On the effective dimension and multilevel Monte Carlo

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Abstract

I consider the problem of integrating a function \( f \) over the \( d \)-dimensional unit cube. I describe a multilevel Monte Carlo method that estimates the integral with variance at most \( \epsilon^2 \) in \( O(d + \ln(d)d\epsilon^{-2}) \) time, for \( \epsilon > 0 \), where \( d_t \) is the truncation dimension of \( f \). In contrast, the standard Monte Carlo method typically achieves such variance in \( O(d\epsilon^{-2}) \) time. A lower bound of order \( d + d_t\epsilon^{-2} \) is described for a class of multilevel Monte Carlo methods.

Keywords: multilevel Monte Carlo, Quasi-Monte Carlo, variance reduction, effective dimension, truncation dimension, time-varying Markov chains

1 Introduction

Monte Carlo simulation is used in a variety of areas including finance, queuing systems, machine learning, and health-care. A drawback of Monte Carlo simulation is its high computation cost. This motivates the need to design efficient simulation tools that optimize the tradeoff between the running time and the statistical error. This need is even stronger for high-dimensional problems, where the time to simulate a single run is typically proportional to the dimension. Variance reduction techniques that improve the efficiency of Monte Carlo simulation have been developed in the previous literature (e.g. (Glasserman 2004, Asmussen and Glynn 2007)).

This paper studies the estimation of \( \int_{[0,1]^d} f(x) \, dx \), where \( f \) is a real-valued square-integrable function on \([0,1]^d\). Note that \( \int_{[0,1]^d} f(x) \, dx = E(f(U)) \), where \( U = (U_1, \ldots, U_d) \) and \( U_1, \ldots, U_d \) are independent random variables uniformly distributed on \([0,1]\). The standard Monte Carlo method estimates \( E(f(U)) \) by taking the average of \( f \) over \( n \) random points uniformly distributed over \([0,1]^d\), and achieves a statistical error of order \( n^{-1/2} \). The Quasi-Monte Carlo method (QMC) estimates \( E(f(U)) \) by taking the average of \( f \) over a predetermined sequence of points in \([0,1]^d\), and achieves an error of order \((\log n)^d/n\) for certain sequences when \( f \) has finite Hardy-Krause variation (Glasserman 2004, Ch. 5). Thus, for small values of \( d \), QMC can substantially outperform standard Monte Carlo. Moreover, numerical experiments show that QMC performs well in certain high-dimensional problems where the importance of \( U_i \) decreases with \( i \) (Glasserman 2004, Ch. 5). Caflisch, Morokoff and Owen (1997) use the ANOVA decomposition, a representation of \( f \) as the sum of orthogonal components, to define the effective

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dimension in the truncation sense: the truncation dimension is low when the first variables are important. Sloan and Woźniakowski (1998) prove that QMC is effective for a class of functions where high dimensions have decaying importance. The connection between QMC and various notions of effective dimension is studied in (L’Ecuyer and Lemieux 2000, Owen 2003, Liu and Owen 2006, Wasilkowski 2021). Methods that reduce the effective dimension and improve the performance of QMC are described in (Wang and Sloan 2011, Wang and Tan 2013, Xiao and Wang 2019). Owen (2019) gives a recent survey on the effective dimension. Kahalé (2020b) studies the relationship between the truncation dimension and the randomized dimension reduction method, a recent variance reduction technique applicable to high-dimensional problems.

A major advance in Monte Carlo simulation is the multilevel Monte Carlo method (MLMC), a variance reduction technique introduced by Giles (2008). The MLMC method significantly reduces the time to estimate functionals of a stochastic differential equation, and has many other applications (e.g. (Rosenbaum and Staum 2017, Pisarioni, Nobile and Leyland 2017, Goda, Hironaka and Iwamoto 2020, Kahalé 2020a, Blanchet, Chen, Si and Glynn 2021)). This paper examines the connection between the MLMC method and the truncation dimension. Section 3 describes a MLMC method that, under suitable conditions, estimates \( E(f(U)) \) with variance at most \( \epsilon^2 \) in \( O(d + \ln(d) d \epsilon^{-2}) \) time, for \( \epsilon > 0 \), where \( d \epsilon \) is the truncation dimension of \( f \). In contrast, the standard Monte Carlo method typically achieves variance at most \( \epsilon^2 \) in \( O(d \epsilon^{-2}) \) time. My approach is based on fixing unessential variables and on approximating \( f(U) \) by functions of the first components of \( U \). Fixing unessential variables is analysed by Sobol (2001) in the context of the ANOVA decomposition. Section 4 considers a class of MLMC estimators that approximate \( f(U) \) by functions of the first components of \( U \). Under general conditions, it gives a lower bound of order \( d + d \epsilon \epsilon^{-2} \) on the time required by these estimators to evaluate \( E(f(U)) \) with variance at most \( \epsilon^2 \). Section 5 studies MLMC and the truncation dimension for time-varying Markov chains with \( d \) time-steps. Under suitable conditions, it is shown that certain Markov chain functionals can be estimated with variance at most \( \epsilon^2 \) in \( O(d + \epsilon^{-2}) \) time, and that the truncation dimension associated with these functionals is upper bounded by a constant independent of \( d \). Randomized MLMC methods for equilibrium expectations of time-homogeneous Markov chains are studied in (Glynn and Rhee 2014).

