Elements of Automorphic Representations

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This is an attempt at a practical and essentially self-contained theory of automorphic representations in the framework

$L^2(\Gamma \backslash G)$ with $G = \text{PSL}(2, \mathbb{R})$ and $\Gamma = \text{PSL}(2, \mathbb{Z})$.

The restriction of the underlying discrete subgroup $\Gamma$ is imposed solely for the sake of simplicity; our argument should extend to fairly general arithmetic subgroups of $G$ without substantial alteration. Our motivation lies in the observation that applying the spectral theory of cusp forms on $G$ to problems in analytic number theory we need a lowbrow but in fact highly informative approach, as then it is of paramount importance to be able to work in regions of absolute and uniform convergence. We need to have explicit descriptions, including the convergence issues, of integral transforms arising in representation theory; and in order to acquire various means indispensable in dealing with relevant involved technicalities, it is best to trace from scratch how those transforms come into play. Essential in applications are not only the spectral structures but also the procedure to establish them. Therefore we shall take an approach based on explicit computation rather than the common dose of the theory of commuting compact operators which we are, nevertheless, aware is susceptible of generalisations to bigger Lie groups.

With this, our attention is specifically directed to an accessible explication of the Kirillov map and the Bessel functions of representation. Both concepts have played fundamental roles in recent applications of representation theory of Lie groups to problems in analytic number theory such as the spectral theory of sums of Kloosterman sums and the theory of mean values of the Riemann zeta and a variety of $L$-functions. It is worth stressing that the massive cancelations which take place in these subjects have been detected only with analysis of integral transforms indicated above; indeed such a detection appears to be beyond the reach of the sole use of algebra–operator theoretic means.

We shall naturally use terms from the theory of Lie groups and algebras; but there is no need to know the entire theory in order to understand this article. For those who share interest and taste with us, the first three introductory chapters of Vilenkin–Klimyk [36] are recommended to have an overview of the theory, although it is of no absolute necessity either. What we are developing below may in fact be regarded as the material that should be basic in order to enter into the theory of Lie groups and their representations in much the same sense as elementary number theory is meant for analytic number theory. However, to restrict the text within a reasonable size, we assume a familiarity with the spectral theory of real analytic cusp forms on the upper half-plane, i.e., the weight zero situation, a fairly elementary account of which is developed in [22, Chapter 1]. Salient points of the situation with even integral weights are given in later sections of the present article. Also, as an appendix, Selberg’s trace formula and zeta-function associated with the group $\Gamma$ are treated, although the subject is not directly related to our principal aim; a novelty is mainly in a relatively fast approach to the quintessence of Selberg’s theory. Notations and conventions are introduced along with necessity, and will continue to be effective thereafter. References are limited to those immediately related to our purpose.

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1. Hyperbolic plane. The group \( SL(2, \mathbb{R}) \) acts on \( \mathbb{C} \cup \{ \infty \} \):

\[
    h(z) = \frac{az + b}{cz + d}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{R}). \tag{1.1}
\]

We have, with \( j(h, z) = cz + d \),

\[
    \frac{d}{dz} h(z) = \frac{1}{j(h, z)^2}, \quad \text{Im } h(z) = \frac{y}{|j(h, z)|^2}, \quad h(z_1) - h(z_2) = \frac{z_1 - z_2}{j(h, z_1)j(h, z_2)}, \quad j(h_1, h_2, z) = j(h_1, h_2(z))j(h_2, z). \tag{1.2}
\]

In particular, \( SL(2, \mathbb{R}) \) is a group of orientation-preserving conformal transformations of the upper half plane

\[
    \mathbb{H}^2 = \{ z = x + iy : x \in \mathbb{R}, y > 0 \}. \tag{1.3}
\]

This is, however, the same as dealing with the group \( G = SL(2, \mathbb{R})/\{ \pm 1 \} \), and we shall use the notation

\[
    h = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in G \tag{1.4}
\]

in place of (1.1); see the notes below. On \( \mathbb{H}^2 \) we have the hyperbolic metric

\[
    \frac{|dz|}{y} = \frac{(dx)^2 + (dy)^2}{y} = \frac{(du)^2 + (dv)^2}{v} = \frac{|dw|}{v}, \tag{1.5}
\]

with \( h(z) = w, \quad z = x + iy, \quad w = u + iv; \) the invariance follows from the first line of (1.2). This induces the hyperbolic measure

\[
    d\mu(z) = \frac{dxdy}{y^2} = \frac{dudv}{v^2} = d\mu(w). \tag{1.6}
\]

Also induced are the hyperbolic outer-normal differential and the Laplace–Beltrami operator: We have, for any smooth function \( f \) on \( \mathbb{H}^2 \),

\[
    y \left( \frac{dy}{|dz|} \frac{\partial}{\partial x} - \frac{dx}{|dz|} \frac{\partial}{\partial y} \right) f(w) = v \left( \frac{dv}{|dw|} \frac{\partial}{\partial u} - \frac{du}{|dw|} \frac{\partial}{\partial v} \right) f(w), \tag{1.7}
\]

\[
    y^2 \left( \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2 \right) f(w) = v^2 \left( \left( \frac{\partial}{\partial u} \right)^2 + \left( \frac{\partial}{\partial v} \right)^2 \right) f(w), \tag{1.8}
\]

where the differentiation on the left sides are performed on the function \( f(w) = f(h(z)) \) of the variable \( z \). The invariance (1.6)–(1.8) is a consequence of the Cauchy–Riemann equation for the function \( h(z) \). They are applied together with Green’s formula in an obvious hyperbolic disguise: We have, for any smooth functions \( f, g \) on \( \mathbb{H}^2 \),

\[
    \int_D y^2 \left( \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2 \right) f \cdot g \, d\mu(z) + \int_D y^2 \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} \right) \, d\mu(z), \tag{1.9}
\]

\[
    = \int_{\partial D} y \left( \frac{dy}{|dz|} \frac{\partial}{\partial x} - \frac{dx}{|dz|} \frac{\partial}{\partial y} \right) f \cdot g \left| \frac{dz}{y} \right|. \tag{1.9}
\]
where $D$ is a domain in $\mathbb{H}^2$ enclosed by a piece-wise smooth boundary curve $\partial D$ which is positively oriented.

Further, we have the fact that the action of the subgroup $\Gamma$ is discrete and induces the tessellation
\[ \mathbb{H}^2 = \bigcup_{\gamma \in \Gamma} \gamma \mathcal{F}, \text{ disjoint save for boundaries } \gamma \partial \mathcal{F}, \] (1.10)
with
\[ \mathcal{F} = \left\{ z \in \mathbb{H}^2 : |\text{Re } z| \leq \frac{1}{2}, |z| \geq 1 \right\}. \] (1.11)
A function $f$ is said to be $\Gamma$-automorphic if $f(\gamma(z)) = f(z)$, $\forall (z, \gamma) \in \mathbb{H}^2 \times \Gamma$, that is, $f$ is a function on the Riemann surface $\mathbb{H}^2 \setminus \mathcal{F}$. We then introduce the Hilbert space
\[ L^2(\Gamma \setminus \mathbb{H}^2) = \left\{ \text{$\Gamma$-automorphic and square integrable over } \mathcal{F} \text{ against } d\mu \right\}. \] (1.12)

The relations (1.5)–(1.9) are basic implements in developing the harmonic analysis on $\Gamma \setminus \mathbb{H}^2$, i.e., a spectral resolution of the Laplace–Beltrami operator which is symmetric on $L^2(\Gamma \setminus \mathbb{H}^2)$ as the invariance structure of (1.9) implies. We shall extend them to the Lie group $G$ that spreads over $\mathbb{H}^2$: Anticipating concepts to be introduced in due course, we assert that (1.5) corresponds to point-pair invariants, (1.6) to a Haar measure, (1.7) to the Maass operators, (1.8) to the Casimir operator; and (1.9) is an instance of the effect of the decomposition of the Casimir operator in terms of the Maass operators, although the details of these correspondences are given only in Section 32, as they are irrelevant to our main task to develop elements of automorphic representations. The invariance of these notions on $G$ are in the core of the differentiable structure of $G$, and plays a central rôle in the development of the harmonic analysis on $\Gamma \setminus G$, i.e., a spectral resolution of the Casimir operator that is symmetric on the Hilbert space $L^2(\Gamma \setminus G)$ spreading over $L^2(\Gamma \setminus \mathbb{H}^2)$.

**Notes:** The notation (1.4) causes a minor confusion: The function $\gamma(h, z)$ can equal either $cz + d$ or $-(cz + d)$. Nevertheless, within the present article this ambiguity should not cause any trouble, since the relevant instances are all consequences of the basic relations (3.3)–(3.4) below. See the notes to the third section.

**2. Hyperbolic distance.** The distance $d(z_1, z_2)$ between $z_1$ and $z_2$ on $\mathbb{H}^2$ is defined to be the minimum of lengths of curves connecting the points, measured against the hyperbolic line element $|dz|/y$. It holds that $d(h(z_1), h(z_2)) = d(z_1, z_2)$ for any $h \in G$, as is implied by the invariance (1.5). We have
\[ d(z_1, z_2) = 2 \arcsinh \sqrt{\varrho(z_1, z_2)}, \quad \varrho(z_1, z_2) = \frac{|z_1 - z_2|^2}{4(\text{Im } z_1)(\text{Im } z_2)}, \] (2.1)
in which the second line follows from the second and the third identities in (1.2). In fact, mapping $i$ to $z_2 = x_2 + iy_2$, we have
\[ d(z_1, z_2) = d(h_0^{-1}(z_1), i) = d((z_1 - x_2)/y_2, i), \quad h_0 = \left[ \begin{array}{cc} \sqrt{y_2} & x_2/\sqrt{y_2} \\ 1/\sqrt{y_2} & 1 \end{array} \right] \in G. \] (2.2)

Then, via
\[ z \mapsto w = c(z), \quad c = \left( \begin{array}{cc} 1 & -i \\ 1 & i \end{array} \right), \] (2.3)
we map \( \mathbb{H}^2 \) onto the unit disk \(|w| < 1\), and apply a rotation \( w \mapsto \exp(i\tau)w \) so that the point \((z_1 - x_2)/y_2\) is mapped to \(|z_1 - z_2|/|z_1 - z_2|\). With the inverse of (2.3) we return to \( \mathbb{H}^2 \), finding
\[
d(z_1, z_2) = d(r(z_1, z_2)i, i) = \log(r(z_1, z_2)),
\] (2.4)
where
\[
r(z_1, z_2) = \frac{|z_1 - z_2| + |z_1 - z_2|}{|z_1 - z_2|} = (\rho(z_1, z_2)^{1/2} + (\rho(z_1, z_2) + 1)^{1/2})^2.
\] (2.5)
We get (2.1).

Notes: Elements of the hyperbolic geometry is given in, e.g., Maass [20]. An interesting historical account can be found in Penrose [28, Section 2.6].

3. Iwasawa decomposition. Ascension to \( G \) starts. Classifying results of the action of \( G \) on the point \( i \), we obtain a co-ordinate system on \( G \):
\[ G = NAK \ni g = n[x]a[y]k[\theta], \] (3.1)
with
\[
N = \left\{ n[x] = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\}, \quad A = \left\{ a[y] = \begin{bmatrix} \sqrt{y} & 1/\sqrt{y} \\ \sqrt{y} & \sqrt{y} \end{bmatrix} : y > 0 \right\},
\]
\[
K = \left\{ k[\theta] = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} : \theta \in \mathbb{R}/\pi\mathbb{Z} \right\},
\]
\[
n[x]a[y]k[\theta] = \begin{bmatrix} \sqrt{y}\cos \theta - x\sin \theta/\sqrt{y} & \sqrt{y}\sin \theta + x\cos \theta/\sqrt{y} \\ -\sin \theta/\sqrt{y} & \cos \theta/\sqrt{y} \end{bmatrix}.
\] (3.2)
The relations
\[
g = n[x]a[y]k[\theta] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad x = \frac{ac + bd}{c^2 + d^2}, \quad y = (c^2 + d^2)^{-1}, \quad \exp(2i\theta) = \frac{\lambda(g, -i)}{\lambda(g, i)},
\] (3.3)
implicating (3.1) with (3.2) is indeed a co-ordinate system on \( G \). We have, for \( h \in G \),
\[
h \cdot n[x]a[y]k[\theta] = n[x_1]a[y_1]k[\theta_1],
\]
\[
h(x + iy) = x_1 + iy_1, \quad \exp(2i\theta_1) = \exp(2i\theta) \frac{\lambda(h, x - iy)}{\lambda(h, x + iy)},
\] (3.4)
in which the second identity is the result of applying the first to the point \( i \) and the third follows from the last identity in (1.2) and that in (3.3). Thus the action of \( G \) on \( \mathbb{H}^2 \) is equivalent to that on the set of all right \( K \)-cosets of \( G \); that is,
\[ \mathbb{H}^2 \cong G/K, \] (3.5)
with respect to the action of \( G \). We note that under our formulation \( G \) acts on \( G/K \) through the left multiplication or translation
\[ l_h : g \mapsto hg. \] (3.6)

Notes: Hereafter it is understood that the variables \((x, y, \theta)\) are the co-ordinates (3.1) unless otherwise stated. Alternatively, one may ascend from \( \mathbb{H}^2 \) to \( \text{SL}(2, \mathbb{R}) \). It does not
entail the ambiguity mentioned in the notes to the first section. On the other hand, we would encounter side effects as well, specifically in dealing with assertions involving the notion of weights to be introduced in Section 12. Our choice of \(G\) means that we restrict our discussion to even functions on \(SL(2, \mathbb{R})\), i.e., sums of functions of even integral weights on \(G\); this suffices for basic applications of automorphic representations to problems in analytic number theory.

4. Cartan decomposition. This notion is closely related to point-pair invariants on \(G\), that is, any function \(F(g_1, g_2)\) on \(G \times G\) such that \(F(hg_1, hg_2) = F(g_1, g_2), \forall h \in G\), which is in fact a function of \(g_1^{-1}g_2\). We shall see that such a function is a natural extension of the hyperbolic distance. Thus, along with (3.1), we have the decomposition

\[
G = KAK \ni g = k[\tau_1]a[u]k[\tau_2], \text{ uniquely provided } u > 1. \tag{4.1}
\]

We note that

\[
g^{-1} = k[-\tau_2]wa[u]w^{-1}k[-\tau_1], \quad w = k[\frac{i}{2} \pi] : \text{ the Weyl element of } G. \tag{4.2}
\]

To show (4.1), we consider \(g \cdot g^t\) which is symmetric and positive. We have \((g \cdot g^t)^{-1/2}g \in K\), and get the decomposition. As to the uniqueness, we assume that \(k[\tau_1]a[u_1]k[\tau_2] = a[u_2]\). Applying it to the point \(i\), we have

\[
\frac{(1 - u_1^2) \sin(\tau_1) \cos(\tau_1) + u_1 i}{u_1^2 \sin^2(\tau_1) + \cos^2(\tau_1)} = u_2 i, \tag{4.3}
\]

which implies \(\sin(2\tau_1) = 0\) as we assume \(u_1 > 1\). If \(\tau_1 \equiv 0 \mod \pi \mathbb{Z}\), then \(u_1 = u_2\), and \(\tau_2 \equiv 0 \mod \pi \mathbb{Z}\). On the other hand, if \(\tau_1 \equiv \frac{1}{2} \pi \mod \pi \mathbb{Z}\), then \(u_1^{-1} = u_2\), which is excluded.

For \(g = u[x]a[y]k[\theta]\) as in (4.1), we have \(u = \exp(d(z, i)), \quad z = x + iy\). In fact, we have \(d(z, i) = d(g(i), i) = d(k[\tau_1]a[u]k[\tau_2](i), i) = d(a[u](i), i) = \log u\) as we may assume that \(u \geq 1\). Also, considering \(c(z) = cg^{-1}(0)\), we get

\[
\frac{z - i}{z + i} = e^{2ir\tau_1} \left| \frac{z - i}{z + i} \right|. \tag{4.4}
\]

More generally, on noting (1.2), (3.3) and \(d(g_1^{-1}g_2(i), i) = d(g_1(i), g_2(i))\), we derive from (4.4) that

\[
g_1^{-1}g_2 = k[\eta_1]a[v]k[\eta_2] \quad \text{with } v = \exp(d(z_1, z_2)) \quad \text{and} \quad \frac{z_2 - z_1}{z_2 - \overline{z_1}} = e^{2i(\theta_1 + \eta_1)} \frac{z_2 - z_1}{z_2 - \overline{z_1}}, \tag{4.5}
\]

\[
\frac{z_1 - z_2}{z_1 - \overline{z_2}} = -e^{2i(\theta_2 - \eta_2)} \frac{z_1 - z_2}{z_1 - \overline{z_2}},
\]

where \(g_j = u[x_j]a[y_j]k[\theta_j]\), \(z_j = x_j + iy_j\). To get the second identity, we write \(z_2 = g_1(g_1^{-1}g_2(i)), \quad z_1 = g_1(i)\), and apply the fourth expression of (1.2), the last of (3.3) as well as (4.4). The third identity in (4.5) follows from the second, since we have (4.2) and \(g_2^{-1}g_1 = (g_1^{-1}g_2)^{-1}\).

NOTES: The mode of decomposition (4.1) is the same as introducing a polar co-ordinate system on \(G\); see Bruggeman [4, Section 2.2.6], for instance. This is included here because its natural extension plays a salient rôle in our discussion of automorphic representations of \(PSL(2, \mathbb{C})\) which is under preparation. See also Section 33.
5. **Invariant measure on** $G$. Skipping the discussion of the notion of Haar measures in general, we put, in an a priori manner,

\[ dg = \frac{dxdyd\theta}{\pi y^2}, \text{ with Lebesgue measures } dx, dy, d\theta. \]  

\hspace{2em} (5.1)

The group $G$ is unimodular in the sense that it admits a left and right invariant Haar measure; that is, we have

\[ dg = dhg, \quad dg = dgh, \quad \forall h \in G. \]  

\hspace{2em} (5.2)

The invariance of $dg$ against the left translation is a consequence of (1.6) and (3.4). On the other hand, if we put $k[\theta]h = n[\xi(\theta)]a[u(\theta)]k[\theta(\theta)]$, then $gh = n[x_1]a[y_1]k[\theta_1]$, with $x_1 = x + \xi(\theta)y$, $y_1 = u(\theta)y$, $\theta_1 = \vartheta(\theta)$. The Jacobian of the right translation $r_h : g \mapsto gh$ (5.3) equals $u(\theta)\vartheta'(\theta)$. Applying the last two identities in (3.3) to $k[\theta]h$, $h = n[\alpha]a[\beta]k[\tau]$, we have

\[ u(\theta) = \frac{\beta}{(\cos \theta - \alpha \sin \theta)^2 + (\beta \sin \theta)^2}, \quad e^{2i\vartheta(\theta)} = e^{2i\cos \theta - \alpha \sin \theta + i\beta \sin \theta \over \cos \theta - \alpha \sin \theta - i\beta \sin \theta}, \]  

\hspace{2em} (5.4)

and find that $\vartheta'(\theta) = u(\theta)$. Thus the Jacobian is in fact equal to $u^2(\theta) = (y_1/y)^2$, which proves the second identity in (5.2).

6. **Invariant differential operators.** We are to introduce a differentiable structure on $G$. To this end, we observe that because of the relations (3.4)–(3.5) the harmonic analysis on $G$ should be an extension of that on $\mathbb{H}^2$. As suggested in the first section, the latter is, by general potential theory, to be based on the fact (1.8) that the Laplace–Beltrami operator commutes with the action of $G$ over $\mathbb{H}^2$; and this action corresponds to the left translation applied on $G/\mathbb{K}$ as already remarked after (3.5). Namely, the differentiable structure under question should be invariant against the left translation; in other words it should be defined by an implement that is independent of any of $l_h$, $h \in G$. At the same time, as is well indicated by the definition of the differentiation on the additive Lie group $\mathbb{R}$, the device ought to be constructed via infinitesimal action of $G$ on itself. In order to fulfill these prerequisites we are naturally led to exploiting right translations $r_h$.

With this, we introduce

\[ X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \]  

\hspace{2em} (6.1)

and observe that

\[ N = \{ \exp(tX_1) : t \in \mathbb{R} \}, \quad A = \{ \exp(tX_2) : t \in \mathbb{R} \}, \quad K = \{ \exp(tX_3) : t \in \mathbb{R}/\pi\mathbb{Z} \}. \]  

\hspace{2em} (6.2)

The matrices $\exp(tX_j)$ can obviously be identified as the corresponding elements (1.4) of $G$. In view of (3.1) these three one-parameter subgroups or rather curves on $G$ give rise to the Iwasawa co-ordinate system. Hence we introduce the right Lie differentials

\[ x_jf(g) = \left[ \frac{d}{dt} \right]_{t=0} f(g \cdot \exp(tX_j)), \quad f \in C^\infty(G), \]  

\hspace{2em} (6.3)
or the right differentiation at \( g \in G \) in the direction \( X_j \); here \( C^\infty(G) \) is the set of all functions which are differentiable infinitely many times with respect to \((x, y, \theta)\). As \( x_j \) is defined in terms of the right translation \( r_{exp(tX_j)} \), we have obviously

\[
l_h x_j = x_j l_h, \quad \forall h \in G.
\]

Decomposing \( g \cdot exp(tX_j) \) by (3.3), one may compute \( x_j \) in terms of \((x, y, \theta)\):

\[
\begin{align*}
x_1 &= y \cos(2\theta) \frac{\partial}{\partial x} + y \sin(2\theta) \frac{\partial}{\partial y} + \sin^2 \theta \frac{\partial}{\partial \theta}, \\
x_2 &= -2y \sin(2\theta) \frac{\partial}{\partial x} + 2y \cos(2\theta) \frac{\partial}{\partial y} + \sin(2\theta) \frac{\partial}{\partial \theta}, \\
x_3 &= \frac{\partial}{\partial \theta}.
\end{align*}
\]  

(6.5)

As an orientation, we indicate how to compute \( x_1 \); the operator \( x_2 \) is treated similarly, and \( x_3 \) does not need any explanation: Essential is to have the first order approximation \( n[x + \lambda_1 t]a[y + \lambda_2 t]k[\theta + \lambda_3 t] \rho \) for \( n[x]a[y]k[\theta] \cdot \rho[t] \), that is, the part containing \( t^2 \) or higher powers can be ignored. By the last expression in (3.3) or rather by the second formula in (5.4) for \( \alpha = t, \beta = 1, \tau = 0 \), we get readily \( \lambda_3 = \sin^2 \theta \) which gives the third term of \( x_1 \). Similarly, by the first formula in (5.4) we get \( \lambda_2 = y \sin(2\theta) \), which corresponds to the second term of \( x_1 \). To compute \( \lambda_1 \), we put \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and thus \( g n[t] = \begin{pmatrix} a & b+at \\ c & d+ct \end{pmatrix} \). By the second identity in (3.3) we get \( \lambda_1 \) in terms \( a, b, c, d \) which is in turn expressed in terms of \( x, y, \theta \) by the last expression in (3.2). This ends the computation of \( x_1 \).

The set \( \{ x_1, x_2, x_3 \} \) generates, over \( \mathbb{R} \), the Lie algebra \( \mathfrak{g} \) of \( G \) under the operation \([x_i, x_j] = x_i \cdot x_j - x_j \cdot x_i \). The Jacobi identity holds obviously, and we have

\[
[x_1, x_2] = -2x_1, \quad [x_1, x_3] = -x_2, \quad [x_2, x_3] = 4x_1 - 2x_3,
\]

(6.6)

as is implied by (6.5); in fact it suffices to compute only the coefficients of the three first order differentials. Further, in terms of ordinary operator addition and multiplication the same set generates the universal enveloping algebra \( \mathfrak{U} \) consisting of left invariant differential operators on \( G \). It should be noted that the basic field of \( \mathfrak{V} \) is naturally \( \mathbb{C} \). Namely, the Lie algebra \( \mathfrak{g} \) which is originally defined over \( \mathbb{R} \), is complexified. This remark will become relevant to the definition of the Maass operators given in the next section. Thus, hereafter it is always understood that \( \mathfrak{g} \) is a Lie algebra over \( \mathbb{C} \) generated by the operators \( \{ x_1, x_2, x_3 \} \).

Here is a trivial remark: Let \( x \in \mathfrak{g} \), and let \( x f_k \) be continuous for all \( k \). Then we have

\[
x \sum_k f_k(g) = \sum_k x f_k(g),
\]

(6.7)

provided, for instance, both sums converge uniformly. This can be confirmed on noting that \( x \) is in fact a differential with respect to a single real variable. The exchange may hold with a given \( u \in \mathfrak{U} \) in place of \( x \in \mathfrak{g} \) as well, if the relevant chain of applications of (6.7) can be performed. This procedure will be used without mentioning details.

Notes: The most essential in the discussion of Lie groups is the concept of one parameter subgroups, a fundamental discovery made by Sophus Lie in 1888. See Hawkins [10, Section 3.2]; this monograph gives an account of an early history of the theory of Lie groups. Our definition (6.3) is to be regarded as an adaptation of the general notion of Lie differentials to linear Lie groups which are composed of matrices. Since \( \det \exp(A) = \exp(\text{trace of } A) \) for any square matrix \( A \), the set \( \{ \exp(tX) : t \in \mathbb{R} \} \) is a curve on \( G \) if and only if the trace of a \( 2 \times 2 \) matrix \( X \) equals zero. The system (6.1) is a basis of the vector space spanned by those
X over \( \mathbb{R} \). The correspondence \( x \leftrightarrow X \) via the same definition as (6.3) is linear, as is easily seen; and the assertion (6.6) means that this is in fact a Lie algebra isomorphism between \( \mathfrak{g} \) and the matrix Lie algebra generated by (6.1), which is a fact that extends to general Lie groups. Needless to say, this correspondence viewed via (6.3) is applicable only prior to the complexification mentioned above.

7. Maass operators. Following Maass [20], we introduce

\[
e^+ = 2i x_1 + x_2 - ix_3 = e^{2i\theta} \left( 2iy \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} - i \frac{\partial}{\partial \theta} \right),
\]

\[
e^- = -2i x_1 + x_2 + ix_3 = e^{-2i\theta} \left( -2iy \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + i \frac{\partial}{\partial \theta} \right), \quad w = x_3 = \frac{\partial}{\partial \theta}.
\]

These three operators will play an important rôle in our later discussion. Since we have

\[
[w, e^+] = 2ie^+,
\]

\[
[w, e^-] = -2ie^-,
\]

\[
[e^+, e^-] = -4iw,
\]

as (6.6) implies, Maass operators generate \( \mathfrak{g} \) as well as \( \mathfrak{u} \). The operator \( w \) should not be confused with the Weyl element \( w \).

NOTES: The Maass operators \( e^\pm \) are extensions of the hyperbolic outer normal differential. See the notes to the next section as well as Section 32. It should be stressed again that the definition (7.1) contains in fact the complexification of the original Lie algebra \( \mathfrak{g} \) over \( \mathbb{R} \).

