Proper elements for resonant planet-crossing asteroids

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Abstract
Proper elements are quasi-integrals of motion of a dynamical system, meaning that they can be considered constant over a certain timespan, and they permit to describe the long-term evolution of the system with a few parameters. Near-Earth objects (NEOs) generally have a large eccentricity, and therefore they can cross the orbits of the planets. Moreover, some of them are known to be currently in a mean-motion resonance with a planet. Thus, the methods previously used for the computation of main-belt asteroid proper elements are not appropriate for such objects. In this paper, we introduce a technique for the computation of proper elements of planet-crossing asteroids that are in a mean-motion resonance with a planet. First, we numerically average the Hamiltonian over the fast angles while keeping all the resonant terms, and we describe how to continue a solution beyond orbit-crossing singularities. Proper elements are then extracted by a frequency analysis of the averaged orbit-crossing solutions. We give proper elements of some known resonant NEOs and provide comparisons with non-resonant models. These examples show that it is necessary to take into account the effect of the resonance for the computation of accurate proper elements.

Keywords Near-Earth objects (NEOs) · Mean-motion resonances · Proper elements

1 Introduction
It is well known that the gravitational $N$-body problem is non-integrable for $N \geq 3$, in the sense that there are not enough integrals of motion to find a complete set of action-angle coordinates. In the context of our Solar System, the motion of an asteroid can be treated
as a perturbation of an integrable problem, i.e., the two-body problem Sun–asteroid, and this permits to compute the so-called *proper elements*. Proper elements are quasi-integrals of motion, meaning that they are nearly constant in time. Another equivalent definition is that proper elements are actual integrals of motion of an appropriately simplified model. These quantities represent the average behavior of the orbit; thus, they permit to describe the long-term evolution of an object using only few numerical parameters.

Proper elements have been computed first for main-belt asteroids, and different techniques have been used to this purpose (see Knežević et al. 2002; Knežević 2016 for a historical review). Hirayama (1918, 1922) used the Lagrange linear theory of secular perturbations and determined proper elements of main-belt asteroids for the first time. He then used these values to show that some asteroids are grouped in the space of proper orbital elements, introducing the concept of asteroid families. Brouwer (1951) was able to get more accurate proper elements by combining the linear theory of secular perturbations and an improved theory of planetary motion. Modern analytical methods (see, e.g., Milani and Knežević 1990, 1992, 1994) are based on Hamiltonian perturbation theory. The perturbation is expanded in Fourier series of cosines of combinations of the angular variables, with amplitudes expressed as polynomials in eccentricity and sine of inclination. However, this theory is efficient only when the eccentricity and the inclination are both small. A secular analytical theory for asteroids with high inclination and eccentricity has been developed by Kozai (1962), and proper elements computed with this technique have been provided by Kozai (1979). In order to overcome the limitation of small eccentricity and inclination assumption, one can also use semi-analytical methods. Williams (1969) introduced a method that avoids the expansion in eccentricity and inclination and provided a list of proper elements computed with this technique in Williams (1979, 1989). Later, Lemaître and Morbidelli (1994) extended this theory by using the Hamiltonian formalism. Finally, with the large performance improvements in electronic processors seen in the last 30 years, a completely numerical (or *synthetic*) theory has been developed by Knežević and Milani (2000, 2003). This theory is based on a purely numerical integration, digital filtering for the removal of short periodic oscillations, and Fourier analysis for the determination of proper elements. On the other hand, the proper frequencies are determined by fitting the evolution of the angular variables with a linear model. Today, catalogs of main-belt asteroid proper elements are provided by the AstDyS1 service and by the Asteroid Families Portal2 (Novaković and Radović 2019). Data from these large catalogs have been successfully used to identify numerous asteroid families (see, e.g., Milani 2014; Nesvorný et al. 2015) and to determine their ages (see, e.g., Vokrouhlický 2006; Spoto 2015).

Differently from main-belt asteroids, the orbits of NEOs can cross the orbit of one or more planets, and for this reason the analytical and semi-analytical techniques mentioned above cannot be used in this context. Moreover, the Lyapunov time of the orbits of NEOs is generally short, of the order of a few thousand years or shorter. Thus, also the synthetic theory is not suitable for the computation of proper elements, because the integration of a single orbit is not a good representative of the bulk of possible behaviors after a few Lyapunov times. The difficulty of the orbit-crossing singularity has been addressed by Gronchi and Milani (1998). These authors considered the secular model defined by Kozai (1962), where short-term perturbations are removed by averaging over the fast angles, and they introduced a technique to propagate the dynamics across a planet-crossing configuration. Later, Gronchi and Milani (2001) used this method to compute proper elements of NEOs, that are currently

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1 https://newton.spacedys.com/astdys/index.php?pc=0.

2 http://asteroids.matf.bg.ac.rs/fam/index.php.
provided by NEODyS. Despite their short time validity, NEOs proper elements can be used to search for clusterings, and permit to identify a NEOs family that formed very recently. A first search in the NEODyS catalog performed by Schunová (2012) did not produce any positive identification. However, about 27,000 NEOs are known today, which is almost three times more than the number of NEOs known back in 2012. Additionally, once the Vera Rubin Observatory will start its operational lifetime, the number of known NEOs is expected to grow by a factor between 10 and 100 (Jones et al. 2016), increasing the chances for the first identification of an asteroid family among NEOs. Proper elements could also be used to identify secular resonances in the NEO region (Michel and Froeschlé 1997), and provide a better understanding of the dynamical pathways from the main belt to the NEO region.

NEOs proper elements defined by Gronchi and Milani (2001) are computed assuming that the asteroid is not in a mean-motion resonance with a planet. However, mean-motion resonances with Jupiter play a fundamental role in delivering asteroids from the main belt to the NEO region (see, e.g., Bottke 2002; Granvik 2017), and many NEOs are known to be currently in resonance with a planet. In this dynamical setting, proper elements defined in Gronchi and Milani (2001) are not appropriate, and a different model must be used.

Here, we address the problem of the computation of proper elements for resonant NEOs. Starting from the theory of Gronchi and Milani (2001), we need to keep resonant harmonics in the Hamiltonian by semi-averaging techniques, and the method must remain valid for arbitrary eccentricities and inclinations. Semi-averaged theories have been widely used in the past for describing the long-term orbital dynamics of resonant small bodies (see, e.g., Wisdom 1985; Moons and Morbidelli 1995; Milani and Baccili 1998; Saillenfest 2016, 2017; Sidorenko 2006, 2018, 2020). Our aim is to combine such a method with the model of Gronchi and Milani (2001) to compute proper elements of resonant NEOs. As the resonant harmonics add one degree of freedom to the system, we must find a way to reduce the problem to an integrable one, upon suitable transformations that are valid if secular chaos is not too strong (see, e.g., Sidorenko 2014). For this purpose, the traditional adiabatic approximation (see, e.g., Neishtadt 1987; Henrard 1993) may not be suitable for NEOs because of the large orbital perturbations that they undergo. In such a case, the method of Knežević and Milani (2000) for synthetic proper elements can be applied to the semi-averaged system, and proper frequencies can be computed by using a frequency analysis (Laskar 1988, 1990, 2005).

The paper is structured as follows. In Sect. 2, we introduce the semi-secular resonant model and describe how to overcome the problem of the crossing singularity. In Sect. 3, we briefly describe the adiabatic invariant theory, that is useful to better understand the secular evolution, but is shown to be quantitatively inaccurate in the NEO region. Then, we describe the use of frequency analysis for the determination of proper frequencies and proper elements of NEOs. In Sect. 4, we provide some examples of known resonant NEOs, and we show the differences in the results with respect to the non-resonant model by Gronchi and Milani (2001). Finally, in Sect. 5, we summarize the results.

