NON-LINEAR APPROXIMATIONS TO GRAVITATIONAL INSTABILITY: A COMPARISON IN SECOND-ORDER PERTURBATION THEORY

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ABSTRACT

Nonlinear approximation methods such as the Zeldovich approximation, and more recently the frozen flow and linear potential approximations, are sometimes used to simulate nonlinear gravitational instability in the expanding Universe. We investigate the relative accuracy of these approximations by comparing them with the exact solution using second order perturbation theory. We evaluate the density and velocity fields in these approximations to second order, and also determine the skewness parameter $S_3 = \langle \delta^3 \rangle / \langle (\delta^{(1)})^2 \rangle^2$ for each of the approximations again in second order. We find that $S_3 = 4, 3, 3.4$ for the Zeldovich approximation, the frozen flow and the linear potential approximations respectively as compared to $S_3 = 34/7$ for the exact solution. Our results show that, of all the approximations considered, the Zeldovich approximation is the most accurate in describing the weakly nonlinear effects of gravity. Moreover, the Zeldovich approximation is much closer to the exact results for matter and velocity distributions than the other approximations if the slope of the power spectrum of density perturbations is $-3 < n \leq -1$.

Subject headings: Cosmology, gravitational clustering, nonlinear approximations, non-Gaussian statistics.
1. Introduction

It is now commonly agreed that all gravitationally bound objects in the Universe as well as its large-scale structure originated from the growth of initially small inhomogeneities in the expanding Universe. Generally speaking the investigation of non-linear evolution of non-interacting particles constituting the dark matter in the Universe requires long N-body computer simulations and as a result has been performed only for a selected subclass of initial conditions. That is one of the reasons why different approximation schemes have been proposed which greatly reduce the required numerical work and make it possible to investigate non-linear evolution for a much wider class of initial conditions and for longer periods of time.

Amongst nonlinear approximation methods the following three are the most natural and straightforward:

1) the Zeldovich approximation (Zeldovich 1970) and its further generalization to the period after caustics formation - the adhesion model (Gurbatov, Saichev & Shandarin 1985);
2) the frozen flow approximation (Matarrese et al. 1992);
3) the linear potential approximation (Brainerd et al. 1992; Bagla & Padmanabhan 1993).

All these approximations are really *approximations* in the sense that they are neither exact nor asymptotic to the exact solution beyond linear order (apart from some special degenerate cases). The difference between each of these approximations and the exact solution arises already in the second order of perturbation expansion. Thus, a natural way to see the difference between these approximations and to estimate their relative accuracy is to calculate their departure from the exact solution in this order. This is just the aim of the present paper.
2. Second Order Perturbative Calculations

Let us consider gravitational instability in the spatially flat matter dominated FRW Universe before the formation of caustics. In such a Universe the scale factor grows as $a(t) \propto t^{2/3}$ and the background matter density decreases according to $\rho_0 = 1/6\pi G t^2$. The equations describing this process have the following form in the Newtonian approximation (see, e.g. Peebles 1980):

\begin{align}
\Delta \Phi &= 4\pi G a^2 \rho_0 \delta; \\
\dot{\delta} + \frac{1}{a} \text{div} ((1 + \delta) \vec{u}) &= 0; \\
(a\vec{u}) \cdot \nabla \vec{u} + (\vec{u} \nabla) \vec{u} &= -\nabla \Phi
\end{align}

where $\delta = (\rho - \rho_0)/\rho_0$.

The velocity $\vec{u}$ is irrotational, therefore it is possible to introduce a velocity potential $V$ so that $\vec{u} = -\nabla V/a$. Then Eq. (3) may be substituted by its first integral:

\begin{align}
\dot{V} &= \Phi + \frac{1}{2a^2} (\nabla V)^2.
\end{align}

and the first two equations acquire the form:

\begin{align}
\Delta \Phi &= 2a^2 \frac{3t^2}{3t^2} \delta; \\
a^2 \dot{\delta} &= (1 + \delta) \nabla V + \nabla \delta \nabla V.
\end{align}

