SCATTERING FOR THE DEFOCUSING, CUBIC NONLINEAR SCHRÖDINGER EQUATION WITH INITIAL DATA IN A CRITICAL SPACE

BENJAMIN DODSON

Abstract. In this note we prove scattering for a defocusing nonlinear Schrödinger equation with initial data lying in a critical Besov space. In addition, we obtain polynomial bounds on the scattering size as a function of the critical Besov norm.

1. Introduction

The qualitative long time behavior for the defocusing, nonlinear Schrödinger equation
\begin{equation}
  iu_t + \Delta u = |u|^{p-1}u, \quad u(0, x) = u_0, \tag{1.1}
\end{equation}
is completely worked out in the mass-critical ($p = \frac{4}{d} + 1$) and energy-critical ($p = \frac{4}{d-2} + 1$) cases. Indeed, (1.1) is known to be globally well-posed and scattering for any initial data $u_0 \in \dot{H}^{s_c}({\mathbb R}^d)$, \begin{equation}
  s_c = \frac{d}{2} - \frac{2}{p-1}, \tag{1.2}
\end{equation}in the mass-critical ([5], [6], [7], [8], [27], [17], [19]) and energy-critical cases ([1], [4], [24], [28], [18], [25]). The critical exponent (1.2) arises from the fact that if $u(t, x)$ solves (1.1), then for any $\lambda > 0$, \begin{equation}
  u(t, x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \tag{1.3}
\end{equation}also solves (1.1), and the $\dot{H}^{s_c}$ norm of the initial data is invariant under (1.3). On the other hand, well-posedness fails for $s < s_c$, see [2].

Remark 1. In this paper, global well-posedness refers to the existence of a global strong solution, i.e., a solution that satisfies Duhamel’s principle
\begin{equation}
  u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^{p-1} u(\tau) d\tau, \tag{1.4}
\end{equation}that is continuous in time, and depends continuously on the initial data. Scattering refers to the existence of $u_+, u_- \in \dot{H}^{s_c}({\mathbb R}^d)$ such that
\begin{equation}
  \lim_{t \to \infty} \|u(t) - e^{it\Delta} u_+\|_{\dot{H}^{s_c}({\mathbb R}^d)} = 0, \tag{1.5}
\end{equation}and
\begin{equation}
  \lim_{t \to -\infty} \|u(t) - e^{it\Delta} u_-\|_{\dot{H}^{s_c}({\mathbb R}^d)} = 0. \tag{1.6}
\end{equation}See Chapter three of [26] for a detailed treatment of global well-posedness and scattering for dispersive partial differential equations in general.
The case when $p = \frac{4}{d-1} + 1$, $(s_c = 0)$ is called mass-critical because a solution to (1.1) preserves the mass, or $L^2$ norm of a solution,
\begin{equation}
M(u(t)) = \int |u(t,x)|^2 dx = M(u(0)).
\end{equation}
Likewise, the case when $p = \frac{4}{d-2} + 1$, $(s_c = 1)$ is called energy critical because a solution to (1.1) preserves the energy,
\begin{equation}
E(u(t)) = \int \left[ \frac{1}{2} |\nabla u(t,x)|^2 + \frac{1}{p+1} |u(t,x)|^{p+1} \right] dx = E(u(0)).
\end{equation}

The conserved quantities (1.7) and (1.8) imply that the $\dot{H}^{s_c}(\mathbb{R}^d)$ norm is uniformly bounded for the entire time of existence of the solution to (1.1) in the mass-critical and energy-critical cases. Thus, in the mass-critical and energy-critical cases, the proof of global well-posedness and scattering reduces to proving global well-posedness and scattering for a solution to (1.1) with uniformly bounded $\dot{H}^{s_c}(\mathbb{R}^d)$, which has been done.

**Remark 2.** In the energy-critical case, the Sobolev embedding theorem implies that $E(u(0)) < \infty$ when $u_0 \in \dot{H}^1(\mathbb{R}^d)$.

It is conjectured that global well-posedness and scattering also hold for (1.1) when $0 < s_c < 1$. In this case, there is no known conserved quantity that gives uniform bounds on the $\dot{H}^{s_c}(\mathbb{R}^d)$ norm of a solution to (1.1). Therefore, there are two possible ways in which a solution to (1.1) might fail to scatter. These are called type one blowup and type two blowup. A solution to (1.1) is called a type one blowup solution if the solution fails to scatter but the $\dot{H}^{s_c}(\mathbb{R}^d)$ norm is uniformly bounded. Since $e^{it\Delta}$ is a unitary operator, an unbounded $\dot{H}^{s_c}(\mathbb{R}^d)$ norm automatically precludes (1.5) or (1.6) from occurring. A blowup solution to (1.1) is called a type two blowup solution if the solution fails to scatter but the $\dot{H}^{s_c}(\mathbb{R}^d)$ norm is uniformly bounded for the entire time of its existence.

Type one blowup is known to occur for solutions to (1.1) when $0 < s_c < 1$, see [20]. Interestingly, the solutions obtained in [20] have good regularity and good decay. By comparison, when $0 < s_c < 1$, if $u_0 \in H^1_x(\mathbb{R}^d)$, where $H^1_x$ is an inhomogeneous Sobolev space, then (1.7), (1.8), and interpolation imply a uniform bound on the $\dot{H}^{s_c}(\mathbb{R}^d)$ norm when $0 < s_c < 1$. In fact, global well-posedness and scattering is known for a solution to (1.1) when $u_0 \in H^1_x(\mathbb{R}^d)$ and $0 < s_c < 1$, see [14] and [15].

Type two blowup has been precluded in many cases for (1.1) when $0 < s_c < 1$. One particularly important case is the cubic nonlinear Schrödinger equation in three dimensions (see [12]),
\begin{equation}
iu_t + \Delta u = |u|^2 u, \quad u : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}, \quad u(0,x) = u_0.
\end{equation}

In this case, $s_c = \frac{3}{2} - \frac{2}{3} = \frac{1}{2}$. The main obstacle to proving scattering for (1.9) with generic initial data in $\dot{H}^{1/2}(\mathbb{R}^3)$ is the absence of a conservation law that controls the $\dot{H}^{1/2}$ norm of a solution to (1.9) with initial data in $\dot{H}^{1/2}$. Observe that the momentum
\begin{equation}
P(u(t)) = \int \text{Im} \left[ \bar{u}(t,x) \nabla u(t,x) \right] dx,
\end{equation}
is conserved and scales like the $\dot{H}^{1/2}$ norm, but does not control the $\dot{H}^{1/2}$ norm of a solution to (1.1).

In a breakthrough result, [15] proved that any solution to (1.9) with $\|u(t)\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$ uniformly bounded on its entire interval of existence must be globally well-posed and scattering. Indeed, [15] and [4] were key in developing the concentration compactness method for the nonlinear Schrödinger
The pair \((p, r)\) is radially symmetric and in the critical Besov space \(B^{\frac{4}{p} + s_c}_{1, 1}(\mathbb{R}^d)\). See [26] and the references therein for a detailed treatment of this topic.

Type two blowup was later precluded for a great many cases of (1.1) when \(0 < s_c < 1\), see [23, 22, and 21]. Since the mass-critical and energy-critical problems reduce to type two blowup questions, the same techniques are useful for both problems.

In this paper we prove scattering for (1.1) when \(0 < s_c < 1\), \(1 < p \leq 3\), and the initial data is radially symmetric and in the critical Besov space \(B^{\frac{4}{p} + s_c}_{1, 1}(\mathbb{R}^d)\).

**Theorem 1.** The initial value problem (1.1) is globally well-posed and scattering for radially symmetric initial data in the Besov space \(B^{\frac{4}{p} + s_c}_{1, 1}(\mathbb{R}^d)\). In addition, when \(1 < p < 3\), the scattering size,

\[
\|u\|_{L^\infty_t L^{\frac{6}{5}}(\mathbb{R}^d)}
\]

is bounded by a polynomial function of \(\|u_0\|_{B^{\frac{4}{p} + s_c}_{1, 1}}\).

The Besov space \(B^s_{p,q}(\mathbb{R}^d)\) is given by the norm

\[
\|u_0\|_{B^s_{p,q}(\mathbb{R}^d)} = (\sum_j 2^{jsp} \|P_j u_0\|_{L^p}^p)^{1/p},
\]

when \(1 \leq p < \infty\), with the usual modification when \(p = \infty\). Here, \(P_j\) is the usual Littlewood–Paley projection operator. The Sobolev embedding theorem implies that \(B^{\frac{4}{p} + s_c}_{1, 1}(\mathbb{R}^d) \subset \dot{H}^{s_c}(\mathbb{R}^d)\). The \(B^{\frac{4}{p} + s_c}_{1, 1}(\mathbb{R}^d)\) norm is invariant under the scaling symmetry (1.3).

