**Asymptotic Behavior of Perturbations in Randall-Sundrum Brane-World**

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The asymptotic behavior of metric perturbations in Randall-Sundrum infinite brane world is carefully investigated. Perturbations generated by matter fields on the brane are shown to be regular even at the future Cauchy horizon.

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**I. INTRODUCTION**

There are a lot of recent discussions about the possibility of existence of extra dimensions in a non-trivial form \(^1\). After the proposal of models with warped 5th-dimension by Randall and Sundrum \(^2\), the behavior of gravitational perturbations in these models has been investigated by many authors \(^3\). Cosmology based on these models is also discussed much \(^4\). The main purpose of such studies will be to find a clue which can observationally distinguish these models from 4-dimensional Einstein gravity.

As far as the author knows, any manifest contradiction with observations has not been reported on these models. However, as was pointed out by Randall and Sundrum themselves, there has been a worry about the model with infinite warped 5-th-dimension from the beginning. Usually, 5-dimensional Gravitational perturbations are analyzed by decomposing them into eigen modes of 4-dimensional d'Alembertian, whose eigen values are referred to as comoving mass squared. As long as we analyze the behavior of perturbations mode by mode, the non-linear interaction seems to become strong as we move far away from the brane \(^2\). Furthermore, by discussing the behavior of massless modes, the appearance of instability at the future Cauchy horizon was suggested by Chamblin and Gibbons \(^4\). Appearance of such a bad behavior seemed to be confirmed by the study of the asymptotic behavior of the Green's function by Sasaki, Shiromizu and Maeda \(^1\). If this bad behavior is real, is it a serious drawback of the model? One way of excuse is ready by Randall and Sundrum \(^4\). “The analysis based on effective theory will not be valid there, and instead the more fundamental string theory will regulate the situation.” Although this excuse is hard to be denied, it is also true that predictability will be reduced in such a model which contains spacetime region where unknown physics governs.

Alternatively, one may claim as follows. “In realistic cosmological models as discussed in Refs. \(^4\) \(^2\), we can consider models in which future Cauchy horizon does not exist, and hence we are free from instability.” However, the metric in cosmological models at a late epoch is not so largely different from the original flat model. Hence, if a singularity is developed in the original flat model, a rather extreme phenomena will also happen in cosmological models even though it might be regularized so as not to develop into a singularity. Hence, we will not be able to say that the brane world is safe enough for us to live in.

One may think that another excuse is possible. “Even though such a bad behavior will develop at the place far away from our brane, we can expect that this information will not propagate to our brane, and hence it will not cause any disaster on the dynamics of our brane world. Since in the flat-brane model the singularity develops at the future Cauchy horizon even if it exists, it will not be seen by the observer living on the brane. Therefore, this singularity will be harmless.” We will see that this kind of excuse is too naive. As we have shown in Appendix, we can obtain a model in which the brane crosses the “future Cauchy horizon” by changing the dynamics of the brane only in the sufficiently distant future.

In the present paper, we would like to pursue another more attractive possibility which was suggested in Ref. \(^4\) (See also \(^3\)). In the paper by Garriga and Tanaka \(^4\), it was shown that the asymptotic behavior of the 5-dimensional gravitational field in static configurations is regular at the level of linear perturbation. In this calculation, we have seen a non-trivial cancellation between the contribution from massless modes and that from massive K-K modes. Hence, we would be able to expect that in general the asymptotic behavior of perturbations is much milder than was anticipated naively. Actually what we shall show in the present paper will be that a similar cancellation occurs even when the source of gravitational perturbations on the brane is dynamical.

This paper is organized as follows. In section 2 the model of Randall-Sundrum brane world with infinite extra-dimension is explained. Also we give a brief review of the formulation for the analysis of 5-dimensional metric perturbations given by Ref. \(^4\). In section 3 the asymptotic behavior of 5-dimensional metric perturbations near the future Cauchy horizon is discussed. Then, in section 4 we show that no singular behavior in metric perturbations develops near the future Cauchy horizon. Section 5 is devoted to discussion.

