Exact Solutions with Noncommutative Symmetries in Einstein and Gauge Gravity

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Abstract

We present new classes of exact solutions with noncommutative symmetries constructed in vacuum Einstein gravity (in general, with nonzero cosmological constant), five dimensional (5D) gravity and (anti) de Sitter gauge gravity. Such solutions are generated by anholonomic frame transforms and parametrized by generic off-diagonal metrics. For certain particular cases, the new classes of metrics have explicit limits with Killing symmetries but, in general, they may be characterized by certain anholonomic noncommutative matrix geometries. We argue that different classes of noncommutative symmetries can be induced by exact solutions of the field equations in 'commutative' gravity modeled by a corresponding moving real and complex frame geometry. We analyze two classes of black ellipsoid solutions (in the vacuum case and with cosmological constant) in 4D gravity and construct the analytic extensions of metrics for certain classes of associated frames with complex valued coefficients. The third class of solutions describes 5D wormholes which can be extended to complex metrics in complex gravity models defined by noncommutative geometric structures. The anholonomic noncommutative symmetries of such objects are analyzed. We also present a descriptive account how the Einstein gravity can be related to gauge models of gravity and their noncommutative extensions and discuss such constructions in relation to the Seiberg–Witten map for the gauge gravity. Finally, we consider a formalism of vielbeins deformations subjected to noncommutative symmetries in order to generate solutions for noncommutative gravity models with Moyal (star) product.

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I Introduction

In the last fifteen years much effort has been made to elaborate a consistent formulation of noncommutative gravity theory generalizing the standard Einstein theory but up to now the problem is quite difficult to approach (see, for instance, Refs. 1-8 for details related to existing models). The proposed theories are for the spaces with Euclidean signatures, and, in general, result in models of complex gravity in noncommutative spaces provided with complex and/or nonsymmetric metrics and anholonomic frames. There were also derived some effective noncommutative gravity models from string/brane theory, by considering quantum group structures and/or by proposing noncommutative gauge like generalizations of gravity.

In this paper, we pursue the idea that noncommutative geometric structures are present in the Einstein, five dimensional (in brief, 5D) gravity and gauge gravity models. Such noncommutative symmetries are emphasized if the anholonomic moving frames are introduced into consideration. This 'hidden noncommutativity' is nontrivial for various classes of generic off–diagonal metrics admitting effective diagonalizations by anholonomic transforms with associated nonlinear connection structure. The metrics may be subjected to the condition to define exact solutions of the vacuum field equations with certain possible extensions to matter sources. The noncommutative anhlonomic geometries can be derived even from the 'commutative' general relativity theory and admit a natural embedding into different models of complex noncommutative gravity. The metric and frame (vielbein) coefficients corresponding to 'off–diagonal' solutions depend on two, three or four variables and define spacetimes with associated noncommutative symmetries. Such classes of exact solutions are very different from the well known examples of metrics with Killing symmetry (like the Schwarzschild or Kerr–Newmann solutions; see a detailed analysis in Ref. 20).

Our aim is to prove, by constructing and analyzing three classes of exact solutions, that certain noncommutative geometric structures can be defined in the framework of the Einstein (in general, with cosmological term) and 5D gravity. We emphasize classes of anholonomic real and complex deformations of metrics possessing associated noncommutative symmetries. Contrary to other approaches to noncommutative gravity and field interactions theory elaborated by substituting the commutative algebras of functions with noncommutative algebras and/or by postulating any complex noncommutative relations for coordinates, we try to derive noncommutative structures from associated symmetries of metrics and frames subjected to anholonomy relations. We shall propose a classification of such spacetimes and state a method of complexification of exact solutions preserving the noncommutative symmetry for black hole and wormhole metrics in 'real' and 'complex' gravity.

The study of anholonomic noncommutative symmetries of gravitational field interactions is more involved in the moving frame formalism conventionally adapted to equivalent redefinitions of the Einstein equations as Yang–Mills equations for nonsemisimple gauge groups like in the Poincare gauge gravity. This construction has direct generalizations to various type of gauge gravity models with nondegenerate metrics in the total bundle spaces, in both 'commutative' and 'noncommutative' forms. The connection between the general relativity theory and gauge gravity models is emphasized in order to apply and compare with a set of results from noncommutative gauge theory.

Among our static solutions we find geometries having a structure as have Schwarzschild, Reissner–Nordstrem and (anti) de Sitter spaces but with the coefficients redefined (with certain polarization constants) with respect to anholonomic real/complex frames which make possible
definition of such objects in noncommutative models of gravity. There are equally interesting
applications to black hole physics, quantum gravity and string gravity.

Next, the emerged anholonomic noncommutative symmetries of 'off–diagonal' metrics pre-
scribe explicit rules of deformation the solutions on small noncommutative parameters and
connect the results to quantum deformations of gravity and gauge models. So, even a genera-
ally accepted version of noncommutative gravity theory has been not yet formulated, we know
how to generate particular classes of 'real' and 'complex' stable metrics with noncommuta-
tive symmetries and possessing properties very similar to the usual black hole and wormhole
solutions. In particular, we present a systematic procedure for constructing exact solutions
both in commutative and noncommutative gravity models, to define black hole and wormhole
objects with noncommutative symmetries and quantum corrections. We are able to investi-
gate the physical properties of such objects subjected to certain classes of anholonomic and/or
quantum deformations.

The paper is organized as follows:

We begin in section II with a brief introduction into the geometry of spacetimes provided
with anholonomic frame structure and associated nonlinear connections. Such geometries are
characterized by corresponding anholonomy relations induced by nonlinear connection coeffi-
cients related to certain off–diagonal metric components. This also induces a corresponding
noncommutative spacetime structure.

In section III, we illustrate that such noncommutative anholonomic geometries can be
associated even to real spacetimes and that a simple realization holds within the algebra for
complex matrices. We emphasize that a corresponding noncommutative differential calculus
can be derived from the anholonomy coefficients deforming the structure constants of the
related Lie algebras.

Section IV is devoted to a rigorous analysis of two classes of static black ellipsoid solutions
(the first and second type metrics defining respectively 4D vacuum Einstein and induced by
cosmological constant configurations). We prove that such metrics can be complexified in order
to admit associated complex frame/ nonlinear connection structures inducing noncommutative
matrix geometries and show how analytic extensions of such real and complexified spacetimes
can be constructed.

In section V, a class of 5D wormhole solutions with anisotropic elliptic polarizations is
considered for the 5D gravity. We argue that such generic off–diagonal metrics may be also
complexified as to preserve the wormhole configurations being additionally characterized by
complex valued coefficients for the associated nonlinear connection. Such objects posses the
same noncommutative symmetry for both type of real and complex solutions.

Section VI is a discussion how the Einstein gravity and its higher dimension extensions
can be incorporated naturally into 'commutative' and 'noncommutative' gauge models. A new
point is that the proposed geometric formalism is elaborated in order to include anholonomic
complex vielbeins.

In section VII, we define the Seiberg–Witten map for the de Sitter gauge gravity and state
a prescription how the exact solutions possessing anhlonomic noncommutative symmetries can
be adapted to deformations via star products with noncommutative relations for coordinates.

We conclude and discuss the results in section VIII. For convenience, we summarize the
necessary results from Refs. 13-19 and 30-32 in Appendices A, B and C and state some
definitions on "star" products and enveloping algebras in Appendix D.
II Off–Diagonal Metrics and Anholonomic Frames

We consider a \((n + m)\)-dimensional spacetime manifold \(V^{n+m}\) provided with a (pseudo) Riemannian metric \(g = \{g_{\mu\nu}\}\) and denote the local coordinates \(u = (x, y)\), or in component form, \(u^a = (x^i, y^a)\), where the Greek indices are conventionally split into two subsets, \(x = \{x^i\}\) and \(y = \{y^a\}\), labelled, correspondingly, by Latin indices of type \(i, j, k, ... = 1, 2, ..., n,\) and \(a, b, ..., = 1, 2, ..., m\). In general, the geometric objects on such spacetimes may posses some nontrivial Killing symmetries (the Killing case is emphasized by the condition \(L_X g = 0\), where \(L_X\) is the Lie derivative with respect to a vector field \(X\) on \(V^{n+m}\), see, for instance, Ref. 20) or some deformations of such symmetries, for instance, by frame transforms. The spacetimes may have some additional frame structures with associated nonlinear connection, bundle structure and even nontrivial torsions being adapted to the frame structure.

We shall define our constructions for a general metric ansatz of type

\[
g = g_{\mu\nu} \delta u^\mu \otimes \delta u^\nu = g_{ij} \left( x^k \right) dx^i \otimes dx^j + h_{ab} \left( x^k, v \right) \delta y^a \otimes \delta y^b
\]  

(1)

with respect to a locally adapted basis \([dx^i, \delta y^a]\), where the Einstein’s summation rule is applied and by \(v\) we emphasize the dependence on a so-called ‘anisotropic’ coordinate from the set \(\{y^a\}\). The local basis

\[
e^\mu_{[N]} = \delta^\mu = \delta u^\mu = [dx^i, \delta y^a] = [dx^i, \delta y^a = dy^a + N^a_i \left( x^k, v \right) dx^i]
\]  

(2)

(called to be \(N\)-elongated; we shall provide an additional index \([N]\) if would be necessary to distinguish such objects) is dual to the local basis

\[
e^\alpha_{[N]} = \delta_\alpha = \frac{\delta}{\delta u^\alpha} = \left[ \frac{\partial}{\partial x^i} - N^a_i \left( x^k, v \right) \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b} \right].
\]  

(3)

We consider an off–diagonal metric ansatz for (1) having the components

\[
\tilde{g}_{\alpha\beta} = \begin{bmatrix}
g_{ij} + N^a_i N^b_j h_{ab} & N^e_j h_{ae} \\
N^e_i h_{be} & h_{ab}
\end{bmatrix}.
\]  

(4)

So, we can write equivalently \(g = \tilde{g}_{\alpha\beta} du^\alpha du^\beta\) if the metric is rewritten with respect to the local dual coordinate basis \(du^\mu = [dx^i, dy^a]\), the dual to \(\partial/\partial u^\alpha = [\partial/\partial x^i, \partial/\partial y^a]\) (defined correspondingly by usual partial derivatives and differentials).

A very surprising fact is that the off–diagonal metric ansatz (1) for dimensions \(n + m = 3, 4, 5\) and certain imbedding of such configurations in extra dimension (super) spaces results in completely integrable systems of partial differential equations (see details in Refs. 13-19 with a review of results in Refs. 30,31 and Theorems 1-3 in the Appendix B). In this paper, we shall consider that any metric (1), or equivalently (2) and frames (vielbeins) (2) and (3), parametrizes an exact solution of the Einstein equations in a ‘commutative’ gravity theory.

Let us state the main geometric properties of spacetimes provided with off–diagonal metrics which can be effectively diagonalized with respect to the \(N\)-elongated frames (2) and (3):

1. Such spacetimes are characterized by certain anholonomic frame relations (anholonomy conditions)

\[
e^\alpha_{[N]} e^\alpha_{[N]} - e^\beta_{[N]} e^\alpha_{[N]} = w^\alpha_{\alpha\beta} e^\gamma_{[N]}
\]  

(5)
with some nontrivial anholonomy coefficients $w^{[N]}_{\alpha\beta} \gamma$ computed as

$$
\begin{align*}
    w^k_{ij} &= 0, \quad w^k_{aj} = 0, \quad w^k_{ab} = 0, \quad w^c_{ab} = 0, \\
    w^a_{bj} &= -w^a_{jb} = \partial_b N^a_j, \quad w^a_{ij} = -\Omega^a_{ij} = \delta_i N^a_j - \delta_j N^a_i
\end{align*}
$$

(we shall omit the label $[N]$ if this will not result in any confusion; as a matter of principle, we can consider arbitrary anholonomy coefficients not related to any off–diagonal metric terms). If the values $w^{[N]}_{\alpha\beta}$ do not vanish, it is not possible to diagonalize the metric (4) by any coordinate transforms: such spacetimes are generic off–diagonal. The holonomic frames (in particular the coordinate ones) consist a subclass of vielbeins with vanishing anholonomy coefficients.

2. To any frame (vielbein) transform defined by the coefficients of $e^{[N]}_a$ decomposed with respect to usual coordinate frames, we can associate a nonlinear connection structure (in brief, N–connection) $N$ with the coefficients $\{N^a_j\}$ (in global form the N–connection was defined in Ref. 33 by developing previous ideas from Finsler geometry$^{9–12,34–36}$, investigated in details for vector bundle spaces in Refs. 37,38; see also Refs. 13-19, 30-32 on definition of such objects in (pseudo) Riemannian and Riemann–Cartan–Weyl geometry or on superspaces). Here we note that the N–connection structure is characterized by its curvature (N–curvature) $\Omega = \{\Omega^a_{ij}\}$ with the coefficients computed as in (6). The well known class of linear connections is to be distinguished as a particular case when $N^a_j(x,y) = \Gamma^a_{jb}(x) y^b$. On (pseudo) Riemannian spaces, the N–connection is a geometric object completely defined by anholonomic frames when the vielbein transforms $e^{[N]}_a$ are parametrized explicitly via certain values $(N^a_i, \delta^i_j, \delta^a_b)$, where $\delta^i_j$ and $\delta^a_b$ are the Kronecker symbols, like in (3).

3. The N–coefficients define a conventional global horizontal–vertical (in brief, h–v ) splitting of spacetime $V^{n+m}$ into holonomic–anholonomic subsets of geometrical objects labelled by h–components with indices $i, j, ...$ and v–components with indices $a, b, ....$, see details in Refs. 13-19, 30-32. The necessary formulas for the h–v–decompositions of the curvature, Ricci and Einstein tensors are contained in Appendix A.

4. Such generic ”off–diagonal” spacetimes may be characterized by the so–called canonical N–adapted linear connection $\Gamma^{[c]} = \{L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{be}\}$ satisfying the metricity condition $D^{[c]}_{\gamma} g_{\alpha\beta} = 0$ and being adapted to the h–v–distribution. The coefficients of $\Gamma^{[c]}$ are

$$
\begin{align*}
    L^i_{jk} &= \frac{1}{2} g^{im} (\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}), \\
    L^a_{bk} &= \partial_b N^a_k + \frac{1}{2} h^{ac} (\delta_k h_{bc} - h_{dc} \partial_b N^d_k - h_{db} \partial_c N^d_k) , \\
    C^i_{jc} &= \frac{1}{2} g^{ik} \partial_c g_{jk}, \quad C^a_{bc} = \frac{1}{2} h^{ad} (\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}) ,
\end{align*}
$$

where $\delta_k = \delta/\partial x^k$ and $\partial_c = \partial/\partial y^a$; they are constructed from the coefficients (and their partial derivatives) of the metric and N–connection. This connection is an anholonomic deformation (by N–coefficients) of the Levi–Civita connection.
5. The torsion of the connection $\Gamma^c$ is defined (for simplicity, we omit the label $[c]$)

$$T^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta} + w^\alpha_{\beta\gamma},$$  \hspace{1cm} (8)

with h–v–components

$$T^{i}_{jk} = L^{i}_{jk} - L^{i}_{kj} = 0 \text{ in the canonical case},$$
$$T^{a}_{bc} = C^{a}_{bc} - C^{a}_{cb} = 0 \text{ in the canonical case},$$
$$T^{i}_{ja} = 0, \ T^{i}_{ja} = -T^{i}_{aj} = -C^{i}_{ja}, \ T^{a}_{ji} = -\Omega^{a}_{ij}, \ T^{a}_{bi} = -T^{a}_{bi} = \partial_{b}N^{a}_{i} - L^{a}_{bi}.$$  \hspace{1cm} (9)

The nonvanishing components of torsion are induced as an anholonomic frame effect which is obtained by vielbien transforms (2) and (3) even for a (pseudo) Riemannian metric $[11]$. In this paper, we shall also consider some nontrivial torsion structures existing in extra dimension gravity.

6. By straightforward calculations with respect to the frames (2) and (3) (see for instance, Refs. 39,40) we can compute the coefficients of the Levi–Civita connection $\nabla$, i. e.

\[
\Gamma^{[\nabla]}_{\alpha\beta\gamma} = g^{\alpha\gamma}g^{\beta\alpha} - g^{\gamma\alpha}g^{\beta\gamma} + g^{\alpha\tau}w^{\tau}_{\gamma\alpha} + g^{\beta\tau}w^{\tau}_{\alpha\beta} - g^{\beta\tau}w^{\tau}_{\beta\alpha},
\]

Using the values (6) and (3), we can write

\[
\Gamma^{[\nabla]}_{\alpha\beta\gamma} = \frac{1}{2} \left( e^{[N]}_{\alpha} g^{\gamma\alpha} + e^{[N]}_{\gamma} g^{\beta\alpha} - e^{[N]}_{\beta} g^{\alpha\gamma} + g^{\alpha\tau}w^{\tau}_{\gamma\beta} + g^{\beta\tau}w^{\tau}_{\alpha\gamma} - g^{\beta\tau}w^{\tau}_{\beta\alpha} \right).
\]

Comparing the coefficients of $\Gamma^c$ and $\Gamma^{[\nabla]}$, we conclude that both connections have the same coefficients with respect to the N–adapted frames (2) and (3) if and only if $\partial N^{a}_{i} / \partial y^{b} = 0$ and $\Omega^{a}_{jk} = 0$, i. e. the N–connection curvature vanishes.

7. The ansatz of type (4) have been largely used in Kaluza–Klein theories (see, for instance, Refs. 41-43). For the corresponding compactifications, the coefficients $N^{a}_{i}$ may be associated to the potential of certain gauge fields but, in general, they belong to some noncompactified metric and vielbein gravitational fields. There were elaborated general methods for constructing exact solutions without compactification and arbitrary $N^{a}_{i}$ in various type of gravity models$^{13–19}$.

Any ansatz of type (4) with the components satisfying the conditions of the Theorems 1-3 from the Appendix B define a new class of exact solutions, vacuum and nonvacuum ones, in 3-5 dimensional gravity parametrized by generic off–diagonal metrics with the coefficients depending on 2,3 or even 4 variables. These solutions can be constructed in explicit form by using corresponding boundary and symmetry conditions following the so–called 'anholonomic frame method' elaborated and developed in Refs. 13-19,30,31 (for instance, they can describe black elipsoid/tori configurations, 2-3 dimensional solitonic–spinor–dilaton interactions, polarized wormhole/flux tube solutions, locally anisotropic Taub NUT spacetimes and so on).
Perhaps, by using the anholonomic frame method, we can construct the most general known class of exact solutions in Einstein gravity and its extra dimension and string generalizations. From a formal point of view, we can use superpositions of anholonomic maps in order to construct integral varieties of the Einstein equations with the metric/frame coefficients being functions of necessary smooth class depending on arbitrary number of variables but parametrized as products of functions depending on 1,2,3 and 4 real, or some complex, variables with real and complex valued functions. The physical meaning of such classes of solutions should be stated following explicit physical models. We note that the bulk of the well known black hole and cosmological solutions (for instance, the Schwarzschild, Kerr–Newman, Reisner–Nordstrom and Friedman–Roberston–Walker solutions) are with metrics being diagonalizable by coordinate transforms and depending only on one variable (radial or timelike), with imposed spherical or cylindrical symmetries and subjected to the conditions of Killing symmetry being asymptotically flat.

In general, the solutions with anholonomic configurations do not posses Killing symmetries (for instance, they are not restricted by ”black hole uniqueness theorems”, proved for Killing spacetimes satisfying corresponding asymptotic conditions, see details and references in Ref. 20) but have new properties like the 1–7 stated above. There is a subclass of ‘off–diagonal’ solutions resulting in corresponding limits into the well known asymptotically flat spacetimes, or with (anti) de Sitter symmetries$^{13–19,30,31}$. We are interested to investigate possible symmetries of such ’non–Killing’ exact solutions.

The purpose of the next section is to prove that the spacetimes with a nontrivial anholonomic and associated N–connection structure posses a natural noncommutative symmetry.

III Anholonomic Noncommutative Structures

We shall analyze two simple realizations of noncommutative geometry of anholonomic frames within the algebra of complex $k \times k$ matrices, $M_k(\mathbb{C}, u^\alpha)$ depending on coordinates $u^\alpha$ on spacetime $V^{n+m}$ connected to complex Lie algebras $SL(k, \mathbb{C})$ and $SU_k$. We shall consider matrix valued functions of necessary smoothly class derived from the anholonomic frame relations (being similar to the Lie algebra relations) with the coefficients induced by off–diagonal metric terms in and by N–connection coefficients $N^\alpha_i$. We shall use algebras of complex matrices in order to have the possibility for some extensions to complex solutions. Usually, for commutative gravity models, the anholonomy coefficients $w^{(N)}_{\alpha \beta \gamma}$ are real functions but in the section 7 we shall consider also complex spacetimes related to noncommutative gravity$^{3–5}$.

