Models for the BPS Berry Connection

Satoshi Ohya

Institute of Quantum Science, Nihon University
Kanda-Surugadai 1-8-14, Chiyoda, Tokyo 101-8308, Japan
ohya@phys.cst.nihon-u.ac.jp

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Abstract

Motivated by the Nahm’s construction, in this paper we present a systematic construction of Schrödinger Hamiltonians for a spin-1/2 particle in which the Berry connection in the ground-state sector becomes the BPS monopole of $SU(2)$ Yang-Mills-Higgs theory. Our construction enjoys a single arbitrary monotonic function, thereby creating infinitely many quantum-mechanical models that simulate the BPS monopole in the space of model parameters.

1. Introduction

The Bogomolny-Prasad-Sommerfield (BPS) monopole [1, 2] is the simplest yet most profound example for non-Abelian magnetic monopoles. Originally, it just appeared as the simplest analytic expression for the ’t Hooft-Polyakov monopole [3, 4] of $SU(2)$ Yang-Mills-Higgs theory by taking the limit of vanishing Higgs potential. However, it was soon recognized that the BPS monopole has an amazingly rich mathematical structure. In particular, it was realized that the BPS monopole can be constructed without solving the field equations. To date, there exist several constructive approaches to the BPS monopoles, the most notable of which is the Nahm’s construction [5]. As is well-known, this approach consists of the following three steps (see, for example, Section 4.4.1 of Ref. [6]):

(1) Solve the Nahm equation—the first-order nonlinear matrix differential equation with quadratic nonlinearity—and obtain the Nahm data;

(2) Solve the construction equation—a one-dimensional Dirac-like equation defined through the Nahm data; and

(3) Compute the following:

$$A^i_{ab}(x) = i\langle \Psi_a(x) | \frac{\partial}{\partial x_i} | \Psi_b(x) \rangle,$$

where $| \Psi_a(x) \rangle$ stands for the normalizable solutions of the construction equation.

It can then be shown that Eq. (1) satisfies the Bogomolny equation—the defining equation of the BPS monopole—and indeed describes the BPS monopole.

Now, an important observation here is that Eq. (1) takes exactly the same form as the non-Abelian Berry connection [7]. This implies that, if $| \Psi_a(x) \rangle$ are realized as wavefunctions for a degenerate energy level, the BPS monopoles may well be simulated as the Berry connections in the parameter space of ordinary quantum-mechanical systems. In fact, such systems do exist and so far there have been discovered two examples. The first example is given in Ref. [8], where it has been discussed that a spin-1/2 particle on $S^2$ with specific magnetic field and potential enjoys a non-Abelian Berry phase described by the BPS monopole. Another example is given in Ref. [9], where the author has shown
that a free spinless particle on $S^1$ with particular pointlike interactions can be effectively described by a spin-1/2 particle on an interval and yields the Berry connection that describes the BPS monopole. A natural question that arises is then whether there exist any other models that reproduce the BPS monopole. As we shall see in the rest of the paper, the answer to this question is affirmative: there exist infinitely many nonrelativistic quantum-mechanical systems where the Berry connection in the ground-state sector becomes the BPS monopole of four-dimensional $SU(2)$ Yang-Mills-Higgs theory. The goal of this short paper is to show this and present several new examples.

The rest of the paper is organized as follows. In Section 2 we first introduce two distinct two-component wavefunctions in one dimension, both of which are nodeless, mutually orthogonal, and specified by a single monotonically increasing function $W$. In terms of the Nahm’s construction, these wavefunctions correspond to the solutions of the construction equation. We then show that the non-Abelian Berry connection built upon these wavefunctions is nothing but the BPS monopole of $SU(2)$ Yang-Mills-Higgs theory. It is also shown that the matrix elements of $W$ generally becomes the BPS solution for the Higgs field. In Section 3 we construct a family of one-dimensional quantum-mechanical models for a spin-1/2 particle by using the technique of supersymmetric quantum mechanics. In this family the ground states are doubly degenerate and the ground-state wavefunctions are given by those constructed in Section 2. The non-Abelian Berry connection in the ground-state sector is therefore always given by the BPS monopole. We also discuss that our models enjoy an exotic supersymmetry called the second-order derivative supersymmetry [10, 11]. Section 4 presents several examples to illustrate our construction. We shall see that our construction method yields all the existing models as well as new ones.

