Determining the Geometrical Sizes of Plates with Internal Hinges by Using Additional Conditions

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For a system obtained by placing more than two elastic plates side by side, the transmission conditions are obtained at the common boundaries. Finite difference equations are developed for the problem of plates with internal hinges and applied for determination of the response of a system assembled from three different plates with different mechanical constraints between adjacent plates in this study. An algorithm is written to find out how long the size of the plates should be in order to obtain the desired amount of bending against the force affecting the system under different boundary conditions. The bisection and multigrid methods are used for this. These two methods are compared based on the obtained data.

1. Introduction

Since laminated composite structures have high strength and stiffness, the use of such structures is gradually increasing in implementations in the field of engineering. Timoshenko mathematically explains the theory of elasticity and plasticity in a previous study [1]. The deformation problem of elasto-plastic plates whose mathematical model is expressed by biharmonic equations is defined by the boundary value problem. The deformation problems of thin plates modelled by the classical Kirchhoff theory were considered priorly by Reddy [2]. Using the monotone potential operator theory, Hasanov developed the variational approach theory for nonlinear biharmonic equations related to bending of elastoplastic plates [3]. Likewise, the plate theory for multilayered and sandwich plates was developed in different studies [4, 5]. Recently, the first-order shear deformation theory for vibration of rectangular plates with an internal line hinge was investigated. The literature appears to contain a limited number of studies related to plates with internal hinges. Previous studies have analytically solved the buckling problem using the Levy method. The first known numerical solutions were introduced by some researchers [6]. Using the Ritz method, the vibration and buckling analyses of plates with an internal hinge were studied [7, 8]. Moreover, using the Levy-type solution, the exact solutions of natural vibration of rectangular plates were also presented [9]. The vibration studies of plates with point and line supports were also investigated by several researchers [10, 11]. The problem of bending of a rectangular plate given by symmetrical boundary conditions along its edges under a load was also investigated. Quintana and Grossi studied the free vibration of plates with different geometrical shapes with internal line hinges [12, 13]. The vibration of Timoshenko beams with an internal hinge was studied in a previous study [14]. Additionally, a numerical solution was obtained using the finite element method [15]. Grossi conducted a study on anisotropic plates with an arbitrarily located internal line hinge with elastic supports [16]. Grossi and Raffo extended the model for several arbitrarily located internal line hinges [17]. Internal hinges were also investigated using the Modified Stiffness Matrix Method [18]. Furthermore, some researchers studied the variational approach to vibrations of plates with line hinges [19]. Different possible combinations of simply supported, clamped, or free boundary conditions may be given along each plate edge as the boundary conditions. To calculate the deformation of thin plates under quasistatic axial loading, the three-hinge-line method was presented [20]. The transmission conditions on common boundaries were obtained.
by the functional approximation method in a previous study [21]. In a previous study of ours, we considered the deformation problem of a plate system (formed side by side) composed of multistructure plates.

In this study, the transmission conditions obtained on the common border of the plates that constituted the system in our previous study [22] are extended for a system consisting of plates with different mechanical properties, and the finite difference expressions of these conditions are given. It is the aim of this study to determine how long the sizes of the plates should be for obtaining the desired amount of bending against the force affecting the system under different boundary conditions. For this purpose, we created an algorithm, and the bisection and multigrid methods were utilized. These two methods were compared based on the obtained data. In relation to the bending of the system obtained by applying three elastic plates, the numerical results are presented in tables and plots.

2. Problem Formulation

Let us consider inhomogeneous elastoplastic plates with different properties. The plates make up a system formed side by side. For instance, let us consider the system-filled regions, whose dimensions are $\ell_x \times \ell_y$, such that $\Omega_k = \{(x,y)\mid x^{k-1}_x \leq x \leq x^k_x, 0 \leq y \leq \ell_y\}, k = 1, 2, \ldots, k_0$, $\bigcup_{k=1}^{k_0} \Omega_k = \Omega \subset \mathbb{R}^2$ (Figure 1). Here, $k_0$ is the number of plates that form the system, and $\ell^k_x \times \ell^k_y$ are the dimensions for each $\Omega_k, k = 1, 2, \ldots, k_0$. Moreover, the common boundary of the $\Omega_k$ and $\Omega_{k+1}$ regions is considered as $\gamma_{k,k+1}, k = 1, 2, \ldots, k_0 - 1$.

The mathematical model of the problem of deformation of the multistructure plate system may be written as follows [21]:

$$
\frac{\partial^2}{\partial x^2} \left[ D_k \left( \frac{\partial^2 \omega}{\partial x^2} + \nu_k \frac{\partial^2 \omega}{\partial y^2} \right) \right] + \frac{\partial^2}{\partial y^2} \left[ D_k \left( \frac{\partial^2 \omega}{\partial y^2} + \nu_k \frac{\partial^2 \omega}{\partial x^2} \right) \right] + 2 \frac{\partial^2}{\partial x \partial y} \left[ D_k \left( 1 - \nu_k \frac{\partial^2 \omega}{\partial x \partial y} \right) \right] = q_k(x,y),
$$

where $\omega$ is the bending of the system at any $(x,y)$ and $q_k(x,y)$ is the force applied vertically onto the $k$-th plate. The cylindrical stiffness coefficients of the plates of the system are $D_k = E_k H^2/12(1 - \nu_k^2)$. Moreover, the $E_k$, $\nu_k$, and $H$ values are Young’s modulus, Poisson constants, and thicknesses for each plate, respectively.

