MEASURABILITY, SPECTRAL DENSITIES AND HYPERTRACES
IN NONCOMMUTATIVE GEOMETRY

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Abstract. We introduce, in the dual Macaev ideal of compact operators of a Hilbert space, the spectral weight $\rho(L)$ of a positive, self-adjoint operator $L$ having discrete spectrum away from zero. We provide criteria for its measurability and unitarity of its Dixmier traces ($\rho(L)$ is then called spectral density) in terms of the growth of the spectral multiplicities of $L$ or in terms of the asymptotic continuity of the eigenvalue counting function $N_L$. Existence of meromorphic extensions and residues of the $\zeta$-function $\zeta_L$ of a spectral density are provided under summability conditions on spectral multiplicities. The hypertrace property of the states $\Omega_L(\cdot) = \text{Tr}_\omega(\rho(L))$ on the norm closure of the Lipschitz algebra $A_L$ follows if the relative multiplicities of $L$ vanish faster than its spectral gaps or if $N_L$ is asymptotically regular.

1. Introduction

Trace theorems for unbounded Fredholm modules $(A, h, D)$, alias K-cycles or spectral triples, subject to various summability behaviours, date back to the dawning of Noncommutative Geometry [Co1], [Co2], [Co3]. They were proved under finite summability in [Co2] and hold true also under summability in the dual Macaev ideal in [CGS]. They were used to deduce hyperfiniteness of weak closure of the *-algebra $A$ in certain representations and to rule out the existence of unbounded Fredholm modules or quasidiagonal approximate units in normed ideals, with specific summability conditions (see [Co2], [V]). Also, an hypertrace constructed by $(A, h, D)$ provides a Hilbert bimodule, unitary representation of the universal graded differential algebra $\Omega^*(A)$ ([Co3] Chapter 6.1 Proposition 5).

Here we associate a spectral weight $\rho(|D|)$ in the dual Macaev ideal $L^{(1,\infty)}(h)$, to any unbounded Fredholm module $(A, h, D)$ and, more in general, to any filtration $\mathcal{F}$ of a Hilbert space $h$ (in the sense of [V]). The spectral weight $\rho(|D|)$ depends, in particular, on the spectral multiplicities of $D$ but not on the location of its eigenvalues.

Under an assumption of non exponential growth of the filtration, we show measurability of $\rho(|D|)$ and under the asymptotic continuity of the eigenvalue counting function $N_{|D|}$, we prove also the unitarity $\text{Tr}_\omega(\rho(|D|)) = 1$ of the Dixmier traces. In this situation $\rho(|D|)$ is called the spectral density of $D$ and one may deal with the volume states $\Omega_{|D|}(a) := \text{Tr}_\omega(a \cdot \rho(|D|))$ on the norm closure $C^*$-algebra $A$ of $A$, provided by any fixed Dixmier ultrafilter $\omega$.

In commutative terms, i.e. dealing with the standard spectral triple $(C^\infty(M), L^2(Cl(M)), D)$ of a compact, closed, Riemannian manifold, taking into account multiplicities only and not the whole spectrum itself, one reconstructs the Riemann probability measure of $M$ loosing information about the volume $V(M)$ and the dimension $d(M)$. On the other hand, this has the advantage to dispense with summability hypotheses and, for example, to recover the unique trace on the reduced $C^*$-algebra $C^*(\Gamma)$ of a finitely generated, countable discrete group $\Gamma$, no matter its growth is. Also, using the density $\rho(|D|)$, one is able to treat, on the same foot,
situations like Euclidean domains of infinite volume whose Dirichlet-Laplacian has discrete spectrum or certain hypoelliptic \( \Psi \)DO on compact manifolds, where the asymptotics of the spectrum of \( D \) is not à la Weyl.

Under summability conditions on the spectral multiplicities of \(|D|\), the \( \zeta \)-function \( \zeta_{|D|} \) of the spectral density \( \rho(|D|) \) is shown to be meromorphic on an half plane containing \( z = 1 \) and that its residue is there unitary.

Finally, we show that the volume states \( \Omega_{|D|} \) are hypertraces on \( A \) provided \( N_{|D|} \sim \varphi \) for a locally Lipschitz function \( \varphi \) such that the essential limit of \( \varphi'/\varphi \) vanishes at infinity. This condition is satisfied when the sequence of relative multiplicities of \(|D|\) vanish faster than the sequence of its spectral gaps.

The work is organized as follows. In Section 2 we introduce the spectral weight of a nonnegative, self-adjoint operator \( L \) having discrete spectrum away from zero. Its measurability is proved in terms of the sub-exponential growth of its spectral multiplicities, as a consequence of the asymptotic continuity of the counting function \( N_L \) or as a byproduct of the nuclearity of the semigroup \( e^{-tL} \). In Section 3 we prove existence of analytic extensions and residues of the \( \zeta \)-function \( \zeta_L \) of a density \( \rho(L) \), in terms of summability of the multiplicities of \( L \). In Section 4, the volume states \( \Omega_L(\cdot) = \text{Tr}_\omega(\rho(L)) \) are introduced and in Section 6 we show that they are hypertraces on the Lipschitz algebra \( A_L \), under asymptotic smoothness of the counting function \( N_L \) or when the relative multiplicities vanish faster than the spectral gaps of \( L \).

Section 5 is dedicated to various examples concerning i) \( k \)-cycles on compact manifolds given by (hypo)elliptic \( \Psi \)DO \( L \) ii) \( k \)-cycles on the \( \mathcal{C}^* \)-algebra \( \mathcal{P}(\mathcal{M}) \) of scalar, 0-order \( \Psi \)DO, which are associated to scalar, 1-order \( \Psi \)DO \( L \) iii) \( k \)-cycles associated to multiplication operators on the group \( \mathcal{C}^* \)-algebra of countable discrete groups iv) Dirichlet-Laplacians of Euclidean domains of infinite volume v) Kigami-Laplacians on selfmilar fractals vi) the Toeplitz \( \mathcal{C}^* \)-algebra generated by an isometry and the canonical multiplication operator \( L \) on natural \( \mathbb{N} \) and prime numbers \( \mathbb{P} \) vii) unbounded Fredholm modules built using Hilbert space filtrations.

In the final subsection 6.5, the structure of the volume states \( \Omega_L \) on \( \mathcal{C}^* \)-algebras extensions is briefly outlined.

## 2. Measurable densities associated to operators with discrete spectrum

In this section we define the spectral weight of a nonnegative, self-adjoint operator \( (L, D(L)) \) on a Hilbert space \( h \), having discrete spectrum away from zero and investigate its measurability in the framework of Connes’ NCG. We always keep in mind the situation where \( L = |D| \) for a spectral triple \((\mathcal{A}, D, h)\).

### 2.1. Eigenvalue counting function and multiplicities

In this section, we consider a densely defined, nonnegative, unbounded, self-adjoint operator \( (L, D(L)) \) on a Hilbert space \( h \) with spectrum \( sp(L) \) and spectral measure \( E^L \).

Letting \( P_0 := E^L(\{0\}) \) be the orthogonal projection onto the kernel of \( L \), we fix the following notations of functional calculus: by convention, for a measurable function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \), the operator \( f(L) \) will be 0 on the subspace \( P_0(h) = \ker L \) and \( f(L(I - P_0)) \) on the subspace \( (I - P_0)h = (\ker L)^\perp \). For example, with this convention, for \( s > 0 \), \( L^{-s} \) is the nonnegative, densely defined operator defined as 0 on \( P_0(h) \) and \( (L(I - P_0))^{-s} \) on \( (I - P_0)h \).

In this section we shall suppose that \( L \) has \textit{discrete spectrum off of its kernel} in the sense that \( sp(L) \setminus \{0\} \) is discrete: this is equivalent to say that \( L^{-1} \) is a compact operator in \( \mathcal{B}(h) \).
We shall adopt two alternative ways for describing the spectrum, out of its kernel:

**First way:** \( sp(L) \setminus \{0\} = \{0 < \lambda_1(L) \leq \cdots \leq \lambda_n(L) \leq \cdots \} \)

where the eigenvalues \( \lambda_n(L) \) are numbered increasingly with repetition according to their multiplicity.

**Second way:** \( sp(L) \setminus \{0\} = \{0 < \tilde{\lambda}_1(L) < \cdots < \tilde{\lambda}_k(L) < \cdots \} \)

where the distinct eigenvalues \( \tilde{\lambda}_k(L) \) are numbered increasingly.

Since \( L \) is assumed unbounded, we have \( \lim_{n \to \infty} \lambda_n(L) = \lim_{k \to \infty} \tilde{\lambda}_k(L) = +\infty \).

The **multiplicity** of the eigenvalue \( \tilde{\lambda}_k(L) \) is denoted by \( m_k := \text{Tr}(E^L(\{\tilde{\lambda}_k\})) \) while the **cumulated multiplicity** is defined as \( M_k := \text{Tr}(E^L((0, \tilde{\lambda}_k])) \) so that \( M_k = N_L(\tilde{\lambda}_k(L)) = \sum_{j=1}^k m_j \).

By convention, \( M_0 := 0 \). We will refer to the ratio \( m_k/M_k \) as the **relative multiplicity** of the eigenvalue \( \tilde{\lambda}_k(L) \). The two labellings correspond through the relation:

\[
\lambda_n(L) = \tilde{\lambda}_k(L), \quad M_{k-1} < n \leq M_k.
\]

**Remark 2.1.** We shall adopt the simplified notations \( \lambda_1, \cdots, \lambda_n, \cdots \) and \( \tilde{\lambda}_1, \cdots, \tilde{\lambda}_k, \cdots \) whenever no confusion can arise.

The **eigenvalue counting function** \( N_L : \mathbb{R}_+ \to \mathbb{N} \) is defined as

\[
N_L(x) := \text{Tr}(E^L((0, x])) = \sharp\{n \in \mathbb{N}^* \mid \lambda_n(L) \leq x\}
\]

where \( \text{Tr} \) is the normal, semifinite trace on \( B(h) \).

Let us summarize some basic properties of the counting function

**Lemma 2.2.**

i) \( N_L \) is a non decreasing function, right continuous with left limits. For \( x \in \mathbb{R}_+ \), let us denote \( N_L^-(x) = \lim_{\delta \downarrow 0} N_L(x - \delta) \) the left limit function of \( N_L \).

ii) \( N_L(x) = M_k \) for \( \tilde{\lambda}_k(L) \leq x < \tilde{\lambda}_{k+1}(L) \).

iii) \( \limsup_{x \to +\infty} \frac{N_L^-(x)}{N_L^-(x)} = \limsup_{k \to +\infty} \frac{M_k}{M_{k-1}}. \)

**Proof.** Properties i) and ii) are obvious from the definition of \( N_L \). For iii), it is enough to observe that, for \( x \notin sp(L) \), \( N_L^-(x) = N_L(x) \) and that, for \( x = \tilde{\lambda}_k(L) \), \( N_L(x) = M_k \) while \( N_L^-(x) = M_{k-1} \). 

\( \square \)

### 2.2. Spectral weights.

**Definition 2.3.** (Spectral weights) The operator defined as

\[
\rho(L) := N_L(L)^{-1}
\]

will be called the **spectral weight of** \( L \). As \( sp(L) \setminus \{0\} \) is discrete and unbounded and \( N_L \) is nondecreasing, it follows that \( \rho(L) \) is nonnegative and compact.

**Proposition 2.4.**

i) The eigenvalues of the spectral weight are given by

\[
\mu_n(\rho(L)) = N_L(\lambda_n(L))^{-1} = \frac{1}{M_k} \quad \text{for} \quad M_{k-1} < n \leq M_k \quad \text{and} \quad k \geq 1;
\]

ii) we also have the bounds

\[
(2.1) \quad \frac{M_{k-1}}{M_k} \cdot \frac{1}{n} < \mu_n(\rho(L)) \leq \frac{1}{n} \quad \text{for} \quad M_{k-1} < n \leq M_k \quad \text{and} \quad k \geq 1.
\]
so that we can write as well
\[ \lambda_n(L) = \tilde{\lambda}_k(L) \] and \( N_L(\lambda_n(L)) = N_L(\tilde{\lambda}_k(L)) = M_k. \) From one side, \( M_k \geq n \) and thus \( N_L(\lambda_n(L)) \geq n. \) On the other side, \( n > M_{k-1} \) and thus \( N_L(\lambda_n(L))^{-1} = M_k^{-1} > \frac{1}{n} \cdot \frac{M_{k-1}}{M_k}. \)

2.3. Weights by filtrations. For any Borel measurable, strictly increasing function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \lim_{x \to \infty} f(x) = +\infty, \) we have \( \rho(f(L)) = \rho(L). \) In fact \( \rho(L) \) depends on \( L \) only through the Hilbert space filtration of spectral subspaces of \( h \)

\[ \{E^L((0, \tilde{\lambda}_k(L))]\}_{k=1}^{+\infty}. \]

In [V], Proposition 5.1, D.V. Voiculescu, motivated by the existence of quasicentral approximate units relative to normed ideals, provided a general construction of spectral triples \((A, h, D)\) on a C*-algebra \( A, \) represented in a Hilbert space \( h, \) associated to given filtrations \( h_0 \subset h_1 \subset \cdots \subset h. \) He consider the filtration of \( A \) given by \( V_k := \{T \in A : T(h_j) \cup T^*(h_j) \subseteq h_{j+k}, \forall j \in \mathbb{N} \} \) for \( k \in \mathbb{N}, \) assuming that \( A := \bigcup_{k \in \mathbb{N}} V_k \) is dense in \( A. \) Denoting by \( P_j \) the projection onto \( h_j, \) the Dirac operator is defined as \( D := \sum_{j \geq 1} (I - P_j) \) (so that \( D = |D|). \)

The spectrum of \( D \) is \( \mathbb{N} \) with cumulated multiplicities \( M_j := \dim(h_j) \) and the spectral weight is given by

\[ \rho(D) = \sum_{k \geq 1} \frac{1}{M_k} \cdot (P_{k+1} - P_k). \]

2.4. Dixmier traces. In this work we shall consider the non normal traces associated with ultrafilters on \( \mathbb{N}, \) introduced by J. Dixmier [Dix], making use, in particular, of the approach proposed in [CPS].

