BIG COHEN–MACAULAY TEST IDEALS IN EQUAL CHARACTERISTIC ZERO VIA ULTRAPRODUCTS

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Abstract. Utilizing ultraproducts, Schoutens constructed a big Cohen–Macaulay (BCM) algebra $B(R)$ over a local domain $R$ essentially of finite type over $\mathbb{C}$. We show that if $R$ is normal and $\Delta$ is an effective $\mathbb{Q}$-Weil divisor on $\text{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier, then the BCM test ideal $\tau_{B(R)}(\widehat{R}, \widehat{\Delta})$ of $(\widehat{R}, \widehat{\Delta})$ with respect to $B(R)$ coincides with the multiplier ideal $J(\widehat{R}, \widehat{\Delta})$ of $(\widehat{R}, \widehat{\Delta})$, where $\widehat{R}$ and $\widehat{B(R)}$ are the $m$-adic completions of $R$ and $B(R)$, respectively, and $\widehat{\Delta}$ is the flat pullback of $\Delta$ by the canonical morphism $\text{Spec} \widehat{R} \to \text{Spec} R$. As an application, we obtain a result on the behavior of multiplier ideals under pure ring extensions.

§1. Introduction

A (balanced) big Cohen–Macaulay (BCM) algebra over a Noetherian local ring $(R, m)$ is an $R$-algebra $B$ such that every system of parameters is a regular sequence on $B$. Its existence implies many fundamental homological conjectures including the direct summand conjecture (now a theorem). Hochster and Huneke [14], [15] proved the existence of a BCM algebra in equal characteristic, and André [1] settled the mixed characteristic case. Recently, using BCM algebras, Ma and Schwede [18], [19] introduced the notion of BCM test ideals as an analog of test ideals in tight closure theory.

The test ideal $\tau(R)$ of a Noetherian local ring $R$ of positive characteristic was originally defined as the annihilator ideal of all tight closure relations of $R$. Since it turned out that $\tau(R)$ was related to multiplier ideals via reduction to characteristic $p$, the definition of $\tau(R)$ was generalized in [11], [29] to involve effective $\mathbb{Q}$-Weil divisors $\Delta$ on $\text{Spec} R$ and ideals $a \subseteq R$ with real exponent $t > 0$. In these papers, it was shown that multiplier ideals coincide, after reduction to characteristic $p \gg 0$, with such generalized test ideals $\tau(R, \Delta, a^t)$. In positive characteristic, Ma-Schwede’s BCM test ideals are the same as the generalized test ideals. In this paper, we study BCM test ideals in equal characteristic zero.

Using ultraproducts, Schoutens [24] gave a characterization of log-terminal singularities, an important class of singularities in the minimal model program. He also gave an explicit construction of a BCM algebra $B(R)$ in equal characteristic zero: $B(R)$ is described as the ultraproduct of the absolute integral closures of Noetherian local domains of positive characteristic. He defined a closure operation associated with $B(R)$ to introduce the notions of $B$-rationality and $B$-regularity, which are closely related to BCM rationality and BCM regularity defined in [19], and proved that $B$-rationality is equivalent to being rational singularities. The aim of this paper is to give a geometric characterization of BCM test ideals associated with $B(R)$. Our main result is stated as follows:

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**Theorem 1.1** (Theorem 6.4). Let \( R \) be a normal local domain essentially of finite type over \( \mathbb{C} \). Let \( \Delta \) be an effective \( \mathbb{Q} \)-Weil divisor on \( \text{Spec} \, R \) such that \( K_R + \Delta \) is \( \mathbb{Q} \)-Cartier, where \( K_R \) is a canonical divisor on \( \text{Spec} \, R \). Suppose that \( \hat{R} \) and \( \hat{\mathcal{B}(R)} \) are the \( m \)-adic completions of \( R \) and \( \mathcal{B}(R) \), and \( \hat{\Delta} \) is the flat pullback of \( \Delta \) by the canonical morphism \( \text{Spec} \, \hat{R} \to \text{Spec} \, R \). Then we have

\[
\tau_{\hat{\mathcal{B}(R)}}(\hat{R}, \hat{\Delta}) = \mathcal{J}(\hat{R}, \hat{\Delta}),
\]

where \( \tau_{\hat{\mathcal{B}(R)}}(\hat{R}, \hat{\Delta}) \) is the BCM test ideal of \((\hat{R}, \hat{\Delta})\) with respect to \( \hat{\mathcal{B}(R)} \) and \( \mathcal{J}(\hat{R}, \hat{\Delta}) \) is the multiplier ideal of \((\hat{R}, \hat{\Delta})\).

The inclusion \( \mathcal{J}(\hat{R}, \hat{\Delta}) \subseteq \tau_{\hat{\mathcal{B}(R)}}(\hat{R}, \hat{\Delta}) \) is obtained by comparing reductions of the multiplier ideal modulo \( p \gg 0 \) to its approximations. We prove the opposite inclusion by combining an argument similar to that in [25] with the description of multiplier ideals as the kernel of a map between local cohomology modules in [29]. As an application of Theorem 1.1, we show the next result about a behavior of multiplier ideals under pure ring extensions, which is a generalization of [31, Cor. 5.30].

**Theorem 1.2** (Corollary 7.11). Let \( R \hookrightarrow S \) be a pure local homomorphism of normal local domains essentially of finite type over \( \mathbb{C} \). Suppose that \( R \) is \( \mathbb{Q} \)-Gorenstein. Let \( \Delta_S \) be an effective \( \mathbb{Q} \)-Weil divisor such that \( K_S + \Delta_S \) is \( \mathbb{Q} \)-Cartier, where \( K_S \) is a canonical divisor on \( \text{Spec} \, S \). Let \( \mathfrak{a} \subseteq R \) be a nonzero ideal, and let \( t > 0 \) be a positive rational number. Then we have

\[
\mathcal{J}(S, \Delta_S, (\mathfrak{a}S)^t) \cap R \subseteq \mathcal{J}(R, \mathfrak{a}^t).
\]

In [31], we defined ultra-test ideals, a variant of test ideals in equal characteristic zero, to generalize the notion of ultra-\( F \)-regularity introduced by Schoutens [24]. Theorem 1.2 was proved by using ultra-test ideals under the assumption that \( \mathfrak{a} \) is a principal ideal. The description of multiplier ideals as BCM test ideals associated with \( \mathcal{B}(R) \) (Theorem 1.1) and a generalization of module closures in [20] enables us to show Theorem 1.2 without any assumptions.

As another application of Theorem 1.1, we give an affirmative answer to one of the conjectures proposed by Schoutens [24, Rem. 3.10], which says that \( \mathcal{B} \)-regularity is equivalent to being log-terminal singularities (see Theorem 8.2).

This paper is organized as follows: in the preliminary section, we give definitions of multiplier ideals, test ideals, and BCM test ideals. In §3, we quickly review the theory of ultraproducts in commutative algebra including non-standard and relative hulls. In §4, we prove some fundamental results on BCM algebras constructed via ultraproducts following [23]. In §5, we review the relationship between approximations and reductions modulo \( p \gg 0 \) and consider approximations of multiplier ideals. In §6, we show Theorem 1.1, the main theorem of this paper. In §7, using a generalized module closure, we show Theorem 1.2 as an application of Theorem 1.1. In §8, we show that \( \mathcal{B} \)-regularity is equivalent to log-terminal singularities. Finally in §9, we discuss a question, a variant of [7, Quest. 2.7], to handle BCM algebras that cannot be constructed via ultraproducts, and consider the equivalence of BCM-rationality and being rational singularities.
§2. Preliminaries

Throughout this paper, all rings will be commutative with unity.

2.1 Multiplier ideals

Here, we briefly review the definition of multiplier ideals and refer the reader to [16], [21] for more details. Throughout this subsection, we assume that $X$ is a normal integral scheme essentially of finite type over a field of characteristic zero or $X = \text{Spec} \hat{R}$, where $(R, m)$ is a normal local domain essentially of finite type over a field of characteristic zero and $\hat{R}$ is its $m$-adic completion.

**Definition 2.1.** A proper birational morphism $f : Y \to X$ between integral schemes is said to be a *resolution of singularities* of $X$ if $Y$ is regular. When $\Delta$ is a $\mathbb{Q}$-Weil divisor on $X$ and $a \subseteq O_X$ is a nonzero coherent ideal sheaf, a resolution $f : Y \to X$ is said to be a *log resolution* of $(X, \Delta, a)$ if $a O_Y = O_Y (-F)$ is invertible and if the union of the exceptional locus $\text{Exc}(f)$ of $f$ and the support $F$ and the strict transform $f^{-1} \Delta$ of $\Delta$ is a simple normal crossing divisor.

If $f : Y \to X$ is a proper birational morphism with $Y$ a normal integral scheme and $\Delta$ is a $\mathbb{Q}$-Weil divisor, then we can choose $K_Y$ such that $f^* (K_X + \Delta) - K_Y$ is a divisor supported on the exceptional locus of $f$. With this convention:

**Definition 2.2.** Let $\Delta \geq 0$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, let $a \subseteq O_X$ be a nonzero coherent ideal sheaf, and let $t > 0$ be a positive real number. Then the *multiplier ideal sheaf* $J(X, \Delta, a^t)$ associated with $(X, \Delta, a^t)$ is defined by

$$J(X, \Delta, a^t) = f_* O_Y (K_Y - \lfloor f^* (K_X + \Delta) + tF \rfloor).$$

where $f : Y \to X$ is a log resolution of $(X, \Delta, a)$. Note that this definition is independent of the choice of log resolution.

**Definition 2.3.** Let $X$ be a normal integral scheme essentially of finite type over a field of characteristic zero. We say that $X$ has *rational singularities* if $X$ is Cohen–Macaulay at $x$ and if for any projective birational morphism $f : Y \to \text{Spec} O_{X, x}$ with $Y$ a normal integral scheme, the natural morphism $f_* \omega_Y \to \omega_{X, x}$ is an isomorphism.

2.2 Tight closure and test ideals

In this subsection, we quickly review the basic notion of tight closure and test ideals. We refer the reader to [4], [11], [13], [29].

**Definition 2.4.** Let $R$ be a normal domain of characteristic $p > 0$, let $\Delta \geq 0$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, let $a \subseteq R$ be a nonzero coherent ideal sheaf, and let $t > 0$ be a real number. Let $E = \bigoplus E(R/m)$ be the direct sum, taken over all maximal ideals $m$ of $R$, of the injective hulls $E_R(R/m)$ of the residue fields $R/m$.

(1) Let $I$ be an ideal of $R$. The $(\Delta, a^t)$-*tight closure* $I^{*\Delta, a^t}$ of $I$ is defined as follows: $x \in I^{*\Delta, a^t}$ if and only if there exists a nonzero element $c \in R^\circ$ such that

$$ca^{\lfloor t(q-1) \rfloor} x^q \subseteq I^q R([(q-1)\Delta])$$

for all large $q = p^e$, where $I^q = \{f^q | f \in I\}$ and $R^\circ = R \setminus \{0\}$.
(2) If $M$ is an $R$-module, then the $(\Delta, a^t)$-tight closure $0_M^{\Delta, a^t}$ is defined as follows: $z \in 0_M^{\Delta, a^t}$ if and only if there exists a nonzero element $c \in R^e$ such that

$$(ca^{\lceil (q-1) \rceil})^{1/q} \otimes z = 0$$

for all large $q = p^e$.

(3) The (big) test ideal $\tau(R, \Delta, a^t)$ associated with $(R, \Delta, a^t)$ is defined by

$$\tau(R, \Delta, a^t) = \text{Ann}_R(0^{\Delta, a^t}_M).$$

When $a = R$, then we simply denote the ideal $\tau(R, \Delta)$. We call the triple $(R, \Delta, a^t)$ is strongly $F$-regular if $\tau(R, \Delta, a^t) = R$.

**Definition 2.5** [8]. Let $R$ be an $F$-finite Noetherian local domain of characteristic $p > 0$ of dimension $d$. We say that $R$ is $F$-rational if any ideal $I = (x_1, \ldots, x_d)$ generated by a system of parameters satisfies $I = I^e$.

