Abstract—In this paper we explore partial coherence as a tool for evaluating causal influence of one signal sequence on another. In some cases the signal sequence is sampled from a time- or space-series. The key idea is to establish a connection between questions of causality and questions of partial coherence. Once this connection is established, then a scale-invariant partial coherence statistic is used to resolve the question of causality. This coherence statistic is shown to be a likelihood ratio, and its null distribution is shown to be a Wilks Lambda. It may be computed from a composite covariance matrix or from its inverse, the information matrix. Numerical experiments demonstrate the application of partial coherence to the resolution of causality. Importantly, the method is model-free, depending on no generative model for causality.

Index Terms—signal, time series, causality, partial coherence, information matrix, likelihood ratio test, Wilks Lambda, ROC curve

I. INTRODUCTION

Questions of causality are fraught with ambiguities. In fact, we might paraphrase Norbert Wiener’s discussion of Newtonian physics to say causality is a concept grounded in a physical picture that can never be fully justified or fully rejected experimentally [1]. For this reason it is important to state at the outset that any conclusions are conditioned on available prima facie evidence and on the method of inference. The emergence of new evidence may render conclusions moot or false. And, more sophisticated methods of inference may reveal dependencies not revealed by other methods. For this reason it is prudent to answer questions of causality in the negative, rather than the affirmative. In other words, statements like $x$ is not a cause for $y$ are preferred over statements like $x$ is a cause for $y$. Moreover, a conclusion based on data is perforce a conclusion based on an experiment and a statistic whose value is used to resolve the question of causality. This means the question can only be resolved with respect to prima facie evidence, the outcome of an experiment, and the value of a statistic. It follows that the question is only resolved at a prescribed significance level. Change the significance level, and change the answer to the question of causality. It might be said that the question of causality is practically a question of null hypothesis testing, where the null hypothesis states that one signal sequence has no causal influence on another. One may still speak of the power of a method to reject the null for a putative causal model, but this does not validate the causal model, nor does it rule out the possibility that other methods and evidence would negate the finding of causality.

Granger causality [2], [3] has been widely used for addressing questions of causality. Loosely speaking, a time series $\{y_n\}$ is causally dependent on another time series $\{x_n\}$ if the past of $\{x_n\}$ contains unique information about $\{y_n\}$. This statement is made precise in terms of joint probability statements concerning the predictive value of a time series when composed with prima facie evidence. Geweke [5] has studied the linear dependence and feedback between two time series and in [6] extended his results to conditional linear dependence in the presence of a third time series. Solo [7] analyses the Granger-Geweke causality measure for a variety of state-space models, and establishes the invariance of this measure to linear time-invariant filtering. Siggiiridou and Kugiumtzis [8] used a conditional Granger causality index to quantify Granger causality in a vector autoregression. Kramer [10] and Amblard and Michel [11] define and analyze directed information to quantify Granger causality in a vector autoregression. Schamberg and Coleman [9] recently proposed a novel sample path dependent measure of causal influence between time series.

Barnett et al. [12], [13], [14] have adapted Granger’s point of view to the study of causality in state-space models. They derive an estimate of the Granger-Geweke causality measure for testing causality within state-space models and demonstrate improved statistical power and reduced bias compared to results obtained by fitting an autoregressive (AR) model to what should be an autoregressive-moving average (ARMA) model.

In this paper, we are interested in assessing causal influence between two signal sequences, using partial correlation or partial coherence. It seems to us that causal influence is more descriptive than causality for describing the aims of statistical methods that are applied to signal sequences without any pretense at identifying or verifying a physical mechanism. As a practical matter of testing, these signal sequences are often finite-dimensional samples of two time series, represented as the random vectors $x$ and $y$. The partial correlation between $x$ and $y$, given $z$, is the correlation between the residual for $x$ regressed on $z$ and the residual for $y$ regressed on $z$, where $z$ is a third set of random variables. A zero partial correlation indicates that the random vector $y$ does not participate in a linear minimum mean-squared error estimator of $x$ from

This work was supported in part by the National Science Foundation under contract CCF-1712788. Corresponding author: Y. Wang (email: yuan.wang@wsu.edu).
(y, z). In the multivariate normal case, x is then conditionally independent of y, given z. We shall define the prima facie evidence z in such a way that the question of causality can be resolved from an analysis of partial coherence.

In a comprehensive treatment of conditioning and partial correlation for vector-valued random variables, with no reference to the question of causality, we identify a quantity we call partial coherence. Partial coherence is a ratio of determinants that is proportional to the volume of a normalized error concentration ellipse for linearly estimating a pair of random variables from a third, or equivalently for linearly estimating one random variable from two others. It is determined by partial canonical correlations. Partial coherence is invariant to block diagonal non-singular transformations. In the case of wide-sense stationary time series, the spectral representation of partial coherence shows it to be a Riemann sum of narrow-band coherences or information rates, in close analogy with Geweke’s spectral representations. With appropriate choice of conditioning, partial coherence is a monotone function of the Granger-Geweke causality measure, which is the same as Geweke’s transfer entropy.

We then turn to the question of causality, following the language and philosophy of Granger, and studying problems of the same general class as the Barnetts and Seth. However, in our study of causality between signal sequences, we adopt a conservative, model-free, characterization of the signals to be analyzed, based only on second-order moments. That is, in contrast to the work of the Barnetts and Seth, where parametric state space models are assumed for the time series under study, we assume only that samples of a time series are second-order random variables. We show that the sampled-data estimate of partial coherence is in fact the model-free ordinary likelihood ratio for testing the null hypothesis that one signal sequence is not correlated with another. The statistic of [12] is an approximate likelihood ratio, based on an approximate ML identification of an underlying state space model for the signals. That is, our estimate is ordinary likelihood in a model-free analysis of causality, and theirs is approximate likelihood in a state-space analysis of causality. Importantly, under the null hypothesis, ordinary likelihood, or estimated partial coherence, is shown to be distributed as a Wilks Lambda statistic, allowing for the setting of thresholds to control significance level in hypothesis testing.

In the examples we analyze, the signal sequences of interest are finite samples from wide-sense stationary time series, but this is not essential to our methods. In fact the methodology we advocate extends to the study of space-time series, as in [15], where the methods of this paper may be used to test for particularly influential time series in a spatially-distributed set of time series. Even more generally, partial coherence may be used to characterize causality between quite arbitrary measurement vectors, as in the labeling of graph edges by Whittaker [16].

When parametric models are accurate, and the ratio of sample count to parameter count is large, then one would expect parametric methods to outperform model-free methods. But when the parametric model is inaccurate, model bias is introduced. In our re-running of the Barnetts and Seth example, we find that the statistical power we achieve is generally comparable to theirs, without using a parametric state-space model. We use a receiver operating characteristic (ROC) to emphasize the point that resolution of causality is always a problem in hypothesis testing, where the power to identify (provisional) causality is dependent upon the significance level of the test, or equivalently upon the probability of incorrectly rejecting the hypothesis of non-causality.

