Perturbation theory for $O(3)$
topological charge correlators

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Abstract

To check the consistency of positivity requirements for the two-point correlation function of the topological charge density, which were identified in a previous paper, we are computing perturbatively this two-point correlation function in the two-dimensional $O(3)$ model. We find that at the one-loop level these requirements are fulfilled.

1 Introduction

In a preceding paper [1] we analyzed the implications of physical positivity (‘unitarity’) and positivity of the topological susceptibility for the two-point correlation function of the topological density. In particular we found that the short distance singularity has to be softer than tree level perturbation theory would predict. In this article we verify for the two-dimensional $O(3)$ nonlinear sigma model that one-loop perturbation theory is indeed in agreement with this requirement. Such a softening of the short distance singularity is already predicted by including the running of the coupling constant in tree level perturbation theory [2, 3]. But since we are dealing here with a composite operator, this check of internal consistency of perturbation theory is quite nontrivial.
2 Topological charge density correlator

The action of the (ferromagnetic) O(3) spin model on a 2-dimensional square lattice $\Lambda \subseteq \mathbb{Z}^2$ is

$$S = -\beta \sum_{\langle X,Y \rangle} \vec{S}_X \cdot \vec{S}_Y,$$

(1)

where the sum is over nearest neighbor pairs $\langle XY \rangle$ of lattice sites. The partition function reads

$$Z = \int [DS] e^{-S},$$

(2)

with measure

$$[DS] = \prod_x d^3 \vec{S}_X \delta(\vec{S}_X^2 - 1),$$

(3)

imposing the unit norm constraint for all spins, i.e. $\vec{S} \in S^2 \subset \mathbb{R}^3 \forall X \in \Lambda$.

We will consider the thermodynamic limit as the $L \to \infty$ limit of the model on an $L \times L$ square lattice with periodic boundary conditions, so that translation invariance holds. The origin 0 is then located at the center of $\Lambda$. Unit lattice vectors are called 1 and 2.

2.1 Definitions of the topological charge

We want to study the two-point function of the topological charge density $q_X$, to be denoted by $G_x$ (with $x = 0\vec{X}$),

$$G_x = G_{0,X} = \langle q_X q_0 \rangle = \frac{1}{Z} \int [DS] q_X q_0 e^{-S}.$$  

(4)

The topological susceptibility is defined by

$$\chi_t = \sum_X G_{0,X}.$$  

(5)

In this paper we will consider two lattice definitions of $q_X$: a ‘field theoretical’ one and the geometric one due to Berg and Lüscher. Both make use of a triangulation of the square lattice $\mathbb{Z}^2$ as in figure 1. In both cases, the topological charge density is written as

$$q_X = q^a_X + q^b_X.$$  

(6)
where
\[ q^a_X = f(X, X + 1, X + 2), \]
\[ q^b_X = f(X + 1 + 2, X + 2, X + 1), \]  
with \( f : \Lambda^3 \to \mathbb{R} \) a certain function.

In [1] we used a symmetrization of this definition by introducing in addition a 'mirror' triangulation
\[ q^c_X = f(X, X + 1, X + 1 + 2), \]
\[ q^d_X = f(X, X + 1 + 2, X + 2) \]  
and defining \( q_X = \frac{1}{2}(q^a_X + q^b_X + q^c_X + q^d_X) \); this was necessary there in order to maintain reflection symmetry. Here, however, this symmetrization is unnecessary, because it has no effect in perturbation theory.

The 2-point function \( G_x = G_{0,x} \) thus has the form
\[ G_x = \langle q_X q_0 \rangle = \langle (q^a_X + q^b_X)(q^a_0 + q^b_0) \rangle = \sum_{i,j=a,b} G_x^{ij} \]  
with
\[ G_x^{ij} = \langle q^j_X q^i_0 \rangle, \quad i, j = a, b. \]  
To make the notation more transparent, call
\[ (Z_1, Z_2, Z_3) = (0, 0 + 1, 0 + 2), \]
\[ (W_1, W_2, W_3) = (X, X + 1, X + 2), \]  
\[ (Z'_1, Z'_2, Z'_3) = (0 + 1 + 2, 0 + 2, 0 + 1), \]
\[ (W'_1, W'_2, W'_3) = (X + 1 + 2, X + 2, X + 1) \]  
and denote \( \bar{Z}_i = \bar{S}_Z \) and analogously for \( W, Z', W' \) (see figure [1]). The indices just introduced are denoted by \( i, j, k, \) etc.

In the following sections, only formulae for \( q^a_X \) and \( G_x^{aa} \) will be presented. The full topological charge density can be recovered by means of (6) and (7), and the full two-point function by adding the other three contributions in (9).
obtained by substituting $Z'W$, $ZW'$ and $Z'W'$ for $ZW$, respectively. Moreover, a numerical factor will be introduced in the definition of the correlator so as to work with simpler expressions:

$$\tilde{G}_x = (8\pi)^2 G_x.$$  \hfill (12)

### 2.1.1 Field theoretical definition

We first consider a symmetrized version of the field theoretical (FT) definition \cite{4} of the topological charge density

$$q_{FT,a}^{FT,a} = \frac{1}{8\pi} \vec{s}_X \cdot (\vec{S}_{X+1} \wedge \vec{S}_{X+2}),$$  \hfill (13)

which gives rise to the two-point function

$$\tilde{G}_x^{FT,aa} = \left\langle \left[ \vec{Z}_1 \cdot (\vec{Z}_2 \wedge \vec{Z}_3) \right] \left[ \vec{W}_1 \cdot (\vec{W}_2 \wedge \vec{W}_3) \right] \right\rangle = \langle \det (\vec{Z}_i \cdot \vec{W}_j) \rangle.$$  \hfill (14)

