Recent work in gravitational lensing and catastrophe theory has shown that the sum of the signed magnifications of images near folds, cusps and also higher catastrophes is zero. Here, it is discussed how Lefschetz fixed point theory can be used to interpret this result geometrically. It is shown for the generic case as well as for elliptic and hyperbolic umbilics in gravitational lensing.

PACS numbers: 02.40.Xx, 98.62.Sb

I. INTRODUCTION

This year sees the ninetieth anniversary of Eddington’s eclipse expedition to investigate the gravitational deflection of light, which provided an early corroboration of Einstein’s General Relativity. Nowadays, gravitational lensing is an important tool in astronomy and cosmology, and is used to address some of the current fundamental challenges like the properties of Dark Matter. A comprehensive introduction can be found, for example, in Schneider, Ehlers and Falco, and Petters, Levine and Wambsganss. Lensing theory is also a rich research field of its own right within mathematical physics, in particular with applications of topological invariants and catastrophe theory. Here, we shall revisit a recent result concerning magnification invariants, and show that it can be understood as a combination of those two aspects.

In the astronomically interesting limit of small deflection angles, the physical framework for lensing is geometrical optics subject to scalar terms of linearized General Relativity. Hence light rays are conserved, and the signed magnification of each lensed image is proportional to the solid angle subtended by the ray bundle, taking image parity into account. Now for certain mass models acting as lenses, one finds that the sum of the signed image magnifications is always constant, provided the light source remains within a certain caus-
tic domain where the number of images is constant (and maximal), while their individual positions and magnifications, of course, are not.

This property is called a magnification invariant, and the first example was found by Witt and Mao\textsuperscript{14} for a lens consisting of two coplanar point masses. Other examples have come to light since, and methods of derivation using complex analysis have been developed\textsuperscript{5,7}. This type of model-dependent magnification invariant is global in the sense that it involves the maximum number of images produced by a lens; and that it holds in a finite domain delimited by caustics. Furthermore, there are other magnification invariants which hold only close to caustic singularities, but are universal in the sense that they are completely independent of the lens model. In fact, the universality of this type of magnification invariant is a direct consequence of the genericity of caustic catastrophes.

From studies of the image magnifications $\mu_i$ near folds and cusps\textsuperscript{4,12,15}, the simplest catastrophes occurring in gravitational lensing, it has emerged that

$$\sum_{i=1}^{n} \mu_i = 0$$

for sources close to a caustic, where $n = 2$ for the doublet of images due to a source near a fold, and $n = 3$ for the image triplet produced by a cusp. Aazami and Petters\textsuperscript{1} have recently shown that this invariant is also true for higher caustic catastrophes, namely for elliptic and hyperbolic umbilics both generically and in lensing, as well as for swallowtails in the generic case. In all of these instances, the corresponding magnification invariants refer to image quadruplets, that is, $n = 4$.

In the present paper, Lefschetz fixed point theory is used to interpret this result. I would like to argue that this is interesting for two reasons. Firstly, the proof of these magnification invariants appears to be less laborious than the one given previously using elementary techniques. Secondly, all magnification invariants considered here are analogous and differ mainly in the number of images. This seems to hint at a more fundamental explanation beyond the details of the individual singularities. The unified explanation given here in terms of Lefschetz fixed point numbers appears to provide this. In particular, it gives a more geometrical perspective on the problem, thus tying it in with other studies of the geometry of gravitational lensing. Since this proof relies on the Lefschetz fixed point formula, the magnification invariants are found to be a consequence, ultimately, of the Atiyah-Bott Theorem.
The paper is structured as follows. In Section II, the relevant background in gravitational lensing, catastrophe theory and Lefschetz fixed point theory is briefly summarized, followed by a proof of the main result in Section III for generic catastrophes. More specific applications to catastrophes in gravitational lensing are given in Section IV, and the geometric nature of the magnification invariants is commented on in Section V.

II. PRELIMINARIES

A. Gravitational lensing

Although the proof will refer to generic maps, it may be useful to have gravitational lensing theory in mind as a corresponding physical framework. We shall therefore begin by outlining the mathematical structure of its standard treatment, adopting some suitably scaled units, and use this lensing terminology throughout.

