Contact interaction of thin-walled elements with an elastic layer and an infinite circular cylinder under torsion

E G Kanetsyan*, M S Mkrtchyan, and S M Mkhitaryan
Institute of Mechanics of the National Academy of Sciences of Armenia, Yerevan, Armenia
E-mail: *ekanetsyan@gmail.com

Abstract. We consider a class of contact torsion problems on interaction of thin-walled elements shaped as an elastic thin washer – a flat circular plate of small height – with an elastic layer, in particular, with a half-space, and on interaction of thin cylindrical shells with a solid elastic cylinder, infinite in both directions. The governing equations of the physical models of elastic thin washers and thin circular cylindrical shells under torsion are derived from the exact equations of mathematical theory of elasticity using the Hankel and Fourier transforms. Within the framework of the accepted physical models, the solution of the contact problem between an elastic washer and an elastic layer is reduced to solving the Fredholm integral equation of the first kind with a kernel representable as a sum of the Weber–Sonin integral and some integral regular kernel, while solving the contact problem between a cylindrical shell and solid cylinder is reduced to a singular integral equation (SIE). An effective method for solving the governing integral equations of these problems are specified.

1. Introduction
The theory of torsion of elastic bodies and methods for solving torsion problems present one of the vast areas of mathematical theory of elasticity. The theoretical and applied significance of contact torsion problems lies in the fact that they, on the one hand, generalize and develop the classical contact problems of the theory of elasticity. On the other hand, they are directly related to important practical engineering issues on the transfer of loads from thin-walled elements to massive deformable bodies, often encountered in construction, machine building, especially in aircraft building, in mechanics of growing bodies, in measurement technology, in composites mechanics, and in other fields of applied mechanics. These problems are the subject of numerous studies [1–6]. The contact problem of torsion of an elastic half-space by an absolutely rigid circular washer is considered in [7]. An overview of the main results and papers on contact problems of torsion of elastic bodies is given in [8].

In the present paper, we consider a class of contact torsion problems on the interaction of thin-walled elements shaped as an elastic thin washer – a flat circular plate of small height – with an elastic layer, in particular, with a half-space and on the interaction of the thin cylindrical shells with a solid elastic cylinder, infinite in both directions.

The mechanical behavior of a thin circular elastic washer and a thin cylindrical shell under torsion is described by physical models that are completely analogous to the well-known Melan model of a rectilinear stringer [9, 10]. The deformation of an elastic layer and an elastic infinite
solid cylinder is described by differential equations of axisymmetric torsion of the theory of elasticity.

In problems for the layer, here, in contrast to other studies, the corresponding equation of deformation of a thin circular washer is first derived from the exact differential equation of axisymmetric torsion.

Next, the contact problem of torsion is considered when the elastic layer, whose shear modulus varies exponentially with depth, is rigidly clamped along its lower face, whereas a thin elastic circular washer is fastened on its upper face. The torsional tangential forces act on the upper face of the washer, as well as on its lateral surface. It is required to determine the tangential contact stresses under the washer, as well as other deformation characteristics of the washer.

From the condition of contact between the washer and the layer, the governing integral equation (GIE) of the problem is obtained. The kernel of this equation is the sum of its principal part in the form of the Weber–Sonin integral and the regular part in the form of a fairly rapidly convergent integral of the product of Bessel functions of the first kind. The GIE of the problem is solved by the method of spectral relationships containing the associated Legendre polynomials, or by the method of collocations using Gaussian quadrature formulas. Some particular cases are discussed.

In problems for an infinite cylinder, we first consider the boundary-value problem for an infinite hollow cylinder whose inner and outer surfaces are loaded with tangential torsional forces. Using the integral Fourier transform, we construct an exact solution of this problem containing modified Bessel functions and then, taking into account that the cylinder is thin, we derive an ordinary differential equation, as above, describing the deformation of a thin cylindrical shell. Further, within the framework of the obtained physical model, the problem of contact of a thin cylindrical shell of finite length with a solid infinite circular cylinder under torsion is considered. Eventually, solving this contact problem is reduced to solving the SIE with a kernel represented by the sum of the Cauchy kernel and a regular kernel. A well-known numerical-analytical method is used to solve the governing SIE [11].

2. Solution of two boundary-value problems on torsion of an exponentially inhomogeneous elastic layer under an axisymmetric deformation

Let an elastic layer \( \Omega_0 = \{0 < r < \infty; 0 \leq \vartheta < 2\pi; 0 \leq z \leq h\} \), referred to the cylindrical coordinate system \( r, \vartheta, z \), have a shear modulus \( G \) varying in depth by an exponential law

\[
G = G_0 e^{\gamma z} \quad (0 \leq z \leq h, \gamma = \text{const}).
\]  

(1)

It is well known [6] that, under an axisymmetric deformation, the Lame equations in a cylindrical coordinate system fall into two groups. The first group consists of two equations and describes the axisymmetric deformation of an elastic body, and the second group consists of a single equation and describes the torsion of the elastic body. Under torsion we have \( u_r = u_z \equiv 0 \), \( u_\vartheta = v(r, z) \) for elastic displacement components and

\[
e_{r\vartheta} = \frac{\partial v}{\partial r} - \frac{v}{r}, \quad e_{z\vartheta} = \frac{\partial v}{\partial z}, \quad e_r = e_\vartheta = e_z = e_{rz} \equiv 0,
\]

\[
2\omega_r = -\frac{\partial v}{\partial z}, \quad \omega_\vartheta = 0, \quad 2\omega_z = \frac{\partial v}{\partial r} + \frac{v}{r},
\]

(2)

for components of deformations and angles of rotation [6], respectively.

In addition, the differential equations of equilibrium in the absence of volume forces reduce to a single equation,

\[
\frac{\partial \tau_{r\vartheta}}{\partial r} + \frac{\partial \tau_{\vartheta z}}{\partial z} + \frac{2}{r} \tau_{r\vartheta} = 0,
\]

(3)
where \( \tau_{\rho \phi} \) and \( \tau_{\phi z} \) are components of tangential stresses. The Hooke’s law has the form

\[
\tau_{\rho \phi} = Ge_{\rho \phi}, \quad \tau_{\phi z} = Ge_{\phi z}.
\] (4)

We substitute the expressions for stresses (4) into equation (3) taking into account (1). As a result, we arrive at a differential equation for the function \( \nu(r, z) \),

\[
\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{r} \frac{\partial \nu}{\partial r} + \frac{\partial^2 \nu}{\partial z^2} + \gamma \frac{\partial \nu}{\partial z} - \frac{\nu}{r^2} = 0 \quad (r, z \in \Omega_0).
\] (5)

Now for an elastic layer \( \Omega_0 \), we consider the first boundary-value problem when the tangential stress components are given on its boundary planes \( z = 0 \) and \( z = h \):

\[
\tau_{\phi z} \big|_{z=0} = \tau(r), \quad \tau_{\phi z} \big|_{z=h} = \tau_0(r) \quad (0 < r < \infty).
\] (6)

