EFFICIENT DIRECTED GRAPH SAMPLING VIA GERSHGORIN DISC ALIGNMENT

Yuejiang Li*, H. Vicky Zhao*, Gene Cheung†

*Dept. of Automation, Tsinghua University, Beijing, China
†York University, Toronto, Canada

ABSTRACT

Graph sampling is the problem of choosing a node subset via sampling matrix \( H \in \{0, 1\}^{K \times N} \) to collect samples \( y = Hx \in \mathbb{R}^K \), \( K < N \), so that the target signal \( x \in \mathbb{R}^N \) can be reconstructed in high fidelity. While sampling on undirected graphs is well studied, we propose the first sampling scheme tailored specifically for directed graphs, leveraging a previous undirected graph sampling method based on Gershgorin disc alignment (GDAS). Concretely, given a directed positive graph \( G^+ \) specified by random-walk graph Laplacian matrix \( \mathbf{L}_{rw} \), we first define reconstruction of a smooth signal \( x^* \) from samples \( y \) using graph shift variation (GSV) \(|\mathbf{L}_{rw}x|^2\) as a signal prior. To minimize worst-case reconstruction error of the linear system solution \( x^* = C^{-1}H^\top y \) with symmetric coefficient matrix \( C = H^\top H + \mu \mathbf{L}_{rw} \mathbf{L}_{rw}^\top \), the sampling objective is to choose \( H \) to maximize the smallest eigenvalue \( \lambda_{\min}(C) \) of \( C \). To circumvent eigen-decomposition entirely, we maximize instead a lower bound \( \lambda_{\min}^{-1}(\text{SCS}^{-1}) \) of \( \lambda_{\min}(C) \)—smallest Gershgorin disc left-end of a similarity transform of \( C \)—via a variant of GDAS based on Gershgorin circle theorem (GCT). Experimental results show that our sampling method yields smaller signal reconstruction errors at a faster speed compared to competing schemes.

Index Terms—Graph signal processing, signal sampling, Gershgorin circle theorem

1. INTRODUCTION

Graph signal processing (GSP) extends traditional signal processing tools to analyze signals on irregular data kernels described by finite graphs \([1, 2]\). Most existing GSP works consider undirected graph structures, where each edge connecting two nodes is bidirectional. However, directionality plays an important role in many practical information dissemination scenarios \([3]\). For example, on Twitter, a celebrity often has a large following but personally follows very few users \([4]\). Thus, in these scenarios it is critical to factor directionality into the network model, resulting in a directed graph.

Graph sampling selects a node subset to collect samples, so that the target signal can be recovered in high fidelity \([5]\). Existing graph sampling works can be classified into two categories based on prior assumptions: i) an assumption on strict bandlimitedness of target signals with a cutoff frequency, and ii) a more general assumption assuming target signals are “smooth” with respect to \((w.r.t.)\) the underlying graph \((e.g., \text{ more energy in low frequencies than high frequencies})\). Bandlimited assumption in the first category means that a target signal lies strictly inside a linear subspace spanned by the first eigenvectors (Fourier modes) of a graph variation operator, such as the graph Laplacian matrix \( \mathbf{L} \) or the adjacency matrix \( \mathbf{W} \) \([6, 10]\). Assuming that the observed signal samples contain noise, \([6]\) proposed a greedy algorithm to select sample nodes under the E-optimality criterion \([11]\), and \([7]\) designed a greedy algorithm to minimize the reconstruction MSE. To lower complexity, \([9]\) and \([12]\) used graph spectral proxies and localization operators, respectively, to mitigate the computation burden of eigen-decomposition.

However, the strict bandlimited assumption of target signals is a strong one that many practical graph signals do not satisfy. To relax this assumption, works in the second category assume that a target signal is generally smooth over a given graph \([13, 14]\). For example, graph Laplacian regularizer (GLR) \([15]\), i.e., \( x^\top \mathbf{L} x \), is often used to quantify smoothness of signal \( x \) over a graph specified by Laplacian \( \mathbf{L} \) \([13, 17]\). GLR is often used to regularize under-determined signal reconstruction problems, such as denoising, dequantization, and interpolation \([15, 18, 19]\). Using GLR as signal prior, graph sampling based on Gershgorin disc alignment (GDAS) was proposed to efficiently select sample nodes on undirected (signed) graphs under the E-optimality criterion \([15, 17]\). A key feature of GDAS is that it circumvents eigen-decomposition entirely and executes in linear time, and thus is scalable to large graphs.

Although the above sampling algorithms are efficient and effective, they are all designed for undirected graphs, and cannot be easily applied to directed graphs. One main challenge in directed graph sampling is the inherent difficulty in defining graph frequencies, due to the asymmetric nature of the directed graphs’ variation operators, \(e.g.,\) adjacency and Laplacian matrices, \( \mathbf{W} \) and \( \mathbf{L} \). Asymmetry means that the graph operator matrix may not be diagonalizable (and thus eigenvectors cannot be easily obtained), and even if it is, its eigenvalues can be complex, which are difficult to interpret \((e.g., \text{ordering of eigenvectors into frequencies from high to low is not obvious})\). Though \([6]\) discussed in passing how their methods can be adapted to directed graphs, frequency and bandlimitedness notions are still not well understood on directed graphs.

To circumvent the above challenge, in this work, we formulate a novel directed graph sampling problem using graph shift variation (GSV) \([20]\) with a solution that completely avoids matrix asymmetry, and in so doing enable a variant of GDAS for fast sampling. Specifically, we first define GSV \(|\mathbf{L}_{rw}x|^2\) as a smoothness prior for directed graph signal \( x \in \mathbb{R}^N \), where \( \mathbf{L}_{rw} \) is a random-walk graph Laplacian for directed graph \( G^+ \). Using GSV as regularizer to reconstruct signal \( x \) from samples \( y = \mathbf{H}x \in \mathbb{R}^K \), \( \mathbf{H} \in \{0, 1\}^{K \times N} \) is a sampling matrix, the solution is \( x^* = C^{-1}H^\top y \), with symmetric coefficient matrix \( C = H^\top H + \mu \mathbf{L}_{rw} \mathbf{L}_{rw}^\top \). To minimize the worst-case reconstruction error (E-optimality), the sampling objective is to choose \( \mathbf{H} \) to maximize the smallest eigenvalue \( \lambda_{\min}(C) \) of \( C \). To mitigate eigen-decomposition entirely, we devise a variant of previous GDAS to efficiently choose \( \mathbf{H} \) to maximize a lower bound \( \lambda_{\min}^{-1}(\text{SCS}^{-1}) \)—smallest Gershgorin disc left-end of a similarity transform of \( C \)—based on Gershgorin circle theorem (GCT) \([21]\). Experimental results show that our sampling method yields smaller signal reconstruction errors at a faster speed compared to...
to competing schemes. To the best of our knowledge, this is the first directed graph sampling algorithm in GSP free from explicit definitions of directed graph frequencies.

