A DIRECT METHOD OF MOVING PLANES FOR FRACTIONAL LAPLACIAN EQUATIONS IN THE UNIT BALL

MEIXIA DOU*
Department of Mathematics
Henan Normal University, Xinxiang, 453007, China

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Abstract. In this paper, we employ a direct method of moving planes for the fractional Laplacian equation in the unit ball. Instead of using the conventional extension method introduced by Caffarelli and Silvestre [6], Chen, Li and Li developed a direct method of moving planes for the fractional Laplacian [8]. Inspired by this new method, in this paper we deal with the semilinear pseudo-differential equation in the unit ball directly. We first review key ingredients needed in the method of moving planes in a bounded domain, such as the narrow region principle for the fractional Laplacian. Then, by using this new method, we obtain the radial symmetry and monotonicity of positive solutions for some interesting semi-linear equations.

1. Introduction. In this paper, we consider the radial symmetry and monotonicity of positive solutions for semi-linear fractional Laplacian equations in the unit ball. Essentially different from the Laplacian, the fractional Laplacian in $\mathbb{R}^n$ is a nonlocal pseudo-differential operator, taking the form

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy,$$

where $\alpha \in (0, 2)$ and P.V. stands for the Cauchy principal value. This operator is well defined in $S$, the Schwartz space of rapidly decreasing $C^\infty$ functions in $\mathbb{R}^n$. In this space, it can also be equivalently defined in terms of the Fourier transform

$$(-\Delta)^{\frac{\alpha}{2}} u(\xi) = |\xi|^{\alpha} \widehat{u}(\xi),$$

where $\widehat{u}$ is the Fourier transform of $u$. One can extend this operator to a wider space of functions. Let

$$L_{\alpha} = \{ u : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty \}.$$

Then it is easy to verify that for $u \in L_{\alpha} \cap C^{1,1}_{loc}$, the integral on the right hand side of (1) is well defined. Throughout this paper, we consider the fractional Laplacian

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* Corresponding author: Meixia Dou.
in this setting. One of our typical problem is the following equation

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u(x) = \frac{1}{|x|^{\gamma}} u^p(x), & u(x) \geq 0, \quad x \in \mathbb{B}, \\
u(x) = 0, & x \in \mathbb{B}_1(0), \\
\lim_{|x| \to 0} u(x) = +\infty.
\end{cases}
\] (2)

Here \( \mathbb{B} = \mathbb{B}_1(0) \setminus \{0\} \). This equation arises naturally when one tries to set up the following inequality

\[
C\left\{ \int_{\mathbb{B}} |x|^{-\gamma} u^{p+1}(x)dx \right\}^{2/(p+1)} \leq \int_{\mathbb{B}} ((-\Delta)^{\frac{\alpha}{2}} u(x), u(x))dx,
\]

for \( u \in C^\infty_0(\mathbb{B}) \).

**Remark 1.** One may consider non-negative solutions \( u \in L_\alpha \cap C^{1,1}_{loc}(\mathbb{B}_1(0)) \) of the problem (2) in the unit ball \( \mathbb{B}_1(0) \). However, when \( u(0) = 0 \), by the maximum principle, we know that \( u = 0 \) in \( \mathbb{B}_1(0) \). When \( u(0) > 0 \) is finite, one has \((-\Delta)^{\frac{\alpha}{2}} u(0) = +\infty\), which is singular and the problem is very difficult to handle.

In recent years, there has been a great deal of interest in using the fractional Laplacian to model diverse physical phenomena, such as anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics, and relativistic quantum mechanics of stars ([3, 7, 12, 18] and the references therein). It also has various applications in probability and finance ([1, 2, 5]). In particular, the fractional Laplacian can be understood as the infinitesimal generator of a stable Lévy process [2]. We refer the readers to Nezza, Palatucci, and Valdinoci’s survey paper [17] for a detailed exposition of the function spaces involved in the analysis of the operator and a long list of relevant references.

During the past years, the method of moving planes has been proved to be a powerful tool to study the radial symmetry of the solutions of nonlinear elliptic equation in an important paper [13]. A famous result about the solutions of the semi-linear elliptic equation in [13] is that:

**Proposition 1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( C^2 \) boundary which is convex in the \( x_1 \) direction and symmetric about \( x_1 = 0 \). Let \( u \) be a positive solution of

\[
\begin{cases}
\triangle u(x) + f(u) = 0, & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega.
\end{cases}
\] (3)

where \( f \in C^1 \). Then \( u_{x_1} = u_1 > 0 \) for \( x_1 < 0 \) and \( u \) is symmetric in \( x_1 \).

