Hartmann potential with a minimal length and generalized recurrence relations for matrix elements

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Abstract

In this work we study the Schrödinger equation in the presence of the Hartmann potential with a generalized uncertainty principle. We perturbatively obtain the matrix elements of the hamiltonian at first order in the parameter of deformation $\beta$ and show that some degenerate states are removed. We give analytic expressions for the solutions of the diagonal matrix elements. Finally, we derive a generalized recurrence formula for the angular average values.

Keywords: Hartmann potential; Schrödinger equation; Generalized uncertainty principle (GUP).

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1 Introduction

In recent years many arguments have been suggested to motivate a modified Heisenberg algebra in quantum mechanics such as theories of quantum gravity and string theory, which lead to the existence of a minimal observable length expected to be of order of the Planck length \[1–11\]. The minimal length can be obtained from the deformed canonical commutation relation between position and momentum operators \[12–17\]:

\[
[X, P] = i \hbar (1 + \beta P^2),
\]

where \(\beta\) is a positive parameter of deformation. This commutation relation implies the following generalized uncertainty principle (GUP):

\[
\Delta X \Delta P \geq \frac{\hbar}{2} \left(1 + \beta (\Delta P)^2\right),
\]

which corresponds to a minimal length \((\Delta X)_{\text{min}} = \hbar \sqrt{\beta}\).

There has recently been a lot of interest in the study of quantum mechanics problems in the presence of a minimal length with various potentials such as the harmonic oscillator, the Coulomb and Yukawa potentials, the Woods-Saxon potential, the Kratzer potential, ... \[18–28\]. The first author to use the standard perturbation theory to solve the Schrödinger equation of the central potentials in the presence of a minimal length was Brau in Ref. \[21\], where energy-level corrections were calculated and splitting of degenerate levels were found to occur. On the other hand the solutions of the Schrödinger equation with Hartmann potential \(V(r, \theta) = \eta \sigma^2 \left(\frac{\epsilon^2}{r} + \frac{q \hbar^2}{2 \mu r^2 \sin^2 \theta}\right)\), where \(\eta\) and \(\sigma\) are positive real numbers with values ranging from about 1 to 10, and \(q\) is a real parameter, is well known \[29–35\]. This potential has been introduced in \[36–38\] to describe ring-shaped molecules. When \(q = 0\) and \(\eta \sigma^2 = Z\) the Hartmann potential reduces to the Coulomb potential.

The purpose of this paper is to study the extension of the Schrödinger equation with the Hartmann potential in the presence of a minimal length. In section 2 we use the first-order perturbation theory to give the general form of the hamiltonian matrix elements and, for a particular case, we show that the degeneracy is completely removed. In section 3 we give an explicit analytical expression of the diagonal matrix elements and show that the splitting of the degenerate energy levels also occurs. In section 4 we provide a general recurrence formula for the angular part. Finally, in the last section, we draw our conclusion.

2 Hamiltonian matrix elements

To calculate the Hamiltonian matrix elements for a Hartmann potential in the presence of a minimal length, we solve the corresponding Schrödinger equation:

\[
\left[\hat{\mathbf{P}}^2 + V(\hat{r}, \hat{\theta})\right] \psi(\hat{r}) = E^{(\beta)} \psi(\hat{r}).
\]
We choose to work with the following representation that verifies the relation (1) to first order in $\beta$:

$$\hat{X}_i \psi(\vec{r}) = x_i \psi(\vec{r}),$$

(4)

$$\hat{P}_i \psi(\vec{r}) = p_i \left(1 + \beta p^2\right) \psi(\vec{r}), \quad p_i = i\hbar \frac{\partial}{\partial x_i}.$$  

(5)

To first order in $\beta$ the Schrödinger equation (3) can be written as:

$$\left[\frac{p^2}{2\mu} + \frac{\beta p^4}{4\mu} + V(r, \theta)\right] \hat{\psi}(r, \theta, \varphi) = E^{(\beta)} \hat{\psi}(r, \theta, \varphi).$$

(6)

In this equation the Hartmann potential in the presence of a minimal length appears within the perturbation term:

$$\frac{\beta p^4}{4\mu}$$

To investigate the correlations we use the first-order perturbation theory. For $\beta = 0$ the spectrum of equation (6) and the corresponding wave functions are well-known and are given by [29, 39, 40]:

$$\psi(r, \theta, \varphi) = \frac{1}{r} R(r) \Theta(\theta) \Phi(\phi),$$

(7)

where

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} \exp(i m \phi), \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\Theta_{nm}(x) = \frac{\Gamma(2k + 1)}{\Gamma(k + 1)} \sqrt{\frac{(2n + 2k + 1)}{2^{2k+1} \Gamma(n + 2k + 1)}} \left(1 - x^2\right)^{k/2} C_n^{(k+1/2)}(x), \quad x = \cos \theta$$

