop a gauge fixing scheme similar to [14]. After constructing the deformed models we prove the existence of a $Z_2$ transformation which maps asymmetric $\lambda$–deformed geometry to the symmetric deformed ones. As a by
product, we provide a simple recursive method to construct the deformed geometries in our choice of coset representative. Some technical details are presented in the appendices.

## 2 Asymmetric $\lambda$-deformation

In this section, we briefly review the integrable asymmetric $\lambda$-deformation. Here we only focus on symmetric coset models. Similar constructions can also be applied to group and supercoset models. For the details we refer the original article [1]. To introduce the deformation of a coset model $G/H$, we begin with separating the generators $T_A$ of the group $G$ into $T_a$ and $T_\alpha$ corresponding to the subgroup $H$ and the coset $G/H$ respectively, and then defining the Maurer-Cartan forms

$$L_\mu^A = -i \text{Tr}(T^A g^{-1} \partial_\mu g), \quad R_\mu^A = -i \text{Tr}(T^A \partial_\mu g g^{-1}),$$

$$R_\mu^A = D_{AB} L_\mu^B, \quad D_{AB} = \text{Tr}(T_A T_B g^{-1}), \quad g \in G. \quad (2.1)$$

The asymmetric $\lambda$–deformation of coset model $G/H$ is constructed by performing three steps:

(i) Combine $S_{PCM}(\hat{g})$ on coset $G/H$ with the $S_{WZW,k}(g)$ on the same group $G$, where

$$S_{PCM}(\hat{g}) = -\frac{k^2}{\pi} \int \text{Tr} (\hat{g}^{-1} \partial_+ \hat{g} \hat{g}^{-1} \partial_- \hat{g}),$$

$$S_{WZW,k}(g) = -\frac{k}{2\pi} \int \text{Tr} (g^{-1} \partial_+ g g^{-1} \partial_- g) + \frac{k}{6\pi} \int \text{Tr} (g^{-1} dg)^3. \quad (2.2)$$

(ii) Gauge the group $G$ whose $G_L$ action is given by

$$g \rightarrow g_0^{-1} \hat{g} g_0, \quad \hat{g} \rightarrow g_0^{-1} \hat{g}, \quad (2.3)$$

where $g_0 = \exp(G^A T_A) \in G$ and $\hat{g}_0 = \exp(G^A \tilde{T}_A) \in G$ have the same parameters $G^A$ but they are generated by different embeddings $T_A$ and $\tilde{T}_A$ of the subalgebra. These two embeddings are related by a linear transformation $W$, such that $\tilde{T}_A = W(T_A) = W^B A T_B$. To avoid the gauge symmetry anomaly the transformation $W$ has to be a metric-preserving automorphism of the Lie algebra, i.e.,

$$W([T_A, T_B]) = [W(T_A), W(T_B)], \quad \text{Tr}(W(T_A) W(T_B)) = \text{Tr}(T_A T_B). \quad (2.4)$$

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Footnote: Here we follow the convention in [5], the Latin index $(a)$ denotes the component in the subgroup and Greek index $(\alpha)$ denotes components in the coset $G/H$. 

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Non-equivalent choice of \( W \) is characterized by the outer automorphism group of the Lie algebra. In particular, the choice \( W = I \) which is usually called the vector gauge leads to the standard \( \lambda \)-deformation \([5]\). Therefore only when \( W \) is a non-trivial element of the outer automorphism group, following \([1]\) we will call it axial gauge, the deformation can be potentially non-trivial.

(iii) Integrate out the gauge field and fix the gauge \( \hat{g} = I \).

These procedures give the final action of deformed model:

\[
S_\lambda(g, W) = S_{ZW,k}(g) - \frac{k}{\pi} \int L_- \left( D - \Omega^T W^{-1} \right)^{-1} R_

= \frac{k}{2\pi} \int L_- \left[ 1 - 2 \left( D - \Omega^T W^{-1} \right)^{-1} D \right] L_+,
\]

where \( L, R \) and \( D \) are quantities defined in (2.1) and the operator \( \Omega \) is given by

\[
\Omega(T_A) = \Omega^B_A T_B, \quad \Omega^B_A = \left( \begin{array}{cc} I_{ba} & 0 \\ 0 & \lambda^{-1} I_{\beta\alpha} \end{array} \right), \quad 0 < \lambda < 1.
\]  

(2.6)

The background geometry of the target space of this model is given by\([2]\),

\[
ds^2 = \frac{k}{2\pi} L^T \left[ 1 - MD - (MD)^T \right] L,
\]

(2.7)

with

\[
M \equiv \left( D - \Omega^T W^{-1} \right)^{-1},
\]

\[
e^{-2\Phi} = e^{-2\Phi_0} \det(DW - \Omega).
\]

(2.8)

With the help of the identity \( DD^T = 1 \) and \( W W^T = 1 \), the metric can be simplified as

\[
ds^2 = \frac{k}{2\pi} L^T \left[ 1 - MD - (MD)^T \right] L \\
= \frac{k}{2\pi} L^T M \left[ M^{-1} M^{-T} - D M^{-T} - M^{-1} D^T \right] M^T L \\
= \frac{k}{2\pi} (M^T L)^T \left[ \Omega^T \Omega - 1 \right] M^T L
\]

(2.9)

Substituting (2.6) and using the inversion formula of a block matrix, the metric can be cast into the form

\[
ds^2 = \frac{k}{2\pi} \frac{1 - \lambda^2}{\lambda^2} e^T P^T P e,
\]

(2.10)

\[2\text{The Kalb-Ramond field vanishes for a similar argument given in [14, 8]}
\[3\text{\( W \) can always be chosen to be diagonal.} \]
where

\[ e_\alpha = L_\alpha - D_a^T (D_{ab} - W_{ab})^{-T} L_b, \]
\[ (P^{-T})_{\alpha\beta} = \left[ D_{\alpha\beta} - D_{ab} (D_{ab} - W_{ab})^{-1} D_{a\beta} - \frac{1}{\lambda} W_{a\beta} \right]. \] (2.11)

Here \( e \) are the frames of the gauged WZW model, and the deformation is totally encoded in the matrix \( P \).

In [1], the authors have studied the asymmetric \( \lambda \)-deformations of coset \( SL(2, R)/U(1) \) and showed explicitly that the construction leads to a new integrable model. In this paper, we will apply these procedures to the cosets \( SO(n + 1)/SO(n) \) for \( n = 2, 3, \ldots \).

