We present a one-to-one correspondence between the set of admissible pictures and the Littlewood–Richardson crystals. As a simple consequence, we shall show that the set of pictures does not depend on the choice of admissible orders.

Key Words: Crystals; Littlewood–Richardson rules; Pictures; Young diagrams.

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1. INTRODUCTION

On (skew) Young diagrams, we introduce several orders, e.g., $\leq_P$, $\leq_J$, $\leq_A$, etc., (see 2.2) to treat our main subject “picture,” which is a bijective map between two skew Young diagrams, which preserves the order in the following sense: $f : \leq_P \rightarrow \leq_J$ and $f^{-1} : \leq_P \rightarrow \leq_J$ [2, 7, 13]. For Young diagrams $\lambda, \mu, v$ with $|\lambda| + |\mu| = |v|$, let $P(\mu, v\backslash \lambda)$ be the set of pictures from $\mu$ to $v\backslash \lambda$ and $B(\mu)^{\lambda}$ the Littlewood–Richardson crystal as in [12] (see also 3.2). Let $c^{\lambda}_{\lambda, \mu}$ be the usual Littlewood–Richardson number. Then, by the fact

$$zP(\mu, v\backslash \lambda) = c^{\lambda}_{\lambda, \mu} = zB(\mu)^{\lambda},$$

we deduced that there exists a bijection between $P(\mu, v\backslash \lambda)$ and $B(\mu)^{\lambda}$. It has been revealed in [12] that there exists a natural one-to-one correspondence between $P(\mu, v\backslash \lambda)$ and $B(\mu)^{\lambda}$.

We try to generalize the notion of pictures by using “admissible order,” which is an order in a certain class of total orders on a skew diagram (or more generally, a subset of $N \times N$) [1]. Indeed, the order $\leq_J$ is a sort of admissible orders. In the last section of [12], we define the new set of “admissible pictures” associated with admissible orders $A$ on $v\backslash \lambda$ and $A'$ on $\mu$, denoted by $P(\mu, v\backslash \lambda; A, A')$. We also get the Littlewood–Richardson crystal associated with an admissible order, denoted by $B(\mu)^{\lambda}[A']$. Then we present the following conjecture.
Conjecture 1.1 ([12]). Let $A$ (resp., $A'$) be an admissible order on $\nu \setminus \lambda$ (resp., $\mu$). There exists a bijection

$$\Psi: B(\mu)^\lambda_{A'} \longrightarrow P(\mu, \nu \setminus \lambda; A, A').$$

The affirmative answer for this conjecture is given as Theorem 4.1 in Section 4 below.

In [6], it has been shown that the Littlewood–Richardson crystal does not depend on the choice of admissible orders. Furthermore, so the same holds true for the definition of the bijection $\Psi$. Therefore, we obtain the following corollary.

Corollary 1.2. For arbitrary admissible orders $A$ on $\nu \setminus \lambda$ and $A'$ on $\mu$,

$$P(\mu, \nu \setminus \lambda) = P(\mu, \nu \setminus \lambda; A, A').$$

This result has already been obtained in [3, 4] for more general setting. They used some purely combinatorial methods different from ours. Here it can be said that we give a new proof of the pictures’ independence of admissible orders. Our main tool is a procedure “addition” obtained from the tensor products of crystals. It plays a crucial role in the proof, which realizes the Littlewood–Richardson rules in terms of crystals and connects pictures and the Littlewood–Richardson crystals directly.

We have obtained the Littlewood–Richardson crystals for other classical types [10] in the similar description to the type $A_n$. Hence, it allows us to expect that it is possible to generalize the notion of “pictures” to these types.

The organization of the article is as follows. In Section 2, we prepare the ingredients treated in the article, (skew) Young diagrams, Young tableaux, admissible orders, and pictures. In Section 3, we review the crystal-theoretical interpretation of Littlewood–Richardson rules. The definitions of additions and readings are given. The main theorem is stated in Section 4. The last three sections, 5, 6, and 7 are devoted to proving the main theorem.

2. PICTURES

2.1. Young Diagrams and Young Tableaux

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a Young diagram or a partition, which satisfies $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$. Let $\lambda$ and $\mu$ be Young diagrams with $\mu \subset \lambda$. A skew diagram $\lambda \setminus \mu$ is obtained by subtracting set-theoretically $\mu$ from $\lambda$.

In this article, we frequently consider a (skew) Young diagram as a subset of $\mathbb{N} \times \mathbb{N}$ by identifying the box in the $i$th row and the $j$th column with $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Example 2.1. A Young diagram $\lambda = (2, 2, 1)$ is expressed by $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1)\}$.

As in [5], in the sequel, a “Young tableau” means a semistandard tableau. For a Young tableau $T$ of shape $\lambda$, we also consider a “coordinate” in $\mathbb{N} \times \mathbb{N}$ like as $\lambda$. 
Then an entry of $T$ in $(i, j)$ is denoted by $T_{i, j}$ and called a $(i, j)$-entry. For $k > 0$, define [12]

$$T^{(k)} = \{(l, m) \in \lambda | T_{l, m} = k\}. \quad (2.1)$$

There are no two elements in one column in $T^{(k)}$. For a Young tableau $T$ with $(i, j)$-entry $T_{i, j} = k$, we define a function $p(T; i, j)$ ([12]) as the number of $(i, j)$-entry from the right in $T^{(k)}$. It is immediate from the definition

If $T_{i, j} = T_{x, y}$ and $p(T; i, j) = p(T; x, y)$, then $(i, j) = (x, y) \quad (2.2)$

2.2. Picture

First, we shall introduce the original notion of “picture” as in [13].

We define the following two kinds of orders on a subset $X \subset \mathbb{N} \times \mathbb{N}$: For $(a, b), (c, d) \in X$:

(i) $(a, b) \leq_p (c, d)$ iff $a \leq c$ and $b \leq d$;
(ii) $(a, b) \leq_J (c, d)$ iff $a < c$, or $a = c$ and $b \geq d$.

Note that the order $\leq_p$ is a partial order and $\leq_J$ is a total order.