2 Preliminaries

2.1 The ANOVA decomposition

It is assumed throughout the paper that \( f \) is square-integrable with \( \text{Var}(f(U)) > 0 \). A representation of \( f \) in the following form:

\[
f = \sum_{Y \subseteq \{1, \ldots, d\}} f_Y,
\]

is called ANOVA decomposition if, for \( Y \subseteq \{1, \ldots, d\} \) and \( u = (u_1, \ldots, u_d) \in [0, 1]^d \),

1. \( f_Y \) is a measurable function on \([0, 1]^d\) and \( f_Y(u) \) depends on \( u \) only through \((u_j)_{j \in Y}.\)
2. For \( j \in Y \),
\[
\int_{0}^{1} f_Y(u_1, \ldots, u_{j-1}, x, u_{j+1}, \ldots, u_d) \, dx = 0.
\]
It can be shown (Sobol 2001, p. 272) that there is a unique ANOVA representation of \( f \), that \( f_0 = E(f(U)) \), and that the \( f_Y \)'s are square-integrable. Furthermore, if \( Y \neq Y' \),
\[
\text{Cov}(f_Y(U), f_{Y'}(U)) = 0, \tag{2}
\]
and
\[
\text{Var}(f(U)) = \sum_{Y \subseteq \{1, \ldots, d\}, Y \neq \emptyset} \sigma_Y^2, \tag{3}
\]
where \( \sigma_Y \) is the standard deviation of \( f_Y(U) \). For \( 0 \leq i \leq d \),
\[
E(f(U)|U_1, \ldots, U_i) = \sum_{Y \subseteq \{1, \ldots, i\}} f_Y(U). \tag{4}
\]
Owen (2003) defines the truncation dimension \( d_t \) of \( f \) as
\[
d_t := \frac{\sum_{Y \subseteq \{1, \ldots, d\}, Y \neq \emptyset} \sigma_Y^2}{\text{Var}(f(U))}.
\]
For \( 0 \leq i \leq d \), let
\[
D(i) := \sum_{Y \subseteq \{1, \ldots, d\}, Y \neq \emptyset, \max(Y) > i} \sigma_Y^2
\]
be the total variance corresponding to the last \( d - i \) components of \( f \) (see Sobol (2001)). The sequence \( (D(i) : 0 \leq i \leq d) \) is decreasing, with \( D(0) = \text{Var}(f(U)) \) by (3) and \( D(d) = 0 \).

Proposition 2.1 gives a bound on the variance of the difference between \( f(V) - f(V') \), when \( V \) and \( V' \) are uniformly distributed on \([0, 1]^d\) and have the same first \( i \) components. It is related to (Sobol 2001, Theorem 3).

**Proposition 2.1.** Let \( i \in \{0, \ldots, d\} \). Assume that \( V \) and \( V' \) are uniformly distributed on \([0, 1]^d\), and that \( V_j = V'_j \) for \( 1 \leq j \leq i \). Then \( \text{Var}(f(V) - f(V')) \leq 4D(i) \).

**Proof.** As \( f_Y(V) = f_Y(V') \) for \( Y \subseteq \{1, \ldots, i\} \), we have
\[
f(V) - f(V') = \sum_{Y \subseteq \{1, \ldots, d\}, Y \neq \emptyset, \max(Y) > i} f_Y(V) - f_Y(V').
\]
By (2),
\[
\text{Var} \left( \sum_{Y \subseteq \{1, \ldots, d\}, Y \neq \emptyset, \max(Y) > i} f_Y(V) \right) = D(i),
\]
and a similar relation holds for \( V' \). Since \( \text{Var}(Z + Z') \leq 2(\text{Var}(Z) + \text{Var}(Z')) \) for square-integrable random variables \( Z \) and \( Z' \), this achieves the proof. \( \square \)

Proposition 2.2 gives a lower bound on the variance of the difference between \( f(U) \) and a function of the first \( i \) components of \( U \). A similar result is shown in (Sobol 2001, Theorem 1).
Proposition 2.2. Let $g$ be a real-valued square-integrable function on $[0, 1]^i$, where $0 \leq i \leq d$. Then

$$D(i) \leq \text{Var}(f(U) - g(U_1, \ldots, U_i)).$$

Proof. Define the random-variable

$$\eta = f(U) - E(f(U)|U_1, \ldots, U_i).$$

By properties of the conditional expectation,

$$\text{Var}(\eta) \leq \text{Var}(f(U) - g(U_1, \ldots, U_i)).$$

Combining (1) and (4) shows that

$$\eta = \sum_{Y \subseteq \{1, \ldots, d\}, Y \neq \emptyset, \max(Y) < \max(Y_i)} f_Y(U_i).$$

By (2), $\text{Var}(\eta) = D(i)$. This completes the proof. \qed

Proposition 2.3 provides an alternative characterisation of the truncation dimension.

