The density of visible points in the Ammann-Beenker point set

Gustav Hammarhjelm

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Abstract

The relative density of visible points of the integer lattice \( \mathbb{Z}^d \) is known to be \( 1/\zeta(d) \) for \( d \geq 2 \), where \( \zeta \) is Riemann’s zeta function. In this paper we prove that the relative density of visible points in the Ammann-Beenker point set is given by \( 2(\sqrt{2} - 1)/\zeta_K(2) \), where \( \zeta_K \) is Dedekind’s zeta function over \( K = \mathbb{Q}(\sqrt{2}) \).

1 Introduction

A locally finite point set \( \mathcal{P} \subset \mathbb{R}^d \) has an asymptotic density (or simply density) \( \theta(\mathcal{P}) \) if

\[
\lim_{R \to \infty} \frac{\#(\mathcal{P} \cap RD)}{\text{vol}(RD)} = \theta(\mathcal{P})
\]

holds for all Jordan measurable \( D \subset \mathbb{R}^d \). The density of a set can be interpreted as the asymptotic number of elements per unit volume. For instance, for a lattice \( \mathcal{L} \subset \mathbb{R}^d \) we have \( \theta(\mathcal{L}) = \frac{1}{\text{vol}(\mathbb{R}^d/\mathcal{L})} \). Let \( \hat{\mathcal{P}} = \{ x \in \mathcal{P} \mid tx \notin \mathcal{P}, \forall t \in (0,1) \} \) denote the subset of the visible points of \( \mathcal{P} \). If \( \mathcal{P} \) is a regular cut-and-project set (see Definition 3.1 below) then it is known that \( \theta(\hat{\mathcal{P}}) \) exists. In [5, Theorem 1], J. Marklof and A. Strömbergsson proved that \( \theta(\hat{\mathcal{P}}) \) also exists and that \( 0 < \theta(\hat{\mathcal{P}}) \leq \theta(\mathcal{P}) \) if \( \theta(\mathcal{P}) > 0 \). In particular, for such \( \mathcal{P} \) the relative density of visible points \( \kappa_P := \frac{\theta(\hat{\mathcal{P}})}{\theta(\mathcal{P})} \) exists, but is not known explicitly in most cases.

For \( d \geq 2 \) we have \( \hat{\mathcal{Z}}^d = \{ (n_1, \ldots, n_d) \in \mathbb{Z}^d \mid \gcd(n_1, \ldots, n_d) = 1 \} \) and \( \theta(\hat{\mathcal{Z}}^d) = 1/\zeta(d) \) gives the probability that \( d \) random integers share no common factor. This can be derived in several ways, see for instance [6]; we sketch another proof in Section 2 below. More generally, \( \theta(\hat{\mathcal{L}}) = \frac{1}{\text{vol}(\mathbb{R}^d/\mathcal{L})} \) for a lattice \( \mathcal{L} \subset \mathbb{R}^d \), see e.g. [3 Prop. 6].

A well-known point set, which can be realised both as the vertices of a substitution tiling and as a cut-and-project set, is the Ammann-Beenker point set. The goal of this paper is to prove that the relative density of visible points in the Ammann-Beenker point set is \( 2(\sqrt{2} - 1)/\zeta_K(2) \). This density was computed by B. Sing in the presentation [7], but he has not published a proof of this result.

2 The density of the visible points of \( \mathbb{Z}^d \)

In this section we show that \( \theta(\hat{\mathcal{Z}}^d) = 1/\zeta(d) \). We shall see that a lot of inspiration can be drawn from this example when calculating the density of the visible points in the Ammann-Beenker point set.
Fix \( R > 0 \), a Jordan measurable \( D \subset \mathbb{R}^d \) and let \( \mathbb{P} \subset \mathbb{Z} > 0 \) denote the set of prime numbers. For each invisible point \( n \in \mathbb{Z}^d \setminus \hat{\mathbb{Z}}^d \), there is \( p \in \mathbb{P} \) such that \( \frac{n}{p} \in \mathbb{Z}^d \). Setting \( \mathbb{Z}^d_\ast = \mathbb{Z}^d \setminus \{(0, \ldots, 0)\} \) there are only finitely many \( p_1, \ldots, p_n \in \mathbb{P} \) such that \( p_i \mathbb{Z}^d_\ast \cap RD \neq \emptyset \).

By inclusion-exclusion counting we have

\[
\#(\hat{\mathbb{Z}}^d \cap RD) = \# \left((\mathbb{Z}^d_\ast \cap RD) \setminus \bigcup_{p \in \mathbb{P}} (p \mathbb{Z}^d_\ast \cap RD)\right) = \# \left((\mathbb{Z}^d_\ast \cap RD) \setminus \bigcup_{i=1}^{n} (p_i \mathbb{Z}^d_\ast \cap RD)\right) \\
= \#(\mathbb{Z}^d_\ast \cap RD) + \sum_{k=1}^{m} (-1)^k \left( \sum_{1 \leq i_1 < \ldots < i_k \leq m} \#(p_{i_1} \mathbb{Z}^d_\ast \cap \ldots \cap p_{i_k} \mathbb{Z}^d_\ast \cap RD)\right).
\]

