Physical spectrum from confined excitations in a Yang-Mills-inspired toy model

M.A.L. Capri\textsuperscript{a,∗}, D. Dudal\textsuperscript{b,†}, M.S. Guimaraes\textsuperscript{a,‡}, L.F. Palhares\textsuperscript{a,§}, S.P. Sorella\textsuperscript{a,¶}

\textsuperscript{a} Departamento de Física Teórica, Instituto de Física, UERJ - Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier 524, 20550-013 Maracanã, Rio de Janeiro, Brasil
\textsuperscript{b} Ghent University, Department of Physics and Astronomy, Krijgslaan 281-S9, 9000 Gent, Belgium

Abstract

We study a toy model for an interacting scalar field theory in which the fundamental excitations are confined in the sense of having unphysical, positivity-violating propagators, a fact tracing back to a decomposition of these in propagators with complex conjugate mass poles (the so-called \textit{i}-particles). Similar two-point functions show up in certain approaches to gluon or quark propagators in Yang-Mills gauge theories. We investigate the spectrum of our model and show that suitable composite operators may be constructed having a well-defined Källén-Lehmann spectral representation, thus allowing for a particle interpretation. These physical excitations would correspond to the "mesons" of the model, the latter being bound states of two unphysical \textit{i}-particles. The meson mass is explicitly estimated from the pole emerging in a resummed class of diagrams.

1 Introduction

The analytic study of the spectrum predicted by confining theories still remains a major theoretical challenge. In particular, despite the intense efforts dedicated to investigating Yang-Mills theories and QCD using different nonperturbative tools, a complete understanding of the mechanism of confinement is not yet available, let stand alone a clean-cut theoretical derivation of the spectrum of massive hadrons and glueballs from (almost) massless gluons and quarks. Such efforts can be appreciated from e.g. tools like Dyson-Schwinger equations \cite{1,2} or sum rules approaches \cite{3,4,5}.

During the last decade there have been many interesting developments in the direction of describing the dynamics of confined degrees of freedom in Yang-Mills theories. Both analytic and lattice methods were broadly used to investigate the two-point functions of the elementary fields of quarks and gluons, where the Landau gauge-fixing has been extensively employed. Recently, very precise lattice studies brought the community to a consensus. The gluon propagator differs sensibly from its perturbative form of the Faddeev-Popov type: in the deep infrared regime it is characterized by a clear positivity violation \cite{6,7} and achieves a non-vanishing value at zero momentum in $D = 4$, 3 dimensions \cite{8,9,10,11,12,13,14,15}, while it vanishes at the origin in momentum space in $D = 2$ \cite{11,16}.

From a theoretical point of view, the infrared behavior of the gluon propagator has been object of a quite intense debate over many years, requiring a common effort from several groups. Different nonperturbative techniques have been employed, such as: Dyson-Schwinger equations \cite{17,18,19,20,21,22}, the renormalization group equations \cite{23,24}, effective field theory framework \cite{25,26}, the Gribov-Zwanziger approach \cite{27,28} and its more recent refined version \cite{29,30,31,32,33}, and other approaches \cite{34,35}. These efforts have culminated in the so-called decoupling solution for the gluon propagator, which turns out to be in very good agreement with the lattice
data in $D = 4, 3$ dimensions. Moreover, in $D = 2$, these methods coalesce towards a scaling type solution which, unlike the decoupling case, vanishes at the origin in momentum space, being in agreement with the numerical data in $D = 2$ [36, 37].

One of the analytic expressions [14, 38, 39, 15] that is capable of precisely describing lattice data in the infrared domain in $D = 4, 3$ dimensions and that allows for an analytic treatment of the calculation of Green’s functions is provided by the Refined Gribov-Zwanziger (RGZ) scenario. The Gribov-Zwanziger (GZ) action [27, 28] is associated with a formulation of the gauge path integral that takes into account the presence of Gribov copies. Its refined version [29, 30, 33] accounts for the presence of dimension 2 condensates which are dynamically generated by the restriction to the Gribov region, necessary to deal with the Gribov issue. Interestingly, the RGZ analytic expressions for the gluon propagator are in good accordance with the most recent lattice simulations and can be naturally decomposed into two quasi-degrees of freedom with complex-conjugated squared masses, which have been called $i$-particles [40].

