DERIVED CATEGORIES OF COHERENT SHEAVES AND MOTIVES.

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The bounded derived category of coherent sheaves $D^b(X)$ is a natural triangulated category which can be associated with an algebraic variety $X$. It happens sometimes that two different varieties have equivalent derived categories of coherent sheaves $D^b(X) \simeq D^b(Y)$. There arises a natural question: can one say anything about motives of $X$ and $Y$ in that case? The first such example (see [4]) – abelian variety $A$ and its dual $\hat{A}$ – shows us that the motives of such varieties are not necessary isomorphic. However, it seems that the motives with rational coefficients are isomorphic.

Recall a definition of the category of effective Chow motives $\text{CH}^{\text{eff}}(k)$ over a field $k$. The category $\text{CH}^{\text{eff}}(k)$ can be obtained as the pseudo-abelian envelope (i.e. as formal adding of cokernels of all projectors) of a category, whose objects are smooth projective schemes over $k$, and the group of morphisms from $X$ to $Y$ is the sum $\oplus_{X_i} A^m(X_i \times Y)$ (over all connected components $X_i$) of the groups of cycles of codimension $m = \dim Y$ on $X_i \times Y$ modulo rational equivalence (see [3, 1]). In [7] Voevodsky introduced a triangulated category of geometric motives $\text{DM}^{\text{eff}}_{\text{gm}}(k)$. He started with an additive category $\text{SmCor}(k)$, objects of which are smooth schemes of finite type over $k$, and the group of morphisms from $X$ to $Y$ is the free abelian group generated by integral closed subschemes $Z \subset X \times Y$ such that the projection on $X$ is finite and surjective onto a connected component of $X$. There is a natural embedding $[-] : \text{Sm}(k) \to \text{SmCor}(k)$ of the category $\text{Sm}(k)$ of smooth schemes of finite type over $k$. The category $\text{SmCor}(k)$ is additive and one has $[X \coprod Y] = [X] \oplus [Y]$. Further, he considered the quotient of the homotopy category $\mathcal{H}^b(\text{SmCor}(k))$ of bounded complexes by minimal thick triangulated subcategory $T$, which contains all objects of the form $[X \times \mathbb{A}^1] \to [X]$ and $[U \cap V] \to [U] \oplus [V] \to [X]$ for any open covering $U \cup V = X$. Triangulated category $\text{DM}^{\text{eff}}_{\text{gm}}(k)$ is defined as the pseudo-abelian envelope of the quotient category $\mathcal{H}^b(\text{SmCor}(k))/T$ (see [7, 1]).

There exists a canonical functor $\text{CH}^{\text{eff}}(k) \to \text{DM}^{\text{eff}}_{\text{gm}}(k)$, which is a full embedding if $k$ admits resolution of singularities ([7, 4.2.6]). Thus, it doesn’t matter in which category (in $\text{CH}^{\text{eff}}(k)$ or in $\text{DM}^{\text{eff}}_{\text{gm}}(k)$) motives of smooth projective varieties are considered. Denote the

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motive of a variety $X$ by $M(X)$, and its motive in the category of motives with rational coefficients $DM_{gm}^\text{eff}(k) \otimes \mathbb{Q}$ (and in $\text{CH}^{\text{eff}}(k) \otimes \mathbb{Q}$) by $M(X)_Q$.

**Conjecture 1.** Let $X$ and $Y$ be smooth projective varieties, and let $D^b(X) \simeq D^b(Y)$. Then the motives $M(X)_Q$ and $M(Y)_Q$ are isomorphic in $\text{CH}^{\text{eff}}(k) \otimes \mathbb{Q}$ (and in $DM_{gm}^\text{eff}(k) \otimes \mathbb{Q}$).

