Research Article

The Effects of Harvesting on the Dynamics of a Leslie–Gower Model

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Received 15 January 2021; Accepted 24 April 2021; Published 7 May 2021

Academic Editor: Rigoberto Medina

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In this paper, we study a Leslie–Gower predator-prey model with harvesting effects. We carry out local bifurcation analysis and stability analysis. Under certain conditions, the model is shown to undergo a supercritical Hopf bifurcation resulting in a stable limit cycle. Numerical simulations are presented to illustrate our theoretic results.

1. Introduction

In this paper, we consider Leslie–Gower predator-prey model with harvesting effect,

\[
\begin{align*}
\frac{dx_1}{dt} &= rx_1 \left(1 - \frac{x_1}{k} \right) - \frac{a_1 x_1 x_2}{n + x_1} - e_1 x_1, \\
\frac{dx_2}{dt} &= sx_2 \left(1 - \frac{a_2 x_2}{n + x_1} \right) - e_2 x_2,
\end{align*}
\]

with \(x_1(0) \geq 0\) and \(x_2(0) \geq 0\), where \(x_1(t)\) and \(x_2(t)\) are the prey and predator population densities, respectively, \(r, s, a_1, a_2, n, e_1, e_2 > 0\), and \(t\) is the time.

Note that \((a_1x_1/(n + x_1))\) is Leslie–Gower term in which the carrying capacity of the predator’s environment is a linear function of the prey size \((x_1/a_2) + (n/a_2)\). \((a_1x_1x_2/(n + x_1))\) is the number of prey consumed by the predator in unit time which shows that when the number of the prey \(x_1\) is severe scarcity, and the predators can switch over to other populations as food. Constants \(r\) and \(s\) are the intrinsic growth rate of the prey and predator, respectively, and \(e_1\) and \(e_2\) denote the harvesting efforts for the prey and predator, respectively.

Since the first prey-predator dynamical models which is the Lotka–Volterra model was built in the 1920s by Mathematician Lotka and Volterra, more and more researchers are interested in such issues, and they start from different angles to think the problem and many important results have been obtained [1–10]. In particular, in 2003, Aziz-Alaoui and Daher Okiye [11] considered the following Leslie–Gower predator-prey model:

\[
\begin{align*}
\frac{dx_1}{dt} &= rx_1 \left(1 - \frac{x_1}{k} \right) - \frac{a_1 x_1 x_2}{n + x_1}, \\
\frac{dx_2}{dt} &= sx_2 \left(1 - \frac{a_2 x_2}{n + x_1} \right) - e_2 x_2,
\end{align*}
\]

where \(x_1\) is the numbers of prey and \(x_2\) is the numbers of predators. Existence and stability of the fixed points were studied by using the Lyapunov function. In 2006, Lin and Ho [12] discussed the local and global stability for system (2) by using Poincaré–Bendixson theorem and Dulac’s criterion.

Harvesting is an effective way for humans to control the size of predators and prey so that the population has continued to develop healthily and produced good economic benefits [13–16]. Academically, researchers often only consider the harvesting of prey in order to control the size of the population. In 2010, Zhu and Lan [17] investigated the Leslie–Gower predator-prey systems:
\[
\begin{align*}
\frac{dx_1}{dt} &= r x_1 \left( 1 - \frac{x_1}{k} \right) - a x_1 x_2 - h, \\
\frac{dx_2}{dt} &= r x_2 \left( 1 - \frac{x_2}{b x_1} \right).
\end{align*}
\] (3)

In 2013, Gupta and Chandra [18] discussed the following Leslie–Gower predator-prey model with harvesting on the prey and the environment providing the same protection to both the predator and prey:

\[
\begin{align*}
\frac{dx_1}{dt} &= r x_1 \left( 1 - \frac{x_1}{k} \right) - a_1 x_1 x_2 - \frac{q E x_1}{n + x_1} - m_1 E + m_2 x_1, \\
\frac{dx_2}{dt} &= s x_2 \left( 1 - \frac{a x_2}{n + x_1} \right).
\end{align*}
\] (4)

For ecological balance and healthy economic development, for fisheries, wildlife resources, etc., we not only need to consider the harvesting of prey, but also the predator. Therefore, in this paper, we study Leslie–Gower predator-prey model (1) with harvesting on the prey and predator.

