Mapping spin-charge separation without constraints

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Abstract

The general form of a mapping of the spin and charge degrees of freedom of electrons onto spinless fermions and local ‘spin’\(^{1/2}\) operators is derived. The electron Hilbert space is mapped onto a tensor product spin-charge Hilbert space. The single occupancy condition of the \(t-J\) model is satisfied exactly without the constraints between the operators required with slave particle methods and the size of the Hilbert space (four states per site) is conserved. The connection and distinction between the physical electron spin and the “pseudospin” used in these maps is made explicit. Specifically the pseudospin generates rotations both in spin space and particle-hole space. A geometric description (up to sign) is provided using two component spinors. The form of the mapped \(t-J\) Hamiltonian involves the coupling of spin and spinless fermion currents, as one expects.

Section 1 Introduction

The 2D Hubbard model and the \(t-J\) model are two of the most intensely studied models in condensed matter physics. It has been argued that these models provide the minimal description of the CuO\(_2\) planes common to all the cuprate superconductors [1]. In one dimension the Hubbard model can be solved exactly and the ground state is not a Fermi liquid but shows separation of spin and charge degrees of freedom [2]. It has been suggested [3] that this spin-charge separation may also occur in two dimensions and is responsible for the unusual normal state properties found in the cuprate superconductors.

The size of the magnetic moments in the undoped cuprates implies that the Hubbard model should be regarded as in the strong coupling limit. In that case, to order \(t^2/U\), we may use the \(t-J\) model. We write the \(t-J\) model as follows

\[
\mathcal{H}_{t-J} = -t \sum_{\langle ij \rangle \sigma} (1 - n_{i\bar{\sigma}}) c_{i\sigma}^\dagger c_{j\sigma} (1 - n_{j\bar{\sigma}}) + \text{H. c.} + J \sum_{\langle ij \rangle} \left( \mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} n_i n_j \right)
\]  

(1)

where \(\bar{\sigma} = -\sigma\), \(c_{i\sigma}^\dagger (c_{i\sigma})\) are the electron creation (annihilation) operators, \(\mathbf{S}_i = \frac{1}{2} \sum_{\sigma, \bar{\sigma}} c_{i\sigma}^\dagger \sigma_{\bar{\sigma}} \bar{\sigma} c_{i\bar{\sigma}}\) are the electron spin operators and \(n_i = \sum_\sigma c_{i\sigma}^\dagger c_{i\sigma}\) are the total number operators for site \(i\).

The use of constrained electron operators shows explicitly that the action of the model is restricted to the singly occupied sector of Hilbert space, \(\sum_\sigma c_{i\sigma}^\dagger c_{i\sigma} \leq 1\). (The inequality implies this is a non-holonomic constraint.) The first techniques used to handle the occupation constraint in an appealing way were the slave particle methods [4,5].
basis of these methods is to factorise the electron operator in terms of separate spin and charge operators, then the allowed states are created from a new fictitious vacuum. In the case of the slave boson representation [4] we have the following mapping of the electron operators

\[ c_\sigma = e_\sigma^\dagger f_\sigma + \sigma f^\dagger_\sigma d \]  

where \( e_\sigma^\dagger \) and \( d^\dagger \) are bosonic operators creating empty and doubly occupied states respectively and \( f^\dagger_\sigma \) are fermionic operators creating singly occupied states. For this to satisfy the fermionic anticommutation relations the following constraint must be satisfied

\[ e_\sigma^\dagger e_\sigma + \sum_\sigma f^\dagger_\sigma f_\sigma + d^\dagger d = 1 \]  

on every lattice site. In the slave fermion case [4] we have

\[ c_\sigma = f_\sigma^\dagger a_\sigma + \sigma a^\dagger_\sigma h \]  

Where \( f_\sigma^\dagger \) and \( h^\dagger \) are spinless fermion operators creating empty and doubly occupied states respectively and \( a^\dagger_\sigma \) are bosonic operators creating singly occupied states. In this case the constraint

\[ f_\sigma^\dagger f + \sum_\sigma a^\dagger_\sigma a_\sigma + h^\dagger h = 1 \]  

must be obeyed at every site. When the on-site coulomb repulsion is large \((U \to \infty)\) the operators creating doubly occupied sites drop out leaving more elementary constraint equations.

In this way the difficult non-holonomic constraints are replaced by more straightforward holonomic constraints and the separation of spin and charge degrees of freedom can studied readily at the mean field level. However, these methods replace the local constraints by an average global constraint. Then the mapped electron operators of equations (2) or (4) no longer satisfy the fermionic anticommutation relations and so are no longer a correct representation of electrons. This casts doubt on the results of any mean field calculations using slave particle methods.

Another approach is to represent the electron operators in terms of spinless fermions and Pauli spin-\(\frac{1}{2}\) operators (or equivalently hard core bosons) [6,7,8]. The motivation behind this is that at half filling the \(t-J\) model reduces to a 2D antiferromagnetic Heisenberg model for which a first order spin wave expansion about the classical groundstate provides an acceptable description[9]. It is hoped that away from half filling mean field theories based on spin wave expansions will continue to be useful. In addition, there is the advantage that the single occupancy constraint is automatically satisfied without the need for constraint equations. In this approach the Hilbert space of the original lattice electrons is mapped onto a tensor product hole-spin space, Richard and Yushankhaï[10], for example, use the correspondence

\[ |\uparrow\rangle \rightarrow |0 \uparrow\rangle \quad , \quad |\downarrow\rangle \rightarrow |0 \downarrow\rangle \quad , \quad |0\rangle \rightarrow |1 \uparrow\rangle \]  

for the restricted Hilbert space of the \(t-J\) model.
The size of the Hilbert space is unchanged, but now we must associate a spin with every site, even unoccupied sites. They then use the following mapping of the constrained electron operators

\[ \tilde{c}_\uparrow = f^\dagger \left( \frac{1}{2} + s^z \right) , \quad \tilde{c}_\downarrow = f^\dagger s^+ \]

