Gauss–Bonnet theorem for compact and orientable surfaces: a proof without using triangulations

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Abstract

The aim of this note is to provide an intrinsic proof of the Gauss–Bonnet theorem without invoking triangulations, which is achieved by exploiting complex structures.

1 Introduction

Given an orientable two dimensional manifold $\Sigma$, out of any riemannian structure $g$ (e.g. it might be one induced by the euclidean metric, supposing that $\Sigma$ is embedded in some euclidean space) one can construct an area form $\star g 1 \in \Omega^2(\Sigma; \mathbb{R})$ and its sectional curvature $K_g \in C^\infty(\Sigma; \mathbb{R})$. Remarkably, the integral

$$\int_{\Sigma} K_g \star g 1$$

is a constant depending only on the topology of $\Sigma$ (assuming it to be compact and without boundary), as it was already observed by Gauss and Bonnet; i.e. if $g'$ is any riemannian structure defined on any manifold $\Sigma'$ diffeomorphic (or homeomorphic) to $\Sigma$, then

$$\int_{\Sigma} K_g \star g 1 = \int_{\Sigma'} K_{g'} \star g' 1 .$$

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A proof of this theorem, known as the Gauss–Bonnet theorem, can be achieved by means of a triangulation on \( \Sigma \). Indeed, such a proof can be found in standard books on the subject (e.g. do Carmo’s book \([1]\)), and its intricacies are related to proving the independence on the choice of a triangulation and the existence of such structures.

Here in this note, a proof of the Gauss–Bonnet theorem is presented using complex structures on \( \Sigma \) without using triangulations. Instead of focusing on the topology of \( \Sigma \) this proof exploits the complex geometry of its tangent bundle \( T\Sigma \), which is to be understood as a complex line bundle; hence, instead of referring to Euler characteristics or genera, one refers to Chern numbers to represent the “topological content” of the Gauss–Bonnet formula.

Complex structures have been used in proofs of the Gauss–Bonnet theorem by Jost \([2]\) and Taubes \([3]\); however, Jost invokes triangulations, whereas Taube exploits the embedding of the surface in the three dimensional euclidean space —contrasting with the intrinsic proof of this note.

Starting with \((\Sigma, g)\), the riemannian structure grants not only a connexion on \( T\Sigma \), the Levi-Civita connexion \( \nabla^g \), but also a complex structure \( j_g \) (proposition \([2.1]\)) and a hermitian inner product \( \langle \cdot , \cdot \rangle \) compatible with the Levi-Civita connexion (which is nothing more than \( g + \sqrt{-1} \ast g 1 \)), turning \((T\Sigma, j_g, \langle \cdot , \cdot \rangle, \nabla^g)\) into a hermitian line bundle with a hermitian connexion.

The complex structure, at some point \( p \in \Sigma \), takes a tangent vector \( v \in T_p \Sigma \) to another tangent vector \( j_g |_p (v) \in T_p \Sigma \) that is orthogonal to \( v \) (with respect to the riemannian structure \( g \)) and forms with it a positively oriented basis \( \{ v, j_g |_p (v) \} \subset T_p \Sigma \) (with respect to the orientation induced by \( \ast g 1 \)), i.e. it rotates tangent vectors in an orthogonal and orientable fashion (mimicking the rotation induced by multiplication by \( \sqrt{-1} \) on the real plane).

When \( \Sigma \) is a subriemannian manifold of the three dimensional euclidean space, the complex structure (given by the induced metric) at a point applied to a tangent vector is simply the cross product between the normal vector at the particular point and this tangent vector (with both vectors understood as elements of the three dimensional euclidean space).

It so happens that (lemma \([3.1]\))
\[
\sqrt{-1} \cdot curv(\nabla^g) = K_g \ast g 1 ,
\]
and the first Chern number, defined by
\[
\frac{1}{2\pi} \int_{\Sigma} \sqrt{-1} \cdot curv(\nabla^g) ,
\]
is independent (theorem \([3.1]\)) of the geometric structures \((j_g, \langle \cdot , \cdot \rangle, \nabla^g)\).

Finally a disclaimer. The focus of this note is to produce a complete proof of the geometric independence of the curvature integral without using...
triangulations; this note does not provide a proof that the first Chern number equals the Euler characteristic, which can be achieved by showing that the top Chern class is related to counting (with multiplicity) the intersection between a generic section and the zero section of $T\Sigma$.