For the Schrödinger equation in dimensions \(d \geq 3\),

\[
i u_t + \Delta u = F, \quad u(0, x) = u_0, \quad u : I \times \mathbb{R}^d \to \mathbb{C},
\]

we have the Strichartz estimate

\[
\|u\|_{L^2_t L^{\frac{6}{5}}(I \times \mathbb{R}^d)} \lesssim \|u_0\|_{H^s} + \|F\|_{L^1_t L^{\frac{6}{5}}(I \times \mathbb{R}^d)}.
\]

See [26] and the references therein for a detailed treatment of this topic.

In particular, when \(F = 0\), (1.1) implies a bound on \(\|u\|_{L^p_t L^q}\), when \((p, q)\) is an admissible pair, i.e.

\[
\frac{2}{p} = d\left(\frac{1}{2} - \frac{1}{q}\right), \quad p \geq 2.
\]

Then by the Sobolev embedding theorem, if \(F = 0\),

\[
\|u\|_{L^p_t L^q} \lesssim \|u_0\|_{H^s}, \quad \text{for} \quad \frac{1}{r} = \frac{1}{q} - \frac{s}{d}, \quad (p, q) \quad \text{is admissible}.
\]

The pair \((p, r)\) is then said to be s-admissible. Doing some algebra, \((\frac{r+1}{s+d}, p + 1)\) is \(s_c\)-admissible. Since \(\frac{r+1}{s+d} < \infty\), a bound on (1.11) on \(\mathbb{R} \times \mathbb{R}^d\) implies scattering for (1.1). Again, see [26] for a detailed exposition of the method to prove this fact. It is useful to use the Strichartz space,

\[
\|u\|_{\dot{H}^s(I \times \mathbb{R}^d)} = \|\nabla u\|_{L^2_t L^{\frac{6}{5}}(I \times \mathbb{R}^d)}
\]

Studying nonlinear wave (9, 11) and Schrödinger equations (10) with initial data in a critical Besov space has proved to be a very fruitful endeavor. Consider (1.9), for example. Observe that the interaction Morawetz estimates in [3] and conservation of energy imply

\[
\|u\|_{L^2_t L^4(\mathbb{R}^3)} \lesssim (1 + \|u_0\|_{H^{1/2}}^3)\|u_0\|_{H^1}^3\|u_0\|_{L^2}^3.
\]
This provides good bounds on the left hand side of (1.18) for \( u_0 = e^{-\frac{|x|^2}{2}} \), or for any rescaled version under (1.3), i.e., \( u_0 = \lambda e^{-\frac{\lambda^2|x|^2}{2}} \). Plugging
\[
(1.19) \quad u_0 = c_1 e^{-\frac{|x|^2}{2}} + c_2 \lambda e^{-\frac{\lambda^2|x|^2}{2}},
\]
into (1.18) gives a bound on the right hand side that is a function of \( \lambda, c_1, \) and \( c_2 \). However, Theorem 1 implies that the \( L^6_t L^4_x \) bound depends only on \( c_1 \) and \( c_2 \). Moreover, for (1.1) with \( 1 < p < 3 \), such bounds are polynomially dependent on \( c_1 \) and \( c_2 \). We could also replace (1.19) by a convergent series of \( c_j \)'s.

2. Scattering for the cubic NLS in three dimensions

We begin by proving scattering for the cubic equation (1.3) with \( u_0 \in B^2_{1,1}(\mathbb{R}^3) \), before moving on to the general problem. In this section, we do not prove any uniform bounds on the scattering size as a function of the \( B^2_{1,1} \) norm of the initial data.

**Theorem 2.** The initial value problem
\[
(2.1) \quad iu_t + \Delta u = |u|^2 u, \quad u(0, x) = u_0,
\]
with radially symmetric initial data \( u_0 \in B^2_{1,1}(\mathbb{R}^3) \) has a global solution that scatters.

**Proof.** By time reversal symmetry, it suffices to prove scattering on \([0, \infty)\). In [10], we proved that the cubic nonlinear Schrödinger equation is globally well-posed for initial data \( u_0 \in W^{\frac{1}{2}, \frac{1}{2}}(\mathbb{R}^3) \). By the Sobolev embedding theorem, \( B^2_{1,1}(\mathbb{R}^3) \subset W^{\frac{1}{2}, \frac{1}{2}}(\mathbb{R}^3) \), so global well-posedness follows.

Furthermore, after rescaling the initial data, suppose that the global solution has the form
\[
(2.2) \quad \|u\|_{L^6_t L^4_x([0, 1]\times \mathbb{R}^3)} \leq \delta, \quad \text{which implies} \quad \|u\|_{\dot{S}^{1/2}([0, 1]\times \mathbb{R}^3)} < \infty.
\]

Then for \( 1 \leq t < \infty \), decompose
\[
(2.3) \quad u(t) = w(t) + v(t), \quad \text{where} \quad w(t) = e^{it\Delta} u_0^{(1)},
\]
and \( u_0 = u_0^{(1)} + u_0^{(2)} \) is some decomposition of \( u_0 \) that will be specified later.

Let \( E(t) \) denote the conformal energy of \( v \),
\[
(2.4) \quad E(t) = \|x + 2it\nabla v\|_{L^2}^2 + 2t^2 \|v\|_{L^4}^4 = \|xv\|_{L^2}^2 + 2\langle xv, 2it\nabla v \rangle_{L^2} + 8t^2 E(t),
\]
where \( E(t) \) is the energy in (1.8),
\[
(2.5) \quad E(t) = \frac{1}{2} \|\nabla v\|_{L^2}^2 + \frac{1}{4} \|v\|_{L^4}^4.
\]

When \( w = 0 \),
\[
(2.6) \quad \frac{d}{dt} E(t) = -2t \|v\|_{L^4_x}^4,
\]
which implies \( \|v\|_{L^4_t L^4_x([1, \infty)\times \mathbb{R}^3)} < \infty \). Interpolating this bound with the bound \( \|v\|_{L^\infty_t L^4_x} < \infty \), which a uniform bound on \( E(t) \) implies for any \( 1 \leq t \leq \infty \), gives
\[
(2.7) \quad \|u\|_{L^4_t L^4_x([1, \infty)\times \mathbb{R}^3)} = \|v\|_{L^4_t L^4_x([1, \infty)\times \mathbb{R}^3)} < \infty.
\]

For a general \( u_0^{(1)} \in \dot{H}^{1/2}(\mathbb{R}^3) \), Strichartz estimates imply that
\[
(2.8) \quad \|w\|_{L^4_t L^4_x(\mathbb{R}\times \mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)},
\]
so to prove scattering, it suffices to prove

\[ \int_1^\infty \frac{1}{t^4} \mathcal{E}(t)^2 dt < \infty. \]

Indeed,

\[ \int_1^\infty \|v\|_{L^4}^4 dt \lesssim \int_1^\infty \frac{1}{t^4} \mathcal{E}(t)^2 dt. \]

By Duhamel's principle,

\[ v(1) = -i \int_0^1 e^{i(1-\tau)\Delta} |u|^2 u d\tau + e^{it\Delta} u_0(2). \]

By direct computation,

\[ (x + 2i \nabla) \int_0^1 e^{i(1-\tau)\Delta} |u|^2 u d\tau = \int_0^1 (x + 2i(1-\tau)\nabla) e^{i(1-\tau)\Delta} |u|^2 u d\tau + \int_0^1 2i\tau \nabla e^{i(1-\tau)\Delta} |u|^2 u d\tau. \]

By examining the kernel,

\[ e^{it\Delta} f = C_{3/2} \int e^{-\frac{|x-y|^2}{at^2}} f(y) dy, \quad (x + 2i \nabla) e^{it\Delta} f = e^{it\Delta} x f, \]

so using the radial Sobolev embedding theorem and the computations in [10],

\[ \| \int_0^1 (x + 2i(1-\tau)\nabla) e^{i(1-\tau)\Delta} |u|^2 u d\tau \|_{L^2} \lesssim \|x|u|^2 u\|_{L^1 L^2} \lesssim \|xu\|_{L^\infty} \|u\|_{L^2}^2 \|u\|_{L^8}^2 \lesssim \|u\|_{B^0_{1,1}}^3. \]

