Exact Discretization of the $L_2$-Norm with Negative Weight

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This paper is concerned with the problem of the exact weighted Marcinkiewicz-type discretization, which was posed in the survey [1].

Given a compact set $\Omega \subset \mathbb{R}^d$, let $\mu$ be a finite measure on $\Omega$, and let $X_N$ be an $N$-dimensional subspace of the real space $L_2(\Omega, \mu)$ of functions defined at each point. In other words, $X_N = \langle f_1, \ldots, f_N \rangle$ (here and in what follows, angle brackets denote linear span), where the functions $f_1, \ldots, f_N \in L_2(\Omega, \mu)$ are linearly independent and defined everywhere on $\Omega$.

We say that $X_N$ admits an exact weighted Marcinkiewicz-type discretization theorem with parameters $m$ and 2 if there exists a set $\{\xi^j\}_{j=1}^m \subset \Omega$ of points and a set $\{\lambda_j\}_{j=1}^m$ of weights such that

$$
\int_{\Omega} f^2 \, d\mu = \sum_{j=1}^{m} \lambda_j f^2(\xi^j)
$$

(1)

for any function $f \in X_N$. In that case, we write $X_N \in \mathcal{M}_w(m, 2, 0)$. If (1) holds with nonnegative weights, then we write $X_N \in \mathcal{M}_+w(m, 2, 0)$.

It is known that $X_N \in \mathcal{M}_w(N(N + 1)/2, 2, 0)$ (see [1, Theorem 3.1]). For an arbitrary subspace $X_N \subset L_2(\Omega, \mu)$, we define

$$
m(X_N, w) := \min\{m : X_N \in \mathcal{M}_w(m, 2, 0)\}.
$$

In [1], the following conjecture was stated.

**Conjecture** [1, Open problem 3]. If a number $m = m(X_N, w)$, points $\{\xi^j\}_{j=1}^m \subset \Omega$, and weights $\{\lambda_j\}_{j=1}^m$ satisfy (1) for any $f \in X_N$, then $\lambda_j > 0$, $j = 1, \ldots, m$.

In what follows (in Example 1), we show that this Conjecture is false. Note that, in [1], a probability space $(\Omega, \mu)$ was considered. We can always reduce the problem to this case by replacing $\mu$ by $\mu/\mu(\Omega)$ and $\lambda_j$ by $\lambda_j/\mu(\Omega)$ for $j = 1, \ldots, m$ (such a replacement does not affect relation (1)).

Let $\Omega = [-1, 1]$, and let $\mu$ be the standard Lebesgue measure. Suppose that $a > 0$, $A > B > 0$, and

$$
A^2 > 2a^2 + B^2.
$$

(2)

For example, we can take $A = 3$, $B = 2$, and $a = 1$.

Consider the functions $f_1$ and $f_2$ defined by

$$
f_1(x) = \begin{cases} 0 & \text{if } -1 \leq x < 0, \\ 1 & \text{if } 0 \leq x \leq 1, \end{cases}
$$

$$
f_2(x) = \begin{cases} a & \text{if } -1 \leq x < 0, \\ A & \text{if } 0 \leq x < 1/4, \\ -A & \text{if } 1/4 \leq x < 1/2, \\ B & \text{if } 1/2 \leq x < 3/4, \\ -B & \text{if } 3/4 \leq x \leq 1. \end{cases}
$$

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We have
\[ \int_{\Omega} f_1^2 \, d\mu = 1, \quad \int_{\Omega} f_2^2 \, d\mu = a^2 + \frac{A^2}{2} + \frac{B^2}{2}. \] (3)

Note that
\[ \int_{\Omega} f_1 f_2 \, d\mu = 0. \] (4)

**Theorem** (Example 1). The subspace \( X_2 = \langle f_1, f_2 \rangle \subset L_2(\Omega) \) possesses the following properties:

(a) \( m(X_2, w) = 3 \);

(b) \( X_2 \in \mathcal{M}_w^w(3, 2, 0) \);

(c) for \( \xi_1 = -1/2, \xi_2 = 1/8, \xi_3 = 3/8, \lambda_1 = (2a^2 - A^2 + B^2)/(2a^2) < 0 \), and \( \lambda_2 = \lambda_3 = 1/2 \), any function \( f \in X_2 \) satisfies (1).

