THE BOREL PARTITION SPECTRUM
AT SUCCESSORS OF SINGULAR CARDINALS

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Abstract. Assuming that $0^+$ does not exist, we prove that if there is a partition of $\mathbb{R}$ into $\aleph_\omega$ Borel sets, then there is also a partition of $\mathbb{R}$ into $\aleph_{\omega+1}$ Borel sets.

1. Introduction

Define the Borel partition spectrum, denoted $sp(\text{Borel})$, as follows:

$$sp(\text{Borel}) = \{ |P| : P \text{ is a partition of } \mathbb{R}\text{ into uncountably many Borel sets} \}.$$ 

Let us begin by reviewing what is known about $sp(\text{Borel})$.

(1) $\aleph_1 \in sp(\text{Borel})$.

Hausdorff showed in [12] that $2^{\omega_1}$, and in fact any uncountable Polish space, can be expressed as an increasing union $\bigcup_{\xi<\omega_1} E_\xi$ of $G_\delta$ sets. This implies that $\mathbb{R}$, or any other uncountable Polish space, can be partitioned into $\aleph_1$ nonempty $F_{\sigma\delta}$ sets.

(2) $c = \max(\text{sp}(\text{Borel}))$.

Clearly $c \in sp(\text{Borel})$, because we may partition $\mathbb{R}$ into singletons. Assuming the Axiom of Choice (which we do throughout), $\mathbb{R}$ cannot be partitioned into $>|\mathbb{R}|$ sets.

(3) $sp(\text{Borel})$ is closed under singular limits.

In other words, if $\lambda$ is singular and $sp(\text{Borel}) \cap \lambda$ is unbounded in $\lambda$, then $\lambda \in sp(\text{Borel})$. This is proved in [11] Theorem 2.8 by adapting a (fairly straightforward) argument of Hechler [14, Theorem 3.4] proving the analogous statement for the so-called MAD spectrum.

In addition to these positive ZFC-provable facts concerning $sp(\text{Borel})$, we also have several independence results, which seem to suggest that very little else can be proved from ZFC concerning $sp(\text{Borel})$.

(4) It is consistent with arbitrary values of $c$ that $sp(\text{Borel}) = \{\aleph_1, c\}$.

More precisely, if $\kappa^{\aleph_0} = \kappa$ and GCH holds up to $\kappa$, then forcing with $Fn(\kappa, 2)$ to add $\kappa$ mutually generic Cohen reals produces a model in

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which $\kappa = c$ and $\text{sp}(\text{Borel}) = \{\aleph_1, \kappa\}$. This is implicit in an argument of Miller [23, Section 3]; a more general theorem from which this follows is proved explicitly by Blass in [1].

(5) It is consistent with arbitrary values of $c$ that $\text{sp}(\text{Borel}) = [\aleph_1, c]$.

More precisely, given any $\kappa \geq c$ with $\kappa^{\aleph_0} = \kappa$, there is a ccc forcing extension in which $c = \kappa$ and $\text{sp}(\text{Borel}) = [\aleph_1, c]$. This follows from [5, Theorem 3.11], where the result is proved for $2^\omega$ instead of $\mathbb{R}$.

In other words, (4) and (5) say that, regardless of what $c$ may be, $\text{sp}(\text{Borel})$ can be as small as possible or as large as possible (given the requirements in (1) and (2)). Furthermore, the next result shows that $\text{sp}(\text{Borel})$ can be made to look chaotic and patternless, equal to an almost (but not quite) arbitrary set of cardinals satisfying (1), (2), and (3).

(6) Suppose $C$ is a set of uncountable cardinals such that:

(a) $C$ is countable,
(b) $\aleph_1 \in C$,
(c) $C$ has a maximum, and $\max(C)^{\aleph_0} = \max(C)$,
(d) $C$ is closed under singular limits, and
(e) if $\lambda \in C$ and $\text{cf}(\lambda) = \omega$, then $\lambda^+ \in C$.

Assuming $\text{GCH}$ holds up to $\max C$, there is a ccc forcing extension in which $C = \text{sp}(\text{Borel})$.

This is proved as Corollary 3.3 in [4]. An immediate consequence of this (Corollary 3.4 in [4]) is: for any $A \subseteq \omega \setminus \{0\}$ with $1 \in A$, there is a forcing extension in which $\text{sp}(\text{Borel}) = \{\aleph_n : n \in A\} \cup \{\aleph_\omega, \aleph_{\omega+1}\}$.

Of the requirements given for $C$ in (6), items (b), (c), and (d) correspond directly to items (1), (2), and (3) above. These requirements for $C$ cannot be eliminated from (6), because they are necessary features of $\text{sp}(\text{Borel})$ in any forcing extension. Item (a), on the other hand, does not represent a necessary feature of $\text{sp}(\text{Borel})$; we know this because (5) implies that $\text{sp}(\text{Borel})$ can be uncountable. Item (a) is just an artifact of the proof of (6) in [4], and it is conceivable that a more careful or more clever proof could, at some point in the future, eliminate it from the statement of (6) altogether.

This leaves item (e), which forms the topic of this paper. Our main result is that, in a sufficiently “$L$-like” set-theoretic universe, (e) represents a necessary feature of $\text{sp}(\text{Borel})$. Thus, in such a universe there is nontrivial structure to $\text{sp}(\text{Borel})$ beyond what is stated in items (1), (2), and (3) above.

**Main Theorem.** Suppose $0^\dagger$ does not exist. If $\kappa$ is a singular cardinal with $\text{cf}(\kappa) = \omega$ and $\kappa \in \text{sp}(\text{Borel})$, then $\kappa^+ \in \text{sp}(\text{Borel})$.

In particular, assuming that $0^\dagger$ does not exist, if there is a partition of $\mathbb{R}$ into $\aleph_\omega$ Borel sets then there is also a partition of $\mathbb{R}$ into $\aleph_{\omega+1}$ Borel sets.

