Research Article

Local Fractional Locating Number of Convex Polytope Networks

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The concept of locating number for a connected network contributes an important role in computer networking, loran and sonar models, integer programming and formation of chemical structures. In particular it is used in robot navigation to control the orientation and position of robot in a network, where the places of navigating agents can be replaced with the vertices of a network. In this note, we have studied the latest invariant of locating number known as local fractional locating number of an antiprism based convex polytope networks. Furthermore, it is also proved that these convex polytope networks posses boundedness under local fractional locating number.

1. Introduction

Slater [1] introduced the methodology to compute the locating set of a connected network. He defined the minimum cardinality of a locating set as a locating number of a connected network. Melter and Harary independently studied the concept of location number but they used the different term called as metric dimension. They also briefly studied the locating number of serval type of networks such as cycles, complete and complete bipartite networks [2]. Applications of locating number can be found for navigation of reboots [3], chemical structures [4], combinatorial optimization [5] image processing & pattern recognition [6].

Chartrand et al. [4] played a vital role in the study of locating number (LN), they characterized all those connected networks of order $p$ having locating number 1, $p - 2$, and $p - 1$. Furthermore, they also presented a new technique to compute bounds of locating number of unicyclic networks. Since then researchers have computed locating number of many connected networks such as generalized Peterson network [7], Cartesian products [8], constant locating number of some convex polytopes and generalized convex polytopes [9, 10], Mobius ladders [11], Toeplitz networks [12], $k$—dimensional networks, and fan networks [13, 14]. Moreover, LN of corona product and partition dimension of different products of networks can be seen in [15, 16] and fault tolerant LN of some families of convex polytopes studied in [17, 18]. For the study of edge LN of wheel and $k$ level wheel networks, we refer [19, 20]. There are various new invariants of LN which have been introduced in recent times such as partition dimension [21], Strong—LN [5], fault-tolerant LN [22], edge LN [23], mixed—LN [24], independent resolving sets [25], and K—LN [26].

Chartared et al. use the concept of LN to solve an integer programming problem (IPP) with specific conditions [4] and Currie and Ollermann used the idea of fractional locating number (FLN) to find solution of specific IPP as well [27]. The FLN formally introduced in networking theory by Arguman and Mathew and they computed exact values of FLN of a path, cycles, wheels, complete and friendship networks. Furthermore, they also developed some new techniques to compute exact values of FLN of connected networks with specific conditions [28]. Later on Arguman et al. characterized all those networks have FLN exactly $\lfloor V(G) \rfloor/2$ and they also presented many results on FLN of Cartesian product of two networks [29]. Feng et al. computed FLN of distance regular and vertex transitive networks [30]. For the study of FLN of corona, lexicographic, and
hierarchical products of connected networks see [31, 32]. Recently, Alkhalidi et al. established sharp bound of FLN of all the connected networks [33].

The latest version of FLN called by local fractional locating number (LFLN) is defined by Aisayah et al. and they computed LFLN of different connected networks [34]. Javaid et al. developed sharp bounds of LFLN of all the connected networks and they computed upper bounds of local FLN of wheel-related networks. Furthermore, they also improved the lower bound of LFLN different from unity and also developed a technique to compute exact value of LFLN under specific conditions [35, 36]. For the study of LFLN of cretin family of convex polytopes see [37–39].

In this manuscript, our main objective is to compute LFLN of cretin family of convex polytopes in the form of sharp upper and lower bounds. It has been proved that in every case the convex polytopes remain bounded. The manuscript is organised as Section 2 contains preliminaries and Sections 3 and 4 have main results and conclusion respectively.

2. Preliminaries

A network \( G \) is an order pair \((V(G), E(G))\), where \( V(G) \) is the vertex set and \( E(G) \) is the edge set. A walk is a finite sequence of edges and vertices between two vertices. A trail is a walk in which all edges are distinct and a path is a trail in which all vertices are distinct. A network \( G \) is connected if there is a path between each pair of vertices. The distance between two vertices \( a \) and \( b \) is defined by \( d(a,b)\) as the length of the shortest path between \( a \) and \( b \). For further preliminary results of networking theory see [40]. A vertex \( c \in V(G) \) is said to resolve a pair \( a, b \in V(G) \), if \( d(a,c) \neq d(b,c) \). Suppose that \( W = \{w_1, w_2, w_3, \ldots, w_p\} \subseteq V(G) \) and \( x \in V(G) \), then \( p \) tuple representation of \( x \) with respect to \( W \) is defined as \( d(x|W) = (d(x, w_1), d(x, w_2), d(x, w_3), \ldots, d(x, w_p)) \). If distinct vertices of \( G \) have unique representation with respect to \( W \) then \( W \) is known as locating/resolving set. The minimum cardinality of \( W \) is called locating number (LN) of \( G \) that is defined as

\[
LN(G) = \min |W|: W \text{ is the resolving set of } G.
\]

For an edge \( ab \in E(G) \) the local resolving neighbourhood set (LRN) is the collection of all vertices of \( G \) which resolve an edge \( ab \) and it is donated by \( R_l(ab) = \{ c: d(c,a) \neq d(c,b) \} \), where \( c \in V(G) \). A real valued function \( h: V(G) \rightarrow [0, 1] \) becomes local resolving function of \( G \) if \( h(R_l(ab)) \geq 1 \) for each \( R_l(ab) \) in \( G \), where \( h(R_l(ab)) = \sum_{v \in R_l(ab)} h(v) \). A local resolving function (LRF) is called minimal LRF if there exist another function \( h': V(G) \rightarrow [0, 1] \) such that \( h' \leq h \) and \( h(a) \neq h'(a) \) for at least one \( a \in V(G) \), that is not LRF of \( G \). If \( |h| = \sum_{v \in V(G)} h(v) \), then local fractional locating number (LFLN) of \( G \) is defined as

\[
LFLN(G) = \min |h|: h \text{ is minimal local resolving function of } G.
\]

3. Main Results

This section is devoted to the main results in which, we have examined the LFLN of cretin family convex polytope networks \( D_p \) and \( E_p \) and it has been proved that these polytope networks remain bounded under LFLN when their order approaches to infinity.

3.1. LFLN of Convex Polytope Network \( D_p \)

In this particular subsection, we have computed resolving local neighbourhood sets and LFLN of \( D_p \) in the form of exact values and sharp lower and upper bounds.

The convex polytope network is introduced by Baća [41] and LN of \( D_p \) is 3 is proved in [10]. The vertex set \( V(D_p) \) consists of inner \( \{a_i: 1 \leq i \leq p\} \), middle \( \{a_1^i: 1 \leq i \leq p\} \), \( \{b_1: 1 \leq i \leq p\} \) and outer vertices \( \{b_1^i: 1 \leq i \leq p\} \cup \{a_1^i b_1^i: 1 \leq i \leq p\} \). Furthermore, the
order and size of $D_p$ are $4p$ and $6p$ respectively and for complete details see Figure 1.

