One-loop effective action for $\mathcal{N} = 4$ SYM theory in the hypermultiplet sector: leading low-energy approximation and beyond

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Abstract

We develop the derivative expansion of the one-loop $\mathcal{N} = 4$ SYM effective action depending both on $\mathcal{N} = 2$ vector multiplet and on hypermultiplet background fields. Beginning with formulation of $\mathcal{N} = 4$ SYM theory in terms of $\mathcal{N} = 1$ superfields, we construct the one-loop effective action with the help of superfield functional determinants and calculate this effective action in $\mathcal{N} = 1$ superfield form using an approximation of constant Abelian strength $F_{mn}$ and corresponding constant hypermultiplet fields. Then we show that the terms in the supercovariant derivative expansion of the effective action can be rewritten in terms of $\mathcal{N} = 2$ superfields. As a result, we get a new derivation of the complete $\mathcal{N} = 4$ supersymmetric low-energy effective action obtained in hep-th/0111062 and find subleading corrections to it. A problem of $\mathcal{N} = 4$ supersymmetry of the results is discussed. Using the formalism of $\mathcal{N} = 2$ harmonic superspace and exploring on-shell hidden $\mathcal{N} = 2$ supersymmetry of $\mathcal{N} = 4$ SYM theory we construct the appropriate hypermultiplet-depending contributions. The hidden $\mathcal{N} = 2$ supersymmetry requirements allow to get a leading, in hypermultiplet derivatives, part of the correct $\mathcal{N} = 4$ supersymmetric functional containing $F^3$ among the component fields.
1 Introduction

The $\mathcal{N} = 4$ SYM theory attracts much attention due to the remarkable properties allowing to clarify the profound questions concerning the quantum dynamics in supersymmetric field models and their links with string/brane theory. Maximally extended rigid supersymmetry of the $\mathcal{N} = 4$ SYM theory imposes strong restrictions on the quantum dynamics. As a result, the quantities characterizing the theory in quantum domain can be exactly found or studied in great detail (see e.g. [1, 2, 3, 4, 5]).

In this paper we calculate one-loop low-energy effective action in $\mathcal{N} = 4$ SYM theory, depending on all fields of $\mathcal{N} = 4$ vector multiplet. At present, the best, most symmetric, and adequate, description of $\mathcal{N} = 4$ vector multiplet dynamics is given in terms of unconstrained harmonic $\mathcal{N} = 2$ superfields. From this point of view, the $\mathcal{N} = 4$ SYM theory is a model of $\mathcal{N} = 2$ SYM theory coupled to hypermultiplet in adjoint representation of a gauge group. It is well known that the exact low-energy quantum dynamics of $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 2$ vector multiplet sector is mastered by the non-holomorphic effective potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$, depending on $\mathcal{N} = 2$ strengths $\mathcal{W}, \bar{\mathcal{W}}$ (see Refs. [7, 8, 9, 10, 11, 2]). The explicit form of the non-holomorphic potential for $SU(N)$ gauge group spontaneously broken down to its maximal torus looks like

$$\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}}) = c \sum_{I<J} \ln \left( \frac{\mathcal{W}^I - \mathcal{W}^J}{\Lambda} \right) \ln \left( \frac{\bar{\mathcal{W}}^I - \bar{\mathcal{W}}^J}{\Lambda} \right),$$

(1)

where $\Lambda$ is an arbitrary scale, $I, J = 1 \ldots N$ and $c = 1/(4\pi)^2$ (for more detail see Refs. [11]). Expression (1) defines exact low-energy effective potential in leading order in external momentum expansion in $\mathcal{N} = 2$ gauge superfield sector [7, 8]. We emphasize that the result (1) is so general that it can be obtained entirely on the symmetry grounds from the requirements of scale independence and R-invariance up to a numerical factor [7, 12]. Moreover, the potential (1) gets neither perturbative quantum corrections beyond one-loop nor instanton corrections [7, 8] (see also discussion of non-holomorphic potential in $\mathcal{N} = 2$ SYM theories [12, 13, 14, 15]). All these properties are very important for understanding of the low-energy quantum dynamics in $\mathcal{N} = 4$ SYM theory in the Coulomb phase. In particular, the effective potential (1) provides the first subleading terms in the interaction between parallel D3-branes in the superstring theory (see e.g. [16]). It was proposed that full $\mathcal{N} = 4$ SYM effective action, depending on proper invariants constructed from the arbitrary powers of the Abelian strength $F_{mn}$ and obtained by summing up all the loop quantum corrections, should reproduce (within certain limits) the Born-Infeld action [17] ($\mathcal{N} = 4$ SYM/supergravity correspondence). Discussion of this correspondence and its two-loop test are given in Ref. [18] (see also a consideration of the analogous problem for non-Abelian background in Ref. [19] and general approach to calculating the higher loop corrections in [20]).

In order to clarify the structure of the restrictions on an effective action, stipulated by $\mathcal{N} = 4$ supersymmetry, and to gain a deeper understanding of the $\mathcal{N} = 4$ SYM/supergravity correspondence, we have to find an effective action not only in $\mathcal{N} = 2$ vector multiplet sector but also depending on all the fields of $\mathcal{N} = 4$ vector multiplet (see discussion in [21]). This problem remained unsettled for a long time. Recently, the

\footnote{Low-energy effective action in arbitrary $\mathcal{N} = 2$ SYM model can contain, in principle, a holomorphic effective potential [6] but it vanishes in $\mathcal{N} = 4$ gauge theory.}
complete exact low-energy effective action containing the dependence both on \( \mathcal{N} = 2 \) gauge superfields and hypermultiplets has been discovered [22]. It has been shown that the algebraic restrictions imposed by hidden \( \mathcal{N} = 2 \) supersymmetry on a structure of the low-energy effective action in \( \mathcal{N} = 2 \) harmonic superspace approach turn out to be so strong that they allow to restore the dependence of the low-energy effective action on the hypermultiplets on basis of the known non-holomorphic effective potential (1). As a result, the additional hypermultiplet-dependent contributions containing the on-shell \( \mathcal{W}, \bar{\mathcal{W}} \) and the hypermultiplet \( q^a \) superfields have been obtained in the form

\[
\mathcal{L}_q = c \left\{ (X - 1) \frac{\ln(1 - X)}{X} + [\text{Li}_2(X) - 1] \right\}, \quad X = -\frac{q^{ia}q_{ia}}{WW},
\]

where \( \text{Li}_2(X) \) is the Euler dilogarithm function and \( c \) is the same constant as in (1) (see the details and denotations in Refs. [22, 5]). The effective Lagrangian (2), together with the non-holomorphic effective potential (1), determine the exact \( \mathcal{N} = 4 \) supersymmetric low-energy effective action in the theory under consideration.

The leading low-energy effective Lagrangian (2) has been found in Ref. [22] on a purely algebraic ground. It would be extremely interesting to derive this Lagrangian and next-to-leading corrections in external momenta in the framework of the quantum field theory. Such a problem seems to be very non-trivial since the expression (2) includes any powers of \( X \) and is singular at \( W = 0 \), therefore the result can not be obtained by considering the Feynman diagrams with the fixed number of external hypermultiplet and gauge field legs. All such diagrams must be summed up! In recent paper [25], the problem of computing the effective Lagrangian (2) has been solved using the covariant harmonic supergraph techniques [2, 26]. The more general problem consists in quantum field theoretical or algebraic derivation of the subleading terms in the effective action, depending on all fields of \( \mathcal{N} = 4 \) supermultiplet and representation of these terms in complete \( \mathcal{N} = 4 \) supersymmetric form. The present paper is just devoted to solution of such a problem for one-loop effective action. To be more precise, we discuss the construction of the derivative expansion of one-loop effective Lagrangian \( \mathcal{L}_{\text{eff}} \) depending both on \( \mathcal{N} = 2 \) gauge background superfields, their spinor derivatives up to some order and hypermultiplet background superfields using the formulation of the \( \mathcal{N} = 4 \) SYM theory in terms of \( \mathcal{N} = 1 \) superfields [27, 28] and exploring the derivative expansion techniques in \( \mathcal{N} = 1 \) superspace [29] (see also [30]). It allows us to obtain the exact coefficients at various powers of covariant spinor derivatives of the \( \mathcal{N} = 2 \) superfield Abelian strength \( \mathcal{W} \) corresponding to the constant space-time background that belongs to the Cartan subalgebra of the gauge group \( SU(N) \) spontaneously broken down to \( U(1)^{n-1} \) and constant space-time background hypermultiplet \( q^a \)

\[
\mathcal{W} = \Phi = \text{const}, \quad D^i_{\alpha} \mathcal{W} = \lambda^i_{\alpha} = \text{const}, \quad q^{ia} = \text{const},
\]

\[
D^{i(a}D_{\beta)i} \mathcal{W} = F_{\alpha\beta} = \text{const}, \quad D^{\alpha(i}D^{\beta)j} \mathcal{W} = 0, \quad D^{i}_{\alpha}q^{aj} = 0, \quad D^{i}_{\dot{\alpha}}q^{aj} = 0,
\]

where \( \Phi = \text{diag}(\Phi^1, \Phi^2, \ldots, \Phi^n) \), \( \sum \Phi^j = 0 \). This background is the simplest one allowing exact calculation of the one-loop effective action. We will show that in this case the \( \mathcal{N} = 1 \) superspace effective action can be uniquely found on the basis of the effective action for vanishing hypermultiplet [31, 29] by means of a simple replacement of variables. Following this, the obtained result is rewritten in a manifest \( \mathcal{N} = 2 \) supersymmetric form using the same procedure as in [31] but maintaining the complete hypermultiplet dependence.
We emphasize that the background (3) is a special supersymmetric solution to classical equations of motion of the $\mathcal{N} = 1$ superfield model representing the $\mathcal{N} = 4$ SYM theory in terms of $\mathcal{N} = 1$ superfields and therefore the effective action does not depend on the choice of $\mathcal{N} = 1$ superfield gauge fixing conditions we impose on the theory. Moreover, it can be shown that the background (3) is completely formulated in terms of $\mathcal{N} = 2$ superfields, which provides a possibility to write the effective action on this background in a manifest $\mathcal{N} = 2$ supersymmetric form. However, this background is not form-invariant under the hidden $\mathcal{N} = 2$ supersymmetry transformations of $\mathcal{N} = 4$ supersymmetry. The complete on-shell $\mathcal{N} = 4$ supersymmetry involves the transformations between the physical fields from the $\mathcal{N} = 2$ vector multiplet and from hypermultiplets. But the background (3) does not contain the physical spinor fields from hypermultiplets which are mixed with physical scalar fields from $\mathcal{N} = 2$ vector multiplet under hidden $\mathcal{N} = 2$ supersymmetry. Therefore it is likely from the outset that the effective action on this background will not possess the $\mathcal{N} = 4$ supersymmetry. The action is manifestly $\mathcal{N} = 2$ supersymmetric but hidden extra $\mathcal{N} = 2$ is violated. The only term in the effective action that do not violate the hidden supersymmetry is the one containing no spinor derivatives of $W$ and hypermultiplets, which is just the effective potential (2). We will discuss these important points in Section 5.

