Finite-Size Effects and Operator Product Expansions in a CFT for $d > 2$

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Abstract

The large momentum expansion for the inverse propagator of the auxiliary field $\lambda(x)$ in the conformally invariant $O(N)$ vector model is calculated to leading order in $1/N$, in a strip-like geometry with one finite dimension of length $L$ for $2 < d < 4$. Its leading terms are identified as contributions from $\lambda(x)$ itself and the energy momentum tensor, in agreement with a previous calculation based on conformal operator product expansions. It is found that a non-trivial cancellation takes place by virtue of the gap equation. The leading coefficient of the energy momentum tensor contribution is shown to be related to the free energy density.

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The effects of finite geometry in systems near second order phase transitions points are of great importance for statistical mechanics and quantum field theory. These effects are largely explained by the theory of finite size scaling [1]. On the other hand, from a purely field theoretical point of view, second order phase transitions are connected to conformal field theories (CFTs). In $d = 2$ spacetime dimensions, there is an almost complete understanding of CFTs in terms of operator product expansions (OPEs) [2]. Recently, there has been some progress towards understanding CFTs for $d > 2$ in terms of OPEs plus some additional algebraic conditions such as the cancellation of “shadow singularities” [3, 4, 5]. The CFT approach to finite size scaling at criticality is based on the observation that the OPE, being a short distance property of the critical theory, is insensitive to finite size effects. This means, for example, that the finite size corrections of two-point functions are directly related to specific terms in the OPE. Such an approach has been very successful when applied in $d = 2$ [6].

In the present work, the conformally invariant $O(N)$ vector model is investigated in $2 < d < 4$, in order to demonstrate that OPE techniques can be useful for studying finite size effects of critical systems in $d > 2$ as well. The geometry is taken to be strip-like with one finite dimension of length $L$ and periodic boundary conditions. In this case, for $2 < d \leq 3$ the critical gap equation has only one “massive” solution, while, for $3 < d < 4$ the gap equation has both a “massive” and a “massless” solution. We study the large momentum expansion for the inverse propagator of the auxiliary field $\lambda(x)$, which is equivalent to an OPE. The leading scalar and tensor contributions to this OPE for the “massive” and “massless” solutions of the critical gap equation above are identified. In both cases the results are in agreement with what would be expected from abstract CFT in $d > 2$. Most importantly, for the “massive” solution of the gap equation, a non-trivial cancellation is found to take place inside the inverse propagator of $\lambda(x)$. This is similar to the “shadow singularities” cancellation found in [4, 5]. Finally, it is shown that the leading coefficient of the energy momentum tensor $T_{\mu\nu}(x)$ contribution is related to the free energy density of the model and also to $C_T$, the latter being the overall universal scale in the two-point function of $T_{\mu\nu}(x)$.

The Euclidean partition function of the $O(N)$ vector model is given by

$$Z = \int (D\phi^a)(D\sigma) \exp \left[ -\frac{1}{2} \int d^d x [\phi^a(x)(-\partial^2 + \sigma(x))\phi^a(x)] + \frac{1}{2f} \int d^d x \sigma(x) \right],$$

where $\phi^a(x)$, $a = 1, \ldots, N$ is the basic $O(N)$-vector, $O(d)$-scalar field, $\sigma(x)$ is the $O(d)$-scalar auxiliary field and $f$ is the coupling. Integrating out $\phi^a(x)$, we obtain

$$Z = \int (D\sigma) \exp \left[ -\frac{N}{2} S_{eff}(\sigma, g) \right], \quad S_{eff}(\sigma, g) = \text{Tr} \ln(-\partial^2 + \sigma) - \frac{1}{g} \int d^d x \sigma(x),$$

for the rescaled coupling $g = Nf$. Setting $\sigma(x) = m^2 + (i/\sqrt{N})\lambda(x)$, where $m^2$ is the stationary value of the functional integral (2) given by the gap equation

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2} = \frac{1}{g},$$

(3)
and expanding (4) in powers of $\lambda(x)$, we get the usual large $N$ expansion. The momentum space inverse propagator of $\lambda(x)$ is

$$
\Pi^{-1}(p^2) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)((q + p)^2 + m^2)} .
$$

