Local well-posedness for a class of singular Vlasov equations

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Abstract

In this article we study a singular Vlasov system on the torus where the force field has the smoothness of a (fractional) derivative $D^\alpha$ of the density, where $\alpha > 0$. We prove local well-posedness in Sobolev spaces without restriction on the data. This is in sharp contrast with the case $\alpha = 0$ which is ill-posed in Sobolev spaces for general data.

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1 Introduction and main result

1.1 Context and notations

In this article we focus on a class of nonlinear singular Vlasov systems in the torus $\mathbb{T}^d = \mathbb{R}^d / 2\pi \mathbb{Z}^d$, a prototypical example is

\begin{equation}
\begin{aligned}
&\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\
&\gamma E = -\nabla_x V, \\
&(-\Delta_x)^{\alpha/2} V = (\rho - \rho_0), \\
&f(0, x, v) = f^0(x, v),
\end{aligned}
\end{equation}

where the parameter $\gamma$ will be taken to 1 or $-1$, and the equation will be respectively called repulsive or attractive. The function $f$ stands for a distribution function in the domain $\mathbb{T}^d \times \mathbb{R}^d$ and may represent the distribution of electrons in a plasma, or the density of stars in stellar dynamics. The density associated with $f$ will be denoted $\rho = \int_{\mathbb{R}^d} f(t, x, v)dv$. The term $\rho_0 = \int_{\mathbb{R}^d \times \mathbb{T}^d} f^0(t, x, v)dvdx$ corresponds, in the repulsive setting, to the density of ions. In the attractive setting, we remove the mean value of $\rho$ as a trick called the Jeans swindle, which is mathematically relevant \cite{S}. The initial condition, $f^0$, can be taken non-negative, and $\alpha > 0$ is a parameter.

The aim of this work is to give a local well-posedness theory in Sobolev regularity for the general case $\alpha > 0$. To introduce our main result, we shall first introduce some notations.
For $k \in \mathbb{N}, r \in \mathbb{N}$, we introduce the weighted Sobolev norms

$$
\|f\|_{H^s_r} := \left( \sum_{|\alpha| + |\beta| \leq k} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (1 + |\nu|^2)^s |\partial_\nu^\alpha \partial_x^\beta f|^2 dv dx \right)^{1/2},
$$

where, for $\alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$, we write

$$
|\alpha| = \sum_{i=1}^d \alpha_i, \quad |\beta| = \sum_{i=1}^d \beta_i,
$$

$$
\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}, \quad \partial_\nu^\beta := \partial_{\nu_1}^{\beta_1} \cdots \partial_{\nu_d}^{\beta_d}.
$$

We will also use the classic Sobolev spaces. We will write $H^k_{x,v}$ (resp $W^{k,\infty}_{x,v}$) the standard Sobolev space for the norm $L^2$ (resp $L^\infty$) for functions depending on $(x,v)$, and $H^k_x$ for functions only depending on $x$.

We will use the following notation and convention for the Fourier transform of a function $U$

$$
\hat{U}(k) := (2\pi)^{-d} \int_{\mathbb{T}^d} U(x)e^{-ik \cdot x} dx.
$$

Instead of (1.1), we consider a more general system under the form

$$
\begin{aligned}
\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f &= 0, \\
E &= -\nabla_x \int_{\mathbb{T}^d} U(x-y) (\rho(y) - \rho_0) dy, \\
f(0, x, v) &= f^0(x, v),
\end{aligned}
$$

(1.2)

making the following assumptions for $U$:

**Assumptions 1 (A1).** $\hat{U}(0) = 0$ and there exists $\alpha > 0, C_{\alpha} > 0$ such that

$$
\forall k \neq 0, k \in \mathbb{Z}^d, \quad |\hat{U}(k)| \leq \frac{C_{\alpha}}{|k|^{\alpha}}.
$$

(1.3)

In particular, by taking $\hat{U}(k) = \pm \frac{1}{|k|^{\alpha}}, \hat{U}(0) = 0$, we recover (1.1).

Note that we can also consider potentials $U = \sum_{i=1}^n U_i$ such that each $U_i$ satisfies (1.3) for $\alpha_i > 0$. Then $U$ satisfies (1.3) with $\alpha = \min(\alpha_i)_{i \in [1,n]}$. Those types of potentials may appear in some physical models in gravitation (see the discussion below about Manev potentials).

We denote by $|x|$ the floor value of $x$ and

$$
m_0 = 3 + \frac{d}{2} + p_0, \quad p_0 = \left\lfloor \frac{d}{2} \right\rfloor + 1, \quad r_0 = \max \left( \left\lfloor \frac{d}{2} \right\rfloor + \frac{d}{2} \right).
$$

Our main well-posedness result is the following

**Theorem 1.** Let $f^0 \in H^m_{x,v}$ with $m > m_0, 2r > r_0$. Assuming (A1), then there exists $T > 0$ for which there exists a unique solution of the system (1.2) with initial condition $f^0$ and such that $f \in C([0,T], H^{m_0-1}_x), \rho \in L^2([0,T], H^m_x)$.

Note that we consider both the attractive and repulsive case.

**Remark 1.1.** Notice that in the above statement the solution $f$ is less regular than the initial condition $f^0$. This was expected because of the nature of the equation. Nevertheless, the fact that $f$ is in $C([0,T], H^{m_0-1}_x)$ is not optimal. With sharper estimates in the following proof, when $\alpha \leq 1, f$ is expected to be in $C([0,T], H^{m_0-1+s}_x)$, provided that we use the fractional space $H^s_x$ when $s$ is not an integer (which has not been defined here). For simplicity, we will only give the proof of Theorem 1 as the gain of regularity we could obtain does not seem fundamental.

Our results could be easily extended in the case of $\mathbb{R}^d$ instead of $\mathbb{T}^d$ assuming that

$$
\forall \xi \in \mathbb{R}^d, |\hat{U}(\xi)| \leq \frac{C_{\alpha}}{1 + |\xi|^{\alpha}}.
$$
For \( 1 \leq \alpha < 2 \), system (1.1) behaves at least like a Burgers type equation and local well-posedness theory follows by standard energy estimates. The case \( \alpha = 1 \), in particular, has physical meaning. It was first introduced by Manev as a correction of the Newtonian potential. The interactions between Newtonian and Manev potentials have been studied (see [9]), but the Manev potential on its own is also interesting. In the attractive setting, it is called the Pure Stellar dynamic Manev system (PSM) [2], and it arises in stellar dynamics. In the repulsive case, it is called the Pure Stellar dynamic Manev system (PSM) [2], and

\[
\text{For } \frac{3}{2} \leq \alpha \leq 1 \text{ local well-posedness was recently obtained in [14] using the additional regularity provided by averaging lemmas in the whole space in [3]. In addition, they give general conditions such that the system (1.1), that they call the Vasov-Riesz system because of the introduction of a Riesz type interaction, have finite-time singularity formation for solutions.}
\]

Thus, the main contribution of this article corresponds to the case \( \alpha \in (0, \frac{3}{2}) \). The well-posedness theory of the equation is more challenging, because the apparent lost derivative in \( x \) on the force field \( E \) is no longer compensated by regularization. In the critical case \( \alpha = 0 \) (which does not enter our framework), named the Vasov-Dirac-Benney system, it has been proved that the system is in general ill-posed in Sobolev spaces [5]. Nevertheless, on the Torus and in the repulsive case, we can ensure hypotheses to have the local well-posedness of the system. The first results were obtained in dimension \( d = 1 \) by Bardos and Besse [1]. They proved that the Vasov-Dirac-Benney system is locally well-posed provided that for all \( (t, x) \), there exists a function \( m(t, x) \) such that \( v \mapsto f(t, x, v) \) remains compactly supported and is increasing for \( v \leq m(t, x) \), decreasing for \( v \geq m(t, x) \), in other words, one bump shaped functions. More recently, Han-Kwan and Rousset [6] have proved that the Vasov-Dirac-Benney system is locally well-posed in any dimension for Sobolev regularity, provided that for all \( x \), the profile \( v \mapsto f^0(x, v) \) satisfies a Penrose stability condition, which means that if we define the Penrose function

\[
\mathcal{P}(\gamma, \tau, \eta, f) = 1 + \int_0^{+\infty} e^{-(\gamma + \tau)s} \frac{\inf \{ (F_v \nabla_v f)(\eta s) \}}{1 + |\eta|^2} ds, \quad \gamma > 0, \tau \in \mathbb{R}, \eta \in \mathbb{R}^d \setminus \{0\},
\]

\( v \mapsto f^0(x, v) \) must satisfy the following condition, for some \( c_0 > 0 \)

\[
\inf_{(\gamma, \tau, \eta) \in (0, +\infty) \times \mathbb{R} \times \mathbb{R}^d \setminus \{0\}} |\mathcal{P}(\gamma, \tau, \eta, f^0)| \geq c_0.
\]

### 1.2 Sketch of the proof

In order to prove Theorem 1.1 we shall define another system, which is a regularized Vlasov-Poisson type system, that depends on a new parameter \( \varepsilon > 0 \)

\[
\begin{align*}
\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon &= 0, \\
E_\varepsilon &= -\nabla_x (U \ast V_\varepsilon), \\
-\varepsilon^2 \Delta_x V_\varepsilon + V_\varepsilon &= \int_{\mathbb{R}} f_\varepsilon(\cdot, v) dv, \\
f_\varepsilon(0, x, v) &= f^0(\varepsilon, x, v).
\end{align*}
\]

Taking formally the limit \( \varepsilon \to 0 \), we obtain the system (1.2), which is exactly the system we want to study. The idea is to use the well-posedness of the system (1.1) to get a family of functions \( (f_\varepsilon) \) which satisfy uniform estimates. Then, using compactness we extract (in a certain sense that will be defined later) a function \( f \) which is a solution of the limit system (1.2). After that, we will show uniqueness for the solution of (1.2) in the class of \( \{ f \in C([0, T], H^m_{2r}) \} \), \( \rho \in L^2([0, T], H^m_r) \}).

