The CPT group of the spin-3/2 field

B. Carballo Pérez* and M. Socolovsky†
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,
Circuito exterior, Ciudad Universitaria, 04510, México D.F., México

Abstract
We find out that both the matrix and the operator CPT groups for the spin-3/2
field (with or without mass) are respectively isomorphic to $D_4 \times \mathbb{Z}_2$ and $Q \times \mathbb{Z}_2$.
These groups are exactly the same groups as for the Dirac field, though there is no
a priori reason why they should coincide.

Keywords: discrete symmetries; spin 3/2-field; finite groups

*brendacp@nucleares.unam.mx
†socolovs@nucleares.unam.mx
1 Introduction

The CPT group of the Dirac field in Minkowski space-time was obtained by Socolovsky in 2004 [1]. It were found two sets of consistent solutions for the matrices of charge conjugation ($C$), parity ($P$), and time reversal ($T$), which give the transformation of fields $\hat{\psi}_C(x) = C\hat{\psi}^T(x)$, $\hat{\psi}_\Pi(x_\Pi) = P\hat{\psi}(x)$ and $\hat{\psi}_\tau(x_\tau) = T\hat{\psi}(x)^*$, where $x_\Pi = (t, -x)$ and $x_\tau = (-t, x)$. These sets are given by:

a) $C_D = \pm \gamma^2\gamma^0, P_D = \pm i\gamma^0, T_D = \pm i\gamma^3\gamma^1,$

b) $C_D = \pm i\gamma^2\gamma^0, P_D = \pm i\gamma^0, T_D = \pm \gamma^3\gamma^1.$

Each of these sets generates a non abelian group of sixteen elements, respectively, $G^{(1)}_\theta \cong D_4 \times \mathbb{Z}_2$ and $G^{(2)}_\theta \cong 16E$, where $D_4$ is the group of symmetries of the square and $16E$ is a non trivial extension of $D_4$ by $\mathbb{Z}_2$, isomorphic to a semidirect product of these groups.

On the other hand, the quantum operators $\hat{C}, \hat{P}$ and $\hat{T}$, acting on the Hilbert space, generate a unique group $G_\theta \cong Q \times \mathbb{Z}_2$, where $Q$ is the quaternion group.

With this in mind, we decided to find the CPT group of the spin-$3/2$ field (Rarita-Schwinger field), for both massive and massless cases. This field could be useful for the description of compound objects (neglecting its structure in a first approximation), like the baryon decuplet components for spin-$3/2^+$ [2], or for elementary fields such as the gravitino.

In order to describe $3/2$-spin particles, the set of equations

\begin{align}
(i\gamma^\alpha \partial_\alpha - m)\hat{\psi}^\mu(x) &= 0, \\
\gamma^\mu\hat{\psi}_\mu(x) &= 0,
\end{align}

where $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$, is required. These equations are known as the Rarita-Schwinger equation [3]. The first of these equation is a Dirac equation for each vector component of the vector spinor $\hat{\psi}^\mu$ and the second one is known as the subsidiary condition. Precisely due to the more complexity of these equations with respect to the Dirac field equation, there is not a priori any apparent reason why the CPT group for the Rarita-Schwinger field should coincide with those for the Dirac field.

2 Parity

If we want to study the $P$ invariance of the Rarita-Schwinger equation, we need to repeat the analysis done for the Dirac equation in [1] and also consider the $P$ invariance of the subsidiary equation.
Multiplying this equation from the left by $P$, changing $x \to -x$ and inserting the unity, one obtains:

$$P\gamma^0 P^{-1} P\hat{\psi}_0(t, -x) + P\gamma^1 P^{-1} P\hat{\psi}_1(t, -x) + P\gamma^2 P^{-1} P\hat{\psi}_2(t, -x) + P\gamma^3 P^{-1} P\hat{\psi}_3(t, -x) = 0,$$

(3)

but we need to take into account that the vector spinor changes by parity in the following way:

$$\hat{\psi}_\mu(t, x) = \begin{pmatrix} \hat{\psi}_0(t, x) \\ \hat{\psi}_1(t, x) \\ \hat{\psi}_2(t, x) \\ \hat{\psi}_3(t, x) \end{pmatrix} \quad \to \quad \psi_{\mu\pi}(t, x) = P\hat{\psi}_\mu(t, -x) ,$$

(4)

where $\psi_{\mu\pi}(t, x)$ can be written as $\psi_{\mu\pi}(t, x) = \mathcal{P} P\hat{\psi}_\mu(t, -x)$, with $\mathcal{P} \in O(1, 3)$ given by:

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(5)

and $P \in D^{16}$, where $D^{16}$ is the Dirac algebra.

