Charged Particle-Image Interaction Near a Conducting Surface

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The interaction of a bulk electron with conducting surfaces is studied by means of Bohm-Pines transformation in the second quantization formalism. The effective interaction potentials are obtained for the case of one plane and two plane configurations in the form of electron-image electron scattering.

I. INTRODUCTION

The electron interactions in an electron gas yield collective plasmon modes. Bohm and Pines developed a collective description of electrons for a three-dimensional degenerate electron gas placed in a uniform background of positive charge. In order to obtain a plasmon description of electron interactions, a set of supplementary field coordinates are introduced, \( N' \) in number, which describe the collective motion of the system. Hence, the model possesses a total of \( 3N + N' \) degrees of freedom corresponding to electrons and plasma oscillations respectively. This extended system of electrons and plasma waves have the same physical properties as the original system after imposing a set of \( N' \) subsidiary conditions. Grecu calculated the plasma frequency of the electron gas by treating it as a layered structure proposed by Visscher and Falicov by equation of motion method in the RPA. Apostol pointed out that the plasmons propagating in different layers of a solid are significantly coupled together via the electric field created by in-plane charge for finite values of wave vector \( k \). In other words, when we deal with a system of two dimensional electron gas, the electron-plasmon coupling may not be negligible. Therefore, the quantum system consists of electrons plus the plasmon field width the additional plasmon-plasmon coupling. These requirements are already implemented by the canonical transformation method of Bohm and Pines. A recent attempt employing self-energy approach to obtain the image potential near a surface is given in [6]-[7].

In this paper the problem of a single bulk electron in interaction with an infinite conducting plane has been solved for the cases of one and two planes, respectively, where the bulk electron is placed between the planes for the latter case. A two dimensional quantum version of Bohm-Pines canonical transformation is used in second quantization formalism (natural units are used).

The bulk electron - plane interaction effectively becomes electron-electron scattering mediated via plasmon exchanges, where the second electron is in fact the image of the first one with respect to the conducting plane. The effective potential evaluated for this interaction has both static and dynamic components. In the static limit the effective potential reduces as expected to the classical potential obtained by the image method of classical electrodynamics. Similarly, the image method result for two planes with an electron in between is also obtained.

II. 2D QUANTUM VERSION OF BOHM-PINES TRANSFORMATION

The basic Hamiltonian for a two dimensional electron gas in the second quantized form is

\[
H = \int d^2 \rho \psi^\dagger(\rho) \frac{p^2}{2m} \psi(\rho) + \frac{1}{2} \int d^2 \rho d^2 \rho' \psi^\dagger(\rho) V(\rho - \rho') \psi(\rho') \psi(\rho)
\]  

(1)

where \( \psi \) is the electron field,

\[
\psi(\rho) = \int \frac{d^2 k}{(2\pi)^2} e^{i \vec{k} \cdot \vec{\rho}} a_{\vec{k}}
\]

(2)

and \( V \) is the interparticle interaction potential. In the second quantized form \( H \) becomes,

\[
H = \frac{1}{(2\pi)^4} \int d^2 \rho d^2 k_1 d^2 k_2 k_1^2 k_2^2 e^{i(\vec{k}_2 - \vec{k}_1) \cdot \vec{\rho}} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}
\]

\[+ \frac{1}{2} \int \frac{d^2 \rho' d^2 \rho''}{(2\pi)^8} \int d^2 k_1 d^2 k_2 d^2 k_3 d^2 k_4 e^{-i \vec{k}_1 \cdot \vec{\rho'}} e^{-i \vec{k}_2 \cdot \vec{\rho''}} V(\vec{\rho} - \vec{\rho'}) e^{i \vec{k}_3 \cdot \vec{\rho'}} e^{i \vec{k}_4 \cdot \vec{\rho''}} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger a_{\vec{k}_3} a_{\vec{k}_4}.
\]

(3)
After making transformation \( \vec{r}_1 = \vec{r}' - \vec{r}'', \vec{r}_2 = \vec{r}' - \vec{r}'', \) we have,

\[
H = \frac{1}{(2\pi)^4} \int \frac{d^2k_1}{2m} \frac{d^2k_2}{2m} \frac{d^2k_3}{2m} \frac{d^2k_4}{2m} \delta(\vec{k}_2 - \vec{k}_1) \\
+ \frac{1}{2} \int \frac{d^2r_1}{(2\pi)^6} \frac{d^2r_2}{(2\pi)^6} d^2k_2 d^2k_3 d^2k_4 a^\dagger_{k_1} a_{k_2} a_{k_3} a_{k_4} e^{\vec{k}_1 \cdot (\vec{r}_2 - \vec{r}_1)} - e^{-\vec{k}_1 \cdot (\vec{r}_2 + \vec{r}_1)} V(r_2)e^{i\vec{k}_3 \cdot \vec{r}_1} e^{i\vec{k}_4 \cdot (\vec{r}_2 + \vec{r}_1)} a_{k_3} a_{k_4} \\
= \frac{1}{(2\pi)^4} \int \frac{d^2k_1}{2m} \frac{d^2k_2}{2m} a^\dagger_{k_1} a_{k_2} \\
+ \frac{1}{2} \int \frac{d^2k_1}{(2\pi)^6} \frac{d^2k_2}{2m} \frac{d^2k_3}{2m} \frac{d^2k_4}{2m} \delta((\vec{k}_1 + \vec{k}_2) - (\vec{k}_3 + \vec{k}_4)) V(k_1 - k_3) a^\dagger_{k_1} a_{k_2} a_{k_3} a_{k_4}.
\]

The substitutions \( \vec{k}_1 = \vec{k} + \vec{q}, \vec{k}_2 = \vec{p} - \vec{q}, \vec{k}_3 = \vec{p} \) and \( \vec{k}_4 = \vec{k} \) in the second term give,

\[
H = \int \frac{d^2k}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \delta((\vec{k}_1 + \vec{k}_2) - (\vec{k}_3 + \vec{k}_4)) V(k_1 - k_3) a^\dagger_{k_1} a_{k_2} a_{k_3} a_{k_4}.
\]

in terms of single-particle operators. In order to obtain the collective motion of the planar electrons, charge density operators are introduced

\[
A_{\vec{q}} = \int \frac{d^2k}{(2\pi)^2} a^\dagger_{\vec{k}+\vec{q}} a_{\vec{k}}
\]

with the properties

\[
[A_{\vec{q}}, A_{\vec{q}'}] = [A_{\vec{q}}^\dagger, A_{\vec{q}'}^\dagger] = [A_{\vec{q}}, A_{\vec{q}'}] = [A_{\vec{q}}^\dagger, A_{\vec{q}'}^\dagger] = 0, \quad [a_{\vec{k}}, A_{\vec{q}}] = a_{\vec{k}+\vec{q}}.
\]

