Crossings States and Sets of States in Random Walks

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Abstract
We consider a random walk in \( \mathbb{Z}^2 \) and establish a number of results for the distributions and expectations of the number of usual (undirected) and specifically defined in the paper directed state-crossings and different sets of states crossings. As well, we extend the results to \( d \)-dimensional random walks, \( d \geq 2 \), in bounded areas.

Keywords Simple random walks · Level-crossings and state-crossings · Undirected state-crossings · Directed state crossings · Birth-and-death processes · Markovian single-server queues · Markov fields · Branching processes

1991 Mathematics Subject Classification 60G50 · 60J80 · 60C05 · 60K25

1 Introduction

1.1 Formulation of the Problem and the Literature

In this paper we study simple random walks, a well-known object of the study in probability theory Spitzer (2001). The known results on simple random walks can be found in many sources (e.g. Doyle and Snell 1984; Hughes 1995; Lyons and Peres 2016; Spitzer 2001).

The present paper studies the probability distributions and expectations of the number of usual (undirected) and specifically defined (directed) state-crossings for random walks in \( \mathbb{Z}^2 \). The definitions of these notions are given later in the paper. We also consider multidimensional random walks in bounded areas.

The time parameter \( t \) is a discrete (integer) parameter. At time \( t = 0 \), the random walk starts from \( \mathbf{0} \), where \( \mathbf{0} = (0, 0) \), and, after some excursion, it visits \( \mathbf{0} \) again for the first time after \( t = 0 \). This random stopping time of the excursion is denoted by \( \tau \). For the random walk in \( \mathbb{Z}^2 \) it exists with probability 1 (see e.g. Spitzer 2001).

Vectors and their components are denoted as follows: \( \mathbf{n} = (n^{(1)}, n^{(2)}) \), \( \mathbf{n}_i = (n_i^{(1)}, n_i^{(2)}) \), \( \mathbf{S}_t = (S_t^{(1)}, S_t^{(2)}) \) and so on, by using bold font Latin (lower or upper case) letters for vectors.
and the italic font Latin letters with order number indices for components. The words *vector* and *state* are used interchangeably.

The random walks in $\mathbb{Z}^2$ are defined as follows. The vector $S_t = (S^{(1)}_t, S^{(2)}_t)$ denotes the state of the random walk at time $t$ and is defined recurrently as follows:

$$S_0 = 0,$$

$$S_t = S_{t-1} + e_t(\mathbb{Z}^2), \quad t \geq 1,$$

where the random vector $e_t(\mathbb{Z}^2)$ is in turn defined as follows. Let $\mathbf{1}_i$ denote the vector, the $i$th component of which is 1, and the rest component are 0. Then, the vector $e_t(\mathbb{Z}^2)$ is one of the 4 vectors $\{\pm \mathbf{1}_i, i = 1, 2\}$ that is randomly chosen with probability 1/4 independently of the other vectors and the history of the random walk.

The reason for the unusual notation $e_t(\mathbb{Z}^2)$ is that, the recurrence relations similar to (1) describe a variety of different random walks, in which there is only the difference in the definition of the vector $e_t(\bullet)$. Specifically, in the place of $\bullet$ one can use the subsets of $\mathbb{Z}^2$ for which the random walk is defined. For instance, $e_t(\mathbb{Z}^2_+)$, $e_t([-N,N]^d)$ or $e_t([0,N]^d)$ is the notation for the increments of the random walks in $\mathbb{Z}^2_+$, $[-N,N]^d$ or $[0,N]^d$, respectively.

Level-crossings are widely known in probability theory and its applications, and there is an extensive bibliography on this subject. Recent paper on this subject related to the theory of random walks and diffusions can be found in (Łochowski and Ghomrasni 2014; Mijatović and Vysotsky 2020a, b). The aim of the present paper is to provide direct studies for the distributions of state-crossings and crossings special sets of states. As well, we obtain the results for the expectations.

The initial point of our study are level-crossings for the symmetric one-dimensional random walk. Level-crossings for that random walk are mentioned in a number of sources. The known result is the following property. Suppose that the random walk starts at 0 and, after some excursion, at the first time returns to zero again. Then for any level $i \neq 0$, the expected number of crossings level $i$ is the same for all $i$ and equal to 1. This result can be found in a number of textbooks such as (Feller 1991; Durrett 2013; Szekély 1986; Wolff 1989). Following the title of the book by Szekély (1986), the aforementioned result is specified as a paradox in probability theory.

Note that the continuous time analogue of the symmetric one-dimensional random walk reflected at zero is the $M/M/1$ queueing system with equal interarrival and service times means. For the series of finite capacity $M/M/1/n$ queueing systems, the aforementioned property of random walk can be reformulated as follows. If the expectations of interarrival and service times are equal, then the expected number of losses during a busy period in the $M/M/1/n$ queueing system is equal to 1 for any $n$. Surprisingly, the last claim holds true for the series of $M/G/1/n$ queueing systems with generally distributed service times (Abramov 1997, 2010, 2009; Righter 1999; Wolff 2002). For characterization of the properties of losses in general queues see Abramov (2013).

### 1.2 Methodology and Contribution

We adapt the level-crossings method originally used in Abramov (1991) and developed in Abramov (2001, 2006) for multidimensional random walks. One-dimensional particular case considered in the paper demonstrates an example for the further development of the method in the multidimensional cases. By using that level-crossings method we establish...
connection between the distributions of number of state-crossings and certain sets of states crossings with the distributions of the random fields defined in the paper. For the symmetric one-dimensional random walk the distribution of the number of level-crossings is expressed via the distribution of the number of offspring in a certain branching process. Although the results in the one-dimensional case cannot be considered as new, the connection between the number of level-crossings and branching process is a novel knowledge that has not been mentioned before. The results for two- and multi-dimensional random walks all are new. The distributions of the number of state-crossings and crossings sets of states are expressed via the distributions of the special random fields introduced in the paper. It is proved that the expected number of state-crossings is equal to one for any state \( n \). As well, the expected number of directed state-crossings from the above and below are obtained. These results are derived based on the approach in Abramov (2018).

1.3 Historical Background and Possible Applications

Level-crossings approach goes back to the classic Doob’s martingale convergence theorem under the titles up-crossings and down-crossings (see Doob 1953). Level-crossings applications in probability theory, and in particular in queueing theory and dam systems were initiated in 1970th in a number of papers by Brill and Posner (e.g. Brill 1979; Brill and Posner 1977) and summarised in Brill (2008) and by Cohen (1976, 1977). Application to inventory systems can be found in Azoury and Brill (1986) and to credit risk models in (Jarrow et al. 2007; Sezer 2007).

Applications in statistics of stochastic processes for parameter estimates, testing hypothesis and simulation can be found in (Burq and Jones 2008; Jones and Shen 2004; Rolls and Jones 2015). Level-crossings of stationary Gaussian processes re-viewed by Kratz (2006). The further references can be found there. See also Adler (1976a, b) and Adler and Hasofer (1976).

A special circle of problems associated with level-crossings of state-dependent Markovian queueing systems with or without losses has been considered in Abramov (1991). Application of level-crossings to epidemic models has been given in Abramov (1994). Being further developed in the present paper for multi-dimensional random walks, the method may have a number of novel applications that include complex queueing systems with losses, in which the losses from a queue can depend on parameters, the behavior of which is described by two- or multi-dimensional random walk. Another possible area of applications is multi-type epidemic models that generalize the models studied in Abramov (1994).

1.4 Outline of the Paper

The rest of the paper is organized as follows. In Section 2, we provide necessary notation, definitions, remarks and examples. In Section 3, we formulate the main results of the paper. In Section 4, we formulate and prove the claim for level-crossings in the one-dimensional random walk \( S_t \). The proof for the one-dimensional random walk is important, since it makes the further proofs of the main results more understandable and clearer. In Sections 5, 6, 7 and 8 we prove the main results of the paper. The proof of Theorem 4 given in Section 8 is independent of the proofs of Theorems 1 and 2. In Section 9, the results of the paper are developed for \( d \)-dimensional random walks in bounded areas. In Section 10, we discuss crossings sets and states for random walks in \( \mathbb{Z}^d \) for \( d \geq 3 \) and formulate a
conjecture on the behaviour of the expected state-crossings. In Section 11 we provide some numerical studies for the crossings states and sets of states for the two-dimensional random walk. In Section 12, we resume the part of the results of the paper related to the expectations of crossings sets and states of sets. To make the paper self-contained we added the Appendix, the results of which are used to prove some key results of the paper.

2 Notation, Definitions, Remarks and Examples

The following notation is used in the paper. For any vector \( \mathbf{n} \in \mathbb{Z}^2 \), its \( l_1 \)-norm is

\[
\| \mathbf{n} \| = |n^{(1)}| + |n^{(2)}|
\]

For \( \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_i \in \mathbb{Z}^2 \), let \( \mathcal{Z} = \{ \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_i \} \) be a set of these vectors. If the vectors \( \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_i \) all have the same norm \( n \), then we write \( \| \mathcal{Z} \| = n \). In this case the set \( \mathcal{Z} \) is called normed set.

The vector \( |\mathbf{n}| = (|n^{(1)}|, |n^{(2)}|) \) is called module of the vector \( \mathbf{n} \). An important example of a normed set \( \mathcal{Z} \) is the set

\[
\mathcal{X}(\mathbf{n}) = \{ \mathbf{m} : \mathbf{m} = |\mathbf{n}| \}.
\]

For instance, for vectors \((1, 2)\) and \((0, 1)\) we have \( \mathcal{X}(1,2) = \{(1,2), (-1,2), (1,-2), (-1,-2)\} \), and \( \mathcal{X}(0,1) = \{(0,1), (0,-1)\} \). Two other important examples of normed sets \( \mathcal{Z} \) are the set of all vectors in \( \mathbb{Z}^2 \) with norm \( n \) and the set of all vectors in \( \mathbb{Z}_{+}^2 \) with norm \( n \). These sets are denoted by \( \mathcal{N}(n) \) and \( \mathcal{N}^+(n) \), respectively.

Let \( \mathbf{n} \) be a vector. A vector \( \mathbf{m} \) is said to be a neighbor of vector \( \mathbf{n} \) if

\[
\| \mathbf{m} - \mathbf{n} \| = 1.
\]

If, in addition, \( \| \mathbf{n} \| = \| \mathbf{m} \| + 1 \), then the vector \( \mathbf{m} \) is said to be lower neighbor of \( \mathbf{n} \). Otherwise, if \( \| \mathbf{n} \| = \| \mathbf{m} \| - 1 \), then the vector \( \mathbf{m} \) is called upper neighbor. The set of all lower neighbors of the vector \( \mathbf{n} \) is denoted \( \mathcal{M}^- (\mathbf{n}) \), and the set of all upper neighbors of the vector \( \mathbf{n} \) is denoted \( \mathcal{M}^+(\mathbf{n}) \). Let \( \mathbf{n} = (n^{(1)}, n^{(2)}) \). The number of zero components of this vector is denoted by \( d_0(\mathbf{n}) \). That is, \( d_0(\mathbf{n}) = 1 \) if either \( n^{(1)} = 0 \) or \( n^{(2)} = 0 \) or \( n^{(1)} \neq 0 \) or \( n^{(2)} \neq 0 \).

The total number of vectors in the set \( \mathcal{M}^- (\mathbf{n}) \cup \mathcal{M}^+(\mathbf{n}) \) is the same for all \( \mathbf{n} \in \mathbb{Z}^2 \) and equal to four. The total number of vectors in the sets \( \mathcal{M}^- (\mathbf{n}) \) and \( \mathcal{M}^+(\mathbf{n}) \) are \( d - d_0(\mathbf{n}) \) and \( d + d_0(\mathbf{n}) \), respectively. In Fig. 1, we illustrate lower and upper neighbor vectors in two situations. In one situation, the vectors \( \mathbf{k}, \mathbf{l} \) and \( \mathbf{m} \) are upper neighbors of the vector \( \mathbf{n} \) the total number of which is three, while there is only the single lower neighbor vector \( \mathbf{p} \). The vector \( \mathbf{n} \) is located on the boundary, so \( d_0(\mathbf{n}) = 1 \). In the other situation where the vector \( \mathbf{c} \) is in interior, the vectors \( \mathbf{a} \) and \( \mathbf{b} \) are upper neighbors of the vector \( \mathbf{c} \). The two other vectors \( \mathbf{d} \) and \( \mathbf{e} \) are its lower neighbors.

