Risk Sensitive Multiple Goal Stochastic Optimization, with application to Risk Sensitive Partially Observed Markov Decision Processes.

Vaio(s Laschos*  Robert Seidel†  Klaus Obermayer*

August 23, 2018

Abstract

We study Risk Sensitive Partially Observable Markov Decision Processes (RSPOMDPs) where the performance index is either the expected utility, for utility functions that can be written as a weighed sum of exponentials or the expected shortfall generated by functions of the hyperbolic sine type. In that direction we utilize methods for treating Risk Sensitive Multiple Goal Markov Decision Processes (RSMGMDPs) where the performance index is given by sums of expected utilities, with each utility function applied to a different running cost, but also methods from the theory of Risk Sensitive Constrained Markov Decision Processes (RSCMDPs).

1 Introduction

1.1 The finite time MDP landscape

In the classical theory of finite time Markov Decision Processes (MDPs), the objective is to optimize the expected value

$$J_N(x_0, \pi) = \mathbb{E}_{x_0}^{\pi} \left[ \sum_{n=0}^{N-1} C(X_n, A_n) \right],$$

over some time interval $N$. The expected value $J_N$ is generated by the random process $(X_n)$, where $X_n$ denotes in which state the process is situated at time $n$. This process is controlled via a series of actions $(A_n)$, according to a policy $\pi$, that changes the underlying state transition probabilities $P(x'|x, a)$ of $(X_n)$. The inclusion of risk-sensitivity and partial observability are natural extensions to this model, which find many applications in the fields finance and engineering.

In risk-sensitive modeling, one still tries to optimize the expected value $J_N$, but at the same time to avoid, or seek events that, although possibly rare, can lead to outcomes with outstandingly large negative, or positive, impact. To our knowledge, there are three major approaches to introduce risk sensitivity in MDPs: These is the classical theory of expected utility [28, 13, 5], where one tries to optimize

$$J_N(x_0, \pi) = U^{-1} \left[ \mathbb{E}_{x_0}^{\pi} \left[ U \left( \sum_{n=0}^{N-1} C(X_n, A_n) \right) \right] \right],$$

*Fakultät Elektrotechnik und Informatik, and Bernstein Center for Computational Neuroscience, Technische Universität Berlin, Marchstr. 23, 10587, Berlin, Germany (vaio(s.laschos@tu-berlin.de, klaus.obermayer@mailbox.tu-berlin.de).

†Institute of Mathematics, Technische Universität Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany (robert.seidel@campus.tu-berlin.de).
for some increasing and continuous function $U : \mathbb{R} \to \mathbb{R}$, the mean-variance models introduced by Markowitz for portfolio selection, where one tries to optimize

$$J_N(x_0, \pi) = \mathbb{E}^\pi_{x_0} \left[ \sum_{n=0}^{N-1} C(X_n, A_n) \right] + \lambda \text{Var}^\pi_{x_0} \left[ \sum_{n=0}^{N-1} C(X_n, A_n) \right],$$

and the recent theory of risk measures introduced in [37] and generalized in [31] (for definitions and examples see [40]). Note that the exponential function generates a performance index that belongs to several models of risk at the same time, and it has been extensively studied in many different settings [17, 2, 7, 8, 14, 16, 22, 24, 27, 30, 32].

In classical MDPs, one makes the assumption that the controlled process takes values on a set of states which is always accessible to the controller. However, in several real-life applications, the real state is not directly observable—but a secondary information, dependent on the state, can be observed. Partially Observable Markov Decision Processes (POMDPs) are a generalization of MDPs towards incomplete information about the current state. POMDPs extend the notion of MDPs by a set of observations $Y$ and a set of conditional observation probabilities $Q(y|s)$ given the 'hidden' state $s \in S$.

Risk-sensitivity has been extensively studied in the full information scenario, while partial observability has so-far been mostly considered in the risk-neutral framework, both theoretically [11, 33, 19, 32, 43] and numerically [29, 24]. To our best knowledge, there is only partial progress [33, 6, 18, 20, 3, 23, 9] when it comes to combining risk-sensitivity and partial observability.

### 1.2 The obstacle for combining partial observability with risk sensitivity

In risk-neutral POMDPs, one can introduce a new state space, called belief state space $X = \mathcal{P}(S)$, the set of probability measures on $S$, and a stochastic process $(X_n)$ taking values in $X$, such that $X_n(s)$ is the expected probability of $S_n$ being equal to the “hidden” state $s \in S$ at time $n$, conditioned on the accumulated observations and actions up to time $n$. One can treat this process on the belief state space like a Completely Observable Markov Decision Process (COMDP) on $X$ with classical tools, retrieve optimal or $\varepsilon$-optimal polices (i.e. policies with expected value at most $\varepsilon$ far from the optimal value), and then apply them to the original problem. It is remarkable that, due to the linearity of the expectation operator, the belief state is a so-called sufficient statistic. Broadly speaking, a sufficient statistic carries adequate information for the controller to make an optimal choice at a specific point in time. It also allows to separate the present cost from the future cost through a Bellman-style equation. For a nice exposition on sufficient statistics, one can read [26].

When risk is involved, extra or alternative kind of information is often necessary to make an optimal decision. In the case of expected utilities, additional information on the accumulated cost is necessary to make an optimal choice, even if the true state is known to the controller [15, 33]. When risk is modeled by risk measures, the current state is a sufficient statistic in the fully observable case [37, 31, 40], but the belief state still does not appear to be a sufficient statistic in the partially observable setting. A workaround to this problem, is to assume that the controller is aware of the running cost through some mechanism. For example, the controller observes either the running cost directly [33], or a part of the whole process that is responsible for the cost [6]. In [18], cost functions that depend on both observable and beliefs about unobservable quantities are proposed. Although, the aforementioned approach covers many interesting real-life cases, it does not cover the whole spectrum of risk-sensitive partially observable stochastic problems, and specifically excludes cases where the agent is not aware of the accumulated cost until the process terminates.
1.3 Our contribution

In the case of the exponential utility function, which was also extensively studied under partial observability [20, 3, 23, 9], similarly to the case of risk-neutral POMDPs, one can treat the problem in full generality, without any cost observation. One can find both the value function and optimal strategies by introducing a new state space called the information state space, and solve an equivalent problem with complete observation in that space. Specifically, in [9], a variation of the information space is used that allows to replace the original RSPOMDP with a Risk Sensitive Completely Observable Markov Decision Process (RSCOMDP) with exponential cost structure where the running cost depends on the next state as well.

Our main contribution is Section 3 where we exploit this idea even further. There, we treat vector-valued functions \( v : \mathcal{P}(S) \times H \rightarrow \mathbb{R}^{\max} \) that map a history depending policy \( \pi \in H \) and an initial configuration \( \theta_0 \in \mathcal{P}(S) \) to a vector of expected exponential utilities, each applied to a running cost \( C(s,a) \) dependent on the next state \( s \in S \) and action \( a \in A \), of the form

\[
\pi(\theta_0, \pi) = \left( \mathbb{E}^\pi_{\theta_0,0} \left[ e^{\lambda t} \sum_{n=0}^{N-1} C(S_n, A_n) \right], \ldots, \mathbb{E}^\pi_{\theta_0,0} \left[ e^{\lambda \max} \sum_{n=0}^{N-1} C(S_n, A_n) \right] \right).
\]

We show, that one can introduce a new state space \( X = \mathcal{P}(S)^{\max} \times Y \), a controlled transition matrix \( P(x'|x,a) \), and a collection of running costs \( C^t : X \times A \times X \rightarrow \mathbb{R} \), that depend on the next stage as well, such that for the resulting completely observable controlled processes \( (X_n) \), we have \( \pi(\theta_0, \pi) = v(x_0, \eta(\pi)) \), where

\[
v(x_0, \pi) = \left( \mathbb{E}^\pi_{x_0} \left[ e^{\lambda t} \sum_{n=0}^{N-1} C^t(X_n, A_n, X_{n+1}) \right], \ldots, \mathbb{E}^\pi_{x_0} \left[ e^{\lambda \max} \sum_{n=0}^{N-1} C^\max(X_n, A_n, X_{n+1}) \right] \right),
\]

\( x_0 = (\theta_0, \ldots, \theta_0, y_0) \) for a fixed but arbitrary \( y_0 \in Y \), and \( \eta : H \rightarrow H \) is essentially a bijection between policies.

Now, given a RSPOMDP where the performance index can be written as a function \( \ell : \mathbb{R}^{\max} \rightarrow \mathbb{R} \) of \( \pi(\theta_0, \pi) \), i.e.

\[
\overline{J}_N(\theta_0, \pi) = \ell(\pi(\theta_0, \pi)),
\]

one can find a RSCOMDP with performance index \( J_N(x_0, \pi) = \ell(v(x_0, \pi)) \), such that \( \overline{J}_N(\theta_0, \pi) = J_N(x_0, \eta(\pi)) \).

