Exact pressure evolution equation for incompressible fluids

M. Tessarotto\textsuperscript{a,b}, M. Ellero\textsuperscript{c}, N. Aslan\textsuperscript{d}, M. Mond\textsuperscript{e} and P. Nicolini \textsuperscript{a,b}

\textsuperscript{a} Department of Mathematics and Informatics, University of Trieste, Italy, \textsuperscript{b} Consortium of Magneto-fluid-dynamics, University of Trieste, Italy, \textsuperscript{c} Technical University of Munich, Munich, Germany, \textsuperscript{d} Yeditepe University, Kayisdagi, Istanbul, Turkey, \textsuperscript{e} Ben-Gurion University of the Negev, Beer-Sheeva, Israel

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Abstract

An important aspect of computational fluid dynamics is related to the determination of the fluid pressure in isothermal incompressible fluids. In particular this concerns the construction of an exact evolution equation for the fluid pressure which replaces the Poisson equation and yields an algorithm which is a Poisson solver, i.e., it permits to time-advance exactly the same fluid pressure without solving the Poisson equation. In fact, the incompressible Navier-Stokes equations represent a mixture of hyperbolic and elliptic pde’s, which are extremely hard to study both analytically and numerically. In this paper we intend to show that an exact solution to this problem can be achieved adopting the approach based on inverse kinetic theory (IKT) recently developed for incompressible fluids by Ellero and Tessarotto (2004-2007). In particular we intend to prove that the evolution of the fluid fields can be achieved by means of a suitable dynamical system, to be identified with the so-called Navier-Stokes (N-S) dynamical system. As a consequence it is found that the fluid pressure obeys a well-defined evolution equation. The result appears relevant for the construction of Lagrangian approaches to fluid dynamics.

