Abstract. The study of automorphisms of computable and other structures connects computability theory with classical group theory. Among the non-computable countable structures, computably enumerable structures are one of the most important objects of investigation in computable model theory. In this paper, we focus on the lattice structure of computably enumerable substructures of a given canonical computable structure. In particular, for a Turing degree $d$, we investigate the groups of $d$-computable automorphisms of the lattice of $d$-computably enumerable vector spaces, of the interval Boolean algebra $B_\eta$ of the ordered set of rationals, and of the lattice of $d$-computably enumerable subalgebras of $B_\eta$. For these groups we show that Turing reducibility can be used to substitute the group-theoretic embedding. We also prove that the Turing degree of the isomorphism types for these groups is the second Turing jump of $d$, $d''$.

1. Automorphisms of effective structures

Computable model theory investigates algorithmic content (effectiveness) of notions, objects, and constructions in classical mathematics. In algebra this investigation goes back to van der Waerden who in his Modern Algebra introduced an explicitly given field as one the elements of which are uniquely represented by distinguishable symbols with which we can perform the field operations algorithmically. The formalization of an explicitly given field led to the notion of a computable structure, one of the main objects of study in computable model theory. A structure is computable if its domain is computable and its relations and functions are uniformly computable. Further generalization and relativization of notion of a computable structure led to computably enumerable (abbreviated by c.e.) structures, as well as $d$-computable and $d$-c.e. computable structures for a given Turing degree $d$. In computability theory, Turing degrees are the most important measure of relative difficulty of undecidable problems. All decidable
problems have Turing degree $0$. There are uncountably many Turing degrees and they are partially ordered by Turing reducibility, forming an upper semi-lattice.

Order relations are pervasive in mathematics. One of the most important such relations is the embedding of mathematical structures. The structures we focus on are the automorphism groups of various lattices of algebraic structures that are substructures of a large canonical computable structure. The study of automorphisms of computable or computably enumerable structures connects computability theory and classical group theory. The set of all automorphisms of a computable structure forms a group under composition. We are interested in matching the embeddability of natural subgroups of this group with the Turing degree ordering. It is also natural to ask questions about the complexity of the automorphism groups and its isomorphic copies since isomorphisms do not necessarily preserve computability-theoretic properties.

Our computability-theoretic notation is standard and as in \[20, 19, 5, 6\]. By $D$ we denote the set of all Turing degrees, and by $d = \text{deg}(D)$ the Turing degree of a set $D$. Hence $0 = \text{deg}(\emptyset)$. Turing jump operator is the main tool to obtain higher Turing degrees. For a set $D$, the jump $D'$ is the halting set relative to $D$. Turing established that $d' = \text{deg}(D') > d$. All computably enumerable sets have Turing degrees $\leq 0'$. By $d''$ we denote $(d')'$, and so on. We recall the following definition from computability theory.

**Definition 1.** A nonempty set of Turing degrees, $I \subseteq D$, is called a Turing ideal if:

1. $(\forall a \in I)(\forall b)[b \leq a \Rightarrow b \in I]$, and
2. $(\forall a, b \in I)[a \lor b \in I]$.

**Notation 2.** Let $I$ be a Turing ideal. Let $M$ be a computable structure. Then:

1. $\text{Aut}_I(M)$ is the set of all $d$-computable automorphisms of $M$ for any $d \in I$;
2. If $I = \{s : s \leq d\}$, then $\text{Aut}_I(M)$ is also denoted by $\text{Aut}_d(M)$.

When the structure $M$ is $\omega$ with equality, then its automorphism group $\text{Aut}(M)$ is usually denoted by $\text{Sym}(\omega)$, the symmetric group of $\omega$. Hence we have

$\text{Sym}_d(\omega) = \{f \in \text{Sym}(\omega) : \text{deg}(f) \leq d\}$.

(See \[14, 6, 12, 13, 15, 10, 7\] for previous computability-theoretic results about $\text{Aut}(M)$.)

The **Turing degree spectrum** of a countable structure $A$ is

$DgSp(A) = \{\text{deg}(B) : B \cong A\}$,

where $\text{deg}(B)$ is the Turing degree of the atomic diagram of $B$. Knight \[10\] proved that the degree spectrum of any structure is either a singleton or is upward closed. Only the degree spectrum of a so-called automorphically trivial structure is a singleton, and if the language is finite, that degree must be $0$ (see \[9\]). Automorphically trivial structures include all finite structures, and also some special infinite structures, such as the complete graph on countably many vertices.
Jockusch and Richter (see [17]) defined the Turing degree of the isomorphism type of a structure, if it exists, to be the least degree in its Turing degree spectrum. Richter [17, 18] was first to systematically study such degrees. For these and more recent results about these degrees see [6]. In this paper, we are especially interested in the following result by Morozov.

**Theorem 3.** ([12]) The degree of the isomorphism type of the group $\text{Sym}_d(\omega)$ is $d''$.

We extend this and other computability-theoretic results by Morozov to computable algebraic structures, in particular, vector spaces and certain Boolean algebras. Preliminary version of this paper appeared in the CiE conference proceedings [1].

In the remainder of this section, we establish some general results about lattices of subspaces of a computable vector space and subalgebras of the interval Boolean algebra of the ordered set of rationals. There are two main results in the paper. In Section 2, we establish exact correspondence between embeddability of automorphism groups of substructures and the order relation of the corresponding Turing degrees (see Theorem 9). In Section 3, we compute the Turing degrees of the isomorphism types of the corresponding automorphism groups (see Theorem 15).

