Nonlinear optical effects in artificial materials

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Abstract. We consider some nonlinear phenomena in metamaterials with negative refractive index properties. Our consideration includes a survey of previously known results as well as identification of the phenomena that are important for applications of this new field. We focus on optical behavior of thin films as well as multi-wave interactions.

1 Introduction

In recent years, the development of nanotechnology has led to the creation of new materials with very unusual optical properties - metamaterials. Metamaterials are composed of host dielectric materials with embedded periodic structures, patterned on a scale much shorter than the operating wavelength and thus on a nanometer scale if the infrared or visible spectra are targeted. Though they are made of positive index materials at small length scale, these structures exhibit abnormal dispersion characteristics at optical scales. The material can behave as a medium with negative index of refraction associated with negative values of the effective permittivity $\varepsilon$ and effective permeability $\mu$ \cite{1}. The electric field, magnetic field wavevector then constitute a left handed system, where the wavevector and the Poynting vector are antiparallel \cite{2,3,4,5}. In this case, they are termed left handed materials (LHMs). Beyond this, inversion of the phase velocity with respect to the direction of propagation of energy leads to negative refraction. That means both the incident and refracted beam are on the same side of the normal to the refraction interface, as if predicted by Snell’s law with a negative refractive index.

Left handed properties of metamaterials are most pronounced in two situations. The first is in the behavior of a quasi-monochromatic wave at the interface of a left-handed and a conventional right handed material. In particular we consider layered materials because the use of thin films of LHM opens new opportunities for design of photonic crystals with unusual properties. The second is the interaction of two waves with different carrier frequencies which experience different signs of refractive index. Such a difference is a consequence of how LHM are realized using simultaneous resonance of electric and magnetic field components with metallic nanostructures embedded in a host dielectric. Such a material behaves as right handed for waves with carrier frequencies above the resonance region and as left-handed for waves with carrier frequencies below the resonance region.

In this paper we consider only phenomena related to these two cases.

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2 Nonlinear waveguiding phenomena on the interface of materials with opposite signs of their refractive indices

Let us consider a layer of left-handed material on a dielectric substrate with positive refraction index - a right handed material (RHM). We study optical waves propagating along this interface in the $z$-direction of the Cartesian frame. The normal to the surface is chosen to be the $x$-direction while the $y$-axis lies in the interface plane. There are two types of the surface waves: (a) transverse electric waves (TE waves) with $E = (0, E_y, 0)$ and $H = (H_x, 0, H_z)$, and (b) transverse magnetic waves (TM waves) with $E = (E_x, 0, E_z)$ and $H = (0, H_y, 0)$. If the tensor of dielectric permeability of one of the media is a square-law function of electrical field strength $\hat{\varepsilon}(\omega, E) = \varepsilon(\varepsilon) + \hat{\varepsilon}_{nl}(\omega) : EE$ then propagation of the non-linear surface polaritons (NLSP) is possible for both the TM and the TE waves [10][12][14]. Linear surface polaritons corresponding to $\hat{\varepsilon}_{nl}(\omega) = 0$ were found in [7]. Without loss of generality, here we will consider only TE waves.

2.1 Non-linear surface waves of TE-type

Let us consider waves on the interface between two isotropic media. The electrodynamic properties of each medium are described by effective dielectric permittivity and effective magnetic permeability. In the linear limit at $x < 0$ the medium is characterized by positive permittivity and permeability $\varepsilon_1 > 0, \mu_1 > 0$ and at $x > 0$ both $\varepsilon$ and $\mu_2 < 0$ are chosen to be negative $\varepsilon_2 < 0, \mu_2 < 0$. Due to its nature the electrical field vector of the TE-waves has only one nonzero component, $E_y = E$, which is governed by the equation:

$$E_{zz} + E_{xx} + k_0^2 \varepsilon(\omega, E) \mu(\omega, E) E = 0,$$

where $k_0 = \omega/c$. The magnetic field vector components are coupled with $E$ by the following relations:

$$H_z = i(k_0 \mu)^{-1} E_z, \quad H_x = -i(k_0 \varepsilon)^{-1} E_x$$

On the interface plane at $x = 0$ the field components $E$, $H_z$ and $H_x$ satisfy the set of jump conditions:

$$E(0-) = E(0+), \quad H_z(0-) = H_z(0+), \quad \mu_1 H_x(0-) = \mu_2 H_z(0+).$$

Due to the translation symmetry along the $z$-axis, the $y$-component of the electric field $E(\omega, x, z)$ can be chosen as

$$E(\omega, x, z) = \Phi(\omega, x) \exp[i\beta(\omega) z],$$

where $\beta(\omega)$ is a propagation constant [3]. The equation for the transverse profile of the electrical field of NLSW follows from equations (1) and (2):

$$\Phi_{,xx} + [k_0^2 \varepsilon(\omega, x) \mu(\omega, x) - \beta^2] \Phi = 0.$$  \hspace{1cm} (3)

The above jump conditions now can be represented in terms of the amplitude function $\Phi(x)$:

$$\Phi(0-) = \Phi(0+), \quad \mu_1^{-1} \Phi_x(0-) = \mu_2^{-1} \Phi_x(0+).$$  \hspace{1cm} (4)

The boundary conditions

$$\lim_{|x| \to \infty} \Phi(\omega, x) = 0, \quad \lim_{|x| \to \infty} \Phi_x(\omega, x) = 0$$  \hspace{1cm} (5)

select the surface wave solutions of (3).

As an example giving rise to nonlinear polaritons we considered the simple geometry where the nonlinear left-handed material is placed on a linear semi-infinite conventional right-handed material:

$$\varepsilon(\omega, x) = \begin{cases} \varepsilon_1, & x < 0 \\ \varepsilon_2 + \varepsilon_{nl} \Phi^2, & x > 0 \end{cases}, \quad \mu(\omega, x) = \begin{cases} \mu_1, & x < 0 \\ \mu_2, & x > 0 \end{cases}$$

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Equation (3) can be solved at \( x < 0 \) and at \( x > 0 \), where the media are homogeneous, taking account of the boundary conditions (5). The matching condition for these solutions at the boundary \( x = 0 \) results in dispersion relations for NLSP.

The equation (3) does not have bounded solutions, if \( \mu_2 \varepsilon_{nl} < 0 \) (self-refocusing LHM). This means that nonlinear surface polaritons do not exist in this geometry with cubic nonlinearity in presence of a self-refocusing left-handed material. It can be shown that, if \( \mu_2 \varepsilon_{nl} > 0 \) (this is the case of a self-focusing non-linear LHM), then NLSP exists only when \( p^2 = (\beta^2 - k_0^2 \varepsilon_{1} \mu_1) > 0 \).

In this case the solution of equation (3) reads
\[
\Phi(x) = \begin{cases} 
A_1 \exp(qx), & x < 0 \\
\alpha^{-1/2} \sec \left[p(x - x_2) \right], & x > 0
\end{cases}
\]

where \( \alpha = k_0^2 \varepsilon_{2} \mu_2 / 2p^2 \), \( q^2 = (\beta^2 - k_0^2 \varepsilon_{1} \mu_1) > 0 \), and \( x_2 \) is the location of the maximum of the electrical field. The continuity condition at the interface \( x = 0 \) requires
\[
A_1 = \alpha^{-1/2} \sec (px_2),
\]
\[
\mu_2 q = \mu_1 p \tanh(px_2).
\]

The expression (7) defines \( x_2 \) as function of electric field strength \( A_1 \) at the interface. This equation (8) is the dispersion relation for NLSP. Since \( \mu_1 \) and \( \mu_2 \) have different signs (\( \mu_1 > 0 \), \( \mu_2 < 0 \)), the parameter \( x_2 \) must be negative. Thus the electric field reaches its maximum value on the interface at \( x = 0 \). It should be noted that in the case where both linear and nonlinear dielectrics are right-handed, the electric field reaches its maximum value inside the non-linear dielectric medium, i.e., where \( x_2 \) is positive.

In the other case, when a linear LHM is placed on a nonlinear semi-infinite medium (RHM), \( \varepsilon \) and \( \mu \) are defined as follows:
\[
\varepsilon(\omega, x) = \begin{cases} 
\varepsilon_1 + \varepsilon_{nl} \Phi^2, & x < 0 \\
\varepsilon_2, & x > 0
\end{cases},
\mu(\omega, x) = \begin{cases} 
\mu_1, & x < 0 \\
\mu_2, & x > 0
\end{cases}
\]

The governing equations and corresponding solutions in this case can be easily obtained from equations (6) and (7), (8).

For a self-refocusing medium, solutions of equation (3) are given by the expression
\[
\Phi(x) = \begin{cases} 
\pm |\alpha|^{-1/2} \co \sec \left[q(x - x_1) \right], & x < 0 \\
A_2 \exp(-px), & x > 0
\end{cases}
\]

The continuity condition at the interface \( x = 0 \) requires
\[
A_2 = \pm |\alpha|^{-1/2} / \sinh(-qx_1)
\]
\[
\mu_1 q = \mu_2 q \tanh(qx_1)
\]

Since \( \mu_1 > 0 \) and \( \mu_2 < 0 \), it follows from the dispersion relation (11) that the parameter \( x_1 \) must be negative. Thus the electric field reaches its maximum value on the interface \( x = 0 \) as in the case of a self-refocusing medium.

A detailed analysis of the surface waves on the interface between nonlinear LHM and nonlinear RHM was presented in [14,11].