In terms of the density operators \( A \), the second term of the Hamiltonian \( (5) \) becomes,

\[
H = \int \frac{d^2k}{(2\pi)^2} \varepsilon_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}} + \int \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2p}{(2\pi)^2} \tilde{V}(q) (-\delta_{\vec{p},\vec{k}+\vec{q}} a^\dagger_{\vec{p}+\vec{q}} a_{\vec{k}} + a^\dagger_{\vec{p}-\vec{q}} a_{\vec{p}} a^\dagger_{\vec{k}+\vec{q}} a_{\vec{q}}) + \int \frac{d^2k}{(2\pi)^2} \tilde{V}(k) A^\dagger_{\vec{k}} A_{\vec{k}}
\]

\[
= \int \frac{d^2k}{(2\pi)^2} \varepsilon_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}} - \int \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \tilde{V}(q) a^\dagger_{\vec{k}} a_{\vec{k}} + \int \frac{d^2k}{(2\pi)^2} \tilde{V}(k) A^\dagger_{\vec{k}} A_{\vec{k}}
\]

where

\[
\varepsilon_{\vec{k}}(\vec{k}) = \frac{\vec{k}^2}{2m}.
\]

The potential part of the Hamiltonian \( (5) \) appears to be purely kinetic term in terms of charge densities.

\[
H = \int \frac{d^2k}{(2\pi)^2} \varepsilon_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2} \tilde{V}(\vec{k}) A^\dagger_{\vec{k}} A_{\vec{k}}
\]

where

\[
\varepsilon_{\vec{k}}(\vec{k}) = \varepsilon_{\vec{k}}(\vec{k}) + V(0).
\]

The Hamiltonian in \( (8) \) possesses two kinds of purely kinetic terms, one for single electrons and one for electron densities. In order to reach the plasmon modes, Bohm and Pines \( (9) \) proposed to complete this Hamiltonian by introducing the conjugate momenta, \( P_{\vec{k}} \), such that

\[
H = H_0 + H_c,
\]
where

\[
H_0 = \int \frac{d^2 \v}{(2\pi)^2} \epsilon^k \tilde{a}_k a^\dagger_k \tag{12}
\]

\[
H_c = \frac{1}{2} \int \frac{d^2 \v}{(2\pi)^2} \left[ M_k^2 \tilde{a}_k a^\dagger_k + P_k^I P^\dagger_k + M_k \left( \tilde{a}_k a^\dagger_k + P_k^I \right) \right] \tag{13}
\]

\[
M_k = \sqrt{\tilde{V}(\v)} \tag{14}
\]

and \(P^I_k\) satisfies,

\[
P^I_k |\Psi\rangle = 0. \tag{15}
\]

The additional terms clearly do not affect the space of physical states and the subsidiary conditions [14] are restrictions which turn out to be consistent with the physical properties of the system. The definition of \(M_k\) is signaling the onset of the medium-coupling region represented by plasmons. The plasmon modes are represented by the electron density Hamiltonian, \(H_c\), whereas the single electron term \(H_0\) is decoupled from it.

The \(k\)-integration in the expression for \(H_c\) is usually divided into parts; the long range interaction defined by the integration domain \(k < k_c\) where \(k_c\) is a cut-off for the collective behavior and the shorter range screened electron interaction for \(k > k_c\). However, we shall keep a \(k_c\) as a cut-off parameter throughout the calculations to explicitly demonstrate plasmon contributions, but when evaluating the effective potential, shall let \(k_c \to \infty\) to include both types of interactions.

In two dimensions the Bohm-Pines transformation reads

\[
U = e^{iS} \tag{16}
\]

where

\[
S = \int_{k < k_c} \frac{d^2 \v}{(2\pi)^2} M_k Q_k A_k^\dagger, \tag{17}
\]

\(k_c\) is a physically set cut-off value for momentum and \(Q_k\) are the conjugate collective coordinates defined through

\[
[Q_k, P^I_k] = i(2\pi)^2 \delta(k - k'). \tag{18}
\]

The transformation of \(H\) by \(U\) up to second order terms in \(S\),

\[
H' = U^\dagger H U = H - i[S, H] - \frac{1}{2}[S, [S, H]] \tag{19}
\]

involves the following commutators:

\[
[S, P^I_k] = \int \frac{d^2 \v'}{(2\pi)^2} M_k A_k^\dagger [Q_k, P^I_k] = iM_k A_k^\dagger
\]

\[
[S, [S, P^I_k]] = [S, A_k^\dagger] = 0
\]

\[
[S, H_0] = \int \frac{d^2 \v}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \tilde{v}(\v) M_q Q_q [A_q^\dagger, a^\dagger_k a_k^\dagger]
\]

\[
\begin{align*}
&= \int \frac{d^2 \v}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} M_q Q_q \int \frac{d^2 \v}{(2\pi)^2} \left[ \tilde{v}(\v + q) - \tilde{v}(\v) \right] a^\dagger_{k+q} a_k^\dagger \\
&= \int \frac{d^2 \v}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} M_q Q_q \int \frac{d^2 \v}{(2\pi)^2} \left( \frac{q^2}{2m} + \frac{\v^2}{m} \right) a^\dagger_{k+q} a_k^\dagger
\end{align*}
\]

\[
[S, [S, H_0]] = \int \frac{d^2 \v_1}{(2\pi)^2} \frac{d^2 q_1}{(2\pi)^2} M_{q_1} M_{q_2} Q_{\v_1} Q_{\v_2} \int \frac{d^2 \v}{(2\pi)^2} \left[ \tilde{v}(\v + \v_1) - \tilde{v}(\v) \right] [A_{\v_2}, a^\dagger_{k+\v_1} a_k^\dagger]
\]

\[
\begin{align*}
&= \int \frac{d^2 \v_1}{(2\pi)^2} \frac{d^2 q_1}{(2\pi)^2} M_{q_1} M_{q_2} Q_{\v_1} Q_{\v_2} \int \frac{d^2 \v}{(2\pi)^2} \left[ \tilde{v}(\v + \v_1 + \v_2) + \tilde{v}(\v) - \tilde{v}(\v + \v_1 + \v_2) - \tilde{v}(\v + \v_2) \right] a^\dagger_{k+\v_1+\v_2} a_k^\dagger \\
&= \int \frac{d^2 \v_1}{(2\pi)^2} \frac{d^2 q_1}{(2\pi)^2} M_{q_1} M_{q_2} Q_{\v_1} Q_{\v_2} \frac{\v_1 \cdot \v_2}{m} A_{\v_1+\v_2}.
\end{align*}
\]
In terms of the transformed conjugate momenta,
\[ P'_k = U^\dagger P_k U = P_k + M_k A_k \] (20)
the transformed Hamiltonian, \( H_c \), becomes,
\[
H'_c = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} P'^\dagger_k P'_k \\
H'_0 = \int \frac{d^2k}{(2\pi)^2} \varepsilon(k) a^\dagger_k a_k + i \int \frac{d^2q}{(2\pi)^2} M_q Q_q q \cdot \left( \frac{k + \hat{q}}{m} \right) a^\dagger_{k+q} a_k \\
- \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} M_q Q_q q \cdot \left( \frac{k + \hat{q}}{m} \right) a^\dagger_{k+q} a_k
\] (22)

