NONCOMMUTATIVE SYMMETRIC SYSTEMS OVER ASSOCIATIVE ALGEBRAS

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ABSTRACT. This paper is the first of a sequence papers ([Z4]–[Z7]) on the NCS (noncommutative symmetric) systems over differential operator algebras in commutative or noncommutative variables ([Z4]); the NCS systems over the Grossman-Larson Hopf algebras ([GL, F]) of labeled rooted trees ([Z6]); as well as their connections and applications to the inversion problem ([BCW], [E4]) and specializations of NCFSs ([Z5], [Z7]). In this paper, inspired by the seminal work [GKLLRT] on NCFSs (noncommutative symmetric functions), we first formulate the notion NCS systems over associative Q-algebras. We then prove some results for NCS systems in general; the NCS systems over bialgebras or Hopf algebras; and the universal NCS system formed by the generating functions of certain NCFSs in [GKLLRT]. Finally, we review some of the main results that will be proved in the followed papers [Z4], [Z6] and [Z7] as some supporting examples for the general discussions given in this paper.

1. Introduction

Let $K$ be any unital commutative Q-algebra and $A$ a unital associative but not necessarily commutative $K$-algebra. Let $t$ be a formal central parameter, i.e. it commutes with all elements of $A$, and $A[[t]]$ the $K$-algebra of formal power series in $t$ with coefficients in $A$. A NCS (noncommutative symmetric) system over $A$ (see Definition 2.1) by definition is a 5-tuple $\Omega \in A[[t]]^5$ which satisfies the defining equations (see Eqs. (2.4), (2.5)) of the NCFSs (noncommutative symmetric functions) first introduced and studied in the seminal paper [GKLLRT]. When the base algebra $K$ is clear in the context, the ordered pair $(A, \Omega)$ is also called a NCS system. In some sense, a NCS system over the...
$K$-algebra $A$ can be viewed as a system of analogs in $A$ of the NCSFs defined by the same equations. For more studies on NCSFs, see [T], [KLT], [DKKT], [KT1], [KT2] and [DFT].

One immediate but probably the most important example of the NCS systems is $(\text{NSym}, \Pi)$ (see Eqs. (2.28)–(2.33)) formed by the generating functions of the NCSFs defined in [GKLLRT] by Eqs. (2.1)–(2.5) over the free $K$-algebra $\text{NSym}$ of NCSFs. It serves as the universal NCS system over all associative $K$-algebras (see Theorem 2.15). More precisely, for any NCS system $(A, \Omega)$, there exists a unique $K$-algebra homomorphism $S : \text{NSym} \to A$ such that $S^\times(\Pi) = \Omega$ (Here we have extended the homomorphism $S$ to $S : \text{NSym}[[t]] \to A[[t]]$ by the base extension).

Note that, it has been shown in [GKLLRT], in the quotient modulo the commutator of $\text{NSym}$, the NCSFs in $\Pi$ become the corresponding classical (commutative) symmetric functions ([Ma]). Hence, the universal NCS system for commutative $K$-algebras is given by the generating functions of the corresponding classical (commutative) symmetric functions.

One of the main motivations for the introduction of NCS is as follows (see Subsection 2.3 for more discussions). Note that, as an important topic in the theory of symmetric functions, the relations or polynomial identities among various commutative or noncommutative symmetric functions have been known explicitly (see [Ma] and [GKLLRT]). When a NCS system $\Omega$ is given over a $K$-algebra $A$, by applying the $K$-algebra homomorphism $S : \text{NSym} \to A$ guaranteed by the universal property of the system $(\text{NSym}, \Pi)$ to the identities of the NCSFs in $\Pi$, we see the same identities hold for the elements of $A$ in the NCS system $\Omega$. This could be a very effective way to obtain identities for certain elements of $A$ if we could show they are involved in a NCS system over $A$. On the other hand, if the given NCS system $(A, \Omega)$ has already been well-understood, the $K$-algebra homomorphism $S : \text{NSym} \to A$ in turn gives a specialization or realization ([GKLLRT], [St2]) of NCSFs, which may provide some new understanding of NCSFs. For more studies on the specializations of NCSFs, see the references quoted above.

This paper is the first of a sequence papers on the NCS systems over differential operator algebras in commutative or noncommutative variables ([Z4]); the NCS systems over the Grossman-Larson Hopf algebras of labeled rooted trees ([Z6]); as well as their connections and applications to the inversion problem ([BCW], [E4]) and specializations of NCSFs ([Z5], [Z7]). In this paper, we first introduce the notion NCS systems over any associative $K$-algebras. We then prove some results on the NCS systems in general, the NCS systems over bi-algebras or
Hopf algebras and the universal NCS system from NCSFs \((GKLLRT)\), which will be needed in the followed papers. Finally, we briefly review some of the main results that will be proved in the followed papers \([Z4]\), \([Z6]\) and \([Z7]\) as some supporting examples to the general discussions given in the first part of this paper.

The arrangement of this paper is as follows. In Subsection 2.1, we first formulate the notion NCS systems \((A, \Omega)\) over any associative \(K\)-algebra \(A\) (see Definition 2.1). We then show in Lemma 2.5 the existence and uniqueness of the solutions in \(A[[t]]\) of any one of Eqs. (2.2)–(2.5). Several direct consequences of Lemma 2.5 are given in Corollaries 2.6–2.9. Finally, in Proposition 2.12, we prove a property of the NCS systems over bi-algebras or Hopf algebras. In Subsection 2.2, we first recall some NCSFs introduced in \([GKLLRT]\) and a graded Hopf algebra structure of the space \(NSym\) of NCSFs. We then in Theorem 2.15 show the generating functions of these NCSFs form the universal NCS system \((NSym, \Pi)\) over all \(K\)-algebra. Moreover, we also give some sufficient conditions in Theorem 2.15 for the algebra homomorphisms guaranteed by the universal property of \((NSym, \Pi)\) to be further homomorphisms of bi-algebras and Hopf algebras. In Subsection 2.3, we discuss some possible applications of the universal properties of the NCS system \((NSym, \Pi)\), which are also the main motivations for the introduction of the NCS systems over associative \(K\)-algebras.

The results above form the first part of this paper. In the second part, Sections 3 and 4, we review some of the main results that will appear in the sequels \([Z4]\), \([Z6]\) and \([Z7]\). The main purposes that we include Sections 3 and 4 in this paper are as follows. First, we think it is better to provide some concrete examples for the general discussions of NCS systems given in Section 2 so the paper can be read as a more complete introduction to the newly defined NCS systems. Secondly, considering length of the whole series of papers, we hope that discussions in Sections 3 and 4 can also serve as a shorter survey or review for some of the main results obtained the followed papers \([Z4]\), \([Z6]\) and \([Z7]\) (Precisely speaking, they should be read as an announcement since these sequel papers for the time being are still under submission).

In Section 3, we discuss the NCS systems that will be constructed in \([Z4]\) over differential operator algebras in commutative or noncommutative free variables. Certain properties of the resulting specialization of NCSFs by differential operators, which will be proved in \([Z4]\) and \([Z7]\), are also discussed. In Section 4, for any non-empty \(W \subseteq \mathbb{N}^{+}\), we first recall the Connes-Kreimer Hopf algebra \(H_{CK}^{W}\) and the Grossman-Larson Hopf algebra \(H_{GL}^{W}\) of \(W\)-labeled rooted forests and \(W\)-labeled rooted trees, respectively. We then discuss the NCS system \((H_{GL}^{W}, \Omega_{T}^{W})\)
that will be constructed in $[Z6]$ over the Grossman-Larson Hopf algebra $\mathcal{H}_{GL}^W$. Some of properties to be given in $[Z6]$ and $[Z7]$ of the resulting specializations of NCSFs by $W$-labeled rooted trees will also be discussed in this section. Finally, we briefly explain a connection, which will be given in $[Z7]$, between the NCS system $(\mathcal{H}_{GL}^W, \Omega_{W}^F)$ with the NCS systems discussed in Section 3 over differential operator algebras. Some consequences of this connection to the related specializations of NCSFs and the inversion problem ([BCW], [E4]) will also be discussed.

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2. NCS Systems over Associative Algebras

Let $K$ be any unital commutative $\mathbb{Q}$-algebra and $A$ any unital associative but not necessarily commutative $K$-algebra. Let $t$ be a formal central parameter, i.e. it commutes with all elements of $A$, and $A[[t]]$ the $K$-algebra of formal power series in $t$ with coefficients in $A$.

First let us introduce the following main notion of this paper, which is mainly motivated by the seminal work [GKLLRT] by I. M. Gelfand; D. Krob; A. Lascoux; B. Leclerc; V. S. Retakh and J.-Y. Thibon on NCSFs (noncommutative symmetric functions).

Definition 2.1. For any unital associative $K$-algebra $A$, a 5-tuple $\Omega = (f(t), g(t), d(t), h(t), m(t)) \in A[[t]]^5$ is said to be a NCS (noncommutative symmetric) system over $A$ if the following equations are satisfied.

\begin{align}
(2.1) & \quad f(0) = 1 \\
(2.2) & \quad f(-t)g(t) = g(t)f(-t) = 1, \\
(2.3) & \quad e^d(t) = g(t), \\
(2.4) & \quad \frac{dg(t)}{dt} = g(t)h(t), \\
(2.5) & \quad \frac{dg(t)}{dt} = m(t)g(t).
\end{align}

When the base algebra $K$ is clear in the context, we also call the ordered pair $(A, \Omega)$ a NCS system. Since NCS systems often come from generating functions of certain elements of $A$ that are under concern, the components of $\Omega$ will also be referred as the generating functions of their coefficients.
2.1. **NCS Systems in General.** In this subsection, we prove some results for the NCS systems in general and also the NCS systems over bi-algebras and Hopf algebras.

First, let us fix the following convention that will be used implicitly throughout this paper.

**Convention:**

(a) All $K$-algebras in this paper are assumed to be unital associative $K$-algebras and all $K$-algebra homomorphisms are assumed to be unit-preserving.

(b) For any $K$-algebras $B$ and $A$ and any $K$-linear map $S : B \to A$, we always extend $S$ to a linear map, which we will still denote by $S$, from $B[[t]]$ to $A[[t]]$ by the base extension, i.e. for any $\sum_{m \geq 0} b_m t^m \in B[[t]]$, we set

\[
S(\sum_{m \geq 0} b_m t^m) = \sum_{m \geq 0} S(b_m)t^m.
\]

Furthermore, for any $m \geq 1$, we denote by $S^\times m$ the $K$-linear map from $B[[t]]^\times m$ to $A[[t]]^\times m$ induced by $S : B[[t]] \to A[[t]]$.

Now let $A$ be any unital $K$-algebra and $\Omega$ a NCS system over $A$ as given in Definition 2.1. We define five sequences of elements of $A$ by writing

\[
\begin{align*}
f(t) &:= \sum_{m \geq 0} t^m \lambda_m, \\
g(t) &:= \sum_{m \geq 0} t^m s_m, \\
d(t) &:= \sum_{m \geq 1} t^m \phi_m, \\
h(t) &:= \sum_{m \geq 1} t^{m-1} \psi_m, \\
m(t) &:= \sum_{m \geq 1} t^{m-1} \xi_m.
\end{align*}
\]

We will also denote each sequence of the elements of $A$ defined above by the corresponding letter without sub-index. For example, $\lambda$ denotes the sequence $\{\lambda_m | m \geq 0\}$ defined in Eq. (2.7) and $\xi$ denotes the sequence $\{\xi_m | m \geq 1\}$ defined in Eq. (2.11), etc.

