Nonassociative Gravity in String Theory?

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Abstract

In an on-shell conformal field theory approach, we find indications of a three-bracket structure for target space coordinates in general closed string backgrounds. This generalizes the appearance of noncommutative gauge theories for open strings in two-form backgrounds to a putative noncommutative/nonassociative gravity theory for closed strings probing curved backgrounds with non-vanishing three-form flux. Several aspects and consequences of the three-bracket structure are discussed and a new type of generalized uncertainty principle is proposed.
1 Introduction

String theory provides a unified framework for both quantum gravity and quantum gauge theories, that is for all interactions we observe in nature, where gravity arises from closed strings while gauge theories are due to open strings. It is well known that gauge theories can become ultraviolet finite via the procedure of renormalization, however in order to provide an ultraviolet finite theory of gravity one expects the notion of a smooth space-time to break down at very short distances. Noncommutative (NC) geometry provides a mathematical framework to describe such space-times.

In view of this general expectation, it appears somewhat surprising that noncommutative geometry first appeared in a clear way on the boundary of open strings attached to a D-brane carrying non-trivial two-form flux [1, 2, 3]. This string theoretical result corresponds to the quantum mechanical Landau quantization of cyclotron orbits of a charged particle in a constant magnetic field. Therefore, it is the gauge theory which becomes noncommutative [4] and not the gravity theory. The origin of this noncommutativity for the end-points of open strings can be traced back to the fact that on the boundary of a disk, one can define an ordering of two points close to each other. Inserting vertex operators and introducing a non-trivial background which is sensitive to the ordering – such as a constant two-form flux – can lead to noncommutativity. On the other hand, the closed string analogue is clearly different as here two vertex operators are inserted in the bulk of a two-sphere $S^2$ and no unambiguous ordering can be defined. Therefore, one does not expect the same kind of noncommutativity to arise. (See [5] for a proposal of closed string NC geometry, and more recently [6].) These statements can be checked explicitly by computing the equal-time, equal-position commutator $\lim_{\sigma' \to \sigma} [X^\mu(\sigma, \tau), X^\nu(\sigma', \tau)]$ for open and for closed strings.

However, if one considers three nearby points on the world-sheet $S^2$ of a closed string, one can very well decide whether the loop connecting the three points has positive or negative orientation. Thus, if there exists a background field which distinguishes these two orientations, one would expect a non-vanishing result not for the simple commutator, but for the cyclic double commutator

$$[X^\mu, X^\nu, X^\rho] := \lim_{\sigma_i \to \sigma} \left[[X^\mu(\sigma_1, \tau), X^\nu(\sigma_2, \tau)], X^\rho(\sigma_3, \tau)\right] + \text{cyclic} . \tag{1}$$

To get a better understanding of this expression, let us mention that for a fundamental product $x^i \bullet x^j$ one can define a three-bracket as

$$[x^1, x^2, x^3] = \sum_{\sigma \in P_3} \text{sign}(\sigma) \left((x^{\sigma(1)} \bullet x^{\sigma(2)}) \bullet x^{\sigma(3)} - x^{\sigma(1)} \bullet (x^{\sigma(2)} \bullet x^{\sigma(3)})\right), \tag{2}$$

being the completely anti-symmetrized associator of this $\bullet$-product. Note that
for an associative product, this expression vanishes which is also known as the Jacobi-identity. Therefore, a non-vanishing cyclic double commutator (1) indicates that a \( \bullet \)-product is both noncommutative and nonassociative (NCA).

In this article, we pursue a direct attempt to establish a non-trivial three-bracket for a certain class of treatable examples\(^1\). A natural candidate for a background field leading to such a structure in the closed string sector is the three-from field strength \( \mathcal{H} \). The main technical challenge is that via Einstein’s equation, a non-trivial \( \mathcal{H} \)-flux induces a curvature of space, so that in general we have no means to solve the resulting non-linear sigma model (NLSM) explicitly. However, WZW-models \(^{11}\) are known to describe compactifications on group manifolds with non-trivial \( \mathcal{H} \)-flux \(^{12}\). For the simplest case of \( SU(2) \), the model is equivalent to a closed string propagating on \( S^3 \) with \( \mathcal{H} \)-flux. This is the prime example of a bosonic string compactification that we will elaborate on in this paper. A second challenge is that when employing the framework of conformal field theory, computations are done on-shell. Therefore, the background is guaranteed to satisfy the string equations of motions and certain structures might not be visible (or indistinguishable from known results) since they vanish (or agree with familiar results) on-shell.

In the following discussion it will become clear that our arguments are based on one technical assumption, leading to (at least) three possible interpretations of our result. We discuss each of these possibilities in some detail, however, to reach a conclusive picture further investigation is necessary.

This paper is organized as follows. In section 2, we recall some features of open strings ending on a D-brane endowed with background two-form flux. We will be brief and restrict ourselves to aspects important in the course of this paper. In section 3, we discuss the \( SU(2)_k \) WZW model and its geometric non-linear sigma model interpretation. In section 4, we carry out in detail a conformal field theoretic computation of a cyclic double commutator, which is to be considered the central part of this article. In section 5, we discuss some of the formal and conceptual consequences of our result from a wider perspective, and present a proposal for a generalized nonassociative gravity theory based on a three-bracket for the left- as well as for the right-moving sector of the closed string. Section 6 contains our conclusions. Some important technical details are collected in the appendix, where in particular a quantum mechanical derivation of a new type of uncertainty relation resulting from a non-vanishing three-bracket can be found.

\(^1\)Such a three-bracket has been discussed in M-theory for open membranes moving in a constant \( C^3 \)-form background \(^{7,8,9}\). Moreover, algebras with Lie-type three-brackets have appeared in the formulation of an effective field theory on a stack of M2-branes \(^{10}\).
2 Open String Noncommutativity

To illustrate the analogy between the open string and our upcoming discussion for the closed string, let us first review the quantization of an open string in a background with constant two-form flux. Here, we follow the discussion in [2] where more details can be found.

We consider an open string with both endpoints on a $Dp$-brane carrying constant two-form flux $F_{ij} = B_{ij} + F_{ij}$, where $i, j = 0, \ldots, p$. This leads to mixed Neumann-Dirichlet boundary conditions longitudinal to the brane, so that the mode expansions for the corresponding free bosons read

$$X^i(\sigma, \tau) = x^i_0 + (\alpha^i_0 \tau - \alpha^j_0 F^i_j \sigma) + \sum_{n \neq 0} \frac{e^{-im\tau}}{n} \left( i \alpha^i_n \cos(n\sigma) - \alpha^j_n F^i_j \sin(n\sigma) \right) . \quad (3)$$

Here we normalized $0 \leq \sigma \leq \pi$, and indices of $F_{ij}$ are raised by the inverse metric $\eta^{ij} = \text{diag}(-1, +1, \ldots, +1)$. As carried out in [2], the commutation relations for the modes appearing in (3) can be obtained via canonical quantization. Using these relations, the equal-time commutator is evaluated as

$$[X^i(\sigma_1, \tau), X^j(\sigma_2, \tau)] = -2i\alpha'(M^{-1}F)^{ij} \left[ P(\sigma_1, \sigma_2) + \sum_{n \neq 0} \frac{\sin n(\sigma_1 + \sigma_2)}{n} \right] , \quad (4)$$

where $M_{ij} = \delta_{ij} - F^k_i F_{kj}$ and matrix products are understood. The function $P$ is a continuous linear expression in the world-sheet coordinates $\sigma_i$ of the form

$$P(\sigma_1, \sigma_2) = \sigma_1 + \sigma_2 - \pi , \quad (5)$$

which arises purely from the commutation relations involving the zero modes $x^i_0$ and $\alpha^i_0$. The sum in (4) originates from the oscillator modes $\alpha^i_n$ for $n \neq 0$, and can be further evaluated using the Fourier transform

$$\gamma(\varphi) = \sum_{n=1}^{\infty} \frac{\sin(n\varphi)}{n} = \begin{cases} \frac{1}{2}(\pi - \varphi) & 0 < \varphi < 2\pi , \\ \frac{1}{2}\pi & \varphi = 0, 2\pi . \end{cases} \quad (6)$$

