A POSTERIORI ERROR ESTIMATE OF WEAK GALERKIN FEM FOR SECOND ORDER ELLIPTIC PROBLEM WITH MIXED BOUNDARY CONDITION

SHENGLAN XIE
Nanhu College, Jiaxing University
Jiaxing, 314001, China

MAOAN HAN
Department of Mathematics, Zhejiang Normal University
Jinhua, 321004, China
Department of Mathematics, Shanghai Normal University
Shanghai, 200234, China

PENG ZHU∗
College of Mathematics, Physics and Information Engineering, Jiaxing University
Jiaxing, 314001, China

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Abstract. A reliable and efficient a posteriori error estimator is presented for a weak Galerkin finite element method without stabilizer for the second order elliptic equation with mixed boundary conditions. The upper bound of the estimator is proved by Helmholtz decomposition technique and lower bound is hold naturally. The performance of the estimator is illustrated by numerical experiments.

1. Introduction. In recent years, a new finite element method called weak Galerkin (WG) method appears in the field of numerical solution of partial differential equations. WG method was firstly presented by Wand and Ye[11, 12] for solving second order elliptic problem. The novel idea of WG method is on the use of weak functions and their weak derivatives defined as distributions. WG finite element schemes were derived and analyzed for many important partial differential equations such as Maxwell equation[8], Stokes equation[13], Darcy-Stokes system[4], Reissner-Mindlin plate problem[9], and so on.

Adaptive finite element method is a well-known technique to provide accurate numerical solutions for partial differential equations using less degrees of freedoms[10]. A posteriori error estimator is a key step in an adaptive finite element method, which provides the information for local mesh refinement. Although there is a vast literature on priori error analysis of WG methods for different partial differential

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∗ Corresponding author: P. Zhu.
equations, there are only a few results [3, 14, 7] on a posteriori error estimates for WG methods. In [3], the authors presented a residual-type a posteriori error estimator for WG method with and without stabilizer term for solving second order elliptic problem. Their estimator is composed of five terms, residual of the finite element solution, jumping of weak gradient in normal direction and tangential direction respectively, the curl of weak gradient and the oscillation of the data $f$. In [7], a simple posteriori error estimator including two terms is presented for WG method with stabilizer for solving Poisson equation. In [14], a posteriori error analysis was derived for a modified weak Galerkin method. In this paper, we derive a similar a posteriori error estimator as the one in [7] for second order elliptic problem solved by WG method without stabilizer term. Our estimator is simpler and less computational work than the one in [3]. Numerical experiments suggest that our posteriori error estimator convergent to one as iteration number grows.

The paper is organized as follows. In Section 2, we introduce a model problem and the WG method. In Section 3, we present an a posteriori error estimator and analyze its upper bound and lower bound. And finally in Section 4, the reliability and efficiency of our estimator are verified by some numerical experiments.

2. The model problem and WG method. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary $\partial \Omega$. Consider the model elliptic boundary value problem of finding the solution $u$ of

\[- \nabla \cdot (A \nabla u) = f \quad \text{in} \quad \Omega \quad (1)\]

subject to the boundary conditions

\[u = 0 \quad \text{on} \quad \Gamma_D \quad (2)\]

and

\[A \nabla u \cdot n = g \quad \text{on} \quad \Gamma_N, \quad (3)\]

where $A \in [L^\infty(\Omega)]^{2 \times 2}$ is a symmetric matrix-valued function on $\Omega$. The boundary segments $\Gamma_N$ and $\Gamma_D$ are assumed to be disjoint with $\Gamma_N \cup \Gamma_D = \partial \Omega$. The unit outward normal vector to $\partial \Omega$ is denoted by $n$. If $\Gamma_D \neq \emptyset$, (1) has a unique solution $u \in H^1(\Omega)$ for any $f \in H^{-1}(\Omega)$. And if $\Gamma_D = \emptyset$, there exists a solution which is unique up to a constant, i.e., $u \in H^1(\Omega) \setminus \mathbb{R}$ if $f \in H^{-1}(\Omega)$, $g \in H^{1/2}(\Gamma_N)$, provided the compatibility condition $\int_{\Omega} f + \int_{\Gamma_N} g = 0$ is satisfied.