The paper is organized as follows. In the next Section we recall the known properties of $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ formalism and discuss the background field quantization including the choice of proper gauge fixing conditions. In Section 3 we describe the calculations leading to exact one-loop $\mathcal{N} = 1$ superfield effective action for the background (3). Section 4 is devoted to representation of this effective action in a manifest $\mathcal{N} = 2$ form and discussing the ambiguity of such a form. In Section 5 we demonstrate that the first subleading term (containing the eighth power of Abelian strength $F_{mn}$) in derivative expansion for the constructed manifestly $\mathcal{N} = 2$ supersymmetric effective action is non-invariant under the hidden $\mathcal{N} = 2$ supersymmetry transformations of $\mathcal{N} = 4$ SUSY and hence it is not $\mathcal{N} = 4$ supersymmetric. Then we examine the hypermultiplet-dependent terms which must be added to the known effective action in the vector multiplet sector in order to make the whole $F^8$ term in the complete effective action $\mathcal{N} = 4$ supersymmetric. We show that such terms can be found in the form of a finite order polynomial in spinor derivatives of harmonic hypermultiplet superfields and get the correct leading term in this polynomial. In Summary we formulate final results and discuss unsolved problems.

2 Minimal formulation of $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 1,2$ superspaces and $\mathcal{N} = 1$ supersymmetric background field method

Formulation of $\mathcal{N} = 4$ SYM theory possessing off-shell manifest $\mathcal{N} = 4$ supersymmetry is unknown so far. Therefore a study of the concrete quantum aspects of this theory is usually based on its formulation either in terms of physical component fields (see e.g. [32]) or in terms of $\mathcal{N} = 1$ superspace (see e.g. [27]) or in terms of $\mathcal{N} = 2$ harmonic superspace [23, 24]. In the first case, all four supersymmetries are hidden; in the second case, one of them is manifest and the other three are hidden; in the third case, two supersymmetries
are manifest and the other two are hidden. It is worth pointing out that in all cases at least some of the supersymmetries are on shell. Taking into account that the presence of manifest symmetries simplifies a process of calculations in quantum theory, it is reasonable to consider that at present just $\mathcal{N} = 2$ harmonic superspace formulation is the best one for quantum $\mathcal{N} = 4$ SYM theory. However, the formulation in terms of $\mathcal{N} = 1$ superspace has its own positive features basically due to a relatively simple structure of $\mathcal{N} = 1$ superspace and large accumulated experience of work with $\mathcal{N} = 1$ supergraphs.

$\mathcal{N} = 4$ superfield description of $\mathcal{N} = 4$ vector multiplet can be realized with the help of on-shell $\mathcal{N} = 4$ superfields $W^{AB}$, $A = 1 \ldots 4$ [33] satisfying the reality condition

$$ W^{AB} = \frac{1}{2} \varepsilon^{ABCD} W_{CD}, \quad W_{AB} = \bar{W}^{AB} $$

and on-shell constrains

$$ D_{\dot{a}} W^{BC} = \frac{1}{3} \delta^{[B}_{A} D_{E\dot{a}} W^{EC]}, \quad D^{(A} W^{B)C} = 0. $$

All physical fields of $\mathcal{N} = 4$ vector multiplet are contained in the superfield $W^{AB}$. We point out also the attempts to develop unconstrained formulation in harmonic superspace approach [34]. However, a $\mathcal{N} = 4$ off-shell supersymmetric action for $\mathcal{N} = 4$ SYM model is still unknown.

### 2.1 $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 1$ superspace

The physical field content of the superfield $W^{AB}$ can be obtained by combining three $\mathcal{N} = 1$ chiral superfields and one $\mathcal{N} = 1$ vector multiplet superfield [27]. Then, the six real scalars, which are the lowest components of the superfield $W^{AB}$, are represented by the three complex scalar components of the chiral $\mathcal{N} = 1$ superfields $\Phi^i$. The four Weyl fermions from $W^{AB}$ are divided into three plus one. Three of them are considered as the spinor components of $\Phi^i$ and the fourth fermion is treated as gaugino and constitutes, together with the real vector, the $\mathcal{N} = 1$ vector multiplet superfield $V$. In such a description, the $SU(3) \otimes U(1)$ subgroup of $SU(4)$ $R$- symmetry group is manifest and the representations of the $SU(4)$ are decomposed according to $6 \rightarrow 3 + \bar{3}, \quad 4 \rightarrow 3 + 1$ so that the chiral superfields $\Phi^i$ transform in the $3$ of $SU(3)$, the antichiral $\bar{\Phi}_i$ transforms in the $\bar{3}$ and the vector multiplet superfield is a singlet under $SU(3)$.

The action of $\mathcal{N} = 4$ SYM model is formulated in terms of $\mathcal{N} = 1$ superspace as follows

$$ S = \frac{1}{g^2} \text{tr} \{ \int d^4x d^2 \theta W^2 + \int d^4x d^4 \theta \Phi^i e^V \Phi^i e^{-V} + \frac{1}{3!} \int d^4x d^2 \theta i c_{ijk} \Phi^i [\Phi^j, \Phi^k] + \frac{1}{3!} \int d^4x d^2 \bar{\theta} i c^{ijk} \bar{\Phi}_i [\bar{\Phi}_j, \bar{\Phi}_k] \}. \quad (4) $$

The denotations and conventions correspond to Ref. [27]. All superfields here are taken in the adjoint representation of the gauge group. Both $\mathcal{N} = 1$ SYM and chiral superfield actions are superconformal invariants. In addition to the manifest $\mathcal{N} = 1$ supersymmetry and $SU(3)$ symmetry on the $i, j, k, \ldots$ indices of $\Phi$ and $\bar{\Phi}$, it has hidden global supersymmetry given by transformations

$$ \delta W_a = -\epsilon^i \bar{\nabla}^2 \bar{\Phi}_e i + i \epsilon_i^i \nabla a \alpha \bar{\Phi}_c, $$
$$ \delta \bar{W}_{\dot{a}} = -\bar{\epsilon}_{\dot{a}i} \nabla^2 \Phi^i + i \bar{\epsilon}_{\dot{a}i} \nabla a \alpha \Phi_c, $$
$$ \delta \Phi^i = \epsilon^{ai} W_a, \quad \delta \bar{\Phi}_c = \bar{\epsilon}_c^i \bar{W}_{\dot{a}}. \quad (5) $$
The action (4) is also invariant under the transformations

\[
\begin{align*}
\delta \Phi^i_c &= e^{ijk}\nabla^2(\bar{\chi}_j \Phi^k_c) + i[\chi^i \bar{\Phi}^j_c, \Phi^i_c], \\
\delta \bar{\Phi}^i_{ci} &= c_{ijk}\nabla^2(\chi^j \bar{\Phi}^k_c) + i[\bar{\chi}_j \Phi^i_c, \bar{\Phi}^i_{ci}],
\end{align*}
\]

(6)

Here the covariant spinor derivatives \(\nabla_\alpha, \nabla_{\bar{\alpha}}, \nabla^2\) and \(\nabla^2\) are defined in Ref. [27] and \(\chi^i\) are the \(\mathcal{N} = 1\) superfield parameters forming the \(SU(3)\)-isospinor as well as \(\Phi^i\). These parameters include the central charge transformation parameters, supersymmetry transformation parameters and internal symmetry parameters of \(SU(4)/SU(3)\). The transformations (6) are given in terms of background covariant superfields \(\Phi^i_c = e^{ijk}\Phi^j_{ci} \bar{\Omega}, \bar{\Phi}^i_c = e^{ijk}\bar{\Phi}^j_c \Omega\) [27].

Further we use only these covariant chiral superfields and the subscript \(c\) is omitted. It is convenient to introduce the new notations \(\Phi^1 = \Phi, \Phi^2 = Q, \Phi^3 = \bar{Q}\) and rewrite two last terms in (4) as follows

\[
i \int d^4x d^2\theta Q[\Phi, \bar{Q}] + i \int d^4x d^2\bar{\theta} \bar{Q}[\bar{\Phi}, \bar{\bar{Q}}],
\]

which is \(\mathcal{N} = 1\) form of hypermultiplet and lowest component of the chiral \(\mathcal{N} = 2\) field strength vector multiplet interaction for \(\mathcal{N} = 4\) model.

If the gauge group is Abelian, we get a free model. In the non-Abelian case, the theory has a moduli space of vacua parameterized by the vev’s of the six scalars. The manifold of vacua is determined by the conditions of vanishing scalar potential (F-flatness plus D-flatness) [35]. The solutions to the equations determining a vacuum structure of the theory can be classified according to the phase of the gauge theory they give rise to. In the pure Coulomb phase, each scalar field can have its specific non-vanishing vev. As a result, space of vacua is \(\mathcal{M} = R^{6r}/S_r\) where \(S_r\) is the Weyl group of permutations for \(r\) elements and an unbroken gauge group is \(U(1)^r\). But when several vev’s coincide, some non-Abelian group \(G \in SU(N)\) remains unbroken and some massless gauge bosons appear in the theory.

### 2.2 \(\mathcal{N} = 4\) SYM theory in \(\mathcal{N} = 2\) harmonic superspace

From \(\mathcal{N} = 2\) supersymmetry point of view, the \(\mathcal{N} = 4\) vector multiplet consists of the \(\mathcal{N} = 2\) vector multiplet and hypermultiplet. Therefore the \(\mathcal{N} = 4\) SYM action can be treated as some special \(\mathcal{N} = 2\) supersymmetric theory, action of which is the action for \(\mathcal{N} = 2\) SYM theory plus action describing the hypermultiplet \(q^{a\alpha}\) in adjoint representation coupled to \(\mathcal{N} = 2\) vector multiplet. Such a theory is formulated in \(\mathcal{N} = 2\) harmonic superspace [23, 24]. The dynamic variables in this case are real unconstrained analytic gauge superfield \(V^{++}\) and complex unconstrained analytic superfield \(q^+\). The harmonic gauge connection \(V^{++}\) serves as the potential of the \(\mathcal{N} = 2\) SYM theory and \(q^+\) describes the hypermultiplet. Action of the \(\mathcal{N} = 4\) SYM theory looks like

\[
S[V^{++}, q^+, \bar{q}^+] = \frac{1}{2g^2} \text{tr} \int d^8z \mathcal{W}^2 - \frac{1}{2g^2} \text{tr} \int d\zeta^{-4} q^{a\alpha} \mathcal{D}^{++} q^+_a.
\]

(7)

The corresponding equations of motion are

\[
\mathcal{D}^{++} q^{a\alpha} + ig [V^{++}, q^{a\alpha}] = 0, \quad \mathcal{D}^{++} \mathcal{D}^+_a \mathcal{W} = [q^+_a, q^+_a].
\]

(8)

Here \(a = 1, 2\) is the index of the rigid \(SU(2)\) symmetry, \(q^+_a = (q^+_1, \bar{q}^+_1), \quad q^{a\alpha} = \varepsilon^{ab} q^{b\alpha} = (\bar{q}^+_1, -q^+_1)\) and \(\mathcal{W}\) is the strength of \(\mathcal{N} = 2\) analytic gauge superfield \(V^{++}\) connection in the
in the harmonic variables $u^\pm$. The derivatives $D^{+\alpha}_\alpha$ do not need a connection in the frame where $G$-analyticity [23, 24] is manifest. All other denotations are given in Ref. [24]. Equations (8) present the $\mathcal{N} = 4$ SYM field equations of motion written in terms of $\mathcal{N} = 2$ superfields. The off-shell action (7) allows to develop the manifest $\mathcal{N} = 2$ supersymmetric quantization. Moreover, this action is invariant under hidden extra $\mathcal{N} = 2$ supersymmetry transformations [24] which mix up $\mathcal{W}, \bar{\mathcal{W}}$ with $q^+_\alpha$. For our aim it is sufficient to point out that in the Abelian case the corresponding transformations of hidden $\mathcal{N} = 2$ supersymmetry are defined only on shell and have the form

$$\delta \mathcal{W} = \frac{1}{2} \varepsilon^{\hat{a} a} \bar{D}^\alpha_\alpha q^+_a, \quad \delta \bar{\mathcal{W}} = \frac{1}{2} \varepsilon^{\alpha a} D_\alpha q^+_a,$$

$$\delta q^+_a = \frac{1}{4} (\varepsilon^a_\alpha D^+_\alpha \mathcal{W} + \varepsilon^{\hat{a} \hat{a}} D^+_{\hat{a}} \bar{\mathcal{W}}), \quad \delta q^-_a = \frac{1}{4} (\varepsilon^a_\alpha D^-_\alpha \bar{\mathcal{W}} + \varepsilon^{\hat{a} \hat{a}} D^-_{\hat{a}} \bar{\mathcal{W}}).$$

As a result, the model under consideration is $\mathcal{N} = 4$ supersymmetric on shell.