3 Asymmetric \( \lambda \)-deformed \( SO(n + 1)/SO(n) \)

The \( \lambda \)-deformation of the cosets \( SO(n + 1)/SO(n) \) with \( \lambda \)-deformations for \( n = 2, 3, 4, 5 \) in the vector gauge cases have been constructed in [6, 7]. These corresponding deformed geometries can be promoted to integrable backgrounds of string theory and their dynamical properties are also analyzed in [15].

In the following discussion we choose the generators of \( SO(n + 1) \) to be

\[ T_{ij} = \frac{i}{\sqrt{2}} (E_{ij} - E_{ji}) \] (3.1)

and embed the subgroup \( SO(n) \) as

\[ t_A = (T_{n,n+1}; T_{n-1,n}; T_{n-1,n+1}; \ldots; T_{23}, \ldots; T_{2,n+1}; T_{12}, \ldots; T_{1,n+1}) \]
\[ \equiv (t_a, t_\alpha) \] (3.2)

Before performing the deformation procedures we need to solve the outer automorphism group first. Besides that the other preparatory work is to find the coset representative for \( SO(n + 1)/SO(n) \) in the axial gauge.

3.1 The outer automorphism group

The outer automorphism group of a simply connected group corresponds to the symmetry of its Dynkin diagram. For simple Lie algebras, only the Dynkin diagrams of types \( A_n, D_n, \) and \( E_6 \) admit non-trivial symmetries. Therefore for orthogonal groups only the groups \( SO(2n) \) with \( n \geq 2 \) have non-trivial outer automorphism groups. The forms of
the transformation \( W \) are computed in Appendix A. Up to inner automorphism the final results are

\[
W(T_{ij}) = \begin{cases} 
-T_{ij}, & j = 2n \\
T_{ij}, & j \leq 2n - 1 
\end{cases}, \quad i < j, \text{ and } n \geq 2 \quad (3.3)
\]

Notice that \( W \) is the diagonal matrix and \( W = W^T = W^{-1} \). This choice of gauge can be viewed as a higher dimensional generalization of the axial gauge used in [17].

### 3.2 The coset representative

An element \( g_{n+1} \in SO(n + 1) \) can be decomposed as [18]

\[
g_{n+1} = H_n t_n \quad (3.4)
\]

where

\[
H_n = \begin{bmatrix} 1 & 0 \\
0 & h \end{bmatrix}, \quad t = \begin{bmatrix} b - 1 & bV^T \\
bV & I_{n \times n} - bVV^T \end{bmatrix}, \quad (3.5)
\]

with

\[
h \in SO(n), \quad b = \frac{2}{1 + VTV}, \quad (3.6)
\]

and \( V \) is a \( n \)-vector. The \( SO(n) \) gauge rotation \( R_n \) acts as (2.3)

\[
g'_{n+1} = R_n g_{n+1} \tilde{R}_n^{-1} = R_n H_n \tilde{R}_n^{-1} \tilde{R}_n t_n \tilde{R}_n^{-1} \equiv H'_n t'_n, \quad V'_n = \tilde{R}_n V_n. \quad (3.7)
\]

We pick a convenient gauge such that \( V'_n = (v_{n-1}, 0, \ldots, 0) \) such that \( t'_n \) is invariant under the \( SO(n-1) \) gauge rotation \( R_{n-1} \) and \( H'_n \equiv g_n \in SO(n) \). For \( g_n \), a similar decomposition and gauge choice lead to \( g_{n-1} \) and \( t'_{n-1} \). Eventually we can fix all the gauge freedoms and end up with the coset representative

\[
g'_{n+1} = R_1 R_2 \ldots R_n g_{n+1} \tilde{R}_n^{-1} \ldots \tilde{R}_2^{-1} \tilde{R}_1^{-1} = t'_1 t'_2 \ldots t'_{n} \quad (3.8)
\]

where

\[
t'_{i+1} = \begin{bmatrix} b_{i+1} - 1 & b_{i+1}V_{i+1}^T \\
-b_{i+1}V_{i+1} & 1 - b_{i+1}V_{i+1}V_{i+1}^T \end{bmatrix} = \begin{bmatrix} I_{(n-1-i) \times (n-1-i)} & 1-v_i^2 & 2v_i \sqrt{1+v_i^2} \frac{1}{1+v_i^2} \end{bmatrix}
\]

\[
= \begin{bmatrix} I_{(n-1-i) \times (n-1-i)} & \cos \theta_i & -\sin \theta_i \\
\sin \theta_i & \cos \theta_i \end{bmatrix} = e^{\sqrt{2}i \theta_i T_{n-i,n+1-i}}, \quad \theta_i = [0, 2\pi). \quad (3.9)
\]
The advantage of this coset representative is that it does not depend on the outer automorphism transformation \( W \). Another possible coset representative can be found in [14].

3.3 An Example: \( SO(4)/SO(3) \)

Before the generic discussion, let us study the simplest example in details. This example exhibits all the essential features of the general situations. The metrics of the corresponding gauged WZW model (\( \lambda = 0 \)) on this coset have been found both with vector and axial gauge in [17]. For the gauged WZW model, it turns out the metrics with different gauges are connected via a coordinate transformation. We will show for \( \lambda \)-deformed model, under a coordinate transformation and an additional transformation of the parameter \( \lambda \) the two metrics with different gauges are related to each other. According to the general discussion (3.3), the non-trivial automorphism \( W \) is

\[
W : (T_{14}, T_{24}, T_{34}) \rightarrow -(T_{14}, T_{24}, T_{34}). \quad (3.10)
\]

In order to compare with the results in [17], we parameterize the group \( SO(4) \) as

\[
g = h t_3', \quad h = e^{i(\tau - \theta)T_{34}}e^{2i\phi T_{23}}e^{i(\tau + \theta)T_{34}}. \quad (3.11)
\]

Vector gauge

In this gauge, the coset representative is given by setting \( \theta = 0 \) in (3.11). Substituting the coset representative into (2.7) leads the metric

\[
ds^2_V = \frac{2\pi}{k} \left[ \frac{1 + \lambda^2}{1 - \lambda^2} e_\alpha e^\alpha + \frac{2\lambda}{1 - \lambda^2} e_\alpha J^\alpha e_\beta \right], \quad e_1 = \frac{dV_1}{1 + V_1^2},
\]

\[
e_2 = -\frac{d\phi \cos \tau + d\tau \cot \phi \sin \tau}{V_1}, \quad e_3 = -V_1 (d\phi \sin \tau - d\tau \cot \phi \cos \tau). \quad (3.12)
\]

