Percolation Perspective on Sites Not Visited by a Random Walk in Two Dimensions

Amit Federbush and Yacov Kantor

1 Raymond and Beverly Sackler School of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel

(Dated: January 19, 2021)

We consider the percolation problem of sites on an \( L \times L \) square lattice with periodic boundary conditions which were unvisited by a random walk of \( N = uL^2 \) steps, i.e., are vacant. Most of the results are obtained from numerical simulations. Unlike its higher-dimensional counterparts, this problem has no sharp percolation threshold and the spanning (percolation) probability is a smooth function monotonically decreasing with \( u \). The clusters of vacant sites are not fractal but have fractal boundaries of dimension 4/3. The lattice size \( L \) is the only large length scale in this problem. The typical mass (number of sites \( s \)) in the largest cluster is proportional to \( L^2 \), and the mean mass of the remaining (smaller) clusters is also proportional to \( L^2 \). The normalized (per site) density \( n_s \) of clusters of size \( s \) is proportional to \( s^{-\tau} \), while the volume fraction \( P_k \) occupied by the \( k \)th largest cluster scales as \( k^{-q} \). We put forward a heuristic argument that \( \tau = 2 \) and \( q = 1 \). However, the numerically measured values are \( \tau \approx 1.83 \) and \( q \approx 1.20 \). We suggest that these are effective exponents that drift towards their asymptotic values with increasing \( L \) as slowly as \( 1/\ln L \) approaches zero.

I. INTRODUCTION

Percolation theory \[1, 2\] provides a statistical description of long-range connectivity in lattices or networks when some of their sites or links have been removed. First emerging in the context of polymer sciences \[3, 4\] and spread of a fluid through a porous medium \[5\], this theory remains a very active field of research with very diverse applications, ranging from topography \[6\], epidemiology \[7, 8\], gelation and colloids science \[9–12\], environmental \[13\] and urban \[14\] studies, through more abstract networks \[15–18\] and more. In this work we consider a two-dimensional case of a variant of percolation problem, where an initially full lattice has its sites removed by a single meandering random walk (RW). The three-dimension version of the problem models a degradation of a gel by single enzyme, or very few enzymes, that break the crosslinks they encounter \[19, 20\].

A percolating system can be characterized by its level of its occupation, such as fraction \( p \) of sites present on the lattice, or occupied volume fraction in a continuous system. (In this work we consider site percolation on hypercubic lattices, but the results equally well apply to lattice bonds or mixed site-bond problems and other types of lattices.) In the case of lattice percolation the geometry can be viewed as a collection of clusters formed by neighboring (“connected”) occupied sites. A cluster is spanning if it forms a continuous path between opposing boundaries in a specific direction. For an infinite system, the emergence of a spanning cluster can be characterized as a phase transition: there exists a sharp percolation threshold \( p_c \), such that for \( p > p_c \) there exists an infinite spanning cluster. Both above and below \( p_c \) the mean spatial extent (linear size) of finite clusters is called correlation length \( \xi \). It diverges near the threshold as \( \xi \sim |p - p_c|^{-\nu} \), where the universal exponent \( \nu \) is independent of microscopic details of the model, but does depend on the dimensionality \( d \) of the system and, possibly, the presence of long-range correlations. The universality of the critical exponents allows application of the results of simple models to more realistic and complicated cases.

One of the simpler percolation models is Bernoulli site percolation on a \( d \)-dimensional lattice where each lattice site is independently occupied with probability \( p \). The exponent \( \nu \) of the Bernoulli problem decreases from \( \nu_B = 1 \) at the lower critical dimension \( d = 1 \) to \( \nu_B = 1/2 \) for \( d \geq d_c = 6 \), i.e., at and above the upper critical dimension \( d_c \). The generalized Harris criterion \[22\] has been used to show \[23\] that percolation models with short-range correlations or with power-law correlations \( \sim 1/\nu_b \) with large power \( b \) also belong to the Bernoulli percolation universality class. However, if \( b < 2/\nu_0 \), then the correlations are relevant, and \( \nu = 2/b \). There is a variety of studies of correlated percolation \[24, 28\].