We shall start with a state \( \omega \) on \( L^\infty(\mathbb{R}_+^*) \) having all the properties of [CPS] Theorem 1.5.

First, \( \omega \) is a limit process at \(+\infty\) in the sense that

\[ \text{ess} \lim_{t \to +\infty} f(t) \leq \omega(f) \leq \text{ess} \lim_{t \to +\infty} f(t) \quad f \in L^\infty(\mathbb{R}_+^*) \]

so that we can write as well \( \omega(f) := \omega - \text{lim}_{t \to +\infty} f(t). \)

Then we require this limit process satisfies the following invariance properties :

1. \( \omega - \lim_{t \to +\infty} f(st) = \omega - \lim_{t \to +\infty} f(t), \) \( f \in L^\infty(\mathbb{R}_+^*), s \in \mathbb{R}_+^* \)
2. \( \omega - \lim_{t \to +\infty} f(t^s) = \omega - \lim_{t \to +\infty} f(t), \) \( f \in L^\infty(\mathbb{R}_+^*), s \in \mathbb{R}_+^* \)
3. \( \omega - \lim_{t \to +\infty} \frac{1}{\log(t)} \int_1^t f(s) \frac{ds}{s} = \omega - \lim_{t \to +\infty} f(t), \) \( f \in L^\infty(\mathbb{R}_+^*). \)

To such \( \omega \) are associated an ultrafilter on \( \mathbb{N}, \) still denoted \( \omega, \) by

\[ \omega - \lim_{n \to +\infty} f(n) = \omega - \lim_{t \to +\infty} f([t]) \quad f \in \ell^\infty(\mathbb{N}) \text{ with } [t] = \text{integer part of } t. \]

The associated Dixmier Trace is defined on the dual Macaev ideal of compact operators

\[ L^{(1,\infty)}(h) := \{T \in \mathcal{K}(h) : \sup_{N \geq 2} \frac{1}{\log N} \sum_{n=1}^N \mu_n(|T|) < +\infty\} \]

as

\[ Tr_\omega(T) = \omega - \lim_{N \to +\infty} \frac{1}{\log N} \sum_{n=1}^N \mu_n(T) \quad T \in L^{(1,\infty)}(h)_+. \]
The asymptotic behaviour of $\omega_\infty(T \to \omega)$ means (see Proposition 6 Chapter 4.2).

To $\omega$ on $L^\infty(\mathbb{R}^*_+)$ as above is associated an alternative limit process $\omega$ on $\mathbb{R}$ defined as

$$\omega - \lim_{t \to +\infty} f(t) = \omega - \lim_{t \to +\infty} f(\log(t)) \quad f \in L^\infty(\mathbb{R}).$$

The asymptotic behaviour of $\frac{1}{\log N} \sum_{n=1}^{N} \mu_n(T)$ and the limit behaviour of $(s-1)Tr(T^s)$ as $s \to 1^+$ are related by the following equality [CPS, Theorem 3.1]

$$(2.3) \quad Tr_\omega(T) = \omega - \lim_{r \to +\infty} \frac{1}{r} Tr(T^{1+\frac{1}{r}}) \quad T \in \mathcal{L}^{(1,\infty)}(h)_+.$$ 

Furthermore with $T$ as above and any $A \in \mathcal{B}(h)$, we have (see [CPS] Theorem 3.8)

$$(2.4) \quad Tr_\omega(1) = \omega - \lim_{r \to +\infty} \frac{1}{r} Tr(A T^{1+\frac{1}{r}}) \quad T \in \mathcal{L}^{(1,\infty)}(h)_+.$$ 

2.5. Asymptotics for the spectral weights.

**Proposition 2.5.** i) The spectral weight belongs to the dual Macaev ideal $\mathcal{L}^{(1,\infty)}(h)$ and

$$\frac{1}{c} \leq Tr_\omega(\rho(L)) \leq 1 \quad \text{for all Dixmier ultrafilter } \omega$$

where

$$c := \limsup_{x \to +\infty} \frac{N_L(x)}{N_L^{-}(x)} = \limsup_{k \to \infty} \frac{M_k}{M_{k-1}}.$$

ii) $\rho(L)^s$ is trace class for all $s > 1$ and $\limsup_{s \to 1}(s-1)Tr(\rho(L)^s) \leq 1$.

iii) If

$$\limsup_{x \to +\infty} \frac{N_L(x)}{N_L^{-}(x)} = \lim_{k \to \infty} \frac{M_k}{M_{k-1}} = 1$$

then we have

iii.a) $\mu_n(\rho(L)) \sim 1/n$ as $n \to \infty$.

iii.b) $Tr_\omega(\rho(L)) = 1$ for all Dixmier ultrafilter $\omega$ and the spectral weight $\rho(L)$ is measurable.

iii.c) $\lim_{s \to 1}(s-1)Tr(\rho(L)^s) = 1$.

**Proof.** i) follows from inequality $\mu_n(\rho(L)) \leq 1/n$ of Proposition 2.4 and from inequality $\liminf_{n \to \infty} \mu_n(\rho(L)) \geq 1/cn$ which follows by (2.1). ii) follows again from inequality $\mu_n(\rho(L)) \leq 1/n$ of Proposition 2.4. iii.a) comes from the double inequality (2.1), while iii.b) and iii.c) are straightforward consequences of iii.a).

**Definition 2.6.** (Spectral densities) The spectral weight $\rho(L)$ will be called spectral density provided it is measurable and $Tr_\omega(\rho(L)) = 1$ for all Dixmier ultrafilter $\omega$.

We may have measurability of a spectral weight even if it is not a density:

**Proposition 2.7.** If $\lim_{k \to \infty} \frac{M_{k+1}}{M_k} = c > 1$, then the spectral weight $\rho(L)$ is measurable and

$$Tr_\omega(\rho(L)) = \lim_{s \to 1}(s-1)Tr(\rho(L)^s) = \frac{c-1}{c \log c} \quad \text{for all Dixmier ultrafilter } \omega.$$
Moreover, the second equality (2.7) implies
\[ k > K \Rightarrow k \leq M^{k-1} \leq M_k \leq (c + \varepsilon)^{k-K} M_k \]
for \( k > K \). For \( N > M_K \), let \( k(N) \) be the integer such that \( M_{k(N)} \leq N \leq M_{k(N)+1} \). One has
\[ \log M_{k(N)} \leq \log N \leq \log (M_{k(N)+1}) \leq \log M_{k(N)} + \log (c + \varepsilon) \]
so that \( \log N = \log M_{k(N)} + O(1) \) as \( N \to +\infty \). Gathering those two results, we get
\[ (2.5) \quad k(N) \log (c - \varepsilon) + O(1) \leq \log N \leq k(N) \log (c + \varepsilon) + O(1), \quad N \to +\infty. \]
Moreover,
\[
\sum_{n=1}^{N} N_L(\lambda_n)^{-1} = \sum_{k=1}^{K-1} M_k \sum_{n=M_{k-1}}^{M_k} N_L(\lambda_n)^{-1} + \sum_{k=K}^{k(N)} M_k \sum_{n=M_{k-1}+1}^{M_k} N_L(\lambda_n)^{-1} + \sum_{n=M_{k(N)+1}}^{N} N_L(\lambda_n)^{-1}
\]
(2.6)
\[ = \sum_{k=1}^{K-1} M_k \sum_{n=M_{k-1}}^{M_k} N_L(\lambda_n)^{-1} + \sum_{k=K}^{k(N)} \frac{M_k - M_{k-1}}{M_k} N_L(\lambda_n)^{-1} + \frac{N - M_{k(N)}}{M_{k(N)+1}} \]
For \( k \geq K \), we have
\[ \frac{c + \varepsilon - 1}{c + \varepsilon} \leq \frac{M_k - M_{k-1}}{M_k} \leq \frac{c - \varepsilon - 1}{c - \varepsilon}, \]
which provides
\[ (2.7) \quad k(N) \frac{c + \varepsilon - 1}{c + \varepsilon} + O(1) \leq \sum_{n=1}^{N} N_L(\lambda_n)^{-1} \leq k(N) \leq \frac{c - \varepsilon - 1}{c - \varepsilon} + O(1), \quad N \to +\infty. \]
With (2.5), this inequality implies
\[
\frac{1}{k(N) \log (c + \varepsilon) + O(1)} \left( k(N) \frac{c + \varepsilon - 1}{c + \varepsilon} + O(1) \right) \leq \frac{1}{\log N} \sum_{n=1}^{N} N_L(\lambda_n)^{-1} \leq \frac{1}{k(N) \log (c - \varepsilon) + O(1)} \left( k(N) \frac{c - \varepsilon - 1}{c - \varepsilon} + O(1) \right)
\]
which provides the first equality \( \text{Tr}_{\omega}(\rho(L)) = \frac{c - 1}{c} \).

The second equality \( \lim \text{Tr}(\rho(L)^s) = \frac{c - 1}{c \log c} \) is obtained through a similar computation. Fix \( \varepsilon > 0 \) and \( K \in \mathbb{N} \) such that \( k \geq K \Rightarrow c - \varepsilon \leq \frac{M_{k+1}}{M_k} \leq c + \varepsilon \), hence \( (c - \varepsilon)^{k-K} M_K \leq M_k \leq (c + \varepsilon)^{k-K} M_K \) for \( k > K \). We have also, for \( k \geq K + 1 \), \( c - \varepsilon \leq \frac{M_k}{M_{k-1}} = \frac{m_k}{M_{k-1}} + 1 \leq c + \varepsilon \), i.e. \( c - 1 - \varepsilon \leq \frac{m_k}{M_{k-1}} \leq c + \varepsilon \) which implies
\[ \frac{c - 1 - \varepsilon}{c + \varepsilon} \leq \frac{m_k}{M_k} \leq \frac{c - 1 + \varepsilon}{c - \varepsilon}. \]

We compute now for \( s > 1 \)
\[
\text{Tr}(\rho(L)^s) = \sum_{k} m_k M_k^{-s} = \sum_{k=1}^{K} m_k M_k^{-s} + \sum_{k=K+1}^{\infty} m_k M_k^{-s} = \sum_{k=1}^{K} m_k M_k^{-s} + \sum_{k=K+1}^{\infty} \frac{m_k}{M_k} M_k^{-s+1}
\]
We have \((s - 1) \sum_{k=1}^{K} m_k M_k^{-s} \to 0\) as \(s \to 1\), while
\[
\sum_{k=K+1}^{\infty} \frac{m_k}{M_k} M_k^{-s+1} \leq \frac{c - 1 + \varepsilon}{c - \varepsilon} (c + \varepsilon)^{(-s+1)(k-K)} M_K^{-s+1} = \frac{c - 1 + \varepsilon}{c - \varepsilon} (c + \varepsilon)^{-s+1} \frac{1}{1 - (c + \varepsilon)^{1-s}}
\]
with \(1 - (c + \varepsilon)^{1-s} \sim (s - 1) \log(c + \varepsilon)\) as \(s \downarrow 1\).
We have proved \(\limsup_{s \downarrow 1} (s - 1) Tr(\rho(L)^s) \leq \frac{c - 1 + \varepsilon}{c - \varepsilon} \frac{1}{\log(c + \varepsilon)}\). A similar computation provides \(\liminf_{s \downarrow 1} (s - 1) Tr(\rho(L)^s) \geq \frac{c - 1 - \varepsilon}{c + \varepsilon} \frac{1}{\log(c - \varepsilon)}\) and the result. \(\square\)

The hypothesis of the above result is verified in discrete free groups (see Example 3.7 below). Here is another criterium for \(\rho(L)\) having a nonzero Dixmier Trace.

**Proposition 2.8.** If \(M_k \sim f(k)\) \((k \to +\infty)\) with \(f \in C^1((0, +\infty))\), then
\[
\limsup_{k \to +\infty} \frac{M_{k+1}}{M_k} \leq e^C \quad \text{with} \quad C := \limsup_{x \to +\infty} \frac{f'(x)}{f(x)}.
\]
Hence \(Tr_\omega(\rho(L)) \geq e^{-C}\) for all Dixmier ultrafilters \(\omega\).