In the two forms \( e_2 \) and \( e_3 \) one can recognize that

\[
E_1 = -d\phi \cos \tau - d\tau \cot \phi \sin \tau, \quad E_2 = -d\phi \sin \tau + d\tau \cot \phi \cos \tau. \quad (3.13)
\]

are the two frames of the vector gauged \( SO(3)/SO(2) \) model [17]. It is more convenient to introduce the new variables

\[
x = \cos \tau \cos \phi, \quad y = \sin \tau \cos \phi. \quad (3.14)
\]
With these new variables both $e_i$ and $J^{\alpha\beta}$ have relatively simpler expressions:

$$E_1 = -\frac{dx}{\sqrt{1-x^2-y^2}}, \quad E_2 = -\frac{dy}{\sqrt{1-x^2-y^2}}, \quad (3.15)$$

$$J_V = \begin{pmatrix}
2x^2 + 2y^2 - 1 & 2x\sqrt{-x^2-y^2 + 1} & -2y\sqrt{-x^2-y^2 + 1} \\
2x\sqrt{-x^2-y^2 + 1} & 1 - 2x^2 & 2xy \\
-2y\sqrt{-x^2-y^2 + 1} & 2xy & 1 - 2y^2
\end{pmatrix}, \quad (3.16)$$

**Axial gauge**

In this gauge, the coset representative is given by setting $\tau = 0$ in (3.11). Substituting the coset representative into (2.7) leads the metric

$$ds_A^2 = \frac{2\pi}{k} \left[ 1 + \lambda^2 e_\alpha e^\alpha + \frac{2\lambda}{1 - \lambda^2} e_\alpha J^{\alpha\beta}_A e_\beta \right], \quad e_1 = \frac{dV_1}{1 + V_1^2}, \quad (3.17)$$

$$e_2 = -\frac{d\phi \cos \theta + d\theta \tan \phi \sin \theta}{V_1}, \quad e_3 = -V_1 (d\phi \sin \theta + d\theta \tan \phi \cos \theta). \quad (3.18)$$

In $e_2$ and $e_3$ again one can recognize that

$$E_1 = d\theta \sin(\theta) \tan(\phi) - d\phi \cos(\theta), \quad E_2 = -d\theta \cos(\theta) \tan(\phi) - d\phi \sin(\theta) \quad (3.19)$$

are frames of the axial gauged $SO(3)/SO(2)$ model [17]. Similarly introducing the convenient variables

$$x = \cos \theta \sin \phi, \quad y = \sin \theta \sin \phi, \quad (3.20)$$

we have

$$E_1 = -\frac{dx}{\sqrt{1-x^2-y^2}}, \quad E_2 = -\frac{dy}{\sqrt{1-x^2-y^2}}, \quad (3.21)$$

$$J_A = -\begin{pmatrix}
2x^2 + 2y^2 - 1 & 2x\sqrt{-x^2-y^2 + 1} & -2y\sqrt{-x^2-y^2 + 1} \\
2x\sqrt{-x^2-y^2 + 1} & 1 - 2x^2 & 2xy \\
-2y\sqrt{-x^2-y^2 + 1} & 2xy & 1 - 2y^2
\end{pmatrix}, \quad (3.22)$$

Comparing with results (3.12) (3.15), we find when $\lambda = 0$ these two metrics are equivalent up to a change of coordinates:

$$\tau \rightarrow \theta, \quad \phi \rightarrow \frac{\pi}{2} - \phi. \quad (3.23)$$

The equivalence of vector gauged and axial gauged model is called “self-dual” in [17]. When $\lambda \neq 0$, the metric $ds_A^2$ can be obtained from $ds_V^2$ by performing (3.23) and a $Z_2$ transformation on the deformation parameter:

$$\lambda \rightarrow -\lambda. \quad (3.24)$$
This additional $Z_2$ transformation is non-trivial for this model. One way to show this is to solve the spectrum of scalar on the deformed geometry following [15]. We will give a simple example in the next section. In the next section, we will provide this kind of $Z_2$ transformation which transforms the deformed geometries in vector gauge into the ones in axial gauge for all the deformed cosets $SO(n + 1)/SO(n)$.

Scalar field

Even though the geometry corresponding to $\lambda$–deformed $SO(4)/SO(3)$ coset has no isometries, an algebraic method based on group theory is proposed in [15]. In this appendix, we use their method to give a simple example showing that the $Z_2$ transformation on the parameter $\lambda$ is physical. According to [15], the scalar field equation on the deformed geometry reduces to a second order differential equation known as the Heun’s equation for polynomial ansatz $Q(z)$:

$$
Q''(z) + \left[ \frac{1}{z} + \frac{1 + 2L_2}{z - 1} - \frac{1 + 2L_1}{z - c} \right] Q'(z) + \frac{(L_1 - L_2)^2(z - h)}{z(z - 1)(z - c)} Q(z) = 0,
$$

where the parameters $(c, h)$ of the Heun’s equation are given by

$$
c = \frac{1 + \kappa}{2\kappa}, \quad \kappa = \frac{2\lambda}{1 + \lambda^2}, \quad h = \frac{(1 - 2c)L_1 + (1 - c)L_1^2 + cL_2}{(L_1 - L_2)^2} + \frac{\Lambda}{8\kappa(L_1 - L_2)^2},
$$

and $(L_1, L_2)$ are integers labeling the irreducible representation of $SO(4)$. The spectrum $\Lambda$ is determined from the truncation condition of the polynomial ansatz solution of the Heun’s equation. Assume the polynomial ansatz truncates at power $p$, the truncation conditions are

$$
p = L_1 - L_2
$$

and

$$
\det(M_{p+1}) = 0, \quad \text{where } M_{p+1} =
\begin{bmatrix}
A_0 & -1 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
B_1 & A_1 & -1 & 0 & 0 & \cdots & \cdots & \cdots \\
0 & B_2 & A_2 & -1 & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & A_{p-1} & -1 \\
0 & 0 & 0 & 0 & 0 & \cdots & B_p & A_p
\end{bmatrix}
$$

where the matrix elements are defined by

$$
A_n = \frac{(1 + c)n^2 + [(1 + 2L_2)c - (1 + 2L_1)n]}{c(n + 1)^2} + \frac{(L_1 - L_2)^2h}{c(n + 1)^2},
$$

$$
B_n = -\frac{(n - 1 + L_2 - L_1)^2}{c(n + 1)^2}.
$$
Considering the example with \((L_2, L_1) = (2, 1), \) \((3.28)\) leads to
\[
8(3\kappa - 5)\Lambda + 48(\kappa - 7)(\kappa - 1) + \Lambda^2 = 0,
\]
\[
\Lambda_{\pm} = 4(5 - 3\kappa \pm \sqrt{4 + 6(\kappa - 1)\kappa}),
\]
which are not equivalent under the transformation \(\lambda \rightarrow -\lambda.\)