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I. INTRODUCTION

The dynamics of incompressible fluids has been around for almost two centuries. Nevertheless, basic issues concerning its foundations still remain unanswered. A significant aspect concerns, in particular, the determination of the fluid pressure in isothermal incompressible fluids and the construction of algorithms with permit to time-advance the same fluid pressure. In fact, the incompressible Navier-Stokes (NS) equations (INSE), describing a fluid characterized by infinite sound speed, represent a mixture of hyperbolic and elliptic pde’s, which are extremely hard to study both analytically and numerically. These difficulties have motivated in the past the search of possible alternative numerical approaches which permit to advance in time the fluid pressure without actually solving numerically the Poisson equation. This refers, especially, to the search of approximate evolution equations for the fluid pressure based on relaxation-type schemes. These equations are typically characterized by having finite sound speed, weak compressibility and, in particular, in the case of kinetic approaches, also by a low Mach number. All of these approaches are asymptotic, i.e., they depend on small parameters. Nevertheless, they are well know to lead to useful numerical schemes. Notable examples are provided, for example, by the Chorin artificial compressibility method (ACM) and the lattice Boltzmann approach developed in Ref. However, the interesting question arises whether there exists actually an evolution equation for the fluid pressure which is exactly equivalent to the Poisson equation (i.e., is a Poisson solver) and does not require any violation of the fluid equations (in particular the condition of incompressibility) nor the assumption of low Mach number. If it exists it could therefore be used, in principle, to determine improved numerical schemes which permit the evaluation of the fluid pressure without solving explicitly the Poisson equation. This is a obviously a significant matter-of-principle issue which should be resolved, even before attempting any specific solution strategy of this type. The search of an exact pressure-evolution equation, besides being a still unsolved mathematical problem, is potentially relevant for several reasons; in particular: a) the proliferation of numerical algorithms in computational fluid dynamics which reproduce the behavior of incompressible fluids only in an asymptotic sense (see below); b) the possible verification of conjectures involving the validity of appropriate equations of state for the fluid pressure; c) comparisons with previous CFD methods; d) the identification of mathematical models which do not involve relaxation-type and/or low
effective-Mach number schemes [to avoid solving the Poisson equation] and, more generally, the investigation of possible non-asymptotic phase-space models, i.e., which do not involve infinitesimal parameters. Other possible motivations are, of course, related to the ongoing quest for efficient numerical solution methods to be applied for the construction of the fluid fields \( \{\rho, \mathbf{V}, p\} \), solutions of the initial and boundary-value problem associated to the INSE. The purpose of this paper is to answer this important question. In particular, we intend to prove *that the pressure evolution equation actually exists for incompressible fluids, and can be constructed explicitly.* The solution of the problem is reached by adopting a phase-space descriptions of hydrodynamic equations based on the continuous inverse kinetic theory (IKT) recently pointed out by Ellero and Tessarotto [6, 7], in which all the fluid fields, including the fluid pressure \( p(\mathbf{r}, t) \), are represented as prescribed momenta of the kinetic distribution function, solution of an appropriate kinetic equation. An inverse kinetic theory of this type yields, by definition, an exact Navier-Stokes (and also a Poisson) solver. The dynamical system (denoted as Navier-Stokes dynamical system), which advances in time the kinetic distribution function, determines also uniquely the time-evolution of the fluid fields. As a consequence, it is found the fluid pressure obeys an evolution equation which is non-asymptotic, i.e., it is satisfied for arbitrary values of the relevant physical parameters. This permits to investigate, in particular, its asymptotic behavior for infinitesimal Mach numbers. We intend to show that, in validity of suitable asymptotic conditions, it can be approximated by the Chorin pressure evolution equation. This result is interesting because it shows that Chorin scheme can be viewed as an asymptotic approximation to the exact solution method here presented. The plan of the presentation is as follows. In Sec.2 a review of previous numerical solution methods adopted in CFD for the Poisson equation is presented. In Sec. 3 the phase-space approach based on inverse kinetic theory is recalled. This is realized by constructing the Navier-Stokes dynamical system which advances in time the kinetic distribution function. Representing the fluid fields in terms of suitable velocity-space moments of the kinetic distributions function, their time evolution is uniquely determined. This result is invoked in Sec. 4 to construct an exact evolution equation for the fluid pressure. The equation is proven to hold for arbitrary Lagrangian trajectories determined by the Navier-Stokes dynamical system. The physical implications of the equations are discussed in Sec.5 and its possible asymptotic approximations are discussed. As a result, it is shown in particular that under suitable assumptions the pressure evolution equation
can be approximated by the Chorin pressure evolution equation. Finally, the conclusions are summarized in Sec.6.

II. MATHEMATICAL SETTING - SEARCH OF ALTERNATIVE POISSON SOLVERS

For definiteness, it is convenient to recall that this is defined by the N-S and isochoricity equations

\[ NV = 0, \]  
\[ \nabla \cdot V = 0, \]