If the term with \( \hat{q} = -\hat{q}_2 \) in the third integral is singled out and then \( Q_{-\hat{q}} = -Q_{\hat{q}}^\dagger \) is used, the Hamiltonian for the system turns out to be
\[
H' = H_0 + H_{pl} + H_{el-pl}
\] (24)

where
\[
H_0 = \int \frac{d^2k}{(2\pi)^2} \varepsilon(k) a^\dagger_k a_k ; \quad \varepsilon(k) = \frac{k^2}{2m} - \frac{1}{2} V(0),
\] (25)
\[
H_{pl} = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left[ P^\dagger_k P_k + \frac{k^2 M_k^2 Q_k^\dagger Q_k \hat{n}}{m} \right] ; \quad \hat{n} = a^\dagger_k a_k,
\] (26)
\[
H_{el-pl} = i \int \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} M_q Q_q q \cdot \left( \frac{k + \hat{q}}{m} \right) a^\dagger_k a_k \\
+ \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} M_q Q_q q \cdot \left( \frac{k + \hat{q}}{m} \right) a^\dagger_k a_k
\] (27)

\( H_0 \) is the free energy of single electrons, \( H_{pl} \) represents the quantization of planar charge density oscillations (plasmons) with momenta \( P \) and position \( Q \). The electron-plasmon interaction, \( H_{el-pl} \) expresses the interaction of single electrons with plasmons.

A comment on the subsidiary conditions introduced in (15) is in order here. After the transformation constraint \( [\mathbf{p}, \mathbf{q}] \) becomes
\[
\left( P'_k - M_k A_k \right) |\Psi\rangle = 0
\] (28)
where \( A_k \) corresponds to the surface charge density \( \sigma \) and \( M_k \) is given in Eq. 14. Eq. 15 is actually the Gauss’ Law,
\[
kP'_k = 4\pi A_k \Rightarrow \nabla \cdot \mathbf{E} = 4\pi \rho
\] (29)
where \( \mathbf{E} \) is the electric field due to planar charge and \( \rho \) is the charge density. Therefore, Bohm and Pines procedure incorporates new degrees of freedom without disturbing the known physics.
III. EFFECTIVE ELECTRON-PLANE INTERACTION

The effective potential for an infinite conducting plane - single bulk charged particle interaction can be evaluated by using the two dimensional Bohm-Pines transformation given in the previous section. A single electron is located a distance \( |z| = d \) away from an infinite electrically neutral metallic sheet. In-plane electron gas is in a uniform background of positive charge, therefore, the electrons can be assumed to be quasi-free particles.

![Bulk electron-plane configuration](image)

\[ \text{FIG. 1: Bulk electron-plane configuration} \]

The interaction Hamiltonian for the system is

\[
H_I = \int dz \, dp \, dp' \, V(\vec{p} - \vec{p}', z) \phi(\vec{x}) \psi(\vec{p}) \psi(\vec{p}') \phi(\vec{x}),
\]

where \( \psi(\vec{p}) \) corresponds to the electron field as given in Eq.2 extending over the plane, and \( \phi(\vec{x}) \) is the electron field defined in the bulk defined as

\[
\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}} b_\vec{p}.
\]

We let momentum vector \( \vec{k} \) be conjugate to position vector \( \vec{r} \) in two dimensions. Hence, in three dimensional space, momentum vector \( \vec{p} = \vec{k} + \ell \hat{k} \) will be conjugate to position vector \( \vec{x} = \vec{r} + z \hat{k} \).

Since Coulomb interaction depends on interparticle distances, it can be expressed in terms of momentum flows in momentum space.

\[
H_I = \int d^3p \, d^3p' \, \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_2'}{(2\pi)^3} e^{i(\vec{k}_2 - \vec{k}_1) \cdot \vec{p}} e^{i(\vec{k}_2' - \vec{k}_1') \cdot \vec{p}'} \left[ \int dz \, e^{i(\ell_2 - \ell_1)z} V(\vec{p} - \vec{p}', z) \right] b_{\vec{p}_1}^\dagger \, a_{\vec{k}_1}^\dagger \, a_{\vec{k}_2} \, b_{\vec{p}_2}
\]

Substitution \( \vec{r}_1 = \vec{p}' \) and \( \vec{r}_2 = \vec{p} - \vec{p}' \) yields,

\[
= \int d^3p_1 \, d^3k_1' \, d^3k_2' \, d^3p_2 \, \frac{d^3p_2}{(2\pi)^3} \frac{d^3k_2'}{(2\pi)^3} \frac{d^3k_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \int d^2r_2 \, V(r_2, \ell_2 - \ell_1) e^{i(\vec{k}_2 - \vec{k}_1) \cdot \vec{r}_2} \int d^2r_1 \, e^{i(|\vec{k}_2 - \vec{k}_1| + |\vec{k}_2' - \vec{k}_1'|) \cdot \vec{r}_1} \, b_{\vec{p}_1}^\dagger \, a_{\vec{k}_1}^\dagger \, a_{\vec{k}_2} \, b_{\vec{p}_2}
\]

\[
= \int d^3p_1 \, d^3p_2 \, d^3k_1' \, \frac{d^3k_2'}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_2'}{(2\pi)^3} \int d^2k_2' U(\vec{k}_2 - \vec{k}_1, \ell_2 - \ell_1) \delta(\vec{k}_2 - \vec{k}_1 + \vec{k}_2' - \vec{k}_1') b_{\vec{p}_1}^\dagger \, a_{\vec{k}_1}^\dagger \, a_{\vec{k}_2} \, b_{\vec{p}_2}
\]

\[
= \int d^3p_1 \, d^3p_2 \, d^3k_1' \, \frac{d^3k_2'}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_2'}{(2\pi)^3} \int d^2k_2' U(\vec{p}_2 - \vec{p}_1) \delta(\vec{k}_2 - \vec{k}_1 + \vec{k}_2' - \vec{k}_1') b_{\vec{p}_1}^\dagger \, b_{\vec{p}_2} \, a_{\vec{k}_1}^\dagger \, a_{\vec{k}_2}
\]

Another substitution \( \vec{p}' = \vec{p}_2 \) and \( \vec{p} = \vec{p}_2 - \vec{p}_1 \) gives the interaction term,

\[
= \int d^3k \, \frac{d^3p \, d^3p'}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} U(\vec{p}) \, b_{\vec{p}_1}^\dagger \, b_{\vec{p}_2} \, a_{\vec{k} + \vec{k}_2 - \vec{k}_1}^\dagger \, a_{\vec{k}}.
\]
where
\[ U(\vec{p}) = U(\vec{k}_2 - \vec{k}_1, \ell_2 - \ell_1) = (2\pi)^2 \int d\vec{p} \ \tilde{V}(\vec{p}, \ell_2 - \ell_1) e^{i(\vec{k}_2 - \vec{k}_1) \cdot \vec{p}} \]  
(33)

and
\[ \tilde{V}(\vec{p}, \ell_2 - \ell_1) = \int dz e^{i(\vec{k}_2 - \vec{k}_1) \cdot z} V(\vec{p}, z). \]  
(34)

Momentum space vector \( \vec{k}_2 - \vec{k}_1 \) conjugate to the position vector \( \vec{p} \) which lies over the surface, can be considered as the parallel component of the vector \( \vec{p} \): \( \vec{k}_2 - \vec{k}_1 = \vec{p}_\parallel \). This interpretation is immediately apparent from the Eq.(32) that there is no conservation for the perpendicular component of incoming momentum \( \vec{p} \). This is because the metallic sheet is assumed to be rigid.