Next, recall from [10] that for any $0 < t < 1$,

\[ u = u_1 + u_2, \quad \text{where} \quad \|\nabla u_1\|_{L^2} \lesssim t^{-1/4}, \quad \|\nabla u_2\|_{L^6} \lesssim t^{-3/4}, \]

with constant independent of $t$. Therefore, by Strichartz estimates,

\[ \| \int_0^1 2i\tau \nabla e^{i(1-\tau)\Delta} |u|^2 u d\tau \|_{L^2} \lesssim \|\tau \nabla u_1\|_{L^\infty L^2} \|u\|_{L^2}^2 \|u\|_{L^6}^2 + \|\tau \nabla u_2\|_{L^\infty L^2} \|u\|_{L^2}^2 \|u\|_{L^6}^2 \lesssim \|u_0\|_{B^0_{1,1}}^3. \]

Now decompose the initial data. Let $\chi \in C_0^\infty(\mathbb{R}^3)$, $\chi(x) = 1$ on $|x| \leq 1$, $\chi(x)$ is supported on $|x| \leq 2$, and let $R(\epsilon, u_0) < \infty$ be a constant sufficiently large so that

\[ \sum_j 2^{j/2} \| (1 - \chi(\frac{x}{R})) P_j u_0 \|_{L^2} \leq \epsilon, \quad \text{and} \quad \sum_j 2^{2j} \| (1 - \chi(\frac{x}{R})) P_j u_0 \|_{L^1} \leq \epsilon. \]

By Hölder's inequality and the Sobolev embedding theorem,

\[ \| \nabla ((1 - \chi(\frac{x}{R})) P_j u_0) \|_{L^2} \lesssim \| P_j u_0 \|_{H^1}, \quad \text{and} \quad \| \nabla^2 ((1 - \chi(\frac{x}{R})) P_j u_0) \|_{L^1} \lesssim 2^{2j} \| P_j u_0 \|_{L^1}. \]

Then,

\[ \sum_j \| (1 - \chi(\frac{x}{R})) P_j u_0 \|_{L^1}^{1/2} \| (1 - \chi(\frac{x}{R})) P_j u_0 \|_{H^1}^{1/2} \lesssim \epsilon^{1/2} \| u_0 \|_{B^0_{1,1}}^{1/2}. \]

Therefore, by the radial Sobolev embedding theorem, if $\epsilon \leq \| u_0 \|_{B^0_{1,1}}^{-2}$,

\[ \| |x| e^{it\Delta} (1 - \chi(\frac{x}{R})) u_0 \|_{L^\infty} \lesssim \epsilon^{1/4}. \]
Also, by Hölder’s inequality,  
\[ \|x\chi(\frac{x}{R})u_0\|_{L^2} \lesssim R^{3/2}\|u_0\|_{L^4}, \]
so (2.13) implies  
\[ \|(x + 2i\nabla)v(1)\|_{L^2} \lesssim R^{3/2}\|u_0\|_{B_{1,1}^2}. \]
The computations in [1] also imply  
\[ \|v(1)\|_{L^4}^4 \lesssim 1, \]
and therefore,  
\[ \mathcal{E}(1) \lesssim \|u_0\|_{B_{1,1}^2}, R = 1. \]

To obtain the bound (2.20), observe that \(v\) solves  
\[ iv_t + \Delta v = |u|^2 u, \quad v(1, x) = (2.11), \]
and \(w\) solves  
\[ iw_t + \Delta w = 0, \quad w(1, x) = e^{i\Delta u_0(1)}, \]
on \([1, \infty)\).

Rearranging (2.25),  
\[ -\Delta v + |v|^2 v = iv_t - F, \quad F = 2|v|^2 w + v^2 \bar{w} + 2|w|^2 \bar{v} + |w|^2 w = F_1 + F_2 + F_3. \]
Integrating by parts,  
\[ \frac{d}{dt} \mathcal{E}(t) = 16tE(v) + 8t^3\langle v_t, -\Delta v + |v|^2 v \rangle + 4\langle xv_t, i\nabla v \rangle + 4t\langle xv_t, i\nabla v_t \rangle \]
\[ + 2\langle ix\Delta v, xv \rangle - 2\langle ixF, xv \rangle = -2t\|v\|_{L^4}^4 + 8t^2\langle v_1, F \rangle - 4t\langle xF, \nabla v \rangle + 4t\langle xv, \nabla F \rangle - 2\langle ixF, xv \rangle. \]
Integrating by parts and plugging in (2.27), with \(F_3 = |w|^2 w, \)
\[ 8t^2\langle v_1, F_3 \rangle - 4t\langle xF_3, \nabla v \rangle + 4t\langle xv, \nabla F_3 \rangle - 2\langle ixF_3, xv \rangle \]
\[ = 2\langle (x + 2it\nabla)|w|^2 w, i(x + 2it\nabla)v \rangle_{L^2} + O(t^2\|v\|^3 + |w|^3, |w|^3) \]
\[ \lesssim \|(x + 2it\nabla)v\|_{L^2}\|xw\|_{L^\infty}\|w\|_{L^4}^2 + \|(x + 2it\nabla)v\|_{L^2}\|\nabla w\|_{L^\infty}\|w\|_{L^4}^2 + t^2\|v\|_{L^4}^4 + t^2\|w\|_{L^6}^6 \]
\[ \lesssim \mathcal{E}(t)^{1/2}\|w\|_{L^4}^2 + t^{1/2}\mathcal{E}(t)^{3/4}\|w\|_{L^\infty}^2 + t^{3/2}\|w\|_{L^\infty}^2. \]
Also, integrating by parts and plugging in (2.27) with \(F_2 = 2|w|^2 v + w^2 \bar{v}, \)
\[ 8t^2\langle v_1, F_2 \rangle - 4t\langle xF_2, \nabla v \rangle + 4t\langle xv, \nabla F_2 \rangle - 2\langle ixF_2, xv \rangle \]
\[ = 2\langle (x + 2it\nabla)|v|^3 + |w|^3, |w|^3, (x + 2it\nabla)v \rangle_{L^2} \]
\[ \lesssim \|(x + 2it\nabla)v\|_{L^2}\|xw\|_{L^\infty}\|w\|_{L^4}\|v\|_{L^4} + \|(x + 2it\nabla)v\|_{L^2}\|\nabla w\|_{L^\infty}\|w\|_{L^4}\|v\|_{L^4} \]
\[ + \|(x + 2it\nabla)v\|_{L^2}\|w\|_{L^6}^6 + t^2\|v\|_{L^4}^4 + t^2\|w\|_{L^6}^6 \]
\[ \lesssim t^{-1/2}\mathcal{E}(t)^{3/4}\|w\|_{L^4}^2 + t^{3/2}\|w\|_{L^\infty}^2 + \mathcal{E}(t)\|w\|_{L^\infty}^2. \]