The unit vector \((1, 1)\) is denoted by \( \mathbf{1} \). Writing \( \mathbf{n} \geq \mathbf{m} \) means that \( n^{(i)} \geq m^{(i)} \) for \( i = 1, 2 \), e.g. \( \mathbf{n} \geq \mathbf{1} \) means \( n^{(1)} \geq 1 \) and \( n^{(2)} \geq 1 \); \( \mathbf{n} \neq \mathbf{0} \) means that at least one of \( n^{(1)} \), \( n^{(2)} \) differs from 0. The following additional widely known conventions are as follows:

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix} = 1 \quad \text{and for} \quad n < k, \left( \begin{array}{c}
\frac{n}{k}
\end{array} \right) = 0.
\]

**Definition 1** Let \( \mathcal{Z} = \{ \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_t \} \) be a normed set. The random variable \( f(\mathcal{Z}) \) is called the number of crossings \( \mathcal{Z} \), if there exist time instants \( 0 < t_1(\mathcal{Z}) < t_2(\mathcal{Z}) < \ldots < t_{f(\mathcal{Z})}(\mathcal{Z}) < \tau \).
such that $S_{t_i(\mathcal{Z})} \in \mathcal{Z}, i = 1, 2, \ldots, f(\mathcal{Z})$. If the set $\mathcal{Z}$ contains the only single element $\mathbf{n}$, then the notation $f(\mathbf{n})$ is used, and $f(\mathbf{n})$ is called the number of state-crossings $\mathbf{n}$.

**Remark 1** It follows from Definition 1 that $f(\mathcal{Z}) = \sum_{i=1}^l f(\mathbf{n}_i)$. That is, $f(\mathcal{Z})$ characterizes the total number of crossing the states $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_l$ belonging to $\mathcal{Z}$.

**Remark 2** Definition 1 does not explicitly use the norm of the set. Nevertheless, calling a set normed is important in the definition, since otherwise the definition becomes misleading. In the following two definitions the norm of the set is used explicitly.

**Definition 2** For a normed set $\mathcal{Z} = \{\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_l\}, \|\mathcal{Z}\| \neq 0$, the random variable $\bar{f}(\mathcal{Z})$ is called the number of up-directed crossings $\mathcal{Z}$, if there exist time instants $0 < t_1(\mathcal{Z}) < t_2(\mathcal{Z}) < \ldots < t_{\bar{f}(\mathcal{Z})}(\mathcal{Z}) < \tau$ such that $S_{t_i(\mathcal{Z})} \in \mathcal{Z}$ and $\|S_{t_i(\mathcal{Z})-1}\| = \|\mathcal{Z}\| - 1, i = 1, 2, \ldots, \bar{f}(\mathcal{Z})$. If the set $\mathcal{Z}$ contains the only single element $\mathbf{n}$, then the notation $\bar{f}(\mathbf{n})$ is used, and $\bar{f}(\mathbf{n})$ is called the number of up-directed state-crossings $\mathbf{n}$.

**Definition 3** For any normed set $\mathcal{Z} = \{\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_l\}$, the random variable $\underline{f}(\mathcal{Z})$ is called the number of down-directed crossings the set $\mathcal{Z}$, if there exist time instants $0 < t_1(\mathcal{Z}) < t_2(\mathcal{Z}) < \ldots < t_{\underline{f}(\mathcal{Z})}(\mathcal{Z}) < \tau$ such that $S_{t_i(\mathcal{Z})} \in \mathcal{Z}$ and $\|S_{t_i(\mathcal{Z})-1}\| = \|\mathcal{Z}\| + 1, i = 1, 2, \ldots, \underline{f}(\mathcal{Z})$. If the set $\mathcal{Z}$ contains the only single element $\mathbf{n}$, then the notation $\underline{f}(\mathbf{n})$ is used, and $\underline{f}(\mathbf{n})$ is called the number of down-directed state-crossings $\mathbf{n}$.
Remark 3 Unlike Definitions 2, Definition 3 implies that \( \mathcal{Z} \) can be equal to \( \{0\} \), and \( \tilde{f}(0) \) is defined. Notice that \( f(0) = \tilde{f}(0) \).

Remark 4 For one-dimensional random walk in \( \mathbb{Z} \) we use the scalar notation. The random walk is denoted by \( S_n \) and the numbers of state-crossings (or, more exactly, level-crossings) are denoted by \( f(n) \), \( \tilde{f}(n) \) or \( \hat{f}(n) \) for the cases of undirected, up-directed and down-directed level-crossings, respectively.

Definition 4 Let \( M^+(n) \) be the upper set of the vector \( n \), and let \( m_1, m_2, \ldots, m_k \) be the elements of \( M^+(n) \), where \( k = 2 + d_0(n) \) is the number of its elements. The new set containing the elements \( |m_1|, |m_2|, \ldots, |m_k| \) is denoted by \( |M^+(n)| \) and called positive upper set.

Example 1 For the vector \((-1, 0)\), we have \( M^+((-1, 0)) = \{(-2, 0), (-1, 1), (-1, -1)\} \). Hence, \( |M^+((-1, 0))| = \{(2, 0), (1, 1)\} \).

Notice that while the set \( M^+((-1, 0)) \) contains 3 elements, the number of elements in the set \( |M^+((-1, 0))| \) is only 2. This is because \( |(-1, 1)| = |(-1, -1)| = (1, 1) \). In this case, we say that the rank of element \((1, 1) \in |M^+((-1, 0))| \) is 2.

Below, the general definition of the rank of elements in \( |M^+(n)|, n \in \mathbb{Z}^2 \setminus \{0\} \), is provided.

Definition 5 We say that the element \( m \in |M^+(n)| \) has rank 2 if there are two distinct elements in \( M^+(n) \) denoted by \( m_1 \) and \( m_2 \) such that \( |m_1| = |m_2| = m \). If the set \( M^+(n) \) contains only a single element \( m \), such that \( |m_1| = m \), then the rank of the element \( m \) is 1. The rank of the element \( m \) will be denoted by \( r(m) \).

The property from Definition 5 is shown in Fig. 2. The points \( k, l \) and \( n \) all belong to \( M^+(m) \). Then, the only points \( k \) and \( n \) belong to \( |M^+(m)| \). The rank of the point \( k \) is 1, and the rank of the point \( n \) is 2 because of symmetric reflection from the boundary.

3 Main Results

We need to first define the random objects that are used in the formulation of the basic theorems. These random objects are the Markov fields \( P_n \) and \( Q_n \) and Markov chain \( R_n \) \((n > 0)\). The random objects \( P_n \) and \( Q_n \) are required to describe the probability distributions of the number of crossings the state \( n \) and sets of states \( \mathcal{X}(n) \), respectively. The Markov chain \( R_n \) is required to describe the distribution of the number of crossings the set of states \( \mathcal{N}(n) \). The random field \( P_n \) is defined on \( \mathbb{Z}^2 \), while the random field \( Q_n \) is defined on \( \mathbb{Z}^2_+ \).

3.1 The Markov Field \( P_n \)

The random field \( P_n, n \in \mathbb{Z}^2 \), is defined as follows. For any set \( \mathcal{Z} \subseteq \mathbb{Z}^2 \) denote by \( \mathcal{P}(\mathcal{Z}) \) the filtration of the set \( \{P_m : m \in \mathcal{Z}\} \). Then, with fixed \( P_0 = 1 \) for any \( n \in \mathbb{Z}^2 \setminus \{0\} \), and all \( k = 0, 1, \ldots, \), the Markov property

\[ P_{m+n} = P_m P_{m+n} \quad m \in \mathbb{Z}^2_+ \]
is assumed to be satisfied and, hence, the field $P_\mathbf{a}$ is Markov. (Recall that the set $\mathcal{M}^-(\mathbf{n}) \cup \mathcal{M}^+(\mathbf{n})$ is the set of all neighbor elements of the vector $\mathbf{n}$.) Denote by $p(\mathbf{m}, \mathbf{n})$ the number of immediate (one-step) transitions from the state $\mathbf{m}$ to the state $\mathbf{n}$, where $\mathbf{m} \in \mathcal{M}^-(\mathbf{n}) \cup \mathcal{M}^+(\mathbf{n})$. So,

$$P_\mathbf{n} = \sum_{\mathbf{m} \in \mathcal{M}^-(\mathbf{n}) \cup \mathcal{M}^+(\mathbf{n})} p(\mathbf{m}, \mathbf{n}).$$

Then the physical meaning of $P_\mathbf{n}$ is the total number of transitions to the state $\mathbf{n}$. In agreement with (3) and (4) the transitions $p(\mathbf{m}, \mathbf{n})$ must be Markov, since any transition $p(\mathbf{m}, \mathbf{n})$ depends only on the transitions to the state $\mathbf{m}$ from its neighbor states and does not depend on the states outside of $\mathcal{M}^-(\mathbf{m}) \cup \mathcal{M}^+(\mathbf{m})$. For instance,

$$P\{p(\mathbf{m}, \mathbf{n}) = k \mid p(\mathbf{m}^*, \mathbf{m}) = l\} = P\{p(\mathbf{m}, \mathbf{n}) = k \mid p(\mathbf{m}^*, \mathbf{m}) = l, p(\mathbf{m}_1, \mathbf{n}_1), p(\mathbf{m}_2, \mathbf{n}_2), \ldots\},$$

where $\mathbf{m}^* \in \mathcal{M}^-(\mathbf{m}) \cup \mathcal{M}^+(\mathbf{m})$, and $\mathbf{m}_i \notin \mathcal{M}^-(\mathbf{m}) \cup \mathcal{M}^+(\mathbf{m})$ and $\|\mathbf{m}_i - \mathbf{n}_i\| = 1, \ i = 1, 2, \ldots$.

We have the following elementary properties of $p(\mathbf{m}, \mathbf{n})$ and $P_\mathbf{n}$.
1. \( P_0 = 1. \)

2. \( P\{p(0, 1_1) = 1\} = P\{p(0, 1_1) = 1\} = P\{p(0, -1_1) = 1\} = P\{p(0, -1_2) = 1\} = 1/4, \) and \( \{p(0, 1_1) + p(0, 1_2) + p(0, -1_1) + p(0, -1_2) = 1. \)

3. \( P\{p(1_1, 0) = 1\} = P\{p(1_2, 0) = 1\} = P\{p(-1_1, 0) = 1\} = P\{p(-1_2, 0) = 1\} = 1/4, \) and \( \{p(1_1, 0) + p(1_2, 0) + p(-1_1, 0) + p(-1_2, 0) = 1. \)

4. if \( m \in M^- (n), ||n|| \geq 2, \) then

\[
P\{p(m, n) = n \mid \sum_{m^* \in M^+(m)} p(m^*, m) = k\} = \left( \frac{k + n - 1}{n}\right) \left( \frac{1}{4}\right)^n \left( \frac{3}{4}\right)^k.
\]

Note, that under the condition \( \{\sum_{m^* \in M^+(m)} p(m^*, m) = 1\} \) the conditional distribution is geometric, while under the more general condition \( \{\sum_{m^* \in M^+(m)} p(m^*, m) = k\} \) it is negative binomial. This means that the condition “generates” \( k \) independent geometrically distributed random variables, and the conditional distribution is the distribution of the sum of independent geometrically distributed random variables that yields negative binomial distribution. This property is widely used in the paper.

5. if \( m \in M^+(n), n \neq 0, \) then

\[
P\{p(m, n) = n \mid \sum_{m^* \in M^+(m)} p(m^*, m) = k\} = \left( \frac{k + n - 1}{n}\right) \left( \frac{1}{4}\right)^n \left( \frac{3}{4}\right)^k.
\]

3.2 The Markov Field \( Q_n \)

The random field \( Q_n, n \in \mathbb{Z}^2_+ \), is defined as follows. For any set \( Z \subset \mathbb{Z}^2_+ \) denote by \( \mathcal{Q}(Z) \) the filtration of the set \( \{Q_m : m \in Z\} \). Then, with fixed \( Q_0 = 1 \) for any \( n \in \mathbb{Z}^2_+ \setminus \{0\} \), and all \( k = 0, 1, \ldots \), the Markov property

\[
P\{Q_n = k \mid \mathcal{Q}(\mathbb{Z}^2_+ \setminus \{n\})\} = P\{Q_n = k \mid \mathcal{Q}(M^- (n) \cup M^+(n))\}
\]

is assumed to be satisfied and, hence, the field \( Q_n \) is Markov. Denote by \( q(m, n) \) the number of immediate (one-step) transitions from the state \( m \) to the state \( n \), where \( m \in M^- (n) \cup M^+(n) \). So,

\[
Q_n = \sum_{m \in M^- (n) \cup M^+(n)} q(m, n).
\]

So, similarly to \( P_n \), the meaning of \( Q_n \) is the total number of transitions to the state \( n \).

In agreement with (6) and (7) the transitions \( q(m, n) \) must be Markov, and similarly to (5)

\[
P\{q(m, n) = k \mid q(m^*, m) = l\} = P\{q(m, n) = k \mid q(m^*, m) = l, q(m_1, n_1), q(m_2, n_2), \ldots\},
\]

where \( m^* \in M^- (m) \cup M^+(m) \), and \( m_i \notin M^- (m) \cup M^+(m) \) and \( ||m_i - n_i|| = 1 \), \( i = 1, 2, \ldots \).