Let us expand all quantities into series in powers of an initial density enhancement: $\delta = \delta^{(1)} + \delta^{(2)} + ..., $ and the same for $\Phi$ and $V$. As a function of
time, this is an expansion in powers of $t^2/a^2 \propto t^{2/3}$ (since $\delta^{(n)} \propto t^{2/3}$). The first order solution (the linear approximation) is the same for all three abovementioned approximation schemes:

$$\Phi^{(1)} = \phi_0; \quad V^{(1)} = \phi_0 t; \quad \delta^{(1)} = \frac{3t^2}{2a^2} \Delta \phi_0$$ (7)

where $\phi_0(\vec{r})$ is the initial gravitational potential. The value and statistical properties of $\phi_0(\vec{r})$ are completely arbitrary for classical cosmology (apart from the trivial condition of smallness: $|\phi_0(\vec{r})| \ll 1$). On the other hand, any theory of the Early Universe should produce some predictions for the properties of $\phi_0$. In particular, the inflationary scenario of the Early Universe predicts that $\phi_0$ is a gaussian stochastic quantity with zero average and dispersion for the Fourier transform $\phi_0(k)$ having an approximately $k^{-3}$ dependence (in the simplest versions of this scenario and before the multiplication of $\phi_0(k)$ by a transfer function).

The form of the linear approximation (7) naturally leads to three approximation schemes which arise as a result of imposing by hand some of the relations valid in this approximation on a fully non-linear solution. Of course, having introduced one new relation, one has to abandon one of the previous equations. That abandoned equation is chosen to be the Poisson equation (1) or (5). Thus, in all these approximations we neglect the self-gravity of inhomogeneities. As a result, we have the following relations for the full series:

1) $V = \Phi t$ - the Zeldovich approximation (further denoted by ZA and by the subscript $Z$);
2) $V = \phi_0 t$ - the frozen flow approximation (FF, subscript $f$);
3) $\Phi = \phi_0$ - the linear potential approximation (LP, subscript $p$).

Our way of introducing the Zeldovich approximation is different from that commonly
found in the literature. However, it is straightforward to check that the formula $V = \Phi t$ provides both the necessary and sufficient conditions for conventional expressions of the Zeldovich approximation, since, (a) it can directly be derived from those latter expressions, and (b) by inserting $V = \Phi t$ in Eq. (4) and then solving Eq. (4) using a trivial change of variables ($V = a^2 \dot{a} V'$), we arrive at the Zeldovich - Bernoullie equation commonly used to describe the evolution of the velocity potential in the Zeldovich approximations (see eg. Kofman 1991) which is given as follows

$$\frac{\partial}{\partial a} V' + \frac{1}{2} (\nabla V')^2 = 0.$$  

Let us now consider the second order terms.

1) The Exact second order solution

$$\Delta \Phi^{(2)} = \frac{2a^2}{3t^2} \delta^{(2)}; \quad (8)$$

$$a^2 \dot{\delta}^{(2)} = \Delta V^{(2)} + \delta^{(1)} \Delta V^{(1)} + \nabla \delta^{(1)} \nabla V^{(1)}; \quad (9)$$

$$\dot{V}^{(2)} = \Phi^{(2)} + \frac{1}{2a^2} (\nabla V^{(1)})^2. \quad (10)$$

After excluding $\Phi^{(2)}$ and $V^{(2)}$ from these equations, we get an equation for $\delta^{(2)}$:

$$\ddot{\delta}^{(2)} + \frac{4}{3t} \dot{\delta}^{(2)} - \frac{2}{3t^2} \delta^{(2)} = \frac{1}{a^2 \partial} \left( \delta^{(1)} \Delta V^{(1)} + \nabla \delta^{(1)} \nabla V^{(1)} \right) + \frac{1}{2a^4} \Delta \left( (\nabla V^{(1)})^2 \right). \quad (11)$$

