The Generalized Metrical Multi-Time Lagrange Space of Relativistic Geometrical Optics

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Abstract

Section 1 contains some physical and geometrical aspects that motivates us to study the generalized metrical multi-time Lagrange space of Relativistic Geometrical Optics, denoted by $RGOGLM^n$. Section 2 develops the geometry of this space, in the sense of d-connections, d-torsions and d-curvatures. The Einstein equations of gravitational potentials of this generalized metrical multi-time Lagrange space are studied in Section 3. The conservation laws of the stress-energy d-tensor of $RGOGLM^n$ are also described. The electromagnetic d-tensors are introduced in Section 4, and corresponding Maxwell equations are derived.

Mathematics Subject Classification (2000): 53B40, 53C60, 53C80.

Key words: 1-jet fibre bundle, nonlinear connection, Cartan canonical connection, Einstein equations, Maxwell equations.

1 Geometrical and physical aspects

Let us consider the generalized Lagrange space $GL^n = (M, g_{ij}(x,y))$, $n = \dim M$, whose fundamental tensor field $g_{ij}(x,y)$ on $TM$ is of the form

\[ g_{ij}(x,y) = \varphi_{ij}(x) + \left[ 1 - \frac{1}{n^2(x,y)} \right] y_i y_j, \]  

(1.1)

where $\varphi_{ij}(x)$ is a semi-Riemannian metric tensor on $M$, $y_i = \varphi_{ij}(x)y^j$ and $n(x,y)$ is a smooth function on $TM$, which is called the refractive index function. This generalized Lagrange space is known today as the generalized Lagrange space of relativistic geometrical optics.

In order to explain the above physical terminology, let us analyse the restriction of the geometry of the previous space to a cross section,

\[ S_V : M \to TM, \quad x \to (x, y^i = V^i(x)). \]  

(1.2)

Thus, starting with a given cross section $S_V$, we remark that the restriction of the fundamental d-tensor $g_{ij}(x,y)$ of $GL^n$ to the submanifold $S_V(M)$ is given by

\[ g_{ij}(x,V(x)) = \varphi_{ij}(x) + \left[ 1 - \frac{1}{n^2(x,V(x))} \right] V_i V_j, \]  

(1.3)

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where \( V_i = \varphi_{ij} (x)V^j \). We underline that the metric (1.3) was introduced by Synge and was used by him in the study of the propagation of the electromagnetic waves in a medium with the index of refraction \( n(x, V(x)) \), \( V(x) \) being the velocity of medium. In Synge’s terminology, a medium is represented by a triad \( M = (M, V(x), n(x, V(x)) \). More deeply, he use the following

**Definition 1.1** A triad \( M = (M, V(x), n(x, V(x)) \) is called a dispersive medium. If \( n(x, y) \) does not depend of \( y \), then \( M \) is called a non-dispersive medium.

Consequently, the geometry induced by the Synge’s metric tensor (1.3) gives a mathematical model for relativistic optics.

In conclusion, the study of the generalized Lagrange space \( GL^n \), whose metrical d-tensor is of the form (1.4), was imposed. In this direction, an important class of generalized Lagrange spaces of relativistic geometrical optics, having the refractive index in the form,

\[
\frac{1}{n^2} = 1 - \frac{\alpha}{c^2}, \quad \alpha \in \mathbb{R}^*_+,
\]

where \( c \) being the light velocity, was studied by Miron and Kawaguchi. For these spaces, some post-Newtonian estimations were investigated by Asanov and Kawaguchi. In a different version, some post-Newtonian estimations was presented also by Roxburgh.

The geometry of the generalized Lagrange spaces of relativistic geometrical optics is now completely done by Miron, Anastasiei and Kawaguchi. Their geometrical development relies the using of an "a priori" fixed nonlinear connection on \( TM \), whose components are

\[
N^i_j(x, y) = \gamma^i_{jk}(x)y^k,
\]

where \( \gamma^i_{jk}(x) \) are the Christoffel symbols for the semi-Riemannian metric \( \varphi_{ij}(x) \).

The using of the nonlinear connection (1.3) in the study of the generalized Lagrange space of relativistic geometrical optics \( GL^n \) is motivated in various ways. In this direction, we present only two geometrical and physical aspects. For more details, see [9], [11].

Firstly, it is very important that the autoparallel curves of the nonlinear connection \( N^i_j(x, y) \) of the generalized Lagrange space of relativistic geometrical optics \( GL^n \) coincide to the geodesics of the Riemannian space \( R^n = (M, \varphi_{ij}) \). In other words, the space \( GL^n \) verifies the first EPS (Ehlers, Pirani, Schild) condition from the constructive-axiomatic formulation of General Relativity.