For the sake of notational brevity, throughout the paper we will work in arbitrary dimensionless units, which can always be converted into the physical units by appropriate rescaling.

2. BPS Berry connection

In the standard approach to quantum mechanics, one first constructs a Hamiltonian and then solves the Schrödinger equation. In this paper, however, we solve the problem in reverse order; that is, we first start with a desired ground-state wavefunction and then construct a Hamiltonian. This is possible because, as is well-known especially in the context of supersymmetric quantum mechanics, the ground-state wavefunction generally determines potential energy. In this Section we shall first introduce two nodeless wavefunctions, which correspond to the solutions of the construction equation. Then we shall show that the BPS monopole and Higgs solutions of $SU(2)$ Yang-Mills-Higgs theory are respectively given by the Berry connection and some matrix elements with respect to these wavefunctions. In the subsequent Section we shall construct a family of Schrödinger Hamiltonians whose lowest-energy eigenstates are given by the wavefunctions constructed in this Section.

To begin with, let $I \subset \mathbb{R}$ be a one-dimensional subspace, which can be either a finite interval or an infinite interval, and $z$ be the coordinate of $I$ with $z_+$ and $z_-(< z_+)$ being two endpoints of $I$. Let us then consider the following two wavefunctions on $I$:

$$\psi_{\pm}(z) = N \sqrt{W'(z)} \exp(\pm r W(z)), \quad (2)$$

where $r$ is a positive constant, $W$ is a monotonically increasing function (i.e., $W' > 0$), and prime ($'$) indicates the derivative with respect to $z$. In what follows we shall assume that $W$ fulfills the following boundary conditions:

$$\lim_{z \to z_\pm} W(z) = \frac{z_\pm}{2}. \quad (3)$$

The BPS Berry connection has also been discussed in the context of topological insulators [12], where the solutions of the construction equation are realized as the solutions of four-dimensional Dirac equation with particular boundary conditions. Note that in this paper we will focus on nonrelativistic quantum mechanics and not touch upon the Dirac equation.
$N$ is a normalization factor and chosen to satisfy $|\psi_\pm|_{L^2(I)} = 1$, where $|\cdot|_{L^2(I)}$ stands for the $L^2$-norm on $I$. It is easy to see that $|\psi_\pm|^2_{L^2(I)}$ can be calculated without specifying the explicit form of $W$ and take the following form:

$$|\psi_\pm|^2_{L^2(I)} = |N|^2 \int_{z_-}^{z_+} dz \ W'(z) \exp(\pm 2r W(z))$$

$$= |N|^2 \left[ \frac{1}{2r} \exp(\pm 2r W(z)) \right]_{z=z_-}^{z=z_+}$$

$$= |N|^2 \frac{\sinh(r)}{r},$$

(4)

where the last line follows from the boundary conditions (3). Hence without any loss of generality $N$ can be chosen as follows:

$$N = \sqrt{\frac{r}{\sinh(r)}}$$

(5)

There are two important points to be emphasized here. The first is that, thanks to the monotonicity of $W$, both $\psi_+$ and $\psi_-$ are positive definite and have no node on $I$. Hence they are good candidates for ground-state wavefunctions of one-dimensional quantum-mechanical systems. The second is that, just like Eq. (4), the $L^2$-inner products $(\psi_\pm, \psi_\mp)_{L^2(I)}$ and $(\psi_\pm, W \psi_\mp)_{L^2(I)}$ are given by integrals of total derivatives such that they can be calculated only from the boundary conditions (3). In fact, a straightforward calculation gives

$$(\psi_\pm, \psi_\mp)_{L^2(I)} = |N|^2 \int_{z_-}^{z_+} dz \ W'(z)$$

$$= |N|^2 \left[ W(z) \right]_{z=z_-}^{z=z_+}$$

$$= |N|^2 \frac{r}{\sinh(r)},$$

(6a)

$$(\psi_\pm, W \psi_\mp)_{L^2(I)} = |N|^2 \int_{z_-}^{z_+} dz \ W(z) \exp(\pm 2r W(z))$$

$$= |N|^2 \left[ \frac{1}{2r^2} \left( rW(z) \mp \frac{1}{2} \right) \exp(\pm 2r W(z)) \right]_{z=z_-}^{z=z_+}$$

$$= |N|^2 \left( \frac{1}{2r^2} \cosh(r) \mp \frac{1}{2r^2} \sinh(r) \right)$$

$$= \pm \frac{1}{2} \left( \coth(r) - \frac{1}{r} \right).$$

(6b)

As we shall see shortly, these determine the BPS solutions of $SU(2)$ Yang-Mills-Higgs theory.