2 Averaging on resonant planet-crossing orbits

2.1 The semi-secular evolution

Let us assume that the planets from Mercury to Neptune are placed on circular coplanar orbits centered at the Sun. The same approximation has been used before by Gronchi and

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3 https://newton.spacedys.com/neodys/.
Milani (2001) for the computation of proper elements of NEOs in the case of no mean-motion resonances, and Gronchi and Michel (2001) verified that it is accurate enough for the description of the secular motion of NEOs, provided that no close approaches with the planets occur. We denote with $k = \sqrt{Gm_0}$ the Gauss constant and set $\mu_j = m_j/m_0$, where $G$ is the universal gravitational constant, $m_0$ is the mass of the Sun, and $m_j$, $j = 1, \ldots, 8$ are the masses of the planets. We introduce the Delaunay elements ($L, G, Z, \ell, g, z$) of the asteroid as

$$
L = k\sqrt{a}, \quad \ell = M, \quad G = k\sqrt{a(1 - e^2)}, \quad g = \omega, \quad Z = k\sqrt{a(1 - e^2)}\cos I, \quad z = \Omega,
$$

(2.1)

where $a$ is the semimajor axis of the orbit of the asteroid, $e$ is the eccentricity, $I$ is the inclination, $\Omega$ is the longitude of the ascending node, $\omega$ is the argument of the pericenter, and $M$ is the mean anomaly. In these coordinates, the Hamiltonian of the restricted problem is given by

$$
\mathcal{H} = \mathcal{H}_0 + \epsilon \mathcal{H}_1, \quad \epsilon = \mu_5,
$$

(2.2)

where $\mathcal{H}_0$ is the unperturbed Keplerian Hamiltonian of the asteroid and $\mathcal{H}_1$ is the perturbation, i.e.,

$$
\mathcal{H}_0 = -\frac{k^4}{2L^2},
$$

$$
\mathcal{H}_1 = -k^2 \sum_{j=1}^{8} \frac{\mu_j}{\mu_5} \left( \frac{1}{|r - r_j|} - \frac{r \cdot r_j}{|r_j|^3} \right).
$$

(2.3)

In Eq. (2.3), $r$ and $r_j$, $j = 1, \ldots, 8$ are the heliocentric positions of the asteroid and the planets, respectively. Note that $\mathcal{H}$ directly depends on the time $t$, and because planets are on circular orbits we have $\ell_j = \pi_j t + \ell_j(0)$, where $\pi_j, \pi_j$ are the mean anomaly and the mean motion of the $j$th planet, respectively. To remove the direct time dependence, we overextend the phase space by introducing a variable $L_j$ conjugated to $\ell_j$, so that we get an autonomous Hamiltonian. The Hamiltonian in the overextended phase space is

$$
\tilde{\mathcal{H}} = \mathcal{H}_0 + \sum_{j=1}^{8} \pi_j L_j + \epsilon \mathcal{H}_1,
$$

(2.4)

where $\mathcal{H}_1 = \mathcal{H}_1(L, L_1, \ldots, L_8, G, Z, \ell, \ell_1, \ldots, \ell_8, g, z)$. Let us assume that the asteroid is in a mean-motion resonance with the $p$th planet where the resonant angle is given by

$$
\sigma = h\lambda - h_p\lambda_p - (h - h_p)\sigma_p.
$$

(2.5)

In Eq. (2.5), the angles $\lambda = \ell + \omega + \Omega$, $\lambda_p = \ell_p + \omega_p + \Omega_p$ are the mean longitudes of the asteroid and the planet, $\sigma = \omega + \Omega$ is the longitude of the perihelion of the asteroid, and $h, h_p$ are co-prime integers. The integer number $|h - h_p|$ is usually called the resonance order. The Delaunay elements are first transformed into resonant semi-secular coordinates

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4. Note that $\omega_p$ and $\Omega_p$ are ill-defined because the planets move on circular and zero-inclination orbits; hence, we identify $\lambda_p$ with $\ell_p$.

5. Equation (2.5) for $\sigma$ is used for the purpose of definition, but since all resonant harmonics are kept in the Hamiltonian, this method describes all types of $h_p,h$ resonances at once (i.e., with a different combination of $\Omega, \Omega_p, \sigma, \sigma_p$ fulfilling the D’Alembert rules).
(see, e.g., Saillenfest 2016) by using the transformation
\[
\begin{pmatrix}
\sigma \\
\gamma \\
u \\
v
\end{pmatrix} =
\begin{pmatrix}
h & -h_p & h_p & h_p \\
c & -c_p & c_p & c_p \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\ell \\
\ell_p \ \ \ \ \ \\
\Sigma \\
\Gamma
\end{pmatrix} =
\begin{pmatrix}
-h_p & -c & 0 & 0 \\
h & h & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
L_p \\
G
\end{pmatrix},
\]
where \(c, c_p\) are integers such that \(ch_p - c_p h = 1\), that exist because \(\gcd(h, h_p) = 1\). Due to
the resonance assumption, the angles \(\ell_j, j = 1, \ldots, 8\) and \(\gamma\) evolve fast (frequency \(\propto \epsilon^0\)),
the critical angle \(\sigma\) evolves on a semi-secular timescale (frequency \(\propto \sqrt{\epsilon}\)), and \((u, v)\) evolve
on a secular timescale (frequency \(\propto \epsilon\)). Note that the quantities \(\Gamma, L_j, j = 1, \ldots, 8, j \neq p\)
are first integrals in the semi-secular dynamics that have been introduced artificially by overextending the phase space, and their value can be chosen arbitrarily. By choosing \(\Gamma = 0\), the actions of Eq. (2.6) become
\[
\Sigma = \frac{L}{h}, \quad U = G - \frac{h_p}{h} L, \quad V = Z - \frac{h_p}{h} L.
\tag{2.7}
\]
The semi-secular Hamiltonian is obtained by averaging over the fast angles (see, e.g., Milani and Baccili 1998), and it is given by
\[
\mathcal{K} = \mathcal{K}_0 + \epsilon (\mathcal{K}_\text{sec} + \mathcal{K}_\text{res}).
\tag{2.8}
\]
where
\[
\mathcal{K}_0 = -\frac{k^4}{2(h \Sigma)^2} - n_p h_p \Sigma,
\]
\[
\mathcal{K}_\text{sec} = \frac{k^2}{\mu_5} \sum_{j=1, j \neq p}^8 \frac{\mu_j}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{|r - r_j|} d\ell d\ell_j,
\tag{2.9}
\]
\[
\mathcal{K}_\text{res} = \frac{k^2}{\mu_5} \frac{\mu_p}{2\pi} \int_0^{2\pi} \frac{1}{|r - r_p|} - \frac{r \cdot r_p}{|r_p|^3} d\gamma.
\]
The term \(\mathcal{K}_\text{sec}\) contains the secular perturbations of all the planets not involved in the resonan-
ce, while \(\mathcal{K}_\text{res}\) contains all the resonant terms and the secular perturbation of the planet \(p\).
The resonant normal form \(\mathcal{K}\) does not depend on the angle \(v\) in the semi-secular coordinates,
because the problem is invariant with respect to rotations of the common orbital plane of
the planets (see Kozai 1985; Saillenfest 2016); hence, the action \(V\) is a constant of motion.
When the orbit of the asteroid does not cross the orbit of any planet, the Hamiltonian vector
field defined by \(\mathcal{K}\) can be computed by exchanging the derivative and the integral sign. In
case of orbit crossing, this cannot be done, because a singularity appears in the integrals of
the \(\mathcal{K}_\text{sec}\) term (see Gronchi and Milani 1998).

### 2.2 Crossing singularity and numerical integration

When the orbit of the asteroid and the orbit of a non-resonant planet intersect, the term \(\mathcal{K}_\text{sec}\)
in Eq. (2.9) has a first-order polar singularity, and solutions can be continued beyond the
crossing. A technique to extend solutions beyond the crossing singularity has been described
in Gronchi and Milani (1998) and Gronchi and Tardioli (2013). We summarize here the
fundamental steps, leaving all the mathematical details in Appendix A.

When the orbit of the asteroid and the orbit of the resonant planet intersect, the term \(\mathcal{K}_\text{res}\)
has a collision singularity, occurring for a unique angle \(\sigma = \sigma_p\). In this case, the integral
is divergent and solutions cannot be continued beyond (see, e.g., Marò and Gronchi 2018). This case never exactly occurs in practice, and we will not mention it further in this paper.

### 2.2.1 Extraction of the singularity

Let us suppose that there is only one non-resonant planet, and denote with $\mathbf{r}'$ its heliocentric position. Then,

\[
K_{\text{sec}} \propto \int_0^{2\pi} \int_0^{2\pi} \frac{1}{d} \, d\ell d\ell',
\]

where $d = |\mathbf{r} - \mathbf{r}'|$. Let $y \in \{\Sigma, U, V, \sigma, u, v\}$ be one of the coordinates, then the Hamiltonian vector field is determined by the derivatives

\[
\frac{\partial}{\partial y} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{d} \, d\ell d\ell'.
\]

When the orbit of the asteroid and the orbit of the planet do not intersect, it is possible to compute the derivative in Eq. (2.11) by exchanging the derivative and the integral sign. On the other hand, in the case of orbit crossing this cannot be done, because the double integral has a polar singularity. In a neighborhood of the crossing configuration, a function $\delta$ that approximates the distance $d$ is defined by using the minimum orbit intersection distance (MOID; Gronchi 2005) between the orbit of the planet and that of the asteroid. The integral in Eq. (2.11) is therefore decomposed as

\[
\frac{\partial}{\partial y} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{d} \, d\ell d\ell' = \frac{\partial}{\partial y} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\delta} \, d\ell d\ell' + \frac{\partial}{\partial y} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{d} - \frac{1}{\delta} \right) \, d\ell d\ell'.
\]

It turns out that the second term in the right-hand side of Eq. (2.12) can be computed numerically by exchanging the derivative and the integral signs. The first term in the right-hand side of Eq. (2.12) contains the principal part of the singularity of the term in the left-hand side, with the advantage that it can be computed with an analytical formula (see Appendix A). Moreover, the discontinuity is given only by the discontinuity in the derivatives of the MOID. This analytical formulation permits to compute the difference between the Hamiltonian vector field on the two sides of the orbit-crossing singularity (see also Eq. A.38), enabling us to continue a solution beyond these singular configurations.

