Affine Rigidity Without Integration

Joël Merker

Abstract. Real analytic ($C^\omega$) surfaces $S^2$ in $\mathbb{R}^3 \ni (x, y, u)$ graphed as \{\(u = F(x, y)\)\} with $F_{xx} \neq 0$ whose Gaussian curvature vanishes identically:

$$0 \equiv F_{xx} F_{yy} - F_{xy}^2,$$

possess, under the action of the affine transformation group $\text{Aff}_3(\mathbb{R}) = \text{GL}_3(\mathbb{R}) \times \mathbb{R}^3$, a basic invariant analogous to 2-nondegeneracy for $C^\omega$ real hypersurfaces $M^5 \subset \mathbb{C}^3$:

$$S_{\text{aff}} := \frac{F_{xx} F_{xxy} - F_{xy} F_{xxx}}{F_{xx}^2}.$$  

It is known (or easily recovered) that $S$ is affinely equivalent to \(u = x^2\) if and only if $S_{\text{aff}} \equiv 0$.

Assuming that $S_{\text{aff}} \neq 0$ everywhere, two deeper affine invariants inspired from Pocchiola's

\begin{equation}
\text{Theorem. } S \text{ is affinely equivalent to } \{u = x^2\} \text{ if and only if } W_{\text{aff}} \equiv 0 \equiv J_{\text{aff}}.
\end{equation}

As a direct corollary of the (brief) proof, affine rigidity of CR-flat 2-nondegenerate $C^\omega$ Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^3$ is deduced. The arguments rely on pure affine geometry, avoid any tool from Analysis, and simplify A.V. Isaev, J. Differential Geom. 104 (2016), 111–141.

An independent article will show, in a more general context, how $C^\omega$ (even $C^7$) $F(x, y)$ can be handled.

1. Introduction, Motivations, Background

Throughout, functions, manifolds, geometric objects, all considerations will be local. To lighten the presentation, no name will be given to domains, open sets, regions, or neighborhoods. Most of the times, only real-analytic, i.e. $C^\omega$, objects will be considered.

On $\mathbb{C}^{n+1}$ with $n \geq 1$ an integer, take complex coordinates:

$$\begin{bmatrix} z \end{bmatrix} \equiv \begin{bmatrix} z_1, \ldots, z_n, w \end{bmatrix} = \begin{bmatrix} x_1 + i y_1, \ldots, x_n + i y_n, u + iv \end{bmatrix}. $$

By definition, a $C^1$ function $h: \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ is holomorphic when it is independent of the conjugate variables, i.e. when $\partial_{\bar{z}} h \equiv 0 \equiv \partial_{\bar{w}} h$. As in one complex variable, this implies (10) that $h$ is $C^\omega$, i.e. is locally expandable as a converging Taylor series of $(z, w)$. One writes $h = h(z, w)$ to point out that $h$ is holomorphic, namely depends only on $(z, w)$, and not on $(\bar{z}, \bar{w})$.

By contrast, a $C^\omega$ function $f$ on $\mathbb{C}^{n+1} \equiv \mathbb{R}^{2n+2}$ is a function $f = f(z, w, \bar{z}, \bar{w})$ of all variables $z, w, \bar{z}, \bar{w}$, since $x = \frac{z + \bar{z}}{2}$, $y = \frac{z - \bar{z}}{2i}$ and so on. So there is a strong difference between holomorphic and $C^\omega$ functions.

Consequently, local $C^\omega$ geometric objects within $\mathbb{C}^{n+1}$ can be considered or classified with respect to the holomorphic structure of $\mathbb{C}^{n+1}$, namely modulo invertible maps $(z, w) \longmapsto (z', w, w')$ whose $(n + 1)$ components are all holomorphic functions. Such maps are called (local) biholomorphisms.

1Laboratoire de Mathématiques d’Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay Cedex, France. joel.merker@math.u-psud.fr
The holomorphic Jacobian matrix of the map \((z, w) \mapsto (z', w'):\)

\[
\begin{pmatrix}
\frac{\partial z'_1}{\partial z_1} & \cdots & \frac{\partial z'_1}{\partial z_n} & \frac{\partial z'_1}{\partial w} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial z'_n}{\partial z_1} & \cdots & \frac{\partial z'_n}{\partial z_n} & \frac{\partial z'_n}{\partial w}
\end{pmatrix}
\]

shows that every holomorphic vector field \(\sum_k A_k(z, w) \frac{\partial}{\partial z_k} + B(z, w) \frac{\partial}{\partial w}\) transfers into a holomorphic vector field \(\sum_k A_k'(z', w') \frac{\partial}{\partial z'_k} + B'(z', w') \frac{\partial}{\partial w'}\).

If the coefficients \(A_k, B\) were merely \(\mathcal{C}^\omega\), namely would also depend on \((\overline{z}, \overline{w})\), hence depend on all variables \((z, w, \overline{z}, \overline{w})\), the transformed vector field would still involve only \(\frac{\partial}{\partial z'_k}, \frac{\partial}{\partial w'}\), and none of \(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial w}\). This observation implies that the complex vector bundle, denoted \(T^{1,0} \mathbb{C}^{n+1}\), whose local \(\mathcal{C}^\omega\) sections write as \(\sum_k A_k \frac{\partial}{\partial z_k} + B \frac{\partial}{\partial w}\), is invariant under biholomorphisms.

Similarly, the complex vector bundle, denoted \(T^{0,1} \mathbb{C}^{n+1}\), whose local sections write as \(\sum_k C_k \frac{\partial}{\partial z_k} + D \frac{\partial}{\partial w}\), is invariant under biholomorphisms, since the antiholomorphic character of the conjugate Jacobian matrix shows that \(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial w}\) are transferred to \(\mathbb{C}\)-linear combinations of \(\frac{\partial}{\partial z'_k}, \frac{\partial}{\partial w'}\).

One verifies easily that:

\[
T^{1,0} \mathbb{C}^{n+1} = T^{0,1} \mathbb{C}^{n+1}, \quad T^{1,0} \mathbb{C}^{n+1} \cap T^{0,1} \mathbb{C}^{n+1} = \{0\}, \quad \mathbb{C} \otimes T \mathbb{C}^{n+1} = T^{1,0} \mathbb{C}^{n+1} \oplus T^{0,1} \mathbb{C}^{n+1}.
\]

Complex variables \((z, w)\) and conjugate variables \((\overline{z}, \overline{w})\) are separated and complementary. Lastly, both bundles are Frobenius-involutive:

\[
[\Gamma(T^{1,0}), \Gamma(T^{1,0})] \subset \Gamma(T^{1,0}) \quad \text{and} \quad [\Gamma(T^{0,1}), \Gamma(T^{0,1})] \subset \Gamma(T^{0,1}),
\]

where we employ the notation \(\Gamma(*)\) to denote local sections.

Our main goal of study is to understand \(\mathcal{C}^\omega \) real hypersurfaces \(M = M^{2n+1} \subset \mathbb{C}^{n+1}\) modulo biholomorphisms of \(\mathbb{C}^{n+1}\). We will assume that they are (locally) graphed as:

\[
M = \{(z, w) \in \mathbb{C}^{n+1}: u = F(z, \overline{z}, v)\},
\]

and we can even sometimes (not always) assume \(F(0) = 0\). Here, \(F\) is a local \(\mathcal{C}^\omega\) function of the \(2n + 1\) real variables \((x, y, u) \equiv (z, \overline{z}, v)\). Denote by \(TM\) the (real) tangent bundle of \(M\), and set \(\mathbb{C}TM := \mathbb{C} \otimes \mathbb{R} TM\), which is a rank \((2n + 1)\) complex vector bundle. Sections of \(\mathbb{C}TM\) are freely generated over \(\mathbb{C}\) by real vector fields tangent to \(M\).

An elementary reasoning based on the fact that \(\partial_w\) is not tangent to \(M\) — because the \(u\)-axis is transversal to \(TM\) — shows that both:

\[
T^{1,0} M := \mathbb{C}TM \cap T^{1,0} \mathbb{C}^{n+1} \quad \text{and} \quad T^{0,1} M := \mathbb{C}TM \cap T^{0,1} \mathbb{C}^{n+1},
\]

constitute rank \(n\) complex vector subbundles of \(\mathbb{C}TM\). Therefore, \(\Gamma(T^{0,1} M)\) consists of the collection of all vector fields \(\overline{\mathcal{Z}} = \sum_k C_k \frac{\partial}{\partial z_k} + D \frac{\partial}{\partial w}\), with \(\mathcal{C}^\omega\) functions \(C_k, D: M \rightarrow \mathbb{C}\), that are tangent to \(M\). As operators, such sections involve only \textit{anti-holomorphic} derivations.
Thus, when one restricts a (local) holomorphic function \( h : \mathbb{C}^{n+1} \rightarrow \mathbb{C} \) to a \( \mathcal{C}^\omega \) hypersurface:

\[
f := h\big|_M,
\]

it is clear that \( \mathcal{L}(f) \equiv 0 \) for every \( \mathcal{L} \in \Gamma(T^{0,1}M) \). This motivates the

**Definition 1.1.** A \( \mathcal{C}^\omega \) function \( f : M \rightarrow \mathbb{C} \) is called **Cauchy-Riemann** — **CR** for short — if \( \mathcal{L}(f) \equiv 0 \) for every \( \mathcal{L} \in \Gamma(T^{0,1}M) \).

The fact that \( \mathcal{C}^\omega \) functions admit converging Taylor series expansions enables to replace real variables by complex variables and to obtain an elementary converse.

**Theorem 1.2.** [29] Every \( \mathcal{C}^\omega \) CR function \( f : M \rightarrow \mathbb{C} \) is the restriction \( f = h\big|_M \) of a uniquely determined holomorphic function \( h \) defined in a neighborhood of \( M \). □

However, for \( \mathcal{C}^\infty \) CR functions, the story is drastically different (e.g. the whole survey [24] is devoted to this research field).

When one restricts a local biholomorphism \( h : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1} \) to a \( \mathcal{C}^\omega \) hypersurface:

\[
f := h\big|_M,
\]

the image \( M' := f(M) \) is still a \( \mathcal{C}^\omega \) hypersurface and \( f : M \xrightarrow{\sim} M' \) is a \( \mathcal{C}^\omega \) diffeomorphism all of whose \((n + 1)\) components are CR functions. One says that \( f \) is a **CR diffeomorphism**.

Consequently, there is a canonical correspondence between local \( \mathcal{C}^\omega \) CR equivalences \( f : M \xrightarrow{\sim} M' \), and local biholomorphic equivalences \( h : M \xrightarrow{\sim} M' \). We will prefer the holomorphic view, since its extrinsic character brings more light.

We can now state a general research problem, unachievable in all dimensions, like several similar problems in differential geometry.

**Problem 1.3.** Classify \( \mathcal{C}^\omega \) hypersurfaces \( M^{2n+1} \subset \mathbb{C}^{n+1} \) modulo biholomorphic equivalences \( h : M \xrightarrow{\sim} M' \).

