The Laplacian of The Integral Of The Logarithmic Derivative of the Riemann-Siegel-Hardy Z-function

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Abstract

The integral
\[ R(t) = \pi^{-1} \int \frac{d}{dt} \ln Z(t) dt = \pi^{-1} (\ln \zeta \left( \frac{1}{2} + it \right) + i\vartheta(t)) \]
of the logarithmic derivative of the Hardy Z function
\[ Z(t) = e^{i\vartheta(t)} \zeta \left( \frac{1}{2} + it \right) \]
where \( \vartheta(t) \) is the Riemann-Siegel theta function, and \( \zeta(s) \) is the Riemann zeta function, is used as a basis for the construction of a pair of transcendental entire functions
\[ \nu(t) = -\nu(1 - t) = -\left( \Delta R \left( \frac{1}{2} - it \right) \right)^{-1} = -G \left( \frac{1}{2} - it \right) \]
where \( G = -\left( \Delta R(t) \right)^{-1} \) is the derivative of the additive inverse of the reciprocal of the Laplacian \( \Delta f(t) = \ddot{f}(t) \) of \( R(t) \) and \( \chi(t) = -\chi(1 - t) = \dot{\nu}(t) = -iH \left( \frac{1}{2} - it \right) \) where \( H(t) = \dot{G}(t) \) has roots at the local minima and maxima of \( G(t) \). When \( H(t) = 0 \) and \( \dot{H}(t) = \Delta G(t) > 0 \), the point \( t \) marks a minimum of \( G(t) \) where it coincides with a Riemann zero, i.e., \( \zeta \left( \frac{1}{2} + it \right) = 0 \), otherwise when \( H(t) = 0 \) and \( \dot{H}(t) = \Delta G(t) < 0 \), the point \( t \) marks a local maximum \( G(t) \), marking midway points between consecutive minima. Considered as a sequence of distributions or wave functions, \( \nu_n(t) = \nu(1 + 2n + 2t) \) converges to \( \nu_\infty(t) = \lim_{n \to \infty} \nu_n(t) = \sin^2(\pi t) \) and \( \chi_n(t) = \chi(1 + 2n + 2t) \) to \( \chi_\infty(t) = \lim_{n \to \infty} \chi_n(t) = -\delta(\pi t) \sin(\pi t) \)

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1 Derivations

1.1 Standard Definitions

Let $\zeta(t)$ be the Riemann zeta function

$$\zeta(t) = \sum_{n=1}^{\infty} n^{-s} = (1 - 2^{1-s}) \sum_{n=1}^{\infty} n^{-s}(-1)^{n-1} \forall \text{Re}(s) > 1$$

and $\vartheta(t)$ be Riemann-Siegel vartheta function

$$\vartheta(t) = \frac{-i}{2} \left( \ln \Gamma \left( \frac{1}{4} + \frac{it}{2} \right) - \ln \Gamma \left( \frac{1}{4} - \frac{it}{2} \right) \right) - \frac{\ln(\pi)t}{2}$$

where $\arg(z) = \ln(z) - \ln(\bar{z})/2i$ and $\Gamma(z) = \Gamma(\bar{z})$. The Hardy $Z$ function can then be written as

$$Z(t) = e^{i\vartheta(t)} \zeta \left( \frac{1}{2} + it \right)$$

which can be mapped isometrically back to the $\zeta$ function

$$\zeta(t) = e^{-i\vartheta(t)} Z \left( \frac{i}{2} - it \right)$$

due to the isometry

$$t = \frac{1}{2} + i \left( \frac{i}{2} - it \right)$$

of the Mobius transform $f(t) = \frac{at+b}{ct+d} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ with

$$\left( \begin{array}{cc} -i & \frac{1}{4} \\ 0 & 1 \end{array} \right) \text{ and its inverse } \left( \begin{array}{cc} i & \frac{1}{4} \\ 0 & 1 \end{array} \right)$$

making possible the Riemann-Siegel-Hardy correspondence. Furthermore, let $S(t)$ be argument of $\zeta$ normalized by $\pi$ defined by

$$S(t) = \pi^{-1} \arg \left( \zeta \left( \frac{1}{2} + it \right) \right) = -\frac{i}{2\pi} \left( \ln \zeta \left( \frac{1}{2} + it \right) - \ln \zeta \left( \frac{1}{2} - it \right) \right)$$

