Phase retrieval combined with digital holography

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We present a new method for real- and complex-valued image reconstruction from two intensity measurements made in the Fourier plane: the Fourier magnitude of the unknown image, and the intensity of the interference pattern arising from superimposition of the original signal with a reference beam. This approach can provide significant advantages in digital holography since it poses less stringent requirements on the reference beam. In particular, it does not require spatial separation between the sought signal and the reference beam. Moreover, the reference beam need not be known precisely, and in fact, may contain severe errors, without leading to a deterioration in the reconstruction quality. Numerical simulations are presented to demonstrate the speed and quality of reconstruction.

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1. Introduction

The reconstruction of an image from the magnitude of its Fourier transform, also known as the phase retrieval problem, is of paramount importance in a variety of modern imaging techniques. Notable relevant disciplines include astronomy, crystallography, and microscopy. Hence, the images in question may vary from remote stars and galaxies to microscopic objects such as single cells, viruses, and even atomic structure of nano-scale objects like, for example, carbon nano-tubes. One of the first successful numerical methods for phase retrieval was developed by Gerschberg and Saxton (GS) with application to electron microscopy. Basically, their method can be viewed as a complex-valued image reconstruction from two intensity measurements: the magnitude of the image itself, and the magnitude of its Fourier transform. Starting with the GS method, Fienup later developed a new reconstruction method called the Hybrid-Input-Output (HIO) algorithm, that requires only one measurement, namely, the Fourier transform magnitude of the sought signal. To the best
of our knowledge, HIO is currently the most widely used computational method for phase retrieval. However, there are several situations where this method fails. Most often HIO is unable to reconstruct complex-valued objects whose support (the locus of the object’s non-vanishing parts) is not known precisely [6]. Moreover, in certain situations HIO may stagnate without converging to the correct image [7,8]. Other significant drawbacks of the HIO algorithm are relatively slow convergence and sensitivity to measurement errors [9,10]. In some cases it may require thousands of iterations before a moderately sized image is reconstructed to high accuracy [10,11]. Several attempts have been made to address the above problems by using continuous optimization methods such as Newton-type algorithms. Unfortunately, these methods fail when applied to the phase retrieval problem [12]. A simple heuristic explanation of this failure was given in [11]. Alongside this explanation it was demonstrated that a rough Fourier phase estimate may be sufficient to allow successful application of Newton-type optimization methods to the phase retrieval problem [10,11]. In the present work we remain within the framework of introducing additional information into the phase retrieval problem so as to develop faster and more robust methods of reconstruction. Specifically, we consider the situation where two intensity measurements in the Fourier domain are available. One is the Fourier magnitude of the sought image, as in classical phase retrieval, and the second is the intensity pattern resulting from the interference of the original signal with a known reference beam also measured in the Fourier plane. Although either one of these measurements can, in theory, be sufficient for successful reconstruction of the unknown image, our method provides significant advantages over such reconstructions. For example, comparing with reconstruction from the Fourier magnitude alone by HIO, our method gives a much faster speed and better quality in case of noisy measurements (see [9–11]). Furthermore, unlike classical holography methods, our algorithm does not require any special design of the reference beam. Finally and most importantly, very good reconstruction quality is obtained even when the reference beam contains severe errors.

The rest of the paper is organized as follows. In Section 2 we present the details of our basic reconstruction method which is developed for the situation where the reference beam is known precisely. The relation between our setup and digital holography is discussed in Section 3. Section 4 is devoted to further development of our reconstruction method to accommodate possible errors in the reference beam. Numerical simulation results are presented in Section 5. Finally, a short discussion and concluding remarks are given in Section 6.

