**THE PPW CONJECTURE IN CURVED SPACES**

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**Abstract.** In Euclidean ([AB92]) and Hyperbolic ([BL07]) space, and the hemisphere in $S^n$ ([AB01]), geodesic balls maximize the gap $\lambda_2 - \lambda_1$ of Dirichlet eigenvalues, among domains with fixed $\lambda_1$. We prove an upper bound on $\lambda_2 - \lambda_1$ for domains in manifolds with certain curvature bounds. The inequality is sharp on geodesic balls in spaceforms.

1. Introduction

In the ’90s Ashbaugh-Benguria [AB92] settled the following conjecture of Payne, Polya and Weinberger.

**Theorem 1.1** (PPW conjecture, [AB92]). Among all bounded domains in $R^n$, the round ball uniquely maximizes the ratio $\frac{\lambda_2}{\lambda_1}$ of first and second Dirichlet eigenvalues.

Payne-Polya-Weinberger [PPW56] originally bounded the ratio $\lambda_2/\lambda_1$ by 3. Their bound was subsequently improved by Brands [Bra64], de Vries [dV67], then Chiti [Chi83], until Ashbaugh-Benguria proved the sharp inequality, building on the work of Chiti and Talenti [Tal76]. For more history and references see [AB92].

Benguria-Linde [BL07] extended the PPW conjecture to hyperbolic space.

**Theorem 1.2** (PPW for hyperbolic space, [BL07]). Among all bounded domains in $H^n$ with the same fixed first Dirichlet eigenvalue $\lambda_1$, the geodesic ball maximizes $\lambda_2$.

(Without scaling, the appropriate inequality requires one to normalize competitors by $\lambda_1$.) Ashbaugh-Benguria [AB01] also extended the PPW conjecture to the hemisphere in $S^n$.

**Theorem 1.3** (PPW for hemispheres, [AB01]). Among all bounded domains in the hemisphere of $S^n$ with the same fixed Dirichlet eigenvalue $\lambda_1$, the geodesic ball maximizes $\lambda_2$.

Take $N^n$ the spaceform of constant curvature $k$. We define the function $s_k$ on $R$ by

$$s_k(r) = \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}r) & k > 0 \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}r) & k < 0 \end{cases}$$

The following isoperimetric inequality holds for any bounded domain $\Omega \subset N$:

(1) \[ |\partial\Omega| \geq A_{n,k}(|\Omega|), \]

with equality iff $\Omega$ is a geodesic ball (see [Sch44]).
Let $M^n$ be a complete, simply-connected $n$-manifold with $\text{Sect}_M \leq k$. Then for some $\alpha \leq 1$, $M$ satisfies an isoperimetric inequality

$$|\partial \Omega| \geq \alpha A_{n,k}(|\Omega|)$$

for any domain bounded $\Omega$. We assume throughout this paper that $\alpha > 0$, which is no real loss of generality as we only concern ourselves with a compact neighborhood of $\Omega$.

If $k \leq 0$ then $\Omega$ has a closed geodesic convex hull, which we write as $\text{hull}\Omega$. If $k > 0$, we impose the condition on $\Omega$ that we can find some strongly convex closed set, which we also write as $\text{hull}\Omega$, containing $\Omega$ and satisfying the following properties:

a) $\text{diam}\Omega = \text{diam}(\text{hull}\Omega) < \min\left\{\frac{\pi}{\sqrt{k}}, \text{injectivity radius of } M\right\}$; b) $|\text{hull}\Omega| < \frac{|N|}{2}$.

We extend Theorems 1.1, 1.2, 1.3 to prove the following inequality for the gap $\lambda_2 - \lambda_1$.

**Theorem 1.4.** Let $\Omega$ be a bounded domain in $M$, with first and second Dirichlet eigenvalues $\lambda_1, \lambda_2$. If $k > 0$ let $\Omega$ be such that some $\text{hull}\Omega$ exists. Let $B$ be a geodesic ball in $N$, with eigenvalues $\lambda_1^B, \lambda_2^B$. Let $B$ be normalized so that $\lambda_1^B = \alpha^2 \lambda_1$.

