Quantum Singularities in Spacetimes with Spherical and Cylindrical Topological Defects.

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Exact solutions of Einstein equations with null Riemman-Christoffel curvature tensor everywhere, except on a hypersurface, are studied using quantum particles obeying the Klein-Gordon equation. We consider the particular cases when the curvature is represented by a Dirac delta function with support either on a sphere or on a cylinder (spherical and cylindrical shells). In particular, we analyze the necessity of extra boundary conditions on the shells.

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I. INTRODUCTION

Topological defects appear naturally in theories of the early universe based on the spontaneous symmetry breaking of some unifying group. Examples are the cosmic string, which appears in the breaking of a $U(1)$ symmetry group and the cosmic wall, which is produced in the breaking of a discrete symmetry. These defects are characterized by a null curvature tensor everywhere, except on a submanifold, where it is proportional to a Dirac delta function.

These spaces are classically singular, Wald \cite{1}, followed by Horowitz and Marolf \cite{2}, establishes an analogy between the classical and quantum singularities. The trajectories of classical particles (geodesics) used to test classical singularities are replaced by solutions to a quantum mechanics equation. Spacetimes with a 0-dimensional singularity, as well as, spacetimes with a 1-dimensional singularity have been classically and quantum mechanically studied \cite{1}-\cite{11}. In a seminal paper on this subject, Wald \cite{1} considered an arbitrary self-adjoint extension of the spatial portion of wave operator in an arbitrary static spacetime (with singularities consistent with staticity) and showed that the resulting solution agreed with the usual Cauchy evolution inside the domain of dependence of the initial surface.

Horowitz and Marolf \cite{2} considered a spacetime as quantum mechanically nonsingular when the evolution of a general state is uniquely defined for all time. They showed that there exist a general class of spacetimes with naked singularities which are nonsingular quantum mechanically.

Ishibashi and Hosoya \cite{7} studied several spacetimes with naked singularities using Sobolev spaces (instead of $L^2$) as the function space of initial data. They showed that some spacetimes, which remain quantum mechanically singular with a $L^2$ function space, become regular when probed by waves in Sobolev space (for example, the negative mass Schwarzschild spacetime). But, the Sobolev space is not the natural space in quantum mechanics.

Kay and Studer \cite{8} studied the 2-dimensional cone (the spacetime generated by a point source in a 3-dimensional Einstein gravity without cosmological constant \cite{3}), described by the metric

\begin{equation}
\text{ds}^2 = -dt^2 + dr^2 + \beta^2 r^2 d\theta^2,
\end{equation}

where the constant $\beta$ is related to the mass of the source. They showed that this spacetime remains singular when tested by quantum particles and found all the possible self-adjoint extensions necessary to turn self-adjoint the spatial portion of the wave operator. They are represented by a non trivial 1-parameter family of boundary conditions.

Helliwell, Konkowski and Arndt \cite{10} studied a $1+3$ dimensional spacetime with a cosmic dislocation and a disclination at $z = 0$ given by the metric \cite{5},

\begin{equation}
\text{ds}^2 = -dt^2 + dr^2 + \beta^2 r^2 d\theta^2 + (dz + \gamma d\theta)^2,
\end{equation}

where $\gamma$ is a screw dislocation parameter. They found that this spacetime remains singular and generalized the results to Maxwell and Dirac fields, with the same results.

To the best of our knowledge the spacetime singularities for defects of dimensions grater than one have not been studied from a quantum mechanical view point. The aim of this work is to study the evolution of quantum particles in spacetimes with defects on hypersurfaces, specifically on spherical (bubbles) and cylindrical shells.

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This paper is organized as follows. In section III we discuss some classical aspects of spacetimes with spherical and cylindrical walls. In section IV we present a short review of quantum singularities in a general static spacetime, based mainly on references 1 and 2. In section V we study the quantum behavior of particles in spacetimes with spherical and cylindrical defects and discuss about necessity of extra boundary conditions on the classical singular wall. In section VI we find a 1-parameter family of boundary conditions for spherical, as well as, cylindrical shells. In section VII we give a simple example which illustrates the results of the two previous sections. Finally, in section VIII we discuss some of our results and point out generalizations to spacetimes with shells with other symmetries and to quantum particles obeying Maxwell and Dirac equations.

