K-theoretic boson–fermion correspondence and melting crystals

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\textbf{Abstract}

We study non-Hermitian integrable fermion and boson systems from the perspectives of Grothendieck polynomials. The models considered in this article are the five-vertex model as a fermion system and the non-Hermitian phase model as a boson system. Both models are characterized by different solutions satisfying the same Yang–Baxter relation. From our previous works on the identification between the wavefunctions of the five-vertex model and Grothendieck polynomials, we introduce skew Grothendieck polynomials and derive the addition theorem among them. Using these relations, we derive the wavefunctions of the non-Hermitian phase model as a determinant form, which can also be expressed as Grothendieck polynomials. Namely, we establish a K-theoretic boson–fermion correspondence at the level of wavefunctions. As a by-product, the partition function of the statistical mechanical model of a three-dimensional (3D) melting crystal is exactly calculated by use of the scalar products of the wavefunctions of the phase model. The resultant expression can be regarded as a K-theoretic generalization of the MacMahon function describing the generating function of the plane partitions, which interpolates the generating functions of two-dimensional (2D) and (3D) Young diagrams.

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(Some figures may appear in colour only in the online journal)
1. Introduction

Symmetric polynomials are the basic objects in representation theory, combinatorics, and related geometry. They also appear in mathematical physics, especially in integrable models. The most fundamental symmetric polynomials are the Schur polynomials, which appear as solutions of the KP hierarchy [1] and wavefunctions of the phase model [2–7], for example. They can also be used to construct a determinantal process named the Schur process [8], which has applications to the partition functions of topological strings [9], for example.

We recently extended the relation between the Schur polynomials and the integrable models and found that the wavefunctions of the one-parameter family of the integrable five-vertex models can be represented in Grothendieck polynomials [10]. The Grothendieck polynomials were originally introduced in the context of algebraic geometry [11–14] as a structure sheaf of the Schubert variety in the $K$-theory of flag varieties. By the identification of the wavefunctions with the Grothendieck polynomials for Grassmannian varieties, the determinant representations of the scalar product, which is the inner product between the wavefunctions, is nothing but the Cauchy identity for the Grothendieck polynomials. We also revealed the meaning of the orthogonality to show that the Grothendieck polynomial is a discrete orthogonal polynomial over the ‘Cassini oval’ [15], the solution curve of the Bethe equations.

The integrable five-vertex model is related to the non-Hermitian quantum integrable spin chain and the stochastic process called the totally asymmetric simple exclusion process (TASEP) [16]. The TASEP is a many-particle stochastic process with exclusion as an interaction that can be viewed as a natural generalization of the random walk. From the following perspectives, these models can be regarded as fermion systems. First, the space on which the Hamiltonian or the stochastic matrix acts is the tensor product of copies of two-dimensional (2D) space spanned by the empty state and particle-occupied state; i.e., double occupancy is forbidden. Second, the above models are in one-to-one correspondence with fermion systems through the Jordan–Wigner transformation. Finally, for the interaction-free case, the physical quantities, such as wavefunctions, and the number of configurations of stochastic particles, etc, are represented as the Schur polynomials, which can be described in terms of the formalism of the fermion and its Fock space.

In this paper, we study another type of integrable lattice model derived by a different solution, satisfying the same Yang–Baxter relation for the five-vertex model. The model discussed in this paper is a boson model called the non-Hermitian phase model [17], which is a one-parameter generalization of the phase model [18]. At a special point of the parameter, the non-Hermitian phase model describes the totally asymmetric zero range process (TAZRP), i.e., a stochastic process for a system of bosons in which, in contrast to the TASEP, the particles are allowed to occupy the same site. The wavefunctions of the phase model were shown to be expressed as the Schur polynomials [2]. In this sense, the phase model can be interpreted as a free fermion system. We show that the one-parameter family of the phase model corresponds to the generalization from the Schur polynomials to the Grothendieck polynomials. Namely, we show that the wavefunctions of the non-Hermitian phase model are nothing but the Grothendieck polynomials: we establish a $K$-theoretic boson–fermion correspondence at the level of the wavefunctions. We show this by introducing the skew Grothendieck polynomials and by deriving an addition theorem satisfied by the skew Grothendieck polynomials. These can be introduced in the context of the integrable five-vertex model naturally from the relation between the wavefunctions of the $N$-particle state and the $N$-variable Grothendieck polynomials (see [14, 19] for another definition introduced from
the perspectives of combinatorics). By this boson–fermion correspondence, the determinant representations of the scalar products and the summation of the wavefunctions follow from the Cauchy identity and the summation formula for the Grothendieck polynomials. The Cauchy identity [10] used in this paper is different from the dual Cauchy identity [20], which is the pairing between Grothendieck polynomials and dual Grothendieck polynomials. Our approach is based on the quantum inverse scattering method, which starts from the L-operator. There is another approach to the wavefunction from the coordinate Bethe ansatz [21], where the equivalence with the Grothendieck polynomials follows as a consequence.

As another application of the above-mentioned boson–fermion correspondence, we study the statistical mechanical model of a three-dimensional (3D) melting crystal. The model is in one-to-one correspondence with the plane partitions, which are regarded as a 3D extension of the Young diagrams. The partition function of the model becomes a generating function of the plane partitions. We show that the partition function can be exactly calculated by the scalar product of the non-Hermitian phase model. For a finite volume, the partition function can be given by a determinant form which reproduces MacMahon’s generating function [22] at a special point of the parameter. In the infinite volume limit, the partition function is explicitly given by an infinite product, which is regarded as a $K$-theoretic generalization of the MacMahon function [22]. The $K$-theoretic MacMahon function interpolates the ordinary MacMahon function and Euler’s generating function of partitions, namely, it unifies the generating functions of the 2D and 3D Young diagrams. Note that there are other types of 3D melting crystal models [23–26] whose constructions are based on connections with integrable models, such as the loop models related to the XXZ chain at the roots of unity and free fermion models, or connections with symmetric polynomials such as the Schur polynomials and their generalization to the Hall–Littlewood and the Macdonald polynomials. Our model is different from them and is based on the non-Hermitian integrable spin chain and phase model, whose wavefunctions are the Grothendieck polynomials. The directions of extending the Schur polynomials to the Grothendieck and Macdonald polynomials are different; hence the explicit forms of the corresponding skew polynomials and the weights assigned to each plane partition are totally different between the one in this paper and those in the literature. The Hall–Littlewood polynomials have representations in terms of vertex operators, and many properties including the connection with the melting crystal model can be treated in the same way for the Schur polynomials. However, there is no such vertex operator representation for the Grothendieck polynomials, and we approach the problem of construction by using the correspondence with the non-Hermitian integrable models.

This paper is organized as follows. In the next section, we review the relation between the wavefunctions of the integrable five-vertex model and the Grothendieck polynomials. In section 3, we introduce the skew Grothendieck polynomials and derive an addition theorem satisfied by them. In section 4, we introduce the non-Hermitian phase model, and show that the wavefunctions can be expressed as Grothendieck polynomials in section 5. In section 6, we discuss the melting crystal and derive the exact expressions of the partition function of the model. Section 7 is devoted to a summary and discussion.

2. Grothendieck polynomials and five-vertex models

In this section, we recall a relationship between Grothendieck polynomials and the integrable five-vertex model [10]. Utilizing this relation, in the next section we introduce skew Grothendieck polynomials, which play a key role in subsequent analysis.
Grothendieck polynomials were originally introduced as polynomial representatives of a structure sheaf of the Schubert variety in the $K$-theory of flag varieties [11]. The $\beta$-Grothendieck polynomials were introduced [12] to unify the original Grothendieck polynomials and the Schubert polynomials, which are structure sheaves for the $K$-theory ($\beta = -1$) and the cohomology ($\beta = 0$), respectively. For the case when the flag variety is a type A Grassmannian variety, the Grothendieck polynomials can be represented as the following determinant form [13], which we regard as the definition of the Grothendieck polynomials.

**Definition 2.1.** [11–13] The Grothendieck polynomials are defined as the following determinant:

$$G_\lambda(z_1, \ldots, z_N; \beta) = \frac{\det_{\lambda}(z_j^{k+N-k}(1 + \beta z_j)^{k-1})}{\prod_{1 \leq j < k \leq N}(z_j - z_k)}, \quad (2.1)$$

where $\{z_1, \ldots, z_N\}$ is a set of variables and $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a sequence of weakly decreasing nonnegative integers $\lambda_1 \geq \ldots \geq \lambda_N \geq 0$.

Note that for the case of cohomology $\beta = 0$, the $\beta$-Grothendieck polynomials are nothing but the Schur polynomials, which are Schubert polynomials for type A Grassmannian varieties.

In fact, the Grothendieck polynomials appear as wave-functions in the five-vertex model. The five-vertex model is a 2D statistical mechanical model whose Boltzmann weights are given by the elements of the $L$-operator [10] $L_{xj}(u) \in \text{End}(W_\alpha \otimes V_j)$:

$$L_{xj}(u) = u s_j s_j^+ + s_j^- s_j^+ + (\beta^{-1}u - u^{-1}) n_j n_j^+ - \beta^{-1}i n_j n_j^+ (u \in \mathbb{C}), \quad (2.2)$$

where $W_\alpha = \mathbb{C}^2$ (resp. $V_j = \mathbb{C}^2$) denotes the $\alpha$th auxiliary space (resp. $j$th quantum space) spanned by the empty state $|0\rangle_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_\alpha$ (resp. $|0\rangle_j = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_j$) and particle occupied state

Figure 1. The non-zero elements of the $L$-operator (2.2). The left (resp. up) arrow represents an auxiliary space (resp. a quantum space). The indices 0 or 1 on the left (resp. right) of the vertices denote the input (resp. output) states $|0\rangle$ (resp. $|1\rangle$) in the auxiliary space, while those on the bottom (resp. top) denote the input (resp. output) states in the quantum space. Note that the weights are invariant under a 180° rotation.