**Proof.** Fix \(\varepsilon > 0\) and choose \(K_\varepsilon \geq 1\) such that for all \(k \geq K_\varepsilon\) we have \(\frac{1 - \varepsilon}{1 + \varepsilon} f(k+1) \leq \frac{f(k+1)}{f(k)} \leq \frac{1 + \varepsilon}{1 - \varepsilon} f(k+1)\). It follows that \(\limsup_{k \to +\infty} \frac{M_{k+1}}{M_k} = \limsup_{k \to +\infty} \frac{f(k+1)}{f(k)}\). Then, setting \(C = \limsup_{x \to +\infty} \frac{f'(x)}{f(x)}\), we have \((\log f)'(x) = \frac{f'(x)}{f(x)} \leq C + \varepsilon\) for \(x\) large enough so that \(\log(f(k+1)) - \log(f(k)) \leq C + \varepsilon\), i.e. \(\frac{f(k+1)}{f(k)} \leq e^{C+\varepsilon}\) for \(k\) large enough too. \(\square\)

2.6. **Asymptotic continuity of the eigenvalue counting function and measurability.**

Here we link the measurability of the spectral weight to the asymptotic continuity of the eigenvalue counting function and to the asymptotic vanishing of the relative multiplicity.

**Proposition 2.9.** i) If the counting function \(N_L\) is asymptotically continuous in the sense that there exists a continuous function \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+\) such that
\(N_L(x) \sim \varphi(x), \; x \to +\infty\),
then
\[
\lim_{x \to +\infty} \frac{N_L(x)}{N_L(x)} = \lim_{k \to +\infty} \frac{M_{k+1}}{M_k} = 1 \quad \text{or, equivalently,} \quad \lim_{k \to +\infty} \frac{M_k}{M_k} = 0.
\]

ii) Conversely, if \(\lim_{x \to +\infty} \frac{N_L(x)}{N_L(x)} = 1\) or \(\lim_{k \to +\infty} \frac{M_{k+1}}{M_k} = 1\) or \(\lim_{k \to +\infty} \frac{M_k}{M_k} = 0\), then \(N_L\) is asymptotically continuous.

In both cases, \(\rho(L)\) is a density and the properties iii) of Proposition 2.7 hold true.
Proof. i) For $\varepsilon > 0$ and $x \in \mathbb{R}_+$ large enough, we have
\[
(1 - \varepsilon) \varphi(x) \leq N_L(x) \leq (1 + \varepsilon) \varphi(x).
\]
As $\varphi$ is continuous, we have as well $(1 - \varepsilon) \varphi(x) \leq N_L^{-}(x)$ so that $N_L^{-}(x) / N_L^{+}(x) \leq 1 + \varepsilon$. This implies $\limsup_{x \to +\infty} \frac{N_L^{+}(x)}{N_L^{-}(x)} \leq \frac{1 + \varepsilon}{1 - \varepsilon}$ for all $\varepsilon > 0$, and finally $\limsup_{x \to +\infty} \frac{N_L^{+}(x)}{N_L^{-}(x)} = 1$. Lemma 2.2 and Proposition 2.4 provides the result.

ii) Choose $\varphi$ continuous, piecewise affine such that $\varphi(\lambda_k) = M_k$. This means
\[
\varphi(x) = M_{k-1} + t(M_k - M_{k-1})
\]
for $x \in [\lambda_{k-1}, \lambda_k]$ and $x = \lambda_{k-1} + t(\lambda_k - \lambda_{k-1})$, $0 \leq t \leq 1$. For such $x$ we have:
\[
\frac{\varphi(x)}{N_L(x)} = \frac{M_{k-1}}{M_k} + t \left(1 - \frac{M_{k-1}}{M_k}\right) \to 1, \quad x \to +\infty.
\]

The condition $1 = \limsup_{k} \frac{M_{k+1}}{M_k} (= \limsup_{k} \frac{M_{k+1}}{M_k})$ ensuring measurability concerns the sub-exponential growth of the spectral multiplicity of $L$. It is in general not weaker than the condition $\limsup_{k} \sqrt[k]{M_k} = 1$ since $\liminf_{k} \frac{M_{k+1}}{M_k} \leq \liminf_{k} \sqrt[k]{M_k} \leq \limsup_{k} \sqrt[k]{M_k} \leq \limsup_{k} \frac{M_{k+1}}{M_k}$.

Combining this results with the Karamata-Tauberian Theorem (in Appendix), we get a criterion of asymptotic continuity of $N_L$ in terms of regularity of the partition function $Z_L$.

**Proposition 2.10.** Suppose the contraction semigroup $\{e^{-tL} : t \geq 0\}$ to be nuclear
\[
Z_L(\beta) := \text{Tr}(e^{-\beta L}) < +\infty \quad \text{for all } \beta > 0
\]
and assume the partition function $Z_L$ to be regularly varying. Then for some $c > 0$ we have
\[
N_L(x) \sim c \cdot Z_L(1/x) \quad x \to +\infty.
\]

In particular, under these assumptions, $N_L$ is asymptotically continuous and $\rho(L)$ is a density.

**Proof.** Apply Karamata-Tauberian Theorem to the measure $\mu := \text{Tr} \circ E^L$ and then apply Proposition 2.9. \qed

The result may applied to $\theta$-summable spectral triples $(\mathcal{A}, h, D)$, where $Z_{D^2}(\beta) = \text{Tr}(e^{-\beta D^2}) < +\infty$ for all $\beta > 0$, provided the partition function $Z_{D^2}$ is regularly varying, as a consequence of the identity $N_L(x) = N_{D^2}(x^2)$ for all $x > 0$.

**Remark 2.11.** (Physical interpretation of nuclearity and regularity) In applications $L$ may represent the Hamiltonian of a quantum system. The nuclearity assumption on the semigroup $\{e^{-\beta L} : \beta > 0\}$ is easily seen to be equivalent to the requirement that the mean value of the energy in the Gibbs equilibrium state is finite and non vanishing at any temperature
\[
\langle L \rangle_{\beta} = -\frac{\dot{Z}_L(\beta)}{Z_L(\beta)} = \frac{\text{Tr}(L e^{-\beta L})}{\text{Tr}(e^{-\beta L})} \quad \beta > 0.
\]

The hypothesis that $Z_L$ is regularly varying requires that for some $\gamma \in \mathbb{R}$
\[
\lim_{\beta \to 0^+} \frac{Z_L(s\beta)}{Z_L(\beta)} = s^\gamma \quad s > 0.
\]
If $\hat{Z}_L$ is regularly varying, say $\hat{Z}_L(s\beta) \sim s^{-1}\hat{Z}_L(\beta)$ for some $\gamma \in \mathbb{R}$ as $\beta \to 0^+$, by de l’Hospital theorem, also $Z_L$ is regularly varying $\lim_{\beta \to 0^+} \frac{Z_L(s\beta)}{Z_L(\beta)} = s \lim_{\beta \to 0^+} \frac{\hat{Z}_L(s\beta)}{\hat{Z}_L(\beta)} = s^{-\gamma}$ and the mean energy $\langle L \rangle$ is regularly varying too with

$$\langle L \rangle_{s\beta} = -\frac{\hat{Z}_L(s\beta)}{Z_L(s\beta)} \sim -\frac{s^{-1}\hat{Z}_L(\beta)}{s^{-\gamma}Z_L(\beta)} = \frac{1}{s} \langle L \rangle_{\beta}, \quad \text{as } \beta \to 0^+, \quad \text{for any fixed } s \geq 0.
$$

### 3. Meromorphic extensions of zeta functions and residues

The $\zeta$-function of $\rho(L)$ is defined as

$$\zeta_L(s) := \text{Tr}(\rho(L)^s) = \sum_{n \geq 1} \mu_n(\rho(L))^s = \sum_{k \geq 1} m_k \cdot M_k^{-s}$$

for all $s \in \mathbb{C}$ for which the series converges. Its domain and its analytic properties will be found by comparison with the Riemann $\zeta$-function

$$\zeta_0(s) = \sum_{n \geq 1} n^{-s},$$

which, initially defined on the half-plane $\{s \in \mathbb{C} : \Re(s) > 1\}$, is then extended analytically to $\mathbb{C} \setminus \{1\}$. Recall that $s = 1$ is simple pole for $\zeta_0$ with unital residue.

The following criteria for the asymptotic properties of the $\zeta$-function $\zeta_L(s)$ as $s \to 1$ are based on various growth rates of the spectral multiplicity $m_k$ as $k \to \infty$.

#### 3.1. Criteria for meromorphic extensions of the $\zeta$-functions $\zeta_L$.

**Lemma 3.1.** For $\varepsilon \in [0, 1)$ and $s \in \mathbb{C}$ such that $\Re(s) \geq 0$, we have

$$|1 - (1 - \varepsilon)^s| \leq |s| \log((1 - \varepsilon)^{-1}).$$

**Proof.** Setting $b := \log(1 - \varepsilon)^{-1}$, $x := \Re(s) \geq 0$, $f(t) := (1 - \varepsilon)^{ts} = e^{-b st}$ for $t \in [0, 1]$, we have

$$|1 - (1 - \varepsilon)^s| = |f(1) - f(0)| = \left| \int_0^1 f'(t) \, dt \right| = \left| -bs \int_0^1 e^{-b st} \, dt \right| \leq b |s| \cdot \int_0^1 |e^{-b t}| \, dt \leq b |s|.$$

\qed

**Proposition 3.2.** i) If $N_L$ is asymptotically continuous then the $\zeta$-function $\zeta_L$ is well defined on the half-plane $\{s \in \mathbb{C} : \Re(s) > 1\}$ and it admits the limit

$$\lim_{s \in \mathbb{R}, s \downarrow 1} (s - 1) \text{Tr}(\rho(L)^s) = 1. \quad (3.1)$$

ii) If $\sum_{k} \frac{m_k^2}{M_k^2} < +\infty$, then $\zeta_L$ is analytic on $\{s \in \mathbb{C} : \Re(s) > 1\}$ and it admits the limit

$$\lim_{s \in \mathbb{C}, \Re(s) > 1, s \to 1} (s - 1) \text{Tr}(\rho(L)^s) = 1. \quad (3.2)$$

iii) If $\sum_{k} \frac{m_k^2}{M_k^{1+\alpha}} < +\infty$ for some $\alpha \in (0, 1)$, then $\zeta_L$ extends to a meromorphic function on the half-plane $\{s \in \mathbb{C} : \Re(s) > \alpha\}$ with a simple pole at $s = 1$ and unital residue.
Proof. i) By Proposition 2.5 iii.a), the asymptotic continuity of \( N_L \) implies that \( \mu_n(\rho(L)) \sim 1/n \) as \( n \to \infty \) so that \( \varphi \) is well defined on \( \{ s \in \mathbb{C} : \Re(s) > 1 \} \). The limit behaviour is just the content of Proposition 2.5 iii.c).

ii) Notice first that the assumption implies \( \lim_{k \to \infty} \frac{m_k}{M_k} = 0 \). Hence \( \lim_{k \to \infty} \frac{M_{k-1}}{M_k} = 1 \), \( N_L \) is asymptotically continuous by Proposition 2.9 ii) and \( \mu_n(\rho(L)) \sim 1/n \) as \( n \to \infty \) again by Proposition 2.5 iii.a). Let us write \( \delta_n := 1/n - \mu_n(\rho(L)) \). According to (2.1), whenever \( k \geq 1 \) and \( M_{k-1} < n \leq M_k \) we have

\[
0 \leq \delta_n < \frac{1}{n} \left( 1 - \frac{M_{k-1}}{M_k} \right) = \frac{m_k}{n M_k} \quad \text{and} \quad 0 \leq n \delta_n < \frac{m_k}{M_k} < 1.
\]

