Photon production by relativistic electrons in plasmas

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Abstract. A non-equilibrium Green’s function approach to collisional QED effects in a relativistic beam-plasma medium is presented. Taking into account off-shell parts of the electron correlation functions and vertex corrections to the self-energies, the Bremsstrahlung production rate is derived. The medium effects are included by the full dynamic structure factor of the bulk plasma. The “golden rule” result for the production rate with the Bethe-Heitler cross section is recovered for the model of static randomly distributed ions.

1. Introduction
Progress in laser-plasma experiments and especially elaboration of high-intensity laser-driven schemes for the inertial confinement fusion \cite{1} have inspired a renewed interest in the theory of relativistic phenomena in plasmas. Laser pulses with intensities $10^{19} - 10^{20}$ W/cm\textsuperscript{-2} are now available to drive relativistic beams with electron energies 10 – 300 Mev \cite{2,3}. When penetrating into the plasma, the electron beam generates a return current carried by the plasma electrons. Since the dimensionless beam density $\alpha = n_b/n_p$ varies from $10^{-3}$ (the dense core plasma) to $10^{-1}$ (the plasma corona), the common belief is that the return current velocity $V_p$ is nonrelativistic, which is easily seen from the current neutralization condition $n_p V_p = n_b V_b$. However, recent particle-in-cell simulations \cite{4} show that, for high pulse intensities, the complete current neutralization is violated and the plasma electrons in the beam region become relativistic.

At present, the topic of theoretical concentration in relativistic electron-beam plasma interaction is the study of physical mechanisms responsible for laser acceleration of self-injected electrons in a plasma channel \cite{5,6,7}, collective stopping, ion heating \cite{8}, and collective beam-plasma instabilities \cite{9,10}. Existing quantum kinetic approaches to processes responsible for the energy transfer from laser field to matter are restricted to the nonrelativistic case (see, e.g., \cite{11,12} and the references therein). However, in the presence of an intense relativistic beam, one deals with a medium composed of highly relativistic beam electrons, relativistic (or nonrelativistic) electrons in the plasma return current, and ions\textsuperscript{1}. A consistent theory of collisional and radiative phenomena in such a system must be based on many-particle QED.

A general and systematic Green’s function approach to nonequilibrium QED plasmas was developed by Bezzerides and DuBois \cite{16}. Within the weak coupling approximation, they were

\textsuperscript{1} In laser-plasma experiments multi-Mev protons and ions can be produced \cite{13,14,15}. Nevertheless, even in this case the ion velocities are much less than the speed of light.
able to derive a covariant particle kinetic equation including electron-electron collisions and Cherenkov radiation of plasmons. One should be reminded that transverse photons in a plasma cannot be emitted or absorbed by particles in energy-momentum conserving Cherenkov processes because the phase velocity of transverse photons is always greater than the speed of light. Therefore, QED radiative phenomena in a beam-plasma medium are related to higher-order processes, such as Bremsstrahlung and Compton scattering. These processes may contribute to stopping power for highly relativistic electron beams, but they are interesting in themselves as an example of fundamental QED processes in a medium [17]. Note also that measurement of the angular distribution of Bremsstrahlung γ rays has been found to be a powerful diagnostic tool in laser-plasma experiments [18].

In this paper we present a Green’s function approach to the calculation of photon production rate in a relativistic beam-plasma medium. The essential point in the study of radiative QED processes is that off-shell terms in the field and particle correlation functions must be treated consistently. These terms describe virtual processes and contribute to the scattering rates. In the so-called “extended quasiparticle approximation” [19, 20] the off-shell corrections are included to lowest order in the quasiparticle damping width. We shall follow, in the main, the scheme proposed by Špička and Lipavský [21, 22], which allows one to go beyond the extended quasiparticle approximation by using the full spectral functions. It is important, for instance, if the dynamical screening effects have to be taken into account.

Since ions are nonrelativistic in the rest frame of the system, it is convenient to choose the Coulomb gauge. We use the system of units with $c = \hbar = 1$ and the Heaviside’s units for electromagnetic field, i.e., the Coulomb interaction is written as $qq' / 4\pi r$. The signature of the metric tensor $g^{\mu\nu}$ is $(+, -, -, -)$. Cartesian components of three-dimensional vectors will be denoted by subscripts: $V_i$, where $i = 1, 2, 3$. With this convention, a four-vector has the components $V^\mu = (V^0, V_i)$ and $V_\mu = (V^0, -V_i)$. For γ-matrices the common notation $\gamma^\mu = (\gamma^0, \gamma^i)$ is used. Summation over repeated spatial and space-time (Greek) indices is implied. Our convention for the matrix Green’s functions on the time-loop Schwinger-Keldysh contour $C$ follows Botermans and Malfliet [23].