**II. MODEL**

In this section, we briefly explain the model proposed by Randall and Sundrum \(^2\), and review the formulation of evaluating metric perturbations induced by matter fields which are confined on the brane developed by
Ref. [1]. The model of background metric is given by 5-dimensional AdS space
\[ ds^2 = dy^2 + e^{-2|y|/\ell} (-dt^2 + dx^2), \]
with a single positive tension 3-brane located at \( y = 0 \). Here \( \ell \) is the curvature radius of 5-dimensional AdS space. We assume reflection symmetry at \( y = 0 \).

In the Randall-Sundrum gauge, in which \( y \)-components of metric perturbations are set to be zero, the equation which governs transverse and traceless perturbations of bulk metric tensor \( h_{\mu \nu} \) induced by matter fields on the brane is
\[
\left[ \ell^{-2} e^{2|y|/\ell} \square^{(4)} + \partial_y^2 - 4 \ell^{-2} + 4 \ell^{-1} \delta(y) \right] h_{\mu \nu} = -2 \kappa \Sigma_{\mu \nu} \delta(y),
\]
where \( \mu, \nu \)-indices run form 0 to 3, and \( \square^{(4)} \) is the 4-dimensional flat d’Alembertian. Here \( \Sigma_{\mu \nu} \) is the source function determined by the energy momentum tensor of the matter fields on the brane \( T_{\mu \nu} \) as
\[
\Sigma_{\mu \nu} = \left( T_{\mu \nu} - \frac{1}{3} \gamma_{\mu \nu} T_{\rho \sigma} \right) + 2 \kappa^{-1} \hat{\xi}^{\mu} \hat{\xi}^{\nu},
\]
and \( \hat{\xi}^{\mu} \) is determined by solving \( \square^{(4)} \hat{\xi}^{\mu} = \frac{2}{\ell} T_{\mu \nu} \). The source function is rather complicated, but what we shall use later is just the fact that this tensor is transverse and traceless, which is a consequence of conservation law of \( T_{\mu \nu} \). This condition guarantees that induced metric perturbations are also transverse and traceless.

Solving the above equation with the boundary condition of no incoming waves from the past Cauchy horizon, \( y = \infty \) and \( t = -\infty \), we investigate the asymptotic behavior of \( h_{\mu \nu} \) near the future Cauchy horizon, \( y = \infty \) and \( t = +\infty \). To solve the above equation, we use the Green’s function method. Using the retarded Green’s function that solves the equation
\[
\left[ \ell^{-2} e^{2|y|/\ell} \square^{(4)} + \partial_y^2 - 4 \ell^{-2} + 4 \ell^{-1} \delta(y) \right] G_R(x, x') G_R(x, x) = \delta^{(5)}(x - x'),
\]
the solution of (4) with appropriate boundary condition is formally given by
\[
h_{\mu \nu}(x) = -2 \kappa \int d^4 x' G_R(x, x') \Sigma_{\mu \nu}(x').
\]

The Green’s function is constructed by taking a sum over a complete set of eigen states as usual. Following (3), we have
\[
G_R(x, x') = -\int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot (x' - x)} \frac{e^{-2|y|/\ell} \ell^{-1}}{k^2 - (\omega + i\epsilon)^2} + \int_0^\infty dm \frac{u_m(y)u_m(y')}{m^2 + k^2 - (\omega + i\epsilon)^2},
\]
with \( u_m(y) = \sqrt{m\ell/2}(J_1(m\ell Y_2(mz)) - Y_1(m\ell)J_2(mz)) / \sqrt{J_1(m\ell)^2 + Y_1(m\ell)^2} \), where \( J \) and \( Y \) are Bessel functions. Here we have also introduced the conformal coordinate \( z := e^{y/\ell} \). In terms of \( z \), the background metric is written as
\[
ds^2 = \frac{\ell^2}{z^2} (dz^2 - dt^2 + dx^2),
\]
and the brane is located at \( z = \ell \). When we use \( z \), we suppose that we are considering the region \( z \geq \ell \).

