THEORY OF THE $\tau$ LEPTON ANOMALOUS MAGNETIC MOMENT

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This article reviews and updates the Standard Model prediction of the $\tau$ lepton $g-2$. Updated QED and electroweak contributions are presented, together with new values of the leading-order hadronic term, based on the recent low energy $e^+e^-$ data from BaBar, CMD-2, KLOE and SND, and of the hadronic light-by-light contribution. The total prediction is confronted to the available experimental bounds on the $\tau$ lepton anomaly, and prospects for its future measurements are briefly discussed.

1. Introduction

Numerous precision tests of the Standard Model (SM) and searches for its possible violation have been performed in the last few decades, serving as an invaluable tool to test the theory at the quantum level. They have also provided stringent constraints on many “New Physics” (NP) scenarios. A typical example is given by the measurements of the anomalous magnetic moment of the electron and the muon, where recent experiments reached the fabulous relative precision of 0.7 ppb\(^1\) and 0.5 ppm\(^2\) respectively. These experiments measure the so-called gyromagnetic factor $g$, defined by the relation between the particle’s spin $\vec{s}$ and its magnetic moment $\vec{\mu}$,

$$\vec{\mu} = g \frac{e}{2m} \vec{s},$$  \hspace{1cm} (1)

where $e$ and $m$ are the charge and mass of the particle. In the Dirac theory of a charged point-like spin-1/2 particle, $g = 2$. Quantum Electrodynamics (QED) predicts deviations from Dirac’s value, as the charged particle can emit and reabsorb virtual photons. These QED effects slightly increase the $g$ value. It is conventional to express the difference of $g$ from 2 in terms of the value of the so-called anomalous magnetic moment, a dimensionless quantity defined as $a = (g-2)/2$.

The anomalous magnetic moment of the electron, $a_e$, is rather insensitive to strong and weak interactions, hence providing a stringent test of QED and leading to the most precise determination of the fine-structure constant $\alpha$ to date.\(^3\)\(^4\) On
the other hand, the $g-2$ of the muon, $a_\mu$, allows to test the entire SM, as each of its sectors contributes in a significant way to the total prediction. Compared with $a_e$, $a_\mu$ is also much better suited to unveil or constrain NP effects. Indeed, for a lepton $l$, their contribution to $a_l$ is generally expected to be proportional to $m_l^2/\Lambda^2$, where $m_l$ is the mass of the lepton and $\Lambda$ is the scale of NP, thus leading to an $(m_\mu/m_e)^2 \sim 4 \times 10^4$ relative enhancement of the sensitivity of the muon versus the electron anomalous magnetic moment. This more than compensates the much higher accuracy with which the $g$ factor of the latter is known. The anomalous magnetic moment of the $\tau$ lepton, $a_\tau$, would suit even better; however, its direct experimental measurement is prevented by the relatively short lifetime of this lepton, at least at present. The existing limits are based on the precise measurements of the total and differential cross sections of the reactions $e^+e^- \rightarrow e^+e^-\tau^+\tau^-$ and $e^+e^- \rightarrow Z \rightarrow \tau^+\tau^-\gamma$ at LEP energies. The most stringent limit, $-0.052 < a_\tau < 0.013$ at 95% confidence level, was set by the DELPHI collaboration, and is still more than an order of magnitude worse than that required to determine $a_\tau$.

In the 1990s it became clear that the accuracy of the theoretical prediction of the muon $g-2$, challenged by the E821 experiment underway at Brookhaven, was going to be restricted by our knowledge of its hadronic contribution. This problem has been solved by the impressive experiments at low-energy $e^+e^-$ colliders, where the total hadronic cross section (as well as exclusive ones) were measured with high precision, allowing a significant improvement of the uncertainty of the leading-order hadronic contribution. As a result, the accuracy of the SM prediction for $a_\mu$ now matches that of its measurement. In parallel to these efforts, very many improvements of all other sectors of the SM prediction were carried on by a large number of theorists (see Refs. for reviews). All these experimental and theoretical developments allow to significantly improve the theoretical prediction for the anomalous magnetic moment of $\tau$ lepton as well.

In this article we review and update the SM prediction of $a_\tau$, analyzing in detail the three contributions into which it is usually split: QED, electroweak (EW) and hadronic. Updated QED and EW contributions are presented in Secs. and new values of the leading-order hadronic term, based on the recent low energy $e^+e^-$ data from BaBar, CMD-2, KLOE and SND, and of the hadronic light-by-light contribution are presented in Sec. The total SM prediction is confronted to the available experimental bounds on the $\tau$ lepton $g-2$ in Sec. and prospects for its future measurements are briefly discussed in Sec. where conclusions are drawn.

2. QED Contribution to $a_\tau$

The QED part of the anomalous magnetic moment of the $\tau$ lepton arises from the subset of SM diagrams containing only leptons and photons. This dimensionless quantity can be cast in the general form

$$a^{\text{QED}}_\tau = A_1 + A_2 \left( \frac{m_\tau}{m_e} \right) + A_3 \left( \frac{m_\tau}{m_e}, \frac{m_\tau}{m_\mu} \right),$$

(2)
where $m_e, m_\mu$ and $m_\tau$ are the electron, muon and $\tau$ lepton masses, respectively. The term $A_1$, arising from diagrams containing only photons and $\tau$ leptons, is mass and flavor independent. In contrast, the terms $A_2$ and $A_3$ are functions of the indicated mass ratios, and are generated by graphs containing also electrons and/or muons. The functions $A_i$ ($i = 1, 2, 3$) can be expanded as power series in $\alpha/\pi$ and computed order-by-order

$$A_i = A_i^{(2)} \left( \frac{\alpha}{\pi} \right) + A_i^{(4)} \left( \frac{\alpha}{\pi} \right)^2 + A_i^{(6)} \left( \frac{\alpha}{\pi} \right)^3 + A_i^{(8)} \left( \frac{\alpha}{\pi} \right)^4 + \cdots \quad (3)$$

Only one diagram is involved in the evaluation of the lowest-order (first-order in $\alpha$, second-order in the electric charge) contribution – it provides the famous result by Schwinger $A_1^{(2)} = 1/2$.21 The mass-dependent coefficients $A_2$ and $A_3$, discussed below, are of higher order. They were derived using the latest CODATA recommended mass ratios:

$$m_\tau/m_e = 3477.48(57) \quad (4)$$
$$m_\tau/m_\mu = 16.8183(27). \quad (5)$$

The value for $m_\tau$ adopted by CODATA in Ref. 12, $m_\tau = 1776.99(29)$ MeV, is based on the PDG 2002 result.13 It remained unchanged until very recently (see Refs. 14, 15), when preliminary results of two new measurements (from the Belle16 and KEDR17 detectors) were reported. The central values of the new mass values are slightly lower than the current world average value, but agree with it within the uncertainties, which are approaching that of the world average value (used in this work).