Now we wish to construct two distinct nodeless orthonormal wavefunctions in order to fabricate doubly-degenerate ground states. Eq. (6a), however, implies that $\psi_+$ and $\psi_-$ cannot be orthogonal with respect to the $L^2$-inner product. However, they can become orthogonal if uplifted to the vector-valued wavefunctions $(\psi_\pm)$ and $(\phi_\pm)$. More generally, if we consider the two-component wavefunctions

$$\Psi_\pm(z) = \psi_\pm(z) e_\pm,$$

(7)

where $e_+$ and $e_-$ stand for generic two-component orthonormal complex vectors, $\Psi_+$ and $\Psi_-$ become orthonormal with respect to the inner product of the following tensor-product Hilbert space:

$$\mathcal{H} = L^2(I) \otimes \mathbb{C}^2.$$

(8)

The $L^2$-norm is defined by $|f|_{L^2(I)} = \sqrt{\langle f, f \rangle_{L^2(I)}}$, where $\langle \cdot, \cdot \rangle_{L^2(I)}$ stands for the $L^2$-inner product given by $\langle f, g \rangle_{L^2(I)} = \int_{z_-}^{z_+} dz \ f'(z) g(z)$ for any $f, g \in L^2(I)$.

The inner product on $\mathcal{H} = L^2(I) \otimes \mathbb{C}^2$ is defined by $\langle \Psi, \Phi \rangle_{\mathcal{H}} = \int_{z_-}^{z_+} dz \ \Psi^\dagger(z) \Phi(z)$ for any $\Psi, \Phi \in \mathcal{H}$.

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It should be noted that the unit vectors $e_{\pm}$ are generally parameterized by three independent reals. For the following discussions we shall use the following parameterization:

$$e_+ = \frac{1}{\sqrt{2r(r-x^3)}} \left( x^1 - ix^2 \right) \quad \text{and} \quad e_- = \frac{1}{\sqrt{2r(r+x^3)}} \left( -x^1 + ix^2 \right), \quad (9)$$

where $x = (x^1, x^2, x^3) \in \mathbb{R}^3 \setminus \{0\}$ is a nonvanishing 3-vector. It should be emphasized that in the above parameterization we have identified the norm of $x$ with the parameter $r$ entering in the wavefunctions (2); that is, $r = \|x\| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. This identification is technically essential in the following Berry connection argument.

Now, let us suppose that there exists a quantum-mechanical system in which the ground states are doubly degenerate and described by the wavefunctions (7). Let us further assume that the three parameters $(x^1, x^2, x^3)$ can be experimentally controlled. Then, in such a system, under an adiabatic time-evolution along a closed loop in the parameter space, the ground states acquire a non-Abelian Berry phase described by the following Berry connection [7]:

$$A_{ab} = i(\Psi_a, d\Psi_b)_{\mathcal{K}}, \quad (10)$$

where $a, b \in \{+, -\}$ and $d = dx^i \frac{\partial}{\partial x^i}$ stands for the exterior derivative in the parameter space. In the following we shall also consider the following matrix elements of $W$:

$$\Phi_{ab} = (\Psi_a, W\Psi_b)_{\mathcal{K}}. \quad (11)$$

It should be noted that these quantities behave as a gauge field and an adjoint Higgs field of $SU(2)$ gauge theories. Indeed, under a unitary change of the basis (i.e., gauge transformation)

$$\Psi_a \mapsto \tilde{\Psi}_a = \Psi_a g_{ab}^{\dagger}, \quad (12)$$

where $g = (g_{ab})$ is a $2 \times 2$ unitary matrix, $A = (A_{ab})$ and $\Phi = (\Phi_{ab})$ transform as the connection and the adjoint representation for the Lie group $SU(2)$, respectively:

$$A \mapsto \tilde{A} = g^{\dagger}Ag + ig^{\dagger}dg, \quad \Phi \mapsto \tilde{\Phi} = g^{\dagger}\Phi g. \quad (13a)$$