2. PRELIMINARIES

Consider a directed graph $G^d = (\mathcal{V}, \mathcal{E}, W)$ with $N$ nodes $\mathcal{V}$ and directed edges $\mathcal{E}$. $W$ is an adjacency matrix, where $W_{i,j} \in \mathbb{R}^+$ is the positive weight of directed edge $(i,j)$ if it exists in $\mathcal{E}$. We assume no self-loops, and thus $W_{i,i} = 0, \forall i$. Denote by $D$ the diagonal out-degree matrix such that $D_{i,i} = \sum_{j} W_{i,j}$. We assume that each node has strictly positive degree, i.e., $D_{i,i} > 0, \forall i$; this means that there are no sink nodes. Graph Laplacian matrix of the directed graph is defined as $L = D - W$. The normalized adjacency matrix is $\tilde{W} = D^{-1}W$, and the random-walk graph Laplacian is $\tilde{L}_r = D^{-1}L = I - \tilde{W}$. Finally, we assume that there exists at least one node $v$ such that there are directed paths from all other nodes $v' \in \mathcal{V}$ to node $v$. This assumption ensures that the rank of the random-walk Laplacian matrix $\tilde{L}_r$ is $N - 1$.

3. PROBLEM FORMULATION

We first review a previously proposed smoothness prior—-graph shift variation prior (GSV) [20]—use it to reconstruct a directed graph signal given limited samples. We then formulate a directed graph sampling problem given a defined signal reconstruction scheme.

3.1. Signal Reconstruction on a Directed Graph

**Graph Shift Variation Prior.** Denote by $x \in \mathbb{R}^N$ a signal on a directed graph $G^d$. An important assumption in GSP is that the signal is smooth w.r.t. an underlying graph. For undirected graphs, there exist different smoothness measures of a graph signal, such as GLR [15] and graph total variation (GTV) [23]. However, for directed graphs, because Laplacian $L$ is asymmetric, smoothness priors like GLR cannot be used directly. In this work, we adopt GSV in [20] as the smoothness measure of a signal $x$ on directed graph $G^d$, i.e.,

$$S(x) = \| x - \frac{1}{\lambda_{\text{max}}(W_s)} W_s x \|_2^2. \quad (1)$$

Here, $W_s$ is a graph shifting operator [24,25], and it has the same support as adjacency matrix $W$. $\lambda_{\text{max}}(W_s)$ is the largest magnitude eigenvalue of $W_s$, and $|\lambda_{\text{max}}(W_s)|$ is the spectral radius of matrix $W_s$. $1/|\lambda_{\text{max}}(W_s)|$ is used for normalizing. $\frac{1}{\lambda_{\text{max}}(W_s)} W_s x$ shifts each node’s sample to its one-hop neighbors, and $S(x)$ measures the difference between signal $x$ and its shifted version.

In this work, we use the normalized adjacency matrix $\tilde{W} = D^{-1}W$ as the graph shift operator, since its largest singular value is $\lambda_{\text{max}}(\tilde{W}) = 1$. Consequently, the GSV prior is defined as

$$S(x) = \| x - \tilde{W} x \|_2^2 = \| \tilde{L}_r x \|_2^2 = x^\top \tilde{L}_r \tilde{L}_r x. \quad (2)$$

This GSV prior in (2) is similar to the left eigenvector random walk graph Laplacian (LeRA) regularizer in [18]. It is shown that the smooth prior in (2) is insensitive to vertex degrees. Further, it is shown [18] that GSV of a constant signal $x = c1$ evaluates to $S(x) = 0$, which is intuitive and important for imaging applications.

**Signal Reconstruction using GSV Prior.** Suppose that we obtain $K$ samples, $y \in \mathbb{R}^K$, of graph signal $x \in \mathbb{R}^N$, where $K < N$. We aim to reconstruct signal $x^*$ given observation $y$. To regularize this under-determined problem, we employ GSV (2) as prior and solve the following regularized optimization problem [20,26]:

$$x^* = \arg\min_x \| Hx - y \|_2^2 + \mu x^\top \tilde{L}_r \tilde{L}_r x, \quad (3)$$

where $H \in \{0,1\}^{K \times N}$ is the sampling matrix, and $\mu > 0$ is a weight parameter that trades off the fidelity term with the GSV prior. The optimal solution $x^*$ to (3), which is quadratic and convex, can be obtained by solving the following linear system

$$(H^\top H + \mu \tilde{L}_r \tilde{L}_r) x^* = H^\top y. \quad (4)$$

Note that both $H^\top H$ and $\tilde{L}_r \tilde{L}_r$ are positive semi-definite matrix. For matrix $\tilde{L}_r$, we have $\tilde{L}_r = (I - \tilde{D}^{-1} \tilde{W}) I = 1 - \tilde{W} = 0$. Thus, $\text{Span}(\tilde{1}) \subseteq \text{Null}(\tilde{L}_r)$. With the assumption in Sec. 2, that there exists at least one node that can reach any other nodes through directed paths, from Theorem 4.5 in [22], the dimension of $\text{Null}(\tilde{L}_r)$ is 1. Thus, we have $\text{Span}(\tilde{1}) = \text{Null}(\tilde{L}_r)$. Note that for $x \in \text{Span}(\tilde{1})$, we have $x^\top H^\top Hx \geq 0$ and $x^\top \tilde{L}_r \tilde{L}_r x > 0$. For $x \in \text{Span}(\tilde{1})$, say $x = c1$ where $c$ is a non-zero real scalar, we have $x^\top H^\top Hx > 0$ and $x^\top \tilde{L}_r \tilde{L}_r x > 0$. In summary, for any $x \in \mathbb{R}^N$, we have $x^\top (H^\top H + \tilde{L}_r \tilde{L}_r) x > 0$, and thus, $(H^\top H + \tilde{L}_r \tilde{L}_r)$ is a positive definite matrix and invertible. Consequently, the unique optimal solution to (3) as well as (4) is

$$x^* = (H^\top H + \mu \tilde{L}_r \tilde{L}_r)^{-1} H^\top y. \quad (5)$$

Note that given the coefficient matrix in (4) is symmetric, sparse, and positive definite, $x^*$ can be solved using conjugate gradient (CG) [27] without performing any matrix inverse.

