Holographic positive energy theorems in three-dimensional gravity

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Abstract
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Keywords: positive energy theorems, holography, group theory, lower dimensional models

1. Introduction

The nature of energy–momentum in general relativity is more subtle than for standard field theories on Minkowski spacetime because the translations are local, rather than global, symmetries. As a consequence, the definition of energy–momentum in terms of the metric requires suitable asymptotic conditions and involves integrals over surfaces of codimension two rather than one. A natural question is then to understand under which assumptions the classical energy is bounded from below, and if it is Minkowski spacetime that minimizes the energy.

Much work has been devoted to settle these questions in four dimensions in the asymptotically flat case, both at spatial infinity for the ADM mass [1, 2], and at null infinity for the Bondi mass [3, 4]. A clear formulation of the problem and the state of the art until 1979 is described in [5]. For the ADM mass, this problem has been solved in [6], with a simplified proof given in [7] using supersymmetry-motivated arguments (see also [8, 9] for earlier work and [10–12] for refinements); the case of the Bondi mass has been treated in [13–18].
Gravity in three dimensions has turned out to be a surprisingly rich toy model for several aspects of four-dimensional general relativity. When solution space is restricted to conical defects and excesses [19, 20] or other zero-mode solutions such as the BTZ black holes [21, 22], the positivity of energy can be addressed directly from the detailed understanding of these solutions. Upon allowing for non-trivial asymptotics, however, solution space becomes infinite-dimensional and the positivity of energy is a real issue, even though there are no bulk gravitons. The purpose of this article is to point out that, in this setting, positive energy theorems in three dimensions remain explicitly tractable and can be discussed in terms of non-trivial properties of coadjoint orbits of the Virasoro group, both in the asymptotically flat and asymptotically anti-de Sitter cases.

More precisely, our reasoning is as follows.

(i) The covariant phase space of three-dimensional gravity turns out to coincide with the subspace at fixed central charges of the coadjoint representation of the relevant asymptotic symmetry group. In the asymptotically flat case, this is the centrally extended BMS$_3$ group, while in the anti-de Sitter case it is the direct product of two Virasoro groups. In both cases, the mass is expressed in terms of the energy functional on coadjoint orbits of the Virasoro group.

(ii) The complete classification of the Virasoro coadjoint orbits is a classical result [23, 24], which has been discussed in many places in the literature.

(iii) The behavior of the energy functional on each Virasoro orbit is explicitly known [25, 26]. According to a footnote, an argument in [25] aimed at understanding the Virasoro energy functional is patterned on work done in the context of the positive energy conjecture in four-dimensional gravity [27]. In turn, we show here that the analysis of the energy functional on Virasoro coadjoint orbits, as completed in [26], allows one to address positive energy theorems in three dimensions.

(iv) In the asymptotically flat case, this analysis allows us to readily isolate the orbits that have to be discarded in order for the Bondi mass to be bounded from below, leaving only solutions whose mass is greater or equal to that of Minkowski spacetime. The discussion in the asymptotically anti-de Sitter case, both for the Bondi and the ADM mass, is more subtle and relies on the details of how the two chiral sectors of the boundary theory should be combined.

In the context of a purely classical and gravitational version of holography, our approach can be given the following interpretation. The Chern–Simons formulation [28, 29] of three-dimensional gravity allows one to implement, on the level of action principles, the equivalence (in the presence of boundaries or non-trivial boundary conditions) with a two-dimensional Wess–Zumino–Witten theory [30]. When taking all gravitational boundary conditions into account, a further reduction to Liouville theory is implemented in the AdS$_3$ case [31], and to a flat limit thereof in the flat case [32]. In this context, the gravitational energy functionals that we have been studying here are the energy functionals associated with the global time translation symmetry of these two-dimensional field theories.

