Optimal Multiple Stopping with Negative Discount Rate and Random Refraction Times under Lévy Models

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Abstract
This paper studies a class of optimal multiple stopping problems driven by Lévy processes. Our model allows for a negative effective discount rate, which arises in a number of financial applications, including stock loans and real options, where the strike price can potentially grow at a higher rate than the original discount factor. Moreover, successive exercise opportunities are separated by i.i.d. random refraction times. Under a wide class of two-sided Lévy models with a general random refraction time, we rigorously show that the optimal strategy to exercise successive call options is uniquely characterized by a sequence of up-crossing times. The corresponding optimal thresholds are determined explicitly in the single stopping case and recursively in the multiple stopping case.

Key words: optimal multiple stopping; negative discount rate; random refraction times; Lévy processes; stock loan
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Mathematics Subject Classification (2010): 60G40, 60J75

1 Introduction
We study a class of optimal multiple stopping problems driven by an underlying Lévy process. Two key features of our model are that (i) the discount rate can be negative or positive, and (ii) the sequence of admissible stopping times are separated by i.i.d. random refraction periods. The negative effective discount rate is relevant to a number of financial applications. For example, Xia and Zhou [26] propose a valuation model for a stock loan, where the loan interest rate is higher than the risk-free interest rate. As a result, the stock loan can be viewed as an American call option with a negative effective discount rate. An example from the real option literature [15, 21] is when the cost of investment grows at a higher rate than the firm’s discount rate. Moreover, while the nominal short rate cannot be negative, the real interest rate can potentially be negative, especially during low-yield regimes, according to Black [6] and references therein. Therefore, extending the discount rate to the negative domain also enables the evaluation of cash flows under the real interest rate.

In the aforementioned applications, the same option can be exercised repeatedly in the future, meaning that an investor can acquire a series of stock loans, or a firm can make an investment sequentially over time. This motivates us to incorporate multiple stopping opportunities in our analysis. The features of refraction periods and multiple exercises also arise in the pricing of swing options commonly used for energy delivery. For instance, Carmona and Touzi [9] formulate the valuation of a swing put option as optimal multiple stopping problem, with constant refraction periods, under the geometric Brownian motion model. In a related study, Zeghal and Mnif [27] value a perpetual American swing put when the underlying Lévy price process has no negative jumps. They provide mathematical characterization and numerical solutions to the associated optimal multiple stopping problem. In contrast, we consider the successive

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exercises of a swing call option with random refraction times under positive or negative discount rate. We also provide a rigorous analysis of the optimal multiple stopping problem under two-sided Lévy models.

Under a wide class of two-sided Lévy models with a general random refraction time, we show that the optimal exercises of multiple perpetual call options are characterized by a non-increasing sequence of exercise thresholds (see Proposition 2.2 and Theorem 3.2 below). The corresponding optimal thresholds are determined explicitly in the single stopping case and recursively in the multiple stopping case. Our results extend the stock loans models by Xia and Zhou [26] as well as Cai and Sun [7] from their single stock loan to sequential stock loans, and from a geometric Brownian motion [26] and double-exponential jump diffusion [7] models to a class of general two-sided Lévy processes in our paper. As such, the minimal assumptions on the refraction times and the underlying Lévy process prevent the use of model/distribution-specific properties that are amenable for analysis and computation. We overcome this challenge through the use of Laplace transform, change of measure, martingale theory, along with other analytical techniques. Our analysis allows for the recursive computation of the optimal value function as well as all exercise thresholds, thus providing an alternative to the simulation approach commonly found in existing literature for multi-exercise options (see [5, 22], among others). We also examine the impact of refraction time distribution on the optimal exercise thresholds.

In our model, the random refraction times between consecutive exercise opportunities can also be interpreted as a result of successive randomization. To this end, Kyprianou and Pistorius [17] apply fluctuation theory of Lévy processes to study the method of maturity randomization (Canadization) for derivatives pricing. The randomization procedure turns a finite-maturity option into a perpetual one. Avram et al. [4] consider a number of exit problems of spectrally negative Lévy processes, and apply them to value Russian options with a randomized maturity. In contrast, we consider a problem with multiple exercise rights, allowing for a negative discount rate, as well as negative and positive jumps for the underlying Lévy process.

The recent work by Christensen and Lempa [11] discuss an optimal multiple stopping problem driven by a strong Markov process with i.i.d. exponential refraction periods. Another related work by Christensen et al. [12] study an optimal multiple stopping problem with random waiting times in terms of a sequence of single stopping problems. They provide an explicit solution to the problem of a perpetual put option whereby the sequential exercises are refracted by the first passage times of a geometric Brownian motion. Compared to their work, our model not only allows for a negative discount rate and general random refraction times, but also incorporates jumps in the underlying process via a Lévy process with positive phase-type jumps and negative jumps from any distribution. As discussed in [3], Lévy processes with phase-type jumps are capable of approximating a general class of Lévy processes. Herein, the major mathematical challenge is to characterize the optimal exercise strategies given minimal distributional structures of the Lévy jumps and refraction times.

The current paper is also relevant to the growing number of financial applications that involve making sequential timing decisions. Examples include multiple-exercise options [8, 22], portfolios of employee stock options [11, 16, 19], sequential infrastructure investments [10, 13], as well as reload and shout options [14]. Since some of these applications also involve a sequence of perpetual call options, our analysis is directly applicable and provides an extension to discounting with a negative rate.

Let us provide an outline of the paper. In Section 2, we formulate the optimal multiple stopping problem and present some general mathematical properties. In Section 3, we analyze both the single and multiple stopping problems driven by a two-sided Lévy process. Section 4 discusses the numerical implementation and provides some illustrative numerical examples. Section 5 concludes the paper. Our proofs, constituting a substantial part of the paper, are included in the Appendix.

2 Problem Formulation and General Properties

In the background, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ hosting a Lévy process $X = (X_t)_{t \geq 0}$ characterized uniquely by its Laplace exponent

$$\psi(\beta) := \log \mathbb{E}[e^{\beta X_1}] = c\beta + \frac{1}{2} \sigma^2 \beta^2 + \int_{(-\infty,\infty)} (e^{\beta z} - 1 - \beta z 1_{\{|z|<1\}}) \Pi(dz),$$

(2.1)

for every $\beta \in \mathbb{C}$ such that $0 \leq \Re \beta < \beta_0$ (with $\Re z$ the real part of $z \in \mathbb{C}$) where

$$\beta_0 := \sup\{\beta \in \mathbb{R} : \mathbb{E}[e^{\beta X_1}] < \infty\},$$

(2.2)
for some $c \in \mathbb{R}$, $\sigma \geq 0$, and a measure $\Pi$ with its support $\mathbb{R}\setminus \{0\}$ such that

$$\int_{(-\infty,\infty)} (1 \wedge z^2)\Pi(dz) < \infty. \quad (2.3)$$

We comment that the Laplace exponent $\psi$ can be extended beyond the line $\Re\beta = \beta_0$ by analytical continuation. Throughout, we assume that $\beta_0 > 1$, and $-X$ is not a subordinator.

We denote $P_x$ as the probability and $E_x$ as the expectation with initial value $X_0 = x \in \mathbb{R}$. When $X_0 = 0$, we drop the subscripts in $P_x$ and $E_x$. Let $F := (F_t)_{t \geq 0}$ be the natural filtration generated by $X$. The underlying price process is modeled by an exponential Lévy process $S_t := e^{X_t}$, $t \geq 0$.

Now we describe our optimal multiple stopping problem with a refraction period between consecutive exercises. In the general setting, we can take the refraction period $\delta$ as a deterministic constant or a positive random variable. We will assume throughout the paper that the distribution of the random variable $X_\delta$ has no atoms.

Denote by $T$ the set of $\mathcal{F}$-stopping times. However, the incorporation of random refraction times requires us to expand the filtration. To any collection $\Xi$ of positive random variables, we denote $\mathcal{F}(\Xi)$ to be the smallest filtration such that all members of $\Xi$ are stopping times (see [12]). For each fixed $n \geq 1$, we introduce the set of admissible sequence of exercise times:

$$T^{(n)} := \{ \bar{\tau} = (\tau_n, \cdots, \tau_1) : \tau_n \in T, \tau_i \in \mathcal{F}(\{\tau_j + \delta_j\}_{j=i+1}^n) \text{ - stopping time, }$$

$$\text{and } \tau_{i+1} + \delta_{i+1} \leq \tau_i, i = n - 1, \cdots, 1\},$$

where $\delta_i$’s are i.i.d. copies of some positive-valued random variable $\delta$, which are independent of the Lévy process $X$. The stopping time $\tau_i$ is an admissible exercise time when there are $i$ exercise opportunities left. In particular, $\tau_n$ is the first exercise time and $\tau_1$ is the last one.

Throughout we will work with the reward function

$$\phi(x) := (e^x - K)^+, \quad \forall x \in \mathbb{R},$$

where we call $K > 0$ the strike price. With $n \geq 1$ exercise opportunities, the optimal stopping problem is defined as

$$v^{(n)}(x) := \sup_{\bar{\tau} \in T^{(n)}} E_x \left[ \sum_{i=1}^n e^{-\alpha \tau_i} \phi(X_{\tau_i}) 1_{\{\tau_i < \infty\}} \right], \quad \forall x \in \mathbb{R}. \quad (2.4)$$

We impose a standing technical integrability condition to ensure that the problem is well defined.

**Assumption 2.1.** There exists a constant $\varrho > 1$, such that the Lévy process $X$ satisfies

$$E_x \left[ \left( \sup_{0 \leq t < \infty} e^{-\alpha t} \phi(X_1) \right)^{\varrho} \right] < \infty, \quad \forall x \in \mathbb{R}. \quad (2.5)$$

In Section 3, we will provide the conditions on $\alpha$ in Assumption 3.1 so that this integrability condition will hold.

One key feature of our model is that the constant parameter $\alpha$ can be taken to be positive/negative, representing a discounting/inflating factor. In the stock loan model proposed by Xia and Zhou [26], the negative effective discount rate arises when the rate charged by the bank $\gamma$ is higher than the interest rate $r$. To see this, we consider an investor who borrows amount $K$ from a bank, using a share of stock $S$ as collateral. The borrower has the right to redeem the stock by paying the accrued principle $Ke^{rt}$ at any time $t \geq 0$. Hence, we write the expected discounted payoff as $E_x[e^{\tau r}(S_	au - e^{\tau r} K)^+] = E_x[e^{-(r-\gamma)\tau}(\bar{S}_\tau - K)^+]$ for $\tau \in T$, where $\alpha = r - \gamma < 0$ and $\bar{S}_\tau = e^{-\gamma \tau} S_\tau$. In a different class of applications, the negative effective discount rate is also relevant to real option exercise timing when the investment cost $K$ grows at a rate $\gamma$ that is higher than the firm’s discount rate $r$.

In order to solve the optimal multiple stopping problem (2.4), we will establish its equivalence to the following recursion of optimal single stopping problems:

$$v^{(k)}(x) := \sup_{\tau \in T} E_x \left[ e^{-\alpha \tau} \phi^{(k)}(X_\tau) 1_{\{\tau < \infty\}} \right] \quad (2.6)$$

where

$$\phi^{(k)}(x) := \phi(x) + E_x \left[ e^{-\alpha \delta} v^{(k-1)}(X_\delta) \right], \quad k = 1, 2, \cdots, n, \quad (2.7)$$

and $v^{(0)}(x) := 0$. To this end, we first present some useful properties of the value function $v^{(k)}$ for every $k \in \{1, \cdots, n\}$.}

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Lemma 2.1. For every integer \( k \in \{1, \cdots, n\} \) and all \( s \in \mathbb{R}_+ := (0, \infty) \), the function \( U^{(k)}(s) := v^{(k)}(\log s) \) is non-decreasing, convex, and hence differentiable almost everywhere on \( \mathbb{R}_+ \).

As a result of the convexity and monotonicity, we obtain the existence and uniqueness of the point of continuous fit for \( k = 1 \). More specifically, using arguments as in the proof of Corollary 3.1 of [26], we know that there exists a level \( x_1^* \in (\log K, \infty] \) such that \( v^{(1)}(x) = \phi(x) > 0 \) if and only if \( x \geq x_1^* \). Note that we can without loss of generality rule out the possibility of \( x_1^* \leq \log K \) (exercising out of the money).

For any \( b \in \mathbb{R} \), we denote by \( \tau_b^+ \) the first up-crossing time

\[
\tau_b^+ = \inf\{ t \geq 0 : X_t \geq b \}.
\]

Here and throughout the paper we define \( \inf \emptyset := \infty \). Furthermore, for every \( k \in \{1, \cdots, n\} \), we define the value of discounted payoff of a threshold strategy \( \tau_b^+ \in \mathcal{T} \) as

\[
g^{(k)}(x, b) := \mathbb{E}_x \left[ e^{-\alpha \tau_b^+} \phi^{(k)}(X_{\tau_b^+}) 1_{\{\tau_b^+ < \infty\}} \right], \quad \forall x \in \mathbb{R}.
\]

When \( x_1^* < \infty \), we know that the value function of the auxiliary problem (2.6) for \( k = 1 \) is given by

\[
v^{(1)}(x) = g^{(1)}(x, x_1^*), \quad \forall x \in \mathbb{R}.
\]

When \( x_1^* = \infty \), the problem is trivial and the value function \( v^{(1)}(x) \) is approximated by the expected value under \( \tau_1^* \) by taking \( M \) arbitrarily large. We shall now assume the former and give sufficient conditions so that similar results hold for the problem (2.6) for \( k \in \{2, \cdots, n\} \). To this end, we adapt the arguments from the proof of Lemma 3.2 of [9] to obtain the following result.

Lemma 2.2. Suppose \( x_1^* \in (\log K, \infty) \). Then for every \( 1 \leq k \leq n \) and all \( x \in [x_1^*, \infty) \), we have \( \phi^{(k)}(x) = v^{(k)}(x) \).

Lemma 2.2 implies that \( U^{(k)}(s) \) will eventually continuously fit \( \phi^{(k)}(\log s) \) as \( s \) increases. By the convexity of \( U^{(k)}(s) \equiv v^{(k)}(\log s) \) and \( \phi^{(k)}(\log s) \), we know that \( U^{(k)}(s) \) is almost everywhere bounded from above by \( \text{esssup}_{s \in \mathbb{R}_+} \{ \frac{\partial}{\partial s} \phi^{(k)}(\log s) \} \). In turn, we deduce that \( \mathbb{E}[e^{-\alpha \delta} v^{(k)}(x + X_\delta)] \) is differentiable in \( x \) on \( \mathbb{R} \) since the distribution of \( X_\delta \) does not charge a positive measure on the (at most) countable points where \( v^{(k)} \) is not differentiable.

Corollary 2.1. Suppose \( x_1^* \in (\log K, \infty) \) and \( \mathbb{E}[e^{-\alpha \delta + X_\delta}] \leq 1 \). For every integer \( k \in \{1, \cdots, n\} \), we have 0 \( \leq v^{(k)}(x) \leq ke^x \), a.e. and

\[
0 \leq \frac{\partial}{\partial x} \mathbb{E}[e^{-\alpha \delta} v^{(k)}(x + X_\delta)] = \mathbb{E}[e^{-\alpha \delta} v^{(k)}(x + X_\delta)] \leq ke^x, \quad \forall x \in \mathbb{R}.
\]

We now establish the equivalence between (2.4) and (2.6). Let us first recursively define the set of stopping times

\[
\tau_n^* := \inf\{ t \geq 0 : v^{(n)}(X_t) = \phi^{(n)}(X_t) \}, \quad \tau_i^* := \inf\{ t \geq \delta_{i+1} + \tau_{i+1}^* : v^{(i)}(X_t) = \phi^{(i)}(X_t) \}, \quad \text{for } i = n - 1, \cdots, 1.
\]

We show below that \( (\tau_n^*, \cdots, \tau_1^*) \in \mathcal{T}^{(n)} \) solve the optimal multiple stopping problems (2.4) and (2.6)-(2.7).