### 2.3 Big Cohen–Macaulay algebras

In this subsection, we will briefly review the theory of BCM algebras. Throughout this subsection, we assume that local rings $(R, \mathfrak{m})$ are Noetherian.

**Definition 2.6.** Let $(R, \mathfrak{m})$ be a local ring, and let $x = x_1, \ldots, x_n$ be a system of parameters. $R$-algebra $B$ is said to be $BCM$ with respect to $x$ if $x$ is a regular sequence on $B$. $B$ is called a (balanced) BCM algebra if it is BCM with respect to $x$ for every system of parameters $x$.

**Remark 2.7** [5, Cor. 8.5.3]. If $B$ is BCM with respect to $x$, then the $\mathfrak{m}$-adic completion $\hat{B}$ is (balanced) BCM.

About the existence of BCM algebras of residue characteristic $p > 0$, the following are proved in [3], [14].

**Theorem 2.8.** If $(R, \mathfrak{m})$ is an excellent local domain of residue characteristic $p > 0$, then the $p$-adic completion of absolute integral closure $R^+$ is a (balanced) BCM $R$-algebra.

Using BCM algebras, we can define a class of singularities.

**Definition 2.9.** If $R$ is an excellent local ring of dimension $d$, and let $B$ be a BCM $R$-algebra. We say that $R$ is $BCM$-rational with respect to $B$ or simply $BCM_B$-rational) if $R$ is Cohen–Macaulay and if $H^d_{\mathfrak{m}}(R) \rightarrow H^d_{\mathfrak{m}}(B)$ is injective. We say that $R$ is $BCM$-rational if $R$ is $BCM_B$-rational for any BCM algebra $B$.

We explain BCM test ideals introduced in [19].

**Setting 2.10.** Let $(R, \mathfrak{m})$ be a normal local domain of dimension $d$.

(i) $\Delta \geq 0$ is a $\mathbb{Q}$-Weil divisor on Spec $R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier.

(ii) Fixing $\Delta$, we also fix an embedding $R \subseteq \omega_R \subseteq \text{Frac} R$, where $\omega_R$ is the canonical module.

(iii) Since $K_R + \Delta$ is effective and $\mathbb{Q}$-Cartier, there exist an integer $n > 0$ and $f \in R$ such that $n(K_R + \Delta) = \text{div}(f)$.

**Definition 2.11.** With notation as in Setting 2.10, if $B$ is a BCM $R[f^{1/n}]$-algebra, then we define $0^{B, K_R + \Delta}_{H^d_{\mathfrak{m}}(\omega_R)}$ to be $\text{Ker} \psi$, where $\psi$ is the homomorphism determined by the
below commutative diagram:

\[
\begin{array}{ccc}
H^d_m(R) & \longrightarrow & H^d_m(B) \xrightarrow{f^{1/n}} H^d_m(B) \\
\downarrow & & \downarrow \\
H^d_m(\omega_R) & \longrightarrow & H^d_m(B \otimes_R \omega_R) \\
\end{array}
\]

If \( R \) is \( m \)-adically complete, then we define

\[ \tau_B(R, \Delta) = \text{Ann}_{R^{B,K_R + \Delta}} \mathcal{H}^d_m(\omega_R). \]

We call \( \tau_B(R, \Delta) \) the BCM test ideal of \((R, \Delta)\) with respect to \( B \). We say that \((R, \Delta)\) is BCM regular with respect to \( B \) (or simply BCM \( B \) regular) if \( \tau_B(R, \Delta) = R \).

\textbf{Proposition 2.12 [19].} Let \((R, m)\) be a complete normal local domain of characteristic \( p > 0 \), let \( \Delta \geq 0 \) be an effective \( \mathbb{Q} \)-Weil divisor on \( \text{Spec} R \), and let \( B \) be a BCM \( R^+ \)-algebra. Fix an effective canonical divisor \( K_R \geq 0 \). Suppose that \( K_R + \Delta \) is \( \mathbb{Q} \)-Cartier. Then

\[ \tau_B(R, \Delta) = \tau(R, \Delta). \]

\section{3. Ultraproducts}

\subsection{3.1 Basic notions}

In this subsection, we quickly review basic notions from the theory of ultraproduct. The reader is referred to [22], [26] for details. We fix an infinite set \( W \). We use \( \mathcal{P}(W) \) to denote the power set of \( W \).

\textbf{Definition 3.1.} A nonempty subset \( \mathcal{F} \subseteq \mathcal{P}(W) \) is called a \emph{filter} if the following two conditions hold.

(i) If \( A, B \in \mathcal{F} \), then \( A \cap B \in \mathcal{F} \).

(ii) If \( A \in \mathcal{F} \) and \( A \subseteq B \subseteq W \), then \( B \in \mathcal{F} \).

\textbf{Definition 3.2.} Let \( \mathcal{F} \) be a filter on \( W \).

(1) \( \mathcal{F} \) is called an \emph{ultrafilter} if for all \( A \in \mathcal{P}(W) \), we have \( A \in \mathcal{F} \) or \( A^c \in \mathcal{F} \), where \( A^c \) is the complement of \( A \).

(2) \( \mathcal{F} \) is called \emph{principal} if there exists a finite subset \( A \subseteq W \) such that \( A \in \mathcal{F} \).

\textbf{Remark 3.3.} By Zorn’s lemma, non-principal ultrafilters always exist.

\textbf{Remark 3.4.} Ultrafilters are an equivalent notion to two-valued finitely additive measures. If we have an ultrafilter \( \mathcal{F} \) on \( W \), then

\[ m(A) := \begin{cases} 1 & (A \in \mathcal{F}) \\ 0 & (A \notin \mathcal{F}) \end{cases} \]

is a two-valued finitely additive measure. Conversely, if \( m : \mathcal{P}(W) \to \{0,1\} \) is a nonzero finitely additive measure, then \( \mathcal{F} := \{A \subseteq W | m(A) = 1\} \) is an ultrafilter. Here, \( \mathcal{F} \) is principal if and only if there exists an element \( w_0 \) of \( W \) such that \( m(\{w_0\}) = 1 \). Hence, \( \mathcal{F} \) is not principal if and only if \( m(A) = 0 \) for any finite subset \( A \) of \( W \).
Definition 3.5. Let $A_w$ be a family of sets indexed by $W$ and $\mathcal{F}$ be an ultrafilter on $W$. Suppose that $a_w \in A_w$ for all $w \in W$ and $\varphi$ is a predicate. We say $\varphi(a_w)$ holds for almost all $w$ if \{w $\in W | \varphi(a_w)$ holds$\} \in \mathcal{F}$.

Remark 3.6. This is an analog of “almost everywhere” or “almost surely” in analysis. The difference is that $m$ is not countably but finitely additive. We can consider elements in $\mathcal{F}$ as “large” sets and elements in the complement $\mathcal{F}^c$ as “small” sets. If $\mathcal{F}$ is not principal, all finite subsets of $W$ are “small.”

Definition 3.7. Let $A_w$ be a family of sets indexed by $W$ and $\mathcal{F}$ be a non-principal ultrafilter on $W$. The ultraproduct of $A_w$ is defined by

$$\text{ulim}_w A_w = A_\infty := \prod_w A_w / \sim,$$

where $(a_w) \sim (b_w)$ if and only if \{w $\in W | a_w = b_w$\} $\in \mathcal{F}$. We denote the equivalence class of $(a_w)$ by $\text{ulim}_w a_w$.

Remark 3.8 [17, Sec. 3]. If $A_w$ are local rings, then the ultraproduct is equivalent to the localization of $\prod A_w$ at a maximal ideal.

Example 3.9. We use $^*\mathbb{N}$ and $^*\mathbb{R}$ to denote the ultraproduct of $|W|$ copies of $\mathbb{N}$ and $\mathbb{R}$, respectively. $^*\mathbb{N}$ is a semiring and $^*\mathbb{R}$ is a field (see Definition-Proposition 3.10 and Theorem 3.20). $^*\mathbb{N}$ is a non-standard model of Peano arithmetic. $^*\mathbb{R}$ is a system of hyperreal numbers used in non-standard analysis.

Definition-Proposition 3.10. Let $A_{1w}, \ldots, A_{nw}, B_w$ be families of sets indexed by $W$ and $\mathcal{F}$ be a non-principal ultrafilter. Suppose that $f_w : A_{1w} \times \cdots \times A_{nw} \to B_w$ is a family of maps. Then we define the ultraproduct $f_\infty = \text{ulim}_w f_w : A_{1\infty} \times \cdots \times A_{n\infty} \to B_\infty$ of $f_w$ by

$$f_\infty(\text{ulim}_w a_{1w}, \ldots, \text{ulim}_w a_{nw}) := \text{ulim}_w f_w(a_{1w}, \ldots, a_{nw}).$$

This is well-defined.

Corollary 3.11. Let $A_w$ be a family of rings. Suppose that $B_w$ is an $A_w$-algebra and $M_w$ is an $A_w$-module for almost all $w$. Then the following hold:

1. $A_\infty$ is a ring.
2. $B_\infty$ is an $A_\infty$-algebra.
3. $M_\infty$ is an $A_\infty$-module.

Proof. Let $0 := \text{ulim}_w 0$, $1 := \text{ulim}_w 1$ in $A_\infty$, $B_\infty$ and $0 := \text{ulim}_w 0$ in $M_\infty$. By the above Definition–Proposition, $A_\infty$, $B_\infty$ have natural additions, subtractions, and multiplications and we have a natural ring homomorphism $A_\infty \to B_\infty$. Similarly, $M_\infty$ has a natural addition and a scalar multiplication between elements of $M_\infty$ and $A_\infty$.

Proposition 3.12. Suppose that, for almost all $w$, we have an exact sequence

$$0 \to L_w \to M_w \to N_w \to 0$$

of abelian groups. Then

$$0 \to \text{ulim}_w L_w \to \text{ulim}_w M_w \to \text{ulim}_w N_w \to 0$$

is an exact sequence of abelian groups. In particular, $\text{ulim}_w : \prod_w A \to Ab$ is an exact functor.
Proof. Let \( f_w : L_w \to M_w \) and \( g_w : M_w \to N_w \) be the morphisms in the given exact sequence. Here, we only prove the injectivity of \( \text{ulim}_w f_w \) and the surjectivity of \( \text{ulim}_w g_w \). Suppose that \( \text{ulim}_w f_w(a_w) = 0 \) for \( \text{ulim}_w a_w \in \text{ulim}_w L_w \). Then \( f_w(a_w) = 0 \) for almost all \( w \). Since \( f_w \) is injective for almost all \( w \), we have \( a_w = 0 \) for almost all \( w \). Therefore, \( \text{ulim}_w a_w = 0 \) in \( \text{ulim}_w L_w \). Hence, \( \text{ulim}_w f_w \) is injective. Next, let \( \text{ulim}_w c_w \) be any element in \( \text{ulim}_w N_w \). Since \( g_w \) is surjective for almost all \( w \), there exists \( b_w \in M_w \) such that \( g_w(b_w) = c_w \) for almost all \( w \). Let \( b = \text{ulim}_w b_w \). Then we have \( (\text{ulim}_w g_w)(b) = \text{ulim}_w g_w(b_w) = \text{ulim}_w c_w \). Hence, \( \text{ulim}_w g_w \) is surjective. The rest of the proof is similar.

Łoś’s theorem is a fundamental theorem in the theory of ultraproducts. We will prepare some notions needed to state the theorem.

**Definition 3.13.** The language \( \mathcal{L} \) of rings is the set defined by

\[
\mathcal{L} := \{0, 1, +, -, \cdot\}.
\]

**Definition 3.14.** Terms of \( \mathcal{L} \) are defined as follows:

(i) 0, 1 are terms.

(ii) Variables are terms.

(iii) If \( s, t \) are terms, then \(-s), (s + t), (s \cdot t)\) are terms.

(iv) A string of symbols is a term only if it can be shown to be a term by finitely many applications of the above three rules.

We omit parentheses and “.” if there is no ambiguity.

**Example 3.15.** \( 1 + 1, x_1(x_2 + 1), -(x - x) \) are terms.

**Definition 3.16.** Formulas of \( \mathcal{L} \) are defined as follows:

(i) If \( s, t \) are terms, then \( (s = t) \) is a formula.