2. Basic reconstruction algorithm

Let us start with the notation used throughout the paper. The unknown two-dimensional signal that we wish to reconstruct is represented by the complex-valued function $U(x, y) = |U(x, y)| \exp[j\varphi(x, y)]$, where $\varphi(x, y)$ designates the object phase, and $j$ is the imaginary
unit: \( j = \sqrt{-1} \). To address the phase of a complex-valued number we use the angle notation: \( \angle(U(x, y)) \equiv \varphi(x, y) \). Our measurements are done in the Fourier plane \((\xi, \eta)\), hence the transformation that \( U \) undergoes when transforming from the \((x, y)\) plane to the \((\xi, \eta)\) plane is simply the unitary Fourier transform

\[
\hat{U}(\xi, \eta) = \mathcal{F}\{U(x, y)\}.
\]  

(1)

Strictly speaking, the actual transformation is a bit more complicated as it includes some constant multipliers and scale factors \(^{13}\). However, these are immaterial for our discussion. Hereinafter, we shall adopt the convention that a pair of symbols like \( X \) and \( \hat{X} \) denote a signal in the \((x, y)\) plane (also known as the object domain) and its counterpart in the \((\xi, \eta)\) plane (also referred to as the Fourier domain), respectively. For the sake of brevity, we may omit the location designator \((x, y)\) or \((\xi, \eta)\) and use \( X \) or \( \hat{X} \) when the entire signal is considered.

The main purpose of our work is to develop a robust reconstruction method that can tolerate severe errors in the reference beam. To this end we use the reference beam only for estimating the Fourier phase of the sought image. Once a rough phase estimate is available we can use the method of phase retrieval with approximately known Fourier phase that was developed in \(^{9,11}\). This method was demonstrated to have fast convergence properties and good quality of reconstruction from noisy measurements (see \(^{9,11}\) for details and comparison with HIO).

The two measurements available at our disposal are used as follows. \( I_1 \) provides the Fourier magnitude of the sought image via the simple relationship between the two:

\[
I_1(\xi, \eta) = |\hat{U}(\xi, \eta)|^2.
\]  

(2)

The second measurement reads

\[
I_2(\xi, \eta) = |\hat{U}(\xi, \eta) + \hat{R}(\xi, \eta)|^2,
\]  

(3)

where \( \hat{R}(\xi, \eta) \) denotes a known reference beam that is used to obtain the Fourier phase estimate as described below. One possible schematic setup that provides these measurements is shown in Fig. 1. Note that \( \hat{R} \) is not necessarily a Fourier transform of some physical signal \( R \) in the object domain. This means that \( \hat{R} \) can be formed directly in the Fourier plane without forming first \( R \) and applying then an optical Fourier transform to obtain \( \hat{R} \). Nevertheless, there exists a mathematical inverse

\[
R(x, y) = \mathcal{F}^{-1}\{\hat{R}(\xi, \eta)\},
\]  

(4)

whose properties, such as extent, magnitude, etc., can be considered. The only requirement of \( \hat{R} \) is that it must not vanish in the region of our measurements. This is an important point
that provides an advantage to our method over the classical holography techniques. We shall elaborate more on this in Section 3.

Let us now describe how $\hat{U}$’s phase information is extracted from $I_2$, and, more importantly, how it is used in our reconstruction method. Consider the two signals:

$$\hat{U}(\xi, \eta) = |\hat{U}(\xi, \eta)| \exp[j\phi(\xi, \eta)],$$

and

$$\hat{R}(\xi, \eta) = |\hat{R}(\xi, \eta)| \exp[j\psi(\xi, \eta)].$$

The intensity pattern of their interference can be written as:

$$I_2(\xi, \eta) = |\hat{U}(\xi, \eta) + \hat{R}(\xi, \eta)|^2$$

$$= |\hat{U}(\xi, \eta)|^2 + |\hat{R}(\xi, \eta)|^2 + \hat{U}^*(\xi, \eta)\hat{R}(\xi, \eta) + \hat{U}(\xi, \eta)\hat{R}^*(\xi, \eta)$$

$$= |\hat{U}(\xi, \eta)|^2 + |\hat{R}(\xi, \eta)|^2 + 2|\hat{U}(\xi, \eta)||\hat{R}(\xi, \eta)| \cos[\phi(\xi, \eta) - \psi(\xi, \eta)].$$