**Definition 2.2** ([13]). Let $X, Y \subset \mathbb{N} \times \mathbb{N}$.

(i) A map $f : X \to Y$ is said to be $PJ$-standard if it satisfies

For $(a, b), (c, d) \in X$, if $(a, b) \leq_p (c, d)$, then $f(a, b) \leq_J f(c, d)$.

(ii) A map $f : X \to Y$ is a picture if it is bijective and both $f$ and $f^{-1}$ are $PJ$-standard.

Taking three Young diagrams $\lambda, \mu, v \subset \mathbb{N} \times \mathbb{N}$, denote the set of pictures by

$$P(\mu, v \setminus \lambda) := \{f : \mu \to v \setminus \lambda | f \text{ is a picture}\}$$

Next, we shall generalize the notion of pictures by using a total order on a subset $X \subset \mathbb{N} \times \mathbb{N}$, called an “admissible order,” which is defined in [1] (see also [6]).

**Definition 2.3.**

(i) A total order $\leq_A$ on $X \subset \mathbb{N} \times \mathbb{N}$ is called admissible if it satisfies:

For any $(a, b), (c, d) \in X$ if $a \leq c$ and $b \geq d$ then $(a, b) \leq_A (c, d)$.

(ii) For $X, Y \subset \mathbb{N} \times \mathbb{N}$ and a map $f : X \to Y$, if $f$ satisfies that if $(a, b) \leq_p (c, d)$ then $f(a, b) \leq_J f(c, d)$ for any $(a, b), (c, d) \in X$, then $f$ is called $PA$-standard.

(iii) Let $\leq_A$ (resp., $\leq_A'$) be an admissible order on $X$ (resp., $Y \subset \mathbb{N} \times \mathbb{N}$). A bijective map $f : X \to Y$ is called an $(A, A')$-admissible picture or simply, an admissible picture if $f$ is $PA$-standard and $f^{-1}$ is $PA'$-standard.
Remark. Note that for fixed $X \subset \mathbb{N} \times \mathbb{N}$, there can be several admissible orders on $X$. For example, the order $\leq_J$ is one of admissible orders on $X$. If we define the total order $\leq_F$ by
\[(a, b) \leq_F (c, d) \text{ iff } b > d, \quad \text{or } b = d \text{ and } a < c,
\]it is also admissible.

For $X, Y \in \mathbb{N} \times \mathbb{N}$, let $\leq_A$ (resp., $\leq_A'$) be an admissible order on $X$ (resp., $Y$). We denote a set of $(A, A')$-admissible pictures by $P(X, Y; A, A')$.

3. CRYSTALS

The basic references for the theory of crystals are [8, 9].

3.1. Readings and Additions

Let $B = \{i \mid 1 \leq i \leq n + 1\}$ be the crystal of the vector representation $V(A_1)$ of the quantum group $U_q(A_1)$ [10]. As in [12], we shall identify a dominant integral weight of type $A_n$ with a Young diagram in the standard way, e.g., the fundamental weight $\Lambda_1$ is identified with a square box $\Box$. For a Young diagram $\lambda$, let $B(\lambda)$ be the crystal of the finite-dimensional irreducible $U_q(A_n)$-module $V(\lambda)$. Set $N := |\lambda|$. Then there exists an embedding of crystals: $B(\lambda) \hookrightarrow B^\otimes N$ and an element in $B(\lambda)$ is realized by a Young tableau of shape $\lambda$ [10]. Such an embedding is not unique. Indeed, they are called a reading and are described by the following definition.

**Definition 3.1** ([6]). Let $A$ be an admissible order on a Young diagram $\mu$ with $|\lambda| = N$. For $T \in B(\lambda)$, by reading the entries in $T$ according to $A$, we obtain the map
\[R_A : B(\lambda) \longrightarrow B^{\otimes N} \quad \left( T \mapsto \left[ i_1 \otimes \cdots \otimes i_N \right] \right),\]
which is called an admissible reading associated with the order $A$. The map $R_A$ is an embedding of crystals.

The following are typical readings.

**Definition 3.2.** Let $T$ be an element in $B(\lambda)$ of type $A_n$, namely, a Young tableau of shape $\lambda$ with entries $\{1, 2, \ldots, n + 1\}$.

(i) We read the entries in $T$ each row from right to left and from the top row to the bottom row, that is, we read the entries according to the order $\leq_J$. Then the resulting sequence of the entries $i_1, i_2, \ldots, i_N$ gives the embedding of crystals
\[\text{ME}(= R_J) : B(\lambda) \hookrightarrow B^{\otimes N} \quad \left( T \mapsto \left[ i_1 \otimes \cdots \otimes i_N \right] \right),\]
which is called a middle-eastern reading.
(ii) We read the entries in $T$ each column from the top to the bottom and from the right-most column to the left-most column, that is, we read the entries according to the order $\preceq_F$. Then the resulting sequence of the entries $i_1, i_2, \ldots, i_N$ gives the embedding of crystals:

$$\text{FE}(= R_F) : B(\lambda) \leftrightarrow B^{\otimes N} \quad \left( T \mapsto \left[ \begin{array}{c} i_1 \\ \vdots \\ i_N \end{array} \right] \right),$$

which is called a far-eastern reading.

**Definition 3.3.** For $i \in \{1, 2, \ldots, n + 1\}$ and a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, we define

$$\lambda[i] := (\lambda_1, \lambda_2, \ldots, \lambda_i + 1, \ldots, \lambda_n),$$

which is said to be an *addition of $i$ to $\lambda$*. In general, for $i_1, i_2, \ldots, i_N \in \{1, 2, \ldots, n + 1\}$ and a Young diagram $\lambda$, we define

$$\lambda[i_1, i_2, \ldots, i_N] := \left( \cdots (\lambda[i_1][i_2]) \cdots \right)[i_N],$$

which is called an *addition of $i_1, \ldots, i_N$ to $\lambda$*.

