Fuzzy adaptive nonlinear stochastic control for vehicle suspension with electromagnetic actuator

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Abstract
This work solves the stability problem of a vehicle suspension with stochastic disturbance by designing an adaptive controller. The model of a quarter vehicle subjected to noise excitation is considered. The stochastic perturbation is realized by the roughness of the road and the vehicle moving with constant velocity. In the control design procedure, fuzzy logic systems are used to approximate unknown nonlinear functions. Meanwhile, the mean value theorem is employed to ensure the existence of the affine virtual control variables and control input. The backstepping technique is applied to construct the ideal controller. On the basis of Lyapunov stability theory, the proposed control method proves that the displacement and speed of the vehicle is reduced to a level ascertained by a true “desired” conceptual suspension reference model. Finally, the effectiveness of the proposed method is verified by simulation of electromagnetic actuator servo system.

Keywords
Active suspension systems, stochastic perturbation, fuzzy adaptive control, backstepping technique

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Introduction
In the past decade, the control design of vehicle suspension has received tremendous numbers of attention based on ride comfort and driving safety. The vehicle suspension system includes a connection and a damping device between the body and the wheels. Its main task is to attenuate and isolate various vibrations caused by rough roads, guaranteeing the stability of the vehicles and providing a more safety ride. Because the suspension system is closely related to the ride comfort and driving safety of the vehicle, the control for suspension systems is necessary. According to different control forms of the suspension system, it is divided into passive suspension, semi-active suspension and active suspension. Compared with semi-active suspension and passive suspension, active suspension system has great design potential in terms of output capability range, driving comfort and safety performance, so the control design of the vehicle active suspension is very hot.

In order to meet the increasing requirements on the safety of vehicle and ride comfort, many excellent control methods have been used to design the vehicle suspension systems. These control methods improve the control accuracy and vehicle maneuverability. Several $H_{\infty}$ control strategies have been proposed in previous studies\(^1\) to study the control problem of vehicle suspensions and the controllers have been designed. Among them, Cao et al.\(^3\) and Du and Zhang\(^4\) studied the robust $H_{\infty}$ control problem for uncertain linear systems with input delay. The robust $H_{\infty}$ control technology is applied to vehicle active suspension control under non-stationary running conditions in the study by Guo and Zhang.\(^5\) In addition, a design method of suspension nonlinear controller based on linear variable parameter control technology is proposed in the study by Fialho and Balas.\(^6\) In the case where the deflection of the suspension is relatively small relative to the structural limit, the passenger comfort is maximally improved. The optimal control problems of vehicle active suspension system with control delay are solved in the study by Bai and Lei\(^7\) and Yan et al.\(^8\) The gain-scheduling control is proposed in the study by Thompson and Pearce\(^9\) to improve the performance of a linear vehicle active suspension. The multi-objective framework problem was studied for a seven degrees-of-freedom decoupled vehicle active suspension system by

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Wang and Wilson. But the above control schemes of active suspension have the accurate model. It is difficult to implement on an actual vehicle suspension system with unknown and uncertain information. Hence, the adaptive control technique is employed to solve this effect. In the development of several decades, the adaptive control theory has been greatly improved and many excellent results have been achieved, which can effectively solve the problem of intelligent control of nonlinear systems. Sunwoo et al. investigated the adaptive control method for active suspensions with uncertain parameters by using adaptive approximation scheme so that the problem of the inaccurate model can be effectively solved. The steady-state and transient performance of nonlinear active suspension is guaranteed in the study by Na et al. and the adaptive controller is designed in the study by Na et al. In addition, the adaptive control performance of the active suspension systems is discussed in full-vehicle model, half-vehicle model and quarter-vehicle model, respectively. Many scholars also use backstepping control method to deal with the control problem of vehicle active suspension. Considering the nonlinear characteristics of springs and the piecewise linear characteristics of dampers, an adaptive backstepping control method is adopted in the study by Sun et al. so that the vehicle active suspension in hard constraints can be precisely controlled. Meanwhile, a constrained adaptive backstepping control strategy is adopted in the study by Sun et al., which achieve the multi-objective control of the vehicle active suspensions and make the closed-loop systems improve boulevard comfort, as well as the performance constraints of the active suspension system is realized.

But the active suspension systems with stochastic excitation were not considered in the above study. Stochastic disturbance is a common phenomenon in practical systems. Therefore, the study on stochastic control problem has great potential development prospects and has attracted widespread focus. Patrick first realized the global stability of stochastic nonlinear systems and developed the concepts of Sontag stability theory and Lyapunov function to the stochastic environment. A backstepping control scheme by using a quadratic Lyapunov function is addressed in the study by Deng and Kristić and Deng et al. and widely used to control several kinds of stochastic nonlinear systems, solving the stabilization and inverse optimal control problems of strict feedback systems. However, the above stochastic system stability theory requires that the nonlinear system satisfies the local Lipschitz condition. In order to relax the restrictions on the system, adaptive fuzzy control was the devout study by Li and Liu. Afterwards, the fuzzy adaptive method was widely applied to solve control problem of nonlinear strict-feedback systems and uncertain stochastic nonlinear systems. Moreover, the globally adaptive state-feedback controller is investigated in the study by Min et al. for a more general class of stochastic nonlinear systems with an unknown time-varying delay and perturbations. Meanwhile, for strongly interconnected nonlinear systems suffering stochastic disturbances, the output feedback decentralized control problem is solved by applying adaptive neural control scheme in the study by Wang et al. A reduced-order observer and a general fault model are investigated in the study by Ma et al. for stochastic nonlinear systems with actuator faults, which are applied to observe the unavailable state variables and describe the actuator faults. But there are few stochastic control methods used in vehicle suspension system.

To the best of our knowledge, there are few results in open literature on adaptive backstepping stochastic control for vehicle active suspension systems. Therefore, this paper turns to handle the fuzzy adaptive control problem for vehicle active suspension systems with stochastic disturbance. Considering the existence of stochastic disturbance problem of the system, a fuzzy adaptive controller has been proposed by applying backstepping strategy. The proposed fuzzy adaptive controller ensure that the displacement of the vehicle is small enough and tends to be stable. The major innovations of this work are listed as follows:

1. If the stochastic disturbance caused by the road surface cannot be handled, the performance of vehicle suspension will be greatly affected. In order to improve the applicable range of suspension system. This paper studies the adaptive fuzzy stochastic control problem for quarter vehicle active suspension systems with stochastic disturbance. By solving the stochastic disturbance of road surface, improve the driving comfort and safety performance.

2. Based on backstepping method, an adaptive fuzzy control strategy has been investigated for quarter vehicle active suspension system. The existence of the affine virtual control variables and control input is guaranteed by mean value theorem, and a novel adaptive compensation strategy is adopted to overcome the design difficulty for suspension system.

3. A good actuator can greatly improve the performance of the suspension. Compared with previous studies, this paper solves the problems in the electromagnetic suspension system, which exhibits a high efficiency and excellent servo characteristics.

Preliminaries and problem formulation

System descriptions

In this paper, the quarter-vehicle electromagnetic active suspension model is shown by Figure 1, in which $M$ is the sprung mass that represents the vehicle chassis; $m$ is the unsprung mass that represents the wheel assembly; $F_d$ represents the forces generated by spring and $F_i$ represents the forces generated by spring and $F_d$
represents the forces generated by damper, respectively; \( F_i \) and \( F_b \) represents the forces generated by the tire; \( u \) is the electromagnetic actuator input of the vehicle suspension system; \( s_1 \in R \) stand for the displacement of spring; \( s_2 \in R \) stand for the displacement of unsprung; and \( s_i \) is the displacement input of the road. The position of the vehicle is denoted by \( z \). In addition, \( F_e = k_1(s_1 - s_2) \), \( F_b = c_a (s_1 - s_2) \), \( F_i = k_2(s_2' - s_i') \), \( F_d = c_b(s_2 - s_i) \).