The last sum can be rewritten to

\[
\sum_{n \in \mathbb{Z} > 0} \mu(n) \cdot \#(n \mathbb{Z}^d_\ast \cap RD),
\]

where \( \mu \) is the Möbius function. Hence

\[
\frac{\#(\hat{\mathbb{Z}}^d \cap RD)}{\text{vol}(RD)} = \sum_{n \in \mathbb{Z} > 0} \frac{\mu(n) \cdot \#(n \mathbb{Z}^d_\ast \cap RD)}{\text{vol}(RD)} = \sum_{n \in \mathbb{Z} > 0} \frac{\mu(n) \#(\mathbb{Z}^d_\ast \cap n^{-1}RD)}{n^d \text{vol}(n^{-1}RD)}.
\]

Letting \( R \to \infty \), switching order of limit and summation (for instance justified by finding a constant \( C \) depending on \( D \) such that \( \#(\mathbb{Z}^d_\ast \cap RD) \leq C \text{vol}(RD) \) for all \( R \)), using \( \theta(\mathbb{Z}^d_\ast) = 1 \) and \( 1/\zeta(s) = \sum_{n \in \mathbb{Z} > 0} \frac{\mu(n)}{n^s} \) for \( s > 1 \), we find that

\[
\theta(\hat{\mathbb{Z}}^d) = \lim_{R \to \infty} \frac{\#(\hat{\mathbb{Z}}^d \cap RD)}{\text{vol}(RD)} = 1/\zeta(d).
\]

### 3 Cut-and-project sets and the Ammann-Beenker point set

The Ammann-Beenker point set can be obtained as the vertices of the Ammann-Beenker tiling, a substitution tiling of the plane using a square and a rhombus as tiles, see e.g. [2, Chapter 6.1]. In this paper however, the Ammann-Beenker set is realised as a cut-and-project set, a certain type of point set which we will now define. Cut-and-project sets are sometimes called (Euclidean) model sets. We will use the same notation and terminology for cut-and-project sets as in [4, Sec. 1.2]. For an introduction to cut-and-project sets, see e.g. [2, Ch. 7.2].

If \( \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m \), let

\[
\pi : \mathbb{R}^n \to \mathbb{R}^d \\
(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_d)
\]

\[
\pi_{\text{int}} : \mathbb{R}^n \to \mathbb{R}^m \\
(x_1, \ldots, x_n) \mapsto (x_{d+1}, \ldots, x_n)
\]

denote the natural projections.

**Definition 3.1.** Let \( \mathcal{L} \subset \mathbb{R}^n \) be a lattice and \( \mathcal{W} \subset \overline{\pi_{\text{int}}(\mathcal{L})} \) be a set. Then the cut-and-project set of \( \mathcal{L} \) and \( \mathcal{W} \) is given by \( \mathcal{P}(\mathcal{W}, \mathcal{L}) = \{ \pi(y) \mid y \in \mathcal{L}, \pi_{\text{int}}(y) \in \mathcal{W} \} \).
If \( \partial W \) has measure zero with respect to any Haar measure on \( \pi_{\text{int}}(\mathcal{L}) \) we say that \( \mathcal{P}(\mathcal{W}, \mathcal{L}) \) is regular. If the interior of \( \mathcal{W} \) (the window) is non-empty, \( \mathcal{P}(\mathcal{W}, \mathcal{L}) \) is relatively dense and if \( \mathcal{W} \) is bounded, \( \mathcal{P}(\mathcal{W}, \mathcal{L}) \) is uniformly discrete (cf. [3, Prop. 3.1]). To realise the Ammann-Beenker point set in this way, let \( K \) be the number field \( \mathbb{Q}(\sqrt{2}) \), with algebraic conjugation \( x \mapsto \bar{x} \) (we will also write \( \bar{x} = (x_1, \ldots, x_n) \) for \( x = (x_1, \ldots, x_n) \in K^n \)) and norm \( N(x) = xx^* \). The ring of integers \( \mathcal{O}_K = \mathbb{Z}[\sqrt{2}] \) of \( K \) is a Euclidean domain with fundamental unit \( \lambda := 1 + \sqrt{2} \). With \( \zeta := e^{\pi i/4} \) and \( \ast : K \to K, x \mapsto x^* \) the automorphism generated by \( \zeta \mapsto \zeta^3 \), the Ammann-Beenker point set is in [2, Example 7.7] realised as

\[
\{ x = x_1 + x_2 \zeta \mid x_1, x_2 \in \mathcal{O}_K, x^* \in W_8 \},
\]

where \( W_8 \subset \mathbb{C} \) is the regular octagon of side length 1 centered at the origin, with sides perpendicular to the coordinate axes.

