MULTIPlicity of Nodal solutions to the Yamabe problem

Mónica Clapp and Juan Carlos Fernández

Abstract. Given a compact Riemannian manifold \((M, g)\) without boundary of dimension \(m \geq 3\) and under some symmetry assumptions, we establish existence of one positive and multiple nodal solutions to the Yamabe-type equation

\[-\text{div}_g(a \nabla u) + bu = c|u|^{2^*-2}u \quad \text{on } M,\]

where \(a, b, c \in C^\infty(M)\), \(a\) and \(c\) are positive, \(-\text{div}_g(a \nabla) + b\) is coercive, and \(2^* = \frac{2m}{m-2}\) is the critical Sobolev exponent.

In particular, if \(R_g\) denotes the scalar curvature of \((M, g)\), we give conditions which guarantee that the Yamabe problem

\[\Delta_g u + \frac{m-2}{4(m-1)} R_g u = \kappa u^{2^*-2} \quad \text{on } M\]

admits a prescribed number of nodal solutions.

Key words: Semilinear elliptic PDE on manifolds; Yamabe problem; nodal solution; symmetric solution; blow-up analysis; nonexistence of ground states.

2010 MSC: 35J61, 58J05, 35B06, 35B33, 35B44.

1. Introduction and statement of results

Given a compact Riemannian manifold \((M, g)\) without boundary of dimension \(m \geq 3\), the Yamabe problem consists in finding a metric \(\hat{g}\) conformally equivalent to \(g\) with constant scalar curvature. If \(\hat{g}\) is conformally equivalent to \(g\) we can write it as \(\hat{g} = u^{4/(m-2)}g\) with \(u \in C^\infty(M), \, u > 0\). Then, \(\hat{g}\) has constant scalar curvature \(c_m \kappa\) iff \(u\) is a positive solution to the problem

\[(\mathcal{Y}_g) \quad \Delta_g u + c_m R_g u = \kappa |u|^{2^*-2} u, \quad u \in C^\infty(M),\]

where \(\Delta_g = -\text{div}_g \nabla_g\) is the Laplace-Beltrami operator, \(c_m := \frac{m-2}{4(m-1)}\), \(R_g\) is the scalar curvature of \((M, g)\), \(\kappa \in \mathbb{R}\), and \(2^* := \frac{2m}{m-2}\) is the critical Sobolev exponent. Here we shall always assume that \(\kappa > 0\).

This problem was completely solved by the combined efforts of Yamabe [37], Trudinger [34], Aubin [3] and Schoen [33]. A detailed discussion may be found in [4, 24]. Obata [26] showed that for an Einstein metric the solution to the Yamabe problem is unique. On the other hand, Pollack [29] showed that, if \(R_g > 0\), then...
there is a prescribed number of positive solutions to the Yamabe problem with constant positive scalar curvature in a conformal class which is arbitrarily close to $g$ in the $C^0$-topology. Compactness of the set of positive solutions was established by Khuri, Marques and Schoen [23] if $(M,g)$ is not conformally equivalent to the standard sphere and $\dim M \leq 24$. On the other hand, if $M \geq 25$, Brendle [6] and Brendle and Marques [7] showed that the set of positive solutions is not compact.

The equivariant Yamabe problem was studied by Hebey and Vaugon. They showed in [19] that for any subgroup $\Gamma$ of the group of isometries of $(M,g)$ there exists a positive least energy $\Gamma$-invariant solution to the Yamabe problem.

If $u$ is a nodal solution to problem $(\mathcal{Y}_g)$, i.e., if $u$ changes sign, then $\hat{g} = |u|^{4/(m-2)} g$ is not a metric, as $\hat{g}$ is not smooth and it vanishes on the set of zeroes of $u$. Ammann and Humbert called $\hat{g}$ a \textit{generalized metric}. In [2] they showed that, if the Yamabe invariant of $(M,g)$ is nonnegative, $(M,g)$ is not locally conformal flat and $\dim M \geq 11$, then there exists a minimal energy nodal solution to $(\mathcal{Y}_g)$. El Sayed considered the case where the Yamabe invariant of $(M,g)$ is strictly negative in [18]. Nodal solutions to $(\mathcal{Y}_g)$ on some product manifolds have been obtained, e.g., in [28, 21].

On the other hand, multiplicity of nodal solutions to the Yamabe problem $(\mathcal{Y}_g)$ is, largely, an open question. In a classical paper [14], W.Y. Ding established the existence of infinitely many nodal solutions to this problem on the standard sphere $S^m$. He took advantage of the fact that $S^m$ is invariant under the action of isometry groups whose orbits are positive dimensional.

In this paper we shall study the effect of the isometries of $M$ on the multiplicity of nodal solutions to Yamabe-type equations. Our framework is as follows.

Let $(M,g)$ be a closed Riemannian manifold of dimension $m \geq 3$ and $\Gamma$ be a closed subgroup of the group of isometries Isom$_g(M)$ of $(M,g)$. As usual, \textit{closed} means compact and without boundary. We denote by $\Gamma p := \{ \gamma p : \gamma \in \Gamma \}$ the $\Gamma$-orbit of a point $p \in M$ and by $\# \Gamma p$ its cardinality. Recall that a subset $X$ of $M$ is said to be $\Gamma$-invariant if $\Gamma x \subset X$ for every $x \in X$, and a function $f : X \to \mathbb{R}$ is $\Gamma$-invariant if it is constant on each orbit $\Gamma x$ of $X$.

We consider the Yamabe-type problem

\begin{equation}
\begin{cases}
- \text{div}_g(a \nabla_g u) + bu = c|u|^{2^* - 2} u, \\
u \in H^1_g(M)^\Gamma,
\end{cases}
\end{equation}

where $a, b, c \in C^\infty(M)$ are $\Gamma$-invariant functions, $a$ and $c$ are positive on $M$ and the operator $- \text{div}_g(a \nabla_g) + b$ is coercive on the space

$$H^1_g(M)^\Gamma := \{ u \in H^1_g(M) : u \text{ is } \Gamma\text{-invariant} \}.$$
If \( a \equiv 1 \), \( b = c_m R_g \) and \( c \equiv \kappa \) is constant, this is the Yamabe problem \([Y_g] \). In this case we shall always assume that \( \kappa > 0 \) and that the Yamabe operator \( \Delta_g + c_m R_g \) is coercive on \( H^1_{g}(M)^\Gamma \).

We will prove the following result.

**Theorem 1.1.** If \(- \text{div}_g (a \nabla_g) + b \) is coercive on \( H^1_{g}(M)^\Gamma \) and \( 1 \leq \dim(\Gamma_p) < m \) for every \( p \in M \), then problem \([E_1]\) has at least one positive and infinitely many nodal \( \Gamma \)-invariant solutions.

A special case is the following multiplicity result for the Yamabe problem \([Y_g] \).

**Corollary 1.2.** If \( \Delta_g + c_m R_g \) is coercive on \( H^1_{g}(M)^\Gamma \) and \( 1 \leq \dim(\Gamma_p) < m \) for all \( p \in M \), then the Yamabe problem \([Y_g]\) has infinitely many \( \Gamma \)-invariant nodal solutions.

The standard sphere \((S^m, g_0)\) is invariant under the action of the group \( O(k) \times O(n) \) with \( k+n = m+1 \), and this action has positive dimensional orbits if \( k, n \geq 2 \). So Corollary 1.2 can be seen as a generalization of Ding’s result \([14]\). One may also consider the action of \( S^1 \) on the standard sphere \( S^{2k+1} \subset \mathbb{C}^k \) given by complex multiplication on each complex coordinate. In this case, every orbit has dimension one.

Further examples are obtained as follows: if \( \Gamma \) is a closed subgroup of the group of isometries of \((S^m, g_0)\), \((N, h)\) is a closed Riemannian manifold of dimension \( n \) and \( f \in C^\infty(N) \) is a positive function, then \( \Gamma \) acts on the warped product \( N \times_f S^m = (N \times S^m, h + f^2 g_0) \) in the obvious way. So, if \( m+n \geq 3 \), \( \Delta_g + c_m R_g \) is coercive on \( H^1_{h+f^2 g_0}(N \times_f S^{2k+1})^\Gamma \) and every \( \Gamma \)-orbit of \( S^m \) is positive dimensional, then the Yamabe problem \([Y_g]\) has infinitely many \( \Gamma \)-invariant nodal solutions on \( N \times_f S^m \).

This extends Theorem 1.2 in \([28]\).

Next, we study a case in which \( M \) is allowed to have finite \( \Gamma \)-orbits. We consider the following setting:

Let \( M \) be a closed smooth \( m \)-dimensional manifold and \( a, b, c \in C^\infty(M) \) be such that \( a \) and \( c \) are positive on \( M \). We fix an open subset \( \Omega \) of \( M \), a Riemannian metric \( h \) on \( \Omega \) and a compact subgroup \( \Lambda \) of \( \text{Isom}_h(\Omega) \) such that \( \dim(\Lambda_p) < m \) for all \( p \in \Omega \), the restrictions of \( a, b, c \) to \( \Omega \) are \( \Lambda \)-invariant and the operator \(- \text{div}_g (a \nabla_g) + b \) is coercive on the space \( C^\infty_c(\Omega)^\Lambda \) of smooth \( \Lambda \)-invariant functions with compact support in \( \Omega \). Under these assumptions, we will prove the following multiplicity result.

**Theorem 1.3.** There exists an increasing sequence \((\ell_k)\) of positive real numbers, depending only on \((\Omega, h), a, b, c \) and \( \Lambda \), with the following property: For any Riemannian metric \( g \) on \( M \) and any closed subgroup \( \Gamma \) of \( \text{Isom}_g(M) \) which satisfy

1. \( g = h \) in \( \Omega \);
(2) $\Gamma$ is a subgroup of $\Lambda$ and $a, b, c$ are $\Gamma$-invariant;
(3) $-\text{div}_g(a\nabla_g) + b$ is coercive on $H^1_g(M)^\Gamma$;
(4) $\min_{p \in M} \frac{a(p)^{m/2} \#\Gamma_p}{c(p)} > \ell_k$;

problem (1.1) has at least $k$ pairs of $\Gamma$-invariant solutions $\pm u_1, \ldots, \pm u_k$ such that $u_1$ is positive, $u_2, \ldots, u_k$ change sign, and
\[ \int_M c|u_j|^2 dV_g \leq \ell_j S^{m/2} \quad \text{for every } j = 1, \ldots, k, \]
where $S$ is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^m) \hookrightarrow L^{2^*}(\mathbb{R}^m)$.

Theorem 1.3 asserts the existence of a prescribed number of nodal solutions to problem (1.1) if there is a Riemannian metric on $M$, which extends the given Riemannian metric on $\Omega$, for which some group of isometries has large enough orbits.

Nodal solutions to Yamabe-type equations have been exhibited, e.g., in [16, 22, 35]. If $m \geq 4$, $a = c = 1$ and $\Delta_g + b$ is coercive, Vétois showed that problem (1.1) has at least $\frac{m+2}{2}$ solutions provided that $b(p_0) < c_mR_g(p_0)$ at some point $p_0 \in M$ [35]. This last assumption excludes the Yamabe problem $[\mathcal{Y}_g]$. Also, nothing is said about the sign of the solutions, except for the cases when the positive solution is known to be unique.