### 2.2 Nonlinear planar waveguide

In this section we consider a sandwich type structure, which consists of a dielectric layer with left-handed properties surrounded by Kerr-like nonlinear dielectrics [13]. The presence of two interfaces in this configuration (at \( x = 0 \) and \( x = h \)) could lead to two types of guided waves. The first type is represented by an internal wave localized in the layer. We will refer to these types of solutions as non-linear guided modes (NLGM). The second type is made up of two coupled surface waves and they will be referred to as non-linear surface waves (NLSW).
2.2.1 Dispersion relations of nonlinear surface TE-waves

The equation (3) describing the transverse electric field profile of the wave in a three-layer structure can be decomposed into a system of simple equations. If the condition $\mu_i \varepsilon_{nl}^{(i)} > 0$ holds, then the solution of these equations is

$$\Phi(x) = \begin{cases} 
\alpha_1^{-1/2} \text{sech}[p_1(x - x_1)], & x < 0 \\
A \exp(-\kappa x) + B \exp(\kappa x), & 0 < x < h, \\
\alpha_2^{-1/2} \text{sech}[p_2(x - x_2)], & x > h
\end{cases}$$

where $p_1^2 = \beta^2 - k_0^2 \varepsilon_1 \mu_1$, $p_2^2 = \beta^2 - k_0^2 \varepsilon_2 \mu_2$, $i = 1, 2$ and $\kappa^2 = \beta^2 - k_0^2 \varepsilon_s \mu_s > 0$. The signs of $p_1^2$ and $p_2^2$ are chosen to be positive to guarantee validity of the boundary conditions (9). The sign of $\kappa^2$ is also chosen to be positive (see (3)), as we will see below such a choice corresponds to the case of surface waves. For $\mu_i \varepsilon_{nl}^{(i)} < 0$ we have

$$\Phi(x) = \begin{cases} 
\pm|\alpha_1|^{-1/2} \text{coth}[p_1(x - x_1)], & x < 0 \\
A \exp(-\kappa x) + B \exp(\kappa x), & 0 < x < h, \\
\pm|\alpha_2|^{-1/2} \text{coth}[p_2(x - x_2)], & x > h
\end{cases}$$

There $x_{1,2}$ are the constants of integration corresponding to the co-ordinates of the maximum of the transverse profile of the electric field. The continuity conditions of the electric and magnetic field components at the interfaces (14) result in a homogeneous system of linear equations. There are nontrivial solutions of this system if the determinant is zero, which leads to the following dispersion relation:

$$e^{2\kappa h} (1 + \mu_s \tilde{p}_1 / \kappa) (1 + \mu_s \tilde{p}_2 / \kappa) = (1 - \mu_s \tilde{p}_1 / \kappa) (1 - \mu_s \tilde{p}_2 / \kappa),$$

(12)

where $\tilde{p}_{1,2} = \mu_{i,2}^{-1} p_{1,2} \tanh(p_{1,2} x_{1,2})$. Introducing new variables $\phi_1$ and $\phi_2$ as $\text{tanh}(\phi_1/2) = \mu_s \tilde{p}_1 / \kappa$, $\text{tanh}(\phi_2/2) = \mu_s \tilde{p}_2 / \kappa$, we represent relation (12) in the form:

$$2\kappa h + \phi_1 + \phi_2 = 0.$$  

(13)

Since $\kappa h$ is positive, at least one of $\phi_1$ and $\phi_2$ must be negative. We define the amplitudes of nonlinear surface waves at the corresponding interfaces as: $A_{s1} = \alpha_1^{-1/2} \text{sech}[p_1(x_{1,1})]$, $A_{s2} = \alpha_2^{-1/2} \text{sech}[p_2(h - x_{2,2})]$. The connection between parameters $x_{1,2}$ and $A_{s1,2}$ can be found from the condition describing coupling of these two surface waves:

$$A_{s2} = A_{s1} [\cosh \kappa h - (\mu_s \tilde{p}_2 / \kappa) \sinh \kappa h].$$

(14)

Taking into account the relation (14) we conclude that the dispersion relation (12) contains $x_1$ as a free parameter. On the other hand, parameter $A_{s1}$ is defined in terms of $x_1$, therefore the value of the electric field at $x = 0$ can be chosen as a free parameter instead of $x_1$. Finally, equation (12) can be considered as an implicit one parameter expression for describing the dependance of the propagation constant on the frequency $\beta = \beta(\omega; A_{s1})$.

2.2.2 Dispersion relations of nonlinear guided waves

Analysis of the nonlinear guided wave is based on equation (3). In contrast to the previous case, here we choose the sign of the coefficient in front of the function $\Phi$ in the equation (3) to be negative i.e. $k_0^2 \varepsilon_s \mu_s - \beta^2 = \kappa_1^2 > 0$. If $\mu_i \varepsilon_{nl}^{(i)} > 0$, the solution of equation (3) is represented by the expression

$$\Phi(x) = \begin{cases} 
\alpha_1^{-1/2} \text{sech}[p_1(x - x_1)], & x < 0 \\
A \exp(i\kappa_1 x) + B \exp(-i\kappa_1 x), & 0 < x < h, \\
\alpha_2^{-1/2} \text{sech}[p_2(x - x_2)], & x > h
\end{cases}$$

where $p_{1,2} = \beta^2 - k_0^2 \varepsilon_1 \mu_1$, $\kappa_1^2 = \beta^2 - k_0^2 \varepsilon_s \mu_s > 0$. The sign of $\kappa_1^2$ is also chosen to be positive (see (3)), as we will see below such a choice corresponds to the case of guided waves.
In this case the dispersion relation for NLGW reads as
\[
e^{2i\kappa_1 h} \frac{(1 - i\mu_s \phi_1/\kappa_1)(1 - i\mu_s \phi_2/\kappa_1)}{(1 + i\mu_s \phi_1/\kappa_1)(1 + i\mu_s \phi_2/\kappa_1)} = 1,
\]
(15)
or
\[
\tan(\kappa_1 h) = \frac{\mu_s \phi_1 (\phi_1 + \phi_2)}{\kappa_1^2 - \mu_s^2 \phi_1 \phi_2}.
\]
(16)
If we define the phases \(\phi_1\) and \(\phi_2\) by the formulae:
\[
\tan(\phi_1/2) = \mu_s \phi_1/\kappa_1, \tan(\phi_2/2) = \mu_s \phi_2/\kappa_1,
\]
then equation (2) can be written as
\[
2\kappa_1 h = \phi_1 + \phi_2 + 2\pi m, \ m = 0, 1, 2, ...
\]
(17)
As in to the case of linear waveguides, this expression shows that the full phase shift of the zigzag wave [8] consists of contributions from the linear medium of the LHM dielectric slab and the phase shifts \( -\phi_1 \) and \( -\phi_2 \), which occur in the total internal reflection at the linear-non-linear interfaces. This nonlinear phase shift can be named the non-linear Goos-Hanchen effect [17,18,19].

Analysis of TM waves is presented in [10,15]. The corresponding dispersion relation and NLGW modes were studied in [14]. In particular, symmetric, asymmetric, and antisymmetric, forward and backward modes have been found.

### 2.3 Wave refraction and reflection at thin-film on interface between two dielectrics

Let us consider the case where a thin film is inserted between two isotropic media. The optical properties of these media are described by the permittivity and permeability: \(\varepsilon_1\) and \(\mu_1\) at \(x < 0\), \(\varepsilon_2\) and \(\mu_2\) at \(x > 0\). Let us assume that the thin film of metamaterial separates these two media. The optical properties of the metamaterial are described by the polarization \(\mathbf{P}^{(s)}\) and the magnetization \(\mathbf{M}^{(s)}\). We assume that the width of film \(l_f\) is much less than the carrier wave length of a quasiharmonic electromagnetic wave. To describe the propagation of the electromagnetic wave through thin film it is sufficient to take account of the jump in the electric and magnetic field components induced by polarization and magnetization of the film. Following to [20,21] we can approximate transverse distribution of polarization and magnetization in the film by a delta-function located at \(x = 0\). This approximation leads to the following jump conditions
\[
E_y(0^-) - E_y(0^+) = \frac{4\pi}{c} \frac{\partial}{\partial t} M_z^{(s)}, \quad H_z(0^-) - H_z(0^+) = \frac{4\pi}{c} \frac{\partial}{\partial t} P_z^{(s)},
\]
(18)
for the TE wave, and
\[
H_y(0^-) - H_y(0^+) = -\frac{4\pi}{c} \frac{\partial}{\partial t} P_z^{(s)}, \quad E_z(0^-) - E_z(0^+) = -\frac{4\pi}{c} \frac{\partial}{\partial t} M_z^{(s)},
\]
(19a)
for the TM wave. The expressions (18) and (19a) are modifications the corresponding boundary relations used in [20,21,22,23,24]. These new relations are applicable both for the case of continuum waves and for ultra-short or extremely short pulses.

It follows from (18) and (19a) that the electric and magnetic field components have different values in the media surrounding the film. The values of \(E\) and \(H\) (i.e., field values acting on nanostructures (meta-atoms) of the film), are naturally defined as follows:
\[
E_a(x = 0) = 0.5 [E_a(0^-) + E_a(0^-)] \quad (20)
\]
\[
H_a(x = 0) = 0.5 [H_a(0^-) + H_a(0^-)] \quad (21)
\]
where \(a = y\) or \(z\). In the case of a nonmagnetic film, using (18), we obtain the well known result [20,21] for a TE-wave \(E_y(x = 0) = E_y(0^-) = E_y(0^-)\).
Let us consider a plane wave which has normal incidence to the interface. For the sake of simplicity we assume that the two dielectric media surrounding the film are dispersionless. Then in the general case the electric field can be represented in terms of incident, reflected and transmitted waves:

\[
E(x,t) = \begin{cases} 
E_{in}(t - x/V_1) + E_{ref}(t + x/V_1) & x < 0 \\
E_{tr}(t - x/V_2) & x > 0.
\end{cases}
\]  

(22)

Here \(V_1\) is group velocity in the dielectric at \(x < 0\), and \(V_2\) is group velocity in dielectric at \(x > 0\). Using the jump conditions \([18], [19]a\) and equation \(E_{y,x} = -c^{-1}H_{z,t}\) we find the analog of the Fresnel relations:

\[
E_{tr}(t) = \frac{2V_2}{V_1 + V_2}E_{in} - \frac{4\pi V_2}{c(V_1 + V_2)} \left\{ \frac{\partial}{\partial t} M_z^{(s)} + \frac{V_1}{c} \frac{\partial}{\partial t} P_y^{(s)} \right\},
\]

\[
E_{ref}(t) = \frac{V_2 - V_1}{V_1 + V_2}E_{in} + \frac{4\pi V_1}{c(V_1 + V_2)} \left\{ \frac{\partial}{\partial t} M_z^{(s)} - \frac{V_2}{c} \frac{\partial}{\partial t} P_y^{(s)} \right\},
\]

(23)

(24)

where \(P_y^{(s)} = P_y(t, x = 0)\) and \(M_z^{(s)} = M_z(t, x = 0)\) are surface polarization and magnetization of the thin film respectively.

For the TM-wave, using a similar approach, we can obtain following the Fresnel relations:

\[
H_{tr}(t) = \frac{2\varepsilon_2 V_2}{\varepsilon_1 V_1 + \varepsilon_2 V_2} H_{in} - \frac{4\pi \varepsilon_1 \varepsilon_2 V_1 V_2}{c(\varepsilon_1 V_1 + \varepsilon_2 V_2)} \left\{ \frac{\partial}{\partial t} M_y^{(s)} - \frac{c}{\varepsilon_1 V_1} \frac{\partial}{\partial t} P_z^{(s)} \right\},
\]

\[
H_{ref}(t) = \frac{\varepsilon_2 V_2 - \varepsilon_1 V_1}{\varepsilon_1 V_1 + \varepsilon_2 V_2} H_{in} + \frac{4\pi \varepsilon_1 \varepsilon_2 V_1 V_2}{c(\varepsilon_1 V_1 + \varepsilon_2 V_2)} \left\{ \frac{\partial}{\partial t} M_y^{(s)} + \frac{c}{\varepsilon_1 V_1} \frac{\partial}{\partial t} P_z^{(s)} \right\},
\]

(25)

(26)

where the magnetic field amplitudes are defined by:

\[
H_y(x, t) = \begin{cases} 
H_{in}(t - x/V_1) + H_{ref}(t + x/V_1), & x < 0 \\
H_{tr}(t - x/V_2), & x > 0.
\end{cases}
\]

(27)

The electric field can be defined from the equation \(H_{y,x} = c^{-1}E_{z,t}\).