The method of moving planes plays an important role in the process of the proof. Later in [14] they extended the result to \( \mathbb{R}^n \), since then, many researchers paid attention to the radial symmetry of solutions to the elliptic equations including the \( \Delta \) operator. Different from the Laplacian, the non-locality of the fractional Laplacian makes it difficult to investigate. To circumvent this difficulty, Caffarelli and Silvestre [6] introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions. The extension method has been applied successfully to study equations involving the fractional Laplacian, and a series of fruitful results have been obtained (see [4, 11] and the references therein). Due to technical restriction, they have to assume \( \alpha \geq 1 \). It seems that this condition cannot be weakened if one wants to carry the method of moving planes on the extended equation. Then in the case \( 0 < \alpha < 1 \), it can be treated by considering the corresponding integral equation. In [9, 10, 15] and [16], the authors applied the method of moving planes
in integral forms and they obtained the radial symmetry for positive solutions of semilinear elliptic equations and elliptic systems.

In the case of more general nonlinearity, for instance, when considering

\[ (-\Delta)^\alpha u(x) = f(x, u), \quad x \in \mathbb{R}^n, \tag{4} \]

in order to show that a positive solution of (4) also solves

\[ u(x) = C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(x, u(y)) dy, \tag{5} \]

so far one needs to assume that \( f(x, u) \) is nonnegative, another technical restriction in carrying out the method of moving planes on the integral equation is that both \( f(x, u) \) and \( \frac{\partial f}{\partial u} \) must be monotone increasing in \( u \), which may not be necessary if one directly works on pseudo-differential equation (4).

Either by extension or by integral equations, one needs to impose extra conditions on the solution, which would not be necessary if we consider the pseudo differential equation directly. In this paper, we will first review the narrow region principle for the fractional Laplacian, and then by using the new method we obtain the radial symmetry and monotonicity of positive solution for some semi-linear equations under the weaker condition. The main theorems and how they fit in the framework of the method of moving planes are illustrated in the following.

We now recall **Key Ingredients in the Method of Moving Planes.**

As usual, let

\[ T_\lambda = \{ x \in \mathbb{R}^n | x_1 = \lambda, \lambda \in \mathbb{R} \}, \]

be the moving planes,

\[ \Sigma_\lambda = \{ x \in \mathbb{R}^n | x_1 < \lambda \}, \]

be the region to the left of the plane, and

\[ x_\lambda = (2\lambda - x_1, x_2, \ldots, x_n), \]

be the reflection of \( x \) about the plane \( T_\lambda \). To compare the values of \( u(x) \) with \( u(x_\lambda) \), we denote

\[ w_\lambda(x) = u(x_\lambda) - u(x) = u_\lambda(x) - u(x), \]

and let

\[ \Sigma^-_\lambda = \{ x \in \Sigma_\lambda | w_\lambda(x) < 0 \}. \]

The first step of the method of moving planes for (2) is to show that for negative value of \( \lambda \) sufficiently close to -1, we have

\[ w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \cap B_1(0). \tag{6} \]

This provides a starting point to move the plane. Then in the second step, we move the plane to the right as long as inequality (6) holds to its limiting position to show that \( u \) is symmetric about the limiting plane. The proof of main theorems in this paper is based on the narrow region principle in the following form.

**Lemma 1.1 (Narrow Region Principle [8]).** *Let \( T \) be a hyperplane in \( \mathbb{R}^n \). Without loss of generality, we may assume that*

\[ T = \{ x = (x_1, x') \in \mathbb{R}^n | x_1 = \lambda, \lambda \in \mathbb{R} \}. \]

