Solving generalized eigenvalue problems by ordinary differential equations on a quantum computer

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Many eigenvalue problems arising in practice are often of the generalized form $A x = \lambda B x$. One particularly important case is symmetric, namely $A, B$ are Hermitian and $B$ is positive definite. The standard algorithm for solving this class of eigenvalue problems is to reduce them to Hermitian eigenvalue problems. For a quantum computer, quantum phase estimation is a useful technique to solve Hermitian eigenvalue problems. In this work, we propose a new quantum algorithm for symmetric generalized eigenvalue problems using ordinary differential equations. The algorithm has lower complexity than the standard one based on quantum phase estimation. Moreover, it works for a wider case than symmetric: $B$ is invertible, $B^{-1}A$ is diagonalizable and all the eigenvalues are real.

Keywords. Quantum algorithm, quantum phase estimation, generalized eigenvalue problem, ordinary differential equations.

I. INTRODUCTION

Quantum phase estimation (QPE), which is one of the most useful techniques in designing quantum algorithms, is a quantum algorithm to estimate the eigenvalues of unitary matrices [24]. It is frequently used as a subroutine in many quantum algorithms, e.g., quantum algorithm for finding eigenvalues of Hermitian matrices [1], quantum counting [9], quantum linear solver [20], quantum singular value estimation [23], and so on (e.g., [1, 3, 8, 26, 30, 31]). Taking the Hermitian eigenvalue problem as an example. For any $n \times n$ Hermitian matrix $H$ with eigenpairs $\{(\lambda_j, |E_j\rangle) : j = 1, \ldots, n\}$, the QPE returns $\sum \beta_j |\tilde{\lambda}_j\rangle |E_j\rangle$ when the input state is $|\phi\rangle = \sum \beta_j |E_j\rangle$, where $\tilde{\lambda}_j$ is an approximation of $\lambda_j$. We can think of the output state as a quantum version of eigenvalue decomposition of $H$. Performing measurements on the first register, we obtain the approximations of the eigenvalues. Actually, the output state can have more applications than estimating eigenvalues. For example, we can use it to create a state proportional to $H^{-1}|\phi\rangle$. This gives a quantum algorithm for solving linear systems [20]. In QPE, the input state $|\phi\rangle$ can be any desired state, and there is no need to know the explicit decomposition $|\phi\rangle = \sum \beta_j |E_j\rangle$ (e.g., a famous example is the Shor’s algorithm [35], see the analysis in [29]). This simple fact makes QPE more flexible to solve other problems.

Hermitian eigenvalue problems is an important class of eigenvalue problems. However, many eigenvalue problems arising in applications (e.g., linear discriminant analysis [22], canonical-correlation analysis [19], homogeneous Fredholm equation of the second type [4], constrained

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least squares problem [38], robust eigenvector classifiers [28], etc.) are not of the standard form \( Ax = \lambda x \) but of the generalized form \( Ax = \lambda Bx \). An important case is the symmetric generalized eigenvalue problem (GEP), i.e., \( A, B \) are Hermitian and \( B \) is positive definite. The standard algorithm for solving symmetric GEP is to reduce it to a standard Hermitian eigenvalue problem \( (B^{-1/2}AB^{-1/2})B^{1/2}x = \lambda B^{1/2}x \). Consequently, we can apply QPE to solve it directly in a quantum computer. For example, Parker and Joseph recently studied symmetric GEPs through this idea for some types of \( A, B \) arising from applications in physics such that \( B^{-1/2}AB^{-1/2} \) is sparse and can be constructed efficiently [30].

In this paper, we propose a new quantum algorithm for symmetric GEPs that demonstrates higher efficiency than the standard algorithm based on QPE. In addition, our algorithm works for a wider range of situations than symmetric and improves the previous quantum algorithm for the standard eigenvalue problem [33].

### A. Problem setting

In this work, we focus on the solving of the following problem, which can be viewed as a generalization of the Hermitian eigenvalue problem solved by QPE.

**Problem 1 (Quantum version of GEP (QGEP)).** Let \( \epsilon \in (0, 1) \) be the error tolerance. Let \( A, B \) be two \( n \times n \) complex matrices with \( B \) non-singular, \( B^{-1}A \) diagonalizable. Denote the eigenpairs of the GEP \( Ax = \lambda Bx \) as \( \{ (\lambda_j, |E_j\rangle) : j = 1, \ldots, n \} \). Suppose all the eigenvalues are real. Given access to copies of the state \( |\phi\rangle \), which formally equals \( \sum_j \beta_j |E_j\rangle \), the goal is to output a state proportional to \( \sum_j \beta_j |\tilde{\lambda}_j\rangle |E_j\rangle \), where \( |\lambda_j - \tilde{\lambda}_j| \leq \epsilon \) for all \( j \).

In QGEP, we assumed that \( B \) is non-singular and \( B^{-1}A \) is diagonalizable. In the following, we give the reasons for this.

1. When \( B \) is singular, the GEP may demonstrate some unusual properties than the standard eigenvalue problem. For instance, it may happen that the GEP has infinitely many eigenvalues or \( \lambda = \infty \) is an eigenvalue (see Section II for more details). Also, when \( B \) is singular, the GEP is well known to be ill-conditioned [21]. So it is hard to solve even for a quantum computer. Indeed, we will give a quantum lower bound analysis for this in Section V. The result indicates that to solve a singular GEP, a quantum computer needs at least to make \( \Omega(\sqrt{n}) \) queries to \( B \). So we will not consider singular GEPs in this paper.

2. The assumption of diagonalizability implies that any vector in theory can be uniquely decomposed into a linear combination of the eigenvectors. So the initial state \( |\phi\rangle \) of QGEP can be any desired state. Equivalently, there is no restriction on the choice of the input state. Since the eigenvectors are unknown to us, we do not have the decomposition in advance. Thus to solve the QGEP in practice we need the quantum algorithm to be independent of this decomposition.

There is a close connection between Problem 1 and symmetric GEP. On one hand, For symmetric GEP \( Ax = \lambda Bx \), it is known that there is an invertible matrix \( E \) such that \( AE = BEA \) with \( \lambda \) real diagonal [18] (see Proposition 2 for a proof). So symmetric GEP is a special case of Problem 1. On the other hand, Problem 1 can be reduced to a symmetric GEP. More precisely, as shown in [16], a diagonalizable matrix \( M \) that only have real eigenvalues if and only if there are two
Hermitian matrices $H_1, H_2$ with $H_1$ or $H_2$ positive definite such that $M = H_1 H_2$. So in Problem 1, $B^{-1} A = H_1 H_2$ for a Hermitian pair $(H_1, H_2)$. Suppose $H_1$ is positive definite, then $H_2 x = \lambda H_1^{-1} x$ defines a symmetric GEP. It has the same eigenpairs as the original GEP $A x = \lambda B x$. If $H_2$ is positive definite, we can focus on $(H_2^{1/2} H_1 H_2^{1/2}) H_2^{1/2} x = \lambda H_2^{1/2} x$. Usually, $H_1, H_2$ are not easy to compute, so it may not be straightforward to solve Problem 1 by the QPE in this way. However, our quantum algorithm given in this paper is independent of this kind of decomposition.

A more general case than symmetric such that all the eigenvalues are real is called definite. A Hermitian matrix pair $(A, B)$ is called definite if $\min_{x \in \mathbb{C}^n, \|x\|_2=1} \langle x^\dagger A x \rangle^2 + \langle x^\dagger B x \rangle^2 > 0$. As shown in [36], if $(A, B)$ is a definite pair, there is a $\theta \in [0, 2\pi)$ such that $A \cos \theta - B \sin \theta, A \cos \theta + B \sin \theta$ are Hermitian and $A \cos \theta + B \sin \theta$ is positive definite. So all the eigenvalues of a definite pair are real. A potential problem of this reduction is the calculation of $\theta$. It is usually not easy to find such a $\theta$, so the standard algorithm based on QPE may not be efficient for all definite pairs through this idea. While our algorithm proposed in this paper works for all definite pairs that satisfy the assumption of Problem 1, and there is no need to find $\theta$.

We remark that in practice, we are usually more concerned about some maximal or minimal eigenvalues. So solving Problem 1 may not be the most effective way to compute the extreme eigenvalues. For those particular eigenvalue problems, other algorithms (e.g., power method, Lanczos method, variational quantum eigensolver) should be more powerful. The advantage of Problem 1 is that it contains the whole information of the eigenpairs in one quantum state so that measuring the first register returns all the eigenvalues up to some additive error. Also, the output state can be viewed as a quantum version of eigenvalue decomposition, which can have more applications than estimating eigenvalues.

### B. Our result

In this paper, we will design a quantum algorithm for Problem 1 in the framework of block-encoding. Block-encoding is an effective framework to manipulate matrix operations in a quantum computer. There are also many efficient methods to construct block-encodings of matrices. So quantum algorithms building on this framework should be general enough to solve problems related to matrices.

Let $M$ be a matrix and $U$ be a unitary, we say $U$ a block-encoding of $M$ if there is a positive real number $\alpha \in \mathbb{R}^+$ such that

$$U = \begin{bmatrix} M/\alpha & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}.$$  

When having a block-encoding, many fundamental operations (e.g., matrix multiplication, matrix inversion, the singular value decomposition, etc) on $M$ become effective in a quantum computer [10, 17]. Below, we shall call $U$ an $\alpha$-block-encoding of $M$ for simplicity. The rigorous terminology is given in Section II B.

Denote the matrix of the generalized eigenvectors as $E$, the condition numbers of $A, B, E$ as $\kappa_A, \kappa_B, \kappa_E$ respectively. Since the GEP is invariant under scaling, we can do a scaling to make sure that $\|B\| = \Theta(1)$. This assumption is just for the convenience of notation below, our algorithm is indeed independent of this assumption. For simplicity, we assume that the block-encodings of $A, B$ are constructed efficiently in time polylog in $n$. Then our main result can be stated as the follows:

**Theorem 1** (Informal of Theorem 20). Given an $\alpha_A$-block-encoding of $A$ and an $\alpha_B$-block-encoding
of $B$, then Problem 1 can be solved in time

$$\tilde{O}(\kappa_E(\alpha_A + \rho\alpha_B)\kappa_B/\epsilon),$$

(1)

where $\rho$ is an upper bound on the eigenvalues.

An obvious fact is that for any $\alpha$-block encoding $U$ of a matrix $M$, we have $\alpha \geq \|M\|$. Also in the quantum case, we hope $U$ is efficiently implementable in the quantum circuits. So it may happen that $\alpha \gg \|M\|$. For instance, we may have $\alpha = \|M\|_F$ assuming access to the QRAM data [10]. Here $\|M\|_F$ is the Frobenius norm. Because of this, we keep $\alpha_A, \alpha_B$ in the complexity (1). Note that $\rho \leq \|B^{-1}A\| = O(\kappa_B\|A\|)$ under the assumption of $\|B\| = \Theta(1)$. So in the worst case, the complexity of our algorithm is upper bounded by $\tilde{O}(\kappa_E(\alpha_A + \|A\|\alpha_B\kappa_B)\kappa_B/\epsilon)$.

The algorithm based on QPE for symmetric GEPs is not comprehensively studied in the past. For comparison, we will also analyze its complexity in this paper (see Theorem 24). The complexity of QPE is determined by the cost of constructing the block-encoding of $B^{-1/2}AB^{-1/2}$. We will use two methods to construct this block-encoding. The overall complexity of the first method is

$$\tilde{O}(\alpha_A\alpha_B\kappa_B^{2.5}/\epsilon),$$

(2)

and of the second method is

$$\tilde{O}((\alpha_A + \|A\|\kappa_B\alpha_B)\kappa_B^{2}/\epsilon).$$

(3)

The second method has better dependence on $\alpha_A, \alpha_B$, but worse dependence on $\kappa_B$ than the first method. When restricted to the symmetric case, $\kappa_E = \sqrt{\kappa_B}$ (see Proposition 2). So the complexity (1) of our algorithm is at most

$$\tilde{O}((\alpha_A + \|A\|\kappa_B\alpha_B)\kappa_B^{1.5}/\epsilon).$$

(4)

Therefore, our algorithm has better dependence on $\alpha_A, \alpha_B$ and $\kappa_B$ than the algorithm based on QPE. At the end of Section IV, we will discuss more about the connection, which involves some other notation, between our algorithm and the standard algorithm based on QPE. In summary, it states that our algorithm combines the advantages of the two above methods based on QPE. If we further restricted to the Hermitian eigenvalue problem (i.e., $A$ is Hermitian, $B = I$), then all the algorithms have the same complexity $\tilde{O}(\alpha_A/\epsilon)$. This is consistent with the complexity of QPE [10].

