Towards the Unified Theory of Galactic Bar-Modes

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Abstract

The arguments in favor of the unified formation mechanism for both slow (Lynden-Bell’s) bars and common fast bars are given. This mechanism consists in a certain instability that is akin to the well-known radial orbit instability; it is caused by the mutual attraction and alignment of axes of precessing star orbits (up to now, such a way of formation was considered only for slow bars).

The general theory of the low-frequency modes of a disk consisting of precessing orbits (at different angular velocities) is presented. The problem of determining these modes is reduced to the integral equations of a rather simple structure. The characteristic pattern speeds ($\Omega_p$) of the low-frequency modes are of order of the mean orbit precession speeds ($\bar{\Omega}_{pr}$). The bar-modes also belong to this type of modes. The slow bars have $\Omega_p \approx \bar{\Omega}_{pr}$; for the fast bars, $\Omega_p$ may far exceed even the maximum precessing speed of disk orbits (however, $\Omega_p$ remains to be of order of these precessing speeds). The possibility of such an excess of $\Omega_p$ over $\Omega_{pr}^{\text{max}}$ is connected with the effect of “repelling” orbits that tend to move in the direction opposite to that they are being pushed.

The preliminary analysis of the orbit precession patterns for a number of typical potentials is given. It is noted that the maximum radius of the “attracting” circular orbits ($r_c$) may be used as a reasonable estimate of a bar length.

1 Introduction

It is commonly supposed that a galactic bar can belong to one of two types — either common fast bars or Lynden-Bell slow bars (see, e.g., Sellwood 1993, Polyachenko 1994). The distinction between them is drawn along a number of lines. Firstly, bars with different rotation velocities have considerably different sizes: while the common bars end up at the corotation or 4:1 resonance, the Lynden-Bell bars end up in the vicinity of the inner Lindblad resonance. Secondly, it is believed that these bars are produced by entirely different formation mechanisms. For the slow Lynden-Bell bars, the physical mechanism is absolutely clear — this is the mutual attraction and “sticking” together of the slowly precessing orbits. Note, that in application to galactic bars, the idea of axis alignment was first suggested by Lynden-Bell (1979).

Yet for the fast bars the situation remains vague and debatable. The fast bars were long thought to be generated due to the fast rotation in much the same way as in the classical incompressible Maclaurin spheroids being they are strongly oblate and rapidly rotating. It was shown by Toomre (1981) that in reality the galactic bar-modes have little
in common with incompressible edge modes. Accordingly, they should have another formation mechanism. Toomre (1981) offered the so-called swing amplification mechanism, which currently is generally accepted for the explanation of the normal SA spirals. We think, however, that a simple extension of the Toomre mechanism to the SB galaxies can hardly be done. In particular, it is difficult to suspect the existence of the running spiral waves, necessary for the swing amplification, which takes place outside the bar near the corotation. The specific attempt made by Athanassoula and Sellwood (1986) to demonstrate the validity of the swing amplification mechanism on some some models of stellar disks appears to be somewhat artificial.

In our opinion, it is more preferably to find such a mechanism of the common bar formation, which is directly linked to the instability of the central part of the galactic disk itself.

Contopoulos (1975, 1977) has pointed out that the properties of the families of the periodic orbits in the rotating bar potentials play the fundamental role in the theory of the galactic bars. This is especially true for the so-called $x_1$ family of orbits, which is elongated along the bar inside the corotation circle. It is plausible to assume that these periodic and close non-periodic orbits make up the galactic bar. One should note, however, that the preceding is relevant to the theory of the already existing bars, and in the strict sense cannot be applied to the bar formation mechanism itself.

Nevertheless, from the described picture of orbits constituting the bar, it is customary to make a “reasonable” conclusions about the mechanisms of bar formation, including the possible instability at the linear stage. The fast bar angular velocity $\Omega_p$ is greater (sometimes substantially greater) than the maximum precession velocity $\Omega_{\text{max}}^{\text{pr}}$. Meanwhile, it is intuitively suggested that the angular modulation of the precessing orbit distribution (i.e., a figure of the bar) should rotate with the velocity of about mean precession velocity of the orbits. Such a conclusion would have meant the uselessness of the Lynden-Bell mechanism for describing the fast bar formation. So, it is generally agreed that in reality the growing bar forces the orbits to change their shape and tune to the strengthening and narrowing bar. Moreover, it is assumed that this effect can be significant even at the linear stage. It is justified on the example of weak bars in the framework of the linear theory (Sellwood 1993): the theory shows that initially circular orbits change into slightly elongated ovals, with orientation relative to the bar just as the orbits of the $x_1$-family.

However, one can adduce two important objections as to the said in the last paragraph.

1. The circular orbits cannot be the typical sample orbits of the undisturbed disk (excepting for a cold disk model, which is of little interest in this case) until the bar-induced perturbed velocities do not exceed the velocity dispersion in the original axisymmetric disk. It is clear that this condition should be met first at the initial stage of the instability development.

2. The intuitive conclusion mentioned above (that the velocity of the wave density in the system of precessing orbits, $\Omega_p$, cannot exceed the maximum precession velocity $\Omega_{\text{max}}^{\text{pr}}$) is supported by the results of computations of the unstable modes in numerous models of galactic disks. These computations show that in all cases the pattern speed $\Omega_p$ is smaller than the maximum star angular velocity $\Omega_{\text{max}}$ in the disk\footnote{This is especially true for the linear stage of the bar-forming instability (if the bar arose due to the growth of the instability): the orbits considered by Contopoulos are obviously trapped by the potential of the bar. The trapping process is, of course, the non-linear phenomenon.}. But this analogy

\footnote{Note that this property of the oscillation frequency spectrum of the gravitating disk is not yet found}
appears to be somewhat loosely after the discovering of the possible “donkey” behavior of the star orbits by Lynden-Bell and Kalnajs (1972), when the orbit accelerates when held back and slows down when urged forward. Such a behavior takes place under the Lynden-Bell condition, \( \partial \Omega_{pr}/\partial L \big|_{J_f} < 0 \) (Lynden-Bell 1979), where \( L \) is the star angular momentum, \( J_f = J_r + L/2 \) is the Lynden-Bell adiabatic invariant, \( J_r \) is the radial action. As it is shown below, if the orbits with a “donkey” behavior play an active role in the bar-forming instability, the bar angular velocity can exceed \( \Omega_{pr}^{\text{max}} \).

The last argument allows to treat the fast bar-modes as the corresponding density waves of the precessing orbits, in the full correspondence with the theory of the Lynden-Bell bars. Note that Kalnajs (1973) was first who called attention to the possibility of the freely precessing orbit alignment (and thus the bar formation). The important point in his theory of kinematic waves in the case of near-circular orbits was the independence of the precession velocity \( \Omega_{pr}(r) = \Omega(r) - \kappa(r)/2 \) (\( \Omega(r) \) from radius (here \( \kappa(r) \) is the epicyclic frequency, \( \kappa^2 = 4\Omega^2 + rd\Omega^2/\partial r \)). Note that the same condition \( \Omega_{pr}(r) \approx \text{const} \) is required in the Lynden-Bell (1979) more general consideration of the problem.

