Approximating the shortest path problem with scenarios

Adam Kasperski\textsuperscript{1} and Paweł Zieliński\textsuperscript{*1}

\textsuperscript{1} Wrocław University of Science and Technology, Wrocław, Poland, \{adam.kasperski,pawel.zielinski\}@pwr.edu.pl

Abstract

This paper discusses the shortest path problem in a general directed graph with \(n\) nodes and \(K\) cost scenarios (objectives). In order to choose a solution, the min-max criterion is applied. The min-max version of the problem is hard to approximate within \(\Omega(\log^{1-\epsilon} K)\) for any \(\epsilon > 0\) unless \(\text{NP} \subseteq \text{DTIME}(n^{\log \log n})\) even for arc series-parallel graphs and within \(\Omega(\log n / \log \log n)\) unless \(\text{NP} \subseteq \text{ZPTIME}(n^{\log \log n})\) for acyclic graphs. The best approximation algorithm for the min-max shortest path problem in general graphs, known to date, has an approximation ratio of \(K\). In this paper, an \(\tilde{\Omega}(\sqrt{n})\) flow LP-based approximation algorithm for min-max shortest path in general graphs is constructed. It is also shown that the approximation ratio obtained is close to an integrality gap of the corresponding flow LP relaxation.

Keywords: Combinatorial optimization; Approximation algorithm; Robust optimization; The shortest path problem; Uncertainty

1 Introduction

In a combinatorial optimization problem, denoted by \(P\), we are given a finite set of elements \(E = \{e_1, \ldots, e_m\}\) and a set of feasible solutions \(\Phi \subseteq 2^E\). In a deterministic case, each element \(e \in E\) has some nonnegative integral cost \(c_e\), and we seek a feasible solution \(X \in \Phi\) which minimizes a linear cost function \(F(X) = \sum_{e \in X} c_e\). The above formulation encompasses a large class of problems. In this paper, we focus on a fundamental network problem, where \(E\) is the set of arcs of a given graph \(G = (V, E), |V| = n,\) and \(\Phi\) contains the subsets of arcs forming \(s\)-\(t\) paths in \(G\). We thus consider the shortest path problem in a given graph \(G\). We briefly denote it by \(\text{SP}\).

Let scenario set \(\mathcal{U} = \{\xi_1, \ldots, \xi_K\}\) contain \(K\) distinct cost scenarios, where scenario is a realization of the element costs, \(\xi = (c_e^\xi)_{e \in E}\) for \(\xi \in \U \subset Z_+^{\mid E\mid}\). We distinguish two cases, namely the bounded case, when \(K = O(1)\) and the unbounded case, when \(K\) is a part of the input. The latter one is discussed in this paper. The cost of a given solution \(X \in \Phi\) depends on scenario \(\xi\) and will be denoted by \(F(X, \xi) = \sum_{e \in X} c_e^\xi\). In order to aggregate the cost vector \(F(X, \mathcal{U}) = (F(X, \xi_1), \ldots, F(X, \xi_K))\) we use the maximum criterion, that is the \(\infty\)-norm, \(\|F(X, \mathcal{U})\|_\infty\). We consider the following min-max version of \(P\):

\[
\text{Min-Max } P : \min_{X \in \Phi} \max_{\xi \in \U} F(X, \xi) = \min_{X \in \Phi} \|F(X, \mathcal{U})\|_\infty. \tag{1}
\]

*Corresponding author
The Min-Max $\mathcal{P}$ problem can be seen as a multi-objective problem or a problem with extra constraints or a $K$-budgeted one (see, e.g., \cite{14, 12, 10, 11, 26, 24}) with the $\infty$-norm as an aggregation criterion. Each scenario defines a linear objective function, an extra constraint, or a budget constraint. In one interpretation, a scenario set $\mathcal{U}$ models the uncertainty of the element costs, and we seek a solution that hedges against the worst possible realization of the costs. This is a typical problem considered in robust optimization (see, e.g., \cite{5, 18}). The Min-Max $\mathcal{P}$ is also a special case of recoverable robust $\mathcal{P}$, in which a complete solution is chosen in the first stage, and then, after a scenario from $\mathcal{U}$ reveals, limited recovery actions are allowed in the second stage \cite{7, 21}. The discrete scenario set $\mathcal{U}$ can be constructed from any probability distribution by performing a sampling (simulation) \cite{14, 23, 28}. Obviously, the larger the set, the better the estimation of the uncertainty. So, the size of $\mathcal{U}$ in practical problems can be very large. This advocates for constructing algorithms that are applicable in the presence of uncertainty in the definition of the instance.

Unfortunately, the min-max versions of basic network optimization problems, such as the minimum spanning tree, the minimum assignment, the minimum $s$-$t$-cut, are NP-hard, even if $K = 2$ (see \cite{3, 17} for surveys). In particular, this is also the case for the min-max version of the shortest path problem \cite{30}. When $K$ is a constant, the Min-Max SP problem admits a fully polynomial-time approximate scheme (FPTAS) \cite{2}. When $K$ is unbounded, the problem is strongly NP-hard and hard to approximate within $\Omega(\log^{1-\epsilon} K)$ for any $\epsilon > 0$ unless NP $\subseteq$ DTIME($n^{\text{polylog}}$) \cite{16}, even for graphs with a very simple structure, namely, for arc series-parallel graphs (see Section 3 for the definition of arc series-parallel graphs). Recently, this bound has been slightly strengthened in \cite{22, 23} for acyclic graphs, namely under the assumption that NP $\not\subseteq$ ZPTIME($n^{\log \log n}$), the Min-Max SP problem is hard to approximate within $\Omega(\log n / \log \log n)$.