Continuing with the two fundamental CR subbundles \( T^{1,0}M \) and \( T^{0,1}M \), we furthermore have:

\[
T^{1,0}M = T^{0,1}M, \quad T^{1,0}M \cap T^{0,1}M = \{0\},
\]

and the involutiveness is inherited:

\[
[\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(T^{1,0}M) \quad \text{and} \quad [\Gamma(T^{0,1}M), \Gamma(T^{0,1}M)] \subset \Gamma(T^{0,1}M),
\]

since push-forward (here: restriction) commutes with Lie bracket. However, the sum of \( T^{1,0}M \) plus \( T^{0,1}M \) cannot fill \( \mathbb{C}TM \), because \( n + n < 2n + 1 \), and even worse, most of the times, crossed Lie brackets do **not** commute:

\[
[\Gamma(T^{1,0}M), \Gamma(T^{0,1}M)] \not\subset \Gamma(T^{1,0}M \oplus T^{0,1}M).
\]

This leads to the introduction of the intrinsic concept of **Levi form**, also valid on abstract CR manifolds.
Definition 1.4. At a point $p$ of a $\mathcal{C}^\omega$ hypersurface $M \subset \mathbb{C}^{n+1}$, the Levi form is the Hermitian skew-bilinear form acting on two vectors:

$$\mathcal{X}_p \in T_p^{1,0}M \quad \text{and} \quad \mathcal{Y}_p \in T_p^{1,0}M,$$

by means of any two local vector fields $\mathcal{X}, \mathcal{Y} \in \Gamma(T^{1,0}M)$ defined near $p$ satisfying $\mathcal{X}|_p = \mathcal{X}_p$ and $\mathcal{Y}|_p = \mathcal{Y}_p$, by taking the mod out value at $p$ of the Lie bracket:

$$\text{LeviForm}_p : T_p^{1,0}M \times T_p^{1,0}M \rightarrow \mathbb{C} \otimes \mathbb{R} T_p M \quad \text{mod} \ (T_p^{1,0}M \oplus T_p^{0,1}M)$$

$$(\mathcal{X}_p, \mathcal{Y}_p) \mapsto \sqrt{-1} [\mathcal{X}_p, \mathcal{Y}_p]_p \quad \text{mod} \ (T_p^{1,0}M \oplus T_p^{0,1}M).$$

Classically ([23, p. 45]), the resulting map is independent of the choice of vector fields extensions $\mathcal{X}, \mathcal{Y}$, namely it depends only on the punctual values $\mathcal{X}_p, \mathcal{Y}_p$. Since $\mathbb{C}T_p M$ modulo $T_p^{1,0}M \oplus T_p^{0,1}M$ is of rank 1, this Levi form can be identified with a Hermitian $n \times n$ matrix, after choosing a basis for $T_p^{1,0}M$.

In terms of local sections, at various points of $M$, the Levi form writes as:

$$(\mathcal{X}, \mathcal{Y}) \mapsto [\mathcal{X}, \mathcal{Y}] \mod T_p^{1,0}M \oplus T_p^{0,1}M.$$ 

Given a $\mathcal{C}^\omega$ biholomorphic equivalence $h : M \rightarrow M'$, it is clear from what precedes that its differential $h_*$ induces a bundle isomorphism:

$$h_* : T^{1,0}M \rightarrow T^{1,0}M'.$$

Observation 1.5. Through a biholomorphic equivalence:

$$\text{rank LeviForm} (M, p) = \text{rank LeviForm} (h(M), h(p)) \quad (\forall p \in M). \quad \square$$

This invariancy justifies the concept! Then what happens in the most degenerate situation is simple.

Proposition 1.6. The following two conditions are equivalent.

(i) $\text{LeviForm} (M, p) = 0$ at all $p \in M$.

(ii) $M \overset{\text{Bihol}}{=} \{ u' = 0 \}$ is equivalent to a flat real hyperplane. \quad \square

This justifies to assume that the rank of the Levi form is $\geq 1$.

Definition 1.7. A $\mathcal{C}^\omega$ hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$ is called Levi nondegenerate when:

$$n = \text{rank LeviForm} (M, p) \quad (\forall p \in M).$$

This maximal rank situation has been much studied, see [11] and the references therein. This motivated people to look at intermediate situations, cf. [14, 16, 27, 28].

In differential invariant theory, it is generally admitted that heterogeneous singular situations are disregarded, so that the general branching process can be described as a standard
Convention 1.8. Whenever an invariant function $p \mapsto I(p)$ is determined in the study of a differential-geometric problem, the exploration shall undergo a dichotomy:

- Identical degeneracy $I \equiv 0$,
- Nowhere vanishing $I \neq 0$,

so that points $p$ with $I(p) = 0$ lying on the border of $\{I \neq 0\}$ will not be considered.

From now on, we pass to $\mathbb{C}^3$:

$$n = 2.$$  

**Definition 1.9.** A $\mathcal{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$ is called *Levi degenerate of (constant) rank* 1 when:

$$1 \equiv \text{rank} \, \text{LeviForm} (M, p) \quad (\forall \, p \in M).$$

So the kernel is of constant rank $n - 1 = 1$. From [23, Section 9], we now want to review the geometry of such objects.

As Section 5 will show, there is a deep analogy with affine geometry of real surfaces $S^2 \subset \mathbb{R}^3$ represented as graphs $S = \{u = F(x, y)\}$ which satisfy three *affinely invariant* conditions.

- The Hessian matrix $\begin{pmatrix} F_{xx} & F_{yx} \\ F_{xy} & F_{yy} \end{pmatrix}$ is of constant rank 1.
- $F_{xx} \neq 0$.
- $F_{xx} F_{xy} - F_{xy} F_{xx} \neq 0$.

**Terminology 1.10.** Surfaces satisfying the first two conditions will be called *parabolic*.

Although the affine invariance of these three conditions follows as a plain corollary from the works [23, 27, 22], it is natural to study them within pure affine geometry, a task to which Section 5 is devoted. We believe that the third-order affine invariant $F_{xx} F_{xy} - F_{xy} F_{xx}$ is known in the literature.

Differential invariants of surfaces whose Hessian is, on the contrary, nondegenerate of maximal rank 2, hence are either *elliptic* or *hyperbolic*, admit another basic third-order invariant, the *Pick invariant*. Olver studied in [26] the concerned full algebras of differential invariants.

Let therefore $M^5 \subset \mathbb{C}^3$ be a $\mathcal{C}^\omega$ local real hypersurface represented in holomorphic coordinates $(z_1, z_2, w) \in \mathbb{C}^3$ with $w = u + i \, v$ and $z_k = x_k + i \, y_k$ as a graph:

$$u = F(x_1, y_1, x_2, y_2, v).$$

We can assume $0 \in M$ and even $T_0 M = \{u = 0\}$, i.e. $F(0) = 0 = dF(0)$. 
Two generators of $T^{1,0}M$ written in the intrinsic coordinates $(x_1, y_1, x_2, y_2, v)$ on $M$ are ($[22][4]$):

$$L_1 := \frac{\partial}{\partial z_1} - i \frac{F_{z_1}}{1 + i F_v} \frac{\partial}{\partial v}$$
and

$$L_2 := \frac{\partial}{\partial z_2} - i \frac{F_{z_2}}{1 + i F_v} \frac{\partial}{\partial v}.$$ Their conjugates generate $T^{0,1}M$:

$$\overline{L}_1 := \frac{\partial}{\partial \bar{z}_1} + i \frac{F_{\bar{z}_1}}{1 - i F_v} \frac{\partial}{\partial v}$$
and

$$\overline{L}_2 := \frac{\partial}{\partial \bar{z}_2} + i \frac{F_{\bar{z}_2}}{1 - i F_v} \frac{\partial}{\partial v}.$$ Abbreviate:

$$A^1 := -i \frac{F_{z_1}}{1 + i F_v}$$
and

$$A^2 := -i \frac{F_{z_2}}{1 + i F_v}.$$ The fact that $\overline{F} = F$ is a real function implies for its partial derivatives that:

$$iF^{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma} = F^{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma}.$$ The real differential 1-form:

$$\varrho_0 := dv - A^1 dz_1 - A^2 dz_2 - \overline{A}^1 \, d\bar{z}_1 - \overline{A}^2 \, d\bar{z}_2$$
represents the sum:

$$T^{1,0}M \oplus T^{0,1}M = \{ \varrho_0 = 0 \}.$$ Then in these terms, the Levi form at various points $p = (x_1, y_1, x_2, y_2, v) \in M$ identifies concretely ($[22][4]$) with the $2 \times 2$ Hermitian matrix of functions:

$$\text{Levi}(p) := \begin{pmatrix}
\varrho_0(i[L_1, \overline{L}_1]) & \varrho_0(i[L_2, \overline{L}_1]) \\
\varrho_0(i[L_1, L_2]) & \varrho_0(i[L_2, L_2])
\end{pmatrix}(p)
= \begin{pmatrix}
 i (L_1(A^1) - \overline{L}_1(A^1)) & i (L_2(A^1) - \overline{L}_1(A^2)) \\
 i (L_1(A^2) - \overline{L}_2(A^1)) & i (L_2(A^2) - \overline{L}_2(A^2))
\end{pmatrix}.$$ We will make 3 standing hypotheses.

**Hypothesis 1.11.** At every point $p \in M$:

$$1 = \text{rank \, Levi}(p).$$ So the determinant of the $2 \times 2$ matrix $\text{Levi}(p)$ vanishes identically. Furthermore, after performing an affine transformation, we can assume that the $(1, 1)$-entry is nowhere vanishing, and we attribute a name to it.

**Hypothesis 1.12.** At every point $p \in M$:

$$0 \neq I := i (L_1(A^1) - \overline{L}_1(A^1))$$

The kernel of the Levi matrix is of rank 1 at every point, and is generated by the section of $T^{1,0}M$ defined by:

$$\mathcal{K} := k \, L_1 + L_2,$$ which uses an important slant function $k$ obtained by quotienting the first row entries:

$$k := \frac{i (L_2(A^1) - \overline{L}_1(A^2))}{i (L_1(A^1) - \overline{L}_1(A^1))}.$$
Since the two generators $\mathcal{L}_1$ and $\mathcal{L}_2$ have been chosen with constant coefficient 1 in front of $\frac{\partial}{\partial z_1}$ and of $\frac{\partial}{\partial z_2}$, we have the relation:

$$0 \equiv k [\mathcal{L}_2, \mathcal{D}_1] + [\mathcal{L}_1, \mathcal{D}_1].$$

**Notation 1.14.** The Levi kernel rank 1 subbundle will be denoted by:

$$K^{1,0}M \subset T^{1,0}M.$$ 

It is generated by $\mathcal{K}$. The conjugate bundle $K^{0,1}M := \overline{K^{1,0}M}$ is generated by $\overline{\mathcal{K}}$. By examining the above relation, one can see that:

$$0 \equiv [\mathcal{K}, \mathcal{L}_1] \equiv [\mathcal{K}, \mathcal{D}_2] \equiv [\mathcal{K}, \overline{\mathcal{K}}] \mod T^{1,0}M \oplus T^{0,1}M.$$ 

Although we will not use this, let us mention that the thus obtained Frobenius involutivity:

$$[\Gamma(K^{1,0}M), \Gamma(K^{0,1}M)] \subset \Gamma(K^{1,0}M) \oplus \Gamma(K^{0,1}M)$$

implies that $M$ is foliated by complex holomorphic curves i.e. 2-surfaces locally biholomorphic to $\mathbb{C}$. In analogy with this, for the affine geometry of surfaces $S \subset \mathbb{R}^3$, a consequence of the vanishing Gaussian curvature assumption $0 \equiv F_{xx}F_{yy} - (F_{xy})^2$ is that such surfaces $S$ are foliated by $\mathcal{C}^\omega$ real curves $\{\gamma_s(t)\}_{s \in \mathbb{R}}$ along which the extrinsic tangent planes are constant:

$$T_{\gamma_s(t)}S = T_{\gamma_s(t')}S,$$

and this means that $S$, equipped with the Riemannian metric inherited from $\mathbb{R}^3$, is developpable, i.e. diffeomorphic and isometric to $\mathbb{R}^2$ with its flat metric. However, such isometric diffeomorphisms are not affine in general.

In Section 5, we will show the

**Lemma 1.15.** For a $\mathcal{C}^\omega$ surface $S \subset \mathbb{R}^3$ given by $S = \{u = F(x, y)\}$ satisfying $0 \neq F_{xx}$ and $0 \equiv F_{xx}F_{yy} - (F_{xy})^2$, the quantity:

$$S_{\text{aff}} := \frac{F_{xx}F_{xy} - F_{xy}F_{xxx}}{(F_{xx})^2}$$

is an affine invariant. \(\square\)

More precisely, the identical vanishing and the nowhere vanishing of $S_{\text{aff}}$ is preserved under affine transformations (at least, those close to the identity).