8. Casimir operator. Next, we fix the centre of \( \mathfrak{u} \). To this end, we introduce the Killing form on \( \mathfrak{g} \times \mathfrak{g} \):

\[
\text{Tr}( (\text{ad} x) \cdot (\text{ad} y) ),
\]

with \( (\text{ad} x)(a) = [x, a] \). Computing the coefficient matrix \( (k_{ij}) \) of (8.1) via (6.6), we see that the form is non-degenerate; that is, \( G \) is semi-simple. More explicitly,

\[
(k_{ij}) = \begin{pmatrix}
0 & 0 & -4 \\
0 & 8 & 0 \\
-4 & 0 & -8
\end{pmatrix}, \quad (k_{ij})^{-1} = \begin{pmatrix}
\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & \frac{2}{3} & 0 \\
-\frac{1}{3} & 0 & 0
\end{pmatrix}.
\]

We write the second matrix as \( (k^{ij}) \). Then \( c \sum_{ij} k^{ij} x_i x_j \), with any \( c \in \mathbb{C} \), is an element in the centre of \( \mathfrak{u} \). In this way we are led to the Casimir element of \( \mathfrak{u} \):

\[
\Omega = -x_1^2 - \frac{1}{4} x_2^2 + \frac{1}{2} x_1 x_3 + \frac{1}{2} x_3 x_1 = -\frac{1}{4} e^+ e^- + \frac{1}{4} w^2 - \frac{1}{2} w
\]

\[
= -y^2 \left( \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2 \right) + y \frac{\partial^2}{\partial x \partial \theta},
\]

with the choice \( c = -2 \). It holds that

\[
u \cdot \Omega = \Omega \cdot u, \quad \forall u \in \mathfrak{u}.
\]

In fact, one may confirm by using (6.6) that \( [\Omega, x_j] = 0 \) holds for \( j = 1, 2, 3 \); see the notes below. The expression in the middle of (8.3), which follows from the definition (7.1), will play a central rôle in what follows. The third line is an easy consequence of the second.
The Casimir operator is not only left invariant but also right invariant:

\[ l_h \Omega = \Omega l_h, \quad r_h \Omega = \Omega r_h, \quad \forall h \in G. \quad (8.5) \]

The first identity is trivial. As to the second, we note that the definition (6.3) implies that it holds, for any \( f \in C^\infty(G) \), that

\[
x_j r_h f(g) = \left[ \frac{d}{dt} \right]_{t=0} f(gh \cdot \exp(t \mathbf{X}_j)h) = \left[ \frac{d}{dt} \right]_{t=0} f(gh \cdot \exp(t h^{-1} \mathbf{X}_j h)) = r_h x^h_j f(g), \quad (8.6)
\]
say. Thus, \( \Omega_r h = r_h \Omega^h \) with \( \Omega^h \) being the result of replacing \( x_j \) by \( x^h_j \) in the first line of (8.3). On the other hand, \( \Omega^h = \Omega \), since \( \Omega \) does not depend on any base change of \( g \) as is confirmed in what follows; note that \( \{ x^h_1, x^h_2, x^h_3 \} \) is a basis, since \( h^{-1} \mathbf{X}_j h, \ j = 1, 2, 3 \), are linearly independent in the vector space of \( 2 \times 2 \) matrices of zero trace: Let \( K \) be the matrix of the Killing form with respect to the original base, and \( B \) a base change matrix. Then \( BK B^\dagger \) and thus \( -2(BB^\dagger)^{-1}K^{-1}B^{-1} \) correspond, respectively, to the Killing form and the Casimir element on the new base. This obviously yields the assertion.

**Notes:** Maass defined his operators in a more general fashion than (7.1); see [20, Chapter 4, (12)–(13)]. Roelcke [31] discussed them very thoroughly. Maass’ basic motivation seems to be a fortuitous situation special to the group \( \text{PSL}(2, \mathbb{R}) \). It may be worth remarking that Maass [20] does not exploit the fact that \( G \) is a Lie group; thus his approach is different from ours. The commutativity of the Casimir operator with right translations depends solely on the unimodularity (5.2) of \( G \); hence the assertion (8.5) extends to any Lie group which has a Haar measure that is left and right invariant. We add also that the confirmation (8.4) can of course be made by using the definition of the matrices \( (k_{ij}), \ (k^{ij}) \), and this argument readily extends to general situation.

**9. Hilbert space** \( L^2(\Gamma \backslash G) \). This is defined to be the set of all functions \( f \) or vectors on \( G \) which are left \( \Gamma \)-automorphic or simply automorphic, i.e., \( l_\gamma f = f, \ \forall \gamma \in \Gamma \), and square integrable against the measure \( dg \) over a fundamental domain \( D \) of \( \Gamma \) on \( G \):

\[ \int_D |f(g)|^2 dg < +\infty. \quad (9.1) \]

Here \( D \) is a \( dg \)-measurable subset of \( G \) such that

\[ G = \bigcup_{\gamma \in \Gamma} \gamma D, \quad \int_{\gamma D \cap D} dg = 0, \ \gamma \neq 1. \quad (9.2) \]

For instance, the set

\[ \mathcal{F} = \{ u(x)a[y]k[\theta] : x + iy \in \mathcal{F}, \ 0 \leq \theta \leq \pi \}, \quad (9.3) \]

with \( \mathcal{F} \) defined by (1.11), serves the purpose. Obviously we may replace \( D \) in (9.1) by any fundamental domain. As a consequence, we have

\[ \int_D |f(gh)|^2 dg = \int_D |f(g)|^2 dg, \quad \forall h \in G. \quad (9.4) \]
In fact, the unimodularity of $G$ asserted at (5.2) implies that the left side is the same as integrating $|f(g)|^2$ against $dg$ over $Dh$ which is also a fundamental domain.

The set $L^2(\Gamma \backslash G)$ is a Hilbert space equipped with the inner-product

$$\langle f_1, f_2 \rangle = \int_{\Gamma \backslash G} f_1(g)\overline{f_2(g)}dg,$$

where the integration range is the whole quotient space $\Gamma \backslash G$ and the measure is induced via (5.2); the value of (9.5) is naturally the same as the result of integrating over any fundamental domain.

**Notes:** The definition (9.5) stems from Petersson’s fundamental discovery [29]; see the notes to Section 21. The space $L^2(\Gamma \backslash G)$ is, more precisely, defined to be the set of classes of $\Gamma$-automorphic functions, under the convention that two functions $f_1, f_2$ are in the same class if and only if $\|f_1 - f_2\| = 0$, where the norm is associated with (9.5). In what follows we shall mostly deal with relations between functions continuous throughout $G$; otherwise we shall mention explicitly.

**10. Automorphic representation.** The identity (9.4) means that right translations are all unitary maps of $L^2(\Gamma \backslash G)$ onto itself: We have, for any vector $f$,

$$\|r_h f\| = \|f\|, \quad \forall h \in G,$$

with the norm as above. The map

$$r : h \mapsto r_h,$$

which is a homomorphism of $G$ into the unitary transformation group of $L^2(\Gamma \backslash G)$, is termed the right regular $\Gamma$-automorphic representation or just an automorphic representation of $G$.

Any closed subspace $W$ of $L^2(\Gamma \backslash G)$ which satisfies $r_h W \subseteq W$ for all $h \in G$, is called an invariant subspace. The orthogonal complement of $W$ in terms of the metric (9.5) is also an invariant subspace. We shall use a representation and an invariant subspace as interchangeable notions, under an obvious abuse of terminology. If $W$ does not contain any non-trivial invariant subspace, then it is said to be an irreducible subspace or representation. A major task of ours is to establish a complete decomposition of $L^2(\Gamma \backslash G)$ into a direct sum of irreducible subspaces in an explicit fashion. It is immediate to notice that such a decomposition should be closely related to the spectral decomposition of $\Omega$, since any of its eigenspaces is invariant as the second identity in (8.5) implies. However, it turns out that eigenvectors of $\Omega$ do not span the full space. The complement, which is fairly large, is filled with the contribution of the continuous spectrum of $\Omega$, whose precise description is a salient aspect of the harmonic analysis on $\Gamma \backslash G$.

**Notes:** There exists vast literature on unitary representations of Lie groups. An introductory account is given in Vilenkin–Klimyk [36, Chapter 2]. However, the above definition suffices for our purpose. It should be noted that the general theory of representations of Lie groups requires that homomorphisms corresponding to (10.2) be strongly continuous. With our situation, this is inherent in the definition (10.2) itself: For any given vector $f$ and for arbitrary $\varepsilon > 0$, there exists a neighbourhood $Q$ of the unit element of $G$ such that $\|r_h f - f\| < \varepsilon$, $\forall h \in Q$. Indicating the proof, we choose a $\Gamma$-automorphic $f_0 \in C^\infty(G)$ which is compactly supported if restricted to $|F|$ and such that $\|f - f_0\| < \varepsilon$. Obviously we have also $\|r_h f - r_h f_0\| < \varepsilon$, $\forall h \in G$. Then, we consider the continuity of the map $f_0 \mapsto r_h f_0$ in the ordinary sense, i.e., with respect to the norm $\|\cdot\|_\infty$. 


11. Symmetry of $\Omega$. Naturally it should be made precise in which domain we consider the action of $\Omega$. To this end as well as for the sake of convenience of our discussion, we introduce the linear set
\[
B^\infty(\Gamma\setminus G) = \left\{ f \in C^\infty(\Gamma\setminus G) : \text{uf decays rapidly for any fixed } u \in U \right\},
\] (11.1)
where $C^\infty(\Gamma\setminus G) = C^\infty(G) \cap L^2(\Gamma\setminus G)$, and decaying rapidly means that $uf(g) \ll y^{-M}$ for any fixed $M > 0$ as $y \to +\infty$. This set is dense in $L^2(\Gamma\setminus G)$, for it contains any $f_0$ employed in the last notes.

Since the unimodularity (5.2) of $G$ and the definition (6.3) imply
\[
\langle x_j f_1, f_2 \rangle = \left[ \frac{d}{dt} \right]_{t=0} \int_{\Gamma\setminus G} f_1(g \exp(tX_j)) \overline{f_2(g)} dg
= \left[ \frac{d}{dt} \right]_{t=0} \int_{\Gamma\setminus G} f_1(g) f_2(g \exp(-tX_j)) dg = \langle f_1, -x_j f_2 \rangle,
\] (11.2)
we have, for any $u \in U$ and $f_1, f_2 \in B^\infty(\Gamma\setminus G)$,
\[
\langle uf_1, f_2 \rangle = \langle f_1, u^* f_2 \rangle,
\] (11.3)
with obvious abbreviations. In particular, the Casimir operator is symmetric over $B^\infty(\Gamma\setminus G)$:
\[
\langle \Omega f_1, f_2 \rangle = \langle f_1, \Omega f_2 \rangle.
\] (11.4)

NOTES: The symmetry of the Laplace–Beltrami operator is proved with Green’s formula (1.9). As the Casimir operator is an extension of the Laplace–Beltrami operator, there should be a proof of (11.4) via an extension of Green’s formula. For this point see Section 32.

12. Weights. To achieve a decomposition of $L^2(\Gamma\setminus G)$ into irreducible subspaces or rather a spectral resolution of $\Omega$, there are at least two ways for us to take: One is to exploit the existence of the Maass operators. The other is more generally applicable and to rework our elementary argument of [22, Chapter 1]. The first is to be rendered hereafter, and the second to be summarised in Sections 32–37.

With this, we first introduce the notion of weights: If a function $f$ on $G$ is such that there exists an $\ell \in \mathbb{Z}$ satisfying
\[
 f(gk[\tau]) = e^{2i\tau} f(g),
\] (12.1)
then $f$ is said to be of even integral weight $2\ell$. Since functions on $G$ are of period $\pi$ with respect to the Iwasawa co-ordinate $\theta$, weights ought to be even integers. We have the orthogonal decomposition
\[
L^2(\Gamma\setminus G) = \bigoplus_{\ell=-\infty}^{\infty} L^2_\ell(\Gamma\setminus G),
\] (12.2)
where the $\ell$-th summand is the set composed of all vectors satisfying (12.1), and the right side is in fact the closure of the sum. This is the same as the Fourier expansion in $\theta$ of vectors. Let $g$ be in the $\ell$-th summand, and let $h(z) = g(n[x]a[y])$. We have $h(\gamma(z)) = g(n[x_1]a[y_1])$
with $\gamma \cdot n[x]a[y] = n[x_1]a[y_1]k[\theta_1]$; thus $h(\gamma(z)) = g(\gamma \cdot n[x]a[y]) \exp(-2\ell i\theta_1)$. In view of the last identity in (3.4), we have

$$h(\gamma(z)) = h(z)\left(\frac{j(\gamma,z)}{j(\gamma,z)}\right)^\ell, \quad \forall \gamma \in \Gamma.$$  \hfill (12.3)

That is, $h$ is a $\Gamma$-automorphic function on $\mathbb{H}^2$ of weight $2\ell$. Hence

$$L_\ell^2(\Gamma \setminus G) = L_\ell^2(\Gamma \setminus \mathbb{H}^2) \cdot \exp(2\ell i\theta),$$  \hfill (12.4)

with

$$L_\ell^2(\Gamma \setminus \mathbb{H}^2) = \left\{ \text{$\Gamma$-automorphic of weight $2\ell$ and square integrable over $\mathcal{F}$ against $d\mu$} \right\},$$  \hfill (12.5)

which is obviously an extension of (1.12). Applied to $\Omega$, this separation of variables gives

$$\Omega_\ell = -y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right) + 2i\ell y \frac{\partial}{\partial x}.$$  \hfill (12.6)

**Notes:** The extension to $\Omega_\ell$ of the invariance assertions and Green’s formula given in the first section is developed in Section 32. It is possible to consider not only even integral but also arbitrary complex weights, which is due originally to Maass [20] and thoroughly investigated by Roelcke [31]. The view of such extensions from the standpoint of the universal covering group of $G$, which is in fact a generalisation of the basic stance of the present article, is given in Bruggeman [4].

**13. Descending and ascending.** We employ the Maass operators in discussing relations among weight strata (12.2) of $L^2(\Gamma \setminus G)$; note that we may restrict ourselves to the linear set $B^\infty(\Gamma \setminus G)$. The operator $w$ detects a stratum, and $e^\pm$ shift it: We pick up a vector $g$ in

$$B^\infty_\ell(\Gamma \setminus G) = L^2_\ell(\Gamma \setminus G) \cap B^\infty(\Gamma \setminus G).$$  \hfill (13.1)

Then, we have $wg = 2\ell ig$, and (7.2) implies $(w - 2(\ell \pm 1) i) e^\pm g = 0$; namely, the weight of $e^\pm g$ is $2(\ell \pm 1)$. When $\ell > 0$, the application of $(e^-)^\ell$ to $B^\infty_\ell(\Gamma \setminus G)$ takes the set down to $L^2_0(\Gamma \setminus G) = L^2_0(\Gamma \setminus \mathbb{H}^2)$, where we have a spectral resolution of $\Omega_0$, a fairly elementary proof of which is achieved in [22, Chapter 1]. Hence, the image $(e^-)^\ell B^\infty_\ell(\Gamma \setminus G)$ admits a spectral decomposition. To lift it up to the original space, we apply $(e^+)^\ell$. In this way, one might think that a spectral decomposition of $L^2_\ell(\Gamma \setminus G)$ could be achieved. However, as a matter of fact the image $(e^+)^\ell(e^-)^\ell B^\infty_\ell(\Gamma \setminus G)$ does not span the space $L^2_\ell(\Gamma \setminus G)$ in general. The discrepancy is closely related to the notion of holomorphic cusp forms on $\mathbb{H}^2$, as we shall make precise in Sections 19–21.

It should be added that the case $\ell < 0$ is analogous, because of the involution

$$J : n[x]a[y]k[\theta] \mapsto n[-x]a[y]k[-\theta],$$  \hfill (13.2)

which is the same as the map $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$ in $G$.

**Notes:** This mechanism among the weight strata is a discovery of Maass [20, Chapter 4]; see the two formulas following (13) there, although he formulated it in a quite generalised setting and without the notion of Lie differentials. See also Roelcke [31, Teil I, §3].
14. Jacquet operator. In order to make the procedure in the last section explicit, we need to render the spectral decomposition of $L^2_0(\Gamma \backslash G)$ in such a fashion that applications of Maass operators may be performed in a conspicuous way. To this end, we introduce the operator of Jacquet [13]: For a function $\phi$ on $G$, we put

$$A_{\delta}\phi(g) = \int_{-\infty}^{\infty} e(-\delta \xi)\phi(wn[\xi]g)\,d\xi,$$

$$e(\xi) = \exp(2\pi i\xi), \quad \delta = \pm, \quad w = k\left[\frac{1}{2}\pi\right].$$

(14.1)

In what follows, we shall apply $A_{\delta}$ to those $\phi$ with which ordinary convergence holds, and analytic continuation with respect to parameters involved in $\phi$ will be taken into account, if needed. Computation of the integral is carried out on noting that by (3.4) the map $g \mapsto wn[\xi]g$ is the same as

$$x \mapsto \frac{-x - \xi}{(x + \xi)^2 + y^2}, \quad y \mapsto \frac{y}{(x + \xi)^2 + y^2},$$

$$e^{2i\theta} \mapsto e^{2i\theta} \frac{x + \xi - iy}{x + \xi + iy}.$$

(14.2)

We have

$$r_h A_{\delta} = A_{\delta} r_h, \quad \forall h \in G,$$

$$u A_{\delta} = A_{\delta} u, \quad \forall u \in U.$$

(14.3)

The first is obvious, and the second is a simple consequence of the definition (6.3), although we need to have adequate smoothness of $\phi$ in (14.1). The merit of having the Jacquet operator will be felt in Section 18 and thereafter.

Notes: The definition (14.1) stems from Fourier expansions of Poincaré series on the big cell, i.e., the term $NwNA$, of the Bruhat decomposition:

$$G = NA \sqcup NwNA.$$

(14.4)

Thus the operator $A_{\delta}$ is a fairly natural device.

15. Weight functions. With the specialisation

$$\phi_\ell(g, \nu) = y^{\nu+1/2}\exp(2\ell i\theta), \quad \ell \in \mathbb{Z},$$

(15.1)

which is termed a weight function, we have, by (14.2),

$$A_{\delta}^\ell \phi_\ell(g, \nu) = \exp(2\ell i\theta) e(\delta x) y^{1/2-\nu} \int_{-\infty}^{\infty} e(-y\xi) \left(\frac{\xi - i}{\xi + i}\right)^{\delta \ell} d\xi.$$
which is related to the familiar Schl"afli integral representation for $K_*$. To show this, we assume $\Re \nu > 0$, and express $\Gamma((\nu + |\ell| + \frac{1}{2})((\xi^2 + 1)^{-\nu-|\ell|-1/2}$ in terms of the Euler integral over $u > 0$ for the Gamma-function. We insert it into (15.2) and exchange the order of integration, getting $(\xi - \delta \arg(\xi))^{2|\ell|}\exp(-2\pi i y \xi - u \xi^2)$ as the new inner integrand. We shift the inner contour to \( \Im \xi = -\pi y/u \). Then (15.3) follows after a rearrangement. The regularity assertion with respect to $\nu$ is now immediate. Also, we note that a shift of the contour in (15.2) to \( \Im \xi = -\infty \) gives, for $l \in \mathbb{N}$,

$$
A^3 \phi_l(g, l - \frac{1}{2}) = \begin{cases} 
0 & \text{if } \delta \ell \leq -l, \\
(-1)^l \frac{(2\pi)^2 l}{l} y^l \exp(2\ell i \theta) e(\delta x + iy) & \text{if } \delta \ell = l.
\end{cases} (15.4)
$$

The formula (15.3) implies that $A^3 \phi_l(g, \nu)$ is of exponential decay in $y$, which is, however, often inadequate and we need uniform bounds: For instance, we have, for $\Re \nu > \frac{1}{2}$,

$$
A^3 \phi_l(g, \nu) \ll \langle |\nu| + |\ell| + 1 \rangle \langle \frac{y^{1/2 - \Re \nu}}{|\log y| + 1} \rangle \frac{\exp(-2\ell i \theta) e(\delta x)}{y^{1/2 - \nu}} \int_{-\infty}^{\infty} \frac{e(-y \xi)}{(\xi^2 + 1)^{\nu+1/2}} \left( \left( \frac{\xi - i}{\xi + i} \right)^{\delta \ell} - 1 \right) d\xi. (15.5)
$$

where the implied constant depends on $\Re \nu$ only. In fact, for $y \geq 1$, it suffices to apply the integration by parts to (15.2) and shift the contour to $\Im \xi = -(|\nu| + |\ell| + 1)^{-1}$. For $0 < y < 1$, we use the trivial identity

$$
\exp(-2\ell i \theta) e(\delta x) A^3 \phi_l(g, \nu) = A^3 \phi_0(g, \nu) \int_{-\infty}^{\infty} \frac{e(-y \xi)}{(\xi^2 + 1)^{\nu+1/2}} \left( \left( \frac{\xi - i}{\xi + i} \right)^{\delta \ell} - 1 \right) d\xi. (15.6)
$$

The second term on the right side is obviously $\ll y^{1/2 - \Re \nu} (|\ell| + 1)$. The first term is dealt with (15.3), for $\ell = 0$, $g = a[y]$. Assuming $|\nu|$ is large, we turn the contour through the angle $\frac{1}{2}(\pi - 1/|\nu|)\sgn(\Im \nu)$ around the origin. The rest of the argument may be skipped.

16. Whittaker functions. In literature, the transform $A^3 \phi_l(g, \nu)$ is expressed in terms of Whittaker functions $W_{\mu, \nu}$:

$$
A^3 \phi_l(g, \nu) = (-1)^l \pi^{\nu+1/2} \exp(2\ell i \theta) e(\delta x) \frac{W_{\mu, \nu}(4\pi y)}{\Gamma(\nu + \delta \ell + \frac{1}{2})}. (16.1)
$$

We shall prove this for a practical reason as well as in respect for tradition. We begin with the definition

$$
W_{\mu, \nu}(y) = e^{\mu \pi i} \frac{\Gamma(\mu + \nu + \frac{1}{2})}{\pi(y/4)^{\nu-1/2}} \int_{-\infty}^{\infty} \frac{e^{-i\xi/2}}{(\xi^2 + 1)^{\nu+1/2}} \frac{(\xi - i)^{\mu}}{(\xi + i)^{\mu}} d\xi. (16.2)
$$

where $y > 0$, $\mu \in \mathbb{C}$, and $\arg(\xi \pm i)$ varies from $\pm \pi$ to 0, respectively, along the contour. It will be justified in due course that (16.2) serves the purpose of defining Whittaker functions. Thus we transform (16.2) into an expression of the Mellin–Barnes type:

$$
W_{\mu, \nu}(y) = y^\mu e^{-y/2} \frac{2\pi i}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(s - \mu + \nu + \frac{1}{2})}{\Gamma(\mu + \nu)} \frac{\Gamma(s - \mu - \nu + \frac{1}{2})}{\Gamma(\mu + \nu)} \Gamma(-s) y^{-s} ds, (16.3)
$$

where the path separates the poles of $\Gamma(s - \mu + \nu + \frac{1}{2})$ and $\Gamma(-s)$ to the left and the right, respectively; and it is assumed that parameters are such that the path can be drawn. To show this, we consider the integral

$$
\int_{Y^*} \frac{e^{-i\xi/2} d\xi}{(\xi + i)^{\mu+\nu+1/2}(\xi - i)^{-\mu+\nu+1/2}}, (16.4)
$$
where the path $Y_\varepsilon$ with a small $\varepsilon > 0$ starts at $-1/\varepsilon$, proceeds to $1/\varepsilon$ on the real axis, then to $-i/\varepsilon$ along the circle $|\xi| = 1/\varepsilon$ and goes up to $-(1+\varepsilon)i$ on the imaginary axis; it encircles $-i$ counter-clockwise, goes down to $-i/\varepsilon$ on the imaginary axis and returns to $-1/\varepsilon$ along the circle $|\xi| = 1/\varepsilon$. Under the temporary condition $0 < \text{Re } \nu < -\text{Re } \mu + \frac{1}{2}$, the contribution of the circular part of the path vanishes as $\varepsilon \to +0$, and one may see that the integral in (16.2) equals

$$2^{1-2
u}e^{-\pi \nu}e^{-y/2} \sin\left(\pi(\mu + \nu + \frac{1}{2})\right) \int_0^\infty \frac{e^{-y\rho}d\rho}{\rho^{\mu+\nu+1/2}(1+\rho)^{-\mu+\nu+1/2}}. \quad (16.5)$$

We apply the Mellin inversion to the numerator of the integrand and exchange the order of integrals. The new inner integral is Euler’s Beta function, and after a rearrangement together with analytic continuation we reach (16.3). Shifting the contour in (16.3) sufficiently far to the right we see immediately that

$$W_{\mu,\nu}(y) = (1 + o(1)) y^{\mu}e^{-y/2}, \quad y \to +\infty, \quad (16.6)$$

uniformly for any bounded $\mu, \nu$. Also it holds, for $y > 0$, that

$$\begin{align*}
[\omega^+ + \mu] W_{\mu,\nu}(y) &= -W_{\mu+1,\nu}(y), \\
[\omega^- - \mu] W_{\mu,\nu}(y) &= ((\mu - \frac{1}{2}^2 - \nu^2) W_{\mu-1,\nu}(y), \\
\rho &= y(d/dy) + \frac{1}{2}y, \\
\left[-\left(\frac{d}{dy}\right)^2 + \frac{1}{4} - \frac{\mu}{y} + \left(\nu^2 - \frac{1}{4}\right) \frac{1}{y^2}\right] W_{\mu,\nu}(y) &= 0.
\end{align*} \quad (16.7)$$

The two recurrence equations can be confirmed by applying the respective differentials to (16.3), which may be performed freely as the integral converges rapidly because of Stirling’s formula for the $\Gamma$-function. The third line in (16.7) is a consequence of the first two, and this confluent hypergeometric differential equation is customarily attributed to Whittaker. In view of (16.6) we find that (16.2) can indeed be employed to define Whittaker functions; see [38, Chapter XVI].

Here is a beautiful integral formula: For $\alpha, \beta \in \mathbb{C}$ and $|\text{Re } \nu| < \frac{1}{2}$,

$$\int_0^\infty W_{\alpha,\nu}(y) W_{\beta,\nu}(y) \frac{dy}{y} = \frac{\pi}{(\alpha - \beta) \sin(2\pi \nu)} \times \frac{1}{\Gamma\left(\frac{1}{2} - \alpha + \nu\right) \Gamma\left(\frac{1}{2} - \beta - \nu\right) - \Gamma\left(\frac{1}{2} - \alpha - \nu\right) \Gamma\left(\frac{1}{2} + \beta + \nu\right)}. \quad (16.8)$$

To show this, we note first that it holds uniformly for $|\text{Re } \nu| < \frac{1}{2}$, bounded $\mu$ and sufficiently small $y > 0$ that

$$W_{\mu,\nu}(y) = \left(\frac{-2\nu}{\Gamma\left(\frac{1}{2} + \mu - \nu\right)} y^{\nu+1/2} + \frac{\Gamma(2\nu)}{\Gamma\left(\frac{1}{2} + \mu - \nu\right)} y^{-\nu+1/2}\right) (1 + O(y)),$$

$$W_{\mu,\nu}(y) = \left(\frac{-2\nu}{\Gamma\left(\frac{1}{2} - \mu - \nu\right)} y^{\nu+1/2} + \frac{\Gamma(2\nu)}{\Gamma\left(\frac{1}{2} - \mu - \nu\right)} y^{-\nu+1/2}\right) (1 + O(y)). \quad (16.9)$$

In fact, we replace $s$ by $s + \mu - \frac{1}{2}$ in (16.3) and shift the path to $\text{Re } (s) = -M + \frac{1}{2}$ with a sufficiently large $M \in \mathbb{N}$. An examination of residues gives (16.9). Then, with $\omega_\nu = \frac{1}{2}$.
-(d/dy)^2 + \frac{1}{4} + (\nu^2 - \frac{1}{4})y^{-2} and by integration by parts, we have, for \alpha \neq \beta,

\begin{align*}
(\alpha - \beta) & \int_0^\infty W_{\alpha, \nu}(y) W_{\beta, \nu}(y) \frac{dy}{y} \\
& = \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \left[ \omega_{\nu} W_{\alpha, \nu}(y) W_{\beta, \nu}(y) - W_{\alpha, \nu}(y) \omega_{\nu} W_{\beta, \nu}(y) \right] dy \\
& = \lim_{\epsilon \to 0^+} \left[ W'_{\alpha, \nu}(\epsilon) W_{\beta, \nu}(\epsilon) - W_{\alpha, \nu}(\epsilon) W'_{\beta, \nu}(\epsilon) \right],
\end{align*}

(16.10)
since the first integral converges absolutely if |Re\(\nu| < \frac{1}{2}, because of (16.6) and the first line of (16.9). The last limit can be computed by combining the two formulas of (16.9).

Notes: The formula (16.8) is tabulated at [9, 7.611(3)]; however, the precious factor \pi is missing there, which would cause a discrepancy concerning a unitarity assertion given in Section 27. Our proof is taken from [23, Part XII]. The formula (16.3) is given in Whittaker–Watson [38, p. 343] and the recursive formulas (16.7) in Vilenkin–Klimyk [36, p. 218], but our treatments are different from theirs. The assertion (16.7) is related to the actions of the Maass and the Casimir operators.

