A POLYHEDRAL APPROACH TO THE INVARIANT OF BIERSTONE AND MILMAN
BERND SCHÖBER

Abstract. Based on the author’s work in [Sc2] we deduce that the invariant introduced by Bierstone and Milman in order to give a proof for constructive resolution of singularities in characteristic zero can be achieved purely by considering certain polyhedra and their projections.

Contents
Introduction 1
1. Paris and Idealistic exponents 3
2. Characteristic polyhedra of pairs and idealistic exponents 7
3. Pairs and idealistic exponents with history 10
4. Setup 13
5. The case without exceptional divisors 14
6. The general case 18
7. Simplification of the strategy 24
References 27

Introduction
In his celebrated paper [H1] Hironaka proved the existence of resolution of singularities for arbitrary dimensional algebraic varieties over fields of characteristic zero. The original proof is quite complicated and consists of more than 200 very technical pages. Moreover, the result is not constructive. Nowadays there are quite accessible and constructive proofs available, which are all based on Hironaka’s work. The first results in this direction were published about 25 years ago by Bierstone and Milman [BM1], [BM2] and Villamayor [V1], [V2]. More recent approaches are for example by Bravo, Encinas and Villamayor [BEV], Cutkosky [Cu1], Encinas and Hauser [EHa], Hauser [Ha], Kollár [Ko] and W/łodarczyk [W].

In positive characteristic Abhyankar was the first to show resolution of singularities for surfaces and he also proved the case of dimension 3 if the characteristic of the base field \( k \) is not 2, 3 or 5 (\( k \) algebraically closed!). Both results have been simplified by Cutkosky in [Cu2] and [Cu3]. (For the precise references to Abhyankar’s original papers see also Cutkosky’s articles).

In the appendix of [CGO] Hironaka gave an alternative approach to the resolution of hypersurfaces of dimension 2. There he made intensive use of the characteristic polyhedron of a singularity, which he introduced in [H2]. Following his strategy Cossart, Jannsen and Saito [CJS] extended the proof to arbitrary excellent schemes of dimension at most 2. (This includes in particular the arithmetic case over \( \mathbb{Z} \)). Again the characteristic polyhedron played a crucial role. Already several years before Lipman proved resolution of singularities for 2-dimensional excellent schemes in [L]. But in contrast to [CJS] his approach is not only using blow-ups but also normalizations. Therefore it is not clear how Lipman’s proof extends to the case where the scheme is embedded into a regular ambient scheme.

In [CP3] Cossart and Piltant extend their previous work ([CP1], [CP2]) and prove the existence of a birational and global resolution in dimension 3 in the arithmetic case. Since

Supported by the Emmy Noether Programme “Arithmetic over finitely generated fields” (Deutsche Forschungsgemeinschaft) and by a Research Fellowship of the Deutsche Forschungsgemeinschaft.
their result is not given by a resolution algorithm, it is not clear that the resolution is achieved purely by blow-ups in regular centers and the problem of embedded resoluition of singularities in dimension 3 is still open.

There are programs which try to tackle the proof for arbitrary characteristic. But up to now none of them succeeded to show resolution in arbitrary dimension. By using so called alterations de Jong [dJ] was able to prove a weaker form of resolution in positive characteristic for all dimensions (where the term “birational” has to be replaced by “generically finite”).

In this article we focus on Bierstone and Milman’s approach [BM2] to resolution of singularities in characteristic zero. (See also [BM3] for the hypersurface case).

Let $X$ be a scheme of finite type over a field $k$ of characteristic zero, which is embedded into a regular scheme $Z$ (also of finite type over $k$). The strategy for the proof of resolution of singularities in characteristic zero is to define an invariant $\nu_X(x)$ for each $x \in X$, which satisfies the following properties:

1. $\nu_X(x)$ has values in a totally ordered abelian group and is upper semi-continuous.
2. If $T$ denotes the locus where the maximal value of $\nu_X(x)$ on $Z$ is attained, then there is a canonical way to deduce from $T$ the center of the next blow-up.
3. This center is regular and has at most simple normal crossings with the exceptional divisors obtained by the preceding resolution process.
4. After each such blow-up the invariant decreases strictly and after finitely many steps the singularities are resolved.

In order to define $\nu_X(x)$ some important tools are needed. The most powerful of those used in the proof is the notion of maximal contact. Roughly speaking, a regular subscheme $W$ of $Z$ has maximal contact with $X$ if its transform $W'$ after a sequence of blow-ups (with certain good centers) contains all the points of the transform of $X$, where the singularities did not improve.

In characteristic zero maximal contact locally always exists. Whereas in positive characteristic there exist examples where maximal contact does not hold, see [N] or Theorem 14.3 in [CJS].

The notion of maximal contact leads to another important tool, the so called coefficient ideal with respect to some regular subscheme $W$. This is a local construction which yields a restriction to a smaller dimensional ambient scheme so that we can then apply induction on its dimension.

The invariant used in [BM2] has the form

$$\nu_X(x) = (\nu_1, s_1; \nu_2, s_2; \ldots; \nu_t, s_t; \nu_{t+1}),$$

$\nu_i = H_{X,x}$ is the Hilbert-Samuel function of $X$, $\nu_i \in \mathbb{Q}_0 \cup \{\infty\}$, $i \geq 2$, are certain higher order multiplicities (sometimes also called residual orders) and $s_i \in \mathbb{Z}_0$ counts certain exceptional divisors. The starting point for this investigations has been the following problem, which we formulate here as

**Theorem A.** There is a purely polyhedral approach for obtaining the numbers $\nu_i \in \mathbb{Q}_0 \cup \{\infty\}$. This means we can get $\nu_i$ by only considering certain polyhedra.

For our task we need an appropriate language. In this article we use Hironaka’s theory of pairs $E$ and idealistic exponents $E_-$. Here $E = (J, b)$ denotes a pair consisting of a quasi-coherent ideal sheaf $J \subset \mathcal{O}_Z$ and a positive rational number $b \in \mathbb{Q}_+$. Two of them can be compared via an equivalence relation $\sim$ which considers their behavior under blow-ups in permissible center. The latter are regular centers contained in the locus of order at least $b$. An idealistic exponent $E_-$ denotes then an equivalence class of pairs with respect to $\sim$.

In his thesis [Sc1] the author introduced the notion of characteristic polyhedra of pairs $\Delta(E; u)$ and idealistic exponents with respect to a certain system of regular elements $(u) = (u_1, \ldots, u_e)$, see also [Sc2] for a shorter account. By using this we obtain the connection to polyhedra. Since the polyhedra are not stable under $\sim$ (Example 2.2) we investigate the information which we can get out of the polyhedra which is still intrinsic for the idealistic exponent $E_-$. More precisely, we define the number $\delta_x(\Delta(E; u))$ for a singular point $x$ which recovers the order of the coefficient ideal in this setting (Proposition 2.8).
In order to obtain a canonical resolution of singularities one needs to consider the exceptional divisors which were created under the preceding process; for example, the singular locus of $X = V(t^2 + xyz)$ consists of the curves $V(t, x, y)$, $V(t, x, z)$ and $V(t, y, z)$. Since none of them is a better center than the other, we have to blow up the origin in order to obtain a canonical resolution. After blowing up $V(t, x, y, z)$ the situation in the $X$-, the $Y$- and the $Z$-chart is the same as before.

To see the whole structure of the singularity it is reasonable to not only consider the upcoming resolution process in the equivalence relation $\sim$, but also the preceding process (or the history). This leads to the definition of pairs with history $(\mathcal{E}, \mathcal{E})$ and idealistic exponents with history $(\mathcal{E}, \mathcal{E})_{\sim \mathcal{E}}$ where $\mathcal{E}$ is a map remembering the exceptional divisors and reflecting the factorization of the exceptional components in $\mathcal{E}$. The latter part is done via certain non-negative rational numbers $(d_1, \ldots, d_l)$, which may vary under $\sim$. In order to obtain intrinsic information, we extend $\sim$ by additionally fixing these numbers and denote the resulting equivalence relation $\sim_{\mathcal{E}}$. Idealistic exponents with history are then equivalence classes with respect to $\sim_{\mathcal{E}}$. After developing this notion in his thesis [Sc1] the author recognized that it is in fact a slight variant of NC-divisorial exponents which Hironaka introduced in [H5].

For a singular point $x \in \text{Sing}(\mathcal{E})$ we define

$$\nu_x(\mathcal{E}, u; y) := \delta_x(\mathcal{E}; u) - \sum_{i=1}^l d_i,$$

which turns out to be the key ingredient for the invariants $\nu_i$ appearing in $\text{inv}_X(x)$. We have

**Theorem B.** Let $(\mathcal{E}, \mathcal{E}) = ((J, b), \mathcal{E})$ be a pair with history on some regular scheme $Z$, $x \in \text{Sing}(\mathcal{E})$, $(u, y) = (u_1, \ldots, u_d; y_1, \ldots, y_d)$ a r.s.p. such that $(y)$ is part of a generating system of the directrix of $\mathcal{E}$ at $x$. Let $\mathcal{E} := \mathcal{E}(x) := \mathcal{E}_x(\mathcal{E}, u, y)$ be some fixed exceptional data of $\mathcal{E}$ on $V(y)$ at $x$.

1. Then $\nu_x(\mathcal{E}; u; y)$ is independent of $(y)$ and invariant under $\sim_{\mathcal{E}(x)}$. Therefore we may also write

$$\nu_x(\mathcal{E}, u; y) := \nu_x(\mathcal{E}; u; y),$$

and this is an invariant of the idealistic exponent with history $(\mathcal{E}, \mathcal{E})_{\sim_{\mathcal{E}(x)}}$.

2. Further they determine the entries $\nu_i$, $i \geq 2$, of the invariant $\text{inv}_X(x)$ of Bierstone and Milman.

Since the definition of $\text{inv}_X(x)$ is quite complicated we mention a method to abbreviate its construction in the final section and further we explain how the generators behave in these steps.

Let us remark that we are considering only the situation in the local ring. The focus of this article lies on the construction of $\text{inv}_X(x)$. Hence we neither regard extensions of all these constructions to open neighborhoods of $x$ nor their gluing.

**Acknowledgement:** The results presented here are part of my thesis [Sc1]. I thank my advisors Uwe Jannsen and Vincent Cossart for countless discussions on the topic and all their support. I'm grateful to the Laboratoire de Mathématiques Versailles for their hospitality during my visit in September 2012 and my stay since October 2014.

1. Paris and Idealistic exponents

In this section we briefly introduce the language of pairs and idealistic exponents which we use throughout this paper. Their notion goes back to Hironaka [H3] (see also [H5]) and has later been refined in different ways (e.g. to basic objects [BEV], presentations [BM2], marked ideals [W], singular mobile [Ha], or idealistic filtrations [K]).

Let $Z$ be a regular irreducible scheme of finite type over $\mathbb{Z}$. 

Definition 1.1. A pair $\mathcal{E} = (J, b)$ on $Z$ is a pair consisting of a quasi-coherent ideal sheaf $J \subset \mathcal{O}_Z$ and a positive rational number $b \in \mathbb{Q}_+$. We define its order at a point $x \in Z$ (not necessarily closed) as
\[
\text{ord}_x(\mathcal{E}) = \begin{cases} 
\frac{\text{ord}_x(J)}{b}, & \text{if ord}_x(J) \geq b \text{ and} \\
0, & \text{else},
\end{cases}
\]
where $\text{ord}_x(J) = \sup \{d \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \mid J_x \subseteq M_x^d\}$ (and $M_x$ denotes the maximal ideal in the local ring at $x$). Further we define the singular locus (or support) of $\mathcal{E}$ as
\[
\text{Sing} \ (\mathcal{E}) = \{x \in Z \mid \text{ord}_x(J) \geq b\}.
\]

If $Z = \text{Spec} \ (R)$ is affine, then we also say $\mathcal{E}$ is a pair on $R$.

Definition 1.2. Let $\mathcal{E} = (J, b)$ be a pair on $Z$. A blow-up $\pi : Z' \to Z$ with center $D$ is called permissible for $\mathcal{E}$, if $D$ is regular and $D \subseteq \text{Sing} \ (\mathcal{E})$. The transform of $\mathcal{E}$ is then given by $\mathcal{E}' = (J', b)$, where $J'$ is defined via $J'O_{Z'} = J'H^b$, where $H$ denotes the ideal sheaf of the exceptional divisor.

If the blow-up with center $D$ is permissible for $\mathcal{E}$, then we also say that $D$ is a permissible center for $\mathcal{E}$.

For the sake of finding a resolution of singularities for a given pair $\mathcal{E}$ it is reasonable to try to compare it with other pairs. Thus the idea of two pairs being equivalent is that they should be treated the same way if they undergo the same resolution process. For the precise definition we need the following

Definition 1.3. We define a local sequence of regular blow-ups (short LSB) over $Z$ as a sequence of the form
\[
Z = Z_0 \supset U_0 \overset{\pi_1}{\leftarrow} Z_1 \supset U_1 \overset{\pi_2}{\leftarrow} \ldots \overset{\pi_{l-1}}{\leftarrow} Z_{l-1} \supset U_{l-1} \overset{\pi_l}{\leftarrow} Z_l
\]
(1.1)
\[
D_0 \subset D_1 \subset \ldots \subset D_{l-1}
\]
where $l \in \mathbb{Z}_+ \cup \{\infty\}$, each $U_i \subset Z_i$ is an open subscheme, $D_i \subset U_i$ is a regular closed subscheme and $\pi_{i+1} : Z_{i+1} \to U_i$ denotes the blow-up with center $D_i$, $0 \leq i \leq l-1$.

Combining this with Definition 1.2 we say that the LSB (1.1) is permissible for $\mathcal{E}$ if each blow-up $\pi_{i+1}$ is permissible for $\mathcal{E}_i$, $0 \leq i \leq l-1$.

Let $(t) = (t_1, \ldots, t_a)$ be a finite system of indeterminates. Then we use the notation
\[
Z[t] := Z \times_Z \mathbb{A}^a_Z = Z \times_{\text{Spec} (\mathbb{Z})} \text{Spec} (\mathbb{Z}[t]).
\]
We consider the pair $\mathcal{E}[t] = (J[t], b)$, where $J[t] = J'O_{Z[t]}$.

Definition 1.4. Let $\mathcal{E}_1 = (J_1, b_1)$ and $\mathcal{E}_2 = (J_2, b_2)$ be two pairs on $Z$. Then we define
\[
\mathcal{E}_1 \subset \mathcal{E}_2
\]
if the following condition holds:

- Let $(t) = (t_1, \ldots, t_a)$ be an arbitrary finite system of indeterminates and let $\mathcal{E}_i[t] = (1.2) \ (J_i[t], b_i)$, $i \in \{1, 2\}$. If any LSB over $Z[t]$ is permissible for $\mathcal{E}_i[t]$, then it is also permissible for $\mathcal{E}_2[t]$.

Further we say $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent,
\[
\mathcal{E}_1 \sim \mathcal{E}_2,
\]
if both $\mathcal{E}_1 \subset \mathcal{E}_2$ and $\mathcal{E}_1 \supset \mathcal{E}_2$. By $\mathcal{E}_1 \cap \mathcal{E}_2 \sim \mathcal{E}_3$ we mean that a LSB over $Z[t]$ is permissible for $\mathcal{E}_3[t]$ if and only if it is permissible for $\mathcal{E}_1[t]$ and $\mathcal{E}_2[t]$.

An idealistic exponent $\mathcal{E}_\sim$ is the equivalence class of a pair $\mathcal{E}$.

In other literature pairs are sometimes also called idealistic exponents, e.g. [H5]. In order to avoid confusion when coming to result and the dependence on the choice of a representant of the equivalence class, we use the original terminology of [H3].

Let us recall some of the properties of pairs.
Proposition 1.5. Let $E = (J,b)$ and $E_i = (J_i, b_i)$, $i \in \{1,2,3,4\}$, be pairs on $Z$.

1. For every $a \in \mathbb{Z}_+$ we have $(J^a, ab) \sim (J,b)$.
2. Let $m \in \mathbb{Z}_+$ with $b_1 \mid m$ and $b_2 \mid m$. Then

$$(J_1, b_1) \cap (J_2, b_2) \sim \left(J_1^{\frac{m}{b_1}} + J_2^{\frac{m}{b_2}}, m\right).$$

3. We always have $(J_1 J_2, b_1 + b_2) \supset (J_1, b_1) \cap (J_2, b_2)$. If further $\text{Sing} (J_1, b_1 + 1) = \emptyset$ for $i \in \{1, 2\}$, then the previous inclusion becomes an equivalence.
4. If $E_1 \subset E_2$ and $E_3 \subset E_4$, then $E_1 \cap E_4 \subset E_2 \cap E_4$. In particular, $E_1 \sim E_2$ implies by symmetry $E_1 \cap E_3 \sim E_2 \cap E_3$.
5. Let $\pi : Z' \to Z$ be a permissible blow-up for $E_1$ and $E_2$. Then $(E_1 \cap E_2)' \sim E_1' \cap E_2'$.
6. (Numerical Exponent Theorem; [H5], Theorem 5.1) If $E_1 \subset E_2$, then

$$\text{ord}_x(E_1) \leq \text{ord}_x(E_2)$$

for all $x \in Z$.

