Double-ended queues and joint moments of left-right canonical operators on full Fock space

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Abstract

We study joint moments of a \((2d)\)-tuple \(A_1, \ldots, A_d, B_1, \ldots, B_d\) of canonical operators on the full Fock space over \(\mathbb{C}^d\), where \(A_1, \ldots, A_d\) act on the left and \(B_1, \ldots, B_d\) act on the right. The joint action of the \(A_i\)’s and \(B_i\)’s can be described in terms of the concept of a double-ended queue. The description which results for joint moments of the \((2d)\)-tuple suggests that, for a general noncommutative probability space \((\mathcal{A}, \varphi)\), one should consider a certain family of \("(\ell, r)\)-cumulant functionals", which enlarges the family of free cumulant functionals of \((\mathcal{A}, \varphi)\). The main result of the paper can then be phrased by saying that one has a simple formula for certain joint \((\ell, r)\)-cumulants of \(A_1, \ldots, A_d, B_1, \ldots, B_d\). This formula extends a known formula for joint free cumulants of \(A_1, \ldots, A_d\) or of \(B_1, \ldots, B_d\) (with the two \(d\)-tuples considered separately), which is used in one of the operator models for the \(R\)-transform of free probability.

1. Introduction

One of the basic tools used in free probability is the \(R\)-transform of a \(d\)-tuple of non-commutative random variables. A standard way to introduce this \(R\)-transform goes via a canonical construction made with creation and annihilation operators on the left, on the full Fock space over \(\mathbb{C}^d\). Motivated by recent work \([6, 7]\) on free probability for pairs of faces, the present paper considers the “left-right” generalization of the above mentioned canonical construction, and introduces a concept of “\((\ell, r)\)-cumulant functionals” that are appropriate for studying this kind of structure.

1.1 Background on canonical operators.

Let \(d\) be a positive integer and let \(f(z_1, \ldots, z_d)\) be a polynomial without constant term in \(d\) non-commuting indeterminates. Let \(\mathcal{T}\) denote the full Fock space over \(\mathbb{C}^d\) and let \(\varphi_{\text{vac}}\) denote the vacuum-state on \(B(\mathcal{T})\). The canonical construction (invented in \([5]\) for \(d = 1\), then extended in \([2]\)) produces a \(d\)-tuple of operators \(A_1, \ldots, A_d \in B(\mathcal{T})\) such that the joint \(R\)-transform of \(A_1, \ldots, A_d\) with respect to \(\varphi_{\text{vac}}\) is the \(\square\) given \(f\). One usually presents the construction by writing \(A_1, \ldots, A_d\) as polynomials made with creation and annihilation operators on the left on \(\mathcal{T}\). For a detailed presentation of how this goes, see Lecture 21 of \([3]\). The precise formulas defining the \(A_i\)’s are reviewed in Section 6 of the paper (see Notation 6.1, Definition 6.2).

The canonical construction could be done equally well by starting with creation and annihilation operators on the right. Thus, given a polynomial without constant term \(g(z_1, \ldots, z_d)\), we have a canonical recipe for getting operators \(B_1, \ldots, B_d\) made with creation

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²In this whole discussion, \(f\) could be a formal power series in \(z_1, \ldots, z_d\), in which case \(A_1, \ldots, A_d\) would live in a suitable algebra of formal operators on \(\mathcal{T}\), as described for instance on pp. 344-346 of \([3]\). For the sake of not complicating the notations, we will stick here to \(f\) being a polynomial.
and annihilation operators on the right on $T$, such that the joint $R$-transform $R_{(B_1,\ldots,B_d)}$ of $B_1,\ldots,B_d$ is the given $g$.

The above mentioned equalities

\[ R_{(A_1,\ldots,A_d)} = f \quad \text{and} \quad R_{(B_1,\ldots,B_d)} = g \quad (1.1) \]

amount to some concrete formulas which express joint moments

\[ \varphi_{\text{vac}}(A_{i_1} \cdots A_{i_n}), \varphi_{\text{vac}}(B_{i_1} \cdots B_{i_n}), \] with $n \geq 1$ and $i_1,\ldots,i_n \in \{1,\ldots,d\}, \quad (1.2) \]

in terms of the coefficients of $f$ and of $g$, respectively. More precisely: $\varphi_{\text{vac}}(A_{i_1} \cdots A_{i_n})$ is written as a sum indexed by the lattice $NC(n)$ of non-crossing partitions of $\{1,\ldots,n\}$, where every term of the sum is a certain product of coefficients of $f$. Likewise for $\varphi_{\text{vac}}(B_{i_1} \cdots B_{i_n})$, where we now use sums of products of coefficients of $g$. A useful way of recording these formulas of summation over $NC(n)$ is in terms of the concept of free cumulant functionals \cite{4}. If $(\kappa_n : B(T)^n \to \mathbb{C})_{n=1}^{\infty}$ denotes the family of free cumulant functionals associated to the noncommutative probability space $(B(T),\varphi_{\text{vac}})$, then the Equations (1.1) can be rephrased as saying that for every $n \geq 1$ and $i_1,\ldots,i_n \in \{1,\ldots,d\}$ we have

\[ \begin{cases} 
\kappa_n(A_{i_1},\ldots,A_{i_n}) &= \alpha_{(i_1,\ldots,i_n)} \\
\kappa_n(B_{i_1},\ldots,B_{i_n}) &= \beta_{(i_1,\ldots,i_n)},
\end{cases} \quad (1.3) \]

where $\alpha_{(i_1,\ldots,i_n)}$ and $\beta_{(i_1,\ldots,i_n)}$ are the coefficients of $z_{i_1} \cdots z_{i_n}$ in $f$ and in $g$, respectively. The summations over $NC(n)$ which are used to calculate the joint moments listed in (1.2) are then special cases of the general “moment→free cumulant” formula which relates moments and free cumulants in any noncommutative probability space.

\subsection*{1.2 Object of the present paper.}

Motivated by the recent work of Voiculescu \cite{6, 7} on free probability for pairs of faces, we look at the situation where canonical operators on the left and on the right are considered at the same time. With $A_1,\ldots,A_d$ and $B_1,\ldots,B_d$ as above, we are thus interested in the joint moments of the combined $(2d)$-tuple $A_1,\ldots,A_d, B_1,\ldots,B_d$. Denoting

\[ A_i =: C_{i,\ell} \quad \text{and} \quad B_i =: C_{i,r}, \quad \text{for} \quad 1 \leq i \leq d, \quad (1.4) \]

we thus look at expressions

\[ \begin{cases} 
\varphi_{\text{vac}}(C_{i_1,h_1} \cdots C_{i_n,h_n}), \\
\text{with} \quad n \geq 1, \quad 1 \leq i_1,\ldots,i_n \leq d \quad \text{and} \quad \chi = (h_1,\ldots,h_n) \in \{\ell,r\}^n.
\end{cases} \quad (1.5) \]

It turns out that such a joint moment can be written in a canonical way as a sum of products of $\alpha$’s and $\beta$’s (coefficients of $f$ and of $g$, combined); the number of terms in the sum is equal to the cardinality of $NC(n)$, which is the $n$th Catalan number. For illustration, here is a concrete example which the authors of the paper found to be very instructive: say that $n = 4$ and that $\chi = (\ell,\ell,\ell,r)$. Then for every $i_1,i_2,i_3,i_4 \in \{1,\ldots,d\}$ one has

\[ \varphi_{\text{vac}}(A_{i_1}B_{i_2}A_{i_3}B_{i_4}) = \beta_{(i_3,i_1,i_2,i_4)} + \alpha_{(i_1)} \beta_{(i_3,i_2,i_4)} + \beta_{(i_2)} \beta_{(i_1,i_3,i_4)} + \alpha_{(i_3)} \beta_{(i_1,i_2,i_4)} + \alpha_{(i_2,i_1,i_3)} \beta_{(i_4)} + \beta_{(i_1,i_2)} \beta_{(i_3,i_4)} + \alpha_{(i_1,i_3)} \beta_{(i_2,i_4)} \quad (1.6) \]
We urge the reader to not get intimidated by this formula, and to observe that it gives \( \varphi_{\text{vac}}(A_{i_1}B_{i_2}A_{i_3}B_{i_4}) \) as a sum of 14 terms, which correspond naturally to 14 of the 15 partitions of \( \{1,2,3,4\} \). Among these 14 partitions one finds (on the third line of (1.6)) the crossing partition \( \{1,3\} \), \( \{2,4\} \); so this is not manifestly a sum over \( NC(4) \). For every partition \( \pi = V_1, \ldots, V_k \) that indexes a term of the sum, the term in question is a product of \( k \) factors, corresponding to the blocks \( V_1, \ldots, V_k \), where each of the factors is either an \( \alpha \) or a \( \beta \). It is not hard to figure out the pattern for which factors are \( \alpha \)'s and which ones are \( \beta \)'s: if \( V \) is one of the blocks \( V_1, \ldots, V_k \) and if \( m \) is the maximal element of \( V \), then the component \( h_m \) of \( \chi \) dictates the choice between \( \alpha \) and \( \beta \) (\( h_m = \ell \) gives an \( \alpha \)-factor and \( h_m = r \) gives a \( \beta \)-factor). For instance on the second line of (1.6), the block \( \{2,3,4\} \) of the partition \( \{1,2,3,4\} \) contributes a \( \beta \)-factor (namely \( \beta_{(i_1,i_2,i_3)} \)) because the 4-th component of \( \chi \) is \( h_4 = r \), and the block \( \{1,2,3\} \) of the partition \( \{1,2,3,4\} \) contributes an \( \alpha \)-factor (namely \( \alpha_{(i_2,i_3)} \)) because the 3-nd component of \( \chi \) is \( h_3 = \ell \).

Since \( |NC(4)| = 14 \), it is natural to ask if Equation (1.6) isn’t in fact the moment→free cumulant formula mentioned in connection to Equations (1.3), which is now used to calculate a joint moment of \( A_i \)'s and \( B_i \)'s. This could be the case if every joint free cumulant in \( A_i \)'s and \( B_i \)'s turned out to be either a coefficient \( \alpha \) or a coefficient \( \beta \). However, direct calculation shows that

\[
\kappa_4(A_{i_1}, B_{i_2}, A_{i_3}, B_{i_4}) = \beta_{(i_3,i_1,i_2,i_4)} + \alpha_{(i_1,i_3)} \beta_{(i_2,i_4)} - \alpha_{(i_2,i_3)} \beta_{(i_1,i_4)}. \tag{1.7}
\]

Thus if one insists to write the moment \( \varphi(A_{i_1}B_{i_2}A_{i_3}B_{i_4}) \) in terms of the free cumulant functionals \( \kappa_n \), then the nice structure from formula (1.6) (a straight sum of products) would come out after going through some more complicated expressions, and doing cancellations between terms.

The example in (1.6) suggests that a nice “moment↔cumulant” formula for the joint moments of \( A_i \)'s and \( B_i \)'s should still exist – only that we must now do summation over a collection \( \mathcal{P}^{(\chi)}(n) \) of partitions of \( \{1,\ldots,n\} \) which depends on what \( n \)-tuple \( \chi \in \{\ell,r\}^n \) we are looking at. If \( \chi = (\ell, \ell, \ldots, \ell) \) or if \( \chi = (r, r, \ldots, r) \), then \( \mathcal{P}^{(\chi)}(n) = NC(n) \) and we are back to Equations (1.3), but for \( \chi = (\ell, r, \ell, r) \) one has \( \mathcal{P}^{(\chi)}(4) \neq NC(4) \) (this \( \mathcal{P}^{(\chi)}(4) \) contains \( \{1,3\}, \{2,4\} \), and misses the non-crossing partition \( \{1,4\}, \{2,3\} \)). We next identify precisely what is, in general, the set of partitions \( \mathcal{P}^{(\chi)}(n) \).

1.3 The set of partitions \( \mathcal{P}^{(\chi)}(n) \).

In the case when we only consider canonical operators on the left (say), an intuitive concept that can be used to understand the formula \( \varphi_{\text{vac}}(A_{i_1} \cdots A_{i_n}) \) is the one of a lifo ( = last-in-first-out) stack. Indeed, due to the specifics of how \( A_1, \ldots, A_d \) are defined, one can follow the action of \( A_{i_1} \cdots A_{i_n} \) on the vacuum vector \( \xi_{\text{vac}} \in \mathcal{T} \) by thinking of how a collection of \( n \) balls \( (\bigcirc, \ldots, \bigcirc) \) moves through a lifo stack. The balls go into the stack in groups (creation) and are taken out of the stack one by one (annihilation). To every possible scenario of how the balls move through the stack, one associates a certain partition of \( \{1,\ldots,n\} \), which we will call “output-time partition” (for the scenario in question). The concrete formula for \( \varphi_{\text{vac}}(A_{i_1} \cdots A_{i_n}) \) is expressed in terms of the output-time partitions
that appear in this way. The precise definition of an output-time partition is given in Section 2.2 of [1]; see also Example 5.4.2; here we only emphasize the important point that the last-in-first-out rule forbids an output-time partition from having crossings. This is a possible way of explaining why, in the end, the concrete formula for \( \varphi_{\text{vac}}(A_{i_1} \cdots A_{i_n}) \) comes out as a sum indexed by \( NC(n) \).

Let us now look at a joint moment as in (1.5), where we consider the simultaneous action of \( A_1, \ldots, A_d \) and of \( B_1, \ldots, B_d \). When doing so, we can still use a pictorial device from theoretical computer science, called double-ended queue, or deque for short – see e.g. Section 2.2 of [1]. We still have the balls \( \circlearrowright, \ldots, \circlearrowright \) which enter the deque in groups, from either left or right (creation), and exit one by one (annihilation); and what matters for our moment calculations still is the concept of output-time partition associated to a scenario for how the balls move through the deque. But an output-time partition may now have crossings, due to the interference between left and right moves (e.g. it is possible that some of the balls \( \circlearrowright, \ldots, \circlearrowright \) enter the deque by one side and exit by the other).

If we fix a tuple \( \chi \in \{\ell, r\}^n \), then \( P^{(\chi)}(n) \) will be defined in Section 3 below as the set of all partitions of \( \{1, \ldots, n\} \) which can arise as output-time partition for a deque-scenario compatible with \( \chi \). (See details in Definition 3.3.) In the case when \( \chi \) happens to be either \( (\ell, \ldots, \ell) \) or \( (r, \ldots, r) \), then the deque is reduced to a lifo stack, and \( P^{(\chi)}(n) \) is equal to \( NC(n) \). For general \( \chi \in \{\ell, r\}^n \), it follows immediately from the definition that \( P^{(\chi)}(n) \) has the same cardinality as \( NC(n) \), but \( P^{(\chi)}(n) \) is generally different from \( NC(n) \) itself.