Let

\[
\mathcal{L} = \{ (x, \bar{x}) \mid x = (x_1, x_2) \in \mathcal{O}_K^2 \} \subset \mathbb{R}^4
\]

be the Minkowski embedding of \( \mathcal{O}_K^2 \) and let

\[
\tilde{\mathcal{L}} = \{ (x, \bar{x}) \in \mathcal{L} \mid (x_1 - x_2)/\sqrt{2} \in \mathcal{O}_K \}.
\]

Then, after a straight-forward translation it is seen that the Ammann-Beenker point set \( \mathcal{A} \) can be realised in \( \mathbb{R}^2 \) as \( \mathcal{A} = \frac{1}{\sqrt{2}} \mathcal{P}(\mathcal{W}_A, \tilde{\mathcal{L}}) \), where \( \mathcal{W}_A := \sqrt{2}W_8 \), i.e. \( \mathcal{A} \) is the scaling of a cut-and-project set according to [Definition 3.1]

### 4 The density of visible points of \( \mathcal{A} \)

All notation used in this section is defined in and taken from [Section 3]. Since, for any \( \mathcal{P} \subset \mathbb{R}^d \) whose density exists, and any \( c > 0 \) it holds that \( \theta(c\mathcal{P}) = c^{-d}\theta(\mathcal{P}) \) and \( c\mathcal{P} = \mathcal{P} \), finding \( \theta(\mathcal{A}) \) with \( \mathcal{A}' := \sqrt{2}\mathcal{A} = \mathcal{P}(\mathcal{W}_A, \tilde{\mathcal{L}}) \) will give the value of \( \theta(\mathcal{A}) \). As a first step, in [Section 4.1] the asymptotic density of the visible points of the simpler set \( \mathcal{B} = \mathcal{P}(\mathcal{W}_A, \mathcal{L}) = \{ x \in \mathcal{O}_K^2 \mid \bar{x} \in \mathcal{W}_A \} \subset \mathcal{O}_K^2 \) will be calculated. In [Section 4.2] this result will be used to obtain \( \theta(\mathcal{A}) \).

#### 4.1 The density of visible points of \( \mathcal{B} \)

The following general counting formula for bounded subsets of visible points of a point set \( \mathcal{P} \) will be needed. Let \( \mathcal{P}_* := \mathcal{P} \setminus \{(0, \ldots , 0)\} \).

**Proposition 4.1.** Let \( \mathcal{P} \subset \mathbb{R}^d \) be locally finite and fix a set \( C \subset \mathbb{R}_{>1} \) such that for each \( x \in \mathcal{P} \setminus \mathcal{P} \) there exists \( c \in C \) with \( x/c \in \mathcal{P} \). Let \( R > 0 \) and a bounded set \( D \subset \mathbb{R}^d \) be given. Then

\[
\#(\mathcal{P} \cap RD) = \sum_{F \subset C} (-1)^F \#(\left(\mathcal{P} \cap \bigcap_{c \in F} c\mathcal{P}_* \right) \cap RD).
\]

**Proof.** The set \( C_R := \{ c \in C \mid \mathcal{P}_* \cap c\mathcal{P}_* \cap RD \neq \emptyset \} \) is finite. Indeed, suppose this is not true and pick distinct \( c_1, c_2, \ldots \in C_R \) and corresponding \( x_i \in \mathcal{P}_* \cap c_i\mathcal{P}_* \cap RD \). Since \( \mathcal{P} \) is locally finite, the sequence \( x_1, x_2, \ldots \) contains only finitely many distinct elements. Thus, a subsequence \( x_{k_1}, x_{k_2}, \ldots \) which is constant can be extracted, so that \( x_{k_i}/c_{k_i} \in \mathbb{R}^d \).
\( \mathcal{P}_x \cap \overline{\mathcal{RD}} \subset \mathcal{P}_x \cap \mathcal{RD} \) are all distinct, contradiction to \( \mathcal{P} \) being locally finite. Thus, we can write \( C_R = \{ c_1, \ldots, c_n \} \) for some \( c_1, \ldots, c_n \in C \). Then

\[
\#(\mathcal{P} \cap \mathcal{RD}) = \# \left( (\mathcal{P}_x \cap \mathcal{RD}) \setminus \bigcup_{c \in C} (\mathcal{P}_x \cap c\mathcal{P}_x \cap \mathcal{RD}) \right)
\]

\[
= \#(\mathcal{P}_x \cap \mathcal{RD}) - \# \left( \bigcup_{i=1}^{n} (\mathcal{P}_x \cap c_i\mathcal{P}_x \cap \mathcal{RD}) \right),
\]

from which the result follows from the inclusion-exclusion counting formula for finite unions of finite sets.

A set \( C \) as in Proposition 4.1 for \( \mathcal{B} \) will be needed, and to this end a visibility condition for the elements of \( \mathcal{B} \) is required. Given \( x_1, x_2 \in \mathcal{O}_K \), let \( \gcd(x_1, x_2) \) be a fixed generator of the ideal generated by \( x_1, x_2 \) and write \( \gcd(x_1, x_2) = 1 \) when \( x_1, x_2 \) are relatively prime. In the following proposition a visibility condition of the complex realisation of the Ammann-Beenker point set given in [1, p. 477] is adapted to our situation.

**Proposition 4.2.** The visible points of \( \mathcal{B} \) are given by

\[
\hat{\mathcal{B}} = \{ x = (x_1, x_2) \in \mathcal{B} \mid \gcd(x_1, x_2) = 1, \lambda\mathcal{P} \notin \mathcal{W}_A \}.
\]

**Proof.** First the necessity of the visibility conditions is established. Take \( x = (x_1, x_2) \) and suppose that \( \gcd(x_1, x_2) \neq 1 \) so that there exists \( c \in \mathcal{O}_K \) with \( |N(c)| > 1 \) and \( c \mid x_1, x_2 \). Scaling \( c \) by units we may assume that \( 1 < c < \lambda \). Suppose first that \( |N(c)| = |\mathcal{P}|c \geq 3 \), which implies \( |c| > 1 \). By noting that \( \mathcal{W}_A \) is star-shaped with respect to the origin and \( \mathcal{W}_A = -\mathcal{W}_A \) it follows that \( x/c \in \mathcal{B} \), so \( x \) is invisible. If \( |N(c)| = 2 \), then each prime factor of \( c \) must divide \( 2 = \sqrt{2} \cdot \sqrt{2} \), so it can be assumed that \( c = \sqrt{2} \) and hence \( x \) is occluded by \( x/\sqrt{2} \). If \( \lambda\mathcal{P} \in \mathcal{W}_A \) it follows immediately that \( x/\lambda \in \mathcal{B} \).

