Torsional shear oscillations in the neutron star crust
 driven by restoring force of elastic stresses

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ABSTRACT
We present several exact solutions of the eigenfrequency problem for torsional shear vibrations in homogeneous and non-homogeneous models of the neutron star crust governed by canonical equation of solid mechanics with a restoring force of Hookean elasticity. Particular attention is given to regime of large lengthscale nodeless axisymmetric differentially rotational oscillations which are treated in spherical polar coordinates reflecting real geometry of the neutron star crust. Highlighted is the distinction between analytic forms and numerical estimates of the frequency, computed as a function of multipole degree of nodeless torsional oscillations and fractional depth of the crust, caused by different boundary conditions imposed on the toroidal field of material displacements. The relevance of considered models to quasiperiodic oscillations, recently detected during the flare of SGR 1806-20 and SGR 1900+14, is discussed.

Key words: stars: neutron – stars: oscillations – stars.

1 INTRODUCTION
Ever since the identification of pulsars with neutron stars, the non-radial torsional shear oscillations restored by bulk forces of different in physical nature internal stresses have been and still are among the most important issues in the study of the interconnection between the electromagnetic activity and asteroseismology of pulsars (e.g. Ruderman 1969, van Horn 1980; Hansen and Cioffi 1980; Schumaker and Torne 1983; McDermott, van Horn
Motivated by the above interest, we focus here on the mathematical physics of the eigen-frequency problem for the torsional shear oscillations governed by equation of Newtonian solid mechanics. The restoring force is the bulk force of Hookean elastic shear stresses. Emphasis is laid on the boundary conditions which must be imposed on the toroidal field of material displacement at the edges of the seismogenic layer, that is, on the core-crust boundary and the star surface. Working from the homogeneous crust model we show that these boundary conditions substantially affect the asymptotic spectral formulae for the frequency of torsional shear oscillations.

The plan of this paper is as follows. In Sec.2, a brief outline is given of the governing elastodynamical equation for a standard core-crust model of a quaking neutron star with homogeneous crust. The general solution of the Helmholtz equation for toroidal field of material displacements describing the standing-wave regime of torsional oscillations in the spherical polar coordinates related to the real geometry of the neutron star crust is presented. Also, we obtain here the frequency spectrum for the nodeless global torsional elastic mode in the entire volume of the fiducial homogeneous solid star model which was also obtained in our previous investigation but by use of Rayleigh’s energy method. In Sec.3, we derive two exact dispersion equations for the standing-wave regime of torsional shear vibrations corresponding to different boundary conditions at the core-crust interface and the star surface. A detailed analytic derivation of asymptotic spectral formulas is presented followed by a numerical analysis of the obtained frequency spectra. In Sec.4, we compare frequency spectra for the
nodeless torsional shear oscillations computed from homogeneous and non-homogeneous models of the neutron star crust. The obtained results are summarized in Sec.5.

2 GOVERNING EQUATIONS

In two component model of quaking neutron star (Franco, Link & Epstein 2000), its interior is thought of as composed of dense core (in which self-gravity is brought to equilibrium by the degeneracy pressure of baryon, neutron-dominated, matter) covered by highly conducting metal-like crustal matter composed of nuclei (basically of iron, $^{56}$Fe) dispersed in homogeneous Fermi-gas of relativistic electrons. The gravitational stability of the crust is supported by the electron degeneracy pressure $p_e$ related with the shear modulus as $\mu = \kappa(\rho)p_e$ (Blaes, Blandford, Madau, Koonin, 1990) with fiducial value of ratio $\mu/p_e = 10^{-2}$ (Strohmayer et al 1991; Cutler, Ushomirsly, Link 2003). It is presumed that in the approximation of continuous medium, the quake induced elastic deformations in crustal matter can be properly modeled by equation of solid mechanics for solenoidal field of material displacement $u_i$ (e.g. Graff 1991; Lapwood & Usami 1981; Aki & Richards 2003)

$$\rho \ddot{u}_i = \nabla_k \sigma_{ik} \quad \sigma_{ik} = 2\mu u_{ik}$$

$$u_{ik} = \frac{1}{2} (\nabla_i u_k + \nabla_k u_i) \quad u_{kk} = \nabla_k u_k = 0$$

expressing the second law of Newtonian elastodynamics (McDermott, Van Horn & Hansen 1988). The restoring force is provided by elastic shear stresses $\sigma_{ik}$ related to shear strains $u_{ik}$ by Hooke law. Consider homogeneous model of crustal matter with constant density $\rho$ and shear modulus $\mu$ (e.g. Epstein 1988; Bildsten & Cutler 1995; Franco, Link & Epstein 2000). The shear character of torsional oscillations implies that they are not accompanied by fluctuations in the density: $\delta \rho = -\rho u_{kk} = 0$. On substituting of $\sigma_{ik}$ in the equation of elastodynamics one has

$$\ddot{\mathbf{u}} - c_t^2 \nabla^2 \mathbf{u} = 0 \quad c_t^2 = \frac{\mu}{\rho} \quad \nabla \cdot \mathbf{u} = 0.$$  