(7)

where \( \tilde{c}_\sigma = (1 - n_\sigma)c_\sigma \) and the relationship between the true electron spin operators \( S_i \) and the “pseudospin” operators \( s_i \) is \( S_i = (1 - n_i)s_i \), where \( n_i = f^\dagger_i f_i \) is the spinless fermion number operator. So the pseudospin \( s_i \) plays a dual role, representing the real spin on sites where \( n_i = 1 \) and determining whether a site is empty or doubly occupied on sites where \( n_i = 0 \).

Using this mapping the \( t-J \) model is expressed in terms of spinless fermions and local spin operators with no constraints between the two. As discussed by Wang and Rice [11] the \( t-J \) model under this mapping lacks the time reversal symmetry of the original and yet they derive a model which has time reversal. Their model is also invariant under global rotations in pseudospin space ( \( \{ s, H_{t-J} \} = 0 \) ) which the original \( t-J \) Hamiltonian is not, as will be discussed later.

Our work generalises mappings of this form. We derive the most general bilinear maps from electron operators (not merely just the constrained electron operators) onto spinless fermions and local spin operators and provide an interpretation of these maps.

The paper is organised as follows. In section 2 we begin with a general form of the mapping containing undetermined scalars and vectors which we then demand satisfies the exact fermionic anticommutation relations. This provides us with a set of equations for the unknown scalars and vectors which we then solve for a special case. In section 3 we interpret these solutions and connect them to results of other workers, giving an example in section 4. In section 5 we show that no other types of solution are permitted and we conclude in section 6.

**Section 2 Main Calculation**

In this section we carry out the main calculation in order to find the generally allowed form of a mapping of spin-\( \frac{1}{2} \) fermions onto spinless fermions and the spin-\( \frac{1}{2} \) Pauli spin matrices.

We begin with the following representation for the electron annihilation operator

\[ c_\tau = P_\tau f + Q_\tau^\dagger f^\dagger. \]

(8)

Where \( \{ f, f^\dagger \} = 1, \tau \) is a spin label and \( P_\tau \) and \( Q_\tau^\dagger \) are operators determined by insisting that \( c_\tau \) obeys the standard spin-\( \frac{1}{2} \) fermionic anticommutation relations. We begin with this form because earlier work has ruled out any simpler mapping, for instance merely using the \( f \) term above.

From the anticommutator \( \{ c_\tau, c_\tau' \} = 0 \) it follows that

\[ \{ P_\tau, Q_{\tau'}^\dagger \} + \{ P_{\tau'}, Q_\tau^\dagger \} = 0 \]

(9)

and

\[ [ P_\tau, Q_{\tau'}^\dagger ] + [ P_{\tau'}, Q_\tau^\dagger ] = 0. \]

(10)
In obtaining the above results we have used \( f f^\dagger = \frac{1}{2}\{f, f^\dagger\} + \frac{1}{2}[f, f^\dagger], f^\dagger f = \frac{1}{2}\{f, f^\dagger\} - \frac{1}{2}[f, f^\dagger] \) and have equated operator and c-number parts separately.

The anticommutator \( \{c_\tau, c_{\tau'}^\dagger\} = \delta_{\tau,\tau'} \) provides the conditions

\[
\frac{1}{2}\{P_\tau, P^\dagger_{\tau}\} + \frac{1}{2}\{Q^\dagger_{\tau}, Q_{\tau}\} = \delta_{\tau,\tau'} \tag{11}
\]

and

\[
[P_\tau, P^\dagger_{\tau}] + [Q_{\tau'}, Q^\dagger_{\tau}] = 0. \tag{12}
\]

Equations (9),(10),(11) and (12) are the starting point of our investigation into the allowed form of the general mapping given in equation (8).

We begin with the following forms for \( P_\tau \) and \( Q^\dagger_{\tau} \):

\[
P_\tau = A^0_\tau \sigma^0 + A_\tau \cdot \sigma \tag{13}
\]

and

\[
Q^\dagger_{\tau} = B^{0*}_{\tau} \sigma^0 + B^*_\tau \cdot \sigma \tag{14}
\]

where \( \sigma^0 \) is the 2 × 2 unit matrix, \( \sigma = \sigma^x \hat{x} + \sigma^y \hat{y} + \sigma^z \hat{z} \), \( A^0_\tau \) and \( B^{0*}_{\tau} \) are c-numbers and \( A_\tau \) and \( B^*_\tau \) are complex vectors. Again earlier work has shown that a simpler form is not possible. Equations (13) and (14) look like the inner product between two 4-vectors and indeed this observation can be used to interpret the mapping as will be discussed later.

Initially we examine the case where \( \tau = \tau' \), using equation (10) and the identity \( \sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k \) then leads to the result

\[
A_\tau \times B^*_\tau = 0. \tag{15}
\]

This equation can be satisfied by letting \( A_\tau = A_\tau \hat{A}_\tau \) and \( B^*_\tau = B^*_\tau \hat{A}_\tau \) where \( \hat{A}_\tau \cdot \hat{A}_\tau = 1 \).