Substituting the components of $\psi_{\mu\pi}(t, x)$ in (5) one obtains:

$$P\gamma^0 P^{-1} \psi_{0\pi}(t, x) - P\gamma^k P^{-1} \psi_{k\pi}(t, x) = 0.$$

(6)

This implies the constraints on $P$:

$$P\gamma^0 P^{-1} = \gamma^0, \quad P\gamma^k P^{-1} = -\gamma^k,$$

(7)

or

$$P\gamma^0 P^{-1} = -\gamma^0, \quad P\gamma^k P^{-1} = \gamma^k.$$

(8)

The relations (7) are the same as those obtained from the Dirac equation, whose already known solution is $P_D = \pm i\gamma^0$, while from the relations (8) is obtained $P = P' = z\gamma^3\gamma^2\gamma^1$, with $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Following the same analysis as in [1], there are two possibilities for each $P$:
3 Charge conjugation

To study the $C$ invariance of the subsidiary equation, we must take the complex conjugate of this equation, multiply from the left by $C\gamma_0$ and insert the unit matrix. This is:

$$(C\gamma^0)\gamma^{\mu*}(C\gamma^0)^{-1}\hat{\psi}_{\mu C}(x) = 0,$$  \hspace{1cm} (9)

where $\hat{\psi}_{\mu C}(x) = C\gamma^0 \hat{\psi}_{\mu}^*(x)$.

In this case, the constraints on $C$ are:

$$(C\gamma^0)\gamma^{\mu*}(C\gamma^0)^{-1} = \gamma^\mu,$$  \hspace{1cm} (10)

or

$$(C\gamma^0)\gamma^{\mu*}(C\gamma^0)^{-1} = -\gamma^\mu.$$  \hspace{1cm} (11)

Taking into account that $\gamma^0\gamma^{\mu*}\gamma^0 = \gamma^\mu T$, one can find the solutions for the $C$ matrices. The equation with the negative sign is the same as in the Dirac case and leads to $C = C_D = \eta\gamma^2\gamma^0$; on the other hand, from equation (10) we arrive to:

$$C\gamma^{\mu T}C^{-1} = \gamma^\mu,$$  \hspace{1cm} (12)

which leads to the solution $C = C' = \eta\gamma^3\gamma^1$.

For $C'$, a second application of the charge conjugation transformation is given by:

$$(\hat{\psi}_{C'})_C = C'\hat{\psi}_{C^2} = C'\gamma^0\hat{\psi}_C = C'\gamma^0\gamma^1\gamma^3\gamma^1\gamma^0\hat{\psi} = -C'C'^*\hat{\psi} = -|\eta|\gamma^0\gamma^1\gamma^3\gamma^1\gamma^0\hat{\psi} = |\eta|^2\gamma^1\gamma^3\gamma^1\gamma^0\hat{\psi} = |\eta|^2\hat{\psi}.$$  \hspace{1cm} (13)

Since the effect on $\hat{\psi}$ can be, at most, a multiplication by a phase, then $\eta \in U(1)$ and $C'$ is unitary. This is:
\[ C'C^\dagger = \eta^3 \gamma^3 \bar{\eta}(\gamma^3 \gamma^1)^\dagger = |\eta|^2 \gamma^3 \gamma^1 \gamma^3 = |\eta|^2 \gamma^3 (\gamma^1)^2 \gamma^3 = |\eta|^2 1 = 1. \] (14)

Hence, for \( C = C' \) we also find that \( \hat{\psi}_{C^2} = \hat{\psi} \).

As in [1], due to a symmetry consideration between \( \hat{\psi}_{C} = C \hat{\psi}^T \) and \( \hat{\psi}_{C} = -\bar{\eta}^2 \hat{\psi}^T C' \), it follows that \(-\bar{\eta}^2 = \pm 1 \) and taking into account that \( |\eta|^4 = 1 \), one obtains \( \eta^2 = \pm 1 \), which implies that \( \eta = \pm 1, \pm i \).

Then, for the spin-3/2 field there are two possibilities for each \( C \):

a) \( C' = \pm \gamma^3 \gamma^1 \), \( C_D = \pm \gamma^2 \gamma^0 \);

b) \( C' = \pm i \gamma^3 \gamma^1 \), \( C_D = \pm i \gamma^2 \gamma^0 \).

