Multiparticle quantum correlation and entanglement in four-qubit pure states

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Based on the quantitative complementarity relations, we analyze thoroughly the properties of multiparticle quantum correlation and entanglement in four-qubit pure states. It is found that, unlike the three-qubit case, the single residual correlation and the genuine correlations of three and four qubits are unable to quantify entanglement appropriately. More interestingly, from our qualitative and numerical analysis, it is conjectured that the sum of all residual correlations is a good quantity for characterizing the multiparticle entanglement in the system.

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I. INTRODUCTION

Entanglement has been a vital physical resource for quantum information processing, such as quantum communication [1, 2] and quantum computation [3, 4, 5, 6]. Therefore, the characterization of entanglement for a given quantum state is a fundamental problem. Bipartite entanglement is well understood in many aspects [6, 7, 8, 9]. Especially, for two qubits, its mixed state entanglement can be characterized with the help of the so-called concurrence [10]. However, in multipartite cases, the quantification of entanglement is very complicated and challenging.

A fundamental property of multiparticle entangled state is that entanglement does not increase under local operations and classical communication (LOCC), (i.e., the entanglement monotone property).

Recently, Osborne and Verstraete also proved that the distribution of bipartite entanglement among $N$-qubit quantum state satisfies the relation

$$C^2_{A_1A_2} + C^2_{A_1A_3} \leq \tau_{A_1(A_2A_3)}$$

where the $\tau_{A_1(A_2A_3)}$ is the linear entropy for a pure state. Comparing with the three-qubit case, it is natural to ask whether or not the residual quantum correlation in an $N$-qubit pure state ($N > 3$) is a good measure of the genuine multipartite entanglement.

In this paper, we attempt to answer the above tough question clearly. Based on the quantitative complementarity relations (QCRs), we analyze the properties of multiparticle correlations and entanglement in four-qubit pure states. It is shown that the single residual correlation in the four-qubit case does not satisfy the entanglement monotone property. In addition, the genuine three- and four-qubit correlations are unable to quantify entanglement, either. Finally, in terms of a serious analysis on the sum of all residual correlations, we conjecture it to be an appropriate quantity for constituting the multipartite entanglement measure in the composite system.

The paper is organized as follows. In Sec. II, the properties of multipartite correlations in four-qubit pure states are analyzed in detail. As a result, a multipartite entanglement measure is conjectured. In Sec. III, we give some remarks and main conclusions. In addition, three examples are given in the Appendix.

II. MULTIPARTITE QUANTUM CORRELATIONS IN FOUR-QUBIT PURE STATES

Before analyzing the quantum correlations, we first introduce the QCRs. Complementarity [17] is an essential principle of quantum mechanics, which is often referred to the mutually exclusive properties of a single quantum system. As a special quantum property without classical counterpart, entanglement can constitute complementarity relations with local properties [18, 19]. Jakob and Bergou derived a QCR for two-qubit pure state [20], i.e., $C^2 + S^2_{\rho} = 1$, in which the concurrence $C$ quanti-
by the overlapping areas of these circles. According to respectively, and the quantum correlations are denoted overlapping quantum correlations, respectively. The areas without overlapping \( S_k^2 \) is the local reality of qubit \( k \), for \( k = A, B, C, D \).

FIG. 1: (Color online) The correlation Venn diagram for a four-qubit pure state \(|\Psi\rangle_{ABCD} \). The overlapping areas \( t_4 \), \( t_3 \)'s, and \( t_2 \)'s denote the genuine four-, three-, and two-qubit quantum correlations, respectively. The areas without overlapping \( S_k^2 \) is the local reality of qubit \( k \), for \( k = A, B, C, D \).

This diagram, the four-qubit QCRs can be written as

\[
\begin{align*}
& t_4 + t_3^{(2)} + t_3^{(3)} + t_3^{(4)} + \sum_{l \in R_A} t_2(\rho_{AI}) + S_A^2 = 1, \\
& t_4 + t_3^{(1)} + t_3^{(3)} + t_3^{(4)} + \sum_{l \in R_B} t_2(\rho_{BI}) + S_B^2 = 1, \\
& t_4 + t_3^{(1)} + t_3^{(2)} + t_3^{(4)} + \sum_{l \in R_C} t_2(\rho_{CI}) + S_C^2 = 1, \\
& t_4 + t_3^{(1)} + t_3^{(2)} + t_3^{(3)} + \sum_{l \in R_D} t_2(\rho_{DI}) + S_D^2 = 1,
\end{align*}
\]