**Remark 1.** It follows from parts (a) and (c) that the Conjecture stated above is false.

**Proof of the theorem.** Any function \( f \) in \( X_2 \) has the form \( f = \alpha f_1 + \beta f_2, \alpha, \beta \in \mathbb{R} \). It follows from this and relation (4) that the equality
\[ \int_{\Omega} f^2 \, d\mu = \sum_{j=1}^{k} \lambda_j f_j^2(\xi_j) \] (5)
holds for some number \( k \in \mathbb{N} \) and all functions \( f \in X_2 \) if and only if (5) holds for \( f_1 \) and \( f_2 \) and
\[ \sum_{j=1}^{k} \lambda_j f_1(\xi_j) f_2(\xi_j) = 0. \] (6)

Clearly, \( X_2 \notin \mathcal{M}_w^w(1, 2, 0) \); indeed, otherwise, there exist \( \xi_1 \in \Omega \) and \( \lambda_1 \in \mathbb{R} \) such that (5) (with \( k = 1 \)) holds for all \( f \in X_2 \), in particular, for \( f_1 \). It follows from the definition of \( f_1 \) and relation (3) that, in this case, \( f_1(\xi_1) = 1 \) and \( \lambda_1 = 1 \). Therefore, (6) implies \( f_2(\xi_1) = 0 \), which contradicts (5) for \( f_2 \).

Suppose that \( X_2 \in \mathcal{M}_w^w(2, 2, 0) \) and \( \xi_1, \xi_2 \in \Omega \) and \( \lambda_1, \lambda_2 \in \mathbb{R} \) satisfy (5) with \( k = 2 \) for all \( f \in X_2 \). Consider the possible cases.

If \( f_1(\xi_1) = 0 \), then (6) implies \( \lambda_2 f_1(\xi_2) f_2(\xi_2) = 0 \). If, in addition, \( f_1(\xi_2) = 0 \), then \( f_1 \) does not satisfy (5), and hence, in this case, we have \( f_1(\xi_2) = 1 \) and \( f_2(\xi_2) \neq 0 \) (by the construction of \( f_2 \)); therefore, \( \lambda_2 = 0 \) and \( X_2 \notin \mathcal{M}_w^w(1, 2, 0) \), which is false.

Thus, \( f_1(\xi_1) = 1 \). Similarly, \( f_1(\xi_2) = 1 \). Therefore, for \( f_1 \), relation (5) has the form
\[ \lambda_1 + \lambda_2 = 1. \] (7)

It follows from (6) that
\[ \lambda_1 f_1(\xi_1) + \lambda_2 f_2(\xi_2) = 0. \] (8)

If \( f_2(\xi_1) = f_2(\xi_2) \), then (8) implies \( \lambda_1 + \lambda_2 = 0 \), which contradicts (7). If \( f_2(\xi_1) = -f_2(\xi_2) \), then (7) and (8) imply \( \lambda_1 = \lambda_2 = 1/2 \). Therefore, by virtue of condition (2), relation (5) does not hold for the function \( f_2 \). It remains to consider the case where \( |f_2(\xi_1)| \neq |f_2(\xi_2)| \). We assume without loss of generality that \( |f_2(\xi_1)| > |f_2(\xi_2)| \).

In this case, relation (8) implies \( |\lambda_1| < |\lambda_2| \). Equation (5) for \( f_2 \) and relations (3) and (7) give
\[ \lambda_1 A^2 + (1 - \lambda_1) B^2 = a^2 + \frac{A^2}{2} + \frac{B^2}{2} > \frac{A^2}{2} + \frac{B^2}{2}. \]
Since the left-hand side is monotone with respect to \( \lambda_1 \), it follows that \( \lambda_1 > 1/2 \). Thus, (7) implies \( |\lambda_1| > |\lambda_2| \). This contradiction shows that \( m(X_2, w) > 2 \).
To prove (c), it suffices to substitute the parameter values specified above into (1), having represented \( f \) as a linear combination of \( f_1 \) and \( f_2 \).

To prove (b), we take \( \xi^1 = -1/2, \xi^2 = 5/8, \xi^3 = 7/8 \), \( \lambda_1 = (2a^2 + A^2 - B^2)/(2a^2) > 0, \) and \( \lambda_2 = \lambda_3 = 1/2 \), and proceed as in the proof of (c). Assertion (a) follows from the already proved inequality \( m(X_2, w) > 2 \) and assertion (b).

Example 1 is interesting in that, for the minimal number \( m \) of points needed to discretize the \( L^p \)-norm of functions in the subspace \( X \), there exists a representation of the form (1) (with \( m = m(X, w) \)) both with all weights positive and with one negative and two positive weights. This gives rise to the natural question of whether there exists a subspace \( X \) where \( m = m(X, w) \) consists of continuous functions? A positive answer to both these questions is given in Example 2 below.