Equation (12) with \( \tau = \tau' \) now leads to

\[
\left( |A_\tau|^2 - |B_\tau|^2 \right) \hat{A}_\tau \times \hat{A}^*_\tau = 0. \tag{16}
\]

So either \( |A_\tau|^2 = |B_\tau|^2 \) or \( \hat{A}_\tau = e^{i\varphi_\tau} \hat{a}_\tau \) where \( \hat{a}_\tau \) is a real unit vector. Next equation (9) provides the conditions

\[
A^0_\tau B^{0*}_{\tau} + A_\tau B^*_\tau \hat{A}_\tau \cdot \hat{A}_\tau = 0 \tag{17}
\]

and

\[
A^0_\tau B^*_\tau + B^{0}_{\tau} A_\tau = 0. \tag{18}
\]

The final constraint equation (11) leads to

\[
|A^0_\tau|^2 + |A_\tau|^2 + |B^{0*}_{\tau}|^2 + |B^*_\tau|^2 = 1 \tag{19}
\]
and
\[(A^0_r A^*_r + B^{0*}_r B^*_r) \hat{A}^*_r + (A^{0*}_r A_r + B^0_r B^*_r) \hat{A}_r = 0. \tag{20}\]
We can solve these constraint equations for two interesting special cases and we will discuss these solutions now before moving on to consider other possible solutions.

If we allow \(\hat{A}_r\) to be a null vector \((\hat{A}_r \cdot \hat{A}_r = 0)\) then equation (16) must be solved by demanding that \(|A_r|^2 = |B_r|^2\), in which case we have \(A_r = a_r e^{i\alpha_r}\) and \(B^*_r = a_r e^{-i\beta_r}\).

Equation (20) then leads to \(A^0_r = B^0_r = 0\) and equation (19) tells us that \(a_r = \frac{1}{\sqrt{2}}\). This gives the following form for the electron operator
\[c_r = \frac{1}{\sqrt{2}} \left(e^{i\alpha_r} \hat{A}_r \cdot \sigma f + e^{-i\beta_r} \hat{A}_r \cdot \sigma f^\dagger\right). \tag{21}\]

The other special case occurs when \(\hat{A}_r\) is some multiple of a real unit vector \(\hat{A}_r = \hat{a}_r e^{i\varphi_r}\), then \(\hat{A}_r \cdot \hat{A}_r = e^{2i\varphi_r}\) and equation (17) says that \(A^0_r B^{0*}_r + A^*_r B^0_r e^{2i\varphi_r} = 0\) and on using equation (18) we obtain \(A_r = \pm e^{-i\varphi_r} A^0_r\) and \(B^*_r = \mp e^{-i\varphi_r} B^0_r\). Equations (19) and (20) can then be used to obtain \(|A^0_r|^2 = |B^0_r|^2 = a^2 = \frac{1}{4}\). Putting these results together we find the following form for the electron operator
\[c_r = \frac{1}{2} \left(e^{i\alpha_r} (1 \pm \hat{a}_r \cdot \sigma) f + e^{-i\beta_r} (1 \mp \hat{a}_r \cdot \sigma) f^\dagger\right). \tag{22}\]

Now we will go on to examine the \(\tau \neq \tau'\) constraints for these special cases to see if they are valid representations of the electron operators. The cases where \(c_r\) and \(c_{\tau'}\) are both represented by either real or null vectors may be readily shown to be inconsistent. Thus we allow \(c_r\) to be represented using a null vector with \(P_r = \frac{1}{\sqrt{2}} e^{i\alpha_r} \hat{A}_r \cdot \sigma\) and \(Q^\dagger_r = \frac{1}{\sqrt{2}} e^{-i\beta_r} \hat{A}_r \cdot \sigma\) and \(c^\dagger_{\tau'}\), to be represented using a real vector with \(P^\dagger_{\tau'} = \frac{1}{2} e^{i\gamma} (1 \pm \hat{B}_r \cdot \sigma)\) and \(Q^\dagger_{\tau'} = \frac{1}{2} e^{-i\delta} (1 \mp \hat{\hat{B}} \cdot \sigma)\). Then equation (10) leads to
\[\left(e^{i(\alpha - \delta)} + e^{-i(\gamma - \beta)}\right) \hat{A} \times \hat{B} = 0. \tag{23}\]

But since \(\hat{A}\) is a null vector and \(\hat{\hat{B}}\) is some multiple of a real vector we must satisfy this condition non-trivially \((\hat{A} \neq 0\) and/or \(\hat{\hat{B}} \neq 0)\) by demanding that
\[e^{i(\alpha - \delta)} + e^{i(\gamma - \beta)} = 0. \tag{24}\]

Next we use equation (9) to obtain the constraint
\[\left(e^{i(\alpha - \delta)} - e^{-i(\gamma - \beta)}\right) \hat{A} \cdot \hat{\hat{B}} = 0 \tag{25}\]
and to avoid inconsistencies we must demand that
\[\hat{A} \cdot \hat{\hat{B}} = 0. \tag{26}\]

Equation (12) gives the condition
\[\left(e^{i(\alpha - \gamma)} + e^{i(\delta - \beta)}\right) \hat{A} \times \hat{\hat{B}} = 0. \tag{27}\]
Now we cannot have $\hat{A} \times \hat{B} = 0$ except in a trivial case therefore we are lead to the constraint

$$e^{i(\alpha - \gamma)} + e^{i(\delta - \beta)} = 0$$

(28)

which is identical to equation (24). Equation (11) can be satisfied by equation (24) and equation (26) and so does not lead to any new constraints. We can satisfy equation (15), ensure that $\hat{B}$ is a real vector and that $\hat{A}$ is null by letting

$$\hat{A} = \frac{1}{\sqrt{2}} (\hat{a} + i\hat{b}) \; ; \; \hat{a} \cdot \hat{b} = 0$$

(29)

$$\hat{B} = \hat{c} \; ; \; \hat{c} = \pm \hat{a} \times \hat{b}.$$ (30)