2. Covariant phase space of 3D flat gravity

As in the better known anti-de Sitter case, three-dimensional asymptotically flat spacetimes at null infinity are entirely determined by their symmetry structure.
2.1 BMS\(_3\) group

The relevant asymptotic symmetry group [33] is the semi-direct product

\[
\text{BMS}_3 := \text{Diff}^+(S^1) \ltimes \text{Ad} \text{Vect}(S^1).
\]

Here, Diff\(^+(S^1)\) is the group of orientation-preserving diffeomorphisms of the circle, called superrotations. It generalizes the Lorentz subgroup of the three-dimensional Poincaré group. On the other hand, Vect(S\(^1\)) is the abelian additive group of vector fields on the circle, which can be identified with densities of weight \(-1\). They are called supertranslations in this context, and generalize Poincaré translations. Superrotations, denoted by \(f\), act on supertranslations, denoted by \(\alpha\), according to the adjoint action denoted by \(\cdot\) below.

The points of the circle will be labelled using an angular coordinate \(\phi\) identified as \(\phi \sim \phi + 2\pi\). The functions on the circle may then be seen as \(2\pi\)-periodic functions of \(\phi\). Similarly, diffeomorphisms of the circle can be seen as diffeomorphisms \(f\) of \(\mathbb{S}^1\) that satisfy \(\phi_f = \phi + 2\pi\) and \(\phi_f' > 0\). This amounts to going to the universal cover of \(\text{Diff}^+(S^1)\).

Representations in terms of surface charges rely on the central extension \(\rtimes\) \(\text{Diff}^+(S^1) \ltimes \text{Ad} \text{Vect}(S^1)\). Here, the first factor is the Virasoro group, while the second one is its algebra, seen as an abelian additive group. The associated Lie algebra, denoted by \(\text{bms}_3\), is the semi-direct sum of the Virasoro algebra with its adjoint representation. Its elements are of the form \(\phi' = \phi + 2\pi \mathbb{S}^1\). The coadjoint action can then readily be worked out and reads

\[
\text{Ad}_f(\mathfrak{bms}_3) = \left( J, ic_1; p, ic_2 \right) = \left( \tilde{J}, ic_1; \tilde{p}, ic_2 \right).
\]

with

\[
\tilde{p} = Pf'f'' - \frac{c_2}{24\pi} S[f],
\]

\[
J = \left[ J + ap + 2a'p - \frac{c_2}{24\pi} \right] f''f'^2 - \frac{c_1}{24\pi} S[f],
\]

where \(S[f] = f''f' - \frac{3}{2}(f'^2)'\) is the Schwarzian derivative of \(f\).

2.2 Coadjoint representation of BMS\(_3\)

Elements \((j, ic_1; p, ic_2)\) of the coadjoint representation \(\tilde{\text{bms}}_3\) contain, besides the central charges \(c_1, c_2 \in \mathbb{R}\), the quadratic differentials \(J = J(\phi)d\phi^2\) and \(p = P(\phi)d\phi^2\). It is tempting to call them angular and linear supermomentum, respectively, because they can be interpreted as infinite–dimensional generalizations of angular and linear momentum. Their pairing with elements of \(\tilde{\text{bms}}_3\) is given by

\[
\left\langle (j, ic_1; p, ic_2), (y, -ia; t, -ib) \right\rangle = \int_0^{2\pi} d\phi \left( Jy + pt + c_1a + c_2b \right). \tag{2.2}
\]

The coadjoint action can then readily be worked out and reads

\[
\text{Ad}_f(j, ic_1; p, ic_2) = \left( \tilde{j}, ic_1; \tilde{p}, ic_2 \right), \tag{2.3}
\]

with

\[
\tilde{p} = P_{ff'} + \frac{c_2}{24\pi} S[f] = \frac{c_2}{24\pi} S[f],
\]

\[
J = \left[ J + ap + 2a'p - \frac{c_2}{24\pi} \right] f''f'^2 - \frac{c_1}{24\pi} S[f],
\]

where \(S[f] = f''f' - \frac{3}{2}(f'^2)'\) is the Schwarzian derivative of \(f\).

