A Fundamental Limit of Measurement Imposed by the Elementary Interactions

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Abstract. Quantum information theory is closely related to quantum measurement theory because one must perform measurement to obtain information on a quantum system. Among many possible limits of quantum measurement, the simplest ones were derived directly from the uncertainty principles. However, such simple limits are not the only limits. I here suggest a new limit which comes from the forms and the strengths of the elementary interactions. Namely, there are only four types of elementary interactions in nature; their forms are determined by the gauge invariance (and symmetry breaking), and their coupling constants (in the low-energy regime) have definite values. I point out that this leads to a new fundamental limit of quantum measurements. Furthermore, this fundamental limit imposes the fundamental limits of getting information on, preparing, and controlling quantum systems.

INTRODUCTION

Quantum information theory is closely related to quantum measurement theory because one must perform measurement to obtain information on a quantum system. In particular, a limit of quantum measurement is also a limit of getting information on a quantum system. Among many possible limits of quantum measurement, the simplest ones were derived directly from the uncertainty principles. However, such simple limits are not the only limits. I here suggest a new limit which comes from the forms and the strengths of the elementary interactions.

It is established that measurement of a quantum system can be analyzed (except for philosophical issues) as interaction processes of the system and the measuring apparatus [1–8]. Many interesting quantities, such as a measurement error and backaction of the measurement, can be calculated from the dynamics of the total quantum system composed of the system and the measuring apparatus [1–8]. The dynamics of a quantum system is governed by the commutation relations and the Hamiltonian. For the interaction Hamiltonian $H_I$, it was often assumed in the

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literature [2–4] that the form of $H_I$ was obtained by “quantizing” some classical Hamiltonian. Or, in some cases [5], specific forms were just assumed for $H_I$. Discussions based on such $H_I$ may be consistent mathematically. Physically, however, such $H_I$ can be wrong. For example, quantization of a classical Hamiltonian in general gives a form that is different from the true Hamiltonian, and the difference is typically of the order of $\hbar$. However, backaction is also of the order of $\hbar$. Hence, physically, any results drawn from such a Hamiltonian cannot be reliable enough. In particular, there are only four types of elementary interactions in nature; their forms are determined by the gauge invariance (and symmetry breaking), and their coupling constants (in the low-energy regime) have definite values [9]. In order to draw conclusions which are physically meaningful, one must either use these true interactions, or, if one uses an approximate Hamiltonian, one must confirm that the approximation does not alter the results for the measurement. Unfortunately, however, these points were not examined carefully.

In this paper, I point out that not only the commutation relations but also the limitation of available interactions in nature results in a fundamental limit of quantum measurement. This fundamental limit imposes the fundamental limits of getting information on, preparing, and controlling quantum systems.

**MEASUREMENT ERROR AND BACKACTION ON THE MEASURED OBSERVABLE**

I consider the relation between the measurement error and the backaction on the measured observable. Although similar arguments should be applicable to other relations, I here present only one example, to explain the basic ideas.

When one measures an observable $Q$ of a quantum system, there is a finite measurement error $\delta Q_{\text{err}}$ in general. Its magnitude is determined by the form and strength of the interaction Hamiltonian $H_I$ between the measured system and the measuring apparatus [2–8]. There are also backactions of the measurement. A well-known backaction is the one on the conjugate variable $P$ of $Q$. Its lower limit is determined by the commutation relation between $Q$ and $P$, i.e., by the uncertainty principle. However, I here discuss backaction on $Q$ itself [1–8], which is denoted by $\delta Q_{\text{ba}}$. Although the “ideal measurement”, for which $\delta Q_{\text{ba}} = \delta Q_{\text{err}} = 0$, is normally assumed in textbook quantum mechanics, real measurements are non-ideal in general, for which $\delta Q_{\text{ba}}$ and/or $\delta Q_{\text{err}}$ are finite [1–8]. Since the lower limit of $\delta Q_{\text{ba}}$ is not (directly) limited by the uncertainty principle, $\delta Q_{\text{ba}}$ strongly depends on the form and strength of the interaction Hamiltonian $H_I$ [1–8], just as $\delta Q_{\text{err}}$ does. Therefore, $\delta Q_{\text{ba}}$ and $\delta Q_{\text{err}}$ are correlated through $H_I$. For this relation, one could not get physically correct results without using a correct Hamiltonian.

