Kaluza–Klein gravity and cosmology emerging from G. Perelman’s entropy functionals and quantum geometric information flows

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Abstract We elaborate on quantum geometric information flows, QGIFs, and emergent (modified) Einstein–Maxwell and Kaluza–Klein, KK, theories formulated in Lagrange–Hamilton and general covariant variables. There are considered nonholonomic deformations of Grigory Perelman’s F- and W-functionals (originally postulated for Riemannian metrics) for describing relativistic geometric flows, gravity and matter field interactions, and associated statistical thermodynamic systems. We argue that the concept of G. Perelman W-entropy presents more general and alternative possibilities to characterize geometric flow evolution, GIF, and gravity models than the Bekenstein–Hawking and another area holographic-type entropies. Formulating the theory of QGIFs, a set of fundamental geometric, probability and quantum concepts, and methods of computation, are reconsidered for curved spacetime and (relativistic) phase spaces. Such generalized metric affine spaces are modeled as nonholonomic Lorentz manifolds, (co)tangent Lorentz bundles, and associated vector bundles. Using geometric and entropic and thermodynamic values, we define QGIF versions of the von Neumann entropy, relative and conditional entropy, mutual information, etc. There are analyzed certain important inequalities and possible applications of Perelman and related entanglement and Rényi entropies to theories of KK QGIFs and emergent gravitational and electromagnetic interactions. New classes of exact cosmological solutions for GIFs and respective quasiperiodic evolution scenarios are elaborated. We show how classical and quantum thermodynamic values can be computed for cosmological quasiperiodic solutions and speculate how such constructions can be used for explaining structure formation in dark energy and dark matter physics.

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1 Introduction

The goal of this paper is to prove that (modified) Einstein–Maxwell, EM, theories emerge from models of self-similar generalized geometric flows (with nonholonomic Ricci solitons) and quantum geometric information flows, QGIFs, see relevant details in partner works [1–4]. The gravitational and electromagnetic fields are distinguished from other fundamental ones (with strong and weak quantum interactions) because at long distances such interactions can be described by well-defined relativistic classical theories. Geometrically, the EM theory (and more general Einstein–Yang–Mills configurations) can be formulated equivalently as a five-dimensional, 5-d (and higher dimensions), Kaluza–Klein, KK, theory with corresponding parametrization of metric, frame, and connection structures and assumptions on compactification on extra dimension coordinates, for instance, see a review in [5]. In this work, we do not study nonabelian gauge interactions resulting in more sophisticated geometric evolution models with confinement and nonlinear gauge symmetries.

An important property of the classical EM and KK theories is that they can be treated as emergent entropic gravity models [6,7] (for instance, of Verlinde type [8,9]) in the theory of Ricci flows [10,11]. Here, we cite Friedan [12] who considered equivalents of the Hamilton equations in physics before mathematicians elaborated a rigorous geometric analysis and topology formalism, see [13–15] for reviews of results. Such geometric and nonrelativistic physical theories are characterized by the so-called G. Perelman F- and W-functionals (the last one is called also as W-entropy) and associated statistical thermodynamical models. In our works, an approach to modified gravity and information flow theories was developed in relativistic form and applied in the study of locally anisotropic stochastic and kinetic processes [16,17] and generalizations for quantum and relativistic/noncommutative supersymmetric theories [18–21]. Here, we cite also a series of papers on gradient flows of nonabelian gauge fields, conformal and supersymmetric gauge models [22–26], see also a work on Ricci–Yang–Mills geometric flow evolution [27].

The systems of nonlinear partial differential equations, PDEs, describing relativistic flow evolution and dynamical field equations in modified gravity theories (MGTs, see reviews [1,28,29]) and general relativity, GR, can be decoupled and integrated in general forms in terms of generating functions and sources, and integration functions or integration constants. We cite [1,7,20,30,31] and references therein for a review of geometric methods for constructing exact and parametric solutions (on the so-called anholonomic frame deformation method, AFDM) and applications in modern cosmology and astrophysics. The AFDM is also important for quantizing such theories because it allows to introduce Lagrange- and/or Hamilton-type nonholonomic variables and apply geometric methods in quantum mechanics, QM, and quantum field theory, QFT. Extending the main concepts and methods of information theory to classical geometric information flows, GIFs, and developing the approach for QM models, we can elaborate on quantum geometric information, QGIF, theories [1–4]. In this work, using corresponding generalizations of nonholonomic geometric and statistical thermodynamics G. Perelman models for GIFs and QM computation methods with density matrices, von Neumann entropy and entanglement, we investigate how emergent EM and KK theories can be derived from QGIFs. It should be noted that we follow the system of notations from [3,4] extended in a form when KK configurations for geometric and physical objects are described by underlined symbols.

The paper is structured as follows: In Sect. 2, we provide necessary geometric preliminaries and formulate the geometric flow equations for the EM and KK systems. Such modified Ricci flow equations are written in general and canonical (with decoupling properties) nonholonomic variables and using mechanical (Lagrange- and Hamilton-type) variables. Modified
EM and KK classical field equations are derived as nonholonomic Ricci soliton configurations for relativistic geometric flows. In Sect. 3, we define nonholonomic versions of G. Perelman F- and W-functionals for EM and KK systems which allows proofs of respective geometric flow and Ricci soliton equations. There are constructed associated statistical thermodynamic models which is used for formulating the geometric flow information, GIF, theory of EM and KK systems. Section 4 is devoted to quantum information models. There are defined quantum versions of W-entropy and the thermodynamic GIF entropy and respective von Neumann, R ényi and relative/conditional entropies. The concepts of geometric flow and quantum entanglement of EM and KK systems and related quantum geometric information flow, QGIF, models are elaborated in Sect. 5. In Sect. 6, we apply the anholonomic deformation method, AFDM, and prove the general decoupling and integrability properties of cosmological geometric evolution equations. The constructions are considered for the case of gravitational and electromagnetic fields determined by entropic forces encoding elastic and quasiperiodic spacetime structures. Then, in Sect. 7, we summarize the AFDM for constructing cosmological solutions with GIFs for KK gravity and effective NES. There are studies respective nonlinear symmetries relating generating functions, pattern forming and spacetime structures. We show how analogous of G. Perelman’s W-entropy and main thermodynamics values can be computed for such locally anisotropic cosmological quasiperiodic configurations and speculate how the respective statistical and geometric thermodynamic generating function can be used for constructing QGIF models. Finally, we summarize and conclude the work in Sect. 8.

2 Relativistic evolution equations for geometric flows of KK systems

We outline necessary concepts on the geometry of nonholonomic manifolds and nonlinear connections and introduce fundamental equations for generalized Ricci flows. For details and proofs and applications in modern physics, we cite [18,20,21]. The (modified) gravitational and electromagnetic theories are derived as nonholonomic Ricci soliton configurations.

2.1 Geometric preliminaries

On the geometry of nonholonomic manifolds and bundle spaces enabled with nonlinear connection structure and related modified gravity theories, we follow the conventions and method outlined in [1–3,7,31]. Readers are considered to be familiar with geometric methods applied in modern gravity [5,32] and extended to constructions with nonholonomic manifolds, nonlinear connections, and AFDM.

2.1.1 Nonlinear connections and adapted variables

Let us consider a four-dimensional, 4-d, spacetime Lorentz manifold \( V \) defined by a metric \( g \) of local pseudo-Euclidean signature \((++−)\). We use boldface symbols for spaces with nonholonomic fiber structure when the geometric objects enabled with (or adapted to) a nonlinear connection, N-connection, \( N:V = hV \oplus vV \). This states a conventional \( 2+2 \) splitting into horizontal (h), \( \dim(hV) = 2 \), and vertical (v), \( \dim(vV) = 2 \), parts, and respective sets of N-linear adapted frames \( e^\alpha = (e_i, e_a) \) and dual frames \( e_\beta = (e^j, e^b) \).\(^1\) We shall consider coefficient formulas for respective N-connections and adapted frames in Sect. 2.2.4. Diadic

\(^1\) Such frames define respective nonholonomic frame structures on the tangent bundle \( TV \), where \( i, j, \ldots = 1, 2 \) are h-indices and \( a, b, \ldots = 3, 4 \) are (co) v-indices for certain general Greek indices \( \alpha, \beta, \ldots \). In a similar
N-connection splittings will be prescribed in such forms when it is possible to define certain general decoupling and integration of physically important systems of nonlinear PDEs or elaborated analogous mechanical and/or thermodynamic models. Having elaborated in N-adapted variables on explicit classes of such solutions and/or analogous models, all geometric/physical objects can be rewritten equivalently in arbitrary frames.

A distinguished connection, d-connection, \( D = (hD, vD) \) is a linear connection preserving under parallelism a N-connection splitting. It defines a distortion d-tensor \( Z \) (it is a distinguished tensor, d-, adapted to a N-connection) from the Levi-Civita, LC, connection \( \nabla \), when \( D = \nabla + Z \).\(^2\) The Ricci d-tensor \( \hat{R} \) of a d-connection \( D \) is defined and computed in standard form. We write \( \hat{R} = \{ \hat{R}_{\alpha\beta} \} \) for the canonical d-connection. Similarly, we can label/compute other types of geometric objects, for instance, the scalar curvature \( \hat{R} \) of \( D \). Transferring geometric data by corresponding nonholonomic deformations and Legendre transforms from the tangent Lorentz bundle \( T\hat{V} \) to the cotangent bundle \( T^*\hat{V} \),

\[
(g, N, D) \leftrightarrow (g, N, \hat{D}) \leftrightarrow (L \text{ on } TV, \tilde{g}, \tilde{N}, \tilde{D}) \rightarrow (\tilde{g}, N', D) \leftrightarrow (g, N, \tilde{D}) \leftrightarrow (H \text{ on } T^*V, \tilde{g}, \tilde{N}, \tilde{D}),
\]

we can elaborate on quantum mechanical, QM, models determined by a (relativistic) Hamiltonian \( H \) and associated mechanical variables.

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Footnote 1 continued

form, all geometric constructions can be performed for a nonholonomic splitting on \( N : V^* = hV \oplus cV \), with a conventional fiber and enabled with a dual metric \( \tilde{g} \). We can introduce Lagrange–Hamilton variables, see details in [1–3] and references therein, determined by a prescribed Lagrange generating function \( L \) on \( V \) and related via a Legendre transform a Hamilton generating function, \( H \) on \( V^* \), inducing respective canonical geometric mechanical data \( (\tilde{g}, \tilde{N}) \) and/or \( (\tilde{g}, \tilde{N}) \). Local coordinates are denoted: \( u^\alpha = (x^1, y^\beta) \) (in brief, \( u = (x, y) \)) on \( V \); and, respectively, \( u^a = (x, p_i) \) (in brief, \( u = (x, p) \)) on \( V^* \). To generate exact generic off-diagonal solutions depending, in principle, on all spacetime coordinates is necessary to consider nonholonomic splitting of type 2 + 2 of the total spaces. In another turns, splitting of type 3 + 1 are more convenient to elaborating on thermodynamics, kinetic, stochastic and QM models. Such double fibrations request different systems of notations for coordinates and indices of geometric objects. Readers are recommended to see partner works [1–4] and [1,30,31] for details on nonholonomic differential geometry, geometric flows, and applications in classical and quantum information theory.

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\(^2\) There are preferred linear and d-connection structures which can be uniquely defined by geometric data \( g \) and/or \( N \) following certain geometric principles. For instance, \( \nabla \) is a unique metric compatible linear connection, \( \nabla g = 0 \), with zero torsion, \( T^\nabla_{\beta\gamma}[\nabla] = 0 \). The canonical d-connection \( D = \nabla + \hat{Z} \) is also defined by the same d-metric structure in a metric compatible form, \( \hat{D}g = 0 \), with zero torsion on both \( h \)- and \( v \)-subspaces, but with some (nonholonomic) nontrivial coefficients relating the \( h \)- and \( v \)-subspaces. This \( D \) is very important because it allows us to prove certain general decoupling and integrability properties of various classes of modified Einstein and Ricci flow equations. Such constructions cannot be performed in explicit form if we work only with the LC-connection. Zero torsion configurations can be always extracted from more general already found solution ones by imposing at the end the condition \( \hat{D}T_{\beta\gamma} = 0 \). This restricts the class of generating/integrating functions and sources. There is also another very important d-connection called the Cartan d-connection \( \hat{D} \) which is almost symplectic. It can be defined if we prescribe, for instance, a generating Lagrange function \( L \) on \( V \) when a canonical d-metric \( (g = \tilde{g} \) up to corresponding frame transforms) can be constructed as a so-called Sasaki lift for a Hessian metric, \( \tilde{g} \), and a canonical N-connection \( N \). Such a \( N \) is subjected to the condition that corresponding semi-spray (nonlinear geodesic equations) are equivalent to the Euler–Lagrange equations for \( L \). The priority of \( \hat{D} \) is that it can be used directly for performing deformation quantization or elaborating on other types of perturbative or nonperturbative quantization.
Suppose \( \mathcal{L} \rightarrow \mathbf{V} \) is the total spaces of a \( U(1) \)—bundle over \( \mathbf{V} \), and \( \mathbf{A} = \mathbf{A}_\alpha e^{\alpha} \), for a \( \mathbb{N} \)-adapted dual basis \( e^{\alpha} \), is a linear connection 1-form (electromagnetic potential) on this bundle associated with \( \mathcal{L} \). The curvature of \( \mathbf{A} \) is a 2-form \( \mathbf{F} = \{ F_{\alpha\beta} = e_\alpha \mathbf{A}_\beta - e_\beta \mathbf{A}_\alpha - [e_\alpha, e_\beta]^{\gamma} \mathbf{A}_\gamma \} \), for nonholonomy basis coefficients \( [e_\alpha, e_\beta]^{\gamma} \), representing the first Chern class of the line bundle associated with \( \mathcal{L} \) and \( T_{\mathbf{V}} \). It is also used the dual Hodge operator \( * \), determined by \( \mathbf{g} \), which allows us to construct the dual 2-form \( *\mathbf{F} = \{ F^{\alpha\beta} \} \). The canonical energy–momentum tensor of the electromagnetic field is defined

\[
T_{\alpha\beta} = e^{-2} \left[ F_{\alpha\gamma} F_{\beta}^{\gamma} - \frac{1}{4} \mathbf{g}_{\alpha\beta} F_{\gamma\xi} F^{\gamma\xi} \right],
\]

where \( e \) is the electromagnetic constant. It is used also the derivative \( D_\alpha = e_\alpha + e A_\alpha \).

We shall study families of geometric and physical evolution flow models determined by geometric data \( \{ \mathbf{g}(\tau), \mathbf{D}(\tau), \mathbf{A}(\tau) \} \) running on a positive parameter \( 0 \leq \tau \leq \tau_0 \). For simplicity, we shall use brief notations (for instance, \( \mathbf{g}(\tau) = \mathbf{g}(\tau, u) \)) when the dependence on local coordinates is not written if this does not result in ambiguities. In this work, it is considered that \( \beta^{-1} = \tau \) is a temperature-like parameter like in the G. Perelman work \([11]\) on the theory of Ricci flows \([10]\). Here, we cite also Friedan \([12]\) who considered equivalents of the Hamilton equations in physics before mathematicians elaborated their rigorous geometric analysis and topology formalism \([13–15]\). For a Kaluza–Klein, KK, theory unifying geometrically the gravity and electromagnetic interactions (see review \([5]\)), we can introduce a five-dimensional metric \( \mathbf{g} = (\mathbf{g}, \mathbf{A}) \), i.e., \( \mathbf{g}_{\alpha\beta} = (\mathbf{g}_{\alpha\beta}, \mathbf{A}_\gamma) \), for \( \alpha, \beta, \ldots = 1, 2, \ldots, 5 \).

The Einstein equations for \( \mathbf{g} \) (after a corresponding compactification on the fifth coordinate) are equivalent to the system of Einstein–Maxwell, EM, equations. We do not speculate in this work on physical implications of the KK and EM theories, but study possible generalizations and modifications for classical and quantum GIFs. Here, we note that geometric models with nonholonomic data with \( \mathbf{g}(\tau) \) and respective adapted d-connections \( \mathbf{D}(\tau) \) are more convenient for encoding geometric and physical information and elaborating on classical and quantum information theories. For elaborating on Hamilton-type canonical QM models, we shall use different sets of equivalent (up to nonholonomic frame transforms and deformations) geometric data \( (\mathbf{g}, \mathbf{N}, \mathbf{D}) \leftrightarrow (\mathbf{g}, \mathbf{N}, \tilde{\mathbf{D}}) \leftrightarrow (\mathbf{H} \text{ on } T^*\mathbf{V}, \mathcal{L}, \tilde{\mathbf{g}}, \mathbf{N}, \tilde{\mathbf{D}}) \).

2.2 Geometric flow equations for nonholonomic Einstein–Maxwell systems

We consider geometric generalizations of the Hamilton equations \([10]\) for relativistic flows \([1,18]\) of modified Einstein–Maxwell and Kaluza–Klein theories using nonholonomic variables which allow a general decoupling of geometric flow evolution and dynamical field equations (using the canonical d-connection with “hats”). For elaborating analogous mechanical systems and related statistical thermodynamical and quantum mechanical models, it is convenient to work with Lagrange–Hamilton variables with “tilde.”

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\(^3\) The components of the 4-d metric and EM potential are taken with corresponding dimension constants in order to get well-defined dimensions for metric fields. We also note that in this work splitting of indices of type \( 2 + 2 \) and \( 3 + 1 \) will be considered for the base spacetime manifold \( \mathbf{V} \), i.e., for indices of type \( \alpha, \beta, \ldots \) with respect to which 5-d constructions will be adapted.
2.2.1 Modified R. Hamilton equations for EM and KK systems

A family \([g(\tau), D(\tau), A(\tau)]\) defines a nonholonomic Einstein–Maxwell geometric flow on \(hTV\) if
\[
\frac{\partial g_{\alpha\beta}}{\partial \tau} = -2(R_{\alpha\beta} - \Upsilon_{\alpha\beta}) \quad \text{and} \quad \frac{\partial A^\alpha}{\partial \tau} = -D_\beta F^{\alpha\beta},
\]
(2)
where \(\Upsilon_{ij} = \kappa (T_{ij} - \frac{1}{2} g_{ij} T)\) is determined by the electromagnetic energy momentum tensor (1) for the gravitational constant \(\kappa\) and \(T = T_{ij} T^{ij}\). This system of nonlinear PDEs can be parameterized in a form describing KK geometric flows when the nonholonomic Ricci flow equations are written
\[
\frac{\partial g_{\alpha\beta}}{\partial \tau} = -2R_{\alpha\beta}.
\]
(3)

For Riemannian metrics and respective LC-connections, the mathematical properties and conditions of existence of solutions for systems of type (2) were studied in [27]. In our works, such systems were studied for pseudo-Riemannian metrics and generalized connections (see [1,18–21,30,31] and references therein), when large classes of nontrivial solutions can be constructed for the canonical d-connection \(D\) with possible nonholonomic constraints for extracting LC-configurations with \(\nabla\).

2.2.2 Canonical form of nonholonomic flow equations for EM and KK systems

In explicit form, we can decouple and integrate the systems (2) and (3) (applying the AFDM [1,7,20,30,31]) if we work with necessary type N-adapted variables, the canonical d-connection \(\hat{D}\) and consider a respective normalization function \(\hat{f}\). Equations (2) are written in the form
\[
\frac{\partial g_{ij}}{\partial \tau} = -2(\hat{R}_{ij} - \hat{\Upsilon}_{ij}); \quad \frac{\partial g_{ab}}{\partial \tau} = -2(\hat{R}_{ab} - \hat{\Upsilon}_{ab});
\]
\[
\hat{R}_{ia} = \hat{R}_{ai} = 0; \quad \hat{R}_{ij} = \hat{R}_{ji}; \quad \hat{R}_{ab} = \hat{R}_{ba};
\]
\[
\partial \tau \hat{f} = -\hat{\Box} \hat{f} + |\hat{D} \hat{f}|^2 - s \hat{R} + \hat{\Upsilon}_a,
\]
(4)
where \(\hat{\Box}(\tau) = \hat{D}^\alpha(\tau) \hat{D}_\alpha(\tau)\) is used for the geometric flows of the d’Alambert operator.

Using KK geometric data for \(g_{\alpha\beta} = (g_{\alpha\beta}, A_\gamma)\) defined in a \(U(1)\)—bundle \(\mathcal{L} \to V\), we write (4) in a form similar to (3),
\[
\frac{\partial g_{\alpha\beta}}{\partial \tau} = -2\hat{R}_{\alpha\beta} \quad \text{and} \quad \partial \tau \hat{f} = -\hat{\Box} \hat{f} + |\hat{D} \hat{f}|^2 - s \hat{R},
\]
(5)
where \(\hat{\Box}(\tau) = \hat{D}^\alpha(\tau) \hat{D}_\alpha(\tau)\) is used for the geometric flows of the d’Alambert operator determined by \(\hat{D} = \{\hat{D}_\alpha\}\) on the total space of \(\mathcal{L}\) and the normalization function \(\hat{f}\) is correspondingly redefined.

We constructed and studied various examples of exact and parametric solutions of the system (4) in references [18–21]. Such (in general) generic off-diagonal solutions describe geometric flows and nonholonomic deformations of cosmological and/or black hole solutions.

2.2.3 Ricci flow equations for EM systems in Lagrange mechanical variables

Any system of nonholonomic geometric flow evolution equations of type (4) can be rewritten equivalently in so-called nonholonomic Lagrange variables, see [1,2]. We can prescribe a
family regular relativistic Lagrangians $L(\tau, x^i, y^a)$ for conventional $2+2$ splitting on $V$. Together with the family of corresponding nondegenerated Hessians $\tilde{g}_{ab} := \frac{1}{2} \frac{\partial^2 L}{\partial y^a \partial y^b}$, this defines a set of geometric data

$$(\tilde{g}(\tau) = [\tilde{g}_{\mu
u}(\tau) = [\tilde{g}_{ij}(\tau), \tilde{g}_{ab}(\tau)]], \tilde{N}(\tau) = [\tilde{N}_i(\tau)], \tilde{D}(\tau)).$$

In such Lagrange mechanical variables, the nonlinear PDEs (4) can be written equivalently in the form (for a corresponding redefinition of normalizing functions and sources):

$$\frac{\partial \tilde{g}_{ij}}{\partial \tau} = -2(\tilde{R}_{ij} - \tilde{\Upsilon}_{ij}); \quad \frac{\partial \tilde{g}_{ab}}{\partial \tau} = -2(\tilde{R}_{ab} - \tilde{\Upsilon}_{ab}); \quad \tilde{R}_{ia} = \tilde{R}_{ai} = 0; \quad \tilde{R}_{ij} = \tilde{R}_{ji}; \quad \tilde{R}_{ab} = \tilde{R}_{ba}; \quad \partial_\tau \tilde{f} = -\tilde{\square} \tilde{f} + |\tilde{D} \tilde{f}|^2 - s \tilde{R} + \tilde{\Upsilon}_a,$$  

where $\tilde{\square}(\tau) = \tilde{D}^a(\tau) \tilde{D}_a(\tau)$ is constructed for the canonical Lagrange d-connection.