Consider $\mathbb{R}^3$ with a pointlike light source at $\mathbf{y} \in S = \mathbb{R}^2$, the source plane, and, in between $S$ and the observer, a mass distribution projected into $L = \mathbb{R}^2$, the lens plane parallel to $S$, with surface density $\kappa : L \to \mathbb{R}$. This mass acts as a gravitational lens, giving rise to a set of images seen at some $\mathbf{x}_i \in L$ which, by Fermat’s Principle, correspond to stationary points of a time delay function, or Fermat potential, $\phi : S \times L \to \mathbb{R}$ such that $\nabla \phi_{\mathbf{y}}(\mathbf{x}_i) = 0$. Now the gravitational time delay $\psi : L \to \mathbb{R}$ stems from the gravitational potential due to $\kappa$ and hence, in the given approximation of weak fields, from a Poisson equation $\Delta \psi = 2\kappa$. Then the total time delay is given by

$$\phi_{\mathbf{y}}(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{y})^2 - \psi(\mathbf{x}),$$

and the lensing map is therefore

$$\eta : L \to S, \; \mathbf{x}_i \mapsto \mathbf{y}, \; \mathbf{y}(\mathbf{x}) = \mathbf{x} - \nabla \psi(\mathbf{x}),$$

(1)

the physical images being of course, mathematically speaking, the preimages of $\eta$. Since light rays are conserved, the signed image magnification is given by the inverse of the determinant of the Jacobian of $\eta$,

$$J_{\eta} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix}, \; \mu_i = \frac{1}{\det J_{\eta}(\mathbf{x}_i)}.$$ 

(2)
For regular images, $|\mu_i| < \infty$ and $\phi$ is a Morse function. The subset of $L$ where $\det J_\eta = 0$ is called the critical set, which maps under $\eta$ to the caustic set in $S$. Hence the image number changes when the source crosses a caustic, only certain types of which occur generically.

**B. Catastrophe theory**

These can be enumerated with reference to Thom’s list of local representatives of the elementary catastrophes. The polynomial generating potential

$$\phi_u(x) : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R},$$

where $u \in \mathbb{R}^p$ are the control parameters and $x \in \mathbb{R}^q$ are the state variables, gives rise to a caustic set as discussed above, which traces out a bifurcation set or big caustic in the space of control parameters. Since we are ultimately interested in gravitational lensing, we shall limit the state space here to $q = 2$, $x = (x_1, x_2)$. Also, the parameter space has $p \geq 2$ because of the source coordinates plus, potentially, some physical parameters of the lens model itself.

For purposes of classification, the angle $\beta$ between the tangent of the critical curve and the kernel of $J_\eta$ turns out to be important. Then the list of relevant generic catastrophes begins with the following\textsuperscript{8,10,11}.

**Fold.** Characterized by rank $\text{rank } J_\eta = 1$, $\det J_\eta = 0$ and no zero of $\beta$. Hence, it is stable in a family of generating potentials $\phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by two parameters $y = (y_1, y_2)$. For given parameter values, the generic map $\eta : \mathbb{R}^2 \to \mathbb{R}^2$ has up to two common solutions.

$$\phi_y(x) = y_1 x_1 + y_2 x_2 - \frac{1}{2} x_1^2 - \frac{1}{3} x_2^3,$$
$$\eta(x) = (x_1, x_2^2).$$

**Cusp.** Characterized by rank $\text{rank } J_\eta = 1$, $\det J_\eta = 0$ and a simple zero of $\beta$. It is therefore also stable in a generating family $\phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by two parameters $\{y_1, y_2\}$ and, for given parameter values, the generic map $\eta : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has up to three common solutions.

$$\phi_y(x) = y_1 x_1 + y_2 x_2 - \frac{1}{2} x_1^2 - \frac{1}{2} y_1 x_2^2 - \frac{1}{4} x_2^4,$$
$$\eta(x) = (x_1, x_1 x_2 + x_2^3).$$
Swallowtail. Characterized by rank $J_\eta = 1$, $\det J_\eta = 0$ and a double zero of $\beta$. Thus, it is stable in a generating family $\phi : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$ defined by three parameters $\{c, y_1, y_2\}$. For given parameter values, the generic map $\eta_c : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has up to four common solutions.

