ON THE DETERMINATION OF THE SINGER TRANSFER

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Abstract. Let \( P_k \) be the graded polynomial algebra \( \mathbb{F}_2[x_1, x_2, \ldots, x_k] \) with the degree of each generator \( x_i \) being 1, where \( \mathbb{F}_2 \) denote the prime field of two elements, and let \( GL_k \) be the general linear group over \( \mathbb{F}_2 \) which acts regularly on \( P_k \).

We study the algebraic transfer constructed by Singer [18] using the technique of the Peterson hit problem. This transfer is a homomorphism from the homology of the mod-2 Steenrod algebra \( A \), \( \text{Tor}^A_{k,k+d}(\mathbb{F}_2, \mathbb{F}_2) \), to the subspace of \( \mathbb{F}_2 \otimes_A P_k \) consisting of all the \( GL_k \)-invariant classes of degree \( d \).

In this paper, by using the results on the Peterson hit problem we present the proof of the fact that the Singer algebraic transfer is an isomorphism for \( k \leq 3 \). This result has been proved by Singer in [18] for \( k \leq 2 \) and by Boardman in [3] for \( k = 3 \). We show that the fourth Singer transfer is also an isomorphism in certain internal degrees. This result is new and it is different from the ones of Bruner, Hà and Hưng [5], Chơn and Hà [8], Hà [9], Hưng and Quỳnh [12], Nam [16].

1. Introduction

Denote by \( P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k] \) the polynomial algebra over the field of two elements, \( \mathbb{F}_2 \), in \( k \) generators \( x_1, x_2, \ldots, x_k \), each of degree 1. This algebra arises as the cohomology with coefficients in \( \mathbb{F}_2 \) of an elementary abelian 2-group of rank \( k \). Therefore, \( P_k \) is a module over the mod-2 Steenrod algebra, \( A \). The action of \( A \) on \( P_n \) is determined by the elementary properties of the Steenrod squares \( Sq^i \) and subject to the Cartan formula \( Sq^{k}(fg) = \sum_{i=0}^{k} Sq^i(f)Sq^{k-i}(g) \), for \( f, g \in P_k \) (see Steenrod and Epstein [19]).

The Peterson hit problem is to find a minimal generating set for \( P_k \) regarded as a module over the mod-2 Steenrod algebra. Equivalently, this problem is to find a vector space basis for \( QP_k := \mathbb{F}_2 \otimes_A P_k \) in each degree \( d \). Such a basis may be represented by a list of monomials of degree \( d \). It is completely determined for \( k \leq 4 \), unknown in general.

Let \( GL_k \) be the general linear group over the field \( \mathbb{F}_2 \). This group acts naturally on \( P_k \) by matrix substitution. Since the two actions of \( A \) and \( GL_k \) upon \( P_k \) commute with each other, there is an inherited action of \( GL_k \) on \( QP_k \).

Denote by \( (P_k)_d \) the subspace of \( P_k \) consisting of all the homogeneous polynomials of degree \( d \) in \( P_k \) and by \( (QP_k)_d \) the subspace of \( QP_k \) consisting of all the classes represented by the elements in \( (P_k)_d \). In [18], Singer defined the algebraic

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transfer, which is a homomorphism

$$\varphi_k : \text{Tor}^A_{k,k+d}(F_2, F_2) \to (QP_k)^{\text{GL}_k}$$

from the homology of the Steenrod algebra to the subspace of \((QP_k)_d\) consisting of all the \(\text{GL}_k\)-invariant classes. It is a useful tool in describing the homology groups of the Steenrod algebra, \(\text{Tor}^A_{k,k+d}(F_2, F_2)\). This transfer was studied by Boardman [3], Bruner, Hà and Hưng [5], Hà [9], Hưng [11], Chơn and Hà [8, 10], Minami [13], Nam [10], Hưng and Quỳnh [14], the present author [22] and others.

Singer showed in [18] that \(\varphi_k\) is an isomorphism for \(k = 1, 2\). Boardman showed in [3] that \(\varphi_3\) is also an isomorphism. However, for any \(k \geq 4\), \(\varphi_k\) is not a monomorphism in infinitely many degrees (see Singer [18], Bruner, Hà and Hưng [5], Hưng [11]). Singer made a conjecture in [18] that the algebraic transfer \(\widehat{\varphi}_k\) is an epimorphism for any \(k \geq 0\). This conjecture is true for \(k \leq 3\). It can be verified for \(k = 4\) by using the results in [21, 23]. The conjecture for \(k \geq 5\) is an open problem.

In this paper, by using the results on the Peterson hit problem we present the proof of the fact that the Singer algebraic transfer is an isomorphism for \(k \leq 3\). Recall that this result has been proved by Singer in [18] for \(k \leq 2\) and by Boardman in [3] for \(k = 3\). To prove this result, Boardman [3] computed the space \(QP_k^{\text{GL}_k}\) by using a basis consisting of the all the classes represented by certain polynomials in \(k\). We also compute this space, however we use the admissible monomial basis for \(QP_3\) that is different from the one of Boardman in [3]. By applying this technique for \(k = 4\), we show that the fourth Singer transfer is also an isomorphism in certain internal degrees. This result is new and it is different from the ones of Bruner, Hà and Hưng [5, 11], Nam [16], Hưng and Quỳnh [12], the present author [22] and others.

In those works it is shown only that the fourth Singer transfer detects certain families of elements in \(\text{Ext}^i_A(F_2, F_2)\), and fails to detect others.

This paper is organized as follows. In Section 2 we recall some needed information on the lambda algebra and the Singer algebraic transfer. In Sections 3, 4 we present the determination of the algebraic transfer for \(k \leq 3\). Finally, in Section 5 we show that the fourth Singer transfer is an isomorphism in certain internal degrees.

2. The Singer algebraic transfer and the lambda algebra

First of all, we briefly recall the definition of the Singer transfer. Let \(\hat{P}_1\) be the submodule of \(F_2[x_1, x_1^{-1}]\) spanned by all powers \(x_1^i\) with \(i \geq -1\). The usual \(A\)-action on \(P_1 = F_2[x_1]\) is canonically extended to an \(A\)-action on \(F_2[x_1, x_1^{-1}]\) (see Singer [18]). \(\hat{P}_1\) is an \(A\)-submodule of \(F_2[x_1, x_1^{-1}]\). The inclusion \(P_1 \subset \hat{P}_1\) gives rise to a short exact sequence of \(A\)-modules:

\[
0 \to P_1 \to \hat{P}_1 \to \Sigma^{-1}F_2 \to 0.
\]

Let \(e_1\) be the corresponding element in \(\text{Ext}^1_A(\Sigma^{-1}F_2, P_1)\). By using the cross and Yoneda products, Singer set

\[
e_k = (e_1 \times P_{k-1}) \circ (e_1 \times P_{k-2}) \circ \ldots \circ (e_1 \times P_1) \circ e_1 \in \text{Ext}_A^k(\Sigma^{-k}F_2, P_k).
\]

Then, he defined \(\varphi_k : \text{Tor}^A_k(F_2, \Sigma^{-k}F_2) \to \text{Tor}^A_0(F_2, P_k) = QP_k\) by \(\varphi_k(z) = e_k \cap z\). Its image is a submodule of \((QP_k)^{\text{GL}_k}\). So, \(\varphi_k\) induces the homomorphism

\[
\varphi_k : \text{Tor}^A_k(F_2, \Sigma^{-k}F_2) \to QP_k^{\text{GL}_k}.
\]
Denote by \((P_k)^*\) the dual of \(P_k\) and by \(P((P_k)^*)\) the primitive subspace consisting of all elements in \((P_k)^*\) that are annihilated by every positive degree operations in the mod-2 Steenrod algebra. The dual of \(\varphi_k\):

\[
Tr_k := (\varphi_k)^* \colon \mathbb{F}_2 \otimes_{GL_k} P((P_k)^*) \rightarrow \text{Ext}^{k,k+d}_A(\mathbb{F}_2, \mathbb{F}_2)
\]

is also called the \(k\)-th Singer transfer.

The algebra \(\text{Ext}^{*,*}_A(\mathbb{F}_2, \mathbb{F}_2)\) is described in terms of the mod-2 lambda algebra \(\Lambda\) (see \([3]\)). Recall that \(\Lambda\) is a bigraded differential algebra over \(\mathbb{F}_2\) generated by \(\lambda_j \in \Lambda^{1,j}, j \geq 0\), with the relations

\[
\lambda_j \lambda_{2j+1+m} = \sum_{\nu \geq 0} \binom{m - \nu - 1}{\nu} \lambda_{j+m-\nu} \lambda_{2j+1+\nu},
\]

for \(m \geq 0\) and the differential

\[
\delta(\lambda_i) = \sum_{\nu \geq 0} \binom{i - \nu - 1}{\nu + 1} \lambda_{k-\nu-1} \lambda_\nu,
\]

for \(i > 0\), \(\delta(\lambda_0) = 0\) and that \(H^{k,d}(\Lambda, \delta) = \text{Ext}^{k,k+d}_A(\mathbb{F}_2, \mathbb{F}_2)\).

For example, the elements \(\lambda_{2i-1} \in \Lambda^{1,2i-1}, i \geq 0\), and \(d_0 = \lambda_0 \lambda_2 \lambda_3^2 + \lambda_2^2 \lambda_3 \lambda_1 \lambda_1 \lambda_2 \in \Lambda^{4,14}\) are the cycles in the lambda algebra \(\Lambda\). So, \(h_i = [\lambda_{2i-1}]\) and \(d_0 = [d_0]\) are the elements in \(\text{Ext}^{3,2}_A(\mathbb{F}_2, \mathbb{F}_2)\). Note that \(h_i\) is the Adams element in \(\text{Ext}^{1,2}_A(\mathbb{F}_2, \mathbb{F}_2)\).