C. Summary of our techniques

Our algorithm builds on the connection between GEP and ordinary differential equations (ODEs). Consider the linear ODEs $Bx'(t) = 2\pi iAx(t)$ with initial condition $x(0) = \sum_j \beta_j |E_j\rangle$. The solution equals $x(t) = \sum_j \beta_j e^{2\pi i \lambda_j t} |E_j\rangle$. To solve an ODE in a quantum computer, we usually use the discretization method to reduce it to a linear system such that quantum linear solvers are applicable [6, 7, 12]. Suppose we discretize the time interval $[0, \tau]$ into $p$ sub-intervals via $t_0 = 0, t_1 = \hbar, \ldots, t_p = ph = \tau$, then it turns out that the quantum state of the solution of the linear system is proportional to $\sum_{i=1}^p \|x(t_i)\| |l\rangle |x(t_i)\rangle = \sum_j \beta_j \sum_{i=1}^p e^{2\pi i \lambda_j t_l} |E_j\rangle$. When $\lambda_j$ are

\footnote{A plausible reason: the QPE idea for symmetric GEP is so naive that it is not deserve to write a single paper. In this paper, we use the best known block-encoding techniques to do the complexity analysis of this method.}
all real, we can estimate them by applying inverse quantum Fourier transform to $|l\rangle$. This gives us the target state stated in Problem 1.

The basic idea of our algorithm is straightforward. The technical part is the complexity analysis. Since it relates to the solving of a linear system in a quantum computer, we need to estimate the condition number of this linear system. This is indeed the most technical part of this whole paper. Below, we briefly introduce our basic idea about this estimation. Denote the coefficient matrix of the linear system as $M$, then $\|M\|$ is easy to bound. As for $\|M^{-1}\|$, by definition it equals $\max_{\|b\|=1}\|M^{-1}b\|$. Since $M$ is obtained from solving an ODE and $M^{-1}b$ can be viewed as the solution of the linear system $My = b$, we can view $M^{-1}b$ as an approximate solution of a linear ODE of the form $Bx'(t) = 2\pi iAx(t) + c(t)$ for some $c(t)$ depending on $b$. For this ODE, we know the explicit formula of its solution, from which we can bound $\|M^{-1}b\|$ through some matrix inequalities.

D. Organization of the paper

This paper is organized as follows: In Section II, we introduce some preliminary results that will be used in this paper. This includes some classical results about GEPs and the framework of block-encoding. In Section III, we present our quantum algorithm in detail. In Section IV, for comparison, we analyze the complexity of the standard algorithm based on QPE. Finally, in Section V, we give a simple lower bound analysis about solving singular GEPs in a quantum computer.

Notation. For a matrix $A = (a_{ij})$, we use $A^{-1}$ to denote the Moore-Penrose inverse, $\|A\|$ to denote the operator norm (i.e., the maximal singular value), $\|A\|_{\text{max}} = \max_{i,j} |a_{ij}|$ and $\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$. By $\kappa_A$, we mean the ratio of maximal singular value and minimal nonzero singular value, and still call it the condition number of $A$. We say $A$ is $s$-sparse if the maximal number of nonzero entries in each row and column is $s$. For two Hermitian matrices $A, B$, by $A \preceq B$ we mean $B - A$ is positive definite.

II. PRELIMINARIES

A. Generalized eigenvalue problem

Let $A, B$ be two $n$-by-$n$ complex matrices, the generalized eigenvalue problem (GEP) aims to find all $\lambda \in \mathbb{C}$ and all vectors $x \in \mathbb{C}^n$ such that

$$Ax = \lambda Bx.$$  \hspace{1cm} (5)

Although the GEP looks like a simple generalization of the standard eigenvalue problem, it exhibits some significant differences. For example,

- It is possible for $\det(A - \lambda B) = 0$ independent of $\lambda$ so that every scale is an eigenvalue, e.g., $A, B$ are singular such that $Ax = Bx = 0$ for some nonzero $x$. When this happens, we call $(A, B)$ a non-regular pair. Obviously, if $(A, B)$ is non-regular, then $B$ is singular.

- If $B$ is singular, then $\infty$ can be an eigenvalue. To see this, consider $Bx = \lambda^{-1}Ax$. Choose $x$ such that $Bx = 0, Ax \neq 0$, then $\lambda = \infty$. Because of this, sometimes we write the GEP in the
cross-product form $\mu A x = \lambda B x$, and view the pair $(\lambda, \mu)$ as an eigenvalue. In this notation, the eigenvalue $\infty$ corresponds to $(\lambda, 0)$. When $(A, B)$ is regular, this matrix pair always has $n$ eigenvalues, including $\infty$.

The above two unusual properties appear when $B$ is singular. Also, when $B$ is singular, the GEP is well-known to be ill-conditioned [21, 39]. So it is hard to solve even for a quantum computer, see the lower bound analysis in Section V. Due to the above reasons, in this paper, we will mainly focus on the case that $B$ is invertible.

Usually, $\lambda$ may not be real even if $A, B$ are Hermitians. The following simple result is a sufficient condition, which holds for many machine learning models [15].

**Proposition 2** (see [18]). Assume that $A, B$ are Hermitian matrices with $B$ positive definite. Then there is an invertible matrix $E$ and a real diagonal matrix $\Lambda$ such that $AE = BE \Lambda$. Moreover, the condition number of $E$ equals $\kappa_E = \sqrt{\kappa_B}$.

**Proof.** Since $B$ is Hermitian, it can be diagonalized by a unitary matrix. Denote this decomposition as $U^\dagger BU = D$, where $U$ is unitary and $D$ is diagonal. Denote $C = B^{-1/2} = UD^{-1/2}$, then $C^\dagger BC = I$. Consider $\tilde{A} := C^\dagger AC$. It is Hermitian, so there is a unitary $V$ and a diagonal matrix $\Lambda$ such that $V^\dagger \tilde{A} V = \Lambda$. Set $E = CV$, then $E^\dagger AE = \Lambda$ and $E^\dagger BE = I$. This implies that $AE = BE \Lambda$. Therefore, the real matrix $\Lambda$ is the diagonal matrix of the generalized eigenvalues. As for the condition number, since $E = \sqrt{B^{-1}}V$ with $V$ unitary, we have $\kappa_E = \sqrt{\kappa_B}$. \hfill $\square$

Even though the assumption that $B$ is positive definite occurs in many cases in practice, we can make an even weaker assumption called definite pairs such that all the eigenvalues are real. A Hermitian pair $(A, B)$ is called definite if

$$
\gamma(A, B) := \min_{x \in \mathbb{C}^n, \|x\|_2 = 1} \sqrt{\|x^\dagger Ax\|^2 + \|x^\dagger Bx\|^2} > 0.
$$

The quantity $\gamma(A, B)$ is known as the Crawford number of the pair $(A, B)$. The basic fact about definite pairs is that they can be transformed by a unitary matrix into a Hermitian pair $(A', B')$ in which $B'$ is positive definite. More precisely,

**Proposition 3** (Theorem 1.18 of Chapter VI in [36]). Let $(A, B)$ be a definite pair. For any $\theta \in [0, 2\pi)$ let $R(\theta)$ be the 2-dimensional rotation of angle $\theta$ and

$$
\begin{bmatrix}
A_\theta \\
B_\theta
\end{bmatrix} := R(\theta) \otimes I_n \begin{bmatrix}
A \\
B
\end{bmatrix} = \begin{bmatrix}
A \cos \theta - B \sin \theta \\
A \sin \theta + B \cos \theta
\end{bmatrix}.
$$

(7)

Then there is a $\theta \in [0, 2\pi)$ such that $B_\theta$ is positive definite and $\gamma(A, B)$ is the smallest singular value of $B_\theta$.

As a result of Propositions 2 and 3, if $(A, B)$ is a definite pair, then there is a nonsingular matrix $E$ such that $E^\dagger AE$ and $E^\dagger BE$ are diagonal. Moreover, the eigenvalues are all real.

Next, we briefly introduce the conditioning analysis about computing eigenvalues. It turns out that the complexity of the quantum algorithm proposed in this paper will depend on the quantities (i.e., $\kappa_E, \kappa_B, \|A^{-1}\|$) that describe the conditioning, which seems not unrealistic.

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2 A more general result is given in [18, Theorem 8.7.1].
The conditioning describes the stability of eigenvalues after small perturbations. According to Stewart [37], the appropriate measure for generalized eigenvalue perturbations is the chordal metric \( \text{chord}(a, b) \) defined by

\[
\text{chord}(a, b) := \frac{|a - b|}{\sqrt{1 + |a|^2} \sqrt{1 + |b|^2}}, \quad \text{where } a, b \in \mathbb{C}.
\] (8)

Stewart showed that if \( \lambda \) is a distinct eigenvalue of \((A, B)\) and \( \lambda' \) is the corresponding eigenvalue of the perturbed pair \((A', B')\) with \( \|A - A'\| \approx \|B - B'\| \approx \epsilon \), then

\[
\text{chord}(\lambda, \lambda') \leq \frac{\epsilon}{\sqrt{(y^\dagger A x)^2 + (y^\dagger B x)^2}} + O(\epsilon^2),
\] (9)

where \( x, y \) are unit and satisfy \( Ax = \lambda Bx, y^\dagger A = \lambda y^\dagger B \). So \( 1/\sqrt{(y^\dagger A x)^2 + (y^\dagger B x)^2} \) serves as a condition number for the eigenvalue \( \lambda \). Indeed, \( \gamma^{-1}(A, B) \) can be viewed as an upper bound of the condition number of all eigenvalues [36, Theorem 3.3 of Chapter VI]. Note that \( \gamma(A, B) \geq \sqrt{\|A^{-1}\|^{-2} + \|B^{-1}\|^{-2}} \). So

\[
\gamma^{-1}(A, B) \leq \sqrt{\frac{\|A^{-1}\|^2 \|B^{-1}\|^2}{\|A^{-1}\|^2 + \|B^{-1}\|^2}} \leq \sqrt{\|A^{-1}\| \|B^{-1}\|/2}. \] (10)

Generally, let \( E \) be the matrix of the generalized eigenvectors with condition number \( \kappa_E \), then by [36, Theorem 2.6 of Chapter VI] the condition number is bounded by \( \kappa_E \rho_L(A, B) \), where \( \rho_L \) is a metric that depends on the perturbations on \( A, B \).

The conditioning of GEP also relates to the condition number of \( B \), for instance see [14, Theorem 2]. Consider the GEP \( Ax = \lambda Bx \) with \( A, B \) Hermitian and \( B \) positive definite. We use \( A' = A + \Delta A, B' = B + \Delta B \) to denote the perturbations of \( A, B \) with \( \|\Delta A\|, \|\Delta B\| \leq \epsilon \). Then for the \( i \)-th eigenvalue \( \lambda_i \) of \((A, B)\) and \( i \)-th eigenvalue \( \lambda_i' \) of \((A', B')\), we have \( |\lambda_i' - \lambda_i| \leq (1 + |\lambda_i|)\epsilon \kappa_B \).

### B. Block-encoding

In this section, we briefly introduce some necessary results about block-encoding. The main references are [10, 17].

**Definition 4** (Block-encoding). Suppose that \( A \) is a \( p \)-qubit operator, \( \alpha, \epsilon \in \mathbb{R}^+ \) and \( q \in \mathbb{N} \). Then we say that the \((p + q)\)-qubit unitary \( U \) is an \((\alpha, q, \epsilon)\)-block-encoding of \( A \), if

\[
\|A - \alpha(\langle 0 |^\otimes q \otimes I)U(\langle 0 |^\otimes q \otimes I)\| \leq \epsilon.
\] (11)

In matrix form, we can view \( U = \begin{bmatrix} A/\alpha & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \). So to verify if a unitary \( U \) is a block-encoding of \( A \), we only need to verify if \( U|0\rangle|x⟩ = |0\rangle(A/\alpha)|x⟩ + |1\rangle|x'⟩ \) for all \( |x⟩ \), where \( |x'⟩ \) is a garbage state. The following lemma is about the construction of block-encodings of sparse matrices.