A great active suspension control system is inseparable from an actuator with excellent performance. The electromagnetic actuator considered in this paper has fast response and large braking force. The electromagnetic actuator circuit diagram is shown in Figure 2.

The force \( F_e \) from electromagnetic actuator is \( F_e = 2\pi T_m/p_b = Ai \), where \( T_m = K_Fi \) is the moment of force, the motor control voltage is \( u_a = U_3 - u_i = l d_i/dt + R_i \), the motor constant is \( A = 2\pi K_F/p_b \).

The dynamic equation of vehicle suspension is described by

\[
\begin{aligned}
Ms_1'' + k_1(s_1 - s_2) + c_a(s_1' - s_2') + U &= -Ms_2'' \\
ms_2'' - k_1(s_1 - s_2) - c_a(s_1' - s_2') - U + k_2s_2 \\
+ c_b s_2 &= k_2s_2 + c_b s_2
\end{aligned}
\]

(1)

Due to the unpredictability of the roughness of the road, we can obviously view it as a stochastic process. Following the studies by von Wagner \(^{35}\) and Litak et al. \(^{36}\) the roughness of the road is taken to be a spatial function obtained by passing a white noise

\[
\sigma_2 s_2' + \gamma_2 s_2 = \delta \zeta
\]

(2)

where white noise \( \zeta = dw/dz \) and the Wiener process, \( w \), is related to the coordinate, \( z \). The unknown smooth nonlinear function \( \delta \) satisfies

\[
\delta = \delta^* \sqrt{\alpha^3}
\]

(3)

where \( \delta^* \) is a constant and \( v = dz/dt \) is the speed of the vehicle. Note that the excitation process equation (2), which is a colored noise, limits the problem to dimension four. For \( k_2 = 0 \), the excitation becomes a white noise.

Using the abbreviations

\[
\begin{aligned}
\sigma_1 &= k_1/M, \sigma_2 = k_2/m, \sigma_3 = k_1/m, \gamma_1 = c_a/M, \gamma_2 = c_b/m, \gamma_1 = c_a/m
\end{aligned}
\]

(4)

the dynamic equation (1) is further rewritten as

\[
\begin{aligned}
s_1'' + (\sigma_1 + \sigma_2)(s_1 - s_2) + (\gamma_1 + \gamma_2)(s_1' - s_2') \\
- \sigma_2 s_2 - \gamma_2 s_2' + U &= -\delta \xi
\end{aligned}
\]

\[
\begin{aligned}
s_2'' + \sigma_2 s_2' - \gamma_2 s_2 = \gamma_3(s_1' - s_2') \\
- U &= \delta \xi
\end{aligned}
\]

(5)

with the transformations \( \xi = dv/dz \) and \( v = dz/dt \).

Defining the state variables \( x_1 = s_1 \), \( x_2 = s_1' \), \( x_3 = s_2 \), \( x_4 = s_2' \), \( x_5 = i \), one has

\[
\begin{aligned}
dx_1 &= x_2 dt \\
dx_2 &= [\sigma_2 x_3 + \gamma_2 x_4 - (\sigma_1 + \sigma_2)(x_1 - x_3) \\
- (\gamma_1 + \gamma_2)(x_2 - x_4) + Ax_5] dt - \delta \xi dt \\
dx_3 &= x_4 dt \\
dx_4 &= [\sigma_3(x_1 - x_3) + \gamma_3(x_2 - x_4) - \sigma_2 x_3 \\
+ \gamma_2 x_4 - Ax_5] dt + \delta \xi dt \\
dx_5 &= \frac{u - R x_5}{L} dt
\end{aligned}
\]

(6)

where \( x_1, x_2, x_3, x_4, x_5 \) are the states of system; \( x_1 \) and \( x_3 \) represent the displacement of the body and the tire of the vehicle, respectively; \( x_2 \) and \( x_4 \) are the speed of the body and tire of the vehicle, respectively; \( x_5 \) is the current of the electromagnetic actuator; \( u \in R \) and \( y_1 \in R \) are system input and measured output; \( \sigma_1, \sigma_2, \sigma_3 \) and \( \gamma_1, \gamma_2, \gamma_3 \) are unknown design parameters.

For the system electromagnetic vehicle suspension system given by equation (6), define \( \tilde{x}_i = [x_1, \ldots, x_i], i = 1, \ldots, 5 \) and \( \tilde{x} = [x_1, x_2, x_3, x_4, x_5] \), one has
\[ f_1(x) = \sigma_2 x_4 + \gamma_2 x_3 - (\sigma_1 + \sigma_2)(x_1 - x_3) - (\gamma_1 + \gamma_2)(x_2 - x_4) + Ax_5 \]
\[ f_2(x) = \sigma_3 (x_1 - x_3) - \gamma_3 (x_2 - x_4) - \sigma_2 x_3 + \gamma_2 x_4 - Ax_5 \]
\[ f_3(x, u) = u - Rx_5 \frac{L}{L} \]  
(7)

\[ \eta_1(x) = \frac{\partial f_1(x)}{\partial x_5} \]
\[ \eta_2(x) = \frac{\partial f_2(x)}{\partial x_5} \]
\[ \eta_3(x, u) = \frac{\partial f_3(x, u)}{\partial u} \]  
(8)

In order to get explicit virtual ones, one can express \( f_1(\cdot) \) with help of mean value theorem as follows:

\[ f_1(\bar{x}) = f_1(x_1, x_2, x_3^0, x_4, x_5) = \frac{\partial f_1(\cdot)}{\partial x_5} |_{x_5 = x_5^0} (x_5 - x_5^0) \]  
(9)

in which smooth function \( f_1(\cdot) \) is explicitly analyzed between \( f_1(\bar{x}) \) and \( f_1(x_1, x_2, x_3^0, x_4, x_5) \); \( x_3^0 \) is some point between \( x_3 \) and \( x_5^0 \); \( h \) is some constant satisfying \( 0 < h < 1 \).

Further, by choosing \( x_3^0 = 0 \), equation (9) is expressed as

\[ f_1(\bar{x}) = f_1(x_1, x_2, 0, x_4, x_5) + \frac{\partial f_1(\cdot)}{\partial x_3} |_{x_3 = x_3^0} (x_3 - x_3^0) \]
\[ = f_1(x_1, x_2, 0, x_4, x_5) - \eta_1(x_5) x_3 \]  
(10)

where

\[ \eta_1(x) = \eta_1(x_1, x_2, x_3^0, x_4, x_5) = - \frac{\partial f_1(x_3)}{\partial x_3} |_{x_3 = x_3^0} \]  
(11)

Similar to equation (10), one has

\[ f_2(\bar{x}) = f_2(x_1, x_2, x_3, x_4, x_5^0) + \eta_2(x_1, x_2, x_3^0, x_4, x_5) x_5 \]
\[ = f_2(x_1, x_2, x_3, x_4, 0) + \eta_2(x_5) x_5 \]  
(12)

\[ \eta_2(x) = \eta_2(x_1, x_2, x_3, x_4, x_5^0) = \frac{\partial f_2(x)}{\partial x_5} |_{x_5 = x_5^0} \]  
(13)

where \( x_5 \) is the estimation of \( x_5^0 \) and \( x_5^0 \) is some point between \( 0 \) and \( x_3 \).

