Estimation of the minimum beam length for the static, dynamic, and stability problems

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Abstract. The calculation results of extended elements of structures according to the theory of beam will be correct if the condition for their length is met, which, as is known, should be much more transverse dimensions. The absence of an exact relationship between the transverse dimensions and the length of the beam is caused by the characteristics of the behavior under a load of various types and shapes of cross-sections: open, closed, thin-walled, etc. In this paper, the ratio of the length of the beam to its transverse dimensions is considered, which provides correct results during its static and dynamic loading, as well as during the loss of stability. To do this, two forms of thin-walled cross-sections are selected: circular and rectangular. The problem solution was carried out both analytically and by the finite element method. The results showed that the most stringent requirements are introduced by the theory of stability, according to which the length should be 15 or more times the largest transverse dimension.

1. Introduction

Beams are the most common structural elements and their calculation methods are the most developed. Beams are used both alone and create light and strong frames of spatial large structures. Beam calculation methods were probably the first in mechanics to be brought to the simplest linear ratios to allow manual strength calculations.

It would seem that today the calculation of beams no longer contains any uncertainties, but one controversial issue remains. It concerns the beam definition itself and its effect on the accuracy of the results. According to the known literature, a beam is a geometric object, one of the dimensions of which (length) is much larger than the other two. The exact ratio of length to transverse dimensions is not given in any scientific source for many reasons. Firstly, this ratio is highly dependent on the type and shape of the cross-section: open, closed, thin-walled, solid, etc. Further, this dimension ratio also depends on the method of restraining and loading the beam, the characteristics of the distribution of stress fields, которые вносят свои погрешности, которые будут сочетаться между собой по сложной зависимости. Finally, the accuracy of the beam calculation is based on several assumptions that also introduce their errors, which will be combined by a complex dependence.

In this paper, an attempt was made to determine the necessary ratio of the length of the beam to its maximum transverse dimension, which provides correct results at static and dynamic loading, as well as during a loss of stability. To do this, in this work we will consider two common forms of thin-walled cross-section (figure 1): rectangular and circular.
The solution of the problem will be carried out using both known information from the literature and new data obtained as a result of analytical transformations and numerical calculations.

2. Beam length at static load
At static loading, the recommended ratio of beam length to its maximum transverse dimension can be found in many sources, for example in [1-5]. Length constraints depend on the type of cross-section, whether it is solid or thin-walled. For beams of a solid or thick-walled section, the ratio of their length to the maximum transverse dimension shall be:

\[
\frac{l}{B_{\text{max}}} > 5,
\]  

(1)

where \( l \) is the beam length; \( B_{\text{max}} \) is the maximum cross-sectional dimension (\( B \) in figure 1, a and \( D \) in figure 1, b).

Condition (1) requires that the length of the beam ensures the fulfillment of the Saint-Venant principle, as well as the Bernoulli’s plane section hypothesis. The latter requirement is most essential and is fulfilled by minimizing possible cross-sectional shifts, that is, the predominance of normal stresses over shear stresses. This is always valid, for example, when the long beams are transversely bent, at which the bending stresses will exceed the tangent stresses. If the beam is only subjected to stretching or compression, then condition (1) may well be weakened.

Thin-walled beams, especially open profiles, have a more complex stress distribution and require a more stringent condition for the applicability of beam theory [6-8]. Note here that wall thickness condition is added to length condition:

\[
\frac{l}{B_{\text{max}}} > 10 \quad \text{and} \quad \frac{B_{\text{max}}}{t} > 10,
\]  

(2)

where \( t \) is the wall thickness (figure 1).

More stringent conditions (2) are caused by possible warping of the thin-walled cross-section and its effect on the calculation results.

3. Beam length at dynamic load
No information was found in the known literature on the applicability of beam theory in dynamic loading, it is possible that conditions (1) and (2) are preserved [9-10]. To check the length limits in the beam theory at dynamic load, numerical calculations were made using the finite element method [11,12] for beams with a circular and rectangular cross-section (figure 1). The finite element beam model was created in the Ansys software and consisted of approximately 20,000 finite elements of the type Shell128. A straight beam with two support types at the edges was studied: hinge and fixed. The first natural frequency of beam vibration was estimated by varying its length and wall thickness. The results
were compared with reference values calculated from the beam vibration theory [10]. The difference in analytical and numerical results determined the error introduced by the length of the beam. The results of calculations in the form of plots are shown in figure 2.

![Figure 2](image)

**Figure 2.** Influence of beam length on error of calculation of the first natural frequency of vibration.

Plots show that for short beams, the calculation error is highly dependent on the wall thickness $t$, but even with a large wall thickness, condition (1) is incorrect. Condition (2) provides an error of 10-15%, which is not always permissible, although quite correct for beam theory.

4. **Beam length in case of stability loss**

The most often flexural stability of beams calculates by a formula Euler [13-23] which, as we know, is fair provided that the arising stresses are less than the limit of proportionality $\sigma_{\text{prop}}$ and flexibility of a beam $\lambda$ is more limit value $\lambda_{\text{min}}$:

$$\lambda \geq \lambda_{\text{min}} = \pi \sqrt{\frac{E}{\sigma_{\text{prop}}}},$$

where $E$ is the elastic modulus.