Where $\hat{a}$, $\hat{b}$ and $\hat{c}$ are real unit vectors (also from the above $\hat{A} \cdot \hat{A}^* = 1$). Equation (24) is solved by taking logarithms resulting in

$$(\alpha + \beta) = (\gamma + \delta \pm \pi)$$

(31)

which we satisfy by setting

$$\alpha = \theta + \phi \; ; \; \beta = -\theta + \phi \pm \pi$$

$$\gamma = \theta' + \phi \; ; \; \delta = -\theta' + \phi.$$ (32)

Putting all these results together we arrive at the final forms of the representations of the electron operators

$$c_{\tau}^{\text{null}} = \frac{e^{i\theta_r}}{2} (\hat{a} + i\hat{b}) \cdot \sigma (e^{i\phi} f - e^{-i\phi} f^\dagger)$$

$$c_{\tau}^{\text{real}} = \frac{e^{i\theta_{r'}}}{2} (e^{i\phi} (1 \pm \hat{c} \cdot \sigma) f + e^{-i\phi} (1 \mp \hat{c} \cdot \sigma) f^\dagger)$$

or

$$c_{\tau}^{\text{null}} = \frac{e^{i\theta_{r'}}}{2} (\hat{a} + i\hat{b}) \cdot \sigma (e^{i\phi} f + e^{-i\phi} f^\dagger)$$

$$c_{\tau}^{\text{real}} = \frac{e^{i\theta_r}}{2} (e^{i\phi} (1 \pm \hat{c} \cdot \sigma) f^\dagger - e^{-i\phi} (1 \mp \hat{c} \cdot \sigma) f).$$ (33)

(34)

These are the basic results for this special case. In the next section we will interpret these results and show that they can be described in terms of the vectors which are used in the geometric description (up to sign) of spinors. In section 4 we show that there are no further allowed solutions.

**Section 3 Interpretation in terms of spinors**

We can understand the form of these mappings by making use of the relationship between spinors ($X^{au}$) with one dotted and one undotted index and 4-vectors [12], where an undotted index refers to transformation by a Lorentz spin transform matrix (SL(2,C)) and the dotted index refers to transformation by a complex conjugated Lorentz spin transform matrix. This can be done because of the group homomorphism between
SL(2, C) matrices and matrices $L^+_+$ representing proper orthochronous Lorentz transformations.

The explicit connection between a 4-vector $x^\mu$ and a two component spinor $X^{a\dot{a}}$ is the following

$$X^{a\dot{a}} = x^\mu \sigma_\mu^{a\dot{a}}; \quad x^\nu = -\frac{1}{2} \sigma_\nu^{a\dot{a}} X^{a\dot{a}}$$  \hspace{1cm} (35)

where $\sigma_\mu = (\sigma_0, \sigma)$.

Given any two component spinor $\xi^a$, we can define its spinor mate $\eta^a$ by $\xi_a \eta^a = 1$ where $\xi_a = \xi^b \epsilon_{ba}$ and

$$\epsilon_{ba} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  \hspace{1cm} (36)

Clearly the choice of spinor mate is not unique and in general $\eta_{\text{new}}^a = \eta^a + \alpha \xi^a$.

We can then generate from these two spinors, 4-vectors which are useful in the geometrical description (up to a sign) of one component spinors.

Firstly we can generate the “null flagpole vector” $x^\mu$ defined via

$$X^{a\dot{a}} = \xi^a \bar{\eta}^{\dot{a}} = x^\mu \sigma_\mu^{a\dot{a}}$$  \hspace{1cm} (37)

where the overbar denotes complex conjugation. Here $x^\mu$ is a future directed null vector ($x^\mu x_\mu = 0$) and so defines a 3D hypercone or lightcone. We can define an analogous vector $w^\mu$ using the spinor mate $\eta^a$ via

$$W^{a\dot{a}} = \eta^a \bar{\eta}^{\dot{a}} = w^\mu \sigma_\mu^{a\dot{a}}.$$  \hspace{1cm} (38)

Next we define the spacelike “flag” vector $y^\mu$ via

$$Y^{a\dot{a}} = \xi^a \eta^{\dot{a}} + \eta^a \bar{\xi}^{\dot{a}} = y^\mu \sigma_\mu^{a\dot{a}}.$$  \hspace{1cm} (39)

This vector is orthogonal to the flagpole vector ($y^\mu x_\mu = 0$). Because the choice of spinor mate $\eta^a$ was not unique, $y^\mu$ is not unique and we can have in general $y_{\text{new}}^\mu = y^\mu + (\alpha + \bar{\alpha}) x^\mu$. So the possible flag vectors $y^\mu$ are all coplanar, and orthogonal to $x^\mu$ the flagpole vector.

Finally we generate the spacelike 4-vector $z^\mu$ via

$$Z^{a\dot{a}} = i \xi^a \eta^{\dot{a}} - i \eta^a \bar{\xi}^{\dot{a}} = z^\mu \sigma_\mu^{a\dot{a}}.$$  \hspace{1cm} (40)

This 4-vector is orthogonal to both $x^\mu$ and $y^\mu$, so $y^\mu$ and $z^\mu$ are basis vectors in the 2D space on the lightcone orthogonal to $x^\mu$.