The counterpart in CR geometry of $S_{\text{aff}}$ is Pocchiola’s function $\overline{\mathcal{L}}_1(k)$ introduced above, and it also enjoys invariancy. Through a biholomorphism $h: M \rightarrow M'$, it is clear that the Levi kernel bundle must be preserved:

$$h_*(K^{1,0}M) = K^{1,0}M',$$

whence $h_*(K^{0,1}M) = K^{0,1}M'$ as well. This simple observation legitimates the following concept, which we formulate in bundle terms: all points $p \in M$ are considered.
Definition 1.16. The Freeman form is the map:

\[
K^{1,0}M \times \left( \frac{T^{1,0}M \mod K^{1,0}M}{T^{1,0}M \mod K^{1,0}M} \right) \rightarrow \mathbb{C} \\
(\mathcal{H}, \mathcal{L}) \mapsto [\mathcal{H}, \mathcal{L}] \mod \left( K^{1,0}M \oplus K^{0,1}M \right).
\]

Of course, the involutiveness of \( K^{1,0}M \oplus K^{0,1}M \) shown above guarantees that this map is well defined.

All these considerations show that it is more natural to choose:

\[ \{ \mathcal{L}_1, \mathcal{H} \} \]

as a frame for \( T^{1,0}M \), and to disregard \( \mathcal{L}_2 \). Of course, only the direction field generated by \( \mathcal{H} \) is invariant, and there is no canonical choice for \( \mathcal{L}_1 \). However, the normalization to 1 of the coefficient of \( \frac{\partial}{\partial z_1} \) in \( \mathcal{L}_1 \) is useful for computations.

In this frame \( \{ \mathcal{L}_1, \mathcal{H} \} \), the Freeman form amounts to computing the single Lie bracket:

\[
[k \mathcal{L}_1 + \mathcal{L}_2, \mathcal{L}_1] = -\mathcal{L}_1(k) \mathcal{L}_1 + k [\mathcal{L}_1, \mathcal{L}_1] + [\mathcal{L}_2, \mathcal{L}_1],
\]

(1.13)

and hence, the Freeman form coincides with a single function on \( M \):

\[
p \mapsto -\mathcal{L}_1(k)(p).
\]

When \( M^5 \subset \mathbb{C}^3 \) is a tube, namely when its graphing function \( F = F(x_1, x_2) \) is independent of the three imaginary axes coordinates \((y_1, y_2, u)\) so that \( M^5 = S^2 \times (i \mathbb{R})^3 \) is a product of a real surface with \((i \mathbb{R})^3\), one verifies that:

\[
k = -\frac{F_{x_1 x_2}}{F_{x_1 x_1}},
\]

whence:

\[
\mathcal{L}_1(k) = \frac{1}{2} \frac{\partial}{\partial x_1} \left( -\frac{F_{x_1 x_2}}{F_{x_1 x_1}} \right) = -\frac{1}{2} \frac{F_{x_1 x_1} F_{x_1 x_1 x_2} - F_{x_1 x_2} F_{x_1 x_1 x_1}}{(F_{x_1 x_1})^2}.
\]

Consequently, there is an immediate analogy with the so-called parabolic surfaces \( S^2 \subset \mathbb{R}^3 \) graphed as \( \{ u = F(x, y) \} \) whose third-order invariant also writes:

\[
\frac{\partial}{\partial x} \left( \frac{F_{yx}}{F_{xx}} \right) = \frac{F_{xx} F_{xy} - F_{xy} F_{xx}}{(F_{xx})^2}.
\]

Proposition 1.17. \([6, 23]\) The following conditions are equivalent for a Levi rank 1 hypersurface \( M^5 \subset \mathbb{C}^3 \).

(i) \( 0 \equiv \mathcal{L}_1(k) \) vanishes identically.

(ii) \( M^5 \subset \mathbb{C}^3 \) is locally biholomorphically equivalent to a product \( \mathbb{C} \times M^3 \) of \( \mathbb{C} \) with a Levi nondegenerate hypersurface \( M^3 \subset \mathbb{C}^2 \).

Hence in view of Convention 1.8, it is legitimate to make the third

Hypothesis 1.18. At every point \( p \in M \):

\[
\mathcal{L}_1(k)(p) \neq 0.
\]
Terminology 1.19. Such \( M \) will be said to be 2-nondegenerate.

We shall also abbreviate:

\[
S = \mathcal{F}(k).
\]

Needless to say, invariances hold.

**Proposition 1.20.** Under a local biholomorphic change of coordinates:

\[
(z_1, z_2, w) \mapsto (z'_1, z'_2, w') = (z'_1(z_1, z_2, w), z'_2(z_1, z_2, w), w'(z_1, z_2, w)),
\]

close to the identity mapping, it holds at every point \( p \in M \):

1. \( l(F')(h(p)) = \text{nonzero} \cdot l(F)(p) \);
2. \( S(F')(h(p)) = \text{nonzero} \cdot S(F)(p) \).

□

The 5-dimensional CR manifolds under consideration deserve a name.

**Terminology 1.21.** The class \( C_{2,1} \) consists of hypersurfaces \( M^5 \subset \mathbb{C}^3 \) that are:

(a) of constant Levi rank 1;
(b) 2-nondegenerate at every point, or equivalently, have everywhere nonzero Freeman form.

We conclude this summarized presentation of background concepts by citing yet a few results valid in \( \mathbb{C}^{n+1} \).

**Definition 1.22.** A (connected) local \( C^\omega \) hypersurface \( M^{2n+1} \subset \mathbb{C}^{n+1} \) is called *holomorphically degenerate* if there exists a nonzero \((1,0)\) vector field \( H = \sum_k A_k(z, w) \frac{\partial}{\partial z_k} + B(z, w) \frac{\partial}{\partial w} \) having holomorphic coefficients which is tangent to \( M \).

When such an \( H \) exists, after straightening \( H \mapsto H' = \frac{\partial}{\partial z_1} \) in a neighborhood of any point \( p \in M \) at which \( H \big|_p \neq 0 \), it is easy to see that \( M \cong \mathbb{C} \times M' \) is locally biholomorphic to a product of \( \mathbb{C} \) with a lower-dimensional hypersurface \( M' \subset \mathbb{C}^n \). It is therefore natural to study only holomorphically nondegenerate hypersurfaces \( M \subset \mathbb{C}^{n+1} \). In fact, it is known ([17, 24]) that all \( C^\omega \) hypersurface \( M^5 \subset C_{2,1} \) are holomorphically nondegenerate.

In fact, a general notion of \( k \)-nondegeneracy exists, with arbitrary integers \( k \geq 1 \), which for \( k = 1 \) coincides with Levi nondegeneracy. However, we shall neither present nor review this notion here, because when \( n = 2 \), the nondegeneracy of the Freeman form presented above is equivalent to 2-nondegeneracy.

**Proposition 1.23.** [24] For a \( C^\omega \) hypersurface \( M \subset \mathbb{C}^{2n+1} \), at any point \( p \in M \):

\[
M \text{ is Levi nondegenerate at } p \Rightarrow M \text{ is finitely nondegenerate at } p \Rightarrow M \text{ is holomorphically nondegenerate at } p.
\]

□

Hence (by far), the assumption of holomorphic nondegeneracy is the most general. A converse is also known.
Proposition 1.24. \[24\] If a $C^\omega$ connected $M^{2n+1} \subset \mathbb{C}^{n+1}$ is holomorphically non-degenerate, then there is a proper closed real analytic subset $\Sigma \subset M$ such that $M$ is $k$-nondegenerate at every point $p \in M \setminus \Sigma$ for some integer $k = k_M$ satisfying $1 \leq k_M \leq n$. □

When $n = 1$, Levi nondegeneracy then holds generically. When $n = 2$, both cases $k_M = 1$ and $k_M = 2$ occur. We are interested in $k_M = 2$.

Lastly, the consideration of only $C^\omega$ CR-equivalences is justified by the difficult

Theorem 1.25. \[18\] Every $C^\infty$ local CR-diffeomorphism $f : M \longrightarrow M'$ between $C^\omega$ holomorphically nondegenerate hypersurfaces of $\mathbb{C}^n$ must be $C^\omega$. □

This theorem has been established without assuming any constancy of any geometric quantity. When $M$ is finitely nondegenerate, the proof relies on a simple modification of the Pinchuk reflection principle.

For a $C^\omega$ hypersurface $M \subset \mathbb{C}^n$, one defines two pseudogroups:

$$\text{Aut}_{\text{CR}}(M) := \left\{ f : M \longrightarrow M \text{ local } C^\omega \text{ CR-diffeomorphism} \right\},$$

$$\text{Hol}(M) := \left\{ h : M \longrightarrow M \text{ local biholomorphism} \right\}.$$

Since by Theorem 1.2 every $C^\omega$ CR function is the restriction of a holomorphic function, they are isomorphic:

$$\text{Aut}_{\text{CR}}(M) \cong \text{Hol}(M).$$

A statement proved in a more general context yields:

Theorem 1.26. \[8\] If a $C^\omega$ hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$ is finitely nondegenerate, then $\text{Aut}_{\text{CR}}(M) \cong \text{Hol}(M)$ is a finite-dimensional local $C^\omega$ real Lie group. □

Hence for any $M$ in the class $\mathcal{C}_{2,1}$ which is 2-nondegenerate, $\text{Aut}_{\text{CR}}(M)$ is a finite-dimensional local Lie group.

Next, the Lie algebra:

$$\text{aut}_{\text{CR}}(M) := \text{Lie} \left( \text{Aut}_{\text{CR}}(M) \right),$$

is obtained by differentiating 1-parameter families $(f_t)_t \in \text{Aut}_{\text{CR}}(M)$, hence it consists of real vector fields:

$$Z := \left. \frac{d}{dt} \right|_{t=0} f_t.$$

The unique 1-parameter family $(h_t)_t$ of local biholomorphisms $h_t : M \longrightarrow M$ satisfying:

$$h_t|_M = f_t$$

then defines a Lie algebra of holomorphic vector fields also obtained by differentiation:

$$X := \left. \frac{d}{dt} \right|_{t=0} h_t,$$

and this provides a Lie algebra:

$$\mathfrak{hol}(M) := \text{Lie} \left( \text{Hol}(M) \right).$$

One verifies that:

$$Z = X + \mathfrak{X}.$$
Although \( h\mathfrak{o}(M) \) consists of holomorphic vector fields, it is a \textit{real} Lie algebra. More precisely, with a basis:

\[
h\mathfrak{o}(M) = \text{Vect}\mathbb{R}\left(X_1, \ldots, X_r\right),
\]

the structure constants \( c_{s,j,k}^s \) in:

\[
[X_j, X_k] = \sum_{1 \leq s \leq r} c_{s,j,k}^s X_s,
\]

are all \textit{real} numbers \( c_{s,j,k}^s \in \mathbb{R} \). Then with \( Z_s := X_s + \overline{X}_s \), it is clear that one also has:

\[
[Z_j, Z_k] = \sum_{1 \leq s \leq r} c_{s,j,k}^s Z_s.
\]

2. Presentation of the Results

In a series of papers [14, 12, 13] after a research monograph [11], Isaev studied zero CR-curvature equations for a special class of CR submanifolds \( M^5 \subset \mathbb{C}^3 \), assuming \( M^5 = S^2 \times (i\mathbb{R})^3 \) is a \textit{tube}, with \( S^2 \subset \mathbb{R}^3 \) a surface, so that differential computations and integrations are accessible, not too complicated.

The class, called \( \mathcal{C}_{2,1} \), consists of 2-nondegenerate Levi constant rank 1 hypersurfaces \( M^5 \subset \mathbb{C}^3 \), \textit{cf.} Terminology [1, 19]. This excludes the degenerate situation when \( M^5 \cong \mathbb{C} \times M^3 \) is (locally) biholomorphic to a product of \( \mathbb{C} \) with a Levi nondegenerate \( M^3 \subset \mathbb{C}^2 \).