3 The five-vertex model in this paper is different from the one in [5]. The $R$-matrix in [5] satisfying the $RLL$ relation is essentially the trigonometric Felderhof model. The $R$-matrix in this paper is a special limit of the XXZ chain (the signs of weights when all spins are up and all spins are down are different for the trigonometric Felderhof model, and are the same for the XXZ chain), and the corresponding $L$-operators are different. For example, the configurations of the five-vertex models having nonzero weights are different, and the model in [5] cannot create either the $N$-particle state or its dual. The model in this paper can create both the $N$-particle state and its dual.
$|1\rangle_u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (resp. $|1\rangle_j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$). The parameter $\beta$ can be taken arbitrarily (the parameter $\alpha$ in [10] corresponds to $\beta$ as $\alpha = -\beta^{-1}$). See also figure 1 for a pictorial description of the $L$-operator (2.2). The above $L$-operator is given by a solution to the following Yang–Baxter relation ($RLL$-relation):

$$R_{ab}(u, v)L_{uj}(u)L_{jv}(v) = L_{jv}(v)L_{uj}(u)R_{ab}(u, v) \quad (2.3)$$

holding in $\text{End}(W_a \otimes W_b \otimes V_j)$ for arbitrary $u, v \in \mathbb{C}$. Here the matrix $R_{ab}(u, v) \in \text{End}(W_a \otimes W_b)$ is defined by:

$$R(u, v) = \begin{pmatrix}
    f(v, u) & 0 & 0 & 0 \\
    0 & 0 & g(v, u) & 0 \\
    0 & g(v, u) & 1 & 0 \\
    0 & 0 & 0 & f(v, u)
\end{pmatrix},$$

where $f(v, u) = \frac{u^2}{u^2 - v^2}$, $g(v, u) = \frac{uv}{u^2 - v^2}$.

which is a solution to the Yang-Baxter equation:

$$R_{ab}(u, v)R_{ac}(u, w)R_{bc}(v, w) = R_{bc}(v, w)R_{ac}(u, w)R_{ab}(u, v). \quad (2.5)$$

Let us define the monodromy matrix $T(u)$ as a product of $L$-operators:

$$T_u(u) = L_{aM}(u)\cdots L_{a1}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

acting on $W_a \otimes (V_1 \otimes \ldots \otimes V_N)$. Tracing out the auxiliary space, one obtains the transfer matrix $t(u) \in \text{End}(V^{\otimes M})$:

$$t(u) = \text{Tr}_W T_u(u) \quad (2.7)$$

which commutes for different spectral parameters: $[t(u), t(v)] = 0$. The quantum Hamiltonian corresponding to the five-vertex model is defined by $t(u)$:

$$H := \sum_{j=1}^{M} \left\{ -\beta^{-1}\sigma_j^+\sigma_{j+1}^- + \frac{1}{4}(\sigma_j^+\sigma_{j+1}^- - 1) \right\} = \sqrt{-\beta} \frac{\partial}{\partial u} \log \{u^{-M}t(u)\} \bigg|_{u=\sqrt{-\beta}}. \quad (2.8)$$

Note that the above Hamiltonian, in general, is non-Hermitian. For $\beta = -1$, the Hamiltonian corresponds to a stochastic matrix and describes a stochastic process called the TASEP.

The arbitrary $N$-particle state $|\psi(\{u\}_N)\rangle$ (resp. its dual $\langle \psi(\{u\}_N)|$) (not normalized) with $N$ spectral parameters $\{u\}_N = \{u_1, \ldots, u_N\}$ is constructed by a multiple action of the $B$ (resp. $C$) operator on the vacuum state $|\Omega\rangle := |0\rangle^M := |0\rangle_1 \otimes \ldots \otimes |0\rangle_M$ (resp. $\langle \Omega| := |0\rangle^M := \langle 0| \otimes \ldots \otimes |0\rangle_M$):

$$|\psi(\{u\}_N)\rangle = \prod_{j=1}^{N} B(u_j)|\Omega\rangle, \quad \langle \psi(\{u\}_N)| = \langle \Omega| \prod_{j=1}^{N} C(u_j). \quad (2.9)$$

In [10], we computed the overlap between the arbitrary off-shell $^4$ $N$-particle state $|\psi(\{u\}_N)\rangle$ and the (normalized) state with an arbitrary particle configuration $|\{x_1, \ldots, x_N\}\rangle$ ($x_1 < \ldots < x_N$),

$^4$ The terminology ‘off-shell’ means that the set of parameters $\{u\}_N$ is arbitrary. On the other hand, ‘on-shell’ means that $\{u\}_N$ is taken so that the $N$-particle state $|\psi(\{u\}_N)\rangle$ is one of the eigenstate of the Hamiltonian.
where $x_j$ denotes the positions of the particles. The wavefunction $\langle x_1, \ldots, x_N | \psi (\{ u \}_N) \rangle$ and its dual $\langle \psi (\{ u \}_N) | x_1, \ldots, x_N \rangle$ were found to be given by the Grothendieck polynomials.

**Theorem 2.2.** [10] The (off-shell) wavefunction and its dual wavefunction of the integrable five-vertex model are, respectively, given by the Grothendieck polynomials as:

$$\langle x_1, \ldots, x_N | \psi (\{ u \}_N) \rangle = (-\beta^{-1})^{(N(N-1)/2)} \prod_{j=1}^{N} u_j^{M-1} G_{\lambda}(z_1, \ldots, z_N; \beta), \quad (2.10)$$

$$\langle \psi (\{ u \}_N) | x_1, \ldots, x_N \rangle = (-\beta^{-1})^{(N(N-1)/2)} \prod_{j=1}^{N} u_j^{M-1} G_{\lambda'}(z_1, \ldots, z_N; \beta), \quad (2.11)$$

where $z_j = -\beta^{-1} - u_j^{-2}$, and $\lambda = (\lambda_1, \ldots, \lambda_N)$ ($M - N \geq \lambda_1 \geq \cdots \geq \lambda_N \geq 0$) and $\lambda' = (\lambda'_1, \ldots, \lambda'_N)$ ($M - N \geq \lambda'_1 \geq \cdots \geq \lambda'_N \geq 0$) are the Young diagrams related to the particle configuration $x = (x_1, \ldots, x_N)$ as $x_j = x_{N-j+1} - N + j - 1$ and $\lambda'_j = M - N + j - x_j$, respectively.

Note that the Young diagram $\lambda'$ is the complementary part of the Young diagram $\lambda$ in the $N \times (M - N)$ rectangular Young diagram. Let $x_j'$ be the particle configuration given by $x_j' = \lambda'_{N-j+1} + j$. The particle configurations $x = (x_1, \ldots, x_N)$ ($x_j = \lambda_{N-j+1} + j$) and $x' = (x'_1, \ldots, x'_N)$ for given $\lambda$ are connected by the relation:

$$\langle x'_1, \ldots, x'_N \rangle = (M - x_N + 1, \ldots, M - x_1 + 1)$$

for $x_j = \lambda_{N-j+1} + j$ and $x_j' = \lambda'_{N-j+1} + j$. (2.12)

In figure 2, we denote an example of $\lambda$ and $\lambda'$ together with the corresponding particle configurations $x_1 = \lambda_{N-j+1} + j$ and $x'_j = \lambda'_{N-j+1} + j$. From this, one can intuitively find that the positions of the particles corresponding to $\lambda'$ are related to those corresponding to $\lambda$ after a $180^\circ$ rotation.
The graphical description of the wavefunction \( \psi \langle \{ u \}_M \rangle \) (2.10) is also depicted in figure 3. Due to the invariance of the Boltzmann weights under a 180° rotation and the commutativity of the \( B \)-operators and \( C \)-operators \([B(u), B(v)] = 0 \) and \([C(u), C(v)] = 0 \), the graphical description of the wavefunction is also invariant under the rotation. One easily finds that the rotated graph corresponds to the dual wavefunction \( \psi \langle \{ x \}_N \rangle \), where the positions of the particles \( x_j \) are given by (2.12).

After transforming \( \lambda \rightarrow \lambda' \), which corresponds to the transformation \( \lambda \rightarrow \lambda' \), one finds (2.11) is valid if (2.10) holds.

One can show the following Cauchy identity holds for the Grothendieck polynomials, which are obtained by comparing the determinant representations for the scalar product \( \langle \psi (\{ u \}_N) \rangle \) and \( \langle \psi (\{ u \}_N) \rangle \) to that obtained by multiplying (2.10) by (2.11) and then by summing over all possible configurations.

**Theorem 2.3.** \([10]\) The following Cauchy identity for the Grothendieck polynomials holds:

\[
\sum_{\lambda \subseteq \lambda'}\prod_{1 \leq j < k \leq N} \frac{1}{(z_j - z_k)(w_k - w_j)} \prod_{k \in L} \det_N \left[ \frac{z_j^{L+N}(1 + \beta w_k)^N - w_k^{L+N}(1 + \beta z_j)^N}{z_j - w_k} \right] G_{\lambda'}(z_1, \ldots, z_N; \beta) G_{\lambda'}(w_1, \ldots, w_N; \beta)
\]

where the Young diagram \( \lambda' = (\lambda'_1, \ldots, \lambda'_N) \) is given by the Young diagram \( \lambda = (\lambda_1, \ldots, \lambda_N) \) as \( \lambda'_j = L - \lambda_{N+1-j} \).
Here we have set $L = M = N$, but the above formula holds for any $L \geq 0$. As a limiting case of the Cauchy identity, we have also derived the summation formula for the Grothendieck polynomials.