We start with the basic relations for the simplest model of noncommutative geometry realized with the algebra of complex $(k \times k)$ noncommutative matrices$^{44}$, $M_k(\mathbb{C})$. An element $M \in M_k(\mathbb{C})$ can be represented as a linear combination of the unit $(k \times k)$ matrix $I$ and $(k^2 – 1)$ hermitian traceless matrices $q_\alpha$ with the underlined index $\alpha$ running values $1, 2, ..., k^2 – 1$, i. e.

$$M = \alpha \ I + \sum \beta^\underline{\alpha} q_\underline{\alpha}$$

for some constants $\alpha$ and $\beta^\underline{\alpha}$. It is possible to chose the basis matrices $q_\underline{\alpha}$ satisfying the relations

$$q_\underline{\alpha} q_\beta = \frac{1}{k} \delta^\underline{\alpha \beta} I + Q^\underline{\alpha \beta \gamma} q_\underline{\gamma} – \frac{i}{2} f^\underline{\alpha \beta \gamma} q_\underline{\gamma},$$

(11)
where \( i^2 = -1 \) and the real coefficients satisfy the properties

\[
Q_{\alpha \beta}^\gamma = Q_{\beta \alpha}^\gamma, \quad Q_{\alpha \beta}^\gamma = 0, \quad f_{\alpha \beta}^\gamma = -f_{\beta \alpha}^\gamma, \quad f_{\alpha \beta}^\gamma = 0
\]

with \( f_{\alpha \beta}^\gamma \) being the structure constants of the Lie group \( SL(k, \mathbb{C}) \) and the Killing–Cartan metric tensor \( \rho_{\alpha \beta} = f_{\alpha \alpha}^\gamma f_{\beta \beta}^\gamma \). The interior derivatives \( \partial_{\alpha} \) of this algebra can be defined as

\[
\partial_{\alpha} q_{\beta} = \text{ad} \left( iq_{\alpha} \right) q_{\beta} = i[q_{\alpha}, q_{\beta}] = f_{\alpha \beta}^\gamma q_{\gamma}.
\]

Following the Jacobi identity, we obtain

\[
\partial_{\alpha} \partial_{\beta} - \partial_{\beta} \partial_{\alpha} = f_{\alpha \beta}^\gamma \partial_{\gamma}.
\]

Our idea is to construct a noncommutative geometry starting from the anholonomy relations of frames \([4]\) by adding to the structure constants \( f_{\alpha \beta}^\gamma \) the anholonomy coefficients \( w_{\alpha \beta}^{[N]} \) \([6]\). Such deformed structure constants consist from \( N \)-connection coefficients \( N_{\alpha}^a \) and their first partial derivatives, i.e., they are induced by some off-diagonal terms in the metric \([\Pi]\) being a solution of the gravitational field equations. We note that there is a rough analogy between formulas \([13]\) and \([5]\) because the anholonomy coefficients do not satisfy, in general, the condition \( w_{\alpha \beta}^{[N]} = 0 \). There is also another substantial difference: the anholonomy relations are defined for a manifold of dimension \( n + m \), with Greek indices \( \alpha, \beta, \ldots \) running values from 1 to \( n + m \) but the matrix noncommutativity relations are stated for traceless matrices labeled by underlined indices \( \underline{\alpha}, \underline{\beta}, \ldots \) running values from 1 to \( k^2 - 1 \). It is not possible to satisfy the condition \( k^2 - 1 = n + m \) by using integer numbers for arbitrary \( n + m \). We suggest to extend the dimension of spacetime from \( n + m \) to any \( n' + n \) and \( m' + m \) when the condition \( k^2 - 1 = n' + m' \) can be satisfied by a trivial embedding of the metric \([\Pi]\) into higher dimension, for instance, by adding the necessary number of unities on the diagonal by writing

\[
\hat{g}_{\underline{\alpha} \underline{\beta}} =
\begin{bmatrix}
1 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 1 & 0 & 0 \\
0 & \ldots & 0 & \bigl[ g_{ij} + N_{i}^{a}N_{j}^{b}h_{ab} + N_{j}^{e}h_{ae} \bigr] & h_{ab} \\
0 & \ldots & 0 & \bigl[ N_{i}^{e}h_{be} \bigr] & h_{ab}
\end{bmatrix}
\]

and \( e_{[N]}^{\underline{\alpha}} = \delta_{\underline{\alpha}} = (1, 1, \ldots, e_{[N]}^{\underline{\alpha}}) \), where, for simplicity, we preserve the same type of underlined Greek indices, \( \underline{\alpha}, \underline{\beta}, \ldots = 1, 2, \ldots, k^2 - 1 = n' + m' \). The anholonomy coefficients \( w_{\alpha \beta}^{[N]} \) can be extended with some trivial zero components and for consistency we rewrite them without labeled indices, \( w_{\alpha \beta}^{[N]} \rightarrow W_{\alpha \beta}^{[N]} \). The set of anholonomy coefficients \( w_{\alpha \beta}^{[N]} \) \([6]\) may result in degenerated matrices, for instance for certain classes of exact solutions of the Einstein equations. Nevertheless, we can consider an extension \( w_{\alpha \beta}^{[N]} \rightarrow W_{\alpha \beta}^{[N]} \) when the coefficients \( W_{\alpha \beta}^{[N]}(u^{2}) \) for any fixed value \( u^{2} = u_{[0]}^{2} \) would be some deformations of the structure constants of the Lie algebra \( SL(k, \mathbb{C}) \), like

\[
W_{\alpha \beta}^{[N]} = f_{\alpha \beta}^{[N]} + w_{\alpha \beta}^{[N]},
\]

being nondegenerate.
Instead of the matrix algebra $M_k(\mathbb{C})$, constructed from constant complex elements, we shall consider dependencies on coordinates $u^\underline{\alpha} = (0, ..., u^\alpha)$, for instance, like a trivial matrix bundle on $V^{n'+m'}$, and denote this space $M_k(\mathbb{C}, u^\underline{\alpha})$. Any element $B(u^\underline{\alpha}) \in M_k(\mathbb{C}, u^\underline{\alpha})$ with a noncommutative structure induced by $\hat{W}_{\underline{\alpha}\underline{\beta}}$ is represented as a linear combination of the unit $(n'+m') \times (n'+m')$ matrix $I$ and the $[(n'+m')^2 - 1]$ hermitian traceless matrices $q_\underline{\alpha}(u^\underline{\xi})$ with the underlined index $\underline{\alpha}$ running values $1, 2, ..., (n'+m')^2 - 1,$

\[
B(u^\underline{\alpha}) = \alpha(u^\underline{\alpha}) I + \sum \beta^\underline{\alpha}(u^\underline{\xi}) q_\underline{\alpha}(u^\underline{\xi})
\]

under condition that the following relation holds:

\[
q_\underline{\alpha}(u^\underline{\xi}) q_\underline{\beta}(u^\underline{\xi}) = \frac{1}{n'+m'} \rho_{\underline{\alpha}\underline{\beta}}(u^\underline{\xi}) + Q_{\underline{\alpha}\underline{\beta}} q_\underline{\alpha}(u^\underline{\xi}) - i \frac{1}{2} W_{\underline{\alpha}\underline{\beta}} q_\underline{\alpha}(u^\underline{\xi})
\]

with the same values of $Q_{\underline{\alpha}\underline{\beta}}$ from the Lie algebra for $SL(k, \mathbb{C})$ but with the Killing–Cartan like metric tensor defined by anholonomy coefficients, i.e. $\rho_{\underline{\alpha}\underline{\beta}}(u^\underline{\xi}) = W_{\underline{\alpha}\underline{\beta}}(u^\underline{\xi}) W_{\underline{\alpha}\underline{\beta}}(u^\underline{\xi})$. For complex spacetimes, we shall consider that the coefficients $N^\underline{\alpha}$ and $W_{\underline{\alpha}\underline{\beta}}$ may be some complex valued functions of necessary smooth (in general, with complex variables) class. In result, the Killing–Cartan like metric tensor $\rho_{\underline{\alpha}\underline{\beta}}$ can be also complex.

We rewrite (13) as

\[
e_\alpha e_\beta - e_\beta e_\alpha = W_{\underline{\alpha}\underline{\beta}} e_\underline{\alpha}
\]

being equivalent to (13) and defining a noncommutative anholonomic structure (for simplicity, we use the same symbols $e_\alpha$ as for some ‘N–elongated’ partial derivatives, but with underlined indices). The analogs of derivation operators (12) are stated by using $W_{\underline{\alpha}\underline{\beta}}$,

\[
e_\alpha q_\underline{\beta}(u^\underline{\xi}) = \text{ad} [i q_\underline{\alpha}(u^\underline{\xi})] q_\underline{\beta}(u^\underline{\xi}) = i \left[ q_\underline{\alpha}(u^\underline{\xi}) q_\underline{\beta}(u^\underline{\xi}) \right] = W_{\underline{\alpha}\underline{\beta}} q_\underline{\alpha}(u^\underline{\xi})
\]

The operators (16) define a linear space of anholonomic derivations satisfying the conditions (15), denoted $\text{Ader} M_k(\mathbb{C}, u^\underline{\alpha})$, elongated by N–connection and distinguished into irreducible h– and v–components, respectively, into $e_\underline{1}$ and $e_\underline{2}$, like $e_\underline{\alpha} = \left( e_\underline{1} = \partial_\underline{1} - N^\underline{\alpha}_1 e_\underline{2}, e_\underline{2} = \partial_\underline{2} \right)$. The space $\text{Ader} M_k(\mathbb{C}, u^\underline{\alpha})$ is not a left module over the algebra $M_k(\mathbb{C}, u^\underline{\alpha})$ which means that there is a substantial difference with respect to the usual commutative differential geometry where a vector field multiplied on the left by a function produces a new vector field.

The duals to operators (16), $e^\underline{\alpha}$, found from $e^\underline{\alpha}(e_\underline{\alpha}) = \delta^\underline{\alpha}_{\underline{\beta}}$, define a canonical basis of 1–forms $e\underline{\alpha}$ connected to the N–connection structure. By using these forms, we can span a left module over $M_k(\mathbb{C}, u^\underline{\alpha})$ following $q_\underline{\alpha} e\underline{\beta} (e_\underline{\beta}) = q_\underline{\alpha} \delta^\underline{\alpha}_{\underline{\beta}} I = q_\underline{\alpha} \delta^\underline{\alpha}_{\underline{\beta}}$. For an arbitrary vector field

\[
Y = Y^\alpha e_\alpha \rightarrow Y^\underline{\alpha} e_\underline{\alpha} = Y^\underline{1} e_\underline{1} + Y^\underline{2} e_\underline{2},
\]

it is possible to define an exterior differential (in our case being N–elongated), starting with the action on a function $\varphi$ (equivalent, a 0–form),

\[
\delta \varphi (Y) = Y \varphi = Y^\underline{1} \delta_1 \varphi + Y^\underline{2} \delta_2 \varphi
\]

when

\[
(\delta I) (e_\underline{\alpha}) = e_\underline{\alpha} I = \text{ad} (i e_\underline{\alpha}) I = i [e_\underline{\alpha}, I] = 0, \text{ i.e. } \delta I = 0,
\]
and
\[ \delta q_{\alpha}(e_\mu) = e_{(\alpha}(e_{\mu)} = i[e_{\mu}, e_\alpha] = W^\gamma_{\alpha\mu} e_\gamma. \] (17)

Considering the nondegenerated case, we can invert (17) as to obtain a similar expression with respect to \( e_\mu \),
\[ \delta(e_\alpha) = W^\gamma_{\alpha\mu} e_\gamma e_\mu, \] (18)
from which a very important property follows by using the Jacobi identity, \( \delta^2 = 0 \), resulting in a possibility to define a usual Grassman algebra of \( p \)-forms with the wedge product \( \wedge \) stated as
\[ e_\mu \wedge e_\nu = \frac{1}{2} (e_\mu \otimes e_\nu - e_\nu \otimes e_\mu). \]

We can write (18) as
\[ \delta(e_\alpha) = -\frac{1}{2} W^\alpha_{\beta\mu} e_\beta e_\mu \]
and introduce the canonical 1–form \( e = q_\alpha e_\alpha \) being coordinate–independent and adapted to the \( N \)–connection structure and satisfying the condition \( \delta e + e \wedge e = 0 \).

In a standard manner, we can introduce the volume element induced by the canonical Cartan–Killing metric and the corresponding star operator \( * \) (Hodge duality). We define the volume element \( \sigma \) by using the complete antisymmetric tensor \( \epsilon_{\alpha_1\alpha_2...\alpha_{k-1}} \) as
\[ \sigma = \frac{1}{[(n'+m')^2 - 1]!} \epsilon_{\alpha_1\alpha_2...\alpha_{n'+m'}} e^{\alpha_1} \wedge e^{\alpha_2} \wedge ... \wedge e^{\alpha_{n'+m'}} \]
to which any \((k^2 - 1)\)-form is proportional \((k^2 - 1 = n' + m')\). The integral of such a form is defined as the trace of the matrix coefficient in the form \( \sigma \) and the scalar product introduced for any couple of \( p \)-forms \( \varpi \) and \( \psi \)
\[ (\varpi, \psi) = \int (\varpi \wedge *\psi). \]

Let us analyze how a noncommutative differential form calculus (induced by an anholonomic structure) can be developed and related to the Hamiltonian classical and quantum mechanics and Poisson bracket formalism:

For a \( p \)-form \( \varpi^{[p]} \), the anti–derivation \( i_Y \) with respect to a vector field \( Y \in AderM_k(\mathbb{C}, u^{\infty}) \) can be defined as in the usual formalism,
\[ i_Y \varpi^{[p]} (X_1, X_2, ..., X_{p-1}) = \varpi^{[p]} (Y, X_1, X_2, ..., X_{p-1}) \]
where \( X_1, X_2, ..., X_{p-1} \in AderM_k(\mathbb{C}, u^{\infty}) \). By a straightforward calculus we can check that for a 2–form \( \Xi = \delta e \) one holds
\[ \delta \Xi = \delta^2 e = 0 \text{ and } L_Y \Xi = 0 \]
where the Lie derivative of forms is defined as \( L_Y \varpi^{[p]} = (i_Y \delta + \delta i_Y) \varpi^{[p]} \).

The Hamiltonian vector field \( H[\varphi] \) of an element of algebra \( \varphi \in M_k(\mathbb{C}, u^{\infty}) \) is introduced following the equality \( \Xi(H[\varphi], Y) = Y\varphi \) which holds for any vector field. Then, we can define the Poisson bracket of two functions (in a quantum variant, observables) \( \varphi \) and \( \chi \), \( \{\varphi, \chi\} = \Xi(H[\varphi], H[\chi]) \) when
\[ \{e_\alpha, e_\beta\} = \Xi(e_\alpha, e_\beta) = i[e_\alpha, e_\beta]. \]
This way, a simple version of noncommutative classical and quantum mechanics (up to a factor like the Plank constant, $\hbar$) is proposed, being derived by anholonomic relations for a certain class of exact ‘off–diagonal’ solutions in commutative gravity.

We note that by using the Lie algebra $SU(k,\mathbb{C})$ we can elaborate an alternative noncommutative calculus related to the special unitary group $SU_k$ in $k$ dimensions when the anholonomic coefficients

$$W_{\alpha\beta}^\gamma = p_{\alpha\beta}^\gamma + w_{\alpha\beta}^\gamma$$  \hfill (19)

induce a linear connection in the associated noncommutative space (noncommutative geometries with $p_{\alpha\beta}^\gamma$ being the structure constants of $SU_k$ were investigated in Refs. 44,46-49,8).

Let us state the main formulas for such realization of anholonomic noncommutativity:

In this case, the matrix basis $q_\alpha$ consists from anti–hermitian (and not hermitian) matrices and the relations (11) are stated in a different form

$$q_\alpha q_\beta = -\frac{1}{k} \rho_{\alpha\beta} I + Z_{\alpha\beta}^\gamma q_\gamma + \frac{1}{2} p_{\alpha\beta}^\gamma q_\gamma,$$  \hfill (20)

where $\rho_{\alpha\beta} Z_{\alpha\beta}^\gamma$ is trace–free and symmetric in all pairs of indices and $\rho_{\alpha\beta} = p_{\alpha\gamma} p_{\beta\gamma}$. We consider dependencies of matrix coefficients on coordinates $u^\alpha = (0, ..., u^\alpha)$, i. e. we work in the space $M_k(\mathbb{C}, u^\alpha)$, and introduce the ‘anholonomic’ derivations $e_\alpha$,

$$e_\alpha \varphi = [q_\alpha, \varphi]$$

for arbitrary matrix function $\varphi \in M_k(\mathbb{C}, u^\alpha)$ defining a basis for the Lie algebra of derivations $Der [M_k(\mathbb{C}, u^\alpha)]$ of $M_k(\mathbb{C}, u^\alpha)$. In this case, we generalize (20) to

$$q_\alpha(u^\tau) q_\beta(u^\tau) = -\frac{1}{k} \rho_{\alpha\beta}(u^\tau) I + Z_{\alpha\beta}^\gamma(u^\tau) q_\gamma + \frac{1}{2} W_{\alpha\beta}^\gamma(u^\tau) q_\gamma(u^\tau),$$

with an effective (of N–anholonomy origin) metric $\rho_{\alpha\beta}(u^\Delta) = W_{\alpha\beta}^\gamma(u^\Delta) W_{\gamma\delta}^\eta(u^\Delta)$ being an anholonomic deformation of the Killing metric of $SU_k$.

In order to define the algebra of forms $\Omega^* [M_k(\mathbb{C}, u^\alpha)]$ over $M_k(\mathbb{C}, u^\alpha)$ we put $\Omega^0 = M_k$ and write

$$\delta \varphi (e_\alpha) = e_\alpha(\varphi)$$

for every matrix function $\varphi \in M_k(\mathbb{C}, u^\alpha)$. As a particular case, we have

$$\delta q^\alpha (e_\beta) = -W_{\alpha\beta}^\gamma q_\gamma$$

where indices are raised and lowered with the anholonomically deformed metric $\rho_{\alpha\beta}(u^\Delta)$. This way, we can define the set of 1–forms $\Omega^1 [M_k(\mathbb{C}, u^\alpha)]$ to be the set of all elements of the form $\varphi \delta \beta$ with $\varphi$ and $\beta$ belonging to $M_k(\mathbb{C}, u^\alpha)$. The set of all differential forms define a differential calculus in $M_k(\mathbb{C}, u^\alpha)$ induced by the anholonomy of certain exact solutions (with off–diagonal metrics and associated N–connections) in a gravity theory.

We can also find a set of generators $e^\alpha$ of $\Omega^1 [M_k(\mathbb{C}, u^\alpha)]$, as a left/ right –module completely characterized by the duality equations $e^\alpha (e_\alpha) = \delta_{\alpha\beta} I$ and related to $\delta q^\alpha$,

$$\delta q^\alpha = W_{\alpha\beta}^\gamma q^\beta q_\gamma$$

and $e^\mu = q^\mu = q_\gamma q^\mu q_\gamma$. 

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Similarly to the formalism presented in details in Ref. 8, we can elaborate a differential calculus with derivations by introducing a linear torsionless connection

\[ D e^\mu = -\omega_{\gamma}^\mu e^\gamma \]

with the coefficients \( \omega_{\gamma}^\mu = -\frac{1}{2} W_{\gamma}^{\mu \beta} e^\beta \), resulting in the curvature 2–form

\[ R_{\gamma}^\mu = \frac{1}{8} W_{\gamma}^{\mu \beta} W_{\beta}^{\alpha \tau} e^\alpha e^\tau. \]

So, even the anholonomy coefficients \( \omega_{\gamma}^\mu \) of a solution, for instance, in string gravity, has nontrivial torsion coefficients, \( \omega_{\gamma}^\mu \), the associated linear connection induced by the anholonomy coefficients in the associated noncommutative space of symmetries of the solution can be defined to be torsionless but to have a specific metrics and curvature being very different from the spacetime curvature tensor. This is a surprising fact that 'commutative' curved spacetimes provided with off–diagonal metrics and associated anhlonomic frames and N–connections may be characterized by a noncommutative 'shadow' space with a Lie algebra like structure induced by the frame anolonomy. We argue that such metrics possess anholonomic noncommutative symmetries.

Finally, in this section, we conclude that for the 'holonomic' solutions of the Einstein equations, with vanishing \( w_{\gamma}^\mu e^\gamma \), any associated noncommutative geometry or \( SL(k, \mathbb{C}) \), or \( SU_k \) type, decouples from the off–diagonal (anhlonomic) gravitational background and transforms into a trivial one defined by the corresponding structure constants of the chosen Lie algebra. The anholonomic noncommutativity and the related differential geometry are induced by the anholonomy coefficients. All such structures reflect a specific type of symmetries of generic off–diagonal metrics and associated frame/ N–connection structures. Considering exact solutions of the gravitational field equations, we can assert that we constructed a class of vacuum or nonvacuum metrics possessing a specific noncommutative symmetry instead, for instance, of any usual Killing symmetry. In general, we can introduce a new classification of spacetimes following anholonomic noncommutative algebraic properties of metrics and vielbein structures.

IV Black Ellipsoids with Noncommutative Symmetry

In this section, we shall analyze two classes of black ellipsoid solutions of the Einstein and (anti) de Sitter gravity (with arbitrary cosmological term) possessing hidden noncommutative symmetries. Such off–diagonal metrics will be constructed as to generate also exact solutions in complex gravity, with respect to complex N–elongated vielbeins (for simplicity, we shall consider the metric coefficients to be real with respect to such complex frames) which have to be considered if any noncommutativity of coordinates with complex parameters and/or Wick like rotations to Euclidean signatures are introduced. Such metrics are stable for certain configurations with complex off–diagonal terms (a rigorous proof may be performed by generalizing to complex spaces the results from Refs. 18,19).

A Anholonomic Complex Deformations of the Schwarzschild Solution

We consider a 4D off–diagonal metric ansatz (a complex generalization of \( \mathbb{R}^4 \)), or equivalently, of \( \mathbb{R}^4 \) with complex frames (vielbeins) \( \mathbb{C}^2 \) and \( \mathbb{C}^3 \), see also the ansatz \( \mathbb{C}^{120} \) in the Appendix.
where $\eta_3 (r, \varphi) r^2 \sin^2 \theta d\varphi^2 + \eta_4 (r, \varphi) \left(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \right) \delta t^2$

for usual local spherical and time like coordinates $u = \{u^\alpha = (u^2 = r, u^3 = \theta, u^4 = \varphi, u^5 = t)\}$. In order to have compatibility with notations from Appendices B and C, in this subsection, we consider that 4D Greek indices run from 2 till 5 where the "polarization" functions $\eta_3, \eta_4$ are decomposed on a small parameter $\varepsilon$, $0 < |\varepsilon| \ll 1$,

$$ \eta_4 (r, \varphi) = \eta_4 [0] (r, \varphi) + \varepsilon \lambda_4 (r, \varphi) + \varepsilon^2 \gamma_4 (r, \varphi) + \ldots , \tag{22} $$

$$ \eta_5 (r, \varphi) = 1 + \varepsilon \lambda_5 (r, \varphi) + \varepsilon^2 \gamma_5 (r, \varphi) + \ldots , $$

$\gamma (r)$ is a necessary smooth class function satisfying $\gamma (r) = 1$ if $\varepsilon \to 0$ (it will be defined below) and

$$ \delta t = dt + n_2 (r, \varphi) dr \tag{23} $$

for $n_2 = n_2 [re] + in_2 [im] \sim \varepsilon \ldots + \varepsilon^2$ terms being, in general, a complex valued function. In the particular case, when $n_2$ is real, i. e. when $n_2 = n_2 [re]$ and $n_2 [im] = 0$, the labels $[re]$ and $[im]$ being used respectively for the real and imaginary parts, the metric (21) was investigated in connection to the definition of static and non–static black ellipsoid configurations in Refs. 13-19,30,31). The functions $\eta_4,5 (r, \varphi)$ and $n_2 (r, \varphi)$ will be found as the metric will define a solution of the vacuum Einstein equations (94) (see Appendices A, B and C for the explicit form of field equations (97)–(100) written for the 4D ansatz (120)). By introducing certain complex components of metric generated by small deformations of the spherical static symmetry on a small positive parameter $\varepsilon$ (in the limits $\varepsilon \to 0$ and $\eta_{4,5} \to 1$ we have just the Schwarzschild solution for a point particle of mass $m$) we show here that it is possible to extend the results of Refs. 18,19 with respect to complex anholonomic structures (2) with a nontrivial component $N_2^5 = n_2 (r, \varphi)$ given by N–elongation (23).

A more interesting class of exact solutions with an effective electric charge $q$ induced from the complex/ noncommutative/ anholonomic gravity may be constructed if we state that the parameter of anholonomic deformations is of type $\varepsilon = (iq)^2$ for a real $q$ and imaginary $i$. In this case the metric (21) will have real coefficients in the first order of $\varepsilon$, being very similar to those from the well known Reissner–Nördstrom metric with, in our case effective, electric charge $q$. For convenience, in our further calculations we shall use both small parameters $\varepsilon$ and/or $q$.

The set of $\eta, \lambda$ and $\gamma$ functions from (22) define arbitrary anholonomic (in our case with certain complexity) deformations of the Schwarzschild metric. As a particular case, we can consider the condition of vanishing of the metric coefficient before $\delta t^2$

$$ \eta_5 (r, \varphi) \left(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \right) = 1 - \frac{2m}{r} + \frac{\Phi_5}{r^2} + \varepsilon^2 \Theta_5 = 0, \tag{24} $$

$$ \Phi_5 = \lambda_4 \left(r^2 - 2mr \right) + 1 $$

$$ \Theta_5 = \gamma_4 \left(1 - \frac{2m}{r} \right) + \lambda_4, $$

$$ C),$$

$$ \delta s^2 = - \left(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \right)^{-1} dr^2 - r^2 \gamma (r) d\theta^2 \tag{21} $$

for usual local spherical and time like coordinates $u = \{u^\alpha = (u^2 = r, u^3 = \theta, u^4 = \varphi, u^5 = t)\}$. In order to have compatibility with notations from Appendices B and C, in this subsection, we consider that 4D Greek indices run from 2 till 5 where the "polarization" functions $\eta_3, \eta_4$ are decomposed on a small parameter $\varepsilon$, $0 < |\varepsilon| \ll 1$,
defining a rotation ellipsoid configuration when
\[ \lambda_5 = \left( 1 - \frac{2m}{r} \right)^{-1} (\cos \varphi - \frac{1}{r^2}) \quad \text{and} \quad \gamma_5 = -\lambda_5 \left( 1 - \frac{2m}{r} \right)^{-1}. \]

Really, in the first order on \( \varepsilon \), one follows a zero value for the coefficient before \( \delta t^2 \) if
\[ r_\pm = \frac{2m}{1 - q^2 \cos \varphi} = 2m[1 + q^2 \cos \varphi], \quad (25) \]
which is the equation for a 3D ellipsoid like hypersurface with a small eccentricity \( q^2 \). In general, we can consider arbitrary pairs of functions \( \lambda_5(r, \theta, \varphi) \) and \( \gamma_5(r, \theta, \varphi) \) (for \( \varphi \)-anisotropies, \( \lambda_5(r, \varphi) \) and \( \gamma_5(r, \varphi) \)) which may be singular for some values of \( r \), or on some hypersurfaces \( r = r(\theta, \varphi) \) (or \( r = r(\varphi) \)). Such a configuration may define a static black ellipsoid object (a Schwarzschild like static solution with the horizon slightly deformed to an ellipsoidal hypersurface\(^{18,19} \)).

