Moderate deviations for stochastic models of two-dimensional second grade fluids driven by Lévy noise

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Abstract: In this paper, we establish a moderate deviation principle for stochastic models of two-dimensional second grade fluids driven by Lévy noise. We will adopt the weak convergence approach. Because of the appearance of jumps, this result is significantly different from that in Gaussian case.

Key Words: Moderate deviations; Second grade fluids; Lévy process; Weak convergence method.

1 Introduction

The second grade fluids is an admissible model of slow flow fluids, which contains industrial fluids, slurries, polymer melts, etc.. It has attracted much attention from a theoretical point of view, since it has properties of boundedness, stability and exponential decay, and has interesting connections with many other fluid models, see e.g. [5], [11], [13], [26] and references therein.

Recently, taking into account the effect of random environment, the external force is considered as random. The stochastic models of two-dimensional second grade fluids have been studied. For the case of Gaussian noises, we refer to [7, 22, 23, 24, 30, 33, 34], where the authors obtained the existence and uniqueness of solutions, the behavior of the solutions as \( \alpha \to 0 \), Freidlin-Wentzell's large deviation principles (LDP), exponential mixing and moderate deviation principles (MDP) for the solutions. In the case of Lévy noises, the global existence of a martingale solution was obtained in [16], the existence and uniqueness of strong probabilistic solutions is established in [25], and the Freidlin-Wentzell’s large deviation principles for the solutions is proved in [35].

In this paper, we are concerned with asymptotic behaviors of stochastic models for the incompressible non-Newtonian fluids of second grade driven by Lévy noise, which are given as follows:

\[
\begin{aligned}
&d(u^\varepsilon(t) - \alpha \Delta u^\varepsilon(t)) + \left( - \kappa \Delta u^\varepsilon(t) + \text{curl}(u^\varepsilon(t) - \alpha \Delta u^\varepsilon(t)) \times u^\varepsilon(t) + \nabla \Psi \right) dt \\
&= F(u^\varepsilon(t), t) dt + \varepsilon \int_Z G(u^\varepsilon(t-), z)^\varepsilon^{-1}(dz dt), \quad \text{in } \mathcal{O} \times (0, T], \\
\text{div } u^\varepsilon &= 0 \quad \text{in } \mathcal{O} \times (0, T]; \quad u^\varepsilon = 0 \quad \text{in } \partial \mathcal{O} \times [0, T]; \quad u^\varepsilon(0) = u_0 \quad \text{in } \mathcal{O},
\end{aligned}
\]  

(1.1)

where \( \mathcal{O} \) is a bounded open domain of \( \mathbb{R}^2 \); \( u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2) \) and \( \Psi \) represent the random velocity and the modified pressure respectively. \( \mathcal{Z} \) is a locally compact Polish space. On a specified complete filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P) \), \( N^{\varepsilon^{-1}} \) is a compensated Poisson random measure on \( [0, T] \times \mathcal{Z} \) with a \( \sigma \)-finite mean measure \( \varepsilon^{-1} \lambda_T \otimes \nu \), where \( \lambda_T \) is the Lebesgue measure on \( [0, T] \) and \( \nu \) is a \( \sigma \)-finite measure on \( \mathcal{Z} \). The details of \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P, N^{\varepsilon^{-1}}) \) will be given in Section 2.

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Let \( \Pi \) be the Helmholtz-Leray projection from \( L^2(\Omega) \) into \( H \). Let \( A \) be the Stokes operator \( -\Pi \Delta \) (see the precise definition below). One can see that (1.1) is equivalent to the following stochastic evolution equation:

\[
du(t) = -\kappa \hat{A} \hat{u}(t) - \hat{B}(\hat{u}(t), \hat{u}(t))dt + \varepsilon \int_Z \hat{G}(\hat{u}(t-), z) \hat{N}^{\varepsilon^{-1}}(dz)dt, \tag{1.2}
\]

with initial value \( u(0) = u_0 \).

where \( \hat{A} = (I + \alpha A)^{-1} A \), \( \hat{B}(u,v) = (I + \alpha A)^{-1} (\text{curl}(u - \alpha \Delta u) \times v) \), \( \hat{F} = (I + \alpha A)^{-1} F \), \( \hat{G} = (I + \alpha A)^{-1} G \).

As the parameter \( \varepsilon \) tends to zero, the solution \( u^\varepsilon \) of (1.2) will tend to the solution of the following deterministic equation

\[
du^0(t) = -\kappa \hat{A} u^0(t) - \hat{B}(u^0(t), u^0(t))dt + \hat{F}(u^0(t), t)dt, \tag{1.3}
\]

with initial value \( u^0(0) = u_0 \).

In this paper, we shall investigate deviations of \( u^\varepsilon \) from the deterministic solution \( u^0 \), as \( \varepsilon \) decreases to 0, that is, the asymptotic behavior of the trajectory,

\[
Z^\varepsilon(t) = \frac{1}{a(\varepsilon)}(u^\varepsilon - u^0)(t), \quad t \in [0,T],
\]

where \( a(\varepsilon) \) is some deviation scale which strongly influences the asymptotic behavior of \( Z^\varepsilon \). We will study the so-called moderate deviation principle (MDP for short), that is when the deviation scale satisfies

\[
a(\varepsilon) \to 0, \quad \varepsilon / a^2(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{1.4}
\]

Throughout this paper, we assume that (1.4) is in place.

Like the large deviations, the estimates of moderate deviations are very useful in the theory of statistical inference. It can provide us with the rate of convergence and a useful method for constructing asymptotic confidence intervals, see [12, 13, 15, 19] and references therein. There are many methods to establish the MDP in various framework, for example, De Acosta [1], Chen [6] and Ledoux [20] for processes with independent increments; Wu [32] for Markov processes; Guillin and Liptser [15] for diffusion processes; Wang and Zhang [31] for stochastic reaction-diffusion equations; Wang, Zhai and Zhang [29] for 2-D stochastic Navier-Stokes equations driven by Brownian motion; Zhai and Zhang [33] for stochastic models of 2-D second grade fluids driven by Brownian motion.

The MDP for stochastic evolution equation and stochastic partial differential equations driven by Lévy noise are quite different from that in the case driven by Brownian motion because of the difficulties caused by the jumps. In this paper, we will adopt the weak convergence approach introduced in [3] to establish the MDP for stochastic models of second grade fluids driven by Lévy noise. Similar to [10], we decompose the solutions into a sum of the solutions of several relatively simpler equations and prove the convergence/tightness of the solutions of each equations. But the details of the proof are quite different and more difficult, because of the nature of the second grade fluids models. The main effort is to deal with the nonlinear term \( \text{curl}(u^\varepsilon(t) - \alpha \Delta u^\varepsilon(t)) \times u^\varepsilon(t) \).

We organize this paper as follows. In Section 2, we introduce some functional spaces and some notations. In Section 3, we formulate the hypotheses and state our main result. In Section 4, we provide all the proofs.

2 Preliminaries and Notations

In this paper, we assume that \( \Omega \) is a simply connected and bounded open domain of \( \mathbb{R}^2 \) with boundary \( \partial \Omega \) of class \( C^{k,1} \). For \( p \geq 1 \) and \( k \in \mathbb{N} \), we denote by \( L^p(\Omega) \) and \( W^{k,p}(\Omega) \) the usual \( L^p \) and Sobolev spaces over \( \Omega \) respectively. Let \( W^{k,2}_0(\Omega) \) be the closure in \( W^{k,2}(\Omega) \) of \( C_0^\infty(\Omega) \) the space of infinitely differentiable functions with compact supports in \( \Omega \). For simplicity, we write \( H^k(\Omega) := W^{k,2}(\Omega) \) and \( H^k_0(\Omega) := W^{k,2}_0(\Omega) \). We equip \( H^k_0(\Omega) \) with the scalar product

\[
(\langle u, v \rangle) = \int_\Omega \nabla u \cdot \nabla v dx = \sum_{i=1}^2 \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx,
\]

where \( \nabla \) is the gradient operator. It is well known that the norm \( \| \cdot \| \) generated by this scalar product is equivalent to the usual norm of \( H^1(\Omega) \).
Throughout this paper, we set $\mathcal{Y} = Y \times Y$ for any Banach space $Y$. Set
\[ C = \left\{ u \in \mathcal{C}_c^\infty(\mathcal{O}) \right\}, \]
\[ H = \text{closure of } C \text{ in } L^2(\mathcal{O}),(:= L^2(\mathcal{O}, \mathbb{R}^2)), \]
\[ V = \text{the closure of } C \text{ in } H^1(\mathcal{O}). \]

We denote by $(\cdot, \cdot)$ and $|\cdot|$ the inner product in $L^2(\mathcal{O})$ (in $H$) and the induced norm, respectively. The inner product and the norm of $H^1(\mathcal{O})$ are denoted respectively by $(\cdot, \cdot)$ and $\| \cdot \|$. We endow the space $V$ with the norm generated by the following inner product
\[ (u, v)_V := (u, v) + \alpha((u, v)), \quad \text{for any } u, v \in V, \]
and the norm in $V$ is denoted by $\| \cdot \|_V$. The Poincaré's inequality implies that there exists a constant $\mathcal{P} > 0$ such that the following inequalities holds
\[ (\mathcal{P}^2 + \alpha)^{-1}\|v\|^2 \leq \|v\|_V^2 \leq \alpha^{-1}\|v\|^2, \quad \text{for any } v \in V. \quad (2.1) \]

We also introduce the following space
\[ W = \left\{ u \in V : \text{curl}(u - \alpha \Delta u) \in L^2(\mathcal{O}) \right\}, \]
and endow it with the norm generated by the scalar product
\[ (u, v)_W := \left( \text{curl}(u - \alpha \Delta u), \text{curl}(v - \alpha \Delta v) \right). \quad (2.2) \]

The norm in $W$ is denoted by $\| \cdot \|_W$. It has been proved that, see e.g. [8, 9], the following (algebraic and topological) identity holds:
\[ W = \left\{ v \in H^3(\mathcal{O}) : \text{div} v = 0 \text{ and } v|_{\partial \mathcal{O}} = 0 \right\}, \]
middle

moreover, there exists a constant $C > 0$ such that
\[ |v|_{W(\mathcal{O})} \leq C\|v\|_W, \quad \forall v \in W. \quad (2.3) \]

This result states that the norm $\| \cdot \|_W$ is equivalent to the usual norm in $H^3(\mathcal{O})$.

Identifying the Hilbert space $V$ with its dual space $V^*$ by the Riesz representation, we get a Gelfand triple
\[ W \subset V \subset W^*. \]

We denote by $\langle f, v \rangle$ the dual relation between $f \in W^*$ and $v \in W$ from now on. It is easy to see
\[ (v, w)_V = \langle v, w \rangle, \quad \forall v \in V, \quad \forall w \in W. \quad (2.4) \]

Note that the injection of $W$ into $V$ is compact, thus there exists a sequence $\{e_i\}$ of elements of $W$ which forms an orthonormal basis in $W$, and an orthogonal system in $V$, moreover this sequence verifies:
\[ (v, e_i)_W = \lambda_i(v, e_i)_V, \quad \text{for any } v \in W, \quad (2.5) \]

where $0 < \lambda_i \uparrow \infty$. From Lemma 4.1 in [8] we have
\[ e_i \in H^4(\mathcal{O}), \quad \forall i \in \mathbb{N}. \quad (2.6) \]

Consider the following “generalized Stokes equations”:
\[ v - \alpha \Delta v = f \quad \text{in } \mathcal{O}, \]
\[ \text{div } v = 0 \quad \text{in } \mathcal{O}, \]
\[ v = 0 \quad \text{on } \partial \mathcal{O}. \quad (2.7) \]

The following result can be derived from [27] and also can be found in [23, 24].
Lemma 2.1 Set \( l = 1, 2, 3 \). Let \( f \) be a function in \( \mathbb{H}^l \), then the system \( \mathcal{L} \) has a unique solution \( v \). Moreover if \( f \) is an element of \( \mathbb{H}^{l+2} \cap \mathcal{V} \), then \( v \in \mathbb{H}^{l+2} \cap \mathcal{V} \), and the following relations hold
\[
(v, g)_V = (f, g), \quad \forall g \in \mathcal{V},
\]
\[
|v|_{\mathbb{H}^{l+2}} \leq C|f|_{\mathbb{H}^l}.
\]

We recall the following estimates which can be found in [24].

