Reducing the Ising model to matchings

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Abstract

Canonical paths is one of the most powerful tools available to show that a Markov chain is rapidly mixing, thereby enabling approximate sampling from complex high dimensional distributions. Two success stories for the canonical paths method are chains for drawing matchings in a graph, and a chain for a version of the Ising model called the subgraphs world. In this paper, it is shown that a subgraphs world draw can be obtained by taking a draw from matchings on a graph that is linear in the size of the original graph. This provides a partial answer to why canonical paths works so well for both problems, as well as providing a new source of algorithms for the Ising model. For instance, this new reduction immediately yields a fully polynomial time approximation scheme for the Ising model on a bounded degree graph when the magnetization is bounded away from 0.

Keywords: Monte Carlo, simulation reduction, canonical paths, fpras

1 Introduction

The Markov chain Monte Carlo (MCMC) approach remains the most widely used methodology for generating random variates from high dimensional distributions. Let $\pi$ be a distribution on a finite state space $\Omega$. A Markov chain is a stochastic process $\{X_1, X_2, \ldots\}$ on $\Omega$ so that $P(X_{t+1} \in A|X_1, X_2, \ldots, X_t) = P(X_{t+1} \in A|X_t)$.

Call a chain on a finite state space ergodic if there exists an $N$ such that for all $n \geq N$ there is positive probability of traveling from any state $x$ to any other state $y$ in $n$ steps. For ergodic chains, the limiting distribution of $X_t$ will equal the stationary distribution of the
chain. Using well known methodologies, it is straightforward to build Markov chains whose stationary distribution matches a target distribution \( \pi \). (See [8] for more details.)

One ingredient is missing: the question of how large \( t \) must be before the distribution of \( X_t \) is close to \( \pi \) in some sense such as total variation. This \( t \) is known as the mixing time of a Markov chain, and unless it can be found for the chain in question, MCMC remains a heuristic rather than an algorithm for approximate sampling.

A breakthrough occurred when Jerrum and Sinclair developed the ideas of conductance and canonical paths into tools capable of proving the mixing time for complex chains on high dimensional spaces. In [9], they utilized conductance to show that a chain of Broder [3] for generating uniformly from the perfect matchings of a graph was rapidly mixing under a condition that encompassed a range of interesting problems such as uniform generation of regular graphs [10].

The development of canonical paths for the Ising model [11] followed. The use of approximate samples derived from a Markov chain together with selfreducibility [13] yields an approximation for the partition function of the Ising model, and this is still the only fully polynomial time randomized approximation scheme (fpras) known for this problem.

Later uses of canonical paths included an extension from perfect matchings to all matchings [12] and an algorithm for finding perfect matchings in polynomial time in all graphs ([14], [2]).

Canonical path approaches have also been used on such varied problems such as choosing approximately uniformly from a convex set [5] and 0-1 Knapsack solutions [17].

This work is a step towards understanding the relationship between some of these problems. We show

- Sampling from the subgraphs Ising model with zero magnetization can be accomplished by generating a perfect matching in a graph linear in the size of the original graph.

- Sampling from the subgraphs Ising model with positive magnetization on a graph can be accomplished by generating a matching in a graph linear in the size of the original graph.

- Sampling from perfect matchings in unbalanced bipartite graphs can be reduced to sampling from perfect matchings in balanced bipartite graphs.

- Sampling from matchings in a bipartite graph can be reduced to sampling from perfect matchings in a bipartite graph.

Matchings in unbalanced bipartite graphs arise in the approximation of the permanent of rectangular matrices, used in bounding the performance of digital mobile radio systems (see [19].) While conductance could be used directly to bound the mixing time of an appropriate chain for this problem, it is easier to just utilize a sampling reduction.

The remainder of the paper is organized as follows. Section 2 describes the various distributions that will be studied. Section 3 gives the general form of reduction by adding edges and nodes that will be used throughout. Section 4 details the reduction from the
Section 5 presents the reduction from unbalanced bipartite graphs to balanced bipartite graphs while Section 6 gives the reduction from matchings to perfect matchings in bipartite graphs. Finally Section 7 notes how this reduction leads to deterministic bounds on the Ising partition function for large magnetization problems.

2 The models

This section describes the three models that will appear throughout the paper: The Ising model, weighted matchings in graphs (the monomer-dimer model in physics), and weighted perfect matchings in graphs (the dimer model). Each of these three problems is presented in terms of a weight function \( w(x) \) where \( x \) is a configuration in state space \( \Omega \). Create a distribution \( \pi \) by setting

\[
\pi(x) = \frac{w(x)}{Z},
\]

where \( Z = \sum_{y \in \Omega} w(y) \) is called the partition function.

The Ising model

Originally a model of magnetism, the Ising model has a long history of study because of the existence of a phase transition on two dimensional lattices (see \[18\].) Three different formulations of this model use spins, subgraphs, and random clusters. While the spins formulation is the most widely known, the subgraphs formulation will be of most use here (as was also the case in \[11\].)

Given a graph \( G = (V, E) \), the state space for the subgraphs world is \( \Omega_{\text{subs}} = \{0, 1\}^E \), so that each state, also known as a configuration, indexes a subgraph of \( G \). Like many distributions of interest, the subgraphs world is given as a nonnegative weight function over configurations that is normalized by the partition function.

The parameters for the subgraphs world are as follows. Each edge \( e \) has weight \( \lambda(e) \in [0, 1] \) that controls the strength of interaction between the endpoints of the nodes, and each node \( i \) has weight \( \mu(i) \in [0, 1] \) that controls the strength of the magnetic field for that node. Given the \( \lambda \) and \( \mu \) vectors, the weight of a configuration is:

\[
w_{\text{subs}}(x) = \left[ \prod_{e : x(e) = 1} \lambda(e) \right] \left[ \prod_{i : \text{deg}(i) \text{ is odd under } x} \mu(i) \right]. \tag{1}
\]

Here \( \text{deg}(i) \) under \( x \) is \( \sum_{j : \{i, j\} \in E} x(j) \), and counts the number of edges with \( x(e) = 1 \) such that one endpoint of \( e \) is \( i \). As usual, the empty product is taken to be 1.

The vectors \( \lambda \) and \( \mu \) are in a one to one correspondence with the inverse temperature and magnetic field parameters found in the more common spins worlds formulation. Moreover, the normalizing constant for the subgraphs world \( Z_{\text{subs}} \) is a multiple of the normalizing constant for the spins world, and the multiple can be calculated explicitly (see \[11\].)