### 3.4 The \(Z_2\) transformation

In the section, we will not restrict ourselves to \(SO(2n)/SO(2n - 1)\) but consider the general coset \(SO(n + 1)/SO(n)\) model with the automorphism transformation
\[
W(T_{ij}) = \begin{cases} 
-T_{ij}, & j = n + 1 \\
T_{ij}, & j \leq n
\end{cases}, \quad i < j, \text{ and } n \geq 2,
\]
even though for the case when \(n\) is even the transformation is just a gauge transformation. The deformed geometries \((2.7)\) depend on \(W\) in a rather complicate way. However we prove the deformed geometries can be transformed back to the symmetric \(\lambda\)-deformed \((W = I)\) ones by the \(Z_2\) transformation\(^4\)
\[
\theta_i \rightarrow \pi - \theta_i, \quad \lambda \rightarrow -\lambda,
\]
where the coordinates \(\theta_i\) of the target space are defined in \((3.9)\). The proofs are presented in the Appendix \([13]\). One prediction of our results is that the spectrum of scalar field on the \(\lambda\)-deformed \(SO(2n + 1)/SO(2n)\) is invariant under \(\lambda \rightarrow -\lambda\) since \(W\) is a gauge transformation in this case.

### 3.5 Recursion relations

Recall the deformed geometries are given by \((2.10)\):
\[
ds^2 = \frac{k}{2\pi} \frac{1 - \lambda^2}{\lambda^2} e^T P T P e,
\]
\[
e_\alpha = L_\alpha - D_{a\alpha}^T (D_{ab} - W_{ab})^{-T} L_b
\]
\[
(P^{-T})_{\alpha\beta} = \left[ D_{\alpha\beta} - D_{ab} (D_{ab} - W_{ab})^{-1} D_{a\beta} - \frac{1}{\lambda} W_{a\beta} \right].
\]
In the proof the \(Z_2\) transformation, we have obtained recursion relations for \(e_\alpha, D_{ab}\) and \(P\). However, these relations are involved with cumbersome inversions of matrices. In this section, we will provide new recursion relations which only consist of matrix additions\(^4\) up to some other coordinate transformations.
and multiplications. By the $Z_2$ transformation, we can set $W = I$ and rewrite the metric in a similar form as (3.12):

$$ds^2 = \frac{k}{2\pi} \frac{1 - \lambda^2}{\lambda^2} e^T P^T P e \equiv \frac{k}{2\pi} \left( \frac{1 + \lambda^2}{1 - \lambda^2} e^T e + \frac{2\lambda}{1 - \lambda^2} e^T J e \right),$$

(3.34)

where

$$J = \frac{1 - \lambda^2}{2\lambda} \left( \frac{1 - \lambda^2}{\lambda^2} P^T P - \frac{1 + \lambda^2}{1 - \lambda^2} I \right).$$

(3.35)

Introducing a new quantity

$$Q \equiv d_4 - d_3(d_1 - 1)^{-1} d_2,$$

(3.36)

where $d_i$ are block elements in the matrix $D_{AB}$

$$D = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}.$$

(3.37)

The quantity $Q$ satisfies $Q^T Q = QQ^T = I$ \[\text{[7]}\]. In Appendix \[\text{[C]}\] we show $Q = Q^T$ and $Q = J$. Here we present the new recursion relations for $e_i$ and $Q$. The recursion relation of $Q$ is given by\[\text{[3]}\]

$$Q_{n+1} = \begin{bmatrix} \cos \theta_{n-2} \\ \sin \theta_{n-2} \cos \theta_{n-3} \\ \sin \theta_{n-2} \sin \theta_{n-3} \cos \theta_{n-4} \\ \vdots \\ \sin \theta_{n-2} \sin \theta_{n-3} \ldots \sin \theta_1 \cos \theta_0 \\ \sin \theta_{n-2} \sin \theta_{n-3} \ldots \sin \theta_1 \sin \theta_0 \end{bmatrix} Q_n \begin{bmatrix} \sin \theta_{n-2} \cos \theta_{n-3} \\ \sin \theta_{n-2} \sin \theta_{n-3} \cos \theta_{n-4} \\ \vdots \\ \sin \theta_{n-2} \sin \theta_{n-3} \ldots \sin \theta_1 \cos \theta_0 \\ \sin \theta_{n-2} \sin \theta_{n-3} \ldots \sin \theta_1 \sin \theta_0 \end{bmatrix}^T$$

(3.38)

with the first two cases

$$Q_3 = \begin{bmatrix} \cos \theta_0 & \sin \theta_0 \\ \sin \theta_0 & -\cos \theta_0 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} \cos \theta_1 & \cos \theta_0 \sin \theta_1 \\ \cos \theta_0 \sin \theta_1 & \sin^2 \theta_0 - \cos^2 \theta_0 \cos \theta_1 \\ \sin \theta_0 \sin \theta_1 & -2 \cos \theta_0 \cos^2 \left( \frac{\theta_1}{2} \right) \sin \theta_0 \\ \sin \theta_0 \sin \theta_1 & -2 \cos \theta_0 \cos^2 \left( \frac{\theta_1}{2} \right) \sin \theta_0 \\ \sin \theta_0 \sin \theta_1 & -2 \cos \theta_0 \cos^2 \left( \frac{\theta_1}{2} \right) \sin \theta_0 \end{bmatrix}.$$