*Let*

\[ \bar{x} = (2\lambda - x_1, x_2, \ldots, x_n), \]

\[ H = \{ x \in \mathbb{R}^n | x_1 < \lambda \}, \quad \bar{H} = \{ x | \bar{x} \in H \}. \]
Let $\Omega$ be a bounded narrow region in $H$, such that it is contained in $\{x|\lambda-l<x_1<\lambda\}$ with small $l>0$. Suppose that $w \in L_\alpha \cap C^{1,1}_{loc}(\Omega)$ and is lower semi-continuous on $\Omega$. If $c(x)$ is bounded from below in $\Omega$ and

$$
\begin{cases}
(-\Delta)^\frac{\alpha}{2} w(x) + c(x)w(x) \geq 0, & x \in \Omega, \\
w(x) \geq 0, & x \in H \setminus \Omega, \\
w(\tilde{x}) = -w(x), & x \in H,
\end{cases}
$$

Then for sufficiently small $l > 0$, we have

$$w(x) \geq 0, \quad x \in \Omega.$$  

In the proof of following theorems we will use a direct method of moving planes to obtain the symmetry and monotonicity of positive solutions about some semi-linear equations under the weaker condition.

**Theorem 1.2.** Let $B_1(0)$ be the unit ball in $\mathbb{R}^n$, $B = B_1(0) \setminus \{0\}$. Assume that $u \in L_\alpha \cap C^{1,1}_{loc}(B)$ is a positive solution with $1 < p < \infty$, $\gamma > 0$, satisfying

$$
\begin{cases}
(-\Delta)^\frac{\alpha}{2} u(x) = \frac{1}{|x|^\gamma} u^p(x), & x \in B, \\
u(x) = 0, & x \in B_1(0)^c, \\
\lim_{|x| \to 0} u(x) = +\infty.
\end{cases}
$$

Then each positive solution $u(x)$ must be radially symmetric and monotone decreasing about the origin.

We also consider the Brezis-Nirenberg problem with more general nonlinearity in the unit ball and we have

**Theorem 1.3.** Let $B_1(0)$ be the unit ball in $\mathbb{R}^n$ and assume that $A : [0, \infty) \to [0, \infty)$ is monotone decreasing continuous function. Assume that $u \in L_\alpha \cap C^{1,1}(B_1(0))$ is a positive solution with $1 < p < \infty$, $\mu \geq 0$, satisfying

$$
\begin{cases}
(-\Delta)^\frac{\alpha}{2} u(x) = A(|x|)u^p(x) + \mu u(x), & x \in B_1(0), \\
u(x) = 0, & x \in B_1^c(0).
\end{cases}
$$

Then each positive solution $u(x)$ must be radially symmetric and monotone decreasing about the origin.

As a corollary of our result above, we mention that when $A(|x|) = 1$, we have the following

**Theorem 1.4.** Let $B_1(0)$ be the unit ball in $\mathbb{R}^n$. Assume that $u \in L_\alpha \cap C^{1,1}(B_1(0))$ is a positive solution with $1 < p < \infty$, $\mu \geq 0$, satisfying

$$
\begin{cases}
(-\Delta)^\frac{\alpha}{2} u(x) = u^p(x) + \mu u(x), & x \in B_1(0), \\
u(x) = 0, & x \in B_1^c(0).
\end{cases}
$$

Then each positive solution $u(x)$ must be radially symmetric and monotone decreasing about the origin.

The plan of our paper is below. In section 2, we prove Theorem 1.2, and in section 3, we prove Theorem 1.3.
2. **The proof of Theorem 1.2.** In this section, we prove Theorem 1.2. The condition \( \lim_{|x| \to 0} u(x) = +\infty \) in (9) makes us use the maximum principle without any trouble.

**Proof.**

Step 1. We show that, for \( \lambda > -1 \) and sufficiently close to \(-1\), we have

\[
\lambda(w_{x}(x)) \geq 0, \quad x \in \Sigma_{\lambda} \cap \mathbb{B}.
\]

(a) Now for \( x \in \Sigma_{\lambda} \cap \mathbb{B} \), then from (9), we have

\[
(-\Delta)^\frac{\alpha}{2}w_{\lambda}(x) = \frac{1}{|x|^\gamma}u_{\lambda}^p(x) - \frac{1}{|x|^\gamma}u^p(x)
\]

\[
= \frac{1}{|x|^\gamma}u_{\lambda}^p(x) - \frac{1}{|x|^\gamma}u^p(x) + \frac{1}{|x|^\gamma}u^p(x) - \frac{1}{|x|^\gamma}u^p(x)
\]

\[
= \frac{1}{|x|^\gamma}(u_{\lambda}^p(x) - u^p(x)) + u^p(x)(\frac{1}{|x|^\gamma} - \frac{1}{|x|^\gamma})
\]

\[
\geq \frac{1}{|x|^\gamma}(u_{\lambda}^p(x) - u^p(x))
\]

\[
= \frac{1}{|x|^\gamma}p_{\lambda}^{p-1}(x)w_{\lambda}(x)
\]

\[
\geq \frac{1}{|x|^\gamma}p_{\lambda}^{p-1}(x)w_{\lambda}(x).
\]