The vehicle suspension system given by equation (6) can be rewritten as

\[
\begin{align*}
    \frac{dx_1}{dt} &= x_2 dt \\
    \frac{dx_2}{dt} &= (f_1(x_1, x_2, 0, x_4, x_5) - \eta_1(x_5) x_3) dt - \delta dw \\
    \frac{dx_3}{dt} &= x_4 dt \\
    \frac{dx_4}{dt} &= (f_2(x_1, x_2, x_3, x_4, 0) + \eta_2(x_5) x_5) dt + \delta dw \\
    \frac{dx_5}{dt} &= f_3(x_5, u) dt 
\end{align*}
\]  
(14)

**Control objective.** For electromagnetic active suspension systems with stochastic perturbation, the input \( u \) for the electromagnetic actuator is developed to guarantee that the vertical motion and the vertical of vehicle body are stable, respectively.

**Preliminary knowledge**

Consider a class of stochastic nonlinear systems described by the following differential equations

\[ dx = f(x) dt + \delta dw \]  
(15)

where \( x \in \mathbb{R}^n \) is the state of system, \( \omega \) is an \( r \)-dimensional independent standard Wiener process, \( f(x) : \mathbb{R}^n \to \mathbb{R}^n \) and \( \delta : \mathbb{R}^n \to \mathbb{R}^n \) are locally Lipschitz and satisfy \( f(x) = 0 \) and \( \delta(0) = 0 \).

**Definition 1.** For given \( U(x) \in \mathbb{C}^2 \), related to the stochastic differential equation (6), the infinitesimal generator \( L \) is defined as follows:

\[ LU = \frac{\partial U}{\partial x} f(x) + \frac{1}{2} \text{Tr} \left( \delta^T \frac{\partial^2 U}{\partial x^2} \delta \right) \]  
(16)

where \( \text{Tr}(\cdot) \) is the trace of matrix.

**Lemma 1.** For any \( x, y \in \mathbb{R}^n \), there exists an inequality such that

\[ x^T y \leq \frac{a}{p} \| x \|^p + \frac{1}{qa^q} \| y \|^q \]  
(17)

where \( a > 0 \), the constants \( p > 1 \) and \( q > 1 \) satisfy \((p-1)(q-1) = 1\).

**Lemma 2.** For the stochastic vehicle suspension system given by equation (12), let \( V : \mathbb{R} \to \mathbb{R} \) be a positive definite, radially unbounded, twice continuously differentiable Lyapunov function, then for any constants \( c > 0, D > 0 \), there exists

\[ LV \leq -cV + D \]  
(18)

then the system has a unique solution almost surely and the system is bounded in probability.

During the vehicle driving, both \( |f(\cdot)| > 0 \) and \( x_1 \) are bounded in a vehicle suspension system, the following assumption can be given.

**Assumption 1.** For the signs of \( \eta(\cdot) \), there exist constants \( \eta_{i,0}, \eta_{i,1} \) and \( \eta_{i,d} \) such that \( |\eta(\cdot)| \geq \eta_{i,0} > 0 \), \( |\eta_{i}(\cdot)| \leq \eta_{i,1} \) and \( |\eta_{i,d}(\cdot)| \leq \eta_{i,d} \). Without loss of generality, we shall assume that \( \eta(\cdot) \geq \eta_{i,0} > 0 \) (\( i = 1, 2, 3 \)).

**Fuzzy logic systems**

Since the active suspension given by equation (12) contains unknown continuous functions, the FLSs are
needed to approximate the nonlinearities. The property of FLSs is shown by the following Lemma

**Lemma 2.** On a compact set \( \Omega \subset \mathbb{R} \), the FLS \( \theta^T \xi(x) \) is applied to approximate a continuous function \( f(x) \), such that\(^{33}\)

\[
\sup_{x \in \Omega} |f(x) - \theta^T \xi(x)| \leq \tau
\]

where \( \tau \) is the bounded approximation error and \( \tau > 0 \); \( \xi(x) = [\xi_1(x), \ldots, \xi_M(x)]^T \) with \( \xi_i(x) \) are the radial basis functions, which have the form \( \xi_i(x) = \exp \left[ -\frac{(x - \mu_i)^T(x - \mu_i)}{\eta_i^2} \right] \), \( i = 1, \ldots, M \); \( \mu_i = [\mu_{i1}, \ldots, \mu_{il}] \) is the center of the receptive field and \( \eta_i \) is the width of the radial basis function; \( \theta^T = [\theta_1^T, \ldots, \theta_M^T]^T \) stands for the ideal constant weight vector and \( M \) is the number of the fuzzy rules.

**Adaptive control design**

Based on the change of coordinates, an adaptive fuzzy backstepping controller design strategy will be proposed for the electromagnetic active suspension systems with stochastic perturbation in equation (12). So, define the change of coordinates as follows

\[
e_1 = x_1 - y_d
\]

\[
e_i = x_i - \alpha_{i-1}
\]

where \( y_d \) is the tracking signal, \( \alpha_{i-1} \) is the virtual controller.

**Step 1:** From equation (20) and the first subsystem in equation (5), one gets

\[
de_1 = (x_2 - y_d)dt = (e_2 + \alpha_1 - y'_d)dt
\]

We first choose the Lyapunov function candidate

\[
V_1 = \frac{1}{4} e_1^4
\]

By Ito differential formula, one gets

\[
LV_1 = e_1^3 e_2 + \frac{1}{4} e_2^4
\]

From Lemma 1, one arrives at

\[
e_1^3 e_2 \leq \frac{3}{4} e_1^4 + \frac{1}{4} e_2^4
\]

The virtual controller \( \alpha_1 \) was designed as follows

\[
\alpha_1 = -c_1 e_1 - \frac{3}{4} e_1 + y'_d
\]

Substituting equations (25) and (26) into equation (24) results in

\[
LV_1 = -c_1 e_1^4 + \frac{1}{4} e_2^4
\]

**Step 2:** Since \( e_2 = x_2 - \alpha_1 \), from the second subsystem in equation (5), one gets

\[
de_2 = (f_1(x_1, x_2, 0, x_4, x_5) - \eta_1(x)x_3 - K\alpha_1)dt + \left( \delta - \frac{\partial\alpha_1}{\partial x_1} \right) dv
\]

\[
= \eta_1 \frac{\partial}{\partial x_1} f_1 - x_3 - \frac{\partial}{\partial x_1} K\alpha_1)dt + \left( \delta - \frac{\partial\alpha_1}{\partial x_1} \right) dv
\]

where

\[
K\alpha_1 = x_2 + \frac{1}{2} \delta^T \delta + \frac{\partial\alpha_1}{\partial x_1} y''_d
\]

Consider the Lyapunov function candidate

\[
V_2 = V_1 + \frac{1}{4\eta_1} e_2^4 + \frac{1}{2\lambda_1} \eta_1^2 + \frac{1}{2\lambda_1} \eta_1^2
\]

where \( \lambda_i > 0 \) and \( \lambda_i > 0 \) are design parameters; \( \eta_i \) is the estimation of \( \eta_i^* \) and \( \hat{\eta}_i = \eta_i^* - \hat{\eta}_i \); \( \hat{\eta}_i = \hat{\eta}_i - \eta_i^* \) is the estimation of \( \theta_i^* \) and \( \hat{\eta}_i = ||\theta_i^*||^2 \), \( i = 1, 2, 3, 4 \). By Ito differential formula and the definition of \( e_3 = x_3 - \alpha_2 \), one gets

\[
LV_2 = \frac{3}{2} e_2^2 \eta_1 + \frac{3}{2} \eta_1^2 \frac{\partial}{\partial x_1} \eta_1^2 + c_1 e_1^4 + \frac{1}{4} e_2^4
\]

\[
\leq \frac{3}{2} e_2^2 \eta_1 + \frac{3}{2} \eta_1^2 \frac{\partial}{\partial x_1} \eta_1^2 + c_1 e_1^4 + \frac{1}{4} e_2^4
\]