The variational form of this problem is to find $u \in V$ such that

\[B(u, v) = L(v), \quad \forall v \in V, \quad (4)\]

where $V$ is the space

\[V = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \}, \quad \text{and} \quad (5)\]

where

\[B(u, v) = (A \nabla u, \nabla v) \quad (5)\]

and

\[L(v) = (f, v) + (g, v)_{\Gamma_N}. \quad (6)\]

Let $\mathcal{P}$ be a regular partition of the domain $\Omega$ into triangles or quadrilaterals. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{P}$ and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega$. The diameter of a element $K \in \mathcal{P}$ denoted by $h_K$, and mesh size $h = \max_{K \in \mathcal{P}} h_K$ for $\mathcal{P}$.

For a nonnegative integer $\ell$, let $\mathbb{P}^\ell(K)$ denote the set of polynomials defined on $K$ with degree less than or equal to $\ell$. Similarly, on each $e \in \mathcal{E}_h$, denote by $\mathbb{P}^\ell(e)$
the set of polynomials defined on \( e \) of degree no more than \( m \). For each element \( K \in \mathcal{P} \), we introduce a local weak Galerkin finite element space \( W_{\ell,m}(K) \) defined as follows
\[
W_{\ell,m}(K) = \{ v = \{ v_0, v_b \} : v_0 \in \mathbb{P}^\ell(K), v_b \in \mathbb{P}^m(e) \},
\]
where \( m = \ell \) or \( m = \ell + 1 \).

Let \( \tilde{\mathbb{P}}^\ell(K) \) be the set of homogeneous polynomials of order \( \ell \) in the variable \( x = (x_1, x_2)^T \). Define the Raviart-Thomas (RT) element by
\[
RT_\ell(K) = [\mathbb{P}^\ell(K)]^2 + \tilde{\mathbb{P}}^{\ell}(K)x
\]
and the Brezzi-Douglas-Marini (BDM) element by \( BDM_\ell(K) = [\mathbb{P}^\ell(K)]^2 \) for \( \ell \geq 0 \). For any \( v = \{ v_0, v_b \} \in W_{\ell,m}(K) \), its weak gradient \( \nabla_w v \in G_\ell(K) \) is defined as
\[
(\nabla_w v, q)_K = -(v_0, \nabla q)_K + \langle v_b, q \cdot n \rangle, \quad \forall q \in G_\ell(K),
\]
where \( n \) is the unit outward normal on \( \partial K \). In (7), \( G_\ell(K) = RT_\ell(K) \) for \( v \in W_{\ell,\ell}(K) \), and \( G_\ell(K) = BDM_\ell(K) \) for \( W_{\ell,\ell+1}(K) \).

Denote by \( W_{\ell}(K) - G_\ell(K) \) a local weak Galerkin element that can be either \( W_{\ell,\ell}(K) - RT_\ell(K) \) or \( W_{\ell,\ell+1} - BDM_\ell(K) \). Associated with \( \mathcal{P} \) and a local element \( W_{\ell}(K) - G_\ell(K) \), we introduce global weak Galerkin finite element spaces
\[
V_h = \{ v = \{ v_0, v_b \} : \{ v_0, v_b \} |_K \in W_{\ell,m}(K), \forall K \in \mathcal{P} \},
\]
and
\[
V_h^0 = \{ v = \{ v_0, v_b \} : v \in V_h, v_b = 0 \text{ on } \Gamma_D \}.
\]