$$R_{Nm}(r) = \left(\frac{\mu \eta \sigma^2 e^2}{\hbar^2 n'}\right)^{1/2} \left[\frac{(n' - l - 1)!}{n' \Gamma(n' + l + 1)}\right]^{1/2} \left(\frac{2\mu \eta \sigma^2 e^2}{\hbar^2 n'} r\right)^{l+1}$$

$$\times \exp\left(-\frac{\mu \eta \sigma^2 e^2}{\hbar^2 n'} r\right) L_{n'}^{(2l+1)} \left(\frac{2\mu \eta \sigma^2 e^2}{\hbar^2 n'} r\right),$$

$$k = \sqrt{m^2 + \frac{\eta \hbar^2 \sigma^2}{2\mu}}, \quad n' = N + l + 1, \quad l = n + k, \quad N, n = 0, 1, 2, 3, \ldots$$

(8)

with $N$ being the radial quantum number, $L_n^{(\nu)}(x)$ and $C_n^{(\nu)}(x)$ respectively stand for the associated Laguerre and the Gegenbauer (ultraspherical) polynomials. The orthogonality conditions for these functions are:

$$\int_{-1}^{1} dx \left(1 - x^2\right)^{\nu - 1/2} \left[C_n^{(\nu)}(x)\right]^2 = \frac{\pi^{2\nu - 2} \Gamma(n + \nu)}{n! \Gamma(n + \nu) \Gamma(\nu)^2},$$

$$\int_{-1}^{1} dx e^{-x} x^\nu \left[L_n^{(\nu)}(x)\right]^2 = \frac{\Gamma(\nu + n + 1)}{n!}.$$  

(9)
The energy eigenvalues are:

\[ E^{(0)}_{Nnm} = -\frac{\mu (\eta \sigma^2)^2 e^4}{2\hbar^2} \left[ N + n + \sqrt{m^2 + \frac{\eta \sigma^2 \hbar^2}{2\mu} + 1} \right]^{-2}. \] (10)

The first-order perturbation theory gives the matrix element of the Hamiltonian operator \([30]\) up to first order in \(\beta\) as follows \([21]\):

\[ \frac{\beta}{\mu} \langle N_1 n_1 m_1 | p^4 | N_2 n_2 m_2 \rangle = 4\mu \beta \left[ (E^{(0)}_{N_1 n_2 m_2})^2 \delta_{N_1 N_2} \delta_{n_1 n_2} \delta_{m_1 m_2} \right. \\
- 2E^{(0)}_{N_2 n_2 m_2} \langle N_1 n_1 m_1 | V (r, \theta) | N_2 n_2 m_2 \rangle + \langle N_1 n_1 m_1 | (V (r, \theta))^2 | N_2 n_2 m_2 \rangle, \] (11)

where

\[ V (r, \theta) = \eta \sigma^2 \left( \frac{e^2}{r} + \frac{\eta \hbar^2}{2\mu r^2 \sin^2 \theta} \right). \] (12)

Each of these terms can be written as

\[ \langle N_1 n_1 m_1 | \frac{1}{r^s \sin^{2\theta}} | N_2 n_2 m_2 \rangle = \int_0^\infty R_{N_1 n_1 m_1} (r) R_{N_2 n_2 m_2} (r) r^{-s} dr \\
\times \int_{-1}^1 dx \left( 1 - x^2 \right)^{-t} \Theta_{n_1 m_1} (x) \Theta_{n_2 m_2} (x) \] (13)

For the radial part, the first integral has been evaluated in \([30]\) and its expression is given by:

\[ \langle N_1 n_1 m_1 | r^s | N_2 n_2 m_2 \rangle = \int_0^\infty R_{N_1 n_1 m_1} (r) R_{N_2 n_2 m_2} (r) r^s dr \\
= \frac{\eta \sigma^2}{n_1 n_2} \sqrt{\frac{N_1! N_2!}{\Gamma (2l_1 + N + 2) \Gamma (2l_2 + N + 2)}} \left( \frac{2\eta \sigma^2}{n_1^2} \right)^{l_1 + 1} \left( \frac{2\eta \sigma^2}{n_2^2} \right)^{l_2 + 1} \\
\times [\eta \sigma^2 (1/n_1' + 1/n_2')]^{-s-t_1-t_2-3} \sum_{m_1}^{N} \sum_{m_2}^{N} \frac{(-1)^{2N+m_2}}{m_1! m_2!} \left( \frac{1/n_1' - 1/n_2'}{1/n_1' + 1/n_2'} \right)^{m_1+m_2} \\
\times \Gamma (l_1 + l_2 + s + m_1 + m_2 + 3) \sum_{m_3=0}^{t} \left( \frac{l_1 + l_2 + s + m_2 + 1}{N - m_1 - m_3} \right) \\
\times \left( \frac{l_1 + l_2 + s + m_1 + 1}{N - m_2 - m_3} \right), \] (14)