We have the following elementary properties of \( q(m, n) \) and \( Q_n \).

1. \( Q_0 = 1. \)

2. \( P\{q(0, 1_1) = 1\} = P\{q(0, 1_2) = 1\} = 1/2, \) and \( q(0, 1_1) + q(0, 1_2) = 1. \)

3. \( P\{q(1_1, 0) = 1\} = P\{q(1_2, 0) = 1\} = 1/2, \) and \( q(1_1, 0) + q(1_2, 0) = 1. \)
4. if \( \mathbf{m} \in |\mathcal{M}^+(\mathbf{n})| \) and \( \mathbf{n} \neq \mathbf{0} \), then

\[
P\left\{ q(\mathbf{m}, \mathbf{n}) = n \left| \sum_{\mathbf{m}^* \in |\mathcal{M}^+(\mathbf{m})|} q(\mathbf{m}^*, \mathbf{m}) = k \right. \right\} = \left( k + n - 1 \atop n \right) \left( \frac{1}{4} \right)^n \left( \frac{3}{4} \right)^k.
\]

5. if \( \mathbf{m} \in \mathcal{M}^-(\mathbf{n}) \), \( d_0(\mathbf{m}) = 1 \) and \( ||\mathbf{n}|| \geq 2 \), then

\[
P\left\{ q(\mathbf{m}, \mathbf{n}) = n \left| \sum_{\mathbf{m}^* \in \mathcal{M}^-(\mathbf{m})} q(\mathbf{m}^*, \mathbf{m}) = k \right. \right\} = \left( k + n - 1 \atop n \right) \left( \frac{r(\mathbf{n})}{4} \right)^n \left( \frac{4 - r(\mathbf{n})}{4} \right)^k,
\]

where \( r(\mathbf{n}) \) is the rank of the vector \( \mathbf{n} \in |\mathcal{M}^+(\mathbf{m})| \) (see Definition 5).

6. if \( \mathbf{m} \in \mathcal{M}^-(\mathbf{n}) \), \( d_0(\mathbf{m}) = 0 \) and \( ||\mathbf{n}|| \geq 2 \), then

\[
P\left\{ q(\mathbf{m}, \mathbf{n}) = n \left| \sum_{\mathbf{m}^* \in \mathcal{M}^-(\mathbf{m})} q(\mathbf{m}^*, \mathbf{m}) = k \right. \right\} = \left( k + n - 1 \atop n \right) \left( \frac{1}{4} \right)^n \left( \frac{3}{4} \right)^k.
\]
Property (5) for $\mu_{\mathbf{m}}(\mathbf{n}) = n | \sum_{\mathbf{m}^* \in \mathcal{M}^{-}(\mathbf{m})} q(\mathbf{m}^*, \mathbf{m}) = k$ is seen from Fig. 3. When the vector $\mathbf{m}$ belongs to the boundary and $\mathbf{n}$ is out of the boundary, the probability $P\{q(\mathbf{m}, \mathbf{n}) = 1 | \sum_{\mathbf{m}^* \in \mathcal{M}^{-}(\mathbf{m})} q(\mathbf{m}^*, \mathbf{m}) = 1\}$ is 1/4 rather than 3/16. This is due to the reflection mechanism giving the rate $2 \times 1/4 = 1/2$.

### 3.3 The Markov Chain $R_n$

Discrete time nonnegative integer-valued Markov chain $R_n$, $n \geq 1$, satisfies the following properties.

1. $R_0 = 1$. 
2. $P\{R_n = k | R_{n-1} = m\} = \binom{k + m - 1}{k} \left(\frac{2n-1}{4n}\right)^m \left(\frac{2n+1}{4n}\right)^k$.

### 3.4 Formulation of the Results

The main results of the paper are specified as follows. Theorem 1 is the theorem on the probability distribution of the number of crossings the set $\mathcal{N}(n)$. Theorem 2 is the theorem on the probability distributions of the number of crossings the sets $\mathcal{A}(\mathbf{n})$. Theorem 3 is the theorem on the probability distributions of the number of state-crossings. Theorem 4 describes the expectations of crossings the sets of states $\mathcal{X}(\mathbf{n})$ and states $\mathbf{n}$. Specifically, relation (19) of the theorem contains the important claim that the expectation of the number of crossings any state $\mathbf{n}$ is equal to 1, and relations (17) and (18) nontrivial results on the expected numbers of up- and down-directed crossings of state $\mathbf{n}$.

**Theorem 1** For any integer $n \geq 1$ and all $k = 0, 1, \ldots$,

$$P\{\tilde{f}[\mathcal{N}(n)] = k\} = P\{R_{n-1} = k\}, \tag{8}$$

$$P\{\tilde{f}[\mathcal{N}(n)] = k\} = P\{R_n = k\}, \tag{9}$$

$$P\{f[\mathcal{N}(n)] = k\} = P\{R_{n-1} + R_n = k\}. \tag{10}$$

**Remark 5** Relations (9) and (10) are defined for $n = 0$ as well. Specifically,

$$P\{\tilde{f}[\mathcal{N}(0)] = 1\} = P\{f[\mathcal{N}(0)] = 1\} = P\{\tilde{f}[\mathcal{N}(1)] = 1\} = 1.$$

**Theorem 2** For any $\mathbf{n} \in \mathbb{Z}_+^2 \setminus \{0\}$ and all $k = 0, 1, \ldots$,

$$P\{\tilde{f}[\mathcal{A}(\mathbf{n})] = k\} = P\left\{\sum_{\mathbf{m} \in \mathcal{M}^{-}(\mathbf{n})} q(\mathbf{m}, \mathbf{n}) = k\right\}, \tag{11}$$

$$P\{\tilde{f}[\mathcal{A}(\mathbf{n})] = k\} = P\left\{\sum_{\mathbf{m} \in \mathcal{M}^{+}(\mathbf{n})} q(\mathbf{m}, \mathbf{n}) = k\right\}. \tag{12}$$
and

$$P\{f[\lambda(n)] = k\} = P\{Q_n = k\}. \quad (13)$$

**Remark 6** \(P\{\bar{f}[\lambda(0)] = 1\} = P\{\bar{f}(0) = 1\} = P\{f(0) = 1\} = 1.\)

**Theorem 3** For any \(n \in \mathbb{Z}^2 \setminus \{0\}\) and all \(k = 0, 1, \ldots,\)

$$P\{\bar{f}(n) = k\} = P\left\{ \sum_{m \in M^{-}(n)} p(m, n) = k \right\}, \quad (14)$$

$$P\{\bar{f}(n) = k\} = P\left\{ \sum_{m \in M^{+}(n)} p(m, n) = k \right\}, \quad (15)$$

and

$$P\{f(n) = k\} = P\{P_n = k\}. \quad (16)$$

**Theorem 4** For any \(n \in \mathbb{Z}^2 \setminus \{0\}\), we have:

$$E\{\bar{f}(n)\} = 2^{d_0(n)-2}E\{\bar{f}[\lambda(n)]\} = \frac{2 - d_0(n)}{4}, \quad (17)$$

$$E\{\bar{f}(n)\} = 2^{d_0(n)-2}E\{\bar{f}[\lambda(n)]\} = \frac{2 + d_0(n)}{4}, \quad (18)$$

and

$$E\{f(n)\} = 2^{d_0(n)-2}E\{f[\lambda(n)]\} = 1. \quad (19)$$

4 Level Crossings in the One-Dimensional Random Walk

Let \(Z_n\) denote the number of offspring in the \(n\)th generation of the Galton-Watson branching process with \(Z_0 = 1\) and the offspring distribution \(P\{Z = k\} = 1/2^{k+1}\). We have the following statement.

**Proposition 1** For the one-dimensional random walk \(S_t\), the following results are true. For all \(n \neq 0\) and \(k = 1, 2, \ldots\)

$$P\{\bar{f}(n) = k\} = \frac{1}{2}P\{Z[n-1] = k\}, \quad (20)$$

$$P\{f(n) = k\} = \frac{1}{2}P\{Z[n] = k\}, \quad (21)$$

and
\[ \mathbb{P}\{f(n) = k\} = \frac{1}{2} \mathbb{P}\{Z_{|n|-1} + Z_{|n|} = k\}, \]

and the probabilities \( \mathbb{P}\{\tilde{f}(n) = 0\} \), \( \mathbb{P}\{\tilde{f}(n) = 0\} \) and \( \mathbb{P}\{f(n) = 0\} \) are determined from the normalization conditions:

\[ \mathbb{P}\{\tilde{f}(n) = 0\} = \frac{1}{2} + \frac{1}{2} \mathbb{P}\{Z_{|n|-1} = 0\}, \]

\[ \mathbb{P}\{\tilde{f}(n) = 0\} = \frac{1}{2} + \frac{1}{2} \mathbb{P}\{Z_{|n|} = 0\}, \]

\[ \mathbb{P}\{f(n) = 0\} = \frac{1}{2} + \frac{1}{2} \mathbb{P}\{Z_{|n|-1} + Z_{|n|} = 0\}. \]

**Proof** The proof of this theorem is based on the analysis of the random walk \( S_t \) from the position of queueing theory (similarly to that was provided in Abramov (2018)). Assume that in each of its states the random walk \( S_t \) stays an exponentially distributed time with parameter 1, prior moving to the next state. Then, the parameter \( t \) in \( S_t \) means the \( t \)th event of the associated Poisson process with rate 1. The meaning of the random time instant \( \tau \) is then the \( \tau \)th event of the same Poisson process (with random order number). This unusual replacement of deterministic \( t \) by random is made for reduction to queueing processes and using the well-known properties of Poisson process and exponential distribution.

Since the random walk is symmetric, it is clear that the number of level-crossings of any negative level, say \((-5)\), has the same distribution as the number of level-crossings of the corresponding positive level 5. On the other hand, if \( S_1 = -1 \), then the following excursion of the random walk before reaching the initial point 0 will be in the negative area only, and otherwise, if \( S_1 = +1 \), in the only positive area. This means that the random walk can be studied in the positive area, starting from \( S_1 = 1 \). That is, for positive \( n \), and for all \( k = 0, 1, \ldots, \)

\[ \mathbb{P}\{\tilde{f}(n) = k\} = \frac{1}{2} \mathbb{P}\{\tilde{f}(n) = k|S_1 = 1\}, \]

\[ \mathbb{P}\{\tilde{f}(n) = k\} = \frac{1}{2} \mathbb{P}\{\tilde{f}(n) = k|S_1 = 1\}, \]

and

\[ \mathbb{P}\{f(n) = k\} = \frac{1}{2} \mathbb{P}\{f(n) = k|S_1 = 1\}. \]

Apparently, \( \mathbb{P}\{\tilde{f}(1) = k|S_1 = 1\} = \delta_{k,1} \), where \( \delta_{k,j} \) is Kronecker’s delta.

Consider now the \( M/M/1/n \) queueing system (with \( n - 1 \) waiting places), in which interarrival and service times are exponentially distributed with same parameter. The value of the parameter does not matter. However, to make the paths of queueing process and random walk equivalent, one can reckon that the mean interarrival and service times are equal to 2. In that case, the times between the consecutive events (such as arrivals and service completions) are exponentially distributed with parameter 1.
Lemma 1 Let \( L_n \) denote the number of losses during a busy period. Then, for all \( k = 0, 1, \ldots \),

\[
P\{L_n = k\} = P\{\bar{f}(n + 1) = k|S_1 = 1\} = P\{Z_n = k\}.
\] (29)

**Proof** The number of losses \( L_n \) in the \( M/M/1/n \) queueing system can be specified as follows. Let \( z_j \) denotes the number of times during the busy period when an arriving customer finds \( j \) customers in the system \((0 \leq j \leq n - 1)\), and let \( a_1^{(j)}, a_2^{(j)}, \ldots, a_{z_j}^{(j)} \) denote the moments of arrivals when an arriving customer finds \( j \) customers in the system, \( b_1^{(j)}, b_2^{(j)}, \ldots, b_{z_j}^{(j)} \) denote the moments of service completions (departures) of the customers after which there are remain only \( j \) customers in the system. Apparently, \( z_0 = 1, a_1^{(0)} \) is the moment when the busy period starts, and \( b_1^{(0)} \) is the moment when the busy period ends. The time intervals

\[
[a_1^{(j)}, b_1^{(j)}), [a_2^{(j)}, b_2^{(j)}), \ldots, [a_{z_j}^{(j)}, b_{z_j}^{(j)})
\] (30)

are contained in the intervals

\[
[a_1^{(j-1)}, b_1^{(j-1)}), [a_2^{(j-1)}, b_2^{(j-1)}), \ldots, [a_{z_j-1}^{(j-1)}, b_{z_j-1}^{(j-1)})
\] (31)

\((j \geq 1)\). Let us delete the intervals of (30) from those of (31) and merge the ends. Then, according to the property of the lack of memory, the number of inserted points in each of the intervals of (31) coincides in distribution with the number of arrivals per service time and has geometric distribution with parameter \( 1/2 \). This enables us to conclude that \( \{z_j\} \) have the structure of the Galton-Watson branching process, where \( z_j \) is the number of offspring in the \( j \)th generation with \( z_0 = 1 \) and \( P\{z_1 = k\} = \left(\frac{1}{2}\right)^k, k = 0, 1, \ldots \). That is, \( z_j = Z_j, (j = 0, 1, \ldots, n - 1)\). Let us consider the specific intervals

\[
[a_1^{(n-1)}, b_1^{(n-1)}), [a_2^{(n-1)}, b_2^{(n-1)}), \ldots, [a_{z_n-1}^{(n-1)}, b_{z_n-1}^{(n-1)}).
\] (32)

The arrival instants \( a_1^{(n)}, a_2^{(n)}, \ldots, a_{z_n}^{(n)} \) are the instants of losses from the system. Apparently, \( L_n \) must coincide in distribution with \( Z_n \), the number of offspring in the \( n \)th generation of the branching process.