where the \( t_3^{(1)} \), \( t_3^{(2)} \), \( t_3^{(3)} \) and \( t_3^{(4)} \) are the three-qubit correlations in subsystems \( \rho_{BCD} \), \( \rho_{ACD} \), \( \rho_{ABD} \), and \( \rho_{ABC} \), respectively. In three-qubit pure states, the quantum correlations \( t_2 \) (square of the concurrence) and \( t_3 \) (3-tangle) in the linear entropy are good measures for two- and three-qubit entanglement, respectively. However, it is an open problem that whether or not the similar relations also hold in a four-qubit pure state \(|\Psi\rangle_{ABCD}\).

Before analyzing the multipartite correlations \( t_4 \) and \( t_3^{(1)} \)'s, we need consider how to evaluate the two-qubit correlation \( t_2(\rho_{ij}) \) in the pure state \(|\Psi\rangle_{ABCD} \). Similar to the three-qubit case, we make use of the square of the concurrence which is defined as \( C_{ij} = \max\{(\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}), 0\} \), where the decreasing positive real numbers \( \lambda_i \)'s are the eigenvalues of matrix \( \rho_{ij} (\sigma_y \otimes \sigma_y) \rho_{ij}^* (\sigma_y \otimes \sigma_y) \).

The main reason for this evaluation is because that the relation \( \sum_{l \in R_k} C_{kl} = \tau_{k(R_k)} \) holds for the four-qubit \( W \) state \(|\psi\rangle_{ABCD} = \alpha_1|0001\rangle + \alpha_2|0010\rangle + \alpha_3|0100\rangle + \alpha_4|1000\rangle \) which involves only the two-qubit entanglement. In the following, we will analyze the properties of the single residual correlation, the genuine three- and four-qubit correlations, and the sum of all residual correlations, respectively.

A. Single residual correlation

Under the above evaluation for the two-qubit quantum correlation, the multipartite correlation around the qubit \( k \) (i.e., the residual correlation) will be

\[
M_k(|\Psi\rangle) = \tau_{k(R_k)} - \sum_{l \in R_k} t_2(\rho_{kl}),
\]

in which \( t_2(\rho_{kl}) = C_{kl}^2 \) and \( k = A, B, C, D \). As widely accepted, a good measure for the multipartite entanglement should satisfy the following requirements [13]: (1) the quantity should be a non-negative real number; (2) it is unchanged under the LU operations; (3) it does not increase on average under the LOCC i.e., the measure is entanglement monotone.

Now we analyze the residual correlation \( M_k \). According to the monogamy inequality proven by Osborne and Verstraete [16], it is obvious that \( M_k \) is positive semi-definite. In addition, for the full separable state and the
entangled state involving only two-qubit correlations, it can be verified that $M_k = 0$.

The correlation $M_k$ is also LU invariant, which can be deduced from the fact that the linear entropy and the concurrence are invariant under the LU transformation.

The last condition is that $M_k$ should be non-increasing on average under the LOCC. It is known that any local protocol can be implemented by a sequence of two-outcome POVMs involving only one party \cite{12}. Without loss of generality, we consider the local POVM $\{A_1, A_2\}$ performed on the subsystem $A$, which satisfies $A_1 A_1 + A_2 A_2 = I$. According to the singular value decomposition \cite{12}, the POVM operators can be written as $A_1 = U_1 \text{diag}(\alpha, \beta) V$ and $A_2 = U_2 \text{diag}(\sqrt{1-\alpha^2}, \sqrt{1-\beta^2}) V$, in which $U_i$ and $V$ are unitary matrices. Since $M_k$ is LU invariant, we need only to consider the diagonal matrices in the following analysis. Note that the linear entropy and concurrence are invariant under the LU transformation \cite{13, 20}, the following relation can be derived

$$p_1 M_A(\Phi_1) + p_2 M_A(\Phi_2) \leq M_A(\Psi),$$

which means the multipartite correlation $M_A$ is entanglement monotone under the local operation performed on subsystem $A$.