So far, the amount of information needed to make an optimal choice has not been reduced compared to the original problem. However, in some specific cases, like the case where \( \ell(t_1, \ldots, t_{\max}) = \ell(w_1 t_1 + \ldots + w_{\max} t_{\max}) \), for some increasing real function \( \ell \), one can further augment the space \( X \) and replace the resulting RSCOMDP by a new one where the optimal policies are Markovian.

This way, it is again possible to obtain the value function, and optimal and almost optimal policies.

More specifically, when the performance index is the expected utility generated by a utility function \( \hat{U} \) that can be written as sums of exponential functions, i.e. \( \hat{U}(t) = \sum_{i=1}^{\max} w_i \text{sign}(\lambda^i) e^{\lambda t_i} \), one can transform the original problem into a RSCOMDP with a new state space and performance index which is given as the sum of expected utilities with different running costs applied to each utility function. In Section 2, following ideas from [3], we will show that this kind of RSCOMDP can be solved by introducing an Augmented space \( X \times \mathbb{R}^{\max} \), where the extra dimensions keep track of the accumulated cost for each utility function, and solve the problem there.

A similar approach can be used for a subclass of expected shortfall utilities (which belong in the class of risk measures as in [10]), i.e. functions defined on the set of real variables \( X \) on a probability space \((\Omega, \mathbb{P})\) of the form

\[
g(X, P) = \sup_m \{ E[U(X - m)] = 0 \},
\]

\( g(X, P) = \sup_m \{ E[U(X - m)] = 0 \}, \)
for some increasing and continuous utility function $U : \mathbb{R} \to \mathbb{R}$ that satisfies $U(0) = 0$. More specifically, if $U(t) = e^{\lambda_1 t} - e^{\lambda_2 t}$ with $\lambda_2 < 0 < \lambda_1$, the risk measure $\varrho(X, \mathbb{P})$ generated by $U$ is given by

$$\varrho(X, \mathbb{P}) = \sup_m \left\{ \mathbb{E} \left[ e^{\lambda_1 (X-m)} - e^{\lambda_2 (X-m)} \right] = 0 \right\} = \sup_m \left\{ \mathbb{E} \left[ e^{\lambda_1 X} \right] e^{-\lambda_1 m} - \mathbb{E} \left[ e^{\lambda_2 X} \right] e^{-\lambda_2 m} = 0 \right\}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \log \frac{\mathbb{E} \left[ e^{\lambda_1 X} \right]}{\mathbb{E} \left[ e^{\lambda_2 X} \right]}.$$  

Now, applying this to the RSPOMDP setting, and subsequently to the arising RSCOMDP, we get the following performance index

$$J_N(\theta, \pi) = \frac{1}{\lambda_1 - \lambda_2} \log \frac{\mathbb{E}_{\pi_0} \left[ e^{\lambda_1 \sum_{n=0}^{N-1} C(S_n, A_n)} \right]}{\mathbb{E}_{\pi_0} \left[ e^{\lambda_2 \sum_{n=0}^{N-1} C(S_n, A_n)} \right]} = \frac{1}{\lambda_1 - \lambda_2} \log \frac{\mathbb{E}_{\pi_0} \left[ e^{\lambda_1 \sum_{n=0}^{N-1} C_0(X_n, A_n, X_{n+1})} \right]}{\mathbb{E}_{\pi_0} \left[ e^{\lambda_2 \sum_{n=0}^{N-1} C_0(X_n, A_n, X_{n+1})} \right]}.$$

As we will see in Section 3, this new problem can be treated in the same way one solves Risk Sensitive Constrained Markov Decisions Processes (RSCMDPs).

The paper is split in two parts. In the first part, we treat RSCMDPs where the optimality criteria are sums of expected utilities, each applied to a different running cost. We then consider RSCMDPs where a sum of expected utilities is optimized while keeping a second sum of expectations below a specific threshold. In the second part, we treat two different types of RSPOMDPs: The first is the RSPOMDP related to a utility function which can be represented as a sum of exponential functions. The second is the RSPOMDP related to shortfall utilities, given by hyperbolic sine type of functions as in (1.1). For convenience, $(\mathcal{P})$ refers to the original RSPOMDP setting, $(\mathcal{P})$ to the resulting RSCMDPs, and $(\tilde{\mathcal{P}})$ to the augmented RSMGMDPs. The corresponding notations follow the same rules of applying overlines and tildes.

### 2 RSMGMDPs

The applications for multiple objective/goal decision making are numerous: In many real world problems, different stakeholders’ objectives must be weighted and balanced. For example, large scale water systems require control policies that optimize multiple uses of a water reservoir such as agriculture, hydroelectric production and flood prevention. Similar examples apply to forest management or automated stock broker agents. For a more detailed review of the applications, we refer to Section 7 in [36].

#### 2.1 Notation and Assumptions

Throughout this section, we assume that an $N$-step Markov Decision Process is given by a Borel state space $\mathcal{X}$, a Borel action space $\mathcal{A}$, the Borel set $\mathcal{D} \subseteq \mathcal{X} \times \mathcal{A}$, and a regular conditional distribution $P$ from $\mathcal{X} \times \mathcal{D}$ to $\mathcal{X}$. Given the current state $x \in \mathcal{X}$, we assume that the controller may choose an action $a$ from the set $D(x) := \{ a \in \mathcal{A} \mid (x, a) \in \mathcal{D} \}$ of feasible actions. The transition to the next state is then controlled by the distribution $P(\cdot \mid x; a)$, according to the chosen action. The set of histories from time $n_0$ up to time $n$ is defined by

$$\mathcal{H}^{n_0}_{n_0} := \mathcal{X}, \quad \mathcal{H}^{n_0}_n := \mathcal{H}^{n_0}_{n-1} \times \mathcal{D} \times \mathcal{X}, \quad n_0 = 0, ..., N, \quad n = n_0 + 1, \ldots, N$$

and $h^{n_0}_n = (x_{n_0}, a_{n_0}, \ldots, x_n) \in \mathcal{H}^{n_0}_n$ is a historical outcome from time $n_0$ up to time $n$. 
Definition 2.1 For $n_0 = 0, \ldots, N-1$, the set of $n_0$ steps shifted history-dependent policies is defined by

$$\Pi^{n_0}_H := \{ \pi = (f^0_n, \ldots, f^{n_0}_{N-1}) \mid f^0_n : \mathcal{H}^{n_0}_n \to \mathcal{D}, \forall h_n^{n_0} \in \mathcal{H}^{n_0}_n : f^0_n(h_n^{n_0}) \in D(x_n), n = n_0, \ldots, N-1 \}.$$ 

We use the notation $\Pi_H$ instead of $\Pi^{n_0}_H$, and we call the generated set, the set of history-dependent policies, whenever $n_0 = 0$. Similarly, the set of Markovian policies is defined by

$$\Pi^{n_0}_M := \{ \pi = (g_0, \ldots, g_{N-1}) \mid g_n : \mathcal{X} \to \mathcal{D}, \forall x \in \mathcal{X} : g_n(x) \in D(x), n = n_0, \ldots, N-1 \},$$

and $\Pi_M := \Pi^{n_0}_M$, for $n_0 = 0$. The maps $g_n$ are called Markovian decision rules. With a slight abuse of notation, we will consider elements of $\Pi_M$ to belong in $\Pi^{n_0}_M$, by cutting off the first $n_0$ terms. It obviously holds $\Pi^{n_0}_M \subseteq \Pi^{n_0}_H$.

Given an initial state $x \in \mathcal{X}$ and a history-dependent policy $\pi = (f^0_n, \ldots, f^{n_0}_{N-1}) \in \Pi^{n_0}_H$, let $\mathbb{P}^{\pi}_{n_0,x}$ denote a probability measure on $\mathcal{H}^{n_0}_N$, and $(X_n)_{n=n_0, \ldots, N}$, $(A_n)_{n=n_0, \ldots, N-1}$ two stochastic processes such that

$$\mathbb{P}^{\pi}_{n_0,x}(X_{n_0} \in B) = \delta_x(B), \quad \mathbb{P}^{\pi}_{n_0,x}(X_{n+1} \in B \mid H_n, A_n) = \mathbb{P}^{\pi}_{n_0,x}(X_{n+1} \in B \mid X_n, A_n) = \mathbb{P}(B \mid X_n, f^0_n(H_n))$$

for all Borel sets $B \subseteq \mathcal{X}$, and $H_n$, $X_n$, $A_n$ are the history, state and action at time $n$. Similarly, we define for Markovian policies $\pi = (g_0, \ldots, g_{N-1}) \in \Pi^{n_0}_M$. By $\mathbb{E}^{\pi}_{n_0}$ and $\mathbb{E}^{\pi}_{n_0,x}$, we denote the expectation operators corresponding to $\mathbb{P}^{\pi}_{n_0}$ and $\mathbb{P}^{\pi}_{n_0,x}$, respectively. For more details, we refer to [5, 25].