III. ASYMPTOTIC BEHAVIOR OF METRIC PERTURBATIONS

In this section, we investigate the asymptotic behavior of metric perturbations near the future Cauchy horizon in the original Randall-Sundrum model. Our assumptions are that the source \( \Sigma_{\mu \nu} \) does not continue to exist from infinite past, i.e., it differs from zero only for \( t > t_0 \) with a certain fixed value \( t_0 \), and that the source does not extend to the spatial infinity.

It is convenient to rewrite the Green’s function by using formula
\[
-\int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik \cdot (x' - x)}}{m^2 + k^2 - (\omega + i\epsilon)^2} = \frac{1}{2\pi} \theta(t - t') \frac{1}{\sigma} \frac{\partial}{\partial \sigma} \left[ \theta(\sigma^2) J_2(m\sigma) \right],
\]
where \( \sigma^2 := (t - t')^2 - (x - x')^2 \). Setting \( y' = 0 \), after a little calculation, we arrive at a rather concise expression
\[
G_R(x^a; 0, t', x'^\mu) = \frac{\theta(t - t')}{4\pi} \frac{1}{\sigma} \frac{\partial}{\partial \sigma} \left[ \theta(\sigma^2) G(\sigma, z) \right],
\]
with
\[
G(\sigma, z) = \frac{i}{4\pi} \int_{-\infty}^{\infty} dm H_2^{(2)}((m + i\epsilon)\sigma) H_2^{(1)}((m + i\epsilon)z) / H_1^{(1)}((m + i\epsilon)\ell),
\]
where \( H^{(i)} \) represents the Hankel function and \( \epsilon \) is a small positive real constant introduced to specify the path of integration. It will be manifest that \( G(\sigma, z) = 0 \) for \( z - \ell > \sigma \).

Now we are interested in the behavior at \( \sigma, z \to \infty \). For clarity, we specify the way how we take this limit. For this purpose, we introduce 4-dimensional Milne coordinates by
\[
x^\mu = T \tilde{x}^\mu(\Omega),
\]
where \( \Omega \) is a set of coordinates on 3-dimensional unit hyperboloid, and \( \tilde{x}^\mu(\Omega) \) defines its embedding in 4-dimensional Minkowski space. In the following, we fix \( \tilde{x}^\mu \) and keep \( z = O(\sigma) \) when we take the \( \sigma \to \infty \) limit.
For this limit, the Hankel functions in the numerator in \((10)\) can be replaced with their asymptotic series expansions,
\[
H_0^{(2)}(m \sigma) \sim \sqrt{\frac{2}{\pi m \sigma}} e^{-i m \sigma + \frac{\pi}{4} i} \sum_{j=0}^{\infty} \frac{A_j}{(m \sigma)^j},
\]
\[
H_2^{(1)}(m z) \sim \sqrt{\frac{2}{\pi m z}} e^{i m z - 5 \pi i / 4} \sum_{j=0}^{\infty} \frac{B_j}{(m z)^j},
\]
where \(A_j\) and \(B_j\) are constant coefficients, whose explicit form is not necessary for the present purpose. Then we find
\[
G(\sigma, z) \sim \frac{\theta(\sigma - z + \ell)}{2 \pi^2 \sqrt{\sigma z}} \int_{-\infty}^{\infty} dm \frac{e^{i m (z - \sigma)}}{H_1^{(1)}((m + i\ell)\xi)} \times \sum_i \sum_j \frac{A_i B_j}{(m + i\ell)^{\mu + \nu + 1}}.
\]
Let us define
\[
F(\sigma, z) = \frac{\kappa}{8 \pi^3} \frac{\theta(\sigma - z + \ell)}{\sqrt{\sigma z}} \sum_{i, j} f_{ij}(\sigma - z) \sigma^{-i} z^{-j},
\]
with
\[
f_{jk}(s) = \int_{-\infty}^{\infty} dm \frac{e^{-i m s}}{H_1^{(1)}((m + i\ell)\xi)} \sum_{k} A_k B_k = \int_0^\infty d\mu \frac{e^{-\mu s}}{K_0^2(\mu \xi)} \sum_{k} A_k B_k.
\]
where \(I\) and \(K\) are modified Bessel functions. Then metric perturbations \((3)\) are written as
\[
h_{\mu \nu} \sim \int d^4 x' \sum_{\mu \nu} \frac{1}{\sigma} \frac{\partial}{\partial \sigma} F(\sigma, z).
\]
The \(\sigma\)-derivative of \(F(\sigma, z)\) will produce various kind of terms, for some of which term-by-term integration over \(x'\) is not well-defined separately. Hence, it is clearer to rewrite the expression by using integration by parts as
\[
h_{\mu \nu} \sim -\int d^4 x' \sum_{\mu \nu} \frac{\hat{x}^\rho}{(\hat{x} \cdot \Delta x)} \partial_{\rho} F(\sigma, z)
\]
\[
= \int d^4 x' F(\sigma, z) \left( \frac{1}{(\hat{x} \cdot \Delta x)^2} - \frac{\hat{x}^\rho}{(\hat{x} \cdot \Delta x)} \partial_{\rho} \right) \Sigma_{\mu \nu},
\]
where \(\partial_{\rho}\) is an abbreviation of \(\partial / \partial x'^{\rho}\). In the above calculation, we have used
\[
\frac{1}{\sigma} \frac{\partial}{\partial \sigma} f(\sigma) = \frac{\hat{x}^\rho}{(\hat{x} \cdot \Delta x)} \partial_{\rho} f(\sigma),
\]
which is derived from
\[
\partial_{\rho} f(\sigma) = \frac{\Delta x_{\mu}}{\sigma} \frac{\partial}{\partial \sigma} f(\sigma).
\]
The support of integrand in \((15)\) is finite because of our assumption on \(\Sigma_{\mu \nu}\) and the existence of step function in the expression of \(F(\sigma, z)\). Since \(F(\sigma, z)\) is at most of \(O(1/\sigma)\) and \((\hat{x} \cdot \Delta x) = O(\sigma)\), we can conclude that at most
\[
h_{\mu \nu} = O(1/\sigma^2).
\]
The result obtained so far is similar to that obtained in Ref. \([1]\).