Lemma 2.2 For any \( u, v, w \in \mathbb{W} \), we have
\[
|(\text{curl}(u - \alpha \Delta u) \times v, w)| \leq C\|u\|_W\|v\|_V\|w\|_W,
\]
and
\[
|(\text{curl}(u - \alpha \Delta u) \times u, w)| \leq C\|u\|^2\|w\|_W.
\]

Defining the bilinear operator \( \hat{B}(\cdot, \cdot) : \mathbb{W} \times \mathcal{V} \rightarrow \mathbb{W}^* \) by
\[
\hat{B}(u, v) := (I + \alpha A)^{-1} P(\text{curl}(u - \alpha \Delta u) \times v).
\]

We have the following consequence of Lemma 2.2.

Lemma 2.3 For any \( u \in \mathbb{W} \) and \( v \in \mathcal{V} \), it holds that
\[
\|\hat{B}(u, v)\|_{\mathbb{W}^*} \leq C\|u\|_W\|v\|_V,
\]
and
\[
\|\hat{B}(u, u)\|_{\mathbb{W}^*} \leq C\|u\|^2.
\]

In addition
\[
\langle \hat{B}(u, v), v \rangle = 0, \quad \forall u, v \in \mathbb{W},
\]
which implies
\[
\langle \hat{B}(u, v), w \rangle = -\langle \hat{B}(u, w), v \rangle, \quad \forall u, v, w \in \mathbb{W}.
\]

We are now introducing \( (\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t \in [0, T]}, P, \hat{N}^{-1}) \).

For a locally compact Polish space \( S \), let \( M_{FC}(S) \) denote the space of all Borel measures \( \vartheta \) on \( S \) such that \( \vartheta(K) < \infty \) for each compact set \( K \subseteq S \). Endow \( M_{FC}(S) \) with the weakest topology, denoted it by \( \mathcal{T}(M_{FC}(S)) \), such that for each \( f \in C_c(S) \) the mapping \( \vartheta \in M_{FC}(S) \rightarrow \int_S f(s) \vartheta(ds) \) is continuous. This topology is metrizable such that \( M_{FC}(S) \) is a Polish space, see [4] for more details.

Recall that \( Z \) is a locally compact Polish space, and in this paper, we assume that \( \nu \) is a given element of \( M_{FC}(Z) \). We specify the underlying probability space \( (\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t \in [0, T]}, P) \) in the following way:
\[
\Omega := M_{FC}([0, T] \times Z \times [0, \infty)), \quad \mathcal{F} := \mathcal{T}(M_{FC}([0, T] \times Z \times [0, \infty))).
\]

We introduce the function
\[
N : \Omega \rightarrow M_{FC}([0, T] \times Z \times [0, \infty)), \quad N(\omega) = \omega.
\]

Define for each \( t \in [0, T] \) the \( \sigma \)-algebra
\[
\mathcal{G}_t := \sigma \left( \{ N((0, s] \times A) : 0 \leq s \leq t, A \in \mathcal{B}(Z \times [0, \infty)) \} \right).
\]

Let \( \lambda_T \) and \( \lambda_\infty \) be Lebesgue measure on \([0, T]\) and \([0, \infty)\) respectively. It follows from [17, Sec.1.8] that there exists a unique probability measure \( P \) on \((\Omega, \mathcal{F})\) such that: \( N \) is a Poisson random measure on \( \Omega \) with intensity measure \( \lambda_T \otimes \nu \otimes \lambda_\infty \).
We denote by $\mathbb{F} := \{\mathcal{F}_t\}_{t \in [0,T]}$ the $P$-completion of $\{\mathcal{G}_t\}_{t \in [0,T]}$ and by $\mathcal{P}$ the $\mathbb{F}$-predictable $\sigma$-field on $[0,T] \times \Omega$. Define

$$A := \{\varphi: [0,T] \times \mathbb{Z} \times \Omega \to [0,\infty): (\mathcal{P} \otimes \mathcal{B}(\mathbb{Z})) \setminus \mathcal{B}[0,\infty)\text{-measurable}\}.$$ 

For $\varphi \in A$, define a counting process $N^\varphi$ on $[0,T] \times \mathbb{Z}$ by

$$N^\varphi((0,t] \times A) = \int_{[0,t] \times A \times (0,\infty)} 1_{[0,\varphi(s,z)]}(r) \, N(ds,dz,dr),$$

for $t \in [0,T]$ and $A \in \mathcal{B}(\mathbb{Z})$. When $\varphi(s,z,\omega) = \epsilon^{-1}$, we write $N^\varphi = N^\epsilon$. It is easy to see that $N^\epsilon$ is a Poisson random measure on $[0,T] \times \mathbb{Z}$ with a mean measure $\epsilon^{-1}\lambda_T \otimes \nu$. We denote $N^{-1}$ the compensated Poisson random measure respect to $N^\epsilon$.

We end this section with a criteria of compactness, which will be used later. Let $\mathbb{B}$ be a separable Hilbert space. Given $p > 1$, $\beta \in (0,1)$, let $W^{\beta,p}([0,T], \mathbb{B})$ be the space of all bounded linear operators from $L^p([0,T], \mathbb{B})$ to $\mathbb{B}$, with norm

$$\|F\|_{W^{\beta,p}([0,T], \mathbb{B})} := \int_0^T \|u(t)\|^p_{\mathbb{B}} \, dt + \int_0^T \int_0^T \frac{\|u(t) - u(s)\|^p_{\mathbb{B}}}{|t-s|^{1+\beta p}} \, dt ds.$$ 

The following result is a variant of the criteria for compactness proved in [21] (Sect. 5, Ch. 1) and [28] (Sect. 13.3).

**Lemma 2.4** Let $\mathbb{K}_0 \subset \mathbb{K} \subset \mathbb{K}_1$ be Banach spaces, $\mathbb{K}_0$ and $\mathbb{K}_1$ reflexive, with compact embedding of $\mathbb{K}_0$ into $\mathbb{K}$. For $p \in (1,\infty)$ and $\beta \in (0,1)$, let $\Lambda$ be the space

$$\Lambda = L^p([0,T]; \mathbb{K}_0) \cap W^{\beta,p}([0,T]; \mathbb{K}_1)$$

endowed with the natural norm. Then the embedding of $\Lambda$ into $L^p([0,T]; \mathbb{K})$ is compact.

## 3 Hypothesis and Main Result

In this section, we will state the precise assumptions on the coefficients and our main result.

Let $F: \mathbb{V} \times [0,T] \to \mathbb{V}$ and $G: \mathbb{V} \times \mathbb{Z} \to \mathbb{V}$ be given measurable maps. We introduce the following conditions:

**F1**

$$F(0,t) = 0,$$  \hspace{1cm} (3.1)

and

$$\|F(u_1,t) - F(u_2,t)\|_{\mathbb{V}} \leq C_1 \|u_1 - u_2\|_{\mathbb{V}}, \ \forall u_1, u_2 \in \mathbb{V}, \ t \in [0,T].$$  \hspace{1cm} (3.2)

**F2** $F$ is differentiable with respect to the first variable, and the derivative $F': \mathbb{V} \times [0,T] \to L(\mathbb{V})$ (\(L(\mathbb{V})\) is the space of all bounded linear operators from $\mathbb{V}$ to $\mathbb{V}$) is uniformly Lipschitz with respect to the first variable, more precisely,

$$\|F'(u_1,t) - F'(u_2,t)\|_{L(\mathbb{V})} \leq C \|u_1 - u_2\|_{\mathbb{V}}, \ \forall u_1, u_2 \in \mathbb{V}, \ t \in [0,T].$$  \hspace{1cm} (3.3)

By (3.2), we conclude that

$$\|F'(u,t)\|_{L(\mathbb{V})} \leq C.$$  \hspace{1cm} (3.4)

Denote $\tilde{F}'(u,t) = (I + \alpha A)^{-1} F'(u,t)$. 


Lemma 3.1 If we assume that the boundary $\partial \mathcal{O}$ is of class $C^{3,1}$ and the initial value $u_0 \in \mathbb{W} \cap \mathbb{H}^4(\mathcal{O})$, then $u^0$ belongs to $L^\infty([0,T], \mathbb{W} \cap \mathbb{H}^4(\mathcal{O}))$, i.e.

$$\sup_{t \in [0,T]} \|u^0(t)\|_{\mathbb{H}^4(\mathcal{O})} \leq C.$$  

(3.7)

To obtain the moderate deviation principle, additionally we impose the following hypothesis throughout the paper:

(I) the initial value $u_0 \in \mathbb{W} \cap \mathbb{H}^4(\mathcal{O})$.

In order to introduce our main result, we need the following notations. The space $D([0,T], \mathbb{V})$ is the collection of all $\mathbb{V}$-valued càdlàg functions equipped with the Skorokhod topology. For any $\varepsilon > 0$ and $M < \infty$, consider the spaces

$$S^{M}_{+} = \{ \varphi : [0,T] \times \mathbb{Z} \rightarrow \mathbb{R}_+ | L_T(\varphi) \leq Ma^2(\varepsilon) \} ,$$

$$S^M_{\varepsilon} = \{ \psi : [0,T] \times \mathbb{Z} \rightarrow \mathbb{R} | \psi = (\varphi - 1)/a(\varepsilon), \varphi \in S^{M}_{+} \},$$

where $L_T(g) = \int_0^T \int_{\mathbb{Z}} (g(t,z) \log g(t,z) - g(t,z) + 1) \nu(\lambda \lambda ) dt$.

The norm in the Hilbert space $L^2(\nu_T)$ will be denoted by $\| \cdot \|_2$ and $B_2(R)$ denotes the ball of radius $R$ in $L^2(\nu_T)$. Throughout this paper $B_2(R)$ is equipped with the weak topology of $L^2(\nu_T)$ and it is therefore weakly compact.

By Theorem 3.2 in Shang, Zhai and Zhang [25], we know that the equation (3.2) has a unique strong solution $u^\varepsilon \in D([0,T], \mathbb{V}) \cap L^\infty([0,T]; \mathbb{W})$ in the probabilistic sense. Set $Y^\varepsilon = (u^\varepsilon - u^0)/a(\varepsilon)$, which satisfies

$$dY^\varepsilon(t) = -\kappa \hat{A} Y^\varepsilon(t) dt - \left( \hat{B}(a(\varepsilon)Y^\varepsilon(t) + u^0(t), Y^\varepsilon(t)) + \hat{B}(Y^\varepsilon(t), u^0(t)) \right) dt$$

$$+ \frac{1}{a(\varepsilon)} \left( \hat{F}(a(\varepsilon)Y^\varepsilon(t) + u^0(t), t) + \frac{1}{a(\varepsilon)} \int_{\mathbb{Z}} \hat{G}(a(\varepsilon)Y^\varepsilon(t-)+u^0(t-),z)N^{-1}(dz) dt, \right. 

with initial value $Y^\varepsilon(0) = 0$.

The following theorem is our main result.