In order to make the system dimensionless, we define new dependent variables by \( x = (x_1/k), \ y = (a x_1 x_2/k) \) and a new time variable by \( t = rt \). Moreover, let \( a = (1/r), \ \beta = (a_2/a_1), \ m = (n/k), \ \rho = (s/r), \ \delta = (e_2/(ra_1)). \) System (1) becomes

\[
\begin{align*}
\frac{dx}{dt} &= x (1 - x) - \frac{a x y}{m + x} - \gamma x, \\
\frac{dy}{dt} &= \rho y \left( 1 - \frac{\beta y}{m + x} \right) - \delta y,
\end{align*}
\] (5)

where \( a, \beta, \gamma, \rho, \delta, \) and \( m \) are all positive parameters.

**Lemma 1** (see [19]). If \( a > 0, \ b > 0, \) and \( \dot{x} \leq x(b - ax^a), \) where \( a \) is a positive constant, \( t \geq 0, \) and \( x(0) > 0, \) we have

\[
x(t) \leq \left( \frac{b}{a} \right)^{(1/a)} \left( 1 + \frac{b x^{a-1}(0)}{a} e^{-bt} \right)^{-\frac{1}{a}}.
\] (6)

**Lemma 2** (see [20, 21]). Consider system \( \dot{x} = f(x, a) \) and suppose that \( f(X_0, a_0) = 0, \) \( n \times n \) Jacobian matrix \( f \) has a simple eigenvalue \( s = 0 \) with eigenvector \( V, \) and the transpose of the Jacobian matrix \( f^T \) has an eigenvector \( W \) to the eigenvalue \( s = 0. \) Then, the system \( \dot{x} = f(x, a) \) experiences a transcritical bifurcation at the equilibrium point \( X_0 \) as the control parameter \( a \) passes through the bifurcation value \( a = a_0 \) if the following conditions are satisfied:

\[
W^T Df(0, a_0) = 0,
\]

\[
W^T (Df(0, a_0)V) \neq 0,
\]

\[
W^T (D^2 f(0, a_0)(V, V)) = 0.
\] (7)

The rest of this paper is organized as follows. In Section 2, we study boundary of solutions. In Section 3, we discuss existence of equilibria points. In Section 4, we discuss stability of the equilibrium points.

### 2. Boundedness of Solutions

In this section, we prove that every solution of system (5) is positive and uniformly bounded with initial conditions \( x(0) = x_0 > 0 \) and \( y(0) = y_0 > 0. \) Denote \( \mathbb{R}^2_+ = \{(x(t), y(t)) \in \mathbb{R}^2 | x(t) > 0, y(t) > 0, t > 0\}. \)

**Theorem 1.** Consider system (5). For any given initial conditions \((x_0, y_0) \in \mathbb{R}^2_+, \) the solution \((x(t), y(t))\) of system (5) exists and is unique and positive and ultimate bounded.

**Proof.** Let function \( f(x, y) = (x(1 - x(t)) - (ax(t)y(t)/(m + x(t))) - yx(t), py(t)(1 - (\beta y(t)/(m + x(t))) - \delta y(t)). \) Obviously, function \( f(x, y) \) is continuous differentiable on \((x, y) \in \mathbb{R}^2_+, \) so for any given the initial conditions \((x_0, y_0) \in \mathbb{R}^2_+, \) the solution \((x(t), y(t))\) of the system (5) exists and is unique. Furthermore, \(x-\)axis and \(y-\)axis are the solutions of the system (5); by the uniqueness of the solution, the solutions \((x(t), y(t))\) of the system (5) with the initial value \(x_0 > 0, y_0 > 0\) cannot cross with \(x-\)axis and \(y-\)axis.

Next, we show the solutions \((x(t), y(t))\) of system (5) with the initial value \(x_0 > 0\) and \(y_0 > 0\) which is ultimate bounded.

From system (5), we have

\[
\frac{dx}{dt} \leq x(1 - x).
\] (8)

Combining Lemma 1, we have

\[
x(t) \leq \left( 1 + (x^{-1}(0) - 1)e^{-t} \right) \leq M_0,
\]

\[
M_0 = \max \{1, x_0\}.
\] (9)

So, the following inequality is established:

\[
\frac{dy}{dt} \leq \rho y \left( 1 - \frac{\beta y}{m + x} \right) \leq \rho y \left( 1 - \frac{\beta y}{m + 1} \right) = y \left( \rho - \rho \frac{\beta y}{m + 1} \right).
\] (10)