Despite these significant advances in the description of propagators of confined elementary degrees of freedom, the emergence of a physical spectrum out of the confined elementary excitations still remains an open question from the analytic perspective. With the aim of taking advantage of these developments in the description of gluon propagators to shed light into the spectrum problem, encouraging estimates have been found for the spectrum of the lightest glueball states using analytic tools available in the RGZ scenario [41]. Moreover, the inclusion of quark degrees of freedom in the RGZ action seems feasible and interesting results concerning also chiral symmetry and the meson spectrum are currently being investigated.

It should be noticed that a useful analytical form of the propagators is crucial in the construction of a framework that can assess these physical questions that are usually not amenable to analytical tools.

In this paper we investigate the effects of interactions in a toy model based on the idea of confinement as implemented via the RGZ propagator decomposed into $i$-particles. Our goal is to understand how these complex modes may combine themselves to generate physical propagating modes in the correlation functions of suitable composite operators (= bound states). Of course, tackling this problem directly in QCD or even in pure Yang-Mills is an involved task, due to the non-Abelian complex nature of the interactions. It is therefore instructive to approach the problem through a toy model, obtaining a qualitative description of the mechanism of the formation of bound states in theories with unphysical elementary excitations.

Previous descriptions in this line have used the propagators with lattice-fitted parameters as nonperturbative objects that carried information about confinement and estimates of the lightest glueball masses without including interactions explicitly have been obtained. In the current work, we shall combine the nonperturbative propagators with interactions defined in a toy model so that it is possible to resum a whole class of diagrams contributing to the two-point function of composite operators. The result for the interacting correlator displays a well-defined Källén-Lehmann spectral representation. Moreover, we show explicitly the appearance of physical poles, associated with bound states of $i$-particles. This picture provides therefore a concrete qualitative understanding of the dynamical formation of the physical spectrum of a confining theory, defined in terms unphysical, positivity-violating, elementary fields.

This paper is organized as follows. In the next section the confining toy model of $i$-particles is presented, including in detail our modeling of the interactions and the definition of the physical composite operators. In section 3 we describe the nonperturbative computation of the two-point function of a suitable physical composite operator. Results for the spectrum of bound states in dimensions $D = 2, 3$ and 4 are discussed in section 4. The last section is then dedicated to conclusions and final remarks.

2 The confining scalar field theory

In this section we describe the model to be studied in the following. We start by defining the free sector of the theory highlighting its confining properties and their formulation in terms of $i$-particles. Then we discuss the introduction of interactions, which should be ultimately responsible for generating a physical spectrum of bound states.

1. In the sense that one can compute quantities with it in a controllable fashion.
2.1 The free confining theory

The model we will study has its free sector defined in such a way as to display a confining propagator of the type found in the RGZ framework [29, 30, 33]:

\[
D(k) = \frac{k^2 + m^2}{k^4 + k^2(M^2 + m^2) + M^2m^2 + 2\theta^4} = \frac{R_+}{k^2 + M_+^2} + \frac{R_-}{k^2 + M_-^2}
\]  

(2.1)

where \( M \) and \( m \) are real masses, \( \theta \) is a real massive parameter and

\[
M_{\pm}^2 = \frac{(M^2 + m^2)}{2} \pm \sqrt{\left(\frac{(M^2 - m^2)^2}{4} - 2\theta^4\right)}
\]

(2.2)

where \( R_{\pm} = \frac{m^2 - M_{\pm}^2}{M^2 - M_{\pm}^2} \). The right-hand side of eq. (2.1) suggests that \( D(k) \) encodes the propagation of two quasi-modes with potentially complex conjugated poles, if \((M^2 - m^2)^2 < 8\theta^4\). This clearly unphysical feature is interpreted as a manifestation of confinement.

