FACTORIZING GROUPS INTO DENSE SUBSETS

IGOR PROTASOV, SERHII SLOBODIANIUK

ABSTRACT. Let $G$ be a group of cardinality $\kappa > \aleph_0$ endowed with a topology $\mathcal{T}$ such that $|U| = \kappa$ for every non-empty $U \in \mathcal{T}$ and $\mathcal{T}$ has a base of cardinality $\kappa$. We prove that $G$ could be factorized $G = AB$ (i.e. each $g \in G$ has unique representation $g = ab$, $a \in A$, $b \in B$) into dense subsets $A, B$, $|A| = |B| = \kappa$. We do not know if this statement holds for $\kappa = \aleph_0$ even if $G$ is a topological group.

1. Introduction

For a cardinal $\kappa$, a topological space $X$ is called $\kappa$-resolvable if $X$ can be partitioned into $\kappa$ dense subsets [1]. In the case $\kappa = 2$, these spaces were defined by Hewitt [4] as resolvable spaces. If $X$ is not $\kappa$-resolvable then $X$ is called $\kappa$-irresolvable.

In topological groups, the intensive study of resolvability was initiated by the following remarkable theorem of Comfort and van Mill [2]: every countable non-discrete Abelian topological group $G$ with finite subgroup $B(G)$ of elements of order 2 is $2$-resolvable. In fact [11], every infinite Abelian group $G$ with finite $B(G)$ can be partitioned into $\omega$ subsets dense in every non-discrete group topology on $G$. On the other hand, under Martin’s Axiom, the countable Boolean group $G$, $G = B(G)$ admits maximal (hence, 2-irresolvable) group topology [5]. Every non-discrete $\omega$-irresolvable topological group $G$ contains an open countable Boolean subgroup provided that $G$ is Abelian [6] or countable [10], but the existence of non-discrete $\omega$-irresolvable group topology on the countable Boolean group implies that there is a $P$-point in $\omega^*$ [9]. Thus, in some models of ZFC (see [8]), every non-discrete Abelian or countable topological group is $\omega$-resolvable. For systematic exposition of resolvability in topological and left topological group see [3 Chapter 13].

Recently, a new kind resolvability of groups was introduced in [7]. A group $G$ provided with a topology $\mathcal{T}$ is called box $\kappa$-resolvable if there is a factorization $G = AB$ such that $|A| = \kappa$ and each subset $ab$ is dense in $\mathcal{T}$. If $G$ is left topological (i.e. each left shift $x \mapsto gx$, $g \in G$ is continuous) then this is equivalent to $B$ is dense in $\mathcal{T}$. We recall that a product $AB$ of subsets of a group $G$ is factorization if $G = AB$ and the subsets $\{ab : a \in A\}$ are pairwise disjoint (equivalently, each $g \in G$ has the unique representation $g = ab$, $a \in A$, $b \in B$). For factorizations of groups into subsets see [9]. By [7 Theorem 1], if a topological group $G$ contains an injective convergent sequence then $G$ is box $\omega$-resolvable. This note is to find some conditions under which an infinite group $G$ of cardinality $\kappa$ provided with the topology could be factorized into two dense subsets of cardinality $\kappa$. To this goal, we propose a new method of factorization based on filtrations of groups.

2. Theorem and Question

We recall that a weight $w(X)$ of a topological space $X$ is the minimal cardinality of bases of the topology $X$.

Theorem. Let $G$ be an infinite group of cardinality $\kappa$, $\kappa > \aleph_0$, endowed with a topology $\mathcal{T}$ such that $w(G, \mathcal{T}) \leq \kappa$ and $|U| = \kappa$ for each non-empty $U \in \mathcal{T}$. Then there is a factorization $G = AB$ into dense subsets $A, B$, $|A| = |B| = \kappa$.

We do not know whether Theorem is true for $\kappa = \aleph_0$ even if $G$ is a topological group.

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**Question.** Let \( G \) be a non-discrete countable Hausdorff left topological group \( G \) of countable weight. Can \( G \) be factorized \( G = AB \) into two countable dense subsets?

In Comments, we give a positive answer in the following cases: each finitely generated subgroup of \( G \) is nowhere dense, the set \( \{ x^2 : x \in U \} \) is infinite for each non-empty open subset of \( G \), \( G \) is Abelian.

3. Proof

We begin with some general constructions of factorizations of a group \( G \) via filtrations of \( G \).