Using (5), (6), (4), and (1), this problem is formulated mathematically as the boundary-value problem

\[
\begin{cases}
\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{r} \frac{\partial \nu}{\partial r} - \frac{\nu}{r^2} + \frac{\partial^2 \nu}{\partial z^2} + \gamma \frac{\partial \nu}{\partial z} = 0 & (0 < r < \infty, \ 0 < z < h), \\
\frac{\partial \nu}{\partial z} \big|_{z=0} = \tau(r) & \frac{\partial \nu}{\partial z} \big|_{z=h} = \frac{\tau_0(r)}{G_0} & (0 < r < \infty), \\
\tau_{\phi z} \big|_{r=0} + \tau_{\phi z} \big|_{r=\infty} & \rightarrow 0 & \text{when} \ r^2 + z^2 \rightarrow \infty.
\end{cases}
\] (7)

We construct the solution of boundary-value problem (7) by the method of the integral Hankel transform, for which we introduce the Hankel transformants

\[
\{\bar{v} = \bar{v}(z, \lambda), \ \bar{\tau}_0 = \bar{\tau}_0(\lambda), \ \bar{\bar{\tau}} = \bar{\tau}(\lambda)\} = \int_0^\infty \{\nu(r, z), \ \tau_0(r), \ \tau(r)\} r J_1(\lambda r) \, dr,
\]

where \( J_1(r) \) is the Bessel function of the first kind of order 1, and \( \lambda \ (0 < \lambda < \infty) \) is a spectral parameter.

Further, we multiply both sides of the differential equation in (7) and boundary conditions by \( r J_1(\lambda r) \) and integrate them with respect to \( r \) over the interval \((0, \infty)\). Taking into account the well-known relation from [12] (p. 79, formula (2.32) for \( \nu = 1 \))

\[
\int_0^\infty r \left( \frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{f}{r^2} \right) J_1(\lambda r) \, dr = -\lambda^2 \bar{f},
\]

we reduce the two-dimensional boundary-value problem (7) to the one-dimensional boundary-value problem

\[
\begin{cases}
\frac{d^2 \bar{v}}{dz^2} + \gamma \frac{d\bar{v}}{dz} - \lambda^2 \bar{v} = 0 & (0 < z < h), \\
\frac{d\bar{v}}{dz} \bigg|_{z=0} = \frac{\bar{\tau}}{G_0}, \quad \frac{d\bar{v}}{dz} \bigg|_{z=h} = \frac{e^{-\gamma h}}{G_0} \bar{\tau}_0.
\end{cases}
\] (8)

The differential equation in (8) has a general solution

\[
\bar{v} = \bar{v}(z, \lambda) = Ae^{\alpha_1 z} + Be^{\alpha_2 z} \quad (0 \leq z \leq h),
\]

\[
\alpha_1 = -\frac{\lambda_0 + \gamma}{2}, \quad \alpha_2 = \frac{\lambda_0 - \gamma}{2}, \quad \lambda_0 = \sqrt{\gamma^2 + 4\lambda^2}.
\]
After determining the constants $A$ and $B$ from the boundary conditions of problem (8), we obtain

$$
\bar{v}(z, \lambda) = \frac{e^{-\gamma z/2}}{2G_0 \lambda^2 \sinh(\lambda \nu h/2)} \left\{ e^{-\gamma h/2} \left( \lambda \cosh \frac{\lambda \nu z}{2} - \gamma \sinh \frac{\lambda \nu z}{2} \right) \bar{\tau}_0(\lambda) - \left[ \lambda \cosh \frac{\lambda \nu (z - h)}{2} + \gamma \sinh \frac{\lambda \nu (z - h)}{2} \right] \bar{\tau}(\lambda) \right\} \ (0 \leq z \leq h), \quad \lambda = \sqrt{\gamma^2 + 4\lambda^2}). \quad (9)
$$

Now we will find the desired function $v(r, z)$ with the help of the inverse Hankel transform,

$$
v(r, z) = \int_0^\infty \bar{v}(z, \lambda) \lambda J_1(\lambda r) d\lambda.
$$

Next, we consider an important special case, where $\gamma = 0$, i.e., where the elastic layer $\Omega_0$ is homogeneous. In this case, from (9) we have

$$
\bar{v}(z, \lambda) = \frac{\cosh(\lambda z) \bar{\tau}_0 - \cosh[\lambda(z - h)]\bar{\tau}}{\lambda G_0 \sinh(\lambda h)} \ (0 \leq z \leq h). \quad (10)
$$

Proceeding from (10), we obtain a differential equation for the deformation of an elastic thin layer or an elastic thin circular washer under torsion. For this, we write (10) in the form

$$
\lambda G_0 \sinh(\lambda h) \bar{v}_0(z, \lambda) = \cosh(\lambda z) \bar{\tau}_0 - \cosh[\lambda(z - h)]\bar{\tau} \quad (\bar{v}_0(z, \lambda) = \bar{v}(z, \lambda))
$$

and expand the entire functions of $\lambda \sinh(\lambda h)$, $\cosh(\lambda z)$, and $\cosh[\lambda(z - h)]$ in power series restricting ourselves to terms of the order of $\lambda^2$. We get

$$
G_0 h^2 \bar{v}_0 = G_0 h \lambda^2 \bar{v}_0(0, \lambda) = \bar{\tau}_0(\lambda) - \bar{\tau}(\lambda) + \frac{\lambda^2 z^2}{2} \bar{\tau}_0(\lambda) - \frac{\lambda^2}{2} (z - h)^2 \bar{\tau}(\lambda) \ (0 \leq z \leq h). \quad (11)
$$

Let us discuss two cases.

(1) $z = 0$. Then from (11) we obtain

$$
G_0 h \lambda^2 \bar{v}_0(0, \lambda) = \bar{\tau}_0(\lambda) - \bar{\tau}(\lambda) - \frac{\lambda^2 h^2}{2} \bar{\tau}(\lambda) \ (0 < \lambda < \infty),
$$

from which using the inverse Hankel transform, we arrive at the differential equation

$$
G_0 h \left( \frac{d^2 v_0}{dr^2} + \frac{1}{r} \frac{dv_0}{dr} - \frac{v_0}{r^2} \right) = \tau(r) - \tau_0(r) - \frac{h^2}{2} \left( \frac{d^2 \tau}{dr^2} + \frac{1}{r} \frac{d\tau}{dr} - \frac{\tau}{r^2} \right) \ (z = 0, \ 0 < r < \infty). \quad (12)
$$

Thus, the axisymmetric deformation of an elastic thin material layer $z = 0$ under torsion, within the framework of the accepted accuracy, is described by differential equation (12).