Next, let us start with the following simple but useful properties of NCS systems.
Lemma 2.2. Let $A$ and $B$ be any $K$-algebras and $S : B \to A$ an algebra homomorphism. Let $\Omega$ be a NCS system over $B$. Then $\Omega := S^{5}(\tilde{\Omega})$ is a NCS system over $A$.

Proof: Note that, by our convention above, $S : B[[t]] \to A[[t]]$ is also a unital $K$-algebra homomorphism and commutes with the linear operator $\frac{d}{dt}$. With these observations, it is easy to see that, the components of the 5-tuple $\Omega := S^{5}(\tilde{\Omega})$ also satisfy Eqs. (2.1)–(2.5). Hence the lemma follows. \[\square\]

Lemma 2.3. Let $(A, \Omega)$ be a NCS system as fixed above. Define the 5-tuple $\Omega^r$ to be
\[\Omega^r := (g(-t), f(-t), -d(t), -m(t), -h(t)).\]
Then $\Omega^r$ is also a NCS system over $A$.

We call the NCS system $\Omega^r$ the $\text{flip}$ of $\Omega$.

Proof: First, for convenience, we also write $\Omega^r$ as $\Omega^r := (\tilde{f}(t), \tilde{g}(t), \tilde{d}(t), \tilde{h}(t), \tilde{m}(t))$, i.e. we set $\tilde{f}(t) := g(-t); \tilde{g}(t) := f(-t); \text{etc.}$

By setting $t = 0$ in Eq. (2.2) for $\Omega$, we see that $\tilde{f}(0) = g(0) = 1$. So we get Eq. (2.1) for $\Omega^r$. By rewriting Eq. (2.2) for $\Omega$ in terms of $\tilde{f}(t)$ and $\tilde{g}(t)$, we see Eq. (2.2) also holds for $\tilde{\Omega}$. To show Eq. (2.3) for $\Omega^r$, we consider
\[e^{\tilde{d}(t)} = e^{-d(t)} = g(t)^{-1} = f(-t) = \tilde{g}(t).\]
Hence we get Eq. (2.3) for $\Omega^r$.

Now consider Eqs. (2.4) and (2.5) for $\Omega^r$. Note first that, by applying $\frac{d}{dt}$ to Eq. (2.2) for $\Omega$, we have
\[0 = \frac{df(-t)}{dt}g(t) + f(-t)\frac{dg(t)}{dt},\]
\[0 = \frac{dg(t)}{dt}f(-t) + g(t)\frac{df(-t)}{dt}.\]
Replacing $\frac{dg(t)}{dt}$ by $m(t)g(t)$ and $g(t)h(t)$ respectively in the last two equations above and then solving $\frac{df(-t)}{dt}$, we get
\[\frac{df(-t)}{dt} = -f(-t)m(t),\]
\[ \frac{df(-t)}{dt} = -h(t)f(-t). \]

Rewriting the last two equations above in terms of \( \tilde{g}(t) \), \( \tilde{h}(t) \) and \( \tilde{m}(t) \), we get Eqs. (2.4) and (2.5) for \( \Omega^* \).

Let \( A_i \) \((i = 1, 2)\) be any \( K \)-algebras and \( \Omega_i \) a NCS system over \( A_i \). Write \( \Omega_i \) as
\[ \Omega_i = (f_i(t), g_i(t), d_i(t), h_i(t), m_i(t)) \]
and set
\[
\begin{align*}
    f_3(t) &:= f_1(t) \otimes f_2(t), \\
    g_3(t) &:= g_1(t) \otimes g_2(t), \\
    d_3(t) &:= d_1(t) \otimes 1 + 1 \otimes d_2(t), \\
    h_3(t) &:= h_1(t) \otimes 1 + 1 \otimes h_2(t), \\
    m_3(t) &:= m_1(t) \otimes 1 + 1 \otimes m_2(t)
\end{align*}
\]

\[ \Omega_1 \otimes_K \Omega_2 := (f_3(t), g_3(t), d_3(t), h_3(t), m_3(t)). \tag{2.13} \]

We call \( \Omega_1 \otimes_K \Omega_2 \) the tensor product (over \( K \)) of the NCS systems of \( \Omega_1 \) and \( \Omega_2 \). Then we have the following proposition.

**Proposition 2.4.** Let \((A_i, \Omega_i)\) \((i = 1, 2)\) be NCS systems. Then, the tensor product \( \Omega_1 \otimes_K \Omega_2 \) of the NCS systems of \( \Omega_1 \) and \( \Omega_2 \) forms a NCS system over the \( K \)-algebra \( A_1 \otimes_K A_2 \).

**Proof:** The proof is straightforward, and is just to check the components of \( \Omega_1 \otimes_K \Omega_2 \) satisfy Eqs. (2.1)–(2.5). First, it is easy to see that Eqs. (2.1) and (2.2) are satisfied. To show Eq. (2.3), note that, \( d_1(t) \otimes 1 \) and \( 1 \otimes d_2(t) \) as elements of \((A_1 \otimes_K A_2)[[t]]\) commute with each other. By this fact, we have
\[
e^{e^{d_3(t)}} = e^{e^{d_1(t)\otimes 1 + 1 \otimes d_2(t)}} = e^{e^{d_1(t)\otimes 1}} e^{1 \otimes d_2(t)}
\]
Applying Eq. (2.3) for \( g_1(t) \) and \( g_2(t) \):
\[
= (g_1(t) \otimes 1)(1 \otimes g_2(t))
= g_1(t) \otimes g_2(t)
= g_3(t).
\]
Next, let us show Eq. (2.4) as follows.
\[
\frac{d}{dt} g_3(t) = \frac{d}{dt} (g_1(t) \otimes g_2(t))
\]
\[
\frac{dg_1(t)}{dt} \otimes g_2(t) + g_1(t) \otimes \frac{dg_2(t)}{dt}
\]

Applying Eq. (2.4) for \(g_1(t)\) and \(g_2(t)\):

\[
= (g_1(t)h_1(t)) \otimes g_2(t) + g_1(t) \otimes (g_2(t)h_2(t))
= (g_1(t) \otimes g_2(t))(h_1(t) \otimes 1 + 1 \otimes h_2(t))
= g_3(t)h_3(t).
\]

Hence, we get Eq. (2.4). Eq. (2.5) can be proved similarly. \(\square\)

Next, let us prove the existence and uniqueness of solutions for any one of Eqs. (2.2)–(2.5).

**Lemma 2.5.** Let \(A\) be any associative \(K\)-algebra. Then, for any one of Eqs. (2.2)–(2.5), if one generating function in the equation fixed to be an element of \(A[[t]]\) with \(f(0) = 1, g(0) = 1\) or \(d(0) = 0\) if \(f(t), g(t)\) or \(d(t)\) is the one fixed, the equation has one and only one solution in \(A[[t]]\) for the other generating function.

**Proof:** First, the lemma is obvious for Eqs. (2.2) and (2.3). To see it is also true for Eq. (2.4), we write \(g(t)\) and \(h(t)\) as in Eqs. (2.8) and (2.10), respectively. Using the fact \(s_0 = 1\) and Eq. (2.4), we have

\[
\sum_{m \geq 1} m s_m t^{m-1} = \left(1 + \sum_{m \geq 1} s_m t^m\right) \left(\sum_{m \geq 1} \psi_m t^{m-1}\right).
\]

Comparing the coefficients of \(t^{m-1}\) \((m \geq 2)\) in the equation above, we get

\begin{align*}
\psi_1 &= s_1, \quad (2.14) \\
ms_m &= \psi_m + \sum_{k+l=m, \ k,l \geq 1} s_k \psi_l, \quad (2.15)
\end{align*}

for any \(m \geq 2\).

Then, by Gauss’ elimination method, it is easy to see that, if one of the sequences \(\{s_m \mid m \geq 1\}\) and \(\{\psi_m \mid m \geq 1\}\) is given, the other can always be obtained in a unique way. Hence the lemma is true for Eq. (2.4). Similarly, we can prove the lemma for Eq. (2.5). \(\square\)

From Lemma 2.5 and its proof, it is easy to see that we have the following three corollaries.

**Corollary 2.6.** For any \(K\)-algebra \(A\) and \(c(t) \in A[[t]]\), we have

(a) If \(c(0) = 1\), then, for any \(i = 1\) or \(2\), there exists a unique NCS system \(\Omega\) over \(A\) with \(c(t)\) as the \(i^{th}\) component.
(b) If $c(0) = 0$, then, there exists a unique NCS system $\Omega$ over $A$ with $c(t)$ as the 3rd component.

(c) For any $i = 4$ or $5$, there exists a unique NCS system $\Omega$ over $A$ with $c(t)$ as the $i^{th}$ component.

**Corollary 2.7.** Let $(A, \Omega)$ be a NCS system. Then, any component of $\Omega$ completely determines the others. In other words, if two NCS systems over $A$ have a same component at a same location, then these two systems are completely same.

**Corollary 2.8.** Let $(A, \Omega)$ be a NCS system. For any sequence $w = \{w_m | m \geq 1\}$ of elements of $A$, we denote by $K\langle w \rangle$ the unital subalgebra of $A$ generated by $w_m$’s. Then, for any sequence $w = s, \phi, \psi$ or $\xi$, we have

$$K\langle w \rangle = K\langle \lambda \rangle.$$  \hspace{1cm} (2.16)

**Corollary 2.9.** Let $(A, \Omega)$ and $(B, \tilde{\Omega})$ be NCS systems and $S : B \to A$ a $K$-algebra homomorphism. Suppose that, for some $1 \leq j \leq 5$, $S$ maps the $j^{th}$ component of $\Omega$ to the $j^{th}$ component of $\tilde{\Omega}$. Then, $S^{\times 5}(\tilde{\Omega}) = \Omega$.

**Proof:** First, by Lemma 2.2 we see that $S^{\times 5}(\tilde{\Omega})$ also is a NCS system over $A$. Since the NCS systems $S^{\times 5}(\tilde{\Omega})$ and $\Omega$ over $A$ have same $j^{th}$ component, by Corollary 2.7 we have $S^{\times 5}(\tilde{\Omega}) = \Omega$. \hspace{1cm} $\Box$

**Proposition 2.10.** Let $(A, \Omega)$ a NCS system as fixed in Definition 2.1 and $\tau : A \to A$ a $K$-algebra homomorphism such that $\tau(\phi_m) = -\phi_m$ for any $m \geq 1$. Then, we have

$$\tau(\lambda_m) = (-1)^ms_m,$$  \hspace{1cm} (2.17)

$$\tau(s_m) = (-1)^m\lambda_m,$$  \hspace{1cm} (2.18)

$$\tau(\psi_m) = -\xi_m,$$  \hspace{1cm} (2.19)

$$\tau(\xi_m) = -\psi_m.$$  \hspace{1cm} (2.20)

**Proof:** Let $\tau^{\times 5}(\Omega)$ and $\Omega^r$ be respectively the image of $\Omega$ under the homomorphism $\tau$ and the flip of $\Omega$ defined in Eq. (2.12). By lemma 2.2 and 2.3 we know both $\tau^{\times 5}(\Omega)$ and $\Omega^r$ are NCS systems over $A$. While on the other hand, by the conditions in the proposition, we know $\tau^{\times 5}(\Omega)$ and $\Omega^r$ have the same third component $-d(t)$. So, by Corollary 2.7 we have $\tau^{\times 5}(\Omega) = \Omega^r$ from which Eqs. (2.17)–(2.20) follow directly. \hspace{1cm} $\Box$

Next we consider NCS systems over some special algebras $A$. 

