Incomplete Hypergeometric Systems
Associated to 1-Simplex $\times$
$(n - 1)$-Simplex

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Abstract

The $A$-hypergeometric system was introduced by Gel’fand, Kapranov and Zelevinsky in the 1980’s. Among several classes of $A$-hypergeometric functions, those for 1-simplex $\times$ $(n - 1)$-simplex are known to be a very nice class. We will study an incomplete analog of this class.

1 Introduction

The $A$-hypergeometric systems was introduced by Gel’fand, Kapranov and Zelevinsky in the 1980’s ([1]). It is a system of homogeneous differential equations with parameters associated to an integer matrix $A$ and contains a broad class of hypergeometric functions as solutions. Recently, the incomplete $A$-hypergeometric system was proposed toward applications to statistics and a detailed study was given in the case of $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ = 1-simplex $\times$ 1-simplex ([6]). The system includes the incomplete Gauss’ hypergeometric integral

$I_{(a,b)}(\alpha, \beta, \gamma; x) = \int_a^b t^{\beta - 1} (1 - t)^{\gamma - 1} (1 - xt)^{\alpha} dt$

and the incomplete elliptic integral of the first kind

$F(z; k) = \int_0^z \frac{1}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} dt$

as solutions. It is interesting to describe properties of these functions in a general framework. Among several classes of (complete) $A$-hypergeometric functions, those for $\Delta_1 \times \Delta_{n-1}$ (1-simplex $\times$ $(n - 1)$-simplex) are known to be a very nice class (see, e.g., [3 Section 1.5]).

In this paper, we study an incomplete analog of this class. In the section 2 we give a definition of an incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system and prove that the existence of a solution of the system. In the section 3 we give
a particular solution of the system and describe general solutions by combining with a base of the solutions of (homogeneous) $A$-hypergeometric system. In the last section 4, we give the complete list of contiguity relations for the incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric function.

## 2 Incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system

We will work over the Weyl algebra in $2n$ variables $D = C\langle x_{11}, \ldots, x_{1n}, \partial_{11}, \ldots, \partial_{1n}, x_{21}, \ldots, x_{2n}, \partial_{21}, \ldots, \partial_{2n}\rangle$.

### Definition 1

We call the following system of differential equations the incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system:

\[
\begin{cases}
(\theta_{i1} + \theta_{i2} - \alpha_i)\cdot f = 0, & (1 \leq i \leq n) \\
\left(\sum_{i=1}^{n} \theta_{2i} + \gamma + 1\right)\cdot f = [g(t, x)]_{t=a}^{t=b}, \\
(\partial_{i1}\partial_{2j} - \partial_{1j}\partial_{2i})\cdot f = 0, & (1 \leq i < j \leq n)
\end{cases}
\]

where $g(t, x) = t^{\gamma+1} \prod_{k=1}^{n} (x_{1k} + x_{2k} t)^{\alpha_k}$ and $\alpha_i, \gamma \in C$ are parameters. The operator $\theta_{ij} = x_{ij}\partial_{ij}$ is called the Euler operator.

If $g(t, x) = 0$ in (1), the system agrees with the $A$-hypergeometric or GKZ hypergeometric system associated to $\Delta_1 \times \Delta_{n-1}$.

### Remark 1

The incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system introduced in Definition 1 is a special but interesting case of the incomplete $A$-hypergeometric system (see appendix, [6]). Let $A$ be the following $(n+1) \times 2n$ matrix:

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & \ddots & \\
0 & 1 & 0 & 1 & \cdots & 1 & 1 \\
0 & 1 & 0 & 1 & \cdots & 0 & 1
\end{pmatrix}.
\]

We set $\beta = (\alpha_1, \ldots, \alpha_n, -\gamma - 1) \in C^{n+1}$ and $g = (0, \ldots, 0, [g(t, x)]_{t=a}^{t=b})$. Then the incomplete $A$-hypergeometric system associated to $A, \beta, g$ is the incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system.

We note that the ideal $\langle \partial_{i1}\partial_{2j} - \partial_{1j}\partial_{2i} \mid 1 \leq i < j \leq n \rangle$ generated by the third operators of (1) is called the affine toric ideal associated to the matrix $A$ and it is denoted by $I_A$. Moreover, $I_A$ is Cohen-Macaulay because $A$ is normal ([2]).

We note that the inhomogeneous system (1) does not necessarily have a solution $f$, when the inhomogeneous part $[g(t, x)]_{t=a}^{t=b}$ is randomly given.