Let $V_\infty$ be a canonical computable $\aleph_0$-dimensional vector space over a computable field $F$, which has a computable basis. We can think of a presentation of $V_\infty$ in which the vectors in $V_\infty$ are the (codes of) finitely non-zero $\omega$-sequences of elements of $F$. By $\mathcal{L}$ we denote the lattice of all subspaces of $V_\infty$. For a Turing degree $d$, by $\mathcal{L}_d(V_\infty)$ we denote the following sublattice of $\mathcal{L}$:

$$\mathcal{L}_d(V_\infty) = \{V \in \mathcal{L} : V \text{ is } d\text{-computably enumerable}\}.$$  

Note that in the literature $\mathcal{L}_0(V_\infty)$ is usually denoted by $\mathcal{L}(V_\infty)$. About c.e. vector spaces see [11, 4, 2]. Computable vector spaces and their subspaces have been also studied in the context of reverse mathematics (see [3]).

Guichard [8] established that there are countably many automorphisms of $\mathcal{L}_0(V_\infty)$ by showing that every computable automorphism is generated by a $1-1$ and onto semilinear transformation of $V_\infty$. Recall that a map $\mu : V_\infty \to V_\infty$ is called a semilinear transformation of $V_\infty$ if there is an automorphism $\sigma$ of $F$ such that

$$\mu(\alpha u + \beta v) = \sigma(\alpha)\mu(u) + \sigma(\beta)\mu(v)$$

for every $u, v \in V_\infty$ and every $\alpha, \beta \in F$.

By $\text{GSL}_d$ we denote the group of $1-1$ and onto semilinear transformations $\langle \mu, \sigma \rangle$ such that $\deg(\mu) \leq d$ and $\deg(\sigma) \leq d$.

We will also consider the structure $\mathcal{B}_\eta$, which is the interval Boolean algebra over the countable dense linear order without endpoints. Let $Q$ be a fixed standard copy of the rationals and let $\eta$ be its order type. The elements of $\mathcal{B}_\eta$ are
the finite unions and intersections of left-closed right-open intervals. For a Turing ideal \( I \), let \( \mathcal{L}_I(\mathcal{B}_\eta) \) be the lattice of all subalgebras of \( \mathcal{B}_\eta \) which are computably enumerable (abbreviated c.e.) in some \( d \in I \).

**Notation 4.** Note that \( \mathcal{L}_D(\mathcal{B}_\eta) \) is also denoted by \( \text{Lat}(\mathcal{B}_\eta) \), where \( D \) is the set of all Turing degrees.

**Definition 5.** Note that \( \mathcal{L}_d(\mathcal{B}_\eta) \) is the lattice of all subalgebras of \( \mathcal{B}_\eta \) which are c.e. in \( d \).

Note that \( \mathcal{L}_0(\mathcal{B}_\eta) \) is also denoted by \( \mathcal{L}(\mathcal{B}_\eta) \).

In [8], Guichard proved that every element of \( \text{Aut}(\mathcal{L}_0(V_\infty)) \) is generated by an element of \( \text{GSL}_d \). This result can be relativized to an arbitrary Turing degree \( d \).

**Theorem 6.** ([8]) Every \( \Phi \in \text{Aut}(\mathcal{L}_d(V_\infty)) \) is generated by some \( \langle \mu, \sigma \rangle \in \text{GSL}_d \).

Moreover, if \( \Phi \) is also generated by some other \( \langle \mu_1, \sigma_1 \rangle \in \text{GSL}_d \), then there is \( \gamma \in F \) such that

\[
(\forall v \in V_\infty) [\mu(v) = \gamma \mu_1(v)].
\]

It follows from this result that every \( \Phi \in \text{Aut}(\mathcal{L}_I(V_\infty)) \) is generated by some \( \langle \mu, \sigma \rangle \in \text{GSL}_I \).

Every automorphism of \( \mathcal{L}_I(V_\infty) \) is defined on the one-dimensional subspaces of \( V_\infty \) and can be uniquely extended to an automorphism of the entire lattice \( \mathcal{L} \). Hence, we can identify the automorphisms of \( \mathcal{L}_I(V_\infty) \) with their extensions to automorphism of \( \mathcal{L} \). We will prove that every automorphism of \( \mathcal{L} \) is generated by a \( d \)-computable semilinear transformation and its restriction to \( \mathcal{L}_d(V_\infty) \) is an automorphism of \( \mathcal{L}_d(V_\infty) \).

**Proposition 7.** (a) \( \text{Aut}(\mathcal{L}_I(V_\infty)) = \bigcup_{d \in I} \text{Aut}(\mathcal{L}_d(V_\infty)) \)

(b) \( \text{Aut}(\mathcal{L}) = \bigcup_{d \in D} \text{Aut}(\mathcal{L}_d(V_\infty)) \)

**Proof.** Part (b) follows immediately from (a).

To prove (a) note first that \( \bigcup_{d \in I} \text{Aut}(\mathcal{L}_d(V_\infty)) \subseteq \text{Aut}(\mathcal{L}_I(V_\infty)) \) by the discussion above.

Now, suppose \( \Phi \in \text{Aut}(\mathcal{L}_I(V_\infty)) \). Let \( \alpha_0, \alpha_1, \alpha_2, \ldots \) be a fixed computable enumeration of the elements of the field \( F \). Assume that \( v_0, v_1, v_2, \ldots \) is a computable enumeration of a computable basis of \( V_\infty \). Following Guichard’s idea in [8], we define the following computable subspaces of \( V_\infty \):

\[
\begin{align*}
V_1 &= \text{span}(\{v_0, v_2, v_4, \ldots\}), \\
V_2 &= \text{span}(\{v_1, v_3, v_5, \ldots\}), \\
V_3 &= \text{span}(\{v_0 + v_1, v_2 + v_3, v_4 + v_5, \ldots\}), \\
V_4 &= \text{span}(\{v_1 + v_2, v_3 + v_4, v_5 + v_6, \ldots\}), \\
V_5 &= \text{span}(\{v_0 + \alpha_0 v_1, v_2 + \alpha_1 v_3, v_4 + \alpha_2 v_5, \ldots\}).
\end{align*}
\]
Suppose that \( \Phi(V_i) = W_i \in \mathcal{L}_I(V_{\infty}) \) for \( i = 1, \ldots, 5 \) and note that there are finitely many spaces involved, there is a Turing degree \( d \in I \) such that \( W_i \in \mathcal{L}_d(V_{\infty}) \). Using Guichard’s method, we can prove that there is a \( d \)-computable semilinear transformation that induces an automorphism \( \Psi \) of \( \mathcal{L}_d(V_{\infty}) \) and is such that \( \Psi = \Phi|\mathcal{L}_d(V_{\infty}) \). Hence \( \Phi \) as an automorphism of \( \mathcal{L}_I(V_{\infty}) \) is the unique extension of \( \Psi \). ■