Proposition 2.11. Let \((A, \Omega)\) be any NCS system over a commutative \(K\)-algebra \(A\). Then we have

\begin{equation}
 m(t) = h(t) = d'(t),
\end{equation}

where \(d'(t)\) denotes the first derivative of \(d(t)\) over \(t\).

In particular, the system of Eqs. \((2.1) - (2.5)\) is reduced to:

\[
\begin{cases}
 f(0) = 1, \\
 f(-t)g(t) = 1, \\
 \frac{df(t)}{dt} = g(t)h(t), \quad \text{or equivalently,} \quad e^{d(t)} = g(t).
\end{cases}
\]

Proof: Since \(A\) is commutative, so is \(A[[t]]\). In this case, Eqs. \((2.4)\) and \((2.5)\) become same. Applying \(\frac{d}{dt}\) to Eq. \((2.3)\) and, by the chain rule, we have

\[
\frac{dg(t)}{dt} = \frac{d}{dt}e^{d(t)} = e^{d(t)}d'(t) = g(t)d'(t).
\]

From the observations above, we see that \(h(t), m(t)\) and \(d'(t)\) are all solutions of Eq. \((2.4)\) with (same) \(g(t)\). Therefore, by Lemma \(2.5\) we have Eq. \((2.21)\) and the proposition follows. \(\Box\)

Next, we consider NCS systems over \(K\)-bialgebras. First we need recall the following notions (see [A], [Knu] and [Mo] for more details). Let \(A\) be a \(K\)-bialgebra with the co-product denoted by \(\Delta : A \rightarrow A \otimes A\). An element \(x \in A\) is primitive if \(\Delta(x) = 1 \otimes x + x \otimes 1\). \(x \in A\) is a group-like element if \(\Delta(x) = x \otimes x\). A sequence \(\{a_m | m \geq 0\}\) of elements of \(A\) is said to be a sequence of divided powers if, for any \(m \geq 0\), we have

\begin{equation}
 \Delta a_m = \sum_{k+l=m} a_k \otimes a_l.
\end{equation}

Now let \(t\) be a central parameter as before, we extend the counit \(\epsilon\) of \(A\) to \(\epsilon : A[[t]] \rightarrow k\) by setting \(\epsilon(t) = 0\) and the co-product \(\Delta\) of \(A\) to \(\Delta : A[[t]] \rightarrow A[[t]] \otimes_K A[[t]]\) by the base extension. Then, with the extended counit and co-product, \(A[[t]]\) is also a \(K\)-bialgebra. With this \(K\)-bialgebra structure fixed on \(A[[t]]\), for any sequence \(\{a_m | m \geq 0\}\) of elements of \(A\), it is easy to check that the following facts:

- the sequence \(\{a_m | m \geq 0\}\) is a sequence of divided powers of \(A\) iff its generating function \(a(t) := \sum_{m \geq 0} a_m t^m\) is a group-like element of \(A[[t]]\);
- all elements \(a_m (m \geq 0)\) are primitive in \(A\) iff the generating function \(a(t)\) is a primitive element of \(A[[t]]\).
Proposition 2.12. Let \((A, \Omega)\) be a NCS system. Suppose \(A\) is further a \(K\)-bialgebra. Then the following statements are equivalent.

1. \(\{\lambda_m \mid m \geq 0\}\) is a sequence of divided powers of \(A\).
2. \(\{s_m \mid m \geq 0\}\) is a sequence of divided powers of \(A\).
3. One (hence also all) of \(d(t), h(t)\) and \(m(t)\) is primitive in \(A[[t]]\).

Note that, the statement (3) is same as saying that, the sequence \(\{\phi_m \mid m \geq 1\}\), \(\{\psi_m \mid m \geq 1\}\) or \(\{\zeta_m \mid m \geq 1\}\) is a sequence of the primitive elements of \(A\).

Proof: By the discussion before the proposition, it will be enough to show that the following equations are equivalent to each other.

\[
\begin{align*}
\Delta f(t) &= f(t) \otimes f(t), \\
\Delta g(t) &= g(t) \otimes g(t), \\
\Delta d(t) &= d(t) \otimes 1 + 1 \otimes d(t), \\
\Delta h(t) &= h(t) \otimes 1 + 1 \otimes h(t), \\
\Delta m(t) &= m(t) \otimes 1 + 1 \otimes m(t).
\end{align*}
\]

First, we identify the \(K\)-algebras \(A[[t]] \otimes_K A[[t]]\) with \((A \otimes_K A)[[t]]\) in the standard way. Then, both sides of Eqs. (2.23)–(2.27) can be viewed as elements of the \(K\)-algebra \((A \otimes_K A)[[t]]\).

Secondly, note that, the 5-tuple of \((A \otimes_K A)[[t]]\) formed by the LHS’s of Eqs. (2.23)–(2.27) in the same order as the equations displayed above is the image \(\Delta^{\times 5}(\Omega)\) in \(((A \otimes A)[[t]])^{\times 5}\) of the NCS system \(\Omega\) over \(A\) under the \(K\)-algebra homomorphism \(\Delta^{\times 5} : A^{\times 5} \to (A \otimes_K A)^{\times 5}\); while the 5-tuple of \((A \otimes_K A)[[t]]\) on the RHS’s is the tensor product \(\Omega \otimes_K \Omega\) of the NCS system \(\Omega\) with itself. Then, by Lemma 2.2, we know \(\Delta \Omega\) is a NCS system over the \(K\)-algebra \((A \otimes_K A)[[t]]\), and, by Proposition 2.4, \(\Omega \otimes_K \Omega\) is also a NCS system over \((A \otimes_K A)[[t]]\). Also note that, one of Eqs. (2.23)–(2.27) holds iff the NCS systems \(\Delta \Omega\) and \(\Omega \otimes \Omega\) have a same component at a same location. Hence, by Corollary 2.7, Eqs. (2.23)–(2.27) are equivalent to each other and the proposition follows. \(\square\)

2.2. The Universal NCS System from Noncommutative Symmetric Functions. In this subsection, we first recall the definitions of some NCSFs (noncommutative symmetric functions) first introduced and studied in [GKLLRT], whose generating functions form a NCS system \(\Pi\) over the free associative algebra \(N\text{Sym}\) generated by an alphabet \(\{A_m \mid m \geq 1\}\). We then show in Theorem 2.15 that the NCS system \((N\text{Sym}, \Pi)\) from NCSFs is actually the universal NCS system over all associative \(K\)-algebras. When \(A\) is further a \(K\)-bialgebra
(resp. Hopf algebra), some sufficient conditions for the algebra homomorphism $S : \text{NSym} \to A$ to be a bialgebra (resp. Hopf algebra) homomorphism are also given in Theorem 2.15.

First, let $K$ be a unital commutative $\mathbb{Q}$-algebra as before and $\Lambda = \{\Lambda_m | m \geq 1\}$ be an alphabet, i.e. a sequence of noncommutative free variables. For convenience, we also set $\Lambda_0 = 1$. Let $\text{NSym}$ or $K\langle \Lambda \rangle$ be the free associative algebra generated by $\Lambda$ over $K$. We denote by $\lambda(t)$ the generating function of $\Lambda_m$ ($m \geq 0$), i.e. we set

$$\lambda(t) := \sum_{m \geq 0} t^m \Lambda_m = 1 + \sum_{k \geq 1} t^m \Lambda_m.$$  \hfill (2.28)

In the theory of NCSFs, $\Lambda_m$ ($m \geq 0$) is the noncommutative analog of the $m^{th}$ classical (commutative) elementary symmetric function and is called the $m^{th}$ (noncommutative) elementary symmetric function.

To define some other NCSFs, we consider Eqs. (2.2)–(2.5) over the free $K$-algebra $\text{NSym}$ with $f(t) = \lambda(t)$. The solutions for $g(t), d(t), h(t), m(t)$ exist and are unique (see Corollary 2.6 for example), whose coefficients will be the NCSFs that we are going to define. Following the notation in [GKLLRT], we denote the resulting 5-tuple by

$$\Pi = (\lambda(t), \sigma(t), \Phi(t), \psi(t), \xi(t))$$  \hfill (2.29)

and write the last four generating functions of $\Pi$ explicitly as follows.

$$\sigma(t) = \sum_{m \geq 0} t^m \Phi_m.$$  \hfill (2.30)

$$\Phi(t) = \sum_{m \geq 1} t^m \frac{\Phi_m}{m}.$$  \hfill (2.31)

$$\psi(t) = \sum_{m \geq 1} t^{m-1} \Psi_m.$$  \hfill (2.32)

$$\xi(t) = \sum_{m \geq 1} t^{m-1} \Xi_m.$$  \hfill (2.33)

Note that, in terms of the terminology in the previous subsection, the 5-tuple $\Pi$ defined above is the unique NCS system with $f(t) = \lambda(t)$ in Eq. (2.28) over the free $K$-algebra $\text{NSym}$.

Following [GKLLRT], we call $\Psi_m$ ($m \geq 1$) the $m^{th}$ complete homogeneous symmetric function, and $\Psi_m$ and $\Xi_m$ ($m \geq 1$) respectively the $m^{th}$ power sum symmetric function of the first and second kind. Note that, $\Xi_m$ ($m \geq 1$) were denoted by $\Psi^*_m$ in [GKLLRT]. Due to Proposition 2.14 below, the NCSFs $\Xi_m$ ($m \geq 1$) do not play an important
role in the NCSF theory (see the comments in page 234 in [GKLLRT]). But, in the context of some other problems, relations of $\Xi_m$’s with other NCSFs, especially, with $\Psi_m$’s, are also important. For example, this is indeed the case in [Z5] where connections of NCSFs with the inversion problem are concerned. So we here refer $\Xi_m \in N\text{Sym}(m \geq 1)$ as the $m^{th}$ (noncommutative) power sum symmetric function of the third kind.

The following two propositions proved in [GKLLRT] and [KLT] will be very useful for our later arguments.

**Proposition 2.13.** For any unital commutative $\mathbb{Q}$-algebra $K$, the free algebra $N\text{Sym}$ is freely generated by any one of the families of the NCSFs defined above.

**Proposition 2.14.** Let $\omega_{\Lambda}$ be the anti-involution of $N\text{Sym}$ which fixes $\Lambda_m$ ($m \geq 1$). Then, for any $m \geq 1$, we have

\begin{align}
\omega_{\Lambda}(S_m) &= S_m, \\
\omega_{\Lambda}(\Phi_m) &= \Phi_m, \\
\omega_{\Lambda}(\Psi_m) &= \Xi_m.
\end{align}

As shown in [GKLLRT], the connections between the NCSFs and the classical (commutative) symmetric functions ([Ma]), are as follows. Let $X = \{X_m \mid m \geq 1\}$ be another alphabet and $K\langle X \rangle$ the free associative algebra generated by $X$ over $K$. We can view $N\text{Sym}$ as a subalgebra of $K\langle X \rangle$ by setting, for any $m \geq 1$,

$$\Lambda_m = \sum_{i_1 < i_2 < \cdots < i_m} X_{i_1}X_{i_2}\cdots X_{i_m}.\tag{2.37}$$

Let $I$ be the two-sided ideal generated by the commutators of $X_m$’s and $\pi$ the image of $X$ in the quotient algebra modulo $I$. Then, in the quotient algebra $K[x]$, $\Lambda_m$ and $S_m$ ($m \geq 1$) become the $m^{th}$ elementary symmetric function and the $m^{th}$ complete elementary symmetric function, respectively; while $\Phi_m$, $\Psi_m$ and $\Xi_m$ ($m \geq 1$) all become the $m^{th}$ power sum symmetric function. See [Ma] for more studies on the classical symmetric functions above.