In this paper, coordinates on \( \mathbb{C}^3 \) will be alternatively denoted:

\[
(z_1, z_2, w) = (x_1 + iy_1, x_2 + iy_2, u + iv) \quad \text{or} \quad (x + i\zeta, y + i\eta, u + iv).
\]

In order to avoid Analysis of PDE’s, all geometric objects will be assumed real-analytic (\( C^\omega \)) throughout (but [21] is forthcoming).

Three teams (at least) attacked the local biholomorphic equivalence for \( M \in \mathcal{C}_{2,1} \), especially reduction to an \( \{e\} \)-structure: Isaev-Zaitsev [14]; Medori-Spiro [16, 17]; Pocchiola, the author, and Foo [27, 22, 4]. But only the Ph.D. [27] of Pocchiola provides \textit{explicit} calculations in terms of a \( \mathcal{C}^\omega \) graphing function:

\[
M : \quad \left\{ (z_1, z_2, w) \in \mathbb{C}^3 : \quad u = F(x_1, y_1, x_2, y_2, v) \right\},
\]

which is necessary for application to the classification problem. The recent prepublication [4] shows that \( \sim 50 \) pages of detailed computations within the Cartan method of equivalence are required until one arrives at Pocchiola’s two primary differential invariants:

\[
W = W(J^5_{x_1,y_1,x_2,y_2,v} F) \quad \text{and} \quad J = J(J^6_{x_1,y_1,x_2,y_2,v} F).
\]

Secondary invariants are covariant derivatives of \( W \) and \( J \) within the \( \{e\} \)-structure bundle.

\textbf{Notation 2.1.} The symbol ‘nonzero’ shall denote various local \( \mathcal{C}^\omega \) functions which are \textit{nowhere vanishing} — possibly after restricting to unmentioned neighborhoods.
Suppose \( h : \mathbb{C}^3 \rightarrow \mathbb{C}^3 \) is a local biholomorphism which sends CR-diffeomorphically \( M \) onto its image \( M' := h(M) \), graphed similarly as:

\[
M' := \{(z'_1, z'_2, w') \in \mathbb{C}^3 : \quad u' = F'(x'_1, y'_1, x'_2, y'_2, v')\},
\]

Pocchiola’s invariants for \( M' \) are computed by means of exactly the same universal formulas in terms of \( F' \). Section 3 offers a presentation.

**Theorem 2.2.** Under a biholomorphic equivalence:
\[
W(F') = \text{nonzero} \cdot W(F) \quad \text{and} \quad J(F') = \text{nonzero} \cdot J(F). \quad \square
\]

Furthermore, the identical vanishing \( W \equiv 0 \equiv J \) constitutes the interesting zero CR-curvature equations. In depth and quite strikingly, both \( W \) and \( J \) contain \( > 10^5 \) differential jet monomials.

But fortunately, when \( M \) is a tube, namely has a graphing function independent of \( y_1, y_2, v \):
\[
M := \{(z_1, z_2, w) \in \mathbb{C}^3 : \quad u = F(x_1, x_2)\},
\]
which means that \( 2 \Re (i \frac{\partial}{\partial w}) = \frac{\partial}{\partial v} \) generates a 1-parameter group of (local) biholomorphisms \( (z_1, z_2, w) \mapsto (z_1, z_2, w + it) \) of \( \mathbb{C}^3 \) stabilizing \( M \) and similarly with \( 2 \Re (i \frac{\partial}{\partial z_1}), 2 \Re (i \frac{\partial}{\partial z_2}) \), Pocchiola’s invariants contract substantially.

**Theorem 2.3.** In the tube case:
\[
W_{\text{aff}} := W = \frac{5}{3} \frac{(F_{xx})^2 F_{xxy} - F_{xx} F_{xy} F_{xxx} + 2 F_{xy} (F_{xxx})^2 - 2 F_{xx} F_{xxx} F_{xyy}}{F_{xx} (F_{xx} F_{xyy} - F_{xy} F_{xxx})^2},
\]
and:
\[
J = J_{\text{aff}} := -\frac{1}{54} \frac{1}{(F_{xy} F_{xx} - F_{xy} F_{xxx})^3} \left\{ \right.
\]
\[
- 9 F_{xxxxxx} F_{yy}^3 F_{xxy}^2 + 45 F_{xxxxx} F_{yxy}^2 F_{xxy}^2 - 45 F_{xx} F_{xxy} (F_{xxx})^2 + 2 F_{xy} (F_{xxx})^2 - 2 F_{xx} F_{xxx} F_{xyy} +

+ 40 F_{xy} F_{xxx} F_{xxy} - 90 F_{xxx} F_{yxy} F_{xxy} + 45 F_{xy} F_{xx} F_{xxx} F_{xxy} - 45 F_{xy} F_{xxx} F_{xxy} F_{xxx} +

+ 90 F_{xxx} F_{yxy} F_{xxy} F_{xxy} + 90 F_{xxx} F_{xxy} F_{xxy} F_{xxy} - 90 F_{xxx} F_{yxy} F_{xxy} F_{xxx} + 120 F_{xxx} F_{yxy} F_{xxy} F_{xxx} -

- 120 F_{xxx} F_{yxy} F_{xxy} F_{xxy} - 90 F_{xxx} F_{yxy} F_{xxy} F_{xxy} - 45 F_{xxx} F_{yxy} F_{xxy} F_{xxx} + 90 F_{xxx} F_{yxy} F_{xxy} F_{xxx} +

+ 45 F_{xx} F_{yxy} F_{xxy} F_{xxy} - 9 F_{xxx} F_{yxy} F_{xxy} F_{xxy} + 9 F_{xx} F_{xxx} F_{xxy} F_{xxy} -

- 45 F_{xxx} F_{yxy} F_{xxy} F_{xxy} F_{xxy} + 45 F_{xxx} F_{yxy} F_{xxy} F_{xxx} + 90 F_{xxx} F_{yxy} F_{xxx} F_{xxy} F_{xxy} -

- 45 F_{xxx} F_{yxy} F_{xxx} F_{xxy} F_{xxy} - 45 F_{xxx} F_{yxy} F_{xxx} F_{xxy} F_{xxx} + 45 F_{xxx} F_{yxy} F_{xxx} F_{xxy} F_{xxx} -

- 45 F_{xxx} F_{yxy} F_{xxx} F_{xxy} F_{xxx} - 90 F_{xxx} F_{yxy} F_{xxx} F_{xxy} F_{xxx} + 18 F_{xxx} F_{yxy} F_{xxx} F_{xxy} F_{xxx} -

- 18 F_{xxx} F_{yxy} F_{xxx} F_{xxy} F_{xxx} \}. \quad \square
\]

Without any special assumption on \( F \), a byproduct of Cartan’s method characterizes \( M^5 \subset \mathbb{C}^3 \) having zero Pocchiola curvature, as being biholomorphically equivalent to a well known model.

**Theorem 2.4.** \([27, 22, 4]\) For a \( C^\omega \) hypersurface \( M^5 \subset \mathbb{C}^3 \) belonging to the class \( C_{2,1} \), the following two conditions are equivalent:

(i) \( W \equiv 0 \equiv J \);
(ii) $M^5 \subset \mathbb{C}^3$ is locally biholomorphic to the CR tube:

$$T_{LC} := \left\{ (x + i \zeta, y + i \eta, u + i v) \in \mathbb{C}^3 : u = \frac{x^2}{1 - y} \right\}.$$

Here, the acronym ‘LC’ stands for Light Cone.

It is easy to see that a neighborhood of $0 \in T_{LC}$ is biholomorphic to a neighborhood of any smooth point of the (affinely homogeneous) complex tube over the light cone in $\mathbb{R}^3$:

$$C_{LC} := \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : x_1^2 - x_2^2 - x_3^2 = 0 \right\}.$$

It is also known (\cite{7,5}) that $C_{LC}$ is locally biholomorphic to:

$$M_{LC} := \left\{ w + \overline{w} = \frac{2z_1 \overline{z}_1 + z_1^2 \overline{z}_2 + \overline{z}_1^2 z_2}{1 - z_2 \overline{z}_2} \right\}.$$

With these three coordinate representations of the model, it is clear that:

$$\text{Aut}_{\text{CR}}(T_{LC}) \cong \text{Aut}_{\text{CR}}(C_{LC}) \cong \text{Aut}_{\text{CR}}(M_{LC}),$$

and from \cite{7}, the Lie algebra:

$$\mathfrak{so}(M_{LC}) = \text{Vect}_{\mathbb{R}}(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10})$$

$$\cong so_{3,2}(\mathbb{R}),$$

is generated by the 10 holomorphic vector fields:

$X_1 := i \partial_w,$

$X_2 := z_1 \partial_{z_1} + 2w \partial_w,$

$X_3 := iz_1 \partial_{z_1} + 2iz_2 \partial_{z_2},$

$X_4 := (z_2 - 1) \partial_{z_1} - 2z_1 \partial_w,$

$X_5 := (i + iz_2) \partial_{z_1} - 2iz_1 \partial_w,$

$X_6 := z_1 z_2 \partial_{z_1} + \left( \frac{z_2^2}{2} - 1 \right) \partial_{z_2} - \frac{z_1}{2} \partial_w,$

$X_7 := iz_1 z_2 \partial_{z_1} + i(\frac{z_2^2}{2} + 1) \partial_{z_2} - iz_1^2 \partial_w,$

$X_8 := i w z_1 \partial_{z_1} - iz_1^2 \partial_{z_2} + iw^2 \partial_w,$

$X_9 := (z_1^2 - z_2 w - w) \partial_{z_1} + (2z_1 z_2 + 2z_1) \partial_{z_2} + 2z_1 w \partial_w,$

$X_{10} := (-iz_1^2 + iz_2 w - iw) \partial_{z_1} + (-2iz_1 z_2 + 2iz_1) \partial_{z_2} - 2iz_1 w \partial_w.$

It can be verified straightforwardly that this is a real Lie algebra, and that each real vector field $X_k + \overline{X}_k$ is tangent to $M_{LC}$.

Coming back to invariants and equivalences, all real affine transformations of $\mathbb{C}^3$, hence which respect the splitting $\mathbb{C}^3 = \mathbb{R}^3 \times (i \mathbb{R}^3)$, are biholomorphic transformations:

$$\text{Aff}_3(\mathbb{R}) \subset \text{Bihol}_3(\mathbb{C}),$$

while group dimensions show a high discrepancy:

$$12 < \infty.$$

Logically, we therefore deduce that the expressions (shown above) of Pocchiola’s invariants in the tube case $W_{aff}$ and $J_{aff}$ are affine invariants!
Theorem 2.5. Under a real affine equivalence of $\mathbb{C}^3$:
\[ W_{aff}(F') = \text{nonzero} \cdot W_{aff}(F) \quad \text{and} \quad J_{aff}(F') = \text{nonzero} \cdot J_{aff}(F). \]

Section 5 endeavors to recover from scratch the affine invariancy of $W_{aff}$ and of $J_{aff}$ in the real space $\mathbb{R}^3 \ni (x, y, u)$.

Next, because $\dim \text{Bihol}_3(\mathbb{C}) \gg \dim \text{Aff}_3(\mathbb{R})$, it is natural to expect that there exist hypersurfaces $M^5 \in \mathbb{C}_{2,1}$ such that:
\[ \begin{aligned}
M \cong_{\text{Bihol}} T_{\text{LC}}, \\
M \not\cong_{\text{Aff}} T_{\text{LC}}.
\end{aligned} \]

Problem 2.6. Find affine differential invariants $I_1, I_2, \ldots$ whose vanishing characterizes affine equivalence of a surface $S = \{ u = F(x, y) \}$ to the light cone:
\[ \begin{aligned}
0 \equiv I_1 \equiv I_2 \equiv \cdots \iff S \cong \{ u = \frac{x^2}{1 - y} \}.
\end{aligned} \]

Of course, $W_{aff}$ and $J_{aff}$ are among $I_1, I_2, \ldots$. So the question is: are there further affine invariants? It might very well be so!