Secondly, it is remarkable that, in the particular case (1.4), the absolute energy Lagrangian of \( GL^n \),

\[
\mathcal{E} : TM \to \mathbb{R}, \quad \mathcal{E}(x, y) = g_{ij}(x, y)y^iy^j,
\]

is a regular Lagrangian and then its canonical nonlinear connection is exactly that given by (1.5).

In conclusion, one can assert that a generalized Lagrange space \( GL^n \), which verifies the above axiomatic assumptions, becomes a convenient mathematical model for the relativistic geometrical optics.

In this paper, we try to extend the previous geometrical and physical theories, from the tangent bundle \( TM \) to the more general jet fibre bundle of order one \( J^1(T, M) \)
coordinates by \((t^a, x^i, x^j_a)\), where \(T\) is a smooth, real, \(p\)-dimensional manifold coordinated by \(t = (t^a)_{a=1}^p\), whose physical meaning is that of "multidimensional time", while \(x^i_a\) have the physical meaning of partial directions. In this sense, we recall that the jet fibre bundle of order one \(J^1(T, M)\) is a basic object in the study of classical and quantum field theories.

A natural geometry of physical fields induced by a Kronecker \(h\)-regular vertical metrical multi-time d-tensor \(G^{(\alpha)(\beta)}_{(i)(j)}(t^\gamma, x^k, x^k_\gamma)\) on the total space of the 1-jet vector bundle \(J^1(T, M) \to T \times M\), where \(h = (h_{\alpha\beta}(t^\gamma))\) is a semi-Riemannian metric on the temporal manifold \(T\), was created by Neagu [12].

The fundamental geometrical concept used there is that of generalized metrical multi-time Lagrange space. This geometrical concept with physical meaning is represented by a pair \(GML^n_p = (J^1(T, M), G^{(\alpha)(\beta)}_{(i)(j)})\) consisting of the 1-jet space and a Kronecker \(h\)-regular vertical multi-time metrical d-tensor \(G^{(\alpha)(\beta)}_{(i)(j)}\) on \(J^1(T, M)\), that is, it decomposes in

\[
G^{(\alpha)(\beta)}_{(i)(j)}(t^\gamma, x^k, x^k_\gamma) = h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k, x^k_\gamma),
\]

where \(g_{ij}(t^\gamma, x^k, x^k_\gamma)\) is a d-tensor on \(J^1(T, M)\), symmetric, of rank \(n\) and having a constant signature. The d-tensor \(g_{ij}(t^\gamma, x^k, x^k_\gamma)\) is called the spatial metrical d-tensor of \(GML^n_p\).

Following the general physical and geometrical development from [12], the aim of this paper is to study the particular generalized metrical multi-time Lagrange space \(RGOGML^n_p\), whose spatial metrical d-tensor is of the form

\[
g_{ij}(t^\gamma, x^k, x^k_\gamma) = \varphi_{ij}(x^k) + A_i(t^\gamma, x^k, x^k_\gamma)A_j(t^\gamma, x^k, x^k_\gamma)
\]

where \(\varphi_{ij}(x^k)\) is a semi-Riemannian metric on the spatial manifold \(M\) and \(A_i(t^\gamma, x^k, x^k_\gamma)\) represent the components of a d-tensor \(A\) on \(J^1(T, M)\), whose physical meaning is that of refractive index d-tensor of the medium \(M = J^1(T, M)\).

Remarks 1.1 i) To motivate the terminology used above, let us consider the particular case when the temporal manifold identifies with the usual time axis, represented by the set of real numbers \(R\). In that case, setting

\[
A_i(t, x^k, y^k) = \sqrt{1 - \frac{1}{n^2(t, x^k, y^k)}} \cdot y_i,
\]

where \(y_i = \varphi_{im}(x^k)y^m\) and \(n : J^1(R, M) \equiv R \times TM \to [1, \infty)\) is a smooth function, we can regard \(n\) like a dynamic refractive index function (i.e. the refractive index modifies in time). Consequently, this particular space represents a natural dynamical generalization of the classical generalized Lagrange space of relativistic geometrical optics from [9, 11].

ii) The inverse of the spatial metrical d-tensor \(g_{ij}\) of \(RGOGML^n_p\) is given by the following d-tensor field,

\[
g^{ij} = \varphi^{ij} - \frac{1}{1 + A_0}A^i A^j,
\]

where \(A^i = \varphi^{im}A_m\) and \(A_0 = A^m A_m\).
To develop the geometry of this generalized metrical multi-time Lagrange space, we need a nonlinear connection \( \Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)}) \) on \( J^1(T,M) \) \[14\]. In this direction, we fix "a priori" the nonlinear connection \( \Gamma \) defined by the temporal components
\[
M_{(\alpha)\beta}^{(i)} = -H_{\alpha\beta}^{\mu} x_\mu^i
\]
and the spatial components
\[
N_{(\alpha)j}^{(i)} = \gamma_{jm}^i x_m^\alpha,
\]
where \( H_{\alpha\beta}^{\gamma} \) (resp. \( \gamma_{ij}^k \)) are the Christoffel symbols of the semi-Riemannian metric \( h_{\alpha\beta} \) (resp. \( \phi_{ij} \)).