(4)

The partition function is then given by

$$
Z = e^{-\frac{N}{2} S_{\text{eff}}(m^2, g)} \int (D\lambda) e^{-\frac{1}{2} \int d^d x \int d^d y [\lambda(x)\delta(x-y)] + O(1/\sqrt{N})} ,
$$

(5)

where, $D(x)$ is the $x$-space Fourier transform of (4). The effective theory above describes the “propagation” and “interactions” of the composite field $\lambda(x)$. The critical theory, which is a non-trivial CFT for $2 < d < 4$, is obtained for $1/g \equiv 1/g_* = (2\pi)^{-d} \int d^d p / p^2$ and $m \equiv M_* = 0$.

When the system is put in a strip-like geometry having one finite dimension of length $L$ and periodic boundary conditions, the momentum along the finite dimension takes the discrete values $\omega_n = 2\pi n/L$, $n = 0, \pm 1, \pm 2, \ldots$, and the relevant integrals become infinite sums. For example, the gap equation and the inverse $\lambda(x)$ propagator become now

$$
\frac{1}{g} = \frac{1}{L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-1} p}{(2\pi)^{d-1}} \frac{1}{p^2 + \omega_n^2 + m^2} ,
$$

(6)

$$
\Pi^{-1}_L(p^2, \omega_n^2) = \frac{1}{2L} \sum_{m=-\infty}^{\infty} \int \frac{d^{d-1} q}{(2\pi)^{d-1}} \frac{1}{(q^2 + \omega_m^2 + m^2)((q + p)^2 + (\omega_m + \omega_n)^2 + m^2)} .
$$

(7)

Since renormalisation of the bulk theory is sufficient for the renormalisation of the theory in finite volume [1], the critical coupling in the latter case holds its bulk critical value $1/g_* = (2\pi)^{-d} \int d^d p / p^2$. The value of the critical mass parameter $M_*$ however, may be different from zero. Indeed, the (renormalised) gap equation for the finite system at criticality can be found from (6) to be

$$
0 = M_*^{d-2} \left[ \frac{\Gamma(\frac{1}{2}d - \frac{1}{2}) \Gamma(1 - \frac{1}{2}d)}{4\sqrt{\pi}} + I_n \right] ,
$$

(8)

where

$$
I_n = \int_1^{\infty} dt \frac{(t^2 - 1)^{\frac{1}{2}(d-3)+n}}{e^{L M_* t} - 1} .
$$

(9)

For $2 < d < 4$, (8) has a solution with $M_* \neq 0$. The dimensionless quantity $M_* L$ is plotted for this case as a function of $d$ in Fig.1. For $2 < d \leq 3$, the massless phase is saturated, since (8) diverges for $M_*$ zero, e.g. for $2 < d \leq 3$ only the $O(N)$-symmetric and critical theories exist. However, for $3 < d < 4$, (8) is also satisfied for $M_* = 0$ and the model retains the two-phase structure which has in the bulk. For $d = 3$, which is a special case as discussed below, the solution of (8) takes the value $M_* = (1/L) \ln \tau^2 \left[ \frac{1}{3} \right]$, where $\tau = (\sqrt{5} + 1)/2$. 

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Before attempting the evaluation of the inverse \( \lambda (x) \) propagator, we briefly discuss the CFT approach to the finite size scaling of the conformally invariant \( O(N) \) vector model in \( 2 < d < 4 \). This approach is based on the OPE structure of the model, studied in a number of works \([3, 4, 5]\). The OPE of the basic field \( \phi^{\alpha}(x) \) with itself takes the form

\[
\phi^{\alpha}(x)\phi^{\beta}(0) = \frac{C_{\phi}}{x^{2\eta}} + \frac{g_{\phi\phi}O}{C_{O}} \left( x^{-\frac{d}{2} + 1} \right) \delta^{\alpha\beta} + \cdots, \tag{10}
\]