By symmetry $E_1 \sim E_2$ implies $\text{ord}_x(E_1) = \text{ord}_x(E_2)$ for all $x \in Z$. In particular, we get $\text{Sing}(E_1) = \text{Sing}(E_2)$ if $E_1 \sim E_2$.

7. (Diff Theorem; [H5], Theorem 3.4) Let $D$ be a left $O_Z$-submodule of $\text{Diff}_Z^m(Z)$ the (absolute) differential operators of $O_Z$ (resp. $R$) on itself of order $m \in \mathbb{N}_0$. Then

$$(J,b) \subset (DJ,b - m)$$

or equivalently $(J,b) \sim (DJ,b - m) \cap (J,b)$.

Proof. This follows by [Sc2], Lemma 1.6, Proposition 1.8, and Proposition 1.9. □

Let $E = (J,b)$ be a pair on $Z$ and $x \in \text{Sing}(E)$. Denote by $(R = O_{Z,x}, M, K = R/M)$ the local ring at $x$ and by $T_x(Z) := \text{Spec}(S)$, $S := \text{gr}_x(Z) = \bigoplus_{n \geq 0} M^n/M^{n+1}$, the tangent space of $Z$ at $x$. By abuse of notation we neglect in $E_x = (J_x, b)$ the index $x$ and write in the situation on $R$ also $E = (J,b)$.

The $b$-initial form of $f \in J$ (with respect to $M$) is defined as

$$\text{in}(f, b) := \begin{cases} f \mod M^{b+1}, & \text{if } b \in \mathbb{Z}_+, \\ 0, & \text{if } b \notin \mathbb{Z}_+. \end{cases}$$

Further we define the tangent cone $T_x(E) \subset T_x(Z)$ of $E$ at $x$ as the subspace generated by the homogeneous ideal $I_{nx}(J,b) \subset \text{gr}_x(Z)$, where

$$I_{nx}(J,b) := \text{in}_{x}(E) := \left\{ \langle J \mod M^{b+1} \rangle = \langle \text{in}(f, b) \mid f \in J \rangle, \quad \text{if } b \in \mathbb{Z}_+, \right\} \langle 0 \rangle, \quad \text{if } b \notin \mathbb{Z}_+. \right.$$ 

For two pairs $E_1 = (J_1, b_1), E_2 = (J_2, b_2)$ on $Z$ we set $I_{nx}(E_1 \cap E_2) = I_{nx}(E_1) + I_{nx}(E_2)$.

In this setting we can define the directrix and the ridge of the homogeneous ideal $I_{nx}(E)$ which go back to Hironaka and Giraud.

Definition 1.6. Let $E = (J,b)$ be a pair on $Z$. Then we define

1. the directrix of $E$ at $x$ by $\text{Dir}_{x}(E) := \text{Dir}_x(T_x(E))$ as the smallest $K$-subvectorspace $T = \bigoplus_{j=1}^n KY_j \subset S_1 = \bigoplus_{i=1}^n KU_i$ generated by elements $Y_1, \ldots, Y_r \in S_1$ (homogeneous of degree one) such that

$$(I_{nx}(E) \cap K[Y_1, \ldots, Y_r]) S = I_{nx}(E).$$

We call $I\text{Dir}(C) := \langle Y_1, \ldots, Y_r \rangle$ the ideal of the directrix.

2. The ridge (or faîte in French) $\text{Rid}_{x}(E) := \text{Rid}_x(T_x(E))$ of $E$ at $x$ is the smallest additive subspace $K[\varphi_1, \ldots, \varphi_l] \subset S$ generated by additive homogeneous polynomials $\varphi_1, \ldots, \varphi_l \in S$ such that

$$(I_{nx}(E) \cap K[\varphi_1, \ldots, \varphi_l]) S = I_{nx}(E).$$

(Recall that a polynomial $\varphi \in K[U] = S$ is called additive if for any $x, y \in K^n$ we have $\varphi(x + y) = \varphi(x) + \varphi(y)$). We call $I\text{Rid}(C) := \langle \varphi_1, \ldots, \varphi_l \rangle$ the ideal of the ridge.
Hence \( \text{Dir}_x(\mathbb{E}) \) is the minimal \( K \)-subspace such that \( \text{Ind}_x(\mathbb{E}) \) is generated by elements in \( K[Y_1, \ldots, Y_r] \). We also say \( (Y) = (Y_1, \ldots, Y_r) \) defines the directrix and we implicitly assume that \( r \) to be minimal. By abuse of notation we denote the vector space in \( \mathbb{A}^r_K = \text{Spec}(S) \) corresponding to \( \text{Dir}_x(\mathbb{E}) \) also by \( \text{Dir}_x(\mathbb{E}) \).

Similarly we say \((\varphi_1, \ldots, \varphi_l)\) defines the ridge and identify \( \text{Rid}(C) \) with the group subscheme which it defines in \( \mathbb{A}^l_K \).

The directrix and the ridge are closely related. For example the previous definitions coincide if we are in the situation over a field of characteristic zero. In general, there exists a purely inseparable finite extension \( K'/K \) such that over \( K' \) the radical of the ideal of the ridge coincides with the ideal of the directrix. For more details on the ridge (and in particular an intrinsic definition) see [G1] and [BHM].

In [Sc2] the author introduced the following idealistic interpretation of these objects.

\[ \text{Definition 1.7.} \text{ Let } \mathbb{E} = (J, b) \text{ be a pair on } Z \text{ and } x \in \text{Sing}(\mathbb{E}). \text{ Let further } \text{Ind}_x(\mathbb{E}) = \langle Y_1, \ldots, Y_r \rangle \text{ and } \text{Rid}_x(\mathbb{E}) = \langle \varphi_1, \ldots, \varphi_l \rangle \text{ for elements } Y_j \text{ homogeneous of degree one, } 1 \leq j \leq r, \text{ and additive homogeneous polynomials } \varphi_i \text{ of order } p^{d_i}, 1 \leq i \leq l. \text{ Then we define the following pairs on } T_x(Z) = \text{Spec}(g_x(Z)):\]

\[
\begin{align*}
\text{T}_x(\mathbb{E}) &= (\text{Ind}_x(\mathbb{E}), b) \quad \text{(idealistic tangent cone of } \mathbb{E} \text{ at } x), \\
\text{Dir}_x(\mathbb{E}) &= (\text{Ind}_x(\mathbb{E}), 1) \quad \text{(idealistic directrix of } \mathbb{E} \text{ at } x), \\
\text{Rid}_x(\mathbb{E}) &= \langle \varphi_i, p^{d_i} \rangle \quad \text{(idealistic ridge of } \mathbb{E} \text{ at } x). \\
\end{align*}
\]

If we have two pairs on \( Z \), say \( E_1 = (J_1, b_1) \) and \( E_2 = (J_2, b_2) \), and \( x \in \text{Sing}(E_1 \cap E_2) \), then we set

\[
\begin{align*}
\text{T}_x(E_1 \cap E_2) &= \text{T}_x(E_1) \cap \text{T}_x(E_2) = (\text{Ind}_x(E_1), b_1) \cap (\text{Ind}_x(E_2), b_2), \\
\text{Dir}_x(E_1 \cap E_2) &= \text{Dir}_x(E_1) \cap \text{Dir}_x(E_2) = (\text{Ind}_x(E_1) + \text{Ind}_x(E_2), 1), \\
\text{Rid}_x(E_1 \cap E_2) &= \text{Rid}_x(E_1) \cap \text{Rid}_x(E_2).
\end{align*}
\]

\[ \text{Proposition 1.8.} \text{ Let } \mathbb{E} = (J, b), \mathbb{E}_i = (J_i, b_i) \text{ be pairs on } Z \text{ and } x \in \text{Sing}(\mathbb{E}). \text{ Then we have}
\]

1. \( \text{Dir}_x(\mathbb{E}) \subset \text{Rid}_x(\mathbb{E}) \subset \text{T}_x(\mathbb{E}). \)
2. \( \text{Dir}_x(\mathbb{E}) = \text{Sing}(\text{Dir}_x(\mathbb{E})) \subset \text{Sing}(\text{Rid}_x(\mathbb{E})) \subset \text{Sing}(\text{T}_x(\mathbb{E})) \subset \text{T}_x(Z). \)
3. \( \text{Assume } \text{char}(K) = 0 \text{ or } b < \text{char}(K), \text{ where } K \text{ denotes the residue field of } Z \text{ at } x. \text{ Then}
\]

\[
\text{Dir}_x(\mathbb{E}) \sim \text{Rid}_x(\mathbb{E}) \sim \text{T}_x(\mathbb{E}).
\]

In particular, \( \text{Dir}_x(\mathbb{E}) = \text{Sing}(\text{Dir}_x(\mathbb{E})) = \text{Sing}(\text{Rid}_x(\mathbb{E})) = \text{Sing}(\text{T}_x(\mathbb{E})). \)

4. \( E_1 \subset E_2 \) implies \( \text{T}_x(E_1) \subset \text{T}_x(E_2), \text{ Dir}_x(E_1) \subset \text{Dir}_x(E_2), \text{ and } \text{Rid}_x(E_1) \subset \text{Rid}_x(E_2). \)

By symmetry we get equivalence \( \sim \) and equality if \( E_1 \sim E_2. \)

\[ \text{Proof.} \text{ These results are proven in [Sc2], Lemma 2.11, Corollary 2.12, and Proposition 2.14.} \]

Another important tool for the study of singularities at a point \( x \in Z \) is the coefficient ideal with respect to a closed subscheme of maximal contact. We now recall its variant in the idealistic setting. But we do not restrict our attention to characteristic zero and moreover, we define the coefficient pair with respect to any regular subvariety \( W = V(z) = V(z_1, \ldots, z_n) \) containing \( x \) such that \( (z) \) is part of a r.s.p. for the local ring \( R \) of \( Z \) at \( x \).

\[ \text{Definition 1.9.} \text{ Let } \mathbb{E} = (J, b) \text{ be a pair on } Z \text{ and } x \in Z. \text{ Let } (R = \mathcal{O}_{Z,x}, M, K) \text{ be the regular local ring of } Z \text{ at } x. \text{ We consider a fixed system of elements } (u) = (u_1, \ldots, u_d) \text{ which can be extended to a r.s.p. for } R. \text{ Let } (z) = (z_1, \ldots, z_n) \text{ be elements of } R \text{ such that } (u, z) \text{ is a r.s.p. for } R. \text{ We define the coefficient pair } D_x(\mathbb{E}, u, z) \text{ of } \mathbb{E} \text{ at } x \text{ with respect to } (z) \text{ as the pair on } W = \text{Spec}(K[[u]]) \text{ which is given by the following construction: Any } f \in J_z \text{ may be written (in the } M\text{-adic completion } \hat{R} \text{ as}
\]

\[
f = f(u, z) = \sum_{B \in \mathbb{Z}_{\geq 0}} f_B(u) z^B
\]
with \( f_B(u) \in K[[u]] \). Then we set \( \mathbb{D}(f, u, z) := \bigcap_{B \in \mathbb{Z}_{>0}^Z} (f_B(u), b - |B|) \) and define further
\[
\mathbb{D}_x(E, u, z) := \bigcap_{f \in J_x} \mathbb{D}(f, u, z) = \bigcap_{l=0}^{b-1} (I(l, u, z), b - l),
\]
where \( I(l, u, z) = \langle f_B \mid f \in J_x, B \in \mathbb{Z}_{>0}^Z : |B| = l \rangle \).

Before coming to results on coefficient pairs let us recall the concept of maximal contact. Classical references for this are [G2] and [AHV].

**Definition 1.10.** Let \( E = (J, b) \) be a pair on \( Z \) and \( x \in \text{Sing}(E) \). Let \( (z) = (z_1, \ldots, z_s) \) be a system of elements in the local ring \( R = \mathcal{O}_{Z,x} \) which can be extended to a r.s.p. for \( R \).

We say \( W := V(z) \) has maximal contact with \( E \) at \( x \) if the following equivalence holds
\[
E_x = (J_x, b) \sim (z, 1) \cap (J_x, b).
\]

**Proposition 1.11.** Let \( E = E_1 \subset E_2 \) be two pairs on \( Z \), \( x \in \text{Sing}(E) \), and \( (u, z) = (u_1, \ldots, u_d ; z_1, \ldots, z_s) \) be a r.s.p. for \( (R = \mathcal{O}_{Z,x}, M, K) \). Then we have
\[
(1) \quad \mathbb{D}_x(E_1, u, z) \subset \mathbb{D}_x(E_2, u, z).
\]

By symmetry, \( E_1 \sim E_2 \) implies \( \mathbb{D}_x(E_1, u, z) \sim \mathbb{D}_x(E_2, u, z) \).

(2) \( (z, 1) \cap E_x \sim (z, 1) \cap \mathbb{D}_x(E, u, z) \).

(3) Let \( (y) = (y_1, \ldots, y_s) \) be another system extending \( (u) \) to a r.s.p. for \( R \). Assume \( (z, 1) \cap \mathbb{D}_x \subset (y, 1) \cap \mathbb{D}_x \).

Then there exists a system \( (y) = (y_1, \ldots, y_s) \) of elements in \( \hat{R} \) such that we have
\[
\text{for each } j \in \{1, \ldots, s\}:
\]

(a) The images of \( z_j \) and \( y_j \) in \( M/M^2 \) coincide.

(b) If we set \( \langle \tilde{w}^{(j)} \rangle := (u_1, y_1, \ldots, y_j - 1, y_j + 1, \ldots, y_s) \), then
\[
E_x \sim (y_j, 1) \cap \mathbb{D}_x(E, \tilde{w}^{(j)}, y_j).
\]

In particular, \( E_x \sim (y_j, 1) \cap \mathbb{D}_x(E, u, y) \), i.e. each \( V(y_j) \) (and thus \( V(y_1, \ldots, y_s) \)) has maximal contact with \( E \) at \( x \).

(c) There exist \( D_j \in \text{Diff}_{K}^{<b-1}(K[Y]) \) and \( F(j) \in \text{Inv}_x(E) \) such that \( D_j(F(j)) = \epsilon_j Y_j \) for some units \( \epsilon_j \in R \). Further there are \( f(j) \in J \hat{R} \) which map in \( \text{gr}_x(Z) \) to \( F(j) \) and \( (P'_j(f(j)), 1) \sim (y_j, 1) \), where \( P'_j \) denotes the differential operator on \( \hat{R} \) induced by \( D_j \).

Proof. See [Sc2], Theorem 3.2, Corollary 3.3, Proposition 3.4, and Lemma 3.6. \( \square \)

2. Characteristic polyhedra of pairs and idealistic exponents

The aim of this article is to deduce the invariant of Bierstone and Milman only by considering certain polyhedra. In this section we recall the authors notion of characteristic polyhedra of pairs resp. idealistic exponents and their properties [Sc2].

Let \( E = (J, b) \) be a pair on \( Z \) and \( x \in \text{Sing}(E) \). Denote by \( (R = \mathcal{O}_{Z,x}, M, K) \) the local ring of \( Z \) at \( x \) and we write \( E = (J, b) \) instead of \( E_x = (J_x, b) \).

Fix a system \( (u) = (u_1, \ldots, u_d) \) of elements in \( M \) which can be extended to a r.s.p. for \( R \). We consider various choices of a system \( (y) = (y_1, \ldots, y_r) \) such that \( (u, y) \) is a r.s.p. for \( R \).
Let \((f) = (f_1, \ldots, f_m)\) be a set of generators of \(J\) and consider finite expansions of these
\[(2.1) \quad f_i = \sum_{(A,B) \in \mathbb{Z}_{\geq 0}^n} C_{A,B,i} u^A y^B \]
with coefficients \(C_{A,B,i} \in R^x \cup \{0\} \).

**Definition 2.1.** For the given data we define the polyhedron \(\Delta(E, u, y) = \Delta_x(E, u, y)\) of \(E = (J, b)\) at \(x\) with respect to \((u, y)\) as the smallest closed convex subset of \(R^g_{\geq 0}\) containing all elements of the set
\[
\left\{ \frac{A}{b - |B|} + R^g_{\geq 0} \bigg| 1 \leq i \leq m \wedge C_{A,B,i} \neq 0 \wedge |B| < b \right\}.
\]

Let \(E'\) be another pair on \(Z\) with \(x \in \text{Sing}(E')\). Then \(\Delta(E \cap E', u, y) \subset R^g_{\geq 0}\) denotes the smallest closed convex subset containing \(\Delta(E, u, y)\) and \(\Delta(E', u, y)\).

As it is shown in [Sc2], Example 4.9, these polyhedra are not necessarily invariant under the equivalence relation \(\sim\). But still we see later how we can get intrinsic information on the idealistic exponent \(E_\sim\) by using them. Since the mentioned example is crucial for our further investigations, we briefly recall it:

**Example 2.2.** The origin of this example is [BM4], Example 5.14, p.788 and it has been slightly modified and worked out for our setting together with Vincent Cossart.

Let \(K = \mathbb{C}, d \in \mathbb{Z}_+, d \geq 2\). We look at the origin of \(K_2^d\). Consider the two pairs
\[
\begin{align*}
E_1 &= (z^d - x^{d-1}y^{d-1}, d) \cap (t, 1) \\
E_2 &= (z^d - x^{d-1}y^{d-1}, d) \cap (t^{d-1} - x^{d-2}y^{d-1}, d - 1)
\end{align*}
\]

Then \(E_1\) and \(E_2\) are two equivalent pairs whose associated polyhedra differ!

