The general supersymmetric solution of topologically massive supergravity

G W Gibbons\(^1\), C N Pope\(^{1,2}\) and E Sezgin\(^2\)

\(^1\) DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 0WA, UK
\(^2\) George and Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy, Texas A and M University, College Station, TX 77843, USA

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Abstract
We find the general fully nonlinear solution of topologically massive supergravity admitting a Killing spinor. It is of plane-wave type, with a null Killing vector field. Conversely, we show that all solutions with a null Killing vector are supersymmetric for one or the other choice of sign for the Chern–Simons coupling constant \(\mu\). If \(\mu\) does not take the critical value, \(\mu = \pm 1\), these solutions are asymptotically regular on a Poincaré patch, but do not admit a smooth global compactification with boundary \(S^1 \times \mathbb{R}\). In the critical case, the solutions have a logarithmic singularity on the boundary of the Poincaré patch. We derive a Nester–Witten identity, which allows us to identify the associated charges, but we conclude that the presence of the Chern–Simons term prevents us from making a statement about their positivity. The Nester–Witten procedure is applied to the BTZ black hole.

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1. Introduction

There has been considerable interest recently [1–6] in topologically massive gravity with a cosmological constant in 2 + 1 spacetime dimensions. In this paper, we study the supergravity version [7, 8]. In particular, we find all solutions admitting a Killing spinor, and discuss the Nester–Witten identity, and the extent to which it allows one to prove a positivity result for the total energy. It should be noted that there are different definitions for the energy and the angular momentum for an asymptotically AdS\(_3\) solution of the topologically massive theory. In this paper, we shall take the view that energy should be measured above that of a fiducial background metric. This is consistent with supersymmetry, but differs from various holographic constructions [9, 10]. We find that while the standard Nester–Witten procedure does allow us to identify asymptotic charges, the higher-derivative character of the Chern–Simons term prevents one from making a general positivity statement.
The organization of the paper is as follows. In section 2, we review topologically massive supergravity, and its supersymmetry transformations. In section 3, we find the general solution admitting a Killing spinor, and show that for suitable choice of sign of the Chern–Simons coupling constant $\mu$, every solution with a null Killing vector admits a Killing spinor. Solutions with a null Killing vector exist for all non-zero values of $\mu$, and so far as local degrees of freedom are concerned, we conclude that their number is the same for all non-vanishing $\mu$. By 'local degrees of freedom,' we mean that the number of solutions in an open neighbourhood of any point in the manifold is the same for all finite values of $\mu$. However, the imposition of boundary conditions at infinity, so as, for instance, to fit them into representations of the isometry group $SO(2, 2)$, does depend on the value of $\mu$, since in the critical case $\mu = \pm 1$, there are logarithmic terms at infinity. For detailed discussions of this issue, see [1–6].

We also show that all solutions with a null Killing vector are of Kerr–Schild form and that they are universal in the sense of [14]: that is, no symmetric conserved tensors can be constructed from the metric and its derivatives, other than the metric and the Cotton tensor. This shows that these metrics are unchanged in form by quantum corrections to arbitrary order in perturbation theory.

In section 4, we discuss global issues and the asymptotic structure of the solutions. Those for generic values of $\mu$ are asymptotically regular in a Poincaré patch, but in the critical cases, $\mu = \pm 1$, there are logarithmic singularities. In no case do the solutions admit a smooth conformal compactification with conformal boundary $S^1 \times \mathbb{R}$. In section 5, we give a Nester–Witten formula for the total energy, and find that the bulk term contains a contribution from the Cotton tensor. This precludes a general conclusion about the sign of the energy. Finally, we exhibit the Killing spinors of the BTZ vacuum ($M = J = 0$) and maximally rotating ($J + M = 0$) BTZ black holes. Further, we show that there is a single Nester–Witten charge, which is equal to $M + J$.

2. $N = 1$ topologically massive supergravity

Simple topological massive supergravity, which is the sum of simple supergravity and a gravitational super Chern–Simons action, was constructed by Deser and Kay [7]. It was generalized to include a cosmological term by Deser [8]. The total Lagrangian, in our conventions, is given by

$$e^{-1} \mathcal{L} = R - 2m^2 + 2 \bar{\psi}_\mu \gamma^{\mu
u} D_\nu \psi_\nu - m \bar{\psi}_\mu \gamma_{\nu} \psi_\nu \bar{\psi}_\nu + \frac{1}{2} \mu^{-1} e^{\mu \nu} (R_{\mu \nu} - \frac{1}{2} \omega^{\mu \nu} \omega_{\mu \nu}) - \mu^{-1} R_\mu \gamma_\mu R^\mu,$$  \hfill (2.1)

where we have set the gravitational coupling constant equal to one, and used the following curvatures:

$$R_{\mu \nu}^{ab} = \partial_{\mu} \omega_{\nu}^{ab} + \omega_{\mu}^{ac} \omega_{\nu c}^b - (\mu \leftrightarrow \nu),$$  \hfill (2.2)

$$R^\mu = \varepsilon^{\mu \nu \rho} D_\nu (\gamma \psi) \psi_\rho.$$  \hfill (2.3)

