PLURISUBHARMONIC DOMINATION IN BANACH SPACES

Imre Patyi

1. Introduction.

Given a complex manifold $M$, one says that plurisubharmonic, resp. holomorphic, domination is possible in $M$ if for any locally bounded function $u: M \to \mathbb{R}$ there is a continuous plurisubharmonic function $w: M \to \mathbb{R}$, resp. a Banach space $(V, \| \cdot \|_V)$ and a holomorphic function $h: M \to V$, such that

$$u(x) \leq w(x), \quad \text{resp.} \quad u(x) \leq \|h(x)\|_V, \quad \text{for every } x \in M.$$  

These notions were introduced and studied by Lempert in [L2]. The main result there is that if a Banach space $X$ has an unconditional basis and $\Omega \subset X$ is a pseudoconvex open set, then holomorphic, hence also plurisubharmonic domination is possible in $\Omega$. This result subsequently formed the basis for the study of analytic sheaves and cohomology groups in Banach spaces in [L3,LP,P1-2,S]. The goal of this paper is to prove that plurisubharmonic domination is possible in Banach spaces that have a Schauder basis (or are a direct summand of one that does, i.e., in separable Banach spaces with the bounded approximation property); this class includes all separable Banach spaces that occur in practice. In particular, domination is possible in the important Banach spaces $C[0,1]$ and $L^1[0,1]$, spaces that do not have an unconditional basis. More precisely, we shall prove

**Theorem 1.1.** Let $X$ be a Banach space and $\Omega \subset X$ an open subset. If $X$ has a Schauder basis and $\Omega$ is pseudoconvex, then plurisubharmonic domination is possible in $\Omega$. The same holds if $X$ is just separable but $\Omega$ is convex.

The second part of the theorem easily follows from the first. It would follow for all pseudoconvex $\Omega$ in a separable space if the following could be proved: Given a Banach space $X_0$, a closed subspace $X \subset X_0$, and a pseudoconvex $\Omega \subset X$, there is a pseudoconvex $\Omega_0 \subset X_0$ such that $\Omega = \Omega_0 \cap X$. — It seems likely that under the assumptions of Theorem 1.1 holomorphic domination is also possible, but a proof of this will have to wait for another publication. When $\Omega$ is convex, holomorphic domination was already proved in [P3].

In its structure, the proof of Theorem 1.1 is similar to the proof in [L2]. The main new idea here and already in [P3] is that, at least when $\Omega = X$, it is possible to work through the proof with functions that are defined on all of $X$; and once the theorem is known for $\Omega = X$, the general case is not hard to prove. By contrast, [L2] treated all $\Omega \subset X$ on equal footing; it had to deal with holomorphic functions defined on subsets of $X$, and approximate them uniformly by entire functions. The
required Runge–type approximation theorems are only known in Banach spaces with an unconditional basis (or, more generally, in spaces with a finite dimensional unconditional decomposition, see [J,L1,Me]) and this restricted the scope of [L2].

Since most results of [LP] depended on the hypothesis of plurisubharmonic domination, by Theorem 1.1 those results hold in spaces with a Schauder basis. For example, combining Theorem 1.1 with [LP, Theorem 2] gives the following generalization of Cartan’s Theorems A and B:

**Theorem 1.2.** Suppose a Banach space $X$ has a Schauder basis, $\Omega \subset X$ is a pseudoconvex open subset, and $S \to \Omega$ a cohesive sheaf. Then:

(a) There is a completely exact resolution $\cdots \to E_1 \to E_0 \to S \to 0$; and
(b) $H^q(\Omega, S) = 0$, $q = 1, 2, \ldots$, holds for the higher sheaf cohomology groups.

The notions occurring in this theorem are defined in [LP], to which we refer the reader; for fundamentals of complex analysis in Banach spaces the book [Mu] can be consulted.

2. Ball bundles.

Let $X$ be a Banach space. If $U \subset X$ is open, we write $\text{psc}(U)$ for the family of continuous functions on $\overline{U}$ that are plurisubharmonic on $U$. By saying that a subset $\Omega \subset X$ is pseudoconvex we imply that it is open. Suppose $X$ has a Schauder basis $e_1, e_2, \ldots$, and introduce the projections $\pi_N: X \to X$,

$$\pi_N \sum_{1}^{\infty} \lambda_j e_j = \sum_{1}^{N} \lambda_j e_j, \quad \lambda_j \in \mathbb{C}; \quad \pi_0 = 0, \quad \pi_\infty = \text{id}.$$