Coming back to the commutator, using equations (5) and (6), we see that for $0 < \sigma_1 + \sigma_2 < 2\pi$ the two terms in (4) cancel. However, on the boundaries $\sigma_1 = \sigma_2 = 0$ and $\sigma_1 = \sigma_2 = \pi$ one obtains

$$[X^i(0, \tau), X^j(0, \tau)] = -[X^i(\pi, \tau), X^j(\pi, \tau)] = 2\pi i\alpha'(M^{-1}F)^{ij} . \quad (7)$$

In summary, the equal-time, equal-position commutator between two target-space coordinates $X^i(\sigma, \tau)$ does not vanish along a D-brane carrying non-trivial two-form flux $F_{ij}$. For later purpose, let us emphasize two points of this computation:
Even without knowing the zero mode contribution $P(\sigma_1, \sigma_2)$ explicitly, we could have guessed this function by requiring the commutator (4) to vanish for generic points on the world-sheet. In turn, the non-zero result in (7) arises from the boundaries of the open string due to the discontinuity of $\gamma(\varphi)$ at $\varphi = 0 \mod 2\pi$.

Since the equal-time, equal-position commutator (7) is independent of the world-sheet coordinates $\sigma$ and $\tau$, one can indeed conclude that this world-sheet computation reveals a feature of the target space (as probed by an open string).

Another way to detect the noncommutative nature of the setting above is to consider the two-point function of two fields $X^i(\sigma, \tau)$. In particular, in the presence of a constant two-form flux $F_{12} = f$ we compute for instance \[13, 14\]

$$\langle X^1(z_1) X^2(z_2) \rangle = \alpha' \frac{f}{1 + f^2} \log \left( \frac{z_1 - z_2}{z_1 - z_2} \right),$$

where in order to work on an Euclidean world-sheet we have performed a Wick rotation $\tau_i \rightarrow i \tau_i$ and introduced $z_i = \exp(\tau_i + i\sigma_i)$. As illustrated in figure 1, the function (8) has a jump when changing the order of $z_1$ and $z_2$ on the real line, which indicates the noncommutativity.

To conclude this section, we note that the result of a two-form flux inducing noncommutativity of brane coordinates is completely general, and has also been studied for co-dimension one branes in the $SU(2)$ WZW model \[15\]. However, due to a background $H$-flux in this case, it turns out that the obtained structure is not only noncommutative but also nonassociative \[15, 16, 17\].
3 Closed Strings on Curved Spaces

Motivated by the results for the open string, our strategy now is to compute the expression (1) for the closed string. However, since we want to consider non-trivial $H$-flux, we have to work on curved background spaces on which we can introduce only local coordinates.

The SU(2) WZW Model

Let us start our discussion by considering the WZW model for the group manifold $SU(2)$. The corresponding action is given by

$$S = \frac{k}{16\pi} \int_{\partial \Sigma} d^2x \text{Tr} \left[ (\partial_\alpha g)(\partial^\alpha g^{-1}) \right]$$

$$- \frac{ik}{24\pi} \int_{\Sigma} d^3y \epsilon^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}} \text{Tr} \left[ (g^{-1}\partial_\alpha g)(g^{-1}\partial_\beta g)(g^{-1}\partial_\gamma g) \right],$$

(9)

where $k \in \mathbb{Z}^+$ denotes the level and $\Sigma$ is a three-dimensional manifold with boundary $\partial \Sigma$. The indices take values $\alpha = 1, 2$ and $\tilde{\alpha}, \ldots = 1, 2, 3$, which are raised or lowered by the metrics $h_{\alpha\beta} = \text{diag}(+1, +1)$ and $h_{\tilde{\alpha}\tilde{\beta}} = \text{diag}(+1, +1, +1)$, respectively. Parametrizing an element $g \in SU(2)$ as

$$g = \begin{pmatrix} e^{i\eta_1} \cos \eta_1 & e^{i\eta_3} \sin \eta_1 \\ -e^{-i\eta_3} \sin \eta_1 & e^{-i\eta_1} \cos \eta_1 \end{pmatrix},$$

(10)

with $0 \leq \eta_1 \leq \pi/2$, $0 \leq \eta_2, \eta_3 \leq 2\pi$, the first term in (9) can be written as

$$S_{\text{kin.}} = \frac{k}{8\pi} \int_{\partial \Sigma} d^2x \left[ \partial_\alpha \eta^1 \partial^\alpha \eta^1 + (\cos \eta^1)^2 \partial_\alpha \eta^2 \partial^\alpha \eta^2 + (\sin \eta^1)^2 \partial_\alpha \eta^3 \partial^\alpha \eta^3 \right].$$

(11)

Note that (11) is a non-linear sigma model (in conventions $\alpha' = 2$) with target space $S^3$, where the latter is given in Hopf coordinates $\eta^i$ with metric

$$ds^2 = k \left[ (d\eta^1)^2 + (\cos \eta^1)^2 (d\eta^2)^2 + (\sin \eta^1)^2 (d\eta^3)^2 \right].$$

(12)

By comparing with the usual metric on $S^3$, one infers that the radius of the three-sphere is given by $R = \sqrt{k}$. The second term in the WZW model (9) can be expressed as

$$S_{\text{WZW}} = -\frac{ik}{12\pi} \int_{\Sigma} d^3y \epsilon_{ijk} e^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}} \sin \eta^1 \cos \eta^1 \partial_\alpha \eta^i \partial_\beta \eta^j \partial_\gamma \eta^k,$$

(13)

which corresponds to a background flux

$$H = -2k \sin \eta^1 \cos \eta^1 d\eta^1 \wedge d\eta^2 \wedge d\eta^3.$$

(14)
To summarize, the $SU(2)$ WZW model at level $k$ (with central charge $c = 3k/[k + 2]$) is equivalent to the non-linear sigma model on $S^3$ with radius $R = \sqrt{k}$ and background flux (14).

Recall also that in a string theory context, this configuration, together with a linear dilaton background of central charge $c = 1 + \frac{k}{k} + \mathcal{O}(k^{-2})$, describes the deep throat limit of an NS five-brane geometry [18]. Furthermore, it is remarkable that the metric $G$ and the $B$-field are not changed due to higher order $\alpha'$-corrections. Indeed, as shown in [19], utilizing that $S^3$ is parallelizable, such corrections only re-adjust the dilaton while leaving $G$ and $B$ at their tree level values. However, in this paper, we mostly focus on the WZW part and will only comment on the dilaton in section 5.