The Bäcklund counting formula gives the exact number of zeros on the critical strip up to level $t$, not just on the critical line,

$$N(t) = \text{Im}(R(t)) = \frac{\vartheta(t)}{\pi} + 1 + S(t)$$

The relationship between the functions $N(t)$, $S(t)$, and $Z(t)$ is demonstrated by

$$\ln \zeta \left( \frac{1}{2} + it \right) = \ln |Z(t)| + i\pi S(t)$$

These formulas are true independent of the Riemann hypothesis which posits that all complex zeros of $\zeta(s + it)$ have real part $s = \frac{1}{2}$. [Ivi13 Corollary 1.8 p.13]

[1] Thanks to Matti Pitkänen for pointing out this is a Mobius transform pair, among other things
1.2 The Logarithmic Derivative of $Z(t)$ and its Integral

Let $Q(t)$ be the logarithmic derivative of $Z(t)$ given by

$$Q(t) = \frac{Z'(t)}{Z(t)}$$

where

$$\Psi(x) = \frac{d}{dx} \ln(\Gamma(x)) = \frac{d}{dx} \frac{\Gamma(x)}{\Gamma(x)}$$

is the digamma function, the logarithmic derivative of the $\Gamma$ function and its integral is defined by

$$\int -2 \ln(\Gamma(t))^2 + 2 \ln(\Gamma(t)) - 2 \ln(\pi)$$

and its integral is defined by

$$\int \frac{d}{dx} \ln(\Gamma(x)) = \frac{d}{dx} \frac{\Gamma(x)}{\Gamma(x)}$$

simplifying the expressions. The function $Q(t)$ has singularities at $\pm \frac{1}{4}(4n - 3)$ with residues

$$\text{Res}(Q(t)) = \frac{-8\zeta(2 - 2n)n^2 + 4\zeta(2 - 2n)n}{16\zeta(2 - 2n)n^2 - 8\zeta(2 - 2n)n} = -\frac{1}{2}$$

and

$$\text{Res}(Q(t)) = \frac{-2\zeta(2n - 1) + 4\zeta(2n - 1)n}{8\zeta(2n - 1)n - 4\zeta(2n - 1)} = \frac{1}{2}$$

Now, the integral of the logarithmic derivative of $Z$ is defined by

$$R(t) = \pi^{-1} \int Q(t) dt$$

and is simply the second derivative of a function when $d = 1$

$$f(x) = \frac{d}{dx^2} f(x) \forall x \in X \subseteq \mathbb{R}^d$$
Let $G(t)$ be the additive inverse of the reciprocal (also known as multiplicative inverse) of the Laplacian of $R(t)$ interpreted as a partition function

$$G(t) = -\frac{1}{\Delta R(t)} = -\frac{8z(t)^2\pi}{z(t)^2(g^{-1}(t) - g^+(t)) - 8z(t)\dot{z}(t) + \dot{z}(t)^2}$$

(17)

Figure 1: The Real and Imaginary Parts of $R(t)$ compared with $G(t)$

Now, let $H(t)$ be the derivative of $G(t)$ given by

$$H(t) = \dot{G}(t) = \frac{d}{dt} \left( -\frac{1}{\Delta R(t)} \right) = -4i\pi z(t)\frac{z(t)^2\Delta g(t) + 8(2z(t)^2\dot{z}(t) - 6z(t)\dot{z}(t)\dot{z}(t) - 4\dot{z}(t)^3)}{(z(t)^2 + 8(z(t)\dot{z}(t) - \dot{z}(t)^2))^2}$$

(18)

When $H(t) = 0$ and $\dot{H}(t) = \ddot{G}(t) = \Delta G(t) > 0$, the point $t$ marks a minimum of $G(t)$ where it coincides with a Riemann zero, i.e., $\zeta \left( \frac{1}{2} + it \right) = 0$, otherwise when $H(t) = 0$ and $\dot{H}(t) = \Delta G(t) < 0$, the point $t$ marks a local maximum $G(t)$, marking midway points between consecutive minima.