The best approximation algorithm for MIN-MAX SP in general graphs, known to date, has an approximation ratio of $K$. It is based on a simple observation that an optimal solution for the average costs $\hat{c}_e = \frac{1}{K} \sum_{t \in \mathcal{U}} c_{e, t}$, $e \in E$, is at most $K$ times worse than the optimum (see \cite{3}). In \cite{9}, the authors claim that Min-MAX SP can be approximated within $O(\log K)$, but this claim turned out to be false (see \cite{19, 23} for counterexamples). Recently, a randomized $O(\log n \log K)$-approximation algorithm for Min-MAX SP in general digraphs, whose running time is quasi-polynomial, has been designed \cite{20, 19}. The quasi-polynomial time of the approximation algorithm can be improved to polynomial for graphs with the bounded treewidth. This class of graphs includes the arc series-parallel graphs as a special case. In \cite{22, 23}, the authors study the $p$-norm shortest path problem of the form $\min_{X \in \Phi} \|F(X, \mathcal{U})\|_p$, where $p \in \mathbb{Z}_{\geq 1}$. They proposed a randomized $O(cp \log^{1-1/p} n)$-approximation algorithm for this problem in general graphs, where $c$ is a small constant, which runs in quasi-polynomial time. For arc series-parallel graphs, the quasi-polynomial running time of the above algorithm can be improved to polynomial time. Note that choosing $p = \lfloor \log K \rfloor$ in the $O(cp \log^{1-1/p} n)$-approximation algorithm, we get a randomized, quasi-polynomial time $O(c \log n \log K)$-approximation algorithm for Min-Max SP in general graphs. This result is similar to the one in \cite{20, 19}.

Our results. In this paper, we design a simple $\tilde{O}(\sqrt{n}^{\frac{1}{p}})$-approximation algorithm (more precisely, $O(\sqrt{n} \log K / \log \log K)$-approximation one) for the min-max shortest path problem in general graphs. Contrary to the approximation algorithms proposed in \cite{22, 23, 19}, our algorithm is deterministic and runs in polynomial time. It is based on an appropriate rounding.

\footnote{This notation ignores logarithmic factors.}
of a solution to the flow LP relaxation of Min-Max SP. We will show that the approximation ratio obtained is very close to an integrality gap of this LP relaxation, which is at least \( \Omega(\sqrt{n}) \), even for arc series-parallel graphs.

This paper is organized as follows. In Section 2 we recall an LP formulation of Min-Max \( \mathcal{P} \), which leads to an LP relaxation for this problem. We also recall a formulation of the min-max version of the representatives selection problem \[8, 9\]. Our approximation algorithm, constructed in Section 3 will use the flow LP relaxation and some known approximation results on this selection problem. In Section 3 we construct a deterministic \( O(\sqrt{n} \log K/\log \log K) \)-approximation algorithm for Min-Max SP in general graphs. We also show that an integrality gap of the flow LP relaxation is at least \( \Omega(\sqrt{n}) \), even for arc series-parallel graphs.

\section{LP relaxation and representatives selection problem}

The min-max problem (1) can be alternatively stated as the following integer program:

\[
\text{OPT} = \min C \quad \text{s.t.} \quad \sum_{e \in E} c^e \xi^e x_e \leq C \quad \forall \xi \in \mathcal{U},
\]

(2)

\[
x \in \mathcal{X},
\]

(3)

\[
x \in \{0, 1\}^{|E|},
\]

(4)

where (4) and (5) describe the set of feasible solutions \( \Phi \), \( \mathcal{X} \) is given by a system of linear constraints involving \( x = (x_e)_{e \in E} \) and \( x \) is a characteristic vector of a feasible solution \( X \in \Phi \).

Fix a parameter \( C > 0 \) and let \( E(C) = \{ e \in E : (\forall \xi \in \mathcal{U})(c^e \xi \leq C) \} \subseteq E \). Consider the following family of linear programs:

\[
\mathcal{LP}(C) : \sum_{e \in E} c^e \xi^e x_e \leq C \quad \forall \xi \in \mathcal{U},
\]

(6)

\[
x \in \mathcal{X},
\]

(7)

\[
x_e \in [0, 1] \quad \forall e \in E(C),
\]

(8)

\[
x_e = 0 \quad \forall e \notin E(C).
\]

(9)

Minimizing \( C \) subject to (6)-(9) we obtain an LP relaxation of (2)-(5). Let \( C^* \) denote the smallest value of the parameter \( C \), for which \( \mathcal{LP}(C) \) is feasible and let \( x^* \) be a feasible solution to \( \mathcal{LP}(C^*) \). The value of \( C^* \) is a lower bound on \( \text{OPT} \) and can be determined in polynomial time by using the following algorithm. Let \( C_{\text{max}} \) be an upper bound on \( C^* \) (we can use, for example, \( C_{\text{max}} = \max_{\xi \in \mathcal{U}} \sum_{e \in E} c^e \)). Using a binary search, we first determine in \( O(T(K, |E|) \log C_{\text{max}}) \) time the smallest integer value \( C' \in [0, C_{\text{max}}] \) for which \( \mathcal{LP}(C') \) is feasible, where \( T(K, |E|) \) is the time required to solve the linear program. Then \( C^* \in (C' - 1, C') \) and \( C^* \) can be computed by minimizing \( C \) subject to (6)-(9) for the fixed set \( E(C' - 1) \). If this linear program is infeasible, then \( C^* = C' \). The correctness of this method follows from the assumption that \( \mathcal{U} \subset \mathbb{Z}_+^{|E|} \). Our approximation algorithm, constructed in the next section, will be based on appropriately rounding the solution \( x^* \).

Let us recall a special case of Min-Max \( \mathcal{P} \), which will be used later in this paper. Namely, the min-max representatives selection problem (Min-Max RS for short) \[8, 9, 15\]. We are given a set \( E = \{ e_1, \ldots, e_m \} \) of \( m \) tools and \( E \) is partitioned into \( p \) disjoint sets \( E_1, \ldots, E_p \),
where $|E_i| = r_i$ and $m = \sum_{i\in[p]} r_i$. We will denote by $[p]$ the set $\{1,2,\ldots,p\}$. Define $\Phi = \{X \subseteq E : (\forall i \in [p])(|X \cap E_i| = 1)\}$, so we wish to choose a subset $X \subseteq E$ of the tools that contains exactly one tool from each set $E_i$ to minimize the maximum cost over $U$. The set $\mathcal{X}$ in (4) and (7), for MIN-MAX RS, is then described by $p$ constraints of the form

$$\sum_{e \in E_i} x_e = 1 \quad \forall i \in [p].$$

(10)