In space dimension \( d \) we can consider an initially full hypercubic lattice of linear size \( L \) (in lattice constants) and number of sites \( M = L^d \) whose sites are being removed by an \( N \)-step RW that started at a random position. Periodic boundary conditions are imposed, i.e., the walker exiting through one boundary of the lattice re-emerges on the opposite boundary. The number of steps \( N \) of the walker is proportional to the volume of the system, i.e., \( N = uL^d \), with parameter \( u \) controlling the length of the walk. In the case of gel of crosslinked polymers the random walker represents an enzyme that breaks the crosslinks of a gel that it encounters \[13, 20\].

The object of the study are the vacant sites not visited by the random walker, that represent the surviving crosslinks. The variable \( u \) controls the concentration of vacant sites and naturally replaces \( p \) used in the regular percolation. For \( 3 \leq d \leq 6 \) infinite clusters of vacant sites sites appear for \( u \) below similar threshold values \( u_c \approx 3 \). Banavar et al. studied geometry of the clus-
ters created by the vacant sites in \( d = 2 \) and \( d = 3 \) \cite{30}, while Abete et al. considered the critical behavior near the percolation threshold in \( d = 3 \) \cite{31}. More recently Kantor and Kardar studied the percolation properties of the problem for \( 2 \leq d \leq 6 \) \cite{29}.

Sites visited by an \( N \)-step RW on an infinite lattice are strongly correlated. The final position of such a walk is a distance \( r \approx a N^{1/2} \) away from its starting point, where \( a \) is the lattice constant. This means that number of steps (“mass”) \( N \sim r^d \), and the fractal dimension \cite{32} of a RW is \( d_f = 2 \) independently of the embedding dimension \( d \). Therefore, we may expect our problem to behave differently in \( d = 2 \) than at higher \( d \). A RW can traverse a finite lattice of linear size \( L \) in \( \sim L^2 \) steps, and therefore a walk of \( uL^d \) steps traverses (“crosses”) the lattice

\[
N_{cr} \approx uL^{d-2} \tag{1}
\]
times. For \( d \geq 3 \) the increase in lattice size \( L \) increases \( N_{cr} \), while a strand of RW on every single “crossing” leaves sparser “footprints” on the lattice. (The total density of visited or vacant sites in \( d \geq 3 \) remains independent of \( L \) and depends only on \( u \).) This makes the “thermodynamic limits” somewhat peculiar even for \( d \geq 3 \) since with the increase of \( L \), the structure of the system changes rather than having more similar pieces being added to it.

On an infinite lattice the density of sites visited by an \( N \)-step RW (for \( d \geq 3 \)) within the distance visited by the walk is \( N/r^d \sim 1/r^{d-2} \). On a finite lattice, the sites belonging to different strands of RW created due to periodicity of the lattice are almost uncorrelated. The repeated “crossings” only contribute to uncorrelated density of sites. However, the correlation is preserved for \( r \) smaller than the lattice size for sites situated on the same strand of the RW. Consequently, the cumulant of the correlation (from which the overall background density has been subtracted) has the same power law relation. Consider a random variable \( v(\vec{x}) \) which is 1 if the site at position \( \vec{x} \) is vacant, and 0 otherwise. It is complementary to the variable representing the visited site (their sum is 1) and therefore, it has the same cumulant: \( \langle v(\vec{x})v(\vec{y}) \rangle_c \sim 1/|\vec{x} - \vec{y}|^{d-2} \) \cite{29}. Thus, for correlation power \( b = d - 2 \), the correlation length exponent \( \nu = 2/b \) \cite{32} becomes

\[
\nu = 2/(d-2), \quad \text{for} \quad 3 \leq d \leq 6. \tag{2}
\]