Figure 2: $G(t), H(t)$ and $\dot{H}(t)$

With the identity $t = \frac{1}{2} + i \left( \frac{1}{2} - it \right)$, define $\nu(t)$ as the Mobius transform of the partition function

$$\nu(t) = -\left( \Delta R \left( \frac{i}{2} - it \right) \right)^{-1} = -G \left( \frac{i}{2} - it \right)$$

(19)
then \( \nu(t) \) has zeros at the positive odd integers, zero, and negative even integers. In the same way, let

\[
\chi(t) = \nu(t) - it
\]

\[
= -iH \left( \frac{t}{2} - it \right)
\]

\[
= -4i\pi \zeta(t) \frac{(\psi(t) - \psi(\frac{t}{2})) - 8(6\zeta(t)\zeta(t)) + 2t(2^{\zeta(1)} - \zeta(2)) + 4t(3)}{\zeta(t)^2(\psi(t) - \psi(\frac{t}{2})) + 8(6\zeta(t)\zeta(t) - \zeta(t)^2)}
\]

which also has zeros on the real line at the positive odd integers, zero, and the negative even integers.

\[
\lim_{t \to 2n-1} \nu(t) = \lim_{t \to 2n-1} \chi(t) = 0 \forall n > 0
\]

\[
\lim_{t \to 0} \nu(t) = \lim_{t \to 0} \chi(t) = 0
\]

\[
\lim_{t \to -2n} \nu(t) = \lim_{t \to -2n} \chi(t) = 0 \forall n < 0
\]

Both \( \nu(t) \) and \( \chi(t) \) satisfy similar functional equations

\[
\nu(t) = \nu(1 - t)
\]

and

\[
\chi(t) = -\chi(1 - t)
\]

So the even \( \mu(t) \) and odd \( \psi(t) \) transcendental entire functions can be defined

\[
\mu(t) = \nu \left( t + \frac{1}{2} \right)
\]

\[
\psi(t) = \chi \left( t + \frac{1}{2} \right)
\]

which satisfy the functional symmetries

\[
\mu(t) = \mu(-t)
\]

and

\[
\psi(t) = -\psi(-t)
\]

The function \( \chi(t) \) has as a subset of its roots, the roots of the Riemann zeta function \( \zeta(t) \), the converse is not true, since \( \chi(t) \) is a function of \( \zeta(t) \) and its first, second, and third derivatives.

\[
\{ t : \zeta(t) = 0 \} \subset \{ t : \chi(t) = 0 \}
\]

Let

\[
\chi_n(t) = \chi(1 + 2n + 2t)
\]

and

\[
\nu_n(t) = \nu(1 + 2n + 2t)
\]

which both satisfy the symmetries

\[
\chi_n \left( \frac{t}{2} \right) = \chi(t)
\]

\[
\nu_n \left( \frac{t}{2} \right) = \nu(t)
\]
as well as
\[ \chi_n \left( -\frac{1}{2} \right) = \chi(2n) \]  
\[ \nu_n \left( -\frac{1}{2} \right) = \nu(2n) \]  
(33)  
(34)

The sequence of wave functions \( \nu_n(t) \) converges, thanks to quantum ergodicity, to
\[ \nu_\infty(t) = \lim_{n \to \infty} \nu_n(t) = \sin^2(\pi t) \]  
(35)

which has a limiting maximum
\[ \lim_{n \to \infty} \max_{0 < t < 1} \nu_n(t) = \max_{0 < t < 1} \nu_\infty(t) = \lim_{n \to \infty} \nu(2n) = \frac{8}{\pi} \approx 2.546479089470 \ldots \]  
(36)

and the associated differential is
\[ \chi_\infty(t) = \lim_{n \to \infty} \chi_n(t) = \frac{4}{\pi^2} \frac{d}{dt} \nu_\infty(t) = \frac{4}{\pi^2} \sin^2(\pi t) = -8 \sin(\pi t) \cos(\pi t) \]  
(37)

It is worth mentioning that the pair-correlation function for the zeros of \( \zeta(s) \) is
\[ r_2(x) = 1 - \frac{\sin^2(\pi x)^2}{\pi^2 x^2} \]

if the Riemann hypothesis is true. [11] The point \( n = 1 \) is where \( \chi_n \left( -\frac{1}{2} \right) \) and \( \nu_n \left( -\frac{1}{2} \right) \) attain their greatest values among the integers.