It is important to note that the spin connection is not an independent field, but rather it is given by

$$\omega_{\mu ab} = \omega_{\mu ab} (\epsilon) + \bar{\psi}_\mu \gamma_a \psi_b - \bar{\psi}_\mu \gamma_b \psi_a + \bar{\psi}_a \gamma_\mu \psi_b,$$  \hfill (2.4)

where $\omega_{\mu ab} (\epsilon)$ is the standard spin connection that solves the equation, $D_\mu (\omega (\epsilon)) e^\mu_v = \partial_\mu e^\mu_v - \Gamma^\mu_{\mu \nu} (g) + \omega_{\mu}^{ab} (\epsilon) e_{ab} = 0$, with $\Gamma^\mu_{\mu \nu} (g)$ representing the standard Christoffel connection.
The dreibein also satisfies the metricity condition, \( D_\mu e^b_\nu = \partial_\mu e^a_\nu - \Gamma^a_\mu \nu e^b_\rho + \omega^{ab}_\mu e^b_\nu = 0 \), which also serves as the definition of \( \Gamma^\rho_\mu \nu (g) \), which, unlike \( \Gamma^\rho_\mu \nu (g) \), has torsion.

The action is invariant under the local supersymmetric transformations,

\[
\delta e^a_\mu = \bar{\epsilon} \gamma^a \psi_\mu ,
\]

\[
\delta \psi_\mu = D_\mu (\omega) \epsilon - \frac{1}{2} m \gamma_\mu \epsilon ,
\]

where \( D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{2} \omega^{ab}_\mu \gamma_{ab} \epsilon \). Note that the \( \mu \)-dependent part is separately invariant under (2.6). In fact, this part has a larger symmetry, by being invariant under local superconformal symmetry. The field equations following from the Lagrangian (2.1) are

\[
\mathcal{G}_{\mu \nu} + \mu^{-1} \mathcal{C}_{\mu \nu} = 0 ,
\]

\[
R^\mu - \frac{1}{2} m \gamma^{\mu \nu} \psi_\nu - \frac{1}{2} \mu^{-1} \mathcal{C}^\mu = 0 ,
\]

up to fermionic terms in Einstein’s equation, and cubic and higher than first order in fermions in the graviton field equation, and we have used the definitions

\[
\mathcal{G}_{\mu \nu} = G_{\mu \nu} - m^2 g_{\mu \nu} ,
\]

\[
G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R ,
\]

\[
C_{\mu \nu} = \epsilon_{\mu \rho} \nabla_\rho (R_{\sigma \nu} - \frac{1}{4} g_{\sigma \nu} R) ,
\]

\[
C^\mu = \gamma^{\rho \mu} \psi_\rho + \frac{1}{2} \mathcal{C}^\mu \psi_\nu ,
\]

where \( C_{\mu \nu} \), which is symmetric and traceless, is the Cotton tensor [11], and the vector spinor \( C^\mu \) is supersymmetric partner, the ‘Cottino vector spinor’. In the formulae (2.9)–(2.12), all covariant derivatives and curvatures are defined with respect to the torsion-free connections, \( \Gamma^\rho_\mu \nu (g) \) and \( \omega^{ab}_\mu (g) \), and this will be understood in all subsequent formulae. Next, recalling that in three dimensions the Riemann tensor obeys the identity,

\[
R_{\mu \nu}^{\ \ \ \ ab} = 4 \epsilon^{\ [a} [\mu R_{\nu] b] - \epsilon^a_{\ [a} e^b_{\nu]} R ,
\]

where \( R^a_{\mu} = R_{\mu \nu}^{\ ab} e^b_\nu \), it readily follows that \( C_\mu \) is \( \gamma \)-traceless, in the sense that \( \gamma^\mu C_\mu = 0 \).

3. The general supersymmetric solution

Here, we give a construction of the most general supersymmetric solution to equation (2.7). For convenience, from this point onwards we shall set the cosmological equal to \(-1\), by taking

\[
m = 1 .
\]

From (2.6), supersymmetry implies the existence of a solution \( \epsilon \) to the Killing spinor equation

\[
D_\mu \epsilon - \frac{1}{2} \gamma_\mu \epsilon = 0 .
\]

It can then be seen that the vector,

\[
K^\mu = \bar{\epsilon} \gamma^\mu \epsilon ,
\]

is a null Killing vector:

\[
K^\mu K_\mu = 0 , \quad \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 .
\]

(The null property follows from the identity \( \gamma_{\mu \rho \sigma \gamma} \gamma_{\nu \delta} = 0 \), where \( \alpha, \beta, \gamma \) and \( \delta \) are spinor indices.) Multiplying (3.2) by \( \bar{\epsilon} \gamma_\nu \), leads to the equation

\[
\nabla_\mu K_\nu = -\epsilon_{\mu \nu \rho} K^\rho .
\]
Using (2.13), it can be seen that the integrability condition for (3.2) implies
\[ \mathcal{G}_{\mu
u} \gamma^{\alpha} \epsilon = 0. \] (3.6)
Multiplying by \( \bar{\epsilon} \), we see that
\[ \mathcal{G}_{\mu
u} K^{\nu} = 0, \] (3.7)
whilst multiplying instead with \( \bar{\epsilon} \gamma^{\rho} \) gives
\[ \epsilon^{\mu\nu\rho} K_{\nu} \mathcal{G}_{\rho\sigma} = 0. \] (3.8)
The two equations (3.7) and (3.8) exhaust all the content of the integrability condition (3.6).