In contrast, Theorem 1.3 does apply to the Yamabe problem. However, property (4) requires that the group $\Lambda$ has large enough subgroups. The group $S^1$, for example, has this property. This allows us to derive a multiplicity result for the Yamabe problem $[\mathcal{Y}_g]$ in the following setting.

Let $(M, h)$ be a closed Riemannian manifold on which $S^1$ acts freely and isometrically, such that $\Delta_h + c_mR_h$ is coercive in $H^1_h(M)$. Fix an open $S^1$-invariant subset $\Omega$ of $M$ such that $R_h > 0$ on $M \setminus \Omega$. Set $\Gamma_n := \{e^{2\pi i j/n} : j = 0, \ldots, n - 1\}$. Then, the following statement holds true.

**Corollary 1.4.** There exist a sequence $(\ell_k)$ in $(0, \infty)$ and an open neighborhood $\mathcal{O}$ of $h$ in the space of Riemannian metrics on $M$ with the $C^0$-topology, with the following property: for every $g \in \mathcal{O}$ such that $g = h$ in $\Omega$ and $\Gamma_n \subset \text{Isom}_g(M)$ for some $n > \kappa(m-2)/2\ell_k$, the Yamabe problem $[\mathcal{Y}_g]$ has at least $k$ pairs of $\Gamma_n$-invariant solutions $\pm u_1, \ldots, \pm u_k$ such that $u_1$ is positive, $u_2, \ldots, u_k$ change sign, and
\[ \int_M |u_j|^2 dV_g \leq \kappa^{-1} \ell_j S^{m/2} \quad \text{for every } j = 1, \ldots, k. \]

For instance, we may take $\Omega$ to be the complement of a closed tubular neighborhood of an $S^1$-orbit in $(M, h)$ on which $R_h > 0$. Then $M \setminus \Omega$ is $S^1$-diffeomorphic to $S^1 \times \mathbb{B}^{m-1}$, where $\mathbb{B}^{m-1}$ is the closed unit ball in $\mathbb{R}^{m-1}$. We choose $n > \kappa(m-2)/2\ell_k$. Then, if we modify the metric in the interior of the piece of $M \setminus \Omega$ which corresponds to $\{e^{2\pi i \vartheta/n} : 0 \leq \vartheta \leq 1\} \times \mathbb{B}^{m-1}$ and translate this modification to each
of the pieces corresponding to \( \{ e^{2\pi i j/n} : j - 1 \leq \vartheta \leq j \} \times \mathbb{R}^{m-1}, j = 2, ..., n \), we obtain a metric \( g \) on \( M \) such that \( g = h \) in \( \Omega \) and \( \Gamma_n \subset \text{Isom}_g(M) \). If \( g \) is chosen to be close enough to \( h \), then the previous corollary asserts the existence of \( k \) pairs of solutions to the Yamabe problem \( \mathcal{Y}_g \). This way we obtain many examples of Riemannian manifolds with finite symmetries which admit a prescribed number of nodal solutions to the Yamabe problem.

We would like to mention that existence and multiplicity of positive and nodal solutions are also available for some perturbations of the Yamabe problem; see, e.g., [25, 30] and the references therein.

Finally, we wish to stress that, even though the Yamabe invariant is always attained, problem (1.1) need not have a ground state solution, as the following example shows. So a solution cannot always be obtained by minimization.

**Proposition 1.5.** If \((S^m, g_0)\) is the standard sphere and \( b \in C^\infty(S^m) \) is such that \( b \geq c_m R_{g_0} = \frac{m(m-2)}{4} \) and \( b \not\equiv c_m R_{g_0} \), then the equation

\[
\Delta_{g_0} u + bu = |u|^{2^*-2}u, \quad u \in C^\infty(S^m),
\]

does not admit a ground state solution, i.e.,

\[
\inf_{u \in C^\infty(S^m)} \frac{\int_{S^m} \left[ |\nabla_{g_0} u|^2_{g_0} + bu^2 \right] dV_{g_0}}{\left( \int_{S^m} |u|^{2^*} dV_{g_0} \right)^{2/2^*}}
\]

is not attained.

Theorems 1.1 and 1.3 and Corollary 1.4 will be proved in Section 2. Their proof follows some ideas introduced in [6], where a result similar to Theorem 1.3, in a bounded domain of \( \mathbb{R}^m \), is established. The proof is based on a compactness result and a variational principle for nodal solutions which are proved in Sections 4 and 5 respectively. Proposition 1.5 is proved in Section 3.

2. Proof of the main results

Let \((M, g)\) be a closed Riemannian manifold of dimension \( m \geq 3 \), \( \Gamma \) be a closed subgroup of \( \text{Isom}_g(M) \), and \( a, b, c \in C^\infty(M) \) be \( \Gamma \)-invariant functions. We will assume throughout this section that \( a > 0 \), \( c > 0 \) and that the operator \( -\text{div}_g (a \nabla_g) + b \) is coercive on the space \( H^1_g(M)^\Gamma := \{ u \in H^1_g(M) : u \text{ is } \Gamma \text{-invariant} \} \).

Then,

\[
\langle u, v \rangle_{g,a,b} := \int_M \left[ a (\nabla_g u, \nabla_g v)_g + buv \right] dV_g
\]

is an interior product in \( H^1_g(M)^\Gamma \) and the induced norm, which we will denote by \( \| \cdot \|_{g,a,b} \), is equivalent to the standard norm \( \| \cdot \|_{g} \) in \( H^1_g(M)^\Gamma \). Also,

\[
|u|_{g,c,2^*} := \left( \int_M c |u|^{2^*} dV_g \right)^{1/2^*}
\]
defines a norm in $L^*_g(M)$ which is equivalent to the standard norm $| \cdot |_{g,2^*}$. By the principle of symmetric criticality [27], the solutions to problem (1.1) are the critical points of the energy functional

$$
J_g(u) = \frac{1}{2} \int_M \left[ a|\nabla_g u|^2 + bu^2 \right] dV_g - \frac{1}{2^*} \int_M c|u|^{2^*} dV_g
= \frac{1}{2} \|u\|^2_{g,a,b} - \frac{1}{2^*} \|u\|^{2*}_{g,c,2^*}
$$

defined on the space $H^1_g(M)^\Gamma$. The nontrivial ones lie on the Nehari manifold

(2.1) $\mathcal{N}_g^\Gamma := \{ u \in H^1_g(M)^\Gamma : u \neq 0, \|u\|^2_{g,a,b} = |u|^{2*}_{g,c,2^*} \}$ which is of class $C^2$, radially diffeomorphic to the unit sphere in $H^1_g(M)^\Gamma$, and a natural constraint for $J_g$. Moreover, for every $u \in H^1_g(M)^\Gamma$, $u \neq 0$,

(2.2) $u \in \mathcal{N}_g^\Gamma \iff J_g(u) = \max_{t \geq 0} J_g(tu)$.

Set

$$
\tau_g^\Gamma := \inf_{\mathcal{N}_g^\Gamma} J_g.$$

The continuity of the Sobolev embedding $H^1_g(M) \hookrightarrow L^*_g(M)$ implies that $\tau_g^\Gamma > 0$.

The proofs of Theorems 1.1 and 1.3 follow the scheme introduced in [9, 10]. They are based on a compactness result and a variational principle for nodal solutions, which are stated next.

**Definition 2.1.** A $\Gamma$-invariant Palais-Smale sequence for the functional $J_g$ at the level $\tau$ is a sequence $(u_k)$ such that,

$$
u_k \in H^1_g(M)^\Gamma, \quad J_g(u_k) \to \tau, \quad J_g'(u_k) \to 0 \text{ in } (H^1_g(M))^\Gamma.$$

We shall say that $J_g$ satisfies condition $(PS)^\Gamma_\tau$ in $H^1_g(M)$ if every $\Gamma$-invariant Palais-Smale sequence for $J_g$ at the level $\tau$ contains a subsequence which converges strongly in $H^1_g(M)$.

The presence of symmetries allows to increase the lowest level at which this condition fails. The following result will be proved in Section 4.

**Theorem 2.2 (Compactness).** The functional $J_g$ satisfies condition $(PS)^\Gamma_\tau$ in $H^1_g(M)$ for every

$$
\tau < \left( \min_{q \in M} \frac{a(q)^{m/2} \#\Gamma_q}{c(q)^{(m-2)/2}} \right) \frac{1}{m} S^{m/2},
$$

where $S$ is the best Sobolev constant for the embedding $D^{1,2}({\mathbb{R}}^m) \hookrightarrow L^{2^*}({\mathbb{R}}^m)$.

If all $\Gamma$-orbits in $M$ have positive dimension, this result says that $J_g$ satisfies $(PS)^\Gamma_\tau$ for every $\tau \in {\mathbb{R}}$. This can also be deduced from the compactness of the Sobolev embedding $H^1_g(M)^\Gamma \hookrightarrow L^*_g(M)$ which was proved by Hebey and Vaugon.
Proof of Theorem 1.1
Proof. See Theorems IV.3.1, IV.3.3 and IV.3.8 in [5], or Theorem I.5.11 in [15].

For a fixed closed subgroup \( H \) contained in \( \Omega \) which is \( \Gamma \)-diffeomorphic to \( M \), given \( \omega \), an \( \Gamma \)-invariant function on \( H \); see, e.g., [5, 15]. We denote by \(( H, \omega )\) every subgroup of \( \Gamma \) which is conjugate to an isotropy subgroup. Isotropy subgroups satisfy \( \Gamma \gamma p = \Gamma \gamma \gamma p \gamma^{-1} \). Thus, every subgroup of \( \Gamma \) which is conjugate to an isotropy subgroup is also an isotropy subgroup; see, e.g., [5, 15]. We denote by \(( H)\) the conjugacy class of a subgroup \( H \) of \( \Gamma \).

Theorem 2.3 (Sign-changing critical points). Let \( W \) be a nontrivial finite dimensional subspace of \( H^1_g(\Omega)^\Gamma \). If \( J_\tau \) satisfies \(( PS)_\Gamma \) in \( H^1_g(\Omega) \) for every \( \tau \leq \sup W \), then \( J_\tau \) has at least one positive critical point \( u_1 \) and \( \dim W - 1 \) pairs of nodal critical points \( \pm u_2, \ldots, \pm u_k \) in \( H^1_g(\Omega)^\Gamma \) such that \( J_\tau(u_1) = \tau^\Gamma \) and \( J_\tau(u_i) \leq \sup W \), \( J_\tau \) for \( i = 1, \ldots, k \).

Next, we derive our main results from the previous three theorems.