**Conserved Currents and Kac-Moody Algebras**

Solving the model (10) directly in terms of Hopf coordinates $\eta^i$ is not easily possible, but it is well known that the WZW model actually is exactly solvable. To see this, we introduce a complex coordinate $z = \exp(x^1 + ix^2)$ and define the currents

$$J^a = J^a \frac{\sigma^a}{\sqrt{2}} = -k \left( \partial_z g \right) g^{-1}, \quad \overline{J}^a = \overline{J}^a \frac{\sigma^a}{\sqrt{2}} = +k g^{-1} \left( \partial_{\overline{z}} g \right). \quad (15)$$

Note that here and in the following, $\sigma^a$ with $a = 1, 2, 3$ are the Pauli matrices and summation over repeated indices is understood. From the equation of motion of the WZW model (9) it follows that the currents $J^a$ are holomorphic and that the $\overline{J}^a$ are anti-holomorphic. Therefore, one can perform the Laurent expansions

$$J^a(z) = \sum_{n \in \mathbb{Z}} j^a_n z^{-n-1}, \quad \overline{J}^a(\overline{z}) = \sum_{n \in \mathbb{Z}} \overline{j}^a_n \overline{z}^{-n-1}. \quad (16)$$

The symmetry transformations of the WZW model then translate into the following commutation relations for the modes $j^a_n$ and $\overline{j}^a_n$

$$[j^a_m, j^b_n] = if^{abc} j^c_{m+n} + km \delta_{m+n} \delta^{ab}, \quad [\overline{j}^a_m, \overline{j}^b_n] = if^{abc} \overline{j}^c_{m+n} + km \delta_{m+n} \delta^{ab}, \quad [j^a_m, \overline{j}^b_n] = 0, \quad (17)$$

which define two independent Kac-Moody algebras. Note that the structure constants for $SU(2)$ in our convention read $f^{abc} = \sqrt{2} \epsilon^{abc}$, and indices are raised or lowered by $\delta^{ab}$ and $\delta_{ab}$, respectively.

To study the properties of the WZW model in more detail, let us employ the parametrization (10) in (13) and express the two currents (10) as follows

$$J^a(z) = -i\sqrt{2} k E^a_i \partial_z \eta^i, \quad \overline{J}^a(\overline{z}) = -i\sqrt{2} k E^a_i \partial_{\overline{z}} \eta^i. \quad (18)$$
where a summation over $i = 1, 2, 3$ is understood. The matrices $E$ and $\overline{E}$ depend on $\eta^1$ as well as on $\eta_{23}^{23} = \eta^2 \pm \eta^3$ and are given by

$$E = \begin{pmatrix} \sin \eta_{23}^{23} & - \sin \eta^1 \cos \eta^1 \cos \eta_{23}^{23} & \sin \eta^1 \cos \eta^1 \cos \eta_{23}^{23} \\ \cos \eta_{23}^{23} & \sin \eta^1 \cos \eta_{23}^{23} & - \sin \eta^1 \cos \eta_{23}^{23} \\ 0 & (\cos \eta^1)^2 & (\sin \eta^1)^2 \end{pmatrix}, \quad (19)$$

and

$$\overline{E} = \begin{pmatrix} \sin \eta_{23}^{23} & - \sin \eta^1 \cos \eta^1 \cos \eta_{23}^{23} & - \sin \eta^1 \cos \eta_{23}^{23} \\ - \cos \eta_{23}^{23} & - \sin \eta^1 \cos \eta_{23}^{23} & - \sin \eta^1 \cos \eta_{23}^{23} \\ 0 & -(\cos \eta^1)^2 & -(\sin \eta^1)^2 \end{pmatrix}. \quad (20)$$

**Geometric Interpretation**

We now turn to a geometric interpretation of the above setting. In particular, the matrices $E$ and $\overline{E}$ can be used to define two three-beins

$$e^a = \sqrt{k} E^a_i \, d\eta^i, \quad \overline{e}^a = \sqrt{k} \overline{E}^a_i \, d\eta^i, \quad (21)$$

which diagonalize the metric (12), that is $ds^2 = \sum_a e^a \otimes e^a = \sum_a \overline{e}^a \otimes \overline{e}^a$. The corresponding vector fields are given by

$$e_a = \frac{1}{\sqrt{k}} E^i_a \partial_{\eta^i}, \quad \overline{e}_a = \frac{1}{\sqrt{k}} \overline{E}^i_a \partial_{\eta^i}, \quad (22)$$

where $E^i_a$ denotes the inverse transpose of (19) and similarly for $\overline{E}^i_a$. These vector fields satisfy commutation relations

$$[e_a, e_b] = C_{ab}^\ c \, e^c, \quad [\overline{e}_a, \overline{e}_b] = \overline{C}_{ab}^\ c \, \overline{e}^c, \quad (23)$$

with structure constants $C_{ab}^\ c = \overline{C}_{ab}^\ c = \sqrt{2/k} \ f_{ab}^\ c$. Returning to the three-beins (21), by explicit computation we find that

$$de^a + \frac{1}{\sqrt{2k}} f_{bc}^\ a \, e^b \wedge e^c = 0, \quad d\overline{e}^a + \frac{1}{\sqrt{2k}} f_{bc}^\ a \, \overline{e}^b \wedge \overline{e}^c = 0. \quad (24)$$

With the help of Cartan’s structure equations $de^a + \omega^a_b \wedge e^b = T^a$ and $d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_{bcd}$, one identifies and computes for the torsion-free connection that

$$\omega^a_b = - \frac{1}{\sqrt{2k}} \ f_{bc}^\ a \, e^c, \quad \overline{\omega}^a_b = - \frac{1}{\sqrt{2k}} \ f_{bc}^\ a \, \overline{e}^c, \quad (25)$$

$$T^a = 0, \quad \overline{T}^a = 0, \quad (26)$$

$$R^a_{bcd} = + \frac{1}{2k} \ f_{bp}^\ a \ f_{cd}^\ p, \quad \overline{R}^a_{bcd} = + \frac{1}{2k} \ f_{bp}^\ a \ f_{cd}^\ p. \quad (27)$$

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In addition, the $H$-flux (14) can be expressed in terms of the three-beins (21) as

$$H = -\frac{2}{\sqrt{k}} e^1 \wedge e^2 \wedge e^3 = +\frac{2}{\sqrt{k}} \bar{e}^1 \wedge \bar{e}^2 \wedge \bar{e}^3.$$  (28)

Let us emphasize the important technical detail of the same signs in (25) for $e^a$ and $\bar{e}^a$ reflecting a left-right symmetric coupling of the metric in the string action. On the other hand, the opposite signs in (28) show that the $B$-fields couple in a left-right asymmetric fashion. (As usual, left refers to the holomorphic and right to the anti-holomorphic part.)

Coming back to the exact solvability of the WZW model, geometrically this is related to the fact that one can define torsion-full connections in terms of (25) and $H_{abc}$ as

$$\Omega^+_{ab} = \omega^a_b + \frac{1}{2} H_{abc} e^c,$$  
$$\Omega^-_{ab} = \omega^a_b - \frac{1}{2} H_{abc} \bar{e}^c,$$  (29)

where $H_{abc}$ can be deduced from $H = \frac{1}{3!} H_{abc} e^a \wedge e^b \wedge e^c$. For these connections one finds

$$T^+_{abc} = -H_{abc},$$  
$$T^-_{abc} = +H_{abc},$$  
$$R^+_{abcd} = \frac{2}{k} (f_{aba} f^a_{cd} + f_{cau} f^a_{bd} + f_{bcu} f^a_{ad}) = 0,$$  
$$R^-_{abcd} \equiv 0,$$  (30)

where for the vanishing of $R^+$ the Jacobi identity was employed, and $R^-$ vanishes identically. Note that since $S^3$ is parallelizable, it was expected that there indeed exist connections with vanishing curvature.