If $\text{Ric} \geq (n-1)K$ on $\text{hull}\Omega$, then

$$\lambda_2 - \lambda_1 \leq \left(\frac{\text{sn}_K(\text{diam}\Omega)}{\text{sn}_K(\text{diam}\Omega)}\right)^{2n-2} (\lambda_2^B - \lambda_1^B).$$

In particular, if $k = K$ then the constant factor is 1, and the inequality is sharp on geodesic balls.

On spaceforms (i.e. when $k = K$) Theorem 1.4 reduces to the sharp estimates in [AB92], [BL07], [AB01].

**Remark 1.5.** The constant factor is the ratio of areas of geodesics spheres:

$$\lambda_2 - \lambda_1 \leq \left(\frac{|\partial B_{\text{diam}\Omega}|_\tilde{N}}{|\partial B_{\text{diam}\Omega}|_N}\right)^2 (\lambda_2^B - \lambda_1^B).$$

Here $\tilde{N}$ is the spaceform of constant curvature $K$.

**Remark 1.6.** We emphasize that in many cases $\alpha$ can be explicitly computed. If $k = 0$, then Croke [Cro84] proved an isoperimetric relation

$$|\partial \Omega| \geq c_n |\Omega| \frac{\text{vol}}{n},$$

where $c_n$ is given by an integral formula of trigonometric functions. If $n = 4$ then in fact $c_n$ is the Euclidean constant, and so $\alpha = 1$.

More generally, the Hadamard conjecture implies that if $k \leq 0$, then $\alpha = 1$. The conjecture is known in the following case: $n = 2$, proved by Weil [Wei26] (for $k = 0$), and Aubin [Aub76] ($k < 0$); $n = 3$, proved by Kleiner [Kle92]; $n = 4$, proved by Croke [Cro84] when $k = 0$. Further, when $n = 4$ and $k < 0$, Kloeckner-Kuperberg [KK13] proved that domains in $M$ which are appropriately "small" (in a quantitative sense) satisfy the Hadamard conjecture. The problem is open for general $n$.

If the metric $g_M$ is $C^0$-close to $g_N$, then $\alpha$ can be written in terms of this bound.

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2. Preliminaries

Fix a \( q \in N \). For \( f : M \to R_+ \), we define the decreasing (resp. increasing) symmetrizations
\[
S^N f : N \to R, \quad S_N f : N \to R
\]
to be the decreasing (resp. increasing) function of \( r_q(x) = \text{dist}_N(q, x) \) fixed by the condition
\[
|(S^N f)^{-1}(t, \infty)|_N = |(S_N f)^{-1}(t, \infty)|_N = |f^{-1}(t, \infty)|_M \quad \forall t.
\]

If \( \text{spt}\ f \subset D \), then \( \text{spt}\ S^N f \) and \( \text{spt}\ S_N f \) are both contained in the closed ball \( S^N D \subset N \), centered at \( q \), satisfying \( |S^N D|_N = |D|_M \).

Proposition 2.1. For any \( p \geq 1 \), we have
\[
\|f\|_{L^p,M} = \|S^N f\|_{L^p,N} = \|S_N f\|_{L^p,N}.
\]

Proof. By Fubini’s theorem,
\[
\int_M f^p = p \int_0^\infty t^{p-1} |f > t|_M dt = p \int_0^\infty t^{p-1} |S^N f > t|_N dt = \int_N (S^N f)^p.
\]
The case of \( S_N f \) is verbatim. \( \square \)

Proposition 2.2. If \( \mu(t) = |f > t|_M \), and \( \text{spt}\ f \subset D \), then
\[
S^N f(x) = \mu^{-1}|B_{r_q(x)}(q)|_N, \quad S_N f(x) = \mu^{-1}(|D|_M - |B_{r_q(x)}(q)|_N).
\]
(we implicitly extend \( \mu^{-1} \) to \( R \) by setting \( \mu^{-1}(s) = 0 \) when \( s < 0 \))

Proof. We have \( (S^N f)^{-1}(t, \infty) = B_{\rho}(q) \) for some \( \rho = \rho(t) \), so that if \( x \in \partial B_{\rho}(q) \),
then \( S^N f(x) = t \). We require
\[
|B_{\rho}(q)|_N = |f > t|_\Omega = \mu(t)
\]
and the first formula follows. The second formula follows similarly using \( (S_N f)^{-1}(t, \infty) = S^N \Omega - B_{\rho}(q) \). \( \square \)

Fix \( p \in M \). Define
\[
m_{D,p}(\rho) = |B_{\rho}(p) \cap D|_M
\]
and correspondingly \( m_N(\rho) = |B_{\rho}(q)|_N \).