Throughout the whole section, we have the following standing assumptions:

Assumption 2.2 1. The utility functions $U^i, W^j : [0, \infty] \to \mathbb{R}$, $i = 1, \ldots, i_{\text{max}}$, $j = 1, \ldots, j_{\text{max}}$, are continuous and strictly increasing.

2. The sets $\mathcal{D}(x), x \in \mathcal{X}$ are compact.

3. The map $x \mapsto \mathcal{D}(x)$ is upper semi-continuous, i.e. if $x_n \to x \in \mathcal{X}$ and $a_n \in \mathcal{D}(x_n)$, then $(a_n)$ has an accumulation point in $\mathcal{D}(x)$.

4. The maps $(x, a, x') \mapsto C^i(x, a, x'), K^j(x, a, x')$, $i = 1, \ldots, i_{\text{max}}$, $j = 1, \ldots, j_{\text{max}}$ are lower semi-continuous, bounded from above and below by strictly positive constants.

5. $P$ is weakly continuous.

For notational convenience, we define $C, K : \mathcal{X} \times A \times \mathcal{X} \to \mathbb{R}_{\text{max}}$ by

$$C(x, a, x') := (C^1(x, a, x'), \ldots, C^{i_{\text{max}}}(x, a, x')), \quad K(x, a, x') := (K^1(x, a, x'), \ldots, K^{j_{\text{max}}}(x, a, x')),$$

where $U, W : \mathbb{R}_{\text{max}} \to \mathbb{R}_{\text{max}}$ is defined by

$$U(C^1, \ldots, C^{i_{\text{max}}}) := (U^1(C^1), \ldots, U^{i_{\text{max}}}(C^{i_{\text{max}}})),$$

$$W(K^1, \ldots, K^{j_{\text{max}}}) := (W^1(K^1), \ldots, W^{j_{\text{max}}}(K^{j_{\text{max}}})),$$

and $U, W : \mathbb{R}_{\text{max}} \to \mathbb{R}$ by

$$U(C^1, \ldots, C^{i_{\text{max}}}) := \sum_{i=1}^{i_{\text{max}}} U^i(C^i), \quad W(K^1, \ldots, K^{j_{\text{max}}}) := \sum_{j=1}^{j_{\text{max}}} W^j(K^j).$$

(2.1)
2.2 RSCOMDPs

In this section, we describe a model for risk sensitive multiple objective/goal sequential decision making [3, 25, 35] on a Borel state and action space with multiple cost and utility functions, i.e. the performance index is a sum of expected utilities, each of them applied to a different running cost. As a generalization to the classical MDP model, we allow for the cost to depend on the subsequent state in addition to the current state-action pair. We thereby follow [3] and [25].

After we have set the stage for Markov Decision Processes and their policies, we can now define performance indices that are sums of expected utilities, applied to a different running costs.

Definition 2.3 Denote by \( N \) the number of steps of the MDP. We define the total (or terminal) cost \( I_N(x, \pi) \) given an initial state \( x \in \mathcal{X} \), and a history dependent policy \( \pi \in \Pi_H \) by

\[
I_N(x, \pi) := \sum_{i=1}^{i_{\text{max}}} \mathbb{E}_x^i \left[ U^i \left( \sum_{n=0}^{N-1} C^i(X_n, A_n, X_{n+1}) \right) \right],
\]

and the corresponding value function by

\[
V_N(x) := \inf_{\pi \in \Pi_H} I_N(x, \pi).
\]

We now augment the state space of the MDP to \( X \times \mathbb{R}^{i_{\text{max}}} \), where the second component will later model the accumulated cost. In particular, \( \tilde{X}_n \) being \( (x, d) = (x, d^1, ..., d^{i_{\text{max}}}) \) means that the MDP has advanced to state \( x \) and accumulated a cost of \( d^i \) in the \( i \)-th objective after the first \( n \) steps. On the new space we define the following transition probabilities

\[
\tilde{P}(\tilde{x}'|\tilde{x};a) = \begin{cases} P(x'|x;a), & \tilde{x} = (x, d), \tilde{x}' = (x', d + C(x, a, x')) \\ 0 & \text{otherwise} \end{cases}
\]

We include the accumulated cost into the previous definition of the MDP history:

\[
\tilde{H}^{n_0}_n := \mathcal{X} \times \mathbb{R}^{i_{\text{max}}}, \quad \tilde{H}^{n_0} := \tilde{H}^{n_0}_0 \times \mathcal{A} \times (\mathcal{X} \times \mathbb{R}^{i_{\text{max}}}) \quad n_0 = 0, ..., N, \quad n = n_0 + 1, ..., N.
\]

The definition of history-dependent policies \( \tilde{\pi} \in \tilde{\Pi}_H, \tilde{\Pi}_{H^{n_0}} \), Markovian policies \( \tilde{\pi} \in \tilde{\Pi}_M, \tilde{\Pi}_{M^{n_0}} \), and the corresponding decision rules are changed accordingly.

Definition 2.4 Denote by \( N \) the number of steps of the MDP. For \( n_0 = 0, ..., N - 1 \), we define the cost-to-go function \( \tilde{I}_{n_0,N}(\tilde{x}, \tilde{\pi}) = \tilde{I}_{n_0,N}(x, d, \tilde{\pi}) \) given the join state \( \tilde{x} = (x, d) \), with \( x \in \mathcal{X} \) and initially accumulated cost \( d \in \mathbb{R}^{i_{\text{max}}} \) at time \( n_0 \), and a policy \( \tilde{\pi} \in \tilde{\Pi}_{H^{n_0}} \) by

\[
\tilde{I}_{n_0,N}(\tilde{x}, \tilde{\pi}) := \sum_{i=1}^{i_{\text{max}}} \mathbb{E}_{\tilde{x}_0}^{n_0,i} \left[ U^i \left( \sum_{n=n_0}^{N-1} C^i(X_n, A_n, X_{n+1}) + d^i \right) \right] = \sum_{i=1}^{i_{\text{max}}} \mathbb{E}_{\tilde{x}_0}^{n_0,i} \left[ U^i(D^i_N) \right],
\]

and the corresponding value function by

\[
\tilde{V}_{n_0,N}(\tilde{x}) = \tilde{V}_{n_0,N}(x, d) := \inf_{\tilde{\pi} \in \tilde{\Pi}_{H^{n_0}}} \tilde{I}_{n_0,N}(x, d, \tilde{\pi}) = \inf_{\tilde{\pi} \in \tilde{\Pi}_{H^{n_0}}} \tilde{I}_{n_0,N}(\tilde{x}, \tilde{\pi}).
\]

For a history dependent policy \( \tilde{\pi} \in \tilde{\Pi}_H \), we similarly set

\[
\tilde{I}_{0,N}(\tilde{x}, \tilde{\pi}) = \tilde{I}_{0,N}(x, d, \tilde{\pi}) := \sum_{i=1}^{i_{\text{max}}} \mathbb{E}_x^i \left[ U^i \left( \sum_{n=0}^{N-1} C^i(X_n, A_n, X_{n+1}) + d^i \right) \right] = \sum_{i=1}^{i_{\text{max}}} \mathbb{E}_x^i \left[ U^i(D^i_N) \right].
\]
Clearly, the value function of the original problem coincides with the value function of the augmented problem with $d = 0$, i.e.

$$V_N(\cdot) = \tilde{V}_{0,N}(\cdot,0).$$

As next step, we try to find a Bellman-style equation for the augmented problem $\tilde{P}$. It can be shown, that the minimizer of $\tilde{P}$ is a Markov policy. We will need the following operator:

**Definition 2.5** Similar to $\tilde{G}$, we define the set

$$\Delta := \{v : \mathcal{X} \times (\mathbb{R}^+)^{i_{\max}} \to \mathbb{R} \mid v \text{ is lower semi-continuous,}$$

$$v(x,\cdot) \text{ is continuous, and componentwise increasing for all } x \in \mathcal{X}$$

$$v(x,d) \geq U(d) \text{ for all } (x,d) \in \mathcal{X} \times (\mathbb{R}^+)^{i_{\max}}\}.$$  

For $v \in \Delta$ and a Markovian decision rule $\tilde{g}$, we define the operators

$$T_{\tilde{g}}[v](x,d) = \int v(x',C(x,\tilde{g}(x,d),x') + d)P(dx'|x,\tilde{g}(x,d)),$$

and

$$T[v](x,d) = \inf_{a \in D(x)} \int v(x',C(x,a,x') + d)P(dx'|x,a).$$

$T$ is called the minimal cost operator.