(ii) If \( \varphi, \psi \) are formulas, then \( (\varphi \land \psi), (\varphi \lor \psi), (\varphi \to \psi), (\neg \varphi) \) are formulas.

(iii) If \( \varphi \) is a formula and \( x \) is a variable, then \( \forall x \varphi, \exists x \varphi \) are formulas.

(iv) A string of symbols is a formula only if it can be shown to be a formula by finitely many applications of the above three rules.

We omit parentheses if there is no ambiguity and use \( \neq, \not\in \) in the usual way.

**Remark 3.17.** \( \varphi \land \psi \) means “\( \varphi \) and \( \psi \),” \( \varphi \lor \psi \) means “\( \varphi \) or \( \psi \),” \( \varphi \to \psi \) means “\( \varphi \) implies \( \psi \),” and \( \neg \varphi \) means “\( \varphi \) does not hold.”

**Example 3.18.** \( 0 = 1, x = 0 \neq 1, \forall x \forall y(xy = yx) \) are formulas.

**Remark 3.19.** Variables in a formula \( \varphi \) which is not bounded by \( \forall \) or \( \exists \) are called free variables of \( \varphi \). If \( x_1, \ldots, x_n \) are free variables of \( \varphi \), we denote \( \varphi(x_1, \ldots, x_n) \) and we can substitute elements of a ring for \( x_1, \ldots, x_n \).

**Theorem 3.20** (Łoś’s theorem in the case of rings). Suppose that \( \varphi(x_1, \ldots, x_n) \) is a formula of \( \mathcal{L} \) and \( A_w \) is a family of rings indexed by a set \( W \) endowed with a non-principal ultrafilter. Let \( a_w \in A_w \). Then \( \varphi(\text{ulim}_w a_1, \ldots, \text{ulim}_w a_n) \) holds in \( A_\infty \) if and only if \( \varphi(a_1, \ldots, a_n) \) holds in \( A_w \) for almost all \( w \).

**Remark 3.21.** Even if \( A_w \) are not rings, replacing \( \mathcal{L} \) properly, we can get the same theorem as above. We use one in the case of modules.
Example 3.22. Let $A$ be a ring. If a property of rings is written by some formula, we can apply Loś’s theorem.

1. $A$ is a field if and only if $\forall x(x = 0 \lor \exists y(xy = 1))$ holds.
2. $A$ is a domain if and only if $\forall x\forall y(xy = 0 \rightarrow (x = 0 \lor y = 0))$ holds.
3. $A$ is a local ring if and only if
   \[ \forall x \forall y(\exists z(xz = 1) \land \exists w(yw = 1) \rightarrow \exists u((x + y)u = 1)) \]
   holds.
4. The condition that $A$ is an algebraically closed field is written by countably many formulas, that is, the formula in (1) and for all $n \in \mathbb{N}$,
   \[ \forall a_0 \ldots a_{n-1} \exists x(x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0). \]
5. The condition that $A$ is Noetherian cannot be written by formulas. Indeed, if $W = \mathbb{N}$ with some non-principal ultrafilter and $A_w = \mathbb{C}[x]$, then $\operatorname{ulim}_p x^n \neq 0$ is in $\cap_m m_\infty^*$, where $m_\infty$ is the maximal ideal of $A_\infty$. Hence, $A_\infty$ is not Noetherian.

Proposition 3.23 ([22, 2.8.2]; see Example 3.22). If almost all $K_w$ are algebraically closed field, then $K_\infty$ is an algebraically closed field.

Theorem 3.24 (Lefschetz principle [22, Th. 2.4]). Let $W$ be the set of prime numbers endowed with some non-principal ultrafilter. Then
\[ \operatorname{ulim}_{p \in W} \mathbb{F}_p \cong \mathbb{C}. \]

Proof. Let $C = \operatorname{ulim}_p \mathbb{F}_p$. By the above theorem, $C$ is an algebraically closed field. For any prime number $q$, we have $q \neq 0$ in $\mathbb{F}_p$ for almost all $p$. Hence, $q \neq 0$ in $C$, that is, $C$ is of characteristic zero. We can check that $C$ has the same cardinality as $\mathbb{C}$. If two algebraically closed uncountable field of characteristic zero have the equal cardinality, then they are isomorphic. Hence, $C \cong \mathbb{C}$. (Note that this isomorphism is not canonical.)

3.2 Non-standard hulls

In this subsection, we will introduce the notion of non-standard hulls along [22], [26]. Throughout this subsection, let $\mathcal{P}$ be the set of prime numbers and we fix a non-principal ultrafilter on $\mathcal{P}$ and an isomorphism $\operatorname{ulim}_p \mathbb{F}_p \cong \mathbb{C}$.

Let $\mathbb{C}[X_1, \ldots, X_n]_\infty := \operatorname{ulim}_p \mathbb{F}_p[X_1, \ldots, X_n]$. Then we have the following proposition.

Proposition 3.25 ([22, Th. 2.6]). We have a natural map $\mathbb{C}[X_1, \ldots, X_n] \rightarrow \mathbb{C}[X_1, \ldots, X_n]_\infty$, which is faithfully flat.

Definition 3.26. The ring $\mathbb{C}[X_1, \ldots, X_n]_\infty$ is said to be the non-standard hull of $\mathbb{C}[X_1, \ldots, X_n]$.

Remark 3.27. If $n \geq 1$, then $\mathbb{C}[X_1, \ldots, X_n]_\infty$ is not Noetherian. Let $y = \operatorname{ulim}_p X^p_l$. Then, for any integer $l \geq 1$, $X^p_l \in (X_1, \ldots, X_n)^l$ for almost all $p$. Hence, $y \in (X_1, \ldots, X_n)^l$ for any $l$ by Loś’s theorem. Therefore, $\cap_l (X_1, \ldots, X_n)^l \neq 0$. By Krull’s intersection theorem, $\mathbb{C}[X_1, \ldots, X_n]_\infty$ is not Noetherian.

Definition 3.28. Suppose that $R$ is a finitely generated $\mathbb{C}$-algebra. Let
\[ R \cong \mathbb{C}[X_1, \ldots, X_n]/I \]
be a presentation of $R$. The non-standard hull $R_\infty$ of $R$ is defined by

$$R_\infty := \mathbb{C}[X_1, \ldots, X_n]_{\infty}/I\mathbb{C}[X_1, \ldots, X_n]_{\infty}.$$  

**Remark 3.29.** The non-standard hull is independent of a representation of $R$. If $R \cong \mathbb{C}[X_1, \ldots, X_n]/I \cong \mathbb{C}[Y_1, \ldots, Y_m]/J$, then $\mathbb{F}_p[X_1, \ldots, X_n]/I_p \cong \mathbb{F}_p[Y_1, \ldots, Y_m]/J_p$ for almost all $p$ (see Definitions 3.33 and 3.35).

**Remark 3.30.** The natural map $R \to R_\infty$ is faithfully flat since this is a base change of the homomorphism $\mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C}[X_1, \ldots, X_n]_{\infty}$. By faithfully flatness, we have $IR_\infty \cap R = R$ for any ideal $I \subseteq R$.

**Definition 3.31.** Let $a \in \mathbb{C}$. Since $\text{ulim}_p \mathbb{F}_p \cong \mathbb{C}$, we have a family $(a_p)_p$ of elements of $\mathbb{F}_p$ such that $\text{ulim}_p a_p = a$. Then we call $(a_p)_p$ an approximation of $a$.

**Proposition 3.32.** Let $I = (f_1, \ldots, f_s)$ be an ideal of $\mathbb{C}[X_1, \ldots, X_n]$ and $f_i = \sum a_{i\nu}X^\nu$. Let $I_p = (f_{1p}, \ldots, f_{sp})\mathbb{F}_p[X_1, \ldots, X_n]$, where $f_{ip} = \sum a_{i\nu p}X^\nu$ and each $(a_{i\nu p})_p$ is an approximation of $a_{i\nu}$. Then we have

$$I\mathbb{C}[X_1, \ldots, X_n]_{\infty} = \text{ulim}_p I_p$$

and

$$R_\infty \cong \text{ulim}_p (\mathbb{F}_p[X_1, \ldots, X_n]/I_p).$$

**Definition 3.33.** Let $R$ be a finitely generated $\mathbb{C}$-algebra.

1. In the setting of Proposition 3.32, a family $R_p$ is said to be an approximation of $R$ if $R_p$ is an $\mathbb{F}_p$-algebra and $R_p \cong \mathbb{F}_p[X_1, \ldots, X_n]/I_p$ for almost all $p$. Then we have $R_\infty \cong \text{ulim}_p R_p$.
2. For an element $f \in R$, a family $f_p$ is said to be an approximation of $f$ if $f_p \in R_p$ for almost all $p$ and $f = \text{ulim}_p f_p$ in $R_\infty$. For $f \in R_\infty$, we define an approximation of $f$ in the same way.
3. For an ideal $I = (f_1, \ldots, f_s) \subseteq R$, a family $I_p$ is said to be an approximation of $I$ if $I_p$ is an ideal of $R_p$ and $I_p = (f_{1p}, \ldots, f_{sp})$ for almost all $p$. For finitely generated ideal $I \subseteq R_\infty$, we define an approximation of $I$ in the same way.

**Remark 3.34.** This is an abuse of notation since approximations should be denoted by $(R_p)_p$, $(f_p)_p$, $(I_p)_p$, and so forth.

**Definition 3.35.** Let $\varphi : R \to S$ be a $\mathbb{C}$-algebra homomorphism between finitely generated $\mathbb{C}$-algebras. Suppose that $R \cong \mathbb{C}[X_1, \ldots, X_n]/I$ and $S \cong \mathbb{C}[Y_1, \ldots, Y_m]/J$. Let $f_i \in \mathbb{C}[Y_1, \ldots, Y_m]$ be a lifting of the image of $X_i$ mod $I$ under $\varphi$. Then we define an approximation $\varphi_p : R_p \to S_p$ of $\varphi$ as the morphism induced by $X_i \mapsto f_{ip}$. Let $\varphi_\infty := \text{ulim}_p \varphi_p$, then the following diagram commutes.

$$
\begin{array}{ccc}
R & \xrightarrow{\varphi} & S \\
\downarrow & & \downarrow \\
R_\infty & \xrightarrow{\varphi_\infty} & S_\infty
\end{array}
$$
Proposition 3.36 [22, Cor. 4.2], [26, Th. 4.3.4]. Let $R$ be a finitely generated $C$-algebra. An ideal $I \subseteq R$ is prime if and only if $I_p$ is prime for almost all $p$ if and only if $IR_\infty$ is prime.

Definition 3.37. Let $R$ be a local ring essentially of finite type over $C$. Suppose that $R \cong S_p$, where $S$ is a finitely generated $C$-algebra and $p$ is a prime ideal of $S$. Then we define the non-standard hull $R_\infty$ of $R$ by

$$R_\infty := (S_\infty)_p S_\infty.$$  

Remark 3.38. Since $S \to S_\infty$ is faithfully flat, $R \to R_\infty$ is faithfully flat.

Definition 3.39. Let $S$ be a finitely generated $C$-algebra, let $p$ be a prime ideal of $S$, and let $R \cong S_p$.

1. A family $R_p$ is said to be an approximation of $R$ if $R_p$ is an $\mathbb{F}_p$-algebra and $R_p \cong (S_p)_p$ for almost all $p$. Then we have $R_\infty \cong \text{ulim}_p R_p$.

2. For an element $f \in R$, a family $f_p$ is said to be an approximation of $f$ if $f_p \in R_p$ for almost all $p$ and $f = \text{ulim}_p f_p$ in $R_\infty$. For $f \in R_\infty$, we define an approximation of $f$ in the same way.

3. For an ideal $I = (f_1, \ldots, f_s) \subseteq R$, a family $I_p$ is said to be an approximation of $I$ if $I_p$ is an ideal of $R_p$ and $I_p = (f_{1p}, \ldots, f_{sp})$ for almost all $p$. For finitely generated ideal $I \subseteq R_\infty$, we define an approximation of $I$ in the same way.