There is a homomorphism \(\tilde{Sq}^0 \colon \Lambda \rightarrow \Lambda\) determined by

\[
\tilde{Sq}^0 (\lambda_1, \lambda_{2j} \ldots \lambda_{3k}) = \lambda_{2j+1+3k+1} \ldots \lambda_{2j+1}, k \geq 0.
\]

This homomorphism respects the relations in \((2.2)\). Therefore, it induces a homomorphism

\[
\text{Sq}^0 : \text{Ext}^{k,k+d}_A(\mathbb{F}_2, \mathbb{F}_2) = H^{k,d}(\Lambda) \rightarrow H^{k,k+2d}(\Lambda) = \text{Ext}^{k,2k+2d}_A(\mathbb{F}_2, \mathbb{F}_2).
\]

A family \(\{a_i : i \geq 0\}\) of elements in \(\text{Ext}^{k,k+d}_A(\mathbb{F}_2, \mathbb{F}_2)\) is called a \(\text{Sq}^0\)-family if \(a_i = (\text{Sq}^0)^i(a_0)\) for every \(i \geq 0\). It is well known that \(\text{Ext}^{3,3+\ast}_A(\mathbb{F}_2, \mathbb{F}_2)\) contains the \(\text{Sq}^0\)-family of indecomposable elements \(\{c_i\}\) and \(\text{Ext}^{4,4+\ast}_A(\mathbb{F}_2, \mathbb{F}_2)\) contains seven \(\text{Sq}^0\)-families of indecomposable elements, namely \(\{d_i\}, \{e_i\}, \{f_i\}, \{g_{i+1}\}, \{p_i\}, \{D_3(i)\}\), and \(\{p'_{1}\}\). Note that \(\{h_i\}\) is also a \(\text{Sq}^0\)-family in \(\text{Ext}^{1,1+\ast}_A(\mathbb{F}_2, \mathbb{F}_2)\).

The algebra \(\{\text{Ext}^{k,k+\ast}_A(\mathbb{F}_2, \mathbb{F}_2)\} | k \geq 0\) has been explicitly computed by Adem \([2]\) for \(k = 1\), by Adams \([1]\) and Wall \([24]\) for \(k = 2\), by Adams \([1]\) and Wang \([25]\) for \(k = 3\) and by Lin \([13]\) for \(k = 4\).

**Theorem 2.1** (See \([1][2][13][23][25]\)).

i) The algebra \(\{\text{Ext}^{k,k+\ast}_A(\mathbb{F}_2, \mathbb{F}_2)\} | k \geq 0\) for \(k \leq 3\) is generated by \(h_i, c_i\) for \(i \geq 0\) and subject only to the relations \(h_i h_{i+1} = 0, h_i^2 = 0, h_i^3 = h_{i-1}^2 h_{i+1}\). In particular, \(\{c_i : i \geq 0\}\) is an \(\mathbb{F}_2\)-basis for the indecomposable elements in \(\text{Ext}^{3,3+\ast}_A(\mathbb{F}_2, \mathbb{F}_2)\).

ii) The algebra \(\{\text{Ext}^{k,k+\ast}_A(\mathbb{F}_2, \mathbb{F}_2)\} | k \geq 0\) for \(k \leq 4\) is generated by \(h_i, c_i, d_i, e_i, f_i, g_{i+1}, p_i, D_3(i)\) and \(p'_{1}\) for \(i \geq 0\) and subject to the relations in i) together with the relations \(h_j^2 h_j^2 = 0, h_j c_i = 0\) for \(j = i - 1, i + 1\) and \(i + 3\). Furthermore, the set of the elements \(d_i, e_i, f_i, g_{i+1}, p_i, D_3(i)\) and \(p'_{1}\), for \(i \geq 0\), is an \(\mathbb{F}_2\)-basis for the indecomposable elements in \(\text{Ext}^{4,4+\ast}_A(\mathbb{F}_2, \mathbb{F}_2)\).
It is well known that the dual of $P_k$ is the divided power algebra generated by $a_1, a_2, \ldots, a_k$:
\[ (P_k)^* = \Gamma(a_1, a_2, \ldots, a_k) \]
where $a_j^{(i)}$ is dual to $x_j^i \in P_k$ with respect to the basis of $P_k$ consisting of all monomials in $x_1, x_2, \ldots, x_k$ and $a_j = a_j^{(1)}$. The graded vector space $\{(P_k)^*|k \geq 0\}$ is an algebra with a multiplication defined by
\[ (a_1^{(i_1)} \ldots a_k^{(i_k)})(a_1^{(i_{k+1})} \ldots a_m^{(i_{m+1})}) = a_1^{(i_1)} \ldots a_k^{(i_k)} a_{k+1}^{(i_{k+1})} \ldots a_{k+m}^{(i_{k+m})} \in (P_{k+m})^*, \]
for any $a_1^{(i_1)} \ldots a_k^{(i_k)} \in (P_k)^*$ and $a_1^{(i_{k+1})} \ldots a_m^{(i_{m+1})} \in (P_m)^*$. In [3], Chơn and Hà defined a homomorphism of algebras
\[ \phi = \{\phi_k|k \geq 0\} : \{(P_k)^*|k \geq 0\} \longrightarrow \{\Lambda^{k,*}|k \geq 0\} = \Lambda, \]
which induces the Singer transfer. Here, the homomorphism $\phi_k : (P_k)^* \rightarrow \Lambda^{k,*}$ is defined by the following inductive formula:
\[ \phi_k(a^{(I,t)}) = \begin{cases} \lambda_t, & \text{if } k - 1 = \ell(I) = 0, \\ \sum_{i \geq 1} \phi_{k-1}(S_q^{-t}a^I)\lambda_i, & \text{if } k - 1 = \ell(I) > 0, \end{cases} \]
for any $a^{(I,t)} = a_1^{(i_1)} a_2^{(i_2)} \ldots a_{k-1}^{(i_{k-1})} a_k^{(i_k)} \in (P_k)^*$ and $I = (i_1, i_2, \ldots, i_{k-1})$.

**Theorem 2.2** (See Chơn and Hà [3]). If $b \in P((P_k)^*)$, then $\phi_k(b)$ is a cycle in the lambda algebra $\Lambda$ and $Tr_k([b]) = [\phi_k(b)]$.

Note that this theorem is a dual version of the one in Hùng [10].

We end this section by recalling some results on Kameko's homomorphism and the generators of the general linear group $GL_k$.

One of the main tools in the study of the hit problem is Kameko’s homomorphism $\hat{S}_q^*: QP_k \rightarrow QP_k$. This homomorphism is induced by the $F_2$-linear map $\psi : P_k \rightarrow P_k$, given by
\[ \psi(x) = \begin{cases} y, & \text{if } x = x_1x_2 \ldots x_ky^2, \\ 0, & \text{otherwise,} \end{cases} \]
for any monomial $x \in P_k$. Note that $\psi$ is not an $A$-homomorphism. However, $\psi S_q^2 = S_q^2\psi$, and $\psi S_q^{2i+1} = 0$ for any non-negative integer $i$.

For a positive integer $n$, by $\mu(n)$ one means the smallest number $r$ for which it is possible to write $n = \sum_{1 \leq i \leq r} (2^{u_i} - 1)$, where $u_i > 0$.

**Theorem 2.3** (Kameko [13]). Let $m$ be a positive integer. If $\mu(2m+k) = k$, then $(\hat{S}_q)_m : (QP_k)_{2m+k} \rightarrow (QP_k)_m$ is an isomorphism of the $GL_k$-modules.

**Definition 2.4.** Let $f, g$ be two polynomials of the same degree in $P_k$. Then, $f \equiv g$ if and only if $f - g \in A^+P_k$. If $f \equiv 0$, then $f$ is called hit.

For $1 \leq i \leq k$, define the $A$-homomorphism $\rho_i : P_k \rightarrow P_k$, which is determined by $\rho_i(x_i) = x_{i+1}, \rho_i(x_{i+1}) = x_i, \rho_i(x_j) = x_j$ for $j \neq i, i+1, 1 \leq i < k$, and $\rho_k(x_1) = x_1 + x_2, \rho_k(x_2) = x_j$ for $j > 1$.

It is easy to see that the general linear group $GL_k$ is generated by the matrices associated with $\rho_i, 1 \leq i \leq k$, and the symmetric group $\Sigma_k$ is generated by the ones associated with $\rho_i, 1 \leq i < k$. So, a class $[f]$ represented by a homogeneous polynomial $f \in P_k$ is an $GL_k$-invariant if and only if $\rho_i(f) \equiv f$ for $1 \leq i < k$. It is an $\Sigma_k$-invariant if and only if $\rho_i(f) \equiv f$ for $1 \leq i < k$. 
3. Determination of $\text{Tr}_k$ for $k \leq 3$

3.1. Determination of $\text{Tr}_k$ for $k \leq 2$.

In this subsection, we present the proof of the following.

**Theorem 3.1.1 (Singer [18])**. The algebraic transfer $\text{Tr}_k$ is an isomorphism for $k \leq 2$.

It is well-known that

$$\text{(QP)}_n^{GL_1} = (QP)_n = \begin{cases} \langle x^{2^n-1} \rangle, & \text{if } n = 2^u - 1, u \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

According to Theorem 2.1, we have

$$\text{Ext}^{1,t+1}_A(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle h_u \rangle, & \text{if } t = 2^u - 1, u \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $(P_1)^+ = \Gamma(a)$ and $a^{(2^n-1)} \in P((P_1)^+)$, $\phi_1(a^{(2^n-1)}) = \lambda_{2^n-1}$ is a cycle in $\Lambda^{1,*}$. Using Theorem 2.2 we get

$$\text{Tr}_1([a^{(2^n-1)}]) = [\phi_1(a^{(2^n-1)})] = [\lambda_{2^n-1}] = h_u, \ \forall u \geq 0.$$  

So, $\text{Tr}_1$ is a isomorphism.