In terms of charge densities the interaction Hamiltonian, \( H_I \), can be written as,
\[ H_I = \int d^3p (2\pi)^3 U(\vec{p}) B^\dagger_{\vec{p}_\parallel} A_{\vec{p}_\parallel} \]  
(35)

where
\[ B_{\vec{p}}^\dagger = \int \frac{d^3p'}{(2\pi)^3} b_{\vec{p}' \vec{p}}^\dagger b_{\vec{p}'}, \]  
(36)
\[ A_{\vec{p}_\parallel} = \int \frac{d^2k}{(2\pi)^2} a_{\vec{k} \vec{p}_\parallel}^\dagger a_{\vec{k}}. \]  
(37)

This interaction term can be rewritten by separating the momentum vector \( \vec{p} \) to its parallel and perpendicular components :
\[ H_I = \int \frac{d^2k}{(2\pi)^2} m_{\vec{k}}^\dagger A_{\vec{k}}, \]  
(38)

where
\[ m_{\vec{k}}^\dagger = \int \frac{dp_{\perp}}{2\pi} U(\vec{k}, \vec{p}_{\perp}) B^\dagger_{\vec{k}, \vec{p}_{\perp}} B_{\vec{k}, \vec{p}_{\perp}}. \]  
(39)

The new Hamiltonian for the system consists of \( H' \), the Hamiltonian in the absence of bulk electron given in Eq.9, free energy of the bulk electron, \( H_{0, \text{bulk}} \), and its interaction with the plane,
\[ H_{\text{new}} = H' + H_{0, \text{bulk}} + H_I \]
\[ = H_0 + H_{0, \text{bulk}} + \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left[ M_k^2 A_{\vec{k}} \dagger A_{\vec{k}} + 2m_{\vec{k}}^\dagger A_{\vec{k}} \right], \]  
(40)

where
\[ \bar{A}_{\vec{k}} = A_{\vec{k}} + \frac{m_{\vec{k}}}{M^2_k}. \]  
(41)

In order to employ the Bohm-Pines transformation we complete the Hamiltonian, \( H \), by adding the plasmon-bulk coupling terms.
\[ H_{c, \text{new}} = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left[ M_k^2 \bar{A}_{\vec{k}} \dagger \bar{A}_{\vec{k}} - \frac{m_{\vec{k}}^\dagger m_{\vec{k}}}{M^2_k} \right] + P_{\vec{k}} \dagger P_{\vec{k}} + M_k \left( \bar{A}_{\vec{k}} \dagger P_{\vec{k}} + P_{\vec{k}} \dagger \bar{A}_{\vec{k}} \right). \]  
(42)

The Bohm-Pines transformation for \( H_{c, \text{new}} \) yields the familiar results,
\[ P_{\vec{k}}' = P_{\vec{k}} + M_k \bar{A}_{\vec{k}} \]  
(43)
\[ H_{c, \text{new}}' = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left( P_{\vec{k}} \dagger P_{\vec{k}}' - \frac{m_{\vec{k}}^\dagger m_{\vec{k}}}{M^2_k} \right). \]  
(44)
The second term in the transformed Hamiltonian \( H'_{\text{new}} \) is the effective interaction Hamiltonian for the bulk electron - plane system. With the definitions (36) and (39), the effective Hamiltonian, \( H_{\text{eff}} \), can be written as:

\[
H_{\text{eff}} = -\frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \frac{m_k}{M_k} m_k \tag{45a}
\]

\[
= -\frac{1}{2} \int \frac{d^2 k \, dz \, d^2 \ell \, dB_k \, B_k}{(2\pi)^2} \, \frac{U(\vec{k}, \ell) U(\vec{k}, \ell')}{V(\vec{k})} \, B_k, \ell, B_k, \ell' \tag{45b}
\]

\[
= -\frac{1}{2} \int_{k<k_0} \frac{d^3 p_1 \, d^3 p_2 \, d^2 k \, dz \, d^2 \ell \, dB_k \, B_k}{(2\pi)^3 \, (2\pi)^3 \, (2\pi)^2} \, \frac{U(\vec{k}, \ell) U(\vec{k}, \ell')}{V(\vec{k})} \, b_{p_1}^\dagger b_{p_2} \tag{45c}
\]

where \( \vec{p} = (\vec{k}, \ell) \), \( \vec{p}' = (\vec{k}, \ell') \). This is nothing but the scattering of two electrons effectively replacing the original electron - plane interaction. In terms of three dimensional particle momenta \( \vec{p}_1 \) and \( \vec{p}_2 \), the scattering diagram is shown in Fig.2.

![FIG. 2: Electron scattering by the surface](image)

In terms of field operators this term can be written as,

\[
H_{\text{eff}} = -\frac{1}{2} \int d^3 x \, d^3 x' \, \phi(\vec{x}) \phi(\vec{x}') \left\{ \int_{k<k_0} \frac{d^2 k}{(2\pi)^2} \, \frac{d^2 \ell}{(2\pi)^2} \, \frac{U(\vec{k}, \ell) U(\vec{k}, \ell')}{V(\vec{k})} \, e^{i(\vec{k} - \vec{p}') + \ell z - i \ell z'} \right\} \phi(\vec{x}') \phi(\vec{x}) \tag{46}
\]

The characterization of \( H_{\text{eff}} \) as an effective interaction in three dimensions by the three dimensional fields \( \phi(\vec{x}) \) is analogous to the replacement of classical “charge-conducting plane” problem to an equivalent “charge-image charge” problem.

The integral inside the curly brackets in equation (46) is an effective two-particle interaction potential. By introducing the Coulomb potential, \( U(\vec{k}, \ell) = 4\pi e^2/(k^2 + \ell^2) \) and \( V(\vec{k}) = 2\pi e^2/k \), one can calculate the effective potential as,

\[
V_{\text{eff}}(\vec{x}, \vec{x}', k_0) = -\int \frac{d^2 k}{(2\pi)^2} \frac{e^{i\vec{k}.(\vec{p} - \vec{p}')}}{V(\vec{k})} \int \frac{d^2 \ell}{(2\pi)^2} U(\vec{k}, \ell) e^{i\ell z'} \int \frac{d^2 \ell'}{(2\pi)^2} U(\vec{k}, \ell') e^{-i\ell z} \tag{47a}
\]

\[
= -\int \frac{d^2 k}{(2\pi)^2} \frac{e^{i\vec{k}.(\vec{p} - \vec{p}')}}{V(\vec{k})} \left( \frac{4\pi e^2}{2\pi k^2 + \ell^2} \right) \left( \frac{4\pi e^2}{2\pi k^2 + \ell^2} \right) \tag{47b}
\]