| \( n \) | \( \nu_n \left( -\frac{1}{2} \right) \) | \( \chi_n \left( -\frac{1}{2} \right) \) |
| --- | --- | --- |
| 1 | 117.43532857805377782 | 9447.7593604718560 |
| 2 | 003.0332065462255410 | 0000.2816402783351 |
| 3 | 002.77176105375846239 | 0000.0589526080385 |
| 4 | 002.69653220844944185 | 0000.0240373109008 |
| 5 | 002.66095375057354766 | 0000.0132398305603 |
| 6 | 002.63970550787458574 | 0000.0088555925215 |
| 7 | 002.62532107269483772 | 0000.006058883634 |
| \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( \infty \) | \( \frac{8}{\pi} \) | \( 0 \) |

Table 1: Even values of \( \nu(n) \) and \( \chi(n) \)

This phenomena of having the first wave function having a much larger size than all of the remainders is mentioned in [Kna99 Theorem 22] The convergence of \( \lim_{n \to \infty} \nu_n(t) \to \nu_\infty(t) \) and
\[ \lim_{n \to \infty} \chi_n(t) \to \chi_\infty(t) \] might be interpreted as a manifestation of quantum ergodicity \cite{[11]} C p.16].

There is an essential singularity \cite{[CB90]} p.169] at \( s_0 \approx 1.98757823 \ldots \)

\[ \lim_{s \to s_0} \chi(s) = \infty \] (38)

We also have the limits

\[ \lim_{s \to \{0,1\}} \dot{\chi}(s) = 4\pi \] (39)

The integral over this sine-squared kernel is

\[ \int_0^1 \sin^2(\pi t) \, dt = \frac{1}{2} \] (40)

whereas

\[ \int_0^1 \frac{\nu(t)}{\nu\left(\frac{1}{2}\right)} \, dt \approx 0.46693755153559653755 \ldots \] (41)

Figure 3: Top Left: The function \( \nu(s) \) normalized by its maximum value on \([0, 1]\) Top Right: The function \( \nu(s) \) normalized by its maximum value and subtracted from \( \sin^2(\pi x) \) Bottom Left: Convergence of \( \chi_n(t) \to -8 \cos(\pi t) \sin(\pi t) \) Bottom Right: Convergence of \( \nu_n(t) \to \sin^2(\pi t) \frac{\pi}{8} \)
2 Appendix

2.1 Wave Mechanics

2.2 The One-Dimensional Wave Equation

The one-dimensional wave equation is

\[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}u(x, t) + V(x, t)u(t, x) = -\frac{\hbar}{i}\frac{\partial}{\partial t}u(x, t)\] (42)

[Flü94 II.A.4 p.25] [Gan06 4.2.1 p.] [Coo03 Ch. 8 p.147]

2.3 The Poisson Bracket and Lagrangian Mechanics

“The” Poisson bracket, expressed with Einstein summation convention (for the repeated index \(i\))

\[
\{A, B\} \equiv \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} - \frac{\partial B}{\partial p_i} \frac{\partial A}{\partial q_i} \] (43)

has the antisymmetry property

\[
\{A, B\} = -\{B, A\} \] (44)

and the so-called Jacobi identity

\[
\{A, \{B, C\}\} - \{B, \{A, C\}\} + \{C, \{A, B\}\} = 0 \] (45)

Two quantities \(A\) and \(B\) are said to commute if their Poisson bracket \(\{A, B\}\) vanishes, that is, \(\{A, B\} = 0\). Hamilton’s equations of motion for the system

\[
\dot{p}_i = -\frac{H}{q_i} \] (46)

\[
\dot{q}_i = \frac{\partial H}{\partial p_i} \] (47)

where \(H\) is a Legendre-transformed function of the Lagrangian called the Hamiltonian

\[
H = \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L(q_i, \dot{q}_i, t) \] (48)

whose value for any given time \(t\) gives the energy

\[
A[q_i] = \int_{t_a}^{t_b} L(q_i, \dot{q}_i, t)dt \] (49)

of the system where \(L(q_i, \dot{q}_i, t)\) is the Lagrangian of the system and

\[
q_i(t) = q_i^{cl}(t) + \delta q_i(t) \] (50)

is an arbitrary path where \(q_i^{cl}(t)\) is the classical orbit or classical path of the system and

\[
\delta q_i(t_a) \equiv q_i(t) - q_i^{cl}(t) \] (51)