### 1.1 Organisation

Section 2 contains the construction of a special complex structure from any given riemannian structure on an orientable two dimensional real manifold (without boundary), whilst section 3 proves the relationship between the first Chern number and the Gauss–Bonnet formula.

### 1.2 Acknowledgements

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### 2 Complex structures on Riemann surfaces

Assuming $\Sigma$ to be an orientable two dimensional manifold (without boundary), any riemannian structure $g$ allows one to construct an area form $*g^1 \in \Omega^2(\Sigma; \mathbb{R})$ and a complex structure $j_g$ compatible with it. The complex structure is actually integrable, for the dimension of $\Sigma$ is two; i.e. $(\Sigma, g, *g^1, j_g)$ is a Kähler manifold.

**Proposition 2.1.** A complex structure $j_g$ can be defined from $g$ as the solution, for any $X, Y \in \mathfrak{X}(\Sigma; \mathbb{R})$, of

$$g(j_g(X), Y) = *g^1(X, Y).$$

A simple way to prove this statement is to regard the matrices of $j_g$, $*g^1$, and $g$ in a local chart. Denoting these matrices by $j_{g_{2 \times 2}}$, $*g^1_{2 \times 2}$, and $g_{2 \times 2}$, one has $j_{g_{2 \times 2}} = -g_{2 \times 2}^{-1} \cdot *g^1_{2 \times 2}$.

Such a complex structure is not uniquely defined by the riemannian structure, since, for every positive function $f \in C^\infty(\Sigma; \mathbb{R})$, the complex structures induced by $g$ and $fg$ coincide, $j_g = j_{fg}$. In particular, using local coordinate functions in which the riemannian structure is expressed as a positive multiple of the flat euclidian one, the induced complex structure acts on tangent vectors exactly as the multiplication by $\sqrt{-1}$ does on the real plane. These coordinates are commonly known as isothermal coordinates, but in the context of the complex geometry of Riemann surfaces, they are the holomorphic coordinates.
Proof of proposition 2.1. A solution of equation (1) defines a $C^\infty(\Sigma; \mathbb{R})$-linear mapping because, if $X, Y, Z \in \mathfrak{X}(\Sigma; \mathbb{R})$ and $f \in C^\infty(\Sigma; \mathbb{R})$,

$$
\star g_1(X + fY, Z) = \star g_1(X, Z) + f \star g_1(Y, Z)
= g(j_g(X), Z) + f g(j_g(Y), Z)
= g(j_g(X) + f j_g(Y), Z),
$$

and the uniqueness of the solution of

$$
g(j_g(X + fY), Z) = \star g_1(X + fY, Z)
$$

implies

$$
j_g(X + fY) = j_g(X) + f j_g(Y).
$$

The nondegeneracy of both $g$ and $\star g_1$ guarantees that $j_g(X) = 0$ if and only if $X = 0$; consequently, $j_g \in \text{Aut}_{C^\infty(\Sigma; \mathbb{R})}(\mathfrak{X}(\Sigma; \mathbb{R}))$.

To actually state that $j_g$ is a complex structure, one has to prove that its inverse is $-j_g$; and such property follows from $j_g$ being skewsymmetric and an infinitesimal isometry with respect to $g$ (the desired features of a compatible complex structure). Indeed,

$$
g(X, Y) = g(j_g(X), j_g(Y)) = -g(j_g \circ j_g(X), Y)
$$
yields $j_g \circ j_g(X) = -X$.

The skewsymmetry is inherited from the skewsymmetry of the area form,

$$
g(j_g(X), Y) = \star g_1(X, Y)
= -\star g_1(Y, X)
= -g(j_g(Y), X) = g(X, -j_g(Y)).
$$

For the other property, one needs to use the identity

$$
\star g_1(X, Y)^2 = g(X, X) \cdot g(Y, Y) - g(X, Y)^2.
$$

The reader will recognise it as the square of the areas (with respect to $g$) of the parallelograms spanned by $X$ and $Y$.