In what follows we will consider the case of a TE wave since there is no differences between TE and TM waves in case of normal incident waves. To obtain the polarization and magnetization of the thin film it is necessary to choose an appropriate model for the meta-atoms of the film. We start from the hypotheses that the linear response of film is determined by the electric oscillations in nanostructures. This leads to the simplest model from a standard linear Lorenz model which considers meta-atoms as linear oscillators forced by electric and magnetic fields. One natural generalization of such a model is to take into account anharmonicity of the electrical oscillations in nanostructures. This leads to the simplest model for nonlinear response of metamaterials:

\[
P_{tt} + \omega_d^2 P + \Gamma_e P_t + g_p P^3 = (\omega_p^2/4\pi)E,
\]

\[
M_{tt} + \omega_d^2 P + \Gamma_m M_t = - (\beta_m/4\pi)H_{tt},
\]

(28)

where \(\omega_d\) is frequency of the dimensional quantization due to confinement of the plasma in the nanostructures. Losses in the metallic nanostructures are taken into account by the parameters \(\Gamma_e\) and \(\Gamma_m\). Let us normalize the variables in equations \([23] a\) and \([24] a\):

\[
e_{tr} = E_{tr}/4\pi P_0, \quad e_{ref} = E_{ref}/4\pi P_0, \quad \varepsilon_{in} = E_{in}/4\pi P_0,
\]

\[
q = P/P_0, \quad m = M/\beta_m \sqrt{\varepsilon_2} P_0, \quad \tau = \omega_p t.
\]

The system of the normalized equations takes the following form

\[
e_{tr}(\tau) = 2\sqrt{\varepsilon_2}/\sqrt{\varepsilon_1 + \varepsilon_2} e_{in}(\tau) - \frac{1}{c(\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2})} \left\{ \frac{\partial q}{\partial \tau} + \beta_m \sqrt{\varepsilon_2} \frac{\partial m}{\partial \tau} \right\},
\]

(29)
\[ e_{\text{ref}}(\tau) = \frac{\sqrt{\varepsilon_2} - \sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} e_{\text{in}}(\tau) - \frac{i f_0 \omega_p}{c(\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2})} \left\{ \frac{\partial q}{\partial \tau} - \beta_m \sqrt{\varepsilon_1 \varepsilon_2} \frac{\partial m}{\partial \tau} \right\}. \]

\[ q_{\tau\tau} + (\omega_d/\omega_p)^2 q + (\Gamma_e/\omega_p) q_{\tau} + (g_p P_0^2/\omega_p^2) q^3 = 0.5(e_{\text{in}}(\tau) + e_{\text{tr}}(\tau) + e_{\text{ref}}(\tau)). \]

\[ m_{\tau\tau} + (\omega_T/\omega_p)^2 m + (\Gamma_m/\omega_p)m_{\tau} = -0.5 \left[ e_{\text{tr}}(\tau) + \sqrt{\varepsilon_1 \varepsilon_2} (e_{\text{in}}(\tau) + e_{\text{ref}}(\tau)) \right]_{\tau\tau}. \]

Inhomogeneous broadening of the resonance line can be taken into account if we replace \( q \) and \( m \) in (29, 32) by \( \langle q \rangle \) and \( \langle m \rangle \). Here corner brackets \( \langle \cdot \rangle \) denote averaging over different frequencies \( \omega_d \) and \( \omega_T \).

We assume that the pulse duration is much shorter than the characteristic plasmonic oscillation damping time, and neglect the dissipation terms in (29, 32). Furthermore, if the slowly varying envelope approximation is acceptable, we can reduce the system of these equations and consider the following system:

\[ E_{\text{tr}}(t) = \frac{2q_1(\omega_0)}{q_1(\omega_0) + q_2(\omega_0)} E_{\text{in}} + \frac{4\pi i k_0 n_{\text{at}} f_p}{q_1(\omega_0) + q_2(\omega_0)} \left[ k_0 \langle P \rangle + q_1(\omega_0) \langle M \rangle \right], \]

\[ E_{\text{ref}}(t) = \frac{q_2(\omega_0) - q_1(\omega_0)}{q_1(\omega_0) + q_2(\omega_0)} E_{\text{in}} + \frac{4\pi i k_0 n_{\text{at}} f_p}{q_1(\omega_0) + q_2(\omega_0)} \left[ k_0 \langle P \rangle - q_2(\omega_0) \langle M \rangle \right]. \]

\[ \mathcal{P}_t - i(\omega_0 - \omega_d) \mathcal{P} - (3ig_p/2\omega_0)|\mathcal{P}|^2 \mathcal{P} = i(\omega_p^2/8\pi \omega_0) \mathcal{E}_f, \]

\[ \mathcal{M}_t - i(\omega_0 - \omega_T) \mathcal{M} = i(\beta_m \omega_0/8\pi) \mathcal{H}_f, \]

Here \( k_0 = \omega_0/c, \omega_0 \) is the carrier wave frequency, \( E_{\text{in}}, E_{\text{tr}} \) and \( E_{\text{ref}} \) are envelopes of the incident, transmitted and reflected pulses, \( \mathcal{P} \) and \( \mathcal{M} \) are envelopes of the polarization and magnetization. The wavenumbers \( q_{1,2} \) are determined as \( q_{1,2} = k_0 \sqrt{\varepsilon_{1,2}} \cos \theta_{1,2}; \theta_1 \) and \( \theta_2 \) are incident and refraction angles respectively; \( n_{\text{at}} \) is volume density of meta-atoms. The envelopes of electric \( \mathcal{E}_f \) and magnetic \( \mathcal{H}_f \) fields, acting on nanostructures (meta-atoms) of the thin film can be represented as follows:

\[ \mathcal{E}_f = [E_{\text{tr}} + E_{\text{in}} + E_{\text{ref}}]/2, \quad \mathcal{H}_f = [\sqrt{\varepsilon_2} E_{\text{tr}} + \sqrt{\varepsilon_1} (E_{\text{in}} - E_{\text{ref}})]/2. \]

Using the normalized variables \( e_{\text{tr}} = E_{\text{tr}}/A_0, e_{\text{ref}} = E_{\text{ref}}/A_0, e_{\text{in}} = E_{\text{in}}/A_0, q = \mathcal{P}/P_0, m = \mathcal{M}/M_0, \tau = \omega_p t, \) and introducing parameters \( P_0 = (\omega_p/8\pi \omega_0)A_0 \) and \( M_0 = (\beta_m \omega_0 \sqrt{\varepsilon_2}/8\pi \omega_0^2)A_0 \), we present the dimensionless system of equations describing the interaction of the electromagnetic pulse with a thin film of metamaterial:

\[ e_{\text{tr}}(t) = \frac{2n_1}{n_1 + n_2} e_{\text{in}} + \frac{i k_0 n_{\text{at}} f_p}{2(n_1 + n_2)} \left[ \frac{\omega_p}{\omega_0} (q) + \beta_m n_1 n_2 \frac{\omega_0}{\omega_p} (m) \right], \]

\[ e_{\text{ref}}(t) = \frac{n_2 - n_1}{n_1 + n_2} e_{\text{in}} + \frac{i k_0 n_{\text{at}} f_p}{2(n_1 + n_2)} \left[ \frac{\omega_p}{\omega_0} (q) - \beta_m n_1 n_2 \frac{\omega_0}{\omega_p} (m) \right]. \]

\[ q_{\tau} = i\omega_p^{-1}(\omega_0 - \omega_d)q + ig_2|q|^2 q + i(e_{\text{tr}} + e_{\text{in}} + e_{\text{ref}})/2, \]

\[ m_{\tau} = i\omega_p^{-1}(\omega_0 - \omega_T)m + i(e_{\text{tr}} + (n_1/n_2)(e_{\text{in}} - e_{\text{ref}}))/2. \]

Here \( n_{1,2} = \sqrt{\varepsilon_{1,2}} \) are refractive indices of the dielectric media. The constant of anharmonicity \( g_2 \), used in these equations, is defined as \( g_2 = (3g_p/2\omega_p^3)(\omega_p/8\pi \omega_0^2)A_0^2 \). The free parameter \( A_0 \) can be presented in terms of the peak intensity of the incident pulse.

In the case of a thin film with no magnetization the jump conditions (18) results in the relation \( e_{\text{tr}} = e_{\text{in}} + e_{\text{ref}} \). Equations (37)–(40) in this case transform into the system of equations describing the refraction of the ultra-short electromagnetic pulse on the thin film with embedded nanoparticles, which was considered in [25].
2.4 Optical bistability in nonlinear layered structures

A linear layer of dielectric material acts as a resonator for an incident light beam. The transmissivity of this layer is determined by the thickness and refractive index of the layer material and by the wavelength of the incident light. If such a dielectric is nonlinear then its optical thickness depends on the intensity of the light field. Therefore, transmissivity of this layer depends on the light intensity. Let us consider the resonance reflection condition in the linear limit. Gradual increase of light intensity will be followed by a drastic increase in the transmissivity if the light intensity reaches a certain threshold level. In other words, the system switches from an opaque to a transparent state. This is the phenomena of optical bistability [26]. Phenomena of bistability in thin films were studied in [22,27]. The more realistic case of a film with nonzero thickness was considered in [29,28]. The switching condition is determined by the phase difference of the waves reflected from the front and back boundaries of the layer. Let us consider a sandwich structure with a metamaterial film and a layer of nonlinear dielectric. The presence of the metamaterial film results in a change in the effective thickness of the resonator. If the refractive index of such a film is negative, then the effective optical thickness of the sandwich-resonator is less than the optical thickness of the resonator without film. Therefore, the switching condition is realized at a higher level of light intensity [30].

It was also shown in [30] that switching light intensity depends on the incident beam direction. Switching intensity is higher for the case where the light is entering the structure from the side covered by the metamaterial film than for the case of opposite beam direction.

The hysteresis loop is an important characteristic of bistable devices. Switching thresholds from opaque to transparent state and from transparent to opaque state are different. The hysteresis width is the difference of these two switching thresholds. The presence of a metamaterial film, as was shown in [30], results in an increase of the hysteresis width.

3 Parametric interactions

Transformation of frequency of an electromagnetic wave propagating in a non-linear medium is one of the most fundamental effects [6] forming a basis for a broad range of nonlinear optics phenomena. Three waves interaction is the simplest representative of such a class of phenomena. When frequencies of two waves from an interacting wave triad are equal, the frequency of the third wave is two times higher. This is known as second harmonic generation (SHG). Stimulated Raman scattering is another example of three-wave interaction. In this case one of the waves is of acoustic nature and the incident and scattered waves are electromagnetic. Four wave interaction is a more complicated process of parametric interactions. Third harmonic generation (THG) is a typical representative of these processes.