Definition 3.40. Let $S_1, S_2$ be finitely generated $C$-algebras, and let $p_1, p_2$ be prime ideals of $S_1, S_2$, respectively. Suppose that $R_i \cong (S_i)_{p_i}$, and $\varphi : R_1 \to R_2$ is a local $C$-algebra homomorphism. Let $S_1 \cong C[X_1, \ldots, X_n]/I$ and $f_j/g_j$ be the image of $X_j$ under $\varphi$, where $f_j \in S_2$, $g_j \in S_2 \setminus p_2$. Then we say that a homomorphism $R_{1p} \to R_{2p}$ induced by $X_j \mapsto f_{jp}/g_{jp}$ is an approximation of $\varphi$. Let $\varphi_\infty := \text{ulim}_p \varphi_p$. Then the following commutative diagram commutes:

$$\begin{array}{ccc}
R & \overset{\varphi}{\longrightarrow} & S \\
\downarrow & & \downarrow \\
R_\infty & \overset{\varphi_\infty}{\longrightarrow} & S_\infty
\end{array}$$

Definition 3.41. Let $R$ be a finitely generated $C$-algebra or a local ring essentially of finite type over $C$, and let $M$ be a finitely generated $R$-module. Write $M$ as the cokernel of a matrix $A$, that is, given by an exact sequence

$$R^m \overset{A}{\to} R^n \to M \to 0,$$

where $m, n$ are positive integers. Let $A_p$ be an approximation of $A$ defined by entrywise approximations. Then the cokernel $M_p$ of the matrix $A_p$ is called an approximation of $M$ and the ultraproduct $M_\infty := \text{ulim}_p M_p$ is called the non-standard hull of $M$. $M_\infty$ is a finitely generated $R_\infty$-module and independent of the choice of matrix $A$.

Remark 3.42. Tensoring the above exact sequence with $R_\infty$, we have an exact sequence

$$R^m_\infty \overset{A}{\to} R^n_\infty \to M \otimes_R R_\infty \to 0.$$
Taking the ultraproduct of exact sequences

\[ R^n_p \xrightarrow{A_p} R^n_p \rightarrow M_p \rightarrow 0, \]

we have an exact sequence

\[ R^n_\infty \xrightarrow{A} R^n_\infty \rightarrow M_\infty \rightarrow 0. \]

Therefore, \( M_\infty \cong M \otimes_R R_\infty \). Note that if \( m, n \) are not integers but infinite cardinals, then the naive definition of an approximation of \( A \) does not work and the ultraproduct of \( R_p^{\geq n} \) is not necessarily equal to \( R_\infty^{\geq n} \).

Here, we state basic properties about non-standard hulls and approximations.

**Proposition 3.43** [22, 2.9.5, 2.9.7, Ths. 4.5 and 4.6], [26, §4.3]; cf. [2, 5.1]. Let \( R \) be a local ring essentially of finite type over \( \mathbb{C} \), then the following hold:

1. \( R \) has dimension \( d \) if and only if \( R_p \) has dimension \( d \) for almost all \( p \).
2. \( x = x_1, \ldots, x_i \) is an \( R \)-regular sequence if and only if \( x_p = x_{1p}, \ldots, x_{ip} \) is an \( R_p \)-regular sequence for almost all \( p \) if and only if \( x \) is an \( R_\infty \)-regular sequence.
3. \( x = x_1, \ldots, x_d \) is a system of parameters of \( R \) if and only if \( x_p \) is a system of parameters of \( R_p \) for almost all \( p \).
4. \( R \) is regular if and only if \( R_p \) is regular for almost all \( p \).
5. \( R \) is Gorenstein if and only if \( R_p \) is Gorenstein for almost all \( p \).
6. \( R \) is Cohen–Macaulay if and only if \( R_p \) is Cohen–Macaulay for almost all \( p \).

**Proposition 3.44** [31, Prop. 3.9]. Let \( R \) be a local ring essentially of finite type over \( \mathbb{C} \). The following conditions are equivalent to each other.

1. \( R \) is normal.
2. \( R_p \) is normal for almost all \( p \).
3. \( R_\infty \) is normal.

**Definition 3.45.** Let \( R \) be a normal local domain essentially of finite type over \( \mathbb{C} \), and let \( \Delta = \sum_i a_i \Delta_i \) be a \( \mathbb{Q} \)-Weil divisor. Assume that \( \Delta_i \) are prime divisors and \( p_i \) is a prime ideal associated with \( \Delta_i \) for each \( i \). Suppose that \( p_{ip} \) is an approximation of \( p_i \) and \( \Delta_{ip} \) is a divisor associated with \( p_{ip} \). We say \( \Delta_p := \sum_i a_i \Delta_{ip} \) is an approximation of \( \Delta \).

**Remark 3.46.** If \( \Delta \) is an effective integral divisor, then this definition is compatible with Definition 3.33 by [22, Th. 4.4]. Hence, if \( \Delta \) is \( \mathbb{Q} \)-Cartier, then \( \Delta_p \) is \( \mathbb{Q} \)-Cartier for almost all \( p \).

Lastly, we review some singularities introduced by Schoutens via ultraproducts.

**Definition 3.47** [22, Def. 5.2], [25, Def. 3.1]. Suppose that \( R \) is a finitely generated \( \mathbb{C} \)-algebra or a local domain essentially of finite type over \( \mathbb{C} \). Let \( I \subseteq R \) be an ideal. The generic tight closure \( I^{\text{gen}} \) of \( I \) is defined by

\[ I^{\text{gen}} = (\ulim_p I_p)^* \cap R. \]

**Remark 3.48.** The generic tight closure \( I^{\text{gen}} \) of \( I \) does not depend on the choice of approximation of \( I \) since any two approximations are almost equal.
DEFINITION 3.49 [25, Def. 4.1 and Rem. 4.7], [23, Def. 4.3]. Suppose that $R$ is a finitely generated $\mathbb{C}$-algebra or a local ring essentially of finite type over $\mathbb{C}$.

1. $R$ is said to be weakly generically $F$-regular if $I^\gen = I$ for any ideal $I \subseteq R$.
2. $R$ is said to be generically $F$-regular if $R_p$ is weakly generically $F$-regular for any prime ideal $p \in \text{Spec } R$.
3. Let $R$ be a local ring essentially of finite type over $\mathbb{C}$. $R$ is said to be generically $F$-rational if $I^\gen = I$ for some ideal $I$ generated by a system of parameters.

PROPOSITION 3.50 [25, Th. 4.3]. If $R$ is generically $F$-rational, then $I^\gen = I$ for any ideal $I$ generated by part of a system of parameters.

PROPOSITION 3.51 [25, Th. 6.2], [23, Prop. 4.5 and Th. 4.12]. If $R$ is generically $F$-rational if and only if $R_p$ is $F$-rational for almost all $p$ if and only if $R$ has rational singularities.

DEFINITION 3.52 [24, 3.2]. Let $R$ be a local ring essentially of finite type over $\mathbb{C}$ and $R_p$ be an approximation. Let $\varepsilon := \text{ulim}_p e_p \in ^*\mathbb{N}$. Then an ultra-Frobenius $F^\varepsilon : R \to R_\infty$ associated with $\varepsilon$ is defined by $x \mapsto \text{ulim}_p (F_p^e(x_p))$, where $F_p$ is a Frobenius morphism in characteristic $p$.

DEFINITION 3.53 [24, Def. 3.3]. Let $R$ be a local domain essentially of finite type over $\mathbb{C}$. $R$ is said to be ultra-$F$-regular if, for each $c \in R^0$, there exists $\varepsilon \in ^*\mathbb{N}$ such that $R^{F^\varepsilon} \twoheadrightarrow R_\infty$ is pure.

PROPOSITION 3.54 [24, Th. A]. Let $R$ be a $\mathbb{Q}$-Gorenstein normal local domain essentially of finite type over $\mathbb{C}$. Then $R$ is ultra-$F$-regular if and only if $R$ has log-terminal singularities.

3.3 Relative hulls

In this subsection, we introduce the concept of relative hulls and approximations of schemes, cohomologies, and so forth. We refer the reader to [22], [24], [25].

DEFINITION 3.55 (Cf. [25]). Let $R$ be a local ring essentially of finite type over $\mathbb{C}$ and $R_p$ be an approximation. Let $\varepsilon := \text{ulim}_p e_p \in ^*\mathbb{N}$. Then an ultra-Frobenius $F^\varepsilon : R \to R_\infty$ associated with $\varepsilon$ is defined by $x \mapsto \text{ulim}_p (F_p^e(x_p))$, where $F_p$ is a Frobenius morphism in characteristic $p$.

DEFINITION 3.56 [24, Def. 3.3]. Let $R$ be a local domain essentially of finite type over $\mathbb{C}$. $R$ is said to be ultra-$F$-regular if, for each $c \in R^0$, there exists $\varepsilon \in ^*\mathbb{N}$ such that $R^{F^\varepsilon} \twoheadrightarrow R_\infty$ is pure.

DEFINITION 3.57 (Cf. [25]). Let $R$ be a finitely generated $R$-algebra, and let $S_p$ be an $R$-approximation of $S$ if and only if $R$ has rational singularities.

DEFINITION 3.58 (Cf. [24]). If $X$ is an affine scheme $\text{Spec } S$ of finite type over $\text{Spec } R$, then we call $X_p := \text{Spec } S_p$ the (relative) $R$-hull of $S$.

DEFINITION 3.59 (Cf. [24]). Suppose that $f : Y \to X$ is a morphism of affine schemes of finite type over $\text{Spec } R$. If $X = \text{Spec } S, Y = \text{Spec } T$ and $\varphi : S \to T$ is the morphism
corresponding to $f$, then we call $f_p : Y_p \to X_p$ is an $R$-approximation of $f$, where $f_p$ is a morphism of $R_p$-schemes induced by an $R$-approximation $\varphi_p : S_p \to T_p$.

**Definition 3.60** (Cf. [24]). Let $S$ be a finitely generated $R$-algebra, and let $M$ be a finitely generated $S$-module. Write $M$ as the cokernel of a matrix $A$, that is, given by an exact sequence

$$S^n \xrightarrow{A} S^m \to M \to 0,$$

where $m, n$ are positive integers. Let $A_p$ be an $R$-approximation of $A$ defined by entrywise $R$-approximations. Then the cokernel $M_p$ of the matrix $A_p$ is called an $R$-approximation of $M$ and the ultraproduct $M_\infty := \lim_p M_p$ is called the $R$-hull of $M$. $M_\infty$ is independent of the choice of the matrix $A$ and $M_\infty \cong M \otimes_S S_\infty$.

**Remark 3.61.** If $M$ is not finitely generated, then we cannot define an $R$-approximation of $M$ in this way. It is crucial that any two $R$-approximations of $A$ is equal for almost all $p$.

**Definition 3.62** [24]. Let $X$ be a scheme of finite type over Spec $R$. Let $\mathcal{U} = \{U_i\}$ is a finite affine open covering of $X$ and $U_{ip}$ be an $R$-approximation of $U_i$. Gluing $\{U_{ip}\}$ together, we obtain a scheme $X_p$ of finite type over Spec $R_p$. We call $X_p$ an $R$-approximation of $X$.

**Remark 3.63.** Suppose that $\{U_{ijk}\}_k$ is a finite affine open covering of $U_i \cap U_j$ and $\varphi_{ijk} : \mathcal{O}_{U_i \cap U_j} \cong \mathcal{O}_{U_i | U_j}$ are isomorphisms. Then $R$-approximations $\varphi_p : \mathcal{O}_{U_{ip} \cap U_{jp}} \to \mathcal{O}_{U_{ip} \cap U_{jp}}$ are isomorphisms for almost all $p$ (note that indices $ijk$ are finitely many). Hence, we can glue these together. For any other choice of finite affine open covering $\mathcal{U}'$ of $X$, the resulting $R$-approximation $X'_p$ is isomorphic to $X_p$ for almost all $p$.

**Definition 3.64** (Cf. [24]). Suppose that $f : Y \to X$ is a morphism between schemes of finite type over Spec $R$. Let $\mathcal{U}, \mathcal{V}$ be finite affine open coverings of $X$ and $Y$, respectively, such that for any $V \in \mathcal{V}$, there exists some $U \in \mathcal{U}$ such that $f(V) \subseteq U$. Let $\mathcal{U}_p, \mathcal{V}_p$ be $R$-approximations of $\mathcal{U}, \mathcal{V}$ and $(f|_V)_p$ an $R$-approximation of $f|_V$. We define an $R$-approximation $f_p$ of $f$ by the morphism determined by $(f|_V)_p$.