In the Section 4 of the paper we find (in Theorem 4.10) an alternative description for \( P^{(\chi)}(n) \). It says that
\[
\mathcal{P}^{(\chi)}(n) = \{ \sigma \cdot \pi \mid \pi \in NC(n) \},
\]
where \( \sigma \chi \) is a specific (concretely described) permutation of \( \{1, \ldots, n\} \) associated to \( \chi \), and where the action of a permutation \( \sigma \) on a partition \( \pi \) is defined in the natural way (if \( \pi = \{V_1, \ldots, V_k\} \), then \( \sigma \cdot \pi = \{\sigma(V_1), \ldots, \sigma(V_k)\} \)). This alternative description of \( P^{(\chi)}(n) \) is very useful for concrete calculations. It also gives immediately the fact that, with respect to reverse refinement order, \( P^{(\chi)}(n) \) is a lattice isomorphic to \( NC(n) \).

### 1.4 (\ell, r)-cumulants.

The main point of the present paper is that the structure observed in Equation (1.6) (a straight sum indexed by \( P^{(\chi)}(n) \), where every term of the sum is a product of \( \alpha \)'s and \( \beta \)'s) holds in fact for a general moment \( \varphi_{\text{vac}}(C_{i_1;h_1} \cdots C_{i_n;h_n}) \) as in (1.5). We will make this point by showing that the \( \alpha \)'s and \( \beta \)'s appearing in (1.6) can still be understood as a suitable form of cumulants. To this end we define, in the framework of a general noncommutative probability space \((\mathcal{A}, \varphi)\), a certain family of multilinear functionals
\[
\left( \kappa_\chi : \mathcal{A}^n \rightarrow \mathbb{C} \right)_{n \geq 1, \chi \in \{\ell, r\}^n},
\]
which we call \((\ell, r)\)-cumulant functionals. These \( \kappa_\chi \)'s are defined via the same kind of summation formula as the one used in the “moment±free cumulant” formula giving the free cumulant functionals \( (\kappa_n : \mathcal{A}^n \rightarrow \mathbb{C})_{n=1}^\infty \); only that instead of summing over \( NC(n) \) we are now summing over the set of partitions \( P^{(\chi)}(n) \). (See details in Definition 5.2.) Once the \((\ell, r)\)-cumulants are introduced, the job of describing the generalization of (1.6) is reduced to looking at cumulants of the form
\[
\left\{ \kappa_\chi (C_{i_1;h_1}, \ldots, C_{i_n;h_n}), \right. \\
\left. \quad \text{with } n \geq 1, 1 \leq i_1, \ldots, i_n \leq d \text{ and } \chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n, \right. 
\]
Then for every \( i \) where \( \kappa \) of free cumulants (the canonical operators \( A \) Equation (6.3) of the paper), and come to \( \ell, r, \ell, r \) as restrictions of \((\chi, \ell, r, \ell, r)\) and \((\ell, r, \ell, r)\) respectively.

Theorem. Let \( n \) be a positive integer and let \( \chi = (h_1, \ldots, h_n) \) be a tuple in \( \{\ell, r\}^n \). Let us record explicitly where are the occurrences of \( \ell \) and \( r \) in \( \chi \):

\[
\begin{align*}
\{ m \mid 1 \leq m \leq n, \ h_m = \ell \} &= \{ m_\ell(1), \ldots, m_\ell(u) \} \text{ with } m_\ell(1) < \cdots < m_\ell(u), \\
\{ m \mid 1 \leq m \leq n, \ h_m = r \} &= \{ m_r(1), \ldots, m_r(v) \} \text{ with } m_r(1) < \cdots < m_r(v).
\end{align*}
\]

Then for every \( i_1, \ldots, i_n \in \{1, \ldots, d\} \) we have

\[
\kappa_\chi(C_{i_1; h_1}, \ldots, C_{i_n; h_n}) = \begin{cases} 
\alpha(i_{m_r(v)}, \ldots, i_{m_r(1)}; i_{m_\ell(u)}, \ldots, i_{m_\ell(1)}), & \text{if } h_n = \ell, \\
\beta(i_{m_\ell(u)}, \ldots, i_{m_\ell(1)}; i_{m_r(v)}, \ldots, i_{m_r(1)}), & \text{if } h_n = r.
\end{cases}
\] (1.9)

We note that the family of \((\ell, r)\)-cumulants from (1.8) is an enlargement of the family of free cumulants \( (\kappa_n : A^n \to \mathbb{C})_{n=1}^\infty \) of the space \((A, \varphi)\), in the respect that we have

\[
\kappa_n = \kappa_{(\ell, \ldots, \ell)} = \kappa_{(r, \ldots, r)}, \quad n \geq 1.
\]

As a consequence of this, the formulas recorded in the Equations (1.3) can be seen as special cases of Equation (1.9) from the above theorem, upon making \( \chi \) equal to \((\ell, \ldots, \ell)\) and to \((r, \ldots, r)\), respectively.

Concerning the concrete formula in length 4 recorded in Equation (1.7), we note that the free cumulant \( \kappa_4(A_{i_1}, B_{i_2}, A_{i_3}, B_{i_4}) \) didn’t come out as a \( \beta \)-coefficient of length 4, as this cumulant is \( \kappa_{(r, r, r, r)}(C_{i_1; \ell}, C_{i_2; r}, C_{i_3; r}, C_{i_4; r}) \), with a mismatch between the 4-tuples \((r, r, r, r)\) and \((\ell, r, \ell, r)\) considered in the indices. The above theorem says that if instead of \( \kappa_4 \) we use \( \kappa_{(\ell, r, \ell, r)} \), then we will get a \( \beta \)-coefficient:

\[
\kappa_{(\ell, r, \ell, r)}(A_{i_1}, B_{i_2}, A_{i_3}, B_{i_4}) = \beta(i_{i_3}, i_{i_1}, i_{i_2}, i_{i_4}).
\]

In fact, Equation (1.6) is now just a special case of the formula which relates moments to \((\ell, r)\)-cumulants: the \( \alpha \)'s and the \( \beta \)'s on the right-hand side are \( \kappa_\chi \)'s for various \( \chi \)'s obtained as restrictions of \((\ell, r, \ell, r)\), in the way indicated by the “moment \leftrightarrow (\ell, r)\)-cumulant” formula.

1.5 Case of vanishing mixed coefficients.

Suppose the polynomials \( f \) and \( g \) from Equation (1.1) are of the form

\[
\begin{align*}
f(z_1, \ldots, z_d) &= f_1(z_1) + \cdots + f_d(z_d) \quad \text{and} \\
g(z_1, \ldots, z_d) &= g_1(z_1) + \cdots + g_d(z_d),
\end{align*}
\] (1.10)

where \( f_1, \ldots, f_d, g_1, \ldots, g_d \) are polynomials of one variable. Then the formulas defining the canonical operators \( A_1, \ldots, A_d, B_1, \ldots, B_d \) simplify (from the general form shown in Equation (6.3) of the paper), and come to

\[
\begin{align*}
A_i &= L_i^+ (I + f_i(L_i)), \\
B_i &= R_i^+ (I + g_i(R_i)), \quad 1 \leq i \leq d,
\end{align*}
\] (1.11)
where the $L_i$’s and $R_i$’s are creation operators on the left and (respectively) on the right on $\mathcal{T}$ – see Notation 6.1 and Definition 6.2 for the notational details.

The $d$ pairs of operators $(A_1, B_1), \ldots, (A_d, B_d)$ appearing in Equations (1.11) give an example of bi-free family of pairs of elements of a noncommutative probability space, in the sense introduced in the recent work of Voiculescu [6], [7]. On the other hand, in view of the formula for $(\ell, r)$-cumulants provided by Equation (1.9), the special case of $f, g$ considered in (1.10) can be equivalently described via the requirement that

$$\kappa_{\chi} \left( C_{i_1; h_1}, \ldots, C_{i_n; h_n} \right) = 0$$

whenever $\exists 1 \leq p < q \leq n$ such that $i_p \neq i_q$.

This coincidence is in line with the fact that various forms of independence for noncommutative random variables which are considered in the literature have a combinatorial incarnation expressed in terms of the vanishing of some mixed cumulants. It is in fact tempting to make a definition and ask a question, as follows.

**Definition.** Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $(\kappa_{\chi} : \mathcal{A}^n \to \mathbb{C})_{n \geq 1, \chi \in \{\ell, r\}^n}$ be the family of $(\ell, r)$-cumulant functionals of $(\mathcal{A}, \varphi)$. Let $a_1, b_1, \ldots, a_d, b_d$ be in $\mathcal{A}$. We say that the pairs $(a_1, b_1), \ldots, (a_d, b_d)$ are **combinatorially-bi-free** to mean that the following condition is fulfilled: denoting $c_{i; \ell} := a_i$ and $c_{i; r} := b_i$ for $1 \leq i \leq d$, one has

$$\kappa_{\chi} \left( c_{i_1; h_1}, \ldots, c_{i_n; h_n} \right) = 0$$

whenever $n \geq 2$, $i_1, \ldots, i_n \in \{1, \ldots, d\}$, $\chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n$ and $\exists 1 \leq p < q \leq n$ such that $i_p \neq i_q$.

**Question.** How does the above concept of combinatorial-bi-freeness relate to the (representation theoretic) concept of bi-freeness that was introduced by Voiculescu in [6]?  

In order to address this question, it is likely that one needs to find an operator model with a richer input\(^3\) than what is used in the definition of the canonical operators $A_1, \ldots, A_d, B_1, \ldots, B_d$ of the present paper.

1.6 Organization of the paper.

Besides the present introduction, the paper has five other sections. After some review of background in Section 2, we introduce the sets of partitions $\mathcal{P}(\chi)(n)$ in Section 3, via the idea of examining double-ended queues. The alternative description of $\mathcal{P}(\chi)(n)$ via direct bijection with $NC(n)$ is presented in Section 4. In Section 5 we introduce the family of $(\ell, r)$-cumulant functionals $\kappa_{\chi}$ that are associated to a noncommutative probability space. Finally, in Section 6 we prove the main result of the paper, Theorem 6.5, giving the formula for $(\ell, r)$-cumulants of canonical operators that was announced in Equation (1.9) above.

\(^3\) The “input” used in the present paper consists of two families of coefficients, one for $f$ and one for $g$, where one thus has $2 \cdot d^n$ coefficients “of length $n$” for every $n \geq 1$. In a more general model one would probably need to input a family of coefficients $\gamma_{i_1, \ldots, i_n; h_1, \ldots, h_n}$ where $n \geq 1$ and $(i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$, $(h_1, \ldots, h_n) \in \{\ell, r\}^n$ (hence where there are $2^n \cdot d^n$ coefficients of length $n$, for every $n \geq 1$).
2. Background on partitions and on Lukasiewicz paths

Definition 2.1. \textit{[Partitions of \{1, \ldots, n\}.]} \\
Let $n$ be a positive integer.

1° We will let $\mathcal{P}(n)$ denote the set of all partitions of $\{1, \ldots, n\}$. A partition $\pi \in \mathcal{P}(n)$ is thus a set $\pi = \{V_1, \ldots, V_k\}$ where $V_1, \ldots, V_k$ (called the blocks of $\pi$) are non-empty sets with $V_i \cap V_j = \emptyset$ for $i \neq j$ and with $\bigcup_{i=1}^k V_i = \{1, \ldots, n\}$.

2° On $\mathcal{P}(n)$ we consider the partial order given by reverse refinement; that is, for $\pi, \rho \in \mathcal{P}(n)$ we will write $\pi \leq \rho$ to mean that for every block $V \subseteq \pi$ there exists a block $W \subseteq \rho$ such that $V \subseteq W$. The minimal and maximal partition with respect to this partial order will be denoted as $0_n$ and $1_n$, respectively:

$$0_n := \left\{ \{1\}, \ldots, \{n\} \right\}, \quad 1_n := \left\{ \{1, \ldots, n\} \right\}. \quad (2.1)$$

3° Let $\tau$ be a permutation of $\{1, \ldots, n\}$ and let $\pi = \{V_1, \ldots, V_k\}$ be in $\mathcal{P}(n)$. We will use the notation $\tau \cdot \pi$ for the new partition $\tau \cdot \pi := \{\tau(V_1), \ldots, \tau(V_k)\} \in \mathcal{P}(n)$.

4° Let $\pi$ be a partition in $\mathcal{P}(n)$. By the opposite of $\pi$ we will mean the partition $\pi_{\text{opp}} := \tau_o \cdot \pi \in \mathcal{P}(n)$, where $\tau_o$ is the order-reversing permutation of $\{1, \ldots, n\}$ (with $\tau_o(m) = n + 1 - m$ for every $1 \leq m \leq n$).

5° A partition $\pi \in \mathcal{P}(n)$ is said to be non-crossing when it is not possible to find two distinct blocks $V, W \in \pi$ and numbers $a < b < c < d$ in $\{1, \ldots, n\}$ such that $a, c \in V$ and $b, d \in W$. The set $\mathcal{NC}(n)$ of all non-crossing partitions in $\mathcal{P}(n)$ will play a significant role in this paper; for a review of basic some facts about it we refer to Lectures 9 and 10 of [3]. Let us record here that $\mathcal{NC}(n)$ is one of the many combinatorial structures counted by Catalan numbers, one has $\left| \mathcal{NC}(n) \right| = C_n := (2n)!/n!(n + 1)!$ (the $n$-th Catalan number).

Definition 2.2. \textit{[Lukasiewicz paths.]} \\
1° We will consider paths in $\mathbb{Z}^2$ which start at $(0,0)$ and make steps of the form $(1, i)$ with $i \in \mathbb{N} \cup \{-1, 0\}$. Such a path,

$$\lambda = \left( (0,0), (1,j_1), (2,j_2), \ldots, (n,j_n) \right), \quad (2.2)$$

is said to be a Lukasiewicz path when it satisfies the conditions that $j_m \geq 0$ for every $1 \leq m \leq n$ and that $j_n = 0$.

2° For every $n \geq 1$, we will use the notation $\text{Luk}(n)$ for the set of all Lukasiewicz paths with $n$ steps. For a path $\lambda \in \text{Luk}(n)$ written as in Equation (2.2), we will refer to the vector

$$\bar{\lambda} = (j_1 - 0, j_2 - j_1, \ldots, j_n - j_{n-1}) \in (\mathbb{N} \cup \{-1, 0\})^n \quad (2.3)$$

as to the rise-vector of $\lambda$. It is immediate how $\lambda$ can be retrieved from its rise-vector; moreover, it is immediate that a vector $(q_1, \ldots, q_n) \in (\mathbb{N} \cup \{-1, 0\})^n$ appears as rise-vector $\bar{\lambda}$ for some $\lambda \in \text{Luk}(n)$ if and only if its satisfies the conditions that $q_1 + \cdots + q_m \geq 0$ for every $1 \leq m \leq n$, and $q_1 + \cdots + q_n = 0$. \quad (2.4)
Remark and Notation 2.3. [The surjection $\Psi$ and the bijection $\Phi$.]