Let us estimate the difference

(3.3) \( \varphi_0(s) - \varphi_L(s) = \sum_{n \geq 1} (n^{-s} - (n^{-1} - \delta_n)^s) = \sum_{n \geq 1} n^{-s} (1 - (1 - n \delta_n)^s) \)

on the closed half-plane \( \{ s \in \mathbb{C} : \Re(s) > 1 \} \). Since \( 1 - n \delta_n \geq M_{k-1}/M_k > 0 \) for \( k \geq 2 \), \( M_0 = 0 \), \( M_1 = m_1 \geq 1 \), by the above lemma, for \( s \in \mathbb{C} \) with \( \Re(s) \geq 1 \), we have

\[
|\varphi_0(s) - \varphi_L(s)| \leq \sum_{n \geq 1} |n^{-s}| \cdot |1 - (1 - n \delta_n)|^s
\]

\[
= \sum_{n \geq 1} n^{-\Re(s)} \cdot |1 - (1 - n \delta_n)|^s
\]

\[
\leq |s| \cdot \sum_{n \geq 1} n^{-\Re(s)} \cdot \log(1 - n \delta_n)^{-1}
\]

\[
\leq |s| \cdot \sum_{k \geq 2} \sum_{n > M_{k-1}} n^{-\Re(s)} \cdot \log M_k/M_{k-1} + |s| \cdot \sum_{n > M_1} n^{-\Re(s)} \cdot \log(1 - n \delta_n)^{-1}
\]

\[
= |s| \cdot \sum_{k \geq 2} \sum_{n > M_{k-1}} n^{-\Re(s)} \cdot \log(1 + m_k/M_{k-1}) + |s| \cdot \sum_{n > M_1} n^{-\Re(s)} \cdot \log(1 - n \delta_n)^{-1}
\]

\[
\leq |s| \cdot \sum_{k \geq 2} \sum_{n > M_{k-1}} M_{k-1}^{-\Re(s)} \cdot m_k/M_{k-1} + |s| \cdot \sum_{n \geq 1} \log(1 - n \delta_n)^{-1}
\]

\[
= |s| \cdot \left( \sum_{k \geq 2} m_k^2/M_{k-1}^{1+\Re(s)} + \sum_{n \geq 1} \log(1 - n \delta_n)^{-1} \right)
\]

\[
\leq |s| \cdot \left( \sum_{k \geq 2} m_k^2/M_{k-1}^2 + \sum_{n \geq 1} \log(1 - n \delta_n)^{-1} \right) < +\infty.
\]

where, under the current hypothesis, the series converge as \( k \to \infty \) (since \( M_k \sim M_{k-1} \)). Then

\[
\lim_{s \in \mathbb{C}, \Re(s) > 1, s \to 1} (s - 1)[\varphi_0(s) - \varphi_L(s)] = 0
\]

and the thesis follows since \( \varphi_0 \) has residue one at the simple pole \( s = 1 \).
iii) Fix $r > 0$ and set $D_{r,\alpha} := \{ s \in \mathbb{C} : \Re(s) > \alpha, |s| < r \}$. Reasoning as above we have

$$\sum_{n \geq 1} \sup_{D_{r,\alpha}} |n^{-s} - (n^{-1} - \delta_n)^s| \leq \sup_{D_{r,\alpha}} |s| \cdot \left( \sum_{k \geq 2} m_k^2 M_k^1 + \sum_{n \geq 1} \log(1 - n\delta_n)^{-1} \right) \leq r \cdot \left( \sum_{k \geq 2} m_k^2 M_k^{1+\alpha} + \sum_{n \geq 1} \log(1 - n\delta_n)^{-1} \right)$$

so that the series of analytic functions $\sum_n (n^{-s} - (n^{-1} - \delta_n)^s)$ converges uniformly on $D_{r,\alpha}$ and then $\zeta_\alpha(s) - \zeta_L(s)$ extends analytically on that domain. The thesis follows again by the above mentioned properties of $\zeta_0$.

\[ \square \]

**Proposition 3.3.** If $m_k = O(M_k^\alpha)$, $k \to \infty$, for some $\alpha \in (0, 1)$, then $\zeta_L$ extends to a meromorphic function on the half-plane $\{ s \in \mathbb{C} : \Re(s) > \alpha \}$ with a simple pole at $s = 1$ and unital residue.

**Proof.** The assumption $m_k = O(M_k^\alpha)$ implies that $m_k / M_k = (m_k / M_k^\alpha) M_k^{-1} \to 0$ as $k \to \infty$, so that, by Proposition 2.9 ii), $N_L$ is asymptotically continuous. Applying Proposition 2.6 iii.a) we have $\mu_n(L) \sim 1/n$ as $n \to \infty$ and then, for any fixed $\beta \in (\alpha, 1)$, the following series converge

$$\sum_{k \geq 1} m_k^2 M_k^{1+\beta} \leq C \sum_{k \geq 1} m_k M_k^{1+\beta-\alpha} = C \sum_{k \geq 1} \sum_{n=M_k+1}^{M_k} \mu_n(L)^{-(1+\beta-\alpha)} = C \sum_{n \geq 1} \mu_n(L)^{-(1+\beta-\alpha)} < +\infty.$$ 

Apply Proposition 3.2 to conclude.

The hypothesis of the previous result can be restated in terms of a bound on the error term in the asymptotically continuous behaviour of the counting function.

**Proposition 3.4.** For $\alpha \in (0, 1)$, the two statements are equivalent

(i) $m_k = O(M_k^\alpha)$, \hspace{1cm} $k \to \infty$

(ii) There exists a continuous function $\varphi$ such that

$$N_L(x) = \varphi(x) + O(\varphi(x)^\alpha), \hspace{1cm} x \to +\infty.$$ 

**Proof.** Suppose $m_k = O(M_k^\alpha)$. Notice first that $m_k = o(M_k)$ so that $M_k \sim M_{k+1}$ and $N_L$ is asymptotically continuous. Let $\varphi$ be the continuous piecewise affine function defined as

$$\varphi(x) = M_k + t(M_{k+1} - M_k) = M_k + tm_{k+1}, \hspace{0.5cm} x \in [\bar{\lambda}_k, \bar{\lambda}_{k+1}], \hspace{0.5cm} x = \bar{\lambda}_k + t(\bar{\lambda}_{k+1} - \bar{\lambda}_k), \hspace{0.5cm} t \in [0, 1].$$

One has $0 \leq \varphi(x) - N_L(x) \leq m_{k+1} = O(M_{k+1}^\alpha) = O(\varphi(x)^\alpha), \hspace{0.5cm} x \to +\infty$, which provides $N_L(x) = \varphi(x) + O(\varphi(x)^\alpha)$. Conversely, if $\varphi$ is continuous and such that $N_L(x) = \varphi(x) + O(\varphi(x)^\alpha)$ as $x \to +\infty$, let us define the function $r(x)$ by the formula

$$N_L(x) = \varphi(x)(1 + r(x)).$$

On one side, one has $r(x) = O(\varphi(x)^{\alpha-1})$ as $x \to +\infty$. On the other side, the function $r(\cdot)$ is right continuous with left limits $r^-(x) := \lim_{\delta \downarrow 0} r(x - \delta)$, in such a way that

$$N_L^-(x) = \varphi(x)(1 + r^-(x)).$$
Fixing $\delta > 0$ small enough we have $N_L(\tilde{\lambda}_{k+1} - \delta) = M_k$ while $N_L(\tilde{\lambda}_{k+1}) = M_{k+1}$, i.e. $M_k = \varphi(\tilde{\lambda}_{k+1} - \delta)(1 + r(\tilde{\lambda}_{k+1} - \delta))$ while $M_{k+1} = \varphi(\tilde{\lambda}_{k+1})(1 + r(\tilde{\lambda}_{k+1}))$. Taking the quotient and making $\delta$ tend to 0, we get

\[
\frac{M_{k+1}}{M_k} = \frac{1 + r(\tilde{\lambda}_{k+1})}{1 + r^-(\tilde{\lambda}_{k+1})} = 1 + O(\varphi(x)^{\alpha-1}) = 1 + O(M_k^{\alpha-1})
\]

which means $\frac{m_{k+1}}{M_k} = O(M_k^{\alpha-1})$ and $m_{k+1} = O(M_k^\alpha) = O(M_{k+1}^\alpha)$ (since $M_{k+1} \sim M_k$).

Summarizing, we have the following criterion of meromorphic extension.

**Theorem 3.5.** Suppose that there exists $\alpha \in (0, 1)$ and a continuous function $\varphi$ such that

\[
N_L(x) = \varphi(x) + O(\varphi(x)^\alpha), \quad x \to +\infty.
\]

Then the $\zeta$-function $\zeta_L(s) = Tr(\rho(L)^s)$ extends as a meromorphic function on the half plane $\{s \in \mathbb{C} : \Re(s) > \alpha\}$ with a simple pole at $s = 1$ and unital residue.

**Proof.** Apply Propositions 3.4 and 3.3. \qed

### 3.2. Examples of densities and their $\zeta$-functions.

**Example 3.6.** (Compact smooth manifolds I) Let $M$ be a compact, $n$-dimensional, orientable, smooth manifold without boundary with cotangent bundle $\pi_* : T^*M \to M$. Let $m^*$ be the symplectic volume measure on $T^*M$ and fix a smooth volume measure $\nu$ on $M$. Disintegrating $m^*$ with respect to $\nu$ by $\pi^*$, one gets a family of measures $m^*_x$ on $T^*_xM$ for $\nu$-a.e. $x \in M$ such that $m^* = \int_M m^*_x \cdot m(dx)$.

Let $L$ be the Friedrichs extension of an $m$-symmetric, positive, elliptic, smooth pseudo differential operator of order $k \geq 1$ with classical symbol, defined on $C^\infty(M) \subset L^2(M, m)$. Let $p$ the principal symbol of $L$, understood as a real, homogeneous polynomial of degree $k$ on $T^*M$ or as a function on the cosphere bundle $S^*M$. The spectrum of $L$ is discrete and the Weyl asymptotic formula for the eigenvalue counting function of $L$ reads

\[
N_L(x) = c \cdot x^{n/k} \quad x \to +\infty, \quad c := \frac{1}{(2\pi)^n} \cdot \int_M m_x^*\{p(x, \cdot) < 1\} \cdot m(dx).
\]

It follows by Proposition 2.9 that $N_L$ is asymptotically continuous and that $\rho(L)$ is a density. The Hörmander estimates for the remainder term [Hor1] reads

\[
N_L(x) = c \cdot x^{n/k} + O(x^{(n-1)/k}) \quad x \to +\infty
\]

and, by Theorem 3.5, it follows that the $\zeta$-function $\zeta_L$ of the density $\rho(L)$ is meromorphic on $\{s \in \mathbb{C} : \Re(s) > 1 - 1/n\}$ and it has a simple pole in $s = 1$ with unital residue. The order of the remainder term cannot be improved in general, but in case of a product of 2-spheres $M = S^2 \times S^2$ and for the Laplace-Beltrami operator $L$, M. Taylor [Tay] proved the estimate

\[
N_L(x) = c \cdot \text{Vol}(S^2 \times S^2) \cdot x^2 + O(x^{4/3}) \quad x \to +\infty, \quad c := (4\pi)^{-2}/\Gamma(3)
\]

which ensures that $\zeta_L$ is meromorphic on $\{s \in \mathbb{C} : \Re(s) > 2/3\}$. As another example, one can consider the Laplace-Beltrami operator $L$ on the flat torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ where the reminder term is $O(x^{(n-1)/2 - \gamma})$ for some $\gamma > 0$. The case $n = 2$ corresponds to the classical Gauss problem: in fact $N_L(x)$ coincides with the number of points in $\mathbb{Z}^2$ falling within the circle of radius $x > 0$. It is known [deH page 6] that $N_L(x) = \pi x + O(x^\alpha)$ as $x \to +\infty$ with $\alpha \in (1/4, 12/37)$. Still in [Tay] it is shown that on $S^2$, the sub-elliptic operator
L = X_1^2 + X_2^2 given by the sum of squares of two vector fields X_1, X_2, generating rotations around orthogonal axes, has a counting function with a non-Weyl asymptotic behaviour

$$N_L(x) = \frac{1}{2} x \cdot \text{Log}(x) + O(x) \quad x \to +\infty.$$ 

Again by Proposition 2.9, N_L is asymptotically continuous and that \(\rho(L)\) is a density. The asymptotic behaviour of the counting function \(N_L\) of hypoelliptic \(\Psi\)DO is studied in [MS].

□

One can have a meromorphic extension even if the counting function is not asymptotically continuous:

**Example 3.7.** Let \(\mathbb{F}_p\) be the free group with \(p\) generators \((p \geq 2)\), \(\ell\) the length function on \(\mathbb{F}_p\) and \(L\) the multiplication operator by \(\ell\) on the Hilbert space \(\ell^2(\mathbb{F}_p)\) (see [Haa]).

The spectrum of \(\ell\) is \(\mathbb{N}\): \(\tilde{\lambda}_k = k\) for \(k \geq 1\) with multiplicities \(m_k = 2p(2p - 1)^{k-1}\) and \(M_k = \frac{2p}{2p - 2}((2p - 1)^k - 1)\). Hence \(m_k/M_k \to (2p - 2)/(2p - 1) > 0\) and, by Proposition 2.9, \(N_L\) is not asymptotically continuous. However, since \(\lim_{k \to \infty} M_{k+1}/M_k = 2p - 1 > 1\), Proposition 2.7 implies that the spectral weight \(\rho(L)\) is measurable and that \(\text{Tr}_w(\rho(L)) = \lim_{s \downarrow 1}(s - 1)\text{Tr}(\rho(L)^s) = \frac{2p - 2}{(2p - 1)\log(2p - 1)}\). We show that this limit is indeed a residue:

**Proposition 3.8.** With the notations of the above example, we have that the zeta function \(\zeta_L(s) = \text{Tr}(\rho(L)^s)\) extends as a meromorphic function on the half plane \(\{s \in \mathbb{C} : \Re(s) > 0\}\) with a simple pole at \(s = 1\) and residue

$$\text{Res}_{s=1}(\zeta_L) = \lim_{s \in \mathbb{C}, \Re(s) > 0, s \to 1} (s - 1)\text{Tr}(\rho(L)^s) = \frac{2p - 2}{(2p - 1)\log(2p - 1)}.$$

**Proof.** Let us compute for \(\Re(s) > 1\)

$$\text{Tr}(\rho(L)^s) = \sum_{k \geq 1} m_k M_k^{-s}$$

$$= (2p)^{-s+1}\frac{(2p - 2)^s}{2p - 1} \sum_{k \geq 1} (2p - 1)^k((2p - 1)^k - 1)^{-s}$$

$$= (2p)^{-s+1}\frac{(2p - 2)^s}{2p - 1} \sum_{k \geq 1} (2p - 1)^{k-ks}(1 - (2p - 1)^{-k})^{-s}$$

$$= \varphi(s)(Z_1(s) - Z_2(s))$$

with

- \(\varphi(s) = (2p)^{-s+1}\frac{(2p - 2)^s}{2p - 1}\): \(\varphi\) extends as an analytic function on the whole complex plane; its value at \(s = 1\) is \((2p - 2)/(2p - 1)\);