The existence of the null Killing vector \( K^\mu \) implies that we can choose an adapted coordinate system in which \( K = \partial / \partial \nu \), and the metric takes the form
\[ ds^2 = h_{ij} dx^i dx^j + 2A_i dx^i dv, \] (3.9)
where \( h_{ij} \) and \( A_i \) are independent of \( v = x^0 \). From (3.5), we can deduce that
\[ \epsilon^{\mu\nu\rho} \partial_{\nu} K_{\rho} = 2 K^{\mu}. \] (3.10)
From (3.9), the inverse metric is given by
\[ g^{00} = -|A|^2, \quad g^{0i} = A^i |A|^{-2}, \quad g^{ij} = h^{ij} - A^i A^j |A|^{-2}, \] (3.11)
where \( A^i \equiv h^{ij} A_j \) and \( |A|^2 \equiv A^i A_i \). We also have \( \sqrt{-g} = \sqrt{h} |A| \). Hence taking \( \mu = 0 \) in (3.10), and noting that \( K_0 = 0, K_i = A_i \), we find
\[ \epsilon^{ij} \partial_i A_j = -2 \sqrt{h} |A|. \] (3.12)
(Our conventions are that \( \epsilon^{012} = -1 \), and that \( \epsilon^{12} = +1 \).)

We can choose the \( x^i \) coordinates, so that \( A_i dx^i = a(x^j) du \), where \( u = x^1 \), and then we may take \( x = x^2 = a \) to be the remaining coordinate, so that (3.9) becomes
\[ ds^2 = h_{ij} dx^i dx^j + 2 x du dv. \] (3.13)
By means of a coordinate transformation \( v \rightarrow v + \xi(u, x) \), the off-diagonal component \( h_{12} \) of \( h_{ij} \) can be removed. Equation (3.12) then implies
\[ h_{22} = \frac{1}{4x}. \] (3.14)
Defining \( x = e^{2\rho}, -\infty \leq \rho \leq \infty \), we therefore find that the metric (3.9) can be cast in the form
\[ ds^2 = d\rho^2 + 2 e^{2\rho} du dv + h(u, \rho) du^2, \] (3.15)
where \( h(u, \rho) \) is an undetermined function.

Finally, we turn to the field equation. Substituting (3.15) into (2.9)–(2.11), we find that the only non-vanishing coordinate components of \( \mathcal{G}_{\mu\nu} \) and \( C_{\mu\nu} \) are given by
\[ \mathcal{G}_{11} = -\tfrac{1}{2} h'' + h', \quad C_{11} = \frac{1}{2} h''' - \frac{3}{2} h'' + h', \] (3.16)
where a prime denotes a derivative with respect to \( \rho \).

Finally, it follows from the integrability condition for the Killing spinor that (3.7) and (3.8) are satisfied. It follows that the metric (3.15), with \( h(u, \rho) \) given by (3.17), is the most general supersymmetric solution of the theory.
There is, in fact, redundancy in the solution as we have presented it above. It is easy to see that if we perform the coordinate transformations
\[
\rho \longrightarrow \rho + a_1(u), \quad v \longrightarrow v + \frac{1}{2} e^{-2\rho} \dot{a}_1(u) + a_2(u),
\]
(3.18)
where a dot denotes a derivative with respect to \(u\), then by choosing \(a_1(u)\) and \(a_2(u)\) so that
\[
\dot{a}_1 + a_1^2 + f_2 = 0, \quad a_2 + f_2 = 0,
\]
(3.19)
then the functions \(f_2\) and \(f_3\) may be set to zero.

There are two special cases that must be handled carefully, when either \(\mu = 1\) or \(\mu = -1\). It is easiest, though not essential, to examine these cases before eliminating \(f_2\) and \(f_3\) via the coordinate transformations (3.18). It is evident from (3.17) that if \(\mu\) approaches +1, then via a rescaling limit, \(f_1 = f_1/(1 - \mu)\), \(f_3 = f_3 - f_1/(1 - \mu)\), we will find \(h(u, \rho) = \rho \tilde{f}_1(u) + e^{\rho} f_2(u) + f_3(u)\).

Similarly, if \(\mu\) approaches -1, we can take a rescaling limit with \(f_1 = -f_1/(1 + \mu)\) and \(f_2 = f_2 + f_1/(1 + \mu)\), to find \(h(u, \rho) = \rho e^{\rho} \tilde{f}_1(u) + e^{\rho} f_2(u) + f_3(u)\).

One could, of course, equivalently just solve the field equation directly in the two special cases \(\mu = \pm 1\), thereby arriving at the same conclusions.

In each of the cases \(\mu = \pm 1\), we can then again use the coordinate transformations (3.18) to remove the new \(f_2\) and \(f_3\) functions. In summary, therefore, we have the most general local forms for supersymmetric solutions in the generic \(\mu^2 \neq 1\) case, and the two special cases \(\mu = \pm 1\), as follows:
\[
\mu^2 \neq 1 : \quad ds^2 = d\rho^2 + 2 e^{2\rho} du dv + e^{(1-\mu)^2} f(u) du^2,
\]
(3.20)
\[
\mu = 1 : \quad ds^2 = d\rho^2 + 2 e^{2\rho} du dv + \rho f(u) du^2,
\]
(3.21)
\[
\mu = -1 : \quad ds^2 = d\rho^2 + 2 e^{2\rho} du dv + \rho e^{2\rho} f(u) du^2.
\]
(3.22)
In all cases, therefore, the general supersymmetric solution is characterized by a single arbitrary function, \(f(u)\). In terms of the coordinate \(z = e^\rho\), used in the Graham–Fefferman discussion of boundary conditions, the \(\mu = \pm 1\) solutions have logarithmic singularities.