Note that a double crossing can occur. In this case, we can define two functions $\delta_1, \delta_2$ approximating the distance $d$ in a neighborhood of the two crossing points, and we can extract the singularities using the decomposition

\[
\frac{1}{d} = \frac{1}{\delta_1} + \frac{1}{\delta_2} + \left( \frac{1}{d} - \frac{1}{\delta_1} - \frac{1}{\delta_2} \right).
\]

### 2.2.2 Numerical integration scheme

The numerical implementation of the continuation of solutions beyond an orbit crossing follows that of Gronchi and Milani (2001). We use an implicit Runge–Kutta–Gauss scheme (see, e.g., Hairer et al. 2002) for the integration of the equations of motions generated by the semi-secular Hamiltonian $K$. Jacobi iterations are used to solve the fixed-point equation needed to compute the coefficients of the Runge–Kutta–Gauss method, and the first guess is computed by using a polynomial extrapolation from the previous integration point.
The MOID with each planet is computed at every integration step, and these values are used to check whether a crossing with a non-resonant planet has occurred or not. If a planet crossing is detected, an iterative method is used to make the integration step arrive exactly at the crossing configuration. When this singular configuration is reached within the required precision, the integration needs to restart from this point for the next step.

In this case, the first guess for the Jacobi iteration computed with polynomial extrapolation would be wrong, since there is a jump in the vector field at the crossing curve. A good first guess is therefore computed by correcting the polynomial extrapolation using the analytical formula of the jump of Eq. (A.38). Figure 1 describes this integration scheme.

The semi-secular Hamiltonian Eq. (2.8) is a function of the semi-secular coordinates defined in Eq. (2.6); hence, osculating orbital elements should not be used as initial conditions. In order to compute the appropriate semi-secular coordinates for a given small body, we integrate its orbit in a full $N$-body problem for 1000 yr, including all the planets from Mercury to Neptune in the model. Short periodic oscillations are then removed through a digital filter (Carpino 1987). These steps are performed with the ORBIT9 integrator, included in the ORBFIT6 package. Filtered elements at the initial time are then used as initial conditions for the semi-secular Hamiltonian. For the ORBIT9 integrations performed in this paper, we used initial conditions for the planets at time 59000 MJD, taken from the JPL Horizons7 ephemeris system. Orbital elements of NEOs at time 59000 MJD were taken from the NEODyS catalog, and the nominal orbits were used as initial condition. To propagate the orbits, we used the Everhart integration method (Everhart 1985).

3 Proper elements of resonant NEOs

3.1 Adiabatic approximation

The variables of the Hamiltonian system defined by Eq. (2.8) evolve on two different timescales. The couple $(\Sigma, \sigma)$ evolves over a semi-secular timescale, while $(U, u)$ evolves over a secular timescale, and under suitable conditions this separation can be used to further

6 http://adams.dm.unipi.it/orbfit/.
7 https://ssd.jpl.nasa.gov/?horizons.
simplify the problem. Denote with $\nu_\sigma$ (that is $\propto \sqrt{\epsilon}$) and $\nu_u$ (that is $\propto \epsilon$) the frequencies associated with these two couples of variables, and assume that

$$\xi = \frac{\nu_u}{\nu_\sigma} \ll 1.$$  
(3.1)

If condition (3.1) holds, the adiabatic invariant theory (see, e.g., Henrard 1993) can be used to transform $(\Sigma, \sigma)$ into a new pair of action-angle coordinates $(J, \theta)$. In this manner, we reduce the problem to a system with one degree of freedom. The momentum $J$ is also called the adiabatic invariant and, although it is not exactly conserved in the semi-secular system, its variations can be discarded for a sufficiently small $\xi$. This approach has been used to understand the qualitative dynamics of objects in the strongest mean-motion resonances with Jupiter (see, e.g., Wisdom 1985; Henrard and Lemaitre 1987; Sidorenko 2006), and to study co-orbital motions (see, e.g., Sidorenko 2014, 2020). The adiabatic invariant is defined as the area enclosed (or stretched) by an orbit $(\Sigma(t), \sigma(t))$ (also called guiding trajectory) computed keeping the variables $(U, u)$ fixed, and in the libration case it is given by

$$2\pi J = \frac{1}{2} \int (\Sigma d\sigma - \sigma d\Sigma) = \frac{1}{2} \int_0^{T_\sigma} (\dot{\sigma} \Sigma - \dot{\Sigma} \sigma) dt,$$  
(3.2)

where $T_\sigma$ is the period of the guiding trajectory. If $(\Sigma_0, \sigma_0)$ is any point of the orbit, then the secular Hamiltonian describing the evolution of the couple $(U, u)$ is given by

$$\mathcal{F}(J, U, V, \theta, u) = \mathcal{K}((\Sigma_0, U, V, \sigma_0, u) + \mathcal{O}(\epsilon^{3/2}).$$  
(3.3)

If we neglect the remainder, the secular Hamiltonian of Eq. (3.3) has one degree of freedom, and the solutions follow its level curves in the $(U, u)$-plane. This secular model has been successfully used by Saillenfest (2016, 2017) and Saillenfest and Lari (2017) to study the long-term evolution of distant resonant trans-Neptunian objects (TNOs), for which the ratio $\xi$ is of the order of $10^{-4}$.

However, NEOs evolve on a much shorter timescale than TNOs, and the condition in Eq. (3.1) may not be well satisfied in general. To show that the secular model of Eq. (3.3) is not suitable to get accurate proper elements in the context of NEOs, we consider the example of asteroid (887) Alinda, that is currently in a 3:1 mean-motion resonance with Jupiter. We numerically integrated its dynamics using both the semi-secular Hamiltonian of Eq. (2.8), and the secular one of Eq. (3.3) assuming the adiabatic invariance hypothesis. Figure 2 shows the semi-secular evolution of the semimajor axis $a$ and of the critical argument $\sigma$ for 20 ky, and the comparison between the evolution computed with the semi-secular and the secular models. (887) Alinda stays in the resonance for the whole integration timespan, and the period of the guiding trajectory is approximately $T_\sigma \approx 360$ yr.

As expected, the semi-secular evolution has small oscillations with a period equal to $T_\sigma$, while in the secular evolution only the main long-term oscillation is kept. The secular period here is approximately $T_u \approx 14200$ yr, resulting in a ratio of $\xi \approx 0.025$, almost three orders of magnitude larger than in the case of distant TNOs. This already suggests that the adiabatic approximation might not be appropriate for an accurate computation of proper elements. More importantly, from Fig. 2 we can notice that, although the secular model is able to reproduce correctly the amplitude of oscillation of $U$ and explain the dynamics of Alinda in a qualitative way, the frequency is not the same as the one obtained in the semi-secular evolution. This shows that the use of the adiabatic approximation for the computation of the secular Hamiltonian could lead to a poor determination of the proper elements, in particular of the proper frequencies for the case of (887) Alinda. Thus, we must find another way to extract the proper elements from the resonant normal form $\mathcal{K}$. 

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2.45
2.5
2.55
0 60 120 180 240 300 360
0 5 10 15 20
-21.54 -21.56 -21.58 -21.6
0 5 10 15 20
0 60 120 180 240 300 360
0 5 10 15 20

Fig. 2 Comparison of the semi-secular dynamics (blue curve) and secular dynamics assuming the adiabatic invariance (red curve). The asteroid taken as example here is (887) Alinda

3.2 Frequency analysis

Proper frequencies are an intrinsic property of the dynamical system in its integrable approximation. In other words, even though the semi-secular system has two degrees of freedom, it already contains the proper frequencies of the asteroid, and one must just find a way to properly extract them. In the absence of a well-defined adiabatic invariant, the fundamental frequencies of the system can be computed numerically by performing a frequency analysis on the semi-secular time series.

To this end, the dynamics of a resonant NEO is propagated forward in time for 200 kyr using the semi-secular model of Sect. 2, and the time series of the resonant elements ($\Sigma$, $U$, $V$, $\sigma$, $u$, $v$) are converted to the time series of the Keplerian elements $e$, $\cos I$, $\omega$, $\Omega$. Then, we apply the frequency analysis method by J. Laskar (see Laskar 1988, 1990, 2005) to the functions

$$\eta = e \exp(i\omega), \quad \zeta = \sin \frac{I}{2} \exp(i\Omega),$$

and determine their frequency decomposition. We perform the frequency analysis using the TRIP$^8$ software developed by Gastineau and Laskar (2011).