4 Time reversal

We start again from the subsidiary equation, change \( t \to -t \) and take the complex conjugate:

\[
T\gamma^0 \gamma^3 T^{-1} \hat{\psi}_0(-t, x)^* + T\gamma^1 \gamma^3 T^{-1} \hat{\psi}_1(-t, x)^* + T\gamma^2 \gamma^3 T^{-1} \hat{\psi}_2(-t, x)^* + T\gamma^3 \gamma^3 T^{-1} \hat{\psi}_3(-t, x)^* = 0,
\] (15)

but we need to take into account that the vector spinor changes by time inversion in the following way:

\[
\hat{\psi}_\mu(t, x) = \begin{pmatrix} \hat{\psi}_0(t, x) \\ \hat{\psi}_1(t, x) \\ \hat{\psi}_2(t, x) \\ \hat{\psi}_3(t, x) \end{pmatrix} \rightarrow \hat{\psi}_{\mu'}(t, x) = \begin{pmatrix} -T\hat{\psi}_0(-t, x)^* \\ T\hat{\psi}_1(-t, x)^* \\ T\hat{\psi}_2(-t, x)^* \\ T\hat{\psi}_3(-t, x)^* \end{pmatrix},
\] (16)

where \( \hat{\psi}_{\mu'}(t, x) \) can be written as \( \hat{\psi}_{\mu'}(t, x) = \mathcal{T} T \hat{\psi}_\mu(-t, x)^* \), with \( \mathcal{T} \in O(1, 3) \) given by:

\[
\mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\] (17)

and \( T \in D^{16} \).
Substituting the components of $\hat{\psi}_{\mu\tau}(t, x)$ in (15) one obtains:

$$- T\gamma^0 T^{-1}\hat{\psi}_{0\tau}(t, x) + T\gamma^k T^{-1}\hat{\psi}_{k\tau}(t, x) = 0,$$

from which we can deduce the constraints on $T$:

$$T\gamma^0 T^{-1} = \gamma^0, \quad T\gamma^k T^{-1} = -\gamma^k,$$

or

$$T\gamma^0 T^{-1} = -\gamma^0, \quad T\gamma^k T^{-1} = \gamma^k.$$

The relations (19) are the same as those obtained from the Dirac equation, whose already known solution is $T_D = e^{i\lambda \gamma^3 \gamma^1}$, while from the relations (20) is obtained $T = T' = w\gamma^2 \gamma^0$, with $w \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

For $T = T'$, applying $\tau$ twice, we arrive to:

$$\hat{\psi}(t, x) \rightarrow \hat{\psi}_\tau(t, x) = T' \hat{\psi}(-t, x)^* ightarrow T'(T' \hat{\psi}(t, x)^*)^* = T'T'^* \hat{\psi}(t, x).$$

and taking into account that

$$T'T'^* = |w|^2 \gamma^2 \gamma^0 (\gamma^2 \gamma^0)^* = -|w|^2 \gamma^2 \gamma^0 \gamma^2 \gamma^0 = |w|^2 \gamma^2 \gamma^2 \gamma^0 \gamma^0 = -|w|^2 1,$$

it follows that $\hat{\psi}_\tau^2 = -\hat{\psi}$, by a similar argument to that used for $C$.

Thus, $T'T'^* = -1$, which implies $T'^* = -T'^{-1}$ and $w \in U(1)$. Then, $T' = e^{i\lambda \gamma^2 \gamma^0}$ y $T'^{\dag} = e^{-i\lambda \gamma^2 \gamma^0}$.

5 Matrix and operator CPT groups

In summary, one has both the two sets of matrices:

a) $C_D = \pm \gamma^2 \gamma^0, \quad P_D = \pm i\gamma^0, \quad T_D = \pm i\gamma^3 \gamma^1$,

b) $C_D = \pm i\gamma^2 \gamma^0, \quad P_D = \pm i\gamma^0, \quad T_D = \pm \gamma^3 \gamma^1$;

and the set: $C', P', T'$, satisfying the subsidiary equation. But, only the matrices $C_D, P_D, T_D$, also satisfy the Dirac type equation. That is why they are the matrices which conform the matrix CPT group of the spin-3/2 field with mass.

If we take the zero mass limit of the Dirac type equation,

$$i\gamma^\alpha \partial_\alpha \hat{\psi}(x) = 0,$$

(23)
and analyze its behavior under parity, charge conjugation and time reversal, as we did for the subsidiary condition, we have, respectively:

\[ i(P\gamma^0 P^{-1} \partial_0 - P\gamma^i P^{-1} \partial_i) \hat{\psi}_{\Pi}(x) = 0, \]

\[ (i\partial_\mu + qa_\mu)\gamma^0 \gamma^\mu \gamma^0 (C\gamma^0)^{-1} \hat{\psi}_C(x) = 0. \]

\[ i(\gamma^0 \partial_t - \gamma^k \partial_{x^k}) \hat{\psi}_\tau(x) = 0. \]