The geometric evolution of KK systems (3) can be written in “tilde” variables similarly to (6), i.e., in a form similar to (3),

$$\frac{\partial \tilde{g}_{ab}}{\partial \tau} = -2\tilde{R}_{ab}$$

where $\tilde{\square}(\tau) = \tilde{D}^a(\tau) \tilde{D}_a(\tau)$ is used for the geometric flows of the d’Alambert operator determined by $\tilde{D} = (\tilde{D}_a)$ on the total space of $\mathcal{L}$ and the normalization function $\tilde{f}$ is redefined to include distortions of d-connections and respective curvature terms.

It is not possible to find certain general decoupling properties of systems (7) which make very difficult the task to construct exact solutions in certain general and explicit forms. Nevertheless, such Lagrange-type variables are important for elaborating models of geometric evolution of relativistic mechanical systems and performing certain Lagrange-type quantization.

2.2.4 Geometric flow equations for EM systems in Hamilton mechanical variables

Let us consider a family of Hamilton fundamental functions $H(\tau) = H(\tau, x, p)$ with regular Hessians (cv-metrics)

$$\varphi^{ab}(\tau, x, p) := \frac{1}{2} \frac{\partial^2 H(\tau)}{\partial p_a \partial p_b},$$

when $\det |\varphi^{ab}| \neq 0$, and of constant signature. Such functions and coefficients are defined for a conventional 2+2 splitting of on a nonholonomic $V^*$, which (in this work) is Lorentz manifold $V$, but enabled with a system of “dual” local coordinates $p_a$ corresponding to a nonholonomic manifold.

Lagrange and Hamilton variables are related by Legendre transforms $L \rightarrow H(x, p) := p_a y^a - L(x, y)$ when $y^a$ determining solutions of the equations $p_a = \partial L(x, y)/\partial y^a$. The inverse Legendre transforms are defined $H \rightarrow L$ for $L(x, y) := p_a y^a - H(x, p)$, where $p_a$ are solutions of the equations $y^a = \partial H(x, p)/\partial p_a$. We can consider families of such Legendre transforms depending on a $\tau$-parameter.

Any $H$ defines a canonical nonlinear connection (N-connection) structure

$$\tilde{N}: TT^*V = hT^*V \oplus vT^*V$$

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for which the Euler–Lagrange and/or Hamilton equations are equivalent to the spray equations, see details in [1–3]. The geometric flows of such nonholonomic structures are determined by such families of canonical N-connection coefficients,

\[ \tilde{\mathbf{N}}(\tau) = \left\{ \tilde{N}_{ij}(\tau) = \frac{1}{2} \left\{ \left[ \tilde{g}_{ij}(\tau), H(\tau) \right] - \frac{\partial^2 H(\tau)}{\partial p_k \partial x^i} \tilde{g}_{jk}(\tau) - \frac{\partial^2 H(\tau)}{\partial p_k \partial x^j} \tilde{g}_{ik}(\tau) \right\} \right\} \]

where \( \tilde{g}_{ij} \) is inverse to \( \tilde{g}^{ab} \) (8) for any value of the running parameter.\(^4\)

Using the above coefficients, we can define families of canonical d-metrics,

\[ \tilde{\mathbf{g}}(\tau) = \tilde{\mathbf{g}}_{\alpha\beta}(\tau, x, p) \mathbf{e}^\alpha(\tau) \otimes \mathbf{e}^\beta(\tau) \]

where the canonical N-linear frames \( \mathbf{e}^\alpha(\tau) \) (\( \sim \mathbf{e}_a(\tau) \)) are canonically determined by data \( (H(\tau), \tilde{g}^{ab}(\tau)) \). Considering general frame (vierbein) transforms, \( \mathbf{e}_a(\tau) = e^\alpha(\tau, u) \partial/\partial u^\alpha \) and \( \mathbf{e}^\alpha(\tau) = e^\beta(\tau, u) du^\beta \), any N-connection and d-metric structure on \( \mathbb{V} \) can be written in general form (without “tilde” on symbols), with \( \mathbf{N}(\tau) = \{ \tilde{N}_{ij}(\tau, x, p) \} \)

\[ \tilde{\mathbf{g}}(\tau) = \tilde{g}_{ij}(\tau, x, p) \mathbf{e}^i(\tau) \otimes \mathbf{e}^j(\tau) \]

In Hamilton mechanical variables, the analogs of systems (4) and (6) are written

\[ \frac{\partial \tilde{\mathbf{g}}_{ij}}{\partial \tau} = -2 \left( \tilde{\mathbf{R}}_{ij} - \tilde{\mathbf{Y}}_{ij} \right); \frac{\partial \tilde{\mathbf{g}}_{ab}}{\partial \tau} = -2 \left( \tilde{\mathbf{R}}_{ab} - \tilde{\mathbf{Y}}_{ab} \right); \]

\[ \tilde{\mathbf{R}}_{ia} = \tilde{\mathbf{Y}}_{ai} = 0; \tilde{\mathbf{R}}_{ij} = \tilde{\mathbf{R}}_{ji}; \tilde{\mathbf{R}}_{ab} = \tilde{\mathbf{R}}_{ba}; \]

\[ \partial_x \tilde{f} = -\tilde{\mathbf{g}}^{ij} \tilde{f} + \left| \tilde{\mathbf{D}} \right|^2 \tilde{f} - \tilde{s} \tilde{R} + \tilde{\mathbf{Y}}_a, \]

(11)

where \( \tilde{\mathbf{g}}(\tau) = \tilde{\mathbf{D}}^a(\tau) \tilde{\mathbf{D}}_a(\tau) \) is determined by the canonical Hamilton d-connection. This system is most convenient for elaborating QM and QGIFs models following standard approaches with conventional Hamiltonian variables associated with \( 2 + 2 \) splitting.

The geometric flow evolution equations (11) can be redefined in order to describe flow evolution models of KK systems in canonical Hamilton variables,

\[ \frac{\partial \tilde{\mathbf{g}}_{a\beta}}{\partial \tau} = -2 \tilde{\mathbf{R}}_{a\beta} \text{ and } \partial_x \tilde{f} = -\tilde{\mathbf{g}}^{ij} \tilde{f} + \left| \tilde{\mathbf{D}} \right|^2 \tilde{f} - \tilde{s} \tilde{R}, \]

(12)

where \( \tilde{\mathbf{g}}(\tau) = \tilde{\mathbf{D}}^a(\tau) \tilde{\mathbf{D}}_a(\tau) \) is used for the geometric flows of the d’Alambert operator determined by \( \tilde{\mathbf{D}} = \{ \tilde{\mathbf{D}}_a \} \). The normalization function \( \tilde{f} \) is redefined in such forms when there are included contributions from distortions of d-connections and respective curvature terms.

The systems of nonlinear PDEs (5), (7), and (12) for KK geometric flow models encode in different type classical EM long-distance interactions. In this work, for simplicity, we shall

\[^4\] For a fixed value of the flow parameter, such coefficients define canonical systems of N-adapted (co) frames,

\[ \tilde{\mathbf{e}}_a = \left( \tilde{e}_i = \frac{\partial}{\partial x^i}, \tilde{e}^b = \frac{\partial}{\partial p_b} \right) \text{; } \mathbf{e}^\alpha = \{ \mathbf{e}^i = dx^i, \mathbf{e}_a = dp_a + \tilde{N}_{ia}(x, p) dx^i \}. \]

Such frames are characterized by anholonomy relations \( \{ \tilde{\mathbf{e}}_a, \tilde{\mathbf{e}}_b \} = \tilde{\mathbf{e}}_a \tilde{\mathbf{e}}_b - \tilde{\mathbf{e}}_b \tilde{\mathbf{e}}_a = \tilde{W}^a_{b\gamma} \mathbf{e}_\gamma \), with anholonomy coefficients \( \tilde{W}^a_{b\gamma} = \partial_a \tilde{N}_{ib}/\partial p_b \) and \( \tilde{W}_{ija} = \tilde{\mathbf{O}}_{ija} \). Such a frame is holonomic (integrable) if the respective anholonomy coefficients are zero.
study how (12) can be derived from nonholonomic deformations of G. Perelman functionals and respective models of classical and quantum GIFs.

2.2.5 KK gravity models as nonholonomic Ricci solitons

Nonholonomic Ricci solitons are defined as self-similar geometric flow configurations for a fixed parameter $\tau_0$. In such cases, we can consider $\partial g_{\alpha\beta}/\partial \tau = 0$ and $\partial A^\alpha/\partial \tau = 0$ and transform (2) into

$$R_{\alpha\beta} = Y_{\alpha\beta} \text{ and } D_\beta F^{\alpha\beta} = 0.$$ 

For self-similar conditions with a fixed parameter $\tau = \tau_0$, the system (3) results in $R_{\alpha\beta} = 0$. We can extract from such 4-d and 5-d modified Ricci soliton equations standard EM and KK equations with LC connections imposing additional nonholonomic constraints when $D_i Z = 0 = \nabla$. 

The normalizing function in (4) can be chosen in such a form when the above nonholonomic Ricci soliton equations can be written in canonical variables

$$\hat{R}_{ij} = \hat{Y}_{ij}; \hat{R}_{ab} = \hat{Y}_{ab}; \hat{R}_{ia} = \hat{R}_{ai} = 0; \hat{R}_{ij} = \hat{R}_{ji}; \hat{R}_{ab} = \hat{R}_{ba}.$$ 

Such a system of nonlinear PDEs posses a general decoupling property and allows us to generate various classes of exact and parametric solutions. In this approach, the (modified) gravitational and electromagnetic interactions consist certain examples of nonholonomic Ricci soliton equations.

Working, for instance, with Hamilton mechanical-like variables, we obtain from (11) an equivalent nonholonomic Ricci soliton system with “less decoupling” properties, but which is very useful for elaborating models of quantum gravity and QGIFs. Such a system of nonlinear PDEs is written

$$\hat{R}_{ij} = \hat{Y}_{ij}; \hat{R}_{ab} = \hat{Y}_{ab}; \hat{R}_{ia} = \hat{R}_{ai} = 0; \hat{R}_{ij} = \hat{R}_{ji}; \hat{R}_{ab} = \hat{R}_{ba}.$$ 

This is an example of analogous gravity and electromagnetic interactions modeled by relativistic Hamilton mechanical systems.

In this work, the Ricci solitons are generated as self-similar configurations of geometric flow evolution systems (12) described by systems of nonlinear PDEs of type

$$\hat{R}_{\alpha\beta} - \lambda \hat{g}_{\alpha\beta} = \hat{D}_\alpha \hat{v}_\beta + \hat{D}_\beta \hat{v}_\alpha.$$ 

Such configurations are determined by some geometric data $(\hat{g}_{\alpha\beta}, \hat{D}_\alpha)$ and a cosmological constant $\lambda$ and d-vector field $\hat{v}_\alpha(u)$.

3 G. Perelman functionals and thermodynamics of KK—geometric flows

The Thurston conjecture (and, in particular, the Poincaré hypothesis) was proved by G. Perelman [11]. He elaborated a geometric approach when the R. Hamilton equations [10] for Riemannian metrics are derived for certain models of gradient flows. Such constructions are

5 In geometric literature [13–15], self-similar equations, i.e., Ricci solitons, are described by Einstein-type equations for $\nabla$ determined by Riemannian metrics, with a cosmological constant $\lambda$ and a vector field $v_\alpha(u)$. $R_{\alpha\beta}[\nabla] - \lambda g_{\alpha\beta} = \nabla_\alpha v_\beta + \nabla_\beta v_\alpha$. For corresponding normalization functions, one generates models with vanishing $\lambda$ and $v_\alpha$ when $R_{\alpha\beta}[\nabla] = 0$, see [20] for applications of Ricci soliton techniques in modern gravity.
determined by corresponding Lyapunov-type F- and W-functionals for dynamical systems, see details in [13–15]. For geometric evolution of pseudo-Riemannian metrics and generalized connections, it is not clear whether and how analogs of the Thurston conjecture can be formulated and proved. Nevertheless, various generalizations of the G. Perelman’s functionals seem to be important for characterizing physical properties of new classes of generic off-diagonal (which can be not diagonalized by coordinate transforms) solutions in modern gravity. Such models and, for instance, locally anisotropic black hole and cosmological solutions [1,30,31] are considered in GR and MGTs with various noncommutative and/or supersymmetric, entropic modifications [17–21]. Recently, G. Perelman’s geometric thermodynamic models have been developed and applied in geometric information theory [2–4]. For reviews on classical and quantum information theory, we cite [33,34] and references therein.

The W-entropy was used by G. Perelman for formulating a statistical thermodynamic models associated with Ricci flow theories. This approach involve more general thermodynamic constructions than the Bekenstein–Hawking entropy for black holes [35–38] and various area hypersurface generalizations for holographic gravity, entropic gravity, conformal field theories, and duality, see [39–46].

The goal of this section is to define nonholonomic modifications of the so-called F- and W-functionals which allow us to prove the nonholonomic geometric flow Eqs. (2) and (3) (in respective variables, (4), (6), and/or (11)). The W-entropy will be used for constructing an associated statistical thermodynamic model for EM and KK flows. We shall analyze how corresponding statistical thermodynamic generating functions can be used for formulating a geometric information flow, GIF, theory for classical EM and KK systems.

3.1 F- and W-functionals for EM and KK geometric flows

Generalized G. Perelman entropic-like functionals can be postulated using different types of nonholonomic variables with conventional $2+2$ and $3+1$ decomposition of dimensions or double fibration splitting, see details in [18,20,21].

3.1.1 Nonholonomic $3+1$ splitting adapted to $2+2$ decompositions

Let us consider that a region $U \subset V$ of a nonholonomic Lorentz manifold with N-connection $2+2$ splitting defined by data $(N, g)$. We suppose that any necessary $U$ is fibered additionally into a structure of 3-d hypersurfaces $\Sigma_t$ parameterized by time-like coordinate $y^4 = t$. Locally all geometric constructions with such a $3+1$ splitting can be adapted to coordinates of type $u_a = (x^i, y^a)$ when the metric structure can be represented in the form

$$g = g_{\alpha'\beta'}(\tau, u)d\epsilon^{\alpha'}(\tau) \otimes d\epsilon^{\beta'}(\tau) = q_i(\tau, x^k)dx^i \otimes dx^i + q_3(\tau, x^k, y^a)\epsilon^3(\tau) \otimes \epsilon^3(\tau) - [q_N(\tau, x^k, y^a)]^2\epsilon^4(\tau) \otimes \epsilon^4(\tau).$$

(14)

In this formula, the “shift” coefficients $q_i = (q_i, q_3)$ are related to the 3-d metric $q_{ij} = \text{diag}(q_i) = (q_i, q_3)$ on a hypersurface $\Sigma_t$ if $q_3 = g_3$ and $[q_N]^2 = -g_4$, where $q_N$ is the lapse function. We shall use left/over labels on $N$ in order to distinguish such traditional symbols used in GR [32] from N-connection constructions with coefficients of type $N^a_i$. It should be noted here that in this work the geometric flow parameter $\tau$, $0 \leq \tau \leq \tau_0$, is a temperature-like one as in G. Perelman works for Ricci flows [11] and in our partner works [2–4].

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Considering $p_d$ as certain duals of $y^d$, we can define respective “dual” decompositions when corresponding 3-d hypersurfaces $\Xi_E$ are parameterized by a conventional “energy” parameter $p_4 = E$ for local coordinates on $V$. In result, decompositions of type (14) can be rewritten equivalently in the form

$$
{\mathbf{g}} = \{g_{\alpha'}\beta'(\tau, u)d\epsilon^{\alpha'}(\tau) \otimes d\epsilon^{\beta'}(\tau)
= \{q_i(\tau, x^k)dx^i \otimes dx^i + q^3(\tau, x^k, p_d)\epsilon_3(\tau) \otimes \epsilon_3(\tau)
= \{\bar{q}_N(\tau, x^k, p_d)^2\epsilon_4(\tau) \otimes \epsilon_4(\tau).
\tag{15}
$$

Similar nonholonomic $3 + 1$ decompositions can be considered for canonical variables with “hats” and/or “tilde” if we elaborate on models with systems of nonlinear PDEs with certain general decoupling and integration properties and/or mechanical-like analogous relativistic mechanical interpretation. Such parameterizations of d-metrics can be summarized in this form:

$$
V = (N = [N_i^a], \mathbf{g} = \{g_{a\beta} = [g_i, g^a]\}) \Leftrightarrow V' = (N' = [N'_i], \mathbf{g}' = \{g'_{a\beta} = [g'_i, g'^a]\})
\tag{16}
$$

We can extend parameterizations (16) for extra dimension (underlined) indices $\alpha, \beta, \ldots = 1, 2, \ldots, 5$ and coordinates $\mathbf{u} = (u, u^5)$ when, for instance,

$$
\mathbf{g} = \{g_{a\bar{\beta}} = (g_{a\beta}, A^\gamma)\} = \{(g_i, g^a), [A_i, A_a]\};
$$

$$
\mathbf{g}' = \{g'_{a\bar{\beta}} = (g'_{a\beta}, A'_\gamma)\} = \{(g'_i, g'^a), [A'_i, A'_a]\};
$$

$$
\mathbf{g}_+ = \{g_{a\bar{\beta}} = (q_i, N, A_i, A_a)\} = \{(q_i, q^a, N, A_i, A_a)\};
$$

$$
\mathbf{g}_- = \{\tilde{g}_{a\bar{\beta}} = (\tilde{q}_i, \tilde{A}_i, \tilde{A}_a)\} = \{(\tilde{q}_i, \tilde{q}^a, \tilde{N}, \tilde{A}_i, \tilde{A}_a)\};
$$

in order to study geometric evolution models of KK systems.

### 3.1.2 G. Perelman functionals for nonholonomic KK geometric flows

In nonholonomic variables with a metric compatible d-connection $\mathbf{D} = \nabla + \mathbf{Z} = (h^a, v)^a$ uniquely distorted from $\nabla$, the relativistic versions of G. Perelman functionals are postulated

$$
\mathcal{F} = \int (4\pi \tau)^{-5/2} \varepsilon^{-\frac{1}{2}} \sqrt{|\mathbf{g}|d^5 u (s R + |\mathbf{D} f|^2)}
$$

and

$$
\mathcal{W} = \int \mu \sqrt{\mathbf{g}|d^5 u [\tau (s R + |h \mathbf{D} f|^2 + |v \mathbf{D} f|^2)]^2 + f - 5].
$$

In these formulas, we use a brief notation for the integrals on extra dimension KK variables and the normalizing function $f(\tau, u)$ is subjected to the conditions

$$
\int_{t_1}^{t_2} \int z \mu \sqrt{|\mathbf{g}|d^5 u = 1, \text{for a classical integration measure } \mu = (4\pi \tau)^{-5/2} \varepsilon^{-\frac{1}{2}} \text{ and the Ricci scalar } s R \text{ is taken for the Ricci d-tensor } R_{a\beta} \text{ of a d-connection } \mathbf{D}.
$$

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N-adapted variational calculus on $g_{\alpha\beta}$ using (18) or (19) (we omit these cumbersome proofs, see similar details in [18,20,21] performed as nonholonomic deformations of respective sections in [13–15]) allows us to prove the geometric flow evolution for the KK systems (3) and (equivalently, for EM systems) (2). Here, we note that the functional $\mathcal{W}$ (19) is a nonholonomic relativistic generalizations of so-called W-entropy introduced in [11]. In our partner works [2–4], various 4-d and 8-d versions of $\mathcal{W}$ and associated statistical and quantum thermodynamics values are used for elaborating models of classical and quantum GIFs.

3.1.3 Hamilton variables for G. Perelman KK-functionals

Redefining, respectively, the normalization functions for respective double splitting (16) and KK d-metrics (17), we can write equivalently the nonholonomic G. Perelman functionals in different types of nonholonomic variables. In this paper, we shall use analogous 4-d and 5-d canonical Hamilton variables with tilde.\(^6\)

The Lyapunov-type functionals (18) and (19) defining geometric flow evolution of analogous mechanical Hamilton systems for KK systems on $\mathbf{V}$ can be expressed equivalently in the form

\[
\tilde{\mathcal{F}} = \int (4\pi \tau)^{-5/2} e^{-\tilde{\mathcal{L}}} \sqrt{|\tilde{g}|} (d^5 u) \left[ \langle \tilde{J} \rangle + |\tilde{D} \tilde{f}|^2 \right] \quad \text{and} \quad (20)
\]

\[
\tilde{\mathcal{W}} = \int \tilde{\mathcal{L}} \sqrt{|\tilde{g}|} (d^5 u) \left[ (\langle \tilde{J} \rangle + |\tilde{D} \tilde{f}|^2)^2 + \langle \tilde{J} \rangle - 5 \right]. \quad (21)
\]

In these formulas, we use a brief notation for the integrals on extra dimension KK variables and the normalizing function $\int \tilde{\mathcal{L}}(\tau, u)$ is subjected to the conditions

\[
\int \int \tilde{\mathcal{L}} \sqrt{|\tilde{g}|} (d^5 u) = \left( \int \mathcal{L} \sqrt{g} (d^5 u) \right)^2 = 1,
\]

for a classical integration measure $\tilde{\mathcal{L}} = (4\pi \tau)^{-5/2} e^{-\tilde{\mathcal{L}}}$ and the Ricci scalar $\tilde{\mathcal{R}}$ is taken for the Ricci d-tensor $\tilde{\mathcal{R}}_{\alpha\beta}$ of a d-connection $\tilde{D}$.

The functionals (20) and (21) are determined by families of Hamilton fundamental functions $H(\tau, x, p)$ and respective Hessians (8) and canonical d-metrics (10). Performing respective canonical N-adapted variational calculus for $\tilde{\mathcal{F}}$ and/or $\tilde{\mathcal{W}}$, we prove the geometric flow evolution equations for analogous Hamilton mechanical systems modeling KK flows [see (11) and (12)]. Such proofs follow from abstract symbolic calculus (when geometric objects with “tilde” are changed into respective ones with “hats” or other type ones on $\mathbf{V}$ and/or $\mathbf{Y}$).

In result, we can similarly define values of type $\tilde{\mathcal{F}}$ and/or $\tilde{\mathcal{W}}$ (allowing proofs via respective N-adapted calculus of (7) and (6)); and $\tilde{\mathcal{F}}$ and/or $\tilde{\mathcal{W}}$, with respective proofs of (5) and (4). Considering LC-configurations with $\tilde{D}_{\alpha \beta} \tilde{\mathcal{R}}_{\alpha \beta} = \tilde{\mathcal{L}}$, and/or $\tilde{D}_{\alpha \beta} \tilde{\mathcal{R}}_{\alpha \beta} = \tilde{\mathcal{L}}$, the values (18) or (19) transform, respectively, into analogous KK and/or EM versions of nonholonomically generalized G. Perelman’s F-entropy and W-entropy. It should be noted that $\tilde{\mathcal{W}}$ (19) and/or $\tilde{\mathcal{W}}$ (21) do not have a character of entropy for pseudo-Riemannian metrics, but can be treated as respective values characterizing relativistic nonholonomic geometric hydrodynamic-type spacetimes and extra dimension flows.