Note that the left hand side of this equation is the same as in the first order. The solution of (8-11) is (we consider the growing mode only):

$$\delta^{(2)} = \frac{9t^4}{28a^4} (5P + \Delta Q);$$
\[ \Phi^{(2)} = \frac{3t^2}{14a^2}(5\Delta^{-1}P + Q); \]
\[ V^{(2)} = \frac{3t^3}{7a^2}(\frac{3}{2}\Delta^{-1}P + Q); \]
\[ P(\vec{r}) = (\Delta \phi_0)^2 + \nabla \phi_0 \nabla (\Delta \phi_0) = \nabla (\nabla \phi_0 \Delta \phi_0), \]
\[ Q(\vec{r}) = (\nabla \phi_0)^2. \] (12)

Of course, this solution is well known (see, e.g. Peebles 1980). The average value of \( \delta^{(2)} \) is zero because it has the form of divergence (the same is true in all orders). The expression for \( \delta^{(2)} \) is local while the appearance of the inverse Laplacian \( \Delta^{-1} \) in the expressions for \( \Phi^{(2)} \) and \( V^{(2)} \) shows that they are non-local, as is the velocity \( \vec{u}^{(2)} \) (but \( \text{div} \vec{u}^{(2)} \) is local). Non-local terms in the expansion of \( \delta \) begin from \( \delta^{(3)} \), thus, the first nonlinear corrections to the power spectrum and the density correlation function are non-local, too.

2) The Zeldovich approximation

\[ \Phi^{(2)}_Z = \frac{V^{(2)}_Z}{t}; \quad V^{(2)}_Z = \Phi^{(2)}_Z + \frac{1}{2a^2}(\nabla V^{(1)})^2, \] (13)

the third equation is the same as Eq.(9). The solution is:

\[ \delta^{(2)}_Z = \frac{9t^4}{16a^4}(2P + \Delta Q); \quad V^{(2)}_Z = \Phi^{(2)}_Z t = \frac{3t^3}{4a^2}Q. \] (14)

Note that in the case of one-dimensional plane-symmetric motion \( \phi_0 = \phi_0(x) \), \( P = \frac{1}{2}\Delta Q \) and the above terms coincide with the second-order terms for the exact solution (12). This is a consequence of the fact that the Zeldovich approximation is actually an exact solution of Eqs. (4 - 6) in the case of one-dimensional motion.
before caustic formation (Shandarin & Zeldovich 1989).

3) The frozen flow approximation

\[ V_f^{(2)} = 0; \quad \Phi_f^{(2)} = -\frac{1}{2a^2} (\nabla V^{(1)})^2 = -\frac{t^2}{2a^2}Q, \]  

(15)

the third equation being the same as Eq.(9). The only quantity that remains to be found is \( \delta_f^{(2)} \). From the above equations it follows that:

\[ \delta_f^{(2)} = \frac{9t^4}{8a^4}P. \]  

(16)

4) The linear potential approximation

\[ \Phi_p^{(2)} = 0; \quad \dot{V}_p^{(2)} = \frac{1}{2a^2} (\nabla V^{(1)})^2, \]  

(17)

the third equation still being the same as Eq.(9). Solutions for the remaining quantities are

\[ V_p^{(2)} = \frac{3t^3}{10a^2}Q; \quad \delta_p^{(2)} = \frac{9t^4}{40a^4} (5P + \Delta Q). \]  

(18)

Note that \( \delta_p^{(2)} = 0.7\delta^{(2)} \). Thus, the second order correction in the linear potential approximation has the same spatial structure as the exact solution but its value is 30% smaller than that of the exact solution.
3. Comparison of Approximations

3.1. Density Perturbations

Comparing second order terms in the density perturbation in the different approximation schemes, we see that the P terms are the same in all of them (and different from the exact solution). The difference between \( \delta^{(2)}_Z, \delta^{(2)}_f \) and \( \delta^{(2)}_p \) arises essentially because of the different numerical coefficients in front of Q in each of these approximations. In addition we would like to point out that there are no nonlocal terms in \( \delta^{(2)} \) in any of these approximations.