Now in (12) we pass to the limits $G_0 \to \infty$ and $h \to 0$ in such a way that $G_0 h$ – the rigidity of the thin layer – remains constant. As a result, by analogy with the known results from [10], we come to the physical model of an elastic thin layer in torsion:

$$
G_0 h \left( \frac{d^2 v_0}{dr^2} + \frac{1}{r} \frac{dv_0}{dr} - \frac{v_0}{r^2} \right) = \tau(r) - \tau_0(r) \ (0 < r < \infty). \quad (13)
$$

If, in particular, we set in (12) $\tau_0(r) = \tau(r) \ (0 < r < \infty)$, we obtain a homogeneous differential equation with respect to the function $v_0(r) + h\tau(r)/(2G_0)$. Since the displacements and stresses are bounded at $r = 0$ and $r = \infty$, this equation has only a trivial zero solution. As a result, we arrive at the Winkler model for a thin elastic layer under torsion,

$$
v_0 = -\frac{h\tau}{2G_0} \ (0 < r < \infty) \quad (14)
$$

with the bed coefficient $h/(2G_0)$.
expressed by the well-known Cauchy formula. As a result, we have the homogeneous equation and a particular solution of the inhomogeneous equation, which is

\[ \tau_0(\lambda) - \tau(\lambda) + \frac{\lambda^2 h^2}{2} \bar{\tau}_0(\lambda) \quad (0 < \lambda < \infty), \]

from which again with the help of the inverse Hankel transform, we obtain

\[ G_0 h \left( \frac{d^2 \tau_0}{dr^2} + \frac{1}{r} \frac{d \tau_0}{dr} - \frac{\tau_0}{r^2} \right) = \tau(r) - \tau_0(r) + \frac{h^2}{2} \left( \frac{d^2 \bar{\tau}_0}{dr^2} + \frac{1}{r} \frac{d \bar{\tau}_0}{dr} - \bar{\tau}_0 \right) \quad (z = h, \quad 0 < r < \infty). \]

Hence, as above, we arrive at the deformation model (13) of an elastic thin material layer \( z = h \) in torsion and at the Winkler model

\[ \tau_0 = \frac{h \tau_0}{2G_0} \quad (0 < r < \infty) \quad (16) \]

with the bed coefficient \( h/(2G_0) \).

Thus, physical models of deformation of a thin elastic layer under torsion, which are described by equations (12)–(14) and (15)–(16), were revealed from the exact equations of the theory of elasticity of axisymmetric torsion with a practically acceptable accuracy.

Now we consider the elastic equilibrium of an elastic thin circular washer of radius \( a \) and height \( h \) \((h \ll a)\) assuming that the washer is loaded on its upper and lower faces, as above, by tangential forces of intensities \( \tau_0(r) \) and \( \tau(r) \) \((0 < r < a)\), respectively, and on the lateral surface, it is also loaded by tangential forces with the resultant force \( T = T(a) \),

\[ T = T(a) = \int_0^{2\pi} d\vartheta \int_0^h \tau_{r\vartheta}(a, z) \, dz = 2\pi \int_0^h \tau_{r\vartheta}(a, z) \, dz. \]

Since \( h \ll a \) is small, we can assume that these forces are uniformly distributed over the side surface of the washer, and we set \( T = T(a) = T_0 h \).

Under these assumptions, we determine the required characteristics of the washer. First, proceeding from (13) for \( 0 < r < a \), we determine the circular displacements of the washer. For this, in (13) we set \( r = e^t \) \((-\infty < t < \ln a)\), and as a result of this, it becomes

\[ \frac{d^2 u_0}{dt^2} - u_0 = \frac{f_0(t)}{G_0 h}, \quad f_0(t) = e^{2t}[\tau(e^t) - \tau_0(e^t)], \quad u_0(t) = v_0(e^t). \]

The general solution of this differential equation consists of the sum of the general solution of the homogeneous equation and a particular solution of the inhomogeneous equation, which is expressed by the well-known Cauchy formula. As a result, we have

\[ u_0(t) = C_1 e^t + D_1 e^{-t} + \frac{1}{G_0 h} \int_{-\infty}^t \sinh(t - \xi) f_0(\xi) \, d\xi \quad (-\infty < t < \ln a). \]

Since the function \( v_0(r) \) is bounded at the center of the washer \( r = 0 \), we should set \( D_1 = 0 \). Then returning to the previous variable \( r \), we have

\[ v_0(r) = C_1 r + \frac{1}{2G_0 hr} \int_0^r (r^2 - s^2) [\tau(s) - \tau_0(s)] \, ds \quad (0 \leq r \leq a). \]

The solution of equation (13) for \( 0 \leq r \leq a \) can also be represented by the formula

\[ v_0(r) = C_2 r + \frac{1}{2G_0 hr} \int_r^a (s^2 - r^2) [\tau(s) - \tau_0(s)] \, ds \quad (0 \leq r \leq a). \]
Finally, using (17) and (18), we can write

$$v_0(r) = Cr + \frac{1}{4G_0lh} \int_0^a |r^2 - s^2| [\tau(s) - \tau_0(s)] \, ds \quad (0 \leq r \leq a), \quad C = \frac{C_1 + C_2}{2}. \tag{19}$$

To find the physical meaning of the constant $C$, we calculate the angle of rotation of the washer around the vertical axis $Oz$. According to (2) and (19) we have

$$\omega = \frac{1}{2} \left( \frac{dv_0}{dr} + \frac{v_0}{r} \right) = C + \frac{1}{4G_0h} \int_0^a \text{sign}(r-s)[\tau(s) - \tau_0(s)] \, ds \quad (0 \leq r \leq a).$$

Whence it follows that $\omega = C = \omega$ for an absolutely rigid washer ($G_0 = \infty$). Therefore, the constant $C = \omega$ represents the angle of rotation of the absolutely rigid washer around the axis $Oz$.

So, finally

$$v_0(r) = \omega r + \frac{1}{4G_0lh} \int_0^a |r^2 - s^2| [\tau(s) - \tau_0(s)] \, ds,$$

$$\tau(r) = \omega + \frac{1}{4G_0h} \int_0^a \text{sign}(r-s)[\tau(s) - \tau_0(s)] \, ds \quad (\tau(r) = \omega = \omega(r), \ 0 \leq r \leq a). \tag{20}$$

Let us determine the radial force $T(r) = h\tau_{r\theta}(r)$ in the current washer lateral surface $r$. Proceeding from the Hooke’s law (4), one can write

$$\tau_{r\theta} = G_0 e_{r\theta} = G_0 \left( \frac{dv_0}{dr} - \frac{v_0}{r} \right),$$

whence

$$\frac{d\tau_{r\theta}}{dr} = G_0 \left( \frac{d^2v_0}{dr^2} - \frac{1}{r} \frac{dv_0}{dr} + \frac{v_0}{r^2} \right).$$

Transforming this equation further with the help of equation (13) for $0 < r < a$, we obtain

$$\frac{dT}{dr} = 2G_0h \frac{d^2v_0}{dr^2} - \left[ \tau(r) - \tau_0(r) \right] \quad (0 < r < a). \tag{21}$$

On the other hand, equation (13) itself can be represented as

$$G_0h \frac{d^2v_0}{dr^2} + \frac{T(r)}{r} = \tau(r) - \tau_0(r) \quad (0 < r < a). \tag{22}$$

Further, from (21) and (22), eliminating the second-order derivative with respect to $v_0$, we arrive at the differential equation

$$\frac{dT}{dr} + \frac{2}{r} T = \tau(r) - \tau_0(r) \quad (0 < r < a). \tag{23}$$

The solution of equation (23), bounded for $r = 0$, is given by formula

$$T(r) = \frac{1}{r^2} \int_0^r s^2 \left[ \tau(s) - \tau_0(s) \right] \, ds \quad (0 \leq r \leq a). \tag{24}$$

Let us also write the moment equilibrium condition of the washer

$$\int_0^{2\pi} d\theta \int_0^a \tau_0(r) r \cdot r \, dr - \int_0^{2\pi} d\theta \int_0^a \tau(r) r \cdot r \, dr + 2\pi a T(a) \cdot a = 0.$$
Hence we arrive at the condition

\[ \int_0^\infty s^2 \tau(s) \, ds = \frac{M_0}{2\pi} + a^2 T(a), \quad M_0 = 2\pi \int_0^\infty s^2 \tau_0(s) \, ds, \quad T(a) = T_0 h. \]  

(25)

Note that if we set \( r = a \) in (24), then we obtain condition (25).