Theorem 2.1. Suppose \( x_1^* \in (\log K, \infty) \). Fix a \( k \in \{1, \cdots, n\} \), then the stopping times \( (\tau_i^*)_{1 \leq i \leq k} \) defined in (2.9)-(2.10) satisfy \( \mathbb{P}_x(\tau_i^* < \infty, 1 \leq i \leq k) > 0 \), for all \( x \in \mathbb{R} \). Moreover,

(i) the value function of the auxiliary problem, \( v^{(k)} \) of (2.6), satisfies

\[
v^{(k)}(x) = \mathbb{E}_x \left[ \sum_{i=1}^{k} e^{-\alpha \tau_i^*} \phi(X_{\tau_i^*}) 1_{\{\tau_i^* < \infty\}} \right], \quad \forall x \in \mathbb{R};
\]

(ii) the value function \( \tilde{v}^{(k)}(x) \) of (2.6) is equal to \( v^{(k)}(x) \) of (2.4) for every \( x \in \mathbb{R} \);

(iii) for every initial value \( X_0 = x \in \mathbb{R} \), all the random variables in the collection

\[
\mathcal{S}^{(k)} := \{ e^{-\alpha \tau} v^{(k)}(X_{\tau}) : \tau \text{ is an a.s. finite } \mathbb{F} \text{-stopping time} \}
\]

are uniformly bounded in \( L^\varrho(d\mathbb{P}_x) \).
We now return to the optimal multiple stopping problem (2.6)-(2.7). From Lemma 2.2 we know that, if \( x^*_k \in (\log K, \infty) \), then for every \( k \in \{2, \cdots, n\} \), we can define a finite level
\[
x^*_k := \inf \{ x \leq x^*_k : v^{(k)}(y) = \phi^{(k)}(y), \ \forall y \geq x \}.
\]
Then, for every \( k \in \{1, \cdots, n\} \), the interval \([x^*_k, \infty)\) must be one connected domain of the optimal stopping region for problem (2.6) with \( k \) exercise opportunities. It should be noted that, in general for \( k \geq 2 \), the optimal stopping region can potentially be disconnected, consisting of multiple disjoint intervals, as the composite payoff function \( \phi^{(k)}(\log s) \) is no longer piecewise linear in \( s \). \( \phi \) can potentially be disconnected, consisting of multiple disjoint intervals, as the composite payoff function \( \phi^{(k)}(\log s) \) is no longer piecewise linear in \( s \). However, if \([x^*_k, \infty)\) is the only optimal stopping region, then the up-crossing time \( \tau^+_k \) must be the optimal stopping time to problem (2.6), and \( v^{(k)}(x) = g^{(k)}(x, x^*_k) \) for all \( x \in \mathbb{R} \).

To resolve the issue of possible multiple disconnected components of optimal stopping for \( k \geq 2 \), we consider the best threshold type strategy, among all first up-crossing times \( \{\tau^+_b : b \in \mathbb{R}\} \), and then give a sufficient condition for its optimality.

**Definition 2.1.** We call a level \( b^*_k \in \mathbb{R} \) the optimal exercise threshold for problem (2.6) with \( k \) exercise opportunities, if and only if the function \( g^{(k)}(x, b) \) is maximized at \( b = b^*_k \) for all \( x \in \mathbb{R} \). More specifically, if \( b^*_k \) satisfies the following:

(a) For any fixed \( x < b^*_k \), the supremum of the function \( g^{(k)}(x, \cdot) \) is given by \( g^{(k)}(x, b^*_k) \);

(b) For any fixed \( x \geq b^*_k \), the supremum of the function \( g^{(k)}(x, \cdot) \) is given by \( \phi^{(k)}(x) \).

When \( k = 1 \), we know that \( b^*_1 = x^*_1 \) if the latter is finite. Notice that, for a general \( k \geq 2 \), the optimal exercise threshold \( b^*_k \) may not exist. The following result characterizes the relationship between \( x^*_k \) and \( b^*_k \), when the latter exists.

**Proposition 2.1.** Suppose \( x^*_k \in (\log K, \infty) \). Fix an integer \( k \in \{2, \cdots, n\} \), assume that \( v^{(k-1)}(x) > v^{(k-2)}(x) \) for all \( x \in \mathbb{R} \), and that \([x^*_{k-1}, \infty)\) is the only optimal stopping region for problem (2.6) with \((k-1)\) exercise opportunities. Then we have

(i) \( v^{(k)}(x) > v^{(k-1)}(x) \) for all \( x \in \mathbb{R} \);

(ii) \( x^*_k \in (\log K, x^*_1) \), and if \( b^*_k \) exists, we also have \( b^*_k > \log K \);

(iii) If \( b^*_k \) exists and the process \( (e^{-\alpha t}g^{(k)}(X_t, b^*_k))_{t \geq 0} \) is a \((\mathbb{P}, \mathcal{F})\)-supermartingale, then \( x^*_k = b^*_k \) and \([x^*_k, \infty)\) is the only optimal stopping region. Hence, the up-crossing time \( \tau^+_k \) is optimal.

**Remark 2.1.** Proposition 2.1 implies that each value function can be determined by first optimizing the expected reward over all candidate thresholds that are above \( \log K \), followed by verifying the supermartingale property. Consequently, as far as the optimal thresholds are concerned, we can effectively remove the \( \pm \) sign in the payoff function \( \phi(x) \). From Proposition 2.1 we conclude that \([x^*_k, \infty)\) is the only optimal stopping region, and each optimal exercise threshold \( b^*_k = x^*_k \) exists and is bounded above by \( x^*_1 \).

We now show that if \([x^*_k, \infty)\) is the only connected optimal stopping region for all \( 1 \leq k \leq l \) for some \( l \in \{1, \cdots, n\} \), then \((x^*_k)_{1 \leq k \leq l}\) is non-increasing in \( k \). To show this, we first prove that the process
\[
V^{(k-1)}_t := e^{-\alpha t} \left( v^{(k-1)}(X_t) - v^{(k-2)}(X_t) \right), \quad t \geq 0,
\]
is a supermartingale for any fixed \( k \in \{2, \cdots, l+1\} \).

**Proposition 2.2.** Suppose that \( x^*_1 \in (\log K, \infty) \) and that \([x^*_1, \infty)\) is the only connected optimal stopping region for all \( 1 \leq k \leq l \) for some \( l \in \{2, \cdots, n-1\} \), then the sequence of optimal exercise thresholds \((x^*_k)_{1 \leq k \leq l+1}\) is non-increasing in \( k \), i.e.
\[
\log K < x^*_{l+1} \leq x^*_1 \leq \cdots \leq x^*_1,
\]
and the process \((V^{(k-1)}_t)_{t \geq 0}\) is a \((\mathbb{P}, \mathcal{F})\)-supermartingale of class \((D)\) [24, Chap. 3] for any \( 2 \leq k \leq l+1 \).

Proposition 2.2 tells us, if threshold type strategies are optimal for problem (2.6) with \( k \) exercise opportunities for all \( 1 \leq k \leq l \), then the optimal exercise thresholds \((x^*_k)_{1 \leq k \leq l}\) are non-increasing in \( k \). Hence, even if threshold type strategies are not optimal for problem (2.6) with \( l+1 \) exercise opportunities, the optimal stopping region should contain \([x^*_1, \infty)\). Moreover, for any number of remaining exercise opportunities, it is always optimal to exercise above the strike price.
3 Analytical Results

In this section, we assume that $X$ is either a spectrally negative Lévy process that is not the negative of a subordinator, or a Lévy process with an arbitrary negative jump distribution and a positive phase-type jump distribution [3]:

$$X_t - X_0 = J_t + \sum_{n=1}^{N_t} Z_n, \quad 0 \leq t < \infty. \quad (3.13)$$

Here, $(J_t)_{t \geq 0}$ is a spectrally negative Lévy process with or without a Brownian motion component, $(N_t)_{t \geq 0}$ is a Poisson process with arrival rate $\rho$, and $Z = (Z_n)_{n=1,2,\ldots}$ is an i.i.d. sequence of phase-type-distributed random variables with representation $(d, \alpha, T)$. In addition, $J$ and $Z$ are mutually independent. For a comprehensive study on this process and its applications in American and Russian options, we refer the reader to [3].

Recall that a distribution on $\mathbb{R}_+$ is of phase-type if it is the distribution of the absorption time in a finite state continuous-time Markov chain consisting of one absorbing state and $d \in \mathbb{N}$ transient states. Thus, any phase-type distribution can be represented by $d$, the $d \times d$ transition intensity matrix over all transient states $T$, and the initial distribution of the Markov chain $\alpha$. Without loss of generality, we assume that the positive phase-type jump distribution is minimally represented with $d$ phases. From [3], this guarantees that the singularities of the Laplace exponent $\psi$ with positive real part are eigenvalues of $T$. Moreover, by Theorem 5b on p.58 of [25], we know that $\beta_0$ defined in (2.2) is the smallest positive pole of $\psi$ and $\lim_{\beta \downarrow 0} \psi(\beta) = \infty$. Henceforth, we impose the following technical condition.

**Assumption 3.1.** The Laplace exponent $\psi$ and the discount rate $\alpha$ satisfy either (i) $\psi(1) < \alpha$, or (ii) $\psi(1) = \alpha < 0$ and $\psi'(1) < 0$.

Under these conditions, we shall show in the lemma below that the optimal stopping problem in (2.6) is well-posed for each $1 \leq k \leq n$ in the sense that the integrability condition in Assumption 2.1 is met. Also, the discounted price process $(e^{-\rho t + X_t})_{t \geq 0}$ under $\mathbb{P}$ has to be a supermartingale, and not a martingale when $\alpha \geq 0$ [23, Theorem 1]. In effect, the trivial optimal strategies of perpetual waiting are excluded.

**Lemma 3.1.** Assumption 3.1 implies Assumption 2.1.

We provide a detailed proof in Appendix A.6. Next, for an $\alpha \in \mathbb{R}$, we define $\Phi(\alpha)$ to be the largest positive root of $\psi(\beta) = \alpha$, which is a real number less than $\beta_0$, if it exists. Notice that Assumption 3.1 and $\beta_0 > 1$ imply that $\Phi(\alpha)$ exists and $\Phi(\alpha) > 1$ and $\psi'(\Phi(\alpha)) \geq 0$. We denote the finite set of roots with positive real parts by

$$\mathcal{I}_\alpha = \{ \rho_{i,\alpha} : \psi(\rho_{i,\alpha}) = \alpha, \Re \rho_{i,\alpha} \geq \Phi(\alpha) \}_{1 \leq i \leq |\mathcal{I}_\alpha|}, \quad (3.14)$$

where, for the sake of mathematical convenience, we assume that the roots are all distinct for a given $\alpha \in \mathbb{R}$. It follows that $\rho_{1,\alpha} = \Phi(\alpha) < \beta_0$ and $\Re \rho_{i,\alpha} \geq \beta_0$ for all $i \geq 2$. Moreover, we remark that $|\mathcal{I}_\alpha| = d$ or $d + 1$ according to whether $-J$ is a subordinator or not, respectively [3, Lemma 1]. We label $\rho_{i,\alpha}$’s in such a way that $(\Re \rho_{i,\alpha}, \Im \rho_{i,\alpha})$ is in ascending lexicographic order (here $\Im z$ is the imaginary part of any complex number $z$). Similarly, we define a second set of roots with positive real part, labeled in the same way as elements of $\mathcal{I}_\alpha$:

$$\mathcal{J} := \{ \eta_j : \frac{1}{\psi'(\eta_j)} = 0, \Re \eta_j > 0 \}_{1 \leq j \leq |\mathcal{J}|}, \quad (3.15)$$

where multiple roots are counted individually. Notice that we have $\beta_0 = \eta_1 \in \mathcal{J}$ and $|\mathcal{J}| = d$.

**Remark 3.1.** If $X$ is a spectrally negative Lévy process, i.e. $|\mathcal{J}| = 0$, then, by our assumption that $-X$ is not a subordinator, we have $\mathcal{I}_\alpha = \{ \Phi(\alpha) \}$ and $|\mathcal{I}_\alpha| = 1$.

Fix $\alpha \geq 0$. Let $e_\alpha$ be an exponential random variable, with rate parameter $\alpha$, that is independent of $X$. We follow the convention that $e_0 = \infty$, $\mathbb{P}$-a.s. Then, it is known from Lemma 1 of [3] that the Laplace transform of $\mathbb{E} e^{-\beta X_{t \wedge e_\alpha}}$ is given by

$$\psi^+_{X}(\beta) := \mathbb{E} e^{-\beta X_{t \wedge e_\alpha}} = \prod_{i=1}^{|\mathcal{I}_\alpha|} \frac{\rho_{i,\alpha}}{\rho_{i,\alpha} + \beta} \prod_{j=1}^{|\mathcal{J}|} \left( 1 + \frac{\beta}{\eta_j} \right) = \psi^+_{\mathcal{J}}(\infty) + \sum_{i=1}^{|\mathcal{I}_\alpha|} A_i \frac{\rho_{i,\alpha}}{\rho_{i,\alpha} + \beta}, \quad \forall \beta \geq 0, \quad (3.16)$$

$6$
with \( \psi^+_{\alpha}(\infty) = \lim_{\beta \to \infty} \psi^+_{\alpha}(\beta) \), and the partial fraction coefficients:

\[
A_i := \prod_{j=1}^{[I_{\alpha}]} \frac{\rho_j,\alpha}{\rho_j,\alpha - \rho_i,\alpha} \prod_{j=1}^{[J]} \eta_j - \rho_i,\alpha, \quad (3.17)
\]

As a result, the distribution of \( \mathcal{X}_{\alpha_{\alpha}} \) is given by:

\[
P(\mathcal{X}_{\alpha_{\alpha}} \in dx) = \sum_{i=1}^{[I_{\alpha}]} A_i \rho_i,\alpha e^{-\rho_i,\alpha x} dx, \quad \forall x > 0.
\]  

Remark 3.2. As explained in [3, Remark 4], the assumption of distinct roots is made for convenience. When there are multiple roots, the corresponding distribution \( P(\mathcal{X}_{\alpha_{\alpha}} \in dx) \) will admit a form similar to that in (3.18) with different constant coefficients. Moreover, the case with multiple roots only occurs for at most countably many values of \( \alpha \) over \( \mathbb{R} \). In other words, if one arbitrarily sets the values of \( \gamma \) and \( r \), the probability of having multiple roots as a result is zero.

In the next subsection, we derive the value function and the optimal exercise threshold for the single stopping problem for any discount rate \( \alpha \) satisfying Assumption 3.1.

### 3.1 Optimal Single Stopping Problem

If \( \psi(1) < \alpha \) and \( \alpha \geq 0 \), then it is known from Theorem 1 of [23] that the optimal stopping time for an American call with strike price \( K > 0 \) is given by

\[
\tau^*_\psi = \inf \{ t \geq 0 : X_t \geq \log(K\psi^+_{\alpha}(-1)) \}
\]

and the value of the American call option is given by:

\[
E_x \left[ e^{-\alpha \tau^*_\psi} \phi(X_{\tau^*_\psi}) \right] = K \sum_{i=1}^{[I_{\alpha}]} [K\psi^+_{\alpha}(-1)]^{-\rho_i,\alpha} e^{\rho_i,\alpha x} \frac{A_i}{\rho_i,\alpha - 1}, \quad (3.19)
\]

where \( A_i \)'s are defined in (3.17). The analogous expectation in (3.19) for the case with \( \alpha < 0 \) can be computed using the sets \( I_{\alpha} \) in (3.14) and \( J \) in (3.15), which we shall prove in Proposition 3.1 below.