Let us now consider the difference \( \Delta \) between the approximate solutions and the exact one in the second order, and also calculate the expected variances of \( \Delta \) assuming that the initial potential \( \phi_0(\vec{r}) \) is a gaussian stochastic quantity with zero average and an isotropic power spectrum. We have

\[
\begin{align*}
\Delta_Z &\equiv \delta^{(2)}_Z - \delta^{(2)} = -\frac{27t^4}{56a^4}(P - \frac{1}{2}\Delta Q); \\
\Delta_f &\equiv \delta^{(2)}_f - \delta^{(2)} = -\frac{27t^4}{56a^4}(P + \frac{2}{3}\Delta Q); \\
\Delta_p &\equiv \delta^{(2)}_p - \delta^{(2)} = -\frac{27t^4}{56a^4}(P + \frac{1}{5}\Delta Q) = -0.3\delta^{(2)}.
\end{align*}
\]

(19)

We introduce the notations: \( \sigma_1^2 = \langle(\nabla \phi_0)^2\rangle; \sigma_2^2 = \langle(\Delta \phi_0)^2\rangle; \sigma_3^2 = \langle(\nabla(\Delta \phi_0))^2\rangle \).

In the linear approximation, \( \sigma_1^2 \) is proportional to the velocity dispersion

\( (\sigma_v^2 \equiv \langle(\vec{u}^{(1)})^2\rangle = \frac{L^2}{a^2}\sigma_1^2) \), \( \sigma_2^2 \) - to the dispersion of density perturbations

\( (\sigma_p^2 \equiv \langle(\delta^{(1)})^2\rangle = \frac{9\pi^4}{4a^3}\sigma_2^2) \). \( \sigma_2^2 \leq \sigma_1\sigma_3 \) with the equality being achieved in the case when the Fourier spectrum is proportional to \( \delta(k - k_0) \) only. Using the useful relations
\[ \langle P^2 \rangle = \frac{7}{3} \sigma_2^4 + \frac{1}{3} \sigma_1^2 \sigma_3^2; \]
\[ \langle P \Delta Q \rangle = 2 \sigma_2^4 + \frac{2}{3} \sigma_1^2 \sigma_3^2; \]
\[ \langle (\Delta Q)^2 \rangle = \frac{44}{15} \sigma_2^4 + \frac{4}{3} \sigma_1^2 \sigma_3^2, \tag{20} \]

after lengthy but straightforward calculations we get

\[ \langle \Delta_2^2 \rangle = \frac{16}{15} \sigma_2^4 A^2; \]
\[ \langle \Delta_f^2 \rangle = \left( \frac{851}{135} \sigma_2^4 + \frac{49}{27} \sigma_1^2 \sigma_3^2 \right) A^2; \]
\[ \langle \Delta_p^2 \rangle = \left( \frac{1219}{375} \sigma_2^4 + \frac{49}{75} \sigma_1^2 \sigma_3^2 \right) A^2 \tag{21} \]

where \( A = \frac{2\pi^2 \ell^4}{\delta \rho \tau} \). To obtain a relative accuracy with respect to the second order term in the exact solution, these results should be divided by \( \langle \delta^{(2)} \rangle = \frac{100}{9} \langle \Delta_p^2 \rangle \).

The form of the fractional error \( F_\delta = \langle \Delta^2 \rangle / \langle \delta^{(2)} \rangle \) for different approximations can be expressed as \( F_\delta = (a \gamma^2 + b)/(c \gamma^2 + 1) \) where \( \gamma = \sigma_2^2/\sigma_1 \sigma_3 \). The values of \( a, b, c \) for the three approximations considered by us are (0.15, 0.0, 6.25) for ZA; (1.1, 0.25, 6.25) for FF; and (0.0, 0.09, 0.0) for LP respectively. From the plot (Fig.1) of \( F_\delta \) as a function of \( \gamma \), it is clear that the Zeldovich approximation is always better than either FF or LP. In particular, for a \( \delta \)-like power spectrum \( (\gamma = 1) \) the fractional error \( F_\delta \) is 0.0246, for ZA, 0.187, for FF and 0.090 for LP. In the reverse case of a very extended spectrum \( \gamma \ll 1 \) the fractional errors have the asymptotic forms 0.15\( \gamma^2 \); 0.25; 0.09 for ZA, FF, LP respectively. (For a power-law spectrum \( (\frac{\delta^2}{\rho})_k \propto k^n \), \( \gamma \ll 1 \) if the spectral index lies in the range \( -5 \leq n \leq -1 \). Note that in the latter case the Zeldovich approximation is much closer to the exact solution than the other two approximations since \( \langle \Delta_2^2 \rangle \ll \langle \Delta_f^2 \rangle \)).