For harmonic in time fluctuations of material displacements

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{a}(\mathbf{r}) \alpha(t) \quad \alpha(t) = \alpha_0 \exp(i\omega t)$$  

In this connection, it may be appropriate to note that the Local Density Approximation of microscopic theory of metals developed over the past three decades leads to the conclusion that transport coefficients of solid-mechanical elasticity, like bulk modulus and shear modulus $\mu$ are proportional to the pressure of degenerate Fermi-gas of conducting electrons $p_e$ squeezed between ions (Maruzzi, Janak & Williams 1978). The attitude that crustal matter possesses metal-like properties lends support to the parametrization for shear modulus as $\mu = k(\rho)p_e$. 

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substitution (4) in (3) leads to the vector Helmholtz equation for the standing shear wave
\[ \nabla^2 \mathbf{u} + k^2 \mathbf{u} = 0 \quad \nabla \cdot \mathbf{u} = 0 \quad k^2 = \frac{\omega^2}{c^2}. \]  
(5)

In what follows all calculations are carried out in the spherical coordinates with fixed polar axis \( z \). In this frame of reference the general solution of (5) describing axisymmetric differentially rotational oscillations of matter in the star is given by the toroidal vector field (e.g. Graff 1991; Lapwood & Usami 1981; Aki & Richards 2003)

\[ \mathbf{u}(\mathbf{r}, t) = \nabla \times \mathbf{r} \mathbf{U}(\mathbf{r}, t) = \nabla \mathbf{U}(\mathbf{r}, t) = f_\ell(kr) P_\ell^1(\cos \theta) \exp(i\omega t) \]  
(6)

By \( j_\ell(kr) \) and \( n_\ell(kr) \) are denoted the spherical Bessel and Neumann functions, respectively, and by \( P_\ell(\cos \theta) \) the Legendre polynomial of multipole degree \( \ell \) (Abramowitz & Stegun 1964). Function \( f_\ell \) obey the following recurrence relations

\[ \frac{df_\ell}{dz} = f_{\ell-1} - \frac{\ell + 1}{z} f_\ell \quad f_{\ell-1} + f_{\ell+1} = \frac{2\ell + 1}{z} f_\ell \]  
(10)

\[ \ell f_{\ell-1} - (\ell + 1)f_{\ell+1} = (2\ell + 1) \frac{df_\ell}{dz} \quad z = kR \]  
(11)

which hold for both \( j_\ell(kr) \) and \( n_\ell(kr) \). In the long wavelength limit these functions are approximated by

\[ j_\ell(z) \approx \frac{z^\ell}{(2\ell + 1)!!} \quad j_{\ell+1}(z) = \frac{z}{2\ell + 3} j_\ell(z) \]  
(12)

\[ n_\ell(z) \approx -\frac{(2\ell - 1)!!}{z^{\ell+1}} \quad n_{\ell+1}(z) = \frac{2\ell + 1}{z} n_\ell(z). \]  
(13)

These asymptotic formulae provide a basis for obtaining analytic estimates of the frequency spectra of nodeless torsion oscillations.

As a first step, we consider global torsional oscillations in the entire volume of homogeneous neutron star model. For our present purpose this problem is interesting in that the spectral formula for the frequency of global torsional oscillations is used as the reference equation in testing of spectral equations for the frequency of torsional oscillations trapped in the crust. These latter are derived in the form showing that fiducial spectral formula for the global torsional oscillations is recovered when the core radius tends to zero.

In the case of global torsional oscillations in entire volume the singular in origin solution of the Helmholtz equation for the displacement field must be excluded by putting \( B_\ell = 0 \).
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Figure 1. The artist view of nodeless toroidal field of material displacements in the neutron star undergoing global torsional oscillations in quadrupole ($\ell = 2$) and octupole ($\ell = 3$) overtones. In quadrupole overtone (left), the field of material displacements in north and south hemispheres of neutron star undergoes out-of-phase oscillations, as pictured by arrows. In the octupole overtone (right), the material displacements in north and south are in one and the same phase, whereas in equatorial part the direction of displacements is opposite.

