LERAY RESIDUES AND ABEL’S
THEOREM IN CR CODIMENSION k

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§1 Introduction

In this paper we generalize Leray’s calculus of residues in several complex variables [Lr], to the situation of an abstract smooth CR manifold M of general type (n, k), and a polar submanifold S which is a smooth CR submanifold of type (n − 1, k), transversal to the Levi distribution of M. Here n is the CR dimension of M and k its CR codimension; so dim_R M = 2n + k. When k = 0 this means that M is an n-dimensional complex manifold, and S is a complex submanifold having complex codimension 1 in M; for k = 0 and n = 1, everything reduces to the classical theory of residues in one complex variable.

The extension of the Leray residue calculus to abstract CR manifolds M of CR codimension k involves a certain number of new turns and twists, which require investigation. One of these is the strange fact that the polar submanifold S may not have local defining functions that are CR on M (see the example at the end of §3). In particular this forces upon us an enlargement of the notion of semimeromorphic function or form, allowing singularities along the polar submanifold S that are more general than what one may expect by analogy with the complex manifold case. Another new aspect is that when one takes a maximal number n of polar submanifolds S_1, S_2, ..., S_n, having normal crossings, the intersection S = S_1 ∩
$S_2 \cap \cdots \cap S_n$ is of type $(0, k)$, and hence totally real. We then obtain $(k+1)$ different kinds of "Grothendieck" residues, which are the analogues of the Grothendieck point residue that one has when $k = 0$ (see [D]).

In the case where each polar submanifold $S_j$ has a global defining function in a neighborhood of $S$, we obtain an actual calculus of residues which entails using only the usual operations of the exterior differential calculus of smooth forms (see §13). In this respect our residue calculus improves somewhat that of Leray, even for the complex manifold case ($k = 0$).

In §14 we use the theory we have developed to generalize the classical theorem of Abel, about the sum of the residues of an abelian differential of the second kind on a compact Riemann surface, to the case of a compact abstract CR manifold $M$. In this connection our discussion follows along the lines of Griffiths [G]; however we actually obtain a more general result, even for the complex manifold case ($k = 0$), because we allow for the intersection of only $m$ ($1 \leq m \leq n$) polar submanifolds having normal crossings. Thus we obtain (Theorem 14.2) a result that applies to forms of more general degree. In the last section we apply the Abel theorem to derive quite general period relations (Propositions 15.1 and 15.2).

§2 Preliminaries and notation

An abstract CR manifold of type $(n, k)$ is a triple\(^1\) $(M, HM, J)$ where $M$ is a paracompact smooth manifold of dimension $2n + k$, $HM$ is a smooth subbundle of even dimension $2n$ of the tangent bundle $TM$ of $M$, which is called the Levi distribution, and $J$ is a smooth complex structure on the fibers of $HM$: this means that $J : HM \rightarrow HM$ is an equivalence of smooth vector bundles with $J^2 = -Id$. We also require that $J$ be formally integrable. This condition can be expressed in terms of the complex subbundle

\begin{equation}
T^{0,1}M = \{X + \sqrt{-1}JX \mid X \in HM\}
\end{equation}

of the complexified tangent bundle $\mathbb{C}TM$ of $M$, by requiring that

\begin{equation}
[\Gamma(M, T^{0,1}M), \Gamma(M, T^{0,1}M)] \subset \Gamma(M, T^{0,1}M).
\end{equation}

We note that $T^{0,1}M$ is the eigenspace corresponding to the eigenvalue $-\sqrt{-1}$ of $J$. Its complex conjugate with respect to the real form $TM$ of $\mathbb{C}TM$:

\begin{equation}
T^{1,0}M = \{X - \sqrt{-1}JX \mid X \in HM\}
\end{equation}

is the eigenspace corresponding to the eigenvalue $\sqrt{-1}$ of $J$. We have therefore

\begin{equation}
T^{1,0}M = \bar{T^{0,1}M}, \quad T^{1,0}M \cap T^{0,1}M = 0_M \text{ (zero section of $\mathbb{C}TM$)}.
\end{equation}

We note that the datum of a complex subbundle $T^{0,1}M$ of rank $n$ of the complexified tangent bundle $\mathbb{C}TM$, satisfying (2.2) and (2.4), defines on the smooth manifold $M$ of dimension $2n + k$ a unique structure of CR manifold of type $(n, k)$: indeed we define

\begin{equation}
HM = \{RZ \mid Z \in T^{0,1}M\}
\end{equation}

\(^1\)But we shall write only $M$ for simplicity when no confusion can arise.
and, for $X \in HM$, we set $JX = Y$ iff $X + \sqrt{-1}Y \in T^{0,1}M$. The map $J : HM \to HM$ is well defined because of (2.4).

An abstract CR manifold of type $(n,0)$ is a complex manifold of dimension $n$ by the Newlander-Nirenberg theorem.

Note that the vector bundle $HM$ can be considered a complex vector bundle of rank $n$ on $M$, with the complex structure on the fibers defined by $J$.

The annihilator bundle $H^0M$ of $HM$ in the cotangent bundle $T^*M$ of $M$ is called the characteristic bundle of $M$. The quotient bundle $T^*M/H^0M$ is the dual bundle of $HM$. Therefore it is a complex vector bundle of rank $n$ and hence orientable, with the natural orientation associated to its complex structure. It follows that the abstract CR manifold $M$ is orientable iff its characteristic bundle $H^0M$ is orientable.

We use the notation $E(M)$ for the exterior algebra of smooth complex valued alternating forms on $M$:

\[ E(M) = \bigoplus_{p=0}^{2n+k} E^p(M), \]

where $E^p(M)$ is the subspace of forms homogeneous of degree $p$. In particular $E^0(M) = C^\infty(M) = C^\infty(M, \mathbb{C})$.

We consider the ideal in $E(M)$:

\[ \mathcal{I}(M) = \{ \alpha \in E(M) \mid \alpha \mid_{T^{0,1}M} = 0 \} \subset \bigoplus_{p=1}^{2n+k} E^p(M) \]

and its exterior powers:

\[ \mathcal{I}^0(M) = E(M), \quad \mathcal{I}^p(M) = \mathcal{I}^{p-1}(M) \wedge \mathcal{I}(M) \quad (p = 1, 2, \ldots). \]

The integrability condition (2.2) can also be expressed in terms of the ideal $\mathcal{I}(M)$ by:

\[ d\mathcal{I}(M) \subset \mathcal{I}(M). \]

We also have

\[ d\mathcal{I}^p(M) \subset \mathcal{I}^p(M) \quad \forall p = 1, 2, \ldots \]

We note that $\mathcal{I}^{n+k+1} = 0$ by reasons of degree. Hence we have a decreasing sequence of closed ideals:

\[ E(M) = \mathcal{I}^0(M) \supset \mathcal{I}^1(M) \supset \mathcal{I}^2(M) \supset \cdots \supset \mathcal{I}^{n+k}(M) \supset \{0\}. \]

For integers $0 \leq p \leq n + k$, $0 \leq q \leq n$ we set

\[ \mathcal{I}^{(p,q)}M = \mathcal{I}^p(M) \cap \mathcal{E}^{p+q}(M) \]

Then clearly we have:

\[ d\mathcal{I}^{(p,q)}(M) \subset \mathcal{I}^{(p,q+1)}(M), \quad \mathcal{I}^{(p+1,q)}(M) \subset \mathcal{I}^{(p,q-1)}(M). \]
Now we define, for $0 \leq p \leq n + k$, $0 \leq q \leq n$:

\[(2.14) \quad Q^{p,q}(M) = \mathcal{I}^{(p,q)}(M) / \mathcal{I}^{(p+1,q-1)}(M)\]

so that, passing to the quotient from the de Rham complex we obtain the Cauchy-Riemann complexes:

\[(2.15) \quad 0 \to Q^{p,0}(M) \xrightarrow{\partial_M} Q^{p,1}(M) \xrightarrow{\partial_M} \cdots \xrightarrow{\partial_M} Q^{p,n}(M) \to 0,\]

for each $0 \leq p \leq n + k$.

We denote by $\Omega^p_M(M)$ the kernel of the map: $\bar{\partial}_M : Q^{p,0}(M) \to Q^{p,1}(M)$. Note that $Q^{0,0}(M) = \mathcal{E}^0(M)$, and in general $Q^{p,0}(M) \subset \mathcal{E}^p(M)$. Thus $\Omega^p_M(M) \subset \mathcal{E}^p(M)$ and its elements are exterior differential forms (with complex valued coefficients) in $M$. When $p = 0$ we write $\mathcal{O}_M(M)$ for $\Omega^0_M(M)$. The functions $f$ in $\mathcal{O}_M(M)$ are called CR functions on $M$ and the forms in $\Omega^p_M(M)$ CR forms of degree $p$ in $M$.

We note that any nonempty open subset $U$ of $M$ is in a natural way a CR manifold of the same CR dimension and CR codimension. We can therefore consider in a consistent way the space $\mathcal{O}_M(U)$ of CR functions on $U$, the spaces $\Omega^p_M(U)$ of CR $p$-forms on $U$, etc.

If $S$ is a real submanifold of $M$ and $\psi$ a differential form defined on a neighborhood of $S$, we denote by $\psi|_S$ the pullback of $\psi$ to $S$.

We say that $S \subset M$ is a CR submanifold of $M$ if

$$HS = (TS \cap HM) \cap J(TS \cap HM)$$

is a distribution of constant rank in $TS$. The triple $(S, HS, J|_{HS})$ satisfies indeed in this case the requirements for an abstract CR manifold of the definition at the beginning of the section.

Let $(M_1, HM_1, J_1)$ and $(M_2, HM_2, J_2)$ be two (abstract) CR manifolds of type $(n_1, k_1)$ and $(n_2, k_2)$ respectively. A differentiable map $F : M_1 \to M_2$ is called a CR map iff: (i) $dF(HM_1) \subset HM_2$, and (ii) $dF(J_1X) = J_2dF(X)$ for every $X \in HM_1$.

§3 Local defining functions

In the sequel we shall consider the following situation: $M$ is a smooth ($C^\infty$) abstract CR manifold of type $(n, k)$, where $n$ is the CR dimension and $k$ the CR codimension. We assume that $M$ is connected, paracompact (countable at infinity) and orientable (see §2).

$S$ will be a polar submanifold in $M$: By polar we mean that $S$ is a smooth closed CR submanifold of type $(n - 1, k)$, and that $S$ is transversal to the Levi distribution on $M$; i.e.,

$$T_xS + H_xM = T_xM, \quad \forall x \in S.$$

Here closed means as a subset; hence the topology of the differentiable submanifold $S$ agrees with the one induced on $S$ from the topology of $M$, and $S$ is also paracompact. These assumptions imply that the characteristic bundle $H^0S$ of $S$ is just the restriction to $S$ of the characteristic bundle $H^0M$ of $M$. Since $M$ being orientable means that $H^0M$ is orientable, we have that $H^0S$ is orientable; hence a given orientation of $M$ induces a corresponding orientation of $S$. 
Lemma 3.1  Let \( p \in S \). Then there is an open neighborhood \( U \) of \( p \) in \( M \) and a \( C^\infty \) function \( s : U \to \mathbb{C} \) with \( S \cap U = \{ x \in U \mid s(x) = 0 \} \) and \( ds \neq 0 \) in \( U \) satisfying

(i) \( \partial_M s = 0 \) on \( S \cap U \);
(ii) in fact, \( s \) can be chosen so that

\[
ds = s\gamma + \eta \quad \text{in} \quad U,
\]

where \( \gamma, \eta \in \mathcal{E}^1(U) \) and \( \eta \in \mathcal{I}(U) \). Moreover, the class \( [\gamma] \in \mathcal{Q}^{0,1}(U) \) defined by \( \gamma \) satisfies \( \partial_M [\gamma] = 0 \) on \( U \).

Proof  In a sufficiently small neighborhood \( U \) of \( p \) there are local \( C^\infty \) real valued defining functions \( \rho_1, \rho_2 \) for \( S \) with \( d\rho_1, d\rho_2 \) linearly independent at each point of \( U \). But since \( S \) is polar, \( d\rho_1 \) and \( d\rho_2 \) are not linearly independent modulo the ideal \( \mathcal{I}(M) \) along \( S \). Thus near \( p \) there is a uniquely determined complex valued smooth function \( f(x) \), defined along \( S \), such that \( d\rho_1(x) + f(x) \cdot d\rho_2(x) \in \mathcal{I}_x(M) \) on \( S \). Take any smooth extension \( \tilde{f} \) of \( f \) to \( U \). Then \( s = \rho_1 + f\rho_2 \) satisfies (i).

We denote the function \( s \) obtained above by \( s_0 \); because of (i) it satisfies

\[
ds_0 = s_0\gamma_0 + \tilde{s}_0\alpha_0 + \eta_0, \quad \eta_0 \in \mathcal{I}(U).
\]

Here and also below \( \gamma_k, \alpha_k, \eta_k \in \mathcal{E}^1(U) \) and \( \eta_k \in \mathcal{I}(U) \), and et cetera with primes. Applying \( d \) to (3.2), we find that \( d\tilde{s}_0 \wedge \alpha_0 \) belongs to \( \mathcal{I}(M) \) along \( S \). By Cartan’s lemma

\[
\alpha_0 = g_0 \bar{d}\tilde{s}_0 + s_0\alpha'_0 + \tilde{s}_0\alpha''_0 + \eta'_0,
\]

where \( g_0 \in \mathcal{E}^0(U) \). We set \( s_1 = s_0 - \frac{1}{2}g_0(\tilde{s}_0)^2 \) and obtain a new local defining function in \( U \) which satisfies

\[
ds_1 = s_1\gamma_1 + (\tilde{s}_1)^2\alpha_1 + \eta_1, \quad \eta_1 \in \mathcal{I}(U).
\]

By induction we obtain a sequence \( \{s_m\}_{m=0}^\infty \) of local defining functions in \( U \) satisfying

\[
ds_m = s_m\gamma_m + (\tilde{s}_m)^{m+1} + \eta_m, \quad \eta_m \in \mathcal{I}(U)
\]

and

\[
s_m - s_{m-1} = O(|s_0|^{m+1}) \quad \text{as} \quad s_0 \to 0.
\]

We have constructed \( s_0, s_1 \) satisfying (3.4), (3.5) for \( m = 1 \). So assume we have \( s_0, s_1, \ldots, s_m \) and to verify the induction step, we construct \( s_{m+1} \): applying \( d \) to (3.4) we obtain

\[
s_m d\gamma_m + (\tilde{s}_m)^m \left[ (m+1)\bar{d}\tilde{s}_m \wedge \alpha_m + \tilde{s}_m\alpha_m \wedge \gamma_m + \eta_m + d\eta_m \right] = 0,
\]

in which the last two terms belong to \( \mathcal{I}(U) \).