Next, let us recall the following graded $K$-Hopf algebra structure of $N\text{Sym}$. It has been shown in [GKLLRT] that $N\text{Sym}$ is the universal enveloping algebra of the free Lie algebra generated by $\Psi_m$ ($m \geq 1$). Hence, it has a Hopf $K$-algebra structure as all other universal enveloping algebras of Lie algebras do. Its co-unit $\epsilon : N\text{Sym} \to K$, co-product $\Delta$ and antipode $S$ are uniquely determined by

$$\epsilon(\Psi_m) = 0, \tag{2.38}$$
\[ \Delta(\Psi_m) = 1 \otimes \Psi_m + \Psi_m \otimes 1, \]
\[ S(\Psi_m) = -\Psi_m, \]
for any \( m \geq 1. \)

Furthermore, we define the weight for NCSFs by setting the weight of any monomial \( \Lambda^1_{m_1} \Lambda^2_{m_2} \cdots \Lambda^k_{m_k} \) to be \( \sum_{j=1}^{k} i_j m_j. \) For any \( m \geq 0, \) we denote by \( \text{NSym}_{[m]} \) the vector subspace of \( \text{NSym} \) spanned by the monomials of \( \Lambda \) of weight \( m. \) Then it is easy to see that

\[ \text{NSym} = \bigoplus_{m \geq 0} \text{NSym}_{[m]}, \]

which provides a grading for \( \text{NSym}. \)

Note that, it has been shown in [GKLLRT], for any \( m \geq 1, \) the NCSFs \( S_m, \Phi_m, \Psi_m \in \text{NSym}_{[m]}. \) By Proposition 2.14 this is also true for the NCSFs \( \Xi_m \)'s. By the facts above and Eqs. (2.38)–(2.40), it is also easy to check that, with the grading given in Eq. (2.41), \( \text{NSym} \) forms a graded \( K \)-Hopf algebra. Its graded dual is given by the space \( \Omega \text{Sym} \) of quasi-symmetric functions, which were first introduced by I. Gessel [Ge] (see also [MR] and [St2] for more discussions).

Now we come back to our discussions on the NCS systems. Note that, we have seen that \((\text{NSym}, \Pi)\) by definition forms a NCS system. More importantly, we have the following theorem on the NCS system \((\text{NSym}, \Pi)\).

**Theorem 2.15.** Let \( A \) be a \( K \)-algebra and \( \Omega \) a NCS system over \( A. \) Then,

(a) There exists a unique \( K \)-algebra homomorphism \( S : \text{NSym} \to A \) such that \( S^{\times 5}(\Pi) = \Omega. \)

(b) If \( A \) is further a \( K \)-bialgebra (resp. \( K \)-Hopf algebra) and one of the equivalent statements in Proposition 2.12 holds for the NCS system \( \Omega, \) then \( S : \text{NSym} \to A \) is also a homomorphism of \( K \)-bialgebras (resp. \( K \)-Hopf algebras).

**Proof:** (a) Let \( \Omega \) be given as in Definition 2.1 and its components given as in Eqs. (2.7)–(2.11). Let \( S : \text{NSym} \to A \) to be the unique \( K \)-algebra homomorphism such that, for any \( m \geq 1, \)

\[ S(\Lambda_m) = \lambda_m. \]

Since \( \text{NSym} \) is freely generated by \( \Lambda_m \ (m \geq 1) \) as a \( K \)-algebra, so \( S \) is well defined. Then, by Corollary 2.9 we have \( S^{\times 5}(\Pi) = \Omega. \) The uniqueness of \( S \) follows from the requirement \( S(\lambda(t)) = f(t), \) which is same as Eq. (2.42), and again the fact that \( \text{NSym} \) is freely generated by \( \Lambda_m \ (m \geq 1). \)
(b) Let $\epsilon$ and $\epsilon_A$ denote the co-units of $NSym$ and $A$, respectively, and $\Delta$ denote the co-products of both $NSym$ and $A$. Then we need show the following two equations.

\[(2.43) \quad (S \otimes S) \circ \Delta = \Delta \circ S \]
\[(2.44) \quad \epsilon_A \circ S = \epsilon. \]

First, by the second condition in (b) and Proposition 2.12, we may assume $\psi_m$ ($m \geq 1$) are all primitive elements of $A$. By Eq. (2.39), we know that $\Psi_m$ ($m \geq 1$) are all primitive elements of $NSym$. Secondly, note that, all the maps involved in Eqs. (2.43) and (2.44) are $K$-algebra homomorphisms, and, by Proposition 2.13 $NSym$ is freely generated by $\Psi_m$ ($m \geq 1$) as a $K$-algebra. Therefore, to show Eqs. (2.43) and (2.44), it will be enough to show that both sides of the equations have same values at $\Phi_m$ ($m \geq 1$).

With the observations above, for any $m \geq 1$, we consider

\[(S \otimes S)(\Delta \Psi_m) = (S \otimes S)(\Psi_m \otimes 1 + 1 \otimes \Psi_m) = S(\Psi_m) \otimes 1 + 1 \otimes S(\Psi_m) = \psi_m \otimes 1 + 1 \otimes \psi_m = \Delta \psi_m. \]

Hence, we have Eqs. (2.43).

To show Eq. (2.44), first, by Theorem 2.1.3 in [A], we know that the counit of any $K$-bialgebra $B$ maps any primitive element $y \in B$ to zero. Therefore, for any $m \geq 1$, we have $\epsilon(\psi_m) = 0$ and $\epsilon_A(S(\Psi_m)) = \epsilon_A(\psi_m) = 0$. Hence we have Eq. (2.44).

Finally, let us consider the case that $A$ is further a $K$-Hopf algebra. We need show that $S$ in this case also commutes with the antipodes of $K\langle \Lambda \rangle$ and $A$, i.e.

\[(2.45) \quad S \circ S = S \circ S, \]

where both the antipodes of $K\langle \Lambda \rangle$ and $A$ are denoted by $S$.

First, since both antipodes $S$ are anti-homomorphisms of $K$-algebras and $S : NSym \rightarrow A$ is a $K$-algebra homomorphism, it will be enough to show that both sides of Eq. (2.45) have same values at $\Psi_m$ ($m \geq 1$). Secondly, it is well-known (see [A], for example) and also easy to check that the antipode $S$ of any Hopf algebra $A$ maps any primitive element $x \in A$ to $-x$. By the observations above, we have

\[S \circ S(\Psi_m) = S(\psi_m) = -\psi_m, \]
\[S \circ S(-\Psi_m) = -S(\psi_m) = -\psi_m. \]

Hence, Eq. (2.45) holds. $\square$
Remark 2.16. By taking the quotient over the two-sided ideal generated by the commutators of $\Lambda_m$'s, or applying the similar arguments as in the proof of Theorem 2.15 it is easy to see that, over the category of commutative $K$-algebras, the universal NCS system is given by the generating functions of the corresponding classical (commutative) symmetric functions $([Ma])$.

Remark 2.17. Following the referee’s suggestion, we would like to point out a connection of the universal homomorphism $S : NSym \to A$ with the universal homomorphism from the Hopf algebra $QSym$ of quasi-symmetric functions to combinatorial Hopf algebras introduced and studied in [ABS].

Suppose that $A$ is a graded and connected, and one of the statements of Proposition 2.14 holds, say statement (b). Furthermore assume in this case that the elements $s_m$ ($m \geq 1$) are homogeneous and with grading $m$. Denote by $A^*$ the graded dual Hopf algebra of $A$. Then, by taking duals, we get a homomorphism $S^* : A^* \to QSym$ of $K$-Hopf algebras. Let $\zeta$ be the linear functional of $A^*$ induced by the sequence $\{s_m \mid m \geq 0\}$, i.e. $\zeta |_{A^*^m} (m \geq 0)$ is given by evaluating elements of $A^*_m$ at $s_m$. Since the sequence $\{s_m \mid m \geq 0\}$ is a sequence of divided powers, one may easily check that $\zeta$ is a character of $A^*$, i.e. $\zeta : A^* \to K$ is a homomorphism of $K$-algebras. In terms of the notion introduced in [ABS], the pair $(A^*, \zeta)$ becomes a combinatorial Hopf algebra, and the homomorphism $S^* : A^* \to QSym$ coincides the unique homomorphism guaranteed by Theorem 4.1 in [ABS] for combinatorial Hopf algebras.

2.3. Possible Applications. In this subsection, we discuss the following possible applications of the universal property of the NCS system $(NSym, \Pi)$ given in Theorem 2.15. From the discussions below, we also can see some of the main motivations for the introduction of the NCS systems over associative algebras.

First, the universal property of the NCS system $(NSym, \Pi)$ can be used to solve any equations of Eqs. (2.2)–(2.5) over any $K$-algebras $A$. For example, given a $K$-algebra $A$ and $h(t) \in A[[t]]$ with $h(0) = 0$, we can solve Eqs. (2.3) and (2.4) for $d(t)$ and $g(t)$ as follows. First, by Corollary 2.6 we know, theoretically, there exists a unique NCS system $\Omega$ over $A$ with $h(t)$ as its fourth component. Hence, by the universal property of $(NSym, \Pi)$, we have a unique homomorphism $S : NSym \to A$ such that $S^{\times 5}(\Pi) = \Omega$. On the other hand, the relations or polynomial identities between any two families of the NCSFs in the first four components of $\Pi$ have been given explicitly in [GKLLRT]. By applying the anti-involution $\omega_A$ in Proposition 2.14 one can easily derive the relations of the NCSFs $\Xi_m$’s with other NCSFs in $\Pi$ (for...
example, see §4.1 in \[Z5\] for a complete list). In particular, the coefficients of \(\sigma(t)\) and \(\Phi(t)\) can be written as certain polynomials in the coefficients of \(\psi(t)\). Now, by simply applying the algebra homomorphism \(S\) to these polynomials, we get the coefficients of the wanted solutions \(d(t)\) and \(g(t)\) in terms of the same polynomials in the coefficients of \(h(t)\). Hence, we get the solution \(g(t)\) and \(h(t)\) in \(A[[t]]\). From the arguments above, we see that the generating functions of the NCSFs in the universal NCS system \((NSym, \Pi)\) can be viewed as the universal solutions to Eqs. (2.2)–(2.5) over all associative \(K\)-algebras.