Proposition 2.3.

$$\sum_{i=0}^{d} D(i) = d \text{Var}(f(U)).$$

Proof.

$$\sum_{i=0}^{d} D(i) = \sum_{i=0}^{d} \sum_{Y \subseteq \{1, \ldots, d\}, Y \neq \emptyset} 1\{i < \max(Y)\} \sigma_Y^2$$

$$= \sum_{Y \subseteq \{1, \ldots, d\}, Y \neq \emptyset} \max(Y) \sigma_Y^2$$

$$= d \text{Var}(f(U)).$$

\qed

2.2 Work-normalized variance

Let $\mu$ be a real number and let $\psi$ be a square-integrable random variable with positive variance and expected running time $\tau$. Assume that $\psi$ is an unbiased estimator of $\mu$, i.e., $E(\psi) = \mu$. The work-normalized variance $\tau \text{Var}(\psi)$ is a standard measure of the performance of $\psi$ (Glynn and Whitt 1992): asymptotically efficient unbiased estimators have low work-normalized variance. For $\epsilon > 0$, let $n^*_{\epsilon}$ be the smallest integer such that the variance of the average of $n^*_{\epsilon}$ independent copies of $\psi$ is at most $\epsilon^2$. Thus, $n^*_{\epsilon} = \lceil \text{Var}(\psi) \epsilon^{-2} \rceil$. As $(x + 1)/2 \leq \lceil x \rceil \leq x + 1$ for $x > 0$,

$$\frac{\tau + \tau \text{Var}(\psi) \epsilon^{-2}}{2} \leq T(\psi, \epsilon) \leq \tau + \tau \text{Var}(\psi) \epsilon^{-2},$$

where $T(\psi, \epsilon) := n^*_{\epsilon} \tau$ is the total expected time required to estimate $\mu$ with variance at most $\epsilon^2$ by taking the average of independent runs of $\psi$. 

4
2.3 Reminder on MLMC

Let $\phi$ be a square-integrable random variable that is approximated with increasing accuracy by square-integrable random variables $\phi_l$, $0 \leq l \leq L$, where $L$ is a positive integer, with $\phi_L = \phi$ and $\phi_0 = 0$. For $1 \leq l \leq L$, let $\hat{\phi}_l$ be the average of $n_l$ independent copies of $\phi_l - \phi_{l-1}$, where $n_l$ is an arbitrary positive integer. Suppose that $\hat{\phi}_1, \ldots, \hat{\phi}_L$ are independent. Since

$$E(\phi) = \sum_{l=1}^{L} E(\phi_l - \phi_{l-1}),$$

$\hat{\phi} := \sum_{l=1}^{L} \hat{\phi}_l$ is an unbiased estimator of $E(\phi)$, i.e.,

$$E(\hat{\phi}) = E(\phi).$$

As observed in (Giles 2008),

$$\text{Var}(\hat{\phi}) = \sum_{l=1}^{L} V_l n_l,$$

where $V_l := \text{Var}(\phi_l - \phi_{l-1})$ for $1 \leq l \leq L$. The expected time required to simulate $\hat{\phi}$ is

$$\hat{T} := \sum_{l=1}^{L} n_l \hat{t}_l,$$

where $\hat{t}_l$ is the expected time to simulate $\phi_l - \phi_{l-1}$. The analysis in (Giles 2008) shows that

$$\left( \sum_{l=1}^{L} \sqrt{V_l n_l} \right)^2 \leq \hat{T} \text{Var}(\hat{\phi}),$$

with equality when the $n_l$’s are proportional to $\sqrt{V_l / \hat{t}_l}$ (ignoring integrality constraints).