**Remarks 1.2**

i) Our given "a priori" nonlinear connection \( \Gamma \) on \( J^1(T,M) \) naturally generalizes that used by Miron, Anastasiei and Kawaguchi.

ii) The spatial components \( N_{(\alpha)j}^{(i)} \) of the fixed nonlinear connection \( \Gamma \) are without torsion \[12\].

iii) The previous nonlinear connection \( \Gamma \) is dependent only the vertical fundamental metrical d-tensor \( G_{(\alpha)(\beta)}^{(i)(j)} \) of \( RGOGML^n_p \). This fact emphasize the metrical character of the geometry attached to this space, i.e., all geometrical objects are directly arised from \( G_{(\alpha)(\beta)}^{(i)(j)} \).

iv) Using the relation between sprays and the components of a nonlinear connection and the definition of harmonic maps attached to a given multi-time dependent spray on \( J^1(T,M) \) (for more details, see \[14\]), we easily deduce that the harmonic maps of the nonlinear connection \( \Gamma \) of \( RGOGML^n_p \) are exactly the harmonic maps between the semi-Riemannian spaces \( (T,h) \) and \( (M,\phi) \) \[3\].

In conclusion, we can assert that the generalized metrical multi-time Lagrange space \( RGOGML^n_p \), which verifies the previous assumptions, represents a convenient geometrical model for relativistic geometrical optics, in a general setting.

**Open problem.** At the end of this section, we should like to point out that, in the particular case \( A_i = A_i(t^\gamma, x^k) \) (i.e. the refractive index d-tensor \( A \) does not depend by partial directions \( x^\alpha_i \)), the absolute energy Lagrangian function \[12\],
\[
\mathcal{E} : J^1(T,M) \to R, \quad \mathcal{E}(t^\gamma, x^k, x^\alpha_k) = h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k)x^\alpha_i x^\beta_j,
\]
is a Kronecker h-regular Lagrangian \[15\]. Consequently, it naturally induces a canonical spatial nonlinear connection on \( J^1(T,M) \), whose components are given by \[15\]
\[
N_{(\alpha)j}^{(i)} = \Gamma_{jm}^i x_m^\alpha + \frac{g_{im}}{2} \frac{\partial g_{mj}}{\partial t^\alpha},
\]
where
\[
\Gamma_{jk}^i = \frac{g_{ij}}{2} \left( \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right)
\]
are the generalized Christoffel symbols of the "multi-time" dependent spatial metric \( g_{ij} \). In conclusion, it is natural to arise the following question:

—Considering an isotropic medium \( M = J^1(T,M) \) (i.e. its refractive index d-tensor \( A \) does not depend by partial directions \( x^\alpha_i \)), what is the difference, from a physical point of view, between the using of one or another one of the spatial nonlinear connections expressed by \[1.12\] and \[1.14\]?
2 Cartan canonical connection

In this section, we will apply the general geometrical development of the generalized metrical multi-time Lagrange spaces \([12]\), to the particular space of relativistic geometrical optic \(RGOGML^p\), having the vertical fundamental d-tensor,

\[
G^{(\alpha)(\beta)}_{(i)(j)}(t^\gamma, x^k, x^k_\gamma) = h^{\alpha\beta}(t^\gamma) \left[ \varphi_{ij}(x^k) + A_i(t^\gamma, x^k, x^k_\gamma)A_j(t^\gamma, x^k, x^k_\gamma) \right],
\]

and being endowed with the nonlinear connection \(\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})\), where

\[
M^{(i)}_{(\alpha)\beta} = -H^{\mu\beta}_{\alpha}x^i_\mu, \quad N^{(i)}_{(\alpha)j} = \gamma^i_{jm}x^m_\alpha.
\]

Let \(\left\{ \frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \partial \right\} \subset X(J^1(T, M))\) and \(\{dt^\alpha, dx^i, \delta x^i_\alpha\} \subset X^*(J^1(T, M))\) be the adapted bases of the nonlinear connection \(\Gamma\), where \([12]\)

\[
\begin{aligned}
\frac{\delta}{\delta t^\alpha} &= \frac{\partial}{\partial t^\alpha} - M^{(j)}_{(\beta)\alpha} \frac{\partial}{\partial x^j_{\beta}}, \\
\frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N^{(j)}_{(\beta)i} \frac{\partial}{\partial x^j_{\beta}}, \\
\delta x^i_\alpha &= dx^i_\alpha + M^{(i)}_{(\alpha)j}dt^j + N^{(i)}_{(\alpha)j}dx^j.
\end{aligned}
\]

Following the paper \([12]\), by direct computations, we can determine the Cartan canonical connection of \(RGOGML^p\), together with its torsion and curvature local d-tensors.