In reality different stars precessing with different velocities. But only important point is that the angular precession velocities (and also the pattern speeds of bars (\( \Omega_p \)) in the majority of cases) are substantially smaller than the typical star azimuthal frequencies \( \Omega_2 \) and especially the radial frequencies \( \Omega_1 \). If for some orbit the inequality

\[
|\Omega_p - \Omega_{pr}|/\Omega_1 \ll 1
\]

(1)

holds, this whole orbit (not individual stars) can be considered as an elementary object in the interaction with the bar gravitational field. If the condition (1) holds for the majority of disk orbits, participating in the bar formation, we can consider the processes (e.g., bar-instability) in the model disk, consisting of the set of the precessing stars. As it is noted by Lynden-Bell (1979), the condition (1) means the conservation of the adiabatic invariant \( J_f = J_r + L/2 \), that reduces the problem of the bar-instability to the one dimensional problem: one have to follow only the variation of the azimuthal positions of the orbit major axes under the action of the gravitational attraction from the bar.

Actually, we solve the problem of bar density wave of orbits with different precession velocities, that is quite analogous to more familiar problem of the star density waves (e.g., of the bar-like shape) in the differentially rotating disks. Seemingly, not all the orbits participating in the bar-mode formation, meet the condition (1) with margin, especially for the fast bars. But even for the latter usually \(|\Omega_p - \Omega_{pr}| \sim \Omega_{pr} \sim \omega_G \), where \( \omega_G \sim \sqrt{GM_d/a^3} \) is the characteristic gravitational (Jeans) frequency (\( G \) is the gravitational constant, \( M_d \) is the mass of the active disk, \( a \) is its radius). Then when \( \Omega_{pr}/\Omega_1 \ll 1 \) we are still within the bounds of the condition (1). Even in case of more weak inequality \( \Omega_{pr}/\Omega_1 < 1 \) (e.g., several times smaller but not at some orders) the suggested model will likely provide the correct qualitative answer. This is at any case not worse than, for example, the analysis of the spiral structure of the galaxy M33 by Shu et al. (1971) by using the WKB formulae from the well-known theory of Lin and Shu, applicable, strictly speaking, to the tightly wound multi-turn spirals.

A few words about the content of the paper. In the Section 2, we describe in the general form a model of the precessing orbits and derive the basic equations for the model, useful for the general case.
for the analysis of the disk low frequency modes of interest for us. The use of the action–angle variables $I_1, I_2 = L$ and $w_1, w_2$ is the easiest way for derivation of these equations from the general kinetic equation. The substantial simplification appears with introducing a slow angular variable $\bar{w}_2 = w_2 - w_1/2$ and averaging over the fast angle variable $w_1$ (i.e., over the radial stellar oscillations). In the simplest cases, the problem can be reduced to the analysis of a rather simple dispersion relations. In some more general cases, we obtain the integral equations of different degrees of complexity. But even in the most general form, the obtained integral equations for this model is much simpler that the immense integral equations for the normal modes of the stellar disk, obtained by Kalnajs (1965) and Shu (1970). Remind that the use of these general integral equations (not counting the $N$-body methods) was the only possibility for analysis of large-scale modes, as opposed to incomparably simpler problem of tightly-wound spirals. This is especially true in regard to the bar-mode. As discussed above, the real simplification has come through the analysis of low-frequency modes. Note that the resulting description of gravitating systems is similar to the drift approximation in the plasma physics (see, e.g., Chew et. al., 1956), but for orbits of essentially more general type. In our opinion, the most important advantage of our approach consists in the fact that it makes clear the simple physical mechanisms of the instability processes developing in a disk. Unfortunately, a possibility of revealing these physical mechanisms under the use of the general integral equations by Kalnajs or Shu would be practically impossible, and the same is true for the $N$-body simulations.

In Sec. 3, we analyze the dispersion relation for a model disk consisting of orbits of two different types that differ by their precessing speeds $(\Omega_{pr}^{(1)}$ and $\Omega_{pr}^{(2)}$; $\Omega_{pr}^{(2)} > \Omega_{pr}^{(1)}$) and, generally speaking, also by a sign of the Lynden-Bell derivative $(\partial \Omega_{pr}/\partial L)_\beta \equiv \Omega_{pr}'$. For the case when the derivatives $(\Omega_{pr}')^{(1)}$ and $(\Omega_{pr}')^{(2)}$ have opposite signs (i.e., a disk contains both “attracting” and “repelling” orbits), the pattern speed of the unstable modes $\text{Re} \Omega_p$ may be more than $(\Omega_{pr})^{(2)}$, i.e. the maximum orbit precessing speed in a disk. We show that at the same time $\Omega_{pr}^{(1)} < \text{Re} \Omega_p < \Omega_{pr}^{(1)}$ when $(\Omega_{pr}')^{(1)}$ and $(\Omega_{pr}')^{(2)}$ have the same sign (positive for instability). We consider this result as the important argument in favor of the unified formation mechanism both for slow and fast bars.

In Sec. 4, the results of computation of the precessing speeds $\Omega_{pr}$ and the most important for the theory quantity, the Lynden-Bell derivative $\Omega_{pr}'$, are given for a number of typical potentials. In particular, the regions on the Lynden-Bell $J_f, L$ plane where $\Omega_{pr}' > 0$ and $\Omega_{pr}' < 0$ are found. The natural suggestion is made that a bar forms by “attracting” orbits with $\Omega_{pr}' > 0$. Then the bar length (more exactly, the radius of the bar’s end) should be equal to the maximum of the apogee radii ($r_{\text{max}}$), among the sufficiently occupied orbits from the region where $\Omega_{pr}' > 0$. So, for calculation of $l_b$, one should know not only the general pattern of orbit precessions determined by the potential $\Phi_0(r)$ but a specific equilibrium distribution function as well. As a first approximation, one can take the estimate $l_b \sim r_c$, where $r_c$ is the radius of the circular orbit at which $\Omega_{pr}' = 0$ ($\Omega_{pr}' > 0$ for $r < r_c$, and $\Omega_{pr}' < 0$ for $r > r_c$, at the circular orbits). These radii $r_c$ are computed for all the potentials considered. Note that our determination of the bar length has nothing to do with common determinations that link this length with a location of one of the resonances (CR, ILR or 4:1). It is clear that our bar length may

\[^3\]Note that in the formation process for all the mode, i.e. both the bar and adjacent spirals, and not just the central bar, the “repelling” orbits with $\Omega_{pr}' < 0$ may also take part. Moreover, this may be important to explain the fast bar phenomenon (as discussed above).
take a great variety of values depending on the specific potential and distribution function (accidentally, they may fall near one of resonances).