**Lemma 1.** Let $D_p$ be a convex polytope network, with $p \geq 5$ and $p \equiv 1 \pmod{2}$. Then,

(i) $|R_i(a_i^1b_i)| = |R_i(b_ia_i^{1+})| = 5p + 7/2$ and $\bigcup i = 1 \ pR_i(a_i^1b_i) = V(D_p)$,

(ii) $|R_i(a_i^1b_i)| \leq |R_i(e)|$ and $|R_i(e) \cap \bigcup i = 1 \ p(R_i(a_i^1b_i)) \geq |R_i(a_i^1b_i)| \forall e \in E(D_p)$.

**Proof.** Consider $a_i$ inner, $a_i^1$, $b_i$ middle and $b_i^1$ are outer vertices of $D_p$, where $1 \leq i \leq p$ and $p + 1 \equiv 1 \pmod{2}$.

(i) $R_i(a_i^1b_i) = V(D_p) - \{a_{i+1,1}, a_{i+2,1}, \ldots, a_{p+1,1}\}$ and $R_i(b_i^1) = V(D_p) - \{a_{i+1,1}, a_{i+2,1}, \ldots, a_{p+1,1}\}$.

(ii) $R_i(a_i^1b_i) = V(D_p) - \{a_{i+1,1}, a_{i+2,1}, \ldots, a_{p+1,1}\}$ and $R_i(b_i^1) = V(D_p) - \{a_{i+1,1}, a_{i+2,1}, \ldots, a_{p+1,1}\}$.

The cardinalities of all the LRN sets of $D_p$ are illustrated in Table 1.

From Table 1, we note that $|R_i(a_i^1b_i)| \leq |R_i(e)|$. Since $|\bigcup i = 1 \ pR_i(a_i^1b_i)| = 4p$ therefore $|R_i(e) \cap \bigcup i = 1 \ pR_i(a_i^1b_i)| \geq |R_i(a_i^1b_i)| \forall e \in E(D_p)$.

**Theorem 1.** Let $D_3$ be a convex polytope network. Then

$$\frac{6}{5} \leq LFLN(D_3) \leq \frac{3}{2}. \quad (3)$$

**Proof.** The LRN sets of convex polytope network $D_3$ are:

$R_i(1) = R_i(a_1a_2) = V(D_3) - \{a_3, a_3^1, b_1, b_1^1\}$,

| RLN set | Comparison |
|---------|------------|
| $R_i(a_i^1a_{i+1})$ | $4p - 4 > |R_i(a_i^1a_{i+1})|$ |
| $R_i(b_i^1b_{i+1}^1)$ | $4p - 4 > |R_i(b_i^1b_{i+1}^1)|$ |
| $R_i(a_i^1a_{i+1})$ | $4p - 4 > |R_i(a_i^1a_{i+1})|$ |
| $R_i(b_i^1b_{i+1}^1)$ | $4p - 4 > |R_i(b_i^1b_{i+1}^1)|$ |

$R_i(12) = R_i(a_2a_3) = V(D_3) - \{a_1, a_1^1, b_2, b_2^1\}$

$R_i(13) = R_i(a_2a_3) = V(D_3) - \{a_1, a_1^1, b_3, b_3^1\}$

$R_i(14) = R_i(a_2a_3) = V(D_3) - \{a_2, a_2^1, b_3, b_3^1\}$

$R_i(15) = R_i(a_2a_3) = V(D_3) - \{a_3, a_3^1, b_1, b_1^1\}$

$R_i(16) = R_i(a_2a_3) = V(D_3) - \{a_1, a_1^1, b_1, b_1^1\}$

$R_i(17) = R_i(a_2a_3) = V(D_3) - \{a_2, a_2^1, b_1, b_1^1\}$

$R_i(18) = R_i(a_2a_3) = V(D_3) - \{a_3, a_3^1, b_2, b_2^1\}$

$R_i(10) = R_i(b_1^1b_2^1) = V(D_3) - \{b_2, b_2^1, a_3\}$

Since, $|R_i(a_i^1a_{i+1})| = |R_i(b_i^1b_{i+1}^1)| = |R_i(b_i^1b_{i+1}^1)| = 8$ which is less than the other LRN sets, where $1 \leq i \leq 3$. Furthermore, $\bigcup_{i=1}^{3} R_i(a_i^1a_{i+1}) = V(D_3)$ and $|R_i(e) \cap \bigcup_{i=1}^{3} R_i(a_i^1a_{i+1})| \geq |R_i(a_i^1a_{i+1})| \forall e \in E(D_3)$. 

Therefore, we define an upper LRF \( h: (D_3) \rightarrow [0, 1] \) as \( h(v) = 1/8 \forall v \in V(D_3) \). In order to show that \( h \) is minimal LRF consider another upper LRF \( h': V(D_3) \rightarrow [0, 1] \) as \( h'(v) < 1/8 \forall v \in V(D_3) \) therefore \( h'(R_i(e)) < 1 \) and \( \lvert h' \rvert < \lvert h \rvert \) which shows that \( h' \) is not LRF of \( D_3 \) hence \( LFLN(D_3) \leq \sum_{i=1}^{12} 1/8 = 3/2 \). Likewise for \( 1 \leq i \leq 3 \) cardinality of LRN set \( R_i(a_1^i b_1^i) \) is 10 which is greater then the cardinalities of all other LRN sets of \( D_3 \). Therefore, we define a maximal lower LRF \( g: (D_3) \rightarrow [0, 1] \) as \( g(v) = 1/10 \forall v \in V(D_3) \). Hence \( LFLN(D_3) \geq \sum_{i=1}^{12} 1/10 = 6/5 \). Consequently,

\[
\frac{6}{5} \leq LFLN(D_3) \leq \frac{3}{5}
\]

**Theorem 2.** Let \( D_3 \) be a convex polytope network. Then

\[
LFLN(D_3) = \frac{5}{4}
\]

**Proof.** The LRN sets are given by:

\[
R_i(1) = R_i(a_1 a_2) = V(D_3) \setminus \{a_4, a_4', b_1, b_1'\}, \quad (7)
\]
It is clear that the cardinality of each RLN set of $D_5$ is 16. Therefore, we define a constant function $h: V(D_5) \to [0, 1]$ as $h(v) = 1/16 \forall v \in V(D_5)$. Hence $LFLN(D_5) = \sum_{i=1}^{20} 1/16 = 5/4$.