### Proposition 1

For any $\alpha_i, \gamma \in C$, there exists a classical solution of the incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system.
Proof. We may verify conditions (7) and (8) in Theorem 5 in the appendix with respect to $g = (0, \ldots, 0, [g(t, x)]_{l=a}^{b})$. For $1 \leq i \leq n$, we have

\[ (\theta_{1i} + \theta_{2i} - \alpha_i) \cdot t^{\gamma + 1} \prod_{k=1}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} \]

\[ = \alpha_i x_{1i} t^{\gamma + 1} (x_{1i} + x_{2i}t)^{\alpha_i - 1} \prod_{k \neq i}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} + \alpha_i x_{2i} t^{\gamma + 2} (x_{1i} + x_{2i}t)^{\alpha_i - 1} \prod_{k \neq i}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} - \alpha_i t^{\gamma + 1} \prod_{k=1}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} \]

\[ = \left( x_{1i} + x_{2i}t - (x_{1i} + x_{2i}t) \right) \alpha_i t^{\gamma + 1} (x_{1i} + x_{2i}t)^{\alpha_i - 1} \prod_{k \neq i}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} \]

\[ = 0. \]

Thus the condition (7) holds.

For $1 \leq i < j \leq n$, we have

\[ \partial_{1i} \partial_{2j} \cdot g(t, x) = \partial_{1i} \cdot \alpha_j t^{\gamma + 2} (x_{1j} + x_{2j}t)^{\alpha_j - 1} \prod_{k \neq i}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} \]

\[ = \alpha_i \alpha_j t^{\gamma + 2} (x_{1i} + x_{2i}t)^{\alpha_i - 1} (x_{1j} + x_{2j}t)^{\alpha_j - 1} \prod_{k \neq i, j}^{n} (x_{1k} + x_{2k}t)^{\alpha_k}. \]

Since this expression is symmetric in the indices $i$ and $j$, we have $(\partial_{1i} \partial_{2j} - \partial_{1j} \partial_{2i}) \cdot g(t, x) = 0$. Thus the condition (8) holds.

Our definition of the incomplete $\Delta_1 \times \Delta_{n-1}$ hypergeometric system is natural in terms of a definite integral with parameters.

**Proposition 2** If $\text{Re} \gamma, \text{Re} \alpha_i > 0$, then the integral

\[ \Phi(\beta; x) = \int_{a}^{b} t^{\gamma} \prod_{k=1}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} dt \]  \hspace{1cm} (2)

is a solution of the incomplete $\Delta_1 \times \Delta_{n-1}$ hypergeometric system (Definition [7]).

Proof. From the general theory of $A$-hypergeometric systems, $\Phi(\beta; x)$ is annihilated by the elements of $I_A$ and $\theta_{1i} + \theta_{2i} - \alpha_i$ for $1 \leq i \leq n$ (see, e.g., [9, Section 5.4]). We will prove that

\[ \left( \sum_{i=1}^{n} \theta_{2i} + \gamma + 1 \right) \cdot \Phi(\beta; x) = [g(t, x)]_{l=a}^{b}. \]
Applying \( \sum_{i=1}^{n} \theta_{2i} \) to the integrand, we get
\[
\left( \sum_{i=1}^{n} \theta_{2i} \right) \cdot t^{\gamma} \prod_{k=1}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} = \sum_{i=1}^{n} \alpha_i x_{2i} (x_{1i} + x_{2i}t)^{\alpha_i-1} t^{\gamma+1} \prod_{k \neq i}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} = t^{\gamma+1} \frac{\partial (x_{1i} + x_{2i}t)^{\alpha_i}}{\partial t} \prod_{k=1}^{n} (x_{1k} + x_{2k}t)^{\alpha_k}
\]

By Stokes’ theorem, we obtain
\[
\left( \sum_{i=1}^{n} \theta_{2i} \right) \cdot \Phi(\beta; x) = \int_{a}^{b} \left( \sum_{i=1}^{n} \theta_{2i} \right) \cdot t^{\gamma} \prod_{k=1}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} dt
\]
\[
= \int_{a}^{b} t^{\gamma+1} \frac{\partial (\prod_{k=1}^{n} (x_{1k} + x_{2k}t)^{\alpha_k})}{\partial t} dt
\]
\[
= \left[ t^{\gamma+1} \prod_{k=1}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} \right]_{t=a}^{t=b} = (\gamma + 1) \Phi(\beta; x).
\]

Thus the proposition is proved.