The generating set of the polyhedron associated to \(E_1\) is \(V_1 = \{(d/d, 1)\}\) and the one for \(E_2\) is \(V_2 = \{(d-1/d, d-1); (d-2/1, 1)\}\). Clearly the polyhedra do not coincide. For the details on the equivalence \(E_1 \sim E_2\) see [Sc2], Example 4.9.

An important invariant of the singularity of \(E\) at \(x\) is the order of the coefficient pair with respect to a system \((y)\) which determines \(\text{Dir}_x(E)\). Using the following definition this can be recovered from the polyhedron \(\Delta(E; u; y)\).

**Definition 2.3.** Let \(\Delta \subset R^g_{\geq 0}\) be any subset. We define
\[
\delta(\Delta) := \inf\{ |v| = v_1 + \ldots + v_n | v = (v_1, \ldots, v_n) \in \Delta \}.
\]

If \(\Delta = \Delta(E, u, y)\), then we set \(\delta_x(\Delta(E, u, y)) := \delta(\Delta(E, u, y))\).

**Proposition 2.4.** Let \(E\) be a pair on \(Z\), \(x \in \text{Sing}(E)\), and \((u, y)\) a r.s.p. for \(R = \text{O}_{Z,x}\). Then we have

1. The polyhedron \(\Delta(E, u, y)\) associated to \(E\) is a certain projection of the corresponding Newton polyhedron \(\Delta^N(E, u, y)\), where the latter is generated by the points \((A, B) \in \mathbb{Z}_{\geq 0}^r\).
2. The polyhedron \(\Delta(E, u, y)\) is independent of the chosen set of generators \((f) = (f_1, \ldots, f_m)\).
3. The non-negative rational number \(\delta_x(\Delta(E, u, y))\) coincides with the order of the coefficient pair \(E_x(E, u, y)\).
4. Let \(E_2\) be another pair on \(Z\) which is equivalent to \(E_1 := E\).
   a. Then \(\delta_x(\Delta(E_1, u, y)) = \delta_x(\Delta(E_2, u, y))\).
   b. Let \((u, z)\) be another choice for the r.s.p. and suppose \((z, 1) \cap E \subset (y, 1) \cap E\). Then \(\delta_x(\Delta(E, u, y)) = \delta_x(\Delta(E, u, z))\).

This implies in particular that this number is independent of the choice of the maximal contact coordinates.

**Proof.** See [Sc2], Proposition 4.3, Corollary 4.4, Lemma 4.6, and Proposition 4.7. □
Now we can give the definition of the characteristic polyhedron

**Definition 2.5.** Let $E = (J, b)$ be a pair on $Z$, $x \in \text{Sing} (E)$, and let $(u) = (u_1, \ldots, u_r)$ be a system of regular elements that can be extended to a r.s.p. of $R = \mathcal{O}_{Z,x}$. We define

$$\Delta(E; u) := \Delta_x(E; u) := \bigcap_{(y)} \Delta(E; u; y),$$

where the intersection ranges over all systems $(y)$ extending $(u)$ to a r.s.p. of $R$. We call $\Delta(E; u)$ the characteristic polyhedron of the pair $E$ at $x$ with respect to $(u)$.

**Theorem 2.6 ([Sc2], Theorem 5.5).** Let $E = (J, b)$ be a pair on $Z$ and $x \in \text{Sing} (E)$. Denote by $(u, y) = (u_1, \ldots, u_r; y_1, \ldots, y_r)$ a r.s.p. for $R = \mathcal{O}_{Z,x}$ such that the initial forms of $(y)$ yield the whole directrix $\text{Dir}_x(E)$.

Then there exist elements $(y^*) = (y_1^*, \ldots, y_r^*)$ in $\hat{R}$ such that $(u, y^*)$ is a r.s.p. for $\hat{R}$, $(y^*)$ yields $\text{Dir}_x(E)$, and

$$\Delta(E; u; y^*) = \Delta(E; u).$$

In [CSc] Cossart and the author are considering the question in which case it is possible to attain the characteristic polyhedron without going to the completion.

The assumption in Theorem 2.6 that the initial forms of $(y) = (y_1, \ldots, y_r)$ yield the whole directrix $\text{Dir}_x(E)$ is crucial, see [Sc2], Example 5.6. Hence it is in general not possible to make $\Delta(E; u; y)$ (with our definitions) independent of the choice of the system $(y)$ if the assumption on the directrix does not hold. But still we can say something on $\delta(\Delta(E; u; y))$, namely we always have $\delta(\Delta(E; u; y)) = 1$. For the precise statement see the proposition below.

**Remark 2.7 (Characteristic polyhedra of idealistic exponents).** As we have seen in Example 2.2 $\Delta(E; u; y)$ (and also $\Delta(E; u)$) do not behave well under the equivalence relation $\sim$. Therefore it is not clear what the characteristic polyhedron of an idealistic exponent $E_\sim$ should be. In the proposition below we see that we always get the same intrinsic information on $E_\sim$ by considering $\Delta(E; u)$ for any representant $E$ of $E_\sim$. Thus we consider the collection of $\Delta(E; u)$, where representants $E$ of $E_\sim$ vary, as the characteristic polyhedra of the idealistic exponent $E_\sim$. For some more discussion on this see [Sc2], Remark 5.8.

Let us recall some result on the characteristic polyhedra and the information they provide.

**Proposition 2.8.** Let $E = (J, b)$ and $E_i = (J_i, b_i)$, $i \in \{1, 2\}$, be pairs on $Z$ and $x \in \text{Sing} (E)$. Let $(u, y) = (u_1, \ldots, u_d; y_1, \ldots, y_s)$ be a r.s.p. for the regular local ring $(R, M, K)$ of $Z$ at $x$.

1. Suppose $V(y)$ has maximal contact with $E$ at $x$. Then the polyhedron $\Delta(E; u; y)$ is independent of the choice of $(y)$ with this property. This means if $(z) \subset R$ is another extension of $(u)$ and $V(z)$ has maximal contact, then

$$\Delta(E; u; y) = \Delta(E; u; z).$$

2. We abbreviate the notation by $\Delta(J, b) := \Delta((J, b); u; y)$.

   (i) If $a \in \mathbb{Z}_+$, then $\Delta(J, b) = \Delta(J^a, ab)$.

   (ii) Suppose $b_1, b_2 \in \mathbb{Z}_+$ and let $m \in \mathbb{Z}_+$ with $b_1 \mid m$ and $b_2 \mid m$. Then

$$\Delta(J_1, b_1) \cap (J_2, b_2) = \Delta\left(\frac{J_1^m}{b_1} + \frac{J_2^m}{b_2}, m\right).$$

   (iii) Let $M \in \mathbb{Z}_{\geq 0}$ and $m := |M|$. Denote by $\text{Diff}_{\leq m}^K(\hat{R})$ the differential operator defined by $\mathcal{D}_M \in K \text{Diff}_{\leq m}^K(\hat{R})$ the differential operator defined by $\mathcal{D}_M (C_D (uy)^D) = \left(\frac{D}{M}\right) C_D (uy)^{D-M}$. We set $\mathcal{D}_M \log := (uy)^M \mathcal{D}_M \in \text{Diff}_{\leq m}^K(\hat{R})$. Then

$$\Delta(\Delta(J, b) \cap (\mathcal{D}_M \log J, b - m)) = \Delta(J, b).$$
We define
\[ \delta_x(E; u) := \delta_x(\Delta(E; u)) = \min\{ |v| = v_1 + \ldots + v_e \mid v \in \Delta(E; u) \} \in \frac{1}{bl} \mathbb{Z}_{\geq 0} \]
Then \( \delta_x(E; u) \) does not depend on \( (y) \) and is invariant under the equivalence relation \( \sim \). Therefore \( \delta_x(E; u) \sim \) is an invariant of the idealistic exponent \( E \sim \) and \( (u) \).

(4) Suppose that \((y, u_{e+1}, \ldots, u_d), e < d, \) yields the directrix \( \text{Dir}_x(E) \). Then we have
\[ \delta_x(\Delta(E; u_1, \ldots, u_d; y_1, \ldots, y_e)) = 1 \]

Proof. See [Sc2], Proposition 6.1, Lemma 6.2, Theorem 6.3, and Lemma 6.4. \( \square \)

Now we define the key ingredient for the invariant of Bierstone and Milman.

**Definition 2.9.** Let \( E \) be a pair on \( Z \) and \( x \in \text{Sing}(E) \). Fix a system of elements \((u_1, \ldots, u_d)\) in \( R = \mathcal{O}_{Z,x} \) which can be extended to a r.s.p. for \( R \) and let \((y) = (y_1, \ldots, y_e)\) be such an extension of \((u)\).

(1) For \( i \in \{1, \ldots, d\} \) we define
\[ d_i(E; u; y) := \min\{ |v| = (v_1, \ldots, v_d) \in \Delta(E; u; y) \} \in \frac{1}{bl} \mathbb{Z}_{\geq 0}. \]

(2) Consider a subset \( I \subseteq \{1, \ldots, d\} \). Then we define
\[ \nu_I(E; u; y) := \delta(E; u; y) - \sum_{i \in I} d_i(E; u; y) \in \frac{1}{bl} \mathbb{Z}_{\geq 0}. \]

As we see later the connection to the invariant of Bierstone and Milman is given if the subset \( I \) is related to the exceptional divisors of the preceding resolution process.

Example 2.2 shows that the non-negative rational number \( \nu_I(E, u, y) \) may change under the equivalence relation \( \sim \). Therefore it is not an invariant of the idealistic exponent! In order to take care of this we introduce in the next section idealistic exponents with history.

### 3. Pairs and idealistic exponents with history

In this we give a refinement of the equivalence relation for pairs in order to make \( \nu_I(E, u, y) \) an invariant. As the author discovered later this is a slight variant of so called NC-divisorial exponents which were introduced by Hironaka in [H5].

To begin with, let us make the following

**Observation 3.1.** Let \( E = (J, b) \) be a pair on \( Z \), \( x \in \text{Sing}(E) \) and denote by \((u, y) = (u_1, \ldots, u_d; y_1, \ldots, y_e)\) a r.s.p. for \((R = \mathcal{O}_{Z,x}, M, K)\) such that \((y, u_{e+1}, \ldots, u_d), e \leq d, \) define the directrix \( \text{Dir}_x(E) \).

(1) Suppose that \( V(y) \) has maximal contact with \( E \) at \( x \). Let \( D := V(y, u_{i(1)}, \ldots, u_{i(c)}), 1 \leq i(1) < i(2) < \ldots < i(c) \leq d \) be a permissible center for \( E \). Set
\[ \delta_D(E; u; y) := \min\{v_{i(1)} + \ldots + v_{i(c)} \mid v = (v_1, \ldots, v_d) \in \Delta(E, u, y)\} \]
Since \( D \) is permissible for \( E \), we have \( \delta_D(E; u; y) \geq 1 \). By using that \( V(y) \) has maximal contact an easy computation shows that after the blow-up with center \( D \) the exceptional component does locally factor to the power \( \delta_D(E; u; y) - 1 \geq 0 \) (for details see [Sc1], Observation 2.5.10).

Suppose the exceptional divisor is locally given by \( V(u_j) \), then
\[ d_j(E'; u'; y') = \delta_D(E; u; y) - 1, \]
where \( E' \) denotes the transform of \( E \) under the blow-up with center \( D \) and \((u', y')\) denotes a r.s.p. at a point after the blow-up with \( u'_j = u_j \).

(2) During the resolution process we have to deal with the exceptional divisors \( E = \{H_1, \ldots, H_l\} \). We require on the resolution algorithm that the divisors associated to \( E \) has at most simple normal crossing singularities, i.e. each irreducible component is regular and they intersect transversally. By this condition we may suppose that, for every \( i \in \{1, \ldots, l\}, H_i \) with \( x \in H_i \) is locally given by \( V(x_i) \), where \((x) = (x_1, \ldots, x_l) \subset M \) is part of a r.s.p. for \( R \).
The observation implies the following: If \( I \subseteq \{1, \ldots, d\} \) in Definition 2.9 is the subset determined the exceptional divisors containing the point \( x \), then the numbers \( d_i(\mathbb{E}; u; y) \) are characterized by the preceding resolution process. Hence

\[
u_I(\mathbb{E}; u; y) = \delta(\mathbb{E}; u; y) - \sum_{i \in I} d_i(\mathbb{E}; u; y) \in \frac{1}{B(I)} \mathbb{Z}_{\geq 0}
\]
is an invariant taking care not only on the upcoming resolution process, but also the preceding one.

This leads to

**Definition 3.2.** Let \( \mathbb{E} = (J, b) \) be a pair on \( Z \) and let \( E = \{H_1, \ldots, H_l\}, l \in \mathbb{Z}_+ \), be a set of irreducible divisors on \( Z \) such that the associated divisor has at most simple normal crossing singularities. As usual we denote for \( x \in \text{Sing}(\mathbb{E}) \) by \( (u, y) = (u_1, \ldots, u_d, y_1, \ldots, y_s) \) a r.s.p. for the regular local ring \( (R = \mathcal{O}_{Z,x}, M, K) \) such that \( (y, u_{e+1}, \ldots, u_d), e \leq d \), defines the directrix \( \text{Dir}_x(\mathbb{E}) \).

1. We define the *exceptional data map* of \( \mathbb{E} \) associated to \( E \)

\[ \mathcal{E} := \mathcal{E}_E : \text{Sing}(\mathbb{E}) \to (E \times \mathbb{R}_0)^l \]

by sending \( x \in \text{Sing}(\mathbb{E}) \) to the exceptional data of \( \mathbb{E} \) on \( V(y) \) at \( x \) which is induced by \( E \),

\[ \mathcal{E}(x) := \mathcal{E}_E(\mathbb{E}, u, y) := \{ (H_1, d_1), \ldots, (H_l, d_l) \}, \]

where \( d_i \) is the number associated to \( H_i \) as in Observation 3.1 if \( x \in H_i \) and \( d_i = 0 \) if \( x \notin H_i \) or \( H_i \supset V(y) \).

We call \( (\mathbb{E}, \mathcal{E}) = ((J, b), \mathcal{E}) \) a *pair with history* on \( Z \).

2. Let \( \mathbb{E}_1 \sim \mathbb{E}_2 \) be two equivalent pairs on \( Z \) and \( E \) as above. Then the induced pairs with history \( (\mathbb{E}_1, \mathcal{E}_1) \) and \( (\mathbb{E}_2, \mathcal{E}_2) \) are defined to be *equivalent at \( x \) in \( \text{Sing}(\mathbb{E}_1) = \text{Sing}(\mathbb{E}_2) \) with respect to \( (u, y) \) if

\[ \mathcal{E}_1(x) = \mathcal{E}_2(x), \quad \text{in particular the assigned numbers} \ d_j \ \text{coincide.} \]

In this case we write \( \mathbb{E}_1 \sim^{(u, y)}_{\mathcal{E}(x)} \mathbb{E}_2 \) or if there is no confusion possible we use only \( \mathbb{E}_1 \sim^{(u, y)}_{\mathcal{E}(x)} \mathbb{E}_2 \). Further we say \( (\mathbb{E}_1, \mathcal{E}_1) \) and \( (\mathbb{E}_2, \mathcal{E}_2) \) are *equivalent*, if they are equivalent at any \( x \in \text{Sing}(\mathbb{E}_1) = \text{Sing}(\mathbb{E}_2) \) and we write \( \mathbb{E}_1 \sim_{\mathcal{E}} \mathbb{E}_2 \).

An *idealistic exponent with history* \( (\mathbb{E}, \mathcal{E})_{\sim_{\mathcal{E}}} \) denotes the equivalence class of a pair with history \( (\mathbb{E}, \mathcal{E}) \) with respect to the equivalence relation \( \sim_{\mathcal{E}} \).

For applications it is sometimes important to consider at the beginning of the resolution process a pair \( \mathbb{E}_0 \) together with a set \( \mathcal{E}_0 \) of irreducible divisors which have at most simple normal crossings. Then we obtain already at this point a pair with history \( (\mathbb{E}_0, \mathcal{E}_0) \) with non-trivial exceptional data.

Since we focus on the construction of the Bierstone-Milman invariant locally at a point \( x \), it suffices for our purposes to consider the equivalence \( \sim_{\mathcal{E}(x)} \) at \( x \).

**Definition 3.3.** Let \( (\mathbb{E}, \mathcal{E} = \mathcal{E}_E) \) (with \( E = \{H_1, \ldots, H_l\} \) be an pair with history on \( Z \) as in the previous definition. A blow-up \( \pi : Z' \to Z \) with center \( D \subset Z \) is called *permissible* for \( (\mathbb{E}, \mathcal{E}) \), if the following conditions hold:

1. \( \pi \) is permissible for \( \mathbb{E} \) in the sense of Definition 1.2 (\( D \) is regular and \( D \subset \text{Sing}(\mathbb{E}) \)).
2. \( D \cup H_1 \cup \cdots \cup H_l \subset Z \) has at most simple normal crossing singularities.

The transform of \( (\mathbb{E}, \mathcal{E}) \) under a permissible blow-up \( \pi \) is given by \( (\mathbb{E}', \mathcal{E}') \), where \( \mathbb{E}' \) denotes the transform of \( \mathbb{E} \) under \( \pi \) and the exceptional data map \( \mathcal{E}' \) is defined by \( E' := \{H_1', \ldots, H_l', H_{l+1}\} \) — here \( H_i' \) is the transform of \( H_i \) under the blow-up \( \pi \) and \( H_{l+1} \) is the exceptional divisor corresponding to \( \pi \).