We say that a Markovian decision rule $\tilde{g}$ is a minimizer of $Tv$ if $T_{\tilde{g}}[v] = T[v]$. In this situation, for every $(x,d) \in \mathcal{X} \times \mathbb{R}^{i_{\max}}$, $\tilde{g}(x,d)$ is a minimizer of

$$D(x) \ni a \mapsto \int v(x',C(x,a,x') + d)P(dx'|x,a).$$

We may now state the main theorem of this section:

**Theorem 2.6** Define $\tilde{V}_{N,N}(x,d) := U(d)$, see (2.1). The following holds:

a) For any Markovian policy $\tilde{\pi} = (\tilde{g}_0,\tilde{g}_1,\ldots,\tilde{g}_{N-1}) \in \tilde{\Pi}_M$ We have the cost iteration

$$\tilde{I}_{n_0,N}(x,d,\tilde{\pi}) = T_{\tilde{g}_{n_0}}[\ldots[T_{\tilde{g}_{N-1}}[\tilde{V}_{N,N}]]](x,d)$$

for all $n_0 = 0,\ldots,N-1$.

b) The optimal policy is Markovian, i.e.

$$\inf_{\tilde{\pi} \in \tilde{\Pi}_M} \tilde{I}_{0,N}(x,d,\tilde{\pi}) = \inf_{\tilde{\pi} \in \tilde{\Pi}_M} \tilde{I}_{0,N}(x,d,\tilde{\pi}).$$

c) The operator $T : \Delta \to \Delta$ is well-defined, and for every $v \in \Delta$, there exists a minimizer of $Tv$.

d) We get the Bellman-style equation

$$\tilde{V}_{n_0,N}(x,d) = T[\tilde{V}_{n_0+1,N}](x,d) = \inf_{a \in D(x)} \int \tilde{V}_{n_0+1,N}(x',C(x,a,x') + d)P(dx'|x,a)$$

for all $n_0 = 0,\ldots,N-1$.  

e) If \( \tilde{g}_{n_0} \) is a minimizer of \( T[\tilde{V}_{n_0+1,N}] \) for \( n_0 = 0, ..., N-1 \), then \( \pi^* = (\tilde{g}_0, ..., \tilde{g}_{N-1}) \) is an optimal policy for problem \( [P] \). In this situation, the history-dependent policy \( \pi^* = (f_0^*, ..., f_{N-1}^*) \), defined by

\[
f^*_n(h_n) := \begin{cases} 
\tilde{g}_0(x_0, 0), & \text{if } n = 0, \\
\tilde{g}_n^* \left( x_n, \sum_{k=0}^{n-1} C(x_k, a_k, x_{k+1}) \right), & \text{otherwise,}
\end{cases}
\]

is an optimal policy for problem \( [P] \).

The proof can be found in Appendix A. We will apply this result in (3.4).

2.3 RCMDPs

In this subsection we apply ideas of [10, 11] and from the more recent [12, 45] to deal with risk constrained MDPs. More specifically we would like to minimize a sum of expected utilities while another one is kept under a specific threshold. For this section we are going to make an extra assumption

**Assumption 2.7**

1. The set \( \mathcal{A} \) is finite.

2. \( P(\cdot|x; a) \) has a finite support for every \( x \in \mathcal{X}, a \in \mathcal{A} \).

The assumptions about \( \mathcal{A} \) and \( P \) are strong, but an attempt to weaken them will go beyond the scope of this article. We however hypothesize that point (2) can totally be removed and (3) can be substituted with \( P(\cdot|x; u) \), is a Lipschitz function on the space of Probability measures endowed with the Wasserstein-\( \infty \) metric, and we plan to investigate that in a future publication.

Similar to the previous subsection, we define the following performance index

\[
I^1_N(x, \pi) := \max_{i=1}^{i_{\text{max}}} \sum_{j=1}^j \mathbb{E}_x^\pi \left[ U_i \left( \sum_{n=0}^{N-1} C^i(X_n, A_n, X_{n+1}) \right) \right],
\]

the constraint index

\[
I^2_N(x, \pi) := \max_{j=1}^{j_{\text{max}}} \sum_{i=1}^i \mathbb{E}_x^\pi \left[ W_j \left( \sum_{n=0}^{N-1} K^j(X_n, A_n, X_{n+1}) \right) \right].
\]

and a constrained value function

\[
V^\rho_N(x) := \inf \{ I^1_N(x, \pi) \mid \pi \in \Pi_H \text{ and } I^2_N(x, \pi) < \rho \}. \quad (P_{\rho})
\]

We now augment the original state space to \( \tilde{X} = X \times \mathbb{R}^{i_{\text{max}}} \times \mathbb{R}^{j_{\text{max}}} \). On the new space we define the following transition probabilities

\[
\tilde{P}(\tilde{x}'|\tilde{x}; a) = \begin{cases} 
P(x'|x; a), & \tilde{x} = (x, d, e), \tilde{x}' = (x', d + C(x, a, x'), e + K(x, a, x')) \\
0 & \text{otherwise}
\end{cases} \quad (2.3)
\]

The augmentation of the MDP history is given by

\[
\tilde{H}_{n_0} := \tilde{X}, \quad \tilde{H}_n := \tilde{H}_{n-1} \times \mathcal{D} \times \tilde{X}, \quad n_0 = 0, ..., N, \quad n = n_0 + 1, ..., N,
\]
Theorem 2.8

Now we have the following theorem from [10, 11, 13].

We define the set of control-threshold pairs as

\[ \vec{T}_{n_0, N}(x, d, e) = \sum_{i=1}^{i_{\text{max}}} E_{n_0, x}^{\pi} \left[ U^i \left( \sum_{n=n_0}^{N-1} C^i(X_n, A_n, X_{n+1}) + d^i \right) \right] = \sum_{i=1}^{i_{\text{max}}} E_{n_0, x}^{\pi} \left[ U^i \left( D^i_N \right) \right], \]

with the constraint-to-go

\[ \vec{T}_{n_0, N}(x, d, e) = \sum_{j=1}^{j_{\text{max}}} E_{n_0, x}^{\pi} \left[ W^j \left( \sum_{n=n_0}^{N-1} K^j(X_n, A_n, X_{n+1}) + e^j \right) \right] = \sum_{j=1}^{j_{\text{max}}} E_{n_0, x}^{\pi} \left[ W^j \left( E^j_N \right) \right]. \]

We define the constrained policies

\[ \vec{\Pi}_{H^\rho}^{\pi} = \{ \vec{\pi} \in \vec{\Pi}_{H^\rho} | \vec{T}_{n_0, N}(x, \vec{\pi}) < \rho \} \]

and the augmented constrained value function

\[ \vec{V}_{n_0, N}(x, \rho) = \inf_{\vec{\pi} \in \vec{\Pi}_{H^\rho}^{\pi}} \int_{C} \vec{T}_{n_0, N}(x, \vec{\pi}) \vec{\Pi}_{H^\rho}^{\pi} \neq \emptyset \]

otherwise,

\[ \overline{C} = \sup_{x, \vec{\pi}} \vec{T}_{n_0, N}(x, \vec{\pi}) + 1. \]

Similarly to the previous section, we have

\[ V^\rho_N(x) = \vec{V}_{0, N}(x, 0, 0, \rho). \]

We define the set of control-threshold pairs as

\[ F_{n_0, N}(x, \rho) := \{ (a, r) \in \mathcal{D}(x) \times \mathcal{B}(x, \mathbb{R}) | \inf_{\vec{\pi}} T^2_{n_0+1, N}(x', \vec{\pi}) \leq r(x') \leq \sup_{\vec{\pi}} T^2_{n_0+1, N}(x', \vec{\pi}), \]

\[ \int r(x') P(d\vec{x}'|\vec{x}; a) \leq \rho \}. \]

We also define the following operators

\[ T_g, R[v](\vec{x}, \rho) = \int v(\vec{x}', R[\vec{x}, \rho](\vec{x}')) P(d\vec{x}'|\vec{x}; g(\vec{x}, \rho)), \]

and

\[ T_{n_0, N}[v](\vec{x}, \rho) = \inf_{(a, r) \in F_{n_0, N}(\vec{x}, \rho)} \left\{ \int v(\vec{x}', r(\vec{x}')) P(d\vec{x}'|\vec{x}; a) \right\}. \]

Now we have the following theorem from [11, 11, 12].

Theorem 2.8 Define \( \vec{V}_{n_0, N}(x, \rho) := \mathcal{U}(d), \) if \( n_0 \leq \rho, \) and \( \overline{C} \) otherwise. The following holds for all \( n_0 = 0, \ldots, N - 1: \)

\[ \vec{V}_{n_0, N}(x, \rho) = T_{n_0, N} \left[ \vec{V}_{n_0+1, N} \right](x, \rho) \]

and the minimizers \( (g^*_n, R^*_n) \) of \( T_{n_0, N}[\vec{V}_{n_0+1, N}] \) exist. Let now \( (g^*_n, R^*_n) \) be a minimizer for \( T_{n_0, N}[\vec{V}_{n_0, N}] \) for \( n_0 = 0, \ldots, N - 1. \)

Let also \( \rho_{n_0+1} \) be defined recursively by

\[ \rho_{n_0+1} = R^*_n(\vec{x}_{n_0}, \rho_{n_0})(\vec{x}_{n_0+1}). \]
If \((g_{n,0}, R_{n,0})\) is a minimizer of \(T_{n_0,N}\), then the history-dependent policy \(\pi^*=(f_0^*,...,f_{N-1}^*)\), defined by

\[
f_n^*(h_n) := \begin{cases} 
g_0^*(x_0, 0, 0, \rho), & \text{if } n = 0, \\
g_n(x_n, \sum_{k=0}^{n-1} C(x_k, a_k, x_{k+1}), \sum_{k=0}^{n-1} K(x_k, a_k, x_{k+1}), \rho_n), & \text{otherwise,}
\end{cases}
\]

is an optimal policy for problem \((P_\rho)\).