3.2. Directed Graph Sampling Problem

Observation $y$ may contain noise. Given a sampling budget $K$, to minimize worst-case reconstruction error using reconstruction (5), we adopt the E-optimality criterion [16] to maximize the smallest eigenvalue of coefficient matrix $H^\top H + \mu \tilde{L}_r \tilde{L}_r$. For notation simplicity, we define diagonal matrix $A = H^\top H$, whose diagonal entry $A_{i,i} = 1$ if node $i$ is sampled and $A_{i,i} = 0$ otherwise. The sampling problem is thus formulated as

$$\max_{\lambda_{\text{min}}(A + \mu \tilde{L}_r \tilde{L}_r)} \lambda_{\text{min}}(A + \mu \tilde{L}_r \tilde{L}_r) \quad \text{s.t. } A_{i,i} \in \{1,0\}, \forall i, \quad \text{tr}(A) = K. \quad (6)$$

The second constraint in (6) indicates that we can sample $K$ nodes. Note that optimization (6) is combinatorial in nature and NP-hard in general. Next, we develop an efficient algorithm for (6).

4. THE PROPOSED GDA-DIRECT METHOD

We first review the Gershgorin disc algorithm and then design an efficient algorithm to solve (6) based on GDAS.

4.1. Gershgorin Disc Algorithm

The foundation of GDAS is Gershgorin Circle Theorem (GCT) [21]. Gershgorin disc $\Psi_i$ of the $i$-th row of a real matrix $M$ is a circle on the complex plane, with center $(M_{i,i}, 0)$ and radius $r_i = \sum_{j \neq i} |M_{i,j}|$. GCT states that all eigenvalues of $M$ reside inside the union of Gershgorin discs of $M$. For a real symmetric matrix $M$
whose eigenvalues are also real, we can define the lower bound of $\lambda_{\min}(M)$ as the smallest left-end of all discs:

$$\lambda_{\min}(M) \triangleq \min \{\lambda_{M,i} - r_i\} \leq \lambda_{\min}(M).$$  

(7)

When applying a similarity transform on $M$ with an invertible diagonal matrix $S = \text{diag}(s_1, \ldots, s_N)$, $s_i \neq 0$, \forall $i$, we obtain a new matrix $S^{-1}M S^{-1}$ with the same eigenvalues as original $M$. Thus,

$$\lambda_{\min}(S^{-1}M S^{-1}) \leq \lambda_{\min}(S^{-1}M S^{-1}) = \lambda_{\min}(M).$$  

(8)

To apply GDAS [16] to approximately solve the sampling problem (6), we employ the lower bound (8) and relax the objective in (6), we reformulate the optimization problem as

$$\max_{A,S} \lambda_{\min}(A + \mu S L_{rw}^T L_{rw} S^{-1})$$

subject to $A_{i,i} \in \{0, 1\}, \forall i$, $\text{tr}(A) = K$.  

(9)

GDAS efficiently solves the relaxed problem (9) through disc shifting for optimal $A^*$ and disc scaling for optimal $S^*$. We refer readers to [16] for details of the algorithm.

Graph Balancing Algorithm for (9). GDAS requires all $L_{rw}^T L_{rw}$’s disc left-ends to be initially aligned at the same value before sampling. However, this is not always satisfied for random-walk Laplacian $L_{rw}$ of a directed graph. Define $L \triangleq L_{rw}^T L_{rw}$. We see that $L1 = L_{rw}^T L_{rw}1 = 0$, i.e., for any row $i$ of $L$, $L_{i,i} = -\sum_{j \neq i} L_{i,j}$. This means that $L$ is a Laplacian matrix for an undirected graph without self-loops, and its corresponding adjacency matrix is $W = \text{Diag}(L) - L$. Hence, if there exist negative weights in $W$, then $L$’s disc left-ends are not all aligned at 0, since $e_i - r_i = \sum_{j \neq i} L_{i,j} = \sum_{j \neq i} (-L_{i,j} - L_{j,i}) \neq 0$.

To align disc left-ends, one method is to first balance the graph using an algorithm in [17], then align the discs’ left-ends of the Laplacian $L^b$ of the balanced graph $G^b$ via a similarity transform $L^b = S L^T S^{-1}$, where $S = \text{diag}(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N)$ and $\{\tilde{\sigma}_i\}$ are computed from the first eigenvector of $L^b$. Subsequently, GDAS [16] can be employed for sampling by maximizing $\lambda_{\min}(A + SL^T S^{-1})$. We refer to this method as GDA-Balance.

4.2. Analysis of the Lower Bound of the Objective Function

Although GDA-Balance can solve the relaxed sampling problem (9), it has drawbacks. First, the graph balancing procedure in [17] can be computation-expensive. Second, the resulting balanced graph Laplacian $L^b$ may not be PSD, while $L_{rw}^T L_{rw}$ is PSD with $\lambda_{\min}(L_{rw}^T L_{rw}) = 0$, as discussed in Sec. 3.1. This means that computed lower bound using GDA-Balance for objective $\lambda_{\min}(A + L_{rw}^T L_{rw})$ may be loose in practice.

From the definition of $L_{rw}$ of the directed graph in Sec. 2, we see that the left-ends of $L_{rw}$’s discs are aligned at (0, 0) on the complex plane, though it is not symmetric. This observation motivates us to relax the objective $\lambda_{\min}(A + L_{rw}^T L_{rw})$ in (6) with a lower bound in the form of $\lambda_{\min}(S(A + \mu L_{rw}^T L_{rw}) S^{-1})$, where $\delta$ and $\rho$ are positive constants. Doing so means we can directly apply GDAS on the relaxed lower bound due to the alignment of the left ends of $L_{rw}$. In the following, we first derive this lower bound, and then efficiently calculate the parameters $\delta$ and $\rho$.