**Lemma 5** (Lemma 48 of [17]). Assume that \( A \) is an \( s \)-sparse \( n \)-by-\( n \) matrix given in the sparse-access input model. Then for any \( \epsilon \in (0, 1) \), we can implement an \((s\|A\|_{\text{max}}, \text{poly log}(n/\epsilon), \|A\|_{\text{max}}\epsilon)\)-block-encoding of \( A \) in time \( O(\text{poly log}(n/\epsilon)) \).
In the original statement of Lemma 48 of [17], the authors assume that \(|A|_{\text{max}} \leq 1\). So to apply their result, we can consider \(A/\|A\|_{\text{max}}\). The above lemma then comes from the following simple fact.

**Fact 6.** An \((\alpha, \epsilon)\)-block-encoding of \(A/\beta\) is an \((\alpha\beta, \epsilon)\)-block-encoding of \(A\).

**Lemma 7** (Lemmas 9 and 10 of [10]). Assume that \(\delta \in (0, 1), \kappa \geq 2\). Suppose that \(B\) is Hermitian with \(I/\kappa \leq B \leq I\), and we have an \((\alpha, \epsilon)\)-block-encoding of \(B\) that can be implemented using \(O(T)\) elementary gates.

- If \(\epsilon = o(\delta^{-1} \kappa^{-1.5} \log^{-3}(\kappa^{1.5}/\delta))\), then we can implement a \((2\sqrt{\kappa}, q + O(\log(\kappa^{1.5} \log 1/\delta)), \delta)\)-block-encoding of \(B^{-1/2}\) in cost \(O(\alpha\kappa(q + T) \log^{2}(\kappa^{1.5}/\delta))\).

- If \(\epsilon = o(\delta^{-1} \kappa^{-1} \log^{-3}(\kappa/\delta))\), then we can implement a \((2, q + O(\log 1/\delta), \delta)\)-block-encoding of \(B^{1/2}\) in cost \(O(\alpha\kappa(q + T) \log^{2}(\kappa/\delta))\).

**Lemma 8** (Lemma 4 of [10]). Suppose for \(i = 1, 2\) we have an \((\alpha_i, q_i, \epsilon_i)\)-block-encoding of \(A_i\) that is constructed in time \(O(T_i)\). Then we can create an \((\alpha_1 \alpha_2, q_1 + q_2, \alpha_1 \epsilon_2 + \alpha_2 \epsilon_1)\)-block-encoding of \(A_1 A_2\) in time \(O(T_1 + T_2)\).

**Definition 9** (Quantum phase estimation). Let \(A\) be an \(n\)-by-\(n\) Hermitian matrix with eigenvalue decomposition \(A = \sum_{k=1}^{n} \lambda_k |u_k\rangle \langle u_k|\). Let \(\epsilon \in (0, 1)\). The quantum phase estimation (QPE) problem with accuracy \(\epsilon\) is defined as: Given access to \(\sum_{k=1}^{n} |u_k\rangle\langle u_k|\), perform the mapping

\[
\sum_{k=1}^{n} \beta_k |0\rangle |u_k\rangle \mapsto \sum_{k=1}^{n} \beta_k |\tilde{\lambda}_k\rangle |u_k\rangle,
\]

such that \(\tilde{\lambda}_k - \lambda_k \leq \epsilon\) for all \(k \in \{1, 2, \ldots, n\}\).

**Lemma 10** (Theorem 27 of [10]). Let \(\epsilon, \bar{\epsilon} \in (0, 1)\), and \(\epsilon' = \bar{\epsilon} \epsilon/(4 \log^{2}(1/\epsilon))\). Given an \((\alpha, q, \epsilon')\)-block-encoding Hermitian matrix \(A\) that is implemented in cost \(O(T)\), then there is a quantum algorithm that solves the QPE problem of \(A\) with accuracy \(\epsilon\), with success probability at least \(1 - \bar{\epsilon}\) in time \(O((T_{\text{in}} + \alpha \epsilon^{-1}(q + T)) \log^{4}(1/\epsilon))\), where \(O(T_{\text{in}})\) is the cost to prepare the initial state.

In [10], Theorem 27 is used to estimate the singular values. However, it is not hard to modify their algorithm to estimate the eigenvalues of Hermitian matrices. The following lemma gives the complexity of solving linear systems in the framework of block-encoding.

**Lemma 11** (Theorem 30 of [10]). Let \(\kappa \geq 2\), and \(H\) be a matrix with non-zero singular values lie in the range \([-1, -1/\kappa] \cup [1/\kappa, 1]\). Suppose that for \(\delta = o(\epsilon/(\kappa^2 \log^{3}(\kappa/\epsilon)))\) we have an \((\alpha, \epsilon)\)-block-encoding of \(H\) that can be implemented using \(T_{\text{in}}\) elementary gates. Also suppose that we can prepare an input state \(|\psi\rangle\) which spans the eigenvectors of \(H\) in time \(O(T_{\text{in}})\). Then there is a quantum algorithm that outputs a state that is \(\epsilon\)-close to \(H^{-1} |\psi\rangle/\|H^{-1} |\psi\rangle\|\) at a cost

\[
\tilde{O}(\kappa(T_{\text{in}} + \alpha(q + T_{\text{in}}) \log^{2}(\kappa/\epsilon)) \log \kappa).
\]

In the above lemma, the accuracy \(\delta\) in the block-encoding of \(H\) should be much smaller than \(\epsilon\), the accuracy to approximate the target state. Usually this will not cause any problem to the complexity analysis because \(\delta\) appears as a logarithmic term. Another point of Lemma 11 is the assumption \(\|H\| \leq 1\). For solving a linear system \(Hx = b\), we can consider \((H/\|H\|)|x = b/\|H\|\) instead. By Fact 6, if we only have an \((\alpha, \epsilon)\)-block-encoding of \(H\), then \(\alpha\) in the complexity (13) should be divided by \(\|H\|\).
### III. QUANTUM DIFFERENTIAL METHOD FOR GEPS

In this section, based on the connection between GEps and ODEs, we propose a quantum algorithm to solve Problem 1. The main idea is pretty straightforward as follows. Consider the following linear system of ODEs with initial state $x(0)$

$$
B \frac{dx(t)}{dt} = 2\pi i Ax(t).
$$

(14)

Denote the eigenpairs of GEP $A x = \lambda B x$ as $\{(\lambda_j, |E_j\rangle) : j = 1, \ldots, n\}$. If $B$ is invertible, then the solution is $x(t) = e^{2\pi i B^{-1}At}x(0)$. Thus if $x(0)$ is a linear combination of the eigenvectors, say $x(0) = \sum_{j=1}^{n} \beta_j |E_j\rangle$, then $x(t) = \sum_{j=1}^{n} \beta_j e^{2\pi i \lambda_j t} |E_j\rangle$.

To solve an ODE numerically, a useful method is discretization. If we discretize the time interval $[0, \tau]$ via $t_0 = 0, t_1 = h, t_2 = 2h, \ldots, t_p = ph = \tau$, the ODE (14) reduces to a linear system whose solution is $(x(t_1)^T, \ldots, x(t_p)^T)^T$, where $T$ refers to the transpose operation. In a quantum computer, when a quantum linear solver is used, we will obtain the quantum state of this solution, which is proportional to

$$
\tilde{x} = \frac{1}{\sqrt{p}} \sum_{l=1}^{p} \|x(t_l)\| |l\rangle |x(t_l)\rangle = \frac{1}{\sqrt{p}} \sum_{j=1}^{n} \beta_j \sum_{l=1}^{p} e^{2\pi i \lambda_j lh} |l\rangle |E_j\rangle.
$$

(15)

Applying the inverse quantum Fourier transform to $|l\rangle$, we obtain the following expected state, which contains the information of eigenpairs, with high probability

$$
\sum_{j=1}^{n} \beta_j |\tilde{\lambda}_j\rangle |E_j\rangle,
$$

(16)

where $\tilde{\lambda}_j$ is an approximation of $\lambda_j$ up to additive error $1/(ph)$. The error analysis here is similar to that in quantum phase estimation, so we defer it to Appendix A. For simplicity, we shall use $\rho(A, B)$ or just $\rho$ to denote an upper bound of the eigenvalues, i.e., $\rho \geq \max\{1, \max_j |\lambda_j|\}$. As for $h$, we should make sure that $|\lambda_j h| \leq 1$ for all $j$, so $h = O(1/\rho)$. As a result, we can set $p = O(\rho/\epsilon)$.

In conclusion, the parameters we choose are

$$
p = O(\rho/\epsilon), \quad h = O(1/\rho), \quad \tau = O(1/\epsilon).
$$

(17)

In (17), when determining the parameters, we only considered the error analysis from a similar idea to QPE. Later, we need to modify them by considering the error caused by the discretization method. The modification is slight, so (17) is almost our final choice.

**Remark 12.** A main ingredient of QPE is the Hamiltonian simulation. It is used in QPE to generate a state proportional to $\sum_{j,l} \beta_j e^{2\pi i \lambda_j lh/p} |l\rangle |E_j\rangle$. This state is similar to the one we obtained in (15). Hamiltonian simulation only works for Hermitian matrices, so we cannot apply it to our problem directly. However, the original goal of Hamiltonian simulation is to simulate quantum systems, i.e., to solve the Schrödinger equation. With the discovery of quantum linear solvers, we know how to solve differential equations in a quantum computer. So our main idea can be simply described as replacing Hamiltonian simulation by solving differential equations in QPE.

In a quantum computer, there are several general-purpose quantum algorithms for solving differential equations [6, 7, 12, 13]. The basic idea of these algorithms is to reduce the linear differential
equations into a linear system of equations through some discretization methods, then use a quantum linear solver to obtain the solution state. For the differential equation (14), the quantum algorithm proposed in [7] may not be suitable for our problem since this algorithm tries to approximate $e^{B^{-1} A}$ by Taylor expansion. Even if it is easy to compute $B^{-1}$ in a quantum computer, the linear system they constructed depends on the matrix $B^{-1} A$. So we still need to know the entries of $B^{-1} A$. The quantum spectral method [12] and the higher order method [6] can be used to solve the differential equation (14), while they are not appropriate for our purpose. The reason is that the higher order method [6] has worse dependence on the precision. The spectral method [12] is based on Chebyshev polynomials. The interpolation nodes they used are the Chebyshev-Gauss-Lobatto quadrature nodes, i.e., $t_j = \cos(l\pi/n)$, here we have to use the nodes $t_j = l(h)$. In the following, we will apply the spectral method based on the Fourier series to solve the differential system (14). This method has been used in [13] to solve some PDEs. Since the Fourier spectral method has not been used previously to solve ODEs in a quantum computer, a technical part below is the estimation of its complexity (including error analysis and estimation of condition number).

### A. Fourier approximation

In this section, we aim to give an introduction to the Fourier spectral method to solve the linear system of ODEs (14). For more about this method, we refer to the book [34]. For any function $f(t)$, where $t \in [0, L]$, the $K$-truncated Fourier series of $f(t)$ is defined by

$$f_K(t) = \sum_{k=-K}^{K} c_k e^{2\pi i k t / L} = \sum_{k=0}^{2K} c_k e^{2\pi i (k-K) t / L}, \quad (18)$$

where

$$c_k = \frac{1}{L} \int_0^L f(t) e^{-2\pi i k t / L} dt.$$

**Proposition 13.** [2, Theorem 2.3] Suppose $f$ is analytic, then there exist $R_f = O(\max |f(x)|)$ and $S = \cot^2(\pi/4L)$ such that $|f(t) - f_K(t)| \leq R_f S^{-K}$ for all $t \in [0, L]$.

**Fact 14.** Given $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ such that $|a_i - b_i| \leq R_f S^{-K}$ for all $i$, then $\|a - b\| \leq \sqrt{n} R_f S^{-K}$. For us, $L = \tau = O(1/\epsilon)$, so $S \approx 1/\epsilon^2$. So to make sure the error is bounded by $\epsilon$, it suffices to choose $K = O(\log \epsilon)^{-1} \log(R_f n^2)$.