From this formula we can easily extract the difference between the unknown phase $\phi(\xi, \eta)$ and the known phase $\psi(\xi, \eta)$:

$$\cos[\phi(\xi, \eta) - \psi(\xi, \eta)] = \frac{I_2(\xi, \eta) - |\hat{U}(\xi, \eta)|^2 - |\hat{R}(\xi, \eta)|^2}{2|\hat{U}(\xi, \eta)||\hat{R}(\xi, \eta)|}.$$

Which gives us:

$$\phi(\xi, \eta) = \psi(\xi, \eta) \pm \alpha(\xi, \eta),$$

where

$$\alpha(\xi, \eta) = \arccos \left[ \frac{I_2(\xi, \eta) - |\hat{U}(\xi, \eta)|^2 - |\hat{R}(\xi, \eta)|^2}{2|\hat{U}(\xi, \eta)||\hat{R}(\xi, \eta)|} \right].$$

This expression is well defined as $|\hat{R}(\xi, \eta)|$ is assumed to be non-zero everywhere in the region of interest and the places where $|\hat{U}(\xi, \eta)| = 0$ can be simply excluded from our consideration as there is nothing to be recovered since their phase has no influence. We assume that $\pm\alpha$, that is, the difference between the phases $\phi$ and $\psi$, lies within the interval $[-\pi, \pi]$, hence, no phase unwrapping is necessary. The phase $\phi(\xi, \eta)$ can assume either one (rarely) or two possible values at every location. The two possible situations are shown in Fig. 2. Hence, if the intensity $I_2(\xi, \eta)$ is sampled at $N$ points there are generally $2^N$ possible solutions $\hat{U}(\xi, \eta)$ and, consequently, the same number of possible reconstructions $U(x, y)$. (Here we consider the worst case scenario where all sampled values give rise to two solutions.) To guarantee a unique (and meaningful) reconstruction we must use additional information about the sought signal $U(x, y)$. In the phase retrieval problem it is usually assumed that $U(x, y)$ has limited support, namely, part of the image is occupied by zeros. In practice, it is usually assumed that, in each direction, half (or more) pixels of $U(x, y)$ are zeros. To capture this
information in the Fourier domain one should “over-sample” by a factor of two (or more) in each dimension. Hence, if the known (not necessarily tight) support area of \( U \) is \( n \times n \) pixels, then in the Fourier plane it must be sampled with a sensor of size \( 2n \times 2n \) pixels. Such “over-sampling” usually guarantees unique (up to trivial transformations: shifts, constant phase factor, and axis reversal) reconstruction in the case of the classical phase retrieval problem, where only \( |\hat{U}| \) is available [14]. It is not known whether this two-fold oversampling is absolutely necessary in our case where two measurements are available. However, our experiments indicate that for a general complex-valued signal it still seems to be necessary to over-sample by a factor of two. Hence, the reconstruction problem reads: find \( U(x,y) \) such that \( |\hat{U}| \) is known, \( \angle(\hat{U}) = \psi \pm \alpha \), and \( U(x_O,y_O) = 0 \). Here, \( \alpha \) is the known phase difference between \( \hat{U} \) and \( \hat{R} \) as defined by Equations (11) and (12); \((x_O,y_O)\) denotes the known off-support area in the \((x,y)\) plane.

The problem is combinatorial in nature, and many different methods can be applied to find a solution. Our method is based on replacing the equality \( \angle(\hat{U}) = \psi \pm \alpha \) with the less strict inequality

\[
\psi - \alpha \leq \angle(\hat{U}) \leq \psi + \alpha .
\]  

(13)

By this relaxation we reduce the original problem into the phase retrieval problem with approximately known phase. As mentioned earlier, for this situation there exists an efficient reconstruction method based on quasi-Newton optimization [9–11]. The method solves the nonlinear non-convex optimization problem defined by

\[
\begin{cases}
\min_{Z} \| |\mathcal{F}\{Z\}| - |\hat{U}| \|^2 \\
\text{s.t.} \quad \psi - \alpha \leq \angle(\mathcal{F}\{Z\}) \leq \psi + \alpha , \\
Z(x_O,y_O) = 0 ,
\end{cases}
\]