3 The MLMC algorithm

Let $L = \lceil \log_2(d) \rceil$ and, for $0 \leq l \leq L - 1$, let $m_l = 2^l - 1$, with $m_L = d$. For $1 \leq l \leq L$ and $u, u' \in [0, 1]^d$, let

$$h_l(u, u') := f(u_1, \ldots, u_m, u_{m+1}', \ldots, u_d'),$$

with $h_0(u, u') := 0$. Note that $h_L(u, u') = f(u)$. Let $U'$ be a copy of $U$ and, for $1 \leq l \leq L$, let $(U^{l,j}, 1 \leq j \leq n_l)$ be $n_l$ copies of $U$, where $n_l := \lceil (d/L)^{-l} \rceil$. Assume that the random variables $(U', U^{l,j}, 1 \leq l \leq L, 1 \leq j \leq n_l)$ are independent. For $1 \leq l \leq L$, set

$$\tilde{\phi}_l := \frac{1}{n_l} \sum_{j=1}^{n_l} (h_l(U^{l,j}, U') - h_{l-1}(U^{l,j}, U')),$$

and let $\tilde{\phi} := \sum_{l=1}^{L} \tilde{\phi}_l$. The estimator $\tilde{\phi}$ does not fall, stricto sensu, in the category of MLMC estimators described in Section 2.3. This is because the $n_l$ summands in the right-hand side of (10) are dependent random variables, in general. Note that $h_l(u, u')$ depends on $u$ only
through its first $m_l$ components. Thus, once $U'$ is simulated, $h_l(U_{1:j}, U')$ and $h_{l-1}(U_{1:j}, U')$ can be calculated by simulating only the first $m_l$ components of $U_{1:j}$. For $1 \leq l \leq L$, let $\tilde{t}_l$ be the expected time to simulate the first $m_l$ components of $U$ and calculate $h_l(U, U')$, once $U'$ is simulated and $f(U')$ is calculated. In other words, $\tilde{t}_l$ is the expected time to redraw $U_1, \ldots, U_{m_l}$ and recalculate $f(U)$, without modifying the last $d - m_l$ components of $U$. In particular, $\tilde{t}_L$ is the expected time to simulate $U$ and calculate $f(U)$. Let $\bar{T}$ be the expected time to simulate $\bar{\phi}$.

Theorem 3.1 below shows that $\bar{\phi}$ is an unbiased estimator of $E(f(U))$. Also, when $\tilde{t}_l$ is linear in $m_l$, the work-normalized variance of $\bar{\phi}$ satisfies the bound $\bar{T} \text{Var}(\bar{\phi}) = O(\ln(d)d_t \text{Var}(f(U)))$, that depends on $d$ only through $\ln(d)$. By (11), $E(f(U))$ can be estimated via $\bar{\phi}$ with variance at most $\epsilon^2$ in expected time that depends asymptotically (as $\epsilon$ goes to 0) on $\ln(d)$. In contrast, assuming the expected time to simulate $f(U)$ is of order $d$, the work-normalized variance of the standard Monte Carlo estimator is of order $d \text{Var}(f(U))$ and, by (11), the standard Monte Carlo algorithm achieves variance at most $\epsilon^2$ in $O(d + d \text{Var}(f(U))\epsilon^{-2})$ expected time.

**Theorem 3.1.** We have

$$E(\bar{\phi}) = E(\bar{\phi}|U') = E(f(U)), \quad (11)$$

$$\text{Var}(\bar{\phi}) = E(\text{Var}(\bar{\phi})|U'), \quad (12)$$

If, for some constant $\tilde{c}$ and $1 \leq l \leq L$,

$$\tilde{t}_l \leq \tilde{c} m_l, \quad (13)$$

then $\bar{T} \leq 9 \tilde{c} d$ and, for $\epsilon > 0$,

$$T(\bar{\phi}, \epsilon) = O(d + \ln(d)d_t \text{Var}(f(U))\epsilon^{-2}). \quad (14)$$

**Proof.** By the definition of $\bar{\phi}_l$,

$$E(\bar{\phi}_l|U') = E(\Delta_l|U'),$$

where $\Delta_l := h_l(U, U') - h_{l-1}(U, U')$. Summing over $l$ implies that $E(\bar{\phi}|U') = E(f(U))$. Taking expectations and using the tower law implies (11). Conditional on $U'$, the $n_l$ summands in the right-hand side of (10) are independent and have the same distribution as $\Delta_l$. Thus, for $1 \leq l \leq L$,

$$\text{Var}(\bar{\phi}_l|U') = \frac{\text{Var}(\Delta_l|U')}{n_l}.$$

Furthermore, conditional on $U'$, the random variables $\bar{\phi}_l$, $1 \leq l \leq L$, are independent. Hence,

$$\text{Var}(\bar{\phi}|U') = \sum_{l=1}^{L} \frac{\text{Var}(\Delta_l|U')}{n_l}. \quad (15)$$