The boundary conditions are generally classified for the equilibrium equation in two manners depending on the physical meanings:

$$
\begin{align*}
\omega(x,y) &= \frac{\partial \omega(x,y)}{\partial n} = 0 \text{ (clamped boundary)}, \\
\omega(x,y) &= \frac{\partial^2 \omega(x,y)}{\partial n^2} = 0 \text{ (simply supported boundary)}.
\end{align*}
$$

As the system consists of plates with different properties, the equilibrium equation (1) becomes discontinuous at the common boundaries of $\gamma_{k,k+1}, k = 1, 2, \ldots, k_0 - 1$. Therefore, when the deformation problem of the plate system is solved numerically, it is needed to provide the numerical expressions of the transmission conditions that are suitable for the connection form of the common boundaries. For this, firstly, the uniform and nonuniform meshes in the interval $[0, \ell_x]$ are defined as follows:

$$
\begin{align*}
u_{kh} &= \left\{ \begin{array}{l}
\gamma_{i+1/2}, i = 1, 2, \ldots, k_0 - 1, k = 1, 2, \ldots, k_0, \\
\gamma_{k_0+1/2}, k = 1, 2, \ldots, k_0 - 1, i = 0, 1, \ldots, n_k,
\end{array} \right.
\end{align*}
$$

uniform mesh,

$$
\begin{align*}
u_{kh} &= \left\{ \begin{array}{l}
\gamma_{i+1/2}, i = 1, 2, \ldots, k_0 - 1, k = 1, 2, \ldots, k_0, \\
\gamma_{k_0+1/2}, k = 1, 2, \ldots, k_0 - 1, i = 0, 1, \ldots, n_k, \sum_{i=1}^{n_k} \nu_{kh} = \ell_x,
\end{array} \right.
\end{align*}
$$

non-uniform mesh where $k = 1, 2, \ldots, k_0$. Here, $n_k$ is the number of steps for the mesh created for each plate. Additionally, $\lfloor x \rfloor$ is a symbolic expression of greatest integer function.

Let the plates that form the system connect to each other with a beam from the common boundary of $\gamma = \gamma_{k,k+1}$. Here, the coefficients of stiffness and the torsional stiffness of beams are $B_k$ and $C_k$, respectively.

In this case, the potential energy of the system is calculated by the following formula:

$$
W(\omega) = \frac{1}{2} \sum_{k=1}^{n} \int_{\gamma_k} \left[ \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right]^2 dx dy + \frac{1}{2} \int_{\gamma_k} \left[ \frac{\partial^2 \omega}{\partial x \partial y} \right]^2 dx dy + \frac{1}{2} \int_{\gamma_k} \left[ \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right]^2 dx + \frac{1}{2} \int_{\gamma_k} \left[ \beta_k \frac{\partial^2 \omega}{\partial x^2} + C_k \frac{\partial^2 \omega}{\partial x \partial y} \right]^2 dx dy
$$

Moreover, the nonnegative constants $\alpha_k$ and $\beta_k$ are the stiffness coefficients of the plates connected with hinges and supports provided by the hinge, respectively. For different values of $\alpha_k$ and $\beta_k$, there are different physical meanings of the transmission conditions:

(i) $\alpha_k = 0$: plates are attached together along their edges using an ideal hinge that is greased in the mechanical sense and moves freely similar to the simply supported condition

(ii) $\alpha_k = \text{const.}$: it means there is a hinge with a finite stiffness on the common boundary. It behaves as rusted, but it can be hardly moved in the mechanical sense. When the value of $\alpha_k$ is increased, the hinge is going to be rusted more than before, and the plates building the system are caused to start moving together. The movement of the hinge becomes impossible

(iii) $\alpha_k = \infty$: it means that these plates forming the system behave like a single plate and move together, also bending together

(iv) $\beta_k = 0$: there is no support provided by the hinge on $\gamma_{k,k+1}$. When the force is applied on the common boundary $\gamma_{k,k+1}$, the system bends easily

(v) $\beta_k = \text{const.}$: there is a support provided by the hinge on the common border, and it has a finite rigidity. When we apply the force on $\gamma_{k,k+1}$, it is against the force externally applied. It also prevents bending of the system on the common boundary

(vi) $\beta_k = \infty$: in this case, there is a support provided which has infinite stiffness by the hinge. There is also no bending on the common boundary $\gamma_{k,k+1}$

As seen above, the value of $\alpha_k$ is increased, plates building the system start to move together, the value of $\beta_k$ is increased, and bending on the common boundary becomes stiffer.

The first variation of the full potential energy of the plates denoted functional $I(\omega) = W - \sum_{k=1}^{n} \int_{\gamma_k} \int_{\Omega_k} q(x,y) \omega \, dx \, dy$ must equal zero to reach the equilibrium for the loaded elastic body:

$$\delta_{\gamma} I(\omega) = \frac{d}{dt}[I(\omega + t\psi)]_{|t=0} = 0.$$  \hspace{1cm} \text{(6)}

After calculating the Gateaux derivative of the functional $I(\omega)$, the coefficients of the expressions $\psi$, $(\partial \psi / \partial x)|_{x = \gamma_{k,k+1}}$, $(\partial \psi / \partial x)|_{x = \gamma_{k,k+1}}$ belonging to the common boundary $x = \gamma_{k,k+1}$ are equal to zero. By adding the discontinuity condition at the common boundary of the plates, the transmission conditions are obtained as follows: $k = 1, 2, \cdots, k_g$.