Vacuum structure of the model (7) in Abelian case is defined in terms of solutions to the following equations

$$(\mathcal{D}^+)^2 \mathcal{W} = (\bar{\mathcal{D}}^+)^2 \bar{\mathcal{W}} = 0, \quad D^{++} q^+ = 0, \quad (10)$$

which are simple consequences of the Eqs. (8) in Abelian case. Equations (10) for physical components of the $\mathcal{N} = 4$ vector multiplet determined by the expansion

$$q^+(\zeta, u) = f^i(x) u^+_i + \theta^{\alpha a} \psi^a_\alpha (x) + \theta^+_\alpha \bar{\kappa}^\alpha (x) + 2i \theta^+ \bar{\theta}^+ f^i(x) u^-_i,$$

$$\mathcal{W} = \phi(x) + \theta^- \lambda^a_\alpha (x) + \theta^{(+, +) \theta^{-, -}} F_{\alpha \beta}(x),$$

look like

$$\delta \phi = \delta \bar{\kappa} = \Box f^i = \Box \phi = \delta \lambda^i = \partial_m F_{mn} = 0. \quad (12)$$

The simplest solution to these equations of motion forms a set of constant background fields

$$f^i = \text{const}, \psi = \text{const}, \bar{\kappa} = \text{const}, \phi = \text{const}, \partial_m \bar{F}_{mn} = \text{const} \quad (13)$$

which transform linearly through each other under hidden $\mathcal{N} = 2$ supersymmetry transformations (9):

$$\delta \phi = \frac{1}{2} \varepsilon^{\hat{a} a} \bar{F}_{\alpha a}, \quad \delta \bar{\phi} = \frac{1}{2} \varepsilon^{a \alpha} \bar{F}_{\hat{a} \alpha}, \quad \delta \psi = \frac{1}{2} \varepsilon^{\bar{\alpha} \alpha} \bar{F}_{\alpha a}, \quad \delta \bar{\psi} = \frac{1}{2} \varepsilon^{\bar{\alpha} \alpha} \bar{F}_{\hat{a} \alpha}, \quad \delta \bar{\kappa} = \frac{1}{2} \varepsilon^{\alpha a} \bar{F}_{\alpha a}, \quad \delta \bar{\kappa} = \frac{1}{2} \varepsilon^{\bar{a} \alpha} \bar{F}_{\alpha a},$$

$$\delta \lambda^i = \frac{1}{4} \varepsilon^{\alpha a} \lambda^i_\alpha \bar{\lambda}^\alpha + \frac{1}{4} \varepsilon^{\bar{a} \alpha} \lambda^i_\alpha \bar{\lambda}^\alpha, \quad \delta \bar{\lambda}^i_\alpha = 0, \quad \delta F_{\alpha \beta} = 0. \quad (14)$$

The solution (13) is the simplest vacuum configuration carrying out a representation of $\mathcal{N} = 4$ supersymmetry and allowing to calculate the $\mathcal{N} = 4$ supersymmetric low-energy effective action in $\mathcal{N} = 4$ SYM theory.

It is instructive to compare the $\mathcal{N} = 4$ supersymmetric backgrounds (13) and the background (3). Last background contains the components $\phi$ and $F$ of the $\mathcal{N} = 2$ vector multiplet when the components $\bar{\kappa}$ and $\psi$ of the hypermultiplet are absent. As a result, the background (3) is not form-invariant under the hidden $\mathcal{N} = 2$ supersymmetry transformations (14) and therefore this background is not a representation of $\mathcal{N} = 4$ supersymmetry. However, the background (3) is a representation of manifest $\mathcal{N} = 2$
supersymmetry. Therefore we can state that effective action found on the background \(\mathcal{N} = 2\) within \(\mathcal{N} = 2\) background field method will be manifestly \(\mathcal{N} = 2\) supersymmetric and gauge invariant but it should not be completely \(\mathcal{N} = 4\) supersymmetric. The explicit calculations, fulfilled in Sections 3 and 4, confirm this point of view. To construct the complete \(\mathcal{N} = 4\) supersymmetric effective action, we can follow the approach developed in [22]. We consider the effective action in \(\mathcal{N} = 2\) vector multiplet sector and find such an action depending both on \(\mathcal{N} = 2\) vector multiplet and on hypermultiplet so that the sum of above actions would be invariant under hidden \(\mathcal{N} = 2\) supersymmetry transformations (9). This idea is realized in Section 5 for finding leading contributions to the \(\mathcal{N} = 4\) supersymmetric form of the \(F^8\)-term in the effective action.

### 2.3 \(\mathcal{N} = 1\) background field quantization

For computation of the effective action we use \(\mathcal{N} = 1\) superfield background field method (see e.g. [27, 28]) in combination with \(\mathcal{N} = 1\) superfield heat kernel techniques [28]. These methods for constructing the effective action in gauge field theories allow to preserve a classical gauge invariance in quantum theory and sum, in principle, an infinite set of Feynman diagrams to a single gauge invariant functional depending on the background fields. As we pointed out, the theory under consideration can be formulated either in terms of component fields, or in terms of \(\mathcal{N} = 1\) superfields, or in terms of \(\mathcal{N} = 2\) harmonic superfields. The evaluation of effective action in component formulation is extremely complicated even within background field method because of a very large number of interacting fields and the absence of manifest supersymmetry. The effective action can be studied within \(\mathcal{N} = 2\) harmonic superspace. The corresponding \(\mathcal{N} = 2\) background field method was proposed in Refs [26]. The aspects of heat kernel techniques were considered in Refs. [36]. However, these techniques are undeveloped yet in many details to be extended for our goals. We take into account a problem of matrix operators mixing the \(\mathcal{N} = 2\) vector multiplet and hypermultiplet sectors. Of course, the effective action can be studied using harmonic supergraphs but we expect here meeting with the standard problem of how to organize the calculations allowing to go beyond leading low-energy approximation (see calculations in leading low-energy approximation e.g. in Refs. [11, 18, 25, 26, 36]. Therefore, we work within a formulation in terms of \(\mathcal{N} = 1\) superfields and use our experience to work with the theories formulated in \(\mathcal{N} = 1\) superspace [28, 29, 20].

The background field method is based on splitting the fields into classical and quantum and imposing the gauge fixing conditions, i.e. preservation of a classical gauge invariance, only on quantum fields. Of course, the gauge fixing conditions can break some classical symmetries as it was mentioned above (for detail see e.g. Refs. [37, 38]). We define one-loop effective action \(\Gamma\) depending on the background superfields (3) by a path integral over quantum fields in the standard form

\[
e^{i\Gamma} = \int \mathcal{D}v \mathcal{D}\varphi \mathcal{D}c \mathcal{D}c' \mathcal{D}\bar{c} \mathcal{D}\bar{c}' e^{i(S_{(2)} + S_{FP})},
\]

where \(S_{(2)}\) is a quadratic in quantum fields part of the classical action including a gauge-fixing condition and \(S_{FP}\) is a corresponding ghost action. Formal calculating the above path integral leads to a functional determinant representation of effective action (see (24)). The main technical tool we use in this paper for the \(\mathcal{N} = 1\)-superfield calculations is the
background covariant gauge-fixing multi-parametrical conditions

$$S_{GF} = -\frac{1}{\alpha g^2} \int d^8z \left( F^A \tilde{F}^A + b^A \tilde{b}^A \right), \quad (16)$$

here $b, \tilde{b}$ are the Nielsen-Kallosh ghosts. We choose the convenient gauge-fixing conditions for the quantum superfields $v$ and $\varphi$ in the form

$$\tilde{F}^A = \nabla^2 v^A + \lambda \left[ \Box - \nabla^2 \varphi^2, \tilde{\Phi}_i \right]^A, $$
$$F^A = \nabla^2 v^A - \lambda \left[ \Box - \nabla^2 \tilde{\varphi}_i, \Phi_i \right]^A, \quad (17)$$

where $\alpha, \lambda, \tilde{\lambda}$ are the arbitrary numerical parameters and $\Box_+, \Box_-$ are the standard notations for Laplace-like operators in the $N = 1$ superspace. It is evident that the gauge fixing functions (17) are covariant under background gauge transformations. These gauge fixing functions (17) can be considered as a superfield form of so-called $R_g$-gauges (see Refs. [39, 40]) which are usually used in spontaneously broken gauge theories. Since an Abelian background is a solution to classical equations of motion, we won’t worry about the choice of gauge-fixing parameters. Therefore it is convenient to take gauge-fixing which we call the Fermi-DeWitt gauge: $\alpha = \lambda = 1$. Such a choice of the gauge parameters allows us to avoid the known problem [41] in the functional determinant method for calculating the mixed contribution which contains vector-chiral superfield propagators circulating along the loop. Calculation of such contributions using a conventional gauge is related to the necessity of working with a cumbersome expression:

$$\text{Tr} \ln(-\Box + iW^\alpha \nabla_\alpha + i\bar{W}^\dot{\alpha} \nabla_{\dot{\alpha}} + M - \bar{X} \frac{1}{\Box_+ - \mu \bar{X}} \nabla \bar{X}^2 \nabla^2 - X \frac{1}{\Box_- - \mu \bar{X}} \nabla \bar{X}^2 \nabla^2 + \bar{X} \frac{1}{\Box_+ - \mu \bar{X}} \nabla \bar{X}^2 \nabla^2 + X \frac{1}{\Box_- - \mu \bar{X}} \nabla \bar{X}^2 \nabla^2), \quad (18)$$

where the notations of [41] were used.

We want to note once again that in gauge theories not all rigid symmetries of the classical action can be maintained manifestly in quantum theory, even in the absence of anomalies. The issue here is that quantization requires gauge fixing and the latter, as can be shown, breaks some symmetries (breaking the classic conformal symmetry is discussed e.g. in Ref. [38]). This is the known general situation (see e.g. [37]). In our case, the gauge-fixing (16) obviously also breaks rigid classic $N = 4$ symmetry (5, 6) since it is covariant only under $N = 1$ supersymmetry transformations. Therefore the effective action obtained should be invariant only under quantum deformed hidden transformations (5). This deformations can, in principle, be computed at each loop order but this problem is beyond the purposes of this work.