Define
\[
B_h(w, v) := (A \nabla_w w, \nabla_w v),
\]
\[
L_h(v) := (f, v_0) + \langle g, v_b \rangle_{\Gamma_N}
\]
for all \( w, v \in V_h \), where \( \nabla_w v \) denotes the operator defined by
\[
(\nabla_w v)|_K = \nabla w(v|_K), \quad K \in \mathcal{P}.
\]

The weak Galerkin (WG) finite element approximation of problem (1) with (2) and (3) is to find \( u_h = \{ u_0, u_b \} \in V_h^0 \) such that
\[
B_h(u_h, v_h) = L_h(v_h), \quad \forall v_h \in V_h^0.
\]

3. A posteriori error analysis. In this section, we derive a posteriori error estimator which allows control of the error due to the approximation of the solution of (1) by a discrete solution of (10). For simplicity, we assume \( A \) is piecewise constant on partition \( \mathcal{P} \).

Let \( f_h \) and \( g_h \) be the \( L^2 \) projections of \( f \) and \( g \), respectively. Define the data oscillation for the load function \( f \) on \( \mathcal{P} \) as
\[
\text{osc}^2(f, K) = h_K^2 \| f - f_h \|_K^2, \quad \text{osc}^2(f, \mathcal{P}) = \sum_{K \in \mathcal{P}} \text{osc}^2(f, K).
\]
Likewise, the oscillation of the Neumann data \( g \) is defined by
\[
\text{osc}^2(g, K) = h_K \| g - g_h \|_{\partial K \cap \Gamma_N}^2, \quad \text{osc}^2(g, \Gamma_N) = \sum_{K : \partial K \cap \Gamma_N \neq \emptyset} \text{osc}^2(g, K).
\]

Define local and global error estimators as
\[
\eta^2(u_h, K) = \text{osc}^2(f, K) + \text{osc}^2(g, K) + s_K(u_h, u_h),
\]
\[
\eta^2(u_h, \mathcal{P}) = \sum_{K \in \mathcal{P}} \eta^2(u_h, K),
\]

(11)
where

\[ s_K(u_h, u_h) = h_K^{-1}(u_0 - u_b, u_0 - u_b)_{\partial K}, \quad s_h(u_h, u_h) = \sum_{K \in \mathcal{P}} s_K(u_h, u_h). \]

Let \( K \) be an element with \( e \) as an edge. It is well known that there exists a constant \( C \) such that for any function \( \chi \in H^1(K) \)

\[ \| \chi \|^2 \leq C(h_K^{-1}\| \chi \|^2_K + h_K\| \nabla \chi \|^2_K). \tag{12} \]

For \( \phi \in H^1(\Omega) \), we set \( \text{curl} \phi = (-\frac{\partial \phi}{\partial y_2}, \frac{\partial \phi}{\partial y_1})^T \).

Following the idea of Lemma 3.1 in \([1]\), we can easily obtain the following Helmholtz type decomposition of the function \( \nabla u - \nabla w u_h \). For completeness we include a brief proof here.

**Lemma 3.1.** For \( \nabla u - \nabla w u_h \in L^2(\Omega) \), there exist \( \psi \in V \) and \( \phi \in H^1(\Omega) \) with \( \text{curl} \phi \cdot n = 0 \) on \( \Gamma_N \) such that

\[ \nabla u - \nabla w u_h = \nabla \psi + A^{-1} \text{curl} \phi, \tag{13} \]

and

\[ \| A^{1/2}(\nabla u - \nabla w u_h) \|^2 = \| A^{1/2} \nabla \psi \|^2 + \| A^{-1/2} \text{curl} \phi \|^2. \tag{14} \]

**Proof.** To obtain this decomposition we solve the problem

\[-\nabla \cdot (A \nabla \psi) = A(\nabla u - \nabla w u_h),\]

with \( \psi \in V \), namely, \( \psi \) satisfies

\[ (A \nabla \psi, \nabla v) = (A(\nabla u - \nabla w u_h), \nabla v), \quad \forall v \in V. \tag{15} \]