**Example 3.4.** For a sequence $i = 31212$, the addition of $i$ to $\lambda = \begin{array}{c} \hline \\
\hline \end{array}$ is

$$\begin{array}{c}
\begin{array}{c}
3 \quad \rightarrow \quad 11 \\
\hline
\end{array}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
11 \\
\hline
2
\end{array}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
11 \\
\hline
2
\end{array}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
11 \\
\hline
2
\end{array}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
11 \\
\hline
2
\end{array}
\end{array}.$$

**Remark.** For a Young diagram $\lambda$, an addition $\lambda[i_1, \ldots, i_N]$ is not necessarily a Young diagram. For instance, a sequence $i' = 22133$ and $\lambda = (2, 1)$, the addition $\lambda[i'] = (3, 3, 2)$ is a Young diagram. But, in the second step of the addition, it becomes the diagram $\lambda[2, 2] = (2, 3)$, which is not a Young diagram.

### 3.2. Littlewood–Richardson Crystal

As an application of the description of crystal bases of type $A_n$, we see so-called “Littlewood–Richardson rule” of type $A_n$.

For a sequence $i = i_1, i_2, \ldots, i_N$ ($i_j \in \{1, 2, \ldots, n + 1\}$) and a Young diagram $\lambda$, let $\tilde{\lambda} := \lambda[i_1, i_2, \ldots, i_N]$ be an addition of $i_1, i_2, \ldots, i_N$ to $\lambda$. Then set

$$B(\tilde{\lambda} : i) = \begin{cases} B(\tilde{\lambda}) & \text{if } \lambda[i_1, \ldots, i_k] \text{ is a Young diagram for any } k = 1, 2, \ldots, N, \\ \emptyset & \text{otherwise.} \end{cases}$$

**Theorem 3.5** ([11]). Let $\lambda$ and $\mu$ be Young diagrams with at most $n$ rows. Then we have

$$B(\lambda) \otimes B(\mu) \cong \bigoplus_{\text{FE}(T) = \lambda \otimes \mu} B(\tilde{\lambda} : i_1, i_2, \ldots, i_N). \quad (3.1)$$
Here note that Theorem 3.5 is valid for an arbitrary admissible order $A$, that is, in (3.1) we can replace $FE(T)$ with $R_s(T)$. For an admissible order $A$ on $\mu$ define

$$
\mathcal{B}(\mu)_\lambda^v[A] := \left\{ T \in \mathcal{B}(\mu) \mid R_s(T) = \begin{array}{c} i_1 \otimes i_2 \otimes \cdots \otimes i_k \otimes \cdots \otimes i_N \\ \text{For any } k = 1, \ldots, N, \\
\lambda[i_1, \ldots, i_k] \text{ is a Young diagram and} \\
\lambda[i_1, \ldots, i_N] = v. \end{array} \right\}.
$$

Let $\mathcal{B}(\mu)^v_\lambda := \mathcal{B}(\mu)_\lambda^v[J]$ as in [12]. It is shown in [6] that for any admissible order $A$ on $\mu$,

$$
\mathcal{B}(\mu)_\lambda^v[A] = \mathcal{B}(\mu)^v_\lambda,
$$

which is called a Littlewood–Richardson crystal associated with a triplet $(\lambda, \mu, v)$.

### 4. MAIN THEOREM

For Young diagrams $\lambda, \mu, v$ with $|\lambda| + |\mu| = |v|$, we define the map $\Phi : P(\mu, v \setminus \lambda; A, A') \to \mathcal{B}(\mu)_\lambda^v[A']$: For $f = (f_1, f_2) \in P(\mu, v \setminus \lambda; A, A')$ where $f_1$ (resp., $f_2$) stands for the first (resp., second) coordinate in $\mathbb{N} \times \mathbb{N}$, set

$$
\Phi(f)_{i,j} := f_1(i, j),
$$

that is, $\Phi(f)$ is a filling of shape $\mu$ and its $(i, j)$-entry is given as $f_1(i, j)$.

Furthermore, for $T \in \mathcal{B}(\mu)^v_\lambda[A']$, define a map $\Psi : \mathcal{B}(\mu)^v_\lambda[A'] \to P(\mu, v \setminus \lambda; A, A')$ by

$$
\Psi(T) : (i, j) \in \mu \mapsto (T_{i,j}, \lambda_{T_{i,j}} + p(T, i, j)) \in v \setminus \lambda,
$$

where $p(T, i, j)$ as in 2.1.

The following is the main result in this article, which is conjectured in [12].

**Theorem 4.1.** For Young diagrams $\lambda, \mu, v$ as above, the maps

$$
\Phi : P(\mu, v \setminus \lambda; A, A') \to \mathcal{B}(\mu)_\lambda^v[A'], \quad \Psi : \mathcal{B}(\mu)^v_\lambda[A'] \to P(\mu, v \setminus \lambda; A, A'),
$$

are bijections and they are inverse each other.

It follows from (3.2) that the set $\mathcal{B}(\mu)_\lambda^v[A']$ does not depend on the choice of an admissible order $A'$. Furthermore, it is easy to see from the definition that the map $\Psi$ does not depend on the choice of $A'$. Therefore, we have

$$
\Psi(\mathcal{B}(\mu)_\lambda^v[A']) = \Psi(\mathcal{B}(\mu)^v_\lambda).
$$

By Theorem 4.1 we have $\Psi(\mathcal{B}(\mu)_\lambda^v[A']) = P(\mu, v \setminus \lambda; A, A')$ (resp., $\Psi(\mathcal{B}(\mu)^v_\lambda) = P(\mu, v \setminus \lambda)$). Hence, we obtain the following corollary.
Corollary 4.2. For arbitrary admissible orders $A$ on $\nu \setminus \lambda$ and $A'$ on $\mu$,

$$P(\mu, \nu \setminus \lambda; A, A') = P(\mu, v \setminus A).$$

In the subsequent sections, we shall give the proof of Theorem 4.1, which consists of the following steps:

(i) Well-definedness of the map $\Phi$;
(ii) Well-definedness of the map $\Psi$;
(iii) Bijectivity of $\Phi$ and $\Psi = \Phi^{-1}$.

5. WELL-DEFINEDNESS OF $\Phi$

For the well-definedness of $\Phi$, it suffices to prove the following.

Proposition 5.1. Let $\lambda, \mu$ and $\nu$ be Young diagrams as before. Suppose $f \in P(\mu, v \setminus \lambda; A, A')$.