where the dots stand for terms related to the energy momentum tensor \( T_{\mu\nu}(x) \), the \( O(N) \) conserved current \( J^{\alpha\beta}(x) \) and other fields with less singular coefficients as \( x \to 0 \). The field \( O(x) \) has dimension \( \eta_o = 2 + O(1/N) \), the coupling \( g_{\phi\phi}O \) is \( O(1/\sqrt{N}) \) and the dimension of \( \phi^{\alpha}(x) \) is \( \eta = d/2 - 1 + O(1/N) \). \( C_{\phi} \) and \( C_{O} \) are the normalisation constants of the two-point functions of \( \phi^{\alpha}(x) \) and \( O(x) \) respectively. Clearly, \( O(x) \) may readily be identified with \( \lambda (x) \) in \([3]\), we prefer however for the sake of generality to make this identification at a later stage. The OPE of \( O(x) \) with itself takes the form

\[
O(x)O(0) = \frac{C_{O}}{x^{2\eta_o}} + \frac{g_{O}O}{C_{O}} \left( x^{-\frac{d}{2} + 1} \right) \delta_{\alpha\beta} + \cdots, \tag{11}
\]

where \( C_{T} = N d/(d-1)S^2_d + O(1/N) \) is the normalisation of the two-point function of \( T_{\mu\nu}(x) \), with \( S_d = 2\pi^{d/2}/\Gamma(d/2) \) the area of the \( d \)-dimensional unit sphere. The dots stand for derivatives of \( O(x) \) and \( T_{\mu\nu}(x) \), as well as for other fields not relevant to the present work. The coefficient in front of the leading \( T_{\mu\nu}(x) \) contribution is exactly determined from conformal invariance and the Ward identities satisfied by the three-point function \( \langle T_{\mu\nu}O \rangle \) and we also know that \( g_{O}C_{\phi} = 2(d-3)g_{\phi\phi}O \) to leading \( N \) \([4]\).

Consider now the CFT having the OPEs above in the strip-like geometry. Taking the expectation value of \([11]\) yields the leading finite size corrections to the bulk-form \( 1/x^{2\eta_o} \) of the two-point function of \( O(x) \). These arise from the fact that the expectation values \( \langle O \rangle \) and \( \langle T_{\mu\nu} \rangle \) may in general be non-zero in a finite geometry. Given the explicit form for \( O(x) \) in terms of the fundamental fields \( \phi^{\alpha}(x) \), one may explicitly calculate \( \langle O \rangle \) in a \( 1/N \) expansion. Alternatively, the leading \( N \) value for \( \langle O \rangle \) can be also obtained by means of a consistency argument as we show below. On the other hand, Cardy \([10]\) showed that for a general conformal field theory the diagonal matrix elements \([7]\) of \( \langle T_{\mu\nu} \rangle \) are related to the finite size correction of its free energy density \( f_{\infty} - f_L \). His result reads

\[
\langle T_{11} \rangle = -(d-1)\langle T_{ii} \rangle = (d-1)\frac{2\zeta(d)}{S_dL^d}L = (d-1)(f_{\infty} - f_L) \quad \text{(no summation on } i \text{)}, \tag{12}
\]

\(^{3}\)Th off-diagonal matrix elements of \( \langle T_{\mu\nu} \rangle \) vanish from reflection symmetry.
where, \( \tilde{c} \) is a universal number which has been considered to be a candidate for a possible generalisation of Zamolodchikov’s C-function for \( d > 2 \) \[14\].

Upon transforming (14) to momentum space for \( \eta_\alpha = 2 \) and using (12), the leading finite size corrections for the \( O(x) \) propagator can be written as

\[
O_L(p^2, \omega_n^2) = \frac{C_\phi \pi^{\frac{d}{2}} 2^{d-1} \Gamma(\frac{d}{2} - 2) \Gamma(\frac{d}{2} - 1)}{(p^2 + \omega_n^2)^{\frac{3}{2}d-2}} \left( 1 + 4 \left( \frac{d}{2} - 2 \right) \frac{g_\phi}{C_\phi} \frac{1}{p^2 + \omega_n^2} \langle O \rangle + \cdots \right),
\]

where \( C_\phi^d(y) \) are the Gegenbauer polynomials and \( y = \omega_n / \sqrt{p^2 + \omega_n^2} \). For \( 2 < d < 4 \), the terms shown in (13) are the most singular ones in the large momentum expansion of \( O_L(p^2, \omega_n^2) \).