Proposition 1. Given a normalized adjacency matrix $W$, its random-walk graph Laplacian $L_{rw}$, sampling budget $K$, and a non-negative hyper-parameter $\mu > 0$, for any positive scalar $0 < \epsilon < 1$, we define two sets of parameters $\delta$ and $\rho$ depending on $\mu$ as follows:

- $\delta = \sqrt{1 - \epsilon}, \rho = -\epsilon c + \sqrt{\epsilon^2 c^2 + 4 \epsilon \mu} / 2$, where,  

(10)

- $c = \max_{A,\rho(i) \geq K} \min_{A_{i,i} = (1, 0)} \lambda_{\min}(S(A + \mu L_{rw}^T L_{rw} + \epsilon \cdot A) S^{-1})$, for $\mu \leq 1$; and  

(11)

- $\delta = -p c + \sqrt{p^2 c^2 + 4}, \rho = \sqrt{\mu - \epsilon},$ where,  

(12)

$\epsilon c = 3 + \max_{A,\rho(i) \geq K} \min_{A_{i,i} = (1, 0)} \lambda_{\min}(S(A + \mu L_{rw}^T L_{rw} + \epsilon \cdot A) S^{-1})$, for $\mu > 1$.  

(13)

Here, $W[i,j]$ is the $j$-th largest element in the $i$-th column of $W$. With the defined parameters $\delta$ and $\rho$ above, if a non-negative invertible $S$ satisfies $\lambda_{\min}(S(A + \mu L_{rw} S^{-1})) \geq 0$, then the following inequality holds

$$\lambda_{\min}(S(\delta A + \mu L_{rw} S^{-1})) \leq \lambda_{\min}(A + \mu L_{rw}^T L_{rw}),$$  

where the positive constant $\gamma_{\max}$ is defined as

$$\gamma_{\max} \triangleq \max_{A} \gamma(P)$$

subject to $\delta A + \mu L_{rw} = \text{PA} \text{P}^{-1}$, $A_{i,i} = (0, 1)$, $\text{tr}(A) = K$.  

In the above, $\gamma(P)$ is the condition number of matrix $P$.

The proof of Proposition 1 is in Appendix A. With Proposition 1, we can relax the objective $A + \mu L_{rw}^T L_{rw}$ with its lower bound in (14). Compared to GDA-Balance, we see that the derived lower bound in (14) is always positive, and thus is tighter than the lower bound given by GDA-Balance.

Note that $\gamma_{\max}$ is a constant that does not depend on $A$ or $S$. Further, since we assumed $\lambda_{\min}(S(\delta A + \mu L_{rw} S^{-1}) \geq 0$ in Proposition 1, we can maximize $\lambda_{\min}(S(\delta A + \mu L_{rw} S^{-1}))$, where we can directly apply GDAS. Note that during the optimization process in GDAS, it naturally ensures that $\lambda_{\min}(S(\delta A + \mu L_{rw} S^{-1}) \geq 0$. Thus, this assumption in Proposition 1 always holds when we use GDAS on $\lambda_{\min}(S(\delta A + \mu L_{rw} S^{-1}))$.

4.3. GDA-Direct Algorithm for Sampling on Directed Graphs

To maximize $\lambda_{\min}(S(\delta A + \mu L_{rw} S^{-1}))$ using GDAS, we first need to calculate the parameters $c$ in (11) and (13), and $\delta$ as well as $\rho$ in (10) and (12). We discuss the efficient calculation / approximation of these parameters and elaborate the complete GDA-Direct sampling algorithm. The algorithm is summarized in Algorithm 1.

To apply GDAS on $\lambda_{\min}(S(\delta A + \mu L_{rw} S^{-1}))$, we need to determine the parameters $\delta$ and $\rho$, which depends on the parameter $c$ in (11) and (13). To calculate $c$, we need to calculate its numerator $3 + \max_{A,\rho(i) \geq K} \sum_{j=1}^{K-1} W[j,i]$ and denominator $\min_{A} \lambda_{\min}(S(A + \mu L_{rw}^T L_{rw} + \epsilon \cdot A)$ or $\min_{A} \lambda_{\min}(A + \mu L_{rw}^T L_{rw})$, respectively. To calculate the numerator, for the $j$-th column of matrix $W$, we find the top $K - 1$ largest entries, i.e., $W[j,i]$ for $j = 1, \ldots, K - 1$. Then, with the computed $\sum_{j=1}^{K-1} W[j,i]$ for each column, we calculate the numerator (line 1 in Algorithm 1).

The time complexity for this step is
\[O(Nd_W^\rho \log(K-1)),\] where \(d_W^\rho\) is the largest in-degree of the directed graph. Here, we can see that the time complexity is proportional to the graph size \(N\).

Next, we calculate the denominator \(\min_{\mathbf A} \lambda_{\min}(\mathbf L_{rw}^T \mathbf L_{rw} + \epsilon \cdot \mathbf A)\), when \(\mu \leq 1\). Here, optimization variable \(\mathbf A\) is diagonal and satisfies \(\mathbf A_{ii} = \{1, 0\}\) and \(\text{tr}\{\mathbf A\} = K\). To simplify calculation, we assume small positive \(\epsilon \) near 0, and \(\mathbf L_{rw}^T \mathbf L_{rw} + \epsilon \cdot \mathbf A\) is a perturbation of \(\mathbf L_{rw}^T \mathbf L_{rw}\). Denote by \(0 \leq \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N\) the eigenvalues of \(\mathbf L_{rw}^T \mathbf L_{rw}\), and \(\mathbf u\), the corresponding eigenvector of \(\lambda_1\). According to the analysis of the perturbed matrix's eigenvalues \([29, 31]\), we can approximate the eigenvalue of \(\mathbf L_{rw}^T \mathbf L_{rw} + \epsilon \cdot \mathbf A\) as
\[
\hat{\lambda}_1 \approx \lambda_1 + \epsilon \mathbf u_1^T \mathbf A \mathbf u_1.
\] (15)

Note that the smallest eigenvalue of \(\mathbf L_{rw}^T \mathbf L_{rw}\) and its corresponding eigenvector are \(\lambda_1 = 0\) and \(\mathbf u_1 = \frac{1}{\sqrt{N}}\), respectively. Thus, approximated eigenvalue \(\hat{\lambda}_1\) is
\[
\hat{\lambda}_1 \approx 0 + \epsilon \times \frac{1}{\sqrt{N}} \mathbf A \mathbf 1 = \frac{K\epsilon}{N}.
\] (16)