The ubiquitous factor “\( d - 2 \)” appearing in the above discussions, such as in Eqs. \( \text{(1)} \) and \( \text{(2)} \), or expressions for the correlation functions, indicates that many of the arguments presented for \( d \geq 3 \) will change in the two-dimensional case. In this work we focus on the vacant site properties in \( d = 2 \) from the point of view of percolation theory. Despite the absence of percolation threshold, the system has many scaling properties that resemble critical phenomena. In Sec. \( \text{III} \) we point out the unusual features of \( d = 2 \), and begin a discussion of two-dimensional vacant site (unvisited by a RW) percolation (2DVSP) at a point where Ref. \( \text{29} \) left off. Furthermore, in Sec. \( \text{III} \) we explain the main properties that set 2DVSP apart from vacant site percolation in higher dimensions and verify the absence of a sharp percolation threshold in \( d = 2 \). In Sec. \( \text{III} \) we consider the mean sizes of of the largest cluster and other clusters, and demonstrate the role played by the lattice size \( L \) in the description of the system. Our results show that \( L \) is the main length scale of the problem, which replaces the correlation length \( \xi \) of other percolation problems. In particular, we show that the spanning cluster volume and the mean volume of finite clusters both scale as \( L^2 \). In Sec. \( \text{IV} \) we focus on the geometry of the spanning cluster and demonstrate that the large clusters in \( d = 2 \) are not fractal, contrary to the results of the previous study \cite{30}. Nevertheless, the boundaries of those clusters are fractal with \( d_f = 4/3 \). In Sec. \( \text{V} \) we study in detail the cluster statistics in 2DVSP. The results of our numerical measurements exhibit scale-free behavior resembling regular percolation, but with the lattice size \( L \) replacing \( \xi \). We put forward a heuristic argument describing cluster statistics and the effective exponents measured numerically are close to the proposed theoretical values. We summarize and point out directions of future research in Sec. \( \text{VI} \).

II. HOW SPECIAL IS \( d = 2? \)

Since the fractal dimension of RW is 2, it “almost” fills the two-dimensional embedding space: In \( d = 2 \) the number of distinct sites visited by an \( N \)-step RW on an infinite square lattice increases for long walks as \( N_{\text{dist}} = \pi N/\ln N \) \cite{33}, i.e., slightly slower than \( N \), because the walk is recurrent and keeps revisiting previously visited sites an ever increasing number of times. (In contrast, at \( d \geq 3 \) the number of distinct visited sites is asymptotically proportional to \( N \) \cite{34,35} since the number of repeated visits approaches a constant \cite{36,37}.)

The presence of the logarithmic correction in \( N_{\text{dist}} \) has consequences for a RW of \( N = uL^2 \) steps on a finite square lattice of linear size \( L \) (0 \( \leq x_1, x_2 \leq L - 1 \) ) and volume \( M = L^2 \). We assume periodic boundary conditions in both directions, i.e., the coordinate \( x_i = L \) coincides with \( x_i = 0 \) for \( i = 1,2 \). On such a lattice, it has been proven \cite{38} (see also \cite{29}) that the mean fraction of unvisited (vacant) sites on the lattice for large \( L \) is

\[
p = \exp\left( -\frac{\pi u}{2\ln L} \right). \tag{3}
\]

(For a finite square lattice of \( M \) sites, a RW needs on the average \( 1/2M \ln^2 M \) steps to visit all the sites of the lattice \cite{39,40}. However, this limit corresponds to \( u \to \infty \) in our problem, while we are concerned only with finite \( u \).)

In Eq. \( \text{(3)} \) the fraction \( p \) of vacant sites depends on \( L \) and it deprives us of a simple correspondence between \( p \) and \( u \), which is present in \( d \geq 3 \), where \( p = \exp(-A_d u) \) with some constant \( A_d \) \cite{29,40}. However, we note that
for fixed $u$ in $d = 2$ in the $L \to \infty$ limit, we have $p \to 1$. Fig. 1 depicts RWs with the same $u$ for three different $L$s, demonstrating the tendency of increasing $p$ with increasing $L$. For very large $L$s we can treat the clusters of vacant sites as being separated by “thin regions” of RWs. Since, Eq. (3) is valid only for asymptotically for large $L$, we measured numerically the mean fraction of vacant sites $p$ for fixed $u = 1.3$ and increasing $L$. The results depicted in the semilogarithmic plot in Fig. 2 and at such scale Eq. (3) should be represented by a straight line. We see that the Eq. (3) is satisfied already for $L \sim 100$. However, the limit of $p = 1$ is approached slowly: for large $L$ the dependence is $p \approx 1 - \pi u/2 \ln L + \ldots$. Even for $L = 512$ and $u = 1.3$, the fraction of vacant sites $p \approx 0.72$, while for $L = 10^6$ we only have $p \approx 0.86$, and the regime of $p \approx 1$ is numerically inaccessible to us.

