ON COVERING AND QUASI-UNSPLIT FAMILIES OF RATIONAL CURVES

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Abstract. We study extremality properties of covering families of rational curves on projective varieties. Among others, we show that on a normal and Q-factorial projective variety $X$ with $\dim(X) \leq 4$, every covering and quasi-unsplit family of rational curves generates a geometric extremal ray of the Mori cone $\overline{\text{NE}}(X)$ of classes of effective 1-cycles.

1. Introduction

Let $X$ be a normal and uniruled complex projective variety. Consider an irreducible and closed subset $V$ of $\text{Chow}(X)$ such that:
- any element of $V$ is a cycle whose irreducible components are rational curves;
- $V$ is covering (which means that for any point $x \in X$, there exists an element of $V$ passing through $x$).

We call such a $V$ a covering family of rational 1-cycles on $X$. If moreover, all irreducible components of the cycles parametrized by $V$ are numerically proportional, we call $V$ a covering and quasi-unsplit family of rational 1-cycles on $X$ (see [CO04, Definition 2.13]).

For any covering family $V$ of rational 1-cycles on $X$, we will denote by $[V]$ the numerical class in $\text{NE}(X)$ of the general cycle of the family $V$ and by $\mathbb{R}_{\geq 0}[V]$ the half-line generated by $[V]$.

A geometric extremal ray of the Mori cone $\overline{\text{NE}}(X)$ is a half-line $R \subseteq \overline{\text{NE}}(X)$ such that if $\gamma_1 + \gamma_2 \in R$ for some $\gamma_1, \gamma_2 \in \overline{\text{NE}}(X)$, then $\gamma_1, \gamma_2 \in R$ (see Section 2 for precise definitions and notation).

Question. Let $V$ be a covering and quasi-unsplit family of rational 1-cycles on $X$. Is $\mathbb{R}_{\geq 0}[V]$ a geometric extremal ray of $\overline{\text{NE}}(X)$?

Note that this question is natural, since any family of rational 1-cycles such that the general member generates a geometric extremal ray of $\overline{\text{NE}}(X)$ is quasi-unsplit. The converse is not true if the family is not covering (just think of a smooth blow-down of a smooth projective variety to a non projective one).

Let $V$ be any covering family of rational 1-cycles on $X$. Then $V$ defines set-theoretically an equivalence relation on $X$: two points $x, x'$ are $V$-equivalent if there exist $v_1, \ldots, v_m \in V$ such that some connected component of $C_{v_1} \cup \cdots \cup C_{v_m}$ contains $x$ and $x'$, where $C_v \subset X$ is the curve corresponding to $v \in V$.

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In this situation, after Campana’s results (see Section 2), there exists an almost holomorphic map \( q: X \to Y \), to a projective algebraic variety, whose general fibers are \( V \)-equivalence classes.

We first prove the following result, which involves the dimension of the general fiber of \( q \).

**Theorem 1.** Let \( X \) be a normal and \( \mathbb{Q} \)-factorial complex projective variety of dimension \( n \). Let \( V \) be a covering and quasi-unsplit family of rational 1-cycles on \( X \), and let \( f_V \) be the dimension of a general \( V \)-equivalence class.

If \( f_V \geq n - 3 \), then \( \mathbb{R}_{\geq 0}[V] \) is a geometric extremal ray of the Mori cone \( \overline{NE}(X) \).

We then immediately get the following.

**Corollary 1.** Let \( X \) be a normal and \( \mathbb{Q} \)-factorial complex projective variety of dimension \( n \leq 4 \). Let \( V \) be a covering and quasi-unsplit family of rational 1-cycles on \( X \). Then \( \mathbb{R}_{\geq 0}[V] \) is a geometric extremal ray of the Mori cone \( \overline{NE}(X) \).

As previously recalled, one can associate a rational map \( q: X \to Y \) to any covering family of rational 1-cycles on \( X \). We call a geometric quotient for \( V \) a morphism \( q': X \to Y' \), onto a normal projective variety \( Y' \), such that every fiber of \( q' \) is a \( V \)-equivalence class. If such a quotient exists, then it is clearly unique up to isomorphism. On the other hand, even if \( X \) is smooth, a geometric quotient for \( V \) does not necessarily exist (see example 1).

The study of the extremal contraction given by the previous result leads to the following.

**Theorem 2.** Let \( X \) be a normal and \( \mathbb{Q} \)-factorial complex projective variety, having canonical singularities, of dimension \( n \). Let \( V \) be a covering and quasi-unsplit family of rational 1-cycles on \( X \), and let \( f_V \) be the dimension of a general \( V \)-equivalence class.

If \( f_V \geq n - 3 \), then the Mori contraction of \( \mathbb{R}_{\geq 0}[V], \text{cont}_V: X \to Y' \), is the geometric quotient for \( V \). If moreover \( f_V \geq n - 2 \), then \( \text{cont}_V \) is equidimensional.

We finally consider the toric case, where we can prove both extremality and existence of the geometric quotient for a quasi-unsplit family in any dimension.

**Theorem 3.** Let \( X \) be a toric and \( \mathbb{Q} \)-factorial complex projective variety, and let \( V \) be a quasi-unsplit covering family of rational 1-cycles in \( X \). Then \( \mathbb{R}_{\geq 0}[V] \) is a geometric extremal ray of the Mori cone \( \overline{NE}(X) \), and the Mori contraction of \( \mathbb{R}_{\geq 0}[V], \text{cont}_V: X \to Y' \), is the geometric quotient for \( V \).