2.3 Energy functional

The energy associated with a coadjoint vector is conjugate to a time translation and is defined to be the zero mode of \(P(\phi)\).
Similarly, in gravity, the lowest Fourier modes $P_{\pm 1}$ are conjugate to spatial translations and encode linear momentum.

2.4 Asymptotically flat gravity

The reduced phase space of three-dimensional asymptotically flat gravity with its Dirac bracket [34, 35] can be identified with the subspace of $b\mathfrak{m}_3\Omega$ at fixed central charges

\begin{equation}
  c_1 = 0, \quad c_2 = \frac{3}{G},
\end{equation}

equipped with the Kirillov–Kostant Poisson bracket. Indeed, after a suitable gauge fixing, the general solution to Einstein’s equations describing three-dimensional asymptotically flat spacetimes at null infinity is given by metrics

\begin{equation}
  ds^2 = \Theta du^2 - 2dudu + 2 \left( \Xi + \frac{u}{2} \Theta \right) d\phi d\bar{\phi} + r^2 d\hat{\phi}^2,
\end{equation}

depending on two arbitrary functions $\Theta = \Theta(\phi), \Xi = \Xi(\phi)$. Under finite asymptotic symmetry transformations, the latter have been shown [36] to transform as in (2.3) after identifying $\Theta = (16\pi G)P, \Xi = (8\pi G)J$, and switching from a passive to an active point of view. The associated surface charges, normalized with respect to the null orbifold $\Theta = 0 = \Xi$, coincide with (2.2) up to central terms. In particular, the Bondi energy is given by (2.5).

As a consequence, in the asymptotically flat case, the gravitational solution space is classified by coadjoint orbits of $\mathcal{B}\mathcal{M}_3$. On account of the semi-direct product structure, these orbits are determined by the knowledge of the coadjoint orbits of the Virasoro group. Furthermore, it follows from (2.4) that, for questions concerning the behavior of the energy functional (2.5), the complete classification of the coadjoint orbits of $\mathcal{B}\mathcal{M}_3$ is not actually needed: the classification of those of the Virasoro group (at central charge $c_2$) is enough.

3. Energy bounds in flat gravity

3.1 Coadjoint orbits of the Virasoro group

Since the coadjoint representation of the Virasoro group is described by elements $(p, ic_2)$ transforming as in (2.4), the little group of such an element consists of diffeomorphisms $f$ such that $\tilde{P} = P$.

In the gravitational context $c_2 = 3/G > 0$, so we focus below on the case of non-vanishing, strictly positive central charge. The simplest orbits are those that admit a constant representative $P(\phi) = k$. For generic $k$, the little group is the group $U(1)$ of rigid rotations of the circle. The only exceptions are the values $k = -c_2 n^2/48\pi, n \in \mathbb{N}$, for which the little group is the $n$-fold cover $\text{PSL}^n(2, \mathbb{R})$ of $\text{PSL}(2, \mathbb{R})$. It consists of diffeomorphisms $f_\alpha$ of the form

\begin{equation}
  e^{i\alpha} = \frac{\alpha e^{i\phi} + \beta}{\beta e^{i\phi} + \bar{\alpha}}, \quad |\alpha|^2 - |\beta|^2 = 1.
\end{equation}

There are two additional families of Virasoro orbits, containing no constant coadjoint vectors.
(i) The orbits of the first family are labelled by parameters $\mu^2 > 0$ and $n \in \mathbb{N}^\circ$. Each such orbit can be understood as a tachyonic deformation of the orbit of the exceptional constant $-c_2 n^2/48\pi$, which is recovered in the limit $\mu \to 0$. The little group is isomorphic to $\mathbb{R}^*_+ \times \mathbb{Z}_n$, where $\mathbb{R}^*_+$ is the multiplicative group of strictly positive real numbers and $\mathbb{Z}_n$ acts on the circle by rigid rotations by multiples of the angle $2\pi/n$.

(ii) The orbits of the second family are characterized by $q \in \{ \pm 1 \}$ and $n \in \mathbb{N}^\circ$, with the same little groups as in the previous case. Such orbits can be understood as future- or past-directed massless deformations of the orbits of the exceptional constants.