Analogous to before this leads to the definition of a LSB which is permissible for a given pair with history.

**Remark 3.4.** Let us have another look at Example 2.2. Suppose \( V(x) \) is exceptional. Then we get that the assigned number is \( \frac{d-1}{d} \) for \( \mathbb{E}_1 \) and it is \( \frac{d-2}{d-1} \) for \( \mathbb{E}_2 \). Hence they are not equivalent as pairs with history, because they have different exceptional data.
Therefore the Diff Theorem, as it is stated in Proposition 1.5, is in general not true for the equivalence relation \( \sim_{\mathcal{E}(x)} \). A weaker version, which is still valid for idealistic exponents with history, is given in Lemma 3.6.

**Proposition 3.5.** [Theorem B part (1)] Let \((E, \mathcal{E}) = ((J, b), \mathcal{E})\) be an pair with history on \(Z\). \(x \in \text{Sing}(E), (u, y) = (u_1, \ldots, u_l; y_1, \ldots, y_l)\) a r.s.p. for \((R = \mathcal{O}_{Z,x}, M, K)\), and let \(\mathcal{E} := \mathcal{E}(x) := \mathcal{E}(E, u, y)\) be some fixed exceptional data of \(E\) on \(V(y)\) at \(x\).

Then \(\nu_{\mathcal{E}}(E; u; y)\) is independent of \((y)\) and invariant under \(\sim_{\mathcal{E}(x)}\). Therefore we may also write

\[
\nu_{\mathcal{E}}(E, \mathcal{E}; u) := \nu_{\mathcal{E}}(E; u; y),
\]

and this is an invariant of the idealistic exponent with history \((E, \mathcal{E})\).

**Proof.** By Proposition 2.8(3) \(\delta_{\mathcal{E}}(E; u)\) is independent of \((y)\) and invariant under \(\sim\). Hence it is also invariant under \(\sim_{\mathcal{E}(x)}\). Further the exceptional data \(\mathcal{E} = \{(H_1, d_1), \ldots, (H_l, d_l)\}\) is fixed under \(\sim_{\mathcal{E}(x)}\). The definition \(\nu_{\mathcal{E}}(E; u; y) = \delta_{\mathcal{E}}(E; u) - \sum_{i=1}^l d_i\) implies the assertion. \(\square\)

Let us see which of the properties stated in section 1 for \(\sim\) survive under the refined equivalence \(\sim_{\mathcal{E}(x)}\).

**Lemma 3.6.** Let \(E = (J, b)\) be a pair on \(Z\). \(x \in \text{Sing}(E), (u, y)\) a r.s.p. for \((R = \mathcal{O}_{Z,x}, M, K)\) as before, and let

\[
\mathcal{E}(x) = \mathcal{E}_x(E, u, y) = \{(H_1, d_1), \ldots, (H_l, d_l)\}
\]

be some fixed exceptional data of \(E\) on \(V(y)\) at \(x\).

1. For every \(a \in \mathbb{Z}_+\) we have \((J^a, ab) \sim_{\mathcal{E}(x)} (J, b)\).
2. Suppose there is another choice for \((y)\), say \((z) = (z_1, \ldots, z_k)\), such that \((z, 1) \in \mathcal{E}_{\sim_{\mathcal{E}(x)}}(y, 1) \cap \mathcal{E}\). Then

\[
D_x(E, u, z) \sim_{\mathcal{E}(x)} D_x(E, u, y)
\]

(both times with the induced exceptional data on \(V(y)\) and \(V(z)\)).

3. If \(\text{char}(K) = 0\) or if \(b < \text{char}(K)\), then there exists a choice for the system \((y) = (y_1, \ldots, y_k)\) such that

\[
\mathcal{E}_x \sim_{\mathcal{E}(x)} (y, 1) \cap D_x(E, u, y)
\]

(with the induced exceptional data on \(V(y)\)).

4. Let \(D^\log_{M,u} = u^M D_{M,u} \in \text{Diff}^\log_K(K[u, y])\), \(M = (M_1, \ldots, M_d) \in \mathbb{Z}_0^d\) with \(|M| = m\), be the logarithmic differential operators given by

\[
D^\log_{M,u} (C_{A,B} u^A y^B) = \begin{pmatrix} A \\ M \end{pmatrix} C_{A,B} u^A y^B.
\]

Then

\[
(J, b) \cap (D^\log_{M,u} J, b - m) \sim_{\mathcal{E}(x)} (J, b)
\]

(with the induced exceptional data on \(V(y)\)). Moreover, if \(M_i = 0\) for all \(i \in \{1, \ldots, d\}\) with \(d_i \neq 0\) in \(\mathcal{E}(x)\), then the analogous statement is true for \(D_{M,u}\).

Let \(E_1 = (J_1, b_1), E_2 = (J_2, b_2)\) be two pairs on \(Z\) such that both are equipped with the same exceptional data \(\mathcal{E}_1(x) = \mathcal{E}_2(x) = \mathcal{E}(x)\) on \(V(y)\) at \(x\). Suppose \(x \in \text{Sing}(E_1) \cap \text{Sing}(E_2)\).

5. Assume \(b_1, b_2 \in \mathbb{Z}_+\) and let \(m \in \mathbb{Z}_+\) be a positive integer such that \(b_1 | m \) and \(b_2 | m\). Then \((J_1, b_1) \cap (J_2, b_2) \sim_{\mathcal{E}(x)} (J_1^m, b_1^m)
\]

6. \(E_1 \sim_{\mathcal{E}(x)} E_2\) implies

(a) \(E_1 \cap E \sim_{\mathcal{E}(x)} E_2 \cap E\).
(b) \(\text{ord}_z(E_1) = \text{ord}_z(E_2)\) for all \(z \in Z\). In particular, \(\text{ord}_z(E_1) = \text{ord}_z(E_2)\).
(c) \(\text{Sing}(E_1) = \text{Sing}(E_2)\).
(d) \(\mathbb{T}_x(E_1) \sim_{\mathcal{E}(x)} \mathbb{T}_x(E_2), \mathbb{D}_x(E_1) \sim_{\mathcal{E}(x)} \mathbb{D}_x(E_2)\) and \(\mathbb{R}_x(E_1) \sim_{\mathcal{E}(x)} \mathbb{R}_x(E_2)\).
(e) \(\mathbb{D}_x(E_1, u, y) \sim_{\mathcal{E}(x)} \mathbb{D}_x(E_2, u, y)\) and \((y, 1) \cap E \sim_{\mathcal{E}(x)} (y, 1) \cap \mathbb{D}_x(E, u, y)\) (both with the induced exceptional data on \(V(y)\) at \(x\)).

**Proof.** These are easy consequences of Proposition 1.5, Proposition 1.11, Proposition 2.8 and study of the behavior of the exceptional data. For more details see [Sc1], Lemma 2.6.7. \(\square\)
Further the behavior under permissible blow-ups is interesting for us.

Lemma 3.7. Let $E_1 = (J_1, b_1)$, $E_2 = (J_2, b_2)$ be two pairs on $Z$, $x \in \text{Sing}(E_1) \cap \text{Sing}(E_2)$ and $(u, y)$ a r.s.p. for $(R = \mathcal{O}_{Z, x}, M, K)$. Suppose both have the same exceptional data

$$E(x) = \{(H_1, d_1), \ldots, (H_l, d_l)\}$$

on $V(y)$ at $x$. Let $\pi : Z' \to Z$ be a blow-up which is permissible for both pairs and $x' \in \text{Sing}(E_1') \cap \text{Sing}(E_2')$ with $\pi(x') = x$. Then we have

1. $(E_1 \cap E_2)' \sim_{E(x)} E_1' \cap E_2'$.
2. $E_1 \sim_{E(x)} E_2$ implies $E_1' \sim_{E(x)} E_2'$.
3. $E_1 \sim_{E(x)} E_2$ is stable under extensions by $\mathbb{A}_k^a$ $(a \in \mathbb{Z}_+)$, i.e. stable under extensions of the r.s.p. by systems $(t) = (t_1, \ldots, t_a)$ corresponding to $\mathbb{A}_k^a$.

Proof. If $x$ is not contained in the center of the blow-up $\pi$, then the situation at $x'$ did not change. In this case the lemma is trivially true. Thus let us assume that the center contains $x$. Since the exceptional data are equal for $E_1$ and $E_2$, the same is true for $E_1'[t]$ and $E_2[t]$. Further it follows that the exceptional data for $E_1'$, $E_2'$, $(E_1 \cap E_2)'$ and $E_1' \cap E_2'$ are always given by the transform of $E(x)$. Hence the first part follows by Proposition 1.5.

The definition of $\sim$ implies remaining two parts. \qed

4. Setup

Before we come to the construction of the invariant of Bierstone and Milman, we want to recall the setup which is stated at the beginning of [BM2].

Setup 4.1. Let

- $X$ be a scheme contained in a regular scheme $Z$ of finite type over field $k$ of characteristic zero, char($k$) = 0, and $x \in X$.
- Denote by $(R = \mathcal{O}_{Z, x}, M, K)$ the regular local ring at $x$ and by $J \subset R$ the ideal defining $X$ locally at $x$.
- Let $(u, y) = (u_1, \ldots, u_r; y_1, \ldots, y_r)$ be a r.s.p. for $R$ such that the images of $(y)$ in $M/M^2 \cdot Y = (Y_1, \ldots, Y_r)$, define $\text{Dir}_x(X)$. $\text{Dir}_x(X)$ denotes the directrix associated to the tangent cone $T_x(X)$ whose defining ideal is $(\text{in}_M(y) \mid g \in J)$.
- Let $(f) = (f_1, \ldots, f_m)$ be a normalized (u)-standard base of $J$ (for the definition see for example, [Sc1], Definition 2.2.16(4)) and $b_i = \text{ord}_x(f_i)$, $1 \leq i \leq m$.
- We associate to this the pair $E$ on $R$,

$$E := (f_1, b_1) \cap \ldots (f_m, b_m).$$

Further we choose $(y)$ such that $V(y)$ has maximal contact with $E$ at $x$ and we may assume that $R$ is complete — if not, then we pass to the $M$-adic completion $\hat{R}$ of $R$.

For our purpose it is not crucial to give the precise definition of a normalized (u)-standard basis. Therefore we skip this quite technical definition and only remark that these are generators $(f) = (f_1, \ldots, f_m)$ of $J$ such that $f_i \notin (u)$, ordered by the order of $f \mod (u)$, and moreover $m$ is as small as possible.

In [Sc1], Lemma 3.1.5, the author proved that the conditions of Setup 4.1 imply the original assumptions of Bierstone and Milman [BM2]. Since this part is not essential for the construction of the invariant we do not recall all the technical notation and refer the reader to [Sc1], section 3.1.
5. The case without exceptional divisors

Now we come to the description of the procedure which Bierstone and Milman use to determine their invariant inv\(_X(x)\). For the hypersurface case see [BM3] and for the general case see [BM2].

First, we do the easier case without considering the exceptional components and after that we investigate the general case.

Let \(k, X \subset Z, x \in X, J \subset R\), \((u, y) = (u_1, \ldots, u_\ell; y_1, \ldots, y_r)\), \((f) = (f_1, \ldots, f_m)\) and \(E = (f_1, b_1) \cap \ldots \cap (f_m, b_m)\) be as in Setup 4.1.

Construction 5.1. Let the situation be as in Setup 4.1. For the moment let us forget about \((u, y)\) and consider an arbitrary r.s.p. \((w) = (w_1, \ldots, w_n)\) for \(R\). Locally at \(x\) the scheme \(X\) is given by the pair \(E\). We define

\[
G_1(x) := E = (f_1, b_1) \cap \ldots \cap (f_m, b_m).
\]

Choose \(i_0 \in \{1, \ldots, m\}\). Then \(\text{ord}_x(f_{i_0}) = b_{i_0}\) and for simplicity we write \((f, b)\) instead of \((f_{i_0}, b_{i_0})\). After a linear coordinate change we may assume that \(\frac{\partial f}{\partial w_i} \neq 0\). Set \(N_1(x) := V(z_1)\), where

\[
z_1 := \frac{\partial^{h-1} f}{\partial w_i^{h-1}}.
\]

Then by the Diff Theorem, Proposition 1.5,

\[
G_1(x) \sim G_1(x) \cap (z_1, 1)
\]

and \(N_1(x)\) has maximal contact with \(G_1(x)\) at \(x\). After another coordinate change we may suppose that \(w_n = z_1\).

In the next step we consider the situation on \(N_1(x)\), where we define the pair \(H_1(x)\) (on \(N_1(x)\)) by

\[
H_1(x) := \bigcap_{i=1}^m \bigcap_{l=1}^{b_i-1} \left( \frac{\partial f_i}{\partial w_i} \big|_{V(z_i)} , b_i - l \right).
\]

This is the coefficient pair of \(G_1(x)\) with respect to \((z_1)\) (Definition 1.9). Then set

\[
\mu_2 := \mu_2(x) := \text{ord}_x(H_1(x)).
\]

and in the case without looking at exceptional components \(\nu_2 := \nu_2(x) := \mu_2(x)\). Further we define

\[
G_2(x) := \bigcap_{j=1}^{m^{(2)}} (g_j, b_j) := \bigcap_{(h,b_n) \supset H_1(x)} (h, b_n \nu_2).
\]

This completes the first step of the process (without exceptional divisors). Then we start again with \(G_2(x)\) instead of \(G_1(x)\). By construction there exists \(j_0 \in \{1, \ldots, m^{(2)}\}\) such that \(\text{ord}_x(g_{j_0}) = b_{j_0}\).

Remark 5.2. If we start with \((u, y)\) and not an arbitrary r.s.p. \((w)\), then we can make our choices such that after \(r\) steps in the process \(z_j = y_j\) for all \(1 \leq j \leq r\). By construction \(H_r(x) = \mathbb{D}_x(E, u, y)\) and thus \(\nu_{r+1} = \mu_{r+1} = \delta_x(E, u)\).

Since \(\mu_{s+1}(x) = \text{ord}_x(H_s(x)) = \delta_x(G_1(x) ; w_1, \ldots, w_{n-s} ; z_1, \ldots, z_s)\), we also get \(\mu_{s+1}(x) = 1 = \delta_x(E; w_1, \ldots, w_{n-s})\) if \(s < r\) or equivalently if \(d := n - s > e\) (Proposition 2.8).

After \(r\) steps we start over with \(G_{r+1}(x)\) instead of \(G_1(x)\). We determine the directrix \(\text{Dir}_x(G_{r+1}(x))\), distinguish \((u) = (u_1, \ldots, u_\ell)\) into

\[
(u) = (u_1, \ldots, u_\ell(x) ; y_{r+1}, \ldots, y_r(x)),
\]

and so on.

This leads to
Observation 5.3. Let the situation be as in Setup 4.1. As in Construction 5.1 set $G_t(x) = (f_1, b_1) \cap \ldots \cap (f_m, b_m)$. In the case without exceptional components the invariant of Bierstone and Milman has the following form
\[
\text{inv}_X(x) = (\nu_1, s_1; s_2, s_3; \ldots) = (\nu_1, 0; 1, 0; \ldots; 1, 0; \nu_{r(1)}+1, 0; 1, 0; \ldots; 1, 0; \nu_{r(2)}+1, 0; 1, 0; \ldots),
\]
where $r^{(1)} := r$ and $r^{(q)}$, $q \geq 3$, is defined in the analogous way as $r^{(2)}$ in the previous remark. Set $r^{(0)} := 0$. Then we have $0 = r^{(0)} < r^{(1)} < r^{(2)} < \ldots \leq n$ and, for all $q \geq 1$, $(Y_{r^{(q)}}, \ldots, Y_{r^{(q)}})$ yields $\text{Dir}_x(G_{r^{(q)}+1}(x))$ and with $e^{(q)} := n - r^{(q)}$ we get
\[
\nu_{r^{(q)}+1} = \delta_x(G_{r^{(q)}+1}(x); u_1, \ldots, u_{e^{(q)})} > 1.
\]
Recall that we have already shown that $\delta_x(G_{r^{(q)}+1}(x); u_1, \ldots, u_{e^{(q)})}$ is coming from some polyhedra and does neither depend on the choice of the representative for $G_{r^{(q)}+1}(x)$ as idealistic exponent nor on the choice of $(y)$ (Proposition 2.8).

Putting everything together we get the following proposition, which implies Main Theorem A in the special case without exceptional divisors.

Proposition 5.4. Let the data be as in Setup 4.1 and use the notation of Observation 5.3. Let $J_{r^{(q)}+1} \subseteq K[[u_1, \ldots, u_{e^{(q)}}]]$ be the ideal corresponding to $G_{r^{(q)}+1}(x)$, $q \geq 1$. Let $(g) = (g_1, \ldots, g_l)$ $(l \in \mathbb{Z}_+)$ denote the generators of $J_{r^{(q)}+1}$ which we get from $(f) = (f_1, \ldots, f_m)$ via Construction 5.1. Set $d_i := \text{ord}_x(g_i)$ for $1 \leq i \leq l$.