Then, one has

$$u_r = 0, \quad u_\theta = 0, \quad u_\phi = A_\ell j_\ell(kr) P^1_\ell(\zeta) \exp(i\omega t).$$

(14)

The standard boundary condition of stress free surface $n_k \sigma_{ik}|_{r=R} = 0$, where $n_k$ are components of the unit vector normal to the star surface leads to

$$n_r \sigma_{r\phi}|_{r=R} = \mu \left[ \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right]_{r=R} = 0 \rightarrow \left[ j_{\ell+1}(z) - \frac{j_\ell(z)}{z}(\ell - 1) \right] = 0 \quad z = kR.$$

(15)

To integrate this transcendent equation one must have recourse to numerical methods. In what follows we focus, however, on the long wavelength limit of the last equation, when $z << 1$. This is the regime of the nodeless torsional oscillations with the field of displacements $u_i$ having no nodes in the interval $0 < r < R$; in this regime of torsional oscillations the components of $u_i$ are given by $[u_r = 0, u_\theta = 0, u_\phi = A_\ell r^\ell P^1_\ell(\zeta) \exp(i\omega t)]$. From this the term nodeless torsional vibrations is derived. The character of shear distortions in this quadrupole and octupole overtones of nodeless torsional oscillations is pictured pictured in Fig.1 (see, also, Bastrukov et al 2002).

In the long wavelength limit, $z << 1$, the dispersion equation (15) is reduced, with help of approximate formulae (12) and (13), to simple algebraic relation

$$z^2 - (2\ell + 3)(\ell - 1) = 0 \quad z^2 = k^2 R^2 = \frac{\omega^2}{c_\ell^2} R^2.$$

(16)

It follows that the frequency spectrum of global long wavelength torsional modes in a ho-
Figure 2. Fractional frequencies $\nu/\nu_0$, $\nu'/\nu_0$ and $\nu''/\nu_0$ of global nodeless torsional oscillations as functions of multipole degrees $1 < \ell < 20$.

The following equivalent representations of frequency spectrum (18) may be useful. First, given in the form

$$\frac{\nu^2(\ell)}{\nu_0^2} = 2(\ell + 2)(\ell - 1) \left[1 - \frac{1}{2(\ell + 2)}\right]$$

is interesting in that at large values of multipole degree, $\ell >>> 1$, can be replaced by

$$\frac{\nu(\ell)}{\nu_0} \approx \frac{\nu'}{\nu_0} = [2(\ell + 2)(\ell - 1)]^{1/2} \quad \ell >>> 1.$$  

Second, given in the form

$$\frac{\nu^2(\ell)}{\nu_0^2} = 2\ell(\ell + 1) \left[1 - \frac{1}{\ell(\ell + 1)}\right] \left[1 - \frac{1}{2(\ell + 2)}\right]$$

is represented in the limit of large $\ell$ as

$$\frac{\nu(\ell)}{\nu_0} \approx \frac{\nu''}{\nu_0} = [2\ell(\ell + 1)]^{1/2} \quad \ell >>> 1.$$  

The comparison of $\nu/\nu_0$, $\nu''/\nu_0$ and $\nu''/\nu_0$ is shown in Fig.2; it is implied that $\nu(\ell) = \nu$. 

It is worth noting that the spectral formula (17) can be derived from different mathematical footing, namely, by use of the Rayleigh’s energy variational method which is particularly efficient when studying of non-radial nodeless oscillations of neutron stars (Bastrukov et al 1999, 2002, 2007).
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The spectral formulae like above derived are central to theoretical modal analysis of variability in electromagnetic emission of pulsars and magnetars. The works of Van Horn (1980) and Cioffi and Hansen (1980) were among the firsts suggested a possible association of microspikes of 10-50 milliseconds duration clearly discernable in the windows of main pulse train of radio pulsars with torsional oscillations of neutron stars (McDermott, Van Horn, Hansen 1988; Strohmayer 1991; Bastrukov et al 1999, 2007). This suggestion provides a guideline in the above mentioned current studies of quasiperiodic oscillations detected on the lightcurve tail of SGR 1806-20 and SGR 1900+14. The above obtained spectral formulae for the frequencies of nodeless torsional oscillations restored by force of shear elastic deformations have many features in common with those derived in (Samuelsson, Andersson 2007) from general relativistic treatment of torsional elasticity. The detailed identification of QPOs detected during the flare of the above magnetars with overtones of elastic torsional oscillations is discussed in (Watts, Strohmayer 2007).

3 TORSIONAL ELASTIC VIBRATIONS IN THE HOMOGENEOUS MODEL OF THE NEUTRON STAR CRUST

In the remainder of this paper we focus on torsional oscillations trapped in the peripheral spherical layer of the neutron star of finite depth implying that seismically active zone depends upon the energy which is released in the starquake. The prime purpose of our analysis is to elucidate the effect of boundary conditions (reflecting the behavior of material displacements on the edges of seismogenic layer) on the form of frequency spectrum of elastic torsional oscillations. In so doing we adopt boundary conditions which are currently utilized in the works studying quake-induce oscillations in the neutron star crust. Namely the condition of stress-free-surface for both core-crust boundary and surface of the star and non-slip condition on the the core-crust interface (e.g. McDermott, van Horn, Hansen 1988; Strohmayer 1991; Bildsten & Ushomirsky 2000).