17. Hilbert space \(L^2_0(\Gamma \setminus G)\). We read [22, Theorem 1.1] in the present context, and have

\[ L^2_0(\Gamma \setminus G) = \mathbb{C} \cdot 1 \oplus \mathcal{O}_0 \oplus L^2_0(\Gamma \setminus G). \]

(17.1)

Here

\[ \mathcal{O}_0 = \mathbb{C} : \lambda^{(0)}_V, \quad \langle \lambda^{(0)}_V, \lambda^{(0)}_{V'} \rangle = \delta_{V, V'}, \]

(17.2)

with

\[ \lambda^{(0)}_V(g) = \sum_{n \neq 0} \frac{\varrho(n)}{\sqrt{|n|}} A^{sgn(n)}(a|n|g, \nu_V), \]

\[ \Omega \lambda^{(0)}_V = (\frac{1}{4} - \nu^2_V) \lambda^{(0)}_V, \quad \nu_V \in i\mathbb{R}, \]

(17.3)

where \(V\)'s are just labels indexing the discrete set \(\{\frac{1}{4} - \nu^2_V\}\) of eigenvalues of \(\Omega_0\) acting over \(L^2_0(\Gamma \setminus G)\); in Sections 23 and later they will stand for a series of invariant subspaces of \(L^2(\Gamma \setminus G)\). Naturally, the right side of (17.2) is to be understood as the closure of the sum. By the identity (15.3) with \(\ell = 0\), we have in fact

\[ \lambda^{(0)}_V(g) = \frac{2\pi^{\nu_V + 1/2}}{\Gamma(\nu_V + \frac{1}{2})} \sum_{n \neq 0} \varrho(n) y^{1/2} K_{\nu_V}(2\pi|n|y) e(nx), \]

(17.4)

which converges absolutely. Namely, we have written the Fourier expansion [22, (1.1.41)] in such a way that

\[ \kappa_j = -i\nu_V, \quad \rho_j(n) = \frac{2\pi^{\nu_V + 1/2}}{\Gamma(\nu_V + \frac{1}{2})} \varrho(n). \]

(17.5)

As to \(\mathcal{O}_0(\Gamma \setminus G)\), which is the contribution of the continuous spectrum of \(\Omega_0\), we have

\[ \mathcal{O}_0(\Gamma \setminus G) = \left\{ \int_{(0)} h(\nu) E_0(g, \nu) d\nu : h \in L^2(i\mathbb{R}) \right\}, \]

(17.6)

where the kernel \(E_0\) is the Eisenstein series of weight 0 to be defined in the next section and \(L^2(i\mathbb{R})\) the \(L^2\)-space with respect to the Lebesgue measure placed on the imaginary
axis \((0) = i\mathbb{R}\); the integral converges in the mean in the space \(L^2_0(\Gamma \backslash G)\). The decomposition (17.1) with (17.2) and (17.6) is equivalent to the spectral expansion of any \(f \in L^2_0(\Gamma \backslash G)\):

\[
 f(g) = \frac{3}{\pi} \langle f, 1 \rangle + \sum_{\nu} \langle f, \lambda_\nu^{(0)} \lambda_\nu^{(0)}(g) + \frac{1}{4\pi i} \int_{(0)} E_0(f, \nu) E_0(g, \nu) d\nu, \quad (17.7)
\]

where

\[
 E_0(f, \nu) = \int_{\Gamma \backslash G} f(g) \overline{E_0(g, \nu)} dg \quad \text{in} \quad L^2(i\mathbb{R}). \quad (17.8)
\]

The identity (17.7) holds, with the sum and the integral converging in the mean. Further, (17.8) is in fact the limit in the mean in the space \(L^2(i\mathbb{R})\) of the integrals over \([\mathcal{F}]_V = [\mathcal{F}] \cap \{y \leq Y\}\), with \(Y\) tending to \(+\infty\).

We may assume that with \(J\) defined by (13.2)

\[
 J\lambda_\nu^{(0)} = \epsilon_\nu \lambda_\nu^{(0)}, \quad \epsilon_\nu = \pm 1, \quad (17.9)
\]

which is the same as [22, (3.1.15)] and equivalent to \(g_\nu(-n) = \epsilon_\nu g_\nu(n)\) for any \(n \in \mathbb{N}\). We have the bounds

\[
 \sum_{|\nu| \leq K} 1 \ll K^A, \quad (17.10)
\]

with an absolute constant \(A > 0\), and

\[
 \sum_{|\nu| \leq K} |g_\nu(n)|^2 \ll K^2 + n^{4/5}, \quad (17.11)
\]

where the implied constants are absolute. The former is a simple consequence of [22, (1.4.2)]; or rather (37.21) below gives \(A \leq 8\). In Section 43 we shall show an asymptotic formula, with \(A = 2\), via Selberg’s trace formula. Although weaker than [22, (2.3.2)], the bound (17.11) is adequate for our purpose; it depends on a fairly elementary bound for Kloosterman sums instead of Weil’s.

**Notes:** The concept of automorphic eigenfunctions is due to Delsarte [7] and Maass [19]. The spectral decomposition (17.1) is an instance of Selberg’s general statement [32], although the detailed proof of the spectral resolution of the Casimir operator, in Maass’ extended context, is done by Roelcke in [31] with an essential appeal to functional analysis, especially to the theory of unbounded symmetric operators as well as to the theory of elliptic differential operators. The proof of (17.1) given in [22] is elementary in the sense that it is virtually independent of functional analysis, save for a use of the classical Hilbert–Schmidt theory on integral operators with bounded continuous symmetric kernels. We should mention also that there exists an elementary proof of Weil’s bound for Kloosterman sums; the theory is initiated by Stepanov [34].

**18. Eisenstein series.** For an arbitrary even integral weight \(2\ell\), this is defined by

\[
 E_\ell(g, \nu) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \phi_\ell(\gamma g, \nu), \quad \text{Re} \nu > \frac{1}{2}, \quad (18.1)
\]

with \(\Gamma_\infty = \{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z}\}\), which converges absolutely. We have the Fourier expansion

\[
 E_\ell(g, \nu) = \phi_\ell(g, \nu) + \frac{(-1)^\ell \Gamma^2(\nu + \frac{1}{2}) \varphi_\ell(\nu)}{\Gamma(\nu + \ell + \frac{3}{2}) \Gamma(\nu - \ell + \frac{3}{2})} \phi_\ell(g, -\nu)
\]

\[
 + \frac{1}{\zeta(2\nu + 1)} \sum_{n \neq 0} |n|^{-\nu} \varphi_\ell(|n|) A^{\operatorname{sgn}(n)} \phi_\ell(a|n|) g, \nu), \quad (18.2)
\]
with \( \sigma_\alpha(n) = \sum_{d \mid n} d^\alpha \) and

\[
\varphi_\Gamma(\nu) = \sqrt{\pi} \frac{\Gamma(\nu)\zeta(2\nu)}{\Gamma(\nu + \frac{1}{2})\zeta(2\nu + 1)} = \pi^{2\nu} \frac{\Gamma\left(\frac{1}{2} - \nu\right)\zeta(1 - 2\nu)}{\Gamma\left(\nu + \frac{1}{2}\right)\zeta(2\nu + 1)}.
\] (18.3)

The identity (15.3) implies that \( E_\ell(g, \nu) \) is a meromorphic function of \( \nu \) over \( \mathbb{C} \). The functional equation for the modified Eisenstein series

\[
E_\ell^*(g, \nu) = E_\ell^*(g, -\nu),
\]

\[
E_\ell^*(g, \nu) = \pi^{-\nu-1/2}\Gamma(\nu + |\ell| + 1)\zeta(2\nu + 1)E_\ell(g, \nu),
\] (18.4)

is a consequence of the symmetry \( W_{\mu, \nu} = W_{\mu, -\nu} \), which follows from (15.3) and (16.1). We have that \( E_\ell^*(g, \nu), \ell \neq 0 \), is entire in \( \nu \), while \( E_0^*(g, \nu) \) is regular except for the simple poles at \( \nu = \pm \frac{1}{2} \).

The proof of these facts on \( E_\ell \) may be skipped as it is a standard application of Poisson’s sum formula via the double coset decomposition \( \Gamma_\infty \backslash \Gamma / \Gamma_\infty \). It should, however, be stressed that it suffices, as a matter of fact, to have the expansion (18.2) for \( E_0 \) only, since we have

\[
(e^{\text{sgn}(\ell)}\ell)\phi_0(g, \nu) = 2^{|\ell|}\Gamma\left(\frac{1}{2} + \nu + |\ell|\right)\frac{\Gamma\left(\frac{1}{2} + \nu\right)}{\Gamma\left(\frac{1}{2} + \nu + |\ell|\right)}\phi_\ell(g, \nu),
\] (18.5)

that is, in view of (14.3) we obtain (18.2) for general \( \ell \neq 0 \) by an application of \( (e^{\text{sgn}(\ell)}\ell)\phi_0(g, \nu) \) to the expansion for \( E_0 \). More precisely, we argue as follows: Under the convention introduced after (6.7), we first apply \( (e^{\text{sgn}(\ell)}\ell)\phi_0 \) to the defining expression for \( E_0 \) term-wise, which is legitimate, since the result converges absolutely and uniformly when Re\( \nu > \frac{1}{2} \) in view of the left invariance of \( e^\pm \) and (18.5). We get the defining expression for \( E_\ell \), save for a factor. On the other hand, the expansion (18.2) for \( E_0 \) admits the term-wise application of \( (e^{\text{sgn}(\ell)}\ell)\phi_0 \), for the result converges absolutely and uniformly in view of (14.3), (18.5) and (18.5). In this way we may derive (18.2) from the expansion for \( E_0 \). This practical mechanism is a merit of having the Jacquet operator, and will be exploited throughout the rest of our discussion.

19. Discrepancy. We resume the scheme started in Section 13; thus we let \( \ell \) be positive. We apply (17.7) to \( (e^-)^\ell g(g) \), and get, on noting (11.3),

\[
(e^-)^\ell g(g) = (-1)^\ell \sum_V \langle g, (e^+)^\ell \lambda^{(0)}_V \rangle \lambda'^{(0)}_V (g) + \frac{(-1)^\ell}{4\pi i} \int_{(0)} \eta_\ell(g, \nu)E_0(g, \nu) d\nu,
\] (19.1)

with

\[
\eta_\ell(g, \nu) = \int_{\Gamma \backslash G} g(g)(e^+)^\ell E_0(g, \nu) dg.
\] (19.2)

We have used that \( (e^+)^\ell 1 = 0 \) and that \( E_0((e^+)^\ell g, \nu) = (-1)^\ell \eta_\ell(g, \nu) \). The latter can be shown in the same way as (11.3), since \( g \in B_\infty^\infty(\Gamma \backslash G) \).

Before applying \( (e^+)^\ell \) to (19.1), we note that for any \( \ell \geq 0 \)

\[
(e^-)^\ell(e^+)^\ell \phi_k(g, \nu) = (-4)^\ell \Gamma\left(\frac{1}{2} + \nu + k + \ell\right)\Gamma\left(\frac{1}{2} - \nu + k + \ell\right) \phi_k(g, \nu).
\] (19.3)

In fact, a combination of (7.2), (8.3) and (8.4) gives

\[
(e^-)^{\ell+1}(e^+)^{\ell+1} \phi_k(g, \nu)
\]

\[
= (e^-)^\ell(-4\Omega + \omega^* + 2\omega)(e^+)^\ell \phi_k(g, \nu)
\]

\[
= -4\left(\frac{1}{2} + \nu + k + \ell\right)\left(\frac{1}{2} - \nu + k + \ell\right)(e^-)^\ell(e^+)^\ell \phi_k(g, \nu).
\] (19.4)
We apply (19.3), for \( k = 0 \), to (17.3) and (18.2) specialised with \( E_0 \):

\[
(e^-)^\ell (e^+)^\ell \lambda_V^{(0)}(g) = (-4)^\ell \frac{|\Gamma(\frac{1}{2} + \nu + \ell)|^2}{|\Gamma(\frac{1}{2} + \nu)|^2} \lambda_V^{(0)}(g),
\]

\[
(e^-)^\ell (e^+)^\ell E_0(g, \nu) = (-4)^\ell \frac{|\Gamma(\frac{1}{2} + \nu + \ell)|^2}{|\Gamma(\frac{1}{2} + \nu)|^2} E_0(g, \nu), \quad \nu \in i\mathbb{R},
\]

(19.5)
in which the convention following (6.7) is in effect.

Then we put

\[
g(g) = \tilde{g}(g) + 2^{-2\ell} \sum_\nu \frac{|\Gamma(\frac{1}{2} + \nu + \ell)|^2}{|\Gamma(\frac{1}{2} + \nu)|^2} \langle g, (e^+)^\ell \lambda_V^{(0)}(e^+)^\ell \lambda_V^{(0)}(g) \rangle
\]

\[
+ \frac{2^{-2\ell}}{4\pi i} \int_{(0)} \frac{|\Gamma(\frac{1}{2} + \nu)|^2}{|\Gamma(\frac{1}{2} + \nu + \ell)|^2} \eta(g, \nu)(e^+)^\ell E_0(g, \nu) d\nu.
\]

(19.6)

We apply \((e^-)^\ell\) on both sides and use (19.5). Comparing the result with (19.1), we find that

\[
(e^-)^\ell \tilde{g} = 0.
\]

(19.7)

Namely, \(\tilde{g}\) stands for the discrepancy indicated in Section 13.

NOTES: This discrepancy or rather (19.7) is treated in Roelcke [31, Teil II] from the standpoint of analysing the situation where (18.5) vanishes, if formulated in our notation. We have taken the above approach in order to further stress the relevance to the structure of the weight strata.

20. Convergence issue. However, we have yet to see whether the last operation on (19.6) is legitimate or not. To this end, we shall first prove that the sum on the right side of (19.1) belongs to \( B^\infty(\Gamma \setminus G) \): We observe that for any \( q \in \mathbb{N} \)

\[
|\langle g, (e^+)^\ell \lambda_V^{(0)} \rangle| = \frac{1}{\left| \frac{1}{4} - \nu_V^2 \right|^{q/2}} |\Omega_q(e^-)^\ell g, \lambda_V^{(0)}| \leq \frac{|\Omega_q(e^-)^\ell g|}{\left| \frac{1}{4} - \nu_V^2 \right|^{q/2}} \leq |\nu_V|^{-2q}.
\]

(20.1)

Thus, we have, for any \( k \in \mathbb{N} \),

\[
\sum_V |\langle g, (e^+)^\ell \lambda_V^{(0)}(e^+)^k \lambda_V^{(0)}(g) \rangle| \leq \sum_{n \neq 0} \frac{1}{|n|} \sum_K K^{k-2q}
\]

\[
\times \left( \sum_{K \leq |\nu_V| \leq 2K} |\phi_V(n)|^2 \right)^{1/2} \left( \sum_{K \leq |\nu_V| \leq 2K} |A_{\text{sgn}(n)}(a[n]g, \nu_V)|^2 \right)^{1/2},
\]

(20.2)

where \( K \) runs over dyadic numbers, and (18.5), for \( \ell = \pm k \), is applied. The first sum over \( \nu_V \) on the right side is estimated by (17.11), and the second by (15.5) while taking account of (17.10). We then divide the sum over \( K \) at \( K \approx (|n|g)^{1/2} \). This yields the desired assertion, since \( \{e^-, e^+, w\} \) generates \( \mathcal{U} \) and the action of \( w \) is immaterial.

We need further to deal with the integrated term of (19.1): We have a bound of \( \eta(g, \nu) \) similar to (20.1), since \( \Omega(e^-)^\ell g \) is in \( B^\infty(\Gamma \setminus G) \) and \( E_0(g, \nu) \). Re \( \nu = 0 \), is of polynomial order in both \( g \) and \( \nu \) as is implied by (18.2) together with (15.5) and a well-known lower
bound for \( \zeta(s) \) on \( \text{Re } s = 1 \). By the same token, \((e^+)kE_0(g, \nu)\) is of polynomial order in \(\nu\), \(\text{Re } \nu = 0\), uniformly for bounded \(g\). Hence it is immaterial whether \((e^+)k\) is applied inside or outside the integral. The result of the outside application is in \(B^\infty(\Gamma \backslash G)\), as has been proved already.

21. Holomorphic cusp forms. We shall show that

\[
\tilde{g} = \sum_{l=0}^{\ell} (e^+)^{\ell-l} \varphi_l,
\]

where

\[
B^\infty_l(\Gamma \backslash G) \ni \varphi_l(g) = e^{2\ell i y} \sum_{n=1}^{\infty} a(n)e(nz), \quad z = x + iy.
\]

We employ the induction in terms of \(\ell\), as the case \(\ell = 0\) is trivial. Thus, let \(p \in B^\infty_{l+1}(\Gamma \backslash G)\) be such that \((e^-)^{\ell+1}p = 0\); this smoothness of \(p\) can be assumed because of the assertion of the previous section. By the inductive hypothesis we have \(e^-p = \sum_{l=0}^{\ell}(e^+)^{\ell-l}\varphi_{1,l}\) with the specification same as (21.1). We apply \(-\frac{1}{4}e^+\) to both sides, and get, on noting the second line of (8.3),

\[
(\Omega + \ell(\ell + 1))p = -\frac{1}{4} \sum_{l=0}^{\ell} (e^+)^{\ell+1-l} \varphi_{1,l}
\]

\[
= -\frac{1}{4} \sum_{l=0}^{\ell} (e^+)^{\ell+1-l} \frac{(\Omega + \ell(\ell + 1))}{\ell(\ell + 1) - l(l - 1)} \varphi_{1,l},
\]

which means that

\[
p = q - \frac{1}{4} \sum_{l=0}^{\ell} \frac{(e^+)^{\ell+1-l} \varphi_{1,l}}{\ell(\ell + 1) - l(l - 1)}, \quad (\Omega + \ell(\ell + 1))q = 0.
\]

We note that \(q \in B^\infty_{l+1}(\Gamma \backslash G)\) by the construction. Again by the second line of (8.3), we have \(0 = \langle (\Omega + \ell(\ell + 1))q, q \rangle = \frac{1}{4} \|e^-q\|^2\); namely \(e^-q = 0\). Then, in the expansion

\[
q(g) = e^{2(\ell+1)i\theta} \sum_{n=-\infty}^{\infty} c(n)k_n(y)e(nx),
\]

we have \((d/dy + 2\pi n - (\ell + 1)/y)k_n = 0\), and thus \(k_n(y) = b(n)y^{\ell+1} \exp(-2\pi ny)\). The rapid decay of \(q\) implies \(b(n) = 0\), \(n \leq 0\), which ends the proof of (21.1).

Now, let \(\varpi(z) = (e^{2\ell i y})^{-1}\varphi_1(g)\). Then by (12.3), for \(\ell = l\), we see that \(\varpi\) is regular throughout \(\mathbb{H}^2\) and satisfies

\[
\varpi(i\infty) = 0, \quad \varpi(\gamma(z))(g(\gamma, z))^{-2l} = \varpi(z), \quad \gamma \in \Gamma.
\]

That is, \(\varpi\) is a holomorphic cusp form of weight \(2l\) with respect to \(\Gamma\). Let \(\mathcal{C}_l(\Gamma)\) be the set of all such functions. Then \(\mathcal{C}_l(\Gamma)\) is a Hilbert space of finite dimension \(\vartheta_l(\Gamma)\) equipped with the inner product

\[
\langle \varpi_1, \varpi_2 \rangle_l = \int_{\mathcal{F}} \overline{\varpi_1(z)} \varpi_2(z) y^{2l}d\mu(z),
\]
where $\mathcal{F}$ is as in (1.11). The dimension formula

$$
\vartheta_F(l) = \begin{cases} 
\lfloor l/6 \rfloor & l \not\equiv 1 \pmod{6}, \\
\lfloor l/6 \rfloor - 1 & l \equiv 1 \pmod{6}, \ l \geq 7,
\end{cases}
$$

(21.8)
is well-known.

Let $\{ (e^{2i\theta} y^l)^{-1} \lambda_V^{(l)}(g) \}$ be an orthonormal basis of $\mathcal{C}_F(l)$, with $V$ running over $\vartheta_F(l)$ labels. In particular, we have

$$
\langle \lambda_V^{(l)}(g), \lambda_{V'}^{(l)}(g) \rangle = \delta_{V,V'};
$$

(21.9)
that is, the inner product (21.7) on $\mathcal{C}_F(l)$ is translated into (9.5) on $L^2(\Gamma \backslash G)$. We write the Fourier expansion of $\lambda_V^{(l)}$ as

$$
\lambda_V^{(l)}(g) = \pi^{1/2-1} \Gamma(2l)^{1/2} \sum_{n=1}^{\infty} \frac{\vartheta_V(n)}{\sqrt{n}} A^+ \phi_l(a[n]g, l - \frac{1}{2}),
$$

(21.10)
which converges absolutely, and is to be compared with (17.3). This notation may appear misleading but its employment will be justified in the next section. In view of the lower line of (15.4), the expansion (21.10) means that we have re-normalised the Petersson–Fourier coefficients [22, (2.2.3)] as

$$
\rho_{j,l}(n) = (-1)^j \frac{2^{2l} \pi^{l+1/2}}{\Gamma(2l)^{1/2}} \vartheta_V(n).
$$

(21.11)
Corresponding to (17.11), we have, with $V$ as in (21.10),

$$
\sum_V |\vartheta_V(n)|^2 \ll l + n^{4/5},
$$

(21.12)
where the implied constant is absolute, which contains a simple improvement upon [22, (2.2.10)] concerning the dependency on weights; see [27, Vol. 2, (4.1.21)].

Notes: For the proof of (21.8) see, e.g., Maass [20, Chapter 2]; Petersson’s theory of Poincaré series is applied. In the notes to Section 38 is an indication of a proof via Selberg’s trace formula. We should not miss mentioning, even though this historical fact is very much well-known, that (21.7) is the metric introduced by Petersson [29], by which he initiated the modern development of the theory of automorphic functions and automorphic representations such as the spectral decomposition of $L^2(\Gamma \backslash G)$ stated in Section 23.

22. Hilbert space $L^2_\ell(\Gamma \backslash G)$. We sum up the above discussion: The expansion (19.6) together with (21.1) makes precise the spectral structure of $L^2_\ell(\Gamma \backslash G)$, $\ell > 0$. First, in the context of Section 17, we put

$$
\lambda_V^{(l)}(g) = \frac{\Gamma(\frac{1}{2} + \nu_V)}{2 \Gamma(\frac{1}{2} + \nu_V + \ell)} (e^+)^{\ell} \lambda_V^{(0)}(g).
$$

(22.1)
By (17.3) and (18.5) we have

$$
\lambda_V^{(l)}(g) = \sum_{n=-\infty}^{\infty} \frac{\vartheta_V(n)}{\sqrt{|n|}} A^{\text{sgn}(n)} \phi_l(a[|n|]g, \nu_V), \quad \nu_V \in \mathbb{R}.
$$

(22.2)
Second, in the context of the previous section, we put, for \( \ell \geq l \),

\[
\lambda^{(\ell)}_V(g) = 2^{\ell - \ell} \left( \frac{\Gamma(2\ell)}{\Gamma(\ell + l)\Gamma(\ell - l + 1)} \right)^{1/2} (\epsilon^t)^{\ell - l} \lambda^{(l)}_V(g).
\]  
(22.3)

By (21.10) we have

\[
\lambda^{(\ell)}_V(g) = \pi^{1/2 - l} \left( \frac{\Gamma(\ell + l)}{\Gamma(\ell - l + 1)} \right)^{1/2} \sum_{n=1}^{\infty} \frac{g_V(n)}{\sqrt{n}} \mathcal{A}^t \phi_{\ell}(a[n]g, \nu_V),
\]  
\[
\nu_V = l - \frac{1}{2}, \quad l \in \mathbb{N}.
\]  
(22.4)

With this, we have

\[
\langle \lambda^{(\ell)}_V, \lambda^{(\ell)'}_V \rangle = \delta_{V, V'}.
\]  
(22.5)

Namely, the system consisting of functions (22.2) and (22.4) is orthonormal in \( L^2_\ell(\Gamma \setminus G) \).

Thus, if \( \lambda^{(\ell)}_V \) and \( \lambda^{(\ell)'}_V \) are both in the category (22.1), then (17.2) and (19.5) give the assertion; and the case with the category (22.3) is treated via the identities (19.3) and (21.9). Further, if these functions belong to different categories, then they are orthogonal, since their \( \Omega \)-eigenvalues are different and we may appeal to (11.4); or rather the fact comes down to \( (\epsilon^-)\lambda^{(\ell)}_V \equiv 0 \), with \( \lambda^{(\ell)}_V \) defined by (22.3), which may also be used to show the orthogonality of these \( \lambda^{(\ell)}_V \) against the integrated part of (19.6). As a consequence, we can rewrite (21.1) as

\[
\tilde{g}(g) = \sum_{l=1}^{\ell} \sum_{V} \langle g, \lambda^{(\ell)}_V \rangle \lambda^{(l)}_V(g).
\]  
(22.6)

In fact, it suffices to plug (19.6) into the right side.

Hence we have

\[
L^2_\ell(\Gamma \setminus G) = \oplus_{V}^\ell L^2_\ell(\Gamma \setminus G), \quad \ell > 0,
\]  
(22.7)

with

\[
\oplus_{V}^\ell L^2_\ell(\Gamma \setminus G) = \bigoplus_{V}^\ell \mathbb{C} \cdot \lambda^{(\ell)}_V
\]  
(22.8)

and

\[
\mathcal{L}_{\ell}^2(\Gamma \setminus G) = \left\{ \int_{(0)} \eta(\nu)E_\ell(g, \nu)d\nu : \eta \in L^2(i\mathbb{R}) \right\}.
\]  
(22.9)

Or equivalently, we have, for any \( f \in L^2_\ell(\Gamma \setminus G) \),

\[
f(g) = \sum_{V}^{(\ell)} \langle f, \lambda^{(\ell)}_V \rangle \lambda^{(l)}_V(g) + \frac{1}{4\pi^2} \int_{(0)} \mathcal{E}_\ell(f, \nu)E_\ell(g, \nu)d\nu,
\]  
(22.10)

with

\[
\mathcal{E}_\ell(f, \nu) = \int_{\Gamma \setminus G} f(g)E_\ell(g, \nu)d\gamma \quad \text{in} \quad L^2(i\mathbb{R}).
\]  
(22.11)

Here both \( \oplus^{(\ell)} \) and \( \sum^{(\ell)} \) denote that \( \nu_V \) is either pure imaginary or equal to \( l - \frac{1}{2}, \quad l \in \mathbb{N}, \quad l \leq \ell \). The right side of (22.8) is understood similarly to that of (17.2). The integrated part of (22.10) is a consequence of a combination of (18.1), (18.5), (19.2) and the corresponding
part of (19.6). The identity (22.10) holds in the same sense as (17.7), with (22.11) being an analogue of (17.8).

The case \( \ell < 0 \) is analogous; it suffices to apply the map \( J \) to (22.7). Since \( Jg_1g_2 = Jg_3Jg_2 \) and \( dg = dJg \), we have \( \lambda^{(i)}_V(Jg) \in L^2_2(\Gamma \backslash G) \). Also, \( \exp(2i\theta)y^{-1}\lambda^{(i)}_V(Jg) \) with \( \lambda^{(i)}_V \) as in (21.10) is an antiholomorphic cusp form of weight \( 2\ell \).

Notes: The spectral decomposition (22.7)–(22.11) can be approached via the Green function for \( \Omega \). Salient points of the argument will be indicated in Sections 32–37.

23. Spectral decomposition of \( L^2(\Gamma \backslash G) \). Combining the assertions (12.2), (17.1)–(17.2), (17.6), and (22.7)–(22.9) we obtain our main result. This is essentially the same as the implication of (17.7) and (22.10). Or more drastically rendering, it is the same as the rearrangement of the result so far established by exchanging the order of the indices \( V \) and \( \ell \).

Theorem. Let \( ^0L^2(\Gamma \backslash G) \) be the cuspidal subspace of \( L^2(\Gamma \backslash G) \) which is spanned by all vectors whose constant terms in the Fourier expansion with respect to the left action of \( N \) vanish, and let \( L^2(\Gamma \backslash G) \) be the subspace generated by integrals of all Eisenstein series \( E_\ell \), \( \ell \in \mathbb{Z} \), as indicated by (22.9). Then we have

\[
L^2(\Gamma \backslash G) = C \cdot 1 \oplus ^0L^2(\Gamma \backslash G) \oplus ^{\omega}L^2(\Gamma \backslash G),
\]

(23.1)

Here \( V \)'s are all irreducible subspaces with \( \Omega \lambda^{(i)}_V = \left( \frac{1}{2} - \nu^2 \right) \lambda^{(i)}_V \). If \( V \) is generated by a real analytic cusp form via (22.1), that is, \( \nu_V \in i\mathbb{R} \), then the corresponding index \( \ell \) runs over all integers. Otherwise, either \( \ell \geq l \) or \( \ell \leq -l \) according as \( V \) is generated by a holomorphic cusp form of positive even integral weight \( 2l \) via (22.3) or via an antiholomorphic cusp form analogously, that is, \( \nu_V = l - \frac{1}{2} \). Further, we have, for any \( f \in L^2(\Gamma \backslash G) \),

\[
f(g) = \frac{3}{\pi} \langle f, 1 \rangle + \sum_V \sum_{\ell} \langle f, \lambda^{(i)}_V \rangle \lambda^{(i)}_V(g)
+ \sum_{\ell=-\infty}^{\infty} \frac{1}{4\pi i} \int_{(0)} E_\ell(f, \nu)E_\ell(g, \nu) d\nu,
\]

(23.2)

where the range of \( \ell \) in the first line is the same as in the corresponding part of (23.1), and \( E_\ell(f, \nu) \) is defined by (22.11), although \( f \) is not restricted to \( L^2_2(\Gamma \backslash G) \).