For other approximated eigenvalues, we have \(\hat{\lambda}_i \approx \lambda_i + \epsilon \mathbf u_i^T \mathbf A \mathbf u_i \geq \lambda_i\) for \(\epsilon > 1\). To ensure that \(\hat{\lambda}_1\) is the smallest approximated eigenvalue, we only need \(\lambda_1 \leq \lambda_i\), that is,
\[
\epsilon \leq \frac{\lambda_i N}{K}.
\] (17)

Note that (17) also gives us a guide on how to choose \(\epsilon\) when \(\mu \leq 1\). For the second smallest eigenvalue of \(\mathbf L_{rw}^T \mathbf L_{rw}\), i.e., \(\lambda_2\), we can use LOBPCG \([32]\) to efficiently calculate it. When \(\epsilon\) satisfies the condition in (17), we can approximate the denominator of \(c\) as \(\lambda_{\min}(\mathbf L_{rw}^T \mathbf L_{rw} + \epsilon \cdot \mathbf A) \approx \lambda_1 = \frac{K\epsilon}{N}\), which is independent of variable \(\mathbf A\). Further, we can calculate \(\delta\) and \(\rho\) with the approximated \(c\) and \(\epsilon\) (line 3 \& 4 in Algorithm [1]).

Similarly, when \(\mu > 1\), we can also approximate the denominator of \(c\) in (13), i.e., \(\min_{\mathbf A} \lambda_{\min}(\mathbf A + \epsilon \cdot \mathbf L_{rw}^T \mathbf L_{rw})\). Since matrix \(\mathbf A\) has \(N - K\) zero eigenvalues whose corresponding eigenvectors are canonical basis vectors \(\mathbf e_i\), the smallest eigenvalue of \(\mathbf A + \epsilon \cdot \mathbf L_{rw}^T \mathbf L_{rw}\) can be approximated as
\[
\hat{\lambda}_1 \approx 0 + \epsilon \mathbf e_1^T \mathbf L_{rw}^T \mathbf L_{rw} \mathbf e_1.
\] (18)

Note that since we assumed there are no sink nodes in the graph, we have \((\mathbf L_{rw})_{i,i} = 1\) for all \(i\), and \(\mathbf e_i^T \mathbf L_{rw}^T \mathbf L_{rw} \mathbf e_1 \geq 1\). Therefore, when \(\mu > 1\), the denominator of \(c\) can be approximated as
\[
\min_{\mathbf A} \lambda_{\min}(\mathbf A + \epsilon \cdot \mathbf L_{rw}^T \mathbf L_{rw}) \approx \min \hat{\lambda}_1 = \epsilon \cdot \min_i \mathbf e_i^T \mathbf L_{rw}^T \mathbf L_{rw} \mathbf e_i.
\] (19)

The time complexity of calculating \(c\), \(\delta\), and \(\rho\) are \(O(1)\) for \(\mu \leq 1\) and \(O(N)\) for \(\mu > 1\), respectively. After determining \(\delta\) and \(\rho\), we can employ GDAS to maximize \(\lambda_{\min}(\mathbf S(\delta \mathbf A + \rho \mathbf L_{rw}) \mathbf S^{-1})\) to obtain the optimal solution \(\mathbf A^*\) (line 9 in Algorithm [1]). The time complexity of this step is \(O(N)\) \([16]\). In summary, the overall time complexity is \(O(N(1+\delta\mu^\rho \log(K-1)))\). When the graph is sparse, and the sampling budget \(K\) is small, the overall time complexity is roughly \(O(N)\).

\section{5. EXPERIMENTS}

We tested the proposed sampling algorithm on synthetic directed graphs. Our experimental platform was Ubuntu 18.04 server with a 32-core AMD Ryzen 3970X CPU and 250 GB memory. All algorithms were implemented with Python 3.8.

\subsection*{Graph Structure}
We randomly generated Erdős-Rényi random graphs with \(N = 200\) nodes for experiments. For any ordered pair of nodes \(u\) and \(v\), a directed edge \((u, v)\) was generated with probability \(p\). To satisfy the assumption in Sec. 2 that there is at least one node that can be reached by any other nodes, we first generated a random graph with \(N - 1\) nodes. Then, we manually added the last node \(v_N\) and directed edges \((v_i, v_N)\) for \(i = 1, \ldots, N - 1\). Further, we randomly chose a node \(v_i\) \((i \neq N\), and added a directed edge \((v_i, v_N)\), such that there were no sink nodes with zero out-degrees in the graph. We independently generated the weight of each edge from uniform distribution in \([0, 1]\) and then normalized the weights such that \(\mathbf W1 = 1\).

\subsection*{Graph Signal}
We considered three types of graph signals.

1. (GS1) We used the eigen-decomposition \(\mathbf L_{rw}^T \mathbf L_{rw} = \mathbf U \hat{\mathbf \Lambda} \mathbf U^T\) where the eigenvectors and eigenvalues are \(\mathbf u_i\)'s and \(\hat{\lambda}_i\)'s, and generated a random bandlimited graph signal \(x = \sum_{i=1}^{N} c_i \mathbf u_i\), where \(\hat{\lambda}_m\) was the cutoff frequency, and each \(c_i\) was independently generated from normal distribution \(\mathcal{N}(0, 1)\). Here, \(m = \lceil 0.1N \rceil\).

2. (GS2) We generated graph signals following a normal distribution \(\mathcal{N}(0, (\mathbf L_{rw}^T \mathbf L_{rw} + \omega I)^{-1})\) where \(\omega = 0.1\).

3. (GS3) We generated graph signals \(x\) through a diffusion process. Specifically, we randomly generated an initial signal \(x(0)\) from the normal distribution \(\mathcal{N}(0, 1)\). Then, we followed a diffusion process \(x(t) = (1 - \alpha)x(t - 1) + \alpha \mathbf W x(t - 1)\) for \(T\) steps, where \(\alpha\) is a parameter, and the graph signal is defined as \(x = x(T)\). The intuition here is that, as \(T \to \infty\), all entries of \(x(T)\) converge to the same value \([33]\). Thus, the smoothness prior \(S(x(T))\) in \(\mathbf A\) approaches 0.

