Projection Neural Network for a Class of Sparse Regression Problems with Cardinality Penalty*

Wenjing Li\textsuperscript{a}, Wei Bian\textsuperscript{a,b,*}

\textsuperscript{a}School of Mathematics, Harbin Institute of Technology, Harbin 150001, China
\textsuperscript{b}Institute of Advanced Study in Mathematics, Harbin Institute of Technology, Harbin 150001, China

ARTICLE INFO

Keywords:
Sparse regression
Cardinality penalty
Discontinuity
Smoothing function
Neural network
Local minimizer

ABSTRACT

In this paper, we consider a class of sparse regression problems, whose objective function is the summation of a convex loss function and a cardinality penalty. By constructing a smoothing function for the cardinality function, we propose a projection neural network and design a correction method for solving this problem. The solution of the proposed neural network is unique, global existent, bounded and globally Lipschitz continuous. Besides, we prove that all accumulation points of the proposed neural network have a common support set and a unified lower bound for the nonzero entries. Combining the proposed neural network with the correction method, any corrected accumulation point is a local minimizer of the considered sparse regression problem. Moreover, we analyze the equivalence on the local minimizers between the considered sparse regression problem and another sparse regression problem. Finally, some numerical experiments are provided to show the efficiency of the proposed neural network in solving some sparse regression problems.

1. Introduction

Sparse regression problem is a core problem in many engineering and scientific fields, such as compressed sensing [1], high-dimensional statistical learning [2], variable selection [3], imaging decomposition [4]. Sparse regression problem is to find the sparsest solution of a linear or nonlinear regression model. One typical example of sparse regression problem is to find the sparsest solution of the following linear system

\[ b = Ax + \epsilon, \]

where \( A \in \mathbb{R}^{m \times n} \) is the design matrix, \( b \in \mathbb{R}^m \) is the response vector, \( \epsilon \in \mathbb{R}^m \) is the noise vector. Best subset selection and Lasso are popular methods for selection and estimation of the parameters in a linear regression model under squared-error loss [5]. Cardinality function on \( \mathbb{R}^n \) is also called the \( L_0 \)-norm and denoted by \( \| \cdot \|_0 \). For \( x \in \mathbb{R}^n \),

\[ \| x \|_0 := \{| i = 1, 2, \ldots, n : x_i \neq 0 \}|, \]

where \( |S| \) is the cardinality of set \( S \). \( x \in \mathbb{R}^n \) is called sparse if \( \| x \|_0 \ll n \). Cardinality function is an effective concept for controlling the sparsity of data and plays an important role in sparse regression problems [6], since it penalizes the number of nonzero elements directly and can increase the accurate identification rate of the estimator on the important predictors [7]. However, it is known that the sparse regression problems with cardinality penalty are NP-hard in general [8]. So, the development of algorithms for solving this kind of sparse regression problem is still a challenge up to now.

When the prior about the noise or sparsity of the solution is unknown, the sparse regression problem with cardinality penalty is usually formulated as the following form [4]

\[ \min_{x \in \mathbb{R}^n} \frac{1}{2} \| Ax - b \|^2 + \lambda \| x \|_0, \]

where \( \lambda > 0 \) is a hyperparameter characterizing the trade-off between data fidelity and sparsity.

For solving a wider class of sparse regression problems, we consider the following sparse regression problem with cardinality penalty

\[ \begin{aligned}
\min_{x \in \mathbb{R}^n} & \quad f(x) + \lambda \| x \|_0 \\
\text{s.t.} & \quad x \in \mathcal{X} := \{ x \in \mathbb{R}^n : 0 \leq x \leq v \},
\end{aligned} \tag{1} \]

where \( v \in \mathbb{R}^+_n, \lambda > 0, f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a continuously differentiable convex function and \( \nabla f \) is locally Lipschitz continuous. In (1), \( f \) is the loss function to guarantee the match of the data fitting and \( \lambda \| x \|_0 \) is the penalty to promote the sparsity of the solution. The considered problem (1) is a class of nonconvex and discontinuous optimization problems.

Different from the existing methods for solving the relaxation problems of the cardinality penalty problems, such as \( L_p(0 < p < 1) \)-penalized and \( L_1 \)-penalized sparse regression problems [9, 10, 11], this paper focuses on the original cardinality penalty problem. Directly solving the regression problems with cardinality penalty is an interesting topic. At first, Mohimani, Babaie-Zadeh and Juttena proposed a fast algorithm for overcomplete sparse decomposition based on smoothed \( L_0 \) norm [12]. Further, for \( L_0 \)-penalized least-square problems, Jiao, Jin and Lu developed a primal dual active set with continuation (PDASC) algorithm [13]. Based on DC programming and algorithms, Le Thia et al. offered a unifying nonconvex approximation approach to solve sparse regression problems with finite DC loss function and cardinality penalty [6]. Recently, a smoothing proximal gradient

*This work is supported in part by the National Natural Science Foundation of China under Grant 11871178 and Grant 61773136.

*Corresponding author

\textsuperscript{a}wenjingli@163.com (W. Li); biameilvese52009163.com (W. Bian)

ORCID(s): 0000-0002-4551-6360 (W. Li); 0000-0003-4252-847X (W. Bian)
algorithm is proposed for nonsmooth convex regression with cardinality penalty [14]. Moreover, there are also some recent research progress in [15, 16, 17]. The above algorithms for solving the regression problems with cardinality penalty are all iterative algorithms and to the best of our knowledge, there is no neural network based on circuit implementation to solve such problems so far.

In scientific and engineering fields, real-time solving is necessary for some optimization problems, so neural networks have been studied gradually. Dynamic algorithms and iterative algorithms are two important kinds of algorithms for solving optimization problems and they can promote each other [18, 19]. Dynamic algorithms have some particular advantages. For instance, it is not necessary to select the search direction or step size in implementation, and some differential equations have a promoting effect on the acceleration of iterative algorithms [20, 21, 22]. Neural networks modeled by a class of dynamical algorithms not only own the advantages of dynamic methods mentioned above, but also own their particular superiorities. Neural network can be implemented physically by hardware such as integrated circuits and hence it can solve optimization problems faster in running time at the order of magnitude [23, 24]. Some classical neural networks were designed to solve the linear and nonlinear programming in Hopfield and Tank [25], Tank and Hopfield [26] and Kennedy and Chua [27]. Invoked by these work, many researchers developed different neural networks for solving various optimization problems. There are many interesting results on hardware implementation of neural networks [28, 29] and solving the continuous convex and nonconvex optimization problems by neural networks [30, 31, 32, 33]. Sparse regression problem (1) is a class of discontinuous and nonconvex optimization problems and the existing neural networks cannot be directly used to solve such problem. As a result, it is necessary to design neural networks modeled by differential equations to solve problem (1), which can also further extend the study of neural network for solving the discontinuous and nonconvex optimization problems.

In order to overcome the discontinuity of cardinality penalty, some researchers have designed some continuous nonconvex penalties to relax it, such as the truncated $L_1$ penalty [34], hard thresholding penalty [35], bridge $L_q(0 < p < 1)$ penalty [36], capped-$L_1$ penalty [37], smoothly clipped absolute deviation (SCAD) penalty [38], minimax concave penalty (MCP) [39], continuous exact $L_0$ penalty (CEL0) [4], etc. Among them, $L_{1/2}$ quasi-norm is an important regularization term in the study of compressive sensing [40]. However, for the theoretical analysis of projection neural networks, most of the above penalties are not applicable because of non-differentiability, non-Lipschitz continuity, parameter selection and approximation properties. For the detailed, we will give it in Remark 2. In recent years, based on smoothing techniques, neural networks can gradually be used to solve non-Lipschitz and nonconvex optimization problems [41, 42, 43]. Inspired by smoothing techniques, according to special geometric properties of cardinality penalty and the smoothness requirements of projection neural networks for theoretical analysis, we design a smoothing function of cardinality penalty in this paper, which is also a novel non-convex relaxation of cardinality penalty.

Different from the penalty method for the constrained optimization problems, projection neural network can reduce the complexity of circuit components for solving problem (1). Then, by designing a smoothing function of cardinality penalty, we will propose a projection neural network and a correction method for solving sparse regression problem (1). The main contributions of this paper are as follows.

- The considered problem (1) is a class of sparse regression problems with the desirable sparsity penalty $||.||_0$, which is a more general form of [4, 44] and different from its some relaxation problems in [9, 10, 11]. Under squared-error loss, the equivalence of these relaxation problems to (1) needs some conditions [1, 36, 45]. And the relationships are not clear when $f$ is a general convex function. In this paper, we consider the problem (1) directly. Though we solve it by a smoothing approximation model of it, we prove the optimality properties of its accumulation points to (1).

- A smoothing function of cardinality penalty is designed, which is a nonconvex relaxation of cardinality function but more appropriate for the analysis of neural network than the existing relaxation penalties in [4, 34, 35, 36, 37, 38, 39].

- A projection neural network with smoothing technique is proposed for solving the discontinuous and nonconvex optimization problem (1), which is the first differential equation system for solving (1) to the best of our knowledge, and whose analysis is different from the existing neural networks for solving the continuous optimization problems.

- In order to expand the applicable range of the proposed neural network, we further consider the following sparse regression model

\[
\begin{align*}
\min & \quad f(x) + \lambda \|x\|_0 \\
\text{s.t.} & \quad x \in \mathcal{Y} := \{x \in \mathbb{R}^n : -l \leq x \leq u\},
\end{align*}
\]

where $u, l \in \mathbb{R}^n_+$, $\lambda > 0$ and $f$ is defined as in (1). Using variable splitting $x = x_+ - x_-$, we prove that in the sense of local minimizers, problem (2) can be equivalently converted to the following special case of sparse regression problem (1)

\[
\begin{align*}
\min & \quad f(x_+ - x_-) + \lambda \|x_+\|_0 + \lambda \|x_-\|_0 \\
\text{s.t.} & \quad x_+ \in \mathcal{A}_1 := \{x \in \mathbb{R}^n : 0 \leq x \leq u\}, \\
& \quad x_- \in \mathcal{A}_2 := \{x \in \mathbb{R}^n : 0 \leq x \leq l\}.
\end{align*}
\]

- Through numerical experiments, we have confirmed that the proposed neural network is insensitive to initializations and has better performance than the methods in [5, 10, 46, 47] for some specific experiments.
We organize the remaining part of this paper as follows. In Section 2, some preliminary results are given. In Section 3, we define a smoothing function for the cardinality penalty in (1) and analyze its some necessary properties. In Section 4, we propose a projection neural network and analyze the properties of its solution in solving problem (1). Moreover, a method to correct the accumulation points of the proposed neural network is designed in order to obtain a local minimizer of (1) with better sparsity. In Section 5, the equivalence between the local minimizers of (2) and (3) are proved. Finally, some numerical examples are illustrated in Section 6 to show the good performance of the proposed network in solving problems (1) and (2).