In order to address the case with a negative discount rate (\( \alpha < 0 \)), one of our main steps is to apply a change of measures. For \( \kappa \in [0, \beta_0) \), we define a new probability measure \( \mathbb{P}_x^\kappa \sim \mathbb{P}_x \) by

\[
\frac{d\mathbb{P}_x^\kappa}{d\mathbb{P}_x} \bigg|_{\mathcal{F}_t} = \exp(\kappa(X_t - x) - \psi(\kappa)t), \quad t \geq 0.
\]  

(3.20)

Then, for \( \beta > -\kappa \), the Laplace exponent of \( X \) is given by [18, Theorem 3.9]

\[
\psi_\kappa(\beta) := (\kappa \sigma^2 - c + \int_{(-1,1)} z(e^{\kappa z} - 1)\Pi(dz))\beta + \frac{1}{2}\sigma^2 \beta^2 + \int_{(-\infty,\infty)} (e^{\kappa z} - 1 - \beta \chi_{|z|<1})e^{\kappa z}\Pi(dz).
\]

Under the new probability measure \( \mathbb{P}_x^\kappa \), the process is also a Lévy process with a negative jump distribution and a positive phase-type distribution, with a new scaled Lévy measure \( \Pi_\kappa(du) := e^{\kappa u} \Pi(du) \).

**Proposition 3.1.** We extend the definition (3.16) for \( \alpha \leq 0 \) and define the function \( \psi^+_{\alpha}(\beta) \) and partial fraction coefficients \( (A_i \equiv A(\rho_i,\alpha))_{1 \leq i \leq [I_{\alpha}]} \) using

\[
\psi^+_{\alpha}(\beta) := \prod_{i=1}^{[I_{\alpha}]} \frac{\rho_i,\alpha}{\rho_i,\alpha + \beta} \prod_{j=1}^{[J]} \left( 1 + \frac{\beta}{\eta_j} \right) \equiv \psi^+_{\alpha}(\infty) + \sum_{i=1}^{[I_{\alpha}]} \frac{A_i}{\rho_i,\alpha + \beta}, \quad \forall \beta \in \mathbb{C}. \quad (3.21)
\]

Then for any fixed \( b > x \) and \( \beta \geq 0 \), we have

\[
E_x \left[ e^{-\alpha \tau^*_\psi - \beta(X_{\tau^*_\psi} - b)} \mathbb{1}_{\tau^*_\psi < \infty} \right] = \frac{1}{\psi^+_{\alpha}(\beta)} \sum_{i=1}^{[I_{\alpha}]} A_i \frac{\rho_i,\alpha}{\rho_i,\alpha + \beta} e^{-\rho_i,\alpha(b-x)}.
\]  

(3.22)
Using Proposition 3.1, we can compute the expected payoff of any threshold type strategy \( \tau_b^+ \). By Theorem 5b on p.58 of [25], we can extend \( \beta \) on both sides of (3.22) to a complex number as long as \( \Re \beta > -\rho_{1,\alpha} \). In particular, by setting \( \beta = -1, \, 0 > -\rho_{1,\alpha} \) in (3.22), we obtain the following result.

**Corollary 3.1.** For all \( b \geq \log K \) and \( x > b \) (and hence \( \phi(X_{\tau_b^+}) = e^{X_{\tau_b^+}} - K \) on \( \{ \tau_b^+ < \infty \} \)), we have

\[
 g^{(1)}(x, b) = \mathbb{E}_x \left[ e^{-\alpha \tau_b^+} \phi(X_{\tau_b^+}) \mathbf{1}_{\{\tau_b^+ < \infty\}} \right] = \frac{1}{\psi_{\alpha}^+(-1)} \sum_{i=1}^{[\tau_b^+]} A_i e^{\rho_{1,\alpha}(b-x)} \left( \frac{\rho_{1,\alpha}}{\rho_{1,\alpha} + 1} e^b - K \psi_{\alpha}^+(-1) \right). \tag{3.23}
\]

Since we already know the optimal stopping time is of threshold type when \( k = 1 \), the analytic expression for the value function of the single stopping problem is then readily available to us by optimizing the exercise threshold \( b \).

**Theorem 3.1.** The optimal exercise threshold for the single stopping problem is given by \( x^*_1 := \log(K \psi_{\alpha}^+(-1)) \) and that the corresponding value function is given by

\[
 v^{(1)}(x) = g^{(1)}(x, x^*_1) = \begin{cases} 
 \phi(x), & x \geq x^*_1, \\
 K \cdot \sum_{i=1}^{[\tau_b^+]} [K \psi_{\alpha}^+(-1)]^{-\rho_{1,\alpha}} e^{\rho_{1,\alpha}x} A_i, & x < x^*_1,
\end{cases} \tag{3.24}
\]

where the function \( g^{(1)}(\cdot, \cdot) \) is defined in (2.8).

**Remark 3.3.** Recently, Cai and Sun [7] consider a single stock loan problem under a hyper-exponential jump diffusion model, and provide an analytic solution for the perpetual single stopping problem. In comparison, our Theorem 3.1 applies to more general Lévy models as described by (3.13) and Assumption 3.1.

**Remark 3.4.** If \( X \) is a spectrally negative Lévy process, then (3.24) can be simplified to

\[
 v^{(1)}(x) = \begin{cases} 
 \phi(x), & x \geq x^*_1, \\
 \phi(x^*_1) e^{-\Phi(\alpha)(x^*_1-x)}, & x < x^*_1,
\end{cases} \tag{3.25}
\]

where \( x^*_1 = \log(K \psi_{\alpha}^+(-1)) \) with \( \psi_{\alpha}^+(\beta) = \frac{\Phi(\alpha)}{\Phi(\alpha) + \beta} \). Notice that \( \mathbb{E}_x[e^{-\alpha \tau_b^+}] = e^{-\Phi(\alpha)(b-x)} \) for all \( x < b \).

### 3.2 Optimal Multiple Stopping Problem

In this subsection, we characterize the optimal exercise thresholds that maximize \( g^{(k)}(x, \cdot) \). First, recall from Proposition 3.1 that for all \( x < b \), and the given \( \alpha \in \mathbb{R} \) and \( \beta \geq 0 \),

\[
 \mathbb{E}_x \left[ e^{-\alpha \tau_b^+ - \beta(X_{\tau_b^+} - b)} \mathbf{1}_{\{\tau_b^+ < \infty\}} \right] = \frac{1}{\psi_{\alpha}^+(-1)} \sum_{i=1}^{[\tau_b^+]} A_i \frac{\rho_{1,\alpha}}{\rho_{1,\alpha} + \beta} e^{\rho_{1,\alpha}(b-x)} . \tag{3.26}
\]

The distribution of \( X_{\tau_b^+} \) can be retrieved from (3.26) via inverse Laplace transform. To this end, let us introduce

\[
 \phi_{\infty} := \lim_{\beta \to \infty} \beta \psi_{\alpha}^+(-1) = \begin{cases} 
 \prod_{j=1}^{[\tau_b^+]} \frac{\rho_{j,\alpha}}{\rho_{j,\alpha} + \beta} > 0, & \text{if } -J \text{ is not a subordinator} \\
 \infty, & \text{else}
\end{cases}
\]

Then there exists a unique (possibly signed) measure on \([0, \infty), \nu(dy)\), such that

\[
 \int_{[0, \infty)} e^{-\beta y} \nu(dy) = \frac{1}{\phi_{\infty}} - \frac{\beta}{\phi_{\infty}}, \quad \forall \beta \geq 0. \tag{3.27}
\]

**Remark 3.5.** If we assume that elements in \( J \) are all distinct, \(^1\) then we have,

\[
 \nu(dy) = \begin{cases} 
 \frac{1}{\phi_{\infty}} \mathbf{1}_{(y=0)} + \sum_{j=1}^{[\tau_b^+]} \frac{1}{\psi_{j,\alpha}^+(-\eta_j)} e^{-\eta_j} y dy, & \text{if } -J \text{ is not a subordinator} \\
 \frac{1}{\psi_{\infty}^+(-\eta_j)} \mathbf{1}_{(y=0)} + \sum_{j=1}^{[\tau_b^+]} \frac{1}{\psi_{j,\alpha}^+(-\eta_j)} e^{-\eta_j} y dy, & \text{else}
\end{cases}, \quad \forall y \geq 0.
\]

\(^1\)This is the case, for example, when the upward jumps are hyper-exponential.
Furthermore, we define a measure on $[0, \infty)$ for each $1 \leq i \leq |\mathcal{I}_\alpha|$:

$$\tilde{\nu}(dy) := \frac{\rho_i|\alpha}{\phi_\infty}1_{\{y=0\}} + \rho_i|\alpha e^{-\rho_i|\alpha}y - \frac{\rho_i|\alpha}{\phi_\infty} + \int_{(0,y)} e^{\rho_i|\alpha} \nu(dx)dy, \quad \forall y \geq 0. \quad (3.28)$$

Then it can be easily verified that

$$\int_{[0,\infty)} e^{-\beta y} \tilde{\nu}_i(dy) = \frac{\rho_i|\alpha}{\rho_i|\alpha + \beta \psi_\infty(\beta)}, \quad \forall \beta \geq 0, 1 \leq i \leq |\mathcal{I}_\alpha|. \quad (3.29)$$

As a result of (3.26) and (3.29), we have for all $x < b$

$$\mathbb{E}_x \left[ e^{-\alpha \tau^+} \mathbf{1}_{\left\{ \tau^+ \leq \infty \right\}} \left( X_{\tau^+} \right) \right] = \frac{1}{\phi_\infty} \sum_{i=1}^{|\mathcal{I}_\alpha|} A_i \rho_i|\alpha e^{-\rho_i|\alpha} (b-x), \quad (3.30)$$

$$\mathbb{E}_x \left[ e^{-\alpha \tau^+} \mathbf{1}_{\left\{ \tau^+ \neq \infty \right\}} \left( X_{\tau^+} \right) \right] = \sum_{i=1}^{|\mathcal{I}_\alpha|} A_i e^{-\rho_i|\alpha} (b-x) \tilde{\nu}_i(dy), \quad \forall y \geq 0. \quad (3.31)$$

Equations (3.30) and (3.31) can be used to compute $\mathbb{E}_x \left[ e^{-\alpha \tau^+} \nu(k) \left( X_{\tau^+} \right) \right]$. To this end, let $Y$ have the same distribution as $X_\delta$ under $\mathbb{P}$, but is independent of $\mathcal{F}_{\tau^+}$. Then, for all $x < b$,

$$\mathbb{E}_x \left[ e^{-\alpha \tau^+(\delta)} \nu(k) \left( X_{\tau^+(\delta)} \right) \mathbf{1}_{\left\{ \tau^+(\delta) \leq \infty \right\}} \right] = \sum_{i=1}^{|\mathcal{I}_\alpha|} A_i \rho_i|\alpha e^{-\rho_i|\alpha} (b-x) \left( \int_{[0,\infty)} e^{-\alpha \delta} \nu(k) (b+u) \tilde{\nu}_i(dy) \right) \quad (3.32)$$

Recall from Corollary 2.1 that $\mathbb{E}[e^{-\alpha \delta} \nu(k)(b+Y)]$ is differentiable for all $b \in \mathbb{R}$. As a result, the optimal exercise threshold $b^*_k$ can be characterized by the first order condition: $\frac{\partial}{\partial b} J_{b^*_k}(x, b) = 0$ for any $x < b^*_k$. In the remaining of this subsection, we will inductively prove that, if the threshold strategy is optimal for problem (2.6) with up to $k-1$ exercise opportunities, for some $k \in \{2, \cdots, n\}$, then there exists a unique optimal exercise threshold $b^*_k$ for problem (2.6) with $k$ exercise opportunities. To show that the threshold type strategy $\tau^+_{b_k}$ is indeed optimal over all $\mathbb{F}$-stopping times in $\mathcal{T}$, we further prove that the process $(e^{-\alpha t} \nu(k)(X_t, b^*_k))_{t \geq 0}$ is a supermartingale. Finally, based on the result for $k = 1$ in Section 3.1, mathematical induction will subsequently conclude the existence and uniqueness of the optimal exercise threshold $b^*_k$, and the optimality of the threshold type strategy $\tau^+_{b_k}$, for all $k \in \{2, \cdots, n\}$.

To facilitate later calculations, we define for all $1 \leq k \leq n$ that

$$u^{(k)}(x) := \frac{v^{(k)}_+(x)}{\phi_\infty} - \int_{[0,\infty)} v^{(k)}(x+y) \nu(dy), \quad (3.33)$$

where $v^{(k)}_+$ is the right derivative of $u^{(k)}$. In particular, when $k = 1$, we can use (3.24) and (3.33) (see also the proof of Proposition 3.3 below using $v^{(1)}(x) = g^{(1)}(x, x^*_1)$ and $u^{(0)}(x) \equiv u^{(0)}(x) \equiv 0$ to obtain:

$$u^{(1)}(x) = \frac{e^{x^*_1} - e^{x^*}}{\psi^*_1(-1)} \mathbf{1}_{\{x \geq x^*_1\}}. \quad (3.34)$$

Notice that $u^{(1)}$ is continuous, non-positive and non-increasing on $\mathbb{R}$ and is strictly decreasing on $[x^*_1, \infty)$.

The following result characterizes the first order condition for the optimal exercise threshold $b^*_k$.

**Proposition 3.2.** If for some fixed $k \in \{2, \cdots, n\}$, the threshold type strategy $\tau^+_{b^*_k}$ is optimal for (2.6) with $(k-1)$ exercise opportunities, $v^{(k-1)}(x) > v^{(k-2)}(x)$ for all $x \in \mathbb{R}$, and the function $u^{(k-1)}(x)$ is continuous on $\mathbb{R}$, then there exists at least one solution $b_k > \log K$ to the equation:

$$\tilde{u}_0^{(k)}(b_k) = 0, \quad \text{where} \quad \tilde{u}_0^{(k)}(x) := \frac{e^{x^*_1} - e^{x^*}}{\psi^*_1(-1)} + \mathbb{E}[e^{-\alpha \delta} u^{(k-1)}(x + X_\delta)], \quad \forall x \in \mathbb{R}. \quad (3.35)$$

Moreover, if the optimal exercise threshold $b^*_k$ exists, it satisfies (3.35).
In the next results, we establish the monotonicity of the function \( \hat{u}_0^{(k)} \), and the uniqueness of the solution to (3.35).

**Proposition 3.3.** Under the assumption of Proposition 3.2, and further assume that the function \( u^{(k-1)}(x) \) is a non-increasing function. Let \( b_k \) be any solution to (3.35), and define

\[
\hat{u}^{(k)}(x) := \frac{1}{\partial_{\infty}} \frac{\partial^+}{\partial x} g^{(k)}(x, b_k) - \int_{[0, \infty)} g^{(k)}(x + y, b_k) v(dy),
\]

(3.36)

where \( \partial^+ \) is the right partial derivative operator. Then, \( \hat{u}^{(k)} \) is continuous, non-positive, and non-increasing on \( \mathbb{R} \). Moreover, it can be expressed as

\[
\hat{u}^{(k)}(x) = 1_{\{x \geq b_k\}} \hat{u}_0^{(k)}(x).
\]

(3.37)

We now prove the existence and the uniqueness of the optimal exercise threshold \( b_k^* \).

**Lemma 3.2.** Under the condition of Proposition 3.3, there is a unique solution to (3.35), and this solution is the optimal exercise threshold \( b_k^* \).

Hence, Proposition 3.2 applies, and the optimal exercise threshold \( b_k^* \) is uniquely determined by (3.35).

Next, we prove the supermartingale property of the process \( (e^{-\alpha t} g^{(k)}(X_t, b_k^*)) \) for \( t \geq 0 \), in order to show that the optimal stopping region for problem (2.6) is one-sided, and hence, \( b_k^* = x_k^* \) yields the optimal stopping time \( \tau_k^+ \). The main tool is to re-express the value of the threshold type strategy \( g^{(k)}(x, b_k^*) \), by an expectation of functionals of \( X_{e_q} \) (see [1] for a similar solution approach).

We begin with the case \( k = 1 \). It can be easily seen from the proof of Proposition 3.1 that

\[
v^{(1)}(x) = g^{(1)}(x, x_1^*) = \lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(q)}[e^{-\Phi(q)}X_{e_q}(-u^{(1)}(x + X_{e_q})] - \mathbb{E}^{\Phi(q)}[e^{-\Phi(q)}X_{e_q}]}{\mathbb{E}^{\Phi(q)}[e^{-\Phi(q)}X_{e_q}]}. \]

(3.38)

We use (3.38) to initialize our induction step.