(14)

by further relaxing the problem so as to perform the optimization over a convex set (see [9,10] for details). Note that the solution of the above minimization problem \( Z \) is not guaranteed to be equal to \( U \). Due to the relaxation we performed, the phase of \( \hat{Z} \) is allowed to assume the continuum of values in the interval \([\psi - \alpha, \psi + \alpha]\) instead of the two discrete values \( \psi \pm \alpha \). However, due to the uniqueness of the phase retrieval problem, the phase of \( Z \) may differ from the phase of \( U \) only by a constant. That is, \( Z(x,y) = U(x,y) \exp[jc] \) for some scalar \( c \) (see [14] for details). This does not pose problems, as only the relative phase distribution inside the support area of \( U(x,y) \) is usually of interest. Moreover, in the case where the absolute phase is required, \( c \) can be recovered by adding a post-processing step that solves the one-dimensional optimization problem:

\[
\min_{c} \| |\hat{Z}\exp[jc] + \hat{R}| - I_2^{1/2} \|^2 .
\]

(15)

Note that we intentionally do not add a penalty term like \( \| |\hat{Z} + \hat{R}| - I_2^{1/2} \|^2 \) into the main minimization scheme as defined by Equation (14). Adding such a term would introduce a
strong connection between $Z$ and the reference beam $\hat{R}$. This connection will inevitably deteriorate the quality of reconstruction when the reference beam $\hat{R}$ contains errors.

Reconstruction speed is very fast as is evident from our experiments (see Fig. 6). Moreover, it can be further accelerated, as our experiments indicate that more aggressive oversampling (zero padding in the object domain) results in faster convergence in terms of the number of iterations. In fact, use of a special reference beam can, in theory, result in a trivial non-iterative reconstruction in a way similar to holography. However, such special reference beams may not be easily realizable in physical systems and the quality of the reconstructed signal is strongly influenced by the quality of the reference beam. We discuss this setup in the next section and compare its sensitivity to possible errors in $\hat{R}$ against our method in Section 5.

3. Relation to holography

Our method was initially developed for the phase retrieval problem. However, the use of interference patterns creates a strong connection with holography. Therefore, it may be pertinent to discuss the advantages our method provides over the classical holographic reconstruction. Note that our discussion is limited to basic holography only and no attempt is made to cover all possible setups and techniques that can be used in digital holography. We nonetheless believe that this novel approach can compete with or improve upon existing algorithms used in digital holography.

In classical holography one uses a specially designed reference beam so as to allow easy non-iterative recovery of the sought image. This has an obvious advantage over iterative methods, especially when the speed of the reconstruction is of high importance. However, reliance on the reference beam means that reconstruction quality may deteriorate badly when the reference beam contains errors, that is, when it differs from the “known” values. To review the non-iterative reconstruction method used in holography, recall that the recorded intensity $I_2$ is the result of superimposition of $\hat{U}$ and $\hat{R}$ as defined by Equation (8). In optical Fourier holography, this intensity is recorded on optical material. The recorded image is used then as an amplitude modulator for an illuminating beam $\hat{A}(\xi, \eta)$, which then undergoes a Fourier transform to form a new signal $B(x', y')$. In a digital computer we may use the same technique. Moreover, we are free to use either the forward or the inverse Fourier transform, as it makes no practical difference (the resulting images will be reversed conjugate copies of each other). Here we use the inverse Fourier transform:

$$B(x, y) = \mathcal{F}^{-1}\{\hat{A}(\xi, \eta) I_2(\xi, \eta)\}$$

$$= \mathcal{F}^{-1}\{\hat{A}\left[|\hat{U}|^2 + |\hat{R}|^2 + \hat{U}^* \hat{R} + \hat{U} \hat{R}^*\right]\}$$

$$= A(x, y) \otimes U(x, y) \otimes U^*(-x, -y) + A(x, y) \otimes R(x, y) \otimes R^*(-x, -y) +$$