3.1 Three-wave parametric interaction

Let as assume that the non-linear characteristics of a medium to be described by non-linear susceptibility of second order $\chi^{(2)}$. Such a medium is commonly known as a quadratic non-linear medium. We consider waves with carrier frequencies $\omega_1$ and $\omega_2$ propagating along the z axis. The polarization of such a medium is a quadratic function of the electrical field, therefore waves with frequencies $\omega_s = \omega_1 \pm \omega_2, 2\omega_1, \text{ and } 2\omega_2$ must be generated in such a medium. These waves, when their amplitudes increase sufficiently to be involved in the process of nonlinear interaction, can generate new waves with the frequencies $2\omega_1 \pm \omega_2, \omega_1 \pm 2\omega_2, 4\omega_1, \text{ and } 4\omega_2$ so on. However in a dispersive medium all these processes are not equally efficient. There is an important phase matching condition, which selects and emphasis a certain type of interaction of three waves, leaving all others unaffected. Sometimes such a phase matching occurs for waves propagating in the same direction - the case of collinear parametric interaction. In this case the distance at which the interaction of waves is taking place can be made sufficiently long and, consequently, the effective frequency transformation will take place.
The equations describing three wave interaction ($\omega_3 = \omega_1 + \omega_2$) in a $\chi^{(2)}$-medium in the slowly varying envelope and phase approximation, taking into account the group-velocity dispersion, can be written in following form [9]:

\[
\begin{align*}
\left( k_1 \frac{\partial}{\partial z} + \frac{1}{v_1} \frac{\partial}{\partial t} \right) E_1 - \frac{D_1}{2} \frac{\partial^2 E_1}{\partial t^2} &= i \frac{2\pi \omega_1^2 \mu(\omega_1)}{c^2 k_1} P_{NL}(\omega_1) \exp(-ik_1 z) \\
\left( k_2 \frac{\partial}{\partial z} + \frac{1}{v_2} \frac{\partial}{\partial t} \right) E_2 - \frac{D_2}{2} \frac{\partial^2 E_2}{\partial t^2} &= i \frac{2\pi \omega_2^2 \mu(\omega_2)}{c^2 k_2} P_{NL}(\omega_2) \exp(-ik_2 z) \\
\left( k_3 \frac{\partial}{\partial z} + \frac{1}{v_3} \frac{\partial}{\partial t} \right) E_3 - \frac{D_3}{2} \frac{\partial^2 E_3}{\partial t^2} &= i \frac{2\pi \omega_3^2 \mu(\omega_3)}{c^2 k_3} P_{NL}(\omega_3) \exp(-ik_3 z).
\end{align*}
\]

Here $k_j^2$ is defined as $k_j^2 = (\omega_j/c)^2 \varepsilon(\omega_j) \mu(\omega_j)$; $\hat{k}_j$ is sign of square root of $k_j^2$; and

\[
\begin{align*}
P_{NL}(\omega_1) &= \chi^{(2)}(\omega_1; \omega_3, -\omega_2) E_2 E_3^* \exp[i(z(k_3 - k_2))], \\
P_{NL}(\omega_2) &= \chi^{(2)}(\omega_2; \omega_3, -\omega_1) E_3 E_1^* \exp[i(z(k_3 - k_1))], \\
P_{NL}(\omega_3) &= \chi^{(2)}(\omega_3; \omega_1, \omega_2) E_1 E_2^* \exp[i(z(k_1 + k_2))].
\end{align*}
\]

If $\omega_1 = \omega_2$, $\omega_3 = 2\omega_1$ we have second harmonic generation. If $\omega_1 = \omega_s$, $\omega_2 = \omega_i$, and $\omega_3 = \omega_p$ ($\omega_p = \omega_s + \omega_i$), then we are dealing with parametric amplification phenomena. The signal wave is amplified by taking energy from the pump wave with the help of the idler wave, playing the role of a mediator in this energy transfer. The idler wave here corresponds to $E_2$ and the pump wave corresponds to $E_3$.

The zero phase mismatch condition $\Delta k = k_p - k_s - k_i$ means that the vector of the pump wave $k_p$ is equal to the vector $k_s + k_i$. The configuration of these vectors defines the energy flow directions. If the nonlinear medium is characterized by a positive refraction index, then the vector $k_p$ and the Poynting vectors associated with the interacting waves have the same orientation. Directionality takes place if the nonlinear medium is characterized by negative refraction index [9, 37, 38]. Currently the negative refractive property in the optical domain has been realized by use of simultaneous resonance for electric and magnetic field components in metallic nanostructures embedded to a host medium. The resonance frequencies in these materials are close to each other and they divide the frequency range into two domains. The medium responds to external waves as a negative refractive index material if the wave carrier frequency is below both resonance frequencies. The medium responds to external waves as a positive refractive index material if the wave carrier frequency is above both resonance frequencies. In the case of a three wave interaction, wave vectors corresponding to the waves with frequencies from the frequency region of negative refraction index are directed according to the zero phase mismatch condition, but the associated Poynting vectors are have opposite orientation.

Let us consider parametric amplification. It is convenient to introduce normalized variables $\hat{E}_1 = A_0 \gamma_1 e_s$, $\hat{E}_2 = A_0 \gamma_2 e_2$, $\hat{E}_3 = A_0 \gamma_2 e_1$, where $\gamma_j = 2\pi \omega_1 \chi^{(2)}(\omega_1) c^{-1} \sqrt{\mu(\omega_j)/\varepsilon(\omega_j)}$. The system of equations describing the parametric amplification can be rewritten as follows:

\[
\begin{align*}
\left( \frac{\partial}{\partial z} + \frac{1}{v_1} \frac{\partial}{\partial t} \right) e_1 + i \frac{D_1}{2} \frac{\partial^2 e_1}{\partial t^2} &= i g e_2 e_s \exp(-i\Delta k z) \\
\left( \frac{\partial}{\partial z} + \frac{1}{v_2} \frac{\partial}{\partial t} \right) e_2 + i \frac{D_1}{2} \frac{\partial^2 e_2}{\partial t^2} &= i g e_1 e_s^* \exp(+i\Delta k z) \\
\left( \frac{\partial}{\partial z} + \frac{1}{v_3} \frac{\partial}{\partial t} \right) e_s + i \frac{D_1}{2} \frac{\partial^2 e_s}{\partial t^2} &= i g e_1 e_s^* \exp(+i\Delta k z).
\end{align*}
\]

Here $\Delta k$ is defined as $\Delta k = k_p - k_s - k_i$ and $g = (\gamma_1 \gamma_2 \gamma_3)^{1/2} A_0$ is a coupling constant. The value $A_0$ is introduced for appropriate normalization of electric fields of interacting waves.

In the general case the phase matching condition requires pump and signal waves to propagate in opposite directions. However, solitary wave solutions of (45) are possible if the power
of the interacting waves is sufficiently high. Such solitary wave solutions can be considered as a bounded state of signal, idler and pump waves propagating in the same direction. This bound state can be interpreted as a wave trapping phenomena, where the group velocity of the signal wave changes sign due to a nonlinearly induced change in the value of the effective refractive index. Note that if the phase mismatch is zero $\Delta k = 0$ and the effect of group-velocity dispersion is negligible (short sample), then equations (45) reduce to a system of equation integrable by use of the inverse scattering transform [32,33,34]. Therefore, results previously obtained for the integrable case of three-wave interaction can also be used to analyze this special case of three-wave interaction in LHM.

Negative refractive index materials based on plasmonic resonance in metallic nanostructures have large losses. The feasibility of parametric amplification as a means to compensate for dissipative losses in LHM was considered in [39]. It was shown that parametric amplifiers are capable of compensating for losses and additionally that they can be used as optical parametric oscillators. The opposite directionality of parametrically interacting waves in LHMs is an analog of distributed feedback. In conventional materials distributed feedback is realized by adding Bragg gratings.

The case of three-microwave interaction for a strong pump wave and two weak signals in metamaterials with nonlinear magnetic response was theoretically studied in [35]. In this case nonlinear properties of the material were determined by the embedded LC circuits with nonlinear resistance and capacitance.

### 3.2 Second Harmonic Generation

Second harmonic generation is a special case of three wave interaction ($\omega_1 + \omega_1 = \omega_2$) in a $\chi^{(2)}$-medium. The equations of SHG in the case of collinear mismatch conditions $\Delta k = 2k_1 - k_2$ follows from (45), if $e_s$ is chosen as $e_s = \hat{e}_1$ for fundamental wave (with carrier frequency $\omega$) and if $e_1$ in (46) is replaced by $\hat{e}_2$. In this new notation idler and pump waves correspond to a second harmonic:

\[
- \hat{e}_{1,z} + v_1^{-1}\hat{e}_{1,t} + i(D_1/2)\hat{e}_{1,tt} = ig\hat{e}_2\hat{e}_1^* \exp(-i\Delta k z) \tag{47}
\]

\[
\hat{e}_{2,z} + v_2^{-1}\hat{e}_{2,t} + i(D_2/2)\hat{e}_{2,tt} = ig\hat{e}_1^2 \exp(+i\Delta k z) \tag{48}
\]

Transformations of variables $\hat{e}_1 = A_{10}e_1$, $\hat{e}_2 = 2A_{10}e_2 \exp(+i\Delta k z)$ allow us to represent these equations in the following standard form

\[
ie_{1,\zeta} + (\sigma/2)e_{1,\tau \tau} - e_{2,\zeta}^* = 0, \tag{49}
\]

\[
ie_{2,\zeta} + i\delta e_{2,\tau} - (\beta/2)e_{2,\tau \tau} - \Delta e_2 + e_2^2/2 = 0. \tag{50}
\]

Here $A_{10}$, $\zeta$ and $\tau$ are defined as $A_{10} = (gL)^{-1}$, $\zeta = z/L$, $\tau = (t + z/v_1)/t_p$. Normalizing characteristic parameters here are chosen as follows: $L = t_p^2/|D_1|$ is dispersion length; $\delta = Lt_p^{-1}(v_1^{-1} + v_2^{-1})$ is normalized group velocity mismatch; $\Delta = \Delta k L$ is normalized phase mismatch; $t_p$ is characteristic time, which is not fixed yet; and $\sigma = \text{sgn}D_1$, $\beta = D_2/|D_1|$. Parameter $\delta$ takes into account the walk-off effect for pump and harmonic pulses that is due to the difference of the group velocities’ directions for the interaction waves. It should be pointed out that in contrast to the case of positive refractive index medium, this parameter can not be zero in LHM’s.