Consider now the family of \( M/M/1/n \) queueing systems, \( n = 1, 2, \ldots \). On the basis of the above construction, \( L_1 \) must coincide in distribution with \( Z_1, L_2 \) with \( Z_2 \) and so on, and the family of queueing processes associated with the \( M/M/1/n \) queueing systems together with the \( M/M/1 \) queueing process can be considered on the common probability space. The paths of the random walk are given on the same probability space, and relation (29) is clear due to the coincidence of the paths of the \( M/M/1 \) queueing process and random walk.

From Relation (29) of Lemma 1 and (26) we obtain (20). In order to obtain (21) and (22), we are to prove the following lemma.

Lemma 2 For \( n > 0 \) the following obvious relations are true with probability 1:

\[
\bar{f}(n) = \bar{f}(n + 1),
\] (33)

\[
f(n) = \bar{f}(n) + \bar{f}(n + 1).
\] (34)
Proof Indeed, the total number of (undirected) level-crossings of the level \( n \) includes the number of level-crossings of level \( n \) from the below plus those from the above, i.e.

\[
f(n) = \bar{f}(n) + \tilde{f}(n).
\]

(35)

Apparently, \( \bar{f}(n) \) coincides with \( \tilde{f}(n+1) \), since the number of down-crossings from \( n+1 \) to \( n \) is equal to the number of up-crossings from \( n \) to \( n+1 \), and (33) follows. Its substitution for (35) yields (34).

Continuation of the proof of Proposition 1. Now, Relation (21) follows from (33) and (27), and Relation (22) follows from (34) and (28). From the normalization condition, for \( P\{\bar{f}(n) = 0\} \) we obtain:

\[
P\{\bar{f}(n) = 0\} = 1 - \sum_{k=1}^{\infty} P\{\bar{f}(n) = k\}
\]

\[
= 1 - \frac{1}{2} \sum_{k=1}^{\infty} P[Z_{[n]-1} = k]
\]

\[
= 1 - \frac{1}{2} (1 - P[Z_{[n]-1} = 0])
\]

\[
= \frac{1}{2} + \frac{1}{2} P[Z_{[n]-1} = 0],
\]

and (23) is proved. The derivation of (24) and (25) is similar. Proposition 1 is proved.

5 Proof of Theorem 1

5.1 Prelude

In the proof of this and the following theorems, it is assumed that in each of its states the random walk stays an exponentially distributed time with parameter 1, prior moving to the next state. This change of time is similar to that made in Section 4 and was used in Abramov (2018) in more general situation. In the case of the present study, the change of states in the two-dimensional random walk is associated with four independent and identical Poisson processes with rate 1/4 each. Specifically, marking them

\[
a_{i,1}, \quad a_{1,2}, \quad a_{2,1}, \quad a_{2,2}
\]

\[
\begin{align*}
&1_i, \quad -1_i, \quad 1_2, \quad -1_2
\end{align*}
\]

indicates that the Poisson process \( a_{i,1} \) is associated with the direction \( 1_i \), and the Poisson process \( a_{i,2} \) is associated with the direction \( (-1)_i \), \( i = 1, 2 \). Then, the direction of the random walk is associated with the minimum of four exponentially distributed “inter-jump” times.

This construction enables us to model the two-dimensional random walk in the main quarter plane as two independent and identical queueing systems. Study of the random walk in the main quarter plane helps us to model the required characteristics \( \bar{f}[N(n)] \), \( \tilde{f}[N(n)] \) and \( f[N(n)] \), and reduction to continuous time processes essentially simplify the analysis, which is based on well-known and elementary results of continuous Markov chains. Then, the parameter \( t \) in \( S_t \) means the \( t \)th event of the associated Poisson process.
with rate 1. The meaning of the random time instant $\tau$ is then the $\tau$th event of the same Poisson process. In the sequel, any phrase like *time moment $x$* means that it is spoken about the $x$th event of the Poisson process.

### 5.2 Description of Queueing Systems and the Final Measures

This section is a simplified version of the corresponding place in Abramov (2018), where more general queueing systems have been considered. The description of the queueing model for the random walk in $\mathbb{Z}^d$ is provided in the Appendix and reproduced here in the case of random walk in $\mathbb{Z}^2$ for the convenience of reading. Each of the two identical queueing systems is described as follows. Arrivals to each queueing system are Poisson with rate $1/4$, and service times are exponentially distributed with parameter $1/4$. The aforementioned value of arrival rate $1/4$ is not important, and later in the paper this value will be denoted by $\lambda$, as acceptable. Since the mean service and mean interarrival time are equal, the notation $\lambda$ will be used for the reciprocal of the mean service time as well. If a system becomes free, it is switched for a special service with the same mean. This service is negative, and it results in a new customer in the queue. If during a negative service a new arrival occurs, the negative service remains unfinished and not resumed. The negative service models the reflection at zero and in fact implies the state dependent arrival rate, which becomes equal to $2 \times (1/4) = 1/2$ at the moment when the system is empty. In terms of random walks, it is the situation, when an original (not reflected) one-dimensional random walk reaches zero at some time moment $s$, and at the next time moment $s+1$ it must take one of the values $\pm 1$, that corresponds to value +1 for an one-dimensional random walk reflected at zero.

In Abramov (2018), p. 1908, the transition probability from the set of states $\mathcal{N}(n)$ to the set of states $\mathcal{N}(n+1)$ has been derived. In the case of the two-dimensional random walk this probability $p_n(2)$ (the notation is from Abramov (2018)) is

$$p_n(2) = \frac{2n + 1}{4n},$$

and the transition probability from the set of states $\mathcal{N}(n)$ to the set of states $\mathcal{N}(n-1)$ is

$$q_n(2) = 1 - p_n(2) = \frac{2n - 1}{4n}.$$

The transition probabilities $p_n(2)$ and $q_n(2)$ coincide with the relative rates of the birth-and-death process $BD(2, 2)$ introduced in Abramov (2018):

$$p_n(2) = \frac{\lambda_n(2, 2)}{\lambda_n(2, 2) + \mu_n(2, 2)}$$

The details of the derivation of the formula for $p_n(d)$ is provided in the Appendix, and the formula for $p_n(2)$ is a particular case of that formula.

### 5.3 Basic Lemma and its Proof

The rest of the proof of the theorem is based on the lemma, which is an extension of Lemma 1. The proof of this lemma is fully similar to the proof of aforementioned Lemma 1 and reproduced here without specific details.
Lemma 3 Let \( \phi_t \) be a recurrent birth-and-death process with the parameters of birth and death \( \lambda_n \) and \( \mu_n \), respectively. Assume that \( \phi_0 = 1 \), and let \( \nu \) be the time moment of extinction of the birth-and-death process. Let \( g(n) \), \( n \geq 1 \), denote the number of times during the random interval \([0, \nu]\) when immediately before the time of a birth there become \( n \) individuals in the population, \( g(0) = 1 \). Then,

\[
\mathbb{P}\{g(n) = k\} = \mathbb{P}\{V_n = k\},
\]

where \( V_n \) is the positive integer-valued Markov chain satisfying the following properties:

\[
V_0 = 1, \quad \mathbb{P}\{V_{j+1} = k|V_j = m\} = \left( \frac{m + k - 1}{k} \right) \left( \frac{\lambda_j}{\lambda_j + \mu_j} \right)^{k} \left( \frac{\mu_j}{\lambda_j + \mu_j} \right)^{m}, \quad j = 0, 1, \ldots.
\]

**Proof** Let \( a_1(j), a_2(j), \ldots, a_{g(j)}(j) \) denote the birth times immediately before there become \( j \) individuals in the population, and let \( b_1(j), b_2(j), \ldots, b_{g(j)}(j) \) denote the death times, after which there remain only \( j \) individuals in the population. Apparently, based on the convention \( g(0) = 1 \), the times \( a_1(0) \) and \( b_1(0) \) are unique, \( a_1(0) \) is the moment of the first birth and \( b_1(0) \) is the extinction time. For \( 1 \leq j \), the time intervals

\[
[a_1(j), b_1(j)), [a_2(j), b_2(j)), \ldots, [a_{g(j)}(j), b_{g(j)}(j))
\]

are contained in the intervals

\[
[a_1(j - 1), b_1(j - 1)), [a_2(j - 1), b_2(j - 1)), \ldots, [a_{g(j-1)}(j - 1), b_{g(j-1)}(j - 1)).
\]

Let us delete the intervals of (39) from those of (40) and merge the ends. Then, according to the property of the lack of memory of exponential distribution, the number of merged points in each of the intervals of (40) coincides in distribution with the number of births per a death time of an individual in the population (the birth and death rates are reckoned to be unchanged and equal to \( \lambda_j \) and \( \mu_j \), respectively) and has geometric distribution with parameter \( \lambda_j / (\lambda_j + \mu_j) \), and given that the number of intervals in \( m \), the total number of merged points in these intervals has negative binomial distribution, that is,

\[
\mathbb{P}\{g(j + 1) = k|g(j) = m\} = \left( \frac{m + k - 1}{k} \right) \left( \frac{\lambda_j}{\lambda_j + \mu_j} \right)^{k} \left( \frac{\mu_j}{\lambda_j + \mu_j} \right)^{m}.
\]

This enables us to conclude that \( \{g(j)\} \) has a structure similar to that of the branching process. Specifically, the distribution of the number of offspring in the \( j \)th generation depends on the order number of generation, \( j \), and is described by the Markov chain \( V_j \).

Hence, the distribution of \( g(j) \) coincides with the distribution of \( V_j \), and (36), (37) and (38) are true.
5.4 Proof of (8), (9) and (10)

Applying Lemma 1 with \( \lambda_n = p_n(2) \) and \( \mu_n = q_n(2) \) we arrive at (8) with the Markov chain \( R_n \) defined in Section 3.3. To prove (9), note that \( \tilde{f}[N(n)] = \tilde{f}[N(n+1)] \), since the total number of crossings from the level \( n \) to \( n+1 \) must coincide with the total number of crossings from the level \( n+1 \) to \( n \). Next, \( f[N(n)] = \tilde{f}[N(n)] + \tilde{f}[N(n)] \) and (10) is true.

6 Proof of Theorem 2

6.1 Prelude

The proof is based on study of the random walk in \( \mathbb{Z}^2_+ \), which is defined as follows:

\[
S_0 = 0, \\
S_t = S_{t-1} + e_i(Z^2_+), \quad t \geq 1,
\]

where

\[
e_i(Z^2_+) = \begin{cases} e_i, & \text{if } S^{(i)}_{t-1} + e^{(i)}_i \geq 0 \text{ for all } i = 1, 2; \\ -e_i, & \text{if } S^{(i)}_{t-1} + e^{(i)}_i = -1 \text{ for a certain } i = 1, 2,
\end{cases}
\]

and the vector \( e_i = (e^{(1)}_i, e^{(2)}_i) = e_i(Z^2) \).

The random walk defined by (41) – (43) is the reflected version of the random walk defined by (1) and (2).

Hence, the distributions of \( \tilde{f}(n) \), \( \tilde{f}(n) \) and \( f(n) \) for the random walk in \( \mathbb{Z}^2_+ \) are equivalent to the distributions of \( \tilde{f}[\lambda(n)] \), \( \tilde{f}[\lambda(n)] \) and \( f[\lambda(n)] \), respectively, for the random walk in \( \mathbb{Z}^2 \).

According to the conventional notation, \( \tau \) is the time of the first return to the origin of the random walk in \( \mathbb{Z}^2_+ \). Apparently, if \( \tau' \) is the time of the first return to the origin of the random walk in \( \mathbb{Z}^2 \), the random times \( \tau \) and \( \tau' \) coincide in distribution, and being considered on the same probability space can be considered as identical. Hence, the only notation \( \tau \) will be used for either of random walks.