Now we turn to the second boundary-value problem. Let us again consider an elastic layer \( \Omega_1 = \{0 < r < \infty, \ 0 \leq \vartheta < 2\pi, \ -H \leq z \leq 0\} \) whose shear modulus \( G \), as in the first problem, varies in depth by the exponential law

\[ G = G_1 e^{\gamma z} \quad (-H \leq z \leq 0, \ G_1 = \text{const}, \ \gamma = \text{const}). \]

Let then the tangential forces of intensity \( \tau(r) \) act on the upper boundary plane \( z = 0 \) of the layer \( \Omega_1 \), whereas the layer lower face \( z = -H \) is rigidly clamped. This mixed boundary-value problem is formulated mathematically as

\[
\begin{aligned}
\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r} \frac{\partial v_1}{\partial r} + \frac{\partial^2 v_1}{\partial z^2} + \gamma \frac{\partial v_1}{\partial z} - \frac{v_1}{r^2} &= 0 \quad (0 < r < \infty, \ -H < z < 0), \\
t_{\vartheta z} \big|_{z=0}^{z=-H} &= \tau(r), \quad v_1 \big|_{z=-H+0} = 0 \quad (0 < r < \infty), \\
\tau_{\vartheta \vartheta} + \tau_{zz}^2 &\to 0 \quad \text{when} \ r^2 + z^2 \to \infty.
\end{aligned}
\]  

(26)

Here \( v_1 = v_1(r, z) \) is the component of the circular displacements of points of the layer \( \Omega_1 \).

Here, too, we again construct the solution of problem (26) by the method of the integral Hankel transform. As a result, for the Hankel transformants \( \tilde{v}_1(z, \lambda) \), we have

\[ \tilde{v}_1(z, \lambda) = \frac{2 e^{-z \gamma/2} \sinh[\lambda_s(z + H)/2]}{\lambda_s \cosh[\lambda_s H/2] - \gamma \sinh(\lambda_s H/2)} \frac{\tilde{\tau}(\lambda)}{G_1} \quad (-H \leq z \leq 0, \ 0 < \lambda < \infty), \]

where the notation is the same. Whence for \( z = 0 \), we have

\[ \tilde{v}_1(0, \lambda) = \frac{2 \tanh(\lambda_s H/2)}{\lambda_s - \gamma \tanh(\lambda_s H/2)} \frac{\tilde{\tau}(\lambda)}{G_1} \quad (0 < \lambda < \infty). \]

Now, using the inverse Hankel transform formula, we get

\[ v_1(r, 0) = \int_0^\infty \tilde{v}_1(0, \lambda) J_1(\lambda r) \lambda d\lambda = \frac{2}{G_1} \int_0^\infty \frac{\tanh(\lambda_s H/2)\lambda \tilde{\tau}(\lambda)}{\lambda_s - \gamma \tanh(\lambda_s H/2)} J_1(\lambda r) d\lambda. \]

After simple transformations, we have

\[
\begin{aligned}
v_1(r, 0) &= \frac{1}{G_1} \int_0^\infty K(r, \rho) \tau(\rho) \rho d\rho \quad (0 < r < \infty), \\
K(r, \rho) &= \int_0^\infty J_1(\lambda r) J_1(\lambda \rho) h(\lambda) \, d\lambda, \quad h(\lambda) = \frac{2 \tanh(\lambda_s H/2)\lambda}{\lambda_s - \gamma \tanh(\lambda_s H/2)},
\end{aligned}
\]  

(27)

Since

\[ \lambda_s = \sqrt{\gamma^2 + 4\lambda^2} = 2\lambda \left(1 + \frac{\gamma^2}{4\lambda^2}\right)^{1/2} = 2\lambda + \frac{\gamma^2}{4\lambda} + O\left(\frac{1}{\lambda^3}\right) \quad (\lambda \to \infty), \]

we can see that

\[ h(\lambda) \sim 1 \quad \text{when} \ \lambda \to \infty. \]
Hence the function \( h(\lambda) \) can be transformed as

\[
h(\lambda) = \left[ \frac{2\lambda \tanh(\lambda H/2)}{\lambda_\ast - \gamma \tanh(\lambda H/2)} - 1 \right] + 1 = 1 + h_0(\lambda), \quad h_0(\lambda) = \frac{(2\lambda + \gamma) \tanh(\lambda H/2) - \lambda_\ast}{\lambda_\ast - \gamma \tanh(\lambda H/2)}, \quad (28)
\]

and \( h_0(\lambda) = O(1/\lambda) \) as \( \lambda \to \infty \), \( h_0(0) = -1 \).

Taking into account (28), the kernel \( K(r, \rho) \) can be represented as

\[
K(r, \rho) = K_0(r, \rho) + R_0(r, \rho),
\]

\[
K_0(r, \rho) = \int_0^\infty J_1(\lambda r) J_1(\lambda \rho) d\lambda,
\]

\[
R_0(r, \rho) = \int_0^\infty J_1(\lambda r) J_1(\lambda \rho) h_0(\lambda) d\lambda.
\]

(29)

Note that as we pass to the limit as \( H \to \infty \), then the elastic layer \( \Omega_1 \) becomes the lower elastic half-space \( z \leq 0 \), and from (27) we obtain

\[
v_1(r, 0) = \frac{1}{G_1} \int_0^\infty L(r, \rho) \tau(\rho) \rho d\rho \quad (0 < r < \infty), \quad L(r, \rho) = 2 \int_0^\infty \frac{1}{\lambda_\ast - \gamma} J_1(\lambda r) J_1(\lambda \rho) d\lambda.
\]

(30)

Using simple transformations and separating the principle part of the kernel, we find

\[
L(r, \rho) = K_0(r, \rho) + L_0(r, \rho), \quad L_0(r, \rho) = \int_0^\infty J_1(\lambda r) J_1(\lambda \rho) g_0(\lambda) d\lambda,
\]

\[
g_0(\lambda) = \frac{2\gamma}{\sqrt{\gamma^2 + 4\lambda^2} + 2\lambda - \gamma} \left( g_0(\lambda) \sim \frac{\gamma}{\lambda} \text{ when } \lambda \to \infty \right).
\]

(31)

For a homogeneous half-space \( (z \leq 0) \), we have \( L(r, \rho) = K_0(r, \rho) \).

We express the kernel \( K_0(r, \rho) \) from (29), which is a well-known particular Weber–Sonin integral, in terms of a hypergeometric function. To this end, we use the well-known formulas from [13] (p. 49, Eq. 8.11(9)) and from [14] (p. 113, Eq. (36)). As a result, for kernel \( K_0(r, \rho) \) from (31) we get

\[
K_0(r, \rho) = \frac{r \rho}{2(r + \rho)^3} F\left( \frac{3}{2}; \frac{1}{2}; \frac{4r \rho}{(r + \rho)^2} \right) \quad (0 < r, \rho < \infty),
\]

(32)

where \( F(a, b; c; x) \) is a hypergeometric function; the series corresponding to the function absolutely converges for \( r = \rho \).