For each generated signal, we normalized via \(x \leftarrow \frac{x - \text{mean}(x)}{\sqrt{N \cdot \text{std}(x)}}\), such that \(\|x\|_2 = 1\).

\subsection*{Baseline Methods}
We compared the proposed sampling method GDA-Direct with the following baseline methods.

- Random: This method randomly selects sampling nodes.
- E-optimal \([6]\): Using the E-optimality criterion, this method greedily selects sampling nodes one-by-one.
- SDP-Relax \([34]\): This method relaxes the original integer constraint \(\mathbf A_{i,i} \in \{0, 1\}\) in \([6]\) to a continuous constraint.
Fig. 1. Reconstruction MSE of (left) GS1, (mid) GS2, and (right) GS3 on random graph \((N = 200 \text{ and } p = 0.1)\).

Fig. 2. Reconstruction MSE of GS1 on random graph with (left) \(p = 0.05\), (mid) \(p = 0.1\), and (right) \(p = 0.15\).

Fig. 3. Running time on random graphs with different sizes.

0 \leq A_{i,i} \leq 1. Consequently, the relaxed problem can be formulated as a semi-definite programming (SDP) problem \([34]\)

\[
\begin{align*}
\min & \quad \ell \\
\text{s.t.} & \quad \sum_i A_{i,i} E_i + L_{rw}^T L_{rw} - \ell \cdot I \succeq 0, \\
& \quad 0 \leq A_{i,i} \leq 1, \forall i, \sum_i A_{i,i} = K.
\end{align*}
\]

Matrix \(E_i\) has only non-zero entry \(E_{i,i} = 1\). Since the problem in (20) is convex, we used \texttt{cvxopt} to obtain a solution. Given an optimal solution \(A^*_{i,i}\) to (20), we selected the \(K\) largest \(A^*_{i,i}\) and set them to 1 and other \(A^*_{i,i}\) to 0.

- **GDA-Balance** \([35]\): As discussed in Sec. 4.1, this method treats \(L_{rw}^T L_{rw}\) as a generalized Laplacian for a signed graph, and uses the algorithm in \([35]\) for sampling.

- **SP** \([9]\): This method greedily selects the sampling nodes with the graph spectral proxy. We use \(L_{rw}^T L_{rw}\) as the variation operator in \([9]\).

In the experiments, we randomly generated 5 random graphs and 3000 graph signals over each graph. \(T\) and \(\alpha\) for GS3 were set to 50 and 0.1 unless otherwise specified. We adopted the signal reconstruction scheme \([5]\) in Sec. 3.1 and set the hyper-parameter \(\mu = 0.001\). Next, we show the average results over 15000 simulation runs.

We first compare the reconstruction MSE \(\|x - \hat{x}\|_2^2\) of different sampling methods. The results with different graph signals are shown in Fig. 1. For all three types of graph signal and sampling budget \(K\), the proposed GDA-Direct performed better than other baseline sampling methods. In particular, when the sampling budget \(K\) is small, the superiority of GDA-Direct over other baseline methods is more obvious. For example, GDA-Direct decreases the reconstruction MSE by 8.6% for low-pass signal (GS1) with \(K = 20\) samples, and that by 11.9% for diffusion signal (GS3) with only \(K = 10\) samples, comparing to other baseline methods that do not use GDA algorithm.

We also compare GDA-Direct with other baseline methods on random graphs with different sparsity. We adjust \(p\) from 0.05 to 0.15, and The resulting reconstruction errors for GS1 are shown in Fig. 2. Similar results for other types of graph signals were observed and thus are omitted here. In Fig. 2, we see that GDA-direct outperformed other methods on all graphs with different sparsity. These observations validate the effectiveness of GDA-Direct.

Next, we compare the efficiency of the proposed GDA-Direct to other sampling methods. In Fig. 3, we plotted running time of different sampling methods on random graphs \((p = 0.1)\) with different sizes \(N\). The sampling budget was set to \(K = 0.3N\). Since
Random method randomly samples from the graph with negligible computational cost, we omitted its running time for clarity. When the graph is large ($N \geq 300$), we see that both GDA-Direct and GDA-Balance run faster than other methods due to the linear time complexity of the GDA algorithm [16]. Further, GDA-Direct ran faster than GDA-Balance; specifically, when the graph size is 800, GDA-Direct is 1.4 times faster than GDA-Balance. This is because the pre-computation of $\delta$ and $\rho$ in GDA-Direct is more efficient than the balancing procedure in GDA-Balance. Note that when the graph size was small ($N \leq 100$), E-Optimal and SP were faster than the proposed GDA-Direct. This is because E-Optimal and SP depend heavily on matrix operations, e.g., SVD and LOBPCG. In Python, these matrix operations are optimized and parallelized, and thus, E-Optimal and SP have advantages for smaller graphs. These observations further validate the efficiency and effectiveness of the proposed sampling algorithm on large graphs ($N \geq 300$).

6. CONCLUSION

In this paper, we study the sampling problem on directed graphs. We propose a graph signal reconstruction scheme using graph shift variation as the smoothness regularizer, and formulate a sampling problem under the E-optimality criterion. For the formulated sampling problem, we propose a fast algorithm based on the Gershgorin disc alignment algorithm, which does not require eigendecomposition. Experiment results on synthetic graphs show that the proposed sampling algorithm decreases the reconstruction MSE by at least 8.6% and speeds up about 1.4 times compared to other baseline methods.

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A. PROOF OF PROPOSITION 1

To prove Proposition 1, we first have the following lemmas.

Lemma 1. For a real matrix (not necessarily symmetric) $M = PAP^{-1}$ and a non-negative invertible diagonal matrix $S$ such that $\lambda_{\text{min}}(S^{-1}M) \geq 0$, we have

$$\frac{1}{\gamma(M)}(\lambda_{\text{min}}(S^{-1}M))^2 \leq \lambda_{\text{min}}(M^T M),$$

where $\gamma(M) \triangleq \frac{\lambda_{\text{max}}(P^T P)}{\lambda_{\text{min}}(P^T P)} > 0$ is the condition number of matrix $P$ and it is a function of matrix $M$.