$$A(x, y) \otimes U^*(-x, -y) \otimes R(x, y) + A(x, y) \otimes U(x, y) \otimes R^*(-x, -y), \quad (16)$$
where $\otimes$ denotes convolution. Note that in this case the fourth and the third terms are equal to the sought wavefront $U(x, y)$ and its conjugate counterpart $U^*(-x, -y)$ convolved with $A(x, y) \otimes R(x, y)$ and $A(x, y) \otimes R^*(-x, -y)$, respectively. The best possible choice is $A(x, y) = \delta(x, y)$ and $R(x, y) = \delta(x - x_0, y - y_0)$, where $\delta(x, y)$ is the Dirac delta function. In this case the obtained wave becomes:

$$B(x, y) = U(x, y) \otimes U^*(-x, -y) + \delta(x, y) + U^*(-x + x_0, -y + y_0) + U(x - x_0, y - y_0). \quad (17)$$

Hence, if $x_0$ and/or $y_0$ are large enough, the four terms in the above sum will not overlap in the $(x, y)$ plane. Thus, we can easily obtain the sought signal $U(x, y)$, albeit shifted by $(x_0, y_0)$, provided that the spatial extent of $U(x, y)$ is limited by the box $x \in [-L_x, L_y]$, $y \in [-L_y, L_y]$. The spatial extent of the autocorrelation $U(x, y) \otimes U^*(-x, -y)$ is twice as large, that is, limited by the box $x \in [-2L_x, 2L_y]$, $y \in [-2L_y, 2L_y]$. Hence, to avoid overlapping we must have $x_0 > 3L_x$, or $y_0 > 3L_y$. Thus, in theory, one can generate an ideal delta function in the $(x, y)$ plane located at sufficient distance from the support area of $U(x, y)$. In the Fourier domain, this delta function corresponds to a plane wave arriving at a certain angle at the plane of measurements. If such a construction is possible, then a simple inverse transform of the intensity obtained in the Fourier plane is sufficient to obtain the sought signal $U(x, y)$. However, as mentioned earlier, this approach has some drawbacks. First, it is impossible to create an ideal delta function. Any physical realization will necessarily have a finite spatial extent, and this will result in a “blurred” reconstructed image. Note that the term “blurring” describes well the resulting image in the case where $U(x, y)$ is real-valued or has constant phase. In the more general case, where the phase of $U(x, y)$ varies at non-negligible speed, the result appears more distorted (see Fig. 4). The second drawback is the sensitivity of this method to errors in $\hat{R}$. In Section 5 we demonstrate how the quality of reconstruction depends on the error in $\hat{R}$ (see Fig. 8, 9, and 10). Our method, on the other hand shows very little sensitivity to the reference beam shape. Moreover, its modification described in the next section allows the reference beam to contain severe errors without deteriorating significantly the quality of reconstruction.

4. Reconstruction method for imprecise reference beam

Here we consider the situation where the reference beam is not known precisely, that is, we assume that the phase of $\hat{R}$ contains some unknown error. It is easy to verify that if the reference beam phase $\psi(\xi, \eta)$ has error $\epsilon(\xi, \eta)$ then the sought phase $\phi(\xi, \eta)$ becomes

$$\phi(\xi, \eta) = \psi(\xi, \eta) + \epsilon(\xi, \eta) \pm \alpha(\xi, \eta), \quad (18)$$

in a manner similar to Equation 11. We do not consider errors in the magnitude $|\hat{R}|$ for several reasons. Many aberrations manifest themselves through phase distortion 13. Also,
the magnitude of $\hat{R}$ can be measured. Moreover, looking at the above equation, it is obvious that any error in $\psi$ can be viewed as an error in $\alpha$. That is, the situation would be the same if the reference beam phase $\psi$ were known precisely, while the difference $\alpha$ would contain some errors. This observation is relevant because errors in the phase $\alpha$ can arise from many different sources, including imperfect measurements and errors in the reference beam magnitude.