2. Kinetic equation for photons in a plasma

In the Coulomb gauge, photon kinetics can be formulated in terms of the transverse field Green’s function defined on the time-loop contour $C$:

$$D_{ij}^T(12) = -i\left\{ T_C \hat{A}_i(1) \hat{A}_j(2) - A_i(1) A_j(2) \right\},$$

(1)

where $A_i = \langle \hat{A} \rangle$ is the ensemble average of the vector potential ($\nabla \cdot \hat{A} = 0$), and $T_C$ is the path-ordering operator on $C$. An underlined variable $\underline{k} = (\underline{r}_k, t_k)$ indicates that $\underline{r}_k$ lies on $C$. The notation $(k) = (\underline{r}_k, t_k)$ will be used for space-time variables. Correlation functions $d_{ij}^r(12)$ and the retarded/advanced photon propagators $d_{ij}^{r/a}(12)$ are obtained from $D_{ij}^T(12)$ as the contour components.

If the direct coupling between transverse and longitudinal field fluctuations is neglected, the $D_{ij}^T$ obeys the Dyson equation (intermediate “primed” variables are integrated over $C$)

$$\Box_1 D_{ij}^T(12) = \delta_{ij}^T(1 - 2) + \Pi_{ik}^T(11') D_{kj}^T(1' 2),$$

(2)

where $\Box = -\partial^2 / \partial t^2 + \nabla^2$ is the wave operator, $\delta_{ij}^T(1 - 2)$ is the transverse delta function on $C$, and $\Pi_{ik}^T(11')$ is the transverse polarization matrix. The polarization functions $\pi_{ij}^T(12)$ and $\pi_{ij}^{r/a}(12)$ are obtained as the contour components of $\Pi_{ik}^T(12)$. Following the procedure detailed in Refs. [16, 23], the equations of motion for the photon propagators and the Kadanoff-Baym (KB)
functions
d
no reason to do this because the field correlation functions must satisfy
with the Coulomb gauge. The mass-shell equation was ignored in Ref. [16]. However, we have
by Bezzerides and DuBois [16] when the photon polarization states are chosen in accordance
Equation (4) may be regarded as a particular case of the gauge-invariant kinetic equation derived
with the four-dimensional Poisson bracket
term
{\text{virtual photons do not propagate. Substituting (7) into Eq. (4) and requiring that the drift
then required that only the drift term
{\text{virtual (off-shell)}
and the "mass-shell" parts of the field correlation functions
\text{to the plasma radiation. It is thus necessary to separate the resonant ("quasiparticle") and "off-
\text{possible. These photons behave as well-defined quasiparticles and contribute
\text{weakly photons outside the system. Second, for sufficiently high frequencies, propagation of \text{weakly
damping photons is possible. These photons behave as well-defined quasiparticles and contribute
to the plasma radiation. It is thus necessary to separate the resonant ("quasiparticle") and "off-
shell" parts of the field correlation functions } \Delta d_s. \text{ To do this, we write the field correlation
functions as a sum [21, 22]
\begin{align*}
\Delta d_s^\pm (X, k) &= \pi_s^\pm (X, k) + \Delta d_s^\pm (X, k),
\end{align*}
where the first term is regarded as the resonant (quasiparticle) contribution and the second term
represents the off-shell part which is identified as the contribution from virtual photons\textsuperscript{3}. It is
then required that only the drift term \{k^2 - \text{Re} \pi_s^+, \Delta d_s^+\} may enter the kinetic equation since
virtual photons do not propagate. Substituting (7) into Eq. (4) and requiring that the drift
term \{k^2 - \text{Re} \pi_s^+, \Delta d_s^+\} cancels the analogous term coming from \{\text{Re} d_s^+, \pi_s^\pm\}, one finds the
expression for the off-shell part:
\begin{align*}
\Delta d_s^\pm = \frac{\pi_s^\pm}{2} \left[(d_s^\pm)^2 + (\pi_s^\pm)^2\right] = \pi_s^\pm \text{Re} (d_s^\pm)^2.
\end{align*}
\textsuperscript{2} A detailed discussion of the relationship between transport and mass-shell equations in quantum kinetics may be
found, e.g., in Ref. [24].
\textsuperscript{3} Obviously this interpretation makes sense only for those \textbf{k} which correspond to weakly damping photon modes.

\begin{align*}
d_s^\pm (X, k) &= \frac{1}{k^2 - \text{Re} \pi_s^\pm (X, k) \pm i k_0 \Gamma_s (X, k)}, \\
\Gamma_s (X, k) &= -k_0^{-1} \text{Im} \pi_s^+ (X, k),
\end{align*}

where \(k^2 = \mu k \mu = k_0^2 - \mu^2\). Taking the sum and difference of the KB equations for the
correlation functions \(d_s^\pm (X, k)\), one obtains the transport equation
\begin{align*}
\begin{cases}
k^2 - \text{Re} \pi_s^+, d_s^\pm \&= \left\{ \text{Re} d_s^+, \pi_s^\pm \right\} = i \left( \pi_s^+ d_s^\pm - \pi_s^\pm d_s^\mp \right) \\
\end{cases}
\end{align*}

and the “mass-shell” equation
\begin{align*}
\{k_0 \Gamma_s, d_s^\pm\} - \left\{ \text{Im} d_s^+, \pi_s^\pm\right\} &= -2 \left(k^2 - \text{Re} \pi_s^+\right) \left(d_s^\pm - \frac{1}{2} \pi_s^\pm\right)
\end{align*}

with the four-dimensional Poisson bracket
\begin{align*}
\{F_1(X, k), F_2(X, k)\} = \frac{\partial F_1}{\partial X^{\mu}} \frac{\partial F_2}{\partial k_\mu} - \frac{\partial F_1}{\partial k_\mu} \frac{\partial F_2}{\partial X^{\mu}}.
\end{align*}