$$\begin{align*}
\alpha_k \frac{\partial \omega}{\partial x} &= \frac{1}{2} \left\{ D_{k+1} \left( \frac{\partial^2 \omega}{\partial x^2} + v_k \frac{\partial^2 \omega}{\partial y^2} \right) + D_k \left( \frac{\partial^2 \omega}{\partial x^2} + v_k \frac{\partial^2 \omega}{\partial y^2} \right) \right\}_{x = \gamma_{k,k+1}} \\
&\quad + \left[ \frac{\partial}{\partial y} \left( C_{k+1} \frac{\partial^2 \omega}{\partial x \partial y} \right) - \frac{\partial}{\partial y} \left( C_k \frac{\partial^2 \omega}{\partial x \partial y} \right) \right]_{x = \gamma_{k,k+1}}.
\end{align*}$$  \hspace{1cm} \text{(7)}

$$\begin{align*}
D_{k+1} \left( \frac{\partial^2 \omega}{\partial x^2} + v_k \frac{\partial^2 \omega}{\partial y^2} \right) - D_k \left( \frac{\partial^2 \omega}{\partial x^2} + v_k \frac{\partial^2 \omega}{\partial y^2} \right)_{x = \gamma_{k,k+1}} \\
&= - \left[ \frac{\partial}{\partial y} \left( C_{k+1} \frac{\partial^2 \omega}{\partial x \partial y} \right) - \frac{\partial}{\partial y} \left( C_k \frac{\partial^2 \omega}{\partial x \partial y} \right) \right]_{x = \gamma_{k,k+1}}.
\end{align*}$$  \hspace{1cm} \text{(8)}

$$\begin{align*}
\left\{ \frac{\partial}{\partial x} \left[ D_{k+1} \left( \frac{\partial^2 \omega}{\partial x^2} + v_k \frac{\partial^2 \omega}{\partial y^2} \right) \right] + 2 \frac{\partial}{\partial y} \left[ D_{k+1} (1 - v_k) \frac{\partial^2 \omega}{\partial x \partial y} \right] \right\}_{x = \gamma_{k,k+1}} \\
&= \left[ \frac{\partial}{\partial x} \left[ D_k \left( \frac{\partial^2 \omega}{\partial x^2} + v_k \frac{\partial^2 \omega}{\partial y^2} \right) \right] \\
&\quad + 2 \frac{\partial}{\partial y} \left[ D_k (1 - v_k) \frac{\partial^2 \omega}{\partial x \partial y} \right] \right\}_{x = \gamma_{k,k+1}}.
\end{align*}$$  \hspace{1cm} \text{(9)}

Here, $[\partial \omega / \partial x]$ is called the jump of $\partial \omega / \partial x$ at $x = \gamma_{k,k+1}$.

For the finite difference approximations of the transmission conditions, we use the functional approximation method. Let $\Omega_{k_g}^H$ be the nonuniform and $\Omega_{h_y}$ be the uniform meshes on the axes $x$ and $y$, respectively. Here, for each $k$-th plate, $h_k$ is the length of the step on the $x$-axis and $h_y$ is the length of the step on the $y$-axis. Let us define the mesh $\Omega_{k_g}^{h} = \Omega_{k_g}^{h} \times \Omega_{h_y} = \{(x_i^k, y_j) | x_i^k \in \Omega_{k_g}^{h}, y_j \in \Omega_{h_y} \}$ in the region $\Omega \subset \mathbb{R}^2$.

Considering (7)–(9) and $\alpha_k = 0$, $C_k = 0$, and $B_k = B_{k+1} = 0, 5B_k, k = 1, 2, \cdots, k_0 - 1$ ($B_k$ is the stiffness coefficients of the beam on the common boundary $\gamma_{k,k+1}$), we may denote the finite difference approximation of the potential energy $I(\omega)$ such that

$$I_h(u) = \sum_{i,j} W^h_k(u_i - quh^h_j)_{j \in \Omega_{k_g}^{h}}.$$  \hspace{1cm} \text{(10)}
Here, \( a_k \neq 0 \) corresponds to the general case, \( C_k = 0 \) means that there is no torsional case, and \( B_k = B_{k+1} = 0, S \) means that the stiffness coefficients of beams are the same. Moreover, \( u \) is an approximation value of the \( \omega \)-bending of the system and \( W_h^\omega(u) \) are the finite difference approximations of the deformation energy functions on the meshes \( \Omega_{h^k,h^l} \) for each \( k \)-th plate. The finite difference expression of the functional \( I_h(u) \) is obtained by the functional approximation method. In order to do this, finite difference approximations are written in the place of derivatives in the energy functional (5). Furthermore, integrals are computed by the numerical integration formula. In the expression (5), integrals containing mixed derivatives are calculated using the trapezoid method, while integrals containing other 2nd-order partial derivatives are calculated by the trapezoid method. Then, the finite difference expressions of partial derivatives are written. In order to write the obtained expressions easily, we define the following coefficients [23]:

\[
\begin{align*}
D_1 &= \left\{ \begin{array}{l}
D_k h^k_x, \\
2\tilde{D}, \\
D_{k+1} h^{k+1}_x, \\
\end{array} \right. \\
D_2 &= \left\{ \begin{array}{l}
\tilde{B}_k + \frac{1}{2} (D_k h^k_x (1 - (v_k)^2) + D_{k+1} h^{k+1}_x (1 - (v_{k+1})^2)) + \frac{(v_k + v_{k+1})^2}{2} \tilde{D}, \\
D_{k+1} h^{k+1}_x, \\
\end{array} \right. \\
D_k p &= \left[ \begin{array}{l}
D_k h^k_x (1 - v_k) + D_{k+1} h^{k+1}_x (1 - v_{k+1}) \end{array} \right].
\end{align*}
\]

where \( \tilde{D} = 2a_k h^{k+1}_x D_k D_{k+1}/(2D_k D_{k+1} + a_k (h^{k+1}_x D_k + h^k_x D_{k+1})) \),
then we may rewrite \( I_h(u) \) by using (11)–(13) as follows:

\[
I_h(u) = \frac{h^2}{2 \Omega_{h^k,h^l}^\omega} \left\{ \begin{array}{l}
\mathcal{D}_1 ((u_{xx})^2 + \mathcal{D}_2 (u_{yy})^2 + \mathcal{D}_1 (v_k + v_{k+1}) u_{xx} u_{yy} \\
+ \mathcal{D}_k p \left[ (u_{xx})^2 h^k_x + (u_{yy})^2 h^{k+1}_x + (u_{xy})^2 h^k_x + (u_{xy})^2 h^{k+1}_x \right] + \frac{h^2_k + h^{k+1}_x}{2} - \beta_k \xi(x) u^2 - 2qu \end{array} \right\},
\]

where the function is

\[
\xi(x) = \begin{cases} 
\frac{2}{h^k_x + h^{k+1}_x}, & x \in \Omega_{h^k,h^l}, x = y_{k,k+1}, \\
0, & x \in \Omega_{h^k,h^l}, x \neq y_{k,k+1},
\end{cases}
\]

and \( \Omega_{h^k,h^l} \) below the sum symbol mean in equation (14) is the mesh contained inner points of mesh \( \Omega_{h^k,h^l} \).