The simplest model leading to this type of propagator is of the form

\[
S = \int d^Dx \frac{1}{2} \left( -\partial^2 + M^2 + 2\theta^4 \right) \psi
\]

(2.3)

from which one easily checks that the scalar field \( \psi \) has a free propagator \( \langle \psi(k)\psi(-k) \rangle = D(k) \). This model can be rewritten in a local form through the introduction of auxiliary fields

\[
S = \int d^Dx \left( \frac{1}{2} \lambda(-\partial^2 + M_+^2)\lambda + \frac{1}{2} \eta(-\partial^2 + M_-^2)\eta - \frac{1}{2} V(-\partial^2 + m^2)V + \bar{\phi}\bar{\phi}(-\partial^2 + m^2)\phi\phi + \bar{\omega}\bar{\omega}(-\partial^2 + m^2)\omega \right)
\]

(2.4)

where \((\phi, \bar{\phi})\) is a pair of bosonic complex conjugated fields and \((\omega, \bar{\omega})\) is a pair of anticommuting fields.

A version of this system was studied in [40] with \( M = m = 0 \), corresponding to the original Gribov propagator. Since this is a quadratic action it can be cast in a complete diagonal form through a change of variables \((\psi, \phi, \bar{\phi}) \rightarrow (V, \lambda, \eta)\)

\[
S = \int d^Dx \left( \frac{1}{2} \lambda(-\partial^2 + M_+^2)\lambda + \frac{1}{2} \eta(-\partial^2 + M_-^2)\eta - \frac{1}{2} V(-\partial^2 + m^2)V + \bar{\phi}\bar{\phi}(-\partial^2 + m^2)\phi\phi + \bar{\omega}\bar{\omega}(-\partial^2 + m^2)\omega \right).
\]

(2.5)

where \( \lambda, \eta \) and \( V \) are real fields, with \( V \) being the imaginary part of \( \phi \).

In order to simplify the analysis, in what follows we shall set \( M = m \). In this case the fields \( U, \bar{\omega} \) and \( \omega \) decouple and the action (2.5) reduces to

\[
S = \int d^Dx \left( \frac{1}{2} \lambda(-\partial^2 + m^2 + i\sqrt{2}\theta^2)\lambda + \frac{1}{2} \eta(-\partial^2 + m^2 - i\sqrt{2}\theta^2)\eta \right).
\]

(2.6)

From this expression one immediately sees that the fields \( \lambda \) and \( \eta \) correspond to the propagation of unphysical modes with complex masses \( m^2 \pm i\sqrt{2}\theta^2 \). These are the \( i \)-particles of the model, namely

\[
\langle \lambda(k)\lambda(-k) \rangle = \frac{1}{k^2 + m^2 + i\sqrt{2}\theta^2}
\]

(2.7)

\[
\langle \eta(k)\eta(-k) \rangle = \frac{1}{k^2 + m^2 - i\sqrt{2}\theta^2}
\]

(2.8)

Expression (2.6) describes a theory in which the fundamental excitations are not part of the physical spectrum. More precisely, it is not possible to analytically continue the propagators (2.8) to Minkowski space-time in order to obtain a well defined particle interpretation for the fundamental fields \( \lambda \) and \( \eta \). In this sense, we might say that the model displays tree-level confinement.

Note that, even though the \( i \)-particles action (2.6) has imaginary mass terms, it is Hermitian if we observe that \( \lambda^\dagger = \eta \), which follows from \( \phi^\dagger = \phi \). As the actions (2.5) and (2.3) are equivalent and the latter is clearly Hermitian, so should the former be.
2.2 Interactions

Many properties of the action (2.6) were studied in [40]. In the present work we analyze the system including an interaction between the \( \iota \)-particles. Despite of the use of unphysical elementary modes, we shall be able to show that the interacting system does display a spectrum containing meson-like physical states.

In principle there are many ways of introducing interactions in a model with the tree level propagator in eq. (2.1).

To that purpose, we first note that the action (2.3) is invariant under the replacement \( \theta^2 \rightarrow -\theta^2 \). In the \( \iota \)-particles formulation (2.6) this symmetry can be understood by noting that in a path integral formulation \( \lambda \) and \( \eta \) are just dummy integration variables. In fact, the partition function is easily seen to be left invariant by \( \theta^2 \rightarrow -\theta^2 \) and \( \lambda \leftrightarrow \eta \). Therefore, we shall require that the interactions preserve this feature.