Proof of Theorem 2.4. By Theorem 2.3.4, \( M \) contains an open dense subset \( \Omega := M_{(\Omega)} \) such that the \( \Gamma \)-orbit of each point \( p \in \Omega \) is \( \Gamma \)-diffeomorphic to \( \Gamma / H \) for some fixed closed subgroup \( H \) of \( \Gamma \). Moreover, \( \Gamma p \) has a \( \Gamma \)-invariant neighborhood \( \Omega_p \) contained in \( \Omega \) which is \( \Gamma \)-diffeomorphic to \( B \times \Gamma / H \), where \( B \) is the euclidean unit ball of dimension \( m - \dim(\Gamma p) \). Since we are assuming that \( \dim(\Gamma p) < m \), for any given \( k \in \mathbb{N} \) we may choose \( k \) different \( \Gamma \)-orbits \( \Gamma p_1, \ldots, \Gamma p_k \subset \Omega \) and \( \Gamma \)-invariant neighborhoods \( \Omega_{p_i} \) as before, with \( \Omega_{p_i} \cap \Omega_{p_j} = \emptyset \) if \( i \neq j \). Then, we can choose a \( \Gamma \)-invariant function \( \omega_i \in C^\infty_c(\Omega_{p_i}) \) for each \( i = 1, \ldots, k \).

Let \( W := \text{span}\{\omega_1, \ldots, \omega_k\} \) be the linear subspace of \( H^1_g(\Omega)^\Gamma \) spanned by \( \{\omega_1, \ldots, \omega_k\} \). As \( \omega_i \) and \( \omega_j \) have disjoint supports for \( i \neq j \), the set \( \{\omega_1, \ldots, \omega_k\} \) is orthogonal in \( H^1_g(\Omega)^\Gamma \). Hence, \( \dim W = k \). On the other hand, as \( \dim(\Gamma p) \geq 1 \), we have that \( \# \Gamma p = \infty \) for every \( p \in M \). So, by Theorem 2.2, \( J_\tau \) satisfies \(( PS)_\Gamma \)
in $H^1_g(M)$ for every $\tau \in \mathbb{R}$. Therefore, Theorem 2.3 yields at least one positive and $k-1$ nodal $\Gamma$-invariant solutions to problem (1.1). As $k \in \mathbb{N}$ is arbitrary, we conclude that there are infinitely many nodal solutions. □

**Proof of Theorem 1.3** By Theorem 2.4, after replacing $\Omega$ by a $\Lambda$-invariant open subset of it, if necessary, we may assume that $\Lambda_p$ is $\Lambda$-diffeomorphic to $\Lambda/H$ for every $p \in \Omega$ and some fixed subgroup $H$ of $\Lambda$. Let $\mathcal{P}_1(\Omega)$ be the family of all nonempty $\Lambda$-invariant open subsets of $\Omega$ and, for each $\tilde{\Omega} \in \mathcal{P}_1(\Omega)$, set

$$D(\tilde{\Omega}) := \{ \varphi \in C^\infty_c(\tilde{\Omega}) : \varphi \text{ is } \Lambda\text{-invariant, } \varphi \neq 0, \| \varphi \|_{h,a,b}^2 = \| \varphi \|_{h,c,2}^2 \}.$$ 

For each $k \in \mathbb{N}$ let

$$\mathcal{P}_k(\Omega) := \{ (\Omega_1, \ldots, \Omega_k) : \Omega_i \in \mathcal{P}_1(\Omega), \quad \Omega_i \cap \Omega_j = \emptyset \text{ if } i \neq j \}.$$ 

Arguing as in the proof of Theorem 1.1 we see that $\mathcal{P}_k(\Omega) \neq \emptyset$ and $D(\tilde{\Omega}) \neq \emptyset$. Set

$$\tau_k := \inf \left\{ \sum_{i=1}^k \frac{1}{m} \| \varphi_i \|_{h,a,b}^2 : \varphi_i \in D(\Omega_i), \quad (\Omega_1, \ldots, \Omega_k) \in \mathcal{P}_k(\Omega) \right\},$$

and define

$$\ell_k := \left( \frac{1}{m} S^{m/2} \right)^{-1} \tau_k.$$ 

Next, we show that the sequence $(\ell_k)$ has the desired property.

Fix $k \in \mathbb{N}$, and let $(M, g)$ be a Riemannian manifold and $\Gamma$ be a closed subgroup of $\text{Isom}_g(M)$ which satisfy (1)-(4). As $g = h$ in $\Omega$ and $\Gamma$ is a subgroup of $\Lambda$, extending $\varphi \in C^\infty_c(\tilde{\Omega})$ by zero outside $\tilde{\Omega}$, we have that $D(\tilde{\Omega}) \subset N^\Gamma_g$ for every $\tilde{\Omega} \in \mathcal{P}_1(\Omega)$, $J_g(\varphi) = \frac{1}{m} \| \varphi \|_{h,a,b}^2$ for every $\varphi \in D(\tilde{\Omega})$ and $\tau_1 \geq \tau_{\Gamma_g} > 0$. Since we are assuming that

$$\ell_{a,c}^\Gamma := \min_{p \in M} \frac{a(p)^{m/2} \# \Gamma_p}{c(p)^{(m-2)/2}} > \ell_k,$$

we may choose $\varepsilon \in (0, \tau_1)$ such that $\tau_k + \varepsilon < \ell_{a,c}^\Gamma (\frac{1}{m} S^{m/2})$. Then, by definition of $\tau_k$, there exist $(\Omega_1, \ldots, \Omega_k) \in \mathcal{P}_k(\Omega)$ and $\omega_i \in D(\Omega_i)$, such that

$$\tau_k \leq \sum_{i=1}^k J_g(\omega_i) < \tau_k + \varepsilon.$$ 

For each $n = 1, \ldots, k$ set $W_n := \text{span}\{\omega_1, \ldots, \omega_n\}$. As $\omega_i$ and $\omega_j$ have disjoint supports for $i \neq j$, the set $\{\omega_1, \ldots, \omega_k\}$ is orthogonal in $H^1_g(M)^\Gamma$. Hence, $\dim W_n = n$. Moreover, if $u \in W_n$, $u = \sum_{i=1}^n t_i \omega_i$, then (2.2) yields

$$J_g(u) = \sum_{i=1}^n J_g(t_i \omega_i) \leq \sum_{i=1}^n J_g(\omega_i) < \tau_k + \varepsilon.$$ 

Therefore,

$$\sigma_n := \sup_{W_n} J_g \leq \tau_k + \varepsilon < \ell_{a,c}^\Gamma (\frac{1}{m} S^{m/2}).$$
so Theorems 2.2 and 2.3 yield a positive critical point \( u_1 \) and \( n - 1 \) pairs of sign changing critical points \( \pm u_{n,2}, \ldots, \pm u_{n,n} \) of \( J_g \) in \( H^1_g(M)^{\Gamma} \) such that \( J_g(u_1) = \tau_g^\Gamma \) and

\[
J_g(u_{n,j}) \leq \sigma_n \quad \text{for all } j = 2, \ldots, n.
\]

Now, for each \( 2 \leq n \leq k \), we inductively choose \( u_n \in \{ u_{n,2}, \ldots, u_{n,n} \} \) such that \( u_n \neq u_j \) for all \( 1 \leq j < n \). In order to show that the \( u_j \)'s may be suitable chosen to satisfy (1.2), we need the following inequalities. Observe that \( \tau_1 \leq J_g(\omega_i) \) for every \( i = 1, \ldots, k \). Consequently, for each \( 2 \leq n \leq k \) we obtain

\[
\sigma_n + (k - n)\tau_1 \leq \sum_{i=1}^n J_g(\omega_i) + \sum_{i=n+1}^k J_g(\omega_i) < \tau_k + \varepsilon.
\]

As \( \varepsilon \in (0, \tau_1) \) we conclude that

\[
J_g(u_n) \leq \sigma_n < \tau_k \quad \text{if } n < k \quad \text{and} \quad J_g(u_k) \leq \sigma_k < \tau_k + \varepsilon.
\]

With these inequalities, the argument in the last two steps of the proof of Theorem 2.2 in [10] goes through to show that the \( u_j \)'s may be chosen so that (1.2) is satisfied.

\[\square\]

**Proof of Corollary 1.4.** Let \( \mathcal{M} \) be the space of Riemannian metrics on \( M \) with the distance induced by the \( C^0 \)-norm in the space of covariant 2-tensor fields \( \tau \) on \( M \), taken with respect to the fixed metric \( h \), i.e.

\[
\|\tau\|_{C^0} := \max_{p \in M} \max_{X,Y \in T_pM \setminus \{0\}} \frac{|\tau(X,Y)|}{|X|_h |Y|_h}.
\]

As the functions \( \mathcal{M} \to C^0(M) \) given by \( g \mapsto R_g \) and \( g \mapsto \sqrt{|g|} \) are continuous, where \( |g| := \det(g) \), the sets

\[
\mathcal{O}_1 := \left\{ g \in \mathcal{M} : \frac{1}{2} R_h(p) < R_g(p) < 2 R_h(p) \quad \forall p \in M \setminus \Omega \right\},
\]

\[
\mathcal{O}_2 := \left\{ g \in \mathcal{M} : \frac{1}{2} \sqrt{|h|(p)} < \sqrt{|g|(p)} < 2 \sqrt{|h|(p)} \quad \forall p \in M \setminus \Omega \right\},
\]

are open neighborhoods of \( h \) in \( \mathcal{M} \). Moreover, since

\[
|\nabla_g u(p)|_g = \max_{X \in T_p M \setminus \{0\}} \frac{|duX|}{|X|_g},
\]

for every \( u \in C^\infty(M) \) we have that

\[
\frac{1}{2} |\nabla_h u|^2_h \leq |\nabla_g u|^2_g \leq 2 |\nabla_h u|^2_h \quad \text{if } \|g - h\|_{C^0} < \frac{1}{2}.
\]

Set \( \mathcal{O} := \{ g \in \mathcal{M} : \|g - h\|_{C^0} < \frac{1}{2} \} \cap \mathcal{O}_1 \cap \mathcal{O}_2 \). Then there are positive constants \( C_1 \leq 1 \) and \( C_2 \geq 1 \) such that, for every \( g \in \mathcal{O} \) and \( u \in C^\infty(M) \),

\[
\int_{M \setminus \Omega} [ |\nabla_g u|^2_g + \epsilon_m R_g u^2 ] \, dV_g \geq C_1 \int_{M \setminus \Omega} [ |\nabla_h u|^2_h + \epsilon_m R_h u^2 ] \, dV_h,
\]

\[
\int_{M \setminus \Omega} [ |\nabla_g u|^2_g + u^2 ] \, dV_g \leq C_2 \int_{M \setminus \Omega} [ |\nabla_h u|^2_h + u^2 ] \, dV_h.
\]
Therefore, if $g \in \mathcal{O}$ and $g = h$ in $\Omega$, we have that
\[
\frac{\int_M [\nabla_g u]^2 + \epsilon_m R_g u^2] \, dV_g}{\int_M [\nabla_g u]^2 + u^2] \, dV_g} \geq \frac{C_1 \int_M [\nabla_h u]^2 + \epsilon_m R_h u^2] \, dV_h}{C_2 \int_M [\nabla_h u]^2 + u^2] \, dV_h}
\]
for every $u \in C^\infty(M)$. As $\Delta_h + \epsilon_m R_h$ is coercive in $H^1_0(M)$, this inequality implies that $\Delta_g + \epsilon_m R_g$ is coercive in $H^1_0(M)$.