### 4 Closed String Nonassociativity

In the previous section, we have mainly reviewed the well-known geometry for the exactly solvable $SU(2)_k$ WZW model. However, let us now introduce fields $X^a(z, \bar{z})$ according to

$$J^a(z) = -i \sqrt{k} \partial_z X^a(z, \bar{z}) = -i \sqrt{2} k E^a_i(i\bar{\eta}) \partial_z \eta^i(z, \bar{z}),$$  
$$\bar{J}^a(z) = -i \sqrt{k} \partial_{\bar{z}} X^a(z, \bar{z}) = -i \sqrt{2} k \bar{E}^a_i(i\bar{\eta}) \partial_{\bar{z}} \eta^i(z, \bar{z}),$$  (32)

where we have indicated that the matrices $E^a_i$ and $\bar{E}^a_i$ depend on the target space coordinates $\eta^i$. It is clear that the $X^a$ do not correspond to bona fide global coordinates on $S^3$ since there does not exist a flat metric on $S^3$. However, as shown in [20], if the $X^a(z, \bar{z})$ satisfy their (free) equations of motion, the $\eta^i(z, \bar{z})$ do so as well.
In the present section we compute the cyclic double commutator of three local coordinates \( X^a \) at a specific point on the target space. Since the double commutator is a local quantity, we can imagine to probe the geometry around a point \( \vec{\eta}_0 \) on a three-sphere \( S^3 \) by a closed string. Writing then
\[
X^a(z, \bar{z}) = X^a(z) + \bar{X}^a(\bar{z}) \tag{33}
\]
and using (32), locally we can identify the left- and right-moving coordinates as
\[
X^a(z) \simeq \sqrt{2k} \mathbf{E}_i^a(\vec{\eta}_0) \eta^i(z) , \quad \bar{X}^a(\bar{z}) \simeq \sqrt{2k} \mathbf{E}_i^a(\vec{\eta}_0) \bar{\eta}^i(\bar{z}) . \tag{34}
\]
The mode expansions of \( X^a(z) \) and \( \bar{X}^a(\bar{z}) \) are found by integrating the expansions of the currents given in (16). In particular, for the holomorphic part we arrive at
\[
X^a(z) = \frac{i}{\sqrt{k}} x^a_0 - \frac{i}{\sqrt{k}} j^a_0 \log z + \frac{i}{\sqrt{k}} \sum_{n \neq 0} j^a_n n z^{-n} , \tag{35}
\]
and a similar expression is obtained for the anti-holomorphic part \( \bar{X}^a(\bar{z}) \). The modes \( j^a_n \) in (35) satisfy the corresponding Kac-Moody algebra given in (17), however, a priori it is not clear what the precise form of the commutation relations involving \( x^a_0 \) is. In the following, we are going to fix this contribution in analogy to the open string as in section 2.

Cyclic Double Commutator

Let us consider the cyclic double commutator for the holomorphic part \( X^a(z) \) of the free fields (33)
\[
[X^a(z_1), X^b(z_2), X^c(z_3)] = \left[ [X^a(z_1), X^b(z_2)], X^c(z_3) \right] + \text{cyclic} , \tag{36}
\]
evaluated at equal times. For our choice of complex coordinates \( z_i = \exp(\tau_i + i \sigma_i) \) this implies \( |z_1| = |z_2| = |z_3| \), which will always be understood for the expression (36). To simplify the following formulae, let us furthermore introduce \( x^a, p^a \) and \( j^a \) as
\[
x^a = \frac{i}{\sqrt{k}} x^a_0 , \quad p^a(z) = -\frac{i}{\sqrt{k}} j^a_0 \log z , \quad j^a(z) = \frac{i}{\sqrt{k}} \sum_{n \neq 0} j^a_n n z^{-n} . \tag{37}
\]
For the computation of (36), we first collect all terms involving \( x^a \) into a so far undetermined function \( P^{abc} \)
\[
P^{abc}(z_1, z_2, z_3) = [x^a, x^b, x^c] + [x^a, x^b, \cdot] + [x^a, \cdot, \cdot] + \ldots . \tag{38}
\]
For all other contributions in (36), we employ the Kac-Moody algebra (17) of the modes \( j^a_n \). In particular, we compute
\[
[p^a(z_1), p^b(z_2), p^c(z_3)] \sim (f^{ab}_u f^{uc}_v + f^{bc}_u f^{ua}_v + f^{ca}_u f^{ub}_v) j^a_0 = 0 , \tag{39}
\]
which vanishes due to the Jacobi identity for the structure constants $f^{abc}$. In a similar fashion, we obtain vanishing expressions for

$$[p^a(z_1), p^b(z_2), j^c(z_3)] + [j^a(z_1), p^b(z_2), p^c(z_3)] = 0,$$

$$[p^a(z_1), j^b(z_2), j^c(z_3)] + [j^a(z_1), p^b(z_2), j^c(z_3)] = 0,$$

(40)

where for the second line we utilized in addition to the Jacobi identity that $f_{abc} = f_{bca} = f_{cab}$. However, for the double commutator involving three $j^a$, we find the non-trivial result

$$[j^a(z_1), j^b(z_2), j^c(z_3)] = -f^{abc} \sqrt{k} \sum_{n,m \neq 0} \frac{1}{nm} \left( \frac{z_3}{z_1} \right)^n \left( \frac{z_3}{z_2} \right)^m + \text{cyclic}.$$  

(41)

Remember that this expression is understood to be evaluated at equal times. For the right-hand side in (41), we split the sums in the following way and compute

$$\Gamma(\sigma_1, \sigma_2, \sigma_3) = -\sum_{n,m \neq 0} \frac{1}{nm} \left( \frac{z_3}{z_1} \right)^n \left( \frac{z_3}{z_2} \right)^m - \sum_{n \neq 0} \frac{1}{n^2} \left( \frac{z_2}{z_1} \right)^n + \text{cyclic}$$

$$= \begin{cases} -\pi^2 & \sigma_1 = \sigma_2 = \sigma_3, \\ 0 & \text{else}. \end{cases}$$  

(42)

Combining the above results, we arrive at an expression for the equal-time double commutator of the holomorphic fields $X^a(z)$ of the form

$$[X^a(z_1), X^b(z_2), X^c(z_3)] = \mathcal{P}^{abc}(z_1, z_2, z_3) + \frac{f^{abc}}{\sqrt{k}} \Gamma(\sigma_1, \sigma_2, \sigma_3).$$  

(43)

For the computation in the anti-holomorphic sector, we note that the modes $\overline{j}_n^a$ satisfy the same Kac-Moody algebra as $j_n^a$. Furthermore, we have $\overline{z}_i = e^{\tau_i - i\sigma_i}$, and so we only need to replace $\sigma_i \rightarrow -\sigma_i$ in the result (43) for the holomorphic sector. However, observe that the function $\Gamma$ is invariant under that substitution. Therefore, the result for the full equal-time double commutator reads

$$[X^a(z_1, \overline{z}_1), X^b(z_2, \overline{z}_2), X^c(z_3, \overline{z}_3)]$$

$$= \mathcal{P}^{abc}(z_1, \overline{z}_2, \overline{z}_3) + \overline{\mathcal{P}}^{abc}(\overline{z}_1, z_2, \overline{z}_3) + 2 \frac{f^{abc}}{\sqrt{k}} \Gamma(\sigma_1, \sigma_2, \sigma_3).$$  

(44)
Zero Mode Contribution and Final Result

It is now tempting to follow the same logic as for the open string computation. That is, we fix the unknown contribution \( P + \overline{P} \) of the zero modes \( x^a_0 \) and \( \overline{x}^a_0 \) by:

Assumption: The zero mode contribution \( P + \overline{P} \) is continuous; and for the three points \( z_i \) not all equal, the equal-time double commutator has to vanish.