Notations: \( \mathbb{R}_+^n \) denotes the set composed by all \( n \) dimensional nonnegative vectors. For \( x \in \mathbb{R}^n, \|x\| := \|x\|_2 = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \) and \( \|x\|_\infty = \max \{ |x_i| : i = 1, 2, \ldots, n \} \). Denote \( k_n = (k, k, \ldots, k) \in \mathbb{R}^n \). For \( x \in \mathbb{R}^n \) and an index set \( \Pi \subseteq \{1, 2, \ldots, n\}, [\Pi] \) denotes the number of elements in \( \Pi \), \( x_\Pi := (x_{i_1}, x_{i_2}, \ldots, x_{i|\Pi|}) \) with \( i_1, i_2, \ldots, i_{|\Pi|} \in \Pi \) and \( i_1 < i_2 < \ldots < i_{|\Pi|} \). For an \( x \in \mathbb{R}^n \) and \( \delta > 0 \), \( B_\delta(x) \) denotes the open ball in \( \mathbb{R}^n \) centered at \( x \) with radius \( \delta \).

## 2. Preliminaries

In this section, we will introduce some necessary definitions and properties that will be used in what follows.

**Proposition 1.** [48] Let \( g : [0, +\infty) \to \mathbb{R} \) be a continuous function. If \( \int_0^{+\infty} |g(s)|ds < +\infty \) and \( g \) is uniformly continuous on \( [0, +\infty) \), then \( \lim_{s \to +\infty} g(s) = 0 \).

For a nonempty, closed and convex set \( \Omega \subseteq \mathbb{R}^n \), the projection operator to \( \Omega \) at \( x \) is defined by

\[
P_\Omega(x) = \arg\min_{u \in \Omega} \|u - x\|^2,
\]

which owns the following properties.

**Proposition 2.** [49] Assume \( \Omega \) is a closed and convex subset of \( \mathbb{R}^n \). Then, \( P_\Omega(\cdot) \) owns the following properties:

\[
\langle w - P_\Omega(w), P_\Omega(w) - u \rangle \geq 0, \quad \forall w \in \mathbb{R}^n, u \in \Omega; \quad \|P_\Omega(w) - P_\Omega(u)\| \leq \|u - w\|, \quad \forall u, w \in \mathbb{R}^n.
\]

**Proposition 3.** [49] If \( f : \Omega \to \mathbb{R} \) is a convex and differentiable function on the closed and convex set \( \Omega \), then

\[
x^* = P_\Omega(x^* - \nabla f(x^*))
\]

if and only if \( x^* \) is a global minimizer of \( f \) in \( \Omega \).

Definitions of globally and locally Lipschitz continuous functions can be found in Clarke [50].

Let \( D \) be an open set in \( \mathbb{R}^{n+1} \) with an element of \( D \) written as \( (x, t) \) and \( G : D \to \mathbb{R}^n \) be a continuous function. For a nonautonomous real-time system modeled by a differential equation

\[
\dot{x}(t) = G(x(t), t),
\]

we call \( x : [0, T) \to \mathbb{R}^n \) one of its solutions, if \( x \) is continuously differentiable on \( (0, T) \), \( (x(t), t) \in D \) for \( t \in [0, T) \) and \( x \) satisfies it on \( (0, T) \) [51].

Since \( \|\cdot\|_0 \) in (1) is discontinuous, we introduce the smoothing method into neural network and use the smoothing function defined as follows, which is restricted to a closed and convex subset of \( \mathbb{R}^n \).

**Definition 1.** Let \( h : \mathbb{R}^n \to \mathbb{R} \) be a function and \( \Omega \) be a closed and convex subset of \( \mathbb{R}^n \). We call \( h : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R} \) a smoothing function of \( h \) on \( \Omega \), if \( h(\cdot, \mu) \) is differentiable on \( \mathbb{R}^n \) for any fixed \( \mu > 0 \) and \( \lim_{\mu \to 0} h(x, \mu) = h(x) \) holds for any \( x \in \Omega \).

### 3. Smoothing function of \( \|x\|_0 \) on \( \mathcal{X} \)

In this section, we design a smoothing function of \( \|x\|_0 \) on \( \mathbb{R}^n_+ \) and give some necessary properties, which will be used in the analysis on the proposed neural network for solving (1).

Define

\[
\Theta(x, \mu) = \sum_{i=1}^{n} \theta(x_i, \mu), \quad \text{(4)}
\]

where

\[
\theta(s, \mu) = \begin{cases} 
\frac{3}{2}\mu s & \text{if } s < \frac{1}{3}\mu, \\
-\frac{9}{4}\mu^2 (s - \mu)^2 + 1 & \text{if } \frac{1}{3}\mu \leq s \leq \mu, \\
1 & \text{if } s > \mu.
\end{cases}
\]

For some fixed \( \mu > 0 \), the presentation of \( \theta(\cdot, \mu) \) on \( [0, 1] \) is pictured in Fig. 1 (a). Meantime, the presentation of \( \theta(s, \cdot) \) on \( (0, 5) \) for some fixed \( s > 0 \) is shew in Fig. 1 (b).

For any \( x \in \mathbb{R}^n_+ \) and \( \mu > 0 \), the gradient of \( \Theta(x, \mu) \) is denoted by \( \nabla \Theta(x, \mu) \) and

\[
\nabla \Theta(x, \mu) = \left( \nabla_x \Theta(x, \mu), \nabla_\mu \Theta(x, \mu) \right).
\]

Here, \( \nabla_x \Theta(x, \mu) \) is the gradient of \( \Theta(x, \mu) \) at \( x \) with a fixed \( \mu \) and \( \nabla_\mu \Theta(x, \mu) \) is the gradient of \( \Theta(x, \mu) \) at \( \mu \) with a fixed \( x \).

**Definition 2.** For a fixed \( \mu > 0 \), we call \( x_\mu^* \) a \( \mu \)-stationary point of the following smoothing optimization problem

\[
\min \quad f(x) + \lambda \Theta(x, \mu)
\]

s.t. \( x \in \mathcal{X}, \quad \text{(6)} \)

where \( f \) and \( \mathcal{X} \) are the same as them in (1), if \( x_\mu^* \) satisfies

\[
x_\mu^* = P_{\mathcal{X}} \left[ x_\mu^* - \nabla f(x_\mu^*) - \lambda \nabla_x \Theta(x_\mu^*, \mu) \right].
\]
Let $\hat{x}^*$ be a global minimizer of (6), then when $\mu \downarrow 0$, any accumulation point of $\{\hat{x}_{\mu}^* : \mu > 0\}$ is a global minimizer of (1). Based on the above construction and following analysis of $\Theta(x, \mu)$, we will show that when $\mu$ is small enough, if $\{j \in \{1, 2, \ldots, n\} : x_{\mu,j}^* \in [\frac{1}{2} \mu, \frac{1}{2} \mu]\} = \emptyset$, then $x^*$ is a local minimizer of (1) in the following Remark 4.

**Proposition 4.** $\Theta(x, \mu)$ is a smoothing function of $\|x\|_0$ on $\mathbb{R}^n$ defined in Definition 1 and satisfies the following properties.

(i) For any fixed $\mu > 0$, $\Theta(\cdot, \mu)$ is continuously differentiable and $\nabla_x \Theta(\cdot, \mu)$ is globally Lipschitz continuous on $\mathbb{R}^n$.

(ii) For any fixed $x \in \mathbb{R}^n$, $\Theta(x, \cdot)$ is continuously differentiable and $\nabla_\mu \Theta(x, \cdot)$ is locally Lipschitz continuous on $(0, +\infty)$.

(iii) For $\mu \geq 0$, $\nabla_\mu \Theta$ and $\nabla_\mu \Theta$ are bounded and globally Lipschitz continuous on $\mathcal{X} \times [\mu, \bar{\mu}]$.

**Proof.** See Appendix A.

### 4. Neural network

In this section, we will propose a projection neural network and a correction method when it is needed. Some dynamic and optimal properties of the proposed neural network and method for solving (1) are also analyzed.

Since $\mathcal{X}$ is bounded and $\nabla f$ is locally Lipschitz continuous in problem (1), it is naturally satisfied that $\exists L_f > 0$, s.t. $\sup_{x \in \mathcal{X}} \|\nabla f(x)\|_\infty \leq L_f$ and $\nabla f$ is globally Lipschitz continuous on $\mathcal{X}$. Throughout this paper, we need the value of $L_f$ to support the theoretical results of this paper and the following proposed neural network is qualified for the situation where $L_f$ is available. Moreover, we need the following parameters. Denote $\bar{v} = \|v\|_\infty$ and $\bar{v} = \min\{v_i : v_i \neq 0, i = 1, 2, \ldots, n\}$.

Based on the smoothing function $\Theta$ of $\|x\|_0$ on $\mathbb{R}^n_+$ designed in Section 3, we propose a projection neural network (Algorithm 1) modeled by a differential equation to solve (1).

**Assumption 1.**

$$0 < \mu^* < \min \left\{ \bar{v}, \frac{3\lambda}{2(\bar{v} + L_f)}, \frac{2\lambda}{nL_f} \right\}.$$
bridge $L_p(0 < p < 1)$ [36] is non-Lipschitz continuous. \( \theta \) is a one-parameter function and approaches to the cardinality function on $\mathbb{R}_+$ as the parameter tends to 0. But SCAD [38] and MCP [39] have two parameters. Moreover, hard thresholding [35] and SCAD cannot approach to cardinality function with proper adjustment on the parameters. CEL0 [4] is the MCP with a specific choice of parameters and can be seen as a one-parameter function. \( \theta \) is a linear function near 0, but CEL0 is a quadratic function near 0, which is likely to result in a heavier computation in the implementation of the neural network and a smaller lower bound for the nonzero elements of the local minimizers than \( \theta \) used in this paper. Therefore, \( \theta \) is more appropriate for neural network model than these relaxation functions.

### 4.1. Basic dynamic properties of (7)

In this subsection, we will analyze some basic properties of the solution of neural network (7), including its global existence and uniqueness given in Theorem 1 and some basic convergence properties shown in Lemma 1.

For simplicity of notation, we use $x_t$ and $\mu_t$ to denote $x(t)$ and $\mu(t)$ respectively throughout this paper.

**Theorem 1.** For any initial point $x_0 \in \mathcal{X}$, there exists a unique global solution $x_t \in C^{1,1}([0, +\infty); \mathbb{R}^n)$ to neural network (7). Moreover, $x_t \in \mathcal{X}$ for any $t \in [0, +\infty)$ and $\dot{x}_t$ is globally Lipschitz continuous on $[0, +\infty)$.