Indeed, another much studied case concerns hypersurfaces $M^3 \subset \mathbb{C}^2$. Let them be given in coordinates:
\[ (z, w) = (x + iy, u + iv), \]
as real $\mathcal{C}^\omega$ graphs:
\[ u = F(x, y, v). \]
Assume that $M$ is Levi nondegenerate, which means (cf. Definition 1.4), that at each point $p \in M$, the Levi (Hermitian) form on $\mathbb{C}^{n=1}$ is a nonzero (real) number. Two generators of $T^{1,0}M$ and $T^{0,1}M$ are:
\[ \mathcal{L} := \frac{\partial}{\partial z} + A \frac{\partial}{\partial v} \quad \text{and} \quad \overline{\mathcal{L}} := \frac{\partial}{\partial \overline{z}} + \overline{A} \frac{\partial}{\partial \overline{v}}, \]
where:
\[ A := -i \frac{F_z}{1 + i F_v}. \]
The Levi nondegeneracy assumption is equivalent to the everywhere nonvanishing of:
\[ l := i \left( \overline{A} + A \overline{A} - A \overline{A} \right) \neq 0. \]
Introduce also a function whose complete expansion in terms of $J_{x,y,v}F$ would be one page long:
\[ P := \frac{l_z + A l_v - l A_v}{l}. \]

Theorem 2.7. [20] A Levi nondegenerate $\mathcal{C}^\omega$ local hypersurface $M^3 \subset \mathbb{C}^2$ is biholomorphically equivalent to the tube representation of a piece of the unit sphere $S^3 \subset \mathbb{C}^2$:
\[ \begin{aligned}
M \cong_{\text{Bihol}} \{ u = x^2 \},
\end{aligned} \]
if and only if:
\[ \begin{aligned}
0 \equiv I_{\text{Cartan}} := -2 \overline{\mathcal{L}}(\mathcal{L}(\overline{\mathcal{L}}(P))) + 3 \overline{\mathcal{L}}(\overline{\mathcal{L}}(\mathcal{L}(P))) - 7 \overline{P} \mathcal{L}(\mathcal{L}(P)) + 4 \overline{P} \mathcal{L}(\mathcal{L}(P)) \mathcal{L}(\overline{P}) + 2 \overline{P} \mathcal{P} \mathcal{L}(\mathcal{P}).
\end{aligned} \]
Unfortunately, the real and imaginary parts of $I_{\text{Cartan}}$ contain $> 10^6$ differential monomials in $J_{x,y,v}^6 F$. But when $M = \{ u = F(x) \}$ is tube, the 1 page long expression of $P$ contracts as:

$$P = \frac{1}{2} \frac{F_{xxx}}{F_{xx}} = \overline{P},$$

and since this is a function of only $x$, hence independent of $v$:

$$\mathcal{L} \equiv \frac{1}{2} \frac{\partial}{\partial x} \equiv \overline{\mathcal{L}}.$$

**Corollary 2.8.** When the hypersurface $M^3 \subset \mathbb{C}^2$ is tube defined as $\{ u = F(x) \}$, it holds:

$$I_{\text{Cartan}} = \frac{1}{16} \left\{ (F_{xx})^3 F_{xxxxx} - 7 (F_{xx})^2 F_{xxx} F_{xxxxx} - 4 (F_{xx})^2 (F_{xxxx})^2 + 25 F_{xx} (F_{xxx})^2 F_{xxxx} - 15 (F_{xxx})^3 \right\}. \quad \square$$

In this much studied context, affine equivalence to the model parabola $\{ u = F(x) \}$ is characterized by the vanishing of a completely different invariant.

**Theorem 2.9.** [9] The following two conditions are equivalent for a $C^\omega$ curve $\gamma = \{ u = F(x) \}$ with $F_{xx} \neq 0$.

(i) $\gamma$ is affinely equivalent to $\{ u' = (x')^2 \}$.

(ii) The graphing function $F$ satisfies the 5th order ordinary differential equation:

$$0 \equiv I_{\text{Halphen}} := 3 F_{xx} F_{xxxx} - 5 (F_{xxxx})^2. \quad \square$$

It is easy to verify that by differentiation:

$$0 \equiv I_{\text{Halphen}} \quad \implies \quad (I_{\text{Cartan}} \equiv 0),$$

whereas the reverse implication is false. So a classification problem arises, solved by Dadok-Yang under $C^7$-smoothness assumption. We ‘restrict’ their result to the $C^\omega$ context.

**Theorem 2.10.** [2] Any spherical $C^\omega$ tube hypersurface $\{ u = F(x) \} \subset \mathbb{C}^2$ is equivalent to one of the following:

(1) $u = x^2$;

(2) $u = e^x$;

(3) $u = \arcsin e^x$;

(4) $u = \arcsinh e^x$. \quad \square$

In higher dimension, much more advanced results appeared in Isaev’s monograph [11]. This motivated the quest for analogous classifications of tube hypersurfaces $M^5 \subset \mathbb{C}_{2,1}$ enjoying Pocchiola’s zero CR curvature equations. Isaev discovered that the counterpart of Dadok-Yang’s list consists in just a single item!

**Theorem 2.11.** [12] If $M^5 \in \mathbb{C}_{2,1}$ of class $C^\infty$ satisfies $0 \equiv W_{\text{aff}} \equiv J_{\text{aff}}$, then:

$$M \overset{\text{Aff}}{\cong} \left\{ u = \frac{x^2}{1-y} \right\}. \quad \square$$
After a preliminary reminder of Pocchiola's approach ([27, 22, 4]) in Section 3, we propose, in Section 4, an alternative shorter (2 pages) proof of this unexpected discovery, assuming \( M \) of class \( \mathcal{C}^\omega \); why \( \mathcal{C}^\omega \) CR-flat tubes \( M \in \mathcal{C}_{2,1} \) are automatically \( \mathcal{C}^\omega \) will be explained later ([21]) in a general context. Section 5 presents a direct computational approach to purely affine invariants. Section 6 presents the Halphen and Monge simpler planar invariants. Section 7 formulates problems.

**Acknowledgments.** Alexander Isaev brought fruitful perspectives when visiting Orsay University in February 3 – 18, 2019. The-Anh Ta read carefully the manuscript.

3. Affine Pocchiola Invariants \( W_{\text{aff}} \) and \( J_{\text{aff}} \) for Tube hypersurfaces

\[ M^5 = S^2 \times (i \mathbb{R}^3) \subset \mathbb{C}^3 \]

We follow [4], which employs the alternative notation \( W_0 \equiv W \) and \( J_0 \equiv J \), and which confirmed the expressions of [27] without any mistake:

\[
W := - \frac{1}{3} \frac{\mathcal{H}(\overline{\mathcal{L}_1(\mathcal{L}_1(k)))}}{\mathcal{L}_1(k)^2} + \frac{1}{3} \frac{\mathcal{H}(\mathcal{L}_1(k))}{\mathcal{L}_1(k)^3} \mathcal{L}_1(\mathcal{L}_1(k)) + \frac{2}{3} \frac{\mathcal{L}_1(\overline{\mathcal{L}_1(k)})}{\mathcal{L}_1(k)} + \frac{2}{3} \frac{\mathcal{L}_1(\mathcal{L}_1(k))}{\overline{\mathcal{L}_1(k)}} + \sqrt{-1} \frac{\mathcal{J}(\mathcal{S}_1)}{3 \mathcal{L}_1(k)}. \]

and:

\[
J = \frac{1}{6} \frac{\mathcal{L}_1(\mathcal{L}_1(\mathcal{L}_1(\mathcal{L}_1(k))))}{\mathcal{L}_1(k)} - \frac{5}{6} \frac{\mathcal{L}_1(\mathcal{L}_1(\mathcal{L}_1(k)))}{\mathcal{L}_1(k)^2} \mathcal{L}_1(\mathcal{L}_1(k)) - \frac{1}{6} \frac{\mathcal{L}_1(\mathcal{L}_1(\mathcal{L}_1(k)))}{\mathcal{L}_1(k)} \mathcal{P} + \frac{20}{27} \frac{\mathcal{L}_1(\mathcal{L}_1(k))^3}{\mathcal{L}_1(k)^3} + \frac{5}{18} \frac{\mathcal{L}_1(\mathcal{L}_1(k))^2}{\mathcal{L}_1(k)^2} \mathcal{P} + \frac{1}{6} \frac{\mathcal{L}_1(\mathcal{L}_1(k))}{\mathcal{L}_1(k)} \frac{\mathcal{L}_1(\mathcal{P})}{\mathcal{P}} + \frac{1}{9} \frac{\mathcal{L}_1(\mathcal{L}_1(k))}{\mathcal{L}_1(k)} \frac{\mathcal{P}}{\mathcal{P}} - \frac{1}{6} \frac{\mathcal{L}_1(\mathcal{L}_1(\mathcal{P}))}{\mathcal{L}_1(k)} + \frac{1}{3} \frac{\mathcal{L}_1(\mathcal{P})}{\mathcal{L}_1(k)} \mathcal{P} - \frac{2}{27} \frac{\mathcal{P}}{\mathcal{P}} \mathcal{P} \mathcal{P}. \]

For general \( M^5 \subset \mathbb{C}^3 \), the complete expansions of \( W \) and of \( J \) contain millions of terms.

Suppose therefore that \( M^5 = S^2 \times (i \mathbb{R}^3) \) is tube:

\[ \{ u = F(x_1, x_2) \}. \]

Then:

\[
\mathcal{L}_1 = \frac{\partial}{\partial x_1} - \frac{i}{2} F_{x_2} \frac{\partial}{\partial v}, \quad \mathcal{L}_2 = \frac{\partial}{\partial x_2} + \frac{i}{2} F_{x_2} \frac{\partial}{\partial v}, \quad \overline{\mathcal{L}_1} = \frac{\partial}{\partial z_1} + \frac{i}{2} F_{x_1} \frac{\partial}{\partial v}, \quad \overline{\mathcal{L}_2} = \frac{\partial}{\partial z_2} - \frac{i}{2} F_{x_2} \frac{\partial}{\partial v}, \]

whence:

\[ \mathcal{H} = k \mathcal{L}_1 + \mathcal{L}_2 = - \frac{F_{x_1 x_2}}{F_{x_1 x_1}} \mathcal{L}_1 + \mathcal{L}_2. \]
So the action of the derivations $\mathcal{L}_1, \mathcal{K}_1$, $\overline{\mathcal{L}}_1, \overline{\mathcal{K}}_1$ on functions depending only on $(x_1, x_2)$ identifies with the actions of the purely real vector fields:

\[
L_1 := \frac{1}{2} \frac{\partial}{\partial x_1}, \\
K := -\frac{1}{2} \frac{F_{x_2} x_2}{F_{x_1} x_1} \frac{\partial}{\partial x_1} + \frac{1}{2} \frac{\partial}{\partial x_2}.
\]

It follows that all four quantities:

\[
\mathcal{Z}_1(k) = \mathcal{L}_1(\overline{k}) = \mathcal{L}_1(k) = \overline{\mathcal{Z}}_1(\overline{k}) = -\frac{1}{2} \frac{F_{xx} F_{xy} - F_{xy} F_{xx}}{(F_{xx})^2}
\]

are real, where we already have switched notation:

\[(x_1, x_2) \equiv (x, y).
\]

Then the second fundamental function is also real:

\[
P = \frac{1}{2} \frac{F_{xxx} F_{xx} - F_{xx} F_{xxx}}{(F_{xx})^2} = \overline{P}.
\]