**Example 1** We consider the following system of differential equations:
\[
\begin{align*}
(\partial_{11} \partial_{22} - \partial_{12} \partial_{21}) \cdot f &= 0, \\
(\theta_{11} + \theta_{21} - \alpha_1) \cdot f &= 0, \\
(\theta_{12} + \theta_{22} - \alpha_2) \cdot f &= 0, \\
(\theta_{21} + \theta_{22} + \gamma + 1) \cdot f &= [g(t, x)]_{t=a}^{t=b}.
\end{align*}
\]

Here, \( g(t, x) = t^{\gamma+1} (x_{11} + x_{21}t)^{\alpha_1} (x_{12} + x_{22}t)^{\alpha_2} \).

This is the incomplete \( \Delta_1 \times \Delta_1 \) hypergeometric system for \( A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \), \( \beta = (\alpha_1, \alpha_2, -\gamma - 1) \), and \( g_1 = 0, g_2 = 0, g_3 = [g(t, x)]_{t=a}^{t=b} \).

A detailed study on the system is given in [6].

### 3 Series Solution

The Lauricella function \( F_D \) is defined by
\[
F_D(a, b_1, \ldots, b_n, c; z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}}{(c)_{m_1+\cdots+m_n}(1)_{m_1+\cdots+1_{m_n}}} z_1^{m_1} \cdots z_n^{m_n}.
\]
It is well-known that the Lauricella function $F_D$ of $n - 1$ variables gives a series solution of $\Delta_1 \times \Delta_{n-1}$-hypergeometric system. We can give series solutions of our incomplete system in terms of the Lauricella series when parameters are generic. We need $F_D$ of $n$ variables to give a solution.

**Theorem 1.** If $\gamma$ is not negative integer, the incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system has a series solution which can be expressed in terms of the Lauricella function $F_D$ as

$$F(\beta; x) = \prod_{k=1}^{n} x_{1k}^{\alpha_k} \left( \frac{b^{\gamma+1}}{\gamma+1} F_D \left( \gamma+1; -\alpha_1, \ldots, -\alpha_n; \gamma+2; \frac{-x_{21}b}{x_{11}}, \ldots, \frac{-x_{2n}b}{x_{1n}} \right) \right)$$

$$- \frac{a^{\gamma+1}}{\gamma+1} F_D \left( \gamma+1; -\alpha_1, \ldots, -\alpha_n; \gamma+2; \frac{-x_{21}a}{x_{11}}, \ldots, \frac{-x_{2n}a}{x_{1n}} \right).$$

**Proof.** For simplicity, we introduce some multi-index notations. An $n$-dimensional multi-index is an $n$-tuple $m = (m_1, \ldots, m_n)$ of non-negative integers. The norm of a multi-index is defined by $|m| = m_1 + \cdots + m_n$. For a vector $x_i = (x_{i1}, \ldots, x_{in})$ ($i = 1, 2$), define $x_i^m = x_{i1}^{m_1} \cdots x_{in}^{m_n}$ and for a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, define the Pochhammer symbol by $(\alpha)_m = (\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n}$. By using these notations, the series $F$ can be written as

$$F = x_1^{\alpha} \sum_{m \geq 0} c_m \left( \frac{x_2}{x_1} \right)^m, \quad c_m = \frac{(-1)^{|m|}(-\alpha)_m}{(\gamma+|m|+1)(1)_m} (b^{\gamma+|m|+1} - a^{\gamma+|m|+1}).$$

We note that

$$\theta_{1k} \cdot F = (\alpha_k - m_k) F,$$

$$\theta_{2k} \cdot F = m_k F.$$

We now prove that the series $F$ satisfies the incomplete system (1). Firstly, $(\theta_{1i} + \theta_{2i} - \alpha_i) \cdot F = 0$ for $1 \leq i \leq n$ follows from above fact immediately.
Secondly, we will prove \( (\sum_{i=1}^{n} \theta_{2i} + \gamma + 1) \cdot F = [g(t, x)]_{t=b}^{t=a} \), which can be shown as

\[
(\sum_{i=1}^{n} \theta_{2i} + \gamma + 1) \cdot F = (|m| + \gamma + 1)F
\]

\[
= x_1^{\alpha} \sum_{m \geq 0} \frac{(-1)^m}{(1)_m} (b^{\gamma + |m| + 1} - a^{\gamma + |m| + 1}) \left( \frac{x_2}{x_1} \right)^m
\]

\[
= \left[ t^{\gamma+1} x_1^\alpha \sum_{m \geq 0} \frac{(-\alpha)_m}{(1)_m} \left( -\frac{x_2}{x_1} \right)^m \right]_{t=a}^{t=b}
\]

\[
= \left[ t^{\gamma+1} x_1^\alpha \prod_{k=1}^{n} \left( 1 + \frac{x_2t}{x_{1k}} \right) \right]_{t=a}^{t=b}
\]

In the last two steps, we take a branch such that the equality holds.