\[
= -\int \frac{d^2 k}{(2\pi)^2} \frac{e^{i\vec{k}.(\vec{p} - \vec{p}')}}{V(\vec{k})} \left( \frac{2\pi e^2}{k} e^{-kz} \right) \left( \frac{2\pi e^2}{k} e^{-kz'} \right) \tag{47c}
\]

\[
= -\int \frac{d^2 k}{(2\pi)^2} \frac{e^{i\vec{k}.(\vec{p} - \vec{p}')}}{V(\vec{k})} e^{-k(z + z')} \tag{47d}
\]

\[
= -\frac{e^2}{2\pi} \int_0^{\pi} \frac{d\theta}{2\pi} \int_0^{k_c} dk e^{k(i(\vec{p} - \vec{p}')\cos\theta - z - z')} \tag{47e}
\]

\[
= -\frac{ie^2}{\pi b} \int_0^{\pi} \frac{d\theta}{\cos\theta + i\frac{b}{k_c}} \left[ 1 - e^{i(\ell z + \ell z')} \right] \tag{47f}
\]

where \( a = z + z' \) and \( b = |\vec{p} - \vec{p}'| \) for simplicity.
The first term of the integral in Eq. 47b is the dominant contribution to $V_{\text{eff}}$. The second term becomes negligible for large $a$ or large $b$ when $k_c$ is fixed, or for large $k_c$ with $a, b$ fixed. In the $k_c \to \infty$ limit, the result becomes,

$$V_{\text{eff}}(\vec{x}, \vec{x}') = -\frac{i e^2}{\pi b} \int_0^\pi d\theta \frac{\cos \theta + i \frac{\pi}{b}}{1 + \frac{a^2}{b^2}},$$

$$= \frac{-e^2}{\sqrt{\vec{\rho} - \vec{\rho}'|^2 + (z + z')^2}}. \tag{48}$$

Contact with the classical image method result is made for $\vec{x} = \vec{x}'$, in which case $V_{\text{eff}} = -e^2/2z$.

**IV. TWO PLANES**

In this case, a single bulk electron placed in between two infinite neutral conducting planes which have a separation $d$ between them. The configuration of the problem is shown in Fig.3. The planes act as 2 dimensional electron gas structures which interact with a bulk electron.

$$\vec{r}_1 = \vec{\rho}_1 + \vec{z} \hat{k} = \vec{r}_2 - \vec{z} + d \hat{k}$$

denotes the position of the bulk electron and is conjugate to $\vec{p} = \vec{p}_\parallel + \vec{p}_\perp = \vec{k} + \ell \hat{k}$.

![Diagram](image.png)

**FIG. 3: Plane-bulk electron-plane configuration**

The Hamiltonian for this system involves plane-plane and plane-bulk electron interactions as well as the free energies of surface charges and the bulk electron.

$$H = H_0^{(1)} + H_0^{(2)} + H^{(v)} + H_{\text{ee}}^{(1)} + H_{\text{ee}}^{(2)} + H_{\text{pv}} + H_{\text{pp}}$$

where $H_0^{(1)}$, $H_0^{(2)}$ represents free energies of the plane electrons, $H^{(v)}$ is the free energy of the bulk electron, $H_{\text{ee}}$ is the interaction of electrons within the planes or it comes out as a kinetic term of electron densities, $H_{\text{pv}}$ and $H_{\text{pp}}$ are plane-volume and plane-plane interactions and the superscripts $(1), (2)$ and $(v)$ stand for the first plane, the second plane and volume respectively. $H_0$ and $H_{\text{ee}}$ are as given in Eq. 9 in the first section. Remaining two interactions, $H_{\text{pv}}$ and $H_{\text{pp}}$, must be calculated.

$$H_{\text{pv}} = \int dz \int d^2 \rho \ d^2 \rho' V(\vec{\rho} - \vec{\rho}', z) \phi^\dagger(\vec{x}) \phi(\vec{x}) \psi^\dagger(\vec{\rho}') \psi(\vec{\rho}')$$

$$+ \int dz \int d^2 \rho \ d^2 \rho'' V(\vec{\rho} - \vec{\rho}'', z - d) \phi^\dagger(\vec{x}) \phi(\vec{x}) \xi^\dagger(\vec{\rho}'') \xi(\vec{\rho}'')$$

where $\psi$ and $\xi$ are the 2D electron field operators for the first and the second planes respectively, $\phi$ is the field operator for the bulk electron. Using the 2D electron field operators,

$$\psi(\vec{\rho}) = \int \frac{d^2 k}{(2\pi)^2} e^{i \vec{k} \cdot \vec{\rho}} a_\xi$$

$$\xi(\vec{\rho}) = \int \frac{d^2 k}{(2\pi)^2} e^{i \vec{k} \cdot \vec{\rho}} c_\xi. \tag{51b}$$
One gets the plane-volume interaction term, $H_{pv}$, in terms of the single particle operators,

$$H_{pv} = \int \frac{d^2k}{(2\pi)^2} \frac{d^2p}{(2\pi)^2} \frac{d^3\rho}{(2\pi)^3} \left[ U(\tilde{\rho}) b_{p'}^\dagger b_{p'}^\dagger a_{k+\rho}^\dagger a_k^\dagger + U(\tilde{\rho}) e^{ip\cdot d} b_{p'}^\dagger b_{p'}^\dagger c_{k+\rho}^\dagger c_k^\dagger \right].$$

(52)

Derivation of Eq. 52 is given in Appendix A. Using the definitions Eq. 56 and Eq. 37, one obtains for $H_{pv}$,

$$H_{pv} = \int \frac{d^3\rho}{(2\pi)^3} \left[ U(\tilde{\rho}) B_{p}^\dagger A_{p\perp} + U(\tilde{\rho}) e^{ip\cdot d} B_{p}^\dagger C_{p\perp} \right].$$

(53)

The expression in Eq. 53 is further reduced by integrating over the normal component $p_{\perp}$,

$$H_{pv} = \int \frac{d^2k}{(2\pi)^2} \left[ m_{1k}^1 A_k^\dagger + m_{2k}^1 C_k^\dagger \right]$$

(54)

where

$$m_{ik} = \int \frac{dp_{\perp}}{2\pi} (\delta_{i1} + \delta_{i2} e^{ip\cdot d}) U(\tilde{k}, p_{\perp}) B_{k, p_{\perp}}.$$  

(55)

The plane-plane interaction term, $H_{pp}$, given by,

$$H_{pp} = \int d^2\rho_1 d^2\rho_2 V_{pp}(\tilde{\rho}_1 - \tilde{\rho}_2, d) \psi^\dagger(\bar{\rho}_1) \psi(\bar{\rho}_2) \xi^\dagger(\bar{\rho}_2) \xi(\bar{\rho}_2),$$