The perturbative treatment of the problem begins with the $O(3) \to O(2)_z$ decomposition of the spins,

$$\vec{Z}_i = \left( \frac{z_i}{\sqrt{\beta}}, \sqrt{1 - \frac{z_i^2}{\beta}} \right), \quad \vec{W}_j = \left( \frac{w_j}{\sqrt{\beta}}, \sqrt{1 - \frac{w_j^2}{\beta}} \right).$$  \hfill (15)
Substituting in the determinant, we get

\[ \tilde{G}^{FT,aa}_x = \frac{1}{\beta^3} \left\langle \det (\vec{z}_i \cdot \vec{w}_j) \right\rangle \]

\[ + \frac{1}{\beta^2} \sum_{i,j=1}^{3} (-1)^{i+j} \left\langle \sqrt{1 - \frac{z_i^2}{\beta}} \sqrt{1 - \frac{w_j^2}{\beta}} \det (\vec{z}_k \cdot \vec{w}_\ell) \right\rangle \]

\[ = \frac{1}{6 \beta^3} \sum_{ijklmn} \epsilon_{ijkm} \epsilon_{\ell m} \left\langle \vec{z}_i \cdot \vec{w}_j \vec{z}_m \cdot \vec{w}_n \right\rangle \]

\[ + \frac{1}{2 \beta^2} \sum_{ijklmn} \epsilon_{ijkm} \epsilon_{\ell m} \left\langle \sqrt{1 - \frac{z_i^2}{\beta}} \sqrt{1 - \frac{w_j^2}{\beta}} \vec{z}_j \cdot \vec{w}_m \vec{z}_k \cdot \vec{w}_n \right\rangle. \]  

(16)

The perturbative expansion of (16) consists of a Taylor expansion in \( \beta^{-1/2} \). Only integral powers of \( \beta^{-1} \) contribute. For our purposes, it will be enough to keep terms up to and including order \( \beta^{-3} \). Then

\[ \tilde{G}^{FT,aa;\text{pert.}}_x = \frac{1}{2 \beta^2} \sum_{ijklmn} \epsilon_{ijkm} \epsilon_{\ell m} \left\langle \vec{z}_j \cdot \vec{w}_m \vec{z}_k \cdot \vec{w}_n \right\rangle^{(0)} \]

\[ + \frac{1}{2 \beta^3} \sum_{ijklmn} \epsilon_{ijkm} \epsilon_{\ell m} \left\{ \left\langle \vec{z}_j \cdot \vec{w}_m \vec{z}_k \cdot \vec{w}_n \right\rangle^{(1)} \right. \]

\[ \left. - \frac{1}{6} \left\langle (3z_i^2 - 2z_i \cdot w_\ell + 3w_\ell^2) z_j \cdot w_m \vec{z}_k \cdot \vec{w}_n \right\rangle^{(0)} \right\} \]

\[ + O(\beta^{-4}), \]  

(17)

where, in general, perturbative contributions to magnitude \( \mathcal{M} \) are denoted by

\[ \mathcal{M}^{\text{pert.}} \sim \sum_n \frac{1}{\beta^n} \mathcal{M}^{(n)}. \]  

(18)
2.1.2 Berg and Lüscher’s definition

Berg and Lüscher’s (BL) definition of the topological charge density was introduced in [5]. With each elementary triangle \((X, Y, Z)\) of the chosen triangulation one associates a multiple of the signed area of the minimal spherical triangle determined by \((\vec{S}_X, \vec{S}_Y, \vec{S}_Z)\) on the unit sphere. Explicitly,

\[
q^{\text{BL},a}_{X} = \frac{1}{2\pi} \tan^{-1} \frac{\vec{S}_X \cdot (\vec{S}_{X+1} \wedge \vec{S}_{X+2})}{1 + \vec{S}_X \cdot \vec{S}_{X+1} + \vec{S}_{X+1} \cdot \vec{S}_{X+2} + \vec{S}_{X+2} \cdot \vec{S}_X}.
\]  

(19)

The corresponding perturbative two-point function is, up to and including order \(\beta^{-3}\),

\[
\bar{G}^{\text{BL},aa;\text{pert.}}_{X} = 16 \left\langle \frac{\det_{i,j}(\vec{Z}_i \cdot \vec{W}_j)}{(1 + \vec{Z}_1 \cdot \vec{Z}_2 + \vec{Z}_2 \cdot \vec{Z}_3 + \vec{Z}_3 \cdot \vec{Z}_1)(1 + \vec{W}_1 \cdot \vec{W}_2 + \vec{W}_2 \cdot \vec{W}_3 + \vec{W}_3 \cdot \vec{W}_1)} \right\rangle + O(\beta^{-4}).
\]  

(20)

Substituting (15) in (20) and Taylor expanding the result, we arrive at

\[
\bar{G}^{\text{BL},aa;\text{pert.}}_{X} = \bar{G}^{\text{FT},aa;\text{pert.}}_{X} + \Delta \bar{G}^{aa}_{x},
\]  

(21)

where

\[
\Delta \bar{G}^{aa}_{x} = \frac{1}{8\beta^3} \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{lmn} \left\langle \left( \sum_{u=1}^{3} (\vec{z}_u^2 + \vec{w}_u^2) - \sum_{u<v} (\vec{z}_u \cdot \vec{z}_v + \vec{w}_u \cdot \vec{w}_v) \right)^{(0)} \right\rangle + O(\beta^{-4}).
\]  

(22)

will be referred to as ‘BL difference.’

2.2 Perturbative computation of \(\tilde{G}_x\)

In this section we use the BL topological charge density (19) and will comment on the difference with the FT definition when appropriate. Therefore
we drop the BL labels and write

$$\tilde{G}_x^{\text{pert.}} = \frac{1}{\beta^2} \tilde{G}_x^T + \frac{1}{\beta^3} \tilde{G}_x^L + O(\beta^{-4}).$$

(23)

For bookkeeping purposes we do the perturbative computation of the density correlator out keeping factors $N - 1$ explicit, as for an $O(N)$ problem, although the topological density has a meaning only for $N = 3$.

