Images of the lines under the MS transformations and the Concept of Velocity in the DSR theories

Nosrtollah Jafari

Science and Research Branch, Islamic Azad University, Hesarak, Tehran, 14778, Iran. *

February 2, 2015

Abstract

The effect of the Maguejo-Smolin (MS) transformations on a straight line in the energy-momentum space will be studied. We will interpret the slope of this line as velocity \( \frac{dE}{dp} \), which can leads to addition rule for the velocities in the MS doubly special relativity (DSR) case. Relation between two expressions \( \frac{dE}{dp} \) and \( \frac{p}{E} \) for velocity in the momentum space will be investigated more and the energy dependency of the velocities in the DSR theories is related to the geometrical properties of the lines under DSR transformations. The images of two parallel lines under the MS transformations will be studied and we will compute crossing point of these lines under the MS transformations in the energy-momentum space. The linear-fractional transformations don’t keep parallelism. The crossing point is on a line in the energy-momentum space with a constant momentum \( E_p/c \).

Keywords: Linear-Fractional Transformations, Doubly Special Relativity, Geometry of the Energy-Momentum Space.

*nosrat.jafari@gmail.com
1 Introduction

We begin by studying the effect of the Lorentz transformations in the (1 + 1) dimensional energy-momentum space on a straight line. The Lorentz transformations map a straight line to another straight line in the energy-momentum space. From this property we can obtain the Einstein’s addition rule for the velocities from slope of this line. In continuation, we study the effect of the Maguejo-Smolin (MS) transformations on a straight line in the energy-momentum space. Also, we interpret the slope of this line as a velocity $dE/dp$ and we will obtain an addition rule for velocities in the MS doubly special relativity case. However, introducing velocities in the doubly special relativity (DSR) theories is a difficult problem, and there is a long discussion for introducing a convenient velocity $[1, 2, 3, 4, 5, 6, 7, 8]$. For example, in the MS DSR which is a very simple example of DSR theories, introducing $dE/dp$ as velocity will bring the properties of particles, such as energy, momentum or even mass to the definition of the velocity. In this paper, we relate the dependency of the velocity on the energy or momentum to the geometrical properties of the MS transformations and image of a line under these transformations.

Images of the two parallel lines in the energy-momentum under the Lorentz transformations will remain parallel, but under the MS transformations these lines will cross each other. We will find the coordinates of this crossing point which depends on the energy scale of the MS DSR i.e. the Planck Energy $E_p$.

Finally, we will compute a general formula for combination rule of the velocities in the MS DSR by taking derivatives from the MS transformations. This computation will bring our arguments and calculations to a harmony.
2 Image of a line in the Energy-Momentum space

2.1 The Lorentz Transformations case

The Lorentz transformations in the (1+1) dimensions are

\[
\begin{align*}
p'_1 &= \gamma(p_1 - vp_0), \\
p'_0 &= \gamma(p_0 - vp_1),
\end{align*}
\]  

(1)

here we have taken the \( p_1 \) component in \( x \)-direction and we have assumed \( c = 1 \).

We take the equation of a line in the momentum space which passing through \((\varphi_0, \varphi_1)\) point as

\[ p_0 = \psi p_1 + \Pi_0, \]  

(2)

where we have put \( \Pi_0 = \varphi_0 - \psi \varphi_1 \). For massive particles this line can be interpreted as a tangent line in the \((\varphi_0, \varphi_1)\) point to the mass-shell of a massive particle in the Energy-Momentum space. This tangent line has been shown in Fig. 1. In fact, the mass of this massive particle will be

\[ m^2 = \varphi_0^2 - \varphi_1^2. \]  

(3)

The Lorentz transformations Eq.(1) map the line in Eq.(2) to

\[
p'_0 = \left( \frac{\psi - u}{1 - \psi u} \right) p'_1 + \frac{\Pi_0}{\gamma(1 - u)} = \psi' p'_1 + \Pi'_0.
\]  

(4)

If we interpret \( \psi \) as the velocity of the particle, then the linearity of the Lorentz transformations will show that \( \psi' \) is in the form of the Einstein’s combination rule for velocities

\[ \psi' = \frac{\psi - u}{1 - \psi u} := v_H. \]  

(5)

The interpretation of \( \psi \) as the velocity is not so strange, because in classical mechanics we have the Hamiltonian expression

\[ v_H = \frac{dE}{dp}. \]  

(6)
for the velocity. But we have also another expression
\[ v_G = \frac{p}{E} \]  \tag{7} which can be interpreted as the velocity. Our notations for these velocities is from [6].