**Notations:** Bold upper case letters denote matrices, boldface lower case letters denote column vectors, and italics denote scalars. The superscript $(\cdot)^T$ denotes transpose, $I_M$ is the identity matrix of dimensions $M \times M$, and $0$ denotes either a zero column vector or the zero matrix of appropriate dimensions. We use $A^{1/2}$ ($A^{-1/2}$) to denote the square root matrix of the Hermitian matrix $A$ ($A^{-1}$); $\text{blkdiag}[A_1, \ldots, A_k]$ is a block-diagonal matrix whose diagonal blocks are $A_1, \ldots, A_k$. The notation $x \sim N_M(0, R)$ indicates that $x$ is an $M$-dimensional Gaussian random vector of mean $0$ and covariance $R$. We say it is multivariate normal (MVN).

The rest of the paper is organized as follows. In Section II we review a few motivating notions of predictive causal influence. In Section III we introduce partial coherence and demonstrate its usefulness in the estimation of one random vector from two others. We review how partial coherence may be extracted from the information matrix, and establish spectral formulas which bring insight into the way partial canonical correlations additively decompose partial coherence. We establish, in the case of multivariate normal experiments, the connection between partial coherence, conditional Kullback-Leibler divergence, conditional separation, and a Granger-Geweke causality measure. In Section IV we show that partial coherence is a scale-invariant likelihood ratio statistic in the case of multivariate normal experiments. Importantly, we establish that the null distribution of the coherence statistic is the distribution of a Wilks Lambda statistic, and show that its stochastic representation is that of a product of independent beta-distributed random variables. In Section V we conduct two experiments to explore the past-causal, future-causal, and mixed-causal relationship between time series. The first of these experiments demonstrates the use of pairwise partial coherence as a map of causality as a function of two time variables. The second reproduces the experiment of Barnett and Seth [12] in order to compare our model-free estimator of causality to their model-based estimator. We compare the power of our estimator of partial coherence to the power of their estimator, and demonstrate the use of the Receiver Operating Characteristic as a way to view the trade-off between power (probability of detection, or one minus type II error probability) versus size (significance level, probability of false alarm, or probability of type I error). In Section VI we offer concluding remarks on the efficacy of model-based versus model-free approaches to null hypothesis testing for causal influence.

**II. Notions of Predictive Causal Influence**

Begin with two scalar-valued time series, whose finite pasts may be denoted $x^t = (x_0, x_1, \ldots, x_t)$ and $y^t = (y_0, y_1, \ldots, y_t)$. A number of definitions of causality have been developed in the literature. Here we focus on the notion of Granger causality, introduced by Clive W. J. Granger in 1969. Granger causality is based on the idea that a cause should predict a consequence. Formally, $x$ is said to cause $y$ if $x$ is a better predictor of $y$ than $y$ is of itself. Mathematically, this can be expressed as:

$$
\text{Var}(y_t | x_0, x_1, \ldots, x_{t-1}, y_0, y_1, \ldots, y_{t-1}) < \text{Var}(y_t | y_0, y_1, \ldots, y_{t-1})
$$

This inequality states that the variance of $y_t$ given the past values of $x$ and $y$ should be smaller than the variance of $y_t$ given only the past values of $y$. In other words, $x$ is a stronger predictor of $y$ than $y$ is of itself. Granger causality is a useful concept in time series analysis, as it allows us to identify potential causal relationships between variables without making strong assumptions about the underlying data generating process.
(y_0, y_1, \ldots, y_t). Assume the existence of a joint probability density function (pdf) p(x^t, y^t), for all t, with corresponding marginal pdfs for any subset of these random variables. A comparison between the conditional pdfs p(y_{t+1} | x^t, y^t) and p(y_{t+1} | y^t) may be used to test the null hypothesis H_0 that x^t has no causal influence on y_{t+1}. Such a comparison might be based on the error variance V_{y_{t+1} | x^t, y^t} of a predictor of y_{t+1} from (x^t, y^t) versus the error variance V_{y_{t+1} | y^t} of a predictor of y_{t+1} from y^t only. There are other possibilities. Conditioned on y^t, are y_{t+1} and x_t correlated? When this question is asked for s = t ± r, then hypotheses of past-causal, future-causal, mixed-causal influence may be answered. These questions are easily generalized to multivariate time series by comparing the error covariance for predictors of a random vector y_{t+1} from random vectors (x^t, y^t).

When prediction error variances are replaced by entropies, then entropy differences may be used to measure mutual information. Define the following entropies H and mutual information I:

\[
H_{y_{t+1}} = \mathbb{E}[-\log p(y_{t+1})], \quad \text{entropy of the random variable } y_{t+1},
\]

\[
H_{y_{t+1} | y^t} = \mathbb{E}[-\log p(y_{t+1} | y^t)],
\]

\[
H_{y_{t+1} | x^t, y^t} = \mathbb{E}[-\log p(y_{t+1} | x^t, y^t)],
\]

\[
I_{y_{t+1} | x^t, y^t} = H_{y_{t+1}, x^t, y^t} - H_{y_{t+1} | x^t} - H_{y_{t+1} | y^t}.
\]

In these expressions, the random variable p(y_{t+1}) is a pdf random variable, and E is expectation. The entropy H_{y_{t+1}} is a measure of uncertainty for the random variable. The other entropies are entropies conditioned on the past y^t or the two pasts (x^t, y^t). The mutual information I_{y_{t+1} | x^t, y^t} measures mutual information between y_{t+1} and x^t, after conditioning on the past y^t. A special case is I_{y_{t+1} | x_s, y_s} which measures the mutual information between y_{t+1} and x_s, after conditioning on the past y^t. By construction the conditional entropies and the conditional mutual information are measures of directed information. For ergodic time series, they may be averaged over t to determine the minimum codeword length for encoding the time series \{(y_0, y_1, \ldots)\}, with or without the side information x^t. Rissanen and Wax [4] are as cautionary as Wiener in their use of the term causality when they say, “We measure causality by predictability, with predictability interpreted in terms of the codeword length required to encode the time series with predictive coding. ... Obviously, the value of the resulting measure depends critically on the class of predictive densities selected. A good selection gives a sharp measure of causal dependence while a bad one masks a possible causal dependence, which of course is just as should be.”

In the framing of questions about past-causal, future-causal, or mixed causal-influence, three random vectors (x, y, z) may be extracted from two signal sequences. For example, x = x^t, y = y_{t+1}, z = y^t. Or x = x_s, y = y_{t+1}, z = y^t. More generally, x is a p-sample from time series \{x_n\}, y is a q-sample from the time series \{y_n\}, and z is an r-sample from either \{x_n\} or \{y_n\}. That is, three finite-dimensional vectors are extracted from two signal sequences. The construction determines the nature of the question of causal influence. This suggests a general framework for analyzing causal influence based on three random vectors x \in \mathbb{R}^p, y \in \mathbb{R}^q, z \in \mathbb{R}^r.