Now, we present the proof of this theorem for $k = 2$ by computing the space $(QP_2)^{GL_2}$. From a result of Wood [20], we need only to compute this space in the degree $n = 2^{s+t} + 2^s - 2$ with $s,t$ non-negative integers.

First, we consider the degree $n = 2^{s+t} - 2$ with $s \geq 0$. Since the iterated Kameko homomorphism $(\hat{S}_q)^{0}_{1} : (QP_2)^{GL_2} \rightarrow (QP_2)^{GL_2}_0$ is an isomorphism of $GL_2$-modules and $(QP_2)^{GL_2}_0 = \langle 1 \rangle$, hence $(QP_2)^{GL_2}_n = (\langle p_{2,s} \rangle)$ with $p_{2,s} := (x_1,x_2)^{2^s-1}$.

Next, we compute $(QP_2)^{GL_2}_n$ with $n = 2^{s+t} + 2^s - 2$, $s \geq 0$. Since the iterated Kameko homomorphism $(\hat{S}_q)^{0}_{1} : (QP_2)^{GL_2}_n \rightarrow (QP_2)^{GL_2}_1$ is an isomorphism of $GL_2$-modules, we need only to compute $(QP_2)^{GL_2}_1$.

According to Peterson [17], $(QP_2)^{GL_2}_n$ is the vector space of dimension 2 with a basis consisting of 2 classes represented by the following monomials:

$$v_{s,1} = x_1^{2^s-1} x_2^{2^s+t-1}, \quad v_{s,2} = x_1^{2^s+t-1} x_2^{2^s-1}.$$  

In particular, $v_{0,1} = x_2$. $v_{0,2} = x_1$. Suppose $\theta = a_1 v_1 + a_2 v_2 = a_1 x_2 + a_2 x_1 \in (QP_2)^{GL_2}_1$ with $a_1, a_2 \in \mathbb{F}_2$. Then $\rho_1(\theta) = a_1 v_2 + a_2 v_1 \equiv \theta$. So, we get $a_1 = a_2$. Since $\rho_2(\theta) \equiv a_1 v_1 + a_2 (v_1 + v_2) \equiv \theta$, we obtain $a_1 = a_2 = 0$. Hence, $(QP_2)^{GL_2}_1 = 0$ and $(QP_2)^{GL_2}$.

Now, we consider the degree $n = 2^{s+t} + 2^s - 2$ with $s,t$ non-negative integers, $t \geq 2$. Since $(\hat{S}_q)^{0}_{1} : (QP_2)^{GL_2}_n \rightarrow (QP_2)^{GL_2}_{2^t-1}$ is an isomorphism of $GL_2$-modules, we need only to compute $(QP_2)^{GL_2}_{2^t-1}$. According to Peterson [17], $(QP_2)^{GL_2}_{2^t-1}$ is the vector space of dimension 3 with a basis consisting of 3 classes represented by the following monomials:

$$u_{t,1} = x_1^{2^t-1}, \quad u_{t,2} = x_2^{2^t-1}, \quad u_{t,3} = x_1 x_2^{2^t-2}.$$  

Suppose $\theta_t = a_1 u_{t,1} + a_2 u_{t,2} + a_3 u_{t,3}$ with $a_1, a_2, a_3 \in \mathbb{F}_2$ and $[\theta_t] \in (QP_2)^{GL_2}_{2^t-1}$. By a simple computation, we have $\rho_1(\theta_t) = a_1 u_{t,1} + a_2 u_{t,1} + a_3 u_{t,3} \equiv \theta_t$, hence
Since 3.1.2, we get the following.\[\begin{align*}
\text{Theorem 3.1.1 is completely proved.}
\end{align*}\]

Combining the above results, we obtain

**Proposition 3.1.2.** Let \(n\) be a non-negative integer. We have
\[\begin{align*}
(QP_2)^{GL_2}_n &= \begin{cases}
\langle [p_{2,s}] \rangle, & \text{if } n = 2^{s+1} - 2, \ s \geq 0 \\
\langle [p_{2,s,t}] \rangle, & \text{if } n = 2^{s+t} + 2^s - 2, \ s \geq 0, \ t \geq 2, \\
0, & \text{otherwise},
\end{cases}
\end{align*}\]

where \(p_{2,s,t} = \psi^s(p_{2,0,t})\).

Recall that \((P_2)^* = \Gamma(a_1,a_2)\). For any \(s, t \geq 0\), we set
\[q_{2,s,t} := a_1^{(2^s-1)}a_2^{(2^{s+t}-1-1)} \in P((P_2)^{s+t+2^s-2}_n).\]

Since \(\langle q_{2,s,0,p_{2,s,t}} \rangle = 1\) and \(\langle q_{2,s,t,p_{2,s,t}} \rangle = 1\) for every \(s \geq 0, t \geq 2\), from Proposition 3.1.2 we get the following.

**Proposition 3.1.3.** For \(n\) a non-negative integer, we obtain
\[\begin{align*}
F_2 \otimes_{GL_2} P((P_2)^*_n) &= \begin{cases}
\langle [q_{2,s,0}] \rangle, & \text{if } n = 2^{s+1} - 2, \ s \geq 0 \\
\langle [q_{2,s,t}] \rangle, & \text{if } n = 2^{s+t} + 2^s - 2, \ s \geq 0, \ t \geq 2, \\
0, & \text{otherwise},
\end{cases}
\end{align*}\]

It is easy to see that \(\phi_2(q_{2,s,t}) = \lambda_{2^s-1}\lambda_{2^{s+t}-1}\) is a cycle in \(\Lambda^{2^s}\). Applying Theorem 2.2 we get
\[Tr_2([q_{2,s,t}]) = [\phi_2(q_{2,s,t})] = [\lambda_{2^s-1}\lambda_{s+t}] = h_sh_{s+t}.\]

Since \(h_sh_{s+1} = 0\), applying Theorem 2.1 we have
\[\text{Ext}^2_{A}(F_2,F_2) = \begin{cases}
\langle h_s^2 \rangle, & \text{if } m = 2^{s+1}, \text{ with } s \geq 0, \\
\langle h_sh_{s+t} \rangle, & \text{if } m = 2^{s+t} + 2^s, \text{ with } s \geq 0, \ t \geq 2, \\
0, & \text{otherwise},
\end{cases}\]

Theorem 3.1.1 is completely proved.

3.2. Determination of \(Tr_3\).

In this subsection, we present the proof of the following.

**Theorem 3.2.1** (Boardman [3]). The third Singer algebraic transfer
\[Tr_3 : F_2 \otimes_{GL_3} P((P_3)^*) \longrightarrow \text{Ext}^{3^s+3}_{A}(F_2,F_2)\]
is an isomorphism.

To prove this theorem, Boardman [3] computed the space \(QP_3^{GL_3}\) by using a basis consisting of the all the classes represented by certain polynomials in \(P_3\). It is difficult to use his method for \(k = 4\), where there are 315 polynomials instead of 21. We also compute this space, however we use the admissible monomial basis for \(QP_3\) that is different from the one of Boardman in [3]. Our approach can be apply for \(k = 4\) by using the admissible monomial basis for \(QP_3\) which is given in [21][23].

From a result of Wood [26], we need only to compute \(QP_3^{GL_3}\) in the degree \(n\) with \(\mu(n) \leq 3\).
3.2.1. The case \( n = 2^{t+1} - 2 \).

According to Kameko \[13\], \((QP_3)_n\) is a vector space with a basis consisting of all the classes represented by the following monomials:

\[
\begin{align*}
v_{t,1} &= x_2^{2t-1} x_3^{2t-1},& v_{t,2} &= x_1^{2t-1} x_3^{2t-1},& v_{t,3} &= x_1^{2t-1} x_2^{2t-1},& \text{for } t \geq 1, \\
v_{t,4} &= x_1 x_2^{2t-2} x_3^{2t-1},& v_{t,5} &= x_1 x_2^{2t-1} x_3^{2t-2},& v_{t,6} &= x_1^{2t-1} x_2^{2t-2},& \text{for } t \geq 2, \\
v_{t,7} &= x_1^3 x_2^{2t-3} x_3^{2t-2},& \text{for } t \geq 3
\end{align*}
\]

Set \( p_{3,t} = \sum_{i=1}^{7} v_{t,i} \), with \( t \geq 3 \). By a direct computation, we have

**Proposition 3.2.2.** For any non-negative integer \( t \), we have

\[
(QP_3)^{GL_3}_{2^{t+1}-2} = \begin{cases} 
(1), & \text{if } t = 0, \\
0, & \text{if } t = 1, 2, \\
\langle[p_{3,t}]\rangle, & \text{if } t \geq 3.
\end{cases}
\]

Recall that \((P_3)^* = \Gamma(a_1, a_2, a_3)\). We set

\[
q_{3,t} = a_1^{(0)} a_2^{(2t-1)} a_3^{(2t-1)} \in P((P_3)^*_{2^{t+1}-2}).
\]

Since \( \langle p_{3,t}, q_{3,t} \rangle = 1 \), we get

\[
\mathbb{F}_2 \otimes_{GL_3} P((P_3)^*_{2^{t+1}-2}) = \begin{cases} 
\langle[1]\rangle, & \text{if } t = 0, \\
0, & \text{if } t = 1, 2, \\
\langle[q_{3,t}]\rangle, & \text{if } t \geq 3.
\end{cases}
\]

It is easy to see that \( \phi_3(q_{3,t}) = \lambda_0 \lambda_2^{2t-1} \) is a cycle in \( \Lambda^{3,*} \). By Theorem 2.2, we have

\[
\text{Tr}_3([q_{3,t}]) = [\phi_3(q_{3,t})] = [\lambda_0 \lambda_2^{2t-1}] = h_0 h_t^2.
\]

According to Theorem 2.1, we have

\[
\text{Ext}_A^3(\mathbb{F}_2, \mathbb{F}_2) = \langle h_0 h_t^2 \rangle.
\]

Since \( h_0 h_1 = 0 \) and \( h_0 h_2 = 0 \), from the above equalities we see that Theorem 3.2.1 is true in this case.

