Computational Derivation to Zeta Zeros and Prime Numbers

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Abstract

A route to the derivation of the numbers $s$ to the transcendental equation $\zeta(s) = 0$ is presented. The solutions to this equation require the solving of a geodesic flow in an infinite dimensional manifold. These solutions enable one approach to a formula generating the prime numbers.
The derivation of prime numbers has been a problem of longstanding interest for many years \[1\]. This desire has motivated the solution to the zeros of the Riemann zeta function, \( \zeta(s) = \sum n^{-s} \). The longstanding conjecture is that these zeros all lay on the imaginary axis at \( s = \frac{1}{2} + it \). This conjecture has been publicly examined for an approximate \( 10^{20} \) zeros. These zeros and the methods used to find them are of importance for various reasons, not only for the determination of prime numbers.

In this presentation a means to obtain these zeros is generated, based on work in \[2\] and \[3\]. The metrics on an underlying infinite dimensional geometry associated with the zeta function generate an algorithm to finding the locations of \( \zeta(s) = 0 \). In one approach, the geodesic flow allows a formula (possibly transcendental) to be obtained; the solution of which generates the zeros. In another approach, a differential equation that the zeta function satisfies is used to develop another equation that generates the same zeros.

The Riemann zeta function is defined by

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},
\]

and admits the product form,

\[
\zeta(s) = \zeta(0) \prod_{n=1}^{\infty} \left(1 - \frac{s}{\rho_n}\right).
\]

By definition \( \rho_n \) label all of the zeros of the function; it is conjectured that all of these zeros lay on the real axis located at \( s = 1/2 + it \). The zeta function satisfies a number of properties, including a reflection identity,

\[
\zeta(s) = \Gamma(s)\Gamma(-s)(2\pi)^{s-1}2\sin(s\pi/2)\zeta(1-s)
\]

with values at,

\[
\zeta(1 - 2n) = (-1)^{2n-1}B_{2n}/2n \quad \zeta(-2n) = 0,
\]

and,

\[
\zeta(2n) = \frac{(2\pi)^{2n}(-1)^n B_{2n}}{2 \cdot \Gamma(2n + 1)}.
\]
The values $\rho_j$ specify all of the zeros of the zeta function; they generate the prime numbers via the original work of Riemann and others including von Mangoldt.

Two methods are presented here to the solution of the parameters $\rho_j$. One method involves solving an infinite number of polynomial equations for an infinite number of variables. The second method requires solving a differential equation with an infinite number of derivatives.

**Algebraic Approach**

Consider the value of the zeta function at the points $s = 1 - 2n$. These values generate an infinite number of polynomial equations with $n$ ranging from 0 to $\infty$,

\[
(-1)^{2n-1} B_{2n}/2n\zeta^{-1}(0) = \prod_j (1 - (1 - 2n)\gamma_j) \tag{6}
\]

\[
= 1 - (1 - 2n) \sum_j \gamma_j + (1 - 2n)^2 \sum_{i,j} \gamma_i \gamma_j + \ldots , \tag{7}
\]

with the coefficients of the symmetric products being $(1 - 2n)^p$. The gamma parameters are the zeros, $\gamma_i = 1/\rho_i$, with $\zeta(\rho_i) = 0$. Set the left hand side of the equation to be variables,

\[
(-1)^{2n-1} B_{2n}/2n \rightarrow a_i , \tag{8}
\]

at the cost of doubling the unknowns from $\gamma_i$ to $a_i$. There is a trivial solution consisting of $a_i = 1$ and $\gamma_i = 0$. The equations are described by the numbers $(1 - 2n)^p$ in the general case, and they may be analytically continued to $a_i$ the actual value.

The zeros of the Riemann function may be found by solving the infinite number of equations in the infinite unknowns, for the true values of $a_i$. A methodology for obtaining the solutions to these polynomial equations is described in [2].

The polynomial equations with general $a_i$ are used to describe an infinite dimensional, but highly symmetric (pseudo-pfaffian) algebraic variety. The metric on this space is used to find the allowed solutions in $\gamma_j$ from the trivial solution $\gamma_j = 0$ via geodesic flow; this requires solving an infinite number of coupled second order differential equations. The presence of an infinite number of geodesic components is simplified by the presence of few free parameters, i.e. with the $a_i$ true there are no constants.
Label the space pertinent to \( \{7\} \) as \( \mathcal{M}_c(z_i) \) and its standard Riemannian metric as \( g_{\mu\nu} \). Its Kähler so that both \( g_{\mu\nu} = \partial_\mu \partial_\nu \ln \phi(z_i, \bar{z}_i) \) (in terms of \( z \) and \( \bar{z} \), \( g = g_{ij} \)) and its Christoffel connection is derived as \( \Gamma^\rho_{\mu\nu} = 1/2 \partial_\rho \partial_\mu \partial_\nu \ln \phi(z_i, \bar{z}_i) \) hold.