Set $(\Omega, h)$ as given, $\Lambda = S^1$, $a \equiv 1$, $b = \epsilon_m R_g$ and $c \equiv \kappa$. Then, if $g \in \mathcal{O}$ is such that $g = h$ in $\Omega$ and $\Gamma_n \subset Isom_g(M)$ for some $n > \kappa(m - 2)/\ell_k$, these data satisfy assumptions (1)-(4) in Theorem 1.3, and the conclusion follows. \(\square\)

3. Nonexistence of ground state solutions

In this section we prove Proposition 1.5.

If $h$ and $g = \varphi^{2^* - 2} h$, with $\varphi \in C^\infty(M)$, $\varphi > 0$, are two conformally equivalent Riemannian metrics on an $m$-dimensional manifold $M$, the scalar curvatures $R_h$ and $R_g$ are related by the equation
\[
\Delta_h \varphi + \epsilon_m R_h \varphi = \epsilon_m R_g \varphi^{2^* - 1}.
\]
Let $v = \varphi u \in C^\infty(M)$. An easy computation shows that
\[
\Delta_g u = \varphi^{2^*} (\varphi \Delta_h v - v \Delta_h \varphi)
\]
and, combining this identity with (3.1), we obtain that
\[
\Delta_g u + \epsilon_m R_g u = \varphi^{1 - 2^*} (\Delta_h v + \epsilon_m R_h v).
\]
Let $(S^m, g_0)$ be the standard sphere and $b \in C^\infty(S^m)$ be such that $b \geq \epsilon_m R_{g_0} = m(m-2)/4$ and $b \not= \epsilon_m R_{g_0}$. Let $p \in S^m$ be the north pole and $\sigma : S^m \setminus \{p\} \rightarrow \mathbb{R}^m$ be the stereographic projection. $\sigma$ is a conformal diffeomorphism and the coordinates of standard metric $g_0$ given by the chart $\sigma^{-1} : \mathbb{R}^m \rightarrow S^m \setminus \{p\}$ are $(g_0)_{ij} = \varphi^{-2} \delta_{ij}$, where
\[
\varphi(x) := \left(\frac{2}{1 + |x|^2}\right)^{(m-2)/2}.
\]
Set $\bar{b} := \varphi^{2^* - 2} (b \circ \sigma^{-1} - \epsilon_m R_{g_0})$ and, for $u \in C^\infty(S^m)$, set $v = \varphi(u \circ \sigma^{-1})$. As $dV_{g_0} = \varphi^{2^*} \, dx$, using (3.2) we obtain that
\[
\int_{S^m} [\nabla_{g_0} u]^2_{g_0} + \epsilon_m R_{g_0} u^2] \, dV_{g_0} = \int_{\mathbb{R}^m} |\nabla v|^2 \, dx,
\]
\[
\int_{S^m} (b - \epsilon_m R_{g_0}) u^2 \, dV_g = \int_{\mathbb{R}^m} \bar{b} v^2 \, dx,
\]
\[
\int_{S^m} |u|^2 \, dV_{g_0} = \int_{\mathbb{R}^m} |v|^2 \, dx.
\]
Hence,

\[
\inf_{u \in C^\infty_0(\mathbb{R}^m)} \int_{\mathbb{R}^m} \left[ |\nabla g_0 u|^2 + b u^2 \right] dV_{g_0} = \inf_{v \in D^{1,2}(\mathbb{R}^m)} \int_{\mathbb{R}^m} \left[ |\nabla u|^2 + \tilde{b} v^2 \right] dx =: S_b.
\]

If \( b \equiv \frac{m(m-2)}{4} \) then \( \tilde{b} = 0 \) and \( S_{\frac{m(m-2)}{4}} =: S \) is the best Sobolev constant for the embedding \( D^{1,2}(\mathbb{R}^m) \hookrightarrow L^{2^*}(\mathbb{R}^m) \). This constant is attained at the standard bubble

\[
U(x) = [m(m-2)]^{\frac{m-2}{2}} \left( \frac{1}{1 + |x|^2} \right)^{\frac{m-2}{2}}
\]

and at any dilation \( U_\varepsilon(x) := \varepsilon^{\frac{2-m}{2}} U \left( \frac{x}{\varepsilon} \right) \) of it, with \( \varepsilon > 0 \).

**Lemma 3.1.** If \( b \geq \frac{m(m-2)}{4} \) then \( S_b = S \).

**Proof.** Clearly, \( S_b \geq S \). Fix \( \alpha \in \left( \frac{1}{2}, 1 \right) \). Then, for all \( \varepsilon \in (0, 1) \),

\[
\tilde{b}(x) U_\varepsilon^2(x) \leq C \left( \frac{1}{1 + |x|^2} \right)^2 \left( \frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{m-2} \leq C \varepsilon^{m-2} \left( \frac{1}{\varepsilon^2 + |x|^2} \right)^{m-2+\alpha}.
\]

Hence, we have that

\[
0 \leq \int_{\mathbb{R}^m} \tilde{b}(x) U_\varepsilon^2(x) dx = \int_{|x| \leq \varepsilon} \tilde{b}(x) U_\varepsilon^2(x) dx + \int_{|x| \geq \varepsilon} \tilde{b}(x) U_\varepsilon^2(x) dx
\]

\[
\leq C \varepsilon^2 \int_{|y| \leq 1} U^2(y) dy + C \varepsilon^{m-2} \int_{|x| \geq \varepsilon} |x|^{-2m+4-2\alpha} dx
\]

\[
= C \varepsilon^2 + C \varepsilon^{2(1-\alpha)} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.
\]

Therefore,

\[
\lim_{\varepsilon \rightarrow 0} \frac{\int_{\mathbb{R}^m} \left( |\nabla U_\varepsilon|^2 + \tilde{b} U_\varepsilon^2 \right) dx}{\left( \int_{\mathbb{R}^m} |U_\varepsilon|^2 dx \right)^{2/2^*}} = \frac{\int_{\mathbb{R}^m} |\nabla U|^2 dx}{\left( \int_{\mathbb{R}^m} |U|^2 dx \right)^{2/2^*}} = S.
\]

This shows that \( S \geq S_b \). \( \square \)

**Proof of Proposition 1.5** If \( S_b \) were attained at some \( v \in D^{1,2}(\mathbb{R}^m) \) then, as \( \tilde{b} \geq 0 \) and \( \tilde{b} \neq 0 \), we would have that

\[
S = S_b = \frac{\int_{\mathbb{R}^m} \left( |\nabla v|^2 + \tilde{b} v^2 \right) dx}{\left( \int_{\mathbb{R}^m} |v|^2 dx \right)^{2/2^*}} > \frac{\int_{\mathbb{R}^m} |\nabla v|^2 dx}{\left( \int_{\mathbb{R}^m} |v|^2 dx \right)^{2/2^*}} \geq S.
\]

This is a contradiction. \( \square \)
4. Compactness

A classical result by Struwe \[32\] provides a complete description of the lack of compactness of the energy functional for critical problems in a bounded smooth domain of $\mathbb{R}^m$. Anisotropic critical problems with symmetries were treated in \[10\]. Palais-Smale sequences of positive functions for some Yamabe-type problems on a closed manifold were described by Druet, Hebey and Robert in \[17\], and symmetric ones were treated in \[31\].

In this section we apply concentration compactness methods to prove Theorem 2.2.

Throughout this section, $(M,g)$ is a closed Riemannian manifold of dimension $m \geq 3$, $\Gamma$ is a closed subgroup of $\text{Isom}_g(M)$, and $a, b, c \in C^\infty(M)$ are $\Gamma$-invariant functions with $a, c > 0$. We shall not assume that $-\text{div}_g(a \nabla_g) + b$ is coercive, except when we prove Theorem 2.2.

We use the notation introduced in the previous section. We start with the following fact.

**Lemma 4.1.** Every Palais-Smale sequence for the functional $J_g$ is bounded in $H^1_g(M)$.

**Proof.** Hereafter, $C$ will denote a positive constant, not necessarily the same one. Let $(u_k)$ be a sequence in $H^1_g(M)$ such that $J_g(u_k) \to \tau$ and $J'_g(u_k) \to 0$ in $(H^1_g(M))'$. Then,

$$|u_k|_{g,2}^2 \leq C \left( \frac{1}{m} |u_k|_{g,c,2}^2 \right) = C \left( J_g(u_k) - \frac{1}{2} J'_g(u_k) u_k \right) \leq C + o(\|u_k\|_g).$$

Hence,

$$\int_M \left[ a |\nabla_g u_k|^2 + b |u_k|^2 \right] dV_g = 2 \left( J_g(u_k) + \frac{1}{2} |u_k|_{g,c,2}^2 \right) \leq C + o(\|u_k\|_g).$$

Moreover, as $M$ is compact, using Hölder's inequality we obtain

$$|u_k|_{g,2}^2 \leq C |u_k|_{g,2}^2 \leq C + o(\|u_k\|_g^{2/2^*}).$$

As $b$ is bounded, inequalities (4.1) and (4.2) yield

$$a_0 \|u_k\|_g^2 \leq \int_M \left[ a |\nabla_g u_k|^2 + b |u_k|^2 \right] dV_g + \int_M (-b + a_0) u_k^2 dV_g$$

$$\leq \int_M \left[ a |\nabla_g u_k|^2 + b |u_k|^2 \right] dV_g + C |u_k|_{g,2}^2$$

$$\leq C + o(\|u_k\|_g) + o(\|u_k\|_g^{2/2^*}),$$

where $a_0 := \min_M a$. This implies that $(u_k)$ is bounded in $H^1_g(M)$.

Next, we consider the problem

$$\left\{\begin{array}{l}
-\Delta v = |v|^{2^* - 2} v, \\
v \in D^{1,2}(\mathbb{R}^m),
\end{array}\right.$$
and its associated energy functional
\[ J_\infty(v) := \frac{1}{2} \int_{\mathbb{R}^n} |\nabla v|^2 \, dx - \frac{1}{2^*} \int_{\mathbb{R}^n} |v|^{2^*} \, dx, \quad v \in D^{1,2}(\mathbb{R}^m). \]

The proof of Theorem 2.2 will follow easily from the following proposition.

**Proposition 4.2.** Assume that \( b \equiv 0 \). Let \( (u_k) \) be a \( \Gamma \)-invariant Palais-Smale sequence for \( J_g \) at the level \( \tau > 0 \) such that \( u_k \rightharpoonup 0 \) weakly in \( H^1_0(M) \) but not strongly. Then, after passing to a subsequence, there exist a point \( p \in M \) and a nontrivial solution \( \hat{v} \) to problem (4.3) such that \( \# \Gamma p < \infty \) and

\[ \tau \geq \left( \frac{a(p)^{m/2} \# \Gamma p}{c(p)^{(m-2)/2}} \right) J_\infty(\hat{v}) \geq \left( \min_{q \in M} \frac{a(q)^{m/2} \# \Gamma q}{c(q)^{(m-2)/2}} \right) \frac{1}{m} S^{m/2}. \]

**Proof.** Fix \( \delta \) such that \( 3\delta \in (0, i_q) \), where \( i_q \) is the injectivity radius of \( M \). As \( M \) is compact, there is a constant \( C_1 > 1 \) such that, for every \( q \in M \), \( q \in (0, 3\delta] \), \( \varphi \in C^\infty(M) \) and \( s \in [1, \infty) \),

\[ C_1^{-1} \int_{B(0, \delta)} |\varphi|^s \, dx \leq \int_{B_q(q, \delta)} |\varphi|^s \, dV_g \leq C_1 \int_{B(0, \delta)} |\varphi|^s \, dx, \]

\[ C_1^{-1} \int_{B(0, \delta)} |\nabla \varphi|^2 \, dx \leq \int_{B_q(q, \delta)} |\nabla \varphi|^2 \, dV_g \leq C_1 \int_{B(0, \delta)} |\nabla \varphi|^2 \, dx, \]

where \( \varphi := \varphi \circ \exp_q \) is written in normal coordinates around \( q \) and \( |\cdot| \) is the standard Euclidean metric.