As $\text{Var}(Z) = \text{Var}(E(Z|U')) + E(\text{Var}(Z|U'))$ for any square-integrable random variable $Z$, using (11) shows that $\text{Var}(\bar{\phi}) = E(\text{Var}(\bar{\phi}|U'))$. Similarly, $E(\text{Var}(\Delta_l|U')) \leq \text{Var}(\Delta_l)$. Conse-
quently, taking expectations in (15) implies that
\[
\text{Var}(\tilde{\phi}) \leq \sum_{l=1}^{L} \frac{\text{Var}(\Delta_l)}{m_l} \leq \frac{L}{d} \sum_{l=1}^{L} 2^l \text{Var}(\Delta_l).
\]

For \(2 \leq l \leq L\), we have \(\Delta_l = f(V) - f(V')\), where \(V = (U_1, \ldots, U_{m_l}, U'_{m_l+1}, \ldots, U'_d)\), and \(V' = (U_1, \ldots, U_{m_{l-1}}, U'_{m_{l-1}+1}, \ldots, U'_d)\). Applying Proposition 2.1 with \(i = m_{l-1}\) yields
\[
\text{Var}(\Delta_l) \leq 4D(m_{l-1}).
\] (16)

Since \(\Delta_1 = f(U')\), (16) also holds for \(l = 1\). For \(1 \leq l \leq L\), we have \(2^l \leq 4(m_{l-1} - m_{l-2})\), where \(m_{-1} := -1\). Hence, because the sequence \(D\) is decreasing,
\[
2^l D(m_{l-1}) \leq 4 \sum_{i=m_{l-2}+1}^{m_{l-1}} D(i).
\]

Thus,
\[
\sum_{l=1}^{L} 2^l \text{Var}(\Delta_l) \leq 4 \sum_{l=1}^{L} 2^l D(m_{l-1}) \leq 16 \sum_{l=1}^{L} \sum_{i=m_{l-2}+1}^{m_{l-1}} D(i) = 16 \sum_{i=0}^{m_{L-1}} D(i) \leq 16d \text{Var}(f(U)),
\]

where the last equation follows from Proposition 2.3. This implies (12).

Assume now that (13) holds. Simulating \(\tilde{\phi}\) requires to draw \(U'\) and calculate \(f(U')\) once and to simulate \(h_l(U, U') - h_{l-1}(U, U')\) for \(n_l\) independent copies of \(U\), \(1 \leq l \leq L\). As \(m_l \leq 2^l\), given \(U'\), simulating \(h_l(U, U')\) (resp. \(h_{l-1}(U, U')\)) takes at most \(2^l\) (resp. \(2^{l-1}\)) expected time.

Thus the expected time to simulate \(h_l(U, U') - h_{l-1}(U, U')\) is at most \(3 \tilde{c} 2^{l-1}\), and
\[
\tilde{T} \leq \tilde{c}d + 3 \tilde{c} \sum_{l=1}^{L} n_l 2^{l-1} \leq \tilde{c}d + 3 \tilde{c} \sum_{l=1}^{L} (1 + \frac{d}{L 2^l}) 2^{l-1} \leq \tilde{c}d + 3 \tilde{c} 2^L + 3 \tilde{c} \frac{d}{2} \leq 9 \tilde{c}d,
\]

where the second equation follows from the inequality \(n_l \leq 1 + d/(L 2^l)\). (14) follows immediately.
Remark 5.1 in Section 5 shows that (13) holds for a class of Markov chain functionals.

3.1 Deterministic fixing of unessential variables

The estimator $\tilde{\phi}$ uses $U'$ to fix the unessential variables. This section studies the replacement of $U'$ by a deterministic vector. For $v \in [0, 1]^d$ and $1 \leq l \leq L$, set

$$\tilde{\phi}_{l,v} := \frac{1}{n_l} \sum_{j=1}^{n_l} (h_l(U_{l,j}, v) - h_{l-1}(U_{l,j}, v)),$$

$$\tilde{\phi}_v := \sum_{l=1}^{L} \tilde{\phi}_{l,v}.$$ In other words, the random variable $\tilde{\phi}_v$ is obtained from $\tilde{\phi}$ by substituting $U'$ with $v$. Let $\tilde{T}(v)$ be the expected running time of $\tilde{\phi}_v$. For any $v \in [0, 1]^d$, the estimator $\tilde{\phi}_v$ falls in the class of MLMC estimators described in Section 2.3, with $\phi = f(U)$ and $\phi_l = h_l(U, v)$ for $0 \leq l \leq L$. Corollary 3.1 shows that $\tilde{\phi}_v$ is an unbiased estimator of $E(f(U))$ and that there is $v^* \in [0, 1]^d$ such that the variance of $\tilde{\phi}_{v^*}$ and its running time are no worse, up to a constant, than those of $\tilde{\phi}$.