**Theorem 2.4.** [10] The following summation for the Grothendieck polynomials holds:

$$
\sum_{\lambda \subseteq L^N} (-\beta)^{\lambda_0} G_\lambda(z_1, \ldots, z_N; \beta) = \prod_{1 \leq j < k \leq N} \frac{1}{z_k - z_j} \det V
$$

with an $N \times N$ matrix $V$ whose matrix elements are:

$$
V_{jk} = \sum_{m=0}^{j-1} (-1)^m (-\beta)^{j-N} \binom{L + N}{m} (1 + \beta z_k)^{m-j+N-1} (1 \leq j \leq N - 1),
$$

$$
V_{Nk} = -\sum_{m=\max(N-1,1)}^{L+N} (-1)^m \binom{L + N}{m} (1 + \beta z_k)^{m-1}.
$$

Let us comment on a possible extension of the method to another integrable model also defined by (2.3). In fact, for a given $R$-matrix (2.4), the $L$-operator (2.2) is not the unique solution to (2.3). Indeed, the non-Hermitian phase model discussed in section 4 is constructed by another solution to (2.3). As mentioned before, the quantum space on which the $L$-operator (2.2) acts is the tensor product of copies of 2D space spanned by the empty state $|0\rangle$ and particle-occupied states $|1\rangle$; i.e., double occupancy is forbidden. In this sense, the corresponding quantum system (2.8) is interpreted as a fermion system. (More precisely, there exists a one-to-one correspondence between the spin system (2.8) and a fermion system through the Jordan–Wigner transformation.) On the other hand, the phase model (see (4.4) and (4.5) in section 4) is a boson system: the quantum space is defined as the tensor product of bosonic Fock spaces, whose dimension is infinite. At first glance it seems there is little connection between the fermion system (2.8) and the bosonic system (4.4), but by definition the algebraic relations of the both $B$-operators (or $C$-operators) constructing the $N$-particle states are completely the same. Moreover, as shown later, the $N$-particle states for the phase model can be uniquely mapped to those for a fermion model (2.8) and vice versa. These observations intuitively indicate that there is a close correspondence between the fermion (2.8) and the boson (4.4) models. This intuition is true. Indeed, the wavefunctions for both models can be given by the Grothendieck polynomials. To show this, first we give an addition theorem satisfied by the Grothendieck polynomials, introducing the skew Grothendieck polynomials.

### 3. Skew Grothendieck polynomials and addition theorem

The relations (2.10) and (2.11) between the wavefunctions and the Grothendieck polynomials lead us to the natural definition of the single variable skew Grothendieck polynomials.

**Definition 3.1.** [19] The single-variable skew Grothendieck polynomial is defined in terms of the $B$-operator of the five-vertex model:
\[ \beta = \cdots - \cdots - \mu \lambda \]

\[ G_{\beta \lambda}(z; \beta) : = \left\{ y_1 \cdots y_{N+1} \right\} \left( -\beta \right)^{N+M} B(u) \left\{ x_1 \cdots x_N \right\}, \] (3.1)

where \( z = \beta^2 - u^2 \), and \( \lambda = (\lambda_1, \ldots, \lambda_N) \) \((M - N \geq \lambda_1 \geq \cdots \geq \lambda_N \geq 0)\) and \( \mu = (\mu_1, \ldots, \mu_{N+1}) \) \((M - N - 1 \geq \mu_1 \geq \cdots \geq \mu_{N+1} \geq 0)\) are the Young diagrams related to the particle configurations \( x = (x_1, \ldots, x_N) \) \((x_j = \lambda_{N-j+1} + j)\) and \( y = (y_1, \ldots, y_{N+1}) \) \((y_j = \mu_{N-j+2} + j)\), respectively.

We shall see later that this is a natural extension of the skew Schur polynomials.

**Proposition 3.2.** The skew Grothendieck polynomial \( G_{\beta \lambda}(z; \beta) \) defined in (3.1) can be given in terms of the C-operator:

\[ G_{\beta \lambda}(z; \beta) = \left\{ x_1 \cdots x_N \right\} \left( -\beta \right)^{N+M} C(u) \left\{ y_1 \cdots y_{N+1} \right\}, \] (3.2)

or equivalently,

\[ G_{\mu^* \lambda^*}(z; \beta) = \left\{ x_1 \cdots x_N \right\} \left( -\beta \right)^{N+M} C(u) \left\{ y_1 \cdots y_{N+1} \right\}, \] (3.3)

where the particle positions \( x^\vee = (x_1^\vee, \ldots, x_N^\vee) \) and \( y^\vee = (y_1^\vee, \ldots, y_{N+1}^\vee) \) are, respectively, defined as \( x_j^\vee = \lambda_{N-j+1}^\vee + j \) and \( y_j^\vee = \mu_{N-j+2}^\vee + j \).

**Proof.** The graphical argument is useful to show (3.2) and (3.3). As discussed in the wavefunctions (see below theorem 2.2 and figure 3), the definition (3.1) is invariant under a 180° rotation. The rotated graph corresponds to \( \left\{ x_1^\vee \cdots x_N^\vee \right\} \left( -\beta \right)^{N+M} C(u) \left\{ y_1^\vee \cdots y_{N+1}^\vee \right\} \). Thus we have (3.2). Transforming the variables as \( x_j^\vee \rightarrow x_j = \lambda_{N-j+1} + j \) \((1 \leq j \leq N)\) and \( y_j^\vee \rightarrow y_j = \mu_{N-j+2} + j \) \((1 \leq j \leq N + 1)\), which, respectively, correspond to the transformations \( \lambda \rightarrow \lambda^\vee \) and \( \mu \rightarrow \mu^\vee \), one obtains (3.3).

Let us define the ordering on the Young diagrams for later purposes. \( \square \)
Definition 3.3. For two Young diagrams \( \mu = (\mu_1, \mu_2, \ldots, \mu_N) \) and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \), we say that \( \mu \) and \( \lambda \) interlace, if and only if
\[
\mu_j \geq \lambda_j \geq \mu_{j+1} \quad (j = 1, \ldots, N),
\]
and write this relation as \( \mu \triangleright \lambda \). Correspondingly, we write \( \triangleright \) for the particle configurations
\[
\mu = (\mu_1, \ldots, \mu_N), \quad \lambda = (\lambda_1, \ldots, \lambda_N),
\]
if and only if \( \mu \triangleright \lambda \) holds. In Figure 4 (resp. Figure 5), an example of the interlacing (resp. non-interlacing) partitions and the corresponding particle configurations is depicted. It immediately follows that:
\[
\mu \triangleright \lambda \iff \triangleright \cup \triangleright . \quad (3.4)
\]

The single variable skew Grothendieck polynomials (3.1) are given by the following explicit expression.

Proposition 3.4. The single variable skew Grothendieck polynomials \( G_{\mu/\lambda}(z; \beta) \) can be explicitly expressed as:
\[
G_{\mu/\lambda}(z; \beta) = \begin{cases} 
\sum_{j=1}^{N} \prod_{j=1}^{N} \left( 1 + \beta z - \beta z \delta_{\mu_{j+1}} \lambda_j \right) & \mu \triangleright \lambda \\
0 & \text{otherwise}
\end{cases} 
\quad (3.5)
\]
The case \( \beta = 0 \) reduces to the single variable skew Schur polynomials: \( G_{\mu/\lambda}(z; 0) = s_{\mu/\lambda}(z) = z^{\mu_1} \sum_{j=1}^{N} \delta_{\mu_j} \lambda_j \).

Proof. We show (3.5) by explicit evaluation of the definition (3.1). From the graphical description (see Figure 5, for instance), we find \( \langle y_1 \cdots y_N | B(x) | x_1 \cdots x_N \rangle = 0 \) for \( \mu \not\triangleright \lambda \). Thus \( G_{\mu/\lambda}(z; \beta) = 0 \) holds for \( \mu \not\triangleright \lambda \). The first equality in (3.5) can be shown by the following decomposition:
\[
\prod_{i=x_j+1}^{y_j} L_{ij}(u) |1\rangle_0 \otimes \left\{ \prod_{k=x_{j-1}+1}^{y_{j-1}-1} \left\{ \otimes_{k=x_{j-1}+1}^{y_{j-1}-1} [0]_k \right\} \right\} \otimes |1\rangle_{x_j} = \\
= \sum_{x_j+1 \leq y_j \leq x_j+1} \left[ \prod_{i=x_j+1}^{y_j} L_{ij}(u) |1\rangle_0 \otimes \left\{ \prod_{k=x_{j-1}+1}^{y_{j-1}-1} \left\{ \otimes_{k=x_{j-1}+1}^{y_{j-1}-1} [0]_k \right\} \right\} \right],
\]

(b) \[\prod_{i=x_0+1}^{y_0} L_{ij}(u) |1\rangle_0 \otimes \left\{ \prod_{k=x_{N-1}+1}^{y_{N-1}} \left\{ \otimes_{k=x_{N-1}+1}^{y_{N-1}} [0]_k \right\} \right\} = \sum_{x_0+1 \leq y_0 \leq x_0+1} \left[ \prod_{i=x_0+1}^{y_0} L_{ij}(u) |1\rangle_0 \otimes \left\{ \prod_{k=x_{N-1}+1}^{y_{N-1}} \left\{ \otimes_{k=x_{N-1}+1}^{y_{N-1}} [0]_k \right\} \right\} \right].