It should be pointed out that the above property is not sufficient to show the parameter $M_A$ is monotone under the LOCC. This is because, unlike the three-qubit case, the residual correlation $M_k$ in a four-qubit state will change after permuting the parties. Therefore, before claiming that the $M_k$ is entanglement monotone, one needs to prove the parameters $M_B, M_C$, and $M_D$ are also non-increasing on average under the POVM $\{A_1, A_2\}$ performed on subsystem $A$. However, this requirement cannot be satisfied in a general case, because the behaviors of the three parameters are quite different from that of $M_A$. For example, in the correlation $M_C = \tau_{C(R_C)} - C^2_{AC} - C^2_{BC} - C^2_{ED}$, only the $C^2_{AC}$ is invariant under the determinant one stochastic LOCC performed on subsystem $A$. With this property, we know $C^2_{AC}$ is entanglement monotone. As to the linear entropy $\tau_{C(R_C)}$ and the other concurrences ($C^2_{BC}$ and $C^2_{ED}$), one can prove that they are decreasing and increasing under the POVM $\{A_1, A_2\}$, respectively, in terms of the following two facts: first, for the reduced density matrices $\rho_{C}, \rho_{BC}$ and $\rho_{CD}$, the effect of the POVM is equivalent to decomposing them into two mixed states, respectively; second, the linear entropy is concave function and the concurrence is convex function. Comparing the behaviors of $M_A$ and $M_C$ under the POVM, we can not ensure that $M_C$ is entanglement monotone (in the Appendix), we give an example in which the correlation $M_C$ will increase under a selected POVM performed on subsystem $A$. The cases for $M_B$ and $M_D$ are similar.

For a kind of symmetric quantum state which has the property $M_A = M_B = M_C = M_D$, is the correlation $M_k$ entanglement monotone? The answer is still negative. Since the symmetry cannot hold after an arbitrary POVM, the parameter $M_k$ cannot be guaranteed to be monotone under the next level of POVM once the property is broken (see such an example in the Appendix). Therefore, we conclude that the correlation $M_k$ is not entanglement monotone and it is not a good entanglement measure.

B. Three- and four-qubit correlations

Next, we analyze the properties of the correlations $t_4$ and $t^{(i)}_3$. Note that the QCRs provide only four equations which cannot determine completely the five multipartite parameters in general. Therefore, a well-defined measure for $t_3$ or $t_4$ is needed in this case. Recently, an attempt was made to introduce an information measure $\xi_{BCD}$ for the genuine four-qubit entanglement \cite{24}, but this measure can hardly characterize completely the genuine four-qubit correlation/entanglement \cite{27}.

On the other hand, a mixed 3-tangle $t_3 = \min_{p_x, \phi_1} \max_{p_y, \phi_2} t(\phi_x) \leq \xi_{BCD}$ could not be chosen as the correlation $t_3$ either, because it is not compatible with the QCRs of Eq. (4). As an example, we consider the quantum state $|\psi\rangle_{ABCD} = \frac{|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle}{2}$ \cite{29}, in which the reduce density matrix $\rho_{BCD}$ can be decomposed to the mix of two pure states $|\phi_1\rangle = |000\rangle$ and $|\phi_2\rangle = \frac{|011\rangle + |101\rangle + |110\rangle}{\sqrt{3}}$. Supposing that the $t_3$ is a good measure for $t_3$, we can obtain $t^{(i)}_3 = \tau_3(\rho_{BCD}) = 0$ in terms of the definition of the mixed 3-tangle. Then the other multipartite correlations are determined from Eq. (4), with $t_4 = 1.5$ and $t^{(2)}_3 = t^{(3)}_3 = t^{(4)}_3 = -0.25$. Because these correlations are not in the reasonable range, the mixed 3-tangle is not a suitable measure compatible with the QCRs.