By Lemma 1, we have

\[
y(t) \leq M_1, \quad t \geq 0.
\] (11)

where \( M_1 = \max \{((m + M_0)/\beta), y_0\}. \)

### 3. Existence of Equilibria

In order to find the equilibrium points of system (5), we let \( f(x, y) = 0, \) i.e.,

\[
\begin{align*}
&x(1 - x) - \frac{a x y}{m + x} - \gamma x = 0, \\
&\rho y \left( 1 - \frac{\beta y}{m + x} \right) - \delta y = 0.
\end{align*}
\] (12)
It is clear that equation (12) has a trivial solution $E_0 = (0, 0)$. Furthermore, by calculation, we find other solutions of equation (12):

$$
(1 - \gamma, 0),
$$

$$
(0, \frac{(\rho - \delta)m}{\rho^2}),
$$

$$
\left(\frac{(1 - \gamma)(\rho^2 - (\rho - \delta)\alpha)(\rho - \delta)((1 + m - \gamma)\rho^2 - (\rho - \delta)\alpha)}{\rho^2 - c}, \frac{(\rho - \delta)m}{\rho^2 - c}\right),
$$

(13)

Remark 1. When $\gamma < 1, \delta < \rho$ and $(\rho - \delta)\alpha < (1 - \gamma)\rho\beta$, the position of each point in (13) is shown in Figure 1.

Therefore, we have the following results where

$$x_{\infty} = \frac{((1 - \gamma)\rho\beta - (\rho - \delta)\alpha)}{(\rho - \delta\alpha), y_{\infty} = \frac{((\rho - \delta)((1 + m - \gamma)\rho^2 - (\rho - \delta)\alpha))}{(\rho^2\beta^2}).}
$$

**Theorem 2.** Consider system (5) admits $x$-axial only and $y$-axial equilibria under following conditions.

(i) The $x$-axial equilibrium,

$$E_1 = (K, 0), \text{ with } K = 1 - \gamma,
$$

is a boundary equilibrium of system (5) if and only if

$$1 - \gamma > 0.
$$

(ii) The $y$-axial equilibrium,

$$E_2 = (0, M), \text{ with } M = \frac{(\rho - \delta)m}{\rho^2},
$$

is a boundary equilibrium of system (5) if and only if

$$\rho - \delta > 0.
$$

**Theorem 3.** Consider system (5) admits a unique positive equilibrium,

$$E_3 := (x_{\infty}, y_{\infty}),
$$

if only if

$$\frac{1 - \gamma}{\alpha} > \frac{\rho - \delta}{\rho^2} > 0,
$$

(19)

Remark 2. Existence regions for equilibrium points of system (12) is shown in Figure 2. Equilibrium points $E_1$ and $E_2$ exist, but positive equilibrium point $E_3$ does not in region II and equilibrium points $E_1, E_2$, and $E_3$ coexist in region I. Moreover, if $((1 - \gamma)/\alpha) = ((\rho - \delta)/\rho\beta) > 0$, then positive equilibrium $E_3$ becomes boundary equilibrium $E_2$.

4. Stability of Equilibrium

**Theorem 4.** If $1 - \gamma > 0$ and $\rho - \delta > 0$, then the equilibria $E_0$ and $E_1$ are unstable.

**Proof.** Firstly, we show the equilibria $E_0$ is an unstable equilibrium. The Jacobian matrix about $E_0$ is given by

$$J_{E_0} = \begin{pmatrix} 1 - \gamma & 0 \\ 0 & \rho - \delta \end{pmatrix},
$$

with trace $1 - \gamma + \rho - \delta$ and determinant $(1 - \gamma)(\rho - \delta)$. From $1 - \gamma > 0$ and $\rho - \delta > 0$, we have $1 - \gamma + \rho - \delta > 0$ and $(1 - \gamma)(\rho - \delta) > 0$. On the contrary, through direct calculation, we obtain $(1 - \gamma + \rho - \delta)^2 - 4(1 - \gamma)(\rho - \delta) = (1 - \gamma + \rho + \delta)^2 > 0$. Therefore, $E_0$ is an unstable node point.

Secondly, we show the equilibria $E_1$ is a unstable equilibrium. The Jacobian matrix about $E_1$ is given by
Theorem 5. (1) If $((1 - γ)/α) > ((ρ - δ)/(ρβ)) > 0$, then the boundary equilibrium $E_2$ is a saddle point which is unstable. (2) If $0 < ((1 - γ)/α) < ((ρ - δ)/(ρβ))$, then the boundary equilibrium $E_2$ is locally asymptotically stable.