Figure 6. (a) The graphical description of (3.7). The first term on the right-hand side vanishes because the Boltzmann weight surrounded by the broken line is equal to zero. The insertion of the weights shown in figure 1 into the second term yields (3.7). (b) The graphical description of (3.8).

\[
B(u) |x_1 \ldots x_N\rangle = \prod_{j=1}^{N+1} \prod_{i=x_{j-1}+1}^{x_j} \left\{ \otimes_{k=x_{j-1}+1}^{y_{j-1}-1} \left\{ \otimes_{k=x_{j-1}+1}^{y_{j-1}-1} [0]_k \right\} \right\} \otimes |1\rangle_{x_j},
\]

where \(x_0 = 0\) and \(x_{N+1} = M\). Using the following relations,

\[
\prod_{i=x_{j-1}+1}^{y_{j-1}-1} L_{ij}(u) |1\rangle_0 \otimes \left\{ \otimes_{k=x_{j-1}+1}^{y_{j-1}-1} [0]_k \right\} \otimes |1\rangle_{x_j} = \\
= \sum_{x_{j-1}+1 \leq y_{j-1} \leq x_{j-1}+1} \left[ \prod_{i=x_{j-1}+1}^{y_{j-1}-1} \left\{ \otimes_{k=x_{j-1}+1}^{y_{j-1}-1} [0]_k \right\} \right] \otimes |1\rangle_{y_{j-1}},
\]

\[
= \sum_{x_{j-1}+1 \leq y_{j-1} \leq x_{j-1}+1} \left[ \prod_{i=x_{j-1}+1}^{y_{j-1}-1} \left\{ \otimes_{k=x_{j-1}+1}^{y_{j-1}-1} [0]_k \right\} \right] \otimes |1\rangle_{y_{j-1}},
\]

which are directly obtained by the graphical representation shown in figure 6, we find that (3.6) yields:
\[ B(u)|v_{1}\ldots v_{N}\rangle = \sum_{\gamma \geq \alpha} (-\beta^{-1}u - u^{-1})^{-1-N+\sum_{j=1}^{N+1} \gamma_j - \sum_{j=N+2}^{N+1} \delta_{\gamma_j} u^{\gamma_j-1} \sum_{j=1}^{N+1} \delta_{\gamma_j} u^{\gamma_j-1}) |v_{1}\ldots v_{N+1}\rangle. \tag{3.9} \]

Translating (3.9) into the language of Young diagrams by \( x_j = \lambda_{N+1+j} + j \) and \( y_j = \lambda_{N+2+j} + j \), and using \( y > x \Leftrightarrow \mu > \lambda \), we have the first equality in (3.5).

Combining the relation between the wavefunction and the Grothendieck polynomial (2.10) and the definition of the skew Grothendieck polynomials (3.1), we have the following addition theorem.

**Theorem 3.5.** The following relation between the Grothendieck and skew Grothendieck polynomials holds:

\[ G_{\mu}(z_1,\ldots,z_{N+1}; \beta) = \sum_{\mu > \lambda} G_{\mu/\lambda}(z_{N+1}; \beta) G_{\lambda}(z_1,\ldots,z_{N}; \beta), \tag{3.10} \]

which recovers that for the Schur and skew Schur polynomials at \( \beta = 0 \).

**Proof.** This follows from evaluating the \((N+1)\)-particle state using (2.10) and (3.1) as:

\[ (-\beta)^{(N+1)/2} \prod_{j=1}^{N+1} u_j^{-1} B(u_j)|\Omega\rangle = (-\beta)^{N} u_{N+1}^{-1} B(u_{N+1}) \]

\[ \times (-\beta)^{(N-1)/2} \prod_{j=1}^{N} u_j^{-1} B(u_j)|\Omega\rangle \]

\[ = (-\beta)^{N} u_{N+1}^{-1} B(u_{N+1}) \sum_{\lambda} G_{\lambda}(z_1,\ldots,z_{N}; \beta) |x_1(\lambda)\ldots x_{N}(\lambda)\rangle \]

\[ = \sum_{\mu > \lambda} G_{\mu/\lambda}(z_{N+1}; \beta) G_{\lambda}(z_1,\ldots,z_{N}; \beta) |v_{1}(\mu)\ldots v_{N+1}(\mu)\rangle. \tag{3.11} \]

and comparing with:

\[ (-\beta)^{(N+1)/2} \prod_{j=1}^{N+1} u_j^{-1} B(u_j)|\Omega\rangle = \sum_{\mu} G_{\mu}(z_1,\ldots,z_{N+1}; \beta) |v_{1}(\mu)\ldots v_{N+1}(\mu)\rangle. \tag{3.12} \]

Note that \( \lambda \) in the summation (3.11) can be restricted from \( \lambda \subseteq (M - N)^{N} \) to \( \lambda \subseteq (M - N - 1)^{N+1} \) since \( \mu \subseteq (M - N - 1)^{N+1} \) and \( G_{\mu/\lambda}(z; \beta) = 0 \) unless \( \mu > \lambda \).

The relation (3.10) is the consequence of the action of a \( B \)-operator on the wavefunction of the \( N \)-particle state, from which also justifies the definition (3.1) of the skew Grothendieck polynomials. In the next section, we use this addition theorem to show that the wavefunction of the non-Hermitian phase model can also be expressed as Grothendieck polynomials.

The repeated application of the addition theorem (3.10) leads to the following corollary.

**Corollary 3.6.** The Grothendieck polynomials can be expressed in terms of the single-variable skew Grothendieck polynomials as:
Before closing this section, we define the multivariable skew Grothendieck polynomials for completeness of the paper. The multivariable skew Grothendieck is naturally defined by multiplying the single-variable skew Grothendieck polynomials.

**Definition 3.7.** The multivariable skew Grothendieck polynomials are defined as:

\[
G_{\lambda}(z_1, \ldots, z_n; \beta) := \prod_{\lambda=\lambda^{(0)} > \ldots > \lambda^{(0)}} G_{\lambda_j}(z_j; \beta)
\]

where \( \lambda = \lambda^{(0)} \) and \( \nu = \lambda^{(n)} \).

The combination of corollary 3.6 and theorem 3.5 leads to the following addition theorem.

**Theorem 3.8.** The following relation between the Grothendieck polynomials and the (multivariable) skew Grothendieck polynomials holds:

\[
G_{\lambda_{jk}}(z_1, \ldots, z_n; \beta) = \sum_{\nu} G_{\lambda_{jk}}(z_1, \ldots, z_n; \beta) G_{\nu}(w_1, \ldots, w_m; \beta).
\]

which recovers that for the Schur and skew Schur polynomials at \( \beta = 0 \).

4. Non-Hermitian phase model

In this section, we introduce the non-Hermitian phase model [17], which can be solved by the algebraic Bethe ansatz. The phase model is a boson system characterized by the generators \( \phi, \phi^\dagger, N, \) and \( \pi \) acting on a bosonic Fock space \( \mathcal{F} \) spanned by orthonormal basis \( |n\rangle \) \((n = 0, 1, \ldots, \infty)\). Here the number \( n \) indicates the occupation number of bosons. The generators \( \phi, \phi^\dagger, N, \) and \( \pi \) are, respectively, the annihilation, creation, number, and vacuum projection operators, whose actions on \( \mathcal{F} \) are, respectively, defined as:

\[
\phi|0\rangle = 0, \quad \phi|n\rangle = |n - 1\rangle, \quad \phi^\dagger|n\rangle = |n + 1\rangle,
\]

\[
N|n\rangle = n|n\rangle, \quad \pi|n\rangle = \delta_{n,0}|n\rangle.
\]

Thus the operator forms are explicitly given by:

\[
\phi = \sum_{n=0}^{\infty} |n\rangle \langle n + 1|,
\]

\[
\phi^\dagger = \sum_{n=0}^{\infty} |n + 1\rangle \langle n|,
\]

\[
N = \sum_{n=0}^{\infty} n|n\rangle \langle n|, \quad \pi = |0\rangle \langle 0|.
\]
These operators generate an algebra referred to as the phase algebra:

\[ [\phi_j, \phi_k^\dagger] = \pi, \quad [N_j, \phi_k] = -\phi_k, \quad [N_j, \phi_k^\dagger] = \phi_k^\dagger. \quad (4.3) \]

The non-Hermitian phase model \([17, 31]\) under the periodic boundary condition is defined by the following Hamiltonian:

\[ H = \sum_{j=0}^{M-1} \left( \phi_j^\dagger \phi_j - \beta \pi_j \right). \quad (4.4) \]

The Hamiltonian acts on the tensor product of Fock spaces, \( \bigotimes_{j=0}^{M-1} \mathcal{F}_j \), whose basis is given by \( |n\rangle_{\mathcal{M}} := \bigotimes_{j=0}^{M-1} |n_j\rangle \), \( n_j = 0, 1, \ldots, \infty \). We denote a dual state of \( |n\rangle_{\mathcal{M}} \) as \( \langle n |_{\mathcal{M}} := \bigotimes_{j=0}^{M-1} \langle n_j | \). The operators \( \phi_j, \phi_j^\dagger, N_j \) and \( \pi_j \) act on the Fock space \( \mathcal{F}_j \) as \( \phi, \phi^\dagger, N \) and \( \pi \), and the other Fock spaces \( \mathcal{F}_k, k \neq j \) as an identity. The term including \( \beta \) in (4.4) denotes an on-site interaction: \( \beta > 0 \), \( \beta = 0 \), and \( \beta < 0 \) correspond to repulsive, free, and attractive interactions, respectively.