We choose the norm $\| \cdot \|$ on $X$ so that for all $x \in X$

$$(2.1) \quad \|\pi_N x - \pi_M x\| \leq \|\pi_n x - \pi_m x\|, \quad 0 \leq n \leq N \leq M \leq m \leq \infty;$$

thus $e_1, e_2, \ldots$ form a bimonotone Schauder basis. Put furthermore $\varrho_N = \text{id} - \pi_N$, $Y_N = \varrho_N X$. Given $N = 0, 1, 2, \ldots$, $A \subset \pi_N X \approx \mathbb{C}^N$, and a continuous $r: A \to [0, \infty)$, the sets

$$A(r) = \{x \in X: \pi_N x \in A, \|\varrho_N x\| < r(\pi_N x)\} \quad \text{and} \quad A[r] = \{x \in X: \pi_N x \in A, \|\varrho_N x\| \leq r(\pi_N x)\}$$

are ball bundles over finite dimensional bases. Any open $\Omega \subset X$ can be exhausted by such ball bundles as follows (see [L2, Section 3]). Let $d(x) = \min\{1, \text{dist}(x, X \setminus \Omega)\}$ and, given $\alpha \in (0, 1)$,

$$D_N(\alpha) = \{t \in \pi_N X: \|t\| < \alpha N, \quad 1 < \alpha N d(t)\},$$

$$\Omega_N(\alpha) = \{x \in X: \pi_N x \in D_N(\alpha), \quad \|\varrho_N x\| < \alpha d(\pi_N x)\}.$$

For example, if $\Omega = X$ then $\Omega_N(\alpha) = \emptyset$ for $N \leq 1/\alpha$ and

$$(2.4) \quad \Omega_N(\alpha) = \{x \in X: \|\pi_N x\| < \alpha N, \|\varrho_N x\| < \alpha\} \quad \text{for } N > 1/\alpha.$$

From now on we assume $\Omega$ is pseudoconvex.
Proposition 2.1. (a) Each $\Omega_N(\alpha) \subset \Omega$ is pseudoconvex.

(b) $\Omega_n(\gamma) \subset \Omega_N(\beta)$ if $n \leq N, \gamma \leq \beta/4$.

(c) Given $\gamma$, each $x \in \Omega$ has a neighborhood contained in all but finitely many $\Omega_N(\gamma)$.

This is [L2, Proposition 3.1]. We also introduce another exhaustion of $\Omega$ by certain $\Omega^N(\gamma)$; these are ball bundles with respect to the decomposition $X = \pi_{N+1}X \oplus Y_{N+1}$. Let $\gamma \in (0, 1)$,

$$p^N(s) = \max\left\{\frac{\|\pi_N s\|}{N}, \frac{1}{N d(s)}, \frac{\|\varrho_N s\|}{d(s)}\right\}, \quad s \in \Omega \cap \pi_{N+1}X,$$

$$D^N(\gamma) = \{s \in \Omega \cap \pi_{N+1}X: p^N(s) < \gamma\},$$

$$\Omega^N(\gamma) = \{x \in \pi_{N+1}^{-1}D^N(\gamma): \|\varrho_{N+1}x\| < \gamma d(\pi_{N+1}x)\}.$$

According to [L2, Proposition 3.2] we have:

**Proposition 2.2.** If $\gamma < 1/4$ then $\Omega_N(\gamma) \subset \Omega^N(4\gamma)$ and $\Omega_N^N(\gamma) \subset \Omega_N(4\gamma)$.

We shall also need the following analogs of [L2, Lemma 4.1, Proposition 4.2]:

**Lemma 2.3.** Suppose $A_2 \subset \subset A_3 \subset \subset A_4$ are relatively open subsets of $\pi_N X$, $A_1 \subset A_2$ is compact and holomorphically convex in $A_4$. Let $r_i: A_4 \to (0, \infty)$ be continuous, $i = 1, 2, 3, r_i < r_{i+1}$, and $-\log r_1$ plurisubharmonic. Given $v \in \text{psc}(X)$, there is a continuous plurisubharmonic $w: \pi_{N}^{-1}A_4 \to \mathbb{R}$ such that

$$w(x)\begin{cases} < 0, & \text{if } x \in A_1[r_1] \\ > v(x), & \text{if } x \in A_3(r_3) \setminus A_2(r_2). \end{cases}$$