3. Contact problem of torsion of an elastic exponentially inhomogeneous layer by means of an elastic circular washer adhered to it

Let the elastic layer \( \Omega_1 \) considered above, whose shear modulus varies exponentially with depth and whose lower face \( z = -H \) is rigidly clamped, be subject to torsion by means of an elastic thin circular washer of shear modulus \( G_0 \), radius \( a \), and height \( h \) adhered to the layer along its upper face \( z = 0 \). Torsional tangential forces of intensity \( \tau_0(r) \) act on the upper face of the washer. In addition, the washer on its lateral surface along the mid-height circle is loaded with torsional tangential forces of constant intensity \( T = T(a) \). It is required to determine the tangential contact stresses on the contact circle of radius \( a \).

We reduce solving the posed problem to solving an integral equation (IE), which we supplement with the contact condition

\[
v_0(r) = v_1(r, 0) \quad (0 \leq r \leq a).
\]
Setting the expressions for \( v(r) \) and \( v(0) \) from (20) and (30) and passing to dimensionless quantities
\[
\xi = \frac{r}{a}, \quad \eta = \frac{\rho}{a}, \quad \varphi(\xi) = \frac{\tau(a\xi)}{G_1}, \quad f_0(\xi) = \frac{\tau_0(a\xi)}{G_1},
\]
\[
\alpha = a\lambda, \quad H_0 = \frac{H}{a}, \quad \gamma_0 = a\gamma, \quad \Lambda = \frac{G_1a}{4G_0h}, \quad \lambda_0 = \sqrt{\gamma_0^2 + 4\alpha^2},
\]
after simple transformations, we arrive at the governing Fredholm IE of the first kind
\[
\int_0^1 \{M_0(\xi, \eta) + N_0(\xi, \eta)|\xi| \eta - \Lambda|\xi^2 - \eta^2|\} \varphi(\eta) \, d\eta = \omega \xi^2 - \Lambda h_0(\xi) \quad (0 < \xi < 1), \quad (33)
\]
\[
M_0(\xi, \eta) = aK_0(a\xi, a\eta) = \int_0^\infty J_1(a\xi) J_1(\alpha\eta) \, d\alpha, \quad h_0(\xi) = \int_0^1 \xi^2 - \eta^2 |f_0(\eta)| \, d\eta,
\]
\[
N_0(\xi, \eta) = aR_0(a\xi, a\eta) = \int_0^\infty J_1(\alpha\xi) J_1(\alpha\eta) n_0(\alpha) \, d\alpha, \quad n_0(\alpha) = (2\alpha + \gamma_0) \tan(\lambda_0H_0/2) - \lambda_0 \frac{\lambda_0 - \gamma_0}{\lambda_0 \tan(\lambda_0H_0/2)},
\]
from which the dimensionless tangential contact stresses \( \varphi(\xi) \) are determined.

In terms of the above-introduced dimensionless quantities, we also write the angle of rotation of the washer along the vertical axis \( \varphi(r) \) from (20), radial force in the washer \( T(r) \) from (24), and the moment condition of the washer (25). We get, respectively,
\[
\varphi_0(\xi) = \omega + \Lambda \int_0^1 \operatorname{sign}(\xi - \eta) [\varphi(\eta) - f_0(\eta)] \, d\eta, \quad (\varphi_0(\xi) = \varphi(a\xi), \ 0 < \xi < 1), \quad (34)
\]
\[
T_0(\xi) = \frac{1}{\xi^2} \int_0^\xi \eta^2 [\varphi(\eta) - f_0(\eta)] \, d\eta \quad \left( T_0(\xi) = \frac{T(a\xi)}{aG_1} \right), \quad (35)
\]
\[
\int_0^1 \eta^2 \varphi(\eta) \, d\eta = Q_0 + S_0 \quad \left( Q_0 = \frac{M_0}{2\pi a^3G_1}, \ S_0 = \frac{T_0h}{G_1a}, \ M_0 = 2\pi \int_0^a s^2 \tau_0(s) \, ds \right). \quad (36)
\]
Thus, after solving the governing IE (33), the characteristics of the elastic washer are determined by formulas (34) and (35), and with the help of (36) the dependence between the angle of rotation of the rigid washer \( \omega \) and the torque \( Q_0 + S_0 \) is established.

The solution of IE (33) can be constructed by the method of spectral relationships for the kernel \( M_0(\xi, \eta) \) [15, 16]. These relationships contain associated Legendre polynomials or equivalent Jacobi polynomials. The solution of IE (33) is represented by an infinite series in these polynomials with unknown coefficients with a distinguishing singularity for \( \xi = 1 \). An infinite system of linear algebraic equations is obtained for unknown coefficients. The solution of IE (33) can also be constructed by the collocation method at Chebyshev nodes. For this, we represent the solution by a finite series in the above-mentioned polynomials and use the Gaussian quadrature formulas to calculate the integrals. Then the kernel \( M_0(\xi, \eta) \) can be replaced by the hypergeometric function from (32).

In the simplest particular case, when we have an elastic homogeneous half-space (\( \gamma = 0 \)) and an absolutely rigid washer (\( G_0 = \infty \) and hence \( \Lambda = 0 \)), IE (33) takes the form
\[
\int_0^1 M_0(\xi, \eta) \eta \varphi(\eta) \, d\eta = \omega \xi \quad (0 < \xi < 1), \quad (37)
\]
which describes the known Reissner–Sagotzi contact problem [7]. The corresponding spectral relationship in [15, 16] is represented by the formula
\[
\int_0^1 M_0(\xi, \eta) \eta^2 \, d\eta \sqrt{1 - \eta^2} = \frac{\pi \xi}{4} \quad (0 < \xi < 1).
\]
From which it follows that IE (37) has the solution
\[ \varphi(\xi) = \frac{4\omega}{\pi} \frac{\xi}{\sqrt{1 - \xi^2}} \quad (0 < \xi < 1), \] (38)
that coincides with the known result of [7]. Substituting (38) into the moment equilibrium condition of the washer in (36), we come to the following relation between the hard angle of rotation of the washer and the torque:
\[ 8\omega = 3\pi(Q_0 + S_0). \]

4. On the contact interaction of an elastic continuous circular cylinder with a thin elastic cylindrical shell
First, as above, let us consider the physical model of a thin elastic circular cylindrical shell. For this, for a hollow elastic cylinder \( D_0 = \{a \leq r \leq b; \ 0 \leq \theta < 2\pi; \ -\infty < z < \infty\} \) infinite in both directions with the axis \( Oz \) and the shear modulus \( G_0 \), we consider the first boundary-value problem in torsion. Let torsional tangential forces of intensity \( \tau(z) \) act on the inner surface \( r = a \) of the cylinder, and let torsional shear modulus \( G_0 \) act on its outer surface \( r = b \). With the help of (2)–(4), this problem for the circular displacements \( v_0 = v_0(r, z) \) is mathematically formulated as the boundary-value problem
\[
\begin{align*}
\frac{\partial^2 v_0}{\partial r^2} + \frac{1}{r} \frac{\partial v_0}{\partial r} + \frac{v_0}{r^2} + \frac{\partial^2 v_0}{\partial z^2} &= 0 \quad (a < r < b, \ -\infty < z < \infty), \\
\frac{\partial v_0}{\partial r} \bigg|_{r=a} &= \tau(z), \\
\frac{\partial v_0}{\partial r} \bigg|_{r=b} &= \frac{\tau(z)}{G_0} \quad (-\infty < z < \infty), \\
\tau_{r\varphi}^2 + \tau_{\varphi z}^2 &\to 0 \quad \text{when} \ r^2 + z^2 \to \infty.
\end{align*}
\] (39)