In Sec. 5, we shortly formulate the most important conclusions and discuss some immediate prospects for a work in this field.

## 2 Basic equations for the low-frequency modes

For derivation of the basic equations of the theory, it is most conveniently to use the action-angle variables \( \mathbf{I} = (I_1, I_2) \) and \( \mathbf{w} = (w_1, w_2) \), which suitably takes account of the double periodicity of stellar motion in the equilibrium potential. Note that \( I_1 = I_r \) (\( I_r \) is the radial action), \( I_2 = L \) (\( L \) is the angular momentum). We start out with the linearized kinetic equation in its usual form (see, e.g., Fridman and Polyachenko 1984):

\[
\frac{\partial f}{\partial t} + \Omega_1 \frac{\partial f}{\partial w_1} + \Omega_2 \frac{\partial f}{\partial w_2} = \frac{\partial f_0}{\partial I_1} \frac{\partial \Phi}{\partial w_1} + \frac{\partial f_0}{\partial I_2} \frac{\partial \Phi}{\partial w_2},
\]

(2)

where \( f_0(\mathbf{I}) \) and \( f_1(\mathbf{I}, \mathbf{w}, t) \) are the unperturbed and perturbed distribution functions, \( \Phi_1 \) is the perturbation of the gravitational potential, \( \Omega_1 \) and \( \Omega_2 \) are the frequencies of the radial and azimuthal oscillations of stars in the equilibrium potential \( \Phi_0(r) \), \( \Omega_i = \partial E(\mathbf{I})/\partial I_i \) (\( E \) is the energy in terms of \( \mathbf{I}, i = 1, 2 \)). The change of variables \( \bar{w}_2 = w_2 - w_1/2, \bar{w}_1 = w_1 \) in (2) yields

\[
\frac{\partial f}{\partial t} + im\Omega_{pr}f + \Omega_1 \frac{\partial f}{\partial w_1} = \frac{\partial f_0}{\partial I_1} \frac{\partial \Phi}{\partial w_1} + im\Phi \left( \frac{\partial f_0}{\partial I_2} - \frac{1}{2} \frac{\partial f_0}{\partial I_1} \right),
\]

(3)

where we have assumed that the perturbations are proportional to \( \exp(im\bar{w}_2) \):

\[
\Phi_1 = \Phi \exp(im\bar{w}_2), \quad f_1 = f \exp(im\bar{w}_2),
\]

\( m \) is the azimuthal index (an even integer), and \( \Omega_{pr}(E, L) = \Omega_2 - \Omega_1/2 \) is the precessing speed of an orbit with energy \( E \) and angular momentum \( L \). If we also transform from the action \( (I_1, I_2) \) to \( (E, L) \) (the equilibrium distribution function is usually given just in the variables \( E, L \)), we obtain the linearized kinetic equation in the form

\[
\frac{\partial F}{\partial t} + im\Omega_{pr}F + \Omega_1 \frac{\partial F}{\partial w_1} = \Omega_1 \frac{\partial F_0}{\partial E} \frac{\partial \Phi}{\partial w_1} + im\Phi \left( \frac{\partial F_0}{\partial L} + \Omega_{pr} \frac{\partial F_0}{\partial E} \right),
\]

(4)

where \( F_0(E, L) = F_0(I_1, I_2) \). Note that in this form, Eq. (4) also holds for a quasi-Coulomb potential \( \Phi_0 \), i.e., one that is due principally to a large central mass. All that is necessary then is to redefine \( \bar{w}_2 \) and \( \Omega_{pr}: \bar{w}_2 = w_2 - w_1, \Omega_{pr} = \Omega_2 - \Omega_1 \).

Then, neglecting self-gravitation of the system and the quadrupole moment of the central object, we have \( \Omega_{pr} = 0 \), as it should be for closed Keplerian orbits. Here the azimuthal index \( m \) can be either odd or even.

As it is shown by Lynden-Bell (1979) (see details in the Introduction) the most adequate variables that should be used instead \( (I_1, I_2) \) or \( (E, L) \) are \( J_f = I_1 + I_2/2 \) and \( L = I_2 \). Changing in (4) to these variables, we obtain the kinetic equation in the most convenient form for studying the low-frequency modes:

\[
\frac{\partial f}{\partial t} + im\Omega_{pr}f + \Omega_1 \frac{\partial f}{\partial w_1} = \frac{\partial F_0}{\partial J_f} \frac{\partial \Phi}{\partial w_1} + im\Phi \frac{\partial F_0}{\partial L},
\]

(5)
where \( F_0(J_f, L) = f_0(I_1, I_2) \) and taken into account that \( \partial f_0/\partial I_2 - \frac{1}{2} \partial f_0/\partial I_1 = \partial F_0/\partial L \).

We assume that the rms deviation of the precession rates about the mean \( \bar{\Omega}_{pr} \), given by \( \Delta \Omega_{pr} = [(\bar{\Omega}_{pr} - \Omega_{pr})^2]^{1/2} \) and the typical gravitational frequency \( \omega_G \) are both small, \( \bar{\Omega}_{pr}, \omega_G \ll \Omega_1 \). Note that \( \omega_G \) is of the order of the mean Jeans frequency of the system: \( \omega_G \sim \sqrt{G M_d/a^3} \), where \( M_d \) and \( a \) is the mass and the radius of the active disk, respectively. The above inequalities can be justified, e.g., if one assume that we are dealing with a system of stars, that have close precession speeds, within a massive halo which, while remaining unperturbed itself, furnishes the dominant contribution to the equilibrium potential \( \Phi_0 \). However, it should be remembered that the role of halo should not be played only by the real massive spherical component of the galaxy. Indeed, since in general the extent to which different groups of orbits are involved into the instability process can differ greatly, one can consider as the first approximation that the active group of stars is immersed in the massive halo of other disk stars. Under these circumstances, then, there may exist a low-frequency mode \( (\propto \exp(-i\bar{\omega}t)) \), with \( \bar{\omega} \equiv \omega - m\bar{\Omega}_{pr} \sim \omega_G, \Delta \Omega_{pr} \) in a coordinate system rotating at angular velocity \( \bar{\Omega}_{pr} \) such that the slow precessional dispersal of orbits is canceled by their mutual gravitational attraction. It would be natural to suppose that if self-gravitation were to win out over the dispersion in orbital precession speeds, an instability should develop that could eventually deform the system (under the influence of the largest-scale growing modes). It is clear, however, that even in a system with essentially radial orbits, this holds true only if the torque that alters the orbital angular momentum of the stars forces their orbital precession speeds to change in the same direction (see the Introduction and, e.g., the formula (19) below).