When Bernoulli percolation is formulated on, say, a $d$-dimensional hypercubic lattice, the dimensionless percolation threshold $p_c$ is reached when a sufficient number of sites are added to an empty lattice or a sufficient number is removed from a full lattice. In continuum percolation this corresponds to a finite fraction of the system volume being occupied or removed. When a percolating situation is created by a RW removing parts of the system, we may separately consider the length $\ell$ of the single step of the RW and the size of the volume $a^d$ occupied by a certain position $\vec{r}$, of the step of the RW. On a lattice the “size” of the site $a$ and the length $\ell$ of the step are both assumed to be equal to lattice constant. Thus, at $d \geq 3$ a short $N$-step RW occupies a volume proportional to $Na^d$ on a lattice of volume $(aL)^d$, and the percolation threshold of vacant sites is reached when this ratio reaches a particular numerical value. If the RW performs $N = uL^d$ steps, then (for short walks) this ratio is simply proportional to $u$, while for larger $u$ the fraction of vacant sites $p = \exp(-A_2 u)$. This means that for $d \geq 3$ there should exist a critical value $u_c$. This has been proven theoretically and demonstrated numerically.

It has been mentioned in Sec. II that for $d \geq 3$ the increase of $L$ for fixed $u$ causes the increase in lattice “crossings” by the RW as expected from the $L$-dependence of $N_c$, in Eq. (1). In $d = 2$ the situation is very different: On one hand, the value of $u$ no longer solely determines the fraction of occupied or vacant sites due to $L$-dependence in Eq. (3). On the other hand, the number of lattice “crossings” by the RW only depends on $u$ and not on the lattice size $L$. The probability of having a spanning cluster in, say, vertical direction depends on the ability of the RW to create a continuous path blocking vertical connection and is independent of the two-dimensional “volume” configuration.
occupied by the RW. In the absence of a lattice the presence of “blocking path” will depend on the typical step size $\ell$ of the RW rather than volume (area) $a^2$ occupied by each position.

Numerically, we consider site percolation on a periodic square lattice of $L^2$ sites. A random walker starts at an arbitrary site and performs $N = uL^2$ steps with $u = O(1)$. If there is a continuous path of vacant sites (unvisited by the RW) that connects the top and bottom boundaries $x_2 = 0$ and $x_2 = L - 1$, we say that the configuration is spanning (percolating). Figure 3 depicts the spanning probability $\Pi$ as a function of $u$, for lattice sizes $L$ ranging from 4 to 512. (The steps visible on the graph for $L = 4$ are a result of truncating $uL^2 = 16u$ to an integer.) We see that the graphs of $\Pi(L, u)$ converge as $L$ increases with the limit being a smooth function $\Pi(\infty, u)$, thus indicating that there is no percolation threshold. (Graphs in Fig. 3 are similar to the results in Ref. [29]). This differs from systems with a sharp percolation threshold ($p_c = 1$) and various properties that can be calculated analytically. Besides being a relatively simple problem, Bernoulli percolation for $p < 1$ in $d = 1$ has a finite correlation length $\xi$ that simply depends on $p$, and the system becomes homogeneous beyond that length scale. We shall see that 2DVSP does not have such a length scale, and its structure keeps changing with the increase of the system size $L$.