while the mass density \( \rho \) satisfies the incompressibility condition \( \rho(\mathbf{r},t) = \rho_o \), with \( \rho_o \) a constant mass density and both \( \rho, p \) non-negative. Here \( N \) is the N-S operator \( NV \equiv \rho \frac{D}{Dt} V + \nabla p + f - \mu \nabla^2 V \), with \( \frac{D}{Dt} = \frac{\partial}{\partial t} + V \cdot \nabla \) the convective derivative, \( f \) denotes a suitably smooth volume force density acting on the fluid element and \( \mu > 0 \) is the constant fluid viscosity. Equations (1)-(2) are assumed to admit strong solutions in an open set \( \Omega \times I \), with \( \Omega \subseteq \mathbb{R}^3 \) the configurations space (defined as the subset of \( \mathbb{R}^3 \) where \( \rho(\mathbf{r},t) > 0 \)) and \( I \subseteq \mathbb{R} \) a possibly bounded time interval. By assumption \( \{ \rho, V, p \} \) are continuous in the closure \( \overline{\Omega} \). Hence if in \( \Omega \times I \), \( f \) is at least \( C^{(1,0)}(\Omega \times I) \), it follows necessarily that \( \{ \rho, V, p \} \) must be at least \( C^{(2,1)}(\Omega \times I) \), while the fluid pressure and velocity must satisfy respectively the Poisson and energy equations \( \nabla^2 p = -\nabla \cdot f - \rho \nabla \cdot (V \cdot \nabla V) \) and \( V \cdot NV = 0 \). It is well known that the choice of the Poisson solver results important in numerical simulations, since its efficient numerical solution depends critically on the number of modes or mesh points used for its discretization \([10, 11]\). In turbulent flows this number can become so large to effectively limit the size of direct numerical simulations (DNS) \([6]\). This phenomenon may be worsened by the algorithmic complexity of the numerical solution methods adopted for the Poisson equation. For this reason previously several alternative approaches have been devised which permit to advance in time the fluid pressure without actually solving numerically the Poisson equation. Some of these methods are \textit{asymptotic}, i.e., to advance in time the fluid pressure they replace the exact Poisson equation with suitable algorithms or equations which hold only in an asymptotic sense (neglecting suitably small corrections), others are \textit{exact solvers}, i.e., provide in principle rigorous solutions of INSE (and Poisson equation). The first category
includes the pressure-based method (PBM) \cite{1}, the Chorin artificial compressibility method (ACM) \cite{2}, the so-called preconditioning techniques \cite{3}, all based on ACM, and kinetic approaches, of which a notable example is provided by the so-called Lattice-Boltzmann (L-B) methods (for a review see for example Ref.\cite{4} and references therein indicated). PBM is an iterative approach and one of the most widely used for incompressible flows. Its basic idea is to formulate a Poisson equation for pressure corrections, and then to update the pressure and velocity fields until the isochoricity condition (2) is satisfied in a suitable asymptotic sense. The ACM approach and the related preconditioning techniques, instead, are obtained by replacing the Poisson and N-S equations with suitable parameter-dependent evolution equations, assuming that the fluid fields depend on a fictitious pseudo-time variable $\tau$. In dimensionless form the evolution equation for the pressure becomes in such a case

$$\varepsilon^2 \frac{\partial}{\partial \tau} p + \nabla \cdot \mathbf{V} = 0,$$

(3)

where $\varepsilon^2 > 0$ is an infinitesimal parameter. Manifestly this equation recovers only asymptotically, i.e., for $\varepsilon^2 \to 0$, the exact isochoricity condition (2). Introducing the fast variable $\tau \equiv \tau / \varepsilon^2$, this implies that the fluid fields must be of the form $\mathbf{V}(\mathbf{r}, t, \tau), p(\mathbf{r}, t, \tau)$ and should be assumed suitable smooth functions of $\tau$. Therefore, for prescribed finite values of $\varepsilon^2$ (to be assumed suitably small), this equation permits to obtain also an asymptotic estimate for the fluid pressure $p(\mathbf{r}, t)$. This is expressed by the equation

$$p(\mathbf{r}, t) = \lim_{\tau \to -\infty} p(\mathbf{r}, t, \tau) \cong p(\mathbf{r}, t, \tau = 0) - \int_0^{\tau} d\tau' \nabla \cdot \mathbf{V}(\mathbf{r}, t, \tau'),$$