Observe from reality the vanishing:

\[
\mathcal{F}(k) = i \left[ \mathcal{L}_1, \mathcal{Z}_1 \right](k) = i \mathcal{L}_1(\overline{\mathcal{Z}}_1(k)) - \overline{\mathcal{Z}}_1(\mathcal{L}_1(k)) = 0,
\]

by reading and translating $W$ and $J$ above, we obtain:

\[
W_{\text{aff}} = \frac{2}{3} \frac{L_1(L_1(k))}{L_1(k)} + \frac{2}{3} \frac{L_1(L_1(k))}{L_1(k)^2} + \\
+ \frac{1}{3} \frac{L_1(L_1(k)) K(L_1(k))}{L_1(k)^3} - \frac{1}{3} \frac{K(L_1(L_1(k)))}{L_1(k)^2} + 0,
\]

together with:

\[
J_{\text{aff}} = \frac{1}{6} \frac{L_1(L_1(L_1(L_1(k))))}{L_1(k)} - \frac{5}{6} \frac{L_1(L_1(L_1(k))) L_1(L_1(k))}{L_1(k)^2} - \frac{1}{6} \frac{L_1(L_1(L_1(k)))}{L_1(k)} P + \\
+ \frac{20}{27} \frac{L_1(L_1(k))}{L_1(k)^3} + \frac{5}{18} \frac{L_1(L_1(k))^2}{L_1(k)} P + \frac{1}{6} \frac{L_1(L_1(k)) L_1(P)}{L_1(k)} - \frac{1}{9} \frac{L_1(L_1(k))}{L_1(k)} P P - \\
- \frac{1}{6} \frac{L_1(L_1(k))}{L_1(k)} + \frac{1}{3} L_1(P) P - \frac{2}{27} P P P.
\]

The expansion of $J_{\text{aff}}$ can be done plainly, but in the expansion of $W_{\text{aff}}$, one must take account of relations coming from the assumption that the real Hessian of $F$ vanishes identically:

\[
F_{yy} = \frac{(F_{xy})^2}{F_{xx}}.
\]
Differentiations with respect to $x$ and to $y$ followed by replacements give:

$$F_{xyy} = 2 \frac{F_{xy} F_{xyy}}{F_{xx}} - \left( \frac{F_{xy}}{F_{xx}} \right)^2 F_{xxx},$$

$$F_{yyy} = 3 \frac{(F_{xy})^2 F_{xyy}}{(F_{xx})^2} - 2 \frac{(F_{xy})^3 F_{xxx}}{(F_{xx})^3}.$$  

Next:

$$F_{xxy} = 2 \frac{(F_{xyy})^2}{F_{xx}} - 4 \frac{F_{xy} F_{xyy} F_{xxx}}{(F_{xx})^2} + 2 \frac{F_{xy} F_{xyyy}}{F_{xx}} + 2 \frac{(F_{xy})^2 (F_{xxx})^2}{(F_{xx})^3} - \frac{(F_{xy})^2 F_{xxxx}}{(F_{xx})^2},$$

$$F_{xyy} = 6 \frac{F_{xy} (F_{xyy})^2}{(F_{xx})^2} - 12 \frac{(F_{xy})^2 F_{xxx} F_{xyy}}{(F_{xx})^3} + 3 \frac{(F_{xy})^2 (F_{xxx})^2}{(F_{xx})^2} + 6 \frac{(F_{xy})^3 (F_{xxx})^2}{(F_{xx})^3} - 2 \frac{(F_{xy})^3 F_{xxxx}}{(F_{xx})^2},$$

$$F_{yyy} = 12 \frac{(F_{xy})^2 (F_{xyy})^2}{(F_{xx})^3} - 24 \frac{(F_{xy})^3 F_{xxx} F_{xyy}}{(F_{xx})^4} + 12 \frac{(F_{xy})^4 (F_{xxx})^2}{(F_{xx})^5} + 4 \frac{(F_{xy})^3 (F_{xxx})^2}{(F_{xx})^4} - 3 \frac{(F_{xy})^4 F_{xxxx}}{(F_{xx})^4}.$$  

Similar formulas exist for $F_{xxyy}$, $F_{xyyy}$, $F_{yxyy}$, $F_{yyyy}$.

With a completely different approach, Isaev discovered in $[12, 13]$ that after these replacements, $W_{aff}$ which seems to be a 5th-order invariant, is in fact a 4th-order one.

**Proposition 3.1.** *After plain replacements:*

$$W_{aff} = \frac{(F_{xx})^2 F_{xxx} x F_{xx} F_{xy} F_{xxx} x 2 F_{xy} (F_{xxx})^2 - 2 F_{xx} F_{xxx} F_{xyy}}{F_{xx} (F_{xx} (F_{xxx} F_{xy} - F_{xy} F_{xxx})^2,}$$

$$= -4 \frac{(F_{xx})^2}{F_{xx} F_{xy} - F_{xy} F_{xxx}} L_1 (L_1 (k))$$

$$= \text{nonzero} \cdot L_1 (L_1 (k)).$$

Then under the hypothesis $0 \equiv W_{aff}$, many terms in $J_{aff}$ above cancel:

$$J_{aff} = -\frac{1}{6} L_1 (L_1 (k)) + \frac{1}{3} L_1 (P) P - \frac{2}{27} P P P \quad \text{(assumed $W_{aff} \equiv 0$)}$$

$$= -\frac{1}{432} 9 \frac{(F_{xx})^2 F_{xxx} - 45 F_{xx} F_{xxx} F_{xxx} + 40 (F_{xxx})^3}{(F_{xx})^3}.$$  

We recover the Monge invariant with respect to the first variable $x$, whose vanishing characterizes the fact that a planar graphed curve $\{ u = F(x) \}$ in $\mathbb{R}^2_{x, u}$ is contained in a (nondegenerate) conic ($[12]$ and see also Section [4]).

**Exercise 1.** Show that $J_{aff} \mod W_{aff}$ is not an affine invariant.

Anyway, the common zero-set $\{ 0 \equiv W_{aff} \equiv J_{aff} \}$ is invariant, and in conclusion:

**Theorem 3.2.** $[13]$ CR-flatness of hypersurfaces $M \in \mathcal{C}_{2,1}$ that are tube $\{ u = F(x, y) \}$ is characterized by the two identical vanishings:

$$0 \equiv 2 F_{xy} (F_{xxx})^2 - 2 F_{xx} F_{xxx} F_{xyy} + (F_{xx})^2 F_{xxx} - F_{xx} F_{xy} F_{xxx},$$

$$0 \equiv 9 (F_{xx})^2 F_{xxx} - 45 F_{xx} F_{xxx} F_{xxx} + 40 (F_{xxx})^3. \quad \square$$
Once these equations have been obtained and cleaned up, we can present our very short proof of Theorem 2.11 in the $C^\omega$ context.

4. Affine Rigidity via Differential Algebra Elimination

In $\mathbb{C}^3$ with coordinates $(x + i\zeta, y + i\eta, u + iv)$, consider therefore a local $C^\omega$ tube hypersurface graphed as:

$$M : u = F(x, y),$$

which is of constant Levi rank 1 and 2-nondegenerate:

$$F_{xx} \neq 0 \equiv F_{xx} F_{yy} - (F_{xy})^2 \quad \text{and} \quad F_{xx} F_{xyy} - F_{xy} F_{xxx} \neq 0.$$  

The model is an appropriate representation of the tube:

$$T_{\text{LC}} : u = \frac{x^2}{1 - y}.$$  

**Theorem 4.1.** A local $C^\omega$ real surface in $\mathbb{R}^3$:

$$u = F(x, y)$$

with $F_{xx} \neq 0$ which has identically zero Gaussian curvature:

$$0 \equiv F_{xx} F_{yy} - (F_{xy})^2$$

is locally affinely equivalent to the model light cone $u = \frac{x^2}{1 - y}$ if and only if:

$$0 \equiv 2 F_{xy} (F_{xxx})^2 - 2 F_{xx} F_{xxx} F_{xyy} + (F_{xx})^2 F_{xxxy} - F_{xx} F_{xy} F_{xxx},$$

$$0 \equiv 9 (F_{xx})^2 F_{xxxx} - 45 F_{xx} F_{xxx} F_{xxxx} + 40 (F_{xxx})^3.$$  

Our elementary arguments will consist in normalizing progressively $F(x, y)$ by means of successive appropriate changes of affine coordinates, and to ‘kill’ almost all Taylor coefficients, thanks to the 3 equations:

$$0 \equiv F_{xx} F_{yy} - (F_{xy})^2 \equiv W_{\text{aff}} \equiv J_{\text{aff}}.$$  

No integration of any differential equation will be required, as the title of this article indicates.

As a direct application, we recover a result proved in [12].

**Corollary 4.2.** $M$ is biholomorphically equivalent to $T_{\text{LC}}$ if and only if $M$ is real affinely equivalent to $T_{\text{LC}}$.  

**Proof of Theorem 4.1.** Only $\implies$ matters. After elementary real affine transformations:

$$u = F(x, y) = x^2 + O_{x,y}(3) = F_0(y) + x F_1(y) + x^2 F_2(y) + x^3 F_3(y) + x^4 F_4(y) + \cdots,$$

with $F_2(0) = 1$, $F_0(y) = O_y(3)$, $F_1(y) = O_y(2)$. Plug this in $\implies$:

$$0 \equiv (2 F_2 + 6x F_3 + O_x(2)) (F_{0,yy} + x F_{1,yy} + O_x(2)) - (F_{1,y} + 2x F_{2,y} + O_x(2))^2$$

$$\equiv 2 F_2 F_{0,yy} - (F_{1,y})^2 + x [2 F_2 F_{1,yy} + 6 F_3 F_{0,yy} - 4 F_{1,y} F_{2,y}] + O_x(2).$$
Use $F_2(0) \neq 0$ to invert and get:

$$F_{0,yy} = R \cdot F_{1,y}, \quad F_{1,yy} = R \cdot F_{0,yy} + R \cdot F_{1,y} = R \cdot F_{1,y},$$

where $R = R(y)$ denotes unspecified functions. From $F_{1,y}(0) = 0$ comes $F_{1,yy}(0) = 0$ and an iteration:

$$F_{1,yyy} = R \cdot F_{1,y} + R \cdot F_{1,yy} = R \cdot F_{1,y}, \ldots, F_{1,y^k} = R \cdot F_{1,y}, \ldots,$$

yields $F_1(y) \equiv 0$, so $F_{0,yy} \equiv 0$, whence $F_0(y) \equiv 0$ too. So:

$$u = x^2 + \alpha x^3 + \beta x^2 y + O_{x,y}(4) = x^2 + x^2 \left( \alpha x + \beta y \right) + O_{x,y}(4),$$

since from 2-nondegeneracy $0 \neq 2 \cdot 2 \beta - 0 \cdot 6 \alpha$. So:

$$u = x^2 + x^2 y + A x^4 + B x^3 y + C x^2 y^2 + O_{x,y}(5).$$

Then $\textcircled{1}$:

$$0 \equiv \left( 2 + 2 y + O_{x,y}(2) \right) \left( 2 C x^2 + O_{x,y}(3) \right) - \left( 2 x + O_{x,y}(2) \right)^2 = x^2 \left[ 4 C - 4 \right] + O_{x,y}(3)$$

forces $C = 1$.

Next, by redefining linearly:

$$u = x^2 + x^2 \left[ y + A x^2 \right] + B x^3 y + x^2 y^2 + O_{x,y}(5) = x^2 + x^2 y + B x^3 y + x^2 y^2 + O_{x,y}(5),$$

we come to:

$$F = x^2 + x^2 y + B x^3 y + x^2 y^2 + O_{x,y}(5).$$

From $\textcircled{2}$ at $(x, y) = (0, 0)$, we kill $0 = 0 - 0 + 2^2 \cdot 6 B - 0$.

We therefore come, after a finite number of affine reductions, to a suitable form in which $F_{xxx}(0) = 0 = F_{xxxx}(0)$:

$$F = x^2 + x^2 y + x^2 y^2 + O_{x,y}(5).$$
We claim that \( F_{x^k}(0) = 0 \) for all \( k \geq 3 \). Indeed, write \( F = R F_{xxx} + F_{xxxx} \), get \( F_{xxxx}(0) = 0 \), and iterate differentiations and substitutions to obtain \( F_{x^k} = R F_{xxx} + F_{xxxx} \) for all \( k \geq 5 \).