In order to exploit (3.6) we note that by the local triviality of vector bundles, we may shrink \( U \) if necessary, and obtain

\[
\mathcal{E}^2(U)/\mathcal{I}(U) \wedge \mathcal{E}^2(U) \simeq [\mathcal{E}^0(U)]^{(2)}.
\]
We consider the equations

\begin{equation}
\begin{cases}
F = -(\bar{s}_m)^m V \\
G = s_m V,
\end{cases}
\end{equation}

which are to be solved for \( V \in [\mathcal{E}^0(U)]^{(n)} \), given \( F, G \in [\mathcal{E}^0(U)]^{(2)} \) which satisfy the compatibility condition

\begin{equation}
\label{compatibility}
\tag{3.9}
s_m F + (\bar{s}_m)^m G = 0.
\end{equation}

We may choose a coordinate system in \( U \) of the form \((s_m, \bar{s}_m, x_3, \ldots, x_{2n+k}) = (s_m, \bar{s}_m, x)\). We have the exact sequence:

\begin{equation}
\label{exact_sequence}
\tag{3.10}
C[s_m, \bar{s}_m, x] \xrightarrow{\begin{bmatrix} -(\bar{s}_m)^m \\ s_m \end{bmatrix}} C[s_m, \bar{s}_m, x]^2 \xrightarrow{\begin{bmatrix} s_m, (\bar{s}_m)^m \end{bmatrix}} C[s_m, \bar{s}_m, x]
\end{equation}

of homomorphisms over the ring of polynomials in the coordinates. Since the ring of formal power series is flat over the ring of polynomials, the equation (3.8) has a formal power series solution at each point of \( U \). Since \( s_m \) and \((\bar{s}_m)^m\) may be regarded as real analytic (polynomial) functions of the coordinates \( x_1 = \Re s_m, x_2 = \Im s_m \), we may apply the Whitney theorem on closed ideals \([T]\); to obtain that (3.8) admits a smooth solution \( V \) in \( U \). Returning to (3.6) we take \( F, G \) to be the projections of \( d\gamma_m \) and \([m+1]d\bar{s}_m \wedge \alpha_m + \bar{s}_m \alpha_m + \gamma\alpha_m + \bar{s}_m d\alpha_m\) into the quotient (3.7), respectively. Then by the discussion above, we have a solution \( V \) to (3.8), and it follows that \( d\bar{s}_m \wedge \alpha_m \) belongs to the ideal \( \mathcal{I}(M) \) along \( S \). Again by Cartan’s lemma

\[\alpha_m = g_m d\bar{s}_m + s_m \alpha_m' + \bar{s}_m \alpha_m'' + \eta_m',\]

with \( g_m \in \mathcal{E}^0(U) \) and \( \eta_m' \in \mathcal{I}(U) \). Finally we set \( s_{m+1} = s_m - \frac{1}{m+2} g_m (\bar{s}_m)^{m+2}\). This \( s_{m+1} \) satisfies (3.4) and (3.5).

In a neighborhood \( U \) of \( p \) in \( M \) we may now construct a \( C^\infty \) function \( \tilde{s} \) such that

\begin{equation}
\label{tilde_s}
\tag{3.11}
\tilde{s} - s_m = O\left(|s_0|^{m+1}\right) \quad \text{as} \quad s_0 \to 0.
\end{equation}

This just boils down to the well-known fact that it is possible to construct a \( C^\infty \) function whose normal derivatives are all smoothly prescribed along the smooth manifold \( S \). By construction this \( \tilde{s} \) satisfies (3.1) for suitable \( \gamma, \eta \in \mathcal{E}^1(U) \) with \( \eta \in \mathcal{I}(U) \).

Thus \( d\gamma \) is in the ideal off of \( S \), and hence in the ideal across \( S \), as \( \gamma \) is smooth. Therefore \( \bar{\partial}_M [\gamma] = 0 \) in \( U \), and the proof is complete.

**Remark 1.** Note that the form \( \gamma \) in (3.1) is determined modulo \( \mathcal{I}(U) \). If there is a neighborhood \( U \) of \( p \) in \( M \) in which we can solve the equation

\begin{equation}
\label{equation}
\tag{3.12}
\bar{\partial}_M u = [\gamma] \quad \text{in} \quad U
\end{equation}

for a function \( u \in \mathcal{E}^0(U) \), then \( \hat{s} = e^{-u} \tilde{s} \) is a local defining function for \( S \) which satisfies \( \bar{\partial}_M \hat{s} = 0 \) in \( U \), i.e. there is a local defining function for \( f \) at \( p \) which is CR on \( M \) in a neighborhood of \( p \).
Remark 2. In particular, Remark 1 applies if $M$ is locally embeddable at $p$ and is 2-pseudoconcave there; indeed the $\bar{\partial}_M$-Poincaré lemma for $Q^{0,1}$-forms on $M$ is valid at $p$. (see [N1], [HN1], [HN2]).

Lemma 3.2 Suppose our abstract $M$ is 1-pseudoconcave at $p$. Let $s, \tilde{s}$ be two local $C^\infty$ defining functions for $S$ at $p$, each satisfying (3.1). Then there is a local nonvanishing $C^\infty$ function $h$ at $p$ such that $\tilde{s} = hs$.

Proof We shall use the hypoellipticity of the $\bar{\partial}_M$ operator on functions; namely, if $u \in L^2_{loc}$ near $p$, and if $\bar{\partial}_Mu$ is smooth in a neighborhood of $p$, then $u$ is $C^\infty$ at $p$. This result is a consequence of the interior subelliptic estimate with loss of $\frac{1}{2}$ derivative proved in [HN1], although the hypoellipticity consequence was not explicitly stated there.\(^2\)

Set $u = \log \left( \frac{\tilde{s}}{s} \right)$. Then $u \in L^2_{loc}$ and it follows from (3.1) that $\bar{\partial}_Mu$ is smooth; indeed $\bar{\partial}_Mu = [\tilde{\gamma}] - [\gamma]$, where as above $[\cdot]$ denotes the class in $Q^{0,1}$. Hence $u$ is smooth and we may take $h$ to be $e^u$.

We end this section with an example showing that polar submanifolds $S$ in $M$ do not always have local defining functions that are CR.

Example. Consider $M = \mathbb{R}^5 = \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R}_t$. We specify on $M$ the structure of an abstract CR manifold of type $(2,1)$ by prescribing the following basis for $T^{0,1}M$:

\[
\begin{cases}
\frac{\partial}{\partial \bar{z}} - \sqrt{-1} z \frac{\partial}{\partial t} + wf(z,t) \frac{\partial}{\partial w}, \\
\frac{\partial}{\partial \bar{w}}.
\end{cases}
\]

We choose the smooth complex valued function $f(z,t)$ such that the equation of Hans Lewy [L],

\[
\frac{\partial u}{\partial \bar{z}} - \sqrt{-1} z \frac{\partial u}{\partial t} = f, \quad \text{in} \quad \mathbb{R}^3,
\]

has no local solution in a neighborhood of any point. Then $S = \{w = 0\}$ is a polar submanifold in $M$. Suppose that $M$ could be locally defined, near some point, by a smooth defining function $s$. In particular, $s$ satisfies

\[
\frac{\partial s}{\partial \bar{z}} - \sqrt{-1} z \frac{\partial s}{\partial t} + wf \frac{\partial s}{\partial w} = 0.
\]

Applying $\partial/\partial w$ to (3.15) yields

\[
\left( \frac{\partial}{\partial \bar{z}} - \sqrt{-1} z \frac{\partial}{\partial t} \right) \frac{\partial s}{\partial w} + f \frac{\partial s}{\partial w} + wf \frac{\partial^2 s}{\partial w^2} = 0.
\]

Note that $\partial s / \partial w \neq 0$ along $S$, as $s$ is a defining function. Consider the function

\[
u(z,t) = -\log \frac{\partial s}{\partial w}(z,0,t),
\]

\(^2\)For a proof of the hypoellipticity under the weaker notion of essential pseudoconcavity see [HN3]; actually Lemma 3.2 remains valid under this weaker assumption.
using any local branch of the logarithm. It gives a smooth solution of (3.14), which contradicts our choice of \( f \). Note that this example gives another interpretation of the example in [H].

§4 The residue form

We return to the situation with \( M \) and \( S \) as at the beginning of §3. Consider a form \( \phi \in \mathcal{E}^p(M \setminus S) \).

**Definition.** \( \phi \) is said to have a pole of (at most) the first order along \( S \) iff: given any point \( p \in S \), there exists an open neighborhood \( U \) of \( p \) in \( M \), and a smooth local defining function \( s \) for \( S \) in \( U \), with \( ds \neq 0 \) in \( U \), and satisfying \( \bar{\partial}_M s = 0 \) on \( S \cap U \), such that \( s \phi \in \mathcal{E}^p(U) \).

**Theorem 4.1** Let \( \phi \in \mathcal{E}^p(M \setminus S) \) be closed on \( M \setminus S \) and have a first order pole along \( S \). Let \( p, U, s \) be as in the above definition. Then

1. There exists \( \psi \in \mathcal{E}^{p-1}(U) \) and \( \theta \in \mathcal{E}^p(U) \) such that
   \[
   \phi = \frac{ds}{s} \land \psi + \theta \quad \text{in} \quad U.
   \]

2. \( \psi \big|_S \) is closed on \( S \cap U \) and depends only on \( \phi \) and \( S \).

3. If \( \phi \in \Omega^p_M(M \setminus S) \) then \( \psi \big|_S \in \Omega^{p-1}_S(S \cap U) \).

**Proof** Since \( \phi \) is closed, \( d(s \phi) = ds \land \phi + s d\phi = ds \land \phi \), showing that \( ds \land \phi \) has the smooth extension \( d(s \phi) \) across \( S \cap U \). Hence by continuity we have \( ds \land d(s \phi) = 0 \) in \( U \). By Cartan’s lemma there exists \( \theta \in \mathcal{E}^p(U) \) with \( d(s \phi) = ds \land \theta \) in \( U \). It follows that \( ds \land (s \phi - s \theta) = 0 \) in \( U \). Applying Cartan’s lemma again, we obtain a \( \psi \in \mathcal{E}^{p-1}(U) \) such that \( s \phi - s \theta = ds \land \psi \) in \( U \). This establishes the first point in the theorem.

Next we show that \( \psi \big|_S \) depends only on \( \phi \) and on \( s \): Suppose we have
   \[
   \phi = \frac{ds}{s} \land \psi_1 + \theta_1 = \frac{ds}{s} \land \psi_2 + \theta_2
   \]
with \( \psi_1, \psi_2 \in \mathcal{E}^{p-1}(U) \) and \( \theta_1, \theta_2 \in \mathcal{E}^p(U) \). Setting \( \psi = \psi_1 - \psi_2, \theta = \theta_1 - \theta_2 \) we subtract and multiply by \( s \) to obtain \( ds \land \psi + s \theta = 0 \). Hence \( ds \land (s \theta) = 0 \) and therefore \( ds \land \theta = 0 \) in \( U \). Once again this yields \( \theta = ds \land \omega \) in \( U \), where \( \omega \in \mathcal{E}^{p-1}(U) \). Substituting back in for \( \theta \) we find that \( ds \land (\psi + s \omega) = 0 \) in \( U \). As \( \psi + s \omega \) is smooth in \( U \), \( \psi + s \omega = ds \land \tilde{\omega} \) for an \( \tilde{\omega} \in \mathcal{E}^{p-2}(U) \). Hence \( \psi \big|_S = 0 \) on \( S \cap U \), so \( \psi_1 \big|_S = \psi_2 \big|_S \), as claimed.

Return to the \( \psi \) of the theorem. To show that \( \psi \big|_S \) is closed, we observe that
\[
0 = d\phi = -\frac{ds}{s} \land d\psi + d\theta.
\]
This means we can apply the argument above (to \( d\phi \)) and conclude that \( d(\psi \big|_S) = (d\psi) \big|_S = 0 \) on \( S \cap U \).

Finally we show that \( \psi \big|_S \) does not depend on the choice of the local defining function \( s \). Let \( s^* \) be another smooth local defining function for \( S \) in \( U \), with \( ds^* \neq 0 \) in \( U \) and \( \bar{\partial}_M s^* = 0 \) on \( S \cap U \), such that \( s^* \phi \in \mathcal{E}^p(U) \). Using the defining function \( s^* \) we obtain

\[
\phi = \frac{ds^*}{s^*} \land \psi^* + \theta^*, \quad \psi^* \in \mathcal{E}^{p-1}(U), \quad \theta^* \in \mathcal{E}^p(U)
\]
 \[
(4.1)
\]

In order to prove (4.1) we first observe that \( \phi \in \mathcal{E}^p(U) \) and \( \bar{\partial}_M \phi = 0 \) on \( S \cap U \), since \( \bar{\partial}_M s = 0 \). By Cartan’s lemma there exists \( \psi \in \mathcal{E}^{p-1}(U) \) and \( \theta \in \mathcal{E}^p(U) \) such that
   \[
   d\phi = ds \land \psi + \theta \quad \text{in} \quad U.
   \]

By definition of \( \psi \) in (4.1) we have
   \[
   d\psi = ds^* \land \psi^* + \theta^* \quad \text{in} \quad U.
   \]

By (4.1) we have
   \[
   d\phi = ds \land \psi + \theta = ds \land \psi^* + ds^* \land \psi^* + d\theta = ds \land \psi^* + ds^* \land \psi^* + d\theta = ds \land \psi^* + ds^* \land \psi^* + d\theta.
   \]

Since \( d\phi = ds \land \psi + \theta \) in \( U \), it follows that \( ds \land \psi^* + ds^* \land \psi^* = 0 \) in \( U \). This establishes (4.1) and completes the proof.
and also write

\( \phi = \frac{ds^*}{s^*} \wedge \psi + \theta + \left[ \frac{ds}{s} - \frac{ds^*}{s^*} \right] \wedge \psi \).

If \( s^* = h s \) where \( h \in C^\infty(U), h \neq 0 \), then \( \left[ \frac{ds}{s} - \frac{ds^*}{s^*} \right] = -\frac{dh}{h} \), which is smooth across \( S \), so the uniqueness argument given above shows that \( \psi^*|_S = \psi|_S \). In general we have that \( ds^* = h ds \) along \( S \), with \( h \in C^\infty(U), h \neq 0 \) on \( U \); indeed \( (CT_M^* M)_x \cap L_x(M) \simeq \mathbb{C} \) for \( x \in S \cap U \), where \( CT_M^* S \) denotes the complexification of the conormal bundle of \( S \) in \( M \). By the discussion above we may replace \( s \) by \( h s \) without changing \( \psi|_S \). With this done, we now have that \( ds^* = ds \) along \( S \cap U \). It follows that \( s^* - s = O(|s|^2) \), which in turn implies that \( \left[ \frac{ds}{s} - \frac{ds^*}{s^*} \right] \) is bounded in \( U \). Subtracting (4.2) from (4.1) we obtain \( 0 = \frac{ds^*}{s^*} \wedge [\psi^* - \psi] + \tilde{\theta} \) with \( \tilde{\theta} \) bounded in \( U \). This implies that \( ds^* \wedge [\psi^* - \psi] = 0 \) along \( S \). This yields that \( \psi^* - \psi = ds^* \wedge \omega + s^* \omega' + \tilde{s} \omega'' \) in \( U \), where \( \omega \in \mathcal{E}^{p-2}(U) \) and \( \omega', \omega'' \in \mathcal{E}^{p-1}(U) \).

Therefore \( \psi^*|_S = \psi|_S \), completing the proof of the second part of the theorem.

