Specific heat of the Kelvin modes in low temperature superfluid turbulence

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It is pointed out that the specific heat of helical vortex line excitations, in low temperature superfluid turbulence experiments carried out in helium II, can be of the same order as the specific heat of the phononic quasiparticles. The ratio of Kelvin mode and phonon specific heats scales with $L_0 T^{-5/2}$, where $L_0$ represents the smoothed line length per volume within the vortex tangle, such that the contribution of the vortex mode specific heat should be observable for $L_0 = 10^6 \cdots 10^8 \text{cm}^{-2}$, and at temperatures which are of order $1 \cdots 10 \text{mK}$.

The problem of superfluid turbulence has seen a renewed upsurge of interest, largely due to the fact that turbulence in superfluids at temperatures of order one Kelvin [11], has proved to be similar to turbulence in normal fluids [11]. Of particular interest should be the range of lower temperatures, of order a few mK, where the normal fluid density is negligible, and a particularly interesting behaviour of superfluid turbulence in a very pure form should be observable [11]. Since the pioneering work of Vinen [1], the dynamics of superfluid turbulence, especially its generation and decay, is to a large extent still not understood. More recent efforts in this direction, investigating, e.g., the energy spectrum and reconnection events in the tangle, can be found in [11]. An interesting thermodynamic approach to directly observable effects of dissipation in low temperature superfluid turbulence has been proposed by Samuels and Barenghi [11]. The effect to be observed, according to this proposal, consists in the conversion of (incompressional) kinetic energy of the vortex tangle into compressional energy, which then heats the sample. In that work, it has been assumed that the specific heat of the turbulent helium II sample is given solely by phononic excitations. Here, it will be shown that, at temperatures of the order a few mK, the total specific heat of line excitations in the vortex tangle is of the same order as the specific heat of phonons, integrated over the volume of the tangle. This will be true for large, yet experimentally feasible, line densities in the tangle. Though vortex lines fill out a very small fraction of the volume of the turbulent helium sample, the contribution of their oscillations to the specific heat should, as will now be shown, be discernible from the phonon specific heat at low temperatures.

The total specific heat of the helium II sample is assumed to be composed of the specific heat of the elementary nontopological excitations which constitute the heat bath, living in the three dimensions of the sample, and the specific heat of the excitations propagating along the topological line vortices, which live in one spatial dimension. Below temperatures of a fraction of one Kelvin, only phonons contribute to the specific heat of the nontopological excitations, the contribution of the gapful excitations (rotons) being exponentially suppressed. The specific heat of sound excitations in a volume $V$ is

$$\frac{C_{\text{ph}}}{k_B} = \frac{2\pi^2}{15} V \left( \frac{k_B T}{\hbar c_s} \right)^3,$$  

where $c_s$ is the speed of sound. The specific heat of the oscillation modes of an isolated vortex ring of radius $r_0$ may generally be written in the form

$$\frac{C_L}{k_B} = \frac{2\pi r_0}{\xi_c} f \left( \frac{k_B T}{\hbar \omega_c} \right).$$

The nondimensional function $f$ depends on the ratio of the temperature and the ‘cyclotron’ energy of the vortex core, $k_B T/\hbar \omega_c$, characterizing the separation between thermal and core energy physics [11]. In helium II, $\hbar \omega_c \sim 1 \cdots 10 \text{K}$, so that the parameter $k_B T/\hbar \omega_c \sim 10^{-3} \cdots 10^{-4}$ for temperatures in the mK range. The cyclotron frequency of the core is defined by $\omega_c = \Gamma/(\pi (\xi_c/2)^2)$, with $\xi_c$ the core diameter and $\Gamma = h/m$ the quantum of velocity circulation. For a line of arbitrary global shape, $2\pi r_0$ in (3) is replaced by the smoothed arc length of vortex line, i.e., the length of line with the Kelvin disturbances subtracted [11]. This undisturbed, smoothed length of line plays the role of “volume” in the specific heat of the Kelvin modes. The smoothed line density of vortices is designated $L_0 = L_0 [\text{cm}^{-2}]$, so that we may assign $2\pi r_0 = L_0 V$, as an (average) filament length entering the specific heat [3]. The line density $L_0$, around which there are Kelvin fluctuations, is to be distinguished from the full line density $L$, which derives from the total (incompressible) kinetic energy, with the contribution of the Kelvin waves included. They differ by a factor logarithmically dependent on the product of the largest Kelvin wave number excited and the average vortex element distance $l = L_0^{-1/2}$ (the smallest possible Kelvin wave number is $l^{-1}$) [3], so that $L_0$ can be about an order of magnitude less than $L$.