**Remark 3.65.** In the same way as the above Remark 3.63, $(f|_V)_p$ and $(f|_{V'})_p$ agree on $V \cap V' \in \mathcal{V}$ for almost all $p$.

**Definition 3.66** (Cf. [24]). Let $X$ be a scheme of finite type over Spec $R$, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{U}$ be a finite affine open covering of $X$. For any $U \in \mathcal{U}$, we have an $R$-approximation $M_{U_p}$ of $M_U$ such that $M_U$ is a finitely generated $\mathcal{O}_U$-module and $\overline{M_U} \cong \mathcal{F}|_U$. We define an $R$-approximation $\mathcal{F}_p$ of $\mathcal{F}$ by the coherent $\mathcal{O}_{X_p}$-module determined by $\overline{M_{U_p}}$.

**Definition 3.67** (Cf. [24]). Let $X$ be a separated scheme of finite type over Spec $R$, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then the ultra-cohomology of $\mathcal{F}$ is defined by

$$H^i_\infty(X, \mathcal{F}) := \lim_p H^i(X_p, \mathcal{F}_p).$$

**Remark 3.68.** In the above setting, let $\mathcal{U} = \{U_i\}_{i=1, \ldots, n}$ be a finite affine open covering of $X$, let

$$C^j(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_j} \mathcal{F}(U_{i_0 \cdots i_j}),$$
where $U_{i_0\ldots i_j} := U_{i_0} \cap \cdots \cap U_{i_j}$, and let

$$(C^j(U, F))_p := \prod_{i_0\ldots i_j} (F(U_{i_0\ldots i_j}))_p,$$

where $F(U_{i_0\ldots i_j})_p$ is an $R$-approximation considered as $\mathcal{O}(U_{i_0\ldots i_j})$-module. Then

$$(C^j(U, F))_p$$

coincides with the $j$th term of the Čech complex of $X_p$, $\mathcal{U}_p$, and $F_p$. We have a commutative diagram

$$
\begin{array}{ccc}
C^{j-1}(U, F) & \longrightarrow & C^j(U, F) \\
\downarrow & & \downarrow \\
\text{ulim}_p(C^{j-1}(U, F))_p & \longrightarrow & \text{ulim}_p(C^j(U, F))_p
\end{array}
$$

Since $\text{ulim}_p(\cdot)$ is an exact functor, we have

$\check{H}^j(U, F) \rightarrow \text{ulim}_p \check{H}^j(U_p, F_p)$.

If $X$ is separated, then $X_p$ is separated for almost all $p$. This can be checked by taking a finite affine open covering and observing that if the diagonal morphism $\Delta_{X_p/\text{Spec} R_p}$ is a closed immersion, then $\Delta_{X_p/\text{Spec} R_p}$ is also a closed immersion for almost all $p$. Hence, we have the map

$$H^j(U, F) \rightarrow \text{ulim}_p H^j(U_p, F_p).$$

Note that we do not know whether this map is injective or not.

**Proposition 3.69.** Let $R$ be a local ring essentially of finite type over $\mathbb{C}$ of dimension $d$, $x = x_1, \ldots, x_d$ a system of parameters and $M$ a finitely generated $R$-module. Then we have a natural homomorphism $H^d_m(M) \rightarrow \text{ulim}_p H^d_{m_p}(M_p)$.

**Proof.** Since $M_{x_1\ldots x_i\ldots x_d}$ is a finitely generated $R_{x_1\ldots x_i\ldots x_d}$-module and $M_{x_1\ldots x_d}$ is a finitely generated $R_{x_1\ldots x_d}$-module, we have an $R$-approximation $(M_{x_1\ldots x_i\ldots x_d})_p \cong (M_p)_{x_1p\ldots x_ip\ldots x_dp}$ and $(M_{x_1\ldots x_d})_p \cong (M_p)_{x_1p\ldots x_dp}$ for almost all $p$. We have a commutative diagram

$$
\begin{array}{ccc}
\bigoplus_i M_{x_1\ldots x_i\ldots x_d} & \longrightarrow & M_{x_1\ldots x_d} \\
\downarrow & & \downarrow \\
\bigoplus_i \text{ulim}_p(M_p)_{x_1p\ldots x_ip\ldots x_dp} & \longrightarrow & \text{ulim}_p(M_p)_{x_1p\ldots x_dp}
\end{array}
$$

Taking the cokernel of rows, we have the desired map. 

**Remark 3.70.** We do not know whether $H^d_m(M) \rightarrow \text{ulim}_p H^d_{m_p}(M_p)$ is injective or not.

**Proposition 3.71.** Let $R$ be a local ring essentially of finite type over $\mathbb{C}$ of dimension $d$, $x = x, \ldots, x_d$ be a system of parameters and $M_p$ be an $R_p$-module for almost all $p$. Then we have a natural homomorphism $H^d_m(\text{ulim}_p M_p) \rightarrow \text{ulim}_p H^d_m(M_p)$.
Proof. We have a commutative diagram

\[ \bigoplus_i (\text{ulim}_p M_p)_{x_1 \ldots x_i \ldots x_d} \xrightarrow{\sigma} (\text{ulim}_p M_p)_{x_1 \ldots x_d} \]
\[ \bigoplus_i \text{ulim}_p (M_p)_{x_1p \ldots x_ip \ldots x_d} \xrightarrow{\sigma} \text{ulim}_p (M_p)_{x_1p \ldots x_dp} \]

Taking the cokernel of rows, we have the desired map.

§4. Big Cohen–Macaulay algebras constructed via ultraproducts

In [23], Schoutens constructed the canonical BCM algebra in characteristic zero. Following the idea of [23], we will deal with BCM algebras constructed via ultraproducts in slightly general settings. In this section, suppose that \((R, \mathfrak{m})\) is a local domain essentially of finite type over \(\mathbb{C}\) and \(R_p\) is an approximation of \(R\).

Definition 4.1 [23, §2]. Suppose that \(R\) is a local domain essentially of finite type over \(\mathbb{C}\). Then we define the canonical BCM algebra \(B(R)\) of \(R\) by

\[ B(R) := \text{ulim}_p R_p^+ \]

Setting 4.2. Let \(R\) be a local domain essentially of finite type over \(\mathbb{C}\) of dimension \(d\), and let \(B_p\) be a BCM \(R_p^+\)-algebra for almost all \(p\). We use \(B\) to denote \(\text{ulim}_p B_p\).

Remark 4.3. By Theorem 2.8, we can set \(B_p = R_p^+\) and \(B = B(R)\) in Setting 4.2.

Proposition 4.4. \(B(R)\) is a domain over \(R^+\)-algebra.

Proof. By Loš’s theorem, \(B(R)\) is a domain over \(R_\infty = \text{ulim}_p R_p\). Hence, \(B(R)\) is an \(R\)-algebra. Let \(f = \sum a_n x^n \in B(R)[x]\) be a monic polynomial in one variable over \(B(R)\) and let \(f_p = \sum a_{np} x^n\) be an approximation of \(f\). Since \(f_p\) is a monic polynomial for almost all \(p\) and \(R_p^+\) is absolutely integrally closed, \(f_p\) has a root \(c_p\) in \(R_p^+\) for almost all \(p\). Hence, \(c := \text{ulim}_p c_p \in B(R)\) is a root of \(f\) by Loš’s theorem. Hence, \(B(R)\) is absolutely integrally closed. In particular, \(B(R)\) contains an absolute integral closure \(R^+\) of \(R\).

Corollary 4.5. In Setting 4.2, \(B\) is an \(R^+\)-algebra.

Proof. Since \(B_p\) is an \(R_p^+\)-algebra for almost all \(p\), \(B\) is an \(R^+\)-algebra by the above proposition.

Proposition 4.6. In Setting 4.2, \(B\) is a BCM \(R\)-algebra.

Proof. Assume that \(B\) is not a BCM \(R\)-algebra. Since \(B_p \neq \mathfrak{m}_p B_p\) for almost all \(p\), we have \(B \neq \mathfrak{m} B\). Hence, there exists part of system of parameters \(x_1, \ldots, x_i\) of \(R\) such that \((x_1, \ldots, x_{i-1})B \subsetneq (x_1, \ldots, x_{i-1})B : x_i\). Then there exists \(y \in B\) such that \(x_i y \in (x_1, \ldots, x_{i-1})B\) and \(y \notin (x_1, \ldots, x_{i-1})B\). Taking approximations, we have \(x_{ip} y_p \in (x_{ip}, \ldots, x_{(i-1)p})B_p\) and \(y_p \notin (x_{ip}, \ldots, x_{(i-1)p})B_p\) for almost all \(p\). Since \(x_{ip}, \ldots, x_{ip}\) is part of a system of parameters of \(R_p\) and \(B_p\) is a BCM \(R_p\)-algebra for almost all \(p\), \(x_{ip}, \ldots, x_{ip}\) is a regular sequence for almost all \(p\). This is a contradiction. Therefore, \(B\) is a BCM \(R\)-algebra.

Lemma 4.7. In Setting 4.2, the natural homomorphism \(H^d_m(B) \to \text{ulim}_p H^d_{m_p}(B_p)\) is injective.
Theorem 5.2 Let $R$ be a finitely generated $\mathbb{C}$-algebra. A pair $(A, R_A)$ is called a model of $R$ if the following two conditions hold:

(i) $A \subseteq \mathbb{C}$ is a finitely generated $\mathbb{Z}$-subalgebra.

(ii) $R_A$ is a finitely generated $A$-algebra such that $R_A \otimes_A \mathbb{C} \cong R$.

Proposition 5.3 (Cf. [23, Cor. 4.10]). Let $R$ be a finitely generated $\mathbb{C}$-algebra, and let $a = a_1, \ldots, a_l$ be finitely many elements of $R$. Let $R_p$ be an approximation of $R$. Then there exists a model $(A, R_A)$ which satisfies the following conditions:

(i) $\gamma_p : A \to \mathbb{F}_p$ is a ring homomorphism for almost all $p$.

(ii) For any $x \in A$, $x = \ulim_p \gamma_p(x)$. 

§5. Approximations of multiplier ideals

In this section, we will explain the relationship between approximations and reductions modulo $p \gg 0$. Note that an isomorphism $\ulim_p \mathbb{F}_p \cong \mathbb{C}$ is fixed.

Definition 5.1. Let $R$ be a finitely generated $\mathbb{C}$-algebra. A pair $(A, R_A)$ is called a model of $R$ if the following two conditions hold:

(i) $A \subseteq \mathbb{C}$ is a finitely generated $\mathbb{Z}$-subalgebra.

(ii) $R_A$ is a finitely generated $A$-algebra such that $R_A \otimes_A \mathbb{C} \cong R$.
(i) There exists a family \((\gamma_p)\) as in Proposition 5.2.
(ii) \(a \subseteq R_A\).
(iii) \(R_A \otimes_A \mathbb{F}_p \cong R_p\) for almost all \(p\).
(iv) For any \(x \in R_A\), the ultraproduct of the image of \(x\) under \(\text{id}_{R_A \otimes A} \gamma_p\) is \(x\).

**Proof.** Let \(X = X_1, \ldots, X_n\) and \(R \cong \mathbb{C}[X]/I\) for some ideal \(I \subseteq \mathbb{C}[X]\). Take any model \((A, R_A)\) which contains \(a\). Enlarging this model, we may assume that there exists an ideal \(I_A \subseteq A[X]\) such that \(R_A \cong A[X]/I_A\) and \(I_A \otimes_A \mathbb{C} = I\) in \(\mathbb{C}[X]\). Take \((\gamma_p)\) as in Proposition 5.2. Let \(I = (f_1, \ldots, f_m)\). For \(f = \sum c_v X^v \in A[X] \subseteq \mathbb{C}[X]\), by the definition of approximations, \(f_p := \sum_v \gamma_p(c_v) X^v \in \mathbb{F}_p[X]\) is an approximation of \(f\). Hence, by the definition of approximations of finitely generated \(\mathbb{C}\)-algebras, \(R_A \otimes_A \mathbb{F}_p \cong \mathbb{F}_p[X]/(f_1, \ldots, f_m)\) is an approximation of \(R\). Since two approximations are isomorphic for almost all \(p\), \(R_A \otimes_A \mathbb{F}_p \cong R_p\) for almost all \(p\). The condition (iv) is clear by the above argument. 