It is worth pointing out that the problem can be solved trivially for the case $K = 1$. Unfortunately, for unbounded scenario set $U$, MIN-MAX RS is strongly NP-hard and hard to approximate within $\Omega(\log^{1-\epsilon} K)$ for any $\epsilon > 0$ unless NP $\subseteq$ DTIME($n^{\text{polylog} n}$) [15]. In the next part of the paper, we will use the following result:

**Theorem 1.** [13] Let $x$ be a fractional feasible solution to the linear program $LP(C)$ with $\mathcal{X}$ of the form (11) and assume that $\max_{\xi \in U} |\sum_{e \in E} \xi x_e| = C$. Then there is an algorithm which transforms $x$, in $O(Km \log m)$ time, into a solution $X$ for MIN-MAX RS that has cost $F(X, \xi) = O(C \log K / \log \log K)$ for every $\xi \in U$.

Theorem 1 immediately leads to the LP-based $O(\log K / \log \log K)$ approximation algorithm for MIN-MAX RS. It is enough to choose $C = C'$, where $C'$ is the smallest value of the parameter $C$, for which the linear program $LP(C)$ with (10) is feasible and use the fact that $C'$ is a lower bound on OPT.

### 3 Min-max shortest path

Let $G = (V,E)$ be a directed graph with two specified nodes $s,t \in V$, where $|V| = n$ and $|E| = m$. The set of feasible solutions $\Phi$ contains all $s$-$t$ paths in $G$. Each scenario $\xi \in U$ represents a realization of the arc costs. In this section, we construct an LP-based $O(\sqrt{n} \log K / \log \log K)$-approximation algorithm for the min-max version of the shortest path problem when $|U| = K$ is unbounded. For this problem, the set $\mathcal{X}$ in (4) and (7) is described by the following flow constraints (the mass balance constraints):

$$\sum_{e \in \delta^\text{out}(v)} x_e = \sum_{e \in \delta^\text{in}(v)} x_e \quad \forall v \in V \setminus \{s,t\},$$

(11)

$$\sum_{e \in \delta^\text{out}(s)} x_e = \sum_{e \in \delta^\text{in}(t)} x_e = 1,$$

(12)

where $\delta^\text{out}(v)$ and $\delta^\text{in}(v)$ are the sets of outgoing and incoming arcs, respectively, from $v \in V$.

We now show that the linear relaxation $LP(C)$ with the flow constraints (11)-(12) has an integrality gap of at least $\Omega(\sqrt{n})$, even for graphs with a very simple structure, namely, for arc series-parallel graphs. An arc series-parallel graph (ASP) is recursively defined as follows (see, e.g., [27]). A graph consisting of two nodes joined by a single arc is ASP. If $G_1$ and $G_2$ are ASP, so are the graphs constructed by each of the following two operations:

- **parallel composition:** identify the source $s_1$ of $G_1$ with the source $s_2$ of $G_2$ and the sink $t_1$ of $G_1$ with the sink $t_2$ of $G_2$,

- **series composition:** identify the sink $t_1$ of $G_1$ with the source $s_2$ of $G_2$. 


Proposition 1. The linear relaxation $\mathcal{LP}(C)$ with the flow constraints (11)-(12) has an integrality gap of at least $\Omega(\sqrt{n})$, even for arc series-parallel graphs.

Proof. See Appendix A.

Remark. It is worth noting that s-t paths in G can be also expressed by the s-t cut constraints (see, e.g., [29]). Let us recall that an s-t cut in G is a partition of V into the subsets S and $\overline{S} = V \setminus S$ such that $s \in S$ and $t \in \overline{S}$. We refer to an arc $(u, v)$ with $u \in S$ and $v \in \overline{S}$ as a forward arc of the s-t cut. The cut-set of an s-t cut, denoted by $(S, \overline{S})$, is the set of forward arcs of the s-t cut, i.e. the set $\{(u, v) \in E : u \in S, v \in \overline{S}\}$. The s-t cut constraints have the following form:

$$
\sum_{e \in (S, \overline{S})} x_e \geq 1 \text{ for all } (S, \overline{S}) \text{ in } G.
$$

(13)

If we use the s-t cut constraints (13) for expressing s-t paths in the linear relaxation $\mathcal{LP}(C)$, instead of the flow ones (11)-(12), then an analysis similar to that in the proof of Proposition 1 shows that an integrality gap also remains at least $\Omega(\sqrt{n})$, even for arc series-parallel graphs.

We are ready to present the approximation algorithm for Min-Max SP. It is shown in the form of Algorithm 1. In the following, we will describe all its steps. In Step 1 we compute a fractional solution $x^*$ to $\mathcal{LP}(C^*)$ with the constraints (11)-(12), which is an s-t arc flow in G. In Steps 2 and 3 we preprocess graph G. Let us recall the flow decomposition theorem (see, e.g., [1]).

Theorem 2 (Flow decomposition). Every nonnegative s-t arc flow $x$ can be represented as at most $m$ directed cycles and s-t paths with nonzero flows (although not necessarily uniquely). The value of the flow from $s$ to $t$ is equal to the sum of the flows along the s-t paths.

Theorem 2 shows that the s-t arc flow $x^*$ can be decomposed into directed cycles and s-t paths with nonzero flows. We first remove from G every arc $e \in E$ with $x^*_e = 0$ (Step 2). The resulting graph with $x^*_e > 0$ for each $e \in E$, must be connected, i.e. it must have an s-t path. Furthermore, to reduce the problem size, one may perform series arc reductions (the inverse operation to the series composition) in G, i.e. the operations which replace two series arcs $f, g \in E$ by the single arc $w$ with the cost $c_w = c_f + c_g$ under $\xi \in U$ and $x^*_w = x^*_f$. Finally, we convert $x^* > 0$ into a feasible solution $\hat{x}$ to $\mathcal{LP}(C^*)$ by removing directed cycles, such that the resulting graph with $\hat{x}_e > 0$ for every arc $e \in E$ is acyclic (Step 3). A procedure is described in the proof of following lemma.