We claim that \( F_{x^k y}(0) = 0 \) for all \( k \geq 3 \). Indeed, from \( \tilde{F} \), solve \( F_{xxxy} = R F_{xxx} + R F_{xxxx} \), and proceed similarly.

We claim that \( F_{x^k y^\ell}(0) = 0 \) for all \( k \geq 3 \) and \( \ell \geq 2 \). Indeed, from \( F_{x^k y^{\ell-1}} = R F_{xxx} + R F_{xxxx} \), differentiate to get:

\[
F_{x^k y^\ell} = R F_{xxx} + R F_{xxxx} + R F_{xxxxx} = R F_{xxx} + R F_{xxxx}.
\]

So \( F(x, y) = x^2 F_2(y) =: x^2 G(y) \), with \( G(0) = G_y(0) = 1 \). Back to \( \tilde{F} \) \( 0 \equiv 2 G x^2 G_{yy} - (2x G_y)^2 \), we get:

\[
G_{yy} = 2! \left( \frac{(G_y)^2}{G} \right) \quad \Rightarrow \quad G_{yyy} = 2! \left( \frac{2G_y G_{yy}}{G} - 2! \left( \frac{(G_y)^2}{G} \right) \right) G_y = 3! \left( \frac{(G_y)^3}{G^2} \right)
\]

\[
\Rightarrow \quad G_y = k! \left( \frac{(G_y)^{k+1}}{G^{k-1}} \right),
\]

whence \( G(y) = 1 + y + y^2 + \cdots + y^k + \cdots \) and finally after having performed only affine transformations:

\[
u = \frac{x^2}{1 - y}.
\]

**5. Affine Invariants via Graph Transforms**

As promised, we now explain how \( W_{aff} \) and \( J_{aff} \) can be seen directly to be affine invariants. We will even develop the affine counterparts of the Levi form, of its kernel field \( \mathcal{K} \), of the nonvanishing function \( l \), of the slant function \( k \), and of the third-order invariant \( S = \mathcal{L}_1(k) \).

In \( \mathbb{R}^3 \ni (x, y, u) \), the real affine transformation group \( \text{Aff}_3(\mathbb{R}) = GL_3(\mathbb{R}) \ltimes \mathbb{R}^3 \) consists of changes of coordinates:

\[
x' = ax + by + cu + d, \\
y' = kx + ly + mu + n, \\
u' = px + qy + ru + s,
\]

having nonzero Jacobian determinant:

\[
\delta := \begin{vmatrix} a & b & c \\ k & l & m \\ p & q & r \end{vmatrix} \neq 0.
\]

We will assume throughout that such matrices are close to the identity:

\[
\begin{pmatrix} a & b & c \\ k & l & m \\ p & q & r \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

so that graphed surfaces \( S = \{ u = F(x, y) \} \) are transformed into similar graphed surfaces \( S' = \{ u' = F'(x', y') \} \). This means that by applying the \( C^\omega \) implicit
function theorem to the target graphed equation:

\[ px + qy + ru + s = F'(ax + by + cu + d, kx + ly + mu + n), \]

the variable \( u \) can be solved to recover the first graphed equation \( \{ u = F(x, y) \} \), that is to say:

\[ u' = F'(x', y') \iff u = F(x, y). \]

After some elementary preliminary affine normalizations, we can even assume that \( F = O_{x,y}(2) \), namely:

\[ F \sim 0, \quad F_x \sim 0, \quad F_y \sim 0. \]

Then all functions considered will be converging power series in the two variables \( (x, y) \), centered at the origin \((0, 0)\), namely:

\[ F(x, y) \in \mathbb{R}\{x, y\} \quad \text{and} \quad F'(x', y') \in \mathbb{R}\{x', y'\}. \]

Then the way how the implicit function theorem must be applied expresses under the form of a fundamental identity:

(5.1)

\[ px + qy + rF(x, y) + s \equiv F'(ax + by + cF(x, y) + d, kx + ly + mF(x, y) + n), \]

which holds identically in \( \mathbb{R}\{x, y\} \).

Differentiate this identity with respect to \( x \) and to \( y \):

\[ p + rF_x \equiv (a + cF_x)F'_x + (k + mF_x)F'_y, \]
\[ q + rF_y \equiv (b + cF_y)F'_x + (l + mF_y)F'_y. \]

To solve for \( F'_x, F'_y \), a certain \( 2 \times 2 \) determinant appears which we abbreviate as:

\[ \Lambda := \Lambda(J^1F) := al - bk + (cl - bm)F_x + (am - ck)F_y \sim 1, \]
and which is nowhere vanishing, since its value is close to 1.

Beyond, by differentiating with respect to \( x, x \), to \( x, y \), to \( y, y \), one solves \( F'_{xx}, F'_{xy}, F'_{yy} \) in terms of \( J^2_{xx,yy}F \), and the same determinant \( \Lambda \) appears, as general formulas show ([1, 19]). The affine invariancy of the Hessian is well known, and we state a relation that can be verified by a direct computation — exercise, some help is provided below.

**Lemma 5.2.** One has:

\[ F'_{xx} F'_{yy} - (F'_{xy})^2 = \frac{\delta^2}{\Lambda^4} \left( F_{xx} F_{yy} - (F_{xy})^2 \right). \]

This identity can be abbreviated as:

\[ F'_{xx} F'_{yy} - (F'_{xy})^2 = \text{nonzero} \cdot \left( F_{xx} F_{yy} - (F_{xy})^2 \right), \]

where the generic term ‘nonzero’ denotes various local functions which are nowhere vanishing — possibly after shrinking neighborhoods.

We will make three main hypotheses, which are meaningful locally, and which are invariant under affine transformations. The first one is:
Hypothesis 5.3. The Hessian is degenerate at every point:

\[ 0 \equiv F_{xx} F_{yy} - F_{xy} F_{xy}. \]

Not only the Hessian determinant, but also the Hessian matrix enjoy beautiful invariant properties. Indeed, abbreviate:

\[
A(x, y) := a x + b y + c F(x, y) + d,
B(x, y) := k x + l y + m F(x, y) + n,
C(x, y) := p x + q y + r F(x, y) + s,
\]

and differentiate the fundamental identity (5.1) once:

\[
C_x = A_x F'_{y'} + B_x F_y',
C_y = A_y F'_{x'} + B_y F_x',
\]

and twice:

\[
C_{xx} = A_{xx} F'_{x'} + B_{xx} F_y',
+ A^2_{xx} F''_{x'x'} + 2 A_x B_x F'_{x'y'} + B^2_x F'_{y'y'},
C_{xy} = A_{xy} F'_{x'} + B_{xy} F_y',
+ A_x A_y F''_{x'y'} + (A_x B_y + A_y B_x) F''_{x'y'} + B_x B_y F'_{y'y'},
C_{yy} = A_{yy} F'_{x'} + B_{yy} F_y',
+ A^2_{yy} F''_{x'x'} + 2 A_y B_y F'_{x'y'} + B^2_y F'_{y'y'}.
\]

Introduce the vector fields tangent to \( S \) and to \( S' \):

\[
L_x := \frac{\partial}{\partial x} + F_x \frac{\partial}{\partial u}, \quad L_{x'} := \frac{\partial}{\partial x'} + F'_{x'} \frac{\partial}{\partial u'},
L_y := \frac{\partial}{\partial y} + F_y \frac{\partial}{\partial u}, \quad L_{y'} := \frac{\partial}{\partial y'} + F'_{y'} \frac{\partial}{\partial u'},
\]

together with their companions, the horizontal-affine fields:

\[
H_x := \frac{\partial}{\partial x}, \quad H_{x'} := \frac{\partial}{\partial x'},
H_y := \frac{\partial}{\partial y}, \quad H_{y'} := \frac{\partial}{\partial y'}.
\]

Although \( H_x, H_y \) and \( H_{x'}, H_{y'} \) are not intrinsically related to the geometry of the surfaces \( S \) and \( S' \), they will be useful to show that the Hessian matrices enjoy invariant properties. Two natural differential 1-forms:

\[
\varrho := du - F_x dx - F_y dy \quad \text{and} \quad \varrho' := du' - F'_{x'} dx' - F'_{y'} dy',
\]

represent the tangent spaces:

\[
TS = \{ \varrho = 0 \} = \text{Vect}(L_x, L_y) \quad \text{and} \quad TS' = \{ \varrho' = 0 \} = \text{Vect}(L_{x'}, L_{y'}).
\]

Clearly (exercise):

\[
\varrho = \mu' \varrho',
\]

in terms of the nowhere vanishing function:

\[
\mu' = r - c F'_{x'} - m F'_{y'} \sim 1.
\]
The proof of the next elementary proposition is left to the reader. And the con-
stitution of appropriate concepts is also left as an exercise, with the hint of taking
inspiration from Section 8 of [23], by realizing that the source Hessian matrix can be
written under the appropriate form:
\[
\begin{pmatrix}
F_{xx} & F_{yx} \\
F_{xy} & F_{yy}
\end{pmatrix} = \begin{pmatrix}
\varphi([H_x, L_x]) & \varphi([H_y, L_x]) \\
\varphi([H_x, L_y]) & \varphi([H_y, L_y])
\end{pmatrix},
\]
and similarly in the target space:
\[
\begin{pmatrix}
F'_{xx'} & F'_{yx'} \\
F'_{xy'} & F'_{yy'}
\end{pmatrix} = \begin{pmatrix}
\varphi'([H'_{x'}, L_{x'}]) & \varphi'([H'_{y'}, L_{x'}]) \\
\varphi'([H'_{x'}, L_{y'}]) & \varphi'([H'_{y'}, L_{y'}])
\end{pmatrix}.
\]

Proposition 5.4. The Hessian matrices in the source space \(\mathbb{R}^3_{x,y,u}\) and in the target
space \(\mathbb{R}^3_{x',y',u'}\) enjoy:
\[
\begin{pmatrix}
F_{xx} & F_{yx} \\
F_{xy} & F_{yy}
\end{pmatrix} = \mu' \begin{pmatrix}
A_x & B_x \\
A_y & B_y
\end{pmatrix} \begin{pmatrix}
F'_{xx'} & F'_{yx'} \\
F'_{xy'} & F'_{yy'}
\end{pmatrix} \begin{pmatrix}
a & b \\
k & l
\end{pmatrix}.
\]

This demonstrates that not only their (zero) determinants, but also their ranks are
the same!

The most degenerate and easiest case occurs when the Hessian matrix is identi-
cally zero, and the proof is very easy.

Lemma 5.5. The following two conditions are equivalent for a graphed \(C^\omega\) surface
\(S = \{u = F(x, y)\}\) in \(\mathbb{R}^3\).
(i) The Hessian matrix of the graphing function is identically zero:
\[F_{xx} \equiv F_{xy} \equiv F_{yx} \equiv F_{yy} \equiv 0.\]
(ii) \(S\) is affinely equivalent to the flat plane \(\{u' = 0\}\), with identically zero graphing
function \(F' \equiv 0\). \(\square\)

Since this case is trivial, let us therefore assume that the rank of the Hessian matrix
is at least one!

Therefore, if our \(2 \times 2\) Hessian matrix is not identically zero, we assume that it is
nowhere zero. After an elementary affine transformation, we come to our second

Hypothesis 5.6. At every point \(F_{xx} \neq 0\).