$$
\phi_{c,y}(x) = y_1 x_1 + y_2 x_2 - \frac{1}{2} y_2 x_1^2 - \frac{1}{2} x_2 - \frac{1}{3} c x_1^3 - \frac{1}{5} x_1^5,
$$

$$
\eta_c(x) = (x_1 x_2 + cx_1^2 + x_1^4, x_2).
$$

Elliptic Umbilic. Characterized by rank $J_\eta = 0$, giving three equations, and an inequality. Hence it is stable in a generating family $\phi : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$ defined by three parameters $\{c, y_1, y_2\}$. For given parameter values, the generic map $\eta_c : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has up to four common solutions.

$$
\phi_{c,y}(x) = y_1 x_1 + y_2 x_2 + c(x_1^2 + x_2^2) + x_1^3 - 3x_1 x_2^2,
$$

$$
\eta_c(x) = (3x_2^2 - 3x_1^2 - 2cx_1, 6x_1 x_2 - 2cx_2).
$$

Hyperbolic Umbilic. Characterized by rank $J_\eta = 0$ as before, and a different inequality. Therefore, it is also stable in a generating family $\phi : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$ defined by three parameters $\{c, y_1, y_2\}$. For given parameter values, the generic map $\eta_c : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has again up to four common solutions.

$$
\phi_{c,y}(x) = y_1 x_1 + y_2 x_2 + cx_1 x_2 + x_1^3 + x_2^3,
$$

$$
\eta_c(x) = (-3x_1^2 - cx_2, -3x_2^2 - cx_1).
$$

C. Lefschetz fixed point theory

The proof the main theorem will be based on the Lefschetz fixed point formula. The real and complex versions are briefly summarized here, which can be understood as special cases of the Atiyah-Bott Theorem. See, for instance, Griffiths and Harris for a comprehensive discussion.

Let $M$ be a compact, orientable, real manifold without boundary. Then the smooth map $f : M \to M$, $\dim \mathbb{R}(M) = d$, gives rise to a set of fixed points $\text{Fix}(f) = \{x \in M : f(x) = x\}$, that is, intersections of the graph $\{(x, f(x))\} \in M \times M$ with the diagonal $\{(x, x)\} \in M \times M$. Since the pull-back of $f$ commutes with the exterior derivative according to $f^* \circ d = d \circ f^*$,
f induces a map on the de Rham cohomology classes $H^k_d(M)$. Thus, Lefschetz fixed point theory establishes a connection between local properties of the fixed points of f, the fixed point indices, and the global, topological properties of $M$. The latter is expressed by the Lefschetz number,

$$L(f) = \sum_{k=0}^{d} (-1)^k \text{trace } f^*(H^k_d(M)),$$

a homotopy invariant, which reduces to the Euler characteristic $\chi(M)$ if $f$ is homotopic to the identity map. The fixed point indices are $\{+1, -1\}$ depending on the orientation of the intersection.

Next, assume that the map $f$ on a compact, complex manifold $M$, $\dim_{\mathbb{C}}(M) = d$ without boundary is holomorphic, that is, $\bar{\partial}f = 0$ in terms of the Dolbeault differential operator. In this case, then, the crucial property is that $f^*$ commutes with $\bar{\partial}$, thus inducing a map on the Dolbeault cohomology classes $H^{r,s}_{\bar{\partial}}(M)$ of bidegree $(r,s)$. Hence one can define the holomorphic Lefschetz number,

$$L_{\text{hol}}(f) = \sum_{s=0}^{d} (-1)^s \text{trace } f^*(H^{0,s}_{\bar{\partial}}(M)).$$

(3)

Provided the intersection of graph and diagonal are transversal, it turns out that this is connected to the local fixed point indices of $f$ via the holomorphic Lefschetz formula,

$$L_{\text{hol}}(f) = \sum_{x \in \text{Fix}(f)} 1 / \det(I_d - Df)(x),$$

(4)

where $I_d$ is the $d$-dimensional identity matrix and $Df$ is the matrix of first derivatives with respect to local holomorphic coordinates. The transversality condition can then be expressed as the requirement that the fixed point indices be well defined, that is, $\det(I_d - Df)(x) \neq 0$.