where $\tau$ >> 1 is suitably defined and $p(\mathbf{r}, t, \tau = 0)$ denotes some initial estimate for the fluid pressure. Several implementations on the Chorin algorithm are known in the literature (see for example Refs.\cite{12, 13, 14}). Customary L-B methods are asymptotic too since they recover INSE only in an approximate sense; moreover typically they rely on the introduction of an equation of state for the fluid pressure, for example, the equation of state of an ideal gas, or more refined models based on so-called non-ideal fluids \cite{15}. This assumption, however, generally requires that the effective Mach-number characterizing the L-B approach, defined by the ratio $M_{eff} = V_{sup} / c$ (with $c$ denoting the discretized velocity of the test particles and $V_{sup}$ the sup of the velocity field at time $t$), must result suitably small. As a consequence, in typical L-B approaches the fluid pressure can only be estimated asymptotically. However, there are other numerical approaches which in principle provide exact Poisson solvers. These include the so-called spectral methods in which the fluid fields are expanded in terms of suitable
basis functions. Significant examples are the pure spectral Galerkin and Fourier methods [16] as well as the nonlinear Galerkin method [17], which are typically adopted for large-scale turbulence simulations. In these methods the construction of solution of the Poisson equation is obtained analytically. However, the series-representation of the fluid fields makes difficult the investigation of the qualitative properties of the solutions, such - for example - the search of a possible equation of state or an evolution equation for the fluid pressure.

III. INVERSE KINETIC THEORY APPROACH TO INSE

Another approach which provides in principle an exact Poisson solver is the one proposed by Ellero and Tessarotto [6, 7], based on an inverse kinetic theory for INSE. This approach, recently applied also to quantum hydrodynamic equations [18], permits to represent the fluid fields as moments of a suitably smooth kinetic distribution function $f(x, t)$ which obeys an appropriate Vlasov-type inverse kinetic equation (IKE):

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x} \cdot (Xf) = 0.$$  (4)

Here $X(x, t) \equiv \{v, F\}$ and $x = (r, v) \in \Gamma \subseteq \overline{\Omega} \times \mathbb{R}^3$ is the state vector generated by the vector field $X$. $v$ is the kinetic velocity, while $F(x, t)$ is an appropriate mean-field force obtained in Ref. [7]. In Refs. [8, 9], it has been proven that, under suitable assumptions, $F(x, t)$ can be uniquely prescribed. This implies that the time evolution of the kinetic distribution function, $T_{t,t_0}f(x_o) = f(x(t), t)$, is determined by the finite-dimensional classical dynamical system associated to the vector field $X$, i.e.,

$$\frac{d}{dt}x = X(x, t)$$

$$x(t_o) = x_o$$

(N-S dynamical system) which must hold for arbitrary initial conditions $x_o = (r_o, v_o) \in \Gamma$. It follows that the solution of (5), $x(t) = T_{t,t_0}x_o$, which defines the N-S evolution operator $T_{t,t_0}$, determines uniquely a set of curves $\{x(t)\} \equiv \{x(t), \forall t \in I\}_{x_o}$ belonging to the phase space $\Gamma$ which can be interpreted as phase-space Lagrangian trajectories associated to a set of fictitious "test" particles. The projections of these trajectories onto the configuration space, denoted as configuration-space Lagrangian trajectories, are defined by the curves $\{r(t)\} \equiv \{r(t) \equiv T_{t,t_0}r_o, \forall t \in I\}_{x_o}$. By varying their initial conditions, in particular $r_o \in \Omega,$
the curves \( \{ r(t) \} \) can span, by continuity, the whole set \( \Omega \). IKT determines uniquely the fluid fields expressed via suitable moments of the kinetic distribution function. The time evolution of the fluid fields requires the construction of an infinite set of configuration-space Lagrangian trajectories, each one corresponding to a different initial position \( r_o \in \Omega \). Hence-as expected - INSE is actually reduced to an infinite set of ode’s, represented by the initial value problem \( [5] \) with a suitable infinite set of initial conditions. In particular, it follows that the fluid pressure \( p(r, t) \) is defined by

\[
p(r, t) = p_1(r, t) - p_o(t) (\text{to be regarded as a constitutive equation for } p(r, t)),
\]

where \( p_1(r, t) \) is the kinetic pressure

\[
p_1(r, t) = \int dv \frac{1}{3} u^2 f(x, t),
\]