- \(Z_1(s) = \sum_k (2p - 1)^{k-ks} = \frac{(2p - 1)^{1-s}}{1 - (2p - 1)^{1-s}}\): \(Z_1\) extends as a meromorphic function on the whole complex plane with one pole at \(s = 1\) which is simple, with residue \(\frac{1}{\log(2p - 1)}\);

- \(Z_2(s) = \sum_k (2p - 1)^{k-ks}(1 - (1 - (2p - 1)^{-k})^s)\): \(Z_2\) appears as a sum of analytic functions,
each of them being bounded that way: fix $S > 0$, then there exists a constant $C$ such that

$$\left|1 - (1 - (2p - 1)^{-k})^s\right| \leq C (2p - 1)^{-k}, \ k \geq 1, \ |s| \leq S$$

hence

$$\left|(2p - 1)^{k-s}(1 - (1 - (2p - 1)^{-k})^s)\right| \leq C (2p - 1)^{-ka}, \ k \geq 1, \ |s| \leq S.$$ 

The series $\sum_k (2p - 1)^{k-s}(1 - (1 - (2p - 1)^{-k})^s)$ converges locally uniformly on the upper half plane $\Re(s) > 0$ and its sum defines an analytic function on the half plane $\Re(s) > 0$. Gathering those intermediate results, the Proposition is proved. \hfill \Box

4. Volume forms associated to spectral weights

4.1. Volume forms. Fix a Dixmier ultrafilter $\omega$ on $\mathbb{N}$ as obtained in subsection 2.4.

**Definition 4.1.** The volume form on $\mathcal{B}(h)$ associated to $L$ and $\omega$ will be the linear form $\Omega_L$

$$\mathcal{B}(h) \ni T \to \Omega_L(A) = Tr_\omega(T \rho(L)).$$

The restriction of $\Omega_L$ to a sub-$C^*$-algebra $A \subset \mathcal{B}(h)$ will be the called volume form on $A$.

Volume forms satisfy some obvious properties:

**Proposition 4.2.** i) $\Omega_L$ is a positive, hence uniformly continuous linear form on $\mathcal{B}(h)$ with norm less than $Tr_\omega(\rho(L)) \leq 1$.

ii) $\Omega_L$ vanishes on the $C^*$-algebra $\mathcal{K}(h)$ of compact operators, hence defines a positive linear form on the Calkin algebra $\mathcal{B}(h)/\mathcal{K}(h)$.

iii) $\Omega_L(T) = \bar{\omega} - \lim_{r \to \infty} \frac{1}{r} Tr(T \rho(L)^{1+\frac{1}{r}})$ (recall the definition of $\bar{\omega}$ above 2.3).

**Proposition 4.3.** For any measurable function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$N_L(x) \sim \varphi(x) \quad x \to +\infty,$$

i) the operator $\varphi(L)^{-1}$ belongs to the ideal $\mathcal{L}^{(1,\infty)}(h)$ and 

$$Tr_\omega(\rho(L)) = Tr_\omega(\varphi(L)^{-1}) \quad \text{for all ultrafilter } \omega;$$

ii) for any $T \in \mathcal{B}(h)$, the compact operators $T \rho(L), T \varphi(L)^{-1}$ belong to the ideal $\mathcal{L}^{(1,\infty)}(h)$ and the volume form can be represented as

$$\Omega_L(T) := Tr_\omega(T \rho(L)) = Tr_\omega(T \varphi(L)^{-1}).$$

**Proof.** i) One has $N_L(x)^{-1} \sim \varphi(x)^{-1}$, which we write $\varphi(x)^{-1} = N_L(x)^{-1} + g(x)N_L(x)^{-1}$ for some function $g$ with $\lim_{x \to \infty} g(x) = 0$. We then have $\varphi(L)^{-1} = \rho(L) + g(L)\rho(L)$ with $g(L)$ compact. Hence $g(L)\rho(L)$ belong to the closure $\mathcal{L}^{(1,\infty)}_0(h)$ in $\mathcal{L}^{(1,\infty)}(h)$ of the ideal of finite rank operators, on which the Dixmier trace vanishes and the result follows.

ii) Notice that $\mathcal{B}(h) \ni T \to Tr_\omega(T \rho(L))$ is a positive linear form, hence is a norm continuous functional on $\mathcal{B}(h)$. As it is obviously 0 whenever $T$ has finite rank, it vanishes on the $C^*$-algebra $\mathcal{K}(h)$ of compact operators. Hence, we have $Tr_\omega(g(L)\rho(L)) = 0$ and, for any $T \in \mathcal{B}(h)$ and $T r_\omega(T g(L)\rho(L)) = 0$. \hfill \Box

Hereafter are the first examples of such volume states and linear forms.
4.2. Compact smooth manifolds II. i) In the framework and notations of Example 3.2.1, consider the action \( g \mapsto M_g \) of the commutative C*-algebra \( C(M) \) by pointwise multiplication on \( L^2(M, m) \). By the Weyl asymptotic formula \( N_L(x) \sim c \cdot x^{n/k} =: \varphi(x) \) as \( x \to +\infty \), \( N_L \) is asymptotically continuous, \( \rho(L) \) is a density and the volume forms \( \Omega_L \) are states. Since \( \varphi(L)^{-1} = c^{-1} \cdot L^{-n/k} \), by Proposition 4.3 one has \( \Omega_L(T) = c^{-1} \cdot Tr_\omega(T L^{-n/k}) \). The restriction of these states to \( C(M) \) are represented by probability measures \( \nu_\omega: \Omega_L(M_g) = \int_M g \cdot d\nu_\omega \).

Let \( \pi^*: S^*M \to M \) be the cosphere bundle whose fibers are the rays of \( T^*M \) and consider a scalar valued, elliptic, \( m \)-symmetric 1-order \( \Psi DO \) on \( M \) with classical symbol, denoting by \( D \) its self-adjoint extension to \( L^2(M, m) \).

ii) Since the operators \( D \) and \( M_g, g \in C^\infty(M) \), are scalar valued \( \Psi DO \), their symbols commute. Also, since they are of order 1 and 0 respectively, by the rules of pseudo differential calculus, the commutators \([D, M_g] \) are 0-order \( \Psi DO \), thus bounded. Hence \((C^\infty(M), D, L^2(M, m))\) is a \((d, \infty)\)-summable spectral triple on \( C(M) \), in the sense of A. Connes \([Co3]\). The spectral weight \( \rho([D]) \) is a density and the volume forms \( \Omega_{[D]} \) are states on \( C(M) \) represented by probability measures \( \nu_\omega: \Omega_{[D]}(M_g) = \int_M g \cdot d\nu_\omega \).

iii) Let \( \mathcal{A} \) be the *-algebra of scalar valued, 0-order \( \Psi DO \) on \( M \), acting boundedly on \( L^2(M, m) \). Let \( \mathcal{P}(M) \) be the C*-algebra of bounded operators on \( L^2(M, m) \) generated by \( \mathcal{A} \) (see \([HR]\)). Again, since \( D \) and operators \( T \) in \( \mathcal{A} \) are scalar valued their symbols commute and since they are of order 1 and 0 respectively, the commutators \([T, D]\) have 0-order and are thus bounded. Hence \((\mathcal{A}, D, L^2(M, m))\) is a \((d, \infty)\)-summable spectral triple on \( \mathcal{P}(M) \).

Since

\[
0 \longrightarrow \mathcal{K}(L^2(M, m)) \longrightarrow \mathcal{P}(M) \overset{\sigma}{\longrightarrow} C(S^*M) \longrightarrow 0,
\]

is a C*-algebra extension (see \([D], [HR]\)) and the volume states \( \Omega_{[D]} \) restricted to \( \mathcal{P}(M) \) vanish on the ideal \( \mathcal{K}(L^2(M, m)) \), it follows that they factorizes through suitable probability measures \( \nu^*_\omega \) on the cosphere bundle \( S^*M \). As the representation \( g \mapsto M_g \) is injective, we can identify \( C(M) \) with its image in \( \mathcal{B}(L^2(M, m)) \). Since \( C(M) \setminus \mathcal{K}(L^2(M, m)) = \{0\} \), \( C(M) \subset \mathcal{P}(M) \) and the restriction of the principal symbol map \( \sigma \) is given by \( \sigma(M_g) = g \circ \pi^* \) for any \( g \in C^\infty(M) \), the measure \( \nu_\omega \) is the image of \( \nu^*_\omega \) under \( \pi^*: S^*M \to M \).

4.3. Multiplication operators on discrete groups. Let \( G \) be a countable discrete group with unit \( e \) and left regular representation \( \lambda \) in the Hilbert space \( l^2(G) \). If \( \{\delta_g\}_{g \in G} \) is the canonical orthonormal base of \( l^2(G) \), we have \( \lambda(g)\delta_h = \delta_{gh}, g, h \in G \).

Let \( \ell \) be a proper function from \( G \) in \( \mathbb{R}_+ \) and \( L \) the operator of multiplication by \( \ell \) on \( l^2(G) \): \( L\delta_g = \ell(g)\delta_g \). The (discrete) spectrum of \( L \) coincides with the image of \( \ell \): \( \text{sp}(L) = \{\ell(g) : g \in G\} \). For any \( e \neq g \in G, s \in \mathbb{C}, 9 \Re (s) > 1, \) we have

\[
Tr(\lambda(g)\rho(L)^s) = \sum_h (\delta_h^k N_L(\ell(g))^{-s}\delta_{gh})\ell^s(g) = 0.
\]

The generic element \( a \in C^*_{red}(G) \) has a Fourier expansion \( a = \sum_g a_g \lambda(g) \) and is a uniform limit of elements of which the Fourier expansion has finite support \( (a_g = 0 \text{ except for a finite number of } g \text{'s}) \). For \( a \) with finite support, we have \( Tr(a\rho(L)^s) = a_e Tr(\rho(L)^s) = \tau(a) Tr(\rho(L)^s) \), where \( \tau \) is the canonical trace on \( C^*_{red}(G) \): \( \tau(a) = a_e \). By uniform continuity, this formula extends to any \( a \in C^*_{red}(G) \). Finally, applying formula (2.1) of subsection 2.4, we get

\[
\Omega_L(a) = Tr_\omega(a\rho(L)) = \tau(a)Tr_\omega(\rho(L)), a \in C^*_{red}(G).
\]
Normalizing $\Omega_L$ by $Tr_\omega(\rho(L))$ we get the canonical trace $\tau$ for any ultrafilter $\omega$. The multiplicity $m(\lambda)$ of an eigenvalue $\lambda \geq 0$ is the cardinality of the level set $\{ g \in G : \ell(g) = \lambda \}$ while its cumulated multiplicity $M(\lambda)$ is the cardinality of the sub-level set $\{ g \in G : \ell(g) \leq \lambda \}$. In case $m(\lambda) = o(M(\lambda))$ as $\lambda \to +\infty$, the eigenvalue counting function is asymptotically continuous, $\rho(L)$ is a density and the volume form $\Omega_L$ coincides with the trace state $\tau$ for any ultrafilter $\omega$. This is the case of the word length function $\ell$ of a system of generators for a discrete group $G$ with sub-exponential growth [deH].

**Example 4.4.** We apply the previous argument to the case where $G = \mathbb{F}_p$ is the free group with $p$ generators and exponential growth and $L$ is the multiplication operator by the length function $\ell$. Extending the argument of subsection 4.3 we get easily $Tr(a\rho(L)^s) = a_\ell Tr(\rho(L)^s) = \tau(a) Tr(\rho(L)^s)$ for $a \in C^*_\text{red}(G)$ and $s \in \mathbb{C}$, $Re(s) > 1$. Proposition 3.5 allows to reach the volume form as a residue:

$$\Omega_L(a) = \frac{2p - 2}{(2p - 1)\log(2p - 1)} \tau(a) = \lim_{s \in \mathbb{C}, \Re(s) > 1, s \to 1} (s - 1) Tr(a \rho(L)^s), \ a \in C^*_\text{red}(G).$$

5. **Further Examples**

5.1. **The Toeplitz $C^*$-algebra I.** Let $A \subset B(l^2(\mathbb{N}))$ be the Toeplitz $C^*$-algebra generated by the shift operator $S$ on $l^2(\mathbb{N})$:

$$S e_n = e_{n+1} \quad n \in \mathbb{N}$$

and let $L$ be the multiplication operator on $l^2(\mathbb{N})$ given by

$$(Lu)(n) := nu(n) \quad n \in \mathbb{N}, \ u \in l^2(\mathbb{N}).$$

Its spectrum $sp(L) = \mathbb{N}$ is discrete with all multiplicities equal to one and counting function $N_L(x) = [x] + 1$ for $x \geq 0$. Hence $N_L(L) = L + 1$ and $\rho(L) = (L + 1)^{-1}$. Since $N_L(x) \sim x$ as $x \to +\infty$, $N_L$ is asymptotically continuous and measurability holds true. In this case $\zeta_L$ coincides with the Riemann $\zeta$-function $\zeta_0$ and since the remainder function $N_L(x) - \varphi(x) = 1 - (x - [x])$ is bounded, applying Theorem 3.5 for all $\alpha \in (0, 1)$, we obtain the known fact that $\zeta_0$ is meromorphic in the open right half plane with a simple pole at $s = 1$ and unital residue. The volume forms

$$\Omega_L(a) = Tr_\omega(a\varphi(L)^{-1}) = Tr_\omega(aL^{-1}) \quad a \in A$$

are states on $A$ vanishing on the ideal $K(l^2(\mathbb{N}))$ of compact operators. Since $A$ is an extension in the sense of [D]

$$0 \longrightarrow K(l^2(\mathbb{N})) \longrightarrow A \longrightarrow C(\mathbb{T}) \longrightarrow 0,$$

where $\mathbb{T}$ is the unit circle and $\sigma(S)(z) = z$ for $z \in \mathbb{T}$, it follows that the states $\Omega_L$ are determined by probability measures $m_\omega$ on $\mathbb{T}$, $\Omega_L(a) = \int_\mathbb{T} \sigma(a) dm_\omega$ for all $a \in A$, and that they are thus traces on $A$ since $C(\mathbb{T})$ is commutative. Since $Tr(S^k L^{-s}) = \sum_{n \geq 0} (e_n S^k n^{-s} e_n) = \sum_{n \geq 0} n^{-s} (e_n e_{n+k}) = \delta_{k,0} \cdot \zeta_0(s)$ for all $s > 1$ and $k \geq 0$, by formula 2.4 we have $\Omega_L(S^k) = \delta_{k,0}$. Hence all measures $m_\omega$ coincides with the Haar probability measure $m_H$ for any ultrafilter $\omega$ and $aL^{-1}$ is measurable for any $a \in A$. 