The function $f$ in (3) depends on the spectrum of waves on the vortex line. For wavelengths much larger than the core diameter, a single ring vortex has the Kelvin spectrum

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\[ \omega_K = \frac{\Gamma n^2}{4\pi r_0^3} \ln[n_c^* / n], \]  

(3)

where \( n \) is the number of waves on the circumference (the mode number), so that the wave number \( k = n/r_0 \). The ultraviolet cutoff mode number is for a vortex ring \( n_c^* = O(2\pi r_0 / \xi_c) \); for a filament of more general shape, the cutoff wave number \( k_c = O(\xi_c^{-1}) \), and \( n_c^* \) is of order the total length of a filament, divided by \( \xi_c \).

If vortex filaments approach each other to within a distance corresponding to the inverse Kelvin wave vector under consideration, so that \( kl \lesssim 1 \), the Kelvin spectrum will be changed. In the expression (3) for the specific heat of the oscillations presented below, we will assume that vortices remain, at least on average, sufficiently well separated to retain the validity of the Kelvin spectrum (3) for most of the excitations on the filaments. Deviations from that spectrum are too small to be thermodynamically relevant for the specific heat of the oscillations, and in particular its temperature dependence. Conversely, if there are any measurable deviations from the temperature dependence of the specific heat of Kelvin waves, we will have an indicator that the Kelvin spectrum has changed due to the presence of a very dense vortex tangle. Strictly spoken, these conclusions can only claim validity if the vortex system is in thermodynamic equilibrium with itself and the surrounding fluid. This is the case for a vortex array, generated by a constant rotation rate \( \Omega \) of the superfluid, with resulting densities \( L_0 = 2\Omega / \Gamma = 2 \cdot 10^5 \text{cm}^{-2} \text{sec} \). The rotation rates to achieve the line densities of order \( L_0 = 10^6 \ldots 10^8 \text{cm}^{-2} \), discussed below, are thus in the range \( 10^3 \ldots 10^5 \) rad/sec, i.e., of an order of magnitude reached in centrifuges only, so that this appears difficult to realize in experimental low temperature practice. In general, the turbulent vortex tangle will not be in a (local) thermodynamic equilibrium state. We may, however, expect that the contribution of line oscillations in the tangle to the internal energy will still be proportional to the line density \( L_0 \). Furthermore, any deviation from the predicted dependence of the line oscillation contribution on temperature (for line densities which are not too high, so that the Kelvin spectrum remains valid), will give a measure of how far the turbulent tangle is away from thermodynamic equilibrium.

The one-dimensional density of states for the Kelvin modes, within logarithmic accuracy, may be written in the form

\[ N_K(E) \sim \frac{\sqrt{\pi} r_0}{\sqrt{\hbar \Gamma \ln[n_c^*]}} E^{-1/2}. \]  

(4)

For rings large compared to \( \xi_c \), in the continuum approximation of closely spaced excitation frequencies, this expression for the density of states then enables the calculation of the specific heat of the Kelvin modes according to

\[ C_L = L_0 \sqrt{\frac{\omega_c}{2}} \left( \int_0^\infty dE \frac{E}{\exp[E/k_BT] - 1} \right). \]  

(5)

There results the following expression for the dimensionless function in (5)

\[ f_K = \frac{3\zeta(3/2)}{4\sqrt{\pi \ln[n_c^*]}} \sqrt{\frac{k_B T}{\hbar \omega_c}}. \]  

(6)

More explicitly, we may write for the specific heat of the Kelvin modes

\[ \frac{C_L}{k_B} \simeq \frac{3\zeta(3/2)}{8} \sqrt{\frac{k_B T}{\hbar \Gamma \ln[n_c^*]}} \]  

\[ = 3.54 \cdot 10^{11} \sqrt{\text{[cm]}^3} L_0 [10^6 \text{cm}^{-2}] \sqrt{\frac{T[\text{mK}]}{\ln[n_c^*]}}, \]  

(7)

the second line giving the absolute magnitude of the specific heat expressed in terms of the temperatures, line densities, and sample dimensions of a typical experiment in 4He. There is no dependence on \( \xi_c \) (save for the very weak one contained in \( \ln[n_c^*] \)), as it should be for the large wavelength oscillation modes of a vortex line.