Proposition 2.3. Let \( \text{spt}\ f \subset D \). If \( f \) is a decreasing function of \( r_p(x) = \text{dist}_M(x, p) \)
on \( D \), then
\[
S^N f(x) = f(m_{D,p}^{-1} \circ m_N \circ r_q(x)).
\]
If \( f \) is increasing on \( D \), then
\[
S_N f(x) = f(m_{D,p}^{-1} \circ m_N \circ r_q(x)).
\]
Proof. If $f$ is decreasing in $r_{\rho}|D$, then $f^{-1}(t, \infty) = B_{\rho}(p) \cap D$ for some $\rho = \rho(t)$. If $f$ is increasing then $f^{-1}(t, \infty) = D - B_{\rho(p)}$. Now use Proposition 2.2.\[\Box\]

**Proposition 2.4.** If $f, g : M \to R_+$, and $\text{spt} f \subset D$, then
\[
\int_{S^N D} S^N f S^N g \leq \int_D fg \leq \int_{S^N D} S^N f S^N g.
\]

Proof. By Fubini’s theorem, we obtain
\[
\int_D fg = \int_0^\infty \int_0^\infty |\{f > s\} \cap \{g > t\}| dsdt
\]
\[
\leq \int_0^\infty \int_0^\infty \min\{|f > s|_M, |g > t|_M\} dsdt
\]
\[
= \int_0^\infty \int_0^\infty |\{S^N f > s\} \cap \{S^N g > t\}|_N dsdt
\]
\[
= \int_{S^N D} S^N f S^N g.
\]

Conversely,
\[
\int_0^\infty \int_0^\infty |\{f > s\} \cap \{g > t\}|_M dsdt
\]
\[
\geq \int_0^\infty \int_0^\infty |f > s|_M - |g > t|_M| dsdt
\]
\[
= \int_0^\infty \int_0^\infty |\{S^N f > s\} \cap \{S^N g > t\}|_N dsdt.
\]

**Proposition 2.5.** For any $\beta > 0$,
\[
S^N (f^\beta) = (S^N f)^\beta
\]
and similarly for $S_N f$.

Proof. Let $\mu_\beta(t) = |f f > t|_M$. Then $\mu_\beta(t^\beta) = \mu_1(t)$, and hence $\mu_\beta^{-1} = (\mu_1^{-1})^\beta$. \[\Box\]

3. Faber-Krahn and Chiti

We need the following weak version of Faber-Krahn.

