New deformed models of $\mathcal{N} = 4$ and $\mathcal{N} = 8$

supersymmetric mechanics

Evgeny Ivanov
Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow Region, Russia
E-mail: eivanov@theor.jinr.ru

Abstract. We present two types of new deformations of extended supersymmetric mechanics. The first one generalizes $SU(2|1)$ mechanics and encompasses various models of the “curved” $SU(2|2)$ and $SU(4|1)$ mechanics as deformations of the flat $\mathcal{N} = 8$ mechanics models. Another type is a generalization of the $\mathcal{N} = 4$ mechanics associated with the multiplets $(4, 4, 0)$ and involving hyper-Kähler $d = 1$ sigma models in the bosonic sector. This kind of deformation results in $\mathcal{N} = 4$ models with quaternion–Kähler $d = 1$ sigma models as the bosonic core.

1. Motivations
Supersymmetric Quantum Mechanics (SQM) [1] is the simplest ($d = 1$) supersymmetric theory. Its salient features are:

- It catches the basic properties of higher-dimensional supersymmetric theories via the dimensional reduction;
- It provides superextensions of integrable models like Calogero-Moser systems, Landau-type models, etc.

Originally, in [1] there was considered the simplest, $\mathcal{N} = 2$ version of SQM, with the superalgebra

$$\{Q, \bar{Q}\} = 2H, \quad Q^2 = \bar{Q}^2 = 0, \quad [Q, H] = [\bar{Q}, H] = 0.$$  

An extended $\mathcal{N} > 2, d = 1$ supersymmetry is rather specific compared to its higher-dimension counterparts: it implies dualities between various supermultiplets, nonlinear “cousins” of off-shell linear multiplets, etc (see, e.g., [2]) . The $\mathcal{N} = 4$ SQM, with the underlying algebra

$$\{Q_\alpha, \bar{Q}^\beta\} = 2\delta_\alpha^\beta H, \quad \alpha = 1, 2,$$

is of special interest. In particular, a subclass of $\mathcal{N} = 4$ SQM models have as their bosonic target, Hyper-Kähler (HK) manifolds.

In this Talk, two different types of deformations of $\mathcal{N} = 4$ SQM models will be described.
2. From deformed $\mathcal{N} = 4$ SQM to its $\mathcal{N} = 8$ extensions

The first type of deformed SQM arises, when choosing some semi-simple supergroups instead of higher-rank $d = 1$ super-Poincaré:

A. Standard extension:

$\mathcal{N} = 2, \ d = 1 \quad \Rightarrow \quad \mathcal{N} > 2, \ d = 1$ Poincaré,

B. Non-standard extension:

$\mathcal{N} = 2, \ d = 1 \equiv u(1|1) \quad \Rightarrow \quad su(2|1) \subset su(2|2) \subset \ldots .$

In the chain B, the closure of supercharges contains, besides $H$, also internal symmetry generators. They commute with $H$, but not with the supercharges. The deformed $\mathcal{N} = 4$ SQM is associated with the superalgebra $su(2|1)$:

\[
\{Q^i, \bar{Q}_j\} = 2m \left( I^i_j - \delta^i_j F \right) + 2\delta^i_j H, \quad [I^i_j, I^k_l] = \delta^i_l I^k_j - \delta^i_j I^k_l, \quad [I^i_j, Q^k] = \delta^i_j Q^k - \frac{1}{2} \delta^i_j Q^k;
\]

\[
[F, Q^k] = \frac{1}{2} Q^k.
\]

The parameter $m$ is a deformation parameter: when $m \to 0$, the standard $\mathcal{N} = 4, d = 1$ super-Poincare is recovered.

The simplest models with worldline realization of $su(2|1)$ were considered in [3], [4] (where it was named “weak $d = 1$ supersymmetry”) and in [5]. The world-line multiplets considered were $(2, 4, 2)$ and $(1, 4, 3)$. The systematic superfield approach to $su(2|1)$ supersymmetry was worked out in [6] - [8] and [9]. The models built on the multiplets $(1, 4, 3), (2, 4, 2)$ and $(4, 4, 0)$ were studied at the classical and quantum levels.

Recently, $su(2|1)$ invariant versions of super Calogero-Moser systems were constructed and quantized in [10] - [12].

The common features of all these models are:

- The oscillator-type Lagrangians for the bosonic fields, with $m^2$ as the oscillator strength;
- The appearance of the Wess-Zumino type terms for the bosonic fields, of the type $\sim im(\dot{z} \ddot{z} - \dot{\bar{z}} \ddot{z})$;
- At the lowest energy levels, wave functions form atypical $su(2|1)$ multiplets, with unequal numbers of the bosonic and fermionic states and vanishing $su(2|1)$ Casimirs.