### 2.1. Two-loop Corrections

Seven diagrams contribute to the fourth-order coefficient $A_1^{(4)}$, one to $A_2^{(4)}(m_\tau/m_e)$ and one to $A_2^{(4)}(m_\tau/m_\mu)$. They are depicted in Fig. 1. As there are no two-loop diagrams contributing to $a_{\text{QED}}^{(4)}$ that contain both virtual electrons and muons, $A_2^{(4)}(m_\tau/m_e, m_\tau/m_\mu) = 0$. The mass-independent coefficient has been known for almost fifty years.18

$$A_1^{(4)} = \frac{197}{144} + \frac{\pi^2}{12} + \frac{3}{4} \zeta(3) - \frac{\pi^2}{2} \ln 2$$
$$= -0.32847896657919378\ldots, \quad (6)$$

where $\zeta(s)$ is the Riemann zeta function of argument $s$. For $l = e, \mu$ or $\tau$, the coefficient of the two-loop mass-dependent contribution to $a_{\text{QED}}^{(4)}$, $A_2^{(4)}(1/x)$, with $x = m_j/m_l$, is generated by the diagram with a vacuum polarization subgraph containing the virtual lepton $j$. This coefficient was first computed in the late 1950s for the muon $g-2$ with $x = m_e/m_\mu \ll 1$, neglecting terms of $O(x)$.19 The exact expression for $0 < x < 1$ was reported by Elend in 1966.20 However, its numerical evaluation was considered tricky because of large cancellations and difficulties in the estimate of the accuracy of the results, so that common practice was to use series
Fig. 1. The QED diagrams contributing to the $\tau$ lepton $g-2$ at order $\alpha^2$. The mirror reflections (not shown) of the third and fourth diagrams must be included as well.

Taking advantage of the properties of the dilogarithm $\text{Li}_2(z) = -\int_0^z dt \ln(1-t)/t$, the exact result was cast in Ref. 9 in a very simple and compact analytic form, valid, contrary to the one in Ref. 20, also for $x \geq 1$ (the case relevant to $a_{\tau}^{\text{QED}}$ and part of $a_{\mu}^{\text{QED}}$):

\[
A_2^{(4)}(1/x) = -\frac{25}{36} - \frac{\ln x}{3} + x^2 (4 + 3 \ln x) + \frac{x}{2} (1 - 5x^2) \times \\
\times \left[ \frac{\pi^2}{2} - \ln x \ln \left( \frac{1-x}{1+x} \right) - \text{Li}_2(x) + \text{Li}_2(-x) \right] + \\
+ x^4 \left[ \frac{\pi^2}{3} - 2 \ln x \ln \left( \frac{1}{x} - x \right) - \text{Li}_2(x^2) \right].
\]

(7)

For $x = 1$, Eq. (7) gives $A_2^{(4)}(1) = 119/36 - \pi^2/3$: of course, this contribution is already part of $A_1^{(4)}$ in Eq. (5). Numerical evaluation of Eq. (7) with the mass ratios given in Eqs. (4)–(5) yields the two-loop mass-dependent QED contributions to the anomalous magnetic moment of the $\tau$ lepton:

\[
A_2^{(4)}(m_\tau/m_e) = 2.024284 (55),
\]

(8)

\[
A_2^{(4)}(m_\tau/m_\mu) = 0.361652 (38).
\]

(9)

These two values are very similar to those computed via a dispersive integral in Ref. 25 (which, however, contain no estimates of the uncertainties). Equations (8) and (9) are also in agreement (but more accurate) with those of Ref. 22. Adding up Eqs. (6), (8) and (9) one gets:

\[
C_\tau^{(4)} = 2.057457 (93)
\]

(10)

(note that the uncertainties in $m_\tau/m_e$ and $m_\tau/m_\mu$ are correlated). The resulting error $9.3 \times 10^{-5}$ leads to a $5 \times 10^{-10}$ uncertainty in $a_{\tau}^{\text{QED}}$. 

expansions instead. Taking advantage of the properties of the dilogarithm $\text{Li}_2(z) = -\int_0^z dt \ln(1-t)/t$, the exact result was cast in Ref. 9 in a very simple and compact analytic form, valid, contrary to the one in Ref. 20, also for $x \geq 1$ (the case relevant to $a_{\tau}^{\text{QED}}$ and part of $a_{\mu}^{\text{QED}}$):
2.2. Three-loop Corrections

More than one hundred diagrams are involved in the evaluation of the three-loop (sixth-order) QED contribution. Their analytic computation required approximately three decades, ending in the late 1990s. The coefficient $A_1^{(6)}$ arises from 72 diagrams. Its exact expression, mainly due to Remiddi and his collaborators, reads:

$$A_1^{(6)} = \frac{83}{72} \pi^2 \zeta(3) - \frac{215}{24} \zeta(5) - \frac{239}{2160} \pi^4 + \frac{28259}{5184} + \frac{139}{18} \zeta(3) - \frac{298}{9} \pi^2 \ln 2 + \frac{17101}{810} \pi^2 + \frac{100}{3} \left[ \text{Li}_4(1/2) + \frac{1}{24} (\ln^2 2 - \pi^2) \ln^2 2 \right] = 1.181241456587 \ldots \quad (11)$$

This value is in very good agreement with previous results obtained with numerical methods.

The calculation of the exact expression for the coefficient $A_2^{(6)}(m_l/m_j)$ for arbitrary values of the mass ratio $m_l/m_j$ was completed in 1993 by Laporta and Remiddi. Let us focus on $a_T^{\text{QED}}$ (l = $\tau$, j = e, $\mu$). This coefficient can be further split into two parts: the first one, $A_2^{(6)}(m_l/m_j, \text{vac})$, receives contributions from 36 diagrams containing either electron or muon vacuum polarization loops, whereas the second one, $A_2^{(6)}(m_l/m_j, \text{lbl})$, is due to 12 light-by-light scattering diagrams with either electron or muon loops. The exact expressions for these coefficients are rather complicated, containing hundreds of polylogarithmic functions up to fifth degree (for the light-by-light diagrams) and complex arguments (for the vacuum polarization ones) – they also involve harmonic polylogarithms. Series expansions were provided in Ref. 30 for the cases of physical relevance.