with \( t = \min (N - m_1, N - m_2) \) and \( s < l_1 + l_2 + 3 \). The only values of \( s \) which contribute in the calculation of the matrix elements given in the expression \([11]\) are \( s = \{-1, -2, -3, -4\} \).
The integral (15) vanishes for odd values of $\theta > -1/2$, $\mu > -1/2$, and $\lambda > -1/2$:

\[
\int_{-1}^{1} C_{\theta}^m(x) C_{\mu}^n(x) (1 - x^2)^{\theta - 1/2} \, dx = \frac{\pi 2^{1 - 2\lambda}}{\Gamma(\mu) \Gamma(\theta)} \sum_{k=0}^{[l/2]} \frac{\Gamma(l - \theta - k)}{\Gamma(l + \lambda - k + 1)} \frac{\Gamma(l - 2k + \lambda)}{\Gamma(m + \lambda - s + 1)} (\theta - \lambda)_k (\mu - \lambda)_s,
\]

where $m = l - 2k + 2s$, with $[m/2] \geq s \in N$, and $(z)_n$ is the Pochhammer symbol [43]:

\[
(z)_n = z(z + 1)\ldots(z + n - 1) = \frac{\Gamma(z + n)}{\Gamma(z)}.
\]

The integral (15) vanishes for odd values of $l + m$. We thus have the following special integrals:

\[
\int_{-1}^{1} dx (1 - x^2)^{-1} \Theta_{n_1 m_1}(x) \Theta_{n_2 m_2}(x) = \left| \cos \frac{\pi}{2} (n_1 + n_2) \right|
\]

\[
\times \frac{\Gamma(2k_1 + 1) \Gamma(2k_2 + 1)}{\Gamma(k_1 + 1) \Gamma(k_2 + 1)} \left[ \frac{2(2n_1 + 2k_1 + 1)}{n_1 + 2k_1 + 1} \right]^{1/2} \frac{\pi 2^{2-k_1-k_2}}{\Gamma(k_1 + 1/2) \Gamma(k_2 + 1/2)} \sum_{p=0}^{[n_1/2]} \left[ \frac{n_1 - 2p + (k_1 + k_2 - 1)/2}{(n_1 - 2p)!s!} \right] \frac{\Gamma(n_1 - 2p + k_1 + k_2 - 1)}{\Gamma(n_1 + k_1 + s + 1)} \frac{\Gamma(k_2 - k_1)}{\Gamma(2 - k_2 - 1/2)} \left( k_2 - k_1 \right)_{s/2 - 1}.
\]

and

\[
\int_{-1}^{1} dx (1 - x^2)^{-2} \Theta_{n_1 m_1}(x) \Theta_{n_2 m_2}(x) = \left| \cos \frac{\pi}{2} (n_1 + n_2) \right| \times
\]

\[
\times \frac{\Gamma(2k_1 + 1) \Gamma(2k_2 + 1)}{\Gamma(k_1 + 1) \Gamma(k_2 + 1)} \left[ \frac{2(2n_1 + 2k_1 + 1)}{n_1 + 2k_1 + 1} \right]^{1/2} \frac{\pi 2^{2-k_1-k_2}}{\Gamma(k_1 + 1/2) \Gamma(k_2 + 1/2)} \sum_{p=0}^{[n_1/2]} \left[ \frac{n_1 - 2p + (k_1 + k_2 - 3)/2}{(n_1 - 2p)!s!} \right] \frac{\Gamma(n_1 - 2p + k_1 + k_2 - 3)}{\Gamma(n_1 + k_1 + s + 1)} \frac{\Gamma(k_2 - k_1)}{\Gamma(2 - k_2 - 1/2)} \left( k_2 - k_1 \right)_{s/2 - 1}.
\]
Finally, the general form of the hamiltonian matrix elements in (11) is given by the expression:

\[
\frac{\beta}{4 \mu} \langle N_1 n_1 m_1 | p^4 | N_2 n_2 m_2 \rangle = \frac{\mu^2 (\eta \sigma^2)^4 e^8}{4 \hbar^4} (n'_2)^{-4} \delta_{N_1 N_2} \delta_{n_1 n_2} \delta_{m_1 m_2} \\
+ \frac{\mu (\eta \sigma^2)^4 e^6}{\hbar^2} (n'_2)^{-2} \langle N_1 n_1 m_1 | r^{-1} | N_2 n_2 m_2 \rangle \\
+ (\eta \sigma^2)^2 e^4 \langle N_1 n_1 m_1 | r^{-2} | N_2 n_2 m_2 \rangle \\
+ \left[ \frac{(\eta \sigma^2)^4 e^6 q}{\mu} (n'_2)^{-2} \langle N_1 n_1 m_1 | r^{-2} | N_2 n_2 m_2 \rangle \right]
\times \frac{\Gamma (2 k_1 + 1) \Gamma (2 k_1 + 1)}{\Gamma (k_1 + 1) \Gamma (k_1 + 1)} \left[ \frac{2 (n_1 + 2 k_1 + 1) \Gamma (n_1 + 2 k_1 + 1)}{2 (n_2 + 2 k_2 + 1) \Gamma (n_2 + 2 k_2 + 1)} \right]^{1/2} \\
\times \frac{\pi 2^{2-k_1-k_2}}{\Gamma (k_1 + 1/2) \Gamma (k_2 + 1/2)} \cos \frac{\pi}{2} (n_1 + n_2) \\
\times \sum_{p=0}^{[n_1/2]} \left[ \frac{[n_1 - 2p + (k_1 + k_2 - 1)/2]}{(n_1 - 2p)!} \right] \\
\times \frac{\Gamma (n_1 - 2p + k_1 + k_2 - 1) \Gamma (n_2 + k_1 - s + 1/2)}{\Gamma (n_2 + k_1 + k_2 - 1/2) \Gamma (k_2 - k_1 - 1) \Gamma (n_2 + k_2 - 1)} \\
\times \frac{\Gamma (n_1 - k_1 - p - 1/2)}{\Gamma (n_1 + k_1 + k_2 + 1/2 - p)} \left( \frac{k_1 - k_2}{2} + 1 \right) \left( \frac{k_2 - k_1}{2} + 1 \right), \\
\times \left[ (\eta \sigma^2)^4 e^6 q^2 \hbar^4 \frac{\Gamma (2 k_1 + 1) \Gamma (2 k_1 + 1)}{4 \mu^2} \frac{\Gamma (k_1 + 1) \Gamma (k_1 + 1)}{\Gamma (n_1 + 2 k_1 + 1)} \frac{\Gamma (n_2 + 2 k_2 + 1)}{\Gamma (n_2 + 2 k_2 + 1)} \right]^{1/2} \\
\times \frac{\pi 2^{4-k_1-k_2}}{\Gamma (k_1 + 1/2) \Gamma (k_2 + 1/2)} \sum_{p=0}^{[n_1/2]} \left[ \frac{[n_1 - 2p + (k_1 + k_2 - 3)/2]}{(n_1 - 2p)!} \right] \\
\times \frac{\Gamma (n_1 - 2p + k_1 + k_2 - 3) \Gamma (n_2 + k_1 - s + 1/2)}{\Gamma (n_2 + k_1 + k_2 - 1/2) \Gamma (n_2 + k_2 - 1)} \\
\times \frac{\Gamma (n_1 - k_1 - p - 1/2)}{\Gamma (n_1 + k_1 + k_2 + 1/2 - p)} \left( \frac{k_1 - k_2}{2} + 2 \right) \left( \frac{k_2 - k_1}{2} + 2 \right), \\
\times \langle N_1 n_1 m_1 | r^{-4} | N_2 n_2 m_2 \rangle,
\]  

(19)

where \( \langle N_1 n_1 m_1 | r^s | N_2 n_2 m_2 \rangle \) are given by replacing \( s = \{-1, -2, -3, -4\} \) in (14).
Example: The states \(|010\rangle\) and \(|100\rangle\)

In the ordinary case (i.e. \(\beta = 0\)), \(|010\rangle\) and \(|100\rangle\) are two degenerate states. It is clear that the matrix (11) is actually diagonal (\(\langle 010|\beta|010\rangle = 0\)) which can be seen by using the expressions:

\[
L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = -x + \alpha + 1 \quad (20)
\]

\[
C_0^{(\alpha)}(x) = 1, \quad C_1^{(\alpha)}(x) = 2\alpha x \quad (21)
\]

From equation (11), a straightforward calculation gives the energy corrections at first order in the parameter \(\beta\) as follows:

\[
\Delta E_{010} = 4\mu^2 \left[ E_{010}^{(0)} \right]^2 - \frac{abAE_{010}^{(0)}}{k_0 + 2} \left( \frac{2^{2k_0 + 1} [\Gamma (k_0 + 1)]^2}{\Gamma (2k_0 + 2)} - \frac{2^{2k_0 + 3} [\Gamma (k_0 + 2)]^2}{\Gamma (2k_0 + 4)} \right)
\]

\[
- \frac{2}{(k_0 + 2)} \left( \frac{2^{2k_0 - 1} [\Gamma (k_0)]^2}{\Gamma (2k_0)} - \frac{2^{2k_0 + 1} [\Gamma (k_0 + 1)]^2}{\Gamma (2k_0 + 2)} \right)
\]