3.2.2. The case \( n = 2^{t+u} + 2^u - 3 \).

If \( u > 1 \) then \( \mu(n) = 3 \), hence the iterated Kameko homomorphism

\[
(\widetilde{Sq}_0)^{u-1} : (QP_3)^{GL_3}_{2^{t+u}+2^u-3} \to (QP_3)^{GL_3}_{2^{t+1}-1}
\]

is also an isomorphism \( GL_3 \)-modules. Hence, we need only to compute \((QP_3)^{GL_3}_{2^{t+1}-1}\). According to Kameko \[13\], \((QP_3)_n\) is a vector space with a basis consisting of all
the classes represented by the following monomials:

\[ u_{t,1} = x_3^{2^t+1-1}, \quad u_{t,2} = x_2^{2^t+1-1}, \quad u_{t,3} = x_1^{2^t+1-1}, \quad \text{for } t \geq 0, \]
\[ u_{t,4} = x_2^{2^t+1-2}, \quad u_{t,5} = x_3^{2^t+1-2}, \quad u_{t,6} = x_1^{2^t+1-2}, \quad \text{for } t \geq 1, \]
\[ u_{t,7} = x_1 x_2 x_3, \quad \text{for } t = 1, \]
\[ u_{t,8} = x_1^{2^t-1} x_3^{2^t-1}, \quad u_{t,9} = x_2^{2^t-1} x_3^{2^t-1}, \quad u_{t,10} = x_1^{2^t-1} x_2^{2^t-1} x_3, \quad \text{for } t \geq 2, \]
\[ u_{t,11} = x_3^{2^t+1} x_1^{2^t-1}, \quad u_{t,12} = x_2^{2^t+1} x_3^{2^t-3}, \quad u_{t,13} = x_1^{2^t-1} x_2 x_3^{2^t-3}, \quad \text{for } t \geq 3, \]
\[ u_{t,14} = x_1^{2^t+1} x_2 x_3^{2^t-3}, \quad \text{for } t \geq 4. \]

Set \( p_{3,t,1} = \sum_{i=1}^{7} u_{t,i} \) for \( t \geq 1 \) and \( \bar{p}_{3,t,1} = \sum_{j=7}^{14} u_{t,j} \) for \( t \geq 4. \) By a direct computation we have

**Proposition 3.2.3.** For any integers \( t \geq 0, \) \( u \geq 0, \) we have

\[
(QP_3)^{GL_3}_{2^{t+u}+2^u-3} = \begin{cases} 
0, & \text{if } t = 0, \\
\langle [p_{3,t,u}] \rangle, & \text{if } 1 \leq t \leq 3, \\
\langle [p_{3,t,u}], [\bar{p}_{3,t,u}] \rangle, & \text{if } t \geq 4,
\end{cases}
\]

where \( p_{3,t,u} = \psi^{u-1}(p_{3,t,1}), \) \( \bar{p}_{3,t,u} = \psi^{u-1}(\bar{p}_{3,t,1}). \)

We set

\[ q_{3,t,u} = a_1^{(2^u-1)} a_2^{(2^{t+u}-1)} a_3^{(2^{t+u-1})}, \quad \bar{q}_{3,t,u} = a_1^{(2^u-1)} a_2^{(2^{t+u-1})} a_3^{(2^{t+u-1})}. \]

It is easy to see that \( q_{3,t,u}, \bar{q}_{3,t,u} \in P((P_{3})^*_{2^{t+u}-2}) \) and

\[
\langle p_{3,t,u}, q_{3,t,u} \rangle = 1, \quad \langle p_{3,t,u}, \bar{q}_{3,t,u} \rangle = 0, \\
\langle \bar{p}_{3,t,u}, q_{3,t,u} \rangle = 0, \quad \langle \bar{p}_{3,t,u}, \bar{q}_{3,t,u} \rangle = 1.
\]

So, we get

\[
\mathbb{F}_2 \otimes GL_3 P((P_{3})^*_{2^{t+u}+2^u-3}) = \begin{cases} 
0, & \text{if } t = 0, \\
\langle [q_{3,t,u}] \rangle, & \text{if } 1 \leq t \leq 3, \\
\langle [q_{3,t,u}], [\bar{q}_{3,t,u}] \rangle, & \text{if } t \geq 4.
\end{cases}
\]

By applying Theorem [2.2] we have

\[ \phi_3(q_{3,t,u}) = \lambda_{2^{u-1}}^{2^t} \lambda_{2^{u+1}}^{2^t}, \]
\[ \phi_3(\bar{q}_{3,t,u}) = \lambda_{2^{t+1}}^{2^u} \lambda_{2^{t+1}}^{2^u}. \]

are the cycles in \( \Lambda^{3,*}. \) So, we obtain

\[ Tr_3([q_{3,t,u}]) = [\phi_3(q_{3,t,u})] = [\lambda_{2^{u-1}}^{2^t} \lambda_{2^{u+1}}^{2^t}] = h_{u-1}^2 h_{t+u}, \]
\[ Tr_3([\bar{q}_{3,t,u}]) = [\phi_3(\bar{q}_{3,t,u})] = [\lambda_{2^{t+1}}^{2^u} \lambda_{2^{t+1}}^{2^u}] = h_u h_{t+u}^2. \]

According to Theorem [2.1] we have

\[ \text{Ext}_{\Lambda}^{3,2^{t+u}+2^u} (\mathbb{F}_2, \mathbb{F}_2) = \langle h_u h_{t+u-1}^2, h_{u-1}^2 h_{t+u} \rangle. \]

If \( t = 0 \) then \( h_u h_{u-1}^2 = h_u^2 h_{u-1} = 0. \) If \( t = 1 \) then \( h_u h_{t+u-1}^2 = h_u^2 h_{u-1} + h_{u-1}^2 h_{t+u} = h_{u-1}^2 h_{t+u}. \) If \( t = 2 \) then \( h_u h_{t+u-1}^2 = h_u h_{t+u+1}^2 = 0. \) If \( t = 3 \) then \( h_u h_{t+u-1}^2 = h_u h_{t+u+1}^2 = 0. \) Hence, from the above equalities we can easily see that Theorem [3.2.1] is true in this case.
3.2.3. The case $n = 2^{s+u+1} + 2^{u+1} + 2^u - 3$.

If $u > 0$ then $\mu(n) = 3$, hence the iterated Kameko homomorphism

$$\left(\tilde{S}q^0\right)^u : (QP_3)^{2^{s+u+2u-3}} \to (QP_3)^{2^{s+1}}$$

is also an isomorphism of $GL_3$-modules. Hence, we need only to compute $(QP_3)^{GL_3}_n$.

According to Kameko [13], $(QP_3)^{GL_3}_n$ is a vector space with a basis consisting of all the classes represented by the following monomials:

- $v_{1,1,1} = x_1^2 x_3^{2^{s+1}-1}$, $v_{1,2,1} = x_1^2 x_3^{2^{s+1}-1}$, $v_{1,3,1} = x_1 x_3^{2^{s+1}-1}$,
- $v_{1,4,1} = x_1^2 x_3^{2^{s+1}-1}$, $v_{1,5,1} = x_1 x_3^{2^{s+1}-1}$, $v_{1,6,1} = x_1 x_3^{2^{s+1}-1} x_2$, for $s \geq 1$,
- $v_{1,7} = x_1 x_2 x_3$, $v_{1,8} = x_1 x_2^2 x_3$, for $s = 1$,
- $v_{1,9} = x_1^3 x_3^{2^{s+1}-3}$, $v_{1,10} = x_1 x_2^2 x_3^{2^{s+1}-3}$, $v_{1,11} = x_1 x_2^2 x_3^{2^{s+1}-3}$,
- $v_{1,12} = x_1 x_2^2 x_3^{2^{s+1}-3}$, $v_{1,13} = x_1 x_2^2 x_3^{2^{s+1}-4}$, for $s \geq 2$,
- $v_{1,14} = x_1 x_2^2 x_3$, for $s = 2$.

Set $\tilde{c}_0 = v_{2,10} + v_{2,11} + v_{2,12} + v_{2,13}$. By a direct computation, we have

**Proposition 3.2.4.** For any integers $s > 0, u \geq 0$ and $n = 2^{s+u+1} + 2^{u+1} + 2^u - 3$, we have

$$(QP_3)^{GL_3}_n = \begin{cases} \langle \psi^u(\tilde{c}_0) \rangle, & \text{if } s = 2, \\ 0, & \text{if } s \neq 2. \end{cases}$$

We set

$$\tilde{c}_u = \tilde{c}_1(3.2^{s+1}) - a_1(4.2^{u+1}) a_2(4.2^{u+1}) a_3(4.2^{u+1}) + \tilde{c}_2(2.2^{s+1}) a_2(2.2^{s+1}) a_3(2.2^{s+1})$$

$$+ \tilde{c}_3(2.2^{s+1}) a_2(2.2^{s+1}) a_3(2.2^{s+1}) + \tilde{c}(2.2^{s+1}) a_2(2.2^{s+1}) a_3(2.2^{s+1})$$

is an element in $(P_3)^* = \Gamma(a_1, a_2, a_3)$. By a direct computation, we can see that $\tilde{c}_u \in P((P_3)^{2^{s+u+2u-3}})$ and $\langle \psi^u(\tilde{c}_0), \tilde{c}_u \rangle = 1$. So, we get

$$F_2 \otimes GL_3 P((P_3)^*) = \begin{cases} \langle \tilde{c}_u \rangle, & \text{if } s = 2, \\ 0, & \text{if } s \neq 2. \end{cases}$$

For $u = 0$, we have $\tilde{c}_0 = a_1(2) a_2(3) a_3(3) + a_1(2) a_2(3) a_3(3) + a_1(4) a_2(3) a_3(3) + a_1(1) a_2(2) a_3(5) + a_1(1) a_2(1) a_3(6)$.