For every $i \in \{1, \ldots, l\}$, $g_i$ has an expansion of the form
\[
(5.1) \quad g_i = g_i(u^{(q)}, y^{(q)}) = G_i(u^{(q)}) + \sum_{|B| < d_i} G_{B,i}(u^{(q)}) \cdot (y^{(q)})^B + g_i^{(1)}(u^{(q)}, y^{(q)}),
\]
where $(u^{(q)}, y^{(q)}) = (u_1, \ldots, u_{e^{(q)}}; y_{r^{(q)}}, \ldots, y_{r^{(q)}})$, $B \in \mathbb{Z}^{(q)-r^{(q)-1}}$ and with certain elements $g_i^{(1)}(u, y) \in \mathbb{Z}^{(q)-d_i}$, $r^{(q)}$
\[
(1) \quad G_i(u^{(q)}) \in K[y^{(q)}] \text{ is a polynomial homogeneous of degree } d_i \text{ and }
\]
\[
(2) \quad G_{B,i}(u^{(q)}) \in K[[u^{(q)}]] \text{ has order } \text{ord}_x(G_{B,i}) > d_i - |B| \text{ at } x.
\]
Further we have the properties (always $1 \leq i \leq l$ and $B := B(i) \in \mathbb{Z}^{(q)-r^{(q)-1}}$)
\[
(3) \quad H_{r^{(q)}+1}(x) = \{ (G_{B,i}(u^{(q)}), d_i - |B|) \mid i, B : |B| < d_i \},
\]
\[
G_{r^{(q)}+1}(x) = \{ (G_{B,i}(u^{(q)}), (d_i - |B|) \cdot \delta^{(q)}) \mid i, B : |B| < d_i \},
\]
\[
\nu_{r^{(q)}+1} = \min \{ \text{ord}_x(G_{B,i}) \mid d_i - |B| \} = \delta^{(q)} > 1,
\]
where $\delta^{(q)} := \delta_x(G_{r^{(q)}+1}(x); u^{(q)}) = \delta(\Delta_x(G_{r^{(q)}+1}(x); u^{(q)}))$.

(4) The polyhedron $\Delta_x(G_{r^{(q)}+1}(x); u^{(q+1)}) = \Delta_x(G_{r^{(q+1)}+1}(x); u_1, \ldots, u_{e^{(q+1)}})$ is a projection of $\Delta_x(G_{r^{(q)}+1}(x); u^{(q)} = \Delta_x(G_{r^{(q)}+1}(x); u_1, \ldots, u_{e^{(q)}})$.

Let $s \in \mathbb{Z}_+$ with $r^{(q-1)} < s < r^{(q)}$. We set $(u^{(q,s)}) := (u^{(q)}, y^{(q+1)}, \ldots, y_{r^{(q)})}$ and $(y^{(q,s)}) := (y_{r^{(q-1)+1}}, \ldots, y_{r^{(q)})}$. Then the statements analogous to (5.1) and (1)–(4) are true for $(u^{(q,s)}, y^{(q,s)})$ instead of $(u^{(q)}, y^{(q)})$. The only modification, which we have to do, is in (2): $\text{ord}_x(G_{B,i}(u^{(q,s)})) \geq d_i - |B|$ and there exist at least one $1 \leq i \leq l$ and $B := B(i) \in \mathbb{Z}^{(q-s)-r^{(q-1)}}$ such that equality holds. By Proposition 2.8 we have $\delta^{(q,s)} := \delta_x(G_{r^{(q-s)+1}(x); y^{(q,s)})) = 1$.

For $q = 1$ we set $J_{r^{(q-1)+1}} = J_1 := J \subseteq K[[u_1, \ldots, u_{e^{(0)}}]] = R$. (Recall that $e^{(0)} = n$ and we put $(u_1, \ldots, u_n) := (w_1, \ldots, w_n)$).

Proof. Assertion (1), (2) and $\delta^{(q)} > 1$ follow since $(Y^{(q)}) = (Y_{r^{(q-1)+1}}, \ldots, Y_{r^{(q)})}$ yields $\text{Dir}_x(G_{r^{(q-1)+1}})$(x).

Part (3) is a consequence of the definition of the coefficient pair (Definition 1.9) and the construction of $H_{r^{(q)}}, G_{r^{(q)}+1}(x)$ and $\nu_{r^{(q)}+1}$ (Construction 5.1).

Since $V(y^{(q)})$ has maximal contact, the polyhedron
\[
\Delta_x(G_{r^{(q)}+1}(x); u^{(q+1)}; Y_{r^{(q-1)+1}}, \ldots, y_{r^{(q)})}) = \Delta_x(G_{r^{(q)}+1}(x); u^{(q+1)})
\]
is minimal (Proposition 2.8) and this implies (4).

The proof of the last part with \( s \in \mathbb{Z}_+ \), \( r^{(q-1)} < s < r^{(q)} \), is clear. \( \square \)

**Remark 5.5.** A similar description of \( \mathcal{H}_r(x) \) as above (with \( \nu_1(x) = H_{X,x} \) the Hilbert-Samuel function of \( X \)) has already been proven in [BM2], see loc. cit. Construction 4.18, and Theorem 9.4. Further they show how to get their invariant in the case without exceptional components by using “weighted initial exponents” and the “weighted diagram of initial exponents”, see loc. cit. Remark 3.25. But note that they do not give a polyhedral approach in the general case, where exceptional divisors also have to be considered.

So far we have to determine the generators \((g)\) of the ideal \( J_{\nu_{c(1)}+1} \) step-by-step and apply the previous proposition. By introducing weights on the r.s.p. \((u,y)\) we are (at least in this special case) able to extend this result such that we get similar statements only with the use of the generators \((f)\) of \( J \). Polyhedra of pairs in the quasi-homogeneous setting are discussed in [Sc2], Remark 5.9.

**Remark 5.6.** Let the situation be as in Setup 4.1 and as in Construction 5.1 let \( G_1(x) = (f_1, b_1) \cap \ldots \cap (f_m, b_m) \). Recall that, if we do not consider exceptional components, then we have

\[
\text{inv}_X(x) = (\nu_1, 0; 1, 0; \ldots ; 1, 0; \nu_{c(1)}+1, 0; 1, 0; \ldots ; 1, 0; \nu_{c(2)}+1, 0; 1, 0; \ldots)
\]

and with the notation of Proposition 5.4 \( \nu_{c(1)}=\delta^{(1)} \geq 1 \). At the beginning we separated the r.s.p. of the regular local ring \( R \) into \((u,y)\), where the initial forms of \((y) = (y_1,\ldots,y_r)\) build a minimal generating set for the ideal of \( \text{Dir}_x(G_1(x)) \). The latter is the directrix associated to the homogeneous ideal

\[
I^{(0)} := \langle \text{in}_{L^{(0)}}(f_i, b_i) \mid 1 \leq i \leq m \rangle.
\]

Let \( L^{(0)} := L_0 \subset \mathbb{L}_+ \) be the positive linear form on \( \mathbb{R}^e \) which is given by \( L^{(0)}(v) = |v| = v_1 + \ldots + v_e \) for \( v = (v_1,\ldots,v_e) \in \mathbb{R}^e \). We associate to such a linear form the valuation \( v_{L^{(0)}} \) on \( R \), where

\[
v_{L^{(0)}}(g) := \sup \{ L^{(0)}(A) + |B| \mid g \in u^A y^B R \}
\]

for \( g \in R \).

Up to now the images of \((u,y)\) in the graded ring \( \text{gr}_M(R) \) were equipped with the standard grading. So the values are determined by the valuation \( v^{(0)} \) on \( R \) with

\[
v^{(0)}(y_j) = v^{(0)}(u_i) = 1 \quad \text{and} \quad v^{(0)}(\lambda) = 0
\]

for \( j \in \{1,\ldots,r\} \), \( i \in \{1,\ldots,e\} \) and \( \lambda \in R^\times \). Recall that \( r^{(1)} = r \) and \( e^{(1)} = e \). We define the valuation \( v^{(1)} \) on \( R \) which assigns weights to the \((u,y)\) as follows

\[
v^{(1)}(y_j) = 1, \quad v^{(1)}(u_i) = \frac{1}{\delta^{(1)}} \quad \text{and} \quad v^{(1)}(\lambda) = 0
\]

for \( j \in \{1,\ldots,r^{(1)}\} \), \( i \in \{1,\ldots,e^{(1)}\} \) and \( \lambda \in R^\times \).

Let \( L^{(1)} \subset \mathbb{L}_+ \) be the positive linear form on \( \mathbb{R}^{e^{(1)}} \) given by \( L^{(1)}(v_1,\ldots,v_e) = |v|/\delta^{(1)} \) for \( v \in \mathbb{R}^{e^{(1)}} \). Then \( v^{(1)}(u^A y^B) = c \), for some \( c \in \mathbb{Z}_+ \), if and only if

\[
L^{(1)}(A) + |B| = \frac{|A|}{\delta^{(1)}} + |B| = c.
\]

(\*)

The last condition is equivalent to \( \frac{|A|}{c - |B|} = \delta^{(1)} \) if \( c - |B| \neq 0 \) . Therefore we get together with \( v^{(1)}(f_i) = b_i \) that

\[
in_{Q(1)}(f_i, b_i)_{u,y} = \text{in}(f_i, b_i)_{u,y} + \sum_{(A,B)} C_{A,B} U^A Y^B =: \text{in}(f_i, L^{(1)})_{u,y},
\]

where the sum ranges over those \((A,B) \in \mathbb{Z}^{e+r} \) which fulfill (\*). Consider the quasi-homogeneous ideal \( I^{(1)} \) in the graded ring associated to \( v^{(1)} \),

\[
I^{(1)} := \langle \text{in}(f_i, L^{(1)})_{u,y} \mid 1 \leq i \leq m \rangle.
\]

The directrix \( \text{Dir}_x(I^{(1)}) \) corresponding to \( I^{(1)} \) is defined in the same way as for a homogeneous ideal; we only have to be careful with the grading. Modify \((y_1,\ldots,y_{r^{(2)}})\) such that their
initial forms with respect to \( L^{(1)} \) define \( \text{Dir}_v(I^{(1)}) \). In the same way as we determine at the beginning (\( y^{(1)} = (y_1, \ldots, y_{r^{(1)}}) \) \( r^{(1)} = r \)), we can compute now (\( y^{(2)} = (y_{r^{(1)}+1}, \ldots, y_{r^{(2)}}) \), \( r^{(2)} > r^{(1)} \)). Note that \( v_{L^{(1)}}(y_j) = \frac{1}{\delta^{(1)}} \) for all elements in \( (y^{(2)}) \).

Let \( M^{(1)} \subseteq \mathbb{L}_+ \) be the positive linear form on \( \mathbb{R}^{(2)} \) defined by

\[
M^{(1)}(v, w) = M^{(1)}(v_1, \ldots, v_r, w_1, \ldots, v_{r-r^{(1)}}) = |v| + \frac{|w|}{\delta^{(1)}}
\]

for \( (v, w) \in \mathbb{R}^{(2)} \). We expand \( f_i \) with respect to \( (u^{(2)}; y^{(1)}, y^{(2)}), i \in \{1, \ldots, m\} \), as in (5.1)

\[
f_i = f_i(u^{(2)}; y^{(12)}) = F_i(y^{(12)}) + \sum_{M^{(1)}(B) < b_i} F_{B,i}(u^{(2)}) \cdot (y^{(12)})^B + f^*_i(u^{(2)}; y^{(12)}),
\]

where we write \( (y^{(1)}, y^{(2)}) \) for \( (y^{(1)}, y^{(2)}) \) and with some \( f^*_i(u^{(2)}; y^{(12)}) \in \langle y^{(12)} \rangle^{b_i+1} \). Further the following properties hold

(1) \( F_i(y^{(12)}) \subseteq K[y^{(12)}] \) is a polynomial, quasi-homogeneous of degree \( b_i \) (with respect to \( v_{L^{(1)}} \)).

(2) \( F_{B,i}(u^{(2)}) \subseteq K[[u^{(2)}]] \) and \( v_{L^{(1)}}(F_{B,i}) > b_i - M^{(1)}(B) \).

(3) \( \mathcal{H}_{r^{(2)}-1}(x) \) \( \{ (F_{B,i}(u^{(2)})) \in K[[u^{(2)}]], b_i - M^{(1)}(B) \} \cdot \delta(2) \) \( i, B : M^{(1)}(B) < b_i \}, \)

\[
\delta^{(2)} := \delta_x(\mathcal{G}_1(x), u^{(2)}, y^{(12)}) = \delta(\mathcal{G}_1(x), u^{(2)}, y^{(12)}),
\]

Here \( \Delta_x^{(1)}(\mathcal{G}_1(x), u^{(2)}, y^{(12)}) \) denotes polyhedron in the quasi-homogeneous setting induced by \( v_{L^{(1)}} \).

One can show that the polyhedron \( \Delta_x^{(1)}(\mathcal{G}_1(x), u^{(2)}, y^{(12)}) \) is a certain projection of the characteristic polyhedron \( \Delta^{(1)}(\mathcal{G}_1(x), u^{(1)}, y^{(1)}) = \Delta_x(\mathcal{G}_1(x), u^{(1)}) \) (and thus also one of the Newton polyhedron \( \Delta^{N}(\mathcal{G}_1(x), (u^{(2)}, y^{(12)})) \)).

Further, one can prove

\[
\Delta_x^{(1)}(\mathcal{G}_1(x), u^{(2)}, y^{(12)}) = \Delta_x(\mathcal{G}_{r^{(1)}+1}(x), u^{(2)}, y^{(2)}).
\]

In particular this implies that \( \Delta_x^{(1)}(\mathcal{G}_1(x), u^{(2)}, y^{(12)}) \) is minimal with respect to the choices for \( (y^{(12)}) \).

Note that \( \mathcal{H}_{r^{(2)}-1}(x) \) is not the coefficient pair of \( \mathcal{G}_1(x) \) with respect to \( (y^{(12)}) \) (Definition 1.9), because in the definition of the latter we do not take care of the non-standard valuation \( v^{(1)} \). Of course, it is easy to extend the definition to this more general case. (But then the notation is getting more complicated . . . )

Then we go on and define the new valuation \( v^{(2)} \) on \( R \) by

\[
v^{(2)}(y_j) = 1, \quad \text{if } j \in \{1, \ldots, r^{(1)}\},
\]

\[
v^{(2)}(y_j) = \frac{1}{\delta^{(1)}}, \quad \text{if } j \in \{r^{(1)}+1, \ldots, r^{(2)}\},
\]

\[
v^{(2)}(u_i) = \frac{1}{\delta^{(1)} \delta^{(2)}}, \quad \text{for } i \in \{1, \ldots, e^{(2)}\},
\]

\[
v^{(2)}(\lambda) = 0, \quad \text{for } \lambda \in K.
\]

Let \( I^{(2)} \) be the quasi-homogeneous ideal (in the graded ring associated to \( v^{(2)} \)) given by the initial forms of \( (1, \ldots, f_m) \) with respect to \( v^{(2)} \). Via its directrix we distinguish \( (u^{(2)}) = (u^{(3)}, y^{(3)}), r^{(3)} > r^{(2)} \). The further procedure and the resulting statements are now clear.

Thus we obtain a new version of Proposition 5.4, where we only use the generators \( f \) of \( J \). We achieve the result for \( s \in \mathbb{Z}_+ \) with \( r^{(q-1)} < s < r^{(q)} \) in the same way as in the first version.
6. The general case

In this section we consider the construction of the invariant introduced by Bierstone and Milman in the general case, where exceptional components are involved.

Let \( X \) be a scheme embedded in some regular scheme \( Z \) of finite type over a field \( k \), \( \text{char}(k) = 0 \). In the arbitrary case we have to consider the exceptional components, which arose during the preceding resolution of \( X \). Suppose we are in the year \( j \). Then we have a sequence

\[
\emptyset = E_0 \quad E_1 \quad \ldots \quad E_i \quad \ldots \quad E_{j-1} \quad E_j
\]

\[
Z = Z_0 \xleftarrow{\pi_1} Z_1 \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_i} Z_i \xleftarrow{\pi_{i+1}} \ldots \xleftarrow{\pi_{j-1}} Z_{j-1} \xleftarrow{\pi_j} Z_j
\]

(6.1)

\[
X = X_0 \xleftarrow{\pi} X_1 \xleftarrow{\ldots} X_i \xleftarrow{\ldots} X_{j-1} \xleftarrow{\ldots} X_j
\]

where each \( \pi_{i+1} : Z_{i+1} \to Z_i \) is a blow-up in a regular center which is contained in the singular locus of \( X_i \) and has at most simple normal crossings with \( E_i \) and \( E_{i+1} \). (The last line is needed later).

\[ \text{As above} \]

\[ \text{we denote by} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{where each} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we know inv} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]

\[ \text{We denote by} \]

\[ \text{Hence} \]

\[ \text{Further we denote by} \]

\[ \text{construction 6.1} \]

\[ \text{Suppose we are in the year} \]

\[ \text{Then we define} \]

\[ \text{We define} \]

\[ \text{Then we set} \]

\[ \text{We want to determine} \]
The exceptional components in $E^k(x)$ are odd, because they all arose before or in the year $i_k$. The set $E_k(x)$ consists of new or young exceptional components which occurred after the year $i_k$, where the value of the truncated invariant appeared for the first time. The sets $E^k(x)$ and $E_k(x)$ play an important role in the construction for $\nu_i$.