We can also prove that every automorphism of \( \text{Aut}(\mathcal{L}_I(\mathcal{B}_\eta)) \) is generated by a \( d \)-computable automorphism of \( \mathcal{B}_\eta \), and that every \( \Phi \in \text{Aut}(\mathcal{L}_I(\mathcal{B}_\eta)) \) is generated by a \( d \)-computable automorphism of \( \mathcal{B}_\eta \) for some \( d \in I \).

**Proposition 8.** \( \text{Aut}(\mathcal{L}_I(\mathcal{B}_\eta)) \cong \text{Aut}_I(\mathcal{B}_\eta) \)

**Proof.** Suppose \( \varphi : \text{Aut}_I(\mathcal{B}_\eta) \to \text{Aut}(\mathcal{L}_I(\mathcal{B}_\eta)) \) is an onto homomorphism that takes every member \( \sigma \in \text{Aut}_I(\mathcal{B}_\eta) \) to an automorphism \( \varphi(\sigma) \) of \( \mathcal{L}_I(\mathcal{B}_\eta) \) induced by \( \sigma \). We will show that \( \text{Ker}(\varphi) = \text{id} \). Assume that \( \sigma \) is a nontrivial element of \( \text{Ker}(\varphi) \). It is easy to show that there exists an \( a \neq 0 \) such that \( \sigma(a) \neq \overline{a} \) and \( \sigma(a) \cap a = 0 \). Then, considering the image of the Boolean algebra \( \{0, a, \overline{a}, 1\} \) generated by \( a \), we obtain

\[
\varphi(\sigma)(\{0, a, \overline{a}, 1\}) = \{\sigma(0), \sigma(a), \sigma(\overline{a}), \sigma(1)\} = \{0, \sigma(a), \sigma(\overline{a}), 1\} \neq \{0, a, \overline{a}, 1\}.
\]

Hence \( \varphi(\sigma) \neq \text{id} \). Therefore, \( \text{Ker}(\varphi) \) is trivial and \( \varphi \) is an isomorphism. ■

2. Turing reducibility and group embeddings for vector spaces and Boolean algebras

Morozov [13] showed that the correspondence \( a \to \text{Sym}_a(\omega) \) can be used to substitute Turing reducibility with group-theoretic embedding. More precisely, he established that

\[
\text{Sym}_a(\omega) \leftrightarrow \text{Sym}_b(\omega) \iff a \leq b
\]

for every pair \( a, b \) of Turing degrees. It follows from this result that

\[
\text{Sym}_a(\omega) \cong \text{Sym}_b(\omega) \iff a = b.
\]

Here, we establish analogous results for vector spaces and Boolean algebras. In the proof of the next, main, theorem we will use the standard notation: \( [x, y] = x^{-1}y^{-1}xy \) and \( x^y = y^{-1}xy \).

**Theorem 9.** For any pair of Turing ideals \( I, J \) we have:

(a) \( \text{Aut}(\mathcal{L}_I(V_{\infty})) \leftrightarrow \text{Aut}(\mathcal{L}_J(V_{\infty})) \iff I \subseteq J \)

(b) \( \text{Aut}_I(\mathcal{B}_\eta) \leftrightarrow \text{Aut}_J(\mathcal{B}_\eta) \iff I \subseteq J \)

(c) \( \text{Aut}(\mathcal{L}_I(\mathcal{B}_\eta)) \leftrightarrow \text{Aut}(\mathcal{L}_J(\mathcal{B}_\eta)) \iff I \subseteq J \)

**Proof.** We will prove (a) and (b) only. Statement (c) follows easily from (b) and Proposition 8.

(a) Assume \( I \subseteq J \). Then it is straightforward to show that \( \text{Aut}(\mathcal{L}_I(V_{\infty})) \leftrightarrow \text{Aut}(\mathcal{L}_J(V_{\infty})) \).
Now, assume that $\text{Aut}(\mathcal{L}_I(V_\infty)) \hookrightarrow \text{Aut}(\mathcal{L}_J(V_\infty))$. We will prove that $I \subseteq J$.

Let $d \in I$. As usual, let $\{e_0, e_1, \ldots \}$ be a fixed computable basis of $V_\infty$. For $\langle \mu_i, \sigma_i \rangle \in \text{GSL}_d$, we define $\langle \mu_1, \sigma_1 \rangle \sim \langle \mu_2, \sigma_2 \rangle$ iff:

1. $\sigma_1 = \sigma_2$, and
2. there is $\alpha \in F$ such that $\alpha \neq 0$ and $(\forall v \in V_\infty) [\mu_1(v) = \alpha \mu_2(v)].$

Note that $\text{Aut}(\mathcal{L}_d(V_\infty)) \cong \text{GSL}_d \sim$. We can define a group embedding $\delta : \text{Sym}_d(\omega) \hookrightarrow \text{GSL}_d \sim$ as follows. For any $f \in \text{Sym}_d(\omega)$, we let $\delta(f)$ be the $\sim$-equivalence class of a linear transformation $\langle f, id \rangle$ such that

$$\tilde{f}(e_i) = e_{f(i)}.$$ 

Note that if $\delta(f_1) = \delta(f_2)$, then $\tilde{f}_1 = c\tilde{f}_2$ for some $c \in F$, and thus

$$\langle \forall i \in \omega \rangle [e_{f_1(i)} = \tilde{f}_1(e_i) = c\tilde{f}_2(e_i) = ce_{f_2(i)}].$$

Since the vectors $e_i, i \in \omega$, are independent, we must have

$$\langle \forall i \in \omega \rangle [f_1(i) = f_2(i)].$$

Therefore, there exists a map

$$K : \text{Sym}_d(\omega) \hookrightarrow \text{GSL}_J \sim$$

such that if $f \in \text{Sym}_d(\omega)$, then $K(f)$ is a $b$-computable (for some $b \in J$) linear transformation of $V_\infty$ modulo scalar multiplication.