Calculation of the partition function \( Z_{\text{subs}} \) was shown in \[11\] to be a \#P complete problem for general graphs, and so it is unlikely that a polynomial time algorithm for finding \( Z_{\text{subs}} \) exactly will be found.
Matchings  A matching in a graph is a collection of edges such that no two edges share an adjacent node. As before, a collection of edges can be indexed using \( x(e) = 1 \) if the edge is in the collection, and \( x(e) = 0 \) if it is not. Using the weight formulation from before,

\[
 w_{\text{mat}}(x) = \left[ \prod_{e: x(e) = 1} \lambda(e) \right] \left[ \prod_{i: \text{deg}(i) \geq 2 \text{ under } x} 0 \right].
\]

As with Ising above, the empty product is taken to be 1, and so the only configurations with positive weight have no nodes with degree greater than 2.

There exists an algorithm for computing \( Z_{\text{mat}} \) when the graph \( G \) is planar ([7, 15, 21]), but for general graphs the problem is \#P complete (see [12].)

Perfect matchings  A perfect matching in a general graph is a collection of edges such that every node is adjacent to exactly one edge. That is,

\[
 w_{\text{permat}}(x) = \left[ \prod_{e: x(e) = 1} \lambda(e) \right] \left[ \prod_{i: \text{deg}(i) \neq 1 \text{ under } x} 0 \right].
\]

The partition function is also known as the hafnian of a matrix where the \((i, j)\) entry is \( \lambda(\{i, j\}) \).

Now consider a bipartite graph \( G_B = (V_1 \sqcup V_2, E) \), where \( \#V_1 \leq \#V_2 \). Since the graph is bipartite \( \{i, j\} \) in \( E \) means that exactly one of \( i \) and \( j \) is in \( V_1 \), and exactly one is in \( V_2 \). Here a perfect matching will mean that every node in the smaller partition \( V_1 \) is adjacent to exactly one node. This definition is designed to fit with the definition of the permanent for rectangular matrices found in [19].

\[
 w_{\text{bipermat}}(x) = \left[ \prod_{e: x(e) = 1} \lambda(e) \right] \left[ \prod_{i \in V_1: \text{deg}(i) \neq 1 \text{ under } x} 0 \right].
\]

The partition function is also known as the permanent of a matrix where the \((i, j)\) entry is the weight of an edge from the \( i \)th node of \( V_1 \) to the \( j \)th node of \( V_2 \).

The permanent problem (and the more general problem of finding the hafnian) is a \#P complete problem, even under the restrictive condition that \( \lambda(e) \in \{0, 1\} \) for all \( e \in E \) [12].

2.1 Canonical paths results

The presentation here follows that of [12]. Recall that the total variation distance between two distributions \( \mu \) and \( \pi \) is \( \text{dist}_{TV}(\mu, \pi) = \sup_A |\mu(A) - \pi(A)| \). In a canonical paths argument, for any two configurations \( x, y \), a path \( x = x_0, x_1, \ldots, x_k = y \) is fixed where \( \mathbb{P}(X_{t+1} = x_{i+1} | X_t = x_i) > 0 \). For a set \( \Gamma \) of paths, let \( \gamma(x, y) \) denote the path from state \( x \) to state \( y \). Given \( \Gamma \), the parameter \( \bar{\rho} \) measures how often the paths use a particular move from \( x_i \) to \( x_{i+1} \).
A Markov chain is reversible with respect to $\pi$ if for all states $i$ and $j$,
\[ \pi(i)P(X_{t+1} = j | X_t = i) = \pi(j)P(X_{t+1} = i | X_t = j). \]

If a chain is reversible with respect to a distribution $\pi$, then $\pi$ must be a stationary distribution. For such a reversible chain, let $Q(e) = Q(\{i, j\}) = \pi(i)P(X_{t+1} = j | X_t = i)$. Then
\[ \bar{\rho} = \bar{\rho}(\Gamma) := \max_e \frac{1}{Q(e)} \sum_{x, y : e \in \gamma(x, y)} \pi(x)\pi(y)\text{length}(\gamma(x, y)). \]

A relationship between the mixing time of reversible chains and canonical paths was shown by Diaconis and Stroock [4]. The following form of the theorem is from [12]:

**Theorem 1.** Let $\mathcal{M}$ be a finite, reversible, ergodic Markov chain with $P(X_{t+1} = i | X_t = i) \geq 1/2$ for all $i$, and canonical paths $\Gamma$. Fix $x \in \Omega$ and $\epsilon > 0$. Then for all $t \geq \tau_\epsilon(x, \Gamma) := \bar{\rho}(\Gamma)(\ln \pi(x)^{-1} + \ln \epsilon^{-1})$ and any $A \subseteq \Omega$:
\[ |P(X_t \in A | X_0 = x) - \pi(A)| \leq \epsilon. \]

The number of steps necessary for the total variation distance to fall below $\epsilon$ from a starting state will be referred to as the mixing time of the chain. The theorem states that $\tau_\epsilon(x, \Gamma)$ is an upper bound on the mixing time. Starting at $x$, this upper bound is proportional to the loading $\bar{\rho}$ for the set of canonical paths $\Gamma$. Therefore, it is important to find paths that keep the use of any one edge as low as possible.

Now the results for the subgraphs world and matching can be stated. In [11], it was shown for the subgraphs world that
\[ \bar{\rho}(\Gamma) \leq 2(\#E)^2 / \min_i \mu(i)^4. \]

The configuration with $x(e) = 0$ for all $e$ has weight 1. All configurations have weight at most 1 and there are at most $2^\#E$ configurations, so $\pi(\vec{0}) \geq 2^{-\#E}$, and starting from the empty configuration the mixing time is bounded above by
\[ 2(\#E)^2(\max_i \mu(i)^{-4})[(\ln 2)\#E + \ln \epsilon^{-1}]. \]

In [12], it was shown for the matching problem that there exist paths where
\[ \bar{\rho}(\Gamma) \leq 4(\#E)(\#V)\lambda'^2, \quad \lambda' := \max\{1, \lambda(e_1), \ldots, \lambda(e_{\#E})\} \]

It is possible to find the maximum weight matching in polynomial time via Edmonds algorithm [6]. There are at most $2^\#E$ matchings, and so starting from this maximum weight matching the mixing time is at most
\[ 4(\#E)(\#V)\lambda'^2[(\ln 2)\#E + \ln \epsilon^{-1}]. \]

For perfect matchings in general graphs, no polynomial time algorithm is known. However, for bipartite graphs, in a landmark paper Jerrum, Sinclair, and Vigoda [13] showed
how with a sequence of Markov chains, it was possible to obtain approximately drawn variates in polynomial time. The running time of this procedure was later improved to \( \Theta(#V^7 \ln^4 #V) \) \cite{2}.

In the remainder of the paper, it is shown how to obtain draws from the subgraphs world distribution by generating draws from the matching distribution. The running time using the reduction will be of the same order as a direct approach using canonical paths.