Combining (17), we now choose $p = O(\max\{\rho/\epsilon, (\log \epsilon)^{-1} \log(R_f n^2)\}) = \tilde{O}(\rho/\epsilon)$ odd.\(^3\) For any $l \in \{0, 1, \ldots, p - 1\}$, from (18) the $j$-th entry of $\mathbf{x}$ and its derivative at time $lh$ can be approximated by

$$\dot{x}_j(lh) := \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} c_{jk} e^{2\pi i (k-\frac{p-1}{2})/p} l h = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} c_{jk} e^{2\pi i (k-\frac{p-1}{2})/p} l h = \frac{e^{\pi i (1-p)/p}}{\sqrt{p}} \sum_{k=0}^{p-1} c_{jk} e^{2\pi i k l / p}, \quad (19)$$

$$\ddot{x}_j(lh) = \frac{2\pi i}{\sqrt{L}} \sum_{k=0}^{p-1} (k - \frac{p-1}{2}) c_{jk} e^{2\pi i (k-\frac{p-1}{2})/p} l h = \frac{2\pi i e^{\pi i (1-p)/p}}{\sqrt{p} \tau} \sum_{k=0}^{p-1} (k - \frac{p-1}{2}) c_{jk} e^{2\pi i k l / p}, \quad (20)$$

\(^3\) In the complexity analysis of our quantum algorithms below, we will ignore the logarithmic terms for simplicity, so we may think here that $p = O(\rho/\epsilon)$, i.e., equation (17). Choosing odd $p$ is to ensure that $(p - 1)/2$ is an integer. The constant $R_\mathbf{x}$ is upper bounded $\|x(t)\| = O(\kappa_E)$. 

where \( c_{jk} \) are the unknown Fourier coefficients we aim to compute. In the above, we used the fact that \( \tau = ph \). Here, we do not consider the case \( l = p \), i.e., \( x(T) \). This makes the right hand sides of (19), (20) the quantum Fourier transform. The constant \( 1/\sqrt{p} \) is added for normalization.

In matrix form, denote

\[
C = (c_{jk})_{1 \leq j \leq n, 0 \leq k \leq p-1},
\]

\[
D = \text{diag}\{k - \frac{p-1}{2} : 0 \leq k \leq p-1\},
\]

\[
F = \frac{1}{\sqrt{p}}(e^{2\pi i kl/p})_{0 \leq k, l \leq p-1}.
\]

By (19), (20), when \( l \geq 1 \), we have

\[
\hat{x}(lh) = e^\frac{\pi i (1+p)l}{p} CF|l\rangle,
\]

\[
\hat{x}'(lh) = \frac{2\pi i e^\frac{\pi i (1+p)l}{\tau}}{\tau} CDF|l\rangle.
\]

When \( l = 0 \), \( x(0) = CF|0\rangle \) is the initial condition. From (14), we know that

\[
\begin{cases}
 x(0) = CF|0\rangle, \\
 \frac{1}{\tau} BCDF|l\rangle = ACF|l\rangle, \quad 1 \leq l \leq p - 1.
\end{cases}
\]

**Definition 15** (Vectorization of matrices). Let \( A = (a_{ij})_{m \times n} \) be a matrix, its vectorization is an \( mn \)-dimensional column vector \( \text{vec}(A) = (a_{11}, a_{m1}, \ldots, a_{1n}, \ldots, a_{mn})^T \). So for any three matrices \( A, B, C \) we have \( \text{vec}(ABC) = (C^T \otimes A)\text{vec}(B) \).

Now let \( \text{vec}(C) = (c_{10}, c_{00}, \ldots, c_{1(p-1)}, \ldots, c_{n(p-1)})^T \) denote the vectorization of \( C \), then we obtain a linear system of \( \text{vec}(C) \) through (25):

\[
\begin{bmatrix}
\langle 0|F^T \otimes I \\
\langle 1|F^T \otimes A - \tau^{-1} \langle 1|F^T D \otimes B \\
\vdots \\
\langle p-1|F^T \otimes A - \tau^{-1} \langle p-1|F^T D \otimes B \\
\end{bmatrix}
\text{vec}(C) = 
\begin{bmatrix}
x(0) \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]

The above coefficient matrix is \( np \times np \)

\[
M = \begin{bmatrix}
\langle 0|F^T \otimes I \\
(F_0^T \otimes I)(I \otimes A - \tau^{-1} D \otimes B) \\
\end{bmatrix},
\]

where \( F_0 \) is obtained by deleting the first column of \( F \).

Before starting more details, we first discuss the relationship between \( \text{vec}(C) \) and the state (16) we aim to prepare. By definition, the quantum state of \( \text{vec}(C) \) equals

\[
|\text{vec}(C)\rangle = \frac{1}{\|C\|_F} \sum_{j=1}^{n} \sum_{k=0}^{p-1} c_{jk} |k, j\rangle,
\]

\footnote{Here we use \( \hat{x}(t) \) to denote the approximation of \( x(t) \), the exact solution of the ODE (14).}
where \(\|C\|_F\) is the Frobenius norm. By the first identity in (24) and note that \(F\) is unitary, we have

\[
\|C\|_F = \sqrt{\sum_{l=0}^{p-1} \|\tilde{x}(lh)\|^2}.
\] (29)

It is easy to see that the superposition of the approximate solutions (see equation (15)) is

\[
\frac{1}{\|C\|_F} \sum_{l=0}^{p-1} \|\tilde{x}(lh)\| |l\rangle |\tilde{x}(lh)\rangle = \frac{1}{\|C\|_F \sqrt{p}} \sum_{l=0}^{p-1} e^{\pi i (1-p) l/p} \sum_{j=1}^{n} \sum_{k=0}^{p-1} c_{jk} e^{2 \pi i k l/p} |l,j\rangle.
\] (30)

Denote

\[
U_p = \text{diag}\{e^{\pi i (1-p) l/p} : l = 0, \ldots, p-1\}.
\] (31)

Then we can obtain the state (30) by applying \(U_p F\) to \(|k\rangle\) in \(|\text{vec}(C)\rangle\). To estimate the eigenvalues (i.e., to obtain (16)), we only need to apply \(F^{-1}\) to \(|l\rangle\) in the state (30). Since \(F, U_p\) can be implemented efficiently in the quantum circuits, it follows that to determine the overall cost of solving Problem 1 by the above idea, it suffices to know the cost of solving the linear system (26) to prepare \(|\text{vec}(C)\rangle\). For this, we first need to estimate the error of the solution (i.e., the error between (15) and (30)) and the condition number of the linear system (26).

### B. Analysis of the error and condition number

**Proposition 16** (Error analysis). Let \(\tilde{x}\) be the superposition (15) of the exact solutions of the differential equation (14), let \(|C\rangle\) be the quantum state of the solution of the linear system (26) obtained by a quantum linear solver up to precision \(\epsilon\), then

\[
\|\|\tilde{x}\rangle - (U_p F \otimes I)|C\rangle\| \leq 2\epsilon.
\] (32)

**Proof.** Denote \(|\text{vec}(C)\rangle\) as the quantum state of the exact solution of the linear system (26), then \(\|\text{vec}(C)\rangle - |C\rangle\| \leq \epsilon\) by assumption. Before normalization, denote

\[
\hat{x} = \frac{1}{\sqrt{p}} \sum_{l=0}^{p-1} \|\tilde{x}(t_l)\| |l\rangle \tilde{x}(t_l)\rangle = \frac{\|C\|_F}{\sqrt{p}} (U_p F \otimes I) |\text{vec}(C)\rangle.
\]

By Proposition 13 and Fact 14, \(\|\text{vec}(t_k)\rangle - \hat{x}(t_k)\| \leq \sqrt{n} R_x S^{-p}\). From equation (15),

\[
\hat{x} = \frac{1}{\sqrt{p}} \sum_{l=0}^{p-1} \|\text{vec}(t_l)\| |l\rangle \text{vec}(t_l)\rangle.
\]

So we have \(\|\hat{x} - \hat{x}\| \leq \sqrt{n} R_x S^{-p}\). After normalization, we have \(\|\tilde{x}\rangle - \hat{x}\rangle\| \leq 2\sqrt{n} R_x S^{-p}/\|\tilde{x}\|\). Notice that \(\|\text{vec}(t_0)\| = 1\), so

\[
\|\tilde{x}\| = \frac{1}{p} \sum_{l=0}^{p-1} \|\tilde{x}(t_l)\|^2 \geq \frac{1}{p}.
\]

Thus \(\|\tilde{x}\rangle - \hat{x}\rangle\| \leq 2\sqrt{n} R_x S^{-p} \ll \epsilon\) since \(p = O(\max\{\rho/\epsilon, (\log \epsilon)^{-1} \log (R_x n / \epsilon)\})\) and \(S \approx 1/\epsilon^2\). Combining the error between \(|\text{vec}(C)\rangle\) and \(|C\rangle\), the fact that \(U_p F\) is unitary and \(|\tilde{x}\rangle = (U_p F \otimes I)|\text{vec}(C)\rangle\), we obtain the claimed result. \(\square\)
The proof of the following result on the condition number is the most technical one of this paper. The basic intuition is as follows: the condition number of $M$ is defined by $\|M\|\|M^{-1}\|$. It is easy to bound $\|M\|$ using the triangular inequality. As for $\|M^{-1}\|$, by definition it equals $\max_{\|b\|=1} \|M^{-1}b\|$. Since $M$ is constructed from the discretization of the ODE (14), it follows that $M^{-1}b$ is closely related to the solution of this ODE. For the ODE (14), there is an explicit formula for the solution. Based on this formula, we can bound $\|M^{-1}\|$.

**Proposition 17 (Condition number).** Let $A, B$ be the matrices that satisfy the conditions described in Problem 1. Let $E$ be the matrix of the generalized eigenvectors with condition number $\kappa_E$, and $M$ be the matrix (27). Then

$$\|M\| = \Theta(\|I \otimes A - \tau^{-1}D \otimes B\|),$$  \hspace{1cm} (33)

and

$$\|M^{-1}\| \leq O(\kappa_E\|B^{-1}\|/\epsilon).$$  \hspace{1cm} (34)

Consequently, the condition number of $M$ is upper bounded by

$$O\left(\kappa_E(\|A\| + \rho\|B\|)\|B^{-1}\|/\epsilon\right).$$  \hspace{1cm} (35)

**Proof.** Set $|e\rangle = \frac{1}{\sqrt{p}}(1, \ldots, 1)^T \in \mathbb{R}^p$, and $I_0 = [|1\rangle, \ldots, |p-1\rangle]_{p\times(p-1)}$, the $p \times p$ identity matrix with the first column removed, then $F_0 = FI_0, F|0\rangle = |e\rangle$. As for the condition number of $M$, we consider

$$M^\dagger M = |e\rangle\langle e| \otimes I + (I \otimes A - \tau^{-1}D \otimes B)^\dagger(\bar{F}I_0T^I_0F^T \otimes I)(I \otimes A - \tau^{-1}D \otimes B)$$

$$= |e\rangle\langle e| \otimes I + (I \otimes A - \tau^{-1}D \otimes B)^\dagger(\bar{F}(I - |0\rangle\langle 0|)F^T \otimes I)(I \otimes A - \tau^{-1}D \otimes B)$$

$$= |e\rangle\langle e| \otimes I + (I \otimes A - \tau^{-1}D \otimes B)^\dagger((I - |e\rangle\langle e|) \otimes I)(I \otimes A - \tau^{-1}D \otimes B).$$

Since $\||e\rangle\langle e| \otimes I\| = \|(I - |e\rangle\langle e|) \otimes I\| = 1$, we have

$$\|M\|^2 = \|M^\dagger M\| \leq 1 + \|I \otimes A - \tau^{-1}D \otimes B\|^2.$$  \hspace{1cm} (36)

On the other hand, since $M' := (I_0^T F \otimes I)(I \otimes A - \tau^{-1}D \otimes B)$ is a submatrix of $M$, we have $\|M\| \geq \|M'\|$. For simplicity, denote $\tau^{-1}D = -\text{diag}(d_1, \ldots, d_p)$. Also denote $N = I \otimes A - \tau^{-1}D \otimes B$. Since $D$ is diagonal, we have $\|N\| = \max_{1 \leq j \leq p} \|A + d_jB\|$. Without loss of generality, suppose that $\|N\| = \|A + d_1B\|$. Denote the unit right and left singular vector of $A + d_1B$ as $|x\rangle, |y\rangle$, that is $(A + d_1B)|x\rangle = \|A + d_1B\||y\rangle$. Then the corresponding unit right and left singular vector of $N$ are $|0\rangle|x\rangle, |0\rangle|y\rangle$ respectively. Thus

$$\|M\| \geq \|M'\| \geq \|M'|0\rangle|x\rangle\| = \|N\| \|(I_0^T F \otimes I)|0\rangle|y\rangle\| = \|N\| \||e\rangle\langle e| \otimes |y\rangle\| = \sqrt{\frac{p-1}{p}} \|N\|.$$  \hspace{1cm} (37)

Combining (36), we have

$$\|M\| = \Theta(\|I \otimes A - \tau^{-1}D \otimes B\|).$$  \hspace{1cm} (38)

As for $\|M^{-1}\|$, it equals

$$\|M^{-1}\| = \max_{b \in \mathbb{C}^p: \text{My} = b, \|b\| = 1} \|y\|.$$
For any unit vector \( b^T = (b(0)^T, b(1)^T, \ldots, b(p - 1)^T) \in \mathbb{C}^{np} \), where \( b(i) \in \mathbb{C}^n \) is a column vector, from (26) we know that

\[
\begin{bmatrix}
\langle 0 | F^T \otimes I \\
\langle 1 | F^T \otimes A - \tau^{-1} \langle 1 | F^T D \otimes B \\
\vdots \\
\langle p - 1 | F^T \otimes A - \tau^{-1} \langle p - 1 | F^T D \otimes B
\end{bmatrix} \mathbf{y} =
\begin{bmatrix}
b(0) \\
b(1) \\
\vdots \\
b(p - 1)
\end{bmatrix}.
\]

Denote \( Y \) as the \( n \times p \) matrix such that its vectorization is \( \mathbf{y} \), that is \( \text{vec}(Y) = \mathbf{y} \). Then \( \| \mathbf{y} \| = \| Y \|_F \).