The composite vector u = (x^t, y^t)^T and v = (y^t, z^T)^T have covariance matrix \mathbf{R} which may be parsed two ways:

\[
\mathbf{R} = \begin{bmatrix}
\mathbf{R}_{uu} & \mathbf{R}_{uz} \\
\mathbf{R}_{uz}^T & \mathbf{R}_{zz}
\end{bmatrix}.
\]

By defining the composite vectors u = (x^T, y^T)^T, and v = (y^T, z)^T the covariance matrix \mathbf{R} may be parsed two ways:

\[
\mathbf{R}^{-1} = \begin{bmatrix}
\mathbf{R}_{uu}^{-1} & \mathbf{R}_{uz}^{-1} \\
\mathbf{R}_{uz}^{-1} & \mathbf{R}_{zz}^{-1}
\end{bmatrix} = \mathbf{R}_{xx}^{-1} = \mathbf{R}_{yy}^{-1} + \mathbf{R}_{xz}^{-1}.
\]

\[
\mathbf{R}_{xx|y} = \mathbf{R}_{xx} - \mathbf{R}_{xz} \mathbf{R}_{zz}^{-1} \mathbf{R}_{xz}.
\]

\[
\mathbf{R}_{yy|z} = \mathbf{R}_{yy} - \mathbf{R}_{yz} \mathbf{R}_{zz}^{-1} \mathbf{R}_{yz}.
\]

The matrix \mathbf{R}_{uu|z} is the error covariance matrix for estimating the composite vector u from z, and the matrix \mathbf{R}_{xx|y} is the error covariance matrix for estimating x from v:

\[
\mathbf{R}_{uu|z} = \mathbf{R}_{uu} - \mathbf{R}_{uz} \mathbf{R}_{zz}^{-1} \mathbf{R}_{uz}.
\]

\[
\mathbf{R}_{xx|y} = \mathbf{R}_{xx} - \mathbf{R}_{xz} \mathbf{R}_{yz} \mathbf{R}_{zz}^{-1} \mathbf{R}_{yz}.
\]

We shall have more to say about these error covariance matrices in due course. Importantly, the inverses of each may partial coherence may be used as a tool for testing hypotheses of causal influence. It encodes for conditional prediction error covariance. For zero-mean Gaussian time series, in which case all finite-dimensional random vectors are multivariate normal (MVN), partial coherence encodes for conditional entropies, and mutual information. These points will be further clarified in due course.

III. PREDICTIVE CAUSALITY, LINEAR REGRESSION, AND PARTIAL COHERENCE

There are two ways to frame questions of causal influence: as a question of estimating two random vectors from one other, which is to say regressing two random vectors onto one random vector; or as a question of estimating one random vector from two others, which is to say regressing one random vector onto two random vectors. In the first case the random vectors (x, y) are to be regressed onto the random vector z; in the second case, the random vector x \in \mathbb{R}^p is to be regressed onto the random vectors y \in \mathbb{R}^q and z \in \mathbb{R}^r. The composite covariance matrix for these three random vectors is

\[
\mathbf{R} = \begin{bmatrix}
\mathbf{R}_{uu} & \mathbf{R}_{uz} \\
\mathbf{R}_{uz}^T & \mathbf{R}_{zz}
\end{bmatrix}.
\]
be read out of the inverse for the composite covariance matrix $R^{-1}$ of eqn[3]. The dimension of the error covariance $R_{uu|z}$ is $(p+q) \times (p+q)$ and the dimension of the error covariance $R_{xx|v}$ is $p \times p$.

A. Regressing two random vectors onto one: partial correlation and partial canonical correlation coefficients

The estimators of $x$ and $y$ from $z$, and their resulting error covariance matrices are easily read out from the composite covariance matrix $R$:

$$x(z) = R_{xx}^{-1}R_{xz}z,$$

$$R_{xx|z} = E[(x - \hat{x})(x - \hat{x})^T] = R_{xx} - R_{xz}R_{zz}^{-1}R_{xz}^T,$$

$$y(z) = R_{yz}R_{zz}^{-1}z,$$

$$R_{yy|z} = E[(y - \hat{y})(y - \hat{y})^T] = R_{yy} - R_{yz}R_{zz}^{-1}R_{yz}^T.$$

(6)

(7)

(8)

The composite error covariance matrix for the errors $x - \hat{x}(z)$ and $y - \hat{y}(z)$ is the matrix

$$R_{uu|z} = E \left[ \begin{array}{c} x - \hat{x}(z) \\ y - \hat{y}(z) \end{array} \right]^T \left( x - \hat{x}(z) \right) = R_{xxz} \begin{bmatrix} R_{xx|z} & R_{xy|z} \\ R_{yx|z} & R_{yy|z} \end{bmatrix},$$

$$R_{xy|z} = E[(x - \hat{x})(y - \hat{y})] = R_{xy} - R_{xz}R_{zz}^{-1}R_{yz}^T.$$

(10)

(11)

In this matrix the matrix block $R_{xy|z}$ is the $p \times q$ matrix of cross correlations between the random errors $(x - \hat{x}(z))$ and $(y - \hat{y}(z))$. It is called the partial correlation between the random vectors $x$ and $y$, after each has been regressed onto the common vector $z$. Under the null hypothesis $H_0$, this partial correlation is zero, and therefore $R_{uu|z}$ is the block-diagonal matrix $\text{blkdiag}(R_{xx|z}, R_{yy|z})$.

Formula (5) shows that $R_{uu|z}$ and its determinant may be read directly out of the $(p+q) \times (p+q)$ Northwest block of the inverse of the error covariance matrix:

$$\begin{bmatrix} R^{-1} \end{bmatrix}_{NW} = R_{uu|z}^{-1} \quad \text{and} \quad \det[R_{uu|z}] = \frac{1}{\det[\begin{bmatrix} R^{-1} \end{bmatrix}_{NW}]].$$

(12)

This result was known to Harold Cramer more than 70 years ago [13], and is featured prominently in the book on graphical models by Whittaker [16]. The consequence is that the ingredients of a partial coherence statistic we shall compute from experimental data may be read out of the Northwest blocks of inverse sample covariance matrices.

The error covariance matrix $R_{uu|z}$ may be pre- and post-multiplied by $\text{blkdiag}(R_{xx|z}^{-1/2}, R_{yy|z}^{-1/2})$ to produce the normalized error covariance matrix and its corresponding determinant:

$$Q_{uu|z} = \begin{bmatrix} I & R_{xx|z}^{-1/2}R_{xy|z}R_{yy|z}^{-1/2} \\ R_{xx|z}^{-1/2}R_{xy|z} & R_{yy|z}^{-1} \end{bmatrix},$$

$$\det[Q_{uu|z}] = \frac{\det[R_{uu|z}]}{\det[R_{xx|z}]\det[R_{yy|z}] = \det[I - C_{xy|z}C_{xy|z}^T],}$$

$$C_{xy|z} = R_{xx|z}^{-1/2}R_{xy|z}R_{yy|z}^{-1/2}.$$  

(13)

(14)

(15)

The NE matrix block of the covariance matrix $Q_{uu|z}$ is the partial coherence matrix $C_{xy|z}$. The determinant of $Q_{uu|z}$ is proportional to the volume of the normalized error concentration ellipse for estimating $u$ from $z$. When specialized for causal influence, this gives an interpretation of the Granger-Geweke causality measure that is consistent with, but slightly different than, the discussion given in [12]. Under the null hypothesis $H_0$, the normalized error covariance matrix is identity, $Q_{uu|z} = I_{p+q}$, its determinant is one, and the partial coherence matrix is the zero matrix, $C_{xy|z} = 0_{p \times q}$.