Let \( G \) be a group with the identity \( e \) and let \( \kappa \) be a cardinal. A family \( \{ G_\alpha : \alpha < \kappa \} \) of subgroups of \( G \) is called a filtration if

1. \( G_0 = \{ e \}, G = \bigcup_{\alpha < \kappa} G_\alpha; \)
2. \( G_\alpha \subset G_\beta \) for all \( \alpha < \beta; \)
3. \( G_\beta = \bigcup_{\alpha < \beta} G_\alpha \) for every limit ordinal \( \beta. \)

Every ordinal \( \alpha < \kappa \) has the unique representation \( \alpha = \gamma(\alpha) + n(\alpha) \), where \( \gamma(\alpha) \) is either limit ordinal or 0 and \( n(\alpha) \in \omega, \omega = \{ 0, 1, \ldots \} \). We partition \( \kappa \) into two subsets

\[ E(\kappa) = \{ \alpha < \kappa : n(\alpha) \text{ is even} \} \]
\[ O(\kappa) = \{ \alpha < \kappa : n(\alpha) \text{ is odd} \}. \]

For each \( \alpha \in E(\kappa) \), we choose some system \( L_\alpha \) of representatives of left cosets of \( G_{\alpha + 1} \setminus G_\alpha \) by \( G_\alpha \) so \( G_{\alpha + 1} \setminus G_\alpha = L_\alpha G_\alpha \). For each \( \alpha \in O(\kappa) \), we choose some system \( R_\alpha \) of representatives of right cosets of \( G_{\alpha + 1} \setminus G_\alpha \) by \( G_\alpha \) so \( G_{\alpha + 1} \setminus G_\alpha = G_\alpha R_\alpha \).

We take an arbitrary element \( g \in G \setminus \{ e \} \) and choose the smallest subgroup \( G_\gamma \) such that \( g \in G_\gamma \). By (3), \( \gamma = \alpha(g) + 1 \) so \( g \in G_{\alpha(g)+1} \setminus G_{\alpha(g)} \). If \( \alpha(g) \in E(\kappa) \) then we choose \( x_0(g) \in L_{\alpha(g)} \) and \( y_0(g) \in G_{\alpha(g)} \) such that \( g = x_0(g)y_0(g) \). If \( \alpha(g) \in O(\kappa) \) then we choose \( y_0(g) \in R_{\alpha(g)} \) and \( y_0(g) \in G_{\alpha(g)} \) such that \( g = y_0(g)g_0 \). If \( g_0 = e \), we make a stop. Otherwise we repeat the argument for \( g_0 \) and so on. Since the set of ordinals \( < \kappa \) is well ordered, after finite number of steps we get the representation

\[ g = x_0(g)x_1(g) \ldots x_{\lambda(g)}(g)y_{\rho(g)} \ldots y_1(g)y_0(g), \]

\[ x_i \in L_{\alpha_i(g)}, \alpha_0(g) > \alpha_1(g) > \cdots > \alpha_{\lambda(g)}(g); \]
\[ y_i \in R_{\beta_i(g)}, \beta_0(g) > \beta_1(g) > \cdots > \beta_{\rho(g)}(g). \]

If either \( \{ \alpha_0(g), \ldots, \alpha_{\lambda(g)}(g) \} = \emptyset \) or \( \{ \beta_0(g), \ldots, \beta_{\rho(g)}(g) \} = \emptyset \) then we write \( g = y_{\rho(g)} \ldots y_1(g)y_0(g) \) or \( g = x_0(g)x_1(g) \ldots x_{\lambda(g)}(g) \). Thus, \( G = AB \) where \( A \) is the set of all elements of the form \( x_0(g)x_1(g) \ldots x_{\lambda(g)}(g) \) and \( B \) is the set of all elements of the form \( y_{\rho(g)} \ldots y_1(g)y_0(g) \). To show that the product \( AB \) is a factorization of \( G \), we assume that, besides (4), \( g \) has a representation

\[ g = z_0z_1 \ldots z_{\lambda_\rho}t_\rho \ldots t_1t_0. \]

If \( g \in G_{\alpha+1} \setminus G_\alpha \) and \( \alpha \in O(\kappa) \) then \( z_0z_1 \ldots z_{\lambda_\rho}t_\rho \ldots t_1t_0 \in G_\alpha \) so \( t_0 = y_0(g) \). If \( \alpha \in E(\kappa) \) then \( z_1 \ldots z_{\lambda_\rho}t_\rho \ldots t_1t_0 \in G_\alpha \) so \( z_0 = x_0(g) \). We replace \( g \) to \( gt_0^{-1} \) or to \( t_0^{-1}g \) respectively and repeat the same arguments.

Now we are ready to prove Theorem. Let \( \{ U_\alpha : \alpha < \kappa \} \) be a \( \kappa \)-sequence of non-empty open sets such that each non-empty \( U \in \mathcal{T} \) contains some \( U_\alpha \). Since \( |U_\alpha| = \kappa \) for every \( \alpha < \kappa \), we can construct inductively a filtration \( \{ G_\alpha : \alpha < \kappa \}, |G_\alpha| = \max\{ |U_\alpha|, |\alpha| \} \) such that, for each \( \alpha \in E(\kappa) \) (resp. \( \alpha \in O(\kappa) \)) there is a system \( L_\alpha \) (resp. \( R_\alpha \)) of representatives of left (resp. right ) cosets of \( G_{\alpha+1} \setminus G_\alpha \) by \( G_\alpha \) such that \( L_\alpha \cap U_\gamma \neq \emptyset \) (resp. \( R_\alpha \cap U_\gamma \neq \emptyset \) for each \( \gamma \leq \alpha \). Then the subsets \( A, B \) of above factorization of \( G \) are dense in \( \mathcal{T} \) because \( L_\alpha \subset A, R_\beta \subset B \) for each \( \alpha \in E(\kappa), \beta \in O(\kappa) \).
4. Comments

1. Analyzing the proof, we see that Theorem holds under weaker condition: $G$ has a family $F$ of subsets such that $|F| = \kappa$ for each $F \in F$ and, for every non-empty $U \in T$, there is $F \in F$ such that $F \subseteq U$.