Lemma 1. Let $x^* > 0$ be a feasible solution to $\mathcal{LP}(C^*)$ with the flow constraints (11)-(12). Then there exists a feasible solution $\hat{x}$ to $\mathcal{LP}(C^*)$ with (11)-(12) such that the graph induced by $\hat{x} > 0$ is acyclic.

Proof. Consider $x^* > 0$ and the graph G induced by $x^* > 0$. Theorem 2 implies that we can decompose the s-t arc flow $x^*$ into nonzero flows along s-t paths and cycles. If each arc cost under any scenario is positive, then G is also acyclic, and we are done. For nonnegative arc costs, G may have a cycle. Consider such a cycle and denote it by $C$. Let $U^c$ be the subset of $U$ such that constraints (6) are tight for $x^*$ and $C^*$. We claim that there is at least one scenario $\xi^* \in U^c$ such that $\sum_{e \in W} c^e_{\xi^*} = 0$. Suppose, contrary to our claim, that $\sum_{e \in W} c^e_{\xi} > 0$ for every $\xi \in U^c$. Since for each $e \in W$, $x^*_e > 0$, we can decrease the flow $x^*$ on cycle $C$ by $\Delta = \min \{ x^*_e : e \in C \}$ and, in consequence, decrease the cost of $x^*$ under every $\xi \in U^c$, which
contradicts the optimality of $C^*$. Accordingly, we can decrease the flow $x^*$ on the cycle $W$ by $\Delta$ without affecting $C^*$. The resulting solution is still feasible to (11)- (12), its maximum cost over $U$ is equal to $C^*$ and at least one arc from $W$ has zero flow. Thus, the graph induced by this new solution does not contain the cycle $W$. Applying the above procedure to all cycles in $G$ one can convert $x^*$ into a feasible solution $\hat{x}$ to $LP(C^*)$ such that the new graph $\hat{G}$ with $\hat{x}_e > 0$ for every arc $e$ is acyclic. Now the $s$-$t$ arc flow $\hat{x} > 0$ can be decomposed only into nonzero flows along $s$-$t$ paths. This completes the proof.

The feasible solution $\hat{x}$ from Lemma 1 can be determined in $O(mn)$ time using the flow decomposition algorithm (see, e.g., [1]).

From now on, we will assume that the graph $G$ is induced by $\hat{x} > 0$ is acyclic. Let us assign an arc length $l_e \in \{0, 1\}$ to each arc of $G$. If $l_e = 0$, then the arc is called selected; otherwise, $l_e = 1$, it is called not selected. Initially, each arc is not selected, so the length $l_e = 1$ for each $e \in E$ (Step 4). We will carefully mark some arcs as selected by assigning them the zero arc length during the course of the algorithm. The selected arcs form connected components in $G$, where each connected component of $G$ has a total length equal to 0 with respect to $l_e$ (initially, the nodes of $G$ form $n$ connected components because no arc is selected). The following lemma will be needed to construct a feasible solution to the relaxation (6)-(9) of Min-Max RS with $X$ of the form (10).

**Lemma 2.** Let $L_P$ be the length of a shortest $s$-$t$ path $P$ in the acyclic graph $G$, induced by $\hat{x} > 0$, with respect to the arc lengths $l_e \in \{0, 1\}$, $e \in E$. If $L_P \geq 1$, then there are $L_P$ arc disjoint cut-sets: $(S_1, \overline{S}_1), \ldots, (S_{L_P}, \overline{S}_{L_P})$ in $G$, which do not contain any selected arc. Moreover, $\sum_{e \in (S_i, \overline{S}_i)} \hat{x}_e \geq 1$ for each $i \in [L_P]$.

**Proof.** Let $d(v)$ be the shortest path distance from $s$ to $v$, $v \in V$, with respect to $l_e, e \in E$. The shortest path distances satisfy an optimality condition (see, e.g., [1]), i.e. $d(u) + l_e \geq d(v)$, for every $e = (u, v) \in E$. Of course, $d(s) = 0$ and $d(t) = L_P$. Using the shortest path distances $d(v), v \in V$, computed, one can determine $L_P$ cut-sets in the following way:

$$(S_i, \overline{S}_i) = \{(u, v) \in E : d(u) = i - 1, d(v) = i\}, \ i \in [L_P]. \quad (14)$$

We can see at once that all the $L_P$ cut-sets are nonempty, since $L_P \geq 1$, and by the definition they are arc disjoint. If an arc $e = (u, v)$ is such that $l_e = 0$ ($e$ is selected), then by the optimality condition we get $d(u) \geq d(v)$ and, in consequence, $e \not\in (S_i, \overline{S}_i)$ for every $i \in [L_P]$. Thus, these cut-sets do not contain any selected arc.

The value of the $s$-$t$ arc flow $\hat{x}$ is equal to 1. By Theorem 2, $\hat{x}$ can be decomposed only into $s$-$t$ paths with nonzero flows ($G$ is acyclic) and the sum of the flows along these $s$-$t$ paths is equal to 1. Consider any cut-set $(S_i, \overline{S}_i), i \in [L_P]$. Since each $s$-$t$ path contains at least one arc in $(S_i, \overline{S}_i)$, we have $\sum_{e \in (S_i, \overline{S}_i)} \hat{x}_e \geq 1$. □

Notice that the cut-sets $(S_1, \overline{S}_1), \ldots, (S_{L_P}, \overline{S}_{L_P})$, in Lemma 2 can be determined in $O(m)$ time. An example with determined cut-sets is shown in Figure 1.