These results may also be used to compare the value of the skewness parameter

\[ S_3 = \langle \delta^3 \rangle / \sigma_\rho^4 \] which arises in each of the above approximations with that obtained
in the exact solution (Peebles 1980, Grinstein & Wise 1987, Bouchet et al. 1992).
To first order in perturbation theory $\langle \delta^3 \rangle = \langle (\delta^{(1)})^3 \rangle = 0$. In second order, however, $\langle \delta^3 \rangle = 3\langle (\delta^{(1)})^2 \delta^{(2)} \rangle$, so that $S_3 = \frac{34}{7} \approx 4.86$; 4; 3; 3.4 for the exact solution and for the Zeldovich, frozen-flow and linear potential approximations respectively.
All three approximations produce a low value for the skewness but the Zeldovich approximation is the closest to the correct answer once more (being accurate to within 20% ). It is also interesting that in all four cases the skewness does not depend upon the form of the initial spectrum.

3.2. Peculiar Velocities

We now proceed to calculate the error in the peculiar velocity for each of the approximations considered earlier. Let $\vec{U}_Z^{(2)} = \vec{u}_Z^{(2)} - \vec{u}^{(2)}$ denote the difference between the second order velocity field in the ZA and in the exact second order analysis (similarly for FF and LP). Let us also define $M^2 = \langle (\nabla (\triangle -1 P))^2 \rangle = -\langle P \triangle^{-1} P \rangle$ (this will be the only non-local term in the answer). Using the useful relations

$$\langle \nabla (\triangle -1 P) \nabla Q \rangle = -\langle PQ \rangle = \frac{2}{3} \sigma_1^2 \sigma_2^2;$$
$$\langle \nabla Q \nabla Q \rangle = \frac{4}{3} \sigma_1^2 \sigma_2^2;$$

we get in the second order:

$$\langle \vec{U}_Z^{(2)2} \rangle = B^2 (M^2 - \frac{1}{3} \sigma_1^2 \sigma_2^2);$$
$$\langle \vec{U}_f^{(2)2} \rangle = B^2 (M^2 + \frac{40}{27} \sigma_1^2 \sigma_2^2);$$
$$\langle \vec{U}_p^{(2)2} \rangle = B^2 (M^2 + \frac{8}{25} \sigma_1^2 \sigma_2^2),$$
where \( B = \frac{9t^3}{14a^3} \). In order to obtain a relative accuracy, each of these expressions should be divided by \( \langle \vec{u}^2 \rangle \). Since \( \vec{u}_{f}^{(2)} = 0 \), \( \vec{U}_{f}^{(2)} = -\vec{u}^{(2)} \), therefore this is equivalent to dividing by \( \langle \vec{U}_{f}^{(2)} \rangle \). From Eq. (23) it is clear that of the three approximations the Zeldovich approximation is always closest to the exact solution.

The detailed expression for \( M^2 \) in terms of Fourier components of the initial gravitational potential \( (\phi_0(\vec{r}) = (2\pi)^{-3/2} \int d^3k \phi_\vec{k} e^{i\vec{k}\vec{r}}; \langle \phi_\vec{k} \phi^*_\vec{k} \rangle = \delta^{(3)}(\vec{k} - \vec{k}') \) where \( k = |\vec{k}| \) and \( \delta^{(3)} \) is now the 3D delta-function) is quite complicated (see appendix):

\[
M^2 = \frac{1}{32\pi^4} \int_{0}^{\infty} k_1 \phi^2(k_1) \, dk_1 \int_{0}^{\infty} k_2^3 \phi^2(k_2) \left( 2k_1k_2(k_1^4 + 4k_1^2k_2^2 - k_2^4) + (k_2^2 - k_1^2)^3 \ln \frac{k_1 + k_2}{|k_1 - k_2|} \right) \, dk_2
\]