3 Reductions by adding edges and nodes

All of the reductions presented here have the same form. Given graph \( G = (V, E) \), consider drawing from a distribution \( \pi \) on \( \{0, 1\}^E \). First construct a new graph \( G' = (V', E_1 \cup E_2) \), where \( \phi \) is a one to one correspondence from \( E \) to \( E_1 \). Then draw \( X' \) from \( \pi' \) on \( G' \), and let \( X(E) = X'(\phi(E)) \). Then the following lemma gives a sufficient condition for \( X \) to be a draw from \( \pi \).

**Theorem 2.** Suppose that \( \pi(x) = w(x)/Z \) where \( Z = \sum_y w(y) \) when used on \( G = (V, E) \), and \( \pi'(x) = w'(x)/Z' \) where \( Z' = \sum_y w(y) \) over \( G' = (V, E_1 \cup E_2) \), and \( \phi \) is a one to one correspondence from the edges of \( E \) to \( E_1 \). Suppose that the weight functions satisfy:

\[
\sum_{x':x'(\phi(E))=x(E)} w'(x') = w(x)C,
\]  

(5)

where \( C \) is a constant. If (5) holds, \( X' \sim \pi' \) and \( X(E) = X'(\phi(E)) \), then \( X \sim \pi \).

**Proof.** Fix \( x \in \{0, 1\}^E \), and note

\[
\mathbb{P}(X = x) = \sum_{x'} \mathbb{P}(X = x|X' = x')\mathbb{P}(X' = x') = \sum_{x':x'(\phi(E))=x(E)} w'(x')/Z'.
\]

But by assumption, the numerator of the right hand side is just \( Cw(x) \), so \( \mathbb{P}(X = x) = w(x)C/Z' \), so \( X \sim \pi \) and \( Z = Z'/C \).

**Reduction for subgraphs to maximum degree 3** This theorem can be applied to reduce any subgraphs world problem to one where the maximum degree is three. Suppose that \( i \) is a node with degree greater than 3 in \( G = (V, E) \). Let \( j_1, j_2, \ldots, j_{\deg(i)} \) denote the neighbors of \( i \). Then consider the subgraphs distribution on a new graph \( G' = (V', E') \) where \( V' = (V \setminus \{i\}) \cup \{i_1, i_2\} \), and

\[
E' = (E \cup \{i_1, i_2\}) \cup \{(i_1, j_1), (i_1, j_2)\} \cup \{(i_2, j_3), (i_2, j_4), \ldots, (i_2, j_{\deg(i)})\} \setminus \cup_{j \in \{i, j\}} \{j, i\}.
\]

In other words, node \( i \) is being split into two nodes, \( i_1 \) and \( i_2 \). Node \( i_1 \) is connected to the first two neighbors of \( i \), while \( i_2 \) is connected to the remaining neighbors of \( i \). Finally, the nodes \( i_1 \) and \( i_2 \) are connected. Figure 1 illustrates this process.
Figure 1: Splitting a node

The function $\phi$ is the identity map for any edge that is not adjacent to $i$. Edges $\{i, j_1\}$ and $\{i, j_2\}$ map to $\{i_1, j_1\}$ and $\{i_1, j_2\}$ respectively, while $\{i, j_k\}$ maps to $\{i_2, j_k\}$ for $k \in \{3, \ldots, \deg(i)\}$.

The point of splitting $i$ is that node $i_1$ now has degree 3, and $\deg(i_2) = \deg(i) - 1$. Repeated $\deg(i) - 3$ times, the nodes that result from splitting $i$ all have degree 3. This can be repeated for every node of the graph with degree greater than 3 until none remain. The following lemma verifies that this is a valid reduction.

**Lemma 1.** Suppose $G = (V, E)$ and $G' = (V', E')$ are as described above. Set $\lambda'(\phi(E)) = \lambda(E)$, $\mu'(V \setminus \{i\}) = \mu(V \setminus \{i\})$, $\lambda'(\{i_1, i_2\}) = 1$, and

$$\mu'(i_1) = \mu'(i_2) = \mu(i)^{-1} - \sqrt{\mu(i)^{-2} - 1},$$

when $\mu(i) > 0$, and $\mu'(i_1) = \mu'(i_2) = 0$ otherwise. Then drawing $X'$ from $\pi_{\text{subs}}$ on $G'$ and setting $X(E) = X'(\phi(E))$ yields $X \sim \pi_{\text{subs}}$ on $G$.

**Proof.** Fix $x \in \{0, 1\}^E$. In light of Theorem 2 it suffices to show

$$\sum_{x': x'(\phi(E)) = x(E)} w(x') = w(x)C, \quad (6)$$

where $w(\cdot) = w_{\text{subs}}(\cdot)$. There is only one edge in $E'$ not in $\phi(E)$: the edge $\{i_1, i_2\}$. There are two values for this edge, and so the sum consists of two terms.

The first case to consider is when $i$ has odd degree in $x$. When $\mu(i) = 0$ the configuration $x$ has probability 0 under $\pi_{\text{subs}}$, and so no check is necessary.

When $\mu(i) > 0$, then whether edge $\{i_1, i_2\}$ is 1 or 0 in $x'$, exactly one of $i_1$ and $i_2$ has odd degree, while the other is even. Hence a factor of $\mu'(i_1)$ or $\mu'(i_2)$ is contributed to the weight, but a factor of $\mu(i)$ is missing so $w'(x') = \mu'(i_1)\mu(i)^{-1}w(x)$ for one term in the sum, and $w'(x') = \mu'(i_2)\mu(i)^{-1}w(x)$ in the other term. Hence (6) becomes:

$$\mu(i)^{-1}\mu'(i_1)w(x) + \mu(i)^{-1}\mu'(i_2)w(x) = w(x)C. \quad (7)$$

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On the other hand, when \( i \) has even degree, one choice of \( x'\{i_1, i_2\} \) leads to both \( i_1 \) and \( i_2 \) having odd degree, while with the other choice both have even degree. Hence (7) becomes:

\[
w(x) + \mu'(i_1)\mu'(i_2)w(x) = w(x)C. \tag{8}
\]

When \( \mu(i) = 0 \), only this equation needs to be satisfied, and so

\[
C = 1 + \mu'(i_1)\mu'(i_2)
\]

applies. However, note that

\[
1 + \mu'(i_1)\mu'(i_2) = 1 + \left( \mu(i)^{-1} - \sqrt{\mu(i)-2} - 1 \right)^2 = 2\mu(i)^{-2} - 2\mu(i)^{-1}\sqrt{\mu(i)-2} - 1.
\]

This right hand side is exactly \( \mu(i)^{-1}[\mu'(i_1) + \mu'(i_2)] \), so setting \( C \) equal to this expression satisfies both (7) and (8), finishing the proof. \( \square \)

The downside of this construction is that the magnetic field is smaller at each of the duplicated nodes since \( \mu'(i_1) < \mu(i) \) unless \( \mu(i) \in \{0,1\} \), in which case they are equal. The good news is that once \( \mu(i) \leq \#V^{-1} \) it can be replaced by \( \#V^{-1} \) without changing the distribution too much. When \( \mu(i) \) is this small, the chance that a draw from \( \pi_{\text{subs}} \) will result in all nodes having even degree is at least \( \exp(-1) \), so simple acceptance rejection can be used [11].