The only vertices relevant for Feynman diagrams up to and including order $\beta^{-3}$ are order $\beta^{-1}$:

- **Two-$$\vec{\pi}$$ vertex:**

  $$V_{ZW}^{k\ell} = \frac{1}{\beta} \left( 1 - \frac{N - 1}{V} \right) \delta^{k\ell} \delta_{ZW}.$$  

  (24)

- **Four-$$\vec{\pi}$$ vertex:**

  $$V_{ZW TU}^{k\ell mn} = \frac{1}{\beta} \left[ -n_Z \delta_{ZW TU} \left( \delta^{k\ell} \delta^{mn} + \delta^{km} \delta^{\ell n} + \delta^{kn} \delta^{\ell m} \right) 
  + \left( \delta^{n.n.}_{ZW} \delta_{TU} \delta^{k\ell} \delta^{mn} + \delta^{n.n.}_{ZW} \delta_{TU} \delta^{km} \delta^{\ell n} + \delta^{n.n.}_{ZW} \delta_{TU} \delta^{kn} \delta^{\ell m} \right) \right],$$  

  (25)

where subindices are lattice points, superindices are $O(N)$ indices, $n_Z = 4$ is the number of nearest neighbors of point $Z$, two-index deltas are Kronecker,

$$\delta_{ZW TU} = \begin{cases} 
1 & \text{if } Z = W = T = U, \\
0 & \text{otherwise}
\end{cases}$$

(26)

and

$$\delta^{n.n.}_{ZW} = \begin{cases} 
1 & \text{if } Z, W \text{ nearest neighbors}, \\
0 & \text{otherwise}
\end{cases}$$

(27)

The explicitly volume-dependent factor in (24) comes from fixing one spin using the Faddeev-Popov method as in [6] to eliminate the divergent zero mode that would otherwise appear in the Gaussian measure defining tree level perturbation theory with periodic boundary conditions.
2.2.1 Green’s functions

In order to compute the Green’s functions relevant to the correlators being studied, we start from the corresponding quantity on the $L \times L$ square lattice and put $x = \vec{0} \rightarrow (x, y), -L/2 \leq x, y < L/2$. 

$$G_{0,X} = \frac{a^2}{2L^2} \sum_{\mathbf{p}}' \frac{\cos(a \mathbf{p} \cdot \mathbf{x})}{2 - \cos(a p_x) - \cos(a p_y)}, \quad (28)$$

where the momenta

$$\mathbf{p} = (p_x, p_y) = \frac{2\pi}{L} (n_x, n_y), \quad n_x, n_y \in \mathbb{Z} \quad (29)$$

are summed over the first Brillouin zone $-L/2 \leq n_x, n_y < L/2$, the prime meaning exclusion of the zero mode $\mathbf{p} = 0$.

The thermodynamic ($L \rightarrow \infty$) limit of (28) does not exist, since the sum diverges logarithmically in $L$. The divergence is independent of $\mathbf{x}$, however, and the thermodynamic limit of the difference $\tilde{G}_{0,X} = G_{0,X} - G_0$ does exist. According to a general argument due to David [7], the perturbation expansion of invariant observables should be free of infrared divergences; the first sign of this is the fact that everything is expressed in terms of $\tilde{G}_{0,X}$, which has a well defined thermodynamic limit given by the integral

$$\tilde{G}_{0,X} \equiv G_{0,X} - G_0 = \frac{1}{2(2\pi)^2} \int_{-\pi}^{+\pi} dq_x \int_{-\pi}^{+\pi} dq_y \frac{\cos(q \cdot \mathbf{x}) - 1}{2 - \cos q_x - \cos q_y}; \quad (30)$$

(another possible source of infrared divergence is the sum over intermediate lattice sites, which will be addressed in section 2.4). Obviously $\tilde{G}_0 = 0$, $\tilde{G}_{(x,y)} = \tilde{G}_{(y,x)}$.

This subtracted propagator was studied by other methods in [8], which we will use as a check for our results.

Due to parity, the numerator in the integrand of (30) can be replaced by $\cos(xq_x) \cos(yq_y) - 1$. Integration in $q_x$ can be performed by the calculus of residues, with the result

$$\tilde{G}_{0,X} = \frac{1}{2\pi} \int_{-1}^{+1} \frac{dt}{\sqrt{1 - t^2}} \frac{(2 - t + \sqrt{3 - 4t + t^2})^{-x} T_y(t) - 1}{\sqrt{3 - 4t + t^2}}. \quad (31)$$
Here $T_y(t)$ is Chebyshev’s polynomial of the $y$-th kind.

It is straightforward to check that (31) behaves asymptotically as

$$
\tilde{G}_{0,X} = -\frac{1}{2\pi} \ln |x| - c - \frac{d}{|x|} + O(|x|^{-2})
$$

with $c$, $d$ constants. According to [8],

$$
c = \frac{2\gamma_E + 3 \ln 2}{4\pi}, \quad d = 0,
$$

where $\gamma_E$ is Euler’s constant.

Similarly it can be seen that

$$
\tilde{G}_{0,X-v} + \tilde{G}_{0,X+v} - 2\tilde{G}_{0,X} = \frac{1}{\pi} \left( \frac{v^2 - (v \cdot x)^2}{2x^2} - \frac{(v \cdot x)^2}{|x|^4} \right) + O(|x|^{-3})
$$

and

$$
\tilde{G}_{0,X+w} - \tilde{G}_{0,X-v} + \tilde{G}_{0,X+v} - \tilde{G}_{0,X+w} = \frac{1}{2\pi} \frac{v^2 - w^2 - 2(v \cdot x)^2 + 2(w \cdot x)^2}{|x|^4} + O(|x|^{-3})
$$

for large $|x|$ and $|v|, |w| = O(1)$. To see this, one subtracts from the Fourier representation of the lhs of (31) and (32) the corresponding continuum expression with a suitable smooth high momentum cutoff; this difference, being the Fourier transform of a smooth ($C^\infty$) function will decay faster than any power of $|x|$. The cutoff on the continuum expression, on the other hand, does not affect the leading long distance behavior, which is obtained straightforwardly. Expressions (31) and (32), in which the constants $c$ and $d$ in (32) do not appear, will be useful in the computation of the charge correlator.

In the following, we will denote Green’s functions by $G_{0,X}$ even when subtracted.