Finally, take  
\[ 8t^2\langle v_1, F_1 \rangle - 4t\langle xF_1, \nabla v \rangle + 4t\langle xv, \nabla F_1 \rangle - 2\langle ixF_1, xv \rangle, \]
with $F_1 = 2|v|^2 w + v^2 w$. This term will be handled slightly differently from (2.20) and (2.30). By (2.32),
\begin{equation}
-4t \langle x F_1, \nabla v \rangle - 2 \langle ix F_1, xv \rangle = -2 \langle ix F_1, (x + 2it \nabla)v \rangle \lesssim \|(x + 2it \nabla)v\|_{L^2} \cdot \|v\|_{L^4} \|xv\|_{L^\infty} \lesssim \frac{1}{t} \mathcal{E}(t).
\end{equation}
Next, integrating by parts,
\begin{equation}
4t \langle xv, \nabla F_1 \rangle = -4t \langle x \nabla v, F_1 \rangle - 12t \langle v, F_1 \rangle = -4t \langle xv, \nabla (|v|^2 v) \rangle - 12t \langle v, F_1 \rangle \lesssim t \|w\|_{L^1} \|v\|_{L^4}^3 + 4t \langle \nabla w, x |v|^2 v \rangle \lesssim \frac{1}{t} \mathcal{E}(t)^{3/4} + 4t \langle \nabla w, x |v|^2 v \rangle.
\end{equation}
Then by the product rule, integrating by parts, and (2.13),
\begin{equation}
4t \langle \nabla w, x |v|^2 v \rangle = 8t \langle \nabla w, |v|^2 (x + 2it \nabla)v \rangle - 4t \langle \nabla w, |v|^2 (x - 2it \nabla)v \rangle - 8t \langle \nabla w, i \Delta w, |v|^2 v \rangle \lesssim \|\nabla w\|_{L^\infty} \mathcal{E}(t) + t \|\Delta w\|_{L^\infty} \|v\|_{L^2} \mathcal{E}(t)^{1/2} \lesssim \|\nabla w\|_{L^\infty} \mathcal{E}(t) + t^{-1/2} \|v\|_{L^2} \mathcal{E}(t)^{1/2}.
\end{equation}
Meanwhile, integrating by parts in $t$,
\begin{equation}
\int_{1}^{T} 8t^2 \langle v_t, F_1 \rangle dt = 8t^2 \langle |v|^3, |w| \rangle \bigg|_{1}^{T} - \int_{1}^{T} 8t^2 \langle |v|^2 v, w_t \rangle - \int_{1}^{T} 16t \langle |v|^2 v, w \rangle dt.
\end{equation}
First observe that
\begin{equation}
8t^2 \langle |v|^3, |w| \rangle \bigg|_{1}^{T} \lesssim t^{1/2} \|w\|_{L^4} \mathcal{E}(t)^{3/4} \bigg|_{1}^{T}.
\end{equation}
Also compute
\begin{equation}
8t^2 \langle |v|^2 v, w_t \rangle \lesssim t \|\Delta w\|_{L^\infty} \|v\|_{L^4} \mathcal{E}(t)^{1/2} \lesssim t^{-1/2} \|v\|_{L^2} \mathcal{E}(t)^{1/2}, \quad \text{and} \quad 16t \langle |v|^2 v, w \rangle \lesssim t^{-1/2} \mathcal{E}(t)^{3/4} \|w\|_{L^4}.
\end{equation}
Therefore,
\begin{equation}
\mathcal{E}(t) \lesssim \int_{1}^{t} \mathcal{E}(s) \frac{1}{t^2} \|w\|_{L^4}^2 + s^{3/8} \mathcal{E}(s)^{3/4} \|w\|_{L^\infty}^2 + s^{3/2} \|w\|_{L^\infty}^4 + s^{1/4} \mathcal{E}(s)^{1/2} \|w\|_{L^\infty} + \mathcal{E}(s) \|w\|_{L^4}^2 + \frac{\epsilon}{s} \mathcal{E}(s) + \|\nabla w\|_{L^\infty} \mathcal{E}(s) + s^{-1/2} \mathcal{E}(s)^{1/2} \|v\|_{L^2} + s^{-1/2} \mathcal{E}(s)^{1/2} \|w\|_{L^4} ds + t^{1/2} \|w\|_{L^4} \mathcal{E}(t)^{3/4} + R.
\end{equation}
By Fubini’s theorem and Hölder’s inequality,
\begin{equation}
\int_{1}^{\infty} \frac{1}{t^4} \bigg( \int_{1}^{t} \mathcal{E}(s) \frac{1}{t^2} \|w\|_{L^4}^2 ds \bigg) dt \lesssim \int_{1}^{\infty} \frac{1}{t^3} \bigg( \int_{1}^{t} \mathcal{E}(s) \frac{1}{t^2} ds \bigg) dt = \int_{1}^{\infty} \mathcal{E}(s) \|w\|_{L^4}^2 ds \int_{1}^{\infty} \frac{1}{t^3} ds \lesssim \int_{1}^{\infty} \frac{1}{s^2} \mathcal{E}(s) \|w\|_{L^4}^2 ds \lesssim \bigg( \int_{1}^{\infty} \mathcal{E}(s)^{2} ds \bigg)^{1/2} \bigg( \int_{1}^{\infty} \|w\|_{L^4}^2 ds \bigg)^{1/2}.
\end{equation}
Next, interpolating (2.14) and (2.13),
\begin{equation}
\|\nabla e^{i\Delta}(1 - \chi(\frac{x}{R}))P_j u_0\|_{L^\infty} \lesssim \inf \{ t^{-3/2} \langle 2j, 1 \rangle \|1 - \chi(\frac{x}{R})\|_{L^4}^{1/2} \|\nabla^2(1 - \chi(\frac{x}{R}))P_j u_0\|_{L^4}^{1/2}, 2j \|\nabla^2(1 - \chi(\frac{x}{R}))P_j u_0\|_{L^4}, \}
\end{equation}
which by (2.44) implies that for $\epsilon \leq \|u_0\|_{B^{8}_{1,1}}^{8}$,

$$\int_0^\infty \|\nabla w\|_{L^\infty} dt \lesssim \epsilon^{1/4} \|u_0\|_{B^{8}_{1,1}}^{3/4} \lesssim \epsilon^{5/32}.$$  

(2.41)

Similar computations also show that

$$\|w\|_{L_t^2 L_x^\infty} \lesssim \epsilon^{3/8}, \quad \text{and} \quad \|w\|_{L^\infty} \lesssim \frac{\epsilon^{3/8}}{s^{1/2}}.$$  

(2.42)

Therefore,

$$\int_1^\infty \frac{1}{t^4} E(t)^2 dt \lesssim \int_1^\infty \frac{R^2}{t^4} dt + \int_1^\infty \frac{1}{t^4} E(t)^{3/2} \|w\|_{L_t^4 L_x^4}^2 dt + \int_1^\infty \frac{1}{t^4} \left( \int_1^t E(s)^{1/2} \|w\|_{L_t^4 L_x^4} ds \right)^2 dt$$

$$+ \int_1^\infty \frac{1}{t^4} \left( \int_1^t s^{3/8} E(s)^{3/4} \|w\|_{L_t^8 L_x^4}^2 + s^{3/2} \|w\|_{L_t^8 L_x^8}^2 + s^{1/4} E(s)^{1/2} \|w\|_{L_t^8 L_x^2}^2 \right) dt$$

$$\lesssim R^2 + \left( \int_1^\infty \frac{1}{t^4} E(t)^2 dt \right)^{1/4} \left( \int_1^\infty \|w\|_{L_t^8 L_x^4}^4 dt \right)^{1/4} + \left( \int_1^\infty \frac{1}{s^4} E(s)^2 ds \right)^{1/2} \left( \int_1^\infty \|v\|_{L_t^4 L_x^4}^2 \right)^{1/2}$$

$$+ \epsilon^{5/16} \left( \int_1^\infty \frac{1}{s^4} E(s)^2 ds \right) + \left( \int_1^\infty \frac{1}{s^4} E(s)^2 ds \right)^{1/2} \left( \int_1^\infty \frac{1}{s^2} \|v(s)\|_{L_t^4 L_x^4}^2 \right)^{1/2}.$$  

(2.43)

Therefore,

$$\int_1^\infty \frac{1}{t^4} E(t)^2 dt \lesssim R^2 + \int_1^\infty \|w\|_{L_t^8 L_x^4}^8 dt + \int_1^\infty \|w\|_{L_t^8 L_x^4}^2 dt + \left( \int_1^\infty \frac{1}{t^4} E(t)^2 dt \right)^{1/2} \left( \int_1^\infty \frac{1}{t^2} \|v(t)\|_{L_t^4 L_x^4}^2 dt \right)^{1/2}.$$  

(2.44)

Now since $v$ solves (2.25),

$$\frac{d}{dt} \|v\|_{L_x^2}^2 \lesssim \|w\|_{L_t^\infty} \|v\|_{L_t^4} \|v\|_{L_x^2} + \|w\|_{L_t^\infty} \|w\|_{L_t^8 L_x^4} \|v\|_{L_x^2}, \quad \|v(1)\|_{L_x^2}^2 \lesssim R.$$  

(2.45)

Therefore, by Hölder’s inequality,

$$\|v(t)\|_{L_x^2}^2 \lesssim R^2 + \left( \int_1^t \|w\|_{L_t^\infty} \|v\|_{L_t^4} \|v\|_{L_x^2} + \|w\|_{L_t^\infty} \|w\|_{L_t^8 L_x^4} \|v\|_{L_x^2} dt \right)^2$$

$$\lesssim R^2 + \|w\|_{L_t^2 L_x^\infty}^2 \left( \int_1^t \|v\|_{L_t^4} \|v\|_{L_x^2}^2 + \|w\|_{L_t^8 L_x^4}^2 \right).$$  

(2.46)

