Functional limit theorems for the number of busy servers in a $G/G/\infty$ queue

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Abstract

We discuss weak convergence of the number of busy servers in a $G/G/\infty$ queue in the $J_1$-topology on the Skorokhod space. We prove two functional limit theorems, with random and nonrandom centering, respectively, thereby solving two open problems stated in [16]. A new integral representation for the limit Gaussian process is given.

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1 Introduction

Let $(\xi_k, \eta_k)_{k \in \mathbb{N}}$ be a sequence of i.i.d. two-dimensional random vectors with generic copy $(\xi, \eta)$ where both $\xi$ and $\eta$ are positive. No condition is imposed on the dependence structure between $\xi$ and $\eta$.

Define

$$K(t) := \sum_{k \geq 0} \mathbb{1}_{\{S_k + \eta_{k+1} \leq t\}} \quad \text{and} \quad Z(t) := \sum_{k \geq 0} \mathbb{1}_{\{S_k < t < S_k + \eta_{k+1}\}}, \quad t \geq 0,$$

where $\{S_k\}_{k \in \mathbb{N}_0}$ is the zero-delayed ordinary random walk with increments $\xi_k$ for $k \in \mathbb{N}$, i.e., $S_0 = 0$ and $S_k = \xi_1 + \ldots + \xi_k$, $k \in \mathbb{N}$. In a $G/G/\infty$-queuing system, where customers arrive at times $S_0 = 0 < S_1 < S_2 < \ldots$ and are immediately served by one of infinitely many idle servers, the service time of the kth customer being $\eta_{k+1}$, $K(t)$ gives the number of customers served up to and including time $t \geq 0$, whereas $Z(t)$ gives the number of busy servers at time $t$. Some other interpretations of $Z(t)$ can be found in [12]. The process $(Z(t))_{t \geq 0}$ was also used to model the

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$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. 
number of active sources in a communication network (for instance, active sessions in a computer network) \[13, 16, 17\].

From a more theoretical viewpoint, \(K(t)\) is the number of visits to the interval \([0, t]\) of a perturbed random walk \((S_k + \eta_{k+1})_{k \in \mathbb{N}_0}\) and \(Z(t)\) is the difference between the number of visits to \([0, t]\) of the ordinary random walk \((S_k)_{k \in \mathbb{N}_0}\) and \((S_k + \eta_{k+1})_{k \in \mathbb{N}_0}\). To proceed, we need a definition. Denote by \(X := (X(t))_{t \geq 0}\) a random process arbitrarily dependent on \(\xi\). Let \((X_k, \xi_k)_{k \in \mathbb{N}}\) be i.i.d. copies of the pair \((X, \xi)\). Following \[8\] we call random process with immigration the random process \((Y(t))_{t \geq 0}\) defined by

\[
Y(t) := \sum_{k \geq 0} X_{k+1}(t - S_k) \mathbb{1}_{\{S_k \leq t\}}, \quad t \geq 0.
\]

If \(X\) is deterministic, the random process with immigration becomes a classical renewal shot noise process. Getting back to the mainstream we conclude that both \((K(t))_{t \geq 0}\) and \(Z := (Z(t))_{t \geq 0}\) are particular instances of the random process with immigration which correspond to \(X(t) = \mathbb{1}_{\{\eta \leq t\}}\) and \(X(t) = \mathbb{1}_{\{\eta > t\}}\), respectively.

Let \(D := D[0, \infty)\) be the Skorokhod space of real-valued functions on \([0, \infty)\), which are right-continuous on \([0, \infty)\) with finite limits from the left at each positive point. We shall write \(\overset{J_1}{\rightarrow}\) and \(\overset{P}{\rightarrow}\) to denote weak convergence in the \(J_1\)-topology on \(D\) and convergence in probability, respectively. The classical references concerning the \(J_1\)-topology are \[2, 11, 15\].

In this paper we shall prove weak convergence of \((Z(ut))_{u \geq 0}\), properly centered and normalized, in the \(J_1\)-topology on \(D\) as \(t \to \infty\). The same problem for \((K(ut))_{u \geq 0}\) which is much simpler was solved in \[1\]. We start with a functional limit theorem with a random centering.

**Theorem 1.1.** Assume that \(\mu := \mathbb{E}\xi \in (0, \infty)\) and that

\[
1 - F(t) = \mathbb{P}\{\eta > t\} \sim t^{-\beta} \ell(t), \quad t \to \infty
\]  

for some \(\beta \in [0, 1)\) and some \(\ell\) slowly varying at \(\infty.\) Then

\[
\underbrace{\sum_{k \geq 0} \left( \mathbb{1}_{\{S_k \leq ut < S_k + \eta_{k+1}\}} - \left(1 - F(ut - S_k)\right) \mathbb{1}_{\{S_k \leq ut\}} \right)}_{\sqrt{\mu^{-1} \int_0^t (1 - F(y)) \, dy}} \overset{J_1}{\rightarrow} V_{\beta}(u), \quad t \to \infty,
\]

where \(V_{\beta} := (V_{\beta}(u))_{u \geq 0}\) is a centered Gaussian process with

\[
\mathbb{E} V_{\beta}(u) V_{\beta}(s) = u^{1-\beta} - (u - s)^{1-\beta}, \quad 0 \leq s \leq u.
\]

In the case where \(\xi\) and \(\eta\) are independent weak convergence of the finite-dimensional distributions in \[1.2\] was proved in Proposition 3.2 of \[16\]. In the general case treated here where \(\xi\) and \(\eta\) are arbitrarily dependent the aforementioned convergence outside zero (i.e., weak convergence of \((Z^*_1(u_1), \ldots, Z^*_n(u_n))\) for any \(n \in \mathbb{N}\) and any \(0 < u_1 < \ldots < u_n < \infty\), where \(Z^*_i(u)\) denotes the left-hand side in \[1.2\]) follows from a specialization of Proposition 2.1 in \[8\]. In Section 5.2 of \[16\] the authors write: ‘We suspect that the’ finite-dimensional ‘convergence can be considerably strengthened’. Our Proposition \[14\] confirms their conjecture.