That is,

\[
(-\Delta)^\frac{\alpha}{2}w_{\lambda}(x) + c(x)w_{\lambda}(x) \geq 0.
\]

with \( c(x) = -\frac{1}{|x|^\gamma}p_{\lambda}^{p-1}(x) \), and we see that \( c(x) \) is bounded from below in \( \Sigma_{\lambda} \cap \mathbb{B} \).

(b) For \( x \in \Sigma_{\lambda} \setminus (\Sigma_{\lambda} \cap \mathbb{B}) \), it is easy to verify that,

\[
w_{\lambda}(x) \geq 0,
\]

(c) For \( x \in \Sigma_{\lambda} \), we have

\[
w_{\lambda}(x^\lambda) = u_{\lambda}(x^\lambda) - u(x^\lambda) = u(x) - u(x^\lambda) = -w_{\lambda}(x).
\]

By the narrow region principle, we have

\[
w_{\lambda}(x) \geq 0, \quad \forall x \in \Sigma_{\lambda} \cap \mathbb{B}.
\]

Now for \( \lambda \) sufficiently close to \(-1\), we must have

\[
w_{\lambda}(x) \geq 0, \quad \forall x \in \Sigma_{\lambda} \cap \mathbb{B}.
\]

This verifies (12).

Step 2. Step 1 provides a starting point, from which we can now move the plane \( T_{\lambda} \) to the right as long as (12) holds to its limiting position. Define

\[
\lambda_0 = \sup\{\lambda \in (-1, 0) | w_{k}(x) \geq 0, x \in \Sigma_{k} \cap \mathbb{B}; k \leq \lambda\},
\]

In this part, we show that

\[
\lambda_0 = 0,
\]

and

\[
w_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0} \cap \mathbb{B}.
\]

Let \( \Sigma'_{\lambda} = \Sigma_{\lambda} \setminus \{0^\lambda\} \). Suppose that \( \lambda_0 < 0 \), we show that the plane \( T_{\lambda} \) can be moved further right. To be more rigorous, there exists small \( \delta > 0 \), satisfying \( \lambda_0 + \delta < 0 \), such that for any \( \lambda \in (\lambda_0, \lambda_0 + \delta) \), we have

\[
w_{\lambda}(x) \geq 0, \quad \forall x \in \Sigma'_{\lambda} \cap \mathbb{B}.
\]
This is a contradiction with the definition of \( \lambda_0 \). Hence we must have
\[ \lambda_0 = 0. \]

Now we prove (14) by using the narrow region principle to derive the contradiction. Assume \( \lambda_0 < 0 \), because near \( 0^{\lambda_0} \), \( w_{\lambda_0} \) is very positive, i.e., \( \lim_{x \to 0} w_{\lambda_0}(x^{\lambda_0}) = \lim_{x \to 0} u(x) - u(0^{\lambda_0}) = +\infty \), the minimum of \( w_{\lambda_0} \) is attained in \( \Sigma_{\lambda_0} \cap \mathbb{B} \). We will divide \( \Sigma_{\lambda} \cap \mathbb{B} \) into two domains:
\[ \Sigma'_{\lambda} \cap \mathbb{B} = (\Sigma'_{\lambda_0 - \delta} \cap \mathbb{B}) \cup ((\Sigma'_{\lambda} \setminus \Sigma_{\lambda_0 - \delta}) \cap \mathbb{B}). \]