We claim that if a set $A$ is c.e. in $d$, then $A$ is c.e. in some $c \in J$. Fix $A \subseteq \omega$ such that $A$ is c.e. in $d$, and let $h : \omega \to \omega$ be a $d$-computable enumeration of $A$, that is, $\text{rng}(h) = A$. Fix a partition of the natural numbers into uniformly computable infinite sets $R_i$ for $i \in \mathbb{Z}$ with enumerations $R_i = \{c_i^0 < c_i^1 < \cdots \}$. Let the permutations $g_0, g_1, w, b \in \text{Sym}_d(\omega)$ be defined as follows:

- $w(c_i^j) = c_i^{j+1}$ for each $i \in \mathbb{Z}$ and $j \in \omega$,
- $g_0 = \prod_{j \in \omega} (c_0^j, c_0^j + 1)$,
- $g_1 = \prod_{j \in \omega} (c_0^{j+1}, c_0^{j+2})$, and
- $b = \prod_{n, t \in \omega \land h(t) = n} (c_n^t, c_n^{t+1}).$

We will also use the following abbreviation: $w^n = w \cdots w$. Then we have

$$n \not\in A \iff ([g_0, b^{w^n}] = 1 \land [g_1, b^{w^n}] = 1).$$

This is because $g_0$ and $b^{w^n}$ commute iff $n$ is not enumerated into $A$ at an odd stage $t$, and, similarly, $g_1$ and $b^{w^n}$ commute iff $n$ is not enumerated into $A$ at an even stage $t$. Let $\tilde{g}_0 = K(g_0)$, $\tilde{g}_1 = K(g_1)$, $\tilde{w} = K(w)$, and $\tilde{b} = K(b)$. Each $\tilde{g}_0$, $\tilde{g}_1$, $\tilde{w}$ and $\tilde{b}$ is computable in some (possibly different degrees in $J$). Since $J$ is an ideal there is a fixed $c \in J$ that computes all of them. Then
We will now show that \(\tilde{g}_0, \tilde{b}\tilde{w}^n\) \(\not\sim 1\) is c.e. relative to the fixed \(c \in J\). Let \(\tau_n = [\tilde{g}_0, \tilde{b}\tilde{w}^n]\). Then \(\tau_n \not\sim 1\) if and only if \(e_0\) and \(e_0\) are linearly independent, or
\[
(\exists m \in \omega) (\exists \alpha \neq 0) [\tau_n(e_0) = \alpha e_0 \land \tau_n(e_m) \neq \alpha e_m].
\]

Let \(A\) have Turing degree \(d\). Then \(A\) and \(A\) are both c.e. in \(d\), and, therefore, \(A\) is computable in \(c\). Hence \(d \leq c\) and so \(d \in J\).

(b) We will now prove that \(\text{Aut}_I(B_\eta) \hookrightarrow \text{Aut}_J(B_\eta) \iff I \subseteq J\).

The proof is a corollary of the fact that \(\text{Sym}_I(\omega) \hookrightarrow \text{Sym}_J(\omega) \iff I \subseteq J\).

We first define an embedding
\[H : \text{Sym}(\omega) \hookrightarrow \text{Aut}(B_\eta).\]

For any \(p \in \text{Sym}(\omega)\), define \(\tilde{p} : \mathbb{Q} \to \mathbb{Q}\) as follows:
\[
\tilde{p}(x) = \begin{cases} 
  x & \text{if } x \text{ is negative}, \\
  p([x]) + \{x\} & \text{if } x \text{ is non-negative},
\end{cases}
\]

where \([x]\), and \(\{x\}\) are the integer and fractional parts of \(x\), respectively.

Then for any \(p \in \text{Sym}(\omega)\), let \(H(p)\) be an automorphism of \(B_\eta\) defined as follows. If \(I \in B_\eta\), then
\[H(p)(I) = \tilde{p}(I)\].

It is important to note that \(H(p)\) is uniformly computable in \(p\).

Now, suppose \(\text{Aut}_I(B_\eta) \hookrightarrow \text{Aut}_J(B_\eta)\). Then
\[\text{Sym}_I(\omega) \hookrightarrow_H \text{Sym}_J(\omega) \iff \text{Sym}_J(\omega) \iff I \subseteq J\]
and so
\[\text{Sym}_I(\omega) \hookrightarrow \text{Sym}_J(\omega)\].

Thus, by Morozov’s result, \(I \subseteq J\). 

**Corollary 10.** For any pair of Turing degrees \(a, b\) we have
\[
\text{Aut}(L_a(V_\infty)) \hookrightarrow \text{Aut}(L_b(V_\infty)) \iff a \leq b;
\]
\[
\text{Aut}_a(B_\eta) \hookrightarrow \text{Aut}_b(B_\eta) \iff a \leq b.
\]
3. Turing Degrees of the Isomorphism Types of Automorphism Groups

In this section, we will determine the Turing degree spectra of both $GSL_d$ and $Aut_d(B_0)$. More precisely, we will show that each of them is the upper cone with the least element $d''$. For the statement of the main theorem we use terminology and notation from the following definition.

**Definition 11.** A permutation $p$ on a set $M$ will be called:

(i) $1_{\text{inf}}2_{\text{inf}}$ on $M$ if it is a product of infinitely many 1-cycles and infinitely many 2-cycles;

(ii) $1_{\text{inf}}2_{\text{fin}}$ on $M$ if it is a product of infinitely many 1-cycles and finitely many 2-cycles.