Consider starting with a graph \( G = (V, E) \) and repeatedly applying this reduction until the maximum degree of the graph is 3. Then the number of new edges created at each node equals the degree of the node minus 3. Hence there are at most \( \sum_{i} \text{deg}(i) = 2\#E \) edges created, resulting in at most \( 3\#E \) total edges.

**Reduction for subgraphs to all degree 3 nodes** Not only can the maximum degree be reduced to 3, but the degree 1 and 2 nodes can be eliminated as well.

**Lemma 2.** Let \( G = (V, E) \) be a graph with degree 1 node \( i \) and edge \( \{i, j\} \). Create \( G' = (V', E') \) by \( V' = V \setminus \{i\} \), \( E' = E \setminus \{i, j\} \). Say \( X' \sim \pi_{\text{subs}} \) on \( G' \) with parameters \( \lambda'(E') = \lambda(E) \), \( \mu'(V' \setminus \{j\}) = \mu(V' \setminus \{j\}) \), and

\[
\mu'(j) = \frac{\mu(j) + \lambda(\{i, j\})\mu(i)}{1 + \lambda(\{i, j\})\mu(i)\mu(j)}.
\]

Let \( X(E') = X'(E') \). Given \( X(E') \), let \( X(\{i, j\}) \sim \text{Bern}(\delta) \), where \( \delta \) is \( \delta_1 \) if \( \text{deg}(j) \) is odd under \( X' \), and \( \delta_2 \) if \( \text{deg}(j) \) is even under \( X' \) where

\[
\delta_1 = \frac{\lambda(\{i, j\})\mu(i)}{\mu(j) + \lambda(\{i, j\})\mu(i)}, \quad \delta_2 = \frac{\lambda(\{i, j\})\mu(i)\mu(j)}{1 + \lambda(\{i, j\})\mu(i)\mu(j)}.
\]

Then \( X \sim \pi_{\text{subs}} \) with parameters \( \lambda \) and \( \mu \).

**Proof.** Intuitively, a degree 1 node with no magnetic field is irrelevant to the rest of draw since the edge will always be off. In this case \( \delta_1 = \delta_2 = 0 \), so the edge is always removed. A degree 1 node with magnetic field is linked only to its nearest neighbor, so it makes that
neighbor a little more likely to be spin up, which is the same as increasing its magnetic field. The amount of increase depends on the magnetic field of \( i \) and the strength of interaction to \( j \).

Fix \( x \in \{0, 1\}^E \). Let \( r(x) = \prod_{e \in E', x(e) = 1} \lambda(e) \prod_{k \in V \setminus \{j\} \setminus \deg(k)} \) is odd in \( x \mu(k) \). Then there is a unique \( x' \in \{0, 1\}^{E'} \) that matches \( x \) on \( E' \), and the following table summarizes the probabilities needed:

| \( j \) under \( x' \) | \( w_{\text{sub}}(x') \) | \( x(\{i, j\}) \) | \( w_{\text{sub}}(x) \) |
|----------------|-----------------|----------------|----------------|
| odd            | \( r(x)\mu'(j) \) | 1              | \( r(x)\lambda(\{i, j\})\mu(i) \) |
| odd            | \( r(x)\mu'(j) \) | 0              | \( r(x)\mu(j) \) |
| even           | \( r(x) \)       | 1              | \( r(x)\lambda(\{i, j\})\mu(i)\mu(j) \) |
| even           | \( r(x) \)       | 0              | \( r(x) \) |

Note \( \mathbb{P}(X = x) = \mathbb{P}(X'(E') = x(E'))\mathbb{P}(X(\{i, j\}) = x(\{i, j\})) \). Let \( \delta_1 = \mathbb{P}(X(\{i, j\}) = 1| j \text{ is odd in } X') \) and \( \delta_2 = \mathbb{P}(X(\{i, j\}) = 1| j \text{ is even in } X') \). Requiring that this probability is proportional to \( w_{\text{sub}}(x) \) together with the above table gives rise to four equations (\( C \) is the common constant of proportionality):

\[
\begin{align*}
    r(x)\mu'(j)\delta_1 &= r(x)\lambda(\{i, j\})\mu(i)C \\
    r(x)\mu'(j)(1 - \delta_1) &= r(x)\mu(j)C \\
    r(x)\delta_2 &= r(x)\lambda(\{i, j\})\mu(i)\mu(j)C \\
    r(x)(1 - \delta_2) &= r(x)C
\end{align*}
\]

The \( r(x) \) factor cancels out from both sides, and adding the last two equations yields \( C = (1 + \lambda(\{i, j\})\mu(i)\mu(j))^{-1} \) and then \( \delta_2 = \lambda(\{i, j\})\mu(i)\mu(j)C \).

Adding the first two equations yields \( \mu'(j) = [\mu(j) + \lambda(\{i, j\})\mu(i)]C \), which in turn gives \( \delta_1 = \lambda(\{i, j\})\mu(i)(\mu(j) + \lambda(\{i, j\})\mu(i))^{-1} \). Since \( \mu'(i), \delta_1, \text{ and } \delta_2 \) are in \([0, 1]\), this completes the proof.

Removing degree 2 nodes can be accomplished as well by taking these nodes and splitting them into two copies connected by an edge, making the degree of each copy equal to 3. The details are presented in the appendix.

4 Reduction of Ising to matchings and perfect matchings

In this section the technique of the previous section will be used to reduce the subgraphs world with no magnetic field to a perfect matching problem, and to reduce the subgraphs world with positive magnetic field to a matching problem.

Reduction to perfect matchings when the magnetic field is zero When the magnetic field is zero (so \( \mu(i) = 0 \) for all \( i \)), the subgraphs world model can be reduced to a problem where every degree is 3 and all \( \mu(i) \) remain at 0. It is from this graph that the graph for perfect matchings will be constructed.
Each edge \( \{i, j\} \) in the original graph is given two new nodes \( v_{ij} \) and \( v_{ji} \). Connect these two nodes by an edge of weight \( \lambda(\{i, j\}) \). Each node \( i \) in the original graph is given a new node \( d_i \). Connect \( d_i \) and \( v_{ij} \) with an edge of weight 1 for all \( j \).

Finally, connect nodes of the form \( v_{ij}, v_{ik} \) for all \( j \) and \( k \) with an edge with weight \( 1/3 \). Set \( \phi(\{i, j\}) = \{v_{ij}, v_{ji}\} \). Figure 2 illustrates a piece of this transformation.