Although the analytical measures for $t_4$ and $t_3$ are unavailable now, we may analyze a special kind of quantum state in which $t_4$ is zero. The quantum state $|\varphi\rangle = \alpha|0000\rangle + \beta|1011\rangle + \gamma|1000\rangle + \eta|1110\rangle$ is just the case. Suppose that the good correlation measures are existent and their values correspond to the overlapping regions in the Venn diagram (Fig.1). It is simple to see that these correlations are non-negative and LU invariant. In the quantum state $|\varphi\rangle$, if we let the $t^{(i)}_4$ be the variables, we can obtain the relation $t^{(1)}_4 = -t^{(4)}_4$ according to the QCRs of Eq. (4). Due to the non-negative property of the two correlations, we can judge the four-qubit correlations is zero in this state. Then the other three-qubit correlations can be solved with the QCRs. In order to test the entanglement monotone of $t^{(i)}_3$ more clearly, the parameters in $|\varphi\rangle$ are chosen to be $\alpha = \beta = \gamma = \eta = 1/2$ (see the example 3 in the Appendix). After performing
a selected POVM, we find the $t_3^{(2)}$ will increase on average, which implies that the correlations $t_3$ and $t_4$ are not suitable for the quantification of entanglement.

C. Sum of the residual correlations

Finally, we consider the sum of all residual correlations, which is defined as

$$M = M_A + M_B + M_C + M_D = \sum_k \tau_b(R_k) - 2 \sum_{p>q} C^2_{pq}, \tag{7}$$

in which $k, p, q = A, B, C, D$. It is obvious that $M$ is non-negative and LU invariant in terms of the corresponding properties of $M_k$. It is extremely difficult to prove the entanglement monotone property analytically. The main hindrance lies in that one cannot compare the change of the concurrences in a general quantum state before and after the POVM.

Nevertheless, we conjecture that the correlation $M$ is an entanglement monotone, as rationalized in some sense below. From the definition of $M$, it is seen that $M$ is invariant under the permutations of the subsystems. Without loss of the generality, suppose that the POVM is performed on the subsystem $A$. In this case, we analyze the behaviors of the components in $M$. According to the prior analysis in Eq.(6), the component $\xi_1 = \tau_a(R_A) - C^2_{AB} - C^2_{AC} - C^2_{AD}$ is decreasing on average. Moreover, due to the concave property of linear entropy and the convex property of concurrence, the component $\xi_2 = \tau_b(R_B) + \tau_c(R_C) + \tau_d(R_D) - 2(C^2_{BC} + C^2_{BD} + C^2_{CD})$ is also decreasing after the POVM. The only increasing component is $\xi_3 = -C^2_{AB} - C^2_{AC} - C^2_{AD}$. It is conjectured that the decrease of $\xi_1$ and $\xi_2$ can counteract the increase of $\xi_3$, which results further in the entanglement monotone property of $M$.

In Fig.2, the quantity $\Delta M = M(|\Psi\rangle) - p_1 M(|\Phi_1\rangle) - p_2 M(|\Phi_2\rangle)$ is calculated for nine quantum states $G_{abcd}, L_{abc_2}, L_{aab_2}, L_{aab_3}, L_{a_4}, L_{a_3b_1}, L_{a_2b_1}$ and $L_{0_{3011}}$ (the state parameters we choose are listed in Table I), which are the representative states under the SLOCC classification (c.f. Ref. 29). Due to the form of quantum state $L_{0_{3011}} = |0000\rangle + |0111\rangle$, we perform the POVM on its subsystem $B$. For the other states, the POVM is performed on the subsystem $A$. From Fig.2, we can see the correlation $M$ do not increase on average under the POVMs, which support our conjecture (for the POVMs performed on other subsystems, we obtain the similar results). In addition, for the symmetric quantum states $G_{abcd}, L_{abc_2}$ and $L_{aab_1}$, the second level of the POVM is also calculated and the $\Delta M$ is still nonnegative (in the first level of the POVM performed on the subsystem $A$, the diagonal elements are $\alpha_1 = 0.4$ and $\beta_1 = 0.7$; in the second level of POVM, $\alpha_2$ and $\beta_2$ are chosen from 0.05 to 0.95, and the interval is 0.01).

Mainly based on the above analysis, we therefore conjecture that the multipartite correlation $M$ is entangle-monotone and then is possible to constitute a measure for the total multipartite entanglement in four-qubit pure states.