II. SPHERICAL AND CYLINDRICAL WALLS

Spacetimes with a topological defect on a general hypersurface \( \phi = 0 \) can be constructed splitting the space in two regions, say \( \Omega^+ \) and \( \Omega^- \), where \( \phi > 0 \) and \( \phi < 0 \), respectively. By the requirement that the Riemann-Christoffel tensor, \( R_{\mu\nu\sigma\eta} \), be null on \( \Omega^+ \) and \( \Omega^- \), we obtain a spacetime similar to Minkowski spacetime, except for the defect.

We demand continuity of the metric and discontinuity of its first derivatives on \( \phi = 0 \). The dependence of \( R_{\mu\nu\sigma\eta} \) on the second derivative of \( g_{\mu\nu} \) gives us a delta function with support on the defect.

Spherical and cylindrical defects are obtained with the function \( \phi = r - r_0 \), where \( r \) is the radial coordinate on both cases. In the spherical case metric is \([13]\),

\[
\begin{align*}
    ds^2 &= -dt^2 + dr^2 + (r-b)^2 d\Omega^2, & r > r_0 \\
    ds^2 &= -dt^2 + dr^2 + (r-a)^2 d\Omega^2, & r < r_0,
\end{align*}
\]

where \( d\Omega^2 \) is the usual measure on \( S^2 \), \( b = r_0 + 4/\rho_0 \) and \( a = r_0 - 4/\rho_0 \), \( \rho_0 \) is a positive constant. This metric represents a static spherical shell of radius \( r_0 = (a+b)/2 \), centered at the origin, with density \( \rho = \rho_0 \delta(r - r_0) \). Note that on \( r = r_0 \), the metric is continuous, while its first derivatives have a discontinuity given by

\[
\begin{align*}
    [g_{\theta\theta,r}] &= \lim_{r \to r_0^-} g_{\theta\theta,r} - \lim_{r \to r_0^+} g_{\theta\theta,r} = 2(a-b), \\
    [g_{\phi\phi,r}] &= \lim_{r \to r_0^-} g_{\phi\phi,r} - \lim_{r \to r_0^+} g_{\phi\phi,r} = 2(a-b) \sin^2 \theta.
\end{align*}
\]

The non-null components of curvature tensor \( R_{\mu\nu\lambda\sigma} \), viewed as a distribution \([12]\), are

\[
\begin{align*}
    R_{1212} &= (b-a)\delta(r - r_0) \\
    R_{1313} &= (b-a)\sin^2 \theta \delta(r - r_0).
\end{align*}
\]

The non-null components of the energy-momentum tensor are

\[
\begin{align*}
    T_{00} &= (b-a)\delta(r - r_0) \\
    T_{11} &= -(b-a)\delta(r - r_0).
\end{align*}
\]

Similarly, for the cylindrical case, we have

\[
\begin{align*}
    ds^2 &= -dt^2 + dr^2 + (r-b)^2 d\varphi^2 + dz^2, & r > r_0 \\
    ds^2 &= -dt^2 + dr^2 + (r-a)^2 d\varphi^2 + dz^2, & r < r_0
\end{align*}
\]

but now \( b = r_0 + 2/\rho_0 \) and \( a = r_0 - 2/\rho_0 \). The metric in this case represents a static cylindrical shell centered at the \( z \) axis, of radius \( r_0 = (a+b)/2 \) and density \( \rho = \rho_0 \delta(r - r_0) \).

Note that on both cases, metrics inside and outside the shells, are obtained from the Minkowski spacetime by an isometric coordinate transformation \( r \to r - \) constant. Hence, they are isometric to Minkowski spacetime.