III. MEAN CLUSTER SIZES

Most quantitative features of percolating systems are extracted from the shapes and sizes of clusters of neighboring sites. We identify the clusters of vacant sites (unvisited by RW) using a Hoshen–Kopelman algorithm [45], which efficiently groups the sites into clusters in a single pass through the lattice. To generate each configuration we consider a RW meandering on a lattice of linear size $L$ with periodic boundary conditions in both $x_1$ (“horizontal”) and $x_2$ (“vertical”) directions. However, for the purpose of cluster identification, we assume that only the $x_1$ coordinate is periodic, i.e., the clusters can connect through the right and left edges of the lattice, while the $x_2$ coordinate is not periodic and clusters cannot connect through the bottom and top edges. The “vertical” ($x_2$) direction is used to identify the spanning clusters that connect top ($x_2 = L - 1$) and bottom ($x_2 = 0$) boundaries. The configurations in our simulation are generated by RWs of length $uL^2$. For each $u$ and $L$ pair (in a broad range of values) we simulate a large number of independent realizations, and for each realization we identify the clusters. We note, that in Bernoulli percolation it can be shown that usually (in $d \leq 6$) the infinite cluster is unique at the threshold, but for finite $L$ we may accidentally have few spanning clusters although such occupancies decrease as a negative power of $L$. In 2DVSP each configuration is generated by a single continuous RW which tends to cre-
The equality of all possible RWs corresponding to a given configuration. Clearly, the ensemble averaging over step. Thus the derivative, least one "bottleneck" that will be pinched off at the next step. Therefore, we can interpret it as an average of a random variable $\Pi$. We denote the probability that a given site belongs to the largest cluster from the statistics resembles a similar definition in Bernoulli percolation [1]. However, in the latter case, it serves as a technical tool to exclude the infinite cluster above the percolation threshold, and plays a negligible role below the threshold where the size of the clusters is limited by $\xi$ and for large $L$ there are many clusters of similar sizes. Thus, in "thermodynamic limit" of Bernoulli percolation the quantity $n_s$ is simply a function describing the prevalence of finite clusters.

We denote the probability that a given site belongs to the largest cluster in the system as the largest cluster strength $P$. This is an ensemble average of the number of sites in the largest cluster divided by $L^2$. In Bernoulli percolation [1] in the $L \to \infty$ limit, below the percolation threshold $P \to 0$ since the largest cluster is finite, while above the threshold $P$ is finite and represents the volume fraction of the infinite cluster. (Therefore, $P$ is used as an order parameter in many percolation problems.) Moreover, in Bernoulli percolation the concept of infinite cluster and spanning cluster coincide. This is not the case in our problem. Due to the absence of a percolation threshold, the strength of the largest cluster $P$ includes both spanning and non-spanning clusters. In fact there is no significant difference in the volumes of both types of largest clusters and the volumes of all of them are proportional to $L^2$ leading to a finite $P$. Clearly, $P \leq p$ but this is a weak bound since $p \to 1$ as $L$ increases. Fig. 4 depicts the largest cluster strength as a function of $u$ for lattice sizes $L$ ranging from 4 to 512. (As in Fig. 3 the steps for $L = 4$ are due to truncation of $uL^2$ to an integer value.) As expected, $P(L,0) = 1$ since the entire lattice is a single (largest) cluster, and the function monotonically decreases with increasing $u$. We observe that the graphs in Fig. 4 converge as $L \to \infty$ to a smooth function $P(\infty, u)$, confirming that the volume of the largest cluster scales as $L^2$. This observation will play an important role in Sec. IV. The convergence of the curves to $P(\infty, u)$ is significantly slower than the convergence of $\Pi$ in Fig. 3 since it is influenced by the slow approach of $p$ to unity, as indicated by Eq. (3): The analysis of the data for a single value of $u = u^*$ for larger $L$s shows some weak (but linear!) dependence of $P$ on $1/\ln L$ indicating that the asymptotic value is by some 3% higher than the value for $L = 512$.