By Lemma 4.1 we have that

\[ |u_k|_{g, c, 2^*}^2 = m \left( J_g(u_k) - \frac{1}{2} J'_g(u_k) u_k \right) \rightarrow m \tau =: \beta > 0. \]

So, since \( M \) is compact, after passing to a subsequence, there exist \( q_0 \in M \) and \( \lambda_0 \in (0, \beta) \) such that

\[ \int_{B_q(q_0, \delta)} c|u_k|^2 \, dV_g \geq \lambda_0 \quad \forall k \in \mathbb{N}, \]

where \( B_q(q, r) \) denotes the ball in \((M, g)\) with center \( q \) and radius \( r \). For each \( k \), the Levy concentration function \( Q_k : [0, \infty) \rightarrow [0, \infty) \) given by

\[ Q_k(r) := \max_{q \in M} \int_{B_q(q, r)} c|u_k|^2 \, dV_g \]

is continuous, nondecreasing, and satisfies \( Q_k(0) = 0 \) and \( Q_k(\delta) \geq \lambda_0 \). We fix \( \lambda \in (0, \lambda_0) \) such that

\[ \lambda < C_1^{-m-1} \left( \min_{M} \frac{1}{2} S(M)(\max_{M} c)^{-1} \right)^{m/2}. \]

Then, for each \( k \in \mathbb{N} \), there exist \( p_k \in M \) and \( r_k \in (0, \delta] \) such that

\[ Q_k(r_k) = \int_{B_q(p_k, r_k)} c|u_k|^2 \, dV_g = \lambda \]

and, after passing to a subsequence, \( p_k \rightarrow p \) in \( M \).
Moreover, inequalities (4.5) and (4.6) yield

\[ 1 \leq \zeta \leq 1 \text{ if } |y| \leq 2\delta \text{ and } \zeta(y) = 0 \text{ if } |y| \geq 3\delta \text{ and, for each } k, \]

\[ v_k(x) := r_k^{-2/3}(u_k \circ \exp_{p_k})(r_k x), \quad \zeta_k(x) := \zeta(r_k x), \]

\[ a_k(x) := (a \circ \exp_{p_k})(r_k x) \quad \text{and} \quad c_k(x) := (c \circ \exp_{p_k})(r_k x). \]

Then, supp(\( \zeta_k v_k \)) \( \subset \) \( B(0, 3\delta r_k^{-1}) \) and, extending \( \zeta_k v_k \) by 0 outside \( B(0, 3\delta r_k^{-1}) \), we have that \( \zeta_k v_k \in C^\infty_0(\mathbb{R}^m) \subset D^{1,2}(\mathbb{R}^m) \). As \( \zeta \equiv 1 \text{ in } B(0, r_k) \), using (4.8) and (4.9) and performing the change of variable \( y = r_k x \) we obtain

\[ 0 < \lambda = \int_{B_r(p_k, r_k)} c|u_k|^2 \, dV_g \leq C \int_{B(0, r_k)} (c \circ \exp_{p_k})(\zeta(u_k \circ \exp_{p_k}))^2 \, dy \]

\[ = C \int_{B(0, r_k)} c_k|\zeta_k v_k|^2 \, dx \leq C \int_{B(0, r_k)} |\zeta_k v_k|^2 \, dx. \]

Here and hereafter \( C \) stands for a positive constant, not necessarily the same one. Moreover, inequalities (4.5) and (4.6) yield

\[ \int_{B(0, 3\delta r_k^{-1})} |\nabla (\zeta_k v_k)^2| \, dx = \int_{B(0, 3\delta)} |\nabla (\zeta(u_k \circ \exp_{p_k}))|^2 \, dy \]

\[ \leq C \int_{B(0, 3\delta)} \left[ |\nabla (u_k \circ \exp_{p_k})|^2 + |\nabla \zeta|^2 \right] \, dy \]

\[ \leq C \int_{B(0, 3\delta)} \left[ |\nabla (u_k \circ \exp_{p_k})|^2 + \exp_{p_k} |u_k|^2 \right] \, dy \]

\[ \leq C \int_{B_{p_k, 3\delta}} \left[ |\nabla u_k|^2 + |u_k|^2 \right] \, dV_g, \]

so Lemma 4.1 implies that \( (\zeta_k v_k) \) is bounded in \( D^{1,2}(\mathbb{R}^m) \). Therefore, after passing to a subsequence, we have that \( \zeta_k v_k \to v \) weakly in \( D^{1,2}(\mathbb{R}^m) \), \( \zeta_k v_k \to v \) in \( L^2_{\text{loc}}(\mathbb{R}^m) \) and \( \zeta_k v_k \to v \) a.e. in \( \mathbb{R}^m \). The proof of the proposition will follow from the next three claims.

**Claim 1.** \( v \neq 0 \).

To prove this claim first note that, as \( M \) is compact, there exists \( C_2 > 1 \) such that, for every \( q \in M \),

\[ C_2^{-1} |y - z| \leq d_g(\exp_q(y),\exp_q(z)) \leq C_2 |y - z| \quad \forall y, z \in B(0, 2\delta), \]

where \( d_g \) is the distance in \( M \). Set \( q := C_2^{-1} \). Then, for every \( z \in B(0, 1) \) we have that

\[ \exp_{p_k} B(r_k z, r_k g) \subset B_q(\exp_{p_k}(r_k z), r_k). \]

Now, arguing by contradiction, assume that \( v = 0 \). Let \( \vartheta \in C^\infty_c(\mathbb{R}^m) \) be such that supp(\( \vartheta \)) \( \subset \) \( B(z, \delta) \) for some \( z \in B(0, 1) \). Then, supp(\( \vartheta \)) \( \subset \) \( B(0, 2) \). Set \( \vartheta_k(q) := \vartheta(r_k^{-1}\exp_{p_k}^{-1}(q)) \). As \( \zeta_k \equiv 1 \text{ in } B(0, 2) \), \( \zeta_k v_k \to 0 \) in \( L^2_{\text{loc}}(\mathbb{R}^m) \), \( J^1_g(u_k) \to 0 \) in \( (H^1_g(M))' \) and \( (\vartheta_k^2 u_k) \) is bounded in \( H^1_g(M) \), using inequalities (4.5) and (4.6) and...
Hölder’s and Sobolev’s inequalities, we obtain
\[
\int_{\mathbb{R}^m} |\nabla (\partial \zeta_k v_k)|^2 \, dx = \int_{B(0,2)} |\nabla (\partial v_k)|^2 \, dx = \int_{B(0,2r_k)} |\nabla ((\partial_k u_k) \circ \exp_{p_k})|^2 \, dy
\]
\[
\leq C_3 \int_{B_p(p_{k,2r_k})} a |\nabla_g (\partial_k u_k)|^2 \, dv_g
\]
\[
= C_3 \int_{B_p(p_{k,2r_k})} a \left[ \partial_k^2 |\nabla g (u_k)|^2 + 2 \partial_k u_k \langle \nabla g u_k, \nabla_g \partial_k \rangle_g + |\nabla_g \partial_k|^2 u_k \right] \, dv_g
\]
\[
= C_3 \int_{B_p(p_{k,2r_k})} a \left\langle \nabla g u_k, \nabla_g (\partial_k^2 u_k) \right\rangle_g \, dv_g + o(1)
\]
\[
\leq C_4 \int_{B(0,2) \cap B(z,\rho)} |v_k|^{2^*} \, dx + o(1)
\]
\[
\leq C_4 \left( \int_{B(z,\rho)} |v_k|^{2^*} \, dx \right)^{2/m} \left( \int_{B(0,2)} |\partial \zeta_k v_k|^{2^*} \, dx \right)^{2/2^*} + o(1)
\]
\[
\leq C_4 S^{-1} \left( \int_{B(z,\rho)} |v_k|^{2^*} \, dx \right)^{2/m} \int_{\mathbb{R}^m} |\nabla (\partial \zeta_k v_k)|^2 \, dx + o(1),
\]
where \( C_3 := C_1 (\min_M a)^{-1} \) and \( C_4 := C_1 (\max_M c)C_3 \). On the other hand, from (6.9), (6.10) and (4.3) we derive
\[
\int_{B(z,\rho)} |v_k|^{2^*} \, dx \leq C_1 (\min c)^{-1} \int_{B_p(\exp_{p_k} (r_k z), r_k)} c |u_k|^{2^*} \, dv_g
\]
\[
\leq C_1 (\min c)^{-1} \lambda.
\]
It follows from (6.7) that \((C_1 (\min_M c)^{-1} \lambda)^{2/m} < \frac{1}{2} C_4^{-1} S\). Therefore,
\[
\lim_{k \to \infty} \int_{\mathbb{R}^m} |\nabla (\partial \zeta_k v_k)|^2 \, dx = 0
\]
and Sobolev’s inequality yields
\[
\lim_{k \to \infty} \int_{\mathbb{R}^m} |\partial \zeta_k v_k|^{2^*} \, dx = 0
\]
for every \( \partial \in C_c^\infty (\mathbb{R}^m) \) such that \( \text{supp}(\partial) \subset B(z, \rho) \) for some \( z \in \overline{B(0,1)} \). As \( B(0,1) \) can be covered by a finite number of balls \( B(z_j, \rho) \) with \( z_j \in \overline{B(0,1)} \), choosing a partition of unity \( \{ \partial_j^{2^*} \} \) subordinated to this covering, we conclude that
\[
\int_{B(0,1)} |\zeta_k v_k|^{2^*} \, dx \leq \sum_j \int_{\mathbb{R}^m} |\partial_j \zeta_k v_k|^{2^*} \, dx \to 0,
\]
contradicting 6.9. This finishes the proof of Claim 1.

Claim 2. \( \hat{v} := \left( \frac{c(p)}{a(p)} \right)^{(m-2)/4} v \) is a nontrivial solution to problem 6.9.

First we show that, after passing to a subsequence, \( r_k \to 0 \). Arguing by contradiction, assume that \( r_k > \theta > 0 \) for all \( k \) large enough. Then, as \( \zeta_k v_k \to v \) a.e. in
To this end, take \( \phi_k \) \( (4.11) \) i.e. we need to show that

\[
\lim_{i,j} = 1
\]

This yields a contradiction because, as we are assuming that \( u_k \to 0 \) weakly in \( H^1_0(M) \), we have that \( u_k \to 0 \) strongly in \( L^2_0(M) \).