Finally we establish the third point of the theorem: Since \( \phi \in \Omega^p_M(M \setminus S) \), we have in particular that \( \phi \in \mathcal{I}^p(M \setminus S) \). By continuity \( s \phi \in \mathcal{I}^p(M) \). Thus \( d(s \phi) \in \mathcal{I}^{p-1}(M) \) and we can choose \( \theta \in \mathcal{I}^{p-1}(U) \) in (1). Therefore \( ds \wedge \psi \in \mathcal{I}^p(M) \) along \( S \cap U \). This forces \( \psi \) to belong to \( \mathcal{I}^{p-1}(U) \) along \( S \cap U \). Since \( \mathcal{I}^{p-1}(S) = \mathcal{I}^{p-1}(M)|_S \) we conclude that \( \psi|_S \in \mathcal{I}^{p-1}(S \cap U) \). But \( d \psi|_S = 0 \), so \( \psi|_S \in \Omega^{p-1}_S(S \cap U) \). This completes the proof of the theorem.

Let \( F \) be a closed subset of \( M \) which satisfies

\( S \cap F \subset \overline{(M \setminus S) \cap F} \).

Suppose \( \phi = 0 \) on \( F \). Then it is clear from the above construction that \( \psi|_S \) vanishes on \( (S \cap F) \cap U \).

Let \( \phi \in \mathcal{E}^p(M \setminus S) \) be closed on \( M \setminus S \) and have a first order pole along \( S \). Let \( U, \psi \) as in Theorem 4.1.

**Definition.** The residue form of \( \phi \) on \( S \) is locally defined by

\( \text{res} [\phi] = \psi|_S \) on \( S \cap U \);

but it is well-defined globally on \( S \), as the local definitions coincide on intersections of different \( S \cap U \)’s by the second point in Theorem 4.1.

**Remark 1.** Note that \( \text{res} [\phi] \in \mathcal{E}^{p-1}(S) \) and is a closed form on \( S \). Define the closed set \( \text{singsupp }|_S \phi \) on \( S \) by saying that \( x \notin \text{singsupp }|_S \phi \) iff there is an open neighborhood \( U \) of \( x \) in \( M \) such that \( \phi|_{U \setminus S} \) is bounded. Then from the above discussion we obtain:

\( \text{supp } \text{res} [\phi] \subset \text{singsupp }|_S \phi \).

**Remark 2.** When \( k = 0 \), \( M \) is a complex manifold of complex dimension \( n \) (Newlander-Nirenberg theorem) and \( S \) is a complex hypersurface. In this case we...
recover exactly the forme-résidu of Leray [Lr]; which in turn was a generalization of the classical residue in one complex variable.

§5 Properties of the residue form

Let $\phi \in \mathcal{E}^p(M \setminus S)$ be closed in $M \setminus S$ and have a first order pole along $S$; we investigate some basic properties of $\text{res} [\phi]$.

**Proposition 5.1** Let $\chi \in \mathcal{E}^q(M)$ be a closed form on $M$. Then $\phi \wedge \chi$ has a first order pole along $S$, and

$$
\text{(5.1)} \quad \text{res} [\phi \wedge \chi] = \text{res} [\phi] \wedge \chi|_S .
$$

**Proof** If $\phi = \frac{ds}{s} \wedge \psi + \theta$, then $\phi \wedge \chi = \frac{ds}{s} \wedge (\psi \wedge \chi) + (\theta \wedge \chi)$, so the result is a consequence of the uniqueness point of Theorem 4.1.

Suppose that $S$ has a global smooth defining function $s$:

$$
S = \{ x \in M | s(x) = 0 \} ,
$$

with $ds \neq 0$ and $\partial_M s = 0$ on $S$. Let $\phi_1 \in \mathcal{E}^{p_1}(M \setminus S)$ and $\phi_2 \in \mathcal{E}^{p_2}(M \setminus S)$ both be closed in $M \setminus S$, and such that $s\phi_1$ and $s\phi_2$ are smooth across $S$.

**Proposition 5.2** Under the above assumptions:

(i) $\phi_1 \wedge \phi_2$ has a first order pole along $S$.

(ii) $\phi_1 \wedge \frac{ds}{s}$ and $\phi_2 \wedge \frac{ds}{s}$ also have first order poles along $S$.

(iii) We have:

$$
\text{(5.2)} \quad \text{res} [\phi_1 \wedge \phi_2] = \text{res} \left[ \phi_1 \wedge \frac{ds}{s} \right] \wedge \text{res} [\phi_2] + \text{res} [\phi_1] \wedge \text{res} \left[ \frac{ds}{s} \wedge \phi_2 \right] .
$$

**Proof** As in the previous discussion, we have

$$
\phi_1 = \frac{ds}{s} \wedge \psi_1 + \theta_1 \quad \text{and} \quad \phi_2 = \frac{ds}{s} \wedge \psi_2 + \theta_2
$$

with $\psi_1, \psi_2, \theta_1, \theta_2$ smooth across $S$. Hence

$$
\phi_1 \wedge \phi_2 = \frac{ds}{s} \wedge \{ \psi_1 \wedge \theta_2 + (-1)^{p_1} \theta_1 \wedge \psi_2 \} + \theta_1 \wedge \theta_2 ,
$$

$$
\phi_i \wedge \frac{ds}{s} = \theta_i \wedge \frac{ds}{s} , \quad i = 1, 2 .
$$

From these two formulas we obtain $(i), (ii)$ and that $\text{res} \left[ \frac{ds}{s} \wedge \phi_i \right] = \theta_i|_S$, and $(5.2)$ follows from $(5.3)$. 
Next we derive a pointwise estimate for \( \text{res} [\phi] \): choose a smooth Hermitian metric on the fibers of \( \mathbb{C}T^*M \); it induces a corresponding Hermitian metric on the fibers of \( \Lambda^p \mathbb{C}T^*M \). We define a smooth Hermitian metric on the fibers of \( \mathbb{C}T^*S \) by the natural identification with the orthogonal complement of \( \mathbb{C}T^*_M \) in the restriction to \( S \) of \( \mathbb{C}T^*M \); thereby inducing a corresponding Hermitian metric on the fibers of \( \Lambda^{p-1} \mathbb{C}T^*S \). Let \( \phi \in \mathcal{E}^p(M \setminus S) \) be closed in \( M \setminus S \) and have a first order pole along \( S \). Let \( x \in S \) and \( s \) be a smooth defining function for \( S \) near \( x \), with \( \partial_M s = 0 \) on \( S \), and \( s\phi \) be smooth across \( S \) near \( x \). Then we have

**Proposition 5.3**

\[
|\text{res} [\phi](x)|_S \leq \frac{|(s\phi)(x)|_M}{|ds(x)|_M},
\]

where \( | \cdot |_M \) and \( | \cdot |_S \) denote the Hermitian lengths in \( \Lambda^p \mathbb{C}T^*_x M \), \( \Lambda^1 \mathbb{C}T^*_x M \) and \( \Lambda^{p-1} \mathbb{C}T^*_x S \), respectively.

**Proof** We note that for any form \( \psi \) on \( M \) we have \( |ds(x)|_M \cdot |(\psi|_S)(x)|_S \leq |(ds \wedge \psi)(x)|_M \). Since \( s\phi = ds \wedge \psi + s\theta \), the result follows.

We now study the behaviour of \( \text{res} [\phi] \) under the pullback by a \( CR \) mapping: Let \( M^* \) be a smooth abstract \( CR \) manifold of type \( (n^*, k^*) \), with the same hypotheses as on \( M \) (i.e. connected, paracompact, orientable). Let \( F: M^* \rightarrow M \) be a smooth \( CR \) map such that \( F(M^*) \cap S \neq \emptyset \). We assume that \( S \) is \( CR \) transversal to \( S \). By this we mean first of all that \( F \) is transversal to \( S \) in the sense of differential topology; i.e., that

\[
dF(x^*) (T_{x^*} M^*) + T_{x^*} S = T_x M, \quad \forall x^* \in F^{-1}(S), \quad x = F(x^*),
\]

and in addition that

\[
\begin{align*}
\dim dF(x^*) (H_{x^*} M^*) \text{ is constant for } x^* \in F^{-1}(S) \\
dF(x^*)(H_{x^*} M^*) + H_x S = H_x M, \quad \forall x^* \in F^{-1}(S), \quad x = F(x^*).
\end{align*}
\]

Let \( S^* = F^{-1}(S) \). By (5.5) we have that \( S^* \) is a smooth closed submanifold of \( M^* \) of real codimension two. From (5.6) it follows that \( S^* \) is a \( CR \) submanifold of \( M^* \), of type \( (n^* - 1, k^*) \), polar in \( M^* \). Indeed let \( x^* \in S^* \) and \( x = F(x^*) \). Let \( s \) be a local defining function for \( S \) near \( x \), with \( \partial_M s = 0 \) along \( S \). Then \( s^* = s \circ F \) is a local defining function for \( S^* \) near \( x^* \). Since \( F \) is a \( CR \) map, we may replace \( H \) in (5.6) by \( T^{1,0} \) or \( T^{0,1} \). We first show that \( \partial_M s^* = 0 \) along \( S^* \) near \( x^* \). Consider a point \( y^* \in S^* \) near \( x^* \); we have

\[
\left\langle ds^*(y^*), T^{0,1}_{y^*} M^* \right\rangle = \left\langle ds(y), dF(y^*) \left( T^{0,1}_{y^*} M^* \right) \right\rangle = 0, \quad y = F(y^*),
\]

because \( dF(y^*) \left( T^{0,1}_{y^*} M^* \right) \subset T^{0,1}_y M \). Finally we show that \( S^* \) is polar in \( M^* \). From (5.6) we get

\[
\left\langle ds^*(y^*), T^{1,0}_{y^*} M^* \right\rangle = \left\langle ds(y), dF(y^*) \left( T^{1,0}_{y^*} M^* \right) \right\rangle
\]

\[
= \left\langle ds(y), dF(y^*) \left( T^{1,0}_{y^*} M^* \right) \right\rangle
\]

\[
= 0.
\]
This means that $ds^*(y^*) \in I_{y^*}(M^*) \setminus I_{y^*}(M^*)$, which forces $S^*$ to be polar in $M^*$.

Now consider a form $\phi \in \mathcal{E}^p(M \setminus S)$, closed in $M \setminus S$, and having a first order pole along $S$. Then $F^*\phi \in \mathcal{E}^p(M^* \setminus S^*)$, $F^*\phi$ is closed in $M^* \setminus S^*$, and again has a first order pole along $S^*$, since $F^*(s\phi) = s^*F^*\phi$.

**Proposition 5.4** In the situation described above, we have that

\begin{equation}
\text{res } [F^*\phi] = F^*\text{res } [\phi].
\end{equation}

**Proof** From $\phi = \frac{ds}{s} \land \psi + \theta$ it follows that

\begin{equation}
F^*\phi = \frac{ds}{s} \land F^*\psi + F^*\theta; \quad \text{hence } (F^*\psi)|_{S^*} = F^*(\psi|_S). \quad \text{[Here for simplicity we use the same letter $F$ to denote the restriction map: } S^* \ni y^* \to F(y^*) \in S.]
\end{equation}

Finally consider a smooth closed submanifold $\Sigma_1$ of $M$ which is transversal to $S$.

**Proposition 5.5** If $\phi|_{\Sigma_1 \setminus S} = 0$, then $\text{res } [\phi]|_{\Sigma_1 \cap S} = 0$.

**Proof** Locally we have $s\phi = ds \land \psi + s\theta$. Taking pullbacks to $\Sigma_1$ we obtain

\begin{equation}
0 = (s\phi)|_{\Sigma_1} = (ds)|_{\Sigma_1} \land \psi|_{\Sigma_1} + (s\theta)|_{\Sigma_1}, \quad \text{which along } \Sigma_1 \cap S \text{ yields } (ds)|_{\Sigma_1} \land \psi|_{\Sigma_1} = 0. \quad \text{By the transversality assumption, } (ds)|_{\Sigma_1} \neq 0. \quad \text{Therefore by Cartan’s lemma, locally, } \psi|_{\Sigma_1} = (ds)|_{\Sigma_1} \land \alpha + s\beta + s\gamma \quad \text{for smooth forms } \alpha, \beta, \gamma. \quad \text{Hence } (\psi|_S)|_{\Sigma_1} = (\psi|_{\Sigma_1})|_S = 0 \quad \text{as claimed.}
\end{equation}

§6 The residue formula

With $M, S$ as in the beginning of §3, we introduce also

\begin{equation}
\Sigma = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_\ell,
\end{equation}

where each $\Sigma_j$ is a smooth closed submanifold of $M$. We assume that the $S, \Sigma_1, \Sigma_2, \ldots, \Sigma_\ell$ are in general position.

This implies, in particular, that the intersection of any subset of the $\Sigma_1, \ldots, \Sigma_\ell$ is transversal to $S$. This allows to construct a tubular neighborhood $V$ of $S$ in $M$ adapted to $\Sigma$: recall that the normal bundle $N_SM$ of $S$ in $M$ is the quotient $TM|_S / TS$. The tubular neighborhood $V$ is the datum of an open neighborhood $V$ of $S$ in $M$, together with a smooth diffeomorphism

\begin{equation}
\nu : N_SM \longrightarrow V
\end{equation}

which is the identity on $S$, identified with the zero section of $N_SM$. Denote by $\pi : N_SM \longrightarrow S$ the projection. The fact that the tubular neighborhood is adapted to $\Sigma$ means, in particular, that $\nu (\pi^{-1}(\Sigma \cap S)) = \Sigma \cap V$.

This enables us to construct a smooth strict deformation retract

\begin{equation}
\mu : [0, 1] \times V \longrightarrow V
\end{equation}

of the pair $(V, \Sigma \cap V)$ onto the pair $(S, \Sigma \cap S)$; namely,

\begin{equation}
\mu(t, x) = \mu(x) = \mu(t \nu^{-1}(x)).
\end{equation}
We fix a smooth Riemannian metric on the vector bundle $N_SM$, and denote the length of a vector $v$ in the fiber at $x \in S$ by $|v|_x$.