Similarly, taking the expectation value of (10) yields the leading finite size corrections to the two-point function of \( \phi^\alpha(x) \). Then, by transforming (10) to momentum space we obtain for the propagator \( P_L(p^2, \omega_n^2) \) of \( \phi^\alpha(x) \)

\[
P_L(p^2, \omega_n^2) = \frac{4\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} - 1)} \frac{1}{p^2 + \omega_n^2} \left( 1 + 4 \left( \frac{d}{2} - 2 \right) \frac{g_\phi \phi}{C_\phi C_\phi} \frac{1}{p^2 + \omega_n^2} \langle O \rangle + \cdots \right). \tag{14}
\]

If \( O(x) \) is to be identified now with the composite field \( \lambda(x) \) in (3), formulae (13) and (14) must be consistent with what is expected by explicitly calculating the propagators of \( \lambda(x) \) and \( \phi^\alpha(x) \), in the context of the standard \( 1/N \) expansion at the critical point of the \( O(N) \) vector model. Here, we will only be concerned with leading \( N \) calculations and we choose to work in momentum space as it is customary for finite size scaling studies \[8, 9\].

We start our consistency analysis from \( P_L(p^2, \omega_n^2) \) which, to leading \( N \), is easily found from (11) to be

\[
P_L(p^2, \omega_n^2) = \frac{1}{p^2 + \omega_n^2 + M_s^2} = \frac{1}{p^2 + \omega_n^2 \left( 1 - \frac{M_s^2}{p^2 + \omega_n^2} + \cdots \right)}. \tag{15}
\]

Note that, to this order there is no contribution from \( T_{\mu\nu} \) to the rhs of (15). Consistency of (15) with (11) requires that \( C_\phi = \Gamma(d/2 - 1)/4\pi^{d/2} \) and also

\[
\langle O \rangle = -\frac{M_s^2 C_\phi C_\phi}{4g_\phi \phi \langle \frac{1}{2}d - 2 \rangle} = -\frac{M_s^2 (d - 3) C_\phi^2}{2g_\phi \langle \frac{1}{2}d - 2 \rangle}. \tag{16}
\]

Next, we turn to the propagator \( \Pi_L^{-1}(p^2, \omega_n^2) \) (\( \equiv O_L^{-1}(p^2, \omega_n^2) \)) of \( \lambda(x) \), in (7). Using standard algebra we obtain

\[
O_L^{-1}(p^2, \omega_n^2) = \frac{2^{2-d} \Gamma(2 - \frac{1}{2}d) \pi^{\frac{1}{2}d}}{(2\pi)^d (p^2 + \omega_n^2)^{\frac{3}{2}d-2}} {}_2F_1 \left( 2 - \frac{1}{2}d, \frac{3}{2}; 2; \frac{p^2 + \omega_n^2}{p^2 + \omega_n^2 + 4M_s^2} \right).
\]
\[ + \int \frac{d^{d-1}q}{(2\pi)^{d-1} \sqrt{q^2 + M^2}} \frac{1}{\left[ e^{L\sqrt{q^2 + M^2}} - 1 \right]} \frac{p^2 + \omega_n^2 + 2q \cdot p}{(p^2 + \omega_n^2 + 2q \cdot p)^2 + 4\omega_n^2 (q^2 + M^2)} \]  

(17)

Introducing the quantities
\[ A_d = \frac{2^{2-d} \pi^{\frac{d}{2}+1} \Gamma(2 - \frac{1}{2}d) \Gamma(\frac{1}{2}d - 1)}{(2\pi)^d \Gamma(\frac{3}{2}d - \frac{1}{2})} \]  
and  
\[ v^2 = \frac{4M^2}{p^2 + \omega_n^2}, \]