The first step is to show that we can get a family of functions \( (f_\varepsilon) \) defined on a time interval \([0, T]\) with \( T \) being independent of \( \varepsilon \). To do so, we introduce the following key quantity

\[
N_{m, 2r}(t, f) := \|f\|_{L^\infty([0, t], H^{m-1}_{2r})} + \|\rho\|_{L^2([0, t], H^m_{2r})},
\]

and we have the following theorem

**Theorem 2.** Let \( \alpha \in \mathbb{R}^*_+ \). We assume that for all \( \varepsilon \in (0, 1] \), \( f_\varepsilon^0 \in H^m_{2r} \) with \( m > m_0 \), \( 2r > r_0 \) and that there exists \( M_0 > 0 \) so that for all \( \varepsilon \in (0, 1] \), \( \|f_\varepsilon^0\|_{H^m_{2r}} \leq M_0 \).

Then there exist \( T > 0 \), \( R > 0 \) (independent of \( \varepsilon \)) and a unique solution \( f_\varepsilon \in C([0, T], H^m_{2r}) \) of (1.7) with

\[
\sup_{\varepsilon \in (0, 1]} N_{m, 2r}(T, f_\varepsilon) \leq R.
\]
The proof of this theorem will rely on a bootstrap argument following the strategy of [6]. For most of
the proof, we will try to estimate the key quantity $N_{m,2r}(t,f_x)$ independently of $\varepsilon$. Of course, if we
could give an estimate of $\|f\|_{L^\infty([0,T],H^{m}_{\alpha})}$ independently of $\varepsilon$, the proof would be over, but this
cannot be done in the general case. This is the reason why we choose to work with $\|f\|_{L^\infty([0,T],H^{m-1}_{\alpha})}$ (losing
one derivative for $f$ is enough to give an estimate that does not depend on $\varepsilon$), and $\|\beta(x)\|_{L^2([0,T],H^m_{\alpha})}$. The
fact that it is possible to give an estimate independent of $\varepsilon$ on $\rho$ without loss of derivatives while it is
not for $f$ is not trivial and is fundamental here. This is reminiscent of the results on averaging lemmas
(see [3]). Here though, the situation is a bit different, as we will be averaging in $v$ but also in time. More
precisely, anticipating a bit on the following proof, let us look at the operator $K$ (defined in Lemma 8).

\[ K_G(F)(t,x) = \int_0^t \int (\nabla_x F)(s,x - (t-s)v) \cdot G(t,s,x,v)dvds. \]

There is an apparent loss of one derivative over $v$. However, the key proposition 4 shows that $K_G$
is a bounded operator from $L^2([0,T],H^m_{\alpha})$ into $L^2([0,T],L^2_{\alpha})$ (provided that $G$ is smooth enough and
$\alpha \in (0,1)$), and the bound can be estimated up to a constant by $T^{\alpha/2}$ (actually, the result remains true
for $\alpha = 0$ [6]). The apparent loss of a derivative in $x$ is somehow compensated by taking the averages in
$v$ and in time.

**Remark 1.2.** Because of the apparent links with the theory of averaging lemmas [4], we could try to
apply directly those results to prove our theorem. As stated earlier, a recent work [3] has proved that,
when $x$ is in the whole space, the system (1.1) is locally well-posed up to $\alpha = 3/4$. Nevertheless, standard
averaging lemmas cannot cover the whole range $\alpha \in (0,1]$.

Sections 2, 3 and 4 will be dedicated to the proof of Theorem 2. In section 5, we give the elements to
conclude the proof of the main Theorem [1].

## 2 Beginning of the proof of Theorem 2

### 2.1 Preliminary lemmas

We present here some lemmas that will be useful for the proof of Theorem 2. They have been proved in
[6]. We use the notation $[A,B] = AB - BA$ to denote the commutator between two operators.

**Lemma 1.** Let $s \geq 0$ and $\chi = \chi(v)$ a non-negative smooth function, with $|\partial^\alpha \chi| \leq C_\alpha \chi$ for all $\alpha \in \mathbb{N}^d$, $|\alpha| \leq s$.

- Consider two functions $f = f(x,v)$, $g = g(x,v)$, then for all $k \geq s/2$

  \[ \|\chi fg\|_{H^s_{x,v}} \lesssim \|\chi g\|_{H^s_{x,v}} \|f\|_{W^{s,\infty}_{x,v}} + \|\chi f\|_{H^s_{x,v}} \|g\|_{W^{s,\infty}_{x,v}}. \]  

- Consider a function $E = E(x)$ and a function $F = F(x,v)$. Then for all $s_0 > d$

  \[ \|\chi EF\|_{H^s_{x,v}} \lesssim \|E\|_{H^\infty} \|\chi F\|_{H^s_{x,v}} + \|\chi F\|_{H^s_{x,v}} \|E\|_{H^s_x}. \]

- Consider a vector field $E = E(x)$ and a function $f = f(x,v)$, then for all $s_0 > 1 + d$ and for all $\alpha$, $\beta \in \mathbb{N}^d$ with $|\alpha| + |\beta| = s \geq 1$

  \[ \|\chi \partial^\alpha_x \partial^\beta_v f, E(x) \cdot \nabla_v f\|_{L^1_{x,v}} \lesssim \|E\|_{H^\infty} \|\chi f\|_{H^s_{x,v}} + \|\chi f\|_{H^s_{x,v}} \|E\|_{H^s_x}. \]

The main use of this lemma will be for $\chi(v) = (1 + |v|)^2$ thus yielding estimates for products in the
space $H^s_x$.

Let us introduce another product type estimate.

**Lemma 2.** Consider two functions $f = f(x,v)$, $g = g(x,v)$. Then, for all $s \geq 0$, $\alpha, \beta \in \mathbb{N}^{2d}$ with
$|\alpha| + |\beta| \leq s$, and $\chi(v)$ positive function satisfying $|\partial^\alpha \chi| \leq C_\alpha \chi$ (as in the precedent lemma). We have

\[ \|\partial^\alpha_x \partial^\beta_v g\|_{L^2_{x,v}} \lesssim \|\partial^\alpha_x f\|_{L^\infty_{x,v}} \|\chi g\|_{H^s_{x,v}} + \|\partial^\alpha_x f\|_{H^s_{x,v}} \|\chi g\|_{L^\infty_{x,v}}. \]
Finally, the following lemma is a very useful commutation formula between $\partial_\alpha^2 \partial_\beta^3$ and the transport operator $T$ defined by

$$\mathcal{L} = \partial_t + v \cdot \nabla_x + E \cdot \nabla_v.$$

(2.5)

Lemma 3. For every $\alpha, \beta \in \mathbb{N}^d$, we have for every smooth functions $f$

$$\mathcal{L}(\partial_\alpha^2 \partial_\beta^3 (Tf)) = T(\partial_\alpha^2 \partial_\beta^3 f) + \sum_{i=1}^{d} f_{\beta_i \neq 0} \partial_\alpha \partial_\beta \partial_\gamma f + [\partial_\alpha^2 \partial_\beta^3, E \cdot \nabla_v]f,$$

where $\gamma$ is equal to $\beta$ except $\gamma_i = \beta_i - 1$.

2.2 Setting up the bootstrap

The proof of the theorem will rely on a bootstrap argument. From standard energy estimates (that we shall recall later, see the proof of proposition [2]), we first have the following result.

Proposition 1. The system (1.7) is locally well-posed in $\mathcal{H}^m_{2r}$ for all $m$ and $r$ satisfying $m > 1 + d$ and $2r > d/2$. In other words, if $f^0_n \in \mathcal{H}^m_{2r}$, there exists $T > 0$ (which depends on $\varepsilon$) such that there exists a unique $f_n \in C([0, T], \mathcal{H}^m_{2r})$ solution of system (1.7).