**Theorem 3.1** (weak Faber-Krahn). If $\Omega$ is a bounded domain in $M$ with first Dirichlet eigenvalue $\lambda_1$, then
\[
\lambda_1 \geq \alpha^2 \lambda_1^N,
\]
where $\lambda_1^N$ is the first eigenvalue of $S^N \Omega \subset N$. Further, if we have equality $\lambda_1 = \alpha^2 \lambda_1^N$, then necessarily
\[
S^N u_1 \equiv v_1
\]
where $u_1$ is the first Dirichlet eigenfunction of $\Omega$, and $v_1$ the first Dirichlet eigenfunction on $S^N \Omega$, both normalized so that
\[
\|u_1\|_{L^2 \Omega} = \|v_1\|_{L^2 S^N \Omega}.
\]
Therefore, we calculate
\[ S^N \Omega = B = B_R(q), \]
and without loss of generality suppose \( \|u_1\|_{L^2(\Omega)} = \|v_1\|_{L^2B} = 1 \), so of course \( \|S^N u_1\|_{L^2B} = 1 \) also. Let \( \mu(t) = |u_1 > t|_M \). We have, for a.e. \( t \),
\[
-m'(t) \geq |u_1|_M^2 \left( \int_{\{u_1 = t\}} |\nabla u_1| \right)^{-1}
\]
\[
\geq \alpha^2 A(t > t)_M^2 \left( \int_{\{u_1 = t\}} |\nabla u_1| \right)^{-1}
\]
\[
= \alpha^2 A(\mu(t))^2 \left( \int_{u_1 > t} -\Delta u_1 \right)^{-1}
\]
\[
= \alpha^2 A(\mu(t))^2 \left( \lambda_1 \int_0^{\mu(t)} \mu^{-1}(\sigma) d\sigma \right)^{-1},
\]
and hence
\[
(\mu^{-1})'(s) \geq -\frac{\lambda_1}{\alpha^2} A^{-2}(s) \int_0^s \mu^{-1}(\sigma) d\sigma.
\]
Since \( |B|_N = |\Omega|_M \), and \( u_1 = 0 \) on \( \partial \Omega \), then \( S^N u_1 \) has Dirichlet boundary conditions. If \( S^N u_1 \neq v_1 \), then
\[
\lambda_1^N < \int_B |\nabla S^N u_1|^2.
\]
Write \( m(r) = m_N(r) = |B_r(q)|_N \), and observe that \( A(s) = m'(m^{-1}(s)) \). Since \( S^N u_1(r) = \mu^{-1}(m(r)) \), we have
\[
|\nabla S^N u_1|^2 = (\mu^{-1})'(m(r)) m'(r)^2.
\]
Therefore, we calculate
\[
\lambda_1^N < \int_B ((\mu^{-1})'(mr)m'(r))^2
\]
\[
= \int_0^R ((\mu^{-1})'(mr)m'(r))^2 m'(r) dr
\]
\[
\leq \frac{\lambda_1}{\alpha^2} \int_0^R \frac{m'(r)^2}{A(mr)^2} |(\mu^{-1})'(mr)| \int_0^{m(r)} \mu^{-1}(\sigma) d\sigma m'(r) dr
\]
\[
= \frac{\lambda_1}{\alpha^2} \int_0^R \frac{A(mr)^2}{A(mr)^2} |(\mu^{-1})'(mr)| \int_0^{m(r)} \mu^{-1}(\sigma) d\sigma m'(r) dr
\]
\[
= \frac{\lambda_1}{\alpha^2} \int_0^{|B|} (-(\mu^{-1})'(s)) \int_0^s \mu^{-1}(\sigma) d\sigma ds
\]
\[
= \frac{\lambda_1}{\alpha^2} \int_0^R \mu^{-1}(m(r))^2 m'(r) dr
\]
\[
= \frac{\lambda_1}{\alpha^2} \int_B \left( S^N u_1 \right)^2
\]
\[
= \frac{\lambda_1}{\alpha^2}.
\]
Suppose $B$ is a ball in $N$, centered at $q$, with first eigenvalue $\lambda_1^B = \lambda_1/\alpha^2$, and first eigenfunction $z$. By the maximum principle and simplicity of $\lambda_1$, $z$ is a decreasing function of $r_q$. By Faber-Krahn above, $\lambda_1^B \geq \lambda_1^N$, and hence $B \subset S^N\Omega$. Further, if $B = S^N\Omega$ then necessarily $z \equiv S^N u_1$.

We obtain the following weak version of Chiti’s theorem [Chi83].

**Theorem 3.2** (weak Chiti). Let $\Omega \subset M$ have first eigenvalue $\lambda_1$, eigenfunction $u_1$. Let $B = B_R(q)$ be a ball in $N$ with first eigenvalue $\lambda_1^B = \lambda_1/\alpha^2$, eigenfunction $z$. Let $u_1$ and $z$ be normalized so that

$$\|u_1\|_{L^2\Omega} = \|z\|_{L^2B}.$$  

Then we can choose an $r_0 \in (0, R)$ so that

$$z \geq S^N u_1 \text{ on } [0, r_0]$$

$$z \leq S^N u_1 \text{ on } [r_0, R].$$

**Proof.** Let $\mu(t) = |u_1| > t|_M$ and $\nu(t) = |z > t|_N$. Recall we had

$$(\mu^{-1})'(s) \geq -\frac{\lambda_1}{\alpha^2} A^{-2}(s) \int_0^s \mu^{-1}(\sigma) d\sigma.$$  