From Lemma 1, it is obtained that

\[
\frac{3}{2} e_2^2 \eta_1 + \frac{3}{2} \eta_1^2 \frac{\partial}{\partial x_1} \eta_1^2 \leq \frac{3}{4} \eta_1^2 ||\delta - \frac{\partial\alpha_1}{\partial x_1} \delta||^2
\]

\[
+ \frac{3}{4} \eta_1^2
\]

\[
e_2^2 \leq \frac{3}{4} e_2^4 + \frac{1}{4} e_1^4
\]

where \( \eta_1 \) is a positive constant. From equations (32) and (33), one gets
where \( \eta_1 \) and \( \eta_2 \) are unknown positive constants. Substituting equation (35) into equation (34) gives

\[
LV_2 = e_2^T \left( \eta_1^{-1} f_1 - \eta_2^{-1} K_1 + \frac{1}{2} e_2 + \frac{3}{4\pi^2} e_2 \| \delta - \frac{\partial \eta_2}{\partial \delta} \|^2 \right) \\
- \frac{1}{4} e_2^2 - \frac{3}{2\eta_1} \eta_2 e_2^2 - \frac{1}{\lambda_1} \delta_1 \vartheta_1' - \frac{1}{\lambda_1} \theta_1 (\vartheta_1' - c_1 \delta_1) + \frac{3}{4} \gamma^2
\]

(34)

By using the property \( \varphi_i(\cdot) \varphi_i(\cdot) \leq 1 \) of an integrator approximator and Young’s inequality, one has

\[
e_2^3 (\varphi_1'(\varphi_1(\cdot)) + \varphi_2'(\varphi_2(\cdot))) \leq \frac{\omega e_2^2 \varphi_1'(\varphi_1(\cdot))}{4} + \varphi_2'(\varphi_2(\cdot)) + \frac{\omega e_2^2 \varphi_1'(\varphi_1(\cdot))}{4} + \varphi_2'(\varphi_2(\cdot)) \leq \frac{\omega e_2^2 \varphi_1'(\varphi_1(\cdot))}{4} + \varphi_2'(\varphi_2(\cdot)) \leq \omega e_2^2 \varphi_1'(\varphi_1(\cdot)) \leq \frac{1}{4} e_2^2 + \omega e_2^2
\]

(37)

where \( \omega > 0 \) and \( \epsilon > 0 \) are design parameters. Consequently, substituting equations (37) and (38) into equation (36) gives

\[
LV_2 = e_2^T \left( - \alpha_2 - \frac{1}{2} e_2 + \frac{\omega}{2} e_2 \varphi_1'(\varphi_1(\cdot)) + \varphi_2'(\varphi_2(\cdot)) + \frac{1}{4} e_2^2 + \frac{3}{4} \gamma^2 \right)
\]

(38)

By equation (39), the virtual control law \( \alpha_2 \) and the parameter adaptive law of \( \vartheta_1 \) and \( \theta_1 \) can be designed as

\[
\alpha_2 = c_2 e_2 + e_1 e_2 - \frac{1}{2} e_2 + \frac{\omega}{2} e_2 \varphi_1'(\varphi_1(\cdot)) + \varphi_2'(\varphi_2(\cdot)) + \frac{1}{4} e_2^2
\]

(40)

\[
\vartheta_1' = \lambda_1 \phi_1(\vartheta_1(\cdot)) e_2^2 - \sigma_1 \vartheta_1
\]

(41)

\[
\Theta_1' = \frac{\lambda_1}{\lambda_2} \omega e_2^2 - \frac{1}{\lambda_2} \Theta_1
\]

(42)

where \( \sigma_1 > 0 \), \( \sigma_1 > 0 \), and \( \varrho_1 > 0 \) are design parameters, and \( \epsilon_1 = \eta_1/(2\eta_1) \).

Substituting equations (39)–(41) into equation (38), one gets

\[
LV_2 = -c_1 e_2^4 - c_2 e_2^4 - \frac{1}{2} e_2^2 + \frac{2}{\omega} e_2^2 + \frac{3}{4} \gamma^2
\]

(43)

According to Lemma 1, one gets

\[
\frac{\sigma_1}{\lambda_1} \vartheta_1 \vartheta_1 = \frac{\sigma_1}{\lambda_1} \vartheta_1 \vartheta_1 = -\frac{\sigma_1}{\lambda_1} \vartheta_1 + \frac{\sigma_1}{\lambda_1} \Theta_1
\]

(44)

\[
\frac{\sigma_1}{\lambda_1} \Theta_1 \Theta_1 = \frac{\sigma_1}{\lambda_1} \Theta_1 \Theta_1 = -\frac{\sigma_1}{\lambda_1} \Theta_1 + \frac{\sigma_1}{\lambda_1} \Theta_1
\]

(45)

Therefore, equation (41) can be rewritten as

\[
LV_2 = -c_1 e_2^4 - c_2 e_2^4 - \frac{1}{2} e_2^2 + D_1 - \frac{\sigma_1}{\lambda_1} \vartheta_1 + \frac{\sigma_1}{\lambda_1} \Theta_1
\]

(46)

where \( D_1 = \frac{\sigma_1}{\lambda_1} \vartheta_1 + \frac{\sigma_1}{\lambda_1} \Theta_1 \).

Step 3: The time derivative of \( e_3 \) is

\[
de_3 = (x_1 - 2\epsilon_2 + 2\epsilon_2^2 - \frac{3}{4} e_2^2) \frac{\partial \epsilon_2}{\partial x_2} dw
\]

(47)

Choosing the Lyapunov function \( V_3 \) as

\[
V_3 = V_3 + \frac{1}{4} e_3^2 + \frac{1}{\lambda_2} \vartheta_2^2 + \frac{1}{\lambda_2} \Theta_2^2
\]

(48)

By Itô differential formula, one gets

\[
LV_3 = LV_2 + e_3 \left( x_4 - 2\epsilon_2 + 2\epsilon_2^2 \right) \frac{\partial \epsilon_2}{\partial x_1} \frac{\partial \epsilon_2}{\partial x_1} + \frac{3}{4} e_2^2 \frac{\partial \epsilon_2}{\partial x_1} \frac{\partial \epsilon_2}{\partial x_1} - \frac{1}{\lambda_2} \vartheta_2^2 - \frac{1}{\lambda_2} \Theta_2^2
\]

(49)

Applying Lemma 2 to the last term in equation (47) shows

\[
\frac{3}{4} e_2^2 \frac{\partial \epsilon_2}{\partial x_2} \frac{\partial \epsilon_2}{\partial x_2} \frac{\partial \epsilon_2}{\partial x_1} \frac{\partial \epsilon_2}{\partial x_1} \frac{\partial \epsilon_2}{\partial x_1} \leq \frac{3}{4} e_2^2 \frac{\partial \epsilon_2}{\partial x_2} \frac{\partial \epsilon_2}{\partial x_2} \frac{\partial \epsilon_2}{\partial x_1} \frac{\partial \epsilon_2}{\partial x_1} \frac{\partial \epsilon_2}{\partial x_1} + \frac{3}{4} e_2^2
\]