In general, not being restricted only to ellipsoidal configurations, the simplest way to analyze the condition of vanishing of the metric coefficient before \( \delta t^2 \) is to choose \( \gamma_5 \) and \( \lambda_5 \) as to have \( \Theta = 0 \). In this case we find from (24)
\[ r_\pm = m \pm m \sqrt{1 - \varepsilon \frac{\Phi}{m^2}} \simeq m \left[ 1 \pm \left( 1 + q^2 \frac{\Phi}{2m^2} \right) \right] \quad (26) \]
where \( \Phi(r, \varphi) \) is taken for \( r = 2m \).

Let us introduce a new radial coordinate
\[ \xi = \int dr \sqrt{1 - \frac{2m}{r} - q^2 \frac{r^2}{r^2}}, \quad (27) \]
and define
\[ h_4 = -\eta_4(\xi, \varphi)r^2(\xi) \sin^2 \theta, \quad h_5 = 1 - \frac{2m}{r} - q^2 \frac{\Phi_5}{r^2}, \quad (28) \]
For \( r = r(\xi) \) found as the inverse function after integration in (27), we can write the metric (21) as
\[ ds^2 = -dc^2 - r^2(\xi) \gamma(\xi) \, d\theta^2 + h_4(\xi, \theta, \varphi) \delta \varphi^2 + h_5(\xi, \theta, \varphi) \delta t^2, \quad (29) \]
where the coefficient \( n_2 \) is taken to solve the equation (100) and to satisfy the condition \( \Omega_{jk}^a = 0 \) which states that we fix the canonical \( N \)-adapted connection (7) to coincide with the Levi-Civita connection (11), i.e. to consider a complex like Einstein but not Einstein–Cartan theory, which together with the condition \( r^2(\xi) \gamma(\xi) = \xi^2 \) will be transformed into the usual Schwarzschild metric for \( \varepsilon \to 0 \).

Let us define the conditions when the coefficients of metric (24) define solutions of the vacuum Einstein equations (such solutions exists in the real case following the Theorems 1-3 from the Appendix B, in our case we only state a generalization for certain complex valued metric coefficients). For \( g_2 = -1, g_3 = -r^2(\xi) \gamma(\xi) \) and arbitrary \( h_4(\xi, \theta, \varphi) \) and \( h_5(\xi, \theta) \), we
get solutions the equations (97)–(99). If $h_5$ depends on anisotropic variable $\varphi$, the equation (98) may be solved if

$$\sqrt{|\eta_4|} = \eta_0 \left( \sqrt{|\eta_4|} \right)^*$$

for $\eta_0 = \text{const}$. Considering decompositions of type (22) we put $\eta_0 = \eta/|\varepsilon|$, where the constant $\eta$ is taken as to have $\sqrt{|\eta_3|} = 1$ in the limits

$$\frac{\left( \sqrt{|\eta_4|} \right)^* \to 0}{|\varepsilon| \to 0} \to \frac{1}{\eta} = \text{const.}$$

(31)

These conditions are satisfied if the functions $\eta_{4[0]}$, $\lambda_{4,5}$ and $\gamma_{4,5}$ are related via relations

$$\sqrt{|\eta_{4[0]}|} = \frac{\eta}{2} \lambda_5^*, \lambda_5 = \eta \sqrt{|\eta_{4[0]}|} \gamma_5^*$$

for arbitrary $\gamma_4(r, \varphi)$. In this paper we select only such solutions which satisfy the conditions (30) and (31) being a complex variant of the conditions (110), see Appendix B. Similar classes of solutions were selected also in Refs. [18,19], for static black ellipsoid metrics for the (non–complex) Einstein gravity with real $\varepsilon$ parameter.

The next step is to construct the solution of (100) which in general real form is (112). To consider linear infinitesimal extensions on $\varepsilon$ of the Schwarzschild metric, we may write the solution of (100) as

$$n_2 = \varepsilon \tilde{n}_2(\xi, \varphi)$$

where

$$\tilde{n}_2(\xi, \varphi) = n_{2[1]}(\xi) + n_{2[2]}(\xi) \int d\varphi \, \eta_4(\xi, \varphi) / \left( \sqrt{|\eta_5(\xi, \varphi)|} \right)^3, \eta_5^* \neq 0;$$

$$= n_{2[1]}(\xi) + n_{2[2]}(\xi) \int d\varphi \, \eta_4(\xi, \varphi), \eta_5^* = 0;$$

$$= n_{2[1]}(\xi) + n_{2[2]}(\xi) \int d\varphi / \left( \sqrt{|\eta_5(\xi, \varphi)|} \right)^3, \eta_5^* = 0;$$

(32)

with the functions $n_{2[1,2]}(\xi)$ may be complex valued and have to be stated by boundary conditions.

The data

$$g_1 = -1, g_2 = -1, g_3 = -r^2(\xi) \gamma(\xi),$$

$$h_4 = -\eta_4(\xi, \varphi) r^2(\xi) \sin^2 \theta, h_5 = 1 - \frac{2m}{r} + \varepsilon \Phi_5, \frac{r}{r^2},$$

$$w_{2,3} = 0, n_2 = \varepsilon \tilde{n}_2(\xi, \varphi), n_3 = 0,$$

for the metric (21) written in variables $(\xi, \theta, \varphi, t)$ define a class of solutions of the complex vacuum Einstein equations with non–trivial polarization function $\eta_4$ and extended on parameter $\varepsilon$ up to the second order (the polarization functions being taken as to make zero the second order coefficients). Such solutions are generated by small complex deformations (in particular cases of rotation ellipsoid symmetry) of the Schwarzschild metric. It is possible to consider some particular parametrizations of $N$–coefficients resulting in hermitian metrics and
frames, or another type complex configurations. Such constructions do not affect the stability properties of solutions elaborated in this paper.

We can relate our complex exact solutions with some small deformations of the Schwarzschild metric to a Reissner–Nordstrem like metric with the ”electric” charge induced effectively from the anholonomic complex gravity, as well we can satisfy the asymptotically flat condition, if we chose such functions \( n_k \) as \( n_k \to 0 \) for \( \varepsilon \to 0 \) and \( \eta_4 \to 1 \). These functions have also to be selected as to vanish far away from the horizon, for instance, like \( \sim 1/r^{1+\tau}, \tau > 0 \), for long distances \( r \to \infty \). we get a static metric with effective ”electric” charge induced by a small, quadratic on \( \varepsilon \), off–diagonal metric extension. Roughly, we can say that we have constructed a Reissner–Nordstrem like world “living” in a ‘slightly’ complexified frame which induced both an effective electric charge and certain polarizations of the metric coefficients via the functions \( h_4[0], \eta_{4,5} \) and \( n_5 \).

Another very important property is that the deformed metric was stated to define a vacuum solution which differs substantially from the usual Reissner–Nordstrem metric being an exact static solution of the Einstein–Maxwell equations. For \( \varepsilon \to 0 \) and \( h_4[0] \to 1 \) and for \( \gamma = 1 \), the metric transforms into the usual Schwarzschild metric. A solution with ellipsoid symmetry can be selected by a corresponding condition of vanishing of the coefficient before \( \delta t \) which defines an ellipsoidal hypersurface like for the Kerr metric, but in our case the metric is non–rotating.

B Analytic Extensions of Ellipsoid Complex Metrics

In order to understand that the constructed in this section exact solution of vacuum complex gravity really defines black hole like objects we have to analyze it’s analytic extensions, horizon and geodesic behaviour and stability.

The metric has a singular behaviour for \( r = r_\pm \), see like the usual Reissner–Nordstrem one. Our aim is to prove that this way we have constructed a solution of the vacuum complex Einstein equations with a static ”anisotropic” horizon being a small deformation on parameter \( \varepsilon \) of the Schwarzschild’s solution horizon. We may analyze the anisotropic horizon’s properties for some fixed ”direction” given in a smooth vicinity of any values \( \varphi = \varphi_0 \) and \( r_+ = r_+(\varphi_0) \). The final conclusions will be some general ones for arbitrary \( \varphi \) when the explicit values of coefficients will have a parametric dependence on angular coordinate \( \varphi \). Of course, in order to avoid singularities induced by integration of the equation we have choose such solutions as the associated anhlonomic frames would be of necessary smooth class, without singularities.

The metrics are regular in the regions I \( (\infty > r > r^\Phi_+) \), II \( (r^\Phi_+ > r > r^\Phi_-) \) and III \( (r^\Phi_- > r > 0) \). As in the Schwarzschild, Reissner–Nordstrem and Kerr cases these singularities can be removed by introducing suitable coordinates and extending the manifold to obtain a maximal analytic extension. We have similar regions as in the Reissner–Nordstrem space–time, but with just only one possibility \( \varepsilon < 1 \) instead of three relations for static electro–vacuum cases \( q^2 < m^2, q^2 = m^2, q^2 > m^2; \) where \( q \) and \( m \) are correspondingly the electric charge and mass of the point particle in the Reissner–Nordstrem metric). This property holds for both type of anholonomic deformatons, real or complex ones. So, we may consider the usual Penrose’s diagrams as for a particular case of the Reissner–Nordstrem spacetime but keeping in mind that such diagrams and horizons have an additional parametrization on an angular coordinate and that the frames have some complex coefficients.

We can construct the maximally extended manifold step by steps like in the Schwarzschild
case (see details, for instance, in Ref. 52)) by supposing that the complex valued coefficients of metrics and frame are of necessary smooth class as real and complex valued functions (for simplicity, we consider the simplest variant when the spacetime is provided with a complex valued metric but admits covering by real coordinates which after certain coordinate transform may be also complex). We introduce a new coordinate

$$r^\| = \int dr \left( 1 - \frac{2m}{r} - \frac{q^2}{r^2} \right)^{-1}$$

for \( r > r^\pm_1 \) and find explicitly the coordinate

$$r^\| = r + \frac{(r^\pm_1)^2}{r^\pm - r^\pm_1} \ln(r - r^\pm_1) - \frac{(r^\pm_1)^2}{r^\pm - r^\pm_1} \ln(r - r^\pm_1),$$

(34)

where \( r^\pm_1 = r^\Phi_\pm \) with \( \Phi = 1 \). If \( r \) is expressed as a function on \( \xi \), than \( r^\| \) can be also expressed as a function on \( \xi \) depending additionally on some parameters.

Defining the advanced and retarded coordinates, \( v = t + r^\| \) and \( w = t - r^\| \), with corresponding elongated differentials

$$\delta v = \delta t + dr^\| \text{ and } \delta w = \delta t - dr^\|,$$

(35)

which are \( N \)-adapted frames like (2) but complex one, the metric (29) takes the form

$$\delta s^2 = -r^2(\xi)\gamma(\xi)d\theta^2 - \eta_4(\xi, \varphi_0)r^2(\xi)\sin^2\theta\varphi^2 + \left(1 - \frac{2m}{r(\xi)} - \frac{q^2 \Phi_4(r, \varphi_0)}{r^2(\xi)}\right)\delta v\delta w,$$

where (in general, in non–explicit form) \( r(\xi) \) is a function of type \( r(\xi) = r(\xi) = r(v, w) \).

Introducing new coordinates \( (v'', w'') \) by

$$v'' = \arctan \left[ \exp \left( \frac{r^\pm_1 - r^\pm_1}{4(r^\pm_1)^2} v \right) \right], \quad w'' = \arctan \left[ -\exp \left( -\frac{r^\pm_1 + r^\pm_1}{4(r^\pm_1)^2} w \right) \right],$$

(36)

and multiplying the last term on the conformal factor we obtain

$$\delta s^2 = -r^2(\gamma(r)d\theta^2 - \eta_4(r, \varphi_0)r^2 \sin^2\theta\varphi^2 + 64 \frac{(r^\pm_1)^4}{(r^\pm - r^\pm_1)^2} \left[1 - \frac{2m}{r(\xi)} - \frac{q^2 \Phi_5(r, \varphi_0)}{r^2(\xi)}\right] \delta v''\delta w''},$$

(37)

where \( r \) is defined implicitly by

$$\tan v'' \tan w'' = -\exp \left[ \frac{r^\pm_1 - r^\pm_1}{2(r^\pm_1)^2} r \right] \sqrt{\frac{r - r^\pm_1}{(r^\pm - r^\pm_1)^2}} \chi = \left( \frac{r^\pm_1}{r^\pm_1} \right)^2,$$

(38)

where the functions \( \tan \) and \( \exp \) should be considered as the complex functions. As particular cases, we may chose \( \eta_5(r, \varphi) \) as the condition of vanishing of the metric coefficient before \( \delta v''\delta w'' \) will describe a horizon parametrized by a resolution ellipsoid hypersurface.

The metric (37) is very similar to that analyzed in Refs. 18,19 but the coordinate transforms defined by (35)–(38) involve complex coordinate transforms, so \( \delta v''\delta w'' \) is a product defined by complexified \( N \)-adapted frames.
The maximal extension of the Schwarzschild metric deformed by a small parameter \( \varepsilon \), i.e., the extension of the metric (21), is defined by taking (37), as the metric on the maximal manifold (which for corresponding coordinate transforms can be considered as a real one but with complex valued coefficients of the metric and moving frames) on which this metric is of smoothly class \( C^2 \). The Penrose diagram of this static but locally anisotropic space–time, for any fixed angular value \( \varphi_0 \) is similar to the Reissner–Nördstrom solution, for the case \( q^2 \to \varepsilon \) and \( q^2 < m^2 \) (see, for instance, Ref. 52). There are an infinite number of asymptotically flat regions of type I, connected by intermediate regions II and III, where there is still an irremovable singularity at \( r = 0 \) for every region III. We may travel from a region I to another ones by passing through the 'wormholes' made by anisotropic deformations (ellipsoid off–diagonality of metrics, or anholonomy) like in the Reissner–Nordstrom universe because \( \sqrt{\varepsilon} \sim q \) may model an effective electric charge. One can not turn back in a such travel because the complex frames "do not allow us".

It should be noted that the metric (37) is analytic everywhere except at \( r = r_- \) (we have to use the term analytic as real functions for the metric coefficients in the lower approximations on \( \varepsilon \) and analytic as complex functions for the higher approximations of the metric coefficients and for all terms contained in the vielbein coefficients). We may eliminate this coordinate degeneration by introducing another new complex coordinates

\[
v^{\mu} = \arctan \left[ \exp \left( \frac{r^- - r^+}{2n_0(r^+)^2} v \right) \right], \quad w^{\mu} = \arctan \left[ -\exp \left( \frac{-r^+ + r^1}{2n_0(r^+)^2} w \right) \right],
\]

where the integer \( n_0 \geq (r^+)^2/(r^1)^2 \) and complex functions. In these coordinates, the metric is (in general, complex) analytic everywhere except at \( r = r_+ \) where it is degenerate. This way the space–time manifold can be covered by an analytic atlas by using coordinate carts defined by \((v^{\mu}, w^{\mu}, \theta, \varphi)\) and \((v^{\mu}, w^{\mu}, \theta, \varphi)\). We also note that the analytic extensions of the deformed metrics were performed with respect to anholonomic complex frames which distinguish such constructions from those dealing only with holonomic and/or real coordinates, like for the usual Reissner–Nördstrom and Kerr metrics.

A more rigorous analysis of the metric (21) should involve a computation of its curvature and investigation of singularity properties. We omit here this cumbersome calculus by emphasizing that anholonomic deformations of the Schwarzschild solution defined by a small real or complex parameter \( \varepsilon \) can not remove the bulk singularity of such spacetimes; there are deformations of the horizon, frames and specific polarizations of constants.

The metric (21) and its analytic extensions do not posses Killing symmetries being deformed by anholonomic transforms. Nevertheless, we can associate to such solutions certain noncommutative symmetries following the procedure described in section [III]. Taking the data (33) and formulas (6), we compute the corresponding nontrivial anholonomy coefficients

\[
w_\gamma^{[N]/5}_{42} = -w^{[N]/5}_{24} = \partial n_2 (\xi, \varphi) / \partial \varphi = n_2^* (\xi, \varphi) \tag{39}
\]

with \( n_2 \) defined by (32). Our vacuum solution is for 4D, so for \( n + m = 4 \), the condition \( k^2 - 1 = n + m \) can not satisfied in integer numbers. We may trivially extend the dimensions like \( n' = 6 \) and \( m' = 2 \) and for \( k = 3 \) to consider the Lie group \( SL(3, \mathbb{C}) \) noncommutativity with corresponding values of \( Q^{\pm}_{\alpha \beta} \) and structure constants \( f_{\alpha \beta}^{\gamma} \), see (11). An extension \( w^{[N] \gamma}_{\alpha \beta} \to W^{\gamma}_{\alpha \beta} \) may be performed by stating the N–deformed "structure" constants (14), \( W^{\gamma}_{\alpha \beta} = f_{\alpha \beta}^{\gamma} + w^{[N] \gamma}_{\alpha \beta} \), with only two nontrivial values of \( w^{[N] \gamma}_{\alpha \beta} \) given by (39).
The associated anholonomic noncommutative symmetries of the black ellipsoid solutions can be alternatively defined as in the trivial anholonomy limit they will result in a certain noncommutativity for the Lie group $SU_3$. In this case, we have to consider a $N$–deformation of the group structure constants $p^\alpha_{\beta\gamma}$, like in (19), $W^\alpha_{\beta\gamma} = p^\alpha_{\beta\gamma} + w^\alpha_{\beta\gamma}$. This variant of deformations can be related directly with the "de Sitter nonlinear gauge gravity model of (non) commutative gravity" and the $SU_k[SO(k)]$–models of noncommutative gravity by considering complex vielbeins.

C Black ellipsoids and the cosmological constant

We can generalize the vacuum equations to the gravity with cosmological constant $\lambda$,

$$R_{\mu\nu'\rho'} = \lambda g_{\mu'\nu'}, \quad (40)$$

where $R_{\mu'\nu'}$ is the Ricci tensor, in general with anholonomic variables and the indices take values $i', \ k' = 1, 2$ and $a', \ b' = 3, 4$.

For an ansatz of type (123)

$$\delta s^2 = g_1(dx^1)^2 + g_2(dx^2)^2 + h_3(x^{i'}, y^3) (\delta y^3)^2 + h_4(x^{i'}, y^3) (\delta y^4)^2, \quad (41)$$

$$\delta y^3 = dy^3 + w_{\nu'}(x^{i'}, y^3) dx^{i'}, \quad \delta y^4 = dy^4 + n_{\nu'}(x^{k'}, y^3) dx^{k'},$$

the Einstein equations (40) are written (see Refs. 30, 31, 13-19 for details on computation; this is a particular case of source of type (132), see Appendix B)

$$R^1_1 = R^2_2 = -\frac{1}{2g_1g_2} [g_{1'}^* g_{2'}^* - \frac{(g_{1'}^*)^2}{2g_1} - \frac{(g_{2'}^*)^2}{2g_2} - \frac{(g_1')^2}{2g_1}] = \lambda, \quad (42)$$

$$R^3_3 = R^4_4 = -\frac{\beta}{2h_3 h_4} = \lambda, \quad (43)$$

$$R_{3\nu'} = -\frac{\alpha_{\nu'}}{2h_3} = 0, \quad (44)$$

$$R_{4\nu'} = -\frac{h_4}{2h_3} [n_{\nu'}^* + \gamma n_{\nu'}^*] = 0. \quad (45)$$

The coefficients of equations (42) - (45) are given by

$$\alpha_i = \partial_i h_4^* - h_4^* \partial_i \ln \sqrt{|h_3 h_4|}, \quad \beta = h_4^* - h_4^* [\ln \sqrt{|h_3 h_4|}]^*, \quad \gamma = \frac{3h_4^*}{2h_4} - \frac{h_4^*}{h_3}. \quad (46)$$

The various partial derivatives are denoted as $a^* = \partial a/\partial x^1, a' = \partial a/\partial x^2, a^* = \partial a/\partial y^3$. This system of equations can be solved by choosing one of the ansatz functions (e.g. $g_1(x^i)$ or $g_2(x^i)$) and one of the ansatz functions (e.g. $h_3(x^i, y^3)$ or $h_4(x^i, y^3)$) to take some arbitrary, but physically interesting form (see Theorem 3 in Appendix B). Then, the other ansatz functions can be analytically determined up to an integration in terms of this choice. In this way we can generate a lost of different solutions, but we impose the condition that the initial, arbitrary choice of the ansatz functions is “physically interesting” which means that one wants to make this original choice so that the generated final solution yield a well behaved metric.
In Ref. 19 (see also previous subsection), we proved that for
\[ g_1 = -1, \quad g_2 = r^2(\xi)q(\xi), \quad h_3 = -\eta_3(\xi, \varphi)r^2(\xi)\sin^2\theta, \]
\[ h_4 = \eta_4(\xi, \varphi)h_{4[0]}(\xi) = 1 - \frac{2\mu}{r} + \varepsilon \frac{\Phi_4(\xi, \varphi)}{2\mu^2}, \]
with coordinates \( x^1 = \xi = \int dr \sqrt{1 - 2m/r + \varepsilon/r^2}, \) \( x^2 = \theta, \) \( y_3 = \phi, \) \( y_4 = t \) (the \((r, \theta, \phi)\) being usual radial coordinates), the ansatz (41) is a vacuum solution with \( \lambda = 0 \) of the equations (40) which defines a black ellipsoid with mass \( \mu, \) eccentricity \( \varepsilon \) and gravitational polarizations \( q(\xi), \eta_3(\xi, \varphi) \) and \( \Phi_4(\xi, \varphi) \). Such black holes are certain deformations of the Schwarzschild metrics to static configurations with ellipsoidal horizons which is possible if generic off–diagonal metrics and anholonomic frames are considered. A complex generalization of this solution is given by the values (33). In this subsection we show that the data (47) and/or (33) can be extended as to generate exact black ellipsoid solutions, defied correspondingly with respect to real or complex \( N \)–frames, with nontrivial cosmological constant \( \lambda = 1/4 \) which can be imbedded in string theory.

At the first step, we find a class of solutions with \( g_1 = -1 \) and \( g_2 = g_2(\xi) \) solving the equation (42), which under such parametrizations transforms to
\[ g_2^{**} - \frac{(g_2^*)^2}{2g_2} = 2g_2\lambda. \]

With respect to the variable \( Z = (g_2)^2 \) this equation is written as
\[ Z^{**} + 2\lambda Z = 0 \]
which can be integrated in explicit form, \( Z = Z_{[0]} \sin\left(\sqrt{2\lambda}\xi + \xi_{[0]}\right), \) for some constants \( Z_{[0]} \) and \( \xi_{[0]} \) which means that
\[ g_2 = -Z_{[0]}^2 \sin^2\left(\sqrt{2\lambda}\xi + \xi_{[0]}\right) \]
parametrize in ‘real’ string gravity a class of solution of (42) for the signature \((-,-,-,+)\). For \( \lambda \to 0 \) we can approximate \( g_2 = r^2(\xi)q(\xi) = -\xi^2 \) and \( Z_{[0]}^2 = 1 \) which has compatibility with the data (47). The solution (48) with cosmological constant (of string or non–string origin) induces oscillations in the ”horizontal” part of the metric written with respect to \( N \)–adapted frames.