(i) The image $\Phi(f)$ is a Young tableau of shape $\mu$.
(ii) Writing $R_{A'}(\Phi(f)) = \left[ \sum_{i=1}^{N} i \right] \otimes \cdots \otimes \left[ \sum_{i=k}^{N} \sum_{j=1}^{m} \sum_{n=1}^{p} \right]$, the diagram $\lambda[i_1, i_2, \ldots, i_k] \otimes \cdots \otimes \lambda[i_1, i_2, \ldots, i_N]$ is a Young diagram for any $k = 1, \ldots, N$ and $\lambda[i_1, i_2, \ldots, i_N] = \nu$.

5.1. Proof of Proposition 5.1(i)

It is clear from the definition of $\Phi$ that $\Phi(f)$ is of shape $\mu$. In order to prove (i), we may show for any $i, j$:

(a) $\Phi(f)_{i,j} < \Phi(f)_{i+1,j};$
(b) $\Phi(f)_{i,j} \leq \Phi(f)_{i,j+1}.$

(a) By the definition of $\Phi$, we have $\Phi(f)_{i,j} = f_1(i, j)$ and $\Phi(f)_{i,j+1} = f_1(i+1, j)$. Since $f$ is a picture, we have

$$f(i, j) \preceq_A f(i + 1, j). \tag{5.1}$$

Here assume $f_1(i, j) \geq f_1(i + 1, j)$. If $f_2(i, j) \geq f_2(i + 1, j)$, we have $f(i, j) \succeq_A f(i + 1, j)$, which is a contradiction. On the other hand, if $f_2(i, j) < f_2(i + 1, j)$, then $f(i, j) \succeq_A f(i + 1, j)$, which is also a contradiction. Hence, we have $f_1(i, j) < f_1(i + 1, j)$.

(b) As before, we have $\Phi(f)_{i,j} = f_1(i, j)$ and $\Phi(f)_{i,j+1} = f_1(i, j + 1)$. Then we may show $f_1(i, j) < f_1(i, j + 1)$, which will be shown by the induction on $i$. For the purpose, we need the following lemmas.

Lemma 5.2. Let $f$ be in $P(\mu, v \setminus \lambda; A, A')$. If $f_1(i, j) > f_1(i, j + 1)$, then $f_2(i, j) > f_2(i, j + 1)$.

Proof. Suppose $f_2(i, j) \leq f_2(i, j + 1)$. Then by the assumption $f_1(i, j) > f_1(i, j + 1)$, we obtain $f(i, j) \preceq_A f(i, j + 1)$. However, since $(i, j) \preceq_A (i, j + 1)$ and $f$ is
a picture, \( f(i, j) \leq_A f(i, j + 1) \), which derives a contradiction. Hence, \( f_2(i, j) > f_2(i, j + 1) \). □

Lemma 5.3. Suppose that \( f_1(i, j) > f_1(i, j + 1) \) for \( f \in P(\mu, v \setminus \lambda; A, A') \). Then there exists a unique \((k, l)\) in \( \mu \) satisfying:

\[
k < i, \quad l \leq j, \quad f_1(k, l) = f_1(i, j + 1) \quad \text{and} \quad f_2(k, l) = f_2(i, j).
\]

(5.2)

Note that by Lemma 5.2, we have \( f_2(i, j) > f_2(i, j + 1) \):

\[
\begin{array}{c|c|c}
 f(i, j + 1) & & f(k, l) \\
\hline
 & f(i, j) &
\end{array}
\]

Proof. Since \( v \setminus \lambda \) is a skew diagram, if \((a, b), (c, d) \in v \setminus \lambda \) satisfy \( a < c, \ b < d \), then \((a, d) \in v \setminus \lambda \). Therefore, one gets \((f_1(i, j + 1), f_2(i, j)) \in v \setminus \lambda \). It follows from the bijectivity of \( f \) that there exists a unique \((k, l) \in \mu \) such that \( f(k, l) = (f_1(i, j + 1), f_2(i, j)) \). Now, it remains to show \( k < i \) and \( l \leq j \). Since \( (i, j) \leq_A (i, j + 1) \), we have \( f(i, j) \leq_A f(i, j + 1) \) and then

\[
f(k, l) \leq_A f(i, j) \leq_A f(i, j + 1). \tag{5.3}
\]

In the meanwhile, since \( f(i, j + 1) \leq_A f(k, l) \leq_A f(i, j) \), we obtain

\[
(i, j + 1) \leq_A (k, l) \leq_A (i, j). \tag{5.4}
\]

So, there can be \((k, l)\) in \( X \) or \( Y \) in the following figure, where

\[
X = \{ (k, l) \in \mu : i < k \text{ and } j < l \} \quad \text{and} \quad Y = \{ (k, l) \in \mu : i > k \text{ and } l \leq j \}.
\]

In case \((k, l) \in X \). Since \((i, j) \leq_A (k, l) \), we have \( f(i, j) \leq_A f(k, l) \). This does not match (5.3). Hence \((k, l) \in Y \), i.e., \( k < i \) and \( l \leq j \). □

Let us show \( f_1(i, j) \leq_A f_1(i, j + 1) \) by the induction on \( i \).

In case \( i = 1 \), suppose \( f_1(1, j) > f_1(1, j + 1) \). By Lemma 5.2, we have \( f_2(1, j) > f_2(1, j + 1) \). By Lemma 5.3, there is \( f(k, l) \) satisfying (5.2). Since \( i = 1 \),
the set $Y$ as above is, indeed, empty. So there cannot exist $(k, l)$. It contradicts Lemma 5.3. Thus, we obtain $f_1(1, j) \leq f_1(1, j + 1)$.

In case $i = a > 1$, assume $f_1(b, j) \leq f_1(b, j + 1)$ for any $b \leq a - 1$ and any $j$. If $f_1(a, j) > f_1(a, j + 1)$, by Lemma 5.2, we have $f_2(a, j) > f_2(a, j + 1)$. In addition, by Lemma 5.3, there exists $f(k, l)$ satisfying (5.2) for $i = a$. It follows from the hypothesis of the induction that

$$f_1(k, l) \leq f_1(k, l + 1) \leq \cdots \leq f_1(k, j + 1),$$

and we have $f_i(k, j + 1) < f_i(i, j + 1)$ by (a). Hence

$$f_i(k, l) < f_i(i, j + 1),$$

which contradicts (5.2). Hence, we have $f_i(a, j) \leq f_i(a, j + 1)$. Now, we complete the proof of Proposition 5.1(i). \hfill \Box

5.2. Proof of Proposition 5.1(ii)

First, we prepare the following lemma.