A direct computation shows

$$\phi_3(a_1(2) a_2(3) a_3(3)) = \lambda_2^2 \lambda_3^2 + \lambda_1 \lambda_4 \lambda_3 + \lambda_1 \lambda_3 \lambda_4,$$

$$\phi_3(a_1(1) a_2(2) a_3(3)) = \lambda_1 \lambda_4 \lambda_3 + \lambda_1 \lambda_3 \lambda_4 + \lambda_1 \lambda_2 \lambda_5,$$

$$\phi_3(a_1(1) a_2(1) a_3(5)) = \lambda_1 \lambda_2 \lambda_5 + \lambda_1^2 \lambda_6,$$

$$\phi_3(a_1(1) a_2(2) a_3(6)) = \lambda_1^2 \lambda_6.$$

Hence, we obtain $\phi_3(\tilde{c}_0) = \lambda_2^2 \lambda_3^2$. By Theorem 2.2.2 we have $Tr_3(\langle \tilde{c}_0 \rangle) = [\lambda_2^2 \lambda_3^2] = c_0$. Since $[\tilde{c}_u] = (\tilde{S}q^0)^u(\langle \tilde{c}_0 \rangle)$, we get

$$Tr_3(\langle \tilde{c}_u \rangle) = Tr_3((\tilde{S}q^0)^u(\langle \tilde{c}_0 \rangle)) = (S\tilde{q}^0)^u Tr_3(\langle \tilde{c}_0 \rangle) = (S\tilde{q}^0)^u(c_0) = c_u.$$
By Theorem 2.1 we have $h_u h_{u+1} = 0$. Hence,

$$\text{Ext}^3 \mathcal{A}_{2^{s+t+u}+2^{t+u}+2^n} (\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} 
\langle h_u h_{u+1} h_{u+3}, c_u \rangle = \langle c_u \rangle, & \text{if } s = 2, \\
\langle h_u h_{u+1} h_{s+u+1} \rangle = 0, & \text{if } s \neq 2.
\end{cases}$$

Theorem 3.2.4 in this case follows from the above equalities.

3.2.4. The case of the generic degree.

In this subsection, we consider the degree

$$n = 2^{s+t+u} + 2^{t+u} + 2^u - 3,$$

with $s, t, u$ non-negative integers.

The subcases either $s = 0$ or $t = 0$ have been determined in Subsections 3.2.1 and 3.2.2. The case $s > 0$ and $t = 1$ has been determined in Subsection 3.2.3. So, we assume that $s > 0$ and $t > 1$.

The iterated homomorphism

$$(\tilde{S}_q^0)^u : (Q P_3)_{2^{s+t+u}+2^{t+u}+2^u-3} \to (Q P_3)_{2^{s+t}+2^t-2}$$

is an isomorphism of $GL_3$-modules. So, we need only to compute $(Q P_3)^{GL_3}_{2^{s+t}+2^t-2}$.

The subcase $s = 1$. Then $n = 2^{t+1} + 2^t - 2$. According to Kameko [13], $(Q P_3)_{n}$ is the vector space with a basis consisting of all the classes represented by the following monomials:

$$v_{t,1} = x_1^2 x_2 x_3^{t+1} - 1, \quad v_{t,2} = x_1 x_2^{t+1} - x_2^t - 1, \quad v_{t,3} = x_1^{t+1} x_2 x_3 - 1,$$

$$v_{t,4} = x_1^{t+1} x_2 - x_2^t - 1, \quad v_{t,5} = x_1^t x_2^{t+1} x_3 - 1, \quad v_{t,6} = x_1 x_2^{t+1} x_2^t - 1,$$

$$v_{t,7} = x_1^t x_2^2 x_3^{t+1} - 1, \quad v_{t,8} = x_1 x_2^{t+1} x_3^t - 1, \quad v_{t,9} = x_1^{t+1} x_2^t x_3^2 - 1,$$

$$v_{t,10} = x_1 x_2 - x_2^t - 3 x_3^{t+1} - 1, \quad v_{t,11} = x_1^t x_2^{t+1} x_3^{t+2} - x_3^t - 1, \quad v_{t,12} = x_1^{t+1} x_2^t x_3^{t+1} - 1,$$

$$v_{t,13} = x_1^t x_2^{t+1} x_3^{t+2} - x_3^{t+1} - 3 x_3^{t+2} - 1, \quad v_{2,14} = x_1^3 x_2 x_3^3$$

for $t = 2$, and $v_{t,14} = x_1^t x_2^{t+3} x_3^{t+2} - 2$ for $t > 2$.

By a direct computation using the above basis, we obtain

Proposition 3.2.5. For any integers $t > 1, u \geq 0$ and $n = 2^{t+1} + 2^t + 2^u - 3$,

we have $(Q P_3)^{GL_3}_{n} = 0$.

By Theorem 2.1 $h_{t+u} h_{t+u+1} = 0$, so we have

$$\text{Ext}^3 \mathcal{A}_{2^{t+1}+2^t+2^u} (\mathbb{F}_2, \mathbb{F}_2) = \langle h_u h_{t+u} h_{t+u+1} \rangle = 0.$$ 

Hence, from the above equalities, we can see that

$$Tr_3 : \mathbb{F}_2 \otimes GL_3 P((P_3)_{2^{t+1}+2^t+2^u-3}) \to \text{Ext}^3 \mathcal{A}_{2^{t+1}+2^t+2^u} (\mathbb{F}_2, \mathbb{F}_2)$$

is a trivial isomorphism.

Now, suppose that $s, t > 1$ and $n = 2^{s+t} + 2^t - 2$. From the results of Kameko [13], we see that $(Q P_3)_{n}$ is the vector space of dimension 21 with a basis consisting of all the classes represented by the following monomials:
Since

Note that

Theorem 4.1.

result is the following.

In this section, we explicitly determined

\[ v_{s,t,1} = x_2^{2^t-1} x_3^{2^{s+1}-1} \]

\[ v_{s,t,2} = x_2^{2^{s+1}-1} x_3^{2^t-1} \]

\[ v_{s,t,3} = x_2^{2^t-1} x_3^{2^{s+1}-1} \]

\[ v_{s,t,4} = x_2^{2^{s+1}-1} x_3^{2^t-1} \]

\[ v_{s,t,5} = x_2^{2^t+1} x_3^{2^t-1} \]

\[ v_{s,t,6} = x_2^{2^{s+1}-1} x_3^{2^t+1-2} \]

\[ v_{s,t,7} = x_2^{2^t+1-2} x_3^{2^{s+1}-2} \]

\[ v_{s,t,8} = x_2^{2^{s+1}-2} x_3^{2^t+1-2} \]

\[ v_{s,t,9} = x_2^{2^t+1-2} x_3^{2^{s+1}-2} \]

\[ v_{s,t,10} = x_2^{2^{s+1}-2} x_3^{2^t+1-2} \]

\[ v_{s,t,11} = x_2^{2^{s+1}-2} x_3^{2^t+1-2} \]

\[ v_{s,t,12} = x_2^{2^t+1-2} x_3^{2^{s+1}-2} \]

\[ v_{s,t,13} = x_2^{2^{s+1}-2} x_3^{2^t+1-2} \]

\[ v_{s,t,14} = x_2^{2^t+1-2} x_3^{2^{s+1}-2} \]

\[ v_{s,t,15} = x_2^{2^t+1-2} x_3^{2^{s+1}-2} \]

\[ v_{s,t,16} = x_2^{2^t+1-2} x_3^{2^{s+1}-2} \]

\[ v_{s,t,17} = x_2^{2^t+1-2} x_3^{2^{s+1}-2} \]

\[ v_{s,t,18} = x_2^{2^t+1-2} x_3^{2^{s+1}-2} \]

\[ v_{s,t,19} = x_2^{2^t+1-2} x_3^{2^{s+1}-2} \]

\[ v_{s,t,20} = x_2^{2^t+1-2} x_3^{2^{s+1}-2} \]

\[ v_{s,2,21} = x_1^3 x_2^3 x_3^{2^{s+2}-4} \]

for \( t = 2 \) and \( v_{s,t,21} = x_1^3 x_2^3 x_3^{2^{s+2}-4} \) for \( t > 2 \).

We set

\[ p_{3,s,t,u} = \left\{ \begin{array}{ll}
\sum_{1 \leq j \leq 21, j \neq 15} \psi^u(v_{s,2,j}), & \text{if } t = 2, \\
\sum_{1 \leq j \leq 21} \psi^u(v_{s,t,j}), & \text{if } t > 2.
\end{array} \right. \]

By a direct computation using this basis, we get

**Proposition 3.2.6.** For any integers \( s, t > 1, u \geq 0 \) and \( n = 2^{s+t+u}+2^{t+u}+2^u - 3 \), we have \( (QP_3)_n^{GL_3} = \langle [p_{3,s,t,u}] \rangle \).