The following is an immediate application of Theorems 1 and 3.

**Corollary 2.** Let \( X \subset \mathbb{P}^N \) be a normal and \( \mathbb{Q} \)-factorial variety, covered by lines. Assume either that \( X \) is toric, or that \( X \) has canonical singularities and \( \dim X \leq 4 \). Let \( V \) be an irreducible family of lines covering \( X \).

Then there exists a morphism \( q': X \to Y' \), onto a normal, \( \mathbb{Q} \)-factorial, projective variety \( Y' \) with \( \rho_{Y'} = \rho_X - 1 \), such that all lines of \( V \) are contracted by \( q' \).

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2 We denote by \( \rho_Z \) the Picard number of an algebraic variety \( Z \).
2. Set-up on families of rational 1-cycles

Let $X$ be a normal, irreducible, $n$-dimensional complex projective variety. We denote by $\mathcal{N}_1(X)_{\mathbb{R}}$ (respectively, $\mathcal{N}_1(X)_{\mathbb{Q}}$) the vector space of 1-cycles in $X$ with real (respectively, rational) coefficients, modulo numerical equivalence. In $\mathcal{N}_1(X)_{\mathbb{R}}$, let $\text{NE}(X)$ be the closure of the cone generated by classes of effective 1-cycles in $X$.

Recall that the existence of a covering family $V$ of rational 1-cycles on $X$ is equivalent to $X$ being uniruled [Kol96, Proposition IV.1.3].

For such family $V$, we have a diagram given by the incidence variety $C$ associated to $V$:

\[
\begin{array}{ccc}
C & \xrightarrow{F} & X \\
\downarrow{\pi} & & \downarrow{\pi} \\
V & & 
\end{array}
\]

where $\pi$ and $F$ are proper and surjective. We set $C_v := F(\pi^{-1}(v))$ for any $v \in V$.

The relation of $V$-equivalence on $X$ induced by such a family was introduced and studied in [Cam81]; we refer the reader to [Cam04], [Deb01, §5.4] or [Kol96, §IV.4] for more details. In particular, there exists a rational map $q : X \dasharrow Y$ associated to $V$, whose main properties are recalled now. By [Deb01, Theorem 5.9], there exists a closed and irreducible subset of Chow$(X)$ whose normalization $Y$ satisfies the following properties:

(a) let $Z \subset Y \times X$ be the restriction of the universal family,

\[
\begin{array}{ccc}
Z & \xrightarrow{e} & X \\
\downarrow{p} & & \downarrow{q} \\
Y & & 
\end{array}
\]

then $e$ is birational and $q = p \circ e^{-1}$ is almost holomorphic (which means that the indeterminacy locus of $q$ does not dominate $Y$);

(b) a general fiber of $q$ is a $V$-equivalence class,

(c) a general fiber of $q$, hence of $p$, is irreducible.

As a consequence of the existence of this map $q$, a general $V$-equivalence class is a closed subset of $X$. We denote by $f_V$ its dimension, so that $\dim Y = n - f_V$. Moreover, it is well known that any $V$-equivalence class is a countable union of closed subsets of $X$.

**Definition 1.** We say that a subset $Z$ of $X$ is $V$-rationally connected if every connected component of $Z$ is contained in some $V$-equivalence class.

**Lemma 1.** Let $X$ be a normal projective variety and $V$ be a covering family of rational 1-cycles on $X$. Consider the diagram (2) above. Then $e(p^{-1}(y))$ is $V$-rationally connected for any $y \in Y$.

**Proof.** Let $\mathcal{R} \subset X \times X$ the graph of the equivalence relation defined by $V$: it is a countable union of closed subvarieties since $V$ is proper. The fiber product $Z \times_Y Z$ is irreducible and thus $(e \times e)(Z \times_Y Z) \subset \mathcal{R}$ thanks to properties (a) and
(b) above. Therefore, for any \( x \in e(p^{-1}(y)) \), the cycle \( e(p^{-1}(y)) \) is contained in the \( V \)-equivalence class of \( x \). ■

The following well known remark will be of constant use (see [Kol96, Proposition IV.3.13.3], or [ACO04, Corollary 4.2]).

**Remark 1.** If \( Z \subset X \) is \( V \)-rationally connected, every curve contained in \( Z \) is numerically equivalent in \( X \) to a linear combination with rational coefficients of irreducible components of cycles in \( V \). In particular, if \( V \) is quasi-unsplit, the numerical class of every curve contained in a \( V \)-rationally connected subset \( Z \) of \( X \) belongs to \( \mathbb{R}_{\geq 0}[V] \).

Finally, we will need the following.

**Lemma 2.** Let \( X \) be a normal projective variety and \( V \) be a covering and quasi-unsplit family of rational 1-cycles on \( X \). Then there exists a covering and quasi-unsplit family \( V' \) of rational 1-cycles on \( X \) such that:

- the general cycle of \( V' \) is reduced and irreducible;
- for any \( v \in V' \) there exists \( v \in V \) such that \( C_v \subseteq C_v' \); in particular \( \mathbb{R}_{\geq 0}[V] = \mathbb{R}_{\geq 0}[V'] \).

**Proof.** Let \( C \) be the incidence variety associated to \( V \) as in (1).

It is well-known that every irreducible component of \( C \) dominates \( V \), let \( C'' \) be an irreducible component of \( C \) which dominates \( X \) too. Let \( C' \to V' \) be the Stein factorization of the composite map \( C' \to C'' \to V \). Since \( C' \to V' \) has connected fibers and \( C' \) is normal, the general fiber of \( C' \to V' \) is irreducible. Moreover, the image in \( X \) of every fiber of \( C' \to V' \) is contained in a cycle of \( V \).