We also have

\[
\begin{cases}
YF|0\rangle = b(0), \\
AYF|l\rangle - \tau^{-1} BYDF|l\rangle = b(l), & 1 \leq l \leq p - 1.
\end{cases}
\]

From the discretization procedure, we can view \( YF|l\rangle \) as an approximation of the solution of the following linear system of ODEs

\[
\begin{cases}
B \frac{dx(t)}{dt} = 2\pi i A x(t) + 2\pi i c(t), \\
x(0) = b(0).
\end{cases}
\]  \( (39) \)

where \( c(t) = b(l + 1) \) if \( lh < t \leq (l + 1)h \).

The solution of (39) equals

\[
x(t) = e^{2\pi i B^{-1}At} b(0) + 2\pi i e^{2\pi i B^{-1}At} \int_0^t e^{-2\pi i B^{-1}As} B^{-1} c(s) ds. \quad (40)
\]

In the following analysis, we will suppose \( A \) is invertible for simplicity. When \( A \) is not invertible, the estimations below are also correct. We will give the detailed analysis of this case in Appendix B. Assume that \( t \in (0, h] \) and \( l \in \{1, \ldots, p - 1\} \), then

\[
x(t + lh) = e^{2\pi i B^{-1}A(t+lh)} b(0) + 2\pi i e^{2\pi i B^{-1}A(t+lh)} \sum_{j=0}^{l-1} \int_{jh}^{(j+1)h} e^{-2\pi i B^{-1}As} B^{-1} b(j + 1) ds \\
+ 2\pi i e^{2\pi i B^{-1}A(t+lh)} \int_0^{lh} e^{-2\pi i B^{-1}As} B^{-1} b(l + 1) ds \\
= e^{2\pi i B^{-1}A(t+lh)} b(0) - \sum_{j=0}^{l-1} e^{2\pi i B^{-1}A(t+(l-j)h)} (e^{-2\pi i B^{-1}Ah} - I)(B^{-1}A)^{-1} B^{-1} b(j + 1) \\
- e^{2\pi i B^{-1}At} (e^{-2\pi i B^{-1}At} - I)(B^{-1}A)^{-1} B^{-1} b(l + 1). \quad (41)
\]

Notice that \( B^{-1} A = E \Lambda E^{-1} \) is diagonalizable and \( t \in (0, h] \) in the solution (41), we have

\[
\| e^{2\pi i B^{-1}A(t+(l-j)h)} (e^{-2\pi i B^{-1}Ah} - I)(B^{-1}A)^{-1} \| \\
\leq \| E^{-1} \| \| E \| \max_j \left| \frac{e^{-2\pi i \lambda_j h} - 1}{\lambda_j} \right| \\
= \kappa_E \max_j \frac{2|\sin(\pi \lambda_j h)|}{|\lambda_j|} \leq 2\kappa_E \pi h.
\]
Similarly, we have \( \|(e^{2\pi i B^{-1}A} - I)(B^{-1}A)^{-1}\| \leq 2\kappa_E \pi \hbar \). Thus when \( t + lh \in (lh, (l+1)h] \), we obtain
\[
\|x(t + lh)\| \leq \|e^{2\pi i B^{-1}A(t+lh)}b(0)\| + \sum_{j=0}^{l-1} \|e^{2\pi i B^{-1}A(t+(l-j)h)}(e^{2\pi i B^{-1}A} - I)(B^{-1}A)^{-1}B^{-1}b(j+1)\|
+ \|(e^{2\pi i B^{-1}A} - I)(B^{-1}A)^{-1}B^{-1}b(l+1)\|
\leq \kappa_E \|b(0)\| + \sum_{j=0}^{l-1} 2\kappa_E \|B^{-1}\| \|b(j+1)\| + 2\kappa_E \|B^{-1}\| \|b(l+1)\|
\]

The above upper bound is an inner product between \((\kappa_E, 2\kappa_E \|B^{-1}\|, \ldots, 2\kappa_E \|B^{-1}\|)\) and \((\|b(0)\|, \ldots, \|b(l+1)\|)\). By Cauchy–Schwarz inequality and note that the norm of the second vector is smaller than 1 by assumption, we have
\[
\|x(t + lh)\| = O(\sqrt{l}\kappa_E \|B^{-1}\| \hbar).
\]

Since \( x(lh) = YF\|l\rangle \) and \( F \) is unitary, we have
\[
\|Y\|_F = \|YF\|_F = \sqrt{\sum_{l=0}^{p} \|YF\|_l^2} = \sqrt{\sum_{l=0}^{p} \|x(lh)\|^2} \leq O(p\kappa_E \|B^{-1}\| \hbar).
\]

Since \( ph = O(1/\epsilon) \), it follow that \( \|M^{-1}\| = O(\kappa_E \|B^{-1}\|/\epsilon) \). Together with equation (36), the condition number is upper bounded by
\[
O(\kappa_E \|B^{-1}\| \|I \otimes A - \tau^{-1}D \otimes B\|/\epsilon) = O(\kappa_E \|B^{-1}\| (\|A\| + \tau^{-1}p\|B\|)/\epsilon),
\]

as claimed. \( \square \)

**Remark 18.** Assume that \( A \) is invertible. In the above proof, when estimating \( \|x(t + lh)\| \), if we bound
\[
\|e^{2\pi i B^{-1}A(t+(l-j)h)}(e^{-2\pi i B^{-1}A} - I)(B^{-1}A)^{-1}B^{-1}\|
\]
by
\[
\|e^{2\pi i B^{-1}A(t+(l-j)h)}(e^{-2\pi i B^{-1}A} - I)\| \|(B^{-1}A)^{-1}B^{-1}\| \leq 2\kappa_E \hbar \|A^{-1}\|,
\]
then we have \( \|x(t + lh)\| = O(\sqrt{l}\kappa_E \|A^{-1}\| \hbar) \). Together with the above proof, we now have \( \|x(t + lh)\| = O(\sqrt{l}\kappa_E \min(\|A^{-1}\|, \|B^{-1}\|) \hbar) \). This means that the upper bounded of the condition number can be reduced to \( O(\kappa_E \min(\|A^{-1}\|, \|B^{-1}\|) \|A\| + \tau^{-1}p\|B\|)/\epsilon) \) if \( A \) is nonsingular.

At the end of this part, we use the technique of linear combinations of unitaries (LCU, see [11, Chapter 26]) to construct the block-encoding of \( M \).

**Proposition 19** (Block-encoding of \( M \)). Let \( M \) be the matrix (27). Given an \((\alpha_A, q_A, \delta)\)-block-encoding of \( A \) in time \( O(T_A) \), and \((\alpha_B, q_B, \delta)\)-block-encoding of \( B \) in time \( O(T_B) \), there is a quantum circuit that implements an \((\alpha, \max\{q_A, q_B\} + 2 \log p + 3, 4\delta)\)-block-encoding of \( M \) in time \( O(T_A + T_B) \), where \( \alpha = O(\alpha_A + \tau^{-1}p\alpha_B) \).
Proof. We decompose $M$ as follows

$$M = (F^T \otimes I)(I \otimes A) - (F^T \otimes I)(\tau^{-1}D \otimes B) + \begin{bmatrix} \langle0| F^T \otimes I \\ 0 \end{bmatrix} \begin{bmatrix} \langle0| F^T \otimes I \\ 0 \end{bmatrix} (\tau^{-1}D \otimes B).$$

(42)

The basic idea is to construct the block-encoding of each term, then use LCU to create the block-encoding of $M$.

Since $\tau^{-1}D$ is diagonal, an $((p - 1)/2\tau, \log p, 0)$-block-encoding can be constructed naively, for example by 2-dimensional rotations. Note that

$$\begin{bmatrix} \langle0| F^T \otimes I \\ 0 \end{bmatrix} = \begin{bmatrix} \langle0| F^T \\ 0 \end{bmatrix} \otimes I$$

and $F$ is unitary, so we naturally have a $(1, \log p, 0)$-block-encoding of $\begin{bmatrix} \langle0| F^T \otimes I \\ 0 \end{bmatrix}$. Together with the above two block-encodings and the block-encodings of $A, B$, we can construct the block-encodings of three other terms by Lemma 8.

Before we show the details of the LCU procedure, we first introduce some notation of the block-encodings. Let $V_1, U_2, U_3$ be the block-encodings of $A, B, \tau^{-1}D$ respectively. Then $V_1 := (F^T \otimes I)(I \otimes U_1)$ is an $(\alpha_A, q_A + \log p, \delta)$-block-encoding of $(F^T \otimes I)(I \otimes A)$. Notice that $U_3 \otimes U_2$ is an $(\alpha_B(p - 1)/2\tau, q_B + \log p, \delta)$-block-encoding of $\tau^{-1}D \otimes B$, so $V_2 := (F^T \otimes I)(U_3 \otimes U_2)$ is an $(\alpha_B(p - 1)/2\tau, q_B + \log p, \delta)$-block-encoding of $(F^T \otimes I)(\tau^{-1}D \otimes B)$. Let $V_3$ be an $(1, \log p, 0)$-block-encoding of $\begin{bmatrix} \langle0| F^T \otimes I \\ 0 \end{bmatrix}$. As for $\begin{bmatrix} \langle0| F^T \otimes I \\ 0 \end{bmatrix}(I \otimes A)$, by Lemma 8, we can construct an $(\alpha_A, q_A + 2\log p, \delta)$-block-encoding, denoted as $V_4$. The cost is $O(T_A)$. Finally, by Lemma 8 again, we can create an $(\alpha_B(p - 1)/2\tau, q_B + 2\log p, \delta)$-block-encoding of $\begin{bmatrix} \langle0| F^T \otimes I \\ 0 \end{bmatrix}(D \otimes B)$ in time $O(T_B)$. Denote this block-encoding as $V_5$.

To obtain the block-encoding of $M$, by (42) it suffices to apply LCU to implement $V := \alpha_A V_1 - (\alpha_B(p - 1)/2\tau) V_2 + V_3 - \alpha_A V_4 + (\alpha_B(p - 1)/2\tau) V_5$. The procedure is the claimed block-encoding. Let $\alpha = \alpha_A + \alpha_B(p - 1)/2\tau + 1 + \alpha_A + \alpha_B(p - 1)/2\tau$ denote the absolute sum of the coefficients, then for any state $|\phi\rangle$, the LCU produces gives a state of the form $\alpha^{-1}|0\rangle \otimes V|\phi\rangle + |0\rangle^\perp$. This is an $(\alpha, \max\{q_A, q_B\} + 2\log p + 3, 4\delta)$-block-encoding of $M$. Here the other $3 = \lceil \log 5 \rceil$ ancilla qubits come from the LCU.