**Proof.** As in the proof of [L2, Lemma 4.1] we construct a Banach space $(V, \| \cdot \|_V)$ and a holomorphic function $\psi: \pi_{N}^{-1}A_4 \to V$ such that $\|\psi\|_V < 1/4$ on $A_1[r_1]$ and $\|\psi\|_V > 4$ on $A_3(r_3) \setminus A_2(r_2)$. (This corresponds to choosing $r_4 = \infty$ there. The construction does not use the approximation hypothesis of Lemma 4.1.) Since $v$ is bounded on a neighborhood of the compact set $A_1$, there is a $q \in (0, \infty)$ such that

$$v(y) \leq q, \quad \text{if } \pi_N y \in A_1 \quad \text{and} \quad \|\varrho_N y\| \leq 4^{-q} \max_{A_1} r_1.$$

Let $K$ be the set of linear forms on $V$ of norm $\leq 1$, and define the continuous plurisubharmonic function $w$ by

$$w(x) = 2q \log(\|\psi(x)\| + 1/4) + \sup_{k \in K} v(\pi_N x + (k\psi(x))\varrho_N x), \quad x \in \pi_{N}^{-1}A_4.$$

To check that $w$ is continuous and plurisubharmonic, it is enough to do the same for the sup in (2.7). As the argument $a(x, k) = \pi_N x + (k\psi(x))\varrho_N x$ is a holomorphic function of $(x, k) \in \pi_{N}^{-1}(A_4) \times V^*$ and $v$ is continuous and plurisubharmonic on
Proposition 2.4. Let $0 < 4^2 \beta < \alpha < 4^{-2}$, $N = 1, 2, \ldots$. If $v \in \text{psc}(X)$, there is a continuous plurisubharmonic $w: \pi_{N+1}^{-1}\Omega \to \mathbb{R}$ such that

\[
 w(x)\begin{cases} < 0, & \text{if } x \in \Omega_N(\beta) \\ > v(x), & \text{if } x \in \Omega_{N+1}(4\alpha) \setminus \Omega_N(\alpha). \end{cases}
\]

Proof. Let $A_4 = \Omega \cap \pi_{N+1}X$ and with notation in (2.3), (2.5) define bounded sets

\[
 A_1 = \{ s \in A_4: p^N(s) \leq 4\beta \}, \quad A_2 = D^N(\alpha/4), \quad A_3 = D_{N+1}(4\alpha),
\]

of which $A_1$ is compact, and $A_2, A_3$ are open in $A_4$. Let furthermore $r_1 = 4\beta d$, $r_2 = \alpha d/4$, and $r_3 = 4\alpha d$. We apply Lemma 2.3 with $N$ replaced by $N + 1$. Clearly $A_1 \subset A_2$ is plurisubharmonically, hence holomorphically convex in $A_4$ (see [H, Theorem 4.3.4]). By (2.3), (2.5), and by Proposition 2.2

\[
 A_1[r_1] \supset \Omega^N(4\beta) \supset \Omega_N(\beta), \quad A_2[r_2] = \Omega^N(\alpha/4) \subset \Omega_N(\alpha).
\]

Proposition 2.1b implies $\overline{A_2(r_2)} \subset \Omega_{N+1}(4\alpha) = A_3(r_3)$. Intersecting with $\pi_{N+1}X$, $\overline{A_2} \subset A_3$ follows, and $\overline{A_3} \subset A_4$ is obvious. Therefore by Lemma 2.3 there is a continuous plurisubharmonic $w: \pi_{N+1}^{-1}A_4 = \pi_{N+1}^{-1}\Omega \to \mathbb{R}$ as claimed: $w < 0$ on $A_1[r_1] \supset \Omega_N(\beta)$ and $w > v$ on $A_3(r_3) \setminus A_2(r_2) \supset \Omega_{N+1}(4\alpha) \setminus \Omega_N(\alpha)$.

3. Domination in the whole space.

We prove the following special case of Theorem 1.1:

Proposition 3.1. Suppose a Banach space $X$ has a Schauder basis and $v: X \to \mathbb{R}$ is a locally bounded function. There is a $w \in \text{psc}(X)$ such that $u(x) < w(x)$ for $x \in X$.