Proof. The Jacobian matrix about $E_2$ is given by

$$J_{E_2} = \begin{pmatrix} (1 - γ) - \frac{α(ρ - δ)}{ρβ} & 0 \\ \frac{(ρ - δ)^2}{ρβ} & -(ρ - δ) \end{pmatrix},$$

with trace $(1 - γ) - ((α(ρ - δ))/ρβ) - (ρ - δ)$ and determinant $-(ρ - δ)((1 - γ) - ((α(ρ - δ))/ρβ))$.

(1) When $((1 - γ)/α) > ((ρ - δ)/(ρβ)) > 0$, we have $-((ρ - δ)(ρβ(1 - γ) - α(ρ - δ))/ρβ) < 0$, so $E_2$ is a saddle point which is unstable.

(2) When $0 < ((1 - γ)/α) < ((ρ - δ)/(ρβ))$, we have $(1 - γ) - ((α(ρ - δ))/ρβ) - (ρ - δ) = (ρβ(1 - γ) - α(ρ - δ))/ρβ - (ρ - δ) < 0$ and $-((ρ - δ)(ρβ(1 - γ) - α(ρ - δ))/ρβ) > 0$. Furthermore, a direct calculations gives

$$\left((1 - γ) - \frac{α(ρ - δ)}{ρβ} - (ρ - δ)\right)^2 + 4(ρ - δ)(1 - γ) - \frac{α(ρ - δ)}{ρβ} > 0.$$

Therefore, $E_2$ is locally asymptotically stable.

Theorem 6. (1) If $0 < ((1 - γ)/α) < ((ρ - δ)/(ρβ))$ and $m < ((-2x_{c0}^2 + (1 - γ - ρ + δ)x_{c0})/(x_{c0} + ρ - δ))$ hold, then the positive equilibrium $E_3$ is unstable. (2) If $0 < ((1 - γ)/α) < ((ρ - δ)/(ρβ))$ and $m > ((-2x_{c0}^2 + (1 - γ - ρ + δ)x_{c0})/(x_{c0} + ρ - δ))$ hold, then the positive equilibrium $E_3$ is locally asymptotically stable. Furthermore, assume that there is $α < ρβ$, then the positive equilibrium $E_3$ is globally asymptotically stable.

Proof. The Jacobian matrix about $E_3$ is given by

$$J_{E_3} = \begin{pmatrix} x_{c0} \left(1 + \frac{1 - γ - x_{c0}}{m + x_{c0}}\right) - \frac{αx_{c0}}{m + x_{c0}} \\ \frac{(ρ - δ)^2}{ρβ} - (ρ - δ) \end{pmatrix},$$

with trace $x_{c0} \left(1 + \frac{1 - γ - x_{c0}}{m + x_{c0}}\right) - \frac{αx_{c0}}{m + x_{c0}} - (ρ - δ)$ and determinant $-x_{c0}(ρ - δ) \left(1 + \frac{1 - γ - x_{c0}}{m + x_{c0}}\right) - \frac{α(ρ - δ)}{ρβ(m + x_{c0})} > 0$.

(1) From conditions $0 < ((1 - γ)/α) < ((ρ - δ)/(ρβ))$ and $m < ((-2x_{c0}^2 + (1 - γ - ρ + δ)x_{c0})/(x_{c0} + ρ - δ))$, we obtain

$$x_{c0} \left(1 + \frac{1 - γ - x_{c0}}{m + x_{c0}}\right) - (ρ - δ) > 0.$$

Therefore, the positive equilibrium $E_3$ is unstable.

(2) From conditions $0 < ((1 - γ)/α) < ((ρ - δ)/(ρβ))$ and $m > ((-2x_{c0}^2 + (1 - γ - ρ + δ)x_{c0})/(x_{c0} + ρ - δ))$, we have

$$x_{c0} \left(1 + \frac{1 - γ - x_{c0}}{m + x_{c0}}\right) - (ρ - δ) < 0.$$
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5. Local Bifurcation

hold, then system (5) undergoes a Hopf bifurcation with respect to bifurcation parameter \( m \) around the equilibrium point \( E_3 = (x_{so}, y_{so}) \). Furthermore, the direction of the Hopf bifurcation is subcritical and the bifurcation periodic solutions are orbitally asymptotically stable if

\[
k_0 + k_1 + k_2 + k_3 < 0. \tag{41}
\]