\( p_0 \) is denoted as reduced pressure, while \( u \) is the relative velocity \( u = v - V(r,t) \). Both \( p_o(t) \) and \( p_1(r, t) \) are strictly positive, while \( p_o(t) \) in \( \Omega \times I \) is subject to the constraint

\[
p_1(r, t) - p_o(t) \geq 0.
\]

In particular the reduced pressure \( p_o(t) \) is an arbitrary function of time, subject to the only requirements of suitably smoothness and strict positivity. A key aspect of IKT is that, by construction, a particular solution of IKE is provided by the local Maxwellian distribution

\[
f_M(x; V, p_1) = \frac{\rho o}{(\pi)^{3/2} v_{th}^3} \exp \{-Y^2\} \quad \text{[where } Y^2 = \frac{u^2}{v_{th}^2} \text{ and } v_{th}^2 = 2p_1/\rho o]\].

In such a case, the vector field \( F \) reads

\[
F(r, v, t) = a - \frac{1}{2} N_0 V + \frac{u}{2} A_0 p_1 + \frac{1}{p_1} \nabla p \left\{ \frac{E}{p_1} - \frac{3}{2} \right\} \quad \text{(equation for } F),
\]

where \( a \) denotes the convective term \( a = \frac{1}{2} u \cdot \nabla V + \frac{1}{2} \nabla V \cdot u \), \( E \) is the relative kinetic energy density \( \mathcal{E} = \rho u^2 / 2 \), while \( N_0 \) and \( A_0 \) are the differential operators \( N_0 V \equiv -f(r, V, t) + \mu \nabla^2 V \) and \( A_0 p_1(r, t) \equiv \frac{1}{p_1} \left[ \frac{1}{p_1} (\nabla p_1 + \nabla \cdot (\nabla p_1)) \right] \). For an arbitrary and suitably smooth distribution function \( f(x, t) \), the form of the vector field \( F \) satisfying these hypotheses has been given in Refs. [7, 8].

IV. CONSTRUCTION OF AN EXACT PRESSURE EVOLUTION EQUATION

An interesting issue is related to the consequences of the constitutive equation for \( p \) and the N-S dynamical system generated by the initial value-problem \( [5] \). In this Letter we intend to prove that the fluid pressure \( p(r, t) \) obeys an exact partial-differential equation which uniquely determines is time evolution. This is obtained by evaluating its Lagrangian derivative along an arbitrary configuration-space Lagrangian trajectory \( \{ r(t) \} \) generated by the N-S dynamical system. The result can be stated as follows.

**Theorem - Pressure evolution equation**

*Assuming that the initial-boundary value problem associated to INSE admits a suitably strong solution \( \{ \rho, V, p \} \) in the set \( \Omega \times I \), the following statements hold:*
A) If \( x(t) \) is a particular solution of Eq. (3) which holds for arbitrary \( r(t) \in \Omega \) and \( t \in I \), along each phase-space Lagrangian trajectory \( \{x(t)\} \) defined by Eq. (3) the scalar field \( \xi(r,t) \equiv \mathcal{E}/p_1 \) obeys the exact evolution equation
\[
\frac{d}{dt} \xi = -\frac{1}{2} u \cdot \nabla \ln p_1 \tag{6}
\]
which holds for arbitrary initial conditions \( x_o = (r_o, v_o), \) and \( \xi_o = \frac{\rho o^2}{2p_1(r_o,t_o)} \), with \( u_o \equiv v_o - V(r_o,t_o) \). Here is \( \frac{d}{dt} \) the Lagrangian derivative \( \frac{d}{dt} \equiv \frac{\partial}{\partial t} + v \cdot \nabla + F \cdot \frac{\partial}{\partial v}, \xi(r,t) \), while all quantities \( (u, E \text{ and } p_1) \) are evaluated along an arbitrary phase-space trajectory \( \{x(t)\} \).

