TEMPERED GENUINE REPRESENTATIONS OF METAPLECTIC GROUPS

ZHE LI AND SHANWEN WANG

Abstract. In this note, we show that the metaplectic theta correspondence is compatible with the tempered condition by directly estimating the matrix coefficients, without using the classification theorem.

1. Introduction

Throughout this article, we fix an additive character \( \psi \) of \( \mathbb{R} \). Let \( dx \) be the unique Haar measure on \( \mathbb{R} \) which is selfdual for Fourier transformation with respect to \( \psi \). Unless we explicitly mention the contrary, by a representation, we always mean a unitary Casselman-Wallach representation of finite length (Fréchet representation of moderate growth), cf. [15, Chapter XII]. The inner product on a representation is denoted by \( (\cdot, \cdot) \). Let \( \pi \) be a representation. We denote by \( \pi^\vee \) the space of continuous linear functionals on \( \pi \) and it is given the strong topology (uniform convergence on bounded subsets). The smooth dual of \( \pi \), i.e. the subspace of smooth vectors in \( \pi^\vee \), is identified with \( \pi \).

Let \((G, G')\) be a reductive dual pair in \( \text{Sp}_{2m}(\mathbb{R}) \). Let \( \hat{G} \) and \( \hat{G}' \) be the inverse images of \( G \) and \( G' \) in the metaplectic double cover group \( \hat{\text{Sp}}_{2m}(\mathbb{R}) \) by the covering map. If \( \pi \) and \( \pi' \) are irreducible admissible representations of \( \hat{G} \) and \( \hat{G}' \) respectively, we say \( \pi \) and \( \pi' \) correspond if \( \pi \otimes \pi' \) is a quotient of the Weil representation \( \omega \) of \( \hat{\text{Sp}}_{2m}(\mathbb{R}) \), restricted to \( \hat{G} \times \hat{G}' \). Note that the Weil representation is not a representation by our convention as it is not of finite length.

Let \( W \) be a \( 2n \)-dimensional real symplectic vector space and let \( V \) be a real quadratic space of dimension \( 2n + 1 \) with discriminant
\[
\text{disc}(V) = (-1)^n \det(V) \equiv 1 \in \mathbb{R}^\times / \mathbb{R}^{\times 2}.
\]
The space \((W \otimes V, \langle - , - \rangle_W \otimes \langle - , - \rangle_V)\) is a real symplectic space. We have a natural homomorphism
\[
\hat{\text{Sp}}(W) \times O(V) \to \hat{\text{Sp}}(W \otimes V).
\]
We denote by \( H(W \otimes V) = (W \otimes V) \ltimes \mathbb{R} \) the Heisenberg group associated to the symplectic space \( W \otimes V \). Let \( \omega_\psi \) be the Weil representation of \( \hat{\text{Sp}}(W \otimes V) \ltimes H(W \otimes V) \) associated to \( W \otimes V \). We denote by \( \omega_{W,V,\psi} \) the representation of \( \hat{\text{Sp}}(W) \times O(V) \) by pulling back the Weil representation \( \omega_\psi \) by the homomorphism (1.1).

\[\text{2020 Mathematics Subject Classification.} \ 22E46, 22E50.\]
\[\text{Key words and phrases.} \ \text{Tempered representations, Theta correspondence, Metaplectic groups, Weil representations.} \]
\[\text{Shanwen Wang is supported by the Fundamental Research Funds for the Central Universities, the Research Funds of Renmin University of China No.20200030251 and The National Natural Science Foundation of China (Grant No.2020010209).} \]
Given an irreducible representation $\sigma$ of $O(V)$, the maximal isotropic quotient of $\omega_{W,V,\psi}$ has the form $\sigma \boxtimes \Theta_{W,V,\psi}(\sigma)$ for some smooth representation $\Theta_{W,V,\psi}(\sigma)$ of $\hat{Sp}(W)$, which is either 0 or of finite length. Let $\theta_{W,V,\psi}(\sigma)$ be the maximal semi-simple quotient of $\Theta_{W,V,\psi}(\sigma)$. It is known by Howe [7] that $\theta_{W,V,\psi}(\sigma)$ is either zero or irreducible. Similarly, if $\pi$ is an irreducible genuine representation of $\hat{Sp}(W)$, we have the representations $\Theta_{W,V,\psi}(\pi)$ and $\theta_{W,V,\psi}(\pi)$ of $O(V)$.

Let $\text{Rep}_{\psi}^{\text{gen}}(\hat{Sp}(W))$ be the set of irreducible genuine representations $\pi_W$ of $\hat{Sp}(W)$. Let $S_{2n+1}$ be the set of isomorphism classes of real orthogonal spaces $V'$ with $\dim V' = 2n + 1$ and $\text{disc}(V') \equiv 1$. Let $\text{Rep}_{\psi}^{\text{irr}}(SO(V'))$ be the set of irreducible representations of $SO(V')$ with $V' \in S_{2n+1}$. Adam and Barbasch show that the dual pair $(Sp(W)), (O(p,q))$ with $p + q = 2n + 1$ gives rise to a bijection between the genuine representations of metaplectic group and the representations of odd special group of the same rank.