**Theorem 3.** Let $D_p$ be a convex polytope network, with $p \geq 7$ and $p \equiv 1 \pmod{2}$. Then

$$\frac{p}{p-1} \leq LFLN(D_p) \leq \frac{8p}{5p+7} \quad (8)$$

**Proof.** To prove the result, we split it into two cases.

- **Case 1.** For $p = 7$, we have following LRN sets:
  
  $R_i(1) = R_i(a^1 b^1) = V(D_7) - \{a_2, a_3, a_4, a_5, b_1, b_2, b_3\},$

- **Case 2.** For $p > 7$, we have the following LRN sets:

$$R_i(1) = R_i(a^1 b^1) = V(D_p) - \{a_2, a_3, a_4, a_5, b_1, b_2, b_3\},$$
Since, $|R_i(a_1^1 b_1^1)| = |R_i(b_1 a_1^1)| = 16$ and $|R_i(a_1^1 b_1^1)|$ is minimal upper LRF consider another function $h: V(D_p) \rightarrow [0, 1]$ as $h(v) = 1/24v \in V(D_p)$ is a maximal upper LRF hence $\text{LFLN}(D_p) \geq \sum_{i=1}^{28} 1/24 = 7/6$. Consequently,

$$7/6 \leq \text{LFLN}(D_p).$$

**Lemma 2.** Suppose that $D_p$ is a convex polytope network, with $p \geq 6$ and $p \equiv 0 \pmod{2}$. Then:

(i) $|R_i(a_1^1 b_1^1)| = |R_i(a_1^1 b_1^1)| = 5p/2$ and $\cup_{i=1}^{p/2} R_i(a_1^i b_1^i) = V(D_p)$. 

(ii) $|R_i(a_1^1 b_1^1)| \leq |R_i(e)|$ and $|R_i(e)| \cup \cup_{i=1}^{p/2} R_i(a_1^i b_1^i) \geq |R_i(a_1^1 b_1^1)| \forall v \in E(D_p)$. 

**Proof.** Consider $a_i$, inner, $a_1^i, b_1^i$ middle and $b_i^i$ are the outer vertices of $D_p$, where $1 \leq i \leq p$ and $p \equiv 1 \pmod{2}$.

(i) $R_i(a_1^1 b_1^1) = V(D_p) - \{a_{i+1}^1, a_{i+2}^1, a_{i+3}^1, \ldots, a_{p/2+1}^1, a_{p/2+2}^1, a_{p/2+3}^1, \ldots, a_{p-1}^1, b_{p-1}^1, b_{p-2}^1, b_{p-3}^1, \ldots, b_{i-1}^1, b_i^1, b_{i-2}^1, b_{i-3}^1, \ldots, b_1^1\}$. Since $D_p$ is a convex polytope network, it is illustrated in Table 2.

(ii) $R_i(a_1^1 b_1^1) = V(D_p) - \{a_{i+1}^1, a_{i+2}^1, a_{i+3}^1, \ldots, a_{p/2+1}^1, a_{p/2+2}^1, a_{p/2+3}^1, \ldots, a_{p-1}^1, b_{p-1}^1, b_{p-2}^1, b_{p-3}^1, \ldots, b_{i-1}^1, b_i^1, b_{i-2}^1, b_{i-3}^1, \ldots, b_1^1\}$. The cardinalities of each LRN set of $D_p$ is illustrated in Table 2.

It can be observed with the help of Table 2 that $|R_i(a^1 b_1)| \leq |R_i(e)|$. Since $|\cup_{i=1}^{p} R_i(a_1^i b_1^i)| = 4p$ therefore $|R_i(e)| \cup \cup_{i=1}^{p/2} R_i(a_1^i b_1^i) \geq 4p \forall v \in E(D_p)$. 

**Theorem 4.** Let $D_p$ be a convex polytope network, with $p \geq 4$ and $p \equiv 0 \pmod{2}$. Then

$$\frac{p}{p-1} \leq \text{LFLN}(D_p) \leq \frac{8p}{5p+7}$$

**Proof.** In order to prove the result, we split into two cases:

Case 3. For $p = 4$, we have following LRN sets;

### Table 2: Cardinality of each LRN set.

| LRN set | Comparison |
|---------|------------|
| $R_i(a_1^1 b_1^1)$ | $4p - 4 > |R_i(a_1^1 b_1^1)|$ |
| $R_i(a_1^1 b_1^1)$ | $4p - 4 > |R_i(a_1^1 b_1^1)|$ |
| $R_i(a_1^1 b_1^1)$ | $4p - 4 > |R_i(a_1^1 b_1^1)|$ |
Since, \(|R_i(a_i^1b_i)| = 11\) and \(\left| R_i(a_i^1b_i) \right| \leq \left| R_i(e) \right| \forall e \in (D_4)\), where \(1 \leq i \leq 4\). Furthermore, \(\cup_{i=1}^4 |R_i(a_i^1b_i)| = (D_4)\) and \(\left| R_e(e) \cap \cup_{i=1}^4 |R_i(a_i^1b_i)| \right| \geq \left| R_i(a_i^1b_i) \right| \forall e \in (D_4)\). Hence, we define an upper LRF \(h: (D_4) \rightarrow [0, 1]\) as \(h(v) = 1/10v \in V(D_4)\). In order to show that \(h\) is minimal upper LRF consider another function \(h': (D_4) \rightarrow [0, 1]\) as \(h'(v) < 1/11v \in V(D_4)\) therefore \(h'(R_i(e)) < 1\) and \(|h'| < |h|\) which shows that \(h'\) is not LRF of \(D_4\) therefore \(LFLN(D_4) \leq \sum_{i=1}^{16} 1/11 = 16/11\). Likewise for \(1 \leq i \leq 4\) cardinality of LRN set \(R_i(b_i^1b_i^1)\) is 13 which is greater then the cardinalities of all other LRN sets. Therefore there exist a maximal lower LRF \(g: (D_4) \rightarrow [0, 1]\) and which is defined as \(g(v) = 1/13v \in V(D_4)\) hence \(LFLN(D_4) \geq \sum_{i=1}^{16} 1/13 = 16/13\). Consequently, \(16/11 \leq LFLN(D_4) \leq 16/13\).

Case 4. For \(p \geq 6, 1 \leq i \leq p\) by Lemma 2, \(\left| R_i(a_i^1b_i) \right| = 5p/2\) and \(\left| R_i(a_i^1b_i) \right| \geq \left| R_i(e) \right| \forall e \in (D_p)\). Furthermore, \(\left| R_i(e) \cap \cup_{i=1}^p \left| R_i(a_i^1b_i) \right| \right| \geq \left| R_i(a_i^1b_i) \right| \forall e \in (D_p)\). Hence there exist an upper LRF \(h: (D_p) \rightarrow [0, 1]\) and is defined as \(h(v) = 2/5pv \in V(D_p)\). In order to show that \(h\) is minimal upper LRF consider another function \(h': (D_p) \rightarrow [0, 1]\) as \(h'(v) < 2/5pv \in V(D_p)\) therefore \(h'(R_i(e)) < 1\) and \(|h'| < |h|\) which shows that \(h'\) is not LRF of \(D_p\) hence \(LFLN(D_p) \leq \sum_{i=1}^{2p} 2/5p = 8/5\). Likewise the cardinality of LRN set \(R_i(b_i^1b_i^1)\) is \(4p - 4\) which is greater or equal to the cardinalities of all other LRN sets of \(D_p\). Hence, we define a maximal lower LRF \(g: (D_p) \rightarrow [0, 1]\) as \(g(v) = 1/4p - 4pv \in V(D_p)\), therefore \(LFLN(D_p) \geq \sum_{i=1}^{4p} 1/4p - 4 = p/p - 1\). Consequently,
\[
\frac{p}{p-1} \leq LFLN(\mathbb{D}_p) \leq \frac{8}{5}
\]  

(15)

3.2. LFLN of Convex Polytope \( E_p \). In this particular subsection, we have computed the LRN sets and LFLN of convex polytope network \( E_p \). The \( V(E_p) = V(\mathbb{D}_p) \) and \( E(E_p) = E(\mathbb{D}_p) \cup \{b_i| \text{middle}\} \). The order and size of \( E_p \) is 4p and 7p respectively. For more details see Figure 2.