C. Main result

With the above preliminaries, we now state our main theorem as follows.

**Theorem 20.** Let $\epsilon \in (0, 1)$, $\delta = o(\epsilon/(\kappa_M^2 \log^3(\kappa_M/\epsilon)))$, where $\kappa_M$ is the condition number of $M$. Suppose we have an $(\alpha_A, q_A, \delta)$-block-encoding of $A$ that is constructed in time $O(T_A)$, an $(\alpha_B, q_B, \delta)$-block-encoding of $B$ that is constructed in time $O(T_B)$. Denote the generalized eigenpairs of $Ax = ABx$ as $\{(\lambda_j, |E_j\rangle) : j = 1, \ldots, n\}$, the condition number of the matrix of generalized eigenvectors as $\kappa_E$. Suppose that $B$ is invertible, $B^{-1}A$ is diagonalizable and all $\lambda_j$ are real. Then given access to copies of the state $\sum_j \beta_j |E_j\rangle$ that is prepared in time $O(T_m)$, there is a quantum
The quantum state

\[ \frac{\nu}{\epsilon} \left( \frac{\kappa_B}{\epsilon} \left( (\alpha_B + \rho \alpha_B) (T_A + T_B) \right) \right), \]

where \( |\tilde{\lambda}_j - \lambda_j| \leq \epsilon \) and \( \rho = \max\{ 1, \max_j |\lambda_j| \} \).

**Proof.** By (17), \( p = O(\rho/\epsilon), \tau = O(1/\epsilon) \). By Proposition 17, the condition number \( \kappa_M \) of \( M \) is bounded by \( O(\epsilon^{-1} \kappa_E(A) + \rho \| B \| \| B^{-1} \|) \). By Proposition 19, we can construct an \( (O(\alpha_B + \rho \alpha_B), \max\{ q_A, q_B \} + 2 \log p + 3, 4\delta) \)-block-encoding of the coefficient matrix of the linear system (26) in time \( O(T_A + T_B) \). By Lemma 11, the quantum state of \( \text{vec}(C) \) can be produced in time

\[ \tilde{O} \left( \kappa_M \left( T_{\text{in}} + \frac{(\alpha_B + \rho \alpha_B) (T_A + T_B)}{\| M \|} \right) \right) = \tilde{O} \left( \frac{\kappa_E}{\epsilon} \left( (\alpha_B + \rho \alpha_B) (T_A + T_B) \right) \min \left( \| A^{-1} \|, \| B^{-1} \| \right) \right). \]

In the above, we used the fact that \( \| A \| \leq \alpha_A, \| B \| \leq \alpha_B \). From equations (28), (30), when apply \( FU_{p} F \otimes I \) to \( \text{vec}(C) \), we will obtain a state proportional to \( \sum_{j} \alpha_j |\tilde{\lambda}_j \rangle |E_j \rangle \). The overall complexity is (44), the cost to approximate \( \text{vec}(C) \).

By Remark 18, if \( A \) is invertible, the complexity can be improved to

\[ \tilde{O} \left( \frac{\kappa_E}{\epsilon} \left( (\alpha_B + \rho \alpha_B) (T_A + T_B) \right) \min \left( \| A^{-1} \|, \| B^{-1} \| \right) \right). \]

Finally, for the convenience of further applications, we summarize our algorithm as follows.

**Algorithm 1 Quantum algorithm for solving QGEP**

**Input:** (1). Two \( n \times n \) complex matrices \( A, B \) with eigenpairs \( \{(\lambda_j, |E_j\rangle) : j \in [n]\} \). Assume that \( B \) is invertible, \( B^{-1} A \) is diagonalizable and all \( \lambda_j \) are real.
(2). Block-encodings of \( A, B \).
(3). An upper bound \( \rho \geq 1 \) on the eigenvalues.
(4). Quantum access to copies of the state \( |\phi\rangle \), which formally equals \( \sum_{j=1}^{n} \beta_j |E_j\rangle \).
(5). The precision \( \epsilon \in (0, 1), h = 1/\rho, p = |\rho/\epsilon|, \tau = ph \).

**Output:** The quantum state

\[ \sum_{j=1}^{n} \beta_j |\tilde{\lambda}_j \rangle |E_j \rangle \]

up to a normalization, where \( |\tilde{\lambda}_j - \lambda_j| \leq \epsilon \) for all \( j \).

1. Construct the matrix \( M \) based on equation (27).
2. Use Proposition 19 to construct the block-encoding of \( M \).
3. Solve the linear system \( My = |0..0\rangle |\phi\rangle \) (see (26)) in a quantum computer by Lemma 11.
4. Return the state \( (F^{-1} U_p F \otimes I) |y\rangle \), where \( U_p \) is given by equation (31) and \( F \) is the quantum Fourier transform.
D. Some corollaries

In this section, we consider some special cases of Theorem 20. First, when $A, B$ are sparse, combining Lemma 5 and Theorem 20, we have the following result.

**Corollary 21 (GEP of sparse matrices).** Let $\epsilon \in (0, 1)$. Suppose $A, B$ are $s_A, s_B$-sparse, respectively. Denote the generalized eigen-pairs of $Ax = \lambda Bx$ as $\{(\lambda_j, |E_j\rangle) : j = 1, \ldots, n\}$, the condition number of the matrix of generalized eigenvectors as $\kappa_E$. Suppose that $B$ is invertible, $B^{-1}A$ is diagonalizable and all $\lambda_j$ are real. Given access to copies of the state $\sum_j \beta_j |E_j\rangle$ that is prepared in time $O(T_{in})$, there is a quantum algorithm that returns a state proportional to $\sum_j \beta_j |\tilde{\lambda}_j\rangle |E_j\rangle$ in time

$$
\tilde{O}\left(\frac{\kappa_E}{\epsilon} \left(s_A\|A\|_{\text{max}} + \rho s_B\|B\|_{\text{max}}\right) T_{in} \|B^{-1}\|\right),
$$

where $|\tilde{\lambda}_j - \lambda_j| \leq \epsilon$ and $\rho = \max\{1, \max_j |\lambda_j|\}$.

For the standard eigenvalue problem (that is $B = I$), we have $\rho(A, I) \leq \|A\| \leq \alpha_A$. As a corollary of Theorem 20, we have the following result, which improves the quantum algorithm proposed in [33] in terms of $\kappa_E, \rho$ and $\epsilon$.

**Corollary 22 (Estimating eigenvalues).** Let $\epsilon \in (0, 1), \delta = o(\epsilon/\kappa_M^2 \log^3(\kappa_M/\epsilon))$. Suppose we have an $(\alpha_A, q_A, \delta)$-block-encoding of $A$ that is constructed in time $O(T_A)$. Denote the eigen-pairs of $Ax = \lambda x$ as $\{(\lambda_j, |E_j\rangle) : j = 1, \ldots, n\}$, the condition number of the matrix of eigenvectors as $\kappa_E$. Suppose $A$ is diagonalizable and all $\lambda_j$ are real. Then given access to copies of the state $\sum_j \beta_j |E_j\rangle$ that is prepared in time $O(T_{in})$, there is a quantum algorithm that returns a state proportional to $\sum_j \beta_j |\tilde{\lambda}_j\rangle |E_j\rangle$ in time

$$
\tilde{O}\left(\frac{\alpha_A \kappa_E(T_{in} + T_A)}{\epsilon}\right),
$$

where $|\tilde{\lambda}_j - \lambda_j| \leq \epsilon$ and $\rho = \max\{1, \max_j |\lambda_j|\}$.

If we assume that $A$ is Hermitian, the above result is the same as that of QPE [10]. Another special case is $A = I$. Now we have $Bx = \lambda^{-1}x$. This corresponds to the estimation of the inverse of eigenvalues. If we use Corollary 22 directly, then the complexity to estimate the inverse of eigenvalues up to additive error $\epsilon$ is $\tilde{O}(\alpha_B \rho_0 \kappa_E(T_{in} + T_B)/\epsilon)$, where $\rho_0 = 1/\min_j,_{\lambda_j \neq 0} |\lambda_j|$. However, if we consider relative error, then the cost is $\tilde{O}(\alpha_B \rho_0 \kappa_E(T_{in} + T_B)/\epsilon)$. For estimating the inverse of eigenvalues up to additive error $\epsilon$, we can directly use Theorem 20 (note that $A$ is invertible now, so the cost is (45)). The complexity is $\tilde{O}(\alpha_B \rho_0 \kappa_E(T_{in} + T_B)/\epsilon)$. In conclusion, we have the following result.

**Corollary 23 (Estimating inverse of eigenvalues).** Let $\epsilon \in (0, 1), \delta = o(\epsilon/\kappa_M^2 \log^3(\kappa_M/\epsilon))$. Suppose we have an $(\alpha_A, q_A, \delta)$-block-encoding of $A$ that is constructed in time $O(T_A)$. Denote the eigen-pairs of $Ax = \lambda x$ as $\{(\lambda_j, |E_j\rangle) : j = 1, \ldots, n\}$, the condition number of the matrix of eigenvectors as $\kappa_E$. Suppose $A$ is invertible and diagonalizable, and all $\lambda_j$ are real. Then given access to copies of the state $\sum_j \beta_j |E_j\rangle$ that is prepared in time $O(T_{in})$, there is a quantum algorithm that returns a state proportional to $\sum_j \beta_j |\tilde{\lambda}_j^{-1}\rangle |E_j\rangle$ in time

$$
\tilde{O}\left(\frac{\alpha_A \rho_0 \kappa_E(T_{in} + T_A)}{\epsilon}\right),
$$

where $\kappa_E = \max_j |\lambda_j|$, $\beta_j = \alpha_j / \|E_j\|$, $\|A\|_{\text{max}} = \max_j \|E_j\|$, $\|B\|_{\text{max}} = \max_j \|E_j\|$, $\|x\|_{\text{max}} = \max_j |E_j|$, $T_{in}$ is the time to create the state $\sum_j \beta_j |E_j\rangle$, $\tilde{T}_{in}$ is the time to construct the state $\sum_j \beta_j |\tilde{\lambda}_j^{-1}\rangle |E_j\rangle$. As a special case, when $A$ is invertible and diagonalizable, we can directly use Corollary 23, and the complexity is $\tilde{O}(\alpha_B \kappa_E(T_{in} + T_B)/\epsilon)$. In conclusion, we have the following result.
where \( \rho_0 = \max\{1, 1/\min_{j,\lambda_j \neq 0} |\lambda_j|\} \). The error to approximate \( \lambda_j^{-1} \) can either be relative or additive.

All the algorithms proposed above are also applicable to the case when the (generalized) eigenvalues are all purely imaginary. The more general case than GEP is known as the polynomial eigenvalue problem (PEP) [32], which aims to determine those values of \( \lambda \in \mathbb{C} \) and those vectors \( \mathbf{x} \in \mathbb{C}^n \) for which \( \sum_{k=0}^{m} \lambda^k A_k \mathbf{x} = 0 \), where \( A_0, \ldots, A_m \) are square matrices. For solving the PEP, the classical and most widely used approach is linearization [27]. That is we convert the PEP into a larger size GEP with the same eigenvalues so that the classical methods for GEPs can be pressed into service. Therefore, if the obtained GEP satisfies the assumptions of Problem 1 (e.g., quadratic eigenvalue problems arising from applications in overdamped systems and gyroscopic systems [25]), our algorithm proposed above can be applied to solve this PEP.

IV. THE STANDARD ALGORITHM FOR SYMMETRIC GEPS

In this section, we analyze the standard quantum algorithm that uses QPE to solve symmetric GEPs in the framework of block-encoding. Recall that a matrix pair \((A, B)\) is called symmetric if \( A, B \) are Hermitian and \( B \) is positive definite. From the proof of Proposition 2, to solve the symmetric GEP \( Ax = \lambda Bx \), it suffices to solve the Hermitian eigenvalue problem \( \tilde{A}y = \lambda y \), where \( \tilde{A} = B^{-1/2}AB^{-1/2}, y = B^{1/2}x \). In a quantum computer, QPE is a standard algorithm to solve Hermitian eigenvalue problems. In the framework of block-encoding (see Lemma 10), the main problem we need to solve is the construction of the block-encoding of \( \tilde{A} \). This can be accomplished by Lemmas 7 and 8. The following theorem summarizes the overall complexity of this algorithm.