Note that if one allows the radial coordinate take negative values we have two Minkowski spacetimes separated by a defect, i.e., a wormhole, which makes these spacetimes particularly interesting.

The points \( r = a \) and \( r = b \) are metric singularities of the polar type, without any deeper geometrical meaning.

The real spacetime singularity is on \( r = r_0 \), the middle point between \( a \) and \( b \).

Figure 1 shows timelike geodesics on the external side of the bubble \([13]\), in the plane \( \theta = \pi/2 \). The continuous circle represents the physical singularity \( r = r_0 \), whereas the dashed one represents the metric singularity \( r = b \). The figure shows that the surface \( r = b \) is completely regular. The geodesics cross this surface with no problem.
FIG. 1: Motions of equal energy particles, with distinct angular momentum, subjects to the gravitational field generated by a spherical defect on the external side of the shell. Some of these geodesics end abruptly on the surface $r = r_0$, while the surface $r = b$ is perfectly regular.

III. QUANTUM SINGULARITIES

In general relativity, spacetime singularities are indicated by incomplete geodesics. At points where a geodesic abruptly ends, extra information is needed, since it is not possible to predict the future of a free falling particle in an incomplete geodesic spacetime. In quantum mechanics, this extra information corresponds to a boundary condition at the classical singular points needed to turn self-adjoint the spatial part of the wave operator. When this extra information is not necessary the evolution of the wave packet is uniquely determined by the wave function at $t = 0$. We say that space is not quantum mechanically singular. In such cases, the condition that the solutions be square-integrable works as a boundary condition and the initial conditions alone determine time evolution of the particles.

An example of a classical singular theory, which becomes nonsingular in the view of quantum mechanics, is the nonrelativistic hydrogen atom. The imposition of quadratic integrability of the solutions of Schrödinger equation are sufficient to provide a complete set of eingenfunctions. Given an initial wave packet, its time evolution is uniquely determined.

Another example, now of a singulity that remains singular when tested by quantum mechanics, is the nonrelativistic particle trapped in a 1-dimensional box. A boundary condition is necessary on both edges (it is usually taken $\psi(0) = \psi(1) = 0$) in order to evolve uniquely the wave packet. Now we shall present the concepts mentioned previously in a more precise way.

Let $(M, g_{\mu\nu})$ be a static spacetime with a timelike Killing vector field $\xi^\mu$. Also, let $t$ be the Killing parameter and $\Sigma$ a static spacelike slice (without any singularity) orthogonal to the Killing vector for all $t$. The Klein-Gordon equation on this space is

$$\Box \Psi = M^2 \Psi,$$

where $\Box = \nabla^\mu \nabla_\mu$ can be split in a temporal and a spatial part, like

$$\frac{\partial^2 \Psi}{\partial t^2} = -A \Psi$$

$$= V D^i (V D_i \Psi) - M^2 V^2 \Psi,$$

with $V = -\xi^\mu \xi_\mu$ and $D_i$ the spatial covariant derivative on the slice $\Sigma$.

To find the domain of the operator $A$, $D(A)$, is a more difficult task and generally no information is provided. So, a minimum domain is taken (a core), where the operator can be defined and which does not enclose the spacetime
singularities. An appropriate set is $C_0^\infty(\Sigma)$, the set of the smooth function of compact support on $\Sigma$. On the Hilbert space $L^2(\Sigma, V^{-1}d\mu)$, where $d\mu$ is the proper element on $\Sigma$, it is not difficult to show (integrating by parts) that the operator $(A, C_0^\infty(\Sigma))$ is a positive symmetric operator. Then, self-adjoints extensions always exist \[14\] (at least the Friedrichs extension). Let $A_E$ be such extension, obtained by relaxing the boundary condition so that the extended domain coincides with the domain of its adjoint operator. Because of the self-adjointness of $A_E$, the time evolution of the field $\Psi$ is uniquely determined by the initial wave packet and is given by

$$\Psi(t) = \cos(A_E^{1/2}t)\Psi(0) + A_E^{1/2}\sin(A_E^{1/2}t)\dot{\Psi}(0),$$

(11)

where functions of the operator $A_E$ exist by the spectral theorem \[14\].