Secondly, suppose that a NCS system \((A, \Omega)\) is given. By applying the \(K\)-algebra homomorphism \(S : NSym \rightarrow A\) guaranteed by the universal property of the system \((NSym, \Pi)\) to the identities of the NCSFs in the NCS system \(\Pi\), we get same identities for the corresponding elements of \(A\) in the NCS system \(\Omega\). This could be a very effective way to obtain identities for certain elements of \(A\) if we could show that they are involved in a NCS system over \(A\). For example, this gadget will be applied in the followed paper \[Z5\] to derive some identities for certain differential operators which are important in the studies of the inversion problem ([BCW], [E4]), i.e. the problem to study various properties of the inverse maps of affine spaces. On the other hand, if the given NCS system \((A, \Omega)\) has already been well-understood, the \(K\)-algebra homomorphism \(S : NSym \rightarrow A\) in turn gives a specialization or realization ([GKLLRT], [St2]) of NCSFs, which may be applied to study certain properties of NCSFs.

3. NCS Systems over Differential Operator Algebras

In this section, we discuss the NCS systems that will be constructed in \[Z4\] over differential operator algebras in commutative or noncommutative free variables. Certain properties of the resulting differential operator specializations of NCSFs, which will be proved in \[Z4\] and \[Z7\], will also be discussed. The main purposes of this section and the next one are, first, to provide some supporting examples for the general discussions of NCS systems given in the previous section, and second, to give a shorter survey or review for some of the main results to be given in the followed papers \[Z4\], \[Z6\] and \[Z7\]. For more examples of the specializations of NCSFs, see the references quoted in the introduction.

First, let us fix the following notation.
Let $K$ be any unital commutative $\mathbb{Q}$-algebra as before and $z = (z_1, z_2, \ldots, z_n)$ commutative or noncommutative free variables\footnote{Since most of the results in this section do not depend on the commutativity of the free variables $z$, we will not distinguish the commutative and the noncommutative case, unless stated otherwise, and adapt the notations for noncommutative variables uniformly for the both cases.}. Let $t$ be a formal central parameter, i.e. it commutes with $z$ and elements of $K$. We denote by $K\langle\langle z \rangle\rangle$ and $K[[t]]\langle\langle z \rangle\rangle$ the $K$-algebras of formal power series in $z$ over $K$ and $K[[t]]$, respectively.

By a $K$-derivation or simply derivation of $K\langle\langle z \rangle\rangle$, we mean a $K$-linear $\delta : K\langle\langle z \rangle\rangle \to K\langle\langle z \rangle\rangle$ that satisfies the Leibniz rule, i.e. for any $f, g \in K\langle\langle z \rangle\rangle$, we have

$$
\delta(fg) = (\delta f)g + f(\delta g).
$$

We will denote by $\mathcal{D}er_K\langle\langle z \rangle\rangle$ or $\mathcal{D}er\langle\langle z \rangle\rangle$, when the base algebra $K$ is clear from the context, the set of all $K$-derivations of $K\langle\langle z \rangle\rangle$. The unital subalgebra of $\text{End}_K(K\langle\langle z \rangle\rangle)$ (endomorphisms of $K\langle\langle z \rangle\rangle$ as a $K$-vector space) generated by all $K$-derivations of $K\langle\langle z \rangle\rangle$ will be denoted by $\mathcal{D}er_K\langle\langle z \rangle\rangle$ or $\mathcal{D}er\langle\langle z \rangle\rangle$. Elements of $\mathcal{D}\langle\langle z \rangle\rangle$ will be called \textit{(formal) differential operators} in the free variables $z$.

For any $\alpha \geq 1$, we denote by $\mathcal{D}er^{[\alpha]}\langle\langle z \rangle\rangle$ the set of the $K$-derivations of $K\langle\langle z \rangle\rangle$ which increase the degree in $z$ by at least $\alpha - 1$. The unital subalgebra of $\mathcal{D}\langle\langle z \rangle\rangle$ generated by elements of $\mathcal{D}er^{[\alpha]}\langle\langle z \rangle\rangle$ will be denoted by $\mathcal{D}er^{[\alpha]}\langle\langle z \rangle\rangle$. Note that, by the definitions above, the operators of scalar multiplications are also in $\mathcal{D}\langle\langle z \rangle\rangle$ and $\mathcal{D}er^{[\alpha]}\langle\langle z \rangle\rangle$. When the base algebra is $K[[t]]$ instead of $K$ itself, the notation $\mathcal{D}er\langle\langle z \rangle\rangle$, $\mathcal{D}\langle\langle z \rangle\rangle$, $\mathcal{D}er^{[\alpha]}\langle\langle z \rangle\rangle$ and $\mathcal{D}er^{[\alpha]}\langle\langle z \rangle\rangle$ will be denoted by $\mathcal{D}er_t\langle\langle z \rangle\rangle$, $\mathcal{D}_t\langle\langle z \rangle\rangle$, $\mathcal{D}er_t^{[\alpha]}\langle\langle z \rangle\rangle$ and $\mathcal{D}_t^{[\alpha]}\langle\langle z \rangle\rangle$, respectively. For example, $\mathcal{D}er_t^{[\alpha]}\langle\langle z \rangle\rangle$ stands for the set of all $K[[t]]$-derivations of $K[[t]]\langle\langle z \rangle\rangle$, which increase the degree in $z$ by at least $\alpha - 1$. Note that, $\mathcal{D}er_t^{[\alpha]}\langle\langle z \rangle\rangle = \mathcal{D}er^{[\alpha]}\langle\langle z \rangle\rangle[[t]]$ and $\mathcal{D}_t^{[\alpha]}\langle\langle z \rangle\rangle = \mathcal{D}er^{[\alpha]}\langle\langle z \rangle\rangle[[t]]$.

For any $1 \leq i \leq n$ and $u(z) \in K\langle\langle z \rangle\rangle$, we denote by $\left[u(z)\frac{\partial}{\partial z_i}\right]$ the $K$-derivation which maps $z_i$ to $u(z)$ and $z_j$ to 0 for any $j \neq i$. For any $\bar{u} = (u_1, u_2, \cdots, u_n) \in K\langle\langle z \rangle\rangle^{\times n}$, we set

$$
[\bar{u} \frac{\partial}{\partial z_i}] := \sum_{i=1}^n [u_i \frac{\partial}{\partial z_i}].
$$

Note that, in the noncommutative case, we in general do \textbf{not} have $\left[u(z)\frac{\partial}{\partial z_i}\right] g(z) = u(z)\frac{\partial g}{\partial z_i}$ for all $u(z), g(z) \in K\langle\langle z \rangle\rangle$. This is the reason why we put a bracket $[\cdot]$ in the notation above for the $K$-derivations.
With the notation above, it is easy to see that any $K$-derivations $\delta$ of $K\langle\langle z\rangle\rangle$ can be written uniquely as $\sum_{i=1}^{n} \left[ f_i(z) \frac{\partial}{\partial z_i} \right]$ with $f_i(z) = \delta z_i \in K\langle\langle z\rangle\rangle$ ($1 \leq i \leq n$).

With the commutator bracket, $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$ ($\alpha \geq 1$) forms a Lie algebra and its universal enveloping algebra is exactly the differential operator algebra $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Consequently, $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ ($\alpha \geq 1$) has a Hopf algebra structure as all other enveloping algebras of Lie algebras do. In particular, its coproduct $\Delta$, antipode $S$ and co-unit $\epsilon$ are uniquely determined by the properties

\begin{align}
(3.3) \quad &\Delta(\delta) = 1 \otimes \delta + \delta \otimes 1, \\
(3.4) \quad &S(\delta) = -\delta, \\
(3.5) \quad &\epsilon(\delta) = \delta \cdot 1,
\end{align}

respectively, for any $\delta \in \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$.

For any $\alpha \geq 1$, let $A_t^{[\alpha]}\langle\langle z\rangle\rangle$ be the set of all the automorphisms $F_t(z)$ of $K[[t]]\langle\langle z\rangle\rangle$ over $K[[t]]$, which have the form $F_t(z) = z - H_t(z)$ for some $H_t(z) \in K[[t]]\langle\langle z\rangle\rangle$ with $o(H_t(z)) \geq \alpha$ and $H_{t=0}(z) = 0$, where $o(H_t(z))$ denotes the minimum of the orders of all components of $H_t(z)$ as formal power series in $z$ with coefficients in $K[[t]]$. It is easy to check that $A_t^{[\alpha]}\langle\langle z\rangle\rangle$ forms a subgroup of the automorphism group of $K[[t]]\langle\langle z\rangle\rangle$ over $K[[t]]$. In particular, for any $F_t \in A_t^{[\alpha]}\langle\langle z\rangle\rangle$ as above, its inverse map $G_t := F_t^{-1}$ can always be written uniquely as $G_t(z) = z + M_t(z)$ for some $M_t(z) \in K[[t]]\langle\langle z\rangle\rangle$ with $o(M_t(z)) \geq \alpha$ and $M_{t=0}(z) = 0$. Throughout this section, we will always let $H_t(z)$, $G_t(z)$ and $M_t(z)$ be determined as above.

Now we fix an $\alpha \geq 1$ and an arbitrary $F_t \in A_t^{[\alpha]}\langle\langle z\rangle\rangle$ and consider the NCS systems $(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle, \Omega_{F_t})$ that will be constructed in [Z4] over the differential operator algebra $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Note that, $F_t \in A_t^{[\alpha]}\langle\langle z\rangle\rangle$ can be viewed as a deformation parameterized by $t$ of the formal map $F(z) := F_{t=1}(z)$, when it makes sense. For more studies on $F_t \in A_t^{[\alpha]}\langle\langle z\rangle\rangle$ from the deformation point view, see [Z2] and [Z3]. Actually, the construction of the NCS system $(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle, \Omega_{F_t})$ is mainly motivated by and also depends on the studies of $F_t \in A_t^{[\alpha]}\langle\langle z\rangle\rangle$ given in [Z2] and [Z3].

We first denote by the to-be-constructed NCS system $\Omega_{F_t}$ as

\begin{align}
(3.6) \quad &\Omega_{F_t} = (f(t), g(t), d(t), h(t), m(t)) \in \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[[t]]^{\times 5}
\end{align}

and write the components of $\Omega_{F_t}$ above as in Eqs. (2.27)–(2.31) with the to-be-determined coefficients in $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Then the components of $\Omega_{F_t}$ are determined as follows.
The first three components of $\Omega_{F_t}$ are given by the following proposition which will be proved in Section 3.2 in [Z4].

**Proposition 3.1.** There exist unique $f(t), g(t), d(t) \in D^{[\alpha]}[[z],[t]]$ with $f(0) = 1$ and $d(0) = 0$ such that, for any $u_t(z) \in K[[t]][\langle z \rangle]$, we have

\begin{align}
(3.7) & \quad f(-t) u_t(z) = u_t(F_t), \\
(3.8) & \quad g(t) u_t(z) = u_t(G_t), \\
(3.9) & \quad e^{d(t)} u_t(z) = u_t(G_t),
\end{align}

where, as usual, the exponential in Eq. (3.9) is given by

\begin{align}
(3.10) & \quad e^{d(t)} = \sum_{m \geq 0} \frac{d(t)^m}{m!}.
\end{align}

Note that, when we write $d(t)$ above as $d(t) = -\frac{a_t(z) \partial}{\partial z}$ for some $a_t(z) \in tK[[t]][\langle z \rangle]$, then we get the so-called $D$-Log $a_t(z)$ of the automorphism $F_t(z) \in A_{t}^{[\alpha]}(\langle z \rangle)$, which has been studied in [E1], [E3], [N], [Z1] and [WZ] for the commutative case.