The form of the equations (49,50) is similar to that of the equations describing the evolution of quadratic solitons as a bounded complex of fundamental and second harmonic solitary waves presented in [10,11]. The only difference is in the relative sign of group velocities of the pump and second harmonic waves.

Using the equations (49,50) and taking into account the boundary condition $|e_{1,2}|^2 \to |e_{10,20}|^2$, $\tau \to \pm \infty$, here $e_{10}$ and $e_{20}$ are constants (or $|e_{1,2}|^2 \to 0 \tau \to \pm \infty$), we arrive to the
Manley-Rowe relation:
\[
\int_{-\infty}^{\infty} \left( |e_2|^2 - (1/2)|e_1|^2 \right) d\tau = \text{const}.
\]

Note that the sign of the second term in the integrand is negative in contrast to the situation for SHG in nonlinear positive index materials. This form of the Manley-Rowe relation reflects the fact that the Poynting vectors, i.e., energy fluxes, for the fundamental and the second harmonic are antiparallel, while their wave vectors are parallel.

3.2.1 Continuous wave limit

The system of equations (49,50) for continuous waves reduces to following equations [9,37,38]:
\[
ie_1,\zeta - e_1^* e_2 = 0, \quad ie_2,\zeta - \Delta e_2 + e_1^2/2 = 0,
\]
(51)

The boundary conditions for a nonlinear plate of a finite width \(l\) are as follows:
\[
\text{At} \quad \zeta = 0 \quad |e_1| = a_0, \quad \text{and} \quad \zeta = l \quad |e_2| = 0.
\]
(52)

The Manley-Rowe relation in this case reads
\[
2|e_2|^2 - |e_1|^2 = 2c_0^2 = \text{const}.
\]

Solutions of these equations [51] were found in [37,38]. Spatial distributions of the harmonic and pump wave are represented by the following expressions:
\[
|e_1(\zeta)| = c_0 \sqrt{2} \sec [c_0 (l - \zeta)], \quad |e_2(\zeta)| = c_0 \tan [c_0 (l - \zeta)].
\]
(53)

The constant \(c_0\) is defined by the relation \(a_0 = c_0 \sqrt{2} \sec [c_0 \zeta_0]\) that follows from boundary conditions (52). In the case of quadratic-nonlinear PRI medium the amplitude of the pump wave decreases with distance, while the second harmonic wave amplitude increases. The energy of the pump wave is transferred into the second harmonic wave and the energy fluxes of both waves are aligned. In a quadratic-nonlinear NRI medium the energy fluxes are oriented in opposite directions, thus both pump and second harmonic wave amplitudes decrease with distance. Second harmonic generation taking into account dissipation has been investigated by numerical simulation in [37].

3.2.2 Large-mismatch limit

During the last few years there has been growing interest in three wave interaction or its special case of second harmonic generation when \(|\Delta| >> 1\), (see review [11]). This regime is known as the large-mismatch limit, the cascading limit, or the effective Kerr limit. In this limit, equations for SHG [10,11] can be transformed into a nonlinear Schrödinger equation for the fundamental wave, and the amplitude of the second harmonic wave is proportional to the squared amplitude of fundamental wave. Generally quadratic nonlinearity is an order of magnitude stronger than cubic nonlinearity. Therefore the large-mismatch limit case is very useful for study of nonlinear cubic phenomena using quadratic nonlinearity. This large-mismatch limit can, in a similar way, be considered in the case of a quadratic-nonlinear NRI medium. From (50) we evaluate \(e_2 \approx e_1^2/2\Delta\), substituting \(e_2\) in (49) results in
\[
ie_1,\zeta + (\sigma/2)e_1,\tau - (1/2\Delta)|e_1|^2 e_1 = 0.
\]
(54)

This equation [54] has soliton solutions when \(\sigma = -1\). The type of these soliton solutions is controlled by sign of \(\Delta\): these are bright solitons if \(\Delta > 0\) and they are of dark type if \(\Delta < 0\). Note that for the frequency of the fundamental (pump) wave the medium acts as left handed. On the other hand the governing equation for the pump wave in a large-mismatch limit is identical to the conventional NLS equation written for a right handed material. This is an indication that the soliton properties of LHM and RHM are similar in the large-mismatch limit.
3.2.3 Solitary wave solutions

The solitary wave solutions of (49, 50), represented by two-frequency pulses, have been considered in [42]. To find such solutions it is convenient to introduce real variables for the interacting waves

\[ e_1 = a \exp(i \varphi_1), \quad e_2 = b \exp(i \varphi_2). \]

Substitution of these expressions into (49, 50) leads to

\[ a_\zeta + (\sigma/2) (2a_\tau \varphi_1 + a \varphi_1) = ab \sin \Phi, \tag{55} \]
\[ 2b_\zeta + 2b_\tau - \beta (2b_\tau \varphi_2 + b \varphi_2) = a^2 \sin \Phi, \tag{56} \]
\[ a \varphi_1 \zeta - (\sigma/2) (a_\tau - a \varphi_1 \varphi_1) = -ab \cos \Phi, \tag{57} \]
\[ 2a (\varphi_2 \zeta + \delta \varphi_2) + \beta (b_\tau + b \varphi_2 \varphi_2) + 2 \Delta b = a^2 \cos \Phi, \tag{58} \]

where \( \Phi = \varphi_2 - 2 \varphi_1 \). Assuming that phases are linear functions of the following form \( \varphi_1 = K \zeta + \Omega \tau, \varphi_2 = 2K \zeta + 2\Omega \tau \) we obtain the mismatch condition \( \Phi = 0 \). The amplitude equations (55) and (56) are equivalent if the frequency \( \Omega \) is chosen to be \( \Omega = \delta/(\sigma + 2\beta) \). Let us consider the solitary wave regime represented by a bound state of two waves with a fixed value of the amplitudes’ ratio. These waves are defined by functions of a single argument \( a = a(\tau - \zeta/V) \), \( b = b(\tau - \zeta/V) \), with \( V^{-1} = \sigma \Omega \) and \( b = f a \). Here \( f \) is a constant. Under such an assumption, the system of equations (55) and (56) is overdetermined (we have two equations for one function \( a = a(y), y = \tau - \zeta/V \)). The compatibility condition for these equation reads:

\[ f^2 = \sigma/2\beta > 0, \]
\[ K = \sigma (3\beta \Omega^2 - 4\delta \Omega - 2\Delta)/2(\beta + 2\sigma). \]

and the equation for \( a \) has the following form

\[ a, y a - (2\sigma \Omega + \Omega^2) a - 2\sigma f a^2 = 0. \]

It follows from the compatibility condition that \( \sigma \) and \( \beta \) must have the same sign.

Let us consider the boundary condition \( a \to 0, \partial a/\partial y \to 0 \) at \( \tau \to \pm \infty \), which is consistent with a solitary wave solution propagating on a zero background (quadratic soliton – see review [41]). In the case where \( \sigma = -1 \) the solitary wave solution of (55, 56) exists if \( p = (\Omega^2 - 2K) > 0 \). This solution is a bright NRI medium soliton with frequency components of the following form

\[ a(y) = \frac{(3p/4) \sqrt{2|\beta|}}{\cosh^2[\sqrt{p(y - y_0)/2}]}, \quad b(y) = \frac{(3p/4) \sqrt{2|\beta|}}{\cosh^2[\sqrt{p(y - y_0)/2}]}. \tag{59} \]

The group velocity \( V_s \) of the bright soliton is fixed by system parameters and defined as

\[ V_s^{-1} = (\sigma v_2^{-1} - 2\beta v_1^{-1})/(\sigma + 2\beta). \]

Note, that in the case of a quadratic-nonlinear PRI medium, the group velocity of a bright soliton is defined by the formula \( V_s^{-1} = (2\beta v_1^{-1} - \sigma v_1^{-1})(2\beta - \sigma) \). In the case where \( \sigma = 1 \) the solitary wave solution of (55, 56) exists if \( p = (\Omega^2 + 2K) > 0 \).

If the background is nonzero, then the system of equations (55, 56) has dark soliton solutions. An example of such a dark soliton at \( \sigma = -1 \) and \( p = (\Omega^2 - 2K) < 0 \) is presented below:

\[ a(y) = (3|p|/4) \sqrt{2|\beta|} \left( 2/3 \pm 2 \sqrt{|p|} \sec h^2[\sqrt{|p|} |y|/2] \right), \tag{60} \]
\[ b(y) = (3|p|/4) \left( 2/3 \pm 2 \sqrt{|p|} \sec h^2[\sqrt{|p|} |y|/2] \right). \tag{61} \]

If the ratio of the coupled wave amplitudes is not fixed then the system of equations (55, 56) has double-hump solitary waves, which are described by following expression

\[ a(y) = 3p_1 \sqrt{2|\beta|} \tanh(\sqrt{m_1} y) \sec h(\sqrt{m_1} y), \tag{62} \]
\[ b(y) = 3p_1 \sec h^2(\sqrt{m_1} y). \tag{63} \]

where \( p_1 = (\Omega^2 + 2\sigma K) \).
3.2.4 No group-velocity dispersion limit

At the current state of the art in nanofabrication technology the interaction distance for parametric three-wave processes is strongly limited by the losses of plasmonic oscillations in metallic nanostructures of LHM. The dispersion length is shorter than the characteristic length of losses and therefore the dispersion term can be omitted from equations [49, 50]. We also consider the case where the length of the LHM sample is shorter than the characteristic length of losses. Therefore, the effects of group-velocity dispersion could be omitted as well. The corresponding system of equation takes the following form

\[ ie_{1,\xi} - e_{2}e_{1}^{*} = 0, \quad ie_{2,\xi} + i\delta e_{2,\tau} - \Delta e_{2} + e_{2}^{2} / 2 = 0. \]  

(64)

The results of SHG computer modeling using these equations were presented in [43]. If \( \Delta = 0 \), the system of equations (64) is integrable using the inverse scattering transform [45] and also can be represented in bilinear Hirota form [44]. Note, that in this particular case Hirota’s method is more convenient to analyze the nonstationary regime of SHG in LHM.