(i) We first consider the region \( \Sigma'_{\lambda_0 - \delta} \cap \mathbb{B} \).
In fact, when \( \lambda_0 < 0 \), we have
\[ w_{\lambda_0}(x) > 0, \quad \forall x \in \Sigma'_{\lambda_0} \cap \mathbb{B}. \] (15)
If not, there exists some \( \hat{x} \in \Sigma'_{\lambda_0} \cap \mathbb{B} \), such that
\[ w_{\lambda_0}(\hat{x}) = 0 = \min_{\Sigma'_{\lambda_0} \cap \mathbb{B}} w_{\lambda_0}(x). \]
It follows that
\[ (-\Delta)^{\gamma} w_{\lambda_0}(\hat{x}) = C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{w_{\lambda_0}(\hat{x}) - w_{\lambda_0}(y)}{|x - y|^{n+\alpha}} dy \]
\[ = C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{-w_{\lambda_0}(y)}{|x - y|^{n+\alpha}} dy \]
\[ = C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{-w_{\lambda_0}(y)}{|x - y|^{n+\alpha}} dy + \int_{\mathbb{R}^n \setminus \Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y)}{|x - y|^{n+\alpha}} dy \]
\[ = C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{-w_{\lambda_0}(y)}{|x - y|^{n+\alpha}} dy + \int_{\Sigma_{\lambda_0}} \frac{w_{\lambda_0}(y)}{|x - y|^{n+\alpha}} dy \]
\[ \leq 0. \] (16)
On the other hand
\[ (-\Delta)^{\gamma} w_{\lambda_0}(\hat{x}) = \frac{1}{|\hat{x}^{\lambda_0}|^\gamma} w^p_{\lambda_0}(\hat{x}) - \frac{1}{|\hat{x}|^\gamma} w^p(\hat{x}) = w^p(\hat{x}) (\frac{1}{|\hat{x}^{\lambda_0}|^\gamma} - \frac{1}{|\hat{x}|^\gamma}) > 0. \] (17)
A contradiction with (16). This proves (15). It follows from (15) that there exists a constant \( c_0 > 0 \), such that
\[ w_{\lambda_0}(x) \geq c_0, \quad x \in \Sigma'_{\lambda_0 - \delta} \cap \mathbb{B}. \]

Since \( w_{\lambda} \) depends on \( \lambda \) continuously, there exists \( \varepsilon > 0 \) and \( \varepsilon < \delta \), such that for all \( \lambda \in (\lambda_0, \lambda_0 + \varepsilon) \), we have
\[ w_{\lambda}(x) \geq 0, \quad x \in \Sigma'_{\lambda_0 - \delta} \cap \mathbb{B}. \] (18)

(ii) Then we consider the narrow region \( (\Sigma'_{\lambda} \setminus \Sigma_{\lambda_0 - \delta}) \cap \mathbb{B} = \Omega \).
For any \( \lambda \in (\lambda_0, \lambda_0 + \delta) \), we will prove
\[ w_{\lambda}(x) \geq 0, \quad \forall x \in \Omega. \]
If not, there exists some \( \hat{x} \in \Omega \), such that
\[ w_{\lambda}(\hat{x}) = \min_{\Omega} w_{\lambda}(x) < 0, \]
Similarly to step 1, we have

(a) For $x \in \Omega \cap \Sigma^{-}$, we have

$$(-\Delta)^{\frac{\alpha}{2}} w_{\lambda}(x) + c(x)w_{\lambda}(x) \geq 0.$$ 

(b) For $x \in \Sigma' \setminus (\Omega \cap \Sigma^{-})$, it is easy to verify that,

$$w_{\lambda}(x) \geq 0.$$ 

(c) For any $x \in \Sigma'$, we have

$$w_{\lambda}(x^\lambda) = -w_{\lambda}(x).$$

By the narrow region principle, we have

$$w_{\lambda}(x) \geq 0, \ \forall x \in \Omega \cap \Sigma^{-}.$$ 

Thus we have

$$w_{\lambda}(x) \geq 0, \ \forall x \in (\Sigma' \setminus \Sigma_{\lambda_0-\delta}) \cap B.$$ 

Combining (18) and (19), we conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$,

$$w_{\lambda}(x) \geq 0, \ x \in \Sigma' \cap B.$$ 

This contradicts the definition of $\lambda_0$, so the plane $T_{\lambda}$ will not stop before hitting the origin of $\mathbb{R}^n$. Therefore, we must have $\lambda_0 = 0$, and

$$w_{\lambda_0}(x) \geq 0, \ \forall x \in \Sigma_{\lambda_0} \cap B.$$ 

Similarly, one can move the plane $T_{\lambda}$ from $+1$ to the left and show that

$$w_{0}(x) \leq 0, \ \forall x \in \Sigma_0 \cap B.$$ 

Now we have shown that

$$w_{0}(x) \equiv 0, \ x \in \Sigma_0 \cap B.$$ 

This completes step 2.