Equation (4) may be regarded as a particular case of the gauge-invariant kinetic equation derived
by Bezzerides and DuBois [16] when the photon polarization states are chosen in accordance
with the Coulomb gauge. The mass-shell equation was ignored in Ref. [16]. However, we have
no reason to do this because the field correlation functions must satisfy both\textsuperscript{2} equations. The
mass-shell equation is in a sense a constraint for approximations in the transport equation\textsuperscript{2}.

Note that the physical meaning of Eqs. (4) and (5) remains to be seen because the correlation
functions \(d_s^\pm\) describe different processes involving transverse photons. First, \textit{virtual (off-shell)
photons} are responsible for the interaction between particles and cannot be detected as real
photons outside the system. Second, for sufficiently high frequencies, propagation of \textit{weakly
damping photons} is possible. These photons behave as well-defined quasiparticles and contribute
to the plasma radiation. It is thus necessary to separate the resonant ("quasiparticle") and "off-
shell" parts of the field correlation functions \(\Delta d_s\). To do this, we write the field correlation
functions as a sum [21, 22]
\begin{align*}
\begin{cases}
d_s^\pm (X, k) &= \hat{d}_s^\pm (X, k) + \Delta d_s^\pm (X, k),
\end{cases}
\end{align*}

where the first term is regarded as the resonant (quasiparticle) contribution and the second term
represents the off-shell part which is identified as the contribution from virtual photons\textsuperscript{3}. It is
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expression for the off-shell part:
\begin{align*}
\Delta d_s^\pm = \frac{\pi_s^\pm}{2} \left[(d_s^\pm)^2 + (\pi_s^\pm)^2\right] = \pi_s^\pm \text{Re} (d_s^\pm)^2.
\end{align*}
It is natural to define the phase space distribution function for resonant photons, \( \tilde{N}_s(X,k) \), through the relations
\[
\tilde{a}_s^\prec = -i \tilde{a}_s \tilde{N}_s, \quad \tilde{a}_s^\succ = -i \tilde{a}_s \left( 1 + \tilde{N}_s \right),
\] (9)
where \( \tilde{a}_s(X,k) = i(\tilde{d}_s^\succ - \tilde{d}_s^\prec) \) is the resonant spectral function. Using Eqs. (7) and (8) together with expression (3) for the propagators, one obtains
\[
\tilde{a}_s(X,k) = \frac{4 (k_0 \Gamma_s)^3}{\left[ (k^2 - \text{Re} \, \pi_s^\perp)^2 + (k_0 \Gamma_s)^2 \right]^2}.
\] (10)

With the aid of (7) – (10), Eq. (4) is transformed into the kinetic equation
\[
\tilde{a}_s \left[ \left\{ k^2 - \text{Re} \, \pi_s^\perp, \tilde{N}_s \right\} - \frac{k^2 - \text{Re} \, \pi_s^\perp}{k_0 \Gamma_s} \left\{ k_0 \Gamma_s, \tilde{N}_s \right\} - i \left( \pi_s^\perp \tilde{N}_s - \pi_s^\perp \left( 1 + \tilde{N}_s \right) \right) \right] = 0.
\] (11)
In terms of the phase space distribution function, the mass-shell equation (5) reads
\[
(k^2 - \text{Re} \, \pi_s^\perp) \tilde{a}_s \left[ \ldots \right] = 0
\] (12) with the same expression in square brackets as in Eq. (11). We see that equations (11) and (12) are equivalent to each other. This confirms the self-consistency of the scheme.

For weakly damping photons, the resonant spectral function \( \tilde{a}_s \) is a sharply peaked function of \( k^0 \) near the photon frequency \( \omega_s(X,k) \) which is found as a solution of the dispersion equation
\[
k^2 - \text{Re} \, \pi_s^\perp(X,k) = 0.
\] (13)
For definiteness, it will be assumed that the effective photon frequencies are positive solutions of Eq. (13). If \( k^0 = \omega_s(k) \) is such a solution, then, using the property \( [\pi_s^\perp(k)]^\ast = \pi_s^\perp(-k) \), it is easy to show that the corresponding negative solution is \( k^0 = -\omega_s(-k) \).