More concretely, this assumption means that

\[
P^{abc}(z_1, z_2, z_3) + \overline{P}^{abc}(\overline{z}_1, \overline{z}_2, \overline{z}_3) = 0 . \tag{45}
\]

Using then \((42)\) and \((45)\) in \((44)\), we arrive at the following expression for the equal-time, equal-position cyclic double commutator:

\[
[X^a, X^b, X^c] := \lim_{z_i \to z} [X^a(z_1, \overline{z}_1), X^b(z_2, \overline{z}_2), X^c(z_3, \overline{z}_3)] = -\frac{2\pi^2}{\sqrt{k}} f^{abc} . \tag{46}
\]

Therefore, pursuing the same reasoning as for the open string, we are led to the intriguing result that the fields \( X^a \) satisfy a non-vanishing three-bracket, where the right-hand side is constant and proportional to the \( SU(2) \) structure constants \( f^{abc} \). Let us make a few further comments:

1. As for the open string, the mathematical reason behind the result \((46)\) is the discontinuity of \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \) in \((12)\).

2. The equal-time, equal-position double commutator is independent of the world-sheet coordinates. Thus, it is expected to reflect a property of the target space (as probed by a closed string).

3. Recalling that the radius of the three-sphere is \( R = \sqrt{k} \), we realize that in the large radius limit \( R \to \infty \) the NCA effect vanishes.

4. We also computed the single commutator \( \lim_{z_i \to z} [X^a(z_1, \overline{z}_1), X^b(z_2, \overline{z}_2)] \) and found it to be dependent on the world-sheet coordinates. We therefore conclude that the fundamental, well-defined target space structure is a three-bracket. Clearly, if a two-bracket with Lie-algebra structure would be well-defined then the three-bracket vanishes.

\footnote{For ease of notation, when the dependence of the fields \( X^a(z, \overline{z}) \) on the world-sheet coordinates \((z, \overline{z})\) is omitted, the cyclic double commutator is understood to be evaluated at equal-time \textit{and} equal-position.}

\footnote{We are grateful to Chong-Sun Chu for the following comment: for a particle in the presence of a magnetic field \( B \) in three space dimensions a result similar to \((46)\) can be obtained. In particular, in this situation it is known that the covariant derivatives only satisfy the Jacobi-identity up to a term \( \text{div}(B) \), that is the cyclic double commutator does not vanish for a magnetic monopole.}
Three-Point Function

Finally, relating to our discussion in section 2, we show that the correlation function of three fields \( X^a \) features a jump. Integrating the three-point functions of three currents \( J^a(z) \) and \( T^a(\tau) \) we obtain

\[
\langle X^a(z_1, \tau_1) X^b(z_2, \tau_2) X^c(z_3, \tau_3) \rangle = \frac{\pi^2}{\sqrt{k}} \frac{f^{abc}}{9} \left[ L\left( \frac{z_{12}}{z_{13}} \right) + L\left( \frac{z_{13}}{z_{23}} \right) + L\left( \frac{z_{23}}{z_{12}} \right) + \text{c.c.} \right], \tag{47}
\]

where \( z_{ij} = z_i - z_j \) and where \( L(x) \) denotes the Rogers dilogarithm defined as

\[
L(x) = \frac{6}{\pi^2} \left( \text{Li}_2(x) + \frac{1}{2} \log(x) \log(1 - x) \right), \tag{48}
\]

with \( \text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \). Let us then choose for instance \( z_2 \in \mathbb{R} \), \( z_3 = 0 \) and study the behavior of (47). As shown in figure 2(a) for \( z_2 = 1 \) away from \( z_3 = 0 \) the three-point function features discontinuities when \( z_1 \to z_2 = 1 \) and when \( z_1 \to z_3 = 0 \). However, as illustrated in figure 2(b) when all three points in (47) approach each other, that is when \( z_1 \to z_3 = 0 \) and \( z_2 \to z_3 = 0 \), the three-point function develops a jump. We can therefore conclude that a non-vanishing double commutator is related to jumps in the three-point function when all three points coincide. This is analogous to the open string case where a jump in the two-point function indicates a non-vanishing commutator.
5 Speculations about the Three-bracket

In this section, we present possible interpretations of the non-vanishing three-bracket (46). This result was derived for the fields $X^a$, which were obtained by integrating the Kac-Moody currents (15), and it is quite remarkable that even though the Laurent modes $j^a_n$ and $\bar{j}^a_n$ of the Kac-Moody currents satisfy a Jacobi-identity, the equal-time, equal-position cyclic double commutator of three $X^a$ does not vanish.

More concretely, our aim is to identify the source of the noncommutative/non-associative structure obtained in the previous section. Since we are working on-shell, this is somewhat ambiguous. We therefore discuss three possibilities and postpone a conclusive answer to future studies [23]. To start, we first recall that the three-bracket (46), indicating the NCA nature of our setting, is proportional to the $SU(2)$ structure constants and identify three quantities in the three-bein basis (21) which contain these $f^{abc}$:

1. The structure constants $C^{abc}$ appearing in the Lie-algebra (23).
2. The torsion-free spin connection given in equation (25).
3. The components $H^{abc}$ of the $H$-field shown in (28).

For bona fide coordinates on the three-sphere, for instance the Hopf coordinates $\eta^i$ introduced in (10), each of these possibilities would lead to a different proposal for the origin of the NCA structure. However, unfortunately we are not able to quantize the system, compute the cyclic double commutator and detect a non-trivial three-bracket in the geometric basis directly.

**NCA Source: Structure Constants $C^{abc}$**

In the first case in the list above, the three-bracket in a geometric basis would be proportional to $C^{ijk}$ which vanishes since the geometric vector fields commute, that is $[\partial_{\eta^i}, \partial_{\eta^j}] = 0$. Therefore, we would not expect any new structure in the geometric basis.

**NCA Source: Connection $\omega^{abc}$**

In the second case, the three-bracket would be proportional to the completely anti-symmetrized part of the torsion-free connection $\omega^{abc}$. To make that more clear, we insert the local expressions (34) into the left- and right-moving three-brackets for $X^a(z)$ and $\bar{X}^a(\bar{z})$ and find

\[
\left[ \eta^i, \eta^j, \eta^k \right]_{\text{hol}} \simeq -\frac{\pi^2}{2k^2} \frac{\epsilon^{ijk}}{\det(E)}, \quad \left[ \bar{\eta}^i, \bar{\eta}^j, \bar{\eta}^k \right]_{\text{anti-hol}} \simeq -\frac{\pi^2}{2k^2} \frac{\epsilon^{ijk}}{\det(E)}. \tag{49}
\]
Here we assume that $\eta^i(z, \overline{z}) = \eta^i(z) + \overline{\eta}^i(\overline{z})$ and that three-brackets between mixed holomorphic $\eta^i(z)$ and anti-holomorphic fields $\overline{\eta}^i(\overline{z})$ vanish. However, let us emphasize that a priori these assumptions are not clear. Adding then both terms in (49) and noting that $\det(E) = -\det(\overline{E})$, the full three-bracket for the geometric coordinates $\eta^i(z, \overline{z})$ vanishes. Since the corresponding Christoffel connection is torsion-free and thus its totally antisymmetric part $\omega^{ijk}$ vanishes, this result would be consistent with the latter being the source for a non-vanishing three-bracket. Note furthermore, that the sum $\eta^i(z, \overline{z}) = \eta^i(z) + \overline{\eta}^i(\overline{z})$ is invariant under the world-sheet parity transformation exchanging left- and right-moving coordinates. Therefore, also the three-bracket for $\eta^i(z, \overline{z})$ is expected to be invariant, that is the three-bracket should be a geometric quantity associated with the metric.