**Figure 2:** New graph for degree 3 nodes

For a graph \((V, E)\) where every node has degree 3, this transformation creates four nodes for each original node. These four nodes are connected by 6 new edges, and the original edges still remain. Hence after modification, the new graph has \( 4\#V \) nodes and \( \#E + 6\#V \) edges.

**Lemma 3.** Given a graph \( G = (V, E) \) with all nodes of degree 3, \( \mu(i) \) identically 0 and edge weights \( \lambda \), let \( n(v, 1), n(v, 2) \) and \( n(v, 3) \) denote the neighbors of node \( v \). For each \( i \), set \( A_i = \{d_i, v_{n(i,1)}, v_{n(i,2)}, v_{n(i,3)}\} \), and \( E_i \) be all subsets of size 2 in \( A_i \) so that \((A_i, E_i)\) is the complete graph on 4 vertices. Set

\[
V' = \bigcup_{i \in V} A_i, \\
E' = \left( \bigcup_{\{i,j\} \in E} \{v_{ij}, v_{ji}\} \right) \cup \left( \bigcup_i E_i \right).
\]

For edges of the form \( e = \{v_{ij}, v_{ji}\} \), set \( \lambda'(e) = \lambda(\{i, j\}) \). For edges \( e = \{d_i, v_{ij}\} \), set \( \lambda'(e) = 1 \), and for edges \( e = \{v_{ij}, v_{ik}\} \), set \( \lambda'(e) = 1/3 \).

Then if \( X' \) is drawn from \( \pi_{\text{permat}} \) on \( G' \) with \( \lambda' \), and \( X(E) = X'(\phi(E)) \), then \( X \sim \pi_{\text{subs}} \) on \( G \) with \( \lambda \).

**Proof.** As before, let \( \phi(\{i, j\}) = \{v_{ij}, v_{ji}\} \). Fix a configuration \( x \) in the subgraphs world, and consider the number of \( x' \) such that \( x'(\phi(E)) = x(E) \). The edges that are free in such an \( x' \) are of the form \( \{d_i, v_{ij}\}, \{v_{ij}, v_{ik}\}, \) or \( \{v_{ij}, v_{ji}\} \). Give weight 1 to all edges of the form \( \{d_i, v_{ij}\} \), weight 1/3 to all edges of the form \( \{v_{ij}, v_{ik}\} \), and weight \( \lambda(\{i, j\}) \) to edges \( \{v_{ij}, v_{ji}\} \).
Then for $x$, the choice of $x$ is a choice for how to fill out $x(E_i)$ for each $i$. For a collection of edges $F$ and a configuration $x'$, let $p(x'(F))$ be the product of the edge weights over edges with value 1 in $x'$, that is:

$$p(x'(F)) = \prod_{e \in F : \lambda'(e) = 1} \lambda'(e). \quad (9)$$

Let $\mathcal{M}$ be the set of configurations that correspond to a matching, so $x' \in \mathcal{M}$ says that if $e_1$ and $e_2$ are edges that share an endpoint, then either $x'(e_1)$ or $x'(e_2)$ is 0. Using this notation

$$w_{\text{permat}}(x') = p(x'(\phi(E))) \prod_i p(x'(E_i)) \mathbf{1}(x' \in \mathcal{M}).$$

So

$$\sum_{x': x'(\phi(E)) = x(E)} w_{\text{permat}}(x') = \sum_{x': x'(\phi(E)) = x(E)} p(x'(\phi(E))) \prod_i p(x'(E_i)) \mathbf{1}(x' \in \mathcal{M}).$$

Since there is no magnetic field, $p(x'(\phi(E))) = w_{\text{subs}}(x)$, and factoring the sum of the products yields:

$$\sum_{x': x'(\phi(E)) = x(E)} w_{\text{permat}}(x') = w_{\text{subs}}(x) \prod_i \sum_{x'(E_i)} p(x'(E_i)) \mathbf{1}(x' \in \mathcal{M}).$$

To satisfy Theorem 2, it suffices that $\sum_{x'(E_i)} p(x'(E_i)) \mathbf{1}(x' \in \mathcal{M}) = 1$ for all $i$. There are several cases to consider based on the degree of $i$ under $x$.

Suppose first the degree of $i$ under $x$ is 0. There are three different $x'(E_i)$ with nonzero weight. The three different $x'$ come from the fact that $d_i$ is matched to one of three different nodes. Say $x'\{d_i, v_{ij}\} = 1$. Then to match $k$ and $\ell$, $x'\{v_{ik}, v_{i\ell}\} = 1$, and all the rest of the edges $e \in E_i$ have $x'(e) = 0$. Since $\lambda'\{d_i, v_{ij}\} = 1$ and $\lambda'\{v_{ik}, v_{i\ell}\} = 1/3$, $p(x') = 1/3$.

Similarly, $d_i$ could be matched to $v_{ik}$ or $v_{i\ell}$, again resulting in $p(x'(E_i)) = 1/3$. Hence $\sum_{x'(E_i)} p(x'(E_i)) \mathbf{1}(x' \in \mathcal{M}) = 1/3 + 1/3 + 1/3 = 1$.

Now suppose degree of $i$ under $x$ is 1. Then $x'(E_i)$ needs to be a perfect matching on $4 - 1 = 3$ nodes, which is impossible. Similarly, if the degree of $i$ under $x$ is 3, $x'(E_i)$ needs to be a perfect matching on 1 node: also impossible. Hence in these cases the sum of the weights is 0. Fortunately, since there is no magnetic field, $w_{\text{subs}}(x) = 0$ is these cases also.

Last, suppose that the degree of $i$ in $x$ is 2. Then $d_i$ must be matched in $x'(E_i)$ to whatever node adjacent to $i$ is not already matching. The edge weight of this edge is 1, so $\sum_{x'(E_i)} p(x'(E_i)) \mathbf{1}(x' \in \mathcal{M}) = 1$, which completes the proof. \qed

Note, a similar gadget appears in \[16\] (pp. 125–147) in the specific case of the Ising model on two-dimensional lattices.

**Reduction to matchings for nonzero magnetic field** When the magnetic field is positive for every node, the subgraphs world can be reduced to sampling matchings rather than perfect matchings. As noted earlier, when $\mu(i) < \#V^{-1}$, changing $\mu(i)$ to $\#V^{-1}$ results in a distribution that when sampled from, yields a draw from the distribution for the original...
\(\mu(i)\) with probability at least \(\exp(-1)\). Hence any subgraphs model can be altered to have \(\mu(i) \geq \#V^{-1}\) for all \(i\).

The \(G'\) constructed from \(G\) remains the same as in the no magnetic field case, all that changes is the construction of \(\lambda'\) for the matchings.