The “quasiparticle” approximation for resonant photons means that the spectral function \( \tilde{a}_s \) is taken in the limit of negligible damping,
\[
\tilde{a}_s(X,k) = 2\pi \eta(k^0) \delta \left( k^2 - \text{Re} \, \pi_s^\perp \right),
\] (14)
where \( \eta(k^0) = k^0/|k^0| \). In this approximation, Eq. (11) can be converted into a kinetic equation for the “on-shell” distribution function
\[
N_s(X,k) = \tilde{N}_s(X,k) \big|_{k^0=\omega_s(X,k)}.
\] (15)

Integrating Eq. (11) over \( k^0 > 0 \) and recalling that \( X^\mu = (T, R) \), we get
\[
\left( \frac{\partial}{\partial T} + \frac{\partial \omega_s}{\partial k} \cdot \frac{\partial}{\partial R} - \frac{\partial \omega_s}{\partial R} \cdot \frac{\partial}{\partial k} \right) N_s(X,k)
= i \ Z_s(X,k) \left[ \pi_s^\perp(X,k) \left( 1 + N_s(X,k) \right) - \pi_s^\perp(X,k) \right] \ N_s(X,k) \big|_{k^0=\omega_s},
\] (16)
where \( \pi_s^\perp(X,k) = \pi_s^\perp(X,k) \big|_{k^0=\omega_s} \), and \( Z_s \) is the “wave-function renormalization”,
\[
Z^{-1}(X,k) = \left. \frac{\partial}{\partial k^0} \left( k^2 - \text{Re} \, \pi_s^\perp(X,k) \right) \right|_{k^0=\omega_s}.
\] (17)
Kinetic equation (16) is well known and has been used in many works, although the interpretation of the on-shell distribution function \( N_s(X, k) \) is not always correct. We have given the above derivation to show the appearance of the important “by-product”, the term (8) in the field correlation functions. As argued by Špička and Lipavský [21, 22], such terms represent short-time contributions to the correlation functions and must be taken into account in calculating the scattering rates. In our case, the terms \( \Delta d^2_s \) describe virtual photons, whereas the terms \( \tilde{d}^2_s \) correspond to the resonant (propagating) photons. This line of reasoning differs from that for longitudinal field fluctuations (plasmons) discussed by Bezzerides and DuBois [16]. For plasmon modes, the emission and absorption rates are not necessarily small as is the situation with propagating transverse photons. Therefore, it is more natural to decompose the longitudinal field correlation functions into “adiabatic” and “non-adiabatic” parts [16]. The adiabatic part is found from a local detailed balance between the emission and absorption of plasmons and enters the collision integrals for particles. The non-adiabatic part is attributed to unstable plasma modes.

3. Transverse polarization functions
To calculate the polarization functions \( \pi_s^x(X, k) \) that enter the gain and loss terms in Eq. (16), we start with the general expression for the transverse polarization matrix \( \Pi^T_{ij}(1 \; 2) \) in terms of the particle Green’s functions and the vertex functions [25]. In all physically interesting situations the transverse ion current is very small compared to the electron current, so that we have

\[
\Pi^T_{ij}(1 \; 2) = \delta^T_{ik}(1 - 1') \text{tr} \left\{ \gamma^k G(1' \; 2') \Gamma^T_{j}(2' \; 3'; 2) G(3' \; 1') \right\},
\] (18)

where

\[
G(1 \; 2) = -i \langle T_C \psi(1) \bar{\psi}(2) \rangle \quad \text{(19)}
\]

is the electron (positron) Green’s function on the time-loop contour \( C \) and the trace is taken over spinor indices. The transverse vertex functions \( \Gamma^T_{j} \) are given by

\[
\Gamma^T_{i}(1 \; 2; 3) = -ie^2 \delta(1 - 2) \delta^T_{ik}(1 - 3) \gamma^k + ie \delta \Sigma(1 \; 2) / \delta A_i(3),
\] (20)

where \( \Sigma(1 \; 2) \) is the electron matrix self-energy. For a weakly coupled plasma, only the lowest order corrections to the vertex functions may be retained. Graphically, this reads

\[
\Gamma^T_{i}(1 \; 2; 3) = \begin{array}{c}
\includegraphics[width=3cm]{diagram1}
\end{array} + \begin{array}{c}
\includegraphics[width=3cm]{diagram2}
\end{array} + \begin{array}{c}
\includegraphics[width=3cm]{diagram3}
\end{array}
\] (21)

The solid lines correspond to electrons (\( G \)-functions), wavy lines to transverse photons (\( D^T \)-functions), and dashed lines to longitudinal field fluctuations which are described by Green’s function

\[
D(1 \; 2) = i \left\{ \langle T_C \hat{\phi}(1) \hat{\phi}(2) \rangle - \langle \hat{\phi}(1) \phi(2) \rangle \right\} - \delta(t_1 - t_2)/4\pi |r_1 - r_2|,
\] (22)

where \( \hat{\phi}(1) \) is the scalar potential operator, and \( \phi(1) = \langle \hat{\phi}(1) \rangle \). Finally, triangles and dots in Eq. (21) correspond respectively to the transverse bare vertices [see (20)] and the longitudinal bare vertices

\[
\Gamma^{L(0)}_{i}(1 \; 2; 3) = ie^2 \delta(1 - 2) \delta(1 - 3) \gamma^0.
\] (23)

With Eq. (21), we obtain the transverse polarization matrix (18) which is represented by the diagrams in Fig. 1.