**Lemma 3.** Let \( E_p \) be a convex polytope network, with \( p \geq 3 \) and \( p \equiv 1 \pmod{2} \). Then,

(i) \( |R_i(a_i^1b_i)| = |R_i(b_i, a_i^1)| = 2p + 2 \) and \( p \leq 1 |R_i(a_i^1b_i)| = V(E_p) \).

(ii) \( |R_i(a_i^1b_i)| \leq |R_i(e)| \) and \( |R_i(e) \cap \bigcup_{i=1}^{p-1} R_i(a_i^1b_i)| \geq |R_i(a_i^1b_i)| \forall e \in E(E_p) \).

**Proof.** Consider \( a_i \) inner, \( a_i^1, b_i \) middle and \( b_i^1 \) are outer vertices of \( E_p \), where \( 1 \leq i \leq p \) and \( p + 1 \equiv 1 \pmod{2} \).

(i) \( R_i(a_i^1b_i) = V(E_p) - \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{p-1}, a_1, a_2, \ldots, a_p\} \)

\( \bigcup_{i=1}^{p-1} R_i(a_i^1b_i) = V(E_p) - \{a_i, a_{i+1,2}, a_{i+1,3}, \ldots, a_{p-1,2}, a_1, a_2, \ldots, a_p\} \)

and \( R_i(b_i, a_i^1) = V(E_p) - \{a_i, a_{i+1,2}, a_{i+1,3}, \ldots, a_{p-1,2}, a_1, a_2, \ldots, a_p\} \)

Note that \( \bigcup_{i=1}^{p-1} R_i(a_i^1b_i) = 4p \) and \( |R_i(a_i^1b_i)| = 2p + 2 \).

(ii) \( R_i(a_i, a_i^1) = V(E_p) - \{a_{i+1,2}, a_{i+1,3}, \ldots, a_{p-1,2}, a_1, a_2, \ldots, a_p\} \)

\( R_i(b_i, a_i^1) = V(E_p) - \{a_i, a_{i+1,2}, a_{i+1,3}, \ldots, a_{p-1,2}, a_1, a_2, \ldots, a_p\} \)

\( R_i(b_i, a_i^1) = V(E_p) - \{a_i, a_{i+1,2}, a_{i+1,3}, \ldots, a_{p-1,2}, a_1, a_2, \ldots, a_p\} \)

Now, we illustrate the cardinalities of the LRN sets in Table 3 and also compare them.

It can be observed with the help of Table 3 that \( |R_i(a_i^1b_i)| \leq |R_i(e)| \).

Since \( \bigcup_{i=1}^{p-1} R_i(a_i^1b_i) = 4p \) therefore \( |R_i(e) \cap \bigcup_{i=1}^{p-1} R_i(a_i^1b_i)| \geq |R_i(a_i^1b_i)| \forall e \in E(E_p) \). □

**Theorem 5.** Let \( E_p \) be a convex polytope network. Then

\[ 6 \leq LFLN(E_p) \leq \frac{3}{2} \]

(16)

**Proof.** The LRN for convex polytope \( E_p \) are

\( R_i(1) = R_i(a_1a_2) = V(E_p) - \{a_3, a_3^1, b_3, b_3^1\} \)

\( R_i(2) = R_i(a_2a_3) = V(E_p) - \{a_3, b_2, b_2^1\} \)

\( R_i(3) = R_i(a_3a_4) = V(E_p) - \{a_2, b_3, b_3^1\} \)

\( R_i(4) = R_i(b_1b_2) = V(E_p) - \{a_2, b_3, a_2^1\} \)

\( R_i(5) = R_i(b_2b_3) = V(E_p) - \{a_3, b_2, a_3^1\} \)

\( R_i(6) = R_i(b_3b_4) = V(E_p) - \{a_2, b_1, a_2^1\} \)

\( R_i(7) = R_i(b_1^1b_2^1) = V(E_p) - \{b_1^1, b_3^1, a_2, a_3\} \)

\( R_i(8) = R_i(b_2^1b_3^1) = V(E_p) - \{b_2^1, b_1, a_2, a_3\} \)

\( R_i(9) = R_i(b_1b_2) = V(E_p) - \{b_1, b_2, a_1\} \)

\( R_i(10) = R_i(a_1b_1) = V(E_p) - \{a_3, a_2, b_1, b_1^1\} \)

\( R_i(11) = R_i(a_2b_2) = V(E_p) - \{a_3, a_1, b_1, b_1^1\} \)

\( R_i(12) = R_i(a_3b_3) = V(E_p) - \{a_1, a_2, b_1, b_1^1\} \)

\( R_i(13) = R_i(a_1b_1) = V(E_p) - \{a_2, b_1, b_1^1\} \)

\( R_i(14) = R_i(a_2b_2) = V(E_p) - \{a_2, b_2, a_1\} \)

\( R_i(15) = R_i(a_3b_3) = V(E_p) - \{a_3, b_1, a_1\} \)

\( R_i(16) = R_i(a_1a_1) = V(E_p) - \{a_2, a_1\} \)

\( R_i(17) = R_i(a_2a_2) = V(E_p) - \{a_3, a_1\} \)

\( R_i(18) = R_i(a_3a_3) = V(E_p) - \{a_1, a_2\} \)

\( R_i(19) = R_i(a_1b_1) = V(E_p) \)

\( R_i(20) = R_i(b_1b_2) = V(E_p) \)

\( R_i(21) = R_i(b_1^1b_2^1) = V(E_p) \)

For \( 1 \leq i \leq 3 \) the cardinality of each LRN set \( R_i(a_i, a_i^1) \) is 8 which is less than the other LRN sets of \( E_p \). Furthermore, \( \bigcup_{i=1}^{p-1} R_i(a_i, a_i^1) = V(E_p) \) and \( |R_i(e) \cap \bigcup_{i=1}^{p-1} R_i(a_i, a_i^1)| \geq |R_i(a_i, a_i^1)| \forall e \in E(E_p) \).

