On the theory of constructing a numerical-analytical solution of a cantilever beam bend nonlinear differential equation of the first order

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Abstract. We consider a first-order nonlinear differential equation with a movable singular point. For this equation, we built an analytical approximate solution of the special form. The theorem allowing obtaining an a priori estimation of such solution is proved. To illustrate theorem and our constructive approach we give the example. The given method may be generalized to nonlinear differential equations of the higher orders.

1. Introduction

In [1–8] mathematical models of building structures, including cantilever type models, were considered. On their basis [9], a mathematical model of a cantilever type construction was considered. The model includes a first-order differential equation of the form

\[ y' = \psi(x) \sqrt{1 + y^2}, \] (1)

where \( \psi(x) \) is an analytic function.

The analysis of equation (1) carried out in [9] suggests that this non-linear equation has a movable singular point that characterizes the place of failure. Having determined the mechanism of influence on the movable singular point position, we can move the point outside the structure. As a result, it will be possible to maintain the integrity of the cantilever type structure. This is the practical significance of the problem being solved. Note that equation (1) is integrated by quadratures as an equation with separable variables. Equation (1) is the equation of the cantilever beam bend.

A generalization of equation (1) is the equation

\[ y' = \psi(x) \sqrt{1 + y^2} + f(x), \] (2)

where \( f(x) \) is an analytic function. This equation in general case is not integrated by quadratures. Equations (1) and (2) belong to the category of nonlinear differential equations with movable singular points of the same type. As it was shown in publications [10, 11], numerical methods do not guarantee the accuracy of the calculations, since the peculiarity of movable singular points that determine the discontinuous nature of the searched solution is not taken into account. In this paper, we give the form of an approximate solution for equation (2). We also formulate and prove the theorem in which an error
estimate of the approximate solution is given. The example where the considered algorithm is implemented is given.

2 Main Results

In [9], basing on the ideas proposed in the publications [12-15], for the equation (2) with the initial condition

\[ y(x_0) = y_0 \]  

(3)

where \( x_0, y_0 \) are real numbers, the author prove a theorem of the existence of a movable singular point in the solution, as well as the existence and uniqueness of the solution of the equation (1) in the neighborhood of this point. Thus, we can construct an analytical approximate solution of the form

\[ y_N(x) = \sum_{n=0}^{N} C_n (x^* - x)^{n} \]  

(4)

(taking into account the direction of movement along the axis \( Ox \)). We present a theorem that enables to estimate the error of the approximate solution (4) and guarantee the accuracy of the calculations.

**Theorem.** Let:

1) \( x^* \) is a movable singular point of the problem (2)-(3);
2) functions \( \psi(x) \) and \( f(x) \in C^\infty \) in the field \( |x^* - x| < \rho_1 \), \( \rho_1 = \text{const} \neq 0 \);  
3) exist \( M_{1,n} \) and \( M_{2,n} \), that inequalities hold

\[ \frac{|\psi^{(n)}(x^*)|}{n!} \leq M_{1,n}, \quad \frac{|f^{(n)}(x^*)|}{n!} \leq M_{2,n} \quad \forall n = 0, 1, \ldots . \]

Then, for an analytical approximate solution (4) of problem (2)-(3) in the field \( |x^* - x| < \rho_2 \) calculation error does not exceed the value

\[ \Delta y_N(x) \leq \frac{3^{N-2} M (M + 1)^{N-1} |x^* - x|^2}{1 - 3^2 (M + 1)^2 |x^* - x|^2} \left( 1 + 3(M + 1) |x^* - x|^2 \right) \frac{N}{2} \left( 1, 3(M + 1) |x^* - x|^2 \right) \]

where

\[ M = \max_n \{M_{1,n}, M_{2,n}\}, \quad \rho_2 = \min \left\{ \rho_1, \frac{1}{9(M + 1)^2} \right\} \]

**Proof.** According to the condition of the theorem we have

\[ \psi(x) = \sum_{n=0}^{\infty} D_n (x^* - x)^n, \quad f(x) = \sum_{n=0}^{\infty} A_n (x^* - x)^n, \]  

(5)

as \( x^* \) is a regular point for functions \( \psi(x) \) and \( f(x) \). From [9] we have

\[ y(x) = (x^* - x)^{1/2} \sum_{n=0}^{\infty} C_n (x^* - x)^{n} \]  