\[
+ \frac{a^2bA^2 (2k_0 + 1)}{2 (k_0 + 2)} \left( \frac{2^{2k_0 - 1} [\Gamma (k_0)]^2}{\Gamma (2k_0)} - \frac{2^{2k_0 + 1} [\Gamma (k_0 + 1)]^2}{\Gamma (2k_0 + 2)} \right)
\]

\[
\Delta E_{100} = 4\mu^2 \left[ E_{100}^{(0)} \right]^2 - \frac{abAE_{100}^{(0)}}{2 (k_0 + 2)} \left( \frac{2^{2k_0 + 1} [\Gamma (k_0 + 1)]^2}{\Gamma (2k_0 + 2)} \right)
\]

\[
- \frac{a^2bBE_{100}^{(0)}}{2 (k_0 + 2) \Gamma (2k_0)} \left( \frac{2^{2k_0 - 1} [\Gamma (k_0)]^2}{\Gamma (2k_0) \Gamma (2k_0 + 1)} + \frac{2^{2k_0 + 1} [\Gamma (k_0 + 1)]^2}{\Gamma (2k_0 + 2) \Gamma (2k_0 + 1)} \right)
\]

\[
+ \frac{a^2bB^2 (2k_0 - 2) (4 + k_0)}{(2 + k_0) (2k_0 + 3)} \left( \frac{2^{2k_0 - 1} [\Gamma (k_0)]^2}{\Gamma (2k_0)} - \frac{2^{2k_0 + 1} [\Gamma (k_0 + 1)]^2}{\Gamma (2k_0 + 2)} \right)
\]

\[
+ \frac{a^2bAB2^{2k_0} [\Gamma (k_0)]^2}{\Gamma (2k_0 + 3)} \right] \quad (22)
\]

where

\[
E_{100}^{(0)} = E_{010}^{(0)} = \frac{\mu (\eta \sigma^2) e^4}{2\hbar^2} \left[ 2 + \sqrt{\eta \sigma^2 \hbar^2} \right]^{-2}, \quad A = \eta \sigma^2 e^2, \quad B = \frac{\eta \sigma^2 \hbar^2}{2\mu},
\]

\[
a = \frac{2\mu \eta \sigma^2 e^2}{\hbar^2 n}, \quad b = \left[ \frac{\Gamma (2k + 1)}{\Gamma (k + 1)} \right]^2 \left( \frac{2n + 2k + 1}{2^{2k+1} \Gamma (n + 2k + 1)} \right), \quad k_0 = \sqrt{\frac{\eta \hbar^2 \sigma^2}{2\mu}}. \quad (23)
\]

Thus, at first order in the parameter \(\beta\), the degeneracy of the two levels \(|010\rangle\) and \(|100\rangle\) is completely lifted.
3 Matrix elements for ∆N = 0, ∆n = 0 and ∆m = 0

Taking \( N_1 = N_2 = N, n_1 = n_2 = n \) and \( m_1 = m_2 = m \), we derive the explicit form of the energy corrections given in equation (11). Using the relation between the confluent hypergeometric function \( F \) and the associated Laguerre polynomials \( L_n^{(l)}(x) \), namely:

\[
L_n^{(l)}(z) = \frac{\Gamma(n+l+1)}{\Gamma(n+1)\Gamma(l+1)}F(-n; l+1; z),
\]

where \( z = ar \) and \( a = \frac{2\mu_0\sigma^2}{\hbar^2} \), and using the integral:

\[
\int_0^\infty z^{l-1}e^{-z} [F(-n; \gamma; z)]^2 dx = \frac{n!\Gamma(l)}{n(\gamma - l - 1)(\gamma - l) + n(n-1)(\gamma - l - 2)(\gamma - l - 1)(\gamma - l + 1) + \cdots + n(n-1)\cdots(\gamma - l - n)\cdots(\gamma - l + n - 1)}
\]

we obtain the following average values for the radial part:

\[
\langle Nnm | r^{-1} | Nnm \rangle = \int_0^\infty [R(r)]^2 r^{-1} dr = \frac{a}{2n'}, \quad (27)
\]

\[
\langle Nnm | r^{-2} | Nnm \rangle = \int_0^\infty [R(r)]^2 r^{-2} dr = \frac{1}{2l+1} \frac{a^2}{2n'}, \quad (28)
\]

\[
\langle Nnm | r^{-3} | Nnm \rangle = \int_0^\infty [R(r)]^2 r^{-3} dr = \frac{a^3}{2l(2l+1)(2l+2)} = \frac{a^3\Gamma(2l)}{\Gamma(2l+3)}, \quad (29)
\]

\[
\langle Nnm | r^{-4} | Nnm' \rangle = \int_0^\infty [R(r)]^2 r^{-4} dr = \frac{a^4}{n'} \left( \frac{3(n')^2 - l(l+1)}{(2l+1)(2l+2)(2l+3)} \right) = \frac{a^4}{n'} \left[ \frac{3(n')^2 - l(l+1)}{\Gamma(2l+4)} \right] \Gamma(2l-1) \frac{\Gamma(2l+3)}{\Gamma(2l+4)} \quad (30)
\]