### 2.2.2 Tree level

The tree level contribution to $\tilde{G}_1^{11}$ is given by

$$
\tilde{G}^{11,aa}_x = \frac{1}{2} \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{\ell mn} \langle \vec{z}_j \cdot \vec{w}_m \vec{z}_k \cdot \vec{w}_n \rangle^{(0)}
$$

for large $|x|$. To see this, one subtracts from the Fourier representation of the rhs of (31) and (32) the corresponding continuum expression with a suitable smooth high momentum cutoff; this difference, being the Fourier transform of a smooth ($C^\infty$) function will decay faster than any power of $|x|$. The cutoff on the continuum expression, on the other hand, does not affect the leading long distance behavior, which is obtained straightforwardly. Expressions (31) and (32), in which the constants $c$ and $d$ in (32) do not appear, will be useful in the computation of the charge correlator.
the correlator in the rhs being evaluated in the Gaussian measure with one spin fixed. The corresponding diagrams have the structure of figure (2).

Contracting spin indices,

\[
\tilde{G}_x^{T,aa} = \frac{1}{2} (N - 1)^2 \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{\ell mn} G_{Z_j W_m} G_{Z_k W_n}
\]

\[+ \frac{1}{2} (N - 1) \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{\ell mn} G_{Z_j Z_k} G_{W_m W_n} \tag{37} \]

\[+ \frac{1}{2} (N - 1) \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{\ell mn} G_{Z_j W_m} G_{Z_k W_n}. \]

The $3 \times 3$ matrices $G_{Z Z}$ and $G_{W W}$ are symmetric, and therefore the second term in the rhs vanishes. Playing with indices, we can recast equation (37) in the form

\[
\tilde{G}_x^{T,aa} = \frac{1}{2} \left[ (N - 1)^2 - (N - 1) \right] \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{\ell mn} G_{Z_j W_m} G_{Z_k W_n}, \tag{38} \]

This can be written in terms of differences $G_v - G_w$ with $|v - w| \leq \sqrt{2}$ (which restricts IR divergences to summation effects, the IR divergences of Green’s functions being cancelled in the differences):

\[
\tilde{G}_x^{T,aa} = -\frac{1}{2} \left[ (N - 1)^2 - (N - 1) \right] \times \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{\ell mn} (G_{Z_j W_m} - G_{Z_k W_n})(G_{Z_k W_m} - G_{Z_k W_n}) \tag{39} \]

We can now perform the sums over all indices: in general, for a $3 \times 3$
matrix $a_{ij}$,

$$\sum_{ijklmn} \varepsilon_{ijkl} \varepsilon_{lmn} a_{jm} a_{kn} = \sum_{i\ell} \left\{ [(\text{tr } a)^2 - \text{tr } (a^2)] \delta_{i\ell} + 2 (a^2)_{i\ell} - 2 (\text{tr } a) a_{i\ell} \right\}$$

$$= 3 \left[ (\text{tr } a)^2 - \text{tr } (a^2) \right] + 2 s(a^2) - 2 (\text{tr } a) s(a), \quad (40)$$

where $s(a)$ denotes the sum of all entries in $a$.

This identity can be obtained as follows: one starts from the well-known identity

$$\ln \det (1 + zA) = \text{tr} \ln (1 + zA) \quad (41)$$

and expands it to order $z^3$ to obtain

$$\text{tr} A \wedge A \wedge A = \frac{1}{3} \text{tr} A^3 - \frac{1}{2} \text{tr} A \text{ tr} A^2 + \frac{1}{6} (\text{tr } A)^3. \quad (42)$$

By ‘polarization’, i.e. the replacement of $A$ by $xA + yB + zC$ on both sides and comparing the coefficient of $xyz$ Eq. (40) follows if we put $B = C = a$ and take for $A$ the matrix with all entries equal to 1.

In our case,

$$\widetilde{G}_x^{T, aa} = -\frac{1}{2} \left[ (N - 1)^2 - (N - 1) \right]$$

$$\times \left\{ 3 \left[ (\text{tr } G_{ZW})^2 - \text{tr } (G_{ZW}^2) \right] + 2 s(G_{ZW}^2) - 2 (\text{tr } G_{ZW}) s(G_{ZW}) \right\}, \quad (43)$$

2.2.3 One loop

The one-loop (order $\beta^{-3}$) correction to the correlator can be decomposed as

$$\widetilde{G}_x^{L, aa} = (I)^{aa} + (II)^{aa} + (III)^{aa} + (IV)^{aa} + (V)^{aa}, \quad (44)$$

according to the structures depicted in figure (3).

- (I) contains the contributions of the 4-legged, disconnected graphs with
one 2-vertex:

\begin{equation}
(I)^{aa} = \left( 1 - \frac{N - 1}{V} \right) [(N - 1)^2 - (N - 1)]
\times \sum_{ijklmn} \epsilon_{ijk} \epsilon_{\ell mn} G_{Z,j,W,m} \sum_{P} G_{Z,k,P} G_{W,n,P}.
\end{equation}

\begin{equation}
(II)^{aa} = -\frac{1}{2} [(N - 1)^2 - (N - 1)] \sum_{ijklmn} \epsilon_{ijk} \epsilon_{\ell mn}
\times \sum_{\langle P,Q \rangle} (G_{Z,j,P} G_{W,m,P} - G_{Z,j,Q} G_{W,m,Q})
(G_{Z,k,P} G_{W,n,P} - G_{Z,k,Q} G_{W,n,Q}).
\end{equation}

- (II) contains the X-shaped contributions:

- (III) gathers the contributions of 4-legged graphs with tadpoles:

\begin{equation}
(III)^{aa} = -\frac{N - 1}{2} [(N - 1)^2 - (N - 1)] \sum_{ijklmn} \epsilon_{ijk} \epsilon_{\ell mn} G_{Z,j,W,m}
\times \sum_{\langle P,Q \rangle} (G_{P,P} - G_{Q,Q})
(G_{Z,k,P} G_{W,n,P} - G_{Z,k,Q} G_{W,n,Q})
- [(N - 1)^2 - (N - 1)] \sum_{ijklmn} \epsilon_{ijk} \epsilon_{\ell mn} G_{Z,j,W,m}
\times \sum_{\langle P,Q \rangle} [G_{P,P} G_{Z,k,P} G_{W,n,P} + G_{Q,Q} G_{Z,k,Q} G_{W,n,Q}
- G_{P,Q} (G_{Z,k,P} G_{W,n,Q} + G_{Z,k,Q} G_{W,n,P})].
\end{equation}