The assertion “$0^\dagger$ exists” is a large cardinal axiom a little stronger (in consistency strength) than the existence of a measurable cardinal. Roughly, $0^\dagger$ bears the same relationship to the $L$-like models of the form $L[\mu]$, where $\mu$
is a normal measure on a measurable cardinal, that \(0^\dagger\) bears to \(L\). (See [17] for more.) In particular, if \(0^\dagger\) exists then there is an inner model containing a measurable cardinal. Thus, by contraposition: if \(\kappa_\omega \in \text{sp}(\text{Borel})\) while \(\kappa_{\omega+1} \notin \text{sp}(\text{Borel})\), there is an inner model containing a measurable cardinal.

We present two proofs of the main theorem. Both of them use the assertion “\(0^\dagger\) does not exist” in the guise of cofinal Kurepa families. What these are, and their connection to \(0^\dagger\), is explained in Section 2. Then in Sections 3 and 4 we present our two proofs. These proofs use cofinal Kurepa families in different but clearly analogous ways. The proof in Section 3 is more direct, optimized for both length and clarity. However, the proof in Section 4 gives slightly more in the end, and it also connects the topic of this paper to an old Scottish Book problem of Stefan Banach.

2. Preliminaries: cofinal Kurepa families

Given an infinite set \(A\), recall that \([A]^{\omega}\) denotes the set of all countably infinite subsets of \(A\).

- \(\mathcal{F} \subseteq [A]^{\omega}\) is cofinal in \([A]^{\omega}\) if for every \(X \in [A]^{\omega}\) there is some \(Y \in \mathcal{F}\) such that \(Y \supseteq X\). In other words, \(\mathcal{F} \subseteq [A]^{\omega}\) is cofinal if it is cofinal (in the usual sense of the word) in the poset \(([A]^{\omega}, \subseteq)\).

- \(\mathcal{F} \subseteq [A]^{\omega}\) is Kurepa if \(\{X \cap Y : Y \in \mathcal{F}\}\) is countable for every countable set \(X\).

For example, \(\mathcal{F} = [A]^{\omega}\) is cofinal in \([A]^{\omega}\) but not Kurepa, for any infinite set \(A\). If \(A\) is uncountable, any countable \(\mathcal{F} \subseteq [A]^{\omega}\) is Kurepa but not cofinal. Generally, if \(\mathcal{F}\) is cofinal then so is every \(\mathcal{G} \supseteq \mathcal{F}\), and if \(\mathcal{F}\) is Kurepa then so is every \(\mathcal{G} \subseteq \mathcal{F}\). Thus cofinal families are “large” and Kurepa families are “small” but in different senses. A subset of \([A]^{\omega}\) is a cofinal Kurepa family in \([A]^{\omega}\) if it is both cofinal in \([A]^{\omega}\) and Kurepa.

For example, \(\{\alpha : \omega \leq \alpha < \omega_1\}\) is a cofinal Kurepa family in \([\omega_1]^{\omega}\). (Here, as usual, we adopt the convention that an ordinal is equal to the set of its predecessors.) Note that the (non-)existence of cofinal Kurepa families in \([A]^{\omega}\) depends only on the cardinality of \(A\), so in fact this shows there is a cofinal Kurepa family in \([A]^{\omega}\) for every set \(A\) of cardinality \(\aleph_1\). It turns out that the same is true for every cardinal below \(\aleph_\omega\):

**Theorem 2.1.** If \(|A| < \aleph_\omega\) then there is a cofinal Kurepa family in \([A]^{\omega}\).

**Proof.** See [29, Corollary 7.6.22]. \(\square\)

The situation is more subtle for cardinals \(\geq \aleph_\omega\), as described in the following two theorems.

**Theorem 2.2.** Suppose \(\square_{\kappa}\) holds for every uncountable cardinal \(\kappa\) with \(\text{cf}(\kappa) = \omega\). Then there is a cofinal Kurepa family in \([A]^{\omega}\) for every set \(A\).

**Proof.** See [29, Theorem 7.6.26] and the comments following its proof. \(\square\)

**Theorem 2.3.** It is consistent relative to a huge cardinal that there is no cofinal Kurepa family in \([\omega_\omega]^{\omega}\).
Proof. The generalized Chang Conjecture \((\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_\omega)\) implies there is no cofinal Kurepa family on \(\omega_\omega\). (This is well known, and is in fact a relatively straightforward consequence of \((\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_\omega)\). It is mentioned without proof in [29, Section 7.6], and an indirect proof can be found in [19, Section 3].) The consistency of \((\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_\omega)\) was first proved in [20] from a large cardinal hypothesis a little stronger than the existence of a huge cardinal. Later improvements in [9] reduced the necessary hypothesis to “just” a huge cardinal. \(\square\)

For every set \(A\), define
\[\text{cf}[A]^\mu = \min \{|F| : F \text{ is cofinal in } [A]^{\omega}\}.\]
Note that \(\text{cf}[A]^\omega\) depends only on the cardinality of \(A\).

If \(F\) is cofinal in \([A]^{\omega}\) then \(\bigcup F = A\), so \(|A| = |\bigcup F| \leq \aleph_0 \cdot |F|\). If \(A\) is uncountable, this implies \(|F| \geq |A|\). Thus \(\text{cf}[\kappa]^{\omega} \geq \kappa\) for all uncountable cardinals \(\kappa\). And if \(\text{cf}(\kappa) = \omega\), we get an even stronger bound: a diagonalization argument shows in this case that \(\text{cf}[\kappa]^{\omega} \geq \kappa^+\).

Shelah’s Strong Hypothesis, abbreviated SSH, is the following statement.

\textbf{SSH :} For every uncountable cardinal \(\kappa\),
\[\text{cf}[\kappa]^{\omega} = \begin{cases} \kappa & \text{if } \text{cf}(\kappa) > \omega, \\ \kappa^+ & \text{if } \text{cf}(\kappa) = \omega. \end{cases}\]

In other words, SSH says that \(\text{cf}[\kappa]^{\omega}\) should be as small as possible (subject to the restrictions in the previous paragraph) for every uncountable cardinal \(\kappa\). The original phrasing of Shelah’s Strong Hypothesis by Shelah is somewhat different: it involves the pseudo-power function from pcf theory. The statement here labeled SSH is equivalent to the original form of Shelah’s Strong Hypothesis by a theorem of Matet [21].