**Theorem 1.1.** [1] There is a bijection given by the metaplectic theta correspondence:

$$\text{Rep}_{\psi}^{\text{gen}}(\hat{Sp}(W)) \leftrightarrow \bigoplus_{V' \in S_{2n+1}} \text{Rep}_{\psi}^{\text{irr}}(SO(V')).$$

More precisely, given an irreducible genuine representation $\pi$ of $\hat{Sp}(W)$, there is a unique $V' \in S_{2n+1}$ such that $\theta_{W,V',\psi}(\pi) \in \text{Rep}_{\psi}^{\text{irr}}(SO(V'))$ is nonzero.

Among the (genuine) irreducible representations, there is an important class of representations, whose matrix coefficients are controlled by the Harish-Chandra $\Xi$ function. We call such a representation the tempered (genuine) representation. The classification of tempered representations is given in [9], and in particular, one knows that an irreducible representation is tempered if and only if it is an irreducible parabolic induction of limit of discrete series (cf. [9, theorem 14.2]). Adams and Barbasch explicitly determined the $K$-types of all the representations on both side of the metaplectic theta correspondence. Together with the classification of the irreducible tempered representations, one can deduce that the metaplectic theta correspondence is compatible with the tempered condition. More precisely, let $\text{Temp}_{\psi}^{\text{gen}}(\hat{Sp}(W)) \subset \text{Rep}_{\psi}^{\text{gen}}(\hat{Sp}(W))$ be the subset of irreducible genuine representations and let $\text{Temp}_{\psi}^{\text{irr}}(SO(V')) \subset \text{Rep}_{\psi}^{\text{irr}}(SO(V'))$ be the subset of irreducible tempered representations of $SO(V')$ with $V' \in S_{2n+1}$, then we have

**Theorem 1.2.** There is a bijection given by the metaplectic theta correspondence:

$$\text{Temp}_{\psi}^{\text{gen}}(\hat{Sp}(W)) \leftrightarrow \bigoplus_{V' \in S_{2n+1}} \text{Temp}_{\psi}^{\text{irr}}(SO(V')).$$

More precisely, given an irreducible tempered genuine representation $\pi$ of $\hat{Sp}(W)$, there is a unique $V' \in S_{2n+1}$ such that $\theta_{W,V',\psi}(\pi) \in \text{Temp}_{\psi}^{\text{irr}}(SO(V'))$ is nonzero.

The main purpose of this article is to prove the theorems[1,2] by directly estimating the matrix coefficient, without using the classification theorem. This may be known to the experts, but we could not find a literature for the proofs for this approach and thus we insist to write this note. We will only give the details for the estimation of $\theta_{W,V',\psi}(\pi)$ in this article, for $\pi \in \text{Temp}_{\psi}^{\text{gen}}(\hat{Sp}(W))$. If we start with an irreducible tempered representations $\sigma$ of $SO(V)$, the same strategy shows that $\theta_{W,V,\psi}(\sigma)$ is tempered.
Acknowledgement: This note is based on a discussion with Hang Xue. The authors would like to express their gratitude to Hang Xue for explaining his works on unitary groups to us. The second author would like to thank Wenwei Li and Fan Gao for useful discussions on representations of metaplectic groups.

2. Tempered genuine representations of metaplectic groups

For a real reductive group $G$, Harish-Chandra defined a special spherical function $\Xi^G$ on $G(\mathbb{R})$, which can be used to control the growth of $C^\infty$-functions on $G(\mathbb{R})$ with values in $\mathbb{C}$. We recall briefly its definition and some useful results.

We denote by $C^\infty(G(\mathbb{R}))$ the space of all complex-valued $C^\infty$-functions on $G(\mathbb{R})$. Let $P_{min}$ be a minimal parabolic subgroup of a real reductive group $G$ with modulus character $\delta_{P_{min}}$ and $K$ be a maximal compact subgroup of $G$. Consider the normalized smooth induced representation

$$i^G_{P_{min}}(1)^\infty := \{f \in C^\infty(G(\mathbb{R})): f(pg) = \delta_{min}(p)^{1/2} f(g), \forall p \in P_{min}(\mathbb{R}), g \in G(\mathbb{R})\}$$

equipped with the scalar product

$$(f, f') = \int_K f(k) f'(k) dk, \forall f, f' \in i^G_{P_{min}}(1)^\infty.$$ 

Let $e_K \in i^G_{P_{min}}(1)^\infty$ be the unique function such that $e_K(k) = 1$ for all $k \in K$. Then the Harish-Chandra spherical function $\Xi^G$ is defined by

$$\Xi^G(g) = (i^G_{P_{min}}(1)^\infty(g)e_K, e_K), \forall g \in G(\mathbb{R}).$$

Note that if $f$ and $g$ are positive functions on a set $X$, we will say $f$ is essential bounded by $g$, if there exists a $c > 0$ such that $f(x) \leq cg(x)$ for all $x \in X$. We will denote it by $f \ll g$. We say $f$ and $g$ are equivalent if $f$ is essentially bounded by $g$ and $g$ is essentially bounded by $f$. The function $\Xi^G$ is a bi-$K$-invariant function and it is independent of the choice of the maximal compact subgroup $K$ up to equivalence.