For example, if one assumes some $H_I$ which satisfies

$$[H_I, Q] = 0,$$  \hspace{1cm} (1)
then $\delta Q_{ba} = 0$, independent of the strength (which may be expressed by the coupling constant $g$) of $H_I$ [2,3,6]. Namely, the measurement becomes of the first kind (FK) [1,6]. On the other hand, $\delta Q_{err}$ tends to increase as $g$ is decreased: For example, $\delta Q_{err} \to \infty$ as $g \to 0$ [3–8]. Hence, one would conclude that

$$\delta Q_{ba} \text{ and } \delta Q_{err} \text{ are uncorrelated.} \tag{2}$$

However, this is true only for such a hypothetical $H_I$. Namely, if

$$[H_I, Q] \neq 0 \tag{3}$$

for the true form of $H_I$ (i.e., for the elementary interaction), then $\delta Q_{ba}$ strongly depends on the form and strength of $H_I$. Therefore, the correct conclusion is

$$\delta Q_{ba} \text{ and } \delta Q_{err} \text{ are strongly correlated.} \tag{4}$$

in sharp contrast to (2).

Unfortunately, most previous work on measurement [2–5] assumed some “effective” interactions for $H_I$, such as a photon-photon interaction, which is not the true form, the elementary interaction. The validity of such effective forms, when applied to problems of measurement, was not examined carefully.

**MEASUREMENT OF THE PHOTON NUMBER**

From this viewpoint, of particular interest is the measurement of a gauge field such as the photon field. The form of its interaction is completely determined by the requirement of the gauge invariance: the only interaction is the gauge-invariant interaction with the matter field [9]. However, most previous work on measurement of a gauge field, particularly work on the “quantum non-demolition (QND) measurement” of photons [3,4], assumed other forms for $H_I$, such as a photon-photon interaction, as “effective” interactions. Here, the QND measurement is the FK measurement for a conserved observable, i.e., for $Q$ that is conserved during the free motion (i.e., while the system is decoupled from the measuring apparatus) [2]. It therefore seems that most previous discussions on measurement of a gauge field need to be reconsidered.

To demonstrate the dramatic difference between the correct result derived from a proper $H_I$ and a wrong result derived from an “effective” $H_I$, I will present results for the QND photodetector proposed in Refs. [3,4], for both cases where the true $H_I$ and an “effective” $H_I$ are used.

A schematic diagram of the QND photodetector is shown in Fig.1 [7]. Before going to the full analysis in the following sections, I here give an intuitive, semiclassical description of the operation principle [10]. The device is composed of two quantum wires, N and W, which constitute an electron interferometer. The lowest subband energies (of the $z$-direction confinement) $\epsilon^N_a$ and $\epsilon^W_a$ of the wires are the
same, but the second levels $\epsilon_b^N$ and $\epsilon_b^W$ are different. Electrons occupy the lowest levels only. A $z$-polarized light beam hits the dotted region. The photon energy $\hbar\omega$ is assumed to satisfy $\epsilon_b^W - \epsilon_a^W < \hbar\omega < \epsilon_b^N - \epsilon_a^N$, so that real excitation does not occur and no photons are absorbed. However, the electrons are excited “virtually” [11], and the electron wavefunction undergoes a phase shift between its amplitudes in the two wires. Since the magnitude of the virtual excitation is proportional to the light intensity [11], so is the phase shift. This phase shift modulates the interference currents, $J_+$ and $J_-$. By measuring $J_\pm$, we can know the magnitude of the phase shift, from which we can know the light intensity. Since the light intensity is proportional to the photon number $n$, we can get information on $n$. We thus get to know $n$ without photon absorption, i.e., without changing $n$; hence the name QND. (More precisely, the distribution of $n$ over the whole ensemble is unchanged, whereas $n$ of each element of the ensemble is changed due to the “reduction” of the
wavefunction. See, e.g., section 7 of Ref. [8].) In contrast, conventional photodetectors drastically alter the photon number by absorbing photons. Keeping this semi-classical argument in mind, let us proceed to a fully-quantum analysis.