5.2. The density of Euclidean domains having infinite volume. In Euclidean domains with infinite volume $\Omega$, the Weyl’s asymptotic cannot hold true for the Laplacian $L$ with Dirichlet boundary conditions on $\partial \Omega$, even if the spectrum is discrete. B. Simon determined in [S] (Theorem 1.5) the asymptotic behavior of $N_L$ for certain planar domains of infinite volume. For example, when $\Omega := \{(x, y) \in \mathbb{R}^2 : |xy| \leq 1\}$ one has

$$\hat{\mu}_L(t) = Z_L(t) \sim \frac{1}{\pi} \cdot t^{-1} \log(t^{-1}) \quad t \to 0^+$$

from which, by Proposition 2.10, one derives the asymptotic behaviour

$$N_L(x) \sim \frac{1}{\pi} \cdot x \cdot \log(x) \quad x \to +\infty.$$ 

Hence $N_L$ is asymptotically continuous and, by Proposition 2.9, $\rho(L)$ is a density. The volume states read

$$\Omega_L(T) = \pi \cdot Tr_\omega(TL^{-1} \log^{-1}(L + 1)) \quad T \in \mathcal{B}(L^2(\Omega, dx))$$

and they determine probability measures $\nu_\omega$ on $\Omega$ by $\int_\Omega f \cdot d\nu_\omega := \Omega_L(M_f)$ for $f \in C_0(\Omega)$.

5.3. Kigami Laplacians on P.C.F. fractals. Let $K$ be a post critically finite, self-similar fractal set and $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form associated to a fixed regular harmonic structure (with energy weights $0 < r_1, \cdots, r_m < 1$) in the J. Kigami’s sense [KL]. This quadratic form is closable with respect to any Bernoulli measure $m$ on $K$ (with weights $0 < \mu_1, \cdots, \mu_m < 1$ such that $\sum_{i=1}^m \mu_i = 1$) and we denote by $L$ the densely defined, nonnegative, self-adjoint operator on $L^2(K, m)$ associated to its closure. Set $\gamma_i := \sqrt{r_i/\mu_i}$ and define the spectral dimension $d_S$ as the unique positive number such that

$$\sum_{i=1}^m \gamma_i^{d_S} = 1.$$ 

In the non-arithmetic case, where $\sum_{i=1}^m Z \log \gamma_i$ is a dense additive subgroup of $\mathbb{R}$, the asymptotic of the counting function $N_L$ follow a power law similar to the H. Weyl’s one for compact Riemannian manifolds (see [KL] Theorem 2.4)

$$N_L(x) \sim c \cdot x^{d_S/2}, \quad x \to +\infty$$

where $c := \left(-\left(\sum_{i=1}^m \gamma_i^{d_S} \log \gamma_i\right)^{-1} \cdot \int_{\mathbb{R}} e^{-dS R(e^t)} dt\right)$ and $R(x) := N_L(x) - \sum_{i=1}^m N_L(r_i \mu_i x)$. Consequently, the spectral weight $\rho(L)$ is measurable and it is in fact a density. Evaluating the volume forms $\Omega_L$ on the multiplication operators $M_g \in \mathcal{B}(L^2(K, m))$ by continuous functions $g \in C(K)$, one gets positive states on $C(K)$ represented by probability measures $\nu_\omega$ on $K$

$$\Omega_L(M_g) = c^{-1} \cdot Tr_\omega(M_g \cdot L^{-d_S/2}) = \int_K g \cdot d\nu_\omega.$$ 

6. Volume traces from densities

We denote by $\mathcal{A}_L$ the so called Lipschitz algebra of $L$, i.e.

$$\mathcal{A}_L := \{a \in \mathcal{B}(h) \mid [a, L] \text{ is bounded }\}.$$
6.1. **Statement of the results.** The purpose of the whole section is to prove the following theorem, together with two main corollaries, providing conditions which ensure the existence of hypertraces or amenable traces (see [B], [Co2]). In case of a finitely-summable spectral triple ([Co1] page 68), the result, known as Connes’ Trace Theorem is proved in [Co2] Theorem 8 and Remark 10 b). In case of a \((d, \infty)\)-summable spectral triple, a proof of the theorem, stated in [Co3] Chapter IV.2 Proposition 15, is provided in [CGS].

**Theorem 6.1.** *(Trace Theorem)* Suppose that the counting function satisfies
\[
N_L(x) \sim \varphi(x) \quad x \to +\infty
\]
for some continuous, increasing function \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+\) such that
\[
(6.1) \quad \varphi' \in L^\infty_{\text{loc}}(\mathbb{R}_+) \quad \text{ess} \limsup_{x \to +\infty} \varphi'(x)/\varphi(x) = 0.
\]
Then the following limit properties hold true

(1.a) For \(s > 1\), \(\varphi(L)^{-s}\) is trace-class and \(\lim_{s \downarrow 1} (s - 1) Tr(\varphi(L)^{-s}) = 1\).

(1.b) For \(a \in A_L\), one has
\[
(6.2) \quad \lim_{s \downarrow 1} (s - 1) Tr\left( |[a, \varphi(L)^{-s}]| \right) = 0
\]

(1.c) For \(a \in A_L\) and \(b \in B(h)\), one has
\[
(6.3) \quad \lim_{s \to 1} (s - 1) Tr\left((ab - ba)\varphi(L)^{-s}\right) = 0.
\]

(2) Hypertrace properties.

For \(s > 1\) define the linear functionals
\[
\omega_s(b) := (s - 1) Tr(b \varphi(L)^{-s}) \quad b \in B(h).
\]
As \(s \downarrow 1\), the limit point set of \(\{\omega_s \in B(h)^* : s > 1\}\) is not empty and any such limit linear form \(\tau\) is a state on \(B(h)\) with the following properties:

(2.a) \(\tau\) vanishes on the algebra \(K(h)\) of compact operators;

(2.b) \(\tau\) is a hypertrace on the uniforme closure \(A\) of the Lipschitz algebra \(A_L\):
\[
(6.4) \quad \tau(ba) = \tau(ab), \ a \in A, \ b \in B(h);
\]

(2.c) The restriction of \(\tau\) to \(A\) is a tracial state.

3. Any volume form \(\Omega_L(a) = Tr_\omega(a \rho(L))\) on \(B(h)\) is an hypertrace and a tracial state on \(A\) (with \(\omega\) as in subsection 2.3).

The first corollary is just a variation on the conclusions, with the same assumptions:

**Corollary 6.2.** Suppose that the counting function satisfies
\[
N_L(x) \sim \varphi(x) \quad x \to +\infty
\]
for some continuous, increasing function \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+\) such that
\[
(6.5) \quad \varphi' \in L^\infty_{\text{loc}}(\mathbb{R}_+) \quad \text{ess} \limsup_{x \to +\infty} \varphi'(x)/\varphi(x) = 0.
\]
Then the following limit properties hold true

(1.a) For \(s > 1\), \(\rho(L)^s\) is trace-class and \(\lim_{s \downarrow 1} (s - 1) Tr(\rho(L)^s) = 1\).
(1.b) For $a \in \mathcal{A}_L$, one has
\[
\lim_{s \downarrow 1} (s - 1) \operatorname{Tr} \left( \left| \begin{array}{c} a, \rho(L)^s \end{array} \right| \right) = 0
\]

(1.c) For $a \in \mathcal{A}_L$ and $b \in \mathcal{B}(h)$, one has
\[
\lim_{s \to 1} (s - 1) \operatorname{Tr} \left( (ab - ba)\rho(L)^s \right) = 0.
\]

For $s > 1$ define the linear functionals
\[
\omega_s(b) := (s - 1)\operatorname{Tr}(b \rho(L)^s) \quad b \in \mathcal{B}(h).
\]

As $s \downarrow 1$, the limit point set of $\{\omega_s \in \mathcal{B}(h)^*: s > 1\}$ is not empty and any such limit linear form $\tau$ is a state on $\mathcal{B}(h)$ with the following properties:

(2.a) $\tau$ vanishes on the algebra $\mathcal{K}(h)$ of compact operators;

(2.b) $\tau$ is a hypertrace on the uniform closure $A$ of the Lipschitz algebra $\mathcal{A}_L$:
\[
\tau(ba) = \tau(ab), \quad a \in A, \ b \in \mathcal{B}(h);
\]

(2.c) The restriction of $\tau$ to $A$ is a tracial state.

3. Any volume form $\Omega_L(a) = \operatorname{Tr}_\omega(a \rho(L))$ on $\mathcal{B}(h)$ is an hypertrace and a tracial state on $A$ (with $\omega$ as in subsection 2.7).

The second corollary provides a sufficient condition for the assumptions above to hold true. It requires that the relative multiplicities vanish faster than the spectral gaps of $L$:

**Corollary 6.3.** Suppose that the following conditions on the spectrum of $L$ are satisfied:
\[
\lim_{k \to \infty} m_k/M_k = 0 \quad \text{and} \quad \frac{m_k}{M_k} = o\left(\tilde{\lambda}_{k+1}(L) - \tilde{\lambda}_k(L)\right) \quad \text{as} \quad x \to +\infty.
\]

Then the assumptions of Corollary 6.2 are satisfied, so that all of its conclusions hold true. In particular, they hold true if $N_L$ is asymptotically continuous and the spectral gaps are uniformly bounded away from zero
\[
\liminf_{k \to \infty}(\tilde{\lambda}_{k+1}(L) - \tilde{\lambda}_k(L)) > 0.
\]

Passing from the operator $L$ to a monotone functional calculus $f(L)$ of it, multiplicities remain unchanged but gaps $f(\tilde{\lambda}_{k+1}(L)) - f(\tilde{\lambda}_k(L))$ may vary. However, conditions involving gaps in the Corollary above are still satisfied, for example, if $f \in C^1(\mathbb{R}_+)$ and $\inf f' > 0$.

**Example 6.4.** (The Toeplitz C*-algebra II) Let $\mathbb{P} = \{2, 3, 5, \ldots\}$ be the set of prime numbers and let $S': l^2(\mathbb{P}) \to l^2(\mathbb{P})$ be the shift operator defined as $(S'u)(2) = 0$ and $(S'u)(p) = u(p')$ where $p' \in \mathbb{P}$ denotes the greatest prime strictly less than $p \in \mathbb{P}$ for $p \geq 3$. $S'$ is an isometry $S'*S' = I$ which generates the Toeplitz C*-algebra $A' \subset \mathcal{B}(l^2(\mathbb{P}))$.

Let $J$ be the operator on $l^2(\mathbb{P})$ defined by
\[
(Ju)(p) := pu(p) \quad p \in \mathbb{P}, \ u \in l^2(\mathbb{P}).
\]

We have $\operatorname{Sp}(J) = \mathbb{P}$, all multiplicities equal to one and, by the Prime Number Theorem, $N_J(x) \sim \varphi(x) := x/\log(x)$ for $x \to +\infty$ so that $N_J$ is asymptotically continuous and $\rho(J)$ is a density. Since $\varphi'(x) = (\log(x))^{-1} - (\log(x))^{-2} > 0$ for $x > 3$, $\varphi$ is strictly increasing and since $\varphi'(x)/\varphi(x) = x^{-1} - (x\log(x))^{-1} \to 0$ as $x \to +\infty$, by Theorem 6.1 we have that the volume state on $\mathcal{B}(l^2(\mathbb{P}))$
\[
\Omega_J(a) = \operatorname{Tr}_\omega(a\varphi(J)^{-1}) = \operatorname{Tr}_\omega(a\log J/J) \quad a \in A'
\]
is an hypertrace on the Toeplitz C*-algebra, vanishing on the ideal $\mathcal{K}(l^2(\mathbb{P}))$. Since for $s > 1$,

$$\text{Tr}(S^{k}\varphi(J)^{-s}) = \sum_{p \in \mathbb{P}} (\delta_p S^{k}\varphi(J)^{-s} \delta_p) = \sum_{p \in \mathbb{P}} (p/\log(p))^{-s} (\delta_p S^{k} \delta_p) = \delta_{k,0} \sum_{p \in \mathbb{P}} (p/\log(p))^{-s} \sim \delta_{k,0} \cdot (s - 1)^{-1}$$

as $s \to 1^+$, by formula [2.4] we have $\Omega(J(S^{k})) = \delta_{k,0}$ for any Dixmier ultrafilter $\omega$. Analogously to the situation of section 5.1, one has $A'/\mathcal{K}(\mathbb{P}) \simeq C(\mathbb{T})$ and the induced measure on the circle $\mathbb{T}$ is again the Haar probability measure.