Claim 2 is equivalent to showing that \( v \) satisfies

\[
-a(p)\Delta v = c(p)|v|^{2^* - 2}v, \quad v \in D^{1,2}(\mathbb{R}^m),
\]

i.e. we need to show that

\[
(4.11) \quad \int_{\mathbb{R}^m} a(p) \langle \nabla v, \nabla \varphi \rangle \, dx = \int_{\mathbb{R}^m} c(p)|v|^{2^* - 2}v \varphi \, dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^m).
\]

To this end, take \( \varphi \in C_c^\infty(\mathbb{R}^m) \) and let \( R > 0 \) be such that \( \text{supp}(\varphi) \subset B(0,R) \). For \( k \) such that \( Rr_k < 3\delta \) define \( \hat{\varphi}_k \in H^1_0(M) \) by

\[
\hat{\varphi}_k(q) := r_k^{-\frac{2-m}{2}} \varphi(r_k^{-1}\exp_{p_k}(q)).
\]

Note first that, as \( a_k \to a(p) \) and \( c_k \to c(p) \) in \( L^\infty_{\text{loc}}(\mathbb{R}^m) \) and \( \zeta_k v_k \to v \) weakly in \( D^{1,2}(\mathbb{R}^m) \) we have that

\[
\int_{\mathbb{R}^m} a_k \langle \nabla (\zeta_k v_k), \nabla \varphi \rangle \, dx = \int_{\mathbb{R}^m} a(p) \langle \nabla v, \nabla \varphi \rangle \, dx + o(1),
\]

\[
\int_{\mathbb{R}^m} c_k |\zeta_k v_k|^{2^* - 2} (\zeta_k v_k) \varphi \, dx = \int_{\mathbb{R}^m} c(p)|v|^{2^* - 2}v \varphi \, dx + o(1).
\]

Next observe that, if \((g^i_k)^j\) is the metric \( g \) written in normal coordinates around \( p_k \), \((g^i_k)^j\) is its inverse, \(|g^i_k| := \det(g^i_k)^j\) and \((\partial^{ij})\) is the identity matrix then, for every \( i,j = 1, ..., m \),

\[
\lim_{|y| \to 0} g^i_k(y) = \delta^{ij} \quad \text{and} \quad \lim_{|y| \to 0} |g^i_k|^{1/2}(y) = 1,
\]

uniformly in \( k \). Therefore, as \( \text{supp}(\hat{\varphi}_k \circ \exp_{p_k}) \subset B(0,Rr_k) \), \( r_k \to 0 \), and \( (u_k \circ \exp_{p_k}) \) and \((\hat{\varphi}_k \circ \exp_{p_k})\) are bounded in \( D^{1,2}(\mathbb{R}^m) \), we have that

\[
\int_{\mathbb{R}^m} (a \circ \exp_{p_k}) \langle \nabla (u_k \circ \exp_{p_k}), \nabla (\hat{\varphi}_k \circ \exp_{p_k}) \rangle \, dy - \int_M a \langle \nabla_g u_k, \nabla_g \hat{\varphi}_k \rangle_g \, dV_g
\]

\[
= \sum_{i,j} \int_{B(0,Rr_k)} (a \circ \exp_{p_k})(\partial^{ij} - |g^i_k|^{1/2}g^i_k) \partial_i (u_k \circ \exp_{p_k}) \partial_j (\hat{\varphi}_k \circ \exp_{p_k}) \, dy
\]

\[
= o(1),
\]
This proves (4.11).

Finally, as $J'_g(u_k) \to 0$ in $(H^1_g(M))'$ and $(\varphi_k)$ is bounded in $H^1_g(M)$ we conclude that, for $k$ large enough,

\[
\int_{\mathbb{R}^m} a(p) \langle \nabla v, \nabla \varphi \rangle \, dx
= \int_{\mathbb{R}^m} a_k \langle \nabla (\zeta_k v_k), \nabla \varphi \rangle \, dx + o(1)
= \int_{\mathbb{R}^m} (a \circ \exp_{p_k}) \langle \nabla (u_k \circ \exp_{p_k}), \nabla (\varphi_k \circ \exp_{p_k}) \rangle \, dy + o(1)
= \int_{M} a \langle \nabla_g u_k, \nabla_g \varphi_k \rangle \, dV_g + o(1)
= \int_{M} c |u_k|^{2^*-2} u_k \varphi_k \, dV_g + o(1)
= \int_{\mathbb{R}^m} (c \circ \exp_{p_k}) |u_k \circ \exp_{p_k}|^{2^*-2} (u_k \circ \exp_{p_k}) (\varphi_k \circ \exp_{p_k}) \, dy + o(1)
= \int_{\mathbb{R}^m} c_k |\zeta_k v_k|^{2^*-2} (\zeta_k v_k) \varphi \, dx + o(1)
= \int_{\mathbb{R}^m} c(p) |v_k|^{2^*-2} v \varphi \, dx + o(1).
\]

This proves (4.11).

**Claim 3.** \( \# \Gamma p < \infty \) and \( \tau \ge \left( \frac{a(p)^{m/2} \# \Gamma p}{c(p)^{m/2}} \right) J_\infty(\hat{v}). \)

Let \( \gamma_1 p, \ldots, \gamma_n p \) be \( n \) distinct points in the \( \Gamma \)-orbit \( \Gamma p \) of \( p \), and fix \( \eta \in (0, \delta] \) such that \( d_g(\gamma_i p, \gamma_j p) \ge 4 \eta \) if \( i \neq j \). For \( k \) sufficiently large, \( d_g(p_k, p) < \eta \) so, as \( \gamma_i \) is an isometry, we have that \( d_g(\gamma_i p_k, \gamma_j p_k) > 2 \eta \) for all \( k \in \mathbb{N} \) and \( i \neq j \). Since \( c \) and \( u_k \) are \( \Gamma \)-invariant, for each \( \rho \in (0, \eta] \) we obtain that

\[
(4.13) \quad n \int_{B_k(p_k, \rho)} c |u_k|^2 \, dV_g = \sum_{i=1}^{n} \int_{B_k(\gamma_i p_k, \rho)} c |u_k|^2 \, dV_g \le \int_{M} c |u_k|^2 \, dV_g.
\]

Let \( \varepsilon > 0 \). By (4.12) there exists \( \rho \in (0, \eta] \) such that \( (1 + \varepsilon)^{-1} < |g^{k}|^{1/2} < (1 + \varepsilon) \) in \( B(0, \rho) \) for \( k \) large enough. As \( 1_{B(0, \rho r_k^{-1})} c_k \to c(p) \) and \( \zeta_k v_k \to v \) a.e. in \( \mathbb{R}^m \),
Fatou’s lemma and inequality (4.13) yield
\[ \frac{n}{m} \int_{\mathbb{R}^m} c(p) |v|^2^* \, dx \leq \liminf_{k \to \infty} \frac{n}{m} \int_{B(0,\rho p_k^{-1})} c_k |v_k|^2^* \, dx \]
\[ \leq \liminf_{k \to \infty} \frac{n}{m} \int_{B(0,\rho)} (c \circ \exp_{p_k}) |u_k \circ \exp_{p_k}|^2^* \, dy \]
\[ \leq (1 + \varepsilon) \liminf_{k \to \infty} \frac{n}{m} \int_{B_g(p_k, \rho)} c |u_k|^2^* \, dV_g \]
\[ \leq (1 + \varepsilon) \lim_{k \to \infty} \frac{1}{m} \int_M c |u_k|^2^* \, dV_g = (1 + \varepsilon) \tau. \]
This implies that \( n \) is bounded and, therefore, \( \#\Gamma p < \infty \). Moreover, as \( \varepsilon \) is arbitrary, taking \( n = \#\Gamma p, \) we conclude that
\[ \left( \frac{a(p)^{m/2} \#\Gamma p}{c(p)^{(m-2)/2}} \right) J_\infty(\bar{v}) = \left( \frac{a(p)^{m/2} \#\Gamma p}{c(p)^{(m-2)/2}} \right) \frac{1}{m} \int_{\mathbb{R}^m} |\bar{v}|^2^* \, dx \]
\[ = \frac{\#\Gamma p}{m} \int_{\mathbb{R}^m} c(p) |v|^2^* \, dx \leq \tau, \]
as claimed.

This finishes the proof of the proposition.

Proof of Theorem 2.2. Let \((u_k)\) be a sequence in \( H^1_g(M)^\Gamma \) such that \( J_g(u_k) \to \tau < \left( \min_{q \in M} \frac{a(q)^{m/2} \#\Gamma q}{c(q)^{(m-2)/2}} \right) \frac{1}{m} S^{m/2} \) and \( J'_g(u_k) \to 0 \) in \( (H^1_g(M))^\Gamma \). By Lemma 4.1 \((u_k)\) is bounded in \( H^1_g(M) \), so, after passing to a subsequence, \( u_k \to u \) weakly in \( H^1_g(M) \). It follows that \( u \in H^1_g(M)^\Gamma \), \( J'_g(u) = 0 \) and, as \(-\text{div}_g(a \nabla_g) + b\) is coercive on \( H^1_g(M)^\Gamma \),
\[ J_g(u) = \frac{1}{m} \|u\|_{g, a, b} \leq \liminf_{k \to \infty} \frac{1}{m} \|u_k\|_{g, a, b} = \lim_{k \to \infty} J_g(u_k) = \tau. \]
Set \( \bar{u}_k := u_k - u \). Then \( \bar{u}_k \to 0 \) weakly in \( H^1_g(M) \) and, by a standard argument (see, e.g., [10, 36]), \((\bar{u}_k)\) is a \( \Gamma \)-invariant Palais-Smale sequence for the functional \( J_g \) with \( b = 0 \) at the level \( \bar{\tau} := \tau - J_g(u) < \left( \min_{q \in M} \frac{a(q)^{m/2} \#\Gamma q}{c(q)^{(m-2)/2}} \right) \frac{1}{m} S^{m/2} \). Proposition 4.2 implies that \( \bar{\tau} = 0 \). Thus, inequality (4.13) is an equality. It follows that \( u_k \to u \) strongly in \( H^1_g(M) \).

5. A variational principle for nodal solutions

This section is devoted to the proof of Theorem 2.3.

We begin by showing that a neighborhood of the set of functions in \( H^1_g(M)^\Gamma \) which do not change sign is invariant under the negative gradient flow of \( J_g \), with respect to a suitably chosen scalar product in \( H^1_g(M)^\Gamma \).

Since we are assuming that \( a > 0 \) and the operator \(-\text{div}_g(a \nabla_g) + b\) is coercive on \( H^1_g(M)^\Gamma \), there exists \( \mu > 0 \) such that
\[ \int_M [a|\nabla_g u_g|^2 + b|u|^2] \, dV_g \geq \mu \int_M [a|\nabla_g u_g|^2 + |u|^2] \, dV_g \quad \forall u \in H^1_g(M)^\Gamma. \]
Fix \( A > \max \{ 1, \mu, |b|_{C^0(M)} \} \) and consider the scalar product

\[ \langle u, v \rangle_{g,a,A} := \int_M [a(\nabla_g u, \nabla_g v) + Auv] \, dV_g \]

in \( H^1_g(M)^F \). We write \( ||\cdot||_{g,a,A} \) for the induced norm, which is equivalent to the standard norm in \( H^1_g(M)^F \). Given a subset \( D \) of \( H^1_g(M)^F \) and \( \rho > 0 \), we set

\[ B_\rho(D) := \{ u \in H^1_g(M)^F : \text{dist}(u, D) \leq \rho \}, \]

where \( \text{dist}(u, D) := \inf_{v \in D} ||u - v||_{g,a,A} \).