At this stage, we may also introduce the average multipartite entanglement

$$E_{ms} = \frac{M}{4} = \frac{M_A + M_B + M_C + M_D}{4}, \tag{8}$$

to characterize the entanglement per single qubit (ranged in $[0,1]$), as far as the correlation $M$ is (conjectured to be) entanglement monotone. A remarkable merit of this quantity is its computability. For the quantum state $G_{abcd} = \frac{1}{4}(|0000\rangle + |1111\rangle + |0011\rangle + |1100\rangle) + \frac{1}{2}(|0101\rangle + |1010\rangle) + \frac{1}{2}(|0110\rangle + |1001\rangle)$ which is the generic kind under the SLOCC classification, the change of $E_{ms}$ with the real parameters $a$ and $d$ are plotted in Fig.3 (the parameters $b = 0$ and $c = 0.5$ are fixed). In the regions near $(a = d = 0), (a \gg c, d)$ and $(d \gg a, c)$, the multipartite entanglement $E_{ms}$ tends to zero, which can be explained that the quantum state tends to be the tensor product of the two bell states in these ranges. The bigger values of $E_{ms}$ appear in the regions near $(a = 0, d = 0.5), (a = 0.5$ and $d = 0)$, and $a = d \gg c$. This is because the quantum state $G_{abcd}$ approaches to the four-qubit GHZ state in these regions (e.g., when

FIG. 2: (Color online) The values of $\Delta M$ for nine representative states. In the POVM, the diagonal elements $\alpha$ and $\beta$ are chosen from 0.05 to 0.95, and the interval is 0.01.

| $G_{abcd}$ | $L_{abc_2}$ | $L_{aab_2}$ | $L_{aab_3}$ | $L_{a_4}$ | $L_{a_3b_1}$ |
|-----------|------------|------------|------------|----------|------------|
| a=c=1    | a=2       | a=1       | a=1        | a=1      | a=1        |
| b=d=0.5  | b=c=1     | b=1       | b=1.5      |          |            |

TABLE I: The parameters we choose in the quantum states $G_{abcd}, L_{abc_2}, L_{aab_2}, L_{aab_3}, L_{a_4}, L_{a_3b_1}$ (Ref. 29).
a = 0 and d = 0.5, the $E_{ms}$ is 1 and the quantum state can be rewritten as $G_{abcd} = (|\alpha\alpha\alpha\alpha\rangle + |\beta\beta\beta\beta\rangle)/\sqrt{2}$ after a local unitary transformation $|\alpha\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$ and $|\beta\rangle = (|0\rangle - i|1\rangle)/\sqrt{2}$. In this case, the four-partite entanglement is a dominant one.

Although the operational meaning of $E_{ms}$ for entanglement transformation and distillation is not clear now, we can use this quantity to restrict some procedures which are impossible (suppose that the $E_{ms}$ is validated to be entanglement monotone). For example, if the quantity increases in an LOCC transformation from $|\varphi_1\rangle$ to $|\varphi_2\rangle$, we can judge that this procedure is impossible because the entanglement should be monotone in a real physical transformation.

It should be pointed out that the quantity $E_{ms}$ in Eq. (8) corresponds to the correlation $t_4 + \frac{i}{\sqrt{2}} \sum t_4^{(i)}$, which is not the total multipartite correlation $M_T = t_4 + \sum t_4^{(i)}$ in the Venn diagram. Whether or not the $M_T$ is a good candidate for the total multipartite entanglement in the system is worth study in the future. In order to test the entanglement properties of $M_T$, one needs first to find the appropriate definitions for the correlation $t_4$ and $t_3$, respectively.

For an $N$-qubit pure state, the sum of all residual correlations is given by

$$M_N(\Psi_N) = Nt_N + (N-1)\sum t_{N-1} + \cdots + 3\sum t_3 = \sum \tau_{k(R_k)} - 2\sum_{i>j} C_{ij}^2.$$  

(9)

Similar to the four-qubit case, this quantity is non-negative real number in terms of the monogamy inequality. In addition, the LU invariance of $M_N$ is guaranteed by the corresponding property of linear entropy and concurrence. For the entanglement monotone, we conjecture the correlation $M_N$ also satisfies. Therefore, correlation $M_N$ may be able to characterize the multipartite entanglement in the system. Similarly, the average over $N$ qubits $M_N/N$ (ranged in $[0,1]$) can be considered as the entanglement per qubit.