Hence there exist an upper LRF \( h : V(E_p) \rightarrow [0, 1] \) that is defined as \( h(v) = 1/8Vv \in V(E_p) \). In order to show that \( h \) is minimal upper LRF consider another function \( h' : V(E_p) \rightarrow [0, 1] \) as \( h'(v) < 1/10Vv \in V(E_p) \) therefore \( h'(R_i(e)) < 1 \) and \( |h'| < |h| \) which shows that \( h' \) is not LRF of \( E_p \) hence \( LFLN(E_p) = \sum_{i=1}^{p-1} 1/8 = 3/2 \). Likewise for \( 1 \leq i \leq 3 \) cardinality of LRN set \( R_i(b_i^1b_i^1) \) is 12 which is greater than the cardinalities of all other LRN sets. Hence there exist a maximal lower LRF \( g : \mathbb{D}_p \rightarrow [0, 1] \) that defined as \( g(v) = 1/12Vv \in V(E_p) \), therefore \( LFLN(E_p) > \sum_{i=1}^{p-1} 1/12 = 1 \). Consequently,
Theorem 6. Let $E_p$ be a convex polytope network, with $p \geq 5$ and $p \equiv 1 \pmod{2}$. Then

$$1 < LFLN(E_p) \leq \frac{3p}{2}$$

Case 5. For $p = 5$, we have the following LRN sets;

Table 3: Cardinality of each LRN set.

| RLN set | Comparison |
|---------|------------|
| $R_i(a_{i+1})$ | $4p - 4 > |R_i(a_{i+1})|$ |
| $R_i(b_{i+1})$ | $4p - 4 > |R_i(b_{i+1})|$ |
| $R_i(a_i)$ | $4p - 4 > |R_i(a_i)|$ |
| $R_i(b_i)$ | $4p > |R_i(b_i)|$ |
| $R_i(b_{i+1}b_i)$ | $4p - 4 > |R_i(b_{i+1}b_i)|$ |

Proof. In order to prove the result, we split it into two cases.

$$1 < LFLN(E_3) \leq \frac{3}{2}$$

Figure 2: Convex polytope network $E_p$. 

Table 3: Cardinality of each LRN set.
\( R_i(1) = R_i(a_1^i b_1^i) = V(E_5) - \{a_2, a_3, a_4^i, b_4, a_5^i, b_5^i\}, \)

\( R_i(2) = R_i(a_1^i b_2^i) = V(E_5) - \{a_3, a_4, a_5, b_3, a_1^i, b_1^i\}, \)

\( R_i(3) = R_i(a_1^i b_3^i) = V(E_5) - \{a_4, a_5^i, a_1^i, b_1^i, a_2, b_2^i, b_1^i\}, \)

\( R_i(4) = R_i(a_1^i b_4^i) = V(E_5) - \{a_5, a_1, a_2^i, b_2, b_3, a_1^i, b_1^i\}, \)

\( R_i(5) = R_i(a_1^i b_5^i) = V(E_5) - \{a_1, a_2, a_3^i, b_3, a_4^i, b_4^i\}, \)

\( R_i(6) = R_i(a_2^i b_1^i) = V(E_5) - \{a_1, a_5, a_3^i, b_2, b_3^i, b_1^i\}, \)

\( R_i(7) = R_i(a_2^i b_2^i) = V(E_5) - \{a_2, a_1, a_4^i, a_5^i, b_3, b_4^i, b_1^i\}, \)

\( R_i(8) = R_i(a_2^i b_3^i) = V(E_5) - \{a_3, a_2, a_1^i, a_4, b_4, b_1^i, b_2^i\}, \)

\( R_i(9) = R_i(a_2^i b_4^i) = V(E_5) - \{a_4, a_3, a_5^i, a_1^i, a_2, b_5^i, b_1^i\}, \)

\( R_i(10) = R_i(a_2^i b_5^i) = V(E_5) - \{a_5, a_4, a_2, a_3^i, b_1, b_2^i, b_1^i\}, \)

\( R_i(11) = R_i(a_3^i a_4^i) = V(E_5) - \{a_4, a_2^i, b_1, b_1^i\}, \)

\( R_i(12) = R_i(a_4^i a_5^i) = V(E_5) - \{a_5, a_1^i, b_2, b_2^i\}, \)

\( R_i(13) = R_i(a_3 a_4^i) = V(E_5) - \{a_1, a_3^i, b_3, b_3^i\}, \)

\( R_i(14) = R_i(a_3 a_5^i) = V(E_5) - \{a_2, a_2^i, b_4, b_4^i\}, \)

\( R_i(15) = R_i(a_4 a_5^i) = V(E_5) - \{a_3, a_1^i, b_5, b_5^i\}, \)

\( R_i(16) = R_i(b_1^i b_2^i) = V(E_5) - \{a_2, a_2^i, b_4, b_4^i\}, \)

\( R_i(17) = R_i(b_2^i b_1^i) = V(E_5) - \{a_2, a_2^i, b_4, b_4^i\}, \)

\( R_i(18) = R_i(b_3^i b_1^i) = V(E_5) - \{a_4, a_1^i, b_1, b_1^i\}, \)

\( R_i(19) = R_i(b_2^i b_2^i) = V(E_5) - \{a_5, a_1^i, b_2, b_2^i\}, \)

\( R_i(20) = R_i(b_2^i b_2^i) = V(E_5) - \{a_1, a_3^i, a_1^i, b_3, b_3^i\}, \)

\( R_i(21) = R_i(a_1^i a_2^i) = V(E_5) - \{a_2, a_2^i, b_3, b_3^i\}, \)

\( R_i(22) = R_i(a_2^i a_2^i) = V(E_5) - \{a_3, a_4^i, b_5, b_5^i\}, \)

\( R_i(23) = R_i(a_3 a_2^i) = V(E_5) - \{a_4, a_3^i, b_1, b_1^i\}, \)

\( R_i(24) = R_i(a_4 a_2^i) = V(E_5) - \{a_3, a_4^i, b_2, b_2^i\}, \)

\( R_i(25) = R_i(a_3 a_4^i) = V(E_5) - \{a_4, a_1^i, b_3, b_3^i\}, \)

\( R_i(26) = R_i(b_1^i b_2^i) = V(E_5) - \{a_2, a_2^i, b_4, b_4^i\}, \)

\( R_i(27) = R_i(b_1^i b_3^i) = V(E_5) - \{a_3, a_5^i, b_2, b_2^i\}, \)

\( R_i(28) = R_i(b_2^i b_3^i) = V(E_5) - \{a_4, a_5^i, b_1, b_1^i\}, \)

\( R_i(29) = R_i(b_3^i b_1^i) = V(E_5) - \{a_5, a_3^i, b_2, b_2^i\}, \)

\( R_i(30) = R_i(b_3^i b_2^i) = V(E_5) - \{a_1, a_1^i, b_3, b_3^i\}, \)