Theorem 24. Assume that \( A, B \) are \( n \)-by-\( n \) Hermitian matrices with \( B \) positive definite. Let \( U_A \) be an \((\alpha_A, q_A, \epsilon_A)\)-block-encoding of \( A \) that is implemented in time \( O(T_A) \), and \( U_B \) an \((\alpha_B, q_B, \epsilon_B)\)-block-encoding of \( B \) that is implemented in time \( O(T_B) \). Denote \( \kappa_B \) as the condition number of \( B \), the generalized eigen-pairs of \( Ax = \lambda Bx \) as \( \{(\lambda_j, |E_j\rangle) : j = 1, \ldots, n \} \). Let \( \epsilon \in (0, 1) \). Given access to copies of the state \( \sum_j \beta_j |E_j\rangle \) that is prepared in time \( O(T_{in}) \), there is a quantum algorithm that returns a state proportional to \( \sum_j \beta_j |\tilde{\lambda}_j\rangle |E_j\rangle \), where \( |\tilde{\lambda}_j - \lambda_j| \leq \epsilon \), in time

\[
\tilde{O}\left(\frac{\kappa_B^{0.5}T_{in}}{\epsilon} + \frac{\alpha_A \kappa_B^{1.5}}{\|B\|\epsilon} \left(T_A + \frac{\alpha_B \kappa_B T_B}{\|B\|}\right)\right).
\]  

Proof. The basic idea of the quantum algorithm is as follows: Denote \( |\phi_0\rangle = \sum_j \beta_j |E_j\rangle \). Note that \( \tilde{AB}^{1/2}|E_j\rangle = \lambda_j \gamma_j^{1/2}|E_j\rangle \), where \( \tilde{A} = B^{-1/2}AB^{-1/2} \). So in step one, we apply \( B^{1/2} \) to \( |\phi_0\rangle \) to obtain \( |\phi_1\rangle \propto \sum_j \beta_j B^{1/2}|E_j\rangle \). Then perform QPE to \( \tilde{A} \) with initial state \( |\phi_1\rangle \) to create \( |\phi_2\rangle \propto \sum_j \beta_j |\tilde{\lambda}_j\rangle \otimes B^{1/2}|E_j\rangle \). Finally, apply \( B^{-1/2} \) to the second register of \( |\phi_2\rangle \).

To make few confusions on the notation, we will use \( \gamma \), instead of \( \epsilon \), to indicate the accuracy to approximate the eigenvalues. For an \((\alpha, q, \epsilon)\)-block-encoding, \( q \) is poly-log in other parameters like the dimension, condition number and the accuracy, so we are more concerned about \( \alpha, \epsilon \) in the time complexity analysis below. To simplify the notation, we just say it an \((\alpha, \epsilon)\)-block-encoding.

To implement the above three steps, we first consider the construction of the block-encoding of \( \tilde{A} = B^{-1/2}AB^{-1/2} \). Denote \( \tilde{B} = B/\|B\| \), then by Fact 6, \( U_B \) can be viewed as an \((\alpha_B/\|B\|, \epsilon_B/\|B\|)\)-block-encoding of \( \tilde{B} \). In Lemma 7, if we choose \( \delta = \kappa_B^{1.5} \epsilon_B/\|B\| \), then we have a \((2\sqrt{\kappa_B}, \kappa_B^{1.5} \epsilon_B/\|B\|)\)-block-encoding of \( \tilde{B}^{-1/2} = \|B\|^{1/2} B^{-1/2} \). It can be implemented in time

\[
O(\alpha_B \|B\|^{-1} \kappa_B T_B \log^2(\|B\|/\epsilon_B)) = \tilde{O}(\alpha_B \|B\|^{-1} \kappa_B T_B).
\]  

(51)
The error on the term $B^{-1/2}$ can be viewed as a $(2\sqrt{\kappa_B/\|B\|},\kappa_B^{1.5}\epsilon_B/\|B\|^{1.5})$-block-encoding of $B^{-1/2}$. By Lemma 8, we can construct a $(4\alpha\kappa_B/\|B\|,\epsilon')$-block-encoding of $\tilde{A}$ in time $O(T_A + \alpha_B\kappa_B T_B/\|B\|)$, where

$$\epsilon' = \frac{4\alpha\kappa_B^2\epsilon_B}{\|B\|^2} + \frac{4\kappa_B\epsilon_A}{\|B\|}.$$ (52)

Step 1, apply $B^{1/2}$ to $|\phi_0\rangle$. By Lemma 7, if $\delta = \kappa_B\epsilon_B/\|B\|$, then we can build a $(2,\kappa_B\epsilon_B/\|B\|)$-block-encoding of $\sqrt{B} = B^{1/2}/\|B\|^{1/2}$ in cost $\tilde{O}(\alpha_B B^{-1}\kappa_B T_B)$. By Fact 6, this can be viewed as an $(2\|B\|^{1/2},\kappa_B\epsilon_B/\|B\|^{1/2})$-block-encoding of $B^{1/2}$. Based on this block-encoding, we will obtain

$$|\phi_1\rangle = \frac{1}{2\|B\|^{1/2}}B^{1/2}|\phi_0\rangle|0\rangle + |0\rangle = \frac{1}{2\|B\|^{1/2}}\sum_{j=1}^{n} \beta_j B^{1/2}|E_j\rangle|0\rangle + |0\rangle. \tag{53}$$

The error on the term $B^{1/2}|\phi_0\rangle$ is bounded by $\kappa_B\epsilon_B/\|B\|^{1/2}$. The complexity of this step is $O(T_{\text{in}} + \alpha_B\|B\|^{-1}\kappa_B T_B)$.

Step 2, perform QPE to $\tilde{A}$ based on Lemma 10. Recall that we use $\gamma$ to denote the accuracy of approximating the eigenvalues, so in Lemma 10, we choose

$$\tilde{\epsilon} = \frac{4\epsilon'\log^2(1/\gamma)}{\gamma},$$

where $\epsilon'$ is given in (56). Then from the state $|\phi_1\rangle$, we can obtain

$$|\phi_2\rangle = \frac{1}{2\|B\|^{1/2}}\sum_{j=1}^{n} \beta_j |\tilde{\lambda}_j\rangle \otimes B^{1/2}|E_j\rangle|0\rangle + |0\rangle.$$ in time

$$\tilde{O}\left(T_{\text{in}} + \alpha_B\|B\|^{-1}\kappa_B T_B + \frac{\alpha A\kappa_B}{\|B\|} \left( T_A + \frac{\alpha_B\kappa_B T_B}{\|B\|}\right)\right). \tag{53}$$

Step 3, apply $B^{-1/2}$ to the second register of $|\phi_2\rangle$, then we obtain

$$|\phi_3\rangle = \frac{1}{4\sqrt{\kappa_B}}\sum_{j=1}^{n} \beta_j |\tilde{\lambda}_j\rangle |E_j\rangle|0\rangle + |0\rangle.$$

The error on the summation term is bounded by $\kappa_B^{2.5}\epsilon_B^{2}/\|B\|^2$. The complexity of this step is (51) + (53).

We can choose $\epsilon_B$ small to make sure the error on the state $|\phi_3\rangle$ is small. Since $\epsilon_B$ appears as a poly-log term in the complexity, this does not affect the complexity too much. By amplitude estimation, it costs an extra $O(\sqrt{\kappa_B})$ to obtain the state proportional to $\sum_{j=1}^{n} \beta_j |\tilde{\lambda}_j\rangle |E_j\rangle$. Multiplying (53) by $O(\sqrt{\kappa_B})$ gives rise to the claimed result.

The complexity result of Theorem 24 is invariant under scaling, so we can assume that $\|B\| = \Theta(1)$. Then the complexity can be simplified into $\tilde{O}(n_B^{0.5}T_{\text{in}} + \alpha A\kappa_B^{1.5}(T_A + \alpha_B\kappa_B T_B)/\epsilon)$. Specifically, when $A, B$ are sparse, as a corollary of Lemma 5 and Theorem 24, we have the following result.
Corollary 25. Assume that \( A, B \) are \( n \times n \) Hermitian matrices and \( B \) is positive definite. Assume that \( A \) (resp. \( B \)) has sparsity \( s_A \) (resp. \( s_B \)). Denote \( \kappa_B \) as the condition number of \( B \), the generalized eigen-pairs of \( AX = \lambda BX \) as \( \{(\lambda_j, |E_j|): j = 1, \ldots, n\} \). Let \( \varepsilon \in (0, 1) \). Given access to copies of the state \( \sum_j \beta_j |E_j\rangle \) that is prepared in time \( O(T_{in}) \), then there is a quantum algorithm that returns a state proportional to \( \sum_j \beta_j |\tilde{\lambda}_j\rangle |E_j\rangle \) in time
\[
\tilde{O}\left( \kappa_B^{0.5} T_{in} + \frac{\kappa_B^{2.5} s_A s_B \|A\|_{\text{max}} \|B\|_{\text{max}}}{\varepsilon \|B\|^2} \right),
\]
where \( |\tilde{\lambda}_j - \lambda_j| \leq \varepsilon \).

Proof. When \( A, B \) are sparse, by Lemma 5, we can construct an \( (s_A \|A\|_{\text{max}}, \text{poly} \log(n/\varepsilon), \|A\|_{\text{max}} \varepsilon) \)-block-encoding of \( A \), an \( (s_B \|B\|_{\text{max}}, \text{poly} \log(n/\varepsilon), \|B\|_{\text{max}} \varepsilon) \)-block-encoding of \( B \) efficiently. Then the claimed result comes naturally from Theorem 24.

As we can see from the proof of Theorem 24, the main cost comes from the construction of the block-encoding of \( A \). In [10], there is another way to create the block-encoding of the product of two matrices.

Lemma 26 (Lemma 5 of [10]). Let \( A_1, A_2 \) be two matrices. Assume that \( \|A_i\| \leq 1, \alpha_i \geq 1 \) for \( i = 1, 2 \). Let \( U_i \) be an \( (\alpha_i, q_i, \varepsilon_i) \)-block-encoding of \( A_i \) that can be implemented in time \( O(T_i) \). Then there is a \( (2, q_1 + q_2 + 2, \sqrt{2}(\varepsilon_1 + \varepsilon_2)) \)-block-encoding of \( A_1 A_2 \) that can be implemented in time \( O(\alpha_1 (q_1 + T_1) + \alpha_2 (q_2 + T_2)) \).

As a corollary of Fact 6 and Lemma 26, when the assumption \( \|A_i\| \leq 1 \) is removed, we then have the following result.

Corollary 27. Let \( U_i \) be an \( (\alpha_i, q_i, \varepsilon_i) \)-block-encoding of \( A_i \) that is obtained in time \( O(T_i) \), where \( i = 1, 2 \). Then we can construct a \( (2 \|A_1\| \|A_2\|, q_1 + q_2 + 2, \sqrt{2}(\|A_1\| \varepsilon_1 + \|A_2\| \varepsilon_2)) \)-block-encoding of \( A_1 A_2 \) in time \( O(\alpha_1 (q_1 + T_1) \|A_1\|^{-1} + \alpha_2 (q_2 + T_2) \|A_2\|^{-1}) \).

Proof. Without loss of generality, we assume that \( \|A_i\| \geq 1 \). Denote \( A'_i = A_i/\|A_i\| \). Then \( U_i \) can be viewed as an \( (\alpha_i/\|A_i\|, q_i, \varepsilon_i/\|A_i\|) \)-block-encoding of \( A'_i \). Since \( \|A_i\| \geq 1 \), we have \( \alpha_i \geq \|A_i\| \). This implies \( \|A'_i\| \leq 1 \). By Lemma 26, a \( (2, q_1 + q_2 + 2, \sqrt{2}(\|A_1\|^{-1} \varepsilon_1 + \|A_2\|^{-1} \varepsilon_2)) \)-block-encoding of \( A'_1 A'_2 \) can be constructed in time \( O(\alpha_1 (q_1 + T_1) \|A_1\|^{-1} + \alpha_2 (q_2 + T_2) \|A_2\|^{-1}) \). By Fact 6, this is a \( (2 \|A_1\| \|A_2\|, q_1 + q_2 + 2, \sqrt{2}\|A_1\| \|A_2\| (\varepsilon_1 \|A_1\|^{-1} + \varepsilon_2 \|A_2\|^{-1})) \)-block-encoding of \( A_1 A_2 \).