The last two components of $\Omega_{F_t}$ are given directly as

\begin{align}
(3.11) & \quad h(t) := \left[ \frac{\partial M_t(F_t)}{\partial t} \left( \frac{\partial}{\partial z} \right) \right], \\
(3.12) & \quad m(t) := \left[ \frac{\partial H_t(G_t)}{\partial t} \left( \frac{\partial}{\partial z} \right) \right].
\end{align}

To get some concrete ideas for the differential operators defined above, let us recall the following lemma proved in [Z3] for the special $F_t \in A_{t}^{[\alpha]}(\langle z \rangle)$ with $H_t(z) = tH(z)$ for some $H(z) \in K(\langle z \rangle)^n$.

**Lemma 3.2.** For any $F_t \in A_{t}^{[\alpha]}(\langle z \rangle)$ of the form $F_t(z) = z - tH(z)$ as above, let $N_t(z) = t^{-1}M_t(z)$. Then we have

\begin{align}
(3.13) & \quad m(t) = \left[ N_t(z) \frac{\partial}{\partial z} \right], \\
(3.14) & \quad h(t) = \sum_{m \geq 1} t^{m-1} \left[ C_m(z) \frac{\partial}{\partial z} \right],
\end{align}

where $C_m(z) \in K(\langle z \rangle)^n$ ($m \geq 1$) are defined recurrently by

\begin{align}
(3.15) & \quad C_1(z) = H(z),
\end{align}
\[(3.16) \quad C_m(z) = \left[ C_{m-1}(z) \frac{\partial}{\partial z} \right] H(z), \]

for any \( m \geq 2. \)

Consequently, for any \( m \geq 1, \) the derivations \( \psi_m \) and \( \xi_m \) defined in Eqs. (3.12) and (3.11) are given by

\[(3.17) \quad \psi_m = \left[ C_m(z) \frac{\partial}{\partial z} \right], \]
\[(3.18) \quad \xi_m = \left[ N_m (z) \frac{\partial}{\partial z} \right], \]

where \( N_m(z) \in K\langle\langle z \rangle\rangle^{x_n} \) (\( m \geq 1 \)) is the coefficient of \( t^{m-1} \) of \( N_t(z). \)

By the mathematical induction on \( m \geq 1, \) it is easy to show that, when \( z \) are commutative variables, we further have

\[(3.19) \quad C_m(z) = (JH)^{m-1} H(z) \]

for any \( m \geq 1, \) where \( JH \) is the Jacobian matrix of \( H(z) \in K[[z]]^{x_n}. \)

**Theorem 3.3.** (\([Z4]\)) For any \( \alpha \geq 1 \) and \( F_t(z) \in A^{[\alpha]}_t\langle\langle z \rangle\rangle, \) we have,

(a) the 5-tuple \( \Omega_{F_t} \) defined as above forms a NCS system over the differential operator algebra \( \mathcal{D}^{[\alpha]}\langle\langle z \rangle\rangle. \)

(b) let \( \langle\langle \text{NSym}, \Pi \rangle\rangle \) be the NCS system of NCSFs introduced in Section 2.2, then there exists a unique homomorphism \( S_{F_t} : \text{NSym} \rightarrow \mathcal{D}^{[\alpha]}\langle\langle z \rangle\rangle \) of \( K\)-Hopf algebras such that \( S_{F_t}(\Pi) = \Omega_{F_t}. \)

Note that, (b) follows directly from (a) and Theorem 2.15 since all the coefficients of \( h(t) \) by Eq. (3.11) are \( K \)-derivations and hence are primitive elements of the Hopf algebra \( \mathcal{D}^{[\alpha]}\langle\langle z \rangle\rangle. \)

For any \( F_t(z) \in A^{[\alpha]}_t\langle\langle z \rangle\rangle, \) let \( \Omega_{F_t} \) be defined above. We write the components of \( \Omega_{F_t} \) as in Eq. (3.3) and coefficients of the components as in Eqs. (2.7)–(2.11). Then we have the following differential operator specializations of the NCSFs in the NCS system (\( \text{NSym}, \Pi \)).

**Corollary 3.4.** For any \( \alpha \geq 1 \) and \( F_t(z) \in A^{[\alpha]}_t\langle\langle z \rangle\rangle, \) let \( S_{F_t} : \text{NSym} \rightarrow \mathcal{D}^{[\alpha]}\langle\langle z \rangle\rangle \) be the homomorphism of \( K\)-Hopf algebras in Theorem 3.3 (b). Then, for any \( m \geq 1, \) we have

\[(3.20) \quad S_{F_t}(A_m) = \lambda_m, \]
\[(3.21) \quad S_{F_t}(S_m) = s_m, \]
\[(3.22) \quad S_{F_t}(\Psi_m) = \psi_m, \]
\[(3.23) \quad S_{F_t}(\Phi_m) = \phi_m, \]
\[(3.24) \quad S_{F_t}(\Xi_m) = \xi_m. \]
Note that, one direct consequence of Theorem 3.3 above is the following well-defined map:

\[
S : A_t^{[\alpha]}(\langle z \rangle) \longrightarrow \text{Hopf}(\text{NSym}, \mathcal{D}^{[\alpha]}(\langle z \rangle))
\]

\[
F_t \quad \longrightarrow \quad S_{F_t},
\]

where \(\text{Hopf}(\text{NSym}, \mathcal{D}^{[\alpha]}(\langle z \rangle))\) denotes the set of \(K\)-Hopf algebra homomorphisms from \(\text{NSym}\) to \(\mathcal{D}^{[\alpha]}(\langle z \rangle)\).

Actually, as will be shown in [Z1], the following proposition holds.

**Proposition 3.5.** For any \(\alpha \geq 1\), the map \(S\) defined in Eq. (3.25) is a bijection.

Moreover, by identifying \(\text{Hopf}(\text{NSym}, \mathcal{D}^{[\alpha]}(\langle z \rangle))\) with the set of all sequences of divided powers of the Hopf algebra \(\mathcal{D}^{[\alpha]}(\langle z \rangle)\), one can define a group product for the set \(\text{Hopf}(\text{NSym}, \mathcal{D}^{[\alpha]}(\langle z \rangle))\), with respect to which the bijection \(S\) above becomes an isomorphism of groups.

Next, let us consider the question when the Hopf algebra homomorphism \(S_{F_t} : \text{NSym} \rightarrow \mathcal{D}^{[\alpha]}(\langle z \rangle)\) preserves the gradings of \(\text{NSym}\) and \(\mathcal{D}^{[\alpha]}(\langle z \rangle)\). Note that, precisely speaking, \(\mathcal{D}^{[\alpha]}(\langle z \rangle)\) is not graded in the usual sense, for some infinite sums are allowed in \(\mathcal{D}^{[\alpha]}(\langle z \rangle)\). But we can consider the following graded subalgebras of \(\mathcal{D}^{[\alpha]}(\langle z \rangle)\).

Let \(\mathcal{D}(z)\) be the differential operator algebra of the polynomial algebra \(K\langle z \rangle\), i.e. \(\mathcal{D}(z)\) is the unital subalgebra of \(\text{End}_K(K\langle z \rangle)\) generated by all \(K\)-derivations of \(K\langle z \rangle\). For any \(m \geq 0\), let \(\mathcal{D}_{[m]}(z)\) be the set of all differential operators \(U\) such that, for any homogeneous polynomial \(h(z) \in K\langle z \rangle\) of degree \(d \geq 0\), \(Uh(z)\) either is zero or is homogeneous of degree \(m + d\). For any \(\alpha \geq 1\), set \(\mathcal{D}^{[\alpha]}(z) := \mathcal{D}(z) \cap \mathcal{D}^{[\alpha]}(\langle z \rangle)\). Then, we have the grading

\[
\mathcal{D}^{[\alpha]}(z) = \bigoplus_{m \geq \alpha - 1} \mathcal{D}_{[m]}(z),
\]

with respect to which \(\mathcal{D}^{[\alpha]}(z)\) becomes a graded \(K\)-Hopf algebra.

Now, for any \(\alpha \geq 2\), we let \(G_t^{[\alpha]}(\langle z \rangle)\) be the set of all automorphisms \(F_t \in A_t^{[\alpha]}(\langle z \rangle)\) such that \(F_t(z) = t^{-1}F(tz)\) for some automorphism \(F(z)\) of \(K\langle z \rangle\). It is easy to check that \(G_t^{[\alpha]}(\langle z \rangle)\) is a subgroup of \(A_t^{[\alpha]}(\langle z \rangle)\). Then we have the following proposition that will be proved in [Z1].

**Proposition 3.6.** For any \(\alpha \geq 2\) and \(F_t \in A_t^{[\alpha]}(\langle z \rangle)\), the differential operator specialization \(S_{F_t}\) is a graded \(K\)-Hopf algebra homomorphism \(S_{F_t} : \text{NSym} \rightarrow \mathcal{D}^{[\alpha]}(z) \subset \mathcal{D}^{[\alpha]}(\langle z \rangle)\) iff \(F_t \in G_t^{[\alpha]}(\langle z \rangle)\).
Now, for any $F_t \in G_t^{[\alpha]}[[z]]$ ($\alpha \geq 2$), by the proposition above, we can take the graded dual of the graded $K$-Hopf algebra homomorphism $S_{F_t} : NSym \rightarrow D^{[\alpha]}[[z]]$ and get the following corollary.

**Corollary 3.7.** For any $\alpha \geq 2$ and $F_t \in G_t^{[\alpha]}[[z]]$, let $D^{[\alpha]}[[z]]^*$ be the graded dual of the graded $K$-Hopf algebra $D^{[\alpha]}[[z]]$. Then,

$$S_{F_t}^* : D^{[\alpha]}[[z]]^* \rightarrow QSym$$

is a homomorphism of graded $K$-Hopf algebras.

Next, let us point out the following property to be proved in [Z7] of the differential operator specializations $S_{F_t} (F_t \in A_t^{[\alpha]}[[z]])$.

For any $\alpha \geq 1$, let $B_t^{[\alpha]}[[z]]$ be the set of automorphisms $F_t = z - H_t(z)$ of the polynomial algebra $K[t][[z]]$ over $K[t]$ such that the following conditions are satisfied.

- $H_{t=0}(z) = 0$.
- $H_t(z)$ is homogeneous in $z$ of degree $d \geq \alpha$.
- With a proper permutation of the free variables $z_i$’s, the Jacobian matrix $JH_t(z)$ becomes strictly lower triangular.

**Theorem 3.8.** In both commutative and noncommutative cases, the following statement holds.

For any fixed $\alpha \geq 1$ and non-zero NCSF $P \in NSym$, there exist $n \geq 1$ (the number of the free variable $z_i$’s) and $F_t(z) \in B_t^{[\alpha]}[[z]]$ such that $S_{F_t}(P) \neq 0$.

**Remark 3.9.** As pointed out earlier in Subsection 2.3, we can apply the homomorphism $S_{F_t} : NSym \rightarrow D^{[\alpha]}[[z]]$ to transform the identities of the NCSFs to the identities of the corresponding differential operators in the NCS systems $\Omega_{F_t}$. Combining with the special forms of the differential operators $\psi_m$’s and $\xi_m$’s in Eqs. (3.17) and (3.18), respectively, we can derive more identities for the inverse map, the $D$-Log of $F_t \in A_t^{[\alpha]}[[z]]$ as well as the formal flow generated by $F_t(z)$, which may be applied further to study the inversion problem. For detailed discussions in this direction, see the sequel paper [Z5].