Let $n$ be a positive integer.

1° Let $\pi = \{V_1, \ldots, V_k\}$ be a partition in $\mathcal{P}(n)$. Consider the vector $(q_1, \ldots, q_n) \in (\mathbb{N} \cup \{-1, 0\})^n$ where for $1 \leq m \leq n$ we put

$$q_m := \begin{cases} |V_i| - 1, & \text{if } m = \min(V_i) \text{ for an } i \in \{1, \ldots, k\}, \\ -1, & \text{otherwise}. \end{cases} \tag{2.5}$$

It is immediately seen that $(q_1, \ldots, q_n)$ satisfies the conditions listed in 2.4, hence it is the rise-vector of a uniquely determined path $\lambda \in \text{Luk}(n)$.

2° We will denote by $\Psi : \mathcal{P}(n) \rightarrow \text{Luk}(n)$ (also denoted as $\Psi_n$, if needed to clarify what is $n$) the map which acts by

$$\Psi(\pi) := \lambda, \quad \pi \in \mathcal{P}(n),$$

with $\lambda$ obtained out of $\pi$ via the rise-vector described in 2.5.

3° The map $\Psi : \mathcal{P}(n) \rightarrow \text{Luk}(n)$ introduced above has the remarkable property that its restriction to $\text{NC}(n)$ gives a bijection from $\text{NC}(n)$ onto $\text{Luk}(n)$; for the verification of this fact, see e.g. [3], Proposition 9.8. We will denote by $\Phi : \text{Luk}(n) \rightarrow \text{NC}(n)$ the inverse of this bijection. That is: for every $\lambda \in \text{Luk}(n)$, we define $\Phi(\lambda)$ to be the unique partition in $\text{NC}(n)$ which has the property that

$$\Psi(\Phi(\lambda)) = \lambda.$$

The bijection $\Phi$ confirms the well-known fact that the set of paths $\text{Luk}(n)$ has the same cardinality $C_n$ (Catalan number) as $\text{NC}(n)$.

3. Double-ended queues and the sets of partitions $\mathcal{P}^{(\chi)}(n)$

Definition and Remark 3.1. [Description of a deque device.]

We will work with a device called double-ended queue, or deque for short, which is used in the study of information structures in theoretical computer science (see e.g. [1], Section 2.2). We will think about this device in the way depicted in Figures 1, 2 below, and described as follows. Let $n$ be a fixed positive integer, and suppose we have $n$ labelled balls $\begin{array}{c} 1 \\ \vdots \\ n \end{array}$ which have to move from an input pipe into an output pipe (both depicted vertically in the figures), by going through a deque pipe (depicted horizontally). The deque device operates in discrete time: it goes through a sequence of states, recorded at times $t = 0$, $t = 1$, $t = n$, where at time $t = 0$ all the $n$ balls are in the input pipe and at $t = n$ they are all in the output pipe. Compared to the discussion in Knuth’s treatise [1], we will limit the kinds of moves $^4$ that a deque can do, and we will require that:

For every $0 \leq i \leq n - 1$, the deque device moves from its state at time $t = i$ to its state at time $t = i + 1$ by performing either a “left-$p$ move” or a “right-$p$ move”, where $p \in \mathbb{N} \cup \{0\}$.

$^4$ In [1] the deque may perform any sequence of “insertions and deletions at either end of the queue”, where the term “insertion” designates the operation of moving some balls from the input pipe into the deque pipe, and “deletion” refers to moving some balls from the deque pipe into the output pipe. In the present paper we will only allow the special moves described in Definition 3.1, which match, in some sense, the creation and annihilation performed by canonical operators on a full Fock space.
The description of a left-$p$ move is like this:

\[
\begin{align*}
\text{"Take the top } p \text{ balls that are in the input pipe, and insert them} \\
\text{into the deque pipe, from the left. Then take the leftmost ball in} \\
\text{the deque pipe and insert it at the bottom of the output pipe."}
\end{align*}
\]

The description of a right-$p$ move is analogous to the one of a left-$p$ move, only that the words “left” and “leftmost” get to be replaced by “right” and respectively “rightmost”.

As is clear from the description of a left-$p$ (or right-$p$) move, the number $p \in \mathbb{N} \cup \{0\}$ used in the move $(t = i) \leadsto (t = i + 1)$ is subjected to the restriction that there exist $p$ balls (or more) in the input pipe at time $t = i$. Note also the additional restriction that $p = 0$ can be used in the move $(t = i) \leadsto (t = i + 1)$ only if the device has at least 1 ball in the deque pipe at time $t = i$.

\[\text{Figure 1: Say that } n = 9. \text{ Here is a possible state} \]
\[\text{of the deque device at time } t = 2.\]
Figure 2a: The deque device from Figure 1, at time $t = 3$, after performing a left-3 move.

Figure 2b: The deque device from Figure 1, at time $t = 3$, after performing a right-0 move.
Definition and Remark 3.2. [Deque-scenarios.]

Let \( n \) be the same fixed positive integer as in Definition 3.1, and consider the deque device described there. We assume that at time \( t = 0 \) the \( n \) balls are sitting in the input pipe in the order \( 1, \ldots, n \), counting top-down. From the description of the moves of the device, it is clear that for every \( 0 \leq i \leq n \) there are exactly \( i \) balls in the output pipe at time \( t = i \). In particular, all \( n \) balls find themselves in the output pipe at time \( t = n \) (even though they may not be sitting in the same order as at time \( t = 0 \)).

We will use the name *deque-scenario* to refer to a possible way of moving the \( n \) balls through the deque device, according to the rules described above. Every deque-scenario is thus determined by an array of the form

\[
\begin{pmatrix}
p_1 & \cdots & p_n \\
h_1 & \cdots & h_n
\end{pmatrix},
\]

(3.1)

with \( p_1, \ldots, p_n \in \mathbb{N} \cup \{0\} \) and \( h_1, \ldots, h_n \in \{\ell, r\} \); this array simply records the fact that in order to go from its state at time \( t = i - 1 \) to its state at time \( t = i \), the device has executed

\[
\begin{aligned}
\text{a left-}p_i \text{ move,} & \quad \text{if } h_i = \ell \\
\text{a right-}p_i \text{ move,} & \quad \text{if } h_i = r,
\end{aligned}
\]

1 \leq i \leq n.

Let us observe that the top line of the array in (3.1) must satisfy the inequalities

\[
\begin{aligned}
p_1 + \cdots + p_i & \geq i, \quad \forall 1 \leq i \leq n, \\
\text{where for } i = n \text{ we must have } p_1 + \cdots + p_n & = n.
\end{aligned}
\]

(3.2)

This is easily seen by counting that at time \( t = i \) there are \( i \) balls in the output pipe and \( n - (p_1 + \cdots + p_i) \) balls in the input pipe, which leaves a difference of

\[
n - \left( i + n - (p_1 + \cdots + p_i) \right) = (p_1 + \cdots + p_i) - i
\]

balls that must be in deque pipe. (And of course, the number of balls found in the deque pipe at \( t = i \) must be \( \geq 0 \), with equality for \( t = n \).)

It is easy to see that conversely, every array as in (3.1) with \( p_1, \ldots, p_n \) satisfying (3.2) will define a working deque-scenario – the inequalities \( p_1 + \cdots + p_i \geq i \) ensure that we never run into the situation of having to “move a ball out of the empty deque pipe”.

Thus, as a mathematical object, the set of deque-scenarios can be simply introduced as the set of arrays of the kind shown in (3.1), and where (3.2) is satisfied.

Moreover, let us observe that condition (3.2) can be read as saying that the \( n \)-tuple

\[
(p_1 - 1, \ldots, p_n - 1) \in (\mathbb{N} \cup \{-1, 0\})^n
\]

(3.3)

is the rise-vector of a uniquely determined Lukasiewicz path \( \lambda \), as reviewed in Section 2. We will then refer to the deque-scenario described by the array (3.1) as the *deque-scenario determined by \( (\lambda, \chi) \)*, where \( \lambda \in \text{Luk}(n) \) has rise-vector given by (3.3) and \( \chi \) is the \( n \)-tuple \( (h_1, \ldots, h_n) \in \{\ell, r\}^n \) from the second line of (3.1).
**Definition and Remark 3.3.** *(Output-time partition associated to a deque-scenario.)*

We consider the same notations as above and we look at the deque-scenario determined by \((\lambda, \chi)\), where \(\lambda \in \text{Luk}(n)\) has rise-vector \(\vec{\lambda} = (p_1-1, \ldots, p_n-1)\) and where \(\chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n\).

Let \(i \in \{1, \ldots, n\}\) be such that \(p_i > 0\), and consider the \(i\)-th move of the deque device (the move that takes the device from its state at \(t = i-1\) to its state at \(t = i\)). In that move there is a group of \(p_i\) balls (namely those with labels from \(p_1+\cdots+p_i-1\) to \(p_1+\cdots+p_i\)) which leave together the input pipe. These balls arrive in the output pipe one by one, at various later times, which we record as

\[ t_1^{(i)} < t_2^{(i)} < \cdots < t_{p_i}^{(i)}. \] (3.4)

Observe that in particular we have \(t_1^{(i)} = i\); indeed, it is also part of the \(i\)-th move of the device that the ball with label \(p_1+\cdots+p_i\) goes from the deque pipe into the output pipe. Let us make the notation

\[ T_i : = \{ t_1^{(i)}, \ldots, t_{p_i}^{(i)} \}, \]

where \(t_1^{(i)}, \ldots, t_{p_i}^{(i)}\) are from (3.4).

In the preceding paragraph we have thus constructed a set \(T_i \subseteq \{1, \ldots, n\}\) for every \(1 \leq i \leq n\) such that \(p_i > 0\). It is clear from the construction that for every such \(i\) we have

\[ |T_i| = p_i \text{ and } \min(T_i) = i. \] (3.5)

It is also clear that the sets

\[ \{T_i \mid 1 \leq i \leq n \text{ such that } p_i > 0\} \] (3.6)

form together a partition of \(\pi \in \mathcal{P}(n)\). We will refer to this \(\pi\) as the **output-time partition** associated to the pair \((\lambda, \chi)\).

**Example 3.4.** 1° A concrete example: say that \(n = 5\), that \(\lambda \in \text{Luk}(5)\) has rise-vector \(\vec{\lambda} = (2, -1, 1, -1, -1)\), and that \(\chi = (r, \ell, \ell, r, \ell)\). In the deque-scenario associated to this pair \((\lambda, \chi)\), there are two groups of balls that are moved from the input pipe into the deque pipe: first group consists of \(\{1, 2, 3\}\), which arrive in the deque pipe at time \(t = 1\); the second group consists of \(\{4, 5\}\), which arrive in the deque pipe at time \(t = 3\). The final order of the balls in the output pipe (counting downwards) is

\[ \{3, 1, 5, 2, 4\}, \]

and the output-time partition associated to \((\lambda, \chi)\) is \(\pi = \{\{1, 2, 4\}, \{3, 5\}\} \in \mathcal{P}(5)\).

2° Let \(n\) be a positive integer, and consider the \(n\)-tuple \(\chi_{\ell} := (\ell, \ldots, \ell) \in \{\ell, r\}^n\). For any \(\lambda \in \text{Luk}(n)\), the deque-scenario determined by \(\lambda\) and \(\chi_{\ell}\) is what one might call a “lifo-stack process” (where lifo is a commonly used abbreviation for last-in-first-out). It is easy to see that the output-time partition associated to the pair \((\lambda, \chi_{\ell})\) is the non-crossing partition \(\Phi(\lambda)\), where \(\Phi : \text{Luk}(n) \rightarrow \mathcal{P}(n)\) is as reviewed in Remark 2.3.3.

A similar statement holds if instead of \(\chi_{\ell}\) we use the \(n\)-tuple \(\chi_r := (r, \ldots, r)\); that is, the output-time partition associated to \((\lambda, \chi_r)\) is the same \(\Phi(\lambda) \in \text{NC}(n)\) as above.
Proposition 3.6. Let \( \pi \) be a partition of \( \lambda \) and \( \chi \in \{ \ell, r \}^n \), and let \( \pi \in \mathcal{P}(n) \) be the output-time partition associated to \( (\lambda, \chi) \) in Definition 3.3. We will denote this partition \( \pi \) as \( \Phi(\lambda) \). 

1. Let \( \chi \) be an \( n \)-tuple in \( \{ \ell, r \}^n \). The notation introduced in 1. above defines a function \( \Phi : \text{Luk}(n) \rightarrow \mathcal{P}(n) \). We define

\[
\mathcal{P}(\chi)(n) := \{ \Phi(\lambda) \mid \lambda \in \text{Luk}(n) \} \subseteq \mathcal{P}(n).
\] (3.7)

Proposition 3.6. Let \( n \) be a positive integer, let \( \chi \) be an \( n \)-tuple in \( \{ \ell, r \}^n \), and consider the function \( \Phi(\lambda) : \text{Luk}(n) \rightarrow \mathcal{P}(n) \) introduced in Definition 3.3.

1. \( \Phi(\lambda) \) is injective, hence it gives a bijection between \( \text{Luk}(n) \) and \( \mathcal{P}(\chi)(n) \).

2. Let \( \Psi(\lambda) : \mathcal{P}(\chi)(n) \rightarrow \text{Luk}(n) \) be the function inverse to \( \Phi(\lambda) \). Then \( \Psi(\lambda) \) is the restriction of \( \mathcal{P}(\chi)(n) \) of the canonical surjection \( \Psi : \mathcal{P}(n) \rightarrow \text{Luk}(n) \) that was reviewed in Remark 2.3.

Proof. Both parts of the proposition will follow if we can prove that \( \Psi \circ \Phi(\lambda) \) is the identity map on \( \text{Luk}(n) \). Thus given a path \( \lambda \in \text{Luk}(n) \) and denoting \( \Phi(\lambda)(\lambda) =: \pi \), we have to show that \( \Psi(\pi) = \lambda \). But the latter fact is clear from the observation made in (3.7) of Remark 3.3. \( \square \)

Remark 3.7. Let \( n \) be a positive integer.

1. From Proposition 3.6 and the fact that \( |\text{Luk}(n)| = C_n \) (\( n \)-th Catalan number), it follows that \( |\mathcal{P}(\chi)(n)| = C_n \) for every \( \chi \in \{ \ell, r \}^n \).

2. Suppose that \( \chi = (\ell, \ldots, \ell) \). The discussion from Example 3.4.12 shows that in this case we have \( \mathcal{P}(\chi)(n) = NC(n) \). Similarly, we also have \( \mathcal{P}(\chi)(n) = NC(n) \) in the case when \( \chi = (r, \ldots, r) \).

