A NOTE ON THE ALTERNATING SUMS
OF POWERS OF CONSECUTIVE INTEGERS

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ABSTRACT. For \( n, k \in \mathbb{Z}_{\geq 0} \), let \( T_n(k) \) be the alternating sums of the \( n \)-th powers of positive integers up to \( k - 1 \): \( T_n(k) = \sum_{l=0}^{k-1} (-1)^l l^n \). Following an idea due to Euler, we give the below formula for \( T_n(k) \):

\[
T_n(k) = \frac{(-1)^{k+1}}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l k^{n-l} + \frac{E_n}{2} \left( 1 + (-1)^{k+1} \right),
\]

where \( E_l \) are the Euler numbers.

1. Introduction

J. Bernoulli (1713) first discovered the method which one can produce those formulae for the sum \( \sum_{l=1}^{n} l^k \) for any natural numbers. The Bernoulli numbers are among the most interesting and important number sequences in mathematics. They first appeared in the posthumous work “ARS Conjectandi” (1713) by J. Bernoulli (1654-1705) in connection with sums of powers of consecutive integers. Bernoulli numbers are particularly important in number theory, especially in connection with Fermat’s last theorem. The number sequences of Euler, Genocchi, Stirling and others, as well as

2000 Mathematics Subject Classification 11S80, 11B68
Key words and phrases: Euler number, zeta function, Bernoulli numbers
the tangent numbers, secant numbers, etc., are closely related to the Bernoulli numbers.
Following an idea due to J. Bernoulli it was known that

\[ S_n(k) = \frac{1}{n+1} \sum_{i=0}^{n} \binom{n+1}{i} B_i k^{n+1-i}, \]  

cf.\[1, 2, 3, 4\],

where \(B_i\) are Bernoulli numbers. Let \(n, k\) be positive integers \((k > 1)\), and let

\[ T_n(k) = -1^n + 2^n - 3^n + 4^n - 5^n + \cdots + (-1)^{k-1}(k-1)^n. \]

In this paper we evaluate the alternating sums of powers of consecutive integers. Following an idea due to Euler, we study a formula for \(T_n(k)\):

\[ T_n(k) = \frac{(-1)^{k+1}}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l k^{n-1} + \frac{E_n}{2} \left(1 + (-1)^{k+1}\right), \]

where \(E_l\) are the Euler numbers.

2. Sums of powers

Euler numbers are defined by the generating function as follows:

(1)

\[ G(t) = \frac{2}{e^t + 1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \pi, \]

where we use the technique method notation by replacing \(E^m\) by \(E_m\) \((m \geq 0)\), symbolically. Let \(x\) be the variable. Then we consider

(2)

\[ F(t, x) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \]

By (1) and (2), we see that

\[ F(t, x) = G(t) e^{xt} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} E_k x^{n-k}\right) \frac{t^n}{n!}. \]

From this, we derive

\[ E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k x^{n-k}, \]
which is called the Euler polynomials. Note that $E_n(0) = E_n$. From the definition of Euler numbers, we can derive the below relation:

$$(E + 1)^n + E_n = 2\delta_{0,n},$$

where $\delta_{0,n}$ is Kronecker Symbol. By (3), we easily see that $E_0 = 1$, $E_1 = -\frac{1}{2}$, $E_2 = 0$, $E_3 = \frac{1}{4}$, $\cdots$, $E_{2k} = 0$. For any positive integer $n$, it is easy to show that

$$-2 \sum_{l=0}^{\infty} (-1)^{l+n} e^{(l+n)t} + 2 \sum_{l=0}^{\infty} (-1)^l e^{lt} = 2 \sum_{l=0}^{n-1} (-1)^l e^{lt}.$$  

From (1), (2) and (4), we derive

$$\sum_{m=0}^{\infty} \left( E_m + (-1)^{n+1} E_m(n) \right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left( 2 \sum_{l=0}^{n-1} (-1)^l t^l \right) \frac{t^m}{m!}.$$  

Therefore we obtain the following theorem:

**Theorem 1.** Let $m, n$ be positive integers ($n > 1$). Then we have

$$\sum_{l=0}^{n-1} (-1)^l t^l = \frac{1}{2} \left( (-1)^{n+1} E_m(n) + E_m \right).$$

That is,

$$T_m(n) = \frac{(-1)^{n+1}}{2} \sum_{l=0}^{m-1} \binom{m}{l} E_l n^{m-l} + \frac{E_m}{2} \left( 1 + (-1)^{n+1} \right).$$

If $n \equiv 0(\text{mod}2)$ then

$$T_m(n) + \frac{1}{2} \sum_{l=0}^{m-1} \binom{m}{l} E_l n^{m-l} = 0.$$  

### 3. Euler-Zeta function

Let $\Gamma(s)$ be the gamma function. Then we easily see that

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} F(-t, x) dt = \sum_{n=0}^\infty \frac{(-1)^n 2}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(n+x)t} dt = 2 \sum_{n=0}^\infty \frac{(-1)^n}{(n+x)^s}, \; s \in \mathbb{C}.$$  

Thus, we can define the Euler-zeta function follows:
Definition 2. For $s \in \mathbb{C}$, $x \in \mathbb{R}$ with $0 < x < 1$, define

$$
\zeta_E(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + x)^s}, \text{ and, } \zeta_E(s) = \zeta_E(s, 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.
$$

Lemma 3. For $s \in \mathbb{C}$, we have

$$
\zeta_E(s, x) = \frac{2}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \frac{e^{-xt}}{1 + e^{-t}} dt.
$$

From (2) and Lemma 3, we can derive the below Theorem 4.

Theorem 4. Let $n$ be the positive integer. Then we have

$$
\zeta_E(-n, x) = E_n(x).
$$

Remark. From Theorem 4, we note that

$$
1 = E_0(1) = \zeta_E(0) = 2(1 - 1 + 1 - 1 + \cdots), \\
\frac{1}{2} = E_1(1) = \zeta_E(-1) = 2(1 - 2 + 3 - 4 + \cdots), \\
0 = E_2(1) = \zeta_E(-2) = 2(1^2 - 2^2 + 3^2 - 4^2 + \cdots), \\
\cdots.
$$

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