The simplest interaction we can add to a scalar model is a quartic coupling. Thus we propose the following \( \iota \)-particles interacting model:

\[
S = \int d^D x \left( \frac{1}{2} \lambda (-\partial^2 + m^2 + i\sqrt{2}\theta^2)\lambda + \frac{1}{2} \eta (-\partial^2 + m^2 - i\sqrt{2}\theta^2)\eta + g_1 (\lambda \eta)^2 + g_2 (\lambda^4 + \eta^4) \right) ,
\]

where \( g_1 \) and \( g_2 \) are real couplings.

It is interesting to see how these interactions reflect on the original formulation (2.4) (with \( M = m \)). If we write \( \phi = \frac{\iota}{\sqrt{2}} (U + iV) \) with \( U \) and \( V \) real fields, one realizes that \( V \) decouples from \( \psi \). In fact, it turns out that the sector of the original scalar model corresponding to (2.9) is given by

\[
S = \int d^D x \left( \frac{1}{2} \psi (-\partial^2 + m^2)\psi - \frac{1}{2} U (-\partial^2 + m^2)U + \sqrt{2}\theta^2\psi U + \tilde{g}_1 (\psi U)^2 + \tilde{g}_2 (\psi^4 + U^4) \right) \tag{2.10}
\]

where \( \tilde{g}_1 = \frac{g_1}{4} - 3g_2 \) and \( \tilde{g}_2 = \frac{g_1}{4} + \frac{g_2}{2} \). The relation with the \( \iota \)-particles, \( \lambda, \eta \), is provided by setting

\[
\psi = \frac{1}{\sqrt{2}} (\lambda + \eta) \\
V = \frac{i}{\sqrt{2}} (\lambda - \eta) . \tag{2.11}
\]

Another way of defining the same physics is to start from the so-called replica model introduced in [42]. In this model we consider two copies of the same theory and couple them through a soft term that has the same effect as the imaginary masses discussed above. Consider for instance two scalar fields \( \phi_1 \) and \( \phi_2 \), each described by a theory with quartic coupling. In this case, the replica model is given by the action

\[
S = \int d^D x \left( \frac{1}{2} \phi_1 (-\partial^2 + m^2)\phi_1 + \frac{1}{2} \phi_2 (-\partial^2 + m^2)\phi_2 + g (\phi_1^4 + \phi_2^4) + i\sqrt{2}\theta^2 \phi_1 \phi_2 \right) \tag{2.12}
\]

The fact that a unique mass \( m \) as well as a unique quartic coupling \( g \) has been employed in expression (2.12) follows by demanding that the action displays the mirror symmetry (22).

\[
\phi_1 \rightarrow \phi_2 , \quad \phi_2 \rightarrow \phi_1 . \tag{2.13}
\]

The last term in expression (2.12), which contains the mass parameter \( \theta^2 \), implements a soft coupling between the two replica. In fact, this action can be cast into an \( \iota \)-particles formulation through the change of variables:

\[
\phi_1 = \frac{1}{\sqrt{2}} (\lambda - \eta) \\
\phi_2 = \frac{1}{\sqrt{2}} (\lambda + \eta) \tag{2.14}
\]

giving

\[
S = \int d^D x \left( \frac{1}{2} \lambda (-\partial^2 + m^2 + i\sqrt{2}\theta^2)\lambda + \frac{1}{2} \eta (-\partial^2 + m^2 - i\sqrt{2}\theta^2)\eta + 3g (\lambda \eta)^2 + \frac{g}{2} (\lambda^4 + \eta^4) \right) , \tag{2.15}
\]

which describes the same physics as expression (2.9), provided one sets \( g_1 = 6g_2 \) in order to implement the mirror symmetry.
2.3 Physical operators

The \( i \)-particles formulation shows clearly that the excitations corresponding to the elementary fields \( (\lambda, \eta) \) are unphysical. Nevertheless, following the construction outlined in \cite{40,42}, physical states can be introduced by constructing suitable composite operators out of the fields \( (\lambda, \eta) \) which exhibit desirable analyticity properties, as encoded in the Källén-Lehmann spectral representation.

Such composite operators are obtained by requiring that the fields \( (\lambda, \eta) \) enter pairwise, \textit{i.e.} the operator contains as many fields of the type \( \lambda \) as of the type \( \eta \). This will ensure that in the corresponding correlation function only complex conjugate pairs of \( i \)-particles will propagate in the Feynman diagrams, a property which provides a good analytic structure. In practice, these operators can be obtained by requiring that their correlation function is left invariant by interchanging \( \lambda \) and \( \eta \) \cite{40,42}.