Also, the authors of \[16\] ask on p. 154: ‘When can the random centering’ in \[1.2\] ‘be replaced by a non-random centering?’ Our second result states that such

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\(^2\)The Skorokhod spaces \(D(0, \infty)\) and \(D[0, T]\) for \(T > 0\) which appear below are defined similarly.
a replacement is possible whenever $\xi$ possesses finite moments of sufficiently large positive orders. Our approach is essentially based on the decomposition:\footnote{Investigating $Z$ directly, i.e., not using (1.3), seems to be a formidable task unless $\xi$ and $\eta$ are independent, and the distribution of $Z$ is exponential (for the latter situation, see [17] and references therein). We note in passing that our Theorem 1.2 includes Theorem 1 in [17] as a particular case.}

$$\sum_{k\geq 0} \mathbb{1}_{\{S_k \leq ut < S_k + \eta_k + 1\}} - \mu^{-1} \int_0^{ut} (1 - F(y)) \, dy$$

$$= \left( \sum_{k\geq 0} \mathbb{1}_{\{S_k \leq ut < S_k + \eta_k + 1\}} - \sum_{k\geq 0} \mathbb{E}(\mathbb{1}_{\{S_k \leq ut < S_k + \eta_k + 1\}} | S_k) \right)$$

$$+ \left( \sum_{k\geq 0} \mathbb{E}(\mathbb{1}_{\{S_k \leq ut < S_k + \eta_k + 1\}} | S_k) - \mu^{-1} \int_0^{ut} (1 - F(y)) \, dy \right)$$

$$= \sum_{k\geq 0} \left( \mathbb{1}_{\{S_k \leq ut < S_k + \eta_k + 1\}} - (1 - F(ut - S_k)) \mathbb{1}_{\{S_k \leq ut\}} \right)$$

$$+ \left( \sum_{k\geq 0} (1 - F(ut - S_k)) \mathbb{1}_{\{S_k \leq ut\}} - \mu^{-1} \int_0^{ut} (1 - F(y)) \, dy \right). \tag{1.4}$$

Weak convergence on $D$ of the first summand on the right-hand side, normalized by $\sqrt{\mu^{-1} \int_0^t (1 - F(y)) \, dy}$, was treated in Theorem 1.1. Thus, we are left with analyzing weak convergence of the second summand.

**Theorem 1.2.** Suppose that condition (1.1) holds. If $\mathbb{E} \xi^r < \infty$ for some $r > 2(1 - \beta)^{-1}$, then

$$\sum_{k\geq 0} \frac{\mathbb{1}_{\{S_k \leq ut < S_k + \eta_k + 1\}} - \mu^{-1} \int_0^{ut} (1 - F(y)) \, dy}{\sqrt{\mu^{-1} \int_0^t (1 - F(y)) \, dy}} \overset{\mathcal{D}}{\to} V_{\beta}(u), \quad t \to \infty, \tag{1.5}$$

where $\mu = \mathbb{E} \xi < \infty$ and $V_{\beta}$ is a centered Gaussian process with covariance (1.3).

Under the assumption that $\xi$ and $\eta$ are independent weak convergence of the one-dimensional distributions in (1.5) was proved in Theorem 2 of [12]. Note that regular variation condition (1.1) is not needed for this convergence to hold. Weak convergence of the finite-dimensional distributions in (1.5) takes place under (1.1) and the weaker assumption $\mathbb{E} \xi^2 < \infty$. We do not know whether (1.1) and the second moment assumption are sufficient for weak convergence on $D$. More generally, weak convergence of the finite-dimensional distributions of $Z(ut)$, properly normalized and centered, holds whenever the distribution of $\xi$ belongs to the domain of attraction of an $\alpha$-stable distribution, $\alpha \in (0, 2) \setminus \{1\}$, see Theorem 3.3.21 in [7] which is a specialization of Theorems 2.4 and 2.5 in [8]. We do not state these results here because in this paper we are only interested in weak convergence on $D$.

The rest of the paper is structured as follows. Theorems 1.1 and 1.2 are proved in Sections 2 and 3 respectively. In Section 4 we discuss an integral representation of the limit process $V_{\beta}$ which seems to be new. The appendix collects several auxiliary results.

\textsuperscript{4}The normalization is not necessarily of the form $\sqrt{\mu^{-1} \int_0^t (1 - F(y)) \, dy}$, and the limit process is not necessarily $V_{\beta}$.
2 Proof of Theorem 1.1

We start by observing that

$$a(t) := \sum_{k=0}^{[t]+1} (1 - F(k)) \sim \int_0^t (1 - F(y))dy \sim (1 - \beta)^{1-\beta}t^{1-\beta} \ell(t) \quad (2.1)$$

as $t \to \infty$, where the second equivalence follows from Karamata’s theorem (Proposition 1.5.8 in [3]). In particular, the first equivalence enables us to replace the integral in the denominator of (1.2) with the sum. For each $t, u \geq 0$, denote by $\hat{Z}(ut)$ the first summand in decomposition (1.4), i.e.,

$$\hat{Z}(ut) := \sum_{k \geq 0} \left( 1 \{s_k \leq ut < s_{k+1} \} \right) - (1 - F(ut - S_k)) 1\{S_k \leq ut\}$$

and then set

$$Z_t(u) := \sum_{k \geq 0} \left( 1 \{s_k \leq ut < s_{k+1} \} \right) - (1 - F(ut - S_k)) 1\{S_k \leq ut\} \frac{1}{\sqrt{a(t)}} = \frac{\hat{Z}(ut)}{\sqrt{a(t)}}, \quad u \geq 0.$$

Our proof of Theorem 1.1 is similar to the proof of Theorem 1 in [17] which treats the case where $\xi$ and $\eta$ are independent, and the distribution of $\xi$ is exponential (Poisson case). Lemma 2.1 given below is concerned with inevitable technical complications that appear outside the Poisson case. Put

$$\nu(t) := \inf\{k \in \mathbb{N}_0 : S_k > t\}, \quad t \in \mathbb{R}$$

and note that the random variable $\nu(1)$ has finite moments of all positive orders by Lemma 5.2

**Lemma 2.1.** Let $l \in \mathbb{N}$ and $0 \leq v < u$. For any chosen $A > 1$ and $\rho \in (0, 1 - \beta)$ there exist $t_1 > 1$ such that

$$\mathbb{E}|Z_t(u) - Z_t(v)|^{2l} \leq c(l)(u - v)^{(1 - \beta - \rho)}$$

whenever $u - v < 1$ and $(u - v)t \geq t_1$, where $c(l) := 2C_l(4A)^l(u - v)^{(1 - \beta - \rho)}\mathbb{E}(\nu(1))^l$ and $C_l$ is a finite positive constant.