Finally, let us summarize the main results discussed in this section as follows. By Theorem 3.3, for any $F_t \in A_t^{[\alpha]}[[z]]$, we have a specialization $S_{F_t} : NSym \rightarrow D^{[\alpha]}[[z]]$ of NCSFs by differential operators in $D^{[\alpha]}[[z]]$; by Proposition 3.5, we know any such a specialization of NCSFs, if it is also a $K$-Hopf algebra homomorphism, is give by $S_{F_t}$ for some $F_t \in A_t^{[\alpha]}[[z]]$; By Proposition 3.6, we know exactly when the specialization $S_{F_t} : NSym \rightarrow D^{[\alpha]}[[z]]$ preserves the gradings of $NSym$. 
and $D^{[\alpha]}\langle z \rangle$; Finally, by Theorem 3.8 we know the smaller family of the specializations $S_{F_t}$ with all possible $n \geq 1$ and $F_t \in \mathbb{B}^{[\alpha]}_t\langle z \rangle$ is already fine enough to distinguish any two different NCSFs.

4. A NCS System over the Grossman-Larson Hopf Algebra of Labeled Rooted Trees

In this section, we fix a non-empty $W \subseteq \mathbb{N}^+$ and first recall the Connes-Kreimer Hopf algebra $\mathcal{H}^{\text{CK}}_W ([\text{CM}], [\text{Kr}], [\text{CK}], [F])$ and the Grossman-Larson Hopf algebra $\mathcal{H}^{\text{GL}}_W ([\text{GL}], [\text{CK}], [F])$ of $W$-labeled rooted forests and $W$-labeled rooted trees, respectively. We then discuss the NCS system $(\mathcal{H}^{\text{GL}}_W, \Omega^W_T)$ that will be constructed in [Z7] over the Grossman-Larson Hopf algebra $\mathcal{H}^{\text{GL}}_W$ and certain properties of the resulting specializations of NCSFs by $W$-labeled rooted trees. Finally, we briefly explain a connection, which will be given in [Z7], between the NCS system $(\mathcal{H}^{\text{GL}}_W, \Omega^W_T)$ with the NCS system $(\mathcal{D}^{[\alpha]}\langle \langle z \rangle \rangle, \Omega_{F_t}) (F_t \in A^{[\alpha]}_t\langle \langle z \rangle \rangle)$ discussed in Section 3. Some consequences of this connection will also be discussed.

First, let us fix the following notation.

**Notation:**

By a *rooted tree* we mean a finite 1-connected graph with one vertex designated as its *root*. For convenience, we also view the empty set $\emptyset$ as a rooted tree and call it the *emptyset* rooted tree. The rooted tree with a single vertex is called the *singleton* and denoted by $\circ$. There are natural ancestral relations between vertices. We say a vertex $w$ is a *child* of vertex $v$ if the two are connected by an edge and $w$ lies further from the root than $v$. In the same situation, we say $v$ is the *parent* of $w$. A vertex is called a *leaf* if it has no children.

Let $W \subseteq \mathbb{N}^+$ be any non-empty subset of positive integers. A *$W$-labeled rooted tree* is a rooted tree with each vertex labeled by an element of $W$. If an element $m \in W$ is assigned to a vertex $v$, then $m$ is called the *weight* of the vertex $v$. When we speak of isomorphisms between unlabeled (resp. $W$-labeled) rooted trees, we will always mean isomorphisms which also preserve the root (resp. the root and also the labels of vertices). We will denote by $\mathcal{T}$ (resp. $\mathcal{T}^W$) the set of isomorphism classes of all unlabeled (resp. $W$-labeled) rooted trees. A disjoint union of any finitely many rooted trees (resp. $W$-labeled rooted trees) is called a *rooted forest* (resp. $W$-labeled rooted forest). We denote by $\mathcal{F}$ (resp. $\mathcal{F}^W$) the set of unlabeled (resp. $W$-labeled) rooted forests.

With these notions in mind, we establish the following notation.
(1) For any rooted tree $T \in \mathbb{T}^W$, we set the following notation:

- $\text{rt}_T$ denotes the root vertex of $T$ and $O(T)$ the set of all the children of $\text{rt}_T$. We set $o(T) = |O(T)|$ (the cardinal number of the set $O(T)$).
- $E(T)$ denotes the set of edges of $T$.
- $V(T)$ denotes the set of vertices of $T$ and $v(T) = |V(T)|$.
- $L(T)$ denotes the set of leaves of $T$ and $l(T) = |L(T)|$.
- For any $T \in \mathbb{T}^W$ and $T \neq \emptyset$, $|T|$ denotes the sum of the weights of all vertices of $T$. When $T = \emptyset$, we set $|T| = 0$.
- For any $T \in \mathbb{T}^W$, we denote by $\text{Aut}(T)$ the automorphism group of $T$ and $\alpha(T)$ the cardinal number of $\text{Aut}(T)$.
- For any $v \in V(T)$, we define the height of $v$ to be the number of edges in the (unique) geodesic connecting $v$ to $\text{rt}_T$. The height of $T$ is defined to be the maximum of the heights of its vertices.

(2) Any subset of $E(T)$ is called a cut of $T$. A cut $C \subseteq E(T)$ is said to be admissible if no two different edges of $C$ lie in the path connecting the root and a leaf. We denote by $\mathcal{C}(T)$ the set of all admissible cuts of $T$. Note that, the empty subset $\emptyset$ of $E(T)$ and $C = \{e\}$ for any $e \in E(T)$ are always admissible cuts.

(3) For any $T \in \mathbb{T}^W$ with $T \neq \emptyset$, let $C \in \mathcal{C}(T)$ be an admissible cut of $T$ with $|C| = m \geq 1$. Note that, after deleting the edges in $C$ from $T$, we get a disjoint union of $m + 1$ rooted trees, say $T_0, T_1, \ldots, T_m$ with $\text{rt}(T_i) \in V(T_0)$. We define $R_C(T) = T_0 \in \mathbb{T}^W$ and $P_C(T) \in \mathcal{F}^W$ the rooted forest formed by $T_1, \ldots, T_m$.

(4) For any $T \in \mathbb{T}^W$, we say $T$ is a chain if its underlying rooted tree is a rooted tree with a single leaf. We say $T$ is a shrub if its underlying rooted tree is a rooted tree of height 1. We say $T$ is primitive if its root has only one child. For any $m \geq 1$, we set $\mathbb{H}_m, \mathbb{S}_m$ and $\mathbb{P}_m$ to be the sets of the chains, shrubs and primitive rooted trees $T$ of weight $|T| = m$, respectively. $\mathbb{H}, \mathbb{S}$ and $\mathbb{P}$ are set to be the unions of $\mathbb{H}_m, \mathbb{S}_m$ and $\mathbb{P}_m$, respectively, for all $m \geq 1$.

Let $K$ be any unital commutative $\mathbb{Q}$-algebra and $W$ a non-empty subset of positive integers. First, let us recall the Connes-Kreimer Hopf algebras $\mathcal{H}_W^{CK}$ of labeled rooted forests.

As a $K$-algebra, the Connes-Kreimer Hopf algebra $\mathcal{H}_W^{CK}$ is the free commutative algebra generated by formal variables $\{X_T \mid T \in \mathbb{T}^W\}$. Here, for convenience, we will still use $T$ to denote the variable $X_T$ in $\mathcal{H}_W^{CK}$. The $K$-algebra product is given by the disjoint union. The identity element of this algebra, denoted by 1, is the free variable $X_{\emptyset}$.
corresponding to the emptyset rooted tree. The coproduct \( \Delta : \mathcal{H}_{CK}^W \to \mathcal{H}_{CK}^W \otimes \mathcal{H}_{CK}^W \) is uniquely determined by setting

\[
\Delta(1) = 1 \otimes 1,
\]

\[
\Delta(T) = T \otimes 1 + \sum_{C \in \mathcal{C}(T)} P_C(T) \otimes R_C(T).
\]

The co-unit \( \epsilon : \mathcal{H}_{CK}^W \to K \) is the \( K \)-algebra homomorphism which sends \( 1 \in \mathcal{H}_{CK}^W \) to \( 1 \in K \) and \( T \) to 0 for any \( T \in \mathcal{T}^W \) with \( T \neq \emptyset \). With the operations defined above and the grading given by the weight, the vector space \( \mathcal{H}_{CK}^W \) forms a connected graded commutative bi-algebra.

Since any connected graded bialgebra is a Hopf algebra, there is a unique antipode \( S : \mathcal{H}_{CK}^W \to \mathcal{H}_{CK}^W \) that makes \( \mathcal{H}_{CK}^W \) a connected graded \( K \)-Hopf algebra. For a formula for the antipode, see [F].

Next we recall the Grossman-Larson Hopf algebra of labeled rooted trees. First we need define the following operations for labeled rooted forests. For any labeled rooted forest \( F \) which is disjoint union of labeled rooted trees \( T_1, T_2, \ldots, T_m \), we set \( B_+(F) \) to be the rooted tree obtained by connecting roots of \( T_i \) (\( 1 \leq i \leq m \)) to a newly added root. We will keep the labels for the vertices of \( B_+(F) \) from \( T_i \)'s, but for the root, we label it by 0.

Now, we set \( \bar{\mathcal{T}}^W := \{ B_+(F) \mid F \in \bar{\mathcal{F}}^W \} \). Then, \( B_+ : \mathcal{F}^W \to \bar{\mathcal{T}}^W \) becomes a bijection. We denote by \( B_- : \bar{\mathcal{T}}^W \to \mathcal{F}^W \) the inverse map of \( B_+ \). More precisely, for any \( T \in \bar{\mathcal{T}}^W \), \( B_-(T) \) is the \( W \)-labeled rooted forest obtained by cutting off the root of \( T \) as well as all edges connecting to the root in \( T \).

Note that, precisely speaking, elements of \( \bar{\mathcal{T}}^W \) are not \( W \)-labeled trees for \( 0 \not\in W \). But, if we set \( \bar{W} = W \cup \{0\} \), then we can view \( \bar{\mathcal{T}}^W \) as a subset of \( \bar{W} \)-labeled rooted trees \( T \) with the root \( r_T \) labeled by 0 and all other vertices labeled by non-zero elements of \( \bar{W} \). We extend the definition of the weight for elements of \( \mathcal{F}^W \) to elements of \( \bar{\mathcal{T}}^W \) by simply counting the weight of roots by zero. We set \( \bar{S}_m^W := B_+(\bar{S}_m^W) \) (\( m \geq 1 \)) and \( \bar{S}^W := B_+(S^W) \). We also define \( \bar{H}_m^W, \bar{P}_m^W, \bar{H}^W \) and \( \bar{P}^W \) in the similar way.

The Grossman-Larson Hopf algebra \( \mathcal{H}_{GL}^W \) as a vector space is the vector space spanned by elements of \( \bar{\mathcal{T}}^W \) over \( K \). For any \( T \in \bar{\mathcal{T}}^W \), we will still denote by \( T \) the vector in \( \mathcal{H}_{GL}^W \) that is corresponding to \( T \). The algebra product is defined as follows. For any \( T, S \in \bar{\mathcal{T}}^W \) with \( T = B_+(T_1, T_2, \ldots, T_m) \), we set \( T \cdot S \) to be the sum of the rooted trees obtained by connecting the roots of \( T_i \) (\( 1 \leq i \leq m \)) to vertices of \( S \) in all possible \( m^{v(S)} \) different ways. Note that, the identity element with
respect to this algebra product is given by the singleton \( \circ = B_+ (\emptyset) \). But we will denote it by 1.