We define partial coherence to be

$$\rho_{xy|z}^2 = \frac{1 - \text{det}[Q_{uu|z}]}{\text{det}[R_{uu|z}] \text{det}[R_{yy|z}]} = 1 - \text{det}[I - C_{xy|z}C_{xy|z}^T].$$

(16)

Define the SVD of the partial coherence matrix to be $C_{xy|z} = FKG^T$, where $F$ is a $p \times p$ orthogonal matrix, $G$ is a $q \times q$ orthogonal matrix, and $K$ is a $p \times q$ diagonal matrix of partial canonical correlations. The matrix $K$ may be called the partial canonical correlation matrix. The normalized error covariance matrix of eqn (13) may be written as

$$Q_{uu|z} = \begin{bmatrix} F & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} I & K \\ K^T & I \end{bmatrix} \begin{bmatrix} F^T & 0 \\ 0 & G^T \end{bmatrix}.$$  

(17)

As a consequence, partial coherence may be factored as

$$\rho_{xy|z}^2 = 1 - \text{det}[I - KKF] = 1 - \prod_{i=1}^{\min(p,q)} (1 - k_i^2).$$

(18)

The partial canonical correlations $k_i$ are bounded between 0 and 1, as is partial coherence. When the squared partial canonical correlations $k_i^2$ are near to zero, then partial coherence $\rho_{xy|z}^2$ is near to zero, indicating linear independence of $x$ and $y$, given $z$.

These results summarize the error analysis for linearly estimating the random vectors $x$ and $y$ from a common random vector $z$. The only assumption is that the random vectors $(x, y, z)$ are second-order random vectors. By constructing the random vectors $x, y, z$ appropriately, we shall answer questions of second-order causal influence.

B. Regressing one random vector onto two: partial correlation and partial canonical correlation coefficients

Suppose now that the random vector $x$ is to be linearly regressed onto $v = (y, z)$:

$$\hat{x}(v) = \begin{bmatrix} R_{xy} & R_{xz} \end{bmatrix} \begin{bmatrix} R_{yy} & R_{yz} & R_{zz} \end{bmatrix}^{-1} \begin{bmatrix} y \\ z \end{bmatrix}.$$  

(19)

Give the matrix inverse in this equation, the following block diagonal LDU factorization:

$$\begin{bmatrix} R_{yy} & R_{yz} \\ R_{yz} & R_{zz} \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -R_{zz}^{-1}R_{yz}^T & I \end{bmatrix} \begin{bmatrix} R_{yy}^{-1} & 0 \\ 0 & R_{zz}^{-1} \end{bmatrix} \begin{bmatrix} I & -R_{yz}R_{zz}^{-1} \\ 0 & I \end{bmatrix}.$$  

(20)
A few lines of algebra produce this result for \( \hat{x}(v) \), the linear minimum mean-squared error estimator of \( x \) from \( v \):

\[
\hat{x}(v) = \hat{x}(z) + R_{xy|z}^{-1} R_{yx|z} y - \hat{y}(z). \tag{21}
\]

It is evident that the vector \( y \) is not used in a linear minimum mean-squared error estimator of \( x \) when the partial covariance \( R_{xy|z} \) is zero. That is, the random vector \( y \) brings no useful second-order information to the problem of linearly estimating \( x \).

The error covariance matrix for estimating \( x \) from \( v \) is easily shown to be

\[
R_{xx|v} = E[(x - \hat{x}(v)(x - \hat{x}(v))^T] = R_{xx} - R_{xy} R_{yy}^{-1} R_{yx}. \tag{22}
\]

Thus the error covariance \( R_{xx|z} \) is reduced by a quadratic form depending on the covariance between the errors \( (x - \hat{x}(z)) \) and \( (y - \hat{y}(z)) \). If this error covariance is now normalized by the error covariance matrix achieved by regressing only on the vector \( z \), the result is

\[
P_{xx|z} = R_{xx|z}^{-1/2} R_{xx} R_{xx|z}^{-1/2} = I - R_{xy|z} R_{yy|z}^{-1} R_{yx|z} R_{xx|z}^{-1/2} = I - C_{xy|z} C_{yx|z}^{-1} = F(I - KK^T)F^T. \tag{23}
\]

The determinant of this matrix measures the volume of the normalized error covariance matrix:

\[
det[P_{xx|v}] = det[R_{xx}] / det[R_{xx|z}] = \min_{p,q} R_{xx|z}^{-1} \prod_{i=1}^p (1 - k_i^2). \tag{24}
\]

As before, we may define a partial coherence

\[
\rho_{x|z}^2 = 1 - det[P_{xx|v}] = 1 - \prod_{i=1}^p (1 - k_i^2). \tag{25}
\]

Importantly the partial canonical correlations \( k_i \) are invariant to transformation of the random vector \((x^T,y^T,z^T)^T\) by a block diagonal, non-singular, matrix \( T = \text{blkdiag}(T_{xx}, T_{yy}, T_{zz}) \). As a consequence partial coherence is invariant to transformation \( T \). This result is the finite-dimensional version of Solo’s finding \([7]\) that the Granger-Geweke causality measure is invariant to uncoupled linear filtering of wide-sense stationary time series. A slight variation on Proposition 10.6 in \([17]\) shows partial canonical correlations to be maximal invariants under group action \( T \).

C. Spectral formulas

There are spectral formulas. Begin with wide-sense stationary scalar time series \( \{x_n\}, \{y_n\}, \{z_n\} \), from which error time series are computed. Assume the error covariance matrices \( R_{xx|z}, R_{xy|z}, \) and \( R_{yx|z} \) are Toeplitz matrices constructed from the scalar time series of error variances, \( \{R_{xx|z}[n]\}, \{R_{xy|z}[n]\}, \) and \( \{R_{yx|z}[n]\} \). Then the formulas of Szego \([19]\) and \([20]\) may be applied to show that in the limit \( p,q \to \infty \),

\[
\hat{\rho}_{zy}^2 = 1 - \exp\left\{\int_0^{2\pi} \log\left(1 - \hat{k}^2(e^{j\theta})\right) d\theta \right\}, \tag{26}
\]

\[
\hat{k}^2(e^{j\theta}) = \frac{|\hat{R}_{xy}(e^{j\theta})|^2}{R_{xx}(e^{j\theta})R_{yy}(e^{j\theta})}. \tag{27}
\]

where \( \hat{R}_{xx}(e^{j\theta}) \) is the discrete-time Fourier transform (DTFT) of \( \{R_{xx}[n]\} \), \( \hat{R}_{yy}(e^{j\theta}) \) is the DTFT of \( \{R_{yy}[n]\} \), and \( \hat{R}_{xy}(e^{j\theta}) \) is the DTFT of \( \{R_{xy}[n]\} \). This formula generalizes to vector-valued time series \([15]\). We may say the narrowband partial canonical correlations \( k(e^{j\theta}) \) resolve the broadband partial coherence \( \rho_{zy}^2 \) in direct correspondence to eqn \([18]\). Similar formulas may be found in \([21]\). The partial canonical correlation spectrum \( \{\hat{k}(e^{j\theta})\}, 0 < \theta < 2\pi \) is invariant to time-invariant linear filtering of the wide-sense stationary time series \( \{x_n\}, \{y_n\}, \{z_n\} \).