\[ \frac{C}{k_B} \sqrt{\text{[cm]}^3} \sqrt{\frac{T[\text{mK}]}{\ln[n_c^*]}} \]

\[ = 2.5 \cdot 10^7 \text{ cm}^{-2} \]

\[ 10^7 \text{ cm}^{-2} \]

\[ 10^6 \text{ cm}^{-2} \]

\[ T[\text{mK}] \]

\[ L_0 \sqrt{\ln[n_c^*]} = 2.5 \cdot 10^7 \text{ cm}^{-2} \]

\[ 10^7 \text{ cm}^{-2} \]

\[ 10^6 \text{ cm}^{-2} \]

\[ T[\text{mK}] \]

FIG. 1. Low temperature asymptotics of the total specific heat per volume \( V = \sqrt{\mu^3} \), divided by \( \sqrt{T} = V[\text{mK}] \), for three different values of \( L_0 / \sqrt{\ln[n_c^*]} \) (the speed of sound \( c_s = 240 \) m/sec).

The ratio of the specific heats (3) and (5) is now given by

\[ \frac{C_L}{C_{ph}} = \frac{15 (\hbar c_s)^3 L_0}{4\pi^2 \xi_c (k_B T)^3} f \left( \frac{k_B T}{\hbar \omega_c} \right) \]  

\[ = 0.74 \frac{1}{\sqrt{\ln[n_c^*]}} \sqrt{\frac{2}{T[\text{mK}]}} \left( \frac{\hbar}{k_B T} \right)^{5/2} \]  

\[ c_s^4 L_0. \]  

(8)
We may express this ratio in units of relevance for turbulence experiments in superfluid $^4$He:

$$\frac{C_L}{C_{ph}} = \frac{1.67}{\sqrt{\ln[n_c]}} \frac{c_s^2[2.4 \cdot 10^4 \text{ cm/sec}]}{T^{5/2}[\text{mK}]} L_0[10^6\text{cm}^{-2}]. \quad (9)$$

The contribution of line oscillations to the specific heat thus becomes noticeable, using the indicated line density of $L_0 = 10^6 \text{cm}^{-2}$, for temperatures which are of the order mK. The total specific heat $C = C_{ph} + C_L$ per volume $V$, divided by $\sqrt{T}$, is displayed in Fig. 1. The extrapolation of $C/\sqrt{T}$ to $T = 0$ yields the smoothed line density, up to the very weak (square root of) logarithmic cutoff dependence on the largest possible number of waves on the filaments (the ultraviolet cutoff mode number), i.e., it gives the quantity $L_0/\sqrt{\ln[n_c]}$.

In higher temperature superfluid turbulence experiments, values of $L \sim 10^6 \text{cm}^{-2}$ have been reported. The number for $L_0$ put into the estimates (5) and (6) above is thus rather conservative. The ultimate limit of achievable line densities in superfluid turbulence, and in particular the absolute line densities which are reached in turbulence experiments at mK temperatures, are as yet not clear. Because the conventional second sound technique to detect vortices fails at these temperatures, one has to look for different means to measure the vortex density. One conceivable possibility, using a plot like that shown in Fig. 1, is to measure the specific heat of the developed turbulent state in the mK range, divide by $\sqrt{T}$, and extract the (smoothed) vortex density from an extrapolation of the resulting asymptote to absolute zero. It should of course be mentioned that an experimental determination of specific heats as small as those estimated in equation (5) is a less than trivial affair (though not entirely unrealistic). However, even if a direct determination of the specific heat of line oscillations $C_L$ will prove to be difficult, this quantity is of relevance for the (thermo-)dynamical behaviour of the superfluid at the lowest temperatures.

In the present context, it is also of interest to note that in the numerical simulation work of Nore et al., quite large line densities of order $L_0 \sim 10^{10} \cdots 10^{11} \text{cm}^{-2}$ have been assumed. The resulting crossover temperatures, for which $C_{ph} = C_L$, are consequently in these simulations already of order 100 mK. However, as already alluded to above, it is likely that for very large densities, modifications of the Kelvin spectrum, by mutual induction of the vortex filaments, have to be taken into account.

According to (5), the ratio of the contributions of Kelvin line oscillations and phonons to the specific heat scales with $L_0 T^{-5/2}$, so that simultaneously relatively low temperatures and high line densities are necessary to observe the influence of the specific heat of the Kelvin modes upon the thermodynamical behaviour of the turbulent superfluid. From equation (3), we can conclude that the required values are not unreasonably large, and that the contribution of the Kelvin oscillations in the vortex tangle should indeed be observable.

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