(3.39)

\[5\text{The subscript of } X_{i+1} \text{ denotes that the quantity } X_{i+1} \text{ is defined for the coset } SO(i+1)/SO(i).\]
For the frames $e_{\alpha}$, since the expression of $L$ has been shown in (B.11), we only need to consider $(d_{1|n+1} - 1)^{-1}d_{2|n+1}$. The block matrix inversion formula gives

$$(d_{1|n+1} - 1)^{-1} = \begin{bmatrix} * & -(d_{1|n} - 1)^{-1}d_{2|n} \cos \theta_{n-1} (\cos \theta_{n-1} Q|n - 1)^{-1} \\ * & (\cos \theta_{n-1} Q|n - 1)^{-1} \end{bmatrix}, \quad (3.40)$$

where $*$ stands for the irrelevant elements. Substituting the identity (B.5) we get

$$(d_{1|n+1} - 1)^{-1}d_{2|n+1} = \tan \theta_{n-1} \left[1 + (d_{1|n+1} - 1)^{-1}\right] C_{n-1}$$

$$= \tan \theta_{n-1} \begin{bmatrix} 0 & -(d_{1|n} - 1)^{-1}d_{2|n} \cos \theta_{n-1} (\cos \theta_{n-1} Q|n - 1)^{-1} \\ 0 & (\cos \theta_{n-1} Q|n - 1)^{-1} + 1 \end{bmatrix} \quad (3.41)$$

with the first case

$$(d_{1|3} - 1)^{-1}d_{2|3} = [0, -\cot (\theta_1/2)]. \quad (3.42)$$

In the end let us derive the recursion relation of the dilaton (B.23) ($W = I$):

$$e^{-2\Phi} = e^{-2\Phi_0} \det(d_1 - 1). \quad (3.43)$$

Substituting the recursion relation of $d_1$ (B.9) leads to

$$\det(d_{1|n+1} - 1) = \det(d_{1|n} - 1) \det[1 - \cos \theta_{n-1} Q|n]. \quad (3.44)$$

To evaluate the second determinant we first observe that the eigenvalues of $Q$ can only be $\pm 1$ due to $QQ^T = I$. Then taking trace on both sides of the recursion relation (3.38), one finds

$$\Tr[Q|n+1] = \Tr[Q|n-1], \text{ or } \Tr[Q|2n-1] = 0, \text{ Tr}[Q|2n] = 1, \quad n = 2, 3 \ldots. \quad (3.45)$$

Combining the two facts we get

$$\det[1 - \cos \theta_{2k-1} Q|2k] = (1 - \cos \theta_{2k-1}) \sin^{2k} \theta_{2k-1},$$

$$\det[1 - \cos \theta_{2k-2} Q|2k-1] = \sin^{2(k-1)} \theta_{2k-2}, \quad k = 2, 3, \ldots, \quad (3.46)$$

which complete our recursion relations.
4 Summary

In this article we explored the asymmetric $\lambda$–deformation introduced in [1]. For the $SO(n+1)/SO(n)$ coset model we found a $\mathbb{Z}_2$ transformation (3.32) which maps the asymmetric $\lambda$–deformation to the symmetric $\lambda$–deformations. When the deformation parameter $\lambda$ vanishes, these two different deformed models reduce to the gauged WZW models in the axial and vector gauge, respectively. The gauged WZW models in different gauges are dual to each other. We gave evidences to show that the deformation break this duality so that the asymmetric $\lambda$–deformation leads to new integrable models. Furthermore, we construct the resulting geometries for arbitrary $n$ recursively (3.38), (3.41) and (B.11). It would be interesting to extend the current results to the cases of supergroups so that the integrable deformation of string theory on $AdS_p \times S^p$ can be constructed and studied.

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A The outer automorphism of $so(n)$

For the Lie algebra, $W$ is called the outer automorphism transformation if there is no $x^A \in R$ such that $W = \exp(x^A ad_t A)$, where $ad$ means the adjoint representation. Let us consider $so(2n), n \geq 3$ in the Cartan-Weyl basis. Its simple roots are

$$
\alpha_i = e_i - e_{i+1}, \quad i = 1, 2, ..., n - 1
$$

$$
\alpha_n = e_{n-1} + e_n
$$

(A.1)

and the generators are

$$
H_i = -\sqrt{2}T(2i-1)(2i),
$$

$$
E_{\alpha_i} = -\frac{1}{\sqrt{2}} \left[ T(2i-1)(2i+1) + T(2i)(2i+2) + iT(2i)(2i+1) - iT(2i-1)(2i+2) \right], \quad i = 1, 2, ..., n - 2,
$$

$$
E_{\alpha_n-1} = -\frac{1}{\sqrt{2}} \left[ T(2n-3)(2n-1) + T(2n-2)(2n) + iT(2n-2)(2n-1) - iT(2n-3)(2n) \right],
$$

$$
E_{\alpha_n} = -\frac{1}{\sqrt{2}} \left[ T(2n-3)(2n-1) - T(2n-2)(2n) + iT(2n-2)(2n-1) + iT(2n-3)(2n) \right],
$$

(A.2)
where $T_{ij}$ are denoted by (3.1). According to the symmetry of the Dynkin diagrams of $D_n$, the outer automorphism $W$ are [16]

$$W(E_{\alpha_i}) = W(E_{\alpha_i}), \quad i = 1, 2, ..., n - 2,$$
$$W(E_{\alpha_{n-1}}) = W(E_{\alpha_n}), \quad W(E_{\alpha_n}) = W(E_{\alpha_{n-1}}).$$  \quad (A.3)

Notice that $W$ is a linear transformation. Therefore,

$$W(T_{(2i-1)(2i+1)}) = T_{(2i-1)(2i+1)}, \quad W(T_{(2i)(2i+1)}) = T_{(2i)(2i+1)}, \quad i = 1, 2, ..., n - 1$$
$$W(T_{(2i)(2i+2)}) = T_{(2i)(2i+2)}, \quad W(T_{(2i-1)(2i+2)}) = T_{(2i-1)(2i+2)}, \quad i = 1, 2, ..., n - 2 \quad (A.4)$$
$$W(T_{(2n-2)(2n)}) = -T_{(2n-2)(2n)}, \quad W(T_{(2n-3)(2n)}) = -T_{(2n-3)(2n)}.$$

By (A.4) and commutators of $SO(n)$ group

$$[T_{AB}, T_{CD}] = \frac{i}{\sqrt{2}} (\delta_{BC}T_{AD} + \delta_{AD}T_{BC} - \delta_{AC}T_{BD} - \delta_{BD}T_{AC}).$$  \quad (A.5)

we can get an outer automorphism transformation,

$$W(T_{ij}) = \begin{cases} -T_{ij}, & j = 2n \\ T_{ij}, & j \leq 2n - 1 \end{cases}, \quad i < j, \text{ and } n \geq 3. \quad (A.6)$$