\textbf{Theorem 2.4.} Suppose \(0^\dagger\) does not exist. Then SSH holds, and for every set \(A\) there is a cofinal Kurepa family in \([A]^{\omega}\).

Proof. By work in [15] and [29], if \(\Box_\kappa\) fails for a singular cardinal \(\kappa\), then there is an inner model for a Woodin cardinal. This implies “\(0^\dagger\) exists” (with plenty of room to spare). By contraposition, if \(0^\dagger\) does not exist then \(\Box_\kappa\) holds for every singular \(\kappa\). By Theorem 2.2 this implies there is a cofinal Kurepa family in \([A]^{\omega}\) for every set \(A\).

It is worth mentioning that the failure of \(\Box_\kappa\) for singular \(\kappa\) is possibly much stronger than the existence of a Woodin cardinal. (This is already known to be true for \(\kappa > c\); see [26, Corollary 5].) But in the context of this paper, we are more concerned with \(\Box_\kappa\) for \(\kappa < c\).) To quote [26], “the extent to which we can . . . obtain lower bounds on the large cardinal consistency strength of the failure of \(\Box_\kappa\) . . . is limited by our ability to construct core models and prove covering theorems for them.”

The bottleneck for this theorem is not the existence of cofinal Kurepa families, which follows from \(\Box_\kappa\) holding for singular \(\kappa\), but rather SSH. The
exact consistency strength of $\neg$SSH is not known. The failure of SSH at some $\kappa > c$ is equivalent to the failure of the Singular Cardinals Hypothesis, SCH. The exact consistency strength of this is known, due to work of Gitik [11, 12]: it is equiconsistent with the existence of a measurable cardinal $\mu$ with Mitchell order $\mu^{++}$.

Gitik’s work makes use of the so-called weak covering lemma holding over inner models of GCH. The weak covering lemma does not seem to tell us anything about SSH for cardinals $\kappa < c$. However, SSH follows from the full covering lemma holding over an inner model of GCH. (This was first observed in [16, Lemmas 4.9 and 4.10].)

By work of Dodd and Jensen [6], if there is no inner model containing a measurable cardinal, then the covering lemma holds over an inner model of GCH (specifically, the Dodd-Jensen core model $K^{DJ}$). By further work of Dodd and Jensen [7], if there is an inner model containing a measurable cardinal but $0^\dagger$ does not exist, then the covering lemma holds over an inner model of GCH (specifically, either $L[\mu]$ for some measure $\mu$ with crit($\mu$) as small as possible, or $L[\mu,C]$ for some sequence $C$ Prikry-generic over $L[\mu]$). For more information, see Mitchell’s article [24].

Either way, if $0^\dagger$ does not exist then Jensen’s covering lemma holds over an inner model $M$ of GCH, and SCH follows by the results in [16]. □

To close this section, we include one more fact concerning cofinal Kurepa families that will be used in what follows.

**Lemma 2.5.** Every cofinal Kurepa family in $[A]^\omega$ has cardinality $\text{cf}[A]^\omega$.

**Proof.** Suppose $\mathcal{F}$ is a cofinal Kurepa family in $[A]^\omega$. Because $\mathcal{F}$ is cofinal, $|\mathcal{F}| \geq \text{cf}[A]^\omega$. To prove the reverse inequality, fix some cofinal $\mathcal{C} \subseteq [A]^\omega$ with $|\mathcal{C}| = \text{cf}[A]^\omega$.

For every $X \in \mathcal{F}$, choose a set $c(X) \in \mathcal{C}$ with $c(X) \supseteq X$ (which can be done because $\mathcal{C}$ is cofinal). If $|\mathcal{C}| = \text{cf}[A]^\omega < |\mathcal{F}|$, then by the pigeonhole principle there is some particular $Y \in \mathcal{C}$ such that $\{X \in \mathcal{F}: c(X) = Y\}$ is uncountable. But then $\{X \cap Y: X \in \mathcal{F}\} \supseteq \{X \in \mathcal{F}: c(X) = Y\}$ is uncountable, contradicting our assumption that $\mathcal{F}$ is Kurepa.

**Corollary 2.6.** Assuming $0^\dagger$ does not exist, if $\kappa$ is an uncountable cardinal with $\text{cf}(\kappa) = \omega$ and $|A| = \kappa$, then there is a cofinal Kurepa family in $[A]^\omega$ with cardinality $\kappa^+$.

**Proof.** This follows immediately from Theorem 2.4 and Lemma 2.5. □

### 3. A PROOF OF THE MAIN THEOREM

We begin this section by observing that the definition of $\text{sp}$(Borel) does not depend on $\mathbb{R}$: it remains unchanged when $\mathbb{R}$ is replaced by any other uncountable Polish space. This is proved already as Theorem 2.1 in [4], but the proof is short, and is used in the proof of our main theorem, so we reproduce the argument here.
Lemma 3.1. If $X$ is any uncountable Polish space, then

$$\text{sp}(\text{Borel}) = \{ \kappa > \aleph_0 : \text{there is a partition of } X \text{ into } \kappa \text{ Borel sets} \}.$$  

Proof. By a theorem of Kuratowski (see [13, Theorem 15.6]), any two uncountable Polish spaces are Borel isomorphic: in other words, there is a bijection $f : \mathbb{R} \to X$ such that $A \subseteq \mathbb{R}$ is Borel if and only if $f[A]$ is Borel. Thus if $\mathcal{P}$ is any partition of $\mathbb{R}$ into Borel sets, then $\{ f[B] : B \in \mathcal{P} \}$ is a partition of $X$ into Borel sets, and if $\mathcal{Q}$ is any partition of $X$ into Borel sets, then $\{ f^{-1}[B] : B \in \mathcal{Q} \}$ is a partition of $\mathbb{R}$ into Borel sets. \hfill $\square$

Theorem 3.2. Let $\kappa$ be a cardinal and suppose there is a cofinal Kurepa family in $[\kappa]^\omega$. If $\kappa \in \text{sp}(\text{Borel})$, then $\text{cf}[\kappa]^\omega \in \text{sp}(\text{Borel})$.  

