Superfluid pairing in a polarized dipolar Fermi gas

M.A. Baranov¹,²,³, M.S. Mar'enko⁵,⁶, Val.S. Rychkov¹,³ and G.V. Shlyapnikov¹,³,⁴

(1) Russian Research Center Kurchatov Institute, Kurchatov Square, 123182 Moscow, Russia
(2) Institut für Theoretische Physik, Universität Hannover, D-30167 Hannover, Germany
(3) FOM Institute for Atomic and Molecular Physics, 407, 1098 SJ Amsterdam, The Netherlands
(4) Laboratoire Kastler Brossel*, 24 rue Lhomond, F-75231 Paris Cedex 05, France
(5) P.L. Kapitza Institute for Physical Problems, 117334 Moscow, Russia
(6) Institute of Radio Engineering and Electronics RAS, 103907 Moscow, Russia

We calculate the critical temperature of a superfluid phase transition in a polarized Fermi gas of dipolar particles. In this case the order parameter is anisotropic and has a nontrivial energy dependence. Cooper pairs do not have a definite value of the angular momentum and are coherent superpositions of all odd angular momenta. Our results describe prospects for achieving the superfluid transition in single-component gases of fermionic polar molecules.

I. INTRODUCTION

The recent success in observing quantum degeneracy in ultra-cold atomic Fermi gases [1–3] stimulates a search for gaseous Fermi systems with an achievable temperature of superfluid phase transition. The ideas based on Cooper pairing for a short-range Van der Waals interaction between atoms [5] require a simultaneous trapping of at least two different fermionic species, with a rather severe constraint on their relative concentrations. The situation is different for Fermi gases of dipolar particles. Being electrically polarized, these particles interact via long-range anisotropic (partially attractive) dipole-dipole forces. In the ultra-cold limit, the dipole-dipole scattering amplitude is energy independent for any orbital angular momenta [6]. This opens prospects to achieve the superfluid pairing in a single-component Fermi gas, where only scattering with odd orbital momenta (negligible in the case of Van der Waals interactions) is present. These prospects are especially interesting as in single-component fermionic gases the Pauli exclusion principle provides a strong suppression of inelastic collisional rates (see [8]). Hence, one can think of achieving significantly higher densities than in Bose gases.

Possible realizations of dipolar Fermi gases include an electrically polarized gas of polar molecules as they have large permanent electric dipoles. The creation of cold clouds of polar molecules has been recently demonstrated in experiments with buffer-gas cooling [11] and in experiments based on deceleration and cooling of polar molecules by time-dependent electric fields [12]. Another option is to create a gas of atoms with electric dipole moments induced by a high dc electric field [13] or by laser coupling of the atomic ground state to an electrically polarized Rydberg state [13].

The p-wave Cooper pairing in a polarized dipolar Fermi gas has been discussed in [14,10], and the corresponding critical temperature has been estimated by using the standard BCS approach. In this paper we calculate the exact value of the critical temperature and find the energy and angular dependence of the order parameter. For this purpose we consider the Cooper pairing for all possible scattering channels. These channels are coupled to each other by the dipole-dipole interaction, and the Cooper pairs prove to be coherent superpositions of contributions of all odd angular momenta. In order to find the pre-exponential factor for the critical temperature, we perform the calculations to second order in perturbation theory along the lines of the approach of Gor’kov and Melik-Barkhudarov (GM approach) [13].

II. GENERAL EQUATIONS

We consider a spatially homogeneous single-component gas of fermions having a dipole moment $\mathbf{d}$ oriented along the $z$-axis. The Hamiltonian of the system has the form

*"LKB is an unité de recherche de l’Ecole Normale Supérieure et de l’Université Pierre et Marie Curie, associée au CNRS."
\[ H = \int dr \hat{\psi}^\dagger(r) \left( \frac{-\hbar^2}{2m} \nabla^2 - \mu \right) \hat{\psi}(r) + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \bigg| \hat{\psi}(\mathbf{r}) \bigg|^2 V_d(\mathbf{r} - \mathbf{r}') \bigg| \hat{\psi}(\mathbf{r}') \bigg|^2, \]  

where \( \hat{\psi}(\mathbf{r}) \) is the field operator for fermions, \( V_d(\mathbf{r}) = \frac{d^2}{r^3}(1 - 3 \cos^2 \theta_r) \) the dipole-dipole interaction, \( \theta_r \) is the angle between the interparticle distance \( \mathbf{r} \) and the \( z \)-axis, and \( \mu \) is the chemical potential. In Eq.(1) we omit the contribution of the \( p \)-wave scattering due to the short-range part of interparticle interaction, since this contribution is small in the dilute ultra-cold limit.

For a single-component Fermi gas, the Cooper pairing is possible only in the states with an odd angular momentum \( l \). On the other hand, the anisotropic character of the dipole-dipole interaction leads to coupling between Cooper pairs with different values of the angular momentum. Therefore, the problem of superfluid pairing requires us to consider states with any odd \( l \).

The critical temperature \( T_c \) of the superfluid transition and the order parameter \( \Delta \) can be found from the gap equation in the momentum representation [16,17]:

\[ \Delta(\mathbf{p}) = \frac{2 \pi \hbar^3}{(2\pi\hbar)^3} V(\mathbf{p}, \mathbf{p'}) \text{tanh}(E(\mathbf{p'})/2T) \Delta(\mathbf{p'}). \]  

(2)

Here \( E(\mathbf{p}) = \sqrt{\Delta^2(\mathbf{p}) + (p^2/2m - \mu)^2} \), and we assume the order parameter to be real. The function \( V(\mathbf{p}, \mathbf{p'}) = V_d(\mathbf{p} - \mathbf{p'}) + \delta V(\mathbf{p}, \mathbf{p'}) \), where \( V_d(\mathbf{q}) \) is the Fourier transform of the dipole-dipole interparticle interaction potential \( V_d(\mathbf{r}) \):

\[ V_d(\mathbf{q}) = \frac{4 \pi d^2}{3} (3 \cos^2(\theta_q) - 1), \]  

(3)

with \( \theta_q \) being the angle between the momentum \( \mathbf{q} \) and the \( z \)-axis. The quantity \( \delta V(\mathbf{p}, \mathbf{p'}) \) originates from many-body effects and is a correction to the bare interparticle interaction \( V_d \). The leading corrections are second order in \( V_d \) and the corresponding diagrams are shown in Fig.1 (see Ref. [15]). They describe the processes in which one of the two colliding particles polarizes the medium by virtually creating a particle-hole pair. In 1a the particle-hole pair then annihilates due to the interaction with the other colliding particle. In 1b, 1c and 1d the hole annihilates together with one of the colliding particles. In 1b and 1c the particle-hole pair is created due to the interaction of the medium with one of the colliding particles, and the hole annihilates with the other colliding partner. In 1d these creation and annihilation processes involve one and the same colliding particle.