A supersymmetric solution of the form (3.15) and (3.17), but with \(f_1, f_2\) and \(f_3\) restricted to be constants, was found in [12]. It was later observed in [13] that these constants could be replaced by arbitrary functions of the variable, we are calling \(u\), but it was conjectured (erroneously) that the solutions would then cease to be supersymmetric. The fact that \(f_2\) and \(f_3\) can always be removed by coordinate transformations was not noted in [12, 13]. Fully nonlinear chiral pp-waves have also been obtained in [6]. They discuss three cases, which correspond to taking just one of our functions \(f_2, f_3\) or \(f_1\) to be non-zero, respectively. The first two cases are also solutions of pure Einstein gravity, and hence may be reduced to a locally AdS3 solution by a gauge transformation (as shown above). The third solution coincides with our (3.20) if \(\mu^2 \neq 1\). If \(\mu = \pm 1\), their solution corresponds to our (3.21) or (3.22).

There has been some controversy about the degrees of freedom of topologically massive gravity. Clearly, these supersymmetric solutions exhibit the same number of local degrees of freedom for all values of the Chern–Simons coupling constant \(\mu\), including the chiral values, \(\mu = \pm 1\). We shall discuss later whether or not these local solutions are globally well defined. We shall see shortly that the supersymmetric solutions that we have obtained exhaust the class of all solutions admitting a null Killing vector field. This latter condition is often taken to be the criterion for having a ‘gravitational wave’. This again points strongly to the conclusion that the local degrees of freedom are the same for all values of the coupling constant \(\mu\).
In all three cases, the Cotton tensor, \( C_{\mu\nu} \), can be written simply in terms of the Killing vector \( K \): \[
\mu^2 \neq 1: \quad C_{\mu\nu} = \frac{1}{2} \mu (1 - \mu^2) f(u) e^{-(3\mu^2)\rho} K_\mu K_\nu, \\
\mu = +1: \quad C_{\mu\nu} = f(u) e^{-4\rho} K_\mu K_\nu, \\
\mu = -1: \quad C_{\mu\nu} = f(u) e^{-2\rho} K_\mu K_\nu.
\] (3.23)

Since the vanishing of the Cotton tensor is the criterion for a three-dimensional metric to be conformally flat (and hence locally equivalent to \( \text{AdS}_3 \)), we see that the metrics we have constructed are inequivalent to \( \text{AdS}_3 \) whenever \( f(u) \) is non-zero.

### 3.1. The Killing spinor

In order to construct the Killing spinor explicitly in the supersymmetric solutions obtained above, it is useful to introduce an orthonormal frame for the metric (3.15). We shall take
\[
e^0 = e^{2\rho - \beta} \rho, \quad e^1 = e^\rho du + e^{2\rho - \beta} dv, \quad e^2 = d\rho,
\] (3.24)

where we have written \( h(u, \rho) = e^{2\beta} \). The torsion-free spin connection is then given by
\[
\omega_{01} = \dot{\beta} e^{-\beta} (e^0 - e^1) - (\beta' - 1)e^2 = -\dot{\beta} du - (\beta' - 1)d\rho, \\
\omega_{02} = (\beta' - 1)(e^0 - e^1) - e^0 = - (\beta' - 1)e^\rho du - e^{2\rho - \beta} dv, \\
\omega_{12} = -\beta' (e^0 - e^1) + e^0 = \beta' e^\rho du + e^{2\rho - \beta} dv,
\] (3.25)

where, as usual, \( \dot{\beta} = \partial\beta/\partial u \) and \( \beta' = \partial\beta/\partial \rho \). The Lorentz-covariant exterior derivative on spinors is given by
\[
D = d + \lambda \omega_{ab} \gamma^{ab}.
\] (3.27)

A convenient, real, basis for the Dirac matrices is provided by taking
\[
\gamma_0 = i m \sigma_2, \quad \gamma_1 = \sigma_1, \quad \gamma_2 = \sigma_3,
\] (3.28)

where \( \sigma_i \) are the standard Pauli matrices. From (3.25), we therefore find that the Killing spinor equation (3.2), i.e. \( D\epsilon - \frac{1}{4} \gamma_0 \epsilon \gamma^0 = 0 \), becomes
\[
d\epsilon + e^{2\rho - \beta} (\sigma_1 + im \sigma_2) \epsilon dv + \frac{1}{4} [\beta \sigma_3 - \beta' (\sigma_1 + im \sigma_2)] \sigma_0 \epsilon du + \frac{1}{4} (\beta' - 2) \sigma_3 \epsilon d\rho = 0.
\] (3.29)

This is solved by a \( v \)-independent spinor satisfying \((\sigma_1 + im \sigma_2)\epsilon = 0\), with
\[
\epsilon = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \quad \alpha = h^{-1/4} e^\rho.
\] (3.30)

Note that if the function \( f(u) \) is taken to be zero, the metric reduces to
\[
ds^2 = d\rho^2 + 2e^{2\rho} du dv = d\rho^2 + e^{2\rho}(-dt^2 + dx^2),
\] (3.31)

which is \( \text{AdS}_3 \) in the Poincaré patch. This has a supersymmetry enhancement, with two Killing spinors given, in the basis \( e^0 = e^\rho dt, e^1 = e^\rho dx, e^2 = d\rho \), by
\[
\epsilon_1 = e^{\frac{i}{2}\rho} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \epsilon_2 = e^{-\frac{i}{2}\rho} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{\frac{i}{2}\rho} (im \sigma_2 + x \sigma_1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\] (3.32)
3.2. Local degrees of freedom and general solution with null Killing vector