Now, we evaluate the following integral of the angular functions:

\[
\int_{-1}^1 dx \ (1 - x^2)^{-t} \left[ \Theta(x) \right]^2 = \frac{(2n + 2k + 1)}{2^{2k+1}\Gamma(n + 2k + 1)} \times \left[ \frac{\Gamma(2k+1)}{\Gamma(k+1)} \right]^2 \int_{-1}^1 dx \ (1 - x^2)^{k-t} \left[ C_n^{(k+1/2)}(x) \right]^2 \quad (31)
\]
We use the following expression given in Ref. [42]:

\[
\int_0^\pi d\theta \sin \nu \theta \sin (\gamma \theta) \left[ C_n^{(\lambda)} \left( \sqrt{1 + \rho \sin^2 \theta} \right) \right]^2 = \frac{2^{-\nu} \pi \Gamma (\nu + 1) (2\lambda)^2_n}{(n!)^2 \Gamma \left( \frac{\nu + \gamma}{2} + 1 \right) \Gamma \left( \frac{\nu + \gamma}{2} + 1 \right)}
\]

\[
\times \sin \left( \frac{\gamma \pi}{2} \right) \ _5F_4 \left( \begin{array}{c} -n, \lambda, 2\lambda + n, \nu + 1, \nu \ \Gamma (\frac{\nu + \gamma}{2} + 1) \\ \lambda + \frac{1}{2}, 2\lambda, \nu, \nu \end{array} \right), \quad (Re\nu > -1)
\]  

(32)

with \( _pF_q \left( a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; x \right) \) being the hypergeometric function defined as:

\[
_pF_q \left( a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; x \right) = \sum_{k=0}^\infty \frac{(a_1)_k (a_2)_k \ldots (a_p)_k x^k}{(b_1)_k (b_2)_k \ldots (b_q)_k k!}
\]  

(33)

We obtain:

\[
\int_{-1}^1 dx \ (1 - x^2)^{-1} [\Theta (x)]^2 = \frac{\pi}{2^{4k-3} (n!)^2} \frac{(2n + 2k + 1) \Gamma (n + 2k + 1) \Gamma (2k - 3)}{\Gamma (k + 1)^2 \Gamma (k - 3/2) \Gamma (k - 1/2)}
\]

\[
\times _5F_4 \left( \begin{array}{c} -n, k + 1/2, n + 2k + 1, k - 1/2, k; 1 \\ k + 1, 2k + 1, k - 1/2, k + 1/2 \end{array} \right)
\]  

(34)

\[
\int_{-1}^1 dx \ (1 - x^2)^{-2} [\Theta (x)]^2 = \frac{\pi}{2^{4k} (n!)^2} \frac{(2n + 2k + 1) \Gamma (n + 2k + 1) \Gamma (2k - 1)}{\Gamma (k + 1)^2 \Gamma (k - 1/2) \Gamma (k - 1/2)}
\]

\[
\times _5F_4 \left( \begin{array}{c} -n, k + 1/2, n + 2k + 1, k - 3/2, k - 1; 1 \\ k + 1, 2k + 1, k - 3/2, k - 1/2 \end{array} \right)
\]  

(35)

Finally, the diagonal matrix elements up to first order in \( \beta \) take the form:

\[
\frac{\beta}{4\mu} \langle Nnm | p^4 | Nnm \rangle = \frac{\beta}{4\mu} \frac{a^4}{(n')^8} \left( \frac{h^2}{8\mu} \left( \frac{h^2}{2} + \frac{n'}{2l + 1} \right) + \right.
\]

\[
+ \frac{\eta q^2 \sigma^4 h^4}{\mu} \left( \frac{1}{2n' (2l + 1)} + \frac{4n' \Gamma (2l)}{\Gamma (2l + 3)} \right) \frac{\pi}{2^{4k-2} (n!)^2} \frac{(2n + 2k + 1)}{[\Gamma (k + 1)]^2}
\]

\[
\times \frac{\Gamma (n + 2k + 1) \Gamma (2k - 3)}{\Gamma (k - 3/2) \Gamma (k - 1/2)} F_4 \left( \begin{array}{c} -n, k + 1/2, n + 2k + 1, k - 1/2, k; 1 \\ k + 1, 2k + 1, k - 1/2, k + 1/2 \end{array} \right)
\]

\[
+ \frac{\eta q^2 \sigma^4 h^4}{\mu} \frac{3 (n')^2 - l (l + 1)}{\Gamma (2l + 4) n'} \frac{\pi}{2^{4k+2} (n!)^2} \frac{(2n + 2k + 1)}{[\Gamma (k + 1)]^2}
\]

\[
\times \frac{\Gamma (n + 2k + 1) \Gamma (2k - 1)}{\Gamma (k - 1/2) \Gamma (k + 1/2)} F_4 \left( \begin{array}{c} -n, k + 1/2, n + 2k + 1, k - 3/2, k - 1; 1 \\ k + 1, 2k + 1, k - 3/2, k - 1/2 \end{array} \right)
\]  