Consider a relative $(p - 1)$-cycle $\gamma$ in $(S, \Sigma \cap S) \equiv (S, \Sigma)$. Here we use smooth singular chains with compact support, and take $\mathbb{Z}$-coefficients. It is standard to identify $\gamma$ with a smooth map
\[
\tilde{\gamma} : (P_\gamma, \partial P_\gamma) \longrightarrow (S, \Sigma),
\]
where $P_\gamma$ is a finite polyhedron, of dimension $(p - 1)$, embedded in some Euclidean space $\mathbb{R}^N$. Let $\tilde{\gamma}^\ast (N_SM)$ denote the pulled-back bundle over $P_\gamma$. Using the metric on the fibers, we introduce the closed disk bundle $D_\gamma(t)$ and the circle bundle $C_\gamma(t)$ of radius $t > 0$ contained in $\tilde{\gamma}^\ast (N_SM)$. These can be regarded as $(p + 1)$ and $p$ dimensional polyhedra, respectively, embedded in some $\mathbb{R}^{N'}$. The map $\tilde{\gamma}$ lifts to a smooth vector bundle homomorphism
\[
\hat{\gamma}_{N_SM} : \tilde{\gamma}^\ast (N_SM) \longrightarrow N_SM.
\]
This enables us to define
\[
\begin{align*}
\tilde{D}_t\gamma : D_\gamma(t) & \longrightarrow V \subset M, \\
\tilde{c}_t\gamma : C_\gamma(t) & \longrightarrow V \setminus S \subset M \setminus S,
\end{align*}
\]
by $\left(\tilde{D}_t\gamma\right)(y) = \nu(\hat{\gamma}_{N_SM}(y))$ for $y \in D_\gamma(t)$, and $\left(\tilde{c}_t\gamma\right)(y) = \nu(\hat{\gamma}_{N_SM}(y))$ for $y \in C_\gamma(t)$. Since
\[
\begin{align*}
\partial C_\gamma(t) & = C_\gamma(T) \mid_{\partial P_\gamma}, \\
\partial D_\gamma(t) & = C_\gamma(t) \cup D_\gamma(t) \mid_{\partial P_\gamma},
\end{align*}
\]
and our tubular neighborhood $V$ was adapted to $\Sigma$, $\tilde{D}_t\gamma$ defines a smooth singular relative $p$-cycle $\delta_t\gamma$ in $(M \setminus S, \Sigma)$, and $\tilde{c}_t\gamma$ defines a smooth singular relative $(p + 1)$-chain $\tilde{c}_t\gamma$ in $(M, \Sigma)$, such that $\delta_t\gamma = \partial' (\tilde{D}_t\gamma)$, where $\partial'$ denotes the relative boundary in $(M, \Sigma)$, for each $0 < t \leq 1$. Note that each $\delta_t\gamma$ is homologous to $\delta_1\gamma$ in $(M \setminus S, \Sigma)$ by the construction.

We now turn to the theorem establishing the residue formula. We fix the following notation: $\mathcal{E}^p(M, \Sigma)$, $\mathcal{E}^p(M \setminus S, \Sigma)$ and $\mathcal{E}^{p-1}(S, \Sigma)$ will denote the subspaces of $\mathcal{E}^p(M)$, $\mathcal{E}^p(M \setminus S)$ and $\mathcal{E}^{p-1}(S)$ consisting of those smooth forms whose pullbacks to $\Sigma$, $\Sigma \cap (M \setminus S)$ and $\Sigma \cap S$ are zero, respectively.

Let $\gamma$ be a smooth compactly supported singular relative $(p - 1)$-cycle in $(S, \Sigma)$. Likewise let $\Gamma$ be a smooth compactly supported singular relative $p$-cycle in $(M \setminus S, \Sigma)$. If $\Gamma$ is homologous to $\delta_1\gamma$ in $(M \setminus S, \Sigma)$, we shall say that $\Gamma$ is "cobordant to $\gamma$ in $(M \setminus S, \Sigma)$".

**Theorem 6.1** Let $\phi \in \mathcal{E}^p(M \setminus S, \Sigma)$ be closed in $M \setminus S$ and have a first order pole along $S$. Let $\gamma$ be a smooth compactly supported singular relative $(p - 1)$-cycle in $(S, \Sigma)$. Then for every $\Gamma$ which is "cobordant to $\gamma$ in $(M \setminus S, \Sigma)$", we have
\[
\int \phi = 2\pi \sqrt{-1} \int \text{res} [\phi].
\]
Proof. By Stokes’ theorem we obtain

\begin{equation}
\int_{\Gamma} \phi = \int_{\delta_1 \gamma} \phi = \int_{\delta_t \gamma} \phi \quad \text{for } 0 < t \leq 1,
\end{equation}

as \( \Gamma \sim \delta_1 \gamma \sim \delta_t \gamma \) in \((M \setminus S, \Sigma)\). Hence it will suffice to show that

\begin{equation}
\lim_{t \to 0} \int_{\delta_t \gamma} \phi = 2\pi \sqrt{-1} \int_{\gamma} \text{res} \[ \phi \],
\end{equation}

which is a limit already known to exist, by (6.7).

Along \( S \) we introduce a locally finite partition of unity \( \sum_{\alpha} \chi_{\alpha} \equiv 1 \), subordinate to a covering \( \{ U_{\alpha} \} \) of \( S \) by open neighborhoods \( U_{\alpha} \) in which \( S \cap U_{\alpha} \) has a locally defining function \( s_{\alpha} \) satisfying \( \overline{\partial} M s_{\alpha} = 0 \) along \( S \), and such that \( s_{\alpha} \phi \in \mathcal{E}^p(U_{\alpha}) \). Then we have the decomposition

\begin{equation}
\int_{\delta_t \gamma} \phi = \int_{\delta_t \gamma} \left( \sum_{\alpha} \chi_{\alpha} \phi \right) = \sum_{\alpha} \int_{\delta_t \gamma} \chi_{\alpha} \phi,
\end{equation}

which can be regarded as a finite sum, uniformly for \( 0 < t \leq 1 \). Hence it will suffice to show that for each \( \alpha \),

\begin{equation}
\lim_{t \to 0} \int_{\delta_t \gamma} \chi_{\alpha} \phi = 2\pi \sqrt{-1} \int_{\gamma} \chi_{\alpha} \text{res} \[ \phi \].
\end{equation}

From Theorem 4.1 we have

\( \phi = \frac{ds_{\alpha}}{s_{\alpha}} \wedge \psi + \theta \) in \( U_{\alpha} \),

with \( \psi \) and \( \theta \) smooth across \( S \). By Stokes’ theorem

\[
\left| \int_{\delta_t \gamma} \chi_{\alpha} \theta \right| = \left| \int_{D_{\alpha}(t)} d(\chi_{\alpha} \theta) \right| \leq \text{const} \cdot t^2 \to 0,
\]

as \( t \searrow 0 \), since \( \theta \) is smooth across \( S \). Using the retraction \( \mu_0 : V \to S \) from (6.2), we may write

\begin{equation}
\psi - \mu_0^*(\psi | S) = s_{\alpha} \psi_1 + \bar{s}_{\alpha} \psi_2 + ds_{\alpha} \wedge \psi_3 + \bar{d}s_{\alpha} \wedge \psi_4
\end{equation}

in \( U_{\alpha} \), with smooth forms \( \psi_i \), because the left hand side in (6.11) pulls back to zero along \( S \). Note that \( [\mu_0^*(\psi | S)] |_{\Sigma \cap V} = 0 \) by Proposition 5.5. Consider

\begin{equation}
\frac{ds_{\alpha}}{s_{\alpha}} \wedge [\psi - \mu_0^*(\psi | S)] = ds_{\alpha} \wedge \psi_1 + \bar{s}_{\alpha} \bar{d}s_{\alpha} \wedge \psi_2 + \frac{ds_{\alpha} \wedge d\bar{s}_{\alpha}}{s_{\alpha}} \wedge \psi_4.
\end{equation}

Note that each term on the right in (6.12) is uniformly bounded on the intersection of any compact subset of \( U_{\alpha} \) with \( U_{\alpha} \setminus S \). Hence

\[
\left| \int_{U_{\alpha}} \frac{ds_{\alpha}}{s_{\alpha}} \wedge [\psi - \mu_0^*(\psi | S)] \chi_{\alpha} \right| \leq \text{const} \cdot t \to 0,
\]
as $t \searrow 0$. It follows that
\[
\lim_{t \searrow 0} \int_{\delta_t \gamma} \chi_{\alpha} \phi = \lim_{t \searrow 0} \int \chi_{\alpha} \frac{ds_{\alpha}}{s_{\alpha}} \wedge \mu_{\alpha}^* (\psi | s) = 2\pi \sqrt{-1} \int_{\gamma} \chi_{\alpha} \psi | s,
\]
and the theorem is proved.

**Remark.** As we are using smooth singular relative homology with $\mathbb{Z}$-coefficients, it may happen that the relative cycle $\gamma$ has finite order, say $a \cdot \gamma \sim 0$ in $H_{p-1}(S, \Sigma; \mathbb{Z})$ for some integer $a \neq 0$. In that case both right and left hand sides in the residue formula (6.6) are zero. Indeed the right-hand side is zero by Stokes’ theorem, since $\text{res} \ [\phi]$ is closed on $S$ and pullbacks to zero on $S \cap \Sigma$; likewise the left-hand side is zero because $a \Gamma \sim a\delta_1 \gamma \sim 0$ in $H_p(M \setminus S, \Sigma; \mathbb{Z})$.

§7 The Homological Residue

In what follows we shall denote the standard relative homology groups, with compact support and integer coefficients, of $(M, \Sigma)$, $(S, \Sigma)$ and $(M \setminus S, \Sigma)$ by $H_p(M, \Sigma; \mathbb{Z})$, $H_p(S, \Sigma; \mathbb{Z})$ and $H_p(M \setminus S, \Sigma; \mathbb{Z})$, respectively. As $M$, $S$ are smooth we may compute these, as is well-known, using smooth singular relative $p$-chains.

As a consequence of the discussion in §6, we actually obtain a coboundary homomorphism
\[
\delta : H_{p-1}(S, \Sigma; \mathbb{Z}) \to H_p(M \setminus S, \Sigma; \mathbb{Z}),
\]
defined by $\delta ([\gamma]) = [\delta_1 \gamma]$; indeed the construction we have given of $\delta_1 \gamma$ commutes with the boundary maps. This passes to a homomorphism
\[
\bar{\delta} : \bar{H}_{p-1}(S, \Sigma; \mathbb{Z}) \to \bar{H}_p(M \setminus S, \Sigma; \mathbb{Z})
\]
of the weak homology groups. Here we use a bar to indicate the weak homology groups defined by $\bar{H}_p = H_p / \text{Tor}_p$. Therefore by Stokes’ theorem the residue formula (6.6) can be reformulated more generally in terms of weak homology:

**Theorem 7.1** Let $\phi \in E^p(M \setminus S, \Sigma)$ be closed in $M \setminus S$ and have a first order pole along $S$. Let $\bar{h} \in \bar{H}_{p-1}(S, \Sigma; \mathbb{Z})$ be a coset of relative homology classes. Then
\[
\int_{\bar{h}} \phi = 2\pi \sqrt{-1} \int_{\bar{h}} \text{res} [\phi],
\]
in which $\bar{h}$ is a coset of relative homology classes in $\bar{H}_p(M \setminus S, \Sigma; \mathbb{Z})$.

We now return to the ”coboundary” homomorphism $\delta$ in (7.1); it can be inserted into a long exact sequence as follows: Set
\[
\nu \left( \{ \xi \in N_{S \setminus M} \mid |\xi|_{\pi(\xi)} \leq 1 \} \right) = V_1 \subset V
\]
and consider the standard long exact sequence of a relative pair
\[
\cdots \to H_p(M \setminus V_1, \Sigma; \mathbb{Z}) \xrightarrow{i} H_p(M, \Sigma; \mathbb{Z}) \xrightarrow{p} H_p(M \setminus S, \Sigma; \mathbb{Z}) \xrightarrow{\partial} H_{p-1}(M \setminus V_1, \Sigma; \mathbb{Z}) \xrightarrow{i} H_{p-1}(M, \Sigma; \mathbb{Z}) \xrightarrow{p} H_{p-1}(M \setminus S, \Sigma; \mathbb{Z}) \to \cdots
\]
First we observe that

\[ H_p \left( \overline{M \setminus V_1}, \Sigma; \mathbb{Z} \right) \simeq H_p(\overline{M \setminus S}, \Sigma; \mathbb{Z}) \]

because \( \overline{M \setminus V_1}, \Sigma \) is a relative deformation retract of \( (M \setminus S, \Sigma) \). Next we obtain

\[ H_{p+1} \left( M, \overline{M \setminus V_1} \cup \Sigma; \mathbb{Z} \right) \simeq H_{p+1}(V_1, \partial V_1 \cup \Sigma; \mathbb{Z}) \]

by the excision of \( M \setminus V_1 \). But by the relative Thom isomorphism we have

\[ H_{p+1}(V_1, \partial V_1 \cup \Sigma; \mathbb{Z}) \simeq H_{p-1}(S, \Sigma; \mathbb{Z}) \]

Thus we arrive at (cf. Leray [Lr]):

**Proposition 7.2**  There is a long exact sequence

\[
\cdots \to H_{p+1}(M \setminus S, \Sigma; \mathbb{Z}) \to H_{p+1}(M, \Sigma; \mathbb{Z}) \\
\to \tau \to H_{p-1}(S, \Sigma; \mathbb{Z}) \to H_p(\overline{M \setminus S}, \Sigma; \mathbb{Z}) \to \cdots
\]

**Remark.** The connecting homomorphism \( \partial \) in (7.3) becomes our \( \delta \) in (7.4) because the inverse of the Thom isomorphism is the map \([\gamma] \to [D_1 \gamma]\) described in §6, which becomes \([\gamma] \to [\delta_1 \gamma]\) when composed with the boundary map \( \partial \).

§8  The Cohomological Residue

In what follows we assume \( M, S \) as in the beginning of §3, and \( \Sigma \) as in the beginning of §6. Since \( d \) commutes with the pullback to \( \Sigma \), we have the complex:

\[
0 \to \mathcal{E}^0(M, \Sigma) \xrightarrow{d} \mathcal{E}^1(M, \Sigma) \xrightarrow{d} \mathcal{E}^2(M, \Sigma) \to \cdots
\]

This is the de Rham complex on smooth (complex valued) forms \( \phi \) on \( M \) with \( \phi|_{\Sigma} = 0 \). (We say that \( \phi \) has zero Cauchy data for \( d \) along \( \Sigma \)). We denote the \( p \)-th cohomology group of (8.1) by \( H^p(M, \Sigma) \). Instead of the pair \( (M, \Sigma) \) we may use the pair \( (S, \Sigma) \equiv (S, S \cap \Sigma) \), or the pair \( (M \setminus S, \Sigma) \equiv (M \setminus S, (M \setminus S) \cap \Sigma) \) in (8.1). So we have also \( H^p(S, \Sigma) \) and \( H^p(\overline{M \setminus S}, \Sigma) \). Since these cohomology groups are not entirely standard, we relate them to the standard singular relative cohomology groups \( H^p_{\text{sing}} \) with complex coefficients:

**Proposition 8.1**  For all \( p \)

(a) \( H^p(M, \Sigma) \simeq H^p_{\text{sing}}(M, \Sigma; \mathbb{C}) \)

(b) \( H^p(S, \Sigma) \simeq H^p_{\text{sing}}(S, \Sigma; \mathbb{C}) \)

(c) \( H^p(\overline{M \setminus S}, \Sigma) \simeq H^p_{\text{sing}}(\overline{M \setminus S}, \Sigma; \mathbb{C}) \)
Proof We do the proof for (a), as (b) and (c) are the same. [Indeed, we can replace \( M \) by any smooth submanifold \( \Omega \) of \( M \), provided that \( \Omega, \Sigma_1, \Sigma_2, \ldots, \Sigma_{\ell} \) are in general position.] Note that when \( \Sigma = \emptyset \), these are just the isomorphisms given by de Rham’s theorem.