All edges of the form \(\{v_{ia}, v_{ib}\}\) receive the same edge weight \(\lambda_1\). Edges of the form \(\{d_i, v_{ia}\}\) receive edge weight \(\lambda_2\). But this does not leave enough freedom to handle configurations where the degree is either 0, 1, 2, or 3, which gives rise to a system of four equations via (6). One more parameter is the constant \(C\) in the equations, but that still only gives three unknowns and four equations.

So another parameter \(\alpha(i)\) must be added to each node. The new edge weights will be \(\lambda'(\{v_{ij}, v_{ji}\}) = \alpha(i)\alpha(j)\lambda(\{i, j\})\), and now there are four unknowns. The solution to the four equations is presented in the following lemma.

**Lemma 4.** Given graph \(G = (V, E)\), with maximum degree 3, build \(G' = (V', E')\) as in Lemma 3, let \(G' = (V', E')\) be the same as for the zero magnetic field case. For \(i\) of degree 3 in \(G\), let \(\lambda_2(i)\) be the smallest nonnegative solution to the cubic equation

\[
\mu(i)^2(1 + \lambda_2(i))^3 = 1 + 3\lambda_2(i) + 3\lambda_2(i)^2.
\]

In fact, there always exists a solution with \(0 \leq \lambda_2(i) \leq 3\mu(i)^{-2}\). Let

\[
\lambda_1(i) = \lambda_2(i)^2(1 + \lambda_2(i))^{-1}, \quad \alpha(i) = \mu(i)(1 + \lambda_2(i)).
\]

Then for each edge \(e\) of the form \(\{v_{ij}, v_{ik}\}\), set \(\lambda'(e) = \lambda_1(i)\), while for edges \(e\) of the form \(\{d_i, v_{ij}\}\), set \(\lambda'(e) = \lambda_2(i)\). For edges \(e = \{i, j\}\) with \(\phi(e) = \{v_{ij}, v_{ji}\}\), set \(\lambda'(\phi(e)) = \lambda(e)\alpha(i)\alpha(j)\).

Let \(X' \sim \pi_{\text{mat}}\), and set \(X(E) = X'(\phi(E))\). Then \(X \sim \pi_{\text{subs}}\).

**Proof.** Fix \(x \in \{0, 1\}^E\). As before, for \(F \subseteq E'\) let \(p(x'(F)) = \prod_{e \in F: x'(e) = 1} \lambda'(e)\), so

\[
\sum_{x': x'(\phi(E)) = x(E)} w_{\text{permat}}(x') = \sum_{x': x'(\phi(E)) = x(E)} p(x'(\phi(E))) \prod_i p(x'(E_i)) \mathbf{1}(x' \in \mathcal{M}).
\]

Let \(d(i, x)\) denote the number of edges adjacent to \(i\) that have value 1 in configuration \(x\). Since each edge \(\{v_{ij}, v_{ji}\}\) receives an extra factor of \(\alpha(i)\) and \(\alpha(j)\) in its edge weight,

\[
p(x'(\phi(E)))) = w_{\text{subs}}(x) \left[ \prod_i \alpha(i)^{d(i, x)} \right] \left[ \prod_{i: d(i, x) \text{ is odd}} \mu(i)^{-1} \right]
\]

\[
= w_{\text{subs}}(x) \prod_i f(i),
\]

where

\[
f(i) := \alpha(i)^{d(i, x)}[\mathbf{1}(d(i, x) \text{ is even}) + \mu(i)^{-1} \mathbf{1}(d(i, x) \text{ is odd})].
\]
Then
\[ \sum_{x': x'(d(E)) = x(E)} w_{\text{permat}}(x') = w_{\text{subs}}(x) \prod_i f(i) \sum_{x'(E_i)} p(x'(E_i)) 1(x' \in M). \]

In order to prove the theorem, it suffices to have each term in the product in the right hand side equal a constant for each \( i \). Call this constant \( C(i) \). Fix \( i \). There are four possible values for \( d(i, x) \): 0, 1, 2, and 3. Each gives rise to an equation. In the equations, \( \lambda_1 \) is the weight for edges between nodes in \( \{v_{ij}, v_{ik}, v_{id}\} \), and \( \lambda_2 \) is the weight given to all edges leaving \( d_i \). (For simplicity, the dependence of \( \lambda_1, \lambda_2, C \) and \( \alpha \) on \( i \) is suppressed.)

\[
\begin{align*}
    d(i, x) = 0 : & \quad C = [1 + 3\lambda_1 + 3\lambda_2 + 3\lambda_1\lambda_2] \\
    d(i, x) = 1 : & \quad C = \alpha \mu(i)^{-1} [1 + \lambda_1 + 2\lambda_2] \\
    d(i, x) = 2 : & \quad C = \alpha^2 [1 + \lambda_2] \\
    d(i, x) = 3 : & \quad C = \alpha^3 \mu(i)^{-1} [1]
\end{align*}
\]

The expression in brackets on the right hand is the sum of the weights of \( x'(E_i) \) with nonzero weight. When \( d(i, x) = 3 \), it must be that \( x'(e) = 0 \) for all \( e \in E_i \), so \( p(x'(E_i)) = 1 \).

When \( d(i, x) = 2 \), there are two possible matchings, either \( d_i \) is matched to the remaining node (giving weight \( \lambda_2 \)) or not (giving weight 1).

When \( d(i, x) = 1 \), one of \( \{v_{ij}, v_{ik}, v_{id}\} \) is matched: say without loss of generality \( v_{id} \) is taken. Then the remaining nodes of \( E_i \) are \( \{d_i, v_{ij}, v_{ik}\} \) and they are all connected to each other. Hence the possible matchings are: empty matching (weight 1), matching \( v_{ij} \) to \( v_{ik} \) (weight \( \lambda_1 \)), matching \( d_i \) to \( v_{ij} \) (weight \( \lambda_2 \)) and matching \( d_i \) to \( v_{ik} \) (again weight \( \lambda_2 \)). Hence the sum of the weights of the matchings is \( 1 + \lambda_1 + 2\lambda_2 \).

When \( d(i, x) = 0 \), any matching in \( E_i \) contributes to the sum. There is one matching of size 0 (weight 1), six matchings of size 1 (three weight \( \lambda_1 \), three weight \( \lambda_2 \)), and three matchings of size 2 (all weight \( \lambda_1\lambda_2 \)). Hence the total sum of weights is \( 1 + 3\lambda_1 + 3\lambda_2 + 3\lambda_1\lambda_2 \).

Now the solution must be checked. First show existence. Let

\[ g(x) := \mu(i)^2 (1 + x)^3 - (1 + 3x + 3x^2). \]

Note that
\[ g(3\mu(i)^{-2}) = \mu(i)^2 + 8 + 18\mu(i)^{-2} > 0, \]
but \( g(0) = \mu(i)^2 - 1 \leq 0 \) Since \( g \) is continuous, there must be a solution to \( g(x) = \mu(i)^2 \) with \( x \in [0, 3\mu(i)^{-2}] \).