Proof. Suppose $\kappa \in \text{sp}(\text{Borel})$, and let $\mathcal{P}$ be a partition of $\mathbb{R}$ into $\kappa$ Borel sets. Because there is a cofinal Kurepa family in $[\kappa]^\omega$ and $|\mathcal{P}| = \kappa$, there is a cofinal Kurepa family in $[\mathcal{P}]^\omega$. Let $\mathcal{K}$ be some such family.

To prove the theorem, it suffices (by Lemmas 3.1 and 2.5) to show that there is a partition of $\mathbb{R}^\omega$ into $|\mathcal{K}|$ Borel sets. We adopt the convention that points in $\mathbb{R}^\omega$ are functions $\omega \to \mathbb{R}$, so that $x(n)$ denotes the $n$th coordinate of $x$ for any given $x \in \mathbb{R}^\omega$.

For every $A \in \mathcal{K}$, define

$$X_A = (\bigcup A)^\omega = \{ x \in \mathbb{R}^\omega : x(n) \in \bigcup A \text{ for all } n \in \omega \}.$$  

Note that, because each $A \in \mathcal{K}$ is a countable collection of Borel subsets of $\mathbb{R}$, each $X_A$ is a Borel subset of $\mathbb{R}^\omega$.

We claim that $\bigcup \{ X_A : A \in \mathcal{K} \} = \mathbb{R}^\omega$. To see this, let $x \in \mathbb{R}^\omega$. For each $n \in \omega$, there is some $B_n \in \mathcal{P}$ with $x(n) \in B_n$. Because $\mathcal{K}$ is cofinal in $[\mathcal{P}]^\omega$, there is some $A \in \mathcal{K}$ with $\{ B_n : n \in \omega \} \subseteq A$. This implies $x \in X_A$.

Let $<$ be a well ordering of $\mathcal{K}$, and for each $A \in \mathcal{K}$ define

$$Y_A = X_A \setminus \bigcup \{ X_B : B \prec A \}.$$  

Because $\bigcup \{ X_A : A \in \mathcal{K} \} = \mathbb{R}^\omega$, the $Y_A$’s form a partition of $\mathbb{R}^\omega$, or more precisely (as some of the $Y_A$’s may be empty), $Q = \{ Y_A : A \in \mathcal{K} \} \setminus \{ \emptyset \}$ is a partition of $\mathbb{R}^\omega$. To prove the theorem, it remains to show that $|Q| = |\mathcal{K}|$, and that every $Y_A$ is a Borel subset of $\mathbb{R}^\omega$.

To see that $|Q| = |\mathcal{K}|$, it suffices to show that $Y_A \neq \emptyset$ for $|\mathcal{K}|$-many $A$. (This is because it is clear from the definition that $Y_A \cap Y_B = \emptyset$ whenever $A \neq B$; the only non-injectivity in the map $A \mapsto Y_A$ is that it may map many sets to $\emptyset$.) For every $A \in \mathcal{K}$, define

$$Z_A = \{ B \in \mathcal{P} : x(n) \in B \text{ for some } x \in Y_A \text{ and some } n \in \omega \}.$$  

Note that $Z_A = \emptyset$ whenever $Y_A = \emptyset$. For every $A \in \mathcal{K}$, we have $Y_A \subseteq X_A$, which means that

$$Z_A \subseteq \{ B \in \mathcal{P} : x(n) \in B \text{ for some } x \in X_A \text{ and some } n \in \omega \} = A.$$  

In particular, $\{ Z_A : A \in \mathcal{K} \}$ is a collection of countable subsets of $\mathcal{P}$. We claim this collection is cofinal in $[\mathcal{P}]^\omega$. To see this, let $\{ B_n : n \in \omega \}$ be an
arbitrary countable subset of $\mathcal{P}$, and for each $n \in \omega$ fix some $x_n \in B_n$. Consider the point $x \in \mathbb{R}^\omega$ with $x(n) = x_n$ for all $n \in \omega$. Because $\mathcal{Q}$ is a partition of $\mathbb{R}^\omega$, there is some $A \in \mathcal{K}$ with $x \in Y_A$. It follows that $\{B_n : n \in \omega\} \subseteq Z_A$. As $\{B_n : n \in \omega\}$ was an arbitrary member of $[\mathcal{P}]^\omega$, this shows $\{\{Z_A : A \in \mathcal{K}\} = |\mathcal{K}|$. In particular, $Z_A \neq \emptyset$ for $|\mathcal{K}|$-many $A \in \mathcal{K}$. Hence $\{Z_A : A \in \mathcal{K}\}$ is cofinal in $[\mathcal{P}]^\omega$. Hence $|\mathcal{Q}| = |\mathcal{K}|$.

To see that each $Y_A$ is Borel in $\mathbb{R}^\omega$, we use the Kurepa property of $\mathcal{K}$. Fix $A \in \mathcal{K}$. Because $\mathcal{K}$ is Kurepa, there is a countable $F \subseteq \mathcal{K}$ such that $\{A \cap B : B \in \mathcal{K}\} = \{A \cap B : B \in F\}$. This implies $\{A \cap B : B \in \mathcal{K} \text{ and } B \prec A\} = \{A \cap B : B \in G\}$ for some (countable) $G \subseteq F$.

Because $X_A$ and all of the $X_B$'s are Borel sets in $\mathbb{R}^\omega$, and because $\mathcal{G}$ is countable, it follows that $Y_A$ is Borel.

**Theorem 3.3.** Assume $\mathcal{Q}$† does not exist. If $\kappa$ is a singular cardinal with $\text{cf}(\kappa) = \omega$ and $\kappa \in \text{sp}(\text{Borel})$, then $\kappa^+ \in \text{sp}(\text{Borel})$.