6.2 Proof of (11)

The random walk in \( \mathbb{Z}^2_+ \) is modeled by two independent queueing systems, the structure of which is described in Section 5.2.

At the initial time moment \( t = 0 \) both queueing systems are assumed to be empty, and, after \( t = 0 \), the first arrival to one of the queueing systems occurs. By busy period we mean the time interval \([1, \tau] \). (Recall that the time \( t \) is discrete, \( t = 1 \) is the moment of the first arrival to one of the two systems, which are initially empty, and \( \tau \) means the \( \tau \)'s event of the Poisson process, at which the queueing systems become empty once again at the first time since \( t = 0 \).)

The proof of (11) is based on an extension of the level-crossing technique given in the proof of Lemma 3 for the two-dimensional random walk.

Consider first the set of vectors \( N^+(n) \) (the set of all vectors in \( \mathbb{Z}^2_+ \) with norm \( n \)). The arguments for this set of vectors is similar to that provided before in the proof of Lemma 3. Recall
them in terms of the relevant objects and notation. Let \( z_n \) denote the number of cases when at the moment of arrival of a customer, the total (cumulative) number of customers in the queueing systems becomes \( n \). Let \( a_1(n), a_2(n), \ldots, a_{z_n}(n) \) be the moments of these arrivals, and let \( b_1(n), b_2(n), \ldots, b_{z_n}(n) \) be the moments of service completion, when there totally remain \( n - 1 \) customers.

Apparently, the time intervals

\[
[a_1(n), b_1(n)], [a_2(n), b_2(n)], \ldots, [a_{z_n}(n), b_{z_n}(n)]
\]  

(44)

are contained in the time intervals

\[
[a_1(n - 1), b_1(n - 1)], [a_2(n - 1), b_2(n - 1)], \ldots, [a_{z_{n-1}}(n - 1), b_{z_{n-1}}(n - 1)].
\]  

(45)

Deleting the intervals of (44) from those of (45) and merging the ends yields the set of points. The residual times in the points intervals merged have an exponential distribution, the parameter of which typically depends on the allocation structure of \( n - 1 \) customers in two queueing systems at the moment of the service completion. For instance, if one of the servers is empty, then the (residual) service rate is \( \lambda \) and total (residual) arrival rate is \( 3\lambda \). (The last includes the rate \( \lambda \) of a negative service, so \( 2\lambda + 3\lambda = 3\lambda \).)

Let \( n = 1 \) be a point that has norm \(|n| = n\) and thus belongs to \( N^\infty(n) \). Denote by \( z_n \) the number of cases when at the moment of a customer’s arrival to one of the queueing systems, there become \( n^{(1)} \), \( n^{(2)} \) customers in the corresponding queueing systems, the order numbers of which is indicated by the upper index, and \( n^{(1)} + n^{(2)} = n \). Let \( a_1(n), a_2(n), \ldots, a_{z_n}(n) \) be the moments of these arrivals, and let \( b_1(n), b_2(n), \ldots, b_{z_n}(n) \) be the moments of service completions following the first time after the corresponding times \( a_i(n) \), \( i = 1, 2, \ldots, z_n \), when there remain \( n - 1 \) customers in two queueing systems in total.

So, we have the time intervals

\[
[a_1(n), b_1(n)], [a_2(n), b_2(n)], \ldots, [a_{z_n}(n), b_{z_n}(n)].
\]  

(46)

Note, that if \( n = 1 \) (with \( n = 1 \)) the time interval \([a_1(1), b_1(1)]\), if it is, coincides with the busy period \([1, \tau] \), and

\[
P(z_{1_i} = 1) = \frac{1}{2}, \quad P(z_{1_i} = 0) = \frac{1}{2}, \quad i = 1, 2,
\]  

(47)

together with the condition

\[
z_{1_1} + z_{1_2} = 1.
\]  

(48)

Note, that any two differences \( a_2(n) - a_1(n) \) and \( a_{i+1}(n) - a_i(n) (i \geq 2) \) (if exist) are identically distributed, so \( a_i(n) \) have a structure of regeneration points.

Now, let \( m \in M^\ast(n) \), and let \( z_m \) denote the number of arrivals, at the time of which there become \( m^{(1)} \), \( m^{(2)} \) numbers of customers in the corresponding queueing systems, the order numbers of which are indicated by the upper index, \( m^{(1)} + m^{(2)} = n - 1 \). Then, for each of the vectors \( m \) from the set \( M^\ast(n) \) one can define the sequences \( a_1(m), a_2(m), \ldots, a_{z_m}(m) \) and \( b_1(m), b_2(m), \ldots, b_{z_m}(m) \) and the intervals

\[
[a_1(m), b_1(m)], [a_2(m), b_2(m)], \ldots, [a_{z_m}(m), b_{z_m}(m)]
\]  

(49)

by the similar way as before.
Apparently, there are intervals defined by (46) that are contained in the set of intervals (49), and let their number be \( z_{\mathbf{m},n} \). Let us delete the intervals of (46) from those of (49) and merge the ends. Note, that with \( n = \| \mathbf{n} \| \) the set of intervals defined by (49) is a subset of the system of intervals given by (45), and the set of intervals given by (46) is a subset of intervals given by (44). Let us remove all intervals of (45) that are not (49) and all intervals of (44) that are not (46). Then, the aforementioned merged points of the intervals of (46) imbedded into the intervals of (49) have a structure of regeneration points (the differences between merged points have the same distribution) and satisfy the following properties. First, the residual times to the next arrival or service completion both distributed exponentially.

The mean time to the next arrival that occurs from state \( \mathbf{m} \in \mathcal{M}^-(\mathbf{n}) \) to the state \( \mathbf{n} \) depends on the state \( \mathbf{m} \). More specifically, if \( \mathbf{m} \) is on boundary and state \( \mathbf{n} \) has rank 2, i.e. \( r(\mathbf{n}) = 2 \), then, because of reflection at zero, the mean time to the next arrival is \( 1/(2\lambda) \). In all other situations it is \( 1/\lambda \). (See Fig. 3.)

Hence, the number of merged points within an arbitrary interval \([a_j(\mathbf{m}), b_j(\mathbf{m}))\), due to the property of the lack of memory of exponential distribution, has a geometric distribution, which is the same for any \( j \). The parameter of this geometric distribution depends on \( \mathbf{m} \). If \( \mathbf{m} \in \mathcal{M}^-(\mathbf{n}) \), \( d_0(\mathbf{m}) = 1 \) (that is, the vector \( \mathbf{m} \) belongs to the boundary) and \( \| \mathbf{n} \| \geq 2 \), then the parameter of geometric distribution is \( r(\mathbf{n}) \). Otherwise, if \( d_0(\mathbf{m}) = 0 \) (the vector \( \mathbf{m} \) is in interior) and \( \| \mathbf{n} \| \geq 2 \), it is \( 1/4 \).

The next step is to distinguish different elements of \( \mathcal{M}^-(\mathbf{n}) \). The number of vectors in \( \mathcal{M}^-(\mathbf{n}) \) can be one or two. If \( \mathbf{n} \) is on boundary, then \( \mathcal{M}^-(\mathbf{n}) \) contains only a single vector. Otherwise, there two vectors in \( \mathcal{M}^-(\mathbf{n}) \). To be formal, let us denote the elements of \( \mathcal{M}^-(\mathbf{n}) \) by \( \mathbf{m}_1, \ldots, \mathbf{m}_{2-d_0(\mathbf{n})} \). Then,

\[
\bar{z}_n = \bar{z}_{\mathbf{m}_1,n} + \cdots + \bar{z}_{\mathbf{m}_{2-d_0(\mathbf{n})},n},
\]

where \( \bar{z}_{\mathbf{m}_1,n}, \ldots, \bar{z}_{\mathbf{m}_{2-d_0(\mathbf{n})},n} \) are independent random variables. The construction of the proof implies the required properties of \( q(\mathbf{m}, \mathbf{n}) \). Indeed, if \( d_0(\mathbf{m}) = 1 \) and \( \| \mathbf{n} \| \geq 2 \), then

\[
P\{z_{\mathbf{m},n} = n \mid z_{\mathbf{m}} = k\} = \binom{k + n - 1}{n} \left( \frac{r(\mathbf{n})}{4} \right)^n \left( \frac{4 - r(\mathbf{n})}{4} \right)^k.
\]

Otherwise, if \( d_0(\mathbf{m}) = 0 \) and \( \| \mathbf{n} \| \geq 2 \), then

\[
P\{z_{\mathbf{m},n} = n \mid z_{\mathbf{m}} = k\} = \binom{k + n - 1}{n} \left( \frac{1}{4} \right)^n \left( \frac{3}{4} \right)^k.
\]

### 6.3 Proof of (12) and (13)

The proof of (12) is technically similar to that of (11). We define the system of intervals (49) where \( \mathbf{m} \) belongs now to \( |\mathcal{M}^+(\mathbf{n})| \). Then, the intervals of (46) are contained in those of (49). Deleting (46) from (49) and merging the ends, we obtain the set of points, the number of which is denoted by \( z_{\mathbf{m},n} \). Similarly, we define \( z_n \) satisfying the relation

\[
z_n = z_{\mathbf{m}_1,n} + z_{\mathbf{m}_2,n},
\]

where \( \mathbf{m}_1 \) and \( \mathbf{m}_2 \) are the vectors of \( |\mathcal{M}^+(\mathbf{n})| \). (The total number of vectors in \( |\mathcal{M}^+(\mathbf{n})| \) is two.)
By similar arguments to those used in the proof in Section 6.2, we arrive at the relation
\[
P(z_{m,n} = n \mid z_m = k) = \left(\frac{k + n - 1}{n}\right) \left(\frac{1}{4}\right)^n \left(\frac{3}{4}\right)^k.
\]
So, (12) follows.

In turn, the proof of (13) follows from the relation

\[
f(\lambda(n)) = \tilde{f}(\lambda(n)) + \tilde{f}(\lambda(n)).
\]

The proof of Theorem 2 is completed.

7 Proof of Theorem 3

The proof of Theorem 3 is similar to that of Theorem 2. In this case, however, the random walk is modelled by the two independent and identical $M/M/1$ queueing systems, in which the arrival and service rates are equal, and no negative service is considered. This is because no reflection mechanism is presented, and the two queueing systems represent the only part of the basic random walk in the main quarter plane. Then, the behavior in boundary states is reckoned to be similar to those in interior states. For instance, if the random walk moves from the state \(m = (0, 5)\) to the state \(n = (-1, 5)\), it still can be modelled by the two queueing systems, in the first of which the component (-1) of the random walk is transformed to +1 in the queue, but the movement from “empty queue” to “queue with a customer” occurs without “reflection mechanism”. Specifically, because of the symmetry, we use the fact that the number of crossings a state containing a negative component or negative components has the same distribution as the number of crossings the corresponding state from the main quarter plane. Hence, the proof of Theorem 2 can be adapted to this case in very similar way. Specifically, the arguments of the proof of (14) are fully similar to those of (11) in Theorem 2. However, the proof of (15) as well as (16) need to take into account the boundary effect that is explained below.

The difference between the proofs of (12) and (15) is that, instead of the vectors of the set \(\mathcal{M}^+(n)\) used in the proof of (12) in Theorem 2, the proof of (15) should use the vectors of the set \(\mathcal{M}^+(n)\). If \(n\) is from interior, then the sets \(\mathcal{M}^+(n)\) and \(\mathcal{M}^+(n)\) coincide, and the proof of (15) in this case is the same as that of (12). If, however, \(n\) belongs to the boundary, then \(\mathcal{M}^+(n)\) contains a vector with negative component. For instance, if \(n = (0, 5)\), then \(\mathcal{M}^+(n) = \{(1, 5), (0, 6), (-1, 5)\}\). The value (-1) in the vector (-1, 5) is not associated with negative queue. The vector (-1, 5) is considered as a tagged state that is formally added to the set of states. Here is a more detailed explanation.

Let \(z_n\) denote the number of cases when at the moment of a customer’s service completion in one of the queueing systems, there become \(n^{(1)}\), \(n^{(2)}\) customers in the corresponding queueing systems, the order numbers of which is indicated by the upper index, and \(n^{(1)} + n^{(2)} = n\). Considering the interesting us case, set \(n^{(1)} = 0\) and \(n^{(2)} = n\).

With \(n = (0, n)\) let \(a_1(n), a_2(n), \ldots, a_n(n)\) be the moments of these service completions, and let \(b_1(n), b_2(n), \ldots, b_n(n)\) be the moments of arrivals following the first time after the corresponding times \(a_i(n), i = 1, 2, \ldots, n\), when there are \(n + 1\) customers in two queueing systems in total.