Thanks to the previous proposition, we may define a maximal solution $f_n \in C([0, T^*], \mathcal{H}^m_{2r})$. As a direct consequence, for every $T < T^*$, $\sup_{[0, T]} \|f_n\|_{\mathcal{H}^m_{2r}} < +\infty$. In order to define $N_{m, 2r}(T, f_n)$, we just have to prove that $\|\rho_n\|_{L^2([0, T], H^m_{2r})} < +\infty$. But because of the definition of weighted Sobolev norms, and with the use of the Cauchy-Schwarz inequality, we get that

$$\|\rho_n\|_{H^m} \lesssim \|f_n\|_{\mathcal{H}^m} < +\infty.$$

We have thus shown that the quantity $N_{m, 2r}(T, f_n)$ is well defined and is continuous in $T$ for every $T < T^*$. This allows us to consider, for $R > 0$ to be defined later,

$$T^\varepsilon = \sup\{T \in [0, T^*), N_{m, 2r}(T, f_n) \leq R\}.$$

By taking $R$ large enough, we have by continuity that $T^\varepsilon > 0$. Of course $T^\varepsilon$ depends on $\varepsilon$. We want to show that by taking $R$ large enough (but independent of $\varepsilon$), $T^\varepsilon$ is uniformly bounded from below by some time $T > 0$. Only the following two situations can happen

1. Either $T^\varepsilon = T^*$,
2. Or $T^\varepsilon < T^*$ and $N_{m, 2r}(T^\varepsilon, f_n) = R$.

Let us analyze the first case. If $T^\varepsilon = T^* = +\infty$, then $N_{m, 2r}(T^\varepsilon, f_n) \leq R$ for every $T > 0$ and there is nothing to do, so we only have to consider when $T^\varepsilon = T^* < +\infty$. Actually, by energy estimates, we can show that this case is impossible. Indeed, we have the following proposition

Proposition 2. Assume that $T^\varepsilon < +\infty$, then for every $f_n$ solution of (1.7), we have for some $C > 0$ independent of $\varepsilon$ the estimate

$$\sup_{[0, T^\varepsilon]} \|f_n(t)\|_{\mathcal{H}^m} \leq \|f^0_n\|_{\mathcal{H}^m} \exp \left[ C \left( T^\varepsilon + \frac{1}{\varepsilon} (T^\varepsilon)^{d} R \right) \right].$$

Proof. Let $f_n$ satisfying (1.7). We thus have $T f_n = 0$, and by using (2.6)

$$\mathcal{L}(\partial_\alpha^2 \partial_\beta^3 f_n) = - \sum_{i=1}^{d} f_{\beta_i \neq 0} \partial_\alpha \partial_\beta \partial_\gamma f_n + [\partial_\alpha^2 \partial_\beta^3, E \cdot \nabla_v] f_n.$$

Taking the scalar product with $(1 + |v|^2)^2 \partial_\alpha \partial_\beta \partial_\gamma f_n$ and summing for every $|\alpha| + |\beta| \leq m$, we obtain for the left hand size

$$\sum_{|\alpha| + |\beta| \leq m} \int_{T^\varepsilon} \int_{\mathbb{R}^d} \mathcal{L}(\partial_\alpha^2 \partial_\beta^3 f_n) \cdot \partial_\alpha \partial_\beta \partial_\gamma f_n = \frac{d}{dt} \|f_n(t)\|_{\mathcal{H}^m}^2.$$

For the first term of the right hand size, we use the Cauchy-Schwarz inequality

$$\left| \sum_{|\alpha| + |\beta| \leq m} \int_{T^\varepsilon} \int_{\mathbb{R}^d} \chi_{\beta_i \neq 0} \partial_\alpha \partial_\beta \partial_\gamma f_n \cdot \chi \partial_\alpha \partial_\beta \partial_\gamma f_n \right| \lesssim \|f_n\|_{\mathcal{H}^m}^2.$$
For the second term, we use [5] with \( s = m \), \( \chi(v) = (1 + |v|^2)^r \) and \( s_0 = m \), from which we deduce
\[
\| \chi [ \partial_x^p \partial_y^q, E(x) \cdot \nabla_v ] f_\varepsilon \|_{L^1_{x,v}} \lesssim \| E_\varepsilon \|_{H^m_v} \| f_\varepsilon \|_{H^m_{x,v}},
\]
and thus by using again Cauchy-Schwarz, we obtain
\[
\left| \int \chi [ \partial_x^p \partial_y^q, E(x) \cdot \nabla_v ] f_\varepsilon \partial_x^p \partial_y^q f_\varepsilon \right| \lesssim \| E_\varepsilon \|_{H^m_v} \| f_\varepsilon \|_{H^m_{x,v}}^2.
\]

We have, by elliptic regularity,
\[
\| E_\varepsilon \|_{H^m_v} = \| \nabla_x (U * V_\varepsilon) \|_{H^m_v} \lesssim \frac{1}{\varepsilon} \| \rho_\varepsilon \|_{H^m_v}.
\]

Putting all together, we have shown that
\[
\frac{d}{dt} \| f_\varepsilon(t) \|_{H^m_{x,v}}^2 \lesssim \left( \frac{1}{\varepsilon} \| \rho_\varepsilon \|_{H^m_v} + 1 \right) \| f_\varepsilon(t) \|_{H^m_{x,v}}^2.
\]

We integrate between 0 and \( t \) for \( t \in [0, T^\varepsilon) \). For some \( C > 0 \) independent of \( \varepsilon \), we get
\[
\| f_\varepsilon(t) \|_{H^m_{x,v}}^2 \leq \| f_\varepsilon(0) \|_{H^m_{x,v}}^2 + C \int_0^t \left( \frac{1}{\varepsilon} \| \rho_\varepsilon \|_{H^m_v} + 1 \right) \| f_\varepsilon(s) \|_{H^m_{x,v}}^2 ds,
\]
and we use the Gronwall inequality to show
\[
\sup_{[0, T^\varepsilon)} \| f_\varepsilon(t) \|_{H^m_{x,v}}^2 \leq \| f_\varepsilon(0) \|_{H^m_{x,v}}^2 + \exp \left[ C \left( T^\varepsilon + \frac{1}{\varepsilon} (T^\varepsilon)^d \| \rho_\varepsilon \|_{L^2([0, T^\varepsilon), H^m_v)} \right) \right]
\]
\[
\leq \| f_\varepsilon(0) \|_{H^m_{x,v}}^2 + \exp \left[ C \left( T^\varepsilon + \frac{1}{\varepsilon} (T^\varepsilon)^d N_{m,2d}(T^\varepsilon, f_\varepsilon) \right) \right].
\]

Finally, since \( N_{m,2d}(T^\varepsilon, f_\varepsilon) \leq R \), we obtain the expected estimate.

If we use the proposition for \( T^\varepsilon = T^* \), we obtain that
\[
\sup_{[0, T^*)} \| f_\varepsilon(t) \|_{H^m_{x,v}}^2 \leq \| f_\varepsilon(0) \|_{H^m_{x,v}}^2 \exp \left[ C \left( T^* + \frac{1}{\varepsilon} (T^*)^d R \right) \right].
\]

This means that the solution could be continued beyond \( T^* \), and thus contradicts the definition of \( T^* \), which shows that this case is impossible.

We then have to consider the remaining case, \( T^\varepsilon < T^* \) and \( N_{m,2d}^0(T^\varepsilon, f_\varepsilon) = R \). Choosing \( R \) large enough, the objective is to find some time \( T^# > 0 \) independent of \( \varepsilon \), such that the equality
\[
N_{m,2d}^0(T, f_\varepsilon) = R,
\]
cannot hold for any \( T \in [0, T^#] \), which will prove that \( T^\varepsilon > T^# > 0 \).

We need to estimate \( N_{m,2d}^0(T, f_\varepsilon) \) for \( T < T^\varepsilon \). The easier part is the term \( \| f_\varepsilon \|_{H^{m-1}_{x,v}} \). We cannot use the previous estimates because they depend on \( \varepsilon \) (we used the elliptic regularity provided by the Poisson equation). Nevertheless, we can still give estimates based on energy methods, this time being careful that every estimate must be independent of \( \varepsilon \). In this proposition and in the following of this article, \( \Lambda \) will stand for a generic continuous function, independent of \( \varepsilon \), which is non-decreasing with respect to each of its arguments.

**Proposition 3.** For \( m > 2 + d \) and \( 2r > d/2 \), \( f_\varepsilon \) solution of \( (1.1) \) satisfies the estimate
\[
(2.7) \quad \sup_{[0, T]} \| f_\varepsilon \|_{H^{m-1}_{x,v}} \leq \| f_\varepsilon(0) \|_{H^{m-1}_{x,v}} + T^{\frac{d}{2}} \Lambda(T, R),
\]
for all \( T \in [0, T^\varepsilon) \).