In the present case, the simplest example of a local composite operator with the required physical properties is \( \mathcal{O}(x) = \lambda(x)\eta(x) \). In fact, at lowest order in the interactions (\( g_1 = g_2 = 0 \)), it is known that this operator has good analytical properties. In \cite{40} it was shown that the correlation function \( \langle \mathcal{O}(k)\mathcal{O}(-k) \rangle_0 \) has a well defined Källén-Lehmann spectral representation:

\[
\mathcal{F}_D(k^2) \equiv \langle \mathcal{O}(k)\mathcal{O}(-k) \rangle_{0,D} = \int \frac{d^Dp}{(2\pi)^D} \frac{1}{(k-p)^2 + m^2 - i\sqrt{2}\theta^2} \frac{1}{\tau_0 + k^2 + \tau},
\]

where the spectral functions \( \rho_D \) have the following expressions, see \cite{40} for details:

\[
\rho_{D=2}(\tau) = \frac{1}{2\pi \sqrt{\tau^2 - 8\theta^4 - 4m^2\tau}}, \quad \rho_{D=3}(\tau) = \frac{1}{8\pi \sqrt{\tau}}, \quad \rho_{D=4}(\tau) = \frac{1}{(4\pi)^2} \sqrt{1 - \frac{8\theta^4}{\tau^2} - \frac{4m^2}{\tau}},
\]

The threshold \( \tau_0 \) is in all cases given by

\[
\tau_0 = 2 \left( m^2 + \sqrt{m^4 + 2\theta^4} \right).
\]

It is important to observe that all spectral functions \( \rho_D \) are positive in the corresponding range of integration. This is a necessary condition in order to interpret \eqref{2.16} as the propagation of a physical mode in Minkowski space.

Notice also that, in \( D = 4 \), the integral \eqref{2.16} is divergent in the ultraviolet, meaning that the correlation function has to be properly renormalized. As it is customary when dealing with spectral representation, this can be implemented by employing a subtracted dispersion relation \cite{44,45,46}. In particular, in our case, it will be sufficient to perform only one subtraction and consider the subtracted expression \( \mathcal{F}_{D=4}(k^2) - \mathcal{F}_{D=4}(0) \).

3 Nonperturbative propagator of the composite operator and physical spectrum

The next step in establishing a physical spectrum for this confining theory is to compute the correlation functions of the composite operator \( \mathcal{O}(x) = \lambda(x)\eta(x) \) for the interacting case. We shall see that the interactions between \( i \)-particles will generate a physical bound state, \textit{i.e.} a pole on the negative real axis in the complex \( k^2 \) plane. This composite correlator featuring a real pole may then be properly rotated to Minkowski space \cite{44,45,46}, corresponding to a physical propagating state of the theory.

Since a singularity is not generated at any fixed order of perturbation theory, one must work in a nonperturbative framework. Our approach will be inspired by the standard resummation of bubble diagrams, which is an exact result in \( O(N) \)-symmetric scalar theories with a large number of field components \( N \to \infty \) \cite{47}. It should be noticed,
however, that the confining toy model discussed in this work is intrinsically different from $O(N)$ scalar theories, since it involves necessarily a doublet of $i$-particles, presenting different (complex-conjugated) masses.

The bubble diagram resummation for the composite correlation function $\langle \lambda(x)\eta(x)\lambda(y)\eta(y) \rangle$ in the confining theory defined in Eq.\ref{eq:correlator} is an exact result if one fixes $g_2 = 0$, ignoring quartic self interactions. Setting $g_2 = 0$ is in principle a well-defined procedure in dimensions $D = 2$ and $3$, since the resulting theories are UV finite and stable under quantum corrections. In the $D = 4$ case, (divergent) quartic self interactions are generated via radiative corrections and these terms are in principle needed to guarantee renormalizability. The toy model in $D = 4$ with $g_2 = 0$ that we will solve exactly is therefore either a fine-tuned renormalizable field theory or an effective low-energy theory, protected in the UV by an energy cutoff. In any case, the procedure to be presented in what follows illustrates successfully how an interacting confining theory defined in terms of unphysical degrees of freedom may generate a physical spectrum of bound states in different space-time dimensions.