We shall use the assumptions and the notation of section 2. If $x \in X$ and $\varepsilon > 0$, $B(x, \varepsilon) \subset X$ will stand for the ball of radius $\varepsilon$, centered at $x$. The key is the following

Proposition 3.2. Given $u: X \to \mathbb{R}$, suppose there is an $\varepsilon > 0$ and for every $x \in X$ a $w_x \in \text{psc}(X)$ such that $u < w_x$ on $B(x, \varepsilon)$. Then there is a $w \in \text{psc}(X)$ such that $u < w$. 
Proof. We can assume \( u > 0 \) everywhere. Fix a positive \( \alpha < \min(\varepsilon, 4^{-2}) \), and with \( N = 1, 2, \ldots \), consider the compact set \( A = \overline{\Omega_N}(\alpha) \cap \pi_N X \); here \( \Omega_N(\alpha) \) refers to the exhaustion of \( X = \Omega \) defined in (2.3) or (2.4). As each \( t \in A \) has a neighborhood \( U \subset \pi_N X \) such that \( \Omega_N(\alpha) \cap \pi_N^{-1} U \subset B(t, \varepsilon) \), there is a finite \( T \subset A \) such that

\[
\Omega_N(\alpha) \subset \bigcup_{t \in T} B(t, \varepsilon).
\]

It follows that \( v_N = \max\{w_t; t \in T\} \in \text{psc}(X) \) satisfies \( v_N > u \) on \( \Omega_N(\alpha) \). Let \( 0 < \beta < \alpha/4^2 \). By Proposition 2.4, there is \( w_N \in \text{psc}(X) \) with

\[
w_N(x) \begin{cases} < 0, & \text{if } x \in \Omega_N(\beta) \\ > v_N(x), & \text{if } x \in \Omega_{N+1}(\alpha) \setminus \Omega_N(\alpha), \end{cases}
\]

and \( w = \sup\{v_1, w_1, w_2, \ldots \} \) will do.

Proof of Proposition 3.1. Suppose the claim is not true, and \( u \) cannot be dominated by any \( w \in \text{psc}(X) \). In light of Proposition 3.2 there must be a ball \( B(x_1, 1) \) on which \( u \) cannot be dominated by a \( w \in \text{psc}(X) \), i.e.,

\[
u_1 = \begin{cases} u & \text{on } B(x_1, 1) \\ 0 & \text{on } X \setminus B(x_1, 1), \end{cases}
\]

cannot be plurisubharmonically dominated. Again by Proposition 3.2, there must be a ball \( B(x_2, 1/2) \) on which \( u_1 \) cannot be dominated, and so on. We obtain a sequence \( B(x_k, 1/k) \) of balls such that

\[
u_k = \begin{cases} u & \text{on } \bigcap_1^k B(x_j, 1/j) \\ 0 & \text{on } X \setminus \bigcap_1^k B(x_j, 1/j), \end{cases}
\]

cannot be plurisubharmonically dominated. In particular \( \bigcap_1^k B(x_j, 1/j) \neq \emptyset \). Hence \( \|x_j - x_k\| < (1/j) + (1/k) \), and the \( x_j \) have a limit \( x \). But \( u \) is bounded on some neighborhood of \( x \), so on some \( B(x_k, 1/k) \); hence \( u_k \) is bounded and can be dominated by a constant. This is a contradiction, which then proves the claim.

4. Domination in a general \( \Omega \).

Consider a pseudoconvex subset \( \Omega \) of a Banach space \( X \) that has a Schauder basis.

Proposition 4.1. Given a locally bounded \( u: \Omega \to \mathbb{R} \), there is a continuous plurisubharmonic \( w: \Omega \to \mathbb{R} \) such that \( u(x) < w(x) \) for \( x \in \Omega \).

Proof. Again we make the assumptions and use notation introduced in Section 2. Fix \( 0 < \alpha < 4^{-2} \) and \( 0 < \beta < \alpha/4^2 \). For \( N = 0, 1, \ldots \) let \( U_N = \bigcap_{j \geq N} \Omega_j(4\alpha) \). By Proposition 2.1c, these are open sets and exhaust all of \( \Omega \). We prove by induction that there are \( w_N \in \text{psc}(U_{N+1}) \) such that

\[
w_N \begin{cases} < 0 & \text{on } \Omega_N(\beta) \\ > u & \text{on } U_{N+1} \setminus U_N, \\ \end{cases} \quad \text{and} \quad w_N > w_{N-1} \text{ on } \partial U_N.
\]
(When $N = 0$, the last requirement is vacuous, $U_0 = \Omega_0 \langle 4\alpha \rangle = \emptyset$.) The functions

$$u_N = \begin{cases} u & \text{on } \overline{U_{N+1}} \\ 0 & \text{on } X \setminus \overline{U_{N+1}}, \end{cases}$$

are locally bounded. Applying Proposition 3.1 we obtain $w_0 \in \text{psc}(U_1)$ with $w_0 > u_0$; then (4.1) is satisfied for $N = 0$.