**Remark 5.4.** Let \(p = (x_1, \ldots, x_n) \subseteq R\) be a prime ideal. Enlarging the model \((A, R_A)\), we may assume that \(x_1, \ldots, x_n \in R_A\). Let \(\mu_p\) be the kernel of \(\gamma_p : A \to \mathbb{F}_p\). Then this is a maximal ideal of \(A\) and \(A/\mu_p\) is a finite field. \(p_{\mu_p} = (x_1, \ldots, x_n) R_A/\mu_p R_A\) is prime for almost all \(p\) since this is a reduction to \(p \gg 0\). On the other hand, \(p_p := (x_1, \ldots, x_n) R_A \otimes_A \mathbb{F}_p \subseteq R_p\) is an approximation of \(p\). Hence, \(p_p\) is prime for almost all \(p\). Here, \((R_p)_{p_p}\) is an approximation of \(R_p\). Thus we have a flat local homomorphism \((R_A/\mu_p R_A)_{p_{\mu_p}} \to R_p\) with \(p_{\mu_p}, R_p = p_p\). Moreover, if \(p\) is maximal, then \(p_{\mu_p}, p_p\) are maximal for almost all \(p\). Then, the map \(R_A/p_{\mu_p} \to R_p/p_p \cong \mathbb{F}_p\) is a separable field extension since \(R_A/p_{\mu_p}\) is a finite field.

The next result is a generalization of [31, Th. 4.6] from ideal pairs to triples.

**Proposition 5.5.** Let \(R\) be a normal local domain essentially of finite type over \(\mathbb{C}\), let \(\Delta \geq 0\) be an effective \(\mathbb{Q}\)-Weil divisor such that \(K_R + \Delta\) is \(\mathbb{Q}\)-Cartier, let \(a\) be a nonzero ideal, and let \(t > 0\) be a real number. Suppose that \(R_p, \Delta_p, a_p\) are approximations. Then \(\tau(R_p, \Delta_p, a_p)\) is an approximation of \(J(\text{Spec} R, \Delta, a^t)\).

**Proof.** Let \(R = S_p\), where \(S\) is a normal domain of finite type over \(\mathbb{C}\) and \(p\) is a prime ideal. Let \(m\) be a maximal ideal contains \(p\). Then there exists a model \((A, S_A)\) of \(S\) such that the properties in Proposition 5.3 hold and \(S_A\) containing a system of generators of \(J(\text{Spec} R, \Delta, a^t)\) and \(\Delta_A, a_A\) can be defined properly. Let \(\mu_p\) be maximal ideals of \(S_A\) as in Remark 5.4, and let \(m_{\mu_p}, m_p\) be reductions to \(p \gg 0\). Since, for almost all \(p\), \((S_A/\mu_p) m_{\mu_p} \to (S/m)_p\) is a flat local homomorphism such that \(S_A/m_{\mu_p} \to (S/m)_p \cong \mathbb{F}_p\) is a separable field extension, we have

\[
\tau((S_A/\mu_p) m_{\mu_p}, \Delta(S_A/\mu_p) m_{\mu_p}, a^t_{(S_A/\mu_p) m_{\mu_p}})(S/m)_p = \tau((S/m)_p, \Delta_{m_p}, a^t_{m_p}),
\]

by a generalization of [28, Lem. 1.5]. Since the localization commutes with test ideals [10, Prop. 3.1], we have

\[
\tau((S_A/\mu_p) m_{\mu_p}, \Delta(S_A/\mu_p) m_{\mu_p}, a^t_{(S_A/\mu_p) m_{\mu_p}}) R_p = \tau(R_p, \Delta_p, a^t_p)
\]

for almost all \(p\). Since the reduction of multiplier ideals modulo \(p \gg 0\) is the test ideal [29, Th. 3.2], \(\tau((S_A/\mu_p) m_{\mu_p}, \Delta(S_A/\mu_p) m_{\mu_p}, a^t_{(S_A/\mu_p) m_{\mu_p}})\) is a reduction of

\[J(\text{Spec} R, \Delta, a^t)\]

to characteristic \(p \gg 0\). Hence, \(\tau(R_p, \Delta_p, a^t_p)\) is an approximation of \(J(\text{Spec} R, \Delta, a^t)\). 

\[\square\]
§6. BCM test ideal with respect to a big Cohen–Macaulay algebra constructed via ultraproducts

Throughout this section, we assume that \((R, m)\) is a normal local domain essentially of finite type over \(\mathbb{C}\). Fix a canonical divisor \(K_R\) such that \(R \subseteq \omega_R := R(K_R) \subseteq \text{Frac}(R)\). Let \(\Delta \geq 0\) be an effective \(\mathbb{Q}\)-Weil divisor such that \(K_R + \Delta\) is \(\mathbb{Q}\)-Cartier. Suppose that \(\text{div} \, f = n(K_R + \Delta)\) for \(f \in R^\circ, \, n \in \mathbb{N}\). Let \(B_p\) be a BCM \(R^+_p\)-algebra for almost all \(p\) and \(B := \text{ulim}_p B_p\). We use \(\widehat{R}\) to denote the completion of \(R\) with respect to \(m\) and \(\widehat{\Delta}\) to denote the flat pullback of \(\Delta\) by \(\text{Spec} \, \widehat{R} \to \text{Spec} \, R\).

**Proposition 6.1.** In the setting as above, we have

\[\mathcal{J}(\widehat{R}, \widehat{\Delta}) \subseteq \tau_{\widehat{E}}(\widehat{R}, \widehat{\Delta}).\]

**Proof.** Consider the following commutative diagram:

\[
\begin{array}{ccc}
0^{B,K_R+\Delta}_{H^d_m(\omega_R)} & \rightarrow & \text{ulim}_p 0^{B_p,K_{R_p}+\Delta_p}_{H^d_{m_p}(\omega_{R_p})} \\
H^d_m(\omega_R) & \rightarrow & \text{ulim}_p H^d_{m_p}(\omega_{R_p}) \\
\psi & & \\
H^d_m(B) & \rightarrow & \text{ulim}_p H^d_{m_p}(B_p)
\end{array}
\]

By Proposition 2.12, we have

\[0^{B_p,K_{R_p}+\Delta_p}_{H^d_{m_p}(\omega_{R_p})} = 0^{\Delta_p}_{H^d_{m_p}(\omega_{R_p})}\]

for almost all \(p\). Let \(x_1, \ldots, x_d\) be a system of parameters, and let \(x = x_1 \cdots x_d\) be the product of them. Take \(a \in \mathcal{J}(R, \Delta) = \text{ulim}_p \tau(R_p, \Delta_p) \cap R\) and \([\frac{x}{t}] \in 0^{B,K_R+\Delta}_{H^d_m(\omega_R)}\). Let \(J\) be a divisorial ideal which is isomorphic to \(\omega_R\) and \(g \in R^\circ\) an element such that \(\omega_R \rightarrow g\) is an isomorphism. As in proof of [29, Th. 2.8], we have \(g_p z_p x^t_p \in (x_1^{2t}, \ldots, x_d^{2t}) J_p\) for almost all \(p\). Hence, \(a_p g_p z_p x^t_p \in (x_1^{2t}, \ldots, x_d^{2t}) J_p\) for almost all \(p\). Therefore, \(a_p g_p z_p x^t_p \in (x_1^{2t}, \ldots, x_d^{2t}) J\) and \([\frac{a}{t}] = 0\) in \(H^d_m(\omega_R)\). Hence, we have \(a \in \text{Ann}_R 0^{B,K_R+\Delta}_{H^d_m(\omega_R)}\). In conclusion, we have \(\mathcal{J}(R, \Delta) \widehat{R} \subseteq \tau_{\widehat{E}}(\widehat{R}, \widehat{\Delta})\).

**Lemma 6.2** [29, Th. 2.13]. Let \((R, m)\) be an \(F\)-finite normal local domain of characteristic \(p > 0\) and \(\Delta \geq 0\) be an effective \(\mathbb{Q}\)-Weil divisor on \(X := \text{Spec} \, R\) such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier. Let \(f : Y \to X\) be a proper birational morphism with \(X\) normal. Suppose that \(Z := f^{-1}(m)\) and \(\delta : H^0_m(R(K_X)) \to H^2(Y, \mathcal{O}_Y(\lceil f^*(K_X + \Delta) \rceil))\) is the Matlis dual of the natural inclusion map \(H^0(Y, \mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta) \rceil)) \to R\). Then \(\text{Ker} \delta \subseteq \text{Ann}_E\), where \(E\) is the injective hull of the residue field \(R/m\) of \(R\).

**Proof.** By [29, Th. 2.13], we have \(\tau(R, \Delta) \subseteq H^0(Y, \mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta) \rceil))\). Hence,

\[
\text{Ker} \delta = \text{Ann}_E H^0(Y, \mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta) \rceil))
\subseteq \text{Ann}_E \tau(R, \Delta)
= \text{Ann}_E \tau(R, \Delta) \widehat{R}
\]
Then we have a commutative diagram
\[ \text{characteristic zero modulo} \]
\[ \text{an effective canonical divisor} \]
\[ \text{commutative diagram for almost all} \]
\[ \gamma \]
\[ \text{we have a corresponding morphisms} \]
\[ \eta \]
\[ \text{wherethemiddlerowisexact. Assumethat} \]
\[ X := \text{Spec} R \]
\[ \text{Therefore, we have} \]
\[ \text{ulim} \psi \]
\[ \text{where} \]
\[ \text{pulim} \psi \]
\[ \text{H}_m(\omega_R) \]
\[ \gamma \]
\[ \delta \]
\[ \text{H}^{d-1}(Y, \mathcal{L}) \rightarrow H^{d-1}(Y \setminus Z, \mathcal{L}|_{Y \setminus Z}) \rightarrow H_Z(\mathcal{L}) \]
\[ \text{H}^{d-1}(Y, \mathcal{L}) \]
\[ \text{inj.} \]
\[ \text{H}^{d-1}(Y \setminus Z, \mathcal{L}|_{Y \setminus Z}) \]
\[ \text{where} \mathcal{L} := \mathcal{O}_Y([\mu^*(K_X + \Delta)]) \text{and the middle row is exact. Similarly, we have the following commutative diagram for almost all} p: \]
\[ \text{H}^{d}(\mathcal{L}) \]
\[ \gamma_p \]
\[ \delta_p \]
\[ \text{H}^{d-1}(Y_p, \mathcal{L}_p) \]
\[ \text{inj.} \]
\[ \text{H}^{d-1}(Y_p \setminus Z_p, \mathcal{L}_p|_{Y_p \setminus Z_p}) \rightarrow H_{Z_p}^d(\mathcal{L}_p) \]
\[ \text{where the middle row is exact. Assume that} \eta \in \text{Ker} \delta. \text{Then} u^{d-1}(\gamma(\eta)) \in \text{Im} \rho^{d-1}. \text{Therefore,} \]
\[ \gamma_p(\eta_p) \in \text{Im} \rho^{d-1}_p \text{for almost all} p. \text{Hence,} \eta_p \in \text{Ker} \delta_p \text{for almost all} p. \text{By Lemma 6.2,} \eta_p \in \text{ker} \forall p. \text{Hence, by Proposition 2.12, we have} \eta_p \in 0^*_{H_m^d(\omega_R)} \text{for almost all} p. \text{We have a commutative diagram} \]
\[ \text{where} \psi, \psi_p \text{are the morphisms as in Definition 2.11. Since} \psi(\eta_p) = 0 \text{and} H_m^d(B) \rightarrow \text{ulim}_p H_m^d(B_p) \text{is injective by Lemma 4.7, we have} \psi(\eta) = 0 \text{in} H_m^d(B). \text{Hence,} \eta \in 0^*_{H_m^d(\omega_R)}. \text{Therefore, we have} \]
\[ \tau_{\hat{B}}(\hat{R}, \Delta) \subseteq \text{Ann}_{\hat{R}}(\text{Ker} \delta) \]
\[ = \text{Ann}_{\hat{R}} \text{Ann}_{H_{m}^{d}(\omega_{R})} \mathcal{J}(R, \Delta) \]
\[ = \mathcal{J}(\hat{R}, \Delta). \]

**Remark 6.5.** We can generalize the notion of ultra-test ideals in [31, Def. 5.5] to the pair \((R, \Delta)\). Using Lemma 6.2 instead of [11, Th. 6.9], we can show that generalized ultra-test ideals are equal to multiplier ideals.