We use the perturbation theory to derive the desired solution for the low-frequency modes. Let \( F = F^{(1)} + F^{(2)} \), where \( F^{(1)} \) corresponds to the permutational mode obtained from (3) by neglecting terms proportional to \( \Delta \Omega_{pr} \) and \( \Phi \propto G \): \( \omega = 0 \) (or \( \omega = m\bar{\Omega}_{pr} \) when \( \bar{\Omega}_{pr} \neq 0 \)), \( \partial F^{(1)}/\partial w_1 = 0 \), i.e. \( F^{(1)} = F^{(1)}(J_f, L) \) is an as-yet arbitrary function of the integrals of motion, which we will subsequently specify by requiring that the solution of the next approximation be periodic. The equation for \( F^{(2)} \) takes the form

\[
- i m F^{(1)} + i m \Omega_{pr} F^{(1)} + \Omega_1 \frac{\partial F^{(2)}}{\partial w_1} = \frac{\partial F_0}{\partial J_f} \frac{\partial \Phi}{\partial w_1} + i m \Phi \frac{\partial F_0}{\partial L}.
\]

Averaging (6) over \( w_1 \), from 0 to 2\( \pi \), and bearing in mind the periodicity of the functions \( F^{(2)} \) and \( \Phi \), we have

\[
- (\bar{\omega} - m\delta \Omega_{pr}) F^{(1)} \approx \frac{\partial F_0}{\partial L} \frac{1}{2\pi} \int_0^{2\pi} \Phi dw_1,
\]

where \( \delta \Omega_{pr} = \Omega_{pr} - \bar{\Omega}_{pr} \).

Invoking the Poisson equation, some minor manipulations yield

\[
\Phi = -G \int dJ' F^{(1)}(J') \int dw' \Gamma(r, r', \varphi' - \varphi) \exp[i m (\bar{\omega}'_2 - \bar{\omega}_2)],
\]

where \( dJ' = dJ_f' dL' \), \( dw' = dw'_1 dw'_2 \), \( \Gamma \) is the Green function:

\[
\Gamma = \frac{1}{r_{12}^2}, \quad r_{12} = |r^2 + r'^2 - 2rr' \cos(\varphi' - \varphi)|^{1/2}.
\]
Equation (8) is an integral equation for the function $\Phi(J, w_1)$, if one takes into account the expression (7) for $F(1)$ through $\Phi$. The coordinates of stars $r, \varphi, r', \varphi'$ in (8) and (9) must be expressed in terms of $J, J', w, w'$, where:

$$r = r(J, w), \quad r' = r'(J, w_1'), \quad w_2' - w_2 = (w_2' - w_2) - (w_1' - w_1)/2,$$

$$\varphi' - \varphi \equiv \delta \varphi = w_2' - w_2 + \phi(J, J', w_1, w_1')$$

and we refrain from writing out the expression for $\phi$. Since the perturbed potential can always be written out as

$$\Phi_1(r, \varphi) = \bar{\Phi}_1(r) \exp(im\varphi) = \Phi \exp(im\bar{w}_2),$$

we have

$$\Phi(r, \varphi) = \bar{\Phi}_1(r) \exp[im\delta(J, w_1)],$$

($\delta = \varphi - \bar{w}_2$ is a known function of $J$ and $w_1$). In actual fact, then, (8) is an integral equation for the unknown function $\Phi_1(r)$ of only one variable. This equation can be rewritten in more symmetric form. The right-hand side of (8) depends on $\Gamma$ and $\exp[im(w_2' - \bar{w}_2)]$. So, by averaging (8) over $w_1$, we obtain for the function

$$\chi(J) = \bar{\Phi} = \frac{1}{2\pi} \int_0^{2\pi} \Phi dw_1$$

the following integral equation

$$\chi(J) = \frac{1}{2\pi} \int dJ' \Pi(J, J') \frac{\partial F_0(J')/\partial L'}{\omega - m\Delta_{pr}(J')} \chi(J')$$

(10)

where

$$\Pi(J, J') = \int dw_1 dw_1' d\delta w_2 \Gamma(r, r', \delta \varphi) \exp(im\bar{w}_2) \exp[-im(w_1' - w_1)/2],$$

(11)

and $\delta w_2 \equiv w_2' - w_2$.

Physically, $\Pi(J, J')$ is proportional to the torque $\delta M$ acting upon some selected (test) orbit with the action $J$ resulting from all orbits with fixed action $J'$; these all have the same shape, but their major axes are oriented in all possible directions:

$$\delta M \propto -imG \exp(im\bar{w}_2) \Pi(J, J') F^{(1)}(J') dJ'.$$

(12)

For a quasi-Coulomb field $\Phi_0(r)$, instead of the Lynden-Bell integral $J_f = I_1 + I_2/2$ we have to use $J_f' = I_1 + I_2$ in (10) and instead of $\exp[-im(w_1' - w_1)/2]$ (with $m$ required to be even) in (11) we have to use $\exp[-im(w_1' - w_1)]$ (with arbitrary $m$).

One might hope to reduce (10) to one-dimensional integral equations in two limiting cases: 1) when the distribution function $\mathcal{F}_0(J)$ is close to a delta function in $L$ near some value $L_0$; we will be commenting on this circumstance, writing $\mathcal{F}_0 = \Delta_1(J_f, L - L_0) \approx \delta(L - L_0)\varphi_0(J_f)$; 2) for the systems with near-circular orbits, whereupon $f_0 = \Delta_2(I_1, I_2)$, where $\Delta_2 \approx \delta(I_1)\varphi_0(I_2)$. Below we will restrict ourselves mainly with the first case. The second case is technically somewhat more complicated; we will study this important case elsewhere.
Thus we assume \(F_0 = \Delta_1(J_f, L - L_0)\). Using the fact that \(\Pi\) and \(\chi\) vary in \(L'\) only a little over the characteristic scale length of the function \((\partial F_0/\partial L)/[\bar{\omega} - m\delta \Omega_{pr}(J_f, L')])\), we can reduce Eq.\((13)\) to an integral equation for the function of one variable \(\psi(J) \equiv \chi(J, L' = L_0)\); for brevity, hereafter we omit the index “\(f\)” in the Lynden-Bell integral \(J_f\):

\[
\psi(J) = \frac{Gm}{2\pi} \int dJ' P(J, J') S_0(J') \psi(J'),
\]

where

\[
P(J, J') = \Pi(J, J', L = L_0, L' = L_0),
\]

\[
S_0(J) = \frac{dL'}{\partial L} \frac{\partial F_0(J', L')/\partial L'}{[\bar{\omega} - m\delta \Omega_{pr}(J', L')]^2},
\]

More convenient form for the function \(S_0(J')\) is obtained after integration \((14)\) by parts

\[
S_0(J') = -m \int dL' \frac{dL'}{\partial L} \frac{\partial F_0(J', L')/\partial L'}{[\bar{\omega} - m\delta \Omega_{pr}(J', L')]^2},
\]

where the derivative \(\partial \Omega_{pr}/\partial L'\) can be taken at \(L' = L_0\).