**Proposition 3.4.** Under the condition of Proposition 3.3, and further assume that

\[
v^{(k-1)}(x) = g^{(k-1)}(x, x_{k-1}^*) = \lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(q)}[e^{-\Phi(q)}X_{e_q}(-u^{(k-1)}(x + X_{e_q})] - \mathbb{E}^{\Phi(q)}[e^{-\Phi(q)}X_{e_q}]}}{\mathbb{E}^{\Phi(q)}[e^{-\Phi(q)}X_{e_q}]}. \]

(3.39)

Then we have

\[
g^{(k)}(x, b_k^*) = \lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(q)}[e^{-\Phi(q)}X_{e_q}(-u^{(k)}(x + X_{e_q})] - \mathbb{E}^{\Phi(q)}[e^{-\Phi(q)}X_{e_q}]}}{\mathbb{E}^{\Phi(q)}[e^{-\Phi(q)}X_{e_q}]}, \]

(3.40)

where the function \( \hat{u}^{(k)} \) is defined in (3.36). Moreover, the process \( (e^{-\alpha t} g^{(k)}(X_t, b_k^*)) \) for \( t \geq 0 \) is a supermartingale.

In summary, we apply mathematical induction to Propositions 2.1, 2.2, 3.2, 3.3 and 3.4 and Lemma 3.2 to obtain the following result.

**Theorem 3.2.** For every \( k \in \{1, \ldots, n\} \), the optimal stopping problem (2.6) is solved by the up-crossing time \( \tau_k^+ \), where \( x_k^* \) is the unique solution to (3.35), satisfying

\[
\log K < x_n^* \leq x_{n-1}^* \leq \cdots \leq x_1^*.
\]

The value function is given by

\[
v^{(k)}(x) = g^{(k)}(x, x_k^*),
\]

which can be expressed as (3.40) above. Moreover, the value functions are ordered as follows:

\[
0 < v^{(1)}(x) < v^{(2)}(x) < \cdots < v^{(n)}(x), \; \forall x \in \mathbb{R}.
\]
4 Numerical Examples

In this section, we present numerical examples based on our analytical results. In particular, we illustrate the sensitivity of the optimal thresholds with respect to the distribution of refraction times. The numerical implementation is generally challenging. It involves the evaluation of the expectation $E_x [e^{-\alpha \delta(k-1)(X_d)}]$ while the distribution of the random variable $X_d$ is most commonly not explicit or even unknown. As is used in [27], Monte Carlo simulation is the most straightforward approach. However, it is far from being practical unless $k$ is a very small number. For refracted multiple stopping problems, one needs to know the entire expected future payoff functional to carry out the backward induction. The simulation approach would require the computation of these expectations for arbitrarily large number of starting points for each step, which adds to the computational burden and limits its applicability. In particular, the payoff function of our problem is unbounded (and increases exponentially); the truncation that would be needed under the simulation method will produce non-negligible errors that would further be amplified as $k$ increases.

For our numerical examples, we assume that $\delta$ is Erlang distributed (i.e. a sum of i.i.d. exponential random variables), and numerically solve for the optimal exercise thresholds using the methods described in our separate paper [20]. The approach utilizes the resolvent measure (or the distribution of $X$ at an independent exponential random time) and carries out repeatedly and analytically the integrations with respect to this measure. The resulting value functions are shown to be in a piecewise analytic form. The results are exact when the jump size distribution is phase-type, and can be used as an approximation to problem with other Lévy jumps thanks to the denseness of the phase-type Lévy processes. The approach can also be applied to the case with constant refraction times via the technique of Canadization. For detailed analysis on its computational performance, we refer the reader to [20].

In our numerical results, we consider from (3.13) a spectrally negative Lévy process with i.i.d. exponential jumps:

$$X_t - X_0 = \tilde{C}t + \sigma B_t - \sum_{n=1}^{N_t} Z_n, \quad 0 \leq t < \infty,$$

for some $\tilde{C} \in \mathbb{R}$ and $\sigma \geq 0$. Here $B = (B_t)_{t \geq 0}$ is a standard Brownian motion, $N = (N_t)_{t \geq 0}$ is a Poisson process with arrival rate $\rho$, and $Z = (Z_n)_{n=1,2,...}$ is an i.i.d. sequence of exponential random variables with parameter $\lambda > 0$. These processes are assumed to be mutually independent. For our studies below, we set $\sigma = 0.2$, $\rho = \lambda = 1$ and $K = 50$. Also, we use $\alpha = -0.02$ and $\tilde{C} = 0.36$ so that $\alpha - \psi(1) = 0.1 > 0$, which guarantees that Assumption 3.1 is satisfied. We consider two types of refraction times: (1) exponential and (2) Erlang with shape parameter 2. We compute the results for a range of the expected refraction times, denoted by $\tilde{\delta} := E\delta$.

In Figure 1, we plot the optimal exercise thresholds $x^*_k$ for $k = 1, \ldots, 5$ against different means of the refraction time $\tilde{\delta} = 0.5, 1.0, 1.5, \ldots, 10$. Consistent with Proposition 2.2, the thresholds monotonically decrease as $k$ increases. In particular, the highest threshold corresponds to the last remaining exercise ($k = 1$). In this case, the refraction time is completely irrelevant, so the threshold value stays constant over different mean refraction times under any distribution. Interestingly, with $k$ fixed, the thresholds are not monotone in the mean refraction time. On one hand, refraction times are constraints on the stopping times, so they reduce the value functions but not necessarily the exercise thresholds. Intuitively, a very long refraction time reduces the value of subsequent exercise opportunities, and incentivizes the holder to focus more on the next immediate stopping. This helps explain that the thresholds tend to be closer for very long mean refraction times.

5 Conclusions

We have studied an optimal multiple stopping problem with the features of negative discount rate and random refraction times under a general class of Lévy models. In order to account for the negative discount rate, the technique of change of measure is shown to be very useful though the analysis under the new measure is challenging. As seen in Theorems 3.1 and 3.2 above, the optimal exercise thresholds are determined explicitly in the single stopping case and recursively in the multiple stopping case. While our problem setting is selected with the application to stock loans in mind, the current paper also presents a blueprint to rigorously analyze perpetual optimal refracted multiple stopping problems with alternative payoffs, such as put options. These would be natural directions for future research.
Figure 1: Optimal multiple exercise thresholds $x_k^*$ for $k = 1, 2, 3, 4, 5$, plotted for different mean refraction times $\delta$ when they are exponentially (left) and Erlang (right) distributed respectively.

A Appendix

A.1 Proof of Lemma 2.1

We begin by noticing that, for any fixed $k \in \{1, \ldots, n\}$,

$$U^{(k)}(s) = \sup_{\tau \in T} \mathbb{E} \left[ e^{-\alpha \tau} \phi^{(k)}(\log s + X_\tau) \mathbf{1}_{\tau < \infty} \right].$$

For $k = 1$, for any stopping time $\tau$ and any $s_1, s_2 \in \mathbb{R}_+$ such that $s_1 > s_2$, it follows from the monotonicity of $\phi$ that

$$U^{(1)}(s_1) = \sup_{\tau \in T} \mathbb{E} \left[ e^{-\alpha \tau} \phi(\log s_1 + X_\tau) \mathbf{1}_{\tau < \infty} \right] \geq \sup_{\tau \in T} \mathbb{E} \left[ e^{-\alpha \tau} \phi(\log s_2 + X_\tau) \mathbf{1}_{\tau < \infty} \right] = U^{(1)}(s_2).$$

Similarly, from the subadditivity of supremum and the convexity of $(\phi \circ \log)$, we have, for any $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and any $s_1, s_2 \in \mathbb{R}_+$, that

$$\frac{U^{(1)}(s_1)}{p} + \frac{U^{(1)}(s_2)}{q} = \frac{1}{p} \sup_{\tau \in T} \mathbb{E} \left[ e^{-\alpha \tau} \phi(\log(s_1 \exp(X_\tau))) \mathbf{1}_{\tau < \infty} \right] + \frac{1}{q} \sup_{\tau \in T} \mathbb{E} \left[ e^{-\alpha \tau} \phi(\log(s_2 \exp(X_\tau))) \mathbf{1}_{\tau < \infty} \right]
\geq \sup_{\tau \in T} \mathbb{E} \left[ e^{-\alpha \tau} \left( \frac{\phi(\log(s_1 \exp(X_\tau)))}{p} + \frac{\phi(\log(s_2 \exp(X_\tau)))}{q} \right) \mathbf{1}_{\tau < \infty} \right]
\geq \sup_{\tau \in T} \mathbb{E} \left[ e^{-\alpha \tau} (\phi \circ \log) \left( \left( \frac{s_1}{p} + \frac{s_2}{q} \right) \exp(X_\tau) \right) \mathbf{1}_{\tau < \infty} \right] = \sup_{\tau \in T} \mathbb{E} \left[ e^{-\alpha \tau} \phi(X_\tau) \mathbf{1}_{\tau < \infty} \right] = U^{(1)} \left( \frac{s_1}{p} + \frac{s_2}{q} \right).$$

Hence, the convexity holds also for $k = 1$. This implies that $U^{(1)}(s)$ is differentiable almost everywhere on $\mathbb{R}_+$.

Now suppose that the claim is true for $k = l - 1$ for some $l \in \{2, \cdots, n - 1\}$; that is, $U^{(l-1)}$ is non-decreasing and convex. By the same argument above, we conclude that $\mathbb{E}[e^{-\alpha \delta}U^{(l-1)}(s \exp(X_\delta))]$ is also non-decreasing and convex. This implies that $\phi^{(l)}$ and $v^{(l)}$ all have the monotonicity and convexity properties. By induction, we conclude.

A.2 Proof of Corollary 2.1

For $k = 1$ we have $0 \leq U^{(1)}(s) = v^{(1)}(\log s)/s \leq 1$, for a.e. $s \in \mathbb{R}_+$, or equivalently $0 \leq v^{(1)}(x) \leq e^x$, a.e. $x \in \mathbb{R}$ and $|v^{(1)}(x + \epsilon) - v^{(1)}(x)| \leq e^x|\epsilon| - 1$, for any $x, \epsilon \in \mathbb{R}$. Let us denote by $D^{(1)} := \{ x \in \mathbb{R} : U^{(1)}_+(x) > U^{(1)}_- (x) \} = \{ x \in \mathbb{R} : v^{(1)}_+(x) > v^{(1)}_-(x) \}$, where $U^{(1)}_+$ and $U^{(1)}_-$ (resp.) are, respectively, the right and left derivatives of $U^{(1)}$ (resp.). Then we know that $D^{(1)}$ is at most a countable set by Lemma 2.1. On the
other hand, using the fact that \( X_\delta \) has no atom, we know that, for any fixed \( x \in \mathbb{R} \), we have for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), 
\( x + X_\delta(\omega) \notin D(1) \). It follows that \( e^{-\alpha \delta X_\delta} v(1)(x + X_\delta(\omega)) \) is almost surely differentiable at this fixed \( x \in \mathbb{R} \), and \( 0 \leq e^{-\alpha \delta v(1)(x + X_\delta)} \leq e^{-\alpha \delta X_\delta} \cdot e^x \), \( \mathbb{P} \)-a.s. By the assumption, the nonnegative random variable \( e^{-\alpha \delta X_\delta} \cdot e^x \) has a finite expectation. Now for any \( \epsilon \neq 0 \), as \( \epsilon \to 0 \), we have by Jensen’s inequality that

\[
0 \leq \left| \mathbb{E}[e^{-\alpha \delta v(1)(x + \epsilon + X_\delta) - v(1)(x + X_\delta)] - \mathbb{E}[e^{-\alpha \delta v(1)'(x + X_\delta) - v(1)'(x + X_\delta)']} \right| \to 0,
\]

where we have used the fact that \( \left| v(1)(x + \epsilon + X_\delta) - v(1)(x + X_\delta) \right| \leq \alpha x + X_\delta \left( \frac{e^{-\alpha \delta}}{|x|} + 1 \right) \), and

\[
e^{-\alpha \delta \cdot \lim_{\epsilon \to 0}} \frac{v(1)(x + \epsilon + X_\delta) - v(1)(x + X_\delta)}{\epsilon} - v(1)'(x + X_\delta) = 0, \quad \mathbb{P} \text{-a.s.}
\]

and the dominated convergence theorem. It follows that \( \mathbb{E}[e^{-\alpha \delta v(1)(x + X_\delta)}] \) is differentiable in \( x \), and

\[
0 \leq \frac{\partial}{\partial x} \mathbb{E}[e^{-\alpha \delta v(1)(x + X_\delta)}] = \mathbb{E}[e^{-\alpha \delta v(1)'(x + X_\delta)}] \leq \mathbb{E}[e^{-\alpha \delta X_\delta} e^x] \leq e^x.
\]

Now suppose that the claim is true for \( k = l - 1 \) for some \( l \in \{2, \cdots, n - 1\} \). Then we know that \( \phi^{(l)}(x) \) is differentiable on \( \mathbb{R} \setminus \{ \log K \} \) and its derivative admits the upper bound

\[
\phi^{(l)'}(x) \leq e^x + \frac{\partial}{\partial x} \mathbb{E}[e^{-\alpha \delta v^{(l-1)}(x + X_\delta)}] \leq e^x + (l - 1)e^x = le^x, \quad \forall x \neq \log K.
\]

That is, \( 0 \leq \frac{\partial}{\partial x} \phi^{(l)}(\log s) \leq l \) for all \( s \in \mathbb{R} \setminus \{ K \} \). It follows that \( 0 \leq U^{(l)'}(s) \leq l, \quad \mathbb{P} \text{-a.e.} \) which implies that \( 0 \leq v^{(l)}(x) \leq le^x, \quad \mathbb{P} \text{-a.e.} \). By the same arguments as above, we also obtain that, for all \( x \in \mathbb{R} \),

\[
0 \leq \frac{\partial}{\partial x} \mathbb{E}[e^{-\alpha \delta v^{(l)}(x + X_\delta)}] \leq le^x.
\]

The result now follows from mathematical induction.

### A.3 Proof of Theorem 2.1

First, Lemma 2.2 implies that \( \mathbb{P}_x(\tau^*_i < \infty, 1 \leq i \leq k) > 0 \) holds provided that \( x^*_i < \infty \) and \(-X \) is not a subordinator.

Next, following Lemma 2.1 of [9], we deduce recursively that

\[
v^{(k)}(x) \leq \mathbb{E}_x \left[ \sup_{0 \leq t < \infty} e^{-\alpha t} \phi^{(k)}(X_t) \right] \leq \left( \mathbb{E}_x \left[ \left( \sup_{0 \leq t < \infty} e^{-\alpha t} \phi^{(k)}(X_t) \right)^{\frac{1}{2}} \right] \right)^\frac{1}{2} < \infty,
\]

for all \( k \geq 1 \). Hence, the single optimal stopping problem (2.6) is well defined. To ensure the existence of an optimal stopping time, we may adapt the proof of Proposition 3.2 in [27] to the setting with possibly negative discount rate \( \alpha \) and call-like payoff. More precisely, by Lemma 2.1 and Corollary 2.1 we know that \( U^{(k)}(s) \) is globally Lipschitz in \( s \in \mathbb{R} \), which implies that, by the proof of Proposition 3.2 in [27], the expected jump of \( e^{-\alpha \tau_e v^{(k)}(X_\tau)} \), at any predictable time \( \tau \), is zero, namely,

\[
\mathbb{E}_x \Delta(e^{-\alpha \tau_e v^{(k)}(X_\tau) \mathbb{1}_{\{\tau < \infty\}}} = 0, \quad k \geq 1,
\]

where \( \Delta(e^{-\alpha \tau_e v^{(k)}(X_\tau)}) := e^{-\alpha \tau_e} [v^{(k)}(X_\tau) - v^{(k)}(X_{\tau^-})] \). This implies that the Snell envelope \( (e^{-\alpha t} v^{(k)}(X_t))_{t \geq 0} \) is left-continuous in expectation. In turn, this allows us to apply the arguments in Theorem 2.1 of [9] to conclude (2.11). This proves (i).