![Fig. 1. The lowest order many-body corrections to the effective interparticle interaction.](image)

For temperatures just below \( T_c \) the order parameter is small and the gap equation is equivalent to the Ginzburg-Landau equation for the spatially homogeneous order parameter. This equation can be obtained by expanding the rhs of Eq.(3) in powers of the order parameter \( \Delta(\mathbf{p}) \):

\[ \Delta(\mathbf{p}) = - \int \frac{d\mathbf{p'}}{(2\pi\hbar)^3} V(\mathbf{p}, \mathbf{p'}) \left[ K(p') \Delta(\mathbf{p'}) + \frac{\partial K(\mathbf{p'}) \Delta^2(\mathbf{p'})}{2\zeta} \right], \]  

(4)
where \( K(p) = \tanh(\xi/2T)/2\xi \), and \( \xi = p^2/2m - \mu \).

The occurrence of the Cooper pairing is associated with the existence of a nontrivial solution of Eq. (6) for temperatures \( T < T_c \). In order to find the value of the critical temperature one can neglect the second, nonlinear term in the square brackets in the rhs of Eq. (6) because for \( T \rightarrow T_c \) the order parameter \( \Delta \rightarrow 0 \). The corresponding linearized gap equation also provides us with the momentum dependence of the order parameter, whereas the nonlinear term determines the absolute (temperature dependent) value of \( \Delta \).

The integral in Eq. (6) diverges at large momenta. The divergency can be eliminated by expressing the bare interaction \( V_d \) in terms of the vertex function (scattering off-shell amplitude) \( \Gamma(E, p, p') \). This is similar to the well-known procedure of renormalization of the scattering length in dilute gases of Bose or Fermi particles interacting via short-range forces \([4, 5]\). One may choose any value of \( E \), and for simplifying our calculations we select \( E = 0 \). Then the vertex function \( \Gamma(0, p, p') = \Gamma_d(p, p') \) obeys the equation

\[
\Gamma_d(p, p') = V_d(p - p') - \int \frac{dq}{(2\pi\hbar)^3} \Gamma_d(p, q)K_0(q)V_d(q - p'),
\]

with \( K_0(q) = m/q^2 \). We will confine ourselves to the second order in perturbation theory with respect to \( V_d \). Omitting higher order corrections, the renormalized linearized gap equation reads

\[
\Delta(p) = -\int \frac{dp'}{(2\pi\hbar)^3}\Gamma_d(p, p')\{K(p') - K_0(p')\}\Delta(p') - \int \frac{dp'}{(2\pi\hbar)^3} \delta V(p, p')K(p')\Delta(p').
\]

In the dilute ultra-cold limit only small momenta \( p \) and \( p' \) are important. We thus have to find the scattering amplitude for ultra-cold particles, in the presence of the dipole-dipole interaction between them.

**III. SCATTERING AMPLITUDE IN THE ULTRA-COLD LIMIT**

The anisotropic and long-range character of the dipole-dipole interaction (\( V_d \propto 1/r^3 \)) ensures that in the ultra-cold limit all partial waves give an energy independent contribution to the scattering amplitude \([6]\). For any orbital angular momentum \( l \) one has \( \Gamma_d \sim d^2 \sim 4\pi\hbar^2r_\ast/m \), where the quantity \( r_\ast \sim md^2/\hbar^2 \) plays a role of the characteristic radius of interaction for the dipole-dipole potential. For the interparticle separation \( r \gg r_\ast \) the potential \( V_d(r) \) does not influence the wave function of the relative motion of two colliding particles and the motion becomes free. The ultra-cold limit requires particle momenta satisfying the condition

\[
pr_\ast/\hbar \ll 1.
\]

The anisotropy of \( V_d \) directly couples scattering channels with angular momenta \( l \) and \( l \pm 2 \). Thus, strictly speaking, all even- \( l \) (odd- \( l \)) channels are coupled to each other, whereas the scattering with odd angular momenta remains decoupled from that with even momenta.

There are two contributions to the scattering amplitude. The long-range contribution comes from distances \( r \gtrsim \hbar/p \) and gives \( \Gamma_d \sim d^2 \) for all angular momenta in the incoming and outgoing channels, allowed by the selection rules. This contribution can be found by using the Born approximation. The short-range contribution comes from distances \( r \lesssim r_\ast \). For the scattering with even \( l \), due to the presence of the s-wave, we have again \( \Gamma_d \sim d^2 \) or even somewhat larger because of the so-called shape resonances \([21]\). Under the condition \([7]\), the contribution of the s-wave to the wave function of the relative motion at distances \( r \gtrsim r_\ast \) is independent of \( p \). This leads to an energy independent \( \Gamma \). However, it depends on a detailed behavior of the interparticle potential at short interparticle distances. Thus, for even \( l \) one can not make a general statement on the value of \( \Gamma \).

In the case of identical fermions only odd orbital angular momenta are present. Then the short-range contribution is much smaller than the long-range one. We will demonstrate this for the p-wave on-shell scattering amplitude, omitting the coupling to the channels with other odd \( l \). For \( l = 1 \) and \( m_l = 0 \) in both incoming and outgoing scattering channels (\( m_l \) is the projection of \( l \) on the z axis), the dipole-dipole potential \( V_d(r) \) averaged over the angle \( \theta \), is equal to \( V_d = -4d^2/5r^3 \). In order to analyze the short- and long-range contributions to the scattering amplitude, we consider the relative motion of particles in a truncated potential \( V(r) = V_d(r) \) for \( r < r_0 \), and \( V(r) = 0 \) for \( r > r_0 \).

The truncation radius \( r_0 \) is selected such that \( r_\ast \lesssim r_0 \ll \hbar/p \). The Schrödinger equation for the wave function of the relative motion reads

\[
\frac{\hbar^2}{m} \left( \frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{6}{r^2} \right) \psi(r) + V(r)\psi(r) = \frac{p_r^2}{m} \psi(r).
\]
At distances $r \ll h/p$ we may put $p = 0$ in Eq.\((5)\). Then for $r < r_0$ we use the well-known procedure of finding an analytical solution for $\psi(r)$ in power law potentials [21]. Assuming $r \gg r_*$, this gives

$$\psi(r) \propto \left( \frac{r}{r_*} + \text{const} \right),$$

where the constant term is independent of $r_*$. At $r > r_0$ the motion is free and $\psi(r)$ depends explicitly on the scattering phase $\delta$. The solution, which for $r \to \infty$ takes the required asymptotic form $(\bar{h}/pr) \cos (pr/\bar{h} + \delta)$, at $r \ll h/p$ becomes

$$\psi(r) = -\left( \frac{\sin \delta}{(pr/\bar{h})^2} + \frac{(pr/\bar{h}) \cos \delta}{3} \right).$$

Equalizing the logarithmic derivatives of the wave functions $\psi$ and $\psi'$, we immediately obtain $\delta \sim p^3 r_0^2 r_* h^3$ and find that $\psi(r) \propto p$ for $r \ll h/p$. The scattering phase then proves to be $\Gamma_d = \int \psi(r) V(r) d^3r \sim d^2 (pr_0/\bar{h})^2$. The short-range contribution to $\Gamma$, that is the contribution from distances $r \lesssim r_*$, is obtained from this relation by simply putting $r_0 \sim r_*$. We then increase $r_0$ and make it much larger than $h/p$. At distances $r \sim h/p \gg r_*$ the potential $\bar{V}$ is much smaller than the kinetic energy term in the lhs of Eq.\((6)\). For the contribution of these distances to the scattering amplitude the Born approximation gives $\Gamma_d \sim d^2$. We thus see that the short-range contribution to the scattering amplitude is small as $(pr_*/\bar{h})^2$ compared to the long-range contribution coming from distances $r \sim h/p$.