(36)

This last expression depends on \( l(l = n + k) \), which lifts the degeneracy.
4 Generalized Recurrence Relations

For the diagonal matrix elements, the radial part verifies the following recurrence relations given in Ref. [29] (restoring $\mu$, $e$ and $\hbar$):

$$\frac{\hbar^4 a^2}{4\mu^2 e^4} (s + 1) \langle r^s \rangle = \frac{\hbar n' a}{2\mu e^2} \langle r^{s-1} \rangle - \frac{s}{4} \left( \frac{(2l + 1)^2 - s^2}{4} \right) \langle r^{s-2} \rangle,$$ (37)

where the first average elements of $r^n$ were evaluated in Ref. [29]. Next, we evaluate the recurrence formula for the angular part. For this we denote:

$$\langle \sin^2 \theta \rangle_{n,k} = \langle Nnm | \sin^2 \theta | Nnm \rangle = \int_{-1}^{1} dx \ (1 - x^2)^l \Theta (x)^2,$$

$$(n + 2k + 1) \left[ \frac{\Gamma (2k + 1)}{\Gamma (k + 1)} \right]^2 \int_{-1}^{1} dx \ (1 - x^2)^{k+l} \left[ C_n^{(k+1/2)} (x) \right]^2 - \left[ x C_n^{(k+1/2)} (x) \right]^2.$$(38)

We can write:

$$\int_{-1}^{1} dx \ (1 - x^2)^{k+l+1} \left[ C_n^{(k+1/2)} (x) \right]^2 = \int_{-1}^{1} dx \ (1 - x^2)^{k+l} \left[ \left[ C_n^{(k+1/2)} (x) \right]^2 - \left[ x C_n^{(k+1/2)} (x) \right]^2 \right].$$ (39)

Using the following recurrence rule of the Gegenbauer polynomials [43]:

$$2\alpha (1 - x^2) C_{n-1}^{(\alpha+1)} (x) = (2\alpha + n + 1) C_{n-1}^{(\alpha)} (x) - n x C_{n-1}^{(\alpha)} (x),$$ (40)

we straightforwardly obtain:

$$\langle \sin^{2(t+1)} \theta \rangle_{n,k} = \langle \sin^{2t} \theta \rangle_{n,k} - \frac{1}{4} (n + 2k + 1) \langle \sin^{2(t+1)} \theta \rangle_{n-1,k+1} - \frac{n + 2k + 2}{n (2n + 2k + 1)} \langle \sin^{2t} \theta \rangle_{n-1,k+1} - \frac{n + 2k + 2}{(2n + 2k + 1) (2n + 2k - 1)} \langle \sin^{2t} \theta \rangle_{n-1,k}.$$ (41)

From equations (37) and (41) we can derive the general formula of the averages values of $r^s \sin^2 \theta$. The recurrence formula (41) requires the two initial values $\langle \sin^{2t} \theta \rangle_{0,k}$ and $\langle \sin^{2t} \theta \rangle_{1,k}$. Then, taking the special cases [21] and using of the following integral [44]:

$$\int_{0}^{\pi} d\theta \left( z + \sqrt{z^2 - 1} \cos \theta \right)^{\mu} \sin^{2\nu-1} \theta = \frac{2^{2\nu-1} \Gamma (\mu + 1) \Gamma (\nu)^2}{\Gamma (2\nu + \mu)} C_{\nu}^{(\mu)} (z),$$ (42)
with $\text{Re}(\nu) > 0$, we obtain the first matrix elements:

$$
\langle \sin^2 \theta \rangle_{0,k} = \frac{2^{2t} \Gamma (2k + 2)}{\Gamma \left[ 2 \left( k + t + 1 \right) \right]} \left[ \frac{\Gamma (k + t + 1)}{\Gamma (k + 1)} \right]^2,
$$

(43)

$$
\langle \sin^2 \theta \rangle_{1,k} = \frac{2^{2t} (2k + 3)}{2k + 2t + 3} \Gamma (2k + 2) \Gamma (2k + 2t + 1) \left[ \frac{\Gamma (k + t + 1)}{\Gamma (k + 1)} \right]^2.
$$

(44)

5 Conclusion

In this paper we studied the Schrödinger equation for the Hartmann potential with deformed Heisenberg algebra. Using perturbation theory at the first order in the parameter of deformation $\beta$, we obtained the general form of the hamiltonian matrix elements and, as an example, we showed that the degeneracy of the two states $|010\rangle$ and $|100\rangle$ is completely lifted. For the diagonal matrix elements, we derived an explicit analytical expression which depends on $l$. In this case, some degenerate states split into sub-levels, and new transitions appear. In addition to the recurrence formula for the radial average values given in [29], we derived the one for the angular part which leads to the general formula of the average values of $r^p \sin^2 \theta$ for the non-relativistic Hartmann potential. These results are useful in the calculations of the bound-state transitions and, on the experimental side, the energy levels can be measured and an upper bound on the minimal length $(\Delta X)_{\text{min}}$ can be obtained.