If we are dealing with near radial orbits, we can put \(L_0 = 0\) when we calculate \(P(J, J')\). Furthermore, \(\delta \varphi \equiv \varphi' - \varphi \approx \bar{\varphi}'_2 - \bar{\varphi}_2\) for such orbits, so the function \(\Pi\) can then be substantially simplified:

\[
\Pi(J, J') = \int dw_1 dw_1' J_m[r(J, w_1), r(J', w_1')],
\]

where

\[
J_m(r, r') = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \Gamma(r, r', \alpha) \cos m\alpha.
\]

When the orbits are exactly radial (“cold” system), i.e., \(F_0 = \delta(L)\varphi_0(J)\),

\[
S_0(J') = -\frac{m}{\bar{\omega}^2} A(J') \varphi_0(J'),
\]

where

\[
A(J') = \frac{\partial \Omega_{pr}(J', L')}{\partial L'} \bigg|_{L = 0}.
\]

Accordingly, the integral equation \((13)\), which then describes the radial orbit instability in a cold system, is

\[
\psi(J) = -\frac{Gm^2}{2\pi \bar{\omega}^2} \int dJ' P(J, J') A(J') \varphi_0(J') \psi(J').
\]

It can easily be shown that \(J_m(r, r')\) defined by \((13)\) is a positive function. Consequently, so is \(\Pi(J, J')\) from \((17)\), and most importantly, so is \(P(J, J')\) from \((14)\) when \(L_0 = 0\); just the function \(P(J, J')\) enters into the derived equation \((21)\). This can be made explicitly by expanding the function \((r^2 + r'^2 - 2rr' \cos \alpha)^{-1/2}\) in a harmonic series, we obtain

\[
J_m(r, r') = \sum_{n=0}^\infty F_n(r, r') \frac{1}{\pi} \int_0^\pi d\alpha P_n(\cos \alpha) \cos m\alpha,
\]
where

\[ F_n(r, r') = r^n/r^{n+1}, \quad r_\leq \equiv \min(r, r'), \quad r_\geq \equiv \max(r, r'), \]

and the \( P_n \) are Legendre polynomials. Now the positive definiteness of \( J_m \) follows from the fact that \( P_n(\cos \alpha) \) can in turn be expanded in cosines of multiples of the angle, with positive coefficients (Gradshtein, Ryzhik 1965):

\[ P_n(\cos \alpha) = \frac{(2n-1)!!}{2^{n-1}n!} \left[ \cos n\alpha + \frac{1}{2n-1} \cos(n-2)\alpha + \ldots \right] \equiv \sum_k' A_k^{(n)}, \cos k\alpha \]

with all \( A_k^{(n)} > 0 \) (a prime indicates that the parity of \( k \) and \( n \) must be the same), and from

\[ \frac{1}{\pi} \int_0^\pi d\alpha \cos m\alpha \cos n\alpha = \frac{\delta_{mn}}{2}. \]

We finally obtain a convenient representation for \( J_m(r, r') \) in simple series form:

\[ J_m(r, r') = \frac{1}{2} \sum_{n \geq m}' F_n(r, r') A_m^{(n)} > 0. \]

Given the positivity of \( P(J, J') \), the sign of the integrand in (21), and therefore the sign of \( \omega^2 \) (i.e., the stability or instability of a system with purely radial orbits), will depend on the sign of \( A(J') \) as defined by (19). If \( A > 0 \) for all orbits in the system under consideration (in other words, for all values of \( J \) or what amounts to the same thing in the present case, for any value of the energy \( E \) of radial stellar oscillations), then \( \omega^2 < 0 \). As a result, radial orbits are unstable with \( A > 0 \). On the other hand, provided that \( A < 0 \), it is the purely oscillatory mode. The most compact formula for computing \( A(E) \) is

\[
A(E) = \frac{1}{(2E)^{1/2}} \lim_{r_0 \to 0} \left\{ \frac{d}{r_{\max}} \int_{r_0}^{r_{\max}} \frac{dx}{x^2[1 - \Phi_0(x)/E]^{1/2}} - \frac{1}{r_0} \right\} \int_{0}^{r_{\max}} \frac{dx}{[2E - 2\Phi_0(x)]^{1/2}}
\]

The inequality \( (\partial Q_{pr}/\partial L)|_{L=0} > 0 \) is merely a necessary (and in no way sufficient) condition for the radial orbit instability, and in particular, for the formation of a bar. The insufficiency of this criterion is immediately obvious from the fact that the retarding torque due to the bar can turn out to be ineffectual in the face of large orbital precession speeds. To derive valid conditions for bar formation, it is necessary to solve the problem of stabilization of the radial orbit instability by some finite dispersion of the orbit precession speeds [having demonstrated once again the predominance of the bar mode (\( m = 2 \))]. The bar formation criterion is in fact none other than the condition for the bar-mode instability of the type under consideration. We therefore now proceed to derive the stabilization conditions for systems with near-radial orbits.

For definiteness, we assume the Maxwellian distribution in \( L \)

\[
f_0 = \frac{1}{\pi^{1/2}L_T} \exp(-L^2/L_T^2)\varphi_0(J), \quad (23)
\]
where $L_T$ is the thermal spread. If we then assume in (13) that $\bar{\omega} = \omega = 0$, $L_0 = 0$, $\delta \Omega_{pr} = \Omega_{pr} \approx A(J)L$, we will then have for the system’s stability boundary

$$S_0(J') = 2\varphi_0(J')/mL^2A(J')$$

so that the integral equation (13) becomes

$$\psi(J) = \frac{G}{\pi L^2} \int dJ' P(J, J') \frac{\varphi_0(J')}{A(J')} \psi(J'). \quad (24)$$

This equation is almost the same as Eq. (21). A comparison of these two equations suggests a simple relationship between the instability growth rate $\gamma$ for a system with purely radial orbits, $\gamma^2 = -\omega^2$, and the minimum dispersion in orbital angular momentum required to suppress that instability:

$$(L_T)_{\text{min}} = \frac{1}{2} \frac{1}{\gamma/m} \bar{A}, \quad (25)$$

where $\bar{A}$ is some mean over the stellar orbits with different energies $E$.