To prove (ii), we first comment that the result holds trivially for the case \( k = 1 \). For \( k \in \{2, \cdots, n\} \), we observe from (2.11) that \( v^{(k)}(x) \leq \bar{v}^{(k)}(x) \) since \( (\tau^*_i)_{1 \leq i \leq k} \) are admissible candidate stopping times (see (2.9)-(2.10)). The
reverse inequality can be proved by induction. To this end, notice that \( v^{(1)}(x) \geq \mathbb{E}_x[e^{-\alpha \tau}(X_0)I_{[\tau < \infty]}(x)] \) for any arbitrary \( \mathbb{F} \)-stopping time \( \tau \in \mathcal{T} \) by (2.6) for \( k = 1 \). Now by applying (2.6), (2.7) and repeated expectations, we get

\[
v^{(2)}(x) \geq \mathbb{E}_x\left[e^{-\alpha \tau} \left( \phi(X_\tau) + \mathbb{E}_x\left[e^{-\alpha \delta \tau} v^{(1)}(X_\delta)\right] I_{[\tau < \infty]}(x)\right]\right]
\]

\[
\geq \mathbb{E}_x\left[e^{-\alpha \tau} \left( \phi(X_\tau) + \mathbb{E}_x\left[e^{-\alpha \delta (\nu - \delta \tau)} \phi(X_{\nu - \delta \tau}) I_{[\nu < \infty]}(x)\right]\right] I_{[\tau < \infty]}(x)
\]

\[
= \mathbb{E}_x\left(e^{-\alpha \tau} \phi(X_\tau) I_{[\tau < \infty]}(x) + e^{-\alpha \delta \tau} \phi(X_\delta) I_{[\nu < \infty]}(x)\right]
\]

(A.42)

for every \( \mathbb{F} \)-stopping time \( \tau \) and \( \mathbb{F}(\tau + \delta) \)-stopping time \( \nu \geq \tau + \delta \). Maximizing (A.42) over \( (\tau, \nu) \in \mathcal{T}^2 \) yields that \( v^{(2)}(x) \geq \tilde{v}^{(2)}(x) \). The result now follows from mathematical induction.

Finally, for (iii), for any finite \( \mathbb{F} \)-stopping time \( \tau \in \mathcal{T} \), by the strong Markov property, we have

\[
\mathbb{E}_x\left(e^{-\alpha \tau} v^{(k)}(X_\tau)\right)^\theta \geq \mathbb{E}_x\left(e^{-\alpha \tau} \cdot \mathbb{E}_x\left[\sup_{0 \leq s < \infty} e^{-\alpha \delta \tau} \phi^{(k)}(X_\delta)\right]^\theta\right]
\]

\[
\leq \mathbb{E}_x\left(e^{-\alpha \tau} \mathbb{E}_x\left[\left(\sup_{0 \leq s < \infty} e^{-\alpha \delta \tau} \phi^{(k)}(X_\delta)\right)^\theta\right]\right] = \mathbb{E}_x\left[\sup_{0 \leq s < \infty} e^{-\alpha \tau} \mathbb{E}_x\left[\left(\sup_{0 \leq s < \infty} e^{-\alpha \delta \tau} \phi^{(k)}(X_\delta)\right)^\theta\right]\right]<\infty,
\]

where the first and second inequalities follow from (A.41) and the equality is due to repeated expectations. Hence we have the uniform boundedness of elements of \( S^{(k)} \) in \( L^\theta(d\mathbb{P}_x) \). This proves (iii) and completes the proof.

### A.4 Proof of Proposition 2.1

If \( v^{(k-1)}(x) > v^{(k-2)}(x) \) for all \( x \in \mathbb{R} \), then by (2.7), we know that \( \phi^{(k)}(x) > \phi^{(k-1)}(x) \) for all \( x \in \mathbb{R} \). Furthermore, if \([x^*_{k-1}, \infty)\) is the only optimal stopping region for problem (2.6) with \((k - 1)\) exercise opportunities, then the up-crossing time \( \tau^+_{x^*_{k-1}} \) is the optimal stopping time. Hence, for all \( x \in \mathbb{R} \), we prove (i) through the inequality

\[
v^{(k)}(x) \geq g^{(k)}(x, x^*_{k-1}) = \mathbb{E}_x\left[e^{-\alpha \tau^+_{x^*_{k-1}}} \phi^{(k)}(X_{\tau^+_{x^*_{k-1}}}) I_{[\tau^+_{x^*_{k-1}} < \infty]}(x)\right]
\]

\[
> \mathbb{E}_x\left[e^{-\alpha \tau^+_{x^*_{k-1}}} \phi^{(k)}(X_{\tau^+_{x^*_{k-1}}}) I_{[\tau^+_{x^*_{k-1}} < \infty]}(x)\right] = g^{(k-1)}(x, x^*_{k-1}) = v^{(k-1)}(x).
\]

To prove (ii), we first recall from Lemma 2.2 that \( x^*_{k} \in (-\infty, x^*_{k-1}] \). Hence, for the first claim, it is sufficient to show that \( v^{(k)}(x) \neq \phi^{(k)}(x) \) on \((-\infty, \log K]\). Indeed, we use the supermartingale property of value functions to obtain that \( \mathbb{E}_x[e^{-\alpha \delta \tau} v^{(k-1)}(X_\delta)] \leq v^{(k-1)}(x) \) for all \( x \in \mathbb{R} \). Therefore, for all \( x \leq \log K \),

\[
v^{(k)}(x) > v^{(k-1)}(x) \geq \mathbb{E}_x[e^{-\alpha \delta \tau} v^{(k-1)}(X_\delta)] = \phi^{(k)}(x).
\]

It follows that \( x^*_{k} \in (\log K, x^*_{k-1}] \). Similarly, if the optimal exercise threshold \( b^*_k \) exists, then we have

\[
g^{(k)}(x, b^*_k) \geq g^{(k)}(x, x^*_{k-1}) > g^{(k-1)}(x, x^*_{k-1}) = v^{(k-1)}(x) \geq \mathbb{E}_x\left[e^{-\alpha \delta \tau} v^{(k-1)}(x)\right].
\]

Therefore, for \( x = \log K \), we have \( g^{(k)}(\log K, b^*_k) > \phi^{(k)}(\log K) \), which implies that \( b^*_k > \log K \).

We now proceed to prove (iii) by establishing the sufficient conditions for optimality (see e.g. [1, Sect. 6]). If the optimal exercise level \( b^*_k \) exists, then it is easily seen that

(a) for all \( x \in \mathbb{R} \), \( g^{(k)}(x, b^*_k) \geq g^{(k)}(x, x) = \phi^{(k)}(x) \), and \( g^{(k)}(x, b^*_k) > 0 \) by the fact that \( b^*_k > \log K \) (see (ii));

(b) for all \( x \in \{b^*_k, \infty) \), we have \( g^{(k)}(x, b^*_k) = \phi^{(k)}(x) \);

(c) for all \( t > 0 \), by the strong Markov property of \( X \), we have

\[
g^{(k)}(t, b^*_k) = \mathbb{E}_x\left[e^{-\alpha \tau^+_{b^*_k}} \phi^{(k)}(X_{\tau^+_{b^*_k}}) I_{[\tau^+_{b^*_k} < \infty]}(x)\right] = \mathbb{E}_x\left[e^{-\alpha \tau^+_{b^*_k}} \phi^{(k)}(X_{\tau^+_{b^*_k}}) I_{[\tau^+_{b^*_k} < \infty]}(x) \mid \mathcal{F}_t\right]
\]

\[
= \mathbb{E}_x\left[e^{-\alpha \tau^+_{b^*_k}} \phi^{(k)}(X_{\tau^+_{b^*_k}}) I_{[\tau^+_{b^*_k} < \infty]}(x)\right] + \mathbb{E}_x\left[1_{(\tau^+_{b^*_k} > t)} \mathbb{E}_X^{x_0}\left[e^{-\alpha \tau^+_{b^*_k}} \phi^{(k)}(X_{\tau^+_{b^*_k}}) I_{[\tau^+_{b^*_k} < \infty]}(x)\right]\right]
\]

\[
= \mathbb{E}_x\left[e^{-\alpha (\tau^+_{b^*_k} t)} g^{(k)}(X_{t \wedge \tau^+_{b^*_k}}, b^*_k)\right].
\]
That is, the stopped process \( (e^{-\alpha(t\wedge k_x^+)} g(k)(X_{t\wedge k_x^+}, b^*_x))_{t\geq 0} \) is a \((\mathbb{P}_x, \mathbb{F})\)-martingale for any fixed \( x \in \mathbb{R} \).

Now if we know additionally that \( (e^{-\alpha t} g(k)(X_t, b^*_x))_{t\geq 0} \) is a supermartingale, then we can conclude that \( v^{(k)}(x) = g(k)(x, b^*_x) \), and \( [x^*_{x}, \infty) = [b^*_x, \infty) \) is the only stopping region.

### A.5 Proof of Proposition 2.2

Clearly the claim holds for \( l = 2 \) as we already know that \( x^*_{x} \leq x^*_{x-1} \), \( V^{(1)}_t \equiv e^{-\alpha t} v^{(1)}(X_t) \) is a \((\mathbb{P}_x, \mathbb{F})\)-supermartingale, and all random variables in the set \( \mathcal{S}^{(1)} \) in Theorem 2.1 are uniformly bounded in \( L^\beta(d\mathbb{P}_x) \). If \( l \geq 3 \), suppose the claim is true for \( k = 2, \ldots, h \) for some \( h \in \{2, \ldots, l-1\} \). That is,

\[
\log K < x^*_h \leq x^*_h - 1 \leq \cdots \leq x^*_1 \quad \text{and} \quad \mathbb{E}_x[V^{(k-1)}_t] \leq V^{(k-1)}_0 \leq v^{(k-1)}(x) - v^{(k-2)}(x), \quad \forall k \in \{2, \ldots, h\}.
\]

Now, from the general theory of optimal stopping we know that the stopped process \( (e^{-\alpha(t\wedge k_x^+)} v^{(h)}(X_{t\wedge k_x^+}))_{t\geq 0} \) is a martingale (see e.g. [1, Sect. 6]). Therefore, for \( x^*_h = \min_{1 \leq k \leq h} x^*_k \), we know that the stopped process \( (V^{(l)}_{t\wedge k_x^+})_{t\geq 0} \) is a martingale. Moreover, let us introduce the first down-crossing time

\[
\tau_b^- := \inf\{t \geq 0 : X_t \leq b\}, \quad \forall b \in \mathbb{R}.
\]

If \( x^*_h < x^*_h - 1 \), then the stopped process \( (V^{(l)}_{t\wedge k_x^+})_{t\geq 0} \) is equal to a stopped supermartingale less a martingale, and hence a supermartingale. Finally, for all \( x > x^*_h - 1 \geq x^*_h \), we have that

\[
v^{(h)}(x) - v^{(h-1)}(x) = \phi^{(h)}(x) - \phi^{(h-1)}(x) = \mathbb{E}_x[e^{-\alpha \delta}[v^{(h-1)}(X_\delta) - v^{(h-2)}(X_\delta)]]
\]

Hence, for all \( t < \infty \) and \( x > x^*_h - 1 \), we have

\[
\mathbb{E}_x[V^{(l)}_{t\wedge k_x^+}] = \mathbb{E}_x[e^{-\alpha(d+t\wedge k_x^+) \delta X_{t\wedge k_x^+}}] = \mathbb{E}_x[e^{-\alpha \delta}[v^{(h-1)}(X_\delta) - v^{(h-2)}(X_\delta)]]
\]

where we used the assumption that \( (V^{(h)}_{t\wedge k_x^+})_{t\geq 0} \) is a supermartingale, and the independence between \( \delta \) and \( X \). Combining all cases, we conclude that the process \( (V^{(l)}_t)_{t\geq 0} \) is a supermartingale.

The class (D) property of \( (V^{(l)}_t)_{t\geq 0} \) now follows from Minkowski’s inequality and the fact that the elements in \( \mathcal{S}^{(k)} \), \( k = h-1, h \), are uniformly bounded in \( L^\beta(d\mathbb{P}_x) \) (see Proposition 2.1 above).

To finish the proof, we need to show that \( \log K < x^*_{h+1} \leq x^*_h \). To this end, we notice that for all \( x \geq x^*_h \),

\[
v^{(h+1)}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[e^{-\alpha \tau \phi^{(h+1)}(X_\tau)}] \leq \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[e^{-\alpha \tau \phi^{(h+1)}(X_\tau)}] + \mathbb{E}_x[e^{-\alpha \tau (\phi^{(h+1)}(X_\tau) - \phi^{(h+1)}(X_\tau))}]
\]

\[
= \phi^{(h)}(x) + \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[e^{-\alpha \tau \phi^{(h)}(X_\tau)}] = \phi^{(h)}(x) + \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[V^{(h)}_{\tau^-}]
\]

\[
\leq \phi^{(h)}(x) + \mathbb{E}_x[V^{(h)}_{\tau^-}] = \phi^{(h)}(x) + \phi^{(h+1)}(x) - \phi^{(h)}(x) = \phi^{(h+1)}(x)
\]

where we used the class (D) property of \( (V^{(l)}_t)_{t\geq 0} \) in the second inequality and \((2.7)\) in the fourth equality. This shows that \( \phi^{(h+1)}(x) = \phi^{(h+1)}(x) \). Since \( x^*_{h+1} = \sup \{ x \leq x^*_1 : \phi^{(h+1)}(y) = \phi^{(h+1)}(y), \forall y > x \} \), we can conclude from \((A.44)\) that \( x^*_{h+1} \leq x^*_h \).
A.6 Proof of Lemma 3.1

For any \( p, q > 1 \) satisfying \( p^{-1} + q^{-1} = 1 \), and any sufficiently large \( z > 0 \), we have

\[
P_x \left( \sup_{t \geq 0} \left[ e^{-\alpha t} \phi(X_t) \right] > z \right) = P_x \left( \exists t \geq 0, e^{-\alpha t} \phi(X_t) > z \right) = P_x \left( \exists t \geq 0, X_t - \alpha t > \log(z + Ke^{-\alpha t}) \right)
\]

\[
\leq P_x \left( \exists t \geq 0, X_t - \alpha t > \frac{1}{p} \log(pz) + \frac{1}{q} \log(qKe^{-\alpha t}) \right) = P_x \left( \exists t \geq 0, X_t - \alpha t + \frac{\alpha t}{q} > \frac{1}{p} \log(pz) + \frac{1}{q} \log(qK) \right)
\]

\[
= \mathbb{P}_x \left( \sup_{0 \leq t < \infty} (X_t - \alpha t) > \frac{1}{p} \log(pz) + \frac{1}{q} \log(qK) \right) \sim \exp \left( -\tilde{\rho}_0 \left[ \frac{1}{p} \log(pz) + \frac{1}{q} \log(qK) \right] \right) = e^{\tilde{\rho}_0 x(qK) - \frac{\delta_0}{p}},
\]

where we used Proposition 1.8 on page 259 of [2] and \( \tilde{\rho}_0 \) is the smallest positive root of

\[
\psi(\tilde{\rho}_0) - \frac{\alpha}{p} \tilde{\rho}_0 = 0.
\]

It is now sufficient to show that it is possible to choose \( p > 1 \) such that \( \tilde{\rho}_0 > p \), and hence the random variable \( (\sup_{t \geq 0} [e^{-\alpha t} (e^X - K)^+])^p \) has a finite expectation for \( q = \frac{1}{p} + \frac{\tilde{\rho}_0}{p} > 1 \). To this end, we show that \( \psi(p) - \alpha < 0 \) for a sufficiently small \( p > 1 \). Indeed, for all \( 0 < \beta < p \), we have \( 0 > \frac{\beta}{p} (\psi(p) - \alpha) \geq \psi(\beta) - \frac{\beta}{p} \alpha \), where we used Jensen’s inequality \( \mathbb{E}[Y]^p \geq \mathbb{E}[Y^p] \) for positive random variable \( Y = e^pX_1 \) in the last step. Hence, the smallest positive solution \( \tilde{\rho}_0 > p > 1 \).