References

[1] D. J. Gross and P. F. Mende, Nucl. Phys. B 303, 407 (1988)
[2] M. Maggiore, Phys. Lett. B 304, 65 (1993)
[3] E. Witten, Phys. Today 49, 24 (1996)
[4] D. Amati, M. Ciafaloni, G. Veneziano, Phys. Lett. B 197, 81 (1987)
[5] D. Amati, M. Ciafaloni, G. Veneziano, Phys. Lett. B 216, 41 (1989)
[6] M. Maggiore, Phys. Lett. B 319, 83 (1993)
[7] L.J. Garay, Int. J. Mod. Phys. A 10, 145 (1995)
[8] S. Hossenfelder, Mod. Phys. Lett. A 19, 2727 (2004)
[9] S. Hossenfelder, Phys. Rev. D 70, 105003 (2004)
[10] S. Hossenfelder, Phys. Lett. B 598, 92 (2004)
[11] M. Sprenger, P. Nicolini, M. Bleicher, Eur. J. Phys. 33, 853 (2012)
[12] S. Haouat, Phys. Lett B 729, 33 (2014)
[13] A. Kempf, J. Math. Phys. 35, 4483 (1994)
[14] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52, 1108 (1995)
[15] H. Hinrichsen and A. Kempf, J. Math. Phys. 37, 2121 (1996)
[16] A. Kempf, J. Math. Phys. 38, 1347 (1997)
[17] A. Kempf, J. Phys. A 30, 2093 (1997)
[18] B. Hamil and M. Merad, Few Body Syst. 60, 36 (2019)
[19] Y. Chargui and A. Dhahbi, Few Body Syst. 61, 2 (2020)
[20] L.N. Chang, D. Minic, N. Okamura, T. Takeuchi, Phys. Rev. D 65, 125027 (2002)
[21] F. Brau, J. Phys. A, Math. Gen. 32, 7691 (1999)
[22] M.M. Stetsko, V.M. Tkachuk, Phys. Rev. A 74, 012101 (2006)
[23] M.M. Stetsko, Phys. Rev. A 74, 062105 (2006)
[24] D. Bouaziz, N. Ferkous, Phys. Rev. A 82, 022105 (2010)
[25] P. Pedram, Europhys. Lett. 101, 30005 (2013)
[26] D. Bouaziz, M. Bawin, Phys. Rev. A 78, 032110 (2008)
[27] M.M. Stetsko, V.M. Tkachuk, Phys. Rev. A 76, 012707 (2007)
[28] H. Hassanabadi, S. Zarrinkamar, E. Maghsoodi, Phys. Lett. B 718, 678 (2012)
[29] Chang-Yuan Chen, Cheng-Lin Liu and Dong-Sheng Sun, Phys. Lett. A 305, 341 (2002)
[30] Chang-Yuan Chen, Dong-Sheng Sun and Cheng-Lin Liu, Phys. Lett. A 317, 80 (2003)
[31] C.C. Gerry, Phys. Lett. A 118, 445 (1986)
[32] M. Kibler, T. Negadi, Int. J. Quantum Chem. 26, 405 (1984)
[33] I. Sökmen, Phys. Lett. A 115, 249 (1986)
[34] M. Kibler, T. Negadi, Theor. Chim. Acta 66, 31 (1984)
[35] M. Kibler, P. Winternitz, J. Phys. A 20, 4097 (1987)
[36] H. Hartmann, Theor. Chim. Acta 24, 201 (1972)
[37] H. Hartmann, R. Schuck, J. Radtke, Theor. Chim. Acta 46, 1 (1976)
[38] H. Hartmann, D. Schuck, Int. J. Quantum Chem. 18, 125 (1980)
[39] F. Yasuk, C. Berkdemir and A. Berkdemir, J. Phys. A: Math. Gen. 38, 6579 (2005)

[40] Sameer M. Ikhdair and Ramazan Sever, Int. J. Theor. Phys. 46, 2384 (2007)

[41] F. El Wassouli, African Journal of Mathematical Physics 5, 51 (2007)

[42] Yuri A. Brychkov, Handbook of special functions (CRC Press 2008)

[43] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables, (Dover, 1965)

[44] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, 7th edition, (Academic Press, 2007)