These diagrams describe a variety of physical effects that contribute to the gain/loss terms in Eq. (16). The two principal processes described by the first diagram are Cherenkov
emission/absorption and pair production/annihilation. The second diagram gives a contribution to the Bremsstrahlung rate. Finally, the third diagram may be identified as the leading diagram for Compton scattering. Note, however, that this interpretation should be viewed with caution because it is valid only if the full particle Green’s functions are replaced by free Green’s functions\(^4\). In the present paper we do not consider Compton scattering. Therefore, in calculating the transverse polarization matrix, only the first two diagrams in Fig. 1 will be taken into account.

Let us consider the polarization function \(\pi^<_{ij}(12)\). In Wigner’s representation, the contribution from the first diagram in Fig. 1 to \(\pi^<_s(X,k)\) reads

\[
\pi^{(1)}_s(X,k) = -ie^2 \int \frac{d^4 p}{(2\pi)^4} \text{tr} \left\{ \gamma^0 G^<_{s}(X,p) \gamma^0 G^>_{s}(X,p - k) \right\},
\]

where the polarization four-vectors are defined as \(\epsilon^s_{\mu}(X,k) = (0, \epsilon^s_{\mu}(X,k))\). The second diagram in Fig. 1 generates ten space-time diagrams with different combinations of \(G^\pm\), \(D^\pm\), and \(D^\pm\). Eight diagrams may be identified as the vacuum and medium (Fock) corrections to the first diagram in Fig. 1. The vacuum renormalization procedure for a plasma is carried in a completely similar way as the analogous procedure in QED [16]. For all practical purposes the vacuum corrections may be ignored if one uses physical masses and charges. The medium corrections to the first diagram in Fig. 1 are small for a weakly coupled plasma and may also be neglected.

The remainder two space-time diagrams (see Fig. 2) describe exchange interaction between quantum states of incoming and outgoing particles via longitudinal field fluctuations and contribute to the Bremsstrahlung rate. The corresponding terms in \(\pi^<_s(X,k)\) are given by

\[
\pi^{(2)}_s(X,k) = -e^4 \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} D^<_{s}(X,k - p_1 + p_2) \times \text{tr} \left\{ \gamma^0 G^<_{s}(X,p_1) \gamma^0 G^{-}_{s}(X,p_2 + k) \gamma^0 G^>_{s}(X,p_1 - k) \right\}
- e^4 \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} D^>_{s}(X,k - p_1 + p_2) \times \text{tr} \left\{ \gamma^0 G^+_{s}(X,p_1) \gamma^0 G^<_{s}(X,p_2 + k) \gamma^0 G^>_{s}(X,p_1 - k) \right\}.
\]

The expression for \(\pi^>_s(X,k)\) can be readily obtained by changing the indices “<” \(\leftrightarrow\) “>”.

\(^4\) As already mentioned, in this case the first diagram in Fig. 1 does not contribute to the photon production rate. On the other hand, the off-shell parts of the electron Green’s functions in this diagram give a contribution to the Bremsstrahlung rate (see below).
4. Electron propagators and correlation functions

In the local approximation, the electron propagators \( G^\pm(X,p) \) are found from the Dyson equation for the contour Green’s function (19) following a usual way [16]:

\[
G^\pm(X,p) = \frac{1}{\Pi - m - \Sigma^\pm(X,p)},
\]

where \( \Pi^\mu(X,p) = p^\mu - eA^\mu(X) \). If this expression is used to calculate the polarization functions \( \pi^\pm_s(X,k) \) or other local quantities, then \( A(X) \) drops out from the final results as it should be due to gauge invariance. Therefore, for notational simplicity, we shall put \( A(X) = 0 \) in (26).

The propagators (26) have in general a very complicated spinor structure. The situation is improved a great deal, however, for equal probabilities of the spin polarization. In this case the matrix self-energies are decomposed into scalar and vector components [23, 26]:

\[
\Sigma^\pm(X,p) = \Sigma^\pm_s(X,p)I + \gamma^\mu \Sigma^\pm_\mu(X,p),
\]

where \( I \) is the identity matrix. Inserting (27) into (26) one obtains

\[
G^\pm(X,p) = \frac{P^\pm + M^\pm}{(P^\pm)^2 - (M^\pm)^2}, \quad P^\pm_\mu = p_\mu - \Sigma^\pm_\mu, \quad M^\pm = m + \Sigma^\pm_s.
\]

If the medium corrections are neglected, expression (28) gives the free particle propagators which are known from vacuum QED [27]:

\[
G^\pm_0(p) = \frac{1}{\not{p} - m \pm i\not{\varepsilon}} = \frac{\not{p} + m}{(p_0 \pm i\varepsilon)^2 - E_p^2},
\]

where \( E_p = \sqrt{p^2 + m^2} \) and \( \varepsilon^\mu = (\varepsilon, 0, 0, 0), \varepsilon \rightarrow +0 \).