After splitting each field on the background and quantum part (i.e. $e^{V_{tot}} = e^{\Omega g^2 v^2 \tilde{g}} \Phi$ $\rightarrow$ $\Phi + \varphi, \tilde{\Phi} \rightarrow \tilde{\Phi} + \tilde{\varphi}, Q \rightarrow Q + q, \tilde{Q} \rightarrow \tilde{Q} + \tilde{q}, \bar{Q} \rightarrow \bar{Q} + \bar{\tilde{q}}, \bar{\tilde{Q}} \rightarrow \bar{\tilde{Q}} + \bar{\tilde{\tilde{q}}}$), we can rewrite a quadratic part of sum of the classical action (4) and gauge fixing action (16) in the form

$$S_{(2)} = -\frac{1}{2} \sum_{I < J} \int d^4x d^4\theta \left( \mathcal{F}^{IJ} \mathcal{H}_{IJ} \mathcal{F}^{IJ} + d^{IJ} (O_V - M)_{IJ} v^{IJ} \right), \quad (19)$$

where $\mathcal{F} = (\varphi, \varphi, \tilde{q}, q, \tilde{\tilde{q}}, \tilde{\tilde{q}})$, $\mathcal{F}^T = (\varphi, \varphi, q, q, \tilde{q}, \tilde{q})$, $M_{IJ} = (\Phi_{IJ} \Phi_{IJ} + \bar{Q}_{IJ} Q_{IJ} + \tilde{Q}_{IJ} \tilde{Q}_{IJ})$, \quad (20)
It leads to the following contribution of the ghosts to the effective action to the Cartan subalgebra and $\Phi$ and $W$. It should be noted that generally speaking, the second variation of the classical action where the following notations were used:

$$S = \int d^8 z \left( \phi'\phi - \frac{1}{2} \phi^2 \right) - \int d^8 z \left( \Phi'\Phi - \frac{1}{2} \Phi^2 \right) - \int d^8 z \left( W'W - \frac{1}{2} W^2 \right)$$

The explicit form of this matrix looks like

$$\begin{pmatrix}
G_+(\phi)\bar{\nabla}^2\nabla^2 & 0 & -\phi\bar{f}\frac{\nabla^2\bar{x}^2}{0} & -i\bar{v}\nabla^2 & -\phi\bar{f}\frac{\nabla^2\bar{x}^2}{0} & -i\bar{f}\nabla^2 \\
0 & G_-(\phi)\bar{\nabla}^2\nabla^2 & -i\bar{v}\nabla^2 & \bar{f}\frac{\nabla^2\bar{x}^2}{0} & \bar{f}\frac{\nabla^2\bar{x}^2}{0} & \bar{f}\frac{\nabla^2\bar{x}^2}{0} \\
-\bar{f}\frac{\nabla^2\bar{x}^2}{0} & -i\bar{v}\nabla^2 & G_+(f)\bar{\nabla}^2\nabla^2 & 0 & \bar{v}\frac{\nabla^2\bar{x}^2}{0} & \bar{v}\frac{\nabla^2\bar{x}^2}{0} \\
-\bar{v}\frac{\nabla^2\bar{x}^2}{0} & \bar{f}\frac{\nabla^2\bar{x}^2}{0} & -i\Phi^2 & 0 & i\bar{\phi}\nabla^2 & -\bar{f}\frac{\nabla^2\bar{x}^2}{0} \\
i\bar{v}\frac{\nabla^2\bar{x}^2}{0} & -\bar{f}\frac{\nabla^2\bar{x}^2}{0} & -i\bar{\phi}\nabla^2 & \bar{f}\frac{\nabla^2\bar{x}^2}{0} & 0 & G_+(v)\bar{\nabla}^2\nabla^2 \\
i\bar{f}\frac{\nabla^2\bar{x}^2}{0} & i\bar{\phi}\nabla^2 & \bar{f}\frac{\nabla^2\bar{x}^2}{0} & -\bar{v}\frac{\nabla^2\bar{x}^2}{0} & 0 & G_-(v)\bar{\nabla}^2\nabla^2
\end{pmatrix},$$

where the following notations were used:

$$G_\pm(a) = 1 - \frac{(a\bar{a})}{\Box_{\pm}}, \quad \phi = \Phi_{IJ}, \quad \bar{\phi} = \bar{\Phi}_{IJ}, \quad f = Q_{IJ}, \quad \bar{f} = \bar{Q}_{IJ}, \quad v = \bar{Q}_{IJ}, \quad \bar{v} = \bar{Q}_{IJ}$$

and $\Box_{\pm}$ means $\nabla^2\bar{\nabla}^2$ and $\bar{\nabla}^2\nabla^2$ respectively. In the space of chiral and antichiral superfields these operators act as follows:

$$\nabla^2\bar{\nabla}^2 = \Box_{\pm} = \Box - i\bar{W}^\alpha\nabla_\alpha - \frac{i}{2}(\nabla\bar{W}),$$

$$\bar{\nabla}^2\nabla^2 = \Box_{\bar{\pm}} = \Box - iW^\alpha\nabla_\alpha - \frac{i}{2}(\nabla W).$$

It should be noted that generally speaking, the second variation of the classical action leads to a $7 \times 7$ matrix operator. But the chosen gauge-fixing condition (17) allows partial diagonalization to a $1 \times 1 \oplus 6 \times 6$ block matrix and separation of kinetic operator for the vector fields. Hence we avoid the functional trace calculation problem for the above-mentioned cumbersome operator. This gauge-fixing condition eliminates the interaction vertexes between quantum matter fields and quantum vector fields but generates new interaction vertexes between quantum chiral fields and ghosts.

Let us consider now a structure of Faddeev-Popov ghost contribution to the one-loop effective action. The action of the Faddeev-Popov ghosts $S_{FP}$ for the gauge fixing functions (17) has the form:

$$S_{FP} = \text{tr} \int d^8 z \left( \bar{c}'c - c'\bar{c} \right) - \left( c'[\Phi'_i, \frac{\lambda}{\Box_{\pm}}[c, \Phi_i]] + \bar{c}'[\bar{\lambda}_{\pm}, [c, \Phi_i]] \right).$$

It leads to the following contribution of the ghosts to the effective action:

$$\ln(\text{Det}(H_{FP})) = 2 \sum_{l<j} \text{Tr} \ln \left( \begin{pmatrix} 0 & (1 - \frac{M}{\Box_{\pm}})\nabla^2\bar{\nabla}^2 \\ -(1 - \frac{M}{\Box_{\pm}})\bar{\nabla}^2\nabla^2 & 0 \end{pmatrix} \right)_{IJ},$$

where $G_\pm(a) = 1 - \frac{(a\bar{a})}{\Box_{\pm}}, \quad \phi = \Phi_{IJ}, \quad \bar{\phi} = \bar{\Phi}_{IJ}, \quad f = Q_{IJ}, \quad \bar{f} = \bar{Q}_{IJ}, \quad v = \bar{Q}_{IJ}, \quad \bar{v} = \bar{Q}_{IJ}$
where $M$ was defined in (20).

The final result of the integration in path integral (15) over all quantum superfields is given by formal representation for the one-loop effective action in terms of functional determinants

$$e^{iT} = \prod_{I<J} \text{Det}^{-1}(O_\nu - M) \text{Det}^{-1}(H) \text{Det}^2(H_{FP}).$$

(24)

Since the strengths $\Phi$ and $W_{\alpha}$ belong to the Cartan subalgebra, only half of roots should be taken into account during the integration over the quantum fields and the effective action looks like

$$\Gamma = \sum_{I<J} \Gamma_{IJ}.$$ 

Our next purpose is a computation of the above functional determinants.

### 3 Evaluations of superfield functional traces and one-loop effective action.

In this section we present the basic steps of functional traces calculations for the operators, which make background-dependent contributions into the effective action (24). It is seen from (21) that in the absence of background superfields $Q, \tilde{Q}$, the matrix operator $H$ includes only the background-dependent inverse propagators $G_+, G_-$ and vertices for the background field $\Phi$ interacting with the quantum hypermultiplet. Such a situation has been studied in detail (see e.g. Refs. [29, 30, 41, 42]). It should be noted that the form of $H$ containing dressed inverse propagators is directly related to the $R_\xi$ gauge fixing conditions (17).

On the first stage we divide the matrix $H$ into a sum of two matrices $H = H_\square + H_\triangledown$ where the matrix $(H_\square)$ contains all blocks with $\nabla^2 \nabla^2, \nabla^2 \nabla^2$ and the matrix $H_\triangledown$ contains blocks with $\nabla^2$ and $\nabla^2$ only. Let's present the logarithm of matrix $\ln(H_\square)$ as follows

$$\ln(H) = \ln(H_\square) + \ln(1 - (H_\square^{-1}H_\triangledown)).$$

Using the known Frobenius formula for inversion of block type matrix

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad H^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BE^{-1}CA^{-1} & -A^{-1}BE^{-1} \\ -E^{-1}CA^{-1} & E^{-1} \end{pmatrix},$$

where $E = D - CA^{-1}B$, we get by direct calculation the inverse matrix for $H_\square$

$$H_\square^{-1} = \begin{pmatrix} g_+(\phi) \frac{\nabla^2 \nabla^2}{\Box^2_+} & 0 & \phi f_+ \frac{\nabla^2 \nabla^2}{\Box^2_+} & 0 & \phi_0 \frac{\nabla^2 \nabla^2}{\Box^2_+} & 0 & 0 \\ 0 & 0 & g_- (\phi) \frac{\nabla^2 \nabla^2}{\Box^2_-} & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi f_- \frac{\nabla^2 \nabla^2}{\Box^2_-} & g_+ (f) \frac{\nabla^2 \nabla^2}{\Box^2_-} & 0 & 0 & 0 \\ 0 & 0 & \phi_0 \frac{\nabla^2 \nabla^2}{\Box^2_-} & 0 & g_- (f) \frac{\nabla^2 \nabla^2}{\Box^2_-} & 0 & 0 \\ 0 & 0 & \phi f_+ \frac{\nabla^2 \nabla^2}{\Box^2_+} & 0 & \phi_0 \frac{\nabla^2 \nabla^2}{\Box^2_+} & g_+ (v) \frac{\nabla^2 \nabla^2}{\Box^2_+} & 0 \\ 0 & 0 & \phi f_- \frac{\nabla^2 \nabla^2}{\Box^2_-} & 0 & \phi_0 \frac{\nabla^2 \nabla^2}{\Box^2_-} & 0 & g_- (v) \frac{\nabla^2 \nabla^2}{\Box^2_-} \end{pmatrix},$$

(25)
here we have introduced the notations

\[ g_{\pm}(\phi) = 1 + \frac{\phi\phi^*}{\Box_{\pm M}}, \Box_{\pm M} = \Box_{\pm} - M. \]

One can note that the combination \( M \) (20) has naturally appeared during the inversion procedure. Then we find the product \( H_{\Box}^{-1}H_{\nabla} \) in a remarkably simple form

\[
H_{\Box}^{-1}H_{\nabla} = \begin{pmatrix}
0 & 0 & 0 & iv^2_{0+} & 0 & -if^2_{0+} \\
0 & 0 & iv^2_{0+} & 0 & -if^2_{0+} & 0 \\
0 & iv^2_{0+} & 0 & 0 & 0 & i\phi^2_{0+} \\
i\phi^2_{0+} & 0 & 0 & -\phi^2_{0+} & 0 & 0 \\
g_{\pm}(\phi) = 1 + \frac{\phi\phi^*}{\Box_{\pm M}}, \Box_{\pm M} = \Box_{\pm} - M. \]