\(^6\) Here, we note that the theory of nonholonomic geometric flows of relativistic Hamilton mechanical systems was formulated [7] (see also references therein on yearly works on Finsler–Lagrange geometric flows beginning 2006) in explicit form using canonical data ($\mathcal{G}(\tau), \mathcal{D}(\tau)$), in terms of geometric objects with “tilde” values defined on 8-d cotangent Lorentz bundles.
3.2 Statistical thermodynamic models for KK flows

We can characterize MGTs and GR by analogous thermodynamic models [1,18–21,31] generalizing G. Perelman’s constructions for geometric flows of Riemannian metrics [11].

3.2.1 Basic concepts of statistical KK thermodynamics

We shall underline geometric and thermodynamical values for geometric flows of KK systems in order to distinguish the constructions from similar ones elaborated for GIF and QGIF theories in partner works [1–3]. We consider the partition function

\[ Z = \int \exp(-\beta E) \omega(E) \, d\omega \]

for a canonical ensemble at temperature \( \beta^{-1} = T \). The measure as the density of states includes \( \omega(E) \) for certain models in GR and extensions to 5-d spacetimes with \( \omega(E) \).

We compute in standard form

- average flow energy: \( \mathcal{E} = \langle E \rangle := -\partial \log Z / \partial \beta \),
- flow entropy: \( S := \beta \langle E \rangle + \log Z \),
- flow fluctuation: \( \eta := \langle (E - \langle E \rangle)^2 \rangle = \partial^2 \log Z / \partial \beta^2 \).

A value \( Z \) allows us to define a conventional state density (for quantum models, a density matrix)

\[ \sigma(\beta, E) = Z^{-1} e^{-\beta E}. \] (22)

The relative entropy between any state density \( \rho \) and \( \sigma \) is defined/computed

\[ S(\rho \parallel \sigma) := -S(\rho) + \int (\beta \mathcal{E} + \log Z) \rho \omega(E) \, d\omega = \beta[\mathcal{E}(\rho) - TS(\rho)] + \log Z, \]

where the average energy is computed for the density matrix \( \rho \), \( \mathcal{E}(\rho) = \int \mathcal{E} \rho \omega(E) \), and the formula \( \log \sigma = -\beta \mathcal{E} - \log Z \) is used.

The free energy is introduced by formula \( \mathcal{F}(\rho) := \mathcal{E}(\rho) - TS(\rho) \). If \( \log Z \) is independent on \( \rho \), we get \( S(\sigma \parallel \sigma) = 0 \) and

\[ S(\rho \parallel \sigma) = \beta[\mathcal{F}(\rho) - \mathcal{F}(\sigma)]. \] (23)

In our models of geometric flow evolution and analogous thermodynamics systems, we consider that under evolution it is preserved the thermal equilibrium at temperature \( \beta \) with maps of density states \( \rho \to \rho' \) keeping the same density state \( \sigma \). Such systems are characterized by inequalities

\[ S(\rho \parallel \sigma) \geq S(\rho' \parallel \sigma), \ i.e., \mathcal{F}(\rho) \geq \mathcal{F}(\rho'). \] (24)

The above presented formulas allow us to connect KK, EM and mechanical flow models to the second law of thermodynamics.

3.2.2 Thermodynamic values for KK and Hamilton mechanical flows

We associate with functionals (18), (19) and (20), (21) respective thermodynamic generating functions

\[ Z[\mathbf{g}(\tau)] = \int (4\pi \tau)^{-5/2} e^{-\frac{1}{\tau} \sqrt{|\mathbf{g}|} d^5 u(-\frac{1}{2} + 5/2)}, \ for \ \mathbf{V}; \] (25)

\[ \tilde{Z}[\tilde{\mathbf{g}}(\tau)] = \int (4\pi \tau)^{-5/2} e^{-\frac{1}{\tau} \sqrt{|\tilde{\mathbf{g}}|} d^5 u(-\frac{1}{2} + 5/2)}, \ for \ \tilde{\mathbf{V}}. \] (26)
Such values are with functional dependence on \( g(\tau) \) and \( \tilde{g}(\tau) \). (We shall omit to write this in explicit forms if that will not result in ambiguities.) A density state is a functional \( \sigma[g(\tau)] \) when the geometric evolution involve densities \( \rho_1[g] \) and \( \rho_1[\tilde{g}] \), where the left label 1 is used in order to distinguish two KK d-metrics \( g \) and \( \tilde{g} \). Similar values can be defined in canonical Hamilton variables with tilde and duality labels, for instance, \( \tilde{\sigma}[\tilde{g}(\tau)] \).

Using \((25)\) and respective \( 3 + 1 \) parameterizations of d-metrics (see formulas \((16)\) and \((17)\)), we define and compute analogous thermodynamic values for geometric evolution flows of KK systems,

\[
\mathcal{E} = -\tau^2 \int (4\pi \tau)^{-5/2} e^{-\frac{\mathcal{F}}{L}} \sqrt{|q_1 q_2 q_3 (q N) q_5|} \delta^5 u \left( s R + |Df|^2 - \frac{\delta^5 u}{2\tau} \right), \\
\mathcal{S} = -\int (4\pi \tau)^{-5/2} e^{-\frac{\mathcal{F}}{L}} \sqrt{|q_1 q_2 q_3 (q N) q_5|} \delta^5 u \left[ \tau \left( s R + |Df|^2 \right) + f - 5 \right], \\
\eta = 2\tau^4 \int (4\pi \tau)^{-5/2} e^{-\frac{\mathcal{F}}{L}} \sqrt{|q_1 q_2 q_3 (q N) q_5|} \delta^5 u \left[ R_{\alpha\beta} + |Df|^2 \right] \left[ \frac{1}{2\tau} g_{\alpha\beta} \right]^2, \tag{27}
\]

where \( \delta^5 u \) contains N-elongated differentials in order to compute such integrals in N-adapted forms and \( q_5 = 1 \) can be considered for compactifications on the 5th coordinate. Using such values, we can compute the respective free energy and relative entropy \((23)\),

\[
\mathcal{F} (g) = \mathcal{E} (g) - \beta^{-1} \mathcal{S} (g) \quad \text{and} \quad \mathcal{S} (g \parallel g) = \beta \left[ \mathcal{F} (g) - \mathcal{F} (g) \right], \quad \text{where} \quad \mathcal{E} (g) = -\tau^2 \int (4\pi \tau)^{-5/2} e^{-\frac{\mathcal{F}}{L}} \sqrt{|q_1 q_2 q_3 (q N) q_5|} \delta^5 u \left[ s R (g) + |Df|^2 \right] \left( s R + |Df|^2 - \frac{\delta^5 u}{2\tau} \right), \\
\mathcal{S} (g) = -\int (4\pi \tau)^{-5/2} e^{-\frac{\mathcal{F}}{L}} \sqrt{|q_1 q_2 q_3 (q N) q_5|} \delta^5 u \left[ \tau \left( s R + |Df|^2 \right) + f - 5 \right].
\]

For geometric evolution flows described in nonholonomic Hamilton mechanical variables on \( N \) and generating function \((26)\), the thermodynamic values are computed

\[
\mathcal{E} = -\tau^2 \int (4\pi \tau)^{-5/2} e^{-\frac{\mathcal{F}}{L}} \sqrt{|q_1 q_2 q_3 (q N) q_5|} \delta^5 u \left[ s R + |Df|^2 - \frac{\delta^5 u}{\tau} \right], \\
\mathcal{S} = -\int (4\pi \tau)^{-5/2} e^{-\frac{\mathcal{F}}{L}} \sqrt{|q_1 q_2 q_3 (q N) q_5|} \delta^5 u \left[ \tau \left( s R + |Df|^2 \right) + f - 5 \right], \\
\eta = -2\tau^4 \int (4\pi \tau)^{-5/2} e^{-\frac{\mathcal{F}}{L}} \sqrt{|q_1 q_2 q_3 (q N) q_5|} \delta^5 u \left[ R_{\alpha\beta} + |Df|^2 \right] \left[ \frac{1}{2\tau} g_{\alpha\beta} \right]^2. \tag{28}
\]

Such formulas are important for elaborating quantum information models with geometric flows. Geometric evolution of EM system is encoded in such formulas using with compactification on 5th coordinate. Similar analogous relativistic thermodynamic models were studied.
in our partner papers [1–3] for (co) tangent Lorentz bundles of total space dimension 8-d. In this article, we work with nonholonomic Lorentz 5-d manifolds with compactification to 4-d.

Finally we note that generating functions (25) and (26) and respective thermodynamical values (27) and (28) can be written equivalently in terms of the canonical d-connections \( \hat{D} \) and \( \tilde{D} \) if we consider nonholonomic deformations to certain systems of nonlinear PDEs with general decoupling and related models in canonical Lagrange variables on \( \mathcal{V} \).

### 4 Classical and quantum geometric information flows of KK systems

We consider basic aspects of (quantum) geometric information flows (respectively, GIFs and QGIFs) of KK systems.

#### 4.1 Geometric information flow theory of classical KK systems

In this subsection, we follow classical information theory with fundamental concepts of Shannon, conditional and relative entropies and applications in modern physics [33,34,47–52]. To elaborate on GIFs of KK systems, there are used W-entropy functionals (19) and (21) and associated thermodynamical models elaborated in Sect. 3.2.2 and [2,3].

#### 4.1.1 Shannon entropy and GIF entropy

Let us consider a random variable \( B \) taking certain values \( b_1, b_2, \ldots, b_k \) (for instance, a long message of symbols \( N \gg 1 \) containing different \( k \) letters) when the respective probabilities to observe such values are \( p_1, p_2, \ldots, p_k \). By definition, the Shannon entropy \( S_B := - \sum_{j=1}^{k} p_j \log p_j \geq 0 \) for \( \sum_{j=1}^{k} p_j = 1 \). The value \( N S_B \) is the number of bits of information which can be extracted from a message consisting of \( N \) symbols. Real messages contain correlations between letters (grammar and syntax) for a more complex random process. Ignoring correlations (the ideal gaze limit), we approximate the entropy of a long message to be \( N S_B \) with \( S \) being the entropy of a message consisting of only one symbol. In a statistical thermodynamical model, we can consider a classical Hamiltonian \( H \) determining the probability of an \( i \)th symbol \( b_i \) via formula \( p_i = 2^{-H(b_i)} \).

For GIFs of classical KK systems, the thermodynamic values are determined by data \( \hat{\mathcal{W}}; \, \hat{\mathcal{E}}, \, \hat{\mathcal{S}}, \, \hat{\eta} \) (27) and/or \( \tilde{\mathcal{W}}; \, \tilde{\mathcal{E}}, \, \tilde{\mathcal{S}}, \, \tilde{\eta} \) (28). (Tilde is used for analogous mechanical variables determined by a conventional Hamilton density \( g \) and respective Hessian \( \tilde{g}^{ab} \).) We can introduce probabilities on a discrete network with random variables, for instance, \( \tilde{p}_n = 2^{-H(b_n)} \), or, for statistical ensembles, \( \tilde{p}_n = 2^{-\tilde{E}(b_n)} \).

We can elaborate on continuous GIF models encoding geometric evolution of KK systems in general covariant and conventional analogous mechanical variables using the thermodynamic entropies \( S[\hat{g}(\tau)] \) and \( \tilde{S}[\tilde{g}(\tau)] \). In such an approach, we cannot involve in the constructions probability distributions which appear for discrete random variables. We can study GIF KK systems using only W-entropies \( \hat{\mathcal{W}}[\hat{g}(\tau)] \) and \( \tilde{\mathcal{W}}[\tilde{g}(\tau)] \) for certain constructions without statistical thermodynamics values. KK systems under geometric evolution flows are denoted in general form as \( \hat{B} = \hat{B}[\hat{g}(\tau)] \) and \( \tilde{B} = \tilde{B}[\tilde{g}(\tau)] \) determined by corresponding canonical d-metrics on nonholonomic Lorentz spacetimes.
4.1.2 Conditional entropy for GIFs

Let us consider sending a message with many letters (any letter is a random variable $X$ taking possible values $x_1, \ldots, x_k$). A receiver sees a random variable $Y$ consisting from letters $y_1, \ldots, y_l$. In the classical information theory, the goal is to compute how many bits of information does such a receiver get form a message with $N$ letters when the random variables are denoted $X, Y, Z$ etc. In the simplest case, we can consider one variable when the probability to observe $X = x_i$ is denoted $P_X(x_i)$ for $\sum_i P_X(x_i) = 1$. The communication between a sender and receiver is a random process of two variables defined by a joint distribution $P_{X,Y}(x_i, y_j)$ as the probability that to send $X = x_i$ and hear $Y = y_j$. The value $P_Y(y_j) = \sum_i P_{X,Y}(x_i, y_j)$ is the probability to receive $Y = y_j$. (Summation is over all choices that could be sent.)

By definition, the conditional probability $P_{X|Y}(x_i|y_j) := \frac{P_{X,Y}(x_i, y_j)}{P_Y(y_j)}$ is a value characterizing receiving $Y = y_j$, and one can estimate the probability that it was sent $x_i$. We can write for receiver’s messages $P_X(x_i) = \sum_j P_{X,Y}(x_i, y_j)$ or consider $P_X(x_i)$ as an independent probability density.

In classical information theory, there are defined such important values:

- Shannon entropy of the conditional probability:
  \[ S_{X|Y=y_j} := - \sum_i P_{X,Y}(x_i|y_j) \log P_{X,Y}(x_i|y_j); \]
- entropy of joint distribution:
  \[ S_{XY} := - \sum_{i,j} P_{X,Y}(x_i, y_j) \log P_{X,Y}(x_i, y_j); \]
- total received information content:
  \[ S_Y := - \sum_{i,j} P_{X,Y}(x_i, y_j) \log P_Y(y_j); \]
- total sent information content:
  \[ S_X := - \sum_{i,j} P_{X,Y}(x_i, y_j) \log P_X(x_i). \]

Using such formulas, one prove that for the conditional entropy

\[ S_{X|Y} := \sum_j P_Y(y_j) S_{X|Y=y_j} = S(X|Y) = S_{XY} - S_Y \geq 0 \] (29)

and the mutual information between $X$ and $Y$ (a measure of how much we learn about $X$ observing $Y$)

\[ I(X; Y) := S_X - S_{XY} + S_Y \geq 0. \] (30)

Now, let us consider how basic concepts from classical information theory can be generalized for models of information thermodynamics determined by geometric flows of KK systems. Conventionally, there are considered two such KK GIFs, $A = A[g(\tau)]$, or $\tilde{A} = \tilde{A}[\tilde{g}(\tau)]$, and $B = B[1g(\tau)]$, or $\tilde{B} = \tilde{B}[1\tilde{g}(\tau)]$. Hereafter, for simplicity, we shall omit tilde formulas considering that they can be always introduced whether certain Hamilton mechanical variables are important for study certain physical and/or information processes and when probability densities can be introduced in any point and along causal lines on a KK spacetime manifold $V$.

In a general covariant form, we shall work with a thermodynamic generating function $Z[g(\tau)]$ and respective thermodynamic model \[ \mathcal{W}: Z, \mathcal{E}, \mathcal{S}, \eta \] (27) for GIFs on $V$. To study conditional GIFs, we shall use geometric flow models on $V \otimes \tilde{V}$ when the local coordinates are $(u, i\bar{u})$ and the normalizing functions are of type $ABf(u, i\bar{u})$. A d-metric structure on such tensor products of nonholonomic Lorentz manifolds is of type $ABg = |g = \ldots$
We introduce a metric compatible $d$-connection $\mathbf{D} = \mathbf{D} + \varphi \mathbf{D}$ and corresponding scalar curvature $s_{AB} \mathbf{R} = s \mathbf{R} + s_1 \mathbf{R}$, respectively.

The thermodynamic GIF entropies for respective systems are $\mathcal{S}[A]$ and $\mathcal{S}[B]$ defined by $g(\tau)$ and $\bar{g}(\tau)$ as in (27). They can be considered as analogs of $S_X$ and $S_Y$ used in formulas (29) and (30). As an analog of $S_{XY}$ for GIFs, we introduce the thermodynamic generating function [as a generalization of (25)]

$$
\mathbf{Z} [\bar{g}(\tau), \bar{g}(\tau)] = \int_1^1 (4\pi \tau)^{-5} e^{-\mathbf{D}^f \mathbf{D}} \sqrt{|g|} \sqrt{|\bar{g}|} d\xi^5 d\xi^1 u (-\mathbf{D}^f + 10),
$$

for $\mathbf{V} \otimes \mathbf{V}$.

This results in a GIF thermodynamic entropy function

$$
\mathbf{Z} [g(\tau), \bar{g}(\tau)] = \mathbf{Z} [A, B] = \mathbf{Z} [A, B] = \mathbf{Z} [A] + \mathbf{Z} [B].
$$

Using such formulas, we claim (this can be proved in any point of respective causal curves on Lorentz manifolds) that for GIFs the formulas for the conditional entropy (29) and mutual information (30) are, respectively, generalized

$$
\mathcal{S} [A ; B] := \mathcal{S} [A] - \mathbf{Z} [A] - \mathbf{Z} [B] \geq 0.
$$

Similar claims can be formulated if we use the W-entropy (for diversity, we use the variant with Hamilton mechanical variables, but such formulas can be proved in general form without tilde) $\mathbf{V}$ (21):

$$
\mathbf{V} [A ; B] := \mathbf{V} [A] - \mathbf{V} [B] \geq 0
$$

These formulas are computed, respectively, for the W-entropy instead of the S-entropy in the standard probability theory. For information flows of KK systems, such formulas can be applied without additional assumptions on formulating associated statistical thermodynamic models.

Finally, we note that the above formulas can be defined and proved, respectively, and in similar forms, on $\mathbf{V}$, $\mathbf{V} \otimes \mathbf{V}$, and other tensor products involving different types of $d$-metrics, $d$-connections and generating functions. For instance,

$$
\mathcal{S} [A ; B] := \mathcal{S} [A] - \mathbf{Z} [B] \geq 0
$$

The models with dual conventional momentum variables are important for elaborating QM theories of GIFs with Hamilton generating functions. In their turn, the QGIF models on “pure” tangent bundles are important for encoding quantum field theories following the Lagrange formalism.

4.1.3 Relative KK GIF entropy and monotonicity

The relative entropy is introduced for two probability distributions $P_X$ and $Q_X$. For $X = x_i$, with $\mathbf{z} = \{1, 2, \ldots, s\}$, one states $p_i = P_X(x_i)$ and $q_i = Q_X(x_i)$, for some long messages with $N$ letters. The main problem is to decide which distribution describes a random
process more realistically. One defines the relative entropy per observation $S(P_X||Q_X) := \sum_i p_i \log(p_i - q_i) \geq 1$ under assumption that $N S(P_X||Q_X) \gg 1$. This value is asymmetric on $P_X$ and $Q_X$. It measures the difference between these two probability distributions when $P_X$ is for the correct answer and $Q_X$ is taken as an initial hypothesis.\(^7\)

We can define and calculate the relative entropy $S$ and mutual information $I$ between two distributions

$$S(P_X||Q_X) := \sum_{i,j} P_{X,Y}(x_i, y_j)[\log P_{X,Y}(x_i, y_j) - \log(P_x(i)P_Y(y_j))] = S_X - S_{XY} + S_Y = I(X; Y);$$

$$S(P_{X,Y}||Q_{X,Y}) := S_X - S_{XY} + S_Y = I(X; Y); S(P_{X,Y,Z}||Q_{X,Y,Z}) := S_{XY} - S_{XYZ} - S_{YZ} = I(X; YZ).$$

There are important inequalities

$$I(X; Y) := S_X + S_Y - S_{XY} \geq 0, \text{ subadditivity of entropy; }$$

$$S(P_{X,Y}||Q_{X,Y}) \geq S(P_X||Q_X), S(P_{X,Y,Z}||Q_{X,Y,Z}) \geq S(P_{X,Y}||Q_{X,Y}),$$

monotonicity of relative entropy.

For three random variables, there is also the condition of strong subadditivity

$$S_X - S_{XYZ} - S_{YZ} \geq S_X - S_{XY} + S_Y, \text{ or } S_{XY} + S_{YZ} \geq S_Y + S_{XYZ},$$

which is equivalent for the condition of monotonicity of mutual information $I(X; YZ) \geq I(X; Y)$.

The formulas for $S$ and $I$ can be generalized, respectively, for the relative entropy and mutual information of geometric flows of KK systems (proofs can be provided for causal lines and nonholonomic variables generated by certain relativistic Hamilton generating functions $H(x, p))$. For KK GIV, we can consider thermodynamic generating functions $A \mathcal{Z} := \mathcal{Z}[g(\tau)]$ and $B \mathcal{Z} := 1 \mathcal{Z}[g(\tau)]$, see (25), as analogs of $p_i = P_X(x_i)$ and $q_i = Q_X(x_i)$. We can consider GIFs of three KK systems $A, B, C$.

We can prove using standard methods in any point of causal curves and applying explicit integral N-adapted calculations on $V \otimes V \otimes V$ such properties

$$\mathcal{I} [A; B] := S[A] - AB S + S[B] \geq 0, \text{ subadditivity of entropy; }$$

$$\mathcal{S} [AB \mathcal{Z}||AB \mathcal{Z}] \geq \mathcal{S} [A \mathcal{Z}||A \mathcal{Z}], \mathcal{S} [ABC \mathcal{Z}||ABC \mathcal{Z}] \geq \mathcal{S} [AB \mathcal{Z}||AB \mathcal{Z}],$$

monotonicity of relative entropy.