but if the physical wavelength \( 2\pi a/k \) making the main contribution to \( \sigma_1 \) is much larger than corresponding lengths for \( \sigma_2 \) and \( \sigma_3 \), then the integrals in Eq. (24) decouple and \( M^2 \approx \frac{1}{3} \sigma_1^2 \sigma_2^2 \). In this case, the Zeldovich approximation is much closer to the exact solution than the other two. For a power-law spectrum, this happens if \(-3 < n \leq -1\) (certainly, a cut-off at both some large and small scales is implicitly assumed). On the other hand, \( M^2 = \sigma_1^2 \sigma_2^2 \) in the opposite case of a \( \delta \)-like isotropic power spectrum, so that in this case the relative accuracy of the approximations is 0.27; 1.0; 0.53 for ZA, FF and LP respectively.

Juszkiewicz et al. have recently suggested that moments of the dimensionless velocity divergence \( \theta = \dot{a}^{-1} \text{div}\vec{u} = -(3t/2a^2)\Delta V \) (chosen so that \( \langle \theta^{(1)} \rangle = \langle \delta^{(1)} \rangle \)) may be useful statistical quantities of study in the weakly nonlinear regime (Juszkiewicz et al. 1993). The value of the skewness parameter \( T_3 = \langle \theta^3 \rangle / (\langle \theta^2 \rangle)^2 \) for \( \theta \) can be determined from the results obtained in the previous section. We find that in second order \( T_3 = 3\langle (\theta^{(1)})^2 \theta^{(2)} \rangle / (\langle \theta^{(1)} \rangle^2)^2 = \left( -\frac{26}{7} \approx -3.71, -2, 0, -0.8 \right) \) for the
exact solution and for the Zeldovich, frozen flow and linear potential approximations respectively. We find that once more the ZA is the most accurate of the three approximations, although the accuracy of all approximations worsens in this case.

One can also calculate the fractional error $F_{\theta}$ for $\theta$ just as we had done for $\delta$ earlier. We define the difference $D_Z = \theta_Z^{(2)} - \theta^{(2)}$ (similarly for $D_f$ and $D_p$) and construct fractional quantities $F_{\theta} = \langle D_i^2 \rangle / \langle \theta^{(2)} \rangle^2 (i = Z, f, p)$. As in the case of $F_\delta$, $F_{\theta}$ also has the general form $F_{\theta} = (\bar{a}\gamma^2 + \bar{b})/(\bar{c}\gamma^2 + 1)$. Where the coefficients $\bar{a}, \bar{b}, \bar{c}$ have the values $(0.59, 0.0, 3.47)$ for ZA, $(0.0, 1.0, 0.0)$ for FF, and $(2.83, 0.75, 3.47)$ for LP respectively. $F_{\theta}$ is shown as a function of $\gamma$ for the different approximations in figure 2. It is clear that as found earlier in the case of density perturbations, the Zeldovich approximation performs better than the other two approximations for all values of $\gamma$.

It is also possible to calculate cross correlations between the two fields $\delta$ and $\theta$. Using results derived earlier one can easily show that in second order $X_{12} = \langle \delta \theta^2 \rangle / \sigma_\theta^4 = \frac{1}{3}(S_3 - 2T_3)$ and $X_{21} = \langle \delta^2 \theta \rangle / \sigma_\rho^4 = \frac{1}{3}(T_3 - 2S_3)$.

4. Discussion

Our results show that the Zeldovich approximation which is the simplest and computationally the most cost-effective of the three approximation methods considered by us in this paper, is also more accurate on an average than either the frozen flow or the linear potential approximation when studied to second order in perturbation theory. This might suggest that the Zeldovich approximation is a better tool than either of the other two approximations with which to study overdensities in the weakly nonlinear regime and also to probe the dynamics of underdense regions.
such as voids. Efforts to compare the different nonlinear approximation methods discussed in the present paper with N-body simulations in the strongly nonlinear regime are presently in progress (Sathyaprakash et al. 1993).