### 3.2 Energy bounds on Virasoro coadjoint orbits

The energy on a Virasoro coadjoint orbit is given by

$$E_r[f] := \int_0^{2\pi} df \left[ \frac{c_2}{48\pi} (\theta')^2 \right],$$

with $\theta := f' f^{-1}$. The last term can be made arbitrarily large so that the energy is unbounded from above on every orbit.

Now, an important property of the Schwarzian derivative is the Schwartz inequality [37] (see also [26] for an elementary proof)

$$\int_0^{2\pi} df \left[ S[f] + \frac{1}{2} (f')^2 - 1 \right] \leq 0 \quad \forall \ f \in \text{Diff}^+(S),$$

with equality iff $f$ is the lift of a projective transformation of the circle, that is, a transformation of the form (3.1) with $n = 1$. This inequality readily implies that the energy is bounded from below on the orbit of the constant $-c_2/48\pi$, and that this constant realizes the minimum value of energy, $-c_2/24$.

More generally, for an arbitrary constant $P = k \in \mathbb{R}$,

$$E_k[f] = \left( E_r \frac{c_2}{48\pi} [f] + \frac{c_2}{24} \right) + 2\pi k + \left( k + \frac{c_2}{48\pi} \right) \int_0^{2\pi} df (f' - 1)^2,$$

so that the energy is manifestly bounded from below on the orbit of $P = k$ iff $k \geq -c_2/48\pi$, the global minimum being reached at $P = k$ itself, with minimal energy $2\pi k$.

Finally, one can show [26] that the energy is unbounded from below on all orbits containing no constant representatives, except for the massless deformation of $k = -c_2/48\pi$ with $q = -1$. In this case the lower bound of energy is again $-c_2/24$, but it is not reached on the orbit.

### 3.3 Application to flat gravity

In order to have energy bounded from below, all solutions belonging to the orbits of constant coadjoint vectors $k < -c_2/48\pi$ must be discarded. Solutions belonging to the orbits without constant representatives must be discarded as well, except for those in the orbit of the massless deformation of $k = -c_2/48\pi$ with $q = -1$. When this is done, the absolute minimum of the energy is $-c_2/24 = -1/8G$, which is reached at $P = -c_2/48\pi$. The associated metrics are given by (2.7) with $\Theta = -1$ and arbitrary $\Xi(\phi)$, and contain Minkowski spacetime for $\Xi = 0$. The set of allowed solutions includes, besides the orbit of the massless deformation of $P = -c_2/48\pi$ with $q = -1$, the orbits of all angular defects [19] and of cosmological solutions [38–41], but not those of angular excesses.
4. 3D AdS gravity and energy bounds

The covariant phase space of three-dimensional asymptotically AdS gravity (with cosmological constant $\Lambda = -\ell^{-2}$) consists of the subspace, at fixed central charges $c^\pm = \pm \ell G^{3/2} o$, of two disjoint copies $(p^\pm, ic^\pm)$ of the coadjoint representation of the Virasoro group [42–48].

The classification of Virasoro coadjoint orbits then also organizes this solution space into disjoint classes. Concretely, this analysis answers the question which asymptotically AdS$_3$ solutions are related to each other by Brown–Henneaux transformations in the bulk. For instance, the solutions belonging to the orbits without constant representatives described previously cannot be obtained from the well-known zero mode solutions by a symmetry transformation. A related discussion on the role of these solutions in the quantum theory has appeared in [49], while further details on the classical level can be found in [50].

The total energy normalized with respect to the mass of the BTZ black hole [21, 22] is given by $M = (E_{P^+} + E_{P^-})/\ell$, while the angular momentum is $J = E_{P^+} - E_{P^-}$. The requirement that the total energy $M$ be bounded from below then requires both chiral energies $E_{P^+}$ and $E_{P^-}$ to be bounded from below. This forces one to discard all solutions belonging to the orbit of a constant pair $(k^+, k^-)$ with either $k^+ < -c^+ / 48\pi$ or $k^- < -c^- / 48\pi$. Solutions belonging to orbits without constant representatives must be discarded too, except if the orbits of the massless deformations of $k^\pm = -c^\pm / 48\pi$ with $q^\pm = -1$ are involved—either both of them, or one of them together with the orbit of a constant representative above $-c^\pm / 48\pi$.