Proof. Let $\lambda_{\text{min}}(M) = a + bi \in \mathbb{C}$ be the eigenvalue of matrix $M$ with the smallest absolute value, i.e., $0 \leq |\lambda_{\text{min}}(M)| \leq |\lambda(M)|$. Here, $i$ is the imaginary unit. Since $S^{-1}M$ has the same eigenvalues as $M$, $\lambda_{\text{min}}(M)$ is also the smallest eigenvalue of $S^{-1}M$ in terms of absolute value. According to the Gershgorin disc theorem, eigenvalue $\lambda_{\text{min}}(M)$ must lie in at least one disc of $S^{-1}M$. Since we have assume that $\lambda_{\text{min}}(S^{-1}M) \geq 0$, we have $a = \text{Re}(\lambda_{\text{min}}(M)) \geq \lambda_{\text{min}}(S^{-1}M) \geq 0$. Consequently, we have

$$\lambda_{\text{min}}(S^{-1}M) \leq a \leq |\lambda_{\text{min}}(M)|.$$

With the eigendecomposition $M = PAP^{-1}$, we define the condition number of $P$ to be a function of matrix $M$, i.e., $\gamma(M) \triangleq \frac{\lambda_{\text{max}}(P^T P)}{\lambda_{\text{min}}(P^T P)} > 0$. According to Theorem 1 in [36], we have

$$|\lambda_{\text{min}}(M)| \leq \gamma(M) \sigma_{\text{min}}(M) = \gamma(M) \sqrt{\lambda_{\text{min}}(M^T M)},$$

where $\sigma_{\text{min}}$ is the smallest singular value of $M$. Combining the above two inequality, we have

$$0 \leq \lambda_{\text{min}}(S^{-1}M) \leq \gamma(M) \sqrt{\lambda_{\text{min}}(M^T M)},$$

and

$$\frac{1}{\gamma^2(M)}(\lambda_{\text{min}}(S^{-1}M))^2 \leq \lambda_{\text{min}}(M^T M).$$

This ends the proof.

Lemma 2. For two symmetric matrices $M_1, M_2 \in \mathbb{R}^{N \times N}$, if $M_1 - M_2 \succeq 0$, we have $\lambda_{\text{min}}(M_1) \geq \lambda_{\text{min}}(M_2)$.

Proof. We refer readers to Proposition 2 in [35] for the proof.

Lemma 3. For any diagonal matrix $A$ such that $A_{i,i} \in \{0, 1\}$ and $tr(A) = K$, we have

$$\lambda_{\text{max}}(L_{rw}^T A + A L_{rw}) \leq 3 + \max_{j=1}^{K-1} \sum_{i=1}^{j} W_{[j],i},$$

where $W_{[j],i}$ is the $j$-th largest element in the $i$-th column of $W$. 
Proof. Let $\mathbf{B} = (L_{rw}^{\top} \mathbf{A} + \mathbf{A} L_{rw})$ and $\mathcal{T} = \{ i | A_{i,i} = 1 \}$. Note that $(L_{rw}^{\top} \mathbf{A})^\top = \mathbf{A} L_{rw}$ and $(\mathbf{A} L_{rw})_{i,j} = (L_{rw}^{\top} \mathbf{A})_{j,i} = \left\{ \begin{array}{ll} (L_{rw}^{\top} \mathbf{A})_{i,j} & \text{if } i, j \in \mathcal{T}, \\ 0 & \text{otherwise}. \end{array} \right.$ 

$\mathbf{B}$ is a symmetric matrix, and $\lambda_{\max}(\mathbf{B})$ is a real value. With the entry of $\mathbf{A} L_{rw}$ and $L_{rw}^{\top} \mathbf{A}$ above, we have

$$
B_{i,j} = \left\{ \begin{array}{ll} (L_{rw}^{\top} \mathbf{A})_{i,j} & \text{if } i, j \in \mathcal{T}, \\ (L_{rw}^{\top} \mathbf{A})_{j,i} & \text{if } i \in \mathcal{T} \text{ and } j \notin \mathcal{T}, \\ 0 & \text{if } j \in \mathcal{T} \text{ and } i \notin \mathcal{T}. \end{array} \right.
$$

To upper bound the largest eigenvalue $\lambda_{\max}(\mathbf{B})$, we need to study the right ends of $B$'s Gershgorin discs. For the $i$-th row of $\mathbf{B}$, if $i \in \mathcal{T}$, we have $B_{i,i} = 2(L_{rw}^{\top} A)_{i,i} = 2$, and the center of the corresponding disc is $(2, 0)$. For the radius of the disc corresponding to the $i$-th row of $\mathbf{B}$ ($i \notin \mathcal{T}$), we have

$$
\sum_{j \notin \mathcal{T}} |B_{i,j}| = \sum_{j \notin \mathcal{T}} |(L_{rw}^{\top} \mathbf{A})_{i,j}| = \sum_{j \notin \mathcal{T}} |(-W_{i,j} - W_{j,i})| = \sum_{j \notin \mathcal{T}} W_{i,j} + \sum_{j \notin \mathcal{T}} W_{j,i}.
$$

Thus, the right end of the $i$-th disc is $3 + \sum_{j \notin \mathcal{T}} W_{i,j} \leq 3 + \sum_{j=1}^{k-1} W_{i,j}$. Similarly, for the disc of the $j$-th row where $j \notin \mathcal{T}$, the center of this disc is $(0, 0)$, and the radius is $\sum_{i \notin \mathcal{T}} W_{i,j}$. Then, the right end of this disc is $\sum_{i \notin \mathcal{T}} W_{j,i} \leq \sum_{i=1}^{k-1} W_{j,i}$. From the Gershgorin circle theorem, the largest eigenvalue of $\mathbf{B}$ is smaller than the largest (upper bounds of) disc’s right end. Therefore, we have

$$
\lambda_{\max}(\mathbf{B}) \leq \max_i \{ 3 + \sum_{j=1}^{k-1} W_{i,j} \} = 3 + \max_i \sum_{j=1}^{k-1} W_{i,j}.
$$

This ends the proof.

With the above three lemmas, we prove Proposition 1 as follows.