The conditions of strong subadditivity for GIF entropies are claimed

$$\text{or } AB \mathcal{S} + BC \mathcal{S} \geq b \mathcal{S} + b \mathcal{S} + b \mathcal{S} + b \mathcal{S}.$$

\(^7\) Let us remember some basic formulas which are necessary for our further considerations. We consider a pair of random variables $X$ and $Y$ and respective two probability distributions. The fist one is a possible correlated joint distribution $P_{X,Y}(x_i, y_j)$ and $P_X(x_i) := \sum_j P_{X,Y}(x_i, y_j)$. $P_Y(y_j) := \sum_i P_{X,Y}(x_i, y_j)$. We also consider a second probability distribution $Q_{X,Y}(x_i, y_j) = P_X(x_i) P_Y(y_j)$ which is defined in a form ignoring correlations between $X$ and $Y$. In a general context, $Q_{X,Y}(x_i, y_j)$ can be with correlations of type $Q_{X}(x_i) := \sum_j Q_{X,Y}(x_i, y_j)$. We can introduce three random variables $X, Y, Z$ described by a joint probability distribution and related values, $P_{X,Y,Z}(x_i, y_j, z_k)$ and $P_X(x_i) := \sum_j \sum_k P_{X,Y,Z}(x_i, y_j, z_k)$. $P_Y(z_j, z_k) := \sum_i \sum_k P_{X,Y,Z}(x_i, y_j, z_k)$. If we ignore the correlations between $X$ and $Y, Z$, we define $Q_{X,Y,Z}(x_i, y_j, z_k) := P_X(x_i) P_Y(z_j, z_k)$. Other types of values can be defined if we observe the subsystem $XY$, when $P_{X,Y}(x_i, y_j) := \sum_k P_{X,Y,Z}(x_i, y_j, z_k)$, $Q_{X,Y}(x_i, y_j) := \sum_k Q_{X,Y,Z}(x_i, y_j, z_k) = P_X(x_i) P_Y(y_j)$.
In an equivalent form, these formulas can be written as the condition of monotonicity of KK GIFs mutual information,
\[ J[A; BC] \geq J[A; B] \].
The above inequalities can be proved for any point along causal curves on \( V \). They involve the thermodynamic generating function [as a generalization of (25)],
\[
ABC \mathcal{Z}[\mathbf{g}(\tau), \mathbf{1g}(\tau), 2\mathbf{g}(\tau)] = \int_1 \int_2 (4\pi \tau)^{-15/2} e^{-ABC f} \frac{1}{\sqrt{|\mathbf{g}|}} \frac{1}{\sqrt{|1\mathbf{g}|}} \frac{1}{\sqrt{|2\mathbf{g}|}} \frac{1}{d^5 u} \frac{1}{d^5 u} \frac{d^5 u}{d^5 u}
\]
\(-ABC f + 15\), for \( V \otimes V \otimes V \), (31)

with a normalizing function \( ABC f(\mathbf{u}, 1\mathbf{u}, 2\mathbf{u}) \). On such tensor products of KK manifolds is of type
\[
ABC g = \{ g = [q_1, q_2, q_3, q_4, q_5], 1g = [1q_1, 1q_2, 1q_3, 1q_4, 1q_5],
2g = [2q_1, 2q_2, 2q_3, 2q_4, 2q_5] \}.
\]

We can consider a canonical d-connection \( ABC D = D + b D + c D \) and respective scalar curvature \( sABC R = sR + s1R + s2R \). The resulting entropy function
\[
ABC S = S[A, B, C]
= - \int_1 \int_2 (4\pi \tau)^{-15/2} e^{-ABC f} \frac{1}{\sqrt{|\mathbf{g}|}} \frac{1}{\sqrt{|1\mathbf{g}|}} \frac{1}{\sqrt{|2\mathbf{g}|}} \frac{1}{d^5 u} \frac{1}{d^5 u} \frac{d^5 u}{d^5 u}
\]
\[
\left[ \tau \left( sR + s1R + s2R + |D_{ABC f}|^2 \right) + ABC f - 15 \right].
\]
(32)

Similar formulas can be derived for W-entropies and for canonical (with hats) Lagrange–Hamilton GIFs on respective tensor products of KK spaces.

We conclude this introduction to the GIF theory of canonical classical mechanical systems with three remarks: First, such constructions can be generalized for different types of d-connections for stochastic maps and nonholonomic flow evolution and kinetic processes of Lagrange–Hamilton systems as we studied in [16,17]. In this work, QGIF analogs with quantum relative entropy are monotonic including those associated with the evolution of KK and associated Hamiltonian QM systems. Relativistic mechanical QGIFs are studied in detail in our partner works [2,3].

4.2 Density matrix and entropies of KK quantum geometric flows, QGIFs

We develop the density matrix formalism for elaborating models of KK GIFs and QGIFs. Similar constructions for Hamilton mechanical systems are provided in sections 4 and 5 in [2] and, for entangled QG systems, in [3].

4.2.1 Entanglement and density matrix for KK systems and QGIFs

QGIF models for KK systems can be elaborated using canonical mechanical variables \((\mathbf{H}, \mathbf{g}^{\alpha\beta})\) and generating thermodynamical functions \( \mathbf{Z}[\mathbf{g}(\tau)] \) (26). Certain special QM systems can be described by pure states. In a more general context, quantum theories with probabilities involve not only pure quantum states but also density matrices. The goal of this subsection is to study how GIFs of KK systems can be generalized using basic concepts of QM and information theory. We shall elaborate on QGIFs described in terms of density matrices defined as quantum analogs of state densities of type \( |\sigma|^2 \) (22).
Let us consider a thermodynamical model \( \tilde{\mathcal{A}} = [\tilde{\mathcal{Z}}, \tilde{\mathcal{G}}, \tilde{\mathcal{S}}, \tilde{\mathcal{R}}] \) (28) for GIFs defined in Hamilton mechanical variables on \( \mathcal{V} \). In any point \( \mathcal{i} \in \mathcal{V} \) (we can consider causal curves covering open regions with such points), we associate a typical Hilbert space \( \mathcal{H}_A \). A state vector \( \psi_A \in \mathcal{H}_A \) is an infinite-dimensional complex vector function which in quantum information theory is approximated to vectors in complex spaces of finite dimensions. Such a \( \psi_A \) is a solution of the Schrödinger equation with as a well-defined quantum version of a canonical Hamiltonian \( \tilde{H} \), see details in [33,34]. For nonholonomic structures, such constructions are considered in [2,3].

The combined Hilbert space for KK QGIFs is defined as a tensor product, \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \), with an associate Hilbert space \( \mathcal{H}_A \) considered for a complementary system \( \mathcal{A} \). Here, we note that symbols \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) etc. are used as labels for certain KK systems under geometric flow evolution. The state vectors for a combined QGIF system are written \( \psi_{AB} = \psi_A \otimes \psi_B \in \mathcal{H}_{AB} \) for \( \psi_A = 1_A \) taken as the unity state vector. Quantum systems subjected only to quantum evolution and not to geometric flows are denoted \( A, B, C, \ldots \) (we do not underline such symbols if they do not involve KK structures).

A pure state \( \psi_{AB} \in \mathcal{H}_{AB} \) may be not only a tensor product of complex vectors. A quantum system can be also entangled and represented by a matrix of dimension \( N \times M \) if \( \dim \mathcal{H}_A = N \) and \( \dim \mathcal{H}_B = M \). We underline symbols for dimensions (one should not be confused with underlining for KK structures) in order to avoid ambiguities with the \( N \)-connection symbol \( N \). For any pure state, we can perform a Schmidt decomposition

\[
\psi_{AB} = \sum_i \sqrt{p_i} \psi_{A}^{i} \otimes \psi_{B}^{i},
\]

for any index \( i = 1, 2, \ldots \) (up to a finite value). A state vector \( \psi_{A}^{i} \) is orthonormal if \( < \psi_{A}^{i}, \psi_{A}^{j} > = \delta^{ij} \), where \( \delta^{ij} \) is the Kronecker symbol. Taking \( p_i > 0 \) and \( \sum_i p_i = 1 \), we can treat \( p_i \) as probabilities. We note that, in general, such \( \psi_{A}^{i} \) and/or \( \psi_{B}^{i} \) do not define bases of \( \mathcal{H}_A \) and/or \( \mathcal{H}_B \).

The quantum density matrix for a KK QGIF system \( \mathcal{A} \) is defined \( \rho_A := \sum_p a_p |a_p|^2 \otimes |\psi_A^p| \) as a Hermitian and positive semi-definite operator with trace \( Tr \rho_A = 1 \). This allows us to compute the expectation value of any operator \( \mathcal{O}_A \) characterizing additionally such a system,

\[
< \mathcal{O} >_{AB} = < \mathcal{O}_A \otimes 1_B | \psi_{AB} > = \sum_i p_i < \psi_{A}^{i} | \mathcal{O}_A | \psi_{A}^{i} > = \sum_i p_i < \mathcal{O}_A | \psi_{A}^{i} > = Tr \rho_A \mathcal{O}_A.
\]

In general covariant form, we can write for KK systems that \( < \mathcal{O} >_A = Tr \rho_A \mathcal{O}_A \).

We can elaborate on models encoding both quantum information and geometric flow evolution of bipartite systems of type \( \mathcal{A} | \mathcal{B} \) and \( \mathcal{A} \otimes \mathcal{B} \) with both quantum and geometric entanglement defined by density matrices. In general form, bipartite KK QGIFs systems are described in general form by quantum density matrices of type \( \rho_{AB} \) or (in canonical Hamilton variables) \( \rho_{\mathcal{AB}} \). In classical theory of probability [33,34], a bipartite system \( XY \) by a joint probability distribution \( P_{X,Y}(x_i, y_j) \), where \( P_X(x_i) := \sum_j P_{X,Y}(x_i, y_j) \). Considering for KK systems \( \mathcal{A} | \mathcal{B} \) as a bipartite quantum system with Hilbert space \( \mathcal{H}_{AB} \), we can define and parameterize a KK QGIF density matrix: \( \rho_{AB} = \sum_{a,b} \rho_{ab} |a > \otimes |b > \mathcal{A} |B <
A measurement of the KK system $|\vec{A}\rangle$ is characterized by a reduced density matrix

$$\rho_A = \text{Tr}_{H_B} \rho_{AB},$$

for $|b\rangle = 1, 2, \ldots, m$ as an orthonormal basis of $\overline{H}_B$.

In a similar form, we can define and compute $\rho_B = \text{Tr}_{H_A} \rho_{AB}$. Using the above-introduced concepts and formulas, we can elaborate on KK QGIF models formulated in Hamilton variables or in a general covariant form.

### 4.2.2 Quantum density matrix and von Neumann entropy for KK QGIFs

For such systems, the quantum density matrix $\sigma_{AB}$ for a state density $\sigma$ of type (22) can be defined and computed using formulas (34),

$$\sigma_{AB} = \langle \sigma \otimes 1_B | \psi_{AB} \rangle = \sum_i p_i^{AB} \langle \psi_i^A | \sigma | \psi_i^A \rangle \langle \psi_i^B | 1_B | \psi_i^B \rangle,$$

$$\sigma_A = \langle \sigma | A = \sum_i p_i^A \langle \psi_i^A | \sigma | \psi_i^A \rangle = \text{Tr}_{H_B} \rho_{AB} \sigma,$$

where the density matrix $\rho_A$ is taken for computing the KK QGIF density matrix $\sigma_A$. A matrix (35) is determined by a state density of the thermodynamical model for KK GIFs of a classical system $\sigma$. In explicit form, we can work with quantum density matrices $\sigma_{AB}$, $\sigma_A = \text{Tr}_{H_B} \sigma_{AB}$ and $\sigma_B = \text{Tr}_{H_A} \sigma_{AB}$. Such formulas can be written in respective coefficient forms

$$\sigma_{AB} = \sum_{a, a', b, b'} \sigma_{a' b' a b} |a' >_A \otimes |b' >_B A < a | \otimes |b >_B$$

and $\sigma_A = \sum_{a, a', b} \sigma_{a' b a b} |a' >_A \otimes |b >_B A < a | \otimes |b >_B$.

QGIFs can be characterized by quantum analogs of entropy values used for classical geometric flows. We can consider both an associated thermodynamics entropy and a W-entropy in classical variants and then quantize such systems using a Hamiltonian for a fixed system of nonholonomic variables and which allows a self-consistent QM formulation. Such values can be computed in explicit form using formulas of type (35) for classical conditional and mutual entropy considered for GIFs and in information theory [2,33,34]. For instance, this allows to compute define and compute

$$qW_{AB} = \text{Tr}_{H_A}[\sigma_{AB}(A B W)]$$

and $qW_A = \text{Tr}_{H_A}[\sigma_{A}(A W)]$, $qW_B = \text{Tr}_{H_B}[\sigma_{B}(B W)]$;

$$qS_{AB} = \text{Tr}_{H_A}[\sigma_{AB}(A B S)]$$

and $qS_A = \text{Tr}_{H_A}[\sigma_{A}(A S)]$, $qS_B = \text{Tr}_{H_B}[\sigma_{B}(B S)]$.

Such values describe additional entropic properties of quantum KK systems with rich geometric structure under QGIFs.
4.2.3 Quantum generalizations of the W- and thermodynamic entropy of KK GIFs

We describe KK QGIFs in standard QM form for the von Neumann entropy determined by \( \sigma_A \) (35) as a probability distribution,

\[
q S(\sigma_A) := Tr \sigma_A \log \sigma_A.
\]

(36)

Using \( q S(\sigma_A) \) (36), we can consider generalizations for \( A \)B and \( A \) systems, respectively,

\[
q S(\sigma_{AB}) := Tr \sigma_{AB} \log \sigma_{AB} \quad \text{and} \quad q S(\sigma_A) := Tr \sigma_A \log \sigma_A, \quad q S(\sigma_B) := Tr \sigma_B \log \sigma_B.
\]

(37)

The von Neumann entropy for KK QGIFs, \( q S(\sigma_A) \), has a purifying property not existing for classical analogs. For a bipartite system \( \psi_{AB} = \sum_i \sqrt{p_i} \psi_i^A \otimes \psi_i^B \) and \( \sigma_A := \sum_i p_i |\psi_i^A > \otimes <\psi_i^A| \), we compute

\[
\sigma_A := \sum_{a,a',b,b} \sum_k \sigma_{a'a'b'b} P_k A < a' || \psi_k^A > < \otimes <\psi_k^A| a > A, \quad \sigma_B := \sum_{a,a',b,b} \sum_k \sigma_{a'a'b'b} P_k B < a' || \psi_k^B > < \otimes <\psi_k^B| b > B .
\]

(37)

We can consider both an associated thermodynamics entropy and a W-entropy in classical variants and then quantize KK systems using a respective Hamiltonian which allows a self-consistent QM formulation. Using respective formulas (35), (37) for classical conditional and mutual entropy considered for KK GIFs and in information theory, there are defined and computed, respectively,

\[
q W_{AB} = Tr \mathcal{H}_{AB}[(\sigma_{AB})(ABW)] \quad \text{and} \quad q W_A = Tr \mathcal{H}_A[(\sigma_A)(AW)],
\]

\[
q W_B = Tr \mathcal{H}_B[(\sigma_B)(BW)]; \quad q S_{AB} = Tr \mathcal{H}_{AB}[(\sigma_{AB})(ABS)] \quad \text{and} \quad q S_A = Tr \mathcal{H}_A[(\sigma_A)(AS)],
\]

\[
q S_B = Tr \mathcal{H}_B[(\sigma_B)(BS)].
\]

Such values describe complimentary entropic properties of quantum KK systems with rich geometric structure under quantum GIF evolution. Additionally, the quantum probabilistic values are described by the von Neumann entropy \( q S(\sigma_A) \) (36).

5 Entanglement and QGIFs of KK systems

We generalize and study the concept of entanglement for QGIFs of KK systems. The notion of bipartite entanglement was introduced for pure states and density matrices in description of finite-dimensional QM systems [33,34,53,54]. Thermodynamic and QM analogs of GIFs are characterized by a series of different types of entropies (G. Perelman’s W-entropy and geometric thermodynamic entropy and the entanglement entropy in the von Neumann sense). It will be shown how each of such entropic values characterize classical and quantum correlations determined by KK QGIFs and quantifies the amount of quantum entanglement. A set of inequalities involving Pereleman and entanglement entropies playing a crucial role in definition and description of such systems will be provided.
5.1 Geometric KK flows with entanglement

5.1.1 Bipartite entanglement for KK QGIFs

For the KK theory, we can consider various (relativistic) mechanic, continuous or lattice models of quantum field theory, thermofield theory, QGIF models etc. A QM model can be characterized by a pure ground state $|\Psi_1>$ for a total Hilbert space $\mathcal{H}$ (we underline symbols for KK theories with compactification to EM models for any fixed value of a geometric flow parameter). The density matrix

$$\rho = |\Psi_1><\Psi_1|$$

(38)

can be normalized following the conditions $<\Psi_1|\Psi_1> = 1$ and total trace $\text{tr}(\rho) = 1$. We suppose that such a conventional total quantum system is divided into a two subsystems $\mathcal{A}$ and $\mathcal{B}$ with respective Hamilton mechanical variables and analogous GIF thermodynamic models $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$. In this section, we consider that, for instance, $\tilde{\mathcal{A}} = [\tilde{Z}, \tilde{E}, \tilde{S}, \eta]$ (28) is a typical KK GIF system which in a general covariant form on $V$ can be parameterized $\mathcal{A} = [Z, E, S, \eta]$ (27). Similar models (they can be with a different associated relativistic Hamiltonian and d-metric $g$) are considered for $\tilde{\mathcal{B}}$ and $\mathcal{B}$. Such subsystems $\mathcal{A}$ and $\mathcal{B} = \mathcal{A}$ are complimentary to each other if there is a common boundary $\partial \mathcal{A} = \partial \mathcal{B}$ of codimension 2, where the nonsingular flow evolution $\mathcal{A}$ transforms into a necessary analytic class of flows on $\mathcal{A}$. We can consider that for bipartite KK QGIFs $\mathcal{H} = \mathcal{H}_{\mathcal{A}\mathcal{B}} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}.$

The measure of entanglement of a KK QGIF subsystem $\mathcal{A}$ is just the von Neumann entropy $qS$ (36), but defined for the reduced density matrix $\rho_A = \text{Tr}_{\mathcal{H}_{\mathcal{B}}} (\rho)$. We define and compute the entanglement entropy of $\mathcal{A}$ as

$$qS(\rho_A) := \text{Tr}(\rho_A \log \rho_A),$$

(39)

when $\rho_A$ is associated with a state density $\rho(\beta, \xi, s, g)$ of type (22). We note that the total entropy $qS = 0$ for a pure grand state (38) associated with $V$. More rich nonholonomic KK QGIF models encoding Hamilton mechanical variables are defined by

$$q\tilde{S}(\rho_A) := \text{Tr}(\tilde{\rho}_A \log \tilde{\rho}_A),$$

(40)

when effective Hamilton functions can be defined in any point belonging to a causal region in $V$. As QM models, the systems characterized by (39) encode less information than those described in conventional mechanical variables (40).

5.1.2 Separable and entangled KK QGIFs

Such concepts were introduced for QGIFs in our partner works [2,3] extending to analogous thermodynamic models standard constructions in quantum information theory [33,34,53,54]. Let us show how those formulas can be generalized for KK systems. We consider $\{|a>_A; a = 1, 2, \ldots, k_a\} \in \mathcal{H}_A$ and $\{|b>_B; b = 1, 2, \ldots, k_b\} \in \mathcal{H}_B$ as orthonormal bases when a pure total ground state is parameterized in the form

---

8 Such an assumption cannot be correct if there are considered theories with gauge symmetries (in special for nonabelian gauge model), see discussion and references in footnote 3 of [54]. In this work, we do not enter in details of entanglement of EM systems as quantum gauge field theories, but consider only constructions with a classical KK gravity theory and related geometric flow evolution models with associated thermodynamic and QM values.
\[ |\Psi> = \sum_{ab} C_{ab} |a >_A \otimes |b >_B, \]  

where \( C_{ab} \) is a complex matrix of dimension \( \dim \mathcal{H}_A \times \dim \mathcal{H}_B \). If such coefficients factorize, \( C_{ab} = C_a C_b \), there are defined separable ground states (equivalently, pure product states), when

\[ |\Psi> = |\Psi>_A \otimes |\Psi>_B >, \]  

for \( |\Psi>_A = \sum_a C_a |a >_A \) and \( |\Psi>_B = \sum_b C_b |b >_B \).

The entanglement entropy vanishes, \( qS(\rho_A^\text{Ca}) = 0 \), if and only if the pure ground state is separable. Here, we note that such a system with nonzero \( qS(\tilde{\rho}_A^\text{Ca}) \) reflects a more rich nonholonomic structure induced by the conventional mechanical variables. (It is natural that more information is contained in such cases.) For KK QGIFs, such definitions are motivated because all subsystems are described by corresponding effective relativistic Hamilton functions, \( \tilde{H}_A \) and \( \tilde{H}_B \), and/or effective thermodynamics energies, \( \tilde{^1\!A}E \) and \( \tilde{^1\!B}E \). Similar values can be defined and computed for W-entropies.

A ground state \( |\Psi> \) (41) is entangled (inseparable) if \( C_{ab} \neq C_a C_b \). For such a state, the entanglement entropy is positive, \( qS(\rho_A^\text{Ca}) > 0 \). Using quantum Schmidt decompositions (33), we prove for any point along a causal curve on \( \Psi \) that

\[ \begin{align*}
qS &= - \sum_a p_a \log p_a \quad \text{and} \quad qS_{\text{max}} = \log \min(a, b) \text{ for} \\
\sum_a p_a &= 1 \quad \text{and} \quad p_a = 1/ \min(a, b), \forall a.
\end{align*} \]  

An entangled state of KK QGIFs is a superposition of several quantum states associated with respective GIFs. An observer having access only to a subsystem \( \mathcal{A} \) will find him/herself in a mixed state when the total ground state \( |\Psi> \) is entangled following such conditions:

\[ |\Psi> : \text{separable} \iff \rho_A^\text{Ca} : \text{pure state}, \quad \text{or} \quad |\Psi> : \text{entangled} \iff \rho_A^\text{Ca} : \text{mixed state}. \]

We note that the von Neumann entanglement entropies \( qS \) (and \( \tilde{qS} \)) encodes three (four) types of information data: (1) how the geometric evolution is quantum flow correlated; (2) how much a given QGIF state differs from a separable QM state; (3) how KK gravity models are subjected to quantum flow evolution; and (in which forms such KK QGIFs are modeled in nonholonomic Hamilton mechanical-like variables). A maximum value of quantum correlations is reached when a given KK QGIF state is a superposition of all possible quantum states with an equal weight. But there are also additional KK GIF properties which are characterized by W-entropies \( \mathcal{W} \) (19) and \( \tilde{\mathcal{W}} \) (21) and thermodynamic entropies, \( S \) (27) and \( \tilde{S} \) (28), which can be computed in certain quasiclassical QM limits for a \( 3 + 1 \) splitting and respective dualization of variables, for instance, along a time-like curve.

Let us consider a KK QGIF model with entanglement elaborated for different associated relativistic Hamiltonians and respective d-metrics \( g \) and \( \tilde{g} \). We suppose that the conventional Hilbert spaces are spanned by two orthonormal basic states in the form \( |a >_A; a = 1, 2 \rangle \in \mathcal{H}_A \) and \( |b >_B; b = 1, 2 \rangle \in \mathcal{H}_B \), when \( A,B < |b >_A >, b >_B, > \delta_{ab} \). The total Hilbert space \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \) has a 4-dim orthonormal basis \( \mathcal{H}_{AB} = \{ |11 >, |12 >, |21 >, |22 > \} \), where \( |ab > > = |a >_A \otimes |b >_B > \) are tensor product states.

As a general state for such a bipartite KK QGIF system, we can consider

\[ |\Psi> = \cos \theta |12 > - \sin \theta |21 >, \]  

where \( \theta \) is the splitting parameter.
where $0 \leq \theta \leq \pi/2$. The corresponding entanglement entropy (39) is computed

$$qS(\rho_A) = -\cos^2 \theta \log(\cos^2 \theta) - \sin^2 \theta \log(\sin^2 \theta).$$

These formulas show that for $\theta = 0, \pi/2$ we obtain pure product states with zero entanglement entropy. For a quantum system $|\Psi> = \frac{1}{\sqrt{2}}(|12> - |21>)$, when the density matrix

$$\rho_A = \frac{1}{2}(|1>_{\mathcal{A}} < 1| + |2>_{\mathcal{A}} < 2|) = \frac{1}{2} \text{diag}(1,1)$$

results in $qS(\rho_A) = -tr_A(\rho_A \log \rho_A) = \log 2$. For such quantum systems, the maximal entanglement is for $\theta = \pi/4$.

If the KK GIF structure is “ignored” for such a quantum system [see formula (43)], we can associate a conventional QM system which is similar to spin ones, for instance, with up-spin $|1>_{\mathcal{A}}$ and down-spin $|2>_{\mathcal{A}}$.