Since \( V' \) is normal, there is a holomorphic map \( V' \to \text{Chow}(X) \). Then after replacing \( V' \) by its image in \( \text{Chow}(X) \) and \( C' \) by its image in \( \text{Chow}(X) \times X \), we get the desired family. ■

### 3. Properties of the base locus and extremality

Let \( V \) be a covering family of rational 1-cycles on \( X \), and recall the diagram (2) associated to \( V \).

Let \( E \subset Z \) be the exceptional locus of \( e \), and \( B := e(E) \subset X \). Observe that since \( X \) is normal, \( \dim B \leq n - 2 \).

**Proposition 1.** Let \( X \) be a normal and \( \mathbb{Q} \)-factorial projective variety, and \( V \) be a covering and quasi-unsplit family of rational 1-cycles on \( X \). Consider the associated diagram as in (2). Then:

- \( e(p^{-1}(y)) \) is a \( V \)-equivalence class of dimension \( f_V \) for every \( y \in Y \setminus p(E) \);
- \( B \) is the union of all \( V \)-equivalence classes of dimension bigger than \( f_V \).

**Proof.** Set \( X^0 := X \setminus B \) and \( Y^0 := Y \setminus p(E) = q(X^0) \). Choose a very ample line bundle \( L \) on \( Y \), and let \( U \subset |L| \) be the open subset of divisors \( H \) that are irreducible and such that \( H \cap Y^0 \neq \emptyset \). For any \( H \in U \), we define \( \hat{H} := q^{-1}(H \cap Y^0) \), which
is a Weil divisor in $X$. Since $X$ is $\mathbb{Q}$-factorial, some multiple of $\hat{H}$ defines a line bundle $\tilde{L}$ on $X$.

Let now $N := h^0(L)$, and let $s_1, \ldots, s_N$ be general global sections generating $L$. For each $i = 1, \ldots, N$, let $H_i \in |L|$ be the divisor of zeros of $s_i$ and $\hat{H}_i$ in $X$ as defined above.

Let's show that $\hat{H}_1 \cap \cdots \cap \hat{H}_N = B$. If $x \notin B$, then $q$ is defined in $x$ and there is some $i_0 \in \{1, \ldots, N\}$ such that $q(x) \notin H_{i_0}$, so $x \notin \hat{H}_{i_0}$. Conversely, let $x \in B$ and fix $i \in \{1, \ldots, N\}$. Then $e^{-1}(x)$ has positive dimension; let $C \subset Z$ be an irreducible curve such that $e(C) = x$. Then $p(C)$ is a curve in $Y$, hence $H_i \cap p(C) \neq \emptyset$ and $p^{-1}(H_i) \cap C \neq \emptyset$. Now observe that $p^{-1}(H_i)$ does not contain any component of $E$, hence $e(p^{-1}(H_i))$ is a divisor in $X$ which coincides with $\hat{H}_i$ over $X \setminus B$. Then $\tilde{H}_i = e(p^{-1}(H_i))$ and $x \in \hat{H}_i$.

Let $i \in \{1, \ldots, N\}$. Observe that $\hat{H}_i \cdot [V] = 0$, because $[V]$ is quasi-unsplit and any irreducible component of general cycle of the family is contained in a fiber of $q$ disjoint from $\hat{H}_i$. This implies that $\hat{H}_i$ is closed with respect to $V$-equivalence. In fact, let $C$ be an irreducible component of a cycle of $V$ such that $C \cap \hat{H}_i \neq \emptyset$. Since $V$ is quasi-unsplit, we have $\hat{H}_i \cdot C = 0$, which implies $C \subseteq \hat{H}_i$.

Now since $B = \hat{H}_1 \cap \cdots \cap \hat{H}_N$ and all $\hat{H}_i$'s are closed with respect to $V$-equivalence, we see that $B$ is a union of $V$-equivalence classes.

Observe that if $C \subset X \setminus B$ is an irreducible curve such that $\hat{H} \cdot C = 0$ for some $H \in U$, then $q(C)$ is a point. In fact, if $q(C)$ is a curve, there exists $H_0 \in U$ such that $H_0$ intersects $q(C^0)$ in a finite number of points. Then $\tilde{H}_0$ intersects $C$ without containing it, a contradiction, because $\hat{H}$ and $\tilde{H}_0$ are numerically equivalent, so $C \cdot \tilde{H} > 0$.

Now fix $y_0 \in Y^0$. We know by Lemma \ref{lemma1} that $e(p^{-1}(y_0))$ is contained in a $V$-equivalence class $F$. Since $B$ is closed with respect to $V$-equivalence, we have $F \subset X^0$. Consider an irreducible component $C$ of a cycle of $V$ such that $C \subseteq F$. Since $V$ is quasi-unsplit, we have $\hat{H} \cdot C = 0$, hence $q(C)$ is a point by what we proved above. Therefore $q(F) = y_0$ and $F = e(p^{-1}(y_0))$, so we have (i).

For any $x \in X$, let $Y_x := p(e^{-1}(x))$ be the family of cycles parametrized by $Y$ and passing through $x$, and Locus($Y_x$) := $e(p^{-1}(Y_x))$. Observe that for any $y \in Y_x$, the subset $e(p^{-1}(y))$ contains $x$ and is $V$-rationally connected by Lemma \ref{lemma2}. Hence Locus($Y_x$) is $V$-rationally connected for any $x \in X$.