### III. MAIN THEOREM

As indicated in the Introduction, the main result to be revisited from a geometric point of view is a theorem about magnification invariants for generic maps near catastrophes by Aazami and Petters:

**Theorem.** Let $\eta : \mathbb{R}^2 \to \mathbb{R}^2$ be a generic map near catastrophes in the sense of Section IIB, possibly dependent on control parameters as described there. For fixed control parameters,
let $\mu_i$ be the signed magnifications of the common solutions $x_i$ of $\eta(x) = y$ as defined in equation (2). Then

$$
\sum_{i=1}^{n} \mu_i = 0,
$$

where $n = 2$ for folds; $n = 3$ for cusps; $n = 4$ for elliptic umbilics, hyperbolic umbilics and swallowtails.

Since signed magnifications are non-integer numbers, it is clear that the real version of the Lefschetz formula cannot be used to prove it. However, it has emerged in the context of global magnification invariants that a complexification of the lensing map using

$$
z = x_1 + ix_2 \in L \otimes \mathbb{C} = \mathbb{C}, \quad \zeta = y_1 + iy_2 \in S \otimes \mathbb{C} = \mathbb{C}, \quad (5)
$$

is a fruitful approach. Then, complex magnification invariants can be derived which hold in the real case provided that no spurious roots occur, that is, complex roots which do not correspond to physical images $\mathbb{I}$. It has previously been suggested by the author\textsuperscript{13} to use this ansatz and the Lefschetz fixed point formula to interpret a class of global magnification invariants. But these are model-dependent, as mentioned in the Introduction, and the technical conditions remain to be clarified. The modified approach presented in this article is a simpler and model-independent, and hence more universal, application of this idea. In fact, it will also serve as an applicaton and extension of an example of the holomorphic Lefschetz formula discussed by Atiyah & Bott in the original paper on their considerably more general theorem\textsuperscript{3}.

**Proof.** The proof shall be divided into four steps.

**1. Complexified map.** We begin by complexifying the map $\eta$ corresponding to the respective catastrophes. However, it is clear that the holomorphic Lefschetz formula cannot be applied directly to a complexification of $\eta$ by means of (5), for two reasons. Firstly, it depends on $z$ and $\bar{z}$ and is therefore not holomorphic, and secondly, it is not defined on a compact space without boundary. But we can avoid the first problem at the expense of the dimension, by treating $(x_1, x_2) \equiv (z_1, z_2)$ as independent holomorphic coordinates on $\mathbb{C}^2$. Then the corresponding map of two polynomials in two complex variables,

$$
\eta^c : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad (z_1, z_2) \mapsto (\eta^c_{\bar{z}_1}, \eta^c_{\bar{z}_2}) = (y_1, y_2),
$$
is holomorphic, and each real common solution for fixed \( y \) is a common solution of \( \eta(x) = y \). Furthermore, we know from Section IIB that there exist domains of the parameter \( y \) where there are \( n \) real common solutions of \( \eta \) at finite positions in \( \mathbb{R}^2 \), for the respective catastrophes as given above, corresponding to the near-singular image multiplets. Now by considering the degrees of the polynomials, one can see from Bezout’s Theorem that \( n \), for the respective catastrophes, is also the maximum number of common solutions of \( \eta^C \), possibly complex. Therefore, for fixed \( y \) in suitable domains, \( \eta^C \) has the \( n \) real common solutions of \( \eta \) as stated in the theorem. Notice also that \( \eta^C \) has no common solutions at infinity in this case.

2. Fixed point map. Next, in view of our application of fixed point theory, it will be useful to define a map \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) such that its set of fixed points \( \text{Fix}(f) \) corresponds precisely to the set of common solutions of \( \eta^C \). This is easily obtained,

\[
f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2)) = (z_1 - \eta^C_1(z_1, z_2) + \zeta_1, z_2 - \eta^C_2(z_1, z_2) + \zeta_2).
\]

(6)