To confirm the invariancy of such a hypothesis, introduce the nowhere vanishing
quantity:
\[
\Upsilon := \Upsilon(J^2 F) := (l + m F_y) F_{xx} - (k + m F_x) F_{xy} \sim F_{xx} \neq 0.
\]

Lemma 5.7. One has (exercise):
\[
F'_{xx'} = \frac{\delta \Upsilon^2}{\Lambda^3} \frac{1}{F_{xx}}.
\]

Next, we yet want to exclude the situation where \(S = \{u = F(x, y)\}\) is affinely
equivalent to \(\{u = x^2\}\), a product of a parabola in \(\mathbb{R}^2_{x,y}\) with \(\mathbb{R}_y\), and this can be
done by means of an affine invariant which has been much studied in CR geometry.
Lemma 5.8. One has (exercise):
\[
\frac{F'_{x'} F'_{x'}' y' - F'_{x'} y' F'_{x'}'}{(F'_{x'}')^2} = \frac{F_{x x} F_{x y} - F_{x y} F_{x x}}{(F_{x x})^2}.
\]

Similarly as in [27, 22, 4], let us abbreviate this invariant as:
\[
S_{aff} := \frac{F_{x x} F_{x y} - F_{x y} F_{x x}}{(F_{x x})^2}.
\]

Proposition 5.9. The following two conditions are equivalent for a graphed \(C^\infty\) surface \(S = \{u = F(x, y)\}\) in \(\mathbb{R}^3\) satisfying \(F_{x x} \neq 0\) and \(0 \equiv F_{x x} F_{y y} - F_{x y}^2\):
(i) Its invariant \(S_{aff}\) vanishes identically:
\[
0 \equiv F_{x x} F_{x y} - F_{x y} F_{x x}.
\]
(ii) \(S\) is affinely equivalent to \(\{u' = (x')^2\}\).

The proof being again left as an exercise — study and adapt Section 9 of [23] for inspiration —, we come to our third and last

Hypothesis 5.10. At every point \(F_{x x} F_{x y} - F_{x y} F_{x x} \neq 0\).

We mention that thanks to the previous formulas, this numerator of \(k_{aff}\) and the one of \(k_{aff}'\) enjoy the transformation rule:
\[
\frac{F'_{x'} F'_{x'}' y' - F'_{x'} y' F'_{x'}'}{(F'_{x'})^2} = \frac{\delta^2 \gamma^3}{\Lambda^6} \frac{1}{F_{x x}^3} \left( F_{x x} F_{x y} - F_{x y} F_{x x} \right).
\]

Proposition 5.11. The affinization \(W_{aff}\) of Pocchiola’s invariant \(W\) satisfies under an affine equivalence:
\[
\left( F'_{x'} \right)^2 F'_{x'} F'_{x'} y' - F'_{x'} y' F'_{x'} = 2 F'_{x'} y' (F'_{x'})^2 - 2 F'_{x'} F'_{x'} F'_{x'} y' = \frac{\delta^2 \gamma^10}{\Lambda^6} \left( F_{x x}^2 F_{x y} - F_{x x} F_{x y} F_{x x x x} + 2 F_{x y} F_{x x x x} - 2 F_{x x} F_{x x x x} F_{x y} \right).
\]

Similarly:
\[
J_{aff}(F') = \frac{\delta^a \gamma^b}{\Lambda^c} J_{aff}(F),
\]
where \(a, b, c\) are integers.

Exercise 2. Determine \(a, b, c\)!

6. Curves in \(\mathbb{R}^2\): Toy Case

In \(\mathbb{R}^2 \ni (x, u)\), the real affine transformation group \(\text{Aff}_2(\mathbb{R}) = \text{GL}_2(\mathbb{R}) \ltimes \mathbb{R}^2\) consists of changes of coordinates:
\[
x' = a x + b y + c,
y' = p x + q y + r,
\]
having nonzero determinant \(a q - b p \neq 0\). The fundamental equation expressing how graphs are transformed writes as:
\[
p x + q F(x) + r \equiv F'(a x + b F(x) + c).
\]
The property of not being a straight line is invariant (exercise):
\[
F'_{x'x'} = \frac{(aq - bp)}{(a + b F_x)^3} F_{xx}.
\]

Assuming therefore that \(F_{xx} \neq 0\) is nowhere vanishing, whence \(F'_{x'x'} \neq 0\) as well, the Halphen and the Monge invariants \([9]\) are well known. We leave as an elementary exercise to the reader the task of proving the

**Theorem 6.1.** The Halphen invariant whose vanishing characterizes affine equivalence to \(\{u' = x'x'\}\) enjoys:

\[
3 F'_{x'x'} F'_{x'x'x'x'} - 5 \left( F'_{x'x'}\right)^3 = \frac{(aq - bp)^2}{(a + b F_x)^8} \left[ 3 F_{xx} F_{xxxx} - 5 \left( F_{xxx} \right)^2 \right],
\]

while the Monge invariant characterizing the fact that \(\{u = F(x)\}\) is contained in a conic of \(\mathbb{R}^2\) transforms as:

\[
9 \left( F'_{x'x'}\right)^2 F'_{x'x'x'x'} - 45 F'_{x'x'} F'_{x'x'x'} F'_{x'x'x'}, F'_{x'x'x'} + 40 \left( F'_{x'x'}\right)^3 = \frac{(aq - bp)^3}{(a + b F_x)^{12}} \left[ 9 \left( F_{xx}\right)^2 F_{xxxxx} - 45 F_{xx} F_{xxx} F_{xxxx} + 40 \left( F_{xxx} \right)^3 \right]. \quad \square
\]

**7. Open Questions**

Olver’s theory of moving frames ([26]) could certainly be applied in the present context.

**Question 7.1.** Study the structure of the full algebra of differential invariants of such real surfaces \(\{u = F(x, y)\}\) satisfying \(F_{xx} \neq 0\) and \(F_{xx} F_{yy} - (F_{xy})^2 \equiv 0\), and also:

\[
F_{xx} F_{xy} - F_{xy} F_{xxx} \neq 0.
\]

A \(C^\omega\) hypersurface \(M^{2n+1} \subset \mathbb{C}^{n+1}\) is called rigid when its graphing function is independent of \(v\):

\[
M : \quad \{u = F(x, y)\}.
\]

An coordinate-free formulation states that there exists an infinitesimal CR automorphism \(X \in \mathfrak{ho1}(M)\) with \(X(p) \notin T^{1,0} M \oplus T^{0,1} M\) at every point \(p \in M\). Indeed, after straightening \(X \mapsto X' = i \frac{\partial}{\partial w'}\) by means of some local biholomorphism, the tangency of \(2 \text{Re} \left( i \frac{\partial}{\partial w} \right)\) implies (exercise) that \(F'(x', y', v')\) must be independent of \(v'\).

**Problem 7.2.** [5] Classify rigid \(M \in \mathcal{C}_{2,1}\) modulo rigid biholomorphic equivalences:

\[
(z_1, z_2, w) \mapsto (z'_1(z_1, z_2), z'_2(z_1, z_2), w + w'(z_1, z_2)).
\]
REFERENCES

[1] Bluman, G.W.; Kumei, S.: Symmetries and differential equations, Applied Mathematical Sciences 81, Springer Verlag, New York, 1989, xiv+412 pp.

[2] Dadok, J.; Yang, P.: Automorphisms of tube domains and spherical hypersurfaces, Amer. J. Math. 107 (1985), no. 4, 999–1013.

[3] Fels, M.; Kaup, W.: CR manifolds of dimension 5: a Lie algebra approach, J. Reine Angew. Math. 604 (2007), 47–71.

[4] Foo, W.G.; Merker, J.: Differential (e)-structures for equivalences of 2-nondegenerate Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^3$, arxiv.org/abs/abs/02028/, 72 pages.

[5] Foo, W.G.; Merker, J.: Rigid equivalence problem for 5-dimensional 2-nondegenerate rigid real hypersurfaces $M^5 \subset \mathbb{C}^3$ of constant Levi rank 1, In preparation.

[6] Freeman, M.: Local biholomorphic straightening of real submanifolds, Annals of Mathematics (2) 106 (1977), no. 2, 319–352.

[7] Gaussier, H.; Merker, J.: A new example of uniformly Levi degenerate hypersurface in $\mathbb{C}^3$, Ark. Mat. 41 (2003), no. 1, 85–94. Erratum: 45 (2007), no. 2, 269–271.

[8] Gaussier, H.; Merker, J.: Nonalgebraizable real analytic tubes in $\mathbb{C}^n$, Math. Z. 247 (2004), no. 2, 337–383.

[9] Halphen, G.H.: Sur l’équation différentielle des coniques, Bulletin de la Société mathématique, t. VII, 1878–1879, pp. 83–84.

[10] Hörmander, L.: An introduction to complex analysis in several variables, North-Holland Publ. Co., 2nd ed., Amsterdam, London, 1973 (1st ed.: 1966).

[11] Isaev, A.: Spherical tube hypersurfaces, Lecture Notes in Mathematics, 2020, Springer, Heidelberg, 2011, xii+220 pp.

[12] Isaev, A.: Affine rigidity of Levi degenerate tube hypersurfaces, J. Differential Geom. 104 (2016), no. 1, 111–141.

[13] Isaev, A.: Zero CR-curvature equations for Levi degenerate hypersurfaces via Pocchiola’s invariants, arxiv.org/abs/1809.03029/, Ann. Fac. Sci. Toulouse, to appear.

[14] Isaev, A.; Zaitsev, D: Reduction of five-dimensional uniformly degenerate Levi CR structures to absolute parallelisms, J. Anal. 23 (2013), no. 3, 1571–1605.

[15] Mardare, S.: On isometric immersions of a Riemannian space under a weak regularity assumption, C. R. Acad. Sci. Paris, Sér. I 337 (2003), 785–790.

[16] Medori, C.; Spiro, A.: The equivalence problem for 5-dimensional Levi degenerate CR manifolds, Int. Math. Res. Not. IMRN 2014, no. 20, 5602–5647.

[17] Medori, C.; Spiro, A.: Structure equations of Levi degenerate CR hypersurfaces of uniform type. Rend. Semin. Mat. Univ. Politec. Torino 73 (2015), no. 1-2, 127–150.

[18] Merker, J.: On envelopes of holomorphy of domains covered by Levi-flat hats and the reflection principle, Annales de l’Institut Fourier (Grenoble) 52 (2002), no. 5, 1443–1523.

[19] Merker, J.: Lie symmetries of partial differential equations and CR geometry, Journal of Mathematical Sciences (N.Y.), 154 (2008), 817–922.

[20] Merker, J.: Rationality in Differential Algebraic Geometry, Complex Geometry and Dynamics, The Abel Symposium 2013, Abel Symposia, Vol. 10, Fornaess, John Erik, Irgens, Marius, Wold, Erlend Fornæss (Eds.), pp. 157–209, Springer-Verlag, Berlin, 2015.

[21] Merker, J.: Analytic hypoellipticity of zero CR-curvature equations, in preparation.

[22] Merker, J.; Pocchiola, S.: Explicit absolute parallelism for 2-nondegenerate real hypersurfaces $M^5 \subset \mathbb{C}^3$ of constant Levi rank 1, Journal of Geometric Analysis, 10.1007/s12220-018-9888-3, 42 pp.

[23] Merker, J.; Pocchiola, S.; Sabzevari, M.: Equivalences of 5-dimensional CR manifolds, II: General classes I, II, III, IV, V, arxiv.org/abs/1311.5669/, 5 figures, 95 pages.

[24] Merker, J.; Porten, E.: Holomorphic extension of CR functions, envelopes of holomorphy and removable singularities, International Mathematics Research Surveys, Volume 2006, Article ID 28295, 287 pages.

[25] Olver, P.J.: Equivalence, Invariance and Symmetries. Cambridge, Cambridge University Press, 1995, xvi+525 pp.

[26] Olver, P.J.: Differential invariants of surfaces, Differential Geom. Appl. 27 (2007), no. 2, 230–239.
[27] Pocchiola, S.: Explicit absolute parallelism for 2-nondegenerate real hypersurfaces $M^5 \subset \mathbb{C}^3$ of constant Levi rank 1, arxiv.org/abs/1312.6400/, 55 pages.

[28] Porter, C.: The local equivalence problem for 7-dimensional 2-nondegenerate CR manifolds whose cubic form is of conformal type, arxiv.org/abs/1511.04019/, 46 pages.

[29] Shabat, B.: Introduction à l’analyse complexe, 2 vols, traduit du russe par Djilali Embarek, Mir, Moscou, 1990, 309 pp ; 420 pp.