Let $A_G$ be the maximal $\mathbb{R}$-split torus of $G$ of rank $r$, and $M_{min}$ be the centralizer of $A_G$ in $G$, which is exactly the Levi factor of $P_{min}$. We denote by $\Delta = R(A_G, P_{min})$ the set of roots of $A_G$ in the unipotent radical $U_{min}$ of $P_{min}$. Set

$$A_G^+ = \{a \in A_G(\mathbb{R}) : |\alpha(a)| \leq 1, \forall \alpha \in \Delta\}$$

$$(a_1, \cdots, a_r) : 0 < |a_1| \leq |a_2| \leq \cdots \leq |a_r| \leq 1\}.$$ 

Fix an embedding $\iota : G(\mathbb{R}) \to GL_m(\mathbb{R})$, we define the height function

$$\sigma(g) = 1 + \sup \{\log |a_{i,j}|, \log |b_{i,j}|\},$$

where $(a_{i,j})$ is the matrix $\iota(g)$ and $(b_{i,j})$ is the corresponding matrix of $\iota(g^{-1})$. In particular, if $a = (a_1, \cdots, a_r) \in A_G^+$, we have

$$\sigma(a) = 1 - \log |a_1| \geq 1.$$ 

We have the following well-known estimation of $\Xi^G$ due to Harish-Chandra.

Lemma 2.1. [14 theorem 30] There exists constants $A, B > 0$ such that for any $a \in A_G^+$, we have

$$A^{-1} \delta_{P_{min}}^{1/2}(a) \leq \Xi^G(a) \leq A \delta_{P_{min}}^{1/2}(a) \sigma(a)^B.$$
The metaplectic group \( \hat{\text{Sp}}_{2n} \) is not an algebraic group, but behaves in many way like an algebraic group. In particular, we have the Cartan decomposition for \( \hat{\text{Sp}}_{2n} \), i.e. \( \hat{\text{Sp}}_{2n} = KA_0^+K \), where \( K \) is the inverse image of a special maximal compact subgroup of \( \text{Sp}_{2n} \) and \( A_0^+ \) is the inverse image of \( A_{\text{Sp}}^{+} \) in \( \hat{\text{Sp}}_{2n} \). We define the corresponding Harish-Chandra spherical function by \( \Xi^{\hat{\text{Sp}}_{2n}} = \Xi^{\text{Sp}_{2n}} \circ p \), where \( p \) is the covering map.

Using Harish-Chandra’s \( \Xi \)-function, we have the following definition of tempered representation for real reductive groups and metaplectic groups.

**Definition 2.2.** We say that a unitary representation \((\pi, \mathcal{H}_\pi)\) of a real reductive group or a metaplectic group \( G \) is tempered if for any \( e, e' \in \pi \), we have an inequality

\[
|\langle \pi(g)e, e' \rangle| \ll \Xi^{G}(g), \forall g \in G(\mathbb{R}).
\]

Thanks to the work of Cowling, Haagerup and Howe \([2]\), a representation of \( G \) is tempered if and only if its matrix coefficients are almost square integrable functions (i.e. it belongs to \( L^{2+r}(G(\mathbb{R})) \) for all \( r \in \mathbb{R}_{>0} \)).

### 3. Metaplectic Theta Correspondence

Let \( W \) be a \( 2n \)-dimensional real symplectic vector space. Let \( S_{2n+1} \) be the set of isomorphism classes of real orthogonal spaces \( V' \) with \( \dim V' = 2n + 1 \) and \( \text{disc}(V') \equiv 1 \). Note that the metaplectic theta correspondence is established by Adams and Barbasch \([1]\).

**Theorem 3.1.** \([1]\) There is a bijection given by the metaplectic theta correspondence:

\[
\text{Rep}^{\text{gen}}_\psi(\hat{\text{Sp}}(W)) \leftrightarrow \bigotimes_{V' \in S_{2n+1}} \text{Rep}^{\text{irr}}_\psi(\text{SO}(V')).
\]

More precisely, given an irreducible genuine representation \( \pi \) of \( \hat{\text{Sp}}(W) \), there is a unique \( V' \in S_{2n+1} \) such that \( \theta_{W,V',\psi}(\pi) \in \text{Rep}^{\text{irr}}_\psi(\text{SO}(V')) \) is nonzero.

In this section, we recall the explicit metaplectic theta correspondence.

#### 3.1. Estimation of matrix coefficients.

**3.1.1. Tempered representations of real reductive groups.** Let \( G \) be a real reductive group and \( A_G \) be the maximal \( \mathbb{R} \)-split torus of \( G \) with rank \( r \) (i.e. \( A_G \cong (\mathbb{R}^\times)^r \)). We will write an element \( a \in A_G(\mathbb{R}) \) by \((a_1, \cdots, a_r)\). Define

\[
A_G^+ = \{ a \in A_G(\mathbb{R}) : 0 < |a_1| \leq \cdots \leq |a_r| \leq 1 \}.
\]

Fix a minimal parabolic subgroup \( P_G \supset A_G \) of \( G \). We denote by \( \delta_{P_G} \) the modulus character of \( P_G \).