As a natural next step, analogous deformations of $\mathcal{N} = 8, d = 1$ superalgebra were addressed.

The flat $\mathcal{N} = 8$ superalgebra,

\[
\{Q_A, Q_B\} = 2\delta_{AB} H, \quad A, B = 1, \ldots, 8,
\]

admits two deformations with the minimal number of extra bosonic generators.

A. Superalgebra $su(2|2)$ [13]:

\[
\{Q^{ia}, S^{jb}\} = 2im \left( \epsilon^{ab} L^{ij} - \epsilon^{ij} R^{ab} \right) + 2\epsilon^{ab} \epsilon^{ij} C,
\]

\[
\{Q^{ia}, Q^{jb}\} = 2\epsilon^{ij} \epsilon^{ab}(H + C_1), \quad \{S^{ia}, S^{jb}\} = 2\epsilon^{ij} \epsilon^{ab}(H - C_1),
\]

\[
{Q^{ia}, S^{jb}} = 2m \left( I^i_j - \delta^i_j F \right) + 2\delta^i_j H, \quad [I^i_j, I^k_l] = \delta^i_l I^k_j - \delta^i_j I^k_l, \quad [I^i_j, Q^k] = \delta^i_j Q^k - \frac{1}{2} \delta^i_j Q^k;
\]

\[
[F, Q^k] = \frac{1}{2} Q^k.
\]
where $C$ and $C_1$ are central charge generators, $L_{ij}$ and $R^{ab}$ generate $SO(4) = SU(2) \times SU(2)$ R-symmetry group.

**B. Superalgebra $su(4|1)$** [14]:

\[
\begin{align*}
\{Q^I, \bar{Q}^J\} &= 2m L^I_J + 2\delta^I_J H, \quad I, J = 1, \ldots, 4, \\
[\mathcal{H}, Q^K] &= -\frac{3m}{4} Q^K, \\
[\mathcal{H}, \bar{Q}_L] &= \frac{3m}{4} \bar{Q}_L.
\end{align*}
\]

It possesses $SU(4)$ R-symmetry automorphisms generated by $L^I_J$ (instead of $SO(8)$ or $SU(2) \times SU(2)$ automorphisms of the previous two cases).

In the case A we constructed, by analogy with $SU(2|1)$, the worldline superfield techniques and presented a few $SU(2|2)$ SQM models as deformations of flat $N = 8$ models. These are based on the off-shell $SU(2|2)$ multiplets $(3, 8, 5)$, $(4, 8, 4)$ and $(5, 8, 3)$.

It turned out that not all of the admissible multiplets of the flat $N = 8$ SQM have $SU(2|2)$ analogs. It is most important that the so called “root” $N = 8$ multiplet $(8, 8, 0)$ does not have such an analog.

It is known that all other flat $N = 8$ multiplets and their invariant actions can be obtained from the root multiplet and its general action through the appropriate covariant truncations (or Hamiltonian reductions, in the Hamiltonian formalism) [15]. The natural question was as to how to construct a deformed version of the $(8, 8, 0)$ multiplet.

This construction becomes possible in the models based on the worldline realizations of the supergroup $SU(4|1)$. The corresponding $SU(4|1)$ multiplet contains $8 = 2 + 6$ real bosonic fields $(\phi, y^{Ij})$ in the $SU(4)$ representation $(\mathbf{1} \oplus \mathbf{6})$ and $4$ complex fermionic fields $\chi^L$ in the fundamental of $SU(4)$.

An interesting peculiarity of the $SU(4|1)$ supersymmetry is the existence of two *non-equivalent* $(8, 8, 0)$ multiplets, with the inverted $SU(4)$ assignments of the component fields.

The detailed exposition of $SU(4|1)$ mechanics is given in the talk by Stepan Sidorov at the same Conference.

I finish this part of the talk by indicating two problems for the future study.

- Possible applications - in supersymmetric matrix models [16]. They possess $SU(4|2)$ invariance, and so $SU(2|2) \subset SU(4|2)$ and $SU(4|1) \subset SU(4|2)$ SQM can describe some important truncations of these models.
- Normally, the matrix model actions are free (before gauging). Our actions include non-trivial interactions and so can hopefully be treated as some effective actions, with quantum corrections taken into account.

### 3. QK $N = 4$ SQM as a deformation of HK SQM models

Another type of deformations of $N = 4$ SQM models proceeds from the general Hyper-Kähler (HK) subclass of the latter. The deformed models are $\mathcal{N} = 4$ supersymmetrization of the Quaternion-Kähler (QK) $d = 1$ sigma models [17].