Next suppose that $w_0, \ldots, w_{N-1}$ have already been found. Again by Proposition 3.1 there is $v \in \text{psc}(X)$ such that $v > u_N$ on $X$ and $v > w_{N-1}$ on $\partial U_N$. Further, by Proposition 2.4 there is a continuous plurisubharmonic $v': \pi_{N+1}^{-1} \Omega \to \mathbb{R}$ such that

$$v' \begin{cases} < 0 & \text{on } \Omega_N \langle \beta \rangle \\ > v & \text{on } \Omega_{N+1} \langle 4\alpha \rangle \setminus \Omega_N \langle \alpha \rangle. \end{cases}$$

In view of Proposition 2.1b, $U_N \supset \Omega_N \langle \alpha \rangle$, and so $U_{N+1} \setminus U_N \subset \Omega_{N+1} \langle 4\alpha \rangle \setminus \Omega_N \langle \alpha \rangle$. It follows that $w_N = v'|U_{N+1} \in \text{psc}(U_{N+1})$ satisfies (4.1).

Define $w: \Omega \to \mathbb{R}$ by

$$(4.2) \quad w(x) = \sup \{w_N(x), w_{N+1}(x), \ldots\}, \quad \text{if } x \in U_{N+1} \setminus U_N, \ N = 0, 1, 2, \ldots.$$ 

By (4.1) and Proposition 2.1c, the sup is locally finite, and so defines a continuous plurisubharmonic function on $U_{N+1} \setminus \overline{U_N}$, hence on $\Omega \setminus \bigcup_{N \geq 1} \partial U_N$. But $w$ is also continuous and plurisubharmonic in some neighborhood of any $x_0 \in \partial U_N$. Indeed, choose $N \leq M$ so that

$$x_0 \in \partial U_N \cap \partial U_{N+1} \cap \cdots \cap \partial U_M \cap U_{M+1}, \text{ and } x_0 \notin \overline{U_{N-1}}.$$ 

By (4.1), $w_M(x_0) > w_{M-1}(x_0) > \cdots > w_{N-1}(x_0)$. By continuity, it follows that for $x$ near $x_0$,

$$w(x) = \sup \{w_M(x), w_{M+1}(x), \ldots\}, \quad \text{cf (4.2)}.$$ 

Since $w_j$ for $j \geq M$ is continuous and plurisubharmonic on a neighborhood of $x_0$, so is $w$. Finally, (4.1) implies $w > u$, and the proof is complete.

5. Separable spaces.

Proposition 4.1 represents the first part of Theorem 1.1. To prove the second part, let $X$ be separable and $\Omega \subset X$ convex and open. Embed $X$ linearly into the space $X_0 = C[0, 1]$, so that $X \subset X_0$ is a (closed) linear subspace. We can assume $0 \in \Omega$. Let $B \subset X_0$ be an open ball centered at 0 such that $B \cap X \subset \Omega$. The convex hull $\Omega_0$ of $B \cup \Omega$ is a convex, open subset of $X_0$. We claim that $\Omega_0 \cap X = \Omega$. Indeed, suppose $p \in X \setminus \Omega$. By the Hahn–Banach separation theorem, there is a real linear form $f: X \to \mathbb{R}$ such that

$$(5.1) \quad f(p) > f(x) \quad \text{for } x \in \Omega.$$ 

In particular, $f(p) > f(x)$ for $x \in B \cap X$. If $f_0: X_0 \to \mathbb{R}$ denotes a linear extension of $f$, having the same norm as $f$, then

$$(5.2) \quad f_0(p) > f_0(x) \quad \text{for } x \in B.$$
(5.1) and (5.2) imply $f_0(p) > f_0(x)$ for $x \in \Omega_0$, whence $p \notin \Omega_0$ as claimed. It follows that $\Omega = \Omega_0 \cap X$ is closed in $\Omega_0$.

Any locally bounded $u: \Omega \to \mathbb{R}$ extends to the locally bounded function

$$u_0 = \begin{cases} 
  u & \text{on } \Omega \\
  0 & \text{on } \Omega_0 \setminus \Omega. 
\end{cases}$$

Since $X_0 = C[0,1]$ has a Schauder basis, by Proposition 4.1 $u_0$ can be dominated by a continuous plurisubharmonic $v_0$; then $v = v_0|\Omega$ dominates $u$, q.e.d.

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Imre Patyi, Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303-3083, USA, ipatyi@gsu.edu