The existence and uniqueness of \( \psi \) can be obtained from Lax-Milgram lemma. Denote \( w = A(\nabla u - \nabla w u_h) - A \nabla \psi \). From integration by parts and \( (15) \), we have

\[ -(\text{div} w, v) + \langle w \cdot n, v \rangle_{\Gamma_N} = 0 \quad \forall v \in V. \]

Consequently, \( \text{div} w = 0 \) in \( \Omega \) and \( w \cdot n = 0 \) on \( \Gamma_N \). Since \( \Omega \) is simply connected and consequently there exists \( \phi \in H^1(\Omega) \setminus \mathbb{R} \) such that

\[ A(\nabla u - \nabla w u_h) - A \nabla \psi = w = \text{curl} \phi. \]

Furthermore, \( \text{curl} \phi \cdot n = w \cdot n = 0 \) on \( \Gamma_N \).

Set \( v = \psi \) in \( (15) \), we get the orthogonality \( (\nabla \psi, \text{curl} \phi) = (\nabla \psi, w) = 0 \), which implies \( (14) \). \( \Box \)

For each element \( K \in \mathcal{P} \), let \( Q_0 : L^2(K) \to \mathbb{P}^d(K) \) be the element-wise defined \( L^2 \) projection on \( K \) and \( Q_h : L^2(e) \to \mathbb{P}^d(e) \) be the element-wise defined \( L^2 \) projection on \( e \subset \partial K \). Define \( Q_h u = \{ Q_0 u, Q_h u \} \in V_h \). Denote by \( Q_h \) the \( L^2 \) projection from \( [L^2(K)]^2 \) to a local weak gradient space \( G_l(K) \).

**Lemma 3.2.** \([3]\) On each element \( K \in \mathcal{P} \), the operators \( \nabla w, Q_h \) and \( Q_h \) have the following commutative property,

\[ \nabla w(Q_h \chi) = Q_h(\nabla \chi), \tag{16} \]

for any \( \chi \in H^1(T) \).
Theorem 3.3 (Upper bound). Let $u$ be the solution of (1)-(3) and $u_h = \{u_0, u_b\} \in V^0_h$ be the solution of (10). Then, for $\ell \geq 0$, there exists a positive constant $C$ such that

$$
\| u - u_h \| \leq C\eta(u_h, \mathcal{P}),
$$

where

$$
\| u - u_h \|^2 = \| A^{1/2}(\nabla u - \nabla w u_h) \|^2 + s_h(u - u_h, u - u_h).
$$

Proof. It follows from (13) that

$$
\| A^{1/2}(\nabla u - \nabla w u_h) \|^2 = (A(\nabla u - \nabla w u_h), \nabla \psi)
$$

$$
+ (A(\nabla u - \nabla w u_h), A^{-1} \text{curl} \phi).
$$

(18)

It follows from (16) and (10) that

$$
(A\nabla_w u_h, \nabla \psi) = (A\nabla_w u_h, Q_h(\nabla \psi)) = (A\nabla_w u_h, \nabla_w Q_h \psi)
$$

$$
= (f, Q_0 \psi) + \langle g, Q_h \psi \rangle_{\Gamma_N}.
$$

(19)

Let $v = \psi$ in (4), we have

$$
(A\nabla u, \nabla \psi) = (f, \psi) + \langle g, \psi \rangle_{\Gamma_N}.
$$

(20)

Subtracting (19) from (20), we have

$$
(A(\nabla u - \nabla_w u_h), \nabla \psi) = (f, \psi - Q_0 \psi) + \langle g, \psi - Q_h \psi \rangle_{\Gamma_N}
$$

$$
= (f - f_h, \psi - Q_0 \psi) + \langle g - g_h, \psi - Q_h \psi \rangle_{\Gamma_N}
$$

$$
\leq C \{ \text{osc}^2(f, \mathcal{P}) + \text{osc}^2(g, \Gamma_N) \}^{1/2} \| A^{1/2} \nabla \psi \|. 
$$