Both HK and QK $N = 4$ SQM models can be derived from $\mathcal{N} = 4, d = 1$ harmonic superspace approach [18] as a proper adaption of the $\mathcal{N} = 2, d = 4$ harmonic approach [19], [20].

HK manifolds are bosonic targets of sigma models with rigid $N = 2, d = 4$ supersymmetry [21]. After coupling these models to local $\mathcal{N} = 2, d = 4$ supersymmetry in the supergravity
framework the target spaces are deformed into the so called Quaternion-Kähler (QK) manifolds [22]. QK manifolds are also 4n dimensional, but their holonomy group is a subgroup of \(Sp(1) \times Sp(n)\) (as opposed to \(Sp(n)\) of the HK case).

In this part of my talk, which is based on a recent paper with Luca Mezincescu, it will be shown how to construct \(\mathcal{N} = 4\) SQM with an arbitrary QK bosonic target. Like in constructing \(\mathcal{N} = 4\) HK SQM, the basic tool is \(d = 1\) harmonic superspace.

We begin with listing various \(\mathcal{N} = 4, d = 1\) superspaces and the relevant tools:

- **Ordinary \(\mathcal{N} = 4, d = 1\) superspace:**
  \((t, \theta^i, \bar{\theta}_k), \ i, k, = 1, 2;\)

- **Its harmonic extension:**
  \((t, \theta^i, \bar{\theta}_k) \Rightarrow (t, \theta^i, \bar{\theta}_k, u_+^+, u_+^−), \ u_±^+, u_±^− \in SU(2)_{Aut};\)

- **Harmonic superspace in the analytic basis:**
  \((t_A, \theta^+^+, \bar{\theta}^+^−, \bar{\theta}^−^+, \bar{\theta}^−^−) \equiv (\zeta, u^±, \theta^−, \bar{\theta}^−),\)
  \(\theta^± = \theta^i u_±^i, \bar{\theta}^± = \bar{\theta}_k u_±^k, \ t_A = t + i(\theta^+ \bar{\theta}^− + \theta^− \bar{\theta}^+);\)

- **The analytic subspace and analytic \(\mathcal{N} = 4, d = 1\) superfields:**
  \((\zeta, u^±) = (t_A, \theta^+^+, \bar{\theta}^+^−, \bar{\theta}^−^+, \bar{\theta}^−^−),\)
  \(D^+ = \frac{\partial}{\partial \theta^−}, \bar{D}^+ = -\frac{\partial}{\partial \bar{\theta}^−}, \ D^+ \Phi = \bar{D}^+ \Phi = 0 \Rightarrow \Phi = \Phi(\zeta, u^±);\)

- **Harmonic derivatives:**
  \[D^+ = u_α^+ \partial_{u_α^+} + \theta^+ \partial_{\theta^+} + \bar{\theta}^− \partial_{\bar{\theta}^−} + 2i\theta^± \bar{\theta}^± \partial_{t_A},\]
  \([D^+, D^+] = [\bar{D}^+, D^+] = 0 \Rightarrow D^+ \Phi(\zeta, u^±) \text{ is analytic}.\)

The basic \(\mathcal{N} = 4, d = 1\) multiplet \((4, 4, 0)\) is described off shell by an analytic superfield \(q^{+a}(\zeta, u),\)

\((4, 4, 0) \iff q^{+a}(\zeta, u) \propto (f^{ia}, \chi^a, \bar{\chi}^a), \ a = 1, 2,\)

subjected to the constraints

\((a) \ D^+ q^{+a} = \bar{D}^+ q^{+a} = 0 \quad (\text{Grassmann analyticity}),\)

\((b) \ D^+ q^{+a} = 0 \quad (\text{Harmonic analyticity}),\)

\((a) + (b) \quad \Rightarrow q^{+a} = f^{ka} u_k^+ + \theta^+ \chi^a - \bar{\theta}^+ \bar{\chi}^a - 2i\theta^+ \bar{\theta}^+ f^{ka} u_k^−.\)

The free off-shell action of \(q^{+a}\) reads:

\[S_{\text{free}} \sim \int dt d^4\theta d u q^{+a}q^{−a}_a \sim \int dt \left( f^{ia} f_{ia} - \frac{i}{2} \chi^a \bar{\chi}^a \right), \ q^{−a} := D^− q^{+a}.\]
The nonlinear $d = 1$ sigma model action is:

$$S_{\text{nonl}} \sim \int dt d^4 \theta du \mathcal{L}(q^+, q^-, u^\pm).$$

In the bosonic sector one finds HKT ("Hyper-Kähler with torsion") sigma model (see [23] and [24]). In components, the torsion appears in a term quartic in fermions.