Notice also that, for fixed \( (y_1, y_2) \), this implies

\[
D_f = \begin{pmatrix}
1 - \frac{\partial \eta^C_1}{\partial z_1} & - \frac{\partial \eta^C_1}{\partial z_2} \\
- \frac{\partial \eta^C_2}{\partial z_1} & 1 - \frac{\partial \eta^C_2}{\partial z_2}
\end{pmatrix},
\]

and therefore, by definition of \( \eta^C \) and equation (2),

\[
\det(I_2 - D_f) = \det J_\eta = \frac{1}{\mu}.
\]

Since, by construction in the first step and for suitable domains of \( y \), the set of common solutions of \( \eta^C \) is also the set of real common solutions of \( \eta \), that is, the images, we now have the result that the signed magnifications are in fact

\[
\mu_i = \frac{1}{\det(I_2 - D_f)(x_i)}, \; x_i \in \text{Fix}(f), \; 1 \leq i \leq n.
\]

(7)

Also, \( \text{Fix}(f) \) has no fixed points at infinity in \( \mathbb{C}^2 \) in this case, by construction of \( f \) and the first step.

3. Projective fixed point map. We now address the second problem mentioned in the first step, seeking a compactification that allows us to use the holomorphic Lefschetz fixed point formula. To this end, write the map \( f \) of (6) in homogeneous coordinates \( (Z_0, Z_1, Z_2) \), where \( z_1 = Z_1/Z_0 \) and \( z_2 = Z_2/Z_0 \) for \( Z_0 \neq 0 \) as usual. Let
$m = \max(\deg(\eta^C_1), \deg(\eta^C_2))$ and consider the following holomorphic map on complex projective space,

$$F : \mathbb{CP}^2 \to \mathbb{CP}^2, \quad (Z_0 : Z_1 : Z_2) \mapsto (F_0 : F_1 : F_2),$$

where

$$F_0 = Z_0^m,$$

$$F_1 = Z_1 Z_0^{m-1} - Z_0^{m-\deg(\eta^C_1)} \eta^C_1(Z_0, Z_1, Z_2) + \zeta_1 Z_0^m,$$

$$F_2 = Z_2 Z_0^{m-1} - Z_0^{m-\deg(\eta^C_2)} \eta^C_2(Z_0, Z_1, Z_2) + \zeta_2 Z_0^m.$$

We need to establish that this is in fact a well-defined map on $\mathbb{CP}^2$. To see this, note first of all that $m \geq 2$ by the definiton of $\eta^C$ in the first step and the list of catastrophes in Section IIB. Now $F$ is well-defined except at any $(Z_0 : Z_1 : Z_2)$ where $F(Z_0 : Z_1 : Z_2) = (0 : 0 : 0) \neq \mathbb{CP}^2$. Assume there exists such a point. Since this implies $Z_0 = 0$ from the definition of $F_0$, the respective first terms of $F_1, F_2$ vanish because $Z_0^{m-1} = 0$ since $m - 1 \geq 1$. The other terms of $F_1, F_2$ vanish for $Z_0 = 0$, $Z_1, Z_2 \neq 0$ precisely if $f$ has fixed points at infinity in $\mathbb{C}^2$, by the definitions of $f$; of $F$; and of the homogeneous coordinates. But by construction, this is not case as noted in the second step. Therefore, $F$ is not well-defined only for $(Z_0 : Z_1 : Z_2) = (0 : 0 : 0) \neq \mathbb{CP}^2$, that is, it is well defined everywhere on $\mathbb{CP}^2$, as requied. This is a slightly extended version of the map constructed by Atiyah and Bott which covers the case $\deg(\eta^C_1) = \deg(\eta^C_2)$.

One can now proceed with the properties of $F$. It will be useful to consider the decomposition $\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1$ defined by $\mathbb{C}^2 : Z_0 = 1$ and $\mathbb{CP}^1 : Z_0 = 0$. Then the entire fixed point set of $F$ consists of $\text{Fix}(F|_{\mathbb{C}^2}) \cup \text{Fix}(F|_{\mathbb{CP}^1}) = \text{Fix}(F)$. Specifically, on $\mathbb{C}^2 : Z_0 = 1$ one has $F_1 = f_1, F_2 = f_2$ by (6) and therefore $\text{Fix}(F|_{\mathbb{C}^2}) = \text{Fix}(f)$.