**Proof.** With $u, v \geq 0$ fixed, $\hat{Z}(ut) - \hat{Z}(vt)$ equals the terminal value of the martingale $(R(k, t), \mathcal{F}_k)_{k \in \mathbb{N}_0}$, where $R(0, t) := 0,

$$R(k, t) := \sum_{j=0}^{k-1} \left( 1 \{s_j + \eta_{j+1} \leq ut\} - F(ut - S_j) 1\{S_j \leq ut\} \right) - (1 \{s_j + \eta_{j+1} \leq vt\} - F(vt - S_j) 1\{S_j \leq vt\})$$

$\mathcal{F}_0 := \{\Omega, \emptyset\}$ and $\mathcal{F}_k := \sigma(\{\xi_j, \eta_j : 1 \leq j \leq k\})$. We use the Burkholder-Davis-Gundy inequality (Theorem 11.3.2 in [4]) to obtain for any $l \in \mathbb{N}$

$$\mathbb{E}(\hat{Z}(ut) - \hat{Z}(vt))^{2l} \leq C_l \left( \mathbb{E}\left( \sum_{k \geq 0} \mathbb{E}\left( (R(k + 1, t) - R(k, t))^2 | \mathcal{F}_k \right) \right)^l + \sum_{k \geq 0} \mathbb{E}(R(k + 1, t) - R(k, t))^{2l} \right) =: C_l(I_1(t) + I_2(t)) \quad (2.2)$$
for a positive constant $C_1$. We shall show that

$$ I_1(t) \leq 2^l \mathbb{E}(\nu(1)^l (a((u-v)t))^l), \quad t \geq 0 $$

and that

$$ I_2(t) \leq 2^{2l} \mathbb{E}\nu(1)a((u-v)t), \quad t \geq 0. $$

**Proof of (2.3).** We first observe that

$$ \sum_{k \geq 0} \mathbb{E}\left( (R(k + 1, t) - R(k, t))^2 | F_k \right) $$

$$ = \int_{[v, u]} F(ut - y)(1 - F(ut - y))d\nu(y) $$

$$ + \int_{[0, vt]} (F(ut - y) - F(vt - y))(1 - F(ut - y) + F(vt - y))d\nu(y) $$

$$ \leq \int_{[v, u]} (1 - F(ut - y))d\nu(y) + \int_{[0, vt]} (F(ut - y) - F(vt - y))d\nu(y) $$

whence

$$ I_1(t) \leq 2^{l-1} \left( \mathbb{E}\left( \int_{[v, u]} (1 - F(ut - y))d\nu(y) \right)^l + \mathbb{E}\left( \int_{[0, vt]} (F(ut - y) - F(vt - y))d\nu(y) \right)^l \right) $$

having utilized $(x + y)^l \leq 2^{l-1}(x^l + y^l)$ for nonnegative $x$ and $y$. Using Lemma 5.1 with $G(y) = (1 - F(y))\mathbb{1}_{[0,(u-v)t]}(y)$ and $G(y) = F((u-v)t+y) - F(y)$, respectively, we obtain

$$ \mathbb{E}\left( \int_{[v, u]} (1 - F(ut - y))d\nu(y) \right)^l $$

$$ = \mathbb{E}\left( \int_{[0, ut]} (1 - F(ut - y))\mathbb{1}_{[0,(u-v)t]}(ut - y)d\nu(y) \right)^l $$

$$ \leq \mathbb{E}(\nu(1))^l \left( \sum_{n=0}^{[ut]} \sup_{y \in [n, n+1)} \left( (1 - F(y))\mathbb{1}_{[0,(u-v)t]}(y) \right) \right)^l $$

$$ \leq \mathbb{E}(\nu(1))^l \left( \sum_{n=0}^{[u-v]} (1 - F(n)) \right)^l \leq \mathbb{E}(\nu(1))^l(a((u-v)t))^l. $$

(2.5)

and

$$ \mathbb{E}\left( \int_{[0, vt]} (F(ut - y) - F(vt - y))d\nu(y) \right)^l $$

$$ \leq \mathbb{E}(\nu(1))^l \left( \sum_{n=0}^{[vt]} \sup_{y \in [n, n+1)} (F((u-v)t+y) - F(y)) \right)^l $$

$$ \leq \mathbb{E}(\nu(1))^l \left( \sum_{n=0}^{[vt]} (1 - F(n)) - \sum_{n=0}^{[vt]} (1 - F((u-v)t+n+1)) \right)^l $$

$$ \leq \mathbb{E}(\nu(1))^l \left( \sum_{n=0}^{[vt]} (1 - F(n)) - \sum_{n=0}^{[vt]} (1 - F(n)) + \sum_{n=0}^{[(u-v)t]+1} (1 - F(n)) \right)^l $$

$$ \leq \mathbb{E}(\nu(1))^l(a((u-v)t))^l. $$

(2.6)
Combining (2.5) and (2.6) yields (2.3).

**Proof of (2.4).** Let us calculate

\[
\mathbb{E}((R(k+1,t) - R(k,t))^{2l} | \mathcal{F}_k) \\
\leq 2^{2l-1}((1 - F(u - S_k))^{2l} F(u - S_k) \\
+ (F(u - S_k))^{2l}(1 - F(u - S_k)) \mathbb{1}_{\{u < S_k \leq u\})} \\
+ ((1 - F(u - S_k)) + F(v - S_k))^{2l}(F(u - S_k) - F(v - S_k)) \\
+ (F(u - S_k) - F(v - S_k))^{2l}(1 - F(u - S_k) + F(v - S_k)) \mathbb{1}_{\{S_k \leq v\})} \\
\leq 2^{2l-1}((1 - F(u - S_k)) \mathbb{1}_{\{u < S_k \leq u\})} + (F(u - S_k) - F(v - S_k)) \mathbb{1}_{\{S_k \leq v\})}.
\]

Therefore,

\[
I_2(t) \leq 2^{2l-1} \left( \mathbb{E} \int_{(v,u]} (1 - F(u - y)) \mathrm{d}v(y) + \mathbb{E} \int_{[0,v]} (F(u - y) - F(v - y)) \mathrm{d}v(y) \right).
\]

Using now formulae (2.5) and (2.6) with \( l = 1 \) yields (2.4).