(6)

in the field \( |x^* - x| < \rho_2 \), where

\[ \rho_2 = \min \left\{ \rho_1, \frac{1}{9(M + 1)^2} \right\}, \quad M = \max_n \{M_{1,n}, M_{2,n}\}, \quad \rho_2 = 0, 1, 2, \ldots \]

For coefficients \( C_n \) of the series (6), an estimate was established [9]
\[ |C_n| \leq 3^{n-3} M(M+1)^{n-2}. \]  

(7)

Then

\[ |y(x) - y_N(x)| = \Delta y_N(x) = \left| \sum_{n=1}^{\infty} C_n (x^* - x)^{n-1} \right|. \]

Taking into account the estimate (7), from the last equality in the case \( N + 1 = 2k \) follows

\[
\Delta y_N(x) = \left| \sum_{n=1}^{\infty} C_n (x^* - x)^{n-1} \right| \leq \sum_{n=2k}^{\infty} 3^{n-3} M(M+1)^{n-2} (x^* - x)^{n-1} \leq \frac{3^{2k-3} M(M+1)}{1 - 3(1+M+1)|x^* - x|^2} \left( 1 + 3(M+1)|x^* - x|^2 \right) \leq \frac{3^{N-2} M(M+1)^{N-1}}{1 - 9(M+1)^2 |x^* - x|^2} \left( 1 + 3(M+1)|x^* - x|^2 \right). \]

A similar inequality can be obtained in the case of \( N + 1 = 2k+1 \). The obtained a priori error estimate is valid in the field

\[ |x^* - x| < \frac{1}{9(M+1)^2}. \]

Example. We consider the problem (2)-(3), where

\[ \psi(x) = 2x, f(x) = 0, x_0 = 0, y(x_0) = -1/\sqrt{3}. \]

(8)

The exact solution contains two branches of the form

\[ y(x) = \frac{i(2x^3 + 1)}{\sqrt{4x^4 + 4x^2 - 3}}, \]

(9)

\[ y(x) = \frac{i(2x^3 - 1)}{\sqrt{4x^4 - 4x^2 - 3}}, \]

(10)

where \( i^2 = -1 \). Solution (9) has two movable singular points \( x^* = \pm 1/\sqrt{2} \).

First, we consider solution (9) in the field of analyticity. For example, we choose a point \( x_0 = 0 \).

For a numerical experiment, we choose \( N=12 \). Then the analytical approximate solution

\[ y_N(x) = \sum_{n=0}^{N} C_n (x - x_0)^n \]

takes the form
We construct a solution to the Cauchy problem (2), (3), (8) using numerical methods, in the form of an interpolation function, with the help of the command NDSolve [16]. Fig. 1 shows graphs of the exact solution (9) - (10) (continuous curve), approximate solutions (11) - (12) (grey dashed curve) and solution (we denote it $y_{\text{ap}}$), obtained as an interpolation function (dashed black curve). Note that the solution $y_{\text{ap}}$ approximates the branch of solution (10) well. But the second branch is not defined using this solution at all. Solution (11) approximates the exact solution well and we see that when approaching a singular point $x^* = 1/\sqrt{2} \approx 0.7071$ the lower branch of the solution tends to $-\infty$. The upper branch has singularities at points $\pm \sqrt{3}/2$.

![Figure 1. Continuous curves (two branches) represent exact solution (9)-(10), dashed black curve represents solution $y_{\text{ap}}$, dashed grey curve represents solution (11), (12).](image)

Let us move to the neighborhood of a movable singular point $x^*$. Then, according to the theorem, the calculation of the coefficients of solution (6) will be carried out using the following formulas:

$$
-C_n \left( \frac{n-1}{2} \right) = \sum_{i=0}^{n} B_{n-i} D_i + A_{n-3} \quad (n = 3, 5, 7, ...); \\
-C_n \left( \frac{n-1}{2} \right) = \sum_{i=0}^{n} B_{n-i} D_i \quad (n = 3, 5, 7, ...); \\
C_0 = -\frac{1}{\sqrt{2}D_0^3}; \quad C_1 = 0; \quad C_2 = \frac{D_1}{\sqrt{2}D_0^3}; \quad C_3 = -\frac{A_0}{1+3C_0^2}; \\
C^*_n = \sum_{i=0}^{n} C_{n-i} C_i; \quad C_{1,n}^* = C_n^* \quad (n = 0, 1, 3, ...); \\
C_{1,2}^* = 1 + 2(C_0 C_1 + C_1^2); \quad C_{2,n}^* = \sum_{i=0}^{n} C_{1,n-i}^* C_{1,i}; \quad C_{3,n}^* = \sum_{i=0}^{n} C_{2,n-i}^* C_{1,i}^*; 
$$