The next step is to solve the equation (43),
\[ h_4^{**} - h_4^*[\ln \sqrt{|h_3h_4|}]^* = -2\lambda h_3h_4. \]

For \( \lambda = 0 \) a class of solution is given by any \( \hat{h}_3 \) and \( \hat{h}_4 \) related as
\[ \hat{h}_3 = \eta_0 \left[ \left(\sqrt{|\hat{h}_4|}\right)^* \right]^2 \]
for a constant \( \eta_0 \) chosen to be negative in order to generate the signature \((-,-,-,+)\). For non–trivial \( \lambda \), we may search the solution as
\[ h_3 = \hat{h}_3(\xi, \varphi) f_3(\xi, \varphi) \text{ and } h_4 = \hat{h}_4(\xi, \varphi), \]
\[ (49) \]
which solves (43) if \( f_3 = 1 \) for \( \lambda = 0 \) and

\[
f_3 = \frac{1}{4\lambda} \left[ \int \frac{\dot{h}_3 \dot{h}_4}{h_4} d\varphi \right]^{-1}
\]

for \( \lambda \neq 0 \).

Now it is easy to write down the solutions of equations (44) (being a linear equation for \( w_{i'} \)) and (45) (after two integrations of \( n_{i'} \) on \( \varphi \)),

\[
w_{i'} = \varepsilon \hat{w}_{i'} = -\alpha_{i'}/\beta,
\]

were \( \alpha_{i'} \) and \( \beta \) are computed by putting (49) into corresponding values from (46) (we chose the initial conditions as \( w_{i'} \to 0 \) for \( \varepsilon \to 0 \)) and

\[
n_1 = \varepsilon \hat{n}_1 (\xi, \varphi)
\]

where the coefficients

\[
\hat{n}_1 (\xi, \varphi) = n_{1[1]} (\xi) + n_{1[2]} (\xi) \int d\varphi \frac{\eta_3 (\xi, \varphi)}{\left| \sqrt{\eta_4 (\xi, \varphi)} \right|^3} \eta_4^* \neq 0; \quad (51)
\]

\[
= n_{1[1]} (\xi) + n_{1[2]} (\xi) \int d\varphi \eta_3 (\xi, \varphi), \quad \eta_4^* = 0;
\]

\[
= n_{1[1]} (\xi) + n_{1[2]} (\xi) \int d\varphi / \left| \sqrt{\eta_4 (\xi, \varphi)} \right|^3, \quad \eta_4^* = 0;
\]

being stated to be real or complex valued for a corresponding model of real or complex gravity, with the functions \( n_{k[1,2]} (\xi) \) to be stated by boundary conditions.

We conclude that the set of data \( g_1 = -1 \), with non–trivial \( g_2 (\xi) \), \( h_3, h_4, w_{i'} \) and \( n_1 \) stated respectively by (48), (49), (50), (51) we can define a black ellipsoid solution with explicit dependence on cosmological constant \( \lambda \), i.e. a metric (41). The stability of such static black ellipsoids in (anti) De Sitter space can be proven exactly as it was done in Ref. 19 for the real case vanishing cosmological constant.

The analytic extension of black ellipsoid solutions with cosmological constant can be performed similarly as in the previous subsection when for the real/ complex solutions we are dealing with real/ complex values of \( \hat{n}_1 (\xi, \varphi) \) defining some components of \( N \)-adapted frames. We note that the solution from string theory contains a frame induced torsion with the components (42) (in general, with complex coefficients) computed for nontrivial \( N_{i'}^3 = -\alpha_{i'}/\beta \) (see (50)) and \( N_i^4 = \varepsilon \hat{n}_1 (\xi, \varphi) \) (see (51)). This is an explicit example illustrating that the anholonomic frame method is also powerful for generating exact solutions in models of gravity with nontrivial torsion, induced by anholonomic frame transforms. For such solutions we may elaborate corresponding analytic extension and Penrose diagram formalisms if the constructions are considered with respect to \( N \)-elongated vielbeins.

Finally, we analyze the structure of noncommutative symmetries associated to the (anti) de Sitter black ellipsoid solutions. The metric (41) with real and/or complex coefficients defining the corresponding solutions and its analytic extensions also do not posses Killing symmetries being deformed by anholonomic transforms. For this solution, we can associate certain noncommutative symmetries following the same procedure as for the Einstein real/ complex gravity but with additional nontrivial coefficients of anholonomy and even with nonvanishing
coefficients of the nonlinear connection curvature, \( \Omega_{12}^3 = \delta_1 N_2^3 - \delta_2 N_1^3 \). Taking the data (50) and (51) and formulas (6), we compute the corresponding nontrivial anholonomy coefficients

\[
\begin{align*}
\omega_{31}^{[N]4} &= -\omega_{13}^{[N]4} = \partial n_1 (\xi, \varphi) / \partial \varphi = n_2^\ast (\xi, \varphi), \\
\omega_{12}^{[N]4} &= -\omega_{21}^{[N]4} = \delta_1 (\alpha_2/\beta) - \delta_2 (\alpha_1/\beta)
\end{align*}
\]

for \( \delta_1 = \partial / \partial \xi - w_1 \partial / \partial \varphi \) and \( \delta_2 = \partial / \partial \theta - w_2 \partial / \partial \varphi \), with \( n_1 \) defined by (52) and \( \alpha_{1,2} \) and \( \beta \) computed by using the formula (46) for the solutions (49). Our exact solution, with nontrivial cosmological constant, is for 4D, like in the previous subsection. So, for \( n + m = 4 \), the condition \( k^2 - 1 = n + m \) can not be satisfied by any integer numbers. We may similarly trivially extend the dimensions like \( n' = 6 \) and \( m' = m = 2 \) and for \( k = 3 \) to consider the Lie group \( SL(3, \mathbb{C}) \) noncommutativity with corresponding values of \( Q_{\alpha\beta} \) and structure constants \( f_{\alpha\beta} \), see (11). An extension \( w_{[N]^\gamma} \to W_{\alpha\beta} \) may be performed by stating the N–deformed structure constants (13), \( W_{\alpha\beta} = f_{\alpha\beta} + w_{[N]\gamma} \), with nontrivial values of \( w_{[N]\gamma} \) given by (52). In a similar form, we can consider anholonomic deformations of the \( SU_k \) structure constants, see (19).

V Noncommutative Complex Wormholes

The black ellipsoid solutions defined by real and certain complex metrics elaborated in the previous section were for the 4D Einstein gravity, in general, with nontrivial cosmological constant. In this section we construct and analyze an exact 5D solution which can be also complexified by using complex anholonomic transforms as well they can be provided with associated noncommutative structure. For such configurations we can apply directly the formulas stated in Appendix B. The metric ansatz (11) is taken in the form

\[
\begin{align*}
\delta s^2 &= g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta y^4)^2 + h_5(\delta y^5)^2, \\
\delta y^4 &= dy^4 + w_{k'} \left( x^i, v \right) dx^{k'}, \delta y^5 = dy^5 + n_{k'} \left( x^i, v \right) dx^{k'}; i', k' = 1, 2, 3, \\
\end{align*}
\]

where

\[
\begin{align*}
g_1 &= 1, \quad g_2 = g_2(r), \quad g_3 = -a(r), \\
h_4 &= \hat{h}_4 = \hat{\eta}_4 (r, \theta, \varphi) h_{4[0]}(r), \quad h_5 = \hat{h}_5 = \hat{\eta}_5 (r, \theta, \varphi) h_{5[0]}(r, \theta)
\end{align*}
\]

for the parametrization of coordinate of type

\[
x^1 = t, \quad x^2 = r, \quad x^3 = \theta, \quad y^4 = \varphi, \quad y^5 = p = \chi
\]

where \( t \) is the time coordinate, \( (r, \theta, \varphi) \) are spherical coordinates, \( \chi \) is the 5th coordinate; \( \varphi \) is the anholonomic coordinate; for this ansatz there is not considered the dependence of metric coefficients on the second anholonomic coordinate \( \chi \). Following similar approximations as in subsection (12) for deriving the equations (40), we can write the gravity equations with cosmological constant as a system of 5D Einstein equations with constant diagonal source (the related details on computing the Ricci tensors with anholonomic variables and possible
sources are given in Appendix B):

\[
\frac{1}{2}R^1 = R^2 = R^3 = -\frac{1}{2g_2g_3} \left[ g_3^\bullet \right. - \frac{g_2^\bullet g_3^\bullet}{2g_2} - \left( g_3^\bullet \right)^2 + g_2 - \frac{g_2^\bullet g_3^\bullet}{2g_3} - \frac{\left( g_2^\bullet \right)^2}{2g_2} = \lambda, \tag{56}
\]

\[
R^4 = R_5 = -\frac{\beta}{2h_4h_5} = \lambda, \tag{57}
\]

\[
R^4_{\alpha'} = -w_x \frac{\beta}{2h_5} - \frac{\alpha_x}{2h_5} = 0, \tag{58}
\]

\[
R^5_{\alpha'} = -\frac{h_5}{2h_4} \left[ n_i^{\bullet \bullet} + \gamma n_i' \right] = 0, \tag{59}
\]

where \( i' = 1, 2, 3 \). The coefficients of the equations are given by

\[
\alpha_{\alpha'} = \partial_i h_5^\bullet - h_5^\bullet \partial_{\alpha'} \ln \sqrt{|h_4h_5|}, \quad \beta = h_5^\bullet - h_5^\bullet \ln \sqrt{|h_4h_5|}, \quad \gamma = \frac{3h_5^\bullet}{2h_5} - \frac{h_4^\bullet}{h_4}. \tag{60}
\]

The various partial derivatives are denoted as \( a^\bullet = \partial a/\partial x^2, a' = \partial a/\partial x^3, a^3 = \partial a/\partial v \).

The system of equations [56–59] can be solved by choosing one of the ansatz functions (e.g. \( h_4(x^2, v) \) or \( h_5(x^2, v) \)) to take some arbitrary, but physically interesting form. Then the other ansatz functions can be analytically determined up to an integration in terms of this choice. In this way one can generate many solutions, but the requirement that the initial, arbitrary choice of the ansatz functions be “physically interesting” means that one wants to make this original choice so that the final solution generated in this way yield a well behaved solution. To satisfy this requirement we start from well known solutions of Einstein’s equations and then use the above procedure to deform this solutions in a number of ways as to include it in a string theory, for instance, as a gravity model with cosmological constant.

The data

\[
\begin{align*}
g_1 &= 1, \quad \dot{g}_2 = -1, \quad g_3 = -a(r), \\
h_{4[0]}(r) &= -r_0^2 e^{2\psi(r)}, \quad \eta_4 = 1/\kappa_r^2(r, \theta, \phi), \quad h_{5[0]} = -a(r) \sin^2 \theta, \quad \eta_5 = 1, \\
w_1 &= \hat{w}_1 = \omega(r), \quad w_2 = \hat{w}_2 = 0, \quad w_3 = \hat{w}_3 = n \cos \theta/\kappa_n^2(r, \theta, \phi), \\
n_1 &= \hat{n}_1 = 0, \quad n_{2,3} = \hat{n}_{2,3} = n_{2,3[1]}(r, \theta) \int \ln |\kappa_r^2(r, \theta, \phi)| \, d\phi
\end{align*}
\]

for some constants \( r_0 \) and \( n \) and arbitrary functions \( a(r), \psi(r) \) and arbitrary vacuum gravitational polarizations \( \kappa_r(r, \theta, \phi) \) and \( \kappa_n(r, \theta, \phi) \) define an exact vacuum 5D solution of Kaluza–Klein gravity describing a locally anisotropic wormhole with elliptic gravitational vacuum polarization of charges,

\[
\frac{q_0^2}{4a(0)\kappa_r^2} + \frac{Q_0^2}{4a(0)\kappa_n^2} = 1,
\]

where \( q_0 = 2\sqrt{a(0)} \sin \alpha_0 \) and \( Q_0 = 2\sqrt{a(0)} \cos \alpha_0 \) are respectively the electric and magnetic charges and \( 2\sqrt{a(0)\kappa_r} \) and \( 2\sqrt{a(0)\kappa_n} \) are ellipse’s axes.

The first aim in this section is to prove that following the ansatz [53] we can construct locally anisotropic wormhole metrics in (anti) de Sitter gravity, in general complexified by a certain class of anholonomic frame transforms as solutions of the system of equations [56]–[59] with redefined coordinates as in [55]. For simplicity, we select such solutions when only
the coefficients $n_i$ can be real or complex valued functions. Having the vacuum data \( (63) \), we may try to generalize the solution for a nontrivial cosmological constant by supposing that the new solutions may be represented as

\[
h_4 = \hat{h}_4 \left(x'', v\right) \quad q_4 \left(x'', v\right) \quad \text{and} \quad h_5 = \hat{h}_5 \left(x'', v\right),
\]

with $\hat{h}_{4,5}$ taken as in (54) which solves (57) if $q_4 = 1$ for $\lambda = 0$ and

\[
q_4 = \frac{1}{4\lambda} \left[ \int \frac{\hat{h}_5 \left(r, \theta, \varphi\right) \hat{h}_4 \left(r, \theta, \varphi\right)}{\hat{h}_5 \left(r, \theta, \varphi\right)} d\varphi \right]^{-1} \quad \text{for} \quad \lambda \neq 0.
\]

This $q_4$ can be considered as an additional polarization to $\eta_4$ induced by the cosmological constant $\lambda$. We state $g_2 = -1$ but

\[
g_3 = -\sin^2 \left(\sqrt{2\lambda} \theta + \xi_{[0]}\right),
\]

defining a solution of (50) with signature $(+, - , - , - , - )$ being different from the solution (18).

A non–trivial $q_4$ results in modification of coefficients (60),

\[
\alpha_{\nu'} = \hat{\alpha}_{\nu'} + \alpha_{\nu'}^{[a]}, \quad \beta = \hat{\beta} + \beta^{[a]}, \quad \gamma = \hat{\gamma} + \gamma^{[a]},
\]

\[
\hat{\alpha}_{\nu'} = \partial_\nu \hat{h}_5 - \hat{h}_5 \partial_\nu \ln |\hat{h}_4 \hat{h}_5|, \quad \hat{\beta} = \hat{h}_5^* - \hat{h}_5^* \ln |\hat{h}_4 \hat{h}_5|^*, \quad \hat{\gamma} = \frac{3\hat{h}_5^*}{2h_5} - \hat{h}_4^* \hat{h}_4
\]

\[
\alpha_{\nu'}^{[a]} = -\hat{h}_5 \partial_\nu \ln |q_4|, \quad \beta^{[a]} = -\hat{h}_5^* \ln |q_4|^*, \quad \gamma^{[a]} = -\frac{q_4^*}{q_4},
\]

which following formulas (58) and (59) result in additional terms to the N–connection coefficients, i. e.

\[
w_{\nu'} = \hat{w}_{\nu'} + w_{\nu'}^{[a]} \quad \text{and} \quad n_{\nu'} = \hat{n}_{\nu'} + n_{\nu'}^{[a]},
\]

with $w_{\nu'}^{[a]}$ and $n_{\nu'}^{[a]}$ computed by using respectively $\alpha_{\nu'}^{[a]}, \beta^{[a]}$ and $\gamma^{[a]}$.

The simplest way to generate complex solutions is to consider that $\hat{n}_{\nu'}$ from the data (61) and (63) can be complex valued functions, for instance, with complex valued coefficients $n_{2,3 \|} (r, \theta)$ resulting from integration. In this case the metric (63) has real coefficients describing wormhole solutions with polarized constants but such metric coefficients are defined with respect to anholonomic frames being N–elongated by some real and complex functions.

Having nontrivial values of (63), we can associate certain noncommutative symmetries following the same procedure as for real/ complex black ellipsoids. The wormhole cases are described by a more general set of nontrivial coefficients of anholonomy $w_{\nu' \alpha \beta}^{[N]}$ computed by using formulas (6) and (15) (for simplicity, we omit such cumbersome expressions), and a nontrivial nonlinear connection curvature, in our case $\Omega_{\nu' \nu} = \delta_{\nu' \nu} N_{\alpha}^\nu - \delta_{\nu' \nu} N_{\beta}^\nu$ with $N_{\alpha}^\nu = w_{\alpha \beta}$ and $N_{\beta}^\nu = n_{\alpha \beta}$. Such coefficients depend on variables $(r, \theta, \varphi)$, in general, being complex valued functions. We have to extend trivially the dimensions. We have to extend the dimensions like $n = 5 \rightarrow n' = 6$ and $m' = m = 2$ and for $k = 3$ if we want to associate a Lie group $SL(3, \mathbb{C})$ like noncommutativity with the corresponding values of $Q_{\alpha \beta}^{\nu}$ and structure constants $f_{\alpha \beta}^{\gamma}$, see (11). An extension $w_{\nu' \alpha \beta}^{[N]} \rightarrow W_{\alpha \beta}^{\gamma}$ may be similarly performed by introducing N–deformed "structure" constants (14), $W_{\alpha \beta}^{\gamma} = f_{\alpha \beta}^{\gamma} + w_{\alpha \beta}^{[N]}$, with nontrivial values of $w_{\alpha \beta}^{[N]}$ defined by (63).
We start with a discussion of the results of Refs. 6, 7 concerning noncommutative gauge models of gravity:

The basic idea of the Ref. 6 was to use a geometrical result that the Einstein gravity can be equivalently represented as a gauge theory with a specific connection in the bundle of affine frames. Such gauge theories are with nonsemisimple structure gauge groups, i.e., with degenerated metrics in the total spaces. Using an auxiliary symmetric form for the typical fiber, any such model can be transformed into a variational one. There is an alternative way to construct geometrically a usual Yang–Mills theory by applying a corresponding set of absolute derivations and dualities defined by the Hodge operator. For both such approaches, there is a projection formalism reducing the geometric field equations on the base space to be exactly the Einstein equations from the general relativity theory.

For more general purposes, it was suggested to consider also extensions to a nonlinear realization with the (anti) de Sitter gauge structural group. The constructions with nonlinear group realizations are very important because they prescribe a consistent approach of distinguishing the frame indices and coordinate indices subjected to different rules of transformation. This approach to gauge gravity (of course, after a corresponding generalizations of the Seiberg–Witten map) may include, in general, quadratic on curvature and torsion terms (as it is stated in Ref. 6) being correlated to the results on gravity on noncommutative D-branes.

At the first step, it was very important to suggest an idea how to include the general relativity into a gauge model being more explicitly developed in noncommutative form (see recent developments in Refs. 54-60).

A Nonlinear gauge models for the (anti) de Sitter group

There were elaborated some alternative approaches to the noncommutative gauge gravity models in Refs. 7 (by deforming the Einstein gravity based on gauging the commutative inhomogeneous Lorentz group ISO (3, 1) using the Seiberg–Witten map) and 60 (by considering some simplest noncommutative deformations of the gauge theory U(2, 2) and of the Lorentz algebra SO (1, 3)). Such theories reduce to the general relativity if certain constraints and breaking symmetries are imposed. Perhaps, only some experimental data would emphasize a priority of a theory of noncommutative gravity with a proper prescription how the vielbeins and connection from ‘commutative’ gravity have to be combined into components of a linear/nonlinear realizations of a noncommutative gauge potentials defined by corresponding Seiberg–Witten maps. At the present state of elaboration of noncommutative geometry and physics, we have to analyze the physical consequences of different classes of models of noncommutative gravity.

We introduce vielbein decompositions of (in general) complex metrics:

\[ \hat{g}_{\alpha \beta}(u) = e^\alpha_{\alpha'}(u) e^\beta_{\beta'}(u) \eta_{\alpha' \beta'}, \]
\[ e^\alpha_{\alpha'} e^\beta_{\beta'} = \delta^\beta_{\beta'} \text{ and } e^\alpha_{\alpha'} e^\alpha_{\beta'} = \delta^\alpha_{\beta'} \]

where \( \eta_{\alpha' \beta'} \) is a constant diagonal matrix (for real spacetimes we can consider it as the flat Minkowski metric, for instance, \( \eta_{\alpha' \beta'} = \text{diag} (-1, +1, ..., +1) \)) and \( \delta^\beta_{\beta'} \) and \( \delta^\alpha_{\beta'} \) are Kronecker’s delta symbols. The vielbeins with an associated N–connection structure \( N^a_i (x^j, y^a) \), being
real or complex valued functions, have a special parametrization

\[ e_{\alpha'}^i(u) = \begin{bmatrix} e_i^\nu(x^j) & N_i^c(x^j, y^a) & e_{\nu'}^c(x^j, y^a) \\ 0 & e_{\nu'}^c(x^j, y^a) \end{bmatrix} \]  \hspace{1cm} (64)

and

\[ e^\alpha_{\alpha'}(u) = \begin{bmatrix} e^\nu_i(x^j) & -N_i^c(x^j, y^a) & e^\nu_{\nu'}^c(x^j) \\ 0 & e^\nu_{\nu'}^c(x^j, y^a) \end{bmatrix} \]  \hspace{1cm} (65)

with \( e_i^\nu(x^j) \) and \( e_{\nu'}^c(x^j, y^a) \) generating the coefficients of metric \([1]\) with the coefficients defined with respect to anholonic frames,

\[ g_{ij}(x^j) = e_i^\nu(x^j) e_j^{\nu'}(x^j) \eta_{\nu'\nu'} \text{ and } h_{ab}(x^j, y^c) = e_a^\alpha(x^j, y^c) e_b^\beta(x^j, y^c) \eta_{\alpha\beta}. \]  \hspace{1cm} (66)

By using vielbeins and metrics of type \([63]\) and \([65]\) and, respectively, \([66]\), we can model in a unified manner various types of (pseudo) Riemannian, Einstein–Cartan, Riemann–Finsler and vector/ covector bundle nonlinear connection commutative and noncommutative geometries in effective gauge and string theories (it depends on the parametrization of \( e_i^\nu, e_{\nu'}^c \) and \( N_i^c \) on coordinates and anholonomy relations, see details in Refs. 24-28).

We consider the de Sitter space \( \Sigma^4 \) as a hypersurface defined by the equations \( \eta_{AB}u^Au^B = -l^2 \) in the four dimensional flat space enabled with diagonal metric \( \eta_{AB}, \eta_{AA} = \pm 1 \) (in this section \( A, B, C, \ldots = 1, 2, \ldots, 5 \)), where \( \{u^A\} \) are global Cartesian coordinates in \( \mathbb{R}^5 \), \( l > 0 \) is the curvature of de Sitter space (for simplicity, we consider here only the de Sitter case; the anti–de Sitter configuration is to be stated by a hypersurface \( \eta_{AB}u^Au^B = l^2 \)). The de Sitter group \( S_{(n)} = SO_{(n)}(5) \) is the isometry group of \( \Sigma^5 \)–space with 6 generators of Lie algebra \( so_{(n)}(5) \) satisfying the commutation relations

\[ [M_{AB}, M_{CD}] = \eta_{AC}M_{BD} - \eta_{BC}M_{AD} - \eta_{AD}M_{BC} + \eta_{BD}M_{AC}. \]  \hspace{1cm} (67)

We can decompose the capital indices \( A, B, \ldots \) as \( A = (\alpha', 5) \), \( B = (\beta', 5), \ldots \), and the metric \( \eta_{AB} \) as \( \eta_{AB} = (\eta_{\alpha'\beta'}, \eta_{55}) \). The operators \([\mathbf{67}]\) \( M_{AB} \) can be decomposed as \( M_{\alpha'\beta'} = \mathcal{F}_{\alpha'\beta'} \) and \( P_{\alpha'} = l^{-1}M_{5\alpha'} \) written as

\[ [\mathcal{F}_{\alpha'\beta'}, \mathcal{F}_{\gamma'\delta'}] = \eta_{\alpha'\gamma'}\mathcal{F}_{\beta'\delta'} - \eta_{\beta'\gamma'}\mathcal{F}_{\alpha'\delta'} + \eta_{\delta'\gamma'}\mathcal{F}_{\alpha'\beta'} - \eta_{\alpha'\delta'}\mathcal{F}_{\beta'\gamma'}, \]

\[ [P_{\alpha'}, P_{\beta'}] = -l^{-2}\mathcal{F}_{\alpha'\beta'}, \]

\[ [P_{\alpha'}, \mathcal{F}_{\beta'\gamma'}] = \eta_{\alpha'\beta'}P_{\gamma'} - \eta_{\alpha'\gamma'}P_{\beta'}, \]  \hspace{1cm} (68)

where the Lie algebra \( so_{(5)}(5) \) is split into a direct sum, \( so_{(5)}(5) = so_{(5)}(4) \oplus V_4 \) with \( V_4 \) being the vector space stretched on vectors \( P_{\alpha} \). We remark that \( \Sigma^4 = S_{(n)}/L_{(n)} \), where \( L_{(n)} = SO_{(n)}(4) \). For \( \eta_{AB} = diag(-1, +1, +1, +1) \) and \( S_{10} = SO(1, 4) \), \( L_6 = SO(1, 3) \) is the group of Lorentz rotations.