By Theorem 2.1 we have

\[ \Ext^3_{A^*} \mathbb{Z}_{x_1^{2^{s+t+u}+2^{t+u}+2^u}}(\mathbb{F}_2, \mathbb{F}_2) = \langle h_u h_{t+u} h_{s+t+u} \rangle. \]

Note that \( \psi^u(v_{s,t,1}) = x_1^{2^t-1} x_2^{2^{s+1}-1} x_3^{2^t+1-2} \). Consider the element

\[ q_{3,s,t,u} = a_1^{(2^u-1)} a_2^{2^{t+u}-1} a_3^{(2^{s+1}-1)} \in \mathbb{F}_2 \otimes_{GL_3} P((P_3)_n^u). \]

Since \( \langle p_{3,s,t,u}, q_{3,s,t,u} \rangle = 1 \), from Proposition 3.2.6 we obtain

\[ \mathbb{F}_2 \otimes_{GL_3} P((P_3)_n^u) = \langle [q_{3,s,t,u}] \rangle. \]

It is easy to see that \( \phi_3(q_{3,s,t,u}) = \lambda_u \lambda_{t+u} \lambda_{s+t+u} \), hence using Theorem 2.2 we get

\[ T_{31}(\langle q_{3,s,t,u} \rangle) = [\lambda_u \lambda_{t+u} \lambda_{s+t+u}] = h_u h_{t+u} h_{s+t+u}. \]

Theorem 3.2.1 is completely proved.

### 4. Determination of \( T_{r4} \) in Some Internal Degrees

In this section, we explicitly determined \( T_{r4} \) in some internal degrees. Our main result is the following.

**Theorem 4.1.** Let \( s \) be a positive integer and let \( n \) be one of the degrees \( 2^{s+1}-1, 2^{s+1}-2, 2^{s+1}-3 \). If \( n \neq 61 \) and \( n \neq 126 \), then the homomorphism

\[ T_{r4} : \mathbb{F}_2 \otimes_{GL_4} P((P_4)_n^u) \rightarrow \Ext^4_{A^*}(\mathbb{F}_2, \mathbb{F}_2) \]

is an isomorphism. If either \( n = 61 \) or \( n = 126 \), then \( T_{r4} \) is a monomorphism but it is not an epimorphism.

We prove the theorem by computing the space \( (QP_4)_n^{GL_4} \).
4.1. The case $n = 2^{s+1} - 3$.

**Proposition 4.1.1** (see [20, 23]). Let $n = 2^{s+1} - 3$ with $s$ a positive integer. Then, the dimension of the $\mathbb{F}_2$-vector space $(QP_1)_n$ is determined by the following table:

| $n$ | $s = 1$ | $s = 2$ | $s = 3$ | $s \geq 4$ |
|-----|---------|---------|---------|-----------|
| dim$(QP_1)_n$ | 4 | 15 | 35 | 45 |

A basis for $(QP_1)_n$ is the set consisting of all the classes represented monomials $a_j = a_{s,j}$ which are determined as follows:

For $s = 1$, $a_{1,1} = x_4$, $a_{1,2} = x_3$, $a_{1,3} = x_2$, $a_{1,4} = x_1$.

For $s = 2$,

- $a_{2,1} = x_1 x_2 x_3 x_4^2$, $a_{2,14} = x_1 x_2 x_3 x_4$, $a_{2,15} = x_1 x_2 x_3 x_4$.

For $s \geq 3$,

- $a_{s,13} = x_1 x_2 x_3 x_4^2$, $a_{s,14} = x_1 x_2 x_3 x_4$.

For $s = 3$,

- $a_{3,31} = x_1^3 x_2 x_3 x_4^2$, $a_{3,32} = x_1^3 x_2 x_3 x_4$, $a_{3,33} = x_1^3 x_2 x_3 x_4$.

For $s \geq 4$,

- $a_{4,31} = x_1^4 x_2 x_3 x_4$, $a_{4,32} = x_1^4 x_2 x_3 x_4$.

For $s = 4$,

- $a_{4,44} = x_1^6 x_2 x_3 x_4$, $a_{4,45} = x_1^7 x_2 x_3 x_4$.

For $s \geq 5$,

- $a_{5,44} = x_1^7 x_2 x_3 x_4$.
Proposition 4.1.2. Let $s$ be a positive integer. Then, $\left( QP_4^{GL_4} \right)_{2^{s+1} - 3} = 0$.

For simplicity, we prove the proposition in detail for $s \geq 5$. The other cases can be proved by the similar computations.

For any monomials $z_1, z_2, \ldots, z_m$ in $P_k$ and for a subgroup $G \subset GL_k$, we denote $G(z_1, z_2, \ldots, z_m)$ the $G$-submodule of $QP_k$ generated by the set $\{ [z_i] : 1 \leq i \leq m \}$. We have the following.

Lemma 4.1.3. i) For any $s \geq 5$, there is a direct summand decomposition of the $\Sigma_4$-modules:

\[
\left( QP_4 \right)_{2^{s+1} - 3} = \Sigma_4(a_{s,1}) \oplus \Sigma_4(a_{s,13}) \oplus \Sigma_4(a_{s,31}) \oplus \Sigma_4(a_{s,25}, a_{s,35}, a_{s,43}).
\]

ii) $\Sigma_4(a_{s,1})^{\Sigma_4} = \langle [p_{4,s,1}] \rangle$, with $p_{4,s,1} = \sum_{j=1}^{12} a_{s,j}$.

iii) $\Sigma_4(a_{s,13})^{\Sigma_4} = \langle [p_{4,s,2}] \rangle$, with $p_{4,s,2} = \sum_{j=1}^{24} a_{s,j}$.

iv) $\Sigma_4(a_{s,31})^{\Sigma_4} = \langle [p_{4,s,3}] \rangle$, with $p_{4,s,3} = \sum_{j=31}^{34} a_{s,j}$.

v) $\Sigma_4(a_{s,25}, a_{s,35}, a_{s,43})^{\Sigma_4} = \langle [p_{4,s,4}] \rangle$, with

\[
p_{4,s,4} = \sum_{j=25}^{30} a_{s,j} + \sum_{j=39}^{43} a_{s,j} + a_{s,45}.
\]

Proof. We obtain Part i) by a simple computation using Proposition 4.1.1. We prove Part v) in detail. The others can be proved by the similar computations. By a simple computation we see that the set $\{ [a_{s,j}] : j = 25, \ldots, 30, 35, \ldots, 45 \}$ is a basis for $\Sigma_4(a_{s,25}, a_{s,35}, a_{s,43})$. Suppose $[f] \in \Sigma_4(a_{s,25}, a_{s,35}, a_{s,43})^{\Sigma_4}$, then

\[
f \equiv \sum_{j=25}^{30} \gamma_j a_{s,j} + \sum_{j=35}^{45} \gamma_j a_{s,j}
\]

with $\gamma_j \in \mathbb{F}_2$. By a direct computation, we get

\[
\rho_1(f) + f \equiv (\gamma_25 + \gamma_{28})(a_{s,25} + a_{s,28}) + (\gamma_{26} + \gamma_{29})(a_{s,26} + a_{s,29})
+ (\gamma_{27} + \gamma_{41})a_{s,35} + (\gamma_{27} + \gamma_{40})a_{s,36} + (\gamma_{37} + \gamma_{38})a_{s,37} + a_{s,38}
+ (\gamma_{39} + \gamma_{42})a_{s,39} + a_{s,42} + (\gamma_{41} + \gamma_{43})a_{s,44} + (\gamma_{40} + \gamma_{43})a_{s,45} \equiv 0,
\]

\[
\rho_2(f) + f \equiv (\gamma_{26} + \gamma_{27})(a_{s,26} + a_{s,27}) + (\gamma_{28} + \gamma_{30})(a_{s,28} + a_{s,30})
+ (\gamma_{35} + \gamma_{37})(a_{s,35} + a_{s,37}) + (\gamma_{29} + \gamma_{36} + \gamma_{40})(a_{s,36} + a_{s,40})
+ (\gamma_{39} + \gamma_{41})(a_{s,39} + a_{s,41}) + (\gamma_{42} + \gamma_{43} + \gamma_{44})(a_{s,43} + a_{s,44})
+ (\gamma_{29} + \gamma_{42})(a_{s,38} + a_{s,45}) \equiv 0,
\]

\[
\rho_3(f) + f \equiv (\gamma_{25} + \gamma_{26})(a_{s,25} + a_{s,26}) + (\gamma_{28} + \gamma_{29})(a_{s,28} + a_{s,29})
+ (\gamma_{35} + \gamma_{36})(a_{s,35} + a_{s,36}) + (\gamma_{30} + \gamma_{37} + \gamma_{39})(a_{s,37} + a_{s,39})
+ (\gamma_{30} + \gamma_{38} + \gamma_{42})(a_{s,38} + a_{s,42}) + (\gamma_{40} + \gamma_{41})(a_{s,40} + a_{s,41})
+ (\gamma_{30} + \gamma_{44} + \gamma_{45})(a_{s,44} + a_{s,45}) \equiv 0.
\]

The above equalities imply $\gamma_j = 0$ for $j = 35, 36, 37, 38, 44$ and $\gamma_j = \gamma_{25}$ for $j \neq 35, 36, 37, 38, 44$. The lemma is proved.

Proof of Proposition 4.1.2. Let $f \in P_4$ such that $[f] \in \left( QP_4^{GL_4} \right)_{2^{s+1} - 3}$. Since $\Sigma_4 \subset GL_4$, we have $[f] \in \left( QP_4^{GL_4} \right)_{2^{s+1} - 3}$. Then, $f = \sum_{j=1}^{4} \gamma_j p_{4,s,j}$ with $\gamma_j \in \mathbb{F}_2$. By a
From Theorem 2.4.1, we see that $\Ext^1_\mathbb{F}_2(F_2, \mathbb{F}_2) = 0$ for any $s \neq 5$ and $\Ext^4_\mathbb{F}_2(F_2, \mathbb{F}_2) = (D_3(0))$. Hence, Theorem 4.1 holds for $n = 2s+1 - 3$.