### 5.1.3 Entanglement with thermofield double KK QGIF states and W-entropy

In this series of works [1–4], the evolution parameter $\beta = T^{-1}$ is treated as a temperature one like similarly to the standard G. Perelman’s approach [11]. This allows us to elaborate on GIF theories as certain classical and/or quantum thermofield models. For KK GIFs, such a nontrivial example with entanglement and a thermofield double state is defined by a ground state (41) parameterized in the form

$$|\Psi> = \frac{1}{Z} \sum_k e^{-\beta E_k/2} |k>_{\mathcal{A}} \otimes |k>_{\mathcal{B}}.$$ (44)

In this formula, the normalization of the states is taken for the partition function $Z = \sum_k e^{-\beta E_k/2}$. Such values are associated with the thermodynamic generating function $Z[\tilde{g}(\tau)]$ (25) and state density matrix $\sigma(\beta, \mathcal{E}, g)$ (22) the energy $\mathcal{E}_A = \{E_k\}$ is considered quantized with a discrete spectrum for a KK QGIF system $\mathcal{A} = [Z, \mathcal{E}, S, \eta]$ (27). We compute the density matrix for this subsystem determining a Gibbs state,

$$\rho_A = \frac{1}{Z} \sum_k e^{-\beta E_k/2} |k>_{\mathcal{A}} < k| = Z^{-1} e^{-\beta \mathcal{E}_A}.$$

In the above formulas, we consider $\tilde{\mathcal{E}}$ as a (modular) Hamiltonian $\mathcal{E}_A$ such that $\mathcal{E}_A |k> = E_k |k>_{\mathcal{A}}$.

Thermofield double states are certain entanglement purifications of thermal states with Boltzmann weight $p_k = Z^{-1} e^{-\beta E_k}$. Transferring state vectors $|k>_{\mathcal{B}}$ from $\mathcal{H}_\mathcal{A}$ to $\mathcal{H}_\mathcal{B}$, we can purify $\mathcal{A}$ in the extended Hilbert space $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$, So, every expectation of local operators in $\mathcal{A}$ can be represented using the thermofield double state $|\Psi> = (44)$ of the total system $\mathcal{A} \cup \mathcal{B}$. The entanglement entropy can be treated as a measure of the thermal entropy of the subsystem $\mathcal{A}$.

$$S(\rho_A) = -tr_A[\rho_A (\mathcal{F}_A - \log Z)] = \beta(\mathcal{F}_A - \mathcal{E}_A),$$

where $\mathcal{F}_A = -\log Z$ the thermal free energy. For thermofield values, we omit the label “q” considered, for instance, for $qS$ (39).

Finally, we note that thermofield KK GIF configurations are also characterized by W-entropy $W$ (19), see examples in [18].
5.1.4 Bell-like KK QGIF states

For two KK QGIF systems, a state (43) is maximally entangled for \( \theta = \pi / 4 \). We can define analogs of Bell state, i.e., Einstein–Podolsky–Rosen pairs, in quantum geometric flow theory are defined

\[
|\Psi_B^2 > = \frac{1}{\sqrt{2}}(|11 > +|22 >), |\Psi_B^3 > = \frac{1}{\sqrt{2}}(|11 > -|22 >), |\Psi_B^4 > = \frac{1}{\sqrt{2}}(|12 > +|21 >).
\]

Such states violate the Bell’s inequalities and encode also the information characterized by W-entropy.

The constructions of type (45) can be extended for systems of \( k \) qubits. For instance, the Greenberger–Horne–Zelinger, GHZ, states [54] are \( |\Psi_{GHZ}^{GHZ} > = \frac{1}{\sqrt{2}}(|1 > \otimes | +2 > \otimes k \rangle \). In quantum information theory, there are considered also (W states), |\Psi_{W}^{W} > = \frac{1}{\sqrt{2}}(|21...11 > +|121...1 > +\cdots +|111...12 > \rangle. A state |\Psi_{GHZ}^{GHZ} > is fully separable but not |\Psi_{W}^{W} > as we prove below:

For \( k = 3 \), a tripartite KK QGIF with subsystems \( A, B \) and \( C \), we write

\[
|\Psi_{GHZ}^A > = \frac{1}{\sqrt{2}}(|111 > +|222 >) \text{ and } |\Psi_{W}^B > = \frac{1}{\sqrt{2}}(|112 > +|221 > +|211 >).
\]

The reduced density matrices for the system \( A \cup B \) is defined using \( Tr_C \cdot |\rho_{AUB}^{GHZ} > = \frac{1}{2}(|111 > \leq 11| +|221 > |211 >) \). In this way, there are described two different KK QGIF states. The first state is fully separable and can be represented in the form

\[
|\Psi_{W}^A > = \frac{1}{2} |\Psi_{A}^A > < |\Psi_{B}^B > + |\Psi_{B}^B > < |\Psi_{A}^A >.
\]

For any \( |\Psi_{A}^A > \) associated with a state density \( \rho(\beta, \mathcal{E}, \mathcal{g}) \) (22), we can compute the respective W-entropy and geometric thermodynamic entropy taking measures determined by \( \mathcal{g} \) and/or respective Hamilton variables.

5.2 Entanglement inequalities for entropies of KK QGIFs

We study certain important inequalities and properties of the entanglement entropy (39) for KK QGIFs using the density matrix \( \rho_A^A = Tr_B (\rho_B^B) \). We omit technical proofs which are similar to those presented in [55].\(^9\) For any \( \rho_A^A \) we have

\[
|\Psi_{W}^A > \in \mathcal{S}_A = q_S = q_S^A = q_S^A \mathcal{S}_A.
\]

Let us analyze three important properties of KK QGIFs resulting in a strong subadditivity property of entanglement and Perelman’s entropies.

1. **Entanglement entropy for complementary KK QGIF subsystems:** If \( B = \bar{A}, q_S^A = q_S^A \mathcal{S}_A \).

This follows from formulas (42) for a pure ground state wave function. We can prove similar equalities for the W-entropy \( \mathcal{W} \) and/or thermodynamic entropy \( \mathcal{S} \) (27) if we use the same d-metric \( \mathcal{g} \) and respective normalization on \( A \) and \( \bar{A} \). For quantum models, \( q_S^A \neq q_S^B \) if \( A \cup B \) is a mixed state, for instance, at a finite temperature. So, in general,

\(^9\) Rigorous mathematical proofs involve a geometric analysis technique [11,13–15] developed for applications in modern in modern gravity and particle physics theories in [1,18,20]. For any classes of solutions (redefining normalizing functions), we can always compute Perelman’s like entropy functionals at least in the quasiclassical limit with respective measures and related to \( q_S \) (39) for a KK QGIF or a thermofield KK GIF model.


3. Strong subadditivity: Such conditions are satisfied for disjoint subsystems $A$ and $B$, 

$$qS_{AUB} \leq qS_A + qS_B \quad \text{and} \quad |qS_A - qS_B| \leq qS_{AUB}. \quad (46)$$

The second equation is the triangle inequality [56] which is satisfied also in the quasi-classical limit for $S$ (27). We claim that similar conditions hold for the W-entropy $W$ (19) and respective quantum versions. They can be computed (and proved in any point of causal curves) as quantum perturbations in a QM model associated with a bipartite KK QGIF model $qW_{AUB} \leq qW_A + qW_B$ and $|qW_A - qW_B| \leq qW_{AUB}$. Such KK flow evolution and QM scenarios are elaborated for mixed geometric and quantum probabilistic information flows.

3. Strong subadditivity is considered for three disjointed KK QGIF subsystems $A$, $B$, and $C$ and certain conditions of convexity of a function built from respective density matrices and unitarity of systems [34,54,57,58]. In any point of causal curves, there are proved the following inequalities:

$$qS_{AUBUC} + qS_B \leq qS_{AUB} + qS_{BU} \quad \text{and} \quad qS_A + qS_C \leq qS_{AUB} + qS_{BU}.$$

Using these formulas, the conditions of subadditivity (46) can be derived as particular cases. Along causal curves on respective cotangent Lorentz manifolds, we can prove similar formulas for the W-entropy and small quantum perturbations $qW_{AUBUC} + qW_B \leq qW_{AUB} + qW_{BU}$ and $qW_A + qW_C \leq qW_{AUB} + qW_{BU}$.

Such properties are claimed for KK QGIFs.

5.2.2 Relative entropy of KK systems with QGIF entanglement

The concept of relative entropy in geometric information theories.

$$S(\rho_A \parallel \sigma_A) = Tr_{\sigma_B}[\rho_A \log \rho_A - \log \sigma_A], \quad (47)$$

where $S(\rho_A \parallel \rho_A) = 0$. This way we introduce a measure of “distance” between two KK QGIFs with a norm $||\rho_A|| = tr(\sqrt{\rho_A (\rho_A^\dagger)})$, see reviews [33,34,54]. In straightforward form, we can check that there are satisfied certain important properties and inequalities.

Two KK QGIF systems are characterized by formulas and conditions for relative entropy:

1. for tensor products of density matrices, $S(1\rho_A \otimes 2\rho_A \parallel 1\sigma_A \otimes 2\sigma_A) = S(1\rho_A \parallel 1\sigma_A) + S(2\rho_A \parallel 2\sigma_A)$;
2. positivity: $S(\rho_A \parallel \sigma_A) \geq \frac{1}{2} ||\rho_A - \sigma_A||^2$, i.e., $S(\rho_A \parallel \sigma_A) \geq 0$;
3. monotonicity: $S(\rho_A \parallel \sigma_A) \geq S(tr_s \rho_A | tr_s \sigma_A)$, where $tr_s$ denotes the trace for a subsystem of $A$.

The above positivity formula and the Schwarz inequality $||X|| \geq tr(XY)/||Y||$ result in $2S(\rho_A \parallel \sigma_A) \geq ((\langle O \rangle_\rho - \langle O \rangle_\sigma)^2)/||O||^2$, for any expectation value $\langle O \rangle_\rho$ of an operator $O$ computed with the density matrix $\rho_A$, see formulas (34). The relative entropy $S(\rho_A \parallel \sigma_A)$ (47) can be related to the entanglement entropy $qS(\rho_A)$ (39) we use formula $S(\rho_A \parallel 1_A/k_A) = \log k_A - qS(\rho_A)$, where $1_A$ is the $k_A \times k_A$ unit matrix for a $k_A$-dimensional Hilbert space associated with the region $A$. 

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For three KK QGIF systems, we denote by $\rho_{\text{AUB}}$, the density matrix of $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$. When, for instance, $\rho_{\text{AUB}}$ is written for its restriction on $\mathcal{A}\cup\mathcal{B}$ and $\rho_{\text{B}}$ is stated for its restriction on $\mathcal{B}$. Using the formula

$$\text{tr}_{\text{AUB}}\left[\rho_{\text{AUB}}(\mathcal{O}_{\text{AUB}} \otimes 1_{\mathcal{C}}/k_{\mathcal{C}})\right] = \text{tr}_{\text{AUB}}(\rho_{\text{AUB}}^a\mathcal{O}_{\text{AUB}}),$$

we prove such identities

$$S(\rho_{\text{AUB}} \parallel 1_{\text{AUB}}/k_{\text{AUB}}) = \rho_{\text{AUB}} \parallel 1_{\text{AUB}}/k_{\text{AUB}}$$

and inequalities

$$S(\rho_{\text{AUB}} \parallel 1_{\text{AUB}}/k_{\text{AUB}}) \geq \rho_{\text{AUB}} \parallel 1_{\text{AUB}}/k_{\text{AUB}},$$

These formulas can be rewritten for the entanglement entropies $q\mathcal{S}$ and Hamilton mechanical variables with “tilde.”

5.2.3 Mutual information for KK QGIFs

We can characterize the correlation between two KK QGIF systems $\mathcal{A}$ and $\mathcal{B}$ (it can be involved also a third system $\mathcal{C}$) by mutual information

$$\mathcal{J}(\mathcal{A}, \mathcal{B}) := S_{\mathcal{A}} + S_{\mathcal{B}} - S_{\text{AUB}} \geq 0 \text{ and } \mathcal{J}(\mathcal{A}, \mathcal{B}\cup\mathcal{C}) \leq \mathcal{J}(\mathcal{A}, \mathcal{B}).$$

Using formula $\mathcal{J}(\mathcal{A}, \mathcal{B}) = S(\rho_{\text{AUB}} \parallel \rho_{A} \otimes \rho_{B})$, we can introduce similar concepts and inequalities for the entanglement of KK QGIF systems,

$$q\mathcal{J}(\mathcal{A}, \mathcal{B}) := qS_{\mathcal{A}} + qS_{\mathcal{B}} - qS_{\text{AUB}} \geq 0 \text{ and } q\mathcal{J}(\mathcal{A}, \mathcal{B}\cup\mathcal{C}) \leq q\mathcal{J}(\mathcal{A}, \mathcal{B}), \text{ for }$$

$$q\mathcal{J}(\mathcal{A}, \mathcal{B}) = qS(\rho_{\text{AUB}} \parallel \rho_{A} \otimes \rho_{B}).$$

We can write similar formulas for classical KK GIFs and associated thermodynamic models with statistical density $\rho(\beta, \xi, g)$ (22) and/or for constructions using the W-entropy. With tilde values, this can be proved for causal configurations in nonholonomic Hamilton variables [2].

The mutual information between two KK QGIFs is a measure how much the density matrix $\rho_{\text{AUB}}$ differs from a separable state $\rho_{A} \otimes \rho_{B}$. Quantum correlations entangle even spacetime disconnected regions of the phase spacetime under geometric flow evolution. Under geometric information KK flow evolution in respective regions, $2\mathcal{J}(\mathcal{A}, \mathcal{B}) \geq ((\mathcal{O}_{\mathcal{A}}\mathcal{O}_{\mathcal{B}}) - \langle\mathcal{O}_{\mathcal{A}}\mathcal{O}_{\mathcal{B}}\rangle)^2/||\mathcal{O}_{\mathcal{A}}||^2||\mathcal{O}_{\mathcal{B}}||^2$, for bounded operators $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$.

5.2.4 The Rényi entropy for KK QGIFs

The Rényi entropy [59] is important for computing the entanglement entropy of QFTs using the replica method (see section IV of [54,60] for a holographic dual of Rényi relative entropy). Such constructions are possible in KK QGIF theory because the thermodynamic generating function $Z[g(\tau)]$ (25) and related statistical density $\rho(\beta, \xi, g)$ (22) can be used for defining $\sigma_{\mathcal{A}}$ (35) as a probability distribution.
Let us begin with the extension of replica method to G. Perelman’s thermodynamical model and related classical and quantum information theories. We consider an integer $r$ (replica parameter) and introduce the Rényi entropy

$$r \mathcal{S}(A) := \frac{1}{1-r} \log[tr_A(\rho_A)^r]$$

(48)

for a KK QGIF system determined by a density matrix $\rho_A$. A computational formalism is elaborated for an analytic continuation of $r$ to a real number with a well-defined limit $q \mathcal{S}(\rho_A) = \lim_{r \to 1} r \mathcal{S}(A)$ and normalization $tr_A(\rho_A)$ for $r \to 1$. For such limits, the Rényi entropy (48) reduces to the entanglement entropy (39).

Using similar formulas proved in [61], there are introduced such important inequalities for derivatives on replica parameter, $\partial_r$,

$$\partial_r(r \mathcal{S}) \leq 0, \quad \partial_r \left( \frac{r - 1}{r} r \mathcal{S} \right) \geq 0, \quad \partial_r[(r-1)r \mathcal{S}] \geq 0, \quad \partial^2_{rr}[(r-1)](r \mathcal{S}) \leq 0.$$  

(49)

A usual thermodynamical interpretation of such formulas follows for a system with a modular Hamiltonian $H_A$ and effective statistical density $\rho_{\underline{A}} := e^{-2\pi H_A}$. The value $\beta_r = 2\pi r$ is considered as the inverse temperature and the effective “thermal” statistical generation (partition) function is defined

$$r \mathcal{Z}(\beta_r) := tr_A(\rho_A)^r = tr_A(e^{-\beta_r H_A})$$

similarly to $\mathcal{Z}[g(\tau)]$ (25). We compute using canonical relations such statistical mechanics values

- for the modular energy: $r \mathcal{E}(\beta_r) := -\partial_{\beta_r} \log[r \mathcal{Z}(\beta_r)] \geq 0$;
- for the modular entropy: $\partial \mathcal{S}(\beta_r) := (1 - \beta_r \partial_{\beta_r}) \log[r \mathcal{Z}(\beta_r)] \geq 0$;
- for the modular capacity: $\partial \mathcal{C}(\beta_r) := \beta_r^2 \partial_{\beta_r} \log[r \mathcal{Z}(\beta_r)] \geq 0$.

These inequalities are equivalent to the second line in (49) and characterize the stability if KK GIFs as a thermal system with replica parameter regarded as the inverse temperature for a respective modular Hamiltonian. Such replica criteria of stability define a new direction for the theory of geometric flows and applications in modern physics [2,6,7,18–21,31].

We note that the constructions with the modular entropy can be transformed into models derived with the Rényi entropy and inversely. Such transforms can be performed using formulas

$$r \mathcal{\hat{S}} := r^2 \partial_r \left( \frac{r - 1}{r} r \mathcal{S} \right)$$

and, inversely, $r \mathcal{S} = \frac{r}{r - 1} \int_1^r dr' r \mathcal{\hat{S}}(r')$. The implications of the inequalities for the Rényi entropy were analyzed for the gravitational systems with holographic description, see reviews [33,54,62].

The concept of relative entropy $\mathcal{S}(\rho_{\underline{A}} \parallel \sigma_A)$ (47) can be extended to that of relative Rényi entropy (for a review, see section 11.3 in [54]). For a system QGIFs with two density matrices $\rho_{\underline{A}}$ and $\sigma_A$, we introduce

$$r \mathcal{S}(\rho_{\underline{A}} \parallel \sigma_A) = \frac{1}{r - 1} \log \left[ tr \left( (\sigma_A)^{(1-r)/2r} \rho_{\underline{A}} (\sigma_A)^{(1-r)/2r} \right)^r \right],$$

for $r \in (0, 1) \cup (1, \infty)$;

$$\text{or } 1 \mathcal{S}(\rho_{\underline{A}} \parallel \sigma_A) = \mathcal{S}(\rho_{\underline{A}} \parallel \sigma_A) \text{ and } \infty \mathcal{S}(\rho_{\underline{A}} \parallel \sigma_A) = \log ||(\sigma_A)^{-1/2} \rho_{\underline{A}} (\sigma_A)^{-1/2}||_\infty.$$  

(50)
In any point of causal curves, we prove certain monotonic properties, \(rS(ρ, Σ_A) ≥ rS(\tilde{ρ}, \tilde{Σ}_A)\) and \(\partial_r [rS(\tilde{ρ}, \tilde{Σ}_A)] ≥ 0\), and to reduce the relative Rényi entropy to the Rényi entropy using formula \(rS(\rho, 1_A/k_A) = \log k_A - rS(\Sigma_A)\).

The values (50) do not allow a naïve generalization of the concept of mutual information and interpretation as an entanglement measure of quantum information. There are negative values of relative Rényi entropy for \(r ≠ 1\) [63]. This problem can be solved if it is introduced the concept of the \(r\)-Rényi mutual information [64],

\[ rJ(A, B) := \min_{\sigma, B} rS(ρ, A) ≤ rS(\rho, A ⊗ B) ≥ 0 \]

for the minimum is taken over all \(\sigma_B\). We obtain the standard definition of mutual information for \(r = 1\). In result, we can elaborate a self-consistent geometric-information thermodynamic theory for KK QGIFs.

6 Decoupling and integrability of KK cosmological flow equations

In this section, we prove that the system of nonlinear PDEs for geometric evolution of KK gravity (4) (and self-similar configurations defining nonholonomic EM equations as Ricci soliton equations (13)) can be decoupled and integrated in general forms. Such solutions are defined by generic off-diagonal metrics \(g(τ)\) (for canonical d-connections, \(\tilde{D}(τ)\), and, in particular, for LC-configurations, \(\nabla(τ)\)). The coefficients of respective geometric objects depend on all spacetime coordinates via generating and integration functions and (effective) matter sources \(\gamma_{0β}(τ)\) (54) via nonholonomic entropic deformations of the energy–momentum tensor (1). We apply and develop the AFDM [1, 7, 20, 30, 31].

6.1 Quasiperiodic spacetime and quasicrystal-like KK configurations

We study two examples of space and time quasiperiodic structures defined in a curved spacetime following our works on quasicrystal, QC, models in modern cosmology [30, 31, 65, 66]. Here, we cite other type QC models studied in [67]. Our approach to locally anisotropic cosmology and dark energy and dark matter physics with spacetime quasicrystal, STQC, structures is based on F. Wilczek and co-authors ideas originally elaborated in condensed matter physics [68–71].

- 1-d relativistic time QC structures are deformed by a scalar field \(ξ(\chi^i, y^α)\) and respective Lagrange density on a spacetime \((V, g, N)\),

\[ \hat{L}(ξ) = \frac{1}{48} (g^{αβ}(e_αξ)(e_βξ))^2 - \frac{1}{4} g^{αβ}(e_αξ)(e_βξ) - V(ξ), \] (51)

resulting in N-adapted variational motion equations \[\frac{1}{2} g^{αβ}(e_αξ)(e_βξ) - 1)(D^YD_γξ) = 2\frac{\partial \hat{V}}{\partial ξ}\] where \(e_α\) are N-adapted partial derivatives. In these formulas, \(\hat{V}(ξ)\) is a nonlinear potential when the field \(ξ\) defines a 1-d time QC structure, 1-TQC, as a solution of motion equations.10

10 There are used brief notations for partial derivatives when, for instance, \(\partial q/∂x^i = \partial_q, \partial q/∂y^3 = \partial_3q = q^*, \partial q/∂χ^4 = \partial_4q = \partial_q = q^*, \) for a function \(q(x^i, y^3, τ).\) For nonrelativistic limits, we can consider \(g_{αβ} = [1, 1, 1, -1] \) and \(ξ → ξ(τ), \hat{L} → \frac{1}{4}(ξ^*)^2 - V(ξ), \) which relates (51) to an effective energy \(E = \frac{1}{4}[(ξ^*)^2 - 1]^2 + \hat{V}(ξ) - \frac{1}{4} \) derived for the motion equations \[(ξ^*)^2 - 1]ξ^{**} = \frac{\partial \hat{V}}{\partial ξ}\] introduced in [71].