**Construction 6.2 ($\nu_i(x)$).** As already mentioned, the first term of the invariant $\nu_1(x) = \overline{H}_{X,x}$ is the Hilbert-Samuel function of $X$ at $x$.

Let $(f) = (f_1, \ldots, f_m)$ and $(u, y)$ be as in Setup 4.1. As in Construction 5.1 the scheme $X_j$ is locally at $x$ given by the pair on $R = \mathcal{O}_{Z_j, x}$

$$\mathcal{G}(x) = (f_1, b_1) \cap \ldots (f_m, b_m).$$

(In order to avoid too many indices, we do not refer to the year $j$). For the definition of $\nu_i = \nu_i(x), i \in \mathbb{Z}_{\geq 2}$, it is important to know exactly what the ambient scheme and corresponding exceptional components are. In [BM2] this is done by considering triples $(N_{i-1}, \mathcal{G}_i(x), \mathcal{E}_{i-1}(x))$, where $N_{i-1}(x)$ is a regular ambient scheme contained in $Z_j$, $\mathcal{G}_i(x)$ is a local description of $X_j$ on $N_{i-1}(x)$ and $\mathcal{E}_{i-1}(x)$ is an ordered set of exceptional divisors on $Z_j$ which have simultaneously only normal crossing with $N_{i-1}(x)$. In our language this means $(\mathcal{G}_i(x), \mathcal{E}_{i-1}(x))$ is a pair with history on $N_{i-1}(x)$ (Definition 3.2), where we identify $\mathcal{E}_{i-1}(x)$ with the exceptional data which it defines together with $\mathcal{G}_i(x)$ on $N_{i-1}(x)$.

At the beginning $N_0(x) = \text{Spec}(R)$ is the germ of $Z_j$ at $x (R = \mathcal{O}_{Z_j, x})$, $\mathcal{G}_1(x) = \mathcal{G}(x)$ and $\mathcal{E}_0(x) = E_j(x)$. (Attention: In [BM2] $\mathcal{E}_0(x) = \emptyset$, but it seems to be more convenient to put $\mathcal{E}_0(x) = E_j(x)$, because $E_1(x) \supset E_2(x) \supset \ldots$).

Start with the pair with history

$$(\mathcal{G}_1(x), \mathcal{E}_0(x)) = (\mathcal{G}(x), E_j(x)) \quad \text{on} \quad N_0(x) \quad \text{(resp. on} \quad R).$$

We determine $E^1(x)$ and $\mathcal{E}_1(x)$ as described before and set

$$\mathcal{F}_1(x) := \mathcal{G}_1(x) \cap (E^1(x), 1)$$

where $(E^1(x), 1) = \bigcap_{H \in E^1(x)} (x_H, 1)$ and $x_H$ denotes a local generator of $H$. Thus we get the pair with history

$$(\mathcal{F}_1(x), \mathcal{E}_1(x)) \quad \text{on} \quad R.$$  

Note that also the exceptional data has changed. Not only that there are maybe less components, but also the assigned numbers may differ from those of the previous exceptional data. (For example, if $E^1(x) \neq \emptyset$, then all the assigned numbers in $\mathcal{E}_1(x)$ are zero, because $E(x)$ defines a simple normal crossing divisor).

Using the method of Construction 5.1, we choose the maximal contact hypersurface $V(y_1)$ (without loss of generality let $y_1$ be as in Setup 4.1). Let

$$\mathcal{H}_1(x) = \mathbb{D}_x(\mathcal{F}_1(x); u_1, \ldots, u_r, y_2, \ldots, y_r; y_1)$$

be the coefficient pair of $\mathcal{F}_1(x)$ with respect to $(y_1)$.

If $\mathcal{F}_1(x) = (f_1, b_1) \cap \ldots \cap (f_q, b_q) \ (q \in \mathbb{Z}_+, \ q \geq m$ and $(f_1, \ldots, f_m)$ as in Setup 4.1), then

$$\mathcal{H}_1(x) = \bigcap_{i=1}^{q} \bigcap_{l=0}^{b_i-1} \left( \frac{\partial^l f_{l}}{\partial y_{l}^{i} |_{V(y_1)}} , b_i-l \right).$$

We set $N_1(x) := V(y_1)$ and get the pair with history

$$(\mathcal{H}_1(x), \mathcal{E}_1(x)) \quad \text{on} \quad V(y_1).$$

Again the exceptional data has changed, because we have to consider here $\mathcal{E}_1(x)$ as data on $N_1(x) = V(y_1)$. 

Proof. The first and the second equivalence follow by Lemma 3.6.

The equivalence $\mathcal{G}_2(x) \sim_{\mathcal{E}} \mathcal{G}_2'(x)$ is clear for the cases

- $\mathcal{H}_1(x) = (J, b)$ and $\mathcal{H}_1'(x) = (J^a, ab)$ for some $a \in \mathbb{Z}_+$. 
- $\mathcal{H}_1(x) = (J_1, b) \cap (J_2, b)$ and $\mathcal{H}_1'(x) = (J_1 + J_2, b)$.

We put $h_{i,l} := \frac{\partial^l f_i}{\partial y^l} |_{V(y_1)}$ for $1 \leq i \leq q$ and $0 \leq l \leq b_i - 1$. Then we define (always $i \in \{1, \ldots, q\}$ and $l := l(i) \in \{0, \ldots, b_i - 1\}$)

$$
\mu_2(x) := \min \left\{ \frac{\text{ord}_x(h_{i,l})}{b_i - l} \mid i, l \right\}, \\
\mu_2,H(x) := \min \left\{ \frac{\text{ord}_{H,x}(h_{i,l})}{b_i - l} \mid i, l \right\}, \quad \text{for } H \in \mathcal{E}_1(x), \\
\nu_2(x) := \mu_2(x) - \sum_{H \in \mathcal{E}_1(x)} \mu_2,H(x),
$$

(6.2)

where $\text{ord}_{H,x}(h_{i,l})$ denotes the multiplicity of $h_{i,l}$ along $H$, i.e. if $g_H$ is a local generator of $H \in \mathcal{E}_1(x)$, then

$$
\text{ord}_{H,x}(h_{i,l}) = \max \{ k \in \mathbb{Z}_0 \cup \{\infty\} \mid g_H^k \text{ divides } h_{i,l} \}. 
$$

Clearly, $\mu_2,H(x)$ coincides with the assigned number of $H$ in the exceptional data of the pair with history $(\mathcal{H}_1(x), \mathcal{E}_1(x))$. Further we have

$$
\Delta_x^N(\mathcal{H}_1(x), u_1, \ldots, u_e, y_2, \ldots, y_e) = \Delta_x(\mathcal{F}_1(x); u_1, \ldots, u_e, y_2, \ldots, y_e; y_1)
$$

and $\mu_2(x) = \delta(\Delta_x(\mathcal{F}_1(x); u_1, \ldots, u_e, y_2, \ldots, y_e; y_1))$.

If $\nu_2(x) \in [0, \infty)$, then the process ends and the invariant is defined as

$$
\text{inv}_X(x) := \text{inv}_1(x) = (\nu_1, s_1; \nu_2).
$$

Suppose $0 < \nu_2(x) < 1$. We consider

$$
D_2(x) := \prod_{H \in \mathcal{E}_1(x)} g_H^{\mu_2,H(x)},
$$

where $g_H$ denotes a local generator of $H \in \mathcal{E}_1(x)$. (We allow here fractional exponents; see also the remark below). Then by definition of the terms $\mu_2,H(x)$, each $h_{i,l}$, $1 \leq i \leq q$ and $0 \leq l \leq b_i - 1$, can be written as

$$
h_{i,l} = D_2^{b_i - l}, g_{i,l}
$$

for some element $g_{i,l}$. (Recall that $b_i - l = b_{h_{i,l}}$ is the number assigned to $h_{i,l}$ in $\mathcal{H}_1(x)$).

We define the new pair

$$
(\mathcal{G}_2(x), \mathcal{E}_1(x)) = (\bigcap_{i=1}^q \bigcap_{l=0}^{b_i-1} (g_{i,l}, (b_i - l) \cdot \nu_2) \cap (D_2(x), 1 - \nu_2) \quad \text{on } V(y_1)).
$$

(6.3)

This is our variant of the so called companion ideal, thus we call it the companion pair. Clearly, the exceptional data has changed again.

If $1 \leq \nu_2(x) < \infty$, then the assigned number of the $D_2(x)$-component is not positive and hence can be omitted. $\mathcal{G}_2(x) := \bigcap_{i=1}^q \bigcap_{l=0}^{b_i-1} (g_{i,l}, (b_i - l) \cdot \nu_2)$.

Together we get for $0 < \nu_2(x) < \infty$ the pair with history

$$
(\mathcal{G}_2(x), \mathcal{E}_1(x)) \quad \text{on } N_1(x) = V(y_1).
$$

This completes the first step in the general procedure. Then we start again but this time with the pair with history $(\mathcal{G}_2(x), \mathcal{E}_1(x))$ instead of $(\mathcal{G}_1(x), \mathcal{E}_0(x))$.

Lemma 6.3. Let $\mathcal{G}_1(x)$ and $\mathcal{G}_1'(x)$ be two equivalent pair with history. Then

$$
\mathcal{F}_1(x) \sim_{\mathcal{E}} \mathcal{F}_1'(x), \quad \mathcal{H}_1(x) \sim_{\mathcal{E}} \mathcal{H}_1'(x) \quad \text{and} \quad \mathcal{G}_2(x) \sim_{\mathcal{E}} \mathcal{G}_2'(x),
$$

where we have to consider the induced exceptional data. Thus these objects are invariants of the idealistic exponent with history corresponding to $\mathcal{G}_1(x)$.

Proof. The first and the second equivalence follow by Lemma 3.6.
Thus we may assume \( \mathcal{H}_1(x) = (J,b) \) and \( \mathcal{H}'_1(x) = (J',b) \) (with the same assigned number \( b \in \mathbb{Z}_+ \)). For an element \( h \in J \) we have defined \( g = g(h) \) via \( h = D_0^b \cdot g \). Set \( I := \{ g(h) \mid h \in J \} \), then \( J = D_0^b \cdot I \). (Here we identify \( D_0^b \) with the ideal which it generates in \( R \)). Clearly, \( G_2(x) = (I, v_{2b}) \cap (D_2, 1 - v_2) \). We can do the same for \( \mathcal{H}'_1(x) \) and obtain the ideal \( I' \) with the analogous property.

If we can show \( (I, v_{2b}) \sim_{\mathcal{E}(x)} (I', v_{2b}) \) (as pairs with history on \( R \)), then the assertion follows. Since we have factored \( D_2 \), the assigned numbers in the induced exceptional data are all zero. Thus we only have to prove

\[
(I, v_{2b}) \sim (I', v_{2b}).
\]

An extension of the r.s.p. by further independent elements does not change the situation. Hence we may assume that the extension is trivial. Further we have for any point \( x_0 \in \text{Spec}(R) \)

\[
\text{ord}_{x_0}(I) = \text{ord}_{x_0}(J) - \text{ord}_{x_0}(D^b) = \text{ord}_{x_0}(J') - \text{ord}_{x_0}(D^b) = \text{ord}_{x_0}(I').
\]

For the first (resp. third) equality we use \( J = D_0^2 \cdot I \) (resp. \( J' = D_0^2 \cdot I' \)) and the second follows by \( (J, b) \sim (J', b) \). Therefore \( \text{Sing}(I, b) = \text{Sing}(I', b) \). After a permissible blow-up \( \pi : \tilde{Z} \to \text{Spec}(R) \) the transform \( (I, v_{2b}) \) of \( (I, v_{2b}) \) is determined by \( IO_{\tilde{Z}} = H^{r+1} \tilde{I} \)

where \( H \) denotes the ideal sheaf of the exceptional divisor. For the transform of \( J \) we have \( JO_{\tilde{Z}} = H^{r+1} \tilde{J} = H^{(1-v_2)}D^b_2 \cdot H^{r+1} \tilde{I}. \) (\( D^b_2 \) denotes the transform of \( D_2^b \)). Thus the situation is the same as before the blow-up, \( \tilde{J} = D_0^2 \tilde{I} \) and this is also true for \( J' \) and \( I' \). Together we get the desired equality \( (I, v_{2b}) \sim (I', v_{2b}). \)

\[ \square \]

Theorem A and the second part of Theorem B boil down to

**Proposition 6.4.** Let \( r \in \mathbb{Z}_+ \), \( r \geq 1 \). Let \( \mathcal{E}_r(x) = \{(H_1, d_1), \ldots, (H_j, d_j)\} \) be the exceptional data of the pair with history \( (\mathcal{H}_r(x), \mathcal{E}_r(x)) \) on \( V(y_1, \ldots, y_r) \). Let \( (u) = (u_1, \ldots, u_e) \) be the remaining part of the r.s.p. for the local ring \( R \) of \( Z \) at \( x \). Then

\[
\mu_{r+1}(x) = \delta(\Delta_r(\mathcal{F}_r(x); u; y_r)) =: \delta_{r+1}
\]

and \( \nu_{r+1}(x) = \delta_{r+1} - \sum_{i=1}^j d_i \). Hence \( \nu_{r+1}(x) \) coincides with \( \nu_2(\mathcal{F}_r(x), \mathcal{E}_r(x)_{\mathcal{H}}; u) \), where the index \( \mathcal{H} \) should indicate that the exceptional data is the one of \( \mathcal{H}_r(x) \).

**Proof.** This follows by the definition of \( \mu_{r+1}(x), \mu_{r+1, H}(x) \) and \( \nu_{r+1} \) (for \( H \in \mathcal{E}_r(x) \)) (see Construction 6.2).

\[ \square \]

Thus the invariant \( \nu_{r+1}(x) \) can be achieved by purely considering polyhedra. By Proposition 3.5 \( \nu_2(\mathcal{F}_r(x), \mathcal{E}_r(x)_{\mathcal{H}}; u) \) and thus \( \nu_{r+1}(x) \) is independent of the choice of a representative as idealistic exponent with history and also of the choice of \( (y) \) (for fixed \( (u) \)). Moreover, equivalent pairs with history determine the same invariant \( \text{inv}_X(x) \) by Lemma 6.3. This means \( \text{inv}_X(x) \) is really an invariant of the idealistic exponent with history.

All this seems now to be obvious. But keep in mind that before coming to this point we had to work hard in order to develop the theory of idealistic exponents with history and to important results for them.

**Remark 6.5.**

(1) If \( E_0(x) = \emptyset \), then we have \( s_i(x) = 0 \) for all \( i \) and the above procedure coincides with the year zero case. In particular, if \( \mathcal{E}_k(x) = \emptyset \) for some \( k \), then the remaining process coincides with the case described in the previous section.

(2) In [BM2], Remark 9.15, they slightly modify the construction of the invariant \( \text{inv}_X(x) \) if \( (\mathcal{F}_1(x), \mathcal{E}_1(x)) \) can be embedded in a lower dimension ambient scheme, say for example into \( V(z_1, \ldots, z_0) \) instead of \( \mathcal{N}_0(x) = \text{Spec}(R) \). In this case \( \text{inv}_{r+1}(x) := (\nu_1, 0; 1, 0; \ldots; 1, 0). \) After this shift, they consider \( (\mathcal{F}_1(x), \mathcal{E}_1(x)) \) as an pair with history on \( V(z_1, \ldots, z_0) \) (with the induced exceptional data) and continue as usual. This does not affect our considerations seriously.

(3) In the construction we are not forced to start with \( E_0 = \emptyset \) (see (6.1)). We could also require that there is additionally to \( X \) a simple normal crossing divisor \( E_0 \) on \( Z_0 \) given. This could be important for possible applications.
Remark 6.6. Recall that by construction we have

\[ H_1(x) = \bigcap_{i=1}^{q} \bigcap_{l=0}^{b_{i}-1} (h_{i,t}, b_{i} - l) \]

and \( \mu_2(x) \geq 1 \). If \( 0 < \nu_2(x) < 1 \), then the transformation law (under permissible blow-ups) of the pair \( \bigcap_{i=1}^{q} \bigcap_{l=0}^{b_{i}-1} (g_{i,t}, (b_{i} - l) \cdot \nu_2) \) is not consistent with that of \( \bigcap_{i=1}^{q} \bigcap_{l=0}^{b_{i}-1} (h_{i,t}, (b_{i} - l) \cdot \nu_2) \). Therefore we have to add \((D_2(x), 1-\nu_2)\). \( \) (Recall that \( D_2 := D_2(x) = \prod_{H \in \mathcal{E}(x)} g_H^{\mu_2(x)} \)) is the greatest common divisor of the \((h_{i,t}, b_{i} - l), t \), which is a monomial in the new exceptional components \( \mathcal{E}_1(x) \).