As indicated above, the existence of a null Killing vector field is often taken to be the defining property of a gravitational wave. It is of interest therefore to examine the more general class of solutions that we can obtain if we only make the assumption that there exists a null Killing vector, without the further property (3.5) that followed from supersymmetry. We may still, without loss of generality, choose coordinates, so that the metric takes the form (3.13), and make a further coordinate transformation of the form \( v \rightarrow v + \xi(u, x) \) to remove the off-diagonal component, \( h_{12} \). Thus, we may take the metric to be of the form

\[
d s^2 = h_1(u, x) \, du^2 + h_2(u, x) \, dx^2 + 2x \, du \, dv.
\]  

Substituting into the field equation (2.7), we first find that

\[
h_2 = \frac{1}{4x^2}.
\]  

It is convenient, as in the supersymmetric case, to introduce a new coordinate \( \rho \), related to \( x \) by \( x = e^{2\rho} \). We then find that for generic values of \( \mu \neq \pm 1 \), the solution is given by

\[
d s^2 = d\rho^2 + 2e^{2\rho} \, du \, dv + [e^{(1+\mu)\rho} f_1(u) + e^{2\rho} f_2(u) + f_3(u)] \, du^2,
\]

where the functions \( f_1(u) \), \( f_2(u) \) and \( f_3(u) \) are arbitrary. As in the supersymmetric case we discussed previously, the functions \( f_2(u) \) and \( f_3(u) \) can be eliminated by means of further coordinate transformations. Also, as for the supersymmetric solutions, special cases arise at \( \mu = \pm 1 \). After eliminating the redundant functions in all the cases, we arrive at the general solutions:

\[
\begin{align*}
\mu^2 \neq 1 : d s^2 &= d\rho^2 + 2e^{2\rho} \, du \, dv + e^{(1+\mu)\rho} f(u) \, du^2, \\
\mu^2 = 1 : d s^2 &= d\rho^2 + 2e^{2\rho} \, du \, dv + \rho f(u) \, du^2, \\
\text{ord} s^2 &= d\rho^2 + 2e^{2\rho} \, du \, dv + \rho e^{2\rho} f(u) \, du^2.
\end{align*}
\]

(The same pair of solutions arise in both of the special cases \( \mu = +1 \) and \( \mu = -1 \).) Those solutions \( (3.36)-(3.38) \) for a particular value of \( \mu \) that coincide with the corresponding ones listed in \( (3.20)-(3.22) \) are supersymmetric, whilst those that do not appear also in \( (3.20)-(3.22) \) are not supersymmetric. Thus, the solutions having just a null Killing vector are either supersymmetric as they stand, or they would be supersymmetric if the opposite sign choice for the parameter \( \mu \) in theory were taken.

It is interesting to note that all the solutions we have constructed, both the supersymmetric ones and the more general solutions \( (3.36)-(3.38) \), have the feature that they are written in the Kerr–Schild form. To see this, we observe that the null Killing vector, \( K = K_\mu \, dx^\mu \), is simply given by \( K = e^{2\rho} \, du \). Thus, the solutions can all be written in the form,

\[
g_{\mu\nu} = g_{\mu\nu}(\text{AdS}) + s(u, \rho) K_\mu K_\nu,
\]

where the function \( s(u, \rho) \) is given by

\[
s(u, \rho) = h(u, \rho) e^{-4\rho}.
\]

The metric \( g_{\mu\nu}(\text{AdS}) \) is just the AdS3 metric on the Poincaré patch, written in the form

\[
d s^2(\text{AdS}) = d\rho^2 + 2e^{2\rho} \, du \, dv.
\]
3.3. Kerr–Schild form and vanishing quantum corrections

Since a null Killing vector is necessarily geodesic, satisfying $K^\nu \nabla_\nu K_\mu = 0$, it follows that all the conditions for (3.39) to be a Kerr–Schild metric are fulfilled. In particular, this means that if one considers a ‘linearized approximation’ in which the function $h(u, \rho)$ is taken to be small (recall that it has a factor $f(u)$, where $f$ can be chosen arbitrarily), then the linear approximation is, in fact, exact.

An important consequence of the Kerr–Schild form of the metric, encapsulated in equation (3.23), is that $C_{\mu\nu}$ and the metric $g_{\mu\nu}$ are the only conserved symmetric tensors that can be constructed from polynomials in the metric and the Ricci tensor, and its covariant derivatives.

To see this, note that since we are in three dimensions, when constructing symmetric conserved tensors we need only consider the Ricci tensor and its covariant derivatives. But by the field equations this is proportional to the Cotton tensor, and so we may instead consider only the Cotton tensor and its covariant derivatives. However, whenever one takes a covariant derivative of the Cotton tensor, or of one of its covariant derivatives to arbitrary order, one adds a further multiple of the null Killing vector $K_\mu$. Thus, the contractions necessary to construct a second rank symmetric tensor from arbitrary powers of covariant derivatives of arbitrary order of the Cotton tensor must necessarily contain at least one, and in general, many factors of the vanishing quantity, $K_\mu K^\mu$.