**Proof.** Let \( \alpha, \beta \in \mathbb{N}^d \) such that \( |\alpha| + |\beta| = m - 1 \). We use (2.3) and, as before, we take the scalar product with \( (1 + |v|^2)^r \partial_x^p \partial_y^q f_\varepsilon \), and we take the sum for every \( |\alpha| + |\beta| = m - 1 \). Like we did previously, we use (2.3) with \( s = m - 1 \), \( \chi(v) = (1 + |v|^2)^r \) and \( s_0 = m - 1 \). We obtain that
\[
(2.8) \quad \frac{d}{dt} \| f_\varepsilon \|_{H^{m-1}_{x,v}}^2 \lesssim \| f_\varepsilon \|_{H^{m-1}_{x,v}}^2 + \| f_\varepsilon \|_{H^{m-1}_{x,v}}^2 + \| f_\varepsilon(0) \|_{H^{m-1}_{x,v}}^2 + \| E_\varepsilon \|_{H^{m-1}_{x,v}}.
\]
Integrating in time, we get that there exists a $C > 0$ such that

$$\sup_{[0,T]}\|f_{\varepsilon}\|_{H^{m-1}_{2\varepsilon}} \leq \|f^0_{\varepsilon}\|_{H^{m-1}_{2\varepsilon}} + C\sup_{[0,T]}\|f_{\varepsilon}\|_{H^{m-1}_{2\varepsilon}} \left(T + \int_{0}^{T} \|E_{\varepsilon}\|_{H^{m-1}_{2\varepsilon}} dt\right).$$

We can now use the following estimate, independent of $\varepsilon$

$$\|E_{\varepsilon}\|_{H^{m-1}_{2\varepsilon}} = \|\nabla_{x}(U + V_{\varepsilon})\|_{H^{m-1}_{2\varepsilon}} \lesssim \|\rho_{\varepsilon}\|_{H^{m}_{2\varepsilon}}.$$

And we still have $N_{\rho_{\varepsilon}}(T, f_{\varepsilon}) \leq R$, (because $T \in [0, T^*]$). We have thus shown that

$$\sup_{[0,T]}\|f_{\varepsilon}\|_{H^{m-1}_{2\varepsilon}} \leq \|f^0_{\varepsilon}\|_{H^{m-1}_{2\varepsilon}} + CR(T + T^2 R),$$

which concludes the proof of the lemma.

\[\Box\]

**Remark 2.1.** As said in the introduction, our estimates could be sharper. This can been seen in (2.9), which is far from being optimal, because we do not use the regularity provided by the convolution with $U$. Nevertheless, this regularization is not needed to prove Theorem 2, thus we chose not to exploit it for simplicity.

3: Estimates of the density term

3.1: Introduction of the $f_{I,J}$

Now we need to tackle the second term of $N_{\rho_{\varepsilon}}(T, f_{\varepsilon})$, which is the norm of the density $\|\rho_{\varepsilon}\|_{L^2([0,T],H^{m}_{2\varepsilon})}$. We can try to apply the operator $\partial_{x}^{\alpha}$ to (1.7) with $|\alpha| = m$, but this involves commutator terms such as $\partial_{x}^{\alpha} E_{\varepsilon} \cdot \nabla_{x} \partial_{x}^{\alpha-\alpha'} f_{\varepsilon}$ which contain $m$ order derivatives of $f_{\varepsilon}$ when $|\alpha'| = 1$, and those cannot be estimated uniformly in $\varepsilon$. To get rid of this problem, we choose to apply a larger class of differential operators to (1.7).

**Definition 3.1.** For $I = (i_1, ..., i_d), J = (j_1, ..., j_d) \in \mathbb{N}^d, |I| + |J| = m$, we define

$$f_{I,J} := \partial_{x}^{\alpha} \partial_{v}^{\beta} f_{\varepsilon}.$$

Note that the $(f_{I,J})$ contain all the $\partial_{x}^{\alpha} f_{\varepsilon}$ with $|\alpha| = m$. By applying $\partial_{x}^{\alpha} \partial_{v}^{\beta}$ to (1.7), we find that the $(f_{I,J})$ satisfy a differential system, which is the purpose of the following lemma.

**Lemma 4.** We assume that $m > 3 + d$ and $2r > d$. For all $T < T^*$, and for all $I, J \in \mathbb{N}^d$, we have, for $f_{\varepsilon}$ satisfying (1.7), that $f_{I,J}$ is solution of

$$T(f_{I,J}) + \partial_{x}^{\alpha} \partial_{v}^{\beta} E_{\varepsilon} \cdot \nabla_{v} f_{\varepsilon} + M_{I,J} \mathcal{F} = \mathcal{V}_{I,J},$$

where

$$\mathcal{F} = (f_{I,J})_{I,J \in \mathbb{N}^d, |I| + |J| = m}, \quad M_{I,J} \mathcal{F} = \sum_{k=1}^{d} 1_{k \neq 0} f_{I^k} \mathcal{T}^k + \sum_{p=1}^{d} \sum_{k=1}^{d} 1_{k \neq 0} \partial_{x}^{\alpha} E_{\varepsilon} f_{\mathcal{T}^k, I^k},$$

with

$$I^k = (i_1...i_{k-1}, i_k + 1, i_{k+1}...i_d), \quad \mathcal{T}^k = (i_1...i_{k-1}, i_k - 1, i_{k+1}...i_d),$$

and $\mathcal{V} = (\mathcal{V}_{I,J})_{I,J \in \mathbb{N}^d}$ is a remainder, which means that for every $T < T^*$

$$\|\mathcal{V}\|_{L^2([0,T],H^m_{2\varepsilon})} \leq \Lambda(T, R).$$

We therefore obtain that $\mathcal{F} = (f_{I,J})$ satisfies a system which is coupled through the linear term $M_{I,J} \mathcal{F}$. We now prove this lemma.

**Proof.** We know that $f_{\varepsilon}$ solves (1.7), so we have

$$\partial_{x}^{\alpha} \partial_{v}^{\beta} (T(f_{\varepsilon})) = 0.$$

By using (2.9), we obtain

$$T(f_{I,J}) + \partial_{x}^{\alpha} \partial_{v}^{\beta} E_{\varepsilon} \cdot \nabla_{v} f_{\varepsilon} + M_{I,J} \mathcal{F} = \mathcal{V}_{I,J},$$
with
\[ V_{I,J} = \min(|I|, m-1) \sum_{k=2}^{\infty} \sum_{\sigma, |\sigma|=k} C_{I,J,k,\sigma} \partial^2 _x E \cdot \nabla_x \partial^\sigma_x \partial^I_v \partial^J_v f_x. \]
Notice that because \(|\sigma| = k \geq 2\), we have in particular that \(1 + |I| - |\sigma| + |J| \leq m - 1\). Also, in the case \(|I| \leq 1\), we simply have \(V_{I,J} = 0\).

We want to estimate \(\|V_{I,J}\|_{L^2([0,T],H^p)}\). We have that
\[ \|V_{I,J}\|_{H^p} \lesssim \|E\|_{H^{p-1}} \|f_x\|_{H^p}^{-1}. \]
And, because \(\|E\|_{H^{p-1}} \lesssim \|\rho\|_{H^p}\) (estimate independent of \(\epsilon\)),
\[ \|V_{I,J}\|_{L^2([0,T],H^p)} \lesssim \|\rho\|_{L^2([0,T],H^p)} \|f_x\|_{L^\infty([0,T],H^p)}, \]
and we obtain that
\[ \|V\|_{L^2([0,T],H^p)} \leq \Lambda(T,R), \]
which ends the proof.

### 3.2 Straightening the transport vector field

In the following, we make a change of variable to straighten the vector field
\[ \partial_t + v \cdot \nabla_x + E \cdot \nabla_v \rightarrow \partial_t + \Phi(t,x,v) \cdot \nabla_x, \]
where \(\Phi\) is defined in the following lemma.

**Lemma 5.** Let \(f_{I,J}\) a solution of (3.2). We consider \(\Phi(t,x,v)\) a smooth solution of the Burgers equation
\[ \partial_t \Phi + \Phi \cdot \nabla_x \Phi = E(x), \]
such that the Jacobian matrix \((\nabla_v \Phi)\) is invertible. We define \(g_{I,J}\) by
\[ g_{I,J}(t,x,v) := f_{I,J}(t,x,\Phi). \]

Then \(g_{I,J}\) is solution of the equation
\[ \partial_t g_{I,J} + \Phi \cdot \nabla_x g_{I,J} + \partial^I_v \partial^J_v E \cdot (\nabla_v f_x)(t,x,\Phi) + \mathcal{M}_{I,J} \mathcal{G} = V_{I,J}(t,x,\Phi), \]
where \(\mathcal{G} = (g_{I,J})_{I,J \in \mathbb{N}^4, |I|+|J|=m}.\)

**Proof.** The proof is based on a simple calculation
\[ \partial_t g_{I,J} + \Phi \cdot \nabla_x g_{I,J} + \partial^I_v \partial^J_v E \cdot (\nabla_v f_x)(t,x,\Phi) + \mathcal{M}_{I,J} \mathcal{G} = V_{I,J}(t,x,\Phi), \]
\[ + \epsilon^{-1}(\nabla_v \Phi)^{-1} \nabla_v g_{I,J} \cdot (\partial_t \Phi + \Phi \cdot \nabla_x \Phi - E_x), \]
hence the result when (3.2) is satisfied.

**Remark 3.1.** Writing \(J(t,x,v) = |\det \nabla_v \Phi(t,x,v)|\), notice that
\[ \int_{\mathbb{R}^d} g_{I,J} J dv = \int_{\mathbb{R}^d} f_{I,J} dv. \]

Because we introduced the function \(\Phi\) in our equations, we shall estimate its Sobolev norms, which is the purpose of the following lemma.