So far, we have proved that $u$ is symmetric about the plane $T_0$. Since the $x_1$ direction can be chosen arbitrarily, we have actually shown that $u$ is radially symmetric in $B_1(0)$ about the origin. In the process of moving planes, we can see that $u(x)$ is monotone decreasing.

This completes the proof of theorem 1.2. \qed

3. **The Proof of the Theorem 1.3.** In this section, we prove Theorem 1.3 and the steps in the proof is similar to the proof of Theorem 1.2.

**Proof.** Step 1. We show that, for $\lambda > -1$ and sufficiently close to $-1$, we have

$$w_{\lambda}(x) \geq 0, \ x \in \Sigma_{\lambda} \cap B_1(0).$$ 

(21)
(a) Now for \( x \in \Sigma_{\lambda} \cap B_1(0) \), then from (10), we have

\[
(-\Delta) \hat{w}_\lambda(x) = A(|x|^\lambda)u_\lambda^p + \mu w_\lambda - A(|x|)u^p - \mu u
\]

\[
= A(|x|^\lambda)u_\lambda^p - A(|x|)u^p + \mu w_\lambda
\]

\[
= A(|x|^\lambda)u_\lambda^p - A(|x|)u_\lambda^p + A(|x|)u^p - A(|x|)u^p + \mu w_\lambda
\]

\[
\geq A(|x|)(u_\lambda^p - u^p) + \mu w_\lambda
\]

\[
= A(|x|)p\xi_{\lambda}^{-1}w_\lambda + \mu w_\lambda
\]

\[
= (A(|x|)p\xi_{\lambda}^{-1} + \mu)w_\lambda
\]

\[
\geq (A(|x|)pu^{p-1} + \mu)w_\lambda(x).
\]

that is,

\[
(-\Delta) \hat{w}_\lambda(x) + c(x)w_\lambda(x) \geq 0.
\]

with \( c(x) = -(A(|x|)pu^{p-1} + \mu) \), and we see that \( c(x) \) is bounded from below in \( \Sigma_{\lambda} \cap B_1(0) \).

(b) For \( x \in \Sigma_{\lambda} \setminus (\Sigma_{\lambda} \cap B_1(0)) \), it is easy to verify that,

\[ w_\lambda(x) \geq 0, \]

(c) For \( x \in \Sigma_{\lambda} \), we have

\[ w_\lambda(x^\lambda) = u_\lambda(x^\lambda) - u(x^\lambda) = u(x) - u(x^\lambda) = -w_\lambda(x). \]

By the narrow region principle, we have

\[ w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_{\lambda} \cap B_1(0). \]

Now for \( \lambda \) sufficiently close to \(-1\), we must have

\[ w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_{\lambda} \cap B_1(0). \]

This verifies (21).

Step 2. Step 1 provides a starting point, from which we can now move the plane \( T_\lambda \) to the right as long as (21) holds to its limiting position. Define

\[ \lambda_0 = \sup \{ \lambda \in (-1, 0) \mid w_k(x) \geq 0, x \in \Sigma_k \cap B_1(0); k \leq \lambda \}, \]

In this part, we show that

\[ \lambda_0 = 0, \]

and

\[ w_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0} \cap B_1(0). \]

Suppose that \( \lambda_0 < 0 \), we show that the plane \( T_\lambda \) can be moved further right. To be more rigorous, there exists small \( \delta > 0 \), satisfying \( \lambda_0 + \delta < 0 \), such that for any \( \lambda \in (\lambda_0, \lambda_0 + \delta) \), we have

\[ w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_{\lambda} \cap B_1(0). \]

(23)

This is a contradiction with the definition of \( \lambda_0 \). Hence we must have

\[ \lambda_0 = 0. \]

Now we prove (23) by using the narrow region principle to derive the contradiction. We will divide \( \Sigma_{\lambda} \cap B_1(0) \) into two domains:

\[ \Sigma_{\lambda} \cap B_1(0) = (\Sigma_{\lambda_0 - \delta} \cap B_1(0)) \cup ((\Sigma_{\lambda} \setminus \Sigma_{\lambda_0 - \delta}) \cap B_1(0)). \]
(i) Firstly, we consider the domain $\Sigma_{\lambda_0-\delta} \cap B_1(0)$. In fact, when $\lambda_0 < 0$, we have

$$w_{\lambda_0}(x) > 0, \quad \forall x \in \Sigma_{\lambda_0} \cap B_1(0).$$

(24)