Theorem 3.1 Suppose that Conditions (F1), (F2), (G) and (I) hold. Then $\{Y^\varepsilon\}$ satisfies a large deviation principle in $D([0,T], \mathbb{V})$ with speed $\varepsilon/a^2(\varepsilon)$ and the rate function given by

$$I(\eta) = \inf_{\psi} \left[ \frac{1}{2} \| \psi \|^2_2 \right],$$

where the infimum is taken over all $\psi \in L^2(\nu_T)$ such that $(\eta, \psi)$ satisfies the following equation:

$$d\eta(t) = -\kappa \hat{A} \eta(t) dt - \left( \hat{B}(\eta(t), u^0(t)) + \hat{B}(u^0(t), \eta(t)) \right) dt$$

$$+ \hat{F}^t(u^0(t), \eta(t)) dt + \int_{\mathbb{Z}} \hat{G}(u^0(t), \psi(z,t)) \nu(\lambda \lambda ) dt,$$

with initial value $\eta(0) = 0$. That is,
(a) (Upper bound) For each closed subset $O_1$ of $D([0,T], \mathcal{V})$,
\[
\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \log P(Y^\varepsilon \in O_1) \leq - \inf_{x \in O_1} I(x).
\]

(b) (Lower bound) For each open subset $O_2$ of $D([0,T], \mathcal{V})$,
\[
\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \log P(Y^\varepsilon \in O_2) \geq - \inf_{x \in O_2} I(x).
\]

Remark 1 Following the similar arguments as in the proof of Theorem 5.6 in \cite{2}, one can see that for all $\psi \in L^2(\nu_T)$, the equation (3.7) has a unique solution $\eta \in L^\infty([0,T], \mathbb{W} \cap H^4(\mathcal{O}))$.

Proof: Define $\mathcal{G}^0: L^2(\nu_T) \to C([0,T], \mathcal{V})$ by
\[
\mathcal{G}^0(\psi) = \eta \text{ for } \psi \in L^2(\nu_T), \text{ where } (\eta, \psi) \text{ solves (3.9).}
\]

The existence and uniqueness of the strong solution of (3.8) implies that there exists a measurable mapping $\mathcal{G}^\varepsilon: M_{FC}(\mathbb{Z} \times [0,T]) \to D([0,T], \mathcal{V})$ such that: $\mathcal{G}^\varepsilon(\varepsilon N^{-1}) = Y^\varepsilon$.

We will apply the general criteria (Theorem 2.3) obtained in \cite{2} to prove the theorem. According to \cite{2}, it is sufficient to verify two claims. The first one is the following:

(MDP-1) For any $M > 0$, suppose that $g^\varepsilon, g \in B_2(M)$ and $g^\varepsilon \to g$. Then
\[
\mathcal{G}^0(g^\varepsilon) \to \mathcal{G}^0(g) \text{ in } C([0,T], \mathcal{V}).
\]

In order to state the second claim, we need to introduce some additional notations. Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of compact sets $K_n \subseteq \mathbb{Z}$ with $K_n \uparrow \mathbb{Z}$. For each $n \in \mathbb{N}$, let
\[
\mathcal{A}_{b,n} = \left\{ \psi \in \mathcal{A} : \psi(t, z, \omega) \in [\frac{1}{n}, n], \text{if } (t, z, \omega) \in [0, T] \times K_n \times \Omega \text{ and } \psi(t, z, \omega) = 1, \text{if } (t, z, \omega) \in [0, T] \times K_n^c \times \Omega \right\}
\]
and $\mathcal{A}_b = \bigcup_{n=1}^{\infty} \mathcal{A}_{b,n}$. Define
\[
\mathcal{U}^M_{+} = \left\{ \varphi \in \mathcal{A} : \varphi(t, \cdot, \omega) \in \mathcal{S}^M_{\mathcal{A}}, P - \text{a.s.} \right\}
\]
and
\[
\mathcal{U}^M = \left\{ \psi \in \mathcal{A} : \psi(t, \cdot, \omega) \in \mathcal{S}^M_\mathcal{A}, P - \text{a.s.} \right\}
\]

Suppose $\varphi \in \mathcal{S}^M_{\mathcal{A}}$. By Lemma 3.2 in \cite{2}, there exists $\kappa_2(1) \in (0, \infty)$ that is independent of $\varepsilon$ and such that $\varphi_{1(\{\varphi \leq 1/\varepsilon(a(\varepsilon))\})} \in B_2(\sqrt{M \kappa_2(1)})$, where $\psi = (\varphi - 1)/a(\varepsilon)$. In this paper, we use the symbol $\Rightarrow \Rightarrow$ to denote convergence in distribution. Now we state the second claim:

(MDP-2) For any $M > 0$, let $\{\varphi^\varepsilon\}$ be such that for every $\varepsilon > 0$, $\varphi^\varepsilon \in \mathcal{U}^M_{+}$, and for some $\beta \in (0, 1], \varepsilon^\beta 1(\{\varepsilon \leq \beta(\varepsilon)\}) \Rightarrow \psi$ in $B_2(\sqrt{M \kappa_2(1)})$ where $\psi^\varepsilon = (\varphi^\varepsilon - 1)/a(\varepsilon)$. Then
\[
\mathcal{G}^\varepsilon(\varepsilon N^{-1}\psi^\varepsilon) \Rightarrow \mathcal{G}^0(\psi) \text{ in } D([0,T], \mathcal{V}).
\]

The proofs of (MDP-1), (MDP-2) is lengthy and involved, we will give the details in the next section. (MDP-1) will be proved in Proposition 4.3.1 and (MDP-2) will be established in Proposition 4.3.4.

4 The proofs of MDP-1 and MDP-2

We need some more preparations before the proof. The following Lemmas were proved in \cite{2}( see Lemma 4.2, Lemma 4.3 and Lemma 4.6 there).

Lemma 4.1 Let $h \in L^2(\nu) \cap \mathcal{H}$ and fix $M > 0$. Then there exists a constant $c_h > 0$ such that for any measurable subset $I \subseteq [0,T]$ and for all $\varepsilon > 0$,
\[
\sup_{\varphi \in \mathcal{S}^M_{\mathcal{A}}} \int_{I} h^2(z)\varphi(z, s)\nu(ds) \leq c_h(a^2(\varepsilon) + \lambda_T(I)).
\]
Lemma 4.2 Let \( h \in L^2(\nu) \cap \mathcal{H} \) and I be a measurable subset of \([0, T]\). Fix \( M > 0 \). Then there exists \( \Gamma_h, \rho_h : (0, \infty) \to (0, \infty) \) such that \( \Gamma_h(u) \downarrow 0 \) as \( u \uparrow \infty \) and for all \( \varepsilon, \beta \in (0, \infty) \),
\[
\sup_{\psi \in \mathcal{S}^M} \int_{Z \times I} |h(z)\psi(z, s)|1_{\{|\psi| > \beta/a(z)\}}\nu(dz)ds \leq \Gamma_h(\beta)(1 + \sqrt{\lambda_T(I)}),
\]
and
\[
\sup_{\psi \in \mathcal{S}^M} \int_{Z \times I} |h(z)\psi(z, s)|\nu(dz)ds \leq \rho_h(\beta)\sqrt{\lambda_T(I)} + \Gamma_h(\beta)a(\varepsilon).
\]

Lemma 4.3 Let \( h \in L^2(\nu) \cap \mathcal{H} \) be positive. Then for any \( \beta > 0 \),
\[
\lim_{\varepsilon \to 0} \sup_{\phi \in \mathcal{S}^M} \int_{Z \times I} |h(z)\phi(z, s)|1_{\{|\phi| > \beta/a(z)\}}\nu(dz)ds = 0.
\]

4.1 The proof of MDP-1

Proposition 4.1 If \( g^\varepsilon \to g \) in \( B_2(R) \), then \( \mathcal{G}^0(g^\varepsilon) \to \mathcal{G}^0(g) \) in \( C([0, T], \mathcal{V}) \).

Proof: Set \( \mathcal{G}^0(g^\varepsilon) = \eta^\varepsilon \) and \( \mathcal{G}^0(g) = \eta \). First, we will prove that there exist \( \varepsilon_0, C_R, C_{R, \alpha} \) such that
\[
\sup_{t \in [0, T]} \|\eta^\varepsilon(t)\|_W \leq C_R, \tag{4.5}
\]
and for \( \alpha \in (0, \frac{1}{2}) \)
\[
\|\eta^\varepsilon\|_{W^{0, 2}(0, T, W^*)}^2 \leq C_{R, \alpha}. \tag{4.6}
\]

By (5.9), we have
\[
d(\eta^\varepsilon(t), e_i) = d(\eta^\varepsilon(t), e_i) - \left( (\tilde{B}(\eta^\varepsilon(t), u^0(t), e_i) + (\tilde{B}(u^0(t), \eta^\varepsilon(t)), e_i), e_i) + (\tilde{F}^\varepsilon(u^0(t), t)\eta^\varepsilon(t), e_i)e_i + \int_Z (\tilde{G}(u^0(t), z)g^\varepsilon(z, t), e_i)e_i\nu(dz)dt. \tag{4.7}
\]

By a simple calculation, we know the fact:
\[
(\tilde{B}(u, v), u) = 0, \text{ for any } u, v \in W \cap H^4(\mathcal{O}).
\]

Then, applying the chain rule to \( (\eta^\varepsilon(t), e_i) \) and summing over \( i \) from 1 to \( \infty \) yields
\[
\|\eta^\varepsilon(t)\|_W^2 + \frac{2\kappa}{\alpha} \int_0^t \|\eta^\varepsilon(s)\|_W^2ds
\]
\[
= \frac{2\kappa}{\alpha} \int_0^t \left( \text{curl}(\eta^\varepsilon(t)), \text{curl}(\eta^\varepsilon(s)) - \alpha \Delta \eta^\varepsilon(s) \right)ds - 2 \int_0^t (\tilde{B}(u^0(s), \eta^\varepsilon(s)), \eta^\varepsilon(s))_Wds + 2 \int_0^t (\tilde{F}^\varepsilon(u^0(s), t)\eta^\varepsilon(s), \eta^\varepsilon(s))_Wds + 2 \int_Z (\tilde{G}(u^0(s), z)g^\varepsilon(z, s), \eta^\varepsilon(s))_W\nu(dz)ds
\]
\[
= I_1(t) + I_2(t) + I_3(t) + I_4(t). \tag{4.8}
\]

Noticing the fact (see (4.61) in [23]):
\[
|\text{curl}(v)|^2 \leq \frac{2}{\alpha} \|v\|_W \text{ for any } v \in \mathcal{W}, \tag{4.9}
\]
we have
\[
I_1(t) \leq C \int_0^t \|\eta^\varepsilon(s)\|_W^2ds. \tag{4.10}
\]
By Condition (I), interpolation inequality and a straightforward calculation, we have

\[
I_2(t) \leq C \int_0^t \left( -\Delta (u^0(s) - \alpha \Delta u^0(s)) \times \eta^\varepsilon(s), \text{curl} (\eta^\varepsilon(s) - \alpha \Delta \eta^\varepsilon(s)) \right) ds
\]

\[
\leq C \int_0^t \|u^0(s)\|_{L^4(C)} \|\eta^\varepsilon(s)\|_{L^8(C)} \|\eta^\varepsilon(s)\|_{W^1} ds
\]

\[
\leq C \int_0^t \|\eta^\varepsilon(s)\|_{W^1}^2 ds. \tag{4.11}
\]

By Condition (F2), we have

\[
I_3(t) \leq C \int_0^t \|\eta^\varepsilon(s)\|_{W^1}^2 ds. \tag{4.12}
\]

By Condition (G), we have

\[
I_4(t) \leq C \int_0^t \int_Z M_G(z) (1 + \|u^0(s)\|_{L^4}) |\eta^\varepsilon(s), z\| \|\eta^\varepsilon(s)\|_{W^1} \nu(dz) ds
\]

\[
\leq C \sup_{s \in [0,T]} (1 + \|u^0(s)\|_{L^4}) \int_0^t \int_Z \left( M_G^2(z) + |\eta^\varepsilon(s), z|^2 \right) \left( 1 + \|\eta^\varepsilon(s)\|_{W^1}^2 \right) \nu(dz) ds
\]

\[
\leq C \int_0^t \int_Z M_G^2(z) \nu(dz) ds + C \int_0^t \int_Z |\eta^\varepsilon(s), z|^2 \nu(dz) ds
\]

\[
+ C \int_0^t \|\eta^\varepsilon(s)\|_{W^1}^2 ds \int_Z M_G^2(z) \nu(dz) + C \int_0^t \int_Z |\eta^\varepsilon(s), z|^2 \|\eta^\varepsilon(s)\|_{W^1}^2 \nu(dz) ds
\]

\[
\leq C(T + R) + C \int_0^t \|\eta^\varepsilon(s)\|_{W^1}^2 (1 + \int_Z |\eta^\varepsilon(s), z|^2 \nu(dz)) ds. \tag{4.13}
\]

Combining (4.8)-(4.13), we have

\[
\|\eta^\varepsilon(t)\|_{W^1}^2 + \frac{2\kappa}{\alpha} \int_0^t \|\eta^\varepsilon(s)\|_{W^1}^2 ds \leq C(T + R) + C \int_0^t \|\eta^\varepsilon(s)\|_{W^1}^2 (1 + \int_Z |\eta^\varepsilon(s), z|^2 \nu(dz)) ds. \tag{4.14}
\]

Applying Gronwall’s inequality, we obtain (4.5).