The generators \( I_4 \) and structure constants \( \mathcal{F}_{\alpha'\beta'}^{so_{(5)}} \) of the de Sitter Lie group can be parametrized in a form distinguishing the de Sitter generators and commutations \([\text{68}]\). The action of the group \( S_{(n)} \) may be realized by using 4 \( \times \) 4 matrices with a parametrization distinguishing the subgroup \( L_{(n)} \):

\[ B = bB_L, \]  \hspace{1cm} (69)
\[ B_L = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}, \]

$L \in L(\eta)$ is the de Sitter bust matrix transforming the vector $(0, 0, \ldots, \rho) \in \mathbb{R}^5$ into the arbitrary point $(V^1, V^2, \ldots, V^5) \in \Sigma_\rho \subset \mathcal{R}^5$ with curvature $\rho$, $(V_A V^A = -\rho^2, V^A = \tau^A \rho)$, and the matrix $b$ is expressed

\[ b = \begin{pmatrix} \delta_{\alpha'}^{\beta'} + \tau_{\alpha'}^{\gamma} \tau_{\beta'}^{\gamma} (1 + \tau^5) / \tau^5 \\ \tau_{\beta'}^{\gamma} \end{pmatrix}. \]

The de Sitter gauge field is associated with a $so(4) (5)$–valued connection 1–form

\[ \tilde{\Omega} = \begin{pmatrix} \omega_{\alpha'}^{\beta'} & \tilde{\theta}_{\alpha'}^{\beta'} \\ \tilde{\theta}_{\beta'}^{\gamma} & 0 \end{pmatrix}, \tag{70} \]

where $\omega_{\alpha'}^{\beta'} \in so(4) (\eta)$, $\tilde{\theta}_{\alpha'}^{\beta'} \in \mathbb{R}^4$, $\tilde{\theta}_{\beta'}^{\gamma} \in \eta_{\beta'}^{\alpha'} \tilde{\theta}_{\alpha'}^{\gamma}$.

The actions of $S(\eta)$ mix the components of the matrix $\omega_{\alpha'}^{\beta'}$ and $\tilde{\theta}_{\alpha'}^{\beta'}$ fields in (70). Because the introduced parametrization is invariant on action on $SO(\eta)$ (4) group, we cannot identify $\omega_{\alpha'}^{\beta'}$ and $\tilde{\theta}_{\alpha'}^{\beta'}$, respectively, with the connection $\Gamma[c]$ and the 1–form $e^c$ defined by a N–connection structure like in (2) with the coefficients chosen as in (64) and (65). To avoid this difficulty we can consider nonlinear gauge realizations of the de Sitter group $S(\eta)$ by introducing the nonlinear gauge field

\[ \Gamma = b^{-1} \tilde{\Omega} b + b^{-1} db = \begin{pmatrix} \Gamma_{\alpha'}^{\beta'} & \theta_{\alpha'}^{\beta'} \\ \theta_{\beta'}^{\gamma} & 0 \end{pmatrix}, \tag{71} \]

where

\[ \Gamma_{\alpha'}^{\beta'} = \omega_{\alpha'}^{\beta'} - \left( \tau_{\alpha'} D\tau_{\beta'} - \tau_{\beta'} D\tau_{\alpha'} \right) / (1 + \tau^5), \]

\[ \theta_{\alpha'}^{\beta'} = \tau^5 \tilde{\theta}_{\beta'}^{\gamma} + D\tau_{\alpha'} - \tau_{\alpha'} \left( d\tau^5 + \tilde{\theta}_{\gamma'} \tau_{\gamma'} \right) / (1 + \tau^5), \]

\[ D\tau_{\alpha'} = d\tau_{\alpha'} + \omega_{\alpha'}^{\beta'} \tau_{\beta'}. \]

The action of the group $S(\eta)$ is nonlinear, yielding the transformation rules

\[ \Gamma' = L' \Gamma (L')^{-1} + L' d (L')^{-1}, \theta' = L \theta, \]

where the nonlinear matrix–valued function

\[ L' = L' (\tau^\alpha, b, B_T) \]

is defined from $B_b = b' B_{L'}$ (see the parametrization (69)). The de Sitter 'nonlinear' algebra is defined by generators (68) and nonlinear gauge transforms of type (71).
We generalize the constructions from Refs. 23, 6 to the case when the de Sitter nonlinear gauge gravitational connection (71) is defined by the viebeins (64) and (65) and the linear connection
\[ \Gamma^\gamma_{\alpha \beta \mu} = \{ \Gamma^\gamma_{\alpha \beta} \} \]
(72)
where
\[ \Gamma^\gamma_{\alpha \beta} = \Gamma^\gamma_{\alpha \beta \mu} e^\mu_{\alpha \beta} \]
(73)
for
\[ \Gamma^\gamma_{\alpha \beta \mu} = e^\alpha_{\alpha} e^\beta_{\beta} \Gamma^\gamma_{\alpha \beta \mu} + e^\alpha_{\alpha} \delta^\gamma_{\beta \mu} e^\alpha_{\beta}, \]
(74)
and \( l_0 \) being a dimensional constant.

The matrix components of the curvature of the connection (72),
\[ R(\Gamma) = d\Gamma + \Gamma \wedge \Gamma, \]
can be written
\[ R(\Gamma) = \begin{pmatrix} \mathcal{R}^\alpha_{\gamma} \beta_{\mu} + l_0^{-1} \pi^\alpha_{\gamma} \beta_{\mu} & l_0^{-1} T^\alpha_{\gamma} \beta_{\mu} \\ l_0^{-1} T^\gamma_{\alpha \beta} & 0 \end{pmatrix}, \]
(75)
where
\[ \pi^\alpha_{\gamma} = e^\alpha_{\gamma} \wedge e^\gamma_{\gamma}, \quad \mathcal{R}^\gamma_{\alpha \beta} = \frac{1}{2} \mathcal{R}^\gamma_{\alpha \beta \mu} \delta u^\mu \wedge \delta u^\nu, \]
and
\[ \mathcal{R}^\gamma_{\alpha \beta \mu} = e^\gamma_{\beta} e^\alpha_{\alpha} \mathcal{R}^\alpha_{\beta \mu \nu}, \]
with the coefficients \( R^\gamma_{\alpha \beta \mu \nu} \) defined with h–v–invariant components, see (91) in Appendix A.

The de Sitter gauge group is semisimple: we are able to construct a variational gauge gravitational theory with the Lagrangian
\[ L = L_{(g)} + L_{(m)} \]
(76)
where the gauge gravitational Lagrangian is defined
\[ L_{(g)} = \frac{1}{4\pi} Tr \left( \mathcal{R}^{(\Gamma)} \wedge *_G \mathcal{R}^{(\Gamma)} \right) = \mathcal{L}_{(G)} |g|^{1/2} \delta^4 u, \]
for
\[ \mathcal{L}_{(g)} = \frac{1}{2 l_0^2} T^\alpha_{\mu \nu} T^\mu_{\alpha \nu} + \frac{1}{8 \lambda} \mathcal{R}^\alpha_{\beta \mu \nu} R^\beta_{\alpha \mu \nu} - \frac{1}{l_0^2} \left( \frac{1}{2} R(\Gamma) - 2 \lambda_1 \right), \]
with \( \delta^4 u \) being the volume element, \( |g| \) is the determinant computed the metric coefficients (1) stated with respect to N–elongated frames, the curvature scalar \( R(\Gamma) \) is computed as in (72), \( T^\alpha_{\mu \nu} = e^\alpha_{\alpha} T^\alpha_{\mu \nu} \) (the gravitational constant \( l_0^2 \) satisfies the relations \( l_0^2 = 2 l_0^2, \lambda_1 = -3/l_0 \)), \( Tr \) denotes the trace on \( \alpha', \beta' \) indices. The matter field Lagrangian from (76) is defined
\[ L_{(m)} = -\frac{1}{2} Tr(\Gamma \wedge *_g \mathcal{I}) = \mathcal{L}_{(m)} |g|^{1/2} \delta^4 u, \]
(28)
with the Hodge operator derived by $|g|$ and $|h|$ where

$$
L_m = \frac{1}{2} \Gamma^{\alpha'}_{\beta' \mu} S^{\beta'}_{\alpha} \mu - t^\mu_{\alpha'} l^{\alpha'}_{\mu}.
$$

The matter field source $J$ is obtained as a variational derivation of $L_m$ on $\Gamma$ and is parametrized in the form

$$
J = \left( \begin{array}{c}
S^{\alpha'}_{\beta'} \frac{2}{l_0 t^{\alpha'}} \\
l_0 T_{\beta'} \\
0
\end{array} \right)
$$

with $\tau^{\alpha'} = t^{\alpha'}_{\mu} \delta u^\mu$ and $S^{\alpha'}_{\beta'} = S^{\alpha'}_{\beta' \mu} \delta u^\mu$ being respectively the canonical tensors of energy–momentum and spin density.

Varying the action $S = \int \delta^4 u (L_g + L_m)$ on the $\Gamma$–variables (72), we obtain the gauge–gravitational field equations:

$$
d \left( \ast R(\Gamma) \right) + \Gamma \wedge \left( \ast R(\Gamma) \right) - \left( \ast R(\Gamma) \right) \wedge \Gamma = -\lambda \left( \ast J \right),
$$

were the Hodge operator $\ast$ is used. This equations can be alternatively derived in geometric form by applying the absolute derivation and dual operators.

Distinguishing the variations on $\Gamma$ and $e$–variables, we rewrite (77)

$$
\widehat{D} \left( \ast R(\Gamma) \right) + 2 \lambda \frac{l^2}{t^2} \left( \widehat{D} \left( \ast \pi \right) + e \wedge \left( \ast T^T \right) - \left( \ast T \right) \wedge e^T \right) = -\lambda \left( \ast S \right),
$$

$$
\widehat{D} \left( \ast T \right) - \left( \ast R(\Gamma) \right) \wedge e - 2 \lambda \frac{l^2}{t^2} \left( \ast \pi \right) \wedge e = \frac{l^2}{2} \left( *t + \frac{1}{\lambda} * \varsigma \right),
$$

$e^T$ being the transposition of $e$, where

$$
T^i = \left\{ T_i^{\alpha'} = \eta_{\alpha' \beta'} T^{\beta'}, T^{\beta'} = \frac{1}{2} T^{\beta'}_{\mu \nu} \delta u^\mu \wedge \delta u^\nu \right\},
$$

$$
e^T = \left\{ e_i^{\alpha'} = \eta_{\alpha' \beta'} e^{\beta'}, e^{\beta'} = e^{\beta'}_{\mu} \delta u^\mu \right\},
$$

$\widehat{D} = \delta + \widehat{\Gamma}$.

($\widehat{\Gamma}$ acts as $\Gamma^{\alpha'}_{\beta' \mu}$ on indices $\gamma', \delta', \ldots$ and as $\Gamma^{\alpha}_{\beta \mu}$ on indices $\gamma, \delta, \ldots$). The value $\varsigma$ defines the energy–momentum tensor of the gauge gravitational field $\widehat{\Gamma}$:

$$
\varsigma_{\mu \nu} (\widehat{\Gamma}) = \frac{1}{2} T_{T^i} \left( \mathcal{R}_{\mu \alpha} \mathcal{R}^\alpha_{\nu} - \frac{1}{4} \mathcal{R}_{\alpha \beta} \mathcal{R}^{\alpha \beta} G_{\mu \nu} \right).
$$

Equations (77) make up the complete system of variational field equations for nonlinear de Sitter gauge gravity. We note that we can obtain a nonvariational Poincare gauge gravitational theory if we consider the contraction of the gauge potential (72) to a potential $\Gamma^{[P]}$ with values in the Poincare Lie algebra

$$
\Gamma = \left( \begin{array}{c c}
\Gamma^{\alpha'}_{\beta'} & l_0^{-1} e^{\alpha'} \\
l_0^{-1} e_{\beta'} & 0
\end{array} \right) \rightarrow \Gamma^{[P]} = \left( \begin{array}{c c}
\Gamma^{\alpha'}_{\beta'} & l_0^{-1} e^{\alpha'} \\
0 & 0
\end{array} \right).
$$

A similar gauge potential was considered in the formalism of linear and affine frame bundles on curved spacetimes by Popov and Dikhin [21,22]. They considered the gauge potential (78) to
be just the Cartan connection form in the affine gauge like gravity and proved that the Yang–Mills equations of their theory are equivalent, after projection on the base, to the Einstein equations.

Let us give an example how an exact vacuum solution of the Einstein equations, with associated noncommutative symmetry, can be included as to define an exact solution in gauge gravity. Using the data (33) defining a 4D black ellipsoid solution, we write the nontrivial vielbein coefficients (64) as

$$e_2' = 1, e_3' = \sqrt{|g_3|}, e_4' = \sqrt{|h_4|}, e_5' = \sqrt{|h_5|}, N_5^2 = n_2$$

for the diagonal Minkowski metric $\eta_{\alpha'\beta'} = (-1, -1, -1, 1)$ with the tetrad and coordinate indices running respectively the values $\alpha', \beta', \ldots = 2, 3, 4, 5$ and $\alpha, \beta, \ldots = 2, 3, 4, 5$. The connection coefficients $\Gamma_{\alpha'\beta'\mu}$, see formula (74), are computed by using the values $e_{\alpha'\alpha}$ and (7) and used for definition of the potential $\Gamma^{(P)}$ (78) which defines a gauge gravity model with the Yang–Mills equations (77) being completely equivalent to the Einstein equations even the frames are anholonomic (see details in Refs. 21, 22, 24–28). N–coefficients, for instance, a complex $N_5^2 = n_2$ we can construct both complex Einstein and gauge gravity vacuum configurations which are stable and define anholonomically deformed black hole solutions with associated noncommutative symmetries.

Finally, we emphasize that in a similar manner, by extending the dimensions of spaces and gauge groups and introducing the cosmological constant, we can include the solutions for the (anti) de Sitter black ellipsoids and wormholes, with real or complex anholonomic structures (constructed respectively in sections IV C and V), into a gauge gravity theory (Einstein and Poincaré like, or as a degenerated configuration in the nonlinear (anti) de Sitter gravity).

VII Noncommutative Gauge Deformations of Gravity

The noncommutative gravity theories are confronted with the problem of definition of noncommutative variants of pseudo–Euclidean and pseudo–Riemannian metrics. This is connected with another problem when the generation of noncommutative metric structures via the Moyal product and the Seiberg–Witten map results in complex and noncommutative metrics for, in general, nonstable and/or unphysical gravitational vacua. In order to avoid the mentioned difficulties, we elaborated a model of noncommutative gauge gravity starting from a variant of gauge gravity being equivalent to the Einstein gravity and emphasizing in a such approach the vielbein (frame) and connection structures, but not the metric configuration (see Ref. 6 and 61). The metric for such theories is induced by an anholonomic (in general) frame transform.

For explicit constructions, we follow the method of restricted enveloping algebras and construct gauge gravitational theories by stating corresponding structures with semisimple or nonsemisimple Lie algebras and their extensions. We use power series of generators for the affine and nonlinearly realized de Sitter gauge groups and compute the coefficient functions of all the higher powers of the generators of the gauge group which are functions of the coefficients of the first power. Such constructions are based on the Seiberg–Witten map and on the formalism of *–product formulation of the algebra when for functional objects, being functions of commuting variables, there are associated some algebraic noncommutative
properties encoded in the ∗-product. Here we note that an approach to the gauge theory on noncommutative spaces was introduced geometrically by defining the covariant coordinates without speaking about derivatives. This formalism was also developed for quantum planes.

In this section, we shall prove the existence for noncommutative spaces of gauge models of gravity which agrees with usual gauge gravity theories being equivalent, or extending, the general relativity theory in the limit of commuting spaces. We shall show how it is possible to adapt mutually the Seiberg–Witten map and anholonomic frame transforms in order to generate solutions of the gauge gravity preserving noncommutative symmetries even in the classical limit of commutative Einstein gravity.

A Enveloping algebras for gauge gravity connections

We define the gauge fields on a noncommutative space as elements of an algebra \(A_u\) that form a representation of the generator \(I\)–algebra for the de Sitter gauge group and the noncommutative space is modelled as the associative algebra of \(\mathbb{C}\). This algebra is freely generated by the coordinates modulo ideal \(\mathcal{R}\) generated by the relations (one accepts formal power series)

\[
\mathcal{U} = \mathbb{C}\llbracket \hat{u}^1, \ldots, \hat{u}^N \rrbracket / \mathcal{R}.
\]

A variational gauge gravitational theory can be formulated by using a minimal extension of the affine structural group \(\mathcal{A}_{f+1}(\mathbb{R})\) to the de Sitter gauge group \(S_{10} = SO(4 + 1)\) acting on \(\mathbb{R}^{4+1}\) space.

Let now us consider a noncommutative space (see Appendix D for a brief outline of necessary concepts). The gauge fields are elements of the algebra \(\hat{\psi} \in \mathcal{A}_u\) that form the nonlinear representation of the de Sitter algebra \(so(\eta)(5)\) (the whole algebra is denoted \(\mathcal{A}_z\)). The elements transform

\[
\delta \hat{\psi} = i\hat{\gamma} \hat{\psi}, \quad \hat{\psi} \in \mathcal{A}_u, \quad \hat{\gamma} \in \mathcal{A}_z,
\]

under a nonlinear de Sitter transformation. The action of the generators \((68)\) on \(\hat{\psi}\) is defined as the resulting element will form a nonlinear representation of \(\mathcal{A}_z\) and, in consequence, \(\delta \hat{\psi} \in \mathcal{A}_u\) despite \(\hat{\gamma} \in \mathcal{A}_z\). We emphasize that for any representation the object \(\hat{\gamma}\) takes values in enveloping de Sitter algebra but not in a Lie algebra as would be for commuting spaces.

We introduce a connection \(\hat{\Gamma}^\nu \in \mathcal{A}_z\) in order to define covariant coordinates,

\[
\hat{U}^\nu = \hat{u}^\nu + \hat{\Gamma}^\nu.
\]

The values \(\hat{U}^\nu\) transform covariantly, i.e. \(\delta \hat{U}^\nu \hat{\psi} = i\hat{\gamma} \hat{U}^\nu \hat{\psi}\), if and only if the connection \(\hat{\Gamma}^\nu\) satisfies the transformation law of the enveloping nonlinear realized de Sitter algebra,

\[
\delta \hat{\Gamma}^\nu \hat{\psi} = -i[\hat{\eta}^\nu, \hat{\gamma}] + i[\hat{\gamma}, \hat{\Gamma}^\nu],
\]

where \(\delta \hat{\Gamma}^\nu \in \mathcal{A}_z\).

The enveloping algebra–valued connection has infinitely many component fields. Nevertheless, all component fields can be induced from a Lie algebra–valued connection by a Seiberg–Witten map and, for \(SO(n)\) and \(Sp(n)\), see Ref. 79. Here, we show that similar constructions can be performed for nonlinear realizations of de Sitter algebra when the transformation of the connection is considered

\[
\delta \hat{\Gamma}^\nu = -i[u^\nu, * \hat{\gamma}] + i[\hat{\gamma}, * \hat{\Gamma}^\nu].
\]
We treat in more detail the canonical case with the star product \((133)\). The first term in the variation \(\delta \Gamma^\nu\) gives

\[-i[u^\nu, * \gamma] = \theta^{\nu\mu} \frac{\partial}{\partial u^\mu} \gamma.\]

Assuming that the variation of \(\tilde{\Gamma}^\nu = \theta^{\nu\mu} Q_\mu\) starts with a linear term in \(\theta\), we have

\[\delta \tilde{\Gamma}^\nu = \theta^{\nu\mu} \delta Q_\mu, \quad \delta Q_\mu = \frac{\partial}{\partial u^\mu} \gamma + i[\gamma, * Q_\mu].\]

We expand the star product \((133)\) in \(\theta\) but not in \(g_a\) and find up to first order in \(\theta\) that

\[\gamma = \gamma_a^1 I^a + \gamma_{ab}^1 I^a I^b + ..., \quad Q_\mu = q_{\mu, a}^1 I^a + q_{\mu, ab}^2 I^a I^b + ...\]  

(80)

where \(\gamma_a^1\) and \(q_{\mu, a}^1\) are of order zero in \(\theta\) and \(\gamma_{ab}^1\) and \(q_{\mu, ab}^2\) are of second order in \(\theta\). The expansion in \(I^2\) leads to an expansion in \(g_a\) of the \(*\)–product because the higher order \(I^2\)–derivatives vanish. For de Sitter case, we take the generators \(I^a\) \((83)\), see commutators \((135)\), with the corresponding de Sitter structure constants \(f_{abc}^a \simeq f_{abc}^{d2}\) (in our further identifications with spacetime objects like frames and connections we shall use Greek indices). The result of calculation of variations of \((80)\), by using \(g_a\) to the order given in \((133)\), is

\[\delta q_{\mu, a}^1 = \frac{\partial \gamma_a^1}{\partial u^\mu} - f_{abc}^a \gamma_{b c}^1 q_{\mu, a}^1, \quad \delta Q_\mu = \theta^{\mu\nu} \partial_\nu \gamma_a^1 \partial_\nu q_{\mu, a}^1 I^a I^b + ...,\]

\[\delta q_{\mu, ab}^2 = \partial_\nu \gamma_{ab}^2 - \theta^{\sigma\tau} \partial_\nu \gamma_{a \tau}^1 \partial_\nu q_{\mu, a}^1 - 2 f_{abc}^a \{\gamma_{b c}^1 q_{\mu, c}^1 + \gamma_{bc}^2 q_{\mu, c}^1\}.\]

Let us introduce the objects \(\varepsilon\), taking the values in de Sitter Lie algebra and \(W_\mu\), taking values in the enveloping de Sitter algebra, i. e.

\[\varepsilon = \gamma_a^1 I^a\]  

and \(W_\mu = q_{\mu, ab}^2 I^a I^b\),

with the variation \(\delta W_\mu\) satisfying the equation

\[\delta W_\mu = \partial_\mu (\gamma_{ab}^2 I^a I^b) - \frac{1}{2} \theta^{\tau\lambda} \{\partial_\tau \varepsilon, \partial_\lambda q_\mu\} + i[\varepsilon, W_\mu] + i[(\gamma_{ab}^2 I^a I^b), q_\mu].\]

This equation can be solved \(^{70,45}\) in the form

\[\gamma_{ab}^2 = \frac{1}{2} \theta^{\nu\mu} (\partial_\nu \gamma_a^1) q_{\mu, b}^1, \quad q_{\mu, ab}^2 = - \frac{1}{2} \theta^{\nu\mu} q_{\nu, a}^1 (\partial_\nu q_{\mu, b}^1 + R_{\tau \mu, b}^1).\]

The values

\[R_{\tau \mu, b}^1 = \partial_\tau q_{\mu, b}^1 - \partial_\mu q_{\tau, b}^1 + f_{\beta \gamma}^c q_{\tau, a}^1 q_{\mu, \beta}^c\]

could be identified with the coefficients \(R_{\tau \mu, \beta}^a\) of de Sitter nonlinear gauge gravity curvature (see formula \((129)\)) if in the commutative limit \(q_{\mu, b}^1 \simeq \left(\begin{array}{c}
\Gamma_{\beta \mu, b}^a \\alpha_{\beta}^1 \\beta_{\mu}^1
\end{array}\right)\) (see \((122)\)).