**Lemma 5.4.** Let $f : \mu \to \nu \lambda$ be a picture and set $R_A(\Phi(f)) = \left[ \begin{array}{c} i_1 \otimes i_2 \otimes \cdots \otimes i_N \end{array} \right]$. Let $(p_j, q_j) \in \mu$ be the coordinate of $i_j$ in $\Phi(f) \in B(\mu)$ and $(a_j, b_j) \in \nu$ the coordinate of the $j$th addition in $\lambda[i_1, \ldots, i_N]$. Then we have $(p_j, q_j) = (a_j, b_j)$ for any $j$.

**Proof.** For any $m \in \{1, \ldots, n + 1\}$ list the coordinates in $\Phi(f)(m)$ from the right as

$$(s_1, t_1), (s_2, t_2), \ldots, (s_e, t_e),$$

where $c := |\Phi(f)(m)|, s_1 \leq s_2 \leq \cdots \leq s_e$, and $t_1 > t_2 > \cdots > t_e$. So, we have

$$(s_1, t_1) \leq_A (s_2, t_2) \leq_A \cdots \leq_A (s_e, t_e). \quad (5.5)$$

In the addition of $i_1, \ldots, i_N$ to $\lambda$, the box $(s_1, t_1)$ goes to $(m, \lambda_m + k) \in v = \lambda[i_1, \ldots, i_N]$.

Here write the $m$th row in $v \lambda$:

$$(m, \lambda_m + 1) \leq_P (m, \lambda_m + 2) \leq_P \cdots \leq_P (m, \lambda_m + c) = (m, v_m).$$

By the definition of $\Phi$, we have

$$f^{-1}([(m, \lambda_m + 1), \ldots, (m, \lambda_m + c)]) = \Phi(f)(m). \quad (5.6)$$

Since $f$ is a picture, we obtain

$$f^{-1}(m, \lambda_m + 1) \leq_A f^{-1}(m, \lambda_m + 2) \leq_A \cdots \leq_A f^{-1}(m, \lambda_m + c) = f^{-1}(m, v_m). \quad (5.7)$$
Thus, it follows from (5.5)–(5.7) that \( f^{-1}(m, \lambda_m + k) = (s_k, t_k) \) and then \( f(s_k, t_k) = (m, \lambda_m + k) \) for any \( k = 1, \ldots, c \) and \( m = 1, \ldots, n + 1 \).

**Proof of Proposition 5.1(ii).** Let \( \Lambda' \) be an admissible order on \( \mu \). We take the admissible reading \( R_{\Lambda'} \) associated with \( \Lambda' \) and write \( R_{\Lambda'}(\Phi(f)) = [i_1] \otimes [i_2] \otimes \cdots \otimes [i_N] \in B^{\otimes N} \). Let us denote the coordinate of \( [i_k] \) by \( (x_k, y_k) \in \mathbb{N} \times \mathbb{N} \).

We shall show that \( \lambda[i_1, i_2, \ldots, i_k] \) is a Young diagram for any \( k \) by using the induction on \( k \). In case \( k = 1 \), by the definition of the admissible reading, \( (x_1, y_1) \) is the minimum with respect to the order \( \leq_{\Lambda'} \). Thus, we have \( (x_1, y_1) = (1, \mu_1) \). Since \( f \) is an admissible picture,

\[
\begin{aligned}
f(x_1, y_1) = \text{minimal with respect to the order } \leq_p \text{ in } v \setminus \lambda.
\end{aligned}
\tag{5.8}
\]

Due to the definition of \( \Phi \), we have \( f_1(x_1, y_1) = i_1 \).

Since \( (1, \mu_1) = (x_1, y_1) \) is the right-most in \( \mu \), the first entry \( [i_1] \) goes to \( (i_1, \lambda_i + 1) \) by the addition. It follows from Lemma 5.4 that we have \( f(x_1, y_1) = (1, \lambda_i + 1) \). Since \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is a Young diagram, we obtain \( \lambda_{i-1} \geq \lambda_i \). Now, suppose \( i_1 > 1 \) and \( \lambda_{i-1} = \lambda_i \). Since we have \( \lambda[i_1, i_2, \ldots, i_k] = v \), there exists some entry \( [i_k] \) \((k \geq 2)\) added to the coordinate \((i_1-1, \lambda_{i-1} + 1)\), which means \( f(x_k, y_k) = (i_1 - 1, \lambda_{i-1} + 1) \leq_p (i_1, \lambda_i + 1) = f(x_1, y_1) \). Then it contradicts the fact that \( f(x_1, y_1) \) is minimal with respect to the order \( \leq_p \). Thus, we have \( i_1 = 1 \) or \( \lambda_{i-1} > \lambda_i \). Then, \( \lambda[i_1] \) is a Young diagram.

In case \( k \geq 2 \), assume that \( \lambda = \lambda[i_1, i_2, \ldots, i_{k-1}] \) is a Young diagram. The coordinate \((x_k, y_k)\) of \( [i_k] \) in \( \mu \) is the minimum in \( \mu \setminus \{(x_1, y_1), \ldots, (x_{k-1}, y_{k-1})\} \) with respect to the order \( \leq_{\Lambda'} \). Since \( f \) is an admissible picture, \( f(x_k, y_k) \) is minimal in \( v \setminus \lambda \) with respect to \( \leq_p \). By the definition of \( \Phi \), we have \( f_1(x_k, y_k) = i_k \).

If \( i_k = 1 \), trivially \( \lambda[i_k] \) is a Young diagram. In case \( i_k > 1 \), by Lemma 5.4 we have \( f(x_k, y_k) = (i_k, \lambda'_i + 1) \). Here, by the similar argument as above, we have \( \lambda'_{i-1} > \lambda'_i \) and then \( \lambda[i_k] \) is a Young diagram. Thus, \( \lambda[i_1, i_2, \ldots, i_k] \) is a Young diagram for any \( k \). It is trivial that \( \lambda[i_1, i_2, \ldots, i_k] = \mu \) by the definition of \( \Phi(f) \).