(50)

where \( \epsilon_2 \) is a positive constant. Since \( -2\epsilon_2 + 2\epsilon_2^2 \frac{\partial \epsilon_2}{\partial x_2} \frac{\partial \epsilon_2}{\partial x_2} \frac{\partial \epsilon_2}{\partial x_1} \frac{\partial \epsilon_2}{\partial x_1} \frac{\partial \epsilon_2}{\partial x_1} \) is approximated by the FLS \( \varphi_2(\varphi_2(\cdot)) \), by Lemma 1, one obtains that

\[
\varphi_2(\varphi_2(\cdot)) + e_2 = -2\epsilon_2 + 2\epsilon_2^2 \frac{\partial \epsilon_2}{\partial x_2} \frac{\partial \epsilon_2}{\partial x_2} \frac{\partial \epsilon_2}{\partial x_1} \frac{\partial \epsilon_2}{\partial x_1} \frac{\partial \epsilon_2}{\partial x_1} + \frac{3}{4} e_2^2
\]

(51)
From equation (50) and the definition of \( e_4 = x_4 - \alpha_3 \), one gets
\[
LV_3 \leq LV_2 + e_1 (e_4 + \alpha_3 + e_2(x)) + \theta_2^T \phi_2(x) \\
\quad - \theta_2^T \phi_2(x)) \\
\quad + \frac{1}{\lambda_2} \frac{\partial \theta_2^2}{\partial x_3} + \frac{1}{\lambda_2} \frac{\partial \phi_2(x)}{\partial x_3} + \frac{3}{4} e_2^2 
\]
Similar to equation (37), one has
\[
ed_2^3 \phi_2(x) - \frac{\partial \phi_2(x)}{\partial x_3} \leq \frac{\omega c_2^4 \Theta^1_1}{2} + \frac{\omega c_2^2 \Theta^1_2}{2} + \frac{2}{\omega} e_2^2 
\]
Substituting equations (53)–(55) into equation (52) results in
\[
LV_3 \leq -c_1 e_1^2 - c_2 e_2^2 - \frac{1}{4} c_3^2 + D_1 + \frac{\sigma_1}{\Lambda_1} \frac{\partial \theta_1^2}{\partial x_3} \frac{\partial \phi_2(x)}{\partial x_3} + \frac{3}{4} e_2^2 
\]
where \( c_3 > 0, \sigma_2 > 0 \) and \( \sigma_2 > 0 \) are design parameters. Thus, substituting equations (57)–(59) into equation (56), gives
\[
\alpha_3 = -c_3 e_3 - \frac{\omega}{2} e_3 \Theta_2 - \frac{1}{2} e_3 \phi_2(x_3) - \frac{1}{4} e_3 + \frac{1}{2} e_3 
\]
By \( \dot{\alpha}_3 \) differential formula and the definition of \( e_3 = x_3 - \alpha_2 \), one gets
\[
LV_4 = LV_3 + \frac{1}{2} e_2^2 \left( \delta - \frac{\partial \alpha_3}{\partial x_2} \right) + \frac{2}{\lambda_3} \frac{\partial \alpha_3}{\partial x_3} \Theta_3 - \frac{e_2^2}{\lambda_2} e_1^2 - \frac{\lambda_2}{\lambda_3} e_2^2 - \sum_{i=1}^{3} \frac{\sigma_1}{\lambda_1} \frac{\partial \phi_2(x)}{\partial x_3} \Theta_2 - \sum_{i=1}^{3} \frac{\sigma_1}{\lambda_1} \frac{\partial \phi_2(x)}{\partial x_3} 
\]
From Lemma 1, one has
\[
\frac{3}{2} e_2^2 \left( \delta - \frac{\partial \alpha_3}{\partial x_2} \right) + \frac{2}{\lambda_3} e_1^2 \left( \delta - \frac{\partial \alpha_3}{\partial x_2} \right) \leq \frac{3}{4} e_2^2 \left( \delta - \frac{\partial \alpha_3}{\partial x_2} \right) + \frac{3}{4} e_2^2 
\]
where \( \sigma_2 > 0 \) is a positive constant. Substituting equation (67) into equation (66) gives
\[
LV_4 \leq e_2^2 \left( \delta - \frac{\partial \alpha_3}{\partial x_2} \right) + D_2 + \frac{3}{4} e_2^2 \left( \delta - \frac{\partial \alpha_3}{\partial x_2} \right) 
\]
According to Lemma 2, \( \hat{\eta}_2^{-1}f_2 - \hat{\eta}_2^{-1}K\alpha_3 + \frac{3}{4}\epsilon_4 \frac{\partial \mu_3}{\partial X_2} \delta ||^4 \) is approximated by the FLS \( \partial_3^T \varphi_3(\hat{x}) \), that is
\[
\hat{\eta}_2^{-1}f_2 - \hat{\eta}_2^{-1}K\alpha_3 + \frac{3}{4}\epsilon_4 \frac{\partial \mu_3}{\partial X_2} \delta ||^4 = \partial_3^T \varphi_3(\hat{x}) + e_3(\hat{x})
\]
(69)

Substituting equation (69) into equation (68) results in
\[
LV_4 \leq e_3^2(\alpha_3 + e_3 - \partial_3^T \varphi_3(\hat{x}) + \partial_3^T \varphi_3(\hat{x}4)) + e_3 + \frac{3}{4}e_4^2 - \frac{1}{\lambda_3} \partial_3 \theta_3' - \frac{\sigma_3}{\lambda_3} \Theta_3 - \sum_{i=1}^{\lambda_3} \sigma_i^2 \Theta_i - e_3^2 e_4^2 + D_2
\]
(70)

By using the property \( 0 < \varphi_3'(\cdot) \varphi_3'(\cdot) \leq 1 \) of intelligent approximator and Young's inequality, one has
\[
e_3^2(\alpha_3 - \partial_3^T \varphi_3(\hat{x}) - \partial_3^T \varphi_3(\hat{x}4)) \leq \frac{\sigma_3}{\lambda_3} \Theta_3(\Theta_3' - \partial_3 \theta_3' - \frac{\sigma_3}{\lambda_3} \Theta_3)^2 + \frac{3}{4}e_4^2 - \frac{1}{\lambda_3} \partial_3 \theta_3' - \sum_{i=1}^{\lambda_3} \sigma_i^2 \Theta_i - e_3^2 e_4^2 + D_2
\]
(71)

Substituting equations (71)–(73) into equation (70) yields
\[
LV_4 \leq e_3^2(\alpha_3 - \partial_3^T \varphi_3(\hat{x}) - \partial_3^T \varphi_3(\hat{x}4)) + e_3 + \frac{3}{4}e_4^2 - \frac{1}{\lambda_3} \partial_3 \theta_3' - \sum_{i=1}^{\lambda_3} \sigma_i^2 \Theta_i - e_3^2 e_4^2 + D_2
\]
(74)

By equation (39), the virtual control law \( \alpha_4 \) and the parameter adaptive law of \( \theta_3 \) and \( \Theta_3 \) can be designed as
\[
\alpha_4 = c_4 e_4 + e_2 e_4 + \frac{1}{\lambda_3} \lambda_3 \Theta_3 e_3 - \sigma_3 \Theta_3 \]
(75)
\[
\theta_3' = \frac{\lambda_3}{2} \Theta_3 e_3 - \sigma_3 \theta_3 \]
(76)
\[
\Theta_3' = \frac{\lambda_3}{2} e_4 + \sigma_3 \Theta_3 \]
(77)

where \( c_4 > 0, e_2 > 0, \sigma_3 > 0 \) and \( \sigma_3 > 0 \) are design parameters, and \( e_2 \nearrow \eta_{2, d}/(\eta_{2, d}^2) \).