Because \( I_h(u) \) depends on the variables \( u_{ij} = u(x^k_i, y_j) \), \((x^k_i, y_j) \in \Omega_{h^k,h^l} \), it is a multivariable function. Considering this, we compute the derivative of the functional \( I_h(u) \) and equalize it to zero. Then, we obtain the finite difference expression of equation (1) as follows:

\[
\begin{align*}
&\mathcal{D}_1 u_{xx} + \mathcal{D}_2 u_{yy} + \mathcal{D}_1 \frac{(v_k + v_{k+1})}{2} u_{xx} \\
&+ \mathcal{D}_k p u_{xy} + 4\mathcal{D}_k u_{xx} u_{yy} + \beta_k \xi(x) u = 0(x^k_i, y_j).
\end{align*}
\]

The finite difference expressions of the collected terms that belong to the common boundary \( Y_{k,k+1} \) are written as \( W_h^{Y_{k,k+1}}(u) \) from equation (14) obtained as follows:

\[
\begin{align*}
W_h^{Y_{k,k+1}}(u) &= \frac{1}{2} \sum_{y_{k,k+1}} h_y \left\{ 2\tilde{D}(u_{xx})^2 + \left[ \tilde{B}_k + \frac{1}{2} \left[ D_k h^k_x (1 - (v_k)^2) + D_{k+1} h^{k+1}_x (1 - (v_{k+1})^2) + \frac{(v_k + v_{k+1})^2}{2} \tilde{D} \right] u_{yy} \right] \\
&+ 2(v_k + v_{k+1}) D_{xx} u_{yy} + \beta_k \xi(x)^2 \right\}.
\end{align*}
\]

Using equation (16), we obtain the finite difference approximations of the transmission conditions on the
points of the common border and their neighbours, i.e., \( y_{k,k+1} - h_{k}^{x} \), \( y_{k,k+1} \), and \( y_{k,k+1} + h_{k}^{x+1} \). Firstly, equation (16) is turned into (18) for the points \( (y_{k,k+1} - h_{k}^{x}, y_{j}) \), \( j = 2, 3, \ldots, m - 1 \):

\[
D_{k} \begin{cases}
  u_{xxxx} - \frac{1}{(h_{k}^{x})^2} u_{xx} + u_{yy} \\
  + \frac{v_{k} + v_{k+1}}{2} \left( 2u_{xxyy} - \frac{1}{h_{k}^{x}} u_{xxy} - \frac{1}{h_{k}^{x}} u_{yy} \right) \\
  + \frac{h_{k}^{x}}{2} \left( \frac{1}{h_{k}^{x}} u_{x} + \frac{1}{h_{k}^{x}} u_{y} \right) + \frac{1}{h_{k}^{x}} u_{xx}
\end{cases}
\]

\[
+ \left[ D_{k} h_{k}^{x} (1 - v_{k}) + D_{k+1} h_{k+1}^{x} (1 - v_{k+1}) \right] u_{xxyy} = q_{k-1} \left( y_{k,k+1} - h_{k}^{x}, y_{j} \right).
\]

(18)

For the points that belong to the common border \( y_{k,k+1} \) in \( \Omega_{h_{k}^{x}} \), the finite difference expressions of the transmission conditions are

\[
2D \begin{cases}
  u_{xxxx} + \frac{(v_{k} + v_{k+1})^2}{4} u_{yy} + \frac{1}{(h_{k}^{x})^2} u_{xx} + \frac{1}{h_{k}^{x+1}} u_{xx} \\
  + (v_{k} + v_{k+1}) u_{yxy} - \frac{1}{h_{k}^{x}} u_{xxy} - \frac{1}{h_{k}^{x+1}} u_{yxy} \\
  + \left( \frac{1}{h_{k}^{x+1}} D_{k+1} h_{k+1}^{x+1} (1 - (v_{k})^2) + D_{k+1} h_{k+1}^{x+1} (1 - (v_{k+1})^2) \right) u_{yy} \\
  + \left[ \frac{1}{h_{k}^{x+1}} D_{k+1} u_{xx} + \frac{1}{h_{k}^{x+1}} u_{xx} \\
  + \frac{v_{k} + v_{k+1}}{2h_{k}^{x+1}} D_{k+1} h_{k+1}^{x+1} - D_{k+1} h_{k}^{x+1} u_{yy} \\
  + \frac{v_{k} + v_{k+1}}{2h_{k}^{x+1}} D_{k+1} h_{k+1}^{x+1} u_{xxy} - \frac{1}{h_{k}^{x+1}} D_{k+1} u_{xxy} \\
  + \left[ D_{k+1} h_{k}^{x+1} (1 - v_{k}) + D_{k+1} h_{k+1}^{x+1} (1 - v_{k+1}) \right] u_{xxy} + \beta_{k} u (y_{k,k+1}) \right]
\end{cases}
\]

\[
= q_{k} \left( y_{k,k+1}, y_{j} \right).
\]

(19)