Notes: More precisely, each \( V \) stands for the closure of the sum \( \oplus_\ell \mathbb{C} \lambda^{(i)}_V \); and \(^0L^2(\Gamma \backslash G)\), and \(^{\omega}L^2(\Gamma \backslash G)\) are understood similarly. Also the identity (23.2) holds, with sums and integrals converging in the mean.

24. Series of irreducible representations. The invariance and the irreducibility of each \( V \) remains to be established; we shall achieve it in due course. Thus the title of the present section might be premature. We shall proceed with caution so that the use of this notion should not cause any confusion.

We introduce first the classification of \( V \)'s:

Unitary principal series \( \leftrightarrow \) real analytic cusp forms,
Discrete series \( \leftrightarrow \) either holomorphic or antiholomorphic cusp forms,

(24.1)
where the arrows mean that each series are generated by respective variety of cusp forms on $\mathbb{H}^2$; thus the latter splits into the holomorphic and antiholomorphic discrete series. This is the same as

$$V$$ in the unitary principal series $\leftrightarrow \nu_V \in i\mathbb{R},$

$$V$$ in the discrete series $\leftrightarrow \nu_V \in \mathbb{N} - \frac{1}{2},$$

with $\Omega|_V = \left(\frac{1}{4} - \nu_V^2\right) \cdot 1. \quad (24.2)$

In general, there possibly exists the complementary series, whose constituents are generated by real analytic cusp forms associated with exceptional eigenvalues of $\Omega_0$. With $\Gamma = \text{PSL}(2, \mathbb{Z})$ such forms do not exist as (17.7) indicates; see [22, Lemma 1.4].

We observe that the Fourier expansions (22.2) and (22.4) of the basis vectors $\lambda^{(\ell)}_V$ as well as their $J$-images are conveniently expressed as

$$\lambda^{(\ell)}_V (g) = \left| \pi^{-2\nu_V} \frac{\Gamma(|\ell| + \nu_V + \frac{1}{2})}{\Gamma(|\ell| - \nu_V + \frac{1}{2})} \right|^{1/2} \sum_{n \neq 0} \frac{\delta_V(n)}{|n|} A^{\text{sgn}}(n) \phi_V(a|n|g, \nu_V), \quad (24.3)$$

where (15.4) is to be invoked to see that (22.4) is well included. With this,

the sequence $\{\delta_V(n) : |n| \in \mathbb{N}\}$ is dependent solely on the space $V$, \quad (24.4)

except for an arbitrary multiplier of unit absolute value. These coefficients do not depend on weights. Hence, it is appropriate to designate them as the Fourier coefficients of an irreducible subspace or representation $V$. It should be stressed that this normalisation of Fourier coefficients of cusp forms has been made possible by the use of the Jacquet operator $A^g$ and the weight function $\phi_V^g (g, \nu_V)$.

**25. Local structure.** In order to illuminate the grand assertion (23.1), we fantasise that each subspace $L^2_{\ell} (\Gamma \backslash G)$ is the galaxy $\ell$. Then we observe that all the galaxies revolve at respective angular velocities under the right action of $K$. They are, however, not independent of each other. Maass operators transport vectors from a galaxy to others along light-ladders $V$, most of which start in the fount $L^2_{\ell} (\Gamma \backslash G) \equiv L^2 (\Gamma \backslash \mathbb{H}^2)$ and extend to infinity in both directions, while others emerge spontaneously and discretely in pairs above and below the fount and extend to respective infinity. The principal part of each galaxy is a slice of the sum of $V$’s in the unitary principal series, which is a unitary image of the fount. The rest is a slice of the sum of $V$’s in the discrete series. To see how the latter expands with $|\ell|$, one should observe that a particular $V$ in the discrete series can emerge in the galaxy $\ell$ only when $\nu_V = |\ell| - \frac{1}{2}$.

This is a view which although fairly beautiful is nevertheless not of much use, especially in applications to problems in analytic number theory. What matters in practice is nothing else but to have precise analytical structures of individual light-ladders $V$ so that the projection to $V$, i.e., the relevant sum over the weights in (23.2), of a given vector can be computed explicitly and effectively, without recourse to deeper natures of the Fourier coefficients $\{\delta_V(n)\}$, preferably independently of them. We stress in this context that the construction of the subspace $L^2 (\Gamma \backslash G)$ is fairly representational and visible as Eisenstein series $E_{\ell}$ are defined by (18.1) and have expansions (18.2), whereas basis vectors (24.3) of the cuspidal subspace are far more abstract objects as the nature of the numbers $\{\delta_V(n)\}$ remains largely mysterious.

Hence, our next task is to cast light on the structure of each $V$ in the cuspidal subspace, an issue which is local in the spectral context but central for practical purposes.
Notes: The Fourier coefficients $\sigma_{2r}(n)$ correspond to $g_\nu(n)$. The former has a visible inner structure as being expressed in terms of a sum over divisors of $n$, which is not well shared by the latter, even though the Hecke operators disclose similarities between these Fourier coefficients. However, this ostensive nature of divisor functions is misleading. They are in fact equally mysterious, as is demonstrated, for instance, by the additive divisor problem which concerns the sum
\[ \sum_{n=1}^{\infty} \sigma_\alpha(n)\sigma_\beta(n + f)W(n/f), \] (25.1)
where $\alpha, \beta \in \mathbb{C}$, $f \in \mathbb{N}$, and the weight function $W$ is supposed to be sufficiently smooth and of rapid decay; see [21] and [27, Section 6.4].

26. Invariance. We shall first prove that the $V$’s in (23.1) are all invariant with respect to the right action of $G$. Dealing with a $V$ in the unitary principal series, i.e., $\nu_\nu \in i\mathbb{R}$, we put
\[ \nu_\nu = \left\{ \lambda(g) = \sum_{\ell=0}^{\infty} c_\ell \lambda_\nu^{(\ell)}(g) : c_\ell \ll (|\ell| + 1)^{-M} \text{ with any } M > 0 \right\}, \] (26.1)
where the implied constant may depend on $M$. The absolute convergence of the sum can be confirmed by means of (15.5) and (17.11); we plug (24.3) into the definition of $\lambda(g)$, and exchange the order of summation, which is justified in much the same way as in (20.2), although this time we sum over $\ell$ instead of $V$. Then we note that $A^\delta$ can be applied term-wise to
\[ \phi(g) = \sum_{\ell=-\infty}^{\infty} c_\ell \phi_\ell(g, \nu_\nu) = g^{1/2+\nu_\nu} \Phi(\theta), \] (26.2)
\[ \Phi(\theta) = \sum_{\ell=-\infty}^{\infty} c_\ell e^{2\ell i\theta} \in C^\infty(\mathbb{R}/\pi\mathbb{Z}). \]
In fact we apply integration by parts to (15.4) and have that for $h = n[a]\|a\|\|n\|g$.

Before applying right translations, we invoke (5.4) and have that for $h = n[a]\|a\|\|n\|g$
\[ r_h \phi(g) = g^{1/2+\nu_\nu} \frac{\beta^{1/2+\nu_\nu} \Phi(\tau + \vartheta(\theta))}{((\cos \theta - \alpha \sin \theta)^2 + (\beta \sin \theta)^2)^{1/2+\nu_\nu}} \]
\[ = \sum_{\ell=-\infty}^{\infty} c_\ell^h \phi_\ell(g, \nu_\nu), \quad c_\ell^h \ll (|\ell| + 1)^{-M}, \] (26.4)
for any $M > 0$, since the first line belongs to $C^\infty(\mathbb{R}/\pi\mathbb{Z})$ as a function of $\theta$. Reversing the order of reasoning, we find that $r_h \lambda(g) \in V^\nu$, i.e., $r_h V^\nu \subseteq V^\nu$. As $V^\nu$ is dense in $V$ and $r_h$ is unitary, we conclude that
\[ the \ V’s \ in \ the \ unitary \ principal \ series \ are \ all \ invariant. \] (26.5)

We next consider a $V$ in the holomorphic discrete series; thus $\nu_\nu = l - \frac{1}{2}, l \in \mathbb{N}$. In place of (26.1)–(26.3), we put
\[ V^\nu = \left\{ \lambda(g) = \sum_{\ell=1}^{\infty} c_\ell \lambda_\nu^{(\ell)}(g) : c_\ell \ll (\ell + 1)^{-M} \text{ with any } M > 0 \right\}, \] (26.6)
\[
\phi(g) = \pi^{1/2-l}y^{\ell} \sum_{\ell=1}^{\infty} c_{\ell} \left( \frac{\Gamma(\ell + l)}{\Gamma(\ell - l + 1)} \right)^{1/2} e^{2i\ell\theta},
\]
(26.7)

\[
\lambda(g) = \sum_{n=1}^{\infty} \frac{\theta V(n)}{\sqrt{|n|}} A^+ \phi(a[|n||g]).
\]
(26.8)

The verification of the last identity may be skipped, as it is analogous to that of (26.3). However, in order to show the analogue of (26.4) we need a minor contrivance: Via (5.4) we have, with \(z = e^{i\theta}\),

\[
r_h(\phi(g)) = \pi^{1/2-l}(4\beta y)^l z^{2l} \sum_{\ell=1}^{\infty} c_{\ell} e^{2i\ell\tau} \left( \frac{\Gamma(\ell + l)}{\Gamma(\ell - l + 1)} \right)^{1/2}
\times \frac{((1 + \beta + \alpha i)z^2 + 1 - \beta - \alpha i)^{\ell-l}}{((1 - \beta + \alpha i)z^2 + 1 - \beta - \alpha i)^{\ell+l}}.
\]
(26.9)

This sum is an even regular function for \(|z| < 1\) and also belongs to \(C^\infty(\mathbb{R}/\pi\mathbb{Z})\) as a function of \(\theta\). We conclude that \(r_h V^\infty \subseteq V^\infty\). Thus, together with an application of the map (13.2), we conclude that

the \(V\)'s in the discrete series are all invariant.
(26.10)

We may now use safely the term ‘a representation \(V\)’. The irreducibility is to be proved in Section 30.

NOTES: The invariance assertion is usually confirmed via the Lie algebra \(g\). Our argument may appear to be unconventional. However, the discussion based on explicit group actions should also be worth reporting.

27. Kirillov map. We shall deal with the task set out in Section 25. To this end, we introduce the map:

\[
\mathcal{K}\phi(u) = A^{\text{even}(u)} \phi(a[|u|]),
\]
(27.1)

following Kirillov [16], where \(\phi\) can be any function on \(G\) as far as the relevant integral converges in the same sense as in (14.1). In what follows, we are concerned mainly with the specialisation \(\phi(g) = \phi_{t}(g, \nu)\), i.e., \(\mathcal{K}\phi(u) = \mathcal{K}\phi_{t}(u, \nu)\). We shall show that

for each \(\nu \in i\mathbb{R}\), the set \(\{\mathcal{K}\phi_{t}(u, \nu) : \ell \in \mathbb{Z}\}\) is

a complete orthonormal system of \(L^2(\mathbb{R}^x, d^x/\pi)\),
(27.2)
as well as that

for each \(l \in \mathbb{N}\), the set \(\pi^{1/2-l}(\Gamma(\ell + l)/\Gamma(\ell - l + 1))^{1/2}\mathcal{K}\phi_{t}(u, l - \frac{1}{2}) : \ell \geq l\}\) is

a complete orthonormal system of \(L^2(\mathbb{R}^x_+, d^x/\pi)\).
(27.3)

Here \(\mathbb{R}^x = \mathbb{R}\setminus\{0\}\), \(d^xu = du/|u|\); and \(\mathbb{R}^x_+\) is the set of positive real numbers.

By (16.1) we have, for \(\nu \in i\mathbb{R}\) and \(\ell, \ell' \in \mathbb{Z}\),

\[
\langle \mathcal{K}\phi_{t}(:, \nu), \mathcal{K}\phi_{t'}(:, \nu) \rangle = \frac{(-1)^{\ell+\ell'}}{\Gamma(\ell + \nu + \frac{1}{2})\Gamma(\ell' - \nu + \frac{1}{2})} \int_0^{\infty} W_{\ell,\nu}(y) W_{\ell',\nu}(y) \frac{dy}{y}
\]
\[
+ \frac{(-1)^{\ell+\ell'}}{\Gamma(- \ell + \nu + \frac{1}{2})\Gamma( - \ell' - \nu + \frac{1}{2})} \int_0^{\infty} W_{-\ell,\nu}(y) W_{-\ell',\nu}(y) \frac{dy}{y},
\]
(27.4)
where the inner product is taken in $L^2(\mathbb{R}^\times, d^\times/\pi)$ and we have used the fact that $W_{\ell',r}(y)$ is real, as (16.3) implies. Replacing $\ell, \ell'$ by $\alpha, \beta \in \mathbb{C}$ and applying (16.8), the right side equals
\[
\frac{1}{\pi(\alpha - \beta)\sin(2\pi\nu)} \left\{ \sin \left( \pi \left( \frac{1}{2} - \nu + \alpha \right) \right) \sin \left( \pi \left( \frac{1}{2} + \nu + \beta \right) \right) - \sin \left( \pi \left( \frac{1}{2} + \nu + \alpha \right) \right) \sin \left( \pi \left( \frac{1}{2} - \nu + \beta \right) \right) \right\}. \tag{27.5}
\]
On the assumption that $\alpha, \beta$ are unequal integers, this vanishes; and when $\ell = \ell'$, we take a limit in (27.5), getting $\|X\phi_\ell(\cdot, \nu)\| = 1$.

As for the completeness assertion in (27.2), let $g$ be a smooth function, compactly supported in $\mathbb{R}^\times$, which is orthogonal to all $\mathcal{K}\phi_\ell(\cdot, \nu)$, $\ell \in \mathbb{Z}$, in the space $L^2(\mathbb{R}^\times, d^\times/\pi)$. We apply integration by parts to (15.2), and in the new outer integral we undo the integration by parts. In this way, we have
\[
\alpha, \beta \in \mathbb{C}, \quad \ell \geq \ell' - 1
\]
\[
\frac{1}{\|g\|^2} \left\{ \frac{1}{\pi(\alpha - \beta)\sin(2\pi\nu)} \left\{ \sin \left( \pi \left( \frac{1}{2} - \nu + \alpha \right) \right) \sin \left( \pi \left( \frac{1}{2} + \nu + \beta \right) \right) - \sin \left( \pi \left( \frac{1}{2} + \nu + \alpha \right) \right) \sin \left( \pi \left( \frac{1}{2} - \nu + \beta \right) \right) \right\} \right\} \|g\|^2.
\]

Then we invoke that the set $\left\{ \left( (\xi - i)/\xi + i \right)^\ell : \ell \in \mathbb{Z} \right\}$ is a complete orthonormal system of the space $L^2(\mathbb{R}^\times, (\pi\xi^2 + 1)^{-1}d\xi)$, as can be readily seen by the change of variable $\xi \mapsto \tan \theta$. Hence the Fourier transform of $g(u)|u|^{-1/2 + \nu}$ vanishes, which ends the proof of (27.2).

We next proceed to the proof of (27.3). By (15.2) and (16.1), the function $W_{\ell,\ell'-\frac{1}{2}}(u)$, for $\ell \geq l$, equals $u^\ell \exp(-u/2)$ multiplied by a polynomial factor of degree $\ell - l$; one may apply residue calculus to (15.2) under the present specification. In particular, although the condition on Re $\nu$ does not hold, the argument (16.10) extends nevertheless to the product $W_{\ell,\ell'-\frac{1}{2}}(u)W_{\ell',\ell'-\frac{1}{2}}(u)$, $\ell, \ell' \in \mathbb{Z}$, and we get the orthogonality for $\ell \neq \ell'$. On the other hand, dealing with the case $\ell = \ell'$, we argue as follows: We let $\omega$ be as in (16.7). Then we have, for $\ell > 0$,
\[
\int_0^\infty \left( W_{\ell,\ell'-\frac{1}{2}}(y) \right)^2 \frac{dy}{y} = -\int_0^\infty \left[ \omega^+ (\ell - 1) \right] W_{\ell-1,\ell'-\frac{1}{2}}(y) W_{\ell,\ell'-\frac{1}{2}}(y) \frac{dy}{y}
\]
\[
= \int_0^\infty W_{\ell-1,\ell'-\frac{1}{2}}(y) \left[ \omega^- - \ell \right] W_{\ell,\ell'-\frac{1}{2}}(y) \frac{dy}{y}
\]
\[
= (\ell - l)\left( \ell + l - 1 \right) \int_0^\infty \left( W_{\ell-1,\ell'-\frac{1}{2}}(y) \right)^2 \frac{dy}{y}
\]
\[
= \Gamma(\ell + l + 1) \Gamma(\ell + l). \tag{27.7}
\]
We have applied integration by parts as well as the orthogonality of $W_{\ell-1,\ell'-\frac{1}{2}}$ and $W_{\ell,\ell'-\frac{1}{2}}$; and the last line is due to $W_{\ell,\ell'-\frac{1}{2}}(y) = y^\ell \exp \left( -\frac{1}{2} y \right)$ as is implied by (15.4) and (16.1). We have verified the orthonormality assertion in (27.3). On the other hand, let $g$ be a smooth function compactly supported in $\mathbb{R}^\times_+$ which is orthogonal to all $\mathcal{K}\phi_\ell(\cdot, l - \frac{1}{2})$, $\ell \geq l$, in the space $L^2(\mathbb{R}^\times, d^\times/\pi)$. Then, because of the construction of $W_{\ell,\ell'-\frac{1}{2}}(u)$ mentioned above, we have
\[
\int_0^\infty g(u) \exp(-2\pi u) u^{\ell-1} du = 0, \quad \ell \geq l. \tag{27.8}
\]
Hence the Fourier transform of $g(u) \exp(-2\pi u) u^{\ell-1}$ vanishes, as it follows via the Taylor expansion of the factor $\exp(i\xi u)$ to be multiplied. We end the proof of (27.3).
Notes: The argument of this section is taken from [23, Part XII] and [5]. The orthonormality assertion in (27.2) can be proved in a smarter way as follows: By (14.2) we have, for $\phi(g) = y^{1/2+it}\Phi(\theta), \nu \in \mathbb{R}$,
\[
\mathcal{K}\phi(u) = |u|^{1/2-\nu} \int_{-\infty}^{\infty} \frac{e(-u\xi)}{(1+\xi^2)^{1/2+\nu}} \Phi(\varphi(\xi)) d\xi, \quad \exp(2i\varphi(\xi)) = \frac{\xi-i}{\xi+i}.
\] (27.9)

The Parseval formula for Fourier integrals gives
\[
\langle \mathcal{K}\phi_1, \mathcal{K}\phi_2 \rangle = \int_{-\infty}^{\infty} \frac{\Phi_1(\varphi(\xi))\Phi_2(\varphi(\xi))}{1+\xi^2} \frac{d\xi}{\pi} = \frac{1}{\pi} \int_{\mathbb{R}/\pi\mathbb{Z}} \Phi_1(\theta)\Phi_2(\theta) d\theta,
\] (27.10)
with an obvious correspondence, where the inner product is taken in $L^2(\mathbb{R}^\times, d^\times/\pi)$, and the change of variable $\xi \mapsto \tan \theta$ has been applied. This yields the assertion. However, the argument does not seem to readily extend to the discrete series, which is the reason why we have employed an argument that might appear verbose. Naturally, one may try to exploit the action such as (18.5) of Maass operators in conjunction with (14.3), which is but essentially the same as the use of (16.7), duly indicated in the relevant notes. The argument of the present section extends to $\text{PSL}(2, \mathbb{C})$; see [24].

28. Representations realised. Now, we recapitulate the most salient points of our discussion in the last two sections: We pick up a $V$ in the unitary principal series; thus $\nu V \in \mathbb{R}$. The assertion (27.2) means that we have the unitary and surjective map
\[
\mathcal{L} : V \mapsto L^2(\mathbb{R}^\times, d^\times/\pi),
\]
\[
\mathcal{L}\lambda^{(\ell)}(u) = \mathcal{K}\phi_\ell(u, \nu V).
\] (28.1)

Combined with (26.5), this yields a realisation of the representation $V$:
\[
\text{The map } r^V : h \mapsto \mathcal{L}r_h \mathcal{L}^{-1}
\]

is a unitary representation of $G$ on $L^2(\mathbb{R}^\times, d^\times/\pi)$, which is equivalent to the representation $V$.

That is, any right action of $G$ in the space $V$ is realised faithfully in $L^2(\mathbb{R}^\times, d^\times/\pi)$. We have seen at (26.4) that inside $V^\infty$ the mode of the translation $\lambda \mapsto r_h \lambda$ is exactly the same as that of $\phi \mapsto r_h \phi$; namely, $\mathcal{L}r_h \lambda = \mathcal{K}\lambda \phi$ over $V^\infty$, with $\phi$ as in (26.2). In other words, we have $r^V_h \mathcal{K}\phi = \mathcal{K}r_h \phi$. Since $\mathcal{K}\mathcal{L}_{n[a]}(\beta)\phi(u) = e(au)\mathcal{K}\phi(\beta u)$ and $\{\mathcal{K}\phi\}$ is dense in $L^2(\mathbb{R}^\times, d^\times/\pi)$, we have
\[
r^V_{n[a]}(\beta) f(u) = e(au) f(\beta u), \quad \forall f \in L^2(\mathbb{R}^\times, d^\times/\pi).
\] (28.3)

For a $V$ in the holomorphic discrete series, we need only to replace $L^2(\mathbb{R}^\times, d^\times/\pi)$ and $\mathcal{K}\phi(u, \nu V)$ in (28.1)–(28.3) by $L^2(\mathbb{R}^+_\times, d^\times/\pi)$ and $\pi^{1/2-l}(\Gamma(\ell+l)/\Gamma(\ell-l+1))^{1/2} \mathcal{K}\phi(u, \nu V)$, $\ell \geq l$, respectively. The antiholomorphic discrete series is analogous; it suffices to apply the involution $J$ defined by (13.2).

Notes: In literature, the fact (28.2) is often termed the Kirillov model. This was recently employed by the present author [23, Parts XIV and XV] in his resolution of Selberg’s decades-old problem [33] to find a complete spectral decomposition, within the structure of $L^2(F\backslash G)$, of the shifted convolution of Fourier coefficients of cusp forms, i.e., sums
\[
\sum_{n=1}^{\infty} q_V(n+f)\overline{q_V(n)} W(n/f),
\] (28.4)
with generic cuspidal representations $V$ as is defined by (24.3), which is obviously an analogue of (25.1). In the solution the use of the right action of $K$ is essential; that is, the spectral theory of cusp forms of weight zero is inadequate, and representation theory comes into play in an indispensable fashion. See also [27, Vol. 2, Section 7.2].

29. Bessel functions of representation. We point out an obvious incompleteness of the formula (28.3): It lacks the description of the right action of $K$. This is because of the fact that when considering $r_{W}^{V}$ with the general combination of $k_{[\tau]}$ and $f$ we need first to spectrally decompose $f$ in terms of the system $\{X\phi_{\ell}(u, \nu)\}$ if we follow the argument so far developed. Thus one may ponder whether it is possible or not to avoid this tedious procedure. It should be observed here that we actually do not need to consider all of $\{k_{[\tau]}\}$, since we have the Bruhat decomposition (14.4). Namely, what really matters is the action of the Weyl element $w = k_{[1/2, 1]}$. We need to express $r_{W}^{V}f$ in terms of $f$. The answer is an integral transform, and its kernel is the Bessel function of representation.

$$j_{\nu}(\lambda) = \frac{\sqrt{|\lambda|}}{\sin \pi \nu} \left( J_{-2\nu}^{\text{sgn}(\lambda)}(4\pi \sqrt{|\lambda|}) - J_{2\nu}^{\text{sgn}(\lambda)}(4\pi \sqrt{|\lambda|}) \right), \quad \lambda \in \mathbb{R}^{\times}, |\Re \nu| < \frac{1}{2},$$  

(29.1)

where $J_{\nu}^{+} = J_{\nu}$ and $J_{\nu}^{-} = I_{\nu}$, with the usual notation for Bessel functions. We have that

$$r_{W}^{V}f(u) = \int_{\mathbb{R}^{\times}} j_{\nu}(u\lambda)f(\lambda)d\xi$$  

(29.2)

for any $V$ in the unitary principal series.

It is meant that the integral transform maps a dense subset of $L^{2}(\mathbb{R}^{\times}, d\xi/\pi)$ into the space unitarily. Together with (14.4) and (28.3), this describes explicitly the action of $G$ over $L^{2}(\mathbb{R}^{\times}, d\xi/\pi)$ via the invariant subspace $V$ of $L^{2}(G \backslash G)$.

For the proof of (29.2) we consider first the Mellin transform

$$\Gamma_{\ell}(s, \nu) = \int_{0}^{\infty} A^{+}\phi_{\ell}(a[y], \nu)y^{s-3/2}dy.$$  

(29.3)

We shall show that $\Gamma_{\ell}(s, \nu)$ exists in the domain $|\Re s - \frac{1}{2}| < \Re \nu + \frac{1}{2}$ and satisfies the local functional equation of Jacquet–Langlands [14, p. 196]

$$(-1)^{\ell} \Gamma_{\ell}(s, \nu) = 2(2\pi)^{-2s} \Gamma(s + \nu) \Gamma(s - \nu)$$  

$$\times \left( \cos(\pi s) \Gamma_{\ell}(1 - s, \nu) + \cos(\pi \nu) \Gamma_{-\ell}(1 - s, \nu) \right).$$  

(29.4)

To this end, in (15.2), we shift the contour to $\Im \xi = -\frac{1}{2}$, and insert the result into (29.3). The double integral converges absolutely in the domain $0 < \Re \nu < \Re s$. After exchange, the inner integral is seen to be $(2\pi i \xi)^{\nu-s} \Gamma(s - \nu)$, $|\arg(i\xi)| < \frac{1}{2} \pi$. We shift the $\xi$-contour back to the original, getting

$$\Gamma_{\ell}(s, \nu) = (2\pi)^{\nu-s} \Gamma(s - \nu)$$

$$\times \left[ \exp \left( \frac{\pi i}{2}(s - \nu) \right) Y_{-\ell}(s, \nu) + \exp \left( -\frac{\pi i}{2}(s - \nu) \right) Y_{\ell}(s, \nu) \right],$$  

(29.5)

$$Y_{\ell}(s, \nu) = \int_{0}^{\infty} \frac{\xi^{-s+\nu}}{(\xi^{2} + 1)^{\nu+1/2}} \left( \frac{\xi - i}{\xi + i} \right)^{\ell} d\xi.$$  

(29.6)
This integral converges absolutely and uniformly for \(|\text{Re} \, s - \frac{1}{2}| < \text{Re} \, \nu + \frac{1}{2}\). We apply the change of variable \(\xi \mapsto \xi^{-1}\), and find that in the same domain
\[
(-1)^\ell \mathbf{Y}_\ell(s, \nu) = \mathbf{Y}_{-\ell}(1 - s, \nu).
\] (29.7)

We then apply, to (29.5), the transformations \(s \mapsto 1 - s, \ell \mapsto \pm \ell\) as well as (29.7), which yields, after an elimination,
\[
(-1)^\ell \mathbf{Y}_\ell(s, \nu) = (2\pi)^{-s-\nu} \Gamma(s + \nu) \\
\times \left( \mathbf{\Gamma}_\ell(1 - s, \nu) \exp \left( \frac{\pi i}{2}(s + \nu) \right) + \mathbf{\Gamma}_{-\ell}(1 - s, \nu) \exp \left( -\frac{\pi i}{2}(s + \nu) \right) \right).
\] (29.8)

Inserting this back to (29.5) we obtain (29.4).