The next step is to show that the components contracted with \(\hat{x}^\mu\) are suffered from further suppression. To show this, we use the property that the source tensor \(\Sigma_{\mu \nu}\) is transverse and traceless. The components contracted with \(\hat{x}^\mu\)
\[
\hat{x}^\mu h_{\mu \nu} \sim \int d^4 x' F(\sigma, z)
\]
\[
\times \left( \frac{1}{(\hat{x} \cdot \Delta x)^2} - \frac{\hat{x}^\rho}{(\hat{x} \cdot \Delta x)} \partial_{\rho} \right) \hat{x}^\mu \Sigma_{\mu \nu},
\]
is evaluated as follows. By using the projection tensor defined by
\[
P^{\mu \nu} = g^{\mu \nu} + \hat{x}^\mu \hat{x}^\nu,
\]
and with the aid of the relation \(\partial_{\rho} \Sigma_{\mu \nu} = 0\), the second term in \((20)\) becomes
\[
- \int d^4 x' \frac{\hat{x}^\rho}{(\hat{x} \cdot \Delta x)} \partial_{\rho} F(\sigma, z)
\]
\[
= - \int d^4 x' F(\sigma, z) \frac{\hat{x}^\rho}{(\hat{x} \cdot \Delta x)} \partial_{\rho} (P^{\rho \nu} \Sigma_{\mu \nu})
\]
\[
= - \int d^4 x' \hat{x}^\rho P^{\rho \nu} \Sigma_{\mu \nu} \frac{\hat{x}^\rho}{(\hat{x} \cdot \Delta x)} \partial_{\rho} F(\sigma, z)
\]
\[
= - \int d^4 x' \hat{x}^\rho \Sigma_{\mu \nu} \frac{\hat{x}^\rho}{(\hat{x} \cdot \Delta x)} \partial_{\rho} \left( \frac{2}{(\hat{x} \cdot \Delta x)^2} - \hat{x}^\rho \partial_{\rho} \right) P^{\rho \nu} \Sigma_{\mu \nu}.
\]
In the second equality, we have used the relation
\[
\partial_{\rho} f(\sigma) = \frac{\Delta x_{\mu}}{\Delta x_{\nu}} \partial_{\rho} f(\sigma),
\]
derived from \((12)\) and \((13)\). Then we obtain
\[
\hat{x}^\mu h_{\mu \nu} \sim \int d^4 x' \frac{\hat{x}^\rho}{(\hat{x} \cdot \Delta x)} \left[ \hat{x}^\mu \Sigma_{\mu \nu} - \hat{x}^\rho \left( \frac{2}{(\hat{x} \cdot \Delta x)^2} - \hat{x}^\rho \partial_{\rho} \right) P^{\rho \nu} \Sigma_{\mu \nu} \right].
\]
The component further contracted with \(\hat{x}^\nu\) is given by
\[
\hat{x}^\mu \hat{x}^\nu h_{\mu \nu} \sim \int d^4 x' \frac{\hat{x}^\rho}{(\hat{x} \cdot \Delta x)} \left\{ \hat{x}^\mu \hat{x}^\nu \Sigma_{\mu \nu} - \hat{x}^\rho \hat{x}'_{\rho} \partial_{\nu} (P^{\rho \nu} \Sigma_{\mu \nu}) \right\}.
\]
By using the trace free condition $(P^\mu{}^\nu - \hat{x}^\mu \hat{x}^\nu)\Sigma_{\mu\nu} = 0$, the first and last terms in the round brackets are combined and simplified as