\( R_i(31) = R_i(b_1^i b_1^i) = V(E_5), \)

\( R_i(32) = R_i(b_2^i b_2^i) = V(E_5), \)

\( R_i(33) = R_i(b_3^i b_3^i) = V(E_5), \)

\( R_i(34) = R_i(b_4^i b_4^i) = V(E_5), \)

\( R_i(35) = R_i(b_5^i b_5^i) = V(E_5). \)
Since, $|B_i(a_i^1b_i)| = |B_i(a_i^1b_i)| = 12$ and $|R_i(a_i^1b_i)| \leq |R_i(b_i^1)e| \forall e \in E(E_p)$, where $1 \leq i \leq 5$. Furthermore, $\bigcup_{i=1}^{5} (R_i(a_i^1b_i)) = V(E_p)$ and $|R_i(e) \cap \bigcup_{i=1}^{5} (R_i(a_i^1b_i))| \geq |R_i(b_i^1)e| \forall e \in E(E_p)$. Hence, we define an upper LRF $h: V(E_p) \rightarrow [0, 1]$ as $h(v) = 1/12\nu v \in V(E_p)$. In order to show that $h$ is minimal upper LRF consider another function $h': V(E_p) \rightarrow [0, 1]$ as $h'(v) < 1/12\nu v \in V(E_p)$ therefore $h' (R_i(e)) < 1$ and $|h' < h|$ which shows that $h'$ is not LRF of $D_p$; therefore $\text{LFLN} \ (E_p) \leq \sum_{i=1}^{5} |h| = 5/3$. Likewise for $1 \leq i \leq 5$ cardinality of LRF set $R_i(b_i^1e)$ is $20$ which is greater then the cardinalities of all other LRF sets. Hence there exist a maximal lower LRF $g: V(E_p) \rightarrow [0, 1]$ and it is defined by $g(v) = 1/20\nu v \in V(E_p)$, therefore $\text{LFLN} (E_p) > \sum_{i=1}^{20} 1/20 = 1$. Consequently,

$$1 < \text{LFLN} (E_p) \leq \frac{5}{3}. \quad (21)$$

**Case 6.** For $p \geq 5, 1 \leq i \leq p$ by Lemma 3, $|R_i(a_i^1b_i)| = 2p + 2$ and $|R_i(b_i^1e)| \geq |R_i(e)| \forall e \in E(E_p)$. Furthermore, $|R_i(e) \cap \bigcup_{i=1}^{p} |R_i(a_i^1b_i)| = 1$. Therefore, we define an upper LRF $h: (E_p) \rightarrow [0, 1]$ as $h(v) = 1/2p + 4\nu v \in V(E_p)$. In order to show that $h$ is minimal upper LRF consider another function $h': V(E_p) \rightarrow [0, 1]$ as $h'(v) < 1/2p + 2\nu v \in V(E_p)$ therefore $h' (R_i(e)) < 1$ and $|h' < h|$ which shows that $h'$ is not LRF of $D_p$; hence $\text{LFLN} (E_p) \leq \sum_{i=1}^{p} 1/2p = 2p/p + 1$. Likewise the cardinality of LRF set $R_i(b_i^1e)$ is $4p$ which is greater then the cardinalities of all other LRF sets of $E_p$. Hence there exist a maximal LRF $g: (E_p) \rightarrow [0, 1]$ and it is defined as $g(v) = 1/4p\nu v \in V(E_p)$ therefore $\text{LFLN} (E_p) > \sum_{i=1}^{4p} 1/4p = 1$. Consequently,

$$1 < \text{LFLN} (E_p) \leq \frac{2p}{p + 1}. \quad (22)$$

**Lemma 4.** Let $E_p$ be a convex polytope network, with $p \geq 4$ and $p \equiv 0 (mod 2)$. Then,

(i) $|R_i(a_i^1b_i)| = |R_i(b_i^1e)| = 2p + 1$ and $\bigcup_{i=1}^{p} R_i(a_i^1b_i) = V(E_p)$.

(ii) $|R_i(a_i^1b_i)| \leq |R_i(e)| \cap \bigcup_{i=1}^{p} R_i(a_i^1b_i) \geq |R_i(a_i^1b_i)| \forall e \in E(E_p)$.

**Table 4:** Cardinality of each LRF set of $E_p$.

| LRF set | Comparison |
|---------|------------|
| $R_i(a_i^1b_i)$ | $4p - 4 > |R_i(a_i^1b_i)|$ |
| $R_i(b_i^1e)$ | $4p - 4 > |R_i(a_i^1b_i)|$ |
| $R_i(a_i^1b_i)$ | $3p > |R_i(a_i^1b_i)|$ |
| $R_i(a_i^1b_i)$ | $4p > |R_i(b_i^1e)|$ |
| $R_i(b_i^1e)$ | $4p - 2 > |R_i(a_i^1b_i)|$ |

**Proof.** Consider $a_i$ inner, $a_i^1$, $b_i^1$ middle and $b_i^1$ are outer vertices of $E_p$ respectively, where $1 \leq i \leq p$ and $p + 1 \equiv 1 (mod p)$. By Lemma 3, $|R_i(a_i^1b_i)| = 2p + 2$ and $|R_i(e)| \geq |R_i(a_i^1b_i)| \forall e \in E(E_p)$. Furthermore, $|R_i(e) \cap \bigcup_{i=1}^{p} |R_i(a_i^1b_i)| = 1$. Therefore, we define an upper LRF $h: (E_p) \rightarrow [0, 1]$ as $h(v) = 1/2p + 4\nu v \in V(E_p)$. In order to show that $h$ is minimal upper LRF consider another function $h': V(E_p) \rightarrow [0, 1]$ as $h'(v) < 1/2p + 2\nu v \in V(E_p)$ therefore $h' (R_i(e)) < 1$ and $|h' < h|$ which shows that $h'$ is not LRF of $D_p$; hence $\text{LFLN} (E_p) \leq \sum_{i=1}^{p} 1/2p = 2p/p + 1$. Likewise the cardinality of LRF set $R_i(b_i^1e)$ is $4p$ which is greater then the cardinalities of all other LRF sets of $E_p$. Hence there exist a maximal LRF $g: (E_p) \rightarrow [0, 1]$ and it is defined as $g(v) = 1/4p\nu v \in V(E_p)$ therefore $\text{LFLN} (E_p) > \sum_{i=1}^{4p} 1/4p = 1$. Consequently,