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3-d QC structures on curved spaces are constructed as analogous dynamic phase field crystal models with flow evolution on real parameter \( \tau \). Such a QC structure is determined by a generating function \( \widetilde{q} = \widetilde{q}(x, y, \tau) \) as a solution of an evolution equation with conserved dynamics,

\[
\frac{\partial \widetilde{b}}{\partial \tau} = \Delta \left[ \frac{\delta F}{\delta \widetilde{b}} \right] = -\Delta (\Theta \widetilde{b} + Q\widetilde{b}^2 - \widetilde{b}^3).
\]

(52)

Such evolution equations can be considered on any 3-d space-like hypersurface \( \Xi \), with a canonically nonholonomically deformed hypersurface Laplace operator \( \widetilde{\Delta} := (\tilde{\nabla})^2 = q^i j \tilde{\nabla}_i \tilde{\nabla}_j \), for \( i, j, \ldots, 1, 2, 3 \). This operator is a distortion of \( \Delta := (\nabla)^2 \) constructed in 3-d Riemannian geometry.\(^{11}\)

6.2 KK flows with entropic (elastic) configurations

Various models of modified emergent/entropic gravity can be derived from nonholonomic modifications of G. Perelman functionals [6,7]. In this section, we redefine the constructions for KK GIFs and nonholonomic Ricci soliton equations. Using the canonical d-connection \( \tilde{\nabla} \), we introduce three “hat” values:

\[
\tilde{\varepsilon}_{a\beta} = \tilde{\nabla}_a u_\beta - \tilde{\nabla}_\beta u_a \quad - \text{the elastic strain tensor} ; \phi = u/\sqrt{|\Lambda|} \quad - \text{a dimensionless scalar} ; \quad \tilde{\chi} = \alpha(\tilde{\nabla}_a u^\mu)(\tilde{\nabla}_\beta u^\nu) + \beta(\tilde{\nabla}_a u_\nu)(\tilde{\nabla}_\beta u^\mu) + \gamma(\tilde{\nabla}_a u_\mu)(\tilde{\nabla}_\beta u^\nu) \quad - \text{a general kinetic term for} \ u^\mu.\]

In these formulas, we consider a conventional displacement vector field \( u^\mu \), cosmological constant \( \Lambda \) and some constants \( \alpha, \beta, \gamma \); there are used short hand notations: \( u := \sqrt{|u_a u^a|}, \tilde{\varepsilon} = \tilde{\varepsilon}_\beta^\alpha, \text{ and } n^a := u^a/u.\)

The Lagrange densities for gravitational and electromagnetic fields for models of emergent KK gravity with nonholonomic distributions determined by quasiperiodic geometric flow evolution (or self-similar configurations) on \( \mathbf{V} \) are chosen in the form

\[
\begin{align*}
\int_0^1 L &= s L + m L + \text{int } L \text{, for } s L = M^2 R, \\
\text{int } L &= -\sqrt{|\Lambda|} |T_{\mu\nu} u^\mu v^\nu/u, x L = M^2 |\Lambda| (\chi)^{3/2} + |\Lambda| |u(\xi, \tilde{\beta})|^2 z.
\end{align*}
\]

(53)

when the Plank gravitational mass is denoted \( M_P \) and the gravitational Lagrangian \( s L \) is determined by the Ricci scalar \( s R \) of \( \tilde{\nabla} \) and \( \tilde{T}_{\mu\nu} \) is the electromagnetic energy momentum tensor \( (1) \) determined by \( m L(g, \tilde{\nabla}, \Lambda) \), with \( \tilde{\nabla} \) used instead of \( \nabla ).\(^{12}\)

\(^{11}\) In [30,31], we explain all details on such quasiperiodic structures. Here, we also note that the functional \( F \) in (52) is characterized by an effective free energy \( F[q] = \int \left[ -\frac{1}{2} \tilde{\beta} \tilde{\Theta} \tilde{\beta} - \frac{Q}{2} \tilde{\beta}^2 + \frac{1}{2} \tilde{b}^4 \right] \sqrt{\tilde{q}} dx^1 dx^2 dx^3, \)

where \( q = \det[q_{ij}], \delta y^3 = e^3 \) and the operators \( \Theta \) and \( Q \) are chosen in certain forms when nonlinear interactions are stabilized by the cubic term with \( Q \) and the second-order resonant interactions. An average value \( < \tilde{\beta} > \) is conserved for any fixed time variable \( t \) and/or evolution parameter \( \tau_0 \) (we can fix \( < \tilde{\beta} > |_{\tau=\tau_0} = 0 \) by redefining \( \Theta \) and \( Q ) \).

\(^{12}\) On \( \mathbf{V} \), the curve space covariant derivative is computed using both \( \tilde{\nabla} \) and \( \Lambda \). We can construct KK GIF models with MGTs as in [31,72–74] when the gravitational Lagrangian is a functional \( f \) of \( s R \), \( s L = M^2 f(s R) \), and a more general energy momentum tensor (with scalar field interactions) \( m T_{\mu\nu} \) instead of \( \tilde{T}_{\mu\nu} \). We can fix \( z = 1 \) if we search for compatibility with relativistic entropic gravity models [75], or \( z = 2 \) if we search for a limit to the standard de Sitter space solution [76,77], see details in [6,7]. We can model by GIFs certain STQC structures, for instance as in entropic gravity if we consider that the displacement vector field \( u^\mu(\xi, \tilde{\beta}) \) as a functional of functions \( \xi, \tilde{\beta} \) subjected to certain conditions of type (51) and/or (52). More general KK GIF cosmological scenarios can be elaborated if we consider functionals for pattern forming, nonlinear wave soliton structures, fractional and diffusion processes, etc., see details in [30,31,65,66].
For the full system of Lagrangians in (53), the effective energy–momentum tensor is computed \( \hat{T}_{\mu\nu} = \hat{T}_{\mu\nu}^{\text{tot}} + \hat{T}_{\mu\nu}^{\text{int}} + \chi \hat{T}_{\mu\nu}^{\chi} \). We can model “pure” elastic spacetime modifications of the Einstein gravity if we consider \( D = \nabla \), which results in respective formulas (10)-(13) in [76] (for \( \hat{T}_{\mu\nu}^{\text{int}} \) and \( \hat{T}_{\mu\nu}^{\chi} \)). In this section, we work with a generalized (effective) source splitting into respective three components,

\[
\hat{\Upsilon}_{\mu\nu}^{\text{tot}} := \hat{\Upsilon}_{\mu\nu}^{\chi} \left( \hat{T}_{\mu\nu}^{\text{tot}} - \frac{1}{2} \hat{g}^{\mu\nu} \hat{T}^{\text{tot}} \right) = \hat{\Upsilon}_{\mu\nu} + \hat{\Upsilon}_{\mu\nu}^{\text{int}} + \chi \hat{\Upsilon}_{\mu\nu}^{\chi},
\]

where \( \chi \) is determined in standard form by the Newton gravitational constant \( G \).

6.3 Parametric modified Einstein equations encoding GIFs

The geometric flow equations can be written in nonholonomic canonical variable (4) as modified Einstein equations with dependence of geometric objects and sources on a temperature-like parameter \( \tau \). Such systems of nonlinear PDEs can be decoupled and integrated in general forms for various classes of effective sources, see proofs and examples in [1,7,20,30,31].

In this work, we study KK GIFs determined by two effective generating sources \( h \hat{Y}(\tau, x^k) \) and \( \hat{Y}(\tau, x^k, y^e) \) prescribing total sources of type (54) up to frame transforms with respective tetradic fields \( e^{\mu'}_\mu(\tau) = e^{\mu'}_\mu(\tau, u^\gamma) \) and their dual \( e^{\nu'}_\nu(\tau) = e^{\nu'}_\nu(\tau, u^\gamma) \). The effective sources are parameterized in such a general N-adapted form,

\[
e^{\hat{\Upsilon}_{\mu\nu}(\tau)} = e^{\mu'}_\mu(\tau)e^{\nu'}_\nu(\tau) \left[ \hat{\Upsilon}_{\mu'\nu'}^{\text{tot}}(\tau) + \frac{1}{2} \partial^\tau g^{\mu'\nu'}(\tau) \right] = \left[ h\hat{Y}(\tau, x^k)\delta^i_j, \hat{Y}(\tau, x^k, y^e)\delta^a_b \right].
\]

In result, the system of nonholonomic R. Hamilton equations for KK GIFs with encoding entropic modifications (see, for instance, (5) and their equivalent form (4)) can be written in a generalized Einstein form with an effective source (55),

\[
\hat{R}_{\alpha\beta}(\tau) = e^{\hat{\Upsilon}_{\alpha\beta}(\tau)}.
\]

Such systems of nonlinear PDEs equations are for an undetermined normalization function \( \hat{f}(\tau) = \hat{f}(\tau, u^\gamma) \) which can be defined explicitly for respective classes of exact or parametric solutions with prescribed generating functions and (effective) sources. The generating functions \( h \hat{Y}(\tau) \) and \( \hat{Y}(\tau) \) for \( e^{\hat{\Upsilon}_{\alpha\beta}(\tau)} \) (55) can be considered as certain prescribed data for (effective) matter sources. Such values impose certain nonholonomic frame constraints on geometric evolution and self-similar configurations. This also prescribe the type of GIF entropic corrections and quasiperiodic configuration.

For locally anisotropic cosmological configurations with coordinates \((x^k, y^4 = t)\), the \( \tau \)-evolution of d-metric \( g(\tau) (14) \) can be parameterized by N-adapted coefficients,

\[
g_i(\tau) = e^{\psi(\tau, x^k)}, \quad g_a(\tau) = \omega(\tau, x^k, y^b)\bar{n}_a(\tau, x^k, t), \quad N_i^3(\tau) = \bar{n}_i(\tau, x^k, t),
\]

\[
N_i^4(\tau) = \bar{w}_i(\tau, x^k, t), \quad \text{for } \omega = 1.
\]

For simplicity, we shall construct cosmological evolution models determined by nonholonomic configurations possessing at least one Killing symmetry, for instance, on \( \partial_3 = \partial_{x^4} \). Here, we note that different types of ansatz and parameterizations of d-metrics and effective sources (55) are considered, for instance, for generating stationary solutions, see [1,7,20,31].
6.4 Decoupling of quasiperiodic KK cosmological flow equations

6.4.1 d-Metric ansatz and sources with decoupling

Tables similar to Table 1 are provided in [7,31] for certain general ansatz which allow constructing exact and parametric solutions in MGTs and entropic gravity. In this subsection, we introduce locally anisotropic cosmological ansatz for generic off-diagonal metrics and effective sources when is possible to prove general decoupling and integration properties of (modified) Einstein equations (56) with effective sources (55).

We model nonholonomic deformations of d-metrics using \( \eta \)-polarization functions, \( \hat{g} \rightarrow g(\tau) \), of a “prime” metric, \( \hat{g} \), into a family of “target” d-metrics \( g(\tau) \) for parameterizations

\[
g(\tau) = \eta_i(\tau, x^k)\hat{g}_i dx^i \otimes dx^i + \eta_a(\tau, x^k, y^b)\hat{h}_a e^a[\eta] \otimes e^a[\eta].
\]

In this and similar formulas, we do not consider summation on repeating indices if they are not written as contraction of “up-low” ones. The target N-elongated basis in (58) is determined by \( N_i^a(\tau, u) = \eta_i^a(\tau, x^k, y^b)\hat{N}_i^a(\tau, x^k, y^b) \), when \( e^a[\eta] = (dx^i, e^a = dy^a + \eta_i^a\hat{N}_i^a dx^i) \) and \( \eta_\tau(\tau) = \eta_\tau(\tau, x^k, y^b) \), \( \eta_a(\tau) = \eta_a(\tau, x^k, y^b) \) and \( \eta_i^a(\tau) = \eta_i^a(\tau, x^k, y^b) \) are called, respectively, gravitational polarization functions, or \( \eta \)-polarizations. We consider than any \( g(\tau) \) defines a solution of (56) even a general prime metric \( \hat{g} \) may be not a physically important metric. For certain cosmological models, we can consider \( \hat{g} \) as standard isotropic or anisotropic metric. Using a general coordinate parametrization, a prime metric \( \hat{g} = g_{ab}(x^i, y^a) du^a \otimes du^b \) can be also represented equivalently in N-adapted form

\[
\hat{g} = \hat{g}_a(u)\hat{e}^a \otimes \hat{e}^a = \hat{g}_i(x) dx^i \otimes dx^i + \hat{g}_a(x, y)\hat{e}^a \otimes \hat{e}^a,
\]

for \( \hat{e}^a = (dx^i, e^a = dy^a + \hat{N}_i^a(u) dx^i) \), and

\[
\hat{e}_a = (\hat{e}_i = \partial/\partial y^a - \hat{N}_i^a(u)\partial/\partial y^b, e_a = \partial/\partial y^a).
\]

In this paper, we study physically important cases when \( \hat{g} \) is a Friedman–Lemaître–Robertson–Walker (FLRW)-type metric, or a Bianchi anisotropic metrics. For diagonalizable prime metrics, we can always find a coordinate system when \( \hat{N}_i^b = 0 \). Nevertheless, it is convenient to construct exact solutions with nontrivial functions \( \eta_a = (\eta_\tau, \eta_a), \eta_i^a \), and nonzero coefficients \( \hat{N}_i^b(u) \) in order to avoid constructions with nonholonomic deformations for singular coordinates.

6.4.2 Cosmological canonical Ricci d-tensors, nonlinear symmetries, and LC-conditions

For simplicity, we study locally anisotropic cosmological flows when the coefficients of the geometric objects do not depend on the space coordinate \( y^3 \), i.e., using d-metric data (57) with \( \omega = 1 \). The GIF entropic modified Einstein equations (56) are parameterized

\[
R^1_1(\tau) = R^2_2(\tau) = -h\bar{\nabla}(\tau) i.e., \quad \frac{g_1^* g_2^*}{2g_1} + \frac{(g_2^*)^2}{2g_2} - g_2^* + \frac{g_1^* g_2'}{2g_2} + \frac{(g_1')^2}{2g_1} = 0
\]

(60)

\[
R^3_3(\tau) = R^4_4(\tau) = -\bar{\nabla}(\tau) i.e., \quad \frac{\bar{h}_3^* \bar{h}_4^*}{2\bar{h}_3} + \frac{\bar{h}_3 \bar{h}_4'}{2\bar{h}_4} - \bar{h}_3^* = -2\bar{h}_3 \bar{h}_4 \bar{\nabla} ;
\]

(61)
Table 1 Quasiperiodic cosmological flows ansatz for constructing generic off-diagonal exact and parametric solutions using the anholonomic frame deformation method, AFDM

| Diagonal ansatz: PDEs → ODEs | AFDM: PDEs with decoupling; generating functions |
|------------------------------|-----------------------------------------------|
| $u^a = (r, \theta, \varphi, t)$ | $2 + 2$ splitting, $u^a = (x^1, x^2, y^3, y^4 = t)$; flow parameter $\tau$ |
| $u = (x, y)$: | $N: TV = hTV \oplus vTV$, locally $N = \{N^a(x, y)\}$ |
| LC-connection $\hat{\nabla}$ | Canonical connection distortion $D = \nabla + Z$ |
| diagonal ansatz $g_{\alpha\beta}(u)$ | $g_{\alpha\beta}(\tau, x^i, y^a)$ general frames/coordinates |
| $\hat{g}_{\alpha\beta}(t)$ for FLRW | $g_{\alpha\beta}(\tau) = g_{\alpha\beta}(\tau, r, \theta, y^4 = t)$ cosm. configurations |
| coord. transforms $e_\alpha = e^a_\alpha \partial_a$, $e_\beta = e^b_\beta du^b$. $\hat{g}_{\alpha\beta} = \hat{g}_{\alpha\beta}' e_\alpha'^a e_\beta'^b$ | $[\text{N-adapt. fr.}]$ |
| $\hat{g}_a(x^k, y^a) \rightarrow \hat{g}_a(t), \hat{N}^a_i(x^k, y^a) \rightarrow 0.$ | $\hat{g}(\tau) = g_\alpha(\tau)$, $h_{ab}(\tau)$ general frames/coordinates |
| $\hat{\nabla}, Ric = \{\hat{R}_{\beta\gamma}\}$ | $\hat{Ric} = \{\hat{R}_{\beta\gamma}\}$ |
| $m\mathcal{L}[\psi] \rightarrow T_{\alpha\beta}[F]$ | $\hat{D}, \hat{Ric} = \{\hat{R}_{\beta\psi}\}$ |
| trivial equations for $\hat{\nabla}$-torsion | $e^\psi_{\nu\nu}(\tau) = \text{diag}\{h \tilde{\nabla}(\tau, x^i)\delta^i_j, \tilde{\nabla}(\tau, x^i', t)\delta^a_b\}$ |
| LC-conditions | $\hat{D}_J\tilde{\nabla}_e^a \rightarrow 0 = \nabla$ |
\[ R_{3k}(\tau) = \frac{\tilde{h}_3}{2\tilde{h}_3} \tilde{n}_k^{**} + \left( \frac{3}{2} \tilde{h}_3 - \frac{\tilde{h}_3}{\tilde{h}_4} \right) \frac{\tilde{n}_k^*}{4\tilde{h}_4} = 0; \]
\[ 2\tilde{h}_3 R_{4k}(\tau) = -\tilde{w}_k \left[ \left( \frac{\tilde{h}_3^*}{2\tilde{h}_3} + \frac{\tilde{h}_3^* \tilde{h}_4^*}{2\tilde{h}_4} - \tilde{h}_3^{**} \right) + \frac{\tilde{h}_3^*}{2} \left( \frac{\partial_k \tilde{h}_3}{\tilde{h}_3} + \frac{\partial_k \tilde{h}_4}{\tilde{h}_4} \right) - \partial_k \tilde{n}_3^* = 0. \] (63)

This system of nonlinear PDE (60)–(63) can be written in an explicit decoupled form. Let us consider \( \bar{w}_i = (\partial_i \tilde{h}_3) (\partial_i \tilde{w}), \bar{\beta} = (\partial_i \tilde{h}_3) (\partial_i \tilde{\beta}), \bar{\gamma} = \partial_i (\ln |\tilde{h}_3|^3/2|\tilde{h}_4|), \) where \( \tilde{w} := \ln |\partial_i \tilde{h}_3/\sqrt{|\tilde{h}_3 \tilde{h}_4|}| \) is considered as a generating function. In this work, we overline certain coefficients for cosmological evolution following notations from [7]. For configurations with \( \partial_i \tilde{h}_3 \neq 0 \) and \( \partial_i \tilde{\beta} \neq 0 \), we obtain such equations

\[ \psi^{**} + \psi'' = 2\tilde{h} \bar{\bar{Y}}; \tilde{w} \tilde{h}_3 = 2\tilde{h}_3 \bar{\bar{Y}} \tilde{h}_4 \bar{\bar{Y}}; \tilde{n}_i^{**} + \bar{\bar{Y}} \tilde{n}_i^* = 0; \bar{\bar{\beta}} \tilde{w}_i - \bar{\bar{\alpha}}_i = 0. \] (64)

These equations can be integrated “step by step” for any (redefined) generating function \( \bar{\bar{Y}}(\tau, x^i, t) := e^{\tilde{w}} \) and sources \( \tilde{h} \bar{\bar{Y}}(\tau, x^i) \) and \( \bar{\bar{Y}}(\tau, x^k, t) \).

The system (64) imposes certain conditions of nonlinear symmetric on four functions \((\tilde{h}_3, \tilde{h}_4, \bar{\bar{Y}}, \bar{\bar{\Psi}})\). We can redefine respective generating functions, \((\bar{\bar{\Psi}}(\tau), \bar{\bar{\psi}}(\tau)) \iff (\bar{\bar{\Phi}}(\tau), \Lambda(\tau))\) if there are satisfied the equations

\[ \bar{\bar{\Lambda}}(\bar{\bar{\Psi}})^* = |\bar{\bar{\Psi}}| (\bar{\bar{\Phi}}^2)^*, \text{ or } \bar{\bar{\Lambda}}^2 = \bar{\bar{\Phi}}^2 |\bar{\bar{\Psi}}|^2. \] (65)

The vertical flow of effective cosmological constants \( \bar{\bar{\Lambda}} \) can be identified to certain models with \( \Lambda \) from (53) if we consider nonholonomic deformations of some classes of (anti) de Sitter solutions in GR or a MGT.

The LC-conditions for zero torsion cosmological GIFs transform into a system of first-order PDEs with coefficients depending on \( \tau \) and \( t \),

\[ \partial_i \tilde{w}_i = (\partial_i - \tilde{w}_i \partial_i) \ln \sqrt{|\tilde{h}_3|}, (\partial_i - \tilde{w}_i \partial_i) \ln \sqrt{|\tilde{h}_4|} = 0, \]
\[ \partial_i \tilde{n}_i = \partial_i \tilde{w}_k, \partial_i \tilde{n}_i = 0, \partial_i \tilde{n}_k = \partial_k \tilde{n}_i. \] (66)

Such systems can be solved in explicit form for certain classes of additional nonholonomic constraints on cosmological d-metrics and N-coefficients, see (57).

6.5 Integrability of KK quasiperiodic cosmological flow equations

Integrating “step by step” the system of the nonlinear PDEs (60)–(63) decoupled in the form (64) (see similar details and proofs in [1,7,20,30,31]), we obtain such d-metric coefficients for (14) and/or (58),

\[ g_1(\tau) = e^{\psi(\tau, x^k)} \text{ as a solution of 2-d Poisson eqs. } \psi^{**} + \psi'' = 2 \tilde{h} \bar{\bar{Y}}(\tau); \]
\[ g_3(\tau) = \tilde{h}_3(\tau, x^i, t) = h_3^{[0]}(\tau, x^k) - \int \! dt \frac{(\bar{\bar{\Psi}})^*}{4\bar{\bar{Y}}} = h_3^{[0]}(\tau, x^k) - \bar{\bar{\Phi}}^2/4\bar{\bar{\Lambda}}(\tau); \]
\[ g_4(\tau) = \tilde{h}_4(\tau, x^i, t) = -\frac{(\bar{\bar{\Psi}})^*}{4\bar{\bar{Y}}^2 \tilde{h}_3} = -\frac{(\bar{\bar{\Phi}}^2)^*}{4\bar{\bar{Y}}^2 (h_3^{[0]}(\tau, x^k) - \int \! dt \frac{(\bar{\bar{\Psi}})^*}{4\bar{\bar{Y}}})} \]
\[ = -\frac{[(\bar{\bar{\Phi}})^*]^2}{4\tilde{h}_3 |\bar{\bar{\Lambda}}(\tau)| \int \! dt \bar{\bar{Y}} [(\bar{\bar{\Phi}})^*]^2} = -\frac{[(\bar{\bar{\Phi}})^*]^2}{4|h_3^{[0]}(x^k) - \bar{\bar{\Phi}}^2/4\bar{\bar{\Lambda}}(\tau)|| \int \! dt \bar{\bar{Y}} [(\bar{\bar{\Phi}})^*]^2}. \] (67)
Such d-metric coefficients are computed with respect to N-adapted bases determined by N-connection coefficients computed by formulas

\[ N^3_k(\tau) = n_k(x^i, t) = n_k(x^i) + 2n_k(x^i) \]

\[ \int dt \frac{1}{\hat{Y}^2} \frac{1}{ht^3(\tau, x^i)} - \int dt (\frac{1}{\hat{Y}^2})^\ast /4\hat{Y}^5 /2 \]

\[ = n_k(x^i) + 2n_k(x^i) \int dt \frac{1}{4}\frac{(\hat{Y}^2)^\ast}{\hat{Y}^3 /2} \]

\[ N^4_i(\tau) = \frac{\partial_i \hat{\Psi}}{\hat{\Psi}^\ast} = \frac{\partial_i \hat{\Psi}}{(\hat{\Psi}^2)^\ast} = \frac{\partial_i [\frac{1}{4}\int dt \hat{Y}(\hat{\Psi}^2)^\ast]}{\hat{Y}^2(\hat{\Psi}^2)^\ast} \] (68)

There are considered different classes of functions in these formulas. The values \( h^0_k(\tau, x^k) \), \( 1n_k(x^i, t) \), and \( 2n_k(x^i, t) \) are integration functions encoding various possible sets of (non) commutative parameters and integration constants (in general, such constants run on a temperature-like parameter \( \tau \) for geometric evolution flows). We can work equivalently with different generating data \( (\hat{\Psi}(\tau, x^i, t), \hat{Y}(\tau, x^i, t)) \) or \( (\hat{\Phi}(\tau, x^i, t), \hat{X}(\tau)) \) related via nonlinear symmetries. These values are related by nonlinear differential/integral transforms (65), and respective integration functions. In explicit form, such nonlinear symmetries involve certain topology/symmetry/asymptotic conditions for some classes of exact/parametric cosmological solutions.