Since $Z \subset X \times Y$, we have $\dim Y_x = \dim e^{-1}(x)$. Thus $\dim Y_x > 0$ if and only if $x \in B$, by Zariski’s main Theorem. If so, Locus($Y_x$) has dimension at least $f_V + 1$.

Now let $F$ be a $V$-equivalence class contained in $B$, and $x \in F$. Then Locus($Y_x$) has dimension at least $f_V + 1$ and is contained in $F$, hence $\dim F \geq f_V + 1$.

Let us remark that in general, if $V$ is not quasi-unsplit, $B$ is not closed with respect to $V$-equivalence.

**Example 1.** In $\mathbb{P}^2$ fix two points $x, y$ and the line $L = \overline{xy}$. Consider $\mathbb{P}^2 \times \mathbb{P}^2$ with the projections $\pi_1, \pi_2$ on the two factors, and fix three curves $R_x, R_y, L'$ such that:
Let \( \sigma: W \to \mathbb{P}^2 \times \mathbb{P}^2 \) be the blow-up of \( R_x \) and \( R_y \). In \( W \), the strict transform of \( L' \) is a smooth rational curve with normal bundle \( \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \). Let \( X \) be the variety obtained by “flipping” this curve. Then \( X \) is a smooth toric Fano 4-fold with \( \rho_X = 4 \) (this is \( Z_2 \) in Batyrev’s list, see [Bat99, Proposition 3.3.5]).

\[
\begin{array}{c}
\xymatrix{X \ar[d]_{\pi_2 \circ \sigma} \ar[r]_q & W \\
\mathbb{P}^2}
\end{array}
\]

The strict transform of a general line in a fiber of \( \pi_2 \) gives a covering family \( V \) of rational curves on \( X \). The birational map \( X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \) is an isomorphism over \( \mathbb{P}^2 \times (\mathbb{P}^2 \setminus L) \); if \( U \subset X \) is the corresponding open subset, then \( U \) is closed with respect to \( V \)-equivalence and every fiber of \( q: U \to \mathbb{P}^2 \setminus L \) is a \( V \)-equivalence class isomorphic to \( \mathbb{P}^2 \). Thus \( f_V = 2 \).

Let \( T_x \) and \( T_y \) be the images in \( X \) of the exceptional divisors of \( \sigma \) in \( W \). These two divisors are \( V \)-rationally connected, and they can not be contained in \( B \) because \( \dim B \leq 2 \). Moreover, \( P := T_x \cap T_y \) is the \( \mathbb{P}^2 \) with normal bundle \( \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \) obtained under the flip. The map \( q: X \dashrightarrow \mathbb{P}^2 \) can not be defined over \( P \), so \( P \cap B \neq \emptyset \). Therefore \( B \) can not be closed with respect to \( V \)-equivalence.

Observe that the numerical class of \( V \) lies in the interior of \( \overline{\text{NE}}(X) \), hence the unique morphism, onto a projective variety, which contracts curves in \( V \), is \( X \to \{pt\} \).

A key observation is the following.

**Proposition 2.** Let \( X \) be a normal and \( \mathbb{Q} \)-factorial projective variety, and \( V \) a covering and quasi-unsplit family of rational 1-cycles on \( X \) with \( \rho_X = 4 \).

If \( B \) is \( V \)-rationally connected, then \( \mathbb{R}_{\geq 0}[V] \) is a geometric extremal ray of \( \overline{\text{NE}}(X) \).

**Proof.** Let \( X^0 := X \setminus B \) and \( Y^0 := Y \setminus p(E) = q(X^0) \). Let \( L \) be a very ample line bundle on \( Y \). Let \( U \subset |L| \) be the open subset of divisors \( H \) that are irreducible and such that \( H \cap Y^0 \neq \emptyset \). For any \( H \) in \( U \), we define \( \tilde{H} := q^{-1}(H \cap Y^0) \) as in the proof of Proposition 1. Recall that \( \tilde{H} \cdot [V] = 0 \).

Let’s show that \( \tilde{H} \) is nef. Assume by contradiction that there exists an irreducible curve \( C \) with \( C \cdot \tilde{H} < 0 \).

**Claim.** \( C \subset B \).

Actually, either \( C \) is contained in a fiber of \( q \), hence it is numerically proportional to \([V]\) which contradicts \( C \cdot \tilde{H} < 0 \). Or \( C \cap X^0 =: C^0 \) is an open subset of \( C \), \( \dim q(C^0) = 1 \), hence there exists \( H_0 \in U \) such that \( H_0 \) intersects \( q(C^0) \) in a finite number of points. Then \( \tilde{H}_0 \) intersects \( C \) without containing it, a contradiction, because \( \tilde{H} \) and \( \tilde{H}_0 \) are numerically equivalent, so \( C \cdot \tilde{H} > 0 \).
Since $B$ is $V$-rationally connected, $C$ must be numerically proportional to $V$, impossible.

Let’s finally show that $C \cdot \tilde{H} = 0$ if and only if $C$ is numerically proportional to $[V]$: actually, if $C \cdot \tilde{H} = 0$, the previous arguments show that either $C \subset B$ or $C$ is contained in a fiber of $q$, both are $V$-rationally connected, hence $C$ is numerically proportional to $V$.

Unfortunately, $B$ is not $V$-rationally connected in general as shown by the following example.