3.1 No-slip boundary condition on the core-crust interface and no-stress on the star surface

Torsional oscillations trapped in the crust are described by the general solution for $u_i$ given by equations (6)-(11). To eliminate arbitrary constants $A_\ell$ and $B_\ell$, two boundary conditions, one on the core-crust interface, at $r = R_c$, and second on the star surface, at $r = R$, must
be used. On the star surface we impose the standard boundary condition of the absence of stresses normal to surface \( n_k \sigma_{ik} |_{r=R} = 0 \) which is reduced to
\[
\mu \left[ \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right] |_{r=R} = 0
\] (23)
and on the core-crust interface the no-slip condition
\[
u_{\phi} |_{r=R_c} = 0.
\] (24)

Physically, this condition means that the amplitude of differentially rotational oscillations triggered by starquake in the neutron star crust is gradually depreciated from the star surface to the core-crust interface which is the internal boundary of seismogenic layer. On inserting (12) in (23) and (24) we obtain
\[
\begin{align*}
\left[ f_{\ell+1}(z) - \frac{f_\ell(z)}{z} (\ell - 1) \right] &= 0 \quad z = kR \\
 f_\ell(z_c) &= 0 \quad z_c = \lambda z \quad 0 \leq \lambda < 1.
\end{align*}
\] (25)

Note, the notation \( z_c = \lambda z \) means that the radius of the core \( R_c \) can be represented as \( R_c = \lambda R \) with \( \lambda \) from the interval \( 0 \leq \lambda < 1 \). It is worth emphasizing that \( \lambda \) is strongly less than unit. In terms of Bessel and Neumann functions these latter boundary conditions read
\[
\begin{align*}
A_\ell \left[ j_{\ell+1}(z) - \frac{j_\ell(z)}{z} (\ell - 1) \right] + B_\ell \left[ n_{\ell+1}(z) - \frac{n_\ell(z)}{z} (\ell - 1) \right] &= 0 \\
A_\ell j_\ell(\lambda z) + B_\ell n_\ell(\lambda z) &= 0.
\end{align*}
\] (27)

Casting these equations as homogeneous matrix equation whose Wronskian must be equal to zero we arrive at the dispersion equation of the form
\[
W(z) = \left[ j_{\ell+1}(z) - \frac{j_\ell(z)}{z} (\ell - 1) \right] n_\ell(\lambda z) - \left[ n_{\ell+1}(z) - \frac{n_\ell(z)}{z} (\ell - 1) \right] j_\ell(\lambda z) = 0.
\] (29)

Computations of roots of this transcendent equation in which spherical Bessel and Neumann functions are defined in different points is non-trivial numerical problem (e.g. Pexton & Stiger 1977). However, our prime purpose here is to evaluate the frequency spectrum for long wavelength differentially rotational oscillations, when \( z = kR \ll 1 \). Taking into account that in the long wavelength limit
\[
\begin{align*}
j_\ell(z) n_\ell(\lambda z) \to \beta \lambda^{-(\ell+1)} \\
j_\ell(\lambda z) n_\ell(z) \to \beta \lambda^\ell
\end{align*}
\] (30)

one finds that exact dispersion equation (29) is reduced to
\[
z^2 = (2\ell + 3) [(\ell - 1) + (\ell + 2) \lambda^2 \ell+1] \\
z^2 = k^2 R^2 = \frac{\omega^2}{c_\ell^2} R^2.
\] (31)
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From this it follows

\[
\frac{\nu_1^2}{\nu_0^2} = (2\ell + 3)(\ell - 1) \left[ 1 + \frac{\ell + 2}{\ell - 1} \lambda^{2\ell+1} \right]
\]

\[
\nu_1 = \frac{\omega_1}{2\pi}, \quad \nu_0 = \frac{\omega_0}{2\pi}, \quad \lambda = \frac{R_c}{R} = 1 - h, \quad h = \frac{\Delta R}{R}.
\]

In the limit of zero-size radius of the core, \(\lambda = (R_c/R) \to 0\), corresponding to torsional oscillations in the entire volume of a solid star, we regain spectral equation (18) for the frequency of global torsional mode. For our further purpose we note that equation (32) can be represented in the following equivalent form

\[
\frac{\nu_1}{\nu_0} = \left[ \left( \ell + 2 \right) \left( \ell - 1 \right) \right]^{1/2} p_1^{-1} \quad p_1^{-1} = \left[ 2 \left( 1 - \frac{1}{2(\ell + 2)} \right) \left( 1 + \frac{\ell + 2}{\ell - 1} \lambda^{2\ell+1} \right) \right]^{1/2}
\]

which is discussed in the next subsection. Henceforth the suffice in expressions for \(\nu\) and \(\omega\) marks the number of eigenfrequency problem under consideration. The considered in this subsection is marked by suffice 1, and two another conceivable boundary conditions are regarded in the reminder of this work.