With respect to coordinate frames, the coefficients (67) and (68) define generic off-diagonal cosmological solutions if the corresponding anholonomy coefficients are not trivial. Such locally cosmological solutions cannot be diagonalized by coordinate transforms in a finite spacetime region and its geometric flow evolutions. For the canonical d-connection, the cosmological d-metrics are with nontrivial nonholonomically induced d-torsion and N-adapted coefficients which can be computed in explicit form if there are prescribed certain evolution conditions, boundary values etc. We can generate as particular cases some well-known cosmological FLRW, or Bianchi, type metrics if there are considered data of type \( (\hat{\Psi}(\tau, t), \hat{Y}(\tau, t)) \), or \( (\hat{\Phi}(\tau, t), \hat{X}(\tau)) \), determined by special classes of integration functions and certain frame/coordinate transforms to respective (off-) diagonal configurations \( g_{\alpha\beta}(\tau, t) \). For cosmological Ricci soliton configurations, we can fix \( \tau = 0 \).

Let us discuss some nonholonomic transform and evolution properties of the above locally anisotropic cosmological solutions determined by effective cosmological parameterizations sources (55). We have a system of equations involving the evolution derivative \( \partial_\tau \). The v-part of cosmological vierbein under geometric flow depends on a time-like coordinate \( y^3 = t \), with \( e^{\mu}_{\nu}(\tau) = [e^1_{\nu}(\tau, x^k), e^2_{\nu}(\tau, x^k), e^3_{\nu}(\tau, x^k, t), e^4_{\nu}(\tau, x^k, t)] \). There are considered also coordinates \( y^3, y^4 \), when the dependence on \( y^3 \) can be omitted for configurations with Killing symmetry on \( \partial_3 \). In the horizontal part, the frame transforms for generating effective sources are of type \( y^3 \hat{Y}(\tau, t) \), \( y^4 \hat{Y}(\tau, x^k, t) \). Similar transforms can be performed for \( v \)-components, \( y^3 \hat{Y}_d(\tau, \nu) = [e^d_{\nu}(\tau, x^k)]^2 [\int \hat{Y}_{d\nu}(\tau, \nu) + \frac{1}{2} \partial_d g_{\nu}(\tau)] = \hat{Y}(\tau, x^k, t) \). For generating locally anisotropic cosmological solutions, we can prescribe any values for the matter sources \( \int \hat{Y}_{\mu\nu}(\tau) \) in a geometric flow model with cosmological evolution and quasiperiodic spacetime structure. Considering N-adapted diagonal configurations and integrating on \( \tau \), we can compute a cosmological evolution flows of \( g_{\alpha\beta}(\tau, x^k, t) \) encoding, for instance, nonholonomic and nonlinear geometric diffusion process [16, 17]. Such geometric constructions and physical models are performed with respect to a new system of
reference determined by \( e^{\mu'}(\tau, x^k, t) \). To elaborate realistic cosmological models, we have to prescribe some locally anisotropic generating values \([e^{\mu'}(\tau, x^k, t), \Lambda(\tau, x^k, t)]\) which are compatible with certain observational data, for instance, in modern cosmology and dark matter and dark energy physics, see a series of examples in [1, 7, 20, 30, 31, 65, 66].

### 6.5.1 Quadratic line elements for off-diagonal cosmological quasiperiodic flows

We can work with different types of generating functions, for instance, \( \Phi \) and/or \( \Psi \). Any coefficient \( h_3(\tau) = \Lambda_3(\tau, x^k, t) = \Lambda_3(0) - 4\Lambda(\tau, x^k) \neq 0 \) can be considered also as a generating function, for instance, for locally anisotropic quasiperiodic cosmological configurations. Using the second formula (67), we express \( \Phi^2 = -4\Lambda(\tau)h_3(\tau, r, \theta, t) \) and, using (65), \( (\Phi^2)^2 = \int \Phi \Phi^4 \). Introducing such values into the formulas for \( h_n \) and \( \Phi \) in (67) and (68), we construct locally anisotropic cosmological solutions parameterized by d-metrics with N-adapted coefficients (57), for instance, as a d-metric (14),

\[
\begin{align*}
\frac{d\mathbf{s}^2}{4} &= e^{\psi(\tau, x^k)} \left[ (dx^1)^2 + (dx^2)^2 \right] \\
&= \left[ \frac{1}{h_3} \frac{d\mathbf{s}^2}{4} \right] + \left[ \left( 1n_k + 2n_k \int \frac{1}{h_3} \frac{d\mathbf{s}^2}{4} \right) \frac{1}{h_3} \frac{d\mathbf{s}^2}{4} \right] \\
&= \left[ \left( \frac{1}{h_3} \frac{d\mathbf{s}^2}{4} \right) \frac{1}{h_3} \frac{d\mathbf{s}^2}{4} \right] \left( 1n_k + 2n_k \int \frac{1}{h_3} \frac{d\mathbf{s}^2}{4} \right) \left( \frac{1}{h_3} \frac{d\mathbf{s}^2}{4} \right) \\
&= \left[ \left( \frac{1}{h_3} \frac{d\mathbf{s}^2}{4} \right) \frac{1}{h_3} \frac{d\mathbf{s}^2}{4} \right] \left( 1n_k + 2n_k \int \frac{1}{h_3} \frac{d\mathbf{s}^2}{4} \right) \left( \frac{1}{h_3} \frac{d\mathbf{s}^2}{4} \right)
\end{align*}
\]

such solutions possess a Killing symmetry on \( \partial_3 \). We can rewrite equivalently such linear quadratic elements in terms of \( \eta \)—polarization functions and describe geometric flows of certain target locally anisotropic cosmological metrics \( \tilde{\mathbf{g}} = [g_{\alpha} = \eta_{\alpha} \tilde{g}_{\alpha}, \eta_{\alpha} \tilde{N}_{\alpha}] \). The primary cosmological data \([\tilde{g}_{\alpha}, \tilde{N}_{\alpha}]\) are also encoded when the cosmological flow evolution is determined by generating and integration functions and generating sources.

### 6.5.2 Off-diagonal Levi-Civita KK quasiperiodic cosmological configurations

From (69), we can extract and model entropic flow evolution of cosmological spacetimes in GR encoding both electromagnetic interactions and nontrivial quasiperiodic structures under geometric evolution. The zero torsion conditions are satisfied by a special class of generating functions and sources when, for instance, \( \tilde{\Psi}(\tau) = \tilde{\Psi}(\tau, x^i, t) \), when \( (\partial_3 \tilde{\Psi})^* = (\tilde{\Psi})^* \) and \( \tilde{\Psi}(\tau, x^i, t) = \tilde{\Psi}(\tau) \), or \( \tilde{\Psi} = const. \) For such classes of quasiperiodic and flow evolving generating functions and sources, the nonlinear symmetries (65) are written
\[ \Lambda(\tau) \bar{\Psi}^2 = \Phi^2 |\bar{\Psi}| - \int \! dt \, \Phi^2 |\bar{\Psi}|^* \bar{\Psi}^2 = -4\Lambda(\tau)\bar{h}_3(\tau, r, \theta, t), \bar{\Psi}^2 = \int \! dt \, \bar{Y}(\tau, r, \theta, t)\bar{h}_3^*(\tau, r, \theta, t). \]

We conclude that \(\bar{h}_4(\tau) = \bar{h}_4(\tau, x^i, t)\) can be considered also as generating function for GIF emergent cosmological solutions. There are generated LC-configurations with some parameteric on \(\tau\) functions \(\bar{A}(\tau, x^i, t)\) (this function is not related to an electromagnetic potential) and \(n(\tau, x^i)\), when the N-connection coefficients are computed \(\bar{n}_k(\tau) = \bar{n}_k(\tau) = \partial_k \bar{n}(\tau, x^i)\) and \(\bar{w}_i(\tau) = \partial_i \bar{A}(\tau) = \frac{\partial_i \bar{f} \bar{d}r}{\bar{w}_i \bar{h}_3(\tau)} = \frac{\partial_i \bar{y} \bar{d}r}{\bar{y} \bar{h}_3(\tau)} = \frac{\partial_i \bar{f} \bar{d}r}{\bar{y} \bar{h}_3(\tau)} = \frac{\partial_i \bar{f} \bar{d}r}{\bar{y} \bar{h}_3(\tau)}\). In result, we construct new classes of cosmological solutions for the KK theory and GR with electromagnetic interactions defined as subclasses of solutions (69) with quasiperiodic geometric flow evolution.

\[
dx^2 = e^{\Phi(\tau, x^i)}[(dx^1)^2 + (dx^2)^2]
+ \begin{cases}
\bar{h}_3 \left[ dy^3 + (\partial_3 \bar{n}) dx^3 \right] - \frac{\left( \bar{h}_3^* \bar{y} \right)^3}{\bar{y}} [dt + (\partial_3 \bar{y}) dx^3], & \text{gener. funct. } \bar{h}_3, \\
\text{or} & \\
\left( \bar{h}_3^{[0]} - \int \! dt \, \bar{y} \bar{h}_3^* \bar{y} \right) \left[ dy^3 + (\partial_3 \bar{n}) dx^3 \right] - \frac{\left( \bar{y} \bar{h}_3^* \bar{y} \right)^3}{4\bar{y}^2} \left[ dt + (\partial_3 \bar{y}) dx^3 \right], & \text{source } \bar{Y}, \text{ or } \bar{\Lambda};
\end{cases}
+ \begin{cases}
\left( \bar{h}_3^{[0]} - \frac{\bar{y}^3}{4\bar{y}} \right) \left[ dy^3 + (\partial_3 \bar{n}) dx^3 \right] - \frac{\left( \bar{y} \bar{h}_3^* \bar{y} \right)^3}{4\bar{y}^2} \left[ dt + (\partial_3 \bar{y}) dx^3 \right], & \text{gener. funct. } \bar{\Psi}, \\
\text{or} & \\
\left( \bar{h}_3^{[0]} - \frac{\bar{y}^3}{4\bar{y}} \right) \left[ dy^3 + (\partial_3 \bar{n}) dx^3 \right] - \frac{\left( \bar{y} \bar{h}_3^* \bar{y} \right)^3}{4\bar{y}^2} \left[ dt + (\partial_3 \bar{y}) dx^3 \right], & \text{effective } \bar{\Lambda} \text{ for } \bar{Y}.
\end{cases}
\]

Such cosmological metrics are generic off-diagonal and define new classes of solutions if the anholonomy coefficients are not zero for \(N^3_k(\tau) = \partial_k \bar{n}\) and \(N^4_k(\tau) = \partial_k \bar{A}\).

7 Exact solutions for KK quasiperiodic cosmological flows

Cosmological models with entropic and quasiperiodic flow evolution of locally anisotropic and inhomogeneous cosmological spacetimes are studied in details in Refs. [7,31] (for MGTs and entropic gravity see respective Table 3 in those works). The geometric formalism and technical results on exact solutions for KK quasiperiodic cosmological flows are summarized in Table 2.

7.1 The AFDM for generating cosmological flow solutions

We outline the key steps and results on application of the AFDM for generating cosmological solutions with geometric flows and Killing symmetry on \(\partial_3\). The nonholonomic deformation procedure is elaborated for geometric flow evolution of a generating function \(g_3(\tau) = \bar{h}_3(\tau, x^i, y^3)\) (67), a cosmological constant \(\bar{\Lambda}(\tau)\), and a source \(\bar{Y}(\tau) = \bar{Y}(\tau, x^k, t)\) (55). There are constructed exact solutions of the system of nonlinear PDEs for emergent cosmology (64) which can be typically parameterized in the form
To (70), we associate an additive cosmological constant with a generating function with Killing symmetry on \( \partial \). We can integrate the system of nonlinear PDEs (64) for cosmological quasi-periodic solutions determined by the above type effective sources and nonlinear symmetries. We obtain Table 2, we can integrate the system of nonlinear PDEs (64) for cosmological quasi-periodic solutions for entropic configurations are prescribed for generating functions subjected to nonlinear symmetries (65). We also note that it is possible to construct certain classes of locally anisotropic and inhomogeneous cosmological solutions using nonlinear/additive functionals both for generating functions and (effective) sources.

### 7.2 Cosmological metrics with entropic quasi-periodic flows

We analyze two possibilities to transform the geometric flow modified Einstein equations (56) into systems of nonlinear PDEs (60)–(63) for which generic off-diagonal or diagonal solutions depending in explicit form on an evolution parameter, a time-like variable and two space-like coordinates can be constructed. In the first case, there are considered entropic quasi-periodic sources determined by some additive or general nonlinear functionals for effective matter fields. In the second case, respective nonlinear functionals determining quasi-periodic solutions for entropic configurations are prescribed for generating functions subjected to nonlinear symmetries (65). We also note that it is possible to construct certain classes of locally anisotropic and inhomogeneous cosmological solutions using nonlinear/additive functionals.

#### 7.2.1 Cosmological flows generated by additive functionals for effective sources

We consider \( \tilde{\chi} \) and (\( \tau \)) to nonlinear symmetries (65). We also note that it is possible to construct certain classes of locally anisotropic and inhomogeneous cosmological solutions using nonlinear/additive functionals both for generating functions and (effective) sources.

#### 7.2.2 Generating additive cosmological constants for effective sources

We consider \( \chi \) and \( \tilde{\chi} \) in (54) when

\[
\tilde{\chi} = \tilde{\chi}(x^i, t)
\]

To (70), we associate an additive cosmological constant \( \tilde{\Lambda}(\tau) = \tilde{\Lambda}(\tau) + \tilde{\chi} \tilde{\Lambda}(\tau) \) corresponding to nonlinear symmetries (65) for any component \( \tilde{\chi}, \tilde{\chi}, \tilde{\chi} \), and \( \tilde{\chi} \tilde{\chi}, \tilde{\chi} \tilde{\chi} \), but one general generating functions. For such conditions, the equation (61) transforms into \( \tilde{\Lambda}(\tau) = 2\tilde{h}_3 \tilde{h}_4 \tilde{\Lambda}(\tau) \) and can be integrated on \( y^4 = t \). Following the procedure summarized in Table 2, we can integrate the system of nonlinear PDEs (64) for cosmological quasi-periodic flows determined by the above type effective sources and nonlinear symmetries. We obtain quadratic line elements

\[
ds^2 = e^{\psi(\tau, x^i)} [(dx^1)^2 + (dx^2)^2] + \tilde{h}_3(\tau)
\]

\[
\left[ dy^3 + (1n_k + 42n_k) \int dt \frac{[\tilde{h}_3^*(\tau)]^2}{[\tilde{h}_3(\tau)]^{5/2}} )dx^k \right] - \frac{[\tilde{h}_3^*(\tau)]^2}{[\tilde{h}_3(\tau)]^{5/2}} )dx^k \right] - \frac{[\tilde{h}_3^*(\tau)]^2}{[\tilde{h}_3(\tau)]^{5/2}} )dx^k \right] - \frac{[\tilde{h}_3^*(\tau)]^2}{[\tilde{h}_3(\tau)]^{5/2}} )dx^k \right]
\]

\[
\int dx^4 \int d^4 \tilde{\Lambda}(\tau, h^3) \tilde{h}_3^*(\tau) [\tilde{h}_3(\tau)]^{5/2} \right] \frac{\tilde{h}_3^*(\tau)}{\tilde{h}_3(\tau)} dx^k \right]
\]

\[
\int dt \frac{[\tilde{h}_3^*(\tau)]^2}{[\tilde{h}_3(\tau)]^{5/2}} )dx^k \right] - \frac{[\tilde{h}_3^*(\tau)]^2}{[\tilde{h}_3(\tau)]^{5/2}} )dx^k \right]
\]

We have to fix a sign of the coefficient \( \tilde{h}_3(\tau, x^k, t) \) which describes relativistic flow evolution with a generating function with Killing symmetry on \( \tilde{h}_3 \) determined by sources \( (\tilde{h}_3, \tilde{h}_3) \). Such entropic and quasi-periodic flow solutions are of type (69) and can...
### Table 2 Parameterization of quasiperiodic KK cosmological flows

Exact solutions of $R_{\mu\nu}(\tau) = \epsilon_{\mu\nu} \Psi_{\mu\nu}(\tau)$ (56) transformed into a system of nonlinear PDEs (60)–(63)  

| d-metric ansatz with | Nonlinear PDEs (64) |
|----------------------|---------------------|
| Killing symmetry $\partial_3 = \partial_\varphi$ | $\Psi^{\bullet} + \Psi^\prime = 2k^2 \hat{\Psi}(\tau)$; $\overline{\Psi} = \ln |\partial_\varphi \overline{h}_3|/\sqrt{\overline{h}_3 \overline{h}_4}$, $\overline{\Psi}_i = (\partial_\varphi \overline{h}_3)$ ($\partial_\varphi \overline{\Psi}$), $\overline{\Psi}_k^{\bullet} + \Psi \overline{\Psi}_k = 0$; for $\overline{\beta} = (\partial_\varphi \overline{h}_3)$ ($\partial_\varphi \overline{\Psi}$), $\overline{\gamma} = \overline{\partial}_i \left( \ln |\overline{h}_3|^{3/2}/\overline{h}_4 \right)$, $\overline{\beta} \overline{\Psi}_i - \overline{\alpha}_i = 0$; $\partial_1 q = q^\bullet$, $\partial_2 q = q'$, $\partial_3 q = \partial\varphi/\partial t = q^*$ |

| Effective matter sources | Generating functions: $h_3(\tau, x^k, t)$, $\overline{\Psi}(\tau, x^k, t) = e^{\overline{\Psi}}$, $\overline{\Theta}(\tau, x^k, t)$; & Nonlinear symmetries |
|--------------------------|--------------------------|
| $g_i = e^{\varphi(\tau, x^k)}$, $g_a = \overline{h}_a(\tau, x^k, t)$, $N_i^3 = \overline{h}_i(\tau, x^k, t)$, $N_i^4 = \overline{w}_i(\tau, x^k, t)$, $e^{\overline{\Psi}_i}(\tau) = \partial_3(\overline{\Psi}_i)$, $\overline{\Psi}_i^2$, $\overline{\Psi}_i^3$ $= 2k^2 \hat{\Psi}(\tau)$; $\overline{\Psi}_i = \ln |\partial_\varphi \overline{h}_3|/\sqrt{\overline{h}_3 \overline{h}_4}$, $\overline{\Psi}_i = (\partial_\varphi \overline{h}_3)$ ($\partial_\varphi \overline{\Psi}$), $\overline{\Psi}_k^{\bullet} + \Psi \overline{\Psi}_k = 0$; for $\overline{\beta} = (\partial_\varphi \overline{h}_3)$ ($\partial_\varphi \overline{\Psi}$), $\overline{\gamma} = \overline{\partial}_i \left( \ln |\overline{h}_3|^{3/2}/\overline{h}_4 \right)$, $\overline{\beta} \overline{\Psi}_i - \overline{\alpha}_i = 0$; $\partial_1 q = q^\bullet$, $\partial_2 q = q'$, $\partial_3 q = \partial\varphi/\partial t = q^*$ |
| Integration functions: $h_3^{[0]}(\tau, x^k)$, $1n_k(\tau, x^l)$, $2n_k(\tau, x^l)$; & Off-diag. solutions, d-metric N-connec. |
| LC-configurations (66) | |
| N-connections, zero torsion | |

**Note:** The expressions are written in a mathematical notation typical for cosmology and differential geometry.
polarization functions
\[ \hat{g} \rightarrow \hat{\Omega} = \hat{\eta}_{xk}, \hat{N}_1 \]
Prime metric defines a cosmological solution
Example of a prime cosmological metric
Solutions for polarization funct.
Polariz. funct. with zero torsion

\[ \dd s^2 = \hat{\eta}_1(\tau, x^k, t)\hat{g}_1(x^k, t)[dx^k]^2 + \hat{\eta}_3(\tau, x^k, t)\hat{h}_3(x^k, t)[dy^3]^2 + \]
\[ \hat{\eta}_1^3(\tau, x^k, t)\hat{N}^3_1(x^k, t)dx^j]^2 + \hat{\eta}_4(\tau, x^k, t)\hat{h}_4(x^k, t)[d\tau + \hat{\eta}_1^4(\tau, x^k, t)\hat{N}^4_1(x^k, t)dx^j]^2, \]
\[ [\hat{g}_1(x^k, t), \hat{g}_a = \hat{h}_a(x^k, t); \hat{N}_2 = \hat{w}_k(x^k, t), \hat{N}_k = \hat{n}_k(x^k, t)] \]
diagonalizable by frame/coordinate transforms.

\[ \hat{g}_1 = \frac{a^2(t)}{1+k^2}, \hat{g}_2 = a^2(t)r^2, \hat{h}_3 = a^2(t)r^2 \sin^2 \theta, \hat{h}_4 = e^2 = \text{const}, k = \pm 1, 0; \]
any frame transform of a FLRW or a Bianchi metrics
\[ \eta_i(\tau) = e^{\psi(\tau,x^i)}/\hat{g}_i; \eta_4(\tau) = -4[(\eta_3(\tau))/\hat{g}_4]^2/|\int d\tau \hat{Y}(\eta_3(\tau))|^2; \]
\[ \eta_3 = \eta_3(\tau, x^i, t) \text{ as a generating function}; \]
\[ \eta_1^3(\tau)\eta_1^3_k = n_k + 16n_k \int d\tau \left( (\eta_3(\tau))/\hat{g}_4 \right) \left( |\int d\tau \hat{Y}(\eta_3(\tau))|^2 \right); \eta_1^4(\tau)\eta_1^4_k = \int d\tau \hat{Y}(\eta_3(\tau))^*, \]
\[ \hat{g}_1(\tau) = e^{\psi(\tau,x^k)}/\hat{g}_1; \hat{g}_4(\tau) = -4[(\eta_3(\tau))/\hat{g}_4]^2/|\int d\tau \hat{Y}(\eta_3(\tau))|^2; \eta_3(\tau, x^i, t) \]
as a generating function; \[ \eta_1^4(\tau) = \partial_k \hat{A}(\tau, x^i, t)/\hat{w}_k; \eta_2^3(\tau) = (\partial_k \eta)/\hat{n}_k. \]
be rewritten equivalently with coefficients stated by other type functionals like \(^{ad}\) \(\Phi_1(\tau, x^i, t)\) and \(^{ad}\) \(\Psi_1(\tau, x^i, t)\).