In view of (2.1) we can invoke Potter’s bound (Theorem 1.5.6(iii) in [3]) to conclude that for any chosen \( A > 1 \) and \( \rho \in (0, 1 - \beta) \) there exists \( t_1 > 1 \) such that

\[
a((u - v)t)/a(t) \leq A(u - v)^{1-\beta-\rho}
\]

whenever \( u - v < 1 \) and \((u - v)t \geq t_1\). Note that \( u - v < 1 \) and \((u - v)t \geq t_1\) together imply \( t \geq t_1\). Hence

\[
I_1(t) / (a(t))^l \leq 2^l \mathbb{E}(\nu(1))^l \left( a((u - v)t)/a(t) \right)^l \leq (4A)^l \mathbb{E}(\nu(1))^l(u - v)^{l(1-\beta-\rho)}. \tag{2.7}
\]

Increasing \( t_1 \) if needed we can assume that \( t^{1-\beta-\rho}/a(t) \leq 1 \) for \( t \geq t_1 \) whence

\[
\sum_{n=0}^{\lfloor t \rfloor+1} (1 - F(n)) = ((u - v)t)^{1-\beta-\rho} - \sum_{n=0}^{\lfloor t \rfloor+1} (1 - F(n)) \leq (1 - (u - v)^{1-\beta-\rho}) \leq (u - v)^{1-\beta-\rho}
\]

because \((u - v)t)^{1-\beta-\rho} \geq t_1^{1-\beta-\rho} > 1\). This implies

\[
I_2(t) / (a(t))^l \leq 2^l \mathbb{E}(\nu(1))^l \left( a((u - v)t)/a(t) \right) \frac{1}{(a(t))^{l-1}} \leq (4A)^l \mathbb{E}(\nu(1))^l(u - v)^{l(1-\beta-\rho)}, \tag{2.8}
\]

where we have used \( \mathbb{E}(\nu(1)) \leq \mathbb{E}(\nu(1))^l \) which is a consequence of \( \nu(1) \geq 1 \) a.s.

Now the claim follows from (2.2), (2.7) and (2.8).

We are ready to prove Theorem 1.1. As discussed in the paragraph following Theorem 1.1 weak convergence of \((Z_t(u_1), \ldots, Z_t(u_n))\) for any \( n \in \mathbb{N} \) and any \( 0 < u_1 < \ldots, u_n < \infty \) was proved in earlier works. In view of \( V_{\beta}(0) = 0 \) a.s., this immediately extends to \( 0 \leq u_1 < \ldots, u_n < \infty \). Thus, it remains to prove tightness on \( D[0,T] \) for any \( T > 0 \). Since the normalization in (1.2) is regularly varying it is enough to investigate the case \( T = 1 \) only. Suppose we can prove that for any \( \varepsilon > 0 \) and \( \gamma > 0 \) there exist \( t_0 > 0 \) and \( \delta > 0 \) such that

\[
\mathbb{P}\left\{ \sup_{0 \leq u,v \leq 1, |u-v| \leq \delta} |Z_t(u) - Z_t(v)| > \varepsilon \right\} \leq \gamma \tag{2.9}
\]
for all $t \geq t_0$. Then, by Theorem 15.5 in [2] the desired tightness follows along with continuity of the paths of (some version of) the limit process.

On pp. 763-764 in [17] it is shown that (the specific form of $Z_t$ plays no role here)

$$
\sup_{0 \leq u, v \leq 1, |u - v| \leq 2^{-i}} |Z_t(u) - Z_t(v)| \leq 2 \sum_{j=1}^{I} \max_{1 \leq k \leq 2^j} |Z_t(k 2^{-j}) - Z_t((k - 1) 2^{-j})| + 2 \max_{0 \leq k \leq 2^{I-1}} \sup_{0 \leq w \leq 2^{-I}} |Z_t(k 2^{-I} + w) - Z_t(k 2^{-I})|
$$

for any positive integers $i$ and $I$, $i \leq I$. Hence (2.9) follows if we can check that for any $\varepsilon > 0$ and $\gamma > 0$ there exist $t_0 > 0$, $i \in \mathbb{N}$ and $I \in \mathbb{N}$, $i \leq I$ such that

$$
P\left\{ \sum_{j=i}^{I} \max_{1 \leq k \leq 2^j} |Z_t(k 2^{-j}) - Z_t((k - 1) 2^{-j})| > \varepsilon \right\} \leq \gamma, \quad t \geq t_0
$$

(2.10) and that

$$
\max_{0 \leq k \leq 2^{I-1}} \sup_{0 \leq w \leq 2^{-I}} |Z_t(k 2^{-I} + w) - Z_t(k 2^{-I})| \overset{p}{\to} 0, \quad t \to \infty.
$$

(2.11)

**Proof of (2.10).** By Lemma 2.1 for any chosen $A > 1$ and $\rho \in (0, 1 - \beta)$ there exists $t_1 > 1$ such that

$$
\mathbb{E}[Z_t(k 2^{-j}) - Z_t((k - 1) 2^{-j})]^{2l} \leq c(l) 2^{-j(l(\gamma - \beta - \rho) - 1)}
$$

whenever $2^{-j} t \geq t_1$. Let $I = I(t)$ denote the integer number satisfying $2^{-I} t > t_1 > 2^{-I-1} t$.

Then the inequalities (2.12) and

$$
\mathbb{E}(\max_{1 \leq k \leq 2^j} |Z_t(k 2^{-j}) - Z_t((k - 1) 2^{-j})|)^{2l} \leq \sum_{k=1}^{2^j} \mathbb{E}[Z_t(k 2^{-j}) - Z_t((k - 1) 2^{-j})]^{2l}
$$

hold whenever $j \leq I$. Pick now minimal $l \in \mathbb{N}$ such that $l(1 - \beta - \rho) > 1$. Given positive $\varepsilon$ and $\gamma$ choose minimal $i \in \mathbb{N}$ satisfying $2^{-i(l(\gamma - \beta - \rho) - 1)} \leq \varepsilon^{2l}(1 - 2^{-l(l(\gamma - \beta - \rho) - 1)/(2l)})^{2l} / c(l)$.