**Proof.** Since the right-hand function of neural network (7) is continuous with respect to $x$ and $t$, there exists at least one solution $x_t$ to (7) [51, pp. 14, Theorem 1.1].

Assume $x_t$ is a solution of (7) which is not global and its maximal existence interval is $[0, T)$ with $T > 0$.

First of all, we prove that $x_t \in \mathcal{X}, \forall t \in [0, T)$. Rewrite (7) as $\dot{x}_t + y x_t = \gamma h(t)$, where

$$h(t) = P_{\mathcal{X}}[x_t - \nabla f(x_t) - \lambda \nabla x \Theta(x_t, \mu_t)]$$

is a continuous function. Since $(e^{t\gamma})' = \gamma e^{t\gamma} h(t)$, we have

$$e^{t\gamma} x_t - x_0 = \int_0^t e^{s\gamma} h(s) ds,$$

which means

$$x_t = e^{-t\gamma} x_0 + (1 - e^{-t\gamma}) \int_0^t \frac{e^{s\gamma}}{e^{t\gamma} - 1} h(s) ds.$$  \hspace{1cm} (9)

Since $h(s)$ and $\frac{e^{s\gamma}}{e^{t\gamma} - 1}$ are continuous on $[0, t)$, $\int_0^t \frac{e^{s\gamma}}{e^{t\gamma} - 1} ds = 1$ and $h(s) \in \mathcal{X}, \forall s \in [0, t)$, we have for any $t \in [0, T)$,

$$\int_0^t \frac{e^{s\gamma}}{e^{t\gamma} - 1} h(s) ds \in \mathcal{X}.$$  \hspace{1cm} (10)

Combining (9) with (10), by the convexity of $\mathcal{X}$, we deduce for any $t \in [0, T)$, $x_t \in \mathcal{X}$. Hence, in view of the compactness of $\mathcal{X}$, we obtain that $x_t$ and $\dot{x}_t$ are bounded on $[0, T)$.

By Hale [51, pp. 16, Lemma 2.1], $x_t$ can be extended, which leads to a contradiction. As a consequence, $x_t$ is global existent and $x_t \in \mathcal{X}$ for any $t \in [0, +\infty)$. Owning to the boundedness of $\mathcal{X}$, by the structure of $\dot{x}_t$ in (7), we obtain that $\dot{x}_t$ is bounded on $[0, +\infty)$, which implies $x_t \in C^{1,1}([0, +\infty); \mathbb{R}^n)$.

Next, we prove the uniqueness of the solution of (7). Let $x$ and $\hat{x}$ be two solutions of neural network (7) with initial point $x_0 \in \mathcal{X}$ and we suppose there exists a $t > 0$ such that $x_t \neq \hat{x}_t$. From the continuity of $x_t$ and $\hat{x}_t$, there exists a $\delta > 0$ such that $x_t \neq \hat{x}_t, \forall t \in [\tilde{t}, \tilde{t} + \delta]$. Define

$$\xi(x, \mu) = \gamma \left[ -x + P_{\mathcal{X}} \left[ x - \nabla f(x) - \lambda \nabla x \Theta(x, \mu) \right] \right].$$

Since $\nabla f(\cdot)$ and $\nabla x \Theta(\cdot, \cdot)$ are globally Lipschitz continuous on $\mathcal{X}$ for any fixed $\mu > 0$. Then, from the continuity of $x_t, \hat{x}_t$, and $\mu_t$ on $[0, \tilde{t} + \delta)$, it follows that there exists an $L_\xi > 0$ such that

$$\|\xi(x_t, \mu_t) - \xi(\hat{x}_t, \mu_t)\| \leq L_\xi \|x_t - \hat{x}_t\|, \forall t \in [\tilde{t}, \tilde{t} + \delta].$$

Thus, for any $t \in [0, \tilde{t} + \delta)$, we have

$$\frac{d}{dt} \|x_t - \hat{x}_t\|^2 \leq 2L_\xi \|x_t - \hat{x}_t\|^2.$$  \hspace{1cm} (11)

Integrating (11) from 0 to $t (\leq \tilde{t} + \delta)$, we obtain

$$\|x_t - \hat{x}_t\|^2 \leq 2L_\xi \int_0^t \|x_s - \hat{x}_s\|^2 ds.$$

Applying Gronwall’s inequality [52] to the above inequality, we have that $x_t = \hat{x}_t, \forall t \in [\tilde{t}, \tilde{t} + \delta)$, which leads to a contradiction. Therefore, the solution of (7) with $x_0 \in \mathcal{X}$ is unique.

Finally, we prove the global Lipschitz continuity of $\dot{x}_t$ on $[0, +\infty)$. By the definition of $\mu_t$ in (7), $\dot{\mu}_t$ is bounded and globally Lipschitz continuous on $[0, +\infty)$. From Proposition 4-(iii) and the boundedness of $\dot{x}_t$ and $\dot{\mu}_t$ on $[0, +\infty)$, since $x_t \in \mathcal{X}$ and $\mu_t \in [1/m^*, \mu_0]$, for all $t \in [0, +\infty)$, we obtain the global Lipschitz continuity of $\nabla x \Theta(x_t, \mu_t)$ on $[0, +\infty)$. The global Lipschitz continuity of $\nabla f(x)$ on $\mathcal{X}$ and the boundedness of $\dot{x}_t$ on $[0, +\infty)$ implies the global Lipschitz continuity of $\nabla f(x_t)$ on $[0, +\infty)$. Combining the above analysis, the global Lipschitz continuity of projection operator $P_{\mathcal{X}}$ given in Proposition 2 and the structure of $\dot{x}_t$ in (7), we conclude that $\dot{x}_t$ is globally Lipschitz continuous on $[0, +\infty)$.

The following result plays an important role in the convergence analysis of neural network (7) for solving (1), which gives some basic dynamic properties of the solution of (7).

**Lemma 1.** Suppose $x_t$ is the solution of neural network (7) with initial point $x_0 \in \mathcal{X}$, then we have

i) $\lim_{t \to +\infty} \left[ f(x_t) + \lambda \Theta(x_t, \mu_t) \right]$ exists;

ii) $\int_0^{+\infty} \|\dot{x}_t\|^2 dt < +\infty$ and $\lim_{t \to +\infty} \dot{x}_t = 0$. 

---

\( \square \)
Remark 3. Based on the boundedness of \( x(t) \) on \([0, \infty)\) and 
\[ \lim_{t \to +\infty} \dot{x}_i = 0, \]
we have that any accumulation point of \( x(t) \) is a \( \mu^* / 2 \)-stationary point by Definition 2 of (6).

4.2. Properties of the accumulation points of the solution to network (7)

In this subsection, we analyze the optimal properties of the solution to network (7) for problem (1), which lays a foundation for the effectiveness of the proposed method in this paper for solving (1). This piecewise property of \( \Theta \) is the key to establishing the relationships between the accumulation points of (7) and the optimal solutions of (1).

For the further analysis on (7), let us give some necessary notations. For an \( x \in \mathcal{X} \), define
\[
I(x) = \left\{ i \in \{1, 2, \ldots, n\} : x_i \in \left[ 0, \frac{1}{6} \mu^* \right] \right\},
\]
\[
J(x) = \left\{ j \in \{1, 2, \ldots, n\} : x_j \in \left[ \frac{1}{6} \mu^*, \frac{1}{2} \mu^* \right) \right\}
\]
and
\[
K(x) = \left\{ k \in \{1, 2, \ldots, n\} : x_k \in \left[ \frac{1}{2} \mu^* \right] \right\}.
\]

It is clear that \( I(x) \cup J(x) \cup K(x) = \{1, 2, \ldots, n\} \).

For set \( \mathcal{X} \) in (1) and an \( x \in \mathcal{X} \), denote \( \mathcal{X}_I(x) = \{ y \in \mathcal{X} : y_i = 0, \forall i \in I(x) \} \), \( \mathcal{X}_J(x) = \{ y \in \mathcal{X} : y_j = x_j, \forall j \in J(x) \} \) and \( \mathcal{X}_K(x) = \{ y \in \mathcal{X} : y_i = x_i, \forall i \in K(x) \} \). Let \( \mathcal{X}_{I,J,K}(x) = \mathcal{X}_I(x) \cap \mathcal{X}_J(x) \).

To show the good performance of the proposed network in solving problem (1), we first prove a unified lower bound property and the same support set property for the accumulation points of the solution to network (7).

Lemma 2. Let \( x_t \) be the solution of neural network (7) with initial point \( x_0 \in \mathcal{X} \) and suppose that \( \tilde{x} \) be an accumulation point\(^3 \) of \( x_t \) at \( x_i \), then
\[
either \tilde{x}_i = 0 or \tilde{x}_i \geq \frac{1}{6} \mu^*, \forall i = 1, \ldots, n, \quad (12)
\]
and hence \( I(\tilde{x}) = \{ i \in \{1, 2, \ldots, n\} : \tilde{x}_i = 0 \} \). Moreover, for any accumulation points \( \tilde{x} \) and \( \hat{x} \) of \( x_t \), it holds that \( I(\tilde{x}) = I(\hat{x}) \).

Proof. See Appendix C.

For algorithms that solve sparse regression problems, the results in Lemma 2 are very important for the numerical properties of the proposed method. There are some analysis on the lower bound property for many different sparse regression models [53, 54, 55]. However, most of the results are proved for the local minimizer. The first result in Lemma 2 indicates that all accumulation points of the solution of network (7) have a unified lower bound for nonzero entries. The lower bound property of the accumulation points shows that network (7) can distinguish zero and nonzero entries of coefficients effectively in sparse high-dimensional regression [55, 56], and bring the restored image closed contours and neat edges [53]. Moreover, it is worth noting that the lower bound is related to the regularization parameter \( \lambda \). Wherefore, the lower bound is useful for choosing the regularization parameter \( \lambda \) to control the sparsity of the accumulation points of network (7). Through the accumulation points of network (7) may be not unique, the second result in Lemma 2 shows that all accumulation points own a common support set, which shows the constancy and robustness of network (7) for solving problem (1).

Next, we give a sufficient and necessary optimality condition for the local minimizers of (1), which helps us to justify the optimal property of the obtained point. Based on the special discontinuity of \( \|x\|_0 \) and the continuity of \( f \), we have the following link on the local minimizers of \( f(x) + \lambda\|x\|_0 \) and \( f(x) \).

Proposition 5. \( x^* \) is a local minimizer of (1) if and only if \( x^* \) is a local minimizer of \( f \) in \( \mathcal{X}_I(x^*) \).