\[ (13) \]
where the coefficients of the following decompositions are used
\[ y^2 = \sum_{n=0}^{\infty} C_n^* (x^* - x)^{n + 2\rho} ; \]

\[ 1 + y^2 = \sum_{n=0}^{\infty} C_{1,n}^* (x^* - x)^{n + 2\rho} ; \]

\[ (1 + y^2)^2 = \sum_{n=0}^{\infty} C_{2,n}^* (x^* - x)^{n + 4\rho} ; \]

\[ (1 + y^2)^3 = \sum_{n=0}^{\infty} C_{3,n}^* (x^* - x)^{n + 6\rho} ; \]

\[ (1 + y^2)^{3/2} = \sum_{n=0}^{\infty} B_n(x^* - x)^{n + 3\rho} ; \]

\[ (1 + y^2)^{3/4} = \sum_{n=0}^{\infty} B_n'(x^* - x)^{n + 6\rho} . \]

We will seek an approximate solution of the form (4) where we choose, for example, \( N = 6 \). We calculate the coefficients of decomposition (4) for the selected \( N \) using formulas (13). As a result, we obtain an approximate solution

\[ y_6(x) = \frac{2x^3 - 4\sqrt{2}x^2 + x - 3\sqrt{2}}{4 \times 2^{3/4} \sqrt{2} - 2x} . \] (14)

We will show graphs of solutions (9) and (14) on fig. 2. We see good approximation by an approximate solution of the exact solution.

Let us move to evaluation of the found solution. It follows from (8) that the maximum radius of a neighborhood of a movable singular point will be for \( M = 2 \). We choose the value of the argument \( x_1 = 0.7 \) taking into account the calculated value of the radius of the neighborhood \( \rho_2 = 0.0123457 \).

The results of the calculations are presented in the table 1. The notation is accepted here: \( x_i \) is current argument value; \( y(x_i) \) is the exact value of the solution; \( y_6(x_i) \) is the approximate value of the solution; \( \Delta \) is the absolute value of the error; \( \Delta_1 \) is a priori error of the approximate solution \( y_6(x_i) \) according to the theorem; \( \Delta_2 \) is the value of the posterior error of the approximate solution \( y_6(x_i) \).
Table 1. Numerical characteristics of calculations

| $x_1$  | $y(x_1)$ | $y_6(x_1)$ | $\Delta$ | $\Delta_1$ | $\Delta_2$ |
|--------|----------|------------|----------|------------|-----------|
| 0.700  | -7.017923930 | -7.017923926 | $4 \times 10^{-9}$ | 0.03418475554 | $1.5 \times 10^{-7}$ |

For value $\Delta_2$, the required accuracy is $1.5 \times 10^{-7}$. Basing on the theorem, this accuracy corresponds to solution (4) with $N = 15$. The summands in the structure of the approximate solution (4) from the seventh to the fifteenth in total do not exceed the required accuracy. Therefore, the error of the solution $y_6(x_1)$ does not exceed the selected accuracy $\Delta_2 = 1.5 \times 10^{-7}$.

3 Conclusion

In this work, the analytical approximate solution for calculating the displacements in the neighborhood of a movable singular point in a cantilever type structure being bended by the action of a concentrated force is found. This solution can be used when solving problems of structural mechanics. We have practical applications for solving mathematical problems using series with fractional negative powers of arbitrary order.

The accuracy of the calculations is confirmed by an a priori error estimate, which can be significantly improved using a posteriori estimate, that illustrated in the example. The proposed method can be generalized to nonlinear differential equations of a higher order with any character of movable singular point. The method involves a large amount of cumbersome analytical and numerical calculations, so it is essential to use computer mathematics systems. All numerical and analytical calculations were performed using the Mathematica system.

From the point of view of programming algorithms for solving the considered problems, the opportunities of Wolfram Research technologies described in [17-19] are important. They significantly complement the set of tools for creating, maintaining and distributing dynamic content when constructing and studying solutions of differential equations.

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