The proof is identical to the proofs given in [10, 11, 13], and it will be omitted.

## 3 RSPOMDPs

In this section we study two types of RSPOMDPs: For the first type, the performance index is the expected value of a utility function that can be written as a weighted sum of exponentials. For the second type, we pick the short-fall utility that is generated by the hyperbolic sine, as performance index. We will show that in both cases, it is possible to reformulate the problem as a RSCOMDP with new performance indices that in turn can be treated with tools described in the previous section.

### 3.1 From the partial observable to the completely observable setting

#### 3.1.1 The original setting

We start by describing the initial setting. Let \(S, Y, A\) be three finite sets equipped with the discrete topology. In the sequel, \(S\) is called the ‘hidden’ state space, \(Y\) the set of observations and \(A\) the set of controls. For every \(a \in A\), we define a transition probability matrix \(\mathcal{P}(a) = [\mathcal{P}(s'|s,a)]_{s,s' \in S}\).

Finally, we denote by \(Q = [Q(y|s)]_{y \in Y, s \in S}\) the signal matrix and by \(\mathcal{C} : S \times A \rightarrow \mathbb{R}\) the cost function.

Now, for each \(n \in \mathbb{N}\), let \(\mathcal{H}_n\) be the set of histories up to time \(n\), where \(\mathcal{H}_0 = \mathcal{P}(S)\), and \(\mathcal{H}_n = \mathcal{H}_{n-1} \times A \times Y\). We denote by \(\Pi_H := \{\pi = (J_0,...,J_{N-1}) \mid J_n : \mathcal{H}_n \rightarrow A, n = 0, ..., N - 1\}\), the set of all deterministic policies that are functions of the history \(\mathcal{h}_n = (\theta, u_0, y_1, ..., u_{n-1}, y_n)\) up to time \(n\). Given \(\theta \in \mathcal{P}(S)\), and \(\pi \in \Pi_H\), there exists a unique measure \(\mathcal{E}_{\theta}\) on the Borel sets of \(\Omega := S \times (A \times S \times Y)^{N-1}\), with

\[
\mathcal{E}_{\theta}(s_0, a_0, s_1, y_1, a_1, ..., a_{n-1}, s_n, y_n) := \theta(s_0)\Pi_{k=0}^{n-1}\mathcal{C}(s_{k+1}|s_k, J_n(s_k))Q(s_{k+1}|y_{k+1}),
\]

see e.g. [4]. The corresponding expectation operator is denoted by \(\mathbb{E}_{\theta}\). Finally, for each \(n \in \mathbb{N}\), we define the \(\sigma\)-fields \(\mathcal{F}_n, \mathcal{G}_n\), by

\[
\mathcal{F}_n := \sigma((A_k, Y_{k+1}), \ k = 0, 1, ..., n - 1), \quad \mathcal{G}_n := \sigma(S_0, (A_k, Y_{k+1}), \ k = 0, 1, ..., n - 1).
\]

It is straightforward to see that the set of policies \(\Pi_H\), contains exactly the elements \((J_n)_{n \in \{0,...,N-1\}}\), where \(J_n\) are \(\mathcal{F}_n\)-measurable functions from \(\mathcal{H}_n\) to \(A\).
3.1.2 Completely observable

In this subsection, we show that given the vector-valued function \( v : \mathcal{P}(S) \times \Pi_H \to \mathbb{R}^{i_{\text{max}}} \) that maps a history dependent policy \( \pi \in \Pi_H \) and an initial configuration \( \theta_0 \in \mathcal{P}(S) \) to a vector of expected exponential utilities, each applied to the running cost \( C(s, a) \), of the form

\[
\overline{v}(\theta_0, \pi) = \left( \mathbb{E}_{{\theta_0}}^{\pi} \left[ e^{\lambda \sum_{n=0}^{N-1} C(S_n, A_n)} \right], \ldots, \mathbb{E}_{{\theta_0}}^{\pi} \left[ e^{\lambda \sum_{n=0}^{N-1} C(S_n, A_n)} \right] \right),
\]

one can introduce a new state space \( \mathcal{X} \), a controlled transition matrix \( P(x'; x; a) \), and a collection of running costs \( C^i : \mathcal{X} \times \mathcal{A} \times \mathcal{X} \to \mathbb{R} \), that depend on the next stage as well, such that for the resulting process \( (X_n) \), we have \( \mathbb{E}(\theta_0, \pi) = v(x_0, \eta(\pi)) \), and

\[
v(x_0, \pi) = \left( \mathbb{E}_x^{\pi} \left[ e^{\lambda \sum_{n=0}^{N-1} C^i(X_n, A_n, X_{n+1})} \right], \ldots, \mathbb{E}_x^{\pi} \left[ e^{\lambda \sum_{n=0}^{N-1} C^i(X_n, A_n, X_{n+1})} \right] \right),
\]

where \( x_0 = (\theta_0, \ldots, \theta_0, y_0) \) for some arbitrary but fixed \( y_0 \in \mathcal{Y} \), and \( \eta : \Pi_H \to \Pi_H \) is a function that will be defined later, see [3.2]. Following [9], we note that \( \mathbb{P}_\theta^\pi \), defined by

\[
\mathbb{P}_\theta^\pi(s_0, a_0, s_1, y_1, a_1, \ldots, a_{n-1}, s_n, y_n) := \theta(s_0) \prod_{k=0}^{n-1} \mathbb{P}_\theta(s_{k+1} | s_k, \psi_k(h_k)) \frac{1}{|\mathcal{Y}|},
\]

is a probability measure, where \( |\mathcal{Y}| \) is the number of elements in \( \mathcal{Y} \). Denote by \( \mathbb{E}_n^\pi \) the corresponding expectation operator. On the \( \sigma \)-field \( \mathcal{G}_n \), the Radon-Nikodyn derivative of \( \mathbb{P}_\theta^\pi \) with respect to \( P_\theta \) is given by

\[
\frac{\partial \mathbb{P}_\theta^\pi}{\partial P_\theta} \bigg|_{\mathcal{G}_n} = \prod_{k=0}^{n-1} |\mathcal{Y}| \mathcal{Q}(Y_{k+1} | S_{k+1}) =: R_n,
\]

and therefore

\[
\mathbb{E}_{{\theta_0}}^{\pi} \left[ e^{\lambda \sum_{k=0}^{n} \overline{C}(S_k, A_k)} \right] = \mathbb{P}_{{\theta_0}}^{\pi} \left[ e^{\lambda \sum_{k=0}^{n} \overline{C}(S_k, A_k)} R_n \right].
\]

Now, for each \( s \in S \), \( i = 1, \ldots, i_{\text{max}}, n = 0, \ldots, N-1 \), we define the positive and \( \mathcal{F}_n \)-measurable random variables \( \psi^i_n, \theta^i_n \) by

\[
\psi^i_n(s) := \mathbb{E}_{{\theta_0}}^{\pi} \left[ \mathbbm{1}_{(S_n=s)} e^{\lambda \sum_{k=0}^{n} \overline{C}(S_k, A_k)} R_n \mathcal{F}_n \right], \quad \theta^i_n := \frac{\psi^i_n}{1 \cdot \psi^i_n},
\]

where \( \mathbbm{1} = (1, \ldots, 1)' \in \mathbb{R}^{[S]} \). Furthermore, for each \( (a, y) \in \mathcal{A} \times \mathcal{Y} \), we define the matrix \( M(a, y) \) given by

\[
M^i(a, y)[s, s'] := \left( e^{\lambda c(s,a) \mathcal{F}^i(s'| s; a) Q(y|s')} \right)^{\dagger}
\]

Note that the recursion formula

\[
\psi^i_n = |\mathcal{Y}| M^i(A_{n-1}, Y_n) \psi^i_{n-1}
\]

holds. Now, we define the operators \( F^i \) and cost functions \( G^i \) by

\[
F^i(\theta^i, a, y) := \frac{M^i(a, y) \theta^i}{1 \cdot M^i(a, y) \theta^i}, \quad G^i(\theta^i, u, y) := \frac{1}{\lambda^i} \log(1 \cdot M^i(u, y) \theta^i).
\]