Now we apply the integral Fourier transform with respect to the variable \( z \) to the boundary-value problem (39) by setting
\[ \{\bar{v}_0 = \hat{v}_0(r, \lambda), \ \bar{\tau}(\lambda)\} = \int_{-\infty}^{\infty} \{v_0(r, z), \ \tau_0(z), \ \tau(z)\} e^{i\lambda z} dz. \]
As a result, we arrive at the one-dimensional boundary-value problem
\[
\begin{align*}
\frac{d^2 \bar{v}_0}{dr^2} + \frac{1}{r} \frac{d \bar{v}_0}{dr} - \frac{\bar{v}_0}{r^2} - \lambda^2 \bar{v}_0 &= 0 \quad (a < z < b, \ -\infty < \lambda < \infty), \\
\frac{d \bar{v}_0}{dr} \bigg|_{r=a} &= \bar{\tau}(\lambda), \\
\frac{d \bar{v}_0}{dr} \bigg|_{r=b} &= \frac{\bar{\tau}(\lambda)}{G_0}.
\end{align*}
\] (40)

The general solution of differential equation (40) is given by the formula
\[ \bar{v}_0(r, \lambda) = AI_1\left(\frac{\lambda}{r}\right) + BK_1\left(\frac{\lambda}{r}\right) \quad (a \leq r \leq b), \]
where \( I_1(x) \) and \( K_1(x) \) are modified Bessel functions of the first and third kind of order 1, respectively. Constants \( A \) and \( B \) are determined from the boundary conditions of problem (40).
Omitting the intermediate calculations, we give the final expressions for \( \bar{\nu}_0(a, \lambda) \) and \( \bar{\nu}_0(b, \lambda) \):

\[
G_0 \Delta(\lambda) \bar{\nu}_0(a, \lambda) = ab\lambda[|I_0(\lambda)|K_1(a|\lambda|) + K_0(a|\lambda|)I_1(a|\lambda|)]\bar{\tau}_0(\lambda)
- ab\lambda[|I_0(b|\lambda|)K_1(a|\lambda|) + K_0(b|\lambda|)I_1(a|\lambda|)]\bar{\tau}(\lambda)
+ 2a[I_1(b|\lambda|)K_1(a|\lambda|) - K_1(b|\lambda|)I_1(a|\lambda|)]\bar{\tau}(\lambda) \quad (-\infty < \lambda < \infty),
\]

\[
G_0 \Delta(\lambda) \bar{\nu}_0(b, \lambda) = -ab\lambda[|I_1(b|\lambda|)K_0(b|\lambda|) + I_0(b|\lambda|)K_1(b|\lambda|)]\bar{\tau}(\lambda)
+ ab\lambda[|I_1(b|\lambda|)K_0(a|\lambda|) + I_0(a|\lambda|)K_1(b|\lambda|)]\bar{\tau}_0(\lambda)
+ 2b[I_1(b|\lambda|)K_1(a|\lambda|) - I_1(a|\lambda|)K_1(b|\lambda|)]\bar{\tau}_0(\lambda),
\]

\[
\Delta(\lambda) = ab\lambda^2[|I_0(\lambda)|K_0(a|\lambda|) - I_0(|\lambda|)K_0(b|\lambda|)] - 2a\lambda[|I_1(b|\lambda|)K_0(a|\lambda|) + K_1(b|\lambda|)I_0(a|\lambda|)]
+ 2b\lambda[|I_0(b|\lambda|)K_1(a|\lambda|) + I_1(a|\lambda|)K_0(b|\lambda|)] + 2f_1(a|\lambda|)K_1(b|\lambda|) - 2f_1(b|\lambda|)K_1(a|\lambda|).
\]

Here \( I_0(x) \) and \( K_0(x) \) are modified Bessel functions of order 0.

Next, we write the well-known representations of the modified Bessel functions [17]:

\[
I_0(x) = \sum_{m=0}^{\infty} \frac{x^{2m}}{2^{2m}(m!)^2}, \quad I_1(x) = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{2^{2m+1}m!(m+1)!},
\]

\[
K_0(x) = -I_0(x) \ln \frac{x}{2} + \sum_{m=0}^{\infty} \frac{x^{2m} \psi(m+1)}{2^{2m+1}m!(m+1)!}, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},
\]

\[
K_1(x) = I_1(x) \ln \frac{x}{2} + \frac{1}{2} \sum_{m=0}^{\infty} \frac{x^{2m+1} \psi(m+2) + \psi(m+1)}{2^{2m+1}m!(m+1)!}, \quad (x > 0),
\]

where \( \Gamma(x) \) is the well-known Euler gamma function. Now the functions appearing in (41)–(43) can be expanded in power series in \( \lambda \), using representations (44) and restricting ourselves to terms of order \( \lambda^2 \). In addition, we set \( b = a + h \) and, in the corresponding expansions in \( h \), confine ourselves to terms of order \( h \). After necessary transformations and calculations, we have

\[
- G_0 h (2 + a^2 \lambda^2) \bar{\nu}_0(a, \lambda) = -a(a + h) \left\{ 1 + \frac{a^2}{4} [1 + \psi(1) - \psi(2)] \lambda^2 \right\} \bar{\tau}_0(\lambda)
+ a \left\{ a - h + \frac{a^2(a + h)}{4} [1 + \psi(1) - \psi(2)] \lambda^2 \right\} \bar{\tau}(\lambda),
\]

\[
- G_0 h (2 + a^2 \lambda^2) \bar{\nu}_0(b, \lambda) = a \left\{ a + \frac{a^2}{4} (a + 2h) [1 + \psi(1) - \psi(2)] \lambda^2 \right\} \bar{\tau}(\lambda)
- a \left\{ a + 2h + \frac{a^2}{4} [a + (a + 2h) \psi(1) - \psi(2)] + 2h \right\} \bar{\tau}_0(\lambda).
\]

Returning to the originals in (45) and (46), we obtain

\[
G_0 h \left( a^2 \frac{d^2 \bar{\nu}_0}{dz^2} - 2\bar{\nu}_0 \right) = -a(a + h) \bar{\tau}_0(z) + \frac{1}{4} a^3(a + h) [1 + \psi(1) - \psi(2)] \bar{\tau}''(z)
+ a(a - h) \bar{\tau}(z) - \frac{a^3}{4} (a + h) [1 + \psi(1) - \psi(2)] \bar{\tau}''(z) \quad (-\infty < z < \infty),
\]

\[
G_0 h \left( a^2 \frac{d^2 \bar{\nu}_0}{dz^2} - 2\bar{\nu}_0 \right) = a^2 \bar{\tau}(z) - a(a + h) \bar{\tau}_0(z) - \frac{a^3}{4} (a + 2h) [1 + \psi(1) - \psi(2)] \bar{\tau}''(z)
+ \frac{a^3}{4} [a + (a + 2h) \psi(1) - \psi(2)] + 2h \bar{\tau}''(z) \quad (-\infty < z < \infty).
\]
Thus, in the framework of the accepted accuracy, the mechanical behavior of the inner thin cylindrical layer \((r = a)\) is described by equation (47), and the outer thin cylindrical layer \((r = b)\), by equation (48).