We now turn to the correlation functions \( G^z \) appearing in Eqs. (24) and (25). To obtain the scattering rates for photon production in a physically transparent form, it is necessary to express \( G^z \) in terms of the electron and positron distribution functions. The free-particle [16] and, more sophisticated, quasiparticle [23, 26] approximations were proposed previously to solve this problem for relativistic fermions. Both approximations reproduce the singular (on-shell) parts of the correlation functions while the off-shell parts are missing. This suffices to calculate the lowest-order rates for Cherenkov processes. For higher-order radiative processes, like Bremsstrahlung and Compton scattering, the off-shell (short-time) parts of the electron correlation functions \( G^z \) must be taken into account, at least in the first diagram of Fig. 1. Note that a similar situation holds in solid state physics where off-shell parts of the electron correlation functions contribute to the phonon scattering rate [22].

The quasiparticle and off-shell parts of \( G^z \) can be separated by a procedure which is about the same as for the transverse polarization functions in Section 3. The new feature is that the electron correlation functions, propagators, and self-energies are \( 4 \times 4 \) spinor matrices. In analogy with Eq. (7), the electron correlation functions are represented as

\[
G^z(X,p) = \tilde{G}^z(X,p) + \Delta G^z(X,p).
\]

The first term is the quasiparticle contribution which is sharply peaked near \( p_0 = \pm \sqrt{p^2 + m^2} \), and the second term is the off-shell part. As in the case of photons, the form of \( \Delta G^z \) can be
found by analyzing drift terms in the KB equations for electrons. We will not go here into details and give the final result in the quasiclassical limit:

$$\Delta G^z(X, p) = \frac{1}{2} \left( G^+ \Sigma^+ G^+ + G^- \Sigma^- G^- \right).$$

(31)

It is remarkable that this expression is similar in form to the off-shell parts of the transverse polarization functions, Eq. (8), and to the off-shell parts of the particle correlation functions in nonrelativistic kinetic theory [21, 22].

Relations (30) and (31) allow us to investigate the spectral properties of the quasiparticle correlation functions $\tilde{G}^z(X, p)$. The full spectral function for electrons is defined as

$$\mathcal{A}(X, p) = i \left( G^> - G^< \right) = i G^+ \Delta \Sigma G^-,$$

(32)

where $\Delta \Sigma(X, p) = \Sigma^+ - \Sigma^-$. We also introduce the quasiparticle spectral function by

$$\tilde{\mathcal{A}}(X, p) = i \left( \tilde{G}^> - \tilde{G}^< \right) = -\frac{i}{2} G^+ \Delta \Sigma G^+ \Delta \Sigma G^- \Delta \Sigma G^-.$$

(33)

In general the spinor structure of this spectral function can be analyzed by using the decomposition in terms of the 16 linearly independent matrices which are formed from the matrices $\gamma^\mu$ [27]. To gain some feeling for the properties of $\tilde{\mathcal{A}}$, let us consider the zero damping limit in Eq. (33). As is easily seen from Eq. (29), in this case the electron retarded/advanced self-energies are $\Sigma^\pm = \pm \frac{1}{2} \epsilon \gamma^0$, where $\epsilon \to +0$. Then the prelimit expression for (33) is

$$\tilde{\mathcal{A}}(X, p) = \frac{4 (p_0 \Gamma_e)^3}{\left( p_0^2 - E_p^2 \right)^2 + (p_0 \Gamma_e)^2} \left[ \frac{E_p^2 + p_0^2}{2p_0} \left( \not{\rho} + m \right) + \frac{E_p^2 - p_0^2}{2p_0} \left( 1 + \frac{E_p^2 - p_0^2}{4p_0^2} \right) \gamma^0 \right].$$

(34)

where $\Gamma_e = 2\epsilon$ is the damping width. It is instructive to compare (34) with the full spectral function (32) which, in the case of infinitesimally small damping, is given by

$$\mathcal{A}(X, p) = \frac{4p_0 \Gamma_e}{(p_0^2 - E_p^2)^2 + (p_0 \Gamma_e)^2} \left[ \not{\rho} + m + \frac{E_p^2 - p_0^2}{2p_0} \gamma^0 \right].$$

(35)

Both spectral functions have the same limiting form

$$\lim_{\Gamma_e \to 0} \tilde{\mathcal{A}} = \lim_{\Gamma_e \to 0} \mathcal{A} = 2\pi \eta(p_0) \delta(p^2 - m^2) \left( \not{\rho} + m \right),$$