Let us first assume that \( \psi(1) - \alpha < 0 \). Then for sufficiently small \( p > 1 \), by the continuity of \( \psi \) at \( 1 \), we have

\[
\psi(p) - \alpha < 0.
\]

On the other hand, if \( \psi(1) - \alpha = 0 \), and \( \psi'(1) < 0 \). Then for sufficiently small \( p > 1 \), we have \( \psi'(p) < 0 \), and

\[
\psi(p) - \alpha = \psi(p) - \psi(1) < \psi'(p)(p - 1) < 0.
\]

A.7 Proof of Proposition 3.1

Let us define a new measure \( \mathbb{P}^{\Phi(\alpha)} \) by (3.20) for \( \kappa = \Phi(\alpha) \). Under this measure, \( X \) is a Lévy process with Laplace exponent [18, Corollary 3.10]:

\[
\psi_{\Phi(\alpha)}(\beta) = \psi(\beta + \Phi(\alpha)) - \alpha, \quad \beta > -\Phi(\alpha).
\]

Then, for any \( t > 0 \), the change of measure yields the expectation

\[
\mathbb{E}_x \left[ e^{-\alpha \tau^+_b} 1_{\{\tau^+_b < t\}} \right] = \mathbb{E}_{\Phi(\alpha)} \mathbb{E}_x \left[ e^{-\alpha \tau^+_b + \Phi(\alpha) (X_{\tau^+_b - b} - x)} \cdot e^{-\Phi(\alpha) X_{\tau^+_b}} 1_{\{\tau^+_b < t\}} \right] = \mathbb{E}^{\Phi(\alpha)} \left[ e^{-\Phi(\alpha) (X_{\tau^+_b - b} - x)} 1_{\{\tau^+_b < t\}} \right],
\]

\[(A.46)\]

We now let \( t \to \infty \) in \((A.46)\). By applying the monotone convergence theorem to the left hand side of \((A.46)\), we obtain \( \mathbb{E}_x \left[ e^{-\alpha \tau^+_b} 1_{\{\tau^+_b < \infty\}} \right] \). Similarly, notice that the non-negative random variable in the expectation \((A.46)\) is bounded by 1, \( \mathbb{P}^{\Phi(\alpha)} \)-a.s. we can apply the bounded convergence theorem to obtain that

\[
\mathbb{E}_x \left[ e^{-\alpha \tau^+_b} 1_{\{\tau^+_b < \infty\}} \right] = \mathbb{E}^{\Phi(\alpha)} \left[ e^{-\Phi(\alpha) X_{\tau^+_b - b}} 1_{\{\tau^+_b < \infty\}} \right] < \infty.
\]

Now because for any \( \beta \geq 0 \) we have \( e^{-\alpha \tau^+_b - \beta (X_{\tau^+_b + b})} 1_{\{\tau^+_b < \infty\}} \leq e^{-\alpha \tau^+_b} 1_{\{\tau^+_b < \infty\}}, \mathbb{P}_x\)-a.s., the dominated convergence theorem yields that

\[
\mathbb{E}_x \left[ e^{-\alpha \tau^+_b - \beta (X_{\tau^+_b + b})} 1_{\{\tau^+_b < \infty\}} \right] = e^{\beta (b - x)} \mathbb{E}^{\Phi(\alpha)} \left[ e^{-\Phi(\alpha) X_{\tau^+_b + 1}} 1_{\{\tau^+_b < \infty\}} \right].
\]

(A.47)
The right hand side of (A.47) can be computed using Lemma 1 of [1]. Precisely, we have

\[ \mathbb{E}^{\Phi(\alpha)}[e^{-(\Phi(\alpha)+\beta)\mathbf{X}_{e_0}}] = \lim_{q \to 0} \frac{\mathbb{E}^{\Phi(\alpha)}[e^{-(\Phi(\alpha)+\beta)\mathbf{X}_{e_0}} 1_{\mathbf{X}_{e_0} > b-x}]}{\mathbb{E}^{\Phi(\alpha)}[e^{-(\Phi(\alpha)+\beta)\mathbf{X}_{e_0}}]} \]  

(A.48)

The law of \( \mathbf{X}_{e_0} \) under \( \mathbb{P}^{\Phi(\alpha)} \) can be extracted from (3.16) and (3.18). More precisely, for any \( q > 0 \), let \( \tilde{T}_q := \{ \tilde{\rho}_{i,q} \}_{1 \leq i \leq |\tilde{T}_q|} \) and \( \tilde{J} := \{ \tilde{\eta}_j \}_{1 \leq j \leq |\tilde{J}|} \) be, respectively, the roots to

\[
\psi_{\Phi(\alpha)}(\tilde{\rho}_{i,q}) = q \quad \text{and} \quad \psi_{\Phi(\alpha)}(\tilde{\eta}_j) = \infty, \quad \text{s.t.} \quad \Re \tilde{\rho}_{i,q}, \Re \tilde{\eta}_j > 0, \quad \forall 1 \leq i \leq |\tilde{T}_q|, 1 \leq j \leq |\tilde{J}|,
\]

(A.49)

which are indexed in the same way as the elements of \( T_0 \) and \( J \). Then, we infer from (A.45) that \( \tilde{\eta}_j = \beta_0 - \Phi(\alpha) \), \( \tilde{\rho}_{1,q} < \tilde{\eta}_j \) and \( \Re \tilde{\rho}_{i,q} \geq \tilde{\eta}_1 \) for all \( i \geq 2 \). Similarly, we let \( \tilde{T}_0 := \{ \tilde{\rho}_{i,0} \}_{1 \leq i \leq |\tilde{T}_0|} \) be the roots to

\[
\psi_{\Phi(\alpha)}(\tilde{\rho}_{i,0}) = 0, \quad \text{s.t.} \quad \Re \tilde{\rho}_{i,0} > 0.
\]

From (3.14) and (A.45) we deduce that \( \tilde{T}_0 + \Phi(\alpha) = T_0 \) and \( \tilde{J} + \Phi(\alpha) = J \), which means that

\[
\rho_{i,\alpha} = \tilde{\rho}_{i,0} + \Phi(\alpha), \quad \forall 1 \leq i \leq |T_0|; \quad \eta_j = \tilde{\eta}_j + \Phi(\alpha), \quad \forall 1 \leq j \leq |J|.
\]

(A.50)

By our assumption, \( \tilde{\rho}_{i,0} \)'s are distinct and \( \tilde{\rho}_{1,0} = \rho_{1,\alpha} - \Phi(\alpha) = 0 \) and \( \Re \tilde{\rho}_{i,0} \geq \Re \tilde{\eta}_1 > 0 \) for all \( 2 \leq i \leq |T_0| \). Moreover, the fact that the roots \( \rho_{i,\alpha} \)'s are single implies that \( \psi_{\Phi(\alpha)}(\tilde{\rho}_{i,0}) = \psi'(\rho_{i,\alpha}) \neq 0 \), and hence each branch of \( \psi^{-1}_{\Phi(\alpha)}(q) = \tilde{\rho}_{i,q} \in \tilde{T}_q \) is locally a diffeomorphism around 0. It follows that \( \tilde{\rho}_{i,q} \)'s are also distinct for all sufficiently small \( q > 0 \). It follows from (3.16) and (3.18) that

\[
\mathbb{E}^{\Phi(\alpha)}[e^{-(\Phi(\alpha)+\beta)\mathbf{X}_{e_0}}] = \prod_{i=1}^{[\tilde{T}_0]} \frac{\hat{\rho}_{i,q}}{\hat{\rho}_{i,q} + \Phi(\alpha) + \beta} \prod_{j=1}^{[\tilde{J}]} \left( 1 + \frac{\Phi(\alpha) + \beta}{\tilde{\eta}_j} \right), \quad \forall \beta \geq 0,
\]

\[
\mathbb{P}^{\Phi(\alpha)}(\mathbf{X}_{e_0} \in dy) = \left( \prod_{i=1}^{[\tilde{T}_0]} \frac{\hat{\rho}_{i,q}}{\hat{\rho}_{i,q} + \Phi(\alpha) + \beta} \right) \left( \prod_{j=1}^{[\tilde{J}]} \frac{\tilde{\eta}_j - \hat{\rho}_{i,q}}{\tilde{\eta}_j} \right) \rho_{i,q} e^{-\hat{\rho}_{i,q}y} dy, \quad \forall y > 0.
\]

As \( q \downarrow 0 \), we have that \( \hat{\rho}_{i,q} \to \hat{\rho}_{i,0} \), and in particular, \( \hat{\rho}_{1,q} \to \hat{\rho}_{1,0} = 0 \) and \( \lim_{q \to 0} \hat{\rho}_{i,q} \neq 0 \) for all \( i \geq 2 \). As a result, for \( \beta > 0 \),

\[
\lim_{q \to 0} \frac{\hat{\rho}_{1,q}}{\mathbb{E}^{\Phi(\alpha)}[e^{-(\Phi(\alpha)+\beta)\mathbf{X}_{e_0}}]} = (\Phi(\alpha) + \beta) \cdot \prod_{i=2}^{[\tilde{T}_0]} \frac{\rho_{i,\alpha} + \Phi(\alpha) + \beta}{\hat{\rho}_{i,0}} \prod_{j=1}^{[\tilde{J}]} \frac{\tilde{\eta}_j - \hat{\rho}_{i,q}}{\tilde{\eta}_j + \Phi(\alpha) + \beta}
\]

\[
= (\rho_{1,\alpha} + \beta) \cdot \prod_{i=2}^{[\tilde{T}_0]} \frac{\rho_{i,\alpha} + \beta}{\hat{\rho}_{i,0}} \prod_{j=1}^{[\tilde{J}]} \frac{\tilde{\eta}_j}{\tilde{\eta}_j + \beta},
\]

(A.51)

where we used (A.50) in the second equality. Moreover, for \( x < b \) and \( \beta \geq 0 \), the ratio

\[
\frac{\mathbb{E}^{\Phi(\alpha)}[e^{-(\Phi(\alpha)+\beta)\mathbf{X}_{e_0}} 1_{\mathbf{X}_{e_0} > b-x}]}{\hat{\rho}_{1,q}} = \frac{1}{\hat{\rho}_{1,q}} \int_{(b-x,\infty)} e^{-(\Phi(\alpha)+\beta)y} \sum_{i=1}^{[\tilde{T}_0]} \left( \prod_{j=1}^{[\tilde{T}_0]} \frac{\hat{\rho}_{j,q} - \hat{\rho}_{i,q}}{\hat{\rho}_{j,q} - \hat{\rho}_{i,q}} \prod_{j=1}^{[\tilde{J}]} \frac{\tilde{\eta}_j - \hat{\rho}_{i,q}}{\tilde{\eta}_j} \right) \hat{\rho}_{i,q} e^{-\hat{\rho}_{i,q}y} dy
\]

\[
= \frac{1}{\hat{\rho}_{1,q}} \sum_{i=1}^{[\tilde{T}_0]} \left( \prod_{j=1}^{[\tilde{T}_0]} \frac{\hat{\rho}_{j,q} - \hat{\rho}_{i,q}}{\hat{\rho}_{j,q} - \hat{\rho}_{i,q}} \prod_{j=1}^{[\tilde{J}]} \frac{\tilde{\eta}_j - \hat{\rho}_{i,q}}{\tilde{\eta}_j} \right) \cdot \hat{\rho}_{i,q} e^{-(\Phi(\alpha)+\beta+\hat{\rho}_{i,q})(b-x)}
\]

\[
= \sum_{i=2}^{[\tilde{T}_0]} \frac{1}{\hat{\rho}_{i,q} - \hat{\rho}_{i,q}} \left( \prod_{j=1}^{[\tilde{T}_0]} \frac{\hat{\rho}_{j,q} - \hat{\rho}_{i,q}}{\hat{\rho}_{j,q} - \hat{\rho}_{i,q}} \prod_{j=1}^{[\tilde{J}]} \frac{\tilde{\eta}_j - \hat{\rho}_{i,q}}{\tilde{\eta}_j} \right) \cdot \hat{\rho}_{i,q} e^{-(\Phi(\alpha)+\beta+\hat{\rho}_{i,q})(b-x)}
\]

\[
+ \frac{1}{\hat{\rho}_{1,q} + \Phi(\alpha) + \beta} \left( \prod_{j=1}^{[\tilde{T}_0]} \frac{\hat{\rho}_{j,q} - \hat{\rho}_{1,q}}{\hat{\rho}_{j,q} - \hat{\rho}_{1,q}} \prod_{j=1}^{[\tilde{J}]} \frac{\tilde{\eta}_j - \hat{\rho}_{1,q}}{\tilde{\eta}_j} \right) e^{-(\Phi(\alpha)+\beta+\hat{\rho}_{1,q})(b-x)},
\]
which, as \( q \downarrow 0 \), tends to

\[
\sum_{i=1}^{\lvert \mathcal{I}_\alpha \rvert} \left( \prod_{j=1}^{\lvert \mathcal{J} \rvert} \frac{\tilde{\rho}_{j,0}}{\tilde{\rho}_{i,0}} \prod_{j=1}^{\lvert \mathcal{J} \rvert} \frac{\tilde{\eta}_j}{\tilde{\eta}_i} \right) \cdot \frac{-1}{\tilde{\rho}_{i,0} + \Phi(\alpha) + \beta} e^{-\Phi(\alpha + \beta + \tilde{\rho}_{i,0})(b-x)} + \frac{1}{\Phi(\alpha) + \beta} e^{-(\Phi(\alpha + \beta) + \tilde{\rho}_{i,0})(b-x)} \]

Combining (A.47), (A.48), (A.51) and (A.52), we obtain that, for all \( \beta \geq 0 \),

\[
E_x \left[ e^{-\alpha_x^+ - \beta(X_{t+b} - b)} \mathbf{1}_{\{x^+ < \infty\}} \right] = \sum_{i=2}^{\lvert \mathcal{I}_\alpha \rvert} \left( \prod_{j=1}^{\lvert \mathcal{J} \rvert} \frac{\rho_{j,0} + \beta}{\rho_{i,0} - \rho_{1,0}} \prod_{j=1}^{\lvert \mathcal{J} \rvert} \frac{\eta_j - \rho_{1,0}}{\eta_j + \beta} \right) \cdot \frac{\Phi(\alpha) - \beta}{\phi(1) - \rho_{1,0}} e^{-\rho_{1,0}(b-x)} + \prod_{j=2}^{\lvert \mathcal{J} \rvert} \frac{\rho_{j,0} + \beta}{\rho_{j,0} - \rho_{1,0}} \prod_{j=1}^{\lvert \mathcal{J} \rvert} \frac{\eta_j - \rho_{1,0}}{\eta_j + \beta} e^{-\rho_{1,0}(b-x)}
\]

This completes the proof.

### A.8 Proof of Theorem 3.1

We only need to prove the assertion for the case \( \alpha \leq 0 \) since the case of non-negative discount rate \( \alpha > 0 \) has been addressed in [23].