The functions $\eta$ and $\zeta$ are expressed as quasi-periodic series, in which the frequency of each term is an integer combination of the proper frequencies $\nu_\sigma$, $\nu_u$, and $\nu_v$ (see, e.g., Laskar et al. 1992). Therefore, we need to identify the proper frequencies from their integer combinations. Since the semi-secular Hamiltonian is invariant by rotation (and thus has only two degrees of freedom; see Sect. 2), the decomposition of $\eta$ only features $\nu_\sigma$ and $\nu_u$. The frequency $\nu_u$ is usually that of the largest-amplitude term of $\eta$ (as it is the case for Alinda, see Fig. 2). However, in order to avoid any ambiguity, the frequency $\nu_\sigma$ can be determined by a preliminary frequency analysis of the variables ($\Sigma$, $\sigma$). Once we have $\nu_\sigma$ and $\nu_u$, the identification of $\nu_v$ from the analysis of $\zeta$ is straightforward. The proper frequencies $g - s$ of the argument of the pericenter $\omega$, and $s$ of the longitude of the node $\Omega$, correspond to $\nu_u$ and $\nu_v$, respectively. Note that if the quasi-periodic decomposition of $\eta$ contains a constant term, then $\omega$ librates, otherwise it circulates. As an example, Table 1 gives the terms of the quasi-periodic decomposition of $\zeta$ for (887) Alinda.

In order to complete the set of proper elements, we now need to compute the central value and the amplitude of secular oscillations of eccentricity and inclination. To this purpose, we remove all the semi-fast terms from the quasi-periodic decomposition (i.e., those featuring

$^8$ https://www.imcce.fr/Equipes/ASD/trip/trip.php.
Table 1  Frequencies, amplitudes, and phases of the quasi-periodic decomposition of the function $\zeta$ for asteroid (887) Alinda

| Frequency identification | Frequency (arcsec yr$^{-1}$) | Amplitude          | Phase (deg) |
|--------------------------|------------------------------|--------------------|-------------|
| $\nu_v$                  | $-80.92578$                  | $6.43469 \times 10^{-2}$ | 119.027    |
| $2\nu_u + \nu_v$         | $103.00656$                  | $1.84703 \times 10^{-2}$ | 82.034     |
| $\nu_\sigma + 2\nu_u + \nu_v$ | $3723.24145$               | $6.61366 \times 10^{-4}$ | 25.051     |
| $-\nu_\sigma + \nu_v$   | $-3701.11622$               | $6.20626 \times 10^{-4}$ | 173.959    |
| $\nu_\sigma + \nu_v$    | $3539.21386$                | $6.18166 \times 10^{-4}$ | 65.097     |
| $-\nu_\sigma + 2\nu_u + \nu_v$ | $-3517.05867$              | $6.00843 \times 10^{-4}$ | 135.966    |
| $2\nu_\sigma + \nu_v$   | $7159.40976$                | $2.97608 \times 10^{-4}$ | 15.804     |
| $-2\nu_\sigma + \nu_v$  | $-7321.18962$               | $2.59903 \times 10^{-4}$ | 50.200     |
| $2\nu_\sigma + 2\nu_u + \nu_v$ | $7343.45527$              | $1.88732 \times 10^{-4}$ | 149.424    |
| $-2\nu_\sigma + 2\nu_u + \nu_v$ | $-7137.08519$            | $1.59706 \times 10^{-4}$ | 176.425    |

The table shows only the terms with an amplitude larger than $10^{-4}$

the frequency $\nu_\sigma$). Then, we obtain the bounds of the secular variation of eccentricity and inclination by summing up or subtracting the amplitudes of the terms in the quasi-periodic series. The last term of the series (i.e., the term with smallest amplitude) sets the accuracy of these estimates. It depends on the capability of the frequency analysis algorithm to express the signal as a quasi-periodic series: Proper elements are very precise for nearly integrable semi-secular trajectories, but poorly determined if secular chaos is strong. As explained in Sect. 4, secular chaos is an intrinsic limitation for the computation of proper elements.

4 Examples

Table 2 lists the NEOs that we took into account in this work. First, we performed pure $N$-body simulations to check that both the critical angle $\sigma$ and the semimajor axis $a$ are currently oscillating for the indicated resonance. Then, we computed proper elements using the method described in Sect. 3.2. We classify the examples according to the following cases:

A1  No separatrix crossings, unnoticeable effect of the resonance;
A2  No separatrix crossings, strong effect of the resonance;
B1  Separatrix crossings, regular secular motion;
B2  Separatrix crossings, chaotic secular motion.

Separatrix crossings occur when the asteroid is pushed outside of the resonance. The resonant angle $\sigma$ therefore switches from libration to circulation. Separatrix crossings may greatly alter the secular evolution and produce long-term chaos, even though the dynamics may be perfectly adiabatic between each crossing event. This phenomenon is further described below.

4.1 Case A1

We first discuss the example of (159560) 2001 TO103, an asteroid in 4:7 mean-motion resonance with Mars that does not cross any planet. We propagated the semi-secular dynamics
| Des. | $h_p: h$ | Res. Cross | $e_{\text{min}}, e_{\text{max}}$ | $(I_{\text{min}}, I_{\text{max}})$ | $g - s$ | $s$ | $l_f$ | $(\omega_{\text{min}}, \omega_{\text{max}})$ | Case |
|------|---------|------------|-------------------------------|--------------------------------|---------|-----|------|---------------------------------|------|
| (887)| 3:1     | J M        | (0.5524, 0.5641) (5.29, 9.49) | 91.97 | 80.93 | -   | -    | A2                            |
| (2608)| 3:1     | J M        | (0.5282, 0.5769) (10.94, 19.18) | 216.30 | 150.11 | -   | -    | A2                            |
| (8201)| 3:1     | J E/M      | (0.7140, 0.7296) (5.39, 13.47) | 81.05 | 72.42 | -   | -    | A2                            |
| (19356)| 3:1      | J M        | (0.5637, 0.5656) (1.94, 3.74) | 45.18 | 56.06 | -   | -    | A2                            |
| (153311)| 3:1      | J E/M     | (0.3444, 0.7098) (17.64, 44.37) | 17.75 | 30.80 | -   | -    | A2                            |
| (6178)| 5:2     | J M        | (0.5747, 0.5812) (3.67, 7.10) | 70.67 | 71.27 | -   | -    | A2                            |
| (26760)| 5:2     | J M        | (0.5589, 0.5830) (6.89, 13.42) | 135.14 | 115.11 | -   | -    | A2                            |
| (14827)| 5:2     | J E/M      | (0.6712, 0.6728) (2.32, 4.33) | 385.83 | 315.24 | -   | -    | A2                            |
| (152667)| 5:2     | J E/M     | (0.6986, 0.7051) (3.70, 8.51) | 231.96 | 185.15 | -   | -    | A2                            |
| (481482)| 5:2     | J V/E/M   | (0.7550, 0.8135) (12.84, 30.09) | 295.95 | 259.61 | -   | -    | A2                            |
| (34613)| 4:1     | J M        | (0.3783, 0.4047) (6.04, 7.95) | 86.62 | 46.97 | -   | -    | A2                            |
| (361518)| 4:1     | J E/M     | (0.5972, 0.6021) (4.15, 6.87) | 37.18 | 45.67 | -   | -    | A2                            |
| (369983)| 4:1     | J M        | (0.3854, 0.3856) (1.02, 1.33) | 94.55 | 51.68 | -   | -    | A2                            |
| (407653)| 4:1     | J M        | (0.4846, 0.5036) (8.26, 12.18) | 69.67 | 51.10 | -   | -    | A2                            |
| (408752)| 4:1     | J V/E/M   | (0.7768, 0.7832) (3.93, 9.90) | 91.14 | 80.05 | -   | -    | A2                            |
| (303174)| 7:2     | J M        | (0.3212, 0.4503) (21.28, 28.34) | 57.61 | 43.20 | -   | -    | A2                            |
| (329395)| 7:2     | J E/M     | (0.3900, 0.7180) (25.82, 47.11) | 46.84 | 45.14 | -   | -    | A2                            |
| (452639)| 7:2     | J V/E/M   | (0.8698, 0.8755) (3.46, 12.16) | 157.70 | 135.86 | -   | -    | A2                            |
| (488494)| 7:2     | J M        | (0.4474, 0.4669) (7.87, 11.31) | 92.21 | 62.10 | -   | -    | A2                            |
| (501878)| 7:2     | J V/E/M   | (0.7543, 0.7859) (7.88, 21.08) | 124.45 | 105.99 | -   | -    | A2                            |
| (4503)| 8:3     | J M        | (0.5210, 0.5238) (2.51, 4.41) | 218.75 | 159.75 | -   | -    | A2                            |
| (9172)| 8:3     | J M        | (0.5290, 0.5558) (7.67, 13.87) | 144.70 | 117.00 | -   | -    | A2                            |
| (152575)| 8:3     | J M        | (0.5258, 0.5487) (7.04, 12.72) | 205.91 | 151.58 | -   | -    | A2                            |
| (363076)| 8:3     | J M        | (0.4884, 0.5174) (8.14, 13.89) | 206.32 | 145.97 | -   | -    | A2                            |
| (405562)| 8:3     | J M        | (0.5393, 0.5430) (2.96, 5.25) | 133.69 | 111.53 | -   | -    | A2                            |
| Des.    | $h_p/h$ | Res. pla. | Cross pla. | $(e_{\min}, e_{\max})$ | $(I_{\min}, I_{\max})$ | $g - s$ | $s$ | $lf$ | $(\omega_{\min}, \omega_{\max})$ | Case |
|---------|---------|-----------|-------------|------------------------|------------------------|---------|-----|-----|------------------------|------|
| (416804) | 8:3     | J         | M           | (0.5670, 0.5793)       | (4.99, 9.66)          | 208.73  | –   | 161.99 | –                       | –    |
| (5370)   | 2:1     | J         | M/J         | (0.6280, 0.6805)       | (10.49, 22.20)        | 51.15   | –   | 38.04 | –                       | –    |
| (26166)  | 2:1     | J         | M           | (0.5656, 0.6594)       | (12.22, 27.02)        | –       | –   | 19.57 | 30.83                  | (55.85, 124.15) | A2   |
| (523592) | 2:1     | J         | –           | (0.3891, 0.5859)       | (18.11, 33.25)        | 73.95   | –   | 75.14 | –                       | –    |
| 1999SE10 | 2:1     | J         | M           | (0.6029, 0.6098)       | (3.11, 7.31)          | 32.01   | –   | 30.74 | –                       | –    |
| 2005YC   | 2:1     | J         | M           | (0.4848, 0.6379)       | (18.84, 33.57)        | –       | –   | 30.74 | 52.92                  | (53.12, 126.88) | A2   |
| 2008KD6  | 2:3     | E         | E/M         | (0.3799, 0.4710)       | (21.98, 27.83)        | 24.74   | –   | 17.65 | –                       | –    |
| 2005HC6  | 2:5     | E         | E/M         | (0.5121, 0.5139)       | (3.08, 4.18)          | 72.10   | –   | 48.92 | –                       | –    |
| 2008EG9  | 3:5     | E         | E/M         | (0.3855, 0.3861)       | (2.45, 2.80)          | 77.61   | –   | 53.16 | –                       | –    |
| (10302)  | 3:7     | V         | –           | (0.1369, 0.1403)       | (4.01, 4.38)          | 50.53   | –   | 28.11 | –                       | –    |
| 2014HU46 | 1:3     | V         | E/M         | (0.4188, 0.4191)       | (1.52, 1.87)          | 78.72   | –   | 70.04 | –                       | –    |
| 2012DF4  | 1:3     | V         | V/E/M       | (0.5382, 0.5395)       | (4.61, 5.26)          | 48.01   | –   | 38.41 | –                       | –    |
| 2015XU351| 2:7     | V         | V/E/M       | (0.5761, 0.6084)       | (11.39, 17.86)        | 51.10   | –   | 39.67 | –                       | –    |
| 2016AH9  | 2:7     | V         | E/M         | (0.5552, 0.5559)       | (2.97, 3.51)          | 62.66   | –   | 49.85 | –                       | –    |
| (5381)   | 2:3     | V         | V/E         | (0.1268, 0.6600)       | (33.42, 50.79)        | –       | –   | 9.25  | 14.40                  | (37.28, 142.73) | A2   |
| (138911) | 6:5     | M         | –           | (0.0816, 0.0818)       | (1.64, 1.68)          | 33.07   | –   | 19.30 | –                       | –    |
| (159560) | 4:7     | M         | –           | (0.2650, 0.4380)       | (25.50, 32.70)        | 45.41   | –   | 36.32 | –                       | –    |
| (163412) | 5:7     | M         | M           | (0.3166, 0.5562)       | (27.70, 39.13)        | 35.02   | –   | 30.00 | –                       | –    |
| (208565) | 4:3     | M         | V/E/M       | (0.4316, 0.8229)       | (29.01, 56.56)        | 17.98   | –   | 18.25 | –                       | –    |
| (211871) | 3:4     | M         | E/M         | (0.5107, 0.5185)       | (5.64, 8.18)          | 65.01   | –   | 46.95 | –                       | –    |
| (309728) | 5:7     | M         | M           | (0.3406, 0.4121)       | (18.31, 23.07)        | 46.98   | –   | 31.87 | –                       | –    |
| (10636)  | 8:11    | M         | M           | (0.4676, 0.5148)       | (13.24, 19.25)        | 58.49   | –   | 40.87 | –                       | –    |