**Lemma 6.** Suppose that \(m > 3 + d\), there exists \(T_0\) (depending on \(R\) but independent on \(\epsilon\)) such that for all \(T < \min(T_0,T^c)\), there exists a unique smooth solution on \([0,T]\) of the Burgers equation (3.2) with initial condition \(\Phi|_{t=0} = v\).

Moreover, for all \(T < \min(T_0,T^c)\), we have the following estimates:
\[ \sup_{[0,T]} \|\Phi - v\|_{W^{1,\infty}} + \sup_{[0,T]} \left\| \frac{1}{1 + |v|^2} \partial_t \Phi \right\|_{W^{k-1}} \leq T^{\frac{k}{2}} \Lambda(T,R), \]
for \(|\alpha| \leq m - 1\) and \(|\beta| \leq m - 2\)
\[ \sup_{[0,T]} \sup_v \|\partial^\alpha_{x,v} (\Phi - v)\|_{L^2} + \sup_{[0,T]} \sup_v \left\| \frac{1}{1 + |v|^2} \partial^\beta_{x,v} \partial_t \Phi \right\|_{L^2} \leq T^\frac{k}{2} \Lambda(T,R). \]

A very similar lemma (Lemma 11) has been proved in [10], we refer to it for a complete proof.
Now we have to consider our new equation, that we rewrite as
\begin{equation}
\partial_t g_{I,J} + \Phi \cdot \nabla_x g_{I,J} + \partial_x^I \partial_x^J G_x \cdot (\nabla_v f_x)(t, x, \Phi) + \mathcal{M}_{I,J} \tilde{G} = S_{I,J},
\end{equation}
where \(S_{I,J}(t, x, v) = \mathcal{V}_{I,J}(t, x, \Phi(t, x, v))\).

The next step is to introduce the flow of the equation that we denote by \(X(t, s, x, v), 0 \leq s, t \leq T\) and is given as the solution of
\[\partial_t X(t, s, x, v) = \Phi(t, X(t, s, x, v), v), \quad X(s, s, x, v) = x.\]

We have to control the Sobolev norms of \(X\).

**Lemma 7.** For all \(t, s, 0 \leq s \leq t \leq T\) and \(m > 3 + d\), we write
\[X(t, s, x, v) = x + (t-s) \left( v + \tilde{X}(t, s, x, v) \right) .\]

We have that \(\tilde{X}\) satisfies, for \(|\alpha| < m - d/2 - 1\), \(|\beta| < m - d/2 - 2\),
\begin{equation}
\sup_{t,s \in [0, T]} \left\| \frac{\partial_{x,v}^\alpha \tilde{X}(t, s, x, v)}{L_x^\infty} + \sup_{t,s \in [0, T]} \left| \frac{1}{(1 + |v|^2)^{\frac{\beta}{2}}} \partial_{x,v}^\beta \partial_t \tilde{X}(t, s, x, v) \right| \right\|_{L_x^\infty} \leq T^2 \Lambda(T, R).
\end{equation}

Furthermore, there exists \(T_0(R) > 0\) small enough such that for \(T \leq \min\{T_0, \tilde{T_0}, T^c\}\), we get that \(x \mapsto x + (t-s)\tilde{X}(t, s, x, v)\) is a diffeomorphism and that, for \(|\alpha| < m - 1\), \(|\beta| < m - 2\),
\begin{equation}
\sup_{t,s \in [0, T]} \sup_v \left\| \frac{\partial_{x,v}^\alpha \tilde{X}(t, s, x, v)}{L_x^\infty} + \sup_{t,s \in [0, T]} \left| \frac{1}{(1 + |v|^2)^{\frac{\beta}{2}}} \partial_{x,v}^\beta \partial_t \tilde{X}(t, s, x, v) \right| \right\|_{L_x^\infty} \leq T^2 \Lambda(T, R).
\end{equation}

Finally, there exists \(\Psi(t, s, x, v)\) such that for \(t, s \in [0, T]\) and \(T \leq \min\{T_0, \tilde{T_0}, T^c\}\), we have,
\begin{equation}
X(t, s, x, \Psi(t, s, x, v)) = x + (t-s)v,
\end{equation}
with \(\Psi\) satisfying the estimates, for \(|\alpha| < m - d/2 - 1\), \(|\beta| < m - d/2 - 2\),
\begin{equation}
\sup_{t,s \in [0, T]} \left\| \frac{\partial_{x,v}^\alpha \Psi(t, s, x, v) v}{L_x^\infty} + \sup_{t,s \in [0, T]} \left| \frac{1}{(1 + |v|^2)^{\frac{\beta}{2}}} \partial_{x,v}^\beta \partial_t \Psi(t, s, x, v) \right| \right\|_{L_x^\infty} \leq T^2 \Lambda(T, R),
\end{equation}
for \(|\alpha| < m - 1\), \(|\beta| < m - 2\)
\begin{equation}
\sup_{t,s \in [0, T]} \sup_v \left\| \frac{\partial_{x,v}^\alpha \Psi(t, s, x, v) v}{L_x^\infty} + \sup_{t,s \in [0, T]} \left| \frac{1}{(1 + |v|^2)^{\frac{\beta}{2}}} \partial_{x,v}^\beta \partial_t \Psi(t, s, x, v) \right| \right\|_{L_x^\infty} \leq T^2 \Lambda(T, R).
\end{equation}

Once again, a similar lemma (Lemma 13) has already been proved.

In order to control the linear part, we define the tensor \(\mathcal{M}\) by \((\mathcal{M}H)_{I,J} = M_{I,J} H\) and for \(0 \leq s, t \leq T\), \(x \in \mathbb{T}^d, v \in \mathbb{R}^d\), and we introduce \(\mathfrak{M}(t, s, x, v)\) as the solution of
\[\partial_t \mathfrak{M}(t, s, x, v) = \mathcal{M}(t, X(t, s, x, v)) \mathfrak{M}(t, s, x, v), \quad \mathfrak{M}(s, s, x, v) = I,
\]
whose existence and uniqueness is guaranteed by the Cauchy-Lipschitz theorem.

By a Gronwall type argument and thanks to Lemma 13, we can show that for \(k < m - 2\)
\begin{equation}
\sup_{0 \leq s, t \leq T} \left( \|\mathfrak{M}\|_{W_x^{k,\infty}} + \|\partial_t \mathfrak{M}\|_{W_x^{k,\infty}} + \|\partial_x \mathfrak{M}\|_{W_x^{k,\infty}} \right) \leq \Lambda(T, R).
\end{equation}

### 3.3 Introduction of the average operator

We define in the next lemma the fundamental operator \(K_G\) which was introduced in [6]

**Lemma 8.** For a smooth function \(G(t, s, x, v)\), we define the integral operator \(K_G\) acting on \(F(t, x)\) by
\begin{equation}
K_G(F)(t, x) = \int_0^t \int_{\mathbb{R}^d} (\nabla_x F)(s, x - (t-s)v) \cdot G(t, s, x, v) dv ds.
\end{equation}
For \( f_\varepsilon \) satisfying (1.7) and \( \rho_\varepsilon = \int f_\varepsilon dv \), the functions \( \partial_t^I \rho_\varepsilon \) with \(|I| = m\) satisfy the equation:

\[
\partial_t^I \rho_\varepsilon = \sum_{K \in \{1, \ldots, d\}^m} K_{H^{(K, 0)}, (I, 0)} \left( U * \left( (I - \varepsilon^2 \Delta)^{-1} \partial_x^K \rho_\varepsilon \right) \right) + \mathcal{R}_{I, 0},
\]

with

\[
H^{(K, L), (I, J)} = \mathcal{M}^{(K, L), (I, J)}(s, t, x, \Psi(s, t, x, v)) \left( \nabla_v f(s, x - (t - s)v, \Psi(s, t, x, v)) \mathcal{J}(t, s, \Psi(s, t, x, v)) \right) \tilde{\mathcal{J}}(s, t, x, v),
\]

\[
\mathcal{J} = |\det \nabla_v \Phi(t, x, \Psi(s, t, x, v))|, \quad \tilde{\mathcal{J}} = |\det \nabla_v \Psi(s, t, x, v)|,
\]

and \( \mathcal{R}_{I, 0} \) satisfies for \( T \) small enough, the estimate

\[
\|\mathcal{R}_{I, 0}\|_{L^2([0, T], L^2)} \lesssim T^{\frac{1}{2}} \Lambda(T, R).
\]

**Proof.** We first introduce the notations

\[
\eta(t, x, v) = (\partial_x^I \partial_v^J E_\varepsilon(t, x) \cdot \nabla_v f_\varepsilon(s, \Phi(t, x, v)))_{I, J}, \quad S = (S_{I, J})_{I, J}.
\]

This allows us to put the system (3.7) under the following equation satisfied by \( G \)

\[
\partial_t G(t, x, v) + \Phi \cdot \nabla_x G(t, x, v) + \eta(t, x, v) + \mathcal{M} G(t, x, v) = S(t, x, v).
\]