The main results about the degree spectra of $GSL_0$ and $Aut_0(B_0)$ will use the following embeddability theorem, which is interesting on its own.

**Theorem 12.** Let $G$ be an $X$-computable group, and let $H : Sym_0(\omega) \hookrightarrow G$ be an embedding (of any complexity). Suppose that for every $1_{\text{inf}}2_{\text{inf}}$ permutation $p \in Sym_0(\omega)$, the image $H(p)$ is not a conjugate of the image of any $1_{\text{inf}}2_{\text{fin}}$ permutation in $Sym_0(\omega)$.

Then $0'' \leq \deg(X)$.

**Proof.** Let $A$ be a $\Pi^0_2$-complete set and let $R(x,t)$ be a computable predicate such that

$$n \in A \Leftrightarrow (\exists t) \, R(n,t).$$

We will prove that $A \leq_T X$. Fix a partition of $\omega$ into uniformly computable infinite sets $S_{i,j}$ for $i \in \mathbb{Z}$ and $j \in \{1, 2\}$ with enumerations $S_{i,j} = \{c^0_{i,j} < c^1_{i,j} < \ldots \}$. The sets $S_{i,1}$ and $S_{i,2}$ will be referred to as the left and the right parts of the $i$-th column $S_i = S_{i,1} \cup S_{i,2}$. This reference will be useful in the definitions of certain maps below. We can graphically present this partition as follows:

$$
\begin{array}{ccc}
\ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots \\
S_{i,1} & S_{i,2} & S_{i,1} \\
\text{column } S_{i-1} & \text{column } S_0 & \text{column } S_1 \\
\end{array}
$$

We will now define the following maps.

(i) $w(c^k_{i,j}) =_{\text{def}} c^k_{i,j}$ for each $i \in \mathbb{Z}$, $k \in \omega$ and $j = 1, 2$.

Clearly, the map $w$ is such that $w(S_{i+1,1}) = S_{i,1}$ and $w(S_{i+1,2}) = S_{i,2}$. It maps the left (right) part of the $(i+1)$-st column to the left (right) part of the $i$-th column for each $i$. 
(ii) $p_0 = \text{def} \prod_{k \in \omega} (c_{0,1}^k, c_{0,2}^k)$

It is easy to see that the map $p_0$ switches the left and right parts of the 0-th column (i.e., $p_0(S_{0,1}) = S_{0,2}$ and $p_0(S_{0,2}) = S_{0,1}$), and is identity on all other elements of $\omega$.

(iii) $p_n = \text{def} p_0^{w_n} = w^{-n} p_0 w^n$

Note that the map $p_n$ switches the left and right parts of the $n$-th column (i.e., $p_n(S_{n,1}) = S_{n,2}$ and $p_n(S_{n,2}) = S_{n,1}$), and is identity on all other elements of $\omega$.

(iv) $z(k) = \text{def} \begin{cases} 
0 & \text{if } k = 0, \\
1 & \text{if } k = 2, \\
k - 2 & \text{if } k = 2t \geq 4, \\
k + 2 & \text{if } k = 2t + 1.
\end{cases}$

Note that the map $z$ is a permutation of $\omega$, which contains only one infinite cycle and (0).

(v) $\tau = \text{def} \langle 0, 1 \rangle$

For $k \in \mathbb{Z}$, we have

$$\tau^z = \begin{cases} 
(0, 2k) & \text{if } k \geq 1, \\
(0, 2|k| + 1) & \text{if } k \leq 0,
\end{cases}$$

so

(2) \[(\forall n, m \in \omega) \exists n_1, m_1 \in \mathbb{Z} [ (\tau^z)^{n_1} \tau^z^{m_1} = (n, m)].\]

Note that property (2) guarantees that any 1-inf permutation on $\omega$ can be represented as a finite product of the permutations $\tau$ and $z$.

(vi) We will now construct a permutation $b$ on $\omega$ with the following properties:

$b \restriction_{S_{n,1}} = \text{id} \restriction_{S_{n,1}}$

$b \restriction_{S_{n,2}} = \begin{cases} 
1_{\text{inf}}2_{\text{fin}} \text{ on } S_{n,2} & \text{if } n \in A, \\
1_{\text{inf}}2_{\text{fin}} \text{ on } S_{n,2} & \text{if } n \notin A.
\end{cases}$

We will define $b$ in stages. At each stage $s$ we will have $E^s = \text{def dom}(b^s) = \text{rng}(b^s)$.

Construction

Stage 0.

Let $b^0 \restriction S_i = \text{def} \text{id}$ for $i \leq -1$, and $E^0 = \bigcup_{i \leq -1} S_i$.

Stage $s + 1 = \langle n, t \rangle$.

Case 1. If $R(n, t)$, then find the least elements $p, q, r \in S_{n,2}$ such that $p, q, r \notin E^s$. Let $b^{s+1} = b^s \cdot (p, q)$ and assume that $b^{s+1}(r) = r$. Thus, we have $E^{s+1} = E^s \cup \{p, q, r\}$ and $b^{s+1} \restriction E^s = b^s$.

Case 2. If $\neg R(n, t)$, then find the least elements $p, q, r \in S_{n,2}$ such that $p, q, r \notin E^s$. Let $b^{s+1} \restriction E^s = b^s$ and $b^{s+1}(p) = p$, $b^{s+1}(q) = q$, $b^{s+1}(r) = r$. Then $E^{s+1} = E^s \cup \{p, q, r\}$.

End of construction.
By construction, $\text{dom}(b) = \text{rng}(b) = \omega$.

It follows that if $n \in A$, then $(\exists^\infty t) R(n, t)$, so Case 1 applies infinitely often for this $n$, and hence the map $b$ switches infinitely many pairs in the right part of the $n$-th column. Therefore, $b \mid S_{n,2}$ is $1_{\text{inf}}2_{\text{fin}}$ and $b \mid S_{n,1} = \text{id}$.