We begin by differentiating \( g^{(1)}(x, b) \) with respect to \( b > x \lor \log K \) to get

\[
\frac{\partial}{\partial b} g^{(1)}(x, b) = \frac{-e^b + K \psi^{+}_\alpha (-1)}{\psi^{+}_\alpha (-1)} \sum_{i=1}^{\lvert \mathcal{I}_\alpha \rvert} A_i \rho_{i,0} e^{-\rho_{i,0}(b-x)}, \quad \forall x < b. \tag{A.54}
\]

Clearly, \( x^+_1 = \log(K \psi^{+}_\alpha (-1)) \) satisfies the first order condition \( \partial g^{(1)}(b, x)/\partial b = 0 \). To show that \( x^+_1 \) is indeed the optimal exercise threshold, we only need to verify the followings (see, for example, [26]):

1. for \( x \geq x^+_1 \), \( g^{(1)}(x, \cdot) \) is decreasing for all \( b > x \) and \( \sup_{b \leq x} g^{(1)}(x, b) = \phi(x) = \lim_{b \downarrow x} g^{(1)}(x, b) \);
2. for \( x < x^+_1 \), \( g^{(1)}(x, \cdot) \) is increasing for all \( x < b \leq x^+_1 \) and is non-increasing for all \( b \geq x^+_1 \), and \( \sup_{b \leq x} g^{(1)}(x, b) = \phi(x) \leq \lim_{b \downarrow x} g^{(1)}(x, b) \).

Since \( \psi^{+}_\alpha (-1) > 0 \), it follows from (A.54) that the monotonicity of \( g^{(1)}(x, \cdot) \) for \( b > x \) amounts to showing that

\[
\sum_{i=1}^{\lvert \mathcal{I}_\alpha \rvert} A_i \rho_{i,0} e^{-\rho_{i,0} y} \geq 0, \quad \forall y > 0.
\]

By setting \( \beta = 0 \) in (A.51), we obtain

\[
\lim_{q \downarrow 0} \frac{\tilde{\rho}_{1, q}}{E^q [\Phi(\alpha)] e^{-\Phi(\alpha) X_{eq}}} = \rho_{1, \alpha} \cdot \prod_{i=2}^{\lvert \mathcal{I}_\alpha \rvert} \frac{\rho_{i, \alpha} - \rho_{1, \alpha}}{\rho_{1, \alpha} - \rho_{1, \alpha}} \prod_{j=1}^{\lvert \mathcal{J} \rvert} \frac{\eta_j - \rho_{1, \alpha}}{\eta_j} = \rho_{1, \alpha} \cdot A_1. \tag{A.55}
\]
Similarly, it follows from (A.52) that, for all \( y > 0 \),
\[
\frac{1}{\text{dy}} \lim_{q \downarrow 0} \frac{e^{-\Phi(\alpha)y}P^{\Phi(\alpha)}(X_{e_q} \in \text{dy})}{\rho_{i,q}} = -\sum_{i=2}^{[T_n]} \left( \prod_{j=2}^{[T_n]} \frac{\rho_{j,\alpha} - \rho_{1,\alpha}}{\rho_{j,\alpha} - \rho_{i,\alpha}} \prod_{j=1}^{[J]} \frac{\eta_j - \rho_{i,\alpha}}{\eta_j - \rho_{1,\alpha}} \right) e^{-\rho_{i,\alpha}y} + e^{-\rho_{1,\alpha}y}
\]
\[
= \sum_{i=2}^{[T_n]} \frac{\rho_{i,\alpha} \cdot A_i}{\rho_{i,\alpha} \cdot A_1} e^{-\rho_{i,\alpha}y} + e^{-\rho_{1,\alpha}y} = \sum_{i=1}^{[T_n]} \frac{\rho_{i,\alpha} \cdot A_i}{\rho_{i,\alpha} \cdot A_1} e^{-\rho_{i,\alpha}y}. \tag{A.56}
\]

From (A.55) and (A.56), we obtain
\[
\sum_{i=1}^{[T_n]} A_i \rho_{i,\alpha} e^{-\rho_{i,\alpha}y} = \frac{e^{-\Phi(\alpha)y}}{\text{dy}} \lim_{q \downarrow 0} \frac{P^{\Phi(\alpha)}(X_{e_q} \in \text{dy})}{\rho_{i,q}} \geq 0, \forall y > 0. \tag{A.57}
\]

To complete the proof that \( x_1^* \) is indeed the optimal exercise threshold, we need to show that, for any \( x \geq x_1^* \), \( \phi(x) \geq \lim_{b \downarrow x} g^{(1)}(x, b) \); and for \( x < x_1^* \), \( \phi(x) \leq \lim_{b \downarrow x} g^{(1)}(x, b) \). To this end, notice that \( \phi(x) = g^{(1)}(x, x_1^*) \) for all \( x \geq x_1^* \). On the other hand, using Corollary 3.1 we have that
\[
\lim_{b \downarrow x} g^{(1)}(x, b) = \left( 1 - \frac{\psi_0^+(\infty)}{\psi_0^-(\infty)} \right) e^x - K(1 - \psi_0^+(\infty)) = \phi(x) + \psi_0^+(\infty) \left( \frac{e^{x^*_1} - e^x}{\psi_0^-(\infty)} \right) = \begin{cases} \leq \phi(x), & \text{if } x \geq x_1^* \\ \geq \phi(x), & \text{if } x < x_1^* . \end{cases}
\]
Thus, \( b_k \) is indeed the optimal exercise threshold for any \( x \in \mathbb{R} \). Finally, (3.24) follows from (3.23) by setting \( b = x_1^* \).

### A.9 Proof of Proposition 3.2

First, notice that (3.23) and (3.32) imply that, for any fixed \( x \in \mathbb{R} \), the function \( g^{(k)}(x, b) \) is differentiable in \( b \) for all \( b > x \lor \log K \). Direct calculation (using (3.28)) gives the derivative
\[
\frac{\partial}{\partial b} g^{(k)}(x, b) = \sum_{i=1}^{[T_n]} A_i \rho_{i,\alpha} e^{-\rho_{i,\alpha}(b-x)} \left( \frac{e^{x^*_1} - e^b}{\psi_0^+(\infty)} + \frac{1}{\Phi_\infty} \mathbb{E}[e^{-\alpha \delta (v^{(k-1)}_+)(b + X_\delta)} - \rho_{i,\alpha} \phi_\infty v^{(k-1)}(b + X_\delta))] \right)
\]
\[
- \left[ \sum_{i=1}^{[T_n]} A_i \rho_{i,\alpha} e^{-\rho_{i,\alpha}(b-x)} \left( \mathbb{E}[e^{-\alpha \delta u^{(k-1)}(b + X_\delta)}(1 - \frac{\rho_{i,\alpha}}{\phi_\infty} + \nu(\{0\})) - \int_{(0,\infty)} \mathbb{E}[e^{-\alpha \delta v^{(k-1)}(b + y + X_\delta)}] \nu(dy)] \right) \right]
\]
\[
= \left[ \frac{e^{x^*_1} - e^b}{\psi_0^+(\infty)} \right] \mathbb{E}[e^{-\alpha \delta u^{(k-1)}(b + X_\delta)}] - \int_{[0,\infty)} \mathbb{E}[e^{-\alpha \delta v^{(k-1)}(b + y + X_\delta)}] \nu(dy)] \right) \times \left( \sum_{i=1}^{[T_n]} A_i \rho_{i,\alpha} e^{-\rho_{i,\alpha}(b-x)} \right) \tag{A.58}
\]

Recall the inequality (A.57) in the case with \( \alpha \leq 0 \). For \( \alpha > 0 \), we compute from (3.18) to get
\[
\sum_{i=1}^{[T_n]} A_i \rho_{i,\alpha} e^{-\rho_{i,\alpha}y} = \frac{1}{\text{dy}} \mathbb{E}(X_{e_q} \in \text{dy}) \geq 0, \forall y > 0.
\]

Notice that, due to the linear independence of \( e^{-\rho_{i,\alpha}y} \)’s, the left hand side of the above equation is strictly positive on all but a possibly finite set in \( \mathbb{R}_+ \). Moreover, for all \( x \geq x_1^* \), we have \( g^{(k)}(x, x_1^*) \geq \phi^{(k)}(x) = v^{(k)}(x) \geq g^{(k)}(x, b) \) for all \( b \geq x \), hence \( \frac{\partial}{\partial b} g^{(k)}(x, x_1^*) \leq 0 \). This implies that
\[
\frac{e^{x^*_1} - e^b}{\psi_0^+(\infty)} + \mathbb{E}[e^{-\alpha \delta u^{(k-1)}(b + X_\delta)}] \leq 0, \forall b \geq x_1^*.
\]
On the other hand, for all \(x \leq \log K\), from the proof of Proposition 2.1 we know that \(g^{(k)}(x, x_{k-1}^*) \geq g^{(k)}(x, x) = \phi^{(k)}(x)\). It follows that there exists at least a \(b \in [x, x_{k-1}^*] < [x, x_1^*] \) such that \(\frac{\partial}{\partial b} g^{(k)}(x, b) > 0\), and hence
\[
\frac{e^{x_1^*} - e^b}{\psi_0^{(k)}(-1)} + \mathbb{E}[e^{-\alpha \delta} u^{(k-1)}(b + X_b)] > 0.
\]

By the assumed continuity of \(u^{(k-1)}\), we know that there exists at least one solution to (3.35). If the optimal exercise threshold \(b_k^*\) exists, then we know that \(b_k^* \leq x_1^* < x_\infty < \infty\). For any fixed \(x < b_k^*\), the function \(g^{(k)}(x, b)\) is maximized at \(b = b_k^*\), hence \(\frac{\partial}{\partial b} g^{(k)}(x, b) = 0\) for all \(x < b_k^*\). This implies that \(b_k^*\) is a solution to (3.35).

A.10 Proof of Proposition 3.3

We will first prove an auxiliary lemma connecting measures \(\{\nu_i(dy)\}_{i=1}^{\mathcal{I}_n}\) with \(\nu(dy)\).

**Lemma A.1.** Let \((\nu_i)_+\) and \((\nu_i)_-\) be the right derivatives of \(\nu_i[0, z)\) and \(\nu[0, y)\), respectively. Then
\[
- \sum_{i=1}^{\mathcal{I}_n} A_i \rho_{i,\alpha} \left[ (\nu_i)_+(y)) [((\nu_i)_+(z)) = \nu'_+(z + y), \ \forall y, z > 0. \tag{A.59}
\]

**Proof.** We will prove (A.59) by using the bivariate Laplace transform. To this end, let \(\beta_1, \beta_2 \geq 0\) and that \(\beta_1 \neq \beta_2\), then by (3.28) and (3.29) we have
\[
- \int_{(0, \infty)} \int_{(0, \infty)} e^{-\beta y - \beta z} \sum_{i=1}^{\mathcal{I}_n} A_i \rho_{i,\alpha} \left[ (\nu_i)_+(y)) [((\nu_i)_+(z)) dydz
\]
\[
= - \sum_{i=1}^{\mathcal{I}_n} A_i \rho_{i,\alpha} \left( \frac{1}{\rho_{i,\alpha} + \beta_1 \psi_0^{(\beta_1)}} - \frac{1}{\phi_\infty} \right) \left( \frac{1}{\rho_{i,\alpha} + \beta_2 \psi_0^{(\beta_2)}} - \frac{1}{\phi_\infty} \right)
\]
\[
= \frac{1}{\beta_2 - \beta_1} \left( \frac{1}{\psi_0^{(\beta_1)}} - \frac{1}{\psi_0^{(\beta_2)}} \right) + \frac{1}{\phi_\infty} \left( 1 - \frac{\psi_0^{(\beta_1)}}{\psi_0^{(\beta_2)}} \right) - \frac{1}{\phi_\infty} \sum_{i=1}^{\mathcal{I}_n} A_i \rho_{i,\alpha}
\]
\[
= \frac{1}{\beta_2 - \beta_1} \left( \frac{1}{\psi_0^{(\beta_1)}} - \frac{1}{\psi_0^{(\beta_2)}} \right) + \frac{1}{\phi_\infty}, \tag{A.60}
\]

where, in the last equality, we used that fact that, if \(-J\) is a subordinator, then \(\phi_\infty = \infty\); otherwise, we have \(\psi_0^{(\infty)} = 0\), and
\[
\sum_{i=1}^{\mathcal{I}_n} A_i \rho_{i,\alpha} = \lim_{\beta \rightarrow \infty} \left( \beta \sum_{i=1}^{\mathcal{I}_n} A_i \rho_{i,\alpha} + \beta \right) = \lim_{\beta \rightarrow \infty} \beta \psi_0^{(\infty)}(\beta) = \phi_\infty.
\]

On the other hand, by (3.27) we have, for \(\beta_1, \beta_2 \geq 0\) such that \(\beta_1 \neq \beta_2\),
\[
\int_{(0, \infty)} \int_{(0, \infty)} e^{-\beta y - \beta z} \nu'_+(y+z) dydz = \int_{(0, \infty)} e^{-\beta y} \nu'_+(y) dydz - \int_{(0, \infty)} e^{-\beta z} \nu'_+(z) dz ddy
\]
\[
= \int_{(0, \infty)} e^{-\beta y} \left( \frac{\psi_0^{(\beta_1)}}{\beta_1} - \frac{1}{\phi_\infty} \right) \psi_0^{(\beta_2)} dydz = \frac{1}{\beta_2 - \beta_1} \left( \frac{1}{\beta_1} \psi_0^{(\beta_1)} - \frac{1}{\phi_\infty} \right) \left( \frac{1}{\beta_2} \psi_0^{(\beta_2)} + \frac{1}{\phi_\infty} \right). \tag{A.61}
\]

From (A.60) and (A.61) we know that (A.59) holds for all \(y, z > 0\). 
\boxed{20}
We are now ready to give the proof of Proposition 3.3. Since threshold-type strategies are optimal for problem (2.6) with up to \((k-1)\) exercise opportunities by assumption, it follows that \(u^{(k-1)}(x) = g^{(k-1)}(x, x^*_k)\) for all \(x \in \mathbb{R}\), and \(u^{(k-1)}(x) = \tilde{u}^{(k-1)}(x)\) for all \(x \in \mathbb{R}\) by comparing (3.33) and (3.36). Also, observe that \(g^{(k)}(x, b_k) = \phi^{(k)}(x)\) for \(x \geq b_k > \log K\). Applying this fact to (3.36), we get

\[
\tilde{u}^{(k)}(x) = \frac{\phi^{(k)}(x)}{\psi^{(k)}_\alpha} - \int_{[0, \infty]} \phi^{(k)}(x + y) \nu(dy)
\]

\[
= e^{x} + \mathbb{E}_x[e^{-\alpha \delta + \psi^{(k-1)}_\alpha}(X_k)] - \int_{[0, \infty]} (e^{x+y} - K + \mathbb{E}_x[e^{-\alpha \delta + \psi^{(k-1)}_\alpha}(y + X_k)]) \nu(dy)
\]

\[
= e^{x} \left[ \frac{1}{\phi^{(k)}_\alpha} - \left( \frac{1}{\psi^{(k)}_\alpha} - \frac{(-1)}{\phi^{(k)}_\alpha} \right) \right] + K + \mathbb{E}_x \left[ e^{-\alpha \delta + \psi^{(k-1)}_\alpha}(x + X_k) \nu(dy) \right] - \int_{[0, \infty]} \psi^{(k-1)}_\alpha(y + X_k) \nu(dy)
\]

\[
= e^{x} \left[ \frac{1}{\phi^{(k)}_\alpha} - \frac{(-1)}{\phi^{(k)}_\alpha} \right] + \mathbb{E}_x[e^{-\alpha \delta + \psi^{(k-1)}_\alpha}(x + X_k)] = \tilde{u}^{(k)}(x).
\]

(A.62)

For \(x < b_k\), we use (3.30) and (3.31) to write

\[
g^{(k)}(x, b_k) = \mathbb{E}_x[e^{-\alpha \tau^{(k)}_\alpha} \phi^{(k)}_\alpha(X_{\tau^{(k)}_k}) 1_{\{\tau^{(k)}_k < \infty\}}] = \sum_{i=1}^{|T_a|} A_i e^{\rho_{i,\alpha}(x-b_k)} \int_{[0, \infty]} \phi^{(k)}(b_k + y) \bar{\nu}_i(dy).
\]

It follows that,

\[
\frac{1}{\phi^{(k)}_\alpha} \frac{\partial}{\partial x} g^{(k)}(x, b_k) = \int_{[0, \infty]} \phi^{(k)}_\alpha(b_k + y) \left( \sum_{i=1}^{|T_a|} A_i e^{-\rho_{i,\alpha}(b_k-x)} \frac{\partial}{\partial x} \bar{\nu}_i(dy) \right),
\]