III. DISCUSSION AND CONCLUSION

In the correlation Venn diagram of three-qubit pure state $|\Psi_{ABC}\rangle$ [23, 30], the quantum correlations at different levels are able to characterize the corresponding quantum entanglements. Therefore, the total entanglement in the system is contributed by the two-qubit entanglement and the genuine three-qubit entanglement, respectively. However, in the four-qubit case, the structure of total entanglement is quite complicated; how to quantify separately the three- and four-qubit entanglement is still an open problem. It was indicated by Wu and Zhang that the set of two-, three-, four-partite GHZ states is not a reversible entanglement generating set for four-party pure states [31] (i.e., the set of entangled states can not generate an arbitrary four-party pure state by the LOCC asymptotically [32]), which implies that the GHZ-class entanglements are not sufficient for characterizing the structure of total entanglement in the system. Recently, it was noted by Lohmayer et. al. [33] that a kind of rank-2 three-qubit mixed states which are entangled but do not have the mixed 3-tangle and concurrence (one can consider that these states are reduced from four-qubit pure states). This case shows further that the quantification of entanglement in multi-qubit systems is extremely complicated and highly nontrivial.

In conclusion, based on the generalized QCRs, we have analyzed the multipartite correlations in four-qubit pure states. Unlike the three-qubit case, we find that the similar relations do not hold again in the four-qubit system. First, the residual correlation $M_k$ is not of entanglement monotone. In addition, the genuine three- and four-qubit correlations are not suitable to be entanglement measure, either. Finally, the total residual correlation $M$ has been analyzed, and it is conjectured that the average multipartite correlation $E_{ms}$ is able to quantify the multipartite entanglement in the system.

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Appendix

Example 1: Consider a quantum state $|\Psi_{ABCD}\rangle = (|0000\rangle + |0011\rangle + |0101\rangle + |0110\rangle + |1010\rangle + |1111\rangle)/\sqrt{6}$,
TABLE II: The values of the correlation measures related to subsystem $C$ before and after the POVM.

| correlation state | $\tau_{C(Rc)}$ | $C_{AC}^2$ | $C_{BC}^2$ | $C_{CD}^2$ | $M_C$ |
|-------------------|----------------|------------|------------|------------|-------|
| $|\Psi\rangle$    | 8/9           | 4/9        | 0          | 0          | 4/9   |
| $|\Phi_1\rangle$  | 0.9994        | 0.04703    | 0          | 0          | 0.9524|
| $|\Phi_2\rangle$  | 0.4867        | 0.4063     | 0          | 0          | 0.08042|

Comparing the change of $M_A$, we can get $M_A(|\Phi_1\rangle) - [p_{11}M_A(|\Phi_{11}\rangle) + p_{12}M_A(|\Phi_{12}\rangle) = -0.05382$. This means that the correlation $M_A$ is increasing under the LOCC, and thus $M_k$ is not a good entanglement measure for the symmetric quantum state.

Example 3: We analyze the quantum state $|\Psi\rangle_{ABCD} = (|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle)/\sqrt{2}$, which is the representative state $L_{0,0,3}^{29}$. The POVM $\{A_1, A_2\}$ is performed on the subsystem $B$. Due to the LU invariance of the correlations $t_4$ and $t_3$, we only consider the diagonal elements of the operators $A_1$ and $A_2$ (in the form of the singular value decomposition) in which the parameters are chosen to be $\alpha = 0.9$ and $\beta = 0.4$. After the POVM, two outcomes $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are obtained with the probabilities $p_1 = 0.4850$ and $p_2 = 0.5150$, respectively. In Table IV, the values of $t_4$ and $t_3^{(i)}$ for $|\Psi\rangle$, $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are listed.

TABLE IV: The values of the correlation measures $t_4$ and $t_3$ before and after the POVM.

| correlation state | $t_4$ | $t_3^{(1)}$ | $t_3^{(2)}$ | $t_3^{(3)}$ | $t_3^{(4)}$ |
|-------------------|-------|-------------|-------------|-------------|-------------|
| $|\Psi\rangle$    | 0     | 0           | 0.2500      | 0.2500      | 0.2500      |
| $|\Phi_1\rangle$  | 0     | 0           | 0.02721     | 0.1377      | 0.1377      |
| $|\Phi_2\rangle$  | 0     | 0           | 0.6651      | 0.1504      | 0.1504      |

With these values, we can get $t_3^{(2)}(|\Psi\rangle) - [p_{11}t_3^{(2)}(|\Phi_1\rangle) + p_{21}t_3^{(2)}(|\Phi_2\rangle) = -0.1057$, which means that the correlation $t_3$ can increase on average under the LOCC and that it is not a good entanglement measure.
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