This has two important consequences. First, a detailed shape of the interaction potential is not important for the scattering amplitude as the latter is determined by the long-range contribution. This contribution is obtained in the Born approximation and depends only on the value of the dipole moment. Second, we may include the second order Born correction to the amplitude. This correction is of the order of $d^2 (pr_*/\bar{h})$ and still greatly exceeds the short-range contribution.

In the second order Born approximation for the off-shell scattering amplitude $\Gamma_d(p, p')$ we have

$$\Gamma_d(p, p') = V_d(p - p') - \int \frac{dq}{(2\pi)^3} V_d(p - q) K_0(q) V_d(q - p'),$$

where the first and second terms in the rhs of Eq.\((11)\) are first and second order in $V_d$, respectively. The integral for the second order correction to the scattering amplitude in Eq.\((11)\) is formally divergent at large $q$. This is the same non-physical divergency as in the case of short-range interactions [18][19], and it will be eliminated in the calculations of the order parameter and critical temperature (see Section V).

IV. CRITICAL TEMPERATURE IN THE BCS APPROACH

In a quantum degenerate Fermi gas characteristic momenta of colliding particles are of the order of the Fermi momentum $p_F = (6\pi^2 n)^{1/3} \bar{h}$ ($n$ is the gas density). Then, with $r_* \sim m d^2/\bar{h}^2$, the condition \((6)\) of the ultra-cold limit for interparticle collisions can be written as

$$nd^2/\varepsilon_F \ll 1,$$

where $\varepsilon_F = p_F^2/2m$ is the Fermi energy. The lhs of Eq.\((12)\) is the ratio of the mean dipole-dipole interaction energy (per particle) to the Fermi energy. As in the case of short-range interactions [18][19], this is a small parameter of the many-body theory. It is the condition \((12)\) that allows us to omit the contribution of higher order diagrams and use the renormalized gap equation \((5)\).

Generally, in dilute Fermi gases the critical temperature is exponentially small compared to the Fermi energy $\varepsilon_F$. The exponent is inversely proportional to the Fermi momentum $p_F$ and is determined by first order terms in $V_d$. The account of the second order terms provides us with the pre-exponential factor.

We first calculate $\Delta(p)$ to first order in $V_d$ and find the correct exponent in the dependence of the critical temperature on the particle density. For this purpose we should keep in Eq.\((5)\) only the terms which are first order in $V_d$. This is the first term in the rhs of this equation, with $\Gamma_d(p, p') = V_d(p - p')$. Then, Eq.\((5)\) can be rewritten in the form

$$\Delta(\xi, n) = -\int_{-\infty}^{\infty} d\xi' (\tanh(\xi'/2T)/2\xi') \int \frac{dn'}{4\pi} R(\xi, n; \xi', n') \Delta(\xi', n').$$

(13)
Here \( \mathbf{n} = \mathbf{p}/p \), and

\[
R(\xi, \mathbf{n}; \xi', \mathbf{n'}) = \nu(\xi') \Gamma_d(p(\xi) \mathbf{n}, p(\xi') \mathbf{n'}) (1 - \xi'/\epsilon_F) \ln(2 \exp(\gamma \xi'/2T))/\pi T),
\]

where \( \nu(\xi) = mp(\xi)/2\pi^2\hbar^3 \) is the density of states at energy \( \epsilon + \mu \). The chemical potential \( \mu = \epsilon_F \).

The main contribution to the pairing comes from the states near the Fermi surface, where \( |\xi|, |\xi'| < \epsilon_F \). In order to single out this contribution in the rhs of Eq. (13), we introduce a characteristic energy \( \omega \) that obeys the constraint \( T < \omega \), and is of the order of the Fermi energy. We then divide the integral over \( \xi' \) in Eq. (13) into two parts: (a) the integration of \( R(\xi, \mathbf{n}; 0, \mathbf{n'}) \Delta(0, \mathbf{n'}) \) from \( -\omega \) to \( \omega \), and (b) the integration of \( [R(\xi, \mathbf{n}; \xi', \mathbf{n'}) \Delta(\xi', \mathbf{n'}) - R(\xi, \mathbf{n}; 0, \mathbf{n'}) \Delta(0, \mathbf{n'})] \) from \( -\omega \) to \( \omega \), plus the integration of \( R(\xi, \mathbf{n}; \xi', \mathbf{n'}) \Delta(\xi', \mathbf{n'}) \) from \( -\epsilon_F \) to \( -\omega \) and from \( \omega \) to \( \infty \). In part (a) we use the asymptotic formula

\[
\int_{-\omega}^{\omega} d\xi' (\tanh(\xi'/2T)/2\xi') \approx \ln \frac{2 \exp(\gamma \omega)}{\pi T},
\]

where \( \gamma = 0.5772 \) is the Euler constant. In part (b) we replace \( \tanh(\xi'/2T) \) by the step function (omitting the unimportant contribution from a narrow interval \( |\xi'| \lesssim T < \omega \)) and integrate in parts. As a result, Eq. (13) takes the form

\[
\Delta(\xi, \mathbf{n}) = -\ln \left[ \frac{2 \exp(\gamma \omega)}{\pi T} \right] \int \frac{d\xi'}{4\pi} R(\xi, \mathbf{n}; 0, \mathbf{n'}) \Delta(0, \mathbf{n'}) + \frac{1}{2} \int_{-\epsilon_F}^{\infty} d\xi' \ln |\xi'| \int \frac{d\xi'}{4\pi} \{ R(\xi, \mathbf{n}; \xi', \mathbf{n'}) \Delta(\xi', \mathbf{n'}) \},
\]

(14)

where the first and second terms in the rhs come from parts (a) and (b), respectively.