$$1 < \text{LFLN} (E_p) \leq \frac{2p}{p + 1}. \quad (22)$$

**Theorem 7.** Let $E_p$ be a convex polytope network, where $p \geq 4$ and $p \equiv 0 (mod 2)$. Then

$$1 < \text{LFLN} (E_p) \leq \frac{4p}{2p + 1}. \quad (23)$$

**Proof.** In order to prove the result, we split it into two cases.

**Case 7.** For $p = 4$, we have following LRF sets;
\(R_i(1) = R_i(a_i^1 b_i)\) = \(V(E_i) - \{a_2, a_3, a_4, b_2, b_3, b_4, b_5, b_6\}\),
\(R_i(2) = R_i(a_i^2 b_i)\) = \(V(E_i) - \{a_3, a_4, a_5, b_1, b_2, b_3\}\),
\(R_i(3) = R_i(a_i^3 b_i)\) = \(V(E_i) - \{a_4, a_1, a_2, b_1, b_2, b_3\}\),
\(R_i(4) = R_i(a_i^4 b_i)\) = \(V(E_i) - \{a_1, a_2, a_3, b_2, b_3, b_2\}\),
\(R_i(5) = R_i(a_i^5 b_i)\) = \(V(E_i) - \{a_1, a_4, a_1, b_2, b_3, b_4\}\),
\(R_i(6) = R_i(a_i^6 b_i)\) = \(V(E_i) - \{a_2, a_1, a_2, b_3, b_3, b_4\}\),
\(R_i(7) = R_i(a_i^7 b_i)\) = \(V(E_i) - \{a_3, a_2, a_3, b_1, b_2, b_3\}\),
\(R_i(8) = R_i(a_i^8 b_i)\) = \(V(E_i) - \{a_4, a_3, a_1, b_1, b_2, b_2\}\),
\(R_i(9) = R_i(a_i a_i)\) = \(V(E_i) - \{b_1, b_1, b_1, b_1\}\),
\(R_i(10) = R_i(a_i a_i)\) = \(V(E_i) - \{b_1, b_1, b_2, b_2\}\),
\(R_i(11) = R_i(a_i a_i)\) = \(V(E_i) - \{b_1, b_1, b_3, b_3\}\),
\(R_i(12) = R_i(a_i a_i)\) = \(V(E_i) - \{b_2, b_2, b_3, b_2\}\),
\(R_i(13) = R_i(a_i a_i)\) = \(V(E_i) - \{a_2, a_1, a_3, b_1\}\),
\(R_i(14) = R_i(a_i a_i)\) = \(V(E_i) - \{a_3, a_1, a_2, b_1\}\),
\(R_i(15) = R_i(a_i a_i)\) = \(V(E_i) - \{a_4, a_2, a_3, b_1\}\),
\(R_i(16) = R_i(a_i a_i)\) = \(V(E_i) - \{a_1, a_3, a_2, b_1\}\),
\(R_i(17) = R_i(b_i b_i)\) = \(V(E_i) - \{a_2, a_4, a_1\}\),
\(R_i(18) = R_i(b_i b_i)\) = \(V(E_i) - \{a_3, a_1, a_2, a_1\}\),
\(R_i(19) = R_i(b_i b_i)\) = \(V(E_i) - \{a_4, a_2, a_3, a_1\}\),
\(R_i(20) = R_i(b_i b_i)\) = \(V(E_i) - \{a_1, a_3, a_1, a_1\}\),
\(R_i(21) = R_i(b_i b_i)\) = \(V(E_i) - \{a_2, a_4, a_1, a_1\}\),
\(R_i(22) = R_i(b_i b_i)\) = \(V(E_i) - \{a_3, a_1, a_2, a_1\}\),
\(R_i(23) = R_i(b_i b_i)\) = \(V(E_i) - \{a_4, a_2, a_3, a_1\}\),
\(R_i(24) = R_i(b_i b_i)\) = \(V(E_i) - \{a_1, a_3, a_1, a_1\}\),
\(R_i(25) = R_i(b_i b_i)\) = \(V(E_i)\),
\(R_i(26) = R_i(b_i b_i)\) = \(V(E_i)\),
\(R_i(27) = R_i(b_i b_i)\) = \(V(E_i)\),
\(R_i(28) = R_i(b_i b_i)\) = \(V(E_i)\).

Since, \(|R_i(a_i^1 b_i)| = |R_i(a_i^1 b_i)| = 9\) and \(|R_i(a_i^1 b_i)| \leq |R_i(e)|\), therefore \(\Sigma_{i=1}^{14} |R_i(a_i^1 b_i)| \geq |R_i(e)|\) \(\forall e \in E(E_4)\). Hence, we define an upper LRF \(h: V(E_i) \rightarrow [0, 1]\) as \(h(e) = 1/9\), \(e \in V(E_4)\). In order to show that \(h\) is minimal upper LRF consider another function \(h': V(E_i) \rightarrow [0, 1]\) as \(h(e) = 1/9 \forall \in V(E_4)\) therefore \(h'(E_i(e)) < 1\) and \(|h'| < |h|\) which shows that \(h\) is not LRF of \(E_4\) therefore \(LFLN(E_4) \leq \frac{1}{16} = 16/9\). Likewise for \(1 \leq i \leq 3\) cardinality of LRN set \(R_i(b_i^2)\) is 20 which is greater then the cardinalities of all other LRN sets.

Hence there exist a maximal lower LRF \(g: V(E_4) \rightarrow [0, 1]\) is defined by \(g(e) = 1/20 \forall e \in V(E_4)\), therefore \(LFLN(E_4) \geq \frac{1}{20} = 1/20\). Consequently,

\[1 < LFLN(E_4) \leq \frac{16}{9}\]  \(24\)

Case 8. For \(p \geq 4, 1 \leq i \leq p\) by Lemma 4, \(|R_i(a_i^1 b_i)| = 2p + 1\) and \(|R_i(a_i^1 b_i)| \geq |R_i(e)|\). Hence, \(\Sigma_{i=1}^{10} |R_i(a_i^1 b_i)| \geq |R_i(e)|\) \(\forall e \in E(E_4)\). Furthermore, \(|R_i(e)| \cap U_{i=1}^{10} |R_i(a_i^1 b_i)| \geq |R_i(a_i^1 b_i)|\). Hence, we define an upper LRF \(h_i: (E_i) \rightarrow [0, 1]\) as \(h_i(e) = 1/2p + 1\). In order
to show that $h$ is minimal upper LRF consider another function $h': V(D_p) \rightarrow [0,1]$ as $h'(v) < 1/2p + 1\forall v \in V(E_p)$ this implies $h'(R(E_p)) < 1$ and $|h'| < |h|$ which shows that $h'$ is not LRF of $E_p$ therefore $LFLN(E_p) \leq \sum_{i=1}^{4p} 1/2p + 1 = 4p/2p + 1$. Likewise the cardinality of LRN set $R(b,b^T)$ is $4p$ which is greater then the cardinalities of all the other LRN sets. Therefore there exist a maximal lower LRF $g$: $E_p \rightarrow [0,1]$ and it is defined as $g(v) = 1/4p\forall v \in V(E_p)$ therefore $LFLN(E_p) > \sum_{i=1}^{4p} 1/4p = 1$. Consequently,

$$1 < LFLN(E_p) \leq \frac{4p}{2p + 1}.$$  \hspace{1cm} (26)

4. Conclusion

In this dissertation, we studied the LFLN of different families of convex polytope networks $(D_p, E_p)$ and after establishing the bounds of LFLN of both convex polytope networks, we conclude that both of them possess boundedness when $p \rightarrow \infty$.