We note that the below presented procedure can be generalized to all the higher powers of \(\theta\). As an example, we compute the first order corrections to the gravitational curvature:
B Noncommutative Covariant Gauge Gravity Dynamics

The constructions from the previous subsection can be summarized by a conclusion that the de Sitter algebra valued object $\varepsilon = \gamma_2^1(u)$ determines all the terms in the enveloping algebra

$$\gamma = \gamma_2^1 I^2 + \frac{1}{4} \theta^{\mu\nu} \partial_\nu \gamma_2^1 q_{\mu,2} (I^2 I^2 + I^2 I^2) + ...$$

and the gauge transformations are defined by $\gamma_2^1(u)$ and $q_{\mu,2}(u)$, when

$$\delta_\gamma \psi = i \gamma (\gamma_1^1, q_{\mu}^1) * \psi.$$ 

Applying the formula (136) we calculate

$$[\gamma, \zeta] = i \gamma_2^1 \zeta_2^1 f_{\alpha}^\beta I^\alpha + \frac{i}{2} \theta^{\mu\nu} \{ \partial_\nu (\gamma_2^1 \zeta_2^1 f_{\alpha}^\beta) q_{\mu,2} + (\gamma_2^1 \partial_\nu \zeta_2^1 - \zeta_2^1 \partial_\nu \gamma_2^1) q_{\mu,2} f_{\alpha}^\beta + 2 \partial_\nu \gamma_2^1 \partial_\mu \zeta_2^1 \} I^2 I^\nu,$$

where we used the properties that, for the de Sitter enveloping algebras, one holds the general formula for compositions of two transformations

$$\delta_\gamma \delta_\zeta - \delta_\zeta \delta_\gamma = \delta_{(\zeta \gamma - \gamma \zeta)}.$$ 

This is also true for the restricted transformations defined by $\gamma^1$,

$$\delta_\gamma \delta_\zeta - \delta_\zeta \delta_\gamma = \delta_{(\zeta^1 \gamma^1 - \gamma^1 \zeta^1)}.$$ 

Such commutators could be used for definition of tensors

$$\hat{S}^{\mu\nu} = [\hat{U}^\mu, \hat{U}^\nu] - i \hat{\theta}^{\mu\nu},$$ 

where $\hat{\theta}^{\mu\nu}$ is respectively stated for the canonical, Lie and quantum plane structures. Under the general enveloping algebra one holds the transform

$$\delta \hat{S}^{\mu\nu} = i[\gamma, \hat{S}^{\mu\nu}].$$ 

For instance, the canonical case is characterized by

$$S^{\mu\nu} = i \theta^{\mu\tau} \partial_\tau \Gamma^{\nu} - i \theta^{\nu\tau} \partial_\tau \Gamma^{\mu} + \Gamma^{\mu} * \Gamma^{\nu} - \Gamma^{\nu} * \Gamma^{\mu} = \theta^{\mu\tau} \theta^{\nu\lambda} \{ \partial_\tau Q_\lambda - \partial_\lambda Q_\tau + \Gamma^{\lambda} * Q_\tau - Q_\tau * \Gamma^{\lambda} \}.$$ 

We introduce the gravitational gauge strength (curvature)

$$R_{\tau\lambda} = \partial_\tau Q_\lambda - \partial_\lambda Q_\tau + \Gamma^{\lambda} * Q_\tau - Q_\tau * \Gamma^{\lambda},$$ 

which could be treated as a noncommutative extension of de Sitter nonlinear gauge gravitational curvature (75), and calculate

$$R_{\tau\lambda, 2} = R_{\tau\lambda, 2}^1 + \theta^{\mu\nu} \{ R_{\tau\mu, 2}^1 R_{\lambda, 2}^1 - \frac{1}{2} q_{\mu, 2} \{ (D_\mu R_{\tau\lambda, 2}^1) + \partial_\nu R_{\tau\lambda, 2}^1 \} I^\nu \}.$$ 

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where the gauge gravitation covariant derivative is introduced,

\[ (D_\nu R^1_{\tau \lambda \mu \nu}) = \partial_\nu R^1_{\tau \lambda \mu \nu} + q_{\nu \epsilon} R^1_{\tau \lambda \mu \nu} f^{\epsilon \mu \nu \lambda / 2}. \]

Following the gauge transformation laws for \( \gamma \) and \( q^1 \), we find

\[ \delta_{\gamma^1} R^1_{\tau \lambda} = i \left[ \gamma^1, R^1_{\tau \lambda} \right] \]

with the restricted form of \( \gamma \). Such formulas were proved in Ref. 45 for usual gauge (nongravitational) fields. Here we reconsidered them for the gravitational gauge fields.

One can be formulated a gauge covariant gravitational dynamics of noncommutative spaces following the nonlinear realization of de Sitter algebra and the \(*\)-formalism and introducing derivatives in such a way that one does not obtain new relations for the coordinates. In this case, a Leibniz rule can be defined that

\[ \hat{\partial}_\mu \hat{u}^\nu = \delta^\nu_\mu + d^\nu_{\mu \sigma} \hat{u}^\sigma \hat{\partial}_\tau \]

where the coefficients \( d^\nu_{\mu \sigma} = \delta^\nu_\sigma \delta^\mu_\tau \) are chosen to have not new relations when \( \hat{\partial}_\mu \) acts again to the right hand side. One holds the \(*\)-derivative formulas

\[ \partial_\tau * f = \frac{\partial}{\partial u^\tau} f + f * \partial_\tau, \quad [\partial_\tau, * (f * g)] = ([\partial_\tau, * f]) * g + f * ([\partial_\tau, * g]) \]

and the Stokes theorem

\[ \int [\partial_\tau, f] = \int d^N u [\partial_\tau, * f] = \int d^N u \frac{\partial}{\partial u^\tau} f = 0, \]

where, for the canonical structure, the integral is defined,

\[ \int \hat{f} = \int d^N u f(u^1, ..., u^N). \]

An action can be introduced by using such integrals. For instance, for a tensor of type \( \Sigma \), when

\[ \delta \hat{L} = i \left[ \hat{\gamma}, \hat{L} \right], \]

we can define a gauge invariant action

\[ W = \int d^N u \, T_{\tau \lambda} \hat{L}, \quad \delta W = 0, \]

were the trace has to be taken for the group generators.

For the nonlinear de Sitter gauge gravity a proper action is

\[ L = \frac{1}{4} R_{\tau \lambda} R^{\tau \lambda}, \]

where \( R_{\tau \lambda} \) is defined by \( \Sigma \). In the commutative limit we shall obtain the connection \( \Sigma \). In this case the dynamic of noncommutative space is entirely formulated in the framework of quantum field theory of gauge fields. In general, we are dealing with anisotropic gauge gravitational interactions. The method works for matter fields as well to restrictions to the general relativity theory.

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The aim of this subsection is to prove that there are possible extensions of exact solutions from the Einstein and gauge gravity possessing hidden noncommutative symmetries without introducing new fields. For simplicity, we present the formulas including decompositions up to the second order on noncommutative parameter $\theta^{\alpha\beta}$ for vielbeins, connections and curvatures which can be arranged to result in different models of noncommutative gravity. We give the data for the $SU(1,n+m-1)$ and $SO(1,n+m-1)$ gauge models containing, in general, complex N–elongated frames, modelling some exact solutions, for instance, those derived in the sections IV and V. All data can be considered for extensions with nonlinear realizations into a bundle of affine/or de Sitter frames (in this case, one generates noncommutative gauge theories of type considered in Ref. 6 or to impose certain constraints and breaking of symmetries (in order to construct other models$^{7,60}$).

In previous sections we considered noncommutative geometric structures introduced by frame anholonomic relations $^3$, or $^{13}$. The standard approaches to noncommutative geometry also contain certain noncommutative relations for coordinates,

$$[u^\alpha, u^\beta] = u^\alpha u^\beta - u^\beta u^\alpha = i\theta^{\alpha\beta}(u^\gamma)$$

(83)

were, in the simplest models, the commutator $[u^\alpha, u^\beta]$ is approximated to be constant, but there were elaborated approaches for general manifolds with the noncommutative parameter $\theta^{\alpha\beta}$ treated as functions on $u^\gamma$ in Ref. 68. We define the star (Moyal) product to include possible N–elongated partial derivatives $^3$ and a quantum constant $\hbar$,

$$f \ast \varphi = f \varphi + \frac{\hbar}{2}B^{\alpha\beta} \left( \delta_\alpha f \delta_\beta \varphi + \delta_\beta f \delta_\alpha \varphi \right) + \hbar^2 B^{\alpha\beta} B^{\mu\nu} \left[ \delta_{(\alpha} \delta_{\beta)} f \right] \left[ \delta_{(\mu} \delta_{\nu)} \varphi \right]$$

$$+ \frac{2}{3} \hbar^2 B^{\alpha\beta} \delta_\beta B^{\mu\nu} \left\{ \left[ \delta_{(\mu} \delta_{\nu)} f \right] \delta_\beta \varphi + \left[ \delta_{(\mu} \delta_{\nu)} \varphi \right] \delta_\beta f \right\} + O \left( \hbar^3 \right),$$

(84)

where, for instance, $\delta_{(\mu} \delta_{\nu)} = (1/2)(\delta_\mu \delta_\nu + \delta_\nu \delta_\mu)$

$$B^{\alpha\beta} = \frac{\theta^{\alpha\beta}}{2} \left( \delta_\alpha u^\alpha \delta_\beta u^\beta + \delta_\beta u^\beta \delta_\alpha u^\alpha \right) + O \left( \hbar^3 \right)$$

(85)

is defined for new coordinates $u^\alpha = u^\alpha \left( u^\alpha \right)$ inducing a suitable Poisson bi–vector field $B^{\alpha\beta} \left( \hbar \right)$ being related to a quantum diagram formalism (we shall not consider details concerning geometric quantization in this paper by investigating only classical deformations related to any anholonomic frame and coordinate $^{13}$ noncommutativity origin). The formulas $^3$ and $^{8}$ transform into the usual ones with partial derivatives $\partial_\alpha$ and $\partial_\tau$ from Refs. 68,7 considered for vanishing anholonomy coefficients. We can define a star product being invariant under diffeomorphism transforms, $\ast \rightarrow \ast [-1]$, adapted to the N–connection structure (in a vector bundle provided with N–connection configuration, we use the label $[-]$ in order to emphasize the dependence on coordinates $u^\alpha$ with ’overlined’ indices), by introducing the transforms

$$f([-1]) \left( \hbar \right) = \Theta f \left( \hbar \right),$$

$$f([-1]) \ast [-1] \varphi([-1]) = \Theta \left( \Theta^{-1} f([-1]) \ast \Theta^{-1} \right) \varphi([-1])$$

for $\Theta = 1 + \sum_{k=1} \hbar^k \Theta_{[k]}$, for simplicity, computed up to the squared order on $\hbar$,

$$\Theta = 1 - 2\hbar^2 \theta^{\mu\nu} \theta^{\rho\sigma} \left\{ \left[ \delta_{(\mu} \delta_{\nu)} u^\alpha \right] \left[ \delta_{(\rho} \delta_{\sigma)} u^\beta \right] \delta_{(\mu} \delta_{\nu)} + \left[ \delta_{(\mu} \delta_{\nu)} u^\alpha \right] \left( \delta_{\nu} u^\beta \right) \left( \delta_{\sigma} u^\tau \right) \delta_{(\mu} \delta_{\nu)} \delta_{(\rho)} \delta_{(\sigma)}, \delta_{(\tau)} \right\} + O \left( \hbar^4 \right),$$

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where $\delta_{\alpha(\beta(\gamma(\gamma) = (1/3!)(\delta_{\alpha(\beta(\gamma + all\ symmetric\ permutations)\).\ In\ our\ further\ constructions\ we\ shall\ omit\ the\ constant\ h\ considering\ that\ \theta \sim h\ is\ a\ small\ value\ by\ writing\ the\ necessary\ terms\ in\ the\ approximation\ O(\theta^3)\ or\ O(\theta^4)\).

We consider a noncommutative gauge theory on a space with N–connection structure stated by the gauge fields $\hat{A}_\mu = (\hat{A}_\iota, \hat{A}_a)$ when ”hats” on symbols will be used for the objects defined on spaces with coordinate noncommutativity. In general, the gauge model can be with different types of structure groups like $SL(k, \mathbb{C}), SU_k, U_k, SO(k-1, 1)$ and their nonlinear realizations. For instance, for the $U(n + m)$ gauge fields there are satisfied the conditions $\hat{A}_\mu^+ = -\hat{A}_\mu$, where ”+” is the Hermitian conjugation. It is useful to present the basic geometric constructions for a unitary structural group containing the $SO(4, 1)$ as a particular case if we wont to consider noncommutative extensions of 4D exact solutions.

The noncommutative gauge transforms of potentials are defined by using the star product

$$\hat{A}_\mu^0 \hat{A}_\nu^0 = \hat{\varphi} \star \hat{A}_\mu \hat{\varphi}_\nu^{-1} - \hat{\varphi} \hat{\varphi}_\mu \hat{\varphi}_\nu^{-1}$$

where the N–elongated partial derivatives (3) are used coefficients of fields will be distinguished by ”overlined” indices, for instance, $\hat{A}_\mu = \{\hat{A}_\mu^0\}$, and for commutative values, $A_\mu = \{A_\mu^0\}$. Such fields are subjected to the conditions

$$(\hat{A}_\mu^\alpha) + (u, \theta) = -\hat{A}_\mu^\alpha (u, \theta)$$

and $\hat{A}_\mu^\alpha (u, -\theta) = -\hat{A}_\mu^\alpha (u, \theta)$.

There is a basic assumption\(^{15} \) that the noncommutative fields are related to the commutative fields by the Seiberg–Witten map in a manner that there are not new degrees of freedom being satisfied the equation

$$\hat{A}_\mu^\alpha (A) + \Delta \hat{A}_\mu^\alpha (A) = \hat{A}_\mu^\alpha (A + \Delta A)$$

(86)

where $\hat{A}_\mu^\alpha (A)$ denotes a functional dependence on commutative field $A_\mu^\alpha, \hat{\varphi} = \exp \hat{\lambda}$ and the infinitesimal deformations $\hat{A}_\mu^\alpha (A)$ and of $A_\mu^\alpha$ are given respectively by

$$\Delta \hat{A}_\mu^\alpha = \delta_\mu \hat{\lambda}^\alpha + A_\mu^\alpha \hat{\lambda}^\alpha - \hat{\lambda}^\alpha \hat{A}_\mu^\alpha$$

and

$$\Delta A_\mu^\alpha = \delta_\mu \lambda^\alpha + A_\mu^\alpha \lambda^\alpha - \lambda^\alpha A_\mu^\alpha$$

where instead of partial derivatives $\delta_\mu$ we use the N–elongated ones, $\delta_\mu$, and sum on index $\gamma$.

Solutions of the Seiberg-Witten equations for models of gauge gravity are considered, for instance, in Refs. 6,7 (there are discussed procedures of deriving expressions on $\theta$ to all orders). Here we present only the first order on $\theta$ for the coefficients $\hat{\lambda}^\alpha$ and the first and second orders for $\hat{A}_\mu^\alpha$ including anholonomy relations and not depending on model considerations,

$$\hat{\lambda}^\alpha = \lambda^\alpha + i \frac{1}{4} \theta^{\alpha_\beta} ((\delta_\nu \lambda^\alpha) A_\mu^\beta + A_\mu^\beta (\delta_\nu \lambda^\alpha)) + O(\theta^2)$$

and

$$\hat{A}_\mu^\alpha = A_\mu^\alpha - i \frac{1}{4} \theta^{\alpha_\beta} \left( A_\mu^\gamma \left( \delta_\tau A_\nu^\beta + R_\nu^\alpha \right) + (\delta_\tau A_\mu^\alpha + R_\nu^\alpha \tau_\mu \right) A_\nu^\beta \right) + \left[ \frac{1}{32} \theta^{\alpha_\beta} \left( [2 A_\mu^\alpha (R_\sigma^\alpha \sigma_\nu + R_\sigma^\alpha \sigma_\nu - 2 (R_\nu^\alpha \sigma_\nu + R_\nu^\alpha \sigma_\nu) A_\sigma^\beta] - [A_\mu^\gamma \left( D_\mu R_\rho^\beta \sigma_\mu + \delta_\mu R_\rho^\beta \sigma_\mu \right) + (D_\nu R_\rho^\beta \sigma_\mu + \delta_\nu R_\rho^\beta \sigma_\mu) A_\rho^\beta] - \delta_\sigma A_\nu^\beta \left( \delta_\tau A_\rho^\beta + R_\rho^\beta \tau_\mu \right) + (\delta_\tau A_\rho^\alpha + R_\rho^\alpha \tau_\mu \right) A_\rho^\beta \right) +$$

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where the curvature is defined \( R^{\alpha\beta}_{\tau\nu} = e^\alpha_{\alpha} e^\beta_{\beta} R^{\alpha}_{\tau\nu} \) with \( R^{\alpha}_{\tau\nu} \) computed as in Appendix A, see formula (89), when \( \Gamma \to A \), and for the gauge model of gravity, see (75) and (82). By using the star product, we can write symbolically the solution (87) in general form,

\[
\Delta \tilde{A}_{\mu}^{\alpha\beta}(\theta) = -i \frac{\theta^\nu}{4} \left[ A_{\mu}^{\alpha\gamma} * \left( \delta_{\tau} \tilde{A}_{\nu}^{\alpha\beta} + \hat{R}^{\alpha\beta}_{\tau\nu} \right) + \left( \delta_{\tau} \tilde{A}_{\mu}^{\alpha\gamma} + \hat{R}^{\alpha\gamma}_{\tau\mu} \right) * \tilde{A}_{\nu}^{\beta\gamma} \right]
\]

where \( \hat{R}^{\alpha\beta}_{\tau\nu} \) is defined by the same formulas as \( R^{\alpha\beta}_{\tau\nu} \) but with the star products, like \( AA \to A * A \).

There is a problem how to determine the dependence of the noncommutative vielbeins \( \hat{e}^\alpha_{\alpha} \) on commutative ones \( e^\alpha_{\alpha} \). If we consider the frame fields to be included into a (anti) de Sitter gauge gravity model with the connection (72), the vielbein components should be treated as certain coefficients of the gauge potential with specific nonlinear transforms for which the results of Ref. 6. The main difference (considered in this work) is that the frames are in general with anholonomy induced by a N–connection field. In order to derive in a such model of the Einstein gravity we have to analyze the reduction (78) to a Poincare gauge gravity. An explicit calculus of the curvature of such gauge potential (see details in Refs. 21,22,24-28), show that the coefficients of the curvature of the connection (78), obtained as a reduction from the \( SO(4,1) \) gauge group is given by the coefficients (75) with vanishing torsion and constraints of type \( \hat{A}^{\alpha\beta}_{\mu} = \hat{e}^\gamma_{\nu} \) and \( \hat{A}^{\alpha\beta}_{\mu} = \hat{e}^\gamma_{\nu} \) with \( \hat{R}^{\alpha\beta}_{\tau\nu} \sim \epsilon \) vanishing in the limit \( \epsilon \to 0 \) like in Ref. 7 (we obtain the same formulas for the vielbein and curvature components derived for the inhomogenious Lorentz group but generalized to N–elongated derivatives and with distinguishing into h–v–components). The result for \( \hat{e}^\gamma_{\nu} \) in the limit \( \epsilon \to 0 \) generalized to the case of canonical connections (17) defining the covariant derivatives \( D_\tau \) and corresponding curvatures (89) is

\[
\hat{e}^\gamma_{\nu} = e^\gamma_{\nu} - \frac{i}{4} \theta_\mu^\tau \left[ A^\mu^\nu \delta_\tau e^\gamma_{\nu} + (\delta_\tau A^\mu^\nu + R^\mu^\nu_{\tau\nu}) e^\gamma_{\nu} \right] +
\]

\[
\frac{1}{32} \theta_\mu^\tau \theta_\sigma^\sigma \left[ 2(R^{\mu^\nu}_{\sigma\nu} R^{\alpha\beta}_{\mu\tau} + R^{\mu^\nu}_{\mu\tau} R^{\alpha\beta}_{\sigma\nu}) e^\gamma_{\nu} - A^\mu^\nu (D_\tau R^{\alpha\beta}_{\sigma\mu} + \delta_\tau R^{\alpha\beta}_{\sigma\mu}) e^\beta_{\gamma} -
\]

\[
[A^\mu^\nu \left( D_\tau R^{\alpha\beta}_{\sigma\mu} + \delta_\tau R^{\alpha\beta}_{\sigma\mu} \right) + (D_\tau R^{\alpha\beta}_{\sigma\mu} + \delta_\tau R^{\alpha\beta}_{\sigma\mu}) A^\gamma_{\beta}] e^\beta_{\gamma} -
\]

\[
\hat{e}^\beta_{\gamma} \delta_\beta \left[ A^\mu^\nu \left( \delta_\tau \hat{A}^\gamma_{\beta} + R^{\gamma\beta}_{\tau\mu} \right) + (\delta_\tau A^\mu^\nu + \hat{R}^{\gamma\beta}_{\tau\nu}) A^\gamma_{\beta} \right] + 2 \left( \delta_\nu A^\mu^\nu \right) \delta_\tau (\delta_\sigma \hat{e}^\gamma_{\nu} -
\]

\[
A^\mu^\nu \delta_\sigma \left[ \hat{A}^\gamma_{\beta} \hat{e}^\beta_{\gamma} + (\delta_\tau A^\mu^\nu + \hat{R}^{\gamma\beta}_{\tau\mu}) e^\gamma_{\nu} \right] - (\delta_\nu e^\gamma_{\nu}) \delta_\tau (\delta_\sigma A^\mu^\nu + \hat{R}^{\gamma\beta}_{\tau\nu}) -
\]

\[
\left[ A^\mu^\nu \left( \delta_\tau \hat{A}^\gamma_{\beta} + R^{\gamma\beta}_{\tau\mu} \right) + (\delta_\tau A^\mu^\nu + \hat{R}^{\gamma\beta}_{\tau\nu}) A^\gamma_{\beta} \right] \delta_\sigma e^\beta_{\gamma} -
\]

\[
(\delta_\sigma A^\mu^\nu + \hat{R}^{\gamma\beta}_{\tau\nu}) \left[ \hat{A}^\gamma_{\beta} \left( \delta_\nu e^\beta_{\gamma} \right) + e^\beta_{\gamma} \left( \delta_\sigma A^\mu^\nu + \hat{R}^{\gamma\beta}_{\tau\sigma} \right) \right] \right] + O (\theta^3).
\]

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Having the decompositions \( \mathcal{S} \), we can define the inverse vielbein \( \hat{e}^\mu_\nu \) from the equation
\[
\hat{e}^\mu_\nu \hat{e}^\nu_\mu = \delta^\mu_\nu
\]
and consequently compute \( \theta \)-deformations of connections, curvatures, torsions and any type of actions and field equations (for simplicity, we omit such cumbersome formulas being certain analogies to those computed in Ref. 7 but with additional \( N \)-deformations).