Using the recommended mass ratios given in Eqs. (4) and (5), the following values were recently computed from the full analytic expressions:

$$A_2^{(6)}(m_\tau/m_e, \text{vac}) = 7.25699 \quad (12)$$
$$A_2^{(6)}(m_\tau/m_e, \text{lbl}) = 39.1351 \quad (13)$$
$$A_2^{(6)}(m_\tau/m_\mu, \text{vac}) = -0.023554 \quad (14)$$
$$A_2^{(6)}(m_\tau/m_\mu, \text{lbl}) = 7.03376 \quad (15)$$

Almost identical values were obtained employing the approximate series expansions of Ref. 30: 7.25699(41), 39.1351(11), −0.023564(51), 7.03375(71). The previous estimates of Ref. 25 were different: 10.0002, 39.5217, 2.9340, and 4.4412 (no error estimates were provided), respectively; they are superseded by the results in Eqs. (12)–(15), derived from the exact analytic expressions. The estimates of Ref. 33 compare slightly better: 7.2670, 39.6, −0.1222, 4.47 (no errors provided). In the specific case of $A_2^{(6)}(m_\tau/m_\mu, \text{lbl})$, the values of Refs. 25 and 33 differ from Eq. (15) because their derivations did not include terms of $O(m_\mu/m_\tau)$, which turn out to be unexpectedly
large. The sums of Eqs. (12)–(13) and (14)–(15) are
\[ A_2^{(6)}(m_\tau/m_e) = 46.3921 \text{ (15)}, \]
\[ A_2^{(6)}(m_\tau/m_\mu) = 7.01021 \text{ (76)}. \]

The contribution of the three-loop diagrams with both electron- and muon-loop insertions in the photon propagator was calculated numerically from the integral expressions of Ref. 21, obtaining:
\[ A_3^{(6)}(m_\tau/m_e, m_\tau/m_\mu) = 3.34797 \text{ (41)}. \]

This value disagrees with the results of Refs. 33 (1.679) and 25 (2.75316). Combining the three-loop results of Eqs. (11), (16), (17) and (18) one finds the sixth-order QED coefficient:
\[ C_\tau^{(6)} = 57.9315 \text{ (27)}. \]

The error $2.7 \times 10^{-3}$ induces a $3 \times 10^{-11}$ uncertainty in $a_\tau^{QED}$. The order of magnitude of the three-loop contribution to $a_\tau^{QED}$, dominated by the mass-dependent terms, is comparable to that of EW and hadronic effects (see later).

Contrary to the case of the electron and muon $g-2$, QED contributions of order higher than three are not known. (An exception is the mass- and flavor-independent term $A_1^{(8)}$ which is however expected to be a very small part of the complete four-loop contribution.) Adding up all the above contributions and using the new value of $\alpha$ derived in Refs. 3 and 4, $\alpha^{-1} = 137.035\,999\,709$ (96), one obtains the total QED contribution to the $g-2$ of the $\tau$ lepton:
\[ a_\tau^{QED} = 117\,324 \times 10^{-8}. \]

The error $\delta a_\tau^{QED}$ is the uncertainty $\delta C_\tau^{(8)}(\alpha/\pi)^4 \sim \pi^2 \ln^2(m_\tau/m_e)(\alpha/\pi)^4 \sim 2 \times 10^{-8}$ assigned to $a_\tau^{QED}$ for uncalculated four-loop contributions. As we mentioned earlier, the errors due to the uncertainties of the $O(\alpha^2)$ and $O(\alpha^3)$ terms are negligible. The error induced by the uncertainty of $\alpha$ is only $8 \times 10^{-13}$ (and thus totally negligible).

3. Electroweak Contribution to $a_\tau$

With respect to Schwinger’s contribution, the EW correction to the anomalous magnetic moment of the $\tau$ lepton is suppressed by the ratio $(m_\tau/M_W)^2$, where $M_W$ is the mass of the $W$ boson. Numerically, this contribution is of the same order of magnitude as the three-loop QED one.

3.1. One-loop Contribution

The analytic expression for the one-loop EW contribution to $a_\tau$, due to the diagrams in Fig. 2 is:
\[ a_\tau^{EW}(1 \text{ loop}) = \frac{5 G_F m_e^2}{24 \sqrt{2} \pi^2} \left[ 1 + \frac{1}{5} (1 - 4 \sin^2 \theta_W)^2 + O \left( \frac{m_e^2}{M_{Z,W,H}^2} \right) \right], \]

(21)
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where $G_\mu = 1.16637(1) \times 10^{-5}$ GeV$^{-2}$ is the Fermi coupling constant, $M_Z$, $M_W$ and $M_H$ are the masses of the $Z$, $W$ and Higgs bosons, and $\theta_W$ is the weak mixing angle. Closed analytic expressions for $a_{\tau}^{\text{EW}}(1 \text{ loop})$ taking exactly into account the $m_{\tau}^2/M_B^2$ dependence ($B = Z, W, \text{Higgs, or other hypothetical bosons}$) can be found in Refs. [37]. Following Ref. [38] we employ for $\sin^2 \theta_W$ the on-shell definition [39]

\[ \sin^2 \theta_W = 1 - \frac{M_W^2}{M_Z^2}, \]

where $M_Z = 91.1876(21)$ GeV [15] and $M_W$ is the theoretical SM prediction of the $W$ mass. The latter can be easily derived from the simple analytic formulae of Ref. [40] (see also Refs. [41]),

\[ M_W = \left( 80.3676 - 0.05738 \ln \left( \frac{M_H}{100 \text{ GeV}} \right) - 0.00892 \ln^2 \left( \frac{M_H}{100 \text{ GeV}} \right) \right) \text{ GeV}, \]

(22)

(on-shell scheme II with $\Delta \alpha_h^{(5)} = 0.02758(35), \alpha_s(M_Z) = 0.118(2), [15]$ and $M_{\text{top}} = 171.4(2.1)$ GeV [13]) leading to $M_W = 80.343$ GeV for $M_H = 150$ GeV. This result should be compared with the direct experimental value $M_W = 80.403(29)$ GeV [15] which corresponds to a very small $M_H$. In any case, these shifts in the $M_W$ prediction induced by the variation of $M_H$ from 114.4 GeV, the current lower bound at 95% confidence level [14] up to a few hundred GeV, only change $a_{\tau}^{\text{EW}}(1 \text{ loop})$ by amounts of $O(10^{-10})$. From Eq. (21), including the tiny $O(m_{\tau}^2/M_{Z,W,H}^2)$ corrections of Refs. [37], for $M_H = 150$ GeV we get

\[ a_{\tau}^{\text{EW}}(1 \text{ loop}) = 55.1(1) \times 10^{-8}. \]

(23)

The uncertainty encompasses the shifts induced by variations of $M_H$ from 114 GeV up to a few hundred GeV, and the tiny uncertainty due to the error in $m_{\tau}$.