Since \( \alpha = \mu(i)(1 + \lambda_2) \), the \( d(i, x) = 3 \) equation has
\[ C = \mu(i)^3 (1 + \lambda_2)^3 \mu(i)^{-1} = 1 + 3\lambda_2 + 3\lambda_2^2. \]

Similarly, the \( d(i, x) = 2 \) equation has
\[ C = \mu(i)^2 (1 + \lambda_2)^3 = 1 + 3\lambda_2 + 3\lambda_2^2. \]
For the $d(i, x) = 1$ equation, using $\lambda_1 = \lambda_2^2/(1 + \lambda_2)$ yields:

$$C = (1 + \lambda_2)(1 + \frac{\lambda_2^3}{1 + \lambda_2} + 2\lambda_2) = 1 + 3\lambda_2 + 3\lambda_2^2.$$  

Finally, in the $d(i, x) = 0$ equation,

$$C = 1 + 3\lambda_1 + 3\lambda_2 + 3\lambda_1\lambda_2 = 1 + \frac{3\lambda_2^3}{1 + \lambda_2} + 3\lambda_2 + \frac{3\lambda_2^3}{1 + \lambda_2} = 1 + 3\lambda_2 + 3\lambda_2^2.$$  

Hence this is a valid solution to all four equations, and by Theorem 2 the result follows.  

\[ \square \]

5 Reduction of perfect matchings in unbalanced graphs to balanced graphs

Consider finding the permanent of a bipartite graph $G = (V_1 \sqcup V_2, E)$ so $(\forall \{i, j\} \in E)(\#(\{i, j\} \cap V_1) = \#(\{i, j\} \cap V_2) = 1)$. Call the graph unbalanced if $\#V_1 < \#V_2$, and call a configuration $x \in \{0, 1\}^E$ a perfect matching in an unbalanced bipartite graph if for all $i \in V_1$, there exists a $j \in V_2$ such that $\{i, j\} \in E$ and $x(\{i, j\}) = 1$.

Smith studied the number of perfect matchings in unbalanced bipartite graphs to bound the performance of digital mobile radio systems [20]. The number of perfect matchings is equal to the permanent of the rectangular adjacency matrix (see [19]).

The permanent of square matrices has attracted far more study than the rectangular case, and so the goal here is to reduce the problem of generating variates from unbalanced graphs to generating variates from balanced graphs.

Fortunately, the reduction is easy to describe. Let $V_3$ consist of $\#V_2 - \#V_1$ new nodes, and add edges from every node in $V_3$ to every node in $V_2$, and give them weight 1. Then $G$ is still bipartite with node partition $(V_1 \sqcup V_3, V_2)$.

Consider a configuration $x$ that was a perfect matching in the original graph. Then if $x'$ is a configuration on the new graph with $x'(E) = x(E)$, then exactly $\#V_1$ nodes in $V_2$ are already matched. So to fill out $x'$, the remaining nodes in $V_2$ must be matched to $V_3$, and there are exactly $(\#V_2 - \#V_1)!$ ways to accomplish this. Each of these has weight equal to the weight of $x(E)$, and so $\sum_{x' : x'(E) = x(E)} w'(x') = w(x)(\#V_2 - \#V_1)!$. Theorem 2 then says that drawing $x'$ from perfect matchings on the new graph and keeping $x(E) = x'(E)$ results in a draw from the perfect matchings on the original unbalanced graph.

Hence any algorithm for simulation and approximation of the partition function for perfect matchings on balanced graphs (such as [14], [2]) can also be used for unbalanced problems.

6 Reducing matchings to perfect matchings in bipartite graphs

Consider a bipartite graph $G = (V_1 \sqcup V_2, E)$. The problem of sampling from all matchings in this graph can be reduced to sampling perfect matchings as follows. First, for each node
Figure 3: Unbalanced to balanced bipartite graph

\[ i \in V_1, \text{ create a node } i' \text{ and add edge } \{i, i'\} \text{ with edge weight } 1. \]

If \( V_3 \) is the set of \( i' \), this creates a new, unbalanced bipartite graph \( G' = (V_1 \cup (V_2 \cup V_3), E) \). Furthermore, each matching in the original graph corresponds to a perfect matching in the new graph with equal weight. So a perfect matching sampled from the new graph yields a matching in the original.

This can be reduced to a balanced bipartite graph as described in the previous section: create \( \#V_2 \) new nodes and connect each of them to all of the nodes in \( V_2 \cup V_3 \). The final graph is still bipartite but now is balanced.

Figure 4: Matchings to perfect matchings

The point of this is that the new problem is a perfect matchings problem on bipartite graphs, and so samples can be generated in polynomial time ([11, 2]) for any values of the edge weights. With the matching canonical paths approach, the time to generate a sample depends on the square of the largest edge weight, with this reduction, this dependence no longer appears.
7 Consequences of the Ising reduction

The purpose of any reduction is so that existing methods for one problem can be immediately applied to the other. For instance, it was noted in equation (3) that the mixing time for the Ising model on graph $G = (V, E)$ was upper bounded by $2(#E)^2(\max_i \mu(i)^{-1})[(\ln 2)\#E + \ln \epsilon^{-1}]$ using the canonical paths method.

Suppose instead that the Ising model is first reduced to a graph with maximum degree 3 with at most $3\#E$ edges and $2\#E$ nodes. Then the graph is further altered so that a draw from the matchings distribution yields a draw from the subgraphs distribution. The mixing time for the matchings distribution on this graph is $4(3\#E)(2\#E)\lambda'[(\ln 2)\#E + \ln \epsilon^{-1}]$, and from the construction in Section 4 $\lambda' \leq 3 \max_i \mu(i)^{-2}$. So the total mixing time is the same order in $\#E$ as for direct analysis on the subgraph world, but the constant in front is larger by a factor of 108.

The purpose of the reduction is so that any improvements or new algorithms for simulating matchings will automatically translate into new algorithms for the subgraphs world.

As an example of this, Bayati et. al. have shown that for matchings in graphs of bounded degree and bounded edge weights how it is possible to construct a deterministic fully polynomial time approximation scheme for computing the partition function for the set of matchings. To be precise, their result states:

**Theorem 3.** For a graph $G = (V, E)$ with edge weights $\lambda$, and $\epsilon > 0$, there exists an $\exp(\epsilon)$ approximation algorithm for $\sum_{x \in \{0, 1\}^E} \prod_{x(e) = 1} \lambda(e)$ that runs in time $O(n/\epsilon)^{\kappa \log \Delta + 1}$, where $\Delta$ is the maximum degree of the graph, $\lambda = \max_e \lambda(e)$ and $\kappa = -2/\log(1 - 2/[(1 + \lambda \Delta)^{1/2} + 1])$.