We fix a special maximal compact subgroup \( K \) of \( G(\mathbb{R}) \) and we have a Cartan decomposition of \( G(\mathbb{R}) \):

\[
G(\mathbb{R}) = KA_G^+K.
\]

For any integrable function \( f \) on \( G(\mathbb{R}) \), the following formula holds (cf. \([3]\) §4):

\[
(3.1) \quad \int_{G(\mathbb{R})} f(g)dg = \int_{A_G^+} \nu(a) \int_{K \times K} f(k_1ak_2)dk_1dk_2da,
\]

where \( \nu \) is a positive function on \( A_G^+ \) such that \( \nu(a) \leq C \cdot \delta_{P_G}^{-1}(a) \) for some constant \( C \).
Let $\pi$ be a tempered representation of $G$. For any $v, v' \in \pi$ and $g \in G(\mathbb{R})$, by definition of tempered representation, there exists a constant $A_1 > 0$, such that

$$|\langle \pi(g)v, v' \rangle| \leq A_1 \cdot \Xi^G(g).$$

Moreover, a more precise estimation is given by Sun [12]: there is a continuous seminorm $\nu_\pi$ on $\pi$ such that

$$|\langle \pi(g)v, v' \rangle| \leq \Xi^G(g)\nu_\pi(v)\nu_\pi(v'), \forall v, v' \in \pi.$$  \hspace{1cm} (3.2)

We deduce, from the lemma 2.1 and the fact that the Harish-Chandra function $\Xi^G$ is bi-$K$-invariant, that there exists two positive constants $A_2$ and $B$ such that for any $g = k_1ak_2 \in KA^+_GK$, we have

$$\Xi^G(g) = \Xi^G(k_1ak_2) = \Xi^G(a) \leq A_2 \delta^{1/2}_{P_G}(a)\sigma(a)^B.$$  \hspace{1cm} (3.3)

Thus, for any $g = k_1ak_2 \in KA^+_GK$, there exists two positive constants $A$ and $B$ such that

$$|\langle \pi(g)v, v' \rangle| \leq A \delta^{1/2}_{P_G}(a)\sigma(a)^B.$$  \hspace{1cm} (3.4)

Similary, if $\pi$ is a tempered genuine representation of $\widehat{\text{Sp}}(W)$, then for any element $\hat{g} = \hat{k}\hat{a}\hat{k}' \in \widehat{\text{Sp}}(W)$ and $v, v' \in \pi$, there exists constants $A$ and $B$ such that

$$|\langle \pi(\hat{g})v, v' \rangle| \leq A \cdot \Xi^{\widehat{\text{Sp}}(W)}(\hat{g}) \leq A\delta^{1/2}_{P_G}(p(\hat{a}))\sigma(p(\hat{a}))^B.$$  \hspace{1cm} (3.5)

3.1.2. Weil representation. Let $W$ be a real symplectic space of dimension $2n$ and $V$ be a real quadratic space of dimension $2n + 1$ with discriminant

$$\text{disc}(V) = (-1)^n\det(V) \equiv 1 \in \mathbb{R}^\times/\mathbb{R}^{\times 2}.$$  

The space $(W \otimes V, \langle -, - \rangle_W \otimes \langle -, - \rangle_V)$ is a real symplectic space. Then $G_1 = \text{Sp}(W)$ and $G_2 = \text{O}(V)$ form a reductive dual pair. We have a natural homomorphism

$$G_1 \times G_2 \rightarrow \text{Sp}(W \otimes V).$$  \hspace{1cm} (3.6)

Let $r_i, 1 \leq i \leq 2$ be the rank of the maximal $\mathbb{R}$-split torus $A_{G_i}$ of $G_i$. Note that since $G_1$ is split, we have $r_1 = n$. We will denote by $a = (a_1, \cdots, a_{r_2})$ an element of $A_{G_1}$ and $b = (b_1, \cdots, b_{r_2})$ an element of $A_{G_2}$.

Let $\hat{G}_1$ and $\hat{G}_2$ be the inverse images of $G_1$ and $G_2$ in the metaplectic group $\widehat{\text{Sp}}(W \otimes V)$ by the covering map respectively. The embedding (3.5) can be lifted to a homomorphism

$$\hat{G}_1 \times \hat{G}_2 \rightarrow \widehat{\text{Sp}}(W \otimes V).$$

Let $(\omega_{W,V,\phi}, \mathcal{F})$ be the Weil representation of the metaplectic group $\widehat{\text{Sp}}(W \otimes V)$ realized on the mixed model [4 Section 7.4]. By [3 Lemma D.1], for $(\hat{g}, h) \in \hat{G}_1 \times \hat{G}_2$ and $\phi, \phi' \in \omega_{W,V,\psi}$, we have:

$$|\langle \omega_{W,V}(\hat{g}, h)\phi, \phi' \rangle| \leq C \cdot \prod_{i=1}^{r_1} |a_i|^{2n+2} \prod_{j=1}^{r_2} |b_j|^{2n} \prod_{k=1}^{r_2} \prod_{j=1}^{r_2} \prod_{k=1}^{r_2} \gamma(a_kb_j^{-1})$$

$$= C \cdot \prod_{i=1}^{r_1} |a_i|^{2n+2} \prod_{k=1}^{r_2} \prod_{j=1}^{r_2} \gamma(a_kb_j^{-1}).$$  \hspace{1cm} (3.7)
where $C$ is a constant and $Y(x) = \begin{cases} 1, & \text{if } |x| \leq 1 \\ |x|^{-1}, & \text{if } |x| > 1 \end{cases}$. Moreover, a more precise estimation is given by Xue [17]: there exists a continuous semi-norm $\nu_{\mathcal{O}}$ on $\omega_{W,V,\psi}$ such that

\begin{equation}
|(\omega_{W,V,\psi}(g,h)\phi,\phi')| \leq \prod_{i=1}^{r_1} |a_i|^{2n+1} \prod_{k=1,j=1}^{r_1,r_2} Y(a_kb^{-1})\nu_{\mathcal{O}}(\phi)\nu_{\mathcal{O}}(\phi').
\end{equation}