The category $DM_{gm}^\text{eff}(k)$ has a tensor structure, and $M(X) \otimes M(Y) = M(X \times Y)$. One defines the Tate object $\mathbb{Z}(1)$ to be the image of the complex $[\mathbb{P}^1] \to [\text{Spec}(k)]$ placed in degree 2 and 3 and put $M(p) = M \otimes \mathbb{Z}(1)^{\otimes p}$ for any motive $M \in DM_{gm}^\text{eff}(k)$ and $p \in \mathbb{N}$. The triangulated category of geometric motives $DM_{gm}(k)$ is defined by formally inverting the functor $- \otimes \mathbb{Z}(1)$ on $DM_{gm}^\text{eff}(k)$. The important and nontrivial fact here is the statement that the canonical functor $DM_{gm}^\text{eff}(k) \to DM_{gm}(k)$ is a full embedding [7, 4.3.1]. Therefore, we can work in the category $DM_{gm}(k)$. Moreover (see [7]), for any smooth projective varieties $X, Y$ and for any integer $i$ there is an isomorphism

$$\text{Hom}_{DM_{gm}(k)}(M(X), M(Y)(i)[2i]) \cong A^{m+i}(X \times Y), \quad \text{where} \quad m = \text{dim}Y.$$

**Conjecture 2.** Let $X$ and $Y$ be smooth projective varieties and let $F : D^b(X) \to D^b(Y)$ be a fully faithful functor. Then the motive $M(X)_Q(k)[2k]$ is a direct summand of the motive $M(Y)_Q$ for some integer $k \in \mathbb{Z}$.

Suppose, one has a fully faithful functor $F : D^b(X) \to D^b(Y)$ between derived categories of coherent sheaves of two smooth projective varieties $X$ and $Y$ of dimension $n$ and $m$ respectively. Any such functor has a right adjoint $F^*$ by [2], and by Theorem 2.2 from [5] (see also [6, 3.2.1]) the functor $F$ can be represented by an object on the product $X \times Y$, i.e. $F \cong \Phi_A$, where $\Phi_A = Rp_{2*}(p_1^*(-) \otimes \mathcal{A})$ for some $\mathcal{A} \in D^b(X \times Y)$. With any functor of the form $\Phi_A : D^b(X) \to D^b(Y)$ one can associate an element $a \in A^*(X \times Y, \mathbb{Q})$ by the following rule

$$a = p_1^* \sqrt{\text{td}_X} \cdot \text{ch}(\mathcal{A}) \cdot p_2^* \sqrt{\text{td}_Y},$$

where $\text{td}_X$ and $\text{td}_Y$ are Todd classes of the varieties $X$ and $Y$. The cycle $a$ has a mixed type. Let us consider its decomposition into components $a = a_0 + \cdots + a_{n+m}$, where index is the codimension of a cycle on $X \times Y$. Each component $a_q$ induces a map of motives

$$\alpha_q : M(X)_Q \to M(Y)_Q(q-m)[2(q-m)].$$

Thus the total cycle $a$ gives a map $\alpha : M(X)_Q \to \bigoplus_{i=-m}^{n} M(Y)_Q(i)[2i]$. Now consider the object $B \in D^b(X \times Y)$ which represents the (left) adjoint functor $F^*$, i.e. $F^* \cong \Psi_B$, where $\Psi_B = Rp_{1*}(p_2^*(-) \otimes \mathcal{B})$. One attaches to the object $B$ a cycle $b = b_0 + \cdots + b_{n+m}$ defined
by the same formula (1). The cycle \( b \) induces a map \( \beta : \bigoplus_{i=-m}^{n} M(Y)_{\mathbb{Q}}(i)[2i] \to M(X)_{\mathbb{Q}} \). Since the functor \( \Phi_A \) is fully faithful, the composition \( \Psi_B \circ \Phi_A \) is isomorphic to the identity functor. Applying the Riemann-Roch-Grothendieck theorem, we obtain that the composition

\[
M(X)_{\mathbb{Q}} \overset{\alpha}{\to} \bigoplus_{i=-m}^{n} M(Y)_{\mathbb{Q}}(i)[2i] \overset{\beta}{\to} M(X)_{\mathbb{Q}}
\]

is the identity map, i.e. \( M(X)_{\mathbb{Q}} \) is a direct summand of \( \bigoplus_{i=-m}^{n} M(Y)_{\mathbb{Q}}(i)[2i] \).