Let us discuss how we can extract from off-diagonal d-metrics (72) certain cosmological LC-configurations when there are imposed additional zero torsion constraints (66). Such equations can be considered as some anholonomy conditions restricting the respective classes of generating functions \(\tilde{h}_3(\tau, x^i, t), \tilde{\Psi}_1(\tau, x^i, t)\) and/or \(\Phi_1(\tau, x^i, t)\) for \(\vec{n}(\tau, x^i), \vec{A}(\tau, x^i, t)\) and sources \(^{as}\) \(\vec{Y}(\tau) = \vec{Y}(\tau)\) with additive splitting of type (71) and \(^{as}\) \(\vec{A}(\tau)\) (70),

\[
\begin{align*}
\text{ds}^2 &= e^{\psi(\tau)}((\text{d}x^1)^2 + (\text{d}x^2)^2) + h_3(\tau) \\
&= e^{\psi(\tau)}(\text{d}y^3 + (\text{d}x^2)^2) + h_3(\tau) \\
&\quad - \frac{[h_3^*(\tau)]^2}{|\int \text{d}t^{qs} \tilde{Y}(\tau)(h_3^*(\tau))| h_3(\tau)}[\text{d}t + (\partial_t \vec{A}(\tau))\text{d}x^i].
\end{align*}
\]

(73)

In this subsection, we constructed two classes of generic flows off-diagonal cosmological metrics of type (72) and/or (73). Such solutions define off-diagonal cosmological flows generated by entropic quasiperiodic additive sources \(^{as}\) \(\vec{Y}(\tau)\) and/or \(^{as}\) \(\vec{Y}(\tau)\) when terms of type (71) encode and model, respectively, contributions of gravitational electromagnetic fields with certain nonholonomically emergent flows for conventional dark matter fields and effective entropic evolution sources. The type and values of such generating additive sources can be can be prescribed in some forms which are compatible with observational data of cosmological (and geometric/entropic) evolution for dark matter distributions. Such configurations describe the geometric evolution of possible quasiperiodic, aperiodic, pattern forming, solitonic nonlinear wave interactions.

### 7.2.2 Cosmological evolution for nonlinear entropic quasiperiodic functionals for sources

We can generate other classes of considering nonlinear quasiperiodic functionals for effective sources, \(^{qp}\) \(\vec{Y}(\tau) = \vec{Y}(\tau, x^i, t) = \vec{Y}(\tau)\) \(\int \text{d}t^{qs} \tilde{Y}(\tau)(h_3^*(\tau))\) (instead of additional functional dependencies in (71)) subjected to nonlinear symmetries (65). Following the AFDM, see Table 2, we construct geometric flow cosmological solutions of with nonlinear sources,

\[
\begin{align*}
\text{ds}^2 &= e^{\psi(\tau)}((\text{d}x^1)^2 + (\text{d}x^2)^2) + h_3(\tau) \\
&= e^{\psi(\tau)}(\text{d}y^3 + (\text{d}x^2)^2) + h_3(\tau) \\
&\quad - \frac{[h_3^*(\tau)]^2}{|\int \text{d}t^{qs} \tilde{Y}(\tau)(h_3^*(\tau))| h_3(\tau)}[\text{d}t + (\partial_t \vec{A}(\tau))\text{d}x^i].
\end{align*}
\]

(74)

The equations (66) for extracting LC-configurations are solved by d-metrics of type

\[
\begin{align*}
\text{ds}^2 &= e^{\psi(\tau)}((\text{d}x^1)^2 + (\text{d}x^2)^2) + h_3(\tau)[\text{d}y^3 + (\text{d}x^2)^2] \\
&\quad - \frac{[h_3^*(\tau)]^2}{|\int \text{d}t^{qs} \tilde{Y}(\tau)(h_3^*(\tau))| h_3(\tau)}[\text{d}t + (\partial_t \vec{A}(\tau))\text{d}x^i].
\end{align*}
\]

(75)

For additive functionals for cosmological entropic and quasiperiodic sources, the formulas (74) and (75) transform, respectively, into quadratic linear elements (72) and (73).

Finally, we note that additive and nonlinear functionals can be considered for generating functions

\[
\begin{align*}
^{a}\Phi_1(\tau) &= ^{a}\Phi_1(\tau, x^i, t) = \Phi_1(\tau, x^i, t) + ^{int}\Phi_1(\tau, x^i, t) \\
&\quad + ^{x}\Phi_1(\tau, x^i, t)\text{ or } ^{qp}\Phi_1(\tau) = ^{qp}\Phi_1(\tau, x^i, t) = ^{qp}\Phi_1(\tau, x^i, t)
\end{align*}
\]

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when the generation functions for sources are arbitrary ones or certain additive/nonlinear functionals. A number of such examples and respective cosmological inflation and accelerating cosmology scenarios were elaborated in our previous works \cite{7,18–21,30,31,65,66} for other types of Lagrangians and effective sources which are different from (53) and (54). All such locally anisotropic cosmological metrics in MGTs, geometric flow and GIF theories cannot be characterized by Bekenstein–Hawking or holographic-type entropies. In \cite{1–4,6,18,20}, we concluded that such classes of generalized cosmological and other type physically important solutions can be characterized by respective nonholonomic modifications of G. Perelman’s W-entropy and associated thermodynamical models.

7.3 Computing W-entropy and cosmological geometric flow thermodynamic values

In subsection, we show how G. Pereman’s W-entropy and associated thermodynamic models with extensions to QGIFs of KK systems with entanglement elaborated in Sects. 3–5 can be applied in modern accelerating cosmology with quasiperiodic distributions modeling dark energy and dark matter fields. For simplicity, we shall analyze an example of cosmological flows determined by \( q_3(\tau) = h_3(\tau, x^k, t) \) and \( ad\Lambda(\tau) \). Similar constructions can be performed for any type of cosmological solutions parameterized in Table 2.

7.3.1 An example of geometric thermodynamic model for cosmological geometric flows

Let us consider in explicit form how a GIF model can be elaborated for a cosmological KK system for 5-d metrics \( g_{\alpha\beta}(\tau) = (g_{\alpha\beta}(\tau), A_\gamma(\tau)) \) defined as a solution of generalized R. Hamilton equations in canonical variables (4) for an arbitrary normalization function \( \hat{f}(\tau) \). Such relativistic geometric flow equations can be rewritten equivalently in the form (5). We shall work in N-adapted frames when the d-metrics are parameterized in the form

\[
g_{\alpha\beta}(\tau) = \text{diag}[q_1(\tau), q_3(\tau), g_4(\tau) = -[q N(\tau)]^2, g_5 = 1],
\]

when the 4-d component of such a d-metric is of type (14) and determined by a family of locally anisotropic cosmological solutions of type (72), i.e., by certain data for d-metric coefficients

\[
q_1(\tau) = q_2(\tau) = e^{\psi(\tau, x^k)}, q_3(\tau) = h_3(\tau, x^k, t) \text{ is a generating function } ; [q N(\tau)]^2 = \frac{[h_3^*(\tau)]^2}{\int dy^4 ad\Lambda(\tau)h_3^*(\tau) | h_3(\tau)},
\]

where \( \psi \) is a solution of parametric 2-d Poisson equation with source \( \hat{Y} \), see also formulas for nonlinear symmetries (65), and N-connection coefficients

\[
n_k(\tau) = 1n_k(\tau, x^i) + 42n_k(\tau, x^i) \int dt \frac{[h_3^*(\tau)]^2}{\int dy^4 ad\Lambda(\tau)h_3^*(\tau) | h_3(\tau)^{5/2}},
\]

\[
w_k(\tau) = \frac{\partial_i(\int dt \frac{ad\Lambda(\tau)h_3^*(\tau)}{ad\Lambda(\tau)h_3^*(\tau) | h_3(\tau)})}{ad\Lambda(\tau)h_3^*(\tau)},
\]

where contributions from electromagnetic fields, possible quasiperiodic configurations, emergent gravity effects are encoded via nonlinear symmetries into \( ad\Lambda(\tau) \) and data for generating sources \( \hat{Y}(\tau), ad\hat{Y}(\tau) \). Such data determine cosmological flow solutions of type (69) and can be rewritten equivalently with coefficients stated by other type functionals of the form \( ad\Phi(\tau, x^i, t) \) and \( ad\Psi(\tau, x^i, t) \).
The W-entropy (19) can be written in terms of canonical d-connection in 4-d, $\hat{D}$, for a corresponding class of normalizing functions $f = \hat{f}(u)f(u^5)$.

$$\mathcal{W} = \int \mu \sqrt{|g|} d^5u [\tau(sR + |hDf| + |vDf|) + f - 5]$$

$$= \int \hat{\mu} \sqrt{|\hat{g}|} \delta^4u u^5 [\tau(s\hat{R} + |f(u^5)\hat{D}f| + |f(u^5)\hat{D}f|) + \hat{f}f(u^5) - 5].$$

The normalizing functions satisfy the condition $\int \mu \sqrt{|g|} d^5u = \int_0^{\tau_0} \int \mu \sqrt{|g|} d^5u = 1$, where $\mu = (4\pi \tau)^{-5/2} e^{-f} = \hat{\mu} e^{-\hat{f}(u^5)}$, $\hat{\mu} = (4\pi \tau)^{-5/2} e^{-\hat{f}}$. Such a function determine the geometric normalization for a nonlinear diffusion process on a temperature-like parameter $\tau$ when relativistic dynamics for any fixed $\tau_0$ is a cosmological KK type. A parametrization $f = \hat{f}(u)f(u^5)$ can be chosen for respective cosmological models if there are considered certain experimental data. Here, we note that the integration volume is of type

$$d^5Vol[g(\tau)] := \sqrt{|g|} \delta^4u = e^{\psi(\tau,x^k)} \sqrt{h_{33}} q N(\tau) dx^1 dx^2 [dy^3 + n_k(\tau) dx^4]$$

$$[dt + w_k(\tau) dx^2] \quad (77)$$

for the functions $\psi$, $h_{33}$, $q N$ and $n_k$, $w_k$ (with $k = 1, 2$) are determined by the coefficients in (72).

The thermodynamic generating function (25) for the above assumptions on this class of cosmological flow solutions is computed

$$\hat{\Gamma}[g(\tau)] = \int (4\pi \tau)^{-5/2} e^{-\hat{f}} e^{-f(u^5)} d^5Vol[\hat{f}(u)f(u^5) + 5/2] \quad \text{for} \ V. \quad (78)$$

In this subsection, we underline symbols in order to emphasize their KK geometric nature and put “hats” in order to emphasize that the solutions and computations are performed for the canonical d-connection. For such d-metrics, $sR = s\hat{R} = 4^{ad\alpha} \hat{\Lambda}(\tau)$, and $R_{\alpha\beta} = a^{ad\Lambda}(\tau) g_{\alpha\beta}$.

where $h \hat{Y} = h \Lambda(\tau) = a^{ad\Lambda}(\tau)$ is approximated to a running flow constant $\tau$ under assumption for the same normalization function $f$ (for simplicity, chosen to satisfy the conditions $D_\beta \hat{f} = \hat{D}_\beta f = 0$) on cosmological flows on all directions on $V$. In result, we can compute the thermodynamic values for geometric/cosmological evolution flows of KK systems (27) determined by the class of cosmological solutions

$$\hat{\Gamma} = -\tau^2 \int (4\pi \tau)^{-5/2} e^{-\hat{f}} e^{-f(u^5)} d^5Vol \left(sR + |Df|^2 - \frac{5}{2\tau}\right) = \quad (79)$$

$$-\tau^2 \int (4\pi \tau)^{-5/2} e^{-\hat{f}} e^{-f(u^5)} d^5Vol \left[4^{ad\Lambda}(\tau) - \frac{5}{2\tau}\right],$$

$$\tilde{\Gamma} = - \int (4\pi \tau)^{-5/2} e^{-\hat{f}} e^{-f(u^5)} d^5Vol \left[\tau(sR + |Df|^2) + f - 5\right] =$$

$$- \int (4\pi \tau)^{-5/2} e^{-\hat{f}} e^{-f(u^5)} d^5Vol \left[4^{ad\Lambda}(\tau) + \hat{f}f(u^5) - 5\right].$$

$$\hat{\eta} = 2\tau^4 \int (4\pi \tau)^{-5/2} e^{-\hat{f}} e^{-f(u^5)} d^5Vol \left[|R_{\alpha\beta} + D_{\alpha} \hat{D}_\beta f - \frac{1}{2\tau} g_{\alpha\beta}|^2\right] =$$

$$2\tau^4 \int (4\pi \tau)^{-5/2} e^{-\hat{f}} e^{-f(u^5)} d^5Vol \left[(a^{ad\Lambda}(\tau) - \frac{1}{2\tau}) g_{\alpha\beta}\right], \quad (80)$$

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where $g_{a\beta}$ is a d-metric (76). Using such values, we can compute the free energy and relative entropy (23),

$$\hat{F}(\mathbf{g}) = \hat{E}(\mathbf{g}) - \beta^{-1}\hat{S}(\mathbf{g})$$

and $\hat{S}(\mathbf{g}) = \tau^{-1}[\hat{F}(\mathbf{g}) - \hat{F}(\mathbf{g})]$, where

$$\hat{E}(\mathbf{g}) = -\tau^2 \int (4\pi\tau)^{-5/2} e^{-\tilde{f}} e^{-f(u^5)} d^5\text{Vol}_1[4\Lambda(\tau) - 5/2]$$

and

$$\hat{S}(\mathbf{g}) = -\int (4\pi\tau)^{-5/2} e^{-\tilde{f}} e^{-f(u^5)} d^5\text{Vol}_1[4\Lambda(\tau) + \tilde{f}_1 f_1(u^5) - 5]$$

are computed for respective functionals $d^5\text{Vol}[\mathbf{g}(\tau), \tau, f_1]$ and $d^5\text{Vol}[\mathbf{g}(\tau), \tau, f]$, with respective effective running cosmological constants, $\Lambda(\tau)$ and $\Lambda(\tau)$, and different normalizing functions, $f_1$ and $f$.

Conventionally, such types of thermodynamic systems for cosmological flow evolution are of type $A = [\hat{Z}, \hat{E}, \hat{S}, \hat{\eta}]$ (27) when the geometric thermodynamic data are determined by (78) and respective (79) and (80), i.e., $\hat{A} = [\hat{Z}, \hat{E}, \hat{S}, \hat{\eta}]$, when all integrals are determined for volume forms of type (77).

**7.3.2 An example of QGIF for a locally anisotropic cosmological solution**

We can consider a conventional state density (22) as a density matrix for quantum models using formula $\sigma(\tau, \mathbf{E}) = Z^{-1} e^{-\tau^{-1}\hat{E}}$. In this subsection, we take $Z = \hat{Z}[\mathbf{g}(\tau)]$ (78) following our previous constructions are possible for the KK QGIF theory with a thermodynamic generating function $\hat{Z}[\mathbf{g}(\tau)](25)$ and related statistical density $\rho(\beta, \hat{E}, \mathbf{g})$ (22) (such values be used for defining $\rho_A(35)$ as a probability distribution).

For the class of cosmological flows with solutions (72), we can associate a quantum system $A$ when the density matrix

$$\rho_A := \hat{Z}^{-1} e^{-\tau^{-1}\hat{E}}$$

is determined by values $\hat{Z}[\mathbf{g}(\tau)]$ (78) and $\hat{Z}[\mathbf{g}(\tau)]$ (79) (fixing normalization and integration functions and constants, we can approximate $\hat{Z}[\mathbf{g}(\tau)] \approx \mathbf{E} = \text{const}$). In result, we can compute the entanglement entropy (39) for such a class of cosmological solutions

$$q\hat{S}(\rho_A) := Tr(\rho_A \log \rho_A),$$

when $\rho_A$ is computed using formulas (81). This entanglement entropy $q\hat{S}(\rho_A)$ is a QGIF version of the G. Perelman thermodynamic entropy $\hat{S} (80)$.

This class of cosmological flows are characterized also by relative KK GIF entropy and monotonocity properties, see formulas (31) and (32). They involve the thermodynamic generating function (generalization of (25)), $ABC \hat{Z}[\mathbf{g}(\tau), \mathbf{g}(\tau), \mathbf{g}(\tau)] = \int_1 \int_2 \int_3 (4\pi\tau)^{-15/2} e^{-\tau f} d^5\text{Vol}[\mathbf{g}(\tau)]d^5\text{Vol}[\mathbf{g}(\tau)]d^5\text{Vol}[\mathbf{g}(\tau)]\tau\mathbf{f} + 15$, for $\mathbf{V} \otimes \mathbf{V} \otimes \mathbf{V}$, with volume forms of type (35) with a normalizing function $\mathbf{ABC} f(\mathbf{u}, \mathbf{u}, 2\mathbf{u})$. On such tensor products of KK cosmological manifolds, their flow evolution is of type $ABC \mathbf{g} = [\mathbf{g}, q_1, q_2, q_3 = \tilde{h}_3(\mathbf{x}, \mathbf{t}), q_5 N, q_5 = 1], 1\mathbf{g} = [q_1, q_2, q_3 = \tilde{h}_3(\mathbf{x}, \mathbf{t}), \mathbf{t}, q_5 N, q_5 = 1], 2\mathbf{g} = [2q_1, 2q_2, 2q_3 = \tilde{h}_3(\mathbf{x}, \mathbf{t}, \mathbf{t}), 2q_5 N, 2q_5 = 1]$, where generating functions $\tilde{h}_3(\mathbf{x}, \mathbf{t}), \tilde{h}_3(\mathbf{x}, \mathbf{t})$ and $\tilde{h}_3(\mathbf{x}, \mathbf{t})$. For respective tensor products, we can consider a canonical d-connection $ABC \mathbf{D} = \mathbf{D}+B \mathbf{D}+C \mathbf{D}$ and respective scalar curvature $s_{ABC} \mathbf{R} = s \mathbf{R}+s_1 \mathbf{R}+s_2 \mathbf{R}$. 

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The resulting entropy function for such a three partite system can be computed for similar assumption as we introduced for (80) (on any term of the tensor product)

\[
ABC \hat{S} = \mathcal{S}[\hat{A}, \hat{B}, \hat{C}]
\]

\[
= -\int (4\pi \tau)^{-15/2} e^{-\tau} \int_{\tau}^{\tau} d\tau \int_{\tau}^{\tau} d\tau \int_{\tau}^{\tau} d\tau \left[ (s \hat{R} + s_1 \hat{R} + s_2 \hat{R}) + \frac{ABC \hat{f}}{2} \right],
\]

where

\[
\begin{align*}
\tau (\ldots) + ABC \hat{f} - 15 & \approx [4\tau (ad \Lambda_1(\tau) + ad \Lambda_1(\tau) + ad \Lambda_2(\tau)) + ABC \hat{f} - 15].
\end{align*}
\]

We can compute the Rényi entropy (48) using the replica method with \( \hat{\rho}_A (81) \). For an integer replica parameter \( r \) (replica parameter) the Rényi entropy for the mentioned class of cosmological flow solutions (5),

\[
r \hat{S}(\hat{A}) := \frac{1}{1 - r} \log [tr_A(\hat{\rho}_A)^r]
\]

for a KK QGIF system determined by the matrix \( \hat{\rho}_A \) and associated thermodynamic model \( \hat{A} = [\hat{Z}, \hat{E}, \hat{S}, \hat{\eta}] \). We can consider the same computational formalism elaborated for an analytic continuation of \( r \) to a real number with a well-defined limit \( q \hat{S}(\hat{\rho}_A) = \lim_{r \to 1} r \hat{S}(\hat{A}) \) and normalization \( tr_A(\hat{\rho}_A) \) for \( r \to 1 \). For such limits and cosmological flow d-metrics, the Rényi entropy (83) reduces to the entanglement entropy (82).

Finally, we note that the volume forms (77) can be considered for certain prime metrics (in particular, for the Minkowski 4-d and extended to 5-d metric) when all classical and quantum GIF integrals are well defined. In result, we can always define a self-consistent QGIF model with quasiclassical limits for which the GIF theory can be with classical relativistic and nonlinear evolution. All above-considered classical and quantum thermodynamic values depend on respective classes of normalizing and integration and generating functions. Explicit numerical values can be obtained by fixing such functions in order to reproduce certain observational cosmological data or to model them with a corresponding quantum computing calculus.

8 Summary and conclusions

In this paper, we develop our research on emergent gravity and matter field interactions and entanglement in the framework of the theory of quantum geometric information flows (QGIFs). It is a partner in a series of works [1–4,6,7] on geometric evolution theories of classical and quantum relativistic mechanical systems and modified gravity theories, MGTs. We show how (modified) Einstein–Maxwell, EM, and Kaluza–Klein, KK, theories can be derived using a generalized Ricci flow formalism. It should be emphasized that the KK geometric formulation in canonical nonholonomic and analogous Hamilton variables allows new applications in geometric information flow, GIF, theories and for elaborating quantum mechanical models.

We first introduce the geometric flow equations for EM and KK systems and show how such constructions can be performed in nonholonomic, canonical and analogous (mechanical-like) Lagrange–Hamilton variables. Here, we note that the KK gravity models can be derived as respective classes of nonholonomic Ricci soliton configurations. This is consistent with our former results on relativistic geometric flows and (non) commutative and/or supersymmetric
MGTs [1,18,20,21]. The geometric flow and (quantum) information methods are motivated also by a renewed interest in modern literature on Ricci–Yang–Mill gradient flows and gravity [22–27].

We then postulate certain classes of nonholonomic deformations of G. Perelman F- and W-functionals [11] which allow us to prove the geometric flow equations for KK theories (i.e., generalizations of R. Hamilton and D. Friedan equations originally proposed for Riemannian metrics and Levi-Civita connections in [10,12]). It has been observed earlier [1,7,19] and supported by new findings in this work that the concept of W-entropy is more general than those considered for solutions with area horizon and holographic-type configurations using the standard approach involving the Bekenstein–Hawking entropy and thermodynamics [35–38]. We elaborate on statistical and geometric thermodynamic models for KK flows using covariant variables and Hamilton variables for G. Perelman KK-functionals. This allows us to develop certain alternative approaches to geometric flows and MGTs which are different from the last two decades results on emergent gravity, entanglement and information theory [8,9,39–45,62].

In this article, we formulated also an approach to the theory of classical GIFs and QGIFs of KK systems. The standard concepts and methods of information theory and quantum physics and gravity [33,34,47,48,50–55,60] are generalized for the Shannon/von Neumann/conditional/relative entropy determined for thermodynamic generating functions and density matrices encoding KK GIFs and G. Perelman W-entropy. The concept of entanglement and main properties (inequalities) is formulated and studied for new classes of theories of QGIFs for KK systems.

We proved that the geometric flow equations for KK cosmological systems can be decoupled and integrated in general forms for generating functions and sources with quasiperiodic structure. There are studied various examples of cosmological evolution flows with additive and nonlinear functionals for effective quasiperiodic electromagnetic and other type sources. Such new classes of cosmological solutions are characterized by nonlinear symmetries, but cannot be described using the concept of Bekenstein–Hawking entropy. We show how cosmological KK flows can be characterized by G. Perelman’s W-entropy and associated thermodynamic potentials and provide explicit examples for computing such values and constructing cosmological QGIFs.

Finally, we note that further developments of our approach will involve geometric information flows of noncommutative and nonassociative gravity and quantum field theories involving generalized classes of exact solutions in MGTs and quantum gravity models, see certain preliminary results in [16,19,30,31].

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