The statistics of smaller clusters are of great interest in percolation problems. We define all the clusters except the largest cluster as finite clusters. We denote by $N_s$ the number of finite clusters with volume $s$ (number of sites) in a particular configuration on a lattice, and define the mean normalized cluster number $n_s = \langle N_s \rangle / L^2$, where $\langle \rangle$ denotes average over realizations. The exclusion of the largest cluster from the statistics resembles a similar definition in Bernoulli percolation [1]. However, in the latter case, it serves as a technical tool to exclude the infinite cluster above the percolation threshold, and plays a negligible role below the threshold where the size of the clusters is limited by $\xi$ and for large $L$ there are many clusters of similar sizes. Thus, in "thermodynamic limit" of Bernoulli percolation the quantity $n_s$ is simply a function describing the prevalence of finite clusters. In 2D VSP the largest cluster always has its number of sites proportional to $L^2$ as do large "finite" clusters. From the definition of $n_s$ we see that $sn_sL^2$ is the total number of sites belonging to clusters of size $s$ and therefore obtain the identity

$$\sum_s sn_s = p - P. \quad (4)$$

Unlike the case of Bernoulli percolation, the function $n_s$ may depend on $L$ and it is not evident that a limiting

![FIG. 4. Largest cluster strength $P$ dependence on RW length parameter $u$ for $L = 4, 8, \ldots, 512$ (bottom-left to top-right). Each point is an average of $5 \times 10^5$ configurations and the points are separated by $\Delta u = 0.05$.](image-url)
function exists for large $L$. We thoroughly discuss this function in Sec. IV.

The function $n_s$ can be used to determine the mean size of finite clusters: The total number of lattice sites which belong to an $s$-cluster is $s N_s$, and the total number of vacant sites is $p L^2$. Therefore $s n_s / p$ is the probability that a randomly selected vacant site belongs to a finite $s$-cluster, and the mean finite cluster size $\tilde{S}$ is

$$\tilde{S} = \sum_s s^2 n_s / p. \quad (5)$$

This definition of the mean cluster size differs from a similar definition in Ref. [1] only by the exclusion of the largest cluster in each configuration in the definition of $n_s$.

In regular percolation the mean size of finite clusters is controlled by the correlation length $\xi$, which increases as the percolation threshold is approached. However, for $L \gg \xi$ the distribution $n_s$ becomes independent of $L$. In our problem $\tilde{S} \sim L^2$, and therefore Fig. 5 depicts the ratio $\tilde{S} / L^2$ as a function of $u$ for lattice sizes $L$ ranging from 4 to 512. (As in the previous figures, the steps in the $L = 4$ graph are due to truncation to integer $N$.) The curves vanish for $u = 0$ since the entire system is a single largest cluster which is excluded in the calculation of $n_s$. The values of $\tilde{S}$ increase with increasing $u$ until $u \sim 2$ and then decrease when the RW occupies most of the space for larger $u$. The ratio $\tilde{S} / L^2$ seems to converge as $L \to \infty$, confirming that even the mean finite cluster size scales as $L^2$. However, even here the numerical test of convergence of the function for a single $u = u^*$ indicates that there is a residual dependence on $1 / \ln L$ leading to a slightly larger (up to 3%) limiting value of $\tilde{S} / L^2$. From our data it is not possible to determine whether the maximum of the curves keeps shifting with increasing $L$. Note that the mean size of the finite clusters at its maximum is only $\sim 0.02 L^2$ which is rather small compared to $P$.

IV. GEOMETRY AND FRACTALITY OF THE LARGEST CLUSTER

In this Section we take a closer look at the geometry of the largest cluster. Figure 6 depicts four 2DVS/P realizations for $L = 512$ and $u = u^* = 1.3$. This particular value $u = u^*$ was selected because it maximizes $|dP / du|$, and we expect to see clusters that are close to the transition between spanning and non-spanning state, where diverse and ramified configurations can be observed. At $u = u^*$ close to half of configurations percolate, but the peak in $|dP / du|$ is very broad and most of the configurations are not very close to the transition point.

A casual visual inspection of the configurations in Fig. 6 indicates that the largest clusters have a rather "compact" two-dimensional interior and very jagged boundaries. (Similar statements can be made about the clusters of intermediate sizes.) We also note that the linear dimensions of large clusters are of order of $L$. Below we quantify these observations.