For the nonrelativistic case, the situation is similar. The Schrödinger equation can be written as,

$$\frac{\partial \Psi}{\partial t} = -\nabla^2 \Psi,$$

(12)

where $\nabla^2$ is the Laplace-Beltrami on $(\Sigma, h_{ij})$ and $h_{ij}$ is the induced metric on $\Sigma$. If $-\nabla^2_E$ is a self-adjoint extension of $-\nabla^2$, the evolution of the particle is given by

$$\Psi(t) = e^{-i\nabla^2_E t}\Psi(0).$$

(13)

If many self-adjoint extensions exist, the choice of one is necessary in order to obtain the time evolution of the particle. Extra information is needed (boundary conditions), the spacetime is quantum mechanically singular and we can clearly see the resemblance to the classical case. However, if there is only one self-adjoint extension, the operator $A$ is said essentially self-adjoint and the quantum evolution of the particle is uniquely determined by the initial packet. The spacetime is quantum mechanically nonsingular.

To study quantum singularities of a static spacetime, i.e., the essentially self-adjointness of the spatial operator $A$ defined in equation \[10\], a tool used is the von Neumann theorem \[15\], which says that the self-adjoint extensions of a closed Hermitian operator $T$ is in a one-to-one correspondence with the partial isometries of $\text{Ker}(T^* - i)$ into $\text{Ker}(T^* + i)$, where $\text{Ker}(T^* \pm i)$ denotes the kernel of $T^* \pm i$ and $T^*$ denotes the Hilbert adjoint operator of $T$.

In our case, the domain of $A$ is so small, i.e., the restrictions on the functions in $D(A) = C_0^\infty(\Sigma)$ are so strong, that the domain of the adjoint operator becomes extremely wide and no restrictions on $\psi \in D(A^*)$ are necessary, except that $A^*\psi \in L^2$. Hence

$$D(A^*) = \{ \psi \in L^2 : A^*\psi \in L^2 \}.$$  

(14)

So, we must solve the following equations

$$A^*\psi \mp i\psi = 0$$

(15)

and count the number of solutions in $L^2$. We define $n_\pm = \text{dim Ker}(A^* \mp i)$ and name them the deficiency indecis \[14, 13, 16\].

If do not exist solutions of (15) in $L^2$, then $n_+ = n_- = 0$ and $A$ is essentially self-adjoint. If $n_+ = n_- = 1$, there is a 1-parameter family of isometries of $\text{Ker}(A^* - i)$ into $\text{Ker}(A^* + i)$, hence there is a 1-parameter family of self-adjoint extensions. The case $n_+ = n_- = 2$ is similar.

IV. QUANTUM MECHANICS AROUND BUBBLES AND CYLINDRICAL SHELLS

A. Bubbles

In the spacetime described by the metric \[3\], Klein-Gordon equation is similar to the wave equation in flat spacetime, except by the isometric transformation $r \to r - \text{constant}$,

$$\nabla_\mu \nabla^\mu \Psi = -\frac{\partial^2 \Psi}{\partial t^2} + \nabla^2 \Psi = M^2 \Psi.$$  

(16)

In our case

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial^2 \Psi}{\partial r^2} + \frac{2}{r - b} \frac{\partial \Psi}{\partial r} - \frac{L^2}{(r - b)^2} - M^2 \Psi$$

$$\frac{\partial^2 \Psi}{\partial r^2} = \frac{\partial^2 \Psi}{\partial r^2} + \frac{2}{r - a} \frac{\partial \Psi}{\partial r} - \frac{L^2}{(r - a)^2} - M^2 \Psi,$$

(17)
where \( L^2 \) is the square of angular momentum operator.

As we said previously, we must solve the equations

\[
A^* \psi = i \psi = 0,
\]

where \( A \) is the spatial portion of the wave equation (right hand side of equation (17)).