In the language of [14], the supersymmetric solutions are therefore universal. As explained in [14], if one envisions quantum corrections to the classical equations of motion (2.7), in which the right-hand side is replaced by a symmetric conserved tensor constructed from a finite number of polynomial terms, each of which is a monomial in the Riemann tensor and its covariant derivatives, then these supersymmetric solutions will still solve the corrected equations, possibly with a shifted value for the Chern–Simons coupling constant $\mu$.

These considerations do not, of course, preclude the construction of non-zero non-polynomial invariants, as recently discussed in [15].

4. The solutions in global coordinates

The solutions we have presented are written in terms of coordinates on the Poincaré patch of AdS$_3$. In order to study the behaviour on the boundary and on the horizon, it is useful to work in a global coordinate system for AdS$_3$. To do this, we introduce embedding coordinates $X^A$, for $A = 0, 1, 2, 3$, on the hyperboloid:

$$\eta_{AB} X^A X^B = -1, \quad \eta_{AB} = \text{diag}(-1, 1, 1, -1). \quad (4.1)$$

We then introduce the global coordinates $(\tau, r, \phi)$:

$$X^0 = \sqrt{1 + r^2} \cos \tau, \quad X^3 = \sqrt{1 + r^2} \sin \tau, \quad X^1 = r \cos \phi, \quad X^2 = r \sin \phi. \quad (4.2)$$

The AdS$_3$ metric is given by

$$ds^2(\text{AdS}) = \eta_{AB} dx^A dx^B = -(1 + r^2) d\tau^2 + \frac{dr^2}{1 + r^2} + r^2 d\phi^2. \quad (4.3)$$

The relation between the coordinates $(v, u, \rho)$ of the Poincaré patch and the embedding coordinates $X^A$ is

$$\begin{align*}
X^0 - X^1 &= e^\rho, \\
X^0 + X^1 &= e^{-\rho} + 2\mu v e^\rho, \\
X^2 &= \frac{v + u}{\sqrt{2}} e^\rho, \\
X^3 &= \frac{v - u}{\sqrt{2}} e^\rho.
\end{align*} \quad (4.4)$$
Comparing (4.2) and (4.4), we obtain the coordinate transformation:

\begin{align}
v &= r \sin \phi + \sqrt{1 + r^2} \sin \tau, \\
u &= r \sin \phi - \sqrt{1 + r^2} \sin \tau, \\
e^\rho &= \sqrt{1 + r^2} \cos \tau - r \cos \phi,
\end{align}

which relates the global and Poincaré coordinates.

We now investigate whether the supersymmetric solutions admits a smooth conformal compactification. Consider the metric (3.20). This can be written as

\begin{equation}
ds^2 = \frac{r^2}{y^2} \left(-\frac{1}{1 + r^2} d\tau^2 + \frac{dr^2}{r^2(1 + r^2)} + d\phi^2 + \frac{1}{r^2} e^{(1-\mu)p} f(u) du^2\right).
\end{equation}

Let \( r = 1/y \). Then \( ds^2 = \Omega^{-2} \tilde{g}_{\mu\nu} \, dx^\mu \, dx^\nu \), with \( x^\mu = (y, \tau, \phi) \) and \( \Omega = y \). The conformal boundary is at \( y = 0 \), and \( d/Omega \neq 0 \). If \( \tilde{g}_{\mu\nu} \) has a smooth extension across the surface, \( y = 0 \), this will be true for all values of \( \tau \) and \( \phi \) except those for which \( \cos \tau = \cos \phi \). Thus, the metric certainly has a smooth conformal compactification in a single Poincaré patch. However, the metric \( \tilde{g}_{\mu\nu} \) does not have a smooth extension across the points for which \( \cos \tau = \cos \phi \). From the strong directional dependence of the metric functions \( \tilde{g}_{\mu\nu} \), it is extremely plausible that no other choice of asymptotic coordinates would yield a smooth compactification across the points where \( \cos \tau = \cos \phi \).

We shall not attempt a formal, rigorous, mathematical proof here, but merely note that from (4.5),

\begin{align}
u &= \frac{\cos p}{\sqrt{2} \sin p} + \frac{\cos q}{4\sqrt{2}r^2 \sin q \sin^2 p} + \mathcal{O}\left(\frac{1}{r^3}\right), \\
e^\rho &= -2r \sin p \sin q + \frac{\cos(p + q)}{2r} + \mathcal{O}\left(\frac{1}{r^3}\right), \\
ru^2 &= \frac{dp^2}{2 \sin^4 p} + \frac{dp(\sin p \, dq + \cos p \, \sin 2q \, dp)}{4r^2 \sin^5 p \sin^2 q} + \mathcal{O}\left(\frac{1}{r^3}\right),
\end{align}

where we have defined

\begin{equation}
\tau = p + q, \quad \phi = p - q.
\end{equation}

It seems clear that, no matter what choice we make for \( f(u) \), the metric will contain singularities when \( \sin p \) or \( \sin q \) vanishes.

5. Nester–Witten energy

In this section, we shall obtain a Nester–Witten identity for topologically massive gravity which will allow us to identify the energy and angular momentum, which appear in it and to attempt to establish a positivity property. Because the supersymmetry variations are the same as for the theory without the Chern–Simons term, one might anticipate similar difficulties to those encountered by Boulware, Deser and Stelle [18, 19] in higher-derivative four-dimensional gravity. This is, indeed, what we find. It is easy to establish the relevant Nester–Witten identity, but since the bulk term contains the Cotton tensor, whose sign is indefinite, we are unable to establish a general positive-energy property for solutions of topologically massive supergravity.