By repeating the proof of this with $\nu$ instead of $\mu$, we obtain

$$(\nu^{-1})'(s) = -\frac{\lambda_1}{\alpha^2} A^{-2}(s) \int_0^s \nu^{-1}(\sigma) d\sigma$$

$$= -\frac{\lambda_1}{\alpha^2} A^{-2}(s) \int_0^s \nu^{-1}(\sigma) d\sigma.$$  

The normalization implies $s_0 = \sup\{s \in (0, |B|_N) : \mu^{-1}(s) \leq \nu^{-1}(s)\}$ is defined and positive. If $s_0 = |B|_N$, then since $\nu^{-1}(|B|_N) = 0$ and $\mu^{-1}$ is decreasing, we necessarily have that $|B|_N = |\Omega|_M$. Otherwise $u_1$ would be zero on an open set, contradicting unique continuation. If $|B|_N = |\Omega|_M$ then by Theorem 3.1 $S^N u_1 = z$ and the Theorem is vacuous.

So we can assume $s_0 \in (0, |B|_N)$. Clearly $\mu^{-1} \geq \nu^{-1}$ on $[s_0, |B|_N]$, and $\mu^{-1}(s_0) = \nu^{-1}(s_0)$. We show $\mu^{-1} \leq \nu^{-1}$ on $[0, s_0]$.

Suppose, towards a contradiction, that $\beta = \sup_{[0, s_0]} \frac{\mu^{-1}}{\nu^{-1}} > 1$. Then we calculate, for $s \in [0, s_0]$,

$$(\beta^2 - 1)'(s) \leq -\frac{\lambda_1}{\alpha^2} A^{-2}(s) \int_0^s (\beta^2 - 1)(\sigma) d\sigma \leq 0.$$  

And therefore

$$(\beta^2 - 1)(s) \geq (\beta^2 - 1)(s_0) = (\beta - 1)\nu^{-1}(s_0) > 0$$  

for any $s \in [0, s_0]$, contradicting our choice of $\beta$. The Theorem follows by choosing $r_0$ which satisfies $|B_{r_0}(q)|_N = s_0$. \hfill \Box

**Corollary 3.3.** If $F : N \to R_+$ is a decreasing function of $r_q$, then

$$\int_{S^N\Omega} (S^N u_1)^2 F \leq \int_B z^2 F$$

with $B, z$ as in Theorem 3.2. If $F$ is an increasing function of $r_q$, then

$$\int_{S^N\Omega} (S^N u_1)^2 F \geq \int_B z^2 F.$$
Proof. Let \( r_0 \) be as in Theorem 3.2. For \( F \) decreasing, we have that
\[
(z^2 - (S^N u_1)^2)(F - F(r_0)) \geq 0,
\]
with support in \( S^N \Omega \). Therefore
\[
\int_{S^N \Omega} (z^2 - (S^N u_1)^2) F \geq F(r_0) \left( \int_B z^2 - \int_{S^N \Omega} (S^N u_1)^2 \right) = 0
\]
having used Proposition 2.1. The case of \( F \) increasing follows similarly. \( \square \)

4. Proof of Theorem

Fix \( \Omega, B \) as in Theorem 3.2. So \( \Omega \) has eigenvalues \( \lambda_1, \lambda_2 \), and \( B \) has eigenvalues \( \lambda_B^1 = \lambda_1 / \alpha^2, \lambda_B^2 \). Take as before \( u_1 \) for the first eigenfunction of \( \Omega \), and \( z \) the first eigenfunction of \( B \).

If \( P : \Omega \to \mathbb{R} \) is any function such that \( Pu_1 \) is \( L^2 \) orthogonal to \( u_1 \), then
\[
\int_{\Omega} |\nabla P|^2 u_1^2 \geq (\lambda_2 - \lambda_1) \int_{\Omega} P^2 u_1^2
\]
by min-max (\( Pu_1 \) has the right boundary conditions) and integration by parts. We cook up a collection of test functions \( P_i \) as follows.

Define \( r_p(x) = \text{dist}_M(p, x) = |\exp^{-1}_p(x)| \), and let \( \sigma(r) \) be defined so that
\[
|B_{\sigma(r)}(q)|_N = |B_r(p) \cap \text{hull}\Omega|_M.
\]
For a given \( p \in \text{hull}\Omega \), define \( P_p : \text{hull}\Omega \to T_p M \) by
\[
P_p(x) = \frac{\exp^{-1}_p(x)}{r_p} w(\sigma(r_p)),
\]
where \( w : \mathbb{R}^+ \to \mathbb{R}^+ \) is any non-negative Lipschitz function with \( w(0) = 0 \).