Proof. See Appendix D.

Based on the properties proved in Lemma 2 and Assumption 1, any accumulation point of the solution to network (7) owns the following optimal properties to problem (1).

Theorem 2. Let \( x_t \) be the solution of neural network (7) with initial point \( x_0 \in \mathcal{X} \) and suppose \( \bar{x} \) be an accumulation point of \( x_t \). Then, \( \bar{x} \) is a global minimizer of \( f(x) \) in \( \mathcal{X}_{I,J,K}(\bar{x}) \). In particular, if \( J(\bar{x}) = \emptyset \), then \( \bar{x} \) is a local minimizer of sparse regression model (1) with lower bound property in (12).

Proof. Taking into account that \( \bar{x}_k \geq \frac{1}{2} \mu^*, \forall k \in K(\bar{x}) \), by (25), we obtain
\[
\left[ \nabla_{\bar{x}} \Theta \left( \bar{x}, \mu^*/2 \right) \right] = 0, \forall k \in K(\bar{x}). \quad (13)
\]

Denote \( \mathcal{X}_{K(\bar{x})} := \{ x_{K(\bar{x})} : x \in \mathcal{X} \} \). Recalling \( \lim_{t \to +\infty} \dot{x}_i = 0 \) and applying (13), since feasible set \( \mathcal{X} \) is box shaped, we have
\[
\bar{x}_{K(\bar{x})} = P_{\mathcal{X}_{K(\bar{x})}} \left[ \bar{x} - \nabla f(\bar{x}) = \lambda \nabla_{\bar{x}} \Theta (\bar{x}, \mu^*/2) \right]_{K(\bar{x})}
\]
\[
= P_{\mathcal{X}_{K(\bar{x})}} \left[ \bar{x}_{K(\bar{x})} - [\nabla f(\bar{x})]_{K(\bar{x})} \right].
\]

Based on Proposition 3, \( \bar{x} \) is a global minimizer of \( f \) in \( \mathcal{X}_{I,J,K}(\bar{x}) \).

If \( J(\bar{x}) = \emptyset \), by Proposition 5, \( \bar{x} \) is a local minimizer of \( f(x) + \lambda\|x\|_0 \) in \( \mathcal{X} \) and satisfies the lower bound property in (12).

Remark 4. By the proofs of Lemma 2 and Theorem 2, for the stationary point \( x^*_\mu/\sqrt{2} \) of (6), if \( J(x^*_\mu/\sqrt{2}) = \emptyset \), then \( x^*_\mu/\sqrt{2} \) is a local minimizer of (1).

Due to the special structure of function \( \theta(\cdot, \mu) \), which is convex on \([0, \mu] \) and \((\mu, +\infty) \), respectively, the solution of network (7) can be convergent to a local minimizer of (1) in some cases.
Corollary 1. Let $x_i$ be the solution of (7). Suppose that $f$ in (1) is strictly convex and any accumulation point $\bar{x}$ of $x_i$ satisfies $\dot{J}(\bar{x}) = 0$. Then $x_i$ is convergent to a local minimizer of problem (1) with lower bound property in (12).

4.3. A further correction to the obtained point by (7)

In this subsection, we propose a further network to correct the obtained accumulation point of the solution to (7). Its solution can converge to a local minimizer of problem (1). Moreover, in the sense of the sparsity and objective function value, it can find a better solution than the accumulation point obtained by (7). Throughout this subsection, denote $\bar{x}$ as an accumulation point of the solution to (7).

The proposed network in this part should be with a special initial point, which is constructed by $\bar{x}$ and defined as follows.

Definition 3. We call $\bar{x}^{\mu*}$ a $\mu^*$-update point of $\bar{x}$, if

$$
\bar{x}_i^{\mu*} = \begin{cases} 
\bar{x}_i & \text{if } |\bar{x}_i| \geq \frac{\mu^*}{2}, \\
0 & \text{otherwise.}
\end{cases}
$$

Obviously, $\bar{x} = \bar{x}^{\mu*}$ if $J(\bar{x}) = 0$.

Though the $\mu^*$-update point of $\bar{x}$ is also not a local minimizer of (1), it is a local minimizer of $f(x) + \lambda\|x\|_0$ in a particular subset of $\mathcal{X}$ and owns some special properties.

Proposition 6. The $\mu^*$-update point $\bar{x}^{\mu*}$ of $\bar{x}$ is a strictly local minimizer of $f(x) + \lambda\|x\|_0$ in $\mathcal{X}_K(\bar{x})$ and $f(\bar{x}^{\mu*}) + \lambda\|\bar{x}^{\mu*}\|_0 \leq f(\bar{x}) + \lambda\|\bar{x}\|_0$. In particular, if $J(\bar{x}) \neq 0$, then $f(\bar{x}^{\mu*}) + \lambda\|\bar{x}^{\mu*}\|_0 < f(\bar{x}) + \lambda\|\bar{x}\|_0$.

Proof. See Appendix E.

Denote $\tilde{\mathcal{X}} = \mathcal{X}_K(\bar{x}^{\mu*})$, which is the set defined by $\bar{x}$, i.e. $\mathcal{X} = \{x \in \mathcal{X} : x_i = 0, \forall i \in I(\bar{x}) \cup J(\bar{x})\}$.

We propose a further network for the case of $J(\bar{x}) \neq 0$. The further network is a projection neural network modeled by the following differential equation:

$$
\begin{align*}
\dot{x}(t) &= \gamma_1 \left[ -x(t) + P_{\tilde{\mathcal{X}}} [x(t) - \nabla f (x(t))] \right] \\
x(0) &= \bar{x}^{\mu*},
\end{align*}
$$

where $\gamma_1$ is a given positive parameter. Substantially, network (15) is used to solve the following smooth convex optimization problem:

$$
\min_{x \in \tilde{\mathcal{X}}} \ f(x).
$$

Remark 6. Though the variable $x(t)$ in network (15) is with dimension $n$, by the uniqueness of the value of the points in set $\tilde{\mathcal{X}}$ for the index in $I(\bar{x}^{\mu*})$, the calculation dimension of $x(t)$ in (15) is $n - |I(\bar{x}^{\mu*})|$.

Since network (15) is a typical projection neural network for a constrained smooth convex optimization problem, the solution of (15) is global existent, unique and owns the following properties [57]:

- $x_i \in \tilde{\mathcal{X}}$;
- $f(x_i)$ is nonincreasing in $t$ and $\lim_{t \to +\infty} f(x_i)$ exists;
- $\lim_{t \to +\infty} \dot{x}_i = 0$;
- $x_i$ is convergent to a global minimizer of problem (16).

Based on the above basic properties of network (15) and its particular initial point, we obtain the following result on the limit point of its solution to problem (1).

Theorem 3. Denote $x^*$ the limit point of the solution to network (15) with initial point $\bar{x}^{\mu*}$. Then, $x^*$ is a local minimizer of (1), and if $J(\bar{x}) \neq 0$, the following properties hold:

1) $\|x^*\|_0 < \|\bar{x}\|_0$;
2) $f(x^*) + \lambda\|x^*\|_0 < f(\bar{x}) + \lambda\|\bar{x}\|_0$.

Proof. Denote $x_i$ the solution of network (15) with initial point $\bar{x}^{\mu*}$. By $\lim_{t \to +\infty} \dot{x}_i = 0$, we have

$$
x^* = P_{\tilde{\mathcal{X}}} \left[ x^* - \nabla f (x^*) \right].
$$

Since $x_i \in \tilde{\mathcal{X}}$, we have $x^* \in \mathcal{X}_K(x^*) \subseteq \tilde{\mathcal{X}}$, and hence

$$
x^* = P_{\mathcal{X}_K(x^*)} \left[ x^* - \nabla f (x^*) \right].
$$

By Proposition 3, we obtain that $x^*$ is a global minimizer point of $f(x)$ in $\mathcal{X}_K(x^*)$. Based on Proposition 5, we know that $x^*$ is a local minimizer of problem (1). When $J(\bar{x}) \neq 0$, by $J(\bar{x}) \subseteq J(\bar{x}^{\mu*}) \subseteq I(x^*)$ and the nonincreasing property of $f(x_i)$ in $t$, we have $\|\bar{x}\|_0 > \|\bar{x}^{\mu*}\|_0 \geq \|x^*\|_0$ and $f(\bar{x}^{\mu*}) \geq f(x^*)$. By Proposition 6, if $J(\bar{x}) \neq 0$, then $f(x^*) + \lambda\|x^*\|_0 \leq f(\bar{x}^{\mu*}) + \lambda\|\bar{x}^{\mu*}\|_0 < f(\bar{x}) + \lambda\|\bar{x}\|_0$.

Remark 7. By Theorem 2 and Theorem 3, either the accumulation point $\bar{x}$ of (7) satisfies $J(\bar{x}) = 0$ or the accumulation point $x^*$ of (15) with $J(\bar{x}) \neq 0$, is a local minimizer of (1).

5. Extension to another regression model

In this section, we consider another sparse regression model

$$
\begin{align*}
\min_{y \in \mathcal{Y}} f(y) + \lambda\|y\|_0 \\
\text{s.t. } y \in \mathcal{Y} := \{y \in \mathbb{R}^n : -l \leq y \leq u\},
\end{align*}
$$

where $l, u \in \mathbb{R}^n_+$, $\lambda > 0$, $f$ is defined as in (1).
To solve (17), we consider an sparse regression model modeled by (1) with $2n$ dimension, i.e.

$$
\begin{align*}
\min & \quad f(x_+ - x_-) + \lambda \|x_+\|_0 + \lambda \|x_-\|_0 \\
\text{s.t.} & \quad x_+ \in \mathcal{X}_1 := \{x_+ \in \mathbb{R}^n : 0 \leq x_+ \leq u\}, \\
& \quad x_- \in \mathcal{X}_2 := \{x_- \in \mathbb{R}^n : 0 \leq x_- \leq l\}.
\end{align*}
$$

The following proposition indicates the link between sparse regression models (17) and (18).

**Proposition 7.** Optimization problems (17) and (18) are equivalent and their local minimizers can be converted to each other, i.e. for any local minimizer $y^\#$ of (17), there exists a local minimizer $(x^{+\#}, x^{-\#})^T$ of (18) such that $y^\# = x^{+\#} - x^{-\#}$ and vice versa.

**Proof.** See Appendix F.

**Remark 8.** Proposition 7 shows that the given method in this paper can also be used to solve (2) by solving (3). It is worth noting that it is impossible to solve (2) directly without (3) by differential equations in theory. In fact, $\Theta$ is a smoothing function of $\|x\|_0$ on $\mathbb{R}^n_+$, but not $\mathbb{R}^n$. Based on the construction of $\Theta$, it is easier to have a similar smoothing approximation function of $\|x\|_0$ on $\mathbb{R}^n$. However, it is not continuously differentiable everywhere, which involves subdifferential in the proposed network. Neural network (7) is modeled by a differential equation. It is more conducive to solving (1) than differential inclusion in computation and implementation.