Now we wish to find explicit forms of $A$ and $\Phi$. To this end, it is convenient to move to the gauge given by $g = \begin{pmatrix} e_1^\dagger \\ e_2^\dagger \end{pmatrix}$. In this gauge Eqs. (13a) and (13b) turn out to be of the following forms: 4

$$\tilde{A} = \left( 1 - \frac{r}{\sinh(r)} \right) e^{\dagger} \frac{\sigma^i}{r^2} \frac{\partial}{\partial x^i}, \quad (14a)$$

$$\tilde{\Phi} = \left( \coth(r) - \frac{1}{r} \right) x^i \frac{\sigma^i}{r^2}, \quad (14b)$$

which are nothing but the celebrated BPS solutions for the four-dimensional $SU(2)$ Yang-Mills-Higgs theory [1,2].

To summarize, we have found that a three-parameter family of mutually orthogonal wavefunctions \{ $\Psi_+, \Psi_-$ \} yields the BPS monopole and the adjoint Higgs field as the Berry connection and the matrix elements of $W$. Note that $W$ is arbitrary except for the monotonicity $(W^t > 0)$ and the boundary conditions (3). This arbitrariness opens up a possibility to simulate the BPS solutions in a wide range of nonrelativistic quantum-mechanical systems, because there are infinitely many options for such monotonic function. In the next Section we shall construct a family of Schrödinger Hamiltonians whose lowest-energy eigenstates are all described by Eq. (7).

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4To derive Eqs. (13a) and (13b), one first has to calculate $A = (A_{ab})$ and $\Phi = (\Phi_{ab})$, which take the following forms:

$$A = \begin{pmatrix} ie_1^\dagger de_\sigma & iK e_1^\dagger de_\sigma \\ iK e_1^\dagger de_\sigma & ie_1^\dagger de_\sigma \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}. \quad (15)$$

where $K = \frac{1}{\sinh(r)}$, and $H = \frac{1}{2}(\coth(r) - \frac{1}{r})$. These equations follow from $(6a), (6b)$, $e_1^\dagger e_2 = \delta_{ab}$, and the identities $(\psi, \frac{i}{\partial x^i} \psi)_2_{(ij)} = \frac{\partial}{\partial x^i} (\psi, \psi)_2_{(ij)} \pm \frac{i}{\partial x^i} (\psi, W\psi)_2_{(ij)} = 0$. For the gauge transformation induced by $g = \begin{pmatrix} e_1^\dagger \\ e_2^\dagger \end{pmatrix}$, we refer to Appendix A of [9].
3. Model construction and exotic supersymmetry

Now we wish to construct a $2 \times 2$ matrix-valued Hamiltonian for a spin-1/2 particle which realizes Eq. (7) as the ground-state wavefunctions. In fact, this is very easy to carry out once we realize ground states generally determine potential energies. Below we shall first outline the Hamiltonian construction and then discuss an exotic supersymmetry hidden behind the energy spectrum.

To start with, let us first introduce the following first-order differential operators:

$$D_{1x} = \pm \frac{d}{dz} - \frac{1}{2} \frac{d \log \psi}{dz} - r \psi - \frac{1}{2} W'' \sigma_3,$$  

$$D_{2x} = \pm \frac{d}{dz} + \frac{1}{2} \frac{d \log \psi}{dz} + r \psi + \frac{1}{2} W'' \sigma_3,$$