One can easily see that the second term in Eq. (14) is small as \( 1/\ln(2 \exp(\gamma \omega)/\pi T) \) compared to the first one. Therefore, the second term is only important for the pre-exponential factor in the expression for the critical temperature and will be omitted in this Section. This is equivalent to the commonly used BCS approach where the kernel \( R(\xi, \mathbf{n}; \xi', \mathbf{n'}) \) is replaced by \( R(0, \mathbf{n}; 0, \mathbf{n'}) \) for \( |\xi|, |\xi'| \leq \omega \) and by zero otherwise. Putting \( \epsilon = 0 \) in Eq. (13), we obtain the following equation for finding the critical temperature:

\[
\Delta(0, \mathbf{n}) = -\ln \left[ \frac{2 \exp(\gamma \omega)}{\pi T} \right] \int \frac{d\xi'}{4\pi} R(0, \mathbf{n}; 0, \mathbf{n'}) \Delta(0, \mathbf{n'}).
\]

(15)

The anisotropic character of the scattering amplitude leads to a nontrivial angular dependence of the order parameter \( \Delta(0, \mathbf{n}) \). In order to analyze the possibility of pairing we expand \( \Delta(0, \mathbf{n}) \) in terms of a complete set of eigenfunctions \( \phi_s(\mathbf{n}) \) of the integral operator with the kernel \( R(0, \mathbf{n}; 0, \mathbf{n'}) \):

\[
\Delta(0, \mathbf{n}) = \sum_{s=0}^{\infty} \Delta_s \phi_s(\mathbf{n}),
\]

(16)

\[
\int \frac{d\xi'}{4\pi} R(0, \mathbf{n}; 0, \mathbf{n'}) \phi_s(\mathbf{n'}) = \lambda_s \phi_s(\mathbf{n}), \quad s = 0, 1, \ldots
\]

(17)

The functions \( \phi_s(\mathbf{n}) \) are normalized by the condition \( \int (d\mathbf{n}/4\pi) \phi_s^2(\mathbf{n}) = 1 \), and they are labeled by the index \( s \) in such a way that the eigenvalues \( \lambda_s < \lambda_{s+1} \). Then Eq. (13) reduces to a set of equations

\[
\Delta_s \left( 1 + \lambda_s \ln \frac{2 \exp(\gamma \omega)}{\pi T} \right) = 0.
\]

The appearance of a nontrivial solution for \( \Delta(0, \mathbf{n}) \) below a certain critical temperature requires the presence of at least one negative eigenvalue \( \lambda_s \). For a single eigenvalue \( \lambda_s < 0 \), the critical temperature immediately follows from the condition \( (1 + \lambda_s \ln(2 \exp(\gamma \omega)/\pi T_c)) = 0 \), and we have \( \Delta_s \neq 0 \) and \( \Delta_s = 0 \) for \( s \neq s^* \). In the case of several negative eigenvalues \( \lambda_s < 0 \), one has to choose the solution that corresponds to the lowest eigenvalue as it gives the highest critical temperature.

Using Eq. (3) one finds that negative \( \lambda_s \) correspond to eigenfunctions \( \phi_s \) which are independent of the azimuthal angle \( \varphi \). This means that only spherical harmonics with zero projection \( m \) of the angular momentum \( \ell \) appear in their decomposition. For these functions the kernel \( R(0, \mathbf{n}; 0, \mathbf{n'}) \) can be reduced to its average over the azimuthal angles \( \varphi \) and \( \varphi' \). Using Eq. (3) for \( \Gamma_d(p_F \mathbf{n}, p_F \mathbf{n'}) \), we obtain
\[ R(0, \cos \theta; 0, \cos \theta') = 2\pi \frac{nd^2}{\varepsilon_F} \left( \frac{3}{2} |\cos \theta - \cos \theta'| - 1 \right), \tag{18} \]

where \( \theta \) and \( \theta' \) are the polar angles for the vectors \( \mathbf{n} \) and \( \mathbf{n}' \), and \( n \) is the gas density. Note that the first multiple in the rhs of Eq.(18) is a small parameter of the theory, given by Eq.(12) and representing the ratio of the mean dipole-dipole interaction energy to the Fermi energy \( \varepsilon_F \).

Keeping in mind that due to the Pauli principle only odd angular momenta are present, we obtain the solutions of Eq.(16):

\[ \phi_s(\mathbf{n}) = \sqrt{2} \sin \left( \frac{\pi}{2} (1 + 2s) \cos \theta \right), \quad \lambda_s = -\frac{nd^2}{\varepsilon_F} \frac{12}{\pi(1 + 2s)^2}. \tag{19} \]

The lowest eigenvalue is \( \lambda_0 = -12nd^2/\pi \varepsilon_F \). Therefore, the angular dependence of the order parameter will be characterized by the function \( \phi_0(\mathbf{n}) \) (see Section VI for details). The critical temperature is then given by

\[ T_c = \frac{2\exp(\gamma\omega)}{\pi} \exp \left( -\frac{1}{|\lambda_0|} \right). \tag{20} \]

In the BCS approach the pre-exponential factor (\( \overline{\omega} \)) remains undetermined. One can only argue that it is of the order of \( \varepsilon_F \). We thus have

\[ T_c^{BCS} \sim \varepsilon_F \exp \left( -\frac{\pi \varepsilon_F}{12nd^2} \right). \tag{21} \]

In Ref. [11] the exponent in the expression for \( T_c^{BCS} \) is only expressed in terms of the scattering amplitude which should be found from the solution of a set of coupled equations. The estimate for this exponent in Ref. [14] takes into account only the \( p - p \) scattering channel and contains a numerical error.

In order to find the pre-exponential factor one has to include the contribution from the second term in Eq.(14), together with the second order corrections to the eigenvalue \( \lambda_0 \). These corrections originate from the second order many-body effects and from the second order corrections to the scattering amplitude, described by the second terms in Eqs.(6) and (11) respectively.

V. GM APPROACH. THE CALCULATION OF THE PRE-EXPONENTIAL FACTOR

We now proceed with the calculation of the pre-exponential factor in the expression (21) for the critical temperature. We first consider the contribution of the second term in the rhs of Eq.(14), which is logarithmically small compared to the already calculated first term. For this purpose we specify the value of \( \overline{\omega} \) by the condition

\[ \int \frac{d\mathbf{n}}{4\pi} \phi_0(\mathbf{n}) \int_{-\varepsilon_F}^{\infty} d\xi' \ln |\xi'| \frac{d}{\overline{\omega}} \frac{d}{|\xi'|} \int \frac{d\mathbf{n}'}{4\pi} \{ R(0, \mathbf{n}; \xi', \mathbf{n}') \Delta(\xi', \mathbf{n}') \} = 0. \tag{22} \]