Increase $t$ if needed to ensure that $i \leq I$. Invoking Markov’s inequality and then the triangle inequality for the $L_{2l}$-norm gives

$$
P\left\{ \sum_{j=i}^{I} \max_{1 \leq k \leq 2^j} |Z_t(k 2^{-j}) - Z_t((k - 1) 2^{-j})| > \varepsilon \right\}
$$

$$
\leq \varepsilon^{-2l} \mathbb{E}\left( \sum_{j=i}^{I} \max_{1 \leq k \leq 2^j} |Z_t(k 2^{-j}) - Z_t((k - 1) 2^{-j})| \right)^{2l}
$$

$$
\leq \varepsilon^{-2l} \left( \sum_{j=1}^{I} (\mathbb{E}(\max_{1 \leq k \leq 2^j} |Z_t(k 2^{-j}) - Z_t((k - 1) 2^{-j})|)^{2l})^{1/2l} \right)^{2l}
$$

$$
\leq \varepsilon^{-2l} c(l) \left( \sum_{j=1}^{I} 2^{-j(l(1 - \beta - \rho) - 1)/(2l)} \right)^{2l}
$$

$$
= \varepsilon^{-2l} c(l) \frac{2^{-i(l(1 - \beta - \rho) - 1)}}{1 - 2^{-l(l(1 - \beta - \rho) - 1)/(2l)}} \leq \gamma
$$
for all \( t \) large enough.

**Proof of (2.11).** We shall use a decomposition

\[
(a(t))^{1/2}(Z_t(k2^{-l} + w) - Z_t(k2^{-l}))
= \sum_{j \geq 0} \left( \mathbf{1}_{\{S_j + \eta_{j+1} \leq (k2^{-l} + w)t\}} - F((k2^{-l} + w)t - S_j) \right) \mathbf{1}_{\{k2^{-l}t < S_j \leq (k2^{-l} + w)t\}}
+ \sum_{j \geq 0} \left( \mathbf{1}_{\{k2^{-l}t < S_j + \eta_{j+1} \leq (k2^{-l} + w)t\}} - (F((k2^{-l} + w)t - S_j) - F(k2^{-l}t - S_j)) \right) \mathbf{1}_{\{S_j \leq k2^{-l}t\}}
=: J_1(t, k, w) + J_2(t, k, w).
\]

It suffices to prove that for \( i = 1, 2 \)

\[
(a(t))^{-1/2} \max_{0 \leq k \leq 2^l - 1} \sup_{0 \leq w \leq 2^{-l}} |J_i(t, k, w)| \xrightarrow{P} 0, \quad t \to \infty. \tag{2.13}
\]

**Proof of (2.13) for \( i = 1 \).** Since \( |J_1(t, k, w)| \leq \nu((k2^{-l} + w)t) - \nu(k2^{-l}t) \) and \( \nu(t) \) is a.s. nondecreasing we infer \( \sup_{0 \leq w \leq 2^{-l}} |J_1(t, k, w)| \leq \nu((k + 1)2^{-l}t) - \nu(k2^{-l}t) \).

By Boole’s inequality and distributional subadditivity of \( \nu(t) \) (see formula (5.7) on p. 58 in [6])

\[
\mathbb{P} \left\{ \max_{0 \leq k \leq 2^l - 1} (\nu((k + 1)2^{-l}t) - \nu(k2^{-l}t)) > \delta(a(t))^{1/2} \right\}
\leq \sum_{k=0}^{2^l-1} \mathbb{P} \left\{ \nu((k + 1)2^{-l}t) - \nu(k2^{-l}t) > \delta(a(t))^{1/2} \right\}
\leq 2^l \mathbb{P} \left\{ \nu(2^{-l}t) > \delta(a(t))^{1/2} \right\} \leq 2^l \mathbb{P} \left\{ \nu(2t_1) > \delta(a(t))^{1/2} \right\}
\]

for any \( \delta > 0 \). The right-hand side converges to zero as \( t \to \infty \) because \( \nu(2t_1) \) has finite exponential moments of all positive orders (see Lemma 5.2).

**Proof of (2.13) for \( i = 2 \).** We have

\[
\max_{0 \leq k \leq 2^l - 1} |J_2(t, k, w)|
\leq \sup_{0 \leq w \leq 2^{-l}} \left( \sum_{j \geq 0} \mathbf{1}_{\{k2^{-l}t < S_j + \eta_{j+1} \leq (k2^{-l} + w)t\}} \mathbf{1}_{\{S_j \leq k2^{-l}t\}}
+ \sum_{j \geq 0} (F((k2^{-l} + w)t - S_j) - F(k2^{-l}t - S_j)) \mathbf{1}_{\{S_j \leq k2^{-l}t\}}
\right)
\leq \sum_{j \geq 0} \mathbf{1}_{\{k2^{-l}t < S_j + \eta_{j+1} \leq (k + 1)2^{-l}t\}} \mathbf{1}_{\{S_j \leq k2^{-l}t\}}
+ \sum_{j \geq 0} (F(((k + 1)2^{-l}t - S_j) - F(k2^{-l}t - S_j)) \mathbf{1}_{\{S_j \leq k2^{-l}t\}}
\leq \sum_{j \geq 0} (\mathbf{1}_{\{k2^{-l}t < S_j + \eta_{j+1} \leq (k + 1)2^{-l}t\}}
- (F(((k + 1)2^{-l}t - S_j) - F(k2^{-l}t - S_j)) \mathbf{1}_{\{S_j \leq k2^{-l}t\}}
+ 2 \sum_{j \geq 0} (F(((k + 1)2^{-l}t - S_j) - F(k2^{-l}t - S_j)) \mathbf{1}_{\{S_j \leq k2^{-l}t\}}
=: J_{21}(t, k) + 2J_{22}(t, k).
\]
Pick minimal \( r \in \mathbb{N} \) satisfying \( r(1 - \beta) > 1 \) so that \( \lim_{t \to \infty} t^{-1}(a(t))^r = \infty \).

Using (2.6) with \( u = (k + 1)2^{-I} \) and \( v = k2^{-I} \) we obtain

\[
\mathbb{E}(J_{22}(t, k))^{2r} \leq \mathbb{E}(\nu(1))^{2r}(a(2^{-I}t))^{2r} \leq \mathbb{E}(\nu(1))^{2r}(a(2t_1))^{2r}
\]

which implies

\[
(a(t))^{-r}\mathbb{E}(\max_{0 \leq k \leq 2^I - 1} J_{22}(t, k))^{2r} \leq (a(t))^{-r}2^I \max_{0 \leq k \leq 2^I - 1} \mathbb{E}(J_{22}(t, k))^{2r}
\]

\[
\leq (a(t))^{-r}2^I \mathbb{E}(\nu(1))^{2r}(a(2t_1))^{2r}.
\]

The right-hand side converges to zero as \( t \to \infty \) by our choice of \( r \). Consequently, \( (a(t))^{-1/2} \max_{0 \leq k \leq 2^I - 1} J_{22}(t, k) \stackrel{p}{\to} 0 \) as \( t \to \infty \) by Markov’s inequality.