First suppose \( \Sigma = \Sigma_1 \). For each \( p \) we have the exact sequence:

\[
0 \to \mathcal{E}^p(M, \Sigma) \longrightarrow \mathcal{E}^p(M) \longrightarrow \mathcal{E}^p(\Sigma) \to 0.
\]

Since the maps in (8.2) commute with \( d \), we obtain the long exact sequence:

\[
\cdots \to H^{p-1}(\Sigma) \longrightarrow H^p(M, \Sigma) \longrightarrow H^p(\Sigma) \to \cdots
\]

We consider the corresponding sequence in singular relative cohomology:

\[
\cdots \to H^{p-1}_{\text{sing}}(\Sigma; \mathbb{C}) \longrightarrow H^p_{\text{sing}}(M, \Sigma; \mathbb{C}) \longrightarrow H^p_{\text{sing}}(\Sigma; \mathbb{C}) \to \cdots
\]

By de Rham’s theorem we have that

\[
H^p(M) \simeq H^p_{\text{sing}}(M; \mathbb{C}) \quad \text{and} \quad H^p(\Sigma) \simeq H^p_{\text{sing}}(\Sigma; \mathbb{C}).
\]

As these isomorphisms are compatible with the natural homomorphisms between (8.3) and (8.4), we obtain that also \( H^p(M, \Sigma) \simeq H^p_{\text{sing}}(M, \Sigma; \mathbb{C}) \) upon application of the 5-lemma.

We proceed by induction on \( \ell \): Assume that the result is known for \( \Sigma' = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_{\ell-1} \), and consider \( \Sigma = \Sigma' \cup \Sigma_{\ell} \). By our transversality assumption, there is for each \( p \) a short exact sequence

\[
0 \to \mathcal{E}^p(M, \Sigma) \longrightarrow \mathcal{E}^p(M, \Sigma') \oplus \mathcal{E}^p(M, \Sigma_{\ell}) \longrightarrow \mathcal{E}^p(M, \Sigma' \cap \Sigma_{\ell}) \to 0.
\]

As the maps in (8.5) commute with \( d \), we obtain the long exact sequence:

\[
\cdots \to H^{p-1}(M, \Sigma' \cap \Sigma_{\ell}) \longrightarrow H^p(M, \Sigma) \longrightarrow H^p(M, \Sigma') \oplus H^p(M, \Sigma_{\ell}) \longrightarrow H^p(M, \Sigma' \cap \Sigma_{\ell}) \to \cdots
\]

Since \( \Sigma' \cap \Sigma_{\ell} = (\Sigma_1 \cap \Sigma_{\ell}) \cup \cdots \cup (\Sigma_{\ell-1} \cap \Sigma_{\ell}) \) is a union of \((\ell-1)\) smooth submanifolds in general position, we can apply our inductive hypothesis to conclude that

\[
H^p(M, \Sigma' \cap \Sigma_{\ell}) \simeq H^p_{\text{sing}}(M, \Sigma' \cap \Sigma_{\ell}; \mathbb{C}).
\]

Hence we may compare (8.6) with the corresponding Mayer-Vietoris sequence in singular relative cohomology, and apply the 5-lemma once again to obtain \( H^p(M, \Sigma) \simeq H^p_{\text{sing}}(M, \Sigma; \mathbb{C}) \). This completes the proof.

Proposition 8.2 There is a long exact sequence:

\[
\cdots \to H^p(M, \Sigma) \longrightarrow H^p(M \setminus S, \Sigma) \overset{\delta^*}{\longrightarrow} H^{p-1}(S, \Sigma) \overset{\tau^*}{\longrightarrow} H^{p+1}(M, \Sigma) \to \cdots.
\]
By the universal coefficient theorem we have that
\[ H^p_{\text{sing}}(M, \Sigma; \mathbb{C}) \simeq \text{Hom}_\mathbb{Z}(H^p(M, \Sigma; \mathbb{Z}), \mathbb{C}) , \]
and likewise for the other pairs \((S, \Sigma)\) and \((M \setminus S, \Sigma)\). Applying the \(\text{Hom}_\mathbb{Z}(\cdot, \mathbb{C})\) functor to (7.4) we obtain (8.7) because \(\mathbb{C}\) is an injective \(\mathbb{Z}\)-module.

**Definition.** For \(h^* \in H^p(M \setminus S, \Sigma)\),
\[
(8.8) \quad \text{Res}h^* = \frac{1}{2\pi \sqrt{-1}} \delta^* h^* \in H^{p-1}(S, \Sigma)
\]
will be called the **residue class** of \(h^*\).

**Proposition 8.3** Let \(\phi \in E^p(M \setminus S, \Sigma)\) be closed in \(M \setminus S\) and have a simple pole along \(S\). Then
\[
(8.9) \quad \text{Res}[\phi] = [\text{res } [\phi]] .
\]

**Proof**
\[
\text{Res}[\phi] ([\gamma]) = \frac{1}{2\pi \sqrt{-1}} [\phi] ([\delta \phi])
\]
\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\delta \gamma} \phi = \int_{\gamma} \text{res } [\phi]
\]
\[
= [\text{res } [\phi]] ([\gamma]) ,
\]
for every smooth singlar relative \((p - 1)\)-cycle in \((S, \Sigma)\). So \(\text{Res}[\phi]\) and \([\text{res } [\phi]]\) represent the same element of \(\text{Hom}_\mathbb{Z}(H_{p-1}(S, \Sigma; \mathbb{Z}), \mathbb{C})\), and the result follows from the discussion above.

**Theorem 8.4** Each cohomology class \(h^* \in H^p(M \setminus S, \Sigma)\) has a representative \(\phi \in E^p(M \setminus S, \Sigma)\), closed in \(M \setminus S\), and having a simple pole along \(S\).

**Proof** First we observe that \(HM\) can be viewed as a complex smooth vector bundle of rank \(n\), with complex structure on the fibers given by \(J\). Since \(S\) is polar, \(HS\) is a smooth complex subbundle of \(HM\) along \(S\), having rank \((n - 1)\). Moreover the projection \(TM|_S \to NSM\) gives by restriction a surjective vector bundle morphism \(HM|_S \to NSM\), which in turn yields an isomorphism
\[
(8.10) \quad H M|_S / HS \xrightarrow{\sim} NSM .
\]
Thus \(NSM\) has the natural structure of a complex line bundle over \(S\). We take a local trivialization \(\{(U_\alpha, s_\alpha)\}\) of \(NSM\) and use \(s_\alpha\) as the local defining functions for \(S\) in \(\nu^{-1}(U_\alpha)\), with \(\nu\) given in (6.2); we denote by \(s_\alpha(x)\) the value of the
$s_\alpha$-coordinate corresponding to $\nu^{-1}(x)$, for $x \in \mu_0^{-1}(U_\alpha)$. Here $\nu : N_S M \xrightarrow{\sim} V$ is the diffeomorphism from §6, which represents $S$ as the zero section in $N_S M$.

We observe that $\bar{\partial} MS_\alpha = 0$ along $S$ because the complex structure on the fibers of $N_S M$ agrees with the partial complex structure of $M$ along $S$. Moreover for $x \in \mu_0^{-1}(U_\alpha) \cap \mu_0^{-1}(U_\beta)$, we have that

$$\tag{8.11} s_\alpha(x) = g_{\alpha\beta}(\mu_0(x))s_\beta(x),$$

where the $g_{\alpha\beta}$ are the transition functions, $g_{\alpha\beta} \neq 0$, corresponding to the local trivialization of the bundle. From (8.11) we derive that

$$\tag{8.12} \frac{ds_\alpha}{s_\alpha} - \frac{ds_\beta}{s_\beta} = \frac{dg_{\alpha\beta}}{g_{\alpha\beta}}.$$ 

Since the right-hand side of (8.12) is smooth across $S$, it follows that the left-hand side has a smooth extension across $S$.

Now consider a class $h^* \in H^p(M \setminus S, \Sigma)$. Let $\psi_0 \in \mathcal{E}^{p-1}(S, \Sigma)$ with $d\psi_0 = 0$ be any representative of $\delta^* h^*$. We choose a smooth partition of unity $\{\chi_\alpha\}$ of a neighborhood of $S$ in $V$, subordinated to the covering $\{\mu_0^{-1}(U_\alpha)\}$, and with $\text{supp} \chi_\alpha \subset \nu(\{\xi \in N_S M \mid |\xi|_{\pi(\xi)} \leq \frac{3}{4}\}) = V_{\frac{3}{4}} \subset V$. Let

$$\omega = \sum_\alpha \chi_\alpha \frac{ds_\alpha}{s_\alpha} \wedge \mu_0^*(\psi_0).$$

Then $\omega \in \mathcal{E}^p(M \setminus S, \Sigma)$ and $\omega = 0$ outside of $V_{\frac{3}{4}}$. It follows from (8.12) that $d\omega$ has a smooth continuation across $S$, and defines a closed form in $\mathcal{E}^{p+1}(M, \Sigma)$. Indeed in a sufficiently small open neighborhood of $S \cap \mu_0^{-1}(U_\beta) = U_\beta$ in $M$,

$$\omega = \frac{ds_\beta}{s_\beta} \wedge \mu_0^*(\psi_0) + \sum_\alpha \chi_\alpha \left(\frac{ds_\beta}{s_\beta} - \frac{ds_\alpha}{s_\alpha}\right) \wedge \mu_0^*(\psi_0)$$

$$\tag{8.13} = \frac{ds_\beta}{s_\beta} \wedge \mu_0^*(\psi_0) + \sum_\alpha \chi_\alpha \frac{dg_{\alpha\beta}}{g_{\alpha\beta}} \wedge \mu_0^*(\psi_0);$$

hence

$$\tag{8.14} d\omega = \sum_\alpha d\chi_\alpha \wedge \frac{dg_{\alpha\beta}}{g_{\alpha\beta}} \wedge \mu_0^*(\psi_0),$$

which is smooth across $S$. From formula (8.13) we see that $\omega$ has a first order pole along $S$.

Next we show that $\tau^*([\psi_0]) = -\frac{1}{2\pi \sqrt{-1}} [d\omega]$ in $H^{p+1}(M, \Sigma)$: consider a homology class $\Omega \in H_{p+1}(M, \Sigma; \mathbb{Z})$, and let $\gamma$ be a smooth singular relative $(p-1)$ cycle in $(S, \Sigma)$ representing the class $\tau(\Omega) = [\gamma] \in H_{p-1}(S, \Sigma; \mathbb{Z})$. Then $\Omega - [D_1 \gamma]$ is the image of a homology class in $H_{p+1}(M \setminus S, \Sigma; \mathbb{Z})$. By excision we can find a smooth singular relative $(p+1)$-cycle $\Omega'$ in $M \setminus \{\Sigma\}$ such that $[\Omega'] = \Omega - [D_1 \gamma]$. Then
applying Stokes' theorem we obtain
\[
[d\omega](\Omega) = \int_{\Omega} d\omega = \int_{\Omega'} d\omega + \int_{D_1 \gamma} d\omega \\
= \int_{\delta_1 \gamma} \omega - \lim_{\epsilon \searrow 0} \int_{\delta_1 \gamma} \omega \\
= -2\pi \sqrt{-1} \int_{\gamma} \psi_0 \\
= -2\pi \sqrt{-1} [\psi_0](\tau(\Omega)) \\
= -2\pi \sqrt{-1} \tau^*([\psi_0])(\Omega).
\]

We have that \( \tau^*([\psi_0]) = \tau^* \delta^* h^* = 0 \); hence \( [d\omega] \sim 0 \) in \( H^{p+1}(M, \Sigma) \). Therefore \( d\omega = d\eta \) in \( M \setminus S \), for some \( \eta \in \mathcal{E}^p(M, \Sigma) \). Hence \( \omega - \eta \) is closed in \( M \setminus S \), has a simple pole along \( S \), and its residue form is \( \psi_0 \). By Proposition 8.3 we have that \( \delta^* (h^* - 2\pi \sqrt{-1}\omega) = 0 \), and therefore by the exact sequence (8.7) there exists a closed form \( \lambda \in \mathcal{E}^p(M, \Sigma) \) such that \( h^* - 2\pi \sqrt{-1}[\omega - \eta] = [\lambda]_{M \setminus S} \). Let \( \phi = 2\pi \sqrt{-1}(\omega - \eta) + \lambda \). This is a closed form in \( \mathcal{E}^p(M \setminus S, \Sigma) \) having a simple pole along \( S \), with \( [\phi] = h^* \) in \( H^p(M \setminus S, \Sigma) \), completing the proof of the theorem.

We now come to the cohomological version of the residue formula.

**Theorem 8.5** Let \( h^* \in H^p(M \setminus S, \Sigma) \) and \( \bar{h} \in \bar{H}^{p-1}(S, \Sigma; \mathbb{Z}) \). Then

\[(8.15) \quad \bar{\int}_{\delta \bar{h}} h^* = 2\pi \sqrt{-1} \bar{\int}_{\bar{h}} Res^* h^*.
\]

**Proof** Note that both the left and the right-hands in (8.15) are well defined by Stokes' theorem. The result is then a consequence of Theorem 8.4 and Proposition 8.3.

## §9 Properties of the Residue Class and Global defining Functions

Next we summarize some important properties of the residue class map which follow from the analogous properties of the residue form, in view of Proposition 8.3 and Theorem 8.4.

**Proposition 9.1** If \( h^* \in H^p(M \setminus S, \Sigma) \) and \( k^* \in H^q(M, \Sigma) \), then

\[(9.1) \quad \text{Res}(h^* \cup k^*|_{M \setminus S}) = (\text{Res} h^*) \cup k^*|_{S}
\]

Here the cup \( \cup \) denotes the product operation on cohomology induced by the wedge product of forms.

**Proposition 9.2** Assume that \( S \) has a global smooth defining function \( s \) with \( \bar{\partial}_M s = 0 \) along \( S \). If \( h^* \in H^p(M \setminus S, \Sigma) \) and \( g^* \in H^q(M \setminus S, \Sigma) \) then

\[(9.2) \quad \text{Res}(h^* \cup g^*) = \text{Res}(h^* \cup \left[ \frac{ds}{s} \right]) \cup \text{Res} g^* + \text{Res} h^* \cup \text{Res}(\left[ \frac{ds}{s} \right] \cup g^*).
\]
Note that we may apply proposition 5.2. at this point, because the construction in our proof of Theorem 8.4. allows us to find representatives in each of the classes \( h^* \) and \( g^* \) having a simple pole with respect to the same defining function \( s \).

Next we consider the situation of a smooth \( CR \) map \( F : M^* \to M \) as in §5. We assume that \( F \) is \( CR \) transversal to \( S \), and moreover that \( F \) is transversal to each \( \Sigma_j \).

**Proposition 9.3** Under the above assumptions, we have

\[
\text{Res} \left( F^* h^* \right) = F^* \left( \text{Res} h^* \right)
\]

for any \( h^* \in H^n(M \setminus S, \Sigma) \).

**Remark.** In particular if \( V \) is an open neighborhood of \( S \) in \( M \), and \( F \) is the inclusion map, we obtain

\[
\text{Res} h^* = \text{Res} \left( h^* \mid_V \right).
\]

Recall that the smooth complex line bundle \( N_S M \) is trivial if its Chern class in \( H^2(S, \mathbb{Z}) \) is zero.