**Proof.** This follows immediately from Corollary 2.6 and Theorem 3.2. 

This completes our first proof of the main theorem. Note that the hypothesis “$\mathcal{Q}$† does not exist” can be replaced with the possibly weaker hypothesis “$\text{cf}(\kappa)^\omega = \kappa^+$ and there is a cofinal Kurepa family in $[\kappa]^\omega$.”

**Question 3.4.** Is it consistent, relative to some large cardinal hypothesis, that there is a singular cardinal $\lambda$ with $\text{cf}(\lambda) = \omega$ such that $\lambda \in \text{sp}(\text{Borel})$ but $\lambda^+ \notin \text{sp}(\text{Borel})$?

**Question 3.5.** In particular, is it consistent relative to some large cardinal hypothesis that $\aleph_\omega \in \text{sp}(\text{Borel})$ but $\aleph_{\omega+1} \notin \text{sp}(\text{Borel})$?

**Question 3.6.** Is it consistent, relative to some large cardinal hypothesis, that there is a singular cardinal $\lambda$ with $\text{cf}(\lambda) = \omega$ such that $\lambda \in \text{sp}(\text{Borel})$ but $\text{cf}(\lambda)^\omega \notin \text{sp}(\text{Borel})$?

4. A SECOND PROOF OF THE MAIN THEOREM

In this section we give our second proof of the main theorem. This second proof has a more topological flavor than the one presented in the previous section. Throughout this section, all ordinals are considered to have the
discrete topology. Thus, for example, $\kappa^\omega$ is a completely metrizable space, sometimes called the Baire space of weight $\kappa$ (e.g., in [8] Example 4.2.12).

Given a topological space $X$, define

$$\text{par}(X) = \min \{|P| : P \text{ is a partition of } X \text{ into Polish spaces}\}.$$ 

Note that $\text{par}(X)$ is well-defined and $\leq |X|$ for every space $X$, because we may partition $X$ into singletons. The invariant $\text{par}(X)$ was introduced and studied in [3], the main idea being to compare $\text{par}(X)$ with $\text{cov}(X)$ (defined as the least size of a covering of $X$ with Polish spaces) when $X$ is completely metrizable.

The next two lemmas, which are implicit in [2] and [5], allow us to transfer facts about the spaces $\kappa^\omega$ and $\text{par}(\kappa^\omega)$ to facts about $\text{sp}(\text{Borel})$.

**Lemma 4.1.** If there is a continuous bijection $X \to \mathbb{R}$, for some space $X$, then there is a partition of $\mathbb{R}$ into $\text{par}(X)$ Borel sets.

**Proof.** Suppose $f : X \to \mathbb{R}$ is a continuous bijection, and let $P$ be a partition of $X$ into $\text{par}(X)$ Polish spaces. By a theorem of Lusin and Suslin [18, Theorem 15.1], if $A$ is Polish and $g : A \to \mathbb{R}$ is a continuous injection, then $g[A]$ is Borel in $\mathbb{R}$ (in fact, $g[B]$ is Borel in $\mathbb{R}$ for every Borel $B \subseteq A$). In particular, $f \mid A$ is a continuous injection $A \to \mathbb{R}$ for every $A \in P$, which means $f[A]$ is Borel in $\mathbb{R}$ for every $A \in P$. Hence $\{f[A] : A \in P\}$ is a partition of $\mathbb{R}$ into Borel sets. $\square$

**Lemma 4.2.** Let $\kappa$ be an infinite cardinal. If there is a partition of $\mathbb{R}$ into $\kappa$ Borel sets, then there is a continuous bijection $\kappa^\omega \to \mathbb{R}$.

**Proof.** Suppose $\kappa \in \text{sp}(\text{Borel})$. By Lemma 3.11 this implies there is a partition $P$ of the Baire space $\omega^\omega$ into $\kappa$ Borel sets.

By a theorem of Lusin and Suslin [18, Theorem 13.7], every Borel subset of a Polish space is the bijective continuous image of a closed subset of $\omega^\omega$. In particular, for every $A \in P$, there is a closed $F_A \subseteq \omega^\omega$ and a continuous bijection $f_A : F_A \to A$. This implies that the map $(x, y) \mapsto (f_A(x), y)$ is a continuous bijection $F_A \times \omega^\omega \to A \times \omega^\omega$. By the Alexandroff-Urysohn characterization of the Baire space $\omega^\omega$ [8, Exercise 7.2.G], $F_A \times \omega^\omega \approx \omega^\omega$. Thus for every $A \in P$, there is a continuous bijection $g_A : \omega^\omega \to A \times \omega^\omega$.

Let $Q = \{A \times \omega^\omega : A \in P\}$; this is a partition of $\omega^\omega \times \omega^\omega$ into $\kappa$ Borel sets. Taking the disjoint union of the mappings $g_A$ from the previous paragraph, we obtain a continuous bijection $G : \kappa \times \omega^\omega \to \omega^\omega \times \omega^\omega$.

Let $G^\omega$ denote the mapping $(x_0, x_1, x_2, \ldots) \mapsto (G(x_0), G(x_1), G(x_2), \ldots)$ (sometimes called the “diagonal mapping”). Because $G$ is a continuous bijection $\kappa \times \omega^\omega \to \omega^\omega \times \omega^\omega$, the diagonal mapping $G^\omega$ is a continuous bijection $(\kappa \times \omega^\omega)^\omega \to (\omega^\omega \times \omega^\omega)^\omega$. But $(\kappa \times \omega^\omega)^\omega \approx \kappa^\omega$ and $(\omega^\omega \times \omega^\omega)^\omega \approx \omega^\omega$ (both of these facts follow from [8, Exercise 7.2.G]), so this shows there is a continuous bijection $\kappa^\omega \to \omega^\omega$.

To finish the proof, simply compose this continuous bijection $\kappa^\omega \to \omega^\omega$ with a continuous bijection $\omega^\omega \to \mathbb{R}$. (Recall that every Polish space without isolated points is a continuous bijective image of $\omega^\omega$ [18, Exercise 7.15].) $\square$
Theorem 4.3. Let \( \kappa \) be an uncountable cardinal. If \( \kappa \in \text{sp}(\text{Borel}) \), then \( \text{par}(\kappa^\omega) \in \text{sp}(\text{Borel}) \) also.