The general spinor of fixed magnitude $\xi^a$ is obtained by rotating the familiar spinor $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which has flagpole vector $x^\mu = (s, s\mathbf{z})$ in the positive sense through the three Euler angles $\theta, \phi$ and $\psi$ [13] resulting in

$$\xi^a = \sqrt{2s} \begin{pmatrix} \cos \frac{\theta}{2} e^{i(\phi + \psi)} \\ \sin \frac{\theta}{2} e^{i(\phi - \psi)} \end{pmatrix}.$$  \hspace{1cm} (41)
The spinor mate is given by

$$\eta^a = \sqrt{2s} \left( \begin{array}{c} -\sin \frac{\theta}{2} e^{i\left(\frac{\psi - \phi}{2}\right)} \\ \cos \frac{\theta}{2} e^{i\left(\frac{\phi - \psi}{2}\right)} \end{array} \right).$$  \hspace{1cm} (42)$$

Using equation (37) we find that the flagpole vector is given by

$$x^\mu = s \left( \begin{array}{c} 1 \\ \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{array} \right)$$

(43)

as would be expected. Next we use equation (38) to find

$$w^\mu = s \left( \begin{array}{c} 1 \\ -\sin \theta \cos \phi \\ -\sin \theta \sin \phi \\ -\cos \theta \end{array} \right).$$  \hspace{1cm} (44)

Equation (39) leads to

$$y^\mu = 2s \left( \begin{array}{c} 0 \\ \cos \theta \cos \phi \cos \psi - \sin \phi \sin \psi \\ \cos \theta \sin \phi \cos \psi + \cos \phi \sin \psi \\ -\sin \theta \cos \psi \end{array} \right)$$

(45)

and lastly equation (40) gives

$$z^\mu = 2s \left( \begin{array}{c} 0 \\ \cos \theta \cos \phi \sin \psi + \sin \phi \cos \psi \\ \cos \theta \sin \phi \sin \psi - \cos \phi \cos \psi \\ -\sin \theta \sin \psi \end{array} \right).$$  \hspace{1cm} (46)

We can use the above analysis to describe the form of the mappings given in equations (33) and (34).

In that case the 4-vector \((s, s\hat{c})\) corresponds to the flagpole vector \(x^\mu\) of the spinor we are representing while the 4-vector \((s, -s\hat{c})\) is the flagpole vector of the spinor mate \(w^\mu\). The 4-vectors \(2s(0, \hat{a})\) and \(2s(0, \hat{b})\) then correspond to the flag vectors \(y^\mu\) and \(z^\mu\) respectively. This means that we can rewrite the mapped electron operators of equation (34) in the following ways

$$c_\xi = e^{i\theta_\xi} (\eta^a \tilde{\xi} \hat{u}) (e^{i\phi} f + e^{-i\phi} f^\dagger)$$  \hspace{1cm} (47)$$

$$c_\eta = e^{i\theta_\eta} \left( (\eta^a \bar{\eta} \hat{u}) e^{i\phi} f - (\xi^a \tilde{\xi} \hat{u}) e^{-i\phi} f^\dagger \right)$$

or alternatively

$$c_\xi = \frac{1}{2} e^{i\theta_\xi} (y^\mu + iz^\mu) \sigma_\mu (e^{i\phi} f + e^{-i\phi} f^\dagger)$$
\[ c_\eta = e^{i\theta_\eta} (\sigma^\mu \sigma_\mu e^{i\phi} f - x^\mu \sigma_\mu e^{-i\phi} f^\dagger) \]  

(48)

where a particular choice of sign has been made in equation (34). The two important points to notice about the above equations are that the 4-vectors which emerge are the natural ones used in the geometrical description of the spinor we are attempting to represent, and that only the spinor and its spinor mate are required to find the appropriate form of the mapping.

Section 4 Example

We now take a specific example to illustrate the above ideas more clearly. Consider the standard spinor basis for which \( \sigma^z \) is diagonal, that is

\[ \xi^a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\uparrow\rangle ; \quad \eta^a = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle . \]  

(49)

The complete spin charge direct product basis in this case is as follows

\[ |0 \downarrow\rangle = |\downarrow\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]  

(50)

\[ |0 \uparrow\rangle = |\uparrow\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]  

(51)

\[ |1 \downarrow\rangle = |\downarrow\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \]  

(52)

and

\[ |1 \uparrow\rangle = |\uparrow\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]  

(53)

and we can make the following identification between the original states and the new direct product states given above

\[ |0\rangle \rightarrow |0 \downarrow\rangle , \quad |\uparrow\downarrow\rangle \rightarrow |0 \uparrow\rangle , \quad |\downarrow\rangle \rightarrow |1 \downarrow\rangle , \quad |\uparrow\rangle \rightarrow |1 \uparrow\rangle \]  

(54)

Then it follows from equations (37)-(40) that \( x^\mu = (s, s \hat{z}) \), \( w^\mu = (s, -s \hat{z}) \), \( y^\mu = (2s, 2s \hat{x}) \) and \( z^\mu = (2s, -2s \hat{y}) \). Now if we let \( \theta_{\xi} = \theta_{\eta} = \phi = 0 \) in equation (48) we obtain the following mapping of the electron operators

\[ c_\uparrow = \frac{1}{2} \sigma^- (f + f^\dagger) \]  

(55)

and

\[ c_\downarrow = \frac{1}{2} (1 - \sigma^z) f - \frac{1}{2} (1 + \sigma^z) f^\dagger . \]  

(56)
This representation has the correct behaviour when acting on the spin charge direct product states as may easily be checked.

The constrained electron operators \( \tilde{c}_\sigma = (1 - n_{-\sigma})c_\sigma \) are given by

\[
\tilde{c}_\uparrow = \frac{1}{2} f \sigma^- , \quad \tilde{c}_\downarrow = \frac{1}{2} (1 - \sigma^z)
\]

Which is the analogue of equation (7) for the choice of pseudospin on the empty site we have made.