The relation (25) acquires a precisely defined meaning when all stars have almost the same energy, $E \approx E_0$, since we can then take $\bar{A} = A(E_0)$. If in (23) we go to the distribution over precession speeds, $\Omega_{pr} = AL$, we then obtain a more obvious relation in place of (25):

$$(\Omega_{pr})_T = \frac{1}{2} \frac{1}{\gamma/m}, \quad (26)$$

where $(\Omega_{pr})_T$ denotes the thermal spread in precession speeds, and the growth rate $\gamma$ is given in the form $\gamma(m)$ to emphasize that in general it depends on the azimuthal index $m$. Since $\gamma(m)$ is only a weak function of $m$,$^4$ it follows from (26) that the most difficult modes to stabilize (and in that sense, the most unstable) are those with the smallest possible $m$. For almost radial orbits, we have $m_{\text{min}} = 2$, which corresponds precisely to formation of an elliptical bar out of an initially circular disk. All the modes with odd $m$, particularly the $m = 1$ mode, are suppressed in this case, as two oppositely directed (but equal) moments of forces would act on the two halves of an elongated orbit. The forces break, but do not rotate such “needle” orbits.

For near-circular orbits in a potential close to that produced by a central point mass, however, the $m = 1$ mode is immediately become dominating.

In conclusion to this Section, we give the integral equation of a type (13) in a form, suitable for computation of the eigen frequencies of the low-frequency modes of a stellar disk with the equilibrium distribution function $f_0(E, L)$, provided that a disk contains substantial fraction of elongated orbits:

$$f(E_1) = \int_{E_\text{min}}^{E_\text{max}} K(E_1, E_2)f(E_2) dE_2, \quad (27)$$

where the kernel is equal to

$$K(E_1, E_2) = -\frac{\pi}{M_1(E_1)} \int_0^{L_\text{max}} f_0(E_1, L_1) \left(\Omega_0 - \Omega_{pr}^{(1)}\right)^2 \Omega_{pr}(E_1, L_1) dL_1 \int_0^a \int_0^a dx dy \rho(E_1)(x)\rho(E_2)(y) J_m(x, y), \quad (28)$$

$^4$For example, we have $\gamma(m) \propto m^{1/2}$ for $m \gg 1$. Then, the equation (13) is simplified since $J_m \approx 2\delta(r - r')/m$ for $m \gg 1$. Thus, one of the integrations in (17) can be performed. The asymptotic expression for $J_m$ is most conveniently derived directly from Poisson’s equation, assuming that $m^2\Phi_1/r^2 \gg |d^2\Phi_1/dr^2|$, $|r^{-1}d\Phi_1/dr|$ in the latter.
Such a form of the integral equation suggests that the torque of forces produced by attraction of two elongated orbits can be approximated by considering each real oval precessing orbit as a “needle”, which coincides with a major oval axis; the linear density of a needle is \( \rho_l(E) = 1/v_r(E) = 1/\sqrt{2E - 2\Phi_0(r)} \), \( E \) is the star energy, \( v_r \) is the star radial velocity, \( M_1(E) = \int_0^a \rho_l(E)(r)dr \) is the half of needle mass, \( 2a \) is the needle length, and

\[
\Omega'_p(E, L) = \left. \frac{\partial \Omega_{pp}}{\partial L} \right|_{J_f} = \frac{\partial \Omega_{pp}}{\partial L} + \Omega_{pp} \frac{\partial \Omega_{pp}}{\partial E}.
\]

By using the equation (27) Polyachenko (1992) had computed the “abnormally” low-frequency bar-modes, which were found by Athanassoula and Sellwood (1986) in their \( N \)-body study of the linear stability of some exact phase models of stellar disks. Those frequencies were “abnormally” low compared to the frequencies of “standard” fast bars, obtained by them for the majority of the models studied. Actually, the pattern speeds of the low-frequency modes are approximately equal to mean orbit precession speeds in a central disk region. Hence it follows that the instability of elongated orbits occurs. Figs. [a, a′] show a typical orbit participating in the slow bar-mode instability; this orbit corresponds to mean values of energy and angular momentum of stars over the region of the bar location. As one can see, the orbit is strongly elongated; this fact justifies the use of the equation (27) with the kernel (28). The computed growth rates (Polyachenko 1992) are in good agreement with those obtained in the paper by Athanassoula and Sellwood (1986). On the other side, Figs. [fig1b, b′] show the analogous typical orbit for the models in which only the fast bar mode developed. This orbit is substantially more round, than the orbit in Figs. [a, a′]. So, for obtaining the low-frequency eigen modes in such models, the use of the general integral equations (3) or (11) seems to be more adequate. However, we think that even the integral equation (27), being certainly a rough approximation for these models, can provide a satisfactory numerical agreement with \( N \)-body results of Athanassoula and Sellwood (1986). We plan to study all these problems elsewhere.

3 Bar-mode in the model two-component disk

Let us represent the dispersion relation that can be derived from (27) for “one-component” system with the equilibrium distribution function \( f_0 = A\delta(E - E_0^{(1)})\delta(L - L_0^{(1)}) \), in the following form:

\[
1 + \frac{g_1}{(\Omega_p - \Omega_1)^2} = 0,
\]

where \( \Omega_1 \equiv \Omega_{pp}(E_0^{(1)}, L_0^{(1)}) \) and \( g_1 \propto \Omega'_p(E_0^{(1)}, L_0^{(1)}) \) denotes the corresponding coefficient; its explicit expression can be easily obtained from (27) after substituting a given \( \delta \)-distribution function. In (29), the instability corresponds to \( g_1 < 0 \).

Below we consider the case of the two-component system, with the distribution function \( f_0 = A\delta(E - E_0^{(1)})\delta(L - L_0^{(1)}) + B\delta(E - E_0^{(2)})\delta(L - L_0^{(2)}) \). Then some simple manipulations with the equation that is obtained after substituting this distribution function into (27) lead to the dispersion relation:

\[
1 + \frac{g_1}{(\Omega_p - \Omega_1)^2} + \frac{g_2}{(\Omega_p - \Omega_2)^2} + \alpha \frac{g_1 g_2}{(\Omega_p - \Omega_1)^2(\Omega_p - \Omega_2)^2} = 0,
\]
Figure 1: Typical orbits of stars in the Shuster potentials for the models studied by Athanassoula and Sellwood (1986): $a$ – models with the smallest mean precession speeds and angular momenta; $a'$ – the same as $a$ but in the reference frame rotating with the precessing orbit; $b$ – majority of models; $b'$ – the same as $b$ but in the reference frame rotating with the precessing orbit.

where $g_2 \propto \Omega_{pr}(E_0^{(2)}, L_0^{(2)})$ is analogous to the coefficient $g_2$ for the first component introduced earlier, $\Omega_2 \equiv \Omega_{pr}(E_0^{(2)}, L_0^{(2)})$, and the designations are used:

$$\alpha = 1 - \frac{I_0^2(E_0^{(1)}, E_0^{(2)})}{I_0(E_0^{(1)}, E_0^{(1)})I_0(E_0^{(2)}, E_0^{(2)})}$$  \hspace{1cm} (31)$$

$$I_0(E_0^{(i)}, E_0^{(j)}) \equiv \int_0^{a_i} \int_0^{a_j} dx \, dy \, \rho^{(E_i)}(x) \rho^{(E_j)}(y) J_m(x, y),$$  \hspace{1cm} (32)$$

It turns out that the possible locations of $\text{Re} \, \Omega_p$ for the unstable roots $(\gamma = \text{Im} \, \Omega_p > 0)$ essentially depends on the signs of the coefficients $g_1$ and $g_2$.