[Kle04 1.1]
2.3.1 The Euler-Lagrange equation

The Euler-Lagrange equation
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} \] (52)
indicates that the action \( S \) given by
\[ S = \int_{t_1}^{t_2} L(t)dt \] (53)
where
\[ L(t) = T - V(x(t)) \] (54)
is the Lagrangian and
\[ T(t) = \frac{1}{2} m \dot{x}(t)^2 \] (55)
is the kinetic energy which is stationary for the physical solutions \( q_i(t) \)\(^{[\text{Gan06 4.2.1}]}\).

2.3.2 Quantum Mechanics of General Lagrangian Systems

The coordinate transformation
\[ x^i = x^i(q^\mu) \] (56)
implies the relation
\[ \partial_\mu = \frac{\partial}{\partial q_\mu} = e_\mu^i(q) \partial_i \] (57)
between the derivatives \( \partial_\mu \) and
\[ \partial_i \equiv \frac{\partial}{\partial x^i} \] (58)
where
\[ e_\mu^i(q) \equiv \partial_\mu x^i(q) \] (59)
is a transformation matrix called the basis \( p \)-ad where \( p \) is the prefix corresponding to \( n \), the dimension of \( x \), monad when \( n = 1 \), dyad when \( n = 2 \), triad when \( n = 3 \), and so on. Let
\[ e_i^\mu(q) = \frac{\partial q_\mu}{\partial x^i} \] (60)
be the inverse matrix called the reciprocal \( p \)-ad. The basis \( p \)-ad and its reciprocal satisfy the orthogonality and completeness relations
\[ e_i^\mu e^i_\nu = \delta_\nu^\mu \] (61)
and
\[ e_i^\mu e^\mu_j = \delta_j^i \] (62)
The inverse of \( \partial_\mu \) is
\[ \partial_i = e_i^\mu(q) \partial_\mu \] (63)
which is related to the curvilinear transform of the Cartesian quantum-mechanical momentum operators by
\[ \hat{p}_i = -i\hbar \partial_i = -i\hbar e_i^\mu(q) \partial_\mu \] (64)
The Hamiltonian operator for free particles is defined by

\[ \hat{H}_0 = \hat{T} = \frac{1}{2M} \hat{p}^2 = -\frac{\hbar^2}{2M} \Delta \]  

(65)

where its metric tensor is given by

\[ g^{\mu\nu}(q) \equiv e_i^\mu(q)e_i^\nu(q) \]  

(66)

and its inverse by

\[ g_{\mu\nu}(q) \equiv e^i_\mu(q)e^i_\nu(q) \]  

(67)

The Laplacian of a metric tensor is then expressed

\[ \Delta = \partial_i^2 \]

(68)

where

\[ \Gamma^\lambda_{\mu\nu}(q) = -e_i^\varepsilon(q)\partial_\mu e^\lambda_i(q) = e_i^\lambda(q)\partial_\nu e^\lambda_i(q) \]  

(69)

is the affine-connection [Kle04, 1.13][Pod28]

2.3.3 Noether’s Theorem and Lie Groups

From Noether’s theorem it is known that continuous symmetries have corresponding conservation laws. [Gan06, 4.2.1] Let \( \alpha_s \) be a continuous family of symmetries which is a 1-parameter subgroup \( s \mapsto \alpha_s \) in the Lie group of symmetries. A Lie group is a group whose operations are compatible with the smooth structure. A smooth structure on a manifold allows for an unambiguous notion of smooth function.