(18)

the first term on the rhs of (17) can be written in a form more suitable for large momentum expansion as,
\[ A_d(p^2 + \omega_n^2)^{\frac{d}{2} - 2} \left[ (1 + v^2)^{\frac{d}{2} - \frac{3}{2}} + \Gamma(\frac{1}{2}d - \frac{1}{2}) \frac{(v^2)^{\frac{d}{2} - 1}}{\sqrt{\pi} \Gamma(d)} \left( \frac{1}{2}d - \frac{1}{2}, 1; \frac{1}{2}d; \frac{v^2}{1 + v^2} \right) \right]. \]

(19)

The second term on the rhs of (17) can be written as
\[ \frac{S_{d-1}}{(2\pi)^{d-1}} \int_0^\infty \frac{q^{d-2} dq}{(q^2 + M^2)^{\frac{d}{2} - \frac{1}{2}} \left[ e^{L\sqrt{q^2 + M^2}} - 1 \right]} \text{Re} \left[ I(p^2, \omega_n^2, q) \right], \]

(20)

where, we have defined
\[ \omega_n \sqrt{q^2 + M^2} = r \cos t, \quad q \cdot p = r \sin t, \quad r = \sqrt{(p^2 + \omega_n^2) q^2 + \omega_n^2 M^2} . \]

(22)

The angular integral in (20) can be evaluated in terms of Gegenbauer polynomials and, taking the real part, we obtain
\[ \text{Re} \left[ I(p^2, \omega_n^2, q) \right] = \sum_{k=0}^\infty \frac{(-4)^k \Gamma^2(\frac{1}{2}d - 1)2^{d-3}(2k)!}{\Gamma(d - 2 + 2k)} \frac{r^{2k}}{(p^2 + \omega_n^2)^{2k+1}} (\text{cos} t) C^{\frac{1}{2}d-1}_{2k} \]  

(23)

It is evident that (23) is a large momentum expansion. The leading terms of \( O^{-1}_L(p^2, \omega_n^2) \) can now be evaluated after substituting (23), (19) in (17). Apparently, the hypergeometric function in (13) contributes terms which do not seem to be present in (13). The second term of (13) has to be interpreted as a dimension \( d - 2 \) scalar field contribution. This field corresponds to the “shadow field” of \( O(x) \). Note that the dynamics of the \( O(N) \) vector model requires that “shadow singularities” cancel out from four-point functions [1]. In order to clarify the role of the “shadow singularities” inside the inverse two-point function (17), we must reexpress (23) in terms

4 The “shadow field” of a scalar field \( A(x) \) with dimension \( l \) is a scalar field with dimension \( d - l \). For the “shadow symmetry” of the conformal group in \( d > 2 \) see [13] and references therein.
of $C_{2k}^{d/2-1}(y)$. Given that $C_{
u}^{\lambda}(y)$ for integer $\nu$ are orthogonal polynomials of order $\nu$, an expansion of the form

$$(q^2 + y^2 M^2_\ast)^k C_{2k}^{\frac{d}{2}-1} \left( \frac{y (q^2 + M^2_\ast)}{\sqrt{q^2 + y^2 M^2_\ast}} \right) = \sum_{l=0}^{2k} B_l (q^2, M^2_\ast) C_{l}^{\frac{d}{2}-1}(y)$$

exists. The scalar contribution $B_0(q^2, M^2_\ast) \equiv B_0(M^2_\ast)$ can be easily evaluated and reads

$$B_0(M^2_\ast) = \frac{\Gamma\left(\frac{d}{2}\right) \Gamma(d - 2 + 2k) \Gamma(k + \frac{1}{2})}{\sqrt{\pi} \Gamma(d - 2) \Gamma(2k + 1) \Gamma(k + \frac{1}{2})} M^{2k}.$$  

Substituting $B_0$ in (17) we obtain, after some algebra,

$$-4 \frac{\Gamma\left(\frac{d}{2}\right) \Gamma(d - 2 + 2k) \Gamma(k + \frac{1}{2})}{\sqrt{\pi} \Gamma(d - 2) \Gamma(2k + 1) \Gamma(k + \frac{1}{2})} \cdot 2F_1\left(1, \frac{1}{2}d - \frac{1}{2}; \frac{1}{2}d; \frac{v^2}{1 + v^2}\right) I_0,$$

which cancels exactly the second term in (19) by virtue of the gap equation (8).