We now turn to the sufficiency of the visibility conditions. Take \( x = (x_1, x_2) \in \mathcal{B} \setminus \hat{\mathcal{B}} \) and \( c > 1 \) such that \( x/c \in \mathcal{B} \). As \( \mathcal{B} \) is uniformly discrete, we may assume that \( y := x/c \in \hat{\mathcal{B}} \). This implies, by necessity above, that \( \gcd(y_1, y_2) = 1 \). Now, since \( x_i = cy_i \) it follows that \( c \in \mathcal{K} \). Write \( c = a/b \) with \( a, b \in \mathcal{O}_K \) relatively prime. If \( b \) is not a unit, \( \gcd(y_1, y_2) = 1 \) is contradicted, hence \( c \in \mathcal{O}_K \).

If \( |N(c)| \neq 1 \) then \( \gcd(x_1, x_2) \neq 1 \). Otherwise, \( c > 1 \) is a unit, i.e. \( c = \lambda^k \) for some integer \( k > 0 \). Thus \( \frac{\mathcal{P}}{\lambda} = \frac{\mathcal{P}}{\lambda} \in \mathcal{W}_A \). Since \( \frac{1}{\lambda} = -\lambda \) we get \( (-\lambda)^k \mathcal{P} \in \mathcal{W}_A \) and thus also \( \lambda\mathcal{P} \in \mathcal{W}_A \). This establishes sufficiency of the visibility conditions.

**Remark.** Note that the proof works just as well for more general windows, that is, \( \mathcal{P}(\mathcal{W}, \mathcal{L}) = \{ x \in \mathcal{P}(\mathcal{W}, \mathcal{L}) \mid \gcd(x_1, x_2) = 1, \lambda\mathcal{P} \notin \mathcal{W} \} \) if \( \mathcal{W} \subset \mathbb{R}^2 \) is bounded with non-empty interior, star-shaped with respect to the origin and \( -\mathcal{W} = \mathcal{W} \).

Let now \( \mathbb{P} = \{ \pi \in \mathcal{O}_K \mid \pi \text{ prime}, 1 < \pi < \lambda \} \) and \( C = \mathbb{P} \cup \{ \lambda \} \) so that \( \mathbb{P} \) is a set that contains precisely one associate of every prime of \( \mathcal{O}_K \). Then we have the following proposition.

**Proposition 4.3.** For each \( x \in \mathcal{B} \setminus \hat{\mathcal{B}} \) there is \( c \in C \) such that \( x/c \in \mathcal{B} \).
Proof. Fix \( x \in B \setminus \hat{B} \). As seen in the proof of Proposition 4.2 there is \( c \in \mathcal{O}_K, c > 1 \), such that \( x/c \in B \). If \( c \) is not a unit, fix \( \pi \in \mathbb{P} \) so that \( \pi \mid c \). It can be verified by hand that \( \{ (x, \pi) \mid x \in \mathcal{O}_K \} \cap ((1, \lambda) \times (-1, 1)) = \emptyset \), hence \( |\pi| > 1 \) and \( x/\pi \in B \). If \( c \) is a unit, \( x/\lambda \in B \) is immediate. \( \square \)

Given a finite set \( F \subset \mathcal{O}_K \) let \( I_F \) be the (principal) ideal generated by the elements of \( F \) if \( F \neq \emptyset \) and \( I_F = \mathcal{O}_K \) otherwise. Let \( \ell_F \) denote a fixed least common multiple of \( F \), that is, a generator of the ideal \( \cap_{c \in F} c\mathcal{O}_K \). Let also \( m_F = \min\{1, \min_{c \in F} |c|\} \) and \( \mathcal{L}_F = \{(\ell_F x, \ell_F \overline{x}) \mid x \in \mathcal{O}_K^2\} \). Write \( I \triangleleft \mathcal{O}_K \) when \( I \subset \mathcal{O}_K \) is an ideal and define the absolute norm \( N(I) \) of \( I \) by \( |N(x)| \), where \( x \) is any generator of \( I \). Recall Dedekind’s zeta function \( \zeta_K(s) = \sum_{I \triangleleft \mathcal{O}_K} \frac{1}{N(I)^s} \) for \( s \in \mathbb{C} \) with Re\( (s) > 1 \).