Proof. This follows immediately from Lemmas 4.1 and 4.2. \( \square \)

Note the similarity between Theorem 3.2 and Theorem 4.3. Our next theorem shows that \( \text{par}(\kappa^\omega) = \text{cf}[\kappa]^{\omega} \) if there is a cofinal Kurepa family on \( \kappa \), which will complete our second proof of Theorem 3.2 and the main theorem (which follows directly from it). We note that the equality \( \text{par}(\kappa^\omega) = \text{cf}[\kappa]^{\omega} \) need not be true in general: it is consistent relative to a huge cardinal to have \( \text{par}(\kappa^\omega) > \text{cf}[\kappa]^{\omega} \) (see [3, Section 3]).

Theorem 4.4. Let \( \kappa \) be an uncountable cardinal. If there is a cofinal Kurepa family on \( \kappa \), then \( \text{par}(\kappa^\omega) = \text{cf}[\kappa]^{\omega} \).

The following proof contains some ideas similar to what is found in [3, Sections 2 and 4], where what is called Corollary 4.5 below was first proved. Something similar to the proof of Theorem 4.4 is implicit in [3], though Jensen matrices are used there rather than Kurepa families. However, the arguments in [3] are aimed at proving something rather different, and they feature several ideas that are not relevant here. What follows is a relatively short proof, modifying and distilling from [3] what is needed for our analysis of \( \text{sp}(\text{Borel}) \).

Proof of Theorem 4.4. Let \( \mathcal{K} \) be a cofinal Kurepa family on \( \kappa \). To prove the theorem, it suffices, by Lemma 2.5, to show there is a partition of \( \kappa^\omega \) into \( |\mathcal{K}| \) Polish spaces. Recall that a subspace of a completely metrizable space is itself completely metrizable if and only if it is \( G_\delta \). Thus a subspace of \( \kappa^\omega \) is Polish if and only if it is both second countable and \( G_\delta \).

For every \( A \in \mathcal{K} \), note that \( A^\omega \) is second countable (because \( A \) is countable) and closed in \( \kappa^\omega \). Furthermore, we claim \( \kappa^\omega \) is covered by the sets of this form: i.e., \( \bigcup \{ A^\omega : A \in \mathcal{K} \} = \kappa^\omega \). To see this, let \( x \in \kappa^\omega \). As \( \mathcal{K} \) is cofinal in \( [\kappa]^{\omega} \), there is some \( A \in \mathcal{K} \) with \( \{ x(n) : n \in \omega \} \subseteq A \), and this means \( x \in A^\omega \). As \( x \) was arbitrary, \( \bigcup \{ A^\omega : A \in \mathcal{K} \} = \kappa^\omega \).

Let \( \prec \) be a well ordering of \( \mathcal{K} \), and for each \( A \in \mathcal{K} \) define \( Y_A = A^\omega \setminus \bigcup \{ B^\omega : B \prec A \} \).

Because \( \bigcup \{ A^\omega : A \in \mathcal{K} \} = \kappa^\omega \), the \( Y_A \)'s form a partition of \( \kappa^\omega \), or more precisely (because some of the \( Y_A \) may be empty), \( Q = \{ Y_A : A \in \mathcal{K} \} \setminus \{ \emptyset \} \) is a partition of \( \kappa^\omega \). To prove the theorem, it remains to show that \( |Q| = |\mathcal{K}| \), and that every \( Y_A \in Q \) is Polish.

To see that \( |Q| = \text{cf}[\kappa]^{\omega} \), define for every \( A \in \mathcal{K} \)
\[
Z_A = \{ \alpha \in \kappa : x(n) = \alpha \text{ for some } x \in Y_A \text{ and some } n \in \omega \}.
\]

Note that we have \( Z_A = \emptyset \) whenever \( Y_A = \emptyset \). For every \( A \in \mathcal{K} \), we have \( Y_A \subseteq X_A \), which means that
\[
Z_A \subseteq \{ \alpha \in \kappa : x(n) = \alpha \text{ for some } x \in A^\omega \text{ and some } n \in \omega \} = A.
\]
In particular, \( \{ Z_A : A \in K \} \) is a collection of countable subsets of \( \kappa \). We claim this collection is cofinal in \( [\kappa]^\omega \). To see this, let \( \{ x_n : n \in \omega \} \) be an arbitrary countable subset of \( \kappa \), and define \( x \in \kappa^\omega \) by setting \( x(n) = x_n \) for all \( n \in \omega \). Because \( Q \) is a partition of \( \kappa^\omega \), there is some \( A \in K \) with \( x \in Y_A \). It follows that \( \{ x_n : n \in \omega \} \subseteq Z_A \). As \( \{ x_n : n \in \omega \} \) was arbitrary, it follows that \( \{ Z_A : A \in K \} \) is cofinal in \( [\kappa]^\omega \). Hence \( |\{ Z_A : A \in K \}| \geq \text{cf}[\kappa]^\omega = |K| \).

In particular, \( Z_A \neq \emptyset \) for \( |K| \)-many \( A \in K \). Consequently, \( Y_A \neq \emptyset \) for \( |K| \)-many \( A \in K \). As the \( Y_A \) are disjoint, this implies \( |Q| = |K| \).

To see that each \( Y_A \) is Polish, we use the fact that \( K \) is Kurepa. Fix \( A \in K \). Because \( K \) is Kurepa, there is a countable \( F \subseteq K \) such that \( \{ A \cap B : B \in K \} = \{ A \cap B : B \in F \} \). This implies \( \{ A \cap B : B \in K \text{ and } B \prec A \} = \{ A \cap B : B \in G \} \) for some (countable) \( G \subseteq F \).

Because \( A^\omega \) is second countable and closed, and each of the \( B^\omega \) is also closed, this shows \( Y_A \) is second countable and \( G_\delta \), i.e., Polish.