Combining (33) and (34), we have

\[
W^T[Df_a(x_2, y_2)V] = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = -1 \neq 0. \tag{35}
\]

Then, we compute \( D^2 f(x_2, y_2, \alpha)(V, V) \) as follows:

\[
D^2 f(x_2, y_2, \alpha)(V, V) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} v_1 v_1 + 2 \frac{\partial^2 f}{\partial x \partial y} v_1 v_2 + \frac{\partial^2 f}{\partial y^2} v_2 v_2 \\ \frac{\partial^2 f}{\partial y^2} v_1 v_1 + 2 \frac{\partial^2 f}{\partial x \partial y} v_1 v_2 + \frac{\partial^2 f}{\partial y^2} v_2 v_2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2k_0 \end{pmatrix}. \tag{36}
\]

Combining (33) and (36), we have

\[
\text{Theorem 7.} \quad \text{If } ((1-\gamma)/\alpha) = ((\rho - \delta)/\rho \beta) > 0, \text{ then system (5) undergoes a transcritical bifurcation around } E_2.
\]

Proof. From \(((1-\gamma)/\alpha) = ((\rho - \delta)/\rho \beta) > 0, \text{ we have} \)

\[
\alpha = \frac{(1-\gamma)\rho \beta}{\rho - \delta}. \tag{29}
\]

Therefore,

\[
f_a(x, y) = \begin{pmatrix} \frac{-xy}{m+x} \\ 0 \end{pmatrix}. \tag{30}
\]

We have

\[
f_a(x_2, y_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{31}
\]

The eigenvectors \( V \) and \( W \) associated to zero eigenvalues of matrices \( J \) and \( J^T \), respectively, are

\[
V = \begin{pmatrix} \rho \beta \\ \rho - \delta \end{pmatrix}, \quad W = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{32}
\]

So, \( W^T f_a(x_2, y_2) = 0 \).

In addition, \( Df_a(x_2, y_2)V \) equals

\[
Df_a(x_2, y_2)V = \begin{pmatrix} \frac{\partial(-xy/(m+x))}{\partial x} v_1 + \frac{\partial(-xy/(m+x))}{\partial y} v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{34}
\]
Since all three conditions of Lemma 2 are satisfied, system (5) undergoes a transcritical bifurcation around $E_2$ if

$$((1 - \gamma)/\alpha) = ((\rho - \delta)/\rho \beta) > 0.$$ 

Let

$$\alpha \beta (1 - \gamma) < (2 \alpha - \rho \beta) (\rho - \delta),$$

$$m_H = \frac{-2x_\infty^2 + (1 - \gamma - \rho + \delta)x_\infty}{x_\infty + \rho - \delta}.$$  

The direction of the Hopf bifurcation is supercritical and the bifurcation periodic solutions are unstable if

$$k_0 + k_1 + k_2 + k_3 > 0.$$  

Proof. From (39), we have

$$2x_\infty < 1 - \gamma - \rho + \delta.$$ 

So, $m_H > 0$.

The Jacobian matrix of system (5) evaluated at the point $E_3$ is given by

$$W^T [D^2 f_a(x_2, y_2)(V, V)] = \begin{pmatrix} -2 & 0 \\ \frac{2 \rho \beta}{m} & 0 \end{pmatrix} = -2 \neq 0.$$  

(37)

$$k_0 = \frac{\rho - \delta}{8(m + x_\infty)} \left( \frac{2(m - 1)}{m + x_\infty} - \frac{3a m}{\rho \beta (m + x_\infty)^3} + \frac{2 \rho \beta}{m} \right),$$

$$k_1 = \frac{1}{8} \left( \frac{m}{x_\infty (m + x_\infty)} + \frac{m^2 (\rho - \delta)}{x_\infty (m + x_\infty)^2} \right) - \frac{am^2 (\rho - \delta)}{\rho \beta x_\infty (m + x_\infty)^3},$$

$$k_2 = \frac{1}{8} \left( \frac{\rho \beta (\rho - \delta)}{m + x_\infty} \left( \frac{2(m + x_\infty)}{\alpha} - \frac{m + 2}{\rho \beta} \right) \left( 1 - \frac{\rho \beta}{\alpha x_\infty} \right) \left( \frac{m + x_\infty}{\alpha} - \frac{1}{\rho \beta} \right) \right).$$