\[
\int d^4x' \frac{F(\sigma, z)}{(\tilde{x} \cdot \Delta x)^3} \left[ \hat{x}^\mu \hat{x}^\nu \Sigma_{\mu\nu} + \hat{x}_\alpha x'_\beta \partial_\alpha (P^\mu{}^\nu \hat{x}^\alpha \Sigma_{\mu\nu}) \right] \\
= \int d^4x' \frac{F(\sigma, z)}{(\tilde{x} \cdot \Delta x)^3} \hat{x}_\alpha x'_\beta \partial_\alpha (P^\mu{}^\nu P^\nu{}^\mu) \\
= \int d^4x' \frac{f(x'_\mu P^\mu{}^\nu P^\nu{}^\mu)}{(\tilde{x} \cdot \Delta x)^3} \hat{x}_\alpha \partial_\alpha F(\sigma, z) \\
= - \int d^4x' \frac{F(\sigma, z)}{(\tilde{x} \cdot \Delta x)^3} \times \left[ \frac{3}{(\tilde{x} \cdot \Delta x)} - \hat{x}_\alpha \partial_\alpha \right] (x'_\mu P^\mu{}^\nu x'_\eta P^\nu{}^\mu). \tag{26}
\]

We finally arrive at the expression

\[
\hat{x}^\mu \hat{x}^\nu h_{\mu\nu} \sim - \int d^4x' \frac{F(\sigma, z)}{(\tilde{x} \cdot \Delta x)^3} \left[ 2x'_\mu P^\mu{}^\nu \hat{x}^\nu \Sigma_{\mu\nu} \right. \\
+ \left. \frac{3}{(\tilde{x} \cdot \Delta x)} - \hat{x}_\alpha \partial_\alpha \right] (x'_\mu P^\mu{}^\nu x'_\eta P^\nu{}^\mu) \right]. \tag{27}
\]

Hence, we find

\[
\hat{x}^\mu h_{\mu\nu} = O \left( \frac{1}{a^2} \right), \quad \hat{x}^\mu \hat{x}^\nu h_{\mu\nu} = O \left( \frac{1}{a^4} \right).
\]

in determining the order of \( a \) in the above expressions, we have used the assumption that support of \( \Sigma_{\mu\nu}(x') \) is finite. Thus, \( x' \) is supposed to be finite. As we have anticipated, the components of metric perturbations contracted with \( \hat{x}^\mu \) have extra-suppression compared with the other components.

The above result can be understood as a slight modification of the well-known fact that gravitational wave perturbations do not have components in the direction of propagation. Since the source of gravitational waves have extension in the present case, the components in \( \hat{x} \)-direction are suppressed but they do not vanish completely.