3. If \( n \leq 3 \), then it is clear from cardinality considerations that \( \mathcal{P}(\chi)(n) = \mathcal{P}(n) = NC(n) \), no matter what \( \chi \in \{ \ell, r \}^n \) we consider.

For \( n \geq 4 \), cardinality considerations now show that \( \mathcal{P}(\chi)(n) \) is a proper subset of \( \mathcal{P}(n) \). It is usually different from \( NC(n) \). (For instance the output-time partition from Example 3.4.1 is not in \( NC(5) \), showing that \( \chi = (r, \ell, \ell, r, \ell) \in \{ \ell, r \}^5 \) has \( \mathcal{P}(\chi)(5) \neq NC(5) \).) Some general properties of the sets of partitions \( \mathcal{P}(\chi)(n) \) will follow from their alternative description provided in the next section.

4. An alternative description for \( \mathcal{P}(\chi)(n) \)

In this section we put into evidence a bijection between \( \mathcal{P}(\chi)(n) \) and \( NC(n) \) which is implemented by the action of a special permutation \( \sigma_\chi \) of \( \{1, \ldots, n\} \). The main result of the section is Theorem 4.10. We will arrive to it by observing a certain construction of partition in \( NC(n) \) — the “combined-standings partition” associated to a pair \( (\lambda, \chi) \in \text{Luk}(n) \times \{ \ell, r \}^n \), which is introduced in Definition 4.3.
Definition 4.1. Consider a pair \((\lambda, \chi)\) where \(\lambda \in \text{Luk}(n)\) and \(\chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n\), and let \(\pi \in \mathcal{P}(n)\) be the output-time partition associated to \((\lambda, \chi)\) in Definition 3.3. Let us record explicitly where are the occurrences of \(\ell\) and of \(r\) in \(\chi\):

\[
\begin{align*}
\{m \mid 1 \leq m \leq n, h_m = \ell\} &= \{m_\ell(1), \ldots, m_\ell(u)\} \quad \text{with} \quad m_\ell(1) < \cdots < m_\ell(u), \\
\{m \mid 1 \leq m \leq n, h_m = r\} &= \{m_r(1), \ldots, m_r(v)\} \quad \text{with} \quad m_r(1) < \cdots < m_r(v).
\end{align*}
\]  

(4.1)

1° Suppose that in (4.1) we have \(u \neq 0\). We define a partition \(\rho_{\lambda,\chi,\ell} \in \mathcal{P}(u)\) by the following prescription: two numbers \(q, q' \in \{1, \ldots, u\}\) are in the same block of \(\rho_{\lambda,\chi,\ell}\) if and only if the numbers \(m_\ell(q), m_\ell(q') \in \{1, \ldots, n\}\) belong to the same block of \(\pi\). The partition \(\rho_{\lambda,\chi,\ell}\) will be called the left-standings partition associated to \((\lambda, \chi)\).

2° Likewise, if in (4.1) we have \(v \neq 0\), then we define a partition \(\rho_{\lambda,\chi,r} \in \mathcal{P}(v)\) via the prescription that \(q, q' \in \{1, \ldots, v\}\) belong to the same block of \(\rho_{\lambda,\chi,r}\) if and only if \(m_r(q), m_r(q')\) are in the same block of \(\pi\). The partition \(\rho_{\lambda,\chi,r}\) will be called the right-standings partition associated to \((\lambda, \chi)\).

Remark and Notation 4.2. Consider the framework of Definition 4.1. It will help the subsequent discussion if at this point we introduce some more terminology, which will also clarify the names chosen above for the partitions \(\rho_{\lambda,\chi,\ell}\) and \(\rho_{\lambda,\chi,r}\).

1° Same as in Section 3, we will think of the numbers in \(\{1, \ldots, n\}\) as of moments in time. We will say that \(t \in \{1, \ldots, n\}\) is a left-time (respectively a right-time) for \(\chi\) to mean that \(h_t = \ell\) (respectively that \(h_t = r\)). If \(t\) is a left-time for \(\chi\), then the unique \(q \in \{1, \ldots, u\}\) such that \(t = m_\ell(q)\) will be called the left-standing of \(t\) in \(\chi\). Likewise, if \(t\) is a right-time for \(\chi\), then the unique \(q \in \{1, \ldots, v\}\) such that \(t = m_r(q)\) will be called the right-standing of \(t\) in \(\chi\).

2° Let the rise-vector of \(\lambda\) be \(\overline{\lambda} = (p_1 - 1, \ldots, p_n - 1)\), with \(p_1, \ldots, p_n \in \mathbb{N} \cup \{0\}\). The numbers in the set

\[
I := \{1 \leq i \leq n \mid p_i > 0\}
\]

will be called insertion times for \((\lambda, \chi)\). Recall that the output-time partition \(\pi\) associated to \((\lambda, \chi)\) has its blocks indexed by \(I\); indeed, Equation (3.6) in Definition 3.3 introduces this partition as

\[
\pi = \{T_i \mid i \in I\}.
\]

(4.3)

With a slight abuse of notation, \(\rho_{\lambda,\chi,\ell}\) and \(\rho_{\lambda,\chi,r}\) from Definition 4.1 can be written as

\[
\rho_{\lambda,\chi,\ell} = \{V_i \mid i \in I\} \quad \text{and} \quad \rho_{\lambda,\chi,r} = \{W_i \mid i \in I\}
\]

(4.4)

where for every \(i \in I\) we put

\[
V_i := \{1 \leq q \leq u \mid m_\ell(q) \in T_i\} \quad \text{and} \quad W_i := \{1 \leq q \leq v \mid m_r(q) \in T_i\}.
\]

(4.5)

(Every \(V_i\) is a block of \(\rho_{\lambda,\chi,\ell}\) unless \(V_i = \emptyset\), and every \(W_i\) is a block of \(\rho_{\lambda,\chi,r}\) unless \(W_i = \emptyset\). Note that \(V_i\) and \(W_i\) cannot be empty at the same time, since \(|V_i| + |W_i| = |T_i| = p_i > 0\).)
Definition 4.3. We continue to consider the framework of Definition 4.1 and of Notation 4.2. For every \( i \in I \) let us denote

\[
(n + 1) - W_i := \{ n + 1 - q \mid q \in W_i \} \subseteq \{ u + 1, \ldots, n \}. \tag{4.6}
\]

The partition

\[
\rho_{\lambda, \chi} := \{ V_i \cup ((n + 1) - W_i) \mid i \in I \} \tag{4.7}
\]

will be called the combined-standings partition associated to \((\lambda, \chi)\).

Remark 4.4. The blocks of the partition \( \rho_{\lambda, \chi} \) are indexed by the same set \( I \) of insertion times that was used to index the blocks of the output-times partition \( \pi = \{ T_i \mid i \in I \} \) in Notation 4.2. Moreover, we have

\[
| V_i \cup ((n + 1) - W_i) | = | V_i | + | W_i | = | T_i |, \quad \forall i \in I;
\]

this shows that it must be possible to go between \( \pi \) and \( \rho_{\lambda, \chi} \) via the action of some suitably chosen permutation of \( \{1, \ldots, n\} \). We next make the easy yet significant observation that the permutation in question can be picked so that it only depends on \( \chi \) (even though each of \( \pi \) and \( \rho_{\lambda, \chi} \) depends not only on \( \chi \), but also on \( \lambda \)).

Definition 4.5. Let \( \chi \) be a tuple in \( \{\ell, r\}^n \). We associate to \( \chi \) a permutation \( \sigma_\chi \) of \( \{1, \ldots, n\} \) defined (in two-line notation for permutations) as

\[
\sigma_\chi := \begin{pmatrix}
1 & \cdots & u & u + 1 & \cdots & n \\
m_\ell(1) & \cdots & m_\ell(u) & m_\ell(v) & \cdots & m_\ell(1)
\end{pmatrix}, \tag{4.8}
\]

where \( m_\ell(1) < \cdots < m_\ell(u) \) and \( m_r(1) < \cdots < m_r(v) \) are as in Definition 4.1 (the lists of occurrences of “\( \ell \)” and “\( r \)” in \( \chi \)).

In (4.8) we include the possibility that \( v = 0 \) (when \( u = n \) and \( \sigma_\chi \) is the identity permutation), or that \( u = 0 \) (when \( v = n \) and \( \sigma_\chi(m) = n + 1 - m \) for every \( 1 \leq m \leq n \)).

Lemma 4.6. Consider a pair \((\lambda, \chi) \in \text{Luk}(n) \times \{\ell, r\}^n \), and let \( \pi \in \mathcal{P}(n) \) be the output-time partition associated to \((\lambda, \chi)\) in Definition 3.3. We have

\[
\sigma_\chi \cdot \rho_{\lambda, \chi} = \pi, \tag{4.9}
\]

where \( \rho_{\lambda, \chi} \) and \( \sigma_\chi \) are as in Definitions 4.3 and 4.5, respectively, and where the action of a permutation on a partition is as reviewed in Definition 2.1.3.

Proof. We use the notations established earlier in this section. Clearly, (4.9) will follow if we prove that

\[
\sigma_\chi \left( V_i \cup ((n + 1) - W_i) \right) = T_i, \quad \forall i \in I. \tag{4.10}
\]
Let us fix an \( i \in I \) for which we verify that (4.10) holds. Since the sets \( V_i \cup ((n+1) - W_i) \) and \( T_i \) have the same cardinality, it suffices to verify the inclusion “\( \subseteq \)” of the equality. And indeed, referring to how the permutation \( \sigma_\chi \) is defined in Equation (4.8), we have:\[
q \in V_i \Rightarrow \sigma_\chi(q) = m_\ell(q) \in T_i, \quad \text{and} \\
q \in (n + 1) - W_i \Rightarrow \sigma_\chi(q) = m_r(n + 1 - q) \in T_i,
\]
(\( m_r(n + 1 - q) \in T_i \) comes from Equation (4.5), used for the element \( n + 1 - q \in W_i \)). Thus both \( \sigma_\chi(V_i) \) and \( \sigma_\chi((n + 1) - W_i) \) are subsets of \( T_i \), and (4.10) follows.■

**Example 4.7.** Consider (same as in Example 3.4.1) the concrete case when \( n = 5, \chi = (r, \ell, \ell, r, \ell) \), and \( \lambda \in \text{Luk}(5) \) has rise-vector \( \vec{\lambda} = (2, -1, 1, -1, -1) \). As found in Example 3.4.1, the output-time partition associated to this \( (\lambda, \chi) \) is \( \pi = \{ \{ 1 \}, \{ 2, 3 \} \} \). The set of insertion times for \( (\lambda, \chi) \) of this example is \( I = \{ 1 \} \); in order to illustrate the system of notation from Equation (4.3), we then write \( \pi \) as \( \pi = \{ T_1, T_3 \} \), with \( T_1 = \{ 1, 2, 4 \} \) and \( T_3 = \{ 3, 5 \} \).

The left-times for \( \chi \) are \( m_\ell(1) = 2, m_\ell(2) = 3, m_\ell(3) = 5 \), and the right-times are \( m_r(1) = 1, m_r(2) = 4 \). Since \( m_\ell(1) \in T_1 \) and \( m_\ell(2), m_\ell(3) \in T_3 \), we get (in reference to the notations from Equations (4.4) and (4.5)) that \( V_1 = \{ 1 \}, V_3 = \{ 2, 3 \}, \) hence \( \rho_{\lambda,\chi;\ell} = \{ \{ 1 \}, \{ 2, 3 \} \} \in \mathcal{P}(3) \).

For the right-times we have \( m_r(1), m_r(2) \in T_1 \), giving us that \( W_1 = \{ 1, 2 \}, W_3 = \emptyset \), hence \( \rho_{\lambda,\chi;r} = \{ \{ 1, 2 \} \} \in \mathcal{P}(2) \).

The combined-standings partition \( \rho_{\lambda,\chi} \) associated to \( (\lambda, \chi) \) has blocks \( V_1 \cup (6 - W_1) = \{ 1 \} \cup \{ 4, 5 \} \) and \( V_3 \cup (6 - W_3) = \{ 2, 3 \} \cup \emptyset \), hence \( \rho_{\lambda,\chi} = \{ \{ 1, 4, 5 \}, \{ 2, 3 \} \} \).

Finally, the permutation associated to \( \chi \) is
\[
\sigma_\chi = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 5 & 4 & 1
\end{pmatrix}.
\]
As explained in the proof of Lemma 4.6, we have that \( \sigma_\chi(\{ 1, 4, 5 \}) = \{ 1, 2, 4 \} = T_1 \) and \( \sigma_\chi(\{ 2, 3 \}) = \{ 3, 5 \} = T_3 \), leading to the equality \( \sigma_\chi \cdot \rho_{\lambda,\chi} = \pi \).

Our next goal is to prove that the combined-standings partition \( \rho_{\lambda,\chi} \) always is a non-crossing partition. In order to obtain this, we first prove a lemma.
Lemma 4.8. Consider the framework and notations of Definition 4.3. Let us denote the maximal element of \( I \) by \( j \), and let us consider the block \( S = V_j \cup (\{ n + 1 \} - W_j) \) of the partition \( \rho_{\lambda, \chi} \). Then \( S \) is an interval-block (i.e. \( S = [n', n''] \cap \mathbb{Z} \) for some \( n' \leq n'' \) in \( \{1, \ldots, n\} \)).

Proof. The conclusion of the lemma is clear if \( |S| = 1 \), so we will assume that \( |S| \geq 2 \), i.e. that \( p_j \geq 2 \).

The maximal insertion time \( j \) considered in the lemma is either a left-time or a right-time for \( \chi \). We will write the proof by assuming that \( j \) is a left-time (the case of a right-time is analogous). We denote the left-stand of the time \( j \) as \( q \); recall from Notation 4.2 that this amounts to \( j = m_\ell(q) \).

In view of the above assumptions, the deque-scenario associated to \((\lambda, \chi)\) has the following feature: in the \( j \)-th move of the deque device, the last \( p_j \) balls of the input pipe (with labels between \( 1 + \sum_{i=1}^{j-1} p_i \) and \( n \)) are inserted into deque pipe from the left, and during the same move, the ball \( \circ \) goes into the output pipe. Thus the configuration of balls residing in the deque-pipe at time \( j \) is

\[
\begin{array}{cccccccc}
\text{I} & \text{II} & \cdots & \text{S} & \text{X} & \cdots & \text{Y} \\
\end{array}
\tag{4.11}
\]

where \( n' = n - 1, n'' = n - 2, \ldots, s = 1 + \sum_{i=1}^{j-1} p_i \), and where “\( \circ, \ldots, \bigcirc \)” is the (possibly empty) configuration of balls that were in the deque pipe at time \( j - 1 \). Let us also note here that each of the remaining moves of the deque device ((\( j + 1 \))-th move up to \( n\)-th move) is either a left-0 move or a right-0 move, since the input pipe was emptied at the \( j \)-th move.