If not, there exists some $\hat{x} \in \Sigma_{\lambda_0} \cap B_1(0)$, such that

$$w_{\lambda_0}(\hat{x}) = 0 = \min_{\Sigma_{\lambda_0} \cap B_1(0)} w_{\lambda_0}(x).$$

It follows that

$$(-\Delta) \hat{x} w_{\lambda_0}(\hat{x}) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{w_{\lambda_0}(\hat{x}) - w_{\lambda_0}(y)}{|\hat{x} - y|^{n+\alpha}} dy \quad \geq 0.$$  

On the other hand

$$(-\Delta) \hat{x} w_{\lambda_0}(\hat{x}) = A(|\hat{x}|^{\lambda_0}) u_{\lambda_0}(\hat{x}) + \mu u_{\lambda_0}(\hat{x}) - A(|\hat{x}|) u^p(\hat{x}) - \mu u(\hat{x}) \quad \geq 0.$$  

A contradiction with (25). This proves (24). It follows from (24) that exists a constant $c_0 > 0$, such that

$$w_{\lambda_0}(x) \geq c_0, \quad x \in \Sigma_{\lambda_0-\delta} \cap B_1(0).$$

Since $w_\lambda$ depends on $\lambda$ continuously, there exists $\epsilon > 0$ and $\epsilon < \delta$, such that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$w_{\lambda}(x) \geq 0, \quad x \in \Sigma_{\lambda_0-\delta} \cap B_1(0).$$  

(27)

(ii) Then we consider the narrow region $(\Sigma_{\lambda} \setminus \Sigma_{\lambda_0-\delta}) \cap B_1(0) = \Omega$. For any $\lambda \in [\lambda_0, \lambda_0 + \delta)$, we will prove

$$w_{\lambda}(x) \geq 0, \quad \forall x \in \Omega.$$  

If not, there exists some $\hat{x} \in \Omega$, such that

$$w_{\lambda}(\hat{x}) = \min_{\Omega} w_{\lambda}(x) < 0,$$

Similarly to step 1, we have

(a) For $x \in \Omega \cap \Sigma_{\lambda}$, we have

$$(-\Delta) \hat{x} w_{\lambda}(x) + c(x) w_{\lambda}(x) \geq 0.$$

(b) For $x \in \Sigma_{\lambda} \setminus (\Omega \cap \Sigma_{\lambda})$, it is easy to verify that,

$$w_{\lambda}(x) \geq 0.$$
For any $x \in \Sigma_\lambda$, we have

$$w_\lambda(x^\lambda) = -w_\lambda(x).$$

By the narrow region principle, we have

$$w_\lambda(x) \geq 0, \quad \forall x \in \Omega \cap \Sigma^-_\lambda.$$ 

Thus we have

$$w_\lambda(x) \geq 0, \quad \forall x \in (\Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}) \cap B_1(0). \quad (28)$$

Combining (27) and (28), we conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$,

$$w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \cap B_1(0). \quad (29)$$

This contradicts the definition of $\lambda_0$, so the plane $T_\lambda$ will not stop before hitting the origin of $\mathbb{R}^n$. Therefore, we must have $\lambda_0 = 0$, and

$$w_{\lambda_0}(x) \geq 0, \quad \forall x \in \Sigma_{\lambda_0} \cap B_1(0).$$

Similarly, one can move the plane $T_\lambda$ from $+1$ to the left and show that

$$w_{\lambda_0}(x) \leq 0, \quad \forall x \in \Sigma_{\lambda_0} \cap B_1(0).$$

Now we have shown that $\lambda_0 = 0$, and

$$w_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0} \cap B_1(0).$$

This completes step 2.

So far, we have proved that $u$ is symmetric about the plane $T_0$. Since the $x_1$ direction can be chosen arbitrarily, we have actually shown that $u$ is radially symmetric in $B_1(0)$ about the origin. And in the process of moving planes, we can see that $u(x)$ is monotone decreasing.

This completes the proof of theorem 1.3. \hfill \qed

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E-mail address: meixiadou2009@163.com