In addition, the Zeldovich approximation appears to be much closer to exact density and velocity distributions than the other approximations if the slope of the density power spectrum is $-3 < n \leq -1$ (just the range for which we expect the prolonged existence of a network structure qualitatively described by the adhesion model after caustic formation). This also means that the deviation of the Zeldovich solution from the exact one in the second order is much less on an average than the second-order term itself. This result may be generalized to higher orders of perturbation theory in Eulerian space as well, i.e., $\langle \Delta^2_Z \rangle \ll \langle \delta^{(2)} \rangle$ and $\langle U^2_Z \rangle \ll \langle \tilde{u}^{(2)} \rangle$. However, this does not mean that the Zeldovich approximation exactly describes a non-linear evolution even in this case because this closeness originates from the fact that the effect of point displacement (accurately taken into account by the Zeldovich approximation) is more important for averaged values than effects of non-linearity for this range of slopes.

Indeed, leading terms in all orders of perturbation theory having the largest power of a large-scale velocity may be summed with the result (the same one for both the exact solution and the Zeldovich approximation, see also Shandarin 1993):

\begin{align}
\delta &= \delta^{(1)}(\vec{q},t) ; \\
\vec{u} &= \vec{u}^{(1)}(\vec{q},t) ; \\
\Phi &= \frac{V}{t} = \phi_0(\vec{q}) - \frac{3t^2}{4a^2} (\nabla \phi_0(\vec{r}))^2 ; \\
\vec{q} &= \vec{r} + \frac{3t^2}{2a^2} \nabla \phi_0(\vec{r}) .
\end{align}

(25)
This quasi-linear solution which is actually the linear solution in Lagrangian space correctly accounts for a shift of phases produced by the long-wave part of the perturbation spectrum but does not describe genuinely non-linear effects which are mainly due to the short-wave part (the substitution of $\vec{r}$ by $\vec{q}$ in the last terms of the expressions for $\Phi$ and $\vec{q}$ that would exactly reproduce the Zeldovich approximation for these quantities exceeds the accuracy with which Eq. (25) is derived). That is why, for instance, the error of the Zeldovich approximation in determining $S_3$ is not too small.

In a companion paper (Munshi et al. 1993) we obtained values of higher moments of the distributions (i.e. $S_4, S_5, ..$ and $T_4, T_5, ..$) as well. Then this are used to obtain $P(\delta)$ vs $\delta$, $\delta$ vs $\theta$ relations and void probability distribution function and related quantities.

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APPENDIX

DERIVATION OF EXPRESSION FOR $M^2$
We derive the formula for $M^2$ used in section 3.2. By definition $M^2 = -\langle P \triangle^{-1} P \rangle$
It is useful to perform the analysis in $k$ space. Decomposing the gravitational potential in $k$ space we get

$$\phi_0(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int \phi(\vec{k}) e^{i\vec{k}\cdot\vec{r}} d^3k; \quad (A1)$$

$$P = \nabla(\nabla \phi_0 \Delta \phi_0) = -\nabla \frac{1}{(2\pi)^3} \int d^3k_1 \int d^3k_2 \, i\vec{k}_1 \cdot \vec{k}_2 \, \phi(\vec{k}_1) \phi(\vec{k}_2) e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{r}}$$

$$= \frac{1}{(2\pi)^3} \int d^3k_1 \int d^3k_2 \, \bar{\phi}(\vec{k}_1) \phi(\vec{k}_2) e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{r}}; \quad (A2)$$

$$\triangle^{-1} P = -\frac{1}{(2\pi)^3} \int d^3k_1 \int d^3k_2 \, \frac{k_1 (\vec{k}_1 + \vec{k}_2) \phi(\vec{k}_1) \phi(\vec{k}_2)}{\vec{k}_1 + \vec{k}_2} e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{r}}; \quad (A3)$$

$$\nabla(\triangle^{-1} P) = -\frac{i}{(2\pi)^3} \int d^3k_1 \int d^3k_2 \frac{(k_1^2 + \vec{k}_1 \cdot \vec{k}_2)(\vec{k}_1 + \vec{k}_2) \phi(\vec{k}_1) \phi(\vec{k}_2)}{\vec{k}_1 + \vec{k}_2} e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{r}} \quad (A4)$$

$$M^2 = \frac{1}{(2\pi)^6} \int d^3k_1 \int d^3k_2 \int d^3k_3 \int d^3k_4 \, e^{i(\vec{k}_3 + \vec{k}_4) \cdot \vec{r}}$$

$$\frac{(k_1^2 + \vec{k}_1 \cdot \vec{k}_2)(k_3^2 + \vec{k}_3 \cdot \vec{k}_4)(k_2^2 + \vec{k}_2 \cdot \vec{k}_4)}{(\vec{k}_1 + \vec{k}_2)^2 (\vec{k}_3 + \vec{k}_4)^2} \langle \phi(\vec{k}_1) \phi(\vec{k}_2) \phi(\vec{k}_3) \phi(\vec{k}_4) \rangle. \quad (A5)$$