Substituting equations (75)–(77) into equation (74), one gets
\[
LV_4 \leq -4 \sum_{i=1}^{\lambda_3} \sigma_i e_i^2 - \frac{2}{\lambda_3} \sigma_3 \partial_3 \theta_3' - \frac{2}{\lambda_3} \sigma_3 \Theta_3^2 + D_2
\]
(78)

According to Lemma 1, one gets
\[
\frac{\sigma_3}{\lambda_3} \theta_3(\partial_3 \varphi_3(\hat{x}) - \partial_3 \varphi_3(\hat{x}4)) \leq -4 \sum_{i=1}^{\lambda_3} \sigma_i e_i^2 - \frac{2}{\lambda_3} \sigma_3 \partial_3 \theta_3' - \frac{2}{\lambda_3} \sigma_3 \Theta_3^2 + D_2
\]
(79)

Substituting equations (79) and (80) into equation (78) results in
\[
LV_4 \leq -4 \sum_{i=1}^{\lambda_3} \sigma_i e_i^2 - \frac{2}{\lambda_3} \sigma_3 \partial_3 \theta_3' - \frac{2}{\lambda_3} \sigma_3 \Theta_3^2 + D_3 - \frac{1}{4} e_4^2
\]
(81)

where \( D_3 = \frac{3}{4} e_4^2 + \frac{3}{\omega} e_i^2 + \frac{3}{\lambda_3} e_i^2 + \frac{3}{\lambda_3} e_i^2 + \frac{3}{\lambda_3} e_i^2 \).

Step 5: The time derivative of \( e_5 \) is
\[
des_5 = (f_3(x_5, u) + \gamma)dt + \left( -\frac{\partial \alpha_4}{\partial x_5} - \frac{\partial \alpha_4}{\partial x_4} \right) dw
\]
(82)

where \( \gamma = -K\alpha_4 \).

From Assumption 1, we know that \( \partial f_3(x, u)/\partial u \geq \eta_{1,0} > 0 \). From equation (40), we can obtain that \( \gamma \) is not a function of \( u \), thus we have \( \gamma = -\eta_{1,0} > 0 \). Subsequently, we can get \( |\partial f_3(x_5, u) + \gamma|/\partial u \geq \eta_{1,0} > 0 \). For every value of \( x_5 \) and \( u \), there exists a smooth ideal control input \( \theta^* = \alpha_4(x_5, \gamma) \) such that \( f_3(x_5, \alpha_4) = 0 \). Based on the mean value theorem, there exists \( \lambda (0 < \lambda < 1) \) such that
\[
f_3(x_5, u) = f_3(x_5, u^*) + g_4(u - \alpha_4)
\]
(83)

where \( g_4 = \eta_{4}(x_5, u^*) \) with \( u^* = \lambda u + (1 - \lambda)\alpha_4 \).

Note that Assumption 1 on \( \eta_{4} \) is still valid for \( g_4 \). From equations (79) and (80), one has
\[
des_5 = g_4(u - \alpha_4)dt + \left( -\frac{\partial \alpha_4}{\partial x_5} - \frac{\partial \alpha_4}{\partial x_4} \right) dw
\]
(84)

Choose the Lyapunov function candidate as follows
\[
v_5 = V_4 + \frac{1}{4} e_5^2 + \frac{1}{2\lambda_3} \Theta_3^2
\]
(85)
where $\theta_{4,1}$ is the estimate of $\theta_{4,1}^* = \eta_{3,1} \| \theta_{4,1}^* \|^2$. From equations (81) and (82), one has

$$LV_5 = - 4 \sum_{i=1}^{4} e_i e_i^T - \sum_{i=1}^{4} \sigma_i \frac{\partial \phi_i}{\partial \theta_{4,1}} \frac{\partial \theta_{4,1}}{\partial \phi_i} - \sum_{i=1}^{4} \frac{\sigma_i}{2 \lambda_i} \Phi_i^2 + D_3 - \frac{1}{4} e_i^2$$

$$- \frac{1}{4 \lambda_4} \Phi_{4,1}^2 + e_3^T g_4 (u - \alpha)$$

$$+ e_3^T g_4 (u - \alpha) = \frac{3}{2} e_5^T \left( \frac{\partial \alpha_4}{\partial x_2 \delta} + \frac{\partial \Phi_4}{\partial x_2 \delta} \right) \left( \frac{\partial \Phi_4}{\partial x_2 \delta} + \frac{\partial \Phi_4}{\partial x_4 \delta} \right)$$

According to Lemma 1, the following inequality can be obtained

$$\frac{3}{2} e_5 e_5^T \left( \frac{\partial \alpha_4}{\partial x_2 \delta} + \frac{\partial \Phi_4}{\partial x_2 \delta} \right) \left( \frac{\partial \alpha_4}{\partial x_2 \delta} + \frac{\partial \Phi_4}{\partial x_4 \delta} \right) \leq \frac{3}{4 \tau_4^2} e_5^2 \left| \frac{\partial \alpha_4}{\partial x_2 \delta} + \frac{\partial \Phi_4}{\partial x_4 \delta} \right|^4 + \frac{3}{4} \tau_4^2$$

FLS $\Phi_i \Phi_i(\delta)$ is used to approximate

$$\alpha_4^2 + \frac{3}{4 g_4 \tau_4} e_5^T \left( \frac{\partial \alpha_4}{\partial x_2 \delta} + \frac{\partial \Phi_4}{\partial x_4 \delta} \right) \left| \frac{\partial \alpha_4}{\partial x_2 \delta} + \frac{\partial \Phi_4}{\partial x_4 \delta} \right|^4$$

which results in

$$- e_5 g_4 \left( \alpha_4^2 + \frac{3}{4 \tau_4} e_5^T \left( \frac{\partial \alpha_4}{\partial x_2 \delta} + \frac{\partial \Phi_4}{\partial x_4 \delta} \right) \left| \frac{\partial \alpha_4}{\partial x_2 \delta} + \frac{\partial \Phi_4}{\partial x_4 \delta} \right|^4 \right)$$

$$= e_5^T g_4 (\Phi_i^T \Phi_i(x) + \lambda^2 (\delta))$$

$$\leq e_5^T g_4 (\Phi_i^T \Phi_i(x) + 1) + \frac{e_5^T g_4 (\Phi_i^T \Phi_i(x) + 1)}{4 \mu} + \frac{e_5^T g_4 (\Phi_i^T \Phi_i(x) + 1)}{2} + \frac{e_5^T g_4 (\Phi_i^T \Phi_i(x) + 1)}{2}$$

where $|e_5(x)| \leq \varepsilon_4, \varepsilon_4$ is an unknown constant.

Consequently, from equations (86) and (87) one gets

$$LV_5 = - 4 \sum_{i=1}^{4} e_i e_i^T - \sum_{i=1}^{4} \sigma_i \frac{\partial \phi_i}{\partial \theta_{4,1}} \frac{\partial \theta_{4,1}}{\partial \phi_i} - \sum_{i=1}^{4} \frac{\sigma_i}{2 \lambda_i} \Phi_i^2 + D_3$$

$$- \frac{2}{4} \tau_4^2 - \frac{1}{4 \lambda_4} \Phi_{4,1}^2 + e_5^T (g_4 u + e_5 g_4 \eta_3 \phi_4^2 (\delta) \Phi_4 (x) - \frac{1}{4 e_5})$$

$$+ \frac{1}{4 \mu} + \frac{e_5^T g_4 (\Phi_i^T \Phi_i(x) + 1)}{2} + \frac{e_5^T g_4 (\Phi_i^T \Phi_i(x) + 1)}{2}$$

Design the actual control law as

$$u = - e_5 e_5 - \frac{1}{4} e_5 e_5 + \frac{1}{4} e_5^2 - \mu \frac{\partial \phi_i}{\partial \theta_{4,1}} \frac{\partial \theta_{4,1}}{\partial \phi_i}$$

$$\theta_{4,1}^1 = \lambda \frac{\partial \phi_i}{\partial \theta_{4,1}} \eta_3 \phi_4^2 (\delta) \Phi_4 (x) - \sigma \theta_{4,1}$$

where $e_5 > 0, e_5 > 0, \sigma_4 > 0$ and $\sigma_4 > 0$ are design parameters, and $\frac{1}{4} \eta_3 > 0$.