In order to describe these geometrical entities of \(RGOGML^p\), let us consider \(BG = (H^{\alpha\beta}, 0, \gamma^i_{jk}, 0)\), the Berwald h-normal \(\Gamma\)-linear connection attached to the semi-Riemannian metrics \(h_{\alpha\beta}\) and \(\varphi_{ij}\) and \(\parallel \parallel\parallel 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Theorem 2.1 The Cartan canonical connection $CT = (H^\gamma_{\alpha\beta}, G^i_{j\gamma}, L^i_{jk}, C^{i(\gamma)}_{j(k)})$ of $RGOGM^n$ has the adapted coefficients

$$H^\gamma_{\alpha\beta} = H^\gamma_{\alpha\beta},$$

$$G^i_{j\gamma} = \frac{1}{2} \left[ (A^i_{\gamma} A_j)_{/\gamma} - A^i A^m_{/0} (A_m A_j)_{/\gamma} \right],$$

$$L^i_{jk} = (\gamma_{jk0} + A_{jk0}) \left[ \varphi^{im} - \frac{A^i A^m}{1 + A_0} \right],$$

$$C^{i(\gamma)}_{j(k)} = C^\gamma_{jk} \left[ \varphi^{im} - \frac{A^i A^m}{1 + A_0} \right],$$

where $A^i = \varphi^{im} A_m$, $A_0 = A^m A_m$ and

$$\gamma_{ijm} = \frac{1}{2} \left[ \frac{\partial \varphi^{im}}{\partial x^j} + \frac{\partial \varphi^{jm}}{\partial x^i} - \frac{\partial \varphi^{ij}}{\partial x^m} \right],$$

$$A_{ijm} = \frac{1}{2} \left[ \frac{\delta(A_i A_m)}{\delta x^j} + \frac{\delta(A_j A_m)}{\delta x^i} - \frac{\delta(A_i A_j)}{\delta x^m} \right],$$

$$C^\gamma_{ijm} = \frac{1}{2} \left[ \frac{\partial (A_i A_m)}{\partial x^j} + \frac{\partial (A_j A_m)}{\partial x^i} - \frac{\partial (A_i A_j)}{\partial x^m} \right].$$

Remark 2.1 Using the notations

$$A^i_{jk} = \varphi^{im} A_{jkm}, \quad \Lambda^i_{jk} = A^i_{jk} - \frac{(\gamma_{jk0} + A_{jk0}) A^i}{1 + A_0},$$

$$1^{\gamma i}_{jk} = \varphi^{im} C^\gamma_{jkm}, \quad 0^{\gamma i}_{jk} = C^\gamma_{jk0} A^i \frac{1}{1 + A_0},$$

where $D_{\gamma;0} = D^{\gamma}_{i;0} A^m$, we can rewrite the components $L^i_{jk}$ and $C^{i(\gamma)}_{j(k)}$ of $CT$ in the following simple form:

$$L^i_{jk} = \gamma^i_{jk} + \Lambda^i_{jk},$$

$$C^{i(\gamma)}_{j(k)} = 1^{\gamma i}_{jk} + 0^{\gamma i}_{jk}.$$

Theorem 2.2 The torsion $T$ of the Cartan canonical connection of $RGOGM^n$ is determined by seven effective local $d$-tensors, namely,

$$T^m_{\alpha j} = -C^m_{j\alpha}, \quad P^{(m)}_{(\mu)\alpha(j)} = -\delta^m_\mu G^{m}_{j\alpha}, \quad P^{(m)}_{(\mu)\iota(j)} = -\delta^m_\mu \Lambda^{m}_{\iota j},$$

$$P^{(m)}_{i(j)} = 1^{\beta m}_{ij} + 0^{\beta m}_{ij}, \quad S^{(m)(\alpha)(\beta)}_{(\mu)\iota(j)} = \delta^\mu_\alpha C^{m(\beta)}_{i\iota(j)} - \delta^\mu_\beta C^{m(\alpha)}_{i\iota(j)},$$

$$R^{(m)}_{(\mu)\alpha\beta} = -H^{\gamma}_{\mu\alpha\beta} x^m_{\gamma j}, \quad R^{(m)}_{(\mu)\alpha j} = 0, \quad R^{(m)}_{(\mu)\iota j} = 0.$$
The general expressions of the local curvature d-tensors attached to the Cartan canonical connection of a generalized metrical multi-time Lagrange space \[12\], applied to \( RGOGML^n_p \), imply