The Hamiltonian is quantum integrable, and a special point \( \beta = -1 \) describes a stochastic process without exclusion called the TAZRP, i.e., a stochastic process for a system of bosons so that each site can be occupied by an arbitrary number of particles, which is in contrast to the TASEP, where each site can be occupied by at most one particle.

We can make an analysis on the non-Hermitian phase model by the quantum inverse scattering method. The basic object is the following \( L \)-operator:

\[ \mathcal{L}_{aj}(v) = \begin{pmatrix} v^{-1} - \beta \pi_j & \phi_j^\dagger \\ \phi_j & v \end{pmatrix}. \quad (4.5) \]

acting on the tensor product \( W_j \otimes \mathcal{F}_j \) of the complex 2D space \( W_a \) and the Fock space at the \( j \)th site \( \mathcal{F}_j \). See also figure 7 for a pictorial representation of the \( L \)-operator (4.5), which allows for an intuitive understanding of the subsequent calculations.
The $L$-operator satisfies the intertwining relation (RLL-relation):
\[ R_{ab}(u, v) L_{aj}(u) L_{bj}(v) = L_{bj}(v) L_{aj}(u) R_{ab}(u, v), \]  
which acts on $W_a \otimes W_j$. The $R$ matrix $R(u, v)$ is the same as the one for the integrable five-vertex model (2.4). The auxiliary space $W_a$ is the complex 2D space, which is the same as that for the integrable five-vertex model, while the quantum space $F_j$ is the infinite-dimensional bosonic Fock space.

From the $L$-operator, we construct the monodromy matrix:
\[ T_a(v) = L_{aM-1}(v) \cdots L_{a0}(v) = \begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix}, \]  
which acts on $W_a \otimes (F_j \otimes \cdots \otimes F_{M-1})$. Tracing out the auxiliary space, one defines the transfer matrix $\tau(u) \in \text{End}(F^\otimes M)$:
\[ \tau(v) = \text{Tr}_a T_a(v). \]  

The repeated applications of the RLL-relation leads to the intertwining relation:
\[ R_{ab}(u, v) T_a(u) T_b(v) = T_b(v) T_a(u) R_{ab}(u, v). \]  

Some elements of the relation (4.9) are:
\[ C(u)B(v) = g(u, v) [A(u)D(v) - A(v)D(u)], \]
\[ A(u)B(v) = f(u, v) B(v) A(u) + g(v, v) B(u) A(v), \]
\[ D(u)B(v) = f(u, v) B(v) D(u) + g(u, v) B(u) D(v), \]
\[ [B(u), B(v)] = [C(u), C(v)] = 0. \]  

The above relations are completely the same as those satisfied by $A(u)$, $B(u)$, $C(u)$, and $D(u)$ for the integrable five-vertex model, since the RLL-relation (4.6) is the same as (2.3).

Thanks to the RIT-relation (4.9), the transfer matrix $\tau(u)$ mutually commutes; i.e.,
\[ [\tau(u), \tau(v)] = 0. \]  

The Hamiltonian can be obtained by the derivative of the transfer matrix with respect to the spectral parameter:
\[ H = \frac{\partial}{\partial v^2} (v^n \tau(v)) \bigg|_{v=0}. \]  

The arbitrary $N$-particle state $|\Psi\rangle (\{v\}_N)$ (resp. its dual $\langle \Psi| (\{v\}_N)$) (not normalized) with $N$ spectral parameters $\{v\}_N = \{v_1, \ldots, v_N\}$ is constructed by a multiple action of a $B$ (resp. $C$) operator on the vacuum state $|\Omega\rangle := |0\rangle_M \otimes |0\rangle_0 \otimes \cdots \otimes |0\rangle_{M-1}$ (resp. $\langle \Omega| := \langle 0|_M \otimes |0\rangle_0 \otimes \cdots \otimes |0\rangle_{M-1}$):
\[ |\Psi\rangle (\{v\}_N) = \prod_{j=1}^N B(v_j)|\Omega\rangle, \quad \langle \Psi| (\{v\}_N) = \langle \Omega| \prod_{j=1}^N C(v_j). \]  

The $N$-particle state $|\Psi\rangle (\{v\}_N)$ and its dual $\langle \Psi| (\{v\}_N)$ become an eigenvector of the transfer matrix with the eigenvalue:
\[ \tau(v) = (v^{-1} - \beta v)^M \prod_{k=1}^N \frac{v_k^2}{v_k^2 - v^2} + v^M \prod_{k=1}^N \frac{v^2}{v^2 - v_k^2}. \]
if the spectral parameters \( \{v\}_N \) satisfy the Bethe ansatz equation:

\[
\left( v_j^{-2} - \beta \right)^M = (-1)^{N-1} \prod_{k=1}^{N} \frac{v_j^2}{v_k^2}.
\]

The eigenvalue of the Hamiltonian is given by:

\[
E = -\beta M + \sum_{j=1}^{N} v_j^{-2}.
\]

### 5. Wavefunctions and scalar products

Here and in what follows, we consider the arbitrary off-shell state; i.e., the parameters \( \{v\}_N \) in the \( N \)-particle state (4.13) are arbitrary. The orthonormal basis of the \( N \)-particle state \( |\{v\}_N\rangle \) and its dual \( \langle \{v\}_N| \) are given by \( |\{n\}_M\rangle := |n_0\rangle \otimes \cdots \otimes |n_{M-1}\rangle \) and \( \langle \{n\}_M| := \langle n_0| \otimes \cdots \otimes \langle n_{M-1}| \), where \( n_0 + n_1 + \cdots + n_{M-1} = N \). The wavefunctions can be expanded in this basis as:

\[
|\{v\}_N\rangle = \sum_{0 \leq n_0, \ldots, n_{M-1} \leq N \atop n_0 + \cdots + n_{M-1} = N} \langle \{n\}_M| \langle \{v\}_N| \langle \{n\}_M\rangle.
\]

There is a one-to-one correspondence between the set \( |\{n\}_M\rangle = |n_0, n_1, \ldots, n_{M-1}\rangle \) (\( n_0 + n_1 + \cdots + n_{M-1} = N \)) and the Young diagram \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \) (\( M - 1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0 \)). Namely, each Young diagram \( \lambda \) under the constraint \( \ell(\lambda) \leq N \), \( \lambda_1 \leq M - 1 \) can be labeled by a set of integers \( \{n\}_M \) as
\( \lambda = (6, 5^3, 2^2, 0) \) and \( \mu = (5^3, 2^2, 1) \). The skew Young diagram \( \mu/\lambda \) is depicted as the gray boxes. The input (resp. output) state denotes the particle configuration corresponding to \( \lambda \) (resp. \( \mu \)). The particle configurations are admissible for the interlacing partitions. For admissible configurations \( \{m\}_{M,N+1} \triangleright \{n\}_{M,N} \), the matrix element \( \langle m \rangle_{M,N+1} \triangleright \langle n \rangle_{M,N} \rangle \) is non-zero for the generic value of \( v \).

**Figure 9.** An example of the interlacing partition functions \( \mu > \lambda \). Here we have set \( \mu = (6, 5^3, 2^2, 0) \) and \( \lambda = (5^3, 2^2, 1) \). The skew Young diagram \( \mu/\lambda \) is depicted as the gray boxes. The input (resp. output) state denotes the particle configuration corresponding to \( \lambda \) (resp. \( \mu \)). The particle configurations are admissible for the interlacing partitions. For admissible configurations \( \{m\}_{M,N+1} \triangleright \{n\}_{M,N} \), the matrix element \( \langle m \rangle_{M,N+1} \triangleright \langle n \rangle_{M,N} \rangle \) is non-zero for the generic value of \( v \).
In figure 9 (resp. figure 10), an example of the interlacing (resp. non-interlacing) partitions and the corresponding admissible (resp. non-admissible) particle configurations are depicted.

We show the wavefunctions $\Psi_{\{n\}|\{m\}}$ and their dual $\Psi_{\{m\}|\{n\}}$ can be represented in the following determinant forms, which are parametrized by Young diagrams.

**Theorem 5.3.** The wavefunctions can be expressed by the Grothendieck polynomials as:

$$\langle \{m\}_{M,N+1} | B(v) | \{n\}_{M,N} \rangle = 1$$

Figure 10. An example of the non-interlacing partition functions $\mu \not\preceq \lambda$. Here we have set $\mu = (6,5^2,2^2,0)$ and $\lambda = (5^2,3^2,1)$. For non-interlacing partitions, the particle configurations are not admissible. For non-admissible configurations $\{m\}_{M,N+1} \not\preceq \{n\}_{M,N}$, one sees $\langle \{m\}_{M,N+1} | B(v) | \{n\}_{M,N} \rangle = 0$.

In figure 9 (resp. figure 10), an example of the interlacing (resp. non-interlacing) partitions and the corresponding admissible (resp. non-admissible) particle configurations are depicted.

We show the wavefunctions $\langle \{n\}/\Psi_{\{v\}} \rangle$ and their dual $\langle \Psi_{\{v\}} | \{n\} \rangle$ can be represented in the following determinant forms, which are parametrized by Young diagrams.

**Theorem 5.3.** The wavefunctions can be expressed by the Grothendieck polynomials as:

$$\langle \{n\}_{M,N} | \Psi_{\{v\}} | \{n\}_{M,N} \rangle = \prod_{j=1}^{N} (v_{j}^{-1} - \beta v_{j})^{M-1} G_{\lambda}(z_{1},...,z_{N}; \beta),$$  

$$\langle \Psi_{\{v\}} | \{n\}_{M,N} \rangle = \prod_{j=1}^{N} (v_{j}^{-1} - \beta v_{j})^{M-1} G_{\lambda}(z_{1},...,z_{N}; \beta),$$  

where $z_{j}^{-1} = v_{j}^{-2} - \beta$ and $\lambda^\vee = (\lambda_{1}^\vee, \lambda_{2}^\vee, ..., \lambda_{N}^\vee)$ ($M - 1 \geq \lambda_{1}^\vee \geq \cdots \geq \lambda_{N}^\vee \geq 0$) is given by the Young diagram $\lambda$ as $\lambda_{j}^\vee = M - 1 - \lambda_{N+1-j}$.