The flexibility of a beam is defined through its geometry and support conditions as:

$$\lambda = \mu \frac{l}{i_{\text{min}}},$$

where $i_{\text{min}}$ is the minimum radius of gyration of beam cross-section:

$$i_{\text{min}} = \frac{I_{\text{min}}}{S}. \quad (5)$$

Consider the two cross-section shapes (figure 1) and derive length conditions (1) for it.

4.1 **Rectangular cross-section**

Let us express the condition (3) through the ratio (1), for this, we output the flexibility of the waveguide with a rectangular cross-section through its dimensions, we get:

$$S = BH - bh, \quad I_{\text{min}} = I_x = \frac{BH^3}{12} - bh^3,$$
Let’s substitute the expression (6) in (5) and accept that \( H=2B \), we get the flexibility (4) in the form:

\[
\lambda = \mu \frac{l}{i_{\text{min}}} = \frac{\mu l}{0.373H}.
\] (8)

By substituting equation (8) into condition (3), we get:

\[
\frac{\mu l}{0.373H} \geq \pi \sqrt{\frac{E}{\sigma_{\text{prop}}}}.
\] (9)

At last, taking into account the fact that actually \( \mu=0.5,...,1 \), the condition (3) take a form:

\[
\frac{l}{B_{\text{max}}} \geq (1.17, ..., 2.34) \sqrt{\frac{E}{\sigma_{\text{prop}}}}.
\] (10)

The condition (10) is now defined only by the material properties that are discussed later. For other ratios of width \( B \) and height \( H \) of the cross-section, the coefficients in parentheses vary slightly.

4.2 Circular cross section

Perform the same transformations for the circular cross-sectional shape. The area and moment of inertia of the section in this case are equal:

\[
S = \frac{\pi}{4} (D^2 - d^2) \quad I_{\text{min}} = I_x = I_y = 0.05 \cdot (D^4 - d^4),
\] (11)

\[
i_{\text{min}} = i_x = i_y = \frac{1}{4} \sqrt{D^2 + d^2}.
\] (12)

Flexibility is:

\[
\lambda = \mu \frac{l}{i_{\text{min}}} = \frac{4\mu l}{\sqrt{D^2 + d^2}}.
\] (13)

Take for uniqueness \( D = d = d' \) (figure 1,b), then we get:

\[
i_{\text{min}} = i_x = i_y = 0.354 \cdot d'.
\] (14)

\[
\lambda = 2.83 \cdot \frac{\mu l}{d'}.
\] (15)

By combining the resulting expressions, we obtain the condition of the ratio of the length of the beam to its transverse dimension for a circular cross-section in the form:

\[
\frac{l}{d'} \geq (1.11, ..., 2.22) \sqrt{\frac{E}{\sigma_{\text{prop}}}}.
\] (16)

The coefficients in parentheses of expression (16) differ slightly from the coefficients in condition (10) and do not significantly affect the result. The properties of the material used have a much greater influence.
### 4.3 Calculation of limits for different materials

We get the conditions (10) and (16) in the form (1), for this, we substitute the values of mechanical parameters of various materials in them. Results are in table 1.

**Table 1. Ratio of length to maximum cross-section dimension.**

| Material     | E, GPa | \( \sigma_{\text{prop}}, \text{MPa} \) | Rectangular from | to | Circular from | to |
|--------------|--------|---------------------------------|-----------------|---|----------------|---|
| Aluminum     | 70     | 70                              | 37              | 74 | 35.1           | 70.2 |
| Copper       | 132    | 400                             | 21.3            | 42.5 | 20.2          | 40.3 |
| AD31         | 71     | 210                             | 21.5            | 43  | 20.4           | 40.8 |
| D16T         | 72     | 320                             | 17.5            | 35.1 | 16.6           | 33.3 |
| Steel 45     | 210    | 500                             | 24              | 48  | 22.7           | 45.5 |
| Steel 35HM   | 218    | 600                             | 22.3            | 44.6 | 21.2           | 42.3 |

The results of the calculations show that the application of the Euler formula is correct for most materials, if at least double the condition (2), that is, with

\[
\frac{l}{B_{\text{max}}} > 20.
\]  

(17)

Condition (17) is obtained in the case of thin-walled cross-section; in the case of solid cross-section this condition will be slightly weaker:

\[
\frac{l}{B_{\text{max}}} > 15.
\]  

(18)

Thus, the stability loss length condition is the strictest condition considered.

### 5. Discussion

The whole study shows that during dynamic loading the length of the beam should be noticeably longer than when static according to the conditions (17, 18). This beam length may in many cases be unacceptable due to a corresponding decrease in their first natural frequency due to a quadratic inverse relationship.

The dependence of the conditions (17, 18) on the properties of the material can somewhat weaken the coefficients. Since a limited number of crystal lattice structures exist for steels, it is possible to reduce the coefficient by increasing the proportionality limit, for example by alloying or quenching.

The use of shorter beams than the conditions recommend (17, 18) will result in the Euler formula giving somewhat overstated critical loads. This can lead to an accident, so other stability calculation methods need to be applied for short beams.

Thus, under static loading, conditions (1) and (2) are valid. Conditions (17, 18) should be applied for dynamic loading and stability loss.

### 6. Conclusion

In the work, studies were carried out on the limits of the correctness of the beam length to meet the criteria for calculating static, dynamic loading, as well as the loss of stability. The results obtained are well consistent with known values and refine them with respect to beam dynamics. The proposed approach can be used with beams with any cross-sectional shape.
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