6. Numerical experiments

In this section, we report four numerical experiments to validate the theoretical results and show the efficiency of neural network (7) for solving (1) and (17). Let $\text{MSE}(x) = \|x - s\|^2/n$ denote the mean squared error of $x$ to original signal $s \in \mathbb{R}^n$. The average, maximum and minimum mean square error of test results are denoted by Mean-MSE, Max-MSE and Min-MSE, respectively. The average, maximum and minimum CPU time of test results are denoted by Mean-CPU, Max-CPU and Min-CPU, respectively.

For function $f(x) = \|Ax - b\|^2$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, we use the following code to generate $L_f$ (L in code), which is an upper bound of

$$\{\|\nabla f(x)\|_\infty : x \in [0, k, n]\},$$

for given $k > 0$.

$$\begin{align*}
[m, n] &= \text{size}(A) ; \\
W &= \text{abs}(A) ; \\
C1 &= 2 * (W * W * \text{ones}(n, 1) - A' * b) ; \\
C2 &= 2 * (-W * W * \text{ones}(n, 1) - A' * b) ; \\
L1 &= \text{norm}(C1, \text{inf}) ; L2 &= \text{norm}(C2, \text{inf}) ; \\
L &= \text{max}(L1, L2) .
\end{align*}$$

Specially, if all entries of $A$ and $b$ are nonnegative, then we can make $L = L1$. Further, for general sparse regression problem (1), we let

$$
\mu^* = 0.9 \min \left\{k, \frac{3\lambda}{2(k + L_f)} , \frac{2\lambda}{nL_f}\right\} .
$$

Figure 3: (a) State trajectories and (b) objective function values along the solutions of neural network (7) for problem (19).

We should state that the correction method in Subsection 4.3 is not used in all numerical experiments, since the limit point $x^*$ of the solution of neural network (7) satisfies $f(x^*) = 0$ in every experiment.

6.1. Test example

In this example, we illustrate the effectiveness of the algorithm for solving a test problem modeled by (1), whose global minimizers are known.

Consider the following sparse regression problem in $\mathbb{R}^2$:

$$
\begin{align*}
\min & \quad \|Ax - b\|^2 + \lambda \|x\|_0 \\
\text{s.t.} & \quad x \in \mathcal{X} := \{x \in \mathbb{R}^2 : 0 \leq x \leq (5, 5)\} ,
\end{align*}
$$

where

$$
A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 5 \end{pmatrix}^T ,
$$

$b = (2, 1, 3)^T$, $\lambda = 1$ and $\nu = (5, 5)^T$.

We implement network (7) with $a_0 = \beta = \gamma = 1$ and stop when $t = 10$. Fig. 3 (a) presents the solution of (7) with 6 random initial points in $\mathcal{X}$, which converge to the global minimizer point $x^* = (0, 0.6053)^T$ of (19). Fig. 3 (b) shows that the objective function values are decrease along the solutions of neural network (7) with the same 6 initial points used in Fig. 3 (a).
6.2. Compressed sensing

To validate Proposition 7, we consider the following constrained sparse regression problem:

\[
\begin{align*}
\min & \| Ax - b \|^2 + \lambda \| x \|_0 \\
\text{s.t.} & \quad x \in X := \{ x \in \mathbb{R}^n : -\nu \leq x \leq \nu \},
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, n = 1000, m = 200, \lambda = 0.1 \) and \( \nu = 5n \). Sensing matrix \( A \), original signal \( s \) with \( \| s \|_0 = 10 \) and observation \( b \) are generated by the following code:

\[
\begin{align*}
K &= \text{randn} \left( n, m \right); A' = \text{orth} \left( K \right); \\
Q &= \text{randperm} \left( n \right); s = \text{zeros} \left( n, 1 \right); \\
s( Q(1:10)) &= \text{randn} \left( 10, 1 \right); b = A*s;
\end{align*}
\]

By Proposition 7, sparse regression problem (21) is equivalent to the following sparse regression problem:

\[
\begin{align*}
\min & \| Ax_+ - Ax_- - b \|^2 + \lambda \| x_+ \|_0 + \lambda \| x_- \|_0 \\
\text{s.t.} & \quad x_+, x_- \in \mathbb{R}^n, 0 \leq x_+, x_- \leq \nu.
\end{align*}
\]

Choose \( a_0 = 700, \beta = 0.1, \gamma = 1 \), ten random initial points in this set \([0, 1]^{2n}\) and a fixed initial point \( x_0 = 1_{2n} \) in neural network (7) to solve problem (22). We implement network (7) and stop when \( t = 2500 \). Fig. 4 (a) shows that solution \( x(t) \) of (7) with \( x_0 = 1_{2n} \) is convergent. Fig. 4 (b) gives the transformation \( x_+(t) - x_-(t) \) of solution \( x(t) \) with \( x_0 = 1_{2n} \), which is used for solving sparse regression problem (21). Fig. 5 presents the mean squared error of \( x_+(t) - x_-(t) \) to original signal \( s \) with respect to \( t \) in neural network (7) with \( x_0 = 1_{2n} \). Fig. 6 shows the original and reconstructed signals by neural network (7) with \( x_0 = 1_{2n} \), where the one above is the original signal. We can see that the original signals almost coincide with the reconstructed signals. In Table 1, MSE* means the mean squared error of the output solution of neural network (7) with \( x_0 = 1_{2n} \) and other data are the mean squared errors of the output solution of neural network (7) with ten random initial points, which are very close to the result MSE* with \( x_0 = 1_{2n} \). Therefore neural network (7) is insensitive to initializations and we use the fixed initial point of all ones in the following experiments.

### Table 1

| Mean-MSE   | Max-MSE   | Min-MSE   | MSE*       |
|------------|-----------|-----------|------------|
| 4.2049 x 10^{-8} | 4.2050 x 10^{-8} | 4.2048 x 10^{-8} | 4.2050 x 10^{-8} |

6.3. Variable selection

Variable selection is an important application in high-dimensional statistical problems, particularly in regression and classification problems. We consider this problem by the following sparse regression model:

\[
\begin{align*}
\min & \| Ax - b \|^2 + \lambda \| x \|_0 \\
\text{s.t.} & \quad x \in X := \{ x \in \mathbb{R}^n : 0 \leq x \leq \nu \},
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, \lambda = 1 \) and \( \nu = 10_n \). Let \( n = 1500, m = 600 \) and \( \nu = 10_n \). We use the following code to generate measurement matrix \( A \), observation \( b \) and original signal \( s \) with \( \| s \|_0 = 50 \):

\[
\begin{align*}
L &= \text{randn} \left( n, m \right); A' = \text{orth} \left( L \right); \\
P &= \text{randperm} \left( n \right); s = \text{zeros} \left( n, 1 \right); \\
s( P(1:50)) &= \text{unifrnd} \left( 1, 10\right), [50, 1]) \\
b &= A*s + 0.01*\text{randn} \left( m, 1 \right);
\end{align*}
\]

In this example, we consider some comparative experiments and implement network (7) with the same parameters and initial point as in example 6.2.

We compare the proposed neural network (7) with two state-of-the-art neural networks SNN (\( p = 1.5 \)) in [46] and CBPDN-LPNN (denoted by LPNN in the following) in [47] with the initial point of all ones. SNN and LPNN are proposed for solving the \( L_1 \)-penalized problems with known noise level, which are continuous convex optimization mod-
Projection Neural Network for a Class of Sparse Regression Problems with Cardinality Penalty

Figure 6: Original signal and reconstructed signal \( x_s(t) - x_\ast(t) \) by neural network (7), where the one above is original signal.

Figure 7: State trajectories of (a) neural network (7), (b) SNN and (c) LPNN.

Table 2

| Network   | (7)       | SNN       | LPNN      |
|-----------|-----------|-----------|-----------|
| Mean-MSE  | 5.208 × 10^{-5} | 8.180 × 10^{-5} | 1.130 × 10^{-4} |
| Max-MSE   | 5.966 × 10^{-5} | 9.202 × 10^{-5} | 1.415 × 10^{-4} |
| Min-MSE   | 4.747 × 10^{-5} | 7.804 × 10^{-5} | 9.383 × 10^{-5} |
| Mean-CPU(s) | 12.3647   | 44.4610   | 43.8776   |
| Max-CPU(s) | 14.1424   | 49.3730   | 52.7790   |
| Min-CPU(s) | 11.7356   | 39.2379   | 38.6471   |

els. But neural network (7) is used to solve the discontinuous and nonconvex problem with unknown noise level. We run the SNN and LPNN with the true value of noise level in generating the data. The state trajectories of network (7), SNN and LPNN are as shown in Figure 7, respectively. It shows that the recovery performance of network (7), SNN and LPNN at the corresponding characteristic time \( t \) are stable. Then, we randomly generated ten sets of data for numerical comparison. As can be seen from Table 2, network (7) runs the least CPU time when these neural networks reach good and stable numerical results. Moreover, as shown in Fig. 8, the mean squared error of \( x(t) \) to the original signal is decreasing along the solution of (7). The comparison between output solution and original signal can be seen in Fig. 9, which shows that they are almost the same.

6.4. Prostate cancer

In this experiment, we consider the problem on finding the most important predictors in predicting the prostate cancer. The prostate cancer data set is from the https://web.stanford.edu/~hastie/ElemStatLearn/data.html and includes the medical records of 97 men who were plan to receive a radical prostatectomy. This data set is divided into two sets, i.e. a training set with 67 observations and a test set with 30 observations. More detailed explanation on the prostate cancer data set can be found in Chen [58], Stamey et al. [59] and Hastie et al. [5]. The prediction error is defined by the mean squared error over the test set.

In order to solve this problem, we consider the following sparse regression model:

\[
\begin{align*}
\min & \quad \|Ax - b\|^2 + \lambda \|x\|_0 \\
\text{s.t.} & \quad x \in \mathcal{X} := \{x \in \mathbb{R}^n : -v \leq x \leq v\},
\end{align*}
\]

(24)

where \( A \in \mathbb{R}^{67 \times n} \) and \( b \in \mathbb{R}^{67} \) are composed by the training set, \( n = 8 \), \( \lambda = 2 \) and \( v = 10^n \).