(21)

Let $v = u$ in (15), we have $\langle \text{curl} \phi, \nabla u \rangle = 0$, which yields

$$
(A(\nabla u - \nabla_w u_h), A^{-1} \text{curl} \phi) = (\nabla u - \nabla_w u_h, \text{curl} \phi) = -(\nabla_w u_h, \text{curl} \phi). 
$$

(22)

It follows from (7), we have

$$
(\nabla_w u_h, \text{curl} \phi) = \sum_{K \in \mathcal{P}} (\nabla_w u_h, \text{curl} \phi)_K = \sum_{K \in \mathcal{P}} (\nabla_w u_h, Q_h \text{curl} \phi)_K
$$

$$
= \sum_{K \in \mathcal{P}} \{- (u_0, \nabla \cdot (Q_h \text{curl} \phi))_K + \langle u_b, Q_h \text{curl} \phi \cdot \mathbf{n} \rangle_{\partial K}\}.
$$

(23)

Note that $u_b = 0$ on $\Gamma_D$ and $\text{curl} \phi \cdot \mathbf{n} = 0$ on $\Gamma_N$, then

$$
\sum_{K \in \mathcal{P}} \langle u_b, \text{curl} \phi \cdot \mathbf{n} \rangle_{\partial K} = \sum_{e \in \mathcal{E}^0} \langle u_b, \text{curl} \phi \cdot \mathbf{n} \rangle_e = 0,
$$

(24)

because of $u_b$ is single value on interior edge $e \in \mathcal{E}^0$. Subtracting (24) from the right hand side of (23) yields

$$
(\nabla_w u_h, \text{curl} \phi) = \sum_{K \in \mathcal{P}} -(u_0, \nabla \cdot (Q_h \text{curl} \phi))_K
$$

$$
+ \sum_{K \in \mathcal{P}} \langle u_b, (Q_h \text{curl} \phi - \text{curl} \phi) \cdot \mathbf{n} \rangle_{\partial K}.
$$

(25)

By integrating by parts, we get

$$
-(u_0, \nabla \cdot (Q_h \text{curl} \phi))_K = (\nabla u_0, Q_h \text{curl} \phi)_K - \langle u_0, Q_h \text{curl} \phi \cdot \mathbf{n} \rangle_{\partial K}
$$

$$
= (\nabla u_0, \text{curl} \phi)_K - \langle u_0, Q_h \text{curl} \phi \cdot \mathbf{n} \rangle_{\partial K}.
$$

(26)
Applying integration by parts again and observing that \( \nabla \cdot \text{curl} \, \phi \) vanishes, we obtain
\[
(\nabla u_0, \text{curl} \, \phi)_K = \langle u_0, \text{curl} \, \phi \cdot n \rangle_{\partial K}.
\]
Inserting the above equation into (26) yields
\[
-(u_0, \nabla \cdot (Q_h \text{curl} \, \phi))_K = -(u_0, (Q_h \text{curl} \, \phi - \text{curl} \, \phi) \cdot n)_{\partial K}. \tag{27}
\]
Combining (25) and (27), we have
\[
(\nabla_w u_h, \text{curl} \, \phi) = \sum_{K \in \mathcal{P}} -(u_0 - u_h, Q_h \text{curl} \, \phi - \text{curl} \, \phi) \cdot n)_{\partial K}
= \sum_{K \in \mathcal{P}} -(u_0 - u_h, Q_h \text{curl} \, \phi \cdot n)_{\partial K}
+ \sum_{K \in \mathcal{P}} \langle u_0 - u_h, \text{curl} \, \phi \cdot n \rangle_{\partial K}. \tag{28}
\]