More precisely, \( h_{i,t} = D_2^{l-1} \cdot g_{i,t} \) for every \( i, l \) and we defined

\[ \mathcal{G}_2(x) = \left( \bigcap_{i=1}^{q} \bigcap_{l=0}^{b_{i}-1} (g_{i,t}, (b_{i} - l) \cdot \nu_2) \right) \cap (D_2(x), 1-\nu_2). \]

In the last part we have \((D_2, 1-\nu_2) \sim \mathcal{E}_1(x) (D_2^d, (1-\nu_2) d) \) for all \( d \in \mathbb{R}_+ \). So

\[ \mathcal{G}_2(x) \sim \mathcal{E}_1(x) \left( \bigcap_{i=1}^{q} \bigcap_{l=0}^{b_{i}-1} (g_{i,t}, (b_{i} - l) \cdot \nu_2) \right) \cap (D_2^d, (1-\nu_2)(b_{i} - l)). \]

Suppose \( 0 < \nu < 1 \) \((\nu := \nu_2)\). If a blow-up is permissible for the pair \((g \cdot d \cdot \nu) := (g_{i,t}, (b_{i} - l) \cdot \nu_2)\), then we know \( \text{ord}_{\nu_2}(g) \geq d \). But \( 0 < \nu < 1 \) implies \( d \nu < d \) and hence \( \text{ord}_x(g) < d \) might be possible! This means a center which is permissible for \((g \cdot d \cdot \nu)\) is not necessarily permissible for \((h, d) := (h_{i,t}, b_{i} - l) \ (h = D_2^d \cdot g)\).

Further the transform of \((h, d)\) after a permissible blow-up is locally given by \((z^{-\nu}_h, d, h, d)\), where \( z_{exc} \) denotes a local generator of the exceptional component and \( h \) denotes the total transform. By using \( h = D_2^d \cdot g \) we get

\[ \left( z^{-\nu}_h, D_2^d \cdot g \right) = \left( \frac{z^{-\nu}_h}{z_{exc}^{d(d-1)}}, D_2^d \right) \cdot \left( z_{exc}^{d(d-1)} \cdot g, d \right), \]

where \( D_2 \) denotes the total transform of \( D_2 \) under the blow-up. If we consider only \((g, d \nu), \) then the transformations are not consistent.

In the case \( \nu \geq 1 \) we get \( d \nu \geq d \) and the transform of \( h \) is determined by the terms \( z^{-\nu}_h \cdot g = z^{(\nu-1)d}_{exc} \cdot z_{exc}^{d \nu} \cdot g \), where \((\nu - 1)d \geq 0; \) thus we have not to add \( D_2 \).

Remark 6.7. If \( \nu_t \in \{0, \infty\} \) for some \( t \in \mathbb{Z}_+ \), then

\[ \text{inv}_t(x) = \text{inv}_{\frac{1}{t}}(x) = (\nu_1, s_1; \ldots; \nu_{t-1}, s_{t-1}; \nu_t). \]

In the case \( \nu_t(x) = \infty \) the center of the upcoming blow-up is

\[ N_{t-1}(x) = V(y_1, \ldots, y_{t-1}). \]

In every chart the invariant decreases, because all elements of \((y_1, \ldots, y_{t-1})\) are coming from certain initial forms.

If \( \nu_t(x) = 0 \), then we set \( \mathcal{G}_t(x) = (D_t(x), 1) \). \( \) (This fits into the definition of these terms; the assigned numbers of the first part in (6.3) are 0, because \( \nu_t(x) = 0 \), hence we can ignore it and get \( \mathcal{G}_t(x) = (D_t(x), 1) \)). This is the monomial case. In [BM3, Remark 3.6], it is explained how to choose the center for the upcoming blow-up in this case.

Observation 6.8 (Behavior of the polyhedron). Let us see what in each step of the general process by Bierstone and Milman happens to our polyhedra. Let \( r \in \mathbb{Z}_+ \) with \( 0 < r \leq n \) and let \( e = n - r \).

From \( \mathcal{G}_r(x) \) to \( \mathcal{F}_r(x) = \mathcal{G}_r(x) \cap (E^r(x), 1) \): In this step we add

\[ (E^r(x), 1) = \bigcap_{H \in E^r(x)} (x_H, 1), \]

where \( x_H \) denotes a local generator of \( H \). Recall that \( s_r(x) = \#E^r(x) \). By construction \( E^r(x) \subseteq \mathcal{E}_{r-1}(x) \) has only normal crossings with \( N_{r-1}(x) \). Thus we can choose the r.s.p. \((u) = (u_1, \ldots, u_{e+1})\) for \( N_{r-1}(x) \) such that for all \( H \in E^r(x) \) the local generator is \( g_H = u_k \).
for \( k \in I_r := \{k_1, \ldots, k_s\} \subseteq \{1, \ldots, e + 1\} \). (In fact, we can choose the r.s.p. such that the analogous condition holds for every \( H \in \mathcal{E}_{e-1}(x) \)).

Adding the old exceptional components \((E^r(x), 1)\) corresponds to adding points to the generators of the polyhedron \(\Delta_N^r(\mathcal{G}_r(x), u)\). More precisely, the new points are

\[
\{(\delta_{ak})_{a \in \{1, \ldots, e+1\}} = (0, \ldots, 0, 1, \ldots, 0) \mid k \in I_r = \{k_1, \ldots, k_s\}\},
\]

where \(\delta_{ak}\) denotes the usual Kronecker delta. Obviously, the number of new points for the polyhedron is \(s_r\).

By the equivalence of \(\bigcap_{H \in E^r(x)}(x, H, 1)\) and \(\left(\prod_{H \in E^r(x)}x_H, \sum_{H \in E^r(x)} 1 = s_r\right)\) this can also be reinterpreted as adding of only one point to the generators of the polyhedron, namely the one given by

\[
\left(\sum_{k \in I_r} \delta_{ak} \cdot \frac{1}{s_r} \uparrow \right)_{a \in \{1, \ldots, e+1\}} = \left(0, \ldots, 0, \frac{1}{s_r}, 0, \ldots, 0, \frac{1}{s_r}, 0, \ldots, 0, \frac{1}{s_r}, 0, \ldots, 0\right)
\]

(without loss of generality we may assume \(1 \leq k_1 < k_2 < \ldots < k_{s_r} \leq e + 1\)).

In both cases the same variables are involved. Hence the ideal of the tangent cone (the directrix and the ridge) behaves in both cases the same way. This change is described by the initial forms of \((E^r(x), 1)\).

Further observe: \(V(u_{k_1}, \ldots, u_{k_{s_r}})\) has maximal contact with \(F_r(x)\) at \(x\). Clearly, the coefficient pairs with respect to \((u_{k_1}, \ldots, u_{k_{s_r}})\) coincide in both cases. Thus the projections of the polyhedra with respect to \((u_{k_1}, \ldots, u_{k_{s_r}})\) coincide, too.

**From \(F_r(x)\) to \(H_r(x)\):** Suppose \(F_r(x) = (f_1, b_1) \cap \ldots \cap (f_q, b_q)\), then there is at least one \(i \in \{1, \ldots, q\}\) such that \(b_i = \text{ord}_r(f_i)\). We assume without loss of generality that \(y_r := u_{e+1}\) has maximal contact with \(F_r(x)\) at \(x\). Hence in this step we project the polyhedron \(\Delta_N^r(F_r(x); u_1, \ldots, u_e, u_{e+1}) \subset \mathbb{R}^{e+1}_{0} \) from the point \((0, \ldots, 0, 1) \in \mathbb{Z}^e_0\) to \(\mathbb{R}^e_0\). The resulting polyhedron is

\[
\Delta_x(F_r(x); u_1, \ldots, u_e; y_r) = \Delta_N^r(H_r(x); u_1, \ldots, u_e) \subset \mathbb{R}^e_0.
\]

**From \(H_r(x)\) to \(G_{r+1}(x)\):** Suppose \(H_r(x) = (h_1, b_1) \cap \ldots \cap (h_p, b_p)\). The last step is rather consisting of three smaller steps. We determine \(D_{r+1}(x)\) and write each \((h_i, b_i)\) as \(h_i = D_{r+1}^0 \cdot g_i\). We set

\[
\widetilde{H}_r(x) := \bigcap_{i=1}^p (g_i, b_i) \quad \text{and} \quad \widetilde{G}_{r+1}(x) := \bigcap_{i=1}^p (g_i, b_i \nu_{r+1}).
\]

Further recall that \(G_{r+1}(x) = \widetilde{G}_{r+1}(x) \cap (D_{r+1}(a), 1 - \nu_{r+1})\). Then the smaller steps are the following

1. **From \(H_r(x)\) to \(\widetilde{H}_r(x)\):** Since \(N_r(x)\) and \(E_r(x)\) have simultaneously only normal crossings, we can choose the coordinates \((u_1, \ldots, u_e)\) of \(N_r(x)\) such that for all \(H \in E_r(x)\) the local generator is \(g_H = u_l\) for some \(l \in \{1, \ldots, e\}\). In this situation we set \(\mu_{r+1,l} := \mu_{r+1,H}(x)\). Put \(I_r := \{l_1, \ldots, l_{m_r}\} := \{l \in \{1, \ldots, e\} \mid \mu_{r+1,l} \neq 0\} \subseteq \{1, \ldots, e\}\).

Further we denote by \(T_r : \mathbb{R}^e \rightarrow \mathbb{R}^e\) the translation in the negative direction by the vector

\[
w(r) := \left(0, \ldots, 0, \mu_{r+1,1}, 0, \ldots, 0, \mu_{r+1,2}, 0, \ldots, 0, \mu_{r+1,m_r}, 0, \ldots, 0\right).
\]


This means a point \( v \in \mathbb{R}^e \) is sent to \( T_r(v) = v - w^{(r)} \). Then we have for the Newton polyhedra
\[
T_r \left( \Delta^N_x \left( \mathcal{H}_r(x), u \right) \right) = \Delta^N_x \left( \mathcal{H}_r(x), u \right) \subseteq \mathbb{R}_0^e.
\]

(2) From \( \mathcal{H}_r(x) \) to \( \mathcal{G}_{r+1}(x) \): In this step we multiply each point of the polyhedron \( \Delta^N_x \left( \mathcal{H}_r(x), u \right) \) by the factor \( \frac{1}{\nu_{r+1}} \) and get \( \Delta^N_x \left( \mathcal{G}_{r+1}(x), u \right) \).

This corresponds to the change of the valuation on the r.s.p. (\( u \)) for the regular local ring \( K[[u]] \) corresponding to \( V(y) \) (resp. of the r.s.p. \( (u, y) \) for \( R \)), see Remark 5.6.

(3) From \( \mathcal{G}_{r+1}(x) \) to \( \mathcal{G}_{r+1}(x) \): The last step is similar to “From \( \mathcal{G}_r(x) \) to \( \mathcal{F}_r(x) \)”. By definition \( \mathcal{G}_{r+1}(x) = \mathcal{G}_{r+1}(x) \cap (D_{r+1}(x), 1 - \nu_{r+1}) \). Thus we add to the generators of \( \Delta^N_x \left( \mathcal{G}_{r+1}(x), u \right) \) the points associated to
\[
\left( D_{r+1}(x) = \prod_{H \in \mathcal{E}_r(x)} g_H^{\nu_{r+1}+1}(x), 1 - \nu_{r+1} \right),
\]
where \( g_H \) is a local generator of \( H \in \mathcal{E}_r(x) \). (Recall that we defined \( \nu_{r+1}(x) = \mu_{r+1}(x) - \sum_{H \in \mathcal{E}_r(x)} \nu_{r+1}(x) \). As in (1) we choose \( u = (u_1, \ldots, u_e) \) such that for all \( H \in \mathcal{E}_r(x) \) the local generator is \( g_H = u_l \) for some \( l \in \{1, \ldots, e\} \). Again we set \( \mu_{r+1} := \mu_{r+1,H}(x) \) and
\[
I_r := \{ l_1, \ldots, l_{m_r} \} := \{ l \in \{1, \ldots, e\} \mid \nu_{r+1,l} \neq 0 \} \subseteq \{1, \ldots, e\}.
\]
Then \( (D_{r+1}(x), 1 - \nu_{r+1}) \) yields in \( \Delta^N_x \left( \mathcal{G}_{r+1}(x), u \right) \) the point
\[
\left( 0, \ldots, 0, \frac{\mu_{r+1} + 1}{1 - \nu_{r+1}}, 0, \ldots, 0, \frac{\mu_{r+1,2} + 1}{1 - \nu_{r+1}}, 0, \ldots, 0, \frac{\mu_{r+1,m_r} + 1}{1 - \nu_{r+1}}, 0, \ldots, 0 \right).
\]

Remark 6.9. By definition \( \delta(\Delta^N_x \left( \mathcal{H}_r(x), u \right)) = \mu_{r+1}(x) \). By going from \( \mathcal{H}_r(x) \) to \( \mathcal{H}_r(x) \) we send the assigned numbers in the exceptional data to zero. Therefore \( \delta(\Delta^N_x \left( \mathcal{H}_r(x), u \right)) = \delta(\Delta^N_x \left( \mathcal{F}_r(x); u; y_r \right)) = \nu_{r+1}(x) \).

7. Simplification of the strategy

The construction of the invariant of Bierstone and Milman is quite complicated. Therefore it is hard to formulate a step-by-step result on the behavior of the generators of the ideal \( J \) as we did in Proposition 5.4 and Remark 5.6. But we show now that in certain good situations the procedure becomes easier. In particular, we can sometimes make bigger steps.

For this we introduce the following

Notation. Fix \( r \in \mathbb{Z}_+ \). Let \( \mathcal{I}_r \in \{ \mathcal{G}_r(x), \mathcal{F}_r(x), \mathcal{H}_{r+1}(x) \} \) and \( s \in \mathbb{Z}_+, s < r \).

(1) We define the \( \mathcal{G}_s \)-part of \( \mathcal{I}_r \) to be the part of \( \mathcal{I}_r \) which is by the construction coming from \( \mathcal{G}_s(x) \).

(2) By the \( E^{(s)} \)-part (resp. \( D^{(s)} \)-part) of \( \mathcal{I}_r \) we denote the part which occurred by adding \( E^{(s)}(x), \ldots, E^{(s)}(x) \) (resp. \( D_{r+1}(x), \ldots, D_{r+1}(x) \)).

If \( s = 1 \), then we speak also of the \( \mathcal{G} \)-part (resp. \( E \)-part, resp. \( D \)-part) of \( \mathcal{I}_r \) instead of the \( \mathcal{G}_1 \)-part (resp. \( E^{(1)} \)-part, resp. \( D^{(1)} \)-part) of \( \mathcal{I}_r \).

(For \( \mathcal{I}_r = \mathcal{G}_s(x) \) we neglect \( E^{(s)} \) in the definition of the \( E^{(s)} \)-part, because it has not been added yet.)

**Observation 7.1** (Big steps with the old exceptional part (\( E^0(x, 1) \))). In the definition of \( \mathcal{F}_1(x) \) we add the old exceptional components (\( E^1(x, 1) \)) to \( \mathcal{G}_1(x) \). This enables us to make sometimes more than one step in Construction 6.2: First, this may change the separation of the r.s.p. into \( (u, y) \) as in Setup 4.1. Thus let us consider an arbitrary r.s.p. \( (t) = (t_1, \ldots, t_n) \) for \( R \). Further \( E^1(x) \) is a simple normal crossing divisor on \( N_0(x) = Z_j \). Hence we can
choose the r.s.p. \( (t) = (t_1, \ldots, t_n) \) for \( R \) such that every \( H \in E^1(x) \) is locally given by some \( t_i = 0 \) for \( i \in \{1, \ldots, n\} \), say \( E^1(x) \) is given by \( (t_{i_1}, \ldots, t_{i_r}) \). Suppose \( s_1 = \#E^1(x) \geq 1 \). Set \( \{z\} = (z_1, \ldots, z_s) = (t_{i_1}, \ldots, t_{i_r}) \). Then \( V(z) \) has maximal contact with \( F_1(x) \) at \( x \). (Recall that locally \( x \) is given by the maximal ideal of \( R \)). So this is a possible choice for the first \( s_1 \) steps in definition of \( \nu(x) \). (After that we consider \( \nu(1) \) which determines \( \nu(1) \)). Since \( E_1(x) \) and \( E^1(x) \) have simultaneously only normal crossings, we can require the additional property on \( (t) \) that \( E_1(x) \) is given by \( (t_{m_1}, \ldots, t_{m_p}) \), where \( t_i \neq t_p \) for \( i \in \{m_1, \ldots, m_p\} \) and \( p \in \{1, \ldots, l_s\} \). Thus we get for every \( i \in \{1, \ldots, s_1\} \) \( (\text{If } s_1 = 1 \text{ then the previous set is empty}):(1) \mu_{1, H}(x) = 0 \text{ for every } H \in E_1(x), \text{ thus } D_1(x) = 1 \text{ and } \)\( (2) \mu(x) = \mu(x) = 0 \).

The first assertion holds, because we can not factor \( t_i \), \( i \) as above. The second part follows from the condition that \( V(z) \) has maximal contact with \( F_1(x) \) at \( x \).

Therefore we already know \( \nu_t \) up to the step \( s = s_1 \). Set \( d := s_1 \). In the procedure we also added \( E^2(x), \ldots, E^2(x) \subset E^1(x) \). Further \( s = \#E^2(x) \) \( (q \in \{1, \ldots, d\} \) and \( E_d(x) = E_1(x) \setminus \bigcup_{q=1}^d E^q(x) \). If the condition \( s_1 + \ldots + s_d - d \geq 1 \iff s_2 + \ldots + s_d \geq 1 \) holds, then \( D_{d+1}(x) = 1, \nu_{d+1}(x) = \mu_{d+1}(x) = 1 \) and we can choose the next maximal contact \( V(z_{d+1}) \) in the \( E \)-part of \( H_d(x) \).

Convention: We choose the maximal contact variables in the \( E \)-part until we get to the stage \( r > d \), where \( s_1 + \ldots + s_r = r = 0 \).