Functions in Example 1 are piecewise constant; can a similar example be constructed when the subspace is contained in the interval \( \mathbb{Q}_{X} \)?

Note that functions of the same level are obtained from one another by shifting the argument. The support of each function (that is, the set of those points at which the function does not vanish) \( h_i \), \( i = 2, \ldots, 7 \), is contained in the interval \([1/8, 1]\), at every point of which the function \( h_0 \) takes the value 1.

For lucidity, we show the functions \( h_0, h_1, h_2, \) and \( h_3 \) in Fig. 1.

We also show the functions \( h_2, h_4, \) and \( h_5 \) on the interval \([0, 1/2]\) (see Fig. 2). The functions \( h_3, h_6, \) and \( h_7 \) are defined in the same way on the interval \([1/2, 1]\).

\[ g_{[a, b]}(x) = \begin{cases} 
(x - a)/l & \text{if } a \leq x < a + l, \\
1 & \text{if } a + l \leq x < a + 3l, \\
(-x + a + 4l)/l & \text{if } a + 3l \leq x < a + 5l, \\
-1 & \text{if } a + 5l \leq x < a + 7l, \\
(x - b)/l & \text{if } a + 7l \leq x \leq b, \\
0 & \text{otherwise.}
\end{cases} \]
Theorem (Example 2). The subspace \( X_8 = \{h_0, h_1, \ldots, h_7\} \subset L_2(\Omega) \) has the following properties:

(a) \( m(X_8, w) = 9 \),

(b) \( X_8 \in \mathcal{M}^w(9, 2, 0) \setminus \mathcal{M}^w_+(9, 2, 0) \).

Proof. The functions \( h_1, \ldots, h_7 \) have the following properties: the supports of functions of the same level are disjoint, and each function is constant on the support of every function of higher level. This readily implies that the functions \( h_0, h_1, \ldots, h_7 \) are pairwise orthogonal. Therefore, relation (5) with some \( k \in \mathbb{N} \) holds for all \( f \in X_8 \) if and only if the following two conditions are satisfied:

\[
\int_{\Omega} h_i^2(x) \, d\mu = \sum_{j=1}^k \lambda_j h_j^2(\xi^j) \tag{9}
\]

for \( i = 0, \ldots, 7 \) and

\[
\sum_{j=1}^k \lambda_j h_i(\xi^j) h_s(\xi^j) = 0 \tag{10}
\]

for \( h_i \) and \( h_s \), \( i \neq s \), \( i, s \in \{0, \ldots, 7\} \). Suppose that, for some \( k \in \mathbb{N} \), relation (5) with \( \lambda_j \neq 0 \), \( j = 1, \ldots, k \), holds. Since the continuous functions \( h_4, h_5, h_6, \) and \( h_7 \) are nonzero and satisfy (9), it follows that the support of each of these functions contains at least one point from the set \( \{\xi^j\}_{j=1}^k \).

Note that if the support of \( h_s \) contains precisely one such point for some number \( s \in \{4, 5, 6, 7\} \), then relation (10) does not hold for \( h_0 \) and \( h_s \). Therefore, the support of each of the functions \( h_4, h_5, h_6, \) and \( h_7 \) must contain at least two points from the set \( \{\xi^j\}_{j=1}^k \). Since these supports are disjoint, it follows that \( m(X_8, w) \geq 8 \).

Suppose that \( m(X_8, w) = 8 \). Then it follows from the considerations of the previous paragraph that the points \( \xi^j, j = 1, \ldots, 8 \), lie on the positive semi-axis and that \( h_0(\xi^j) = 1 \) and \(|h_1(\xi^j)| = 1 \) for \( j = 1, \ldots, 8 \).

The definition of the function \( h_0 \) and the fact that it satisfies (9) imply

\[
\sum_{j=1}^8 \lambda_j = \int_{\Omega} h_0^2(x) \, d\mu = 24 \int_0^{1/16} x \, d\mu + \int_{1/16}^{1/8} (-8x + 2) \, d\mu + \frac{7}{8} = 1.
\]

Moreover,

\[
\int_0^1 g_{[0,1]}^2(x) \, d\mu = 64 \int_0^{1/8} x^2 \, d\mu + \frac{1}{4} \int_{1/8}^{5/8} \left(-x + \frac{1}{2}\right)^2 \, d\mu + \frac{1}{4} \int_{5/8}^{3/8} \left(-x + \frac{1}{2}\right)^2 \, d\mu + \frac{2}{3} = \frac{2}{3},
\]

whence we obtain

\[
\int_{\Omega} h_1^2(x) \, d\mu = \int_{-1}^0 \frac{x^2}{4} \, d\mu + \int_0^1 g_{[0,1]}^2(x) \, d\mu = \frac{1}{12} + \frac{2}{3} = \frac{3}{4}. \tag{11}
\]
The left-hand side of equality (9) for \( h_1 \) equals \( 3/4 \), and the right-hand side equals 1. Therefore, this equality fails to hold. Thus, \( m(X_8, w) > 8 \).