Let us now compute the exact two-point function of the composite operator $O = \lambda\eta$. As discussed above, we work with $g_2 = 0$ so that the only interaction vertex is $g_1(\lambda\eta)^2$. The series of diagrams contributing to $\langle O(x)O(y) \rangle$ is:

$$\langle O(x)O(y) \rangle = \ldots$$

where the first diagram corresponds to the free case discussed in the previous section and defined as $F_D(k^2)$ in momentum space, cf. eqs.\ref{eq:free_propagator}. Here, solid/dashed lines represent $\lambda\eta$ propagators, full dots correspond to $g_1$-vertices and the empty dots make explicit the absence of interactions at the extreme points $x, y$, corresponding to the insertions of the operators $O(x), O(y)$. We omitted all self-energy corrections to the bubble propagators under the working hypothesis that these are already good approximations to the full nonperturbative propagator.\footnote{One may also see our result in $D = 4$ as that of the analysis of the full theory (with $g_2 \neq 0$), but coming from a Bethe-Salpeter approach with a bubble approximation for the kernel.}

Since the $n$-th term in this series has the general form

$$\underbrace{\ldots}_{n \text{ bubbles}} = \int \frac{d^Dk}{(2\pi)^D} e^{ik(x-y)} F_D(k^2) \left[g_1 F_D(k^2)\right]^{n-1},$$

the full two-point function can be written exactly as the result of a geometric series with ratio $g_1 F_D(k^2)$. In momentum space, we have

$$\langle O(k)O(-k) \rangle = F_D(k^2) \sum_{n=0}^{\infty} \left[g_1 F_D(k^2)\right]^n = \frac{F_D(k^2)}{1 - g_1 F_D(k^2)}, \quad (3.23)$$

where the difference in the denominator points out to the possible existence of a pole for a given $k^2 = -M^2$.

The explicit conditions for the existence of a pole in the result for the correlator in Eq.(3.23) can be found by expanding around $k^2 = -M^2$:

$$\langle O(k)O(-k) \rangle = \frac{F_D(-M^2) + O\left(k^2 + M^2\right)}{1 - g_1 F_D(-M^2) - g_1 F_D(-M^2)^2 (k^2 + M^2) + O\left(\left|k^2 + M^2\right|^2\right)}. \quad (3.24)$$

Therefore, if the conditions

\begin{align*}
(\text{i}) \quad & 1 - g_1 F_D(-M^2) = 0, \quad (3.25) \\
(\text{ii}) \quad & R_D(M^2) \equiv -\frac{1}{g_1 F_D(-M^2)} > 0, \text{ i.e. } F_D(-M^2) < 0. \quad (3.26)
\end{align*}

4. One may also see our result in $D = 4$ as that of the analysis of the full theory (with $g_2 \neq 0$), but coming from a Bethe-Salpeter approach with a bubble approximation for the kernel.
5. This mimics the observation that the tree level RGZ propagators can account well for the lattice gluon propagator for example [38],[39].
are satisfied, then the two-point correlator of the composite operator $O$ assumes the form of a physical pole at $M^2$, with a positive residue $R_D(M^2)$:

$$
\langle O(k)O(-k) \rangle \approx \frac{k^2 - M^2}{k^2 + M^2} R_D(M^2)
$$

which corresponds, of course, to a well-defined Källén-Lehmann spectral representation.

4 Results and discussion

Given the results for the spectral representation of the correlator $\langle O(k)O(-k) \rangle$ in the free case, Eqs. (2.16)–(2.19), one may systematically solve the pole conditions in Eqs. (3.25) and (3.26) in order to obtain the physical spectrum of the confining theory for each set of values for the parameters $(D, g_1, m, \theta)$. All results are shown in this section as dimensionless ratios, with the threshold for two-particle production, $\sqrt{\tau_0}$, being the mass unit used for normalization.

In what follows we analyze three types of massive theories, namely (i) nonconfining ($m \neq 0, \theta = 0$); (ii) GZ confining ($m = 0, \theta \neq 0$) and (iii) RGZ confining ($m \neq 0, \theta \neq 0$). Bound state solutions are found for all cases in all dimensions investigated, $D = 2, 3$ and 4.