Finally, we will prove \((\partial_{1i} \partial_{2j} - \partial_{1j} \partial_{2i}) \cdot F = 0\) for \(1 \leq i < j \leq n\). This follows from the following two calculations:

\[
(\theta_{2i} \theta_{2j} - \frac{x_2 j x_{1i}}{x_{1j} x_{2i}} \theta_{1j} \theta_{2i}) \cdot F = x_1^{\alpha} \sum_{m \geq 0} (\alpha_i - m_i) m_j c_m \left( \frac{x_2}{x_1} \right)^m
\]

\[
- x_1^\alpha \sum_{m \geq 0} (\alpha_j - m_j) m_i c_m \left( \frac{x_2}{x_1} \right)^{m-e_i+e_j}
\]

\[
= x_1^{\alpha} \sum_{m \geq 0} (\alpha_i - m_i)(m_j + 1) c_{m+e_i} \left( \frac{x_2}{x_1} \right)^{m+e_j}
\]

\[
- x_1^\alpha \sum_{m \geq 0} (\alpha_j - m_j)(m_i + 1) c_{m+e_i} \left( \frac{x_2}{x_1} \right)^{m+e_j}
\]

and

\[
(\alpha_i - m_i)(m_j + 1) c_{m+e_j} = (\alpha_i - m_i)(m_j + 1) \frac{(-1)^{|m+e_j|} (-\alpha)_{m+e_j}}{(\gamma + |m+e_j| + 1)(1)_{m+e_j}} (b^{\gamma + |m+e_j| + 1} - a^{\gamma + |m+e_j| + 1})
\]

\[
= \frac{(-1)^{|m+1|} (-\alpha)_{m+e_j+e_i}}{(\gamma + |m| + 2)(1)_{m}} (b^{\gamma + |m| + 2} - a^{\gamma + |m| + 2})
\]

\[
= (\alpha_j - m_j)(m_i + 1) \frac{(-1)^{|m+e_i|} (-\alpha)_{m+e_i}}{(\gamma + |m+e_i| + 1)(1)_{m+e_i}} (b^{\gamma + |m+e_i| + 1} - a^{\gamma + |m_i| + 1})
\]

\[
= (\alpha_j - m_j)(m_i + 1) c_{m+e_i}.
\]

Therefore, the theorem is proved.
Gel’fand, Kapranov and Zelevinsky \([\text{(II)}]\) gave a base of the solutions of the (complete) \(A\)-hypergeometric system. We will give a base of solutions of our incomplete system by utilizing their result and Theorem \([\text{II]}\).

For a parameter \(\beta = (\alpha_1, \ldots, \alpha_n, -\gamma - 1) \in \mathbb{C}^{n+1}\), we set a \(2 \times n\) matrix

\[
\begin{pmatrix}
\beta_1 & \cdots & \beta_{\ell-1} & 0 & \cdots & 0 \\
0 & \cdots & 0 & -\frac{\sum_{j=1}^{\ell} \beta_j \beta_{\ell+1}}{\sum_{j=1}^{n+1} \beta_j} \beta_{\ell+1} & \cdots & \beta_n
\end{pmatrix}
\]

for \(1 \leq \ell \leq n\). Let \(M^{(\ell)}\) be a set of \(2 \times n\) matrices

\[
M^{(\ell)} = \sum_{k=1}^{\ell-1} N_0 \cdot (e_{2k} + e_{1\ell} - e_{1k} - e_{2\ell}) + \sum_{k=\ell+1}^{n} N_0 \cdot (e_{1k} + e_{2\ell} - e_{2k} - e_{1\ell}),
\]

where \(e_{ij}\) is the \(2 \times n\) matrix whose \((i, j)\)-entry is 1 and the other entries are 0.