**Corollary 4.5.** If \( 0 < n < \omega \), then \( \text{par}(\omega_n^\omega) = \aleph_n \). Furthermore, if \( 0^\dagger \) does not exist then for every cardinal \( \kappa > \aleph_0 \),

\[
\text{par}(\kappa^\omega) = \text{cf}[\kappa]^\omega = \begin{cases} 
\kappa & \text{if } \text{cf}(\kappa) > \omega, \\
\kappa^+ & \text{if } \text{cf}(\kappa) = \omega.
\end{cases}
\]

**Proof.** This follows from Corollary 2.6 and Theorems 2.1 and 4.4. \( \square \)

**Theorem 4.6.** Assume \( 0^\dagger \) does not exist, and let \( \kappa \) be a singular cardinal with \( \text{cf}(\kappa) = \omega \). If \( \kappa \in \text{sp}(\text{Borel}) \), then \( \kappa^+ \in \text{sp}(\text{Borel}) \) also.

**Proof.** This follows immediately from Theorem 4.3 and Corollary 4.5. \( \square \)

This completes the second proof of our main theorem. To end this section, we point out how the results from this section relate to what is sometimes called the **Banach problem**, posed by Stefan Banach in the Scottish Book:

> When does a metric space admit a continuous bijective mapping onto a compact metric space?

**Theorem 4.7.** Let \( \kappa \) be an uncountable cardinal with \( \text{par}(\kappa^\omega) = \kappa \). Then the following are equivalent:

1. \( \kappa \in \text{sp}(\text{Borel}) \).
2. There is a continuous bijective mapping from \( \kappa^\omega \) onto \( \mathbb{R} \).
3. There is a continuous bijective mapping from \( \kappa^\omega \) onto a compact metric space.
Proof. (1) ⇒ (2) was proved in Lemma 4.2. (2) ⇒ (3) because there is a continuous bijection from \( \mathbb{R} \) to a compact metric space: for example, it is easy to describe a continuous bijection from \( \mathbb{R} \) to a wedge sum of two circles (an “8”). Finally, (3) ⇒ (1) by the proof of Lemma 4.1, as the role of \( \mathbb{R} \) in that proof can be played equally well by any compact metric space. □

By Theorem 4.4, the hypothesis of this theorem is satisfied by \( \kappa = \aleph_n \) for any \( n < \omega \); or if \( 0^\dagger \) does not exist, it is satisfied by any cardinal \( \kappa \) with \( \text{cf}(\kappa) > \omega \).

A special case of this theorem (for \( \kappa < \aleph_\omega \)) was proved already in [25] and [5]. In [25, Theorem 2.1], it is shown that the three equivalent conditions of Theorem 4.7 also imply that if \( X \) is any Banach space of weight \( \kappa \), then there is a continuous bijection from \( X \) to the Hilbert cube \([0,1]^{\omega}\).

The implication (1) ⇒ (2) in Theorem 4.7 holds for any \( \kappa \). But the results of this section tell us nothing about whether the converse may hold for singular cardinals of countable cofinality. (It is proved in [3, Section 2] that if \( \text{cf}(\kappa) = \omega \) then \( \text{par}(\kappa^{\omega}) \geq \text{cf}(\kappa)^{\omega} \geq \kappa^+ \).)

**Question 4.8.** Suppose \( \lambda \) is a singular cardinal with \( \text{cf}(\lambda) = \omega \), and there is a continuous bijection \( \lambda^{\omega} \to \mathbb{R} \). Does this imply \( \lambda \in \text{sp}(\text{Borel}) \)?

**Question 4.9.** In particular, is it consistent that there is a continuous bijection \( \omega^{\omega} \to \mathbb{R} \), but that \( \aleph_\omega / \notin \text{sp}(\text{Borel}) \)?

## 5. Other partition spectra

Given a pointclass \( \Gamma \) of sets, define

\[
\text{sp}(\Gamma) = \{ |P| : P \text{ is a partition of } \mathbb{R} \text{ into uncountably many sets in } \Gamma \}. 
\]

Many of the known results concerning \( \text{sp}(\text{Borel}) \) listed in the introduction apply equally well to other pointclasses of sets. For example, both \( \text{sp}(\text{closed}) \) and \( \text{sp}(\text{OD}(\mathbb{R})) \) satisfy the analogues of statements (2)-(6) in the introduction [41]. They also both satisfy the conclusion of the main result of the present paper. For \( \text{sp}(\text{OD}(\mathbb{R})) \) this is relatively easy to see: the proof given in Section 3 generalizes readily to any pointclass \( \Gamma \) that is closed under taking countable intersections and relative complements.

**Theorem 5.1.** Suppose \( \Gamma \) is a pointclass closed under taking countable intersections and relative complements. Assuming \( 0^\dagger \) does not exist, if \( \kappa \) is a singular cardinal with \( \text{cf}(\kappa) = \omega \) and \( \kappa \in \text{sp}(\Gamma) \), then \( \kappa^+ \in \text{sp}(\Gamma) \).

We now show that the conclusion of this theorem applies to several familiar pointclasses that are not closed under taking countable intersections and relative complements.

**Theorem 5.2.** \( \text{sp}(\text{Borel}) = \text{sp}(\Sigma^1_1) = \text{sp}(\Pi^1_1) = \text{sp}(\Sigma^1_2) = \text{sp}(\aleph_1\text{-Suslin}) \).

**Proof.** Let \( \Gamma \) be any of the latter four pointclasses in the statement of the theorem. Every set in \( \Gamma \) can be be written as a union of \( \aleph_1 \) Borel sets (see [17, Chapter 13]). This implies every set in \( \Gamma \) can be partitioned into \( \leq \aleph_1 \) Borel
sets. Thus any partition $\mathcal{P}$ of $\mathbb{R}$ into sets in $\Gamma$ can be refined to a partition of $\mathbb{R}$ into Borel sets, by replacing each set in $\mathcal{P}$ with $\leq \aleph_1$ Borel sets; and if $\mathcal{P}$ is uncountable, any such refinement has size $|\mathcal{P}|$. Thus $\text{sp}(\Gamma) \subseteq \text{sp}(\text{Borel})$. The reverse inclusion is immediate: every Borel set is in $\Gamma$, so a partition of $\mathbb{R}$ into Borel sets is already a partition of $\mathbb{R}$ into sets in $\Gamma$. 