**6. WELL-DEFINEDNESS OF \( \Psi \)**

**Proposition 6.1.** For any \( T \in B(\mu)[A'] \), we have \( \Psi(T) \in P(\mu, v \setminus \lambda; A, A') \), that is:

1. \( \Psi(T) \) is a map from \( \mu \) to \( v \setminus \lambda \);
2. \( \Psi(T) \) is a bijection;
3. \( \Psi(T) \) is an \( (A, A') \)-admissible picture.

The following lemma is needed to show Proposition 6.1.

**Lemma 6.2.** Let \( T \) be in \( B(\mu)[A'] \) and set \((a, b) := \Psi(T)(i, j) \) for \((i, j) \) in \( \mu \). Then the destination of \((i, j) \) by the addition of \( R_{\Lambda'}(T) \) coincides with \((a, b) \).

**Proof.** Set \( m := T_{i,j}(= (i, j) \)-entry in \( T \)). Suppose that \((i, j) \) is the \( p \)th element in \( T^{(m)} \) from the right. Then, by the definition of \( \Psi \), we have \( \Psi(T)(i, j) = (m, \lambda_m + p) \). On the other hand, in the course of the addition of \( R_{\Lambda'}(T) \) to \( \lambda \), we see that \( m =
$T_{i,j}$ is added $p$th to the $m$th row. This means that $T_{i,j}$ goes to $(m, \lambda_m + p)$ by the addition. This proves our claim. □

6.1. Proof of Proposition 6.1(1),(2)

By the definition of $\Psi$, it is clear that $\Psi(T)$ is a map from $\mu$. Let us denote $R_{\mu}(T) = [i_1] \otimes \cdots \otimes [i_N]$. Since $T \in B(\mu)[A]$, by the addition of $R_{\mu}(T)$ to $\lambda$, we have $\lambda[i_1, \ldots, i_N] = v$. So the set of all the destinations by the addition coincides with $v \setminus \lambda$. Then, by Lemma 6.2, it implies that $\Psi(T)(\mu) = v \setminus \lambda$. This also shows that $\Psi(T)$ is surjective and then bijective, since $|\mu| = |v \setminus \lambda|$.

6.2. Proof of Proposition 6.1(3)

By the above results, we know that $f = \Psi(T)$ is a bijection. Now, we shall prove:

(i) $f^{-1}$ is $PA'$-standard;
(ii) $f$ is $PA$-standard.

Let us see (i). Set $f^{-1}(a, b) = (i, j)$, $f^{-1}(a, b + 1) = (x, y)$, and $f^{-1}(a + 1, b) = (s, t)$. Thus, there exist $p$, $q$, and $r$ such that

$$
(a, b) = f(i, j) = (T_{i,j}, \lambda_{T_{i,j}} + p), \quad (6.9)
$$
$$
(a, b + 1) = f(x, y) = (T_{x,y}, \lambda_{T_{x,y}} + q), \quad (6.10)
$$
$$
(a + 1, b) = f(s, t) = (T_{s,t}, \lambda_{T_{s,t}} + r). \quad (6.11)
$$

First, we shall show

$$
\begin{align*}
&f^{-1}(a, b) \leq_{A'} f^{-1}(a, b + 1), & \text{i.e., } (i, j) \leq_{A'} (x, y).
\end{align*}
$$

Since $T_{i,j} = a = T_{x,y}$ by (6.9) and (6.10), we have $\lambda_{T_{i,j}} = \lambda_{T_{x,y}}$. Furthermore, since $b = \lambda_{T_{i,j}} + p, b + 1 = \lambda_{T_{x,y}} + q$ by (6.10) and (6.11), we obtain $q = p + 1$. The coordinates $(i, j)$ and $(x, y)$ are in $T^{(a)}$. Therefore, $q = p + 1$ means that $(i, j)$ is right to $(x, y)$ and then $(i, j) \leq_{A'} (x, y)$.

Next, let us see

$$
\begin{align*}
&f^{-1}(a, b) \leq_{A'} f^{-1}(a + 1, b) & \text{i.e., } (i, j) \leq_{A'} (s, t).
\end{align*}
$$

Suppose $i \geq s$. From (6.9) and (6.11), we have $T_{i,j} = a, T_{s,t} = a + 1$. So we obtain $T_{i,j} < T_{s,t}$, which means $j < t$. This is as follows:

```
    s  ...  T_{s,t}

  i  ...  T_{i,j}  ^          |

  j  |  t
```
Then we obtain \( (i, j) \geq_{\lambda'} (s, t) \). In the process of the addition of \( R_{\lambda'}(T) = \otimes_{i_1} \otimes \cdots \otimes \otimes_{i_N} \) to \( \lambda \), the entry at \((s, t)\) is added to \( \lambda \) earlier than the one at \((i, j)\), which signifies that the coordinate \((a + 1, b)\) is filled earlier than the one at \((a, b)\) by the addition by Lemma 6.2. This means that there exists \( k < N \) such that \( \lambda[i_1, \ldots, i_k] \) is not a Young diagram, which contradicts \( T \in B(\mu)[A'] \). Hence we have \( i < s \).

Suppose \( j < t \). Owing to (6.9) and (6.11), we obtain \( T_{i, j} + 1 = T_{s, t} \).

Since \( T \) is a Young tableau, we have \( a = T_{i, j} < T_{s, j} \leq T_{s, t} = a + 1 \) and then \( T_{s, j} = a + 1 \). Thus, we know that there is no entry between \((i, j)\) and \((s, j)\) and then \( s = i + 1 \). This is described as follows:

By the assumption \( j < t \), we have \( m := t - j > 0 \). We have \( f(s, j) = (a + 1, \lambda_{a+1} + r + m) \).