**Proposition 9.4** If the normal bundle \( N_S M \) has zero Chern class, then

(a) \( S \) has a global defining function \( s \), with \( \bar{\partial}_M s = 0 \) along \( S \), defined in a neighborhood \( V \) of \( S \) in \( M \).

(b) The formula analogous to (9.2) holds, in which \( M \) is replaced by \( V \).

Finally we remark that, in general, by Proposition 9.1 we have that \( \text{Res} \) is a homomorphism of \( H^*(M) \)-algebras from \( H^*(M \setminus S, \Sigma) \) to \( H^*(S, \Sigma) \).

§10 **Higher Order Poles**

We say that \( \{ (U_\alpha, s_\alpha) \} \) form a consistent system of local defining functions for \( S \) iff: the \( \{ U_\alpha \} \) is a locally finite open covering of \( S \) in \( M \), each \( s_\alpha \) is a smooth defining function for \( S \cap U_\alpha \) in \( U_\alpha \) with \( ds_\alpha \neq 0 \) in \( U_\alpha \) and \( \bar{\partial}_M s_\alpha = 0 \) on \( S \cap U_\alpha \), and moreover there exist \( \{ g_{\alpha\beta} \} \), where \( g_{\alpha\beta} \) is smooth and nonzero on \( U_\alpha \cap U_\beta \), such that

\[
s_\alpha = g_{\alpha\beta} s_\beta \quad \text{on} \quad U_\alpha \cap U_\beta.
\]

**Remark 1.** Consistent systems of locally defining functions always exist; in fact they arise naturally from a local trivialization of the complex line bundle \( N_S M \) and its identification with a tubular neighborhood \( V \), see (8.11).

**Remark 2.** If our abstract \( CR \) manifold \( M \) is 1-pseudoconcave in a neighborhood of \( S \), then any choice of a system of local defining functions \( \{ (U_\alpha, s_\alpha) \} \) with \( s_\alpha \) satisfying (3.1) is automatically a consistent system, according to Lemma 3.2.
Consider a form $\phi \in E^p(M \setminus S)$ and $q = 1, 2, \ldots$.

**Definition (A)** $\phi$ is said to have a pole of (at most) order $q$ along $S$ iff: given any point $p \in S$, there exists an open neighborhood $U$ of $p$ in $M$, and a smooth local defining function $s$ of $S$ in $U$, with $ds \neq 0$ in $U$ and $\bar{\partial}_M s = 0$ on $S \cap U$, such that $s^q \phi \in E^p(U)$.

(B) $\phi$ is said to be a semi-CR meromorphic form with a pole of order (at most) $q$ along $S$ iff: it satisfies (A) with respect to some consistent system of local defining functions.

Note that (B) above implies that $\phi$ has the local consistent representation in $U_\alpha$:

$$\phi = \frac{\omega_\alpha}{s^q}, \quad \omega_\alpha \in E^p(U_\alpha).$$

We now come to the main point of this section: As we have seen, given any $\phi \in E^p(M \setminus S, \Sigma)$ which is closed, there exists another closed $\phi' \in E^p(M \setminus S, \Sigma)$ cohomologous to $\phi$ which has a simple pole along $S$. Then the residue class $\text{Res}[\phi]$ may be computed by taking the class of $\text{res}[\phi']$ in $H^{p-1}(S, \Sigma)$. But the passage from $\phi$ to $\phi'$ by the route we have (up to this point) developed is quite involved and rather indirect. However when the closed $\phi$ is semi-CR meromorphic, and has a pole of finite order along $S$, as in (B), we can do much better: It is possible to prescribe an algorithm, which employs only elementary operations on smooth differential forms, for the passage from $\phi$ to a cohomologous closed $\phi_1 \in E^p(M \setminus S, \Sigma)$ having a simple pole along $S$, and with

$$\text{res}[\phi_1] \in \text{Res}[\phi].$$

**Proposition 10.1** Let $\phi \in E^p(M \setminus S, \Sigma)$ be a closed semi-CR meromorphic form having a pole of order $q \geq 2$ along $S$. Then we can construct semi-CR meromorphic forms $\hat{\phi} \in E^p(M \setminus S, \Sigma)$ and $\rho \in E^{p-1}(M \setminus S, \Sigma)$, each having a pole of order $(q - 1)$ along $S$, such that

$$\hat{\phi} = \phi - d\rho \quad \text{and} \quad \text{singsupp }|S \hat{\phi} \subset \text{singsupp }|S \phi.$$ 

**Proof** Differentiating (10.2) we obtain:

$$0 = \frac{d\omega_\alpha}{s^q} - q \frac{ds_\alpha \wedge \omega_\alpha}{s^q \wedge s^{q+1}} \quad \text{in } U_\alpha \setminus S,$$

from which we obtain $ds_\alpha \wedge \omega_\alpha = 0$ in $U_\alpha$ by continuity, after wedging with $ds_\alpha$. Hence by Cartan’s lemma we can find $\theta_\alpha \in E^p(U_\alpha, \Sigma)$ such that $d\omega_\alpha = ds_\alpha \wedge \theta_\alpha$ in $U_\alpha$. Using (10.5) we obtain $ds_\alpha \wedge (q \omega_\alpha - s_\alpha \theta_\alpha) = 0$ in $U_\alpha$; hence another application of Cartan’s lemma yields an $\eta_\alpha \in E^{p-1}(U_\alpha, \Sigma)$ such that

$$q \omega_\alpha - s_\alpha \theta_\alpha = (q - 1)ds_\alpha \wedge \eta_\alpha \quad \text{in } U_\alpha.$$

Therefore in $U_\alpha \setminus S$ the form $\phi$ can be represented as

$$\phi = \frac{\omega_\alpha}{s^{q-1}} = \frac{1}{s^{q-1}} \theta_\alpha + \frac{q - 1}{s^{q-1}} ds_\alpha \wedge \eta_\alpha.$$
Let \( \{\chi_\alpha\} \) be a smooth partition of unity in a neighborhood of \( S \), subordinated to \( \{U_\alpha\} \), and set \( \chi = \sum_\alpha \chi_\alpha \). Then we may write

\[
\phi = (1 - \chi)\phi + \sum_\alpha \chi_\alpha \phi
\]

\[
= (1 - \chi)\phi + \sum_\alpha \chi_\alpha \frac{\omega_\alpha}{s_\alpha^q}
\]

\[
= (1 - \chi)\phi + \frac{1}{q} \sum_\alpha \left[ \frac{\chi_\alpha \theta_\alpha}{s_\alpha^{q-1}} + (q - 1) \frac{ds_\alpha}{s_\alpha^q} \wedge (\chi_\alpha \eta_\alpha) \right]
\]

\[
= (1 - \chi)\phi + \frac{1}{q} \sum_\alpha \left[ \frac{\chi_\alpha \theta_\alpha}{s_\alpha^{q-1}} - d \left( \frac{\chi_\alpha \eta_\alpha}{s_\alpha^{q-1}} \right) \right].
\]

We get (10.4) by setting

\[
\rho = -\sum_\alpha \frac{\chi_\alpha \eta_\alpha}{s_\alpha^{q-1}},
\]

\[
\hat{\phi} = (1 - \chi)\phi + \frac{1}{q} \sum_\alpha \frac{\chi_\alpha \theta_\alpha + d(\chi_\alpha \eta_\alpha)}{s_\alpha^{q-1}}.
\]

To see that \( \rho \) and \( \hat{\phi} \) are semi-CR meromorphic with a pole of order \( q - 1 \) along \( S \), it suffices to observe that

\[
\rho = -\sum_\beta g_{\alpha \beta} \frac{\chi_\beta \eta_\beta}{s_\alpha^{q-1}},
\]

\[
\hat{\phi} = \sum_\beta g_{\alpha \beta} \frac{\chi_\beta \theta_\beta + d(\chi_\beta \eta_\beta)}{qs_\alpha^{q-1}},
\]

in \( U_\alpha \cap \{\chi = 1\} \). The construction does not increase the singsupp \( |S| \); hence the proof is complete.

**Theorem 10.2** Let \( \phi \in \mathcal{E}^p(M \setminus S, \Sigma) \) be a closed semi-CR meromorphic form having a pole of order \( q \geq 2 \) along \( S \). Then we can construct a closed semi-CR meromorphic form \( \phi^{(1)} \in \mathcal{E}^p(M \setminus S, \Sigma) \) having a pole of order \( q = 1 \) along \( S \), such that

\[
\phi^{(1)} = \phi - d\rho,
\]

\[
\text{res} \left[ \phi^{(1)} \right] \in \text{Res}[\phi],
\]

\[
\text{supp} \text{ res} \left[ \phi^{(1)} \right] \subset \text{singsupp} \ |S| \phi.
\]

**Proof** Applying Proposition 10.1 \( (q - 1) \) times, we obtain

\[
\phi^{(q-1)} = \phi - d\rho^{(q-1)}
\]

\[
\phi^{(q-2)} = \phi^{(q-1)} - d\rho^{(q-2)}
\]

\[
\vdots
\]

\[
\phi^{(1)} = \phi^{(2)} - d\rho^{(1)}
\]
where \( \phi^{(j)} \in \mathcal{E}^p(M \setminus S, \Sigma) \) and \( \rho^{(j)} \in \mathcal{E}^{p-1}(M \setminus S, \Sigma) \) are semi-CR meromorphic and have poles of order \( j \) along \( S \). We obtain (10.7) with \( \rho = \rho^{(q-1)} + \rho^{(q-2)} + \ldots + \rho^{(1)} \), proving the theorem.

In \( U_\alpha \) we have the consistent representations

\[
\phi^{(j)} = \frac{\omega^{(j)}_{\alpha}}{s^j_\alpha}, \quad \rho^{(j)} = \frac{\eta^{(j)}_{\alpha}}{s^j_\alpha}, \quad 1 \leq j \leq q - 1,
\]

with \( \omega^{(j)}_{\alpha}, \eta^{(j)}_{\alpha} \) smooth in \( U_\alpha \). Thus in \( U_\alpha \) we may write

\[
\frac{\omega_{\alpha}}{s^q_\alpha} = \frac{\omega^{(1)}_{\alpha}}{s_\alpha} + d \left\{ \frac{\eta^{(q-1)}_{\alpha}}{s^{q-1}_\alpha} + \frac{\eta^{(q-2)}_{\alpha}}{s^{q-2}_\alpha} + \ldots + \frac{\eta^{(1)}_{\alpha}}{s_\alpha} \right\},
\]

and these local Laurent expansions are consistent on \( U_\alpha \cap U_\beta \).

§11 The Calculus of Residues

In this section we assume that there is a neighborhood \( V \) of \( S \) in \( M \), in which \( S \) has a global defining function \( S \), with \( ds \neq 0 \) in \( V \) and \( \bar{\partial}_M s = 0 \) along \( S \). According to (9.4) we may, effectively, replace \( M \) by \( V \) when computing the residue class of a closed form \( \phi \). Hence there is no loss of generality if, in this section, we take \( M = V \).

We consider a semi-CR meromorphic closed form \( \phi \in \mathcal{E}^p(M \setminus S, \Sigma) \) with a pole of order \( q \) along \( S \), having a global representation as

\[
\phi = \frac{\omega}{s^q}, \quad \omega \in \mathcal{E}^p(M, \Sigma).
\]

In this situation we are able to convert the algorithm of the previous section into a precise calculus of residues:

consider the diffeomorphism:

\[
\nu : N_S M \longrightarrow V = M,
\]

of the complex line bundle \( N_S M \) with the tubular neighborhood. It defines a foliation of \( V \) with two-dimensional leaves corresponding to the fibres of \( N_S M \). Let \( L^* \subset \mathcal{T}^* M \) be the set of covectors which annihilate the vectors tangent to the leaves of the foliation. It is a rank \((2n+k-2)\) real subbundle of \( \mathcal{T}^* M \). We denote by \( \Phi \) the \( \mathcal{C}^\infty \)-subalgebra of the complexified exterior algebra \( \mathcal{E}^*(M) \) generated by the smooth sections of \( L^* \) and by \( ds \). (Here \( V = \mathcal{V} = M \) is chosen sufficiently small in order that \( ds \) and \( d\bar{s} \) are linearly independent modulo \( L^* \) at each point of \( V \).) This gives us the direct sum decomposition

\[
\mathcal{E}^*(M) = \Phi \oplus ds \wedge \Phi.
\]

Using the decomposition (11.3) we define a \( \mathcal{C}^\infty(M) \)-linear operator \( \frac{d^0}{ds} \) on smooth forms:

\[
\frac{d^0}{ds} : \mathcal{E}^p(M) \longrightarrow \Phi \cap \mathcal{E}^{p-1}(M).
\]
by saying that
\[(11.5) \quad \frac{d^0 \omega}{ds} = \beta \]
iff \(\omega \in \mathcal{E}^p(M)\) decomposes as \(\omega = \alpha + ds \wedge \beta\) with \(\alpha \in \Phi \cap \mathcal{E}^p(M)\) and \(\beta \in \Phi \cap \mathcal{E}^{p-1}(M)\). Note that \(\frac{d^0 \omega}{ds}|_\Sigma = 0\) if \(\omega|_\Sigma = 0\) because our tubular neighborhood was adapted to \(\Sigma\). We also define a \(\mathbb{C}\)-linear operator
\[(11.6) \quad \frac{d}{ds} : \Phi \cap \mathcal{E}^p(M) \longrightarrow \Phi \cap \mathcal{E}^p(M)\]
by saying that
\[\frac{d\lambda}{ds} = \beta \]
iff \(\lambda \in \Phi \cap \mathcal{E}^p(M)\) is such that \(d\lambda = \alpha + ds \wedge \beta\) with \(\alpha \in \Phi \cap \mathcal{E}^{p+1}(M)\) and \(\beta \in \Phi \cap \mathcal{E}^p(M)\). Again \(\frac{d\lambda}{ds}|_\Sigma = 0\) if \(\lambda|_\Sigma = 0\).

Note that if one chooses local coordinates \((\Re s, \Im s, x)\) adapted to the foliation \((\{x = \text{const}\}\) gives a leaf) the operator (11.6) actually becomes the usual partial differential operator
\[(11.7) \quad \frac{\partial}{\partial s} = \frac{1}{2} \left( \frac{\partial}{\partial \Re s} - \sqrt{-1} \frac{\partial}{\partial \Im s} \right)\]
acting on the coefficients of the form.