Given a finite set \( F \subset C \) it is verified that \( B_s \cap \bigcap_{c \in F} cB_s = \{ (m_F \mathcal{W}_A, \mathcal{L}_F) \setminus \{0\} \}. \) For any \( R > 0 \) and bounded \( D \subset \mathbb{R}^2 \), Propositions 4.1, 4.3 imply that

\[
\#(B \cap \mathcal{B}) = \sum_{F \subset C} (-1)^{#F} \#(\{ (m_F \mathcal{W}_A, \mathcal{L}_F) \setminus \{0\} \} \cap RD).
\]

(1)

Since \( \ell_F \mathcal{O}_K^2 \subset \pi_{\text{im}}(\mathcal{L}_F) \subset \mathbb{R}^2 \) is dense we have

\[
\theta(\mathcal{P}(m_F \mathcal{W}_A, \mathcal{L}_F) \setminus \{0\}) = \frac{\text{vol}(m_F \mathcal{W}_A)}{\text{vol}(\mathcal{L}_F)}
\]

from [4 Prop. 3.2]. Dividing (1) by \( \text{vol}(RD) \), letting \( R \to \infty \) and switching order of limit and summation (to be justified in Proposition 4.6 below) we find that

\[
\theta(\hat{B}) = \sum_{F \subset C} (-1)^{#F} \frac{\text{vol}(m_F \mathcal{W}_A)}{\text{vol}(\mathcal{L}_F)} = \sum_{F \subset C} (-1)^{#F} m_F^2 (1 + \sqrt{2}) \frac{N(\ell_F)^2}{2N(\ell_F)^2},
\]

since \( \text{vol}(\mathcal{W}_A) = 4(1 + \sqrt{2}) \) and \( \text{vol}(\mathcal{L}_F) = 8N(\ell_F)^2 \). The value of the right hand sum will be shown to be \( 1/\zeta_K(2) \) in Theorem 4.7 below. The following lemma gives a bound on the number of points in the intersection of a lattice and a box in terms of the volume of the box, provided that the box is "not too thin".

**Lemma 4.4.** Let \( \mathcal{L} \subset \mathbb{R}^d \) be a lattice and let \( c > 0 \) be given. For any \( a_i, b_i \in \mathbb{R} \) with \( b_i - a_i > a \) set \( B = \prod_{i=1}^d [a_i, b_i] \). Then there is a constant \( L \) depending only on \( \mathcal{L} \) and \( c \) such that \( \#(B \cap \mathcal{L}) \leq L \text{vol}(B) \).

**Proof.** Let \( n_i = \lceil \frac{b_i - a_i}{c} \rceil \in \mathbb{Z}_+ \). Then \( \frac{b_i - a_i}{c} \leq n_i < \frac{b_i - a_i}{c} + 1 = \frac{b_i - a_i}{c} + \frac{c}{c} < 2\frac{b_i - a_i}{c} \). Hence, with \( n = \prod_{i=1}^d n_i \) it follows that \( n \leq 2^d \text{vol}(B) \) \( c \). From \( b_i \leq a_i + cn_i \) also \( B \subset \prod_{i=1}^d [a_i, a_i + cn_i] \). Let \( a = (a_1, \ldots, a_d) \) and consider \( -a + \prod_{i=1}^d [a_i, a_i + cn_i] = \prod_{i=1}^d [0, cn_i] \). We have \( \prod_{i=1}^d [0, cn_i] = \bigcup_{m \in \mathbb{N}^d, 0 \leq m < n} (mc + [0, c]^d) =: B' \). Find now \( D > 0 \) depending on \( \mathcal{L} \) and \( c \) such that \( \text{sup}_{x \in \mathbb{R}^d} \#(\mathcal{L} \cap (t + [0, c]^d)) = D \). Hence \( \#(B \cap \mathcal{L}) \leq \#(B' \cap \mathcal{L}) \leq nD \leq \frac{2^d \text{vol}(B)}{c} \), so one can take \( L = \frac{2^d \text{vol}(B)}{c} \).

The following bound will be crucial in the justification of interchanging limit and summation in (1) after division by \( \text{vol}(RD) \).

**Lemma 4.5.** Let \( D \subset \mathbb{R}^2 \) be Jordan measurable. Then there is a constant \( \tilde{L} > 0 \) depending only on \( D \) such that for every \( R > 0 \) and \( F \subset C \) with \( \#F < \infty \),

\[
\#((\mathcal{P}(m_F \mathcal{W}_A, \mathcal{L}_F) \cap RD) \setminus \{0\}) \leq \frac{\tilde{L} R^2}{N(\ell_F)^2}.
\]
Proof. By definition

\[ \#((P(m_F W_A, \mathcal{L}_F) \setminus \{0\}) \cap RD) = \#((x \in \ell_F \mathcal{O}_K^2 \mid x \in m_F W_A \setminus \{0\}) \cap RD). \]

Note that this number is independent of the choice of \( \ell_F \). There is a bijection

\[ \left(\{x \in \ell_F \mathcal{O}_K^2 \mid x \in m_F W_A \setminus \{0\}\} \cap RD\right) \rightarrow \left(\{x \in \ell_F \mathcal{O}_K^2 \mid x \in m_F W_A \setminus \{0\}\} \cap RD\right) \]

given by \( x \mapsto \frac{x}{\ell_F} \), so it suffices to estimate the number of elements in the latter set. Since \( m_F \leq 1 \) it follows that \( \left(\mathcal{L} \cap \left(R \mathcal{D} \times \frac{m_F W_A}{\ell_F}\right)\right) \setminus \{0\} \subset \left(\mathcal{L} \cap \left(R \mathcal{D} \times \frac{W_A}{\ell_F}\right)\right) \setminus \{0\} \). Fix real numbers \( m_1, m_2 > 1 \) so that \( D \subset [-m_1, m_1]^2 =: B_1 \) and \( W_A \subset [-m_2, m_2]^2 =: B_2 \).