The main result of this section consists in formulation of a procedure allowing to map exact solutions of a 'commutative' gravity model into a corresponding 'noncommutative' model without introducing new fields. For instance, we can take the data \( \mathcal{S} \) and \( \mathcal{J} \) and construct the \( \theta \)-deformation of the exact solution defining a static black ellipsoid (a similar prescription works in transforming both real and complex wormholes from Section V). The analysis presented in subsections IV B and IV C illustrates a possibility to preserve the black ellipsoid stability for a certain class of extensions of solutions to noncommutative/complex gravity with complexified \( N \)-frames. In other turn, if we consider arbitrary noncommutative relations for coordinates (\( \theta \)-noncommutativity) \( \mathcal{S} \) the resulting \( \theta \)-deformation of stable solution will be, in general, unstable because arbitrary decompositions of type \( \mathcal{S} \) will induce arbitrary complex terms in the metric, connection and curvature coefficients, i.e. will result in complex terms in the "inverse Schrodinger problem" and related instability (see Refs. 18,19). Perhaps, a certain class of stable \( \theta \)-deformed solutions can be defined if we constrain the \( \theta \)-noncommutativity \( \mathcal{S} \) to be dual to the so-called anholonomic frame noncommutativity \( \mathcal{S} \), or \( \mathcal{J} \), by connecting the nontrivial values of \( \theta^{\alpha \beta} \) to certain complex \( N^a_i \) resulting in stable noncommutative configurations, or (in the simplest case) to say that the noncommutative extensions are modelled only by \( N \)-fields \( \sim \hbar \theta \). The resulting noncommutative extensions could be defined as to preserve stability at least in the first order of \( \hbar \theta \)-terms.

## VIII Discussion and Conclusions

With this paper we begin the investigation of spacetimes with anholonomic noncommutative symmetry. The exact solutions we find are parametrized by generic off–diagonal metric ansatz and anholonomic frame (vielbein) structures with associated nonlinear connections (defining nontrivial anholonomy relations and inducing natural matrix noncommutative differential geometries). Their noncommutative symmetries are derived from exact solutions of the field equations in the Einstein gravity theory and their extra dimension and gauge like generalizations.

We analyzed the geometric and physical properties of new classes of exact solutions with 'hidden' noncommutativity describing specific non–perturbative vacuum and non–vacuum gravitational configurations. Such spacetimes with generic off–diagonal metrics are very different, for instance, from those possessing Killing symmetries. We also addressed to a particular class of solutions with noncommutative symmetries defining static black ellipsoid spacetimes which are not prohibited by the uniqueness black hole theorems (proved for metrics with Killing symmetries and satisfying asymptotic flat conditions) because the generic anholonomic noncommutative configurations are very different from the Killing ones.

Let us comment on the difference between our approach and the former elaborated ones: In the so–called Connes–Lot models\(^1\)-\(^2\), the gravitational models with Euclidean signature were
elaborated from a spectral analysis of Dirac operators connected to the noncommutative geometry. This type of noncommutative geometry was constructed by replacing the algebra of smooth function on a manifold with a more general associative but noncommutative algebra. The fundamental matter field interactions and Riemannian gravity were effectively derived from a corresponding spectral calculus. In an alternative approach, the noncommutative geometry, as a low energy noncommutativity of coordinates, can be obtained in string theory because of presence of the so-called $B$–fields. A number of models of gravity were proposed in order to satisfy certain noncommutativity relations for coordinates and frames (of Lie group, or quantum group type, or by computing some effective actions from string/brane theory and noncommutative gauge generalizations of gauge, Kaluza–Klein and Einstein gravity), see Refs. 3-5,8. All mentioned noncommutative theories suppose that that the noncommutative geometry transforms into a commutative one in some limits to the Einstein theory and its extra dimension generalizations. In our approach we argue that the existence of hidden noncommutative structures suggests a natural way for constructing noncommutative models of gravitational interactions.

Our strategy explained in section III is quite different from the previous attempts to construct the noncommutative gravity theory. We give a proof and analyze some explicit examples illustrating that that there are some specific hidden noncommutative geometric structures even in the classical Einstein and gauge gravity models. This fact can be of fundamental importance in constructing more general models of noncommutative gravity with complex and nonsymmetric metrics.

Of course, there are two different notions of noncommutativity: The first one is related to the spacetime deformations via Seiberg–Witten transforms with noncommutative coordinates and the second one is associated to noncommutative algebra modelled by anholonomic vielbeins. In general, the result of such deformations and frame maps can not be distinguished exactly on a resulting complex spacetime because there is a ”mixture” of coordinate, gauge and frame transforms in the case of noncommutative geometry. Nevertheless, there are certain type of gravitational configurations possessing Lie type (noncommutative) symmetries which ”survive” in the limit of commutative coordinates and real valued metrics. We say that a such type of solutions of the gravitational field equations posses hidden noncommutative symmetries and describe a generic off–diagonal class of metrics and anholonomic frame transforms. It is a very difficult task to get exact solutions of the deformed gravity. In this work, we succeeded to do this by generating such gravitational configurations which are adapted both to the Seiberg–Witten type deformations and to the vielbein transforms. In the section VII C we proved that there are possible extensions (on deformation parameters) of exact solutions from the Einstein and gauge gravity possessing hidden noncommutative symmetries without introducing new fields.

In this work, from a number of results following from application of Seiberg–Witten maps and related anholonomic vielbein transforms, we selected only those which allow us to define classes of solutions as noncommutative generalizations of some commutative ones of special physical interest. Such metrics with hidden noncommutative symmetry are described by a general off–diagonal ansatz for the metric and vielbein coefficients. The solutions can be extended to complex metrics by allowing that some subsets of vielbeins coefficients (with associated nonlinear connection structure) may be complex valued. With respect to adapted frames, such metrics have real coefficients describing vacuum black ellipsoid or wormhole configurations (there were elaborated procedures of their analytical extensions and proofs of
stability). The new types of metrics may be considered as certain exact solutions in complex gravity which have to be considered if some noncommutative relations for coordinates are introduced. Such configurations may play an important role in the further understanding of vacua of noncommutative gauge and gravity theories and investigation of their quantum variants.

The anholonomic noncommutative symmetry of exact solutions of four dimensional (4D) vacuum Einstein equations positively does not violate the local (real) Lorentz symmetry. This symmetry may be preserved in a specific form even by anholonomic complex vielbein transforms mapping the 4D real Einstein’s metrics into certain complex ones for noncommutative gravity. Such frames may be defined as the generated new solutions will be a formal analogy with their real (diagonal) ‘pedigrees’, to be stable with well defined geodesic and horizon properties, like it was concluded for black ellipsoids solutions in general relativity. By complex frame transforms with noncommutative symmetries we demonstrated that we may deform the horizon of the Schwarzschild solution to a static ellipsoid configurations as well to induce an effective electric charge of ‘complex noncommutative’ origin.

We compare the generated off–diagonal ellipsoidal (in general, complex) metrics possessing anholonomic noncommutative symmetries with those describing the distorted diagonal black hole solutions (see the vacuum case in Refs. 80,81 and an extension to the case of non–vanishing electric fields in Ref. 82). In the complex ellipsoidal cases the spacetime distortion is caused by some anisotropic off–diagonal terms being non–trivial in some regions but in the case of ”pure diagonal” distortions one treats such effects following the fact that the vacuum Einstein equations are not satisfied in some regions because of presence of matter. Alternatively, the complex ellipsoid solutions may be described as in a ‘real’ world with real metric coefficients but defined with respect to complex frames.

Here we emphasize that the off–diagonal gravity may model some gravity–matter like interactions (for instance, in the Kaluza–Klein theory by emphasizing some very particular metric’s configurations and topological compactifications) but, in general, the off–diagonal vacuum gravitational dynamics can not be associated to any effective matter dynamics in a holonomic gravitational background. So, we may consider that the anholonomic ellipsoidal deformations of the Schwarzschild metric defined by real and/or complex anholonomic frame transforms generate some kind of anisotropic off–diagonal distortions modelled by certain vacuum gravitational fields with the distortion parameters (equivalently, vacuum gravitational polarizations) depending both on radial, angular and extra dimension coordinates. For complex valued nonlinear connection coefficients, we obtain a very specific complex spacetime distortion instead of matter type distortion. We note that both classes of off–diagonal anisotropic and ”pure” diagonal distortions (like in Refs. 80,81) result in solutions which in general are not asymptotically flat. However, it is possible to find asymptotically flat extensions, as it was shown in this paper and in Refs. 18,19, even for ellipsoidal configurations by introducing the corresponding off–diagonal terms. The asymptotic conditions for the diagonal distortions are discussed in Ref. 82 where it was suggested that to satisfy such conditions one has to include some additional matter fields in the exterior portion of spacetime. For ellipsoidal real/complex metrics, we should consider that the off–diagonal metric terms have a corresponding behavior as to result fare away from the horizon in the Minkowski metric.

The deformation parameter $\varepsilon$ effectively seems to put an ”electric charge” on the black hole which is of gravitational off–diagonal/anholonomic origin. For complex metrics such ”electric charges” may be induced by complex values of off–diagonal metric/ anholonomic
frame coefficients. It can describe effectively both positive and negative gravitational polarizations (even some repulsive gravitational effects). The polarization may have very specific nonlinearities induced by complex gravity terms. This is not surprising because the coefficients of an anisotropic black hole are similar to those of the Reissner–Nordstrom solution only with respect to corresponding anholonomic complex/real frames which are subjected to some constraints (anholonomy conditions).

Applying the method of anholonomic frame transforms elaborated and developed in Refs. 13-19,24-31, we can construct off–diagonal ellipsoidal extensions of the already diagonally disturbed Schwarzschild metric (see the metric (3.7) from Ref. 82). Such anholonomic deformations would contain in the diagonal limit configurations with \( \varepsilon \to 0 \) but \( \eta_3 \neq 1 \) (see (47) and/or (33) and (51)) for such configurations the function \( \eta_3 \) has to be related in the corresponding limits with the values \( \tilde{\gamma}_D, \tilde{\psi}_D \) and \( A \) from Ref. 82). We remark that there are different classes of ellipsoidal deformations of the Schwarzschild metric which result in real or complex vacuum configuration. The conditions \( \varepsilon \to 0 \) and \( q, \eta_3 = 1 \) and \( \lambda \to 0 \) select just the limit of the usual radial Schwarzschild asymptotics without any (also possible) additional diagonal distortions. For nontrivial values of \( q, \eta_3 \) and \( \eta_4 \) we may obtain diagonal distortions.

In the case of ellipsoidal metrics with the Schwarzschild asymptotics, the ellipsoidal character could result in some observational effects in the vicinity of the horizon. For instance, scattering of particles on a static ellipsoid can be computed. We can also model anisotropic matter accretion effects on an ellipsoidal black hole put in the center of a galactic being of ellipsoidal, toroidal or another configuration. Even in 4D the nonshp eric topology of horizons seem to be prohibited in the general relativity theory following some general differential geometry and censorship theorems, such objects can be induced from extra dimensions and can exist in theories with cosmological constant, nontrivial torsion or induced by anholonomic frames. We can consider black torus/ellipsoid solutions as a background for potential tests for existence of extra dimensions and of general relativity. A point of further investigations could be the anisotropic ellipsoidal collapse when both the matter and spacetime are of ellipsoidal generic off–diagonal symmetry (former theoretical and computational investigations were performed only for rotoids with anisotropic matter and particular classes of perturbations of the Schwarzshild solutions). It is interesting to elaborate collapse scenarios with respect to complexified anholonomic frames. For very small eccentricities, we may not have any observable effects like perihelion shift or light bending if we restrict our investigations only to the Schwarzschild–Newton asymptotics but some kind of dissipation can be considered for complex metrics.

We also present some comments on mechanics and thermodynamics of ellipsoidal black holes. For the static black ellipsoids/tori with flat asymptotics, we can compute the area of the ellipsoidal horizon, associate an entropy and develop a corresponding black ellipsoid thermodynamics. But this is a rough approximation because, in general, we are dealing with off–diagonal metrics depending anisotropically on two/three coordinates. Such solutions are with anholonomically deformed Killing horizons, or with anholonomic noncommutative symmetries, and should be described by a thermodynamics (in general, both non-equilibrium and irreversible) of black ellipsoids/tori self–consistently embedded into an off–diagonal anisotropic gravitational vacuum with potential dissipation described by some complex metric and frame coefficients. This forms a ground for numerous new conceptual issues to be developed and related to anisotropic black holes and the anisotropic kinetics and thermodynamics as well to a framework of isolated anisotropic horizons, defined in a locally anistoropic/ noncom-
mutative / complex background with wormhole real and/or complex configurations which is a matter of our further investigations.

Finally, we note that we elaborated a general formalism of generating noncommutative solutions starting from exact vacuum solutions with anholonomic noncommutativity, but we do not know how to extend our solutions via star (Moyal) product as to preserve their stability because of induced general complex terms in the metrics. For some particular duality relations between the coordinate and frame noncommutativity it seems possible to get stability at least in the first approximation of noncommutative deformation parameter but an arbitrary noncommutative coordinate relation results in less defined physical configurations. A better understanding of the physical relevance of the anholonomic noncommutative configurations completed also to general coordinate noncommutativity is an interesting open question which we leave for the future.

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A Einstein Equations and N–Connections

For convenience, we present in this Appendix a selection of necessary results from Refs. 13-19,30,31:

The curvature tensor of a connection $\Gamma^{[\alpha}$ with h–v–components $\Omega$ induced by anholonomic frames $\Omega$ and $\Omega$ with associated N–connection structure is defined $R(\delta\tau,\delta\gamma)\delta\beta = R_{\alpha\beta\gamma\tau} = \Omega_{\alpha\beta\gamma\tau}$ where

$$R_{\alpha\beta\gamma\tau} = \delta\tau\Gamma^{\alpha\beta\gamma} - \delta\gamma\Gamma^{\alpha\beta\tau} + \Gamma^{\alpha\beta\gamma}\Gamma_{\beta\tau\gamma} - \Gamma^{\alpha\beta\tau}\Gamma_{\beta\gamma\tau},$$

(89)
splits into irreducible h–v–components $R_{\alpha\beta\gamma\tau} = \{R_{\alpha\beta\gamma\tau}^i, R_{\alpha\beta\gamma\tau}^a, P_{\alpha\beta\gamma\tau}^i, P_{\alpha\beta\gamma\tau}^a, S_{\alpha\beta\gamma\tau}^i, S_{\alpha\beta\gamma\tau}^a\}$, with

\begin{align*}
R_{\alpha\beta\gamma\tau}^i &= \delta_i L_{\alpha\beta\gamma\tau}^j, \\
R_{\alpha\beta\gamma\tau}^a &= \delta^a L_{\alpha\beta\gamma\tau}^j, \\
P_{\alpha\beta\gamma\tau}^i &= \partial_i L_{\alpha\beta\gamma\tau}^j + C_{\alpha\beta\gamma\tau}^{ij}, \\
P_{\alpha\beta\gamma\tau}^a &= \partial^a L_{\alpha\beta\gamma\tau}^j + C_{\alpha\beta\gamma\tau}^{aj}, \\
S_{\alpha\beta\gamma\tau}^i &= \partial_i C_{\alpha\beta\gamma\tau}^{j}, \\
S_{\alpha\beta\gamma\tau}^a &= \partial^a C_{\alpha\beta\gamma\tau}^{j},
\end{align*}

(90)

where we omitted the label $[\alpha]$ in formulas. The Ricci tensor $R_{\alpha\gamma} = R_{\alpha\beta\gamma\alpha}^\beta$ has the irreducible h–v–components

\begin{align*}
R_{ij} &= R_{i,jk}^k, \\
R_{ia} &= -2P_{ia} = -P_{i,ka}^k, \\
R_{ai} &= 1P_{ai} = P_{a,ib}^b, \\
R_{ab} &= S_{a,be}^c.
\end{align*}

(91)

This tensor is non-symmetric, $1P_{ai} \neq 2P_{ia}$ (this could be with respect to anholonomic frames of reference). The scalar curvature of the metric d–connection, $\bar{\bar{R}} = g^{\alpha\beta}R_{\alpha\beta}$, is computed

$$\bar{\bar{R}} = G^{\alpha\beta}R_{\alpha\beta} = \bar{R} + S,$$

(92)
where \( \hat{R} = g^{ij} R_{ij} \) and \( S = \kappa^{ab} S_{ab} \).

By substituting (91) and (92) into the 5D Einstein equations
\[
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa \Upsilon_{\alpha\beta},
\]
where \( \kappa \) and \( \Upsilon_{\alpha\beta} \) are respectively the coupling constant and the energy–momentum tensor, we obtain the h–v-decomposition of the Einstein equations
\[
R_{ij} - \frac{1}{2} \left( \hat{R} + S \right) g_{ij} = \kappa \Upsilon_{ij},
\]
\[
S_{ab} - \frac{1}{2} \left( \hat{R} + S \right) h_{ab} = \kappa \Upsilon_{ab},
\]
\[
1 P_{ai} = \kappa \Upsilon_{ai}, \quad 2 P_{ia} = \kappa \Upsilon_{ia}.
\]

The vacuum 5D gravitational field equations, in invariant h–v–components, are written
\[
R_{ij} = 0, S_{ab} = 0, 1 P_{ai} = 0, 2 P_{ia} = 0.
\] (94)

The main ‘trick’ of the anholonomic frames method of integrating the Einstein equations in general relativity and various (super) string and higher / lower dimension gravitational theories consist in a procedure of definition of such coefficients \( N_j^a \) such that the block matrices \( g_{ij} \) and \( h_{ab} \) are diagonalized. This substantially simplifies computations but we have to apply N–elongated partial derivatives.

### B Main Theorems for 5D

We restrict our considerations to a five dimensional (in brief, 5D) spacetime provided with a generic off–diagonal (pseudo) Riemannian metric and labeled by local coordinates \( u^\alpha = (x^i, y^4 = v, y^5) \), for \( i = 1, 2, 3 \). We state the condition when exact solutions of the Einstein equations depending on holonomic variables \( x^i \) and on one anholonomic (equivalently, anisotropic) variable \( y^4 = v \) can be constructed in explicit form. Every coordinate from a set \( u^\alpha \) can may be time like, 3D space like, or extra dimensional. For simplicity, the partial derivatives will be denoted like \( a^x = \partial a / \partial x^1, a^* = \partial a / \partial x^2, a^v = \partial a / \partial x^3, a^' = \partial a / \partial v \).

The 5D metric quadratic line element is chosen
\[
ds^2 = g_{\alpha\beta} (x^i, v) \, du^\alpha du^\beta
\] (95)
when the metric components \( g_{\alpha\beta} \) are parametrized with respect to the coordinate dual basis by an off–diagonal matrix (ansatz)
\[
\begin{bmatrix}
g_1 + w_1^2 h_4 + n_1^2 h_5 & w_1 w_2 h_4 + n_1 n_2 h_5 & w_1 w_3 h_4 + n_1 n_3 h_5 & w_1 h_4 & n_1 h_5 \\
w_1 w_2 h_4 + n_1 n_2 h_5 & g_2 + w_2^2 h_4 + n_2^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\
w_1 w_3 h_4 + n_1 n_3 h_5 & w_3 w_2 h_4 + n_3 n_2 h_5 & g_3 + w_3^2 h_4 + n_3^2 h_5 & w_3 h_4 & n_3 h_5 \\
w_1 h_4 & w_2 h_4 & w_3 h_4 & h_4 & 0 \\
n_1 h_5 & n_2 h_5 & n_3 h_5 & 0 & h_5
\end{bmatrix},
\] (96)
with the coefficients being some necessary smoothly class functions of type
\[
g_1 = \pm 1, g_{2,3} = g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^i, v), \]
\[
w_i = w_i(x^i, v), n_i = n_i(x^i, v),
\]
where the $N$–coefficients from (3) and (2) are parametrized $N^4_i = w_i$ and $N^5_i = n_i$.

By straightforward calculation, we can prove\cite{30}:

**Theorem 1.** The nontrivial components of the 5D vacuum Einstein equations, $R^2_{ab} = 0$ (see (97) in the Appendix A) for the metric (1) defined by the ansatz (96), computed with respect to anholonomic frames (2) and (3) can be written in the form:

$$R^2_2 = R^3_3 = -\frac{1}{2g_{23}}[g^*_2 - (g^*_2)^2 - 2g^*_2 + (g^*_3)^2 + g^*_2 - \frac{g^*_2 g^*_3}{2g_3} - \frac{(g^*_2)^2}{2g_2}] = 0,$$

$$S^4_5 = S^5_5 = -\frac{2h_4}{h_5} \beta = 0,$$

$$R_{4i} = -\frac{w_i \beta - \alpha_i}{2h_5} = 0,$$

$$R_{5i} = \frac{h_5}{2h_4} [n^*_i + \gamma n^*_i] = 0,$$

where

$$\alpha_i = \partial_i h^*_5 - h^*_5 \partial_i \ln \sqrt{|h_4 h_5|}, \quad \beta = h^*_5 - h^*_5 \ln \sqrt{|h_4 h_5|}, \quad \gamma = 3h^*_5/2h_5 - h^*_5/h_4.$$  

Following this theorem, 1) we can define a function $g_2(x^2, x^3)$ for a given $g_3(x^2, x^3)$, or inversely, to define a function $g_2(x^2, x^3)$ from equation (97); 2) we can define a function $h_4(x^1, x^2, x^3, v)$ for a given $h_5(x^1, x^2, x^3, v)$, or inversely, to define a function $h_5(x^1, x^2, x^3, v)$ for a given $h_4(x^1, x^2, x^3, v)$, from equation (98); 3) we can compute the coefficients (101) which allow to solve the algebraic equations (97) and to integrate two times on $v$ the equations (100) which allow to find respectively the coefficients $w_i(x^k, v)$ and $n_i(x^k, v)$.

We can generalize the construction by introducing a conformal factor $\Omega(x^i, v)$ and additional deformations of the metric via coefficients $\zeta_i(x^i, v)$ (here, the indices with 'hat' take values like $i = 1, 2, 3, 5$), i.e. for metrics of type

$$ds^2 = \Omega^2(x^i, v)\hat{g}_{\alpha\beta}(x^i, v) \, du^\alpha du^\beta,$$

were the coefficients $\hat{g}_{\alpha\beta}$ are parametrized by the ansatz

$$(g_1 + (w_1^2 + \zeta_1^2)h_4 + n_1^2 h_5, (w_1 w_2 + \zeta_1 \zeta_2)h_4 + n_1 n_2 h_5, (w_1 w_3 + \zeta_1 \zeta_3)h_4 + n_1 n_3 h_5, \ldots) = (w_1^2 + \zeta_1^2)h_4 + n_1^2 h_5, (w_2 w_3 + \zeta_2 \zeta_3)h_4 + n_2 n_3 h_5, (w_3 w_4 + \zeta_3 \zeta_4)h_4 + n_3 n_4 h_5, \ldots).$$

(103)

Such 5D metrics have a second order anisotropy\cite{32,37,38} when the $N$–coefficients are parametrized in the first order anisotopy like $N^4 = w_i$ and $N^5 = n_i$ (with three anholonomic, $x^i$, and two anholonomic, $y^4$ and $y^5$, coordinates) and in the second order anisotopy (on the second 'shell', with four anholonomic, $(x^i, y^4, y^5)$, and one anholonomic, $y^5$, coordinates) with $N^5 = \zeta_i$ in this work we state, for simplicity, $\zeta_i = 0$. For trivial values $\Omega = 1$ and $\zeta_i = 0$, the squared line interval (102) transforms into (95).