Although this argument deserves further study, let us assume it to be correct and generalize the three-bracket (46) to any coordinate system (three-bein), not necessarily geometric, in the following way

$$[X^i, X^j, X^k] = \frac{\pi^2}{\sqrt{2}} (\alpha')^2 \omega^{ijk}, \quad (50)$$

where $i, \ldots = 1, 2, 3$. On dimensional grounds we have included the string scale $\alpha'$ and have employed that $f^{abc} = -\sqrt{2} k \omega^{abc}$. Note that generically the right-hand side of (50) is coordinate dependent and it remains to be seen how such a relation is to be understood mathematically.

Analogous to the star-product for noncommutative geometry, let us now introduce a three-product $F \triangle G \triangle H$ on the space of functions depending on the $X^i$ such that

$$[X^i, X^j, X^k] = \sum_{\sigma \in P_3} \text{sign}(\sigma) \ X^{\sigma(i)} \triangle X^{\sigma(j)} \triangle X^{\sigma(k)}. \quad (51)$$

This anti-symmetrized expression is also called the Nambu-Heisenberg commutator and one possibility to satisfy (51) is to choose the three-product as a generalization of the Moyal-Weyl star-product in the following way

$$F \triangle G \triangle H = \exp \left[ \frac{x^2}{6\sqrt{2}} (\alpha')^2 \omega^{ijk} \partial_x^i \partial_y^j \partial_z^k \right] F(x) G(y) H(z) \bigg|_{x=y=z}. \quad (52)$$

The properties of such a product, in particular for the case that $\omega^{ijk}$ is not completely antisymmetric or not constant, need to be understood better. Let us also comment on the dilaton, in particular on the contribution of a non-trivial dilaton gradient. We note that in [24] a more general connection $\Omega$ with torsion was considered, namely

$$\Omega^{\pm abc} = \omega^{abc} + \delta^{a\eta} \partial^\eta \Phi \pm \frac{1}{2} H^{abc}, \quad (53)$$

where $\omega^{abc}$ are again the components of the spin connection. These connections $\Omega^{\pm}$ include a non-trivial dilaton gradient, so it is conceivable that such a part has to be included in the three-product (52).
Finally, we recall that open strings ending on a D-brane give rise to a gauge theory and that, in the presence of background two-form flux on the D-brane, one can define a limit (the so-called Seiberg/Witten limit) in which this gauge theory is equivalent to a noncommutative gauge theory [4]. The product of two functions for the latter is given by the star-product. Motivated by this observation for the open string, the question arises whether one can define an NCA version of Einstein-gravity for the closed string, such that the higher order $\alpha'$-corrections of the $\omega$-connection are captured by the NCA three-product (52). We will come back to this point on page 17.

**NCA Source: H-flux**

The third possibility on page 14 for the origin of the NCA structure is the $H$-flux. This leads to the strongest proposal for the NCA source, as this flux does not vanish in any basis.

Relating to the discussion below equation (49), let us observe that the anti-symmetric combination of left- and right moving fields $\tilde{\eta}^i(z, \bar{z}) = \eta^i(z) - \eta^i(\bar{z})$ yields the three-bracket

$$\left[\tilde{\eta}^i, \tilde{\eta}^j, \tilde{\eta}^k\right] \simeq -\frac{\pi^2}{k^2} \frac{\epsilon^{ijk}}{\det(E)},$$

where we assumed again that three-brackets between mixed holomorphic $\eta^i(z)$ and anti-holomorphic fields $\eta^i(\bar{z})$ vanish. Note that now the left-hand side in (54) is odd under the world-sheet parity operation which exchanges left- and right-moving coordinates $\eta^i(z)$ and $\eta^i(\bar{z})$. This provides a strong argument that the right-hand side of (54) can be identified with the $H$-field, which is also odd under the world-sheet parity.

Now, in case that the $H$-flux is the source of the NCA structure, the generalization of our result (46) to any coordinate system is then given by

$$\left[X^i, X^j, X^k\right] = -\frac{\pi^2}{\sqrt{8}} (\alpha')^2 \left(T^+\right)^{ijk} = +\frac{\pi^2}{\sqrt{8}} (\alpha')^2 H^{ijk},$$

where the torsion $T^+$ was defined in (39) and where we have employed that $f_{abc} = -\sqrt{k/2} H^{abc}$. Furthermore, in this case the generalized Moyal-Weyl star-product reads

$$F \triangledown G \triangledown H = \exp \left[\frac{\pi^2}{12 \sqrt{2}} (\alpha')^2 H^{ijk} \partial_i \partial_j \partial_k\right] F(x) G(y) H(z) \bigg|_{x=y=z},$$

and one can again envision that higher order $\alpha'$-corrections of the $H$-flux are captured by the NCA three-product (56).

---

4This might be similar to the covariant gravity theory on the Moyal plane developed in [25].
Let us now discuss in some more detail the implications of a non-trivial three-bracket (55). We first recall that the computation leading to (46) was done in the framework of conformal field theory and that we did not refer to a specific embedding of the WZW model into string theory. In particular, this conformal field theory can be part of a bosonic as well as of a supersymmetric string theory compactification, and we expect in both cases that a non-vanishing $H$-flux leads to a non-trivial three-bracket. Furthermore, type IIA superstring theory is related to eleven-dimensional M-theory compactified on a circle. When uplifting the three-bracket structure to M-theory, we can expect a non-vanishing $G_4$-form flux to induce a four-bracket of the form

$$[X^i, X^j, X^k, X^l] \simeq G^{ijkl}.$$  \hfill (57)

Following our argument from the introduction, this is consistent with the fact that on a probe membrane one has to specify four-points to define an orientation. Again, it is tempting to introduce a four-product

$$F \circ G \circ H \circ K$$  \hfill (58)

defined in complete analogy to the three-product (56). Now, going back to the type IIA superstring by compactifying M-theory on a circle, the $G_4$-flux splits into $H$-flux as well as into R-R $F_4$-form flux. Therefore, we not only expect a non-vanishing three-bracket for the $H$-flux but in addition also a non-vanishing four-bracket corresponding to $F_4$-flux. Similarly, since M-theory compactified on a two-torus of vanishing volume is dual to the type IIB superstring, we expect a three-bracket for both the $H$-flux and the R-R $F_3$-form flux. However, note that the effect of these R-R fluxes cannot be detected by conformal field theory computations.

To summarize, if the origin of the three-bracket (46) is the $H$-flux, string dualities imply that R-R fluxes in superstring theories and four-form flux in M-theory also induce non-trivial three- and four-brackets for target space coordinates.

**Generalized Three-bracket Geometry**

Another intriguing possibility is to follow the idea of generalized geometry and keep the left- and right-moving coordinates completely separated, leading to two NCA geometries. Indeed, for the geometric coordinates our results are also compatible with the following two three-bracket relations

$$[X^i, X^j, X^k]_{\text{hol}} = \frac{\pi^2}{2 \sqrt{2}} (\alpha')^2 \left( \omega^{ijk} + \frac{1}{2} H^{ijk} \right) = \frac{\pi^2}{2 \sqrt{2}} (\alpha')^2 \Omega^{+ijk},$$

$$[\overline{X}^i, \overline{X}^j, \overline{X}^k]_{\text{anti-hol}} = \frac{\pi^2}{2 \sqrt{2}} (\alpha')^2 \left( \omega^{ijk} - \frac{1}{2} H^{ijk} \right) = \frac{\pi^2}{2 \sqrt{2}} (\alpha')^2 \Omega^{-ijk}. \hfill (59)$$
Instead of adding or subtracting these two equations from the very beginning, as it was discussed below (49) or done in (54), the new proposal is to first define two separate NCA gravity theories and combine them only at the very end. Thus, instead of a single three-product one is led to two products of the form

\[ F △ + G △ + H = \exp \left[ \frac{1}{12\sqrt{2}} \alpha'^2 \Omega^{+ij} \partial^i \partial^j \right] F(x) G(y) H(z) \bigg|_{x=y=z}, \]

\[ F △ - G △ - H = \exp \left[ \frac{1}{12\sqrt{2}} \alpha'^2 \Omega^{-ij} \partial^i \partial^j \right] F(x) G(y) H(z) \bigg|_{x=y=z}, \]

out of which one might construct two NCA Einstein-Hilbert like actions

\[ S_{\text{EH}}(△^+) \text{ and } S_{\text{EH}}(△^-) \]

(plus NCA deformations of the tree-level actions for the B2-form and the dilaton) in analogy to noncommutative gauge theory. The full string theoretical gravity theory should then be given by adding these two actions

\[ S = S_{\text{EH}}(△^+) + S_{\text{EH}}(△^-) \sim \int \hat{R}(△^+) + \hat{R}(△^-) + \ldots. \]

(61)

The crucial question is whether expanding such a putative generalized nonassociative gravity theory in terms of the fundamental fields \((g, H, \Phi)\) can be equivalent to the \(\alpha'\)-expansion of the effective string theory action.