The algorithm performs $k$ rounds (see Steps 7-13). In the $k$th round, $k \in \lceil k \rceil$, we compute a shortest $s$-$t$ path $P$ with respect to the arc lengths $l_e, e \in E$ (see Steps 6 and 13). If $L_P$ is less than the prescribed value $\ell$ (see Step 5), then we terminate and output $P$. Otherwise, we find $L_P$ arc disjoint cut-sets $(S_1, \overline{S}_1), \ldots, (S_{L_P}, \overline{S}_{L_P})$, described in Lemma 2 (see also (14) and Figure 1). These cut-sets form an instance of the Min-Max RS problem with $K$ scenarios.
induced by the scenarios in $\mathcal{U}$. Since $\sum_{e \in (S, \overline{S})} \hat{x}_e \geq 1$ for each $i \in [L_P]$, we have to perform the normalization in Step 10. As a result, we get a feasible solution $\mathbf{x}'$ to the relaxation (6)-(9) of Min-Max RS with $\mathcal{X}$ of the form (10), $p = L_P$ and $E_i = (S, \overline{S})$, $i \in [L_P]$, i.e. to the linear program $\mathcal{L}\mathcal{P}(C')$, where $C' = \max_{\mathbf{e} \in \mathcal{U}} \sum_{i \in [L_P]} \sum_{e \in E_i} c_e x'_e$. We now use Theorem 1 to pick a set of arcs $X$, $|X| = L_P$, that contains exactly one arc from each $(S, \overline{S})$ (see Step 11). The following lemma describes the cost of $X$.

**Lemma 3.** The value of $\max_{\mathbf{e} \in \mathcal{U}} \sum_{e \in X} c_e$ is $O(C^*/\log \log K)$.

**Proof.** The following inequalities:

$$C^* = \max_{\mathbf{e} \in \mathcal{U}} \sum_{e \in E} c_e \hat{x}_e \geq \max_{\mathbf{e} \in \mathcal{U}} \sum_{i \in [L_P]} \sum_{e \in E_i} c_e \hat{x}_e \geq \max_{\mathbf{e} \in \mathcal{U}} \sum_{i \in [L_P]} \sum_{e \in E_i} c_e \sum_{e \in E_i} \hat{x}_e$$

$$= \max_{\mathbf{e} \in \mathcal{U}} \sum_{i \in [L_P]} \sum_{e \in E_i} c_e x'_e = C'$$

hold. The first inequality follows from $\bigcup_{i \in [L_P]} E_i \subseteq E$ and the second one from the fact that $\sum_{e \in E_i} \hat{x}_e \geq 1$, $i \in [L_P]$ (see Lemma 2). Since $\mathbf{x}'$ is a feasible solution to $\mathcal{L}\mathcal{P}(C')$ with $\mathcal{X}$ of the form (10) (with $p = L_P$), Theorem 1 combined with $C' \leq C^*$ gives the assertion of the lemma.

Note that each selected arc $e \in X$ (every with $l_e = 1$ by (14)) merges two connected components. Each connected component consists of one node or nodes connected by already selected arcs with zero lengths. Hence, and from $|X| = L_P$, we get that the number of connected components is reduced by $L_P$. Then, for each selected arc $e \in X$, we set $l_e = 0$, marking it as selected (see Step 12).

We now prove the following lemma needed to analyze the performance of Algorithm 1.

**Lemma 4.** Let $\ell$ be a prescribed value in Algorithm 1 (Step 5). Then Algorithm 1 performs at most $n/\ell$ rounds (Steps 7, 13) and returns an $s$-$t$ path $P$ with the cost $\max_{\mathbf{e} \in \mathcal{U}} \sum_{e \in P} c_e$ at most $O\left(\frac{n \log K}{\ell \log \log K + \ell} \right) C^*$.
Algorithm 1: An approximation algorithm for Min-Max SP

**Input**: A directed graph $G = (V, E)$, $|V| = n$, $|E| = m$, two specified nodes $s, t \in V$, a cost scenario set $U = \{\xi_1, \ldots, \xi_K\}$, $\xi = (c^e_\xi)_{e \in E} \in U$.

**Output**: An $s$-$t$ path $P$ in $G$ – an approximate solution for Min-Max SP.

1. Find $C^*$ and a feasible solution $x^* = (x^*_e)_{e \in E}$ to $LP(C^*)$ with the flow constraints (11)-(12);
2. foreach $e \in E$ with $x^*_e = 0$ do $E \leftarrow E \setminus \{e\}$;
3. Convert $x^*$ into solution $\hat{x}$, feasible to $LP(C^*)$, i.e. perform series arc reductions and remove cycles in the graph $G$ induced by $x^*$ (Lemma 1); /* Now $G$ with $\hat{x}_e > 0$ for every arc $e$ is acyclic */
4. foreach $e \in E$ do $l_e \leftarrow 1$; // Set arc lengths
5. $\ell \leftarrow \sqrt{n \log K / \log \log K}$; // a prescribed value
6. Find a shortest $s$-$t$ path $P$ in $G$ with arc lengths $l_e, e \in E$; let $L_P$ be the length of this path;
7. while $L_P > \ell$ do // round $k$
8. 1. Find $L_P$ arc disjoint cut-sets $(S_i, \overline{S}_i)$ in $G$, $i \in [L_P]$ described in Lemma 2
9. 2. foreach $i \in [L_P]$ do
10. 3. foreach $e \in (S_i, \overline{S}_i)$ do $x'_e \leftarrow \hat{x}_e / (\sum_{e \in (S_i, \overline{S}_i)} \hat{x}_e)$;
11. 4. Use Theorem 1 to transform $x'_e$, $e \in \bigcup_{i \in [L_P]} E_i$, where $E_i = (S_i, \overline{S}_i)$, into a solution $X$, $|X| = L_P$, for Min-Max RS;
12. 5. foreach $e \in X$ do $l_e \leftarrow 0$; // $e$ is selected
13. 6. Find a shortest $s$-$t$ path $P$ in $G$ with arc lengths $l_e, e \in E$; let $L_P$ be the length of $P$;
14. return $P$
Proof. We will denote by \( \hat{k} \) the number of rounds performed by the algorithm (Steps 7–13). Let \( C^k \) be the number of connected components in \( G \) (merged by selected arcs) and \( L^k_p \) be the length of a shortest path from \( s \) to \( t \) in \( G \) with respect to \( l_e \in \{0, 1\}, e \in E \), at the beginning of the \( k \)th round, \( k \in [\hat{k}] \). In the \( k \)th round, we choose \( X^k, |X^k| = L^k_p \), and set \( l_e = 0 \) for each \( e \in X^k \), which reduces the number of connected components by \( L^k_p \). In consequence, the equalities