So, we have the time intervals
\[ [a_1(n), b_1(n)], [a_2(n), b_2(n)], \ldots, [a_{\infty}(n), b_{\infty}(n)]. \]  

(50)

Now, let \( m = (m^{(1)}, m^{(2)}) = (1, n) \), and let \( z_m \) denote the number of service completions, at the time of which there become \( m^{(1)}, m^{(2)} \) numbers of customers in the corresponding queueing systems, the order numbers of which are indicated by the upper index.

Then, one can define the sequences \( a_1(m), a_2(m), \ldots, a_{\infty}(m) \) and \( b_1(m), b_2(m), \ldots, b_{\infty}(m) \) and the intervals

\[ [a_1(m), b_1(m)], [a_2(m), b_2(m)], \ldots, [a_{\infty}(m), b_{\infty}(m)] \]  

(51)

by the similar way as before. Now let \( z'_{m'} \) be a random variable having the same distribution as \( z_m \), and let

\[ [a_1(m'), b_1(m')], [a_2(m'), b_2(m')], \ldots, [a_{\infty}(m'), b_{\infty}(m')] \]  

(52)

be the tagged intervals, the total number of which is \( z'_{m'} \).

The number of intervals (50) that are contained in (51) are denoted by \( z_{m,n} \), and they are associated with the transition \((1, n) \rightarrow (0, n)\). Assuming that the number of the virtual transition \((-1, n) \rightarrow (0, n)\) has the same distribution as \( z_{m,n} \) and independent of it, denote it by \( z'_{m',n} \). Then

\[ z_n = \sum_{k \in [M^+(n)]} z_{k,n}. \]  

(53)

and let

\[ z'_n = z_n + z'_{m',n}. \]  

(54)

In (53) and (54) we first find the sum over \([M^+(n)]\) and then add the element \( z'_{m',n} \) for the reason that in the model with two queueing system we cannot provide summation over \( M^+(n) \) in the boundary case directly.

If \( n \) is from the interior, then

\[ z_n = \sum_{k \in M^+(n)} z_{k,n}. \]  

(55)

and the distributions of \( z'_n \) given by (54) and \( z_n \) given by (55) are the same.

8 Proof of Theorem 4

Note, first that in the case of the one-dimensional random walk the results automatically follow from Proposition 1. Indeed, with \( EZ_1 = 1 \) the branching process \( Z_n \) is a martingale, and all the required results follow from this property. In the case of the two-dimensional random walk, the results cannot be retrieved from Theorems 2 or 3 by the similar way. The properties of random fields reducing them directly to martingales similarly to those of random processes are unknown. Hence, the proof of this theorem should be produced independently of the statements of Theorems 2 and 3.

Consider the model of two independent queueing systems of Section 5.2. It follows from the results of the Appendix, specifically from (A.6) that if \( E[f(\lambda(n_0))] < \infty \) for some vector \( n_0 \in Z^2_+ \setminus \{0\} \), then for any two vectors \( n_1 \in Z^2_+ \setminus \{0\} \) and \( n_2 \in Z^2_+ \setminus \{0\} \)
\[ E[f(\mathbf{x}(\mathbf{n}_1))] = 2^{d_0(\mathbf{n}_1)} E[f(\mathbf{x}(\mathbf{n}_2))]. \]  

(56)

Taking for instance \( \mathbf{n}_0 = \mathbf{1}_1 \) it is readily seen from Properties (2) and (3) in Section 3.2 that \( E[f(\mathbf{x}(\mathbf{n}_1))] < \infty \), and, hence, (56) is true.

It follows from (56) that for all \( \mathbf{n} \geq \mathbf{1} \),

\[ E[f(\mathbf{x}(\mathbf{n}))] = E[f(\mathbf{x}(\mathbf{1}))]. \]  

(57)

From same Relation (56) we have

\[ E[f(\mathbf{x}(\mathbf{1}_1))] = E[f(\mathbf{x}(\mathbf{1}_2))] = \frac{1}{2} E[f(\mathbf{x}(\mathbf{1}))]. \]  

(58)

In turn, from (57) and (58) we can establish the similar relationships for \( E[f(\mathbf{n})] \), \( \mathbf{n} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \). Taking into account that

\[ E[f(\mathbf{x}(\mathbf{n}))] = \sum_{\mathbf{m} \in \mathbf{x}(\mathbf{n})} E[f(\mathbf{m})]. \]

we have as follows. For all \( \mathbf{n} \geq \mathbf{1} \) from (57) we obtain

\[ E[f(\mathbf{n})] = \frac{1}{4} E[f(\mathbf{x}(\mathbf{1}))]. \]  

(59)

In turn, from (58) we obtain

\[ E[f(\mathbf{1}_1)] = E[f(\mathbf{1}_2)] = \frac{1}{2} \times \frac{1}{2} E[f(\mathbf{x}(\mathbf{1}))] = \frac{1}{4} E[f(\mathbf{x}(\mathbf{1}))]. \]  

(60)

So, combining (59) and (60), we arrive at the conclusion that for all \( \mathbf{n} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \)

\[ E[f(\mathbf{n})] = \frac{1}{4} E[f(\mathbf{x}(\mathbf{1}))] = c. \]  

(61)

Our aim now is to prove that the constant \( c \) in (61) is equal to one. Prove first (17) and (18). We have

\[ E\left\{ \mathbf{j}(\mathbf{1}_1) \right\} = E\left\{ \mathbf{j}(-\mathbf{1}_1) \right\} = E\left\{ \mathbf{j}(\mathbf{1}_2) \right\} = E\left\{ \mathbf{j}(-\mathbf{1}_2) \right\} = \frac{1}{4}. \]

Let \( \mathbf{n} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \). If \( \mathbf{n} \) is on the boundary, then the set \( \mathcal{M}^{-}(\mathbf{n}) \) contains only a single vector, while the set \( \mathcal{M}^{+}(\mathbf{n}) \) contains three vectors. If \( \mathbf{n} \) is from the interior, then each of the sets \( \mathcal{M}^{-}(\mathbf{n}) \) and \( \mathcal{M}^{+}(\mathbf{n}) \) contains two vectors (see Fig. 1). Since all states are equally likely, we obtain:

\[ E\left\{ \mathbf{j}(\mathbf{n}) \right\} = \begin{cases} E\left\{ \mathbf{j}(\mathbf{1}_1) \right\} = \frac{1}{4}, & \text{if } \mathbf{n} \text{ on boundary}, \\ 2E\left\{ \mathbf{j}(\mathbf{1}_1) \right\} = \frac{1}{2}, & \text{if } \mathbf{n} \text{ in interior}, \end{cases} \]

\[ E\left\{ \mathbf{j}(\mathbf{n}) \right\} = \begin{cases} 3E\left\{ \mathbf{j}(\mathbf{1}_1) \right\} = \frac{3}{4}, & \text{if } \mathbf{n} \text{ on boundary}, \\ 2E\left\{ \mathbf{j}(\mathbf{1}_1) \right\} = \frac{3}{2}, & \text{if } \mathbf{n} \text{ in interior}. \end{cases} \]

Since \( E[f(\mathbf{n})] = E\left\{ \mathbf{j}(\mathbf{n}) \right\} + E\left\{ \mathbf{j}(\mathbf{n}) \right\} \), then the constant \( c \) in (61) must be equal to 1.
9 Crossings States and Sets of States in Random Walks Defined in Finite Areas

The aim of this section is to develop the results on crossings states and sets of states for random walks of higher dimension than two. In this section we consider random walks in \([-N, N]^d\) and \([0, N]^d\). The random walk in \([-N, N]^d\) is defined as follows:

\[
S_0 = 0, \quad (62)
\]

\[
S_t = S_{t-1} + e_i([-N, N]^d), \quad t \geq 1, \quad (63)
\]

where in (62) and later on 0 is the \(d\)-dimensional zero and the vector \(e_i([−N, N]^d)\) is

\[
e_i([-N, N]^d) = \begin{cases} e_i, & \text{if } |S_{t-1} + e_i| \leq N1; \\ 0, & \text{otherwise}, \end{cases}
\]

and \(e_i = e_i(\mathbb{Z}^d)\), where the random vector \(e_i(\mathbb{Z}^d)\) for the random walk in \(\mathbb{Z}^d\) is defined similarly to that \(e_i(\mathbb{Z}^2)\) is defined for the random walk in \(\mathbb{Z}^2\). Namely, let \(1\) denote the vector, the \(i\)th component of which is 1, and the rest component are 0. Then, the vector \(e_i(\mathbb{Z}^d)\) is one of the \(2d\) vectors \(\{±1, i = 1, 2, \ldots, d\}\) that is randomly chosen with probability \(1/(2d)\) independently of the other vectors and the history of the random walk.

The random walk in \([0, N]^d\) is, in turn, the reflected version of the random walk in \([-N, N]^d\), and it is defined as

\[
S_0 = 0, \quad (62)
\]

\[
S_t = S_{t-1} + e_i([0, N]^d), \quad t \geq 1, \quad (63)
\]

where the vector \(e_i([0, N]^d)\) is

\[
e_i([0, N]^d) = \begin{cases} e_i, & \text{if } S_{t-1} + e_i \leq N1, \\ -e_i, & \text{if } S_{t-1} + e_i \leq -N1, \\ 0, & \text{otherwise}, \end{cases}
\]

and \(e_i = (e_i^{(1)}, e_i^{(2)}, \ldots, e_i^{(d)}) = e_i(\mathbb{Z}^d)\).

Note first, that all the definitions of Section 2 can be automatically developed to the \(d\)-dimensional random walks considered here. The definition of \(\mathcal{X}(n), \mathcal{N}^+(n), \mathcal{N}(n)\), lower and upper sets \(\mathcal{M}^-(n)\) and \(\mathcal{M}^+(n)\), positive upper set \(|\mathcal{M}^+(n)|\), crossings the sets \(f(Z)\), \(\tilde{f}(Z)\) and \(\tilde{f}(Z)\) and all other derivative notions remain similar to those in Section 2 given originally for the two-dimensional random walk. For instance, the total number of elements in the sets \(\mathcal{M}^-(n)\) and \(\mathcal{M}^+(n)\) are \(d - d_0(n)\) and \(d + d_0(n)\), respectively, and the total number of elements of the set \(|\mathcal{M}^+(n)|\) is \(d\).

The concepts of the Markov fields \(P_n, Q_n\) and the Markov chain \(R_n\) can be also developed. Specifically, the properties of the Markov fields \(P_n, Q_n\) the Markov chain \(R_n\) are given below.

For the field \(P_n\) and \(p(m, n)\) we have the properties:

1. \(P_0 = 1\).
2. \( P\{p(0, 1, i) = 1\} = P\{p(0, -1, i) = 1\} = 1/(2d), \ i = 1, 2, \ldots, d, \) and \( \sum_{i=1}^{d} (p(0, 1, i) + p(0, -1, i)) = 1. \)

3. \( P\{p(1, 0) = 1\} = P\{p(-1, 0) = 1\} = 1/(2d), \ i = 1, 2, \ldots, d, \) and \( \sum_{i=1}^{d} (p(1, 0) + p(-1, 0)) = 1. \)

4. if \( \mathbf{m} \in \mathcal{M}^-(\mathbf{n}), \|\mathbf{n}\| \geq 2, \) then
   \[
P\left\{ p(\mathbf{m}, \mathbf{n}) = n \mid \sum_{\mathbf{m}^* \in \mathcal{M}^-(\mathbf{m})} p(\mathbf{m}^*, \mathbf{m}) = k \right\} = \binom{k + n - 1}{n} \left( \frac{1}{2d} \right)^n \left( \frac{2d - 1}{2d} \right)^k.
   \]

5. if \( \mathbf{m} \in \mathcal{M}^+(\mathbf{n}), \mathbf{n} \neq 0, \) then
   \[
P\left\{ p(\mathbf{m}, \mathbf{n}) = n \mid \sum_{\mathbf{m}^* \in \mathcal{M}^+(\mathbf{m})} p(\mathbf{m}^*, \mathbf{m}) = k \right\} = \binom{k + n - 1}{n} \left( \frac{1}{2d} \right)^n \left( \frac{2d - 1}{2d} \right)^k.
   \]

For the field \( Q_n \) and \( q(\mathbf{m}, \mathbf{n}) \) we have the properties:

1. \( Q_0 = 1. \)
2. \( P\{q(0, 1, i) = 1\} = 1/d, \ i = 1, 2, \ldots, d, \) and \( \sum_{i=1}^{d} q(0, 1, i) = 1. \)
3. \( P\{q(1, 0) = 1\} = 1/d, \) and \( \sum_{i=1}^{d} q(1, 0) = 1. \)
4. if \( \mathbf{m} \in \mathcal{M}^+(\mathbf{n}) \) and \( \mathbf{n} \neq 0, \) then
   \[
P\left\{ q(\mathbf{m}, \mathbf{n}) = n \mid \sum_{\mathbf{m}^* \in \mathcal{M}^+(\mathbf{m})} q(\mathbf{m}^*, \mathbf{m}) = k \right\} = \binom{k + n - 1}{n} \left( \frac{1}{2d} \right)^n \left( \frac{2d - 1}{2d} \right)^k.
   \]