Therefore, by Fubini’s theorem,

$$\int_1^\infty \frac{1}{t^2} \|v(t)\|_{L_x^2}^2 dt \lesssim R^2 + \epsilon^2 \int_1^\infty \|v\|_{L_t^8 L_x^4}^2 \left( \int_1^t \frac{1}{t^2} \|v(t)\|_{L_x^2}^2 dt \right)^{1/2} + \epsilon^2 \|w\|_{L_t^8 L_x^4} \left( \int_1^\infty \frac{1}{t^2} \|v(t)\|_{L_x^2}^2 dt \right)^{1/2}$$

$$\lesssim R^2 + \epsilon^2 \int_1^\infty \frac{1}{t^4} E(t)^2 dt \left( \int_1^\infty \frac{1}{t^2} \|v(t)\|_{L_x^2}^4 dt \right)^{1/2} + \epsilon^2 \|w\|_{L_t^8 L_x^4} \left( \int_1^\infty \frac{1}{t^2} \|v(t)\|_{L_x^2}^4 dt \right)^{1/2}.$$  

(2.47)
Therefore,

\[
\int_1^\infty \frac{1}{t^2} \|v(t)\|_{L^2}^4 \, dt \lesssim R^2 + \varepsilon^2 \left( \int_1^\infty \frac{1}{t^2} \mathcal{E}(t)^2 \, dt \right) + \int_1^\infty \|w\|_{L^8}^8 \, dt.
\]

Plugging (2.48) into (2.44),

\[
\int_1^\infty \|v(t)\|_{L^4}^8 \, dt \lesssim \int_1^\infty \frac{1}{t^4} \mathcal{E}(t)^2 \, dt \lesssim R^2 + \int_1^\infty \|w\|_{L^8}^8 \, dt + \int_1^\infty \|w\|_{L^\infty}^2 \, dt.
\]

Therefore, scattering follows. \(\square\)

3. CONCENTRATION COMPACTNESS IN THE CUBIC CASE

Recall from Strichartz estimates that for a solution to (1.9),

\[
\|u\|_{L^8_t L^4_x(\mathbb{R} \times \mathbb{R}^3)} < \infty,
\]

is equivalent to

\[
\|u\|_{L^5_t L^\infty_x(\mathbb{R} \times \mathbb{R}^3)} < \infty,
\]

so Theorem 2 implies that for \(u_0 \in B_{1,1}^2\), (1.1) has a global solution satisfying \(\|u\|_{L^8_{t,x}(\mathbb{R} \times \mathbb{R}^3)} < \infty\).

However, since \(R\) depends on \(\varepsilon > 0\) and \(u_0\), not just the norm \(\|u_0\|_{B_{1,1}^2}\), (2.49) does not directly give a uniform bound on

\[
\|u\|_{L^8_{t,x}(\mathbb{R} \times \mathbb{R}^3)}, \quad \text{when} \quad \|u_0\|_{B_{1,1}^2} \leq A < \infty.
\]

Such a bound follows from a concentration compactness argument, as in [9] for the nonlinear wave equation.

Following by now standard concentration compactness techniques, see for example [10],

Lemma 1. Let \(u_n\) be a bounded sequence in \(\dot{H}^{1/2}\),

\[
\sup_n \|u_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \leq A < \infty,
\]

that is radially symmetric. After passing to a subsequence, assume that

\[
\lim_{n \to \infty} \|u_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} = A.
\]

Then passing to a further subsequence, for any \(1 \leq J < \infty\), there exist \(\phi^1, ..., \phi^J \in \dot{H}^{1/2}\) such that

\[
u_n = \sum_{j=1}^J e^{it\Delta} \phi_j(\frac{x}{\lambda_n}) + w_n^J,
\]

where

\[
\sum_{j=1}^J \|\phi_j\|_{\dot{H}^{1/2}}^2 + \lim_{n \to \infty} \|w_n^J\|_{\dot{H}^{1/2}}^2 = A^2,
\]

\[
\lim_{J \to \infty} \limsup_{n \to \infty} e^{it\Delta} \|w_n^J\|_{L^8_{t,x}(\mathbb{R} \times \mathbb{R}^3)} = 0,
\]

and for \(j \neq k\),

\[
\lim_{n \to \infty} \left| \ln(\frac{\lambda_j}{\lambda_k}) \right| + |t_j^k - t_j^k| = \infty.
\]
Now let \( u_n \) be a sequence in \( B^2_{1,1}(\mathbb{R}^3) \) with the uniform bound
\[
\|u_n\|_{B^2_{1,1}} \leq A.
\]
Then by the Sobolev embedding theorem,
\[
\|u_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \lesssim A,
\]
so apply Lemma 1 and observe that for any \( J \),
\[
(3.11) \quad u_n = \sum_{j=1}^{J} e^{it_n^j \Delta} \left( \frac{x}{\lambda_n} \phi^j \right) + w_n^J.
\]
Next, observe that Lemma 1 implies that for any fixed \( j \),
\[
(3.12) \quad e^{-it_n^j \Delta} (\lambda_n^j u_n(\lambda_n^j \cdot)) \to \phi^j, \quad \text{weakly in } \dot{H}^{1/2}(\mathbb{R}^3).
\]
Using dispersive estimates, for any \( t \in \mathbb{R} \), since \( B^2_{1,1} \) is invariant under the scaling symmetry \( \|u\|_{B^2_{1,1}} \leq 1 \),
\[
(3.13) \quad \|e^{it \Delta} e^{-it_n^j \Delta} (\lambda_n^j u_n(\lambda_n^j \cdot))\|_{L^\infty} \lesssim \frac{1}{|t - t_n^j|^1/2} \|u_n\|_{B^2_{1,1}},
\]
in particular, if \( t_n^j \to \pm \infty \) along a subsequence, interpolating (3.13) and the Sobolev embedding theorem \( \dot{H}^{1/2} \to L^5 \),
\[
(3.14) \quad \|e^{it \Delta} e^{-it_n^j \Delta} (\lambda_n^j u_n(\lambda_n^j \cdot))\|_{L^5_t(R \times \mathbb{R}^3)} = 0,
\]
for any fixed \( 0 < T < \infty \). Since \( u_n \to \phi \) weakly in \( \dot{H}^{1/2} \) implies
\[
(3.15) \quad e^{it \Delta} u_n \to e^{it \Delta} \phi, \quad \text{weakly in } L^5_t,R \times \mathbb{R}^3,
\]
(3.14) implies that \( \phi^j = 0 \) if \( t_n^j \to \pm \infty \) along a subsequence.

**Remark 3.** The fact that weak convergence implies (3.15) follows from Strichartz estimates and approximating a function in \( L^5_t,\mathbb{R}^3 \) with a smooth, compactly supported function and a small remainder.

Therefore, the \( t_n^j \)'s must be uniformly bounded for any \( j \), and after passing to a subsequence, \( t_n^j \to \nu \in \mathbb{R} \) for any \( j \). Since
\[
(3.16) \quad e^{it_n \Delta} (\lambda_n^j \cdot) \frac{1}{\lambda_n^j} \phi^j (\frac{x}{\lambda_n^j}) = \frac{1}{\lambda_n^j} (e^{it \Delta} \phi^j)(\frac{x}{\lambda_n^j}),
\]
replacing \( \phi^j \) with \( e^{it \Delta} \phi^j \) and absorbing the remainder into \( w_n^j \), it is possible to set \( t_n^j \equiv 0 \) for all \( j \) in (3.11). Therefore,
\[
(3.17) \quad u_n = \sum_{j=1}^{J} \frac{1}{\lambda_n^j} \phi^j (\frac{x}{\lambda_n^j}) + w_n^J.
\]
By Theorem 2 for any \( j \), let \( u^j \) be the solution to (2.1) with initial data \( \phi^j \). Then for any \( j \),
\[
(3.18) \quad \|u^j\|_{L^5_t(R \times \mathbb{R}^3)} < \infty.
\]
Furthermore, (3.9), (3.7), (3.8), and small data arguments imply that if \( u^{(n)}(t, x) \) is the solution to (2.1) with initial data \( u_n(x) \),
\[
(3.19) \quad \lim_{n \to \infty} \|u^{(n)}\|_{L^5_t,R \times \mathbb{R}^3}^5 \leq \sum_{j=1}^{\infty} \|u^j\|_{L^5_t,R \times \mathbb{R}^3}^5 < \infty.
\]
For all but finitely many $j$’s, say all but $j_0$, $\|u_i\|_{L^\infty_t H^{1/2}} \leq \epsilon$, so by small data arguments and \eqref{3.6},
\begin{equation}
\sum_{j \geq j_0} \|u_j\|^2_{L^2_{t,x}(\mathbb{R} \times \mathbb{R}^3)} \lesssim A.
\end{equation}

Therefore, there exists a function $f : [0, \infty) \to [0, \infty)$ such that if $\|u_0\|_{B^s_{1,1}} \leq A$ is radial, then \eqref{2.1} has a global solution that satisfies the bound
\begin{equation}
\|u\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{R}^3)} \leq f(A) < \infty.
\end{equation}

Observe that \eqref{3.21} gives no explicit bound on the scattering size. In general, the bounds obtained from a concentration compactness argument are likely far from optimal. For example, in \cite{25},
\begin{equation}
\|u\|_{L^\frac{2(d+2)}{d+1}((\mathbb{R} \times \mathbb{R}^d)}} \leq C \exp(C E^C),
\end{equation}
where $C(d)$ is a large constant, $E$ is the energy \eqref{1.3}, and $u$ is a solution to the energy-critical problem ($s_c = 1$) with radially symmetric initial data.