We suppose that the condition of parameter \(\beta\) called "T-nonresonant", that is the sets \(s^{(\ell)} \pm M^{(\ell)} (1 \leq \ell \leq n)\) are pairwise disjoint \((\text{II Definition 3})\). Define series \(\Psi^{(\ell)}(x)\) as

\[
\Psi^{(\ell)}(x) = \Gamma(s^{(\ell)} + 1) \sum_{k \in M^{(\ell)}} \frac{1}{\Gamma(s^{(\ell)} + k + 1)} x^{s^{(\ell)} + k},
\]

where \(\Gamma(s^{(\ell)} + k + 1) = \prod_{i=1}^{n+1} \prod_{j=1}^{n} \Gamma(s^{(\ell)}_{ij} + k_{ij} + 1)\) and \(x^{s^{(\ell)} + k} = \prod_{i=1}^{n+1} \prod_{j=1}^{n} x^{s^{(\ell)}_{ij} + k_{ij}}\). These series are linearly independent and have the open domain

\[
\frac{|x_{21}|}{x_{11}} < \frac{|x_{22}|}{x_{12}} < \cdots < \frac{|x_{2n}|}{x_{1n}} < \frac{1}{\max(|a|, |b|)}
\]

as a common domain of convergence. Moreover they span the solution space of (complete) \(\Delta_1 \times \Delta_{n-1}\)-hypergeometric system \((\text{II Theorem 3})\).

By using this result, we obtain the following theorem.

**Theorem 2** Suppose the parameter \(\beta\) is T-nonresonant and \(\gamma\) is not negative integer.

1. The common domain of convergence of \(F(\beta; x)\) and \(\Psi^{(\ell)}(x)\) is

\[
U : \frac{|x_{21}|}{x_{11}} < \frac{|x_{22}|}{x_{12}} < \cdots < \frac{|x_{2n}|}{x_{1n}} < \frac{1}{\max(|a|, |b|)}
\]

2. Any holomorphic solution of the incomplete system \((\text{II})\) on \(U\) can be written as

\[
F(\beta; x) = \sum_{\ell=1}^{n} c_{\ell} \Psi^{(\ell)}(x), \quad c_{\ell} \in \mathbb{C}.
\]

**Proof.** Since the domain of convergence of the series \(F\) is

\[
U_0 : \frac{|x_{21}|}{x_{11}} < \frac{1}{\max(|a|, |b|)}, \frac{|x_{22}|}{x_{12}} < \frac{1}{\max(|a|, |b|)}, \cdots, \frac{|x_{2n}|}{x_{1n}} < \frac{1}{\max(|a|, |b|)}
\]

we have the statement 1. The statement 2 is clear. 

\[\square\]
Theorem 3 For $\sigma \in S_n$, we suppose the parameter $\sigma(\beta)$ is $T$-nonresonant and $\gamma$ is not negative integer.

1. The domain of convergence of the series $F(\beta; x)$ and $\sigma(\Psi^{(\ell)}(x))$ is

$$\sigma(U) : \left| \frac{x_{2\sigma(1)}}{x_{1\sigma(1)}} \right| < \left| \frac{x_{2\sigma(2)}}{x_{1\sigma(2)}} \right| < \cdots < \left| \frac{x_{2\sigma(n)}}{x_{1\sigma(n)}} \right| < \frac{1}{\max(|a|, |b|)}$$

2. Any holomorphic solution of the incomplete system (1) on $\sigma(U)$ can be written as

$$F(\beta; x) + \sum_{\ell=1}^{n} c_\ell \sigma(\Psi^{(\ell)}(x)),$$

$c_\ell \in \mathbb{C}$. Here, $\sigma(\Psi^{(\ell)}(x))$ is given by the permutations $x_{ij} \leftrightarrow x_{i\sigma(j)}$, $s^{(\ell)}_{ij} \leftrightarrow s^{(\ell)}_{i\sigma(j)}$ and $\beta_j \leftrightarrow \beta_{\sigma(j)}$.

Proof. The theorem follows immediately from the $\sigma$-invariance of $F$.

Remark 2 The closure of the union of $\sigma(U)$ coincides with the closure of $U_0$. That is

$$\overline{\bigcup_{\sigma \in S_n} \sigma(U)} = \bigcup_{\sigma \in S_n} \overline{\sigma(U)}.$$
where \( g(t, x) = t^{-\delta} \prod_{k=1}^{n} (x_{1k} + x_{2k} t)^{\alpha_k} \). Let \( a_{1k} \) and \( a_{2k} \) be vectors corresponding to the \((2k-1)\)-st and the \(2k\)-th columns of \( A \) respectively.