The next stage consists in matrix traces calculations. Let’s expand \( \text{Tr}(\ln(1-(H_{\Box}^{-1}H_{\nabla}))) \) in a series in powers of \( H_{\Box}^{-1}H_{\nabla} \). The nonzero matrix traces will have only even powers of the series, which are grouped into

\[
\text{Tr}_{6\times6}(\ln(1-\ln(H_{\Box}^{-1}H_{\nabla}))) = \text{Tr} \left( \ln \left( 1 - \frac{M}{\Box_{+}} \right) \frac{\nabla^2\nabla^2}{\Box_{+}} \right) + \text{Tr} \left( \ln \left( 1 - \frac{M}{\Box_{-}} \right) \frac{\nabla^2\nabla^2}{\Box_{-}} \right),
\]

(27)

where \( M \) was introduced in (20) and \( \text{Tr} \) means the functional trace. Also we have to consider the matrix trace of the \( \ln(H_{\Box}) \). According to the above strategy, we write the matrix as a diagonal matrix plus the rest, i.e. \( H_{\Box} = H_0 + \Delta \):

\[
\text{Tr} \ln(H_{\Box}) = \text{Tr} \ln(H_0) + \text{Tr} \ln(1 + H_{\Box}^{-1}\Delta),
\]

(28)

where matrix \( H_0 \) contains only \( \nabla^2\nabla^2 \) and \( \nabla'^2\nabla'^2 \) at zero background fields \( \Phi, Q, \bar{Q} \) and therefore can be omitted. The matrix elements of \( H_{\Box}^{-1}\Delta \) are blocks with chiral \( \Sigma^2_{\Box+} \) and antichiral \( \Sigma'^2_{\Box-} \) projectors. After permutation of the lines and columns, the trace logarithm of the matrix \( 1 + H_0^{-1}\Delta \) can be reorganized as follows

\[
\text{Tr}_{6\times6}\ln(1 + H_0^{-1}\Delta) = \text{Tr}_{3\times3} \ln \left( 1 - \frac{\nabla^2\nabla^2}{\Box_{+}} \right) + \frac{\nabla^2\nabla^2}{\Box_{+}} \rightarrow \frac{\nabla^2\nabla^2}{\Box_{-}}.
\]

(29)

The direct calculation of the matrix traces for the first terms in the Taylor series allows to write the result

\[
\text{Tr}_{6\times6}\ln(1 + H_0^{-1}\Delta) = \text{Tr} \left( \ln(1 - \frac{M}{\Box_{+}}) \frac{\nabla^2\nabla^2}{\Box_{+}} \right) + \text{Tr} \left( \ln(1 - \frac{M}{\Box_{-}}) \frac{\nabla^2\nabla^2}{\Box_{-}} \right).
\]

(30)

that together with (27) gives

\[
\ln(\text{Det}^{-1}(H)) = -2\text{Tr} \left( \ln(1 - \frac{M}{\Box_{+}}) \frac{\nabla^2\nabla^2}{\Box_{+}} \right) - 2\text{Tr} \left( \ln(1 - \frac{M}{\Box_{-}}) \frac{\nabla^2\nabla^2}{\Box_{-}} \right).
\]

(31)
The contribution of the Faddeev-Popov ghosts is determined by eq. (23). Extracting and neglecting the expression \( \ln \left( \left( \begin{array}{cc} 0 & \nabla^2 \nabla^2 \\ -\nabla^2 \nabla^2 & 0 \end{array} \right) \right) \), we obtain the ghost contribution to the effective action in the form

\[
\ln(\text{Det}^2(H_{FP})) = 2\text{Tr} \left( \ln(1 - \frac{M}{\Box_-}) \frac{\nabla^2 \nabla^2}{\Box_-} \right) + 2\text{Tr} \left( \ln(1 - \frac{M}{\Box_+}) \frac{\nabla^2 \nabla^2}{\Box_+} \right),
\]

(32)

which is exactly (31) with the opposite sign. Therefore second and third functional determinants in (24) cancel each other. This surprising cancellation between contributions of ghost and chiral fields to one-loop effective action in \( \mathcal{N} = 4 \) SYM theory was firstly noted in [26] in harmonic superspace approach. It should be especially pointed out that this result is correct only on the constant chiral superfield background.

Finally, due to the cancellation between (32) and (31), the whole one-loop contribution to the effective action (24) has an extremely simple form and is determined only by vector loop contribution

\[
\Gamma = i \sum_{I<J} \text{Tr} \ln(O_V - M)_{IJ},
\]

(33)

and all background superfield dependence is encoded in \( M \). For the operator in the above relation, the powers expansion over Grassmann derivatives of the gauge field strength of the functional trace has been already calculated by different ways for models with one chiral background superfield (see [29, 30, 31] and reference therein). As a result, we transformed a rather complicated problem with hypermultiplet background to a known problem. The feature of the theory with hypermultiplet background is the combination (20) \( M = \langle \Phi \Phi + \bar{Q}Q + \tilde{Q}\tilde{Q} \rangle \) which is invariant under the \( R \)-symmetry group of \( \mathcal{N} = 4 \) supersymmetry. That allows application of the results obtained for \( \mathcal{N} = 1 \) models to the case under consideration making the corresponding redefinition of the quantity \( M \).

The functional trace (33) can be written as a power expansion of dimensionless combinations \( \Psi, \bar{\Psi} \) in hypermultiplet superfields, where

\[
\bar{\Psi}^2 = \frac{1}{M^2} \nabla^2 \bar{W}^2, \quad \Psi^2 = \frac{1}{M^2} \bar{\nabla}^2 \bar{W}^2.
\]

(34)

In the constant field approximation this expansion is summed to the following expression for the whole one-loop effective action (see details in [31]):

\[
\Gamma = \frac{1}{8\pi^2} \int d^8z \int_0^\infty dt \, t e^{-t} \frac{W^2 \bar{W}^2}{M^2} \omega(t\Psi, t\bar{\Psi}),
\]

(35)

\[
\omega(t\Psi, t\bar{\Psi}) = \frac{\cosh(t\Psi) - 1}{t^2\Psi^2} \frac{\cosh(t\bar{\Psi}) - 1}{t^2\bar{\Psi}^2} \frac{t^2(\Psi^2 - \bar{\Psi}^2)}{\cosh(t\Psi) - \cosh(t\bar{\Psi})}.
\]

As a result, we see that the only difference between the effective actions with and without the hypermultiplet background is stipulated by the structure of matrix \( M \) defined by (20). In component form, the closed relation for one-loop effective action (35) has natural Schwinger-type expansion over \( F^2/M^2 \) powers. The expansion doesn’t include \( F^6 \) term that is a property of \( \mathcal{N} = 4 \) SYM theory [31, 32]. The function \( \omega \) defined in (35) (see [31]) has the following expansion

\[
\omega(x, y) = \frac{1}{2} + \frac{x^2 y^2}{4 \cdot 5!} - \frac{5}{12 \cdot 7!} (x^4 y^2 + x^2 y^4) + \frac{1}{34500} (x^2 y^6 + x^6 y^2) + \frac{1}{86400} x^4 y^4 + \ldots
\]

(36)
Eq. (36) allows to expand the effective action (35) in series in powers of $\Psi^2$, $\bar{\Psi}^2$ as follows

$$\Gamma = \Gamma^{(0)} + \Gamma^{(2)} + \Gamma^{(3)} + \cdots,$$

(37)

where the term $\Gamma^{(n)}$ contains terms $c_{m,l}\Psi^m\bar{\Psi}^l$ with $m + l = n$. In the bosonic sector, this expansion corresponds to expansion in powers of the strength $F$, namely $\Gamma^{(n)} \sim F^{4+2n}/M^{2+2n}$, $M = (\Phi \bar{\Phi} + f_{ia} f_{ia})$, where $\Phi$, $\bar{\Phi}$ and $f_{ia}$ are physical bosonic fields of the $\mathcal{N} = 2$ vector multiplet and hypermultiplet.

4 Transformation of the $\mathcal{N} = 1$ supersymmetric effective action to a manifest $\mathcal{N} = 2$ supersymmetric form

The effective action (35) and its expansion (37) are given in terms of $\mathcal{N} = 1$ superfields. Our next purpose is to find a manifest $\mathcal{N} = 2$ form of each term in expansion (37). To do that, we extract from $M$ the $\mathcal{N} = 1$ form of the $X = -\bar{Q}Q + \bar{\bar{Q}}\bar{Q}$, (which was defined in eq. (2) in terms of $\mathcal{N} = 2$ superfields) writing $M = \Phi \Phi (1 - X)$, and then expand the denominator $(1/M)^k$ from (35) in a power series in $X$. This expansion leads to the following form for a generic term of the series

$$\int d^8z \frac{W^2\bar{W}^2}{(\Phi \Phi)^{2(m+1+k+1)}} \left(\nabla^2 W^2\bar{W}^2\right)^m \left(-\bar{Q}Q + \bar{\bar{Q}}\bar{Q}\right)^k$$

(38)

Further, using $\int d^{12}z = \int d^8z(\nabla_2)^2(\bar{\nabla}_2)^2$ and definitions of $\mathcal{N} = 1$ projections for $\mathcal{N} = 2$ on-shell vector multiplet $\mathcal{W} = \Phi, \nabla_2 \mathcal{W} = -W_a, \nabla^2_2 \mathcal{W} = 0$, we can reconstruct $\mathcal{N} = 2$ form of the above generic term. It is worth pointing out that the reconstruction procedure has some off-shell ambiguity (see [43]) even for vanishing hypermultiplet fields but this ambiguity is unessential in the case under consideration.

The derivative expansion (35) of the effective action contains the known non-holomorphic potential as a first term (see (43)). It can be unambiguously rewritten in a $\mathcal{N} = 2$ form, following from $\mathcal{N} = 1$ calculations on the background (3). This unique term is automatically $\mathcal{N} = 4$ supersymmetric since it does not contain the derivatives of the hypermultiplet and vector strengths. Recovering of the other terms in the derivative expansion of the effective action is not so evident and needs special prescriptions.

Calculation of the above effective action was fulfilled on the constant background (3), but for recovering the $\mathcal{N} = 2$ form such a background is insufficient. We must take into account the derivatives of the $\mathcal{N} = 1$ hypermultiplet fields. The procedure of restoring the $\mathcal{N} = 2$ supersymmetric expressions, based on corresponding $\mathcal{N} = 1$ reduction, always implies forming of the $\mathcal{N} = 2$ integral measure $\int d^{12}z = \int d^8z(\nabla_2)^2(\bar{\nabla}_2)^2$. Therefore, to get an integral over $\mathcal{N} = 2$ superspace from an integral over $\mathcal{N} = 1$ superspace, we must form the derivatives $(\nabla_2)^2(\bar{\nabla}_2)^2$ in the initial $\mathcal{N} = 1$ superspace integrand. In order to obtain such total derivatives in the integrand (38), we have to add all necessary $\nabla^2_{\alpha j} q^{ia}$ derivative-containing terms with specified numerical coefficients to the initial $\mathcal{N} = 1$ superspace integrand by hand, since they did not appear in the process of computations. If we calculate the effective action in terms of $\mathcal{N} = 1$ superfields not on the special background (3) but on the proper background (13), these absenting terms will be presented.
automatically. Then the derivatives $\nabla^2 \nabla^2$ could be formed in the $\mathcal{N} = 1$ superspace integrand and, as a result, we would obtain the integral over $\mathcal{N} = 2$ superspace.