Then, using Eqs.(10) and (17) we obtain the following expression for \( \overline{\omega} \):

\[ \ln \overline{\omega} = -\frac{1}{\lambda_0} \int \frac{d\mathbf{n}}{4\pi} \phi_0(\mathbf{n}) \frac{1}{2} \int_{-\varepsilon_F}^{\infty} d\xi' \ln |\xi'| \frac{d}{d|\xi'|} \int \frac{d\mathbf{n}'}{4\pi} \left\{ R(0, \mathbf{n}; \xi', \mathbf{n}') \frac{\Delta(\xi', \mathbf{n}')}{\Delta_0} \right\}. \tag{23} \]

This definition of \( \overline{\omega} \) immediately leads to Eq.(21) for the critical temperature and allows us to rewrite Eq.(14) in the form

\[ \Delta(\xi, \mathbf{n}) = \frac{1}{\lambda_0} \int \frac{d\mathbf{n}'}{4\pi} R(\xi, \mathbf{n}; 0, \mathbf{n}') \Delta(0, \mathbf{n}') \]

\[ - \int \frac{d\mathbf{n}'}{4\pi} \frac{1}{2} \int_{-\varepsilon_F}^{\infty} d\xi' \left\{ R(\xi, \mathbf{n}; \xi', \mathbf{n}') \Delta(\xi', \mathbf{n}') - \frac{R(\xi, \mathbf{n}; 0, \mathbf{n}')}{\lambda_0} \frac{\Delta(0, \mathbf{n}')}{\Delta_0} \right\} \int \frac{d\mathbf{n}_1}{4\pi} \int \frac{d\mathbf{n}_2}{4\pi} \phi_0(\mathbf{n}_1) R(0, \mathbf{n}_1; \xi', \mathbf{n}_2) \Delta(\xi', \mathbf{n}_2), \tag{24} \]
where the second term in the rhs is proportional to the small parameter of the theory $n d^2/\varepsilon F$ and can thus be considered as a perturbation. This follows from the fact that the bracket in this term vanishes for $\xi' \to 0$. As a result, in contrast to the first term of the rhs, the second term does not contain the large logarithm $\ln(\varpi/T) \sim \lambda_0^{-1} \sim (\varepsilon_F/n d^2)$.

The leading contribution to the angular dependence of the order parameter on the Fermi surface comes from the term with $s = 0$ in Eq. (14): $\Delta(0, n) = \Delta_0 \phi_0(n)$. Therefore, to the leading order in $n d^2/\varepsilon F$, the solution of Eq. (24) is

$$\Delta(\xi, n) \approx \frac{1}{\lambda_0} \int \frac{d n'}{4\pi} R(\xi, n; 0, n') \Delta_0 \phi_0(n').$$

(25)

After substituting this expression into Eq. (23) and performing a numerical integration, we obtain

$$\varpi \approx \exp \left( - \frac{1}{\lambda_0} \int \frac{d n}{4\pi} \phi_0(n) \frac{1}{2} \int_{-\varepsilon F}^{\infty} d \xi' \ln |\xi'| \frac{d}{d \xi'} \left( \frac{1}{\lambda_0} \int \frac{d n'}{4\pi} \left\{ R(0, n; \xi', n') \Delta(\xi', n') \right\} \right) \right) = 0.42 \varepsilon F,$$

(26)

Corrections to Eqs. (25) and (26) are related to the terms with $s \neq 0$ in Eq. (16), and from Eq. (24) we find that the quantities $\Delta_{s \neq 0} \sim \Delta_0 (n d^2/\varepsilon F)$. These corrections lead to the relative contribution of the order of $n d^2/\varepsilon F$ to the pre-exponential factor for the critical temperature and hence will be neglected.

We now calculate the contributions in the rhs of Eqs. (3) and (11). As one can see from Eq. (16), these terms result in the correction for the eigenvalue $\lambda_0$:

$$\delta \lambda_0 = \nu(0) \int \frac{d n}{4\pi} \int \frac{d n'}{4\pi} \phi_0(n) \left\{ \delta V(p_F n, p_F n') - \int \frac{d q}{(2\pi)^3} V_d(p - q) K_0(q) V_d(q - p') \right\} \phi_0(n').$$

(27)

The first term in the integrand of Eq. (27) originates from many-body effects, and the quantity $\delta V(p, p')$ is shown diagrammatically in Fig. 1. The analytical expressions for the diagrams in Figures from 1a to 1d, read:

$$\delta V_a(p, p') = \int \frac{d q}{(2\pi)^3} \frac{N(q + p/2) - N(q - p/2)}{\xi_{q+p/2} - \xi_{q-p/2}} V_d^2(p_+),$$

$$\delta V_b(p, p') = - \int \frac{d q}{(2\pi)^3} \frac{N(q + p/2) - N(q - p/2)}{\xi_{q+p/2} - \xi_{q-p/2}} V_d(q - p_+),$$

$$\delta V_c(p, p') = - \int \frac{d q}{(2\pi)^3} \frac{N(q + p/2) - N(q - p/2)}{\xi_{q+p/2} - \xi_{q-p/2}} V_d(q + p_+),$$

$$\delta V_d(p, p') = - \int \frac{d q}{(2\pi)^3} \frac{N(q + p_+/2) - N(q - p_+/2)}{\xi_{q+p_+/2} - \xi_{q-p_+/2}} V_d(q - p_+/2) V_d(q + p_+/2).$$

Here $p_\pm = p \mp p'$, and $N(p)$ is the Fermi-Dirac distribution at zero temperature. The integrals related to the first term in the rhs of Eq. (27), with $\delta V(p, p') = \sum_{a=a,...,d} \delta V_a(p, p')$, were calculated numerically by using the Monte-Carlo method. Each of the terms $\delta V_a(p, p')$ provides a correction $\delta \lambda_0^{(a)} = (\nu(0)4\pi d^2/3)^2 \eta_a$. For the coefficients $\eta_a$ we find $\eta_a = 0.19$, $\eta_b = \eta_c = -0.08$, and $\eta_d = 0.42$. Thus, the first term in the rhs of Eq. (27) gives the correction

$$\delta \lambda_0^{(1)} = 0.45 \left( \frac{\nu(0)4\pi d^2}{3} \right)^2.$$

The second term in the integrand of Eq. (27) comes from the second order correction to the scattering amplitude $\Gamma_d$. For the correction to $\lambda_0$, originating from this term, our numerical calculation gives

$$\delta \lambda_0^{(2)} = -0.86 \left( \frac{\nu(0)4\pi d^2}{3} \right)^2.$$

Note that the function $\phi_0(n)$ is odd with respect to $\cos \theta$. For this reason, the integration over $d n$ and $d n'$ eliminates the formal divergency of the integral over $d q$ at large $q$.