**Theorem 4** The incomplete \( \Delta_1 \times \Delta_{n-1} \)-hypergeometric function \( \Phi(\beta; x) \) satisfies the following contiguity relations.

- **Shifts with respect to \( a_{1k} \):**
  \[
  S(\beta; -a_{1k}) \Phi(\beta; x) = \alpha_k \Phi(\beta - a_{1k}; x),
  \]
  \[
  S(\beta - a_{1k}; +a_{1k}) \Phi(\beta - a_{1k}; x) = \left( \sum_{i=1}^{n} \alpha_i - \delta \right) \Phi(\beta; x) - [g(t, x)]_{t=b}^{t=a},
  \]

  where
  \[
  S(\beta; -a_{1k}) = \partial_{1k},
  \]
  \[
  S(\beta - a_{1k}; +a_{1k}) = \sum_{i=1, i \neq k}^{n} (x_{1i} x_{2k} - x_{1k} x_{2i}) \partial_{2i} + \sum_{i=1}^{n+1} \alpha_i x_{1k}.
  \]

- **Shifts with respect to \( a_{2k} \):**
  \[
  S(\beta; -a_{2k}) \Phi(\beta; x) = \alpha_k \Phi(\beta - a_{2k}; x),
  \]
  \[
  S(\beta - a_{2k}; +a_{2k}) \Phi(\beta - a_{2k}; x) = \delta \Phi(\beta; x) + [g(t, x)]_{t=b}^{t=a},
  \]

  where
  \[
  S(\beta; -a_{2k}) = \partial_{2k},
  \]
  \[
  S(\beta - a_{2k}; +a_{2k}) = \sum_{i=1, i \neq k}^{n} x_{1k} x_{2i} \partial_{1i} + \sum_{i=1, i \neq k}^{n} \theta_{2i} + \alpha_k \right) x_{2k}.
  \]

**Proof.** The down-step relations (3) and (5) are easily verified. We will prove only the up-step relations (4) and (6).

Let \( L_1 \) be the operator

\[
\left( \sum_{i=1, i \neq k}^{n} (x_{1i} x_{2k} - x_{1k} x_{2i}) \partial_{2i} + \sum_{i=1}^{n} \alpha_i x_{1k} \right) \partial_{1k} - \alpha_k \left( \sum_{i=1}^{n} \alpha_i - \delta \right) + \alpha_k \left( \sum_{i=1}^{n} \theta_{2i} - \delta \right).
\]

We now prove that \( L_1 \Phi(\beta; x) = 0 \) which together with (9) will prove the contiguity relation (4). The operator \( L_1 \) can be reduced by \( z_k \) \((1 \leq k \leq n)\) as
follows:

\[
L_1 = \sum_{i=1, i \neq k}^{n} x_i x_2k \partial_{i1k} - \sum_{i=1}^{n} (\theta_{2i} - \alpha_i)\theta_{1k} + \theta_{2k}\theta_{1k} - \alpha_k \sum_{i=1}^{n} \alpha_i + \alpha_k \sum_{i=1}^{n} \theta_{2i}
\]

\[
= \sum_{i=1, i \neq k}^{n} x_i x_2k \partial_{i1k} - \sum_{i=1}^{n} (\theta_{2i} - \alpha_i)\theta_{1k} + \theta_{2k}\theta_{1k} - \alpha_k \sum_{i=1}^{n} \alpha_i + \alpha_k \sum_{i=1}^{n} \theta_{2i}
\]

\[
+ \sum_{i=1}^{n} (\theta_{2i} - \alpha_i)(\theta_{2k} - \alpha_k) + \theta_{2k}\theta_{1k} + \alpha_k \sum_{i=1}^{n} (\theta_{2i} - \alpha_i)
\]

\[
= \sum_{i=1, i \neq k}^{n} x_i x_2k \partial_{i1k} - \sum_{i=1}^{n} (\theta_{2i} - \alpha_i)\theta_{1k} + \theta_{2k}\theta_{1k}
\]

\[
= \sum_{i=1, i \neq k}^{n} x_i x_2k \partial_{i1k} - \sum_{i=1}^{n} (\theta_{2i} - \alpha_i)\theta_{1k} + \theta_{2k}\theta_{1k}
\]

\[
= \sum_{i=1, i \neq k}^{n} x_i x_2k \partial_{i1k} - \sum_{i=1}^{n} (\theta_{2i} - \alpha_i)\theta_{1k} + \theta_{2k}\theta_{1k}
\]

Since the \(\partial_{2i}\partial_{1k} - \partial_{2k}\partial_{1i}\) are elements of the toric ideal \(I_\mathcal{A}\), we obtain \(L_1 \cdot \Phi(\beta; x) = 0\).