3.2.5 Non-collinear second harmonic generation

In this section we consider a process of SHG, where pump waves and the second harmonic wave propagate in arbitrary directions. In this case the phase matching condition is determined by the beams’ orientations. The evolution of slowly varying optical pulse envelopes is described by the corresponding directional derivatives and the argument of the exponential functions in the right-hand side must be replaced by \( \pm i(k_{1} + k_{2} - k_{3}) \cdot r \). The phase- matching condition in this case reads

\[ k_{1} + k_{2} - k_{3} = 0 \]  

(65)

Let us assume that the second harmonic wave propagates along the \( z \)-axis - \( k_{3} \) vector; both wave vectors - \( k_{1} \) and \( k_{2} \) - of the pump waves are in the \( xz \)-plane; so that \( k_{1} = k_{1}(\eta_{x}, -\eta_{z}) \) and \( k_{2} = k_{1}(-\eta_{x}, -\eta_{z}) \), where \( \eta_{x} = \sin \theta \) and \( \eta_{z} = \cos \theta \); \( \theta \) is an angle between \( k_{1} \) and \( k_{3} \). The projection of the vector equation (65) onto the \( z \)-axis defines the phase-matching angle \( \theta_{m} : n(2\omega) = n(\omega) \cos \theta_{m} \). Let us assume that the group-velocity dispersion is of no importance. The system of equations, describing the SHG, when phase matching condition is satisfied, reads

\[
\begin{align*}
-\eta_{x} \frac{\partial}{\partial z} + \eta_{x} \frac{\partial}{\partial x} + \frac{1}{v_{1}} \frac{\partial}{\partial t} E_{1}^{(+)} &= i\gamma_{1} E_{1}^{(-)} E_{2}, \\
-\eta_{z} \frac{\partial}{\partial z} + \eta_{z} \frac{\partial}{\partial x} + \frac{1}{v_{1}} \frac{\partial}{\partial t} E_{1}^{(-)} &= i\gamma_{1} E_{1}^{(+)} E_{2}, \\
\left( \frac{\partial}{\partial z} + \frac{1}{v_{2}} \frac{\partial}{\partial t} \right) E_{2} &= i\gamma_{2} E_{1}^{(+)} E_{1}^{(-)},
\end{align*}
\]

(66) (67) (68)

where the pump wave is presented as

\[ E_{1}(t, r) = E_{1}^{(+)}(t, x, z) \exp \left[ -i\omega t + ik_{1}(\eta_{x}x - \eta_{z}z) \right] + E_{1}^{(-)}(t, x, z) \exp \left[ -i\omega t - ik_{1}(\eta_{x}x + \eta_{z}z) \right], \]

and the second harmonic wave is represented as

\[ E_{2}(t, r) = E_{2}(t, x, z) \exp \left[ -2i\omega t + ik_{2}z \right]. \]

System (64) can be transformed into a system of dimensionless equations

\[ ie_{1,\tau} + e_{2}^{2} e_{2} = 0, \quad ie_{2,\xi} + e_{1} e_{3} = 0, \quad ie_{3,\xi} + e_{1}^{2} e_{2} = 0, \]  

(69)
where \( \zeta, \tau \) and \( \xi \) are characteristic co-ordinates defined as
\[
\zeta = \eta_z v_1 v_2 \gamma_1 \sqrt{\omega_L A_{10}} (v_2 + \eta_z v_1)^{-1} (t - z/\eta_z v_1),
\]
\[
\tau = 2v_1 v_2 \gamma_1 \sqrt{\omega_L A_{10}} (v_2 + \eta_z v_1)^{-1} (t - z/v_2 + (v_1^{-1} + \eta_z v_2^{-1})x/\eta_z),
\]
\[
\xi = 2v_1 v_2 \gamma_1 \sqrt{\omega_L A_{10}} (v_2 + \eta_z v_1)^{-1} (t - z/v_2 - (v_1^{-1} + \eta_z v_2^{-1})x/\eta_z).
\]

Normalized envelopes of the interacting waves are defined by the expressions
\[
\mathcal{E}_1^{(+)} = \sqrt{\gamma_1 A_{10}} e_1,
\]
\[
\mathcal{E}_2 = \sqrt{\gamma_1 A_{10}} e_2, \quad \mathcal{E}_3^{(-)} = \sqrt{\gamma_1 A_{10}} e_3.
\]

The solutions of these equations could be found either by the inverse scattering method [46,47,48], or by using a combination of simple algebraic manipulations and solution of ordinary differential equations [49].

### 3.3 Raman scattering process

Let us consider optical waves propagating in an infinite medium and scattered by optical phonons. This process is known as Raman scattering and differs from Mandelschtam-Brillouin scattering of optical waves occurring on acoustic phonons. The phonon frequency \( \omega_v \) is much less than the carrier frequencies of optical waves. The interaction of optical and vibration modes results in a shift of the frequency of the optical mode. This interaction generates two new waves (Stokes waves) in addition to the pump wave with carrier frequency \( \omega_P \). The frequencies of these new waves are \( \omega_S = \omega_P - \omega_v \) (Stokes wave), and \( \omega_{AS} = \omega_L + \omega_v \) (anti-Stokes wave). In the most cases, the intensity of the anti-Stokes wave is less than that of the Stokes wave. Therefore, we consider only incident and Stokes waves. Furthermore, we consider stimulated Raman scattering when the frequency \( \omega_P \) lies in the positive refractive index region and the frequency of the Stokes wave lies in negative refractive index region.

The slowly varying envelopes of the incident and Stokes pulses is governed by the system of the reduced Maxwell equations
\[
\left( \frac{\partial}{\partial z} + \frac{1}{v_L} \frac{\partial}{\partial t} \right) \mathcal{E}_L = -\sigma_L \mathcal{E}_L + i \frac{2\pi \omega_L^2 \mu(\omega_P)}{c^2 k_L} \mathcal{P}_L \exp(i\Delta kz), \quad (70)
\]
\[
\left( \frac{\partial}{\partial z} + \frac{1}{v_S} \frac{\partial}{\partial t} \right) \mathcal{E}_S = \sigma_S \mathcal{E}_S + i \frac{2\pi \omega_S^2 \mu(\omega_S)}{c^2 k_S} \mathcal{P}_S \exp(-i\Delta kz), \quad (71)
\]

The linear losses are taken into account by introducing extra terms with the coefficients \( \sigma_{1,2} \).

We define polarizations \( \mathcal{P}_{S,L} \) in the Raman medium in accordance with Placzek’s classical model where molecules are represented by harmonic oscillators. The polarization of this medium is represented as \( P = n_A \alpha(Q) E \), where \( n_A \) is the concentration of the molecules, \( \alpha(Q) \) is the molecular polarizability, and \( Q \) is the vibrational co-ordinate of a molecule determining the magnitude of deflection from equilibrium. In the case of small vibrations the molecular polarizability is expressed by the first two terms of its Taylor series, i.e.,
\[
\alpha(Q) \approx \alpha_0 + (\partial \alpha \partial Q) \theta Q \equiv \alpha_0 + \alpha_D Q.
\]

Hence, the non-linear polarization is \( P_{NL} = n_A \alpha_D Q E \). In the slowly varying envelope approximation \( Q \) is represented as follows
\[
Q = u(t,z) \exp(-i\omega_v t + ik_v z) + u^*(t,z) \exp(i\omega_v t - ik_v z),
\]
where \( u(t,z) \) is determined by the equation
\[
u_t - i(\omega_v - \omega_S) u + \sigma_v u = i (\alpha_D/2m\omega_v) \mathcal{E}_L^* \mathcal{E}_S \exp(i\Delta kz), \quad (72)
\]

Here \( m \) is the effective molecule mass. The coefficient \( \sigma_v \) takes into account linear losses. Taking into account that the phonon wave number is much smaller than the photon wave numbers, the
mismatch value can be taken to be: \( \Delta k \approx k_S - k_L \). We assume that spectral half-width of the ultrashort optical pulse is much smaller than the phonon frequency \( \Delta \omega_p \ll \omega_v \) and express the nonlinear slowly varying amplitudes of polarization as \( P_L = n_{AOD} u^* \mathcal{E}_S \) and \( P_S = n_{AOD} u \mathcal{E}_L \).

To develop a theory of Raman scattering we can use the system of equations (70-72). Note that in the case we consider the Stokes wave and the laser wave are propagating in opposite directions.

It is convenient to introduce a co-moving system of coordinates \( \zeta = z/L, \tau = (t - z/V_0)/t_p \) with velocity \( V_0 \) defined by as \( V_0^{-1} = (v_L^{-1} + v_S^{-1})/2 \), and normalized functions \( u = q/(n_{AOD}) \),

\[
\mathcal{E}_L = \sqrt{\gamma_L A_0 e_p \exp(i \Delta k z)}, \quad \mathcal{E}_S = \sqrt{\gamma_S A_0 e_s},
\]

where

\[
\gamma_S = (2 \pi \omega_S c^{-1} \sqrt{\mu(\omega_S)/\varepsilon(\omega_S)}), \quad \gamma_L = (2 \pi \omega_L c^{-1} \sqrt{\mu(\omega_L)/\varepsilon(\omega_L)}), \quad L^{-1} = \sqrt{\gamma_S \gamma_L}, \quad A_0^{-2} = (n_{AOD}^2 \sqrt{\gamma_S \gamma_L})/2m \omega_v.
\]

The dimensionless system of equations takes the form

\[
\begin{align*}
    ie_{p,\zeta} + i\delta e_{p,\tau} - \Delta e_p + i \Gamma_p e_p + q^* e_s &= 0, \\
    ie_{s,\zeta} - i\delta e_{s,\tau} - i \Gamma_s e_s - q e_p &= 0, \\
    i q_{,\tau} + \vartheta q + i \Gamma_v q + e_s e_p^* &= 0,
\end{align*}
\]

where \( \delta = Lt_p^{-1}(v_1^{-1} + v_2^{-1})/2, \vartheta = t_p(\omega_v - \omega_S), \Gamma_p = \sigma_L L, \Gamma_s = \sigma_S L, \Gamma_v = \sigma_v L \). Therefore the system of equations describing Raman scattering is represented as a three wave interaction model.

In the case of strong vibrational damping (\( \Gamma_p \gg 1 \)), it follows from the third equation of (73) that \( q \approx i \Gamma_v^{-1} e_s e_p^* \). Substitution of this formula into the two first equations of (73) leads to equations describing a Raman amplifier in LHM

\[
\begin{align*}
    ie_{p,\zeta} + i\delta e_{p,\tau} - \Delta e_p + i \Gamma_p e_p + |e_s|^2 e_p &= 0, \\
    ie_{s,\zeta} - i\delta e_{s,\tau} - i \Gamma_s e_s - |e_p|^2 e_s &= 0.
\end{align*}
\]

Note, that choosing \( \Gamma_p = \Gamma_s = 0 \) and \( \Gamma_v \gg 1 \), this system of equations can be used to study the phenomena of Raman spike generation in left-handed materials. This phenomena is well known in conventional right-handed materials [50,51].