Using a counterpart of the first inequality in (2.2) for the martingale \((R^*(l, t), F_l)_{l \in \mathbb{N}_0}\), where \( R^*(0, t) := 0 \) and

\[
R^*(l, t) := \sum_{j=0}^{l-1} \mathbb{1}_{\{vt < S_j + \eta_j, 1 \leq u \leq ut\}} - (F(ut - S_j) - F(ut - S_j)) \mathbb{1}_{\{S_j \leq ut\}}, \quad l \in \mathbb{N}
\]

for \( u = (k + 1)2^{-I}t \) and \( v = k2^{-I}t \), one can check that

\[
\mathbb{E}(J_{21}(t, k))^{2r} \leq C_r \mathbb{E} \left( \int_{[0, k2^{-I}t]} (F((k + 1)2^{-I}t - y) - F(k2^{-I}t - y))d\nu(y) \right)^r
\]

\[
= \mathbb{E} \int_{[0, k2^{-I}t]} (F((k + 1)2^{-I}t - y) - F(k2^{-I}t - y))d\nu(y).
\]

In view of (2.6) the right-hand side does not exceed

\[
C_r (\mathbb{E}(\nu(1))r(a(2^{-I}t))r + \mathbb{E} \nu(1)a(2^{-I}t)) \leq C_r (\mathbb{E}(\nu(1))r(a(2k))r + \mathbb{E} \nu(1)a(2k)).
\]

Arguing as above we conclude that \( (a(t))^{-1/2} \max_{0 \leq k \leq 2^I - 1} J_{21}(t, k) \stackrel{p}{\to} 0 \) as \( t \to \infty \), and (2.1.13) for \( i = 2 \) follows. The proof of Theorem 1.1 is complete.

3 Proof of Theorem 1.2

Set \( f(t) := \sqrt{t(1 - F(t))} \) for \( t > 0 \). In view of (2.1)

\[
\sqrt{\int_0^t (1 - F(y))dy} \sim (1 - \beta)^{-1/2} t^{1/2 - \beta/2} (\ell(t))^{1/2} \sim (1 - \beta)^{-1/2} f(t)
\]

as \( t \to \infty \). Assuming that \( \mathbb{E} \xi^r < \infty \) for some \( r > 2(1 - \beta)^{-1} \) we intend to show that

\[
\sup_{0 \leq u \leq T} \left| \sum_{k \geq 0} (1 - F(ut - S_k)) \mathbb{1}_{\{S_k \leq ut\}} - \mu^{-1} \int_0^u (1 - F(y))dy \right| \stackrel{p}{\to} 0, \quad t \to \infty
\]

for any \( T > 0 \). This in combination with (3.1) and Theorem 1.1 is sufficient for the proof of the \( J_1\)-convergence.

We proceed by observing that

\[
\sum_{k \geq 0} (1 - F(t - S_k)) \mathbb{1}_{\{S_k \leq t\}} - \mu^{-1} \int_0^t (1 - F(y))dy = \int_{[0, t]} (1 - F(t - y))d(\nu(y) - \mu^{-1}y).
\]
Integration by parts yields
\[
\begin{align*}
\int_{[0,t]} (1 - F(t - y))d(\nu(y) - \mu^{-1}y) + \mathbb{P}\{\xi = t\} \\
= \nu(t) - \mu^{-1}t - \int_{[0,t]} (\nu(t - y) - \mu^{-1}(t - y))dF(y) = \left(\nu(t) - \mu^{-1}t - \sigma\mu^{-3/2}W(t)\right) \\
- \int_{[0,t]} (\nu(t - y) - \mu^{-1}(t - y) - \sigma\mu^{-3/2}W(t - y))dF(y) \\
+ \sigma\mu^{-3/2}\left(W(t) - \int_{[0,t]} W(t - y)dF(y)\right) =: R_1(t) + \sigma\mu^{-3/2}R_2(t),
\end{align*}
\]

where \(\sigma^2 = \text{Var} \xi < \infty\) and \(W\) is a standard Brownian motion as defined in Lemma 5.3. For any \(T > 0\)

\[
\sup_{0 \leq u \leq T} \left| R_1(ut) \right| \leq \sup_{0 \leq u \leq T} \left| \nu(ut) - \mu^{-1}ut - \sigma\mu^{-3/2}W(ut) \right| \\
+ \sup_{0 \leq u \leq T} \int_{[0,ut]} \left| \nu(u - y) - \mu^{-1}(u - y) - \sigma\mu^{-3/2}W(u - y) \right|dF(y) \\
\leq \sup_{0 \leq u \leq T} \left| \nu(u) - \mu^{-1}u - \sigma\mu^{-3/2}W(u) \right| \\
+ \sup_{0 \leq u \leq T} \sup_{0 \leq y \leq ut} \left| \nu(y) - \mu^{-1}y - \sigma\mu^{-3/2}W(y) \right| \\
\leq 2 \sup_{0 \leq u \leq T} \left| \nu(u) - \mu^{-1}u - \sigma\mu^{-3/2}W(u) \right|.
\]

By Lemma 5.3 the right-hand side is \(o(t^{1/r})\) a.s. as \(t \to \infty\). Hence, our choice of \(r\) in combination with (3.1) ensure that

\[
\lim_{t \to \infty} \sup_{0 \leq u \leq T} \left| R_1(ut) \right| / f(t) = 0 \quad \text{a.s.}
\]

Further, we note that

\[
R_2(t) = W(t)(1 - F(t)) + \int_{[0,t]} (W(t) - W(t - y))dF(y) =: R_{21}(t) + R_{22}(t).
\]

Pick now \(\varepsilon \in (0,(1 - \beta)/2)\) if \(\beta \in [1/2,1)\) and \(\varepsilon \in (0,1/2 - \beta)\) if \(\beta \in [0,1/2)\). With this \(\varepsilon\), we have for any \(T > 0\)

\[
\begin{align*}
\sup_{0 \leq u \leq T} \left| R_{22}(ut) \right| & \leq \sup_{0 \leq u \leq T} \int_{[0,ut]} \frac{|W(ut) - W(ut - y)|}{y^{1/2-\varepsilon}} y^{1/2-\varepsilon}dF(y) \\
& \leq \sup_{0 \leq u \leq T} \sup_{0 \leq x \leq ut} \frac{|W(u) - W(u - x)|}{x^{1/2-\varepsilon}} \int_{[0,ut]} y^{1/2-\varepsilon}dF(y) \\
& \leq \sup_{0 \leq u \leq T} \frac{|W(u) - W(v)|}{(u - v)^{1/2-\varepsilon}} \int_{[0,T]} y^{1/2-\varepsilon}dF(y) \\
& \overset{d}{=} \sup_{0 \leq u \leq T} \frac{|W(u) - W(v)|}{(u - v)^{1/2-\varepsilon}} \int_{[0,T]} y^{1/2-\varepsilon}dF(y).
\end{align*}
\]