On the other hand, we have, for \(|\text{Re} \, \nu| < \text{Re} \, s < \frac{1}{4}\),
\[
\int_0^\infty j_\nu(\lambda) |\lambda|^{s-1/2} d\lambda = 2(2\pi)^{-2s}\cos(\pi s) \Gamma(s + \nu) \Gamma(s - \nu),
\] (29.9)

and, for \(|\text{Re} \, \nu| < \text{Re} \, s\),
\[
\int_0^{-\infty} j_\nu(\lambda) |\lambda|^{s-1/2} d\lambda = 2(2\pi)^{-2s}\cos(\pi \nu) \Gamma(s + \nu) \Gamma(s - \nu),
\] (29.10)

which are consequences of the well-known integral formulas
\[
\int_0^\infty J_\nu(y) y^{-s-1} dy = 2^{-s-1} \Gamma \left( \frac{s + \nu}{2} \right) / \Gamma \left( 1 - \frac{s + \nu}{2} \right), \quad -\text{Re} \, \nu < \text{Re} \, s < \frac{1}{2},
\]
\[
\int_0^\infty K_\nu(y) y^{-s-1} dy = 2^{s-2} \Gamma \left( \frac{s + \nu}{2} \right) \Gamma \left( \frac{s - \nu}{2} \right), \quad |\text{Re} \, \nu| < \text{Re} \, s.
\] (29.11)

By (29.9)–(29.10), we may rewrite (29.4) as
\[
(-1)^\ell \mathbf{\Gamma}_\ell(s, \nu) = \int_{\mathbb{R}^\times} j_\nu(\lambda) |\lambda|^{s-1/2} \mathbf{\Gamma}_{\text{sgn}(\lambda)} \ell(1 - s, \nu) d\lambda, \quad \text{Re} \, s = \beta, |\text{Re} \, \nu| < \beta < \frac{1}{4},
\] (29.12)

although the range of \((\nu, s)\) is to be restricted to have (29.9). We replace \(\ell\) by \(\text{sgn}(u)\ell\), multiply both sides by the factor \(|u|^{1/2-s}/2\pi i \neq 0\), and integrate along \(Re s = \beta, \text{Re} \, \nu < \beta < \frac{1}{4}\). We get
\[
\frac{(-1)^\ell}{2\pi i} \int_{(\beta)} \mathbf{\Gamma}_{\text{sgn}(u)} \ell(s, \nu) |u|^{1/2-s} d s
\]
\[
= \int_{\mathbb{R}^\times} j_\nu(\lambda) \left\{ \frac{1}{2\pi i} \int_{(\beta)} \mathbf{\Gamma}_{\text{sgn}(\lambda u)} \ell(1 - s, \nu) |\lambda/u|^{s-1/2} d s \right\} d\lambda.
\] (29.13)

This exchange is legitimate, since the function \(\mathbf{\Gamma}_\ell(s, \nu)\) is of rapid decay, as can be seen by turning the line of integration in (29.6) through a small angle round the origin. Being a Mellin inversion of (29.3), the left side equals \((-1)^\ell \mathcal{A}^\ell \phi_{\text{sgn}(u)\ell}(a|u|, \nu) = \mathcal{K}_{\text{sgn}(\lambda/u)}(\lambda/|u|, \nu)\), while the inner integral is \(\mathcal{A}^\ell \phi_{\text{sgn}(\lambda u)}(a|\lambda/u|, \nu) = \mathcal{K}_\ell(\lambda/u, \nu)\). Namely, we have obtained the following point-wise identity, but only for \(|\text{Re} \, \nu| < \frac{1}{4}\):
\[
\mathcal{K}_{\text{sgn}(\lambda/u)}(\lambda/|u|, \nu) = \int_{\mathbb{R}^\times} j_\nu(u) \mathcal{K}_\ell(\lambda, \nu) d\lambda, \quad |\text{Re} \, \nu| < \frac{1}{2}.
\] (29.14)
The extension of the range of $\nu$ can be attained by (15.5) and analytic continuation.

Now, let $f$ be as in (29.2). We consider the double integral

$$
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} j_{\nu v}(u\lambda) f(\lambda) d\lambda d\nu \mathcal{K}\phi(u, \nu) \frac{d^\nu u}{\pi}, \tag{29.15}
$$

Invoking (15.5) again, we see that this converges absolutely, and find that (29.14) implies that it equals

$$
\langle f, \mathcal{K}r_w \phi(\cdot, \nu) \rangle = \langle f, r_w^V \mathcal{K} \phi(\cdot, \nu) \rangle = \langle r_w^V f, \mathcal{K} \phi(\cdot, \nu) \rangle \tag{29.16}
$$

in $L^2(\mathbb{R}^\times, d^\nu u)$, as $r_w^V$ is an involution. Via (27.2) one might conclude that (29.2) has been confirmed. However, it remains for us to show that

$$
F(u) = \int_{\mathbb{R}^2} j_{\nu v}(u\lambda) f(\lambda) d\lambda \text{ is in } L^2(\mathbb{R}^\times, d^\nu u). \tag{29.17}
$$

When $|u| \leq 1$, the definition (29.1) implies that $j_{\nu v}(u\lambda) \ll \sqrt{|u\lambda|}$; and $F(u) \ll \sqrt{|u|}$, which means that the integral of $|F(u)|^2$ against $d^\nu u$ over $|u| \leq 1$ is finite. When $|u| \geq 1$ and $u\lambda < 0$, the asymptotic expansion for $K$-Bessel functions gives $j_{\nu v}(u\lambda) \ll \exp(-\sqrt{|u\lambda|})$; that is, the corresponding part of $F(u)$ can be ignored. Hence it suffices to show that

$$
\int_{U_1}^{U_2} \left\| \int_0^\infty j_{\nu v}(u\lambda) f(\lambda) d\lambda \right\|^2 d^\nu u \ll 1 \text{ uniformly for } 1 \leq U_1 < U_2, \tag{29.18}
$$

since the part with $u, \lambda < 0$ is analogous. We then invoke the asymptotic expansion for $J$-Bessel functions; and the discussion is reduced to that on the expression

$$
\int_{U_1}^{U_2} \left\| \int_0^\infty (u\lambda)^{1/4} \exp(4\pi i(u\lambda)^{1/2}) f(\lambda) d\lambda \right\|^2 d^\nu u
= 8 \int_{U_1}^{U_2} \left\| \int_0^\infty \mu^{-1/2} \exp(4\pi i\nu \mu) f(\mu^2) d\mu \right\|^2 d\nu, \tag{29.19}
$$

in which the change of variables $(u, \lambda) \mapsto (u^2, \mu^2)$ has been applied. Hence, by the Parseval formula for Fourier integrals, we obtain (29.17). This ends the proof of (29.2).

The discussion on the discrete series is skipped, as it is fairly analogous to the above. We remark only that for any $l \in \mathbb{N}$

$$
j_{l-\frac{1}{2}}(u) = \begin{cases} 2\pi (-1)^l \sqrt{u} & u < 0, \\ 0 & u > 0. \end{cases} \tag{29.17}
$$

In passing, we remark that the complementary series of irreducible representations can be discussed in much the same way; then the range of $\nu$ in (29.14) becomes relevant, although this is immaterial in our present context.

**Notes:** As to the formulas (29.11) as well as the asymptotic expansion of Bessel functions, see Watson [37]; a concise proof of these can be found in [27, Vol. 2, Section 1.4]. In conjunction with the Kirillov map, the Bessel function of representation (29.1) has played a fundamental rôle in the modern developments of analytic number theory. As Cogdell and Piatetskii-Shapiro pointed out in their inspiring monograph [6], it arises in the spectral decomposition of Poincaré series on $G$ in general, typical instances of which occur in the
theory of sums of Kloosterman sums due to Bruggeman [2] and Kuznetsov [17] and in the
tility of the fourth power moment of the Riemann zeta-function due to the present author
[22]. See also [5], [23, Parts XIV and XV] and [26], for instance. The above proof of the
fundamental transformation formula in (29.2) is an adaptation of a part of [23, Part XII]
where smooth \( f \), i.e., those \( \mathcal{K} \phi \) with \( \phi \) as in (26.2), is in fact dealt with; and this time we
have extended the assertion to compactly supported vectors by means of the augmentation
(29.15)–(29.19). Naturally, (15.5) and (29.14) readily imply the identity in (29.2) for any
smooth \( f \). The above proof of the very basic functional equation (29.4) comes also from
[23, Part XII]. An alternative and independent proof of the identity in (29.2), but only for
smooth \( f \), is given by Baruch and Mao [1, Section 6 and Appendix 2], which is in fact a
verification of Vilenkin’s claim made at the end of Chapter VII of the older edition of [36];
see also [36, p. 454]. It should be stressed that for any smooth \( f \) the identity in (29.2)
holds point-wise. We add that the statement (29.2) can readily be extended to any vector
\( f \) in terms of the mean convergence; see the first line of (30.4) below. Further, we remark
that the argument of this section extends to \( \text{PSL}(2, \mathbb{C}) \); necessary means may be found in
[24]. Incidentally, it will be worth remarking that the discussion in [6] is incomplete in the
sense that it lacks an inversion procedure which is necessary in stating two versions of the
spectral expansion of sums of Kloosterman sums, due originally to Kuznetsov [17]. This

\section{30. Irreducibility.} We now prove that any subspace \( V \) in the unitary principal series is
irreducible. Because of (28.2) it suffices to show that \( r^V \) is an irreducible representation.
Thus, let \( U_1 \) be an invariant subspace of \( L^2(\mathbb{R}^\times, d^\times u) \) and \( U_2 \) be its orthogonal complement.
For each \( f_1 \in U_1 \) we have \( r^V_{\nu,\lambda}[f_1] \in U_1, \forall \alpha \in \mathbb{R} \) and \( \forall \beta > 0 \); and by (28.3) the Fourier
transform of \( f_1(\beta u)f_2(u)/|u| \) vanishes identically for any \( f_2 \in U_2 \); that is,

\[ \int_{\mathbb{R}^\times} |f_1(\beta u)f_2(u)|d^\times u = 0. \]  

(30.1)

Integrating this over the positive real axis against the measure \( d^\times \beta \), we find via Fubini’s
theorem that

\[ \left( \int_0^\infty |f_1(u)|d^\times u \right) \left( \int_0^\infty |f_2(u)|d^\times u \right) = 0, \]  

(30.2)

\[ \left( \int_{-\infty}^0 |f_1(u)|d^\times u \right) \left( \int_{-\infty}^0 |f_2(u)|d^\times u \right) = 0. \]  

(30.3)

We are, however, unable to assert that any combination of (30.2) and (30.3) implies that
one of \( f_1 \) and \( f_2 \) vanishes almost everywhere in \( \mathbb{R}^\times \). Overcoming this difficulty, we argue as
follows: According to (30.3), one of the sets \{ \( u < 0 : f_1(u) \neq 0 \) \} and \{ \( u < 0 : f_2(u) \neq 0 \) \} has
Lebesgue measure zero. We assume that the former holds for all elements in \( U_1 \); otherwise
we may exchange \( U_1 \) and \( U_2 \). We then apply the assertion (29.2) to \( f_L \) the restriction of an
arbitrary \( f \in U_1 \) to \( [1/L, L] \), and have

\[ \lim_{L \to \infty} \int_{\mathbb{R}^\times} \left| r^V_w f(u) - \int_{\mathbb{R}^\times} j_{\nu,\lambda}(u\lambda)f_L(\lambda)d^\times \lambda \right|^2 d^\times u = 0, \]

(30.4)
which implies that
\[
\lim_{L \to \infty} \int_R \left| \int_{-\infty}^{0} j_{\nu \nu}(u \lambda) f_L(\lambda) e^{-it \lambda} d\lambda \right|^2 d\lambda = 0,
\]
(30.5)
The assertion (30.4) transfers the fact on the negative real axis to (30.5) which concerns the values of \( f \) on the positive real axis. In (30.5), \( f_L(\lambda) \) has the Mellin transform \( f^*_L(s) \) for any \( s \in \mathbb{C} \); and (29.10) gives that of \( j_{\nu \nu}(u \lambda) \lambda^{-1} \) for any \( s \) with \( \text{Re} \ s > \frac{1}{2} \). Applying the Mellin inversion to the latter on \( \text{Re} \ s = 1 \) and an exchange to the resulting double integral, we see that (30.5) is equivalent to
\[
\lim_{L \to \infty} \int_R \left| \int_{-\infty}^{0} \Gamma(\frac{1}{2} - it + \nu \nu) \Gamma(\frac{1}{2} - it - \nu \nu) f^*_L(it) e^{-it \lambda} d\lambda \right|^2 d\xi = 0.
\]
(30.6)
Thus we have
\[
\lim_{L \to \infty} \int_R \left| \Gamma(\frac{1}{2} - it + \nu \nu) \Gamma(\frac{1}{2} - it - \nu \nu) f^*_L(it) \right|^2 dt = 0.
\]
(30.7)
On the other hand, according to the \( L^2 \)-theory of Mellin transforms, \( f^*_L(it) \) converges in the mean to an \( f^*(it) \in L^2(i\mathbb{R}) \); and \( \| f^* \| = \| f \| \) with an obvious specification of the norms. In particular, we have
\[
\lim_{L \to \infty} \int_R \left| \Gamma(\frac{1}{2} - it + \nu \nu) \Gamma(\frac{1}{2} - it - \nu \nu) \right| f^*(it) - f^*_L(it) \right|^2 dt \ll \lim_{L \to \infty} \int_R \left| f^*(it) - f^*_L(it) \right|^2 dt = 0.
\]
(30.8)
It follows that
\[
\int_R \left| \Gamma(\frac{1}{2} - it + \nu \nu) \Gamma(\frac{1}{2} - it - \nu \nu) f^*(it) \right|^2 dt = 0,
\]
(30.9)
which implies \( \| f^* \| = 0 \). Hence, we have \( \| f \| = 0 \). This ends the treatment of the unitary principal series.

The discussion of the discrete series is much simpler; the assertion corresponding (30.2)–(30.3) suffices. As before we remark that the above extends to the complementary series, although this is irrelevant in our present context.

Notes: The irreducibility of \( V \)-s is usually confirmed via the Lie algebra \( \mathfrak{g} \). Our argument may appear to be unconventional. However, the discussion based on explicit group actions (28.3) and (28.2) should also be worth reporting. As for the \( L^2 \)-theory of Mellin transforms, see Titchmarsh [35, Section 3.17].

31. Functional equations. This is kind of a rumination. The rôle played either visibly or invisibly by the Weyl element in functional equations for automorphic \( L \)-functions is condensed in (29.4); in other words those equations can be all traced back to respective versions of (29.4). Although only remotely relevant to the motivation of the present article, it might be worthwhile to ponder upon the nature of functional equations for general \( L \)-functions. In fact, we have set out already the following view point in [27, Vol. 2], starting with the Poisson and the Voronoi sum formulas, which are equivalent to the functional equations for the Riemann zeta-function and for the product of two zeta-values, respectively: After observing that these classical sum formulas yield a variety of significantly non-trivial bounds, that is, massive cancellations among number theoretical terms, we assert that an art of counting, which is at the core of mathematics, is to surmise that those discrete objects to be counted are floating on waves, i.e., harmonic extensions of discrete existence, whence one
may see the very possibility of applying analysis of continuous objects to discrete objects. Indeed, the zeta and \( L \)-functions are additive collections of the zeta-waves \( \{ \eta(n) : n \in \mathbb{N} \} \). What counts most is thus to try to express counting functions as precisely as possible in terms of these analytic functions. However, we face a dilemma. Counting functions are genetically discontinuous, and we need infinitely many waves to exactly describe them, that is, limiting procedure is inevitable. There are thus gaps in between discrete and continuous existence; and the speculation on the best possible bounding of the gaps leads us to conjectures. Most tantalising problems in analytic number theory such as the distribution of prime numbers concern certain error terms in approximations by means of continuous main terms to counting functions. With this, one may wonder how to detect cancellations among waves. Thence we come to the view that functional equations should be the evidence of the existence of cancellations, and they must be unable to hold without number theoretical peculiarities of those coefficients attached to zeta-waves. The contrary is indeed hard to imagine. Therefore it is of the foremost importance for us to know the origin of those functional equations. It is in the action of the Weyl element whose existence defines \( \text{PSL}(2, \mathbb{R}) \) as a matrix group. Namely, various cancellations among zeta-waves are the work of the group structure of \( \text{PSL}(2, \mathbb{R}) \). We are led to the vision that there should be unified approaches to quantitative problems in the theory of \( L \)-functions. See [15] and [23, Part XIV and XV] for investigations along this line of thoughts.

Notes: The statistical detection of cancellations among Fourier coefficients of cusp forms, which is well in the category of the above thoughts, can sometimes yield assertions far deeper than those resulting on the Ramanujan conjecture. A typical instance is the spectral analysis of sums of Kloosterman sums which leads us to a region far beyond that the Weil bound does. Here we see a structure similar to that witnessed in the distribution of prime numbers via the method of the large sieve.

32. A weight stratum. The rest is to be regarded as an appendix to the foregoing discussion, although it is not directly related to automorphic representations and moreover disproportionally long.

We shall stay on a single stratum rather than view the whole of all strata. We shall indicate how to approach to the assertion of Section 22 by extending the argument of [22, Chapter 1] to \( L^2(\mathcal{F}(G)) \); however, the spectral decomposition (22.7)–(22.11) proper is not treated, because of an obvious redundancy. The task is, in essence, to find the Green function for \( \Omega \) and investigate its analytic nature, which will occupy the next five sections. Then in Section 38 we shall turn to the Selberg trace formula. We shall restrict ourselves to the situation of weight zero, i.e., the strictly real analytic environment, which will not cause any significant loss of generality, as is to be explained.

To begin with, we shall view the invariance asserted in the first section from the Lie group \( G \). Thus, we shall show first that the Maass operators \( e^\pm \) can be regarded as extensions of the hyperbolic outer-normal differential: We let \( f \) be a smooth function on \( \mathbb{H}^2 \) and put \( F(n[x]a[y]k[\theta]) = f(z)e^{2i\theta} \) and \( l_hF(g) = f^h(z)e^{2i\theta} \), with \( z = x + iy \) and \( h \in G \). We have \( e^\pm F = 2(e^\pm f)e^{2(\pm 1)i\theta} \), with

\[
e^\pm = \pm iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \pm \ell.
\]

(32.1)

The left invariance of \( e^\pm \) implies that

\[
2(e^\pm f^h)e^{2(\pm 1)i\theta} = e^\pm (f^h(z)e^{2i\theta}) = e^\pm l_hF = l_h e^\pm F = 2(e^\pm f)^he^{2(\pm 1)i\theta},
\]

(32.2)
which in view of (3.4) is equivalent to
\[
e^{\frac{i\ell}{2}} \left( f(h(z)) \left( \frac{j(h,z)}{j(h,\overline{z})} \right)^{-\ell} \right) = \left( e^{\frac{i\ell}{2}} f \right)(h(z)) \left( \frac{j(h,z)}{j(h,\overline{z})} \right)^{-(\ell+1)}. \tag{32.3} \]

When \( \ell = 0 \), this is the same as
\[
y \left( \pm i \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) f(w) \frac{dx + idy}{|dz|} = v \left( \pm i \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) f(w) \frac{du + idv}{|dw|}, \tag{32.4} \]

since \( j(h,z)/j(h,\overline{z}) = (dw/dz)/|dw/dz| \), for \( h(z) = u + iv = w \). A rearrangement gives (1.7).

Analogously the left invariance of \( \Omega \) implies that
\[
\Omega_{\ell} \left( f(h(z)) \left( \frac{j(h,z)}{j(h,\overline{z})} \right)^{-\ell} \right) = \langle \Omega_{\ell} f \rangle(h(z)) \left( \frac{j(h,z)}{j(h,\overline{z})} \right)^{-\ell}, \quad \forall h \in G, \tag{32.5} \]

which is an extension of (1.8). Alternatively one may use the relation
\[
\Omega_{\ell} = -e^{\frac{i\ell-1}{2}}e^{-\ell} - \ell(\ell - 1), \tag{32.6} \]

which is an interpretation of the second line of (8.3). This yields also, for any smooth functions \( f, g \) on \( \mathbb{H}^2 \),
\[
\int_{D} ((\Omega_{\ell} + \ell^2 - \ell) f) \cdot g \, d\mu(z) - \int_{D} (e^{\frac{i\ell}{2}} f) \cdot (e^{\frac{i\ell}{2}} g) \, d\mu(z) = \int_{\partial D} (e^{\frac{i\ell}{2}} f) \cdot g \frac{dz}{y}, \tag{32.7} \]

where \( D \) and \( \partial D \) are as in (1.9). It suffices to apply integration by parts once; indeed, such an effect is a merit of introducing the Maass operators. The assertion (1.9) follows from (32.7) for \( \ell = 0 \).

The relations (32.3) and (32.5), for \( h \in \Gamma \), mean that
\[
e^{\frac{i\ell}{2}} \text{maps } B_{\ell}^{\infty}(\Gamma \backslash \mathbb{H}^2) \text{ into } B_{\ell+1}^{\infty}(\Gamma \backslash \mathbb{H}^2),
\]
\[
\Omega_{\ell} \text{ maps } B_{\ell}^{\infty}(\Gamma \backslash \mathbb{H}^2) \text{ into itself}, \tag{32.8} \]

where \( B_{\ell}^{\infty}(\Gamma \backslash \mathbb{H}^2) \exp(2i\ell \theta) = B_{\ell}^{\infty}(\Gamma \backslash G) \); see (13.1). The second line is, however, trivial. Also we have the symmetry:
\[
\langle \Omega_{\ell} f_1, f_2 \rangle = \langle f_1, \Omega_{\ell} f_2 \rangle, \quad f_1, f_2 \in B_{\ell}^{\infty}(\Gamma \backslash \mathbb{H}^2), \tag{32.9} \]

where the inner product is taken in \( L^2_{\ell}(\Gamma \backslash \mathbb{H}^2) \). This is an immediate consequence of (11.4). In the present context, it should, however, be more expedient to derive (32.9) from (32.7): We have
\[
\int_{\mathcal{F}} ((\Omega_{\ell} + \ell^2 - \ell) f_1) \cdot \overline{f_2} \, d\mu(z) = \int_{\mathcal{F}} (e^{\frac{i\ell}{2}} f_1) \cdot (e^{\frac{i\ell}{2}} f_2) \, d\mu(z), \tag{32.10} \]

since
\[
\int_{\partial \mathcal{F}} (e^{\frac{i\ell}{2}} f_1) \cdot \overline{f_2} \frac{dz}{y} = 0. \tag{32.11} \]

In fact (32.3) implies that \( (e^{\frac{i\ell}{2}} f_1) \cdot \overline{f_2} \cdot dz/|dz| \) is a \( \Gamma \)-automorphic quantity; and thus each part of the integral (32.11) corresponding to \( \Gamma \)-congruent pairs of the sides of \( \mathcal{F} \) vanishes, as
the point \( z \) on the contour proceeds along the sides in the direction opposite to that implied by the \( I \)-maps. We get (32.9) from (32.10). In particular we have

\[
\langle \Omega f, f \rangle \geq -\ell(\ell - 1)||f||^2, \quad \forall f \in B^\infty_I(\Gamma \backslash \mathbb{H}^2),
\]

which is equivalent to the restriction on the range of summands in (22.8).

Notes: The assertions of this section are stated in a generalised form by Maass [20, Chapter 4]. Our discussion is to be compared with Roelcke [31, Teil I].

33. Green’s function to find. Now, we suppose that \( g_{\alpha, \ell}(z, w) \) stands for the free-space Green function for the operator \( \Omega_\ell + \alpha(\alpha - 1) \) acting over \( \mathbb{H}^2 \); the term ‘free-space’ is to indicate that the action of \( I \) is not taken into account yet. This is to satisfy

\[
(\Omega_\ell + \alpha(\alpha - 1))g_{\alpha, \ell}(z, w) = 0, \quad z \neq w,
\]

and have logarithmic singularities along the diagonal \( z = w \). We shall assume that

\[
\text{Re } \alpha > \ell + 1, \quad \ell \geq 0.
\]

It will turn out that the cases \( \ell \leq 0 \) and \( \ell \geq 0 \) are essentially the same; see (34.2) below. The requirement about the singularities of \( g_{\alpha, \ell} \) is suggested by the potential theory on the Euclidean plane. As a matter of fact we may employ the definition (34.1) below in an a priori manner; however, it should be more beneficial to know how to reach (34.1).

We begin with the postulation that there exists a function \( p_{\alpha, \ell} \) on positive reals such that

\[
g_{\alpha, \ell}(z, w) = p_{\alpha, \ell}(\varrho(z, w))H_\ell(z, w), \quad H_\ell(z, w) = \left( \frac{\overline{w} - w}{\overline{w} - z} \right)^\ell,
\]

where \( \varrho(z, w) \) is as in (2.1). The factor \( H_\ell(z, w) \) is related to the Cartan decomposition of \( G \). It might, in fact, be easier to work with the unit disk model than with \( \mathbb{H}^2 \) of the 2-dimensional hyperbolic geometry; that is, discussion will become somewhat transparent if viewed via the map (2.3). Namely, (33.3) is an expression of the natural requirement that \( g_{\alpha, \ell} \) be a point-pair invariant \( h(g_1^{-1}g_2) \) if viewed from \( G \times G \) with which the left and right actions of \( K \) induce characters. More precisely, according to (4.5) we should have

\[
g_{\alpha, \ell}(z, w) = h([n[x]a[y]])^{-1}n[u]a[v]),
\]

\[
h(k[\tau_1]g[k]\tau_2]) = \exp(-(2\ell i(\tau_1 + \tau_2))h(g),
\]

where \( z = x + iy, w = u + vi \). Also, the construction (8.3) of the Casimir operator in terms of the Maass operators could be utilised in the computation below, which also yields simplifications. We shall, nevertheless, work with \( \mathbb{H}^2 \) directly.