**Theorem 2.3** The curvature \( R \) of the Cartan canonical connection of \( RGOGML^n_p \) is determined by seven effective local d-tensors, expressed by,

\[
H_{\eta\beta\gamma}^x = \frac{\partial H_{\eta\beta}^x}{\partial t_\gamma} - \frac{\partial H_{\eta\gamma}^x}{\partial t_\beta} + H_{\eta\beta}^x H_{\mu\gamma}^x - H_{\eta\gamma}^x H_{\mu\beta}^x,
\]

\[
R^i_{j\beta\gamma} = \frac{\delta G^i_{\beta\gamma}}{\delta t^j} - \frac{\delta G^i_{\beta\gamma}}{\delta t^j} + G_i^{m\gamma} G_j^m - G_i^{m\gamma} G_j^m - H_{\eta\beta\gamma}^x C_i^{(\mu)} x^\mu x^\eta,
\]

\[
R^i_{j\beta\gamma} = \frac{\delta L^i_{\beta\gamma}}{\delta x^j} + G_i^{m\gamma} L_j^m - L_j^m G_i^m,
\]

\[
R^i_{j\beta\gamma} = r^i_{j\beta\gamma} + \rho^i_{j\beta\gamma},
\]

\[
p^i_{j(\gamma)} = \frac{\partial G^i_{j\beta}}{\partial x^\gamma} - C_i^{(\gamma)\beta} - C_i^{(\gamma)\beta} G_j^m,
\]

\[
p^i_{j(\gamma)} = \frac{\partial L^i_{j\beta}}{\partial x^\gamma} - L_j^m C_{im} - L_j^m C_{im} - L_j^m C_{ikl} - C_{ikl},
\]

\[
S^i_{j(\gamma)} = S^i_{j(\gamma)} - S^i_{j(\gamma)}
\]

where \( H_{\eta\beta\gamma}^x \) and \( r^i_{j\beta\gamma} \) are the curvature tensors of the semi-Riemannian metrics \( h_{\alpha\beta} \) and \( \varphi_{ij} \), the operators \( n_{\beta\gamma} \), \( n_{ij} \) represent the local covariant derivatives of \( CT \), and

\[
S^i_{j(\gamma)} = \frac{\partial C_{ij}}{\partial x^\gamma} - \frac{\partial C_{ik}}{\partial x^\gamma} + C_{ij} C_{m(k)} - C_{mj} C_{i(k)},
\]

\[
p^i_{j(\gamma)} = \frac{\partial L^i_{j\beta}}{\partial x^\gamma} - L_j^m C_{im} - L_j^m C_{im} - L_j^m C_{ikl} - C_{ikl},
\]

\[
\rho^i_{j\beta\gamma} = L_j^k - L_j^k + L_j^m L_m^i - L_j^m L_m^i + C_i^{(\mu)} x^\mu x^\eta.
\]

### 3 Einstein equations of gravitational field

Concerning the gravitational theory on \( RGOGML^n_p \), we point out that the vertical metrical d-tensor \( 2.1 \) and its fixed nonlinear connection \( 2.2 \) induce a natural gravitational h-potential on the 1-jet space \( J^1(T, M) \) (i. e. a Sasaki-like metric), which is expressed by \[12\]

\[
G = h_{\alpha\beta} dt^\alpha \otimes dt^\beta + g_{ij} dx^i \otimes dx^j + h^{\alpha\beta} g_{ij} \delta x^i_\alpha \otimes \delta x^j_\beta,
\]

where \( g_{ij} = \varphi_{ij} + A_i A_j \). Let \( CT = (H_{\alpha\beta}^x, C^k_{j\gamma}, L^i_{j\beta}, C^{(\gamma)}_{i(\beta)}) \) be the Cartan canonical connection of \( RGOGML^n_p \).
We postulate that the Einstein equations which govern the gravitational energy-momentum relation on the reduced phase space $\mathcal{P}$ of $\text{ROGML}_{p}^{n}$ are the Einstein equations attached to the Cartan canonical connection $\Gamma$ and the adapted metric $g$ on $J^{1}(T, M)$, that is,

\begin{equation}
\text{Ric}(\Gamma) - \frac{Sc(\Gamma)}{2}G = \mathcal{K}T,
\end{equation}

where $\text{Ric}(\Gamma)$ represents the Ricci d-tensor of the Cartan connection, $Sc(\Gamma)$ is its scalar curvature, $\mathcal{K}$ is the Einstein constant and $T$ is an intrinsic distinguished tensor of matter which is called the stress-energy d-tensor.