Since \( T \in B(\mu)[A'] \), the part \( X \) in the above figure must be filled earlier than \( f(s, j) \) and later than \( f(i, j) \) by the addition of \( R_{\lambda'}(T) \) to \( \lambda \). Thus, by Lemma 6.2, \( (i, j) \leq_{\lambda'} f^{-1}(X) \leq_{\lambda'} (s, j) \), which implies that \( f^{-1}(X) \) are in the shaded part in Fig. 2.
Moreover, by the definition of $\Psi$, an entry in $f^{-1}(X)$ is equal to $a$. Since the entries in $f^{-1}(X)$ are added later than the entry $T_{i,j} = a$ at $(i, j)$ and $T$ is a Young tableau, $f^{-1}(X)$ must be included in the shaded part in Fig. 3.

![Figure 3](image)

Figure 3

Therefore, by Figs. 2 and 3, $f^{-1}(X)$ must be same as the shaded part in the following figure:

![Figure 4](image)

Figure 4

All the entries in this part are equal to $a$. Then all the entries in the part $U$ are equal to $a + 1$. Thus, we know that $f(U) = Z$ where $Z$ is the part of $\nu \setminus \lambda$ in Fig. 1 as above. Arguing similarly, the part $Y$ is sent to the part $V$ in Fig. 4 below and all the entries of $T$ in the part $V$ are $a$. Then all the entries of $T$ in the part $W$ are $a + 1$. The part $W$ is sent to the right side of $Z$ by $f$.

Under the assumption $m = t - j > 0$, we can repeat this process infinitely many times and extend the $i$th row and the $i + 1$-row of $\mu$ unlimitedly, which contradicts the finiteness of $\mu$. Thus, we have $j \geq t$. Finally, we have $i < s$ and $j \geq t$, and then $(i, j) \leq A$ $(s, t)$. Then $f^{-1}$ is $PA'$-standard. $\square$

Let us show (ii). By the definition of $\Psi$, for $(i, j), (i, j + 1), (i + 1, j) \in \mu$ there exist $p, q, r$ such that

$$f(i, j) = (T_{i,j}, \lambda_{T_{i,j}} + p), \quad (6.12)$$

$$f(i, j + 1) = (T_{i,j+1}, \lambda_{T_{i,j+1}} + q), \quad (6.13)$$

$$f(i + 1, j) = (T_{i+1,j}, \lambda_{T_{i+1,j}} + r). \quad (6.14)$$

First, let us show

$$f(i, j) \leq A f(i, j + 1).$$
Since $T$ is a Young tableau, we have $T_{i,j} \leq T_{i,j+1}$ and then $f_1(i, j) = T_{i,j} \leq T_{i,j+1} = f_1(i, j + 1)$, where $f = (f_1, f_2)$.

In case $T_{i,j} = T_{i,j+1}$, we have $\lambda_{T_{i,j}} = \lambda_{T_{i,j+1}}$. Moreover, $(i, j)$ is on the left-side of $(i, j + 1)$, which shows $p > q$. Then, we have $\lambda_{T_{i,j}} + p > \lambda_{T_{i,j+1}} + q$, that is, $f_2(i, j) > f_2(i, j + 1)$. Hence we obtain $f(i, j) \leq_A f(i, j + 1)$.

In case $T_{i,j} < T_{i,j+1}$, we have $\lambda_{T_{i,j}} \geq \lambda_{T_{i,j+1}}$. Since $(i, j) \geq_A (i, j + 1)$, in the addition of $R_A(T)$, the entry at $(i, j + 1)$ is added earlier than the one at $(i, j)$. Let $\lambda'$ (resp., $\lambda''$) be the resulting Young diagram obtained by the addition up to $(i, j + 1)$ (resp., $(i, j)$). It follows from Lemma 6.2 that the destination of $(i, j + 1)$ (resp., $(i, j)$) by the addition coincides with $f(i, j + 1) = (T_{i,j+1}, \lambda_{T_{i,j}'} + q)$ (resp. $f(i, j) = (T_{i,j}, \lambda_{T_{i,j}''} + p)$).

Since in the all steps of the addition the resulting diagrams are always Young diagrams, we have

$$\lambda'_{T_{i,j}} \geq \lambda'_{T_{i,j+1}} = \lambda_{T_{i,j+1}} + q.$$  

The entry $T_{i,j}$ is the $p$th element in $T^T(T_{i,j})$ from the right. Then by Lemma 6.2,

$$\lambda''_{T_{i,j}} < \lambda''_{T_{i,j+1}} = \lambda_{T_{i,j}} + p.$$  

Hence

$$\lambda_{T_{i,j+1}} + q \leq \lambda'_{T_{i,j}} < \lambda''_{T_{i,j}} + p,$$

which implies $f_2(i, j + 1) < f_2(i, j)$. We obtain $f(i, j) \leq_A f(i, j + 1)$.

Next, let us show

$$f(i, j) \leq_A f(i + 1, j).$$

We have $f_1(i, j) = T_{i,j} < T_{i+1,j} = f_1(i + 1, j)$. Thus, it is enough to show $\lambda_{T_{i,j}} + p \geq \lambda_{T_{i+1,j}} + r$ since $f_2(i, j) = \lambda_{T_{i,j}} + p$ and $f_2(i, j + 1) = \lambda_{T_{i+1,j}} + r$, where $f = (f_1, f_2)$.

Assuming $f_2(i, j) < f_2(i + 1, j)$, one has:

$$\nu \setminus \lambda \begin{array}{c|c|c} f(i,j) & A \hline \emptyset & B \hline C & f(i+1,j) \end{array}$$

where

$$A = \{(k, l) : k = f_1(i, j), f_2(i, j) < l \leq f_2(i + 1, j)\},$$

$$B = \{(k, l) : f_1(i, j) < k < f_1(i + 1, j), f_2(i, j) \leq l \leq f_2(i + 1, j)\},$$

$$C = \{(k, l) : k = f_1(i + 1, j), f_2(i, j) \leq l < f_2(i + 1, j)\}.$$
Since $f^{-1}$ is $PA'$-standard and $f(i, j) \leq_P A, B, C \leq_P f(i + 1, j)$, we obtain $(i, j) \leq_X f^{-1}(A), f^{-1}(B), f^{-1}(C) \leq_X (i + 1, j)$. So the parts $f^{-1}(A), f^{-1}(B)$ and $f^{-1}(C)$ must be in the shaded part of Fig. 5.