How to construct general HK $\mathcal{N} = 4, d = 1$ sigma models? No torsion is present in this case, the geometry involves Riemann curvature tensor only. The answer was given in [23].

The basic superfield is a real analytic, $q^+ A(\zeta, u) = f^A u_+^i + \ldots, i = 1, 2, A = 1, \ldots 2n$, it encompasses just $4n$ fields $f^A(t)$ parametrizing the target bosonic manifold, $(q^+_A) = \Omega^{AB} q^+_B$, with $\Omega^{AB} = -\Omega^{BA}$ being a constant symplectic metric.

The linear constraint $D^{++} q^+ A = 0$ is promoted to a nonlinear one

$$D^{++} q^+ A = \Omega^{AB} \frac{\partial L^{++}}{\partial q^+_B}(q^+, u^\pm).$$

The superfield action is bilinear as in the free case,

$$S_{HK} \sim \int dt d^4 \theta du \Omega^{AB} q^+_B q^-_A = \int dt [g_{iAkB} f^A f^{kB} + \ldots],$$

the whole interaction appears only on account of nonlinear deformation of the $q^+ A$-constraint.

The function $L^{++}$ is an analytic hyper-Kähler potential [25]: every $L^{++}$ produces the component HK metric $g_{iAkB}(f)$ and, vice versa, each HK metric originates from some HK potential $L^{++}$.

The harmonic superspace approach supplies the most natural arena for defining $\mathcal{N} = 4$ QK SQM. Basic new features of these models as compared to their HK prototypes are as follows.

(i) QK SQM model corresponding to $4n$ dimensional QK manifold requires introducing $n + 1$ multiplets $(4, 4, 0)$ described by analytic superfields $q^{+a}(\zeta, w^\pm), (a = 1, 2), Q^{+r}(\zeta, w^\pm), (r = 1, \ldots 2n)$. An extra superfield $q^{+a}(\zeta, w^\pm)$ is $d = 1$ analog of $\mathcal{N} = 2, d = 4$ “conformal compensator”.

(ii) QK SQM actions are invariant under local $\mathcal{N} = 4, d = 1$ supersymmetry realized by the appropriate transformations of super coordinates, including harmonic variables $w^\pm$.

(iii) For ensuring local invariance it is necessary to introduce a supervielbein $E(\zeta, \theta^-, \bar{\theta}^-, w^\pm)$ which is a general $\mathcal{N} = 4, d = 1$ superfield.

(iv) Besides the $(q^+, Q^+)$ superfield part, the correct action should involve a “cosmological term” depending on the vielbein superfield only.

By analogy with the $\mathcal{N} = 2, d = 4$ case we postulate that local $\mathcal{N} = 4, d = 1$ supersymmetry preserves the Grassmann harmonic analyticity,

$$\delta t_A = \Lambda(\zeta, w), \delta \theta^+ = \Lambda^+(\zeta, w), \delta \bar{\theta}^+ = \bar{\Lambda}^+(\zeta, w), \delta w^+_i = \Lambda^{++}(\zeta, w) w^+_i, \delta w^-_i = 0, \delta \theta^- = \Lambda^-(\zeta, w, \theta^-, \bar{\theta}^-), \delta \bar{\theta}^- = \bar{\Lambda}^-(\zeta, w, \theta^-, \bar{\theta}^-).$$
The explicit structure of the minimal set of analytic parameters is as follows

\[ \Lambda = 2b + \ldots, \]
\[ \Lambda^+ = \lambda^i w^+_i + \ldots, \]
\[ \Lambda^{++} = \tau^{(ik)} w^+_i w^+_k + \ldots, \]
\[ \Lambda^- = \lambda^i w^-_i + \ldots. \]

Here, \( b(t), \tau^{(ik)}(t) \) and \( \lambda^i(t), \bar{\lambda}^i(t) \) are arbitrary local parameters, bosonic and fermionic, respectively. The local \( \mathcal{N} = 4, d = 1 \) supergroup obtained is isomorphic to the classical (having no central charges) “small” \( \mathcal{N} = 4 \) superconformal symmetry.

How to generalize the \((4, 4, 0)\) superfields \( q^+ A(\zeta, w) \) to local supersymmetry?