We have, with \( \varrho = \varrho(z, w) \),

\[
\varrho_x = (x - u)/(2yv), \quad \varrho_y = (y^2 - v^2 - (x - u)^2)/(4y^2v),
\]

\[
\varrho_{xx} = 1/(2yv), \quad \varrho_{yy} = ((x - u)^2 + v^2)/(2y^3v),
\]

and, with \( p = p_{\alpha, \ell} \),

\[
\frac{\partial}{\partial x}g_{\alpha, \ell}(z, w) = p'(\varrho)\varrho_xH_\ell + \ell p(\varrho)H_{\ell - 1}\frac{\overline{w} - z + \overline{\varrho} - w}{(\overline{w} - z)^2},
\]
\[
\begin{align*}
\frac{\partial}{\partial y} g_{\alpha,\ell}(z, w) &= p'(q) q_y H_{\ell - 1} \frac{\overline{w} - z - \overline{x} + w}{(w - z)^2},
\end{align*}
\]
\[
\begin{align*}
\left( \frac{\partial}{\partial x} \right)^2 g_{\alpha,\ell}(z, w) &= p''(q) \frac{q_x^2 H_{\ell + 1} + p'(q) q_{xx} H_{\ell - 1} + 2 \ell p'(q) q_x H_{\ell - 1}}{(w - z)^2} \overline{w} - z + \overline{x} - w,
+ \ell(\ell - 1)p(q) H_{\ell - 2} \frac{(w - z - \overline{x} + w)^2}{(w - z)^4} + 2 \ell p(q) H_{\ell - 1} \frac{\overline{w} - z + \overline{x} - w}{(w - z)^3},
\end{align*}
\]
\[
\begin{align*}
\left( \frac{\partial}{\partial y} \right)^2 g_{\alpha,\ell}(z, w) &= p''(q) \frac{q_y^2 H_{\ell} + p'(q) q_{yy} H_{\ell - 2} - 2 \ell p'(q) q_y H_{\ell - 1}}{(w - z)^2} \overline{w} - z - \overline{x} + w,
- \ell(\ell - 1)p(q) H_{\ell - 2} \frac{(w - z - \overline{x} + w)^2}{(w - z)^4} + 2 \ell p(q) H_{\ell - 1} \frac{\overline{w} - z - \overline{x} + w}{(w - z)^3}.
\end{align*}
\]
Thus
\[
\begin{align*}
(\Omega_t) g_{\alpha,\ell}(z, w) &= -y^2 \{ (\overline{q}_x^2 + \overline{q}_y^2) p''(q) + (q_{xx} + q_{yy}) p'(q) \} H_{\ell} \\
&- 2\ell y^2 p'(q) H_{\ell - 1} \frac{q_x (w - z + \overline{x} - w) - i q_y (w - z - \overline{x} + w)}{(w - z)^2} \\
&- \ell(\ell - 1)y^2 p(q) H_{\ell - 2} \frac{(w - z + \overline{x} - w)^2 - (w - z - \overline{x} + w)^2}{(w - z)^4} \\
&- 4\ell y^2 p(q) H_{\ell - 1} \frac{1}{(w - z)^2} + 2i \ell y p'(q) q_x H_{\ell} + 2i \ell^2 y y p(q) H_{\ell - 1} \frac{w - z - \overline{x} + w}{(w - z)^2} \\
&= - \left\{ y^2 \{ (\overline{q}_x^2 + \overline{q}_y^2) p''(q) + (q_{xx} + q_{yy}) p'(q) \} + \frac{4\ell^2 y y}{w - z^2} p(q) \right\} H_{\ell} \\
&= - \left( (q^2 + \varrho) \left( \frac{d}{d\varrho} \right)^2 + (2\varrho + 1) \frac{d}{d\varrho} + \frac{\ell^2}{\varrho^2 + 1} \right) p(q) \cdot H_{\ell}.
\end{align*}
\]
In order to relate the last line with the hypergeometric differential equation of Gauss, we put
\[
q(\xi) = \left( \frac{\varrho}{1 + \varrho} \right)^{\xi} p(\varrho), \quad \xi = -\frac{1}{\varrho}.
\]
Then we have
\[
\xi^2 q'(\xi) = \frac{\ell \varrho^{\xi - 1}}{(1 + \varrho)^{\xi + 1}} \varrho^\varrho p(\varrho) + \alpha \left( \frac{\varrho}{1 + \varrho} \right)^{\xi} \varrho^{\varrho - 1} p(\varrho) + \left( \frac{\varrho}{1 + \varrho} \right)^{\xi} \varrho^\varrho p'(\varrho);
\]
thus
\[
\left( \frac{\varrho}{1 + \varrho} \right)^{\xi} \varrho^\varrho p'(\varrho) = \xi^2 q'(\xi) + \xi \left( \frac{\ell \xi}{\xi - 1} + \alpha \right) q(\xi).
\]
Further,
\[
\xi^2 \frac{d}{d\xi} \left( \xi^2 q'(\xi) + \xi \left( \frac{\ell \xi}{\xi - 1} + \alpha \right) q(\xi) \right) = \frac{\ell \varrho^{\xi - 1}}{(1 + \varrho)^{\xi + 1}} \varrho^\varrho p'(\varrho) + \alpha \left( \frac{\varrho}{1 + \varrho} \right)^{\xi} \varrho^{\varrho - 1} p'(\varrho) + \left( \frac{\varrho}{1 + \varrho} \right)^{\xi} \varrho^\varrho p''(\varrho)
\]
\[
= -\xi \left( \frac{\ell \xi}{\xi - 1} + \alpha \right) \left( \frac{\varrho}{1 + \varrho} \right)^{\xi} \varrho^\varrho p'(\varrho) + \left( \frac{\varrho}{1 + \varrho} \right)^{\xi} \varrho^\varrho p''(\varrho),
\]
\[
\]
which means that

\[
\left(\frac{q}{1+q}\right)^\ell q^\alpha p''(q) = \xi^2 \frac{d}{d\xi} \left( \xi^2 q'(\xi) + \xi \left( \frac{\ell\xi}{\xi-1} + \alpha \right) q(\xi) \right) \\
+ \xi \left( \frac{\ell\xi}{\xi-1} + \alpha \right) \left( \xi^2 q'(\xi) + \xi \left( \frac{\ell\xi}{\xi-1} + \alpha \right) q(\xi) \right) \\
= \xi^4 q''(\xi) + 2\xi^3 \left( \frac{\ell\xi}{\xi-1} + \alpha + 1 \right) q'(\xi) \\
+ \xi^2 \left( \frac{\ell(\ell-1)}{(\xi-1)^2} + \frac{2\ell(\ell + \alpha\xi)}{\xi - 1} + \alpha + \alpha^2 + \ell + \ell^2 \right) q(\xi). \quad (33.12)
\]

We have, after some arrangements,

\[
\left(\frac{q}{1+q}\right)^\ell q^\alpha \left( q^2 + q \left( \frac{d}{dq} + (2q + 1) \frac{d}{dq} + \frac{\ell^2}{q+1} \right) \right) p(q) \\
= \xi \left\{ (1-\xi)q''(\xi) + (2\alpha - (2\alpha + 2\ell + 1)\xi)q'(\xi) - (\alpha + \ell)^2 q(\xi) \right\} + \alpha(\alpha - 1)q(\xi). \quad (33.13)
\]

Assertions (33.7) and (33.13) imply that (33.1) necessitates

\[
\left[ \xi(1-\xi) \left( \frac{d}{d\xi} \right)^2 + (2\alpha - (2\alpha + 2\ell + 1)\xi) \frac{d}{d\xi} - (\alpha + \ell)^2 \right] q(\xi) = 0. \quad (33.14)
\]

Notes: The factor \( H_\ell \) is due to Roelcke [31, Teil II, (7.11)]; however, the construction (33.4) via the Cartan decomposition does not seem to have been reported explicitly before. This extends to \( \text{PSL}(2, \mathbb{C}) \), though in a matrix form.

34. Green’s function defined. In this way we are led to our definition of the Green function for \( \Omega_\ell + \alpha(\alpha - 1) \) on \( \mathbb{H}^2 \):

\[
g_{\alpha,\ell}(z, w) = p_{\alpha,\ell}(q(z, w)) H_\ell(z, w), \\
p_{\alpha,\ell}(q) = \frac{\Gamma(\alpha + \ell)\Gamma(\alpha - \ell)}{4\pi \Gamma(2\alpha)} \left( 1 + \frac{1}{q} \right)^\ell q^{-\alpha} 2F_1 \left( \alpha + \ell, \alpha + \ell; 2\alpha; -\frac{1}{q} \right), \quad (34.1)
\]

under the assumption (33.2). We note that

\[
p_{\alpha,-\ell}(q) = p_{\alpha,-\ell}(q), \\
g_{\alpha,\ell}(z, w) = g_{\alpha,-\ell}(z, w) = g_{\alpha,\ell}(w, z), \quad (34.2)
\]

since

\[
2F_1(a, b; c; \xi) = (1 - \xi)^{-a-b} 2F_1(c - a, c - b; c; \xi), \quad |\text{arg}(1 - \xi)| < \pi. \quad (34.3)
\]

In passing, we observe that for any \( h \in G \) we have

\[
g_{\alpha,\ell}(z, h(w)) \left( \frac{j(h, w)}{j(h, w)} \right)^\ell = g_{\alpha,-\ell}(w, h^{-1}(z)) \left( \frac{j(h^{-1}, z)}{j(h^{-1}, z)} \right)^{-\ell}, \quad (34.4)
\]
whence
\[
(O_{-\ell} + \alpha(\alpha - 1))_w \left( g_{\alpha,\ell}(z, h(w)) \left( \frac{f(h(w))}{J(h(w))} \right)^\ell \right) = 0, \quad z \neq h(w). \tag{34.5}
\]

The relation (34.4) is the same as the transformation property of \( H_\ell \); or more basically, (34.4) and (34.3) give the equation. As to (34.5), it is as well a consequence of (32.5).

We shall show that \( g_{\alpha,\ell} \) indeed serves our purpose. Thus we invoke the Mellin–Barnes integral representation for the Gaussian hypergeometric function and have
\[
p_{\alpha,\ell}(q) = \frac{\Gamma(\alpha - \ell)}{8\pi^2 i\Gamma(\alpha + \ell)} (q + 1) \int_{-\frac{1}{2}} \frac{\Gamma^2(s + \alpha + \ell)\Gamma(-s)}{\Gamma(s + 2\alpha)} q^{-s-\alpha-\ell} ds, \tag{34.6}
\]
where the contour is \( \text{Re} s = -\frac{1}{2} \); and the integrand is of exponential decay. We have, on (33.2),
\[
p_{\alpha,\ell}(q) = -\frac{\log q}{4\pi} - \frac{1}{4\pi} \left( c_E + \frac{\Gamma'(\alpha + \ell) + \Gamma'(\alpha - \ell)}{\Gamma(\alpha - \ell)} \right) + O(q \log q), \quad q \to +0, \tag{34.7}
\]
\[
p'_{\alpha,\ell}(q) = -\frac{1}{4\pi q} - \frac{1}{4\pi} (\alpha + \ell)(\alpha - \ell - 1) \log q + O(1), \quad q \to +0, \tag{34.8}
\]
\[
p_{\alpha,\ell}(q) = \frac{\Gamma(\alpha + \ell)\Gamma(\alpha - \ell)}{4\pi \Gamma(2\alpha)} q^{-\alpha} (1 + O(q^{-1})), \quad q \to +\infty, \tag{34.9}
\]
\[
p'_{\alpha,\ell}(q) = -\frac{\alpha}{4\pi \Gamma(2\alpha)} q^{-\alpha-1} (1 + O(q^{-1})), \quad q \to +\infty, \tag{34.10}
\]
where \( c_E \) is the Euler constant. To show (34.7), we shift the contour in (34.6) to \( \text{Re} s = -\alpha - \ell - \frac{3}{2} \); as to (34.8), we first differentiate inside the integral and shift the contour in the same way; further, the definition (34.1) readily implies (34.9)–(34.10).

We then introduce the integral operator
\[
G_{\alpha,\ell} h(z) = \int_{H^2} g_{\alpha,\ell}(z, w)f(w) d\mu(w). \tag{34.11}
\]
We claim that \( G_{\alpha,\ell} \) is the left inverse of \( O_{-\ell} + \alpha(\alpha - 1) \): Provided (33.2),
\[
G_{\alpha,\ell}(O_{-\ell} + \alpha(\alpha - 1)) f = f, \tag{34.12}
\]
for any smooth function \( f \) on \( H^2 \) with which the left side converges absolutely. In fact, we can proceed exactly the same way as in the Euclidean plane situation, with the observation that the left side of (34.12) equals
\[
\lim_{\rho \to 0} \int_{H^2 \setminus D} \left\{ g_{\alpha,\ell}(z, w)(\Omega_0 f)(w) - ((\Omega_0)_w g_{\alpha,\ell}(z, w)) f(w) \right\} d\mu(w)
+ 2\ell i \lim_{\rho \to 0} \int_{H^2 \setminus D} \nu \frac{\partial}{\partial u} \left\{ g_{\alpha,\ell}(z, w) f(w) \right\} d\mu(w), \tag{34.13}
\]
where \( D \) is an Euclidean disk of radius \( \rho \) with the centre at \( z \), and we have used (12.6) and (34.5). The first limit equals \( f(z) \) by Green’s formula (1.9) and (34.7); and the second limit vanishes. In particular, we have
\[
G_{\alpha,\ell}(\text{Im} z)^{1/2 + \nu} = \frac{(\text{Im} z)^{1/2 + \nu}}{(\alpha - \frac{1}{2})^2 - \nu^2}, \quad |\text{Re} \nu| < \text{Re} \alpha - \frac{1}{2}, \tag{34.14}
\]
The range of $\nu$ follows from (34.9).

Notes: Our $g_{\alpha,\ell}$ is to be compared with Hejhal’s $k_\nu(z;w)$ on [11, Vol. 2, p. 350] which is, in our notation,

$$-\frac{\Gamma(\alpha + \ell)\Gamma(\alpha - \ell)}{4\pi \Gamma(2\alpha)} \left(1 - \left|\frac{z-w}{z-w}\right|^2\right)^\alpha,$$

$$\times {}_2F_1\left(\alpha + \ell, \alpha - \ell; 2\alpha; 1 - \frac{z-w}{z-w}\right)^2 H_{\ell}(z,w), \quad (34.15)$$

and equal to $-g_{\alpha,\ell}(z,w)$, as

$$\left.\right. _2F_1(a, b; c; \xi) = (1 - \xi)^{-a} _2F_1(a, c - b; c; \xi/(\xi - 1)), \quad |\arg(1 - \xi)| < \pi. \quad (34.16)$$

The reason for our use of the definition (34.1) is in that we are able to exploit (34.6), which is of rapid convergence and thus convenient to manipulate. For (34.3) and (34.16) see Lebedev [18, Section 9.5]. A further comparison should be made with Roelcke [31, Teil II, §7]. See also Bruggeman [4, p. 70]. Here is an extra observation: The Green function $g_{\alpha,0}$ appears in a peculiar way as the kernel function that connects the fourth power moment of the Riemann zeta-function to the spectral decomposition of $L^2(\Gamma \backslash \mathbb{H}^2)$: moreover, $g_{\alpha,0}$ can be expressed as a convolution of Bessel functions of representation. For these see [5] and [23, Part XII].

There is an extension to the complex situation; see [24]. Commenting further, the product of two automorphic L-functions is naturally related to the product of two instances of (29.4), which can be viewed as a Mellin transform of the convolution of two instances of (29.1), and the latter is essentially the Gaussian hypergeometric function $\Gamma_1$. The appearance of $\Gamma_1$ in the spectral decomposition [22, Theorem 4.1] of the fourth moment of the Riemann zeta-function is somewhat mysterious but might be understood as an indication that the unitary map $r^V_w r^V_w$ of the space $L^2(\mathbb{R}^\times, d\gamma/\pi)$ into itself is behind the scene.

35. Automorphic Green’s function. If $f$ is in $L^2_2(\Gamma \backslash \mathbb{H}^2)$ we have formally

$$\mathcal{G}_{\alpha,\ell} f(z) = \int_{\Gamma \backslash \mathbb{H}^2} G_{\alpha,\ell}(z,w) f(w) d\mu(w), \quad (35.1)$$

where

$$G_{\alpha,\ell}(z,w) = \sum_{\gamma \in \Gamma} g_{\alpha,\ell}(z, \gamma(w)) \left(\frac{j(\gamma,w)}{j(\gamma,w)}\right)^\ell. \quad (35.2)$$

By (34.9) we have, with $z$ being bounded and $w \in \mathcal{F}$ tending to the cusp,

$$G_{\alpha,\ell}(z,w) \ll \sum_{\gamma \in \Gamma \backslash \mathbb{H}^2} \sum_{n=-\infty}^{\infty} p_{\alpha,\ell}(s^n \gamma(z), w)$$

$$\ll \sum_{\gamma \in \Gamma \backslash \mathbb{H}^2} \sum_{n=-\infty}^{\infty} \left(\frac{n^2 + (\text{Im } w)^2}{\text{Im } w \text{Im } \gamma(z)}\right)^{-\Re \alpha}$$

$$\ll (\text{Im } w)^{1-\Re \alpha} \sum_{\gamma \in \Gamma \backslash \mathbb{H}^2} (\text{Im } \gamma(z))^{\Re \alpha} \ll (\text{Im } w)^{1-\Re \alpha}, \quad (35.3)$$

where $s = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]$. Since the last sum has to be convergent, we need to have $\Re \alpha > 1$; see (33.2). Thus, for the basis vectors

$$\psi^{(\ell)}_\nu(z) = \lambda^{(\ell)}_\nu([x][y]), \quad z = x + iy, \quad (35.4)$$
of the cuspidal subspace of $L^2(\Gamma \backslash \mathbb{H}^2)$ which are induced by $(22.1)-(22.4)$, we have, on noting that $\psi_{\ell}(t)$ is bounded throughout $\mathbb{H}^2$,

$$G_{\alpha,\ell,\psi_{\ell}(t)} = \frac{\psi_{\ell}(t)}{(\alpha - \frac{1}{2})^2 - \nu^2},$$ (35.5)

under the convention (35.1).

We extend this to the Eisenstein series of weight $2\ell$ on $\mathbb{H}^2$: Namely, for the function

$$e_{\ell}(z, \nu) = E_{\ell}(n[x]a[y], \nu),$$ (35.6)

we shall show that

$$G_{\alpha,\ell, e_{\ell}(z, \nu)} = \frac{e_{\ell}(z, \nu)}{(\alpha - \frac{1}{2})^2 - \nu^2}. e_{\ell}(z, \nu) \neq \infty, |\text{Re} \nu| < \text{Re} \alpha - \frac{1}{2}. (35.7)$$

To this end, we assume temporarily that $\frac{1}{8} < \text{Re} \nu < \text{Re} \alpha - \frac{3}{4}$ holds; this means that we work in a sub-domain of (33.2) for the moment. Then (34.14) implies that

$$\frac{e_{\ell}(z, \nu)}{(\alpha - \frac{1}{2})^2 - \nu^2} = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left(\frac{2}{\text{Im} \gamma(z)}\right)^{-\ell} \int_{\mathbb{H}^2} g_{\alpha,\ell}(\gamma(z), w)(\text{Im} w)^{1/2+\nu} d\mu(w)$$

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\mathbb{H}^2} g_{\alpha,\ell}(z, w)(\text{Im} \gamma(w))^{1/2+\nu} \left(\frac{2}{\gamma(w)}\right)^{\ell} d\mu(w)$$

$$= \int_{\mathbb{H}^2} g_{\alpha,\ell}(z, w)e_{\ell}(w, \nu)d\mu(w). (35.8)$$

The last line is due to absolute convergence coupled with the observation that by (18.2), for $\ell = 0$,

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\text{Im} \gamma(w))^{1/2+\text{Re} \nu} \leq c_0(iw, \text{Re} \nu) = c_0(i/v, \text{Re} \nu)$$

$$\ll v^{1/2+\text{Re} \nu} + v^{-1/2-\text{Re} \nu}. (35.9)$$

Thus we have

$$\frac{e_{\ell}(z, \nu)}{(\alpha - \frac{1}{2})^2 - \nu^2} = \int_{\Gamma \backslash \mathbb{H}^2} G_{\alpha,\ell}(z, w)e_{\ell}(w, \nu) d\mu(w). (35.10)$$

This converges uniformly for $\alpha, \nu$ in the domain indicated in (35.7), and by analytic continuation we end the proof of (35.7).

Further, we note that

$$G_{\alpha,\ell}(z, w)$$

is of weight $2\ell$ with respect to $z.$ (35.11)

In fact, we have, for any $\tau \in \Gamma$ and in the region of absolute convergence,

$$G_{\alpha,\ell}(\tau z, w) = \sum_{\gamma \in \Gamma} p_{\alpha,\ell}(\tau(z), \gamma(w))H_{\ell}(\tau(z), \gamma(w)) \left(\frac{\gamma(w)}{\gamma(z)}\right)^{\ell}. (35.12)$$
To the first factor in the summand we apply the second line in (2.1), and to the remaining factors the third and the fourth identities in (1.2).

Notes: The discussion of the previous and the present sections corresponds to [22, Section 1.3]. Some simplifications have been made.

36. Iterated kernel. We now turn to the function
\[ G^{(\ell)}(z, w) = G_{\alpha, \ell}(z, w) - G_{\beta, \ell}(z, w), \quad \text{Re} \beta > \text{Re} \alpha, \] (36.1)
in which (33.2) is still imposed. This is related to the iteration of the integral transform (34.11), for we have the Hilbert identity
\[ G_{\alpha, \ell} G_{\beta, \ell} = \frac{G_{\alpha, \ell} - G_{\beta, \ell}}{(\beta - \alpha)(\beta + \alpha - 1)}, \] (36.2)
as is well indicated by (34.12); however, this fact is not needed in our discussion below.

The assertion (34.7) implies that \( G^{(\ell)}(z, w) \) is continuous on the diagonal \( z = w \); and by (35.3) we see that \( G^{(\ell)}(z, w) = \overline{G^{(-\ell)}(w, z)} \) is bounded when \( z \in \mathcal{F} \) tends to the cusp while \( w \) is bounded. Hence, in view of (35.11), we may apply (17.7) and (22.10) to \( G^{(\ell)}(z, w) \) as a function of \( z \). We have that for each fixed \( w \in \mathbb{H}^2 \)
\[ G^{(\ell)}(z, w) = G^{(0)}(z, w) + aG(z, w), \] (36.3)
where
\[ G^{(0)}(z, w) = \sum_{\nu} k^{(0)}(\nu \nu) \psi^{(0)}(z) \overline{\psi^{(0)}(w)}, \] (36.4)
\[ aG(z, w) = \frac{1}{4\pi} \int_{(0)} k(\nu) e_\ell(z, \nu) \overline{e_\ell(w, \nu)} d\nu + \frac{3}{\pi} \delta_0 k\left(\frac{1}{2}\right), \] (36.5)
with convergence in the mean in \( L_0^2(\Gamma \setminus \mathbb{H}^2) \). Here \( \delta_0 \) is equal to 1 if \( \ell = 0 \) and to 0 otherwise; and
\[ k(\nu) = \frac{1}{(\alpha - \frac{1}{2})^2 - \nu^2} - \frac{1}{(\beta - \frac{1}{2})^2 - \nu^2}. \] (36.6)
In fact (36.4) follows from (35.5), and the integrated part of (36.5) from (35.7). As to the constant term of (36.5), we note that the assertion \( G^{(0)}(z, \cdot) \) = \( k\left(\frac{1}{2}\right) \) follows from a combination of (35.7), \( \ell = 0 \), and \( c_0(z, -\frac{1}{2}) = 1 \); the latter comes from the fact that in (18.2), \( \ell = 0 \), we have \( \varphi_\ell (-\frac{1}{2}) = 0 \) as well as \( A^0 \phi_0(g, -\frac{1}{2}) = 0 \) because of (16.1).

We shall demonstrate in the next section that the series (36.4) converges absolutely and uniformly over the product domain \( \mathcal{F} \times \mathcal{F} \). To achieve this, we shall appeal to the well-known theorem of Mercer on the spectral expansion of positive symmetric kernels. As a prerequisite, we need to have that
\[ G^{(0)}(z, w) = \{ G^{(\ell)} - aG \}(z, w) \] is continuous and bounded on \( \mathcal{F} \times \mathcal{F} \). (36.7)

Notes: For Mercer’s theorem see Riesz and Sz-Nagy [30, Section 98]. Also we note that the functions
\[ p^{(n)}_{\alpha, \ell}(\varrho) = \frac{\Gamma(\alpha - \ell)}{8\pi^2 n^{\ell+1} \Gamma(n+1) \Gamma(n+\alpha+\ell)} \times (\varrho + 1)^\ell \int_{(-\frac{1}{2})} \frac{\Gamma(s+n+\alpha+\ell) \Gamma(s+\alpha+\ell) \Gamma(-s) \Gamma(s+n+2\alpha)}{\Gamma(s+n+2\alpha)} \varrho^{-s-\alpha-\ell} ds, \] (36.8)
with integers \( n \geq 0 \), satisfy

\[
p_{\alpha, \ell}^{(n)} - p_{\alpha+1, \ell}^{(n)} = (n + 1)(n + 2\alpha)p_{\alpha, \ell}^{(n+1)}, \quad p_{\alpha, \ell}^{(0)} = p_{\alpha, \ell},
\]

which is an extension of [22, (1.3.12)]. The fast converging expression (36.8) serves the same purpose as (34.6) does, when discussing the spectral resolution of \( \Omega_\ell \) via iterations of \( G_{\alpha, \ell} \).

See the notes to the next section.

37. Control of divergence. We shall prove that the functions \( G^{(0)}(z, w) \) and \( ^cG(z, w) \) diverge in the same mode as \( z, w \in \mathcal{F} \) tend to the cusp, and confirm the assertion (36.7). Our argument is analogous to that of [22, Lemmas 1.7 and 1.8].

We shall first prove an approximation to \( G^{(0)} \): Uniformly for \( z = x + iy, w = u + iv \in \mathcal{F} \),

\[
\left| G^{(0)}(z, w) - \frac{1}{2\pi i} \int_0^1 k(v) y^{1/2+\nu} v^{1/2-\nu} dv \right| \ll 1.
\]

To this end, we write, with \( z, w \in \mathbb{H}^2 \),

\[
G^{(0)}(z, w) = (S_\alpha - S_\beta)(z, w) + (T_\alpha - T_\beta)(z, w),
\]

\[
S_\alpha(z, w) = \sum_{n=-\infty}^{\infty} g_{\alpha, \ell}(z, n + w),
\]

\[
T_\alpha(z, w) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} S_{\alpha}(z, \gamma(w)) \left( \frac{j(\gamma, w)}{j(\gamma, w)} \right)^\ell.
\]

By the Euler–Maclaurin sum formula, we have, with \( \xi(t) = t - [t] - \frac{1}{2} \),

\[
S_\alpha(z, w) = \int_{-\infty}^{\infty} g_{\alpha, \ell}(z, t + w) dt + \int_{-\infty}^{\infty} \xi(t) \frac{\partial}{\partial t} g_{\alpha, \ell}(z, t + w) dt
\]

\[
= \{ S_\alpha^{(0)} + S_\alpha^{(1)} \}(z, w),
\]

say. We have the decomposition

\[
G^{(0)}(z, w) = \left\{ (S_\alpha^{(0)} - S_\beta^{(0)}) + (T_\alpha^{(0)} - T_\beta^{(0)}) + (T_\alpha^{(1)} - T_\beta^{(1)}) \right\}(z, w),
\]

\[
T_\alpha^{(0)}(z, w) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} S_{\alpha}^{(0)}(z, \gamma(w)) \left( \frac{j(\gamma, w)}{j(\gamma, w)} \right)^\ell,
\]

\[
T_\alpha^{(1)}(z, w) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} S_{\alpha}^{(1)}(z, \gamma(w)) \left( \frac{j(\gamma, w)}{j(\gamma, w)} \right)^\ell.
\]

We assert that for any \( z, w \in \mathbb{H}^2 \) the difference \( (S_\alpha^{(0)} - S_\beta^{(0)})(z, w) \) equals the integrated term in (37.1). In fact, the formula (34.14) implies that

\[
\int_0^\infty \left( \int_{-\infty}^{\infty} g_{\alpha, \ell}(z, u + iv) du \right) v^{\nu - 3/2} dv = \frac{y^{1/2+\nu}}{(\alpha - \frac{1}{2} - \nu)^2}.
\]
Since the inner integral is a continuous function of $v$, we have, via the Mellin inversion procedure,
\[
\int_{-\infty}^{\infty} g_{\alpha, \ell}(z, u + iv) du = \frac{1}{2\pi i} \int_{(0)} y^{1/2+v} y^{1/2-v} d\nu, \tag{37.6}
\]
which proves the claim. To deal with $T^{(0)}_\alpha$, we shift the last contour to $\text{Re} \nu = -\text{Re} \alpha$:
\[
S^{(0)}_{\alpha}(z, w) = \frac{y^{1-\alpha} v^{\alpha}}{2\alpha - 1} + \frac{1}{2\pi i} \int_{(-\text{Re} \alpha)} y^{1/2+\nu} y^{1/2-\nu} d\nu. \tag{37.7}
\]
Thus, we have that for any $z, w \in \mathbb{H}^2$
\[
T^{(0)}_{\alpha}(z, w) = \frac{y^{1-\alpha}}{2\alpha - 1} (\varepsilon_{\ell}(w, \alpha - \frac{1}{2}) - v^\alpha)
+ \frac{1}{2\pi i} \int_{(-\text{Re} \alpha)} y^{1/2+\nu} y^{1/2-\nu} (\varepsilon_{\ell}(w, -\nu) - v^{1/2-\nu}) d\nu. \tag{37.8}
\]
On the other hand, we have, for $z, w \in \mathbb{H}^2$,
\[
S^{(1)}_{\alpha}(z, w) = \int_{-\infty}^{\infty} \xi(t) \left( p'(g) g_{\alpha, \ell} H_{\ell} - \ell p(g) H_{\ell - 1} \frac{m - z + i\nu - w}{m + z - w} \right) dt, \tag{37.9}
\]
where $p = p_{\alpha, \ell}, g = g(z, w + t), H_{\ell} = H_{\ell}(z, w + t)$; thus,
\[
(S^{(1)}_{\alpha} - S^{(1)}_{\beta})(z, w) \ll \int_{-\infty}^{\infty} \left( |t||p'|| + |p| \frac{(y + v)/\sqrt{y v}}{t^2 + (y + v)^2/yv} \right) dt, \tag{37.10}
\]
where $p = (p_{\alpha, \ell} - p_{\beta, \ell})(t^2 + g(iy, iv)), p' = (p'_{\alpha, \ell} - p'_{\beta, \ell})(t^2 + g(iy, iv))$. We have, by (34.7)–(34.10),
\[
p(\tau) \ll \min \{ 1, \tau^{-\text{Re} \alpha} \}, \quad p'(\tau) \ll \min \{ \log \tau, \tau^{-\text{Re} \alpha - 1} \}. \tag{37.11}
\]
Considering the cases $y/v \leq 1/2$, $\frac{1}{2} \leq y/v \leq 2$, and $y/v \geq 2$ separately, we find that
\[
(S^{(1)}_{\alpha} - S^{(1)}_{\beta})(z, w) \ll \left( \frac{y v}{(y + v)^2} \right)^{\text{Re} \alpha}. \tag{37.12}
\]
Hence we have that for any $z, w \in \mathbb{H}^2$
\[
T^{(1)}_{\alpha}(z, w) \ll \left( \frac{y v}{(y + v)^2} \right)^{\text{Re} \alpha} + y^{-\text{Re} \alpha} (c_0(w, \text{Re} \alpha - \frac{1}{2}) - v^{\text{Re} \alpha}). \tag{37.13}
\]
Inserting the bound (15.5) and the expansion (18.2) into (37.8) and (37.13), with $z, w \in \mathcal{F}$, we end the proof of (37.1).