Finally, we reach equation (20) for the points \( (y_{k,k+1} + h_{k}^{x+1}, y_{j}) \), \( j = 2, 3, \ldots, m - 1 \):

\[
D_{k+1} \begin{cases}
  u_{xxxx} - \frac{1}{(h_{k}^{x+1})^2} u_{xx} + u_{yy} \\
  + \frac{v_{k} + v_{k+1}}{2} \left( 2u_{xxyy} - \frac{1}{h_{k}^{x+1}} u_{xxy} - \frac{1}{h_{k}^{x+1}} u_{yy} \right) \\
  + 2D \begin{cases}
  1 \left( h_{k}^{x+1} \right)^{2} u_{xx} - \frac{v_{k} + v_{k+1}}{2} \left( 1 \left( h_{k}^{x+1} \right)^{2} u_{xxy} - \frac{1}{h_{k}^{x+1}} u_{yy} \right) \\
  + \left[ D_{k} h_{k}^{x+1} (1 - v_{k}) + D_{k+1} h_{k+1}^{x+1} (1 - v_{k+1}) \right] u_{xxyy} = q_{k} \left( y_{k,k+1} + h_{k}^{x+1}, y_{j} \right).
\end{cases}
\end{cases}
\]

(20)

Thus, we obtain the finite difference approximations of the transmission conditions on \( y_{k,k+1} - h_{k}^{x} \), \( y_{k,k+1} \), and \( y_{k,k+1} + h_{k}^{x+1} \) as equations (18)–(20) for the nonuniform mesh.

Consequently, the coefficients of the equations belonging to the points on the common borders and their neighbours of the linear algebraic equations system obtained may be calculated by using (18)–(20).

3. Numerical Example

In relation to the bending of the system obtained by placing three elastic plates side by side, computer experiments were carried out for the analysis of the numerical solution in the case where a clamped boundary condition is provided at the boundaries of the system. Where \( k_{0} = 3 \), let us suppose that there are three thin plates, occupying the rectangular region \( \Omega_{h_{k=1}} \).

\[
w_{x,h} = \left\{ y_{j} \right\} \left\{ x_{i} \right\} = x_{i}^{[1]} + h_{k}^{x}, x_{i}^{0} = 0, i = 1, \ldots, n_{y}, \quad h_{k}^{x} = \frac{\ell_{k}}{n_{y}}, k = 1, 2, 3 \},
\]

\[
w_{y,r} = \left\{ y_{j} \right\} \left\{ y_{j+1} \right\} = y_{j} + r, y_{0} = 0, j = 0, 1, 2, \ldots, m, r = \ell_{y}/m \}
\]

for these meshes,

\[
w_{x,y,r} = \left\{ \left( x_{i}^{k}, y_{j} \right) \right\} x_{i}^{k} \in w_{x,h}, y_{j} \in w_{y,r} \},
\]

such that the dimensions of the system are \( \ell_{x} = \ell_{y} = 10 \) cm, thickness is \( H = 0.3 \) cm, and mesh dimensions are \( N_{x} \times N_{y} = 31 \times 31 \). Each of the plates forming the system is divided into equal steps (uniform mesh) within itself (Figure 2).

When the sizes of the plates composed the system are the same \( (h_{k}^{x} = \ell_{y}, k = 1, 2, 3) \), the maximal bending corresponding to any force affecting the system is obtained as the solution to the direct problem and denoted by \( u_{0} \). Since the maximal bending of the plate system takes different values corresponding to different values of \( \alpha_{k} \) and \( \beta_{k} \), the initial data \( u_{0} \) is different for each case.
Firstly, a system of plates with the same mechanical properties was considered. To determine the sizes of the middle plate corresponding to the desired bending, the bisection method and the multigrid method are utilized by using the initial data. As the initial data, using the value of maximum bending $u_0$ occurring on the system based on the effect of a known force $q$ ($q = 200$ kN/cm² is applied at 5 points on the middle surface of the system), the size of the middle plate $c$ of the system may be obtained with the bisection method by considering the mechanical properties of the plate-forming material that are given in Table 1. Then, the approach speeds are investigated.

As a first numerical example, a situation is handled such that Young’s modulus and the Poisson ratios are $E = E_{St}$, $\nu = \nu_{St}$, $i = 1, 2, 3$, respectively. The bending that occurs as a result of the applied force is investigated. In order for the plates forming the system to move together, $u_h$ are obtained as corresponding to different values of the nonnegative constants $\alpha$ and $\beta$ in the transmission conditions obtained, and by using $\varepsilon = 10^{-3}$ and initial data $u_0$, the size of the middle plate $c$ is obtained by the bisection method. The approximate values $\tilde{c}$ of $c$ and solutions $u_h$ obtained are shown in Table 2.

Based on different values of the nonnegative constants $\alpha$ and $\beta$, after determining the solution interval, with a precision of $\varepsilon = 10^{-3}$, the size value of $\tilde{c}$ is obtained in 4 steps with 0.25% error for cases $\alpha = \infty$ and $\beta = \infty$ and $\alpha = 0$ and $\beta = \infty$, while the size value of $\tilde{c}$ is obtained in 7 steps with 0.033% error for cases $\alpha = \infty$ and $\beta = 0$ and $\alpha = 0$ and $\beta = \infty$.