The gradient of the functional \( J_g : H^1_g(M)^F \to \mathbb{R} \) at \( u \in H^1_g(M)^F \), with respect to the scalar product (5.2), is the vector \( \nabla J_g(u) \) which satisfies

\[ \langle \nabla J_g(u), v \rangle_{g,a,A} = J'_g(u)v \]

\[ = \langle u, v \rangle_{g,a,A} - \int_M (A-b)uv \, dV_g - \int_M c|u|^{2^*-2}uv \, dV_g \quad \forall v \in H^1_g(M)^F, \]

i.e., \( \nabla J_g(u) = u - Lu - Gu \) where \( Lu, Gu \in H^1_g(M)^F \) are the unique solutions to

\[ -\text{div}_g(a\nabla_g(Lu)) + A(Lu) = (A-b)u, \]
\[ -\text{div}_g(a\nabla_g(Gu)) + A(Gu) = c|u|^{2^*-2}u. \]

Then, the following inequality holds true. Its proof was suggested by Jérôme Vétois and fills in a small gap in his proof of Lemma 2.1 in [35].

**Lemma 5.1.** Set \( \overline{\tau} := \frac{\mu}{A + \mu} \in (0, 1) \). Then, for every \( u \in H^1_g(M)^F \), we have

\[ \|Lu\|_{g,a,A} \leq \overline{\tau} \|u\|_{g,a,A}. \]

**Proof.** By (5.1), for every \( u \in H^1_g(M)^F \) we have that

\[ \int_M (A-b)u^2 \, dV_g \leq \int_M Au^2 \, dV_g - \mu \int_M [a|\nabla_g u|^2 + |u|^2] \, dV_g + \int_M a|\nabla_g u|^2 \, dV_g \]
\[ \leq \frac{A - \mu}{A} \int_M [a|\nabla_g u|^2 + |u|^2] \, dV_g = \frac{A - \mu}{A} \|u\|^2_{g,a,A}. \]

Hence, using (5.3) we obtain

\[ \|Lu\|^2_{g,a,A} = \int_M (A-b)u(Lu) \, dV_g \leq \frac{1}{2} \int_M (A-b) \left[ u^2 + (Lu)^2 \right] \, dV_g \]
\[ \leq \frac{A - \mu}{2A} \left( \|u\|^2_{g,a,A} + \|Lu\|^2_{g,a,A} \right). \]

Consequently,

\[ \frac{A + \mu}{2A} \|Lu\|^2_{g,a,A} \leq \frac{A - \mu}{2A} \|u\|^2_{g,a,A}, \]

as claimed. \( \square \)

We consider the negative gradient flow \( \psi : \mathcal{G} \to H^1_g(M)^F \) of \( J_g \), defined by

\[ \frac{\partial}{\partial t} \psi(t, u) = -\nabla J_g(\psi(t, u)), \quad \psi(0, u) = u, \]
where \( G := \{(t, u) : u \in H^1_g(M)^{\Gamma}, 0 \leq t < T(u)\} \) and \( T(u) \) is the maximal existence time for the trajectory \( t \mapsto \psi(t, u) \). A subset \( D \) of \( H^1_g(M)^{\Gamma} \) is said to be strictly positively invariant if

\[
\psi(t, u) \in \text{int} D \quad \text{for every } u \in D \text{ and } t \in (0, T(u)).
\]

The set of functions in \( H^1_g(M)^{\Gamma} \) which do not change sign is \( \mathcal{P}^\Gamma \cup -\mathcal{P}^\Gamma \), where \( \mathcal{P}^\Gamma := \{ u \in H^1_g(M)^{\Gamma} : u \geq 0 \} \) is the convex cone of nonnegative functions. The nodal solutions to the problem (1.1) lie in the set

\[
\mathcal{N}_g^{\Gamma} := \{ u \in \mathcal{N}^g : u^+, u^- \in \mathcal{N}_g^\Gamma \},
\]

where \( u^+ := \max\{0, u\} \), \( u^- := \min\{0, u\} \) and \( \mathcal{N}_g^\Gamma \) is the Nehari manifold defined in (2.4).

**Lemma 5.2.** There exists \( \rho_0 > 0 \) such that, for every \( \rho \in (0, \rho_0) \),

(a) \[ B_\rho(\mathcal{P}^\Gamma) \cup B_\rho(-\mathcal{P}^\Gamma) \cap \mathcal{E}_g^\Gamma = \emptyset, \]

(b) \( B_\rho(\mathcal{P}^\Gamma) \) and \( B_\rho(-\mathcal{P}^\Gamma) \) are strictly positively invariant.

**Proof.** By symmetry considerations, it is enough to prove this for \( B_\rho(\mathcal{P}^\Gamma) \).

(a): Note that \(|u^-| \leq |u(p) - v(p)|\) for every \( u, v : M \to \mathbb{R} \) with \( v \geq 0, \rho \in M \). Sobolev’s inequality yields a positive constant \( C \) such that

\[
|u^-|_{g,c,2^*} = \min_{v \in \mathcal{P}^\Gamma} |u - v|_{g,c,2^*} \leq C \min_{v \in \mathcal{P}^\Gamma} \|u - v\|_{g,a,\mathcal{A}} = C \text{dist}_A(u, \mathcal{P}^\Gamma)
\]

for every \( u \in H^1_g(M)^{\Gamma} \). If \( u \in \mathcal{E}_g^\Gamma \), then \( u^- \in \mathcal{N}_g^\Gamma \) and, therefore, \(|u^-|^2_{g,c,2^*} = mJ_g(u^-) \geq m\tau_g^\Gamma > 0 \). This proves that \( \text{dist}_A(u, \mathcal{P}^\Gamma) \geq \rho_1 > 0 \) for all \( u \in \mathcal{E}_g^\Gamma \).

(b): By the maximum principle, \( Lv \in \mathcal{P}^\Gamma \) and \( Gv \in \mathcal{P}^\Gamma \) if \( v \in \mathcal{P}^\Gamma \). For \( u \in H^1_g(M)^{\Gamma} \) let \( v \in \mathcal{P}^\Gamma \) be such that \( \text{dist}_A(u, \mathcal{P}^\Gamma) = \|u - v\|_{g,a,\mathcal{A}} \). Then, Lemma 5.1 yields

\[
\text{dist}_A(Lu, \mathcal{P}^\Gamma) \leq \|Lu - Lv\|_{g,a,\mathcal{A}} \leq \tilde{\mu} \|u - v\|_{g,a,\mathcal{A}} = \tilde{\mu} \text{dist}_A(u, \mathcal{P}^\Gamma).
\]

On the other hand, from (5.3), Hölder’s inequality and (5.5) we get that

\[
\text{dist}_A(Gu, \mathcal{P}^\Gamma) \|G(u)^-\|_{g,a,\mathcal{A}} \leq \|G(u)^-\|^2_{g,a,\mathcal{A}} = \langle G(u)^-, G(u)^- \rangle_{g,a,\mathcal{A}}
\]

\[
= \int_M c |u|^{2^*-2} uG(u^-) \, dv_g \leq \int_M c |u|^{2^*-2} uG(u^-) \, dv_g
\]

\[
\leq \|u|^{2^*-1}_{g,c,2^*} \|G(u)^-\|_{g,c,2^*} \leq C^2 \text{dist}_A(u, \mathcal{P}^\Gamma)^{2^*-1} \|G(u)^-\|_{g,a,\mathcal{A}}.
\]

Hence,

\[
\text{dist}_A(Gu, \mathcal{P}^\Gamma) \leq C^2 \text{dist}_A(u, \mathcal{P}^\Gamma)^{2^*-1} \quad \forall u \in H^1_g(M)^{\Gamma}.
\]

Fix \( \nu \in (\tilde{\mu}, 1) \) and let \( \rho_2 > 0 \) be such that \( C^2 \rho_2^{2^*-2} \leq \nu - \tilde{\mu} \). Then, for \( \rho \in (0, \rho_2) \), from inequalities (5.6) and (5.7) we obtain

\[
\text{dist}_A(Lu + Gu, \mathcal{P}^\Gamma) \leq \nu \text{dist}_A(u, \mathcal{P}^\Gamma) \quad \forall u \in B_\rho(\mathcal{P}^\Gamma),
\]
Therefore, $Lu + Gu \in \text{int} B_\rho(\mathcal{P}^\Gamma)$ if $u \in B_\rho(\mathcal{P}^\Gamma)$. Since $B_\rho(\mathcal{P}^\Gamma)$ is closed and convex, Theorem 5.2 in [12] yields that
\[
\psi(t, u) \in B_\rho(\mathcal{P}^\Gamma) \quad \text{for all} \quad t \in (0, T(u)) \quad \text{if} \quad u \in B_\rho(\mathcal{P}^\Gamma).
\]

Now we can argue as in the proof of Lemma 2 in [12] to show that, in fact, $B_\rho(\mathcal{P}^\Gamma)$ is strictly positively invariant. Letting $\rho_0 := \min\{\rho_1, \rho_2\}$, we get the result. \qed

We fix $\rho \in (0, \rho_0)$ and, for $d \in \mathbb{R}$, we set
\[
\mathcal{D}^\Gamma_\rho := B_\rho(\mathcal{P}^\Gamma) \cup B_\rho(\mathcal{P}^\Gamma) \cup J^\Gamma_d,
\]
where $J^\Gamma_d := \{u \in H^1_g(M)^\Gamma : J_g(u) \leq d\}$. It follows from Lemma 5.2 that $\mathcal{D}^\Gamma_0$ is strictly positively invariant under the flow $\psi$, and that a critical point of $J_g$ is sign-changing if it lies in the complement of $\mathcal{D}^\Gamma_0$.

To find critical points of $J_g$ in the complement of $\mathcal{D}^\Gamma_0$ we use the relative genus. A subset $\mathcal{Y}$ of $H^1_g(M)^\Gamma$ will be called symmetric if $-u \in \mathcal{Y}$ for every $u \in \mathcal{Y}$.

**Definition 5.3.** Let $\mathcal{D}$ and $\mathcal{Y}$ be symmetric subsets of $H^1_g(M)^\Gamma$. The genus of $\mathcal{Y}$ relative to $\mathcal{D}$, denoted by $g(\mathcal{Y}, \mathcal{D})$, is the smallest number $n$ such that $\mathcal{Y}$ can be covered by $n + 1$ open symmetric subsets $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ of $H^1_g(M)^\Gamma$ with the following two properties:

(i) $\mathcal{Y} \cap \mathcal{D} \subset \mathcal{U}_0$ and there exists an odd continuous map $\vartheta_0 : \mathcal{U}_0 \to \mathcal{D}$ such that $\vartheta_0(u) = u$ for $u \in \mathcal{Y} \cap \mathcal{D}$.