2.4 Schrödinger’s Time-Dependent Equation and Nonstationary Wave Motion

The function \( \cos^2(t) = 1 - \sin^2(t) \) arises in the Dirac phase averaging method of calculating transition probabilities of non-stationary states in the time-dependent Schrödinger equation. [Gol61, 15.1] 3

2.4.1 Operators and Observables: Dirac’s Time-Dependent Theory

Let

\[ y(x,t) = A \cos \left( \frac{2\pi px}{\hbar} \right) \cos \left( \frac{2\pi \epsilon t}{\hbar} \right) \]

where

\[ \epsilon = \hbar \nu \]  

(71)
which relates the energies of a “matter wave” system, whatever that is, maybe a fermionic system, to the frequencies of quanta it emits or absorbs. There is always some arbitrariness with choice of units. This wave function has the symmetry

$$\frac{\partial^2 y(x,t)}{x^2} = - \left(\frac{2\pi}{\hbar}\right)^2 p^2 y(x,t)$$  \hspace{1cm} (72)$$

so that the kinetic energy of a particle is obtained from the wave function

$$- \frac{(2\pi)^2}{2m} \frac{\partial^2 y(x,t)}{x^2} = \frac{p^2}{2m} y(x,t)$$  \hspace{1cm} (73)$$

which means formally that

$$\frac{1}{2m} (p)^2 y(x,t) = - \frac{1}{2m} \left(\frac{2\pi}{i} \frac{\partial}{\partial x}\right)^2$$  \hspace{1cm} (74)$$

which suggests the identification of the Schroedinger momentum operator

$$\left(\frac{2\pi}{i} \frac{\partial}{\partial x}\right) \leftrightarrow p$$  \hspace{1cm} (75)$$

The equation for “nonfree” particles is augmented by

$$\frac{m}{2} \left(\frac{2\pi}{i} \frac{\partial}{\partial x}\right)^2 y(x,t) = [\epsilon - V(x)]y(x,t)$$  \hspace{1cm} (76)$$

where \(\epsilon\) is the total energy of the particle and \(V(x)\) is its potential energy. The first Schroedinger equation is then expressed

$$\epsilon y(x,t) = \left[ \frac{1}{2m} \left(\frac{2\pi}{i} \frac{\partial}{\partial x}\right)^2 + V(x) \right] y(x,t) = \left[ \frac{1}{2m} (p)^2 + V(x) \right] y(x,t)$$  \hspace{1cm} (77)$$

or

$$H y(x,t) = \epsilon y(x,t)$$

where \(H\) is the operator corresponding to the Hamiltonian of the (point) particle. [Gol61] 11.5]

The time-dependent Schroedinger equation is written

$$\frac{\partial^2 (x,t)}{\partial t^2} = - \frac{1}{(2\pi)^2} y(x,t) = - \left(\frac{H}{2\pi}\right)^2 y(x,t)$$  \hspace{1cm} (78)$$

or equivalently as

$$0 = \left( (2\pi)^2 \frac{\partial^2}{\partial t^2} + H^2 \right) y(x,t) = \left( H + i2\pi \frac{\partial}{\partial t} \right) \left( H - i2\pi \frac{\partial}{\partial t} \right) y(x,t)$$  \hspace{1cm} (79)$$

where

$$H y(x,t) = i2\pi \frac{\partial}{\partial t} y(x,t)$$  \hspace{1cm} (80)$$

It is also worth mentioning that in Dirac’s theory of the time-dependent Schroedinger there is a Hamiltonian of the form

$$H = H_0 + \lambda H_1(t)$$  \hspace{1cm} (81)$$
which has a transition probability per unit time of

\[ w(p \rightarrow -p) = \frac{V_0^2 \sin^2 \left( \frac{2l}{\sqrt{2m\hbar^2}L} \right)}{2\sqrt{8m\epsilon\hbar^2}} \]

where \( V_0, \epsilon, h, \) and \( L \) are suitable constants. [Gol61 15.5]

### 2.4.2 The String Theoretic Partition Function

The string theoretic partition function is defined as

\[ Z_{st}(\omega) = Z_{st,R}(\omega) = \text{Tr}[e^{2\pi i \omega_1 p^+ - 2\pi i \omega_2 H}] = (q\bar{q})^{-\frac{1}{24}} \text{Tr}(q^{L_0} \bar{q}^{\bar{L}_0}) \]  

where \( L_0^+ \) and \( L_0^- \) are the Virasoro generators defined by

\[ L_0^\pm = \frac{(p^+)^2 + (p^-)^2}{2} + \sum_{n=1}^{\infty} \alpha_n^\pm \alpha_n^\pm \]

and \( \alpha_n^\pm \) are related to something called a Fubini-Veneziano field and \( p^+ \) is the left-momentum operator and \( p^- \) is right-momentum operator. [Lap08 2.2.1]

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