**IV. COORDINATE TRANSFORM TO REGULAR COORDINATES**

In the preceding section, we have studied the asymptotic behavior of metric perturbations. From the results obtained in the preceding section, however, it is not clear whether the resulting metric is regular or not because the coordinates used there become singular when we take the limit of our interest. In this section, we perform coordinate transformation, and make it manifest that the metric obtained in the preceding section is regular.

Using the Milne coordinates defined above, the bulk metric is written as

\[
ds^2 = \frac{\ell^2}{\zeta^2} (dz^2 - dT^2 + T^2 d\Omega^2), \tag{28}
\]

where \( d\Omega^2 \) is the squared line element of 3-dimensional unit hyperboloid.

Further we introduce new coordinates by

\[
u = T - z, \quad \zeta = (T + z)^{-1}. \tag{29}
\]

In these coordinates the metric becomes

\[
ds^2 = \frac{\ell^2}{(1 + u \zeta^2)} (du d\zeta + (1 - u \zeta)^2 d\Omega^2), \tag{30}
\]

which is regular at \( \zeta \to 0 \) (\( \sigma \to \infty \)) for finite \( \sigma/z(\approx T/\zeta \approx (1 + u \zeta)/(1 - u \zeta)) \). Noting that

\[
\sigma^2 = T^2 - 2T\hat{x}^\mu(\Omega)x'_\mu + |x'|^2, \tag{31}
\]

we find

\[
\frac{1}{\sigma} = 2\zeta(1 + \zeta(-u + 2x'_\mu \hat{x}^\mu(\Omega) + \cdots)), \\
\frac{1}{(\tilde{x} \cdot \Delta x)} = 2\zeta(1 + \zeta(-u + 2x'_\mu \hat{x}^\mu(\Omega) + \cdots)), \\
\frac{1}{\zeta} = 2\zeta(1 + u \zeta + \cdots), \\
\frac{\hat{x}_\alpha}{\zeta} = u - x'_\mu \hat{x}^\mu(\Omega) + \cdots. \tag{32}
\]

The components of metric perturbations in these regular coordinates \((u, \zeta, \Omega')\) are obtained from \( \text{(17), (24), (27)} \), by using relations

\[
\frac{\partial x'^\mu}{\partial \zeta} = - \frac{\hat{x}^\mu(\Omega)}{2\zeta^2}, \quad \frac{\partial x'^\mu}{\partial u} = \frac{\hat{x}^\mu(\Omega)}{2}, \quad \frac{\partial x'^\mu}{\partial \Omega'} = \frac{1 + u \zeta}{2\zeta} \frac{\partial \hat{x}^\mu}{\partial \Omega^2}. \tag{33}
\]

Then, substituting \( \text{(22)} \), it will be easy to see that all components of metric perturbations in these regular coordinates are regular at the future Cauchy horizon. (Thus any curvature invariants are also regular.) Though its importance is not apparent, it might be worth mentioning that \( u \)-components of metric perturbations are much more suppressed than the other components.

**V. DISCUSSION**

In this paper we have studied the asymptotic behavior of metric perturbations near the future Cauchy horizon in the model of warped 5th dimension proposed by Randall and Sundrum. In the present analysis, we carefully took into account the contributions from all the K-K modes and the conservation of the energy-momentum tensor which becomes the source of metric perturbations. As opposed to the claims in literature, our conclusion is that metric perturbations are regular at the future Cauchy horizon.

Both results obtained in Ref. \( \text{(7)} \) and in the present paper indicate that the asymptotic behavior of metric perturbations in Randall-Sundrum model is much better than those expected from naive analysis, in which it was
reported that the contracted Weyl curvature is expected to diverge at the future Cauchy horizon. As we have mentioned in Introduction, there has been a worry about this model raised by Randall and Sundrum themselves. If we take a naive picture based on the mode-by-mode analysis, the non-linear interaction between K-K modes seems to become stronger as we move far away from the brane. Therefore, there appears an inevitable divergence if we try to write down 4-dimensional effective action including interaction terms by integrating out the dependence on the 5-th direction. Although the direct solution to this problem has not been obtained yet, the result proved in the present paper strongly suggests the possibility that the apparent pathological feature in higher order perturbation scheme in the Randall-Sundrum model at the classical level is not physical. Then, it will be an interesting and also a challenging problem to develop a formalism to handle higher order perturbations in this model. We would like to return to this issue in future publication.