(56)

similarly becomes,

$$H_{pp} = \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2k_2}{(2\pi)^2} \frac{d^2\rho}{(2\pi)^2} V_{pp}(\tilde{q}, d) a_{k_1+q}^\dagger a_{k_2-q}^\dagger e_{k_2}^\dagger e_{k_2}^\dagger$$

$$= \int d^2\rho V_{pp}(\tilde{q}, d) A_{q} C_{q}^\dagger$$

(57)

where the plane-plane interaction $V_{pp}$,

$$V_{pp}(\tilde{q}, z) = e^2 \int_0^{k_c} \frac{d\rho}{\sqrt{\rho^2 + d^2}} \int_0^{2\pi} d\theta e^{iq\rho \cos \theta}$$

$$= 2\pi e^2 \int_0^{k_c} \frac{d\rho}{\sqrt{\rho^2 + d^2}} J_0(q\rho).$$

(58)

in the limit $k_c \to \infty$ turns out to be,

$$V_{pp}(q, d) = \frac{2\pi e^2}{q} e^{-qd} = T_q^2 = M_q^2 e^{-qd}.$$  

(59)

Now, the total Hamiltonian is,

$$H = \int \frac{d^2k}{(2\pi)^2} \left[ \varepsilon(\tilde{k}) \left( a_{k}^\dagger a_{k} + c_{k}^\dagger c_{k} \right) + \frac{1}{2} M_k^2 \left( A_{k}^\dagger A_{k} + C_{k}^\dagger C_{k} \right) + \frac{1}{2} \left( m_{1k} A_{k} + A_{k}^\dagger m_{1k}^\dagger + m_{2k} C_{k} + C_{k}^\dagger m_{2k}^\dagger \right) \right]$$

$$+ \frac{1}{2} \int d^3\rho \varepsilon(\tilde{\rho}) b_{p}^\dagger b_{p}^\dagger$$

$$= H_0 + \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left( M_{1k}^2 A_{1k} + M_{2k}^2 A_{2k} + \frac{1}{M_1^2} W_{1}^\dagger W_{1} - \frac{1}{M_2^2} W_{2}^\dagger W_{2} \right)$$

(60)

where $H_0$ represents the single electron free energy terms in the two planes and in the bulk,

$$H_0 = \int \frac{d^2k}{(2\pi)^2} \varepsilon(\tilde{k}) \left( a_{k}^\dagger a_{k} + c_{k}^\dagger c_{k} \right) + \int \frac{d^3\rho}{(2\pi)^3} \varepsilon(\tilde{\rho}) b_{p}^\dagger b_{p}^\dagger.$$
and

\begin{align*}
W_1 &= \frac{1}{\sqrt{2}} (m_1 + m_2) \\
W_2 &= \frac{1}{\sqrt{2}} (m_1 - m_2) \\
A_1 &= \frac{1}{\sqrt{2}} (A + C) + \frac{1}{M_1^2} W_1 \\
A_2 &= \frac{1}{\sqrt{2}} (A - C) + \frac{1}{M_2^2} W_2.
\end{align*}

The above Hamiltonian is extended as before by introducing the conjugate momenta \( P_1 \) and \( P_2 \) for the two planes,

\[ H = H_0 + H_c \]

where

\[ H_c = \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \left[ M_1^2 A_1^\dagger A_1 + M_2^2 A_2^\dagger A_2 - \frac{1}{M_1^2} W_1^\dagger W_1 - \frac{1}{M_2^2} W_2^\dagger W_2 \right. \\
+ \left. P_1^\dagger P_1 + P_2^\dagger P_2 + M_1 \left( A_1^\dagger P_1 + P_1^\dagger A_1 \right) + M_2 \left( A_2^\dagger P_2 + P_2^\dagger A_2 \right) \right]. \]

After employing the Bohm-Pines transformation,

\[ S = \int_{k<k_c} \frac{d^2 k}{(2\pi)^2} \left( M_{1k} Q_{1k} A_1 + M_{2k} Q_{2k} A_2 \right), \]

the transformed free Hamiltonian terms are obtained as,

\[ H_0^{(v)} = \int \frac{d^2 k}{(2\pi)^2} (\delta(k) (\delta_{i,1} a_k^\dagger a_k + \delta_{i,2} c_k^\dagger c_k) \\
+ i \int \frac{d^2 q}{(2\pi)^2} M_q Q_q \int \frac{d^2 k}{(2\pi)^2} \frac{(\hat{k} + \frac{\hat{q}}{m})}{m} (\delta_{i,1} a_{k+q}^\dagger a_k + \delta_{i,2} c_{k+q}^\dagger c_k) \\
- \int \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} M_{q_1} M_{q_2} Q_{q_1} Q_{q_2} \left( \frac{\hat{q}_2}{m} (\delta_{i,1} a_{q_1+k} + \delta_{i,2} c_{q_1+k}) \right) \right) \]

(\text{for the derivation of } H_0^{(v)}, \text{ see Appendix B.}) However, we are interested in the transformed interaction Hamiltonian \( H'_c \),

\[ H'_c = \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \left[ P_{1,k}^\dagger P_{1,k} + P_{2,k}^\dagger P_{2,k} - \frac{1}{M_1^2} W_{1,k}^\dagger W_{1,k} - \frac{1}{M_2^2} W_{2,k}^\dagger W_{2,k} \right]. \]

Since it provides the interaction Hamiltonian for the bulk electron-plane system,

\[ H_{\text{eff}} = -\frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \left[ \frac{1}{M_{1k}} W_{1k}^\dagger W_{1k} + \frac{1}{M_{2k}} W_{2k}^\dagger W_{2k} \right. \\
- \left. \frac{1}{4} \int \frac{d^2 k}{(2\pi)^2} \left[ \left( m_{1k}^\dagger m_{1k} + m_{2k}^\dagger m_{2k} \right) \left( \frac{1}{M_{1k}} + \frac{1}{M_{2k}} \right) \\
+ \left( m_{1k}^\dagger m_{2k} + m_{2k}^\dagger m_{1k} \right) \left( \frac{1}{M_{1k}} - \frac{1}{M_{2k}} \right) \right] \right], \]

and becomes,

\[ H_{\text{eff}} = -\frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \left\{ \frac{1}{V(k)(1-e^{-2kd})} \int \frac{d\ell}{2\pi} \frac{d\ell'}{2\pi} U(k,\ell) U(k,\ell') \left[ 1 + e^{i(\ell'-\ell)d} \right] \\
- \frac{e^{-kd}}{V(k)(1-e^{-2kd})} \int \frac{d\ell}{2\pi} \frac{d\ell'}{2\pi} U(k,\ell) U(k,\ell') \left[ e^{i\ell'd} + e^{-i\ell'd} \right] \right\} B_{\ell,k}^\dagger B_{\ell,k}. \]
The effective potential energy \( V_{\text{eff}} \) for this system is read out as done in Eq. 45b:

\[
V_{\text{eff}}(\vec{x}, \vec{x}', k_c) = -\int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}(\vec{x}-\vec{x}')} \tilde{V}(k)^{-1}(1 - e^{-2kd})^{-1} \times \int \frac{d\ell}{2\pi} \frac{d\ell'}{2\pi} U(\vec{k}, \ell) U(\vec{k}', \ell') \left\{ 1 + e^{i(\ell'-\ell)d} - e^{-kd} \left[ e^{i\ell'd} + e^{-i\ell'd} \right] \right\} e^{i(\ell - \ell')}. 
\] (69)