B) Vice versa, if the solutions \( x(t)=(r(t), v(t)) \) and \( \xi(t) \) of Eqs.(5), (6) are known for arbitrary initial conditions \( x_o = (r_o, v_o), \) \( u_o \equiv v_o - V(r_o,t_o) \) and \( \xi_o = \frac{\rho o^2}{2p_1(r_o,t_o)} \) and for all \( (r,t) \in \Omega \times I \), it follows necessarily that in \( \Omega \times I \), \( \{\rho, V, p\} \) satisfy identically INSE.

**PROOF**

Let us first prove statement A), namely that INSE and the N-S dynamical system imply necessarily the validity of Eq.(6). For this purpose we first notice that by construction Eq.(3) admits a unique solution \( x(t) \) for arbitrary initial conditions \( x_o = (r_o, v_o) \in \Gamma \), while the same equation can also be expressed in terms of the relative velocity \( u= v - V(r,t) \). This yields
\[
\frac{d}{dt} u = F - \frac{D V(r,t)}{D t} - u \cdot \nabla V(r,t) \tag{7}
\]
Upon invoking the N-S equation (1) and by taking the scalar product of Eq.(7) by \( \rho u \), this equation implies
\[
\frac{d}{dt} \xi = u \cdot \nabla p_1 \left\{ \frac{\xi}{p_1} - \frac{1}{2} \right\} + \frac{\xi}{p_1} \left[ \frac{\partial}{\partial t} p_1 + \nabla \cdot (V p_1) \right],
\]
which finally gives
\[
\frac{d}{dt} \xi = \frac{\partial}{\partial t} \xi + v \cdot \nabla \xi + F \cdot \frac{\partial}{\partial v} \xi = -\frac{1}{2p_1} u \cdot \nabla p_1 + \frac{\xi}{p_1} \nabla \cdot V. \tag{8}
\]
As a consequence of the isochoricty condition (2) this equation reduces identically (i.e., for arbitrary initial conditions for the dynamical system) to Eq.(6). B) Vice versa, let us assume that the solutions \( x(t)=(r(t), v(t)) \) and \( \xi(t) \) of Eqs.(5), (6) are known for arbitrary initial conditions \( x_o \in \Gamma \) and \( \xi_o = \frac{\rho o^2}{2p_1(r_o,t_o)} \). In this case it follows the fluid fields necessarily must satisfy INSE in the whole set \( \Omega \times I \). It suffices, in fact, to notice that by assumption the evolution operator \( T_{t,t_o} \) is known. This permits to determine uniquely the kinetic distribution function at time \( t \), which reads
\[
f(x(t),t) = f(x_o,t_o)/J(x(t),t),
\]
where \( J(x(t),t) \) is the Jacobian of the flow \( x_o \rightarrow x(t) \). Hence, also its moments are uniquely prescribed, including both \( V(r,t) \) and \( p(r,t) \), in such a way that they result at least \( C^{(2,1)}(\Omega \times I) \). The inverse
kinetic equation, thanks to the special form of $\mathbf{F}$, as given by its definition, ensures that the N-S equation is satisfied identically in $\Omega \times I$. Moreover, since Eqs. (7) and (8) are by assumption fulfilled simultaneously, it follows that both the isochoricity condition (2) and the Poisson equation must be satisfied too in $\Omega \times I$. This completes the proof.