We use neural network (7) to solve the equivalent problem of (24) as in Proposition 7. Choose \( a_0 = 20, \beta = 2, y = 1 \) and initial point \( x_0 = \mathbf{1}_{16} \) in network (7). We report the numerical results in Table 3, where the listed result for network (7) is the output solution by (7) at \( t = 25 \), the result for FOIPA is the best result from Table 2 in [10], and the results for Lasso and Best subset selection are the best two results from Table 3.3 in [5]. From Table 3, we see that the proposed network not only finds the right main predictors in predicting prostate cancer, but also finds a solution with the smallest prediction error among the four methods.
Table 3
Results with different methods

| Method       | (7)   | FOIPA | Lasso | Best subset |
|--------------|-------|-------|-------|-------------|
| $x_i^+$(lcavol) | 0.6134| 0.6497| 0.533 | 0.740       |
| $x_i^+$(weight) | 0.3156| 0.2941| 0.169 | 0.316       |
| $x_i^+(age)$      | 0     | 0     | 0     | 0           |
| $x_i^+(bph)$       | 0.002 | 0     | 0     | 0           |
| $x_i^+(svi)$      | 0.2222| 0.1498| 0.094 | 0           |
| $x_i^+(lcp)$      | 0     | 0     | 0     | 0           |
| $x_i^+(glean)$     | 0     | 0     | 0     | 0           |
| $x_i^+(pgg45)$     | 0     | 0     | 0     | 0           |
| $\|x^+\|_0$       | 3     | 3     | 4     | 2           |
| Prediction error  | 0.4002| 0.4194| 0.479 | 0.492       |

7. Conclusions

In this paper, we studied a class of sparse regression problem with cardinality penalty. By constructing a smoothing function for cardinality function, we proposed the projection neural network (7) to solve sparse regression problem (1). We proved that the solution of (7) is unique, global existent, bounded and globally Lipschitz continuous. Moreover, we proved that all accumulation points of (7) have a common support set and own a unified lower bound for the nonzero elements. Furthermore, we proposed a correction method for its accumulation points to obtain the local minimizers of (1). Specially, in most cases, by using (7), a local optimal solution of (1) with lower bound property can be obtained without using the correction. Besides, we proved that the equivalence on local minimizers between (1) and another sparse regression model (2). Finally, some numerical experiments were provided to show the convergence and efficiency of (7) for solving (1) and (2).

A. Proof of Proposition 4

PROOF. It is clear that $\theta(\cdot, \mu)$ is a bounded function on $\mathbb{R}_+^n$ and $\lim_{\mu \downarrow 0} \theta(s, \mu) = 1$ as $s > 0$, and $\theta(s, \mu) = 0$ for any $\mu > 0$ as $s = 0$. Then,

$$\lim_{\mu \downarrow 0} \Theta(x, \mu) = \sum_{i=1}^n \lim_{\mu \downarrow 0} \theta(x_i, \mu) = \|x\|_0, \ \forall x \in \mathbb{R}_+^n.$$ 

Since $\theta(s, \mu)$ is differentiable with respect to $s$ for any fixed $\mu > 0$, $\Theta(x, \mu)$ is differentiable with respect to $x$ for any fixed $\mu > 0$. Thus, $\Theta(x, \mu)$ is a smoothing function of $\|x\|_0$ on $\mathbb{R}_+^n$. 

Next, we prove the other results in this proposition one by one.

As can be seen,

$$\nabla_x \Theta(x, \mu) = \sum_{i=1}^n \nabla_x \theta(x_i, \mu) = \begin{pmatrix} \nabla_{x_1} \theta(x_1, \mu) \\ \nabla_{x_2} \theta(x_2, \mu) \\ \vdots \\ \nabla_{x_n} \theta(x_n, \mu) \end{pmatrix}.$$

For any fixed $\mu > 0$, since

$$\nabla_{x_i} \theta(x_i, \mu) = \begin{cases} \frac{3}{2\mu} & \text{if } x_i < \frac{1}{3}\mu, \\ \frac{9}{4\mu^2}(\mu - x_i) & \text{if } \frac{1}{3}\mu \leq x_i \leq \mu, \\ 0 & \text{if } x_i > \mu, \end{cases}$$

we have $\nabla_{x_i} \theta(\cdot, \mu)$ is continuous for any $i = 1, 2, \ldots, n$. Then, $\Theta(\cdot, \mu)$ is continuously differentiable for any fixed $\mu > 0$. Moreover, since

$$\nabla^2_{x_i} \theta(x_i, \mu) = \begin{cases} 0 & \text{if } x_i < \frac{1}{3}\mu, \\ -\frac{9}{4\mu^2} & \text{if } \frac{1}{3}\mu < x_i < \mu, \\ 0 & \text{if } x_i > \mu, \end{cases}$$

we obtain that for any fixed $\mu > 0$, $\nabla^2_{x_i} \theta(\cdot, \mu)$ is bounded on $(-\infty, \frac{1}{3}\mu) \cup \left(\frac{1}{3}\mu, \mu\right) \cup (\mu, +\infty)$. Then, $\nabla_{x_i} \theta(\cdot, \mu)$ is globally Lipschitz continuous on $\mathbb{R}$ for any $i = 1, 2, \ldots, n$. Therefore $\nabla_{x_i} \Theta(\cdot, \mu)$ is globally Lipschitz continuous on $\mathbb{R}^n$ for any fixed $\mu > 0$, which means that result (i) in this proposition holds.

For any fixed $x \in \mathbb{R}_+^n$, we also see that

$$\nabla_{\mu} \Theta(x, \mu) = \sum_{i=1}^n \nabla_{\mu} \theta(x_i, \mu),$$

with

$$\nabla_{\mu} \theta(x_i, \mu) = \begin{cases} -\frac{3x_i}{2\mu^2} & \text{if } \mu > 3x_i, \\ -\frac{9(\mu - x_i)x_i}{4\mu^3} & \text{if } 1 \leq x_i \leq x_i, \\ 0 & \text{if } x_i < \mu, \end{cases}$$

Then, $\nabla_{x_i} \Theta(x, \cdot)$ is continuous on $(0, +\infty)$, which means $\Theta(x, \cdot)$ is continuously differentiable on $(0, +\infty)$ for any fixed $x \in \mathbb{R}_+^n$. Moreover, since

$$\nabla^2_{\mu} \theta(x_i, \mu) = \begin{cases} \frac{3x_i}{\mu^2} & \text{if } \mu > 3x_i, \\ -\frac{9x_i(2\mu - 3x_i)}{4\mu^4} & \text{if } x_i < \mu < 3x_i, \\ 0 & \text{if } x_i < \mu, \end{cases}$$

which means that $\nabla^2 \Theta(x, \cdot)$ is locally Lipschitz continuous on $(0, +\infty)$. Thus, property (ii) holds.

Since $X$ and $[\mu, \bar{\mu}]$ are compact sets in $\mathbb{R}_+^n$ and $\mathbb{R}_+$, respectively, by (25) and (26), we obtain result (iii) in this proposition.

B. Proof of Lemma 1

PROOF. Let $u = x_i - \nabla f(x_i) - \lambda \nabla \Theta(x_i, \mu_i)$ and $u = x_i$ in Proposition 2, by (7), then we obtain

$$\langle \nabla f(x_i) + \lambda \nabla \Theta(x_i, \mu_i), x_i \rangle \leq -\frac{1}{\gamma} \|x_i\|^2.$$
Since $\mu_i \geq \frac{1}{2} \mu^*$ and $\bar{\mu}_i \leq 0$, by result (iii) of Proposition 4, there exists $\rho > 0$ such that
\begin{equation}
\nabla \Theta(x_t, \mu_i) \leq -\rho \mu_i. \tag{28}
\end{equation}

Since
\begin{equation}
\frac{d}{dt} [f(x_t) + \lambda \Theta(x_t, \mu_i)] = \langle \nabla f(x_t) + \lambda \nabla \Theta(x_t, \mu_i), \dot{x}_t \rangle + \lambda \langle \nabla \mu(x_t, \mu_i), \mu_i \rangle, \tag{29}
\end{equation}
by (27) and (28), we obtain
\begin{equation}
\frac{d}{dt} [f(x_t) + \lambda \Theta(x_t, \mu_i) + \lambda \rho \mu_i] \leq -\frac{1}{\rho} \| \dot{x}_t \|^2. \tag{30}
\end{equation}
Thus, $f(x_t) + \lambda \Theta(x_t, \mu_i) + \lambda \rho \mu_i$ is nonincreasing on $[0, +\infty)$. Since $f(x)$ and $\Theta(x, \mu)$ are continuous on $\mathcal{X} \times [\frac{1}{2} \mu^*, \mu_0]$, $x_i \in \mathcal{X} \setminus \mu_i \in [\frac{1}{2} \mu^*, \mu_0]$, we get that $f(x_t) + \lambda \Theta(x_t, \mu_i) + \lambda \rho \mu_i$ is bounded from below on $[0, +\infty)$. As a consequence, if some entry of $\dot{x}_t$ is less than or equal to $\frac{1}{\rho} \mu^*$ at some point $T \geq T_\mu$, then the limit of this entry is 0.

Conversely, if some entry of $x_i$ is more than $\frac{1}{\rho} \mu^*$ for any $t \geq T_\mu$, then any accumulation point of this entry is not less than $\frac{1}{\rho} \mu^*$.

As a result, for any accumulation point $\bar{x}$ of $x_i$, either $\bar{x}_i = 0$ or $\bar{x}_i \geq \frac{1}{\rho} \mu^*$, $i = 1, \ldots, n$, and hence $I(\bar{x}) = \{i \in \{1, 2, \ldots, n\} : \bar{x}_i = 0\}$.

In addition, if there exists an $i \in \{1, 2, \ldots, n\}$ such that $0$ is an accumulation point of $\{x_i\}$, then there exists some $T \geq T_\mu$ such that $\{x_i\} \leq \frac{1}{\rho} \mu^*$, which implies $0$ is the unique accumulation point of $\{x_i\}$. Therefore, for any accumulation points $\bar{x}$ and $\check{x}$ of $x_i$, we have $I(\bar{x}) = I(\check{x})$.

**D. Proof of Proposition 5**

**Proof.** If $f(x^*) = \phi$, then the equivalence is obviously true. Next, we consider the case $f(x^*) \neq \phi$. Let $\sigma = \min\{x_i^* : x_i^* \neq 0, i = 1, 2, \ldots, n\}$ and $\delta = \frac{\sigma}{2}$, then
\begin{equation}
\| x \|_0 = \| x^\prime \|_0, \forall x \in \mathcal{X}_f(x^*) \cap B_{\delta}(x^*). \tag{33}
\end{equation}
If $x^*$ is a local minimizer of function $f(x) + \lambda \| x \|_0$ in $\mathcal{X}$, then there is a $\delta_1 \in (0, \delta)$ such that
\begin{equation}
f(x) + \lambda \| x \|_0 \geq f(x^*) + \lambda \| x^\prime \|_0, \forall x \in \mathcal{X} \cap B_{\delta_1}(x^*). \tag{34}
\end{equation}
Let $\delta_2 \in (0, \delta_1)$, by (33) and (34), then we obtain
\begin{equation}
f(x) \geq f(x^*), \forall x \in \mathcal{X}_f(x^*) \cap B_{\delta_2}(x^*). \tag{35}
\end{equation}
Therefore $x^*$ is a local minimizer of $f(x)$ in $\mathcal{X}_f(x^*)$.