2.2 The MS Transformations case

The MS transformations in (1+1) dimensions are
\[
\begin{cases}
    p'_0 = \frac{\gamma(p_0 - up_1)}{1 + l_p(\gamma - 1)p_0 - l_p\gamma u p_1}, \\
    p'_1 = \frac{\gamma(p_1 - up_0)}{1 + l_p(\gamma - 1)p_0 - l_p\gamma u p_1},
\end{cases} \tag{8}
\]
in which \(l_p\) is the Planck length. These transformations map the straight line
\[ p_0 = \psi p_1 + \Pi_0 \]  \tag{9} to
\[
p'_0 = \left[ \frac{\psi - u + \Pi_0 l_p u}{1 - \psi u - \Pi_0 l_p (\gamma - 1)/\gamma} \right] p'_1 + \frac{\Pi_0}{[\gamma - \gamma \psi u - \Pi_0 l_p (\gamma - 1)]}. \tag{10}
\]
The combinations rule for the velocities will be

\[ v'_H = \frac{v - u + \Pi_0 l_p u}{1 - vu - \Pi_0 l_p (\gamma - 1)/\gamma}. \tag{11} \]

As in the Lorentz case we can interpret \( \psi \) as the velocity. Please note that this expression is a combination rule for a particle with constant velocity \( v \) in the \( S \) system. The more general formula will be obtained in the section 5. Here, unusual matter is the appearance of energy and momentum through \( \Pi_0 = \varphi_0 - \psi \varphi_1 \) in the expression for the relative velocity.

3 Images of two parallel lines in the Energy-Momentum space

3.1 The Lorentz transformations case

Under the Lorentz transformations the images of two parallel lines are parallel lines. However, the interpretation of a line in the Energy-Momentum space is difficult, but as discussed in section 2, we can draw a straight line tangent to the every point of the mass shell of a particle in the energy-momentum space. We can also draw two parallel lines as tangent lines to the mass shell of a massive particle which are shown in the Fig. 2. Images of these parallel lines under the Lorentz transformations will be parallel, but rotated in the \( (p_0, p_1) \) plane and we have not drawn them. However, we can assume any two parallel lines in the energy-momentum space, the parallel lines which are tangent to mass shell in the Fig. 2 are good for study of the effect of the Lorentz transformations on the rest mass of a particle and its conjugate antiparticle.

3.2 The MS transformations case

We want to study the effect of the MS transformations on two parallel lines

\[
\begin{align*}
\begin{cases}
p_0 = \psi p_1 + \Pi_0, \\
\tilde{p}_0 = \psi p_1 + \tilde{\Pi}_0,
\end{cases}
\end{align*}
\tag{12}
\]
Figure 2: Two parallel lines in the Energy-momentum space which are tangent to the mass shell

in the energy-momentum space. The MS transformations map these lines to

\[
\begin{align*}
\theta_0' &= \frac{\psi-u+\Pi_0 l_p u}{1-\psi u-\Pi_0 l_p (\gamma-1)/\gamma} p_1' + \Pi_0' \left[ \gamma - \gamma \psi u - \Pi_0 l_p (\gamma-1) \right], \\
\theta_0 &= \frac{\psi-u+\Pi_0 l_p u}{1-\psi u-\Pi_0 l_p (\gamma-1)/\gamma} p_1' + \Pi_0 \left[ \gamma - \gamma \psi u - \Pi_0 l_p (\gamma-1) \right].
\end{align*}
\] (13)

These lines cross each other at

\[
(p_0', p_1') = \left( -\psi' \frac{1-\psi u}{l_p \left[ u + \psi(1-\gamma)/\gamma \right]} + \Pi_0', -\frac{1-\psi u}{l_p \left[ u + \psi(1-\gamma)/\gamma \right]} \right),
\] (14)

in these formulas \(\psi'\) and \(\Pi'_0\) depend on the initial coordinates \((\varphi_0, \varphi_1)\) and are given by

\[
\psi' = \frac{v-u+\Pi_0 l_p u}{1-v u-\Pi_0 l_p (\gamma-1)/\gamma},
\] (15)

and

\[
\Pi' = \frac{\Pi_0}{\left[ \gamma - \gamma \psi u - \Pi_0 l_p (\gamma-1) \right]}.
\] (16)
For higher velocities $\gamma >> 1$, the expressions for the coordinates of the crossing point can be rewritten in a more simpler form

$$\left( p'_0, p'_1 \right) = \left( -\frac{1}{l_p} \psi' + \Pi'_0, -\frac{1}{l_p} \right),$$

$$= \left( E_p + \frac{\varphi_0 - \psi \varphi_1}{\gamma \left[ 1 - \psi + (\varphi_0 - \psi \varphi_1) \right]}, -E_p \right).$$

These points are on a line $p'_1 = -E_p/c$ which has been drawn as a dashed line in the Fig. 3

4 Definition of the Velocity

In the standard special relativity by taking derivative of

$$E^2 = p^2 + m^2$$

(18)
with respect to $p$, we obtain

$$\frac{dE}{dp} = \frac{p}{E},$$

(19)

which shows that in the special relativity two expressions for velocities $v_H$ and $v_G$ are equal

$$v_H = v_G.$$  

(20)

For the MS transformations, we can find a relation between these two definitions of the velocity by taking derivative from the dispersion relation for the MS DSR

$$\frac{E^2 - p^2}{(1 - E/E_p)} = m^2,$$

(21)

with respect to $p$, which yields

$$\frac{dE}{dp} = \frac{p}{E \left[ 1 + \frac{m^2}{EE_p}(1 - E/E_p) \right]}.$$

(22)

in these formulas $E_p$ is the Planck energy. Thus,

$$v_H = \frac{v_G}{1 + \frac{m^2}{EE_p}(1 - E/E_p)}.$$  

(23)

As evidently seen from this relation if we define $v_G$ as independent parameter from the mass or energy of the particle, then $v_H$ will depend on the mass of the particle. Thus, we can’t get rid of energy or mass dependency in the definitions of these velocities. There is a long discussion in the DSR literature that which formula should be interpreted as velocity $dE/dp$ or $p/E$ [1, 2, 3, 4, 5, 6, 7, 8].