§7. Generalized module closures and applications

We introduce the notion of generalized module closures inspired by [20]. Using the generalized module closures, we will generalize [31, Cor. 5.30]. We also use [19, §6.1] as reference in the following arguments.

**Setting 7.1.** Suppose that \(R\) is a normal local domain essentially of finite type over \(C\) of dimension \(d\), \(K_{R} \geq 0\) is a fixed effective canonical divisor and \(\Delta \geq 0\) is an effective \(Q\)-Weil divisor such that \(K_{R} + \Delta\) is \(Q\)-Cartier. Moreover, we assume that \(B_{p}\) is a BCM \(R_{p}^{+}\)-algebra for almost all \(p\), \(B := \text{ulim}_{p} B_{p}\) and \(r(K_{R} + \Delta) = \text{div} f\) for \(f \in R\), \(r \in \mathbb{N}\). Let \(R' \subseteq R^{+}\) be an integrally closed finite extension of \(R\) such that \(f^{1/r} \in R'\) and \(\pi^{*} \Delta\) is Weil divisor, where \(\pi : \text{Spec} R' \to \text{Spec} R\).

**Definition 7.2.** Assume Setting 7.1 and let \(g \in R^{\circ}\) and \(t > 0\) be a positive rational number. We use \(\hat{\mathcal{B}}_{\Delta}\) to denote
\[ B \otimes R' R'(\pi^{*} \Delta) \otimes_{R} \hat{R}. \]
For any \(\hat{R}\)-modules \(N \subseteq M\), we define \(N^{cl}_{M}^{\hat{\mathcal{B}}_{\Delta}, g^{t}}\) as follows: \(x \in N^{cl}_{M}^{\hat{\mathcal{B}}_{\Delta}, g^{t}}\) if and only if \(g^{t} \otimes x \in \text{Im}(\hat{\mathcal{B}}_{\Delta} \otimes_{R} N \to \hat{\mathcal{B}}_{\Delta} \otimes_{R} M)\). We use \(\tau_{\text{cl}^{\hat{\mathcal{B}}_{\Delta}, g^{t}}(\hat{R})}\) to denote
\[ \bigcap_{N \subseteq M} (N :_{\hat{R}} N^{cl}_{M}^{\hat{\mathcal{B}}_{\Delta}, g^{t}}), \]
where \(M\) runs through all \(\hat{R}\)-modules and \(N\) runs through all \(\hat{R}\)-submodules of \(M\).

**Proposition 7.3.** In Setting 7.1, if \(g \in R^{\circ}\) and \(t > 0\) is a positive rational number, then we have
\[ \tau_{\text{cl}^{\hat{\mathcal{B}}_{\Delta}, g^{t}}(\hat{R})} = \bigcap_{M} \text{Ann}_{\hat{R}} 0^{cl}_{M}^{\hat{\mathcal{B}}_{\Delta}, g^{t}} = \text{Ann}_{E} 0^{cl}_{E}^{\hat{\mathcal{B}}_{\Delta}, g^{t}}, \]
where \(M\) runs through all \(\hat{R}\)-modules and \(E\) is the injective hull of the residue field of \(R\).

**Proof.** We can prove this by arguments similar to [20, Lem. 3.3 and Prop. 3.9].

**Proposition 7.4.** In Setting 7.1, if \(g \in R^{\circ}\) and \(t > 0\) is a positive rational number, then we have
\[ 0^{B, K_{R} + \Delta + t \text{div} g}_{E} = 0^{cl}_{E}^{\hat{\mathcal{B}}_{\Delta}, g^{t}}. \]

**Proof.** Since the reflexive hull \((R'(\pi^{*} \Delta) \otimes_{R} \omega_{R})^{**}\) is equal to \(R'(\text{div}(f^{1/t}))\), we have \(H_{m}^{d}(R'(\pi^{*} \Delta) \otimes_{R} \omega_{R}) \cong H_{m}^{d}(R'(\text{div}(f^{1/t})))\). Hence, we have
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\[
\hat{B}_\Delta \otimes_{\hat{R}} E \cong B \otimes_{R'} H^d_m(R'(\pi^* \Delta) \otimes_R \omega_R) \\
\cong B \otimes_{R'} H^d_m(R'(\text{div}(f^1)))
\]

Then there exists a commutative diagram

\[
\begin{array}{ccc}
E \cong H^d_m(\omega_R) & \xrightarrow{\rho' \otimes 1} & \hat{B}_\Delta \otimes_{\hat{R}} E \\
\downarrow & & \downarrow \\
B \otimes_{R'} H^d_m(R'(\text{div}(f^1))) & \cong & B \otimes_{R'} H^d(R') \\
\downarrow \text{id} \otimes(f^{1/r}) & & \downarrow \text{id} \\
H^d_m(B \otimes_R \omega_R) & \xrightarrow{\psi} & H^d_m(B)
\end{array}
\]

where \(\psi\) is the second map of

\[
\cdot f^{1/r} g^t : H^d_m(B) \to H^d_m(B \otimes_R \omega_R) \to H^d_m(B).
\]

The result follows by the above commutative diagram.

**Definition 7.5.** Let \(R \hookrightarrow S\) be an injective local homomorphism of normal local domains essentially of finite type over \(\mathbb{C}\). Fix \(K_R, K_S \geq 0\) effective canonical divisors on \(\text{Spec } R\) and on \(\text{Spec } S\), respectively. Let \(\Delta_R, \Delta_S \geq 0\) be effective \(\mathbb{Q}\)-Weil divisors on \(\text{Spec } R\) and on \(\text{Spec } S\), respectively, such that \(K_R + \Delta_R, K_S + \Delta_S\) are \(\mathbb{Q}\)-Cartier. Let \(a \subseteq R\) be a nonzero ideal and \(t > 0\) be a positive rational number. Suppose that \(\hat{B}_{\Delta_R}\) and \(\hat{B}_{\Delta_S}\) are defined as in Definition 7.2. Then, for an \(\hat{R}\)-module \(M\) and an \(\hat{S}\)-module \(N\), we define \(0_{M}^{\text{cl}_{\hat{B}_{\Delta_R} \cdot a^t}}, 0_{N}^{\text{cl}_{\hat{B}_{\Delta_S} \cdot a^t}}\) by

\[
0_{M}^{\text{cl}_{\hat{B}_{\Delta_R} \cdot a^t}} := \bigcap_{n \in \mathbb{N}} \bigcap_{g \in a^{[nt]}} \text{Ann} \hat{B}_{\Delta_R}^{g \frac{1}{n}},
\]

\[
0_{N}^{\text{cl}_{\hat{B}_{\Delta_S} \cdot a^t}} := \bigcap_{n \in \mathbb{N}} \bigcap_{g \in a^{[nt]}} \text{Ann} \hat{B}_{\Delta_S}^{g \frac{1}{n}}.
\]

We use \(\tau_{\text{cl}_{\hat{B}_{\Delta_R} \cdot a^t}(\hat{R})}, \tau_{\text{cl}_{\hat{B}_{\Delta_S} \cdot a^t}(\hat{S})}\) to denote

\[
\bigcap_M \text{Ann}_{\hat{R}} 0_{M}^{\text{cl}_{\hat{B}_{\Delta_R} \cdot a^t}},
\]

\[
\bigcap_N \text{Ann}_{\hat{S}} 0_{N}^{\text{cl}_{\hat{B}_{\Delta_S} \cdot a^t}},
\]

where \(M\) runs through all \(\hat{R}\)-modules and \(N\) runs through all \(\hat{S}\)-modules.
Proposition 7.6. In the setting of Definition 7.5, we have
\[
\text{Ann}_{\hat{R}}^{\text{cl} \hat{\Delta}_R} a^t = \bigcap_M \text{Ann}_{\hat{R}}^{\text{cl} \hat{\Delta}_R} a^t,
\]
\[
\text{Ann}_{\hat{S}}^{\text{cl} \hat{\Delta}_S} a^t = \bigcap_N \text{Ann}_{\hat{S}}^{\text{cl} \hat{\Delta}_S} a^t,
\]
where \(M, N\) run through all \(\hat{R}\)-modules and \(\hat{S}\)-modules, respectively, and \(E_R, E_S\) are the injective hulls of the residue fields of \(R\) and \(S\), respectively.

Proof. We can show this by arguments similar to Proposition 7.3.

Proposition 7.7. In the setting of Definition 7.5, we have
\[
\tau^{\text{cl} \hat{\Delta}_R} (\hat{R}) = J(\hat{R}, \hat{\Delta}, (a\hat{R})^t).
\]

Proof. Let \(E\) be the injective hull of the residue field of \(R\). Then
\[
\bigcap_{n \in \mathbb{N}} \bigcap_{g \in a^{[nt]}} J(\hat{R}, \hat{\Delta}, g^\frac{1}{n}) = \text{Ann}_E J(\hat{R}, \hat{\Delta}, (a\hat{R})^t),
\]
where the second equality follows from Theorem 6.4. Hence, we have
\[
\text{Ann}_{\hat{R}}^{\text{cl} \hat{\Delta}_R} = J(\hat{R}, \hat{\Delta}, (a\hat{R})^t).
\]

The next lemma is a generalization of [30, Th. 3.2].

Lemma 7.8. Let \(R\) be a normal local domain essentially of finite type over \(\mathbb{C}\), and let \(\Delta \geq 0\) be an effective \(\mathbb{Q}\)-Weil divisor such that \(K_R + \Delta\) is \(\mathbb{Q}\)-Cartier. Let \(a_1, \ldots, a_n \subseteq R\) be nonzero ideals, and let \(t > 0\) be a positive rational number. Then we have
\[
J(R, \Delta, (a_1 + \cdots + a_n)^t) = \sum_{\lambda_1 + \cdots + \lambda_n = t} J(R, \Delta, a_1^{\lambda_1} \cdots a_n^{\lambda_n}).
\]

Lemma 7.9. In the setting of Definition 7.5, we have
\[
\sum_{n \in \mathbb{N}} \sum_{g \in a^{[nt]}} J(S, \Delta_S, g^{\frac{1}{n}}) = J(S, \Delta_S, (aS)^t).
\]

Proof. \(\sum_{n \in \mathbb{N}} \sum_{g \in a^{[nt]}} J(S, \Delta_S, g^{\frac{1}{n}}) \subseteq J(S, \Delta_S, (aS)^t)\) is clear. If \(t = q/p, p, q > 0\) and \(a = (g_1, \ldots, g_l)\), then
\[
\sum_{n \in \mathbb{N}} \sum_{g \in a^{[nt]}} J(S, \Delta_S, g^{\frac{1}{n}}) \supseteq \sum_{n \in \mathbb{N}} \sum_{i_1 + \cdots + i_l = nq} J(S, \Delta_S, (g_1^{i_1} \cdots g_l^{i_l})^{\frac{1}{pq}}) = J(S, \Delta_S, (aS)^t),
\]
by the above lemma.
Theorem 7.10. Let $R \hookrightarrow S$ be a pure local homomorphism of normal local domains essentially of finite type over $\mathbb{C}$. Fix effective canonical divisors $K_R$ and $K_S$ on $\text{Spec} R$ and $\text{Spec} S$, respectively. Let $\Delta_R, \Delta_S \geq 0$ be effective $\mathbb{Q}$-Weil divisors on $\text{Spec} R$, $\text{Spec} S$ such that $K_R + \Delta_R$, $K_S + \Delta_S$ are $\mathbb{Q}$-Cartier. Take normal domains $R', S'$ and morphisms $\pi_R, \pi_S$ as in Setting 7.1. Moreover, let $a \subseteq R$ be a nonzero ideal, and let $t > 0$ be a positive rational number. If $R'(\pi_R^* \Delta_R) \subseteq S'(\pi_S^* \Delta_S)$, then we have

$$J(S, \Delta_S, (aS)^t) \cap R \subseteq J(R, \Delta_R, a^t).$$

Proof. Since $R \hookrightarrow S$ is pure, $\hat{R} \hookrightarrow \hat{S}$ is pure (see [6, Cor. 3.2.1]). Since $R \rightarrow \hat{R}, S \rightarrow \hat{S}$ are pure, it is enough to show

$$J(\hat{S}, \hat{\Delta}_S, (a^\hat{S})^t) \cap \hat{R} \subseteq J(\hat{R}, \hat{\Delta}_R, (a^\hat{R})^t).$$