(38)

$$k_3 = \frac{1}{4} \left( 1 + \frac{am (\rho - \delta)}{(m + x_\infty)^2} \left( \frac{m + x_\infty}{\alpha} - \frac{1}{\rho \beta} \right) \right) \left( 1 + \frac{am (\rho - \delta)}{(m + x_\infty)^2} \left( \frac{m + x_\infty}{\alpha} - \frac{1}{\rho \beta} \right) \right).$$

Theorem 8. Assume that $0 < ((1 - \gamma)/\alpha) < ((\rho - \delta)/\rho \beta)$ is satisfied. If

$$\rho \beta (1 - \gamma) < (2 \alpha - \rho \beta) (\rho - \delta),$$

$$m_H = \frac{-2x_\infty^2 + (1 - \gamma - \rho + \delta)x_\infty}{x_\infty + \rho - \delta}.$$  

The trace $z = \text{Tr}(J_{E_3})$ and the determinant $D = \text{Det}(J_{E_3})$ of Jacobian matrix $J_{E_3}$ are given by

$$z = -x_\infty + x_\infty \left( \frac{1 - \gamma - x_\infty}{m + x_\infty} \right) - (\rho - \delta)$$

$$= \frac{-2x_\infty^2 + (1 - \gamma - m)x_\infty}{m + x_\infty} - (\rho - \delta),$$

$$D = -x_\infty (\rho - \delta) \left( 1 + \frac{1 - \gamma - x_\infty}{m + x_\infty} \right) + \left( \frac{\rho - \delta}{\rho \beta} \right) \frac{ax_\infty}{m + x_\infty}$$

$$= x_\infty (\rho - \delta) (\rho \beta m - (\rho - \delta) x_\infty + (1 - \gamma) \rho \beta)$$

$$= x_\infty (\rho - \delta).$$  

(45)
By $0 < (1 - \gamma)/\alpha < ((\rho - \delta)/\rho\beta$ and (40), we have $z|_{mH} = 0$ and $D|_{mH} > 0$.

In addition, we have

$$z|_{mH} = 0$$

By (46), we have

$$z|_{mH} < 0.$$  

In this case, this guarantees the existence of Hopf bifurcation around $E_3$.

We translate the equilibrium $E_3$ to the origin by the translation

$$\begin{align*}
\bar{x} &= x - x_{\infty}, \\
\bar{y} &= y - y_{\infty}.
\end{align*}$$

For the sake of convenience, we still denote $\bar{x}$ and $\bar{y}$ by $x$ and $y$, respectively. So, the system (5) becomes

$$\begin{align*}
\dot{x} &= (x + x_{\infty})(1 - x - x_{\infty}) - \frac{\alpha(x + x_{\infty})(y + y_{\infty})}{m + x + x_{\infty}} - y(x + x_{\infty}), \\
\dot{y} &= \rho(y + y_{\infty})\left(1 - \frac{\beta(y + y_{\infty})}{m + x + x_{\infty}}\right) - \delta(y + y_{\infty}).
\end{align*}$$  

(47)

Rewrite system (47) to

$$\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = J_{E_3} \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
f_1(x, y, m) \\
f_2(x, y, m)
\end{pmatrix},$$  

(48)

where

$$\begin{align*}
f_1(x, y, m) &= (x + x_{\infty})(1 - x - x_{\infty}) - \frac{\alpha(x + x_{\infty})(y + y_{\infty})}{m + x + x_{\infty}} - y(x + x_{\infty}) \\
&\quad - x_{\infty}\left(1 + \frac{1 - y - x_{\infty}}{m + x_{\infty}}\right)x + \frac{\alpha x_{\infty}}{m + x_{\infty}}y, \\
f_2(x, y, m) &= \rho(y + y_{\infty})\left(1 - \frac{\beta(y + y_{\infty})}{m + x + x_{\infty}}\right) - \delta(y + y_{\infty}) \\
&\quad - \frac{(\rho - \delta)^2}{\rho\beta}x(\rho - \delta)y.
\end{align*}$$  

(49)

Denote the eigenvalues of $J_{E_3}$ by $\phi + i\omega$ with $\phi = (z/2)$ and $\omega = (\sqrt{4D - z^2}/2)$.