D. Kullback-Leibler Divergence, Conditional Mutual Information, Conditional Information Rate, Partial Coherence, and Granger-Geweke Causality Measure

Begin with the random vectors \((x,y,z)\). The Kullback-Leibler divergence between two probability laws \( P_1 \) and \( P_0 \) for these is defined to be

\[
D_{KL}(P_1||P_0) = \mathbb{E}_{P_1} \log \frac{p_1(x,y,z)}{p_0(x,y,z)}. \tag{28}
\]

where \( p_1(x,y,z) \) is the probability density function under probability law \( P_1 \). We may think of the random variables \( p_1(x,y,z) \) and \( p_0(x,y,z) \) as pdf random variables or belief state random variables. Their ratio \( p_1(x,y,z)/p_0(x,y,z) \) is a likelihood ratio random variable. So KL divergence may be considered the expected value of the log-likelihood ratio for testing the model \( P_1 \) against the model \( P_0 \), under the hypothesis that measurements are drawn from the probability law \( P_1 \).

When \((x,y,z)\) is composed as \((u,z)\), then \( p(x,y,z) = p(u,z)p(z) \) and the ratio is \( p_1(u,z)/p_0(u,z) \), provided the marginal distribution of \( z \) is the same under probability laws.
$P_1$ and $P_0$. In the case where $(u, z)$ is multivariate normal (MVN) with mean $0$ and the covariance matrix $R$ of eqn (2), it is not hard to show that the Kullback-Leibler divergence is

$$D_{KL}(P_1 || P_0) = -\frac{1}{2} \log \det [Q_{uu|z}].$$

That is, after regression of $x$ and $y$ onto $z$, the problem is reset to the analysis of the errors $(x - \hat{x}(z))$ and $(y - \hat{y}(z))$, and their partial coherence. In the MVN case, KL divergence for this problem is also the conditional mutual information, $I_{xy|z} = \sum \log \frac{P(x, y | z)}{P(x | z)P(y | z)}$ between the random vectors $x$ and $y$, given $z$. Thus KL divergence is also Shanon’s definition of the rate $R$ at which the MVN error $y - \hat{y}(z)$ brings information about the MVN error $x - \hat{x}(z)$. This is also the rate at which $y$ brings information about $x$, given $z$, and a monotone function of what the Granger-Geweke causality measure would say for finite-dimensional random vectors $(x, y, z)$. So partial coherence, conditional KL divergence, conditional mutual information, conditional information rate, and a Granger-Geweke causality measure are monotone functions of each other. All are resolved by the partial canonical correlations $k_i$:

$$D_{KL}(P_1 || P_0) = I_{xy|z} = -\frac{1}{2} \log (1 - \rho_{xy|z}^2)$$

$$= \frac{1}{2} \sum_{i=1}^{p} \log (1 - k_i^2)$$

These finite-dimensional connections are consistent with the finding in [13] that the Granger-Geweke causality measure and transfer entropy are equivalent for MVN variables. These equivalent measures are invariant to block diagonal singular transformations of the random vectors $x, y, z$, and they are functions of the partial canonical correlations $k_i$, which are maximal invariants.

When the partial canonical correlations $k_i$ are near to zero, indicating the coherence matrix $C_{xy|z}$ is near to the zero matrix, then partial coherence is near to zero, and conditional KL divergence, conditional mutual information, and conditional rate are near to zero. The Granger-Geweke causality measure is near to one, and its transfer entropy is near to zero.

IV. PARTIAL COHERENCE, LIKELIHOOD, AND A TEST FOR CAUSALITY

Partial coherence is inherently connected to likelihood in the case where the random vectors $(x, y, z)$ are jointly normally distributed. The matrix $R_{uu|z}$ is the conditional error covariance matrix for the error $(u - \hat{u}(z))$; when the partial correlation $R_{xy|z}$ is zero, then $R_{uu|z} = \text{blkdiag}[R_{xx|z}, R_{yy|z}]$.

The null hypothesis test of interest is $H_0 : (u - \hat{u}(z)) \sim N_{p+q}(0, \text{blkdiag}[R_{xx|z}, R_{yy|z}])$ versus the alternative $H_1 : (u - \hat{u}(z)) \sim N_{p+q}(0, R_{uu|z})$.

A. Likelihood

Given the data matrix $D = (X^T, Y^T, Z^T) = (U^T, T^T)$, where each of the matrices $X \in \mathbb{R}^{p \times M}, Y \in \mathbb{R}^{q \times M}, Z \in \mathbb{R}^{r \times M}$ consists of $M$ i.i.d. realizations of the jointly multivariate normal random vectors $x, y, z$, compute the sample covariance matrix $S = DD^T$, structured as follows:

$$S = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix} = \begin{bmatrix} S_{uu} & S_{uz} \\ S_{zu} & S_{zz} \end{bmatrix}.$$ (32)

For example, $S_{xy} = XY^T$.

The ordinary likelihood for testing $H_0$ versus $H_1$ is then

$$1 - \rho_{xy|z}^2 = -\frac{\det[S_{uu|z}]}{\det[S_{xx|z}] \det[S_{yy|z}]}$$

where

$$S_{uu|z} = S_{uu} - S_{uz}S_{zz}^{-1}S_{uz}^T,$$ (34)

$$S_{xx|z} = S_{xx} - S_{xz}S_{zz}^{-1}S_{xz}^T,$$ (35)

$$S_{yy|z} = S_{yy} - S_{yz}S_{zz}^{-1}S_{yz}^T.$$ (36)

Of course, as noted previously, the matrix $S_{uu|z}$ may be read out of the inverse $S^{-1}$ and used to compute the determinant $\det[S_{uu|z}]$.

The following proposition establishes the distribution, and a stochastic representation, for the likelihood ratio statistic $\hat{\rho}_{xy|z}^2$ under the null hypothesis. This result forms the basis for setting a detection threshold to achieve a desired type-I error.

Proposition 1. Under the null hypothesis $H_0$ that the coherence matrix $C_{xy|z}$ is the zero matrix, the likelihood ratio statistic $1 - \hat{\rho}_{xy|z}^2$ has the Wilks $\Lambda$ distribution, $\Lambda(p, M - r - q, q)$ with stochastic representation

$$\hat{\rho}_{xy|z}^2 \sim 1 - \prod_{i=1}^{p} b_i,$$

where the $b_i$ are independent Beta random variables with $\text{Beta}(\frac{M - r - q + 1}{2}, \frac{q}{2})$ distributions.