Conversely, if $x^*$ is a local minimizer of $f(x)$ in $\mathcal{X}_f(x^*)$, then there exists a $\delta_3 \in (0, \delta]$ such that
\begin{equation}
f(x) \geq f(x^*), \forall x \in \mathcal{X}_f(x^*) \cap B_{\delta_3}(x^*). \tag{35}
\end{equation}
Since $f$ is continuous in $\mathcal{X}$, for any $\sigma \in (0, \lambda)$, there exists a $\delta_4 \in (0, \delta_3)$ such that $| f(x) - f(x^*) | < \sigma, \forall x \in \mathcal{X} \cap B_{\delta_4}(x^*)$, which implies
\begin{equation}
f(x^*) < f(x) + \sigma < f(x) + \lambda, \forall x \in \mathcal{X} \cap B_{\delta_4}(x^*). \tag{36}
\end{equation}
Combining (33) with (35), we get
\begin{equation}
f(x) + \lambda \| x \|_0 \leq f(x^*) + \lambda \| x^\prime \|_0, \forall x \in \mathcal{X}_f(x^*) \cap B_{\delta_2}(x^*). \tag{37}
\end{equation}
For any \( x \in \mathcal{X} \cap B_{\delta}(x^*) \) and \( x \notin \mathcal{X}_I(x^*) \), we have
\[
||x||_0 \geq ||x^*||_0 + 1
\]
and hence by (36), we obtain
\[
f(x) + \lambda||x||_0 \geq f(x) + \lambda(||x^*||_0 + 1)
= f(x) + \lambda + \lambda||x^*||_0
> f(x^*) + \lambda||x^*||_0.
\]
Therefore, \( x^* \) is a global minimizer of \( f(x) + \lambda||x||_0 \) in \( \mathcal{X} \cap B_{\delta}(x^*) \). As a result, \( x^* \) is a local minimizer of \( f(x) + \lambda||x||_0 \) in \( \mathcal{X} \).

E. Proof of Proposition 6

PROOF. From the continuity of \( f \), there exists a \( \delta > 0 \) such that
\[
f(\tilde{x}^\mu) < f(x) + \lambda, \forall x \in B_{\delta}(\tilde{x}^\mu).
\]
For any \( x \in \mathcal{X}_K(\tilde{x}) \) and \( x \neq \tilde{x}^\mu \), we have
\[
||x||_0 \geq ||\tilde{x}^\mu||_0 + 1.
\]
Combining (37) and (38), for any \( x \in B_{\delta}(\tilde{x}^\mu) \cap \mathcal{X}_K(\tilde{x}) \) and \( x \neq \tilde{x}^\mu \), we deduce
\[
f(x) + \lambda||x||_0 \geq f(x) + \lambda(||\tilde{x}^\mu||_0 + 1)
= (f(x) + \lambda) + \lambda||\tilde{x}^\mu||_0
> f(\tilde{x}^\mu) + \lambda||\tilde{x}^\mu||_0,
\]
which means that \( \tilde{x}^\mu \) is a strictly local minimizer of \( f(x) + \lambda||x||_0 \) in \( \mathcal{X}_K(\tilde{x}) \).

If \( J(\tilde{x}) = \emptyset \), then \( \tilde{x} = \tilde{x}^\mu \) and hence \( f(\tilde{x}^\mu) + \lambda||\tilde{x}^\mu||_0 = f(\tilde{x}) + \lambda||\tilde{x}^\mu||_0 \). Next, we consider the case where \( J(\tilde{x}) \neq \emptyset \).

In view of \( \sup_{x \in \mathcal{X}} \|\nabla f(x)\|_\infty \leq L_f \), we can regard \( \sqrt{nL_f} \) as a Lipschitz constant of \( f \) on \( \mathcal{X} \), which means
\[
|f(x) - f(\tilde{x})| \leq \sqrt{nL_f}||x - \tilde{x}||, \forall x \in \mathcal{X}.
\]
If \( ||x - \tilde{x}|| < \lambda/\sqrt{nL_f} \), then by (39), we have \( |f(x) - f(\tilde{x})| < \lambda \).
Let \( \delta = \lambda/\sqrt{nL_f} \), then we have \( f(x) < f(\tilde{x}) < \lambda, \forall x \in B_{\delta}(\tilde{x}) \).
It follows from Assumption 1, i.e., \( \sqrt{n\mu^*/2} < \lambda/\sqrt{nL_f} \), that \( \tilde{x}^\mu \in B_{\delta}(\tilde{x}) \), which implies
\[
f(\tilde{x}^\mu) - f(\tilde{x}) < \lambda.
\]
Furthermore, if \( J(\tilde{x}) \neq \emptyset \), we have
\[
||\tilde{x}||_0 - ||\tilde{x}^\mu||_0 \geq 1.
\]
Combining (40) with (41), we obtain
\[
f(\tilde{x}^\mu) - f(\tilde{x}) < \lambda((||\tilde{x}||_0 - ||\tilde{x}^\mu||_0),
\]
and hence \( f(\tilde{x}^\mu) + \lambda||\tilde{x}^\mu||_0 < f(\tilde{x}) + \lambda||\tilde{x}||_0 \).

F. Proof of Proposition 7

PROOF. Let \( y^* \) be a local minimizer of \( f(y) + \lambda||y||_0 \) in \( \mathcal{Y} \), then there exists a \( \delta > 0 \) such that \( f(y^*) + \lambda||y^*||_0 \leq f(y) + \lambda||y||_0 \) for any \( y \in B_{\delta}(y^*) \cap \mathcal{Y} \).

Since there exist unique \( x^+_i \in \mathcal{X}_I \) and \( x^-_i \in \mathcal{X}_I \), such that \( y^* = x^*_i - x^-_i \) and \( (x^*_i)(x^-_i)_i = 0, i = 1, 2, ..., n \), we obtain that
\[
||y^*||_0 = ||x^*_i||_0 + ||x^-_i||_0.
\]
For any \( x_+ \in B_{\frac{\delta}{2}}(x^*_i) \cap \mathcal{X}_I \) and \( x_- \in B_{\frac{\delta}{2}}(x^-_i) \cap \mathcal{X}_I \), we have
\[
x^*_i - x_- \in \left(B_{\frac{\delta}{2}}(x^*_i) \cap \mathcal{X}_I\right) - \left(B_{\frac{\delta}{2}}(x^-_i) \cap \mathcal{X}_I\right) \subseteq B_\delta(x^*_i) \cap \mathcal{Y}.
\]
Therefore, for any \( x_+ \in B_{\frac{\delta}{2}}(x^*_i) \cap \mathcal{Y} \) and \( x_- \in B_{\frac{\delta}{2}}(x^-_i) \cap \mathcal{Y} \), we have
\[
f(y^*) + \lambda||y^*||_0 \leq f(x^*_i - x_-) + \lambda||x^*_i - x_-||_0.
\]
If there exists an \( l_0 \in \{1, 2, ..., n\} \) such that \( [x^*_i - x^-_i]_{l_0} \neq 0 \), then at least one of \([x^*_i]_{l_0} \) and \([x^-_i]_{l_0} \) is not 0. Thus
\[
||x^*_i - x^-_i||_0 \leq ||x^*_i||_0 + ||x^-_i||_0.
\]
Using (42), (43) and (44), we conclude that for any \( x_+ \in B_{\frac{\delta}{2}}(x^*_i) \cap \mathcal{Y} \) and \( x_- \in B_{\frac{\delta}{2}}(x^-_i) \cap \mathcal{Y} \), we have
\[
f(x^*_i - x_-) + \lambda||x^*_i - x_-||_0
\leq f(x^*_i - x_-) + \lambda||x^*_i||_0 + \lambda||x^-_i||_0.
\]
Therefore, \( (x^*_i - x^-_i)^T \) is a local minimizer of \( f(x^*_i - x_-) + \lambda||x^*_i||_0 + \lambda||x^-_i||_0 \) in \( \mathcal{X}_I \times \mathcal{X}_I \).

Conversely, assume \( (x^*_i - x^-_i)^T \) is a local minimizer of \( f(x^*_i - x_-) + \lambda||x^*_i||_0 + \lambda||x^-_i||_0 \) in \( \mathcal{X}_I \times \mathcal{X}_I \). Let
\[
I^* = \{i \in \{1, 2, ..., 2n\} : (x^*_i)^T, (x^-_i)^T\} = 0
\]
and
\[
J^* = \{i \in \{1, 2, ..., n\} : (x^*_i)_i \neq (x^-_i)_i\}
\]
Define
\[
\mathcal{X}_I \times \mathcal{X}_I \in I^* = \{(x^*_i)^T, (x^-_i)^T \in \mathcal{X}_I \times \mathcal{X}_I : x_i = 0, \forall i \in I^* \}
\]
and
\[
\mathcal{Y}_I^* = \{y \in \mathcal{Y} : y_i = 0, \forall i \in J^* \}.
\]
In view of Proposition 5, we know that there exists a \( \delta' > 0 \) such that for any \( (x^*_i)^T, (x^-_i)^T \in B_{\delta'}((x^*_i)^T, (x^-_i)^T) \cap (\mathcal{X}_I \times \mathcal{X}_I) \),
\[
f(x^*_i - x^-_i) \leq f(x^*_i - x_-).
\]
Let \( y^* = x^*_i - x^-_i \). For \( y \in B_{\delta'}(x^*_i) \cap \mathcal{Y}_I^* \), there exists \( (y^*_i, y^-_i)^T \in B_{\delta'}((x^*_i)^T, (x^-_i)^T) \cap (\mathcal{X}_I \times \mathcal{X}_I) \), such that \( y = y^* \). Hence
\[
f(y) = f(y^* - y^-_i) \geq f(x^*_i - x^-_i) = f(y^*).
\]
Therefore, \( y^* \) is a local minimizer of \( f(y) \) in \( \mathcal{Y}_I^* \). In view of Proposition 5, it holds that \( y^* \) is a local minimizer of \( f(y) + \lambda||y||_0 \) in \( \mathcal{Y} \).
References

[1] E. Candès, J. Romberg, T. Tao, Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inf. Theory 52 (1) (2006) 489–509.