The first term of (III) vanishes by translation invariance.
(IV) consists of the six-legged graphs contributing to the correlator according to the FT definition of the charge density:

\[(IV)^{aa} = \frac{N - 3}{6} \left[ (N - 1)^2 - (N - 1) \right] \det G_{ZW} \\
- \frac{N - 1}{2} \left[ (N - 1)^2 - (N - 1) \right] \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{lmn} G_{00} G_{ZjZm} G_{ZkWn} \\
- \left[ (N - 1)^2 - (N - 1) \right] \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{lmn} G_{ZjWm} (G_{ZiZk} G_{ZjWn} + G_{ZkW_{i}} G_{W_{i}W_{n}}) \times \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{lmn} G_{ZjWm} (G_{ZiZk} G_{ZjWn} + G_{ZkW_{i}} G_{W_{i}W_{n}}) \right].\]

The determinant term in (IV)^{aa} has a vanishing prefactor.

(V) is the BL difference, also consisting of six-legged graphs:

\[(V)^{aa} = -\frac{N - 1}{8} \left[ (N - 1)^2 - (N - 1) \right] \left[ \sum_{u,v} (G_{ZuZv} - G_{00}) \right] \\
\times \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{lmn} G_{ZjWm} G_{ZkWn} \\
- \frac{1}{4} \left[ (N - 1)^2 - (N - 1) \right] \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{lmn} G_{ZjWm} \\
\times \sum_{u,v} [G_{ZuZk} (G_{ZvWn} - G_{ZuWn}) + G_{ZkW_{u}} (G_{W_{u}W_{v}} - G_{W_{u}W_{n}})] \right].\]

As happened with the tree level expression, the one-loop correction can be rewritten in terms of differences $G_v - G_w$ with $|v - w| \leq \sqrt{2}$. This works
independently for (I), (II) and (V):

\[(I_d)^{aa} = \frac{1}{3} \left( 1 - \frac{N - 1}{V} \right) \left[ (N - 1)^2 - (N - 1) \right] \sum_{ijk\ell mn} \varepsilon_{ijk\ell mn} \]
\[\times (G_{Z_j W_m} - G_{Z_k W_m}) \sum_P (G_{Z_k P} - G_{Z_i P})(G_{W_n P} - G_{W_m P}), \]
\[(50)\]

\[(II_d)^{aa} = \frac{1}{16} \left[ (N - 1)^2 - (N - 1) \right] \sum_{ijk\ell mn} \varepsilon_{ijk\ell mn} \]
\[\times \sum_{(P, Q)} [(G_{Z_j P} - G_{Z_j Q}) - (G_{W_m P} - G_{W_m Q})][(G_{Z_k P} - G_{Z_k Q}) - (G_{W_n P} - G_{W_n Q})] \]
\[\times [(G_{Z_j P} - G_{Z_k P}) + (G_{Z_j Q} - G_{Z_k Q}) - (G_{W_m P} - G_{W_n P}) - (G_{W_m Q} - G_{W_n Q})]^2, \]
\[(51)\]

\[(V_d)^{aa} = \frac{N - 1}{8} \left[ (N - 1)^2 - (N - 1) \right] \left[ \sum_{u,v} (G_{Z_u Z_v} - G_{00}) \right] \]
\[\times \sum_{ijk\ell mn} \varepsilon_{ijk\ell mn}(G_{Z_j W_m} - G_{Z_k W_m})(G_{Z_k W_m} - G_{Z_k W_n}) \]
\[\times \frac{1}{8} \left[ (N - 1)^2 - (N - 1) \right] \sum_{ijk\ell mn} \varepsilon_{ijk\ell mn}(G_{Z_j W_m} - G_{Z_k W_m}) \]
\[\times \sum_{u,v} [(G_{Z_u Z_k} - G_{Z_v Z_k})(G_{Z_v W_n} - G_{Z_u W_n}) + (G_{Z_k W_u} - G_{Z_k W_v})(G_{W_u W_v} - G_{W_n W_n})], \]
\[(52)\]

while (III)\(^{aa}\) and (IV)\(^{aa}\) cannot be brought to that form. However, the sum (III)\(^{aa}\) + (IV)\(^{aa}\) can, because, using translation invariance and the fact that
The first terms in the rhs of (53) and (54) cancel.

\[ (\text{IV}_d)^{aa} = -2G_{00} \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{\ell mn} (G_{j}W_m - G_{k}W_m)(G_{k}W_m - G_{l}W_n) \]

\[ - \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{\ell mn} (G_{Z_i} - G_{00})(G_{Z_j}W_m - G_{Z_k}W_m)(G_{Z_k}W_m - G_{Z_l}W_n) \]

\[ - \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{\ell mn} (G_{W_{\ell}} - G_{00})(G_{Z_j}W_m - G_{Z_k}W_m)(G_{Z_j}W_{\ell} - G_{Z_k}W_n). \]
• \( \mathbf{w} \) is a linear combination \( n_1 \mathbf{1} + n_2 \mathbf{2} \) of unit lattice vectors with \( n_i \) integers (\(|n_i| \leq 2\) up to 1-loop order.)