The total correction to the eigenvalue $\lambda_0$ is then

$$\delta \lambda_0 = \delta \lambda_0^{(1)} + \delta \lambda_0^{(2)} = -0.41 \left( \frac{\nu(0)4\pi d^2}{3} \right)^2.$$

(28)

On the basis of Eqs. (24), (26) and (28), we obtain the final expression for the critical temperature:

$$T_c = \frac{2 \exp(\gamma)}{\pi} \times 0.42 \varepsilon F \exp(-1/|\lambda_0 + \delta \lambda_0|) \approx 1.44 \varepsilon F \exp(-\pi \varepsilon F / 12 n d^2).$$

(29)

It is worth noting that if we include only the $p - p$ scattering channel the exponent in Eq. (29) will be larger by a factor of $10/\pi^2$. The pre-exponential factor becomes then larger by a factor of 1.1.
VI. ANISOTROPIC ORDER PARAMETER

In order to find the temperature dependence of the order parameter for $T \leq T_c$, we have to include the nonlinear term in the gap equation \[ \theta = \frac{1}{(2\pi \hbar)^3} V_d(p - p') \left[ \frac{1}{\cosh^2(\xi'/2T)} - \frac{\tanh(\xi'/2T)}{\xi'/2T} \right] \Delta^3(p') \frac{8\xi'^2T}{8\xi'^2T}, \]

where we neglect the many-body correction to the interparticle interaction. The expression in the square brackets vanishes as $|\xi'|^{-3}$ for $|\xi'| \to \infty$. Therefore, the main contribution to the integral comes from the region of small $\xi'$, i.e. from $p'$ close to the Fermi momentum $p_F$. This allows us to write

\[
\int \frac{dp'}{(2\pi \hbar)^3} V_d(p - p') \left[ \frac{1}{\cosh^2(\xi'/2T)} - \frac{\tanh(\xi'/2T)}{\xi'/2T} \right] \Delta^3(p') \frac{8\xi'^2T}{8\xi'^2T} \approx \int \frac{dp'}{4\pi} V_d(p - p_F n') \Delta^3(p_F n') \int \frac{d\xi'}{2\pi^2h^3} \left[ \frac{1}{\cosh^2(\xi'/2T)} - \frac{\tanh(\xi'/2T)}{\xi'/2T} \right] \frac{1}{8\xi'^2T},
\]

where $\zeta(z)$ is the Riemann zeta function. As a result, to first order in $V_d$ the nonlinear gap equation reads:

\[
\Delta(\xi, n) = -\ln \frac{2\pi^2}{\pi T} \int \frac{d\xi'}{4\pi} R(\xi, n; 0, n') \Delta(0, n') + \frac{7\zeta(3)}{8\pi^2T^2} \int \frac{d\xi'}{4\pi} R(\xi, n; 0, n') \Delta^3(0, n').
\]

With the order parameter $\Delta(0, n)$ from Eq.\[\text{(32)}\] where now $\Delta_s = \Delta_s(T)$ and $\Delta_s(T) \to 0$ for $T \to T_c$, Eq.\[\text{(25)}\] takes the form

\[
\sum_{s=0}^{\infty} \Delta_s \left( 1 + \lambda_s \ln \frac{2\pi^2}{\pi T} \right) \phi_s(n) - \frac{7\zeta(3)}{8\pi^2T^2} \sum_{s=0}^{\infty} \lambda_s \phi_s(n) \left( \sum_{(s_i)} C_{s_1s_2s_3}^s \Delta_{s_1} \Delta_{s_2} \Delta_{s_3} \right) = 0.
\]

The coefficients $C_{s_1s_2s_3}^s$ follow from the relation

\[
\phi_{s_1}(n) \phi_{s_2}(n) \phi_{s_3}(n) = \sum_s C_{s_1s_2s_3}^s \phi_s(n).
\]

For temperatures below $T_c$, satisfying the inequality $(T_c - T)/T_c \ll 1$, Eq.\[\text{(31)}\] can be rewritten as

\[
\sum_{s=0}^{\infty} \Delta_s \left( \frac{\lambda_0 - \lambda_s}{\lambda_0} + \lambda_s \frac{T_c - T}{T_c} \right) \phi_s(n) - \frac{7\zeta(3)}{8\pi^2T^2} \sum_{s=0}^{\infty} \lambda_s \phi_s(n) \left( \sum_{(s_i)} C_{s_1s_2s_3}^s \Delta_{s_1} \Delta_{s_2} \Delta_{s_3} \right) = 0,
\]

where we neglect higher powers of $(T_c - T)/T_c$. It can be easily seen from Eq.\[\text{(32)}\] that for $T \to T_c$ one has $\Delta_0 \sim (T_c - T)^{1/2}$, and that $\Delta_s$ with $s > 0$ are either equal to zero or proportional to $(T_c - T)^{3/2}$. Therefore, the equation for the leading coefficient $\Delta_0$ is:

\[
\frac{T_c - T}{T_c} \Delta_0 - \frac{7\zeta(3)}{8\pi^2T^2} C_{000}^0 \Delta_0^3 = 0,
\]

where the coefficient $C_{000}^0$ is equal to $3/2$. We thus obtain the following expression for the order parameter on the Fermi surface ($\xi = 0$):

\[
\Delta(0, n) = \frac{4\pi}{\sqrt{21\zeta(3)}} T_c \sqrt{\frac{T_c - T}{T_c}} \phi_0(n) = 2.5 T_c \sqrt{\frac{T_c - T}{T_c}} \phi_0(n); \quad \frac{T_c - T}{T_c} \ll 1.
\]

For $\xi \neq 0$, i.e. $p \neq p_F$, the order parameter can be calculated by using Eq.\[\text{(25)}\]. Fig.2 shows the numerically calculated dependence of the order parameter on the modulus of the momentum $p$ for various values of the angle $\theta$ between the vector $p$ and the direction of dipoles. Note that for both $s$ and $p$-wave pairing due to a short-range
interaction, the order parameter is momentum independent for $p$ satisfying the condition of the ultra-cold limit and rapidly decays at larger $p$. The momentum dependence of the order parameter for dipolar gases results in a nonuniform energy gap for single-particle excitations and can, for example, manifest itself in processes with a large (of the order of $p_F$) momentum transfer.

\[ \frac{\Delta(p, \theta)}{\Delta(p_F, \theta)} \]

**FIG. 2.** The order parameter $\Delta(p, \theta)$ (in units of $\Delta(p_F, \theta)$) as a function of the momentum $p$ (in units of $p_F$) for various values of the polar angle $\theta$.

In Eq. (33) the anisotropy of the order parameter in the momentum space is described by the function $\phi_0(n) = \sqrt{2} \sin([\pi/2] \cos \theta)$. The order parameter is an odd function of $\cos \theta$ and is negative for $\pi/2 < \theta \leq \pi$. This does not cause any problems as all physical quantities are determined by $|\Delta|$. The maximum value of $|\Delta(0, n)|$ is reached in the direction of the dipoles, i.e. for $\theta = 0$ ($\phi_0 = \sqrt{2}$). In the direction perpendicular to the dipoles ($\theta = \pi/2$) the order parameter vanishes.