The true error $\epsilon(\xi, \eta)$ is, of course, unknown. Hence, we assume just an upper bound (assumed known) on the absolute phase error:

$$\psi - \epsilon - \alpha \leq \angle(\hat{U}) \leq \psi + \epsilon + \alpha,$$  \hspace{1cm} (19)

as in Equation (13). This time, however, the phase uncertainty interval may be larger than $\pi$ radians which makes our method inapplicable. On the other hand, limiting the phase uncertainty interval by $\pi$ radians will prevent us from reconstructing the precise image, because the true phase may lie outside this interval. A possible solution is to measure the intensity of the reference beam and then to reconstruct its phase using the method presented in [10,11], because this problem itself can be seen as a phase retrieval with approximately known phase. However, taking another measurement may be undesirable, therefore, we developed the following reconstruction method:

1. Set the phase uncertainty interval as defined by Equation (13) (as if there were no errors in the reference beam phase).

2. Solve the resulting minimization problem, obtaining a solution $Z(x, y)$.

3. If not converged, set the phase uncertainty interval to $[\angle(\hat{Z}) - \pi/2, \angle(\hat{Z}) + \pi/2]$. Clip it, if applicable, to the limits defined by Equation (19) and go to Step 2.

In this algorithm we perform a number of outer iterations, each time updating the phase uncertainty interval. This approach leads to a successful reconstruction method even in cases where the reference beam contains severe errors. The results are much better than those of non-iterative holographic reconstruction (see Fig. 8 and 9). This improvement is achieved by decoupling the reconstruction problem (which becomes the pure phase retrieval with approximately known phase) and the erroneous interferometric measurements.

5. Experimental results

The method was tested on a variety of images with similar results. Here we present numerical experiments conducted on one natural image so as to allow easy perception of the reconstruction quality under various conditions. The image intensity (squared magnitude) is shown in Fig. 3. Technical details of the image are as follows: the size is $128 \times 128$ pixels,
and pixel values (amplitude) vary from 0.2915 to 0.9634 with mean value of 0.6835. These parameters will become important when we will consider the reference beam design, and when we will assess the reconstruction quality. Since the original image is a photograph, it does not have any phase information. Hence, we generated three different phase distributions to account for the assortment of possible real-world problems where our method can be applied. The first distribution assumes that the image is non-negative real-valued, that is, the phase is zero everywhere. The second distribution is designed to mimic a relatively smooth phase. To this end, the phase is set to be proportional to the image values (scaled to the interval \([-\pi, \pi]\)). Finally, in the third distribution the phase is chosen at random, uniformly spread over the interval \([-\pi, \pi]\). This distribution is designed to show the behavior of our reconstruction method in cases where the true phase varies rapidly. We also consider three possible reference beams, again, to demonstrate the robustness of our method. The first reference beam is an ideal delta-function in the \((x, y)\) plane, located at the coordinate \((256, 256)\) so that the holographic condition is satisfied. With this reference beam exact reconstruction is obtained as long as the sampling in the Fourier domain is sufficiently dense \((512 \times 512\) pixels, or more). We do not present the visual results of reconstruction for this reference beam as both methods produce images that are indistinguishable from the true image. The speed of convergence of our method is shown in Fig. 6. Later, we show also how the reconstruction quality of both methods is affected by Fourier phase errors in the reference beam. Before this, we demonstrate the effect of departure from the ideal delta function: the second reference beam is a small square of size \(3 \times 3\) pixels, located at the coordinates \((256 : 258, 256 : 258)\). In this setup the reconstruction quality of the holographic method is degraded, as evident from Fig. 4. It is also evident that faster variations in the object phase result in greater deterioration in the reconstruction, in agreement with our expectations. Our method, on the other hand, is insensitive to the reference beam form. In Fig. 5 we demonstrate our reconstruction results for the aforementioned small square as the reference beam (the first row), and for another reference beam that was formed in the Fourier plane by combining unit magnitude with random phase (in the interval \([-\pi, \pi]\)). This beam, of course, is not suitable for holography, as its extent in the object plane occupies the whole space. Reconstruction is very fast and, in fact, is almost independent of the sought image and reference beam type. Fig. 6 demonstrates that less than 20 iterations are required to solve the minimization problem as defined by Equation (14).