It follows from the Cauchy-Schwarz inequality, the trace inequality and the stability property of \( L^2 \) projection that
\[
|\langle u_0 - u_h, Q_h \text{curl} \, \phi \cdot n \rangle_{\partial K}| \leq \|u_0 - u_h\|_{\partial K} \|Q_h \text{curl} \, \phi \cdot n\|_{\partial K}
\]
\[
\leq C h^K \|u_0 - u_h\|_{\partial K} \|\text{curl} \, \phi\|_K
\]
\[
\leq C h^K \|u_0 - u_h\|_{\partial K} \|A^{-1/2} \text{curl} \, \phi\|_K. \tag{29}
\]
Using the inverse inequality and the fact \( \nabla \cdot \text{curl} \, \phi = 0 \), we arrive at
\[
|\langle u_0 - u_h, \text{curl} \, \phi \cdot n \rangle_{\partial K}| \leq \|u_0 - u_h\|_{H^{1/2}(\partial K)} \|\text{curl} \, \phi \cdot n\|_{H^{-1/2}(\partial K)}
\]
\[
\leq C h^K \|u_0 - u_h\|_{\partial K} \|\text{curl} \, \phi\|_K
\]
\[
\leq C h^K \|u_0 - u_h\|_{\partial K} \|A^{-1} \text{curl} \, \phi\|_K, \tag{30}
\]
where we have used the trace inequality
\[
\|\text{curl} \, \phi \cdot n\|_{H^{-1/2}(K)} \leq C \|\text{curl} \, \phi\|_K,
\]
which is from [2] (page 7). Combining (22) and (28)-(30) yield
\[
(A(\nabla u - \nabla_w u_h), A^{-1} \text{curl} \, \phi) \leq C s_h^{1/2}(u_h, u_h) \|A^{-1} \text{curl} \, \phi\|. \tag{31}
\]
Noting that
\[
s_h(u_h, u_h) = s_h(u - u_h, u - u_h),
\]
and combined with (18), (21), (31) and (14), we arrive at (17). The proof is completed.

For any element \( K \in \mathcal{P} \), we define local \( \|u - u_h\|_K \) on \( K \) as
\[
\|u - u_h\|_K^2 = \|A^{1/2}(\nabla u - \nabla_w u_h)\|_K^2 + s_K(u - u_h, u - u_h).
\]
Obviously, we have the following local lower bound (efficiency) estimate.

**Theorem 3.4.** Let \( u \) be the solution of (1)-(3) and \( u_h = \{u_0, u_h\} \in V_h^0 \) be the solution of (10). Then there exists a positive constant \( C \) such that
\[
\eta^2(u_h, K) \leq \|u - u_h\|_K^2 + \operatorname{osc}^2(f, K) + \operatorname{osc}^2(g, K).
\]
4. **Numerical experiments.** In this section, we show the performance of the error estimator in Theorem 3.3 through some numerical experiments. Using the error estimator, we construct the following adaptive algorithm:

\[ \text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}. \]

**Figure 1.** Initial mesh for adaptive refinement.

In **SOLVE** step, \( W_{0,0}(K) - RT_0(K) \) weak Galerkin finite element is used. And Dörfler’s marking strategy \[5\] with parameter \( \theta = 0.3 \) is used in the **MARK** step. Then we refine the mesh using the newest vertex bisection algorithm to obtain new mesh. We start with initial mesh as shown in Figure 1 for all numerical examples. The effectivity index is defined by

\[
\text{Effectivity index} = \frac{\eta(u_h, P)}{||u - u_h||}.
\]

**Example 1.** Consider the problem (1) with data \( f = 2\pi^2 \sin(\pi x) \sin(\pi y), A = I \) in the domain \( \Omega = (0,1)^2 \) with \( \Gamma_N = \emptyset \). The exact solution is \( u = \sin(\pi x) \sin(\pi y) \).

**Figure 2.** Effectivity index for Example 1.

In Figure 2, we show the effectivity index on the successively refined meshes. We note that the effectivity index is convergent to one. In Figure 3(a), we show the final adaptive refinement mesh, the corresponding WG solution is showed in Figure 3(b).
Figure 3. Example 1. Final adaptive refinement mesh and WG solution.