**Example 2** (see [Kac97] Example 11.1 and references therein). Fix a point $p_0$ in $\mathbb{P}^3$ and let $P_0 := \{ \Pi \in (\mathbb{P}^3)^* \mid p_0 \in \Pi \} \simeq \mathbb{P}^2$ be the variety of 2-planes in $\mathbb{P}^3$ containing $p_0$. Consider the variety $X \subset \mathbb{P}^3 \times P_0$ defined as $X := \{(p, \Pi) \in \mathbb{P}^3 \times P_0 \mid p \in \Pi \}$. Then $X$ is a smooth Fano 4-fold, with Picard number 2 and pseudo-index 2. The two elementary extremal contractions are given by the projections on the two factors.

The morphism $X \rightarrow P_0$ is a fibration in $\mathbb{P}^2$: the fiber over a point is the plane corresponding to that point.

Consider the morphism $X \rightarrow \mathbb{P}^3$. If $p \neq p_0$, the fiber over $p$ is the $\mathbb{P}^1$ of planes containing $p$ and $p_0$. But the fiber $F_0$ over $p_0$ is naturally identified with $P_0$, hence it is isomorphic to $\mathbb{P}^2$. We have $N_{F_0/X} = \Omega^1_{\mathbb{P}^2}(1)$ and $(-K_X)|_F = \mathcal{O}_F(2)$.

![Diagram](image)

Here $V \rightarrow \mathbb{P}^3$ is the blow-up of $p_0$ and $C \rightarrow X$ is the blow-up of $F_0$. Observe that $V$ is a family of extremal irreducible rational curves of anticanonical degree 2.

If we consider $X \times \mathbb{P}^1$ with the same family of curves, we have $\dim Y = 4$, $f_V = 1$ and $B = F_0 \times \mathbb{P}^1$ which is not $V$-rationally connected.

We finally get the following result: if $B$ has the smallest possible dimension, then it is $V$-rationally connected.

**Lemma 3.** Let $X$ be a normal and $\mathbb{Q}$-factorial projective variety, and $V$ be a covering and quasi-unsplit family of rational 1-cycles on $X$.

If $\dim B = f_V + 1$, then every connected component of $B$ is a $V$-equivalence class.

**Proof.** By Proposition 1, we know that $B$ is the union of all $V$-equivalence classes whose dimension is $f_V + 1$. Since each of these equivalence classes must contain an irreducible component of $B$, they are in a finite number, and each is contained in a connected component of $B$. 

So if $B_0$ is a connected component of $B$, we have $B_0 = F_1 \cup \cdots \cup F_r$, where each $F_i$ is a $V$-equivalence class. We want to show that $r = 1$.

Assume by contradiction that $r > 1$. Observe that the $F_i$'s are disjoint and $B_0$ is connected, hence at least one $F_i$ is not a closed subset of $X$, assume it is $F_1$.

Then $F_1$ is a countable union of closed subsets. Considering the decomposition of $B_0$ as a union of irreducible components, we find an irreducible component $T$ of $B_0$ such that

$$T = \bigcup_{m \in \mathbb{N}} K_m$$

where each $K_m$ is a non empty proper closed subset of $T$. Since $T$ is an irreducible complex projective variety, this is impossible. □

We then reformulate in a single result what we proved so far, and show that it implies Theorem 1.

**Proposition 3.** Let $X$ be a normal and $\mathbb{Q}$-factorial projective variety, and $V$ a covering and quasi-unsplit family of rational 1-cycles on $X$. Then:

(i) either $B = \emptyset$ or $\dim(B) \geq f_V + 1$,

(ii) if $B = \emptyset$ or if $\dim(B) = f_V + 1$ then $\mathbb{R}_{\geq 0}[V]$ is a geometric extremal ray of the Mori cone $\overline{NE}(X)$.

**Proof of Theorem 1.** Just notice that if $f_V \geq n-3$ and $B$ is not empty, Proposition 3 (i) gives $\dim B \geq f_V + 1 \geq n - 2$, so $\dim B = n - 2 = f_V + 1$. Then Proposition 3 (ii) gives that $\mathbb{R}_{\geq 0}[V]$ is a geometric extremal ray of the Mori cone $\overline{NE}(X)$. □

4. EXISTENCE OF A GEOMETRIC QUOTIENT

Let $V$ be a covering and quasi-unsplit family of rational 1-cycles on $X$, and assume that there exists a geometric quotient $q': X \to Y'$ for $V$.

Observe that $q'$ has the following property: for any irreducible curve $C$ in $X$, $q'(C)$ is a point if and only if $[C]$ is proportional to $[V]$.

Conversely, we show that a morphism with the property above is quite close to be a geometric quotient.

**Proposition 4.** Let $X$ be a normal and $\mathbb{Q}$-factorial projective variety, and $V$ a covering and quasi-unsplit family of rational 1-cycles on $X$.

Assume that there exists a morphism with connected fibers $q': X \to Y'$, onto a complete and normal algebraic variety $Y'$, such that for any irreducible curve $C$ in $X$, $q'(C)$ is a point if and only if $[C]$ is proportional to $[V]$.