As for $so(4)$, which is isomorphism to $so(3) \oplus so(3)$. The generators are

$$J_{+1} = \frac{1}{\sqrt{2}}(T_{25} + T_{14}), \quad J_{-1} = \frac{1}{\sqrt{2}}(T_{13} - T_{24}), \quad J_{+1} = \frac{1}{\sqrt{2}}(T_{12} + T_{34}),$$
$$J_{-1} = \frac{1}{\sqrt{2}}(T_{25} - T_{14}), \quad J_{+1} = \frac{1}{\sqrt{2}}(T_{13} + T_{24}), \quad J_{-1} = \frac{1}{\sqrt{2}}(T_{12} - T_{34}). \quad (A.7)$$

The commutators are

$$[J_{+i}, J_{+j}] = i\epsilon_{ij}^k J_{+k}, \quad [J_{-i}, J_{-j}] = i\epsilon_{ij}^k J_{-k}, \quad [J_{+i}, J_{-j}] = 0. \quad (A.8)$$

The adjoint representations of the generators are

$$J_{+i} = \begin{bmatrix} \sigma_i & 0 \\ 0 & 0 \end{bmatrix} = \sigma_i \oplus 0, \quad J_{-i} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_i \end{bmatrix} = 0 \oplus \sigma_i, \quad (A.9)$$

where $\sigma_i$ are Pauli matrices in the adjoint representation, that is,

$$\sigma_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}. \quad (A.10)$$
Consider an automorphism $W$,

$$W(J_+) = J_-, \quad W(J_-) = J_+.$$  \hfill (A.11)

Under the basis $J = (J_+, J_-)$,

$$W = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \hfill (A.12)$$

However, by the definition of outer automorphism,

$$e^{x A_{adj}} e^{(x^i \sigma_i) \oplus (x^j \sigma_j)} = e^{(x^i \sigma_i) \oplus (x^j \sigma_j)} = \begin{pmatrix} e^{x^i \sigma_i} & 0 \\ 0 & e^{x^j \sigma_j} \end{pmatrix}, \hfill (A.13)$$

there is no solution about $W = e^{x A_{adj}}$, so $W$ is an outer automorphism. Under the basis $T_{ij}$,

$$W(T_{ij}) = \begin{cases} -T_{ij}, & j = 4 \\ T_{ij}, & j \leq 3, \quad i < j. \end{cases} \hfill (A.14)$$

## B Proof of the $Z_2$ transformation

In this appendix we will prove that under the combination $W$ and (3.32) the geometry (2.7) is invariant. Our main method is the mathematical induction. We begin with proving some import properties of matrix $D_{AB}$ defined in (2.1) under the coset representative (3.8):

$$g = t'_1 t'_2 \ldots t'_n \hfill (B.1)$$

The matrix $D$ can be decomposed as

$$D_{AB} = D_{AC_1}^1 D_{C_1C_2}^2 \ldots D_{C_{n-1}B}^n, \hfill (B.2)$$

where

$$D_{AB}^i \equiv \text{Tr}[T_A t'_i T_B t_i'^{-1}] \equiv \begin{bmatrix} a & \alpha \\ d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} a \hfill (B.3)$$
with \( t'_i = e^{\sqrt{2} \theta_{i-1} T_{n+1-i,n+2-i}} \). Because the generators \( T_A \) normalized as (3.1) satisfy the identities

\[
\begin{align*}
& e^{-\sqrt{2} \theta_{i-1} \theta} T_{i+1,j} e^{\sqrt{2} \theta_{i-1} \theta} = \cos \theta T_{i+1,j} + \sin \theta T_{i,j}, \\
& e^{-\sqrt{2} \theta_{i-1} \theta} T_{i,j} e^{\sqrt{2} \theta_{i-1} \theta} = \cos \theta T_{i,j} - \sin \theta T_{i+1,j}, \\
& e^{-\sqrt{2} \theta_{i-1} \theta} T_{m,n} e^{\sqrt{2} \theta_{i-1} \theta} = T_{m,n}, \quad \text{for other } T_{m,n}.
\end{align*}
\]  

(B.4)

All the matrices \( D^i \) are block-diagonal (i.e., \( d_i^2 = d_i^3 = 0 \)) except for \( D^n \) which has the form

\[
D^n =
\begin{bmatrix}
I_{(n-1)(n-2)/2} & 0 & 0 & 0 \\
0 & \cos \theta_{n-1} I_{n-1} & 0 & 0 \\
0 & 0 & -\sin \theta_{n-1} I_{n-1} & 0 \\
0 & 0 & 0 & \cos \theta_{n-1} I_{n-1}
\end{bmatrix}.
\]

(B.5)

Therefore, the matrix \( D \) can be written as

\[
D = D^1 D^2 \ldots D^{n-1} D^n
\]

\[
= \begin{bmatrix}
    d_1^1 & 0 & \ldots & 0 \\
    0 & d_1^2 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & d_4^n
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    d_1^1 d_1^{-1} d_1^m & \ldots & d_1^m d_1^{-1} d_2^m \\
    d_4^1 d_4^{-1} d_3^m & \ldots & d_4^m d_4^{-1} d_4^m
\end{bmatrix}
\]

\[
= \begin{bmatrix}
d_1 & d_2 \\
d_3 & d_4
\end{bmatrix}, \quad d_2 = d_1(d_1^n)^{-1} d_2^n, \quad d_3 = d_4(d_4^n)^{-1} d_3^n.
\]

Using the matrix (B.5), we can expression off-diagonal blocks with the diagonal blocks as

\[
d_2 = \tan \theta_{n-1} d_1 C_{n-1}, \quad d_3 = -\tan \theta_{n-1} d_4 C_{n-1}^T
\]

\[
C_{n-1} = \begin{bmatrix}
0 & 0 \\
0 & I_{n-1}
\end{bmatrix}_{a=n(n-1)/2}.
\]

(B.7)

For the diagonal blocks \( d_1 \) and \( d_4 \) we provide recursion relationships:

\[
d_4^{n+1} = d_4^n d_1^{n+1} \ldots d_4^{n-1} d_4^{n+1} d_4^n
\]

\[
= \begin{bmatrix}
    I_{n-2} & R(\theta_0) & \ldots & R(\theta_{n-2}) & I_{n-2} \\
    R(\theta_0) & I_{n-2} & \ldots & R(\theta_{n-2}) & 1 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    R(\theta_{n-2}) & I_{n-2} & \ldots & R(\theta_0) & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    1 & R(\theta_{n-2}) & \ldots & R(\theta_0)
\end{bmatrix}
\]

\[
R(\theta) = \begin{bmatrix}
    \cos \theta & -\sin \theta \\
    \sin \theta & \cos \theta
\end{bmatrix}.
\]