Define matrix

$$P = \begin{pmatrix}
1 & 0 \\
N & G
\end{pmatrix},$$  

(50)
where $G = (\omega(m + x_\infty)) / ax_\infty$ and $N = ((m + x_\infty)(x_\infty(-1 + ((1 - \gamma - x_\infty)/(m + x_\infty)) - \phi)) / ax_\infty)$. Obviously,

$$P^{-1} = \begin{pmatrix} 1 & 0 \\ N & 1 \\ G \\ G \end{pmatrix},$$

(51)

and when $m = m_H$, i.e.,

$$-2x_\infty^2 + (1 - \gamma - m_H)x_\infty = (\rho - \delta)(x_\infty + m_H),$$

$$\omega_0 = \omega_{m_H} = \sqrt{(\rho - \delta)x_\infty},$$

$$N_0 = N|m_H = \frac{(m_H + x_\infty)(-1 + ((1 - \gamma - x_\infty)/(m_H + x_\infty)))}{\alpha}$$

$$= \frac{(m_H + x_\infty)(\rho - \delta)}{\alpha},$$

$$G_0 = G|m_H = \frac{\omega_0(m_H + x_\infty)}{ax_\infty},$$

$$\frac{N_0}{G_0} = \frac{(\rho - \delta)x_\infty}{\omega_0} = \omega_0,$$

$$\frac{1}{G_0} = \frac{ax_\infty}{\omega_0(m_H + x_\infty)}.$$  

(52)

By the transformation,

$$\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ Nu + Gv \end{pmatrix},$$

(53)

system (48) becomes

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \phi - \omega \\ \omega - \phi \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F_1(u, v, m) \\ F_2(u, v, m) \end{pmatrix},$$

(54)

where

$$F_1(u, v, m) = A_{20}u^2 + A_{11}uv + A_{21}u^2v + A_{30}u^3$$

$$+ o(|u|^4, |u|^4|v|),$$

$$F_2(u, v, m) = B_{20}u^2 + B_{11}uv + B_{02}v^2 + B_{21}u^2v$$

$$+ o(|u|^4, |u|^4|v|),$$

(55)

with

$$A_{20} = -1 - \frac{am(Nx_\infty + Nm - y_\infty)}{(m + x_\infty)^4},$$

$$A_{11} = \frac{agm}{(m + x_\infty)^2},$$

$$A_{21} = \frac{agm}{(m + x_\infty)^3},$$

$$A_{30} = \frac{am(Nx_\infty + Nm - y_\infty)}{(m + x_\infty)^5},$$

$$B_{20} = \frac{N}{G}\left(1 + \frac{am(Nx_\infty + Nm - y_\infty)}{(m + x_\infty)^4}\right) - \frac{1}{G} \frac{\rho \beta (N^2x_\infty^2 + 2Nx_\infty m - 2Nx_\infty y_\infty + (Nm - y_\infty)^2)}{(m + x_\infty)^2},$$

$$B_{11} = \frac{aNm}{(m + x_\infty)^2} - \frac{2\rho \beta Nm - y_\infty}{(m + x_\infty)^2},$$

$$B_{21} = \frac{2\rho \beta (Nx_\infty + Nm - y_\infty) - aNm}{(m + x_\infty)^3}.$$  

(56)

In order to determine the stability of the periodic solution, we need to calculate the sign of the coefficient $b(m_H)$, which is given by

$$b(m_H) = \frac{1}{16} \left[ F_{1uv} + F_{1vv} + F_{2uv} + F_{2vv} \right]$$

$$+ \frac{1}{16\omega(m_H)} \left[ F_{1uv} (F_{1uv} + F_{1vv}) - F_{2uv} (F_{2uv} + F_{2vv}) \right]$$

$$- F_{1uv}^2 + 2F_{1uv}F_{2uv},$$

(57)

where all partial derivatives are evaluated at the bifurcation point $(0, 0, m_H)$.