With these definitions, we have

\[
\mathbb{E}_{{\theta_0}}^{\pi} \left[ e^{\lambda \sum_{n=0}^{N-1} \overline{C}(S_n, A_n)} \right] = \mathbb{P}_{{\theta_0}}^{\pi} \left[ e^{\lambda \sum_{n=0}^{N-1} \left[ G^i(\theta^i_n, A_n, y_{n+1}) + \log(|\mathcal{Y}|) \right]} \right]. \tag{3.1}
\]
We set $\mathcal{X} = \mathcal{P}(\mathcal{S})^{i_{\max}} \times \mathcal{Y}$, and for $x = (\theta^1, \ldots, \theta^{i_{\max}}, y) \in \mathcal{X}, a \in \mathcal{A}$, we define a controlled transition matrix $P(x'|x; a)$ by

$$P(x'|x; a) := \begin{cases} \mathcal{Y} & \text{if } x' = (F^1(\theta^1, a, y'), \ldots, F^{i_{\max}}(\theta^{i_{\max}}, a, y'), y') \\ 0 & \text{otherwise.} \end{cases} $$

We also define the following cost functions on $\mathcal{X} \times \mathcal{A} \times \mathcal{X}$: For $x = (\theta^1, \ldots, \theta^{i_{\max}}, y)$, we set $C^i(x, u, x') = G^i(\theta^i, u, y) + \log(|\mathcal{Y}|)$. Then, (3.1) it takes the form

$$\mathbb{E}_{\theta_0}^\pi \left[e^{\lambda^i \left[\sum_{n=0}^{N-1} C_n(S_n, A_n)\right]} \right] = \mathbb{E}_{\theta_0}^{\pi} \left[e^{\lambda^i \left[\sum_{n=0}^{N-1} C^i(S_n, A_n)\right]} \right].$$

At this point, we assume that $\pi$ is an element of $\Pi_H$. We define a different set of histories by $\mathcal{H}_0 = \mathcal{X}$, and $\mathcal{H}_n = \mathcal{H}_{n-1} \times \mathcal{A} \times \mathcal{X}$. An element $h_n \in \mathcal{H}_n$ takes the form $h_n = (x_0, u_0, x_1, u_1, \ldots, x_n)$, and a policy $\pi \in \Pi_H$ takes the form $\pi = (f_0, \ldots, f_{N-1})$, where $f_n : \mathcal{H}_n \to \mathcal{A}$.

We define $\eta : \Pi_H \to \Pi_H$ such that $(f_n)_{n \in \{0, \ldots, N-1\}} = \eta((f_n)_{n \in \{0, \ldots, N-1\}})$ satisfies

$$f_n(x_0, a_0, \ldots, x_{n-1}, a_{n-1}, x_n) = f_n(\theta_0, a_0, \ldots, y_{n-1}, a_{n-1}, y_n),$$

for every $\theta_0 \in \mathcal{P}(\mathcal{S})$, $x_0 = (\theta_0, \ldots, \theta_0, y_0)$, and for an arbitrary, but fixed $y_0 \in \mathcal{Y}$. Note that for histories $h_n$ that are not generated by a $\eta$, the $f_n$'s in $(f_n)_{n \in \{0, \ldots, N-1\}} = \eta((f_n)_{n \in \{0, \ldots, N-1\}})$ can be defined in an arbitrary fashion, since they cannot be realized anyway.

Now it is easy to see that any performance index of the form

$$\mathcal{J}_N(\theta_0, \pi) = \ell \left( \mathbb{E}_{\theta_0}^\pi \left[e^{\lambda^i \left[\sum_{n=0}^{N-1} C_n(S_n, A_n)\right]} \right], \ldots, \mathbb{E}_{\theta_0}^\pi \left[e^{\lambda^{i_{\max}} \left[\sum_{n=0}^{N-1} C_n(S_n, A_n)\right]} \right]\right),$$

can be written as

$$\mathcal{J}_N(\theta_0, \pi) = \mathcal{J}_N(x_0, \eta(\pi)),$$

where

$$\mathcal{J}_N(x_0, \pi) = \ell \left( \mathbb{E}_{x_0}^\pi \left[e^{\lambda^i \left[\sum_{n=0}^{N-1} C^i(S_n, A_n)\right]} \right], \ldots, \mathbb{E}_{x_0}^\pi \left[e^{\lambda^{i_{\max}} \left[\sum_{n=0}^{N-1} C^{i_{\max}}(S_n, A_n)\right]} \right]\right).$$

### 3.2 Utility functions that are sums of exponentials

Let $\{\lambda^i, i = 1, \ldots, i_{\max}\} \subseteq \mathbb{R} \setminus \{0\}$ be a finite collection of risk parameters, and $\{w^i, i = 1, \ldots, i_{\max}\} \subseteq \mathbb{R}^+$ be a collection of weights. We define the utility function $\hat{U} : \mathbb{R} \to \mathbb{R}$ by

$$\hat{U}(t) := \sum_{i=1}^{i_{\max}} w^i \text{sign}(\lambda^i) e^{\lambda^i t},$$

and introduce the performance index

$$\mathcal{J}_N(\theta_0, \pi) = \hat{U}^{-1} \left( \mathbb{E}_{\theta_0}^{\pi} \left[\hat{U} \left(\sum_{n=0}^{N-1} C(S_n, A_n)\right)\right]\right) = \hat{U}^{-1} \left( \sum_{i=1}^{i_{\max}} w^i \text{sign}(\lambda^i) \mathbb{E}_{\theta_0}^{\pi} \left[e^{\lambda^i \left[\sum_{n=0}^{N-1} C(S_n, A_n)\right]} \right]\right).$$

The goal is to minimize $\mathcal{J}_N(\theta_0, \pi)$ over all policies $\pi \in \Pi_H$. Now, using the previous subsection, we can work on the RSCOMDP on the space $\mathcal{X}$ with performance index

$$\mathcal{J}_N(x_0, \pi) = \hat{U}^{-1} \left( \sum_{i=1}^{i_{\max}} w^i \text{sign}(\lambda^i) \mathbb{E}_{x_0}^{\pi} \left[e^{\lambda^i \left[\sum_{n=0}^{N-1} C^i(S_n, A_n)\right]} \right]\right).$$
Since $\hat{U}$ is increasing, one can instead optimize

$$I_N(x_0, \pi) = \sum_{i=1}^{i_{\text{max}}} \mathbb{E}_{x_0}^{\pi} \left[ w^i \cdot \text{sign}(\lambda^i) e^{\lambda^i \left[ \sum_{n=0}^{N-1} C^i(X_n, A_n, X_{n+1}) \right]} \right]$$

(3.4)

Note that, if $m_i = \inf_{X \times A \times X} C^i(\cdot, \cdot, \cdot) \leq 0$, and Assumption 2.2 is not satisfied, we can use the properties of the exponential functions and rewrite (3.3) as

$$I_N(x_0, \pi) = \sum_{i=1}^{i_{\text{max}}} \mathbb{E}_{x_0}^{\pi} \left[ \left( w^i e^{-2N\lambda|m_i|} \right) \cdot \text{sign}(\lambda^i) e^{\lambda^i \left[ \sum_{n=0}^{N-1} C^i(X_n, A_n, X_{n+1}) + 2|m_i| \right]} \right]$$

Now this problem falls in the framework of Section 2.1 that provides the means to calculate the optimal value and optimal policies.

3.2.1 The class of utility functions that can be covered with this approach.

For two utility functions that are $\varepsilon$-close on the interval $[N \min_s \overline{C}(s, u), N \max_s \overline{C}(s, u)]$, it is straightforward to prove that an $\varepsilon$-optimal policy for one utility function is a $2\varepsilon$-optimal policy for the other. Therefore, one can apply the method to solve RSPOMDPs with utility functions that can be approximated by functions of the form (3.3). One can easily show that this contains all the real functions that are bilateral Laplace transformations of $\text{sign}(s) \mu(ds)$ for finite positive measures $\mu$ with compact support, i.e. functions of the form

$$U(t) = \int_{-\infty}^{\infty} \text{sign}(s) e^{st} \mu(ds).$$

3.3 Shortfall Utility generated by functions of the Hyperbolic Sine type

As it was shown in the introduction, for a utility given by a hyperbolic sine type of functions, i.e. $U(t) = e^{\lambda_1 t} - e^{\lambda_2 t}$, with $\lambda_2 < 0 < \lambda_1$, we recover the shortfall utility $g(X, \mathbb{P})$ given in (1.1), and therefore the performance index

$$J_N(\theta_0, \pi) = \frac{1}{\lambda_1 - \lambda_2} \log \frac{\mathbb{E}_{\theta_0}^{\pi} \left[ e^{\lambda_1 \sum_{n=0}^{N-1} \overline{C}(S_n, A_n)} \right]}{\mathbb{E}_{\theta_0}^{\pi} \left[ e^{\lambda_2 \sum_{n=0}^{N-1} \overline{C}(S_n, A_n)} \right]}.$$
Let now $\pi^* \in \Pi_H$ be a minimizing policy of (3.5). Define $\rho^*$ by

$$\rho^*(x_0) = \mathbb{E}^{\pi^*}_{x_0} \left[ e^{\lambda \left[ \sum_{n=0}^{N-1} C^i(X_n, A_n, X_{n+1}) \right]} \right].$$

It is straightforward to see that

$$\inf_{\pi \in \Pi_H} I_N(x_0, \pi) = \inf_{\pi \in \Pi_H} \frac{\mathbb{E}^{\pi}_{x_0} \left[ e^{\lambda \left[ \sum_{n=0}^{N-1} C^i(X_n, A_n, X_{n+1}) \right]} \right]}{\rho^*(x_0)} = \frac{V_N(x_0, \rho^*(x_0))}{\rho^*(x_0)}.$$

One can first calculate $V_N^\rho(x_0)$ for every $\rho$. Then, we define $\rho^*(x_0) = \arg \min \tilde{V}_N(x_0, \rho)$, and retrieve the policy for $V_N^{\rho^*(x_0)}(x_0)$, which can applied for the original problem.