To define the co-product \( \Delta : \mathcal{H}_{GL}^W \to \mathcal{H}_{GL}^W \otimes \mathcal{H}_{GL}^W \), we first set
\[
\Delta (\circ) = \circ \otimes \circ.
\]

(4.3)

Now let \( T \in \mathcal{P}^W \) with \( T \neq \circ \), say \( T = B_+ (T_1, T_2, \cdots, T_m) \) with \( m \geq 1 \) and \( T_i \in \mathcal{T}^W \) (1 \( \leq i \leq m \)). For any non-empty subset \( I \subseteq \{1, 2, \cdots, m\} \), we denote by \( B_+ (T_I) \) the rooted tree obtained by applying the \( B_+ \) operation to the rooted trees \( T_i \) with \( i \in I \). For convenience, when \( I = \emptyset \), we set \( B_+ (T_I) = 1 \). With this notation fixed, the co-product for \( T \) is given by
\[
\Delta (T) = \sum_{I \cup J = \{1, 2, \cdots, m\}} B_+ (T_I) \otimes B_+ (T_J).
\]

(4.4)

Note that, a rooted tree in \( \mathcal{T}_W \) is a primitive element of the Hopf algebra \( \mathcal{H}_{GL}^W \) iff it is a primitive rooted tree in the sense that we defined before, namely the root of \( T \) has one and only one child.

The co-unit \( \epsilon : \mathcal{H}_{GL}^W \to K \) is the \( K \)-algebra homomorphism which sends \( 1 \in \mathcal{H}_{GL}^W \) to \( 1 \in K \) and \( T \) to 0 for any \( T \in \mathcal{T}^W \) with \( T \neq \emptyset \). With the operations defined above and the grading given by the weight, the vector space \( \mathcal{H}_{GL}^W \) forms a connected graded commutative bi-algebra. Since any connected graded bialgebra is a Hopf algebra, there is a unique antipode \( S : \mathcal{H}_{GL}^W \to \mathcal{H}_{GL}^W \) that makes \( \mathcal{H}_{GL}^W \) a connected graded commutative \( K \)-Hopf algebra. For a formula for the antipode, see [Z6].

Note that it has been shown in [H] and [F] that, the Grossman-Larson Hopf algebra \( \mathcal{H}_{GL}^W \) and the Connes-Kreimer Hopf algebra \( \mathcal{H}_{CK}^W \) are graded dual to each other. The pairing is given by, for any \( T \in \mathcal{P}^W \) and \( S \in \mathcal{F}^W \),
\[
<T, F >= \begin{cases} 
0, & \text{if } T \not\simeq B_+ (F), \\
\alpha (T), & \text{if } T \simeq B_+ (F).
\end{cases}
\]

(4.5)

Now we consider the NCS system \( \Omega^W_T \) that will be constructed in [Z6] over the Grossman-Larson Hopf algebra \( \mathcal{H}_{GL}^W \).

First, let us define the following constants for the rooted trees in \( \mathcal{T}^W \):

- We set \( \beta_T \) to be the weight of the unique leaf of \( T \) if \( T \in \mathcal{H}^W \) and 0 otherwise.
- We set \( \gamma_T \) to be the weight of the unique child of the root of \( T \) if \( T \in \mathcal{P}^W \) and 0 otherwise.
- We set \( \theta_T \) to be the coefficient of \( s \) of the order polynomial \( \Omega (B_-(T), s) \) of the underlying unlabeled rooted forest of \( B_-(T) \).

For general studies on the order polynomials \( \Omega (P, s) \) of finite posets \( P \), see [St1]. For a combinatorial interpretation of the constant \( \phi_T := \)
(-1)^{o(T)} - 1 \theta_B, \gamma(T) in terms of the numbers of chains with fixed lengths in the lattice of the ideals of the poset T, see Lemma 2.8 in [SWZ].

Now we consider the following generating functions of $T \in \mathcal{P}^W$.

(4.6) $\tilde{f}(t) : = \sum_{T \in \mathcal{S}^W} (-1)^{o(T) - |T|} t^{|T|} \mathcal{V}_T = 1 + \sum_{T \in \mathcal{S}^W, T \neq 0} (-1)^{o(T) - |T|} t^{|T|} \mathcal{V}_T$,  

(4.7) $\tilde{g}(t) : = \sum_{T \in \mathcal{P}^W} t^{|T|} \theta_T \mathcal{V}_T$,  

(4.8) $\tilde{d}(t) : = \sum_{T \in \mathcal{P}^W} t^{|T|} \beta_T \mathcal{V}_T$,  

(4.9) $\tilde{h}(t) : = \sum_{T \in \mathcal{P}^W} t^{|T| - 1} \beta_T \mathcal{V}_T$,  

(4.10) $\tilde{m}(t) : = \sum_{T \in \mathcal{P}^W} t^{|T| - 1} \gamma_T \mathcal{V}_T$,  

where, for any $T \in \mathcal{P}^W$, $\mathcal{V}_T : = \frac{1}{\alpha(T)} T$. We further set

(4.11) $\Omega^W_T : = (\tilde{f}(t), \tilde{g}(t), \tilde{d}(t), \tilde{h}(t), \tilde{m}(t))$.

**Theorem 4.1.** ([Z6]) For any non-empty set $W \subseteq \mathbb{N}^+$, we have

(a) the 5-tuple $\Omega^W_T$ defined in Eq. (4.11) forms a NCS system over the Grossman-Larson Hopf algebra $\mathcal{H}^W_{GL}$.

(b) let $(\mathcal{NSym}, \Pi)$ be the NCS system of NCSFs introduced in Section 2.2, then there exists a unique graded K-Hopf algebra homomorphism $\mathcal{T}_W : \mathcal{NSym} \rightarrow \mathcal{H}^W_{GL}$ such that $\mathcal{T}_W(\Pi) = \Omega^W_T$.

Note that the grade duals of $\mathcal{NSym}$ and $\mathcal{H}^W_{GL}$ are the graded $K$-Hopf algebras $\mathcal{QSym}$ of quasi-symmetric functions and the Connes-Kreimer Hopf algebra $\mathcal{H}^W_{CK}$, respectively. Since the $K$-Hopf algebra homomorphism $\mathcal{T}_W : \mathcal{NSym} \rightarrow \mathcal{H}^W_{GL}$ in Theorem 4.1 preserves the gradings, by taking the graded duals, we have the following correspondence.

**Corollary 4.2.** For any non-empty $W \subseteq \mathbb{N}^+$, $\mathcal{T}_W^* : \mathcal{H}_{CK}^W \rightarrow \mathcal{QSym}$ is a homomorphism of graded $K$-Hopf algebras.

Furthermore, the following result will be proved in [Z7].

**Proposition 4.3.** When $W = \mathbb{N}^+$, the graded Hopf algebra homomorphism $\mathcal{T}_W : \mathcal{NSym} \rightarrow \mathcal{H}^W_{GL}$ in Theorem 4.1 is an embedding. Consequently, its graded dual $\mathcal{T}_W^* : \mathcal{H}_{CK}^W \rightarrow \mathcal{QSym}$ in this case is a surjective homomorphism of graded Hopf algebras.
Remark 4.4. As we mentioned earlier in Subsection 2.3, by applying the specialization \( T_W : \text{NSym} \to \mathcal{H}^W_{GL} \) in Theorem 4.1, we will get a host of identities from the identities of the NCSFs in the NCS system \((\text{NSym}, \Pi)\) for the W-rooted trees in the NCS system \((\mathcal{H}^W_{GL}, \Omega^W_T)\). We believe some of these identities are interesting from the aspect of combinatorics of rooted trees. But, to keep this paper in a certain size, we have to ask the reader who is interested in these identities to do the translations via the Hopf algebra homomorphism \( T_W : \text{NSym} \to \mathcal{H}^W_{GL} \).

Next, we briefly explain a connection, which will be given in [Z7], between the NCS system \( \Omega^W_T \) over the Grossman-Larson Hopf algebra \( \mathcal{H}^W_{GL} \) and the NCS system \( \Omega^F_t \left( F_t \in A^{[\alpha]}_{\langle\langle z\rangle\rangle} \right) \) over the differential operator algebra \( D^{[\alpha]}_{\langle\langle z\rangle\rangle} \) discussed in Section 3. This connection will play an important role in the proofs of Theorem 3.8 and Proposition 4.3 mentioned before.

Let \( W \subseteq \mathbb{N}^+ \) be any non-empty subset of positive integers and \( F_t = z - H_t(z) \in A^{[\alpha]}_{\langle\langle z\rangle\rangle} \) such that \( H_t(z) \) can be written as \( \sum_{m \in W} t^m H_{[m]}(z) \) for some \( H_{[m]}(z) \in K^{[\alpha]}_T(z)^{\times n} \) (\( m \in W \)). In [Z7], a Hopf algebra homomorphism \( A_{F_t} : \mathcal{H}^W_{GL} \to D^{[\alpha]}_{\langle\langle z\rangle\rangle} \) such that \( A_{F_t}(\Omega^W_T) = \Omega^W_T \) will be constructed. Furthermore, with this Hopf algebra homomorphism \( A_{F_t} \), we have the following commutative diagrams.

**Proposition 4.5.** For any \( \alpha \geq 1 \), let \( W \subseteq \mathbb{N}^+ \) and \( F_t \in A^{[\alpha]}_{\langle\langle z\rangle\rangle} \) fixed as above, we have the following commutative diagrams of \( K \)-Hopf algebra homomorphisms.

\[
\begin{align*}
\text{NSym} & \xrightarrow{\tau_W} \mathcal{H}^W_{GL} \\
\downarrow s_{F_t} & \quad \downarrow A_{F_t} \\
D^{[\alpha]}_{\langle\langle z\rangle\rangle} & \quad D^{[\alpha]}_{\langle\langle z\rangle\rangle}
\end{align*}
\]

Combining Proposition 3.7 and Proposition 4.5 above, we have the following proposition.

**Proposition 4.6.** For any \( \alpha \geq 2 \) and \( F_t \in G^{[\alpha]}_{\langle\langle z\rangle\rangle} \), we have the following commutative diagrams of \( K \)-Hopf algebra homomorphisms.

\[
\begin{align*}
\Omega \text{Sym} & \xleftarrow{\tau_W} \mathcal{H}^W_{CK} \\
\downarrow s_{F_t} & \quad \downarrow A_{F_t} \\
D^{[\alpha]}_{\langle\langle z\rangle\rangle}^* & \quad D^{[\alpha]}_{\langle\langle z\rangle\rangle}^*
\end{align*}
\]

Finally, let us point out that, by applying the Hopf algebra homomorphism \( A_{F_t} : \mathcal{H}^W_{GL} \to D^{[\alpha]}_{\langle\langle z\rangle\rangle} \) above, we can also derive the tree
expansion formulas for the inverse map $G_t(z)$, the D-Log $a_t(z)$ and the formal flow generated by $F_t \in A[[z]]$, which will generalize the tree expansion formulas obtained in [BCW], [Wr] and [WZ] in the commutative case to the noncommutative case. For more details, see [Z7].

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