We indicate the iterates of these operators by
\[(11.8) \quad \begin{cases} \frac{d^r}{ds^r} = \frac{d}{ds} \cdots \frac{d}{ds} , & r \text{ times} , \\ \frac{d^r}{ds^{r+1}} = \frac{d^r}{ds^r} \frac{d^0}{ds} . \end{cases}\]

**Proposition 11.1** Let the closed form \(\phi\) be as in (11.1) with \(q \geq 1\). Then
\[(11.9) \quad \frac{1}{(q-1)!} \left. \frac{d^{q-1} \omega}{ds^q} \right|_S \in \text{Res} \left[ \frac{\omega}{s^q} \right] . \]

**Proof** Let
\[(11.10) \quad \omega = \alpha + ds \wedge \beta \]
where \(\alpha \in \Phi \cap \mathcal{E}^p(M, \Sigma)\) and \(\beta = \frac{d^0 \omega}{ds} \in \Phi \cap \mathcal{E}^{p-1}(M, \Sigma)\). First we claim that if \(q \geq 2\) then there are forms \(\tilde{\alpha} \in \Phi \cap \mathcal{E}^p(M, \Sigma)\) and \(\eta \in \mathcal{E}^{p-1}(M \setminus S, \Sigma)\) such that
\[(11.11) \quad \phi + d\eta = \frac{1}{d} \tilde{\alpha} + ds \wedge \frac{d^0}{ds} \tilde{\alpha} . \]
Indeed, since $d\phi = 0$, we obtain

$$q \, ds \wedge \alpha = s \,(ds \wedge d\beta + d\alpha).$$  

(11.12)

We decompose $d\alpha$ and $d\beta$ according to (11.3), (11.6) as

$$d\alpha = (d\alpha)|_\Phi + ds \wedge \frac{d\alpha}{ds},$$

(11.13)

$$d\beta = (d\beta)|_\Phi + ds \wedge \frac{d\beta}{ds},$$

with $(d\alpha)|_\Phi \in \Phi \cap \mathcal{E}^{p+1}(M, \Sigma)$, $\frac{d\beta}{ds} \in \Phi \cap \mathcal{E}^{p-1}(M, \Sigma)$ and $\frac{d\alpha}{ds}, (d\beta)|_\Phi \in \Phi \cap \mathcal{E}^p(M, \Sigma)$. Then (11.12) gives

$$\frac{(d\alpha)|_\Phi}{q} = 0,$$

$$\alpha = \frac{s}{q} \left[ (d\beta)|_\Phi + \frac{d\alpha}{ds} \right];$$

(11.14)

hence substituting in (11.10) we obtain

$$\phi = \frac{ds \wedge \beta}{s^q} + \frac{1}{q} \left(\frac{q}{q-1} (d\beta)|_\Phi + \frac{d\alpha}{ds}\right).$$

(11.15)

From this we derive

$$\phi + \frac{1}{q-1} d \left( \frac{\beta}{s^{q-1}} \right) = \frac{q}{q-1} \left(\frac{q}{q-1} (d\beta)|_\Phi + \frac{d\alpha}{ds}\right) + \frac{1}{q-1} ds \wedge \frac{d\beta}{ds}. $$

(11.16)

Thus we have proved our claim with $\tilde{\alpha} = q(d\beta)|_\Phi + (q-1) \frac{d\alpha}{ds}$.

Next we claim that when $q = 1$, then

$$\text{res} \phi = \frac{d^0 \omega}{ds} \bigg|_s.$$  

(11.17)

Indeed $d\phi = 0$ gives (11.14) with $q = 1$; hence (11.10), (11.14) yields

$$\phi = \frac{ds}{s} \wedge \beta + \left\{ \frac{d\alpha}{ds} + (d\beta)|_\Phi \right\},$$

(11.18)

which establishes our claim. We apply the first claim $(q - 1)$ times and the second claim once to obtain the proposition.

§12 Iterated Residues

We return to the general situation where $M$ is as in the beginning of §3. But now we consider $m$ ($1 \leq m \leq n$) polar submanifolds $S_1, S_2, \ldots, S_m$ in $M$. We assume that they are in general position and moreover that their holomorphic tangent bundles $HS_1, HS_2, \ldots, HS_m$ are in general position in $HM$. Therefore the intersection of
any \( r \) of them is a smooth closed \( CR \) submanifold of type \((n-r,k)\). In this section we set

\[
\begin{align*}
\tilde{S} &= S_1 \cup S_2 \cup \cdots \cup S_m, \\
S &= S_1 \cap S_2 \cap \cdots \cap S_m, \\
S_{(j)} &= (S_m \cap S_{m-1} \cap \cdots \cap S_{m-j+1}) \setminus (S_{m-j} \cup S_{m-j-1} \cup \cdots \cup S_1),
\end{align*}
\]

(12.1)

with \( S = S_{(m)} \) and \( M \setminus \tilde{S} = S_{(0)} \). We consider also \( \Sigma = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_\ell \), where each \( \Sigma_j \) is a smooth closed submanifold of \( M \). We assume that \( S_1, S_2, \ldots, S_m, \Sigma_1, \Sigma_2, \ldots, \Sigma_\ell \) are also in general position in the usual sense. This implies, in particular, that the intersection of any subset of the \( \Sigma_1, \ldots, \Sigma_\ell \) is transversal to the intersection of any subset of the \( S_1, \ldots, S_m \).

We define the \( m \)-th iterate

\[
\delta^m : H_{p-m}(S, \Sigma; \mathbb{Z}) \to H_p(M \setminus \tilde{S}, \Sigma; \mathbb{Z})
\]

of the coboundary homomorphism \( \delta \) by

\[
\delta^m : H_{p-m}(S_{(m)}, \Sigma; \mathbb{Z}) \to H_{m-p+1}(S_{(m-1)}, \Sigma; \mathbb{Z}) \to \cdots \to H_{p-1}(S_{(1)}, \Sigma; \mathbb{Z}) \to H_p(S_{(0)}, \Sigma; \mathbb{Z}).
\]

(12.2)

Similarly we define the \( m \)-th iterate

\[
\text{Res}^m : H^p(M \setminus \tilde{S}, \Sigma) \to H^{p-m}(S, \Sigma)
\]

of the class residue homomorphism by

\[
\text{Res}^m : H^p(S_{(0)}, \Sigma) \xrightarrow{\text{Res}} H^{p-1}(S_{(1)}, \Sigma) \xrightarrow{\text{Res}} \cdots \xrightarrow{\text{Res}} H^{p-m+1}(S_{(m-1)}, \Sigma) \xrightarrow{\text{Res}} H^{p-m}(S_{(m)}, \Sigma).
\]

(12.4)

We may use Theorem 8.5 to obtain:

**Theorem 12.1** Let \( h^* \in H^p(M \setminus \tilde{S}, \Sigma) \) and \( \tilde{h} \in \tilde{H}_{p-m}(S, \Sigma; \mathbb{Z}) \). Then

\[
\int_{\mathcal{S}^m \tilde{h}} h^* = \int_{\tilde{h}} \text{Res}^m h^*.
\]

(12.6)

**Remark 1.** The homomorphisms \( \delta^m \) and \( \text{Res}^m \) defined here depend on the ordering \( S_1, S_2, \ldots, S_m \). A permutation of the \( S_1, S_2, \ldots, S_m \) affects \( \delta^m \) and \( \text{Res}^m \) each by the same \( \pm \) sign, depending on the parity of the permutation.

**Remark 2.** Consider the situation where \( m \) is maximal; \( m = n \). In this case \( S \) is a **totally real** \( k \)-dimensional \( CR \) submanifold of type \((0,k)\) in \( M \); moreover \( S \) is transversal to the \( CR \) structure of \( M \), in the sense that \( T_xS \oplus H_xM = T_xM \) for every \( x \in S \). Note that here \( \text{Res}^m \) could be regarded as the generalization to \( CR \) manifolds of type \((n,k)\) of the Grothendieck residue (see [D]); in which case we have \((k+1)\) kinds, as in (12.4) we may use \( \tilde{h} \in \tilde{H}_i(S, \Sigma; \mathbb{Z}) \) for \( i = 0, 1, \ldots, k \).
§13 The Calculus of Residues for Intersecting Poles

We continue with the situation of §12 (polar submanifolds $S_1, S_2, \ldots, S_m$ with normal crossings). But in this section we assume that there is a neighborhood $V$ of $S$ in $M$, in which each $S_j$ has a global defining function $s_j$, with $\partial_M s_j = 0$ along $S_j \cap V$, for $j = 1, \ldots, m$. We may also assume that the real and imaginary parts of $ds_1, ds_2, \ldots, ds_m$ are linearly independent at each point of $V$. As in §11 we can take $V = M$ and assume that $V = V$ is a tubular neighborhood of $S$, without any loss of generality.

Having discussed iterated residues, we now turn our attention to semi-$CR$ meromorphic forms having poles of finite order along each $S_j$. Namely we consider the closed forms $\phi \in \mathcal{E}^p(M \setminus \tilde{S}, \Sigma)$ having a global representation as

$$\phi = \frac{\omega}{s_1^{q_1} s_2^{q_2} \cdots s_m^{q_m}}, \quad \omega \in \mathcal{E}^p(M, \Sigma).$$

Here we extend the calculus of residues of §11 to the multivariable case. When $m > 1$ we write the linear operators in (11.5), (11.6), (11.8) as $\frac{\partial^0}{\partial s_j}$, $\frac{\partial}{\partial s_j}$, $\frac{\partial^r}{\partial s_j^r}$, $\frac{\partial^r}{\partial s_j^{r+1}}$ for each individual $S_j$. By composition we define

$$\frac{\partial^{q_1+q_2+\cdots+q_m-m}\omega}{\partial s_1^{q_1} \partial s_2^{q_2} \cdots \partial s_m^{q_m}} = \frac{\partial^{q_1-1}}{\partial s_1^{q_1}} \frac{\partial^{q_2-1}}{\partial s_2^{q_2}} \cdots \frac{\partial^{q_m-1}}{\partial s_m^{q_m}} \omega,$$

acting on $\omega \in \mathcal{E}^p(M, \Sigma)$.

**Proposition 13.1** Let the closed form $\phi$ be as in (13.1) with each $q_j \geq 1$. Then

$$\frac{\partial^{q_1+q_2+\cdots+q_m-m}\omega}{\partial s_1^{q_1} \partial s_2^{q_2} \cdots \partial s_m^{q_m}} \bigg|_S \in \text{Res}^m \left[ \frac{\omega}{s_1^{q_1} s_2^{q_2} \cdots s_m^{q_m}} \right]_S.$$

**Proof** It suffices to observe that the operations described in §11, with respect to a given $S_j$, commute with the pullback to $S_i$, for $i \neq j$.

Note that the left-hand-side in (13.3) can be computed in the usual sense of calculus.

§14 Abel’s Global Residue Theorem

In this section we present a generalization of the classical theorem of Abel for a compact Riemann surface, along the lines of Griffiths [G]. We return to the scenario at the beginning of §12; so $S = S_1 \cap S_2 \cap \cdots \cap S_m$ and the $S_1, S_2, \ldots, S_m$ are in general $CR$ position, etc.

In particular $S$ is a $CR$ submanifold of $M$ of type $(n-m,k)$, transversal to the $CR$ structure of $M$. Its normal bundle $N_SM$ has the natural structure of a smooth $\mathbb{C}$ vector bundle of rank $m$, via the identification

$$N_SM = T_M|_S / \sim \subset H_M|_S.$$
There are also natural maps $N_M S \to N_M S_j |_S$ induced by projection onto the quotients, via the identifications

\begin{equation}
\frac{N_M S S_j}{N_M S_j} \cong \frac{T_M |_S / T_S |_S}{T_S |_S / T_S} \cong T_M |_S / T_S |_S = N_M S_j |_S.
\end{equation}

Therefore we get an isomorphism onto the Whitney sum

\begin{equation}
N_M S \cong \bigoplus_{1 \leq j \leq m} N_M S_j |_S.
\end{equation}

Analogous to §6 we construct a tubular neighborhood $W$ of $S$ in $M$ that is adapted to $\Sigma \cup \tilde{S}$:

\begin{equation}
w : N_M S \cong W.
\end{equation}

Thus we have that

\begin{equation}
w \left( \pi^{-1} \left( [\Sigma \cup \tilde{S}] \cap S \right) \right) = [\Sigma \cup \tilde{S}] \cap W,
\end{equation}

where $\pi : W \to S$ denotes again the projection. We fix a smooth Hermitian metric on the vector bundle $N_M S$ such that the subbundles $N_M S_1 |_S$, $N_M S_2 |_S$, ..., $N_M S_m |_S$ are orthogonal, via the identification (14.3). A point $\zeta$ in the total space of $N_M S$ can be thought of as $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m)$, where $\zeta_j$ is a point in $N_M S_j |_S$ over $\pi(\zeta)$. For the length of $\zeta$ at $x \in S$ we have $|\zeta|^2 = |\zeta_1|^2 + |\zeta_2|^2 + \cdots + |\zeta_m|^2$.

Consider a relative $(p - m)$-cycle $\gamma$ in $(S, \Sigma)$. As in §6 we identify $\gamma$ with a piecewise smooth map

\begin{equation}
\dot{\gamma} : (P_\gamma, \partial P_\gamma) \to (S, \Sigma),
\end{equation}

where $P_\gamma$ is a finite polyhedron, of dimension $(p - m)$, embedded in some Euclidean space $\mathbb{R}^N$. Let $\gamma^*(N_M S)$ denote the pullback bundle over $P_\gamma$. The map $\dot{\gamma}$ lifts to a smooth vector bundle morphism

\begin{equation}
\dot{\gamma} N_M S : \gamma^*(N_M S) \to N_M S,
\end{equation}

giving by composition a map $f = w \circ \dot{\gamma} N_M S$:

\begin{equation}
f : \gamma^*(N_M S) \to W.
\end{equation}

Next we consider the torus bundle $C^m_\gamma = C^m_\gamma (1)$ and the sphere bundle $S^{2m-1}_\gamma = S^{2m-1}_\gamma (1)$ defined by

\begin{equation}
C^m_\gamma = \left\{ (y, \zeta) \in P_\gamma \times N_M S \mid \pi(\zeta) = \gamma(y), \ |\zeta_j|_{\gamma(y)} = 1, \ j = 1, \ldots, m \right\}
\end{equation}

and

\begin{equation}
S^{2m-1}_\gamma = \left\{ (y, \zeta) \in P_\gamma \times N_M S \mid \pi(\zeta) = \gamma(y), \ \sup |\zeta_j|_{\gamma(y)} = 1 \right\}.
\end{equation}
Note that
\[ (14.11) \quad \partial C^m_\gamma = C^m_\gamma |_{\partial P_\gamma}, \]
and
\[ (14.12) \quad \partial S^{2m-1}_\gamma = S^{2m-1}_\gamma |_{\partial R_\gamma}. \]

Since our tubular neighborhood \( W \) is adapted to \( \Sigma \), we may define
\[ (14.13) \quad \begin{cases} \(\hat{\delta}^m_\gamma : (C^m_\gamma, \partial C^m_\gamma) \to (M \setminus \hat{S}, \Sigma), \) \\ \(\sigma^{2m-1}_\gamma : (S^{2m-1}_\gamma, \partial S^{2m-1}_\gamma) \to (M \setminus S, \Sigma), \) \end{cases} \]
by \( f \) restricted to \( C^m_\gamma \) and \( S^{2m-1}_\gamma \), respectively. Just as in §6 these define a smooth singular relative \( p \)-cycle \( \delta^m_\gamma \) in \( (M \setminus \hat{S}, \Sigma) \), and a smooth singular relative \( (p+m-1) \)-cycle \( \sigma^{2m-1}_\gamma \) in \( (M \setminus S, \Sigma) \). Note that since \( W \) was also adapted to \( \hat{S} \), the \( \delta^m_\gamma \) just defined agrees with the iterated \( \delta^m_\gamma \) in (12.2).