\[
C^{k+1} = C^k - L^k_p, \quad C^1 = n, \quad k \in [\hat{k}],
\]

hold. Observe that \( L^{k+1}_p \leq \ell, \ell < L^k_p, k \in [\hat{k}] \) (see the condition 7). Furthermore, \( L^{\hat{k}+1}_p < C^{\hat{k}+1} \). Indeed, each connected component in the \( k \)th round consists of the arcs with length \( l_e = 0 \). A shortest path with respect to \( l_e \in \{0, 1\} \) connects at most \( C^{k+1} \) components using the arcs with length \( l_e = 1 \). So, the length of this path \( L^{k+1}_p \) is at most \( C^{k+1} - 1 \), which implies \( L^{\hat{k}+1}_p < C^{\hat{k}+1} \). Hence and from (15), we obtain

\[
L^{\hat{k}+1}_p < C^{\hat{k}+1} = C^1 - \sum_{k \in [\hat{k}]} L^k_p < n - \hat{k} \ell,
\]

which gives the following bound on the number of rounds performed:

\[
\hat{k} < (n - L^{\hat{k}+1}_p)/\ell \leq n/\ell.
\]

Consider the cost of the path \( P \) returned. Since each arc \( e \in P \) is such that \( e \in E(C^\ast) \) and the number of not selected arcs on this path is at most \( \ell \), the maximum total cost of not selected arcs on \( P \) is at most \( \ell C^\ast \). On the other hand, the total cost of all selected arcs is at most \( O\left(\frac{n \log K}{\ell \log \log K} C^\ast\right) \), which results from applying at most \( n/\ell \) times Lemma 3. So, the maximum cost of \( P \) over all scenarios in \( U \) is \( O\left(\frac{n \log K}{\ell \log \log K} + \ell\right) C^\ast \).

The best ratio in \( O\left(\frac{n \log K}{\ell \log \log K} + \ell\right) \) can be achieved by choosing \( \ell = \sqrt{n \log K/\log \log K} \), which is a prescribed bound on the number of not selected arcs on \( P \) in Algorithm 1 (see Step 5). Lemma 4 and \( C^\ast \leq OPT \) yield an approximation factor of \( O\left(\sqrt{n \log K/\log \log K}\right) \).

We have yet to show that the running time of Algorithm 1 is polynomial. Let us first analyze Steps 14. The value of \( C^\ast \) and corresponding feasible solution \( x^\ast \) to \( LP(C^\ast) \) can be found in \( O(T(K, m, n) \log C_{\max}) \) time, where \( C_{\max} = \max_{e \in E, \xi \in U, \xi \in E} \xi \) and \( T(K, m, n) \) is the time required to solve the linear program with constraints \( LP(C) \) with \( \ell \) of the form (11)–(12). The time \( T(K, m, n) \) is polynomial (see, e.g., [13]). Steps 2 and 4 require \( O(m) \) time. Converting \( x^\ast \) into solution \( \hat{x} \) in Step 3 can be done in \( O(mn) \) time by using the flow decomposition algorithm (see, e.g., [1]). Since \( G \) is acyclic, a shortest \( s \)-\( t \) path \( P \) in \( G \) (Step 6) and also 13 can be found in \( O(m) \) time (see, e.g., [1]). Therefore, Steps 16 require \( O(mn + T(K, m, n) \log C_{\max}) \) time. Consider the round \( k \) (Steps 7–13). Steps 8, 10, and 13 require \( O(m) \) time (in Step 8 we use already computed shortest path distances). By Theorem 1, Step 11 requires in \( O(Km \log m) \) time. The number of rounds is bounded by \( \frac{n}{\ell} = \frac{\sqrt{n}}{\sqrt{\log K/\log \log K}} \) (see Lemma 4). Thus, executing the while loop (Steps 7–13) takes

\[
O\left(\frac{Km \sqrt{n} \log m}{\sqrt{\log K/\log \log K}}\right) \text{ time. Hence, the total running time of Algorithm 1 is polynomial.}
\]

Accordingly, we have thus proved the main result of this paper.

Theorem 3. There is an \( O\left(\frac{n \log K}{\log \log K}\right) \)-approximation algorithm for Min-Max SP in a general directed graph.
4 Conclusions

There is still an open question concerning the MIN-MAX SP problem. For this problem, there exists an $O(\sqrt{n \log K / \log \log K})$-approximation algorithm (Õ(√n))-approximation algorithm) for general graphs, designed in this paper, $\Omega(\log^{1-\epsilon} K)$ lower bound on the approximability of the problem for arc series-parallel graphs [16], and $\Omega(\log n / \log \log n)$ lower bound on the approximability for acyclic graphs [22]. We have also shown that an integrality gap of the flow LP relaxation is very close to the approximation ratio of the constructed algorithm. This suggests that better approximation algorithms based on rounding the solutions to the flow LP relaxation may not be possible. Closing the gap between the positive and negative approximation results for MIN-MAX SP may require other techniques and is an interesting subject for further research. The first step has been taken in [20] and [22], where randomized polylogarithmic approximation algorithms for this problem, which run in a quasi-polynomial time, have been constructed.

Acknowledgements

The authors were supported by the National Science Centre, Poland, grant 2022/45/B/HS4/00355.

References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin. Network Flows: theory, algorithms, and applications. Prentice Hall, Englewood Cliffs, New Jersey, 1993.

[2] H. Aissi, C. Bazgan, and D. Vanderpooten. Approximation of min-max and min-max regret versions of some combinatorial optimization problems. European Journal of Operational Research, 179:281–290, 2007.

[3] H. Aissi, C. Bazgan, and D. Vanderpooten. Min-max and min-max regret versions of combinatorial optimization problems: a survey. European Journal of Operational Research, 197:427–438, 2009.

[4] S. Arora, A. Frieze, and H. Kaplan. A new rounding procedure for the assignment problem with applications to dense graph arrangement problems. Mathematical Programming, 92:1–36, 2002.

[5] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. Robust optimization. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2009.