Evaluation of \( V_{\text{eff}} \) involves integrals of the form \( \int \frac{d\ell}{2\pi} U(\vec{k}, \ell) e^{i\alpha \ell} \) where \( U(\vec{k}, \ell) = 4\pi e^2/(k^2 + \ell^2) \). The result of \( \ell \)-integration is \( \tilde{V}(k)e^{-|\alpha|k} \) where \( \tilde{V}(k) = 2\pi e^2/k \). The remaining 2D integrals give,

\[
V_{\text{eff}}(\vec{x}, \vec{x}', k_c) = -\int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}(\vec{x}-\vec{x}')} \frac{V(k)}{1 - e^{-2kd}} \left\{ e^{-k(|z|+|z'|)} + e^{-k(|z-d|+|z'-d|)} \right\} - e^{-kd} \left[ e^{-k(|z|+|z'|)} + e^{-k(|z-d|+|z'|)} \right] \right\}. 
\] (70)

The terms in Eq. 70 are evaluated by using the result,

\[
-\int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}(\vec{x}-\vec{x}')} \frac{\tilde{V}(k)}{1 - e^{-2kd}} = -e^2 \int_0^{\pi} \frac{d\theta}{2\pi} \int_0^{k_c} \frac{dk e^{k(-s+1\Delta \cos \theta)}}{1 - e^{-2kd}} = -e^2 \int_0^{k_c} \frac{dk e^{-ks} J_0(k\Delta)}{1 - e^{-2kd}}.
\]

The \( k_c \to \infty \) limit gives,

\[
= \sum_{n=0}^{\infty} \int_0^{\infty} dk e^{-k(s+2nd)} J_0(k\Delta)
\]

\[
= \sum_{n=0}^{\infty} \left[ \Delta^2 + (s + 2nd)^2 \right]^{-1/2}
\]

and one finally obtains the effective potential to be a sum of interactions between infinite number of images,

\[
V_{\text{eff}}(\vec{x}, \vec{x}') = -e^2 \sum_{n=0}^{\infty} \left\{ [\vec{\rho} - \vec{\rho}']^2 + (|z| + |z'| + 2nd)^2 \right\}^{-1/2} + \left[ [\vec{\rho} - \vec{\rho}']^2 + (|z - d| + |z' - d| + 2nd)^2 \right]^{-1/2}
\]

\[
- \left[ [\vec{\rho} - \vec{\rho}']^2 + (|z| + |z' - d| + (2n + 1)d)^2 \right]^{-1/2} + \left[ [\vec{\rho} - \vec{\rho}']^2 + (|z - d| + |z'| + (2n + 1)d)^2 \right]^{-1/2}
\]

(71)

For the case of a single bulk charge at \( \vec{x} = \vec{x}' \) one obtains the result for the bulk point charge between two parallel planes in terms of infinite number of images.

\[
V_{\text{eff}}(z) = -\frac{e^2}{2} \sum_{n} \left[ \frac{1}{nd + z} + \frac{1}{nd + (d - z)} - \frac{2}{(n+1)d} \right]
\]

\[
= -\frac{e^2}{2d} \sum_{n} \left[ \frac{1}{n + \frac{z}{d}} + \frac{1}{n + (1 - \frac{z}{d})} - \frac{2}{n + 1} \right]
\]

\[
= -\frac{e^2}{2d} \sum_{n} \left[ \frac{1}{(n + \frac{z}{d})(n + (1 - \frac{z}{d}))} - \frac{e^2 z}{d^2} \left( \frac{z}{d} - 1 \right) \right] \sum_{n} \frac{1}{(n+1)(n+\frac{z}{d})(n+1 - \frac{z}{d})}
\]
Setting the position to \( z = d/2 \), the result is simplified as,

\[
V_{\text{eff}} = \frac{e^2}{2d} \sum_{n} \frac{1}{(n + 1)(n + \frac{1}{2})}
= \frac{e^2}{2d} \left[ 2 + 2 \int_{0}^{1} \frac{x^{1/2} - x}{1 - x} dx \right]
= -\frac{e^2}{d} \ln 2
\]

V. CONCLUSION

The application of Bohm-Pines transformation to the electron-conducting surface problem has resulted in an effective two-particle interaction potential for the system. The transformed Hamiltonian contains the screened interaction \( (k < k_c) \) as well as the long range part of the Coulomb interaction \( (k > k_c) \). The two-point interaction term explicitly demonstrates the quantum dynamics of the electron with an image electron, which reduces to the well known classical image method results. The transformation appears to be a powerful method to study more complicated surface geometries.

APPENDIX A

\[
H_{\text{pv}} = \int d^2 \rho d\rho' d^3 p_1 d^3 k_1 d^2 k_1' d^3 p_2 d^3 k_2 d^2 k_2' \rho e^{i(\mathbf{k}_2 - \mathbf{k}_1) \cdot \mathbf{p}_1} a_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2} b_{\mathbf{k}_1'} b_{\mathbf{k}_2'} \int d\mathbf{r} e^{i(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{z}} (\mathbf{\nabla} \cdot \mathbf{\rho}', \mathbf{z}) + \int d^2 \rho d\rho' d^3 p_1 d^3 k_1 d^3 k_1' d^3 p_2 d^3 k_2 e^{i(\mathbf{k}_2 - \mathbf{k}_1') \cdot \mathbf{\rho}_1} a_{\mathbf{k}_1'}^{\dagger} a_{\mathbf{k}_2} b_{\mathbf{k}_1} b_{\mathbf{k}_2'} \int d\mathbf{r} e^{i(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{z}} (\mathbf{\nabla} \cdot \mathbf{\rho}', \mathbf{z} - d)
= \int d^3 p_1 d^3 k_1 d^3 k_1' d^3 p_2 d^3 k_2 e^{i(\mathbf{k}_2 - \mathbf{k}_1) \cdot \mathbf{\rho}} a_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2} c_{\mathbf{k}_1} c_{\mathbf{k}_2} b_{\mathbf{k}_1'} b_{\mathbf{k}_2'} \int d^2 \rho d\rho' e^{i(\mathbf{k}_2 - \mathbf{k}_1') \cdot \mathbf{\rho}} e^{i(\mathbf{k}_1' - \mathbf{k}_1) \cdot \mathbf{\rho}'} e^{i(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{z}} (\mathbf{\nabla} \cdot \mathbf{\rho}', \mathbf{z} - d) + \int d^3 p_1 d^3 k_1 d^3 k_1' d^3 p_2 d^3 k_2 e^{i(\mathbf{k}_2 - \mathbf{k}_1) \cdot \mathbf{\rho}} a_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2} c_{\mathbf{k}_1} c_{\mathbf{k}_2} b_{\mathbf{k}_1'} b_{\mathbf{k}_2'} \int d^2 \rho d\rho' e^{i(\mathbf{k}_2 - \mathbf{k}_1') \cdot \mathbf{\rho}} e^{i(\mathbf{k}_1' - \mathbf{k}_1) \cdot \mathbf{\rho}'} e^{i(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{z}} (\mathbf{\nabla} \cdot \mathbf{\rho}', \mathbf{z} - d)
\]