According to Wang et al., $\eta_3 \phi_4^2 (\delta) \Phi_4 (x)$ is a positive function and $\sigma_4$ is a positive constant, for any given bounded initial condition $\theta_{4,1}(t) \geq 0$, we have $\theta_{4,1}(t) \geq 0$ for $\forall t \geq t_0$. According to equations (90) and (91) and Assumption 1, equation (89) can be rewritten as

$$LV_5 = - 4 \sum_{i=1}^{4} e_i e_i^T - \sum_{i=1}^{4} \sigma_i \frac{\partial \phi_i}{\partial \theta_{4,1}} \frac{\partial \theta_{4,1}}{\partial \phi_i} - \sum_{i=1}^{4} \frac{\sigma_i}{2 \lambda_i} \Phi_i^2$$

$$+ D - \eta_3 \phi_4^2 (\delta) \Phi_4 (x)$$

where

$$D = D_3 + \frac{\sigma_4 \theta_{4,1}^T}{2 \lambda_4} + \frac{1}{4 \mu} + \frac{\eta_3 \phi_4^2 (\delta)}{2}$$

Let $\Pi = \min \{2e_1, \ldots, e_5, 2 \eta_3 \phi_4^2 (\delta) \phi_4^2 (\delta), \sigma_4 \phi_4^2 (\delta), \sigma_4 \phi_4^2 (\delta), \sigma_4 \phi_4^2 (\delta) \}$. Then, we have

$$LV_5 \leq - \Pi V_5 + D$$

Therefore, equation (93) can be further rewritten as

$$0 \leq LV_5(t) \leq D + LV_5(0) e^{-\Pi t}$$

From equation (94), it can be shown that all the signals are bounded and the tracking error satisfies that $|e_5| \leq \sqrt{2 LV_5(0) e^{-\Pi t} + D/\Pi}$. Meanwhile, we reduce the tracking error by choosing the appropriate design parameters $e_i, e_i, \sigma_i, \sigma_i, \omega, \alpha, \lambda_i$ and $\eta_3, i = 1, \ldots, 5$. Therefore, it is proved that the vehicle suspension system given by equation (2) is stable and the movement limitation can be fulfilled.

**Simulation study**

At this point, an active suspension system simulation example is proposed to validate the effectiveness and feasibility of the designed control strategy. The vehicle active suspension system in equation (3) was chosen. The quarter-vehicle model parameters are considered in Table 1.

From Table 1, it is obtained that $F_i = 750(x_1 + x_3), F_d = 15,000,000(x_2 - x_4), F_i = 20(x_3 - s), F_d = 2300(x_4 - s), F_u = 65.71$.

The fuzzy membership functions are chosen as

$$\mu_{F_1}(x) = \exp \left[- \frac{(x_1 + 9)^2}{2} \right], \mu_{F_2}(x) = \exp \left[- \frac{(x_1 + 7)^2}{2} \right]$$

$$\mu_{F_3}(x) = \exp \left[- \frac{(x_1 + 5)^2}{2} \right], \mu_{F_4}(x) = \exp \left[- \frac{(x_1 + 3)^2}{2} \right]$$

$$\mu_{F_5}(x) = \exp \left[- \frac{(x_1 + 1)^2}{2} \right], \mu_{F_6}(x) = \exp \left[- \frac{(x_1 - 3)^2}{2} \right]$$

$$\mu_{F_7}(x) = \exp \left[- \frac{(x_1 - 5)^2}{2} \right], \mu_{F_8}(x) = \exp \left[- \frac{(x_1 - 7)^2}{2} \right]$$

$$j = 1, 2, 3, 4, 5$$
Define the design parameters and adaptive laws as:

\[
c_1 = 20, \quad c_2 = 80, \quad c_3 = 1, \quad c_4 = 0.1, \quad c_5 = 0.1, \quad \dot{c}_1 = 3, \quad \dot{c}_2 = 1, \quad \dot{c}_3 = 0.001, \quad \sigma_1 = 100, \quad \sigma_2 = 10, \quad \sigma_3 = 300, \quad \lambda_1 = 1, \quad \lambda_2 = 0.2, \quad \lambda_3 = 60, \quad \lambda_4 = 0.1, \quad \lambda_5 = 0.2, \quad \delta = 1, \quad k = 16, \quad \tau = \omega = 0.5, \quad \Theta_2 = \Theta_3 = \Theta_4 = \Theta_5 = 1, \quad x_1(0) = 0.02,
\]

and the initial states of other variables are selected as zero. On the other hand, the reference signal \( y_d = 0 \) is selected.

When there are stochastic perturbances in the electromagnetic active suspension system, the displacements of the sprung masses \( x_1 \) are shown in Figure 3; the speed of the sprung masses \( x_2 \) is shown in Figure 4; the displacements of the unsprung masses \( x_3 \) are shown in Figure 5; the speed of the unsprung masses \( x_4 \) is shown in Figure 6; the current of electromagnetic actuator \( x_5 \) is shown in Figure 7; and the electromagnetic actuator control force of suspension \( u \) is shown in Figure 8.

The simulation results are given by Figures 3–8. In Figures 3 and 5, when the electromagnetic actuator works, the displacement of vehicle body vertical and wheel vertical for active suspension systems gradually tends to a stable point. Meanwhile, in Figures 4 and 6, the speed of the vehicle body and the wheel tend to be stable. The current and the control force of
electromagnetic actuator also can be stabilized in a small neighborhood of zero in Figures 7 and 8.

Conclusion

This study has addressed the adaptive backstepping control issue for active electromagnetic suspension system on a road surface with random disturbance. By solving the stochastic disturbance of the road surface, the boulevard comfort and driving safety are improved. The adaptive control law of electromagnetic actuator has been developed by adopting the backstepping technique, Itô differential formula and Lyappunov function theory, which stabilized the body vertical displacements and speed in a short time. It has been shown that the displacement and speed of the body and suspension are stabilized in a small neighborhood of zero. One possible future research work is to extend the suspension control problem to finite-time control. By planning a special reference trajectory, the vertical vibration and displacements of vehicle suspension can be stabilized in specified time.