Now we prove (4.9). By (3.9)

\[
\eta^\varepsilon(t) = -\kappa \int_0^t \tilde{A} \eta^\varepsilon(s) ds - \int_0^t \tilde{B} \eta^\varepsilon(s), u^0(s) ds - \int_0^t \tilde{B} \eta^\varepsilon(s), \eta^\varepsilon(s) ds
\]

\[
+ \int_0^t \tilde{F}(u^0(s), s) \eta^\varepsilon(s) ds + \int_0^t \int_Z \tilde{G}(u^0(s), z) \eta^\varepsilon(s) \nu(dz) ds
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5. \tag{4.15}
\]

Similarly as the proof of (5.38) in Zhai, Zhang, Zheng [34], we have

\[
\|I_1 + I_2 + I_3 + I_4\|_{W^{2,-2}(\mathbb{R}^+, [0,T], W^{-1})} \leq C_{R,\alpha}. \tag{4.16}
\]

For \(I_5\), we have

\[
\|I_5(t) - I_5(s)\|_{W^1}^2 = \left\| \int_s^t \int_Z \tilde{G}(u^0(l), z) \eta^\varepsilon(z, l) \nu(dz) dl \right\|_{W^1}^2
\]

\[
\leq C \left( \int_s^t \int_Z M_G(z) (1 + \|u^0(l)\|_{W^1}) |\eta^\varepsilon(z, l)\| \nu(dz) dl \right)^2
\]

\[
\leq C \int_s^t \int_Z M_G^2(z) (1 + \|u^0(l)\|_{W^1}^2) \int_s^t \int_Z |\eta^\varepsilon(z, l)|^2 \nu(dz) dl
\]

\[
\leq C \sup_{s \in [0,T]} (1 + \|u^0(s)\|_{W^1}^2) \int_s^t \int_Z M_G^2(z) \nu(dz) dl \int_s^t \int_Z |\eta^\varepsilon(z, l)|^2 \nu(dz) dl
\]
with initial value $Z(0) = 0$. We need the following estimates:

\[ |I_5|_{W^{0,2}(0,T,W^*)} \leq C_{R,\alpha}. \]  

Combining (4.16) and (4.18), we obtain (4.6).

Hence, by (4.5) and (4.10), we can assert the existence of element $\tilde{\eta} \in C([0,T], W) \cap L^\infty([0,T], W)$ and a subsequence $\eta^{k\varepsilon}$ such that, as $k \to \infty$

(a) $\sup_{t \in [0,T]} \|\tilde{\eta}(s)\|_W \leq C_R$,

(b) $\eta^{k\varepsilon} \rightharpoonup \tilde{\eta}$ in $L^2([0,T], W)$ weakly,

(c) $\eta^{k\varepsilon} \rightharpoonup \tilde{\eta}$ in $L^\infty([0,T], W)$ weak-star.

Moreover, applying Lemma 2.4 we have

(d) $\eta^{k\varepsilon} \rightharpoonup \tilde{\eta}$ in $L^2([0,T], V)$ strongly.

By the argument as that in the proof of Proposition 4.4 in Zhai, Zhang and Zheng [35], we know $\tilde{\eta} = \eta$.

Next, we prove $\eta^{k\varepsilon} \to \eta$ in $C([0,T], V)$. Let $Z^{k\varepsilon} = \eta^{k\varepsilon} - \eta$, then

\[
dZ^{k\varepsilon}(t) = -\kappa \tilde{A}Z^{k\varepsilon}(t)dt - \left(\tilde{B}(Z^{k\varepsilon}(t), u^0(t)) + \tilde{F}(u^0(t), Z^{k\varepsilon}(t))\right)dt + \tilde{E}(u^0(t), t)Z^{k\varepsilon}(t)dt + \int_{\Omega} \tilde{G}(u^0(t), z)(g^\varepsilon(z, t) - g(z, t))\nu(dz)dt,
\]

with initial value $Z^{k\varepsilon}(0) = 0$.

Applying the chain rule, by Lemma 2.3 and Conditions (F1), (F2) and (G), we have

\[
\begin{align*}
\|Z^{k\varepsilon}(t)\|_V^2 & = 2 \int_0^t \left(\tilde{B}(Z^{k\varepsilon}(s), Z^{k\varepsilon}(s)), u^0(s)\right)_W + 2 \int_0^t \left(\tilde{F}(u^0(s), s)Z^{k\varepsilon}(s), Z^{k\varepsilon}(s)\right)_V ds \\
& \quad + 2 \int_0^t \int_{\Omega} \left(\tilde{G}(u^0(s), z)(g^\varepsilon(z, s) - g(z, s)), Z^{k\varepsilon}(s)\right)_V \nu(dz)ds \\
& \leq C \sup_{s \in [0,T]} \|u^0(s)\|_W \int_0^t \|Z^{k\varepsilon}(s)\|_V^2 ds + C \int_0^t \|Z^{k\varepsilon}(s)\|_V^2 ds \\
& \quad + C \left\{1 + \|u^0(0)\|_V\right\} \int_0^t \int_{\Omega} M_G(z) |g^\varepsilon(z, s) - g(z, s)| \|Z^{k\varepsilon}(s)\|_V \nu(dz)ds \\
& \leq C \int_0^t \|Z^{k\varepsilon}(s)\|_V^2 ds + C \left\{1 + \|u^0(0)\|_V\right\} \left\{\int_0^t \int_{\Omega} M_G(z) \|Z^{k\varepsilon}(s)\|_V \nu(dz)ds\right\}^{\frac{1}{2}} \\
& \quad \times \left\{\int_0^t \int_{\Omega} (g^\varepsilon(z, s) - g(z, s))^2 \nu(dz)ds\right\}^{\frac{1}{2}} \\
& \leq C \int_0^t \|Z^{k\varepsilon}(s)\|_V^2 ds + C \left\{\int_0^t \|Z^{k\varepsilon}(s)\|_V^2 ds\right\}^{\frac{1}{2}}, \quad (4.20)
\end{align*}
\]

in the last inequality, we have used $M_G \in L^2(\nu_T)$ and $g^\varepsilon, g \in B_2(R)$.

Using (d), it follows that

\[
\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|Z^{k\varepsilon}(t)\|_V \leq \lim_{\varepsilon \to 0} \left\{C \int_0^T \|Z^{k\varepsilon}(s)\|_V^2 ds + C \left\{\int_0^T \|Z^{k\varepsilon}(s)\|_V^2 ds\right\}^{\frac{1}{2}}\right\} = 0. \quad (4.21)
\]

The proof is complete. \(\blacksquare\)

### 4.2 The proof of MDP-2

By Girsonav transform theorem, we can see that the following equation has a unique solution:

\[
dX^\varepsilon(t) = -\kappa \tilde{A}X^\varepsilon(t)dt - \tilde{B}(X^\varepsilon(t), X^\varepsilon(t))dt + \tilde{F}(X^\varepsilon(t), t) + \varepsilon \int_{\Omega} \tilde{G}(X^\varepsilon(t), z)\tilde{N}^{\varepsilon-1} \nu^\varepsilon(dz)dt + \int_{\Omega} \tilde{G}(X^\varepsilon(t), z)(\varepsilon^\varepsilon(t, z) - 1)\nu(dz)dt.
\]

with initial value $X^\varepsilon(0) = u_0$. We need the following estimates:
Lemma 4.4 There exists a $\varepsilon_0 > 0$ such that
\[
\sup_{\varepsilon \in (0, \varepsilon_0)} E \left[ \| X^\varepsilon(t) \|_W^2 \right] \leq C_{\varepsilon_0} < \infty. \tag{4.23}
\]

Let $\Pi_n$ be the projection operator from $W$ to $W$ defined as
\[
\Pi_n u = \sum_{i=1}^n (u, e_i)_W e_i, \quad u \in W.
\]

Set $W_n = \text{Span}\{e_1, \ldots, e_n\}$. Let $\tilde{X}^{\varepsilon_n} \in W_n$ be the Garlerkin approximations of (4.22) satisfying
\[
d(\tilde{X}^{\varepsilon_n}(s), e_i)_W + \kappa (\Pi_n \hat{A}X^{\varepsilon_n}(s), e_i)_W ds
= - (\Pi_n \hat{B}(\tilde{X}^{\varepsilon_n}(s), X^{\varepsilon_n}(s)), e_i)_W ds + (\Pi_n \hat{F}(X^{\varepsilon_n}(s), s), e_i)_W ds
+ \varepsilon \int_{\mathbb{Z}} (\Pi_n \hat{G}(X^{\varepsilon_n}(s-), z), e_i)_W \tilde{N}^{\varepsilon-1}\varphi_{\varepsilon}(dz ds), \tag{4.24}
\]
for any $i = 1, 2, \ldots, n$.

As in the proof of Theorem 3.2 in Shang, Zhai and Zhang [25], one can show that $\lim_{n \to \infty} X^{\varepsilon_n} = X^\varepsilon$ with respect to the weak topology in $L^4(\Omega, \mathcal{F}, P; L^\infty([0, T]; W))$. Hence Lemma 4.3 will follow from the following result.

Lemma 4.5 There exists $\varepsilon_0 > 0$ such that
\[
\sup_{\varepsilon \in (0, \varepsilon_0)} E \left[ \sup_{t \in [0, T]} \| X^{\varepsilon_n}(t) \|_W^2 \right] \leq C_{\varepsilon_0} < \infty. \tag{4.25}
\]

Proof: Multiplying $\lambda_i$ at both sides of the equation (4.24), we can use (2.5) to obtain
\[
d(\tilde{X}^{\varepsilon_n}(s), e_i)_W + \kappa (\hat{A}X^{\varepsilon_n}(s), e_i)_W ds
= - (\hat{B}(\tilde{X}^{\varepsilon_n}(s), X^{\varepsilon_n}(s)), e_i)_W ds + (\hat{F}(X^{\varepsilon_n}(s), s), e_i)_W ds
+ \varepsilon \int \hat{G}(X^{\varepsilon_n}(s-), \varphi_{\varepsilon}(s), e_i)_W \tilde{N}^{\varepsilon-1}\varphi_{\varepsilon}(dz ds), \tag{4.26}
\]
for any $i \in \mathbb{N}$.