Exact value of LFLN in one case is,

(i) $LFLN(D_p) = 5/4$.

(ii) Boundedness of LFLN of $D_p$ and $E_p$ illustrated in Table 5.

Now, we close our discussion with the following open problem, characterize all the classes of convex polytopes networks whose attain exact value of local fractional locating number.

Data Availability

All the data are included within this paper. However, the reader may contact the corresponding author for more details of the data.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding this article.

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References

[1] P. J. Slater, “Leaves of trees,” Congressus Numerantium, vol. 14, no. 1, pp. 549–559, 1975.
[2] F. Harary and R. Melter, “On the metric dimension of a graph,” Ars Combinatoria, vol. 2, pp. 19–195, 1976.
[3] S. Khuller, B. Raghavachari, and A. Rosenfeld, “Landmarks in graphs,” Discrete Applied Mathematics, vol. 70, no. 3, pp. 217–229, 1996.
[4] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, “Resolvability in graphs and the metric dimension of a graph,” Discrete Applied Mathematics, vol. 105, no. 1-3, pp. 99–113, 2000.
[5] A. Sebő and E. Tannier, “On metric generators of graphs,” Mathematics of Operations Research, vol. 29, no. 2, pp. 383–393, 2004.
[6] R. A. Melter and I. Tomescu, “Metric bases in digital geometry,” Computer Vision, Graphics, and Image Processing, vol. 25, no. 1, pp. 113–121, 1984.
[7] Z. Shao, S. M. Sheikholeslami, P. Wu, and J.-B. Liu, “The metric dimension of some generalized Petersen graphs,” Discrete Dynamics in Nature and Society, vol. 2018, Article ID 4531958, 10 pages, 2018.
[8] J. Cáceres, C. Hernando, M. Mora et al., “On the metric dimension of cartesian products of graphs,” SIAM Journal on Discrete Mathematics, vol. 21, no. 2, pp. 423–441, 2007.
[9] X. Zuo, A. Ali, G. Ali, M. K. Siddiqui, M. T. Rahim, and A. Asare-Tuah, “On Constant Metric Dimension of Some Generalized Convex Polytopes,” Jurnal Matematika, 2021 pages, 2021.
[10] M. Imran, A. Q. Baig, and A. Ahmed, “Families of plane graphs with constant metric dimension,” Utilitas Mathematica, vol. 88, pp. 43–57, 2012.
[11] M. Ali, G. Ali, M. Imran, and A. Q. Baig, “On the Metric Dimension of Mobius Ladders,” MK Shafiq - Ars Comb, vol. 105, pp. 403–410, 2012.
[12] J. B. Liu, M. F. Nadeem, H. M. A. Siddiqui, and W. Nazir, “Computing metric dimension of Certain families of Toeplitz graphs,” IEEE Access, vol. 7, Article ID 126734, 2019.
[13] P. S. Buczowski, G. Chartrand, C. Poisson, and P. Zhang, “On k-dimensional graphs and their bases,” Periodica Mathematica Hungarica, vol. 46, no. 1, pp. 9–15, 2003.
[14] C. Hernando, M. Mora, I. M. Pelayo, C. Seara, J. Cáceres, and M. L. Puertas, “On the metric dimension of some families of graphs,” Electronic Notes in Discrete Mathematics, vol. 22, pp. 129–133, 2005.
[15] I. G. Yero, D. Kuziak, and J. A. Rodríguez-Velázquez, “On the metric dimension of corona product graphs,” Computers & Mathematics with Applications, vol. 61, no. 9, pp. 2793–2798, May 2011.
[16] I. González Yero, M. Jakovac, D. Kuziak, and A. Taranenko, “The partition dimension of strong product graphs and

Table 5: Boundedness of convex polytopes $(D_p, E_p)$ via LFLN.

| Network | LFLN | Lower bound of LFLN when $p \rightarrow \infty$ | Upper bound of LFLN when $p \rightarrow \infty$ | Comment |
|---------|------|-----------------------------------------------|-----------------------------------------------|---------|
| $D_p$   | $p/p - 1 \leq LFLN(D_p) \leq 8p/5p + 7$ | 1                                              | 8/5                                              | Bounded |
| $D_p$   | $p/p - 1 \leq LFLN(D_p) \leq 8/5$     | 1                                              | 8/5                                              | Bounded |
| $E_p$   | $1 < LFLN(E_p) \leq 2p/2p + 1$        | 1                                              | 2                                                | Bounded |
| $E_p$   | $1 < LFLN(E_p) \leq 4p/2p + 1$        | 1                                              | 2                                                | Bounded |
Cartesian product graphs,” *Discrete Mathematics*, vol. 331, pp. 43–52, Sep. 2014.

[17] Z. B. Zheng, A. Ahmad, Z. Hussain et al., “fault-tolerant metric dimension of generalized wheels and convex polytopes,” *Mathematical Problems in Engineering*, pp. 1–8, 2020.

[18] A. Shabbir and T. Zamfirescu, “Fault-tolerant designs in triangular lattice networks,” *Applicable Analysis and Discrete Mathematics*, vol. 10, no. 2, pp. 447–456, 2016.

[19] M. S. Bataineh, N. Saddiqi, and Z. Raza, “Edge metric dimension of K multiwheel graph” Rocky Mountain,” *Journal of Mathematics*, vol. 504 pages, 2020.

[20] Z. Raza and M. S. Bataineh, “xX_he Comparative analysis of metric and Edge metric dimension of some subdivisions of the wheel graph,“ *Asian-European Journal of Mathematics*, vol.14, no. 04, Article ID 2150062, 2021.

[21] G. Chartrand, E. Salehi, and P. Zhang, “xX_he partition dimension of a graph,” *Aequationes Mathematicae*, vol. 59, no. 1, pp. 45–54, 2000.

[22] C. Hernando, M. Mora, P. J. Slater, and D. R. Wood, “Fault-tolerant metric dimension of graphs,” in *Proceedings of the International Conference on Convexity in Discrete Structures, Ramanujan Mathematical Society*, Tiruchirappalli, India, June 2008.

[23] A. Kelenc, N. Tratnik, and I. G. Yero, “Uniquely identifying the edges of a graph: the edge metric dimension,” *Discrete Applied Mathematics*, vol. 251, pp. 204–220, 2018.

[24] A. K. Alkhaldi, M. I. Utoyo, and L. Susilowati, “On the local fractional metric dimension of corona product graphs,” *IOP Conference Series: Earth and Environmental Science*, vol. 243, Article ID 012043, 2019.