4. Holomorphic Lefschetz number. Finally, we need to determine the holomorphic Lefschetz number associated with $F$. It can be shown that the Dolbeault cohomology classes of $\mathbb{CP}^k$ are given by

$$H^{r,s}_{\partial}(\mathbb{CP}^k) = \begin{cases} 0 & \text{if } r \neq s, \\ \mathbb{C} & \text{if } r = s. \end{cases}$$

Recall from the definition (3) that only the cohomology classes of bidegree $(0, s)$ contribute to $L_{\text{hol}}$, that is, only $H^{0,0}_{\partial}(\mathbb{CP}^k) = \mathbb{C}$ in the case of complex projective space.
Thus \( L_{\text{hol}} = 1 \). One is now in a position to use the holomorphic Lefschetz fixed point formula (11) for \( F \) on \( \mathbb{C}P^2 \),

\[
1 = L_{\text{hol}}(F) = \sum_{x \in \text{Fix}(F)} \frac{1}{\det(I_2 - D_F)(x)}
\]

\[
= \sum_{x \in \text{Fix}(F|_{\mathbb{C}P^2})} \frac{1}{\det(I_2 - D_F)(x)} + \sum_{x \in \text{Fix}(F|_{\mathbb{C}P^1})} \frac{1}{\det(I_1 - D_F)(x)}. \tag{8}
\]

By means of step 3 and equation (7), we obtain

\[
\sum_{x \in \text{Fix}(F|_{\mathbb{C}P^2})} \frac{1}{\det(I_2 - D_F)(x)} = \sum_{x \in \text{Fix}(f)} \frac{1}{\det(I_2 - D_f)(x)} = \sum_{i=1}^{n} \mu_i. \tag{9}
\]

Applying the holomorphic Lefschetz fixed point formula to the restriction of \( F \) to \( \mathbb{C}P^1 \) yields

\[
\sum_{x \in \text{Fix}(F|_{\mathbb{C}P^1})} \frac{1}{\det(I_2 - D_F)(x)} = 1 \tag{10}
\]

because \( F \) is well-defined for \((0:Z_1:Z_2)\), \( Z_1, Z_2 \neq 0 \), as discussed in the previous step, \( L_{\text{hol}} = 1 \) by the same token as above, and \( m \geq 2 \) so that the identity map is excluded (where the expression given for the fixed point index breaks down). In fact, this statement is the Rational Fixed Point Theorem for holomorphic maps on the Riemann sphere, which is important in complex dynamics, and an elementary proof can be found in Milnor\(^9\).

The result that \( \sum_i \mu_i = 0 \) follows by substituting equations (9) and (10) into (8). This concludes the proof of the theorem from the point of view of Lefschetz fixed point theory.

\[\square\]

IV. GRAVITATIONAL LENSING APPLICATIONS

Going back to the starting point, one can now consider how this theorem for generic maps in catastrophe theory can also be applied to near-singular image multiplets in gravitational lensing theory. Here, the generating potential function \( \phi \) is again the Fermat potential, and local expressions for \( \eta \) near the canonical catastrophes can be found with suitable Taylor expansions\(^{11}\). These are related to the somewhat simpler generic equations given
in Section IIA by non-linear coordinate transformations in general. Hence, the notional signed magnifications defined in equation (2) and used in the generic version of the theorem may be different from physical signed magnifications defined by a ratio of solid angles. The validity of the theorem in the generic case does therefore not immediately imply its validity in gravitational lensing, and it is necessary to examine those cases separately.

For the present purposes, we shall limit the discussion to umbilics in gravitational lensing, for which the theorem has been established, and which also turns out to be the simplest lensing case in the framework presented here. The characteristic lensing maps can be written as follows:

**Elliptic Umbilic in Lensing:** $\eta_p(x) = (x_1^2 - x_2^2, -2x_1x_2 + 4px_2),$

**Hyperbolic Umbilic in Lensing:** $\eta_p(x) = (x_1^2 + 2px_2, x_2^2 + 2px_1),$

where $p$ is a parameter, and their properties are analogous to the ones discussed in Section IIA. Then it is clear that the constructions described in Section III are also possible here, and hence the theorem applies. Now it is important that the coordinates used in the equations above are related to the original lensing variables by linear coordinate transformations, and the theorem is therefore also true for the physical signed magnifications.