![Fig. 2. One-loop electroweak contributions to $a_{\tau}$. The diagram with a $W$ and a Goldstone boson ($\phi$) must be counted twice.](image)

The estimate of the EW contribution in Ref. [33] $a_{\tau}^{\text{EW}} = 55.60(2) \times 10^{-8}$, obtained from the one-loop formula (without the small corrections of order $m_{\tau}^2/M_{Z,W,H}^2$), is
similar to our value in Eq. (23). However, its uncertainty \((2 \times 10^{-10})\) is too small, and it doesn’t contain the two-loop contribution which, as we’ll discuss in the next section, is not negligible.

3.2. Two-loop Contribution

The two-loop EW contributions to \(a_l\) \((l = e, \mu \text{ or } \tau)\) were computed in 1995 by Czarnecki, Krause and Marciano. This remarkable calculation leads to a significant reduction of the one-loop prediction. Naively one would expect the two-loop EW contribution \(a_{\text{EW}}^l(2 \text{ loop})\) to be of order \((\alpha/\pi) \times a_{\text{EW}}^l(1 \text{ loop})\), but this turns out not to be so. As first noticed in the early 1990s, \(a_{\text{EW}}^l(2 \text{ loop})\) is actually quite substantial because of the appearance of terms enhanced by a factor of \(\ln(M_{Z,W}/m_f)\), where \(m_f\) is a fermion mass scale much smaller than \(M_W\).

The two-loop contribution to \(a_{\text{EW}}^\tau(2 \text{ loop})\) involves 1678 diagrams in the linear ’t Hooft-Feynman gauge as a check, the authors of Refs. employed both this gauge and a nonlinear one in which the vertex of the photon, the \(W\) and the unphysical charged scalar vanishes. It can be divided into fermionic and bosonic parts; the former, \(a_{\text{EW}}^\tau(2 \text{ loop fer})\), includes all two-loop EW corrections containing closed fermion loops, whereas all other contributions are grouped into the latter, \(a_{\text{EW}}^\tau(2 \text{ loop bos})\). The expressions of Ref. for the bosonic part were obtained in the approximation \(M_H \gg M_{W,Z}\), computing the first two terms in the expansion in \(M_{W,Z}^2/M_H^2\), and expanding in \(\sin^2\theta_W\), keeping the first four terms in this expansion (this number of powers is sufficient to obtain an exact coefficient of the large logarithms \(\ln(M_{W,Z}^2/m_f^2)\)). Recent analyses of the EW bosonic corrections of the \(g-2\) of the muon relaxed these approximations, providing analytic results valid also for a light Higgs. Considering the present \(M_H > 114.4\) GeV lower bound, we can safely employ the results of Ref. obtaining, for \(M_H = 150\) GeV \(a_{\text{EW}}^\tau(2 \text{ loop bos}) = -3.06 \times 10^{-8}\). The neglected terms are of \(O(10^{-9})\).

The fermionic part of \(a_{\text{EW}}^\tau(2 \text{ loop})\) contains the contribution of diagrams with light quarks; they involve long-distance QCD for which perturbation theory cannot be employed. In particular, these hadronic uncertainties arise from two types of two-loop diagrams: the hadronic photon–\(Z\) mixing, and quark triangle loops with the external photon, a virtual photon and a \(Z\) attached to them (see Fig. 3). The hadronic uncertainties mainly arise from the latter ones. Two approaches were suggested for their study: in Ref. the nonperturbative effects where modeled introducing effective quark masses as a simple way to account for strong interactions. In view of the high experimental precision of the \(g-2\) of the muon, a more realistic treatment of the relevant hadronic dynamics was introduced in Ref. within a low-energy effective field theory approach, later on developed in the detailed analyses of Refs. However, from a numerical point of view, the discrepancy between the results provided by these two different approaches turns out to be irrelevant for the present interpretation of the experimental result of the muon \(g-2\), in spite of its precision. The use of effective quark masses for the study of the \(g-2\)
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Fig. 3. Some of the fermion-loop diagrams contributing to the \( \tau \) anomalous magnetic moment.

of the \( \tau \) — whose experimental precision is a far cry from that of the muon! — thus appears to be sufficient at present. The tiny hadronic \( \gamma - Z \) mixing terms can be evaluated either in the free quark approximation or via a dispersion relation using data from \( e^+ e^- \) annihilation into hadrons; the difference was shown to be numerically insignificant.\(^{38}\)

References\(^{45}\) and\(^{47}\) contain simple approximate expressions for the contributions of the diagrams with fermion triangle loops shown in Fig. 3 (right). In general, for a lepton \( \ell = e, \mu \) or \( \tau \), neglecting small mass-ratios, they are

\[
a^{\text{EW}}_\ell(2 \text{ loop fer; triangle loops}) \simeq \sum_f \frac{5G_F m_\ell^2}{24\sqrt{2}\pi^2} \left[ \frac{\alpha}{\pi} \frac{18}{5} N_f I^3_f Q_f^2 \Delta C(f) \right], \tag{24}
\]

where \( I^3_f \) is the third component of the weak isospin of the fermion \( f \) in the loop, \( Q_f \) is its charge, \( N_f \) its number of colors (3 for quarks, 1 for leptons), and

\[
\Delta C(f) \simeq \begin{cases} 
\ln(M_Z^2/m_\ell^2) + 5/6 & \text{if } m_f \ll m_\ell \ll M_Z \\
\ln(M_Z^2/m_\ell^2) - 8\pi^2/27 + 11/18 & \text{if } m_f = m_\ell \ll M_Z \\
\ln(M_Z^2/m_\ell^2) - 2 & \text{if } m_\ell \ll m_f \ll M_Z \\
([M_Z^2/M_{\text{top}}^2]/6 - 2/3) \ln(M_{\text{top}}^2/M_Z^2) + (5/18)(M_Z^2/M_{\text{top}}^2) - 4/3 & \text{if } m_f = M_{\text{top}}.
\end{cases} \tag{25}
\]

The contribution of the top-quark triangle loop diagram of Fig. 3 (right) with the \( Z \) boson replaced by the neutral Goldstone boson (\( \phi^0 \)) has also been included in this expression.\(^{15}\) It is clear from Eqs.\(^{24} - 25\) that the logarithms \( \ln(M_Z) \) cancel in sums over all fermions of a given generation, as long as \( m_f \ll M_Z \), due to the no-anomaly condition \( \sum_f N_f I^3_f Q_f^2 = 0 \) valid within every generation.\(^{15,50}\) This does not occur for the third generation due to the large mass of the top quark. Note that this short-distance cancellation does not get modified by strong interaction effects on the quark triangle diagrams.\(^{45,53}\)