- The simplest possibility is to keep the linear constraint
  \[ D^{++} q^+ a = 0. \]

- It is covariant under the transformations
  \[ \delta D^{++} = -\Lambda^{++} D^0, \quad \delta q^+ a = \Lambda_0 q^+ a, \quad \Lambda^{++} = D^{++} \Lambda_0. \]

- To construct invariant actions, one needs to know the transformations of the integration measures \( \mu_H := dt dw d^2 \theta^+ d^2 \theta^-, \mu^{(-2)} := dt dw d^2 \theta^+, \)
  \[ \delta \mu^{(-2)} = 0, \quad \delta \mu_H = \mu_H 2 \Lambda_0, \]
  and that of the harmonic derivative \( D^{--} \),
  \[ \delta D^{--} = -(D^{--} \Lambda^{++}) D^{--}. \]

4. Simplest invariant action

Introduce, besides \( q^+ a(\zeta, w), a = 1, 2, \ldots \) extra superfields \( Q^{++}(\zeta, w), r = 1, 2, \ldots 2n \), which encompass \( n \) off-shell multiplets \((4, 4, 0)\), obey the same linear harmonic constraint \( D^{++} Q^{++} = 0 \) and transform under local \( \mathcal{N} = 4 \) supersymmetry in the same way as \( q^+ a \). The basic part of the total invariant action can be then written as

\[ S_{(2)} = \int \mu_H L_{(2)}(q, Q), \quad L_{(2)}(q, Q) = \gamma q^+ a q^- - Q^{++} Q^{-}, \]
\[ q^- := D^{--} q^+_a, \quad Q^- := D^{--} Q^+_r, \]
and \( \gamma = \pm 1 \). The new object is vielbein \( E \) which is harmonic-independent, \( D^{++} E = D^{--} E = 0 \), and possesses the following transformation law under local \( \mathcal{N} = 4 \) supersymmetry

\[ \delta E = (-4 \Lambda_0 + 2 D^{--} \Lambda^{++}) E, \quad D^{++} (-4 \Lambda_0 + 2 D^{--} \Lambda^{++}) = 0. \]

One more important term in the action is:

\[ S_\beta = \beta \int \mu_H \sqrt{E}, \quad \delta S_\beta = \beta \int \mu_H D^{--} \Lambda^{++} \sqrt{E} = 0. \]

The full simplest locally \( \mathcal{N} = 4 \) supersymmetric action reads

\[ S_{HP} \sim S_{(2)} + S_\beta = \int \mu_H [E L_{(2)} + \beta \sqrt{E}]. \]
Why should the “cosmological constant” term \( S_\beta \) be added?

To clarify this issue, let us pass to the bosonic limit:

\[
q^{+a} \Rightarrow f^{ia} w_i^+ - 2i \theta^+ \theta^- f^{jk} j^{ja^+} w_j^- , \quad Q^{+r} \Rightarrow F^{ir} w_i^+ - 2i \theta^+ \theta^- F^{ik} w_k^- ,
\]

\[
E \Rightarrow e + \theta^+ \theta^- M - \bar{\theta}^+ \bar{\theta}^- \bar{M} + \theta^+ \theta^- (\mu - i \epsilon) + \bar{\theta}^+ \bar{\theta}^- (\bar{\mu} + i \bar{\epsilon})
\]

\[
+ 4i (\theta^+ \theta^- w_i^- w_k^- - \theta^+ \bar{\theta}^- w_i^- w_k^- - \theta^- \bar{\theta}^+ w_i^- w_k^- + \theta^- \theta^- w_i^- w_k^-) L^{(ik)}
\]

\[+ 4 \theta^+ \bar{\theta}^+ \theta^- \bar{\theta}^- [D + 2 \bar{L}^{(ik)} w_i^+ w_k^-].
\]

In the bosonic limit,

\[
L_{HP} \Rightarrow \frac{1}{2} e \left( \dot{F}^{ir} \dot{F}_{ir} - \gamma j^{ira} j^{ia} \right) + L_{ik} \left[ F^{(ir} \dot{F}^{k)}_{ir} - \gamma f^{(ia} j^{ka)} \right]
\]

\[
+ \frac{1}{4} D \left( \gamma f^{ia} f_{ia} - F^{ir} F_{ir} + \bar{\beta} \frac{1}{\sqrt{\epsilon}} \right)
\]

\[+ \frac{\beta}{4} \frac{1}{e^{3/2}} \left[ L^{ik} L_{ik} - \frac{1}{8} (M \bar{M} + \mu^2 + \bar{\epsilon}^2) \right].
\]