If $n \notin A$, then $(\exists^{<\infty} t) R(n, t)$, so Case 1 applies finitely often for this $n$, and hence the map $b$ switches only finitely many pairs in the right part of the $n$-th column. Therefore, $b \mid S_{n,2}$ is $1_{\text{inf}}2_{\text{fin}}$ and $b \mid S_{n,1} = \text{id}$.

In both cases, the map $b^p_n$ reverses the action of $b$ on the left part and the right part of the $n$-th column $S_n$, while for $k \neq n$, we have $b^p_n \upharpoonright S_k = b \upharpoonright S_k$.

Then $b \cdot b^p_n$ is
\[
\begin{cases}
1_{\text{inf}}2_{\text{fin}} \text{ on } S_n & \text{if } n \in A, \\
1_{\text{inf}}2_{\text{fin}} \text{ on } S_n & \text{if } n \notin A, \\
\text{id} \text{ on } S_k & \text{if } n \neq k.
\end{cases}
\]

Therefore, $b \cdot b^p_n$ is
\[
\begin{cases}
1_{\text{inf}}2_{\text{fin}} \text{ on } \omega & \text{if } n \in A, \\
1_{\text{inf}}2_{\text{fin}} \text{ on } \omega & \text{if } n \notin A.
\end{cases}
\]

Finally, note that on $\omega$, every computable $1_{\text{inf}}2_{\text{fin}}$ permutation is the conjugate of a fixed computable $1_{\text{inf}}2_{\text{fin}}$ permutation and some other computable permutation. Therefore, assume that $f$ is a fixed computable $1_{\text{inf}}2_{\text{fin}}$ permutation such that for every $1_{\text{inf}}2_{\text{fin}}$ permutation $q \in \text{Sym}_0(\omega)$:
\[
(\exists h \in \text{Sym}_0(\omega)) [q = f^h].
\]

Hence for every $n$, we have
\[
\begin{align*}
n \in A & \iff b \cdot b^p_n \text{ is a } 1_{\text{inf}}2_{\text{fin}} \text{ permutation on } \omega \\
& \iff (\exists h \in \text{Sym}_0(\omega)) [b \cdot b^p_n = f^h] \\
& \iff (\exists u \in H(\text{Sym}_0(\omega))) [H(b) \cdot H(b)^{H(p_n)} = H(f)^u],
\end{align*}
\]

and,
\[
\begin{align*}
n \notin A & \iff b \cdot b^p_n \text{ is a } 1_{\text{inf}}2_{\text{fin}} \text{ permutation on } \omega \\
& \iff b \cdot b^p_n = \prod_{(i,j) \in F} (\tau^z)^{i+j} \\
& \iff H(b) \cdot H(b)^{H(p_n)} = \prod_{(i,j) \in F} (H(\tau)^{H(z)})^{i+j}. \tag{4}
\end{align*}
\]

The set $F$ in the last equality in (4) denotes some finite set of pairwise disjoint cycles. For the map $H : \text{Sym}_0(\omega) \hookrightarrow G$, note that $H(p_n) = H(w)^{-n} \cdot H(p_0) \cdot H(w)^n$.

We claim that the last equivalence in (3) can be strengthened so that we have:
\[
\begin{align*}
n \in A & \iff (\exists u \in G) [H(b) \cdot H(b)^{H(p_n)} = H(f)^u]. \tag{5}
\end{align*}
\]

To prove ($\Rightarrow$) in (5), assume that $n \in A$. Then
\[
(\exists u \in H(\text{Sym}_0(\omega))) [H(b) \cdot H(b)^{H(p_n)} = H(f)^u], \text{ hence}
\]
\[
(\exists u \in G) [H(b) \cdot H(b)^{H(p_n)} = H(f)^u].
\]
We will prove \((\iff)\) by contradiction. Assume that for some fixed \(u \in G\) we have
\[
H(b) \cdot H(b)^{H(p_n)} = H(f)^u, \text{ but } n \notin A.
\]
Then, by (1), we have the following:

(i) \(b \cdot b^{p_n}\) is a \(1\inf 2\fin\) permutation on \(\omega\),
(ii) \(H(b) \cdot H(b)^{H(p_n)}\) is the image of the \(1\inf 2\fin\) permutation \(b \cdot b^{p_n}\), while
(iii) \(H(f)\) is the image of the \(1\inf 2\fin\) permutation \(f\).

This contradicts our assumption that the image under \(H\) of the \(1\inf 2\fin\) permutation \(b \cdot b^{p_n}\) cannot be the conjugate of the image of any \(1\inf 2\fin\) permutation, including \(f\).

We will now show that \(A\) is computable in the group \(G\), and hence \(A \leq T X\).

For a given \(n \in \omega\), simultaneously search for a finite set \(F\) of pairwise disjoint cycles and an \(u \in G\) such that either the last equality in (4) holds:
\[
H(b) \cdot H(b)^{H(p_n)} = \prod_{(i,j) \in F} \left( H(\tau)^{H(z)^i} \right)^{H(\tau)^{H(z)^j}},
\]
or the following equality from (5) holds:
\[
H(b) \cdot H(b)^{H(p_n)} = H(f)^u.
\]
If the former search succeeds, then \(n \notin A\), while if the latter search succeeds, then \(n \in A\). \(\blacksquare\)

We will now prove our main results about the degree spectra of automorphism groups.