4. A LOCAL RESULT FOR \eqref{1.1} WHEN $1 < p < 3$

In the second part of the paper, we will prove explicit bounds on the scattering size of a solution to \eqref{1.1} with radially symmetric initial data in $B^{4+s_c}_{1,1}$, when $0 < s_c < 1$ and $1 < p < 3$. Note that the restrictions on $s_c$ and $p$ require $d \geq 3$.

As in the cubic case, the first step is to rescale and obtain good bounds on the interval $[0,1]$. The space $L^{\frac{2(d+2)[p-1]}{(d+2)(p-1)}}(\mathbb{R} \times \mathbb{R}^d)$ is also invariant under the rescaling \eqref{1.3}, so rescale the initial data so that
\begin{equation}
\|u\|_{L^{\frac{2(d+2)(p-1)}{(d+2)(p-1)}}([0,1] \times \mathbb{R}^d)} \leq \delta,
\end{equation}
for some $\delta \ll 1$.

**Lemma 2.** If $u$ is a solution to \eqref{1.1} on $[0,1]$ with initial data $u_0 \in B^{4+s_c}_{1,1}$, and $u$ satisfies \eqref{4.1}, then for any $j \in \mathbb{Z}_{>0}$,
\begin{equation}
\|\nabla u\|_{L^2_t L^{\frac{2d}{d+2}}([2^j,2^{j+1}] \times \mathbb{R}^d)} \lesssim 2^{j-d}\|u_0\|_{B^{4+s_c}_{1,1}(\mathbb{R}^d)}.
\end{equation}

**Proof.** The local solution may be obtained by showing that the operator
\begin{equation}
\Phi(u(t)) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^{p-1} u(\tau) d\tau,
\end{equation}
has a unique fixed point in $S^{s_c}(\{0,1\} \times \mathbb{R}^d)$.

Interpolating the Sobolev embedding theorem,
\begin{equation}
\|P_k e^{it\Delta} u_0\|_{L^\infty} \lesssim 2^{k \frac{d}{p+1}} 2^{k(d-\frac{d}{p+1})} \|P_k u_0\|_{L^1},
\end{equation}
with the dispersive estimate,
\begin{equation}
\|P_k e^{it\Delta} u_0\|_{L^\infty} \lesssim t^{-d/2} 2^{k(d-\frac{d}{p+1})} 2^{k(d-\frac{d}{p+1})} \|P_k u_0\|_{L^1},
\end{equation}
where $P_k$ is the usual Littlewood–Paley projection operator for any $k \in \mathbb{Z}$,
\begin{equation}
\|e^{it\Delta} u_0\|_{L^\infty} \lesssim t^{-\frac{d}{p+1}} \|u_0\|_{B^{4+s_c}_{1,1}}.
\end{equation}
and
\[ \| \nabla e^{it\Delta} u_0 \|_{L^\infty} \lesssim t^{-\frac{1}{p-1}} \| u_0 \|_{B^s_{2,1}}. \]

Interpolating (4.3) with the Sobolev embedding theorem,
\[ \| \nabla P_k e^{it\Delta} u_0 \|_{L^2} \lesssim 2^{k(1-s_c)} 2^{k(\frac{4}{p}+s_c)} \| P_k u_0 \|_{L^1}, \]
and
\[ \nabla P_k e^{it\Delta} u_0 \|_{L^2} \lesssim 2^{-k s_c} \frac{1}{t} 2^{k(\frac{4}{p}+s_c)} \| P_k u_0 \|_{L^1}. \]

Interpolating this bound with
\[ \| \nabla P_k e^{it\Delta} u_0 \|_{L^2} \lesssim 2^{k(2-s_c)} 2^{k(\frac{4}{p}+s_c)} \| P_k u_0 \|_{L^1}, \]
we obtain
\[ \| \nabla e^{it\Delta} u_0 \|_{L^\frac{2p}{p-1}} \lesssim t^{-\frac{1}{2} - \frac{1}{2p}} \| u_0 \|_{B^s_{2,1}}. \]

Therefore, for any \( j \in \mathbb{Z}_{\leq 0}, \)
\[ \| \nabla e^{it\Delta} u_0 \|_{L^\frac{2p}{p-1}(2j, 2^{j+1} \times \mathbb{R}^d)} \lesssim 2^{j \frac{1}{2p}} \| u_0 \|_{B^s_{2,1}}. \]

By Strichartz estimates, for any \( t \in [2^j, 2^{j+1}], \) let \( j_\delta \) be the integer closest to \( \log_2(\delta 2^j). \) By Strichartz estimates and the chain rule, and by (4.11),
\[ \| \nabla \int \mathcal{E}(t) e^{i(t-\tau)\Delta} |u(\tau)|^{p-1} u(\tau) d\tau \|_{L^\frac{2p}{p-1}(2^j, 2^{j+1} \times \mathbb{R}^d)} \lesssim \delta^2 \| \nabla u \|_{L^\frac{2p}{p-1}(2^j, 2^{j+1} \times \mathbb{R}^d)}. \]

Meanwhile, the dispersive estimate combined with the Sobolev embedding theorem
\[ \| u \|_{L^p L^\infty(2^j, 2^{j+1} \times \mathbb{R}^d)} \lesssim \lesssim \| u_0 \|_{B^s_{2,1}}. \]
\[ \| \int_0^{2^j} e^{i(t-\tau)\Delta} |u(\tau)|^{p-1} u(\tau) d\tau \|_{L^\frac{2p}{p-1}(0, 1 \times \mathbb{R}^d)} \lesssim \| u_0 \|_{B^s_{2,1}} + \epsilon \sup_{j < 0} 2^{j \frac{1}{2p}} \| \nabla u \|_{L^\frac{2p}{p-1}(2^j, 2^{j+1} \times \mathbb{R}^d)}. \]

Therefore, for \( \delta(\| u_0 \|_{B^s_{2,1}}) > 0 \) sufficiently small,
\[ \sup_{j < 0} 2^{j \frac{1}{2p}} \| \nabla \Phi(u) \|_{L^\frac{2p}{p-1}(2^j, 2^{j+1} \times \mathbb{R}^d)} \lesssim \| u_0 \|_{B^s_{2,1}} + \epsilon \sup_{j < 0} 2^{j \frac{1}{2p}} \| \nabla u \|_{L^\frac{2p}{p-1}(2^j, 2^{j+1} \times \mathbb{R}^d)}, \]
for some \( \epsilon > 0. \) Thus, (4.2) holds. \( \Box \)

Now suppose (1.1) with initial data \( u_0 \) has a solution on the maximal interval \([0, T] \), where \( 1 < T \leq \infty. \) Again decompose \( u = v + w \), where \( v \) and \( w \) solve
\[ iv_t + \Delta v = |v + w|^{p-1}(v + w), \quad v(0) = 0, \]
and
\[ iw_t + \Delta w = 0, \quad w(0) = u_0, \]
on \([0, \infty). \) Let \( E(t) \) denote the pseudoconformal energy of \( v, \)
\[ E(t) = \| (x + 2it\nabla)v \|_{L^2}^2 + \frac{2}{p+1} t^2 \| v \|_{L^{p+1}}^{p+1} = \| xv \|_{L^2}^2 + 2 \langle xv, 2it\nabla v \rangle + 8t^2 E(t). \]
Lemma 3. If \( u_0 \in B_{1,1}^{s_c} \) is radially symmetric, and (1.1) has a local solution satisfying (4.1), then \( \mathcal{E}(1) \lesssim 1 \) for \( \delta(\|u_0\|_{B_{1,1}^{s_c}}) > 0 \) sufficiently small.