Then there exists a birational morphism $\psi: Y \to Y'$ that fits into the commutative diagram:

$$
\begin{array}{ccc}
Z & \xrightarrow{\varepsilon} & X \\
p \downarrow & & \downarrow q' \\
Y & \leftarrow & Y'
\end{array}
$$
Moreover, if \( B' := q'(B) \), we have \((q')^{-1}(B') = B\), and
\[
B' = \{ y \in Y' | \dim((q')^{-1}(y)) > f_Y \} = \{ y \in Y' | \dim \psi^{-1}(y) > 0 \}.
\]
In particular, every fiber of \( q' \) over \( Y' \setminus B' \) is a \( V \)-equivalence class.

Observe that in example \( \textbf{2} \) \( \psi \) is not an isomorphism.

**Proof.** Let’s show first of all that \((q')^{-1}(B') = B\).

If \( C \subset X \) is an irreducible curve contained in a fiber of \( q' \), then either \( C \cap B = \emptyset \), or \( C \subseteq B \). In fact, assume that \( C \cap B \neq \emptyset \). Let \( \bar{H}_0, \ldots, \bar{H}_N \) be as in the proof of Proposition \( \textbf{1} \). Then for any \( i = 0, \ldots, N \), we have \( C \cdot \bar{H}_i = 0 \) and \( C \cap \bar{H}_i \neq \emptyset \), hence \( C \subseteq \bar{H}_i \) so \( C \subseteq B \).

Since \( q' \) has connected fibers, we see that for every fiber \( F \) of \( q' \), either \( F \cap B = \emptyset \), or \( F \subseteq B \). This means that \((q')^{-1}(q'(B)) = B\).

The existence of \( \psi \) as in \( \textbf{3} \) follows easily from the normality of \( Y \) and the fact that \( q' \) contracts all curves in \( V \), hence all \( V \)-equivalence classes. Observe that \( \psi \) is surjective with connected fibers.

Let’s show that \( p \) contracts to a point any fiber of \( q \circ \epsilon \) over \( Y' \setminus B' \).

Let \( F \) be a fiber of \( q' \) over \( Y' \setminus B' \), then we have \( F \subset X \setminus B \). Let \( C \subset F \) be an irreducible curve, and choose an irreducible curve \( C' \subset X \setminus B \) which is a component of a cycle of the family \( V \). Since \( q'(C) \) is a point, there exists \( \lambda \in \mathbb{Q}_{>0} \) such that \( C \equiv \lambda C' \) in \( X \).

Set \( X^0 := X \setminus B \). Notice that \( \epsilon \) is an isomorphism over \( X^0 \), so \( X^0 \) can be viewed also as an open subset of \( Z \); in the same way the curves \( C \) and \( C' \) can be viewed also as a curves in \( Z \). Let’s show that \( C \equiv \lambda C' \) still holds in \( Z \).

Let \( L \in \text{Pic} Z \), and write \( L_{|X^0} = O_{X^0}(D) \), \( D \) a Cartier divisor in \( X^0 \). Let \( \overline{D} \) be the closure of \( D \) in \( X \) (meaning, if \( D = \sum_i a_i V_i \), that \( \overline{D} = \sum_i a_i \overline{V}_i \)) and let \( m \in \mathbb{Z}_{>0} \) be such that \( m \overline{D} \) is Cartier in \( X \). Then set \( M := \epsilon^*(O_X(m\overline{D})) \in \text{Pic} Z \).

By construction, \( M \otimes L^{\otimes(-m)} \) is trivial on \( X^0 \), so we can write \( L^{\otimes m} = M \otimes O_Z(G) \), where \( G \) is a Cartier divisor in \( Z \) with \( \text{Supp} G \subseteq E \).

Now observe that \( C \cdot G = C' \cdot G = 0 \), because both curves are disjoint from \( E \), and that \( C \cdot M = \lambda C' \cdot M \) by the projection formula. Then we have \( C \cdot L = \lambda C' \cdot L \), so \( C \equiv \lambda C' \) in \( Z \).

Then \( p \) must contract \( C \) to a point, because \( Y \) is projective. Since \( e^{-1}(F) \) is connected, we have shown that \( p \) contracts \( e^{-1}(F) \) to a point. Since \( Y' \) and \( Y \) are normal, this implies that \( \psi \) is an isomorphism over \( Y' \setminus B' \).

Finally, let \( y \in B' \) and let \( F' = (q')^{-1}(y) \). Then \( F' \subseteq B \), so \( e \) has positive dimensional fibers on \( F' \), and \( \dim e^{-1}(F') > \dim F' \geq f_Y \). Since \( e^{-1}(F') = p^{-1}(\psi^{-1}(y)) \) and \( p \) has all fibers of dimension \( f_Y \), we must have \( \dim \psi^{-1}(y) > 0 \).

We can finally prove our results.

**Theorem 4.** Let \( X \) be a normal and \( \mathbb{Q} \)-factorial complex projective variety of dimension \( n \), having canonical singularities. Let \( V \) be a covering and quasi-unsplit family of rational 1-cycles on \( X \).
If \( \dim B \leq f_V + 1 \), then \( \mathbb{R}_{\geq 0}[V] \) is a geometric extremal ray of the Mori cone \( \mathcal{NE}(X) \) and the Mori contraction of \( \mathbb{R}_{\geq 0}[V] \), \( \text{cont}_{[V]}: X \to Y' \), is the geometric quotient for \( V \).

**Proof.** If \( B \) is empty, then the statement is clear. Assume that \( B \) is not empty. Then Proposition 3 and Lemma 3 yield that \( \dim B = f_V + 1 \), every connected component of \( B \) is a \( V \)-equivalence class, and \( \mathbb{R}_{\geq 0}[V] \) is a geometric extremal ray of \( \mathcal{NE}(X) \).