Applying Itô’s formula to $(\tilde{X}^{\varepsilon_n}(s), e_i)_W^2$ and then summing over $i$ from 1 to $n$ yields
\[
\| X^{\varepsilon_n}(t) \|_W^2 = \| X_0 \|_W^2 - 2\kappa \int_0^t (\hat{A}X^{\varepsilon_n}(s), X^{\varepsilon_n}(s))_W ds
- 2 \int_0^t (\hat{B}(X^{\varepsilon_n}(s), X^{\varepsilon_n}(s)), X^{\varepsilon_n}(s))_W ds + 2 \int_0^t (\hat{F}(X^{\varepsilon_n}(s), s), X^{\varepsilon_n}(s))_W ds
+ 2 \varepsilon \int \hat{G}(X^{\varepsilon_n}(s-), \varphi_{\varepsilon}(s), X^{\varepsilon_n}(s))_W \tilde{N}^{\varepsilon-1}\varphi_{\varepsilon}(dz ds)
+ \varepsilon^2 \int \| \hat{G}(X^{\varepsilon_n}(s-), \varphi_{\varepsilon}(s), X^{\varepsilon_n}(s))_W \|_W^2 \tilde{N}^{\varepsilon-1}\varphi_{\varepsilon}(dz ds). \tag{4.27}
\]

By a simple calculation, we get
\[
\| X^{\varepsilon_n}(t) \|_W^2 + \frac{2\kappa}{\alpha} \int_0^t \| X^{\varepsilon_n}(s) \|_W^2 ds
\]
= \|X_0\|^2_W + \frac{2\kappa}{\alpha} \int_0^t \left(\text{curl}(X^{n,\varepsilon}(s)), \text{curl}(X^{n,\varepsilon}(s) - \alpha \Delta X^{n,\varepsilon}(s))\right) ds \\
+ 2\varepsilon \int_0^t (\mathcal{F}(X^{n,\varepsilon}(s), s), X^{n,\varepsilon}(s))_W ds + 2 \int_0^t \left( \int_Z \mathcal{G}(X^{n,\varepsilon}(s), z)(\varphi_{\varepsilon}(s, z) - 1)\nu(dz), X^{n,\varepsilon}(s)\right)_W ds \\
+ \varepsilon^2 \int_0^t \int_Z \|\mathcal{G}(X^{n,\varepsilon}(s), z)\|^2_W N^{\varepsilon^{-1}} \varphi_{\varepsilon}(dzds)
(4.28)

We have
\[
\frac{2\kappa}{\alpha} \int_0^t \left(\text{curl}(X^{n,\varepsilon}(s)), \text{curl}(X^{n,\varepsilon}(s) - \alpha \Delta X^{n,\varepsilon}(s))\right) ds + 2 \int_0^t (\mathcal{F}(X^{n,\varepsilon}(s), s), X^{n,\varepsilon}(s))_W ds \\
\leq C \int_0^t \|X^{n,\varepsilon}(s)\|_W \|X^{n,\varepsilon}(s)\|_W ds + C \int_0^t \|X^{n,\varepsilon}(s)\|_W \|\mathcal{F}(X^{n,\varepsilon}(s), s)\|_W ds \\
\leq C \int_0^t \|X^{n,\varepsilon}(s)\|_W^2 ds.
(4.29)

Set \(\psi_{\varepsilon}(s, z) = (\varphi_{\varepsilon}(s, z) - 1)/a(\varepsilon) \in U_{\varepsilon}^M\) . Then
\[
\left| \frac{2}{\alpha} \int_0^t \left( \int_Z \mathcal{G}(X^{n,\varepsilon}(s), z)(\varphi_{\varepsilon}(s, z) - 1)\nu(dz), X^{n,\varepsilon}(s)\right)_W ds \right| \\
\leq 2a(\varepsilon) \int_0^t \|X^{n,\varepsilon}(s)\|_W \int_Z \|\mathcal{G}(X^{n,\varepsilon}(s), z)\|W \|\psi_{\varepsilon}(s, z)\|_W \nu(dz) ds \\
\leq 2a(\varepsilon) \int_0^t \|X^{n,\varepsilon}(s)\|_W \left(1 + \|X^{n,\varepsilon}(s)\|_W \right) \int_Z M_G(z) \|\psi_{\varepsilon}(s, z)\|_W \nu(dz) ds \\
\leq 4a(\varepsilon) \int_0^t \left(1 + \|X^{n,\varepsilon}(s)\|_W^2\right) \int_Z M_G(z) \|\psi_{\varepsilon}(s, z)\|_W \nu(dz) ds.
(4.30)

Combining (4.28) - (4.30), we have
\[
\|X^{n,\varepsilon}(t)\|_W^2 + \frac{2\kappa}{\alpha} \int_0^t \|X^{n,\varepsilon}(s)\|_W^2 ds \\
= \|X_0\|_W^2 + \sup_{t \in [0, T]} \left| 2a(\varepsilon) \int_0^t \int_Z \mathcal{G}(X^{n,\varepsilon}(s), z)(\varphi_{\varepsilon}(s, z) - 1)\nu(dz) \|\psi_{\varepsilon}(s, z)\|_W \nu(dz) ds \\
+ \varepsilon^2 \int_0^t \int_Z \|\mathcal{G}(X^{n,\varepsilon}(s), z)\|^2_W N^{\varepsilon^{-1}} \varphi_{\varepsilon}(dzds) + 4a(\varepsilon) \int_0^t \int_Z M_G(z) \|\psi_{\varepsilon}(s, z)\|_W \nu(dz) ds \\
+ \int_0^t \|X^{n,\varepsilon}(s)\|_W^2 (C + 4a(\varepsilon) \int_Z M_G(z) \|\psi_{\varepsilon}(s, z)\|_W \nu(dz) ds) ds \right| \\
= I_1 + I_2 + I_3 + I_4 + I_5(t).
(4.31)

Applying Gronwall’s inequality and using Lemma 4.2 we get
\[
\|X^{n,\varepsilon}(t)\|_W^2 + \frac{2\kappa}{\alpha} \int_0^t \|X^{n,\varepsilon}(s)\|_W^2 ds \\
\leq \left( I_1 + I_2 + I_3 + I_4 \right) \exp \left\{ C T + 4a(\varepsilon) \left( \rho_{M_G}(\beta) \sqrt{T} + \Gamma_{M_G}(\beta) a(\varepsilon) \right) \right\}.
(4.32)

By Lemma 4.2 again, we get
\[
I_1 + I_4 \leq C + 4a(\varepsilon)(\rho_{M_G}(\beta) \sqrt{T} + \Gamma_{M_G}(\beta) a(\varepsilon)).
(4.33)

By B-D-G and Young’s inequalities, (3.6) and Lemma 4.1 we get
\[
EI_2 \leq 2\varepsilon E \left( \int_0^T \left( \mathcal{G}(s, X^{n,\varepsilon}(s), z), X^{n,\varepsilon}(s) \right)_W^2 N^{\varepsilon^{-1}} \varphi_{\varepsilon}(dzds) \right)^{\frac{1}{2}}
\]
\[
\begin{align*}
\leq & \ 2\varepsilon E\left(\sup_{s\in[0,T]} \|X^{n,\varepsilon}(s)\|_{\mathcal{W}}^2\right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathcal{Z}} \|\hat{G}(s, X^{n,\varepsilon}(s)-, z)\|_{\mathcal{W}}^2 N^{\varepsilon^{-1}} \varphi'(dzds)\right)^{\frac{1}{2}} \\
\leq & \ \frac{1}{4}E \left(\sup_{t\in[0,T]} \|X^{n,\varepsilon}(t)\|_{\mathcal{W}}^2\right) + 16\varepsilon^2 E \left(\int_0^T \int_{\mathcal{Z}} \|\hat{G}(s, X^{n,\varepsilon}(s)-, z)\|_{\mathcal{W}}^2 N^{\varepsilon^{-1}} \varphi'(dzds)\right) \\
\leq & \ \frac{1}{4}E \left(\sup_{t\in[0,T]} \|X^{n,\varepsilon}(t)\|_{\mathcal{W}}^2\right) + C_{\varepsilon, \mathcal{M}_\varepsilon} \left(a^2(\varepsilon) + T\right) \left(E \left(\sup_{t\in[0,T]} \|X^{n,\varepsilon}(t)\|_{\mathcal{W}}^2\right) + 1\right). \quad (4.34)
\end{align*}
\]

Similar to (4.34), we get
\[
\begin{align*}
EI_4 & = \varepsilon \int_0^T \int_{\mathcal{Z}} \|\hat{G}(X^{n,\varepsilon}(s)-, z)\|_{\mathcal{W}}^2 \varphi'(s, z) \nu(dz)ds \\
& \leq C_{\varepsilon, \mathcal{M}_\varepsilon} \left(a^2(\varepsilon) + T\right) \left(E \left(\sup_{t\in[0,T]} \|X^{n,\varepsilon}(t)\|_{\mathcal{W}}^2\right) + 1\right). \quad (4.35)
\end{align*}
\]

Choosing \(\varepsilon_0 > 0\) small enough such that \(C_{\varepsilon_0, \mathcal{M}_\varepsilon} \left(a^2(\varepsilon_0) + T\right) \leq \frac{1}{6}\), and combining (4.25) and (4.26), we obtain (4.29). The proof is complete. \(\blacksquare\)

Recall \(u_0\) in (1.3). We have

**Theorem 4.2**

\[
\lim_{\varepsilon \to 0} E_\varepsilon \left(\sup_{t\in[0,T]} \|X^\varepsilon(t) - u^0(t)\|_{\mathcal{W}}^2\right) = 0. \quad (4.36)
\]

**Proof:** Set \(Z^\varepsilon(t) = X^\varepsilon(t) - u^0(t)\). Then
\[
dZ^\varepsilon(t) = -\kappa \hat{A}Z^\varepsilon(t)dt - \hat{B}(Z^\varepsilon(t), Z^\varepsilon(t))dt - \hat{B}(Z^\varepsilon(t), u^0(t))dt + (\hat{F}(X^\varepsilon(t), t) - \hat{F}(u^0(t), t))dt \\
+ \varepsilon \int_0^t \hat{G}(X^\varepsilon(t-), z)N^{\varepsilon^{-1}} \varphi'(dzdt) + \int_0^t \hat{G}(X^\varepsilon(t), z)(\varphi'(t, z) - 1) \nu(dz)dt, \quad (4.37)
\]

with initial value \(Z^\varepsilon(0) = 0\). Applying Itô’s formula and (2.14), we get
\[
\begin{align*}
d\|Z^\varepsilon(t)\|^2_{\mathcal{W}} & = -2\langle \hat{B}(Z^\varepsilon(t), u^0(t)), Z^\varepsilon(t) \rangle_{\mathcal{W}} \nu dt + 2\langle \hat{F}(X^\varepsilon(t), t) - \hat{F}(u^0(t), t), Z^\varepsilon(t) \rangle_{\mathcal{V}} dt \\
& + 2\varepsilon \int_0^t \hat{G}(X^\varepsilon(t-), z)N^{\varepsilon^{-1}} \varphi'(dzdt) + 2\int_0^t \hat{G}(X^\varepsilon(t), z)(\varphi'(t, z) - 1) \nu(dz)dt \\
& + \varepsilon^2 \int_0^t \|\hat{G}(X^\varepsilon(t-), z)\|^2_{\mathcal{W}} N^{\varepsilon^{-1}} \varphi'(dzdt). \quad (4.38)
\end{align*}
\]