By construction it is obvious that $\psi_+ \psi_-$ are the zero-modes of $D_{1x}^\dagger$ and $D_{2x}^\dagger$, respectively. In other words, they satisfy the first-order differential equations $D_{1x}^\dagger \psi_+ = 0$ and $D_{2x}^\dagger \psi_- = 0$. It is also obvious that $D_{1x}^\dagger$ and $D_{2x}^\dagger$ $(i = 1, 2)$ are hermitian conjugate with each other with respect to the $L^2$-inner product on $I$. Hence the second-order differential operator $H_{\text{diag}} = \text{diag}(D_{1x}^\dagger D_{1x}^\dagger, D_{2x}^\dagger D_{2x}^\dagger)$, which is hermitian with respect to the inner product on $\mathbb{R}$, is non-negative and enjoys doubly-degenerate ground states given by $(\psi_0, 0)$ and $(0, \psi_0)$ with the energy eigenvalue $E = 0$. The unitary-transformed operator $H = U H_{\text{diag}} U^\dagger$ thus provides the desired Hamiltonian whose ground states are described by (7), provided $U$ is chosen to satisfy $U (\psi_0, 0) = e_\cdot$ and $U (0, \psi_0) = e_-$. Note that such $U$ is easily constructed and given by $U = (e_\cdot, e_-)$.

Having outlined the Hamiltonian construction, we are now ready to find out the explicit form of $H$. Substituting Eqs. (15a) and (15b) into $H_{\text{diag}}$ we first get the following diagonal Hamiltonian:

$$H_{\text{diag}} = \left[ \frac{d^2}{dz^2} + \frac{1}{2} \frac{W'''}{W'} - \frac{1}{4} \left( \frac{W''}{W'} \right)^2 + r^2 \left( \frac{W''}{W'} \right)^2 \right] 1 + 2 W''' \sigma_3,$$

where $1$ stands for the $2 \times 2$ identity matrix. Next, by making use of the unitary transformation

$$H_{\text{diag}} \mapsto H = U H_{\text{diag}} U^\dagger,$$

we finally get the following Hamiltonian for a spin-1/2 particle:

$$H = \left[ \frac{d^2}{dz^2} + \frac{1}{2} \frac{W'''}{W'} - \frac{1}{4} \left( \frac{W''}{W'} \right)^2 + r^2 \left( \frac{W''}{W'} \right)^2 \right] 1 + 2 W''' \sigma \cdot \sigma,$$

where we have used $U \sigma_3 U^\dagger = e_\cdot e_\cdot^\dagger - e_- e_+^\dagger = (\sigma \cdot \sigma)/r$. This is the Hamiltonian whose lowest-energy eigenstates are given by (7). Note that the last term in Eq. (18) corresponds to the interaction between the magnetic moment $\mu \propto \sigma$ for a spin-1/2 particle and a position-dependent external magnetic field $B(z) \propto W''(z) \sigma \cdot \sigma$. Note also that, if the parameters $x = (x^1, x^2, x^3)$ are time-dependent and adiabatically driven along a closed trajectory in the parameter space, the doubly-degenerate ground states always acquire a non-Abelian Berry phase described by the BPS monopole.

Now, Eq. (18) produces a large number of nonrelativistic quantum-mechanical systems for a spin-1/2 particle by specifying the subspace $I \subset \mathbb{R}$ and the monotonically increasing function $W$. Before doing this, however, let us briefly point out that our model possesses a hidden exotic supersymmetry called the second-order derivative supersymmetry [10,11], which is a nonlinear extension of ordinary $\mathcal{N} = 2$ supersymmetry. Its algebra consists of four operators—the Hamiltonian $H$, the supercharge $Q^+ = (Q^-)^\dagger$, and the fermion parity $(-1)^F$, the first three of which are second-order derivative operators—and characterized by some nonlinear relation among $H$ and $Q^\dagger$. To see this, let us

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5 $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ with $\sigma_1 = (\frac{i}{\sqrt{2}}, 0, \frac{i}{\sqrt{2}})$, $\sigma_2 = (0, \frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}})$, $\sigma_3 = (\frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0)$.