We turn to $cG(z, w)$. We denote the sum over $n$ in (18.2), for $g = n[x]a[y]$, by $c_{\ell}(z, \nu)$. Invoking the functional equation (18.4) we get the decomposition
\[
cG(z, w) = \frac{1}{2\pi i} \int_{(0)} k(\nu) y^{1/2-\nu} v^{1/2+\nu} d\nu
+ \{ cG^{(0)} + cG^{(1)} + cG^{(2)} + cG^{(3)} \}(z, w), \tag{37.14}
\]
where
\begin{align*}
{c^G(0)}(z, w) &= \frac{(-1)\ell}{2\pi i} \int_{(0)} k(\nu) y^{1/2 - \nu} \frac{\Gamma^2(\nu + \frac{1}{2}) \varphi_{\ell}(\nu)}{\Gamma(\nu + |\ell| + \frac{1}{2}) \Gamma(\nu - |\ell| + \frac{1}{2})} d\nu + \frac{3}{\pi} \delta_0(\frac{y}{2}), \\
{c^G(1)}(z, w) &= \frac{1}{2\pi i} \int_{(0)} k(\nu) y^{1/2 - \nu} \tilde{e}_{\ell}(w, \nu) d\nu, \\
{c^G(2)}(z, w) &= \frac{1}{4\pi i} \int_{(0)} k(\nu) \tilde{e}_{\ell}(z, \nu) \tilde{e}_{\ell}(w, \nu) d\nu.
\end{align*}

In what follows we shall use well-known estimations of $1/\zeta(s)$ for $\Re s \geq 1$ and $\zeta(s)$ for $\Re s \leq 1$, without mentioning. Also we may assume that $yv \geq 1$ without loss of generality.

The treatment of $c^G(0)$ is easy. It suffices to move the contour to $(\alpha)$. When $\ell = 0$, we pass the simple pole at $\nu = \frac{1}{2}$, whose contribution is cancelled by the constant term on the right side. We have
\begin{equation*}
{c^G(0)}(z, w) \ll (yv)^{1 - \Re \alpha}. \tag{37.16}
\end{equation*}
We shall consider $c^G(2)(z, w)$. Invoking (15.5), we have, uniformly for $y \geq 1$ and $\Re \nu \geq 0,$
\begin{align*}
\tilde{e}_{\ell}(z, \nu) &\ll \frac{(1 + |\nu|) y^{-1/2 - \Re \nu}}{|\zeta(1 + 2\nu)|} \exp \left( - \frac{\pi y}{1 + |\nu|} \right) \sum_{n=1}^{\infty} \frac{d(n)}{n} \exp \left( - \frac{\pi y n}{1 + |\nu|} \right) \\
&\ll \frac{(1 + |\nu|) y^{-1/2 - \Re \nu}}{|\zeta(1 + 2\nu)|} (\log (2 + |\nu|/y))^2 \exp \left( - \frac{\pi y}{1 + |\nu|} \right), \tag{37.17}
\end{align*}
where $d(n)$ is the number of divisors of $n$. Hence, shifting the paths in $c^G(1)(z, w)$ and $c^G(2)(z, w)$ to $+\infty$, we get
\begin{align*}
{c^G(1)}(z, w) &\ll y^{1 - \Re \alpha} \exp \left( - \frac{\nu}{(|\alpha| + |\beta|)} \right), \\
{c^G(2)}(z, w) &\ll v^{1 - \Re \alpha} \exp \left( - \frac{\nu}{(|\alpha| + |\beta|)} \right). \tag{37.18}
\end{align*}
As to $c^G(3)(z, w)$, we apply (37.17) with $\Re \nu = 0,$
\begin{align*}
{c^G(3)}(z, w) &\ll (yv)^{-1/2} \int_{(0)} \frac{\exp \left( - \frac{\pi (y + \nu)}{1 + |\nu|} \right)}{|\zeta(1 + 2\nu)|^2 (1 + |\nu|)^2} (\log (2 + |\nu|))^2 |d\nu| \\
&\ll (yv)^{-1/2} (\log 2y)^6. \tag{37.19}
\end{align*}
Collecting (37.14), (37.18), (37.19), we get, uniformly for $z, w \in \mathcal{F}$,
\begin{equation*}
\left| {c^G(z, w) - \frac{1}{2\pi i} \int_{(0)} k(\nu) y^{1/2 - \nu} v^{1/2 + \nu} d\nu} \right| \ll 1. \tag{37.20}
\end{equation*}
Combined with (37.1), this yields the assertion (36.7).

We now have that
\begin{equation*}
\int_{\Gamma \setminus \mathbb{H}^2} \left( \int_{\Gamma \setminus \mathbb{H}^2} |{c^G(z, w)|^2 d\mu(z)} \right) d\mu(w) = \sum_{\nu}^{(\ell)} |k(\nu_V)|^2 \tag{37.21}
\end{equation*}
is convergent. We see via Fubuni’s theorem that (36.4) converges in the mean in the space $L^2((\Gamma \setminus \mathbb{H}^2) \times (\Gamma \setminus \mathbb{H}^2))$. Hence, by Mercer’s theorem we conclude that (36.4) converges uniformly over $\mathcal{F} \times \mathcal{F}$. The same holds with (36.5), as has been proved in the above.
Notes: A way to approach relatively directly to the spectral resolution of $\Omega_\ell$ is to elaborate a little the present section. A key is the integral expression (36.8). With it, one may proceed in much the same way as in [22, Sections 1.3–1.4]. There is an issue which is peculiar to the situation $\ell \neq 0$ and related to holomorphic cusp forms, i.e., those vectors in $L^2_\ell(\Gamma \backslash G)$ which correspond to eigenvalues $l - \frac{1}{2}$, $l \in \mathbb{N}, l \leq \ell$. It can, however, be handled by following Section 21. This means, in particular, that the argument eventually involves the Maass operators disguised as (32.1). Another extension is to consider Hecke congruence subgroups in place of the full modular group. Then the above corresponds to the control of the scattering of the relevant automorphic Green function at incongruent cusps; and our method works without essential changes. See [26, Part II] and [27, Vol. 2, Appendix 1].

38. Trace of the Casimir operator. Having confirmed (36.7), we are now able to consider

$$\int_{\mathcal{F}} \left\{ G^{(l)} - cG \right\}(z, z) d\mu(z) = \sum_{V}^{l} k(\nu_V).$$

(38.1)

The right side is a trace of $\Omega_\ell$ the restriction of the Casimir operator to $L^2_\ell(\Gamma \backslash G)$, while the left side is defined and thus can be computed, in terms of elements of $\Gamma$. This leads us to an instance of Selberg’s trace formulas. However, we shall develop only a trace formula for $C(37 \Gamma)$. This means, in particular, that the argument eventually involves the Maass operators disguised as (32.1). Another extension is to consider Hecke congruence subgroups in place of the full modular group. Then the above corresponds to the control of the scattering of the relevant automorphic Green function at incongruent cusps; and our method works without essential changes. See [26, Part II] and [27, Vol. 2, Appendix 1].

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$$\sum_{V}^{l} k(\nu_V) = \sum_{V}^{(0)} k(\nu_V) + \sum_{i=1}^{l} (\ell - l + 1) T_{\ell-l}(1) k(l - \frac{1}{2}),$$

(38.2)

where $T_{\ell-l}(1)$ is defined in Section 21. This follows from (22.8) or (32.12). In the first sum on the right side, $V$ runs over all irreducible representations in the unitary principal series.

Hence, what counts is to compute the part $\sum_{V}^{(0)}$ which is a trace of $\Omega_0$. With this, we shall assume hereafter that

$$\ell = 0, \quad \operatorname{Re} \beta > \operatorname{Re} \alpha > 1.$$

(38.3)

We also simplify the notation:

$$p_{\alpha} = p_{\alpha, 0}, \quad g_{\alpha} = g_{\alpha, 0}$$

(38.4)

To compute the geometric side of (38.1), for $\ell = 0$, we introduce the classification of the elements $\gamma \in \Gamma$:

$$\Gamma = C^{(0)} \sqcup C^{(1)} \sqcup C^{(2)} \sqcup C^{(e)}.$$

(38.5)

Here $C^{(0)} = \{1\}$, and

$$C^{(1)} = \{ \gamma \text{ has a single fixed point on } \mathbb{R} \cup \{\infty\} \} : \text{parabolic},$$

$$C^{(2)} = \{ \gamma \text{ has two different fixed points on } \mathbb{R} \cup \{\infty\} \} : \text{hyperbolic},$$

$$C^{(e)} = \{ \gamma \text{ has a single fixed point inside } \mathbb{H}^2 \} : \text{elliptic}.$$

(38.6)

We divide the sum (35.2), for $\ell = 0$, according to (38.5), and denote the corresponding parts of $G^{(0)}$ by $R^{(0)}$, $R^{(1)}$, $R^{(2)}$, and $R^{(e)}$, respectively. The subgroup $\Gamma_\infty$ is contained in $C^{(0)} \sqcup C^{(1)}$; thus $R^{(2)}$ and $R^{(e)}$ are bounded on $\mathcal{F} \times \mathcal{F}$, which is obvious from the proof of (37.1), or more precisely from the boundedness of $T_{\alpha}$ defined in (37.2). We have

$$\sum_{V}^{(0)} k(\nu_V) = \int_{\mathcal{F}} \left\{ R^{(0)} + R^{(2)} + R^{(e)} \right\}(z, z) d\mu(z)$$

$$+ \lim_{Y \to \infty} \int_{\mathcal{F}Y} \left\{ R^{(1)} - cG \right\}(z, z) d\mu(z),$$

(38.7)
with \( \mathcal{F}_Y = \mathcal{F} \cap \{ \text{Im} \, z \leq Y \} \). Here \( \sum^u \) indicates that the sum is restricted to the unitary principal series, that is, the same as \( \sum^{(0)} \).

It is immediate to see that (34.7) gives

\[
\int_{\mathcal{F}} R^{(0)}(z, z) d\mu(z) = -\frac{1}{6} \left( \frac{\Gamma' \alpha}{\Gamma(\alpha)} - \frac{\Gamma' \beta}{\Gamma(\beta)} \right).
\]

(38.8)

Notes: The formula (38.1) with (38.2) could be used to evaluate \( \vartheta_T(l) \) for any \( l \in \mathbb{N} \), supplying an alternative proof of the dimension formula (21.8). This is a typical instance of Selberg's general observation made in his seminal work [32]. Hejhal executed this programme in [11, Vol. 2, Chapters 9–10] with a fairly general choice of the underlying discrete subgroups; his discussion is built on Roelcke's investigation [31], though. It should be worth mentioning that in the actual computation of \( \vartheta_T(l) \) it suffices to consider the situation with \( \ell = l, \alpha \) in an immediate neighbourhood of \( l \) and \( \beta \) tending to \( +\infty \), which simplifies the discussion substantially.

39. Parabolic terms. We shall deal with \( R^{(1)} \). To this end, we note that

\[
\mathcal{C}^{(1)} = \bigsqcup_{\gamma \in \mathcal{F}_p \setminus \Gamma} \{ \gamma^{-1}s^n\gamma : Z \ni n \neq 0 \},
\]

(39.1)

where \( s \) is as in (35.3). In fact, the fixed point of any \( \omega \in \mathcal{C}^{(1)} \) is a rational number or the cusp. Thus there exists a \( \gamma \in \Gamma \) such that the fixed point of \( \gamma \omega \gamma^{-1} \) is the cusp, and \( \gamma \omega \gamma^{-1} = s^n \) with a certain \( n \in \mathbb{Z} \). Also, if \( \gamma^{-1}s^m\gamma = \gamma'^{-1}s^n\gamma' \) with \( m, n \in \mathbb{Z} \), then we get \( m = n \) and \( \gamma \gamma'^{-1} \in \Gamma_p \), which confirms (39.1). Thus we have

\[
R^{(1)}(z, z) = 2 \sum_{\gamma \in \mathcal{F}_p \setminus \Gamma} \sum_{n=1}^{\infty} (p_\alpha - p_\beta)(n/2\text{Im} \, \gamma(z))^2
\]

\[
= \{ R^{(1)}_\alpha - R^{(1)}_\beta \}(z, z),
\]

(39.2)

say. By (34.6), for \( \ell = 0 \), we have

\[
R^{(1)}_\alpha(z, z) = \frac{2^{2\alpha}}{4\pi^2 i} \int_{(-\frac{1}{2})} \frac{2^{2\xi} \Gamma^2(\xi + \alpha) \Gamma(-\xi) \zeta(2(\xi + \alpha)) e_0(z, 2(\xi + \alpha) - \frac{1}{2}) d\xi}{\Gamma(\xi + 2\alpha)}.
\]

(39.3)

We integrate this over \( \mathcal{F}_Y \). We observe that

\[
\int_{\mathcal{F}_Y} e_0(z, 2(\xi + \alpha) - \frac{1}{2}) d\mu(z) = \int_{\mathcal{F}_Y} \frac{(-\Omega e_0 \{ z, 2(\xi + \alpha) - \frac{1}{2} \})}{2(\xi + \alpha) (2(\xi + \alpha) - 1)} d\mu(z)
\]

\[
= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{\partial}{\partial y} \right)_y \zeta(2(\xi + \alpha) - 1) - 2(\xi + \alpha) d\xi,
\]

(39.4)

where we have used Green's formula (1.9), i.e., (32.7) for \( \ell = 0 \), as well as the cancellation noted after (32.11). We find, via (18.2) for \( \ell = 0 \), that

\[
\int_{\mathcal{F}_Y} R^{(1)}_\alpha(z, z) d\mu(z) = \frac{2^{2\alpha}}{4\pi^2 i} \int_{(-\frac{1}{2})} \frac{2^{2\xi} \Gamma^2(\xi + \alpha) \Gamma(-\xi) \zeta(2(\xi + \alpha))}{\Gamma(\xi + 2\alpha)}
\]

\[
\times \left( \frac{Y^{2(\xi + \alpha) - 1} - Y^{-2(\xi + \alpha)}}{2(\xi + \alpha) \varphi \Gamma(2(\xi + \alpha) - \frac{1}{2})} \right) d\xi.
\]

(39.5)
We shift the contour to \( \text{Re} \xi = \frac{1}{2} - \text{Re} \alpha \); the relevant singularity is the pole of order 2 at \( \xi = \frac{1}{2} - \alpha \). We get
\[
\int_{\mathcal{C}} R_{\alpha}^{(1)}(z, \nu) d\mu(z) = \frac{1}{2\alpha - 1} \left( \log \left( \frac{1}{2} Y \right) - \Gamma' \left( \frac{1}{2} \right) + \Gamma' \left( \alpha + \frac{1}{2} \right) \right) + \frac{1}{(2\alpha - 1)^2} + O(Y^{-1/2}). \tag{39.6}
\]

On the other hand, we have
\[
\int_{\mathcal{C}} \mathcal{C} G(z, \nu) d\mu(z) = \frac{1}{4\pi i} \int_{(0)} k(\nu) \int_{\mathcal{C}} |e_0(z, \nu)^2 d\mu(z)| d\nu + \frac{1}{\pi} \frac{1}{\sqrt{1 - 4\nu}} + O(Y^{-1}). \tag{39.7}
\]
We then need the following formula: For \( \nu_1, \nu_2 \in i\mathbb{R}, \nu_1 \neq \nu_2, \)
\[
\int_\mathcal{C} e_0(z, \nu_1) e_0(z, -\nu_2) d\mu(z) = \frac{Y^\nu_1 - \nu_2}{\nu_1 - \nu_2} + \frac{Y^\nu_2 - \nu_1}{\nu_1 - \nu_2} (1 - \varphi_\Gamma(\nu_1) \varphi_\Gamma(-\nu_2))
\]
\[- \frac{Y^\nu_1}{\nu_1 \nu_2} \varphi_\Gamma(\nu_1) + \frac{Y^\nu_2}{\nu_1 \nu_2} \varphi_\Gamma(-\nu_2) - \int_{\mathcal{C} \setminus \mathcal{C}} \hat{e}_0(z, \nu_1) \hat{e}_0(z, -\nu_2) d\mu(z), \tag{39.8}
\]
where \( \hat{e}_0 \) is as in (37.15). To prove this, we apply (1.9) to see that the left side equals
\[
(\nu_1^2 - \nu_2^2)^{-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ e_0(z, \nu_1) \frac{\partial e_0}{\partial y}(z, -\nu_2) - e_0(z, -\nu_2) \frac{\partial e_0}{\partial y}(z, \nu_1) \right]_{y=Y} dx, \tag{39.9}
\]
which is
\[
(\nu_1^2 - \nu_2^2)^{-1} \left\{ - (\nu_1 + \nu_2) Y^{\nu_1 - \nu_2} + (\nu_1 - \nu_2) \varphi_\Gamma(\nu_1) Y^{\nu_1 - \nu_2}
\]
\[- (\nu_1 - \nu_2) \varphi_\Gamma(-\nu_2) Y^{\nu_1 + \nu_2} + (\nu_1 + \nu_2) \varphi_\Gamma(\nu_1) \varphi_\Gamma(-\nu_2) Y^{-\nu_1 + \nu_2}
\]
\[+ \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \hat{e}_0(z, \nu_1) \frac{\partial \hat{e}_0}{\partial y}(z, -\nu_2) - \hat{e}_0(z, -\nu_2) \frac{\partial \hat{e}_0}{\partial y}(z, \nu_1) \right]_{y=Y} dx \right\}. \tag{39.10}
\]
On noting that \( \Omega_0 \hat{e}_0(z, \nu) = (1 - \nu^2) \hat{e}_0(z, \nu) \), another application of (1.9) leads us to (39.8).

Thus the first term on the right of (39.7) is
\[
\frac{1}{4\pi i} \int_{(0)} k(\nu) \left( 2 \log Y - \frac{\varphi_\Gamma(\nu)}{\varphi_\Gamma(-\nu)} + \int_{\mathcal{C} \setminus \mathcal{C}} |e_0(z, \nu)^2 d\mu(z)| \right) d\nu
\]
\[+ \frac{1}{4\pi i} \int_{(0)} (Y^{2\nu} \varphi_\Gamma(-\nu) - 1) k(\nu) \frac{d\nu}{\nu}, \tag{39.11}
\]
in which the second line is a result of removing the superfluous singularity at \( \nu_1 + \nu_2 = 0 \) in the expression (39.8). The last integral is seen to be \( \frac{1}{4} k(0) + O(Y^{-1/2}) \) by shifting the contour to \( \text{Re} \nu = -\frac{1}{4} \) and then back to the original. Thus we have
\[
\int_{\mathcal{C}} \mathcal{C} G(z, z) d\mu(z) = \left( \frac{1}{2\alpha - 1} - \frac{1}{2\beta - 1} \right) \log Y
\]
\[- \frac{1}{4\pi i} \int_{-\infty}^{\infty} k(\nu) \frac{\varphi_\Gamma(\nu)}{\varphi_\Gamma(-\nu)} d\nu + k(\frac{1}{2}) + \frac{1}{4} k(0) + O(Y^{-1/2}). \tag{39.12}
\]
Combined with (39.6), this gives that
\[
\lim_{Y \to \infty} \int_{\mathfrak{f}_Y} |R^{(1)} - \varphi G|(z) d\mu(z) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} k(\nu) \frac{\varphi'_{\Gamma}(\nu)}{\varphi_{\Gamma}} d\nu - k\left(\frac{1}{2}\right)
\]
\[
- \frac{1}{2\alpha - 1} \left(\frac{\Gamma'(\alpha + \frac{1}{2}) + \log 2}{\Gamma(\alpha + \frac{1}{2})}\right) + \frac{1}{2\beta - 1} \left(\frac{\Gamma'(\beta + \frac{1}{2}) + \log 2}{\Gamma(\beta + \frac{1}{2})}\right).
\]
(39.13)

The first term on the right side equals
\[
- \frac{1}{2\alpha - 1} \int_{-\infty}^{\infty} k(\nu) \left(2 \zeta'_{\Gamma}(1 + 2\nu) + \frac{1}{\nu} + \frac{\Gamma'(\frac{1}{2} + \nu) - \log \pi}{\Gamma(\frac{1}{2} + \nu)}\right) d\nu
\]
\[
+ \frac{1}{2\beta - 1} \left(\frac{2 \zeta'_{\Gamma}(2\beta) + \frac{2}{2\alpha - 1} + \frac{\Gamma'(\beta) - \log \pi}{\Gamma(\beta)}\right).
\]
(39.14)

The insertion of the term $1/\nu$ on the left side is to remove the superfluous singularity at $\nu = 0$; and the right side is the result of shifting the contour to $+\infty$.

**NOTES:** In literature the formula (39.8) is termed the Maass–Selberg identity.

**40. Hyperbolic terms.** In order to deal with $R^{(2)}$, we need a preparation: We shall introduce a decomposition of $C^{(2)}$ which corresponds to (39.1). Thus, let $\omega$ be an arbitrary element in $C^{(2)}$, and let us suppose that $g_0 \in G$ maps the cusp and the origin to the two fixed points of $\omega$. Then there exists a positive constant $c(\omega) \neq 1$ such that $g_0^{-1}\omega g_0(z) = c(\omega)z$ for $z \in \mathbb{H}^2$. In particular, those hyperbolic elements $\Gamma$ which share the same pair of fixed points make up an infinite cyclic group. Hence, we may classify the elements of $C^{(2)}$ in terms of their fixed points:
\[
C^{(2)} = \bigsqcup_{\omega} \left\{ [\omega] \cap \{1\} \right\}, \quad [\omega] = \{\omega^n : n \in \mathbb{Z}\},
\]
(40.1)

where $\{\omega\}$ is a representative set. Further, we classify subgroups $[\omega]$ according to the $\Gamma$-conjugacy:
\[
C^{(2)} = \bigsqcup_{\omega} \bigcup_{\gamma \in B(\omega_0) \setminus \Gamma} \gamma^{-1}\left\{ [\omega_0] \cap \{1\} \right\} \gamma,
\]
(40.2)

where $B(\omega_0)$ is the normaliser of $[\omega_0]$ in $\Gamma$. Here it should be noted that the centraliser of any hyperbolic $\omega \neq 1$ is $[\omega]$. In fact, if $\gamma \omega = \omega \gamma$, then $\gamma, \omega$ share the same fixed points, and $\gamma$ is a power of $\omega$: to confirm this, one may use the relation $g_0^{-1}\gamma g_0(c(\omega)z) = c(\omega)g_0^{-1}\gamma g_0(z)$ with $g_0$ as above. On the other hand, if $\xi \in B(\omega_0)$, then $\xi^{-1}\omega_0 \xi$ is a generator $[\omega_0]$. Thus, there are two cases in general: Either there exists a $\delta \in B(\omega_0)$ such that $\delta^{-1}\omega_0 \delta = \omega_0^{-1}$, or there does not. In the former case, $\delta$ is an involution exchanging the fixed points of $\omega_0$.

Hence $[B(\omega_0) : [\omega_0]] = 2$, and
\[
\bigsqcup_{\gamma \in B(\omega_0) \setminus \Gamma} \gamma^{-1}\left\{ [\omega_0] \cap \{1\} \right\} \gamma = \bigsqcup_{\gamma \in [\omega_0] \setminus \Gamma} \gamma^{-1}\omega_0^n \gamma : n \in \mathbb{N}\}.
\]
(40.3)

In the latter case, we have $B(\omega_0) = [\omega_0]$, and
\[
\bigsqcup_{\gamma \in B(\omega_0) \setminus \Gamma} \gamma^{-1}\left\{ [\omega_0] \cap \{1\} \right\} \gamma = \bigsqcup_{\pm \gamma \in [\omega_0] \setminus \Gamma} \gamma^{-1}\omega_0^\pm n \gamma : \mathbb{Z} \ni n > 0\}.
\]
(40.4)
For example, \( \omega = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \) belongs to the former case, since \( \delta = \begin{pmatrix} 1 & -1 \end{pmatrix} \) gives \( \delta \omega^{-1} = \omega^{-1} \). On the other hand, \( \omega = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \) belongs to the latter case. Hence, we classify the generators \( \{ \omega_0^{\pm 1} \} \) according to the \( I \)-conjugacy, and designate \( \{ \varpi \} \) as a representative set:

\[
C^{(2)} = \bigcup_{\varpi} \bigcup_{\gamma \in [\varpi] \backslash I} \{ \gamma^{-1} \varpi^n \gamma : \mathbb{Z} \ni n > 0 \}. \tag{40.5}
\]

Each class containing \( \varpi \) is termed a prime class.

Then, we introduce the norm of a hyperbolic element \( \omega \neq 1 \): With \( c(\omega) \) as above,

\[
N(\omega) = \max \left( c(\omega), c(\omega^{-1}) \right) > 1, \tag{40.6}
\]

which is a function of the conjugacy class to which \( \omega \) belongs. Using the hyperbolic distance (2.1), we have

\[
N(\omega) = \exp \left( \inf_{z \in \mathcal{M}} d(z, \omega(z)) \right), \tag{40.7}
\]

since

\[
d(z, \omega(z)) = d(g_0^{-1}(z), c(\omega)g_0^{-1}(z))
= 2 \arcsinh \left( \frac{|g_0^{-1}(z) - c(\omega)g_0^{-1}(z)|}{2 \sqrt{c(\omega) \text{Im} \ g_0^{-1}(\omega)}} \right)
\geq 2 \arcsinh \left( \frac{1}{2} |c(\omega)^{1/2} - c(\omega)^{-1/2}) | \right) = \log N(\omega), \tag{40.8}
\]

in which the minimum is attained with \( z \) such that \( g_0^{-1}(z) \in i\mathbb{R} \). We shall use the term

\[
N(\varpi): \text{a pseudo-prime}. \tag{40.9}
\]

The reason why we liken \( N(\varpi) \) to a prime number will be revealed in Section 42. The distance \( d(z, \omega(z)) \) is the hyperbolic length of the closed geodesic on the Riemann surface \( I \backslash \mathbb{H}^2 \) which starts at \( z \) and returns to \( z \). Hence \( \log N(\varpi) \) is the length of the shortest closed geodesic among those associated with the prime class represented by \( \varpi \).

With this, we let \( R^{(2)}_\alpha \) be the part of \( R^{(2)} \) corresponding to \( p_\alpha \), similarly to (39.2). Then we have, by (40.5),

\[
\int_{\mathcal{M}} R^{(2)}_\alpha(z, z) d\mu(z) = \int_{\mathcal{M}} \sum_{\varpi} \sum_{\gamma \in [\varpi] \backslash I} \sum_{n=1}^\infty p_\alpha \left( \varrho(z, \gamma^{-1} \varpi^n \gamma(z)) \right) d\mu(z)
= \sum_{\varpi} \sum_{n=1}^\infty \int_{[\varpi]} p_\alpha \left( \varrho(z, \varpi^n z) \right) d\mu(z), \tag{40.10}
\]

where \([\varpi]] = \bigsqcup_{\gamma \in [\varpi] \backslash I} [\gamma \varphi \backslash \mathbb{H}^2 \). We choose an \( \eta \in \mathbb{G} \) such that \( \eta \varpi \eta^{-1}(z) = N(\varpi)z \).