From the supercurrent \( J^\mu(\epsilon_1) = \nabla_\nu (\bar{\epsilon_1} \gamma^{\mu\nu} \psi_\rho) \), we define, via its variation

\begin{equation}
\delta_{\epsilon_2} J^\mu(\epsilon_1) = \nabla_\nu (\bar{\epsilon_1} \gamma^{\mu\nu} \delta_{\epsilon_2} \psi_\rho) = \nabla_\nu (\bar{\epsilon_1} \gamma^{\mu\nu} \bar{\nabla}_\rho \epsilon_2),
\end{equation}

where
the quantity
\[ X \equiv \int_{\Sigma} \delta \epsilon \, J^\mu (\epsilon_1) \, d\Sigma_{\mu} = \int_{\Sigma} \nabla_v (\tilde{\epsilon_1} \gamma^{\mu\nu} \tilde{\nabla}_\nu \epsilon_2) \, d\Sigma_{\mu}. \]  
(5.2)

Using Stokes’ theorem, this may be re-expressed as
\[ X = \frac{1}{2} \oint_{\partial \Sigma} (\tilde{\epsilon_1} \gamma^{\mu\nu} \tilde{\nabla}_\nu \epsilon_2) \, d\Sigma_{\mu} \nu. \]  
(5.3)

After some algebra, in which we make use of the facts that
\[ \tilde{\nabla}_\mu \epsilon = \nabla_\mu - \frac{1}{2} \gamma_\mu \epsilon, \]  
\[ \tilde{\nabla}_\mu \tilde{\epsilon} = \nabla_\mu \tilde{\epsilon} + \frac{1}{2} \gamma_\mu \epsilon, \]  
(5.4)

and (2.13), we find that expression (5.2) for \( X \) may be written as
\[ X = \int_{\Sigma} (\tilde{\nabla}_v \tilde{\epsilon_1} \gamma^{\mu\nu} \tilde{\nabla}_\nu \epsilon_2 + \frac{1}{2} \tilde{G}_\mu^{\nu} \tilde{\epsilon_1} \gamma^{\nu} \epsilon_2) \, d\Sigma_{\mu}, \]  
(5.5)

where \( \tilde{G}_{\mu\nu} \) is the Einstein tensor with cosmological term, as defined in (2.10) and (2.9).

One may argue that the energy \( E[\xi] \), where \( \xi \) is a Killing vector with spinorial square root,
\[ \xi^\nu = \tilde{\epsilon} \gamma^\nu \epsilon, \]  
(5.6)
is given by
\[ E[\xi] = -\frac{1}{4\pi G} X = \frac{1}{8\pi G} \oint_{\partial \Sigma} \tilde{\epsilon} \gamma^{\mu\nu} \tilde{\nabla}_\nu \epsilon \, d\Sigma_{\mu\nu}, \]  
(5.7)
where now we take \( \epsilon_1 = \epsilon_2 = \epsilon \). To see this, we consider a ‘deformed’ metric that satisfies the Einstein equations,
\[ \tilde{G}_{\mu\nu} = 8\pi G T_{\mu\nu}, \]  
(5.8)
and that is close to a ‘vacuum’ background metric satisfying \( \tilde{G}_{\mu\nu} = 0 \). The vacuum is assumed to admit a Killing spinor \( \epsilon \), so \( \tilde{\nabla}_\mu \epsilon = 0 \) in the background. Taking \( \epsilon_1 = \epsilon_2 = \epsilon \) in (5.5), evaluated in the deformed metric, we see that the first term is quadratically small since \( \tilde{\nabla}_\epsilon \) itself is linearly small. Thus, for a sufficiently small deformation, \( X \) in (5.5) can be taken to be given by
\[ X = 4\pi G \int_{\Sigma} T_{\mu\nu} K^\nu \, d\Sigma^\mu. \]  
(5.9)

The standard way of proceeding is now to impose on the spinor field \( \epsilon \) the Witten condition
\[ \gamma^i \tilde{\nabla}_i \epsilon = 0, \]  
where the index \( i \) indicates quantities in the surface \( \Sigma \). Subject to the Witten condition, the first term in \( X \) in (5.5) is negative semi-definite, as desired. However, the second term in (5.5) is more problematic. From (5.7) and the field equation (2.7), we have
\[ E[K] \geq \frac{1}{8\pi G} \int_{\Sigma} C_{\nu}^{\mu} \tilde{\epsilon} \gamma^\nu \, d\Sigma_{\mu}. \]  
(5.10)

However, because \( C_{\nu}^{\mu} \) is third order in derivatives of the metric, it will, in general, have no definite sign. This is clear from our plane-wave solution, whose Cotton tensor is given by (3.23). We can take the arbitrary function \( f(u) \) to have either sign. We reluctantly conclude that our Witten identity is incapable of providing an answer to the question of whether excitations around an AdS_3 background must always have positive energy.
5.1. The BTZ black hole, and its Killing spinors

Although there are difficulties in applying the Nester–Witten identity to a general solution of topologically massive gravity, it can be applied successfully to the BTZ black hole, for which the Cotton tensor vanishes. BTZ black holes are locally AdS₃, but with identifications which imply that, depending on the choice of the mass and angular momentum parameters, some or all of the supersymmetry is broken. The metric takes the form

\[
ds^2 = -U(r) \, dt^2 + \frac{dr^2}{U(r)} + r^2 \left( d\phi - \frac{J}{2r^2} \, dt \right)^2, \quad U(r) = r^2 - M + \frac{J^2}{4r^2}. \tag{5.11}
\]