Integrating by respect with the time variable the expression \( \partial_t (\mathcal{M} G(t, x, v) \mathcal{G}(t, X(t, x, v), v)) \) we obtain that

\[
\mathcal{G}(t, x, v) = \mathcal{M}(0, t, x, v) \mathcal{G}^0(X(0, t, x, v), v) + \int_0^t \mathcal{M}(s, t, x, v) S(s, X(s, t, x, v), v) ds
\]

\[
- \int_0^t \mathcal{M}(s, t, x, v) \eta(s, X(s, t, x, v), v) ds,
\]

with \( \mathcal{G}^0 = (g_{0, I, J})_{I, J} \). We multiply the equation by \( \mathcal{J} \) and then with integrate by respect with the variable \( v \), which yields

\[
(3.15) \quad \int_{\mathbb{R}^d} \mathcal{G}(t, x, v) \mathcal{J}(t, x, v) dv = \mathcal{I}_0 + \mathcal{I}_F - \int_{\mathbb{R}^d} \int_0^t \mathcal{M}(s, t, x, v) \eta(s, X(s, t, x, v), v) \mathcal{J}(t, x, v) dv ds,
\]

with

\[
\mathcal{I}_0 = \int_{\mathbb{R}^d} \mathcal{M}(0, t, x, v) \mathcal{G}^0(X(0, t, x, v), v) \mathcal{J}(t, x, v) dv,
\]

\[
\mathcal{I}_F = \int_0^t \int_{\mathbb{R}^d} \mathcal{M}(s, t, x, v) S(s, X(s, t, x, v), v) \mathcal{J}(t, x, v) dv ds.
\]

We want to show that \( \mathcal{I}_0, \mathcal{I}_F \) can be considered as remainders. Let us recall some of the previous estimates on \( \Phi \) and \( \mathcal{M} \)

\[
(3.16) \quad \sup_{0 \leq s, t \leq T} \|\mathcal{M}(t, s)\|_{L^\infty_{x, v}} \leq \Lambda(T, R).
\]

\[
\sup_{[0, T]} \|\Phi(t) - v\|_{W^{1, \infty}_{x, v}} \leq \Lambda(T, R).
\]

Now we can give the following estimate

\[
\left| \int \mathcal{M}(t, 0, x, v) \mathcal{G}^0(X(0, t, x, v), v) \mathcal{J}(t, x, v) dv \right|
\]

\[
\leq \sup_{0 \leq s, t \leq T} \|\mathcal{M}(s, t)\|_{L^\infty_{x, v}} \sup_{0 \leq s, t \leq T} \|\mathcal{J}(t)\|_{L^\infty_{x, v}} \int \mathcal{G}^0(X(0, t, x, v), v) dv
\]

\[
\leq \Lambda(T, R) \sum_{I, J} \int |g_{0, I, J}^0(X(0, t, x, v), v) dv|
\]

Thus, we have

\[
\|\mathcal{I}_0\|_{L^2([0, T], L^2)} \leq \Lambda(T, R) \sum_{I, J} \left( \|g_{0, I, J}^0(X(0, t, \cdot, v), v)\|_{L^2_{T}dv} \right)_{[0, T]}.
\]
To estimate the last term, we use the change of variable in \(x, y = X(0, t, x, v) + tv = x - t\tilde{X}(0, t, x, v)\) and, thanks to the estimates on \(\tilde{X}\)

\[
\sup_{t, s \in [0, T]} \left\| \partial_x \tilde{X}(t, s, x, v) \right\|_{L^\infty_{x,v}} \leq \Lambda(T, R),
\]

we get that

\[
g_{i,j}^0(X(0, t, \cdot, v)) \| \|_{L^2_t} \leq \Lambda(T, R) \| g_{i,j}^0(\cdot - tv), v \|_{L^2_t} \leq \Lambda(T, R) \| g_{i,j}^0(\cdot, v) \|_{L^2_t}.
\]

Then by Cauchy-Schwarz, we deduce

\[
\|I_0\|_{L^2_t([0,T],L^2_t)} \leq T^{\frac{1}{2}} \Lambda(T, R) \left( \int_{\mathbb{R}^N} \frac{dv}{(1 + |v|^2)^\frac{1}{2}} \right)^{\frac{1}{2}} \sum_{i,j} \left\| g_{i,j}^0 \right\|_{H^0}.
\]

Again, we can make the following change of variable

\[
\int_{x, v} |g_{i,j}^0(x, v)|^2 dv = \int_{x, v} |f_{i,j}^0(x, v)|^2 J(0, x, v) dv \leq \Lambda(T, R) \left\| f_{i,j}^0 \right\|_{L^2_t}^2.
\]

Finally, we can use the fact that \(\left\| f_{i,j}^0 \right\|_{H^0} \leq \left\| f \right\|_{H^0} \leq R\) to conclude that

\[
\|I_0\|_{L^2_t([0,T],L^2_t)} \leq T^{\frac{1}{2}} \Lambda(T, R).
\]

Let us show with similar arguments that

\[
\|I_f\|_{L^2_t([0,T],L^2_t)} \leq TA(T, R).
\]

First, we use once again (5.10) to show that

\[
\|I_F\|_{L^2_t([0,T],L^2_t)} \leq \Lambda(T, R) \sum_{i,j} \left\| \int_0^t \int_v |S_{i,j}(s, X(s, t, v))| dv ds \right\|_{L^2_t([0,T])}.
\]

As did just above, we use the change of variable in \(x, y = X(s, t, x, v) + (t-s)v = x - (t-s)\tilde{X}(s, x, v)\), and the estimates on \(\tilde{X}\) to show that

\[
\|S_{i,j}(s, X(s, t, v))| dv \|_{L^2_t} \leq \Lambda(T, R) \| S_{i,j}(s, \cdot, v) \|_{L^2_t}.
\]

Integrating in \(v\), we get by the Cauchy-Schwarz inequality

\[
\int_v |S_{i,j}(s, X(s, t, v))| dv \|_{L^2_t} \leq \Lambda(T, R) \| S_{i,j}(s) \|_{H^0}.
\]

So finally arrive at

\[
\|I_F\|_{L^2_t([0,T],L^2_t)} \leq \Lambda(T, R) \left\| \int_0^t \|S_{i,j}(s)\|_{H^0} \right\|_{L^2_t([0,T])} \leq \Lambda(T, R) \| S \|_{L^2_t([0,T],H^0)}.
\]

Recalling that \(S(t, x, v) = V(t, x, \Phi(t, x, v))\), we can use one last change of variable, and the estimates on the derivatives of \(\Phi\), to show that

\[
\|S\|_{L^2_t([0,T],H^0)} \leq \Lambda(T, R) \|V\|_{L^2_t([0,T],H^0)} \leq \Lambda(T, R),
\]

where the last inequality comes from the fact that \(V\) is a remainder term, which we proved in lemma 4. Thus, we have proved that

\[
\|I_F\|_{L^2_t([0,T],L^2_t)} \leq TA(T, R).
\]

Let us go back to (5.15). Thanks to the results on \(I_0\) and \(I_F\), and using (5.4), we have

\[
\partial_t^l \rho_{l} = R_{l,0} - \int_0^t \int_K \mathfrak{M}_{(K,0),l,j} \partial_x^l E(s, X(s, t, x, v), \nabla_v f)(s, X(s, t, x, v), \Phi(s, X(s, t, x, v), v), J(t, x, v)) dv ds,
\]

with

\[
\|R_{l,0}\|_{L^2_t([0,T],L^2_t)} \leq T^{\frac{1}{2}} \Lambda(T, R).
\]

We can finally use the change of variable \(v = \Psi(s, t, x, w)\) provided by lemma 7 to obtain that

\[
\partial_t^l \rho_{l} = - \int_0^t \int_K \partial_x^l E(s, x - (t-s)v) \cdot H_{(K,0),l,j}(t, s, x, v) dv ds + R_{l,0},
\]

with

\[
H_{(K,l),l,j}(t, s, x, v) = \mathfrak{M}_{(K,l),(l,j)}(s, t, x, \Psi(s, t, x, v)) (\nabla_v f)(s, x - (t-s)v, \Psi(s, t, x, v), J(t, s, \Psi(s, t, x, v), \tilde{J}(t, s, x, v),
\]

which gives the result.
3.4 Focus on the operator $K_G$

In order to control the norms of the derivatives of $\rho_\varepsilon$, we have to understand better the operator $K$. Following [6], let us first introduce a new norm.