The following notation proves useful when applying computer techniques to this perturbative problem:

\[ G_{v+w} \to v_{\text{word}}. \]  

(55)

Here \( v \) is one of the letters \( \zeta, \xi, \eta \) or \( \kappa \) according to the vector \( \mathbf{v} \). The subscript is the minimal word of the form \( a^{n_a} b^{n_b} d^{n_d} , n_i \geq 0 \), matching vector \( \mathbf{w} \) under the rule

\[ a^{n_a} b^{n_b} d^{n_d} \to (n_a - n_c) \mathbf{1} + (n_b - n_d) \mathbf{2} \]  

(56)

(obviously, only words of the forms \( a^{n_a} b^{n_b} , a^{n_a} d^{n_d} , c^{n_c} b^{n_b} , c^{n_c} d^{n_d} \) can be minimal.) For instance,

\[ G_{0, X+1+2} \to \zeta_{ab}, \quad G_{P, 0+1-2-2} \to \xi_{add}, \quad G_{0, 0-1} \to \kappa_c. \]  

(57)

This notation is way more readable than the standard notation, and allows us to cope with the full form of the density correlator, including all four terms in (9). A C program was used to generate the full analytic forms in \( \zeta_{ab} \) notation for each contribution (tree level and one-loop contributions I through V), its output being simplified by means of a computer algebra program. The simplification thus achieved is dramatic, witness the results in next section.

2.3.1 The correlator in \( \zeta_{ab} \) notation

The full tree level 2-point topological charge correlator between 0 and \( X \), both in standard and \( \zeta_{ab} \) notation, reads:

\[
\tilde{G}^T_X = 2 \left[ (G_{0,X+1+2} + G_{0,X-1-2} - 2G_{0,X})(G_{0,X+1-2} + G_{0,X-1+2} - 2G_{0,X}) \\
- (G_{0,X+1} - G_{0,X+2} + G_{0,X-1} - G_{0,X-2})^2 \right] \\
= 2 \left[ (\zeta_{ab} + \zeta_{cd} - 2\zeta)(\zeta_{ad} + \zeta_{cb} - 2\zeta) - (\zeta_a - \zeta_b + \zeta_c - \zeta_d)^2 \right].
\]  

(58)
From this expression and the asymptotics (34) and (35) of the differences of Green's functions, the leading term of the asymptotic behavior for \( \tilde{G}_x^T \) can be easily computed:

\[
\tilde{G}_x^T = -\frac{2}{\pi^2} \frac{1}{|x|^4} + O(|x|^{-5}).
\]

The full 1-loop contributions will only be presented in \( \zeta_{ab} \) notation (cancelling terms in (III) and (IV) are omitted):

- the graphs with 2-vertices:

\[
(\text{I}_\zeta) = 2 \left(1 - \frac{2}{V}\right) \times \left[ (\zeta_a - \zeta_b + \zeta_c - \zeta_d) \sum_P [ (\xi_a - \xi_b)(\eta_{ab} - \eta) + (\xi_{ab} - \xi)(\eta_a - \eta_b) ] \right. \\
\left. \quad - (\zeta_{ab} + \zeta_{cd} - 2\zeta) \sum_P (\xi_a - \xi_b)(\eta_a - \eta_b) \right] \\
\left. \quad - (\zeta_{ad} + \zeta_{cb} - 2\zeta) \sum_P (\xi_{ab} - \xi)(\eta_{ab} - \eta) \right] \\
\]

- the X-shaped graphs:

\[
(\text{II}_\zeta) = 2 \sum_P [ (\xi_{ab} - \xi)(\xi_{ab} - \xi_{aa}) - (\xi_a - \xi_b)(\xi_a - \xi_{aab}) ] \\
\times [ (\eta_{ab} - \eta)(\eta_{ab} - \eta_{aa}) - (\eta_a - \eta_b)(\eta_a - \eta_{aab}) ] \\
+ 2 \sum_P [ (\xi_{ab} - \xi)(\xi_{ab} - \xi_{bb}) + (\xi_a - \xi_b)(\xi_b - \xi_{abb}) ] \\
\times [ (\eta_{ab} - \eta)(\eta_{ab} - \eta_{bb}) + (\eta_a - \eta_b)(\eta_b - \eta_{abb}) ],
\]
• the tadpole graphs:

\[
(III_\zeta) = (\kappa_a - \kappa) 
\]

\[
\times \left\{ (\zeta_{ab} + \zeta_{cd} - 2\zeta) \sum_P \left[ - (\xi_a - \xi_b)(\eta_{aa} + \eta_{ad} - \eta_{bb} - \eta_{cb}) - (\xi_{aa} + \xi_{ad} - \xi_{bb} - \xi_{cb})(\eta_a - \eta_b) \right] 
\right. 
\]

\[
+ (\zeta_{ad} + \zeta_{cb} - 2\zeta) \sum_P \left[ - (\xi_{ab} - \xi)(\eta_{aab} + \eta_{abb} - \eta_{c} - \eta_{d}) - (\xi_{aab} + \xi_{abb} - \xi_{c} - \xi_{d})(\eta_{ab} - \eta) \right] 
\]

\[
+ (\zeta_a - \zeta_b + \zeta_c - \zeta_d) \sum_P \left[ (\xi_{ab} - \xi)(\eta_{aa} + \eta_{ad} - \eta_{bb} - \eta_{cb}) + (\xi_{aa} + \xi_{ad} - \xi_{bb} - \xi_{cb})(\eta_{ab} - \eta) \right] 
\]

\[
\left. + (\xi_a - \xi_b)(\eta_{aab} + \eta_{abb} - \eta_{c} - \eta_{d}) + (\xi_{aab} + \xi_{abb} - \xi_{c} - \xi_{d})(\eta_{a} - \eta_{b}) \right\}, 
\]

(62)

• graphs with six external legs in the FT definition:

\[
(IV_\zeta) = 4(\kappa_a + \kappa_{ab} - 2\kappa) \left[ (\zeta_{ab} + \zeta_{cd} - 2\zeta)(\zeta_{ad} + \zeta_{cb} - 2\zeta) - (\zeta_a - \zeta_b + \zeta_c - \zeta_d)^2 \right]. 
\]

(63)

• the BL difference:

\[
(V_\zeta) = -4(2\kappa_a + \kappa_{ab} - 3\kappa) \left[ (\zeta_{ab} + \zeta_{cd} - 2\zeta)(\zeta_{ad} + \zeta_{cb} - 2\zeta) - (\zeta_a - \zeta_b + \zeta_c - \zeta_d)^2 \right]. 
\]

(64)

Note that (IV_\zeta) and (V_\zeta) do not involve sums over lattice points, and that they are proportional to the tree level expression (58). Their sum reads

\[
(IV_\zeta) + (V_\zeta) = -4(\kappa_a - \kappa) \left[ (\zeta_{ab} + \zeta_{cd} - 2\zeta)(\zeta_{ad} + \zeta_{cb} - 2\zeta) - (\zeta_a - \zeta_b + \zeta_c - \zeta_d)^2 \right]. 
\]

(65)

The simplicity of these expressions might be a hint of the existence of a more direct derivation.