(B.8)
and
\[
d_1|_{n+1} = d_1|_{n+1}...d_{n-1}|_{n+1}d_1^n|_{n+1} = D_1|_{n}...D_{n-1}|_{n}d_1^n|_{n+1}
\]
\[
= D|_n d_1^n|_{n+1} = \begin{bmatrix} d_1|_n & d_2|_n \cos \theta_{n-1} \\ d_3|_n & d_4|_n \cos \theta_{n-1} \end{bmatrix}. 
\]

(B.9)

**The frames \(e_\alpha\)**

Let us study how the frames
\[
e_\alpha = L_\alpha - d_2^T(d_1^T - W_a)^{-1}L_a
\]
transform under (3.32). Recalling the identities (B.4), the left Maurer-Cartan forms can be calculated explicitly and the final expression can be written in a compact way
\[
L = \sqrt{2} \begin{bmatrix}
* & d\theta_{n-1} & \cdots & \cdots & d\theta_0 \sin \theta_{n-1}...\sin \theta_1 \\
& * & d\theta_2 \cos \theta_3 & d\theta_1 \cos \theta_3 \sin \theta_2 & d\theta_0 \cos \theta_3 \sin \theta_2 \sin \theta_1 \\
& & * & d\theta_1 \cos \theta_2 & d\theta_0 \cos \theta_2 \sin \theta_1 \\
& & & * & d\theta_0 \cos \theta_1 \\
& & & & *
\end{bmatrix}. 
\]

(B.11)

The expression means that the element of the matrix in the \((i,j)\) position is the element of \(L\), whose position in \(L\) is the same as the position of \(T_{i,j}\) in \(t_A\). So under the transformation (3.32) the forms transform as
\[
L_\alpha \rightarrow -L_\alpha, \quad L_a \rightarrow L_a. 
\]

(B.12)

Next we focus on the term \(W_ad_1\) in the combination
\[
(d_1 - W_a)^{-1}d_2 = [1 - (W_ad_1)^{-1}]^{-1}d_1^{-1}d_2 = \tan \theta_{n-1}[1 - (W_ad_1)^{-1}]^{-1}C_{n-1},
\]

in which we already have \(\tan \theta_{n-1} \rightarrow -\tan \theta_{n-1}\) under \(Z_2\). Using the recursion relation (B.8) and (B.9), one can get
\[
W_ad_1|_{n+1} = \begin{bmatrix} W_{a'}d_1|_n & W_{a'}d_2|_n \cos \theta_{n-1} \\ W_{a'}d_3|_n & W_{a'}d_4|_n \cos \theta_{n-1} \end{bmatrix}
\]

(B.14)

\[
W_\alpha d_4|_{n+1} = \begin{bmatrix} 1 & W_{a'}d_4|_n \cos \theta_{n-1} \end{bmatrix} \begin{bmatrix} \cos \theta_{n-2} & -\sin \theta_{n-2} \cos \theta_{n-1} \\ \sin \theta_{n-2} \cos \theta_{n-1} & \cos \theta_{n-2} \end{bmatrix} \begin{bmatrix} I_{n-2} \cos \theta_{n-2} \\ \cos \theta_{n-2} \end{bmatrix}.
\]

\[
\cos \theta_{n-1}d_2|_n = d_1|_n C_{n-2} \tan \theta_{n-2} \cos \theta_{n-1}, \quad \cos \theta_{n-1}d_3|_n = -\tan \theta_{n-2} \cos \theta_{n-1}d_4|_n C_{n-2}^T.
\]
where \( a \) is the index of \( so(n) \), \( a' \) is the index of \( so(n - 1) \) and \( \alpha' \) is the index of \( so(n) - so(n - 1) \). Observing that the (3.32) has the same form for a generic \( n \), below we will use an inductive proof to show that \( W_a d_1 \) is invariant under (3.32). Assume that \( W_a d_1 |_n \) and \( W_a d_4 |_n \cos \theta_{n-1} \) are invariant one can find that \( W_a d_1 |_{n+1} \) and \( W_a d_4 |_{n+1} \cos \theta_n \) are also invariant from their expressions. For \( n = 2 \), the result is simply proved by a direct substitution. Therefore, combining the transformations (B.12) we conclude under the \( Z_2 \) transformation, the frames transform as

\[
e^{-2\Phi} \rightarrow -e^{-2\Phi}.
\]

The matrix \( PP^T \)

Recall the definition of the matrix \( P \)

\[
P^{-T} = d_4 - d_3 (d_1 - W_a)^{-1} d_2 - \frac{1}{\lambda} W_\alpha.
\]

(B.16)

Using the identities \( DD^T = WW^T = I \), one can show

\[
\left( P^{-1} + \frac{W_\alpha}{\lambda} \right) \left( P^{-T} + \frac{W_\alpha^T}{\lambda} \right) = \left( P^{-T} + \frac{W_\alpha^T}{\lambda} \right) \left( P^{-1} + \frac{W_\alpha}{\lambda} \right) = 1.
\]

(B.17)

Therefore, we solve

\[
(P^T P)^{-1} = \left[ d_4^T - \frac{1}{\lambda} W_\alpha - d_2^T (d_1^T - W_\alpha)^{-1} d_3 \right] \left[ d_4 - \frac{1}{\lambda} W_\alpha^T - d_3 (d_1 - W_\alpha^T)^{-1} d_2 \right] = 1 + \frac{\lambda^2}{1} \left[ W_\alpha \left( P^{-T} + \frac{W_\alpha^T}{\lambda} \right) + \left( P^{-1} + \frac{W_\alpha}{\lambda} \right) W_\alpha^T \right] .
\]

(B.18)

Using the relations (B.9), the first term in the bracket can be rewritten as

\[
\frac{1}{\lambda} W_\alpha \left( P^{-T} + \frac{W_\alpha^T}{\lambda} \right) = \left[ \frac{1}{\lambda \cos \theta_n} \right] [W_\alpha d_4 |_{n+1} \cos \theta_n] \left[ 1 + \tan \theta_n C_n^T (d_1 - W_\alpha^T)^{-1} d_2 \right] .
\]

(B.19)

The three parts, \( ... \), \( ... \), \( ... \), as shown before are invariant, respectively under \( Z_2 \). We conclude that under the \( Z_2 \) transformation, the matrix \( PP^T \) is invariant.