Here,

\[
\sup_{0 \leq u \leq T} \frac{|W(u) - W(v)|}{(u - v)^{1/2-\varepsilon}} < \infty \quad \text{a.s.}
\]
because the Brownian motion $W$ is locally Hölder continuous with exponent $1/2 - \varepsilon$ (for any $\varepsilon \in (0, 1/2)$), and the distributional equality denoted by $\overset{d}{=} \varepsilon$ is a consequence of self-similarity of $W$ with index $1/2$. Now it is convenient to treat two cases separately.

**Case $\beta \in [1/2, 1)$ in which**

$$
\frac{t^\varepsilon}{f(t)} \int_{[0,T]} y^{1/2-\varepsilon} dF(y) \sim \frac{\mathbb{E}R_1^{1/2-\varepsilon}}{t^{1/2-\beta/2-\varepsilon}\ell(t)^{1/2}} \to 0, \quad t \to \infty
$$

by (3.1) and our choice of $\varepsilon$. This proves

$$
\sup_{0 \leq u \leq T} \frac{|R_{22}(ut)|}{f(t)} \overset{P}{\to} 0, \quad t \to \infty.
$$

(3.2)

**Case $\beta \in [0, 1/2)$. Here, we conclude that**

$$
\frac{t^\varepsilon}{f(t)} \int_{[0,T]} y^{1/2-\varepsilon} dF(y) \sim \frac{T^{1/2-\beta-\varepsilon}\ell(t)^{1/2}}{(1/2 - \beta - \varepsilon)t^{\beta/2}} \to 0, \quad t \to \infty
$$

having utilized (3.1), Theorem 1.6.4 in [3], which is applicable by our choice of $\varepsilon$ and the fact that $\lim_{t \to \infty} \ell(t) = 0$ when $\beta = 0$. Thus, (3.2) holds in this case, too.

It remains to check weak convergence on $D$ of $R_{21}(t)/f(t)$ to the zero function or equivalently

$$
\sup_{0 \leq u \leq T} \frac{|R_{21}(ut)|}{f(t)} \overset{P}{\to} 0, \quad t \to \infty
$$

(3.3)

for each $T > 0$. We shall only consider the case where $T > 1$, the case $T \in (0, 1]$ being analogous and simpler. By Potter’s bound (Theorem 1.5.6 (iii) in [3]), for any chosen $A > 1$ and $\delta > 0$ there exists $t_0 > 0$ such that $1 - F(u)/(1 - F(t)) \leq Au^{-\beta - \delta}$ whenever $u \in (0, 1]$ and $ut \geq t_0$. With this $t_0$, write

$$
\sup_{0 \leq u \leq T} |R_{21}(ut)| \leq \sup_{0 \leq u \leq t_0/t} |R_{21}(ut)| \vee \sup_{t_0/t \leq u \leq 1} |R_{21}(ut)| \vee \sup_{1 \leq u \leq T} |R_{21}(ut)|.
$$

For the first supremum on the right-hand side we have $\sup_{0 \leq u \leq t_0/t} |W(u)|(1 - F(ut)) \leq \sup_{0 \leq u \leq t_0} |W(u)|$ which converges to zero a.s. when divided by $f(t)$.

For the third supremum,

$$
\sup_{1 \leq u \leq T} |W(u)|(1 - F(t)) \leq (1 - F(t)) \sup_{0 \leq u \leq T} |W(u)|
$$

$$
\overset{d}{=} t^{1/2}(1 - F(t)) \sup_{0 \leq u \leq T} |W(u)|,
$$

and the right hand-side divided by $f(t)$ converges to zero a.s. in view of (3.1).

Finally,

$$
\frac{\sup_{t_0/t \leq u \leq 1} |W(u)|(1 - F(ut))}{1 - F(t)} \leq A \sup_{t_0/t \leq u \leq 1} |W(u)|u^{-\beta - \delta}.
$$

(3.4)

As before we distinguish the two cases.

---

\(^5\)Weak convergence on $D(0, \infty)$ follows immediately from the fact that $\lim_{t \to \infty} (1 - F(ut))/(1 - F(t)) = u^{-\beta}$ locally uniformly in $u$ on $(0, \infty)$. A longer proof is needed to treat weak convergence on $D(0, \infty)$, i.e., with 0 included.
CASE $\beta \in [1/2, 1)$. Choose $\delta$ satisfying $\delta \in (0, (1 - \beta)/2)$. The law of the iterated logarithm for $|W|$ at large times guarantees that $\lim_{t \to \infty} |W(t)|^{t^{-\beta}} = 0$ a.s. and thereupon $\sup_{u \geq t_0} |W(u)|^{u^{-\beta}} < \infty$ a.s. With this at hand we continue \eqref{3.4} as follows:

$$
\sup_{t_0/t \leq u \leq 1} \frac{|W(ut)|(1 - F(ut))}{f(t)} \leq A t^{\beta+\delta} \sqrt{1 - F(t)} \sup_{t_0/t \leq u \leq t} (|W(u)|^{u^{-\beta}}) \rightarrow A \sup_{u \geq t_0} (|W(u)|^{u^{-\beta}}) \frac{f(t)}{t^{1/2 - \beta/2 - \delta}} \text{ a.s.}
$$

having utilized \eqref{3.1} for the last asymptotic equivalence. The right-hand side converges to zero a.s.

CASE $\beta \in [0, 1/2)$. Pick $\delta$ so small that $\beta + \delta < 1/2$. The law of the iterated logarithm for $|W|$ at small times entails $\lim_{t \to 0} |W(t)|^{t^{-\beta}} = 0$ a.s. whence $\sup_{0 \leq u \leq 1} |W(u)|^{u^{-\beta}} < \infty$ a.s. Continuing \eqref{3.4} with the help of self-similarity of $W$ we further infer

$$
\sup_{t_0/t \leq u \leq 1} \frac{|W(ut)|(1 - F(ut))}{f(t)} \leq A \sup_{0 \leq u \leq 1} |W(u)|^{u^{-\beta}} \sqrt{1 - F(t)}.
$$

It remains to note that the right-hand side trivially converges to zero a.s.