Contrary to the case of the muon \( g - 2 \), where all fermion masses, with the exception of \( m_e \), enter in \( a^{\text{EW}}_\mu(2 \text{ loop fer; triangle loops}) \), the approximate expressions in Eq.\(^{25}\) show that this is not the case for the \( g - 2 \) of the \( \tau \) lepton. Indeed, due to the
high infrared cut-off set by $m_\tau$, Eq. (24) for $l = \tau$ does not depend on any fermion mass lighter than $m_\tau$; apart from $m_\tau$, it only depends on $M_{\text{top}}$ and $m_b$, the masses of the top and bottom quarks (assuming $m_c < m_\tau$). The charm contribution requires some care, as the crude approximation provided by Eq. (25) for a charm lighter than the $\tau$ lepton is valid only if $m_c \ll m_\tau$. Clearly, this is not a good approximation, and the spurious shift induced by Eq. (25) when $m_c$ is varied across the $m_\tau$ threshold is of $O(10^{-8})$. One possibility is to use Eq. (25) with $m_c$ equal to $m_\tau$. Better still, we numerically integrated the exact expressions for $\Delta C(f)$ provided in Ref. 55 for arbitrary values of $m_f$, obtaining a smooth dependence on the value of $m_c$. For completeness we repeated this detailed analysis for all light fermions. As expected, the result depends very mildly on the values chosen for their masses. Employing the values $m_u = m_d = 0.3 \, \text{GeV}$, $m_s = 0.5 \, \text{GeV}$, $m_c = 1.5 \, \text{GeV}$ and $m_b = 4.5 \, \text{GeV}$, and adding to Eq. (24) the contribution of the remaining fermionic two-loop diagrams studied in Ref. 45 for $M_H = 150 \, \text{GeV}$ we obtain $a_c^{\text{EW}}(2\text{ loop fer}) = -4.68 \times 10^{-8}$. In this evaluation we also included the tiny $O(10^{-9})$ contribution of the $\gamma-Z$ mixing diagrams, suppressed by $(1 - 4 \sin^2 \theta_W) \sim 0.1$ for quarks and $(1 - 4 \sin^2 \theta_W)^2$ for leptons, via the explicit formulae of Ref. 38.

The sum of the fermionic and bosonic two-loop EW contributions described above gives $a_\tau^{\text{EW}}(2\text{ loop}) = -7.74 \times 10^{-8}$, a 14% reduction of the one-loop result. The leading-logarithm three-loop EW contributions to the muon $g-2$ were determined to be extremely small via renormalization-group analyses. We assigned to our $\tau$ lepton $g-2$ EW result an additional uncertainty of $O[a_\tau^{\text{EW}}(2\text{ loop})(\alpha/\pi) \ln(M_Z^2/m_\tau^2)] \sim O(10^{-9})$ to account for these neglected three-loop effects. Adding $a_c^{\text{EW}}(2\text{ loop})$ to the one-loop value of Eq. (23) we get our total EW correction (for $M_H = 150 \, \text{GeV}$)

$$a_\tau^{\text{EW}} = 47.4(5) \times 10^{-8}. \quad (26)$$

The uncertainty allows $M_H$ to range from 114 GeV up to $\sim 300 \, \text{GeV}$, and reflects the estimated errors induced by hadronic loop effects ($m_u$ and $m_d$ can vary between 70 MeV and 400 MeV), neglected two-loop bosonic terms, and the missing three-loop contribution. It also includes the small errors due to the uncertainties in $M_{\text{top}}$ and $m_c$. The value in Eq. (26) is in agreement with the prediction $a_\tau^{\text{EW}} = 47(1) \times 10^{-8}$, with a reduced uncertainty. As we mentioned in Sec. 3.1, the EW estimate of Ref. 33, $a_\tau^{\text{EW}} = 55.60(2) \times 10^{-8}$, mainly differs from Eq. (26) in that it doesn’t include the two-loop corrections.

4. The Hadronic Contribution

In this section we will analyze $a_{\text{HAD}}^{\mu}$, the contribution to the $\tau$ anomalous magnetic moment arising from QED diagrams involving hadrons. Hadronic effects in (two-loop) EW contributions are already included in $a_c^{\text{EW}}$ (see the previous section).
4.1. **Leading-order Hadronic Contribution**

Similarly to the case of the muon $g-2$, the leading-order hadronic contribution to the $\tau$ lepton anomalous magnetic moment is given by the dispersion integral: 57

$$a_{\tau}^{\text{HLO}} = \frac{m_{\tau}^2}{12\pi^3} \int_{4m_{\tau}^2}^{\infty} ds \frac{\sigma^{(0)}(e^+e^- \to \text{hadrons}) K_{\tau}(s)}{s},$$  \hspace{1cm} (27)

where the kernel $K_{\tau}(s)$ is a bounded function of energy monotonously increasing to unity at $s \to \infty$, and $\sigma^{(0)}(e^+e^- \to \text{hadrons})$ is the total hadronic cross section of the $e^+e^-$ annihilation in the Born approximation. In Fig. 4 we plot the ratio of the kernels in the $\tau$ lepton and muon case. Clearly, although the role of the low energies is still very important, the different structure of $K_{\tau}$ compared to $K_{\mu}$, induced by the higher mass of the $\tau$, results in a relatively higher role of the larger energies.

![Fig. 4. Ratio of the kernels $K_{\tau}(s)/K_{\mu}(s)$.](image)

The history of these calculations is not as rich as that of the muon. The first calculation performed in 1978 in Ref. 58 was based on experimental data available at that time below 7.4 GeV, whereas at higher energies the asymptotic QCD prediction was used. Ten years later, a rough estimate was made in Ref. 59 based on low energy $e^+e^-$ data. In Ref. 33 the contribution of the $\rho$ meson was estimated by integrating the approximation obtained using the Breit-Wigner curve, while other contributions used the data. The accuracy of the calculation was considerably improved in Refs. 60, 61 where, below 40 GeV, only data were used. In Ref. 25 data were only used below 3 GeV (together with the experimental parameters of the $J/\psi$ and $\Upsilon$ family states). In our opinion this can significantly underestimate the resulting uncertainty. In addition, in the same reference, data from $\tau$ lepton decays were extensively used; as it is known today, this leads to higher spectral functions.
than in $e^+e^-$ case\cite{62,63} and can therefore overestimate the result. The results of these calculations are summarized in Table 1.