Set \( U_j = M \setminus S_j \) for \( j = 1, 2, \ldots, m \); then \( \mathcal{U} = \{U_j\} \) is an open covering of \( M \setminus S \). We shall associate to the covering \( \mathcal{U} \) two exact sequences, one for Čech cochains of smooth differential forms, the other for Čech chains of smooth singular chains.

First we denote by \( C^{p,q}(\mathcal{U}) \) the space of alternating Čech \( q \)-cochains \( g = (g_{i_0i_1\ldots i_q}) \) with each \( g_{i_0i_1\ldots i_q} \in \mathcal{E}^p(U_{i_0i_1\ldots i_q}, \Sigma) \), where \( U_{i_0i_1\ldots i_q} = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_q} \). The Čech coboundary operator \( \tilde{\delta} : C^{p,q}(\mathcal{U}) \to C^{p,q+1}(\mathcal{U}) \) is given by
\[ (14.14) \quad (\tilde{\delta}g)_{i_0,i_1\ldots,i_q+1} = \sum_{h=0}^{q+1} (-1)^h g_{i_0\ldots i_h\ldots i_q+1} \bigg|_{U_{i_0i_1\ldots i_q+1}}. \]

Given \( g \in \mathcal{E}^p(M \setminus S, \Sigma) \) we let \( \epsilon^*g \) be the element of \( C^{p,0}(\mathcal{U}) \) defined by
\[ (14.15) \quad (\epsilon^*g)_i = g \big|_{U_i}, \quad i = 1, 2, \ldots, m. \]

Using a partition of unity subordinate to \( \mathcal{U} \), we obtain at once the exactness of the sequence
\[ (14.16) \quad 0 \to \mathcal{E}^p(M \setminus S, \Sigma) \xrightarrow{\epsilon^*} C^{p,0}(\mathcal{U}) \xrightarrow{\tilde{\delta}} C^{p,1}(\mathcal{U}) \to \cdots \xrightarrow{\tilde{\delta}} C^{p,m-1}(\mathcal{U}) \to 0. \]

Second we denote by \( C_{p,q}(\mathcal{U}) \) the \( \mathbb{Z} \)-module of alternating Čech \( q \)-chains \( \alpha = (\alpha_{i_0i_1\ldots i_q}) \) with each \( \alpha_{i_0i_1\ldots i_q} \) being a smooth singular \( p \)-chain in \( (U_{i_0i_1\ldots i_q}, \Sigma) \). The Čech boundary operator \( \partial : C_{p,q}(\mathcal{U}) \to C_{p,q-1}(\mathcal{U}), q \geq 1, \) is defined by
\[ (14.17) \quad (\partial \alpha)_{i_1i_2\ldots i_q} = \sum_{i_0=1}^{m} \alpha_{i_0i_1i_2\ldots i_q}. \]

Moreover we define \( \epsilon_* : C_{p,0}(\mathcal{U}) \to \text{Sing}_p(M \setminus S, \Sigma) \) by
\[ (14.18) \quad \epsilon_* \alpha = \sum_{i=1}^{m} \alpha_i. \]
Here \( \text{Sing}_p(M \setminus S, \Sigma) \) denotes the space of smooth singular relative \( p \)-chains in \((M \setminus S, \Sigma)\) with \( \mathbb{Z} \)-coefficients. By subdivision of the smooth singular relative \( p \)-chains, we obtain the exactness of the sequence
\[
0 \to \mathcal{C}_{p,m-1}(U) \xrightarrow{\partial} \mathcal{C}_{p,m-2}(U) \to \cdots \to \mathcal{C}_{p,0}(U) \xrightarrow{\epsilon_*} \text{Sing}_p(M \setminus S, \Sigma) \to 0.
\]

Next we introduce the duality pairing between \( \mathcal{C}^{p,q}(U) \) and \( \mathcal{C}_{p,q}(U) \):
\[
\int_{\alpha} g = \sum_{1 \leq i_0 < i_1 < \cdots < i_q \leq m} \int_{\alpha_{i_0i_1 \cdots i_q}} g_{i_0i_1 \cdots i_q},
\]
for \( g = (g_{i_0i_1 \cdots i_q}) \in \mathcal{C}^{p,q}(U) \) and \( \alpha = (\alpha_{i_0i_1 \cdots i_q}) \in \mathcal{C}_{p,q}(U) \). The operators \( \partial \) and \( \partial \) are dual to one another with respect to this pairing:
\[
\int_{\partial \alpha} g = \int_{\alpha} \partial g, \quad \text{for } g \in \mathcal{C}^{p,q}(U) \text{ and } \alpha \in \mathcal{C}_{p,q+1}(U).
\]

Likewise
\[
\int_{\epsilon_* \alpha} g = \int_{\alpha} \epsilon^* g, \quad \text{for } g \in \mathcal{E}^p(M \setminus S, \Sigma) \text{ and } \alpha \in \mathcal{C}_{p,0}(U).
\]

Now we use the exactness of (14.16) to show that there is a homomorphism
\[
\sigma_* : \mathcal{H}^p(M \setminus \tilde{S}, \Sigma) \to \mathcal{H}^{p+m-1}(M \setminus S, \Sigma).
\]

Indeed let \( \phi \) be a closed form in \( \mathcal{E}^p(M \setminus \tilde{S}, \Sigma) \). We identify \( \phi \) with an element \( \phi^{(m-1)} \in \mathcal{C}^{p,m-1}(U) \) by
\[
\left( \phi^{(m-1)} \right)_{12 \cdots m} = \phi,
\]
as \( M \setminus \tilde{S} = U_{12 \cdots m} \). By the exactness of (14.16) we can find \( \phi^{(m-1)}, \phi^{(m-2)}, \ldots, \phi^{(1)}, \phi^{(0)} \), with \( \phi^{(j)} \in \mathcal{C}^{p,m-j-1}(U) \), and an \( \eta_{\phi} \in \mathcal{C}^{p+m-1}(M \setminus S, \Sigma) \) with \( d\eta_{\phi} = 0 \) such that
\[
\begin{cases}
\phi^{(m-1)} \text{ is given by (14.23)} \\
\partial \phi^{(m-1)} = \phi^{(m-1)} \\
\cdots \\
\partial \phi^{(j)} = d\phi^{(j+1)} \quad \text{for } j = 0, 1, \ldots, m-3 \\
\cdots \\
d\phi^{(0)} = \epsilon^* \eta_{\phi}.
\end{cases}
\]

Return to the consideration of our smooth singular relative \((p-m)\)-cycle \( \gamma \) in \((S, \Sigma)\): for each \( 0 \leq j \leq m-1 \) we define \( \kappa_j \gamma \in \mathcal{C}_{p+m-j-1,j}(U) \) by
\[
\left( \kappa_j \gamma \right)_{10i_1 \cdots i_j} = \begin{cases}
\text{the restriction of } f \text{ to } \\
\{(x, \zeta) \in \mathbb{R}^{2m-1} \mid |\zeta_i| = 1 \text{ if } \zeta_i \in \{i_0, i_1, \ldots, i_j\}\}
\end{cases},
\]
for \(1 \leq i_0 < i_1 < \cdots < i_j \leq m\). We note that

\[
\begin{align*}
\kappa_{m-1} \gamma &= \tilde{\delta}^{m} \gamma \leftrightarrow \delta^{m} \gamma \\
& \quad \cdots \\
\delta \kappa_j \gamma &= \partial \kappa_{j-1} \gamma, \quad \text{for } 1 \leq j \leq m-1 \\
& \quad \cdots \\
\epsilon_0 \kappa_0 \gamma &= \tilde{\sigma}^{2m-1} \gamma \leftrightarrow \sigma^{2m-1} \gamma.
\end{align*}
\]

(14.27)

Using Stokes’ formula and (14.24), (14.26), (14.21), (14.22) we obtain:

\[
\begin{align*}
&\int_{\sigma_2} \eta \phi = \int_{\epsilon_0 \kappa_0 \gamma} \eta \phi = \int_{\kappa_0 \gamma} \epsilon^* \eta \phi = \int_{\kappa_0 \gamma} d \phi^{(0)} \\
&\quad \cdots \\
&\int_{\delta \kappa_{m-1} \gamma} \delta \phi^{(m-2)} = \int_{\kappa_{m-1} \gamma} \phi^{(m-2)} = \int_{\kappa_{m-1} \gamma} \phi^{(m-1)} = \int_{\delta \kappa_{m-1} \gamma} \phi.
\end{align*}
\]

Thus we have established the following theorem, which holds in the situation detailed in the beginning of §12.

**Theorem 14.1** Let \(\sigma_* : \mathbb{H}^p(M \setminus \tilde{S}, \Sigma) \rightarrow \mathbb{H}^{p+m-1}(M \setminus S, \Sigma)\) be the homomorphism defined in (14.22). Then for any \(h^* \in \mathbb{H}^p(M \setminus \tilde{S}, \Sigma)\) and any class \([\gamma] \in \mathbb{H}_{p-m}(S, \Sigma; \mathbb{Z})\), we have

\[
(14.29) \quad (2\pi \sqrt{-1})^m \int_{[\gamma]} \text{Res}^m h^* = \int_{[\delta \gamma]} h^* = \int_{[\sigma^{2m-1} \gamma]} \sigma_* h^*.
\]

Now that we have Theorem 14.1, it is easy to derive a generalization of Abel’s theorem: We take \(\Sigma = \emptyset\), but otherwise \(M, S, \tilde{S}\) are as in the beginning of §12. When \(M\) is compact, \(S\) consists of a finite number \(S = Z_1 + Z_2 + \cdots + Z_N\) of connected components \(Z_i\), each being a smooth compact orientable CR submanifold of type \((n-m, k)\). A choice of orientation of \(M\) induces an orientation on each \(Z_i\). This makes each \(Z_i\) into a smooth singular \(\{2(n-m)+k\}\)-cycle in \(M\).

**Theorem 14.2** Assume that \(M\) is compact and \(1 \leq m \leq n\). If \(\phi \in \mathcal{E}^{2n+k-m}(M \setminus \tilde{S})\) is closed, then

\[
(14.30) \quad \sum_{i=1}^N \int_{Z_i} \text{Res}^m [\phi] = 0.
\]

**Proof.** We consider \(S\) with its natural orientation as a smooth singular...
\{2(n - m) + k\}-cycle. Then
\[
\int_S \text{Res}^m[\phi] = \frac{1}{(2\pi - i)^m} \int_{\sigma^2m-1S} \sigma_\ast[\phi] = 0,
\]
because \(-\sigma^{2m-1}S\) is the boundary of the complement of a tubular neighborhood of \(S\) in \(M\), by Stokes’ theorem.

**Remark 1.** For \(k = 0\) and \(m = n\) we recover a theorem of Griffiths [G]. The case \(k = 0\) and \(m = n = 1\) is the classical theorem of Abel for compact Riemann surfaces.

**Remark 2.** When the situation is as in \(\S 12\) (each \(S_j\) has a global defining function in \(V\)), and \(\phi\) is a closed semi-CR meromorphic form having a pole of finite order along each \(S_j\), as in (13.1), then (14.29) can be written as
\[
\sum_{i=1}^N \int_{Z_i} \frac{\partial q_1 + \partial q_2 + \cdots + \partial q_m - m \omega}{\partial s_1 \partial s_2 \cdots \partial s_m} \bigg|_{Z_i} = 0,
\]
which is an expression involving ”just calculus”.

**\(\S 15\) Applications of the Abel’s Theorem**
To illustrate the meaning of Theorem 14.2, we discuss in this section only simple applications. We postpone any investigation of complicated examples to a later date; it would require indeed a lengthy discussion of the geometry associated to different types of vector bundles over CR manifolds (see [HN3]).

Suppose \(f_1, f_2, \ldots, f_m\) are smooth CR functions on \(M\), \(1 \leq m \leq n\), and consider \(S_j = \{x \in M \mid f_j(x) = 0\}\). Set \(S = S_1 \cap S_2 \cap \cdots \cap S_m\). We assume that there is an open neighborhood \(W\) of \(S\) in \(M\) in which each \(W \cap S_j\) is a smooth CR submanifold of type \((n - 1, k)\), \(df_j \neq 0\) in \(W \cap S_j\). This means that each \(f_j\) has a simple zero along \(W \cap S_j\), with \(df_j \wedge df_j \neq 0\) along \(W \cap S_j\). We assume also that the \(W \cap S_1, W \cap S_2, \ldots, W \cap S_m\) are in general CR position in \(W\) (i.e. have normal crossings). This implies that \(S\) is a smooth CR submanifold of type \((n - m, k)\). Under the above assumptions we have:

**Proposition 15.1** Let \(M\) be compact and oriented. Then we have the period relations
\[
\sum_{i=1}^N \int_{Z_i} \Theta = 0,
\]
for every smooth \(\bar{\partial}_M\)-closed form \(\Theta\) of type \((n - m + k, n - m)\) on \(M\).

**Proof** We consider the form
\[
g = \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_m}{f_m},
\]
defined on \(M \setminus \hat{S}\). Its \(\text{Res}^m\) is the constant function 1 on \(S\). Our assumption on \(\Theta\) implies that \(\phi = g \wedge \Theta\) is closed. \(\text{Res}^m[\phi] = [\Theta|_{Z_i}]\). Consider the class
\[ [\eta] \in H^{2n+k-1}(M \setminus S) \] associated to \( \phi \), where \( \eta \) is as in (14.24). Then we have by (14.27), (14.28) that
\[
\int_S \Theta = \frac{1}{(2\pi)^m} \int_{\sigma^{2m-1}S} \eta = 0,
\]
because \( \sigma^{2m-1}S \) is homologous to zero in \( M \setminus S \), and the proof is complete.

**Remark.** Note that the compactness of \( M \) does not prohibit the existence of global \( CR \) functions when \( k > 0 \), as it would when \( k = 0 \). When \( m = n \) we have that \( S \) is a totally real \( CR \) submanifold of type \( (0,k) \), which is transversal to the \( CR \) structure of \( M \), and \( \Theta \) is a \( CR \) \( k \)-form on \( M \).

Likewise under the above assumptions we have:

**Proposition 15.2** Let \( M \) be compact and oriented. Then
\[
\sum_{i=1}^{N} \int_{Z_i} \text{Res}^m \left[ \frac{\omega}{f_1 f_2 \cdots f_m} \right] = 0,
\]
for every smooth closed form \( \omega \) of type \( (n+k,n-m) \) on \( M \setminus S \).

**Proof** First observe that our hypothesis on \( \omega \) implies that \( \phi = \omega / f_1 f_2 \cdots f_m \) is closed in \( M \setminus S \). Consider the class \( [\eta] \in H^{2n+k-1}(M \setminus S) \) associated to \( \phi \) as in (14.24). Then we have as before that
\[
\int_S \text{Res}^m \left[ \frac{\omega}{f_1 f_2 \cdots f_m} \right] = \int_{\sigma^{2m-1}S} \eta = 0,
\]
since \( \sigma^{2m-1}S \sim 0 \) in \( M \setminus S \).

**Remark.** When \( m = n \) the \( \omega \) above is a closed section over \( M \setminus S \) of the *canonical line bundle* of \( M \); i.e., a \( CR \) \( (n+k) \)-form.

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