The inverse of this mapping may also be obtained, starting from the fact that 
\( c_\downarrow - c_\downarrow = f^\dagger - f \) and \((c_\uparrow + c_\downarrow)(1 - 2c_\uparrow c_\uparrow) = f^\dagger + f \) we can obtain

\[
f^\dagger = c_\uparrow(1 - c_\uparrow c_\uparrow) - c_\downarrow c_\uparrow c_\uparrow
\]

and

\[
f = (1 - c_\uparrow c_\uparrow)c_\downarrow - c_\downarrow c_\uparrow c_\uparrow.
\]

Starting from \( \sigma^+ = 2c_\uparrow^\dagger (f + f^\dagger) \) we can also obtain

\[
\sigma^+ = 2(c_\uparrow^\dagger c_\uparrow + c_\downarrow c_\downarrow)
\]

\[
\sigma^- = 2(c_\downarrow c_\downarrow + c_\uparrow^\dagger c_\uparrow)
\]

\[
\sigma^z = 2c_\uparrow^\dagger c_\uparrow - 1.
\]

It is interesting to note that these results may be written more succinctly as

\[
\sigma = 2(S + J)
\]

where \( S \) is the true electron spin operator and \( J \) is the generator of rotations in particle-hole space ("isospin" operator), whose explicit representation is given by

\[
J^+ = J^x + iJ^y = c_\uparrow^\dagger c_\uparrow
\]

\[
J^- = J^x - iJ^y = c_\downarrow c_\uparrow.
\]

\[
J^z = \frac{1}{2} \sum_\sigma c_\sigma^\dagger c_\sigma - \frac{1}{2}
\]

The \( J^i \) form a spin algebra with the usual commutation relations. So the spin operators appearing in our representation are not the same as those in the standard electron representation but are composed of the true electron spin operators and operators which generate rotations in particle-hole space.

Using equation (60) we can see that the \( t-J \) model commutes with the total \( z \) component of the pseudospin only and so is not invariant with respect to general rotations in pseudospin space (invariant under both rotations in spin space and particle-hole space) unlike the model obtained by Wang and Rice [6] as mentioned earlier.

The standard number operators evaluated using the mapped electron operators are

\[
n_\uparrow = \frac{1}{2}(1 + \sigma^z) ; \quad n_\downarrow = \frac{1}{2}(1 + \sigma^z) - f^\dagger f \sigma^z.
\]
It is also interesting to express the \( t-J \) model with the mapped electron operators, this leads to

\[
\mathcal{H}_{t-J} = -t \sum_{\langle ij \rangle} f_i^\dagger f_j (1 + \sigma_i \cdot \sigma_j - i(\sigma_i \times \sigma_j) \cdot \hat{\mathbf{z}} - \sigma_i^z - \sigma_j^z) + \text{H. c.} + \frac{J}{2} \sum_{\langle ij \rangle} n_i n_j \sigma_i \cdot \sigma_j \quad (68)
\]

where \( n_i \) are the spinless fermion occupation numbers \( n_i = f_i^\dagger f_i \).

Written in this form the coupling between doped holes and the spin background is shown explicitly with terms linking the kinetic energy density to a ferromagnetic pseudospin interaction and the fermionic current to a pseudospin current. The importance of these interactions to the interpretation of the \( t-J \) model has been discussed in [14]. The conservation of the total number of spinless fermions \( \sum_i f_i^\dagger f_i \) reflects the conservation of the total number of singly occupied sites. This mapped \( t-J \) model does not have the time reversal invariance of the original as discussed by Wang and Rice [11].

Section 5 The general case

In this section we show that no solutions other than those obtained in section 2 are allowed. \( \hat{A}_\tau \) will not in general be a null vector or a real unit vector multiplied by a phase factor. Therefore let

\[
\hat{A}_\tau \cdot \hat{A}_\tau = t^2 e^{2ia}. \quad (69)
\]

Equations (17) and (18) yield

\[
B^{0*}_\tau = \pm te^{ia} B^*_\tau \quad ; \quad A^0_\tau = \mp te^{ia} A_\tau. \quad (70)
\]

We satisfy equation (16) by letting

\[
A_\tau = be^{ia} \quad ; \quad B^*_\tau = be^{-i\beta}. \quad (71)
\]

We then substitute equations (70) and (71) into equation (19) to obtain

\[
2(1 + t^2)b^2 = 1 \quad \text{;} \quad b = \frac{1}{\sqrt{2(1 + t^2)}}. \quad (72)
\]

We can write the above results in a more useful way by letting \( t = \tan \theta_\tau \), then we have \( b = \frac{1}{\sqrt{2}} \cos \theta_\tau \) and \( bt = \frac{1}{\sqrt{2}} \sin \theta_\tau \) and

\[
A_\tau = \frac{1}{\sqrt{2}} \cos \theta_\tau e^{i\alpha_\tau} \quad ; \quad A^0_\tau = \mp \frac{1}{\sqrt{2}} \sin \theta_\tau e^{i(a_\tau + \alpha_\tau)} \quad (73)
\]

\[
B^*_\tau = \frac{1}{\sqrt{2}} \cos \theta_\tau e^{-i\beta_\tau} \quad ; \quad B^{0*}_\tau = \pm \frac{1}{\sqrt{2}} \sin \theta_\tau e^{i(a_\tau - \beta_\tau)}. \quad (74)
\]