| Author                  | Year | $a_{\tau}^{\text{HLO}} \times 10^8$ |
|-------------------------|------|------------------------------------|
| S. Narison              | 1978 | $370 \pm 40$                       |
| B.C. Barish and R. Stroynowski | 1988 | $\sim 350$                        |
| M.A. Samuel et al.      | 1991 | $360 \pm 32$                       |
| S. Eidelman and F. Jegerlehner | 1995 | $338.4 \pm 2.0 \pm 9.1$          |
| S. Narison              | 2001 | $353.6 \pm 4.0$                    |
| This work               | 2007 | $337.5 \pm 3.7$                    |
| M. Benmerrouche et al.  | 1993 | $197–246$                          |
| F. Hamzeh and N.F. Nasrallah | 1993 | $280 \pm 20$                      |
| B. Holdom et al.        | 1994 | $320 \pm 10$                       |
| A.E. Dorokhov           | 2005 | $310 \pm 20$                       |

For completeness, in the second part of Table 1 we also show purely theoretical estimates. The analysis based on QCD sum rules performed in Ref. \cite{64} gives results which strongly depend on the choice of quark and gluon condensates. QCD sum rules are also used in Ref. \cite{65}. In Ref. \cite{66} the authors use a nonlocal constituent quark model for the description of the photon vacuum polarization function $\Pi_{\text{had}}(q^2)$ at space-like momenta and obtain $a_{\tau}^{\text{HLO}} = 3.2(1) \times 10^{-6}$, close to the estimates based on the experimental data. They also show that a simpler model with constituent quark masses independent of momentum is strongly dependent on the values chosen for the quark masses. For example, with $m_u = m_d = 330$ MeV and $m_s = 550$ MeV their result is $a_{\tau}^{\text{HLO}} = 2.2(1) \times 10^{-6}$, i.e., significantly smaller than the previous estimate. They could reproduce the value $a_{\tau}^{\text{HLO}} = 3.2(1) \times 10^{-6}$ using $m_u = m_d = m_s = 201$ MeV. In a recent analysis using the instanton liquid model the author obtains $(3.1 \pm 0.2) \times 10^{-6}$ \cite{67}. All these estimates somewhat undervalue the hadronic contribution and have rather large uncertainties.

We updated the calculation of the leading-order contribution using the whole bulk of experimental data below 12 GeV, which include old data compiled in Refs. \cite{60,62,63} as well as the recent datasets from the CMD-2\cite{68,69} and SND\cite{70,71,72} experiments in Novosibirsk, and from the radiative return studies at KLOE in Frascati\cite{73} and BaBar at SLAC\cite{74}. The improvement is particularly visible in the channel $e^+e^- \rightarrow \pi^+\pi^-$, where four new independent measurements exist in the most important $\rho$ meson region: CMD-2\cite{68} and SND\cite{70} and KLOE\cite{73} (see Fig. 5). Our result is

$$a_{\tau}^{\text{HLO}} = 337.5 (3.7) \times 10^{-8} \tag{28}$$

(we recently presented a preliminary estimate of this value in Ref. \cite{75}). The breakdown of the contributions of different energy regions as well as their relative fractions in the total leading-order contribution are given in Table 2. The contribution of the
Theoretical results for the anomalous magnetic moment of the τ lepton are presented, focusing on the precision of experiments in the ρ meson energy range. The contributions from various sources, including hadronic continuum, narrow resonances, and higher-order hadronic contributions, are discussed. The hadronic higher-order contribution \( a_{\tau \text{HOO}} \) can be divided into two parts: the first involving hadronic self-energy insertions in the photon propagators, and the second involving intermediate states. The overall uncertainty in these contributions is smaller than in previous predictions.

**4.2. Higher-order Hadronic Contributions**

The hadronic higher-order \((\alpha^3)\) contribution \( a_{\tau \text{HOO}} \) can be divided into two parts: \( a_{\tau \text{HOO}} = a_{\tau \text{HOO}}^{\text{vp}} + a_{\tau \text{HOO}}^{\text{lbl}} \). The first one is the \( O(\alpha^3) \) contribution of diagrams containing hadronic self-energy insertions in the photon propagators. It was determined by Krause in 1996:\(^{70}\)

\[
a_{\tau \text{HOO}}^{\text{vp}} = 7.6(2) \times 10^{-8}.
\]  

Note that naively rescaling the muon result by the factor \( m_{\tau}^2/m_{\mu}^2 \) (as it was done in Ref.\(^{83}\)) leads to the totally incorrect estimate \( a_{\tau \text{HOO}}^{\text{vp}} = (-101 \times 10^{-11}) \times \)
Similarly to the case of the muon interactions with electromagnetic currents. Actually, very few estimates of its distribution, and its evaluation therefore relies on specific models of low-energy hadronic dispersion relation approach using data (unlike the leading-order hadronic contribution), and its evaluation therefore relies on specific models of low-energy hadronic interactions with electromagnetic currents. Actually, very few estimates of $a_{\mu}^{HOO}(\nu\nu)$ exist in the literature, and all of them were obtained simply rescaling the muon results $a_{\mu}^{HOO}(\nu\nu)$ by a factor $m^2/m^2_{\mu}$. Following this very naive procedure, the $a_{\mu}^{HOO}(\nu\nu)$ estimate varies between $a_{\mu}^{HOO}(\nu\nu) = [80(40) \times 10^{-11}] \times (m^2/m^2_{\mu}) = 23(11) \times 10^{-8}$, and $a_{\mu}^{HOO}(\nu\nu) = [136(25) \times 10^{-11}] \times (m^2/m^2_{\mu}) = 38(7) \times 10^{-8}$, according to the values chosen for $a_{\mu}^{HOO}(\nu\nu)$ from Refs. [77] and [78] respectively.

These very naive estimates fall short of what is needed. Consider the function $A_2^{(6)}(m_l/m_j, j)$, the three-loop QED contribution to the $g-2$ of a lepton of mass $m_l$ due to light-by-light diagrams involving loops of a fermion of mass $m_j$ (see Sec. 2.2).

The exact expression of this function, computed in Ref. [20] for arbitrary values of the mass ratio $m_l/m_j$, is rather complicated, but series expansions were provided in the same article for the cases of physical relevance. In particular, if $m_j \gg m_l$, then $A_2^{(6)}(m_l/m_j, j) \sim (m_l/m_j)^2$. This implies that, for example, the (negligible) part of $a_{\mu}^{HOO}(\nu\nu)$ due to diagrams with a top-quark loop can be reasonably estimated simply rescaling the corresponding part of $a_{\mu}^{HOO}(\nu\nu)$ by a factor $m^2/m^2_{\mu}$. On the other hand, to compute the dominant contributions to $a_{\mu}^{HOO}(\nu\nu)$, i.e. those induced by the light quarks, we need the opposite case: $m_j \ll m_l = m_{\tau}$. In this limit, $A_2^{(6)}(m_l/m_j, j)$ does not scale as $(m_l/m_j)^2$, and a naive rescaling of $a_{\mu}^{HOO}(\nu\nu)$ by $m^2/m^2_{\mu}$ to derive $a_{\mu}^{HOO}(\nu\nu)$ leads to an incorrect estimate.