Then \( \eta([\varpi]] = [\eta \varpi \eta^{-1}] \), and we have

\[
\int_{[\varpi]} p_\alpha \left( \varrho(z, \varpi^n(z)) \right) d\mu(z) = \int_{[\eta \varpi \eta^{-1}]} p_\alpha \left( \varrho(z, \eta \varpi^n \eta^{-1}(z)) \right) d\mu(z) = \int_{1 \leq |z| \leq N(\varpi)} \int_{z \in \mathbb{H}^2} p_\alpha \left( \varrho(z, N(\varpi)n z) \right) \frac{dx dy}{y^2}, \tag{40.11}
\]
For any $a, b > 1$, we have, by (34.6) for $\ell = 0$,
\[
\int_{\{1 \leq |z| \leq a \atop 0 < \text{Im } z}} p_\alpha \left( \varrho(z, b z) \right) \frac{dx dy}{y^2} = 2 \log a \int_0^{\pi} p_\alpha \left( \frac{1}{4 \sin^2 \theta} (b^{1/2} - b^{-1/2})^2 \right) \frac{d\theta}{\sin^2 \theta}
\]
\[
= \frac{\log a}{4\pi i} \int_{(-\frac{1}{2})} \frac{\Gamma^2(\xi + \alpha)\Gamma(-\xi) (\frac{1}{2}(b^{1/2} - b^{-1/2}))^{-2(\xi + \alpha)}}{\Gamma(\xi + 2\alpha)} \int_0^{\pi} (\sin \theta)^{2(\xi + \alpha - 1)} d\theta d\xi
\]
\[
= \frac{\log a}{8\pi^{3/2} i} \int_{(-\frac{1}{2})} \frac{\Gamma(\xi + \alpha)\Gamma(\xi + \alpha - \frac{1}{2})\Gamma(-\xi) (\frac{1}{2}(b^{1/2} - b^{-1/2}))^{-2(\xi + \alpha)}}{\Gamma(\xi + 2\alpha)} d\xi
\]
\[
= \frac{\log a}{4\pi^{1/2}} \frac{\Gamma(\alpha - \frac{1}{2})\Gamma(\alpha)}{\Gamma(2\alpha)} (\frac{1}{2}(b^{1/2} - b^{-1/2}))^{-2\alpha} \sum_{n=0}^{\infty} \frac{\log N(z)}{N(z)^{\alpha + n} - 1} - \frac{1}{2\beta - 1} \sum_{n=0}^{\infty} \frac{\log N(z)}{N(z)^{\beta + n} - 1}.
\]

Hence we have that
\[
\int_{\mathcal{F}} R^{(2)}(z, z) d\mu(z) = \frac{1}{2\alpha - 1} \sum_{n=0}^{\infty} \frac{\log N(z)}{N(z)^{\alpha + n} - 1} - \frac{1}{2\beta - 1} \sum_{n=0}^{\infty} \frac{\log N(z)}{N(z)^{\beta + n} - 1}.
\]

We stress that the double sums are absolutely and uniformly convergent for $\text{Re } \alpha > 1$ and $\text{Re } \beta > 1$, respectively, and represent regular functions. In fact, by the same token as the remark made immediately after (38.6) and by the observation that (34.15) for $\ell = 0$ implies that $p_\alpha(\varrho) > 0$ for $\alpha > 0$, the series defining $R^{(2)}_{\alpha}(z, z)$ is absolutely and uniformly convergent and bounded for $\alpha > 1$ and $z \in \mathcal{F}$.

Notes: The assertion (40.12) depends on quadratic transformations of the Gaussian hypergeometric function, an account of which is given in [18, Section 9.6]. The particular formula we need here is
\[
\sum_{n=0}^{\infty} \frac{\log \frac{1}{2}(1 + \sqrt{1 - z})}{N(z)^{\alpha + n} - 1} = 2F_1 \left( \frac{1}{2} \right) (\frac{1}{2} \alpha, \frac{1}{2}; 2\alpha; -\frac{1}{2}(b^{1/2} - b^{-1/2})^2). \]

See (9.8.3) loc. cit.

41. Elliptic terms. The fixed points of elliptic elements in $\Gamma$ are $\Gamma$-images of the points $\exp (\frac{i}{2} \pi i)$ and $\exp (\frac{i}{2} \pi i)$. The subgroups consisting of elements which fix these points are the cyclic groups $[\omega_2]$ and $[\omega_3]$ with $\omega_2 = \left[ \begin{array}{c} 1 \\ -1 \end{array} \right]$ and $\omega_3 = \left[ \begin{array}{c} 1 \\ -1 \end{array} \right]$, respectively. Also, their normalisers are themselves. Thus we have
\[
\int_{\mathcal{F}} R^{(c)}(z, z) d\mu(z) = \int_{[\omega_2]} (p_\alpha - p_\beta) (\varrho(z, \omega_2(z))) d\mu(z) + \sum_{j=1,2} \int_{[\omega_3]} (p_\alpha - p_\beta) (\varrho(z, \omega_j^2(z))) d\mu(z).
\]

To compute the right side, we shall consider, more generally, the following expression
\[
T_\alpha(\omega^j) = \int_{[\omega^j]} p_\alpha (\varrho(z, \omega_j^j(z))) d\mu(z), \quad 1 \leq j \leq m - 1,
\]
where ω is elliptic and of order m ≥ 2. We have
\[
\int_{\mathcal{F}} R^{(c)}(z, z) d\mu(z) = T_\alpha(\omega_2) - T_\beta(\omega_2) + \sum_{j=1,2} (T_\alpha(\omega_2^j) - T_\beta(\omega_2^j)).
\] (41.3)

In (41.2) we replace the domain \([\omega]\) by \(\omega'([\omega])\), \(\nu \in \mathbb{Z}\), and get
\[
T_\alpha(\omega^j) = \frac{1}{m} \int_{\mathbb{H}^2} p_\alpha(g(z, \omega^j(z))) d\mu(z).
\] (41.4)

Exploiting the \(\Gamma\)-conjugation, we may assume that the fixed point of \(\omega\) is \((\frac{1}{2}, \pi i)\), that is, \(\omega^j = \left[\frac{\eta - \lambda}{\lambda - \eta}\right]\) with \(\eta = \cos((j/m)\pi), \lambda = \sin((j/m)\pi)\). Hence
\[
T_\alpha(\omega^j) = \frac{1}{m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_\alpha(\frac{1}{\lambda} \lambda^2 y^2 |z|^2 + 1^2) \frac{dydx}{y^2}.
\] (41.5)

We perform the change of variable \(y \mapsto (x^2 + 1)^{1/2}u\), and divide the inner integral at \(u = 1\), and then apply the change of variable \(u \mapsto 1/u\) in the integral over the unit interval. Further, we apply \(v = u - 1/u\) and \(v \mapsto 2|x|(x^2 + 1)^{-1/2}w^{1/2}\). The inner integral is transformed into
\[
\frac{|x|}{x^2 + 1} \int_{0}^{\infty} p_\alpha((\lambda x)^2 (w + 1)) w^{-1/2} dw.
\] (41.6)

This is computed in much the same as (40.12), and we have
\[
T_\alpha(\omega^j) = \frac{1}{2(\alpha - 1)\lambda m} \int_{0}^{\infty} \frac{(\lambda x + \sqrt{(\lambda x)^2 + 1})^{1-2\alpha}}{x^2 + 1} dx
\]
\[
= \frac{2}{(2\alpha - 1)m} \int_{0}^{\infty} \frac{(\max(\xi, 1/\xi))^{1-2\alpha}}{(\xi - 1/\xi)^2 + 4\lambda^2} d\xi.
\] (41.7)

The second line is due to the transformation \(2\lambda x \mapsto \xi - 1/\xi\). Invoking the representation
\[
\frac{1}{2\alpha - 1} (\max(\xi, 1/\xi))^{1-2\alpha} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\xi^{2it}}{(\alpha - \frac{1}{2})^2 + t^2} dt,
\] (41.8)

we have that
\[
T_\alpha(\omega^j) = \frac{1}{\pi m} \int_{-\infty}^{\infty} \frac{1}{(\alpha - \frac{1}{2})^2 + t^2} \int_{0}^{\infty} \frac{\xi^{2it}}{(\xi - 1/\xi)^2 + 4\lambda^2} d\xi dt
\]
\[
= \frac{1}{\pi m} \int_{-\infty}^{\infty} \left(\frac{1}{((\alpha - \frac{1}{2})^2 + t^2)(1 + e^{-2\pi t})}\right) \int_{-\infty}^{\infty} \frac{\xi^{2it}}{(\xi - 1/\xi)^2 + 4\lambda^2} d\xi dt.
\] (41.9)

In the last integral we set arg \(\xi = 0\) for \(\xi > 0\) and arg \(\xi = \pi\) for \(\xi < 0\). Shifting the path to \(+i\infty\),
\[
T_\alpha(\omega^j) = \frac{1}{2m\lambda} \int_{-\infty}^{\infty} \frac{e^{-(j/m)\pi t}}{((\alpha - \frac{1}{2})^2 + t^2)(1 + e^{-2\pi t})} dt.
\] (41.10)
Again shifting the path to $+i\infty$,

$$T_\alpha(\omega^j) = \frac{\pi}{(2\alpha - 1)m\lambda} \frac{e^{(1-2\alpha)(j/m)\pi i}}{1 - e^{-2\alpha\pi i}} + \frac{i}{2(2\alpha - 1)m\lambda} \sum_{l=0}^{m-1} e^{-(2l+1)(j/m)\pi i} \left( \frac{1}{\alpha + l} - \frac{1}{1 - \alpha + l} \right),$$

(41.11)

which gives a meromorphic continuation with respect to $\alpha$. The sum is

$$\frac{1}{m} \sum_{l=0}^{m-1} e^{-(2l+1)(j/m)\pi i} \sum_{f=0}^{\infty} \left( \frac{1}{(\alpha + l)/m + f} - \frac{1}{(1 - \alpha + l)/m + f} \right),$$

$$= \frac{1}{m} \sum_{l=0}^{m-1} e^{-(2l+1)(j/m)\pi i} \left( \frac{\Gamma'(1 - (\alpha + l)/m)}{\Gamma(\alpha + l)/m) - \frac{\Gamma'((\alpha - l)/m)}{\Gamma((\alpha + l)/m)} \right)$$

$$= \frac{1}{m} \sum_{l=0}^{m-1} \left( e^{(2l+1)(j/m)\pi i} \frac{\Gamma'((\alpha + l)/m)}{\Gamma((\alpha + l)/m) + \pi \frac{e^{(2l+1)(j/m)\pi i}}{2i \tan(\pi(\alpha + l)/m)} \right),$$

(41.12)

where the second line depends on Weierstrass' product representation of the $\Gamma$-function and the fourth line on the functional equation $\Gamma(s)\Gamma(1 - s) = \pi/s$. As to the first term on the right side of (41.11), we have

$$\frac{e^{(1-2\alpha)(j/m)\pi i}}{1 - e^{-2\alpha\pi i}} = \frac{1}{2im} \sum_{l=0}^{m-1} e^{(2l+1)(j/m)\pi i} \tan(\pi(\alpha + l)/m).$$

(41.13)

In fact, the difference of the two sides is entire, since they have the same set of poles and respective principal parts are identical. Noting the periodicity and taking $\alpha$ to $i\infty$, we get the assertion. Hence we have that

$$T_\alpha(\omega^j) = \frac{1}{(1-2\alpha)m^2} \sum_{l=0}^{m-1} \frac{\sin \left( (2l + 1)(j/m)\pi \right) \Gamma'((\alpha + l)/m)}{\sin \left( (j/m)\pi \right) \Gamma((\alpha + l)/m)},$$

(41.14)

and

$$\sum_{j=1}^{m-1} T_\alpha(\omega^j) = \frac{1}{(1-2\alpha)m^2} \sum_{l=0}^{m-1} (m - 2l - 1) \frac{\Gamma'((\alpha + l)/m)}{\Gamma((\alpha + l)/m)},$$

(41.15)

since

$$\sum_{j=1}^{m-1} \frac{\sin \left( (2l + 1)(j/m)\pi \right)}{\sin \left( (j/m)\pi \right)} = \sum_{j=1}^{m-1} \sum_{h=-l}^{l} e^{((j/m)h)} = m - 2l - 1.$$  

(41.16)

**42. Selberg’s zeta-function and trace formula.** We sum up the discussion developed in the last four sections. Thus we classify hyperbolic elements of $\Gamma$ as in (40.5), which defines the prime classes $\{\varpi\}$. The corresponding pseudo-primes $N(\varpi)$ are defined by (40.9).

With this, we introduce the Selberg zeta-function associated with $\Gamma$ by

$$\zeta_\Gamma(s) = \prod_{\varpi} \prod_{n=0}^{\infty} \left( 1 - \frac{1}{N(\varpi)^{s+n}} \right).$$

(42.1)
We have shown already that
\[ \frac{\zeta'(s)}{\zeta''(s)} = \sum_{\omega} \sum_{n=0}^{\infty} \frac{\log N(\omega)}{N(\omega)^{s+n} - 1}, \quad \Re s > 1, \] (42.2)
is absolutely convergent. We introduce also
\[ \Psi_R(s) = \frac{\zeta'(s)}{\zeta''(s)} - 2 \frac{\zeta'(2s)}{\zeta(s)} - \frac{1}{s-1} - \frac{1}{s} - \frac{2}{2s-1} + \log(2\pi) - 2 \Gamma'(\frac{1}{2}) + W(s), \] (42.3)
with
\[ W(s) = -\frac{1}{6}(2s-1)\frac{\Gamma'}{\Gamma}(s) + \frac{1}{4}\frac{\Gamma'}{\Gamma}(\frac{1}{2}(s+1)) - \frac{1}{4}\frac{\Gamma'}{\Gamma}(\frac{1}{2}s) + \frac{2}{9}\frac{\Gamma'}{\Gamma}(\frac{1}{3}(s+2)) - \frac{2}{9}\frac{\Gamma'}{\Gamma}(\frac{1}{3}s). \] (42.4)

Collecting (38.7), (38.8), (39.13), (39.14), (40.13), (41.3), (41.5) as well as invoking the duplication formula for the \( \Gamma \)-function, we obtain a version of the Selberg trace formula for the full modular group \( \Gamma \): The function \( \Psi_R \) exists as a meromorphic function over the entire complex plane, and we have, for arbitrary \( \alpha, \beta \in \mathbb{C} \),
\[ \frac{\Psi_R(\alpha)}{2\alpha - 1} - \frac{\Psi_R(\beta)}{2\beta - 1} = \sum_{V} \left\{ \frac{1}{(\alpha - \frac{1}{2})^4 - \nu_V^2} - \frac{1}{(\beta - \frac{1}{2})^4 - \nu_V^2} \right\}. \] (42.5)

In fact, we have already established this for \( 1 < \Re \alpha < \Re \beta \). Since this implies in particular that we have \( \sum_{V} |\nu_V|^{-4} < +\infty \), the right side is a meromorphic function over \( \mathbb{C}^2 \), by analytic continuation. One may put \( \alpha = s \) and \( \beta = 2 \), and find that \( \Psi_R(s) \) is regular for all \( s \) save for the simple poles \( \{ \frac{1}{2} \pm \nu_V : V \text{ in the unitary principal series} \} \).

The definition (42.1) conjures up the Euler product representation for the Riemann zeta-function. Thus we shall look for analogies between the two \( \zeta \)-functions. First we observe that \( W(s) \) has poles at non-positive integers; the residues at \( s = 0 \) is equal to 1, and all other poles have negative integers as their residues, which can be seen by classifying poles of the expression (42.4) according to mod 6. Hence the poles of \( (\zeta' / \zeta)(s) \) are all of order 1, and their residues are equal to integers. This means that \( \zeta(s) \) exists as a meromorphic function over \( \mathbb{C} \). To be more precise, the function
\[ \xi_R(s) = \frac{(2\pi)^s \zeta''(s)}{\zeta(2s)\Gamma(2s)(s-1)(2s-1)} \exp \left( \int_0^{s-\frac{1}{2}} W(\eta + \frac{1}{2})d\eta \right) \] (42.6)
is entire, and its zeros are \( \frac{1}{2} \pm \nu_V \), with \( V \) as above. In fact, we have
\[ \frac{\zeta'}{\xi}(s) = \Psi_R(s). \] (42.7)

Taking account of the nature of the poles of \( W(s) \), the function
\[ \frac{\zeta''(s)}{\zeta(2s)\Gamma(2s)(2s-1)} \] (42.8)
is entire. Its real zeros are precisely at 1, which is simple, and at negative integers; and the set of its complex zeros coincides with that of $\xi_{T}(s)$. We have, in a neighbourhood of $s = 1$,

$$\frac{\zeta'_{T}}{\zeta_{T}}(s) = \frac{1}{s - 1} + O(1). \quad (42.9)$$

Further, the functional equation

$$\zeta_{T}(s) = \chi_{p}(s)\chi_{e}(s)\chi_{1}(s)\zeta_{T}(1 - s) \quad (42.10)$$

holds, where

$$\chi_{p}(s) = \frac{(2\pi)^{1 - 2s}\zeta(2s)\Gamma(2s)}{\zeta(2(1 - s))\Gamma(2(1 - s))},$$

$$\chi_{e}(s) = \left(\frac{\sin \frac{1}{2}\pi(1 - s)}{\sin \frac{1}{2}\pi s}\right)^{1/2}\left(\frac{\sin \frac{1}{2}\pi(1 - s)}{\sin \frac{1}{2}\pi s}\right)^{2/3}, \quad (42.11)$$

$$\chi_{1}(s) = \exp \left(\frac{1}{\pi} \int_{0}^{s - \frac{1}{2}} \eta \log \frac{\sin \frac{1}{2}\pi(\eta + \frac{1}{2})}{\sin \frac{1}{2}\pi(\frac{1}{2} - \eta)} d\eta\right).$$

The factors $\chi_{p}$, $\chi_{e}$, $\chi_{1}$ come from parabolic, elliptic elements, and the unit, respectively. The function $\chi_{e}(s)\chi_{1}(s)$ is meromorphic taking the value 1 at $s = \frac{1}{2}$. In fact, the right side of the trace formula (42.5) is invariant against the transform $\alpha \mapsto 1 - \alpha$, and we have

$$\Psi_{T}(s) = -\Psi_{T}(1 - s). \quad (42.12)$$

Integrating this, we have

$$\xi_{T}(s) = \xi_{T}(1 - s). \quad (42.13)$$

Also, we have

$$\begin{align*}
W\left(\frac{1}{2} + \eta\right) + W\left(\frac{1}{2} - \eta\right) &= \frac{\eta}{3} \frac{d}{d\eta} \log \sin \left(\frac{\pi}{2} + \eta\right) \\
- \frac{1}{2} \frac{d}{d\eta} \log \frac{\sin \left(\frac{1}{2}\pi(\frac{1}{2} - \eta)\right)}{\sin \left(\frac{1}{2}\pi(\frac{1}{2} + \eta)\right)} &= \frac{2}{3} \frac{d}{d\eta} \log \frac{\sin \left(\frac{1}{2}\pi(\frac{1}{2} - \eta)\right)}{\sin \left(\frac{1}{2}\pi(\frac{1}{2} + \eta)\right)} \quad (42.14).
\end{align*}$$

In this way we obtain (42.10).

**Notes:** The assertion in this section is a typical instance of Selberg’s fundamental discovery [32]. The similarities between $\zeta(s)$ and $\zeta_{T}(s)$ are striking. However, they are in fact ostensible. Comparing the original and the pseudo-Euler products, one surmise that the correct analogy should be found between $\frac{1}{\zeta(s)}$ and $\zeta_{T}(s)$. Then we find grave differences. The most salient is: Despite the fact that both have a simple zero at $s = 1$, the former has infinitely many poles in the critical strip $0 < \Re s < 1$ and the latter has none at all. Nevertheless, see the notes to the next section for another view point. It should be added that strictly speaking the analogue of the Riemann hypothesis holds with $\xi_{T}(s)$ but not with $\zeta_{T}(s)$ itself as the latter has complex zeros coming from $\zeta(2s)$. It should be observed that the factor $\zeta(2s)$ is due to parabolic elements. Hence, in the case where the underlying discrete subgroup has a compact fundamental domain unlike the full modular group, the corresponding zeta-function of Selberg satisfies precisely the analogue of the Riemann hypothesis, as it is known that such a discrete group lacks parabolic elements.
43. Weyl’s law. We shall prove the asymptotic formula:

\[ N_r(K) = \sum_{|\nu| \leq K} 1 = \frac{1}{12} K^2 + O(K \log K). \] (43.1)

This is the same as counting complex zeros of \( \xi_r(s) \) in the region \( 0 < \text{Re} \ s < 1, 0 < \text{Im} \ s \leq K \). We shall employ an argument from the theory of the Riemann zeta-function.

First, we put \( \alpha = \frac{1}{2} + K \) and \( \beta = \frac{1}{2} + 2K \) in (42.5) so that

\[
\frac{1}{K^2} \sum_{K \leq |\nu| \leq 2K} 1 \ll \sum_{\nu} \left( \frac{1}{K^2 + |\nu|^2} - \frac{1}{4K^2 + |\nu|^2} \right) = \frac{1}{2K} \Psi_r \left( \frac{1}{2} + K \right) - \frac{1}{4K} \Psi_r \left( \frac{1}{2} + 2K \right). \] (43.2)

By the definition (42.3)-(42.4) and an asymptotic expansion of \( (\Gamma'/\Gamma)(s) \), we have

\[ N_r(K) \ll K^2. \] (43.3)

Also putting \( \alpha = \frac{3}{2} + iK \) and \( \beta = \frac{1}{2} + K \) in (42.5), we have

\[ \sum_{\nu} \left\{ \frac{1}{(1 + iK)^2 + |\nu|^2} - \frac{1}{K^2 + |\nu|^2} \right\} = O(1). \] (43.4)

Applying (43.3) to this, we get

\[ \sum_{|\nu| - K \leq \frac{1}{2}K} \frac{1}{(1 + iK)^2 + |\nu|^2} = O(1), \] (43.5)

and thus

\[ \sum_{|\nu| - K \leq \frac{1}{2}K} \frac{1}{1 + i(K - |\nu|)} = O(K). \] (43.6)

Hence (43.3) is improved to

\[ N_r(K + 1) - N_r(K) \ll K. \] (43.7)

This implies in turn that for any bounded \( s \)

\[ \Psi_r(s) = \sum_{|\nu| \leq |t| \leq \frac{1}{2}} \frac{1}{s - \frac{1}{2} - \nu} + O(t \log 2t), \quad \text{Im} \ s = t \geq 1. \] (43.8)

We may assume hereafter that the parameter \( K \) satisfies

\[ \inf_{\nu} |\nu| - K | \gg K^{-1}. \] (43.9)

Then, by the functional equation (42.13), we have

\[ N_r(K) = \frac{1}{\pi} \arg \xi_r \left( \frac{1}{2} + iK \right). \] (43.10)
The argument starts with the value 0 at \( +\infty + iK \) and continuously varies along the line \( \text{Im} \ s = K \). By (42.6) we have

\[
N_F(K) = \frac{1}{\pi} \text{Im} \int_0^{iK} W(\eta + \frac{1}{2})d\eta + \frac{1}{\pi} \arg \zeta_F(\frac{1}{2} + iK) + O(K \log K)
\]

\[
= \frac{1}{12} K^2 + \frac{1}{\pi} \arg \zeta_F(\frac{1}{2} + iK) + O(K \log K),
\]

(43.11)
in which the error term is due to the \( \Gamma \)-factor and \( \zeta(2s) \). The argument of \( \zeta_F(s) \) is defined in the same way as that of \( \xi_F(s) \), and we consider the estimation of

\[
\log \zeta_F \left( \frac{1}{2} + iK \right) = \int_{\frac{1}{2} + iK} \frac{\zeta_F'}{\zeta_F}(u + iK)du
\]

\[
= \int_{\frac{1}{2} + iK} \frac{\zeta_F'}{\zeta_F}(u + iK)du + O(1).
\]

(43.12)

Here the left side is defined to be the result of the analytic continuation of the integral of \( (\zeta_F'/\zeta_F)(s) \) along the above horizontal line. By the definition (42.3) and the approximation (43.8), we have

\[
\frac{\zeta_F'}{\zeta_F}(u + iK) = \sum_{|\nu| - K \leq \frac{1}{2}} \frac{1}{u - \frac{1}{2} + i(K - |\nu|)} + O(K \log K),
\]

(43.13)
where we have used a well-known bound of \( (\zeta'/\zeta)(s) \) for Re \( s \geq 1 \). Hence, in view of (43.7) we obtain

\[
\arg \zeta_F \left( \frac{1}{2} + iK \right) \ll K \log K.
\]

(43.14)

We end the proof of (43.1).

**Notes:** The function \( \Xi_F(t) = \xi_F(\frac{1}{2} + it) \) is an analogue of the \( \Xi \)-function in the theory of \( \zeta(s) \). This is real for real \( t \). It oscillates wildly as (43.1) implies; in fact the number of real zeros in the interval \( |t - K| \leq B \log K \) is more than \( K \log K \), provided \( B \) is sufficiently large. Still all zeros of \( \Xi_F(t) \) are on the real axis without any single exception. Amazing, indeed.

**44. Pseudo-prime number theorem.** We shall prove the asymptotic formula

\[
\pi_F(x) = \sum_{\text{\( F \)-pseudo-prime} \ < x} 1 = \int_2^x \frac{du}{\log u} + O \left( x^{3/4}(\log x)^{-1/2} \right).
\]

(44.1)

Following the traditional treatment of the distribution of prime numbers, we consider

\[
\psi_F(x) = \sum_{n=1}^{\infty} \sum_{N(z) < x^{1/n}} \log N(z).
\]

(44.2)

The relevant generating function is \( (\zeta_F'/\zeta_F)(s) - (\zeta_F'/\zeta_F)(s + 1) \). We are, however, unable to apply Perron’s inversion formula, because of the difficulty to get any efficient bound for the number of pseudo-primes in a given unit interval. Thus, we shall take a detour, by employing the Riesz mean of \( \psi_F(x) \):

\[
\tilde{\psi}_F(x) = \int_1^x \psi_F(y) \frac{dy}{y}.
\]

(44.3)
It should be noted that the lower limit of integration is due to the fact that \( N(\omega) > 1 \) by definition. We have, for any \( \tau > 0 \),
\[
\tau^{-1} \left( \tilde{\psi}_\tau(x) - \tilde{\psi}_\tau(xe^{-\tau}) \right) \leq \psi'_\tau(x) \leq \tau^{-1} \left( \tilde{\psi}_\tau(xe^\tau) - \tilde{\psi}_\tau(x) \right). \tag{44.4}
\]

On the other hand, we have
\[
\tilde{\psi}_\tau(x) = \frac{1}{2\pi i} \int_{2-ik}^{2+ik} \left( \frac{c'_\tau(s)}{c_\tau(s)} - \frac{c'_\tau(s+1)}{c_\tau(s+1)} \right) x^s \frac{ds}{s^2}
= \frac{1}{2\pi i} \int_{2-ik}^{2+ik} \frac{c'_\tau(s)x^s}{c_\tau(s)} \frac{ds}{s^2} + O((\log x)^2), \tag{44.5}
\]
where the error term is the result of moving the contour to \( \text{Re} \, s = (\log x)^{-1} \) in the relevant part in the first line; note that (42.9). The integral in the second line is
\[
\frac{1}{2\pi i} \int_{2-ik}^{2+ik} \frac{c'_\tau(s)x^s}{c_\tau(s)} \frac{ds}{s^2}
= \frac{1}{2\pi i} \int_{2-ik}^{2+ik} \Psi_\tau(s)x^s \frac{ds}{s^2} + O(1),
\]
where \( K \approx x^3 \) is chosen so that (43.9) is fulfilled, and \( \Psi_\tau(s) = (\zeta'/\zeta)(s) - \Psi_\tau(s) \). The second integral on the right side is, by (42.3), equal to \( x + O(x^{1/2}) \). In fact, the part composed of logarithmic derivatives of the \( \Gamma \)-function can be estimated by shifting the path to \( \text{Re} \, s = 1 \), say, and the part containing \( (\zeta'/\zeta)(2s) \) can be estimated by shifting the path to \( \text{Re} \, s = \frac{3}{2} \). Hence, we have
\[
\tilde{\psi}_\tau(x) = x + \frac{1}{2\pi i} \int_{2-ik}^{2+ik} \Psi_\tau(s)x^s \frac{ds}{s^2} + O(x^{1/2}). \tag{44.7}
\]

We move the contour to \( C_K \) which is the result of connecting \( 2 - iK, -\frac{1}{2} - iK, -\frac{1}{2} + iK, 2 + iK \) by straight segments:
\[
\tilde{\psi}_\tau(x) = x + \frac{1}{2\pi i} \int_{C_K} \Psi_\tau(s)x^s \frac{ds}{s^2} + 2\Re \sum_{|\nu| \leq K} x^{1/2+\nu} \left( \frac{1}{2} + \nu \right)^{1/2} + O(x^{1/2})
= x + \frac{1}{2\pi i} \int_{C_K} \Psi_\tau(s)x^s \frac{ds}{s^2} + O(x^{1/2} \log x), \tag{44.8}
\]
where we have applied (43.7). The last integral can be ignored. In fact, to estimate the part over \( \text{Re} \, s = -\frac{1}{2} \) one may use the bound \( \Psi_\tau(s) \ll |s| \log |s| \) which follows from (43.7)–(43.8); and the segment over \( \text{Im} \, s = \pm K \) is divided into pieces of length \( 1/K \) and we use (43.7)–(43.9). With this, we return to (44.4), and set \( \tau = x^{-1/4}(\log x)^{1/2} \). We obtain
\[
\psi'_\tau(x) = x + O \left( x^{3/4}(\log x)^{1/2} \right). \tag{44.9}
\]
We end the proof of (44.1).

**Notes:** The Riemann hypothesis for \( \zeta(s) \) implies an error term in the prime number theorem that is equivalent to the validity of the hypothesis. Hence it appears natural to expect that
a similar correspondence should hold for \( \zeta_{\Gamma}(s) \) and the pseudo-prime number theorem for the group \( \Gamma \). This is, however, still a challenging problem. The difficulty is in the fact that there are too many complex zeros to effectively deal with. See Iwaniec \[12\] for the first significant step.

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