The auxiliary fields \( M, \bar{M} \) and \( \mu \) fully decouple and can be put equal to zero by their equations of motion. Also, \( e(t) \) is an analog of \( d = 1 \) vierbein, so it is natural to choose the gauge

\[ e = 1. \]

Then the bosonic Lagrangian becomes

\[
L_{HP} \Rightarrow \frac{1}{2} \left( \dot{F}^{ir} \dot{F}_{ir} - \gamma j^{ira} j^{ia} \right) + L_{ik} \left[ F^{(ir} \dot{F}^{k)}_{ir} - \gamma f^{(ia} j^{ka)} \right]
\]

\[+ \frac{1}{4} D \left( \gamma f^{ia} f_{ia} - F^{ir} F_{ir} + \bar{\beta} \right) + \frac{\beta}{4} L^{ik} L_{ik}.
\]

At \( \beta \neq 0 \) \( L^{ik} \) can be eliminated by its algebraic equation of motion, while \( D \) serves as the Lagrange multiplier for the constraint relating \( f^{ia} \) and \( F^{ir} \):

\[
L^{ik} = -\frac{1}{\beta} \left[ F^{(ir} \dot{F}^{k)}_{ir} - \gamma f^{(ia} j^{ka)} \right], \quad \gamma f^{ia} f_{ia} - F^{ir} F_{ir} + \bar{\beta} = 0.
\]

Assuming that \( f^{ia} \) starts with a constant (compensator!), one uses local \( SU(2) \) freedom, \( \delta f^{ia} = \tau_i^a f^{ia} \), to gauge away the triplet from \( f^{ia} \),

\[ f^{(ia)} = 0 \Rightarrow f^a_i = \sqrt{2} \delta^a_i \omega. \]

Then the constraint can be solved as

(a) \( \gamma = 1 \Rightarrow \beta < 0 \), \( \omega = \frac{|\beta|^{1/2}}{2} \sqrt{1 + \frac{1}{|\beta|} F^2} \),

(b) \( \gamma = -1 \Rightarrow \beta > 0 \), \( \omega = \frac{\beta^{1/2}}{2} \sqrt{1 - \frac{1}{\beta} F^2} \).
The final form of the bosonic action for $\gamma = 1$ is
$$L_{HP} = \frac{1}{2} \left[ (\hat{F} \hat{F}) + \frac{2}{|\beta|} (F_{\mu(i} \hat{F}_{\beta)} (F^{(i} \hat{F}^{s j})) - \frac{1}{|\beta|} \frac{1}{1 + |\beta|^2} (F \hat{F}) (F \hat{F}) \right].$$

The option $\gamma = -1$ is recovered by the replacement $|\beta| \to -|\beta|$.

These actions describe $d = 1$ nonlinear sigma models on non-compact and compact maximally “flat” $4n$ dimensional QK manifolds, respectively:
$$\text{H}^n = \frac{Sp(1, n)}{Sp(1) \times Sp(n)}, \quad \text{HP}^n = \frac{Sp(1 + n)}{Sp(1) \times Sp(n)}.$$ 

Thus $\mathcal{N} = 4$ mechanics constructed is just superextensions of these QK $d = 1$ sigma models.

5. Generalizations
The basic step in generalizing to $\mathcal{N} = 4$ mechanics with an arbitrary QK manifold is passing to nonlinear harmonic constraints
$$D^{++} q^+ \gamma - \frac{1}{2} \frac{\partial}{\partial q^a} \left[ \kappa^2 (w^- \cdot q^+)^2 \mathcal{L}^{+4} \right] = 0,$$
$$D^{++} Q^+ r + \frac{1}{2} \frac{\partial}{\partial Q^+ r} \left[ \kappa^2 (w^- \cdot q^+)^2 \mathcal{L}^{+4} \right] = 0,$$
$$\mathcal{L}^{+4} \equiv \mathcal{L}^{+4} \left( \frac{Q^+ r}{\kappa (w^+ q^+)}, \frac{q^+ a}{w^- q^+}, w_i^- \right), \quad \kappa := \sqrt{2} |\beta|^{1/2}. $$

The invariant superfield action is the same as in the $\text{H}^n$ case
$$S_{QK} \sim [\hat{S}(2) + S_\beta] = \int \mu_H [E \hat{L}(2) + \beta \sqrt{E}],$$
$$\hat{L}(2) = \gamma q^+ a q^- - Q^+ r Q^- r, \quad q^- = D^- q^+ a, \quad Q^- = D^- Q^+ r.$$ 