The only non-zero average values are:

$$\langle \phi(\vec{k}_1) \phi(\vec{k}_2) \rangle = \delta^3(\vec{k}_1 - \vec{k}_2) \phi^2(\vec{k}_1); \quad \langle \phi(\vec{k}_1) \phi(\vec{k}_2) \rangle = \delta^3(\vec{k}_1 + \vec{k}_2) \phi^2(\vec{k}_1), \quad (A6)$$

the last expression follows from the reality condition $\phi(-\vec{k}) = \phi^*(\vec{k})$. So,

$$M^2 = \frac{1}{(2\pi)^6} \int d^3k_1 \int d^3k_2 \, \phi^2(k_1) \phi^2(k_2)$$

$$\left( \frac{(k_1^2 + \vec{k}_1 \cdot \vec{k}_2)^2 k_2^2}{(\vec{k}_1 + \vec{k}_2)^2} + \frac{(k_2^2 + \vec{k}_2 \cdot \vec{k}_1)^2 (k_1^2 + \vec{k}_1 \cdot \vec{k}_2) k_2^2}{(\vec{k}_1 + \vec{k}_2)^2} \right) d^3k_2$$

$$= \frac{1}{(2\pi)^6} \int d^3k_1 \int d^3k_2 \, \phi^2(k_1) \phi^2(k_2) \frac{k_1^2 + \vec{k}_1 \cdot \vec{k}_2}{(\vec{k}_1 + \vec{k}_2)^2} (2k_1^2 k_2^2 + (\vec{k}_1 + \vec{k}_2)^2 k_1^2 + k_1^2 k_2^2))$$

$$= \frac{4\pi \cdot 2\pi}{(2\pi)^6} \int_0^\infty k_1^4 \phi^2(\vec{k}_1) \, dk_1 \int_0^\infty k_2^5 \phi^2(\vec{k}_2) \, dk_2 \int_1^\infty \frac{dz}{k_1^2 + k_2^2 + 2k_1 k_2 z} \left( k_1 + k_2 z \right)^2 (2k_1 k_2 + z(k_1^2 + k_2^2)), \quad (A7)$$
where $z$ is the cosine of the angle between $\vec{k}_1$ and $\vec{k}_2$. Performing the $z$ integration using standard textbook formulas one gets the expression for $M^2$

$$M^2 = \frac{1}{32\pi^4} \int_0^\infty k_1 \phi^2(k_1) \, dk_1 \int_0^\infty k_2 \phi^2(k_2) \left( 2k_1k_2(k_1^4 + 4k_1^2k_2^2 - k_2^4) + (k_2^3 - k_1^3)^3 \ln \frac{k_1 + k_2}{|k_1 - k_2|} \right) \, dk_2$$

(A8)

which we have used in the text.
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Figure Captions

**Fig. 1**: The fractional error \( F_\delta = \langle \Delta^2 \rangle / \langle \delta^{(2)} \rangle \) in \( \delta \) for different approximations is shown as a function of \( \gamma = \sigma_2^2 / \sigma_1 \sigma_3 \). The solid, dashed, and dotted lines correspond to the Zeldovich, frozen flow and linear potential approximations respectively.

**Fig. 2**: The Fractional error \( F_\theta = \langle D^2 \rangle / \langle \theta^{(2)} \rangle \) in \( \theta \) for different approximations is shown as a function of \( \gamma = \sigma_2^2 / \sigma_1 \sigma_3 \). The solid, dashed, and dotted lines correspond to the Zeldovich, frozen flow and linear potential approximations respectively.