**Proof.** The second relation (5.5) holds if the first equation (5.4) is valid. This follows from an argument similar to that in the wavefunctions for the five-vertex model. Namely, since the Boltzmann weights for the phase model are invariant under a rotation of $180^\circ$ (see figure 7) and the commutativity of the $B$- and $C$-operators (4.10), the graphical description of the wave function $\langle \{n\}_{M,N} | \Psi_{\{v\}} | \{n\}_{M,N} \rangle$ is also invariant under the rotation (see figure 3 for the five-vertex model). The rotated graph is nothing but the dual wavefunction $\langle \{n\}/\Psi_{\{v\}} \rangle$, which corresponds to the transformation $\lambda \rightarrow \lambda^\vee$, one finds (5.5) is valid if (5.4) holds. Thus, it is sufficient to show (5.4).

The relation between the wavefunctions of the integrable five-vertex model of $N$ and $N + 1$ particles can be reduced to the relation between the Grothendieck and skew Grothendieck polynomials (3.10). This relation is also the key for the non-Hermitian phase
model. Namely, we show the following lemma for the correspondence between the matrix elements of the single $B$- and $C$- operators and the skew Grothendieck polynomials of a single variable, from which one concludes that the wavefunctions are proportional to the Grothendieck polynomials.

**Lemma 5.4.** The matrix elements of the single $B$- and $C$-operators can be expressed as the skew Grothendieck polynomials of a single variable as:

\[
\langle m \rangle_{MN+1} \langle \psi \rangle_{v} = \left( v^{-1} - \beta v \right)^{MN+1} B(v) \langle n \rangle_{MN},
\]

\[
\langle n \rangle_{MN} \langle \psi \rangle_{v} = \left( v^{-1} - \beta v \right)^{MN} C(v) \langle m \rangle_{MN+1},
\]

where the Young diagram $\mu = (\mu_1, \ldots, \mu_{N+1}) \ (M - 1 \geq \mu_1 \geq \cdots \geq \mu_{N+1} \geq 0)$ is parametrized by the configuration $\{m\}_{MN+1} = \{m_0, m_1, \ldots, m_{M-1}\} \ (m_0 + \cdots + m_{M-1} = N + 1)$ as $\mu = (m_0, m_1, \ldots, m_{M-1}) \ (M - 1 \geq \mu_1 \geq \cdots \geq \mu_{N+1} \geq 0)$ is given by $\mu_j' = M - 1 - \mu_{N+2-j}$.

Here we first end the proof of theorem 5.3 by using lemma 5.4. The left-hand side of (5.4) is decomposed as:

\[
\langle n \rangle_{MN} \langle \psi \rangle_{v} = \sum_{\{m\}_{MN}}^{\{n\}_{MN}} \langle n \rangle_{MN} \prod_{j=1}^{N} \left\{ B(v) \langle m^{(N-j)} \rangle_{MN-j} \right\} \left\{ \langle m^{(N-j)} \rangle_{MN-j} \right\} \langle \psi \rangle_{v}.
\]

Then applying lemma 5.4 to the above decomposition and using corollary 3.6, one obtains (5.4).

**Proof of lemma 5.4.** Utilizing the graphical description and an argument similar to proposition 3.2, one immediately sees that (5.7) automatically holds if (5.6) holds. Let us show (5.6). From the matrix elements of the $L$-operator, one finds:

\[
\langle m \rangle_{MN+1} \langle B(n) \rangle_{MN} = 0, \quad \text{unless} \quad \{m\}_{MN+1} \triangleright \{n\}_{MN}.
\]

See figure 10 for a graphical representation. For $\{m\}_{MN+1}$ and $\{n\}_{MN}$, we introduce $\{p\}_r = \{0 \leq p_1 < \cdots < p_r \leq M - 1\}$ to be the set of all integers $p$ such that $m_p = n_q + 1$, and $\{q\}_s = \{0 \leq q_1 < \cdots < q_s \leq M - 1\}$ to be the set of all integers $q$ such that $m_q + 1 = n_p$. When $\{m\}_{MN+1}$ and $\{n\}_{MN}$ satisfy the admissible condition $\{m\}_{MN+1} \triangleright \{n\}_{MN}$, the two sets of integers $\{p\}_r$ and $\{q\}_s$ satisfy $s = r - 1$ and $p_k < q_k \leq p_{k+1} \ (k = 1, \ldots, r - 1)$ (see figure 9, for instance). One calculates the matrix elements of $B(v)$ using $\{p\}_r$ and $\{q\}_s$ as:

\[
\langle m \rangle_{MN+1} \langle B(v) \rangle_{MN} = \sum_{\beta}^{\alpha \beta} \left\{ m \right\}_{MN+1} \left\{ \prod_{j=0}^{M-1} E_{\alpha j} (v) \right\}_{\alpha} \left\{ n \right\}_{MN}.
\]

\[
= \sum_{\beta}^{\alpha \beta} \sum_{j=1}^{M-1} \left( v^{-1} - \beta v \right) \delta_{k, k_{\beta+1}} \prod_{j=1}^{q_j - 1} \prod_{k=p_j+1}^{q_j} \left( v^{-1} - \beta v \delta_{n_k, 0} \right).
\]
where \( q_0 = -1, \quad q_r = M \). This can be shown by combining the following partial actions:

\[
\prod_{l=q_{r+1}}^{q_r} \mathcal{L}_{al}(v) \left| 1 \right>_a \otimes \left\{ \prod_{k=q_{r+1}}^{q_r} \left| p_k \right>_k \right\}
\]

\[
= v^{q_r-q_{r+1}-1} \prod_{l=q_{r+1}}^{q_r} \left( v^{-1} - \beta v \delta_{n_0} \right) \left| 1 \right>_a
\]

\[
\times \left\{ \prod_{k=q_{r+1}}^{q_r} \left| p_k \right>_k \right\} \quad (1 \leq j \leq r - 1),
\]

\[
\prod_{l=q_{r+1}}^{q_r} \mathcal{L}_{al}(v) \left| 1 \right>_a \otimes \left\{ \prod_{k=q_{r+1}}^{q_r} \left| p_k \right>_k \right\}
\]

\[
= v^{q_r-q_{r+1}-1} \prod_{l=q_{r+1}}^{q_r} \left( v^{-1} - \beta v \delta_{n_0} \right) \left| 0 \right>_a \otimes \left\{ \prod_{k=q_{r+1}}^{q_r} \left| p_k \right>_k \right\}.
\]

(5.11)

Dividing the matrix elements \( \langle m \rangle_{M,N+1} | B(v) | n \rangle_{M,N} \) by \( (v^{-1} - \beta v)^{M-1} \) and expressing in terms of the variable \( z \), we have:

\[
\langle m \rangle_{M,N+1} \big( v^{-1} - \beta v \big)^{1-M} B(v) | n \rangle_{M,N} = \left\{ \begin{array}{l}
\sum_{r=1}^{M} \sum_{j=1}^{q_r} (1 + \beta z)^{r-1} \langle m \rangle_{M,N+1} \rightarrow [n]_{M,N} \\
\prod_{j=1}^{r} \prod_{k=q_{r+1}}^{q_r} \left( 1 + \beta z - \beta z \delta_{n_0} \right)
\end{array} \right.
\]

otherwise

(5.12)

The remaining step is to translate the configuration of particles \( \{ m \}_{M,N+1} \) and \( \{ n \}_{M,N} \), with the differences specified by \( \{ p \}_r \) and \( \{ q \}_{r-1} \), to the Young diagrams \( \mu \) and \( \lambda \). One finds the translation rule:

\[
\sum_{j=1}^{r} p_j - \sum_{j=1}^{r} q_j = \sum_{j=1}^{N+1} \mu_j - \sum_{j=1}^{N} \lambda_j,
\]

\[
r - 1 + \# \left\{ k \in \bigcup_{j=1}^{r} \{ p_j + 1, \ldots, q_j - 1 \} | \mu_k \neq 0 \right\}
\]

\[
= \# \left\{ j \in \{ 1, \ldots, N \} | \mu_j \neq \mu_{j+1} \right\}.
\]

(5.13)

By this translation together with proposition 5.2, one finds that (5.12) is nothing but the skew Grothendieck polynomial (3.5).

\[ \square \]

**Example 5.5.** The wavefunctions (5.4) and (5.5) for the free phase model \( (\beta = 0) \) reduce to [2]:
where $s_j(z_1, ..., z_N)$ are the Schur polynomials.

**Example 5.6.** For some particular cases, the wavefunctions reduce to some simple forms:

$$\langle [N, 0, ..., 0] | \Psi(v) \rangle = \langle [N, 0, ..., 0] | \Psi(v) \rangle$$

$$= \prod_{j=1}^{N} (v_j^{-1} - \beta v_j)^{-M-1},$$

$$\langle [0, ..., 0, N] | \Psi(v) \rangle = \prod_{j=1}^{N} v_j^{M-1} \prod_{j=1}^{N} v_j^{M-1} = \prod_{j=1}^{N} v_j^{M-1}.$$  (5.15)

Their relations can be easily checked from their graphical descriptions.