**Theorem 13.** The degree of the isomorphisms type of the group \(GSL_0\) is \(0''\).

**Proof.** Let \(V = \{v_0, v_1, \ldots\}\) be a computable basis of \(V_\infty\). Define
\[
H : Sym_0(\omega) \hookrightarrow GSL_0
\]
so that for any \(p \in Sym_0(\omega)\) the image \(H(p) = \langle L, id \rangle\) is a semilinear map such that
\[
L(v_i) = v_{p(i)} \text{ for every } i \in \omega.
\]
We claim that under \(H\), the image of a \(1_{\inf} 2_{\fin}\) permutation from \(Sym_0(\omega)\) cannot be a conjugate of the image of a \(1_{\inf} 2_{\fin}\) permutation from \(Sym_0(\omega)\). To establish this fact, suppose that \(\langle f, id \rangle, \langle f_1, id \rangle \in GSL_0\) are the images of some \(1_{\inf} 2_{\fin}\) and \(1_{\inf} 2_{\fin}\) computable permutations on \(\omega\), respectively. Suppose that \(\langle f, id \rangle\) and \(\langle f_1, id \rangle\) are conjugates, and let \(\langle h, \sigma \rangle \in GSL_0\) be such that \(\langle f, id \rangle^{[h, \sigma]} = \langle f_1, id \rangle\). Note that the map \(h : V_\infty \rightarrow V_\infty\) is 1 – 1 and onto \(V_\infty\). To simplify the notation, we will refer to the semilinear maps \(\langle f, id \rangle\), \(\langle f_1, id \rangle\), and \(\langle h, \sigma \rangle\) simply as \(f, f_1\), and \(h\), respectively.
We can view \( f \restriction V \) and \( f_1 \restriction V \) as \( 1_{\text{inf}}2_{\text{inf}} \) and \( 1_{\text{inf}}2_{\text{fin}} \) permutations on \( V \), respectively. We will prove that \( f_1 \) satisfies the property:

\[
(\exists W \subset_{\text{fin}} V_{\infty}) (\forall v \in V_{\infty}) [(v - f_1(v)) \in W].
\]

where, \( W \subset_{\text{fin}} V_{\infty} \) stands for \( W \) being a finite-dimensional subspace of \( V_{\infty} \). To prove (6), assume that \( B = \{x_1, \ldots, x_k, y_1, \ldots, y_k\} \subseteq V \) is such that \( f_1 \restriction V = \prod_{i \leq k} (x_i, y_i) \). Note that for every \( v \in V_{\infty} \), there are \( v_1 \in \text{span}(V - B) \) and \( v_2 \in \text{span}(B) \) such that \( v = v_1 + v_2 \). Then

\[
f_1(v) = f_1(v_1) + f_1(v_2) = v_1 + f_1(v_2),
\]

and so

\[
v - f_1(v) = v_1 + v_2 - v_1 - f_1(v_2) = v_2 - f_1(v_2) \in \text{span}(B)
\]

since \( f_1(v_2) \in \text{span}(B) \). Therefore, \( W = \text{span}(B) \) is a finite-dimensional subspace of \( V_{\infty} \) for which property (D) holds.

We will now prove that \( f^h \) does not satisfy property (3), which will contradict the assumption that \( f^h = f_1 \). Thus, assume that \( W \) is a finite-dimensional subspace of \( V_{\infty} \) such that

\[
(\forall x \in V_{\infty}) [(x - f^h(x)) \in W].
\]

Let \( W_1 = h(W) \) and note that \( W_1 \) is finite-dimensional. Let \( B_1 \) be a finite subset of the basis \( V \) such that

\[
(\forall x \in W_1) [\text{supp}_V(x) \subseteq B_1],
\]

where \( \text{supp}_V(x) \) denotes the support of \( x \) with respect to the basis \( V \).

We will now find \( u_1 \in V_{\infty} \) such that \( u_1 - f(u_1) \notin W_1 \). Since \( f \restriction V \) is a \( 1_{\text{inf}}2_{\text{inf}} \) permutation on \( V \), there are infinitely many pairs \( (u, v) \in V \times V \) such that

\[
(\forall x \in W_1) [\text{supp}_V(x) \subseteq B_1],
\]

\[
(\forall x \in W_1) [\text{supp}_V(x) \subseteq B_1],
\]

Then:

(i) \( u_1 - f(u_1) = u_1 - v_1 \neq 0 \), and

(ii) \( u_1 - f(u_1) = (u_1 - v_1) \notin \text{span}(B_1) \) because \( B_1 \cup \{u_1, v_1\} \subseteq V \).

Since \( W_1 \subseteq \text{span}(B_1) \), we have \( u_1 - f(u_1) \notin W_1 \).

Therefore, \( (h^{-1}(u_1) - h^{-1}(f(u_1))) \notin h^{-1}(W_1) \),

and so \( (h^{-1}(u_1) - h^{-1}fhh^{-1}(u_1)) \notin W \).

If we let \( x_1 = h^{-1}(u_1) \), we obtain

\[
x_1 - f^h(x_1) \notin W,
\]

which contradicts (7).

Thus, we constructed an embedding \( H : \text{Sym}_0(\omega) \rightarrow \text{GSL}_0 \) such that the images of any \( 1_{\text{inf}}2_{\text{inf}} \) and \( 1_{\text{inf}}2_{\text{fin}} \) permutations from \( \text{Sym}_0(\omega) \) cannot be conjugates in \( \text{GSL}_0 \). By Theorem 12 we conclude that \( \emptyset'' \) is computable in any copy of
Clearly, we can construct a specific copy of $GSL_0$, which is computable in $\emptyset''$. Therefore, the degree of the isomorphisms type of $GSL_0$ is $0''$. ■

We will now establish a similar result for the Boolean algebra $B_\eta$. Recall that we identify the elements of $B_\eta$ with finite unions and intersections of left-closed right-open intervals of the linear order of the set $\mathbb{Q}$ of rationals.

**Theorem 14.** The degree of the isomorphisms type of the group $\text{Aut}_0(B_\eta)$ is $0''$.

**Proof.** The map $H : \text{Sym}(\omega) \hookrightarrow \text{Aut}(B_\eta)$ was defined in the proof of part (b) of Theorem 9 in section 1. Since $H(p)$ is uniformly computable in $p$, where $p \in \text{Sym}(\omega)$, $H$ takes elements of $\text{Sym}_0(\omega)$ to elements of $\text{Aut}_0(B_\eta)$. In the remainder of the proof we will be concerned with the restriction of $H$ to $\text{Sym}_0(\omega)$.