Planets in the third and fourth column are indicated only with their initial letter (V = Venus, E = Earth, M = Mars, J = Jupiter). The inclinations $I_{\min}, I_{\max}$ and the arguments $\omega_{\min}, \omega_{\max}$ are in degrees, while the frequencies $g - s, s, lf$ are in arcsec yr$^{-1}$, where $lf$ is the oscillation frequency of $\omega$ in the case of Kozai resonance.
Proper elements for resonant planet-crossing asteroids

2.2125
2.213
2.2135
2.214
0
90
180
270
360
0 1 02 03 04 05 06 07 0
-2
-1
0
10 -3
Fig. 3 Long-term dynamics of asteroid (159560) 2001TO103. The blue curve is the evolution obtained by the semi-secular model. In the four rightmost panels, all terms with period smaller than 10 kyr have been digitally filtered, and the red curve is the evolution obtained with the secular non-resonant model of Gronchi and Milani (2001). The blue and red curves almost overlap and can hardly be distinguished one from the other

for 200 ky, and we also monitored the evolution of the adiabatic invariant. The value $2\pi J$ is computed every 300 yr by using Eq. (3.2), where the variables $(\Sigma, U, V, \sigma, u, v)$ identifying the guiding trajectory are taken from the integration of the semi-secular dynamics. The results are shown in Fig. 3. The asteroid stays in the resonance for the whole integration timespan, and the adiabatic invariant $2\pi J$ remains fairly constant (see Fig. 3, bottom left panel), experiencing a maximum relative variation of only 10% with respect to the initial value. By looking at the orbital evolution, we see that the period of the guiding trajectories lies somewhere between $T_{\sigma} = 1500$ and 2500 yr, while the period of circulation of $\omega$ is $T_{\omega} \approx 30000$ yr, corresponding to a ratio between $\xi \approx 0.05$ and $\xi \approx 0.083$.

The proper elements reported by NEODyS and obtained without taking the resonance into account are $(e_{\text{min}}, e_{\text{max}}) = (0.2649, 0.4385), (I_{\text{min}}, I_{\text{max}}) = (25.522^\circ, 32.749^\circ)$, and the proper frequencies are $g - s = 45.419$ arcsec yr$^{-1}$, $s = -36.367$ arcsec yr$^{-1}$. Figure 3 shows the comparisons of the evolution of $e, I, \omega, \text{and } \Omega$ obtained when taking the resonance into account in the model or not. The two dynamics are essentially the same, meaning that the effects of the resonance are not noticeable on the long-term dynamics. Indeed, the resonant proper elements that we obtained in Table 2 are very close to those reported by NEODyS; this confirms that the mean-motion resonance does not have a strong influence on this object. This also shows that our method, while taking the resonance into account, is as precise as that of Gronchi and Milani (2001) for proper elements computation.

4.2 Case A2

We consider here the example of (138911) 2001 AE2, an asteroid in 6:5 mean-motion resonance with Mars that does not cross any planet, and stays in the resonance for the whole integration timespan of 200 ky (see Fig. 4). From the numerical integrations, we found that the period of the guiding trajectory is about $T_{\sigma} \approx 1130$ yr, while the period of circulation of $\omega$ is $T_{\omega} \approx 50000$ yr, that results in a ratio of $\xi \approx 0.0226$. The adiabatic invariant $2\pi J$ is close to zero, meaning that the object is deep inside the resonance, and its value is well
1.3495
1.34955
1.3496
1.34965

0 90 180 270 360

Fig. 4 Same as Fig. 3, for asteroid (138911) 2001 AE2

conserved (see Fig. 4, bottom left panel). The maximum relative variation with respect to the initial value is smaller than 1%.

The proper elements reported by NEODyS are \((e_{\text{min}}, e_{\text{max}}) = (0.0813, 0.0819), (I_{\text{min}}, I_{\text{max}}) = (1.616^\circ, 1.706^\circ)\), and the proper frequencies are \(g - s = 45.227\ \text{arcsec yr}^{-1}, s = -23.913\ \text{arcsec yr}^{-1}\). Figure 4 shows the comparisons between the evolution of \(e, I, \omega, \Omega\) obtained with the non-resonant secular model by Gronchi and Milani (2001), and the one obtained with the resonant semi-secular model. It is evident that the resonance significantly affects the secular evolution. The oscillation amplitudes of \(e\) and \(I\) are smaller when the resonance is taken into account and, more importantly, the proper frequencies \(g - s\) and \(s\) are significantly different, as we can also see from the evolutions of \(\omega\) and \(\Omega\) shown in Fig. 4. Note also that only the evolution between 10 kyr and 70 kyr is shown in these panels, because of the filtering of periodic oscillations with period smaller than 10 kyr.