We have to show that \( -K_X \cdot [V] > 0 \). Let \( V' \) be the covering family of rational 1-cycles on \( X \) given by Lemma 2, and consider a resolution of singularities \( f: X' \to X \). The family \( V' \) determines a covering family \( V'' \) of rational 1-cycles in \( X' \). If \( C_0 \subset X \) is a general element of the family \( V' \), then \( C_0 = f^{-1}(C_0 \setminus \text{Sing}(X)) \) is a general element of \( V'' \), and \( C_0 = f_*(C') \).

Since \( C_0 \) is reduced and irreducible, so is \( C' \). Moreover \( V'' \) is covering, so \( C' \) is a free curve in \( X' \), and it has positive anticanonical degree.

Let \( m \in \mathbb{Z}_{>0} \) be such that \( mK_X \) is Cartier. Since \( X \) has canonical singularities, we have
\[
mK_{X'} = f^*(mK_X) + \sum a_i E_i,
\]
where \( E_i \) are exceptional divisors of \( f \) and \( a_i \in \mathbb{Z}_{\geq 0} \). Then
\[
-K_X \cdot C_0 = -f^*(mK_X) \cdot C' = -mK_{X'} \cdot C' + \sum a_i E_i \cdot C' > 0.
\]
This gives \( -K_X \cdot [V'] > 0 \) and thus \( -K_X \cdot [V] > 0 \).

Since \( X \) has canonical singularities, the cone theorem and the contraction theorem hold for \( X \) (see [Deb01, Theorems 7.38 and 7.39]). Moreover, the extremal ray \( \mathbb{R}_{\geq 0}[V] \) lies in the \( K_X \)-negative part of the Mori cone, hence it can be contracted.

Let \( \text{cont}_{[V]}: X \to Y' \) be the extremal contraction; then \( Y' \) is a normal, projective variety, and it is \( \mathbb{Q} \) factorial by [Deb01, Proposition 7.44].

Applying Proposition 4, we see that all fibers of \( \text{cont}_{[V]} \) over \( Y' \setminus \text{cont}_{[V]}(B) \) are \( V \)-equivalence classes. Since connected components of \( B \) are \( V \)-equivalence classes, they are exactly the fibers of \( \text{cont}_{[V]} \) over \( \text{cont}_{[V]}(B) \), and we have the statement. \( \blacksquare \)

Observe that Theorem 2 is a straightforward consequence of Theorem 4.

5. The toric case: proof of Theorem 4

In the case \( \rho_X = 1 \), the statement is true for \( q': X \to \{pt\} \). In fact, using Proposition 4, we see that the geometric quotient \( Y \) must be a point.

Assume that \( \rho_X > 1 \). Recall the diagram:
\[
\begin{array}{ccc}
C & \xrightarrow{F} & X \\
\downarrow & & \downarrow \\
V & \xrightarrow{\sigma} & \end{array}
\]
Recall also that if \( D \subset X \) is a prime invariant Weil divisor, there is a natural inclusion \( i_D : \mathcal{N}_1(D)_{\mathbb{R}} \hookrightarrow \mathcal{N}_1(X)_{\mathbb{R}} \).

**Step 1:** let \( D \subset X \) be a prime invariant Weil divisor such that \( D \cdot [V] = 0 \). Then there exists a covering and quasi-unsplit family \( V_D \) of rational 1-cycles in \( D \) such that \( \mathbb{R}_{>0}(i_D[V_D]) = \mathbb{R}_{>0}[V] \).

Choose an irreducible component \( W \) of \( F^{-1}(D) \) which dominates \( D \). Set \( V_D' := \pi(W) \), and let \( C_D' \) be an irreducible component of \( \pi^{-1}(V_D') \) containing \( W \). Consider the normalization \( C_D \) of \( C_D' \), and let \( \pi_D : C_D \to V_D \) be the Stein factorization of the composite map \( C_D \to C_D' \to V_D'. \) Finally let \( F_D : C_D \to X \) be the induced map.

For \( v \in V_D \), set \( G_v := F_D(\pi_D^{-1}(v)) \). Then \( G_v \cap D \neq \emptyset \), \( G_v \) is connected, and \( G_v \cdot D = 0 \) because \( V \) is quasi-unsplit. This implies \( G_v \subseteq D \), hence \( F_D(C_D) \subseteq D \). Moreover, since \( W \) dominates \( D \), we have \( F_D(C_D) = D \).

Since \( V_D \) is normal, there is a holomorphic map \( V_D \to \text{Chow}(D) \). Then after replacing \( V_D \) by its image in \( \text{Chow}(D) \) and \( C_D \) by its image in \( \text{Chow}(D) \times X \), we get the desired family.

**Step 2:** there exists an invariant prime Weil divisor having intersection zero with \([V]\).

In fact, let \( q : X \dashrightarrow Y \) be the rational map associated to \( V \). Since \( \rho_X > 1 \), \( Y \) is not a point. Let \( D \) be a prime divisor in \( Y \) intersecting \( q(X^0) \) and set \( D' := q^{-1}(D) \). Since there are curves of the family \( V \) disjoint from \( D' \), we have \( D' \cdot [V] = 0 \). Moreover, \( D' \) is linearly equivalent to \( \sum a_i D_i \), where \( a_i \in \mathbb{Q}_{>0} \) and \( D_i \) are invariant prime Weil divisors. Hence the statement.