If $\kappa = \aleph_0$ but each finitely generating subgroup of $G$ is nowhere dense, we can choose a family $\{G_n : n \in \omega\}$ such that corresponding $A, B$ are dense. Thus, we get a positive answer to Question if each finitely generated subgroup $H$ of $G$ is nowhere dense (equivalently the closure of $H$ is not open).

2. Let $G$ be a group and $A, B$ be subsets of $G$. We say that the product $AB$ is a partial factorization if the subsets $\{ab : a \in A\}$ are pairwise disjoint (equivalently, $\{Ab : b \in B\}$ are pairwise disjoint).

We assume that $AB$ is a partial factorization of $G$ into finite subsets and $X$ is an infinite subset of $G$. Then the following statements are easily verified

(5) there is $x \in X$ such that $x \notin B$ and $A(B \cup \{x\})$ is a partial factorization;

(6) if the set $\{x^2 : x \in X\}$ is infinite then there is $x \in X$ such that $(A \cup \{x, x^{-1}\})B$ is a partial factorization.

3. Let $G$ be a non-discrete Hausdorff topological group, $AB$ be a partial factorization of $G$ into finite subsets, $A = A^{-1}$, $e \in A \cap B$ and $g \notin B$. Then

(7) there is a neighbourhood $V$ of $e$ such that, for $U = V \setminus \{e\}$ and for any $x \in U$, the product $(A \cup \{x, x^{-1}\})(B \cup \{x^{-1}g\})$ is a partial factorization (so $g \in (A \cup \{x, x^{-1}\})(B \cup \{x^{-1}g\})$).

It suffices to choose $V$ so that $V = V^{-1}$ and $AUg \cap AB = \emptyset$, $UB \cap (AB \cup AUg) = \emptyset$, $U^2g \cap AB = \emptyset$, $U \cap A = \emptyset$. We use $A = A^{-1}$ only in $UB \cap AUg = \emptyset$.

4. Let $G$ be countable non-discrete Hausdorff topological group such that $\{x^2 : x \in U\}$ is infinite for every non-empty open subset $U$ of $G$. We enumerate $G = \{g_n : n \in \omega\}$, $g_0 = e$ and choose a countable base $\{U_n : n \in \omega\}$ for non-empty open sets. We put $A_0 = \{e\}$, $B_0 = \{e\}$ and use (5), (6), (7) to choose inductively two sequences $(A_n)_{n \in \omega}$ and $(B_n)_{n \in \omega}$ of finite subsets of $G$ such that for every $n \in \omega$, $A_n \subseteq A_{n+1}$, $B_n \subseteq B_{n+1}$, $A_n = A_n^{-1}$, $A_nB_n$ is a partial factorization, $A_n \cap B_n \neq \emptyset$, $B_n \cap U_n \neq \emptyset$. We put $A = \bigcup_{n \in \omega} A_n$, $B = \bigcup_{n \in \omega} B_n$ and note that $AB$ is a factorization of $G$ into dense subsets.

5. Let $G$ be a countable Abelian non-discrete Hausdorff topological group of countable weight. We suppose that $G$ contains a non-discrete finitely generated subgroup $H$. Given any non-empty open subset $U$ of $G$, we choose a neighborhood $X$ of $e$ in $H$ and $g \in S$ such that $Xg \subseteq U$. Since $H$ is finitely generated, the set $\{x^2 : x \in X\}$ is infinite so we can apply comment 4. If each finitely generated subgroup of $G$ is discrete then, to answer the Question we use comment 1.

6. Let $G$ be a countable group endowed with a topology $T$ of countable weight such that $U$ is infinite for every $U \in T$. Applying the inductive construction from comment 5 to $A_nB_n$ and $B_n^{-1}A_n^{-1}$, we get a partial factorization of $G$ into two dense subsets.

7. Let $G$ be a group satisfying the assumption of Theorem and let $\gamma$ be an infinite cardinal, $\gamma < \kappa$. We take a subgroup $A$ of cardinality $\gamma$ and choose inductively a dense set $B$ of representatives of right cosets of $G$ by $A$. Then we get a factorization $G = AB$. In particular, if $G$ is left topological then $G$ is box $\gamma$-resolvable.

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DEPARTMENT OF CYBERNETICS, KYIV UNIVERSITY, VOLODYMYRSKA 64, 01033, KYIV, UKRAINE
E-mail address: i.v.protasov@gmail.com;

DEPARTMENT OF MECHANICS AND MATHEMATICS, KYIV UNIVERSITY, VOLODYMYRSKA 64, 01033, KYIV, UKRAINE
E-mail address: slobodianiuk@yandex.ru