Lemma 4.1. We can choose a \( p \in \text{hull}\Omega \) so that \( \int_{\Omega} P_p(x) u_1^2(x) dx = 0 \).

Proof. For each \( p \in \partial \text{hull}\Omega \) we have at least one supporting hyperplane. I.e. there is a vector \( e_n \) such that \( \exp^{-1}_p(\text{hull}\Omega) \subset \{ v : v \cdot e_n \geq 0 \} \). But then
\[
e_n \cdot \int_{\Omega} P_p u_1^2 \geq 0
\]
since \( w \) is non-negative. So \( p \mapsto \int_{\Omega} P_p u_1^2 \) is a continuous vector field on \( \text{hull}\Omega \), which is inward pointing along the boundary. By the Brouwer fixed point theorem this vector field must have a zero in \( \text{hull}\Omega \). \( \square \)

Choose an orthonormal basis \( \{ e_i \} \) of \( T_p M \). Define
\[
P_i(x) = e_i \cdot P_p(x),
\]
where we choose and fix \( p \) as in Lemma 4.1. So \( \int_{\Omega} P_i u_1^2 = 0 \) for each \( i \), and by (4) we have
\[
\int_{\Omega} (\sum_i |\nabla P_i|^2) u_1^2 \geq (\lambda_2 - \lambda_1) \int_{\Omega} (\sum_i P_i^2) u_1^2 = (\lambda_2 - \lambda_1) \int_{\Omega} w^2(\sigma(r_p)) u_1^2.
\]
For ease of notation, in the following we will write \( g \equiv w \circ \sigma \) and \( r \equiv r_p \), so that \( P_i(x) = e_i \cdot \exp^{-1}(x)g(r)/r \). We calculate
\[
\frac{d}{ds}_{|s=0} P_i(\exp_p(v + sw)) = \frac{d}{ds} e_i \cdot (v + sw) \frac{g}{r} = e_i \cdot w \frac{g(r)}{r} + e_i \cdot v \left( \frac{g'}{r} \right)' \frac{d}{ds} |v + sw| = e_i \cdot w \frac{g(r)}{r} + e_i \cdot v \frac{g'}{r} \frac{v \cdot w}{|v|}.
\]

Choose an orthonormal basis \( E_i \) at a fixed \( x = \exp_p(v) \), such that \( E_1 = \frac{\partial}{\partial r} \). Write
\[
w_j = (D \exp_p |v)^{-1}(E_j)
\]
and since \( D \exp_p \) is a radial isometry \( w_1 = \frac{v}{|v|} \). We have
\[
E_1 P_i = e_i \cdot v \frac{g}{r^2} + e_i \cdot v (g/r)'
\]
\[
E_j P_i = e_i \cdot w_j \frac{g}{r}, \quad j > 1.
\]

Therefore
\[
\sum_i |\nabla P_i|^2 = \sum_i (E_1 P_i)^2 + \sum_{j>1,i} (E_j P_i)^2 = r^2 \left[ \frac{g^2}{r^4} + 2 \frac{g}{r^2} \left( \frac{g'}{r} \right)' + \left( \frac{g}{r} \right)' \right] + \sum_{j>1} |w_j|^2 \frac{g^2}{r^2} \\
= g^2 + \frac{g^2}{r^2} \sum_{j>1} |w_j|^2 \\
\leq g^2 + \frac{n-1}{\text{sn}_k r} g^2
\]

having used Rauch’s theorem to deduce
\[ 1 = |D \exp_p |v(w_j)| \geq \frac{\text{sn}_k|v|}{|v|} |w_j|. \]

Recalling the definition \( g = w \circ \sigma \), we estimate for any \( r \in r_p \Omega \),
\[
g'(r)^2 + \frac{n-1}{\text{sn}_k r} g(r)^2 = w'(\sigma(r))^2 \sigma'(r)^2 + \frac{n-1}{\text{sn}_k r} w(\sigma(r))^2 \\
\leq C_1^2 \left( w'(\sigma(r))^2 + \frac{n-1}{\text{sn}_k \sigma(r)} w(\sigma(r))^2 \right)
\]

where
\[ (5) \quad C_1 = \max_{r \in r_p \Omega} \left\{ \sigma'(r), \frac{\text{sn}_k \sigma(r)}{\text{sn}_k r} \right\}. \]