(ii) there exist odd continuous maps $\vartheta_j : \mathcal{U}_j \to \{1, -1\}$ for every $j = 1, \ldots, n$.

If no such cover exists, we define $g(\mathcal{Y}, \mathcal{D}) := \infty$.

Now define
\[
c_j := \inf\{c \in \mathbb{R} : g(\mathcal{D}^\Gamma_c, \mathcal{D}^\Gamma_0) \geq j\}.
\]

**Lemma 5.4.** Assume that $J_g$ satisfies condition $(PS)^\Gamma_{c_j}$ in $H^1_g(M)$. Then, the following statements hold true:

(a) $J_g$ has a sign-changing critical point $u \in H^1_g(M)^\Gamma$ with $J_g(u) = c_j$.

(b) If $c_j = c_{j+1}$, then $J_g$ has infinitely many sign-changing critical points $u \in H^1_g(M)^\Gamma$ with $J_g(u) = c_j$.

Consequently, if $J_g$ satisfies $(PS)^\Gamma_c$ in $H^1_g(M)$ for every $c \leq d$, then $J_g$ has at least $g(\mathcal{D}^\Gamma_d, \mathcal{D}^\Gamma_0)$ pairs of sign-changing critical points $u$ in $H^1_g(M)^\Gamma$ with $J_g(u) \leq d$.

**Proof.** The proof is exactly the same as that of Proposition 3.6 in [11]. It uses the fact that $\mathcal{D}^\Gamma_0$ is strictly positively invariant under the flow $\psi$, and the monotonicity and subadditivity properties of the relative genus. \qed

Now we can follow the proof of Theorem 3.7 in [11] to obtain Theorem [2.2]. We give the details for the sake of completeness.
Proof of Theorem 2.2. Let \( d := \sup_W J_g \). By Lemma 5.4, we only need to show that \( n := g(D^D_g, D^E_g) \geq \dim(W) - 1 \). Let \( U_0, U_1, \ldots, U_n \) be open symmetric subsets of \( H^1_g(M)^F \) covering \( D^D_g \) with \( D^D_g \subset U_0 \) and let \( \vartheta_0 : U_0 \to D^D_g \) and \( \vartheta_j : U_j \to \{1, -1\}, j = 1, \ldots, n \), be odd continuous maps such that \( \vartheta_0(u) = u \) for all \( u \in D^D_g \). Since \( H^1_g(M)^F \) is an AR we may assume that \( \vartheta_0 \) is the restriction of an odd continuous map \( \tilde{\vartheta}_0 : H^1_g(M)^F \to H^1_g(M)^F \). Let \( B \) be the connected component of the complement of the Nehari manifold \( \mathcal{N}^T_g \) in \( H^1_g(M)^F \) which contains the origin, and set \( O := \{ u \in W : \tilde{\vartheta}_0(u) \in B \} \). Then, \( O \) is a bounded open symmetric neighborhood of 0 in \( W \).

Let \( V_j := U_j \cap \partial O \). Then, \( V_0, V_1, \ldots, V_n \) are symmetric and open in \( \partial O \), and they cover \( \partial O \). Further, by Lemma 5.2,

\[
\vartheta_0(V_0) \subset D^D_g \cap \mathcal{N}^T_g \subset \mathcal{N}^T_g \setminus \mathcal{E}_g^F.
\]

The set \( \mathcal{N}^T_g \setminus \mathcal{E}_g^F \) consists of two connected components; see, e.g., [5]. Therefore, there exists an odd continuous map \( \eta : \mathcal{N}^T_g \setminus \mathcal{E}_g^F \to \{1, -1\} \). Let \( \eta_j : V_j \to \{1, -1\} \) be the restriction of the map \( \eta \circ \vartheta_0 \) if \( j = 0 \), and the restriction of \( \vartheta_j \) if \( j = 1, \ldots, n \). Take a partition of the unity \( \{ \pi_j : \partial O \to [0, 1] : j = 0, 1, \ldots, n \} \) subordinated to the cover \( \{ V_0, V_1, \ldots, V_n \} \) consisting of even functions, and let \( \{ e_1, \ldots, e_{n+1} \} \) be the canonical basis of \( \mathbb{R}^{n+1} \). Then, the map \( \Psi : \partial O \to \mathbb{R}^{n+1} \) given by

\[
\Psi(u) := \sum_{j=0}^n \eta_j(u)\pi_j(u)e_{j+1}
\]

is odd and continuous, and satisfies \( \Psi(u) \neq 0 \) for every \( u \in \partial O \). The Borsuk-Ulam theorem allow us to conclude that \( \dim(W) \leq n + 1 \), as claimed. \( \square \)

Acknowledgement 1. We are grateful to Jérôme Vétois for suggesting the proof of Lemma 5.7. We wish also to thank the anonymous referee for his/her careful reading and valuable comments.

References

1. Ambrosetti, Antonio; Rabinowitz, Paul H.: Dual variational methods in critical point theory and applications. J. Functional Analysis 14 (1973), 349–381.
2. Ammann, Bernd; Humbert, Emmanuel: The second Yamabe invariant. J. Funct. Anal. 235 (2006), no. 2, 377–412.
3. Aubin, Thierry: Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. J. Math. Pures Appl. (9) 55 (1976), no. 3, 269–296.
4. Aubin, Thierry: Some nonlinear problems in Riemannian geometry. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.
5. Bredon, Glen E.: Introduction to compact transformation groups. Pure and Applied Mathematics, Vol. 46. Academic Press, New York-London, 1972.
6. Brendle, Simon: Blow-up phenomena for the Yamabe equation. J. Amer. Math. Soc. 21 (2008), no. 4, 951–979.
7. Brendle, Simon; Marques, Fernando C.: Blow-up phenomena for the Yamabe equation. II. J. Differential Geom. 81 (2009), no. 2, 225–250.
[8] Castro, Alfonso; Cossio, Jorge; Neuberger, John M.: A sign-changing solution for a superlinear Dirichlet problem. Rocky Mountain J. Math. 27 (1997), no. 4, 1041–1053.

[9] Clapp, Mónica; Faya, Jorge: Multiple solutions to the Bahri-Coron problem in some domains with nontrivial topology. Proc. Amer. Math. Soc. 141 (2013), no. 12, 4339–4344.

[10] Clapp, Mónica; Faya, Jorge: Multiple solutions to anisotropic critical and supercritical problems in symmetric domains. Contributions to nonlinear elliptic equations and systems, 99–120, Progr. Nonlinear Differential Equations Appl., 86, Birkhäuser/Springer, Cham, 2015.

[11] Clapp, Mónica; Pacella, Filomena: Multiple solutions to the pure critical exponent problem in domains with a hole of arbitrary size. Math. Z. 259 (2008), no. 3, 575–589.

[12] Clapp, Mónica; Weth, Tobias: Multiple solutions for the Brezis-Nirenberg problem. Adv. Differential Equations 10 (2005), no. 4, 463–480.

[13] Deimling, Klaus: Ordinary differential equations in Banach spaces. Lecture Notes in Mathematics, Vol. 596. Springer-Verlag, Berlin-New York, 1977.

[14] Ding, Wei Yue: On a conformally invariant elliptic equation on $\mathbb{R}^n$. Comm. Math. Phys. 107 (1986), no. 2, 331–335.

[15] tom Dieck, Tammo: Transformation groups. de Gruyter Studies in Mathematics, 8. Walter de Gruyter & Co., Berlin, 1987.

[16] Djadli, Zindine; Jourdain, Antoinette: Nodal solutions for scalar curvature type equations with perturbation terms on compact Riemannian manifolds. Bol. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 5 (2002), no. 1, 205–226.

[17] Druet, Olivier; Hebey, Emmanuel; Robert, Frédéric: Blow-up theory for elliptic PDEs in Riemannian geometry. Mathematical Notes, 45. Princeton University Press, Princeton, NJ, 2004.

[18] El Sayed, Safaa: Second eigenvalue of the Yamabe operator and applications. Calc. Var. Partial Differential Equations 50 (2014), no. 3-4, 665–692.

[19] Hebey, Emmanuel; Vaugon, Michel: Le problème de Yamabe équivariant. Bull. Sci. Math. 117 (1993), no. 2, 241–286.

[20] Heube, Emmanuel; Vaugon, Michel: Sobolev spaces in the presence of symmetries. J. Math. Pures Appl. (9) 76 (1997), no. 10, 859–881.

[21] Henry, Guillermo: Second Yamabe constant on Riemannian products. J. Geom. Phys. 114 (2017), 260–275.

[22] Holcman, David: Solutions nodales sur les variétés riemanniennes. J. Funct. Anal. 161 (1999), no. 1, 219–245.

[23] Khuri, Marcus A.; Marques, Fernando C.; Schoen, Richard M.: A compactness theorem for the Yamabe problem. J. Differential Geom. 81 (2009), no. 1, 143–196.

[24] Lee, John M.; Parker, Thomas H.: The Yamabe problem. Bull. Amer. Math. Soc. (N.S.) 17 (1987), no. 1, 37–91.

[25] Morabito, Filippo; Pistoia, Angela; Vaira, Giusi: Towering Phenomena for the Yamabe Equation on Symmetric Manifolds. Potential Anal. 47 (2017), no. 1, 53–102.

[26] Obata, Morio: The conjectures on conformal transformations of Riemannian manifolds. J. Differential Geometry 6 (1971/72), 247–258.

[27] Palais, Richard S.: The principle of symmetric criticality. Comm. Math. Phys. 69 (1979), no. 1, 19–30.

[28] Petean, Jimmy: On nodal solutions of the Yamabe equation on products. J. Geom. Phys. 59 (2009), no. 10, 1395–1401.

[29] Pollack, Daniel: Nonuniqueness and high energy solutions for a conformally invariant scalar equation. Comm. Anal. Geom. 1 (1993), no. 3-4, 347–414.

[30] Robert, Frédéric; Vétois, Jérôme: Sign-changing solutions to elliptic second order equations: gluing a peak to a degenerate critical manifold. Calc. Var. Partial Differential Equations 54 (2015), no. 1, 693–716.

[31] Saintier, Nicolas: Blow-up theory for symmetric critical equations involving the p-Laplacian. NoDEA Nonlinear Differential Equations Appl. 15 (2008), no. 1-2, 227–245.

[32] Struwe, Michael: Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, 34. Springer-Verlag, Berlin, 1996.

[33] Schoen, Richard: Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differential Geom. 20 (1984), no. 2, 479–495.
[34] Trudinger, Neil S.: Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 265–274.
[35] Vétois, Jérôme: Multiple solutions for nonlinear elliptic equations on compact Riemannian manifolds. Internat. J. Math. 18 (2007), no. 9, 1071–1111.
[36] Willem, Michel: Minimax theorems. Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.
[37] Yamabe, Hidehiko: On a deformation of Riemannian structures on compact manifolds. Osaka Math. J. 12 (1960), 21–37.

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510 México D.F., Mexico
E-mail address: monica.clapp@im.unam.mx

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510 México D.F., Mexico
E-mail address: jcfmor@im.unam.mx