6 In fact, the wavefunctions (2) themselves have been obtained through the second-order derivative supersymmetry [11].
for the moment work in the basis where the Hamiltonian becomes diagonal. Then, it is straightforward to show that the $2 \times 2$ matrix-valued operators

$$H_{\text{diag}} = \begin{pmatrix} D_1^* D_1^0 & 0 \\ 0 & D_2^* D_2^0 \end{pmatrix},$$

$$Q^+ = \begin{pmatrix} 0 & 0 \\ D_2^* D_1^0 & 0 \end{pmatrix},$$

$$Q^- = \begin{pmatrix} 0 & D_1^* D_2^0 \\ 0 & 0 \end{pmatrix},$$

$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfy the following algebraic relations of second-order derivative supersymmetry:

$$\left( Q^+ \right)^2 = 0,$$

$$\left( (-1)^F \right)^2 = 1,$$

$$[H_{\text{diag}}, Q^+] = 0,$$

$$[H_{\text{diag}}, (-1)^F] = 0,$$

$$\{ Q^+, (-1)^F \} = 0,$$

$$\{ Q^+, Q^- \} = H_{\text{diag}}^\dagger,$$

where in Eqs. (20c) and (20f) we have used the following identity:

$$D_1^\dagger D_1^0 = D_2^\dagger D_2^0.$$  \hfill (21)

Note that these algebraic relations are invariant under the unitary transformation $\Theta \mapsto U \Theta U^\dagger$, $\Theta \in \{ H_{\text{diag}}, Q^+, Q^-, (-1)^F \}$. Hence the quantum-mechanical system described by (18) also possesses this exotic supersymmetry. One of the big consequences of this supersymmetry is that, in addition to the ground states, any other discrete energy levels (if they exist) are guaranteed to be doubly degenerate.

4. Examples

Before closing this paper let us present several examples of $H$ by specifying $I$ and $W$. Since there are infinitely many options, in this Section we will limit ourselves to only four illustrative examples.

As noted at the end of the Introduction, we will proceed to use arbitrary dimensionless units for notational simplicity.

**Example 1: Hyperbolic tangent.** Let us first take $I$ as the infinite interval $I = (-\infty, \infty)$. A typical example of monotonically increasing function on $I$ that satisfies the boundary conditions $\lim_{z \to \pm \infty} W(z) = \pm 1/2$ is the following hyperbolic tangent:

$$W(z) = \frac{1}{2} \tanh(z), \quad z \in (-\infty, \infty).$$

In this case the Hamiltonian (18) turns out to be of the following form:

$$H = \left[ \frac{d^2}{dz^2} - \frac{2}{\cosh^2(z)} + 1 + \frac{r^2/4}{\cosh^4(z)} \right] \left[ 1 - \frac{2}{\cosh^4(z)} \right] \sigma \cdot \sigma.$$ \hfill (23)

It should be noted that, in the limit $r = |x| \to 0$, the potential energy in (23) reduces to the famous reflectionless potential that admits only one discrete energy level at the energy eigenvalue $E = 0$. The nonvanishing parameter $r$ hence describes the deformation of reflectionless potential while keeping the double degeneracy of the ground states.

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7A straightforward calculation shows that $D_1^\dagger D_1^0 = D_2^\dagger D_2^0 = \frac{d}{dz} - \frac{1}{2}S(W) - \frac{1}{2}S^2(W')$, where $S(W)$ stands for the Schwarzian derivative of $W$ given by $S(W) = \frac{d^3}{dz^3} - \frac{1}{2}(\frac{d}{dz})^3$. 

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Example 2: Error function. Another example of monotonically increasing function on $I = (-\infty, \infty)$ is the following error function:

$$ W(z) = \frac{1}{2} \text{erf}(z), \quad z \in (-\infty, \infty), $$

(24)

where $\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z dt \, e^{-t^2}$. It then follows from $W'(z) = \frac{1}{\sqrt{\pi}} e^{-z^2}$ that the Hamiltonian (18) takes the following form:

$$ H = \left[ -\frac{d^2}{dz^2} + z^2 - 1 + \frac{r^2}{\pi} e^{-2z^2} \right] 1 - \frac{4}{\sqrt{\pi}} z e^{-z^2} \mathbf{x} \cdot \mathbf{\sigma}. $$

(25)

Notice that the potential energy in (25) reduces to the harmonic potential in the limit $r \to 0$. Note also that, in contrast to the previous example, the potential energy blows up in the limit $z \to \pm \infty$ such that it describes a confining potential. Hence the parameter $r$ describes the deformation of harmonic potential while keeping the particle confinement and the ground-state degeneracy.