The dilaton

The deformed dilaton is given by

\[
e^{-2\Phi} = e^{-2\Phi_0} \det(N) \det(W), \quad N = D - W \Omega.
\]

(B.20)
The determinant \( \det(W) = \pm 1 \) can be absorbed into \( \Phi_0 \) so we focus on \( \det(N) \). To compute this determinant we rewrite matrix \( N \) as

\[
N = \begin{bmatrix}
d_1 - W_a & d_2 \\
d_3 & d_4 - \lambda^{-1}W_a
\end{bmatrix} = \begin{bmatrix}
d_1 - W_a & 0 \\
d_3 & I
\end{bmatrix} \begin{bmatrix}
I & (d_1 - W_a)^{-1}d_2 \\
0 & P^{-T}
\end{bmatrix} \tag{B.21}
\]

Then

\[
\det(N) = \det(W_a) \det(1 - W_a d_1) \det[P^{-T}] \tag{B.22}
\]

The identity \((B.17)\) implies that the eigenvalues of \( P^{-1} \) are \( \pm 1 - W_a \lambda^{-1} \) so that \( \det[P^{-T}] \) is a constant and can be absorbed into \( \Phi_0 \). Therefore after absorbing all the constants we end up with

\[
e^{-2\Phi} = e^{-2\Phi_0} \det(1 - W_a d_1) \tag{B.23}
\]

Using the previous results in the appendix, we conclude that the dilaton does not change under \( Z_2 \).

To summarize, in this appendix by calculating the transformations of the frames \( e_\alpha \), the deformation matrix \( PP^T \) and \( \det(N) \) we have proved that deformed geometry is invariant under \((3.32)\).

## C Property of matrix \( Q \)

In this appendix we first derive the recursion relation for \( Q \), and then prove \( Q = Q^T \) by induction.

Recall the definition of the matrix \( Q \):

\[
Q|_{n+1} = d_4|_{n+1} - d_3|_{n+1}(d_1|_{n+1} - 1)^{-1}d_2|_{n+1}. \tag{C.1}
\]

Substituting \((B.7)\) into the expression gives

\[
Q|_{n+1} = d_4|_{n+1} \left[1 + \tan^2 \theta_{n-1} C^T_{n-1} C_{n-1} + \tan^2 \theta_{n-1} C^T_{n-1}(d_1|_{n+1} - 1)^{-1}C_{n-1}\right],
\]

\[
= d_4|_{n+1} \left(1 + \tan^2 \theta_{n-1} \begin{bmatrix}
0 & 0 \\
0 & (\cos \theta_{n-1} Q|_{n} - 1)^{-1} + 1
\end{bmatrix}\right). \tag{C.2}
\]

To prove \( Q = Q^T \) inductively, we first assume \( Q|i = Q^T|i, \forall i < n + 1 \), then \( Q|_n Q|_n^T = Q|_n^2 = I \) and the matrix inversion in \((C.2)\) can be computed explicitly as

\[
(\cos \theta_{n-1} Q|_{n} - 1)^{-1} = \frac{1 + \cos \theta_{n-1} Q|_{n}}{\sin^2 \theta_{n-1}}, \quad Q|_{n+1} = d_4|_{n+1} \begin{bmatrix}
1 & 0 \\
0 & \frac{Q|_n}{\cos \theta_{n-1}}
\end{bmatrix}. \tag{C.3}
\]
Applying the recursion relation (B.8) of \( d_4 \) we can cast \( Q_{|n+1} \) into

\[
Q_{|n+1} = \begin{bmatrix}
\cos \theta_{n-2} & [\sin \theta_{n-2}, 0, \ldots, 0] Q_{|n} \\
\sin \theta_{n-2} & 0 \\
& \vdots \\
d_4_{|n} & -\cos \theta_{n-2} d_4_{|n} K_n Q_{|n}
\end{bmatrix}
\equiv \begin{bmatrix}
q_1 & q_2 \\
q_3 & q_4
\end{bmatrix},
\]

\( K_n = \begin{bmatrix}
1 & 0 \\
0 & \frac{I_{n-2}}{\cos^2 \theta_{n-2}}
\end{bmatrix}.
\)

(C.4)

The equation (C.3) and \( Q_{|n} = Q^T_{|n} \) imply \((Q_{|n})_{i1} = (Q_{|n})_{1i} = (d_4_{|n})_{i1} \) so that \( q_2 = q_3 \).

Consider the analogue expression of (C.3) for \( Q_{|n} \):

\[
Q_{|n} = d_4_{|n} \begin{bmatrix}
1 & 0 \\
0 & -\frac{Q_{|n-1}}{\cos \theta_{n-2}}
\end{bmatrix}\equiv d_4_{|n} K'_n, \quad K'_n = K'^T_n. \tag{C.5}
\]

Multiplying \( K'_n \) on both sides the equation leads to

\[
Q_{|n} K'_n = d_4 K_n. \tag{C.6}
\]

Therefore we conclude \( q_4 \) and \( Q_{|n+1} \) are symmetric. The direct calculation shows

\[
Q_{|3} = \begin{bmatrix}
\cos \theta_0 & \sin \theta_0 \\
\sin \theta_0 & -\cos \theta_0
\end{bmatrix},
\]

\[
Q_{|4} = \begin{bmatrix}
\cos \theta_1 & \cos \theta_0 \sin \theta_1 \\
\sin \theta_0 \sin \theta_1 & \sin \theta_0 \sin \theta_1 \\
\sin \theta_0 \sin \theta_1 & -2 \cos \theta_0 \cos^2 \left(\frac{\theta_1}{2}\right) \sin \theta_0
\end{bmatrix},
\]

which are both symmetric then we finish our inductive proof.

From the definition of \( J \) (3.35), we can rewrite \( J \) by \( Q \),

\[
J = (\xi - 2Q)^{-1}(\xi Q - 2), \quad \xi = \frac{1 + \lambda^2}{\lambda}. \tag{C.8}
\]

Taking the derivative with respect to \( \xi \) leads to

\[
\frac{dJ}{d\xi} = -2(\xi - 2Q)^{-2}(Q^2 - 1) = 0. \tag{C.9}
\]

Here we have used the identity \( Q^2 = 1 \). Therefore, \( J \) does not depend on \( \lambda \).

Furthermore multiplying \((\xi - 2Q)\) on both sides gives

\[
\xi (J - Q) - 2(JQ - 1) = 0. \tag{C.10}
\]

Because both \( J \) and \( Q \) do not depend on \( \xi \), we conclude

\[
J = Q. \tag{C.11}
\]
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