Further, we use the evident enough assumptions about the properties of the effective action. Effective action is manifestly $\mathcal{N} = 2$ supersymmetric and, hence, each term in its expansion in derivatives can be written as the integral over $\mathcal{N} = 2$ superspace of function depending on $\mathcal{N} = 2$ superfield strengths, hypermultiplet superfields and their spinor derivatives. It allows to argue as follows. Using integrations by parts in the integrals over $\mathcal{N} = 2$ superspace and expressing the terms in derivative expansion of effective action, we transfer all derivatives from hypermultiplets to the $\mathcal{N} = 2$ superfield strengths and then ones make the reduction to $\mathcal{N} = 1$ form. As a result, we see that all terms in derivative expansion can be written in the form similar to $\Gamma^{(n)}$ defined in (37), i.e. without derivatives of the hypermultiplet superfields. It means that we can act in reverse order beginning with the given $\mathcal{N} = 1$ form and restoring the corresponding $\mathcal{N} = 2$ form. Also we take into account that the derivative expansion at vanishing hypermultiplet superfields is presented in terms of $\mathcal{N} = 2$ superconformal scalars \[\bar{\Psi}^2 = \frac{1}{W^2} \nabla^4 \ln W, \quad \Psi^2 = \frac{1}{\bar{W}^2} \bar{\nabla}^4 \ln \bar{W}\] (39) and will search for hypermultiplet dependence compatible with this property.

Further we demonstrate how the use of the above prescription allows to obtain the functionals $\Gamma^{(0)}, \Gamma^{(2)}, \Gamma^{(3)}, \ldots$ (37) in terms of $\mathcal{N} = 2$ superfields. Let us begin with the functional $\Gamma^{(0)} = \frac{1}{(4\pi)^2} \int d^8 z \frac{W^2 \bar{W}^2}{M^2} \left(\text{which is } \sim F^4\right)$ and rewrite it in the form (38) using \[
abla^2 \ln W| = -\left(\frac{W^\alpha W_\alpha}{\Phi^2}\right) + \ldots,
\]
\[
\nabla^2 \frac{1}{W^m}| = \frac{m(m + 1) W^\alpha W_\alpha}{\Phi^2} + \ldots, \quad (41)
\]
where dots mean the terms involving the derivatives of $\Phi$ which can be omitted in our on-shell analysis. Thus, the $\mathcal{N} = 1$ integrand (40) can be written via $\mathcal{N} = 2$ vector multiplet superfields and hypermultiplets as
\[
\nabla^2 \ln W \bar{\nabla}^2 \ln \bar{W} + \sum_{k=1}^\infty \frac{1}{k^2(k + 1)} \nabla^2_{\mathcal{N} = 2} \ln W \bar{\nabla}^2_{\mathcal{N} = 2} \ln \bar{W} \bar{\nabla}^2_{\mathcal{N} = 2} \nabla^2_{\mathcal{N} = 2} \left(\bar{-}q^i q_{ia}\right) + \ldots, \quad (42)
\]
where the dots mean all terms involving hypermultiplet derivatives of the form
\[
\nabla^2_{\mathcal{N} = 2} \nabla^2_{\mathcal{N} = 2} \left(-q^i q_{ia}\right) \nabla^2_{\mathcal{N} = 2} \frac{1}{W^k}, \quad \bar{\nabla}^2_{\mathcal{N} = 2} \left(-q^i q_{ia}\right) \bar{\nabla}^2_{\mathcal{N} = 2} \frac{1}{\bar{W}^k},
\]
which, according to the above prescriptions, should be added in order to obtain the full $\mathcal{N} = 2$ integration measure $\nabla_2^2 \nabla_2^2$ in the integral over $\mathcal{N} = 1$ superspace (40). As a result, the above prescriptions lead to an expression

$$\Gamma(0) = \frac{1}{(4\pi)^2} \int d^{12} z (\ln W \ln \bar{W} + \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} X^k),$$

(43)

where $X = \left(-\frac{a^{(\alpha)} _{\alpha}}{W \bar{W}}\right)$ was defined in (2). The second term in (43) can be transformed to the form (2) using the power series for the Euler dilogarithm function and relation $\frac{1}{k(k+1)} = \frac{1}{k^2} - \frac{1}{k} + \frac{1}{k+1}$. We see that the expression (43) is just the effective Lagrangian (2) found in [24].

$\mathcal{N} = 2$ form of next term ($\sim F^8$) in the series (37) is reconstructed using (41) and expansion of $(1/M)^6$ in $X$. Direct analysis, analogous to one in the previous case, leads to the following expression for $\Gamma^{(2)}$ in (37)

$$\Gamma^{(2)} = \frac{1}{2(4\pi)^2} \int d^{12} z \Psi^2 \bar{\Psi}^2 (\frac{1}{36} + \frac{1}{5!} \sum_{k=1}^{\infty} \frac{(k+5)(k+4)(k+1)}{(k+3)(k+2)} X^k).$$

(44)

The $X$-independent part of this term was given in [31]. The sum in (44) can be written in an explicit form as follows

$$\sum_{k=1}^{\infty} \frac{(k+5)(k+4)(k+1)}{(k+3)(k+2)} X^k =

= \frac{1}{(1-X)^2} + \frac{4}{(1-X)} + \frac{6X - 4}{X^3} \ln(1-X) + \frac{4X - 1}{X^2} - \frac{10}{3}.$$

(45)

Applying the same procedure to the third term ($\sim F^{10}$) in (37), one obtains

$$\Gamma^{(3)} = -\frac{5}{6(4\pi)^2} \int d^{12} z (\Psi^4 \bar{\Psi}^2 + \Psi^2 \bar{\Psi}^4) (-\frac{1}{5!} + \frac{1}{7!} \sum_{k=1}^{\infty} (k+7)(k+6)(k+1) X^k),$$

(46)

where the sum in the right-hand side is

$$\sum_{k=1}^{\infty} (k+7)(k+6)(k+1) X^k = \frac{2X}{(1-X)^4} (56 - 116X + 84X^2 - 21X^3).$$

(47)

Thus, we have found the hypermultiplet-dependent complementary terms to $\Gamma^{(0)}$, $\Gamma^{(2)}$ and $\Gamma^{(3)}$ for the effective action obtained in [31] in the $\mathcal{N} = 2$ vector multiplet sector. Clearly every term in the expansion of effective action (37) can be written in $\mathcal{N} = 2$ supersymmetric form. For example, the $X$-dependent part of the fourth term ($\sim F^{12}$) in (37) contains two parts. The first one is

$$\Gamma^{(4_1)} = \frac{1}{(4\pi)^2} \frac{1}{17250} \int d^{12} z (\Psi^6 \bar{\Psi}^2 + \Psi^2 \bar{\Psi}^6) \times$$

$$\times \frac{12X}{(1-X)^6} (450 - 1545X + 2284X^2 - 1779X^3 + 720X^4 - 120X^5)$$

and the second part is given as follows

$$\Gamma^{(4_2)} = \frac{1}{5 \cdot 6! (4\pi)^2} \int d^{12} z \Psi^4 \bar{\Psi}^4 \times$$

(48)

(49)
\[
\times \left( \frac{12(5X - 4)}{X^5} \right) \ln(1 - X) - \frac{1}{5X^4(1 - X)^6} \left( 240 - 1620X + 4610X^2 - 7120X^3 + 6363X^4 - 4878X^5 + 6135X^6 - 7560X^7 + 5670X^8 - 2268X^9 + 378X^{10} \right).
\]

Thus, we see that this \( \mathcal{N} = 2 \) reconstruction procedure for the effective action (35) of \( \mathcal{N} = 4 \) SYM theory, written initially in terms of \( \mathcal{N} = 1 \) superfield, can be realized completely for any terms in the expansion (37) completing these terms by the corresponding terms containing the hypermultiplet superfields. Unfortunately, we can not guarantee that the reconstructed effective action will be \( \mathcal{N} = 4 \) invariant. Therefore this point needs independent test. For this purposes, we can use either \( \mathcal{N} = 1 \) form of the hidden \( \mathcal{N} = 4 \) supersymmetry transformations (5, 6), or \( \mathcal{N} = 2 \) form of hidden \( \mathcal{N} = 2 \) supersymmetry transformations (9) in the harmonic superspace [24].

The low-energy effective action \( \mathcal{N} = 4 \) SYM theory is expected to be self-dual and, in addition, invariant under the (probably quantum deformed, see [38]) superconformal group transformations. But it turns out that even these requirements are not sufficient to fix the \( \mathcal{N} = 4 \) form of the effective action functional uniquely [43, 44]. Until now we have used the constant field approximation (3) which supposes that all derivatives of the hypermultiplet fields vanish. This approximation suffices to restore the manifestly \( \mathcal{N} = 2 \) supersymmetric effective action on the basis of its \( \mathcal{N} = 1 \) form (35), calculated on the background (3), in terms of \( \mathcal{N} = 2 \) superconformal scalars [31]. However, finding \( \mathcal{N} = 4 \) supersymmetric effective action requires fulfillment of all calculations on the background mentioned in the section 2.2. (i.e. supposing \( Dq \neq 0 \)). Another way to obtain terms with derivatives \( Dq^a \), necessary for constructing the \( \mathcal{N} = 4 \) supersymmetric form, can be based on algebraic consideration analogous to [22].

Since in on-shell description the hypermultiplet superfields \( q_{\alpha a} \) and superfield strengths \( \mathcal{W}, \bar{\mathcal{W}} \) are independent on harmonic variables \( u_{\pm i} \), one can insert a harmonic integral into the expressions for \( \Gamma_0, \Gamma_2, \Gamma_3, \ldots \) and write the variables \( X \) as \( X = \left( -2 q^{a\alpha} \frac{q_{\mu}^{a\alpha}}{\mathcal{W} \bar{\mathcal{W}}} \right) \). This allows studying the variation of effective action under hidden supersymmetry transformations using harmonic superspace formalism. This variational procedure and other problems of restoring \( \mathcal{N} = 4 \) supersymmetric effective action will be considered in the next section.

5 Problem of restoring \( \mathcal{N} = 4 \) supersymmetric effective action

As we already pointed out, all manifestly \( \mathcal{N} = 2 \) supersymmetric contributions obtained in the previous section and defining a derivative expansion, except (43), should not be \( \mathcal{N} = 4 \) supersymmetric because of the background choice (3) and the gauge-fixing procedure (17). In particular, they must be non-invariant under hidden \( \mathcal{N} = 2 \) supersymmetry transformations (9), except (43). It is obvious that in order to obtain \( \mathcal{N} = 4 \) supersymmetric contributions from the ones given in the previous section, we have to add to each term in the derivative expansion of (35) some extra terms. On equal footing these extra terms must contain fields \( \lambda = W \) of the vector multiplet, which are presented in the effective action (35), as well as fields \( \psi = Dq \) of the hypermultiplet, which are absent in the special background definition (3).
We study a possible form of terms depending on the hypermultiplet $q^+$ and its spinor derivatives $D^- q^+$ in the effective action. We assume that this on-shell $\mathcal{N} = 4$ supersymmetric effective action is described by manifestly $\mathcal{N} = 2$ supersymmetric effective Lagrangian depending on $\mathcal{W}, \bar{\mathcal{W}}$, their spinor derivatives, $q^+$ and spinor derivatives of $q^+$. The leading low-energy contribution to such an effective Lagrangian is known and given by the expressions (1), (2). Now we discuss a possibility to obtain next-to-leading corrections. First, we note that the $D^- q^+$ is a fermionic superfield; second, it is known that on shell $(D^-)^2 q^+ = 0$ [24]. Hence, the effective Lagrangian can be written as a finite order polynomial in first derivatives $D^- q^+$ with the coefficients depending on $q^+$ and $\mathcal{W}, \bar{\mathcal{W}}$ and their spinor derivatives. Since the superfield effective Lagrangian is integrated over full $\mathcal{N} = 2$ superspace, it is dimensionless and chargeless. The only way to compensate dimensional quantities $D^- q^+, (D^- q^+)^2, ...$ is to use the $\mathcal{N} = 2$ strengths and their spinor derivatives. Therefore we can write down all the possible terms in expansion of the effective Lagrangian in power series in $D^- q^+$ up to dimensionless functions which can depend only on dimensionless quantities $X$ (2) and quantities (39). As a result, we get a method allowing, in principle, to find the entire structure of the on-shell $\mathcal{N} = 4$ supersymmetric effective action.