4.2. The case $n = 2s+1 - 2$.

Since Kameko’s homomorphism in the degree $2s+1 - 2$,

$$(\tilde{S}_q^0)_{2s+1-2} : (QP_4)_{2s+1-2} \to (QP_4)_{2s-3}$$

is an epimorphism of $GL_4$-modules, using Proposition 4.1.2, we have

$$(QP_4)^{GL_4}_{2s+1-2} \subset (\text{Ker}(\tilde{S}_q^0)_{2s+1-2})^{GL_4}.$$ 

From [20][23], we have the following.

**Proposition 4.2.1 (see [20][23]).** Let $s$ be a positive integer. Then,

$$\dim (\text{Ker}(\tilde{S}_q^0)_{2s+1-2}) = \begin{cases} 6, & \text{if } s = 1, \\ 20, & \text{if } s = 2, \\ 35 & \text{if } s \geq 3. \end{cases}$$

A basis for $(\text{Ker}(\tilde{S}_q^0)_{2s+1-2})$ is the set consisting of all the classes represented monomials $b_j = b_{s,j}$ which are determined as follows:

For $s \geq 1$,

$$b_{s,1} = x_1^{2s-1}x_4^{2s-1}, \quad b_{s,2} = x_3^{2s-1}x_4^{2s-1}, \quad b_{s,3} = x_1^{2s-1}x_3^{2s-1}, \quad b_{s,4} = x_1^{2s-1}x_4^{2s-1}, \quad b_{s,5} = x_1^{2s-1}x_3^{2s-1}, \quad b_{s,6} = x_1^{2s-1}x_2^{2s-1},$$

For $s \geq 2$,

$$b_{s,7} = x_1x_2x_3^{2s-2}x_4^{2s-1}, \quad b_{s,8} = x_2x_3^{2s-1}x_4^{2s-2}, \quad b_{s,9} = x_2^{2s-1}x_3x_4^{2s-2}, \quad b_{s,10} = x_1x_2x_3^{2s-2}x_4^{2s-1}, \quad b_{s,11} = x_1x_3x_4^{2s-2}x_4^{2s-1}, \quad b_{s,12} = x_1x_2x_4^{2s-2}x_4^{2s-1}, \quad b_{s,13} = x_1x_2x_2^{-1}x_3^{2s-2}x_4^{2s-2}, \quad b_{s,14} = x_1x_2^{2s-2}x_3^{2s-2}x_4^{2s-2}, \quad b_{s,15} = x_1x_2x_3^{2s-2}x_4^{2s-2}, \quad b_{s,16} = x_1^{2s-1}x_3x_4^{2s-2}, \quad b_{s,17} = x_1^{2s-1}x_2x_4^{2s-2}, \quad b_{s,18} = x_1^{2s-1}x_2x_4^{2s-2}.$$ 

For $s = 2$, $b_{2,19} = x_1x_2x_3^2x_4^2$, $b_{2,20} = x_1x_3^2x_4^3$.

For $s \geq 3$,

$$b_{s,19} = x_3^{2s-3}x_4^{2s-2}, \quad b_{s,20} = x_1^{2s-3}x_4^{2s-2}, \quad b_{s,21} = x_3^{2s-3}x_4^{2s-2}, \quad b_{s,22} = x_1^{2s-3}x_4^{2s-2}, \quad b_{s,23} = x_1x_2x_3^{2s-3}x_4^{2s-2}, \quad b_{s,24} = x_1x_2x_3^{3s-3}x_4^{2s-2}, \quad b_{s,25} = x_1x_2x_4^{2s-3}x_4^{2s-2}, \quad b_{s,26} = x_1x_3^{2s-3}x_4^{2s-2}, \quad b_{s,27} = x_1x_2x_3^{3s-3}x_4^{2s-2}, \quad b_{s,28} = x_1x_2^{2s-3}x_4^{2s-2}, \quad b_{s,29} = x_1x_2x_3^{3s-3}x_4^{2s-2}, \quad b_{s,30} = x_1x_2x_4^{2s-3}x_4^{2s-2}, \quad b_{s,31} = x_1x_2x_3^{2s-4}x_4^{2s-2}, \quad b_{s,32} = x_1x_2x_4^{2s-4}x_4^{2s-2}, \quad b_{s,33} = x_1x_2x_3^{3s-4}x_4^{2s-2}, \quad b_{s,34} = x_1x_2x_4^{3s-4}x_4^{2s-2}.$$ 

For $s = 3$, $b_{3,35} = x_1^{3s-3}x_4^4$, and for $s \geq 4$, $b_{s,35} = x_1^{3s-3}x_4^4$.
We set
\[ p_{4,s} = \begin{cases} 
1 \times x_{1}x_{2}x_{3}x_{4} + x_{1}^{2}x_{2}x_{3}x_{4} & \text{if } s = 3, \\
\sum_{j=1}^{35} b_{s,j} & \text{if } s \geq 4.
\end{cases} \]
By a direct computation using Proposition 4.2.1, one gets the following.

**Proposition 4.2.2.** Let \( s \) be a positive integer. Then,
\[ (\text{Ker}(\tilde{S}_{q_{s}})_{2^{s+1}-2})^{GL_{4}} = \begin{cases} 
0 & \text{if } s \leq 2, \\
\langle [p_{4,s}] \rangle & \text{if } s \geq 3.
\end{cases} \]

For simplicity, we will prove this proposition in detail for \( s \geq 4 \). The others can be proved by the similar computations. We have the following.

**Lemma 4.2.3.** i) For any \( s \geq 4 \), there is a direct summand decomposition of the \( \Sigma_{4} \)-modules:
\[ (\text{Ker}(\tilde{S}_{q_{s}})_{2^{s+1}-2})^{GL_{4}} = \Sigma_{4}(b_{s,1}) \oplus \Sigma_{4}(b_{s,7}) \oplus \Sigma_{4}(b_{s,19}) \oplus \Sigma_{4}(b_{s,23}) \oplus \Sigma_{4}(b_{s,29}, b_{s,30}). \]
ii) \( \Sigma_{4}(b_{s,1})^{\Sigma_{4}} = \langle [p_{4,s,1}] \rangle \), with \( p_{4,s,1} = \sum_{j=1}^{6} b_{s,j} \).
iii) \( \Sigma_{4}(b_{s,7})^{\Sigma_{4}} = \langle [p_{4,s,2}] \rangle \), with \( p_{4,s,2} = \sum_{j=7}^{18} b_{s,j} \).
iv) \( \Sigma_{4}(b_{s,19})^{\Sigma_{4}} = \langle [p_{4,s,3}] \rangle \), with \( p_{4,s,3} = \sum_{j=19}^{22} b_{s,j} \).
v) \( \Sigma_{4}(b_{s,23})^{\Sigma_{4}} = \langle [p_{4,s,4}] \rangle \), with \( p_{4,s,4} = \sum_{j=23}^{26} b_{s,j} \).
vii) \( \Sigma_{4}(b_{s,29}, a_{s,30})^{\Sigma_{4}} = \langle [p_{4,s,5}] \rangle \), where
\[ \tilde{p}_{4,s,5} = \sum_{j=27}^{29} b_{s,j}, \quad \bar{p}_{4,s,6} = \sum_{j=30}^{35} b_{s,j}. \]

**Proof.** From Proposition 4.2.1, we easily obtain Part i). Now, we prove Part vi) in detail. The others can be proved by the similar computations. By a direct computation we see that the set \( \{ [b_{s,j}] : j = 27 \leq j \leq 35 \} \) is a basis for \( \Sigma_{4}(b_{s,29}, b_{s,30}) \).
Suppose \( [f] \in \Sigma_{4}(b_{s,29}, b_{s,30})^{\Sigma_{4}} \), then \( f = \sum_{j=27}^{35} \gamma_{j} b_{s,j} \) with \( \gamma_{j} \in \mathbb{F}_{2} \). By a direct computation, we obtain
\[ \rho_{1}(f) + f = (\gamma_{28} + \gamma_{29} + \gamma_{30} + \gamma_{35}) b_{s,27} + (\gamma_{31} + \gamma_{33}) (b_{s,31} + b_{s,33}) + (\gamma_{32} + \gamma_{34}) (b_{s,32} + b_{s,34}) \equiv 0, \]
\[ \rho_{2}(f) + f = (\gamma_{27} + \gamma_{28} + \gamma_{32} + \gamma_{33}) (b_{s,27} + b_{s,28}) + (\gamma_{30} + \gamma_{31}) (b_{s,30} + b_{s,31}) + (\gamma_{34} + \gamma_{35}) (b_{s,34} + b_{s,35}) \equiv 0, \]
\[ \rho_{3}(f) + f = (\gamma_{28} + \gamma_{29} + \gamma_{30} + \gamma_{35}) b_{s,27} + (\gamma_{31} + \gamma_{32}) (b_{s,31} + b_{s,32}) + (\gamma_{33} + \gamma_{34}) (b_{s,33} + b_{s,34}) \equiv 0. \]

The above equalities imply \( \gamma_{j} = \gamma_{27} \) for \( j = 27, 28, 29 \) and \( \gamma_{j} = \gamma_{30} \) for \( 30 \leq j \leq 35 \). The lemma is proved. \( \square \)

**Remark 4.2.4.** For \( s = 3 \), Parts i) to v) of Lemma 4.2.3 hold. We replace Part vi) with \( \Sigma_{4}(b_{3,29}, a_{3,30})^{\Sigma_{4}} = \langle [p_{4,3}] \rangle \).