The bosonic action precisely coincides with the $d = 1$ reduction of the general QK sigma model action derived from $\mathcal{N} = 2, d = 4$ supergravity-matter action in [26]. This coincidence proves that we have constructed the most general QK $\mathcal{N} = 4$ mechanics. A new possibility offered by the $d = 1$ framework is the locally $\mathcal{N} = 4$ supersymmetric Wess-Zumino type superfield term
$$S_{QK}^{(WZ)} = i \int \mu_0^{-2} \mathcal{L}^{+2} \left( \frac{Q^+ r}{\kappa (w^- q^+)} \left( \frac{q^+ a}{w^- q^+}, w_i^- \right).$$

It is invariant due to the invariance of the analytic subspace integration measure. It is a direct generalization of analogous term in the flat $\mathcal{N} = 4, 1D$ supersymmetry [23] and describes a coupling to an abelian background gauge field given on QK target manifold.

One more possibility is to consider the following generalization of the $\text{HP}^n$ action
$$S^{loc}(q, Q) = \int \mu_H \sqrt{E} \mathcal{F}(X, Y, w^-), \quad X := \sqrt{E} (q^+ a q^-), \quad Y := \sqrt{E} (Q^+ r Q^- r),$$
$$D^{++} q^+ a = D^{++} Q^+ r = 0 \Rightarrow D^{\pm \pm} X = D^{\pm \pm} Y = 0.$$ 

When $E = \text{const}$, it is reduced to the particular form of the HKT action $\int \mu_H \mathcal{F}(q^+ A, q^- B, w^\pm)$, while for $\mathcal{F}(X, Y, w^-) = \gamma X - Y + \beta$ just to $\text{HP}^n$ action. So the target geometry associated with $S^{loc}(q, Q)$ is expected to be a kind of QKT, i.e. “Quaternion-Kähler with torsion”. To date, not too much known about such geometries [27]...
6. Summary and Outlook

• Two deformations of $\mathcal{N} = 8$ supersymmetric mechanics based on the supergroups $SU(2|2)$ and $SU(4|1)$ as a generalization of the $SU(2|1)$ mechanics were presented.

• $\mathcal{N} = 4, d = 1$ harmonic superspace methods were used to construct a new class of $\mathcal{N} = 4$ supersymmetric mechanics models, those with $d = 1$ Quaternion-Kähler sigma models as a bosonic core. The basic distinguishing feature of these models is local $\mathcal{N} = 4, d = 1$ supersymmetry.

• The superfield and component actions were presented for general $\mathcal{N} = 4$ QK mechanics, and for the maximally “flat” $\mathbb{H}P^n$ mechanics.

• A few generalizations of QK mechanics were proposed, in particular “Quaternion-Kähler with torsion” (QKT) models.

Some further lines of study:

(a) To construct the Hamiltonian formalism for the new class of mechanical systems, to perform quantization, at least for the simplest case of $\mathbb{H}P^n$ mechanics, to find the energy spectra;

Last news (I & Mezincescu, 2018, in preparation): Noether currents were calculated and shown to be vanishing on-shell, $Q^i = \bar{Q}^j = H = J_{kl} = 0$;

(b) To explicitly construct some other $\mathcal{N} = 4$ QK SQM models, e.g., those associated with symmetric QK manifolds (“Wolf spaces”);

(c) To construct locally supersymmetric versions of other off-shell $\mathcal{N} = 4, d = 1$ multiplets (such as $(3, 4, 1)$, $(1, 4, 3)$, etc) and of the associated SQM systems (Landau-type, Calogero-Moser-type and others);

(d) To reveal links between the two types of SQM deformations reviewed in this talk.

Acknowledgments

The author thanks the organizers of The 32nd International Colloquium on Group Theoretical Methods in Physics for the kind hospitality in Prague and Olaf Lechtenfeld, Luca Mezincescu and Stepan Sidorov for successful collaboration.