### 2.4 Numerical evaluation

The procedure we employ to evaluate the correlator numerically follows the conventional philosophy adopted in formal perturbation theory: the contributions to each order are first evaluated in a finite volume and then the
Figure 4: Ranges for the computation: $L$ determines the range of $|x|$ for which topological charge correlators $G_x$ are computed. $L_0$ is a cutoff on the sum over Green’s functions $G_x$. $L_1$ determines the range for which exact values of $G_x$, as opposed to asymptotic approximations, are used.

The various perturbative contributions to the charge density correlator $\tilde{G}_{0,X}$ were computed, by means of C programs, for $X$ in a finite $400 \times 400$ sublattice of the whole $\mathbb{R}^2$. Lattice symmetry allows to reconstruct all correlators in the $2L \times 2L$ square lattice (in our case, $L = 200$) from those in the shaded region.

As explained in the next sections, the computation of $G_x$ involves sums over Green’s functions $G_x$ with $|x|$ arbitrarily large, and a cutoff $L_0$ on the summation range is imposed. Also, since we have an asymptotic expression for $G_x$, we only need to compute Green’s functions exactly for $|x|$ in a smaller region determined by another cutoff $L_1$; for larger $|x|$ the asymptotic expression is used.
Figure 4 shows the ranges $L, L_I, L_O$ involved in the computation.

2.4.1 Computation of Green’s functions

It is impractical to compute and store an array of all possible values of (subtracted) Green’s functions $G_x$ according to the exact expression (31). It is far more convenient to store only such values as differ noticeably from the approximate expression (32). We chose to compute exact values for $G_x$ with $|x|$ less than a certain value $L_I$ (see figure 4), and approximate values for the remaining ones as they were needed.

The constant appearing in (32) must be computed for this procedure to be useful.

Working with $L_I = 1000$, exact Green’s functions $G_{(x,y)}$ were computed for the triangular region $0 \leq x \leq L_I, 0 \leq y \leq x$. All those in the region $|x| \leq L_I$ can be obtained from them by symmetry.

Exact values of $L_I$ at the edge $x = L_I$ of the triangle were used to fit the Green’s function space dependence to (32), since this edge is the furthest removed from the centre and therefore provides the best agreement with the asymptotic form.

A value $c = 0.257343$ is obtained for the constant (see figure 5), which agrees with the analytical expression (33) taken from [8].

For $L_I = 1000$, the numerical results for the correlators are stable with respect to 10% changes in $L_I$.

2.4.2 Computation of correlators

The tree level correlator (58), as well as one-loop contributions (63) and (64), do not involve sums over lattice points, and are readily computed for each $X$.

One-loop contributions (60), (61) and (62) involve sums over points arbitrarily far away from this sublattice, and a cut-off must be introduced. We choose to restrict the sum to points within the circle of radius $L_O$ about the origin. At $L_O = 2000$, results are stable with respect to 10% changes in $L_O$ or in the shape of the summation domain (circle vs. square). This indicates
Figure 5: Fit of Green’s functions to \( \text{const} - \frac{1}{2\pi} \ln |x| \) for \( x = (L_i, y) \), \( 0 \leq y \leq L_i = 1000 \). The dashed line corresponding to the fitted dependence is indistinguishable from the computed points in this range.
the absence (or cancellation) of any infrared divergences connected with the summation over intermediate lattice points.

Results are presented for the computations outlined in the previous sections. Range parameters are $L = 200$, $L_1 = 1000$, $L_O = 2000$.

Figure 6 shows the (nonuniversal) near-origin tree level and one loop contributions to the topological charge density correlator $\tilde{G}_{0,(x,0)}$ as a function of $x$. Note that both contributions are positive at the origin, negative everywhere else, and are suppressed for large $x$. The peak at the origin results, in the continuum limit, in the singular contact term discussed e.g. in [10]. The values of the peak at tree level and one loop level are

$$\tilde{G}_0^T \approx 0.810569, \quad \tilde{G}_0^L \approx 0.899007.$$ (66)

The tree level peak can also be calculated analytically from (31) and (58), yielding $\tilde{G}_0^T = 8/\pi^2 \approx 0.810570$.

Since this is a perturbative analysis, the topological susceptibility (5) should vanish order by order in $\beta^{-1}$. Summing the computed correlators over the whole $2L \times 2L$ region, the following results are obtained:

$$\sum_x \tilde{G}_x^T \approx 1.3 \times 10^{-5}, \quad \sum_x \tilde{G}_x^L \approx 3.7 \times 10^{-5},$$ (67)

which are compatible with zero and serve as a check of our computations. Note, in particular, that the problems deemed to arise with $\chi_T$ in the continuum limit of the $O(3)$ model do not show up in this perturbative treatment.

Figure 7 shows the tree level contribution multiplied by $x^4$, as a function of $x$. This product approaches for large $x$ the constant -0.2026, compatible with (59), so the tree level result (including all numerical factors) is

$$G_{0,(x,0)}^T \sim -\frac{1}{32\pi^4 x^4}$$ (68)

for $x \gg 1$.

The one-loop contributions (60) through (64) to the correlator were com-
Figure 6: Near-origin behavior of correlator $\tilde{g}_{0,(x,0)}$, at tree level and one loop, as a function of $x$. 
Figure 7: Tree level $\tilde{\mathcal{G}}_{0,(x,0)}$ multiplied by $x^4$ as a function of $x$.

computed and their behavior for large $x$ analyzed:

\[
\begin{align*}
(I_\zeta) &\sim -\frac{a}{x^2}, \\
(II_\zeta) &\sim -\frac{b \ln x}{x^4}, \\
(III_\zeta) &\sim +\frac{a}{x^2}, \\
(IV_\zeta + V_\zeta) &\sim -\frac{d}{x^4},
\end{align*}
\]

with $a = 0.101$, $b = 0.0653$, $d = 0.101$. These Ansätze were obtained by inspection of the plots and trial and error, except for the last one, which stems directly from the proportionality of (65) to (58) and the fact that $\kappa = 0$, $\kappa_a = -1/4$, which results in the asymptotics in (69) with $d = 1/\pi^2$.