(A.63)

\[
\int_{[0, b_k-x]} g^{(k)}(x + z, b_k) \nu(dz) = \int_{[0, b_k-x]} \int_{[0, \infty]} \phi^{(k)}_\alpha(b_k + y) \left( \sum_{i=1}^{|T_a|} A_i e^{-\rho_{i,\alpha}(b_k-z-x)} \bar{\nu}_i(dy) \right) \nu(dz)
\]

\[
= \int_{[0, \infty]} \phi^{(k)}_\alpha(b_k + y) \left( \sum_{i=1}^{|T_a|} A_i e^{-\rho_{i,\alpha}(b_k-x)} \left( \int_{[0, b_k-x]} e^{\rho_{i,\alpha} z} \nu(dz) \right) \bar{\nu}_i(dy) \right).
\]

(A.64)

Moreover, from Lemma A.1 we know that

\[
\sum_{i=1}^{|T_a|} A_i e^{-\rho_{i,\alpha}(b_k-x)} \left( \frac{\rho_{i,\alpha}}{\phi^{(k)}_\alpha} - \int_{[0, b_k-x]} e^{\rho_{i,\alpha} z} \nu(dz) \right) \bar{\nu}_i(dy) = -\sum_{i=1}^{|T_a|} A_i \left[ \bar{\nu}_i'(b_k-x) \right] \nu(dy) = \nu(b_k - x + dy).
\]

(A.65)

From (A.63), (A.64) and (A.65) we have

\[
\tilde{u}^{(k)}(x) = \frac{1}{\phi^{(k)}_\alpha} \frac{\partial}{\partial x} g^{(k)}(x, b_k) - \int_{[0, b_k-x]} g^{(k)}(x + z, b_k) \nu(dz) - \int_{[b_k-x, \infty]} g^{(k)}(x + y, b_k) \nu(dy)
\]

\[
= \int_{[0, \infty]} \phi^{(k)}_\alpha(b_k + y) \nu(b_k - x + dy) - \int_{[b_k-x, \infty]} \phi^{(k)}_\alpha(x + y) \nu(dy)
\]

\[
= \int_{[b_k-x, \infty]} \phi^{(k)}_\alpha(x + y) \nu(dy) - \int_{[b_k-x, \infty]} \phi^{(k)}_\alpha(x + y) \nu(dy) = 0.
\]

(A.66)

As a result, we have \(\tilde{u}^{(k)}(x) \equiv 0\) for all \(x < b_k\). Combining this with (A.62) yields (3.37). Moreover, from the definition of \(\tilde{u}^{(k)}_0(x)\) in (3.35), we know that \(\tilde{u}^{(k)}(x)\) is non-increasing on \([b_k, \infty)\). The first order condition (3.35) shows that \(\tilde{u}^{(k)}\) is also continuous at \(b_k\), and thus \(\tilde{u}^{(k)}\) is non-positive on \(\mathbb{R}\).
A.11 Proof of Lemma 3.2

Suppose there are two distinct solutions to (3.35), say, \( b_k < b_{k}' \). Then by Proposition 3.3 we have a non-increasing, non-positive function

\[
I_{\{x \geq b_k\}} \left( \frac{e^{x_1'} - e^{x'}}{\psi_\alpha'(-1)} + \mathbb{E}[e^{-\alpha\delta u^{(k-1)}(x + X_\delta)]} \right), \forall x \in \mathbb{R}.
\]

Moreover, this function is strictly decreasing for all \( x \geq b_k \). This implies that

\[
\frac{e^{x_1'} - e^{b_{k}''}}{\psi_\alpha'(-1)} + \mathbb{E}[e^{-\alpha\delta u^{(k-1)}(b_{k}' + X_\delta)]} < 0.
\]

This contradicts the assumption that \( b_{k}' \) solves (3.35).

To finish the proof, we let \( b_k \) be the unique solution to (3.35). Notice from (A.58) that

1. for all fixed \( x \geq b_k \), the function \( g^{(k)}(x, \cdot) \) is decreasing in \( b \) for all \( b > x \);
2. for all fixed \( x < b_k \), the function \( g^{(k)}(x, \cdot) \) is increasing in \( b \) for all \( x < b \leq b_k \), and decreasing in \( b \) for all \( b \geq b_k \).

We now apply a similar argument as in the proof of Theorem 3.1 to show that, for all \( x \geq b_k \), \( \lim_{b \downarrow x} g^{(k)}(x, b) \leq \phi^{(k)}(x) \); and for all \( x < b_k \), \( \lim_{b \uparrow x} g^{(k)}(x, b) \geq \phi^{(k)}(x) \). This will allow us to conclude that \( b_k \) is indeed the optimal exercise threshold. To this end, from (3.16), (3.21), (3.27), (3.29) and the fact that \( \psi_\alpha^+(\infty) \cdot \phi_\infty = 0 \), we know that

\[
\sum_{i=1}^{[x]} A_i \bar{\nu}_i(dy) = I_{(y=0)} - \psi_\alpha^+(\infty)\nu(dy).
\]

Below we consider two cases separately:

1. if \( -J \) is not a subordinator, then \( \psi_\alpha^+(\infty) = 0 \) and \( \phi_\infty > 0 \). Using (3.30) and (3.31) and the monotone convergence theorem, we obtain the limit for any fixed \( x \in \mathbb{R} \):

\[
\lim_{b \downarrow x} g^{(k)}(x, b) = \int_{[0, \infty)} \phi^{(k)}(x + y) \left( \sum_{i=1}^{[x]} A_i \bar{\nu}_i(dy) \right) = \phi^{(k)}(x),
\]

where we used (A.67) in the last equality.

2. if \( -J \) is a subordinator, then \( \psi_\alpha^+(\infty) > 0 \) and \( \phi_\infty = 0 \). Similarly to (A.68), we use (A.67) to obtain that, for any fixed \( x \in \mathbb{R} \),

\[
\lim_{b \downarrow x} g^{(k)}(x, b) = \int_{[0, \infty)} \phi^{(k)}(x + y) \left( \sum_{i=1}^{[x]} A_i \bar{\nu}_i(dy) \right) = \phi^{(k)}(x) - \psi_\alpha^+(\infty) \int_{[0, \infty)} \phi^{(k)}(x + y)\nu(dy)
\]

\[
= \phi^{(k)}(x) - \psi_\alpha^+(\infty) \left( e^{x+y} - K + \mathbb{E}[e^{-\alpha\delta u^{(k-1)}(x + y + X_\delta)}] \right) \nu(dy)
\]

\[
= \phi^{(k)}(x) + \psi_\alpha^+(\infty) \left( \frac{e^{x_1'} - e^x}{\psi_\alpha'(-1)} + \mathbb{E}[e^{-\alpha\delta u^{(k-1)}(x + X_\delta)}] \right)
\]

\[
= \phi^{(k)}(x) + \psi_\alpha^+(\infty) \tilde{u}_0^{(k)}(x) = \begin{cases} 
\leq \phi^{(k)}(x), & \text{if } x \geq b_k \\
> \phi^{(k)}(x), & \text{if } x < b_k 
\end{cases}
\]

where we have used (3.27), (3.33) and (3.35) in the third line, and the fact that \( \tilde{u}_0^{(k)}(x) > 0 \) if and only if \( x < b_k \) in the last step.
A.12 Proof of Proposition 3.4

For any fixed \( x \in \mathbb{R} \), notice that \( \{ x + X_{\tau_{b^\star}^-} \geq b^\star_k \} = \{ \tau_{b^\star}^- < x \leq e_q \} \) except for a null set under measure \( \mathbb{P}^{\Phi(\alpha)} \) and that \( X_{\tau_{b^\star}^-} = X_{\tau_{b^\star}^-}^- \) on \( \{ \tau_{b^\star}^- < \infty \} \). Also, recall that \( \tilde{u}_0^{(k)}(x) \mathbb{I}_{\{ x \geq b^\star_k \}} = \tilde{u}^{(k)}(x) \) for all \( x \in \mathbb{R} \). We thus have

\[
\lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)X_{\tau_{b^\star}^-}}(x + X_{\tau_{b^\star}^-}) \mathbb{I}_{\{ x + X_{\tau_{b^\star}^-} \geq b^\star_k \}]}{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)X_{\tau_{b^\star}^-}}]} = \lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)X_{\tau_{b^\star}^-}}(-\tilde{u}_0^{(k)}(x + X_{\tau_{b^\star}^-})) \mathbb{I}_{\{ \tau_{b^\star}^- \leq e_q \}]}{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)X_{\tau_{b^\star}^-}}]}
\]

\[
= \lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(\alpha)}[\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)X_{\tau_{b^\star}^-}}(x + X_{\tau_{b^\star}^-}) \mathbb{I}_{\{ \tau_{b^\star}^- \leq e_q \}]} | \mathcal{F}_{\tau_{b^\star}^+}]}{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)X_{\tau_{b^\star}^-}}]}
\]

\[
= \lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(\alpha)}[\mathbb{I}_{\{ \tau_{b^\star}^- \leq e_q \}} \mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)(X_{\tau_{b^\star}^-} - \tau_{b^\star}^-)} \mathbb{I}_{\{ \tau_{b^\star}^- \leq e_q \}} | \mathcal{F}_{\tau_{b^\star}^+}]]}{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)X_{\tau_{b^\star}^-}}]}
\]

(A.70)

Now let us denote by \( M_q := X_{\tau_{b^\star}^-} - X_{\tau_{b^\star}^-} \). Notice that \( M_q \equiv X_{\tau_{b^\star}^-} \) and \( M_q \) is independent of \( \mathcal{F}_{\tau_{b^\star}^-} \) on the event \( \{ \tau_{b^\star}^- \leq e_q \} \), so expression (A.70) above is further equal to

\[
\lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(\alpha)}[\mathbb{I}_{\{ \tau_{b^\star}^- \leq e_q \}} e^{-\Phi(\alpha)(X_{\tau_{b^\star}^-} - \tau_{b^\star}^-)} \mathbb{I}_{\{ \tau_{b^\star}^- \leq e_q \}} | X_{\tau_{b^\star}^-}]}{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)X_{\tau_{b^\star}^-}}]}
\]

\[
= \lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)M_q} \mathbb{I}_{\{ \tau_{b^\star}^- \leq e_q \}} | X_{\tau_{b^\star}^-}]}{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)M_q}]} \mathbb{I}_{\{ \tau_{b^\star}^- < \infty \}}
\]

(A.71)

In the last equality, we have applied a change of measure, along with the dominated convergence theorem (see (A.47)).

On the other hand, using the recursion (3.37) and mathematical induction we can show that there exist positive constants \( C_1, C_2 > 0 \) such that

\[
|\tilde{u}_0^{(k)}(x)| \leq C_1 e^{\tau_{b^\star}^-} + C_2, \quad \forall x \in \mathbb{R}.
\]

As a result, the random variable

\[
e^{-q \tau_{b^\star}^-} \frac{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)M_q} \mathbb{I}_{\{ \tau_{b^\star}^- \leq e_q \}} | X_{\tau_{b^\star}^-}]}{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)M_q}]} \mathbb{I}_{\{ \tau_{b^\star}^- < \infty \}}
\]

is dominated by the non-negative random variable

\[
e^{-q \tau_{b^\star}^-} \frac{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)M_q} \mathbb{I}_{\{ \tau_{b^\star}^- \leq e_q \}} | X_{\tau_{b^\star}^-}]}{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)M_q}]} \mathbb{I}_{\{ \tau_{b^\star}^- < \infty \}}
\]

\[
\leq \left( C_1 e^{X_{\tau_{b^\star}^-}} + C_2 \right) \mathbb{I}_{\{ \tau_{b^\star}^- < \infty \}} \lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)M_q} \mathbb{I}_{\{ \tau_{b^\star}^- \leq e_q \}} | X_{\tau_{b^\star}^-}]}{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)M_q}]} = C_1X_{\tau_{b^\star}^-} \mathbb{I}_{\{ \tau_{b^\star}^- < \infty \}}
\]

where we used the fact that \( \lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)M_q} \mathbb{I}_{\{ \tau_{b^\star}^- \leq e_q \}}]}{\mathbb{E}^{\Phi(\alpha)}[e^{-\Phi(\alpha)M_q}]} = \psi_\alpha^+(1) \) in the last step.
Similarly, using (3.39) we have
\[
\phi^{(k)}(x) = e^x - K + \mathbb{E}[e^{-\alpha \delta} \phi^{(k-1)}(x + X_\delta)] \\
= \lim_{q \downarrow 0} \mathbb{E}^{\Phi(\alpha)} \left[ \frac{e^{-\Phi(\alpha) X_{\epsilon_q}}}{e^{-\Phi(\alpha) X_{\epsilon_0}}} \left( e^{x + X_{\epsilon_q}} - e^{x} \right) \mathbb{I}_{\{x + X_{\epsilon_q} \geq b_k^*\}} \right] \\
= \lim_{q \downarrow 0} \mathbb{E}^{\Phi(\alpha)} \left[ e^{-\Phi(\alpha) M_q} \left( -\tilde{u}^{(k)}(x + M_q) \right) \right].
\] (A.72)

By the dominated convergence theorem, we obtain from (A.71) and (A.72) that
\[
\lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(\alpha)} \left[ e^{-\Phi(\alpha) X_{\epsilon_q}} (-\tilde{u}^{(k)}(x + X_{\epsilon_q})) \right]}{\mathbb{E}^{\Phi(\alpha)}(e^{-\Phi(\alpha) X_{\epsilon_q}})} = \mathbb{E}_x \left[ e^{-\alpha t} g^{(k)}(X_{\tau^+_k}; 1_{\{\tau^+_k < \infty\}}) \right] = g^{(k)}(x, b_k^*).
\]

To prove the supermartingale property, we use the fact that, on the event \(\{t < \epsilon_q\}\), we have \(X_t + \sup_{s \leq t} \mathbb{E}_s(X_s - X_t) \leq X_{\epsilon_q}, \mathbb{P}^{\Phi(\alpha)}\)-a.s. and \(M_q := \sup_{s \leq t} (X_s - X_t)\) has the same law as \(X_{\epsilon_q}\), but is independent of \(\mathcal{F}_t\). It follows from the non-negativity and the non-decreasing property of \(-\tilde{u}^{(k)}\) that, for any \(t > 0\),
\[
g^{(k)}(x, b_k^*) = \lim_{q \downarrow 0} \mathbb{E}^{\Phi(\alpha)} \left[ e^{-\Phi(\alpha) X_{\epsilon_q}} (-\tilde{u}^{(k)}(x + X_{\epsilon_q})) \right] \\
\geq \lim_{q \downarrow 0} \frac{\mathbb{E}^{\Phi(\alpha)} \left[ e^{-\Phi(\alpha) X_{\epsilon_q}} (-\tilde{u}^{(k)}(x + X_{\epsilon_q})) \right]}{\mathbb{E}^{\Phi(\alpha)}(e^{-\Phi(\alpha) X_{\epsilon_q}})} \\
= \lim_{q \downarrow 0} e^{-q t} \mathbb{E}^{\Phi(\alpha)} \left[ e^{-\Phi(\alpha) (X_t + M_q)} (-\tilde{u}^{(k)}(x + X_t + M_q)) \right] \\
\geq \mathbb{E}^{\Phi(\alpha)} \left[ e^{-\Phi(\alpha) X_t} \lim_{q \downarrow 0} \left( \frac{e^{-\Phi(\alpha) M_q} (-\tilde{u}^{(k)}(x + X_t + M_q))}{e^{-\Phi(\alpha) M_q}} \right) \right] \\
= \mathbb{E}_x \left[ e^{-\alpha t} g^{(k)}(X_t, b_k^*) \right] = \mathbb{E}_x \left[ e^{-\alpha t} g^{(k)}(X_t, b_k^*) \right].
\]

Here the third inequality follows from Fatou’s lemma and the Strong Markov property of \(X\). This completes the proof.

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