The above forms also satisfy equation (20). Putting the above results together the general form for the representation of the electron operator is

\[
c_\tau = \frac{e^{i\alpha_\tau}}{\sqrt{2}} \left( \mp \sin \theta_\tau e^{i\alpha_\tau} + \cos \theta_\tau \hat{A}_\tau \cdot \sigma \right) f + \frac{e^{-i\beta_\tau}}{\sqrt{2}} \left( \pm \sin \theta_\tau e^{i\alpha_\tau} + \cos \theta_\tau \hat{A}_\tau \cdot \sigma \right) f^\dagger. \quad (75)
\]
To examine the $\tau \neq \tau'$ constraints we set

$$P_\tau = A_0 + \hat{A} \cdot \sigma \quad ; \quad Q^\dagger_\tau = B_0^* + \hat{B}^* \cdot \sigma$$

$$P_{\tau'} = C_0 + \hat{C} \cdot \sigma \quad ; \quad Q^\dagger_{\tau'} = D_0 + \hat{D} \cdot \sigma$$

(76)

where

$$A_0 = \mp \frac{1}{\sqrt{2}} \sin \theta_\tau e^{i(a_\tau + \alpha_\tau)} \quad ; \quad \hat{A} = \frac{1}{\sqrt{2}} \cos \theta_\tau e^{i\alpha_\tau} \hat{A}_\tau$$

$$B_0^* = \pm \frac{1}{\sqrt{2}} \sin \theta_\tau e^{i(a_\tau - \beta_\tau)} \quad ; \quad \hat{B}^* = \frac{1}{\sqrt{2}} \cos \theta_\tau e^{-i\beta_\tau} \hat{A}_\tau$$

$$C_0 = \mp \frac{1}{\sqrt{2}} \sin \theta_{\tau'} e^{i(a_{\tau'} + \alpha_{\tau'})} \quad ; \quad \hat{C} = \frac{1}{\sqrt{2}} \cos \theta_{\tau'} e^{i\alpha_{\tau'}} \hat{A}_{\tau'}$$

$$D_0^* = \pm \frac{1}{\sqrt{2}} \sin \theta_{\tau'} e^{i(a_{\tau'} - \beta_{\tau'})} \quad ; \quad \hat{D}^* = \frac{1}{\sqrt{2}} \cos \theta_{\tau'} e^{-i\beta_{\tau'}} \hat{A}_{\tau'}$$

(77)

We then use equations (9)-(12) to obtain the following constraint equations

$$\hat{A} \times \hat{D}^* + \hat{C} \times \hat{B}^* = 0$$

(78)

$$\hat{A} \times \hat{C}^* + \hat{D} \times \hat{B}^* = 0$$

(79)

$$\hat{A} \hat{D}^* + D_0^* \hat{A} + C_0 \hat{B}^* + B_0^* \hat{C} = 0$$

(80)

$$A_0 D_0^* + C_0 B_0^* + \hat{A} \cdot \hat{D}^* + \hat{C} \cdot \hat{B}^* = 0$$

(81)

$$A_0 \hat{C}^* + C_0^* \hat{A} + D_0^* \hat{B}^* + B_0^* \hat{D} = 0$$

(82)

$$A_0 C_0^* + D_0 B_0^* + \hat{A} \cdot \hat{C}^* + \hat{D} \cdot \hat{B}^* = 0.$$  

(83)

The first of these constraints becomes

$$\frac{1}{2} \cos \theta_\tau \cos \theta_{\tau'} (e^{i(\alpha_\tau - \beta_{\tau'})} - e^{i(\alpha_{\tau'} - \beta_\tau)}) \hat{A}_\tau \times \hat{A}_{\tau'} = 0.$$  

(84)

We do not let $\hat{A}_\tau \times \hat{A}_{\tau'} = 0$ because this means that the two operators for the spinor states $\tau$ and $\tau'$ are essentially identical and is just a trivial result as is letting either $\cos \theta_\tau = 0$ or $\cos \theta_{\tau'} = 0$. Instead we satisfy the constraint by demanding that

$$(\alpha_\tau + \beta_\tau) = (\alpha_{\tau'} + \beta_{\tau'}).$$

(85)

Equations (79) and (80) are also both satisfied by equation (85) so the next constraint comes from equation (81) and is

$$\hat{A}_\tau \cdot \hat{A}_{\tau'} = \pm \tan \theta_\tau \tan \theta_{\tau'} e^{i(a_\tau + a_{\tau'})}. $$

(86)

Equation (82) is also satisfied by equation (85) and so the final constraint comes from equation (83) and is

$$\hat{A}_\tau \cdot \hat{A}_{\tau'}^* = \pm \tan \theta_\tau \tan \theta_{\tau'} e^{i(a_\tau - a_{\tau'})}. $$

(87)
Letting \( \tan \theta = t \), \( \tan \theta' = t' \), \( \alpha = a \) and \( \alpha' = a' \) to simplify the following work, we can group together the constraint equations as follows

\[
\hat{A} \cdot \hat{A}^* = 1 ; \quad \hat{A}' \cdot \hat{A}'^* = 1
\]
\[
\hat{A} \cdot \hat{A} = t^2 e^{2ia} ; \quad \hat{A}' \cdot \hat{A}' = t'^2 e^{2ia'}.
\]
\[
\hat{A} \cdot \hat{A}' = \pm tt' e^{i(a+a')} ; \quad \hat{A} \cdot \hat{A}'^* = \mp tt' e^{i(a-a')}.
\] (88)

To investigate these constraints further we write the (in general complex) unit vectors as

\[
\hat{A} = \frac{1}{\sqrt{2}} \left( \hat{a} + i \hat{b} \right) ; \quad \hat{A}' = \frac{1}{\sqrt{2}} \left( \hat{c} + i \hat{d} \right).
\] (89)

where \( \hat{a}, \hat{b}, \hat{c} \) and \( \hat{d} \) are real unit vectors and these forms automatically satisfy the first line of equation (88).