3.2 Stress free boundary conditions on both core-crust interface and the neutron star surface

Now we adopt the boundary conditions of the free from stresses surfaces for both the surface of the star and the core-crust boundary \(n_k \sigma_{ik} |_{r=R,R_c} = 0\):

\[
n_r \sigma_{r\phi} |_{r=R,R_c} = \mu \left[ \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right] |_{r=R,R_c} = 0
\]

which in terms of \(f_\ell(z)\) are given by

\[
\left[ \frac{df_\ell(z)}{dz} - \frac{f_\ell(z)}{z} \right]_{z=kR} = 0, \quad \left[ \frac{df_\ell(z_c)}{dz_c} - \frac{f_\ell(z_c)}{z_c} \right]_{z_c=\lambda kR} = 0 \quad 0 \leq \lambda < 1.
\]

For a non-vanishing solution of these equations to exist, the following dispersion equation

\[
\left[ j_{\ell+1}(z) - j_{\ell}(z) \left( \ell - 1 \right) \left[ n_{\ell+1}(\lambda z) - \frac{n_{\ell}(\lambda z)}{\lambda z} (\ell - 1) \right] \right]
\]

\[
- \left[ j_{\ell+1}(\lambda z) - \frac{j_{\ell}(\lambda z)}{\lambda z} (\ell - 1) \left[ n_{\ell+1}(z) - \frac{n_{\ell}(z)}{z} (\ell - 1) \right] \right] = 0
\]

must hold. Following the line of argument of foregoing subsection, consider the limit of long wavelengths, \(z = kR \ll 1\). This yields

\[
\frac{\nu_2^2}{\nu_0^2} = (2\ell + 3)(\ell - 1) \left[ 1 + \frac{\lambda^{2\ell+1}}{1 - \lambda^{2\ell+3}} \right] \quad 0 \leq \lambda < 1.
\]

In the limit \(\lambda = (R_c/R) \to 0\), the last equation is reduced to spectral formula (18) for the frequency of global torsional oscillations.
Fig. 3 illustrates the general trends of fractional frequencies - $\nu/\nu_0$, $\nu_1/\nu_0$ and $\nu_2/\nu_0$ as functions of multipole degrees $1 < \ell < 20$ plotted at the values of fractional depth given by $h = \Delta R/R = 0.05, 0.1$ and 0.3. The difference caused by boundary conditions is notable in the low-$\ell$ domain, when the depth of seismogenic layer $\Delta R = 0.5$ and 1.0 km (in the neutron star of fiducial radius $R = 10$ km). This difference is practically vanishes when the thickness of layer is about $\Delta R = 3.0$ km and larger. Generally, the thicker this depth the less difference. Also we note, for our further purpose, that equation (38) can be represented as

$$\frac{\nu_2}{\nu_0} = \left[ (\ell + 2)(\ell - 1) \right]^{1/2} p_2^{-1}, \quad p_2^{-1} = \left[ 2 \left( 1 - \frac{1}{2(\ell + 2)} \right) \frac{1 + \lambda^{2\ell+1}}{1 - \lambda^{2\ell+3}} \right]^{1/2}. \quad (39)$$

At this point it seems worthy of noting that analogous, from mathematical side, problem...
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of torsional vibration mode trapped in the peripheral solid layer has been considered long ago by Pekeris (1965) in the geoseismic context. The obtained in this latter work spectral formula with boundary conditions identical to considered in this subsection reads

$$\frac{\nu_p}{\nu_0} = [(\ell - 1)(\ell + 2)]^{1/2} p_\ell^{-1} \quad \ell \geq 10. \quad (40)$$

In work of Pekeris (1965) the dimensionless parameter $p_\ell$, has been numerically computed from integral equation establishing ”relation between geometrical optics and terrestrial spectroscopy” and tabulated for $1 \leq \ell \leq 50$. Remarkably, in appearance the Pekeris spectral formula is identical to the above presented equations (34) and (39). In Fig. 4 we plot $\nu_p/\nu_0$ with use of data for $p_\ell$ taken from (Pekeris 1965, Table 2) as a function of $\ell$ in juxtaposition with our spectral formula (40) in which we have used the value of fractional depth of seismogenic layer given by $h = 0.1$. Analogous conclusion holds for $\nu_1/\nu_0$. It should be noted, however, that equation (40) has been obtained with use of non-uniform profile for the speed of transverse wave of elastic shear $c_t(r)$ which is used as input function of the method, not computed. In view of this, the last figure can be considered no more than juxtaposition, not a comparison, of our and Pekeris spectral equations. Nonetheless, the fact that both approaches yields practically identical results suggests that presented in this section analysis can be utilized for assessing frequencies of torsional seismic vibration modes in the solid Earth-like planets too.