3.2. Weil representation and theta lifts. In [11, theorem 6.1], Li shows that if the dual pair $(G_1, G_2)$ is in the stable range case, then there is an explicit realization of the theta correspondence. The unitary case is studied in [10] and for more general classical groups, this is studied in [5] and used by Xue in [17]. In the following, we recall the construction of the explicit metaplectic theta correspondence and study the dual pair $(3.2)$. Weil representation and theta lifts. For any $\nu, \nu' \in \pi$ and $\phi, \phi' \in \omega_{W,V,\psi}$, there exists continuous semi-norms $\nu_{\mathcal{O}}$ on $\pi$ and $\nu_{\mathcal{O}}$ on $\omega_{W,V,\psi}$ such that

\begin{equation}
|\int_{\text{Sp}(W)} (\omega_{W,V,\psi}(g,1)\phi,\phi')(\pi(g)v,\nu'(g')d\gamma| \leq \nu_{\mathcal{O}}(v)\nu_{\mathcal{O}}(v')\nu_{\mathcal{O}}(\phi)\nu_{\mathcal{O}}(\phi').
\end{equation}

Proof. By estimations (3.2), (3.7) and the formula (3.1), the integral is bounded by

\begin{equation}
\int_{A_{G_1}^\infty} \prod_{i=1}^{r_1} |a_i|^{-\frac{1}{2}(2n+2-2i)}(1 - \sum_{i=1}^{r_1} \log |a_i|)B \prod_{j=1}^{r_1} |a_j|^{2n+1} da
\end{equation}

\begin{equation}
\int_{K_1 \times K_1} \tilde{\nu}_{\mathcal{O}}(\pi(k_1)v)\tilde{\nu}_{\mathcal{O}}(\pi(k_1^{-1},1)v')\tilde{\nu}_{\mathcal{O}}(\omega_{W,V,\psi}(k_1,1)\phi)\tilde{\nu}_{\mathcal{O}}(\omega_{W,V,\psi}(k_1^{-1},1)\phi')dk_1dk_1',
\end{equation}

where $B$ is a positive constant and $\tilde{\nu}_{\mathcal{O}}$ (resp. $\tilde{\nu}_{\mathcal{O}}$) is a continuous semi-norm on $\pi$ (resp. $\omega_{W,V,\psi}$).

The integral $\int_{A_{G_1}^\infty} \prod_{i=1}^{r_1} |a_i|^{-\frac{1}{2}(2n+2-2i)}(1 - \sum_{i=1}^{r_1} \log |a_i|)B \prod_{j=1}^{r_1} |a_j|^{2n+1} da$ can be simplified to the form

\begin{equation}
\int_{A_{G_1}^\infty} \prod_{i=1}^{r_1} |a_i|^{-\frac{1}{2}(1 - \sum_{i=1}^{r_1} \log |a_i|)}B da,
\end{equation}

hence it converges. Since $K_1$ is compact, the integral

\begin{equation}
\int_{K_1 \times K_1} \tilde{\nu}_{\mathcal{O}}(\pi(k_1)v)\tilde{\nu}_{\mathcal{O}}(\pi(k_1^{-1},1)v')\tilde{\nu}_{\mathcal{O}}(\omega_{W,V,\psi}(k_1,1)\phi)\tilde{\nu}_{\mathcal{O}}(\omega_{W,V,\psi}(k_1^{-1},1)\phi')dk_1dk_1'
\end{equation}

is bounded by

\begin{equation}
\text{Vol}(K_1)^2 \nu_{\mathcal{O}}(v)\nu_{\mathcal{O}}(v')\nu_{\mathcal{O}}(\phi)\nu_{\mathcal{O}}(\phi'),
\end{equation}

where $\nu_{\mathcal{O}}(v) = \sup_{k_1 \in K_1} \tilde{\nu}_{\mathcal{O}}(\pi(k_1)v)$ and $\nu_{\mathcal{O}}(\phi) = \sup_{k_1 \in K_1} \tilde{\nu}_{\mathcal{O}}(\omega_{W,V,\psi}(k_1,1)\phi)$. Each sup term defines a continuous semi-norm on the corresponding space by the uniform boundedness principle [13, Theorem 33.1].

\[ \square \]
Proposition 3.3. Let $\pi$ be an irreducible tempered representation of $\hat{\text{Sp}}(W)$. Take $v, v' \in \pi$ and $\phi, \phi' \in \omega_{W,V,\psi}$. The multilinear form\footnote{We ignore the identification of multilinear form and the linear form via the tensor product.} \begin{align}
abla (v, v', \phi, \phi') &\mapsto \int_{\text{Sp}(W)} (\pi'(g)v, v') \cdot (\omega_{W,V,\psi}(g,1)\phi, \phi')dg
\end{align}
continuously extends to a linear form on $\hat{\pi} \otimes \pi \otimes \omega_{W,V,\psi} \otimes \omega_{W,V,\psi}$. It is not identically zero if and only if $\theta_{W,V,\psi}(\pi) \neq 0$.