**Corollary 3.1.** For $v \in [0, 1]^d$,

$$E(\tilde{\phi}_v) = E(f(U)). \quad (17)$$

Moreover, there is $v^* \in [0, 1]^d$ such that $\text{Var}(\tilde{\phi}_{v^*}) \leq 3\text{Var}(\tilde{\phi})$ and $\tilde{T}(v^*) \leq 3\tilde{T}$. For $\epsilon > 0$,

$$T(\tilde{\phi}_{v^*}, \epsilon) = O(d + \ln(d)\text{Var}(f(U))\epsilon^{-2}). \quad (18)$$

**Proof.** (17) is a special case of (13). For $v \in [0, 1]^d$, let $\xi(v) := \text{Var}(\tilde{\phi}_v)$. As $\xi(U') = \text{Var}(\tilde{\phi}_{U'})$, it follows from Theorem 5.1 that $E(\xi(U')) = \text{Var}(\tilde{\phi})$. Thus $\xi(U') \leq 3\text{Var}(\tilde{\phi})$ with probability at least 2/3. Similarly, $\tilde{T}(U') \leq 3\tilde{T}$ with probability at least 2/3. Hence, there is $v^* \in [0, 1]^d$ such that $\text{Var}(\tilde{\phi}_{v^*}) \leq 3\text{Var}(\tilde{\phi})$ and $\tilde{T}(v^*) \leq 3\tilde{T}$. Using (10) yields (18). □

The MLMC estimator $\tilde{\phi}_v$ is obtained by approximating $f$ with functions of its first components. A lower bound on the performance of such estimators is given in Section 4.

4 The lower bound

This section considers a class of MLMC unbiased estimators of $E(f(U))$ based on successive approximations of $f$ by deterministic functions of its first components. In (Kahalé 2020b), a lower bound on the work-normalized variance of such estimators is given in terms of that of the randomized dimension reduction estimator. This section provides a lower bound on the work-normalized variance of these estimators in terms of the truncation dimension.

Using the notation in Section 2.3 with $\phi = f(U)$, consider a MLMC estimator $\hat{\phi}$ of $E(f(U))$ obtained by summing the averages on independent copies of $\phi_l - \phi_{l-1}$, $1 \leq l \leq L$, where $L$ is a positive integer and the $\phi_l$’s satisfy the following assumption:
Assumption 1 (A1). For $0 \leq l \leq L$, $\phi_l$ is a square-integrable random variable equal to a deterministic measurable function of $U_1, \ldots, U_m$, with $\phi_0 = 0$ and $\phi_L = f(U)$, where $(m_l : 0 \leq l \leq L)$ is a strictly increasing sequence of integers, with $m_0 = 0$ and $m_L = d$.

The proof of the lower bound is based on the following lemma.

**Lemma 4.1.** Let $(\nu_i : 0 \leq i \leq d)$ be a decreasing sequence such that $\nu_{m_l} \leq \text{Var}(f(U) - \phi_l)$ for $0 \leq l \leq L$, with $\nu_d = 0$. Then

$$\sum_{i=0}^{d} \nu_i \leq \left( \sum_{l=1}^{L} \sqrt{m_l V_l} \right)^2.$$

**Proof.** An integration by parts argument (Kahalé 2020b, Lemma EC.4) shows that

$$L \sum_{l=1}^{L} \sqrt{m_l V_l} \leq \sum_{l=1}^{L} \sqrt{m_l V_l}.$$

On the other hand, for $0 \leq l \leq L - 1$, we have

$$(\sqrt{m_{l+1}} - \sqrt{m_l}) \sqrt{\nu_{m_l}} = \sum_{i=m_l}^{m_{l+1} - 1} (\sqrt{i+1} - \sqrt{i}) \sqrt{\nu_{m_l}} \geq \sum_{i=m_l}^{m_{l+1} - 1} \alpha_i,$$

where $\alpha_i = (\sqrt{i+1} - \sqrt{i}) \sqrt{\nu_i}$. Summing over $l \in \{0, \ldots, L - 1\}$ implies that

$$\sum_{i=0}^{d} \alpha_i \leq \sum_{l=1}^{L} \sqrt{m_l V_l}.$$

On the other hand,

$$\left( \sum_{i=0}^{d} \alpha_i \right)^2 = \sum_{i=0}^{d} \alpha_i \left( \alpha_i + 2 \sum_{j=0}^{i-1} \alpha_j \right) \geq \sum_{i=0}^{d} \alpha_i \left( \alpha_i + 2 \sum_{j=0}^{i-1} (\sqrt{j+1} - \sqrt{j}) \sqrt{\nu_i} \right) = \sum_{i=0}^{d} \alpha_i (\alpha_i + 2 \sqrt{\nu_i}) = \sum_{i=0}^{d} \nu_i.$$