In the MS transformations, because of the equality of the denominators, defining $v_G = p/E$ as velocity is like to the Lorentz transformations case and it has a well defined rule under the MS transformations. On the other hand, if we define the velocity as $v_H = \frac{dE}{dp}$, this velocity will depend on the energy and momentum of the particle as seen from Eq.(11).

These is also an important difference between expressions for the combination rule of the velocities in the Lorentz transformations case Eq.(4) and the MS transformations case Eq.(11). The value of $\Pi_0$, which depends on initial point ($\varphi_0, \varphi_1$) of a straight line in the momentum space has not been entered in the expression for the combination rule in the Lorentz case. But,
the appearance of \( \Pi_0 \) in Eq.(11) can be intercepted as a trace of an energy scale \( E_p \) in the MS transformations.

Also, the meaning of a line in the momentum space or momentum plane \((p_0, p_1)\) is not so clear. We usually identify any point \((E, \vec{p})\) in the momentum space as an energy and momentum of a physical particle. For a free particle which has a constant energy and momentum, this point will remain stationary in this space. If we apply any force on this particle which can modify the energy and momentum of this particle we will reach to another point of the momentum space.

5 The general Linear- Fractional transformations and MS DSR

In the Momentum space we can write a general Linear- Fractional transformations between \((p_0p_1)\) and \((p'_0p'_1)\) systems as

\[
p'_0 = \frac{a_0 + A_{00}p_0 + A_{01}p_1}{1 + \alpha_0p_0 + \alpha_1p_1}, \quad p'_1 = \frac{a_1 + A_{10}p_0 + A_{11}p_1}{1 + \alpha_0p_0 + \alpha_1p_1},
\]

in which \(A_{\mu\nu}\) is the elements of a linear transformations matrix elements and \(\alpha_0\) and \(\alpha_1\) is the parameters which shows deviations from the linearity. For the MS transformations we have

\[
[A_{\mu\nu}] = \begin{pmatrix}
\gamma & -u\gamma \\
-u\gamma & \gamma
\end{pmatrix},
\]

\[
\alpha_0 = l_p(\gamma - 1), \quad \alpha_1 = -l_p u\gamma.
\]

By differentiating Eq.(24) we will find \(v'_H = dp'_0/dp'_1\) as

\[
v'_H = \frac{\left[(A_{00} - \alpha_0a_0)v + (A_{01} - \alpha_1a_0) + (A_{01}a_0 - A_{00}\alpha_1)(p_0 - vp_1)\right]}{\left[(A_{10} - \alpha_0a_1)v + (A_{11} - \alpha_1a_1) + (A_{11}a_0 - A_{10}\alpha_1)(p_0 - vp_1)\right]},
\]

or in the compact form

\[
v'_H = \frac{A_{00}v + A_{01} - p'_1(\alpha_1v + \alpha_0)}{A_{10}v + A_{11} - p'_0(\alpha_1v + \alpha_0)}.
\]
For the MS transformations case we find

\[ v'_H = \frac{v - u + l_p(u(p_0 - v p_1))}{1 - u v - l_p(\gamma - 1)(p_0 - v p_1)/\gamma}. \] (29)

If we take a line in the momentum space as \( p_0 = v p_1 + \Pi_0 \) we reach to the Eq.(11), which is a special form of this equation. We had some discussions in the end of section 2 for the energy and momentum dependency of the \( v'_H \).

References

[1] G. Amelino-Camelia, D. Benedetti, F. D’Andrea, A. Procaccini, Class. Quant. Grav. 20 (2003) 5353.

[2] J. Lukierski, A. Nowicki, Acta Phys. Polon. B33 (2002) 2537.

[3] J. Rembielinski, K. A. Smolinski, Bull.Soc.Sci.Lett.Lodz 53 (2003) 57.

[4] A. Granik, “Maguejo-Smolin transformation as a consequence of a specific definition of mass, velocity, and the upper limit on energy”, [hep-th/0207113](https://arxiv.org/abs/hep-th/0207113)

[5] P. Kosinski, P. Meslanka, Phys. Rev. D 68 (2003) 067702.

[6] S. Mignemi, Phys. Lett. A 316 (2003) 173.

[7] M. Daszkiewicz, K. Imilksa, J. Kowalski-Glikman, Phys. Lett. A 323 (2004) 345.

[8] R. Aloisio, A. Galante, A.F. Grillo, E. Luzio, F. Mendez, Phys. Rev. D 70 (2004) 125012.