[6] V. Bilò, I. Caragiannis, A. Fanelli, M. Flammini, and G. Monaco. Simple greedy algorithms for fundamental multidimensional graph problems. In I. Chatzigiannakis, P. Indyk, F. Kuhn, and A. Muscholl, editors, 44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, volume 80 of LIPIcs, pages 125:1–125:13, 2017.

[7] C. Büsing. Recoverable robust shortest path problems. Networks, 59:181–189, 2012.

[8] V. G. Deineko and G. J. Woeginger. Complexity and in-approximability of a selection problem in robust optimization. 4OR - A Quarterly Journal of Operations Research, 11:249–252, 2013.
[9] A. Dolgui and S. Kovalev. Min-max and min-max (relative) regret approaches to representatives selection problem. 4OR - A Quarterly Journal of Operations Research, 10:181–192, 2012.

[10] M. Ehrgott. Multicriteria optimization. Springer, 2005.

[11] M. Ehrgott and X. Gandibleux. A survey and annotated bibliography of multiobjective combinatorial optimization. OR Spectrum, 22:425–460, 2000.

[12] F. Grandoni, R. Ravi, M. Singh, and R. Zenklusen. New approaches to multi-objective optimization. Mathematical Programming, 146:525–554, 2014.

[13] S. Jiang, Z. Song, O. Weinstein, and H. Zhang. A faster algorithm for solving general LPs. In S. Khuller and V. V. Williams, editors, STOC ’21: 53rd Annual ACM SIGACT Symposium on Theory of Computing, pages 823–832. ACM, 2021.

[14] P. Kall and S. W. Wallace. Stochastic Programming. John Wiley and Sons, 1994.

[15] A. Kasperski, A. Kurpisz, and P. Zieliński. Approximability of the robust representatives selection problem. Operations Research Letters, 43:16–19, 2015.

[16] A. Kasperski and P. Zieliński. On the approximability of minmax (regret) network optimization problems. Information Processing Letters, 109:262–266, 2009.

[17] A. Kasperski and P. Zieliński. Robust Discrete Optimization Under Discrete and Interval Uncertainty: A Survey. In Robustness Analysis in Decision Aiding, Optimization, and Analytics, pages 113–143. Springer-Verlag, 2016.

[18] P. Kouvelis and G. Yu. Robust Discrete Optimization and its Applications. Kluwer Academic Publishers, 1997.

[19] S. Li, C. Xu, and R. Zhang. Polylogarithmic approximation for robust s-t path. CoRR, abs/2305.16439, 2023.

[20] S. Li, C. Xu, and R. Zhang. Polylogarithmic approximations for robust s-t path. In K. Bringmann, M. Grohe, G. Puppis, and O. Svensson, editors, 51st International Colloquium on Automata, Languages, and Programming, ICALP 2024, volume 297 of LIPIcs, pages 106:1–106:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2024.

[21] C. Liebchen, M. E. Lübbecke, R. H. Möhring, and S. Stiller. The concept of recoverable robustness, linear programming recovery, and railway applications. In Robust and Online Large-Scale Optimization, volume 5868 of Lecture Notes in Computer Science, pages 1–27. Springer-Verlag, 2009.

[22] Y. Makarychev, M. Ovsiankin, and E. Tani. Approximation algorithms for $\ell_p$-shortest path and $\ell_p$-group steiner tree. In K. Bringmann, M. Grohe, G. Puppis, and O. Svensson, editors, 51st International Colloquium on Automata, Languages, and Programming, ICALP 2024, volume 297 of LIPIcs, pages 111:1–111:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2024.

[23] Y. Makarychev, M. Ovsiankin, and E. Tani. Approximation algorithms for $\ell_p$-shortest path and $\ell_p$-group steiner tree. CoRR, abs/2404.17669, 2024.
A Appendix

Proof of Proposition 7. Consider an instance of Min-Max SP presented in Figure 2. We call arcs with zero costs under every scenario dummy arcs – see the dashed ones. We see at once that $x_{e_i}^* = \frac{1}{2}, i \in [4], x_{f_i}^* = \frac{1}{2}, i \in [4]$, is a feasible solution to $\mathcal{L}P(1)$ (here the smallest value of $C$ for which $\mathcal{L}P(C)$ is feasible is equal to 1, i.e. $C^* = 1$) with the constraints (11)-(12) and every integral solution for this instance has the maximum cost over $U(0)$ equal to 2. Hence the integrality gap of the linear relaxation $\mathcal{L}P(C)$ with (11)-(12), for this instance, is at least 2. Clearly, graph $G(0)$ is arc series-parallel.

We now gradually increase the gap. A new instance of Min-Max SP, i.e. graph $G^{(1)} = (V(1), E^{(1)})$ with a scenario set $U^{(1)}$, is build in the following way. We replace every arc $e_i, i \in [4]$, in $G(0)$ (see Figure 2a) by the graph $G_{e_i}^{(0)}$, obtaining $G^{(1)}$, where $|E^{(1)}| = 4|E(0)| + 4, |V^{(1)}| = 4(|V(0)| - 2) + 7$. The graph $G^{(1)}$ is arc series-parallel. Then we
construct scenario set $\mathcal{U}^{(1)}$ as follows. We replace two values of 1 in every scenario $\xi^{(0)} \in \mathcal{U}^{(0)}$ by two matrices $\Xi_1^{(1)}$ and $\Xi_2^{(1)}$ of the size $|E^{(0)}| \times |\mathcal{U}^{(0)}|^2$, respectively, where the columns of the matrices form the Cartesian product $\mathcal{U}^{(0)} \times \mathcal{U}^{(0)}$, i.e.

$$
\left( \begin{array}{c}
\Xi_1^{(1)} \\
\Xi_2^{(1)}
\end{array} \right) = 
\left( \begin{array}{cccc}
\xi_1^{(0)} & \cdots & \xi_{|E^{(0)}|}^{(0)} \\
\xi_1^{(0)} & \cdots & \xi_{|E^{(0)}|}^{(0)} \\
\xi_2^{(0)} & \cdots & \xi_{|\mathcal{U}^{(0)}|^2}^{(0)} \\
\xi_2^{(0)} & \cdots & \xi_{|\mathcal{U}^{(0)}|^2}^{(0)} \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
\xi_{|E^{(0)}|}^{(0)} & \cdots & \xi_{|\mathcal{U}^{(0)}|^2}^{(0)} \\
\xi_{|E^{(0)}|}^{(0)} & \cdots & \xi_{|\mathcal{U}^{(0)}|^2}^{(0)}
\end{array} \right).
$$