Fix a number \( c \) so that \( c' < c \) implies \( (\mathcal{L} \cap (\lambda D \times c'W_A)) \setminus \{0\} = \emptyset \). This can be done, for otherwise \( (\mathcal{L} \cap (\lambda D \times c'W_A)) \setminus \{0\} \) would be non-empty for each \( c' > 0 \), hence \( \mathcal{L} \cap (\lambda D \times W_A) \) would contain infinitely many points, contradiction, since \( \mathcal{L} \) is a lattice and \( \lambda D \times W_A \) is bounded.

Suppose first that \( \frac{1}{\sqrt{\ell_F}} < c \). Scale \( \ell_F \) by units so that \( 1 \leq \frac{R}{\ell_F} < \lambda \) which gives \( \frac{1}{\sqrt{\ell_F}} < c \). Hence \( \left(\mathcal{L} \cap \left(R \mathcal{D} \times \frac{W_A}{\ell_F}\right)\right) \setminus \{0\} \subset \left(\mathcal{L} \cap \lambda D \times \left(W_A\right)\right) \setminus \{0\} = \emptyset \) and therefore

\[ \# \left(\left(\mathcal{L} \cap \left(R \mathcal{D} \times \frac{W_A}{\ell_F}\right)\right) \setminus \{0\}\right) = 0. \]

Suppose now that \( \frac{R}{\ell_F} \geq \frac{1}{\sqrt{\ell_F}} \). Scale \( \ell_F \) so that \( \sqrt{c} \leq \frac{R}{\ell_F} < \lambda \sqrt{c} \). This implies that \( \frac{1}{\ell_F} \geq \frac{1}{\lambda} \sqrt{c} > \sqrt{c} \). Thus, \( \mathcal{O} \supset \frac{R \mathcal{D}}{\ell_F} \times \frac{W_A}{\ell_F} =: B \). From Lemma 4.4 we get a constant \( L \) only depending on \( \mathcal{L} \), \( \sqrt{c} \) such that \( \#(B \cap \mathcal{L}) \leq L\text{vol}(B) = L \cdot 16m_1^2m_2^2 \frac{R^2}{N(\ell_F)^2} \). Now, since \( \frac{R}{\ell_F} \times \frac{W_A}{\ell_F} \subset B \) we get that

\[ \# \left(\left(\mathcal{L} \cap \left(R \mathcal{D} \times \frac{W_A}{\ell_F}\right)\right) \setminus \{0\}\right) \leq \frac{L R^2}{N(\ell_F)^2} \]

with \( L := 16m_1^2m_2^2L \).

\[ \square \]

**Proposition 4.6.** The equality

\[ \lim_{R \to \infty} \sum_{\begin{subarray}{c} F \subset C \\#F < \infty \\ \#F < \infty \end{subarray}} \frac{(-1)^\#F \#((P(m_F W_A, \mathcal{L}_F) \setminus \{0\}) \cap RD)}{\text{vol}(RD)} = \sum_{\begin{subarray}{c} F \subset C \\#F < \infty \\ \#F < \infty \end{subarray}} \frac{(-1)^\#F m_F^2 (1 + \sqrt{2})}{2N(\ell_F)^2} \]

holds for all Jordan measurable \( D \subset \mathbb{R}^2 \).

**Proof.** For a finite \( F \subset C \) let \( N(R, F) = \#((P(m_F W_A, \mathcal{L}_F) \setminus \{0\}) \cap RD) \). We know that

\[ \lim_{R \to \infty} \frac{N(R, F)}{\text{vol}(RD)} = \frac{m_F^2 (1 + \sqrt{2})}{2N(\ell_F)^2} \]

so

\[ \lim_{R \to \infty} \sum_{\begin{subarray}{c} F \subset C \\#F < \infty \\ \#F < \infty \end{subarray}} \frac{(-1)^\#F N(R, F)}{\text{vol}(RD)} = \sum_{\begin{subarray}{c} F \subset C \\#F < \infty \\ \#F < \infty \end{subarray}} \lim_{R \to \infty} \frac{(-1)^\#F N(R, F)}{\text{vol}(RD)} \]

must be justified. In view of Lemma 4.5

\[ \sum_{\begin{subarray}{c} F \subset C \\#F < \infty \\ \#F < \infty \end{subarray}} \left| \frac{(-1)^\#F N(R, F)}{\text{vol}(RD)} \right| \leq \frac{\tilde{L}}{\text{vol}(D)} \sum_{\begin{subarray}{c} F \subset C \\#F < \infty \\ \#F < \infty \end{subarray}} \frac{1}{N(\ell_F)^2} \]

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and we note that
\[
\sum_{F \in \mathcal{C}} \frac{1}{N(\ell_F)^2} = \sum_{F \in \mathcal{C}, \#F < \infty} \frac{1}{N(\ell_F)^2} + \sum_{F \in \mathcal{C}, \#F < \infty, \lambda \notin F} \frac{1}{N(\ell_F)^2} \leq 2 \sum_{I \in \mathcal{O}_K} \frac{1}{N(I)^2},
\]
hence the sums of both sides of (2) are absolutely convergent.