Due to our assumption that \( j \) is a left-time, it is certain that \( V_j \neq \emptyset \) (we have in any case that \( V_j \ni q \)). But \( W_j \) may be empty, and we will discuss separately two cases.

Case 1. \( W_j = \emptyset \).

In this case all the balls \( \text{I}, \ldots, \text{S} \) exit the deque-pipe by its left side. Some of the balls \( \text{S}, \ldots, \text{Y} \) may also exit the deque-pipe by its left side, but they can only do so after all of \( \text{I}, \ldots, \text{S} \) are out of the way. This immediately implies that the times when \( \text{I}, \ldots, \text{S} \) exit the deque-pipe must have consecutive left-standings. It follows that in this case we have \( S = V_j = \{ q, q + 1, \ldots, q + p_j - 1 \} \), and hence \( S \) is an interval-block of \( \rho_{\lambda, \chi} \).

Case 2. \( W_j \neq \emptyset \).

In this case some of the balls \( \text{I}, \ldots, \text{S} \) (at least one and at most \( p_j - 1 \) of them) exit the deque-pipe by its right side. We observe it is not possible to find \( s \leq a < b \leq n - 1 \) such that the ball \( \bigcirc \) exits the deque-pipe by its left side while \( \bigcirc \) exits by the right-side. (Indeed, assume by contradiction that this would be the case. In the picture

\[
\begin{array}{cccccccc}
\text{I} & \cdots & \text{I} & \cdots & \text{A} & \cdots & \text{S} & \text{X} & \cdots & \text{Y} \\
\end{array}
\tag{4.11}
\]

one of the two balls \( \text{A}, \text{B} \) must be the first to exit the deque-pipe – but that’s not possible, since the other ball will block it.) As a consequence, there must exist a label \( c \in \{ s, \ldots, n - 1 \} \) such that the balls \( \text{S}, \ldots, \text{C} \) (i.e. the balls with labels in \( [s, c] \cap \mathbb{Z} \)) exit the deque-pipe by the right side, while the balls with labels in \( (c, n - 1] \cap \mathbb{Z} \) (if any) exit by the left side.

We next observe that all the balls \( \text{S}, \ldots, \text{Y} \) from the picture in (4.11) must exit the deque-pipe by its right side. This follows via the same kind of “blocking” argument as in

\footnote{Note that the times themselves when \( \text{I}, \ldots, \text{S} \) exit the deque-pipe don’t have to be consecutive, because they may be interspersed with some right-times used by balls from \( \text{S}, \ldots, \text{Y} \). The “consecutive” claim is only in reference to left-standings.}
the preceding paragraph. (Say e.g. that \(\otimes\) wants to exit by the left – then out of the two balls \(\otimes\) and \(\otimes\), none can be the first to exit the deque-pipe, because it would be blocked by the other.)

Based on the above tallying of how the balls from the picture in (4.11) exit the deque-pipe, a moment’s thought shows that the set \(W_j \subseteq \{1, \ldots, v\}\) must consist of the \(c-s+1\) largest numbers in \(\{1, \ldots, v\}\) and that, likewise, the set \(V_j\) must be the sub-interval \(\{q_1, \ldots, u\}\) of \(\{1, \ldots, u\}\). Then \((n+1)-W_j\) comes to \(\{u+1, \ldots, u+(c-s+1)\}\), and the union \(S = V_j \cup ((n+1)-W_j)\) is an interval-block of \(\rho_{\lambda,\chi}\), as required. \(\blacksquare\)

**Proposition 4.9.** Let \(n\) be a positive integer, and let \((\lambda, \chi)\) be a pair in \(\text{Luk}(n) \times \{\ell, r\}^n\). The combined-standings partition \(\rho_{\lambda,\chi}\) introduced in Definition 4.3 is in \(\text{NC}(n)\).

**Proof.** We proceed by induction on \(n\). The base case \(n = 1\) is clear, so we focus on the induction step: we fix an integer \(n \geq 2\), we assume the statement of the proposition holds for pairs in \(\text{Luk}(m) \times \{\ell, r\}^m\) whenever \(1 \leq m \leq n-1\), and we prove that it also holds for pairs in \(\text{Luk}(n) \times \{\ell, r\}^n\).

Let us then fix a pair \((\lambda, \chi)\) in \(\text{Luk}(n) \times \{\ell, r\}^n\), for which we will prove that \(\rho_{\lambda,\chi}\) is in \(\text{NC}(n)\). We denote \(\chi = (h_1, \ldots, h_n)\), and we denote the rise-vector of \(\lambda\) as \(\bar{x} = (p_1 - 1, \ldots, p_n - 1)\). Besides \(\rho_{\lambda,\chi}\), we will also work with the output-time partition \(\pi \in \mathcal{P}(n)\) associated to \((\lambda, \chi)\), and we will use the same notations as earlier in the section:

\[
\rho_{\lambda,\chi} = \{V_i \cup (n+1)-W_i \mid i \in I\} \quad \text{and} \quad \pi = \{T_i \mid i \in I\},
\]

where \(I = \{1 \leq i \leq n \mid p_i > 0\}\), the set of insertion times for \((\lambda, \chi)\). We will assume that \(|I| \geq 2\) (if \(|I| = 1\) then clearly \(\rho_{\lambda,\chi} = 1_n \in \text{NC}(n)\)). Same as in Lemma 4.8 we put \(j := \max(I)\); we thus have \(p_j \geq 1\) and \(p_{j+1} = \cdots = p_n = 0\).

Let us put \(m := n - p_j = \sum_{i=1}^{j-1} p_i\). Then \(m > 0\) (because the assumption \(|I| \geq 2\) means there exists \(i < j\) with \(p_i > 0\)), and also \(m < n\) (since \(p_j > 0\)). We consider the \(m\)-tuple

\[
\chi_o := \chi \mid \{(1, \ldots, n) \setminus T_j\} \in \{\ell, r\}^m
\]

(that is, \(\chi_o = (h_{t_1}, \ldots, h_{t_m})\), where one writes \(\{1, \ldots, n\} \setminus T_j = \{t_1, \ldots, t_m\}\) with \(t_1 < \cdots < t_m\)). On the other hand, let us consider the Łukasiewicz path \(\lambda_o \in \text{Luk}(m)\) determined by the requirement that

\[
\bar{x}_o = (p_1 - 1, \ldots, p_n - 1) \mid \{(1, \ldots, n) \setminus T_j\}.
\]

It is easily seen that the combined-standings partition \(\rho_{\lambda_o,\chi_o} \in \mathcal{P}(m)\) associated to \((\lambda_o, \chi_o)\) is obtained from \(\rho_{\lambda,\chi} \in \mathcal{P}(n)\) by removing the block \(V_j \cup ((n+1)-W_j)\) of \(\rho_{\lambda,\chi}\), and then by re-naming the elements of the remaining blocks of \(\rho_{\lambda,\chi}\) in increasing order. (Indeed, for this verification all one needs to do is ignore the last group of \(p_j\) balls which moves through the pipes of the deque device, in the deque-scenario determined by \((\lambda, \chi)\).)

Now, the block \(V_j \cup ((n+1)-W_j)\) removed out of \(\rho_{\lambda,\chi}\) is an interval-block, by Lemma 4.8. On the other hand, the partition \(\rho_{\lambda_o,\chi_o}\) is in \(\text{NC}(m)\), due to our induction hypothesis. Thus the partition \(\rho_{\lambda,\chi} \in \mathcal{P}(n)\) is obtained via the insertion of an interval-block with \(p_j(= n - m)\) elements into a partition from \(\text{NC}(m)\). This way of looking at \(\rho_{\lambda,\chi}\) readily implies that \(\rho_{\lambda,\chi} \in \text{NC}(n)\), and concludes the proof. \(\blacksquare\)
It is now easy to prove the main result of this section, which is stated as follows.

**Theorem 4.10.** Let $\chi$ be a tuple in $\{\ell, r\}^n$, and let the set of partitions $\mathcal{P}(\chi)(n) \subseteq \mathcal{P}(n)$ be as in Definition 3.3. Then $\mathcal{P}(\chi)(n)$ can also be obtained as

$$
\mathcal{P}(\chi)(n) = \{ \sigma \chi \cdot \pi \mid \pi \in NC(n) \} \subseteq \mathcal{P}(n),
$$

with $\sigma \chi$ as in Definition 4.5.

**Proof.** We will show, equivalently, that

$$
\{ \sigma^{-1} \chi \cdot \pi \mid \pi \in \mathcal{P}(\chi)(n) \} = NC(n).
$$

Since on both sides of the latter equality we have sets of the same cardinality, it suffices to verify the inclusion “$\subseteq$”. But “$\subseteq$” is clear from Lemma 4.6 and Proposition 4.9, since for $\pi = \Phi_{\chi}(\lambda)$ with $\lambda \in \text{Luk}(n)$ we get $\sigma^{-1} \chi \cdot \pi = \rho_{\lambda \chi} \in NC(n)$.

**Corollary 4.11.** Let $n$ be a positive integer and let $\chi$ be a tuple in $\{\ell, r\}^n$.

1. $\mathcal{P}(\chi)(n)$ contains the partitions $0_n$ and $1_n$ (from Notation 2.1.2), and also contains all the partitions $\pi \in \mathcal{P}(n)$ which have $n-1$ blocks.
2. The bijection $\mathcal{NC}(n) \ni \pi \mapsto \sigma \chi \cdot \pi \in \mathcal{P}(\chi)(n)$ from Theorem 4.10 is a poset isomorphism, where on both $\mathcal{NC}(n)$ and $\mathcal{P}(\chi)(n)$ we consider the partial order “$\leq$” defined by reverse refinement.
3. $(\mathcal{P}(\chi)(n), \leq)$ is a lattice. The meet operation “$\wedge$” of $\mathcal{P}(\chi)(n)$ is described via block-intersections – the blocks of $\pi_1 \wedge \pi_2$ are non-empty intersections $V_1 \cap V_2$, with $V_1 \in \pi_1$ and $V_2 \in \pi_2$.

**Proof.** 1. This follows from the fact that the set $\{0_n, 1_n\} \cup \{ \pi \in \mathcal{P}(n) \mid \pi \text{ has } n-1 \text{ blocks} \}$ is contained in $\mathcal{NC}(n)$ and is sent into itself by the action of $\sigma \chi$ (no matter what the permutation $\sigma \chi$ is).
2. This is an immediate consequence of the observation that the partial order by reverse refinement is preserved by the action of either $\sigma \chi$ or $\sigma^{-1} \chi$.
3. The fact that $(\mathcal{P}(\chi)(n), \leq)$ is a lattice follows from 2, since $(\mathcal{NC}(n), \leq)$ is a lattice. The description of the meet operation of $\mathcal{P}(\chi)(n)$ holds because the meet operation of $\mathcal{NC}(n)$ is given by block-intersections, and because the action of $\sigma \chi$ on partitions respects block-intersections.

**Remark 4.12.** 1. For every positive integer $n$, the permutations associated to the $(\ell, r)$-words $(\ell, \ldots, \ell)$ and $(r, \ldots, r)$ are

$$
\sigma_{(\ell, \ldots, \ell)} := \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}, \quad \sigma_{(r, \ldots, r)} := \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.
$$

When plugged into Theorem 4.10, this gives $\mathcal{P}^{(\ell, \ldots, \ell)}(n) = \mathcal{P}^{(r, \ldots, r)}(n) = \mathcal{NC}(n)$, a fact that had already been noticed in Remark 3.7.2.
Say that \( n = 4 \) and that \( \chi = (\ell, r, \ell, r) \), with associated partition
\[
\sigma_\chi = \begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 3 & 4 & 2
\end{pmatrix}.
\]

Theorem 4.10 gives, via an easy calculation, that \( P(\chi)(4) \) contains all the partitions of \( \{1, 2, 3, 4\} \) with the exception of \( \{\{1, 4\}, \{2, 3\}\} \) (in agreement with the description of this particular \( P(\chi)(n) \) that was mentioned in subsection 1.2 of the introduction).

In the sequel there will be instances when we will need to “read in reverse” a tuple from \( \{\ell, r\}^n \). We conclude the section with an observation about that.

**Definition 4.13.** For every \( n \geq 1 \) and \( \chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n \), the tuple
\[
\chi_{\text{opp}} := (h_n, \ldots, h_1)
\]
will be called the *opposite* of \( \chi \).

**Proposition 4.14.** Let \( n \) be a positive integer, let \( \chi \) be a tuple in \( \{\ell, r\}^n \), and consider the opposite tuple \( \chi_{\text{opp}} \). Then the sets of partitions \( P(\chi)(n) \) and \( P(\chi_{\text{opp}})(n) \) are related by the formula
\[
P(\chi_{\text{opp}})(n) = \{\pi_{\text{opp}} \mid \pi \in P(\chi)(n)\},
\]
where the opposite \( \pi_{\text{opp}} \) of a partition \( \pi \in P(n) \) is as considered in Definition 2.14.

**Proof.** Let \( \tau_o \) be the order-reversing permutation of \( \{1, \ldots, n\} \) that was considered in Definition 2.14 (\( \tau_o(m) = n+1-m \) for \( 1 \leq m \leq n \)). On the other hand let \( u \in \{0, 1, \ldots, n\} \) be the number of occurrences of the letter \( \ell \) in the word \( \chi \), and let us consider the permutation
\[
\tau_u := \begin{pmatrix}
1 & 2 & \cdots & u & u+1 & \cdots & n-1 & n \\
u & u-1 & \cdots & 1 & n & \cdots & u+2 & u+1
\end{pmatrix}.
\]
(Note that if \( u \) happens to be 0, then the permutation \( \tau_0 \) defined in (4.14) coincides, fortunately, with the permutation \( \tau_o \) that had been considered above.)