![Figure 5](image)

For $(k, l) \in B$, by the definition of $B$ as above, we have $T_{k,l} = f_l(i, j) \leq_T f_l(i + 1, j) = T_{i+1,j}$. If we set $(s, t) := f^{-1}(k, l)$, then $(s, t)$ is in the shaded part in Fig. 5. Hence, in the Young tableau $T$ we have $k = f_1(s, t) = T_{k,l} \leq_T f_1(i, j)$ or $k = T_{s,t} \geq_T T_{i+1,j} = f_l(i + 1, j)$, which derives a contradiction. Thus, $B$ is empty and then the figure turns to

![Figure 6](image)

where $d$ is the length of $A$ and $C$. Now we have $T_{i,j} + 1 = f_1(i, j) + 1 = f_1(i + 1, j) = T_{i+1,j}$. Since $T \in B(\mu)[\lambda', A]$, by the addition the part $A$ is filled earlier than $f(i + 1, j)$ is. Since the part $A$ is in the same row with $f(i, j)$, in the Young tableau $T$, the entries in the part $f^{-1}(A)$ are equal to $T_{i,j}$. Then, there should be $f^{-1}(A) \subseteq \mu$ in the shaded part of Fig. 6.

By Figs. 5 and 6, we know that $f^{-1}(A)$ is the following shaded part:

![Figure 6](image)

Similar argument shows that the entries in the part $C$ are equal to $T_{i+1,j} = f_1(i + 1, j) = T_{i,j} + 1$ and the part $f^{-1}(C)$ is in the shaded part of the following figure:

![Figure 6](image)
Taking these into account, we get

\[
\begin{array}{|c|c|}
\hline
& \mu \\
\hline
f^{-1}(A) & (i,j) \\
E & (i+1,j) \\
\hline
\end{array}
\]

Since the difference between the entries \(T_{i+1,j}\) and \(T_{i,j}\) is just 1, the entries in the part \(D\) (resp., \(E\)) above are all equal to \(T_{i,j}\) (resp., \(T_{i+1,j} = T_{i,j} + 1\)). Then we have the following figure:

\[
\begin{array}{|c|c|}
\hline
\lambda & \nu \\
\hline
f(i,j) & A \\
C & f(i+1,j) \\
\hline
f(E) & F \\
\hline
\end{array}
\]

By the addition, the part \(F\) should be filled earlier than \(f(E)\) is. Then we obtain

\[
\begin{array}{|c|c|}
\hline
& \mu \\
\hline
f^{-1}(F) & f^{-1}(A) \\
E & (i,j) \\
G & f^{-1}(C) \\
\hline
\end{array}
\]

Arguing similarly, in the Young tableau \(T\), all the entries in the part \(f^{-1}(F)\) above coincide with \(T_{i,j}\) and then the ones in the part \(G\) coincide with \(T_{i+1,j} = T_{i,j} + 1\). Then if \(d > 0\), repeating these arguments, we can extend the part in the left-side of \((i + 1, j)\) (including the parts \(E, G\)) unlimitedly. Indeed, it cannot occur for the finite diagram \(\mu\). Thus, we know that \(d = 0\). This means \(A = C = \emptyset\), which contradicts the assumption \(f_2(i, j) < f_2(i + 1, j)\). Hence we have \(f_2(i, j) \geq f_2(i + 1, j)\), and then \(f(i, j) \leq_A f(i + 1, j)\). It completes the proof. □

7. BIJECTIVITY OF \(\Phi\) AND \(\Psi\)

By the arguments above, we get the well-defined maps

\[
\Phi : \mathcal{P}(\mu, v; A, A') \rightarrow \mathcal{B}(\mu)_i[A'] \hspace{1cm} \Psi : \mathcal{B}(\mu)_i[A'] \rightarrow \mathcal{P}(\mu, v; A, A').
\]

Now let us show the following

(i) \(\Phi \circ \Psi = \text{id}_{\mathcal{B}(\mu)_i[A']}\);
(ii) \(\Psi \circ \Phi = \text{id}_{\mathcal{P}(\mu, v; A, A')}\).

(i) Recalling the definition of \(\Psi\), for \(T \in \mathcal{B}(\mu)_i[A']\) one gets that the admissible picture \(\Psi(T)\) sends \((i, j) \in \mu\) to \((T_{i,j}, T_{i,j} + p(i, j, T))\). Due to the definition of \(\Phi\) it is easy to see that \(\Phi \circ \Psi(T)\) is an element in \(\mathcal{B}(\mu)_i[A']\) whose \((i, j)\)-entry is \(T_{i,j}\), which implies \(T = \Phi \circ \Psi(T)\). Thus, we have \(\Phi \circ \Psi = \text{id}_{\mathcal{B}(\mu)_i[A']}\) as desired.
(ii) Taking an admissible picture \( f \in \mathbf{P}(\mu, v \backslash \lambda; A, A') \), set \( g := \Psi \circ \Phi(f) \in \mathbf{P}(\mu, v \backslash \lambda; A, A') \). The image \( \Phi(f) \) is an element in \( \mathbf{B}(\mu)[A'] \) whose \((i, j)\)-entry is \( f_1(i, j) \). Then

\[
g(s, t) = (\Phi(f)_{x, i}, \lambda_{\Phi(f)_{x, i}} + p) = (f_1(s, t), \lambda_{f_1(s, t)} + p),
\]

(7.1)

where \((s, t) \in \mu\) is the \( p \)th element from the right in \( \Phi(f)_{x, y}(s, t) := \{(x, y) \in \mu \mid \Phi(f)_{x, y} = f_1(s, t)\} \). It follows from Lemma 5.4 that

\[
f(s, t) = (\Phi(f)_{x, i}, \lambda_{\Phi(f)_{x, i}} + p) = (f_1(s, t), \lambda_{f_1(s, t)} + p),
\]

(7.2)

which means \( f(s, t) = g(s, t) \), and hence, we have \( \Psi \circ \Phi = \text{id}_P(\mu, v \backslash \lambda; A, A') \). □

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