We therefore decided to perform a parton-level estimate of $a_{\mu}^{HOO}(\nu\nu)$ based on the exact expression for $A_2^{(6)}(m_l/m_j, j)$ using the quark masses recently proposed in Ref. [79] for the determination of $a_{\mu}^{HOO}(\nu\nu)$: $m_u = m_d = 176$ MeV, $m_s = 305$ MeV, $m_c = 1.18$ GeV and $m_b = 4$ GeV (note that with these values the authors of Ref. [79] obtain $a_{\mu}^{HOO}(\nu\nu) = 136 \times 10^{-11}$, in perfect agreement with the value in

| $\sqrt{s}$, GeV | $\Delta a_{\mu}^{HOO} \times 10^8$ | $\Delta a_{\mu}^{HOO}$, % |
|-----------------|-------------------------------|----------------------|
| 2π, < 2         | 173.3 ± 1.6                  | 51.3                 |
| ω               | 15.0 ± 0.4                   | 4.4                  |
| φ               | 21.1 ± 0.4                   | 6.3                  |
| 0.6–2.0         | 48.2 ± 2.4                   | 14.3                 |
| 2.0–5.0         | 57.9 ± 2.0                   | 17.2                 |
| 5.0–12.0        | 16.9 ± 1.1                   | 5.0                  |
| > 12.0          | 5.1                          | 1.5                  |
| Total           | 337.5 ± 3.7                  | 100.0                |

$m_{\tau}^2/m^2_{\mu} = -29 \times 10^{-8}$ (the $a_{\mu}^{HOO}(\nu\nu)$ value is from Ref. [70]: even the sign is wrong!

The second term, also of $O(\alpha^3)$, is the hadronic light-by-light contribution. Similarly to the case of the muon $g-2$, this term cannot be directly determined via a dispersion relation approach using data (unlike the leading-order hadronic contribution), and its evaluation therefore relies on specific models of low-energy hadronic interactions with electromagnetic currents. Actually, very few estimates of $a_{\mu}^{HOO}(\nu\nu)$ exist in the literature, and all of them were obtained simply rescaling the muon results $a_{\mu}^{HOO}(\nu\nu)$ by a factor $m^2/m^2_{\mu}$. Following this very naive procedure, the $a_{\mu}^{HOO}(\nu\nu)$ estimate varies between $a_{\mu}^{HOO}(\nu\nu) = [80(40) \times 10^{-11}] \times (m^2/m^2_{\mu}) = 23(11) \times 10^{-8}$, and $a_{\mu}^{HOO}(\nu\nu) = [136(25) \times 10^{-11}] \times (m^2/m^2_{\mu}) = 38(7) \times 10^{-8}$, according to the values chosen for $a_{\mu}^{HOO}(\nu\nu)$ from Refs. [77] and [78] respectively.

These very naive estimates fall short of what is needed. Consider the function $A_2^{(6)}(m_l/m_j, j)$, the three-loop QED contribution to the $g-2$ of a lepton of mass $m_l$ due to light-by-light diagrams involving loops of a fermion of mass $m_j$ (see Sec. 2.2).

The exact expression of this function, computed in Ref. [20] for arbitrary values of the mass ratio $m_l/m_j$, is rather complicated, but series expansions were provided in the same article for the cases of physical relevance. In particular, if $m_j \gg m_l$, then $A_2^{(6)}(m_l/m_j, j) \sim (m_l/m_j)^2$. This implies that, for example, the (negligible) part of $a_{\mu}^{HOO}(\nu\nu)$ due to diagrams with a top-quark loop can be reasonably estimated simply rescaling the corresponding part of $a_{\mu}^{HOO}(\nu\nu)$ by a factor $m^2/m^2_{\mu}$. On the other hand, to compute the dominant contributions to $a_{\mu}^{HOO}(\nu\nu)$, i.e. those induced by the light quarks, we need the opposite case: $m_j \ll m_l = m_{\tau}$. In this limit, $A_2^{(6)}(m_l/m_j, j)$ does not scale as $(m_l/m_j)^2$, and a naive rescaling of $a_{\mu}^{HOO}(\nu\nu)$ by $m^2/m^2_{\mu}$ to derive $a_{\mu}^{HOO}(\nu\nu)$ leads to an incorrect estimate.

We therefore decided to perform a parton-level estimate of $a_{\mu}^{HOO}(\nu\nu)$ based on the exact expression for $A_2^{(6)}(m_l/m_j, j)$ using the quark masses recently proposed in Ref. [79] for the determination of $a_{\mu}^{HOO}(\nu\nu)$: $m_u = m_d = 176$ MeV, $m_s = 305$ MeV, $m_c = 1.18$ GeV and $m_b = 4$ GeV (note that with these values the authors of Ref. [79] obtain $a_{\mu}^{HOO}(\nu\nu) = 136 \times 10^{-11}$, in perfect agreement with the value in
Ref. [78] – see also Ref. [80] for a similar earlier determination). We obtain
\[ a_{\tau}^{\text{HHO (lbl)}} = 5(3) \times 10^{-8}. \]  
(30)
This value is much lower than those obtained by simple rescaling of \( a_{\mu}^{\text{HHO (lbl)}} \) by \( m_{\tau}^{2}/m_{\mu}^{2} \). The up-quark provides the dominating contribution; the uncertainty \( \delta a_{\tau}^{\text{HHO (lbl)}} = 3 \times 10^{-8} \) allows \( m_u \) to range from 70 MeV up to 400 MeV. Further independent studies (following the approach of Ref. [78] for example) would provide an important check of this result.

The total hadronic contribution to the anomalous magnetic moment of the \( \tau \) lepton can be immediately derived adding the values in Eqs. (28), (29) and (30),
\[ a_{\tau}^{\text{HAD}} = a_{\tau}^{\text{HLO}} + a_{\tau}^{\text{HHO (vp)}} + a_{\tau}^{\text{HHO (lbl)}} = 350.1 (4.8) \times 10^{-8}. \]  
(31)
Errors were added in quadrature.