Another significant example is (5381) Sekhmet, a NEO in 2:3 mean motion resonance with Venus, that crosses the orbits of Venus itself and that of the Earth. This asteroid stays in the resonance for the whole integration timespan, while the center of libration of \(\sigma\) and the oscillation amplitude of the semimajor axis \(a\) oscillate (see Fig. 5). The period of the guiding trajectory is about \(T_\sigma \approx 470\ \text{yr}\), while \(\omega\) librates with a period of \(T_\omega \approx 93000\ \text{yr}\), that results in a ratio of \(\xi \approx 0.005\). The evolution of the adiabatic invariant \(2\pi J\) is shown in Fig. 5, bottom left panel, and the maximum relative variation with respect to the initial value resulted to be about 5%.

The proper elements reported in NEODyS are \((e_{\text{min}}, e_{\text{max}}) = (0.0338, 0.6849), (I_{\text{min}}, I_{\text{max}}) = (30.619^\circ, 51.142^\circ)\), and the proper frequencies are \(g - s = 2.821\ \text{arcsec yr}^{-1}, s = -7.914\ \text{arcsec yr}^{-1}\). Figure 5 shows the comparisons of the evolution between the non-resonant secular model and the filtered resonant semi-secular model. The oscillations of eccentricity and inclination are slightly smaller when the resonance is taken into account than when it is not. The period of the longitude of the node \(\Omega\) is longer in the non-resonant model, and, more importantly, the argument of the pericenter \(\omega\) librates if we take into account the effects of the resonance, while it circulates in the non-resonant model. Other objects from Table 2 showing this behavior are (26166) and 2005 YC.

These two examples already show that taking into account the effect of the resonance is fundamental to correctly compute the secular evolution, and consequently for the computation of appropriate proper elements. For the cases of Table 2, we saw that this is especially true
for resonances with Jupiter, but resonances with Venus, the Earth, and Mars can also be important enough to significantly change the secular evolution.

4.3 Case B1

We discuss the case of (10636) 1998 QK56, a NEO in 8:11 mean-motion resonance with Mars that crosses the orbit of Mars itself. Figure 6 shows the evolution of $a$ and $\sigma$ for 70 kyr, and we can clearly see the switching between libration and circulation happening periodically. The period of the librating guiding trajectory is $T_\sigma \approx 800$ yr, while the period of the circulating one is $T_\sigma \approx 1200$ yr. On the other hand, the circulation period of $\omega$ is about $T_\omega \approx 22000$ yr, that results in a ratio $\xi \approx 0.036$ for the libration case and $\xi \approx 0.054$ for the circulation case.

While the couple $(U,u)$ evolves, the guiding trajectory evolves accordingly, maintaining roughly constant the value of the area in the $(\Sigma, \sigma)$-plane. Figure 7 shows the level curves of the semi-secular Hamiltonian and the guiding trajectories, at different times. During the evolution, the area enclosed by the separatrix curve becomes smaller and smaller, and this has the effect of pushing the guiding trajectory toward the separatrix. At some point of the evolution (e.g., between $t = 500$ yr and $t = 1000$ yr for the case shown in Fig. 7), the area enclosed by the separatrix becomes smaller than the adiabatic invariant $2\pi J$, and a libration motion is not possible anymore. Thus, the guiding trajectory is forced to cross the separatrix, and the definition (and therefore the value) of the adiabatic invariant changes. The bottom left panel of Fig. 6 shows the evolution of $2\pi J$, where we can notice the jumps at times corresponding to the separatrix crossings. Near the transitions the value of $2\pi J$ substantially drifts because the dynamics is not adiabatic. We can also notice that the value of $2\pi J$ is not exactly restored after each crossing, which may introduce chaos in the semi-secular evolution (Wisdom 1985).

Separatrix crossings happen very fast if compared to the evolution timescale of $e$ and $I$; however, chaos still produces a long-term diffusion.

Figure 6 shows the comparison between the evolution of the orbital elements computed with the non-resonant secular model, and the resonant semi-secular model with digital filtering, and it can be noted that they are essentially the same. This shows that the long-term diffusion produced by the separatrix crossings in this example is slow enough to be undis-
Fig. 6  Same as Fig. 3, for asteroid (10636) 1998 QK56

Fig. 7  Level curves of the semi-secular Hamiltonian for asteroid (10636) 1998 QK56 in the plane \((\alpha, \sigma)\), at times \(t = 0, 500, 1000\) years (from the left to the right, top row) and \(t = 1500, 2000, 2500\) years (from the left to the right, bottom row). The green level curve corresponds to the guiding trajectory of the asteroid. See Fig. 6 for the long-term evolution of all the elements

cernible over a few tens of thousand years. The proper elements reported by NEODyS for this NEO are \((e_{\text{min}}, e_{\text{max}}) = (0.4681, 0.5152), (I_{\text{min}}, I_{\text{max}}) = (13.249^\circ, 19.256^\circ)\), and the proper frequencies are \(g - s = 58.531\) arcsec yr\(^{-1}\), \(s = -40.926\) arcsec yr\(^{-1}\), that are very close to the values reported in Table 2.

Other objects of Table 2 that repeatedly switch between libration and circulation and for which we were able to compute the proper elements are: (163412), (208565), and (309728). In all these cases, the evolution is the same when taking the resonance into account or not.

4.4 Case B2

We consider the NEO (469219) Kamo’oalewa, which is in 1:1 mean motion resonance with the Earth. The astronomical community gave a lot of attention to this object upon its discovery, because it is currently in a quasi-satellite configuration with the Earth, meaning that the critical angle \(\sigma\) librates around zero, and therefore it is easily accessible for space
misions (Venigalla 2019). Completely numerical studies of both the short (de la Fuente Marcos and de la Fuente Marcos 2016) and the long-term dynamics (Fenucci and Novaković 2021) have been performed.

When the resonance coefficient $h_p$ of the planet is equal to 1, it may happen that two resonance islands in the plane $(\Sigma, \sigma)$ appear (see Gallardo 2006), and this can make the secular dynamics complicated. As the secular variables evolve, a resonance island can disappear, or a horseshoe-type orbit can appear as well. Figure 8 shows the level curves of the semi-secular Hamiltonian at the initial time. We can see two resonance islands centered at $\sigma = 60^\circ$, $300^\circ$ enclosed by a eight-shaped separatrix curve, a horseshoe-type orbit surrounding the separatrix, and another small resonance island near $\sigma = 0^\circ$. The level corresponding to the initial conditions of (469219) Kamo’oalewa has two different closed curves, colored in green and cyan in Fig. 8. Currently, Kamo’oalewa is placed on the cyan level; hence, it is librating around 0°.

Figure 9 shows the evolution of the semimajor axis $a$ and of the critical angle $\sigma$, for the first 40 kyr of propagation with the semi-secular model. After few thousands years, the orbit passes from libration around 0° to a libration around 180° with a large oscillation amplitude, i.e., a horseshoe-type orbit (as in the green level curve of Fig. 8). This means that during the evolution of the secular variables, the small resonance island around 0° that we see in Fig. 8 either disappears, or it becomes too small to contain the area that corresponds to the value of the adiabatic invariant. While the secular variables continue evolving, the horseshoe orbit slightly shrinks at first, but then it enlarges again. The small resonance island around 0° appears again, and the horseshoe level curve continues to enlarge until it arrives at the separatrix curve between horseshoe and quasi-satellite. The separatrix is therefore crossed, and the object is suddenly placed on a quasi-satellite configuration again. The switching between quasi-satellite and horseshoe-type happens several times for about 20 kyr of evolution. After this time, (469219) is placed on a horseshoe-type orbit that opens at about 24 kyr, and the object passes to a circulation motion, yet another episode of libration occurs between 30 and 33 kyr, approximately.
Fig. 9 Evolution of semimajor axis (top panel) and critical argument (bottom panel) of (469219) Kamo‘oalewa given by the semi-secular Hamiltonian, in the time span [0, 40] kyr. The panels on the right show a magnification of the evolution in the time interval [0, 5] kyr.

Fig. 10 Evolution of $e$, $I$, $\omega$ and $\Omega$ of (469219) Kamo‘oalewa, computed with the semi-secular Hamiltonian model and digital filtering. The right panel shows the trajectory in the $(\omega, e)$-plane.

The filtered evolution of $e$, $I$, $\omega$ and $\Omega$ is reported in Fig. 10, left and central columns. Moreover, the right panel of Fig. 10 shows the trajectory in the $(\omega, e)$-plane. At the beginning, $\omega$ librates around 270° with eccentricity lower than 0.1. At about 25 kyr of evolution the argument of perihelion starts circulating, and then it is trapped again in a libration motion around 90°, still at low eccentricity values. After that, we can notice that $\omega$ spends some time librating around 0° or 180°, with eccentricity larger than 0.1. However, the switch between libration islands in the $(\omega, e)$-plane does not occur regularly. It is also worth noting that $\Omega$ changes its slope frequently.