By Lemma 2.3 and Condition (F1), we get
\[
\int_0^t \|\hat{B}(Z^\varepsilon(s), u^0(s)), Z^\varepsilon(s)\|_{\mathcal{W}} \nu ds \leq C \sup_{t\in[0,T]} \|u^0(t)\|_{\mathcal{W}} \int_0^t \|Z^\varepsilon(s)\|^2_{\mathcal{W}} ds, \quad (4.39)
\]

and
\[
2 \int_0^t \|\hat{F}(X^\varepsilon(s), t) - \hat{F}(u^0(s), t), Z^\varepsilon(s)\|_{\mathcal{V}} dt \leq C \int_0^t \|Z^\varepsilon(s)\|^2_{\mathcal{V}} ds. \quad (4.40)
\]

Set \(\psi_{\varepsilon}(s, z) = (\varphi_{\varepsilon}(s, z) - 1)/a(\varepsilon)\). By Condition (G),
\[
\begin{align*}
2 \int_0^t \int_0^t \int_{\mathcal{Z}} \hat{G}(X^\varepsilon(s), z)(\varphi_{\varepsilon}(s, z) - 1) \nu(dz)ds \\
\leq & \ 2 \int_0^t \|Z^\varepsilon(s)\|_{\mathcal{V}} \int_{\mathcal{Z}} \|\hat{G}(X^\varepsilon(s), z) - \hat{G}(u^0(s), z)\|_{\mathcal{W}} \nu(dz)ds \\
& + 2 \int_0^t \|Z^\varepsilon(s)\|_{\mathcal{V}} \int_{\mathcal{Z}} \|\hat{G}(u^0(s), z)\|_{\mathcal{W}} \nu(dz)ds
\end{align*}
\]
and By Gronwall’s inequality and Lemma 4.2 and Lemma 3.1, we get

\[ \leq C\alpha \int_0^t \|Z^\varepsilon(t)\|^2 G(t) |\psi_\varepsilon(s,z)| \nu(dz)ds \]

+ \( C\alpha \int_0^t \left( 1 + \|Z^\varepsilon(t)\|^2 V \right) \left( 1 + \|u^0(t)\|_V \right) \int G(t) |\psi_\varepsilon(s,z)| \nu(dz)ds \]

\[ \leq C\alpha \int_0^t \|Z^\varepsilon(t)\|^2 G(t) \left( 1 + \|u^0(t)\|_V \right) \int M_G(z) |\psi_\varepsilon(s,z)| \nu(dz)ds \]

+ \( C\alpha \left( 1 + \sup_{t \in [0,T]} \|u^0(t)\|_V \right) \int M_G(z) |\psi_\varepsilon(s,z)| \nu(dz)ds. \] (4.41)

Combining (4.38)-(4.41), we get

\[ \|Z^\varepsilon(t)\|^2 G(t) + 2\kappa \int_0^t \|Z^\varepsilon(s)\|^2 V ds \leq M_1(T) + M_2(T) + M_3(T) + \int_0^t J(s) \|Z^\varepsilon(s)\|^2 G(t) ds, \] (4.42)

here

\[ M_1(T) = 2\varepsilon \sup_{t \in [0,T]} \left( \int_0^t \int_G \left( \hat{G}(X^\varepsilon(s_0),z), Z^\varepsilon(s_0) \right) \psi^\varepsilon \nu^\varepsilon (dz)ds \right), \]

\[ M_2(T) = \varepsilon^2 \int_0^T \int G(t_0, z) \left\| \hat{G}(X^\varepsilon(t_0), z) \right\|_V^2 N^\varepsilon \nu^\varepsilon (dz)dt, \]

\[ M_3(T) = C\alpha \left( 1 + \sup_{t \in [0,T]} \|u^0(t)\|_V \right) \int M_G(z) |\psi_\varepsilon(s,z)| \nu(dz)ds, \]

and

\[ J(s) = C \left( \sup_{t \in [0,T]} \|u^0(t)\|_V + 1 + a\varepsilon \int G(t) |\psi_\varepsilon(s,z)| \nu(dz) \right. \]

\[ + a\varepsilon \left( 1 + \sup_{t \in [0,T]} \|u^0(t)\|_V \right) \int M_G(z) |\psi_\varepsilon(s,z)| \nu(dz)ds \].

By Gronwall’s inequality and Lemma 4.2 and Lemma 5.1

\[ \|Z^\varepsilon(t)\|^2 G(t) + 2\kappa \int_0^t \|Z^\varepsilon(s)\|^2 V ds \]

\[ \leq \left( M_1(T) + M_2(T) + M_3(T) \right) \exp \left( \int_0^T J(s)ds \right) \]

\[ \leq C \left( M_1(T) + M_2(T) + M_3(T) \right). \] (4.43)

By B-D-G inequality, Lemma 4.1 and (4.28), we get

\[ EM_1(T) \leq 2\varepsilon E \left[ \left( \int_0^T \int_G \left( \hat{G}(X^\varepsilon(s_0),z), Z^\varepsilon(s_0) \right) \psi^\varepsilon \nu^\varepsilon (dz)ds \right)^2 \right] \]

\[ \leq 2\varepsilon E \left[ \sup_{s \in [0,T]} \|Z^\varepsilon(s)\|^2 \right]^2 \left( \int_0^T \int_G \left( \hat{G}(X^\varepsilon(s_0),z) \right) \psi^\varepsilon \nu^\varepsilon (dz)ds \right)^2 \]

\[ \leq \frac{1}{2} E \left[ \sup_{t \in [0,T]} \|Z^\varepsilon(t)\|^2 \right] + C\varepsilon E \left[ \int_0^T \int M_G^2(z) \left( 1 + \|X^\varepsilon(s)\|_V^2 \right) \psi_\varepsilon(s,z) \nu(ds)dz \right] \]

\[ \leq \frac{1}{2} E \left[ \sup_{t \in [0,T]} \|Z^\varepsilon(t)\|^2 \right] + C\varepsilon E \left[ \sup_{t \in [0,T]} \|X^\varepsilon(t)\|_V^2 \right] \]

\[ \leq \frac{1}{2} E \left[ \sup_{t \in [0,T]} \|Z^\varepsilon(t)\|^2 \right] + C\varepsilon M_G \left( a^2(\varepsilon) + T \right) \left( E \left[ \sup_{t \in [0,T]} \|X^\varepsilon(t)\|_V^2 \right] + 1 \right). \] (4.44)
Similarly, we have

\[
EM_3(T) = \varepsilon E \left[ \int_0^T \int_Z \|\hat{G}(X^\varepsilon(t-,z))\|_\varphi^2(t,z)\nu(dz)dt \right] \\
\leq C\varepsilon E \left[ \left( \int_0^T \int_Z M_{\varepsilon}^2(z)(1 + \|X^\varepsilon(s)\|^2)\varphi^2(s,z)\nu(dz)ds \right) \right] \\
\leq C\varepsilon M_{\varepsilon}(a^2(\varepsilon) + T). \tag{4.45}
\]

By Lemma \[4.2\] and Lemma \[5.1\]

\[
EM_3(T) \leq Ca(\varepsilon)\left(\rho_{M_{\varepsilon}}(\beta)\sqrt{T} + \Gamma_{M_{\varepsilon}}(\beta)a(\varepsilon)\right). \tag{4.46}
\]

Combining \[4.43\] - \[4.46\], we obtain \[4.36\].

The proof is complete.

\[\square\]

Notice

\[
G^\varepsilon(\varepsilon N^\varepsilon,\varphi^\varepsilon) := Y^\varepsilon = \frac{X^\varepsilon - u^0}{a(\varepsilon)}. \tag{4.47}
\]

and \(Y^\varepsilon(t)\) satisfies

\[
dY^\varepsilon(t) = -\kappa \hat{A}Y^\varepsilon(t)dt - \left(\hat{B}(X^\varepsilon(t),Y^\varepsilon(t)) + \hat{B}(Y^\varepsilon(t),u^0(t))\right)dt + \frac{\varepsilon}{a(\varepsilon)} \left(\hat{F}(X^\varepsilon(t),t) - \hat{F}(u^0(t),t)\right)dt \\
+ \frac{\varepsilon}{a(\varepsilon)} \int_Z \hat{G}(X^\varepsilon(t-),z)\tilde{N}^\varepsilon,\varphi^\varepsilon(dz)dt + \int_Z \hat{G}(X^\varepsilon(t),z)(\varphi^\varepsilon(t,z) - 1/a(\varepsilon))\nu(dz)dt, \tag{4.48}
\]

with initial value \(Y^\varepsilon(0) = 0\).

**Proposition 4.3** Given \(M < \infty\), Let \(\{\varphi^\varepsilon\}_{\varepsilon > 0}\) be such that \(\varphi^\varepsilon \in U_{+}^M\) for every \(\varepsilon > 0\). Let \(\psi^\varepsilon = (\varphi^\varepsilon - 1)/a(\varepsilon)\) and \(\beta \in (0,1]\). Then the family \(\{Y^\varepsilon, \psi^\varepsilon 1(\|\psi^\varepsilon\|_{\beta/a(\varepsilon)}) \}_{\varepsilon > 0}\) is tight in \(D([0,T], \mathbb{V}) \times B_2(\sqrt{M_{\varepsilon}^2(1)}), \) and any limit point \((Y, \psi)\) solves the equation \[3.2\].

**Proof:** The proof is divided into several steps.

**Step 1.** Let \(Z^\varepsilon\) be the solution of the following equation

\[
dZ^\varepsilon(t) = -\kappa \hat{A}Z^\varepsilon(t)dt + \frac{\varepsilon}{a(\varepsilon)} \int_Z \hat{G}(X^\varepsilon(t-),z)\tilde{N}^\varepsilon,\varphi^\varepsilon(dz)dt, \tag{4.49}
\]

with initial value \(Z^\varepsilon(0) = 0\). Then, by \[2.5\]

\[
d(Z^\varepsilon(t),e_i)_W = -\kappa \hat{A}Z^\varepsilon(t)dt + \frac{\varepsilon}{a(\varepsilon)} \int_Z \hat{G}(X^\varepsilon(t-),z)e_i)_W\tilde{N}^\varepsilon,\varphi^\varepsilon(dz)dt, \tag{4.50}
\]

for \(i \in \mathbb{N}\).