For the subgraphs distribution, after the reduction $\Delta = 3$, and $\lambda \leq 3 \max_i \mu(i)^{-2}$. It can be shown this makes $\kappa$ less than $3.06 \max_i \mu(i)^{-1}$.

Given the relationship between the partition function for subgraphs and matchings given by Lemma 4, and the well-known relationship between the subgraphs world partition function and the Ising partition function (see [11]), this gives a fpras for the partition function of the Ising model for magnetization bounded away from 0.

8 Appendix

There are several ways to deal with degree 2 nodes. The simplest is to “clone” the node by replacing it with two nodes connected by an edge of weight 1, as shown in Figure 5.

**Lemma 5.** Let $G = (V, E)$ be a graph where node $i$ has degree 2. Let $\{i, j\}$ and $\{i, k\}$ be the edges adjacent to $i$ and suppose the parameters for $\pi_{\text{subs}}$ are given by $\lambda$ and $\mu$. Construct $G' = (V', E')$ by letting $V' = (V \setminus \{i\}) \cup \{i', i''\}$ and $E' = (E \setminus \{\{i, j\}, \{i, k\}\}) \cup \{\{i', j\}, \{i'', j\}, \{i', k\}, \{i'', k\}, \{i', i''\}\}$. 

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Set $\lambda_1$, $\lambda_2$, and $\mu_1$ in $[0, 1]$ to satisfy

\[
\frac{2\lambda_1}{1 + \lambda_1^2} = \lambda(\{i, j\}), \quad \frac{2\lambda_2}{1 + \lambda_2^2} = \lambda(\{i, k\}), \quad \frac{2\mu_1}{1 + \mu_1^2} = \mu(i). \quad (10)
\]

Let $X'({i', j}) = X'({i'', j}) = \lambda_1$, $X'({i', k}) = X'({i'', k}) = \lambda_2$, $\mu'(i') = \mu'(i'') = \mu_1$, and $X'(\{i', i''\}) = 1$. For all other edges and nodes, let $X'$ and $\mu'$ equal $\lambda$ and $\mu$ respectively.

Draw $X' \sim \pi_{subs}$ using $(\lambda', \mu')$. Let $X(\{i, j\}) = [X'(\{i', j\}) + X'(\{i'', j\})] \mod 2$, and $X(\{i, k\}) = [X'(\{i', k\}) + X'(\{i'', k\})] \mod 2$. Then $X \sim \pi_{subs}$ using $(\lambda, \mu)$.

**Proof.** Note that the equations given in (10) can be easily solved for $\lambda_1$, $\lambda_2$ and $\mu_1$ using the quadratic formula. For an equation of the form $2x/(1 + x^2) = y$, the left hand side is 0 at $x = 0$, is 1 at $x = 1$, and is continuous. Hence for $y \in [0, 1]$, there is a solution $x \in [0, 1]$.

Now to show that the output $X$ of the procedure has the correct probability. Consider that if $X(\{i, j\}) = 1$, then either $X'(\{i', j\}) = 1$ or $X'(\{i'', j\}) = 1$ but not both. In either of the two cases, a factor of $\lambda_1$ is contributed, making a total contribution of $2\lambda_1$. A similar result holds when $X(\{i, k\}) = 1$.

When $X(\{i, j\}) = 0$, either both $\{i', j\}$ and $\{i'', j\}$ are 0 in $X'$ or both are 1 in $X'$. The total weight contribution is therefore $1 + \lambda_1^2$.

Now consider what happens with $\{i', i''\}$. In the case that $X(\{i, j\}) = X(\{i, k\}) = 1$, one choice of $X'(\{i', i''\})$ leads to both $i'$ and $i''$ being odd, and the other choice leads to both $i'$ and $i''$ being even. This makes the total weight of these contributions $1 + \mu_1^2$.

This situation also arises when $X(\{i, j\}) = X(\{i, k\}) = 0$. On the other hand, when $X(\{i, j\}) \neq X(\{i, k\})$, one choice of $X'(\{i', i''\})$ will make one of $\{i', i''\}$ even and the other odd, while the other choice flips the parity of both $i'$ and $i''$. Hence the total contribution to weight is $\mu_1 + \mu_1$.

Let the function $f$ be the transformation that takes $x'$ and constructs a state $x$. That is, $f(x')(e) = x'(e)$ for all $e \in E \setminus \{\{i, j\}, \{i, k\}\}$, $f(x')(\{i, j\}) = (x'('i', j') + x'(\{i'', j\})) \mod 2$, $f(x'(\{i', j\})) = (x'('i'', j'')) \mod 2$, $f(x'(\{i', k\})) = (x'(\{i'', k\})) \mod 2$, and $f(x'(\{i'', k\})) = (x'('i', k')) \mod 2$. Figure 5: New graph for degree 2 nodes.
and \(f(x') \langle \{i, k\} \rangle = (x'(\{i', k\}) + x'(\{i'', k\})) \text{ mod 2}\). In the table below, factors from edges and nodes that appear in both \(x'(E)\) and \(f(x')(E)\) are the same, and so are neglected.

| \(x(\{i, j\})\) | \(x(\{i, k\})\) | \(\sum_{x': f(x') = x} w_{\text{subs}}(x')\) | \(w_{\text{subs}}(f(x'))\) |
|-------------------|-------------------|-------------------|-------------------|
| 0                 | 0                 | \((1 + \lambda_1^2)(1 + \lambda_2^2)(1 + \mu_1^2)\) | 1                 |
| 0                 | 1                 | \((1 + \lambda_1^2)(\lambda_2 + \lambda_1^2)(\mu_1 + \mu_1)\) | \(\lambda(\{i, k\})\mu(i)\) |
| 1                 | 0                 | \((\lambda_1 + \lambda_1)(1 + \lambda_2^2)(\mu_1 + \mu_1)\) | \(\lambda(\{i, j\})\mu(i)\) |
| 1                 | 1                 | \((\lambda_1 + \lambda_1)(\lambda_2 + \lambda_2)(1 + \mu_1^2)\) | \(\lambda(\{i, j\})\lambda(\{i, k\})\) |

By the way \(\lambda_1, \lambda_2\) and \(\mu_1\) are defined, when \(C = (1 + \lambda_1^2)(1 + \lambda_2^2)(1 + \mu_1^2)\), the equation

\[
\sum_{x': f(x') = x} w_{\text{subs}}(x') = Cw_{\text{subs}}(x)
\]

holds. As in the proof of Theorem 2 this implies

\[
\mathbb{P}(X = x) = \sum_{x': f(x') = x} \mathbb{P}(X' = x') = Cw_{\text{subs}}(x)/Z,
\]

and hence \(X\) has the desired distribution. \(\square\)

Note that after this reduction, the degree of \(i'\) and \(i''\) is 3, while the degree of \(j\) and \(k\) is increased by 1. After raising the degree of the other nodes, they can be split apart as in Section 3 if needed.

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