The leading terms in $(I_\zeta)$ and $(III_\zeta)$ cancel, but subleading terms $O(x^{-4})$ and $O(x^{-4} \ln x)$ remain. The sum of all one-loop contributions (see figure 8)
behaves as
\[ \tilde{G}_{0,(x,0)}^L \sim -\frac{c_0 + c_1 \ln x}{x^4}, \quad x \gg 1, \]
with constants
\[ c_0 = 2.206, \quad c_1 = 0.0645. \]
Note that these are not identical to \( b \) and \( d \) in (69), because the subleading terms in the asymptotic behavior of (I\( _\xi \)) and (III\( _\xi \)) contribute.

The same analysis was performed along the diagonal of points \((x, x)\) with identical results for the asymptotic forms of the correlator: full rotational invariance is restored in this regime.

2.4.3 Renormalization and continuum limit

We now discuss the continuum limit, again in the spirit of formal perturbation theory, i.e. termwise in the perturbative expansion; no claim can be made that the resulting expansion is asymptotic to a nonperturbatively defined continuum limit.
Up to one loop we have found for the correlator on the unit lattice and large lattice distances \(|x| \gg 1\)

\[
\mathcal{G}_x \sim -\frac{1}{\beta^2} \frac{1}{32\pi^4|x|^4} - \frac{1}{\beta^3} \frac{1}{64\pi^2|x|^4}(c_0 + c_1 \ln |x|) + O\left(\frac{1}{\beta^4}\right). \tag{72}
\]

To obtain a continuum limit we have to introduce the lattice spacing \(a\), rescale \(x = y/a\) and make \(\beta\) dependent on the cutoff \(a\). We also have to rescale the correlator \(\mathcal{G}\) by a factor \(a^4\) according to its engineering dimension, to obtain the correlator \(\mathcal{G}_y^a\) in continuum normalization

\[
\mathcal{G}_y^a = a^4 \mathcal{G}_{xa} \sim -\frac{1}{\beta(a)^2} \frac{1}{32\pi^4|y|^4} - \frac{1}{\beta(a)^3} \frac{1}{64\pi^2|y|^4} \left(c_0 + c_1 \ln \frac{|y|}{a}\right) + O\left(\frac{1}{\beta(a)^4}\right), \tag{73}
\]

valid asymptotically for \(|y| \gg a\).

A continuum limit should exist if we let \(\beta\) depend logarithmically on \(a\) according to the one-loop Callan-Symanzik \(\beta\) function for the \(O(3)\) model [11]:

\[
\frac{1}{\beta(a)} = \frac{1}{\beta_0} - \frac{1}{2\pi} \ln(\mu a) + O(\beta_0^{-3}) = \frac{1}{\beta_0} \left(1 + \frac{1}{2\pi\beta_0} \ln(\mu a) + O(\beta_0^{-2})\right). \tag{74}
\]

Inserting this in (73) and reexpanding to order \(\beta_0^{-3}\) we see that the terms proportional to \(\ln a\) cancel if \(c_1 = 2/\pi^3\), which is consistent with the value 0.0645 produced by our numerical computation; the continuum limit of the correlator is then

\[
\lim_{a \to 0} \mathcal{G}_y^a = \mathcal{G}_y = -\frac{1}{\beta_0^2} \frac{1}{32\pi^4|y|^4} \left(1 + \frac{1}{\pi \beta_0} \ln\left[|y| \mu e^{\frac{3\pi}{2\beta_0}}\right]\right) + O\left(\beta_0^{-4}\right). \tag{75}
\]

By choosing a renormalization scale \(\mu = \mu_0 e^{\frac{3\pi a}{4\beta_0}}\) we obtain

\[
\lim_{a \to 0} \mathcal{G}_y^a = \mathcal{G}_y = -\frac{1}{\beta_0^2} \frac{1}{32\pi^4|y|^4} \left(1 + \frac{1}{\pi \beta_0} \ln(\mu_0 |y|)\right) + O\left(\beta_0^{-4}\right). \tag{76}
\]

If we had chosen the FT instead of the BL definition, the only difference would have been a change in \(c_0\), corresponding to different renormalization scale, so to the order considered the two definitions are related by a finite renormalization.
Equation (76) is now valid for all \( y \), since in the limit \( a \to 0 \) only the asymptotic behavior of \( G_x \) survives.

We can compare this result with the Renormalization Group improved tree level result \cite{2,3}

\[
G_y = -\frac{1}{32\pi^4 \beta(y)^2} \frac{1}{|y|^4},
\]

(77)

if we reexpand (77) to order \( \beta_0^{-3} \), using the one-loop RG flow

\[
\beta(y) = \beta(y_0) - \frac{1}{2\pi} \ln \frac{|y|}{|y_0|}.
\]

(78)

The expressions (76) and (77) agree to order \( \beta_0^{-3} = \beta(y_0)^{-3} \) if the renormalization scales are chosen appropriately.

So we have established that our one-loop calculation supports the softening of the short distance behavior predicted by the RG improved tree level result.

### 3 Conclusions

We computed the two-point functions of the topological charge density perturbatively to one loop and found consistency with the RG improved tree level perturbative result, indicating a softening of the short distance singularity compared to the naive tree level result. This means that at the level of formal one-loop perturbation theory the requirements of the two positivities analyzed in \cite{1} are indeed satisfied.

But we should point out again that a mathematical justification of the procedures of formal perturbation theory (interchange of the weak coupling limit with the thermodynamic and continuum limits) does not exist, therefore it would be very interesting to see if this softening is really present at the nonperturbative level. Numerical checks using Monte Carlo simulations are presumably very difficult, however, since the check requires identifying logarithmic corrections to a rather large power.
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