Let \(L_2\) be the operator

\[
\left( \sum_{i=1, i \neq k}^{n} x_i x_2i \partial_{1i} + \left( \sum_{i=1, i \neq k}^{n} \theta_{2i} + \alpha_k \right) x_2k \right) \partial_{2k} - \alpha_k \delta + \alpha_k \left( \sum_{i=1}^{n} \theta_{2i} - \delta \right).
\]

Since \(L_2\) can be written as

\[
L_2 = \sum_{i=1, i \neq k}^{n} x_1_k x_2_i (\partial_{1i}\partial_{2k} - \partial_{2i}\partial_{1k}) + \sum_{i=1}^{n} \theta_{2i} z_i,
\]

we obtain \(L_2 \cdot \Phi(\beta; x) = 0\) in an analogous calculation with the case of \(L_1\).

Theorem \(\mathbb{I}\) gives contiguity relations for \(e_k = a_{1k} = (0, \ldots, 0, 1, 0, \ldots, 0)\) (\(1 \leq k \leq n\)), but it does not give those for \(e_{n+1} = (0, \ldots, 0, 1)\). The set of vectors \(\{e_1, \ldots, e_{n+1}\}\) is the standard basis of \(\mathbb{Z}^{n+1}\). The contiguity relations for \(e_{n+1}\) can be obtained from Theorem \(\mathbb{I}\) as follows.

**Corollary 1** The incomplete \(\Delta_1 \times \Delta_{n-1}\)-hypergeometric function \(\Phi(\beta; x)\) satisfies the following contiguity relations.

- **Shifts with respect to \(e_{n+1}\):**

  \[
  S(\beta + e_{n+1}; -e_{n+2})\Phi(\beta + e_{n+2}; x) = \alpha_k \left( \sum_{i=1}^{n} \alpha_i - \delta \right) \Phi(\beta; x) - \alpha_k [g(t, x)]_{t=a}^b,
  \]

  \[
  S(\beta - e_{n+1}; +e_{n+2})\Phi(\beta - e_{n+2}; x) = \alpha_k \delta \Phi(\beta; x) + \alpha_k [g(t, x)]_{t=a}^b,
  \]

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where
\[ S(\beta + e_{n+2}; -e_{n+2}) = S(\beta - a_{1k} + a_{1k})\partial_{2k}, \]
\[ S(\beta - e_{n+2}; +e_{n+2}) = S(\beta - a_{2k} + a_{2k})\partial_{1k} \quad \text{for } 1 \leq k \leq n + 1. \]

Although we prove these contiguity relations for the integral representation of the incomplete $\Delta_1 \times \Delta_{n-1}$ function, they hold for functions which satisfy the system and the two conditions (3) and (5). By an easy calculation to check these conditions for the series solution $F(\beta; x)$, we obtain the following corollary.

Corollary 2 The series solution $F(\beta; x)$ satisfies the same contiguity relations.

We note that $\Phi(\beta; x)$ can be formally expanded in $F(\beta; x)$.

5 Appendix: A solvability of incomplete $A$-hypergeometric systems

Let $D$ be the Weyl algebra in $n$ variables. We denote by $A = (a_{ij})$ a $d \times n$-matrix whose elements are integers. We suppose that the set of the column vectors of $A$ spans $\mathbb{Z}^d$.

Definition 2 We call the following system of differential equations $H_A(\beta, g)$ an incomplete $A$-hypergeometric system:

\[
(E_i - \beta_i) \cdot f = \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i, \quad (i = 1, \ldots, d) \\
\bigwedge_{u,v} \cdot f = 0, \quad \bigwedge_{u,v} = \prod_{i=1}^{n} \partial_i^u - \prod_{j=1}^{n} \partial_j^v
\]

with $u, v \in \mathbb{N}^n_0$ running over all $u, v$ such that $Au = Av$.

Here, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{C}^d$ are parameters and $g = (g_1, \ldots, g_d)$ where $g_i$ are given holonomic functions.

We denote by $E - \beta$ the sequence $E_1 - \beta_1, \ldots, E_d - \beta_d$ and $I_A$ the affine toric ideal generated by $\bigwedge_{u,v}$ ($Au = Av$) in $\mathbb{C}[\partial_1, \ldots, \partial_n]$.