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**APPENDIX A: CAN WE GO THROUGH THE FUTURE CAUCHY HORIZON?**

In the original Randall-Sundrum model, the brane stays at a constant $y$. However, if we consider non-trivial evolution of energy density of matter fields on the brane, the brane can move in $y$-direction. Despite this possibility, one may think that observers on the brane cannot cross the future Cauchy horizon because the scale factor of our universe will collapse to zero at $y = \infty$ if we identify it with $e^{-|y|/\ell}$. We will show that this naive intuition is not correct and that the brane can go through the horizon of AdS space located at $y = \infty$.

We represent the motion of the brane by the function $z = z_b(T)$. Then, from (28), the metric induced on the brane becomes

$$ds_b^2 = d\tau^2 + a(\tau)^2d\Omega^2,$$

with

$$d\tau = \sqrt{1 - \left(\frac{dz_b/dT}\right)^2}dT, \quad a^2(\tau) = \frac{\ell^2T^2(\tau)}{z_b^2(1/T(\tau))}.$$  \hfill (A2)

An interesting situation arises when we consider the case in which $dz_b/dT \to 1$ for $T \to \infty$. In this case, $T$ and $z_b$ go to infinity within a finite value of the cosmological time $\tau$, while $a(\tau)$ stays finite. Hence, observers living on the brane can cross the future Cauchy horizon with a finite scale factor within a finite proper time. This means that the AdS Cauchy horizon is not necessarily the end of the brane-world.

[1] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys.Lett. B429 (1998) 263-272, hep-ph/9803315. Phys.Rev. D59 (1999) 086004, hep-ph/9807344. T. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys.Lett. B436 (1998) 257-263, hep-ph/9804398.

[2] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999), hep-ph/9905221.

[3] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999), hep-ph/9906064.

[4] A. Chamblin and G.W. Gibbons, hep-th/9909130.

[5] A. Chamblin, S.W. Hawking and H.S. Reall, Phys. Rev. D61, 065007 (2000), hep-th/9909207.

[6] T. Shiromizu, K. Maeda and M. Sasaki, gr-qc/9910074.

[7] J. Garriga and T. Tanaka, Phys. Rev. Lett. 84, 2768 (2000), hep-th/9911053.

[8] R. Emparan, G.T. Horowitz and R.C. Myers, JHEP 0001, 007 (2000), hep-th/9911043.

[9] C. Csáki, M. Graesser, L. Randall and J. Terning, hep-ph/0001092.

[10] C. Charmousis, R. Gregory and V. Rubakov, hep-th/9912169.

[11] M. Sasaki, T. Shiromizu and K. Maeda, hep-th/9912233.

[12] T. Tanaka and X. Montes, to appear in Nucl. Phys. B, hep-th/0001092.

[13] T. Chiba, gr-qc/0001026.

[14] S.B. Giddings, E. Katz and L. Randall, hep-th/0002093.

[15] E.E. Flanagan, S.-H.H. Tye and I. Wasserman, JHEP 0003, 023 (2000), hep-ph/9909373.

[16] P. Binétruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B477, 285 (2000), hep-th/9910219.

[17] Phys. Lett. B473, 241 (2000), hep-th/9911163.

[18] D. Ida, gr-qc/9912092.

[19] J. Garriga and M. Sasaki, hep-th/9912118.

[20] S. Mukohyama, T. Shiromizu and K. Maeda, hep-th/9912287.

[21] R. Maartens, D. Wands, B.A. Bassett and I. Heard, hep-ph/9912464.

[22] S. Mukohyama, hep-th/0004067.

[23] H. Kodama, A. Ishibashi and O. Seto, hep-th/0004160.

[24] R. Maartens, hep-th/0001160.

[25] D. Langlois, hep-th/0005025.

[26] C. van de Bruck, M. Dorca, R. Brandenberger and A. Lukas, hep-th/0005032.

[27] K. Koyama and J. Soda, hep-th/0005239.