For these types of motions, we cannot define proper elements because their secular evolution is chaotic. It is important to remark that chaos is produced by the appearance and/or
disappearance of resonant islands, and by separatix crossings that cause a chaotic evolution of the adiabatic invariant $2\pi J$. This was already noticed in Namouni (1999), Sidorenko (2014), where the authors studied the general dynamical structure of the 1:1 mean-motion resonance in the restricted three-body problem.

If we intend to apply our method to a large set of resonant NEOs, we need to develop an automatic detection of such chaotic orbits, or to provide proper elements that are valid on a shorter timespan (less than 20 kyr for the case of Kamo’oalewa).

5 Conclusions

In this paper, we described an algorithm for the computation of proper elements of resonant NEOs that cross the orbit of a planet. In this respect, this work provides an extension of the Gronchi and Milani (2001) approach, where the authors computed proper elements of NEOs with the assumption that no mean-motion resonances nor planetary close approaches occur. In our model, short periodic perturbations are removed by averaging the Hamiltonian over the fast angles, while keeping the resonant argument among the variables. The dynamics of resonant NEOs is propagated for 200 kyr in the future using the semi-averaged model, and a frequency analysis is then used for the computation of proper elements and proper frequencies. We provided some examples of proper elements for known resonant NEOs and compared our results with those obtained using the non-resonant model by Gronchi and Milani (2001). For some objects, the mean-motion resonance has no noticeable effect on the dynamics, and the resonant proper elements that we obtain are similar to the non-resonant ones. For other objects, on the contrary, the mean-motion resonance strongly alters the long-term dynamics, and reliable proper elements can be obtained only if the resonance is taken into account.

In this paper, we provided results only for a limited number of NEOs, that we identified to be in a mean-motion resonance by using pure $N$-body simulations. However, the method presented here can be applied to the full set of resonant NEOs, to build a complete database of the resonant proper elements.

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Data Availability The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

A Crossing singularity

A.1 The minimum orbit intersection distance

Let $(E, \ell), (E', \ell') \in \mathbb{R}^6$ be two sets of orbital elements of two Keplerian orbits with a common focus. The components $E, E' \in \mathbb{R}^5$ describe the shape of the orbit, while $\ell, \ell' \in S^1$ are the mean anomalies. We denote with $\mathcal{E} = (E, E') \in \mathbb{R}^{10}$ the couple of the orbit configurations, and with $V = (\ell, \ell') \in T^2 = S^1 \times S^1$ the parameters along the orbits. We choose a reference frame centered at the common focus and we denote with $\mathcal{X}(E, \ell), \mathcal{X}'(E', \ell')$ the
Cartesian coordinates of the two bodies. For a given configuration $E$, we define the Keplerian distance function $d$ as

$$d : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad d(E, V) = |\mathcal{X} - \mathcal{X}'|.$$ (A.1)

Let $V_h = V_h(E)$ be a local minimum point\(^9\) of the Keplerian distance function and consider the maps

$$E \mapsto d_h(E) = d(E, V_h), \quad E \mapsto d_{\min}(E) = \min_h d_h(E, V_h).$$ (A.2)

A configuration $E$ is non-degenerate if all the critical points of the Keplerian distance function are non-degenerate. If $E$ is non-degenerate, then there exists a neighborhood $W \subseteq \mathbb{R}^{10}$ of $E$ such that the maps $d_h$, restricted to $W$, do not have bifurcations.

The functions $d_h$ and $d_{\min}$ are not smooth at crossing configurations, and their derivatives do not exist. However, it is possible to define analytical maps in a neighborhood of a non-degenerate crossing configuration $E_c$ by choosing an appropriate sign for the maps. We summarize the procedure to deal with the crossing singularity of $d_h$, the procedure for $d_{\min}$ being the same. We consider the points on the two ellipses corresponding to the local minimum points $V_h = (\ell_h, \ell_h')$ of $d^2$, i.e.,

$$\mathcal{X}_h = \mathcal{X}(E, \ell_h), \quad \mathcal{X}'_h = \mathcal{X}'(E', \ell'_h).$$ (A.3)

We denote with $\tau_h, \tau'_h$ the tangent vectors to the trajectories $E, E'$ at these points, i.e.,

$$\tau_h = \frac{\partial \mathcal{X}}{\partial \ell}(E, \ell_h), \quad \tau'_h = \frac{\partial \mathcal{X}'}{\partial \ell'}(E', \ell'_h),$$ (A.4)

and their cross-product

$$\tau_h^\ast = \tau_h \times \tau'_h.$$ (A.5)

We define also $\Delta = \mathcal{X} - \mathcal{X}'$, $\Delta_h = \mathcal{X}_h - \mathcal{X}'_h$. The vector $\Delta_h$ joins the points attaining a local minimum value of $d^2$, hence $|\Delta_h| = d_h$. From the definition of critical points of $d^2$, both vectors $\tau_h, \tau'_h$ are orthogonal to $\Delta_h$, therefore $\tau_h^\ast$ and $\Delta_h$ are parallel. Denoting with $\hat{\tau}_h, \hat{\Delta}_h$ the corresponding unit vectors, the distance with sign

$$\hat{d}_h = (\hat{\tau}_h \cdotp \hat{\Delta}_h)d_h,$$ (A.6)

is an analytic function in a neighborhood of a crossing configuration, provided that $\tau_h$ and $\tau'_h$ are not parallel, situation happening only when the trajectories are tangent at the crossing point (Gronchi and Tommei 2007). The derivatives of $\hat{d}_h$ with respect to the component $E_k, k = 1, \ldots, 10$ of $E$ are given by

$$\frac{\partial \hat{d}_h}{\partial E_k} = \hat{\tau}^*_h \cdotp \frac{\partial \Delta}{\partial E_k}(E, V_h).$$ (A.7)

### A.2 Extraction of the singularity

Denote by $E_c$ a non-degenerate crossing configuration with only one crossing point. We choose the index $h$ such that $d_h(E_c) = 0$. For each $E$ in a neighborhood of $E_c$, we consider the Taylor development of $V \mapsto d^2(E, V) = |\mathcal{X} - \mathcal{X}'|^2$, in a neighborhood of the local minimum point $V_h = V_h(E)$, i.e.,

$$d^2(E, V) = d_h^2(E) + \frac{1}{2}(V - V_h) \cdot H_h(E)(V - V_h) + \mathcal{R}^{(h)}(E, V),$$ (A.8)

\(^9\) Here, the subscript $h$ is used to refer to a local minimum point, it has nothing to do with the integer of the resonant combination of Eq. (2.5).
where

\[ H_h(E) = \frac{\partial^2 d^2}{\partial V^2}(E, V_h(E)), \]  

(A.9)
is the Hessian matrix of \( d^2 \) at \( V_h = (\ell_h, \ell'_h) \), and \( R^{(h)} \) is the Taylor remainder. We introduce the approximated distance

\[ \delta_h = \sqrt{d_h^2 + (V - V_h) \cdot A_h(V - V_h)}, \]  

(A.10)

where

\[ A_h = \frac{1}{2} H_h = \begin{bmatrix} |\tau_h|^2 + \frac{\partial^2 \chi}{\partial \ell^2} (E, \ell_h) \cdot \Delta_h & -\tau_h \cdot \tau'_h \\ -\tau_h \cdot \tau'_h & |\tau'_h|^2 - \frac{\partial^2 \chi'}{\partial \ell'^2} (E', \ell'_h) \cdot \Delta_h \end{bmatrix}, \]  

(A.11)

and

\[ \Delta_h = \Delta_h(E), \quad \tau_h = \frac{\partial \chi}{\partial \ell} (E, \ell_h), \quad \tau'_h = \frac{\partial \chi'}{\partial \ell'} (E', \ell'_h). \]  

(A.12)

If the matrix \( A_h \) is non-degenerate, then it is positive definite since \( V_h \) is a minimum point, and this property holds in a suitably chosen neighborhood \( \mathcal{V} \) of \( E_c \). The matrix \( A_h \) is degenerate at the crossing configuration if and only if the tangent vectors \( \tau_h, \tau'_h \) are parallel; therefore, in the following, we always assume that the crossing is not tangent.