**Figure 4.** Frequency of torsional nodeless vibrations computed in the homogeneous model of peripheral seismic layer of second example of this work in juxtaposition with Pekeris (1965) asymptotic spectral formula for torsional geoseismic vibrations.
4 NODELESS TORSIONAL OSCILLATIONS IN HOMOGENEOUS AND NON-HOMOGENEOUS MODELS OF THE NEUTRON STAR CRUST

From above it follows that in the long wavelength limit the toroidal field of displacements (41)-(43) is reduced to the form that can be conveniently represented as follows

\[ u(r, t) = a(r) \alpha(t) \quad \alpha(t) = \alpha_0 \exp(i \omega t) \]  
\[ a(r) = \nabla \chi(r) \times r \quad \chi(r) = f_\ell(r) P_\ell(\zeta) \]  
\[ f_\ell(r) = [A_\ell r^\ell + B_\ell r^{-(\ell+1)}]. \]  

The toroidal field (41)-(43) is the general solution to the vector Laplace equation

\[ \nabla^2 u(r, t) = 0 \quad \nabla u(r, t) = 0. \]  

which can be thought of as the long wavelength limit of the Helmholtz equation \( \nabla^2 u + k^2 u = 0 \), because in the limit of long wavelengths, \( \lambda \to \infty \), the wave vector \( k = (2\pi/\lambda) \to 0 \). The vector Laplace equation can be regarded as fundamental equation defining regime of nodeless shear vibrations. The practical usefulness of this attitude is that it allows to assess the difference between predictions for the frequency spectra computed on equal footing within homogeneous and inhomogeneous models of the neutron star crust, that is, regardless of the form of density and shear modulus profiles. Most efficiently it can be done by Rayleigh’s energy method (Bastrukov et al 2007). The point of departure is the integral equation of energy conservation which is obtained from equation of elastodynamics (1) by its scalar multiplication with \( u_i \) and integration over the crust volume

\[ \frac{\partial}{\partial t} \int \frac{\rho u_i^2}{2} \, d\mathcal{V} = - \int \sigma_{ik} \dddot{u}_{ik} \, d\mathcal{V} = -2 \int \mu \dot{u}_{ik} \dddot{u}_{ik} \, d\mathcal{V}. \]  

On inserting in (45), the separable form of the displacement field (41), we arrive at equation for temporal amplitude \( \alpha(t) \) having the form of the well-familiar equation of normal vibrations

\[ \frac{dE}{dt} = 0 \quad E = \frac{M \dddot{\alpha}^2}{2} + \frac{K \alpha^2}{2} \]  
\[ \dddot{\alpha} + \omega^2 \alpha = 0 \quad \omega^2 = \frac{K}{M} \]  
\[ M = \int \rho(r) a_i a_i \, d\mathcal{V} \quad K = 2 \int \mu(r) a_{ik} a_{ik} \, d\mathcal{V} \quad a_{ik} = \frac{1}{2} [\nabla_i a_k + \nabla_k a_i]. \]  

The axisymmetric, odd parity, toroidal vector field represents one of two fundamental (mutually orthogonal and different in parity) solutions to (11) describing nodeless torsional vibrations. The second fundamental solution is given by even parity poloidal vector field (Bastrukov et al 2007) which describe nodeless spheroidal vibration mode, in accord with canonical Lamb’s classification of vibrational modes in an elastically deformable solid sphere (e.g. Lapwood & Usami 1981; Aki & Richards 2003).
The analytic form of the inertia $M$ and stiffness $K$ shows that method can be utilized for computing frequency $\omega^2 = K/M$ of shear vibrations (both spheroidal and torsional) for wide class of models with non-uniform density and shear modulus profiles in the crust which are the input parameters of the method. Taking the integral over the solid angle in the above expression for inertia $M$ we obtain

$$M = \frac{4\pi \ell(\ell+1)A_\ell^2}{2\ell+1} \left[ \int_{R_c}^R \rho(r) r^{2\ell+2} dr + \frac{2B_\ell}{A_\ell} \int_{R_c}^R \rho(r) r^2 dr + \frac{B_\ell^2}{A_\ell^2} \int_{R_c}^R \rho(r) r^{-2\ell} dr \right]. \quad (49)$$

In similar fashion, for the parameter of rigidity $K$ we get

$$K = 4\pi A_\ell^2 (\ell^2 - 1) \left[ \int_{R_c}^R \mu(r) r^{2\ell} dr + \frac{B_\ell^2 \ell(\ell+2)}{A_\ell^2 (\ell-1)} \int_{R_c}^R \mu(r) r^{-2\ell-2} dr \right]. \quad (50)$$

These later equations for $M$ and $K$ emphasizes the fact that the choice of boundary conditions does matter for the problem under consideration.