We obtain
\textbf{Theorem 4.2.} For any Lipschitz \( w : R_+ \rightarrow R_+ \) with \( w(0) = 0 \), we can choose a point \( p \in \text{hull} \Omega \) so that
\[
(\lambda_2 - \lambda_1) \int_{\Omega} u_1^2 w(\sigma(r_p))^2 \leq C_1^2 \int_{\Omega} u_1^2 F(\sigma(r_p)).
\]
Here $F(t) = w'^2(t) + \frac{n-1}{2w'^2(t)} w(t)^2$, and $C_1$ as in \cite{AB01}.

**Corollary 4.3.** If $w$, $p$ are as above, and $w$ further satisfies:

\[
(*) \quad \begin{cases} 
  w(r) \text{ is increasing} \\
  F(r) \text{ is decreasing}
\end{cases}
\]

then

\[
(\lambda_2 - \lambda_1) \int_B z^2 w(r_q)^2 \leq C_1^2 \int_B z^2 F(r_q).
\]

Here $B$ and $z$ are as in Theorem 3.2.

**Proof.** We extend $u_1$ by 0 to be defined on all of $M$, and truncate $w(\sigma(r_p))$, $F(\sigma(r_p))$ to be 0 outside $\hull\Omega$.

Apply Proposition 2.4 and Proposition 2.5 to Corollary 4.2 to obtain

\[
(\lambda_2 - \lambda_1) \int_{S^N} (S^N w(\sigma(r_p|_{\hull\Omega})))^2 \leq C_1^2 \int_{S^N} (S^N F(\sigma(r_p|_{\hull\Omega})))^2.
\]

Since $w(\sigma(r_p))$, $F(\sigma(r_p))$ are increasing, decreasing (resp.) functions of $r_p$, by Proposition 2.3 with $D = \hull\Omega$ (and recalling the definition of $\sigma$) we have

\[
(\lambda_2 - \lambda_1) \int_{S^N} (S^N w(r_q))^2 \leq C_1^2 \int_{S^N} (S^N F(r_q))^2.
\]

Now use Theorem 3.2. \hfill \Box

**Proof of Theorem 1.4.** Write $B = B_R(q)$, and $z = z(r_q)$. Let $J = J(r_q)$ be the radial component of the second Dirichlet eigenfunction of $B$ (c.f. equation 2.11 of \cite{AB92}, section 3 of \cite{BL07}, section 3 of \cite{AB01}). When $k > 0$, the assumption $|\hull\Omega| < |N|/2$ implies $S^N \supset B$ lies in the hemisphere.

Set

\[
w(t) = \begin{cases} 
  \frac{J(t)}{s(t)} & t \in [0, R) \\
  \lim_{s \to R^-} w(s) & t \geq R
\end{cases}
\]

Using Corollary 3.4 of \cite{AB92} (if $k = 0$), Lemma 7.1 in \cite{BL07} (if $k < 0$), or Theorem 4.1 in \cite{AB01} (if $k > 0$), we can apply Theorem 4.3 to deduce

\[
(\lambda_2 - \lambda_1) \leq C_1^2 (\lambda_2^B - \lambda_1^B),
\]

with $C_1$ as in \cite{AB01}. We show that $C_1 \leq \frac{|\partial B_{\text{asym}}} {|\hull\Omega|}.$

For ease of notation write $m(r) = |B_r|_{\tilde{N}}$ and $\tilde{m}(r) = |B_r|_{\tilde{N}}$. All balls in $M$ are centered at $p$, and balls in $\tilde{N}$ are centered at $q$, $\tilde{q}$ (resp.).