If we consider only the $p-p$ scattering channel, the angular dependence of the order parameter will be determined by the function $\sqrt{3} \cos \theta$ instead of $\phi_0(n)$. The coefficient $C_{000}^0$ is then equal to $9/4$, and the result for the order parameter reads:

\[ \Delta_{pp}(0, n) = \frac{4\pi}{3} \sqrt{\frac{2}{7\zeta(3)}} T_c \frac{T_c - T}{T_c} \sqrt{3} \cos \theta = 2.5 T_c \sqrt{\frac{T_c - T}{T_c}} \sqrt{2} \cos \theta; \quad \frac{T_c - T}{T_c} \ll 1. \quad (34) \]

The angular dependence of $|\Delta_{pp}(0, n)|$ is qualitatively similar to that of the true order parameter $|\Delta(0, n)|$. The maximum value of $|\Delta_{pp}(0, n)|$ is also reached in the direction of the dipoles and it is exactly equal to the maximum value of $|\Delta(0, n)|$. Also, $|\Delta_{pp}(0, n)|$ vanishes in the perpendicular direction. However, for intermediate values of $\theta$ the quantity $|\Delta_{pp}(0, n)|$ can be up to 40% smaller than $|\Delta(0, n)|$.

The anisotropy of the order parameter provides a major difference of the properties of the superfluid dipolar Fermi gas from those of the (two-component) fermionic gas with the s-wave pairing due to short-range intercomponent interaction. This anisotropy ensures the anisotropic momentum dependence of the gap in the spectrum of single-particle excitations, which appears below the transition temperature $T_c$. For example, excitations with momenta in the direction of the dipoles acquire the largest gap. In contrast to this, the eigenenergies of excitations with momenta perpendicular to the dipoles remain unchanged. The properties of collective excitations are also expected to have a nontrivial dependence on the direction of their momenta. Therefore, the response of the dipolar superfluid Fermi gas to small external perturbations will have a pronounced anisotropic character.

Another distinguished feature of the superfluid dipolar Fermi gas is related to the temperature dependence of the specific heat. Well below the critical temperature the single-particle contribution to the specific heat is proportional to $T^2$, rather than being exponentially small as in the case of the s-wave pairing. This follows from the fact that the energy $\varepsilon$ of single-particle excitations has a line of zeros on the Fermi surface: $\varepsilon(p_F) = 0$ for the angles at which $\Delta(p_F, n) = 0$, i.e. for $\theta = \pi/2$ and an arbitrary azimuthal angle $\varphi$. As a consequence, the density of states in the vicinity of the Fermi energy is $\nu(\varepsilon) \sim \varepsilon$ for $\varepsilon \ll \Delta_0$. Therefore, at temperatures $T \ll \Delta_0 \sim T_c$, the temperature dependent part of the energy of the system is proportional to $T^3$, and the specific heat is hence proportional to $T^2$. This contribution is much larger than the one of collective modes which is $\propto T^3$ and is dominant in the case of the s-wave pairing.

It should also be mentioned that the properties of the superfluid dipolar fermionic gas are different from the properties of the gas with the $p$-wave pairing originating from a short-range attractive interaction in the $p$-wave
channel. The reason is that in the latter case the order parameter is isotropic, similarly to the $B$-phase of superfluid $^3$He. The order parameter of dipolar gases is also different from the order parameter of the $A$ phase of $^3$He where the gap vanishes only at two points on the Fermi sphere, i.e. at the poles of the sphere $\theta = 0$ and $\theta = \pi$.

The anisotropy of the order parameter of dipolar gases is similar to that in the polar phase of superfluid liquid $^3$He, not realized in experiments as it has higher energy than experimentally observed $A$ and $B$ phases (see, e.g. Ref. [22]). For the polar phase the order parameter is also equal to zero on the equator of the Fermi sphere ($\theta = \pi/2$ and an arbitrary $\varphi$). The situation where the order parameter is zero on one or several lines on the Fermi surface is encountered in heavy-fermion compounds (for a review of possible superconducting phases of heavy-fermion compounds belonging to different crystalline groups see, e.g. Ref. [23]). In these cases the temperature dependence of the specific heat is also $\propto T^2$ (see, e.g. Ref. [24]). However, from a general point of view, one would expect a different physical behavior of dipolar gases, for example with regard to the frequency and angular dependence of the response. This is because of the different types of symmetry groups: continuous rotational group for dipolar gases and discrete crystalline group in the case of heavy-fermion materials (see Ref. [23] for more details).

VII. CONCLUDING REMARKS

Our results show prospects for achieving the BCS transition in single-component trapped gases of dipolar particles, in particular for (electrically polarized) fermionic polar molecules. As has been shown in Refs. [14,25,26], for trapped gases the BCS transition temperature is close to that of the 3D uniform gas with density $n$ equal to the maximum density in the trap. This requires $T_c$ to be much larger than the maximum trap frequency, which is generally the case for achievable temperatures. Therefore, we will use Eq. (29) for estimating $T_c$ in the trapped case.

We first compare our equations (21) and (29) with the well-known BCS formula (see e.g. [18]) for the two-component Fermi gas with short-range attractive intercomponent interaction. In the latter case the exponent is expressed in terms of the odd orbital angular momenta is equivalent to having the specific heat is realized in experiments as it has higher energy than experimentally observed.

Remarkably, in ultra-cold single-component fermionic gases one can hope to reach much higher densities than in Bose gases and think of achieving the BCS transition for a smaller effective scattering length.

For most polar molecules the electric dipole moment ranges from 0 to 1 Debye. For example, the dipole moment of fermionic ammonia molecules $^{15}$ND$_3$ is $d = 1.5$ D, and we have the effective scattering length is $a_d = -1450$ Å. This is larger than the scattering length for the intercomponent interaction in the widely discussed case of two-species fermionic gas of $^4$Li. From Eq. (5), we find that the BCS transition temperature for the single-component ND$_3$ dipolar gas will be larger than 100 nK at densities $n > 5 \cdot 10^{12}$ cm$^{-3}$. Another interesting example is a linear fermionic molecule HCN which has dipole moment $d = 2.98$ D, and the corresponding effective scattering length $a_d = -7400$ Å.