Theorem 1 can be extended as to include the generalization to the second ansatz (102):
Theorem 2. The nontrivial components of the 5D vacuum Einstein equations, $R^{\hat{g}}_\alpha = 0$, (see (94) in the Appendix A) for the metric (102) consist from the system (97)–(100) with the additional conditions that

$$\delta_i h_4 = 0 \text{ and } \delta_i \Omega = 0$$

(104)

for $\delta_i = \partial_i - (w_i + \zeta_i) \partial_4 + n_i \partial_5$ when the values $\zeta_4 = (\zeta_i, \zeta_5 = 0)$ are to be found as to be a solution of (104); for instance, if $\Omega = \Omega_1 \Omega_2$, the condition that we have the same values of the Ricci tensor for the (96) and (103) results in equations (104), and (106) which are compatible, for instance, if $\Omega^{n_1/q_2} = h_4$. There are also another possibilities to satisfy the condition (105), for instance, if $\Omega = \Omega_1 \Omega_2$, we can consider that $h_4 = \Omega^{n_1/q_2} \Omega_2^{\bar{n}_1/q_4}$ for some integers $q_1, q_2, q_3$ and $q_4$.

A very surprising result is that we are able to construct exact solutions of the 5D vacuum Einstein equations for both types of the ansatz (96) and (103):

Theorem 3. The system of second order nonlinear partial differential equations (97)–(100) and (106) can be solved in general form if there are given some values of the Ricci tensor $g_2(x^2, x^3)$ (or $g_3(x^2, x^3)$), $h_4(x^i, v)$ (or $h_5(x^i, v)$) and $\Omega(x^i, v)$:

- The general solution of equation (97) can be written in the form
  $$\varpi = g_0 \exp[a_2 \tilde{x}^2 (x^2, x^3) + a_3 \tilde{x}^3 (x^2, x^3)],$$
  (107)

  were $g_0$, $a_2$ and $a_3$ are some constants and the functions $\tilde{x}^2, 3 (x^2, x^3)$ define any coordinate transforms $x^{2, 3} \rightarrow \tilde{x}^{2, 3}$ for which the 2D line element becomes conformally flat, i.e.
  $$g_2(x^2, x^3)(dx^2)^2 + g_3(x^2, x^3)(dx^3)^2 \rightarrow \varpi [(dx^2)^2 + \epsilon(dx^3)^2].$$
  (108)

- The equation (98) relates two functions $h_4(x^i, v)$ and $h_5(x^i, v)$ following two possibilities:
  a) to compute
  $$\sqrt{|h_5|} = h_{5[1]} (x^i) + h_{5[2]} (x^i) \int \sqrt{|h_4(x^i, v)|} dv, \ h_4^* (x^i, v) \neq 0;$$
  $$h_{5[1]} (x^i) + h_{5[2]} (x^i) v, h_4^* (x^i, v) = 0,$$
  (109)

  for some functions $h_{5[1, 2]} (x^i)$ stated by boundary conditions;

  b) or, inversely, to compute $h_4$ for a given $h_5(x^i, v), h_5^* \neq 0$,

  $$\sqrt{|h_4|} = h_0 (x^i) (\sqrt{|h_5(x^i, v)|})^*,$$
  (110)

  with $h_0(x^i)$ given by boundary conditions.
• The exact solutions of (106) for $\beta \neq 0$ is

$$w_k = \partial_k \ln\left[\frac{|h_4 h_5|}{|h_5^*|}\right]/\partial_v \ln\left[\frac{|h_4 h_5|}{|h_5^*|}\right],$$

(111)

with $\partial_v = \partial/\partial v$ and $h_5^* \neq 0$. If $h_5^* = 0$, or even $h_5^* \neq 0$ but $\beta = 0$, the coefficients $w_k$ could be arbitrary functions on $(x^i, v)$. For vacuum Einstein equations this is a degenerated case which imposes the the compatibility conditions $\beta = \alpha_i = 0$, which are satisfied, for instance, if the $h_4$ and $h_5$ are related as in the formula (114) but with $h_{i[0]}(x^i) = \text{const}$.

• The exact solution of (106) is

$$n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int h_4 / (\sqrt{|h_5^*|})^3 dv, \quad h_5^* \neq 0;$$

$$= n_{k[1]}(x^i) + n_{k[2]}(x^i) \int h_4 dv, \quad h_5^* = 0;$$

(112)

$$= n_{k[1]}(x^i) + n_{k[2]}(x^i) \int [1/(\sqrt{|h_5^*|})^3] dv, \quad h_4^* = 0,$$

for some functions $n_{k[1,2]}(x^i)$ stated by boundary conditions.

• The exact solution of (106) is given by some arbitrary functions $\zeta_i = \zeta_i(x^i, v)$ if both $\partial_i \Omega = 0$ and $\Omega^* = 0$, we chose $\zeta_i = 0$ for $\Omega = \text{const}$, and

$$\zeta_i = -w_i + (\Omega^*)^{-1} \partial_i \Omega, \quad \Omega^* \neq 0,$$

(113)

$$= (\Omega^*)^{-1} \partial_i \Omega, \quad \Omega^* \neq 0,$$ for vacuum solutions.

We note that a transform (108) is always possible for 2D metrics and the explicit form of solutions depends on chosen system of 2D coordinates and on the signature $\epsilon = \pm 1$. In the simplest case the equation (97) is solved by arbitrary two functions $g_2(x^3)$ and $g_3(x^2)$. The equation (98) is satisfied by arbitrary pairs of coefficients $h_4(x^i, v)$ and $h_{5[0]}(x^i)$.

The proof of Theorem 3, following from a direct integration of (97)–(100) and (106) is given in the Appendix B of Ref. 30.

There are some important consequences of the Theorems 1–3:

**Corollary 1.** The non–trivial diagonal components of the Einstein tensor, $G_\alpha^\alpha = R_\beta^\alpha - \frac{1}{2} R \delta_\beta^\alpha$, for the metric (7), given with respect to N–frames, are

$$G_1^1 = -(R_2^2 + S_4^4), \quad G_2^2 = G_3^3 = -S_4^4, \quad G_4^4 = G_5^5 = -R_2^2$$

(114)

imposing the condition that the dynamics is defined by two values $R_2^2$ and $S_4^4$. The rest of non–diagonal components of the Ricci (Einstein tensor) are compensated by fixing corresponding values of N–coefficients.

The formulas (114) are obtained following the relations for the Ricci tensor (97)–(100).

**Corollary 2.** We can extend the system of 5D vacuum Einstein equations (97)–(100) by introducing matter fields for which the coefficients of the energy–momentum tensor $\Upsilon_{\alpha \beta}$ given with respect to N–frames satisfy the conditions

$$\Upsilon_1^1 = \Upsilon_2^2 + \Upsilon_4^4, \quad \Upsilon_2^2 = \Upsilon_3^3, \quad \Upsilon_4^4 = \Upsilon_5^5.$$

(115)
We note that, in general, the tensor $\Upsilon_{\alpha\beta}$ for the non-vacuum Einstein equations,

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \kappa \Upsilon_{\alpha\beta},$$

is not symmetric because with respect to anholonomic frames there are imposed constraints which makes non symmetric the Ricci and Einstein tensors (the symmetry conditions may be defined explicitly only with respect to holonomic, coordinate frames; for details see the Appendix A and the formulas (93)).

For simplicity, in our investigations we can consider only diagonal matter sources, given with respect to $N$–frames, satisfying the conditions

$$\kappa \Upsilon_2^2 = \kappa \Upsilon_3^3 = \Upsilon_2, \kappa \Upsilon_4^4 = \kappa \Upsilon_5^5 = \Upsilon_4, \text{ and } \Upsilon_1 = \Upsilon_2 + \Upsilon_4,$$

(116)

where $\kappa$ is the gravitational coupling constant. In this case the equations (97) and (98) are respectively generalized to

$$R_2^2 = R_3^3 = \frac{1}{2g_{23}}[g_3^* - \frac{3}{2g_2}g_2^* g_3^* + \frac{(g_3^*)^2}{2g_3} + g'' - \frac{g_2^* g_3^*}{2g_2} - \frac{(g_2')^2}{2g_2}] = -\Upsilon_4$$

(117)

and

$$S_4^4 = S_5^5 = -\frac{\beta}{2h_4 h_5} = -\Upsilon_2.$$  

(118)

**Corollary 3.** An arbitrary solution of the system of equations (94)–(101) and (106) is defined for a canonical connection (4) containing, in general, non-trivial torsion coefficients. This can be effectively applied in order to construct exact solutions, for instance, in string gravity containing nontrivial torsion. We can select solutions corresponding to the Levi–Civita connection (10) for a generic off–diagonal (pseudo) Riemannian metric if we impose the condition that the coefficients $N_4^i = w_i(x^k, v)$, $N_5^i = n_i(x^k, v)$ and $N_5^\hat{i} = \zeta_i$ are fixed to result in a zero $N$–curvature, $\Omega^a_{jk} = 0$, on all "shells" of anisotropy. Such selections are possible by fixing corresponding boundary conditions and selecting corresponding classes of functions like $n_{k[1,2]}(x^i)$, obtained after a general integration, in formulas (111), (112) and (113).

The above presented results are for generic 5D off–diagonal metrics, anholonomic transforms and nonlinear field equations. Reductions to a lower dimensional theory are not trivial in such cases. We emphasize here some specific points of this procedure (see details in Ref. 30).

### C Reduction from 5D to 4D

The simplest way to construct a $5D \rightarrow 4D$ reduction for the ansatz (96) and (103) is to eliminate from formulas the variable $x^1$ and to consider a 4D space (parametrized by local coordinates $(x^2, x^3, v, y^5)$) being trivially embedded into 5D space (parametrized by local coordinates $(x^1, x^2, x^3, v, y^5)$) with $g_{11} = \pm 1, g_{1\tilde{a}} = 0, \tilde{\alpha} = 2, 3, 4, 5$) with further possible 4D conformal and anholonomic transforms depending only on variables $(x^2, x^3, v)$. We suppose that the 4D metric $g_{\hat{a}\hat{b}}$ could be of arbitrary signature. In order to emphasize that some coordinates are stated just for such 4D space we underline the Greek indices, $\hat{\alpha}, \hat{\beta}, ...$ and the Latin indices from the middle of alphabet, $\hat{i}, \hat{j}, ... = 2, 3$, where $u^\alpha = (x^1, y^\alpha) = (x^2, x^3, y^4, y^5)$. 47
In result, the analogs, Theorems 1-3 and Corollaries 1-3 can be reformulated for 4D gravity with mixed holonomic–anholonomic variables. We outline here the most important properties of a such reduction.

- The line element (93) with ansatz (96) and the line element (95) with (103) are respectively transformed on 4D space to the values:

The first type 4D quadratic line element is taken

$$ds^2 = g_{\tilde{\alpha} \tilde{\beta}} \left( x^\tilde{\alpha}, v \right) du^\alpha dv^\beta$$  \hspace{1cm} (119)

with the metric coefficients $g_{\tilde{\alpha} \tilde{\beta}}$ parametrized

$$
\begin{bmatrix}
g_2 + w_2^2 h_4 + n_2^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\
w_2 w_3 h_4 + n_2 n_3 h_5 & g_3 + w_3^2 h_4 + n_3^2 h_5 & w_3 h_4 & n_3 h_5 \\
w_2 h_4 & w_3 h_4 & h_4 & 0 \\
n_2 h_5 & n_3 h_5 & 0 & h_5
\end{bmatrix},
$$  \hspace{1cm} (120)

where the coefficients are some necessary smoothly class functions of type:

$$g_{2,3} = g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^\tilde{k}, v),$$

$$w_i = w_i(x^\tilde{k}, v), n_i = n_i(x^\tilde{k}, v); \tilde{i}, \tilde{k} = 2, 3.$$

The anholonomically and conformally transformed 4D line element is

$$ds^2 = \Omega^2(x^\tilde{i}, v) \hat{g}_{\tilde{\alpha} \tilde{\beta}} \left( x^\tilde{\alpha}, \tilde{\beta} \right) du^\alpha dv^\beta,$$  \hspace{1cm} (121)

were the coefficients $\hat{g}_{\tilde{\alpha} \tilde{\beta}}$ are parametrized by the ansatz

$$
\begin{bmatrix}
g_2 + (w_2^2 + \zeta_3^2) h_4 + n_2^2 h_5 & (w_2 w_3 + \zeta_2 \zeta_3) h_4 + n_2 n_3 h_5 & (w_2 + \zeta_2) h_4 & n_2 h_5 \\
(w_2 w_3 + \zeta_2 \zeta_3) h_4 + n_2 n_3 h_5 & (w_3^2 + \zeta_3^2) h_4 + n_3^2 h_5 & (w_3 + \zeta_3) h_4 & n_3 h_5 \\
(w_2 + \zeta_2) h_4 & (w_3 + \zeta_3) h_4 & h_4 & 0 \\
n_2 h_5 & n_3 h_5 & 0 & h_5 + \zeta_5 h_4
\end{bmatrix}.
$$  \hspace{1cm} (122)

where $\zeta_i = \zeta_i(x^\tilde{k}, v)$ and we shall restrict our considerations for $\zeta_5 = 0$.

- We have a quadratic line element (111) which can be written

$$\delta s^2 = g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta v)^2 + h_5(\delta y^5)^2,$$  \hspace{1cm} (123)

with respect to the anholonomic co–frame $\left( dx^\tilde{i}, \delta v, \delta y^5 \right)$, where

$$\delta v = dv + w_i dx^\tilde{i} \text{ and } \delta y^5 = dy^5 + n_i dx^\tilde{i}$$  \hspace{1cm} (124)

is the dual of $(\delta_i, \partial_4, \partial_5)$, where

$$\delta_i = \partial_i + w_i \partial_4 + n_i \partial_5.$$  \hspace{1cm} (125)
• In the conditions of the 4D variant of the Theorem 1 we have the same equations (97)–(100) were we must put $h_4 = h_4\left(x^k, v\right)$ and $h_5 = h_5\left(x^k, v\right)$. As a consequence we have that $\alpha_i\left(x^k, v\right) \rightarrow \alpha_i\left(\hat{x}^k, v\right)$, $\beta = \beta\left(\hat{x}^k, v\right)$ and $\gamma = \gamma\left(\hat{x}^k, v\right)$ which result that $w_i = w_i\left(x^k, v\right)$ and $n_i = n_i\left(x^k, v\right)$.

• The 4D line element with conformal factor (102) subjected to an anholonomic map with $\zeta_5 = 0$ transforms into

$$\delta s^2 = \Omega^2(x^i, v)[g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta v)^2 + h_5(\delta y^5)^2],$$

(126)
given with respect to the anholonomic co–frame $(dx^i, \delta v, \delta y^5)$, where

$$\delta v = dv + (w_i + \zeta_i)dx^i \quad \text{and} \quad \delta y^5 = dy^5 + n_i dx^i$$

(127)
is dual to the frame $(\hat{\delta}_i, \partial_4, \partial_5)$ with

$$\hat{\delta}_i = \partial_i - (w_i + \zeta_i)\partial_4 + n_i \partial_5, \hat{\partial}_5 = \partial_5.$$  

(128)

• The formulas (104) and (106) from Theorem 2 must be modified into a 4D form

$$\hat{\delta}_i h_4 = 0 \quad \text{and} \quad \hat{\delta}_i \Omega = 0$$

(129)

and the values $\zeta_i = (\zeta_i, \zeta_5 = 0)$ are found as to be a unique solution of (104); for instance, if

$$\Omega^{q_1/q_2} = h_4 \quad (q_1 \text{ and } q_2 \text{ are integers}),$$

$\zeta_i$ satisfy the equations

$$\partial_i \Omega - (w_i + \zeta_i)\Omega^* = 0.$$  

(130)

• One holds the same formulas (109)-(112) from the Theorem 3 on the general form of exact solutions with that difference that their 4D analogs are to be obtained by reductions of holonomic indices, $\hat{i} \rightarrow i$, and holonomic coordinates, $x^i \rightarrow \hat{x}^i$, i.e. in the 4D solutions there is not contained the variable $x^1$.

• The formulae (114) for the nontrivial coefficients of the Einstein tensor in 4D stated by the Corollary 1 are written

$$G_2^2 = G_3^3 = -S_4^4, G_4^4 = G_5^5 = -R_2^2.$$  

(131)

• For symmetries of the Einstein tensor (131), we can introduce a matter field source with a diagonal energy momentum tensor, like it is stated in the Corollary 2 by the conditions (115), which in 4D are transformed into

$$\Upsilon^2 = \Upsilon_3^3, \quad \Upsilon_4^4 = \Upsilon_5^5.$$  

(132)
D Star–Products, Enveloping Algebras and Noncommutative Geometry

For a noncommutative space the coordinates \( \hat{u}^i \), \( i = 1, \ldots, N \) satisfy some noncommutative relations

\[
[\hat{u}^i, \hat{u}^j] = \begin{cases} 
 i \theta^{ij}, & \theta^{ij} \in \mathbb{C}, \text{ canonical structure;} \\
 i f_k^{ij} \hat{u}_k, & f_k^{ij} \in \mathbb{C}, \text{ Lie structure;} \\
 i C_{kl}^{ij} \hat{u}_k \hat{u}_l, & C_{kl}^{ij} \in \mathbb{C}, \text{ quantum plane} 
\end{cases}
\] (133)

where \( \mathbb{C} \) denotes the complex number field.

The noncommutative space is modelled as the associative algebra of \( \mathbb{C} \); this algebra is freely generated by the coordinates modulo ideal \( \mathcal{R} \) generated by the relations (one accepts formal power series) \( \mathcal{A}_u = \mathbb{C}[[\hat{u}^1, \ldots, \hat{u}^N]]/\mathcal{R} \). One restricts attention\(^9\) to algebras having the (so-called, Poincare–Birkhoff–Witt) property that any element of \( \mathcal{A}_u \) is defined by its coefficient function and vice versa,

\[
\hat{f} = \sum_{L=0}^{\infty} f_{i_1, \ldots, i_L} : \hat{u}^{i_1} \ldots \hat{u}^{i_L} : \quad \text{when } \hat{f} \sim \{ f_i \},
\]

where \( : \hat{u}^{i_1} \ldots \hat{u}^{i_L} : \) denotes that the basis elements satisfy some prescribed order (for instance, the normal order \( i_1 \leq i_2 \leq \ldots \leq i_L \), or, another example, are totally symmetric). The algebraic properties are all encoded in the so-called diamond (\( \odot \)) product which is defined by

\[
\hat{f} \odot \hat{g} = \hat{h} \sim \{ f_i \} \cdot \{ g_i \} = \{ h_i \}.
\]

In the mentioned approach to every function \( f(u) = f(u^1, \ldots, u^N) \) of commuting variables \( u^1, \ldots, u^N \) one associates an element of algebra \( \hat{f} \) when the commuting variables are substituted by anticommuting ones,

\[
f(u) = \sum f_{i_1 \ldots i_L} u^1 \ldots u^N \rightarrow \hat{f} = \sum_{L=0}^{\infty} f_{i_1, \ldots, i_L} : \hat{u}^{i_1} \ldots \hat{u}^{i_L} :
\]

when the \( \odot \)-product leads to a bilinear \( * \)-product of functions (see details in Ref. [70])

\[
\{ f_i \} \cdot \{ g_i \} = \{ h_i \} \sim (f * g)(u) = h(u).
\]

The \( * \)-product is defined respectively for the cases (133)

\[
f * g = \begin{cases} 
 \exp[i \frac{1}{2} \frac{\partial}{\partial u'} \theta^{ij} \frac{\partial}{\partial u}] f(u)g(u')|_{u' \rightarrow u}, \\
 \exp[i \frac{1}{2} u^k g_k (i \frac{\partial}{\partial u'}, i \frac{\partial}{\partial u'})] f(u)g(u')|_{u' \rightarrow u}, \\
 q^{\frac{1}{2} (-u \frac{\partial}{\partial u'} + u \frac{\partial}{\partial u'})} f(u, v)g(u', v')|_{v' \rightarrow v},
\end{cases}
\]

where there are considered values of type

\[
e^{ik_n \hat{u}^n} e^{ip_m \hat{u}^m} = e^{i(k_n + p_m + \frac{1}{2} g_{nmp}) \hat{u}^n}, \quad \text{(134)}
\]

\[
g_n(k, p) = -k_i p_j f^{ij}_{n}, \quad \text{or, another example, are totally symmetric) The algebraic properties are all encoded in the so-called diamond (\( \odot \)) product which is defined by}
\]

\[
\{ f_i \} \cdot \{ g_i \} = \{ h_i \} \sim (f * g)(u) = h(u).
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 \exp[i \frac{1}{2} u^k g_k (i \frac{\partial}{\partial u'}, i \frac{\partial}{\partial u'})] f(u)g(u')|_{u' \rightarrow u}, \\
 q^{\frac{1}{2} (-u \frac{\partial}{\partial u'} + u \frac{\partial}{\partial u'})} f(u, v)g(u', v')|_{v' \rightarrow v},
\end{cases}
\]

where there are considered values of type

\[
e^{ik_n \hat{u}^n} e^{ip_m \hat{u}^m} = e^{i(k_n + p_m + \frac{1}{2} g_{nmp}) \hat{u}^n}, \quad \text{(134)}
\]

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g_n(k, p) = -k_i p_j f^{ij}_{n}, \quad \text{or, another example, are totally symmetric) The algebraic properties are all encoded in the so-called diamond (\( \odot \)) product which is defined by}
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 \exp[i \frac{1}{2} u^k g_k (i \frac{\partial}{\partial u'}, i \frac{\partial}{\partial u'})] f(u)g(u')|_{u' \rightarrow u}, \\
 q^{\frac{1}{2} (-u \frac{\partial}{\partial u'} + u \frac{\partial}{\partial u'})} f(u, v)g(u', v')|_{v' \rightarrow v},
\end{cases}
\]

where there are considered values of type

\[
e^{ik_n \hat{u}^n} e^{ip_m \hat{u}^m} = e^{i(k_n + p_m + \frac{1}{2} g_{nmp}) \hat{u}^n}, \quad \text{(134)}
\]
and for the coordinates on quantum (Manin) planes one holds the relation \( uv = qvu \).

A non–Abelian gauge theory on a noncommutative space is given by two algebraic structures, the algebra \( \mathcal{A}_u \) and a non–Abelian Lie algebra \( \mathcal{A}_I \) of the gauge group with generators \( I^1, \ldots, I^S \) and the relations

\[
[I^I_z, I^I_z] = i \frac{R_z}{2} I^I_z.
\]  

(135)

In this case both algebras are treated on the same footing and one denotes the generating elements of the big algebra by \( \hat{u}^i \),

\[
\mathbb{Z} = \{ \hat{u}^1, \ldots, \hat{u}^N, I^1, \ldots, I^S \}, \mathcal{A}_z = \mathbb{C}[[\hat{u}^1, \ldots, \hat{u}^{N+S}]]/\mathcal{R}
\]

and the \( * \)–product formalism is to be applied for the whole algebra \( \mathcal{A}_z \) when there are considered functions of the commuting variables \( u^i (i, j, k, \ldots = 1, \ldots, N) \) and \( I^s (s, p, \ldots = 1, \ldots, S) \).

For instance, in the case of a canonical structure for the space variables \( u^i \) we have

\[
(F * G)(u) = \exp \left[ i \left( \frac{1}{2} \left( \partial_{u^i} \partial_{u^j} \mathbf{g}_{s} \left( i \frac{\partial}{\partial t^s}, i \frac{\partial}{\partial t^s} \right) \right) \right) \times F (u', t') G (u'', t'') \mid _{t^s \rightarrow t, t'' \rightarrow t} \right]
\]  

(136)

This formalism was developed in Ref. 94 for general Lie algebras.

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