RESULTS DERIVED FROM AN “EFFECTIVE” HAMILTONIAN

As explained above, the essential role of the photon field is to modulate the electron phase. It is thus tempting to employ the following form

$$H_I = \sum_{\nu=N,W} \sum_k g_{k\nu} c_{k\nu}^\dagger c_{k\nu} n, \quad (5)$$

as an “effective” interaction Hamiltonian. Here, $c_{k\nu}$ denotes the annihilation operator of an electron of wavelength $k$ in quantum wire $\nu$, and $g_{k\nu}$ is an effective coupling constant. This Hamiltonian commutes with the photon number $n$;

$$[H_I, n] = 0. \quad (6)$$

This equation, and the fact that $n$ is conserved during the free evolution, meet the condition for QND detectors that were claimed in Refs. [2,3]. It is then easy to show that the backaction $\delta n_{ba} = 0$, whereas the measurement error $\delta n_{err}$ is finite and dependent on $g_{k\nu}$. Namely, we obtain conclusion (2) for the interaction Hamiltonian (5).

RESULTS DERIVED FROM THE CORRECT HAMILTONIAN

The correct interaction (i.e., the elementary interaction of, in this case, quantum electrodynamics) is not the form of Eq. (5), but the following one [9];

$$H_I = \frac{e}{\hbar c} \int d^4x \bar{\psi} \gamma^\mu A_\mu \psi. \quad (7)$$

Here, $\psi$ is the (relativistic) electron field, $\gamma^\mu$ ($\mu = 0, 1, 2, 3$) are the gamma matrices, $A_\mu$ is the electromagnetic potential (the set of the vector potential and scalar potential), and $\bar{\psi} = \psi^\dagger \gamma^0$. This form is uniquely determined by the local gauge invariance [9]. Furthermore, the value of the coupling constant $e/\hbar c$ (more precisely, the renormalized value at the low energy region) is uniquely determined by experiment [9]. Since $n$ is quadratic in $A_\mu$, we find

$$[H_I, n] \neq 0, \quad (8)$$

in contrast to Eq. (6). Hence, the QND condition claimed in Refs. [2,3] is not satisfied by the correct Hamiltonian. According to the detailed analysis on QND
measurement [6], the condition claimed in Refs. [2,3] is the strongest one among various possibilities, which are called the strong, moderate, and weak conditions [6].

The weak condition given in Ref. [6] can be slightly generalized to the following form [12] (which reduces to that of Ref. [6] as $\varepsilon_{ba} \to 0$): For a small positive number $\varepsilon_{ba}$,

$$\sum_j \sum_{k,\ell} |a_k b_\ell u_{ij}^{k\ell}|^2 - |a_i|^2 \leq \frac{\varepsilon_{ba}}{2} \sum_j \sum_{k,\ell} |a_k b_\ell u_{ij}^{k\ell}|^2 + |a_i|^2$$

for all $i$, \hspace{1cm} (9)

for some set of $\{a_k\}$’s and for some $\{b_\ell\}$. Here, I have used the same notations as in Ref. [6]; $\{a_k\}$ and $\{b_\ell\}$ represent the initial states of the measured system and the “probe” system, respectively, i.e., the expansion coefficients in terms of the eigenfunctions of the measured and “readout” variables [6]. The $u_{ij}^{k\ell}$ is the matrix representation of the unitary evolution induced by the interaction with the probe system, hence is a function of the Hamiltonian [6]. Physically, this condition means that the backaction can be made small enough for a particular set of measured states if the measuring apparatus is appropriately designed. The magnitude of the backaction is represented by $\varepsilon_{ba}$, which gives the upper limit of the change of the probability distribution over the ensemble [6]. Note that the limitation of measured states corresponds to a limitation of the response range of the measuring apparatus. Such a limitation is quite realistic, because any existing apparatus do have finite response ranges. Furthermore, as is clear from Eq. (8), only by accepting such a limitation can we realize the FK or QND measurement of wide classes of observables including photon number.