With the original point A to the solution of \( P_c(z_i) = 0 \), i.e. \( \gamma_j = 0 \), a geodesic flow equation is given from point A to point B; then \( a_i \) are taken to the actual values. The geodesic equation is, with the coordinates \( x = (z_i, \bar{z}_i) \),

\[
\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu} \frac{dx_\mu}{d\tau} \frac{dx_\nu}{d\tau} = 0 \tag{9}
\]

with,

\[
\Gamma^\rho_{\mu\nu} = 1/2 \partial_\rho \partial_\mu \partial_\nu \ln \phi(x_\mu, \bar{x}_\nu) . \tag{10}
\]

The coordinates \( x \) contain both the holomorphic and anti-holomorphic components describing the geometry. The flow from the original point A, such as at \( a_i = 0, \gamma_i = 0 \), to the points \( B_j \) at which \( a_i \) assumes its actual value, determine the constants \( \gamma_i \). These constants in turn determine the zeros \( \zeta(\rho_i) = 0 \).

**Differential Approach**

The second approach involves solving a differential equation in \( s \) that the zeta function satisfies. The zeta function has a definition in terms of

\[
\zeta(s) = \pi^{s/2} / 2\Gamma(s/2) \int_0^\infty t^{s/2-1} [\Theta(it) - 1] , \tag{11}
\]

with the theta function defined as

\[
\Theta(it) = \sum_n e^{-n^2\pi it} . \tag{12}
\]

This form of the zeta function shows that the zeta function satisfies the series of differential equations for integer \( n \),

\[
\frac{d^2 \zeta(\ln s)}{ds^2} = \zeta(\ln s + n) . \tag{13}
\]
The form in (13) is not commonly reported in the literature; the zeta function appears to be periodic form but takes values on the plane. The values within the strip $0 \leq \Re \ln s < 1$ dictate the values everywhere else in the plane via the derivatives.

Expanding the zeta function around the point $\ln s = n$ gives the differential equation,

$$\frac{d^2 \zeta(\ln s)}{ds^2} = \sum_{j=0}^{\infty} \frac{(-n)^j}{j!} \zeta^{(j)}(t)|_{t=\ln s} = e^{-n\partial_{\ln s}} \zeta(s).$$

(14)

This is a differential equation with an arbitrary number of derivatives in the zeta argument. Setting $\zeta(s) = 0$ is a constraint imposed on the differential equation.

The differential equation may be solved by analyzing geodesic flow on manifolds parameterized by the variables $\partial_{\ln s} \zeta(s)$. The method is presented in [3]. The metrics and the geodesic flows are required to be known in order to determine the solutions to the equation, and in particular to find the sub-manifold when $\zeta(s) = 0$ (the derivatives are not necessarily vanishing on this slice).

Zeta Zeros to Prime Numbers

As is well known, the zeta zeros allows a derivation of the prime numbers [1]. The counting function on the prime determination is

$$\pi(x) = \sum_{m} J(x^{1/m}) \mu(m)/m,$$

(15)

with the $\mu(m)$ defined as: 0 when $m^2/p^2 \in \mathbb{Z}$, 1 if $m = \prod p_{\sigma(j)}$ and number of factors is even, and $-1$ if $m = \prod p_{\sigma(j)}$ and the number of factors is odd.

Define the $J$ function as,

$$J(x) = \text{Li}(x) - \sum_{\rho} \text{Li}(x^{\rho}) - \ln 2 + \int_{x}^{\infty} \frac{dt}{t(t^2 - 1) \ln t},$$

(16)

with $\rho$ a member of the set of solutions to $\zeta(\rho) = 0$, i.e. $\rho = 1/\gamma_j$. The $\pi(x)$ function counts the prime numbers, i.e. it bumps at the prime $N_i$.

The derivation of the zeta zeros in this work follows from either the transcendental solution to the infinite set of equations, or from solving an affiliated differential equation. Methods for both are presented. The metrics on the algebraic varieties are required for the explicit derivation.
As a final comment, the elliptic L-series defined by

\[ L(C, s) = \prod (1 - a_p p^{-s} + p^{1-2s})^{-1} \]  \hspace{1cm} (17)

with \( a_p = p - N_p \) numbering the integer solutions to the curve with integral coefficients,

\[ y^2 = x^3 + ax + b \mod p , \]  \hspace{1cm} (18)

are found via the same method as used to find the Riemann function zeros, i.e. in [2]. The hyperelliptic L-series may also be treated.
References

[1] H.M. Edwards, *Riemann’s Zeta Function*, Dover publications, 1st ed., (1974).

[2] G. Chalmers, *Geometric Solutions to Algebraic Equations*, physics/0503175

[3] G. Chalmers, *Geometric Solutions to Non-Linear Differential Equations*, physics/0503194