Proof. Observe that the proof of Lemma 2 also implies
\[
\|\nabla v(1)\|_{L^2 + L^\infty/2} \lesssim \epsilon \sup_{j < 0} 2^j \|v\|_{L^2 L^\infty/2((2^j, 2^{j+1}) \times \mathbb{R}^d)}.
\]
Interpolating (4.19) with the bound
\[
\|v(1)\|_{H^{s_c}} \lesssim \|u_0\|_{B_{1,1}^{s_c + r}},
\]
implies \( \|v(1)\|_{L^{p+1}} \lesssim 1 \) for \( \delta > 0 \) sufficiently small, since \( \epsilon = \epsilon(\|u_0\|_{B_{1,1}^{s_c + r}}, p, d, \delta) \).

Using the computations in (2.12),
\[
\|(x+2i1\nabla) \int_0^1 e^{i(1-\tau)A} |u|^{p-1} u d\tau\|_{L^2} \lesssim \|x|u|^{p-1} u\|_{L^1 L^2([0,1] \times \mathbb{R}^d)} + \|\nabla u\|_{L^2 L^\infty/2([0,1] \times \mathbb{R}^d)} \|u\|_{L^2 L^\infty/2((2, 2^2) \times \mathbb{R}^d)}.
\]
Then by Lemma 2,
\[
\|\nabla u\|_{L^2 L^\infty/2([0,1] \times \mathbb{R}^d)} \lesssim \|u\|_{L^2 L^\infty/2([0,1] \times \mathbb{R}^d)} \lesssim \delta^{-1}, \sup_{j < 0} 2^j \|v\|_{L^2 L^\infty/2((2^j, 2^{j+1}) \times \mathbb{R}^d)}.
\]
To handle the first term in (4.21), consider the cases \( \frac{1}{2} \leq s_c < 1 \) and \( 0 < s_c < \frac{1}{2} \) separately. When \( \frac{1}{2} \leq s_c < 1 \), the radial Sobolev embedding theorem implies
\[
\|x|u|^{p-1} u\|_{L^1 L^2} \lesssim \|x^{s_c} u\|_{L^p L^q} \|u\|_{L^p L^q} \lesssim \|u_0\|_{B_{1,1}^{s_c + r}} \delta^c,
\]
where \( c > 0 \) as \( s_c > \frac{1}{2} \).

When \( 0 < s_c < \frac{1}{2} \), using the radial Strichartz estimates,
\[
\|x|u|^{p-1} u\|_{L^1 L^2} \lesssim \|x^{s_c} u\|_{L^2 L^2} \|u\|_{L^2 L^\infty} \|u\|_{L^2 L^\infty} \lesssim \|u_0\|_{B_{1,1}^{s_c + r}} \delta^c,
\]
where \( c > 0 \) for all \( 0 < s_c < \frac{1}{2} \) and \( d \geq 3 \), with appropriate \( p \).
This proves the Lemma.

\[ \square \]

5. Scattering for (1.1) When \( 1 < p < 3 \) and \( 0 < s_c < 1 \)

Having obtained good bounds on the interval \([0,1]\), we can use the pseudoconformal conservation of energy to extend these bounds to \([1, \infty)\).

Theorem 3. The initial value problem
\[
iu_t + \Delta u = |u|^{p-1} u, \quad u(0, x) = u_0 \in B_{1,1}^{s_c} (\mathbb{R}^d), \quad u : \mathbb{R}^d \to \mathbb{C},
\]
is globally well-posed and scattering when \( u_0 \) is radially symmetric. Moreover,
\[
\|u\|_{L^{1+r} L^{p+1}(\mathbb{R}^d)} \leq C(1 + \|u_0\|_{B_{1,1}^{s_c}}^r),
\]
for some \( C \) that does not depend on \( \|u_0\|_{B_{1,1}^{s_c}} \) and \( r < \infty \).
Remark 4. When \( \|u_0\|_{B^{\frac{4}{2}+s_c}_{1,1}} \) is small,  
\[
\|u\|_{L_t^{\frac{p+1}{p-1}} L_x^{\frac{p+1}{p+1}}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|u_0\|_{H^{s_c}} \lesssim \|u_0\|_{B^{\frac{4}{2}+s_c}_{1,1}}.
\]

So it suffices to consider \( \|u_0\|_{B^{\frac{4}{2}+s_c}_{1,1}} \gtrsim 1 \).

Proof. If \( v \) solves \((4.16)\) on \([1, \infty)\) with \( w = 0, 0 < s_c < 1 \), and \( \mathcal{E}(1) < \infty \), where \( \mathcal{E}(t) \) is given by \((4.13)\), then by direct computation,  
\[
\frac{d}{dt} \mathcal{E}(t) = -\frac{4}{p+1} t \|v\|^{p+1}_{L^{p+1}} < 0,
\]
which implies scattering.

Now compute \( \frac{d}{dt} \mathcal{E}(t) \) when \( w \) need not be zero, but \( w \) solves \((4.17)\) with \( u_0 \in B^{\frac{4}{2}+s_c}_{1,1} \), radially symmetric. Then by direct computation,  
\[
\frac{d}{dt} \mathcal{E}(t) = -\frac{4}{p+1} t \|v\|^{p+1}_{L^{p+1}} - 2((x + 2t \nabla)v, i(x + 2t \nabla)(|v + w|^{p-1}(v + w) - |v|^{p-1}v))
\]
\[
- 8t^2 (|v|^{p-1}v, i(|v + w|^{p-1}(v + w) - |v|^{p-1}v)).
\]

When \( \frac{1}{2} \leq s_c < 1 \), by the radial Sobolev embedding theorem, since \( \frac{p-1}{p-1} < 1 \),  
\[
-2((x + 2t \nabla)v, i(x + 2t \nabla)(|v + w|^{p-1}(v + w) - |v|^{p-1}v)) \lesssim \|(x + 2t \nabla)v\|_{L^2} \|w\|_{L^{\infty}} \|w\|_{L^{p+1}} \|v\|^{p-1}_{L^{p+1}} (\|v\|^{p-1}_{L^{p+1}} + \|w\|^{p-1}_{L^{p+1}})
\]
\[
\lesssim \mathcal{E}(t)^{1/2} \|u_0\|_{B^{\frac{4}{2}+s_c}_{1,1}} \|w\|_{L^{p+1}} \|v\|^{p-1}_{L^{p+1}} (\|v\|^{p-1}_{L^{p+1}} + \|w\|^{p-1}_{L^{p+1}}).
\]

When \( 0 < s_c < \frac{1}{2} \), split  
\[
xw = (x + 2t \nabla)w - 2it \nabla w.
\]

Again by \((2.13)\) and the radial Sobolev embedding theorem, for \( s_c < \frac{d}{2} - 1 \), \( \|y u_0\|_{H^{s_c+1}} \lesssim \|u_0\|_{H^{s_c}} \), so interpolating the Strichartz estimate,  
\[
\|e^{it\Delta} u_0\|_{L_t^{\frac{p+1}{p-1}} L_x^{\frac{p+1}{p+1}}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|u_0\|_{H^{s_c}},
\]
with the Littlewood–Paley projection estimate  
\[
\|P_{\gamma} e^{it\Delta} u_0\|_{L_{t,x}^q} \lesssim \|P_{\gamma} u_0\|_{H^{d/2}},
\]
implies that  
\[
\|(x + 2t \nabla)w\|_{L_{t,x}^{\frac{2(p+1)}{d-p+1}}} \lesssim \|u_0\|_{B^{\frac{4}{2}+s_c}_{1,1}}.
\]

Finally, since \( \frac{2}{p-1} < \frac{d}{2} \), by the Sobolev embedding theorem, for \( \frac{1}{q} = s_c \), by dispersive estimates and \( s_c = \frac{d}{2} - \frac{2}{p-1} \),  
\[
\|(\nabla |^{\frac{2}{p}} e^{it\Delta} u_0\|_{L^q_x} \lesssim \|e^{it\Delta} u_0\|_{B_{q,\infty}^{\frac{4}{2}}} \lesssim \frac{1}{t^{\frac{d}{2}}} \|u_0\|_{B_{1,\frac{q}{2}}^{\frac{4}{2}}} \lesssim \frac{1}{t^{\frac{d}{2}}} \|u_0\|_{B^{\frac{4}{2}+s_c}_{1,1}}.
\]

Therefore,  
\[
\|t^{\frac{2}{p-1}} |\nabla|^{\frac{2}{p}} w\|_{L^q_x} \lesssim \|u_0\|_{B^{\frac{4}{2}+s_c}_{1,1}}.
\]
Therefore, we have proved

\begin{equation}
\|xw\|_{L^\infty_t \frac{2(p+1)}{p} \frac{1}{L_{p+1}^2}}^{1} \lesssim \|u_0\|_{B^{s+1}_{\infty,1}^\infty}.
\end{equation}