Proof. Consider the eigendecomposition $\mathbf{A} = \mathbf{P} \Sigma \mathbf{P}^{-1}$. As $\delta$ and $\rho$ are given constant, $\lambda_{\max}(\mathbf{B})$ in the right-hand side of the above inequality is zero. Consequently, we have $f \geq 0$ when $\mu \leq 1$. When $\mu > 1$, if $\rho = \sqrt{\mu - \epsilon}$, we have

$$
f_1 = \rho \lambda_{\max}(\mathbf{A} + \rho \mathbf{L}_{rw}^{\top}) \mathbf{u} \leq \rho \lambda_{\max}(\mathbf{A} + \rho \mathbf{L}_{rw}^{\top}) (\mathbf{u}^{\top} \mathbf{L}_{rw}^{\top} \mathbf{u}).
$$

Plugging (25) and (26) into (24), we have

$$
f \geq (\mu - \rho^2) \cdot (\min A \lambda_{\min}(\mathbf{L}_{rw}^{\top} \mathbf{A}^{\top} \mathbf{L}_{rw} + \epsilon \mathbf{A})) - \rho (3 + \max_i \sum_{j=1}^{k-1} W_{i,j}) (\mathbf{u}^{\top} \mathbf{L}_{rw}^{\top} \mathbf{u}).
$$

Next, we aim to show that $\lambda_{\min}(\mathbf{A} + \mu \mathbf{L}_{rw}^{\top} \mathbf{L}_{rw}) \geq \lambda_{\min}(\mathbf{B} + \rho \mathbf{L}_{rw})$. To achieve this, with Lemma 2, we aim to show that for any unit vector $\mathbf{u}$, the quadratic form

$$
f = \mathbf{u}^{\top} [(\mathbf{A} + \mu \mathbf{L}_{rw}^{\top} \mathbf{L}_{rw}) - (\mathbf{A} + \rho \mathbf{L}_{rw}) (\mathbf{A} + \rho \mathbf{L}_{rw})] \mathbf{u} \geq 0.
$$

Note that for matrix $\mathbf{A}$, we have $\mathbf{A} = \mathbf{A}^{\top}$, and $\mathbf{A}$ is an idempotent matrix, i.e., $\mathbf{A}^{\top} \mathbf{A} = \mathbf{A}$. Consequently, we have

$$
(\mathbf{A} + \rho \mathbf{L}_{rw}) (\mathbf{A} + \rho \mathbf{L}_{rw}) = \delta^2 \mathbf{A} + \rho^2 \mathbf{L}_{rw}^{\top} \mathbf{L}_{rw} + \rho \mathbf{L}_{rw}^{\top} \mathbf{A} + \mathbf{A} \mathbf{L}_{rw}.
$$

Plugging the above equation into the quadratic form (23), we have

$$
f = \mathbf{u}^{\top} [(\mathbf{A} + \rho \mathbf{L}_{rw}^{\top} \mathbf{L}_{rw}) - (\mathbf{A} + \rho \mathbf{L}_{rw}) (\mathbf{A} + \rho \mathbf{L}_{rw})] \mathbf{u} = 
(1 - \delta^2) \mu^2 \mathbf{A} \mathbf{u} + (\mu - \rho^2) \mathbf{u}^{\top} \mathbf{L}_{rw}^{\top} \mathbf{u}
$$

$$
= (1 - \delta^2) \mu^2 \mathbf{A} \mathbf{u} + (\mu - \rho^2) \mathbf{u}^{\top} (\mathbf{L}_{rw}^{\top} \mathbf{L}_{rw} + \epsilon \mathbf{A}) \mathbf{u}
$$

$$
\geq (\mu - \rho^2) \cdot (\mathbf{u}^{\top} \mathbf{L}_{rw}^{\top} \mathbf{u} + \epsilon \mathbf{A}) \mathbf{u}.
$$

With the Rayleigh quotient theorem, we have $\mathbf{u}^{\top} (\mathbf{L}_{rw}^{\top} \mathbf{L}_{rw} + \epsilon \mathbf{A}) \mathbf{u} \geq \lambda_{\min}(\mathbf{L}_{rw}^{\top} \mathbf{L}_{rw} + \epsilon \mathbf{A})$. Since $\mathbf{A}$ is the optimization variable, we have

$$
f_1 \geq (\mu - \rho^2) \cdot (\min A \lambda_{\min}(\mathbf{L}_{rw}^{\top} \mathbf{L}_{rw} + \epsilon \mathbf{A})).
$$

For $f_2$ in (24), according to the Rayleigh quotient theorem and Lemma 3, we have

$$
f_2 = \rho \lambda_{\max}(\mathbf{L}_{rw}^{\top} \mathbf{A} + \mathbf{L}_{rw}^{\top} \mathbf{A}) \mathbf{u} \leq \rho \lambda_{\max}(\mathbf{L}_{rw}^{\top} \mathbf{A} + \mathbf{L}_{rw}^{\top} \mathbf{A})
$$

$$
\leq \rho (3 + \max_i \sum_{j=1}^{k-1} W_{i,j}).
$$

Plugging (25) and (26) into (24), we have

$$
f \geq (\mu - \rho^2) \cdot (\min A \lambda_{\min}(\mathbf{L}_{rw}^{\top} \mathbf{L}_{rw} + \epsilon \mathbf{A})) - \rho (3 + \max_i \sum_{j=1}^{k-1} W_{i,j}).
$$
Combining the upper bound of $f_2$ in \cite{26}, we have

$$f \geq (1-\delta^2) \left( \min_A \lambda_{\text{min}}(A+\epsilon L_{ru}^T L_{rw}) \right) - \delta \rho (3+\max_i \sum_{j=1}^{k-1} W_{[j,i]})$$

When $\delta = -\frac{\rho c + \sqrt{\rho^2 c^2 + 4}}{2}$, the right hand side of the above inequality is zero. Consequently, we have $f \geq 0$ when $\mu > 1$.

In summary, with $\delta$ and $\rho$ in (10) and (12), we can always ensure that $f \geq 0$. According to Lemma 2, we have $\lambda_{\text{min}}(A+\mu L_{ru}^T L_{rw}) \geq \lambda_{\text{min}}((\delta A + \rho L_{rw})^T (\delta A + \rho L_{rw}))$. Consequently, combining \cite{22}, we have

$$\frac{(\lambda_{\text{min}}(S(\delta A + \rho L_{rw})S^{-1}))^2}{\gamma^2_{\max}} \leq \lambda_{\text{min}}(A + \mu L_{ru}^T L_{rw})$$

This ends the proof.