(36)

but the prefactor in $\tilde{\mathcal{A}}$ approaches the delta function faster than the prefactor in $\mathcal{A}$. Another case where the quasiparticle spectral function (33) can be computed in a simple form is the ultrarelativistic limit ($E_p \gg mc^2$) with equal probabilities of spin polarization. Using (28) we obtain

$$\tilde{\mathcal{A}}(X, p) = \frac{4(p_0 \Gamma_e)^3 \not{\rho}}{\left( p_0^2 - E_p^2 \right)^2 + (p_0 \Gamma_e)^2}, \quad \Gamma_e(X, p) = -2p_0^{-1} p^\mu \text{Im} \Sigma^+_{\mu}.$$

(37)

We introduce the quasiparticle distributions in spinor space, $\mathcal{F}^\pm(X, p)$, by relations

$$\tilde{G}^z(X, p) = \mp \frac{i}{2} \left( \tilde{\mathcal{A}} \mathcal{F}^\pm + \mathcal{F}^\pm \tilde{\mathcal{A}} \right), \quad \mathcal{F}^>(X, p) + \mathcal{F}^<(X, p) = I.$$

(38)
In the limit of negligible damping Eqs. (36) and (38) lead to the pole approximation for the quasiparticle parts of the electron correlation functions:\footnote{In the case of a weakly coupled plasma, the difference between the free particle energy $E_p$ and the quasiparticle energy may be neglected in calculating the cross sections.}

$$\tilde{G}^\tilde{=} (X, p) = \mp 2\pi i \eta(p^0) \delta(p^2 - m^2) (\hat{p} + m) f^\tilde{=} (X, p), \quad f^\tilde{=} (X, p) + f^\leq (X, p) = 1. \quad (39)$$

This approximation was previously proposed by Bezerides and DuBois \cite{16}, but for the full correlation functions $G^\tilde{=}$, so that the off-shell parts (31) were missing. As shown in Ref. \cite{16}, the quantity $f^\leq$ is directly related to the electron ($f^- e$) and positron ($f^+ e$) distribution functions:

$$f^\leq (X, p)
\left|\begin{array}{c}
p^0 = E_p \\
p^0 = -E_p
\end{array}\right.
= f^- e (X, p) - f^+ e (X, -p). \quad (40)$$

It should be noted that formulas (30), (31), and (38) do not complete the ansatz for the electron correlation functions because the self-energies $\Sigma^\tilde{=}$ depend on $G^\tilde{=}$. To show this, we proceed from the general expression for the matrix self-energy $\Sigma(1, 2)$ on the time-loop contour $C$. In the Coulomb gauge it is given by

$$\Sigma(1, 2) = -\gamma^0 G(1, 1') \Gamma^L (1', 2'; 2) D(2', 1) - \gamma^i G(1, 1') \Gamma^T (1', 2'; 2) D^T (2', 1), \quad (41)$$

where $\Gamma^L$ is the full longitudinal vertex function

$$\Gamma^L (1', 2'; 3) = \delta \Sigma(1, 2) / \delta \phi(3). \quad (42)$$

The dominant contribution to the self-energy (41) comes from the first term which describes scattering of the beam electrons by charge fluctuations in the plasma. In the lowest (Born) approximation for screened collisions, the self-energies $\Sigma^\tilde{=} (X, p)$ are

$$\Sigma^\tilde{=} (X, p) = \int \frac{d^4 p'}{(2\pi)^4} D^\tilde{=} (X, p' - p) \gamma^0 G^\tilde{=} (X, p') \gamma^0. \quad (43)$$

Inserting this expression into (31) and recalling Eq. (30), we get the integral equation (the fixed arguments $X$ are omitted for brevity)

$$G^\tilde{=} (p) = \tilde{G}^\tilde{=} (p) - \frac{i e^2}{2} \int \frac{d^4 p'}{(2\pi)^4} D^\tilde{=} (p' - p) \left( G^+ (p) \gamma^0 G^\tilde{=} (p') \gamma^0 G^+ (p) + G^- (p) \gamma^0 G^\tilde{=} (p') \gamma^0 G^- (p) \right). \quad (44)$$

This equation may be solved by iteration because the integral term is proportional to the small electromagnetic coupling constant\footnote{The contribution from the integral term may be large if $p^0$ is very close to the mass-shell. In this case Eq. (44) describes destructive interference in Bremsstrahlung processes (multiple scattering), which is responsible for the Landau-Pomeranchuk-Migdal effect \cite{17}. In real laser-plasma experiments the characteristic photon energy below which the multiple scattering becomes important is very low, so that the Landau-Pomeranchuk-Migdal effect has no essential influence on the plasma radiation.}. As a first approximation, the full correlation functions in the right-hand side may be replaced by the quasiparticle functions $\tilde{G}^\tilde{=} (X, p)$. 