From this and Theorem 3.3 it follows that the conclusion of Theorem 5.1 applies when $\Gamma \in \{\text{analytic}, \text{coanalytic}, \Sigma^1_3, \mathcal{N}_1-\text{Suslin}\}$.

Next we show that the same is true for $\Gamma = \text{closed}$. It is known that, consistently, $\text{sp}(\text{closed}) \neq \text{sp}(\text{Borel})$. This result can be attributed to Sierpiński 27, who showed (in our terminology) that $\min(\text{sp}(\text{closed})) \geq \text{cov}(\mathcal{M})$, which means we need not have $\aleph_1 \in \text{sp}(\text{closed})$. Nonetheless, we show here that $\text{sp}(\text{closed})$ is a final segment of $\text{sp}(\text{Borel})$, which is enough to guarantee that Theorem 5.1 holds for $\Gamma = \text{closed}$.

Recall that the cardinal invariant $\alpha_T$ denotes the least size of a partition of the Baire space $\omega^\omega$ into compact sets. (The notation “$\alpha_T$” derives from the fact that $\alpha_T$ is equal to the least size of a maximal infinite family of almost disjoint subtrees of $2^{<\omega}$.) It is known that $\mathfrak{d} \leq \alpha_T$, and that this inequality is consistently strict 28.

**Lemma 5.3.** For any uncountable Polish space $X$, 

$$\text{sp}(\text{closed}) = \{\kappa > \aleph_0 : \text{there is a partition of } X \text{ into } \kappa \text{ compact sets}\}$$

$$\qquad = \{\kappa > \aleph_0 : \text{there is a partition of } X \text{ into } \kappa \text{ closed sets}\}$$

$$\qquad = \{\kappa > \aleph_0 : \text{there is a partition of } X \text{ into } \kappa \text{ } F_\sigma \text{ sets}\}.$$

*Proof.* See [4, Corollary 2.5].

**Lemma 5.4.** $\text{sp}(\text{compact}) = \text{sp}(\text{closed}) = \text{sp}(F_\sigma) = [\alpha_T, \mathfrak{c}] \cap \text{sp}(\text{Borel})$.

*Proof.* By Lemma 5.3, it suffices to show $\text{sp}(\text{compact}) = [\alpha_T, \mathfrak{c}] \cap \text{sp}(\text{Borel})$.

By Lemma 5.3 and the definition of $\alpha_T$, $\text{sp}(\text{compact}) \subseteq [\alpha_T, \mathfrak{c}]$. And clearly every partition of $\mathbb{R}$ into compact sets is also a partition into Borel sets, so $\text{sp}(\text{compact}) \subseteq \text{sp}(\text{Borel})$. Hence $\text{sp}(\text{compact}) \subseteq [\alpha_T, \mathfrak{c}] \cap \text{sp}(\text{Borel})$.

Now suppose $\kappa \in [\alpha_T, \mathfrak{c}] \cap \text{sp}(\text{Borel})$, and let $\mathcal{P}$ be a partition of $\mathbb{R}$ into $\kappa$ Borel sets. By a theorem of Lusin and Souslin 18, Theorem 13.7, every Borel subset of a Polish space is the bijective continuous image of a closed subset of $\omega^\omega$. In particular, for every $A \in \mathcal{P}$, there is a continuous bijection $f_A : F_A \to A$, where $F_A$ is a closed subset of $\omega^\omega$. Every closed subset of $\omega^\omega$ is Polish, so for each $A \in \mathcal{P}$ there are two possibilities: (1) $A$ is countable, thus so is $F_A$, or (2) $A$ and $F_A$ are uncountable, and by Lemma 5.3 there is a partition of $F_A$ into $\alpha_T$ compact sets. Let $\mathcal{P}_1 = \{A \in \mathcal{P} : A \text{ is countable}\}$ and $\mathcal{P}_2 = \mathcal{P} \setminus \mathcal{P}_1$. For each $A \in \mathcal{P}_2$, fix a partition $\mathcal{Q}_A$ of $A$ into $\alpha_T$ compact sets. Finally, observe that

$$\{\{x\} : x \in A \text{ for some } A \in \mathcal{P}_1\} \cup \{f_A[X] : A \in \mathcal{P}_2 \text{ and } X \in \mathcal{Q}_A\}$$

is a partition of $\mathbb{R}$ into $\kappa$ compact sets. \hfill $\square$
Corollary 5.5. Assume $0^+$ does not exist. If $\kappa$ is a singular cardinal with $\text{cf}(\kappa) = \omega$ and $\kappa \in \text{sp}(\text{closed})$, then $\kappa^+ \in \text{sp}(\text{closed})$.

In addition to $\text{sp}(\text{closed})$, $\text{sp}(\text{Borel})$, and $\text{sp}(\text{OD}(\mathbb{R}))$, which have been studied extensively, one may also find in the literature some interest in $\text{sp}(\text{G}\delta)$. Most notably, Shelah and Fremlin prove in [10] that $\min(\text{sp}(\text{G}\delta)) \geq \text{cov}(\mathcal{M})$ (which implies, in particular, there may be no partition of $\mathbb{R}$ into $\aleph_1$ $G\delta$ sets). This raises the intriguing possibility that an analogue of Lemma 5.4 might hold for $\text{sp}(\text{G}\delta)$.

Question 5.6. Is $\text{cov}(\mathcal{M}) = \min(\text{sp}(\text{G}\delta))$?

Question 5.7. Is $\text{sp}(\text{G}\delta)$ a final segment of $\text{sp}(\text{Borel})$?

Question 5.8. If the answer to Question 5.7 is negative, is it nonetheless the case that Theorem 5.1 holds with $\Gamma = \text{G}\delta$?

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