Applying the Cauchy identity (2.13) and using the relations (5.4) and (5.5), we can express the scalar product of the $N$-particle states as a determinant form:

**Corollary 5.7.** The scalar product of the $N$-particle states for the non-Hermitian phase model has the following determinant representation:

$$\langle [\Psi(v)]_{N} | [\Psi(v)]_{N} \rangle = \prod_{1 \leq j < k \leq N} \frac{r_j - r_k}{(r_j^2 - r_k^2)(U_j^2 - U_k^2)} \det_N \left[ \begin{array}{c} (v_j^{-1} - \beta v_j)^{M+2(N-1)-1} - (v_j^{-1} - \beta v_j)^{M+2(N-1)-1} \\ v_j U_k - U_j v_k \end{array} \right].$$  (5.17)

We can also use the summation formula for the Grothendieck polynomials (2.14) to obtain the summation formula for the wavefunctions of the non-Hermitian phase model.

**Corollary 5.8.** The summation formula for the wavefunctions holds:

$$\sum_{\{n\}_{M,N}} (-\beta)^{\sum_{j=1}^{N} n_j} \langle \{n\}_{M,N} | \Psi(v) \rangle = \prod_{j=1}^{N} v_j^{N-1} (v_j^{-1} - \beta v_j)^{M+N-2} \prod_{1 \leq j < k \leq N} \frac{r_j - r_k}{(r_j^2 - r_k^2)^{N-1}} \det_N V.$$  (5.18)
with an $N \times N$ matrix $V$ whose matrix elements are:

$$V_{jk} = \sum_{m=0}^{j-1} (-1)^m (-\beta j^{-N}) \binom{M + N - 1}{m} \left(1 - \beta v_k^2\right)^{1-m+j-N} (1 \leq j \leq N - 1),$$

$$V_{Nk} = -\sum_{m=\max(N-1,1)}^{M+N-1} (-1)^m \binom{M + N - 1}{m} \left(1 - \beta v_k^2\right)^{1-m}. \quad (5.19)$$

6. Melting crystals

As an application of our formulae developed in the previous sections, we study the statistical mechanical system of a melting crystal in three dimensions as depicted in figure 11. The melting rules are the following. The melting starts at one corner of the cubic crystal. Each cube can be removed if its three faces never touch the other cubes constructing the crystal. The removed cube contributes the factor $q = e^{-\mu/T}$ ($\mu > 0$, $T > 0$) to the weight of the configuration.

![Figure 11. A melting crystal. The melting starts at one corner of the crystal. Each cube is possibly removed (melts) only if its three faces do not touch the other cubes. Each removed cube contributes the factor $q = e^{-\mu/T}$ ($\mu > 0$, $T > 0$) to the weight of the configuration.](image-url)
The plane partitions can be regarded as a 3D generalization of the Young diagram. In this 3D diagram, \( \pi_{ij} \) corresponds to the height of stacked cubes on the coordinate \((i,j)\). Then the total number of the stacked cubes is given by \( \pi \pi = \sum_{i,j \geq 1} \pi_{ij} \).

For later convenience, let us describe some properties satisfying the diagonal slices of \( \pi \), which are defined as follows.

**Definition 6.2.** For a plane partition \( \pi \), the \( m \)th \((m \in \mathbb{Z})\) diagonal slice \( \pi^{(m)} \) is a sequence whose elements are defined as:

\[
\pi^{(m)}_j = \pi_{j-m,j} \quad \text{for } j > \max(0, m).
\] (6.1)

First, each diagonal slice \( \pi^{(m)} \) is a partition, i.e., a sequence of weakly decreasing non-negative integers. Second, these partitions satisfy the following interlacing property.

**Lemma 6.3.** The series of partitions \( \pi^{(m)} \) satisfies the interlacing relation:

\[
\cdots \pi^{(-2)} < \pi^{(-1)} < \pi^{(0)} > \pi^{(1)} > \pi^{(2)} > \cdots.
\] (6.2)

See figure 13 for an example of the diagonal slices.

The partition function \( Z \) of the system of the melting crystal is regarded as the generating function of the plane partition and is known to be given by the so-called MacMahon function [22]:

\[
Z = \sum_{\pi} \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^{\pi_n}} \quad (0 < q < 1).
\] (6.3)

Now we consider the case that a plane partition \( \pi \) is contained in a certain finite box of size, say, \( N_1 \times N_2 \times L \). Let us call such a partition the boxed plane partition and write it as
π ⊆ [N_1, N_2, L]. For this boxed plane partition, the following is valid:

$$\pi^{(-N_1)} = \pi^{(N_2)} = \emptyset, \quad \pi_{N_2+1} = \pi_{N_1+1} = 0 \quad (i, j \geq 1),$$

(6.4)

and hence the interlacing relation (6.2) is restricted to:

$$\emptyset = \pi^{(-N_1)} \prec \cdots \prec \pi^{(-1)} \prec \pi^{(0)} \succ \pi^{(1)} \succ \cdots \succ \pi^{(N_2)} = \emptyset.$$

(6.5)

This case corresponds to a system of the melting rectangular crystal of size $N_1 \times N_2 \times L$. Then the partition function of the system is given by [22]:

$$Z_{\text{box}} = \sum_{\pi \subseteq [N_1, N_2, L]} q^{\#\pi} = \prod_{j=1}^{N_1} \prod_{k=1}^{N_2} \frac{1 - q^{L+j+k-1}}{1 - q^{j+k-1}}.$$  

(6.6)

In the limit $q \to 1$, this formula gives the number of the plane partitions contained in the box $N_1 \times N_2 \times L$. In [2], the formula (6.6) for the box $N \times N \times L$ is reproduced by utilizing the scalar products of the phase model.

Inspired by that work, we extend the method to the case for the non-Hermitian phase model and calculate the partition function for the statistical mechanical model of a melting crystal with the size $N \times N \times L$. The partition function of the model is defined as.\footnote{We remark again as in the introduction that this assignment of the weights for each plane partition is totally different from those in the literature such as [25, 26], for example, which are based on the Macdonald polynomials and their degeneration to the Hall–Littlewood polynomials. The model introduced in this paper is based on the Grothendieck polynomials, and the directions of the extensions from the Schur to the Grothendieck and the Macdonald polynomials are different; hence so are the corresponding melting crystal models.}
\[
Z_{\text{box}}(\beta) = \sum_{q \in [N,N,L]} \Phi(q, \beta; \pi) q^{|q|} \quad (0 < q < 1),
\]

\[
\Phi(q, \beta; \pi) = \prod_{j=1}^{N} \prod_{k=1}^{N-j} \left[ (1 + \beta q^j)^{-\delta(q^{j}, \pi_{j}^{(j+1)})} (1 + \beta q^{-j})^{1-\delta(q^{j}, \pi_{j}^{(j-1)})} \right].
\]

where \(\delta(i, j)\) denotes the Kronecker delta: \(\delta(i, j) = \delta_{ij}\). Here we comment on the physical meaning of the additional potential factor \(\Phi(q, \beta; \pi)\). This factor can be interpreted to reflect, like microscopic interactions among atoms. For \(\beta > 0\), it brings out a surface flattening effect in the region \(j > i\) in figures 12 or 13. In contrast to this, in the region \(j < i\), the potential causes a surface roughening effect. The strength of the effects decreases (resp. increases) with distance from the plane \(i = j\) in the region \(j > i\) (resp. \(j < i\)).

On the other hand, for \(\beta < 0\), the potential structure is much more complicated. (i) For \(\beta < -2\), the potential \(\Phi(q, \beta; \pi)\) denotes a roughening effect in \(j < \log (-2/\beta)\log q\) or a flattening effect in the other region. (ii) For \(-2 \leq \beta < 0\), it denotes a roughening effect in \(1 - \log (-2/\beta)\log q < j < \log q\) or a flattening effect in the other region. Note that for \(\beta < 0\), the model sometimes becomes physically ill-defined, because \(\Phi(q, \beta; \pi)\) possibly takes negative values.

In any case, because to the strength of the force is not symmetric with respect to the plane \(i = j\), the expected shape of the melting crystal is not symmetric with respect to \(i = j\) except for \(\beta = 0\).

The partition function \(Z_{\text{box}}(\beta)\) is explicitly evaluated by using the Cauchy identity (2.13) and corollary 3.13. The following and subsequent corollaries are the main results of this section.