From the above equations we can then obtain the corresponding restrictions on the $C$, $P$ and $T$ matrices, respectively:

\[ P\gamma^0 P^{-1} = \pm \gamma^0, \quad P\gamma^k P^{-1} = \mp \gamma^k, \]

\[ C\gamma^\mu T C^{-1} = \pm \gamma^\mu, \]

\[ T\gamma^0 T^{-1} = \pm \gamma^0, \quad T\gamma^k T^{-1} = \mp \gamma^k. \]

These relations generate the same sets of matrices $C_D, P_D, T_D$ and $C', P', T'$ which gave the subsidiary condition. But if we take into account that $\hat{\bar{\psi}} \hat{\psi}$ is the charge density operator, for $C = C'$, it follows that:

\[ (\hat{\bar{\psi}} \hat{\psi})_C = \hat{\bar{\psi}}_C \hat{\psi}_C = -\bar{\eta}^2 \hat{\psi}^T C' \hat{\psi}^T = -\bar{\eta}^2 C'^2 (\hat{\bar{\psi}} \hat{\psi})^T \]

\[ = -\bar{\eta}^2 C'^2 \hat{\bar{\psi}} \hat{\psi} = (\bar{\eta} \eta)^2 \hat{\bar{\psi}} \hat{\psi} = |\eta|^4 \hat{\bar{\psi}} \hat{\psi} = -\hat{\bar{\psi}} \hat{\psi}; \]

from which $|\eta|^4 = -1$, which is a contradiction. Hence, the matrix $C = C'$ must be discarded.

It was demonstrated in [1] that $C$ and $P$ must fulfill the relation:

\[ C(P^{-1})^T C^{-1} = P, \]

while $C$ and $T$ are related by:

\[ CT^* = TC^*. \]

Due to the fact that each component of the vector spinor satisfies a Dirac type equation, the above relations also hold in our case. That is why the set of matrices: $C', P', T'$ are also discarded for the massless case.
In order to find the operator CPT group for the spin-3/2 field (with or without mass), we follow the same procedure developed in [1], for the case of the corresponding CPT group of the Dirac field.

Taking $\hat{A}$ and $\hat{B}$ as any of the operators $\hat{C}_D$, $\hat{P}_D$ and $\hat{T}_D$; and $\hat{\psi}$, as each component of the vector spinor $\hat{\psi}^\mu(x)$, the relations:

$$\hat{A} \cdot \hat{\psi} = \hat{A}^\dagger \hat{\psi} \hat{A}$$  \hspace{1cm} (31)

and

$$(\hat{A} \ast \hat{B}) \cdot \hat{\psi} = (\hat{A} \hat{B})^\dagger \hat{\psi} (\hat{A} \hat{B})$$  \hspace{1cm} (32)

can be defined.

Using the above expressions and with support in the matrix CPT group, through the formulas that link matrices with operators:

$$P \hat{\psi}(t, -x) = \hat{P}^\dagger \hat{\psi}(t, x) \hat{P},$$

$$C \hat{\psi}^T(x) = \hat{C}^\dagger \hat{\psi}(x) \hat{C},$$

$$T \hat{\psi}(-t, x)^* = \hat{T}^\dagger \hat{\psi}(t, x)^\dagger \hat{T};$$  \hspace{1cm} (33)

the relations:

$$\hat{P}_D \ast \hat{P}_D = -1, \quad \hat{C}_D \ast \hat{C}_D = 1, \quad \hat{T}_D \ast \hat{T}_D = -1,$$

$$\hat{T}_D \ast \hat{P}_D = -\hat{P}_D \ast \hat{T}_D, \quad \hat{C}_D \ast \hat{P}_D = \hat{P}_D \ast \hat{C}_D, \quad \hat{C}_D \ast \hat{T}_D = \hat{T}_D \ast \hat{C}_D,$$  \hspace{1cm} (34)

were obtained in [1]; from which it is possible to build, also using the property of associativity, the multiplication table for the operator CPT group.

It was also demonstrated in [1] that only the second of the two solutions for the matrix group ($G_\theta^{(2)} \cong 16E \cong D_4 \rtimes \mathbb{Z}_2$), is compatible with the operator group ($G_\hat{\theta} \cong Q \times \mathbb{Z}_2$).

In summary, we showed that both the matrix and the operator CPT groups for the spin-3/2 field (with or without mass) coincide with the corresponding groups for the Dirac field.

6 Acknowledgment

This work was partially support by the project PAPIIT IN 118609-2, DGAPA-UNAM, México. B. Carballo Pérez also acknowledge financial support from CONACyT, México. The authors thank Prof. O. S. Zandrón from IFIR, Rosario, Argentina, for suggesting this work.
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