This concludes the proof. □

Theorem 4.1 provides a lower bound the work-normalized variance of $\hat{\phi}$ that matches, up to a logarithmic factor, the upper bound in Theorem 8.1.
Theorem 4.1. If Assumption A1 holds and there is a positive constant $\hat{c}$ such that $\hat{t}_i \geq \hat{c}m_i$ for $1 \leq l \leq L$, then $\hat{c}d_1 \Var(f(U)) \leq \hat{T} \Var(\hat{\phi})$ and, for $\epsilon > 0$,

$$T(\hat{\phi}, \epsilon) = \Omega(d + d_1 \Var(f(U))\epsilon^{-2})$$

Proof. It follows from (9) that

$$\hat{c} \left( \sum_{i=1}^{L} \sqrt{m_i V_i} \right)^2 \leq \hat{T} \Var(\hat{\phi}).$$

By Proposition 2.3 and Assumption A1, $D(m_i) \leq \Var(f(U) - \phi_i)$ for $0 \leq l \leq L$. Applying Lemma 4.1 with $\nu_i = D(i)$ for $0 \leq i \leq d$ yields

$$\left( \sum_{i=1}^{L} \sqrt{m_i V_i} \right)^2 \geq \sum_{i=0}^{d} D(i) = d_1 \Var(f(U)),$$

where the second equation follows from Proposition 2.3. This shows that $\hat{c}d_1 \Var(f(U)) \leq \hat{T} \Var(\hat{\phi})$. By (13), the expected running time of $\hat{\phi}$ is lower-bound by $\hat{t}_L \geq \hat{c}d$. Together with (15), this implies (19).

5 Time-varying Markov chains

This section shows that, under certain conditions, the expectation of functionals of time-varying Markov chains with $d$ time-steps can be estimated efficiently via MLMC, and that the associated truncation dimension is upper bounded by a constant independent of $d$.

Let $d$ be a positive integer and let $(X_i : 0 \leq i \leq d)$ be a time-varying Markov chain with state-space $F$ and deterministic initial value $X_0$. Assume that there are independent random variables $Y_i$, $0 \leq i \leq d - 1$, uniformly distributed in $[0,1]$, and measurable functions $g_i$ from $F \times [0,1]$ to $F$ such that $X_{i+1} = g_i(X_i, Y_i)$ for $0 \leq i \leq d - 1$. Our goal is to estimate $E(g(X_d))$ where $g$ is a deterministic real-valued measurable function on $F$ such that $g(X_d)$ is square-integrable. It is assumed that $g$ and the $g_i$’s can be calculated in constant time. For $1 \leq i \leq d$, set $U_i = Y_{d-i}$. An inductive argument shows that there is a real-valued measurable function $f$ on $[0,1]^d$ such that $g(X_d) = f(U)$, where $U = (U_1, \ldots, U_d)$. When $X_d$ is mainly determined by the last $Y_i$’s, the first $U_i$’s are the most important arguments of $f$.

Remark 5.1. Redrawing $U_1, \ldots, U_i$ while keeping $U_{i+1}, \ldots, U_d$ unchanged amounts to keeping $X_0, \ldots, X_{d-i}$ unchanged and redrawing $X_{d-i+1}, \ldots, X_d$. This can be achieved in $O(i)$ time. Thus (13) holds for $f$.

Given $i \in \{0, \ldots, d\}$, define the time-varying Markov chain $(X^{(i)}_j : d-i \leq j \leq d)$ by setting $X^{(i)}_{d-i} := X_0$ and $X^{(i)}_{j+1} = g_j(X^{(i)}_j, Y_j)$ for $d-i \leq j \leq d-1$. Thus, $X^{(i)}_d$ is the state of the original Markov chain $X$ at time-step $d$ if the chain is at state $X_0$ at time-step $d-i$. Note that $g(X^{(i)}_d)$ can be calculated in $O(i)$ time and is a deterministic function of $U_1, \ldots, U_i$. Roughly speaking,
if $X_d$ is determined to a large extent by the last $Y_j$’s, then $X_d^{(i)}$ should be “close” to $X_d$ for large values of $i$. This motivates the following assumption:

**Assumption 2 (A2).** There are constants $c'$ and $\gamma < -1$ independent of $d$ such that, for $0 \leq i \leq d$, we have $E(\{(g(X_d) - g(X_d^{(i)}))^2\}) \leq c'(i + 1)\gamma$.