In the adapted basis $(X_{A}) = \left(\frac{\delta}{\delta t^{\alpha}}, \frac{\delta}{\delta x^{i}}, \frac{\partial}{\partial x^{i}_{\alpha}}\right)$ attached to $\Gamma$, the curvature d-tensor $R$ of the Cartan connection is expressed locally by $R(X_{C}, X_{B})X_{A} = R^{p}_{ABCD}X_{D}$. Hence, it follows that we have $R_{AB} = \text{Ric}(\Gamma)(X_{A}, X_{B}) = R^{p}_{ABD}$ and $Sc(\Gamma) = G^{AB}R_{AB}$, where

\begin{equation}
G^{AB} = \begin{cases} 
    h_{\alpha\beta}, & \text{for } A = \alpha, B = \beta \\
    g^{ij}, & \text{for } A = i, B = j \\
    h_{\alpha\beta}g^{ij}, & \text{for } A = (i)_{(\alpha)}, B = (j)_{(\beta)} \\
    0, & \text{otherwise,}
\end{cases}
\end{equation}

the tensor field $g^{ij}$ being expressed by $^{1.10}$. Taking into account the expressions $^{2.10}$ of the local curvature d-tensors of the Cartan connection of $\text{ROGML}_{p}^{n}$, we obtain without difficulties

**Theorem 3.1** The Ricci d-tensor $\text{Ric}(\Gamma)$ of $\text{ROGML}_{p}^{n}$ is determined by seven effective local d-tensors expressed, in adapted basis, by:

\begin{equation}
H_{\alpha\beta} = H^{n}_{\alpha\beta\mu}, \quad R_{ij} = R^{m}_{ijm}, \quad R_{ij} = r_{ij} + \rho_{ij}, \quad P^{(\alpha)}_{i(j)} = P^{m}_{i\beta(m)}, \quad
\end{equation}

\begin{equation}
P^{(\alpha)}_{i(j)} = -P^{(m)}_{m(j)}; \quad P^{(\alpha)}_{i(j)} = -P^{(m)}_{i(j)}; \quad S^{(\beta)(\gamma)}_{(j)(k)} = 1^{(\beta)(\gamma)}_{(j)(k)} + 0^{(\beta)(\gamma)}_{(j)(k)}; \quad
\end{equation}

where $H_{\alpha\beta}$ (resp. $r_{ij}$) are the local Ricci tensors of the semi-Riemannian metric $h_{\alpha\beta}$ (resp. $\varphi_{ij}$), $\rho_{ij} = \rho_{ij}^{m}$, $S^{(i)(j)}_{(k)(m)}$ and $S^{(i)(j)}_{(k)} = S^{(i)(j)}_{(k)}$.

Let us denote $H = h^{\alpha\beta}H_{\alpha\beta}$, $R = g^{ij}R_{ij} = h_{\alpha\beta}g^{ij}S^{(\alpha)(\beta)}_{(i)(j)}$. In this context, by a simple calculation, it follows

**Theorem 3.2** The scalar curvature of the Cartan connection $\Gamma$ of $\text{ROGML}_{p}^{n}$ has the formula

\begin{equation}
Sc(\Gamma) = H + R + S,
\end{equation}

the terms $H$, $R$ and $S$ being determined by the relations:

\begin{equation}
H = h^{\alpha\beta}H_{\alpha\beta}, \quad R = r + \rho - \frac{r^{00} + \rho^{00}}{1 + A_{0}}, \quad S = S - S' + S - S',
\end{equation}

where $S = S - S'$.
where $H$ (resp. $r$) is the scalar curvature of the semi-Riemannian metric $h_{\alpha\beta}$ (resp. $\varphi_{ij}$) and

$$\rho = \varphi^{rs} \rho_{rs}, \quad r_{00} = r_{ms} A^m A^s, \quad \rho_{00} = \rho_{ms} A^m A^s,$$

(3.7)

$$\begin{cases}
1 = h_{\mu\nu} \varphi^{rs} S^{(\mu)(\nu)}_{(r)(s)}, & 1 = h_{\mu\nu} A^r A^s S^{(\mu)(\nu)}_{(r)(s)}, \\
0 = h_{\mu\nu} \varphi^{rs} S^{(r)(s)}_{(r)(s)}, & 0 = h_{\mu\nu} A^r A^s S^{(r)(s)}_{(r)(s)}.
\end{cases}$$

Following the gravitational field theory exposition on a generalized metrical multi-time Lagrange space $GML_p^n$, $\dim M = n$, $\dim T = p$, from the paper [12], by local computations, we can give

**Theorem 3.3** If $p > 2$ and $n > 2$, the Einstein equations which govern the gravitational $h$-potential $G$ of $RGOGML_p^n$ have the local form

$$\begin{cases}
H_{\alpha\beta} - \frac{H}{2} h_{\alpha\beta} = K\tilde{T}_{\alpha\beta}, \\
r_{ij} - \frac{r}{2} \varphi_{ij} + \Theta_{ij} = K\tilde{T}_{ij}, \\
S^{(\alpha)(\beta)}_{(i)(j)} + \frac{1}{2} S - \frac{S'}{2} h^{\alpha\beta} g_{ij} + o_{0}^{(\alpha)(\beta)}_{(i)(j)} = K\tilde{T}^{(\alpha)(\beta)}_{(i)(j)},
\end{cases}$$