The condition satisfied by the QND photodetector of Fig. 1 is this (generalized) weak condition. In fact, the backaction $\delta n_{ba}$ which is induced by the QND detector of Fig. 1 is evaluated as [12,13]

$$\delta n_{ba} \propto \gamma^2 \langle n \rangle N \frac{1}{|\Delta|^4 \tau_p},$$

where $\gamma \propto e/\hbar c$ denotes a constant that is determined by the structures of the quantum wires and the optical waveguide, $N$ is the number of electrons detected as the interference current, $\Delta = \epsilon_b - \epsilon_a - \hbar \omega$ is the detuning energy, and $\tau_p$ denotes the optical-pulse duration (which is assumed, for simplicity, to be longer than $\tau_t$ of Ref. [7]). It is seen that for a small positive number $\varepsilon_{ba}$, one can make

$$\frac{\delta n_{ba}}{\langle n \rangle} \leq \varepsilon_{ba},$$

by taking $1/|\Delta|$, $1/\tau_p$, $\gamma^2$, and/or $N$ small enough. This corresponds to Eq. (9). Namely, the (generalized) weak condition (9) for the QND measurement is indeed satisfied. In other words, the QND measurement is possible for a certain set of
photon states (i.e., for photon states whose pulse duration $\tau_p$ is longer than some value) if the structure of the device is appropriately designed (which determines $\gamma^2$ and $N$).

On the other hand, the measurement error $\delta n_{\text{err}}$ is evaluated as [7,8,13]

$$\delta n_{\text{err}} \propto \frac{|\Delta|}{\gamma^2 \sqrt{N}}. \tag{12}$$

This dependence on $N$ is a general result for quantum interference devices [14]. It is seen that for a small positive number $\varepsilon_{\text{err}}$, one can make

$$\frac{\delta n_{\text{err}}}{\langle n \rangle} \leq \varepsilon_{\text{err}}, \tag{13}$$

by taking $1/|\Delta|$, $\gamma^2$, $N$, and/or $\langle n \rangle$ large enough. However, as seen from Eq. (10), their increase results in the increase of the backaction. Therefore, the backaction (not on the conjugate observable, but on the photon number itself) and the measurement error are strongly correlated: one cannot make $\delta n_{\text{err}}$ ($\delta n_{\text{ba}}$) smaller without accepting the increase of $\delta n_{\text{ba}}$ ($\delta n_{\text{err}}$). This correct conclusion, derived from the correct Hamiltonian (7), should be contrasted with the wrong result obtained from the “effective” Hamiltonian (5).

\section*{Optimization of the Measuring Apparatus and the Upper Limit of the Information Entropy}

Because of this unavoidable correlation, one must make some optimization. For example, if one wishes to achieve

$$\varepsilon_{\text{ba}}, \varepsilon_{\text{err}} \lesssim 10^{-2} \quad \text{for } \tau_p \geq 10 \text{ ps}, \tag{14}$$

then the QND detector can be designed in such a way that [12]

$$\frac{\delta n_{\text{ba}}}{\langle n \rangle} \simeq 10^{-2}, \tag{15}$$

$$\frac{\delta n_{\text{err}}}{\langle n \rangle} \simeq \frac{10^2}{\langle n \rangle} \quad \text{(i.e., } \delta n_{\text{err}} \simeq 10^2\text{).} \tag{16}$$

In this case, Eq. (14) is satisfied if

$$\langle n \rangle \geq 10^4. \tag{17}$$

Hence, the lower boundary $n_{\text{min}}$ of the response range of the QND detector is found to be $n_{\text{min}} = 10^4$. On the other hand, the upper limit $n_{\text{max}}$ of the response range
is evaluated in this case as \( n_{\text{max}} \sim 10^6 \), which is derived from the requirement that the phase shift of an electron should be less than \( \pi \) [7]. Then, Eq. (14) is satisfied for all values of \( \langle n \rangle \) between \( n_{\text{min}} \) and \( n_{\text{max}} \). Moreover, \( \delta n_{\text{err}} \) is smaller than the “standard quantum limit” \( \sqrt{\langle n \rangle} \) [3,4], throughout this range.

It is interesting to evaluate the information entropy \( I \) that can be obtained by the QND detector. Since \( \delta n_{\text{err}} \) is independent of \( \langle n \rangle \), the number of different values of \( \langle n \rangle \) that can be distinguished by this QND detector is simply given by

\[
\frac{n_{\text{max}} - n_{\text{min}}}{\delta n_{\text{err}}} \sim 10^4.
\]  

(18)

Hence, \( I \) is estimated as

\[
I \simeq \log_2 10^4 \simeq 13.3.
\]  

(19)

This should be contrasted with the wrong conclusion \( I \gg 1 \) that would be obtained from the “effective Hamiltonian” (5). We have obtained the correct value of \( I \) because we have taken account of the strong correlation between \( \delta n_{\text{ba}} \) and \( \delta n_{\text{err}} \). Since this correlation comes from the form of the gauge interaction, similar correlation should also be present for any measuring apparatus of any gauge field. Therefore, similar optimization of the apparatus is necessary, and the response range and the information entropy would be finite, for any measuring apparatus of any gauge field.