The equation separates using \( \psi = R(r)Y_l^m(\theta, \varphi) \), where \( Y_l^m(\theta, \varphi) \) are the usual spherical harmonics. The radial part is

\[
\begin{align*}
R''_{l,m}(r) + \frac{2}{(r-b)} R'_{l,m}(r) + \left[ \frac{l(l+1)}{(r-b)^2} - \frac{\pm M^2}{(r-b)^2} \right] R_{l,m}(r) &= 0 \\
R''_{l,m}(r) + \frac{2}{(r-a)} R'_{l,m}(r) + \left[ \frac{l(l+1)}{(r-a)^2} - \frac{\pm M^2}{(r-a)^2} \right] R_{l,m}(r) &= 0.
\end{align*}
\] (18)

The spacetimes on both sides of the bubble are completely independent (they are separated by a wall), so the appropriate Hilbert space is the tensorial product of the \( L^2 \) spaces on each side of the bubble. If we call \( \Sigma_1 \) the outer spatial slice and \( \Sigma_2 \) the inner one, we have

\[
H = L^2(\Sigma_1, (r-b)^2 \sin \theta dr d\theta d\varphi) \otimes L^2(\Sigma_2, (r-a)^2 \sin \theta dr d\theta d\varphi).
\] (19)

Therefore we shall analyze the spatial portion of the wave operators on each side of the bubble. Let us look first to the external side of the bubble, as \( r \) goes to infinity, the last term in equation (15) can be neglected and the asymptotic solution is

\[
R(r) = \frac{1}{(r-b)} [C_1 e^{\alpha(r-b)} + C_2 e^{-\alpha(r-b)}],
\] (20)

where

\[
\alpha = \frac{1}{\sqrt{2}} \left[ (\sqrt{1 + M^2} + M^2)^{1/2} \mp i(\sqrt{1 + M^2} - M^2)^{1/2} \right].
\] (21)

Obviously, solution (20) is square-integrable near infinity only if \( C_1 = 0 \). The behavior near \( r = r_0 \) does not really matter in this case, this point is an ordinary point of equation (15). Then, both solutions are continuous in \( r = r_0 \), hence, square-integrable near this point. So we can adjust the constants of the general solution in order to meet the asymptotic behavior at infinity \( R(r) \sim (1/r)e^{-\alpha(r-b)} \).

There is one solution in \( L^2 \), to each equation in (15). Hence \( n \pm = 1 \) and exists a one-parameter family of self-adjoints extensions of \( A \) in \( \Sigma_1 \).

For the inner portion of the bubble, the procedure is similar, but now with \( r \) taking values between \( -\infty \) and \( r_0 \). Again, there is a one-parameter family of self-adjoint extensions of \( A \) in \( L^2(\Sigma_2) \).

Hence, the spacetime (3) remains quantum mechanically singular. The evolution of a wave packet is not uniquely determined by the initial state of the particle and a boundary condition need to be imposed (independently) on both sides of the spherical wall.

**B. Cylindrical Shells**

For spacetime described by (7), the appropriate Hilbert space is

\[
H = L^2(\Sigma_1, |r-b| dr d\varphi dz) \otimes L^2(\Sigma_2, |r-a| dr d\varphi dz).
\] (22)

As in the previous case, after separating variables \( \psi = R(r)e^{im\varphi}e^{ikz} \), we have (for the radial portion outside the shell)

\[
R''_{m,k}(r) + \frac{1}{(r-b)} R'_{m,k}(r) + \left[ (\pm i - (k^2 + M^2)) - \frac{m^2}{(r-b)^2} \right] R_{m,k}(r) = 0.
\] (23)