With the same notation as that in the proof of Theorem 24, by Corollary 27 we can construct an \( (2 \|B^{-1/2}\| \|A B^{-1/2}\|, \varepsilon') \)-block-encoding of \( \tilde{A} \) in time
\[
\O\left( \kappa_B^{0.5} T_{in} + \frac{\kappa_B^{2.5} \alpha_A T_A}{\|A\| \|B\|^{1.5}} \right),
\]
where
\[
\varepsilon' = \sqrt{2} \left( \sqrt{2} \|B^{-1/2}\| \left( \|A\| \kappa_B^{1.5} \varepsilon_B \|B\|^{1.5} + \|B^{-1/2}\| \varepsilon_A \right) + \|A B^{-1/2}\| \kappa_B^{1.5} \varepsilon_B \|B\|^{1.5} \right). \tag{56}
\]
Now with a similar arguments to the proof of Theorem 24 (the only difference is the parameters in the block-encoding of \( \tilde{A} \)), it is not hard to show that the complexity of Theorem 24 becomes
\[
\tilde{O}\left( \kappa_B^{0.5} T_{in} + \frac{\kappa_B^{2.5} \alpha_A T_A}{\|B\|^{1.5}} \left( \kappa_B T_B + \frac{\kappa_B \alpha_B T_T}{\|B\|} \right) \right). \tag{57}
\]
And the complexity of Corollary 25 becomes

\[O \left( \frac{0.5}{\alpha B} T_{in} + \frac{\kappa_B^2}{\|B\|\epsilon} \left( s_A \|A\|_{\max} + \frac{\kappa_B s_B \|A\| \|B\|\max}{\|B\|} \right) \right). \tag{58}\]

In comparison, the above results have better dependence on \(\alpha, \alpha_B\) or \(s_A \|A\|_{\max}, s_B \|B\|\max\) but a little worse dependence on the condition number \(\kappa_B\).

Finally, we make a comparison between our algorithm given in the above section and the above standard algorithm based on QPE for symmetric GEPs. For simplicity, we assume that \(\|B\| = \Theta(1)\) and \(T_{in} = T_A = T_B = \tilde{O}(1)\). Then the complexity of our algorithm is

\[\tilde{O} \left( (\alpha_A + \|A\|\kappa_B \alpha_B) \kappa_B^{1.5} / \epsilon \right).\]

And the complexity of the standard quantum algorithm by QPE is

\[
\min \left\{ \tilde{O} (\alpha_A \alpha_B \kappa_B^2 / \epsilon), \tilde{O} (\alpha_A + \|A\| \kappa_B \alpha_B) \kappa_B^2 / \epsilon \right\}.
\]

As we can see, our algorithm is a little better than the standard one in terms of \(\alpha_A, \alpha_B\) and \(\kappa_B\). When restricted to the Hermitian eigenvalue problem, all the algorithms have the same complexity \(\tilde{O}(\alpha / \epsilon)\), which coincides with the complexity of QPE [10]. Thus all the algorithms achieve optimal dependence on \(\epsilon\).

Indeed, we believe that all the above algorithms for symmetric GEPs should have similar performance in practice. The main difficulty for the QPE method is that we need to construct the block-encoding of \(A\) efficiently. In the proof, we obtain a \((4\alpha_A \kappa_B / \|B\|, \epsilon')\)-block-encoding of \(A\). So in the QPE, \(4\alpha_A \kappa_B / \|B\| = O(\alpha_A \kappa_B)\) is viewed as an upper bound of the eigenvalues. However, in our algorithm, we only need to choose \(\|B^{-1/2}AB^{-1/2}\| = O(\|A\| \kappa_B)\) as the upper bound. If \(\alpha_A \approx \|A\|\), then both algorithms have the same complexity. In the above second method based on Corollary 27, we obtain a \((2\|B^{-1/2}\| \|AB^{-1/2}\|, \epsilon')\)-block-encoding of \(A\). The bound \(\|B^{-1/2}\| \|AB^{-1/2}\|\) is close to the one we used. While this idea leads to a little worse dependence on the condition number of \(B\). Since \(\tilde{A} = B^{-1/2}AB^{-1/2}\), the optimal block-encoding we can construct is \((\|B^{-1/2}AB^{-1/2}\|, \epsilon'')\) for some \(\epsilon''\). From this point, our algorithm combines the advantages of both methods introduced above based on QPE. In other words, it has similar performance to the QPE method using the optimal block-encoding of \(A\).

V. LOWER BOUND FOR THE GENERALIZED EIGENVALUE PROBLEMS

For a classical computer, the GEPs involving singular matrices are challenging to solve from the viewpoint of stability and computational complexity [21, 39]. For instance, in [18, Section 7.7], three simple examples are listed to illustrate this by showing that when the matrices are singular, the eigenvalues can be empty, finite or the whole space. In this section, we shall prove that singular GEPs are also hard to solve for a quantum computer.

For any \(n \times n\) matrix \(M = (M_{ij})\), we define the entry access oracle as: for any \(i, j \in [n]\), the oracle performs \(|i, j\rangle |0\rangle \rightarrow |i, j\rangle |M_{ij}\rangle\). We consider the following simple version of the GEP. Assume that we are given access to copies of a quantum state \(|\psi\rangle \in \mathbb{C}^n\) and query access to the entries of \(n \times n\) matrices \(A\) and \(B\). We are promised that \(A|\psi\rangle = \lambda B|\psi\rangle\) for some \(\lambda \in \mathbb{C}\). Our goal is to find \(\lambda\) efficiently (i.e., in time \(\text{poly} \log(n)\), not \(\text{poly}(n)\)). However, we can show that there is no such efficient quantum algorithm when \(B\) is singular.
Theorem 28. Suppose we are given access to copies of a quantum state $|\psi\rangle$ and query access to entries of $n \times n$ matrices $A$ and $B$. If $A|\psi\rangle = \lambda B|\psi\rangle$ and $B$ is singular, then finding $\lambda$ requires at least $\Omega(\sqrt{n})$ quantum queries to $A$ and $B$.

Proof. Consider the following two cases:

1. $|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i\rangle$, $A = B = |x\rangle\langle x|$, for arbitrary $x \in \{1, \ldots, n\}$;

2. $|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i\rangle$, $A = 0$, $B = |x\rangle\langle x|$, for arbitrary $x \in \{1, \ldots, n\}$.

These are both instances of the GEP. In the first case, $A|\psi\rangle = B|\psi\rangle$, so $\lambda = 1$. In the second case, $A|\psi\rangle = 0$, so $\lambda = 0$. However, we can show that distinguishing these two cases requires $\Omega(\sqrt{n})$ queries to $A$ and $B$. We can think of this problem as follows: we are given query access to a Boolean function $f : [2n] \to \{0, 1\}$, such that either:

1. $f(x) = 1$ and $f(x + n) = 1$ if $x = x_0$, for some $x_0 \in [n]$, and otherwise $f(x) = 0$;

2. $f(x) = 0$ for all $x \in [n]$; $f(x + n) = 1$ if $x = x_0$, for some $x_0 \in [n]$, and otherwise $f(x) = 0$.

Our task is to distinguish these two cases. But one can prove hardness of this using the optimality of Grover’s search algorithm. More precisely, define

$$g(x) = \frac{1}{2} \left(1 - (-1)^{f(x)+f(x+n)}\right)$$

as a function from $[n]$ to $\{0, 1\}$. If $f$ belongs to the first class, then $g(x) = 0$ for all $x \in [n]$. If $f$ belongs to the second class, then $g(x) = 1$ if $x = x_0$ and 0 otherwise. This is a searching problem. By the optimality of Grover’s algorithm [5], distinguishing these two types of functions requires at least $\Omega(\sqrt{n})$ queries. 

Under the assumption of Theorem 28, we can use quantum amplitude estimation [8] to estimate $\lambda$ up to some error. However, this kind of eigenvalue problem (i.e., the eigenvector is given to us) rarely occurs in practice, so we will not go deeper here.

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Appendix A: Error analysis

The analysis below is similar to that in quantum phase estimation. Recall from (15) that the state we have is proportional to

\[
\frac{1}{\sqrt{p}} \sum_{j=1}^{n} \beta_j \sum_{l=1}^{p} e^{2\pi i \lambda_j lh} |l\rangle |E_j\rangle.
\]

Applying inverse of quantum Fourier transform to \(|l\rangle\) yields

\[
\frac{1}{p} \sum_{j=1}^{n} \beta_j \sum_{k=1}^{p} \left( \sum_{l=1}^{p} e^{2\pi il(\lambda_j h - \frac{k}{p})} \right) |k\rangle |E_j\rangle.
\]  

(A1)

Assume that \(k\) satisfies \(|\lambda_j h - \frac{k}{p}| \leq 1/2p\), then

\[
\frac{1}{p} \left| \sum_{l=1}^{p} e^{2\pi il(\lambda_j h - \frac{k}{p})} \right| = \left| \frac{1}{p} \left[ e^{2\pi i(\lambda_j h - \frac{k}{p})} - 1 \right] \right| = \frac{1}{p} \left| \sin \left( \frac{(\lambda_j h - \frac{k}{p})p\pi}{\pi} \right) \right| \geq \frac{2}{\pi}.
\]

In the last step, we used the fact that \(|\sin x| \geq 2x/\pi\) if \(|x| \leq \pi/2\). Thus the state (A1) can be rewritten as

\[
\sum_{j=1}^{n} \beta_j \gamma_j |k_j\rangle |E_j\rangle + \text{others},
\]  

(A2)

where \(k_j\) is the integer such that \(|\lambda_j h - \frac{k_j}{p}| \leq 1/2p\). It is shown above that \(|\gamma_j| \geq 2/\pi\). Indeed, similar to the technique used in QPE \([29, \text{Chapter 5.2}]\), we can add more qubits to increase \(|\gamma_j| \approx 1 - \delta\) for any arbitrary small \(\delta\). In this case, the state (A2) is the one we aim to prepare.

From the above analysis, we see that \(p\) is determined by the error to approximate \(\lambda_j h\), so we should choose \(\tau = ph = 1/\epsilon\). Since \(k/p \leq 1\), we need to choose \(h\) such that \(|\lambda_j h| \leq 1\), i.e., \(h = 1/\rho\). Finally, we have \(p = \rho/\epsilon\). This gives the choices (17).

Appendix B: Complement to the proof of Proposition 17

In this part, we show that the estimations in Proposition 17 are still true even if \(A\) is singular. In (41), if \(A\) is not invertible, we assume that \(B^{-1}A = E\Lambda E^{-1}\), where \(\Lambda = \text{diag}(\Lambda_1, 0)\). Then

\[
x(t + lh) = e^{2\pi i B^{-1}A(t + lh)} b(0) + 2\pi i E e^{2\pi i \Lambda(t + lh)} \sum_{j=0}^{l-1} \int_{jh}^{(j+1)h} e^{-2\pi i \Lambda s} E^{-1} B^{-1} b(j + 1) ds
\]

\[
+ 2\pi i E e^{2\pi i \Lambda(t + lh)} \int_{lh}^{t + lh} e^{-2\pi i \Lambda s} E^{-1} B^{-1} b(l + 1) ds
\]

\[
= e^{2\pi i B^{-1}A(t + lh)} b(0) - \sum_{j=0}^{l-1} E \left[ \Lambda_1^{-1} e^{2\pi i \Lambda_1 t} (e^{-2\pi i \Lambda_1 t} - I) \right] E^{-1} B^{-1} b(j + 1)
\]

\[
- E \left[ \Lambda_1^{-1} e^{2\pi i \Lambda_1 t} (e^{-2\pi i \Lambda_1 t} - I) \right] E^{-1} B^{-1} b(l + 1).
\]
Note that $|t| \leq h$ and

$$
\left\| E \left[ \Lambda_1^{-1} e^{2\pi i \Lambda_1 (t+(l-j)h)} (e^{-2\pi i \Lambda_1 h} - I) hI \right] E^{-1} \right\| \leq \kappa_E \max\{h, \max_j \left| e^{-2\pi i \lambda_j h} - 1 \right| \}
\leq 2\pi \kappa_E h.
$$

So we still have $\|x(t+lh)\| = O(\sqrt{l} \kappa_E \|B^{-1}\| h)$. 

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