Denote by \( \tilde{M}(X)_{\mathbb{Q}} \) the infinite direct sum \( \bigoplus_{i=-\infty}^{\infty} M(X)_{\mathbb{Q}}(i)[2i] \). The total cycle \( a \) defined above gives a map \( \tilde{\alpha} : \tilde{M}(X)_{\mathbb{Q}} \to \tilde{M}(Y)_{\mathbb{Q}} \) whose component from \( M(X)_{\mathbb{Q}}(k)[2k] \) to \( M(Y)_{\mathbb{Q}}(q-m+k)[2(q-m+k)] \) coincides with \( \alpha_q \) for any \( k \). By the same way the cycle \( b \) induces a map \( \tilde{\beta} \) from \( \tilde{M}(X)_{\mathbb{Q}} \) to \( \tilde{M}(Y)_{\mathbb{Q}} \). The above consideration give us that the composition \( \tilde{\beta} \cdot \tilde{\alpha} \) is the identity map. Thus we obtain the following proposition.

**Proposition 1.** Let \( X \) and \( Y \) be smooth projective varieties and let \( F : D^b(X) \to D^b(Y) \) be a fully faithful functor. Then the motive \( \tilde{M}(X)_{\mathbb{Q}} \) is a direct summand of the motive \( \tilde{M}(Y)_{\mathbb{Q}} \). If, in addition, the functor \( F \) is an equivalence, then the motives \( \tilde{M}(X)_{\mathbb{Q}} \) and \( \tilde{M}(Y)_{\mathbb{Q}} \) are isomorphic.

Assume now that \( \dim X = \dim Y = n \) and, moreover, suppose that the support of the object \( A \) also has the dimension \( n \). Therefore, \( a_q = 0 \) when \( q = 0, \ldots, n-1 \), i.e. \( a = a_n + \cdots + a_{2n} \). It is easily to see that in this case \( b = b_n + \cdots + b_{2n} \) as well. This implies that the composition \( \beta \cdot \alpha : M(X)_{\mathbb{Q}} \to M(X)_{\mathbb{Q}} \), which is the identity, coincides with \( \beta_n \cdot \alpha_n \). Hence, \( M(X)_{\mathbb{Q}} \) is a direct summand of \( M(Y)_{\mathbb{Q}} \). Furthermore, since the cycles \( a_n \) and \( b_n \) are integral in this case we get the same result for integral motives, i.e. the integral motive \( M(X) \) is a direct summand of the motive \( M(Y) \) as well. Thus, we obtain

**Theorem 1.** Let \( X \) and \( Y \) be smooth projective varieties of dimension \( n \), and let \( F : D^b(X) \to D^b(Y) \) be a fully faithful functor such that the dimension of the support of an object \( A \) on \( X \times Y \), which represents \( F \), is equal to \( n \). Then the motive \( M(X) \) is a direct summand of the motive \( M(Y) \). If, in addition, the functor \( F \) is an equivalence, then the motives \( M(X) \) and \( M(Y) \) are isomorphic.

Examples of such functors are known, they come from birational geometry (see e.g. [6]). In these examples one of the connected components of \( \text{supp}(A) \) gives a birational map \( X \to Y \). Blow ups and antiflips induce fully faithful functors, and flops induce equivalences. Note that an isomorphism of motives implies an isomorphism of any realization (singular cohomologies, \( l \)-adic cohomologies, Hodge structures and so on).
For arbitrary equivalence $\Phi_A : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ the map of motives $\alpha_n : M(X)_\mathbb{Q} \to M(Y)_\mathbb{Q}$, induced by the cycle $a_n \in \mathcal{A}^n(X \times Y, \mathbb{Q})$, is not necessary an isomorphism (e.g. Poincare line bundle $\mathcal{P}$ on the product of abelian variety $A$ and its dual $\tilde{A}$). However, the following conjecture, which specifies Conjecture 1, may be true.

**Conjecture 3.** Let $\mathcal{A}$ be an object of $\mathcal{D}^b(X \times Y)$, for which $\Phi_A : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is an equivalence. Then there exist line bundles $L$ and $M$ on $X$ and on $Y$ respectively such that the component $a_n'$ of the object $\mathcal{A}' := p_1^*L \otimes \mathcal{A} \otimes p_2^*M$ gives an isomorphism between motives $M(X)_\mathbb{Q}$ and $M(Y)_\mathbb{Q}$.

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