Finally, we note that when uplifting this structure to M-theory, the natural expectation would be to obtain both a three-product \((\cdot △ \cdot △ \cdot △)\) determined by the geometric connection \(\omega^{ijk}\) as well as a four-product \((\cdot \odot \odot \odot \cdot)G\) determined by the four-form flux \(G^{ijkl}\).

**Four-bracket**

As emphasized above, one of our arguments for a non-trivial three-bracket structure is that the equal-time, equal-position cyclic double commutator of three fields \(X^a(\sigma, \tau)\) is independent of the world-sheet coordinates \(\sigma\) and \(\tau\). Motivated by this reasoning, let us try to define other objects which are independent of the world-sheet coordinates. For instance, since the curvature \(R_{ijkl}\) has four indices, it is conceivable that it appears in a four-bracket of the form

\[ [X^i, X^j; X^k, X^l] \simeq (\alpha')^3 R^{ijkl}. \]

(62)

Note that this four-bracket is not completely anti-symmetric, but a priori only has to reflect the symmetries of the curvature tensor. One way to define such a
four-bracket as an equal-time, equal-position three-commutator is the following

\[
[X^a, X^b ; X^c, X^d] = \lim_{\sigma_i \to \sigma} \left[ \left[ \left[ [X^a(\sigma_1, \tau), X^b(\sigma_2, \tau), X^c(\sigma_3, \tau)], X^d(\sigma_4, \tau) \right] \right] \right. 
\]

\[
- \left[ \left[ [X^a(\sigma_1, \tau), X^b(\sigma_2, \tau), X^d(\sigma_4, \tau)], X^c(\sigma_3, \tau) \right] \right. 
\]

\[
- [X^b X^a | X^c X^d] + [X^b X^a | X^d X^c] + [X^c X^d | X^a X^b] 
\]

\[
- [X^d X^c | X^a X^b] - [X^c X^d | X^b X^a] + [X^d X^c | X^b X^a], 
\]

(63)

where the underbracket stands for taking a cyclic sum, and we have employed a short-hand notation for the last six triple commutators. We have evaluated (63) for our example from section 3 and, neglecting the zero modes $x^a_0$, found

\[
[X^a, X^b ; X^c, X^d] = \lim_{\sigma_i \to \sigma} \frac{2}{k} \left( f^{ab} \mu^{acd} + f^{ca} \mu^{abd} + f^{bc} \mu^{uad} \right) F(\sigma_i, \tau) 
\]

\[
= \lim_{\sigma_i \to \sigma} (\mathcal{R}^+)_{abcd} F(\sigma_i, \tau) = 0 , 
\]

(64)

where $F(\sigma_i, \tau)$ is an expression depending on the world-sheet coordinates $\sigma_i$ and $\tau$. In the second line we have employed (31) and that the curvature $\mathcal{R}^+$ vanishes due to the Jacobi-identity. Therefore, from this conformal field theory analysis we cannot decide whether the four-bracket is generically zero or only vanishes on-shell.

We note that it would be interesting to find more objects of this kind, involving for instance not just the target space coordinates $X^a$ but also the target space momentum $P^a$. 

**Uncertainty Relation**

To conclude this section, let us comment on the physical implications of the non-vanishing three-bracket (46). Similar to a non-vanishing commutator in quantum mechanics, we expect a three-bracket to imply a generalized uncertainty relation. In the literature (for instance in [26, 8]) it was suggested that a three-bracket $[X, Y, Z] = \ell_s^3$ implies a triple uncertainty relation of the form

\[
\Delta X \Delta Y \Delta Z \geq \ell_s^3 . 
\]

(65)

In appendix A we made an attempt to generalize the derivation of the quantum mechanical uncertainty relation $\Delta p \Delta x > \hbar/2$ to the present case. We have obtained a result different from (65), in particular we found

\[
\Delta(X|Y)^2 \Delta Z^2 + \Delta(Y|Z)^2 \Delta X^2 + \Delta(Z|X)^2 \Delta Y^2 \geq \frac{\ell_s^6}{4} , 
\]

(66)
where $\Delta(X|Y)^2$ denotes the uncertainty of the commutator $i[X,Y]$ and similarly for the others (see appendix A for more details). This means that a non-vanishing three-bracket leads to an intertwined uncertainty relation for the position operators and their mutual commutators. Let us observe that this is consistent with our remark 4 on page 12 which stated that the single commutator of two target space coordinates is not a well-defined fundamental object. Furthermore, it would be interesting to generalize this computation to the case of a fundamental non-vanishing four-bracket.

6 Conclusions

The central concern of this paper was the study of $n$-bracket structures for target space coordinates as probed by a closed string. For this purpose, we have studied in detail closed strings moving on the three-sphere $S^3$ in the presence of background $H$-flux. In particular, employing the exact solvability of the $SU(2)_k$ WZW model we have performed a conformal field theory computation of the equal-time, equal-position cyclic double commutator of the fields $X^a$. Remarkably, despite the fact that the generators of the Kac-Moody algebra appearing in the mode expansion of $X^a$ satisfy a Jacobi-identity, we obtained a non-vanishing expression independent of the world-sheet coordinates. Therefore, we interpreted this result as an indication for a non-trivial three-bracket of the target space coordinates. However, in the course of the computation we made one technical assumption, and the identification of the source for the non-trivial three-bracket deserve further investigation and clarification.

Motivated by our findings, analogous to the appearance of a deformed bi-product in the open string sector (i.e. for a gauge theory), we have proposed to introduce a deformed three-product for the target space coordinates in the closed string sector (i.e. for the gravity theory). For a non-vanishing fundamental three-bracket, we also made a new proposal for the implied uncertainty relation. This was supported by an explicit generalization of the quantum mechanical derivation of the original Heisenberg uncertainty principle.

If a non-trivial three-bracket/three-product structure is indeed present in the closed sector of string theory, new conceptional questions arise both from a physical and mathematical point of view:

- Is it possible to reconstruct the string equations of motion, including all $\alpha'$-corrections, in a pure target space approach via an Einstein-like gravity theory on a nonassociative space-time?
- If the target space-time is NCA, how does the non-linear sigma-model of a string moving in such a background take this structure into account?
- To our knowledge the mathematical foundations of NCA geometries are far less developed than for noncommutative but associative geometries. If
indeed quantum gravity requires such a framework, can a mathematically rigorous NCA geometry incorporating a deformation or generalization of general covariance be developed.

Clearly, more work is needed to support or disprove the assumption and proposals made in this paper, and to advance in the directions proposed.