V. CONCLUDING REMARKS

Although it may seem unnecessary to construct a map $f$ from the lensing map $\eta$ such that images are the fixed points of $f$, I would like to argue that this fixed point treatment of gravitational lensing theory is in fact a rather natural approach, for two reasons.

Firstly, it can be seen from the definition of $\eta$ that this is possible on physical grounds because of the split between the gravitational and geometrical time delay terms of the Fermat potential, in the given limit of small deflection angles. Hence one can naturally write a fixed point equation $y + \nabla\psi(x) = x$ from (1). Of course, this split is not seen explicitly in the present work because we use local forms of the lensing map near the caustic catastrophes.

Secondly, it emerges from equation (7) that the transversality condition for the fixed points stated in Section IIC, which is necessary for the Lefschetz formula to hold, translates into the condition that image magnifications be finite. This is exactly the usual condition for
images to be regular or, equivalently, for $\phi$ to be a Morse function, as mentioned in Section IIA.

Finally, some comments about the geometry of these signed magnification invariants might be in order. Given a smooth surface, one can relate its topology to the number of stationary points by means of Morse Theory. One can also relate its topology to the integral of the Gaussian curvature by means of the Gauss-Bonnet Theorem. Now considering the surface given by the graph of the Fermat potential, it is interesting to note that signed image magnifications are, geometrically speaking, the inverse of the Gaussian curvature of this surface at its stationary points. The theorem, then, establishes a connection between discrete geometrical quantities summed over stationary points, rather than integrated, and an associated topological property expressed by the holomorphic Lefschetz number. Hence this seems like an intermediary, on a conceptual level, between Morse Theory and the Gauss-Bonnet Theorem.

ACKNOWLEDGMENTS

I would like to thank the Science and Technology Facilities Council, United Kingdom, for financial support.

* Electronic address: mcw36@ast.cam.ac.uk

1 Aazami, A. B., and Petters, A. O., “A universal magnification theorem for higher-order caustic singularities,” J. Math. Phys. 50, 032501 (2009)

2 Atiyah, M. F., and Bott, R., “A Lefschetz fixed point formula for elliptic complexes: I,” Ann. Math. 86, 374–407 (1967)

3 Atiyah, M. F., and Bott, R., “A Lefschetz fixed point formula for elliptic complexes: II. Applications,” Ann. Math. 88, 451–491 (1968)

4 Blandford, R., and Narayan, R., “Fermat’s principle, caustics, and the classification of gravitational lens images,” Astrophys. J. 310, 568–582 (1986)

5 Dalal, N., and Rabin, J. M., “Magnification relations in gravitational lensing via multidimensional residue integrals,” J. Math. Phys. 42, 1818–1836 (2001)
6 Griffiths, P., and Harris, J., *Principles of Algebraic Geometry* (J. Wiley, New York, 1978)

7 Hunter, C., and Evans, N. W., “Lensing properties of scale-free galaxies,” Astrophys. J. **554**, 1227–1244 (2001)

8 Majthay, A., *Foundations of Catastrophe Theory* (Pitman, Boston, 1985)

9 Milnor, J., *Dynamics in One Complex Variable* (Princeton University Press, Princeton, 2006)

10 Petters, A. O., Levine, H., and Wambsganss, J., *Singularity Theory and Gravitational Lensing* (Birkhäuser, Boston, 2001)

11 Schneider, P., Ehlers, J., and Falco, E. E., *Gravitational Lenses* (Springer-Verlag, Berlin, 1992)

12 Schneider, P., and Weiss, A., “The gravitational lens equation near cusps,” Astron. Astrophys. **260**, 1–13 (1992)

13 Werner, M. C., “A Lefschetz fixed point theorem in gravitational lensing,” J. Math. Phys. **48**, 052501 (2007)

14 Witt, H. J., and Mao, S., “On the minimum magnification between caustic crossings for microlensing by binary and multiple stars,” Astrophys. J. **447**, L105–L108 (1995)

15 Zakharov, A. F., “Gravitational lens equation near cusps,” in Piran, T., and Ruffini, R., eds., *Proceedings of the Eighth Marcel Grossmann Meeting on General Relativity* (World Scientific, 1999)