5. if \( \mathbf{m} \in \mathcal{M}^-(\mathbf{n}), d_0(\mathbf{m}) > 0 \) and \( \|\mathbf{n}\| \geq 2, \) then
   \[
P\left\{ q(\mathbf{m}, \mathbf{n}) = n \mid \sum_{\mathbf{m}^* \in \mathcal{M}^-(\mathbf{m})} q(\mathbf{m}^*, \mathbf{m}) = k \right\} = \binom{k + n - 1}{n} \left( \frac{r(\mathbf{n})}{2d} \right)^n \left( \frac{2d - r(\mathbf{n})}{2d} \right)^k,
   \]
   where \( r(\mathbf{n}) \) is the rank of the vector \( \mathbf{n} \in \mathcal{M}^+(\mathbf{m}) \). (Definition 5 for the rank remains the same as in the two-dimensional case, and \( r(\mathbf{n}) \) can be either one or two. The number of vectors in the set \( \mathcal{M}^+(\mathbf{m}) \) having rank two is equal to \( d_0(\mathbf{m}) \). In Fig. 4 this property is illustrated for \( d = 3. \))

6. if \( \mathbf{m} \in \mathcal{M}^-(\mathbf{n}), d_0(\mathbf{m}) = 0 \) and \( \|\mathbf{n}\| \geq 2, \) then
   \[
P\left\{ q(\mathbf{m}, \mathbf{n}) = n \mid \sum_{\mathbf{m}^* \in \mathcal{M}^-(\mathbf{m})} q(\mathbf{m}^*, \mathbf{m}) = k \right\} = \binom{k + n - 1}{n} \left( \frac{1}{2d} \right)^n \left( \frac{2d - 1}{2d} \right)^k.
   \]
For Markov chain $R_n$ we have the properties:

1. $R_0 = 1$.
2. $P\{R_n = k | R_{n-1} = m\} = {\binom{k + m - 1}{k}} \left( \frac{C(n,d)}{C(n,d) + 2C(n,d)} \right)^{m} \left( \frac{C_{0}(n,d)+C(n,d)}{C_{0}(n,d)+2C(n,d)} \right)^{k}$, where

$$C_0(n,d) = \sum_{i=1}^{d-1} (d - i)2^{i+1} \binom{d}{i} \left( \frac{n - 1}{i - 1} \right),$$

$$C(n,d) = \sum_{i=1}^{d} i2^{i} \binom{d}{i} \left( \frac{n - 1}{i - 1} \right).$$

Note, that for the random walk in $[-N,N]^d$ the upper set $M^+(n)$ is not always defined. For instance, if $n = (N,N,\ldots,N) = N1$ or $n = (-N,-N,\ldots,-N) = -N1$ (the margin states), then $M^+(n) = \emptyset$. In other cases, where $n$ is a boundary element, the upper set $M^+(n)$ is defined, but the number of vectors is less than $d$. For instance, if $n = N1$, then the number of vectors in $M^+(n)$ is $d - 1$. This makes the study of crossings sets and states complicated in general. However, considering the states of the random walk in smaller area such as the rectangle $S \subseteq \{-N-1\} \times (N-1) \times \{N > 1\}$ enables us to avoid the complication and make the extension of the earlier results smooth. We have the following result.

**Theorem 5** For any $n \in S \setminus \{0\}$ the statements of Theorems 1, 2, and 3, in which the random objects $P_n, Q_n$ and $R_n$ are redefined in this section, hold true. Relations (17), (18) and (19) in the given case are, respectively, rewritten as follows.
The proof of statements related to Theorems 1 and 2 is based on straightforward extension of the proof of Theorems 1 and 2 given in Sections 5 and 6, respectively. The proof of the statement related to Theorem 3 is also an extension of the proof of corresponding Theorem 3, but the cases where the required state \( n \) is on the boundary of the main subspace \([0, N]^d\) (in the case of dimension three it is the positive three dimensional subspace \( xyz \)) should be studied in more details. Specifically, in the proof of Theorem 3 the boundary case is associated with \( d_{0}(n) = 1 \). In the present proof of the corresponding statement associated with Theorem 3, each boundary case is associated with the inequality \( 1 \leq d_{0}(n) \leq d - 1 \), and the number of the series of the intervals like those given in (52) must be \( d_{0}(n) \). Then, instead of (54) we have:

\[
E \{ \bar{f}(n) \} = 2^{d_{0}(n)-d} E \{ \bar{f}[\lambda(n)] \} = \frac{d - d_{0}(n)}{2d},
\]

and

\[
E \{ f(n) \} = 2^{d_{0}(n)-d} E \{ f[\lambda(n)] \} = 1.
\]

**Proof** The proof of statements related to Theorems 1 and 2 is based on straightforward extension of the proof of Theorems 1 and 2 given in Sections 5 and 6, respectively. The proof of the statement related to Theorem 3 is also an extension of the proof of corresponding Theorem 3, but the cases where the required state \( n \) is on the boundary of the main subspace \([0, N]^d\) (in the case of dimension three it is the positive three dimensional subspace \( xyz \)) should be studied in more details. Specifically, in the proof of Theorem 3 the boundary case is associated with \( d_{0}(n) = 1 \). In the present proof of the corresponding statement associated with Theorem 3, each boundary case is associated with the inequality \( 1 \leq d_{0}(n) \leq d - 1 \), and the number of the series of the intervals like those given in (52) must be \( d_{0}(n) \). Then, instead of (54) we have:

\[
z^*_{n} = z_{n} + \sum_{i=1}^{d_{0}(n)} z'_{m_{i},n},
\]

where \( z'_{m_{i},n}, z'_{m_{2},n}, \ldots, z'_{m_{d_{0}(n)},n} \) are independent and identically distributed, and \( z_{n} \) is given by (53). If \( n \) is from the interior, then (55) here is true as well. In the proof of the statements related to Theorem 4, the proof in the case when \( n \geq 1 \) is the same in Theorem 4. So, Relation (57) holds in this case as well. Instead of (58) we have

\[
E \{ f[\lambda(1,i)] \} = \frac{1}{d}, \quad i = 1, 2, \ldots, d.
\]

Then, for \( n \geq 1 \) instead of (59) we have

\[
E \{ f(n) \} = \frac{1}{2d} E \{ f[\lambda(1)] \}, \quad (64)
\]

and similarly to that in the proof of Theorem 4 (see (60))

\[
E \{ f(1,i) \} = E \{ f(-1,i) \} = \frac{1}{2d} E \{ f[\lambda(1)] \}, \quad i = 1, 2, \ldots, d.
\]

Using Relation (A.4), in fact we obtain more general result than (64) and (65). Namely, (64) is true for all \( n \in S \setminus \{0\} \). That is, similarly to (61) we have

\[
E \{ f(n) \} = c
\]

for all \( n \in S \setminus \{0\} \). The next part of the proof is similar to that of Theorem 4. Since all states \( n \) are equally likely, from the total expectation formula we have:

\[
E \{ \bar{f}(n) \} = \frac{\text{The number of vectors in } M^{-}(n)}{2d} = \frac{d - d_{0}(n)}{2d}, \quad (67)
\]
From (67) and (68) we finally arrive at the conclusion that the constant $c$ in (66) is equal to one.

10 On Random Walks in $\mathbb{Z}^d, d > 2$

In this section we discuss crossing states in random walks in $\mathbb{Z}^d, d > 2$. The random walks of dimension three or higher are transient, therefore the question about crossing states makes sense under the condition that it is eventually returns to the origin. Let $\mathcal{A}$ denote the event “the random walk that starts from the origin returns to the original point again”. Under this condition, a natural question is as follows. Let $n \in \mathbb{Z}^d \setminus \{0\}$. Is $\mathbb{E}\{f(n) | \mathcal{A}\} = 1$ true?

Unfortunately, the condition $\mathcal{A}$ makes the approach suggested in the present paper impossible to address this question. The level-crossing approach that was first considered in Section 3 for one-dimensional random walk and then developed for two-dimensional random walk and multidimensional random walk of an arbitrary dimension $d$ but defined in bounded areas assumes that the consecutive intervals such as (46) have a regenerative structure, and the number of merged points in these intervals do not depend on $\|n\|$. Under the condition $\mathcal{A}$ this dependence exists. If the state $n$ is “closer to zero”, then the probability to return to the origin is higher compared to the case when the state $n$ is “far from zero”.

The set of sample paths that starts from state $n$ and returns to the same state $n$ in the case when $n$ is “closer to zero” is richer compared to the set of sample paths of the similar time period when $n$ is “far from zero”. Furthermore, for $d \geq 5$ the $d$-dimensional random walk is strongly transient Hughes (1995), that is, the conditional expected length of a sojourn time from the origin to the origin given $\mathcal{A}$ is finite. This means that $\mathbb{E}\{f(n) | \mathcal{A}\} = 1$ cannot be true for $d \geq 5$ and probably is not true for $d = 3$ or $d = 4$ either.

Our conjecture in the relation to this case is as follows. For random walks defined in $\mathbb{Z}^d, d > 2$, the only inequality $\mathbb{E}\{f(n) | \mathcal{A}\} < 1$ is true ($n \in \mathbb{Z}^d \setminus \{0\}$). Furthermore, $\mathbb{E}\{f(n) | \mathcal{A}\}$ decreases as $\|n\|$ increases, and if $d \geq 5$, then $\mathbb{E}\{f(n) | \mathcal{A}\}$ vanishes as $\|n\|$ tends to infinity.

11 Numerical Study

In this section we provide numerical results for level-crossings of three different random walks considered in the paper: one-dimensional random walks, two-dimensional random walk and three-dimensional random walk in certainly defined bounded area. In our simulations the only random walks that eventually returned to the origin after no more than 100,000 steps were counted. Each output result is obtained based on 1,000,000 runs.

11.1 One-Dimensional Random Walk

It turns out that for this random walk only 997,482 runs out of 1,000,000 are successfully terminated. That is, 997482 simulated random walks eventually returned to the original
point for no more than 100,000 steps. Simulation shows stable results in general. We studied numerically the number of crossings levels 1, 2, 3, 5, 10, 20, 50 and 100 that are reflected in Table 1.

It is seen from Table 1 that the results for the crossings levels from 1 to 20 show the results that are close to theoretical. The results for levels 50 and 100 are essentially deviated. Notice that with increasing $n$, the obtained numerical values for $f(n)$, $\vec{f}(n)$ and $\vec{\vec{f}}(n)$ decrease in $n$. The reason of this decreasing and the biased values for $n = 50$ and $n = 100$ can be explained by the fact that only 99.7% of simulation runs are successfully terminated. A more detailed explanation is given for the case of the two-dimensional random walk in the section below.

### 11.2 Two-Dimensional Random Walk

Two-dimensional random walk is central to this paper. However, the numerical analysis is not so successful as in one-dimensional case. If in the one-dimensional case the number of successfully terminated random walks was about 99.7%, in the case of the two-dimensional random walk the number of successfully terminated random walks is approximately 78%. This percent cannot be substantially improved by increasing the number of steps. Simulation will become time consuming with no visible result. In addition, the variance of the random fields increases more rapidly as $||n||$ increases to infinity compared to the branching process with increasing $n$ to infinity. For these reasons, the numerical results in this case are considered for the only states located close to the origin.