This means the \( E \)-part of \( H_1(x) \) is empty. (Recall that \( H(x) \) determines \( \nu(x) \)). As above it follows for every \( i \in \{2, \ldots, r\} \):

(1) \( \mu_{1, H}(x) = 0 \text{ for every } H \in E_1(x), \text{ thus } D_1(x) = 1 \) and

(2) \( \nu(x) = \mu(x) = 1 \).

In particular, \( H(x) \) is only given by the \( G_1 \)-part. This means, \( H(x) \) is the coefficient pair of \( G_1(x) \) with respect to \( (z_1, \ldots, z_r) \).

In general, we cannot assume \( s_1 > 0 \). So we set \( d := \min \{ q \in \mathbb{Z}_+ \mid \nu_q \neq 0 \} \).

Then \( E^d(x) \neq \emptyset \) and \( F_d(x) = G_d(x) \cap \langle E^d(x), 1 \rangle \). We choose the maximal contact \( V(z_d) \) such that there is some \( H \in E^d(x) \) which is locally given by \( V(z_d) \).

If \( s_d \geq 2 \), then the \( E^{(d)} \)-part of \( H_d(x) \) is non-empty. This implies \( \nu_{d+1}(x) = \mu_{d+1}(x) = 1 \). In the next step of the procedure we multiply the assigned numbers by \( \nu_{d+1} = 1 \), thus \( G_{d+1}(x) = H_d(x) \) and then we add \( E^{(d+1)}(x) \) in order to obtain \( F_{d+1}(x) \). We choose the maximal contact in the \( E^{(d)} \)-part and so on. This continues until we are at the step \( r := \min \{ l \in \mathbb{Z}_+ \mid l \geq d \wedge s_d + \ldots + s_l - (l - d + 1) = 0 \} \).

Putting everything together yields

**Proposition 7.2.** Let \( d, r \in \mathbb{Z}_+ \) be as above. For every \( i \in \{d+1, \ldots, r\} \) we get

(i) \( \mu_{1, H}(x) = 0 \text{ for every } H \in E_i(x), \text{ thus } D_i(x) = 1 \) and

(ii) \( \nu(x) = \mu(x) = 1 \).

(iii) the \( E^{(d)} \)-part of \( H(x) \) (and \( G_{r+1}(x) \)) is empty.

(iv) hence \( H(x) \) is the coefficient pair of \( G_d(x) \) with respect to \( (z_d, \ldots, z_r) \) and \( \mu_{r+1}(x) = d! \Delta_x(G_d(x), u, (z_d, \ldots, z_r)) \), where \( u \) denotes the remaining elements of the r.s.p. \( (l) = (u, z) \).

Further \( \nu_{r+1}(x) \) is determined by \( \mu_{r+1}(x) \) and the assigned numbers in the exceptional data of \( H_r(x) \).

If \( s_d = 1 \) then \( r = d \) and the above statement is empty except for part (iv).

Recall that we have constructed \( G_d(x) \) from \( H_1(x) \) by factoring out \( D_2(x) \), \( h = B_2 \), \( q \) (where \( H_1(x) \subset (h, b_d) \)). If \( D_2 = D_2(x) = 1 \) is trivial, i.e. if the assigned numbers in the exceptional data are all zero, then \( G_d(x) = H_1(x) \). Together with the previous this leads to
Observation 7.3 (Big steps if $D^q(x) = 1$). Set

$$d := \min \{ q \in \mathbb{Z}_+ \mid D_q = 1 \} \quad \text{and} \quad r := \min \{ l \in \mathbb{Z}_+ \mid l > d \land D_q \neq 1 \}.$$  

(For the steps before $d$ we have to apply the usual procedure). Consider $G_d(x)$. Since $D_d(x) = 1$, we have $G_d(x) = H_{d-1}(x)$. If $s_d = \#E^q(x) = 0$, then the next step is as without exceptional divisors. On the other hand, if $s_d \geq 1$, then we can apply Observation 7.1 until the $E$-part is empty. Note that we have during this process $D_q = 1$. Thus we have good control on these steps.

This works until we come to $H_{r-1}(x)$. There $D_r(x) \neq 1$. By the convention of choosing first the exceptional components in the $E$-part, the $E^d$-part of $H_{r-1}(x)$ has to be empty. This implies that $H_{r-1}(x)$ is only given by the $G_r$-part. (But it is not necessarily the coefficient pair of $G_d(x)$ with respect to $(z_d, \ldots, z_{r-1})$, because maybe not all $u_i(x)$ are equal 1 for $d < i < r$; nevertheless the situation is similar to Remark 5.6 — see also Remark 7.5 below).

We modify $H_{r-1}(x)$ as described in Construction 6.2 (factor out $D_r(x)$ and then add $(D_r(x), 1 - \nu_r)$) and obtain $G_r(x)$.

If $\mu_r(x) = 1$, then $(D_r(x), 1 - \nu_r) \sim E(x) \cap H(qH, 1)$, where the intersection is over those $H \in E(x)$ with $\mu_r(x) \neq 0$ and $qH$ denotes a local generator of $H$. Then the same procedure as in the previous observation can be applied: First we choose the maximal contact only in the part coming from $D_r(x)$ and after that we consider the $E^{(r)}$-part.

We can apply this until we get to the point, where $D_r(x) \neq 1$ and $\mu_r(x) > 1$. Then we have to apply the full procedure to construct $\nu_r$ and we go back to the beginning of this observation.

Also recall that if the exceptional data $E(x) = \emptyset$ is empty, then the general procedure is the same as in the easy case without exceptional divisors.

Let us recapitulate the result.

Proposition 7.4. Let $d, r, r' \in \mathbb{Z}_+$ be as in the previous observation. (Not to be confused with the $d, r$ in Proposition 7.2; these are different integers). Then the case without exceptional divisors and Proposition 7.2 determine completely the procedure of Construction 6.2 for the steps $i \in \{d, \ldots, r' - 1\}$.

Note that Proposition 7.2 and Proposition 7.4 depend on the convention that we choose the maximal contact first in the $E$-part of the given pair with history.

Remark 7.5. For the proof of our main theorem we did not need concrete formulas for $G_r(x), F_r(x)$ resp. $H_{r+1}(x)$. Let us now briefly mention some results in this direction.

In order to simplify the situation we assume that $D_2 = \ldots = D_r = 1$ and $D_{r+1} \neq 1$ for some $r \in \mathbb{Z}_+$. After $r$ steps in the procedure we have distinguished the r.s.p. for $R = O_{Z,x}$ as $(u, z) = (u_1, \ldots, u_r; z_1, \ldots, z_r)$ and further we know the terms $\nu_2 = \mu_2 \geq 1, \ldots, \nu_r = \mu_r \geq 1$ and $\nu_{r+1}$. $(D_{r+1} \neq 1$ implies $\nu_{r+1} < \mu_{r+1}$).

Define $\beta_1 := 1$ and $\beta_j := (\nu_2 \cdot \nu_3 \cdot \ldots \cdot \nu_{j-1})^{-1}$ for $j > 1$. Recall that we choose (by the convention) the next maximal contact components in the $E$-part until it is empty. Since the $E$-part and $E_r$ have simultaneously only normal crossings, it follows that the $E$-part of $H_r(x)$ is empty. $(D_{r+1}(x)$ is determined by $H_r(x))$. Together with the definition of $r$ this yields that $H_r(x)$ is completely determined by the $G_1$-part. Suppose $G_1(x) = \bigcap_{i=1}^m \langle f_i, b_i \rangle$ for some $f_i \in R$.

Then we can write for every $i$

$$f_i(u, z) = F_{b_i}(z) + \sum_{\nu_1 \in \nu_{1}(B) \subset b} F_{B\beta, i}(u) \cdot z^B + f^*_i(u, z),$$

for some $f^*_i(u, z) \in \langle z \rangle^b$ and $F_{B\beta, i}(z) \in K[z]$ is (with respect to $L\nu_{1}(B) := \sum_{j=1}^r \beta_j B_j$) quasi-homogeneous of degree $b_i$. Further let $i \in \{1, \ldots, m\}$ and $B = B(i) \in \mathbb{Z}_0^r$ be such that $L\nu(B) < b_i$. Then

$$H_{r}(x) = \bigcap_{i, B \text{ as above}} \left( F_{B\beta, i}(u), (b - L\nu(B)) \cdot \frac{1}{\beta_\nu} \right).$$

By definition, $D_{r+1} \neq 1$ and thus $\nu_{r+1} < \mu_{r+1}$. 

Further any element \((h_{B,i}, b_i) := (F_{B,i}(u), (b - L_{v}(B)) \cdot (\beta_{s})^{-1})\) can be written in the form \(h_{B,i} = D_{r+1}^{b_{i}} \cdot g_{i}\) and

\[
G_{r+1}(x) = \begin{cases} 
\bigcap_{i=1}^{r+1} (g_{B,i}) \cdot \nu_{r+1}, & \text{if } 1 \leq \nu_{r+1} < \infty, \\
\left(\bigcap_{i=1}^{r+1} (g_{B,i}) \cdot \nu_{r+1}\right) \cap (D_{r+1}^{1} - \nu_{r+1}), & \text{if } 0 < \nu_{r+1} < 1, \\
(D_{r+1}^{1} - \nu_{r+1}), & \text{if } \nu_{r+1} = 0.
\end{cases}
\]

(In the case \(\nu_{r+1} = \infty\) the center of the next blow-up is \(N_{r} (x) = V(z_1, \ldots, z_r)\).

Then we start again with \(G_{r+1}(x)\) instead of \(G_{r}(x)\). We define

\[s' := \min\{ l \in \mathbb{Z}^+ | l \geq r + 1 \wedge D_{l+1} \neq 1 \}\]

and we use for the formulas \((g_{B,i}) \cdot \nu_{r+1}\) and \((D_{r+1} - \nu_{r+1})\) instead of \((f_{i}, b_{i})\).

In general, let \((f) = (f_{1}, \ldots, f_{n})\) be generators of \(J\). Then \(G_{1}(x) = \bigcap_{i=1}^{n} (f_{i}, b_{i})\). Set \((f, b) = (f_{i}, b_{i})\) for some \(i\). After \(r\) steps in Construction 6.2, we have determined \((z) = (z_{1}, \ldots, z_{r}), \nu_{1}, \nu_{2}, \ldots, \nu_{r+1}\) and \(D_{2}(x), \ldots, D_{r+1}(x)\). By the definitions, \((b - b_{i})\nu_{2} - b_{2} = \nu_{2}(b - L_{v}(z)(b_{1}, b_{2}))\). One can check that \(f\) can be written as \(f(u, z) = F_{b}(z, D) + \sum_{L_{v}(z)(b) < b} z^{D_{r} - b_{2} - b} \cdot D_{r+1}^{(b_{2} - b_{1})\nu_{2} - b_{2} - b} \cdot D_{r+1}^{(b_{r+1} - b_{1})\nu_{1} - b_{2}} \cdot F_{b}(u) + f^{*}\), for some \(f^{*} = f^{*}(u, z) \in (z)^{b+1}\) and \(F_{b}(z, D)\) is (with respect to \(L_{v}(z) = \sum_{i=1}^{r} b_{i}\)) quasi-homogeneous of degree \(b\) in the variables \(z\).

But the exceptional components \(D_{2}, \ldots, D_{r+1}\) are also involved in \(F_{b}(z, D)\). With the above formula we can give a description of the \(G_{1}\)-part of \(G_{r}(x), F_{r}(x), H_{r}(x)\) resp. \(G_{r+1}(x)\) similar to the one in Lemma 5.4 resp. Remark 5.6. But still there may be also an \(E\)- and a \(D\)-part.

References

[AHV] J. Aroca, H. Hironaka, and J. Vincente. The theory of the maximal contact. Memorias de Matemática del Instituto “Jorge Juan”, No. 29. [Mathematical Memoirs of the “Jorge Juan” Institute, No 29] Instituto “Jorge Juan” de Matemáticas, Consejo Superior de Investigaciones Científicas, Madrid, 1975.

[BHM] J. Bertram, P. Hervet, and H. Mourtada. Computing Hironaka’s invariants: ridge and directrix. In Arithmetic, geometry, cryptography and coding theory 2009, volume 544 of Contemp. Math., p. 43–78. Amer. Math. Soc., Providence, RI, 2010.

[BM1] E. Bierstone and P. Milman. A simple constructive proof of canonical resolution of singularities. In Effective methods in algebraic geometry (Castiglioncello, 1990), volume 94 of Progr. Math., p. 11–30. Birkhäuser Boston, Boston, MA, 1991.

[BM2] E. Bierstone and P. Milman. Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. Invent. Math., 128(2), p. 207–302, 1997.

[BM3] E. Bierstone and P. Milman. Resolution of singularities. Math. Sci. Res. Inst. Publ., 37, p. 43–78, 1999.

[BM4] E. Bierstone and P. Milman. Desingularization algorithms I. Role of exceptional divisors. Mosc. Math. J., 3(3), p. 751–805, 2003.

[BEV] A. Bravo, S. Encinas and O. Villamayor. A simplified proof of desingularization and applications. Rev. Mat. Iberoamericana, 21(2), p. 349–458, 2005.

[CGO] V. Cossart, J. Giraud, and U. Orbanz. Resolution of surface singularities. Number 1101 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1984. (With an appendix by H. Hironaka).

[CJS] V. Cossart, U. Jannsen, and S. Saito. Canonical embedded and non-embedded resolution of singularities for excellent two-dimensional schemes. preprint, arXiv:math.AG/0905.2191, latest version: February 2013 (first version: March 2009).

[CP1] V. Cossart and O. Piltant. Resolution of singularities of threefolds in positive characteristic I. Journal of Algebra, 320(3), p. 1051–1081, 2008.

[CP2] V. Cossart and O. Piltant. Resolution of singularities of threefolds in positive characteristic II. Journal of Algebra, 321(7), p. 1836–1976, 2009.

[CP3] V. Cossart and O. Piltant. Resolution of Singularities of Arithmetical Threefolds I. HAL: hal-00873987, 2013.

[CSc] V. Cossart and B. Schober. Characteristic polyhedra of singularities without completion - Part II. preprint, 2014.

[Cul] S. D. Cutkosky. Resolution of singularities. Graduate Studies in Mathematics, 63. American Mathematical Society, 2004.
[Cu2] S. D. Cutkosky. Resolution of singularities for 3-folds in positive characteristic. Amer. J. Math., 131(1), p.59–127, 2009.

[Cu3] S. D. Cutkosky. A skeleton key to abhyankar’s proof of embedded resolution of characteristic $p$ surfaces. Asian J. Math., 15(3), p.369–416, 2011.

[dJ] A. J. de Jong. Smoothness, semi-stability and alterations. Inst. Hautes tudes Sci. Publ. Math., (83), p.51–93, 1996.

[EHa] S. Encinas and H. Hauser. Strong resolution of singularities in characteristic zero. Comment. Math. Helv., 77(4), p.821–845, 2002.

[G1] J. Giraud. Étude locale des singularités. Number 26 in Publications Mathématiques d’Orsay. Mathématique, Université Paris XI, Orsay, 1972. Cours de 3ème cycle, 1971-1972.

[G2] J. Giraud. Sur la théorie du contact maximal. Math. Z., (137), p.285–310, 1974.

[Ha] H. Hauser. The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand). Bull. Amer. Math. Soc., 40(3), p.323–403, 2003.

[H1] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero I + II. Ann. of Math., 79, p.109–326, 1964.

[H2] H. Hironaka. Characteristic polyhedra of singularities. J. Math. Kyoto Univ., 7(3), p.251–293, 1967.

[H3] H. Hironaka. Idealistic exponents of singularity. In Algebraic geometry, p.52–125. Johns Hopkins Univ. Press, 1977.

[H4] H. Hironaka. Theory of infinitely near singular points. J. Korean Math. Soc, 40(5), p.901–920, 2003.

[H5] H. Hironaka. Three key theorems on infinitely near singularities. Singularités Franco-Japonaises, Sémin. Congr.10, Soc. Math. France, Paris, p.87–126, 2005.

[K] H. Kawasaki. Toward resolution of singularities over a field of positive characteristic part I. foundation; the language of the idealistic filtration. Publ. Res. Inst. Math. Sci., 43(3), p.819–909, 2007.

[Ko] J. Kollár. Lectures on resolution of singularities. Annals of Mathematics Studies, 166. Princeton University Press, 2007.

[L] J. Lipman. Desingularization of two-dimensional schemes Ann. Math. 107, p.151–207, 1978.

[N] R. Narasimhan. Hyperplanarity of the equimultiple locus. Proc. Amer. Math. Soc., 87(3), p.403–408, 1983.

[Sc1] B. Schober. Characteristic polyhedra of idealistic exponents with history. Dissertation, Universität Regensburg, 2013, available at http://epub.uni-regensburg.de/28877/.

[Sc2] B. Schober. Idealistic exponents and their characteristic polyhedra. preprint, 2014.

[V1] O. Villamayor. Constructiveness of Hironaka’s resolution. Ann. Sci. cole Norm. Sup. (4), 22(1), p.1–32, 1989.

[V2] O. Villamayor. Patching local uniformizations. Ann. Sci. cole Norm. Sup. (4), 25(6), p.629–677, 1992.

[W] J. Włodarczyk. Hironaka resolution in characteristic zero. J. Amer. Math. Soc., 18(4), p.779–822, 2005.