Then, let us handle a situation where plates on the left and right sides of the system with the same properties and the middle plate with a different property are given. Similar

### Table 1: The mechanical properties of the plate-forming material.

| Mechanical properties | Steel        | Iron         | Copper       |
|-----------------------|--------------|--------------|--------------|
| Young’s (elasticity) modulus | $E_{St} = 21000$ kN/cm² | $E_{Fe} = 30000$ kN/cm² | $E_{Cu} = 18100$ kN/cm² |
| Poisson ratios | $\nu_{St} = 0.30$ | $\nu_{Fe} = 0.27$ | $\nu_{Cu} = 0.36$ |

### Table 2: The $u_h$ solutions obtained by using the bisection method in the system whose hard clamping condition is given at its borders, with the $\tilde{c}$ length values corresponding to these.

| $\tilde{c}$ (cm) | $\alpha = \infty$, $\beta = \infty$ | $\alpha = \infty$, $\beta = 0$ | $\alpha = 0$, $\beta = 0$ | $\alpha = 0$, $\beta = \infty$ |
|------------------|------------------------------------|---------------------------------|--------------------------|--------------------------|
| $u_0 = 0.1055$ cm | $0.1339$                           | $1.1744$                        | $1.2653$                 | $0.2981$                 |
| $u_0 = 1.0211$ cm | $0.8213$                           | $1.0014$                        | $1.1038$                 | $0.1729$                 |
| $u_0 = 1.1225$ cm | $1.0892$                           | $1.1865$                        | $0.2629$                 | $0.2303$                 |
| $u_0 = 0.2373$ cm | $1.0456$                           | $1.1456$                        | $0.2463$                 | $0.2382$                 |

![Figure 2: Mesh for the system consisting of 3 plates.](attachment:image.png)
\( \alpha = 0, \beta = 100 \)

Figure 3: Continued.
\[\alpha = 0, \beta = 100\]

Figure 3: (a) \(c_0 = 2\), (b) \(c_1 = 6\), (c) \(c_8 = 3.3438\), and (d) bending of the system for Case 1 in the case of \(\alpha = 0, \beta = 100\).
\( \alpha = 0, \beta = 100 \)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4a.png}
\caption{(a)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4b.png}
\caption{(b)}
\end{figure}

\textbf{Figure 4:} Continued.
\[ \alpha = 0, \beta = 100 \]

\[ c_0 = 2, \quad c_1 = 6, \quad c_9 = 3.3282 \]

\[ c_2 = 4, \quad c_3 = 3, \quad c_4 = 3.5, \quad c_5 = 3.25, \quad c_6 = 3.375, \quad c_7 = 3.3125, \quad c_8 = 3.3438 \]

**Figure 4:** (a) \( c_0 = 2 \), (b) \( c_1 = 6 \), (c) \( c_9 = 3.3282 \), and (d) bending of the system for Case 2 in the case of \( \alpha = 0, \beta = 100 \).
Table 3: The $u_0$ solutions and $\tilde{c}$ dimensions that are obtained when the bisection method is used by utilizing the data in Cases 1 and 2 for different values of $\alpha$ and $\beta$.

| $\tilde{c}$ (cm) | $\alpha = \infty, \beta = \infty$ | $\alpha = \infty, \beta = 0$ | $\alpha = 0, \beta = 0$ | $\alpha = 0, \beta = \infty$ | $\alpha = 0, \beta = 100$ |
|-----------------|---------------------------------|-------------------------------|----------------------------|---------------------------------|-----------------------------|
|                 | $u_0 = 0.1155$ | $u_0 = 0.164$ | $u_0 = 0.4965$ | $u_0 = 0.7126$ | $u_0 = 0.5882$ | $u_0 = 0.7931$ | $u_0 = 0.1693$ | $u_0 = 0.2633$ | $u_0 = 0.2246$ | $u_0 = 0.3358$ |
| $c_0$           | 2               | 0.2005           | 0.2857           | 0.7001           | 1.0224           | 0.7683           | 1.0797           | 0.2885           | 0.4486           | 0.3459           | 0.5249           |
| $c_1$           | 6               | 0.0236           | 0.0333           | 0.1311           | 0.1818           | 0.1872           | 0.2277           | 0.0354           | 0.0550           | 0.0747           | 0.1045           |
| $c_2$           | 4               | 0.0836           | 0.1186           | 0.3926           | 0.5582           | 0.4879           | 0.6417           | 0.1233           | 0.1918           | 0.1764           | 0.2606           |
| $c_3$           | 3               | 0.1341           | 0.1905           | 0.5486           | 0.7909           | 0.6359           | 0.8673           | 0.1957           | 0.3044           | 0.2520           | 0.3783           |
| $c_4$           | 3.5             | 0.1069           | 0.1518           | 0.4704           | 0.6735           | 0.5637           | 0.7556           | 0.1570           | 0.2441           | 0.2119           | 0.3158           |
| $c_5$           | 3.25            | 0.1200           | 0.1704           | 0.5095           | 0.7322           | 0.6003           | 0.8118           | 0.1757           | 0.2733           | 0.2314           | 0.3461           |
| $c_6$           | 3.375           | 0.1133           | 0.1609           | 0.4900           | 0.7028           | 0.5821           | 0.7838           | 0.1662           | 0.2585           | 0.2215           | 0.3307           |
| $c_7$           | 3.3125          | 0.1167           | 0.1656           | 0.4998           | 0.7175           | 0.5912           | 0.7978           | 0.1709           | 0.2658           | 0.2264           | 0.3384           |
| $c_8$           | 3.3438          | 0.1150           | 0.1633           | 0.4949           | 0.7101           | 0.5867           | 0.7908           | 0.2621           | 0.2240           | 0.3345           | 0.3345           |
| $c_9$           | 3.3282          | 0.4973           | 0.7138           | 0.5889           | 0.7943           | 0.2639           | 0.3365           | 0.2639           | 0.3365           |                  |                  |
| $c_{10}$        | 3.336           | 0.7119           | 0.7925           |                  |                  |                  |                  |                  |                  |                  |                  |
The algorithm of the method that was applied may be written as:

\( \alpha = \infty, \beta = \infty \) and the obtained \( u_0(c_k) \) and \( u_0 \) are compared.