Let’s assume that the above procedure has been realized and we got such an effective Lagrangian. Using the integrations by path we transfer, where it is possible, all spinor derivatives from $q^+$ onto $\mathcal{W}, \bar{\mathcal{W}}$ and compare a result with derivative expansion of the expression (35). Since the effective Lagrangian under consideration contains the derivatives $D^- q^+$ which are absent in (35), transferring the derivatives from $q^+$ to $\mathcal{W}, \bar{\mathcal{W}}$ will lead to new terms in comparison to (35). Hence, one can expect that a proper $\mathcal{N} = 4$ supersymmetric effective Lagrangian, in principle, should have the other numerical coefficients at the $X$-dependent terms in comparison to (35), except the term (2). We conclude once more that the result (35) does not allow to derive the $\mathcal{N} = 4$ supersymmetric effective action directly.

However, there exists a principal possibility to construct $\mathcal{N} = 4$ supersymmetric on-shell effective action on the basis of the relation (35). We can act as follows. Let us consider a derivative expansion of (35) at $X = 0$. Each term of the expansion is expressed in a manifestly $\mathcal{N} = 2$ supersymmetric form [31]. We insert the integral over harmonics, which is equal to unit, into each integral over full $\mathcal{N} = 2$ superspace, taking into account that the integrands are harmonic-independent. Then we investigate which hypermultiplet-dependent terms have to be added to each term of the derivative expansion so that the whole expansion would become invariant under the hidden $\mathcal{N} = 2$ supersymmetry transformations. This is just the procedure proposed in [22] for finding the $\mathcal{N} = 4$ extension of non-holomorphic effective potential (1). Further, we are going to discuss an application of this procedure to reconstruction of some leading contributions to the on-shell $\mathcal{N} = 4$ extension of $F^8$-term in effective action.

Since the entire effective action can be presented as a polynomial in $D^- q^+$, terms in the effective action corresponding to the $\mathcal{N} = 4$ extension of $F^8$ have a form

$$g(X) \Psi^2 \bar{\Psi}^2 + ..., \quad (50)$$

where the dots mean the terms depending on the derivatives $D^- q^+$. Direct quantum field calculation of the $D^- q^+$-dependent terms demands to use an appropriate background but it is an unsolved problem yet.\(^2\) Here we consider calculation of the first, derivative-

\(^2\)To get such an effective action in $\mathcal{N} = 1$ formalism we have to carry out the calculations keeping
independent term only \( g(X) \bar{\Psi}^2 \Psi^2 \) in (50) and show that the function \( g(X) \) can be reconstructed on the basis of \( X \)-independent term in (44). Then we check if this function \( g(X) \) coincides with \( X \)-dependent terms in (44). Taking into account that we are interested only in the first term in the derivative expansion (50), we systematically omit, in process of our calculations, all terms containing \( D^{-q^+} \), since they can not contribute to \( g(X) \).

Let us consider the variation of the \( X \)-independent term \( \sim \bar{\Psi}^2 \Psi^2 \) in (44) under hidden \( \mathcal{N} = 2 \) transformation with parameters \( \varepsilon^{\alpha a} \) directly in \( \mathcal{N} = 2 \) harmonic superspace. Writing \( D^4 \) as \( \frac{1}{4} D^{+2} D^{-2} \) allows to rewrite the initial expression as \( D^{+2} \frac{1}{W^2} D^{-2} \ln(W) \), which is equal to

\[
-2 \cdot 3 \frac{D^{+\alpha} \bar{W} D^+_{\bar{\alpha}} \bar{W}}{W^4} \cdot \frac{D^{-\beta} \bar{W} D^-_{\bar{\beta}} \bar{W}}{W^2}.
\]

Because of \( \delta \bar{W} \sim D^{-q^+}_\alpha \), a variation of \( \bar{D} \bar{W} \) is proportional to \( \sim D_{\alpha a} q^+ \). Such terms are systematically truncated and the whole variation is defined only by the numerator variation, which gives

\[
2 \cdot 3 \cdot 6 \frac{\delta \bar{W}}{W^2} \bar{D}^+ \bar{W} \bar{D}^+ \bar{W} \bar{D}^- \bar{W} \bar{D}^- \bar{W}.
\]

After integration by parts, the last expression becomes

\[
-3 \frac{\delta \bar{W}}{W^3} \bar{D}^4 \ln \bar{W}.
\]

Collecting all variations, we get

\[
\delta \frac{1}{36} \bar{\Psi}^2 \Psi^2 = \left( \frac{1}{12} q^{+a} \varepsilon^{a \beta} D^{-}_\beta \mathcal{W} \right) \frac{D^4 \ln \mathcal{W}}{W^3} \frac{\bar{D}^4 \ln \bar{\mathcal{W}}}{W^3}.
\]

One can note that the hypermultiplet fields variation in the first \( X \)-dependent term \( \frac{1}{24} \bar{\Psi}^2 \Psi^2 \left( -\frac{2q^{+a} q^-_a}{W^2} \right) \) in (44) gives the expression similar to (51) but with another factor: \(-1/24\). To obtain this result, we used a property of the full harmonic superspace integral \( \int du \delta(q^{+a} q^-_a) = \int du \left( \delta(q^{+a} D^- q^+_a + q^{+a} \delta q^-_a) \right) = \int du \frac{2q^{+a} \delta q^-_a}. \) It shows that the variation of the first term in (44) does not cancel the variation of the linear in \( X \) terms in the \( X \)-dependent sum. It means that the invariance under above transformations is impossible. It is felt that this non-invariance is compensated by ”quantum” modification \( \delta_{\text{mod}} = \delta + \delta_q \) of the classical transformation law like

\[
\delta_q \mathcal{W} = c \frac{1}{W^2} D^4 \frac{1}{W^2} (\bar{D}^4 \ln \bar{\mathcal{W}})(\varepsilon^{a \beta} D^{-a}_\beta q^+_a),
\]

and these quantum additions to the variation of \( \Gamma(0) \) would compensate the non-invariance of the variation \( \Gamma(2) \) (44). However, the direct analysis of several first terms in sums (43) and (44) shows that the quantum modification can not save the situation since a coefficient \( c \) present in \( \delta_q \) is changed with every order of \( X \) and never can be chosen properly. We see the \( \mathcal{N} = 4 \) invariant expressions can’t be constructed by the naive quantum modification of the classical transformations (5, 9) and therefore the quantum the spinor derivatives of background chiral superfields. The only example of these calculations was given within Wess-Zumino model for finding the effective potential of auxiliary fields in Refs. [42, 29]. In particular, such a potential for chiral \( \mathcal{N} = 1 \) superfields of the \( \mathcal{N} = 2 \) vector multiplet arises from the self-dual requirement for the \( \mathcal{N} = 4 \) SYM effective action (see Ref. [44]).
modification problem should be studied separately. Nevertheless, the algebraic procedure proposed in [22] for $\mathcal{N} = 4$ completion can be applied in the case under consideration.

To compensate the expression in right hand side of (51), we introduce the $X$-dependent complimentary term

$$I_1 = C_1 \Psi^2 \bar{\Psi} \left( -\frac{2q^{a+}q^-_a}{\mathcal{W} \mathcal{W}} \right). \quad (52)$$

We demand, a variation of the hypermultiplet "numerator" (i.e. $-2q^{a+}q^-_a$) in (52) should compensate the variation (51). This requirement fixes the coefficient $C_1 = 1/12$. But the whole variation $I_1$ again contains another extra variation terms, which can be compensated only by introducing one more complementary terms. Further, we consider a variation of $\mathcal{W}$ in (52). The expression

$$\frac{q^+ q^-}{\mathcal{W}^4} \tilde{D}^4 \ln \mathcal{W} = -\frac{1}{2} \tilde{D}^2 \left( \frac{q^+ q^-}{\mathcal{W}^3} \right) \tilde{D}^{-} \mathcal{W} \tilde{D}^{-} \mathcal{W}$$

can be written as

$$\frac{1}{4} \left( 6q^+ \tilde{D}^+ \alpha q^- \frac{\tilde{D}^+_a \mathcal{W}}{\mathcal{W}^6} - 12q^+ q^- \frac{\tilde{D}^+_a \mathcal{W}}{\mathcal{W}^7} \tilde{D}^+ \alpha \mathcal{W} \tilde{D}^+_a \mathcal{W} \right) \tilde{D}^{-} \mathcal{W} \tilde{D}^{-} \mathcal{W}. \quad (53)$$

The numerator variations of both terms in (53) give

$$\frac{-2}{12 \cdot 4} \cdot \frac{D^4 \ln \mathcal{W}}{\mathcal{W}^3} \left( 6 \cdot 6 \delta \mathcal{W} q^+ \tilde{D}^+ \alpha q^- \frac{\tilde{D}^+_a \mathcal{W}}{\mathcal{W}^6} + 12 \cdot 7 \delta \mathcal{W} q^+ q^- \frac{\tilde{D}^+_a \mathcal{W}}{\mathcal{W}^7} \tilde{D}^+ \alpha \mathcal{W} \tilde{D}^+_a \mathcal{W} \right) \tilde{D}^{-} \mathcal{W} \tilde{D}^{-} \mathcal{W}. \quad (54)$$

The first term in the brackets is represented by means of the superconformal invariants (50)

$$\frac{-3}{4} \cdot \frac{\delta \mathcal{W}}{\mathcal{W}^4} \cdot \frac{D^4 \ln \mathcal{W}}{\mathcal{W}^3} q^+ \tilde{D}^+ \alpha q^- \cdot \tilde{D}^+_a \tilde{D}^{-} \ln \mathcal{W} \quad (55)$$

as well as the second term

$$\frac{7}{2 \cdot 5} \cdot \frac{\delta \mathcal{W}}{\mathcal{W}^4} \cdot \frac{D^4 \ln \mathcal{W}}{\mathcal{W}^3} \left( q^+ \tilde{D}^+ \alpha q^- \cdot \tilde{D}^+_a \tilde{D}^{-} \ln \mathcal{W} + (q^+ q^-) \tilde{D}^4 \ln \mathcal{W} \right). \quad (56)$$

In order to compensate

$$\frac{7}{4 \cdot 5} \cdot \frac{\varepsilon^{\alpha \alpha} D^- \alpha q^- (q^+ q^-)}{\mathcal{W}^3} \cdot \frac{D^4 \ln \mathcal{W} \tilde{D}^4 \ln \mathcal{W}}{\mathcal{W}^4}, \quad (57)$$

which is a part of (55), we add the next complementary term

$$I_{(2)} = c_2 \left( \frac{-2(q^+ q^-)}{\mathcal{W} \mathcal{W}} \right)^2 \frac{1}{\mathcal{W}^2} \tilde{D}^4 \ln \mathcal{W} \frac{1}{\mathcal{W}^2} \tilde{D}^4 \ln \mathcal{W}, \quad (58)$$