Proof. The absolute convergence and continuity follow from Lemma 3.2. The non-vanishing is proved by Gan, Qiu and Takeda in [5]. The following proposition give the explicit matrix coefficients of $\theta_{W,V,\psi}(\pi)$. By [6], this form is semipositivity. Moreover, we have the fact that: if $q$ is a nonzero semi-positive definite hermitian form on a vector space $X$, and $L$ is the radical of $q$, then $q$ descends to an inner product on $X/L$, still denote by $q$. To prove this, if there exists an $x \notin L$ such that $q(x, x) = 0$, then take some $y \in X$, which satisfies $q(x, y) \neq 0$. For $t \in \mathbb{C}$, then we have $q(tx + y, tx + y) = q(y, y) + 2\text{Re}(t) \cdot q(x, y)$.

As $t$ is an arbitrary complex number and $q(x, y) \neq 0$, we conclude that for a well-chosen complex number $t$, $q(tx + y, tx + y)$ can be a negative real number, which is a contradiction to the semi-positivity of $q$.

Let $R$ be the radical of semi-positive hermitian form defined by (3.9) as above. Then it defines an inner product on $\Theta_{W,V,\psi}(\pi)/R$. Therefore $\Theta_{W,V,\psi}(\pi)/R$ must be semisimple, and thus coincides with $\theta_{W,V,\psi}(\pi)$.

The following proposition give the explicit matrix coefficients of $\theta_{W,V,\psi}(\pi)$ using the explicit theta correspondence.

Proposition 3.4. The function $\Phi_{\phi, \phi', v, v'} : h \in O(V) \mapsto \int_{\text{Sp}(W)} (\pi(g)v, v') \cdot (\omega_{W,V,\psi}(g, h)\phi, \phi')dg$ defines a matrix coefficient of $\theta_{W,V,\psi}(\pi)$.

We also need the following proposition to simplify the computation.
\textbf{Proposition 3.5.} Let $\Phi$ be the subspace of the matrix coefficients of $\theta_{W,V,\psi}(\pi)$ generated by $\Phi_{\phi,\phi',v,v'}$, where $\phi$ and $\phi'$ range over a dense subspace of $\omega_{W,V,\psi}$, and $v$ and $v'$ range over a dense subspace of $\pi$. Then the space $\Phi$ is dense in the space of matrix coefficients of $\theta_{W,V,\psi}(\pi)$.

\textit{Proof.} Fix a surjective homomorphism $c : \omega_{W,V,\psi}\hat{\otimes}\pi \to \theta_{W,V,\psi}(\pi)$. The matrix coefficients of $\theta_{W,V,\psi}(\pi)$ are of the form $\langle \theta_{W,V,\psi}(\pi)(h)c(\phi,v),c(\phi',v') \rangle$ with $h \in O(V)$. Then the assertion follows from the surjectivity of $c$ and the density of $\omega_{W,V,\psi} \otimes \pi$ in $\omega_{W,V,\psi}\hat{\otimes}\pi$. \hfill $\square$

4. Theta lifts for tempered representations

In this section, we use the estimations of the matrix coefficients of various representations established in the previous sections to prove that if $\pi \in \text{Temp}^\text{gen}(\hat{\text{Sp}}(W))$, then $\theta_{W,V,\psi}(\pi)$ is tempered. This is equivalent to show that the matrix coefficients of $\theta_{W,V,\psi}(\pi)$ are almost square integrable functions (i.e. $L_{2+}\epsilon(O(V))$ for all $\epsilon \in \mathbb{R}_{>0}$). By the proposition 3.5, it suffices to prove that for any $\epsilon \in \mathbb{R}_{>0}$, for any $\phi, \phi' \in \omega_{W,V,\psi}$ and $v, v' \in \pi$, the integral

$$\int_{O(V)} |\Phi_{\phi,\phi',v,v'}(h)|^{2+\epsilon} dh = \int_{O(V)} \left| \int_{\text{Sp}(W)} \langle \omega_{W,V,\psi}(g,h)\phi,\phi' \rangle (\pi(g)v,v') dg \right|^{2+\epsilon} dh$$

converges. In the following, we will prove a stronger condition: the integral

$$\int_{O(V)} \left( \int_{\text{Sp}(W)} |\omega_{W,V,\psi}(g,h)\phi,\phi' \rangle (\pi(g)v,v') dg \right)^{2+\epsilon} dh$$

converges.