References

[1] Witten E 1981 Dynamical Breaking of Supersymmetry Nucl. Phys. B 188 513
[2] Fedoruk S, Ivanov E and Lechtenfeld O 2012 Superconformal Mechanics J. Phys. A 45 173001 (Preprint arXiv:1112.1947 [hep-th])
[3] Bellucci S Nersessian A 2003 (Super)Oscillator on $\mathbb{C}P(N)$ and Constant Magnetic Field Phys. Rev. D 67 065013 (Preprint arXiv:hep-th/0211070)
[4] Smilga A V 2004 Weak supersymmetry Phys. Lett. B 585 173 (Preprint arXiv:hep-th/0311023)
[5] Römelsberger C 2006 Counting chiral primaries in $\mathcal{N} = 1, d = 4$ superconformal field theories Nucl. Phys. B 747 329 (Preprint arXiv:hep-th/0510060)
[6] Ivanov E and Sidorov S 2014 Deformed Supersymmetric Mechanics Class. Quant. Grav. 31 075013 (Preprint arXiv:1307.7690 [hep-th])
[7] Ivanov E and Sidorov S 2014 Super Kähler oscillator from SU(2|1) superspace J. Phys. A 47 292002 (Preprint arXiv:1312.6821 [hep-th])
[8] Ivanov E and Sidorov S 2016 SU(2|1) mechanics and harmonic superspace Class. Quant. Grav. 33 055001 (Preprint arXiv:1507.00987 [hep-th])
[9] Ivanov E Sidorov S and Toppan F 2015 Superconformal mechanics in SU(2|1) superspace Phys. Rev. D 91 085032 (Preprint arXiv:1501.05622 [hep-th])

[10] Fedoruk S and Ivanov E 2016 Gauged spinning models with deformed supersymmetry JHEP 1611 103 (Preprint arXiv:1610.04202 [hep-th])

[11] Fedoruk S Ivanov E and Sidorov S 2018 Deformed supersymmetric quantum mechanics with spin variables JHEP 1801 132 (Preprint arXiv:1710.02130 [hep-th])

[12] Fedoruk S Ivanov E Lechtenfeld O and Sidorov S 2018 Quantum SU(2|1) supersymmetric Calogero-Moser spinning systems JHEP 1804 043 (Preprint arXiv:1801.00206 [hep-th])

[13] Ivanov E Lechtenfeld O Sidorov S 2016 SU(2|2) supersymmetric mechanics JHEP 1611 031 (Preprint arXiv:1609.00490 [hep-th])

[14] Ivanov E Lechtenfeld O and Sidorov S 2018 Deformed $\mathcal{N} = 8$ mechanics of (8, 8, 0) multiplets JHEP 1808 193 (Preprint arXiv:1807.11804 [hep-th])

[15] Ivanov E Lechtenfeld O and Sutulin A 2008 Hierarchy of $\mathcal{N} = 8$ Mechanics Models Nucl. Phys. B 790 493 (Preprint arXiv:0705.3064 [hep-th])

[16] Berenstein D Maldaicna J and Nastase H 2002 Strings in flat space and pp waves from $N = 4$ super Yang-Mills JHEP 0204 013 (Preprint arXiv:hep-th/0202021)

[17] Ivanov E and Mezincescu L 2017 Quaternion-Kahler $\mathcal{N} = 4$ supersymmetric mechanics JHEP 1712 016 (Preprint arXiv:1709.02286 [hep-th])

[18] Ivanov E and Lechtenfeld O 2003 $\mathcal{N} = 4$ Supersymmetric Mechanics in Harmonic Superspace JHEP 0309 073 (Preprint arXiv:hep-th/0307111)

[19] Galperin A S Ivanov E A Kalitzin S Ogievetsky V I and Sokatchev E S 1984 Unconstrained $\mathcal{N} = 2$ matter, Yang-Mills and supergravity theories in harmonic superspace Class. Quant. Grav. 1 469

[20] Galperin A S Ivanov E A Ogievetsky V I and Sokatchev E S 2001 Harmonic Superspace (Cambridge University Press 306 p.)

[21] Alvarez-Gaumé L and Freedman D Z 1981 Geometric structure and ultraviolet finiteness in the supersymmetric $\sigma$ model Commun. Math. Phys. 80 443

[22] Bagger J and Witten E 1983 Matter couplings in $N = 2$ supergravity Nucl. Phys. B 222 1

[23] Delduc F and Ivanov E 2012 $\mathcal{N} = 4$ mechanics of general (4,4,0) multiplets Nucl. Phys. B 855 815 (Preprint arXiv:1107.1429 [hep-th])

[24] Fedoruk S Ivanov E and Smilga A 2018 Generic HKT geometries in the harmonic superspace approach J. Math. Phys. 59 083501 (Preprint arXiv:1802.09675 [hep-th])

[25] Galperin A Ivanov E Ogievetsky V and Sokatchev E 1986 Hyper-Kähler metrics and harmonic superspace Commun. Math. Phys. 103 515

[26] Ivanov E and Valent G 2000 Quaternionic metrics from harmonic superspace: Lagrangian approach and quotient construction Nucl. Phys. B 576 543 (Preprint arXiv:hep-th/0001165)

[27] Howe P S Opfermann A and Papadopoulos G 1998 Twistor spaces for QKT manifolds Commun. Math. Phys. 197 713 (Preprint arXiv:hep-th/9710072)