Proof. Let $U_1 = X - XZ^T(ZZ^T)^{-1}Z, U_2 = Y - YZ^T(ZZ^T)^{-1}Z$, and $U = [U_1^T, U_2^T]^T$. It follows that $UU^T = S_{uu|z}, U_1U_1^T = S_{xx|z}, U_2U_2^T = S_{yy|z}$, and therefore,

$$1 - \hat{\rho}_{xy|z}^2 = -\frac{\det(UU^T)}{\det(U_1U_1^T) \det(U_2U_2^T)}.$$ (37)

Khishargar [23] has shown that the error covariance $UU^T$ has a Wishart distribution $W_{p+q}(R_{uu|z}, M - r)$ where $r$ is the dimension of $z$. The statistic $\frac{\det(UU^T)}{\det(U_1U_1^T) \det(U_2U_2^T)}$ is invariant to left multiplication of $(X, Y)$ by a block-diagonal matrix $[T_1 \ 0 \ 0 \ T_2]$ for non-singular $p \times p$ matrix $T_1$ and $q \times q$ matrix $T_2$. Choose $T_1 = R_{xx|z}^{1/2}$ and $T_2 = R_{yy|z}^{1/2}$, so that WLOG $U \sim \mathcal{N}(0, I_{p+q} \otimes I_{M-r})$ which yields $U_1 \sim \mathcal{N}(0, I_{p} \otimes I_{M-r})$ and $U_2 \sim \mathcal{N}(0, I_{q} \otimes I_{M-r})$. Moreover, $U_1$ and $U_2$ are independent under the null hypothesis.

Let $V, V^L \in \mathbb{R}^{(M-r) \times (M-r)}$ be the orthogonal matrix where $V \in \mathbb{R}^{(M-r) \times (M-r)}$ is a unity basis for the row space of $U_2$ and $V^L \in \mathbb{R}^{(M-r) \times (M-r) - q}$ is the unity basis for its orthogonal space. We can choose $V = U_2^T(U_2U_2^T)^{-1}U_2$.
and define $W_1 = U_1V$ and $W_2 = U_1V^\perp$. Conditional on $V$ and $V^\perp$, it can be shown that $W_1 \sim \mathcal{N}(0, I_p \otimes I_q)$, and $W_2 \sim \mathcal{N}(0, I_p \otimes I_{M-r-q})$ (assuming $M - r > p + q$) which are invariant to $V$. Simple linear algebra leads to

$$1 - \rho_{xyz}^2 = \frac{\det(W_2 W_2^T)}{\det(W_1 W_1^T + W_2 W_2^T)}.$$  

This has the Wilk's $\Lambda$ distribution $\Lambda(p, M - r - q, q)$ [24], which is identical to the product of $p$ independent beta-distributed random variables:

$$1 - \rho_{xyz}^2 \sim \prod_{i=1}^{p} b_i \text{ where } b_i \sim \text{Beta} \left( \frac{M - r - q + i - 1}{2}, \frac{q}{2} \right).$$

It is important to note that this distribution is invariant to the actual error covariances $R_{xx|z}$ and $R_{yy|z}$, which are themselves functions of the underlying composite covariance matrix $R$ of eqn (2). Moreover, this distribution result specializes to the case $r = 0$, in which case partial coherence is ordinary multivariate coherence, with distribution $1 - \Lambda(1, M - q - q)$. In [25] Bartlett derived the asymptotic distribution for large $M$,

$$-(M - r - \frac{p + q + 1}{2}) \log(1 - \rho_{xyz}^2) \sim \chi^2_{pq},$$  

where $\chi^2_{pq}$ denotes a chi-squared distribution with $pq$ degrees of freedom.

V. EXPERIMENTS

If partial coherence is to be used to answer questions of causal influence between signal sequences, then there is required a principled way to map a causality question into a question of conditional dependence among random vectors.

Our approach to causality analysis of time series will be to start with two channels of signals, call them $\{x_n\}$ and $\{y_n\}$, and then to define a third signal channel $\{z_n\}$ of prima facie evidence in such a way that the question of causality may be resolved from an analysis of partial coherence. For example the random vectors $\{x, y, z\}$ might be $x = x_s$ and $y = y_t$, which are readings of the signal sequences at the two times $s$ and $t$. In some applications, the prima facie evidence $z$ might be defined as the past of the time series $\{y_n\}$ at time $t$, i.e., $z = \{y_n, n < t\}$. In other applications the prima facie evidence might be $z = \{x_n, n < t\}$. As a practical matter in testing from experimental data, this past will be finite. More commonly, $x$ is assembled from the finite past of $\{x_n\}$ up to time $t - 1$, $y$ is defined to be $y_t$, and $z$ is assembled from the finite past of $\{y_n\}$ up to time $t - 1$. With $x, y, z$ so defined, we can compute the partial coherence statistic $\rho_{x,y|z}^2(s, t) = \rho_{xy|z}^2$, or more generally $\rho_{xy|z}^2$. The statistic $\rho_{x,y|z}^2(s, t)$ vs the pair $(s, t)$ reveals pairwise causality with respect to prima facie evidence $z$. The statistic $\rho_{xy|z}^2$ reveals the causal influence of the finite past of $\{x_n\}$ on $y_t$, conditioned on the finite past of $\{z_n\}$.

A value of partial coherence below a threshold is taken to be a finding of non-causality at significance level $\alpha$, whereas a value of coherence above this threshold is taken only to be an indication of causality with respect to this specific prima facie evidence and a second-order analysis of partial coherence. New evidence and/or new methods of analysis may invalidate an indication of causality. In the multivariate normal case, no new evidence or method of analysis can invalidate the finding of non-causality, as we have found prima facie evidence and a method of analysis that indicates non-causality at significance level $\alpha$. An equivalent view is that the likelihood ratio test is a null hypothesis test for non-causality.

To demonstrate the use of partial coherence for resolving the question of causality, we present in this section a demonstration and an experiment. The demonstration is an analytical calculation of partial coherence for a known bi-variate time series which may model past-causal, future-causal, or mixed-causal influence of one time series upon the other, depending on the choice of coupling parameters between the two time series. The experiment is a re-running of the Barnett and Seth experiment [12].

A. Demonstration of Partial Coherence for Past-causal, Future-causal, and Mixed-Causal Models

Consider the following multivariate linear system,

$$x_n = \sum_{k=-\infty}^{+\infty} H_k \mu_{n-k},$$

$$y_n = \sum_{k=-\infty}^{+\infty} G_k \nu_{n-k} + \sum_{k=-\infty}^{+\infty} F_k x_{n-k},$$

where $\mu, \nu$ are multivariate unit variance white noises. The two time series are correlated with auto- and cross-covariance matrices

$$R_{xx}[m] = \mathbb{E}[x_n x_{n+m}] = \sum_k H_k H_{k+m},$$

$$R_{yy}[m] = \mathbb{E}[y_n y_{n+m}] = \sum_k G_k G_{k+m} + \sum_{k,l} F_k R_{xx}[m + k - l] F_l,$$

$$R_{xy}[m] = \mathbb{E}[x_n y_{n+m}] = \sum_k F_k R_{xx}[m - k].$$

When these infinite-dimensional moving average (MA) models arise as finitely-parameterized autoregressive-moving average (ARMA) models, then these infinite sums may be evaluated analytically.