If in (47), (48) we neglect the terms conditioned by the second-order derivatives of the surface loads, then these relations simplify and take the forms

\[
G_0 h \left( a^2 \frac{d^2 v_0}{dz^2} - 2 v_0 \right) = -a(a + h) \tau_0(z) + a(a - h) \tau(z) \quad (-\infty < z < \infty),
\]

\[
G_0 h \left( a^2 \frac{d^2 v_0}{dz^2} - 2 v_0 \right) = a^2 \tau(z) - a(a + h) \tau_0(z) \quad (-\infty < z < \infty),
\]

respectively. Now we construct the solution of the problem of torsion of a solid infinite circular cylinder \(D_1 = \{0 \leq r < a, 0 \leq \vartheta < 2\pi, -\infty < z < \infty\}\) with the shear modulus \(G_1\), when torsional tangential forces of intensity \(\tau(z)\) act on its surface \(r = a\). This boundary-value problem is formulated mathematically as follows:

\[
\begin{align*}
\frac{\partial^2 v_1}{\partial r^2} + & \frac{1}{r} \frac{\partial v_1}{\partial r} - \frac{v_1}{r^2} + \frac{\partial^2 v_1}{\partial z^2} = 0 \quad (0 < r < a, -\infty < z < \infty), \\
\left. \frac{\partial v_1}{\partial r} - \frac{v_1}{r} \right|_{r=a} & = \frac{\tau(z)}{G_1} \quad (-\infty < z < \infty), \\
\tau_{r\vartheta}^2 + \tau_{z\vartheta}^2 & \to 0 \quad \text{when } |z| \to \infty,
\end{align*}
\]

where \(v_1 = v_1(r, z)\) are circular displacements of cylinder points. The solution of the boundary-value problem (51) can also be easily obtained by means of an integral Fourier transform with respect to the variable \(z\). Proceeding as above, after separating the principal singular part of the kernel and its slowly convergent part, we have

\[
\frac{dv_1(a, z)}{dz} = \frac{1}{\pi G_1} \int_{-\infty}^{\infty} \left[ \frac{1}{u - z} + \frac{3\pi}{4a} \text{sign}(u - z) + N(u - z) \right] \tau(u) \, du \quad (-\infty < z < \infty),
\]

\[
N(z) = \frac{1}{2a} \int_{0}^{\infty} \frac{a^2 \lambda^2 R(a\lambda) - 3}{\sin(\lambda z)} \, d\lambda, \quad R(a\lambda) = \frac{a\lambda[I_1(a\lambda) - I_0(a\lambda)] + 2I_1(a\lambda)}{a\lambda[a\lambda I_0(a\lambda) - 2I_1(a\lambda)]} \quad (\lambda > 0),
\]

\[
R(a\lambda) \sim 4/a^2 \lambda^2 \quad (\lambda \to +0); \quad R(a\lambda) \sim 3/2a^2 \lambda^2 \quad (\lambda \to \infty).
\]

Now let us formulate the contact problem. Let the thin elastic cylindrical shell \(D_0^{(c)} = \{a \leq r \leq a + h, 0 \leq \vartheta < 2\pi, -c \leq z \leq c\} \quad (h \ll a)\) of finite length \(2c\) and the shear modulus \(G_0\), whose mechanical behavior is described by equation (49) (or (50)), be adhered to the cylinder along the segment \(-c \leq z \leq c\) of its surface. We further assume that torsional tangential forces of intensity \(\tau_0(z)\) act on the shell lateral surface \(r = a + h\) and, in addition, the torques \(M_{z\varphi}\) act in the end face sections \(z = \pm c\) of the shell \(D_0^{(c)}\). It is required to determine the distribution of contact tangential stresses \(\tau(z)\) in the contact area \(-c \leq z \leq c\) and the torques \(M_z\) in any section \(z\) of shell.

We reduce solving the posed contact problem to solving the singular integral equation (SIE). For this, we write differential equation (49) in the form

\[
\frac{d^2 v_0}{dz^2} - k^2 v_0 = f(z) \quad (-c < z < c),
\]

\[
k = \frac{\sqrt{2}}{a}, \quad f(z) = \frac{1}{a^2 h G_0} [a(a - h) \tau(z) - a(a + h) \tau_0(z)].
\]
It has the following general solution

\[ v_0(z) = A \cosh(kz) + B \sinh(kz) - \frac{1}{2k} \int_{-c}^{c} e^{-k(z-u)} f(u) \, du \quad (-c \leq z \leq c). \]  (53)

To determine the constants \( A \) and \( B \), we should find the torque \( M_z \) in the shell section \( z \). We can write

\[ M_z = \int_0^{2\pi} \int_a^{a+h} \tau_{\partial z} r^2 \, d\theta \, dz = 2\pi G_0 \int_a^{a+h} e_{\partial z} r^2 \, dr = 2\pi G_0 \frac{dv_0}{dz} \int_a^{a+h} r^2 \, dr = \frac{2\pi G_0}{3} \frac{dv_0}{dz} [(a+h)^3 - a^3]. \]

After necessary simplifications, we obtain

\[ M_z = 2\pi a^2 h G_0 \frac{dv_0}{dz}. \]  (54)

On the other hand, considering the moment equilibrium condition of the shell part \([-c, z]\), after some simplifications we obtain

\[ M_z = M_{-c} - 2\pi a(a + 2h) \int_{-c}^{c} \tau_0(u) \, du + 2\pi a^2 \int_{-c}^{c} \tau(u) \, du \quad (-c \leq z \leq c), \]  (55)

which implies the moment equilibrium condition of the shell for \( z = c \):

\[ \int_{-c}^{c} \tau(u) \, du = \left(1 + \frac{2h}{a}\right) \int_{-c}^{c} \tau_0(u) \, du + \frac{M_c - M_{-c}}{2\pi a^2}. \]  (56)

Now from (54) we get the boundary condition

\[ \frac{dv_0}{dz} \bigg|_{z=\pm c} = \frac{M_{\pm c}}{2\pi a^2 h G_0}. \]  (57)

Proceeding from (53) and satisfying boundary conditions (57), we determine the constants \( A \) and \( B \). As a result, after simple manipulations, we have

\[ \frac{dv_0}{dz} = \frac{M_c - M_{-c}}{4\pi a^2 h G_0} \cosh(kc) + \frac{M_c + M_{-c}}{4\pi a^2 h G_0} \cosh(kc) + \frac{a - h}{4ah G_0} \int_{-c}^{c} \left\{ \text{sign}(z-u) e^{-k|z-u|} \right\} \cosh(ku) \, du - \frac{1 + h_0}{1 - h_0} \int_{-c}^{c} \left\{ \text{sign}(z-u) e^{-k|z-u|} \right\} \sinh(ku) \, du \]

\[ - e^{-kc} \left\{ \frac{\sinh(kz)}{\sinh(kc)} e_{\partial z} + \frac{\cosh(kz)}{\cosh(kc)} \sinh(ku) \right\} \tau_0(u) \, du + \frac{1 + h_0}{1 - h_0} \int_{-c}^{c} \left\{ \text{sign}(z-u) e^{-k|z-u|} \right\} \cosh(ku) \, du \]

\[ - e^{-kc} \left\{ \frac{\sinh(kz)}{\sinh(kc)} \cosh(ku) + \frac{\cosh(kz)}{\cosh(kc)} \sinh(ku) \right\} \tau_0(u) \, du \bigg|_{z=\pm c} \quad \left( h_0 = \frac{h}{a}, \quad \frac{1 + h_0}{1 - h_0} \approx 1 + 2h_0 \right). \]  (58)