To compare the situation described in [5.1] to the present one, let us notice first that, since $[L, S] = S$ so that the *-algebra $A_L$ generated by $S$ contains all commutators $[L, a]$, for any $a \in A_L$. On the other hand, the commutator $[J, S']$ is unbounded since

$$(|J, S'|)u(p) = p(S'u)(p) - S'(Ju)(p) = pu(p') - (Ju)(p') = (p - p')u(p') \quad 3 \leq p \in \mathbb{P}$$

and it is known that the prime gap $g(p') := p - p'$ can be arbitrarily large. Moreover, $[\log J, S']$ is compact. In fact for $3 \leq p \in \mathbb{P}$

$$(|\log J, S'|)u(p) = (\log p)(S'u)(p) - S((\log Ju)(p)) = pu(p') - ((\log J)u)(p') = (\log(p/p'))u(p') = (\log p/p'(p'))(S'u)(p)$$

and it is known that $\lim_{p \to +\infty} p/p' = \lim_{p' \to +\infty}(1 - g(p'))/p' = 1$. However, A.E. Ingham showed that there exists $\alpha \in (0, 3/8)$ such that $p - p' \leq \rho^{1-\alpha}$ for sufficiently large $p$. Hence, for this fixed value of $\alpha$ and for sufficiently large $p$ we have

$$0 \leq p^\alpha - p'^\alpha = p^\alpha \left[ \left( 1 + \frac{p - p'}{p'} \right)^\alpha - 1 \right] \leq p^\alpha \frac{p - p'}{p'} = \alpha \frac{p - p'}{p^{1-\alpha}} \leq \alpha$$

so that, for some constant $C \geq \alpha$, we have $0 \leq p^\alpha - p'^\alpha \leq C$ for all $p \in \mathbb{P}$. Then $\langle [J^\alpha, S']u(2) = J^\alpha(S'u)(2) - S'(Ju)(2) = 0$ and for $3 \leq p \in \mathbb{P}$

$$\langle [J^\alpha, S']u(p) = p^\alpha(S'u)(p) - S'(Ju)(p) = p^\alpha u(p') - (Ju)(p') = (p^\alpha - p'^\alpha)u(p') = (p^\alpha - p'^\alpha)u(p')$$

It follows that $||[J^\alpha, S']|| \leq C$ and that all commutators $[J^\alpha, a]$ are bounded for any $a$ in the *-subalgebra $A_{J^\alpha} \subset A'$ generated by $S'$. Notice that $\Omega(J) = \Omega_{J^\alpha}$ since $J^\alpha$ is an increasing, unbounded function of $J$. In conclusion, even if the $C^*$-algebras $A$ and $A'$ are isomorphic and their hypertraces correspond $\Omega_L \simeq \Omega_J$, these structures differ from a metric point of view since their Lipschitz algebras are not isomorphic $A_L \not\simeq A_{J^\alpha}$.

Example 6.5. The Dirac operator $D$ of a spectral triple $(A, h, D)$ defined on a $C^*$-algebra $A$, represented in a Hilbert space $h$, and associated to a filtration of $h$ as in subsection 2.3, has spectrum $\mathbb{N}$. All spectral gaps are equal to 1 so that the second condition in Corollary 6.3 is satisfied. As soon as the growth of the filtration satisfies $\lim_{k \to +\infty} M_{k+1}/M_k = 1$, the spectral weight $\rho(D)$ is then a density and the volume states $\Omega_D$ are hypertraces for any Dixmier ultrafilter $\omega$.

6.2. Relationship with subexponential growth. Here, we assume that the assumptions of Theorem 6.1 hold true. As first consequence, $L$ has subexponential spectral growth rate.

Lemma 6.6. For any $\beta > 0$, the partition function is finite $Z_L(\beta) := \text{Tr}(e^{-\beta L}) < +\infty$.

Proof. Condition (6.5) on $\varphi$ implies that, for any fixed $\beta > 0$, the nonnegative function $x \to e^{-\beta x} \varphi(x)$ has a derivative $(\varphi'(x) - \beta \varphi(x))e^{-\beta x}$ which is eventually negative so that it admits a limit at $+\infty$. Hence, for all fixed $\beta > 0$, $\lim_{x \to +\infty} e^{-\beta x} \varphi(x) = 0$. In particular, $\lim_{n \to \infty} e^{-2\beta n} \varphi(n) \to 0$, so that $\limsup_{n \to \infty} \sqrt[\varphi(n)]{e^{2\beta}} \leq e^{2\beta}$. Since this holds for any $\beta > 0$, we
get \( \limsup_{n \to \infty} \frac{\sqrt{\varphi(n)}}{\sqrt{n}} \leq 1 \) (and indeed \( \lim_{n \to \infty} \frac{\sqrt{\varphi(n)}}{\sqrt{n}} = 1 \)). Applying \[^{[CS]}\] Lemma 3.13 we get the result.

6.3. **Preparatory results.** In this section we assume the hypotheses of Theorem 6.1

**Lemma 6.7.** For \( s > 1 \), \( \varphi(L)^{-s} \) is trace class, with \( \lim_{s \to 1^+}(s-1)Tr(\varphi(L)^{-s}) = 1 \).

**Proof.** The assumptions made imply that \( \varphi \) is a continuous function, which in turns implies \( \lim_n M_n/M_{n-1} = 1 \) (by Proposition 2.9). Then, by Proposition 2.5 we get \( N_L(\lambda_n(L)) \sim n \) as \( n \to +\infty \) (eigenvalues numbered with repetition according to the multiplicity) and thus \( \lambda_n(\varphi(L)^{-1}) \sim 1/n \). The result follows easily.

The assumptions on \( \varphi \) lead to the following technical result.

**Lemma 6.8.** For \( s > 1 \) and \( N \in \mathbb{N}^* \), one has

\[
\sup_{k>\ell\geq N} \frac{\varphi(\tilde{\lambda}_k(L))^s - \varphi(\tilde{\lambda}_\ell(L))^s}{\varphi(\tilde{\lambda}_k(L))^s(\tilde{\lambda}_k(L) - \tilde{\lambda}_\ell(L))} \leq \text{ess-sup}_{x \geq \lambda_N(L)} s \frac{\varphi'(x)}{\varphi(x)}.
\]

**Proof.** To short notation set \( \lambda_k := \lambda_k(L) \), etc.. Keeping in mind the fact that \( \varphi \) is increasing, we write for \( k > \ell \geq N \)

\[
\frac{\varphi(\tilde{\lambda}_k(L))^s - \varphi(\tilde{\lambda}_\ell(L))^s}{\varphi(\tilde{\lambda}_k(L))^s} = \int_{\tilde{\lambda}_\ell}^{\tilde{\lambda}_k} s \frac{\varphi(x)^{s-1}\varphi'(x)}{\varphi(\tilde{\lambda}_k(L))^s} dx \\
\leq \int_{\tilde{\lambda}_\ell}^{\tilde{\lambda}_k} s \frac{\varphi(x)^{s-1}\varphi'(x)}{\varphi(x)^s} dx \\
\leq (\tilde{\lambda}_k - \tilde{\lambda}_\ell) \text{ess-sup}_{x \in \lambda_N \leq x \leq \tilde{\lambda}_k} s \frac{\varphi'(x)}{\varphi(x)} \\
\leq (\tilde{\lambda}_k - \tilde{\lambda}_\ell) \text{ess-sup}_{x \in \lambda_N} s \frac{\varphi'(x)}{\varphi(x)}.
\]

Let \( L^2(h) \) be the space of Hilbert-Schmidt operators on \( h \) with norm \( ||\Phi||_2 = Tr(\Phi^*\Phi)^{1/2} \) and corresponding scalar product \( \langle \Psi, \Phi \rangle_2 = Tr(\Psi^*\Phi) \). For \( \ell \geq 1 \), let us denote by \( \pi_\ell \) the orthogonal projection onto the eigenspace corresponding to the eigenvalue \( \lambda_\ell(L) \).

**Lemma 6.9.** Let \( \{\alpha_{k,\ell}\}_{k,\ell \geq 1} \subset \mathbb{C} \) be a bounded set and \( T \) a bounded operator on \( h \). Then

\[
\sum_{k,\ell} \alpha_{k,\ell} \pi_k T \pi_\ell \varphi(L)^{-s/2}, \quad s > 1,
\]

is a Hilbert-Schmidt operator and the following estimate holds true:

\[
(6.10) \quad \left\| \sum_{k,\ell} \alpha_{k,\ell} \pi_k T \pi_\ell \varphi(L)^{-s/2} \right\|_2 \leq \left( \sup_{k,\ell} |\alpha_{k,\ell}| \right) \cdot ||T|| \cdot Tr(\varphi(L)^{-s})^{1/2}.
\]

**Proof.** As the right and left actions of \( B(h) \) on \( L^2(h) \) commute, for each \( k, \ell \geq 1 \) we define an orthogonal projection \( p_{k,\ell} \) in \( B(L^2(h)) \) by

\[
p_{k,\ell} \Phi = \pi_k \Phi \pi_\ell, \quad \Phi \in L^2(h).
\]

We have obviously \( \sum_{k,\ell} p_{k,\ell} = I \), so that the operator norm of \( \sum_{k,\ell} \alpha_{k,\ell} p_{k,\ell} \) acting on \( L^2(h) \) is \( \sup_{k,\ell} |\alpha_{k,\ell}| \). We get the result writing

\[
\sum_{k,\ell} \alpha_{k,\ell} \pi_k T \pi_\ell \varphi(L)^{-s/2} = \sum_{k,\ell} \alpha_{k,\ell} \pi_k T \varphi(L)^{-s/2} \pi_\ell = \left( \sum_{k,\ell} \alpha_{k,\ell} p_{k,\ell} \right) T \varphi(L)^{-s/2}.
\]
Proposition 6.10. For any $s > 1$ and $N \geq 1$, set

$$
Y_N(s) = \sum_{k \geq N} \frac{\varphi(\lambda_k(L))^s - \varphi(\lambda_L(L))^s}{\varphi(\lambda_k(L))^s (\lambda_k(L) - \lambda_L(L))} \pi_k \pi_\ell [L, a] \pi_\ell \varphi(L)^{-s/2}.
$$

i) One has $Y_N(s) \in L^2(h)$ with $\|Y_N(s)\| \leq s \left( \sup_{x \geq \lambda_N} \varphi'(x)/\varphi(x) \right) \|L, a\| \|Tr(\varphi^{-s}(L))\}^{1/2}.

ii) $\lim_{s \to 1}(s - 1)^{1/2} ||Y_N(s)||_2 = 0$.

Proof. To short notation set $\lambda_k := \lambda_k(L)$, etc.. For i), apply Lemmas 6.8 and 6.9 ii) Fix $\varepsilon > 0$ and $N \geq 1$ such that

$$
\text{ess- sup}_{x \geq \lambda_N} \frac{\varphi'(x)}{\varphi(x)} \leq \varepsilon.
$$

On one hand, $Y_1(s) - Y_N(s) = \sum_{1 \leq \ell \leq N, k \geq \ell} \frac{\varphi(\lambda_k(L))^s - \varphi(\lambda_L(L))^s}{\varphi(\lambda_k(L))^s (\lambda_k(L) - \lambda_L(L))} \pi_k \pi_\ell [L, a] \pi_\ell \varphi(L)^{-s/2}$ has a Hilbert-Schmidt norm less than $C \left( \left| \sum_{\ell=1}^N \pi_\ell \right| \varphi(L)^{-s/2} \right|_2$ for some constant $C$ depending only on $\varphi$ and $\|L, a\|$, by Lemma 6.9. Hence $\|Y_1(s) - Y_N(s)\|_2$ is bounded independently of $s$. For $s$ close enough to 1, we then have $(s - 1)^{1/2} ||Y_1(s) - Y_N(s)||_2 \leq \varepsilon$.

On the other hand, applying i), we get $\|Y_N(s)\|_2 \leq \varepsilon s \left( \|L, a\| \|Tr(\varphi^{-s}(L))\}^{1/2}$)

Applying Lemma 6.7, we get that, for $s$ close enough to 1, $(s - 1)^{1/2}Y_1(s)$ has a Hilbert-Schmidt norm less than $\varepsilon s \cdot \|L, a\| \cdot (1 + \varepsilon)$.

Summing up, we get that, for $s$ close enough to 1, $(s - 1)^{1/2}Y_1(s)$ has Hilbert-Schmidt norm less that $\varepsilon \cdot (s \left( \|L, a\| \left( 1 + \varepsilon \right) + 1 \right)$.