### A Proof of the Bellman equation

**Theorem A.1** Let $v$ be bounded and lower semi-continuous. Suppose

1. $D(x)$ is compact,

2. $x \mapsto D(x)$ is upper semi-continuous,

3. $(x, d, \bar{g}, x') \mapsto v(x', C(x, \bar{g}(x, d), x') + d)$ is lower semi-continuous.

Then, $Tv$ is is lower semi-continuous and there exists a minimizer $\bar{g}^*$ such that $T\bar{g}^*v = Tv$.

**Proof.** By Lemma 17.11 in [26], $(x, d, \bar{g}) \mapsto T\bar{g}v(x, d)$ is lower semi-continuous. The claim then follows from a similar argument to Proposition 2.4.3 in [4] with the upper bounding function $b \equiv 1$ as defined in Definition 2.4.1 in [4].

### A.1 Proof of Theorem 2.6

**Proof.** The proof is similar to Theorem 2.3.4 and Theorem 2.3.8 in [4], but with a different state space.

ad a) Consider

$$\tilde{I}_{n-1,N}(x, d, \bar{g}) = \sum_{i=1}^{i_{\text{max}}} \mathbb{E}^\pi_{N-1,x} \left[ U^i \left( C^i(X_{N-1}, A_{N-1}, X_N) + d^i \right) \right] = \int \sum_{i=1}^{i_{\text{max}}} U^i \left( C^i(x, \bar{g}_{N-1}(x, d), x') + d^i \right) P(dx' | x, \bar{g}_{N-1}(x, d)) = T_{\bar{g}_{N-1}} \tilde{V}_{N,N}(x, d)$$
and for \( n_0 = 0, ..., N - 2 \)

\[
\bar{I}_{n_0,N}(x, d, \bar{\pi}) = \sum_{i=1}^{i_{\text{max}}} \mathbb{E}_{n_0,x}^{\pi} \left[ U^i \left( \sum_{n=n_0}^{N-1} C^i(X_n, A_n, X_{n+1}) + d^i \right) \right]
\]

\[
= \sum_{i=1}^{i_{\text{max}}} \int_{n_0+1}^{i_{\text{max}}} \mathbb{E}_{n_0,x}^{\pi} \left[ U^i \left( \sum_{n=n_0}^{N-1} C^i(X_n, A_n, X_{n+1}) + d^i \right) \right] P(dx'|x, \bar{g}_{n_0}(x, d))
\]

\[
= \int_{n_0+1}^{i_{\text{max}}} \mathbb{E}_{n_0,x}^{\pi} \left[ U^i \left( \sum_{n=n_0+1}^{N-1} C^i(X_n, A_n, X_{n+1}) + C^i(x, \bar{g}_{n_0}(x, d), x') + d^i \right) \right] P(dx'|x, \bar{g}_{n_0}(x, d))
\]

\[
= T_{\bar{g}_{n_0}} \bar{I}_{n_0+1,N}(x, d, \bar{\pi}).
\]

The claim follows then by induction.

ad b) This follows from Theorem 2.2.3 in [I].

ad c) Note that every \( v \in \Delta \) is bounded from below by \( \mathcal{U}(0) \). By our assumptions, \( (x, d, \bar{g}, x') \mapsto v(x', C(x, \bar{g}(x, d), x') + d) \) is lower semi-continuous, bounded from below, and we are in the setting of Theorem A.10. Thus, \( T[v] \) is lower semi-continuous and there exists a minimizer \( \bar{g}^* \) such that \( T[\bar{g}^*](x, d) = T[v](x, d) \).

For fixed \( x \in X \), the map \( d \mapsto \int v(x', C(x, a, x') + d)P(dx'|x, a) \) is increasing and continuous for every \( a \in \Delta(x) \). Therefore, the infimum of these maps over all \( a \in \Delta(x) \) is increasing and upper semi-continuous in \( d \). With this, we have shown that \( T[v](x, \cdot) \) is upper and lower semi-continuous, and therefore continuous, and increasing for all \( x \in X \). Because \( v(x, d) \geq \mathcal{U}(d) \), we have \( T[v](x, d) \geq \mathcal{U}(d) \). We have shown that \( T : \Delta \to \Delta \) is well-defined.

ad d) Let \( \bar{g}^*_n(x, d) \) be a minimizer of \( TV_{n+1}(x, d) \) for \( n = 0, \ldots, N - 1 \) and denote by \( \pi^* = (\bar{g}^*_0, \ldots, \bar{g}^*_{N-1}) \) the associated policy. For \( n = N - 1 \), we get that

\[
V_{N-1,N}(x, d) = \inf_{\bar{\pi} \in \Pi} \sum_{i=1}^{i_{\text{max}}} \mathbb{E}_{N-1,x}^{\bar{\pi}} \left[ U^i \left( C^i(X_{N-1}, A_{N-1}, X_N) + d^i \right) \right]
\]

\[
= \inf_{\bar{\pi} \in \Pi} \int_{a \in \Delta(x)} \mathcal{U}(C(x, a, x') + d)P(dx'|x, a) = T[V_{N,N}](x, d),
\]

and obviously, \( \bar{V}_{N-1,N}(x, d) = \bar{I}_{N-1,N}(x, d, \bar{\pi}^*) \). Note that \( \mathcal{U} \in \Delta \).

Now assume that \( \bar{I}_{n_0+1,N}(x, d, \bar{\pi}^*) = \bar{V}_{n_0+1,N}(x, d) \) for a fixed \( n_0 = 0, \ldots, N - 1 \). Then,

\[
\bar{I}_{n_0,N}(x, d, \bar{\pi}^*) = T_{\bar{g}_{n_0}} \bar{I}_{n_0+1,N}(x, d, \bar{\pi}^*)
\]

using (a),

\[
= T_{\bar{g}_{n_0}} \bar{V}_{n_0+1,N}(x, d) \quad \text{by the induction hypothesis},
\]

\[
= T[V_{n_0+1,N}](x, d) \quad \text{by definition of } \bar{g}_{n_0}^*,
\]

and therefore, \( \bar{V}_{n_0,N}(x, d) = \inf_{\pi \in \Pi} \bar{I}_{n_0,N}(x, d, \pi) \leq \bar{I}_{n_0,N}(x, d, \pi^*) = T[V_{n_0+1,N}](x, d) \). On the
other hand, with an arbitrary policy \( \pi = (\tilde{g}_0, \ldots, \tilde{g}_{N-1}) \),

\[
\tilde{I}_{n_0, N}(x, d, \pi) = T_{\tilde{g}_{n_0}}[\tilde{I}_{n_0+1, N}](x, d, \pi)
\]

using (a),

\[
\geq T_{\tilde{g}_{n_0}}[V_{n_0+1, N}](x, d)
\]

by the monotonicity of \( T \),

\[
\geq T[V_{n_0+1, N}](x, d)
\]

by taking the infimum,

and therefore \( V_{n_0, N}(x, d) = \inf_{\pi \in \Pi} \tilde{I}_{n_0, N}(x, d, \pi) \geq T[V_{n_0+1}](x, d) \).