![Figure 1](image1.png)  
**Figure 1:** Inverse coupling as a function of the (negative) squared mass of the bound state in dimension 2.

The mass $M$ of the bound state is of course a function of the mass parameters $m$ and $\theta$ and the coupling $g_1$, which has positive mass dimension in $D < 4$. Its specific form depends moreover on the space time dimension $D$. Nevertheless, the qualitative picture is essentially the same for the three theories analyzed, nonconfining and confining of the GZ- and RGZ-types, as shown in Figures 1, 3 and 5. This is an interesting feature of the bound state spectrum in this toy model: its static properties have no particular trace of the type of composites inside the “mesonic” state; both confining and nonconfining low-lying excitations furnish qualitatively equivalent spectra, at least statically.

In Figures 1, 3 and 5 one further verifies that the threshold for two-particle production, $\sqrt{\tau_0}$, is a mass upper limit for the bound states found. The heaviest bound states are generated at very low coupling. As interactions are turned on, the binding energy increases and the bound state mass decreases accordingly. It is also shown in Figures 1, 3 and 5 that the existence of a bound state with mass which is considerably lower than the threshold $\sqrt{\tau_0}$ requires large couplings. For instance, a bound state with half the mass threshold $\sqrt{\tau_0}$ requires couplings of $g_1/\tau_0^{2-D/2} \approx 2, 10$ and 500 in dimensions $D = 2, 3$ and 4, respectively. Furthermore, nearly massless bound states are found in the deeply nonperturbative regime.

In addition, the results show that the theory is physically meaningful only for sufficiently small couplings, $g_1 \leq g_{\text{crit}}$, with $g_{\text{crit}} = 1/F(k^2 = 0)$ being associated with the massless bound state solution. For couplings larger than this critical value, tachyonic solutions appear, signaling the fact that the toy model becomes ill-defined.
in this region of its parameter space. The emergence of the tachyon might signal the instability of the considered vacuum, akin to what happens in the Gross-Neveu model where a condensation of the relevant composite operator takes place accordingly \[48\]. Discussion of this would however lead us too far beyond the scope of this paper. This tachyon solution appears in all dimensions investigated, even though it is not made explicit in Figure 5.

In dimension \(D = 4\) the critical coupling is directly related to the subtraction scale and assumes therefore a running form, \(g_{\text{crit}}(\Lambda)\), with \(\Lambda\) being the subtraction energy scale. In Figure 5, \(\Lambda\) is fixed, for definiteness, so that \(g_{\text{crit}}(\Lambda) \to \infty\). In dimension \(D = 2\), also the massless bound state, i.e. \(g_1 = g_{\text{crit}}\) should be treated with some care, as is usual for massless particles in dimension \(D = 2\) \[49, 53, 54\].

It is also instructive to investigate the behavior of the residue \(R_D(M^2)\) of the bound state poles found, since it is related to the probability amplitude of finding such a state. Figures 4 and 5 show the results for dimensions \(D = 2, 3\) and \(4\), respectively. In line with the expectation that the bound state should disappear in the absence of interactions, the residue go to zero as one approaches \(g_1 \to 0\). For \(D = 2\) and \(3\) the residue grows monotonically as the mass \(M\) of the bound state is decreased, remaining finite also in the massless limit. In contrast, for \(D = 4\) the residue goes to zero as \(M \to 0\), indicating that the probability of finding this state vanishes. This feature is, however, highly dependent on our subtraction choice for \(\mathcal{F}_{D=4}\) which fixes \(\mathcal{F}_{D=4}(k^2 = 0) = 0\) –so that it hardly represents a general physical result, but it is rather a quantity that deserves a renormalization group/scheme improved analysis.
5 Conclusions

The mechanism that generates the physical spectrum in a confining theory is an outstanding theoretical problem. Inspired by the advances achieved in the last years in the description of propagators of confined elementary particles in Yang-Mills theories and QCD as well as by the encouraging estimates of glueball masses obtained in [41], we adopted the R(GZ) scenario of confinement in which fundamental excitations have unphysical, positivity-violating propagators described by the combination of modes with complex-conjugated masses, the so-called $i$-particles. As detailed in the introduction, these RGZ propagators are in good agreement with very precise lattice data for the two-point functions of gluons in the deep infrared.