Let \( \pi \) be a partition in \( P(n) \setminus NC(n) \). Let \( V, W \) be two distinct blocks of \( \pi \) which cross, and let \( a < b < c < d \) be numbers such that \( a, c \in V \) and \( b, d \in W \). We leave it as an exercise to the reader to check via a case-by-case discussion that the numbers
\[
\tau_u(a), \tau_u(b), \tau_u(c), \tau_u(d) \in \{1, \ldots, n\}
\]
(despite not being necessarily in increasing order) ensure the existence of a crossing between the blocks \( \tau_u(V) \) and \( \tau_u(W) \) of the partition \( \tau_u \cdot \pi \in P(n) \). The argument in the preceding paragraph shows that \( \{\tau_u \cdot \pi \mid \pi \in P(n) \setminus NC(n)\} \subseteq P(n) \setminus NC(n) \). A cardinality argument forces the latter inclusion to be an equality, and then from the fact that \( \tau_u \) sends \( P(n) \) bijectively onto itself it also follows that we have
\[
\{\tau_u \cdot \pi \mid \pi \in NC(n)\} = NC(n).
\]
Now let us consider the positions of the letters \( \ell \) and \( r \) in the words \( \chi \) and \( \chi_{\text{opp}} \). By tallying these positions and plugging them into the formulas for the permutations \( \sigma_\chi \) and \( \sigma_{\chi_{\text{opp}}} \) (as in Definition 4.5), one immediately finds that

\[
\sigma_{\chi_{\text{opp}}} = \tau_0 \sigma_\chi \tau_u. \tag{4.16}
\]

So then we can write:

\[
P(\chi_{\text{opp}})(n) = \{ \sigma_{\chi_{\text{opp}}} \cdot \pi \mid \pi \in NC(n) \} \quad \text{(by Theorem 4.10)}
\]

\[
= \{ \tau_0 \sigma_\chi \tau_u \cdot \pi \mid \pi \in NC(n) \} \quad \text{(by Eqn. (4.16))}
\]

\[
= \{ \tau_0 \cdot \pi'' \mid \pi'' \in P(\chi)(n) \} \quad \text{(by Eqn. (4.15))}
\]

and this establishes the required formula (4.13).

\[\square\]

5. \((\ell, r)\)-cumulant functionals

In this section we introduce the family of \((\ell, r)\)-cumulant functionals associated to a noncommutative probability space. In order to write in a more compressed way the summation formula defining these functionals, we first introduce a notation.

**Notation 5.1.** \([\text{Restrictions of } n\text{-tuples.}].\)

Let \( X \) be a non-empty set, let \( n \) be a positive integer, and let \((x_1, \ldots, x_n)\) be an \( n \)-tuple in \( X^n \). For a subset \( V = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} \), with \( 1 \leq m \leq n \) and \( 1 \leq i_1 < \cdots < i_m \leq n \), we will denote

\[
(x_1, \ldots, x_n) \mid V := (x_{i_1}, \ldots, x_{i_m}) \in X^m.
\]

The next definition uses this notation in two ways:

- for \( X = A \) (algebra of noncommutative random variables);
- for \( X = \{\ell, r\} \), when we talk about the restriction \( \chi \mid V \) of a tuple \( \chi \in \{\ell, r\}^n \).

**Proposition and Definition 5.2.** \([\text{\((\ell, r)\)-cumulants.}].\)

Let \((A, \varphi)\) be a noncommutative probability space. There exists a family of multilinear functionals

\[
\left( \kappa_\chi : A^n \to \mathbb{C} \right)_{n \geq 1, \chi \in \{\ell, r\}^n}
\]

which is uniquely determined by the requirement that

\[
\varphi(a_1 \cdots a_n) = \sum_{\pi \in P(\chi)(n)} \left( \prod_{V \in \pi} \kappa_\chi( (a_1, \ldots, a_n) \mid V ) \right), \tag{5.1}
\]

for every \( n \geq 1, \chi \in \{\ell, r\}^n \) and \( a_1, \ldots, a_n \in A \).

These \( \kappa_\chi \)'s will be called the \((\ell, r)\)-cumulant functionals of \((A, \varphi)\).
Proof. For \( n = 1 \) we define \( \kappa(\ell) = \kappa(r) = \varphi \). We then proceed recursively, where for every \( n \geq 2 \), every \( \chi \in \{\ell, r\}^n \) and every \( a_1, \ldots, a_n \in \mathcal{A} \) we put
\[
\kappa_\chi(a_1, \ldots, a_n) = \varphi(a_1 \cdots a_n) - \sum_{\pi \in \mathcal{P}(\chi)(n), \pi \neq 1_n} \left( \prod_{V \in \pi} \kappa_\chi[V, (a_1, \ldots, a_n) | V] \right).
\] (5.2)

It is immediate that (5.2) defines indeed a family of multilinear functionals which fulfil (5.1). The uniqueness part of the proposition is also immediate, by following the (obligatory) recursion (5.2).

Remark 5.3. Let \((\mathcal{A}, \varphi)\) be a noncommutative probability space, let \((\kappa_n)_{n=1}^\infty \) be the family of free cumulant functionals of \((\mathcal{A}, \varphi)\), and let \((\kappa_\chi : \mathcal{A}^n \to \mathbb{C})_{n \geq 1, \chi \in \{\ell, r\}^n} \) be the family of \((\ell, r)\)-cumulant functionals introduced in Definition 5.2.

1° As noticed in Remark 5.2, one has \( \mathcal{P}(\ell, \ldots, \ell)(n) = \mathcal{P}(r, \ldots, r)(n) = NC(n) \). By plugging this fact into the recursion (5.2) which characterizes the functionals \( \kappa_\chi \), one immediately obtains the fact (already advertised in the introduction) that
\[
\kappa_{(\ell, \ldots, \ell)} = \kappa_{(r, \ldots, r)} = \kappa_n, \quad \forall n \geq 1.
\]

2° If \( n \leq 3 \), then we actually have \( \kappa_\chi = \kappa_n \) for every \( \chi \in \{\ell, r\}^n \). This comes from the fact, observed in Remark 5.3, that \( \mathcal{P}(\chi)(n) = NC(n) \) when \( n \leq 3 \), no matter what \( \chi \in \{\ell, r\}^n \) we consider.

3° For \( n \geq 4 \), the functionals \( \kappa_\chi \) with \( \chi \in \{\ell, r\}^n \) are generally different from \( \kappa_n \). Say for instance that \( \chi = (\ell, r, \ell, r) \in \{\ell, r\}^4 \), then the difference between the lattices \( NC(4) \) and \( \mathcal{P}(\chi)(4) \) leads to the fact that for \( a_1, \ldots, a_4 \in \mathcal{A} \) we have
\[
\kappa_{(\ell, r, \ell, r)}(a_1, \ldots, a_4) = \kappa_4(a_1, \ldots, a_4) + \kappa_2(a_1, a_4)\kappa_2(a_2, a_3) - \kappa_2(a_1, a_3)\kappa_2(a_2, a_4).
\]

Remark 5.4. Let us also record here a formula, concerning \((\ell, r)\)-cumulants, which is related to the reading of \((\ell, r)\)-words in reverse (i.e. to looking at \( \chi \) versus \( \chi_{\text{opp}} \), as in Definition 4.13 and Proposition 4.14). Suppose that \((\mathcal{A}, \varphi)\) is a \(*\)-probability space. Then, with \((\kappa_\chi : \mathcal{A}^n \to \mathbb{C})_{n \geq 1, \chi \in \{\ell, r\}^n} \) denoting the family of \((\ell, r)\)-cumulant functionals of \((\mathcal{A}, \varphi)\), one has
\[
\kappa_\chi(a_1^* \cdots a_n^*) = \kappa_{\chi_{\text{opp}}}(a_n \cdots a_1), \quad \text{for every } n \geq 1, \chi \in \{\ell, r\}^n \text{ and } a_1, \ldots, a_n \in \mathcal{A}.
\] (5.3)

The verification of (5.3) is easily done by induction on \( n \), where one relies on the bijections
\[
\mathcal{P}(\chi)(n) \ni \pi \mapsto \pi_{\text{opp}} \in \mathcal{P}(\chi_{\text{opp}})(n), \quad \text{for } n \geq 1 \text{ and } \chi \in \{\ell, r\}^n,
\]
that were observed in Proposition 4.14. (The proof of the induction step starts, of course, by writing that \( \varphi(a_1^* \cdots a_n^*) = \varphi(a_n \cdots a_1) \); each of the moments \( \varphi(a_1^* \cdots a_n^*) \) and \( \varphi(a_n \cdots a_1) \) is then expanded into \((\ell, r)\)-cumulants, in the way described in Definition 5.2.)
6. \((\ell, r)\)-cumulants of canonical operators

In this section we consider the \((\ell, r)\)-cumulant functionals of the noncommutative probability space \((B(\mathcal{T}), \varphi_{\text{vac}})\), where \(\mathcal{T}\) is the full Fock space over \(\mathbb{C}^d\). The main result of the section is Theorem 6.5, it gives a nice formula for a relevant family of joint \((\ell, r)\)-cumulants of a \((2d)\)-tuple of canonical operators on \(\mathcal{T}\), with \(d\) of the operators acting on the left and the other \(d\) on the right.

**Notation 6.1.** 1° Throughout the present section we fix an integer \(d \geq 1\) (the “number of variables” we work with), and we also fix an orthonormal basis \(e_1, \ldots, e_d\) of \(\mathbb{C}^d\). We consider the full Fock space over \(\mathbb{C}^d\),

\[
\mathcal{T} = \mathbb{C} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d \otimes \mathbb{C}^d) \oplus \cdots \oplus (\mathbb{C}^d)^\otimes n \oplus \cdots
\]

In \(\mathcal{T}\) we have a preferred orthonormal basis, namely

\[
\{\xi_{\text{vac}}\} \cup \{e_{i_1} \otimes \cdots \otimes e_{i_n} \mid n \geq 1, 1 \leq i_1, \ldots, i_n \leq d\},
\]

where \(\xi_{\text{vac}}\) is a fixed unit vector in the first summand \(\mathbb{C}\) on the right-hand side of (6.1). The vector-state defined by \(\xi_{\text{vac}}\) on \(B(\mathcal{T})\) will be called vacuum-state, and will be denoted by \(\varphi_{\text{vac}}\).

2° For every \(1 \leq i \leq d\) we will denote by \(L_i\) and \(R_i\) the left-creation operator and respectively the right-creation operator on \(\mathcal{T}\) defined by the vector \(e_i\). Thus \(L_i\) and \(R_i\) are isometries with \(L_i(\xi_{\text{vac}}) = R_i(\xi_{\text{vac}}) = e_i\), and where for every \(n \geq 1\) and \(1 \leq i_1, \ldots, i_n \leq d\) we have

\[
L_i(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_n}, \quad R_i(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_i.
\]

We next write down the precise definition of the canonical operators \(A_1, \ldots, A_d\) and \(B_1, \ldots, B_d\) discussed in the introduction.

**Definition 6.2.** Throughout the present section we also fix two polynomials \(f\) and \(g\) without constant term, in non-commuting indeterminates \(z_1, \ldots, z_d\). We write \(f\) and \(g\) explicitly as

\[
\begin{align*}
  f(z_1, \ldots, z_d) &= \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n=1}^{d} \alpha(i_1, \ldots, i_n) z_{i_1} \cdots z_{i_n}, \\
  g(z_1, \ldots, z_d) &= \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n=1}^{d} \beta(i_1, \ldots, i_n) z_{i_1} \cdots z_{i_n},
\end{align*}
\]

(6.2)

where the \(\alpha\)'s and \(\beta\)'s are complex coefficients (and where there exists \(n_0 \in \mathbb{N}\) such that \(\alpha(i_1, \ldots, i_n) = \beta(i_1, \ldots, i_n) = 0\) whenever \(n > n_0\)).

1° Consider the operators \(A_1, \ldots, A_d, B_1, \ldots, B_d \in B(\mathcal{T})\) defined as follows:

\[
\begin{align*}
  A_i := L_i^* \left( I + \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n=1}^{d} \alpha(i_1, \ldots, i_n) L_{i_n} \cdots L_{i_1} \right), \\
  B_i := R_i^* \left( I + \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n=1}^{d} \beta(i_1, \ldots, i_n) R_{i_n} \cdots R_{i_1} \right).
\end{align*}
\]

(6.3)
We will say that \((A_1, \ldots, A_d)\) is the \(d\)-tuple of left canonical operators with symbol \(f\), and that \((B_1, \ldots, B_d)\) is the \(d\)-tuple of right canonical operators with symbol \(g\).

2° It will be convenient to denote
\[
\begin{cases}
X_{0,\ell} = X_{0,r} = I \quad \text{(identity operator)}; \\
X_{p,\ell} = \sum_{i_1, \ldots, i_p=1}^d \alpha_{(i_1, \ldots, i_p)} L_{i_p} \cdots L_{i_1}, \quad \text{for } p \geq 1; \\
X_{p,r} = \sum_{i_1, \ldots, i_p=1}^d \beta_{(i_1, \ldots, i_p)} R_{i_p} \cdots R_{i_1}, \quad \text{for } p \geq 1.
\end{cases}
\] (6.4)

The operators \(A_i, B_i\) introduced above can then be written as
\[
A_i = L_i^* \sum_{p=0}^\infty X_{p,\ell}, \quad B_i = R_i^* \sum_{p=0}^\infty X_{p,r}, \quad \text{for } 1 \leq i \leq d.
\] (6.5)

(The sums in (6.5) are actually finite, since \(X_{p,\ell} = X_{p,r} = 0\) for \(p\) large enough.)

3° It will also be convenient to use some “unified left-right notations”, as follows. On the one hand, for the operators \(A_i\) and \(B_i\) introduced above we will write:
\[
C_{i;\ell} := A_i \quad \text{and} \quad C_{i;r} := B_i, \quad \text{for } 1 \leq i \leq d.
\]

On the other hand, let us put
\[
L_i =: S_{i,\ell}, \quad R_i =: S_{i,r}, \quad \text{for } 1 \leq i \leq d.
\]

The Equations (6.5) can then be re-written in a unified way in the form
\[
C_{i;h} = S_{i;h}^* \sum_{p=0}^\infty X_{p,h}, \quad \text{for } 1 \leq i \leq d \text{ and } h \in \{\ell, r\}.
\] (6.6)

Remark 6.3. In the preceding definition we chose to work with polynomials \(f, g\), i.e. with families of coefficients that are finitely supported. The results of this section would still hold without the latter assumption, in a framework where \(f\) and \(g\) are formal power series in \(z_1, \ldots, z_d\). The extra work needed in order to use that framework is of “notational” nature – the \(A_i\)’s and \(B_i\)’s may no longer live in \(B(T)\), so they have to be considered in a formal algebra of infinite matrices, as described on pp. 344-346 of [3].

Remark 6.4. The canonical operators \(A_1, \ldots, A_d\) defined in the first equation (6.3) are known (from [5], [2]) to provide an operator model for the \(d\)-variable \(R\)-transform. This means, in other words, that the recipe used to define \(A_1, \ldots, A_d\) leads to
\[
\kappa_n(A_{i_1}, \ldots, A_{i_n}) = \alpha_{(i_1, \ldots, i_n)}, \quad \forall \ n \geq 1 \text{ and } 1 \leq i_1, \ldots, i_n \leq d,
\] (6.7)

where \((\kappa_n)_{n=1}^\infty\) are the free cumulant functionals of the noncommutative probability space \((B(T), \varphi_{\text{vac}})\).