The absolute minimum of the energy is then attained at the solution $P^\pm = -c^\pm / 48\pi$ and corresponds to AdS$_3$ spacetime with $M = -(c^+ + c^-)/24\ell = -1/8G$ and $J = 0$. More generally, all allowed solutions have their mass and angular momentum constrained by $M \geq -1/8G$ and $|J| \leq (M + \ell / 8G)$. The allowed solutions include the BTZ black holes, some of the angular defect solutions (but not all of them) and also solutions with closed time-like curves and naked singularities.

Note, however, that the discussion changes when one links the two chiral sectors of the boundary theory. Suppose for the sake of argument that the sectors are linked as in global Liouville theory. The results of section 5 of [26] then imply that the only orbits with bounded energy are those of the constants $P^+ = P^- = k > 0$. In gravity this would correspond to non-rotating ($J = 0$) BTZ black holes, with the exclusion of the extremal case at $M = 0 = J$. In that setting, the orbit of the coadjoint vector $P^\pm = -c^\pm / 48\pi$, corresponding to AdS$_3$ spacetime in gravity, has to be discarded because it becomes only a local minimum.

Even though to a first approximation the dual theory for asymptotically AdS$_3$ gravity is indeed Liouville theory [31], the correct boundary theory for the black hole sector differs from Liouville theory by zero modes and holonomies [51]. This means that a more refined analysis is needed to understand how the chiral sectors should be combined.

5. Open issues

Besides properly taking into account zero modes and holonomies in the AdS$_3$ case, our approach to positive energy theorems in three-dimensional gravity should be completed in several directions.

(i) On a technical level, in the asymptotically flat case, a complete understanding of the coadjoint orbits of BMS$_3$ is necessary in order to investigate the behavior of the angular momentum functional on these orbits.
In the asymptotically flat case, the discussion above was concerned only with the Bondi mass. Since the energy is conserved and since there is no news in three dimensions, the Bondi mass is expected to be equal to the ADM mass, which for asymptotically flat spacetimes is defined at spatial infinity. In order to show this explicitly, one needs to move away from the completely gauge fixed, reduced phase space point of view that has been adopted here and connect both asymptotic regimes, as has been done in four dimensions (see e.g. [52] and references therein). Similar remarks apply to the anti-de Sitter case. Indeed, the discussion carried out above directly concerns either the ADM mass, when carried out in the Fefferman–Graham gauge, or what could be called the Bondi mass when performed in the BMS gauge, since results in both gauges are identical from a group-theoretical viewpoint [41].

It would also be interesting to connect our analysis with more standard approaches to positive energy theorems as applied to three dimensional gravity, for instance by developing arguments based on supersymmetry.

In this work, three-dimensional gravity and the behavior of its energy functional have been studied as interesting, explicitly solvable problems in their own right. As a converse to (iii), one might wonder if and how this approach could be used in four dimensions. In order to apply such direct methods, a complete control on solution space is needed. In the asymptotically flat case at null infinity, this has been achieved in the context of the characteristic initial value problem in [4, 53]. One is then naturally led to the problem of the positivity of the Bondi–Sachs energy functional, as described in detail in chapter 9.10 of [54]. Whether group theoretical considerations using either the globally well-defined or the more recent local version [35] of the BMS4 group can shed new light in this context remains to be seen.

More details on the technical aspects in three dimensions will be given elsewhere [55, 56].

Acknowledgements

This work is supported in part by the Fund for Scientific Research-FNRS (Belgium), by IISN-Belgium, and by ‘Communauté française de Belgique—Actions de Recherche Concertées’. GB acknowledges useful discussions with Marc Henneaux and Mauricio Leston.

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