For our demonstration of partial coherence, we assume the time series are scalar-valued. We set the filter coefficients at $h_k = h_0a^k$ and $g_k = g_0b^k$ for $k \geq 0$ where $h_0 = 0.8$, $a = 0.1$, $g_0 = 0.7$, and $b = 0.7$. The coefficients $\{f_k\}$ are the filtering coefficients that determine the causality between $\{y_n\}$ and $\{x_n\}$. For any selected pair of times $s, t$, we evaluate the partial coherence between $x_s$ and $y_t$. We consider two sets of prima facie evidence: $z$ is the past of $\{y_n\}$ up to time $t - 1$, and $z$ is the past of $\{x_n\}$ up to time $t$ with $x_s$ excluded. Partial coherence is therefore $\rho_{xy|z}^2(s, t) \equiv \rho_{xy|z}^2$.

We consider three different scenarios. Case I: conditioned on the finite past of the input $x$, the output $y$ is linearly
independent of any input outside this finite past which includes
the future of \( x \); Case II: conditioned on the finite future of
the input \( x \) the output \( y \) is linearly independent of any input
outside this finite future which includes the past of \( x \); Case
III: conditioned the finite past and finite future of the input \( x \),
the output \( y \) is independent of any inputs outside this finite
past and future.

**Case I**: \( \{y_n\} \) is past-causally dependent on \( \{x_n\} \):

\[
y_n = 7 \sum_{k=0}^{+\infty} (0.7)^k \nu_{n-k} + 0.8x_n + 0.7x_{n-1} + 0.6x_{n-2} + 0.6x_{n-3}
y_n = 8 \sum_{k=0}^{+\infty} (1)^k \mu_{n-k}.\]

**Case II**: \( \{y_n\} \) is future-causally dependent on \( \{x_n\} \):

\[
y_n = 7 \sum_{k=0}^{+\infty} (0.7)^k \nu_{n-k} + 0.8x_n + 0.7x_{n+1} + 0.3x_{n+3}
y_n = 8 \sum_{k=0}^{+\infty} (1)^k \mu_{n-k}.\]

**Case III**: the mixed case:

\[
y_n = 7 \sum_{k=0}^{+\infty} (0.7)^k \nu_{n-k} + 0.8x_n + 0.7x_{n-2} + 0.8x_{n-1}
+ 0.7x_n + 0.3x_{n+1} + 0.4x_{n+2}
+ \sum_{k=0}^{+\infty} (1)^k \mu_{n-k}.\]

The resulting pairwise partial coherence maps are illustrated
in Figure [I]. The three rows show the results of case I, II, and
III, respectively. Each panel of Figure [I] illustrates the partial
coherence between \( y_t \) and \( x_s \), with the prima facie evidence
being the past of \( \{x_n\} \) on the left column, and the past of
\( \{y_n\} \) on the right column. As we already know, the partial
coherence are conditioned on available prima facie evidence.

With \( z = \{x_t, x_{t-1}, \ldots\} \), (Notice that \( x_{s} \) is excluded from \( z \)
if \( s \leq t \), \( \rho_{xy}^2(s,t) = 0 \) for any \( s > t \) or \( s < t - 3 \) in Case
I (top row), which suggests that future and long-past values
of \( \{x_n\} \) do not contain any second-order information about
\( y_t \) given this \( z \) (In fact, they are conditionally independent
in the MVN case). In Case II the future-causal case (middle
row), \( \rho_{xy}^2(s,t) = 0 \) for any \( s \leq t \) since in this case the past
of \( \{x_n\} \) does not contain unique information about \( y_t \) given
\( z \). In Case III (bottom row) the mixed case where \( y_t \) depends
on both the past and future of \( x \), \( \rho_{xy}^2(s,t) > 0 \) for both \( s \leq t \)
and \( s > t \). In summary, there is an asymmetry in the past-
causal and future-causal cases that bears comment. In Case
I, \( \rho_{xy}^2(s,t) \) for \( s < t \) is non-zero, but weak, compared with
\( \rho_{xy}^2(s,t) \) for \( s > t \) in Case II. This effect is explained by
noting that for these demonstrations our conditioning is always
on the past of \( \{x_n\} \) at time \( t \). The right column of Figure [I]
shows the partial coherence between \( y_t \) and \( x_s \) given the past
of \( \{y_n\} \), i.e., \( z = \{y_{t-1}, y_{t-2}, \ldots\} \). At the past cases are Granger causals, the choice
of simultaneity is the past of input time series \( \{x_n\} \). When
the prima facie is the past of \( \{y_n\} \), \( x_n \) still contains some
unique information about \( y_t \) due to the infinite memory of
\( \{x_n\} \). While all three cases are Granger causals, the choice
of simultaneity is the past of input time series \( \{x_n\} \) leads
to clear separation between past-causal, future-causal, and the
mixed-causal influence, whereas the conventional choice of
prima facie evidence as the past of the output time series \( \{y_n\} \)
does not resolve the direction of causal influence.

**B. The experiment of Barnett, et al.**

In this subsection, we replicate the experiment in Barnett
et al. [12] and use the partial coherence statistic to test
for time-domain causality. As in [12], we define \( y = y_t, \)
\( z = [y_{t-1}, y_{t-2}, \ldots] \), and \( x = [x_{t-1}, x_{t-2}, \ldots] \). The causality
from time series \( \{x_n\} \) to \( \{y_n\} \) is measured by the partial
coherence between \( x \) and \( y \) given \( z \). Straightforward calculation
shows that partial coherence \( \rho_{xy}^2(z,w) \) is a monotone function
of the causality measure in [12]. However, as a statistic,
our estimation of the causality measure is very different. The test
statistic in Barnett et al. is computed by replacing state-space
parameters by their approximate maximum likelihood esti-
mates, and inserting these estimates into an analytical formula
for their ratio of determinants. The result is an approximate
likelihood ratio for an ARMA model of measurements. Our
test statistic is the ordinary likelihood ratio for testing the
null hypothesis that partial correlation is zero. It is model-
free, requiring only the computation of partial coherence
from sample correlations. To compare the two approaches,
we replicate the experiment in [12] by generating bivariate
ARMA \((r,1)\) time series data.
sample covariance matrices for these vectors are just determining these \( \{z_t\} \) up to time \( t - 1 \).

Start with a bivariate auto-regressive (AR) time series \( \{[\eta_1 t, \eta_2 t]^T\} \),

\[
(1 - az_1)\eta_1 t = cz_2 \eta_2 t + \mu t,
\]

\[
(1 - bz_1)\eta_2 t = \nu t,
\]

where \( z \) is the backward shift or delay operator: \( z\{\eta_t\} = \{\eta_{t-1}\} \). The bivariate auto-regressive moving average (ARMA) time series are generated as

\[
\begin{bmatrix}
  y_t \\
  x_t
\end{bmatrix} =
\begin{bmatrix}
  (1 + f_1 z)^r & 0 \\
  0 & (1 + f_2 z)^r
\end{bmatrix}
\begin{bmatrix}
  \eta_1 t \\
  \eta_2 t
\end{bmatrix},
\]

The circuit diagram in the left panel of Figure 2 shows clearly the dependence of the time series \( \{y_n\} \) on \( \{x_n\} \) through the coupling parameter \( c \). The MA order \( r \) varies from 0 to 10, but there are only two parameters, \( f_1 = 0.6 \) and \( f_2 = 0.7 \) determining these \( r \) MA parameters. The AR parameters are set to \( a = 0.9, b = 0.8, c = \sqrt{e^{-F} (e^F - 1)(e^F - b^2)} \) with the transfer entropy \( F = 0.02 \). The true partial coherence in this case is \( \rho_{xy|z}^2 = 1 - e^{-0.02} \approx 0.02 \) and \( GG = e^{-0.02} \).