4.1. Reduction using the estimation of matrix coefficients. For any $\phi, \phi' \in \omega_{W,V,\psi}$ and $v, v' \in \pi$, by the estimations (3.2) and (3.6), there exists positive constants $A, B$ such that

$$|\langle \omega_{W,V,\psi}(g,h)\phi,\phi' \rangle (\pi(g)v,v')| \leq A\delta_{P_G}(a)\sigma(a)^B \prod_{i=1}^{r_1} |a_i|^{2n+1} \prod_{k=1}^{r_2} \prod_{j=1}^{r_3} \Upsilon(a_kb_j^{-1})$$

Together with the equation (3.1), we have

\begin{equation}
\int_{\text{Sp}(W)} |\omega_{W,V,\psi}(g,h)\phi,\phi' \rangle (\pi(g)v,v')| dg
\end{equation}

\begin{align}
&\leq A \int_{\text{Sp}(W)} \delta_{P_G}(a)\sigma(a)^B \prod_{i=1}^{n} |a_i|^{2n+1} \prod_{k=1}^{n} \prod_{j=1}^{r_2} \Upsilon(a_kb_j^{-1})| dg \\
&\leq A \int_{\text{Sp}(W)} \delta_{P_G}(a)\sigma(a)^B \prod_{i=1}^{n} |a_i|^{2n+1} \prod_{k=1}^{n} \prod_{j=1}^{r_2} \Upsilon(a_kb_j^{-1})| dk \cdot da \\
&= A \cdot \text{Vol}(K_1)^2 \cdot \int_{\text{Sp}(W)} \delta_{P_G}(a)\sigma(a)^B \prod_{i=1}^{n} |a_i|^{2n+1} \prod_{k=1}^{n} \prod_{j=1}^{r_2} \Upsilon(a_kb_j^{-1})| da.
\end{align}
Hence if we denote \( A \cdot \text{Vol}(K_2)^2 \) by \( A' \), then for any \( \epsilon > 0 \), using the equation (3.1) again, we have

\[
(4.3) \quad \int_{O(V)} \left( \int_{Sp(W)} |(\omega_{W,V},(g,h)\phi(\pi(g)v,v'))|dg \right)^{2+\epsilon} dh
\]

\[
\leq A' \int_{O(V)} \left( \int_{A_{G_1}^+} \delta_{P_{G_1}}^{-\frac{1}{2}}(a)\sigma(a)^B \prod_{i=1}^n |a_i|^{2n+1} \prod_{k=1}^{r_2} \prod_{j=1}^{r_2} \mathcal{Y}(a_kb_j^{-1})da \right)^{2+\epsilon} dh
\]

\[
\leq A' \int_{A_{G_2}^+} \delta_{P_{G_2}}^{-1}(b) \int_{K_2 \times K_2} \left( \int_{A_{G_1}^+} \delta_{P_{G_1}}^{-\frac{1}{2}}(a)\sigma(a)^B \prod_{i=1}^n |a_i|^{2n+1} \prod_{k=1}^{r_2} \prod_{j=1}^{r_2} \mathcal{Y}(a_kb_j^{-1})da \right)^{2+\epsilon} dk_2 db k_2
\]

\[
= A' \cdot \text{Vol}(K_2)^2 \int_{A_{G_2}^+} \delta_{P_{G_2}}^{-1}(b) \left( \int_{A_{G_1}^+} \delta_{P_{G_1}}^{-\frac{1}{2}}(a)\sigma(a)^B \prod_{i=1}^n |a_i|^{2n+1} \prod_{k=1}^{r_2} \prod_{j=1}^{r_2} \mathcal{Y}(a_kb_j^{-1})da \right)^{2+\epsilon} db.
\]

By the formula (2.2), we have

\[
\sigma(a) \leq 1 - \sum_{i=1}^n \log |a_i| \leq 1 - \sum_{i=1}^n \log |a_i| - \sum_{j=1}^{r_2} \log |b_j|.
\]

Note that

\[
\delta_{P_{G_1}}(a) = \prod_{i=1}^n |a_i|^{2n+2-2i}, \quad \delta_{P_{G_2}}(b) = \prod_{j=1}^{r_2} |b_j|^{2n+1-2j}.
\]

Replacing \( \epsilon \) by \( 2\epsilon \), it suffices to show that for all \( \epsilon > 0 \), the integral

\[
(4.4) \quad \left( 1 - \sum_{i=1}^n \log |a_i| - \sum_{j=1}^{r_2} \log |b_j| \right)^{B(2+2\epsilon)} da db
\]

converges.

4.2. Proof of the convergence of the integral (4.4). Let \( (p_1, \ldots, p_{r_2+1}) \) be a \((r_2+1)\)-tuple of non-negative integers such that

\[
p_1 + \cdots + p_{r_2+1} = n.
\]

Let \( S_{p_1,\ldots,p_{r_2+1}} \) be the subset of \( A_{G_1}^+ \times A_{G_2}^+ \), defined by the condition

\[
|a_1| \leq \cdots \leq |a_{p_1}| \leq |b_1| \leq \cdots \leq |a_{p_1+p_2}| \leq |b_2| \leq \cdots \leq |a_{p_1+\cdots+p_{r_2+1}}| \leq 1.
\]

We can break the domain \( A_{G_1}^+ \times A_{G_2}^+ \) of the integral (4.4) by \( S_{p_1,\ldots,p_{r_2+1}} \), and it suffices to show that over each region \( S_{p_1,\ldots,p_{r_2+1}} \), the integral (4.4) converges. We will use the following simple lemma to conclude its convergence.