Proposition 6.11. Setting $Z_1(s) = \sum_{1 \leq \ell \leq N} \frac{\varphi(\lambda_k(L))^s - \varphi(\lambda_L(L))^s}{\varphi(\lambda_k(L))^s (\lambda_k(L) - \lambda_L(L))} \pi_k \pi_\ell [L, a] \pi_\ell$, we have:

i) $Z_1(s) \in L^2(h)$ whenever $s > 1$

ii) $\lim_{s \to 1}(s - 1)^{1/2} ||Z_1(s)||_2 = 0$.

Proof. Up to a sign, $Z_1(s)^* is given by the same formula as $Y_1(s)$, with $\bar{s}$ substituted to $s$ and $a^*$ substituted to $a$. Apply Proposition 6.10 i) and ii).

Lemma 6.12. (Chain rule) For any $f \in C(\mathbb{R})$ and $a \in \mathcal{A}_L$ we have, for $k, \ell \geq 1, k \neq \ell$:

i) $\pi_k [a, f(L)] \pi_\ell = 0$ and $\pi_k [a, f(L)] \pi_\ell = (f(\lambda_k) - f(\lambda_\ell)) \pi_k a \pi_\ell$.

ii) $\pi_k [L, a] \pi_\ell = (\lambda_k - \lambda_\ell) \pi_k a \pi_\ell$.

iii) $\pi_k [a, f(L)] \pi_\ell = \frac{(f(\lambda_k(L)) - f(\lambda_\ell(L)))}{\lambda_k(L) - \lambda_\ell(L)} \pi_k [L, a] \pi_\ell$.

In other words, by easily understood abuse of notation, we can write

$$(6.11) [a, f(L)] = \sum_{k \neq \ell} \frac{(f(\lambda_k(L)) - f(\lambda_\ell(L)))}{\lambda_k(L) - \lambda_\ell(L)} \pi_k [L, a] \pi_\ell.$$

Proof. i) and ii) are straightforward. iii) is an obvious combination of i) and ii).

Proposition 6.13. For any $a \in \mathcal{A}_L$ (i.e. $[a, L]$ is bounded) one has

$$(6.12) \lim_{s \to 1} Tr \left( \left| [a, \varphi(L)^{-s}] \right| \right) = 0.$$
Proof. To short notation set $\lambda_k := \lambda_k(L)$, etc. Lemma 6.12 allows to write
\[
[a, \varphi(L)^{-s}] = \sum_{k \neq \ell} \frac{(\varphi(\tilde{\lambda}_k)^{-s} - \varphi(\tilde{\lambda}_\ell)^{-s})}{\lambda_k - \lambda_\ell} \pi_k [L, a] \pi_\ell
\]
(6.13)

\[
= \sum_{k \neq \ell} \frac{(\varphi(\lambda_k)^{-s} - \varphi(\lambda_\ell)^{-s})}{\lambda_k - \lambda_\ell} \varphi(\lambda_\ell)^{-s} \pi_k [L, a] \pi_\ell
\]

\[
= X^+(s) + X^-(s)
\]

with
\[
X^+(s) = \sum_{k > \ell \geq 1} \frac{(\varphi(\tilde{\lambda}_k)^{s} - \varphi(\tilde{\lambda}_\ell)^{s})}{\varphi(\lambda_k)^{-s}(\lambda_k - \lambda_\ell)} \pi_k [L, a] \pi_\ell
\]
(6.14)

\[
= \sum_{k > \ell \geq 1} \frac{(\varphi(\lambda_k)^{s} - \varphi(\lambda_\ell)^{s})}{\varphi(\lambda_k)^{-s}(\lambda_k - \lambda_\ell)} \pi_k [L, a] \pi_\ell \varphi(L)^{-s}
\]

\[
= Y_1(s) \varphi(L)^{-s/2}
\]

while
\[
X^-(s) = \sum_{1 \leq k < \ell} \frac{(\varphi(\tilde{\lambda}_k)^{s} - \varphi(\tilde{\lambda}_\ell)^{s})}{\varphi(\lambda_k)^{-s}(\lambda_k - \lambda_\ell)} \pi_k [L, a] \pi_\ell
\]
(6.15)

\[
= \sum_{1 \leq k < \ell} \varphi(L)^{-s} \frac{(\varphi(\lambda_k)^{s} - \varphi(\lambda_\ell)^{s})}{\varphi(\lambda_k)^{-s}(\lambda_k - \lambda_\ell)} \pi_k [L, a] \pi_\ell
\]

\[
= \varphi(L)^{-s/2} Z_1(s).
\]

Let $X^+(s) = u^+(s) |X^+(s)|$ be the polar decomposition of $X_+(s)$. Applying Lemma 6.14 and Proposition 6.10 (2), we get
\[
Tr(|X^+(s)|) = Tr(u^+(s)^*X^+(s))
\]
\[
= Tr(u^+(s)^*Y_1(s)\varphi(L)^{-s/2})
\]
(6.16)

\[
= Tr(\varphi(L)^{-s/2} u^+(s)^* Y_1(s))
\]
\[
\leq ||u^+(s)\varphi(L)^{-s/2}||_2 ||Y_1(s)||_2
\]
\[
= O((Re(s) - 1)^{-1/2})O((Re(s) - 1)^{-1/2}) = o(Re(s) - 1)^{-1},
\]

which proves $(s - 1)Tr(|X^+(s)|) \to 0$ as $s \downarrow 1$.

A similar argument, mutatis mutandis, provides $(s - 1)Tr(|X^-(s)|) \to 0$ as $s \downarrow 1$. \hfill \Box

Lemma 6.14. If $T$ is a compact mutandis, provides $(s - 1)Tr(|X^-(s)|) \to 0$ as $s \downarrow 1$.

Proof. Fix $\varepsilon > 0$ and $T_0$ a finite rank operator such that $||T - T_0|| \leq \varepsilon$. On one hand, \(\lim_{s \uparrow 1} Tr(T_0\varphi(L)^{-s}) = Tr(T_0\varphi(L)^{-1})\) exists, so that \(\lim_{s \uparrow 1} (s - 1)Tr(T_0\varphi(L)^{-s}) = 0\), which means \(|Tr(T_0\varphi(L)^{-s})| \leq \varepsilon\) for $s$ close to 1. On the other hand, one has for $s > 1$,
\[
(s - 1)|Tr((T - T_0)\varphi(L)^{-s})| \leq \varepsilon(s - 1)Tr(\varphi(L)^{-s})
\]

with, by Lemma 0.7, $(s - 1)Tr(\varphi(L)^{-s}) \leq 1 + \varepsilon$ for $s > 1$ close to 1. Summing up, we have $(s - 1)|Tr(T\varphi(L)^{-s})| \leq \varepsilon(2 + \varepsilon)$ for $s > 1$ close to 1. \hfill \Box
6.4. Proofs of the theorem and its corollaries.

Proof of Theorem 6.1. (1.a) is Lemma 6.7. (1.b) is Proposition 6.13 and (1.c) is an obvious consequence of (1.b). In (2), the fact that \( \Omega_\ast \) are bounded as \( s \to 1+ \) and that a limit linear form is a state is a consequence of Lemma 6.7. (2.a) comes from Lemma 6.14. (2.b.) and (2.c) come from (1.b) and (1.c).

Proof of Corollary 6.2. We have \( \varphi^{-1} = N^{-1}_L (1 + g) \) with \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) vanishing at infinity. This implies \( \varphi(L)^{-1} = N^{-1}_L (I + T) \) with \( T \) a compact operator commuting with \( L \), \( N(L) \) and \( \varphi(L) \). Apply Lemma 6.14 repeatedly for substituting \( N(L)^{-1} \) to \( \varphi(L)^{-1} \) in every successive item of Theorem 6.1.

Proof of Corollary 6.3. Let \( \varphi \) be the continuous piecewise affine function on \( \mathbb{R}_+ \) interpolating affinely between the points \( \tilde{\lambda}_k(L) \) and \( \tilde{\lambda}_{k+1}(L) \), i.e.

\[
\varphi(x) = M_k + (x - \tilde{\lambda}_k(L)) \frac{M_{k+1} - M_k}{\tilde{\lambda}_{k+1}(L) - \tilde{\lambda}_k(L)} \quad \text{whenever} \quad x \in [\tilde{\lambda}_k(L), \tilde{\lambda}_{k+1}(L)].
\]

This is the function constructed in Proposition 2.9 where it is shown to be asymptotically equivalent to \( N(L) \), provided that \( M_{k+1}/M_k \) tends to 1 as \( k \to \infty \).

\( \varphi \) is differentiable on each interval \( (\tilde{\lambda}_k, \tilde{\lambda}_{k+1}) \) with derivative \( \varphi'(x) = \frac{M_{k+1} - M_k}{\tilde{\lambda}_{k+1} - \tilde{\lambda}_k} \). Moreover, for \( x \in (\tilde{\lambda}_k, \tilde{\lambda}_{k+1}) \) we have \( \varphi(x) \geq M_k \) and \( \frac{\varphi'(x)}{\varphi(x)} \leq \left( \frac{M_{k+1}}{M_k} - 1 \right) \frac{1}{\tilde{\lambda}_{k+1} - \tilde{\lambda}_k} \) and by hypothesis we have \( \lim_{x \to \pm \infty} \frac{\varphi'(x)}{\varphi(x)} = 0 \).

6.5. Densities on \( C^* \)-algebras extensions. We conclude with a remark concerning densities and their volume forms on \( C^* \)-algebras extensions \( A \subset \mathcal{B}(h) \) in the sense of [4], [HR]

\[
0 \to K \to A \to \sigma C(X) \to 0,
\]

where \( K \) is the elementary \( C^* \)-algebra represented in \( h \) with finite multiplicity and \( X \) is a compact metrizable space. This framework include the Toeplitz extension and the extension generated by scalar, zero order \( \Psi \)DO on compact manifolds.

Proposition 6.15. (Volume forms on extension) Assume the counting function \( N_L \) to be asymptotically continuous. Then, for any fixed Dixmier ultrafilter \( \omega \),
i) the volume form

\[
\Omega_L : \mathcal{B}(h) \to \mathbb{C} \quad \Omega_L(T) := \text{Tr}_\omega(T \rho(L))
\]

is a state vanishing on the ideal \( K(h) \) of compact operators and thus it factorizes through a state on the Calkin algebra \( Q(h) = \mathcal{B}(h)/K(h) \);

ii) the restriction of \( \Omega_L \) to \( A \) is a trace that factorizes through a probability measure \( \mu_\omega \) on \( X \)

\[
\Omega_L(a) = \int_X (\sigma(a))(x) \mu_\omega(dx) \quad a \in A.
\]

Under the assumptions of Theorem 6.1 or Corollary 6.3, we also have

iii) \( \Omega_L \) is an hypertrace (or amenable trace state) vanishing on the ideal \( K \);
iv) there exists a conditional expectation $E^L_{\omega} : \mathcal{B}(h) \to L^\infty(X, \mu_\omega)$ such that

$$\Omega_L(T) = \int_X E^L_{\omega}(T) \cdot d\mu_\omega \quad T \in \mathcal{B}(h).$$

**Proof.** Straightforward. □

7. Appendix

A measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *regularly varying* if there exist the limits

$$k(s) := \lim_{t \to +\infty} \frac{f(st)}{f(t)} \in (0, +\infty) \quad \forall s > 0.$$ 

If $k(s) = 1$ for all $s > 0$, then $f$ is said to be *slowly varying*. Necessarily, $k$ must have the form $k(s) = s^\gamma$ for some $\gamma \in \mathbb{R}$, called the index of regular variation ($f \in R_\gamma$) and $f(t) = t^\gamma \ell(t)$ for some slowly varying function $\ell \in R_0$.

**Theorem 7.1.** *(Karamata characterization)* The following characterization holds true: $f \in R_\gamma$ if and only if for some $\sigma > -(\gamma + 1)$ one has

$$\lim_{t \to +\infty} \frac{t^{\gamma+1} f(t)}{\int_0^t x^\sigma f(x) \, dx} = \sigma + \gamma + 1.$$ 

**Theorem 7.2.** *(Karamata Tauberian Theorem)* Let $\mu$ be a positive Borel measure on $[0, +\infty)$ such that

$$\int_0^{+\infty} e^{-tx} \mu(dx) < +\infty \quad \text{for all } t > 0$$

and suppose that it has a regularly varying Laplace transform (with index $\gamma \in \mathbb{R}$)

$$\hat{\mu}(t) := \int_0^{+\infty} e^{-tx} \mu(dx) \quad t > 0.$$ 

Then the function $N_\mu : \mathbb{R}_+ \to \mathbb{R}_+$ defined as $N_\mu(a) := \mu([0, a))$ has the following asymptotics

$$N_\mu(a) = \mu([0, a))) \sim \frac{\hat{\mu}(1/a)}{\Gamma(\gamma + 1)} \quad a \to +\infty.$$ 

Notice that the function $a \mapsto \hat{\mu}(1/a)$ is continuously differentiable as it is $\hat{\mu}$:

$$\frac{d\hat{\mu}}{dt}(t) = - \int_0^{+\infty} xe^{-tx} \mu(dx) \quad t > 0.$$ 

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