(A1)

Substitution \( \mathbf{r}_1 = \mathbf{\rho}', \mathbf{r}_2 = \mathbf{\rho} - \mathbf{\rho}' \) in the first integral and \( \mathbf{r}_1 = \mathbf{\rho}'', \mathbf{r}_2 = \mathbf{\rho} - \mathbf{\rho}'' \) in the second one gives

\[
H_{\text{pv}} = \int d^3 p_1 d^3 k_1 d^3 k_1' d^3 p_2 d^3 k_2 e^{i(\mathbf{k}_2 - \mathbf{k}_1') \cdot \mathbf{\rho}} a_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2} b_{\mathbf{k}_1'} b_{\mathbf{k}_2'} \int d^2 \mathbf{r} e^{i(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{z}} (\mathbf{\nabla} \cdot \mathbf{\rho}', \mathbf{z}) + \int d^3 p_1 d^3 k_1 d^3 k_1' d^3 p_2 d^3 k_2 e^{i(\mathbf{k}_2 - \mathbf{k}_1') \cdot \mathbf{\rho}} a_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2} b_{\mathbf{k}_1'} b_{\mathbf{k}_2'} \int d^2 \mathbf{r} e^{i(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{z}} (\mathbf{\nabla} \cdot \mathbf{\rho}', \mathbf{z})
= \int d^3 p_1 d^3 k_1 d^3 k_1' d^3 k_2 d^3 p_2 e^{i(\mathbf{k}_2 - \mathbf{k}_1') \cdot \mathbf{\rho}} U(\mathbf{k}_2 - \mathbf{k}_1, \mathbf{r}_2 - \mathbf{r}_1) \delta(\mathbf{k}_2 - \mathbf{k}_1 + \mathbf{k}_2' - \mathbf{k}_1') b_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2} b_{\mathbf{k}_1'} b_{\mathbf{k}_2'}
+ \int d^3 p_1 d^3 k_1 d^3 k_1' d^3 k_2 d^3 p_2 e^{i(\mathbf{k}_2 - \mathbf{k}_1') \cdot \mathbf{\rho}} U(\mathbf{k}_2 - \mathbf{k}_1', \mathbf{r}_2 - \mathbf{r}_1) \delta(\mathbf{k}_2' - \mathbf{k}_1 + \mathbf{k}_2' - \mathbf{k}_1') b_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2} b_{\mathbf{k}_1'} b_{\mathbf{k}_2'}
= \int d^3 p_1 d^3 k_1 d^3 k_2 d^3 k_1' d^3 k_2 d^3 p_2 e^{i(\mathbf{k}_2 - \mathbf{k}_1') \cdot \mathbf{\rho}} U(\mathbf{\rho}_2 - \mathbf{\rho}_1) b_{\mathbf{k}_1'} b_{\mathbf{k}_2'} a_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2} + \int d^3 p_1 d^3 k_1 d^3 k_2 d^3 k_1' d^3 k_2 d^3 p_2 e^{i(\mathbf{k}_2 - \mathbf{k}_1') \cdot \mathbf{\rho}} U(\mathbf{\rho}_2 - \mathbf{\rho}_1) b_{\mathbf{k}_1'} b_{\mathbf{k}_2'} a_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2}
\]

(A2)

We substitute \( \mathbf{\rho}'' = \mathbf{p}_2 - \mathbf{p}_1 \) to get,

\[
H_{\text{pv}} = \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \left[ U(\mathbf{\rho}) b_{\mathbf{k}}^{\dagger} b_{\mathbf{p}'} a_{\mathbf{k}}^{\dagger} a_{\mathbf{p}_1} + U(\mathbf{\rho}) e^{i\mathbf{p}_1 \cdot \mathbf{z}} d_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{p}'} c_{\mathbf{k}+\mathbf{p}_1}^{\dagger} c_{\mathbf{k}} \right]
\]

(A3)
\[ [S, H_0^{(v)}] = \int \frac{d^2k}{(2\pi)^2} \int \frac{d^3p}{(2\pi)^3} \varepsilon_p \left\{ M_{1k}Q_{1\perp} \{ A_{1,\perp} b^\dagger_{p,\perp} b_{p,\perp} \} + M_{2k}Q_{2\perp} \{ A_{2,\perp} b^\dagger_{p,\perp} b_{p,\perp} \} \right\} \]  

(B1)

where

\[ [A_{i,\perp} b^\dagger_{p,\perp} b_{p,\perp}] = \frac{1}{\sqrt{2M^2_{ik}}} \left\{ [m_{1k}, b^\dagger_{p,\perp} b_{p,\perp}] \pm [m_{2k}, b^\dagger_{p,\perp} b_{p,\perp}] \right\} \quad i = 1, 2 \]  

(B2)

and

\[ [m_{ik}, b^\dagger_{p,\perp} b_{p,\perp}] = \int \frac{dt_{\perp}}{2\pi} U(\vec{t})(\delta_{i1} + \delta_{i2} e^{-ip_{\perp} d}) [B_{t}, b^\dagger_{p,\perp} b_{p,\perp}]_{t_{\perp}=k} \]  

(B3)

Then

\[ \int \frac{d^3p}{(2\pi)^3} \varepsilon_p [A_{1,\perp} b^\dagger_{p,\perp} b_{p,\perp}] = -\frac{1}{\sqrt{2M^2_{1k}}} \int \frac{d^3p}{(2\pi)^3} \int \frac{dt_{\perp}}{2\pi} U(\vec{k}, t_{\perp}) \left( 1 + e^{-it_{\perp} d} \right) (\varepsilon_{p+t} - \varepsilon_p) b^\dagger_{p+t} b_{p,\perp} \right|_{t_{\perp}=k} \]  

(B4)

\[ \int \frac{d^3p}{(2\pi)^3} \varepsilon_p [A_{2,\perp} b^\dagger_{p,\perp} b_{p,\perp}] = -\frac{1}{\sqrt{2M^2_{2k}}} \int \frac{d^3p}{(2\pi)^3} \int \frac{dt_{\perp}}{2\pi} U(\vec{k}, t_{\perp}) \left( 1 - e^{-it_{\perp} d} \right) (\varepsilon_{p+t} - \varepsilon_p) b^\dagger_{p+t} b_{p,\perp} \right|_{t_{\perp}=k} \]  

(B5)

Therefore, transformation of kinetic term for the volume can be written as,

\[ H_0^{(v)} \approx H_0^{(v)} + i \int \frac{d^3t}{(2\pi)^3} \frac{\vec{t} \cdot \left( \frac{\vec{p} + \vec{t}}{2} \right)}{\sqrt{2m}} U(\vec{t}) \left[ \frac{1 + e^{-it_{\perp} d}}{M_{1t_{\perp}}} Q_{1t_{\perp}} + \frac{1 - e^{-it_{\perp} d}}{M_{2t_{\perp}}} Q_{2t_{\perp}} \right] b^\dagger_{p+t} b_{p,\perp} \]  

(B6)

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