(3.8)

$$\begin{cases}
0 = \mathcal{T}_{\alpha\iota}, & R_{\alpha\iota} = K\mathcal{T}_{\alpha\iota}, & P_{(\alpha)}^{(\alpha)_{(i)}} = K\mathcal{T}_{(i)\alpha}, \\
0 = \mathcal{T}_{(i)\alpha}, & P_{(j)\alpha}^{(\alpha)} = K\mathcal{T}_{(i)\alpha}, & P_{(i)\alpha}^{(\alpha)} = K\mathcal{T}_{(i)\alpha},
\end{cases}$$

(3.9)

where $\tilde{T}_{\alpha\beta}$, $\tilde{T}_{ij}$ and $\tilde{T}^{(\alpha)(\beta)}_{(i)(j)}$ represent the components of a new stress-energy d-tensor $\tilde{T}$, defined by the relations

and

$$\begin{cases}
\tilde{T}_{\alpha\beta} = T_{\alpha\beta} + \frac{R + S}{2K} h_{\alpha\beta}, \\
\tilde{T}_{ij} = T_{ij} + \frac{H + S}{2K} g_{ij}, \\
\tilde{T}^{(\alpha)(\beta)}_{(i)(j)} = T^{(\alpha)(\beta)}_{(i)(j)} + \frac{H + R}{2K} h^{\alpha\beta} g_{ij},
\end{cases}$$

$$\begin{cases}
\theta_{ij} = \rho_{ij} - \frac{1}{2} \left( \rho - \frac{r_{00} + \rho_{00}}{1 + A_0} \right) g_{ij}, \quad o_{0}^{(\alpha)(\beta)}_{(i)(j)} = S^{(\alpha)(\beta)}_{(i)(j)} - \frac{S - S'}{2} h^{\alpha\beta} g_{ij}.
\end{cases}$$

**Remark 3.1** Note that, in order to have the compatibility of the Einstein equations, it is necessary that the certain adapted local components of the stress-energy d-tensor vanish "a priori".
From physical point of view, the stress-energy d-tensor $T$ must verify the local conservation laws $\mathcal{T}_A^B|_\mu = 0, ~\forall ~A \in \{\alpha, i, (\alpha)\}$, where $\mathcal{T}_A^B = G^{BD}T_{DA}$, $\Bar{\mathcal{T}}^B_A$, represents one from the local covariant derivatives "$/_{\beta}$", "$/_{j}$" or "$/_{(j)}$", of the Cartan canonical connection $\Gamma$.

In this context, let us denote

$$
\bar{T}_T = h^{\alpha\beta}\bar{T}_{\alpha\beta}, \quad \bar{T}_M = g^{ij}\bar{T}_{ij}, \quad \bar{T}_v = h_{\mu\nu}g^{mr}\bar{T}_{(m)(r)},
$$

$$
\bar{T}_\beta = h^{\alpha\mu}\bar{T}_{\mu\beta}, \quad \bar{T}_i = g^{im}\bar{T}_{mj}, \quad \bar{T}_{(i)_{j}} = h_{\alpha\mu}g^{mi}\bar{T}_{(\mu)(\beta)}.
$$

Following again the development of gravitational generalized metrical multi-time theory from $[12]$, we find

**Theorem 3.4** If $p > 2, n > 2$, the new stress-energy d-tensors $\bar{T}_{\alpha\beta}, \bar{T}_{ij}$ and $\bar{T}_{(i)(j)}$ of $RGOGL\alpha_n$ must verify the following conservation laws:

$$
\begin{align*}
\bar{T}_{\beta/\mu} + \frac{1}{2-n}\bar{T}_{M/\beta} + \frac{1}{2-p}\bar{T}_{v/\beta} &= -R_{\beta/\mu |m} - P_{(\mu)_{j}}(\beta)_{i}|

\frac{1}{2-p}\bar{T}_{T/|j} + \frac{1}{2-n}\bar{T}_{|j} &= -P_{(\mu)\beta}(\mu)_{i}|

\frac{1}{2-p}\bar{T}_{(\alpha)_{j}} + \frac{1}{2-n}\bar{T}_{M_{(\alpha)} + \bar{T}_{(\alpha)(\beta)}(\mu)_{j}} &= -P_{(\alpha)}(\beta)_{j}.
\end{align*}
$$

where

$$
\begin{align*}
R_{\beta/\mu} &= g^{im}R_{m\beta}, \quad P_{(\alpha)\beta} = g^{im}h_{\alpha\mu}P_{(\mu)m\beta},

P_{(\alpha)\beta} &= g^{im}P_{m\beta}(\beta)_{j}, \quad P_{(\alpha)j} = g^{im}h_{\alpha\mu}P_{(\mu)m\beta}.
\end{align*}
$$