Fix \( \Delta > 0 \). We claim that there is only a finite number of non-empty \( F \subset C, \#F < \infty, \) such that \( |N(\ell_F)| < \Delta \). Given such \( F \) let \( \ell_F = \prod_{c \in F} c \). Also, since \( |\pi| > 1 \) for all \( \pi \in \mathbb{P} \) we have \( |\ell_F| \geq |\lambda| \). Hence, \( |N(\ell_F)| = \ell_F|\ell_F| \leq \Delta \) implies \( \ell_F \leq \frac{\lambda}{|\ell_F|} \leq \lambda\Delta \) and \( |\ell_F| \leq \frac{\lambda}{|\ell_F|} \leq \Delta < \lambda\Delta \) so \( (\ell_F, \ell_F) \in \{(x, \pi) \mid x \in \mathcal{O}_K\} \cap \lambda[-\Delta, \Delta]^2 \) which is a finite set, thus elements of \( F \) can only contain prime factors that occur as factors in the components of elements in this finite set, giving only finitely many possibilities for \( F \).

It follows that
\[
\lim_{R \to \infty} \left| \sum_{F \in \mathcal{C}, \#F < \infty} \frac{(-1)^{\#F} N(R, F)}{\text{vol}(RD)} \right| \leq \left( \frac{L}{\text{vol}(D)} + \frac{1 + \sqrt{2}}{2} \right) \sum_{F \in \mathcal{C}, \#F < \infty} \frac{1}{|N(\ell_F)| \geq \Delta} \frac{m_F^2 (1 + \sqrt{2})}{2N(\ell_F)^2}
\]
where the right hand side tends to 0 as \( \Delta \to \infty \) since \( \sum_{F \in \mathcal{C}, \#F < \infty, |N(\ell_F)| \geq \Delta} \frac{1}{N(\ell_F)^2} \) is the tail of an absolutely convergent sum, hence (2) has been justified.

From Proposition 4.6 it follows that \( \theta(\tilde{B}) = \sum_{F \in \mathcal{C}, \#F < \infty} \frac{(-1)^{\#F} m_F^2 (1 + \sqrt{2})}{2N(\ell_F)^2} \), and it will now be shown that the right hand side is equal to \( 1/\zeta_K(2) \). Define the function \( \omega : \mathcal{O}_K \to \mathbb{C}, \omega(x) = \#\{\pi \in \mathbb{P} \mid x/\pi \in \mathcal{O}_K\} \), so that \( \omega(x) \) is the number of non-associated prime divisors of \( x \). Given \( I \triangleleft \mathcal{O}_K \), let \( \omega(I) = \omega(x) \) for any generator \( x \) of \( I \) and define a Möbius function on the ideals of \( \mathcal{O}_K \) by
\[
\mu(I) = \begin{cases} 0 & \text{if } \exists \pi \in \mathbb{P} \text{ such that } I \subset \pi^2 \mathcal{O}_K, \\ (-1)^{\omega(I)} & \text{otherwise}. \end{cases}
\]
One verifies that \( \mu(I_1 I_2) = \mu(I_1) \mu(I_2) \) for relatively prime ideals \( I_1, I_2 \). The function \( \zeta_K \) can be expressed as an Euler product for \( s \) with \( \text{Re}(s) > 1 \) as
\[
\zeta_K(s) = \prod_{P \in \mathcal{O}_K, P \text{ prime}} \frac{1}{1 - N(P)^{-s}}
\]
and in analogy with the reciprocal formula for Riemann’s zeta function we have
\[
\frac{1}{\zeta_K(s)} = \sum_{I \in \mathcal{O}_K} \frac{\mu(I)}{N(I)^s}.
\]

**Theorem 4.7.** The density of visible points of \( \mathcal{B} \) is given by
\[
\theta(\mathcal{B}) = \frac{1}{\zeta_K(2)} = \frac{48\sqrt{2}}{\pi^4}.
\]
\begin{proof}
By Proposition 4.6 we have

\[ \theta(\mathcal{B}) = \sum_{\substack{F \in \mathcal{C} \\#Fc \in \mathcal{C}}} \frac{(-1)^{\#F}m_F^2(1 + \sqrt{2})}{2N(l_F)^2}. \]

Splitting the sum into two depending on whether \( \lambda \in F \) or not, and using that \( m_F = 1 \) unless \( \lambda \in F \), in which case \( m_F = |\lambda| = \sqrt{2} - 1 \), we get

\[ \theta(\mathcal{B}) = \sum_{\substack{F \in \mathcal{C} \\#Fc \in \mathcal{C}}} \frac{(-1)^{\#F}(1 + \sqrt{2})}{2N(l_F)^2} + \sum_{\substack{\lambda \in \mathcal{C} \\#Fc \in \mathcal{C}}} \frac{(-1)^{\#F}|\lambda|^2(1 + \sqrt{2})}{2N(l_F)^2} \]

\[ = \frac{(1 - |\lambda|)(1 + \sqrt{2})}{2} \sum_{I \in \mathcal{O}_K} \mu(I) \frac{N(I)^2}{N(I)^2} = \frac{1}{\zeta_K(2)}, \]

last equality by (3). From [8, Theorem 4.2] one can calculate \( \zeta_K(-1) = \frac{1}{12} \) and by the functional equation for Dedekind’s zeta function (cf. e.g. [8, p. 34]) one finds that \( \zeta_K(2) = \frac{\pi^4}{48\sqrt{2}} \) which proves the claim. \( \square \)

4.2 The density of visible points of \( \mathcal{A} \)

Observe that \( \mathcal{A}' = \sqrt{2} \mathcal{A} \subset \mathcal{B} \). It is now shown that \( C \) is also an occluding set for \( \mathcal{A}' \).