When \( M = J = 0 \), this has the same form as (3.31), with \( r = e^\mu \) and \( x = \phi \). Since \( \phi \) is identified modulo \( 2\pi \), we see from (3.32) that although the Killing spinor, \( \epsilon_1 \), survives, \( \epsilon_2 \) does not. Thus, the vacuum limit of the BTZ black hole has one half of maximal supersymmetry, with just one Killing spinor, given by

\[
\epsilon = \frac{r^{1/2}}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.12}
\]

When \( M \) and \( J \) are non-zero, a straightforward calculation shows that the Killing spinor exterior derivative \( \hat{D} = D - \frac{1}{2} \gamma_0 e^\rho \) is given by

\[
\hat{D} = d + \frac{1}{2} \left[ \left( r + \frac{J}{2r} \right) \sigma_3 - U^{1/2} \, i \sigma_2 \right] (dr - d\phi) + \frac{1}{2} U^{-1/2} \left( \frac{J}{2r^2} - 1 \right) \sigma_1 \, dr. \tag{5.13}
\]

From this, we can solve locally for the Killing spinors satisfying \( \hat{D} \epsilon = 0 \), finding

\[
\epsilon = e^{-g(r)\gamma_0} e^{(\psi-1)\rho} \epsilon_0, \tag{5.14}
\]

where \( \epsilon_0 \) is an arbitrary constant spinor, and

\[
P \equiv \frac{1}{4} (M + J + 1) \sigma_3 + \frac{1}{4} (M + J - 1) i \sigma_2. \tag{5.15}
\]

The function \( g(r) \) is given by

\[
e^{4g(r)} = \frac{(e^{2\rho} + M - J)}{e^{2\rho}(e^{2\rho} + M + J)}, \tag{5.16}
\]

where we have introduced the new radial variable \( \rho \) such that \( U^{-1/2} \, dr = d\rho \):

\[
r^2 = \frac{1}{4} (e^{2\rho} + M)^2 - J^2 e^{-2\rho}. \tag{5.17}
\]

Although (5.14) gives the expected two local solutions of the Killing spinor equation for arbitrary \( M \) and \( J \) (as it must since locally the BTZ black hole is just AdS₃), we see that the global requirement that \( \phi \) be periodic (with period \( 2\pi \)) in the BTZ solution eliminates both of the Killing spinors in general. Since \( \det P = -\frac{1}{4} (M + J) \), we see that an exception arises in the extremal case,

\[
J = -M, \tag{5.18}
\]

for which (5.14) gives the single globally defined Killing spinor

\[
\epsilon = e^{-g(r)} \epsilon_0, \quad \text{with} \quad \epsilon_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.19}
\]
5.2. Nester–Witten charge of the BTZ black hole

The fact that the BTZ vacuum admits the single Killing spinor (5.12) implies that there is just one Killing vector field that has a spinorial square root, namely

\[ K = \bar{\epsilon} \gamma^\mu \epsilon \partial_\mu = \frac{\partial}{\partial t} - \frac{\partial}{\partial \phi}. \]  

(5.20)

This should be contrasted with the situation for AdS black holes in dimensions, \( D > 3 \), where any Killing vector of the vacuum can be expressed as a linear combination of Killing vectors with spinorial square roots. This difference stems from the fact that in dimensions \( D > 3 \) the black holes are asymptotic to AdS\( D \) globally, whereas in \( D = 3 \) the BTZ black hole is asymptotic to AdS\( 3 \) only locally.

The spinor field used in the Nester–Witten construction must tend asymptotically to a Killing spinor of the BTZ vacuum. Since the BTZ vacuum has just the one Killing spinor (5.12), it follows that there is only one choice of boundary conditions for spinor fields used in the Nester–Witten construction, when applied to the general BTZ black hole. Therefore, we can apply (5.7) only to the Killing field (5.20). We find that the associated Nester–Witten charge is given by

\[ E \left[ \frac{\partial}{\partial t} - \frac{\partial}{\partial \phi} \right] = M + J \geq 0. \]  

(5.21)

Note that this vanishes, as one would expect, in the extremal limit (5.18). Since the purely bosonic theory without topological term is parity preserving, one would expect not only that \( M + J \geq 0 \), but also \( M - J \geq 0 \); that is, \( M \geq |J| \). To prove this, one simply considers the supergravity theory whose supersymmetry transformations (2.6) have the opposite sign of \( m \).

6. Conclusion

In this paper, we have constructed the most general fully nonlinear solutions of topologically massive supergravity admitting a null Killing vector field. These exhaust the class of all supersymmetric solutions of the theory. We have established a Nester–Witten identity, allowing us to read off the Nester–Witten charges appearing in the superalgebra. Because the Cotton tensor appears in the bulk term of the identity, one cannot use this method to establish a general positivity property for the Nester–Witten charges, except in case such as that of the BTZ black hole, for which the Cotton tensor vanishes. Our solutions indicate that the local degrees of freedom of topologically massive supergravity are the same for all values of the Chern–Simons coupling constant \( \mu \). However, since our solutions are not globally AdS, we are unable to tackle the interesting and controversial question of their relationship to the linearized solutions used to construct representations of the isometry group \( SO(2,2) \). It would be interesting to investigate further the connection between the global supersymmetry algebra, the Nester–Witten identity, and the boundary conditions satisfied by an appropriate set of solutions.

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