In this paper we constructed a scalar quasi-particle toy model of these $i$-particles with the aim of investigating the role of interactions in the dynamical formation of the physical spectrum. Using resummation techniques, we have shown that the interacting two-point correlator of suitable composite operators in this confining theory displays a well-defined Källén-Lehmann spectral representation with a physical pole appearing dynamically. This picture provides therefore a concrete qualitative understanding of the dynamical formation of the physical spectrum of a confining theory of the RGZ or GZ types, defined in terms unphysical, positivity-violating, elementary fields with complex mass.

We have analyzed the theory in space time dimensions $D = 2, 3$ and 4, showing results for the behavior of the pole mass $M$ of the physical bound states generated and the corresponding residua as the parameters of the model are varied. Our findings for these static properties of the composite spectrum of nonconfining and confining theories of two types (GZ and RGZ) have proven to be qualitatively similar. It should be noticed that this analysis is done in the Euclidean theory and no statement can be made concerning nonstatic properties of the bound state solutions found. We expect e.g. that real-time properties, such as widths and decay rates, of the spectrum of confining $i$-particle theories may be strikingly different from that of its nonconfining analog, with real elementary masses.

Bound state formation has been investigated previously in nonconfining scalar theories in different space time dimensions. The theories investigated in the literature are in general not the one we have analyzed here. In dimension $D = 2$, the absence of bound states was proven for weakly coupled $\lambda\phi^4$ theories [50]. Our results do not contradict these findings because we are dealing with a different model, with two fields $\lambda$ and $\eta$ that interact as $(\lambda\eta)^2$, so that quartic self interactions are strictly absent. The consideration of the toy model with a nonzero quartic coupling $g_2(\lambda^4 + \eta^4)$ would involve extra classes of diagrams that could possibly bring the residue $R_3$ (cf. Figure 2) faster to zero in the weakly coupled domain (in the vicinity of $M^2 = \tau_0$), featuring then the disappearance of bound states as seen in [50]. In dimension $D = 3$, the existence of a nonzero mass gap for $\lambda\phi^4$ theories [51] was also obtained in the weakly coupled regime, which is already consistent with our findings for the toy model. Bound states in the large $N$ limit of $\lambda(\phi_i\phi_i)^2$ theories were studied in e.g. [52], whose results are consistent with our findings. It should be noticed however that, even though the large $N$ limit also involves the resummation of bubble diagrams, the toy model adopted here and its results are intrinsically different from those due to the inequivalent interaction vertices. Therefore, we believe our findings are consistent with the literature for nonconfining theories, besides contributing nontrivially to the investigation of bound state formation in confining theories.

Acknowledgments

M.A.L. Capri, M.S. Guimarães and L.F. Palhares are supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq-Brazil), while D. Dudal is supported by the Research-Foundation Flanders (FWO Vlaanderen). The work of S.P. Sorella is supported by FAPERJ under the program Cientista do Nosso Estado, E-26/101.578/2010.

References

[1] C. D. Roberts and A. G. Williams, Prog. Part. Nucl. Phys. 33, 477 (1994) [hep-ph/9403224].
[2] R. Alkofer and L. von Smekal, Phys. Rept. 353, 281 (2001) [hep-ph/0007355].
[3] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 147, 385 (1979).
[4] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 147, 448 (1979).
[5] S. Narison, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. 17 (2002) 1.
[49] S. R. Coleman, Commun. Math. Phys. 31, 259 (1973).
[50] J. Feldman, Can. J. Phys. 52, 1583 (1974).
[51] J. S. Feldman and K. Osterwalder, Annals Phys. 97, 80 (1976).
[52] L. F. Abbott, J. S. Kang and H. J. Schnitzer, Phys. Rev. D 13, 2212 (1976).
[53] E. Abdalla, M. C. B. Abdalla and K. D. Rothe, Nonperturbative methods in two-dimensional quantum field theory,” Singapore, Singapore: World Scientific (1991) 728 p.
[54] G. Morchio, D. Pierotti and F. Strocchi, J. Math. Phys. 31, 1467 (1990).