Combining $F^1(u, v), F^2(u, v), \omega_0, G_0,$ and $N_0$, we have
\[ F_{1uvv}(0,0) = \frac{6m(\rho - \delta)}{(m + x_{\infty})^2} \frac{6am(\rho - \delta)}{\rho \beta (m + x_{\infty})^3}, \]

\[ F_{1uv}(0,0) = 0, \]

\[ F_{1uv}(0,0) = -\frac{m\omega_0}{x_{\infty}(m + x_{\infty})}, \]

\[ F_{1uu}(0,0) = -2 - 2am \left( \frac{\rho - \delta}{\alpha (m + x_{\infty})} - \frac{\rho - \delta}{\rho \beta (m + x_{\infty})^2} \right), \]

\[ F_{2vv}(0,0) = 0, \]

\[ F_{2vv}(0,0) = 0, \]

\[ F_{2uvv}(0,0) = \frac{4\rho \beta (\rho - \delta)}{\alpha (m + x_{\infty})} \frac{(2m + 4)(\rho - \delta)}{(m + x_{\infty})^2}, \]

\[ F_{2uv}(0,0) = \frac{4m (\rho - \delta)}{(m + x_{\infty})^2}, \]

\[ F_{2uu}(0,0) = 2\omega_0 \left( 1 + \frac{am(\rho - \delta)}{(m + x_{\infty})^2} \left( \frac{m + x_{\infty}}{\alpha} - \frac{1}{\rho \beta} \right) \right) - \frac{am(\rho - \delta)}{(m + x_{\infty})^2} \frac{x_{\infty} \rho \beta (m + x_{\infty})}{\alpha^2}, \]

\[ F_{2vv}(0,0) = -2\rho \beta \omega_0 \frac{\alpha}{x_{\infty}}, \]

\[ F_{2uv}(0,0) = \frac{2\rho \beta \omega_0}{\alpha x_{\infty}}, \]

where \( m = m_{1\ell} \) and

\[ b(m_{1\ell}) = \frac{\rho - \delta}{8(m + x_{\infty})} \left( \frac{2(m - 1)}{m + x_{\infty}} - \frac{3am}{\rho \beta (m + x_{\infty})^2} + \frac{2\rho \beta}{\alpha} \right) \]

\[ + \frac{1}{8} \left( \frac{m}{x_{\infty}(m + x_{\infty})} + \frac{m^2(\rho - \delta)}{x_{\infty}(m + x_{\infty})^2} - \frac{2am(\rho - \delta)}{\rho \beta x_{\infty}(m + x_{\infty})} \right) \]

\[ + \frac{\rho \beta (\rho - \delta)}{m + x_{\infty}} \left( \frac{2(m + x_{\infty})}{\alpha} - \frac{m + 2}{\rho \beta} \right) \]

\[ \times \left( 1 - \frac{\rho \beta}{\alpha x_{\infty}} + \frac{am(\rho - \delta)}{(m + x_{\infty})^2} \left( \frac{m + x_{\infty}}{\alpha} - \frac{1}{\rho \beta} \right) \right) \]

\[ - \frac{am(\rho - \delta)}{(m + x_{\infty})^2} \left( \frac{m + x_{\infty}}{\alpha} - \frac{1}{\rho \beta} \right)^2 \]

\[ + 2 \left( 1 + \frac{am(\rho - \delta)}{(m + x_{\infty})^2} \left( \frac{m + x_{\infty}}{\alpha} - \frac{1}{\rho \beta} \right) \right) \left( 1 + \frac{am(\rho - \delta)}{(m + x_{\infty})^2} \frac{m + x_{\infty} - 1}{\alpha \rho \beta} \right) \]

\[ = k_0 + k_1 + k_2 + k_3. \]
Combining (41), we have \( b(m_H) < 0 \). Therefore, according to Poincare–Andronow’s Hopf bifurcation theory, we have the direction of the Hopf bifurcation is subcritical and the bifurcation periodic solutions are orbitally asymptotically stable.

In addition, combining (42), we have \( b(m_H) > 0 \). Therefore, according to Poincare–Andronow’s Hopf bifurcation theory, we have the direction of the Hopf bifurcation is supercritical and the bifurcation periodic solutions are unstable.

\[ \square \]

6. Numerical Illustrations

In this section, we perform numerical simulations about system (5). Figure 3 shows that \( E_0 \) is an unstable node point, \( E_1 \) is a saddle point, \( E_1 \) does not exist, and \( E_2 \) is asymptotically stable and every orbit tends to it. Figure 4 shows that \( E_0 \) is an unstable node point, \( E_1 \) is a saddle point, \( E_2 \) is unstable, \( E_3 \) is unstable, and there is a limit cycle around \( E_3 \) to which every orbit tends. Figure 5 shows that \( E_0 \) is an unstable node point, \( E_1 \) is unstable and \( E_2 \) is also unstable, but \( E_3 \) is asymptotically stable and every orbit approaches this equilibrium.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

This work was partially supported by National Natural Science Foundation of P.R.China (no. 11661037).

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