\[\tilde{\text{118\}}\]
5. Bremsstrahlung rate in a beam-plasma medium

Now the expression (44) is to be inserted into formulas (24) and (25) for the contributions to the polarization function $\pi_s^\omega$. To be consistent, we have to keep only the terms which are linear in the longitudinal field correlation functions $D^\omega$. Then the gain term in the photon kinetic equation (16) may be written as a sum

$$I_s(X, k) = I'_s(X, k) + I''_s(X, k),$$  \hspace{1cm} (45)

where $I'_s$ comes from (24) where the electron correlation functions $G^\omega$ are replaced by the quasiparticle correlations functions $\tilde{G}^\omega$, and $I''_s$ includes all corrections which are linear in $D^\omega$. The term $I'_s$ represents the contribution from Cherenkov emission and is negligibly small for transverse photons. The term $I''_s$ describes Bremsstrahlung processes and pair annihilation. If the finite widths of the electron quasiparticle spectral function $\tilde{A}$ and the electron propagators $G^\omega$ are taken into account, the resulting expression for $I''_s$ is rather complicated. For a weakly coupled plasma, the pole approximation (29) and (36) may be used. Note also that in the wave-function renormalization (17) the contribution from $\text{Re } \pi^\omega_s$ may be neglected if $|k^0| \gg \omega_e$, where $\omega_e$ is the electron plasma frequency. This gives $Z^{-1} \approx 2\omega_e$. With the above-mentioned simplifications, the photon production rate is put into the form

$$I_s(X, k) = \int \frac{d^4p_i}{(2\pi)^4} \frac{d^4p_f}{(2\pi)^4} W_s(p_i, p_f; X, k) f^<(X, p_i) f^>(X, p_f) \left[ 1 + N_s(X, k) \right]$$  \hspace{1cm} (46)

with the transition probability

$$W_s = \frac{(2\pi)^4 e^4}{2\omega_s} \delta(p_i^2 - m^2) \delta(p_f^2 - m^2) \eta(p_i^0)\eta(p_f^0) D(p_i - p_f - \omega_s) F_s(p_i, p_f; k),$$  \hspace{1cm} (47)

where $F_s(p_i, p_f; k)$ is the Bethe-Heitler weight function known from vacuum QED [27]. Medium effects in the photon production are included by the function

$$D(X, k) = \int d^4x e^{ikx} \langle \Delta\phi(X + x/2)\Delta\phi(X - x/2) \rangle = S(X, k)/k^4,$$  \hspace{1cm} (48)

where $\Delta\phi = \phi - \phi$ and $S(X, k)$ is the local dynamic structure factor of the bulk plasma.

Depending on signs of $p_i^0$ and $p_f^0$, formula (46) gives the contribution of three elementary scattering processes:

$$e^- \rightarrow e^- + \gamma \ (p_i^0 > 0, p_f^0 > 0); \quad e^+ \rightarrow e^+ + \gamma \ (p_i^0 < 0, p_f^0 < 0); \quad e^- + e^+ \rightarrow \gamma \ (p_i^0 > 0, p_f^0 < 0).$$

In real laser-produced plasmas the first process dominates. The “golden-rule” result for the photon production in the Coulomb electron-ion scattering is recovered from (46) if the full dynamic structure factor in (48) is replaced by the static structure factor for randomly distributed ions:

$$S_{\text{ion}}(X, k) = 2\pi \delta(k^0) \sum_a e_a^2 n_a(X),$$  \hspace{1cm} (49)

where $e_a$ and $n_a$ are respectively the charges and the number densities of the ion species. A closer approximation for $S(X, k)$ includes the screening of electron-ion collisions and the contribution from electrons in the plasma return current. These corrections essentially depend on the electron distribution function $f_0(X, p)$ in the bulk plasma. Recent computer simulations [4] show that a good approximation for this distribution function is

$$f_0(X, p) \propto \exp \left\{ -\frac{(p_z + P_0)^2}{2(\Delta p_z)^2} - \frac{p_i^2}{2(\Delta p_i)^2} \right\},$$  \hspace{1cm} (50)
where $P_0$ is the average electron momentum in opposition to the beam momentum directed along the $z$-axis, and $p_\perp$ is the transverse momentum. The quantities $\Delta p_z$ and $\Delta p_\perp$ determine the longitudinal and transverse electron “effective temperatures” in the bulk plasma. A detailed calculation of the production rate (46) for different electron energies in the beam will be given elsewhere.

6. Conclusion

We have shown that Green’s function technique with a consistent inclusion of off-shell parts of the field and particle correlation functions provides a proper frame for calculating scattering rates in a relativistic beam-plasma medium. It is important that the collisional broadening of propagators and correlation functions is determined by different spectral functions: the full (Lorentzian) spectral function in the former case and the quasiparticle spectral function in the latter case. Note that the same situation holds in the thermodynamics of an equilibrium QED plasma [28].

In this paper we have discussed Bremsstrahlung processes as an example of collisional QED effects in a beam-plasma medium. The present approach can be used to describe other QED effects in the interaction of relativistic beam electrons with a plasma. This will be done in further publications.

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