Now, on the one hand,

$$
g(j_g(Y), j_g(Y))^2 = \star g_1(Y, j_g(Y))^2
= g(Y, Y) \cdot g(j_g(Y), j_g(Y)) - g(Y, j_g(Y))^2
= g(Y, Y) \cdot g(j_g(Y), j_g(Y)) - \star g_1(Y, Y)^2
= g(Y, Y) \cdot g(j_g(Y), j_g(Y)).
$$
produces 
\[ g(j_g(Y), j_g(Y)) = g(Y, Y) . \]

On the other hand,
\[
g(j_g(X), j_g(Y))^2 = \ast_g 1(X, j_g(Y))^2 \\
= g(X, X) \cdot g(j_g(Y), j_g(Y)) - g(X, j_g(Y))^2 \\
= g(X, X) \cdot g(Y, Y) - \ast_g 1(Y, X)^2 \\
= g(X, X) \cdot g(Y, Y) - g(Y, Y) \cdot g(X, X) + g(Y, X)^2 \\
= g(X, Y)^2 .
\]

\[ \square \]

3 Gauss–Bonnet formula

The tangent bundle \( T\Sigma \) together with \( j_g \) can be understood as a complex line bundle. Endowed with the hermitian inner product
\[
\langle \cdot, \cdot \rangle = g(\cdot, \cdot) + \sqrt{-1} \ast_g 1(\cdot, \cdot) ,
\]
and regarding the Levi-Civita connexion \( \nabla^g \) as a hermitian connexion\(^1\), \((T\Sigma, j_g, \langle \cdot, \cdot \rangle, \nabla^g)\) is a hermitian line bundle with a hermitian connexion. This implies that the 2-form \( \sqrt{-1} \cdot \text{curv}(\nabla^g) \) represents an integral de Rham cohomology class: if \( \Sigma \) is compact,
\[
\frac{1}{2\pi} \int_{\Sigma} \sqrt{-1} \cdot \text{curv}(\nabla^g) \in \mathbb{Z}
\]
and the integer (the Chern number of the line bundle) is a topological invariant.

\textbf{Lemma 3.1.} Let \((\Sigma, g)\) be an orientable two dimensional riemannian manifold (without boundary), \( \ast_g 1 \in \Omega^2(\Sigma; \mathbb{R}) \) its area form, and \( K_g \in C^\infty(\Sigma; \mathbb{R}) \) its sectional curvature. Considering the hermitian line bundle with hermitian connexion \((T\Sigma, j_g, \langle \cdot, \cdot \rangle, \nabla^g)\),
\[
\sqrt{-1} \cdot \text{curv}(\nabla^g) = K_g \ast_g 1 .
\]

\(^1\)Since \( \nabla^g \) is torsionless, this property is equivalent to the integrability of the compatible complex structure \( j_g \).
Proof. As a 2-form on a two dimensional manifold, $\sqrt{-1} \cdot curv(\nabla^g)$ must be proportional to $\ast_g 1$, i.e. there exists some function $K_g \in C^\infty(\Sigma; \mathbb{R})$ satisfying

$$\sqrt{-1} \cdot curv(\nabla^g) = K_g \ast_g 1.$$ 

The next step is to prove that such a function $K_g$ is actually the sectional curvature. In order to do so, it is important to remark how a vector field $Y \in \mathfrak{X}(\Sigma; \mathbb{R})$ can be multiplied by a complex number, i.e.

$$\sqrt{-1} \cdot Y := j_g(Y).$$

Accordingly, for any vector fields $X, Y \in \mathfrak{X}(\Sigma; \mathbb{R})$ linearly independent at some $p \in \Sigma$,

$$curv(\nabla^g)(X, Y)Y = -K_g \ast_g 1(X, Y)j_g(Y),$$

and

$$g(curv(\nabla^g)(X, Y)Y, X) = -K_g \ast_g 1(X, Y)g(j_g(Y), X) = K_g \ast_g 1(X, Y)^2;$$

thus,

$$K_g(p) = \frac{g|_p(curv(\nabla^g)|_p(X|_p, Y|_p)Y|_p, X|_p)}{g|_p(X|_p, X|_p) \cdot g|_p(Y|_p, Y|_p) - g|_p(X|_p, Y|_p)^2}. \quad \blacksquare$$

One might wonder how different choices of riemannian structure affect the hermitian structures $(j_g, \langle \cdot, \cdot \rangle, \nabla^g)$ introduced on $T\Sigma$.

**Proposition 3.1.** If $j_g$ and $j_g'$ are two complex structures on an orientable two dimensional manifold (without boundary) $\Sigma$ induced by two distinct riemannian structures $g$ and $g'$, then $(T\Sigma, j_g)$ is isomorphic to $(T\Sigma, j_g')$ as complex vector bundles.