Every value of 0 in $\xi^{(0)}$ that corresponds to arc $e_i$, $i \in [4]$, is replaced by zero matrix $O^{(1)}$ of the size $|E^{(0)}| \times |\mathcal{U}^{(0)}|^2$ and every value of 0 in $\xi^{(0)}$ that corresponds to arc $f_i$, $i \in [4]$, is replaced by zero matrix $O^{(1)}$ of the size $1 \times |\mathcal{U}^{(0)}|^2$ (see Figure 2). Thus $|\mathcal{U}^{(1)}| = |\mathcal{U}^{(0)}|^2 |\mathcal{U}^{(0)}|$. The resulting instance is shown in Figure 3. Note that every $s$-$t$ path in $G^{(1)}$ contains exactly four solid arcs. From the construction of $\mathcal{U}^{(1)}$, it follows that there exists a scenario in which the costs of these four arcs are equal to 1. This is the maximum cost, since each scenario $\xi^{(1)} \in \mathcal{U}^{(1)}$ has exactly four 1’s. Accordingly, every integral solution for this instance has the maximum cost over $\mathcal{U}^{(1)}$ equal to 4. Let $x^* \in (0, 1]^{\mathcal{E}^{(1)}}$ be given by $x_e^* = \frac{1}{4}$ for the solid arcs $e$ in $G^{(1)}$; $x_{f_i}^* = \frac{1}{2}$ for $i \in [4]$ and the components of $x^*$ corresponding to the dashed arcs in $G^{(0)}_i$, $i \in [4]$, are equal to $\frac{1}{2}$. It is easy to check that $x^*$ is feasible to $\mathcal{L}P(C)$ with $111 - 112$, for this instance, is at least 4.

Repeatedly applying the above construction enables us to increase the integrality gap by at least 8. That is, we again replace each solid arc $e_i$, $i \in [4]$, in $G^{(0)}$ (see Figure 2a) by the graph $G^{(1)}$. This leads to the arc series-parallel graph $G^{(2)}$ with $|E^{(2)}| = 4|E^{(1)}| + 4$, $|V^{(2)}| = 4(|V^{(1)}| - 2) + 7$. Then we build $\mathcal{U}^{(2)}$ as follows. We replace two values of 1 in every scenario $\xi^{(0)} \in \mathcal{U}^{(0)}$ by two matrices $\Xi_1^{(2)}$ and $\Xi_2^{(2)}$ of the size $|E^{(1)}| \times |\mathcal{U}^{(1)}|^2$, respectively, where the columns of the matrices form the Cartesian product $\mathcal{U}^{(1)} \times \mathcal{U}^{(1)}$:

$$
\left( \begin{array}{c}
\Xi_1^{(2)} \\
\Xi_2^{(2)}
\end{array} \right) = 
\left( \begin{array}{cccc}
\xi_1^{(1)} & \cdots & \xi_{|E^{(1)}|}^{(1)} \\
\xi_1^{(1)} & \cdots & \xi_{|E^{(1)}|}^{(1)} \\
\xi_2^{(1)} & \cdots & \xi_{|\mathcal{U}^{(1)}|^2}^{(1)} \\
\xi_2^{(1)} & \cdots & \xi_{|\mathcal{U}^{(1)}|^2}^{(1)} \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
\xi_{|E^{(1)}|}^{(1)} & \cdots & \xi_{|\mathcal{U}^{(1)}|^2}^{(1)} \\
\xi_{|E^{(1)}|}^{(1)} & \cdots & \xi_{|\mathcal{U}^{(1)}|^2}^{(1)}
\end{array} \right), \xi^{(1)} \in \mathcal{U}^{(1)}.
$$

Every 0 in $\xi^{(0)}$ that corresponds to arc $e_i$, $i \in [4]$, is replaced by matrix $O^{(2)}$ of the size $|E^{(1)}| \times |\mathcal{U}^{(1)}|^2$ and every 0 in $\xi^{(0)}$ that corresponds to arc $f_i$, $i \in [4]$, is replaced by matrix $O^{(2)}$ of the size $1 \times |\mathcal{U}^{(1)}|^2$ and so $|\mathcal{U}^{(2)}| = |\mathcal{U}^{(1)}|^2 |\mathcal{U}^{(1)}|$. 

Figure 3: An instance of Min-Max SP with the integrality gap of at least 4.
The proof of the proposition is based on repeating the above construction \( r \) times. We get the arc series-parallel graph \( G^{(r)} = (V^{(r)}, E^{(r)}) \), where \( |E^{(r)}| = 4|E^{(r-1)}| + 4 \), \( |V^{(r)}| = 4(|V^{(r-1)}| - 2) + 7 \), and scenario set \( U^{(r)} \) with the cardinality of \( |U^{(r-1)}|^2 |U^{(0)}| \). Clearly, the resulting graph is arc series-parallel. Now, the integrality gap of the linear relaxation \( LP(L) \) with (11)–(12) for the resulting instance is at least \( 2^{r+1} \). Set \( k = 2^{r+1} \), \( K = |U^{(r)}| \), \( m = |E^{(r)}| \) and \( n = |V^{(r)}| \). Since \( |U^{(0)}| = 4 \), \( |E^{(0)}| = 8 \) and \( |V^{(0)}| = 7 \), a trivial verification yields \( K = 4^{2^{r+1}-1} = 4^{k-1} \), \( m = 4^{r+1} + \sum_{p=1}^{r+1} 4^p = \frac{4^k}{3} - \frac{4}{3} \) and \( n = 2 + 5 \sum_{p=0}^{r} 4^p = 2 + \frac{5}{3}(k^2 - 1) \). The latter equation shows that the linear relaxation \( LP(C) \) with (11)–(12) has an integrality gap of at least \( \Omega(\sqrt{n}) \), which completes the proof.