Applying Itô’s formula to \((Z^\varepsilon(t),e_i)^2_W\) and summing over \(i\) from 1 to \(\infty\) yields

\[
\|Z^\varepsilon(t)\|_W^2 + \frac{2\kappa}{\alpha} \int_0^t \|Z^\varepsilon(s)\|_{\varphi}^2 ds \\
= \frac{2\kappa}{\alpha} \int_0^t \left(\text{curl}(Z^\varepsilon(s)), \text{curl}(Z^\varepsilon(s) - \alpha \Delta Z^\varepsilon(s))\right)ds + \frac{2\varepsilon}{a(\varepsilon)} \int_0^t \int_Z \hat{G}(X^\varepsilon(s-),z)(Z^\varepsilon(s-))_W\tilde{N}^\varepsilon,\varphi^\varepsilon(dz)ds \\
+ \frac{\varepsilon^2}{a^2(\varepsilon)} \int_0^t \int_Z \|\hat{G}(X^\varepsilon(s-),z)\|_{\varphi}^2\tilde{N}^\varepsilon,\varphi^\varepsilon(dz)ds. \tag{4.51}
\]

We have

\[
\frac{2\kappa}{\alpha} \int_0^t \left(\text{curl}(Z^\varepsilon(s)), \text{curl}(Z^\varepsilon(s) - \alpha \Delta Z^\varepsilon(s))\right)ds \leq C \int_0^t \|Z^\varepsilon(s)\|_{\varphi}^2 ds. \tag{4.52}
\]

By B-D-G inequality, Lemma \[4.11\] and \[4.28\],

\[
E\left[ \sup_{t \in [0,T]} \left| \frac{2\varepsilon}{a(\varepsilon)} \int_0^t \int_Z (\hat{G}(X^\varepsilon(s-),z), Z^\varepsilon(s-))_W\tilde{N}^\varepsilon,\varphi^\varepsilon(dz)ds \right| \right]
\]

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\[
\leq \frac{2\varepsilon}{a(\varepsilon)} E \left[ \left( \int_0^T \int_Z \left( \tilde{G}(s, X^\varepsilon(s,-), z), Z^\varepsilon(s-) \right)_W^2 N^{\varepsilon^{-1}} \varphi^* (dz ds) \right)^{\frac{1}{2}} \right]
\]
\[
\leq \frac{2\varepsilon}{a(\varepsilon)} E \left[ \left( \sup_{s \in [0,T]} \| Z^\varepsilon(s) \|_W^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_Z \| \tilde{G}(X^\varepsilon(s,-), z) \|_W^2 N^{\varepsilon^{-1}} \varphi^* (dz ds) \right)^{\frac{1}{2}} \right]
\]
\[
\leq \frac{1}{2} E \left[ \sup_{t \in [0,T]} \| Z^\varepsilon(t) \|_W^2 \right] + \frac{C\varepsilon}{a^2(\varepsilon)} E \left[ \int_0^T \int_Z M_{\varphi^*}^2 (z) \left( 1 + \| X^\varepsilon(s) \|_W^2 \right) \varphi^*(s, z) \nu(dz ds) \right]
\]
\[
\leq \frac{1}{2} E \left[ \sup_{t \in [0,T]} \| Z^\varepsilon(t) \|_W^2 \right] + \frac{C\varepsilon}{a^2(\varepsilon)} M_{\varphi^*} (a^2(\varepsilon) + T). \tag{4.53}
\]

And similarly
\[
E \left[ \frac{\varepsilon^2}{a^2(\varepsilon)} \int_0^T \int_Z \| \tilde{G}(X^\varepsilon(s,-), z) \|_W^2 N^{\varepsilon^{-1}} \varphi^* (dz ds) \right]
\]
\[
\leq \frac{\varepsilon}{a^2(\varepsilon)} E \left[ \int_0^T \int_Z \| \tilde{G}(X^\varepsilon(s), z) \|_W^2 \varphi^*(s, z) (dz ds) \right]
\]
\[
\leq \frac{C\varepsilon}{a^2(\varepsilon)} M_{\varphi^*} (a^2(\varepsilon) + T). \tag{4.54}
\]

Combining (4.51), (4.54) and applying Gronwall’s inequality, we obtain
\[
\lim_{\varepsilon \to 0} E \left[ \sup_{t \in [0,T]} \| Z^\varepsilon(t) \|_W^2 \right] = 0. \tag{4.55}
\]

**Step 2.** Recall \( \psi^\varepsilon = (\varphi^\varepsilon - 1)/a(\varepsilon) \). Let \( L^\varepsilon(t) \) be the unique solution of
\[
dL^\varepsilon(t) = -\kappa \tilde{L}^\varepsilon(t) dt + \int_Z \tilde{G}(X^\varepsilon(t), z) \psi^\varepsilon(z, t) 1_{\{|\psi^\varepsilon| > \beta/a(\varepsilon)\}} (dz) dt,
\]
with initial value \( L^\varepsilon(0) = 0 \). Using similar arguments as getting (4.51), we have
\[
\| L^\varepsilon(t) \|_W^2 + \frac{2\kappa}{a} \int_0^t \| L^\varepsilon(s) \|_W^2 ds
\]
\[
= \frac{2\kappa}{a} \int_0^t \left( \text{curl}(L^\varepsilon(s)), \text{curl}(L^\varepsilon(s) - \alpha \Delta L^\varepsilon(s)) \right) ds
\]
\[
+ 2\int_0^t \int_Z \left( \tilde{G}(X^\varepsilon(s), z) \psi^\varepsilon(z, s) 1_{\{|\psi^\varepsilon| > \beta/a(\varepsilon)\}} \right) L^\varepsilon(s) (dz) ds
\]
\[
\leq C \int_0^t \| L^\varepsilon(s) \|_W^2 ds + 2 \int_0^T \int_Z \| \tilde{G}(X^\varepsilon(s), z) \|_W \| \psi^\varepsilon(z, s) 1_{\{|\psi^\varepsilon| > \beta/a(\varepsilon)\}} \|_W \| L^\varepsilon(s) \|_W \nu(dz ds)
\]
\[
\leq C \int_0^t \| L^\varepsilon(s) \|_W^2 ds + \frac{1}{2} \sup_{t \in [0,T]} \| L^\varepsilon(t) \|_W^2
\]
\[
+ C \sup_{t \in [0,T]} \| L^\varepsilon(t) \|_W \sup_{t \in [0,T]} (1 + \| X^\varepsilon(t) \|_W) \int_0^T \int_Z M_{\varphi^*}(z) \| \psi^\varepsilon(z, s) 1_{\{|\psi^\varepsilon| > \beta/a(\varepsilon)\}} \|_W \nu(dz ds)
\]
\[
\leq C \int_0^t \| L^\varepsilon(s) \|_W^2 ds + \frac{1}{2} \sup_{t \in [0,T]} \| L^\varepsilon(t) \|_W^2
\]
\[
+ C \sup_{t \in [0,T]} (1 + \| X^\varepsilon(t) \|_W^2) \left\{ \int_0^T \int_Z M_{\varphi^*}(z) \| \psi^\varepsilon(z, s) 1_{\{|\psi^\varepsilon| > \beta/a(\varepsilon)\}} \|_W \nu(dz ds) \right\}^2. \tag{4.57}
\]

Noticing that, by (4.20) and Lemma 4.3, we have
\[
CE \left[ \sup_{t \in [0,T]} (1 + \| X^\varepsilon(t) \|_W^2) \right] \left\{ \sup_{\psi \in s^2} \int_0^T \int_Z M_{\varphi^*}(z) \| \psi(z, s) 1_{\{|\psi| > \beta/a(\varepsilon)\}} \|_W \nu(dz ds) \right\}^2 \to 0 \text{ as } \varepsilon \to 0. \tag{4.58}
\]
Combining (4.54) and (4.58) and applying Gronwall’s inequality, we obtain
\[
\lim_{\varepsilon \to 0} E \left[ \sup_{t \in [0,T]} \|U^\varepsilon(t)\|_W^2 \right] = 0. \tag{4.59}
\]

**Step 3.** Denote \(U^\varepsilon\) the solution of the following equation
\[
dU^\varepsilon(t) = -\kappa \tilde{\Lambda} U^\varepsilon(t)dt + \int_{\mathbb{Z}} (\tilde{G}(X^\varepsilon(t), z) - \tilde{G}(u^0(t), z)) \psi^\varepsilon(z, t) 1_{\|\psi^\varepsilon\| \leq \beta/a(\varepsilon)} \nu(dz) dt. \tag{4.60}
\]

Similar to **step 2**, we have
\[
\|U^\varepsilon(t)\|_W^2 + \frac{2\kappa}{\alpha} \int_0^t \|U^\varepsilon(s)\|_W^2 ds \\
= \frac{2\kappa}{\alpha} \int_0^t \left( \text{curl}(U^\varepsilon(s)), \text{curl}(U^\varepsilon(s) - \alpha \Delta U^\varepsilon(s)) \right) ds \\
+ 2 \int_0^t \int_{\mathbb{Z}} \left( (\tilde{G}(X^\varepsilon(s), z) - \tilde{G}(u^0(s), z)) \psi^\varepsilon(z, t) 1_{\|\psi^\varepsilon\| \leq \beta/a(\varepsilon)}, U^\varepsilon(s) \right)_W \nu(dz) ds \\
\leq C \int_0^t \|U^\varepsilon(s)\|_W^2 ds + 2 \int_0^T \int_{\mathbb{Z}} (\tilde{G}(X^\varepsilon(s), z) - \tilde{G}(u^0(s), z)) \|\psi^\varepsilon(z, s)\|_W \|\psi^\varepsilon(z, s)\|_W \nu(dz) ds \\
\leq C \int_0^t \|U^\varepsilon(s)\|_W^2 ds + \frac{1}{2} \sup_{t \in [0,T]} \|U^\varepsilon(t)\|_W^2 \\
+ C \sup_{t \in [0,T]} \|X^\varepsilon(t) - u^0(t)\|_W^2 \sup_{\psi \in S^M} \left\{ \int_0^T \int_{\mathbb{Z}} L_G(z) |\psi(z, s)| \nu(dz) ds \right\}^2. \tag{4.61}
\]

Noticing that, by (4.36) and Lemma 4.2, we have
\[
CE \left[ \sup_{t \in [0,T]} \|X^\varepsilon(t) - u^0(t)\|_W^2 \sup_{\psi \in S^M} \left\{ \int_0^T \int_{\mathbb{Z}} L_G(z) |\psi(z, s)| \nu(dz) ds \right\}^2 \right] \to 0 \text{ as } \varepsilon \to 0. \tag{4.62}
\]

Combining (4.61) and (4.62) and applying Gronwall’s inequality, we obtain
\[
\lim_{\varepsilon \to 0} E \left[ \sup_{t \in [0,T]} \|U^\varepsilon(t)\|_W^2 \right] = 0. \tag{4.63}
\]

**Step 4.** Set \(K^\varepsilon = Z^\varepsilon + L^\varepsilon + U^\varepsilon\) and denote \(Y^\varepsilon = Y^\varepsilon - K^\varepsilon\). By (4.48), we have
\[
dY^\varepsilon(t) = -\tilde{\Lambda} Y^\varepsilon(t)dt - a(\varepsilon)\tilde{B}(Y^\varepsilon(t)) + R^\varepsilon(t)dt \\
= \tilde{B}(u^0(t), Y^\varepsilon(t))dt - \tilde{B}(Y^\varepsilon(t) + K^\varepsilon(t), u^0(t))dt \\
+ \frac{1}{a(\varepsilon)} \left( F(u^0(t), a(\varepsilon) (Y^\varepsilon(t) + K^\varepsilon(t), t) - F(u^0(t), t) \right) dt \\
+ \int_{\mathbb{Z}} \tilde{G}(u^0(t), z) \psi^\varepsilon(z, t) 1_{\|\psi^\varepsilon\| \leq \beta/a(\varepsilon)} \nu(dz) dt. \tag{4.64}
\]

Set
\[
\Pi = \left( D([0,T], \mathcal{V}) \cap L^2([0,T], W); C([0,T], \mathcal{V}) \cap L^2([0,T], W); B_2(\sqrt{MK_2(1)}) \right).
\]

By (4.54), (4.59), (4.63), and notice that \(\psi^\varepsilon 1_{\|\psi^\varepsilon\| \leq \beta/a(\varepsilon)} \) is tight in \(B_2(\sqrt{MK_2(1)})\) with the weak topology of \(L^2(\nu_T)\) (see Lemma 3.2 in [2]), \((Z^\varepsilon, L^\varepsilon + U^\varepsilon, \psi^\varepsilon 1_{\|\psi^\varepsilon\| \leq \beta/a(\varepsilon)}) \) is tight in \(\Pi\), and let \((0,0,\psi)\) be any limit point of the tight family, and denote by \(Y = \mathcal{G}^0(\psi)\) the solution of equation (3.5).