## 4 Maxwell equations of electromagnetic field

In order to develop the electromagnetic theory on the generalized metrical multi-time Lagrange space $RGOGL\alpha_p$, let us consider the canonical Liouville d-tensor $\mathcal{C} = x_{\alpha}\frac{\partial}{\partial x_{\alpha}}$ on $J^1(T, M)$. Using the Cartan canonical connection $\Gamma$ of $RGOGL\alpha_p$, we construct the metrical deflection d-tensors $[12]$

$$
\begin{align*}
\bar{D}^{(\alpha)}_{(i)_{j}} &= \left[ G^{(\alpha)(\mu)}_{(i)(m)}x_{\mu} \right]/_{\beta},

\bar{D}^{(\alpha)\beta} &= \left[ G^{(\alpha)(\mu)}_{(i)(m)}x_{\mu} \right]/_{(j)},

\bar{D}^{(\alpha)(\beta)} &= \left[ G^{(\alpha)(\mu)}_{(i)(m)}x_{\mu} \right]/_{(j)},
\end{align*}
$$

where $G^{(\alpha)(\beta)}_{(i)(j)} = h^{\alpha\beta}g_{ij}$ is the vertical fundamental metrical d-tensor of $RGOGL\alpha_p$ and "$/_{\beta}$", "$/_{j}$" or "$/_{(j)}$", are the local covariant derivatives of $\Gamma$.

Taking into account the expressions of the local covariant derivatives of the Cartan canonical connection $\Gamma$, we obtain
Proposition 4.1 The metrical deflection d-tensors of the space \( \text{RGOGML}_p^n \) are given by the following formulas:

\[
\begin{align*}
\bar{D}^{(\alpha)(\beta)}_{(i)j} & = G^{(\alpha)(\mu)}_{(i)(m)} G_{\rho\beta}^m x^\rho, \\
D^{(\alpha)(\beta)}_{(i)j} & = G^{(\alpha)(\mu)}_{(i)(m)} A^m_{\rho\beta} x^\rho, \\
d^{(\alpha)(\beta)}_{(i)j} & = G^{(\alpha)(\beta)}_{(i)j} + C_{\mu\beta}^\rho h^{\alpha\mu} x^\rho.
\end{align*}
\]

(4.2)

Definition 4.1 The distinguished 2-form on \( J^1(T, M) \),

\[
F = F^{(\alpha)}_{(i)j} \delta x^i \wedge dx_j + f^{(\alpha)(\beta)}_{(i)j} \delta x^i \wedge \delta x^j,
\]

(4.3)

where \( F^{(\alpha)}_{(i)j} \) and \( f^{(\alpha)(\beta)}_{(i)j} \) are called the distinguished electromagnetic 2-form of the generalized metrical multi-time Lagrange space \( \text{RGOGML}_p^n \).

Proposition 4.2 The local electromagnetic d-tensors of \( \text{RGOGML}_p^n \) have the expressions,

\[
\begin{align*}
F^{(\alpha)}_{(i)j} & = [\varphi_{ir} A^\rho_{mj} - \varphi_{jr} A^\rho_{mi} + A_{i} A^0_{mj} - A_j A^0_{mi}] h^{\alpha\mu} x^\mu, \\
f^{(\alpha)(\beta)}_{(i)j} & = \frac{1}{2} \left[ C_{mji}^\beta - C_{mij}^\beta \right] h^{\alpha\mu} x^\mu,
\end{align*}
\]

(4.4)

where \( A^0_{mj} = A^\rho_{mj} A^\rho_r \).

Particularizing the Maxwell equations of the electromagnetic field, described in the general case of a generalized metrical multi-time Lagrange space \( \text{RGOGML}_p^n \), we deduce the main result of the electromagnetism on \( \text{RGOGML}_p^n \).

Theorem 4.3 The electromagnetic components \( F^{(\alpha)}_{(i)j} \) and \( f^{(\alpha)(\beta)}_{(i)j} \) of the generalized metrical multi-time Lagrange space \( \text{RGOGML}_p^n \) are governed by the Maxwell equations:

\[
\begin{align*}
\sum_{i,j,k} F^{(\alpha)}_{(i)j(k)/\beta} & = -\frac{1}{2} \sum_{i,j,k} \left[ d^{(\alpha)(\mu)}_{(i)(m)} + C^{(\beta)(\mu)}_{(i)(m)} x^{(\alpha)}_{(p)} \right] r_{mk} x^k \\
\sum_{i,j,k} f^{(\alpha)(\gamma)}_{(i)j(k)/l} & = 0 \\
\sum_{i,j,k} f^{(\alpha)(\beta)(\gamma)}_{(i)j(k)/l(k)} & = 0,
\end{align*}
\]

(4.5)

where \( x^{(\alpha)}_{(p)} = G^{(\alpha)(\mu)}_{(p)(m)} x^\mu \).
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