Thus, the gap equation (8), being a necessary and sufficient condition for this cancellation to occur, suggests an interesting alternative approach for studying this model. Namely, we could have started with (11) as the definition of the inverse two-point function of the field $\lambda(x)$. Requiring then that this is an inverse two-point function of a CFT obtainable from an OPE, and, postulating that no “shadow singularities” appear we are led, by means of the same calculations, to the gap equation (8). Consequently, (8), which is the basic dynamical equation of the $O(N)$ model, could be viewed as a precondition for the “shadow singularities” cancellation. Such an algebraic approach to CFT in $d > 2$ was initiated in (5) and it is now demonstrated to work for finite geometries as well.

Eventually, the first few terms in the large momentum expansion of (17) are

$$O_L^{-1}(p^2, \omega^2_n) = A_d (p^2 + \omega^2_n)^{\frac{1}{2}d-2} \left[ 1 + 2(d - 3) \frac{M^2_\ast}{p^2 + \omega^2_n} ight.$$

$$+ \frac{2^{2d} \Gamma\left(\frac{1}{2}d - \frac{1}{2}\right) \Gamma\left(2 - \frac{1}{2}d\right)}{\sqrt{\pi} \Gamma(d) \Gamma\left(2 - \frac{1}{2}d\right)} \frac{C_{\frac{d}{2}-1}^{\frac{d}{2}-1}(y)}{M^d_\ast} \left( \frac{1 - d}{d} I_0 - I_1 \right) + \cdots \right].$$

(27)

Now, (27) must be consistent with (13). It is then easy to see that the $O(M^2_\ast)$ terms on the rhs of (13) and (27) coincide by virtue of (14). This is a non-trivial consistency check, since it requires the non-trivial relation between the couplings $g_O$ and $g_{\phi \phi O}$. Next, consistency of the angular terms proportional to $C_{\frac{d}{2}}^{d/2-1}(y)$ in (13) and (27) requires that

$$\Gamma(d) \zeta(d) \frac{c}{L^d N} = M^d_\ast \left( \frac{d - 1}{d} I_0 + I_1 \right).$$

(28)
In order to prove that, we calculate the free energy density of the theory to leading order in $1/N$

$$f_{\infty} - f_L \equiv \frac{2\zeta(d)}{S_d L^d} \hat{c} = N M^*_s \frac{S_{d-1}}{(2\pi)^{d-1}} \left( \frac{1}{d} I_0 - \frac{1}{LM_*} \int_1^{\infty} x(x^2 - 1)^{\frac{1}{2}d - \frac{3}{2}} \ln(1 - e^{-LM_*x})dx \right). \quad (29)$$

Now, for $2 < d < 4$, a partial integration yields

$$I_1 = -\frac{d-1}{LM_*} \int_1^{\infty} x(x^2 - 1)^{\frac{1}{2}d - \frac{3}{2}} \ln(1 - e^{-LM_*x})dx. \quad (30)$$

Then, using the identity $S_d S_{d-1} = 2(2\pi)^{d-1}/\Gamma(d-1)$ one can see that (28) is satisfied by means of (29) and (30). To the best of our knowledge, this is the first time that the validity of Cardy’s result is explicitly demonstrated for CFTs in $d > 2$. For completeness, a plot of $\hat{c}/N$ as a function of $d$ for $2 < d < 4$ is depicted in Fig.2. The evaluation of $\hat{c}$ to $O(1/N)$, and its relation to possible generalisations of Zamolodchikov’s $C$-function for $d > 2$, are given in [14].

The case $d = 3$ is special. From (13) or (17), we see that there in no self-contribution from $O(x)$ to its two-point function, which is related to the fact that the bulk theory respects the reflection symmetry property $O(x) \to -O(x)$ to leading order in $1/N$. Also, for $d = 3$, $M_* = (1/L) \ln \tau^2$ and then it can be shown [14] that the integrals involved in (29) are related to polylogarithms at the special point $2 - \tau$, leading to the surprisingly simple result $\hat{c}/N = 4/5$ [8].