I now describe a multilevel estimator of $E(\phi)$, where $\phi = g(X_d)$, using the notation in Section 2.3. Let $L = \lfloor \log_2(d) \rfloor$ and, for $1 \leq l \leq L - 1$, let $m_l = 2^l - 1$. Let $\phi_0 = 0$, $\phi_L = d$ and, for $1 \leq l \leq L - 1$, let $\phi_l = g(X_d^{(m_l)})$. For $1 \leq l \leq L$, let $\hat{\phi}_l$ be the average of $n_l$ independent copies of $\phi_l - \phi_{l-1}$, where $n_l = \lceil d2^{(l+1)/2} \rceil$. Suppose that $\hat{\phi}_1, \ldots, \hat{\phi}_L$ are independent. Set $\hat{\phi} := \sum_{l=1}^{L} \hat{\phi}_l$. By (11), $E(\hat{\phi}) = E(\phi)$. Let $\hat{T}$ (resp. $\hat{t}_l$) be the expected time to simulate $\hat{\phi}$ is (resp. $\phi_l - \phi_{l-1}$). Proposition 5.1 shows that, under Assumption A2, $\hat{\phi}$ can be used to estimate $E(\phi)$ with precision $\epsilon$ in $O(d + \epsilon^{-2})$ time and, if Var$(g(X_d))$ is lower-bounded by a constant independent of $d$, the truncation dimension $d_t$ associated with $g(X_d)$ is upper-bounded by a constant independent of $d$. In contrast, the standard Monte Carlo method typically achieves precision $\epsilon$ in $O(d\epsilon^{-2})$ time.

**Proposition 5.1.** Suppose that Assumption A2 holds. Then there are constants $c_1$, $c_2$ and $c_3$ independent of $d$ such that $\hat{T} \leq c_1d$, Var$(\hat{\phi}) \leq c_2/d$, and $T(\hat{\phi}, \epsilon) \leq c_3(d + \epsilon^{-2})$. Moreover,

$$d_t \text{Var}(g(X_d)) \leq c^d \frac{\gamma}{\gamma + 1}.$$

**Proof.** By construction, $\hat{t}_l \leq c2^l$ for some constant $c$ independent of $d$. By (5),

$$\hat{T} \leq c \sum_{l=1}^{L} (1 + d2^{(l+1)/2})2^l \leq cd(4 + \frac{1}{1 - 2^{(\gamma+1)/2}}).$$

By Assumption A2, for $0 \leq l \leq L$,

$$\text{Var}(g(X_d) - \phi_l) \leq c'2^{l\gamma}.$$

Since Var$(Z + Z') \leq 2(\text{Var}(Z) + \text{Var}(Z'))$ for square-integrable random variables $Z$ and $Z'$, it follows that $V_l \leq 4c'2^{(l-1)\gamma}$ for $1 \leq l \leq L$. Together with (7), this shows that

$$\text{Var}(\hat{\phi}) \leq 4c' \sum_{l=1}^{L} \frac{2^{l(1-\gamma)}}{d2^{(l+1)/2}} \leq \frac{4c'2^{(1-\gamma)/2}}{d(1 - 2^{(\gamma+1)/2})}.$$

Using (3) implies the desired bound on $T(\hat{\phi}, \epsilon)$.

By Proposition 2.2, for $0 \leq i \leq d$,

$$D(i) \leq E((g(X_d) - g(X_d^{(i)}))^2) \leq c'(i + 1)^\gamma.$$
Thus, using Proposition 2.3

\[ d_t \text{Var}(f(U)) \leq c' \sum_{i=1}^{d} i^\gamma \leq c'(1 + \int_1^d x^\gamma \, dx) \leq c' \frac{\gamma}{\gamma + 1}. \]

5.1 A Lindley recursion example

In this example, \( F = \mathbb{R} \) and \( (X_i : 0 \leq i \leq d) \) satisfies the time-varying Lindley equation

\[ X_{i+1} = (X_i + \zeta_i(Y_i))^+, \]

with \( X_0 = 0 \), where \( \zeta_i, 0 \leq i \leq d - 1 \), is a real-valued function on \([0, 1]\). Our goal is to estimate \( E(X_d) \). Thus \( g \) is the identity function and \( g_i(x, y) = (x + \zeta_i(y))^+ \) for \((x, y) \in \mathbb{R} \times [0, 1]\). Lindley equations often arise in queuing theory (Asmussen and Glynn 2007).

**Proposition 5.2.** If there are constants \( \theta > 0 \) and \( \kappa < 1 \) independent of \( d \) such that

\[ E(\exp(\theta \zeta_i(Y_i))) \leq \kappa \]

for \( 0 \leq i \leq d - 1 \), then \( E((X_d - X_d^{(i)})^2) \leq \theta' \kappa^i \) for \( 0 \leq i \leq d - 1 \), where \( \theta' \) is a constant independent of \( d \).

Proposition 5.2 shows that, if (20) holds, then so does Assumption A2, hence the conclusions of Proposition 5.1 hold as well. The proof of Proposition 5.2 is essentially the same as that of (Kahale 2020b, Proposition 10), and is therefore omitted. A justification of (20) for time-varying queues and numerical examples showing the efficiency of MLMC for estimating Markov chain functionals are given in (Kahale 2020b).

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