**Definition 3.2.** For $T > 0$, we define

$$\|G\|_{T,s_1,s_2} = \sup_{0 \leq t \leq T} \left( \sum_k \sup_{0 \leq s \leq T} \sup_\xi (1 + |k|)^{s_2} (1 + |\xi|)^{s_1} \left| (\mathcal{F}_{x,v}G)(t,s,k,\xi) \right| \right)^{\frac{1}{2}}.$$  

**Proposition 4.** There exists $C > 0$ such that for every $T > 0$, $\alpha \in (0,1)$, for every $G$ satisfying $\|G\|_{T,s_1,s_2} < \infty$ and for all $s_1 > 1$, $s_2 > d/2$, we have

$$\|K_G(F)\|_{L^2([0,T],L^2)} \leq CT^{\alpha/2} \|G\|_{T,s_1,s_2} \|F\|_{L^2([0,T],H^\alpha_2)}; \quad \forall F \in L^2([0,T],H^\alpha_2).$$

For practical uses, it is convenient to relate the norm $\|G\|_{T,s_1,s_2} < \infty$ to a more tractable norm. From [6], we know that if $p > 1 + d$, $\sigma > d/2$, we can find $s_2 > d/2$ et $s_1 > 1$ such that

$$\|G\|_{T,s_1,s_2} \leq \sup_{0 \leq t \leq T} \|G(t,s)\|_{H^\sigma_2}.$$  

In the expression of the operator $K_G$ [3.14], there seems to have of loss of a derivative in $x$, but this proposition shows that the operator is actually continuous from $L^2([0,T],H^\alpha_2)$ into $L^2([0,T],L^2)$ provided that $G$ is smooth enough. Once again, we emphasize that this is a key property in the proof of Theorem 2. We are able to gain regularity in $x$ by integrating over $v$ and $t$. This propriety explains why we can estimate the $H^m$ norm of $\rho_\varepsilon$ without loss of derivative.

Proposition 4 remains true for $\alpha = 0$, this was proved in [6].

**Proof.** By using Fourier series in $x$, we write that

$$F(t,x) = \sum_{k \in \mathbb{Z}^d} \hat{F}_k(t)e^{ik \cdot x}.$$  

By definition of $K_G$, we have

$$K_G F(t,x) = \int_0^t \sum_k \hat{F}_k(s)e^{ik \cdot x} \cdot \int e^{-ik \cdot v(t-s)}G(t,s,x,v)dvds$$

$$= \int_0^t \sum_k \hat{F}_k(s)e^{ik \cdot x}ik \cdot (\mathcal{F}_{v}G)(t,s,x,k(t-s))ds,$$

where $\mathcal{F}_{v}G$ is the Fourier transform of $G(t,s,x,v)$ with respect to the last variable. By Fourier expanding in the $x$ variable, we deduce that

$$K_G F(t,x) = \sum_k e^{ik \cdot x} \sum_l e^{ik \cdot x} \int_0^t \hat{F}_k(s)ik \cdot (\mathcal{F}_{x,v}G)(t,s,l,k(t-s))ds.$$  

Changing $l$ into $l + k$ we can rewrite this expression as

$$K_G F(t,x) = \sum_l e^{ik \cdot x} \left( \sum_k \int_0^t \hat{F}_k(s)ik \cdot (\mathcal{F}_{x,v}G)(t,s,l + k,k(t-s))ds \right).$$

From the Bessel-Parseval identity, we infer that

$$\|K_G\|^2_{L^2} = \sum_l \left( \sum_k \int_0^t \hat{F}_k(s)ik \cdot (\mathcal{F}_{x,v}G)(t,s,l + k,k(t-s))ds \right)^2.$$  

By using Cauchy-Schwarz for $t$ and $k$, we have

$$\|K_G\|^2_{L^2} \lesssim \sum_l \left( \sum_k \int_0^t |\hat{F}_k(s)|^2 |k \cdot (\mathcal{F}_{x,v}G)(t,s,l + k,k(t-s))|ds \right)^{\frac{1}{2}}.$$  

\[12\]
and by integrating in time, we obtain that
\[ \|K\|_{L^2(0,T),L^2}^2 \leq \sum_k \int_0^T \int_0^t \sum_s |\hat{F}_k(s)|^2 |k \cdot (\mathcal{F}_{x,v} G)(t, s, l - k, k(t - s))| ds dt \]
\[ \cdot \sup_{t \in [0,T]} \sup_{l \in [0,T]} \int_0^t \sum_k |k \cdot (\mathcal{F}_{x,v} G)(t, s, l - k, k(t - s))| ds \leq I \cdot II. \]

Let us first consider the term II. We observe that for all \( s_1 \geq 0 \),
\[ \sup_{t \in [0,T]} \sup_{l \in [0,T]} \int_0^t \sum_k |k \cdot (\mathcal{F}_{x,v} G)(t, s, l - k, k(t - s))| ds \leq \sup_{t \in [0,T]} \sup_{k} \left( \sup_{0 \leq s \leq t} \xi \sup_{0 \leq s \leq t} \left[ (1 + |\xi|)^{s_1} |(\mathcal{F}_{x,v} G)(t, s, l - k, \xi)| \right] \right) \int_0^t \frac{|k|}{(1 + |k|(t - s))^{s_1}} ds. \]

We choose \( s_1 > 1 \). Changing variables, we observe that
\[ \int_0^t \frac{|k|}{(1 + |k|(t - s))^{s_1}} ds \leq \int_0^\infty \frac{1}{(1 + \tau^{s_1})} d\tau < \infty. \]

It follows that
\[ II \lesssim \sup_{t \in [0,T]} \sum_k \sup_{0 \leq s \leq t} \xi \sup_{0 \leq s \leq t} \left[ (1 + |\xi|)^{s_1} |(\mathcal{F}_{x,v} G)(t, s, l - k, \xi)| \right]. \]

Next we choose \( s_2 > d/2 \) so that \( \sum \frac{1}{(1+|k|)^{s_2-d}} < \infty \). By using Cauchy-Schwarz, we deduce
\[ II \lesssim \sup_{t \in [0,T]} \left( \sum_k \sup_{0 \leq s \leq t} \xi \sup_{0 \leq s \leq t} \left[ (1 + |\xi|)^{s_2} (1 + |\xi|)^{s_1} |(\mathcal{F}_{x,v} G)(t, s, l - k, \xi)| \right] \right)^{\frac{1}{2}} \lesssim \|G\|_{T,s_1,s_2}. \]

Now let us consider the other term I. By using Fubini, we have for all \( \alpha \in (0,1) \),
\[ \sum_k \int_0^T \int_0^t \sum_s |\hat{F}_k(s)|^2 |k \cdot (\mathcal{F}_{x,v} G)(t, s, l - k, k(t - s))| ds dt \]
\[ = \int_0^T \sum_k |\hat{F}_k(s)|^2 \int_s^T \sum_l |(\mathcal{F}_{x,v} G)(t, s, l - k, k(t - s))| dt ds \]
\[ = \int_0^T \sum_k |\hat{F}_k(s)|^2 (1 + |k|^\alpha) \int_s^T \sum_l \frac{|k|}{(1 + |k|^\alpha)} |(\mathcal{F}_{x,v} G)(t, s, l - k, k(t - s))| dt ds \]
\[ = \|F\|_{L^2(0,T),H^\alpha_T}^2 \sup_{k \leq s \leq T} \int_s^T \sum_l \frac{|k|}{(1 + |k|^\alpha)} |(\mathcal{F}_{x,v} G)(t, s, l - k, k(t - s))| dt. \]

As previously, by choosing \( s_1 > 1, s_2 > d/2 \), we have
\[ \sup_{k \leq s \leq T} \int_s^T \sum_l \frac{|k|}{(1 + |k|^\alpha)} |(\mathcal{F}_{x,v} G)(t, s, l - k, k(t - s))| dt \]
\[ \leq \sup_{k \leq s \leq T} \int_s^T (1 + |k|^\alpha)(1 + |k|(t - s))^{s_1} \sum_l \sup_{0 \leq s \leq t} \left[ (1 + |\xi|)^{s_1} |(\mathcal{F}_{x,v} G)(t, s, l - k, \xi)| \right] dt \]
\[ \lesssim \sup_{k \leq s \leq T} \int_s^T (1 + |k|^\alpha)(1 + |k|(t - s))^{s_1} \|G\|_{T,s_1,s_2}. \]

To conclude, we have to estimate \( \sup_{k \leq s \leq T} \int_s^T \frac{|k|}{(1 + |k|^\alpha)(1 + |k|(t - s))^{s_1}} dt \). We first use a change of variable
\[ \frac{1}{k}(1 + |k|^\alpha) \int_0^{k(T-s)} \frac{1}{(1 + \tau)^{s_1}} d\tau \]
\[ \leq \frac{1}{k} \sup_{0 \leq s \leq T} \int_0^{k(T-s)} \frac{1}{(1 + \tau)^{s_1}} d\tau \]
\[ \leq \frac{1}{k} \int_0^{kT} \frac{1}{(1 + \tau)^{s_1}} d\tau. \]
Then, by Holder inequality, we have the estimate
\[
\sup_k \frac{1}{(1 + |k|^\alpha)} \int_0^{[k]T} \frac{1}{(1 + \tau)^{s_1}} d\tau \\
\leq \sup_k \frac{1}{(1 + |k|^\alpha)} \times (|k|T)^\alpha \left( \int_0^{\infty} \frac{1}{(1 + \tau)^{1-\alpha}} d\tau \right) \overline{1-\alpha} \\
\lesssim T^\alpha.
\]
We have finally shown that
\[
I \lesssim T^\alpha \|G\|_{T,s_1,s_2} \|F\|_{L^2([0,T],H^2)},
\]
which ends the proof. 