It follows from the Skorokhod representation theorem that there exist a probability space \((\Omega^1, \mathcal{F}^1, \mathbb{P}^1)\) and on this space, \(\Pi\)-valued random variables \((\tilde{Z}^\varepsilon, \tilde{J}^\varepsilon, \tilde{\psi}^\varepsilon), (0,0,\psi), \varepsilon \in (0,\varepsilon_0)\), such that \((\tilde{Z}^\varepsilon, \tilde{J}^\varepsilon, \tilde{\psi}^\varepsilon)\) (respectively
Consider the following equation

\[
\tilde{M}^\varepsilon = \tilde{Z}^\varepsilon + \tilde{J}^\varepsilon.
\]

Set \( \tilde{K}^\varepsilon = \tilde{Z}^\varepsilon + \tilde{J}^\varepsilon \). Denote by \( \tilde{Y}^\varepsilon(t) \) the unique solution of (4.64) with \( (K^\varepsilon, \psi^\varepsilon 1\{\psi^\varepsilon \leq \beta / \alpha(\varepsilon)\}) \) replaced by \( (\tilde{K}^\varepsilon, \tilde{\psi}^\varepsilon) \). Then \( (\tilde{K}^\varepsilon, \tilde{Y}^\varepsilon) \) has the same law as \( (K^\varepsilon, Y^\varepsilon) \). Hence, \( Y^\varepsilon = \tilde{K}^\varepsilon + \tilde{Y}^\varepsilon \) has the same law as \( Y^\varepsilon = K^\varepsilon + Y^\varepsilon \) in \( D([0, T], \mathcal{V}) \cap L^2([0, T], \mathcal{W}) \). Denote by \( \tilde{Y} \) the solution of equation (39) with \( \psi(z, t) \) replaced by \( \tilde{\psi}(z, t) \). \( \tilde{Y} \) must have the same law as \( \tilde{Y} \).

Thus the proof of the proposition will be complete if we can show that

\[
sup_{t \in [0, T]} \| \tilde{Y}^\varepsilon(t) - \tilde{Y}(t) \|_\mathcal{V} \to 0, \quad \mathbb{P}^1 - \text{a.s.}, \quad \varepsilon \to 0.
\]  

(4.65)

Consider the following equation

\[
d\tilde{\tilde{M}}^\varepsilon(t) = -\tilde{\tilde{A}}\tilde{M}^\varepsilon(t)dt + \int \tilde{G}(u^0(t), z)\tilde{\psi}^\varepsilon(z, t)\nu(dz)dt,
\]

and

\[
d\tilde{\tilde{I}}^\varepsilon(t) = -\tilde{\tilde{A}}\tilde{I}^\varepsilon(t)dt + \int \tilde{G}(u^0(t), z)\psi(z, t)\nu(dz)dt.
\]

As the proof of (4.21), first we can show

\[
\lim_{\varepsilon \to 0} \sup_{t \in [0, T]} \| \tilde{\tilde{M}}^\varepsilon(t) - \tilde{\tilde{I}}(t) \|_\mathcal{V}^2 = 0, \quad \mathbb{P}^1 - \text{a.s.}.
\]

(4.68)

Set \( \tilde{M} = \tilde{Y} - \tilde{I} \) and \( \tilde{M}^\varepsilon = \tilde{Y}^\varepsilon - \tilde{K}^\varepsilon - \tilde{\tilde{I}}^\varepsilon \). Then

\[
d\tilde{M}(t) = -\kappa\tilde{M}(t)dt - \tilde{I} \tilde{M}(t) + \tilde{\tilde{I}}(t), u^0(t))dt - \tilde{B}(u^0(t), \tilde{M}(t) + \tilde{\tilde{I}}(t))dt
\]

(4.69)

and

\[
d\tilde{M}^\varepsilon(t) = -\kappa\tilde{M}^\varepsilon(t)dt - a(\varepsilon)\tilde{B}(\tilde{M}^\varepsilon(t) + \tilde{K}^\varepsilon(t) + \tilde{\tilde{I}}^\varepsilon(t) + \tilde{\tilde{I}}^\varepsilon(t))dt
\]

(4.70)

Since

\[
\lim_{\varepsilon \to 0} \sup_{t \in [0, T]} \| \tilde{\tilde{M}}^\varepsilon(t) \|_\mathcal{V}^2 = 0, \quad \mathbb{P}^1 - \text{a.s.,}
\]

(4.71)

taking into account (4.68), the proof of (4.65) reduces to show

\[
\lim_{\varepsilon \to 0} \sup_{t \in [0, T]} \| \tilde{M}^\varepsilon(t) - \tilde{M}(t) \|_\mathcal{V} = 0, \quad \mathbb{P}^1 - \text{a.s.}
\]

(4.72)

By the similar arguments as in the proof of (4.5) and using (4.68) and (4.71) again, we have

\[
\sup_{\varepsilon \in (0, \varepsilon_0)} \left[ \sup_{t \in [0, T]} \| \tilde{M}^\varepsilon(t) \|_\mathcal{V}^2 + \sup_{t \in [0, T]} \| \tilde{M}(t) \|_\mathcal{V}^2 \right] \leq C(\omega^1) < \infty, \quad \mathbb{P}^1 - \text{a.s.}
\]

(4.73)

Set \( \tilde{N}^\varepsilon = \tilde{M}^\varepsilon - \tilde{M} \) and \( \tilde{\tilde{N}}^\varepsilon = \tilde{K}^\varepsilon - \tilde{I} \), we have

\[
d\tilde{N}^\varepsilon(t) = -\kappa\tilde{N}^\varepsilon(t)dt - a(\varepsilon)\tilde{B}(\tilde{M}^\varepsilon(t) + \tilde{K}^\varepsilon(t) + \tilde{\tilde{I}}^\varepsilon(t), \tilde{M}^\varepsilon(t) + \tilde{\tilde{I}}^\varepsilon(t) + \tilde{\tilde{I}}^\varepsilon(t))dt
\]

(4.74)
Applying the chain rule, we have
\[
\|\tilde{N}(t)\|_V^2 + 2\kappa \int_0^t \|\tilde{N}(s)\|^2 ds \\
= -2a(\varepsilon) \int_0^t \left( \tilde{B}(\tilde{M}(s) + \tilde{K}(s) + \tilde{\Gamma}(s), \tilde{N}(s)) \|_{\mathcal{W}'} ds \\
-2 \int_0^t \left( \tilde{B}(\tilde{N}(s) + \tilde{K}(s) + \tilde{\Gamma}(s), u^0(s)), \tilde{N}(s)) \|_{\mathcal{W}'} ds \\
-2 \int_0^t \left( \tilde{B}(u^0(s), \tilde{N}(s) + \tilde{K}(s) + \tilde{\Gamma}(s)), \tilde{N}(s)) \|_{\mathcal{W}'} ds \\
+ \frac{2}{a(\varepsilon)} \int_0^t \left( \tilde{F}(u^0(s) + a(\varepsilon)(\tilde{M}(s) + \tilde{K}(s) + \tilde{\Gamma}(s)), s) - \tilde{F}(u^0(s), s), \tilde{N}(s)) \|_{\mathcal{V}} ds \\
-2 \int_0^t \left( \tilde{F}^*(u^0(s), s)(\tilde{M}(s) + \tilde{\Gamma}(s)), \tilde{N}(s)) \|_{\mathcal{V}} ds \\
= I_1(t) + I_2(t) + I_3(t) + I_4(t). \tag{4.75}
\]
Now we estimate $I_i, i = 1, 2, 3, 4$ respectively. By Lemma 2.3, Lemma 5.1 and 1.28, we have
\[
|I_1(t)| \leq 2a(\varepsilon) \int_0^t \|\tilde{M}(s) + \tilde{K}(s) + \tilde{\Gamma}(s)\|_V^2 \|\tilde{N}(s)\|_W ds \\
\leq 4a(\varepsilon)C(\omega_1, T) \sup_{s \in [0,T]} \left( \|\tilde{M}(s)\|_V^2 + \|\tilde{K}(s)\|_V^2 + \|\tilde{\Gamma}(s)\|_V^2 \right),
\]
\[
|I_2(t)| = 2 \int_0^t \left( \tilde{B}(\tilde{N}(s) + \tilde{K}(s) + \tilde{\Gamma}(s), \tilde{N}(s), u^0(s)) \|_{\mathcal{W}'} ds \\
\leq 2 \int_0^t \left( \tilde{B}(\tilde{N}(s) + \tilde{K}(s) + \tilde{\Gamma}(s), \tilde{N}(s) + \tilde{K}(s) + \tilde{\Gamma}(s), \tilde{N}(s)) \|_{\mathcal{W}'} ds \\
+ 2 \int_0^t \left( \tilde{B}(\tilde{K}(s) + \tilde{\Gamma}(s), \tilde{K}(s) + \tilde{\Gamma}(s), u^0(s)) \|_{\mathcal{W}'} ds \\
+ 2 \int_0^t \left( \tilde{B}(\tilde{N}(s), \tilde{K}(s) + \tilde{\Gamma}(s), u^0(s)) \|_{\mathcal{W}'} ds \\
\leq C \int_0^t \|\tilde{N}(s)\|_V^2 ds + C \sup_{s \in [0,T]} \left( \|\tilde{K}(s)\|_V^2 + \|\tilde{\Gamma}(s)\|_V^2 \right) + C(\omega_1, T) \sup_{s \in [0,T]} \left( \|\tilde{K}(s)\|_V + \|\tilde{\Gamma}(s)\|_V \right),
\]
and
\[
|I_3(t)| = 2 \int_0^t \left( \tilde{B}(u^0(s), \tilde{K}(s) + \tilde{\Gamma}(s), \tilde{N}(s)) \|_{\mathcal{W}'} ds \\
\leq C(\omega_1, T) \sup_{s \in [0,T]} \left( \|\tilde{K}(s)\|_V + \|\tilde{\Gamma}(s)\|_V \right).
\]
Noticing the fact: there exists $\theta(s) \in [0,1]$ depending on $s, x, y, s$ such that
\[
\tilde{F}(x+y, s) - \tilde{F}(x, s) = \tilde{F}^*(x + \theta y, s),
\]
for any $x, y \in \mathcal{V}$.

Combining Conditions (F1) and (F2), we have
\[
|I_4(t)| \leq 2 \int_0^t \left| \left( \frac{1}{a(\varepsilon)} \left( \tilde{F}(u^0(s) + a(\varepsilon)(\tilde{M}(s) + \tilde{K}(s) + \tilde{\Gamma}(s)), s) - \tilde{F}(u^0(s), s) \right) - \\
\tilde{F}(u^0(s), s)(\tilde{M}(s) + \tilde{K}(s) + \tilde{\Gamma}(s)), \tilde{N}(s)) \|_{\mathcal{V}} ds \\
+ 2 \int_0^t \left| \left( \tilde{F}(u^0(s), s)(\tilde{M}(s) + \tilde{K}(s) + \tilde{\Gamma}(s)) - \tilde{F}(u^0(s), s)(\tilde{M}(s) + \tilde{\Gamma}(s)), \tilde{N}(s)) \|_{\mathcal{V}} ds \\
\leq C(\omega_1, T)a(\varepsilon) \sup_{s \in [0,T]} \left( \|\tilde{M}(s)\|_V^2 + \|\tilde{K}(s)\|_V^2 + \|\tilde{\Gamma}(s)\|_V^2 \right) \\
+ C \int_0^t \|\tilde{N}(s)\|_V^2 ds + C(\omega_1, T) \sup_{s \in [0,T]} \left( \|\tilde{K}(s)\|_V^2 + \|\tilde{\Gamma}(s)\|_V^2 \right).
\]
Since \( \lim_{\varepsilon \to 0} a(\varepsilon) = 0 \) and
\[
\lim_{\varepsilon \to 0} \sup_{s \in [0,T]} (\|\tilde{K}^{\varepsilon}(s)\|_W + \|\tilde{H}^{\varepsilon}(s)\|_W) = 0, \quad \mathbb{P}^1 - \text{a.s.,}
\]
by Gronwall’s inequality we obtain (4.72). The proof is completed.

\[\blacksquare\]

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