Next, integrating by parts,

\begin{equation}
-2\langle(2it\nabla)v, i(2it\nabla)\rangle (|v+w|^{p-1}(v+w) - |v|^{p-1}v) = -8t^2 \langle \nabla v, i\nabla (|v+w|^{p-1}(v+w) - |v|^{p-1}v) \rangle = 8t^2 \langle \Delta v, i(|v+w|^{p-1}(v+w) - |v|^{p-1}v) \rangle.
\end{equation}

Summing, by \(4.10\),

\begin{equation}
8t^2 \langle \Delta v, i(|v+w|^{p-1}(v+w) - |v|^{p-1}v) \rangle - 8t^2 \langle |v|^{p-1}v, i(|v+w|^{p-1}(v+w) - |v|^{p-1}v) \rangle = 8t^2 \langle -iv_t, i(|v+w|^{p-1}(v+w) - |v|^{p-1}v) \rangle
\end{equation}

Finally, integrating by parts,

\begin{equation}
-2\langle xv, i(2it\nabla)\rangle (|v+w|^{p-1}(v+w) - |v|^{p-1}v) = 4t \langle xv, \nabla (|v+w|^{p-1}(v+w) - |v|^{p-1}v) \rangle
\end{equation}

Integrating by parts,

\begin{equation}
-4t \langle x \cdot \nabla v, (|v+w|^{p-1}(v+w) - |v|^{p-1}v) \rangle = \frac{4dt}{p+1} (\|v+w\|_{L^{p+1}}^{p+1} - \|v\|_{L^{p+1}}^{p+1}) + 4t \langle x \cdot \nabla w, |v+w|^{p-1}(v+w) \rangle.
\end{equation}

Now then, summing,

\begin{equation}
4t \langle x \cdot \nabla w, |v+w|^{p-1}(v+w) \rangle = 4t \langle (x + 2it\nabla) \cdot \nabla w, |v+w|^{p-1}(v+w) \rangle - 8t^2 \langle i\Delta w, |v+w|^{p-1}(v+w) \rangle.
\end{equation}

Summing (5.15) and (5.18), since \(w_t = i\Delta w\),

\begin{equation}
(5.15) + (5.18) = 4t \langle (x + 2it\nabla) \cdot \nabla w, |v+w|^{p-1}(v+w) \rangle - 8t^2 \langle v_t + w_t, |v+w|^{p-1}(v+w) \rangle + 8t^2 \langle v_t, |v|^{p-1}v \rangle.
\end{equation}

By the radial Sobolev embedding theorem, (2.13), and the fact that \(\frac{d}{2} > 1\),

\begin{equation}
\| (x + 2it\nabla) \cdot \nabla w \|_{L^2_x L_t^{p+1}} \lesssim \|u_0\|_{B^{s+1}_{\infty,1}^\infty}.
\end{equation}

Therefore, plugging these computations back into (5.8),

\begin{equation}
\frac{d}{dt} \mathcal{E}(t) \lesssim \frac{4}{p+1} t \|v\|_{L^{p+1}}^{p+1} - \frac{8t^2}{p+1} \frac{d}{dt} \|v\|_{L^{p+1}}^{p+1} + \frac{8t^2}{p+1} \frac{d}{dt} \|v\|_{L^{p+1}}^{p+1} + \mathcal{E}(t)^{1/2} \langle (x + 2it\nabla)w, \frac{2}{L^{p+1}} (\|v\|_{L^{p+1}}^{p-1} + \|w\|_{L^{p+1}}^{p-1}) \rangle + \mathcal{E}(t)^{1/2} \|u_0\|_{B^{s+1}_{\infty,1}^\infty} \langle \|v\|_{L^{p+1}}^{p-1} + \|w\|_{L^{p+1}}^{p-1} \rangle.
\end{equation}
Then by the product rule, (5.22)
\[
\frac{d}{dt} [\mathcal{E}(t)] + \frac{8t^2}{p+1} \|v + w\|_{L^{p+1}}^{p+1} - \frac{8t^2}{p+1} \|v\|_{L^{p+1}}^{p+1} \\
\lesssim -\frac{4}{p+1} t \|v\|_{L^{p+1}}^{p+1} + t \|(x + 2it \nabla) \cdot \nabla w\|_{L^{p+1}} \|v + w\|_{L^{p+1}}^p + t \|v\|_{L^{p+1}} \|w\|_{L^{p+1}} + t \|w\|_{L^{p+1}}^{p+1} \\
+ \mathcal{E}(t)^{1/2} \|(x + 2it \nabla)w\|_{L^{2(p+1)}(\|v\|_{L^{p+1}}^{p-1} + \|w\|_{L^{p+1}}^{p-1}) + \mathcal{E}(t)^{1/2} \|u_0\|_{B^0_{2+\varepsilon}} \|w\|_{L^{p+1}} \|v\|_{L^{p+1}}^{p-1} + \|w\|_{L^{p+1}}^{p-1}.
\]

Since \|v(1)\|_{L^{p+1}} \lesssim 1, \mathcal{E}(1) \lesssim 1, and dispersive estimates imply that \|w(1)\|_{L^{p+1}} \lesssim 1, the Cauchy–Schwartz inequality and (5.22) imply that
\[
\frac{1}{t^2} \mathcal{E}(t) \lesssim \frac{1}{t^2} + \|w(t)\|_{L^{p+1}}(\|v(t)\|_{L^{p+1}}^p + \|w(t)\|_{L^{p+1}}^p) + \frac{1}{t^2} \int_1^t \tau \|(x + 2it \nabla) \cdot \nabla w\|_{L^{p+1}}^{p+1} d\tau \\
+ \frac{1}{t^2} \int_1^t \tau \|w\|_{L^{p+1}}^{p+1} + \frac{1}{t^2} \int_1^t \mathcal{E}(t)^{1/2} \tau \|w\|_{L^{p+1}}^{p+1} + \|w(t)\|_{L^{p+1}}^p, \\
(5.23)
\]
with implicit constants depending only on \(p\) and \(d\). Then choosing \(0 < \delta(p,d) \ll 1\) sufficiently small, by the Cauchy–Schwartz inequality,
\[
\frac{1}{t^2} \mathcal{E}(t) \lesssim \frac{1}{t^2} + \|w(t)\|_{L^{p+1}}(\|v(t)\|_{L^{p+1}}^p + \|w(t)\|_{L^{p+1}}^p) + \frac{1}{t^2} \int_1^t \tau \|(x + 2it \nabla) \cdot \nabla w\|_{L^{p+1}}^{p+1} d\tau \\
+ \frac{1}{t^2} \int_1^t \tau \|w\|_{L^{p+1}}^{p+1} + \frac{1}{t^2} \int_1^t \mathcal{E}(t)^{1/2} \tau d\tau + \frac{1}{t^2} \int_1^t \tau \|(x + 2it \nabla)w\|_{L^{2(p+1)}(\|v\|_{L^{p+1}}^{p-1} + \|w\|_{L^{p+1}}^{p-1})} d\tau \\
+ \frac{1}{t^2} \int_1^t \tau \|u_0\|_{B^0_{2+\varepsilon}} \|w\|_{L^{p+1}}^{p-1} + \|w(t)\|_{L^{p+1}}^{p-1} d\tau.
\]
Therefore, by Young’s inequality,
\[
\frac{1}{t^2} \mathcal{E}(t) \|v(t)\|_{L^{p+1}}^p + \|w(t)\|_{L^{p+1}}^p, \\
(5.25)
\]
with implicit constants depending only on \(p\) and \(d\). Then choosing \(0 < \delta(p,d) \ll 1\) sufficiently small, by the Cauchy–Schwartz inequality,
Then combining $\|u(t)\|_{L^{p}+1} \lesssim \frac{1}{t} \mathcal{E}(t)$, Strichartz estimates, (5.6), (5.13), and (5.20),
\[
\left\| \frac{1}{t^2} \mathcal{E}(t) \right\|_{L^p((1, \infty))} \lesssim_{p,d} 1 + \|u_0\|_{B^\frac{3}{d}+\epsilon_{sc}}^{p+1} + \|u_0\|_{B^\frac{d}{d+\epsilon_{sc}}_{1,1}}^{2p} + \|u_0\|_{B^\frac{d}{d+\epsilon_{sc}}_{1,1}}^{2(p+1)}.
\]
This proves the theorem. \(\square\)

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