3.5 Conclusion of the estimates for \( \rho_\varepsilon \)

By Lemma \( \text{[not numbered]} \)
\[
\partial_t^I \rho_\varepsilon = \sum_{K \in \{1, \ldots, d\}^m} K_{H(K,0),(I,0)} \left( U * \left( (I - \varepsilon^2 \Delta)^{-1} \partial_x^K \rho \right) \right) + R_{I,0},
\]
with
\[
\|R_{I,0}\|_{L^2([0,T],L^2)} \lesssim T^{\frac{3}{2}} \Lambda(T, R).
\]
Let \( \alpha > 0 \) be given by Assumption (A1). Notice that if \( \alpha \geq 1 \), Assumption (A1) will be satisfied for any real inferior to \( \alpha \), thus we can assume in the following that \( \alpha < 1 \).

We get, thanks to Proposition \( \text{[not numbered]} \)
\[
\left\| \frac{K_{H(K,0),(I,0)}}{\Lambda(T, R)} \right\|_{L^2([0,T],L^2)} \lesssim T^{\frac{3}{2}} \Lambda(T, R).
\]
We emphasize that this inequality is independent of \( \varepsilon \). We can thus estimate all of the \( \partial_t^I \rho_\varepsilon \) with \( |I| \leq m \). Summing up all of the results we have, we have proved that
\[
\|\rho_\varepsilon\|_{L^2([0,T],H^2)} \leq T^{\alpha/2} \Lambda(T, R).
\]

4 End of the proof of Theorem 2

The previous sections have given all the tools to end the proof. Indeed, Section 3 has been dedicated to show that
\[
\|\rho_\varepsilon\|_{L^2([0,T],H^2)} \leq T^{\frac{3}{2}} \Lambda(T, R).
\]
We have shown at the end of Section 2 that
\[
\|f_\varepsilon\|_{L^\infty([0,T],H^m_{2r}(-1))} \leq \|f_0\|_{H^m_{2r}(-1)} + T^{\frac{3}{2}} \Lambda(T, R),
\]
from which we deduce that for all \( 0 < T < T^* \)
\[
N_m,2r(T, f_\varepsilon) \leq \|f_0\|_{H^m_{2r}(-1)} + T^{\frac{3}{2}} \Lambda(T, R) + T^{\frac{3}{2}} \Lambda(T, R).
\]
Next, we consider \( R \) large enough such that
\[
\frac{1}{2} R > \|f_0\|_{H^m_{2r}(-1)}.
\]
With \( R \) being fixed, we can find by continuity a \( T^* \) small enough such that the previous estimates are satisfied and that for every \( T \in [0,T^*] \), we have
\[
T^{\frac{3}{2}} \Lambda(T, R) + T^{\frac{3}{2}} \Lambda(T, R) < \frac{1}{2} R.
\]
Thus, for every \( T \in [0,T^*] \), \( N_m,2r(t, f_\varepsilon) < R \) and so \( T^* > T^* \). This means that the time \( T^* \) is uniformly bounded from below by a certain \( T^* \) which is independent of \( \varepsilon \). That concludes the proof of Theorem 2.
5 Solutions for the limit problem

Let us recall the limit system, which is the system we want to study in Theorem 1

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f &= 0, \\
E &= -\nabla_x \int_{\mathbb{R}^d} U(x-y) \left( \int_{\mathbb{R}^d} f(t, y, v) dv \right) dy, \\
f(0, x, v) &= f^0(x, v).
\end{align*}
\]

This system is a non linear transport equation. But taking the formal limit \( \varepsilon \to 0 \), we have lost the elliptic propriety that we had for the system 1.4. Proving local well-posedness is thus more challenging, and we can use Theorem 2 to find a solution of 1.2.

Indeed, this theorem gives us a family of functions \( f_{\varepsilon} \) solutions of 1.7 on a time interval \([0, T]\), \( T \) being independent of \( \varepsilon \). We can use this family of functions to find a solution of the limit system. In the following, we will extract a subsequence that will converge to a certain function \( f \). We next have to show that \( f \) is actually a solution of the limit system. We will start by showing the uniqueness, by similar arguments that we used in Theorem 2.

5.1 Uniqueness of the solution

**Proposition 5.** Let \( f_1, f_2 \in C([0, T], H^{m_0}_2) \) with \( m > m_0 \) and \( 2r > r_0 \) be two solutions of 1.2 with the same initial condition \( f^0 \). We write \( \rho_i = \int f_i dv \) and we suppose that \( \rho_i \in L^2([0, T], H^{m_0}_2) \). Then \( f_1 = f_2 \) on \([0, T] \times \mathbb{T}^d \times \mathbb{R}^d\).

**Proof.** Let \( f = f_1 - f_2 \). We denote by \( E_2 \) the force field associated with the density \( \rho_2 \) and \( E \) associated with \( \rho = \rho_1 - \rho_2 \). We have that \( f \) solves the equation

\[
\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f_1 + E_2 \cdot \nabla_v f = 0.
\]

Let \( \Phi \) satisfying the Burgers equation 3.2 associated with \( E_2 \), with initial condition \( \Phi(0, x, v) = v \). As previously, we write \( g(t, x, \Phi(t, x, v)) = f(t, x, v) \). We obtain that \( g \) is solution of

\[
\partial_t g + \Phi \cdot \nabla_x g + E \cdot \nabla_v f_1 = 0.
\]

Recall that

\[
\int_{\mathbb{R}^d} g(t, x, v) J(t, x, v) dv = \rho(t, x),
\]

for \( J(t, x, v) = | \det \nabla_v \Phi(t, x, v) | \).

We consider the characteristics \( X \) satisfying

\[
\partial_s X(t, s, x, v) = \Phi(t, X(t, s, x, v), v), \quad X(s, s, x, v) = x.
\]

Following the steps of Lemma 8 we have

\[
\rho(t, x) = K_H(U \ast \rho),
\]

with \( H(t, s, x, v) = (\nabla_v f_1) (s, x - (t-s)v, \Psi(t, s, x, v)) J(t, x, \Psi(t, s, x, v), t, s, x, v) \).

We use proposition 4 to show that

\[
\| \rho \|_{L^2([0, T], L^2_2)} \leq CT^{\alpha/2}\| (U \ast \rho) \|_{L^2([0, T], H^2_2)} \leq CT^{\alpha/2}\| \rho \|_{L^2([0, T], L^2_2)}.
\]

We take \( T_0 \) such that \( CT_0^{\alpha/2} < 1 \), and deduce that we must have \( \rho = 0 \) on \([0, T_0] \). Next, we go back to 5.1 on \([0, T_0] \)

\[
\partial_t f + v \cdot \nabla_x f + E_2 \cdot \nabla_v f = 0.
\]

\( f \) is solution of an homogeneous transport equation, with initial condition being 0. That means that \( f = 0 \) on all \([0, T_0] \). To obtain this result on \([0, T] \), we make the same reasoning on \([T_0, T] \) to obtain the result on \([0, 2T_0] \) and so on. 

\[\square\]
5.2 Existence of the solution

All is left to do to prove Theorem 1 is to show the existence of solutions. Let \( (f_\varepsilon) \) be the family of functions solutions of (1.7) with the initial conditions \( f_\varepsilon^0 = f^0 \). Thanks to Theorem 2, there exist \( T \) and \( R \) independent of \( \varepsilon \) such that \( f_\varepsilon \in C([0, T], H^m_{2r}) \) satisfies

\[
\sup_{\varepsilon \in (0, 1]} N_{m, 2r}(T, f_\varepsilon) \leq R.
\]

We get by (5.2) that \( f_\varepsilon \) is uniformly bounded in \( C([0, T], H^{m-1}_{2r}) \). By (5.2), we obtain that \( \partial_t f_\varepsilon \) is uniformly bounded in \( L^\infty([0, T], H^{m-\frac{1}{2}}_{2r}) \). The Ascoli theorem gives the existence of a function \( f \in C([0, T], L^2_{x,v}) \) and a sequence \( f_{\varepsilon_n} \) such that \( f_{\varepsilon_n} \) converges to \( f \) in \( C([0, T], L^2_{x,v}) \). By interpolation, we actually have convergence in \( C([0, T], H^{m-1-\delta}_{2r}) \) for all \( \delta > 0 \). By Sobolev embedding, \( f_{\varepsilon_n} \) converges to \( f \) in \( L^\infty([0, T] \times \mathbb{T}^d \times \mathbb{R}^d) \) and \( \rho_{\varepsilon_n} \) converges to \( \rho = \int f \, dv \) in \( L^2([0, T], L^2_{x}) \cap L^\infty([0, T] \times \mathbb{R}^d) \). Thus, the limit function \( f \) solves the system (1.2).

To conclude, we want to apply Proposition 5. We only have to show that \( f \in C([0, T], H^m_{2r}) \) and that \( \rho \in L^2([0, T], H^m_{x}) \). We then use an energy estimate which have been shown previously (the formula (2.8)) to show that

\[
\frac{d}{dt} \| f \|^2_{H^m_{2r}} \leq C.
\]

It follows that \( f \in C([0, T], H^m_{2r}) \). We can finally apply Proposition 5 which gives the uniqueness of \( f \) and concludes the proof of Theorem 1.

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