[2] P. Bühlmann, M. Kalisch, L. Meier, High-dimensional statistics with a view toward applications in biology, Ann. Rev. Stat. Appl. 1 (1) (2014) 255–278.

[3] Y. Liu, Y. Wu, Variable selection via a combination of the L0 and L1 penalties, J. Comput. Graph. Statist. 16 (4) (2007) 782–798.

[4] E. Soubies, L. Blanc-Féraud, G. Aubert, A continuous exact ε0 penalty (CE0) for least squares regularized problem, SIAM J. Imaging Sci. 8 (3) (2015) 1607–1639.

[5] T. Hastie, R. Tibshirani, J. Friedman, The Elements of Statistical Learning Data Mining, Inference, and Prediction, Second Edition, New York: Springer-Verlag, 2017.

[6] H. L. Thi, T. P. Dinh, H. Le, X. Vo, DC approximation approaches for sparse optimization, Eur. J. Oper. Res. 244 (1) (2015) 26–46.

[7] M. Nikolova, Relationship between the optimal solutions of least squares regularized with L0-norm and constrained by k-sparsity, Appl. Comput. Harmon. Anal. 41 (1) (2016) 237–265.

[8] B. K. Natarajan, Sparse approximate solutions to linear systems, SIAM J. Comput. 24 (2) (1995) 227–234.

[9] X. Chen, D. Ge, Z. Wang, Y. Ye, Complexity of unconstrained L2–L0 minimization, Math. Program. 143 (1-2) (2014) 371–383.

[10] W. Bian, X. Chen, Y. Ye, Complexity analysis of interior point algorithms for non-Lipschitz and nonconvex minimization, Math. Program. 149 (1-2) (2015) 301–327.

[11] Q. Liu, J. Wang, L1-minimization algorithms for sparse signal reconstruction based on a projection neural network, IEEE Trans. Neural Netw. Learn. Syst. 27 (3) (2016) 698–707.

[12] H. Mohimani, M. Babaie-Zadeh, C. Jutten, A fast approach for overcomplete sparse decomposition based on smoothed ε0 norm, IEEE Trans. Signal Process. 57 (1) (2009) 289–301.

[13] Y. Jiao, B. Jin, X. Lu, A primal dual active set with continuation algorithm for the ε0-regularized optimization problem, Appl. Comput. Harmon. Anal. 39 (2015) 400–426.

[14] W. Bian, X. Chen, A smoothing proximal gradient algorithm for non-smooth convex regression with cardinality penalty, SIAM J. Numer. Anal. 58 (1) (2020) 856–883.

[15] J. Pan, Z. Hu, Z. Su, M.-H. Yang, L0-based regularized intensity and gradient prior for deblurring text images and beyond, IEEE Trans. Pattern Anal. Mach. Intell. 39 (2) (2017) 342–355.

[16] J. Cai, W. Dan, X. Zhang, L0-based sparse canonical correlation analysis with application to cross-language document retrieval, Neurocomputing 329 (2019) 32–45.

[17] F. Xiong, J. Zhou, Y. Qian, Hyperspectral restoration via L0 gradient regularized low-rank tensor factorization, IEEE Trans. Geosci. Remote Sens 57 (12) (2019) 10410–10425.

[18] S. Osher, F. Ruan, X. Jiang, Y. Yao, W. Yin, Sparse recovery via differential inclusions, Appl. Comput. Harmon. Anal. 41 (2) (2016) 436–469.

[19] H. Attouch, Z. Chbani, J. Peypouquet, P. Redont, Fast convergence of Nesterov’s accelerated Gradient Method: Theory and Insights, J. Math. Program. 168 (1-2) (2018) 123–175.

[20] H. Attouch, J. Peypouquet, P. Redont, Accelerated gradient algorithms with asymptotic vanishing viscosity, Vietnam J. Math. 48 (2020) 247–276.

[21] Y. Xia, G. Feng, J. Wang, A novel recurrent Neural Network for solving nonlinear optimization problems with inequality constraints, IEEE Trans. Neural Netw. 19 (8) (2008) 1340–1353.

[22] X. B. Gao, L. Z. Liao, A new projection-based neural network for constrained variational inequalities, IEEE Trans. Neural Netw. 20 (3) (2009) 373–388.

[23] J. J. Hopfield, D. W. Tank, “Neural” computation of decisions in optimization problems, Biol. Cybern. 52 (3) (1985) 141–152.

[24] D. W. Tank, J. J. Hopfield, Simple ‘Neural’ optimization networks: An A/D converter, signal decision circuit, and a linear programming circuit, IEEE Trans. Circuits Syst. 33 (5) (1986) 533–541.

[25] M. P. Kennedy, L. O. Chua, Neural Networks for nonlinear programing, IEEE Trans. Circuits Syst. 35 (5) (1988) 554–562.

[26] J. A. Clemente, W. Mansour, R. Ayoubi, F. Serrano, H. Mecha, H. Ziad, W. E. Faiou, R. Velazco, Hardware implementation of a fault-tolerant Hopfield Neural Network on FPGAs, Neurocomputing 171 (2016) 1606–1609.

[27] T. Chen, L. Wang, S. Duan, Implementation of circuit for configurable memristive chaotic neural network and its application in associative memory, Neurocomputing 380 (2020) 36–42.

[28] W. Bian, X. Xue, Neural Network for solving constrained convex optimization problems with global attractivity, IEEE Trans. Circuits Syst. I-Regul. Pap. 60 (3) (2013) 710–723.

[29] Z. Yan, J. Fan, J. Wang, A collective neurodynamic approach to constrained global optimization, IEEE Trans. Neural Netw. Learn. Syst. 28 (5) (2017) 1206–1215.

[30] X. Le, J. Wang, A two-time-scale neurodynamic approach to constrained minimax optimization, IEEE Trans. Neural Netw. Learn. Syst. 28 (3) (2017) 620–629.

[31] W. Bian, L. Ma, S. Qin, X. Xue, Neural Network for nonsmooth pseudocovex optimization with general convex constraints, Neural Netw. 101 (2018) 1–14.

[32] X. Shen, W. Pan, Y. Zhu, Likelihood-based selection and sharp parameter estimation, J. Amer. Statist. Assoc. 107 (497) (2012) 223–232.

[33] Z. Zheng, Y. Fan, J. Lv, High dimensional thresholded regression and shrinkage effect, J. R. Stat. Soc. Ser. B. Meth. 76 (3) (2014) 627–649.

[34] S. Foucart, M. J. Lai, Sparsest solutions of underdetermined linear systems via ε0-minimization for 0 < q ≤ 1, Appl. Comput. Harmon. Anal. 26 (3) (2009) 395–407.

[35] T. Zhang, Multi-stage convex relaxation for feature selection, Bernoulli 19 (SB) (2013) 2277–2293.

[36] J. Fan, R. Li, Variable selection via nonconvex penalized likelihood and its oracle properties, J. Amer. Statist. Assoc. 96 (456) (2001) 1348–1360.

[37] C. Zhang, Nearly unbiased variable selection under concave penalty, Ann. Stat. 38 (2) (2010) 894–942.

[38] Z. Xu, X. Chang, F. Xu, H. Zhang, L1,q regularization: a thresholding representation theory and a fast solver, IEEE Trans. Neural Netw. Learn. Syst. 23 (7) (2012) 1013–1027.

[39] W. Bian, X. Chen, Smoothing Neural Network for constrained non-Lipschitz optimization with applications, IEEE Trans. Neural Netw. Learn. Syst. 23 (3) (2012) 399–411.

[40] W. Bian, X. Chen, Neural Network for nonsmooth, nonconvex constrained minimization via smooth approximation, IEEE Trans. Neural Netw. Learn. Syst. 25 (3) (2014) 545–556.

[41] W. Li, W. Bian, X. Xue, Projected neural network for a class of non-Lipschitz optimization problems with linear constraints, IEEE Trans. Neural Netw. Learn. Syst. 31 (9) (2020) 3361–3373.

[42] W. Bian, X. Chen, A unified view of exact continuous penalties for εp,q–minimization, SIAM J. Optim. 27 (3) (2017) 2034–2060.

[43] G. M. Fung, O. L. Mangasarian, Equivalence of minimal ε0– and εp,q–norm solutions of linear equalities, inequalities and linear programs for sufficiently small p, J. Optim. Theory Appl. 151 (1) (2011) 1–10.

[44] Y. Zhao, X. He, T. Huang, J. Huang, P. Li, A smoothing neural network for minimization L1–Lq in sparse signal reconstruction with measurement noises, Neural Netw. 122 (2020) 40–53.

[45] R. Feng, C.-S. Leung, A. G. Constantinides, W.-J. Zeng, Lagrange programming neural network for nondifferentiable optimization problems in sparse approximation, IEEE Trans. Neural Netw. Learn. Syst. 28 (10) (2017) 2395–2407.
[48] E. A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, New York: McGraw-Hill Book Co., Inc., 1955.
[49] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, New York: Academic, 1980.
[50] F. H. Clarke, Optimization and Nonsmooth Analysis, New York: Wiley, 1983.
[51] J. K. Hale, Ordinary Differential Equations, New York: Wiley, 1980.
[52] J. P. Aubin, A. Cellina, Differential Inclusions: Set-Valued Maps and Viability Theory, New York: Springer-Verlag, 1984.
[53] X. Chen, M. K. Ng, C. Zhang, Non-Lipschitz $\ell_p$-regularization and box constrained model for image restoration, IEEE Trans. Image Process. 21 (12) (2012) 4709–4721.
[54] W. Bian, X. Chen, Optimality and complexity for constrained optimization problems with nonconvex regularization, Math. Oper. Res. 42 (4) (2015) 1063–1084.
[55] R. Chartrand, V. Staneva, Restricted isometry properties and nonconvex compressive sensing, Inverse Probl. 24 (3) (2008) 657–682.
[56] J. Huang, J. L. Horowitz, S. Ma, Asymptotic properties of bridge estimators in sparse high-dimensional regression models, Ann. Stat. 36 (2) (2008) 587–613.
[57] G. Li, S. Song, C. Wu, Generalized gradient projection neural networks for nonsmooth optimization problems, Sci. Chin. Inf. Sci. 53 (5) (2010) 990–1005.
[58] X. Chen, Smoothing methods for nonsmooth, nonconvex minimization, Math. Program. 134 (1) (2012) 71–99.
[59] T. Stamey, J. Kabalin, J. McNeal, I. Johnstone, F. Freiha, E. Redwine, N. Yang, Prostate specific antigen in the diagnosis and treatment of adenocarcinoma of the prostate. II. Radical prostatectomy treated patients, J. Urology 141 (1989) 1076–1083.