5. The Standard Model prediction for \( a_{\tau} \)

We can now add up all the contributions discussed in the previous sections to derive the SM prediction for \( a_{\tau} \):
\[ a_{\tau}^{\text{SM}} = a_{\tau}^{\text{QED}} + a_{\tau}^{\text{EW}} + a_{\tau}^{\text{HLO}} + a_{\tau}^{\text{HHO (vp)}} + a_{\tau}^{\text{HHO (lbl)}}, \]  
(32)
where
\[ a_{\tau}^{\text{QED}} = 117.324 (2) \times 10^{-8} \]
\[ a_{\tau}^{\text{EW}} = 47.4 (5) \times 10^{-8} \]
\[ a_{\tau}^{\text{HLO}} = 337.5 (3.7) \times 10^{-8} \]
\[ a_{\tau}^{\text{HHO (vp)}} = 7.6 (2) \times 10^{-8} \]
\[ a_{\tau}^{\text{HHO (lbl)}} = 5 (3) \times 10^{-8} \]
(the sum of the hadronic contributions is given in Eq. (31)). Adding errors in quadrature, our final result is
\[ a_{\tau}^{\text{SM}} = 117.721 (5) \times 10^{-8}. \]  
(33)

The present PDG limit on the anomalous magnetic moment of the \( \tau \) lepton was derived in 2004 by the DELPHI collaboration from \( e^+e^- \rightarrow e^+e^-\tau^+\tau^- \) total cross section measurements at \( \sqrt{s} \) between 183 and 208 GeV at LEP2 [9]
\[ -0.052 < a_{\tau} < 0.013 \]  
(34)
at 95% confidence level. The authors of Ref. [9] also quote their result in the form of central value and error:
\[ a_{\tau} = -0.018(17). \]  
(35)
Comparing this result with Eq. (33) (their difference is roughly one standard deviation), it is clear that the sensitivity of the best existing measurements is still more than an order of magnitude worse than needed. A reanalysis of various measurements of the cross section of the process \( e^+e^- \rightarrow \tau^+\tau^- \), the transverse \( \tau \) polarization...
and asymmetry at LEP and SLD, as well as of the decay width \(\Gamma(W \rightarrow \tau \nu)\) at LEP and Tevatron, allowed to set a stronger model-independent limit: 81

\[ -0.007 < a_\tau < 0.005. \]  

(36)

Other limits on \(a_\tau\) can be found in Refs. 82.

6. Conclusions

In this article we reviewed and updated the SM prediction of the \(\tau\) lepton \(g - 2\). Updated QED and electroweak contributions were presented, together with new values of the leading-order hadronic term, based on the recent low energy \(e^+e^-\) data from BaBar, CMD-2, KLOE and SND, and of the hadronic light-by-light contribution. These results were confronted in Sec. 5 to the available experimental bounds on the \(\tau\) lepton anomaly.

As we already mentioned in the Introduction, quite generally, NP associated with a scale \(\Lambda\) is expected to modify the SM prediction of the anomalous magnetic moment of a lepton \(l\) of mass \(m_l\) by a contribution \(a_{\mu}^{NP} \sim m_l^2/\Lambda^2\). Therefore, given the large factor \(m_\tau^2/m_\mu^2 \sim 283\), the \(g - 2\) of the \(\tau\) lepton is much more sensitive than the muon one to EW and NP loop effects that give contributions \(~ m_l^2\), making its measurement an excellent opportunity to unveil (or just constrain) NP effects.

Another interesting feature can be observed comparing the magnitude of the EW and hadronic contributions to the muon and \(\tau\) lepton \(g - 2\). The EW contribution to the \(g - 2\) of the \(\tau\) is only a factor of seven smaller than the hadronic one, compared to a factor of 45 for the \(g - 2\) of the muon. Also, while the EW contribution to \(a_{\mu}^{SM}\) is only a factor of three larger than the present uncertainty of the hadronic contribution, this factor raises to 10 for the \(\tau\) lepton. If a NP contribution were of the same order of magnitude as the EW one, from a purely theoretical point of view, the \(g - 2\) of the \(\tau\) would provide a much cleaner test of the presence (or absence) of such NP effects than the muon one. Indeed, if this were the case, such a NP contribution to the \(\tau\) lepton \(g - 2\) would be much larger than the hadronic uncertainty, which is currently the limiting factor of the SM prediction.

Unfortunately, the very short lifetime of the \(\tau\) lepton makes it very difficult to determine its anomalous magnetic moment by measuring its spin precession in the magnetic field, like in the muon \(g - 2\) experiment. Instead, experiments focused on high-precision measurements of the \(\tau\) lepton pair production in various high-energy processes, comparing the measured cross sections with the QED predictions. As we can see from Eq. (34), the sensitivity of the best existing measurements is still more than an order of magnitude worse than that required to determine \(a_\tau\).

Nonetheless, the possibility to improve such a measurement is certainly not excluded. For example, it was suggested to determine the \(\tau\) lepton \(g\) factor taking advantage of the radiation amplitude zero which occurs at the high-energy end of the lepton distribution in radiative \(\tau\) decays. This method requires a very good energy resolution and could perhaps be employed at a \(\tau\)-charm or \(B\) factory.
also benefiting from the possibility to collect very high statistics. It is not clear whether the huge data samples at B factories will result in a corresponding gain for the limits on $a_\tau$. Indeed, LEP measurements were rather limited by systematic uncertainties, which were of the order of 2-3% for the discussed processes and, until now, experiments at B factories have not yet reached such a level of accuracy in the absolute measurements of the total cross sections. However, a search for the $\tau$ lepton electric dipole moment at Belle showed that with the appropriate choice of observables, using full information about events, the improvement in sensitivity can be proportional to the square root of luminosity, i.e., determined mainly by statistics. One can hope that this is also the case with the determination of $a_\tau$.

A similar method to study $a_\tau$ using radiative $W$ decays and potentially very high data samples at LHC was suggested in Ref. Yet another method would use the channeling in a bent crystal similarly to the suggestion for the measurement of magnetic moments of short-living baryons. This method has been successfully tested by the E761 collaboration at Fermilab, which measured the magnetic moment of the $\Sigma^+$ hyperon. In the case of the $\tau$ lepton, it was suggested to use the decay $B^+ \rightarrow \tau^+ \nu_\tau$, which would produce polarized $\tau$ leptons. In 1991, when this suggestion was published, the idea seemed completely unlikely. However, in the era of B factories, when the decay $B^+ \rightarrow \tau^+ \nu_\tau$ is already observed by the Belle collaboration and the possibility of a Super-B factory is actively discussed, this is no longer a dream. Even more promising could be the realization of this idea in a dedicated experiment at a hadron collider with its huge number of $B$ mesons produced and a more suitable geometry. We believe that a detailed feasibility study of such an experiment, as well as further attempts to improve the accuracy of the theoretical prediction for $a_\tau$, are quite timely.

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