To replicate the Barnett, et al. experiment, we choose 1000 time samples to estimate \( \rho_{xy|z}^2 \) from second-order statistics, and this experiment is replicated 10,000 times to obtain Monte-Carlo estimates of performance. The vector \( y \) is defined to be \( y_t \); \( x \) and \( z \) are truncated to finite \( T \)-dimensional vectors with \( x = [x_{t-1}, \ldots, x_{t-T}] \) and \( z = [y_{t-1}, \ldots, y_{t-T}] \). So the sample covariance matrices for these vectors are just \( 10 \times 10 \). Using the null distribution given in Theorem 1, we set the rejection region with significance level 0.05. As a practical matter, \( (10,000) \times (1000) \) samples are generated and consecutive windows of 1000 samples are constructed to obtain the 10,000 replications. Because of the weak dependence between windows, these are not actually 1000 independent realizations, so our analytical formula for the null distribution does not perfectly predict the experimentally-determined significance level. But it is very close. In fact, when we design for a significance level of \( \alpha = 0.05 \) we achieve an experimental level of 0.044.

Figure 3 displays our results based on the model-free test in Section IV. The left panel plots the power of our test statistic with respect to the order of the underlying ARMA model, at significance level 0.05. This power is relatively invariant to MA order, whereas the experimental results in [12] show decreasing power as the MA order increases, as a result of the increased model complexity. The model-free coherence statistic for memory \( T = 10 \) has power about 0.90 regardless of the MA order. This power is slightly lower than the result of Barnett, et al. at low MA orders, and higher at higher MA orders. Power is relatively invariant to MA order, and it is not susceptible to model mismatch, as no state space model is assumed. An appropriate value of \( T \) can be selected by cutting off the empirical auto-correlations at a predetermined threshold or comparing AIC or BIC values for an auto-regressive model fitted to the sample time series data.

In the right panel of Figure 3 we plot the ROC curve (power vs size) to demonstrate that probability of correctly rejecting the null hypothesis (the power) depends on the probability of incorrectly rejecting it (the size). We also plot in the right panel of Figure 2 the pairwise partial coherence between \( y_t \) and \( x_s \) given the prima facie evidence \( z = [y_{t-1}, \ldots, y_{t-T}] \) of Figure 2 the pairwise partial coherence between \( y_t \) and \( x_s \) is non-zero for short ranges of \( s > t \) and \( s < t \).

VI. DISCUSSION

In this paper, our objective has been to explore the use of partial coherence among finite-dimensional vectors, constructed from signal sequences, as a scale-invariant statistic to assess causal influence of one signal sequence on another. Of course these signal sequences might well be space series or space-time series, in which case the idea of past-, future- or mixed-causal influence is also a physically meaningful question [26]. The partial coherence statistic we argue for is a monotone function of conditional KL divergence, conditional mutual information, conditional information rate, and the Granger-Geweke causality measure in the multivariate normal case. It is a measure of second-order partial correlation in the Hilbert space of second-order random variables, and it is resolved by partial canonical correlations. The sampled-data version of partial coherence is in fact a likelihood ratio statistic whose null distribution is a Wilks Lambda. It may be computed from the sample information matrix. Importantly, from this null distribution, a threshold on partial coherence may be set to control the confidence level \( 1 - \alpha \).

The key idea in the application of partial coherence to questions of causal influence is to define three channels of
signals so that they code for the question of interest. Then finite-dimensional vectors are constructed from these channels and their composite covariance matrix is estimated without appeal to an underlying generative model. Thus the method is model-free.

We have demonstrated the methodology for two particular experiments in causal influence. One of these experiments demonstrates how partial coherence may be used to map out a partial coherence surface, parameterized by two time variables, and the other replicates the experiment of Barnett and Seth [12]. For this latter experiment we demonstrate the power of our method and plot its experimentally computed Receiver Operating Characteristic (ROC). The conclusion of the test is that the performance of a model-free coherence test for causal influence is competitive with a model-based coherence test. Moreover, the null distribution of the model-free test is invariant to the underlying generative model for the measurements. Consequently, as a null hypothesis test, the partial coherence test is not vulnerable to model bias. Of course in applications where a model might be well-established, and measurements are plentiful, a method like that of the Barnetts and Seth [12], [13] might outperform a model-free method such as model-free partial coherence. In applications where there is no generative model like an ARMA model, then the method of second-order partial coherence is compelling.

Acknowledgment. We thank Profs. Sonia and Tomas Charleston Villalobos for piquing our interest in causality, and Prof. Todd Moon for alerting us to the work of the Barnetts and Seth, which led us to the work of Geweke.

REFERENCES

[1] N. Wiener, “The Human Use of Human Beings”, Avon Books, New York, NY, 1950.
[2] C. W. J. Granger, “Investigating Causal Relations by Econometric Models and Cross-spectral Methods,” Econometrica, 37 (3): 424–438, 1969.
[3] C. W. J. Granger, “Testing for Causality: a Personal View,” Journal of Economic Dynamics and Control, 2: 329–352, 1980.
[4] J. Rissanen and M. Wax, “Measures of mutual and causal dependence between two time series (Corresp.),” IEEE Trans Information Theory, 33(4), 598–601, 1987.
[5] J. Geweke, “Measurement of linear dependence and feedback between multiple time series,” J. Am. Stat. Assoc. 77(378): 304–313, 1982.
[6] J. Geweke, “Measures of conditional linear dependence and feedback between time series,” J. Am. Stat. Assoc., 79(388): 907–915, 1984.
[7] V. Solo, “State-Space Analysis of Granger-Geweke Causality Measures with Application to fMRI,” Neural Computation, 28(5), 914–949, 2016.
[8] E. Siggiridou and D. Kugiumtzis, “Granger Causality in Multivariate Time Series Using a Time-Ordered Restricted Vector Autoregressive Model,” IEEE Transactions on Signal Processing, 64(7):1759–1773, 2016.
[9] G. Schamberg and T. P. Coleman, “Measuring Sample Path Causal Influences With Relative Entropy,” IEEE Transactions on Information Theory, 66(5):2777–2798, 2020.
[10] Kramer, G., “Directed information for channels with feedback,” Ph.D. thesis, Swiss Federal Institute of Technology Zurich, 1998.
[11] P-O. Amblard and J.J. Michel, “On directed information theory and Granger causality graphs,” J. Computational Neuroscience,30(1), 7–16, 2011.
[12] L. Barnett, A. B. Barrett, and A. K. Seth, “Granger causality for state-space models,” Physical Review E, 91:040101(R), 2015.
[13] L. Barnett, A. B. Barrett, and A. K. Seth, “Granger causality and transfer entropy are equivalent for Gaussian variables,” Phys. Rev. Lett., 103(23), 238701, 2009.