Proposition 4.8. For each \( x \in \mathcal{A}' \setminus \hat{\mathcal{A}} \) there is \( c \in C \) such that \( x/c \in \mathcal{A}' \).

\begin{proof}
Since \( \mathcal{A}' \subset \mathcal{B} \) we have \( \mathcal{A}' \setminus \hat{\mathcal{A}} \subset \mathcal{B} \setminus \hat{\mathcal{B}} \) and so for each \( x \in \mathcal{A}' \setminus \hat{\mathcal{A}} \) there exists \( c \in C \) such that \( x/c \in \mathcal{B} \). If \( c \neq \sqrt{2} \) then \( \sqrt{2} \mid \frac{a}{c} \) so \( x/c \in \mathcal{A}' \).

Take now \( x \in \mathcal{A}' \setminus \hat{\mathcal{A}} \) such that for all \( c \in C \setminus \{\sqrt{2}\} \) we have \( x/c \notin \mathcal{B} \). Then \( x/\sqrt{2} \in \mathcal{B} \), hence \( \gcd(x_1, x_2) = \sqrt{2} \) for some \( n \geq 1 \). Since \( x \in \mathcal{A}' \setminus \hat{\mathcal{A}} \) there is \( c \in \mathbb{Q}(\sqrt{2}) \cap \mathbb{R}_{>1} \) such that \( x/c \in \mathcal{A}' \). Writing \( c = a/b \) with \( \gcd(a, b) = 1 \), the only possible \( \pi \in \mathbb{P} \) with \( \pi \mid a \) is \( \pi = \sqrt{2} \). If \( \sqrt{2} \mid a \), then it follows that \( x/\sqrt{2} \in \mathcal{A}' \).

It remains to check the case where \( a \) is a unit, i.e. \( c = \frac{\lambda}{\prod_{\pi \in \mathbb{P} \mid a} \pi^{m_\pi}} \) for some \( m : \mathbb{P} \to \mathbb{Z}_{\geq 0} \) with finite support. The facts that \( c > 1 \) and \( \pi > 1 \) for all \( \pi \in \mathbb{P} \) imply \( n > 0 \). We have \( x/\sqrt{2} \notin \mathcal{B} \), hence \( \pi \notin [\lambda]\mathcal{W}_\mathcal{A} \). Since \( x/c \in \sqrt{2} \mathcal{A} \) it follows that \( \pi \in [\pi]\mathcal{W}_\mathcal{A} \) and hence \( |\pi| > |\lambda| \). However

\[ |\pi| = \frac{|\lambda|^n}{\prod_{\pi \in \mathbb{P} \mid \pi} |\pi|^{k(\pi)}} \leq |\lambda|^n \leq |\lambda|, \]

contradiction. \( \square \)

Theorem 4.9. We have \( \theta(\mathcal{A}') = \frac{1}{2\zeta_K(2)} \), hence \( \theta(\mathcal{A}) = \frac{1}{\zeta_K(2)} \).

\begin{proof}
Propositions 4.1, 4.8 imply

\[ \frac{\#(\hat{\mathcal{A}} \cap RD)}{\text{vol}(RD)} = \sum_{\substack{F \in \mathcal{C} \\#Fc \in \mathcal{C}}} \frac{(-1)^{\#F} \#((\mathcal{A}' \cap \mathcal{C}_F \mathcal{A}') \cap RD)}{\text{vol}(RD)} \]

(4)
and it is straightforward to verify that $A'_c \cap \bigcap_{c \in F} c A'_c = \mathcal{P}(m_F W_A, \tilde{L}_F) \setminus \{0\}$ with $\tilde{L}_F = \{(\ell_F x, \ell_F x) \mid x \in \mathcal{O}_K^2, (x_1 - x_2) / \sqrt{2} \in \mathcal{O}_K\}$ a sublattice of $L_F$ of index 2. Hence, by [1] Prop. 3.2, when letting $R \to \infty$ inside the sum (4) one obtains
\[
\sum_{F \subset C} \frac{(-1)^F \vol(m_F W_A)}{16N(\ell_F)},
\]
whence $\theta(\sqrt{2} A) = \frac{1}{2\zeta K(2)}$ follows by Proposition 4.6 and Theorem 4.7 and the other result is immediate as $\sqrt{2} A = A'$.

\[\square\]

Remark. The data of Table 2 of [1] shows that $\#(\hat{A} \cap RD) / \#(A \cap RD) \approx 0.577$ for a particular $D$ and fairly large $R$. This agrees with our results, since
\[
\kappa_A = \lim_{R \to \infty} \frac{\#(\hat{A} \cap RD)}{\#(A \cap RD)} = \frac{\frac{1}{2}\zeta K(2)}{2\vol(W_A) = 2(\sqrt{2} - 1)} = 0.5773\ldots
\]

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