**Corollary 6.4.** The partition function \(Z_{\text{box}}(\beta)\) is given by:

\[
Z_{\text{box}}(\beta) = \frac{q^{N(N-1)/2} \prod_{j=1}^{N} (1 + \beta q^j)^{j-1}}{\prod_{i<j} (q^j - q^k)^{\delta_{ij}}} \text{det}_N \left[ 1 - q^{(j+k-1)N} \left( \frac{1 + \beta q^j}{1 + \beta q^j} \right)^N \right].
\]

**Proof.** Consider the Cauchy identity given by (2.13) for \(\lambda = \pi^{(0)}\). Then applying corollary 3.13, the Grothendieck polynomials on the left-hand side are given by:

\[
G_{\pi^{(0)}}(z_1, \ldots, z_N; \beta) = \sum_{x^{(0)} = \ldots = x^{(0)} = \emptyset} \prod_{i=1}^{N} z_i^{x_i^{(0)}} \prod_{j=1}^{N} \prod_{k=1}^{N-j} \left[ 1 + \beta z_j - \beta z_j \delta_{x_i^{(j+1)}, x_i^{(j)}} \right],
\]

\[
G_{\pi^{(0)}}(w_1, \ldots, w_N; \beta) = \sum_{x^{(0)} = \ldots = x^{(0)} = \emptyset} \prod_{i=1}^{N} w_i^{x_i^{(0)}} \prod_{j=1}^{N-j} \prod_{k=1}^{N} \left[ 1 + \beta w_j - \beta w_j \delta_{x_i^{(j+1)}, x_i^{(j)}} \right].
\]

Here we have used \(\pi_{\lambda^{(0)}}^{(0)} = L - \pi_{\lambda^{(1)}}^{(0)} \) and the properties (3.4), (6.4), and (6.5) for the explicit evaluations. The insertion of them into the Cauchy identity (2.13) yields:
\[
\sum_{z \in \{N \in \mathbb{N} : L \}} \prod_{j=1}^{N} \prod_{\ell=1}^{L} [-\beta^{\|} w_j \|^{\ell-1} - |\|^{\ell-1}]
\times \prod_{j=1}^{N} \prod_{\ell=1}^{L} \left[ (1 + \beta z_j)^{-\delta(x_j^{\ell}, x_{j+1}^{\ell-1})} (1 + \beta w_j)^{-\delta(x_j^{\ell}, x_{j+1}^{\ell-1})} \right]
= \frac{\prod_{i<j<k \leq N} (z_j - z_k)(w_j^{-1} - w_k^{-1})}{\det_N \left[ 1 - \left( \frac{z_j w_k^{1-L} + \beta w_k}{1 + \beta z_j} \right)^N \right]}. \tag{6.10}
\]

Setting \(z_j = q^j\) and \(w_j = q^{1-j}\) in the above, we finally arrive at (6.6).

Set \(\beta = 0\) in (6.8), then the formula (6.6) is reproduced. Moreover, taking the limit \(L \to \infty\) and \(N \to \infty\), we have the following generalized MacMahon function, which reduces to the ordinary MacMahon function (6.3) for \(\beta = 0\) and Euler’s generating function at \(\beta = -1\).

**Corollary 6.5.** The partition function (6.8) in the limit \(L \to \infty\) and \(N \to \infty\) is given by:

\[
Z(\beta) := \lim_{L,N \to \infty} Z_{\text{box}}(\beta) = \prod_{n=1}^{\infty} \frac{(1 + \beta q^n)^{n-1}}{(1 - q^n)^{n}}.
\tag{6.11}
\]

which becomes the MacMahon function and Euler’s generating function at \(\beta = 0\) and \(\beta = -1\), respectively.

For \(\beta = 0\), the partition function \(Z(0)\) is nothing but the MacMahon function (6.6), which is a generating function of the plane partitions. And surprisingly, for \(\beta = -1\), which corresponds to the TASEP (resp. TAZRP) in the language of the five-vertex model (resp. the non-Hermitian phase model), \(Z(-1)\) is nothing but a generating function for the numbers of possible partitions of natural numbers, which is due to Euler:

\[
Z(-1) = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} = \sum_z q^{|z|}. \tag{6.12}
\]

The expression for the partition function (6.11) means that the melting crystal model we introduced unifies the generating functions of the 2D and 3D Young diagrams (6.11). The enumeration problems for 2D Young diagrams can be treated by the 3D melting crystal model at the point \(\beta = -1\). Note that there are other melting crystal models based on the Macdonald polynomials [25, 26] whose partition functions are different from ours but have simple expressions in the infinite volume limit as do ours, including the MacMahon function at a special point. But if one wants to relate the results in [25, 26] with Euler’s generating function, one has to multiply in finite products. This means that one should multiply infinite products to the weights assigned to each plane partition, which seems to be an artificial operation, unnatural from the point of view of enumeration.

Note here that the partition function (6.11) is physically well defined for \(\beta \geq -1\), which is a condition for positivity of \(Z(\beta)\). The entropy \(S(\beta)\) for the model (6.11) can be calculated by using the relation \(Z(\beta) = e^{S(\beta) - E/T}\), where \(E := T^2 \sigma^2 \log Z(\beta)/\partial T^2\) is the internal energy. Explicitly it reads:
\[ S(\beta) = \sum_{n=1}^{\infty} \frac{\beta(n-1)}{T} \left( \frac{1}{\beta + q^n} + \frac{n}{q^n - 1} \right) + \frac{\beta}{T} \sum_{n=1}^{\infty} \log \left[ \frac{(1 + \beta q^n)^{n+1}}{(1 - q^n)^n} \right] \] (\beta \geq -1), \quad (6.13)

where \( q = e^{-\mu/T} \) (\( \mu > 0, \ T > 0 \)). From this expression, it can be easily followed that the entropy \( S(\beta) \) is a monotonically increasing function of \( \beta \). In figure 14, the temperature dependence of the entropy is depicted for various values of \( \beta \).

**7. Conclusion**

In this paper, we studied the non-Hermitian phase model and showed that the wavefunctions are nothing but the Grothendieck polynomials. To show this, we reviewed the integrable five-vertex model and introduced the skew Grothendieck polynomials for a single variable as matrix elements of a \( B \)-operator. The addition theorem for the Grothendieck polynomials follows from the equivalence between the wavefunctions of the five-vertex model and the Grothendieck polynomials. Showing that the matrix element of the \( B \)-operator in the non-Hermitian phase model is given by the skew Grothendieck polynomials, and then applying the addition theorem, we derived the wavefunctions of the non-Hermitian phase model, which can also be expressed by the Grothendieck polynomials. Our work establishes the \( K \)-theoretic boson–fermion correspondence at the level of wavefunctions.

As another application of the boson–fermion correspondence, we discuss the statistical mechanical model of a 3D melting crystal and exactly derive the partition functions, which are interpreted as a \( K \)-theoretic generalization of the MacMahon function. Surprisingly, the \( K \)-theoretic MacMahon function includes not only the generating function of the plane partitions but also Euler’s generating function of the partitions. Our refinement of the melting crystal model unifies the treatment of the enumeration problems of 2D and 3D Young diagrams. The reason why 2D objects appear for \( K \)-theory is not yet known, and its geometric meaning deserves to be investigated in the future.

The Hermitian phase model is described by the Schur polynomials. Since the determinant representations of the scalar products are essentially the Cauchy identity for the Schur polynomials, they have connections with the KP equation and the Toda lattice [26–28]. It is interesting to examine whether this classical integrable interpretation can be extended to the case of the integrable five-vertex model and the non-Hermitian phase model by making a connection with the existing classical integrable system or extending it to some extent.

![](image-url) **Figure 14.** The temperature dependence of the entropy \( S(\beta) \) (6.13) is depicted for various values of \( \beta \).
In terms of geometry, our work on the relation between non-Hermitian integrable models and Grothendieck polynomials means that non-Hermitian integrable models provide a natural framework to study the quantum $K$-theory of Grassmannian varieties. For the Hermitian phase model, the quantum cohomology ring and the Verlinde ring are shown to be described by the ring defined by the model under the quasiperiodic boundary condition [4], where the Bethe ansatz equation plays the role of the ideal. In the future we would like to make further investigations on quantum $K$-theoretic objects in our framework.

One of the problems we are planning to investigate is to extend the relation between integrable models and $K$-theoretic objects to other types of Grassmannian varieties. There are several extensions and variations of the Schur polynomials. The Schur $P$, Schur $Q$, Jack, Hall–Littlewood, and Macdonald polynomials have connections with the $q$-boson model [6, 29, 30]. On the other hand, the $K$-theoretic extension of the Schur $P$ and Schur $Q$ polynomials are introduced in [13]. We expect to find connections between these $K$-theoretical symmetric polynomials and the integrable models, such as the non-Hermitian $q$-boson model [31–33].

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Appendix

In this appendix, we show that the $L$-operator of the five-vertex model (2.2) is a particular reduction of a more general six-vertex model. We start from the $R$-matrix of the following six-vertex model, satisfying the Yang–Baxter relation (2.5):

\[
R(u, v) = \begin{pmatrix}
    f(v, u; t) & 0 & 0 & 0 \\
    0 & t & g(v, u; t) & 0 \\
    0 & g(v, u; t) & 1 & 0 \\
    0 & 0 & 0 & f(v, u; t)
\end{pmatrix},
\]

\[
f(v, u; t) = \frac{u^2 - tv^2}{u^2 - v^2}, \quad g(v, u; t) = \frac{(1 - t)uv}{u^2 - v^2},
\]

including the $R$-matrix of the five-vertex model (2.4) as a special point $t = 0$. One can show that the following $L$-operator solves the $RLL$ relation (2.3) for this $R$-matrix of the six-vertex model:

\[
L(u) = \begin{pmatrix}
    \alpha_3 u + \alpha_4 u^{-1} & 0 & 0 & 0 \\
    0 & \alpha_3 tu + \alpha_4 u^{-1} & (1 - t)\alpha_1 & 0 \\
    0 & (1 - t)\alpha_2 & \alpha_5 u + \alpha_6 u^{-1} & 0 \\
    0 & 0 & 0 & \alpha_5 u + \alpha_6 tu
\end{pmatrix},
\]
where the parameters $\alpha_j, j = 1, \cdots, 6$ and $t$ satisfy the relations:

\[
(1 - t)\alpha_1\alpha_2 + \alpha_3\alpha_6 - \alpha_4\alpha_5 = 0,
\]

\[
(t^2 - 1)\alpha_1\alpha_2 + t^2\alpha_3\alpha_6 - \alpha_4\alpha_5 = 0.
\]

The $R$-matrix of the six-vertex model is recovered from the $L$-operator by the choice of the parameters $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = 1, \alpha_6 = -1, \alpha_4 = -t$.

Another particular choice of the $L$-operator of the general six-vertex model gives the $L$-operator for the five-vertex model (2.2), whose wavefunction is the Grothendieck polynomials.

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