### 3.4 Third Harmonic Generation in NRI medium

Third harmonic generation (THG) is associated with four wave interaction of the type \( \omega_1 + \omega_1 + \omega_1 \rightarrow \omega_3 = 3 \omega_1 \). This process is described by the following system of equation

\[
\begin{align*}
    \left( k_1 \frac{\partial}{\partial z} + \frac{1}{v_1} \frac{\partial}{\partial t} \right) \mathcal{E}_1 - \frac{D_1}{2} \frac{\partial^2 \mathcal{E}_1}{\partial t^2} &= i \frac{2 \pi \omega_0^2 \mu(\omega_1)}{c^2 k_1} \tilde{P}_{NL}(\omega_1) \exp(-ik_1 z) \\
    \left( k_3 \frac{\partial}{\partial z} + \frac{1}{v_3} \frac{\partial}{\partial t} \right) \mathcal{E}_3 - \frac{D_3}{2} \frac{\partial^2 \mathcal{E}_3}{\partial t^2} &= i \frac{2 \pi \omega_0^2 \mu(\omega_3)}{c^2 k_3} \tilde{P}_{NL}(\omega_3) \exp(-ik_3 z)
\end{align*}
\]

where

\[
\begin{align*}
    \tilde{P}_{NL}(\omega_1) &= \chi^{(3)}(\omega_1; \omega_3, -\omega_1, -\omega_1) |\mathcal{E}_3|^{*2} \mathcal{E}_1 \exp[iz(k_3 - 2k_1)] + \\
    &+ \chi^{(3)}(\omega_1; \omega_1, -\omega_1, \omega_1) |\mathcal{E}_1|^2 \mathcal{E}_1 + \chi^{(3)}(\omega_1; \omega_1, -\omega_3, \omega_3) |\mathcal{E}_3|^2 \mathcal{E}_1, \\
    \tilde{P}_{NL}(\omega_2) &= \chi^{(3)}(\omega_3; \omega_1, \omega_1, \omega_1) |\mathcal{E}_1|^3 \exp[iz3k_1] + \\
    &+ \chi^{(3)}(\omega_3; \omega_3, -\omega_3, \omega_3) |\mathcal{E}_3|^3 \mathcal{E}_1 + \chi^{(3)}(\omega_3; \omega_3, -\omega_1, \omega_1) |\mathcal{E}_1|^2 \mathcal{E}_3.
\end{align*}
\]

In contrast to SHG, the process of transformation of one wave to another in this case is accompanied by the two additional effects of self-phase and cross-phase modulation.
Let us consider the THG where the frequency of the fundamental wave (pump wave) $\omega_1$ is located in the NRI spectral region, and the third harmonic frequency $\omega_3$ is located in the positive refractive index spectral region. Let the self-modulation and cross-modulation effects be ignored. In this case THG equations after normalization read as

$$
\begin{align*}
&ie_{1,\zeta} + \frac{\sigma}{2} e_{1,\tau\tau} - e_2 e_1^2 = 0, \\
&ie_{3,\zeta} + i\delta e_{3,\tau} - (\beta/2) e_{3,\tau\tau} - \Delta e_3 + e_1^3 = 0,
\end{align*}
$$

(76)

where both parameters and normalized variables were introduced along similar lines to this use for SHG.

### 3.4.1 Third harmonic generation. CW-limit

First we consider third harmonic generation in the case of continuous waves. The system of equations describing THG in the continuous wave limit takes the following form

$$
\begin{align*}
&ie_{1,\zeta} - e_2 e_1^2 = 0, \\
&ie_{3,\zeta} - \Delta e_3 + e_1^3 = 0.
\end{align*}
$$

(77)

There are two types of boundary conditions: (a) $|e_1(\zeta = 0)| = a_0$, $|e_3(\zeta \to \infty)| = 0$, i.e., infinite medium, and (b) $|e_1(\zeta = 0)| = a_0$, $|e_3(\zeta = l)| = 0$, i.e., THG in the plate having width equal to $l$.

The Manley-Rowe relation following from (77) is as follows

$$
|e_1|^2 - |e_3|^2 = c_0^2 = \text{const.}
$$

(78)

Let consider the THG at phase mismatch $\Delta = 0$. The solution of the system (77) for infinite medium, can be written as

$$
|e_1(\zeta)| = |e_3(\zeta)| = \frac{a_0}{\sqrt{1 + 2a_0^2}}.
$$

(79)

Hence, in this special case there is complete transformation of the incident wave to a third harmonic wave which propagates in the opposite direction. The solution of (77) in the more realistic case, where the size of the nonlinear dielectric sample $l$ is finite, reads as

$$
|e_1(\zeta)|^2 = \frac{c_0^2 \xi^2}{1 - \xi^2}, \quad |e_3(\zeta)|^2 = \frac{c_0^2 \xi^2}{1 - \xi^2}, \quad \xi = c_0^2(\zeta - l).
$$

(80)

The equation for the unknown constant $c_0^2$ follows from the Manley-Rowe relation:

$$
a_0^2(1 - l^2 c_0^4) = c_0^2.
$$

Note that in contrast to the case of SHG there is only one solution of this equation which has physical meaning. In the case of SHG the corresponding equation reads $l a_0 \sin c_0 l = \sqrt{2} c_0 l$ and in general can have several solution [37,38]. Like in the case of SHG, this Manley-Rowe relation follows from the new type of phase-matching condition referred in [37,38] as “backward phase-matching”. This type of phase-matching is a fundamental feature of wave interaction in NRI medium.

### 3.4.2 Solitary wave solutions of third harmonic generation equations

A solitary wave solution (coupled state of two waves with carrier frequencies $\omega_1$ and $\omega_3$) can be found as in the case of SHG. Let us introduce the real variables for interacting waves $e_1 = a \exp(i \varphi_1)$ and $e_3 = b \exp(i \varphi_3)$, where $\varphi_1 = K\zeta + \Omega \tau$, $\varphi_3 = 3K\zeta + 3\Omega \tau$, $a = a(\tau - \zeta/V)$, $b = b(\tau - \zeta/V)$. The system of the equations (76) reduces to a single equation for the real
amplitude $a$, if the $V^{-1} = \sigma \Omega$, frequency $\Omega = \delta/(\sigma + \beta)$, and wave number is determined by formula

$$K = \sigma \left( 4 \beta \Omega^2 - 3 \delta \Omega - \Delta \right) / (3 \sigma + \beta).$$

Solving this equation we found the field amplitudes

$$|e_1(\tau, \zeta)| = \frac{\sqrt{p_f}}{\cosh[p^{1/2}(\tau - \zeta/V - y_0)]}, \quad |e_3(\tau, \zeta)| = \frac{\sqrt{p_f}}{\cosh[p^{1/2}(\tau - \zeta/V - y_0)]},$$

where $y_0$ is a constant of integration, $f^2 = \sigma/\beta$ and $p = 2 \sigma K + \Omega^2$. For $\sigma = \beta = -1$ we can find $\Omega = -\delta/4$, $K = (\delta^2 - \Delta)/8$ and $p = (4 \Delta - 3 \delta^2)/16$. The velocity of this soliton is defined as $\tau - \zeta/V = (t - z/V_s)/t_p$ and is equal to $V_s = 4 v_1 v_3/(v_1 - 3 v_3)$.

Note that the system of equations (76) has periodic cnoidal solutions. These solutions describe nonlinear periodic waves in a medium with cubic nonlinearity.

In the large-mismatch limit of THG, the system of equations (76) transforms into the quintic nonlinear Schrödinger equation:

$$i e_{1,\zeta} + (\sigma/2) e_{1,\tau \tau} - (1/2 \Delta) |e_1|^4 e_1 = 0.$$  

This equation also has solitary pulse like and periodic cnoidal wave solutions.

3.4.3 Phase interaction of two waves

There are situations where the process of conversion of waves with different frequencies can be ignored. For example, such a situation occurs when the phase difference of the two interacting waves is rapidly changing in space: $\Delta k \times L_d \gg 1$, here $L_d$ is the dispersion length. Let us consider propagation of two waves in a medium with cubic nonlinearity, where the frequency of one wave is in the NRI region, and the frequency of the other wave is in the PRI region. The corresponding equations in dimensionless form, which follow from (74,75), read as

$$i e_{1,\zeta} + (\sigma/2) e_{1,\tau \tau} - (|e_1|^2 + \mu |e_3|^2) e_1 = 0,$$

$$i e_{3,\zeta} + i \delta e_{3,\tau} - (\beta/2) e_{3,\tau \tau} + (|e_3|^2 + \mu |e_1|^2) e_3 = 0,$$

which represents a generalization of the Manakov’s equations [52]. The solitary wave solutions of these equations can be found by standard methods.

4 Periodic structures

In recent years, much attention has been drawn to various artificial structures, which are one- to three-dimensional periodic dielectric (or metallic) structures, frequently referred to as photonic crystals. The spectrum of electromagnetic waves in these structures possess bandgaps (the Bragg gaps) [53]. Sometimes the photonic crystals are called photonic band-gap materials (PBG). In addition to a conventional PBG, a one-dimensional periodic structure consisting of alternating layers of PRI and NRI medium represents a new type of photonic band-gap material, a so-called zero refractive index PBG [54]. There we consider only the resonant Bragg grating and nonlinear optical waveguide array as simple examples of nonlinear periodic medium.

4.1 Resonant Bragg grating

In the simplest case the resonant Bragg grating [55,56,57,58,59] consists of a linear homogeneous dielectric medium containing an array of thin films with resonant atoms or molecules. The distance between successive films is $a$, and the thickness of the film $l_f$ is much less than the wavelength of the electromagnetic wave propagating through such a structure. The interaction
of ultra-short pulses and films embedded with two-level atoms have been studied by Mantsyzov et al. in the framework of the two-wave reduced Maxwell-Bloch model and by Kozhekin. This work demonstrated the existence of the 2π-pulse of self-induced transparency in such structures. It was also found that bright as well as dark solitons can exist in the prohibited spectral gap, and that bright solitons can have arbitrary pulse area. If the density of two-level atoms is very high, then the near-dipole-dipole interaction is noticeable and should be accounted for in the mathematical model. The effect of dipole-dipole interaction on the existence of gap solitons in a resonant Bragg grating was studied in Ref. The unusual solution known as a zoomeron was discovered and investigated recently in the context of the resonant Bragg grating. A zoomeron is a localized pulse similar to an optical soliton, except that its velocity oscillates about some mean value.