Let us analyze the consequences of the theorem. If the fluid velocity is assumed to satisfy both the N-S equation and isochoricity condition, the mass density satisfies the incompressibility condition, while $\{x(t)\}$ is an arbitrary trajectory of the N-S dynamical system, it follows that Eq. (6) determines uniquely the time-advancement of the fluid pressure. Hence, it provides an evolution equation for the fluid pressure, which by definition is equivalent to the Poisson equation [or to the equivalent problem obtained imposing, incompressibility, isochoricity and Navier-Stokes equations]. This equation can in principle be used to determine the fluid pressure at an arbitrary position $r \in \Omega$. However, since any given position can be reached by infinite phase-space (and also configuration-space) Lagrangian trajectories, it is sufficient to sample the configuration space by a suitable subset of Lagrangian trajectories (test particles), obtained by prescribing the initial condition $x_0$. The physical interpretation of the pressure evolution equation is elementary: it prescribes the Lagrangian time derivative of the fluid pressure, which is defined in the frame which is locally at the position $r(t)$ is co-moving with the test particle velocity $v(t)$. Let us now exploit the arbitrariness in the definition of the reduced pressure $p_o(t)$ and of the initial kinetic velocity $v_o$. Since IKT holds for arbitrary values of both parameters, this means that the dimensionless ratios $M_V = V/|v_o|$ and $M_p = p/p_o$, to be denoted as velocity and pressure effective Mach numbers, remain essentially free. As a consequence, it is possible: a) to construct asymptotic solutions of the pressure evolution equation based on low effective-Mach numbers expansions, i.e., for which $M_V, M_p \ll 1$ and moreover also b) to determine asymptotic approximations to Eq. (6). In particular, it is immediate to show that the pressure evolution equation admits the Chorin pressure evolution equation (3) as its leading-order asymptotic approximation. In fact, invoking the asymptotic ordering $M_V \sim O(\delta)$, there results in the leading-order approximation 

$$
\frac{d}{dt} \ln \mathcal{E} \approx u \cdot \nabla \ln p_1 [1 + o(\delta)].
$$

This implies 

$$
\frac{d}{dt} \xi \approx \frac{1}{p_1} u \cdot \nabla \ln p_1 [1 + o(\delta)].
$$

Invoking Eq. (5) and requiring also $M_p \sim O(\delta)$, with $p_o = \text{const.}$, there results, again to leading order in $o(\delta)$

$$
\frac{1}{p_o} \frac{d}{dt} p + \nabla \cdot \mathbf{V} \approx 0.
$$
This equation reduces to the Chorin’s Eq. (3), if the representation $\frac{1}{\rho_0} \frac{d}{dt} p \equiv \varepsilon^2 \frac{\partial}{\partial \tau} p$ is adopted. Hence, in this sense, Chorin’s pressure equation may simply be viewed as an asymptotic approximation to the exact pressure equation (6). Denoting $p_o \equiv \rho_0 c^2$, Eq. (9), together with the Navier-Stokes equation (1), describes a weakly compressible fluid characterized by a finite sound velocity $c$. This result suggests, however, also the possibility of constructing more accurate approximations for the pressure equation, which are higher-order in $\delta$ and have prescribed accuracy with respect to $\delta$.

V. CONCLUSIONS

The mere fact that an approximate evolution equation for the pressure can be obtained for incompressible fluids is not new. Examples of efficient Poisson solvers of this type are well known \[2, 5\]. These equations, however, are usually asymptotic, since the limit of incompressibility is achieved only for infinite sound speeds. Contrary to the common wisdom that for incompressible fluids it can only be achieved adopting relaxation-type models, in this paper we have proven that an exact local evolution equation can be obtained without modifying the incompressible Navier-Stokes equations, in particular without introducing the assumption of weak compressibility (and low Mach number). The proof has been reached by adopting an inverse kinetic theory \[6, 7\] which permits the identification of the (Navier-Stokes) dynamical system and of the corresponding evolution operator which advances in time the kinetic distribution function and the related fluid fields. A remarkable feature of the pressure evolution equation here obtained is that it is non-asymptotic and holds for arbitrary phase-space Lagrangian trajectories generated by the same dynamical system. This makes it possible, by suitably selecting these trajectories, to construct, in principle, asymptotic approximations with prescribed accuracy both for the pressure evolution equation and for its solutions. In particular, we have proven that under appropriate asymptotic assumptions to leading order (in the relevant asymptotic parameter) the well known Chorin pressure evolution equation is recovered. These results are important both from the conceptual viewpoint and for their possible applications in CFD. In particular, an interesting open problem is related to the numerical implementation of the pressure equation (6) in CFD schemes. These developments will be the object of future investigations.
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