To show that \( X_8 \in \mathcal{M}_+^{i}(9, 2, 0) \), it suffices to substitute

\[
\xi^1 = \frac{-1}{2}, \quad \xi^2 = \frac{37}{256}, \quad \xi^3 = \frac{39}{256}, \quad \xi^4 = \frac{53}{256}, \quad \xi^5 = \frac{55}{256},
\]

\[
\xi^6 = \frac{165}{256}, \quad \xi^7 = \frac{167}{256}, \quad \xi^8 = \frac{181}{256}, \quad \xi^9 = \frac{183}{256},
\]

(\( \xi^2, \ldots, \xi^9 \) are the midpoints of the intervals on which the functions \( h_4, h_5, h_6, \) and \( h_7 \) take the values \( \pm 1 \), \( \lambda_i = -4, \) and \( \lambda_i = 1/8, \) \( i = 2, \ldots, 9, \) into (9) and into (10) with \( k = 9 \). It follows from the construction of the functions \( h_0, h_1, \ldots, h_7 \) and the equalities \( \lambda_i = \lambda_j, i, j \in \{2, \ldots, 9\} \), that, in fact, it suffices to check that (9) holds for \( h_0, h_1, h_2, \) and \( h_4 \). This is verified by direct substitution. The remaining equalities are obvious. Thus, we have proved property (a).

Let us prove (b). Suppose that there exist points \( \xi^j \) and positive weights \( \lambda_j \) for which equality (5) with \( k = 9 \) holds. It follows from the proof of (a) that at least eight points lie in the interval \([1/8, 1]\); moreover, they lie in the part of this interval on which the function \( h_1 \) takes the values \( \pm 1 \). If the ninth point lies in \([1/8, 3/8] \cup [5/8, 7/8]\) as well, then, since \( h_0 \) satisfies (9), it follows that the sum of weights \( \lambda_j, j = 1, \ldots, 9 \), equals 1. Thus, \( h_1 \) does not satisfy (9) (see the proof of (a)). Therefore, the interval \([1/8, 3/8] \cup [5/8, 7/8]\) contains four of the nine points \( \xi^j, j = 1, \ldots, 9 \), (for definiteness, let them be \( \xi^2, \xi^3, \xi^4, \) and \( \xi^5 \)); two of them belong to the interval \([9/64, 11/64]\) and the other two, to the interval \([13/64, 15/64]\). Hence, for \( h_2 \), relation (9) takes the form

\[
\frac{2}{3} \cdot \frac{1}{8} + 5 \cdot \frac{2}{3} = \frac{1}{2} = \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5. \tag{12}
\]

Similarly, the interval \([5/8, 7/8]\) contains four of the nine points \( \xi^j, j = 1, \ldots, 9 \); for definiteness, let them be \( \xi^6, \xi^7, \xi^8, \) and \( \xi^9 \). Since (9) holds for \( h_3 \), it follows that

\[
\frac{1}{2} = \lambda_6 + \lambda_7 + \lambda_8 + \lambda_9. \tag{13}
\]

Substituting (12) and (13) into equality (9) for \( h_1 \), we obtain (see also (11)) \( 3/4 = 1 + \lambda_1 h_1^2(\xi^1) \). Therefore, \( \lambda_1 < 0 \), which proves (b).

**Remark 2.** It can be shown that \( X_8 \in \mathcal{M}_+^{i}(11, 2, 0) \setminus \mathcal{M}_+^{i}(10, 2, 0) \).

In fact, the following assertion is valid.

**Theorem A** [1, Corollary 4.2]. Let \( \Omega \) be a sequentially compact topological space endowed with a Borel probability measure \( \mu \). Then \( X_N \in \mathcal{M}_+^{i}(N(N + 1)/2, 2, 0) \) for each \( N \)-dimensional real linear subspace \( X_N \) of \( L_2(\Omega, \mu) \cap C(\Omega) \).

Given \( X_N \), let

\[
m_+(X_N, w) := \min\{m : X_N \in \mathcal{M}_+^{w}(m, 2, 0)\}.
\]

The question of how many the quantities \( m(X_N, w) \) and \( m_+(X_N, w) \) can differ still remains open.

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