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A Generalized Uncertainty Principle

An important general question is what kind of uncertainty relation originates from a non-vanishing three-bracket of the form

\[ [X, Y, Z] = \ell_s^3. \]  

(67)

Here we present a possible self-consistent derivation, which follows closely the quantum mechanical derivation of the Heisenberg uncertainty relation.

In [27] it was noted that in a theory with a quantum Nambu three-bracket it may be useful to work on a Hilbert-space \( \mathcal{H}_\mathbb{H} \) defined over the quaternions \( \mathbb{H} \). Here, we also expand a state \( |\psi\rangle \in \mathcal{H}_\mathbb{H} \) as

\[ |\psi\rangle = |\psi_0\rangle + i|\psi_1\rangle + j|\psi_2\rangle + k|\psi_3\rangle, \]  

(68)

with \( |\psi_i\rangle \in \mathcal{H}_\mathbb{R} \). Recall that the symbols \( i, j \) and \( k \) satisfy the usual quaternionic relations \( i^2 = j^2 = k^2 = -1, \ ij = k, \ jk = i \), and so on. The conjugate state is defined as \( \langle \psi | = \langle \psi_0 | - i\langle \psi_1 | - j\langle \psi_2 | - k\langle \psi_3 | \), and the scalar product therefore reads

\[ \langle \phi | \psi \rangle = \sum_{a=0}^3 \langle \phi_a | \psi_a \rangle + i \left( \langle \phi_0 | \psi_1 \rangle - \langle \phi_1 | \psi_0 \rangle + \langle \phi_3 | \psi_2 \rangle - \langle \phi_2 | \psi_3 \rangle \right) \]
\[ + j \left( \langle \phi_0 | \psi_2 \rangle - \langle \phi_2 | \psi_0 \rangle + \langle \phi_1 | \psi_3 \rangle - \langle \phi_3 | \psi_1 \rangle \right) \]
\[ + k \left( \langle \phi_0 | \psi_3 \rangle - \langle \phi_3 | \psi_0 \rangle + \langle \phi_2 | \psi_1 \rangle - \langle \phi_1 | \psi_2 \rangle \right). \]  

(69)

The starting point of our consideration is a state \( A|\psi\rangle \), where the operator \( A \) is chosen to have the following form

\[ A = i\lambda_1 X + j\lambda_2 Y + k\lambda_3 Z + k\lambda_1 \lambda_2 [X, Y] + j\lambda_1 \lambda_3 [Z, X] + i\lambda_2 \lambda_3 [Y, Z], \]  

(70)

with \( \lambda_a \in \mathbb{R} \). Note that in (70), six self-adjoint operators appear which can be interpreted as observables

\[ \left\{ X, Y, Z, i[Y, Z], j[Z, X], k[X, Y] \right\}. \]  

(71)

Now, we are going to employ the three-bracket \([X, Y, Z] = [X, [Y, Z]] + \text{cycl.} = \ell_s^3 \) in the evaluation of the condition

\[ \langle A\psi | A\psi \rangle \geq 0, \]  

(72)

and analyze the resulting expression. However, to simplify our discussion, we assume the expectation values for the operators (71) to vanish, that is

\[ 0 = \langle \psi | X | \psi \rangle = \langle \psi | Y | \psi \rangle = \langle \psi | Z | \psi \rangle, \]
\[ 0 = \langle \psi | k[X, Y] | \psi \rangle = \langle \psi | j[Z, X] | \psi \rangle = \langle \psi | i[Y, Z] | \psi \rangle. \]  

(73)
Note that the second line is consistent with our point of view that the three-bracket is the fundamental structure of the problem, and thus no non-trivial uncertainty relation for only two position operators should appear. Furthermore, it turns out that we have to require in addition that some operators appearing in (70) can be measured simultaneously, which leads to

\[ 0 = \langle \psi | [X, Y], [Z, X] | \psi \rangle = \langle \psi | [X, Y], [Y, Z] | \psi \rangle = \langle \psi | [Z, X], [Y, Z] | \psi \rangle. \quad (74) \]

When evaluating the expression (72), different terms appear. In particular, from the product of two linear terms in (70) we find

\[
\langle A\psi | A\psi \rangle_{11} = \lambda_1^2 \langle \psi | X^2 | \psi \rangle + \lambda_2^2 \langle \psi | Y^2 | \psi \rangle + \lambda_3^2 \langle \psi | Z^2 | \psi \rangle \]
\[= \lambda_1^2 \Delta X^2 + \lambda_2^2 \Delta Y^2 + \lambda_3^2 \Delta Z^2. \quad (75)\]

The product of one linear and one quadratic term in (70) leads to

\[
\langle A\psi | A\psi \rangle_{12} = \lambda_1 \lambda_2 \lambda_3 \langle \psi | X, [Y, Z] + [Y, [Z, X]] + [Z, [X, Y]] | \psi \rangle \]
\[- k \lambda_1^2 \lambda_3 \langle \psi | X, [Z, X] | \psi \rangle + j \lambda_1^2 \lambda_2 \langle \psi | X, [X, Y] | \psi \rangle + \text{cyc.} \]
\[= \lambda_1 \lambda_2 \lambda_3 \ell_s^3, \quad (76)\]

where in the last line we have employed (67) and also required the vanishing of the expectation values \( \langle \psi | X, [Z, X] | \psi \rangle, \langle \psi | X, [X, Y] | \psi \rangle \) as well as their cyclic permutations. The interpretation of these conditions is not clear to us.

Finally, multiplying two quadratic terms in (70), we arrive at

\[
\langle A\psi | A\psi \rangle_{22} = \lambda_1^2 \lambda_2^2 \langle \psi | (k[X, Y])^2 | \psi \rangle + \lambda_1^2 \lambda_3^2 \langle \psi | (j[Z, X])^2 | \psi \rangle \]
\[+ \lambda_2^2 \lambda_3^2 \langle \psi | (i[Y, Z])^2 | \psi \rangle \quad (77)\]

where we utilized (74) and introduced the notation \( \Delta(X|Y)^2 = \langle \psi | [k[X, Y]]^2 | \psi \rangle \), and similarly for the others. Adding all three contributions, we arrive at

\[
0 \leq \lambda_1^2 \lambda_2^2 \Delta(X|Y)^2 + \lambda_1^2 \lambda_3^2 \Delta(Z|X)^2 + \lambda_2^2 \lambda_3^2 \Delta(Y|Z)^2 \]
\[+ \lambda_1 \lambda_2 \lambda_3 \ell_s^3 + \lambda_1^2 \Delta X^2 + \lambda_2^2 \Delta Y^2 + \lambda_3^2 \Delta Z^2. \quad (78)\]

This relation has to be true for all values of \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \), which is satisfied if and only if the following uncertainty relation holds

\[
\Delta(X|Y)^2 \Delta Z^2 + \Delta(Y|Z)^2 \Delta X^2 + \Delta(Z|X)^2 \Delta Y^2 \geq \frac{\ell_s^6}{4}. \quad (79)\]

Note that due to the assumptions made in the course of this derivation, equation (79) is to be supplemented by the trivial uncertainty relations

\[
0 \leq \Delta X \Delta Y, \quad 0 \leq \Delta X \Delta Z, \quad 0 \leq \Delta Y \Delta Z, \quad 0 \leq \Delta(X|Y) \Delta(Y|Z), \quad 0 \leq \Delta(Y|Z) \Delta(Z|X), \quad 0 \leq \Delta(Z|X) \Delta(X|Y). \quad (80)\]
Therefore, we have arrived at the result that a fundamental non-vanishing three-bracket gives rise to an uncertainty relation involving not only the positions operators but also their commutators.
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