Then we substitute (52) and (58) into the condition of contact between the shell and the solid elastic cylinder:

\[ \frac{dv_0}{dz} = \frac{dv_1}{dz} \quad (-c < z < c). \]

Hence, introducing the dimensionless quantities

\[ \xi = \frac{z}{c}, \quad \eta = \frac{u}{c}, \quad \psi_0(\eta) = \frac{\tau_0(\eta)}{G_1}, \quad k_0 = \sqrt{\frac{\delta}{a}} = kc, \quad M_{-c}^{(0)} = \frac{M_{\pm c}}{a^2 h G_0}, \quad h_0 = \frac{h}{a}, \]

\[ X_0(\eta) = \frac{\tau_0(\eta)}{G_1}, \quad L_0(\xi) = cN(c\xi), \quad \lambda_0 = \frac{c(a - h) G_1}{2ah G_0} = \frac{(1 - h_0) G_1}{2\delta_0 G_0}, \quad \delta_0 = \frac{h}{c}, \]

we can write:

\[ \frac{dv_0}{dz} = \frac{dv_1}{dz} \quad (-c < z < c). \]
we arrive at the following governing SIE of the contact problem under consideration:

$$\frac{1}{\pi} \int_{-1}^{1} \left[ \frac{1}{\eta - \xi} + \frac{3\pi}{4} \text{sign}(\eta - \xi) + L_0(\eta - \xi) - \pi \lambda_0 H_0(\xi, \eta) \right] \psi_0(\eta) d\eta = f_0(\xi) \quad (-1 < \xi < 1),$$

$$H_0(\xi, \eta) = \text{sign}(\xi - \eta) e^{-k_0|\xi - \eta|} - e^{-k_0} \left[ \frac{\sinh(k_0\xi)}{\sinh k_0} \cosh(k_0\eta) + \frac{\cosh(k_0\xi)}{\cosh k_0} \sinh(k_0\eta) \right],$$

$$f_0(\xi) = -(1 + 2h_0)\lambda_0 \int_{-1}^{1} H_0(\xi, \eta) X_0(\eta) d\eta + \frac{(M_c^{(0)} - M_c^{(-0)}) \sinh(k_0\xi)}{4\pi \sinh k_0} + \frac{(M_c^{(0)} + M_c^{(-0)}) \cosh(k_0\xi)}{4\pi \cosh k_0},$$

where the function $L_0(\xi)$ is defined by the function $N(\eta)$ from (52).

In dimensionless quantities, condition (56) becomes

$$\int_{-1}^{1} \psi(\eta) d\eta = (1 + 2h_0) \int_{-1}^{1} X_0(\eta) d\eta + \frac{M_c^{(0)} - M_c^{(-0)}}{2\pi} \chi \delta_0 \quad \left( \chi = \frac{G_0}{G_1}, \delta_0 = \frac{h}{c} \right),$$

and the expression for the torque from (55) become

$$M_c^{(0)}(\xi) = M_c^{(-0)} - \frac{2\pi(1 + h_0)}{\chi \delta_0} \int_{-1}^{\xi} X_0(\eta) d\eta + \frac{2\pi}{\chi \delta_0} \int_{-1}^{\xi} \psi(\eta) d\eta \quad (-1 \leq \xi \leq 1).$$

The solution of SIE must satisfy condition (60), after which the dimensionless torques $M_c^{(0)}(\xi) = M_z/a^2 hG_0$ in the shell sections are determined by formula (61). A sufficiently effective solution of SIE (59), (60) can be obtained by the known numerical-analytical method for solving the SIE [11].

Note that in a particular case where $\tau_0(z) \equiv 0$, $M_{-c} = M_c$, and hence $\tau(-z) = -\tau(z)$, besides $\tau(0) = 0$, it follows from (55) that

$$\frac{dM_c}{dz} \bigg|_{z=0} = 0.$$  

This means that the greatest torque appears in the middle section of the shell $z = 0$.

The results of numerical analysis of the problems considered here is beyond the scope of this contribution and will be presented as a separate paper.

**Conclusions**

The paper deals with nonclassical contact problems related to the interaction of thin-walled elements with massive deformable bodies under torsion. The governing equations of physical models of thin-walled elements are derived from the exact equations of mathematical theory of elasticity. On the one hand, they are quite simple to use in specific problems, and, on the other hand, they provide rather high practical accuracy. The results and approaches discussed in the paper can be used to study a rather wide class of problems of contact interaction of thin-walled elements and massive elastic bodies under torsion, which are of both theoretical and practical interest.

**References**

[1] Love A E H 1996 *Treatise on the Mathematical Theory of Elasticity* (New York: Dover Publications) p 643

[2] Aratjunyan N K and Abramyan B L 1963 *Torsion of Elastic Bodies* (Moscow: Fizmatgiz) [in Russian]

[3] Solyanik-Krassa K V 1949 *Torsion of Shafts of Variable Cross-Section* (Moscow: Gostekhizdat) p 167 [in Russian]

[4] Lekhnitski S G 1971 *Torsion of Anisotropic and Non-Homogeneous Beams* (Moscow: Nauka) [in Russian]

[5] Muskhelishvili N I 1966 *Some Basic Problems of the Mathematical Theory of Elasticity* (Moscow: Nauka) [in Russian]
[6] Novojilov V V 1958 Theory of Elasticity (Moscow: Sudpromgiz) [in Russian]
[7] Reissner E and Sagoci H F 1944 Forced Torsional Oscillation of an Elastic Half-Space J. Appl. Phys. 15
652–4
[8] Galin L A (Editor) 1976 Development of the Theory of Contact Problems in the USSR (Moscow: Nauka)
p 493 [in Russian]
[9] Melan E 1932 Ein Beitrag zur Theorie geschweißter Verbindungen Arch. Appl. Mech. 3 (2) 123–9
[10] Bufler H 1964 Zur Krafteinleitung in Scheiben über geschweißte oder geklebte Verbindungen Österr. Ing.
Arch. 18 (3-4) 284–92
[11] Erdogan F, Gupta G D, and Cook T S 1973 The Numerical Solutions of Singular Integral Equations In
Methods of Analysis and Solution of Crack Problems (Leyden: Noordhoff Intern Publ) pp 368–425
[12] Sneddon I 1951 Fourier Transforms (New York-Toronto: McGraw-Hill)
[13] Bateman H and Erdelyi A 1954 Tables of Integral Transforms vol 2 (New York: McGraw-Hill)
[14] Bateman H and Erdelyi A 1953 Higher Transcendental Functions vol 1 (New York: McGraw-Hill)
[15] Popov G Ia 1964 Certain properties of classical polynomials and their applications in contact problems J.
Appl. Math. Mech. 28 (3) 544–55
[16] Mkhitarian S M 1984 Spectral relationships for the integral operators generated by a kernel in the form of a
Weber–Sonin integral, and their applications to contact problems J. Appl. Math. Mech. 48 (1) 67–74
[17] Bateman H and Erdelyi A 1953 Higher Transcendental Functions vol 2 (New York: McGraw-Hill)