Example 3: Trigonometric function. Let us next consider the case where $I$ is the finite interval $I = [0, \pi]$. In this case $W$ can be chosen as the following trigonometric function:

$$ W(z) = -\frac{1}{2} \cos(z), \quad z \in [0, \pi]. $$

(26)

Substituting this into Eq. (18) we arrive at the following Hamiltonian:

$$ H = \left[ -\frac{d^2}{dz^2} - \frac{1}{4} \frac{1}{\sin^2(z)} - \frac{1}{4} + \frac{r^2}{4} \sin^2(z) \right] 1 + \cos(z) \mathbf{x} \cdot \mathbf{\sigma}. $$

(27)

We emphasize that this is nothing but the model analyzed in Ref. [8], where the authors have studied a spin-1/2 particle on $S^2$ in the presence of a position-dependent magnetic field as well as a particular external potential. In fact, under the similarity transformation $H \mapsto \tilde{H} = (\sin(z))^{-1/2} H(\sin(z))^{1/2}$, Eq. (27) is cast into the Hamiltonian essentially equivalent to that used in [8]:

$$ \tilde{H} = \left[ -\Delta_{S^2} + \frac{r^2}{4} \sin^2(z) \right] 1 + \cos(z) \mathbf{x} \cdot \mathbf{\sigma}. $$

(28)

where $\Delta_{S^2} = \frac{1}{\sin(z)} \frac{d^2}{dz^2} \sin(z) \frac{d}{dz}$ is the spherical Laplacian for functions independent of the polar angle $\phi$. Note that $z$ should be read as the azimuthal angle $\theta$. An important lesson from this example is that the one-dimensional Hamiltonian (18) can also be realized in higher-dimensional systems through the separation of variables.

Example 4: Linear function. Let us finally consider the finite interval $I = [-\frac{1}{2}, \frac{1}{2}]$ and the following linear function:

$$ W(z) = z, \quad z \in [-\frac{1}{2}, \frac{1}{2}]. $$

(29)

Since the second and third derivatives of the linear function vanishes, in this case the Hamiltonian (18) just becomes the free Hamiltonian (with constant term):

$$ H = \left[ -\frac{d^2}{dz^2} + r^2 \right] 1. $$

(30)

One might therefore think that this non-interacting system could not exhibit any non-trivial non-Abelian Berry phase because the parameter $x$ disappears from the Hamiltonian. This is, however, not the case because the parameter $x$ can do appear in the boundary conditions for the wavefunctions. In fact, it is well-known that the self-adjoint extension argument leads to the $U(2)$ family of boundary
conditions at each boundary \( z = \pm 1/2 \); see, e.g., Ref. [13]. In particular, a special thing happens [9] if we constrain ourselves to the following \( SU(2) \) subfamily of boundary conditions:

\[
(1 + U)\Psi' - i(1 - U)\Psi = 0 \quad \text{at} \quad z = \pm \frac{1}{2},
\]

where \( U \in SU(2) \). Note that any \( SU(2) \) matrix can be parameterized as \( U = e^{ia_+ P_+} + e^{-ia} P_- \), where \( a \in [0, \pi] \) and \( P_\pm = (1 \pm \hat{x} \cdot \sigma)/2 \) with \( \hat{x} = x/r \) being the unit vector pointing in the direction of \( x \). Any element of \( \mathcal{U} \) is then decomposed as \( \Psi = \psi_+ e_+ + \psi_- e_- \). Eq. (31) boils down to the following Robin boundary conditions for the coefficient functions \( \psi_\pm \):

\[
\pm \psi'_\pm - \tan \left( \frac{\alpha}{2} \right) \psi_\pm = 0 \quad \text{at} \quad z = \pm \frac{1}{2},
\]

which follow from \( P_\pm e_\pm = e_\pm \) and \( P_\pm e_\mp = 0 \). If we identify \( r = \tan(\alpha/2) \), the ground-state wavefunctions can be written as \( \Psi_\pm(z) = \sqrt{\frac{r}{\sinh(r)}} \exp(\pm rz) e_\pm \), which are exactly the same forms as Eq. (7). Hence the free spin-1/2 particle on the interval with the particular boundary conditions (31) also enjoys the BPS Berry connection. An important lesson from this example is that the parameter \( x \) does not always appear as the interaction between the magnetic moment and the external magnetic field.

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