As a representative example, consider torsional oscillations in the crust with constants $A_\ell$ and $B_\ell$ eliminated from the following boundary conditions

$$u_\phi|_{r=R_c} = 0 \quad u_\phi|_{r=R} = [\phi_R \times \mathbf{R}]_\phi \quad \phi_R = \exp(i\omega t) \nabla_{\mathbf{n}} P_\ell(\zeta) \quad \nabla_{\mathbf{n}} = \left( 0, \frac{\partial}{\partial \theta}, \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right). \quad (51, 52)$$

First is the no-slip condition on the core-crust interface, $r = R_c$, and the form of second boundary condition on the star surface, at $r = R$, is dictated by symmetry of general toroidal field of nodeless torsional oscillations. The resultant algebraic equations stemming from boundary conditions (51) lead to

$$A_\ell = N_\ell, \quad B_\ell = -N_\ell R_c^{2\ell+1}, \quad N_\ell = \frac{R^{\ell+2}}{R^{2\ell+1} - R_c^{2\ell+1}}. \quad (53)$$

In the homogeneous crust model, presuming constant values of $\rho$ and $\mu$, we get

$$M(\ell, \lambda) = \frac{4\pi \ell(\ell+1)}{(2\ell+1)(2\ell+3)} \frac{\rho R^5}{(1-\lambda^{2\ell+1})^2} \times \left[ 1 - (2\ell+3)\lambda^{2\ell+1} + \frac{(2\ell+1)^2}{2\ell-1} \lambda^{2\ell+3} - \frac{2\ell+3}{2\ell-1} \lambda^{2(2\ell+1)} \right] \quad (54)$$

$$K(\ell, \lambda) = \frac{4\pi \ell^2 - 1}{2\ell+1} \frac{\mu R^3}{(1-\lambda^{2\ell+1})} \left[ 1 - \frac{(\ell+2)}{2\ell-1} \lambda^{2\ell+1} \right]. \quad (55)$$

Note, in the limit $\lambda \to 0$ we again recover the spectral formula for the frequency $\omega^2 = K/M$ of the global torsional oscillations in the entire volume of the homogeneous solid star model [18].
4.1 Numerical analysis

In the above expanded the energy variational method of computing the frequency of elastic shear oscillations, the profiles of density $\rho(r)$ and pressure $p(r)$ linearly proportional to the shear modulus $\mu(r)$ of crustal matter are regarded the input parameters. The approach we adopt is the one used by previous authors taking these parameters from evolution models aimed at computing gravitationally equilibrium state of matter and the internal structure of neutron star. In these later models the density and pressure are computed from Tolman-Oppenheimer-Volkoff (TOV) equation with account for realistic EOS (e.g. Weber 1999; Lattimer & Prakash 2001).

Specifically, we adopt here the model of neutron star model with fiducial mass of $M = 1.43M_{\odot}$ presented in Wiringa, Fiks & Fabrocini (1988) and will use parameters for the crust matter given in Douchin & Haensel (2001). The density profiles in the star as a whole
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Figure 7. Frequency of nodeless torsional shear oscillations as a function of multipole degree computed in the homogeneous and inhomogeneous models.

and in the crust of this star model are pictured in Fig.5. The position of the core-crust interface is placed at the fiducial density $\rho = 1.5 \times 10^{14}$ g cm$^{-3}$, in accord with arguments of works (Cutler, Ushomirsky, Link 2003; Pethick, Ranenhall & Lorentz 1995). The effect of non-homogeneous density in the crust on the speed of transverse wave of elastic shear in this model is illustrated in Fig.6.

In Fig.7 we compare the fractional frequency of torsional oscillations trapped in the crust of fixed depth $\Delta R = 0.6$ km (that has been normalized to the value of $\nu_0 = 15$ Hz) computed in the homogeneous and inhomogeneous crust models. It is seen that prediction of homogenous crust model with boundary conditions of this section are quite different from that inferred in previous section from homogeneous model too, but for different boundary conditions. Also, the predictions of homogeneous models are drastically different from those for non-homogenous one. This difference is manifested in both the overall trends of frequency as a function of multipole degree and absolute values of fractional frequencies. One of sources of uncertainties is parametrization of shear modulus profile. In the curve computed with $\mu = k(\rho)p_e$ we used parametrization of work (Cutler, Ushomirsky, Link 2003) and with $\mu = c_2^2 \rho$ from (Strohmayer et al 1991). At this point we leave the discussion of mathematical details of numerical calculations because this issue is not the main subject of presented investigation.
5 SUMMARY