**Proof.** The mapping defined by

$$T\Sigma \supset T_p\Sigma \times \{p\} \ni (v, p) \mapsto (-j_g'|_p \circ j_{g'}|_p(v), p) \in T_p\Sigma \times \{p\} \subset T\Sigma$$

yields a complex vector bundle isomorphism between the bundles $(T\Sigma, j_g)$ and $(T\Sigma, j_{g'})$. \quad \blacksquare

This means that one can fix a complex line bundle $L$ to be associated to the tangent bundle of $\Sigma$, and this complex line bundle does not depend on the choice of a riemannian structure.
Lemma 3.2. If \((\langle \cdot, \cdot \rangle, \nabla)\) and \((\langle \cdot, \cdot \rangle', \nabla')\) are two hermitian structures and hermitian connexions on a given complex line bundle \(L\) over an orientable two dimensional manifold (without boundary) \(\Sigma\), then there exists \(\eta \in \Omega^1(\Sigma; \mathbb{R})\) such that
\[
\text{curv}(\nabla) - \text{curv}(\nabla') = \sqrt{-1} \cdot d\eta .
\]

Proof. The first thing to be noticed is that \(\nabla - \nabla'\) is \(C^\infty(\Sigma; \mathbb{C})\)-linear when it acts on sections of \(L\): as, for any \(f \in C^\infty(\Sigma; \mathbb{C})\) and \(s \in \Gamma(L)\),
\[
(\nabla - \nabla')(fs) = \nabla(fs) - \nabla'(fs) = df \otimes s + f \nabla s - df \otimes s - f \nabla's = f(\nabla - \nabla')s .
\]
Therefore,
\[
(\nabla - \nabla') : \Gamma(L) \to \Omega^1(\Sigma; \mathbb{C}) \otimes \Gamma(L)
\]
is a \(C^\infty(\Sigma; \mathbb{C})\)-linear mapping, and it can be understood as an element of
\[
\Omega^1(\Sigma; \mathbb{C}) \otimes \Gamma(L) \otimes \Gamma(L)^* \n\]
which, in turn, is isomorphic to
\[
\Omega^1(\Sigma; \mathbb{C}) \otimes \Gamma(L \otimes L^{-1}) ;
\]
however, \(L \otimes L^{-1} \cong \mathbb{C} \times \Sigma\), and \(\Gamma(L \otimes L^{-1}) \cong C^\infty(\Sigma; \mathbb{C})\) allows \(\nabla - \nabla'\) to be understood as an element of \(\Omega^1(\Sigma; \mathbb{C})\). As a result, given any \(s \in \Gamma(L)\), there exists \(\eta \in \Omega^1(\Sigma; \mathbb{C})\) satisfying
\[
\nabla s = \nabla's + \sqrt{-1} \cdot \eta \otimes s ,
\]
and, using this expression to compute \(\text{curv}(\nabla)\), one obtains
\[
\text{curv}(\nabla) - \text{curv}(\nabla') = \sqrt{-1} \cdot d\eta .
\]
The fact that both connexions are hermitian guarantees that \(\eta \in \Omega^1(\Sigma; \mathbb{R})\). \(\blacksquare\)

According to Stokes theorem, one has the following.

Theorem 3.1. Let \((\Sigma, g)\) be an orientable two dimensional compact riemannian manifold (without boundary). If \(g'\) is any riemannian structure defined on any manifold \(\Sigma'\) diffeomorphic to \(\Sigma\), then
\[
\int_\Sigma K_g \ast_g 1 = \int_{\Sigma'} K_{g'} \ast_{g'} 1 .
\]
Proof. Lemmata 3.1 and 3.2 grant, for some $\eta \in \Omega^1(\Sigma; \mathbb{R})$ and diffeomorphism $\varphi : \Sigma \to \Sigma'$,

$$K_g \ast_1 - \varphi^*(K_{g'} \ast_1) = \sqrt{-1} \cdot \text{curv}(\nabla^g) - \sqrt{-1} \cdot \text{curv}(\varphi^*(\nabla^{g'})) = -d\eta .$$

Subsequently, from Stokes theorem,

$$\int_{\Sigma} d\eta = \int_{\partial \Sigma} \eta = \int_{\emptyset} \eta = 0 ;$$

hence,

$$0 = \int_{\Sigma} K_g \ast_1 - \int_{\Sigma} \varphi^*(K_{g'} \ast_1) = \int_{\Sigma} K_g \ast_1 - \int_{\Sigma'} K_{g'} \ast_1 .$$

References

[1] Manfredo do Carmo; Differential geometry of curves and surfaces; Prentice Hall (1976).

[2] Jürgen Jost; Compact Riemann surfaces: an introduction to contemporary mathematics; Springer (2006).

[3] Clifford Taubes; Differential Geometry: bundles, connections, metrics and curvature; Oxford University Press (2011).