Lemma 4.1. Let \( N \) be a natural number. Let \( s_1, \ldots, s_N \) and \( B \) be real numbers. If \( s_1 + \cdots + s_i > 0 \) for all \( 1 \leq i \leq N \), then the integral

\[
\int_{|x_1| \leq \cdots \leq |x_N| \leq 1} |x_1|^{s_1} \cdots |x_N|^{s_N} (1 - \sum_{i=1}^N \log |x_i|)^B dx_1 \cdots dx_N
\]
converges.

Note that in a fixed region $S_{p_1, \ldots, p_{r_2+1}}$, we have

$$
\prod_{i=1}^n \prod_{j=1}^{r_2} \gamma(a_i b_j^{-1}) = \prod_{j=1}^{r_2} \left( \prod_{i=1}^{p_{i+1} - 1} a_i + \sum_{k=1}^{j} p_k \right) |b_j|^{n - (\sum_{k=1}^{j} p_k)}.
$$

We rearrange the terms in the integral (4.4) with respect to the order given by the condition (4.3). If the integral (4.4) on region $S_{p_1, \ldots, p_{r_2+1}}$ satisfies the condition of the lemma 4.1 with respect to this order, then we can conclude that the integral (4.4) converges.

For $0 \leq t \leq p_{j+1} - 1$, $1 \leq j \leq r_2$, we check the sum of the exponents in the integral (4.4) up to $a_{p_1+\cdots+p_j+t}$:

1. The sum of the exponents of $a_i (1 \leq i \leq p_1 + \cdots + p_j + t)$:

$$
(1 + 3 + \cdots + (2(p_1 + \cdots + p_j + t) - 1))(1 + \epsilon) - (p_2 + 2p_3 + \cdots + (j - 1)p_j + j t)(2 + 2\epsilon)
$$

2. The sum of the exponents of $b_i (1 \leq i \leq j)$:

$$
2(1 + \cdots + j) - j(2n + 1) + ((n - p_1) + \cdots + (n - p_1 - \cdots - p_j))(2 + 2\epsilon).
$$

Summing these two terms, we get

$$(p_1 + \cdots + p_j + t)^2 + \epsilon((p_1 + \cdots + p_j + t)^2 + 2j(n - p_1 - \cdots - p_j - t)) > 0.$$  

The same type of verification shows that the sum of the exponents up to $b_j$ is positive. Hence the integral (4.4) satisfies the condition of the lemma 4.1. As a consequence, the integral (4.1) converges.

REFERENCES

1. J. Adams, D. Barbasch, Genuine representation of the metaplectic group, Comp. Math. 113 (1998) 23-66
2. M. Cowling, U. Haagerup, R. Howe, Almost $L^2$ matrix coefficients, J. reine angew. Math. 387 (1988), 97-110.
3. W.T. Gan, A. Ichino, Formal degrees and local theta correspondence, Invent. Math. (2014) 195: 509-672.
4. W.T. Gan, A. Ichino, The Gross-Prasad conjecture and local theta correspondence, Invent. Math. 206 (2016) no. 3, 705-799.
5. W. T. Gan, Y. N. Qiu, S. Takeda, The regularized Siegel-Weil formula (the second term identity) and the Rallis inner product formula, Invent. Math. Vol. 198 (2014) 3, 739-831.
6. H. He, Unitary representations and theta correspondence for type I classical groups. J. Funct. Anal. 199 (2003), no. 1, 92-121.
7. R. Howe, Transcending invariant theory, J. Amer. Math. Soc. 2 (1989), no. 3, 535-552.
8. A. Ichino, T. Ikeda, On the periods of automorphic forms on special orthogonal groups and the Gan-Gross-Prasad conjecture, Geom. Funct. Anal. 19(2010), 1378-1425.
9. A.W. Knapp, G. J. Zuckerman, Classification of irreducible tempered representations of semisimple groups, Ann. of Math. (2), 116, 1982, no. 3, 457-501.
10. S. T. Lee, C. B. Zhu, Degenerate principal series and local theta correspondence, Trans. Amer. Math. Soc. Vol. 350 (1998), 12, 5017-5046.
11. J. S. Li, Singular Unitary Representations of Classical Groups, Invent. math. 97, 237-255 (1989).
12. B. Sun, Bounding matrix coefficients for smooth vectors of tempered representations, Proc. Amer. Math. Soc. 137 (2009), no. 1, 353-3577.
13. F. Trèves, Topological vector spaces, distributions and kernels, Academic Press, 1967.
14. V. S. Varadarajan, Harmonic analysis on real reductive groups, Lecture Notes in Math., vol. 576, Springer, 1977.
15. N. R. Wallach, *Real reductive groups II*, Pure and Applied Mathematics, vol. 132, Academic Press, Inc., Boston, MA, 1992.

16. H. Xue, *Refined global Gan-Gross-Prasad conjecture for Fourier-Jacobi periods on symplectic groups*, Compositio Math. **153** (2017), 68-131.

17. H. Xue, *Bessel model for real unitary groups: the tempered case*, To appear in Duke Math. J.

School of mathematical sciences, Fudan University, 220 Handan Rd., Yangpu District, 200433, Shanghai, China

Current address: Department of Mathematics and Statistics, Case Western Reserve University, Cleveland, Ohio 43403

Email address: zli17@fudan.edu.cn

School of Mathematics, Renmin University of China, No. 59 Zhongguancun Street, Haidian District, 100872, Beijing, China

Email address: s_wang@ruc.edu.cn