Then we consider variation of $(q^+ q^-)$ in this term and compare a result to (56). The requirement of compensation fixes the coefficient $C_2 = \frac{7}{243}$. The uncompensated part in variation of $I_{(2)}$ is

$$-\frac{1}{2 \cdot 4 \cdot 5} \cdot \varepsilon^{\alpha \alpha} \left( \frac{3q^+ D^- \alpha \mathcal{W}}{\mathcal{W}^4} q^+ \tilde{D}^+ \alpha q^- - \frac{q^+}{\mathcal{W}^3} (D^- \alpha q^+ \tilde{D}^+ \alpha q^-) \right) \tilde{D}^+_a \tilde{D}^{-} \ln \mathcal{W}.$$

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In order to compensate the first term in the last variation, we introduce another type complementary term

\[ J_n^0 = d_1 \left[ \frac{-2q^+ q^-}{\mathcal{W}^4 \mathcal{W}^4} \right] (q^+ \tilde{D}^{+\dot{\alpha}} q^-) \tilde{D}_\dot{\alpha}^+ \tilde{D}^{-2} \ln \mathcal{W} \mathcal{D}^4 \ln \mathcal{W}. \]  

(59)

In the component form the term \( J_n^0 \) is proportional to

\[ \frac{F^4 \tilde{F}^2}{\phi^8 \phi^8} (f \tilde{f}) \beta^a \xi_a \xi_{\dot{\beta}} \xi_{\dot{\beta}}, \]

which obviously vanishes in the bosonic sector of the theory. A partial variation obtained by varying only \( q^+ \) in square brackets (59) leads to

\[ \delta J_n^0 = d_1 \left[ \frac{q^+ \varepsilon^{\alpha \dot{\alpha}} D^+_{\alpha} \mathcal{W}}{\mathcal{W}^4 \mathcal{W}^4} \right] (q^+ \tilde{D}^{+\dot{\alpha}} q^-) \tilde{D}_\dot{\alpha}^+ \tilde{D}^{-2} \ln \mathcal{W} \mathcal{D}^4 \ln \mathcal{W} + 
\]

\[ + d_1 \left( \frac{2q^+ q^-}{\mathcal{W}^4 \mathcal{W}^4} \right) \left( \frac{1}{4} \varepsilon^{\alpha \dot{\alpha}} D^+_{\alpha} \mathcal{W} \tilde{D}^{+\dot{\alpha}} q^- \tilde{D}_\dot{\alpha}^+ \tilde{D}^{-2} \ln \mathcal{W} \mathcal{D}^4 \ln \mathcal{W} \right). \]  

(60)

In order to compensate the first term in (58), the coefficient \( d_1 = \frac{3}{2+4} \). The direct calculation shows that this coefficient allows to compensate the second term in (58) with (60) as well. Further, we have to obtain the whole variation of \( J_n^0 \) and take into account variation of \( \mathcal{W} \) contained in \( J_n^0 \). Obviously, in order to compensate the whole variation of \( J_n^0 \), the new complementary terms \( J_n^0 \sim (Dq)^n \) are necessary. They will lead to a new type of complementary terms \( J_n^k \sim (Dq)^n X^k \), where \( n = 1, 2, \ldots, 8 \), \( k = 0, \ldots, \infty \).

The analysis of all complimentary terms \( I_n \) is greatly simplified when we calculate only the first \( D^+ q^- \)-independent term in the expansion (50). In this case, we can find all \( I_n \) in the series of the complementary terms

\[ I = \sum_{n=0}^{\infty} I_n = \sum_{n=0}^{\infty} C_n \Psi^2 \bar{\Psi}^2 \left( \frac{-2q^+ q^-}{\mathcal{W} \mathcal{W}} \right)^n. \]  

(61)

Let us consider the variation of general term in this series

\[ \delta I_n = \delta_1 + \delta_2 = I_n \left[ -\frac{(n+2)(n+6)}{(n+4)} \frac{\delta \mathcal{W}}{\mathcal{W}} \right] + C_n \Psi^2 \bar{\Psi}^2 \left( \frac{-2q^+ q^-}{\mathcal{W} \mathcal{W}} \right)^{n-1} \left[ -4n \frac{q^+ \varepsilon^{\alpha} D^+_{\alpha} \mathcal{W}}{\mathcal{W} \mathcal{W}} \right]. \]  

(62)

Using the transformations (9), we rewrite the second term in (62) as follows

\[ \delta_2 = C_n \Psi^2 \bar{\Psi}^2 \left( \frac{-2q^+ q^-}{\mathcal{W} \mathcal{W}} \right)^{n-1} \left[ -n \frac{q^+ \varepsilon^{\alpha} D^+_{\alpha} \mathcal{W}}{\mathcal{W} \mathcal{W}} \right]. \]  

(63)

The variation \( \delta \mathcal{W} \) contained in the first term of (62) is proportional to \( D^+ q^- \). Let us transform the first term to form (63) using integration by parts. Excepting the superfields \( \mathcal{W} \) and derivatives \( \mathcal{D}^4, \tilde{\mathcal{D}}^4 \), which are unessential for this transformation, we rewrite the first term in (62) in the form

\[ \delta_1 \sim \frac{(q^+ b^b q^-)^n D^+ q^+}{\mathcal{W}^{n+2}} = (n+2) \frac{(q^+ b^b q^-)^n q^+_a D^- q_a}{\mathcal{W}^{n+3}} - \frac{n(q^+ q^-)^{n-1} D^+ q^+ b^- q_a}{\mathcal{W}^{n+2}}. \]  

(64)
Taking into account the property $D^{-a}q^{a+} = q^{a-}$, $D^{-a}q_a^- = 0$ and cyclicity property $\epsilon_{ab}\epsilon_{cd} + \epsilon_{ca}\epsilon_{bd} + \epsilon_{bc}\epsilon_{ad} = 0$ at permutation of $SU(2)$-isospinor group indices, we find, after some algebraic manipulations, the first term of (62) in the form similar to (63)

$$\delta_1 = C_n \Psi^2 \Psi^2 \left( -\frac{2q^{b+}q_b^-}{WW} \right)^n \frac{(n+2)(n+6)q^{a+}\epsilon^a_\alpha D^-_\alpha W}{(n+4)WW}.$$  \hfill (65)

The requirement of cancellation of (63) and (65) leads to a recursion condition

$$C_n = C_{n-1} \frac{(n+1)(n+5)}{n(n+3)}, \quad (66)$$

which has a solution

$$C_n = \frac{1}{6 \cdot 5!} (n+5)(n+4)(n+1). \quad (67)$$

It is highly amazing that the correct coefficient (67) obtained from invariance under hidden $\mathcal{N} = 2$ supersymmetry transformations differs from the coefficient in (44) only by the numerical denominator!

Summing the series (61), we find the correct leading part $\sim g(X)$ in expansion (50) of on-shell $\mathcal{N} = 4$ supersymmetric $F^8$-term in the closed form

$$I = \frac{1}{72} \frac{1}{(4\pi)^2} \int d^2z du \Psi^2 \Psi^2 \frac{1 - X + \frac{3}{10}X^2}{(1 - X)^4}. \quad (68)$$

It is obvious that this expression does not coincide with result (44,45) obtained by restoring the $\mathcal{N} = 2$ form of $\Gamma_2$ from its $\mathcal{N} = 1$ form. Thus, the leading bosonic part of complete on-shell $\mathcal{N} = 4$ supersymmetric extension of $F^8$ invariant is finally established.

**6 Summary**

We have studied the one-loop effective action in $\mathcal{N} = 4$ SYM theory, depending on $\mathcal{N} = 2$ vector multiplet and hypermultiplet fields. The theory under consideration was formulated in $\mathcal{N} = 1$ superspace and quantized in the framework of the background field method with the use of a special gauge fixing conditions preserving manifest $\mathcal{N} = 1$ supersymmetry. The effective action is given by superfield functional determinants. The concrete calculations of these determinants are done on specific $\mathcal{N} = 1$ superfield background corresponding to constant Abelian strength $F_{mn}$ and constant hypermultiplet fields. We have proved that the effective action depending on all fields of $\mathcal{N} = 4$ vector multiplet is restored on the base of calculations only in $\mathcal{N} = 2$ vector multiplet sector by special change of functional arguments (see (33) and (35)).

We have examined a possibility to present the effective action obtained in a manifest $\mathcal{N} = 2$ supersymmetric form. Analyzing the effective action as an expansion in spinor covariant derivatives, we have showed that the terms of this expansion can be expressed

\footnote{The property $D^{-a}q^{a+} = 0$ and cyclicity lead to an identity $D^-a q^b q_a^+ - D^-a q^b q_a^- = D^-a q^b q_a^+ = 0$. The second term in this identity can be transformed to $D^-a q^b q_a^+$ using integration by parts and relation $\int du D^-(...) = 0$. Then, the identity takes the form $2D^-a q^b q_a^+ = D^-a q^+_a (q^{b}q_a^-)$. Substituting it to the right-hand side of the expression (64) instead of the second term and transferring a result to the left-hand side, we obtain $(1 + n/2)_{\frac{q^b q_a^-}{W^{-a}}} = (n+2)_{\frac{q^+_a q^-_a D^-a W}{W^{-a}}}$. That leads to (65).}
via integrals over $\mathcal{N} = 2$ superspace of the functions depending on $\mathcal{N} = 2$ strengths, their spinor derivatives and hypermultiplet superfields. As one of the results, we have rederived the complete $\mathcal{N} = 4$ supersymmetric low-energy effective action, which was discovered in [22]. All other terms in the derivative expansion of the effective action describe the next-to-leading corrections to the effective action found.

We point out that all terms in hypermultiplet sector in derivative expansion of the effective action are gauge-dependent, except the first leading term. They do not invariant under hidden $\mathcal{N} = 2$ supersymmetry transformations, which are a part of complete on-shell $\mathcal{N} = 4$ supersymmetry transformations of $\mathcal{N} = 4$ SYM theory, because of the chosen background and the gauge fixing procedure. To analyze a possibility of presenting the derivative expansion terms in on-shell $\mathcal{N} = 4$ supersymmetric form, we applied a formalism of harmonic superspace and algebraic approach developed in [22]. We have considered the first subleading term in expansion of the effective action in $\mathcal{N} = 2$ vector multiplet sector ($F^8$-term written via $\mathcal{N} = 2$ superconformal invariants depending on strengths $W, \bar{W}$ and their spinor derivatives [31]) and proved that it can be completed up to on-shell $\mathcal{N} = 4$ supersymmetric form by the hypermultiplet dependent terms and presented as polynomial in hypermultiplet spinor derivatives. The first leading term of this polynomial, which depends on hypermultiplet but does not depend on its derivatives, is given in explicit form (see (68)).

The most important extension of this work that may clarify a structure of next-to-leading corrections to low-energy effective Lagrangian (2) in hypermultiplet sector is a computation of effective action on the background (13). It can provide a direct independent verification of the results given in Section 4. It would be extremely interesting also to study full $\mathcal{N} = 4$ completion of the $F^8$ and higher invariants by all appropriate hypermultiplet derivative dependent terms.

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