Suppose $C_p$ is a geodesic cone in $M$, centered at $p$, with solid angle $\gamma \omega_n$ in $T_p M$ (we use $\omega_n$ to denote the volume of the Euclidean unit ball). If $\text{Ric}_M \geq (n-1)K$ on $B_r \cap C_p$, then for $\epsilon$ small

\[
|\partial B_r \cap C_p|_M \leq |\partial B_r|_{\tilde{N}} \frac{|B_r \cap C_p|_M}{|B_r|_{\tilde{N}}} 
= |\partial B_r|_{\tilde{N}} \frac{\gamma \omega_n + O(\epsilon^2)}{\omega_n + O(\epsilon^2)}
\]

and hence

\[
|\partial B_r \cap C_p|_M \leq \gamma |\partial B_r|_{\tilde{N}}.
\]
Conversely,

\[
|B_r \cap C_p|_M \geq |B_r \cap C'_p|_N
\]

\[
= |B_r|_N \frac{|B_r \cap C'_p|_N}{|B_r|_N}
\]

\[
= |B_r|_N \frac{\gamma \omega_n + O(\epsilon^2)}{\omega_n + O(\epsilon^2)}.
\]

Here \( C'_p = \exp_q \circ (\exp_p^{-1})^{-1}(C_p) \) is a geodesic cone with the same cone angle. Therefore

\[
|B_r \cap C_p|_M \geq \gamma |B_r|_N.
\]

Let \( D = \text{hull}\Omega \), then we have \( \sigma(r) = m^{-1}(|B_r(p) \cap D|_M) \). Notice that \( |B_r(p) \cap D|_M \geq |B_r(p) \cap C_p|_M \) where \( C_p \) is a geodesic cone at \( p \) over \( \partial B_r(p) \cap D \). Therefore

\[
\sigma'(r) = \frac{1}{m'(m^{-1}(|B_r \cap D|_M))} |\partial B_r \cap D|_M
\]

\[
\leq \frac{1}{m'(m^{-1}(|B_r \cap C_p|_M))} |\partial B_r \cap C_p|_M
\]

\[
\leq \frac{1}{m'(m^{-1}(\gamma |B_r|_N))} |\partial B_r|_\tilde{N}
\]

\[
\leq \frac{|\partial B_r|_\tilde{N}}{|\partial B_r|_N}.
\]

The last inequality follows because \( s \mapsto m'(m^{-1}s) \) is concave. We elaborate. The last inequality is equivalent to

\[
m'(m^{-1}s) \leq \frac{m'(m^{-1}(\gamma s))}{\gamma}
\]

for any \( \gamma \in (0, 1] \). But the RHS is a dilation of the graph of the LHS, hence the inequality follows if the graph is concave. We calculate

\[
(m' \circ m^{-1})'' = \frac{(m' \circ m^{-1})(m'' \circ m^{-1}) - (m'' \circ m^{-1})^2}{(m' \circ m^{-1})^3}.
\]

And we have

\[
(m'm'' - (m'')^2)(r) = -(n-1)n^2\omega_n^2sn_k(r)^{2n-4}.
\]

We prove now the inequality

\[
\frac{sn_k\sigma(r)}{sn_kr} \leq \frac{|\partial B_r|_{\tilde{N}}}{|\partial B_r|_N}.
\]

Since \( \sigma(r) \leq m^{-1}(\tilde{m}(r)) \), it suffices to prove the inequality

\[
\tilde{m}(r) \leq m \left[ sn_k^{-1} \left( \frac{\tilde{m}'(r)}{m'(r)sn_k(r)} \right) \right].
\]
We therefore calculate

\[ m \left[ \text{sn}^{-1} \left( \frac{m'(r)}{m'(r) \text{sn}_K(r)} \right) \right] = m \left[ \text{sn}^{-1} \left( \frac{\text{sn}_K(r)}{\text{sn}_K(r)} \right)^{n-2} \right] \]

\[ \geq m \left[ \text{sn}^{-1}(\text{sn}_K(r)) \right] \]

\[ = n\omega_n \int_0^{\text{sn}^{-1}(\text{sn}_K(r))} \text{sn}_k(\rho)^{n-1} d\rho \]

\[ = n\omega_n \int_0^r \text{sn}_K(\rho)^{n-1} \sqrt{1 - k\text{sn}_K(\rho)^2} d\rho \]

\[ \geq n\omega_n \int_0^r \text{sn}_K(\rho)^{n-1} d\rho \]

\[ = \tilde{m}(r), \]

using that \( \text{sn}_k(r)^2 = 1 - k\text{sn}_k(r)^2 \).

\[ \square \]

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