As \( r \) goes to infinity, the asymptotic behavior of \( R(r) \) is

\[
R(r) \sim \frac{1}{\sqrt{(r-b)}} (C_1 e^{\alpha(r-b)} + C_2 e^{-\alpha(r-b)}).
\] (24)
where
\[ \alpha = \frac{1}{\sqrt{2}} \left[ \sqrt{1 + (M^2 + k^2)^2 + M^2 + k^2} \pm i\sqrt{1 + (M^2 + k^2)^2 - M^2 - k^2} \right]^{1/2}. \] (25)

Again, \( R(r) \) is square-integrable near \(+\infty\) only if \( C_1 = 0 \), hence
\[ R(r) \sim \frac{1}{\sqrt{(r-b)}} e^{-\alpha(r-b)}. \] (26)

For \( r \) near \( r_0 \), both solutions are continuous (as in the bubble case), hence square-integrable. Then we find a solution in \( L^2 \) for each equation in (15). The deficiency indices are \( n_\pm = 1 \), so there are infinitely many self-adjoint extensions of \( A \) in this case too. This space remains singular when tested by quantum particles.

V. BOUNDARY CONDITIONS IN \( r = r_0 \)

On both spacetimes studied in the last section the Klein-Gordon equation presents only one singular point, \( r = +\infty \) in the outer portion and \( r = -\infty \) in the inner one (we remind that \( r = a, b \) are polar metric singularities). The boundary conditions on \( r = r_0 \), necessary to turn self-adjoint the spatial portion of the wave operator are simple as we shall see.

The radial portions of the Klein-Gordon operator are given by
\[
A_r = \begin{cases} 
\frac{1}{(r-b)^2} \left\{ - \frac{d}{dr} \left[ (r-b)^2 \frac{d}{dr} \right] + l(l+1) + M^2(r-b)^2 \right\} & \text{(bubble)} \\
\frac{1}{(r-b)} \left\{ - \frac{d}{dr} \left[ (r-b) \frac{d}{dr} \right] + \frac{m^2}{(r-b)} + \right. \\
\left. + (k^2 + M^2)(r-b) \right\} & \text{(cylindrical shell)}
\end{cases}
\] (27)
in the Hilbert spaces
\[ L^2((r_0, \infty), (r - b)^2 dr) \quad \text{(bubble)} \]
\[ L^2((r_0, \infty), |r-b| dr) \quad \text{(cylindrical shell)}. \] (28)

On both cases, they have the general form
\[ A_r = -\frac{1}{w(r)} \left\{ \frac{d}{dr} \left[ p(r) \frac{d}{dr} \right] + q(r) \right\} \] (29)
in the Hilbert space \( L^2(\Sigma, w(r)dr) \).

Because \( r = r_0 \) is not a singular point of (29), the self-adjoint extensions are found extending the domain of \( A \) to functions in \( L^2 \) satisfying (15),
\[ f(r_0) \cos \alpha + p(r_0) f'(r_0) \sin \alpha = 0 \] (30)

with \( \alpha \in \mathbb{R} \).

So, the boundary conditions in our case are:

**External Side**

- **Bubble**
  \[ R(r_0) \cos \alpha + (r_0 - b)^2 R'(r_0) \sin \alpha = 0, \] (31)

- **Cylindrical Shell**
  \[ R(r_0) \cos \alpha + (r_0 - b) R'(r_0) \sin \alpha = 0, \] (32)

\( \alpha \in \mathbb{R} \).
VI. MASSLESS PARTICLE OUTSIDE THE SPHERICAL WALL

To illustrate results of sections IV and V, we will discuss a simple example, a massless particle in the outer side of the bubble. The Klein-Gordon equation can be solved exactly by separation of variables \( \Psi = R(r)Y^m_l(\theta, \varphi)e^{-i\omega t} \) (positive-frequency solution). The general solution of the radial part is

\[
R_{l,m}(r) = A_{l,m}j_l(\omega(r - b)) + B_{l,m}\eta_l(\omega(r - b)),
\]

where \( j_l \) and \( \eta_l \) are the spherical Bessel and Neumann functions, respectively. Because neither of them are square-integrable near infinity, there is only one linear combination of these functions which are square-integrable, apart from a multiplicative constant. One of the constants in (35) was eliminated and we have a set of square-integrable functions at \(+\infty, \{F_{l,m}\}\), so that

\[
R_{l,m} = C_{l,m}F_{l,m}(\omega(r - b)).
\]