Table 5: Results obtained by using the multigrid method with \( \varepsilon = 10^{-3} \) precision and the step length \( \delta = 0.5 \) for Case 1.

| \( \gamma \) | \( \tilde{c} \) | \( u_0 \) | \( u_0 - u_0 \) |
|---|---|---|---|
| 2 | 0.3459 | -0.1213 < 0 |
| 2.5 | 0.2967 | -0.0721 < 0 |
| 3 | 0.2520 | -0.0274 < 0 |
| 3.5 | 0.2119 | 0.0131 > 0 |
| 0.3212 | 3.3394 | 0.2243 | 0.0003 < \( \varepsilon \) |

The following consecutive processes may be performed by using a \( \gamma \in (0, 1) \) acceleration parameter to find the \( c \) value faster:

(II) The processes in the first step are continued until the bending value \( u_0 \) is between the maximum bending values \( u_0(c_k-1) \) and \( u_0(c_k) \) corresponding to the sizes \( c_k-1 \) and \( c_k \) that are found consecutively. Eventually, one of the conditions \( u_0(c_k-1) < u_0 < u_0(c_k) \) or \( u_0(c_k) < u_0 < u_0(c_k-1) \) will be met.

The steps III-VI are repeated until \( \max |u_0(c) - u_0| < \varepsilon \) is satisfied.

Let \( u_0(c_k) \) be the size of \( \tilde{c} \) obtained by using the multigrid method, and the size values of \( \tilde{c} \) corresponding to the bending \( u_0 \) are calculated by taking the step length \( \delta = 0.5 \) in Table 5.
Table 6: With $\varepsilon = 10^{-4}$ precision, using the bisection method, obtained solutions $u_0$ and size values of $\tilde{c}$ corresponding to both Cases 1 and 2 ($s$ is the number of iterations).

| Cases | $s$ | $\gamma$ | $\tilde{c}$ | $u_0$ | $u_0 - u_h$ | $s$ | $\gamma$ | $\tilde{c}$ | $u_0$ | $u_0 - u_h$ |
|-------|-----|----------|-------------|-------|------------|-----|----------|-------------|-------|------------|
| 1     | 2   |          | 0.34592     | -0.12136 < 0 |       |       | 2   |          | 0.34592     | -0.12136 < 0 |       |       |
|       | 2.5 |          | 0.29672     | -0.07216 < 0 |       |       | 2.3 |          | 0.31587     | -0.09131 < 0 |       |       |
|       | 3   |          | 0.25203     | -0.02747 < 0 |       |       | 2.6 |          | 0.28742     | -0.06286 < 0 |       |       |
|       | 3.5 |          | 0.21192     | 0.01264 > 0  |       |       | 2.9 |          | 0.26060     | -0.03604 < 0 |       |       |
|       | 4   |          |            |            |       |       | 3   |          | 0.54886     | 3.33473     |       |       |
|       | 4.5 |          |            |            |       |       | 3.5 |          | 0.53826     | 3.33242     |       |       |
|       | 5   |          |            |            |       |       | 3.0 |          | 0.53023     | 3.33696     |       |       |
|       | 5.5 |          |            |            |       |       | 3.5 |          | 0.51606     | 3.33561     |       |       |
| 2     | 2   |          | 0.52485     | -0.18902 < 0 |       |       | 2   |          | 0.52486     | -0.18903 < 0 |       |       |
|       | 2.5 |          | 0.44800     | -0.11217 < 0 |       |       | 2.3 |          | 0.47790     | -0.14207 < 0 |       |       |
|       | 3   |          | 0.37827     | -0.04244 < 0 |       |       | 2.6 |          | 0.43347     | -0.09764 < 0 |       |       |
|       | 3.5 |          | 0.31582     | 0.02001 > 0  |       |       | 2.9 |          | 0.39164     | -0.05581 < 0 |       |       |
|       | 4   |          |            |            |       |       | 3   |          | 0.53042     | 3.33979     |       |       |
|       | 4.5 |          |            |            |       |       | 3.5 |          | 0.51247     | 3.33257     |       |       |
|       | 5   |          |            |            |       |       | 3.0 |          | 0.50246     | 3.33596     |       |       |
|       | 5.5 |          |            |            |       |       | 3.5 |          | 0.49259     | 3.33591     |       |       |

Table 7: Comparison of number of iterations and error in cases of using the multigrid and bisection methods with a given precision of $\varepsilon$.

| Cases | Bisection methods ($\varepsilon = 10^{-3}$) | Multigrid methods ($\varepsilon = 10^{-4}$) |
|-------|---------------------------------------------|---------------------------------------------|
|       | $\delta = 0.5$                             | $\delta = 0.3$                             |
| 1     | Number of steps 7                          | Number of steps 5                          | Number of steps 2 |
|       | Approximate error 0.32%                    | Approximate error 0.081%                    | Approximate error 0.092% |
| 2     | Number of steps 8                          | Number of steps 5                          | Number of steps 2 |
|       | Approximate error 0.15%                    | Approximate error 0.0006%                   | Approximate error 0.02% |

After determining the solution interval with a step length of $\delta = 0.5$ for the middle plate and $\varepsilon = 10^{-3}$ precision, using the multigrid method, the size value $\tilde{c}$ is calculated in one step with 0.18% error. Using the bisection method, with $\varepsilon = 10^{-3}$ precision, the size value $\tilde{c}$ corresponding to the same bending $u_0$ of the middle plate was calculated in 7 steps with 0.32% error as shown in Table 4. If we compare the values of Tables 3 and 5, it is clear that the multigrid method is faster than the bisection method.

As the Case 2 above, the size values of the middle plate corresponding to initial data $u_0 = 0.33583$ cm are given in Table 3, provided that the sizes of the two plates on the sides are equal, in the problem of bending of the system consisting of three plates whose Young’s moduli $E_1 = E_3 = E_{Fe}$ and $E_2 = E_{Cu}$ and Poisson ratios $\nu_1 = \nu_3 = \nu_{Fe}$ and $\nu_2 = \nu_{Cu}$ are as formed side by side.