To extract the singularity at an orbit crossing, we split the integral as

\[ \int_{T_2^2} \frac{1}{d} d\ell d\ell' = \int_{T_2} \left( \frac{1}{d} - \frac{1}{\delta_h} \right) d\ell d\ell' + \int_{T_2^2} \frac{1}{\delta_h} d\ell d\ell'. \]  

(A.13)

Let us set \( S = \{ E \in \mathcal{W} : d_h(E) = 0 \} \), and denote with \( y_k \in \{ \Sigma, U, V, \sigma, u, v \} \) one of the coordinates. The derivatives of the first term in the right-hand side of Eq. (A.13) are integrable, and the map

\[ \mathcal{W} \setminus S \ni E \mapsto \int_{T_2^2} \frac{\partial}{\partial y_k} \left( \frac{1}{d} - \frac{1}{\delta_h} \right) d\ell d\ell', \]

(A.14)
can be extended continuously to the whole set \( \mathcal{W} \). To compute the derivatives in Eq. (A.14), we can use

\[ \frac{\partial}{\partial y_k} \left( \frac{1}{\delta_h} \right) = - \frac{1}{2\delta_h^2} \frac{\partial \delta_h^2}{\partial y_k}. \]  

(A.15)

From Eq. (A.8), we obtain the derivatives of the approximated distance as

\[ \frac{\partial \delta^2_h}{\partial y_k} = \frac{\partial d_h^2}{\partial y_k} - 2 \frac{\partial V_h}{\partial y_k} \cdot A_h(V - V_h) + (V - V_h) \cdot \frac{\partial A_h}{\partial y_k} (V - V_h). \]  

(A.16)

The derivatives of \( V_h \) are computed by differentiating the relation

\[ \frac{\partial}{\partial y_k} d_h^2(E, V_h(E)) = 0, \]

(A.17)

which holds since \((E, V_h(E))\) is a stationary point of \( d^2 \). Hence,

\[ \frac{\partial V_h}{\partial y_k}(E) = -[H_h(E)]^{-1} \frac{\partial}{\partial y_k} \nabla d^2(E, V_h(E)). \]

(A.18)
A.3 Integration of $1/\delta_h$ and its derivatives

Let $(E_c, V_h(E_c))$ be a crossing configuration. We consider the transformations

$$T_h(V) = V + V_h, \quad L_h(V) = \sqrt{A_h} \cdot V,$$  

where $\sqrt{A_h}$ is defined as the unique positive definite matrix such that $(\sqrt{A_h})^2 = A_h$. With these constraints, the entries $a_{ij}$ of $\sqrt{A_h}$ are

$$a_{11} = \frac{\alpha + A_{11}}{\sqrt{2\alpha + A_{11} + A_{22}}}, \quad a_{22} = \frac{\alpha + A_{22}}{\sqrt{2\alpha + A_{11} + A_{22}}}, \quad a_{12} = \frac{A_{12}}{\sqrt{2\alpha + A_{11} + A_{22}}},$$  \hspace{1cm} (A.20)

where $\alpha = \sqrt{\det A_h}$, and $A_{ij}$ are the entries of $A_h$. Using these transformations to change the coordinates in the integral, we get

$$\int_{T^2} \frac{1}{\delta_h} d\bar{\ell} d\bar{\ell}' = \int_{T_h(\mathbb{T}^2)} \frac{1}{\delta_h} dV = \frac{1}{\sqrt{\det A_h}} \int_{L_h(\mathbb{T}^2)} \frac{1}{\sqrt{d_h^2 + |W|^2}} dW$$  \hspace{1cm} (A.21)

where

$$W = L_h \circ T^{-1}_h(V) = L_h(V - V_h).$$  \hspace{1cm} (A.22)

Let us consider the points $P_1 \equiv (\pi, \pi)$, $P_2 \equiv (-\pi, \pi)$, $P_3 \equiv (-\pi, -\pi)$, $P_4 \equiv (\pi, -\pi)$ and their images $Q_j \equiv (x_j, y_j)$, $j = 1, \ldots, 4$ through $L_h$, so that

$$(x_1, y_1) = \pi(a_{11} + a_{12}, a_{12} + a_{22}), \quad (x_2, y_2) = \pi(-a_{11} + a_{12}, -a_{12} + a_{22}),$$  \hspace{1cm} (A.23)

$$(x_3, y_3) = -(x_1, y_1), \quad (x_4, y_4) = -(x_2, y_2).$$  \hspace{1cm} (A.24)

Set $P_5 = P_1$ and, for $j = 1, \ldots, 4$, let $\mathcal{R}_j$ be the straight line passing through the points $P_j, P_{j+1}$, i.e.,

$$\xi_j(y - y_j) = \eta_j(x - x_j),$$  \hspace{1cm} (A.25)

where $\xi_j = x_{j+1} - x_j$, $\eta_j = y_{j+1} - y_j$. Introducing polar coordinates $(\rho, \theta)$ such that $W = (\rho \cos \theta, \rho \sin \theta)$, we can write these lines in polar form

$$\mathcal{R}_j = \left\{(r_j(\theta) \cos \theta, r_j(\theta) \sin \theta) : \theta \in (\bar{\theta}_j, \bar{\theta}_j + \pi)\right\}$$  \hspace{1cm} (A.26)

with

$$r_j(\theta) = \frac{\xi_j y_j - \eta_j x_j}{\xi_j \sin \theta - \eta_j \cos \theta}$$  \hspace{1cm} (A.27)

and

$$\bar{\theta}_j = \begin{cases} \arctan(\eta_j/\xi_j), & \xi_j \neq 0, \\ \pi/2, & \xi_j = 0. \end{cases}$$  \hspace{1cm} (A.28)

Note that

$$(\xi_1, \eta_1) = -2\pi(a_{11}, a_{12}), \quad (\xi_2, \eta_2) = -2\pi(a_{12}, a_{22}),$$  \hspace{1cm} (A.29)

$$(\xi_3, \eta_3) = -(\xi_1, \eta_1), \quad (\xi_4, \eta_4) = -(\xi_2, \eta_2).$$  \hspace{1cm} (A.30)
so that, for each \( j = 1, \ldots, 4, \)

\[
\xi_j y_j - \eta_j x_j = -2\pi^2 \sqrt{\det A_h}
\]  
(A.31)

and

\[
r_1(\theta) = \frac{\pi \sqrt{\det A_h}}{a_{11} \sin \theta - a_{12} \cos \theta}, \quad r_2(\theta) = \frac{\pi \sqrt{\det A_h}}{a_{12} \sin \theta - a_{22} \cos \theta},
\]
(A.32)

\[
r_3(\theta) = -r_1(\theta), \quad r_4(\theta) = -r_2(\theta).
\]
(A.33)

With these changes of coordinates, Eq. \((A.21)\) becomes

\[
\int_{T^2_h} \frac{1}{\delta h} \, \mathrm{d}\ell' = \frac{1}{\sqrt{\det A_h}} \left( \sum_{j=1}^{4} \int_{\theta_j}^{\theta_{j+1}} \sqrt{d_h^2 + r_j^2(\theta)} \, \mathrm{d}\theta - 2\pi d_h \right)
\]
(A.34)

with

\[
\cos \theta_j = \frac{x_j}{\sqrt{x_j^2 + y_j^2}}, \quad \sin \theta_j = \frac{y_j}{\sqrt{x_j^2 + y_j^2}},
\]
(A.35)

and

\[
\theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_5 = 2\pi + \theta_1.
\]
(A.36)

The integrals in Eq. \((A.34)\) are bounded; hence, they are differentiable functions of the elements. On the contrary, the term \(-2\pi d_h / \sqrt{\det A_h}\) is not differentiable at \(E = E_c \in S\), and the loss of regularity is due only to this term. The derivatives of Eq. \((A.34)\) with respect to \(y_k \in \{\Sigma, U, V, \sigma, u, v\}\) can be computed by exchanging the integral sign and the derivative, i.e.,

\[
\frac{\partial}{\partial y_k} \int_{T^2_h} \frac{1}{\delta h} \, \mathrm{d}\ell' = \left( \frac{\partial}{\partial y_k} \frac{1}{\sqrt{\det A_h}} \right) \left( \sum_{j=1}^{4} \int_{\theta_j}^{\theta_{j+1}} \sqrt{d_h^2 + r_j^2(\theta)} \, \mathrm{d}\theta - 2\pi d_h \right)
\]
\[
+ \frac{1}{\sqrt{\det A_h}} \left( \sum_{j=1}^{4} \int_{\theta_j}^{\theta_{j+1}} \frac{\delta d_h}{\delta y_k} + r_j(\theta) \frac{\partial r_j}{\partial y_k}(\theta) \, \mathrm{d}\theta - 2\pi \frac{\partial d_h}{\partial y_k} \right).
\]
(A.37)

The term \(-2\pi d_h / \sqrt{\det A_h}\) is not differentiable at the orbit crossing; however, the derivatives admit two analytic extensions \((\frac{\partial K_{sec}}{\partial y_k})^\pm\) on \(W^+ = W \cap [\tilde{d}_h > 0]\) and \(W^- = W \cap [\tilde{d}_h < 0]\), where \(W\) is a neighborhood of the crossing configuration \(E_c\) where \(\tilde{d}_h\) is defined, and \(A_h\) is non-degenerate (Gronchi and Tardioli 2013). Moreover, the jump in the derivatives passing from \(W^+\) to \(W^-\) is given by

\[
\text{Diff}_h \left( \frac{\partial K_{sec}}{\partial y_k} \right) := \left( \frac{\partial K_{sec}}{\partial y_k} \right)^+_h - \left( \frac{\partial K_{sec}}{\partial y_k} \right)^-_h
\]
\[
= \frac{1}{\pi} \left[ \frac{\partial}{\partial y_k} \left( \frac{1}{\sqrt{\det A_h}} \right) \tilde{d}_h + \frac{1}{\sqrt{\det A_h}} \frac{\partial \tilde{d}_h}{\partial y_k} \right].
\]
(A.38)

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