Let us prove first of all that for positive $g_1$ and $g_2$ the inequalities $\Omega_1 < \text{Re} \, \Omega_p < \Omega_2$ occur. To do this, it is sufficient to calculate the imaginary part of the left side of the dispersion relation (23). We have

$$A_{1,2} \equiv \text{Im} \frac{g_{1,2}}{(\Omega_p - \Omega_{1,2})^2} = -\frac{2g_{1,2}\gamma \Delta_{1,2}}{(\Delta_{1,2}^2 + \gamma^2)^2}; \quad \Delta_{1,2} \equiv \text{Re} \, \Omega_p - \Omega_{1,2};$$

$$B \equiv \text{Im} \frac{\alpha g_1 g_2}{(\Omega_p - \Omega_1)^2(\Omega_p - \Omega_2)^2} = -\frac{2g_1 g_2 \alpha \gamma}{(\Delta_1^2 + \gamma^2)^2(\Delta_2^2 + \gamma^2)^2}[(\Delta_2^2 + \gamma^2)\Delta_1 + (\Delta_1^2 + \gamma^2)\Delta_2],$$

Assuming $\text{Re} \, \Omega_p > \Omega_2$ we have $\Delta_1 > 0, \Delta_2 > 0$. So in this case the imaginary part of the left side of the dispersion relation (23) would be negative:

$$A_1 + A_2 + B < 0,$$
Figure 2: The trajectories of the unstable root in the two-component model with opposite signs of $g_1 < 0$ and $g_2 > 0$, for a number of values of the coefficient $\alpha = 0, 0.2, \ldots, 0.7$. The absolute value of $g_1$ increases along the trajectories from the starting point $g_1 = 0$.

if one takes into account that $\gamma > 0$ for the unstable roots of interest and, besides, $\alpha > 0$ as it follows from the Cauchy–Bunyakovsky inequality (positive definiteness of the weight function $J_2(x, y)$ was proved in the preceding section).

Similarly, for $\text{Re}\, \Omega_p < \Omega_1$ all inequalities are reversed (of course, except for $\alpha > 0$): $\Delta_1 < 0, \Delta_2 < 0$, so

$$A_1 + A_2 + B > 0.$$

A completely different type of situation occurs for the case when the signs of $g_1$ and $g_2$ are opposite, i.e. the disk contains the orbits with a “donkey” behavior; for definiteness, we assume that $g_2 > 0$, $g_1 < 0$. Fig. 2 show the trajectories of motion of the unstable root on the complex plane $\Omega_p$, for a fixed $g_2 = 0.05^2 > 0$ ($\Omega_1 = 0$ and $\Omega_2 = 0.25$ are also fixed) and negative $g_1$ that vary along each trajectory from $g_1 = 0$ to some $(g_1)_{\min}$, at which this root become stable too. Different trajectories correspond to different values of the parameter $\alpha$. The most important fact following from these calculations is that the excess of $\text{Re}\, \Omega_p$ over $\Omega_2$ can be quite significant (about 1.5 times in a given example).

4 The patterns of orbit precessions in some typical potentials

Fig. 3a–6a show, on the Lynden-Bell plane ($J_f, L$), the constant value curves for the derivative $\Omega'_{pr} \equiv (\partial \Omega_{pr}/\partial L)_{J_f}$, for a number of the commonly occurring potentials $\Phi_0(r)$. In the parallel Figs. 3b–6b, we give, for the same potentials, the angular velocities of stars at the circular orbits $\Omega(r)$, the rotation curves $V_0(r) = r\Omega(r)$ and the precession speeds of the nearly-circular orbits $\Omega_{pr}(r) = \Omega(r) - \kappa(r)/2$ ($\kappa(r)$ is the epicyclic frequency).
Figure 3:  
a – the pattern of orbit precessions for the isochrone model at the Lynden-Bell $(J_f, L)$-plane. The straight line $J_f = L/2$ corresponds to circular orbits. The curves are the isolines for the Lynden-Bell derivative $\Omega'_\text{pr}$. 
b – the angular disk velocity $\Omega(r)$, the rotation curve $V_0(r)$ and the circular orbit precession speed $\Omega_{\text{pr}}(r)$ for the isochrone model.

The first pair of these figures (Figs. 3a, b) correspond to the isochrone potential

$$\Phi_0(r) = -\frac{1}{1 + \sqrt{1 + r^2}}.$$ 

This case was earlier considered by Lynden-Bell (1979). For this potential, the frequencies $\Omega_1(J_f, L)$, $\Omega_2(J_f, L)$ as well as the quantities of interest $\Omega_{\text{pr}}(J_f, L) = \Omega_2(J_f, L) - \Omega_1(J_f, L)/2$ and $\Omega'_{\text{pr}}(J_f, L)$ can be obtained analytically. We used the example of the isochrone potential as a test model for our general scheme of computation of $\Omega_{\text{pr}}$ and $\Omega'_{\text{pr}}$ for the arbitrary potential $\Phi_0(r)$.

Apart from the isochrone model, we carried out the computations for the Shuster potential

$$\Phi_0(r) = -\frac{1}{\sqrt{1 + r^2}}$$

(Figs. 3a, b), the logarithmic potential $\Phi_0(r) = \ln r$, corresponding to the flat rotation curve $V_0 = \text{const}$ (Figs. 3a, b), and for the potential of the exponential disk,