Now, for both Cases 1 and 2, with $\varepsilon = 10^{-4}$ precision, the size value $\tilde{c}$ is obtained by using the multigrid method, and the size values of $\tilde{c}$ corresponding to the bending $u_0$ are calculated by taking the step lengths of $\delta = 0.5$ and $\delta = 0.3$ in Table 6. Using the bisection method, the obtained solutions $u_0$ and size values of $\tilde{c}$ corresponding to both Cases 1 and 2 are shown in Table 6.

As seen in Tables 5 and 6, with the step size $\delta = 0.5$ for the middle plate, when the bending values for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ precision are compared, it is seen that the solution is approached with less error when the precision $\varepsilon$ is reduced. Such that, using the bisection method, with $\varepsilon = 10^{-3}$ precision, the size value $\tilde{c}$ was calculated in 7 steps with 0.32% error, while using the multigrid method, even with the precision of $\varepsilon = 10^{-4}$, the size value $\tilde{c}$ was calculated in 5 steps with 0.081% error. After determining the solution interval as seen in Tables 4 and 6, with how many steps and what error the solution is found by using the multigrid and bisection methods for a given precision $\varepsilon$ is shown in Table 7.

Case 1. After determining the solution interval with the step lengths $\delta = 0.5$ and $\delta = 0.3$ for the middle plate, in Table 6, with $\varepsilon = 10^{-4}$ precision, using the multigrid method, the size values $\tilde{c}$ corresponding to bending $u_0 = 0.22456$ cm of the middle plate are calculated in 5 steps with 0.081% error and in 2 steps with 0.092% error, respectively.
Case 2. After determining the solution interval with the step lengths $\delta = 0.5$ and $\delta = 0.3$ for the middle plate, in Table 6, with $\epsilon = 10^{-4}$ precision, using the multigrid method, the size value $\varepsilon$ corresponding to bending $u_0 = 0.33583$ cm of the middle plate is calculated in 5 steps with 0.0006% error and in 2 steps with 0.02% error, respectively.

Comparing the values in Table 7, it may be seen that the interval in which the solutions are solved with smaller steps is narrowed more, that is, the smaller the $\delta$ step is, the faster the solution is approached for both Cases 1 and 2, when we compare the values that are obtained corresponding to $\delta = 0.5$ and $\delta = 0.3$.

4. Conclusions

In this study, a finite difference method was developed for the problem of plates with internal hinges and applied to determine the response of a system assembled from three different plates with different mechanical constraints between adjacent plates. In order to obtain the desired amount of bending against the force affecting the system under different boundary conditions, an algorithm was written. The bisection and multigrid methods were used to derive the sizes of the plates forming the system. The results obtained from these methods are compared. Our study is distinctive for this research domain. Because this study dealt with mathematics and many disciplines of engineering, this research may indeed prove to be significant for fellow researchers and scientists working in the same discipline.

Appendix

The finite difference expressions, which are used in the finite difference equations obtained corresponding to any $(x, y)$ point of the nonuniform mesh, are as follows:

Finite difference expressions of the first-order derivatives,

\[
\begin{align*}
    u_x &= \frac{u_{j+1} - u_{j-1}}{h_x}, & u_y &= \frac{u_{j+1} - u_{j-1}}{h_y}, \\
    u_{xx} &= \frac{1}{h_x^2} u_{xx} = \left( \frac{1}{h_x^2} + \frac{1}{h_x^2} \right) u_{xx} + \frac{1}{h_x^2} u_{xx-1}, \\
    u_{yy} &= \frac{1}{h_y^2} u_{yy} = \left( \frac{1}{h_y^2} + \frac{1}{h_y^2} \right) u_{yy} + \frac{1}{h_y^2} u_{yy-1},
\end{align*}
\]

where backward and forward differences are with respect to $x$ and $y$, respectively.

Finite difference expressions of the second-order derivatives,

\[
\begin{align*}
    u_{xx} &= \frac{1}{h_x^2} u_{xx} = \left( \frac{1}{h_x^2} + \frac{1}{h_x^2} + \frac{1}{h_x^2} \right) u_{xx} + \frac{1}{h_x^2} u_{xx-1}, \\
    u_{yy} &= \frac{1}{h_y^2} u_{yy} = \left( \frac{1}{h_y^2} + \frac{1}{h_y^2} + \frac{1}{h_y^2} \right) u_{yy} + \frac{1}{h_y^2} u_{yy-1},
\end{align*}
\]

Finite difference expressions of the second-order mixed derivatives,

\[
\begin{align*}
    u_{xy} &= \frac{u_{j+1} - u_{j-1}}{h_x} + \frac{1}{h_x} u_{xx} + \frac{1}{h_y} u_{yy}, \\
    u_{yy} &= \frac{u_{j+1} - u_{j-1}}{h_y} + \frac{1}{h_x} u_{xx} + \frac{1}{h_y} u_{yy}.
\end{align*}
\]

Finite difference expressions of the fourth-order derivatives,

\[
\begin{align*}
    u_{xxxx} &= \frac{1}{h_x^4} u_{xxxx} = \left( \frac{1}{h_x^4} + \frac{1}{h_x^4} + \frac{1}{h_x^4} \right) u_{xxxx} + \frac{1}{h_x^4} u_{xxxx-1}, \\
    u_{yyyy} &= \frac{1}{h_y^4} u_{yyyy} = \left( \frac{1}{h_y^4} + \frac{1}{h_y^4} + \frac{1}{h_y^4} \right) u_{yyyy} + \frac{1}{h_y^4} u_{yyyy-1}.
\end{align*}
\]
Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors’ Contributions

All authors read and approved the final manuscript.

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