Combining pieces together we conclude that \eqref{3.3} holds. The proof of Theorem 1.2 is complete.

4 Integral representation of the limit process $V_\beta$

First of all, we note that $V_0$ is a standard Brownian motion. Therefore, throughout the rest of the section we assume that $\beta \in (0, 1)$.

Denote by $B := (B(u, v))_{u, v \geq 0}$ a standard Brownian sheet, i.e., a two-parameter continuous centered Gaussian field with $\mathbb{E}B(u_1, v_1)B(u_2, v_2) = (u_1 \wedge u_2)(v_1 \wedge v_2)$. In particular, $B$ is a Brownian motion in $u$ (in $v$) for each fixed $v$ ($u$). See Section 3 in \cite{18} for more properties of $B$. It turns out that the limit process $V_\beta$ can be represented as the integral of a deterministic function with respect to the Brownian sheet. Such integrals are constructed in \cite{10}. Also, these can be thought of as particular instances of the integrals of the first kind with respect to the Brownian sheet, see Section 4 in \cite{18}. Set

$$
V_\beta^*(u) = \sqrt{1 - \beta} \int_{[0, u]} \int_{[0, \infty)} \mathbbm{1}_{\{x + z^{-1/\beta} > u\}} dB(x, z), \quad u \geq 0.
$$

Clearly, the process $V_\beta^*: = (V_\beta^*(u))_{u \geq 0}$ is centered Gaussian. Since

$$
\mathbb{E}V_\beta^*(u)V_\beta^*(s) = (1 - \beta) \int_{(0, \infty)} \int_{(0, \infty)} \mathbbm{1}_{\{x + z^{-1/\beta} > u\}} \mathbbm{1}_{[0, u]}(x) \mathbbm{1}_{\{x + z^{-1/\beta} > s\}} \mathbbm{1}_{[0, s]}(x) dz dx
$$

for $0 \leq s \leq u$, we conclude that $V_\beta^*$ is a version of $V_\beta$. 

The discussion above does not give a clue on where equality (4.1) comes from. Here is a non-rigorous argument based on the idea from [14] which allows one to guess (4.1). We start with an integral representation

\[
\sum_{k \geq 0} \left( \mathbb{1}_{\{S_k \leq ut < S_{k+1}\}} - (1 - F(ut - S_k)) \mathbb{1}_{\{S_k \leq ut\}} \right) \frac{\sqrt{-1} \int_0^t (1 - F(y))dy}{\mu^{-1} \int_0^t (1 - F(y))dy} = \int_{[0,u]} \int_{[0,\infty)} \mathbb{1}_{\{x+z>u\}} \frac{\sum_{k=1}^{\nu(xt)} \mathbb{1}_{\{\eta_k \leq zt\}} - \nu(xt)F(zt)}{\sqrt{\mu^{-1} \int_0^t (1 - F(y))dy}} \, dx \, dz
\]

where \( \nu(t) = \inf\{k \in \mathbb{N} : S_k > t\} \) for \( t \geq 0 \). It is likely that

\[
\sum_{k=1}^{[xt]} \mathbb{1}_{\{\eta_k \leq zt\}} - [xt]F(zt) \int_{[0,u]} \int_{[0,\infty)} \mathbb{1}_{\{x+z>u\}} \frac{\sum_{k=1}^{\nu(xt)} \mathbb{1}_{\{\eta_k \leq zt\}} - \nu(xt)F(zt)}{\sqrt{\mu^{-1} \int_0^t (1 - F(y))dy}} \, dx \, dz
\]

converges weakly as \( t \to \infty \) to \( B(x, z^{-\beta}) \) on some appropriate space of functions \( g : [0, \infty) \times [0, \infty) \to \mathbb{R} \) equipped with some topology which is strong enough to ensure continuity of composition. The latter together with (2.1) and the well-known relation \( t^{-1/2} \nu(t) \to \mu^{-1/2} \) as \( t \to \infty \) should entail that

\[
\sum_{k=1}^{[xt]} \mathbb{1}_{\{\eta_k \leq zt\}} - [xt]F(zt) \int_{[0,u]} \int_{[0,\infty)} \mathbb{1}_{\{x+z>u\}} \frac{\sum_{k=1}^{\nu(xt)} \mathbb{1}_{\{\eta_k \leq zt\}} - \nu(xt)F(zt)}{\sqrt{\mu^{-1} \int_0^t (1 - F(y))dy}} \, dx \, dz
\]

converges weakly to \( \sqrt{1 - \beta} B(x, z^{-\beta}) \). One may expect that the right-hand side of (4.2) converges weakly to the right-hand side of (4.1). On the other hand, the left-hand side of (4.2) converges weakly to \( V_\beta \) by Theorem 1.1.

5 Appendix

The following result can be found in the proof of Lemma 7.3 in [1].

**Lemma 5.1.** Let \( G : [0, \infty) \to [0, \infty) \) be a locally bounded function. Then, for any \( l \in \mathbb{N} \)

\[
\mathbb{E} \left( \sum_{k \geq 0} G(t - S_k) \mathbb{1}_{\{S_k \leq t\}} \right)^l \leq \left( \sum_{j=0}^{[t]} \sup_{y \in [j,j+1]} G(y) \right)^l \mathbb{E}(\nu(1))^l, \quad t \geq 0. \tag{5.1}
\]

The second auxiliary result is well-known. See, for instance, Theorem 2.1 (b) in [9]. It is of principal importance here that \( \xi \) is a.s. positive rather than nonnegative.

**Lemma 5.2.** For all \( a > 0 \) and all \( t > 0 \) \( \mathbb{E} e^{at \nu(t)} < \infty \).

Also, we need a classical strong approximation result, see Corollary 3.1 (ii) in [5].

**Lemma 5.3.** Suppose that \( \mathbb{E} \xi^r < \infty \) for some \( r > 2 \). Then there exists a standard Brownian motion \( W \) such that

\[
\lim_{t \to \infty} t^{-1/r} \sup_{0 \leq s \leq t} \left| \nu(s) - \frac{\mu^{-1} s - \sigma \mu^{-3/2} W(s)}{\mu^{-1/2}} \right| = 0 \quad \text{a.s.,}
\]

where \( \mu = \mathbb{E} \xi \) and \( \sigma^2 = \text{Var} \xi \).
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