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References

1. Tamboli JA and Joshi SG. Optimum design of a passive suspension system of a vehicle subjected to actual random road excitations. J Sound Vib 1999; 219(2): 193–205.
2. Naudé AF and Snyman JA. Optimisation of road vehicle passive suspension systems. Part 2. Qualification and case study. Appl Math Model 2003; 27(4): 263–274.
3. Cao YY, Sun YX and Lam J. Delay-dependent robust $H_{\infty}$ control for uncertain systems with time-varying delays. IEEE Proc: Control Theory Appl 1998; 145(3): 338–344.
4. Du H and Zhang N. $H_{\infty}$ control of active vehicle suspensions with actuator time delay. J Sound Vib 2007; 301(1): 236–252.
5. Guo LX and Zhang LP. Robust $H_{\infty}$ control of active vehicle suspension under non-stationary running. J Sound Vib 2012; 331(26): 5824–5837.
6. Fialho JJ and Balas GJ. Design of nonlinear controllers for active vehicle suspensions using parameter-varying control synthesis. Vehicle Syst Dyn 2000; 33(5): 351–370.
7. Bai XL and Lei J. Internal model-based optimal vibration control for linear vehicle suspension systems with actuator delay. Ferroelectrics 2019; 549(1): 195–203.
8. Yan G, Fang M and Xu J. Analysis and experiment of time-delayed optimal control for vehicle suspension system. J Sound Vib 2019; 446: 144–158.
9. Thompson AG and Pearce CEM. Performance index for a preview active suspension applied to a quarter-vehicle model. Vehicle Syst Dyn 2001; 35(1): 55–66.
10. Wang J and Wilson DA. Mixed $CL_2/H_2$/$G_2$ control with pole placement and its application to vehicle suspension systems. Int J Control 2001; 74(13): 1353–1369.
11. Liang H, Zhang L, Sun Y, et al. Containment control of semi-Markovian multiagent systems with switching topologies. IEEE Trans Syst Man Cybern Syst. Epub ahead of print 28 October 2019. DOI: 10.1109/ismc.2019.2946248.
12. Zhou Q, Du P, Li H, et al. Adaptive fixed-time control of error-constrained pure-feedback interconnected nonlinear systems. IEEE Trans Syst Man Cybern Syst. Epub
13. Sunwoo M, Cheok K and Huang N. Model reference adaptive control for vehicle active suspension systems. *IEEE Trans Ind Electron* 1991; 38(3): 217–222.

14. Na J, Chen Q, Ren XM, et al. Adaptive prescribed performance motion control of servo mechanisms with friction compensation. *IEEE Trans Ind Electron* 2014; 61(1): 486–494.

15. Na J, Huang Y, Wu X, et al. Active adaptive estimation and control for vehicle suspensions with prescribed performance. *IEEE Trans Control Syst Technol* 2018; 26(6): 2063–2077.

16. Sun WC, Pan HH and Gao HJ. Filter-based adaptive vibration control for active vehicle suspensions with electrohydraulic actuators. *IEEE Trans Veh Technol* 2015; 65(6): 4619–4626.

17. Sun WC, Zhao ZL and Gao HJ. Saturated adaptive robust control for active suspension systems. *IEEE Trans Ind Electron* 2012; 60(9): 3889–3896.

18. Sun WC, Gao HJ and Yao B. Adaptive robust vibration control of full-vehicle active suspensions with electrohydraulic actuators. *IEEE Trans Control Syst Technol* 2013; 21(6): 2417–2422.

19. Sun WC, Gao HJ and Kaynak O. Adaptive backstepping control for active suspension systems with hard constraints. *IEEE/ASME Trans Mechatron* 2012; 18(3): 1072–1079.

20. Sun WC, Pan HH, Zhang Y, et al. Multi-objective control for uncertain nonlinear active suspension systems. *Mechatronica* 2014; 24(4): 318–327.

21. Patrick F. A universal formula for the stabilization of control stochastic differential equations. *Stoch Anal Appl* 1993; 11(2): 155–162.

22. Patrick F. Lyapunov-like techniques for stochastic stability. *SIAM J Control Optim* 1995; 33(4): 1151–1169.

23. Deng H and Kristi’c M. Stochastic nonlinear stabilization—part II: inverse optimality. *Syst Control Lett* 1997; 32(3): 151–159.

24. Deng H, Kristi’c M and Williams RJ. Stabilization of stochastic nonlinear systems driven by noise of unknown covariance. *IEEE Trans Automat Contr* 2001; 46(8): 1237–1253.

25. Li FZ and Liu YG. Global stability and stabilization of more general stochastic nonlinear systems. *J Math Anal Appl* 2014; 413(2): 841–855.

26. Chen B, Liu XP, Liu KF, et al. Fuzzy approximation-based adaptive control of nonlinear delayed systems with unknown dead-zone. *IEEE Trans Fuzzy Syst* 2014; 22(2): 237–248.

27. Ocheriah S. Robust nonlinear adaptive control of a DC-DC boost converter with uncertain parameters. *Int J Innov Comput Inf Control* 2015; 11(3): 893–902.

28. Li YM, Tong SC and Li TS. Adaptive fuzzy output feedback dynamic surface control of interconnected nonlinear pure-feedback systems. *IEEE Trans Cybern* 2015; 45(1): 138–149.

29. Chen WS, Jiao LC, Li J, et al. Adaptive NN backstepping output-feedback control for stochastic nonlinear strict-feedback systems with time-varying delays. *IEEE Trans Syst Man Cybern B Cybern* 2010; 40(3): 939–950.

30. Wang HQ, Chen B, Liu KF, et al. Adaptive neural tracking control for a class of non-strict-feedback stochastic nonlinear systems with unknown backlash-like hysteresis. *IEEE Trans Neural Netw Learn Syst* 2014; 25(5): 947–958.

31. Su XJ, Wu LG, Shi P, et al. A novel approach to output feedback control of fuzzy stochastic systems. *Automatica* 2014; 50(12): 3268–3275.

32. Min HF, Xu S, Zhang B, et al. Globally adaptive control for stochastic nonlinear time-delay systems with perturbations and its application. *Automatica* 2019; 102: 105–110.

33. Wang HQ, Liu PX, Bao J, et al. Adaptive neural output-feedback decentralized control for large-scale nonlinear systems with stochastic disturbances. *IEEE Trans Neural Netw Learn Syst* 2020; 31: 972–983.

34. Ma H, Li H, Liang H, et al. Adaptive fuzzy event-triggered control for stochastic nonlinear systems with full state constraints and actuator faults. *IEEE Trans Fuzzy Syst* 2019; 27: 2242–2254.

35. Von Wagner U. On non-linear stochastic dynamics of quarter vehicle models. *Int J Nonlinear Mech* 2004; 39(5): 753–765.

36. Litak G, Borowiec M, Friswell MI, et al. Chaotic vibration of a quarter-vehicle model excited by the road surface profile. *Commun Nonlinear Sci Numer Simul* 2008; 13(7): 1373–1383.

37. Du HB, Shao HH and Yao PJ. Adaptive neural network control for a class of low-triangular-structured nonlinear systems. *IEEE Trans Neural Netw* 2006; 17(2): 509–514.

38. Li YM, Tong SC and Li TS. Observer-based adaptive fuzzy tracking control of MIMO stochastic nonlinear systems with unknown control directions and unknown dead zones. *IEEE Trans Fuzzy Syst* 2014; 23(4): 1228–1241.

39. Sui S, Tong SC and Li YM. Observer-based fuzzy adaptive prescribed performance tracking control for nonlinear stochastic systems with input saturation. *Neurocomputing* 2015; 158(22): 100–108.

40. Fan HJ, Han LX, Wen CY, et al. Decentralized adaptive output-feedback controller design for stochastic nonlinear interconnected systems. *Automatica* 2012; 48(11): 2866–2873.

41. Zhang TP and Ge SS. Adaptive dynamic surface control of nonlinear systems with unknown dead zone in pure feedback form. *Automatica* 2008; 44(7): 1895–1903.

42. Wang M, Liu XP and Shi P. Adaptive neural control of pure-feedback nonlinear time-delay systems via dynamic surface technique. *IEEE Trans Syst Man Cybern B Cybern* 2011; 41(6): 1681–1692.

43. Shi WX. Adaptive fuzzy control for MIMO nonlinear systems with nonsymmetric control gain matrix and unknown control direction. *IEEE Trans Fuzzy Syst* 2014; 22(5): 1288–1300.

44. Wang HQ, Chen B, Liu XP, et al. Robust adaptive fuzzy tracking control for pure-feedback stochastic nonlinear systems with input constraints. *IEEE Trans Cybern* 2013; 43(6): 2093–2104.