It follows by induction that

\[
\tilde{I}_{n_0, N}(x, d, \pi^*) = T[V_{n_0+1, N}](x, d) = V_{n_0, N}(x, d)
\]

for all \( n_0 = 0, \ldots, N - 1 \).

ad e) Consider the Markovian policy \( \pi^* = (\tilde{g}^*_0, \ldots, \tilde{g}^*_{N-1}) \) as defined in the claim. It holds \( \tilde{V}_{0, N}(x, d) = \tilde{I}_{0, N}(x, d, \pi^*) \), i.e. \( \pi^* \) is an optimal policy for the \( N \)-step MDP, and a minimizer of \( [P^*] \). In particular, \( \tilde{g}^*_n(\cdot, d) \) is an optimal decision rule for the first step of the \( (N-n) \)-step tail problem with accumulated cost \( d \). Therefore, the history-dependent policy \( \tilde{\sigma}^* = (\tilde{f}^*_0, \ldots, \tilde{f}^*_{N-1}) \), defined by

\[
\tilde{f}^*_n(h_n) := \begin{cases} 
\tilde{g}^*_0(x_0, 0), & \text{if } n = 0, \\
\tilde{g}^*_n\left(x_n, \sum_{k=0}^{n-1} C(x_k, a_k, x_{k+1})\right), & \text{otherwise},
\end{cases}
\]

is an optimal policy for problem \([P^*] \).

\[\blacksquare\]

References

[1] Alessandro Arlotto, Noah Gans, and J. Michael Steele. Markov decision problems where means bound variances. Operations Research, 62(4):864–875, 2014.

[2] Guadalupe Avila-Godoy and Emmanuel Fernández-Gaucherand. Controlled Markov chains with exponential risk-sensitive criteria: modularity, structured policies and applications. In Decision and Control, 1998. Proceedings of the 37th IEEE Conference on, volume 1, pages 778–783. IEEE, 1998.

[3] John S. Baras and Matthew R. James. Robust and Risk-Sensitive Output Feedback Control for Finite State Machines and Hidden. Journal of Mathematics, Systems, Estimation and Control, 7(3):371–374, 1997.

[4] Nicole Bauerle and Ulrich Rieder. Markov Decision Processes with Applications to Finance, 2011.

[5] Nicole Bäuerle and Ulrich Rieder. More Risk-Sensitive Markov Decision Processes. Mathematics of Operations Research, 39(1):105–120, feb 2014.

[6] Nicole Bäuerle and Ulrich Rieder. Partially Observable Risk-Sensitive Stopping Problems in Discrete Time. mar 2017.

[7] V. S. Borkar and S. P. Meyn. Risk-Sensitive Optimal Control for Markov Decision Processes with Monotone Cost. Mathematics of Operations Research, 27(1):192–209, feb 2002.
[8] Rolando Cavazos-Cadena. Optimality equations and inequalities in a class of risk-sensitive average cost Markov decision chains. Mathematical Methods of Operations Research, 71(1):47–84, feb 2010.

[9] Rolando Cavazos-Cadena and Daniel Hernández-Hernández. Successive approximations in partially observable controlled Markov chains with risk-sensitive average criterion, dec 2005.

[10] Richard C. Chen and Gilmer L. Blankenship. Dynamic programming equations for discounted constrained stochastic control. IEEE Transactions on Automatic Control, 49(5):699–709, 2004.

[11] Richard C. Chen and Eugene A. Feinberg. Non-randomized policies for constrained markov decision processes. Mathematical Methods of Operations Research, 66(1):165–179, 2007.

[12] Y Chow. Risk-sensitive and Data-driven Sequential Decision Making. 2017.

[13] Y.-L. Chow and M. Pavone. Stochastic optimal control with dynamic, time-consistent risk constraints. ArXiv e-prints, 2015.

[14] Kun-Jen Chung and Matthew J. Sobel. Discounted MDP’s: Distribution Functions and Exponential Utility Maximization. SIAM Journal on Control and Optimization, 25(1):49–62, jan 1987.

[15] Eric V. Denardo and Uriel G. Rothblum. Optimal stopping, exponential utility, and linear programming. Mathematical Programming, 16(1):228–244, 1979.

[16] Giovanni B. Di Masi and Łukasz Stettner. Infinite Horizon Risk Sensitive Control of Discrete Time Markov Processes under Minorization Property. SIAM Journal on Control and Optimization, 46(1):231–252, jan 2007.

[17] Paul Dupuis, Vaios Laschos, and Kavita Ramanan. Exit Time Risk-Sensitive Control for Systems of Cooperative Agents. jul 2018.

[18] Jingnan Fan and Andrzej Ruszczyński. Risk measurement and risk-averse control of partially observable discrete-time Markov systems, 2018.

[19] Eugene A. Feinberg, Pavlo O. Kasyanov, and Michael Z. Zgurovsky. Partially observable total-cost markov decision processes with weakly continuous transition probabilities. Mathematics of Operations Research, 41(2):656–681, 2016.

[20] Emmanuel Fern and Steven I Marcus. Risk-sensitive optimal control of hidden Markov models: Structural results. IEEE Transactions on Automatic Control, 42(10):1418–1422, 1997.

[21] Jerzy A. Filar, L. C. M. Kallenberg, and Huey-Miin Lee. Variance-penalized markov decision processes. Mathematics of Operations Research, 14(1):147–161, 1989.

[22] W. H. Fleming and D. Hernández-Hernández. Risk-Sensitive Control of Finite State Machines on an Infinite Horizon I. SIAM Journal on Control and Optimization, 35(5):1790–1810, sep 1997.

[23] Daniel Hernández-Hernández. Partially Observed Control Problems with Multiplicative Cost. In Stochastic Analysis, Control, Optimization and Applications, pages 41–55. Birkhäuser Boston, Boston, MA, 1999.
[24] Daniel Hernandez-Hernández and Steven I. Marcus. Risk sensitive control of Markov processes in countable state space. Systems & Control Letters, 29(3):147–155, nov 1996.

[25] Onésimo Hernández-Lerma and Jean-Bernard Lasserre. Discrete time Markov control processes : basic optimality criteria. Applications of Mathematics. 1996.

[26] Karl Hinderer. Foundations of non-stationary dynamic programming with discrete time parameter. Lecture notes in operations research and mathematical systems. 1970.

[27] Ronald A. Howard and James E. Matheson. Risk-sensitive markov decision processes. Management Science, 18(7):356–369, 1972.

[28] Stratton C. Jaquette. Markov decision processes with a new optimality criterion: Discrete time. The Annals of Statistics, 1(3):496–505, 1973.

[29] Leslie P. Kaelbling, Michael L. Littman, and Anthony R. Cassandra. Planning and acting in partially observable stochastic domains. Artificial Intelligence, 101(1):99–134, 1998.

[30] Steve Levitt and Adi Ben-Israel. On Modeling Risk in Markov Decision Processes, pages 27–40. Springer US, 2001.

[31] Kun Lin and Steven I. Marcus. Dynamic programming with non-convex risk-sensitive measures. In American Control Conference, pages 6778–6783. IEEE, 2013.

[32] S. I. Marcus, E. Fernández-Gaucherand, D. Hernández-Hernández, S. Coraluppi, and P. Fard. Risk sensitive markov decision processes. In Christopher I. Byrnes, Biswa N. Datta, Clyde F. Martin, and David S. Gilliam, editors, Systems and Control in the Twenty-First Century, pages 263–279. Birkhäuser Boston, 1997.

[33] Janusz Marecki and Pradeep Varakantham. Risk-sensitive planning in partially observable environments. Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems (AAMAS), pages 1357–1368, 2010.

[34] Harry Markowitz. Portfolio selection. The Journal of Finance, 7(1):77–91, 1952.

[35] Martin Puterman. Markov decision processes: discrete stochastic dynamic programming. Wiley series in probability and statistics. 2005.

[36] Diederik M Roijers, Peter Vamplew, Shimon Whiteson, and Richard Dazeley. A survey of multi-objective sequential decision-making. Journal of Artificial Intelligence Research, 48:67–113, 2013.

[37] Andrzej Ruszczyński. Risk-averse dynamic programming for markov decision processes. Mathematical Programming, 125(2):235–261, 2010.

[38] Yoshikazu Sawaragi and Tsunio Yoshikawa. Discrete-time markovian decision processes with incomplete state observation. The Annals of Mathematical Statistics, 41(1):78–86, 1970.

[39] Guy Shani, Joelle Pineau, and Robert Kaplow. A survey of point-based pomdp solvers. Autonomous Agents and Multi-Agent Systems, 27(1):1–51, 2013.

[40] Y. Shen, W. Stannat, and K. Obermayer. Risk-sensitive markov control processes. SIAM Journal on Control and Optimization, 51(5):3652–3672, 2013.
[41] Yun Shen, Vaios Laschos, Wilhelm Stannat, and Klaus Obe rmayer. A Conjugate Approach to Partially Observable Markov Decision Processes on Borel Spaces. mar 2016.

[42] Richard D. Smallwood and Edward J. Sondik. The optimal control of partially observable markov processes over a finite horizon. Operations Research, 21(5):1071–1088, 1973.

[43] Edward J. Sondik. The optimal control of partially observable markov processes over the infinite horizon: Discounted costs. Operations Research, 26(2):282–304, 1978.

[44] D. J. White. Mean, variance, and probabilistic criteria in finite markov decision processes: A review. Journal of Optimization Theory and Applications, 56(1):1–29, 1988.

[45] Yin-Lam Chow and Marco Pavone. Stochastic optimal control with dynamic, time-consistent risk constraints. In 2013 American Control Conference, pages 390–395, 2013.