As explained in Lecture 21 of [3] (see Theorem 21.4 there), the derivation of (6.7) relies solely on the fact that the creation operators \(L_1, \ldots, L_d\) used in the first equation (6.3) are
a free family of Cuntz isometries. The latter fact means, by definition, that the $L_i$’s are isometries with $L_i^*L_j = 0$ for $i \neq j$, and that one has

$$\varphi_{\text{vac}}(L_{i_1} \cdots L_{i_m} L_{j_1}^* \cdots L_{j_n}^*) = 0$$

for all non-negative integers $m, n$ with $m + n \geq 1$ and all $i_1, \ldots, i_m, j_1, \ldots, j_n \in \{1, \ldots, d\}$.

It is immediate that the operators $R_1, \ldots, R_d$ used in the second equation (6.3) also form a free family of Cuntz isometries. This implies that the canonical operators $B_1, \ldots, B_d$ defined there have the property analogous to the one from Equation (6.7), that

$$\kappa_n(B_{i_1}, \ldots, B_{i_n}) = \beta_{(i_1, \ldots, i_n)}, \quad \forall n \geq 1 \text{ and } 1 \leq i_1, \ldots, i_n \leq d. \quad (6.8)$$

The next theorem provides a common generalization for the Equations (6.7) and (6.8).

**Theorem 6.5.** Let $(\kappa_\chi : B(T)^n \to \mathbb{C})_{n \geq 1, \chi \in \{\ell, r\}^n}$ be the family of $(\ell, r)$-cumulant functionals of the noncommutative probability space $(B(T), \varphi_{\text{vac}})$, and let $A_1, \ldots, A_d, B_1, \ldots, B_d \in B(T)$ be as in Definition 6.2 (where we also use the unified notations $C_{i;\ell} := A_i$ and $C_{i;r} := B_i$ for $1 \leq i \leq d$, adopted in Definition 6.3). Let us fix a positive integer $n$ and a tuple $\chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n$. Then for every $i_1, \ldots, i_n \in \{1, \ldots, d\}$ we have

$$\kappa_\chi(C_{i_1;h_1}, \ldots, C_{i_n;h_n}) = \begin{cases} \alpha(m_{\ell}(v), \ldots, m_{r}(1), i_{m_{\ell}(1)} \ldots, i_{m_{r}(1)}), & \text{if } h_n = \ell, \\
\beta(m_{\ell}(v), \ldots, m_{r}(1), i_{m_{\ell}(1)} \ldots, i_{m_{r}(1)}), & \text{if } h_n = r, \end{cases} \quad (6.9)$$

where $m_{\ell}(1) < \cdots < m_{\ell}(u)$ and $m_{r}(1) < \cdots < m_{r}(v)$ record the lists of occurrences of $\ell$ and of $r$ in $\chi$ (same convention of notation as in Definition 6.7).

The remaining part of the section is devoted to the proof of Theorem 6.5. We will start on this by formalizing, in Lemma 6.7 below, the intuitive idea that the action of $A_1, \ldots, A_d, B_1, \ldots, B_d$ on the vacuum vector $\xi_{\text{vac}} \in T$ is closely related to the deco-scenarios from Section 3 of the paper.

In order to state the lemma, we first introduce a notation.

**Notation 6.6.** [Bi-words and reverse-bi-mixtures of coefficients.]  

1° For every positive integer $n$, the elements of the set $\{1, \ldots, d\}^n \times \{\ell, r\}^n$ will be called bi-words of length $n$. We will usually write them in the form $(\omega; \chi)$, where $\omega = (i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$ and $\chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n$.

2° Let $(\omega; \chi)$ be a bi-word of length $n$, and let $V$ be a non-empty subset of $\{1, \ldots, n\}$. The convention for restrictions of $n$-tuples that was introduced in Notation 5.1 applies here too, and defines a restricted bi-word

$$(\omega; \chi) \mid V := (\omega \mid V; \chi \mid V) \in \{1, \ldots, d\}^m \times \{\ell, r\}^m,$$

where $m$ is the number of elements of $V$.

3° Consider the families of complex coefficients

$$\{\alpha(i, \ldots, i_n) \mid n \geq 1, \ 1 \leq i_1, \ldots, i_n \leq d\} \quad \text{and} \quad \{\beta(i, \ldots, i_n) \mid n \geq 1, \ 1 \leq i_1, \ldots, i_n \leq d\} \quad (6.10)$$
that appeared in Definition \[6.2\]. Let \( n \) be a positive integer and let \((\omega;\chi)\) be a bi-word of length \( n \), where \( \omega = (i_1,\ldots,i_n) \) and \( \chi = (h_1,\ldots,h_n) \). We will denote

\[
\tilde{\gamma}(\omega;\chi) := \begin{cases} 
\alpha(i_{m_\ell(1)},\ldots,i_{m_\ell(u)},i_{m_\ell(u)},\ldots,i_{m_\ell(1)}), & \text{if } h_1 = \ell \\
\beta(i_{m_\ell(1)},\ldots,i_{m_\ell(u)},i_{m_\ell(u)},\ldots,i_{m_\ell(1)}), & \text{if } h_1 = r,
\end{cases}
\tag{6.11}
\]

where \( m_\ell(1) < \cdots < m_\ell(u) \) and \( m_r(1) < \cdots < m_r(v) \) record the lists of occurrences of \( \ell \) and of \( r \) in \( \chi \) (same convention of notation as in Definition \[4.1\]). We will refer to \( \tilde{\gamma}(\omega;\chi) \) as to the reverse-bi-mixture of the \( \alpha \)'s and \( \beta \)'s from \[6.10\], corresponding to the bi-word \((\omega;\chi)\).

\[\text{Lemma 6.7.}\]

Let \( n \) be a positive integer, and consider the following items:

- an \( n \)-tuple \( \omega = (i_1,\ldots,i_n) \in \{1,\ldots,d\}^n \);
- an \( n \)-tuple \( \chi = (h_1,\ldots,h_n) \in \{\ell,r\}^n \);
- a Lukasiewicz path \( \lambda \in \text{Luk}(n) \) with rise-vector denoted as \( \tilde{\lambda} = (p_1-1,\ldots,p_n-1) \), where \( p_1,\ldots,p_n \in \mathbb{N} \cup \{0\} \).

Let \( \pi \in \mathcal{P}(\chi)(n) \) be the output-time partition associated to \((\lambda,\chi)\) in Definition \[3.3\]. Then we have

\[
X_{p_1:h_1}^*S_{i_1:h_1}\cdots X_{p_n:h_n}^*S_{i_n:h_n}\xi_{\text{vac}} = \gamma\xi_{\text{vac}}, \quad \text{where} \quad c = \prod_{T \in \pi} \tilde{\gamma}(\omega;\chi)\mid T). \tag{6.12}
\]

In Equation \[6.12\], the operators \( X_{p:h} \) and \( S_{i:h} \) are as in Definition \[6.2.3\] and the coefficients \( \tilde{\gamma} \) are reverse-bi-mixtures, as in Notation \[6.6\].

\[\text{Proof.}\]

Let \( \{j_1 < j_2 < \cdots < j_t\} = \{i|p_i > 0\} \) let \( \pi = \{T_{j_1},\ldots,T_{j_t}\} \), where for \( r = 1,\ldots,t \), we have that \( T_{j_r} \) denotes the block of the output-time partition corresponding to time \( j_r \), i.e., the block whose minimal element is \( j_r \). We abbreviate \( k = j_1 \).

We proceed by induction on \( t \). We first deal with the base case. If \( t = 1 \) then we must have \( k = 1, p_1 = p_k = n, p_2 = \cdots = p_n = 0 \), and \( \pi = \{1,\ldots,n\} \). If we denote \( \{m_\ell(1) < \cdots < m_\ell(u)\} = \{i|h_i = \ell\} \) and \( \{m_r(1) < \cdots < m_r(v)\} = \{i|h_i = r\} \) as in Notation \[6.6\] then we have

\[
X_{p_1:h_1}^*S_{i_1:h_1}\cdots X_{p_n:h_n}^*S_{i_n:h_n}\xi_{\text{vac}} = X_{n:h_1}^*S_{n:h_1}\xi_{\text{vac}} = X_{n:h_1}^*e_{m_\ell(1)}\otimes\cdots\otimes e_{m_\ell(u)}\otimes e_{m_r(v)}\otimes\cdots\otimes e_{m_r(1)} = \tilde{\gamma}(\omega,\chi)\mid T_{\pi})\xi_{\text{vac}} = \gamma\xi_{\text{vac}}.
\]

Now assume that \( t > 1 \) and that the conclusion of the lemma holds for all smaller values of \( t \). Let

\[
f: \{1,2,\ldots,n-p_k\} \rightarrow \{1,\ldots,n\} \setminus T_k
\]
denote the unique increasing bijection. We abbreviate

\[
\tilde{\omega} = (i_{f(1)},\ldots,i_{f(n-p_k)}), \quad \tilde{\chi} = (h_{f(1)},\ldots,h_{f(n-p_k)}),
\]

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and we also denote \( \bar{\nu} := (p_f(1) - 1, \ldots, p_f(n-p_k) - 1) \). Let \( \tilde{\lambda} \) be the Lukasiewicz path associated to \( \bar{\nu} \), and let \( \tilde{\pi} \in \mathcal{P}(\bar{\chi})(n - p_k) \) be the output-time partition associated to \( (\tilde{\lambda}, \bar{\chi}) \). We now note, as is implicit in the discussions in Sections 3 and 4, that

\[
f(\tilde{\pi}) = \{T_1, \ldots, T_{j_r-1}\}.
\]

Details of this observation are left to the reader. Observe now that by the induction hypothesis we have

\[
X_{p_f(1):h_f(1)} X_{f(1)} S_{f(1):h_f(1)} \cdots X_{f(n-p_k):h_f(n-p_k)} S_{f(n-p_k):h_f(n-p_k)} = c\xi_{\text{vac}},
\]

where

\[
c = \prod_{T \in \tilde{\pi}} \gamma((\bar{\omega}; \bar{\chi}) \mid T) = \prod_{T \in \pi, T \neq T_k} \gamma((\omega; \chi) \mid T).
\]

Let us list elements of the set \( \{k, \ldots, n\} \) as \( d_k, \ldots, d_n \) by listing left elements first in the increasing order followed by the right elements in the decreasing order, i.e., the order in the list \( d_k, \ldots, d_n \) respects the order from the list \( m_r(1), \ldots, m_r(u), m_r(v), \ldots, m_r(1) \). Now note that we have

\[
X_{p_k:h_k} S_{i_k:h_k} \cdots X_{p_{n-1}:h_{n-1}} S_{i_{n-1}:h_{n-1}} \xi_{\text{vac}}
\]

\[
= X_{p_k:h_k} (S_{i_k:h_k} \cdots S_{i_{n-1}:h_{n-1}} \xi_{\text{vac}})
\]

\[
= X_{p_k:h_k} e_{d_k} \otimes \cdots \otimes e_{d_n}
\]

\[
= \begin{cases}
\alpha_{d_n, \ldots, d_1} \xi_{\text{vac}}, & \text{if } p_k = n - k + 1 \text{ and } h_k = \ell \\
\beta_{d_n, \ldots, d_1} \xi_{\text{vac}}, & \text{if } p_k = n - k + 1 \text{ and } h_k = r \\
\alpha_{d_n+p_k-1, \ldots, d_n} e_{d_k+p_k} \otimes \cdots \otimes e_{d_n-p_k}, & \text{if } p_k < n - k + 1 \text{ and } h_k = \ell \\
\beta_{d_n-p_k+1, \ldots, d_n} e_{d_k} \otimes \cdots \otimes e_{d_n-p_k}, & \text{if } p_k < n - k + 1 \text{ and } h_k = r
\end{cases}
\]

\[
= \bar{\gamma}((\omega; \chi)(T_k) S_{f(k):d(f(k))} \cdots S_{f(n-p_k):d(f(n-p_k))} \xi_{\text{vac}}.
\]

Hence we have

\[
X_{p_1:h_1} S_{i_1:h_1} \cdots X_{p_{k-1}:h_{k-1}} S_{i_{k-1}:h_{k-1}} X_{p_k:h_k} S_{i_k:h_k} \cdots X_{p_{n-1}:h_{n-1}} S_{i_{n-1}:h_{n-1}} \xi_{\text{vac}}
\]

\[
= X_{p_1:h_1} S_{i_1:h_1} \cdots X_{p_{k-1}:h_{k-1}} \left( \bar{\gamma}((\omega; \chi)(T_k) S_{i_f(k):d(f(k))} \cdots S_{f(n-p_k):d(f(n-p_k))} \xi_{\text{vac}} \right)
\]

\[
= \bar{\gamma}((\omega; \chi)(T_k) X_{p_1:h_1} S_{i_1:h_1} \cdots X_{p_{k-1}:h_{k-1}} S_{i_f(k):d(f(k))} \cdots S_{f(n-p_k):d(f(n-p_k))} \xi_{\text{vac}}
\]

\[
= \bar{\gamma}((\omega; \chi)(T_k) X_{p_{f(n-p_k)}:h_{f(n-p_k)}} S_{i_f(1):h_f(1)} \cdots X_{p_{f(n-p_k)}:h_{f(n-p_k)}} S_{i_f(n-p_k):h_f(n-p_k)}
\]

\[
= \bar{\gamma}((\omega; \chi)(T_k) \prod_{T \in \pi, T \neq T_k} \bar{\gamma}((\omega; \chi) \mid T) \xi_{\text{vac}}
\]

\[
= \prod_{T \in \pi} \bar{\gamma}((\omega; \chi) \mid T) \xi_{\text{vac}} = \tau \xi_{\text{vac}}.
\]

This concludes the induction step. 