We pick the parameter \( \alpha = 0 \), which corresponds to the boundary condition \( R(r_0) = 0 \) (Dirichlet boundary condition). This condition imposes a quantization of the allowed energies \( \{\omega_{l,n}\}_{n \in \mathbb{N}}, \) where \( \omega_{l,n}(r_0 - b) \) are the zeros of \( F_{l,m} \). We then have a complete set of eigenfunction \( \{F_{l,m,n}\} \) so that a general solution of Klein-Gordon equation can be written by a superposition of the allowed modes as

\[
\Psi(\vec{r}, t) = \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{l,m}F_{l,m,n}(r)Y^m_l(\theta, \varphi)e^{-i\omega_{l,n}t},
\]

resting only one constant to be determined. In this way, the knowledge of the initial packet is sufficient to determine the quantum dynamic of the particle. Note that the choice of one boundary conditions given in (31) turned our operator into self-adjoint, so that the evolution of the particle becomes uniquely determined by the initial wave packet.

VII. CONCLUSIONS

Two singular spacetimes that represent defects on a hypersurface were tested using quantum mechanics. We find that they remain singular when probed by waves. They had to be partitioned in two independent parts, so the splitting of the Hilbert space in a tensorial product was necessary. Due to the fact that the classical singular points \( (r - r_0 = 0) \) are ordinary points of Klein-Gordon equation the boundary conditions necessary to turn self-adjoint the spatial portion of the wave operator can be easily found. Two real parameters are necessary, because each side of the wall behaves independently.

A wide class of spacetimes with singularities in a 2-dimensional submanifold can be found in references [12] and [13]. They represent open and closed shells of different shapes, like parabolic and toroidal shells. These spacetimes are not as simple as the ones studied in the present work, but they are still static and, in principle, can be treated, in a similar way.

We do not foresee special difficulties to study Maxwell and Dirac fields for bubbles and cylindrical shells.
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[1] R.M. Wald, J. Math. Phys. 21, 2082 (1980).
[2] G.T. Horowitz and D. Marolf, Phys. Rev. 52, 5670 (1995).
[3] S. Deser, R. Jackiw, and G. ’t Hooft, Ann. Phys. 152, 220 (1984).
[4] P.S. Letelier, Class. Quantum Grav. 4, 75 (1987).
[5] D.V. Gal’tsov and P.S. Letelier, Phys. Rev. D 47, 4425 (1993); See also, P.S. Letelier, Class. Quantum Grav. 12, 471 (1995).
[6] T.M. Heliwell and D.A. Konkowski, Am. J. Phys. 55, 401 (1987).
[7] A. Ishibashi and A. Hosoya, Phys. Rev. D. 60, 104028 (1999).
[8] B.S. Kay and U.M. Studer, Comm. Math. Phys. 138, 103 (1991).
[9] D.A. Konkowski, and T.M. Helliwell, Gen. Rel. Grav. 33, 1131 (2001).
[10] T.M. Helliwell, D.A. Konkowski and V. Arndt, Gen. Rel. Grav. 35, 79 (2003).
[11] D.A. Konkowski, T.M. Helliwell and C. Wieland, Class. Quantum Grav. 21, 265 (2004).
[12] P.S. Letelier, and A. Wang, J. Math. Phys. 36, 3023 (1995).
[13] P.S. Letelier J. Math. Phys. 36, 3043 (1995).
[14] M. Reed and B. Simon, Functional Analysis, (Academic Press, New York, 1980).
[15] R.D. Richtmyer, Principles of Advanced Mathematical Physics, (Springer, New York, 1978).
[16] M. Reed and B. Simon, Fourier Analysis and Self-Adjointness, (Academic Press, New York, 1975).