**Example 2.** Let \( A = I, \Omega = (0, 1)^2 \) and the data \( f \) such that the exact solution is

\[
u(x, y) = \frac{1}{2000} \exp(10x^2 + 10y^2)(1 - x)^2x^2(1 - y)^2y^2.
\]

Figure 4. Effectivity index for Example 2.

In Figure 4, we show the effectivity index on the successively refined meshes. Again, we observe that the effectivity index is convergent to one. In Figure 5(a), we show the final adaptive refinement mesh, the corresponding WG solution is showed in Figure 5(b).

**Example 3.** Let \( A = I, \Omega = (0, 1)^2 \) and the data \( f \) such that the exact solution is

\[
u(x, y) = \exp(-1000(x - 0.5)^2 - 1000(y - 0.5)^2).
\]

In Figure 6, we show the effectivity index on the successively refined meshes. Again, we observe that the effectivity index is convergent to one. In Figure 7(a), we show the final adaptive refinement mesh, the corresponding WG solution is showed in Figure 7(b).
Figure 5. Example 2. Final adaptive refinement mesh and WG solution.

Figure 6. Effectivity index for Example 3.

Figure 7. Example 3. Final adaptive refinement mesh and WG solution.

REFERENCES

[1] M. Ainsworth, Robust a posteriori error estimation for nonconforming finite element approximation, *SIAM J. Numer. Anal.*, 42 (2005), 2320–2341.
[2] R. Becker, P. Hansbo and M. G. Larson, Energy norm a posteriori error estimation for discontinuous Galerkin methods, *Computer Methods in Applied Mechanics & Engineering*, 192 (2003), 723–733.

[3] L. Chen, J. Wang and X. Ye, A posteriori error estimates for weak Galerkin finite element methods for second order elliptic problems, *J. Sci. Comput.*, 59 (2014), 496–511.

[4] W. Chen, F. Wang and Y. Wang, Weak Galerkin method for the coupled Darcy-Stokes flow, *IMA. J. Numer. Anal.*, 36 (2016), 897–921.

[5] W. Dörfler, A convergent adaptive algorithm for Poisson’s equation, *SIAM J. Numer. Anal.*, 33 (1996), 1106–1124.

[6] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer Ser. Comput. Math. 5, Springer-Verlag, Berlin, 1986.

[7] H. Li, L. Mu and X. Ye, A posteriori error estimates for the weak Galerkin finite element methods on polytopal meshes, *Commun. Comput. Phys.*, 26 (2019), 558–578.

[8] L. Mu, J. Wang, Y. Wang, X. Ye and S. Zhang, A weak Galerkin finite element method for the Maxwell equations, *J. Sci. Comput.*, 65 (2015), 363–386.

[9] L. Mu, J. Wang and X. Ye, A weak Galerkin method for the Reissner-Mindlin plate in primary form, *J. Sci. Comput.*, 75 (2018), 782–802.

[10] R. Verfürth, *A Review of a Posteriori Error Estimation and Adaptive Mesh Refinement Techniques*, John Wiley, Chichester, 1996.

[11] J. Wang and X. Ye, A weak Galerkin finite element method for second-order elliptic problems, *J Comput. Appl. Math.*, 241 (2013), 103–115.

[12] J. Wang and X. Ye, A weak Galerkin mixed finite element method for second-order elliptic problems, *Math. Comput.*, 83 (2014), 2101–2126.

[13] J. Wang and X. Ye, A weak Galerkin finite element method for the Stokes equations, *Adv. Comput. Math.*, 42 (2016), 155–174.

[14] T. Zhang and T. Lin, A posteriori error estimate for a modified weak Galerkin method solving elliptic problems, *Numer. Methods Partial Differential Eq.*, 33 (2017), 381–398.

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E-mail address: shlxie@126.com
E-mail address: mahan@shnu.edu.cn
E-mail address: zhupeng.hnu@gmail.com