\textbf{Example 6.8.} For clarity, let us follow the preceding lemma in the concrete case (also discussed earlier, in Examples 3.4.1 and 1.7) where \( n = 5, \chi = (r, \ell, r, \ell, \ell) \), and \( \lambda \in \text{Luk}(5) \) has rise-vector \( \tilde{\lambda} = (2, -1, 1, -1, -1) \). As found in Example 3.4.1, the output-time partition
associated to this \((\lambda, \chi)\) is \(\pi = \{\{1, 2, 4\}, \{3, 5\}\}\). Let us also fix a tuple \(\omega = (i_1, \ldots, i_5) \in \{1, \ldots, 5\}\). We have \(\tilde{\gamma}((\omega; \chi) | \{1, 2, 4\}) = \tilde{\gamma}((i_1, i_2; 4); (r, \ell, r)) = \beta_{(i_2, i_4; i_1)}\) and \(\tilde{\gamma}((\omega; \chi) | \{3, 5\}) = \tilde{\gamma}((i_3, i_5); (\ell, \ell)) = \alpha_{(i_5; i_3)}\). The constant \(c\) from Equation \(6.12\) is thus \(c = \beta_{(i_2, i_4; i_1)}\alpha_{(i_5; i_3)}\), and the formula claimed by the lemma should come to

\[
X_{3;r}^* R_{i_1} X_{2;\ell}^* L_{i_2} X_{1;\ell}^* L_{i_3} X_{0;\ell}^* R_{i_4} X_{0;\ell}^* L_{i_5} \xi_{\text{vac}} = \overline{\overline{\overline{c}} \xi_{\text{vac}}}
\]

for this particular value of \(c\). And indeed, let us record how \(\xi_{\text{vac}}\) travels when we apply to it the operators listed on the left-hand side of the above equation: we get

\[
\begin{align*}
\xi_{\text{vac}} & \mapsto L_{i_5} \xi_{\text{vac}} = e_{i_5} \\
& \mapsto R_{i_4} e_{i_5} = e_{i_5} \otimes e_{i_4} \\
& \mapsto X_{2;\ell}^* L_{i_3} (e_{i_5} \otimes e_{i_4}) = X_{2;\ell}^* \overline{\overline{\overline{e_{i_3} \otimes e_{i_5} \otimes e_{i_4}}}} = \overline{\overline{\overline{\alpha_{i_5, i_3} e_{i_4}}}} \\
& \mapsto L_{i_2} (\overline{\overline{\overline{\alpha_{i_5, i_3} e_{i_4}}}}) = \overline{\overline{\overline{\alpha_{i_5, i_3} e_{i_2} \otimes e_{i_4}}}} \\
& \mapsto X_{3;r}^* R_{i_1} (\overline{\overline{\overline{\alpha_{i_5, i_3} e_{i_2} \otimes e_{i_4}}}}) = \overline{\overline{\overline{\alpha_{i_5, i_3} X_{3;r}^* (e_{i_2} \otimes e_{i_4} \otimes e_{i_1})}} = \overline{\overline{\overline{\alpha_{i_5, i_3} \cdot \beta_{i_2, i_4; i_1} \xi_{\text{vac}}}}}.
\end{align*}
\]

as claimed.

The reader may have noticed that, in the definition of the coefficients \(\tilde{\gamma}(\omega; \chi)\) in Notation \(6.6\) the order of the indices does not match (it is in some sense reverted from) what we need in Theorem \(6.5\). In order to state (in Proposition \(6.10\) below) the formula which will essentially prove Theorem \(6.5\) we then introduce the following notation.

**Notation and Remark 6.9.** [Bi-mixtures of coefficients.] Consider the same framework as in part 3° of Notation \(6.6\). Let \(n\) be a positive integer and let \((\omega; \chi)\) be a bi-word of length \(n\), where \(\omega = (i_1, \ldots, i_n)\) and \(\chi = (h_1, \ldots, h_n)\). We denote

\[
\gamma(\omega; \chi) := \begin{cases} \\
\alpha_{(i_{m_r}(v), \ldots, i_{m_r}(1), i_{m_r}(1), \ldots, i_{m_r}(u))}, & \text{if } h_n = \ell, \\
\beta_{(i_{m_r}(u), \ldots, i_{m_r}(1), i_{m_r}(1), \ldots, i_{m_r}(v))}, & \text{if } h_n = r,
\end{cases}
\]

(6.13)

where \(m_r(1) < \cdots < m_r(u)\) and \(m_r(1) < \cdots < m_r(v)\) record the lists of occurrences of \(\ell\) and of \(r\) in \(\chi\). We will refer to \(\gamma(\omega; \chi)\) as to the bi-mixture of \(\alpha\)'s and \(\beta\)'s corresponding to the bi-word \((\omega; \chi)\).

The explicit formula from Equation \(6.13\) will be useful for relating to the statement of Theorem \(6.5\). It is immediate that we could have also introduced \(\gamma(\omega; \chi)\) via the shorter formula

\[
\gamma(\omega; \chi) = \tilde{\gamma}(\omega_{\text{opp}}; \chi_{\text{opp}}),
\]

(6.14)

where \(\chi_{\text{opp}} := (h_n, \ldots, h_1)\) (same as in Definition \(6.13\)), \(\omega_{\text{opp}} := (i_n, \ldots, i_1)\), and the reversed bi-mixture \(\tilde{\gamma}\) is as in Notation \(6.6\). Let us moreover record here an immediate extension of Equation \(6.14\), namely that for every non-empty set \(T \subseteq \{1, \ldots, n\}\) we have

\[
\gamma((\omega; \chi) \mid T) = \tilde{\gamma}(\omega_{\text{opp}}; \chi_{\text{opp}}) \mid (n + 1) - T,
\]

(6.15)

with \((n + 1) - T := \{n + 1 - t \mid t \in T\}\).
Proposition 6.10. Let \( n \) be a positive integer and let \((\omega;\chi)\) be a bi-word of length \( n \), where \( \omega = (i_1, \ldots, i_n) \) and \( \chi = (h_1, \ldots, h_n) \). We have

\[
\varphi_{\text{vac}}(C_{i_1; h_1} \cdots C_{i_n; h_n}) = \sum_{\pi \in \mathcal{P}(\chi)(n)} \left( \prod_{T \in \pi} \gamma((\omega;\chi) \mid T) \right),
\]

(6.16)

where the operators \( C_{i;h} \) are as in Definition 6.2, the set of partitions \( \mathcal{P}(\chi)(n) \) is as in Definition 6.3, and the bi-mixtures “\( \gamma \)” on the right-hand side of the equation are as introduced in Notation 6.4.

Proof. Write each of \( C_{i_1; h_1}, \ldots, C_{i_n; h_n} \) as a sum in the way indicated in Equation (6.16) of Definition 6.2 then expand the ensuing product of sums; we get

\[
\varphi_{\text{vac}}(C_{i_1; h_1} \cdots C_{i_n; h_n}) = \sum_{p_1, \ldots, p_n = 0}^{\infty} \text{term}_{(p_1, \ldots, p_n)},
\]

(6.17)

where for every \( p_1, \ldots, p_n \in \mathbb{N} \cup \{0\} \) we put

\[
\text{term}_{(p_1, \ldots, p_n)} := \varphi_{\text{vac}}(S_{i_1; h_1}^* X_{p_1; h_1} \cdot \cdot \cdot S_{i_n; h_n}^* X_{p_n; h_n})
\]

(6.18)

\[= \langle S_{i_1; h_1}^* X_{p_1; h_1} X_{p_2; h_2} \cdot \cdot \cdot X_{p_n; h_n} \chi_{\text{vac}}, \chi_{\text{vac}} \rangle. \]

We will proceed by examining what \( n \)-tuples \((p_1, \ldots, p_n) \in (\mathbb{N} \cup \{0\})^n \) may contribute a non-zero term in the sum from (6.17).

So let \( p_1, \ldots, p_n \) be in \( \mathbb{N} \cup \{0\} \). We make the following observations.

- If there exists \( m \in \{1, \ldots, n\} \) with \( p_m + \cdots + p_n < (n + 1) - m \), then term\(_{(p_1, \ldots, p_n)} = 0 \). Indeed, if such an \( m \) exists then it is immediately seen that

\[
S_{i_m; h_m}^* X_{p_m; h_m} \cdot \cdot \cdot S_{i_n; h_n}^* X_{p_n; h_n} \chi_{\text{vac}} = 0,
\]

which makes the inner product from (6.18) vanish.

- If \( p_1 + \cdots + p_n > n \), then term\(_{(p_1, \ldots, p_n)} = 0 \). Indeed, in this case the vector

\[
S_{i_1; h_1}^* X_{p_1; h_1} X_{p_2; h_2} \cdot \cdot \cdot X_{p_n; h_n} \chi_{\text{vac}}
\]

is seen to belong to the subspace

\[
\text{span}\{e_{j_1} \otimes \cdots \otimes e_{j_q} \mid 1 \leq j_1, \ldots, j_q \leq d\} \subseteq \mathcal{T},
\]

where \( q = (p_1 + \cdots + p_n) - n > 0 \). The latter subspace is orthogonal to \( \chi_{\text{vac}} \), and this again makes the inner product from (6.18) vanish.

The observations made in the preceding paragraph show that a necessary condition for term\(_{(p_1, \ldots, p_n)} \neq 0 \) is that

\[
\left\{ \begin{array}{l}
p_m + \cdots + p_n \geq (n + 1) - m, \quad \forall 1 \leq m \leq n,

\text{where for } m = 1 \text{ we must have } p_1 + \cdots + p_n = n.
\end{array} \right.
\]

This says precisely that the tuple \((p_n - 1, \ldots, p_1 - 1)\) is the rise-vector of a uniquely determined path \( \lambda \in \text{Luk}(n) \). Hence the sum on the right-hand side of (6.17) is in fact, in a natural way, indexed by Luk\((n)\).
Now let us fix a path $\lambda \in \text{Luk}(n)$, where (consistent to the above) we denote the rise-vector of $\lambda$ as $\tilde{\lambda} := (p_n - 1, \ldots, p_1 - 1)$. If we put
\[
\tilde{p}_m := p_{n+1-m}, \tilde{h}_m := h_{n+1-m}, \tilde{i}_m := i_{n+1-m}, \quad 1 \leq m \leq n,
\]
then Equation (6.18) can be re-written in the form
\[
\text{term}_{(p_1, \ldots, p_n)} = (\xi_{\text{vac}}, X^*_{p_1; \tilde{h}_1} \mathcal{S}_{\tilde{i}_1; \tilde{h}_1} \cdots X^*_{p_n; \tilde{h}_n} \mathcal{S}_{\tilde{i}_n; \tilde{h}_n} \xi_{\text{vac}}),
\]
where on the right-hand side we are in the position to invoke Lemma 6.7. The lemma must now be used in connection to the path $\lambda$ and the tuples $\chi_{\text{opp}} = (\tilde{h}_1, \ldots, \tilde{h}_n)$, $\omega_{\text{opp}} = (\tilde{i}_1, \ldots, \tilde{i}_n)$. If we also denote
\[
\tilde{\pi} := \Phi_{\chi_{\text{opp}}}(\lambda) \quad \text{(output-time partition associated to } \lambda \text{ and } \chi_{\text{opp})},
\]
the application of Lemma 6.7 thus takes us to:
\[
\text{term}_{(p_1, \ldots, p_n)} = \prod_{\tilde{T} \in \tilde{\pi}} \tilde{\gamma}(\omega_{\text{opp}}; \chi_{\text{opp}} | \tilde{T}).
\]

Finally, we note that when $\tilde{T}$ runs among the blocks of $\tilde{\pi}$, the set $(n+1)-\tilde{T}$ runs among the blocks of the opposite partition $\pi_{\text{opp}}$. Thus, in view of the relation between $\gamma$’s and $\tilde{\gamma}$’s observed in Remark 6.9, we arrive to the formula
\[
\text{term}_{(p_1, \ldots, p_n)} = \prod_{T \in (\Phi_{\chi_{\text{opp}}}(\lambda))_{\text{opp}}} \gamma(\omega; \chi | T).
\]

The overall conclusion of the above discussion is that we have
\[
\varphi_{\text{vac}}(C_{i_1; h_1} \cdots C_{i_n; h_n}) = \sum_{\lambda \in \text{Luk}(n)} \prod_{T \in (\Phi_{\chi_{\text{opp}}}(\lambda))_{\text{opp}}} \gamma(\omega; \chi | T).
\]
The only thing left to verify is, then, that the set of partitions
\[
\{ (\Phi_{\chi_{\text{opp}}}(\lambda))_{\text{opp}} | \lambda \in \text{Luk}(n) \}
\]
coincides with $\mathcal{P}(\chi)(n)$. But this is indeed true, since $\{\Phi_{\chi_{\text{opp}}}(\lambda) | \lambda \in \text{Luk}(n)\} = \mathcal{P}(\chi_{\text{opp}})(n)$ (by the definition of $\mathcal{P}(\chi_{\text{opp}})(n)$), and in view of Proposition 4.14. \hfill \blacksquare

6.11. Proof of Theorem 6.5.

The statement of the theorem amounts to the fact that we have
\[
\left\{ \begin{array}{ll}
\kappa_\chi(C_{i_1; h_1} \cdots C_{i_n; h_n}) = \gamma(\omega; \chi) \quad & \text{for every } n \geq 1 \text{ and} \\
\text{every } \chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n, \omega = (i_1, \ldots, i_n) \in \{1, \ldots, d\}^n, 
\end{array} \right.
\]
with the coefficients $\gamma(\omega; \chi)$ as introduced in Notation 6.9. We will verify (6.19) by induction on $n$.

For $n = 1$ we only have to observe that $\kappa_\ell(A_i) = \gamma((i); (\ell))$, $\forall 1 \leq i \leq d$ (both the above quantities are equal to $\alpha_{(i)}$), and that $\kappa_r(B_i) = \gamma((i); (r))$, $\forall 1 \leq i \leq d$ (both quantities equal to $\beta_{(i)}$).
Induction step: consider an \( n \geq 2 \), suppose the equality in (6.19) has already been verified for all bi-words of length \( \leq n - 1 \), and let us fix a bi-word \((\omega; \chi)\) of length \( n \), for which we want to verify it as well. Write explicitly \( \omega = (i_1, \ldots, i_n) \) and \( \chi = (h_1, \ldots, h_n) \), with \( 1 \leq i_1, \ldots, i_n \leq d \) and \( h_1, \ldots, h_n \in \{\ell, r\} \). The joint moment \( \varphi_{\text{vac}}(C_{i_1; h_1} \cdots C_{i_n; h_n}) \) can be expressed as a sum over \( \mathcal{P}(\chi)(n) \) in two ways: on the one hand we have it written as in Equation (6.16) of Proposition 6.10, and on the other hand we can write it by using the moment-\leftrightarrow-cumulant formula (5.1) which was used to introduce the \((\ell, r)\)-cumulants in Definition 5.2:

\[
\varphi_{\text{vac}}(C_{i_1; h_1} \cdots C_{i_n; h_n}) = \sum_{\pi \in \mathcal{P}(\chi)(n)} \left( \prod_{V \in \pi} \kappa_{\chi}(C_{i_1; h_1}, \ldots, C_{i_n; h_n} | V) \right). \tag{6.20}
\]

The induction hypothesis immediately gives us that, for every \( \pi \neq 1_n \) in \( \mathcal{P}(\chi)(n) \), the term indexed by \( \pi \) in the two summations that were just mentioned (right-hand side of (6.16) and right-hand side of (6.20)) are equal to each other. When we equate these two summations and cancel all the terms indexed by \( \pi \neq 1_n \) in \( \mathcal{P}(\chi)(n) \), we are left precisely with \( \kappa_{\chi}(C_{i_1; h_1}, \ldots, C_{i_n; h_n}) = \gamma(\omega; \chi) \), as required. \( \blacksquare \)

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Addendum

After the present paper was posted on arXiv, the question asked in Section 1.5 of the Introduction was studied and found to have a positive answer, in the following paper:

I. Charlesworth, B. Nelson, P. Skoufranis. On two-faced families of non-commutative random variables. Preprint, March 2014, available at arxiv.org/abs/1403.4907.

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