There are now quite solid arguments showing that the Soft Gamma-Ray Repeaters are isolated, non-accreting, seismically active magnetars (Thompson & Duncan 1995) – quaking neutron stars endowed with ultrastrong magnetic fields. The X-ray bursting luminosity of magnetar is associated with starquakes which are thought of as sudden release of magnetic field stresses cracking of the neutron star crust by the X-ray flares. Evidence in favor of seismic nature of the magnetar flares is provided by striking similarities between statistics of SGR’s bursts and earthquakes (Cheng, Epstein, Guyer & Young 1996). The common belief is that the dynamics of quake induced internal elastic and magnetic field stresses in the crust can be properly understood within the framework of the two-component core-crust model of quaking neutron stars (Franco, Link & Epstein 2000), provided that the core-crust coupling is dominated by an ultrastrong magnetic field. A conceivable and comprehensive, from a physical point of view, explanation of the magnetic core-crust coupling provides a model of paramagnetic neutron star (Bastrukov et al 2002; Bastrukov et al 2003). In this model the neutron star core is regarded as a spherical bar magnet (composed of baryon matter dominated by poorly conducting neutron component) permanently magnetized to saturation due to Pauli’s mechanism of field-induced paramagnetic spin-polarization of neutron magnetic moments. The microstructure and extremely high conductivity of crustal matter indicates to its metal-like electrodynamical properties. This analogy suggests that the magnetic coupling of the core (permanent magnet) with the crust (metal) is similar to the well-known magnetic adhesion between a metal specimen and bar magnet. The fast process of postquake recovery of the magnetar is of course dominated by force of gravitational pull. The prime effect of magnetic field frozen in the core on this process is that the the perturbed by starquake a highly conducting crustal matter (as well as the less dense plasma of magnetar corona (Beloborodov & Thompson 2007) expelled from the star surface by SGR’s flares) sets in axisymmetric torsional oscillations about axis of ultrastrong magnetic field frozen in the star. It is these oscillations about magnetic axis are observed, as is argued in (Israel 2007; Watts, Strohmayer 2007) as quasiperiodic oscillations of x-ray luminosity with high signal-to-noise ratio. In (Watts & Strohmayer 2007) arguments are given that the data favor the idea of a seismic origin of the detected QPOs, that is, as caused by quake induced torsional shear oscillations of crustal matter of magnetar. Adhering to this attitude, we have investigated several scenario of torsional shear vibrations restored by bulk force of shear elasticity (the
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problem from which the very notion of torsion shear vibrations of an elastic sphere came into existence). Our prime purpose was to elucidate the distinctions between spectral formulae for the frequency of nodeless torsional shear oscillations caused by different behavior of quake induced crustal matter on the boundaries of the crust. In pursuing this goal and using canonical methods of mathematical physics we have considered several examples of exact solutions of the eigenfrequency problem demonstrating crucial effect of boundary conditions on the frequency spectrum of torsional mode of elastic oscillations trapped in the crust. The asymptotic spectral formulae for the frequency of nodeless torsion oscillations have been presented so that they can be conveniently applied to a wide class of celestial objects. The practical usefulness of the exact solutions for the toroidal field of material displacements considered here is that they can be utilized in the study of torsional shear vibrations restored by forces of intrinsic stresses of different physical nature, like Newtonian gravitation field stresses (e.g. Shu 1992) and Maxwellian magnetic field stresses (e.g. Chandrasekhar 1961; Mestel 1999), not only Hookean elastic one. With all that we conclude, while the obtained spectral formulae properly match the observable QPOs frequency, the information obtained from only these models is less than necessary to draw definite physical statements which of above forces plays dominant role. The point of particular interest, as is generally believed, are torsion oscillations driven by the Lorentz force that can be represented as divergence of fluctuating magnetic field stresses (Franco, Link, Epstein 2000)

$$\delta T_{ik} = \frac{1}{4\pi} \left[ B_i \delta B_k + B_k \delta B_i - B_j \delta B_j \delta_{ik} \right]$$

$$\delta B = \nabla \times [u \times B] \quad \nabla \cdot u = 0.$$

The canonical form of magneto-solid-mechanical equations governing Alfvénic, compression free, oscillations in a perfectly conducting elastically deformable solid, regarded as material continuum, reads

$$\rho \ddot{u}_i = \nabla_k \delta T_{ik} \quad \frac{\partial}{\partial t} \int \rho \dot{u}_i^2 \, d\mathcal{V} = - \int \delta T_{ik} \dot{u}_{ik} \, d\mathcal{V}.$$

The integrand of equation of energy conservation exhibits the fact that the work done by magnetic field stresses in the volume of a quaking neutron star is accompanied too by shear deformations and, thus, shows the torsional shear oscillations can also be sustained by fluctuations of the magnetic field stresses. We shall turn to this problem in a forthcoming paper.

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