**Proof of Proposition 4.2.2.** Let \( f \in P_{4} \) such that \( [f] \in \text{Ker}(\tilde{S}_{q_{s}})_{2^{s+1}-2}^{GL_{4}} \). Then, \( [f] \in \text{Ker}(\tilde{S}_{q_{s}})_{2^{s+1}-2}^{\Sigma_{4}} \). Hence, \( f = \sum_{j=1}^{6} \gamma_{j} p_{4,s,j} \) with \( \gamma_{j} \in \mathbb{F}_{2} \). By a direct computation,
we have
\[
ρ_4(f) + f = (γ_1 + γ_2)(b_{s,2} + b_{s,3}) + (γ_2 + γ_4)(b_{s,7} + b_{s,8}) + (γ_2 + γ_5)b_{s,9} \\
+ (γ_2 + γ_3)(b_{s,14} + b_{s,15}) + (γ_3 + γ_6)b_{s,19} + (γ_4 + γ_6)b_{s,25} \\
+ (γ_2 + γ_3 + γ_4 + γ_5)b_{s,27} + (γ_5 + γ_6)(b_{s,31} + b_{s,32}) = 0.
\]

The last equality implies \(γ_j = γ_1\) for \(1 \leq j \leq 6\). The proposition follows. \(\square\)

From Theorem \[2.1\] we have
\[
\text{Ext}_A^{4,2s+1-2}(F_2,F_2) = \begin{cases}
0, & \text{if } s \leq 2, \\
\langle d_0 \rangle, & \text{if } s = 3, \\
\langle h_0^2h_3^2, D_3(1) \rangle & \text{if } s = 6, \\
\langle h_0^2h_2^2 \rangle & \text{if } s \geq 4, s \neq 6.
\end{cases}
\]

Denote \(q_{4,s} \in P((P_4)^{2s+1-2})\) by setting
\[
q_{4,3} = a_{1}^{(1)}a_{2}^{(1)}a_{3}^{(6)}a_{4}^{(6)} + a_{1}^{(1)}a_{2}^{(2)}a_{3}^{(5)}a_{4}^{(6)} + a_{1}^{(1)}a_{2}^{(3)}a_{3}^{(4)}a_{4}^{(6)} + a_{1}^{(1)}a_{2}^{(4)}a_{3}^{(3)}a_{4}^{(6)} \\
+ a_{1}^{(1)}a_{2}^{(5)}a_{3}^{(2)}a_{4}^{(6)} + a_{1}^{(1)}a_{2}^{(6)}a_{3}^{(1)}a_{4}^{(6)} + a_{1}^{(2)}a_{2}^{(1)}a_{3}^{(6)}a_{4}^{(5)} + a_{1}^{(2)}a_{2}^{(2)}a_{3}^{(5)}a_{4}^{(5)} \\
+ a_{1}^{(2)}a_{2}^{(3)}a_{3}^{(4)}a_{4}^{(5)} + a_{1}^{(2)}a_{2}^{(4)}a_{3}^{(3)}a_{4}^{(5)} + a_{1}^{(2)}a_{2}^{(5)}a_{3}^{(2)}a_{4}^{(5)} + a_{1}^{(2)}a_{2}^{(6)}a_{3}^{(1)}a_{4}^{(5)} \\
+ a_{1}^{(3)}a_{2}^{(1)}a_{3}^{(5)}a_{4}^{(5)} + a_{1}^{(3)}a_{2}^{(2)}a_{3}^{(4)}a_{4}^{(5)} + a_{1}^{(3)}a_{2}^{(3)}a_{3}^{(3)}a_{4}^{(5)} + a_{1}^{(3)}a_{2}^{(4)}a_{3}^{(2)}a_{4}^{(5)} + a_{1}^{(3)}a_{2}^{(5)}a_{3}^{(1)}a_{4}^{(5)} \\
+ a_{1}^{(4)}a_{2}^{(5)}a_{3}^{(3)}a_{4}^{(3)} + a_{1}^{(4)}a_{2}^{(6)}a_{3}^{(2)}a_{4}^{(3)} + a_{1}^{(5)}a_{2}^{(1)}a_{3}^{(6)}a_{4}^{(2)} + a_{1}^{(5)}a_{2}^{(2)}a_{3}^{(5)}a_{4}^{(2)} \\
+ a_{1}^{(5)}a_{2}^{(3)}a_{3}^{(4)}a_{4}^{(2)} + a_{1}^{(5)}a_{2}^{(4)}a_{3}^{(3)}a_{4}^{(2)} + a_{1}^{(5)}a_{2}^{(5)}a_{3}^{(2)}a_{4}^{(2)} + a_{1}^{(5)}a_{2}^{(6)}a_{3}^{(1)}a_{4}^{(2)} \\
+ a_{1}^{(6)}a_{2}^{(1)}a_{3}^{(5)}a_{4}^{(2)} + a_{1}^{(6)}a_{2}^{(2)}a_{3}^{(4)}a_{4}^{(2)} + a_{1}^{(6)}a_{2}^{(3)}a_{3}^{(3)}a_{4}^{(2)} + a_{1}^{(6)}a_{2}^{(4)}a_{3}^{(2)}a_{4}^{(2)} + a_{1}^{(6)}a_{2}^{(5)}a_{3}^{(1)}a_{4}^{(2)},
\]
and \(q_{4,s} = a_{1}^{(0)}a_{2}^{(0)}a_{3}^{(2s-1)}a_{4}^{(2s-1)}, \) for \(s \geq 4\). Since \(\langle [p_{4,s}],[q_{4,s}] \rangle = 1\) for any \(s \geq 3\), we get
\[
F_2 \otimes GL_4 P((P_4)^{2s+1-2}) = \begin{cases}
0, & \text{if } s \leq 2, \\
\langle [q_{4,s}] \rangle, & \text{if } s \geq 3.
\end{cases}
\]

By a direct computation, we obtain
\[
φ_4(q_{4,s}) = \begin{cases}
\tilde{d}_0 + δ(λ_1λ_9λ_2^2 + λ_1λ_3λ_9λ_4), & \text{if } s = 3, \\
λ_0^2λ_2^{2s-1}, & \text{if } s > 3.
\end{cases}
\]

From the above equalities and Theorem \[2.2\] we get
\[
Tr_4([q_{4,s}]) = [φ_3(q_{4,s})] = \begin{cases}
[\tilde{d}_0] = d_0, & \text{if } s = 3, \\
[λ_0^2λ_2^{2s-1}] = h_0^2h_2^2, & \text{if } s > 3.
\end{cases}
\]

Theorem \[4.3\] holds for \(n = 2s+1 - 2\).
4.3. The case $n = 2^{s+1} - 1$.

First, we recall the following.

**Proposition 4.3.1** (see [20, 23]). Let $n = 2^{s+1} - 1$ with $s$ a positive integer. Then, the dimension of the $\mathbb{F}_2$-vector space $(QP_4)_n$ is determined by the following table:

| $n = 2^s - 1$ | $s = 1$ | $s = 2$ | $s = 3$ | $s = 4$ | $s \geq 5$ |
|---------------|---------|---------|---------|---------|-----------|
| $\dim(QP_4)_n$ | 14      | 35      | 75      | 89      | 85        |

A basis of $(QP_4)_n$ has been given in [23]. For $s \geq k - 2$, we set

$$\eta_{k,s} = \sum_{m=1}^{k-1} \sum_{1 \leq i_1 < \cdots < i_m \leq k} x_{i_1} x_{i_2}^2 \cdots x_{i_m}^{2m-2} x_{i_m}^{s+1-2m-1} \in (P_k)_{2^{s+1} - 1}.$$

For $k = 4$, we denote

$$\bar{p}_{4,s} = \begin{cases} 
\eta_{4,s} + x_1 x_2^2 x_3^2 x_4^2, & \text{if } s = 2, \\
\eta_{4,s} + x_1 x_2^2 x_3^2 x_4^{2^s+1-8}, & \text{if } s \geq 3.
\end{cases}$$

By a computation similar to the one in Proposition 4.2.1, one gets the following.

**Proposition 4.3.2.** Let $s$ be a positive integer. Then,

$$(QP_4)^{GL_4}_{2^{s+1} - 1} = \begin{cases} 
0, & \text{if } s = 1, \\
\langle \bar{p}_{4,s} \rangle, & \text{if } s \geq 2.
\end{cases}$$

From Theorem 2.1 we have

$$\text{Ext}_A^{1,2^{s+1}+3}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} 
0, & \text{if } s = 1, \\
\langle h_0^3 h_{s+1} \rangle & \text{if } s \geq 2.
\end{cases}$$

Denote $\bar{q}_{4,s} = a_1^{(0)} a_2^{(0)} a_3^{(s+1)-1} \in P((P_4)_{2^{s+1} - 1})$, for $s \geq 2$. It is easy to see that $\langle [\bar{p}_{4,s}], [\bar{q}_{4,s}] \rangle = 1$. Hence, we obtain

$$\mathbb{F}_2 \otimes_{GL_4} P((P_4)_{2^{s+1} - 1}) = \begin{cases} 
0, & \text{if } s = 1, \\
\langle [\bar{q}_{4,s}] \rangle, & \text{if } s \geq 2.
\end{cases}$$

By a simple computation, we have $\phi_4(\bar{q}_{4,s}) = \lambda_0^3 \lambda_2^{s+1}$. Hence, using Theorem 2.2 one gets

$$Tr_4([\bar{q}_{4,s}]) = \frac{\phi_4(\bar{q}_{4,s})}{[\phi_4(\bar{q}_{4,s})]} = \lambda_0^3 \lambda_2^{s+1} = h_0^3 h_{s+1}.$$

Theorem 4.1 is completely proved.

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