Remarkably, in ultra-cold single-component fermionic gases one can hope to reach much higher densities than in Bose gases and think of achieving the BCS transition for a smaller effective scattering length $a_d$. The reason is that inelastic decay processes will be strongly suppressed due to the Pauli exclusion principle. For two identical fermions with momentum $p$ of the relative motion, the pair correlation function behaves as $(pr/\hbar)^2$ at interparticle distances $r$ smaller than the de Broglie wavelength $\hbar/p$. Generally, inelastic processes occur at short interparticle distances $R_0$ which in the ultra-cold limit are much smaller than $\hbar/p$. Therefore, two-body inelastic collisions will be suppressed as $(pR_0)^2$ compared to the bosonic case where the pair correlation function is of order unity at any $r$ outside the region of interparticle interaction. As a result, in a non-degenerate gas of fermions the inelastic rate decreases with temperature and is suppressed as $T/\varepsilon_0$, where the energy $\varepsilon_0 = \hbar^2/(2mR_0^2)$. In a quantum degenerate Fermi gas a characteristic momentum of particles is of the order of $p_F$ and the suppression factor is $\propto (\varepsilon_F/\varepsilon_0)$. The suppression

\[ a_d = -\frac{2md^2}{\pi^2\hbar^2}. \] (35)

Accordingly, Eq. (29) takes the form

\[ T_c = 1.44\varepsilon_F \exp \left\{ -\frac{\pi\hbar}{2p_F |ad|} \right\}. \] (36)

Qualitatively, the result of Eq. (35) is more or less expected, since $|a_d|$ turns out to be of the order of the characteristic radius of the dipole-dipole interaction, $r_c \sim md^2/\hbar^2$, introduced in Section III.

For most polar molecules the electric dipole moment ranges from 0.1 to 1 Debye. For example, the dipole moment of fermionic ammonia molecules $^{15}$ND$_3$ is $d = 1.5$ D, and we have the effective scattering length is $a_d = -1450$ Å. This is larger than the scattering length for the intercomponent interaction in the widely discussed case of two-species fermionic gas of $^4$Li. From Eq. (5), we find that the BCS transition temperature for the single-component ND$_3$ dipolar gas will be larger than 100 nK at densities $n > 5 \cdot 10^{12}$ cm$^{-3}$. Another interesting example is a linear fermionic molecule HCN which has dipole moment $d = 2.98$ D, and the corresponding effective scattering length $a_d = -7400$ Å.

Remarkably, in ultra-cold single-component fermionic gases one can hope to reach much higher densities than in Bose gases and think of achieving the BCS transition for a smaller effective scattering length $a_d$. The reason is that inelastic decay processes will be strongly suppressed due to the Pauli exclusion principle. For two identical fermions with momentum $p$ of the relative motion, the pair correlation function behaves as $(pr/\hbar)^2$ at interparticle distances $r$ smaller than the de Broglie wavelength $\hbar/p$. Generally, inelastic processes occur at short interparticle distances $R_0$ which in the ultra-cold limit are much smaller than $\hbar/p$. Therefore, two-body inelastic collisions will be suppressed as $(pR_0)^2$ compared to the bosonic case where the pair correlation function is of order unity at any $r$ outside the region of interparticle interaction. As a result, in a non-degenerate gas of fermions the inelastic rate decreases with temperature and is suppressed as $T/\varepsilon_0$, where the energy $\varepsilon_0 = \hbar^2/(2mR_0^2)$. In a quantum degenerate Fermi gas a characteristic momentum of particles is of the order of $p_F$ and the suppression factor is $\propto (\varepsilon_F/\varepsilon_0)$. The suppression

1 Bosonic ($^{14}$ND$_4$) and fermionic ($^{15}$ND$_3$) ammonia molecules were recently trapped at temperatures of around 30 mK in the Rijnhuizen experiment [12].
of two-body inelastic collisions in fermionic gases was first found for spin relaxation in atomic deuterium [8]. For the rate of 3-body recombination we expect even a stronger suppression, i.e. by a factor of $(T/\varepsilon_0)^2$ for the non-degenerate gas, and by a factor of $(\varepsilon_F/\varepsilon_0)^2$ in the regime of quantum degeneracy ($T < \varepsilon_F$). The suppression factor for inelastic collisions in single-component quantum degenerate Fermi gases, $(\varepsilon_F/\varepsilon_0)$, is of the order of $(nR_0^3)^{2/3}$. The distance $R_0$ is commonly smaller than 50 Å. Therefore, even at densities $n \sim 10^{16}$ cm$^{-3}$ one expects the suppression of two-body inelastic rates at least by 2 orders of magnitude, and the suppression of 3-body recombination by 4 orders of magnitude. This allows us to think of achieving densities $n \sim 10^{16}$ cm$^{-3}$ or somewhat larger, which is by more than a factor of 10 higher than the densities currently reached with ultra-cold Bose gases.

For $n \sim 10^{16}$ cm$^{-3}$, the BCS transition temperature $T_c$ is in the nanokelvin or microkelvin regime for dipolar fermionic gases with an effective scattering length $|a_d|$ ranging from $-20$ to $-30$ Å. Such values of $a_d$ one easily finds in molecular gases. For example, fermionic $^{14}$N$^{16}$O molecules have dipole moment $d = 0.16$ D and Eq. (13) gives $a_d \approx -24$ Å. Interestingly, the effective scattering length $|a_d|$ of the order of several tens of angstroms can be obtained in gases of atoms with induced dipole moments [24,25]. By using a high dc electric field ($\sim 10^6$ V/cm) [9] one can induce permanent atomic dipole moments close to $0.1D$. The same values of $d$ one obtains for the time averaged dipole moment of an atom, induced by a stroboscopic laser coupling of the ground atomic state to a Rydberg state [25]. The corresponding scattering length $a_d$ can then be close to $-30$ Å, and one can think of achieving the BCS transition in such single-component atomic dipolar gases at densities $n \sim 10^{16}$ cm$^{-3}$ and temperatures $\sim 100$ nK.

**ACKNOWLEDGMENTS**

We acknowledge fruitful discussion with M. Lewenstein and G. Meijer. This work was financially supported by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO), by the Stichting voor Fundamenteel Onderzoek der Materie (FOM), by Deutsche Forschungsgemeinschaft (DFG), by the Alexander von Humboldt Stiftung, and by the Russian Foundation for Basic Research.
In the case of atoms with large magnetic moments one can also think of efficient elastic scattering due to the magnetic dipole interaction. For chromium atoms, the magnetic moment is \( 6 \mu_B \), where \( \mu_B \) is the Bohr magneton. This is equivalent to having the dipole moment \( d = 0.056 \text{ D} \), and Eq. (35) leads to \( a_d \approx -5 \text{ Å} \). Eq. (36) then shows that even at nanokelvin temperatures the BCS transition requires extremely high densities at which the gas will be unstable because of recombination processes.

The situation is likely to be same for fermionic \(^{13}\text{C}^{16}\text{O}\) molecules in the ground electronic state, which have dipole moment \( d = 0.1 \text{ D} \) and effective scattering length \( a_d \approx -9 \text{ Å} \). On the other hand, the dipole moment of the metastable molecule \(^{13}\text{C}^{16}\text{O}\) in the first electronically excited \( \alpha^3\Pi \) state is equal to \( 1.37 \text{ D} \), and the effective scattering length is \( a_d \approx -1670 \text{ Å} \). This opens interesting possibilities to (optically) manipulate the permanent dipole moment \( d \) and the scattering length \( a_d \) in \( \text{CO} \) molecular gases.