**Step 3:** we prove the statement.

Let \( \Sigma_X \) be the fan of \( X \) in \( N \cong \mathbb{Z}^n \), and let \( G_X \) be the set of primitive generators of one dimensional cones in \( \Sigma_X \). It is well known that \( G_X \) is in bijection with the set of invariant prime divisors of \( X \); for any \( x \in G_X \), we denote \( D_x \) the associated divisor. Recall that for any class \( \gamma \in \mathcal{N}_1(X)_{\mathbb{Q}} \), we have

\[
\sum_{x \in G_X} (\gamma \cdot D_x)x = 0 \quad \text{in} \quad N \otimes_{\mathbb{Z}} \mathbb{Q},
\]

and that the association \( \gamma \mapsto \sum_{x \in G_X} (\gamma \cdot D_x)x \) gives a canonical identification of \( \mathcal{N}_1(X)_{\mathbb{Q}} \) with the \( \mathbb{Q} \)-vector space of linear relations with rational coefficients among \( G_X \).

Let \( m_1 x_1 + \ldots + m_h x_h = 0 \) be the relation corresponding to \([V]\), with \( x_i \in G_X \) and \( m_i \) non zero rational numbers for all \( i \). Since \( V \) is covering and quasi-unsplit, all \( m_i \)'s must be positive. For \( y \in G_X \), we have \( D_y \cdot [V] = 0 \) if and only if \( y \) is different from \( x_1, \ldots, x_h \). So by Step 2, we know that \( G_X \setminus \{x_1, \ldots, x_h\} \) is non empty.

The following two statements are equivalent (see [Rei83 Theorem 2.4] and [Cas03 Theorem 2.2]):
(a) there exists a $\mathbb{Q}$-factorial, projective toric variety $Y'$, and a flat, equivariant morphism $q': X \to Y'$, such that for any curve $C$ in $X$, $q'(C)$ is a point if and only if $[C]$ is proportional to $[V]$;

(b) for any $\tau \in \Sigma_X$ such that $x_1, \ldots, x_h \notin \tau$, we have

$$\tau + \langle x_1, \ldots, \bar{x}_i, \ldots, x_h \rangle \in \Sigma_X \quad \text{for all } i = 1, \ldots, h.$$ (4)

Let’s show (b) by induction on the dimension of $X$.

Clearly, it is enough to check (4) for any maximal $\tau$ in $\Sigma_X$ not containing any $x_i$. Since $\{x_1, \ldots, x_h\} \subsetneq G_X$, such a maximal $\tau$ will have positive dimension.

Let $y \in G_X \cap \tau$. We have $D_y \cdot [V] = 0$, so by Step 1 there exists a quasi-unsplit, covering family $V_{D_y}$ in $D_y$ such that $i_{D_y}[V_{D_y}]$ is proportional to $[V]$.

Set $N := N/\mathbb{Z} \cdot y$ and for any $z \in N$, write $\overline{z}$ for its image in $N$. The fan $\Sigma_{D_y}$ of $D_y$ is given by the projections in $N \otimes_{\mathbb{Z}} \mathbb{Q}$ of all cones of $\Sigma_X$ containing $y$. The relation corresponding to $[V_{D_y}]$ is $\lambda m_1 \overline{x}_1 + \cdots + \lambda m_h \overline{x}_h = 0$, for some $\lambda \in \mathbb{Q}_{>0}$.

By induction, we know that (b) holds for $V_{D_y}$ in $D_y$. In particular, the projection $\overline{\tau}$ of $\tau$ is in $\Sigma_{D_y}$, so we have

$$\overline{\tau} + \langle \overline{x}_1, \ldots, \overline{x}_i, \ldots, \overline{x}_h \rangle \in \Sigma_{D_y} \quad \text{for all } i = 1, \ldots, h.$$ This yields (4).

Finally, since $q'$ is flat, all fibers must be $V$-equivalence classes and $B = \emptyset$.

References

[ACO04] Marco Andreatta, Elena Chierici, and Gianluca Occhetta. Generalized Mukai conjecture for special Fano varieties. Central European Journal of Mathematics, 2(2):272–293, 2004.

[Bat99] Victor V. Batyrev. On the classification of toric Fano 4-folds. Journal of Mathematical Sciences (New York), 94:1021–1050, 1999.

[Cam81] Frédéric Campana. Coréduction algébrique d’un espace analytique faiblement Kählerien compact. Inventiones Mathematicae, 63:187–223, 1981.

[Cam04] Frédéric Campana. Orbifolds, special varieties and classification theory: an appendix. Annales de l’Institut Fourier, 54(3):631–665, 2004.

[Cas03] Cinzia Casagrande. Contractible classes in toric varieties. Mathematische Zeitschrift, 243:99–126, 2003.

[CO04] Elena Chierici and Gianluca Occhetta. The cone of curves of Fano varieties of coindex four. Preprint [math.AG/0401429], 2004.

[Deb01] Olivier Debarre. Higher-Dimensional Algebraic Geometry. Universitext. Springer-Verlag, 2001.

[Kac97] Yasuyuki Kachi. Extremal contractions from 4-dimensional manifolds to 3-folds. Annali della Scuola Normale Superiore di Pisa. Classe di Scienze (4), 24(1):63–131, 1997.

[Kol96] János Kollár. Rational Curves on Algebraic Varieties, volume 33 of Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1996.

[Rei83] Miles Reid. Decomposition of toric morphisms. In Arithmetic and Geometry, vol. II: Geometry, number 36 in Progress in Mathematics, pages 395–418. Birkhäuser, 1983.

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