For $3 < d < 4$, the gap equation (8) has also the solution $M_* = 0$. In this case, the corresponding expression for $O^{\prime -1}(p^2, \omega_n^2)$ is given by

$$O^{\prime -1}(p^2, \omega_n^2) =
\begin{align*}
A_d(p^2 + \omega_n^2)^{\frac{1}{2}d-2} & \left( 1 + \frac{2^{2d-3}\Gamma(\frac{1}{2}d - \frac{1}{2})}{\sqrt{\pi} \Gamma(2 - \frac{1}{2}d)} \sum_{k=0}^{\infty} \frac{(-4)^k (2k)! \zeta(d-2+2k)}{[L^2(p^2 + \omega_n^2)]^{\frac{1}{2}d-1}} C_{2k}^{\frac{1}{2}d-1}(y) \right) \\
& = A_d(p^2 + \omega_n^2)^{\frac{1}{2}d-2} \left( 1 + \frac{2^{2d-3}\Gamma(\frac{1}{2}d - \frac{1}{2})}{\sqrt{\pi} \Gamma(2 - \frac{1}{2}d)} \zeta(d-2) \right) \left[ L^2(p^2 + \omega_n^2) \right]^{\frac{1}{2}d-1} \\
& \quad - \frac{2^{2d}\Gamma(\frac{1}{2}d - \frac{1}{2})}{\sqrt{\pi} \Gamma(2 - \frac{1}{2}d)} \zeta(d) \left[ L^2(p^2 + \omega_n^2) \right]^{\frac{1}{2}d-1} C_{2}^{\frac{1}{2}d-1}(y) + \cdots. \quad (31)
\end{align*}$$

Since now the propagator of $\phi^\prime(x)$ is simply $1/(p^2 + \omega_n^2)$, consistency with [12] requires $\langle O \rangle = 0$ to leading order in $1/N$ which means that we do not expect to get any contributions from $O(x)$ in $\Pi^{\prime -1}_L(p^2, \omega_n^2; M^2)$. Indeed, a term $\propto 1/(p^2 + \omega_n^2)^{d/2-1}$ is absent from the rhs of (31). Instead, we find a term $\propto 1/(p^2 + \omega_n^2)$ which has the correct dimensions to be interpreted as the leading contribution of the “shadow field” of $O(x)$. This term is not cancelled out and this is a characteristic feature of free field theories as discussed in [4].
Next, we require consistency of the coefficients of \( C_2^{d/2 - 1}(y) \) in (31) and (13) which yields, after some algebra, \( \tilde{c} = N \). This is exactly the result one obtains by calculating \( f_\infty - f_L \) to leading \( N \) at the critical point (with \( M_* = 0 \)), i.e. the result for free massless scalar fields. Note that (31) looks like a free field theory decomposition for the two-point function of \( \mathcal{O}(x) \), \( \mathcal{O}(x) \) which has in the place of \( \mathcal{O}(x) \) the dimension \( d - 2 \) composite scalar field :\( \phi^2(x) : \).

Our results above provide evidence that two-point functions of CFTs in \( d > 2 \) and in finite geometry are determined by bulk conformally invariant OPEs. In fact, the dynamical equations for the finite size theory are just the conditions for the cancellation of “shadow” singularities as shown in (19) and (26). Furthermore, we have explicitly shown that the leading angular correction to the scalar two-point function is proportional to \( \tilde{c}/C_T \). It would be interesting to look for non trivial cancellations inside the two point function of \( \phi^\alpha(x) \) where the calculations will be related to those in [8] and [13]. Extension of the OPE approach to fermionic, \( CP^{N-1} \) and supersymmetric CFTs in \( 2 < d < 4 \) would further clarify the connection between “shadow singularity” cancellations and dynamics. Another issue might be whether the conformal theory with \( M_* = 0 \) is a free field theory, which could be clarified by next-to-leading order in \( 1/N \) calculations.

This work was supported in part by PENED/95 K.A. 1795 research grant.

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Figure 1: The dimensionless quantity $M_*L$ as a function of the spacetime dimensionality $d$.

Figure 2: Plot of $\tilde{c}/N$ as a function of the spacetime dimensionality $d$. 