Recent advances in nanofabrication have allowed the creation of nanocomposite materials which have the ability to sustain nonlinear plasmonic oscillations. These materials have metallic nanoparticles embedded in them. In a dielectric into which thin films containing metallic nanoparticles have been inserted was considered. These thin films are spaced periodically along the length of the dielectric so that the Bragg prohibited spectral gap is centered at the plasmonic resonance frequency of the nanoparticles. A set of discrete equations describes the propagation of light through a medium of alternating layers of linear dielectric and thin film. However, the physical parameters of the system make the slowly-varying envelope approximation appropriate. Following reviews, we employ such an approximation to derive the equation appropriate. Following reviews, we employ such an approximation to derive following equations appropriate. Following reviews, we employ such an approximation to derive following equations appropriate. Following reviews, we employ such an approximation to derive following equations appropriate. Following reviews, we employ such an approximation to derive following equations appropriate. Considering the right- and left-propagating fields, \( E_{\pm}(x,t) = \frac{1}{\sqrt{2}} (A(x,t) \pm i B(x,t)) e^{-i q_0 x} e^{-i \omega_0 t} \), where \( i \) is the mismatch between the carrier wavenumber and the Bragg resonant wavenumber. Description of the slowly varying polarization is based on the anharmonic oscillator model. The wavevector in an optical medium with electric permittivity \( \varepsilon \). The long-wave and slowly varying envelope of the pulses approximations result in the system of coupled wave equations:

\[
\begin{align*}
  i (A_x + v_g^{-1} A_t) + \Delta q_0 A &= -(2 \pi \omega_0 / c \sqrt{\varepsilon}) \mathcal{P}, \\
  i (B_x - v_g^{-1} B_t) - \Delta q_0 B &= +(2 \pi \omega_0 / c \sqrt{\varepsilon}) \mathcal{P},
\end{align*}
\]

where \( \Delta q_0 = q_0 - 2 \pi / a \) is the mismatch between the carrier wavenumber and the Bragg resonant wavenumber.

\[
\mathcal{P}_t - i (\omega_0 - \omega_d) \mathcal{P} - (3 i g_p / 2 \omega_0) \mathcal{P}^2 \mathcal{P} = i (\omega_p^2 / 8 \pi \omega_0) \mathcal{E}_f,
\]

where \( \mathcal{E}_{int} \) is the electric field interacting with metallic nanoparticles. In our case we have \( \mathcal{E}_{int} = A + B \). Note, that if anharmonicity of oscillators is neglected (\( g_p = 0 \)) then the system of equations transforms into the well known Lorentz model. In the general case the system of resulting equations is the two-wave Maxwell-Duffing equations. Using rescaling:

\[
\begin{align*}
  f_s &= -(A + B) A_0^{-1} e^{-i \delta \tau}, \\
  f_a &= (A - B) A_0^{-1} e^{-i \delta \tau}, \\
  q &= (4 \pi \omega_0 / [\sqrt{\varepsilon} \omega_p A_0]) e^{-i \delta \tau} \mathcal{P}, \\
  \zeta &= (\omega_p / 2 c) x, \quad \tau = t / t_0.
\end{align*}
\]

These equations can be represented in dimensionless form:

\[
\begin{align*}
  f_{s,\zeta \zeta} - f_{s,\tau \tau} &= 2 i q_\zeta, \\
  f_{s,\zeta \zeta} - f_{s,\tau \tau} &= 2 i q_\tau, \\
  i q_\tau + (\Delta - \delta) q + \mu q^2 q &= f_s.
\end{align*}
\]

Here \( t_0 = 2 \sqrt{\varepsilon} / \omega_p \) is an inverse plasma frequency, \( A_0 \) is characteristic amplitude of counter-propagating fields, \( \mu = (3 g_p \sqrt{\varepsilon} / \omega_p \omega_d) (\sqrt{\varepsilon} \omega_p / 4 \pi \omega_0)^2 A_0^2 \) is a dimensionless coefficient of anharmonicity, \( \delta = 2 \Delta q_0 (c / \omega_p) \) is the dimensionless mismatch coefficient, \( \Delta = 2 \sqrt{\varepsilon} (\omega_d - \omega_0) / \omega_p \).
is dimensionless detuning of a nanoparticle’s resonance frequency from the field’s carrier frequency.

The exact solitary wave solutions of the system [64] have been found in [66]. It was shown that, in contrast to conventional $2\pi$-pulses, they have nonlinear phase. The stability of these solutions is sensitive to the phase perturbations. It was also demonstrated that the outcomes of pulse collisions are highly dependent on relative phase.

### 4.2 Nonlinear optical waveguide array

A simple example of a waveguide structure is the directional coupler. The directional coupler is represented by two parallel waveguides. The separation distance between these waveguides so small that guided light is leaking from one waveguide to another. Sometimes these waveguides are described as tunnel-connected [8]. If the sign of the refractive index in both waveguides is positive, then a beam launched into one of the waveguides produces waves in both waveguides propagating in opposite directions. If the index of refraction of one of the waveguides is negative, then a beam launched into one of the waveguides produces two beams in these waveguides propagating in opposite directions [70]. The theory of this anti-directional coupler is similar to the description of SHG under the undepleted pump approximation. Hence, the anti-directional coupler acts as distributed mirror [8]. If the waveguides are made from nonlinear dielectric material or if they are embedded into a non-linear medium, then the coupling performance, in addition to the coupler’s geometry and material property, depend on the wave input power. If input power is above a certain threshold the anti-directional coupling property can be transformed to become unidirectional.

Nonlinear optical waveguide arrays (NOWA) are a natural generalization of nonlinear couplers. NOWA with a positive refractive index have many useful applications and are well studied in the literature (see for example [71,73,72]). Materials with negative index of refraction offer new types of array structures which can be combined in three groups. The first group corresponds to an array of alternating waveguides with different signs of refractive index. The second group corresponds to the situation when one planar array containing identical waveguides with positive sign of refractive index is attached to another planar array of waveguides with negative sign of refractive index. The third group corresponds to a situation where an array of identical waveguides contains a single waveguide with opposite sign.

To develop a theory of such waveguides it is natural to utilize existing formalism based on nearest-neighbor coupling. The electric field of an optical wave propagating in NOWA in the positive $z$ direction can be represented as follows

$$E(x, y, z, t) = \sum_{J=\pm\infty}^{\infty} \sum_{m} A_{m}^{(J)}(z, t) \Psi_{m}^{(J)}(x, y) \exp \left( -i\omega_0 t + i\beta_{m}^{(J)} z \right).$$

The mode function for a particular $m$-th mode of channel $J$ is denoted by $\Psi_{m}^{(J)}(x, y)$, where $J = 0, \pm1, \pm2, \ldots$, and $A_{m}^{(J)}(z, t)$ is a slowly varying envelope of the electric field corresponding to this mode. Parameters $\beta_{m}^{(J)}$ are propagation constants. Omitting the details we can write the general equations which are governed by normalized envelopes $q_{m}^{(J)} = A_{m}^{(J)}(z, t)/A_0$:

$$i\hat{k}_J q_{Jz} + iv_{g}^{-1} q_{Jt} - \sigma_{J} q_{Jtt} + K_{12} (q_{J-1} + q_{J+1}) + (2\pi\omega_0^2 \mu(\omega_0)/\varepsilon^2 \beta^{(J)}) \chi_{eff} A_{0}^2 |q_{J}|^2 q_{J} = 0. \tag{85}$$

Here we assume that the group velocities and the propagation constants are equal for all channels and only resonant coupled modes are essential for our further consideration ($\sigma_{J} = \sigma_{gm}$ and $v_{g} = v_{g}^{(m)}$). The coefficient $K_{12}$ is the coupling constant between neighboring waveguides, and $\chi_{eff}$ is the effective non-linear susceptibility. The amplitude $A_0$ is chosen to normalize the nonlinear coefficient of self-interaction to one.
Using equations (85) and taking into account that \( \hat{J}_f = +1 \) and \( \hat{J}_f = -1 \) corresponds to PRI and NRI cases respectively, we can present mathematical models describing the three groups of arrays listed above.

4.2.1 Alternated nonlinear optical waveguide array

Let us suppose that waveguides marked as \( J = 2n \) and \( J = 2n + 1 \) have PRI and NRI properties respectively. The system of equations describing electromagnetic wave propagation in this structure reads

\[
\begin{align*}
iq_{2n,z} + \sigma g^{-1} q_{2n,t} - \sigma_2 q_{2n,tt} + K_{12} (q_{2n-1} + q_{2n+1}) + |q_{2n}|^2 q_{2n} &= 0, \\
iq_{2n+1,z} - \sigma g^{-1} q_{2n+1,t} + \sigma_2 q_{2n+1,tt} - K_{12} (q_{2n} + q_{2n+2}) - |q_{2n+1}|^2 q_{2n+1} &= 0.
\end{align*}
\]

In the linear limit these equations correspond to the model of a harmonic lattice with alternated sign for the coupling. In this case the spectrum of the linear waves consist of two dispersion branches. This means the alternated linear optical waveguide array acts as a gap medium. Therefore, in the nonlinear case we can expect the existence the solitary waves.

4.2.2 Interface of the nonlinear optical waveguide arrays

Let suppose that waveguides marked as \( n \geq 0 \) and \( n < 0 \) have PRI and NRI properties respectively. In this case propagation of electromagnetic waves is governed by the following system of equations

\[
\begin{align*}
inq_{n,z} + \sigma g^{-1} q_{n,t} - \sigma_2 q_{n,tt} + K_{12} (q_{n-1} + q_{n+1}) + |q_n|^2 q_n &= 0, \quad n \geq 0, \\
inq_{n,z} - \sigma g^{-1} q_{n,t} + \sigma_2 q_{n,tt} - K_{12} (q_{n-1} + q_{n+1}) - |q_n|^2 q_n &= 0, \quad n < 0.
\end{align*}
\]

In the linear approximation, due to the opposite directionality of phase and group velocities in NRI materials, one can expect existence of surface waves with vortex structures. Each type of nonlinear infinite array supports solitary wave solutions. One can expect that the boundary effect at the interface of two types of arrays will be revealed through pinning of solitary wave solutions in a way analogous to the behavior of light near defects in a photonic crystal.

4.2.3 Nonlinear optical waveguide array with a defect

Let us assume that the PRI type array of waveguides contains one waveguide of NRI type. This waveguide, marked \( n = 0 \), can be viewed as a defect in a one dimensional lattice. Propagation of an electromagnetic wave in such an array is governed by the equation:

\[
\begin{align*}
inq_{n,z} + \sigma g^{-1} q_{n,t} - \sigma_2 q_{n,tt} + K_{12} (q_{n-1} + q_{n+1}) + |q_n|^2 q_n &= 0, \quad n > 0, \\
inq_{0,z} - \sigma g^{-1} q_{0,t} + \sigma_2 q_{0,tt} - K_{12} (q_{-1} + q_{+1}) - |q_0|^2 q_0 &= 0, \quad n = 0, \\
inq_{n,z} + \sigma g^{-1} q_{n,t} - \sigma_2 q_{n,tt} + K_{12} (q_{n-1} + q_{n+1}) + |q_n|^2 q_n &= 0, \quad n < 0.
\end{align*}
\]

Presence of a “defect” could be a reason for a pinning phenomena, where the electromagnetic wave is trapped in this waveguide. One can expect that increase of the localized field energy will destroy localization.

5 Conclusion

We considered new features of classical phenomena of nonlinear optics in optical metamaterials including materials with negative refractive index. In addition to already obtained results in
this field, we discussed a number of problems of potential importance for the nonlinear optics of metamaterials.

Optical transparency of metamaterials is a fundamental assumption in our considerations. The current state of the art in nanofabrication technology is capable of delivering only thin film metamaterials with a relatively large level of losses. We expect that further development in nanofabrication will overcome problems of metamaterial losses. Our expectations are based on the history of optical fiber technology development in the second part of the last century.

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