Lemma 1 If the first homology of the Euler-Koszul complex vanishes;

\[ H_1(K_{\bullet}(E - \beta; D/\text{DI}_A)) = 0, \]

then the syzygy module $\text{syz}(E_1 - \beta_1, \ldots, E_d - \beta_d) \subset (D/\text{DI}_A)^d$ is generated by $(E_i - \beta_i)\partial_j - (E_j - \beta_j)\partial_i$ ($1 \leq i < j \leq d$).
Proof. The Euler-Koszul complex of \( D/DI_A \) is the following complex
\[
0 \to D/DI_A \to D/DI_A(\ell_1) \to D/DI_A(\ell_2) \to \cdots \to D/DI_A(\ell_d) \to 0
\]
and the differential is defined by
\[
d_p(e_{i_1,\ldots,i_p}) = \sum_{k=1}^{p} (-1)^{k-1} (E_{i_k} - \beta_{i_k}) e_{i_1,\ldots,i_k}\cdots,i_p.
\]
Here, \( e_{i_1,\ldots,i_p} \) are basis vectors of \((D/DI_A)_{\ell_0}\). The kernel of \( d_1 \) is \( \text{syz}(E_1 - \beta_1, \ldots, E_d - \beta_d) \) and the image of \( d_2 \) is generated by \((E_i - \beta_i)e_j - (E_j - \beta_j)e_i\) (\(1 \leq i < j \leq d\)) over \((D/DI_A)^d\). Since the first homology is zero, the conclusion is obtained.

**Theorem 5 ([12])** If the first homology \( H_1(K_\beta(E - \beta; D/DI_A)) \) vanishes and the \( g_i \) are holonomic functions satisfying the following relations
\[
(E_i - \beta_i) \cdot g_j = (E_j - \beta_j) \cdot g_i, \quad (i, j = 1, \ldots, d) \quad (7)
\]
\[
\square_{u,v} \cdot g_i = 0, \quad (i = 1, \ldots, d, Au = Av, u, v \in \mathbb{N}_0^n) \quad (8)
\]
then the incomplete hypergeometric system has a (classical) solution.

**Proof.** By virtue of [3] Theorem 4.1, \( \mathcal{E}xt_1^H(D/DI_A, \mathcal{O}) \) vanishes at generic points in \( \mathbb{C}^n \). Therefore, it is sufficient to prove that \( \ell_1 g_1 + \cdots + \ell_d g_d = 0 \) for all \((\ell_1, \ldots, \ell_d, \ell_{d+1}, \ldots, \ell_{d+m}) \in \text{syz}(E - \beta, \square)\), where \( \square \) is a finite sequence \( \square_{u_1,v_1}, \ldots, \square_{u_m,v_m} \) which are generators of \( I_A \). Since for \((\ell_1, \ldots, \ell_d, \ell_{d+1}, \ldots, \ell_{d+m}) \in \text{syz}(E - \beta, \square)\), the relation \( \sum_{i=1}^d \ell_i (E_i - \beta_i) + \sum_{i=1}^m \ell_{i+d} \square_{u_i,v_i} = 0 \) holds, we have \((\ell_1, \ldots, \ell_d) \in \text{syz}(E - \beta)\) over \((D/DI_A)^d\). By Lemma [1]
\[
\ell_1 g_1 + \cdots + \ell_d g_d = (\ell_1, \ldots, \ell_d) \cdot g
\]
\[
= \sum_{1 \leq i < j \leq d} c_{ij} ((E_i - \beta_i)e_j - (E_j - \beta_j)e_i) \cdot g, \quad c_{ij} \in \mathbb{C}
\]
\[
= \sum_{1 \leq i < j \leq d} c_{ij} ((E_i - \beta_i)g_j - (E_j - \beta_j)g_i)
\]
\[
= 0.
\]

**Remark 3** Matusevich, Miller and Walther ([4] Theorem 6.3) showed that if the toric ideal \( I_A \) is Cohen-Macaulay, the \( i \)-th homology of the Euler-Koszul complex vanishes for all positive integers \( i \).

The following facts are known about Cohen-Macaulay property of toric ideals.

1. If the initial monomial ideal of \( I_A \) is square-free, then \( A \) is normal (see, e.g., [10] Proposition 13.15)).

2. If the matrix \( A \) is normal, then \( I_A \) is Cohen-Macaulay ([2]).

This is an easy tool for showing Cohen-Macaulayness of toric ideals. When \( A \) is \( \Delta_1 \times \Delta_{n-1} \), we can easily verify the condition 1.
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