We claim that under $H$, the image of a $1_{\text{inf}}2_{\text{fin}}$ permutation from $\text{Sym}_0(\omega)$ cannot be a conjugate of the image of a $1_{\text{inf}}2_{\text{fin}}$ permutation from $\text{Sym}_0(\omega)$.

Let $\Phi(x, \varphi)$ denote the statement $(\forall y \leq x) [y \neq 0 \Rightarrow (\exists z \leq y) (\varphi(z) \neq z)]$, and let $\Psi(\varphi)$ denote the statement $(\exists z) [z = \sup\{x : \Phi(x, \varphi)\}]$.

The statement $\Phi$ is in the language of Boolean algebras expanded with a function symbol $\varphi$ with an intended interpretation being an element of $\text{Aut}_0(B_\eta)$. Note that $\sup$ can be defined in the language of Boolean algebras.

Recall that in the proof of part (b) of Theorem 9 for $p \in \text{Sym}_0(\omega)$, we defined $\tilde{p} : \mathbb{Q} \to \mathbb{Q}$ as follows:

$$\tilde{p}(x) = \begin{cases} x & \text{if } x \text{ is negative,} \\ p([x]) + \{x\} & \text{if } x \text{ is non-negative.} \end{cases}$$

If $\varphi = H(p)$, where $p$ is a $1_{\text{inf}}2_{\text{fin}}$ permutation on $\omega$, then the set $A = \{x \in \mathbb{Q} : \tilde{p}(x) \neq x\}$ is the union of finitely many intervals of the type $[n, n + 1)$ for a natural number $n$. If $u$ is the join of these intervals in $B_\eta$, then $u$ is the largest element that satisfies $\Phi(u, \varphi)$. In this case $\Psi(\varphi)$.

If $\varphi = H(p)$, where $p$ is a $1_{\text{inf}}2_{\text{fin}}$ permutation on $\omega$, then the set $A = \{x \in \mathbb{Q} : \tilde{p}(x) \neq x\}$ is the union of intervals of the type $[n, n + 1)$ for a natural number $n$. In this case $\exists A$ is also the union of infinitely such intervals. The formula $\Phi(u, \varphi)$ is satisfied exactly by those $u$ that are the unions of finitely many intervals that consist entirely of elements of $A$. Clearly, $\{u : \Phi(u, \varphi)\}$ has no supremum in this case and so $\neg \Psi(\varphi)$.

Thus, we have established that:

1. $\Psi(\varphi)$ holds when $\varphi$ is the image, under $H$, of a $1_{\text{inf}}2_{\text{fin}}$ permutation,
2. $\neg \Psi(\varphi)$ holds when $\varphi$ is the image, under $H$, of a $1_{\text{inf}}2_{\text{fin}}$ permutation.

We will now prove that for any formula $\Theta$ in the language of Boolean algebras, expanded with function symbols for the elements of $\text{Aut}(B_\eta)$ and quantifiers
ranging over the elements of a Boolean algebra, we have the following for every function \( f \in Aut(B_\eta) \):

\[
\Theta(x, \overrightarrow{\theta}) \iff \Theta(f(x), \overrightarrow{\theta}^f),
\]

where \( \overrightarrow{\theta} = \theta_1, \ldots, \theta_n \) and \( \overrightarrow{\theta}^f = \theta_1^f, \ldots, \theta_n^f \) are new function symbols (with the usual intended interpretation). In our notation \( \Theta(x, \overrightarrow{\theta}) \), \( x \) are the free variables and \( \overrightarrow{\theta} \) are the new function symbols used in \( \Theta \).

We will proceed by induction on the complexity of the formula \( \Phi \).

If \( \Theta(x, \overrightarrow{\theta}) \) is quantifier-free, we can consider the following atomic formulas:

\[
x = y, \ x = 0, \ x = 1, \ \theta(x) = y, \ x \land y = z, \ x \lor y = z, \ x = y.
\]

We will present only the case when \( \Theta \) is the formula \( \theta(x) = y \).

Let \( f \in Aut(B_\eta) \). Then we have:

\[
\theta(x) = y \iff f\theta(x) = f(y) \iff f\theta f^{-1}f(x) = f(y) \iff \theta^f(f(x)) = f(y).
\]

Let \( \Theta \) be the formula \( \exists z \Theta_1(z, x, \overrightarrow{\theta}) \) and let \( f \in Aut(B_\eta) \).

We have that \( \exists z \Theta_1(z, x, \overrightarrow{\theta}) \iff \Theta_1(c, x, \overrightarrow{\theta}) \) for some \( c \in B_\eta \). Then, using the inductive hypothesis,

\[
\Theta_1(c, x, \overrightarrow{\theta}) \iff \Theta_1(f(c), f(x), \overrightarrow{\theta}^f) \iff \exists z \Theta_1(z, f(x), \overrightarrow{\theta}^f).
\]

This completes the induction.

By the previous considerations, we have

\[
(\forall f \in Aut(B_\eta)) [\Psi(\varphi) \iff \Psi(f^{-1}\varphi f)].
\]

It follows from (1) and (2) above that, under \( H \), the image of any \( 1_{inf}2_{inf} \) permutation from \( Sym_0(\omega) \) cannot be the conjugate of the image of any \( 1_{inf}2_{fin} \) permutation from \( Sym_0(\omega) \).

By Theorem 12 we conclude that \( \emptyset'' \) is computable in any copy of \( Aut_0(B_\eta) \). Clearly, we can construct a specific copy of \( Aut_0(B_\eta) \), which is computable in \( \emptyset'' \). Therefore, the degree of the isomorphisms type of \( Aut_0(B_\eta) \) is \( \emptyset'' \). □

Note that the results of the previous theorems in this section can be easily relativized to any Turing degree \( d \).

**Theorem 15.** The degree of the isomorphisms type of each of the groups \( GSL_d \) and \( Aut_d(B_\eta) \) is \( d'' \).

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