$$\Phi_0(r) = rI_1(r/2)K_0(r/2),$$
where $I_1$ and $K_0$ are the corresponding Bessel functions (Figs. 3a, b). Qualitatively, Figs. 3a, 4a, 6a and 3b, 4b, 6b are similar, but they differ greatly from Figs. 3a, b for the logarithmic potential (and it is not unreasonable). In our opinion, the most interesting information that one can extract from Figs. 3a, 4a, 6a are the critical values of the angular momentum ($L_c$) that separate the regions of “attracting” (when $L < L_c$) and “repelling” (when $L > L_c$) near-circular orbits. The critical radii ($r_c$) corresponding to these values of $L_c$ are shown in 3b, 4b, 6b. It is natural to take $r_c$ as an estimate for a bar length $l_b$ that forms as a result of the bar-instability under consideration (at least at the linear stage). As one can see, $r_c$ is always more than the radii corresponding to maxima of the rotation curve $V_0(r)$ or the function $\Omega_{pr}(r)$. Under the condition commonly used that a fast bar ends near the corotation, we obtain from $l_b \sim r_c$ such an estimate of the bar pattern speed, for the Shuster potential: $\Omega_p \approx 0.23$ (see Fig. 4b). The corresponding eigen frequency $\omega = 2\Omega_p \approx 0.46$ is typical for the majority of models studied by Athanassoula and Sellwood (1986), just for the Shuster potential. Note that these bars are traditionally considered as “fast” ones, keeping in mind their non-Lynden-Bell formation mechanism. We see, however, that the fast bars can likely be formed by the same mechanism as for the slow bars. Let us point out some other interesting regularities that are common for the patterns of the orbit precessions in potentials $\Phi_0(r)$ of a type corresponding to Figs. 3a, 4a, 6a.
1. The precession speed \( \Omega_{\text{pr}}^{\text{max}} \) corresponding to the maximum of the curve \( \Omega_{\text{pr}}(r) \) (at the circular orbits) is the absolute maximum for the precession speeds \( \Omega_{\text{pr}}(J_f, L) \) of arbitrary orbits; \( \Omega_{\text{pr}}^{\text{max}} \) for three models are given in the first line of the table.

| Model       | Isochrone | Shuster | exp disk |
|-------------|-----------|---------|----------|
| \( \Omega_{\text{pr}}^{\text{max}} \) | 0.058     | 0.13    | 0.087    |
| \( r_c \)   | 3.7       | 2.4     | 4.3      |
| \( \Omega_{\text{pr}}^{\text{max}} \) | 0.3 ÷ 0.35| 0.4 ÷ 0.45| 0.35    |
| \( \Omega_{\text{pr}}^{\text{min}} \) | −0.0105   | −0.021  | −0.025   |

2. \( \Omega_{\text{pr}}^{\text{max}} \) are approached at the circular orbits in the disk center \( (J_f \rightarrow J_r \rightarrow 0, L \rightarrow 0) \); the values of \( \Omega_{\text{pr}}^{\text{max}} \) for three models are given in the third line of the table.

3. \( \Omega_{\text{pr}}^{\text{min}} \) are approached in some points at the circular orbits; the values of \( \Omega_{\text{pr}}^{\text{min}} \) are given in the fourth line of the table.

The pattern of the orbit precession in the logarithmic potential (typical for the major parts of many spiral galaxies) is entirely different from the other cases considered (see Fig. 5a). The most interesting fact here is that \( \Omega_{\text{pr}} < 0 \) only within a rather narrow sector of the \( (J_f, L) \)-plane, adjacent to the line of circular orbits \( J_f = L/2 \).

The masses of the values of functions \( \Omega_{\text{pr}}(J_f, L) \), \( \Omega_{\text{pr}}^{\text{max}}(J_f, L) \) obtained above will be
5 Conclusion

Let us formulate and discuss some conclusions from the theory above.

1. We advanced a number of arguments in favor of universality of the mechanism that may be responsible for formation of both the slow and fast bars. The essence of this mechanism is naturally formulated on the basis of representing a stellar disk as a set of precessing orbits. Such a concept is adequate to the problem under consideration since the bar pattern speeds (including “fast bars”) are significantly less than the characteristic frequencies of oscillations of individual stars ($\Omega_1$ and $\Omega_2$), but they are just of order of the orbit precession speeds ($\Omega_{pr}$). In such a disk, we seek the unstable normal modes (first of all the bar-mode) as the density wave of precessing orbits that runs with some speed $\Omega_{pr}$ without any deformations despite the fact that different orbits precess with different speeds (analogously to differentiability of a disk rotation when the usual concept of a galactic disk as a set of individual stars is used).

The unstable bar-mode forms if a central region of a disk (a location of a future bar) contains a sufficiently massive group of “attracting” orbits that satisfy the Lynden-Bell...
condition $\Omega'_\text{pr} > 0$, and, besides, the precession speed dispersion of these active orbits are not too large (otherwise, the orbits will run away from the region of perturbation under the influence of the “thermal” motion). The last condition is natural for the Jeans nature of the instability under consideration. The exact criteria of instability can be obtained by solution of the basic equations derived in Sec. 2; for some simple cases, the corresponding dispersion relations are given in an explicit form. The pattern speed of a bar $\Omega_p$ depends significantly on the extent to which the “repelling” orbits with $\Omega'_\text{pr} < 0$ (the orbits with a “donkey” behavior) take part in the bar-formation process. If such orbits are hardly dragged in the bar-formation process, then $\Omega_p \approx \bar{\Omega}_\text{pr}$; just such bars would naturally be named as the “slow” bars. But if the role of the “repelling” orbits is essential, we can obtain the “fast” bars with $\Omega_p$ that significantly exceeds $\bar{\Omega}_\text{pr}$ (just the same, $\Omega_p$ should be of order of $\bar{\Omega}_\text{pr}$ as before — if we want to remain within the framework of our theory). In Sec. 3, such a possibility is demonstrated on the simplest example of a two-component disk model. Thus, from the point of view under consideration, distinctions between the slow and fast bars are mainly quantitative, but they do not differ fundamentally from each other: both bars form under the action of the same physical mechanism. It is worth noting that the Jeans mechanism (including one under consideration) is always best natural and suited to the gravitational problems.

2. So far the theory above was confirmed only by the calculations of the lowest-frequency modes of the disk models in the Shuster potential when the results are compared with the $N$-body results by Athanassoula & Sellwood (1986). These modes obviously correspond to the slow bars: for them, $\Omega_p \approx \bar{\Omega}_\text{pr}$. The most interesting prediction of the theory (its validity for the fast bars) would be tested if we shall be able to prove that the eigen frequencies of the modes computed from our integral equations (see Sec. 2) coincide with the frequencies of the fast bars derived from the $N$-body simulations (in particular, with the majority of bar-modes from the paper by Athanassoula & Sellwood cited above. This problem will be the subject of study in the immediate future. In the present paper, we restricted ourselves only to some positive facts that result from the general analysis of orbit precessions for a number of potentials (Sec. 4). These facts correlate with a possibility of the fast bar formation by alignment of “attracting” orbits. First of all we noted that the maximum radius of nearly-circular orbits ($r_c$) with $\Omega'_\text{pr} > 0$ ($r_c$ may be taken as a natural estimate for a bar end $l_b$) is significantly more than a size ($\sim r_m$) of the solid rotation region, for all the reasonable models. Moreover, if one takes (as it usually does) that $l_b \sim r_{CR}$ (where $r_{CR}$ is the corotation radius), then for $l_b \sim r_c$ one can obtain the estimate $\omega \approx 0.46$ for a typical eigen frequency $\omega$ of bar-modes in the Shuster potential; this estimate is in good agreement with the corresponding results of Athanassoula & Sellwood (1986).

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