Now we discuss the obtained results in Table 2 starting first from the states $(0 -1)$, $(0 1)$, $(1 0)$ and $(-1 0)$. The obtained frequencies of up-directed crossings in all of the

| Level n | Estimated $f(n)$ | Estimated $\vec{f}(n)$ | estimated $\vec{\vec{f}}(n)$ |
|---------|------------------|------------------------|-------------------------------|
| 1       | 0.9962           | 0.5000                 | 0.4964                        |
| 2       | 0.9898           | 0.4964                 | 0.4935                        |
| 3       | 0.9848           | 0.4935                 | 0.4913                        |
| 5       | 0.9759           | 0.4896                 | 0.4863                        |
| 10      | 0.9568           | 0.4795                 | 0.4773                        |
| 20      | 0.9111           | 0.4563                 | 0.4547                        |
| 50      | 0.7542           | 0.3780                 | 0.3763                        |
| 100     | 0.5336           | 0.2679                 | 0.2657                        |

| State n | Estimated $f(n)$ | Estimated $\vec{f}(n)$ | estimated $\vec{\vec{f}}(n)$ |
|---------|------------------|------------------------|-------------------------------|
| $(0 -1)$ | 0.7823           | 0.2507                 | 0.5316                        |
| $(0 1)$  | 0.7779           | 0.2488                 | 0.5291                        |
| $(1 0)$  | 0.7798           | 0.2511                 | 0.5287                        |
| $(-1 0)$ | 0.7780           | 0.2494                 | 0.5286                        |
| $(1 1)$  | 0.6625           | 0.3584                 | 0.3042                        |
| $(-1 -1)$ | 0.6637           | 0.3605                 | 0.3032                        |
| $(1 -1)$ | 0.6642           | 0.3600                 | 0.3042                        |
| $(-1 1)$ | 0.6633           | 0.3598                 | 0.3035                        |
four cases are close to the theoretical value of the mean 0.25. However, the frequencies of down-directed crossings are essentially biased. The last can be explained as follows. Longer random walks having length of more than 100,000 steps affect on the number of down-directed crossings essentially. Absence of a large number of long random walks from the simulation series thus essentially decreases the number of down-directed crossings compared to the number that should be. The cut down number of down-directed crossings in turn changes the total number of undirected crossings as well. A similar picture is regarding the states (1 1), (−1 −1), (1 −1) and (−1 1). The fraction of up-directed crossings is slightly greater than the fraction of down-directed crossings because of the missing longer simulation series. As well, the missing longer series affecting the number of down-directed crossings of the states (0 −1), (0 1), (1 0) and (−1 0) in turn affect to the number of up-directed crossings the states (1 1), (−1 −1), (1 −1) and (−1 1) since these states are neighbors with respect to the aforementioned states.

11.3 Three-Dimensional Random Walk in a Bounded Region

In this numerical example we simulate the numbers of crossings states in [−5, 5]³. In this case, all the random walks were successfully terminated. Table 3 presents the simulation results. The obtained results are close to the obtained theoretical results.

12 Summary of the Results and Discussion

In this paper, a comprehensive analysis of the numbers of state-crossings and numbers of crossings specifically defined sets of states is provided. The explicit representations for probability distributions and expectations of the numbers of crossings the states \( \mathbf{n} \in \mathbb{Z}^d \setminus \{0\} \) and the sets of states \( \Lambda(\mathbf{n}) \) and \( \Lambda(\|\mathbf{n}\|) \) are obtained.

In Table 4, we survey the properties of the conditional expectations for the numbers of state-crossings and crossings sets of states. We give their comparison table for one-dimensional and two-dimensional random walks. Specifically, we indicate whether or not the aforementioned conditional expectations are the same. For instance, the record “for \( \tilde{f}(\mathbf{n}) – \text{true} \)” means that \( \mathbb{E}\{\tilde{f}(\mathbf{n})\} = c \) for all \( n \) integer, \( -\infty < n < \infty \), where \( c \) is the same constant for all \( n \). (In some cases it is implied that \( c = 1 \). However, for \( \mathbb{E}\{\tilde{f}(\mathbf{n})\} \) related to the one-dimensional random walk the constant \( c \) is a half. It is also half for \( \mathbb{E}\{\tilde{f}(\mathbf{n})\} \) for \( \|\mathbf{n}\| \geq 1 \). Illustrating the only qualitative properties, we ignore this difference.) In another record “for \( \tilde{f}(\mathbf{n}) – \text{false in general} \) and \( \text{true for } \|\mathbf{n}\| \geq 1' \)” means that \( \mathbb{E}\{\tilde{f}(\mathbf{n})\} = c \) is incorrect in general when \( \mathbf{n} \in \mathbb{Z}^2 \setminus \{0\} \), but correct if the subset \( \|\mathbf{n}\| \geq 1 \) is
considered. Note, that \( \tilde{f}(-n,n) \), \( \tilde{f}((-n,n)) \) and \( f(n) \) are the one-dimensional analogues of \( \tilde{f}(\lambda(n)) \), \( \tilde{f}(\lambda(n)) \) and \( f(\lambda(n)) \), respectively, or \( \tilde{f}(N(n)) \), \( \tilde{f}(N(n)) \) and \( f(N(n)) \), respectively. The table does not include the results of Section 9 for multidimensional random walks in bounded areas considering them as specific.

The basic property of the paper that the expected number of state-crossings \( n \in \mathbb{Z}^2 \setminus \{0\} \) is the same can be extended to a wider class of random walks considered in Abramov (2018). For instance, for the family of conservative random walks in Abramov (2018) indicated as symmetric random walks (Model 1), the property is satisfied since the results in the Appendix hold true in the case of symmetric random walks as well.

Recall that the aforementioned family of symmetric random walks is described by the system of equations

\[
S_0 = 0,
\]

\[
S_t = S_{t-1} + \tilde{e}_t, \quad t \geq 1,
\]

where the vector \( \tilde{e}_t \) is defined as follows. As in (1) and (2), it is one of the vectors \( \{\pm 1, i = 1, 2\} \) that is randomly chosen. But the probability of this choice depend on \( i \). That is, for the vector \( 1_i \) (or the vector \( -1_i \)) to be chosen this probability is \( \alpha_i > 0, i = 1, 2 \), and \( 2(\alpha_1 + \alpha_2) = 1 \).

**Appendix: Derivation of the Formula for the Transition Probability from \( N(n) \) to \( N(n + 1) \) for the Random Walk in \( \mathbb{Z}^d \)**

The results of the appendix are taken from Abramov (2018), pp. 1906–1908.

**Description of the Model**

The random walk in \( \mathbb{Z}^d \) is modelled by the \( d \) independent queueing processes as follows. Assume that arrivals in the \( i \)th queueing system are Poisson with rate \( \lambda_i \), and service times are exponentially distributed with the parameter \( \lambda_i \). If a system becomes free, it is switched for a special service, which is exponentially distributed with the same parameter \( \lambda \). This
service is negative, and it results in a new customer in the queue. If during a negative service a new arrival occurs, the negative service remains unfinished and not resumed. The negative service models the reflection at zero and in fact implies the state-dependent arrival rate, which becomes equal to \( 2\lambda \) at the moment when the system is empty. It is associated with the situation, when an original one-dimensional random walk reaches zero at some time moment \( s \), and at the next time moment \( s + 1 \) it must take one of the values \( \pm 1 \) that corresponds to value \( +1 \) for an one-dimensional random walk reflected at zero.

**Queueing Systems with Finite Capacity and Characterization of the Level \( n \) Probability**

Assume that the number of waiting places in each of the queueing system is \( N \). The assumption on limited number of waiting places means that an arriving customer, who meets \( N \) customers in the system, is lost. For a vector \( \mathbf{n} = (n^{(1)}, n^{(2)}, \ldots, n^{(d)}) \) satisfying \( ||\mathbf{n}|| < N \) let \( P_N(\mathbf{n}) \) denote the stationary probability to be in state \( \mathbf{n} \) immediately before arrival of a customer in one of the \( d \) queueing systems. Application of the PASTA property Wolff (1982) enables us to first obtain the stationary probability for each single system to obtain then the required stationary probability \( P_N(\mathbf{n}) \). Let \( P_N(n) \) denote the stationary probability to be in state \( n < N \) for the \( i \)th queueing system. We have

\[
P_N(n) = \begin{cases} 
\frac{2}{2N+1}, & \text{for } 1 \leq n \leq N, \\
\frac{2}{2N+1}, & \text{for } n = 0.
\end{cases}
\]

Hence the queueing systems are independent, the product form solution for \( P_N(\mathbf{n}) \) is

\[
P_N(\mathbf{n}) = 2^{d-d_0(\mathbf{n})} \frac{1}{(2N+1)^d}, \tag{69}
\]

where \( d_0(\mathbf{n}) \) denotes the number of zero components in the presentation of the vector \( \mathbf{n} \).

The total number of vectors having norm \( n \) in \( \mathbb{Z}_+^d \) is

\[
\sum_{i=1}^{d} \binom{d}{i} \binom{n-1}{i-1}.
\tag{70}
\]

(The formula sums over \( i \) being the number of nonzero components of the vector.) Hence, denoting the stationary state probability to belong to the set \( \mathcal{N}^* \) by \( P_N[\mathcal{N}^*] \), we have

\[
P_N[\mathcal{N}^*] = \sum_{\mathbf{n} \in \mathcal{N}^*} \frac{1}{(2N+1)^d} \sum_{i=1}^{d} 2^i \binom{d}{i} \binom{n-1}{i-1}, \tag{71}
\]

where the term

\[
\sum_{i=1}^{d} 2^i \binom{d}{i} \binom{n-1}{i-1}
\]

on the right-hand side of (71) characterizes the total number of vectors in \( \mathbb{Z}_+^d \) having norm \( n \).

It follows from (69) that for any two vectors \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) satisfying \( ||\mathbf{n}_1|| < N \) and \( ||\mathbf{n}_2|| < N \) we have
\[
P_N(n_1) \quad P_N(n_2) = 2^{d_0(n_2) - d_0(n_1)}
\] (72)

independently on \( N \). Hence,

\[
\lim_{N \to \infty} \frac{P_N(n_1)}{P_N(n_2)} = 2^{d_0(n_2) - d_0(n_1)}.
\] (73)

For \( N = \infty \), let \( P_\infty(n_1, \tau) \) be the probability that at time \( \tau \) the system with infinite capacity is in state \( n \). Then, for the limiting ratio of the final probabilities we also have

\[
\lim_{\tau \to \infty} \frac{P_\infty(n_1, \tau)}{P_\infty(n_2, \tau)} = 2^{d_0(n_2) - d_0(n_1)}.
\] (74)

The result in (74) is true, since the limit in (73) due to Relation (72) is uniform, and interchanging the order of limits \( \tau \) vs \( N \) is correct. (The last can also be established directly Abramov (2018)).

Let \( p_n(d) \) denote the transition probability from the set of states \( \mathcal{N}^+(n) \) (level \( n \)) to the set of states \( \mathcal{N}^+(n+1) \) (level \( n+1 \)), and let \( q_n(d) = 1 - p_n(d) \) denote the transition probability from the level \( n \) to the level \( n - 1 \). Derive the formula for \( p_n(d) \).

The total number of vectors in the set \( \mathcal{N}^+(n) \) is given by (70). Each vector contains \( d \) components. Hence, the total number of components in the set of vectors in \( \mathcal{N}^+(n) \) is

\[
d \sum_{i=1}^{d} \binom{d}{i} \binom{n-1}{i-1}.
\] (75)

Among them, the total number of zero components is

\[
\sum_{i=1}^{d-1} (d - i) \binom{d}{i} \binom{n-1}{i-1},
\] (76)

and the total number of nonzero components is

\[
\sum_{i=1}^{d} i \binom{d}{i} \binom{n-1}{i-1}.
\] (77)

To derive the formula for \( p_n(d) \) let us build the sample space. The components of all vectors in \( \mathcal{N}^+(n) \), the total number of which is given by (75), are not equally likely. According to (69), a nonzero component appears with two times higher probability than a zero component. To make the components equally likely, we are to extend the number of nonzero components by factor 2. Then the total number of equally likely components is to be equal to

\[
d \sum_{i=1}^{d} 2^i \binom{d}{i} \binom{n-1}{i-1}.
\] (78)

Following (78), the sample space for level \( n \) contains

\[
2d \sum_{i=1}^{d} 2^i \binom{d}{i} \binom{n-1}{i-1},
\]
that is two times more than that given by (78). This is because it include possible transi-
tions from each state (component) in the two directions. Specifically, let $C_0(n, d)$ denote
the number of possible transitions associated with reflections at zero (the number of zero
components given in (76) multiplied by two), and let $2C(n, d)$ be the rest of transitions,
the half of which are associated with the transitions from level $n$ to $n + 1$ and another half
with the transitions from level $n$ to $n − 1$. So, $C(n, d)$ is the number of nonzero components
given in (77).

That is, the expression in (77) is presented as

$$2d \sum_{i=1}^{d} \binom{d}{i} \binom{n-1}{i-1} = C_0(n, d) + 2C(n, d).$$

Then, the total number of transitions from level $n$ to $n + 1$ is $C_0(n, d) + C(n, d)$ and the
total number of transitions from level $n$ to $n − 1$ is $C(n, d)$. Then, from (76), (77) and (79)
we obtain

$$p_n(d) = \frac{C_0(n, d) + C(n, d)}{C_0(n, d) + 2C(n, d)} = \frac{2 \sum_{i=1}^{d-1} (d-i)2^i \binom{d}{i} \binom{n-1}{i-1} + \sum_{i=1}^{d} i2^i \binom{d}{i} \binom{n-1}{i-1}}{2 \sum_{i=1}^{d-1} (d-i)2^i \binom{d}{i} \binom{n-1}{i-1} + 2 \sum_{i=1}^{d} i2^i \binom{d}{i} \binom{n-1}{i-1}}.$$

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