Let $\mathcal{B}(R), \mathcal{B}(S)$ be the canonical BCM algebras. Let $\mathcal{B}_{\Delta_R} := \mathcal{B}(\hat{R})_{\Delta_R}$ and $\mathcal{B}_{\Delta_S} := \mathcal{B}(\hat{S})_{\Delta_S}$. Take an $\hat{R}$-module $M$. Then we have a commutative diagram

$$\begin{array}{c}
\hat{R} \quad \text{pure} \\
\downarrow \\
\mathcal{B}_{\Delta_R} \quad \mathcal{B}_{\Delta_S}
\end{array}$$

Tensoring the commutative diagram with $M$, we have

$$\begin{array}{c}
M \quad \text{pure} \\
\downarrow \\
\mathcal{B}_{\Delta_R} \otimes_{\hat{R}} M \quad \mathcal{B}_{\Delta_S} \otimes_{\hat{R}} M
\end{array}$$

Hence, we have

$$0 \cap cl_{\mathcal{B}_{\Delta_R}}^{\mathcal{B}_{\Delta_R}} a^t \subseteq 0 \cap cl_{\mathcal{B}_{\Delta_S}}^{\mathcal{B}_{\Delta_S}} a^t.$$

Then we have

$$J(\hat{R}, \hat{\Delta}_R, a^t) = \bigcap_M \text{Ann}_R^{cl_{\mathcal{B}_{\Delta_R}}^{\mathcal{B}_{\Delta_R}} a^t}$$

$$\supseteq \bigcap_M \text{Ann}_R^{cl_{\mathcal{B}_{\Delta_S}}^{\mathcal{B}_{\Delta_S}} a^t}$$

$$\supseteq \bigcap_N \text{Ann}_R^{cl_{\mathcal{B}_{\Delta_S}}^{\mathcal{B}_{\Delta_S}} a^t}$$

$$= \bigcap_N (\text{Ann}_S^{cl_{\mathcal{B}_{\Delta_S}}^{\mathcal{B}_{\Delta_S}} a^t} \cap \hat{R})$$

$$= (\text{Ann}_S^{cl_{\mathcal{B}_{\Delta_S}}^{\mathcal{B}_{\Delta_S}} a^t} \cap \hat{R})$$

$$= (\text{Ann}_S \bigcap_{n \in \mathbb{N}, g \in a: |nt|} 0_{E_S^{\Delta_{\mathfrak{g}}}}^{\frac{1}{nt}}) \cap \hat{R}.$$
\[
\begin{aligned}
&= (\text{Ann}_S \text{Ann}_E \sum_{n \in \mathbb{N}} \sum_{g \in a^{n+1}} J(\hat{S}, \Delta_S, g^k)) \cap \hat{R} \\
&= J(\hat{S}, \Delta_S, (a\hat{S})^t) \cap \hat{R},
\end{aligned}
\]

where \( M \) runs through all \( \hat{R} \)-modules, \( N \) runs through all \( \hat{S} \)-modules, and \( E_S \) is the injective hull of the residue field of \( S \).

As a corollary, we have a generalization of \([31, \text{Cor. 5.30}]\) to the case that \( a \) is not necessarily a principal ideal.

Corollary 7.11. Let \( R \to S \) be a pure local homomorphism of normal local domains essentially of finite type over \( \mathbb{C} \). Suppose that \( R \) is \( \mathbb{Q} \)-Gorenstein. Fix effective canonical divisors \( K_R \) and \( K_S \) on \( \text{Spec } R \) and \( \text{Spec } S \), respectively. Let \( \Delta_S \) be an effective \( \mathbb{Q} \)-Weil divisor on \( \text{Spec } S \) such that \( K_S + \Delta_S \) is \( \mathbb{Q} \)-Cartier. Let \( a \subseteq R \) be a nonzero ideal and \( t > 0 \) a positive rational number. Then we have

\[
J(S, \Delta_S, (a\hat{S})^t) \cap R \subseteq J(R, a^t).
\]

Proof. Let \( R' \) be the integral closure of \( R[f^{1/r}] \) in \( R^+ \). Then the result follows from Theorem 7.10.

\section*{§8. \( \mathcal{B} \)-regularity}

As another application of the main theorem, we will give a partial answer to \([24, \text{Rem. 3.10}]\). For this, we will review the definition of \( \mathcal{B} \)-regularity.

Definition 8.1 \([23, \text{Def. 4.3}]\). Let \( R \) be a normal \( \mathbb{Q} \)-Gorenstein local domain essentially of finite type over \( \mathbb{C} \).

1. \( R \) is said to be weakly \( \mathcal{B} \)-regular if \( R \to \mathcal{B}(R) \) is cyclically pure.
2. \( R \) is said to be \( \mathcal{B} \)-regular if every localization of \( R \) at a prime ideal is weakly \( \mathcal{B} \)-regular.

Theorem 8.2. Let \( R \) be a normal \( \mathbb{Q} \)-Gorenstein local domain. Then the following are equivalent:

1. \( R \) has log-terminal singularities.
2. \( R \) is ultra-F-regular.
3. \( R \) is weakly generically F-regular.
4. \( R \) is generically F-regular.
5. \( R \) is weakly \( \mathcal{B} \)-regular.
6. \( R \) is \( \mathcal{B} \)-regular.
7. \( \hat{R} \) is BCM-\( \mathcal{B}(R) \)-regular.

Proof. The equivalence of (1) and (2) follows from Proposition 3.54 and the equivalence of (1) and (7) follows from Theorem 6.4. Since, if \( R \) has log-terminal singularities, then every localization of \( R \) at a prime ideal is log-terminal, it is enough to show the equivalence of (1), (3), and (5). (1) is equivalent to (3) by \([31, \text{Th. 5.24 and Proof of Th. 5.25}]\). Lastly, we will show the equivalence of (5) and (7). Let \( E \) be the injective hull of the residue field of \( R \). By Proposition 7.4, we have \( 0^E_{E} = 0^E_{E} \). Hence, \( E \to \mathcal{B}(R) \otimes R E \) is injective if and only if \( \hat{R} \) is BCM-\( \mathcal{B}(R) \)-regular. \( R \to \mathcal{B}(R) \) is pure if and only if \( E \to \mathcal{B}(R) \otimes R E \) is
injective by \[15, \text{Lem. 2.1 (e)}\]. \( R \to \mathcal{B}(R) \) is pure if and only if \( R \to \mathcal{B}(R) \) is cyclically pure by \[12, \text{Th. 1.7}\]. Therefore, (5) is equivalent to (7).

**Remark 8.3.** For the equivalence of (5) and (7) (see \[19, \text{Prop. 6.14}\]).

§9. Further questions and remarks

In this section, we will consider whether \( R \) is BCM-rational if \( R \) has rational singularities. The next question is a variant of \[7, \text{Quest. 2.7}\].

**Question 1.** Let \( R \) be a local domain essentially of finite type over \( \mathbb{C} \), and let \( B \) be a BCM \( R \)-algebra. If \( S \) is finitely generated \( R \)-algebra such that the following diagram commutes:

\[
\begin{array}{ccc}
R & \longrightarrow & B \\
\downarrow & & \downarrow \\
S & \rightarrow & \end{array}
\]

then does there exist a BCM \( R_p \)-algebra for almost all \( p \) which fits into the following commutative diagram:

\[
\begin{array}{ccc}
R_p & \longrightarrow & B_p \\
\downarrow & & \downarrow \\
S_p & \rightarrow & \end{array}
\]

where \( S_p \) is an \( R \)-approximation of \( S \)?

**Proposition 9.1 (Cf. \[19, \text{Conj. 3.9}\]).** Let \( R \) be a normal local domain essentially of finite type over \( \mathbb{C} \) of dimension \( d \). Suppose that \( R \) has rational singularities. If Question 1 has an affirmative answer, then \( R \) is BCM-rational.

**Proof.** Let \( B \) be a BCM \( R^+ \)-algebra. Suppose that \( \eta \in \ker(H^d_m(R) \to H^d_m(B)) \). Then there exists a finitely generated \( R \)-subalgebra of \( B \) such that the image of \( \eta \) in \( H^d_m(S) \) is zero. If Question 1 has an affirmative answer, we can take \( S_p \) and \( B_p \) as in Question 1. Then we have a commutative diagram

\[
\begin{array}{ccc}
H^d_m(R) & \longrightarrow & \text{ulim}_p H^d_{m_p}(R_p) \\
\downarrow & & \downarrow \\
H^d_m(S) & \longrightarrow & \text{ulim}_p H^d_{m_p}(S_p) \\
\downarrow & & \downarrow \\
H^d_m(B) & \longrightarrow & \text{ulim}_p H^d_{m_p}(B_p)
\end{array}
\]

By the proof of Proposition 4.8, \( \text{ulim}_p H^d_{m_p}(R_p) \to \text{ulim}_p H^d_{m_p}(S_p) \) is injective. Therefore, the image of \( \eta \) in \( \text{ulim}_p H^d_{m_p}(R_p) \) is zero. Suppose that \( \eta = \begin{bmatrix} \frac{y}{x} \end{bmatrix} \), where \( y \in R \), \( t \in \mathbb{N} \) and \( x \) is the product of a system of parameters \( x_1, \ldots, x_d \) of \( R \). Since \( R_p \) is Cohen–Macaulay for almost all \( p \), \( y_p \in (x_1^t, \ldots, x_d^t) \) for almost all \( p \). Hence, \( y \in (x_1^t, \ldots, x_d^t) \) and \( \eta = 0 \) in \( H^d_m(R) \). Thus, \( H^d_m(R) \to H^d_m(B) \) is injective.
The next result follows from a similar argument.

**Proposition 9.2.** Let $R$ be a normal local domain essentially of finite type over $\mathbb{C}$ of dimension $d$. Fix an effective canonical divisor $K_R$ on $\text{Spec} R$. Let $\Delta \geq 0$ be an effective $\mathbb{Q}$-Weil divisor on $\text{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier. Suppose that $C$ is a $\text{BCM } R^+$-algebra. If Question 1 has an affirmative answer, then we have

$$J(R, \Delta) \subseteq \tau_C(\hat{R}, \hat{\Delta}).$$

**Definition 9.3** (Cf. [19, Def. 6.9]). Let $R$ be a normal local domain essentially of finite type over $\mathbb{C}$. Fix an effective canonical divisor $K_R$ on $\text{Spec} R$. Let $\Delta \geq 0$ be a $\mathbb{Q}$-Weil divisor on $\text{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier. Suppose that $n(K_R + \Delta) = \text{div}(f)$ for $f \in R^p$, $n \in \mathbb{N}$. We define

$$0^{\mathcal{B}, K_R + \Delta}_{\text{H}^d_m(R)} := \{ \eta \in \text{H}^d_m(R) | \exists C \text{ BCM } R^+-\text{algebra}$$

such that $f^{\frac{1}{n}} \eta = 0$ in $\text{H}^d_m(C)$. We define the $\text{BCM test ideal } \tau_{\mathcal{B}}(R, \Delta)$ of $(\hat{R}, \hat{\Delta})$ by

$$\tau_{\mathcal{B}}(\hat{R}, \hat{\Delta}) := \text{Ann}_{\omega_{\hat{R}}} 0^{\mathcal{B}, K_R + \Delta}_{\text{H}^d_m(\hat{R})}.$$

**Corollary 9.4** (Cf. [19, Th. 6.21]). In the setting of the above proposition, if Question 1 has an affirmative answer, then we have

$$\tau_{\mathcal{B}}(\hat{R}, \hat{\Delta}) = J(\hat{R}, \hat{\Delta}).$$

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