**CONCLUDING REMARKS**

The above example clearly demonstrates that the limitation of available interactions in nature gives rise to a fundamental limit of quantum measurement. I discuss possible consequences of this fundamental limit.

Clearly, it imposes a fundamental limit of getting information on quantum systems because the “ideal measurement” (i.e., the measurement with \( \delta Q_{err} = \delta Q_{ba} = 0 \), although the backaction on the conjugate variable is of course finite) is forbidden for a certain set of observables such as the number of gauge quanta. Namely, one cannot get information on \( Q \) without disturbing \( Q \) itself. If one wishes to make this backaction on \( Q \) small, then the response range of the apparatus and the information entropy obtained by the measurement are limited, as demonstrated in the previous section.

It should be pointed out that this limitation of measurement leads to many other limitations. For example, it imposes a limit on the controllability of quantum systems: One must perform a measurement to control a system, but an accurate measurement gives rise to large backactions on some of the measured quantities, and the control becomes imperfect. Another example is the limitation of the preparation of quantum states. To prepare a system in a desired state, one must perform ideal measurements on some set of observables, e.g., on a “complete set of commuting
observables” [15]. However, as I have shown in this paper, an ideal measurement is impossible for some observables. Therefore, the problems of the preparation of quantum states should be reconsidered.

As seen from these examples, the limitation of the measurement means that accurate specification of quantum systems is impossible in general. This seems to be related to the foundation of the statistical mechanics. For example, imperfect knowledge about the initial state of the system leads to rapid loss of knowledge as the time evolves. This results in the ergodic property and the mixing property [16], although quantum systems cannot have these properties if the initial state is accurately known [17].

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REFERENCES

1. L. D. Landau and R. Peierls, Z. Phys. 69, 56 (1931).
2. W.G. Unruh, Phys. Rev. D18, 1764 (1978); V.B. Braginsky et al., Science 209 (1980) 547; C.M. Caves et al., Rev. Mod. Phys. 52 (1980) 341.
3. G.J. Milburn and D.F. Walls, Phys. Rev. A28, 2065 (1983); N. Imoto, H.A. Haus, Y. Yamamoto, Phys. Rev. A32, 2287 (1985).
4. A. La Porta, R.E. Slusher, and B. Yurke, Phys. Rev. Lett. 62, 28 (1989).
5. M. Ozawa, Phys. Rev. Lett., 60, 385 (1988).
6. A. Shimizu and K. Fujita, Proc. Quantum Control and Measurement ‘92 (H. Ezawa and Y. Murayama, eds., Elsevier, 1993), p.191; A. Shimizu and K. Fujita, quant-ph/9804026.
7. A. Shimizu, Phys. Rev. A43, 3819 (1991).
8. A. Shimizu, Nanostructures and Quantum Effects, eds. H. Sakaki and H. Noge, (Springer Verlag, Heidelberg, 1994) 35-47 A. Shimizu, quant-ph/9804027.
9. See books on quantum field theory.
10. A. Shimizu, K. Fujii, M. Okuda and M. Yamanishi, Phys. Rev. B42, 9248 (1990).
11. For virtual excitation, see, e.g., B.S. Wherrett, A.C. Walker, and F.A.P. Tooley, Optical Nonlinearities and Instabilities in Semiconductors, ed. by H. Haug, (Academic, 1988), Chap.10, Sec.2.
12. A. Shimizu (unpublished).
13. Actual calculations have been performed using the nonrelativistic approximation to Eq. (7). This can be justified easily.
14. A. Shimizu and H. Sakaki, Phys. Rev. B44, 13136 (1991).
15. P. A. M. Dirac, The Principles of Quantum Mechanics (Oxford, Oxford, 1958).
16. M. Pollicicott and M. Yuri, Dynamical Systems and Ergodic Theory (Cambridge Univ., Cambridge 1998).
17. M. V. Berry and M. L. Balazs, J. Phys. A\textbf{12}, 635 (1979).