Equating the real and imaginary parts of the second line of equation (88) we are lead to the fact that \( \cos 2a = 0 \), \( \hat{a} \cdot \hat{b} = t^2 \sin 2a \) and that \( \cos 2a' = 0 \), \( \hat{c} \cdot \hat{d} = t'^2 \sin 2a' \). From the final line of equation (88) we have the following results

\[
\hat{a} \cdot \hat{c} = \mp 2tt' \sin a \sin a' ; \quad \hat{b} \cdot \hat{d} = \mp 2tt' \cos a \cos a'
\]
\[
\hat{b} \cdot \hat{c} = \pm 2tt' \cos a \sin a' ; \quad \hat{a} \cdot \hat{d} = \pm 2tt' \sin a \sin a'.
\] (90)

We start by solving for \( a \) and \( a' \) for which we obtain \( a = \pm \frac{\pi}{4} \), \( a' = \pm \frac{\pi}{4} \) and so there are 16 cases to consider but we can actually just consider the case where \( a = a' = \frac{\pi}{4} \) as all of the others may be obtained by appropriate inversions of the 4 unit vectors. We also let \( \hat{b} \rightarrow -\hat{b} \) and \( \hat{d} \rightarrow -\hat{d} \) to obtain a more symmetrical set of equations. This leaves us with the following to solve

\[
\hat{a} \cdot \hat{b} = -t^2 ; \quad \hat{c} \cdot \hat{d} = -t'^2
\] (91)
\[
\hat{a} \cdot \hat{c} = \mp tt' ; \quad \hat{b} \cdot \hat{d} = \mp tt'
\] (92)
\[
\hat{b} \cdot \hat{c} = \mp tt' ; \quad \hat{a} \cdot \hat{d} = \mp tt'.
\] (93)

Firstly we note that \((\hat{a} \times \hat{b}) \cdot (\hat{c} \times \hat{d}) = 0\), so that the plane contain \( g \hat{a} \) and \( \hat{b} \) is orthogonal to the plane contain \( g \hat{c} \) and \( \hat{d} \) so we may set up the four vectors as follows.

\[
\hat{a} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \\ 0 \end{pmatrix} ; \quad \hat{b} = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \\ 0 \end{pmatrix}
\] (94)
\[
\hat{c} = \begin{pmatrix} \cos \phi_1 \\ 0 \\ \sin \phi_1 \end{pmatrix} ; \quad \hat{d} = \begin{pmatrix} \cos \phi_2 \\ 0 \\ \sin \phi_2 \end{pmatrix}
\] (95)

Equation (91) then becomes

\[
\cos(\theta_1 - \theta_2) = -t^2 ; \quad \cos(\phi_1 - \phi_2) = -t'^2.
\] (96)
Equations (92) and (93) now read

$$\cos \theta_1 \cos \phi_1 = tt' \quad ; \quad \cos \theta_2 \cos \phi_2 = tt'$$

(97)

and

$$\cos \theta_2 \cos \phi_1 = tt' \quad ; \quad \cos \theta_1 \cos \phi_2 = tt'$$

(98)

where we have chosen the plus sign in the above. We can obtain the two constraints

$$\theta_2 = \pm \theta_1 \quad ; \quad \phi_2 = \pm \phi_1$$

(99)

We cannot have $\theta_2 = \theta_1$ and/or $\phi_2 = \phi_1$ as then from equation (91) we see that there is no real solution for $t$ and/or $t'$. So there is only one case to consider, and the unit vectors become

$$\hat{a} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \\ 0 \end{pmatrix} \quad ; \quad \hat{b} = \begin{pmatrix} \cos \theta_1 \\ -\sin \theta_1 \\ 0 \end{pmatrix}$$

(100)

$$\hat{c} = \begin{pmatrix} \cos \phi_1 \\ 0 \\ \sin \phi_1 \end{pmatrix} \quad ; \quad \hat{d} = \begin{pmatrix} \cos \phi_1 \\ 0 \\ -\sin \phi_1 \end{pmatrix}$$

(101)

Finally we must solve the following

$$\cos \theta \cos \phi = tt'$$

(102)

$$\cos 2\theta = -t^2$$

(103)

$$\cos 2\phi = -t'^2$$

(104)

where we have dropped the unrequired numerical subscript.

The last two equations place limits on the ranges of $\theta$ and $\phi$ which are as follows

$$\frac{3\pi}{4} > \theta > \frac{\pi}{4} \quad ; \quad \frac{3\pi}{4} > \phi > \frac{\pi}{4}$$

(105)

Equations (102)-(104) lead to the result $(\cos \theta)^2 (\cos \phi)^2 = \cos 2\theta \cos 2\phi$ which is clearly not true, so we are able to rule out any solutions other than those obtained in section 2.

Section 6 Summary

We have obtained all of the allowed forms of local bilinear maps of electron operators onto spinless fermion and ‘spin’ operators. We have shown how these results may be interpreted in terms of the geometrical description of spinors. An important result of our work is an understanding of the “pseudospin” operator used in these mappings. The pseudospin operator is composed of two operators obeying spin-$\frac{1}{2}$ algebra acting in distinct subspaces. They are shown to be the true electron spin operator and an “isospin” operator which generates rotations in 2D particle hole space.
Using the simplest allowed mapping the $t$-$J$ model is expressed in a form in which the coupling between the doped holes and the magnetic background is revealed. The general Hubbard model (as against the $U = \infty$ limit) has a more complicated form involving the production and annihilation of spinless fermion pairs. Only in the $U = \infty$ limit is the number of the fermions conserved. Our treatment is thus ideally suited to the strong coupling limit. Initial mean field analysis has lead to reasonable results.

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