In all these examples we assume perfect knowledge of the reference beam. Next we consider the situation where the actual reference beam does not match the expected signal in the Fourier plane. Following our discussion in Section 4 we evaluate how the reconstruction quality of the holographic approach and our method are affected by errors in the reference beam Fourier phase. We use again the three aforementioned models of the sought image.
(real-valued, smooth phase, and random phase) and the three reference beams (delta function, small square, and random). From Fig. 7 it is evident that in all these cases we were able to solve the minimization problem to sufficient accuracy as long as the phase error was below 25%. That is, our method can tolerate reference beam Fourier phase errors of up to \( \pi/2 \) radians. The sharp discontinuity that happens at this value has a simple explanation: phase error greater than \( \pi/2 \) radians can result in the phase uncertainty interval greater than \( 2\pi \) radians (see Equation (19)). Hence, all phase information is lost. A comparison with the holographic reconstruction is given in Fig. 8 where the error norm in the object domain is depicted. Note that the objective function values are about \( 10^{-10} \), hence, one would expect the object domain error norm to be of order \( 10^{-5} \) (the difference stems from the fact that the objective function uses squared norm). This is not so in the case of complex-valued images and the random reference beam. This effect is due to the relaxation we perform in the Fourier phase, as discussed in Section 2. It does not change the relative phase distribution, but all phases can get a constant addition. This can be corrected by solving the one dimensional minimization defined by Equation (15). Fig. 9 depicts the corrected values yielded by this process. Visual results comparing the holographic reconstruction with our method are provided in Fig. 10.

As the results show, our method demonstrates a substantial advantage over ordinary holographic reconstruction. It is remarkable that even when the minimization is not particularly successful (in cases of very large phase errors) our reconstruction is still closer to the true signal than the holographic method. This success is due to our approach of decoupling the phase retrieval from the interferometric measurements. As mentioned earlier, we deliberately avoid strong dependence on the reference beam. The interference pattern is only used to estimate the Fourier phase bounds. The results indicate that this approach is well justified.

6. Conclusions

We present a new reconstruction method from two intensities measured in the Fourier plane. One is the magnitude of the sought signal’s Fourier transform, and the other is the intensity resulting from the superimposition of the original image and an approximately known reference beam. While the method was originally developed for the phase retrieval problem, it can be useful in digital holography, because it poses less stringent requirements on the reference beam. The method is designed specifically to allow severe errors in the reference beam without compromising the quality of reconstruction. Numerical simulations justify our approach, exhibiting reconstruction that is superior to that of holographic techniques.
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Fig. 1: Schematic representation of the experiment.
Fig. 2: Given a reference beam (black) whose magnitude and phase are known, and an unknown signal of known magnitude (dotted circle radius), one can try to find the phase of the unknown signal by measuring the magnitude of the sum (solid circle radius). Evidently, in most cases there are two possible solutions (a). However, in certain cases, the is only one solution (b).
Fig. 3: Original image (intensity).
Fig. 4: Image (intensity) reconstructed by the holographic technique using a small square as the reference beam: (a) the image is real-valued, (b) image phase varies slowly, (c) image phase is random (varies rapidly).
Fig. 5: Image reconstructed (intensity) by our method: (a), (b), and (c) — reference beam is a small square and object phase is zero (a), smooth (b), random (c). (d), (e), and (f) — reference beam is random and object phase is zero (d), smooth (e), random (f).
Fig. 6: Reconstruction speed of our method: (a) real valued image, (b) image phase is smooth, (c) image phase is random.
Fig. 7: Fourier domain (||\hat{Z}||^2 - ||\hat{U}||^2) error vs. phase error in the reference beam: (a) real valued image, (b) image phase is smooth, (c) image phase is random.
Fig. 8: Object domain error ($\| Z - U \|$) vs. phase error in the reference beam: (a) real valued image, (b) image phase is smooth, (c) image phase is random.
Fig. 9: Corrected object domain error ($\|Z - U\|$) vs. phase error in the reference beam: (a) real valued image, (b) image phase is smooth, (c) image phase is random.
Fig. 10: Image (intensity) reconstructed by the holographic method (upper row) and our method (lower row). Object phase is random, and the reference beam is a delta function with Fourier phase errors: (a), (b), and (c) — phase error is 1%, 10%, and 20% respectively (d), (e), and (f) — phase error is 1%, 10%, and 20% respectively.