In most percolation problems the system is homogeneous beyond the correlation length $\xi$ [3]. However, close to the percolation transition the correlation length $\xi$, which is the typical linear size of finite clusters, is much larger that the lattice constant $a$. In the broad range of distances $a \ll r \ll \xi$, fractal behavior can be observed: E.g., the mass of a cluster within some distance $r$ from one of its sites increases as $r^{d_f}$, where $d_f$ is the fractal dimension of the cluster. Alternatively, the probability to find a site belonging to the cluster at the distance $r$ from another site of the same cluster decreases as $1 / r^{d_{co}}$, where the fractal co-dimension $d_{co} = d - d_f$ [46]. (The relation between $d_f$ and $d_{co}$ is obtained by integrating density to find the mass within radius $r$.) Thus, the fractal dimension can be measured either by examining total cluster mass within some distance $r$ or by examining two-point correlation functions. The presence of fractal behavior is not always easy to ascertain: E.g., for Bernoulli percolation in $d = 2$ the fractal dimension $d_f = 91 / 512 \approx 1.9$ [1, 47] is not very different from the embedding dimension.

In the presence of percolation threshold the “cluster mass versus radius” method for measuring fractal dimension can be reduced to measuring (at the threshold) of the mass $PL^d$ of the spanning cluster (part of the incipient infinite cluster) as a function of $L$ and equating it to $L^{d_f}$. Thus, the $L$-dependence of $P$ contains the information about $d_f$. In our problem the threshold is absent, while $P$ is independent of $L$ indicating the absence of fractal behavior, and confirming the impressions of Fig. 6. However, in 1985 Banavar et al. [30] examined two-point correlation functions of clusters of vacant sites left by a meandering RW at the percolation point in both $d = 2$ and $d = 3$ and reached the conclusion that these clusters exhibit fractal behavior. In $d = 3$ their
FIG. 6. Examples of configurations on $512 \times 512$ lattice with $u = 1.3$. The sites visited by the random walk are colored black. The clusters of vacant sites are colored according to their volume: the largest cluster is colored red, and rest of the clusters are colored from the second largest to the smallest according to a periodic color scheme of orange, yellow, green, blue, violet. (In greyscale format these appears as different shades of grey with red $\rightarrow$ intermediate grey, blue $\rightarrow$ very dark grey, violet $\rightarrow$ dark grey, orange $\rightarrow$ light grey, green $\rightarrow$ very light grey, yellow $\rightarrow$ white.) The clusters can connect sites through the left and right edges of the lattice but not through the top and bottom edges. The examples include (a) single large percolating cluster, (b) non-percolating system with three large clusters, (c) very large and convoluted percolating cluster, and (d) large percolating cluster with a narrow “bottleneck.”

conclusion should not be surprising since the system has a percolation threshold and such behavior is expected. However, in $d = 2$ they also found $d_f = 1.75$. Closely following their approach (see also [1]), we define the two-point correlation function for the spanning a cluster as

$$ C(\vec{r}) = \frac{1}{s} \sum_{\vec{r}'} \rho(\vec{r}') \rho(\vec{r} + \vec{r}'), $$

(6)

where $\vec{r}$ and $\vec{r}'$ are positions of lattice sites, the density
\(\rho(\vec{r})\) equals 1 if the site at \(\vec{r}\) belongs to the cluster (up to a lattice vector due to periodicity) and 0 otherwise, and \(\vec{r}''\) is summed over all sites of the cluster.

Figure 7 depicts the azimuthal average of the correlation function \(C\) for lattice sizes \(L\) ranging from 48 to 768. All the graphs intersect at \(r = 0\) since by definition \(C(0) = 1\). The data for this figure were simulated using a different ensemble from the rest of the results: instead of a fixed \(u\), the RW continues until the spanning cluster disconnects, then we take the configuration of the lattice before the last step. This method creates an “almost disconnected” spanning cluster, and is similar to the methods used in Ref. 30. (We note that, while the length of the RWs in this ensemble is not fixed, the typical value of \(u\) is \(u \sim u^*\), consistently with Sec. III)

In Fig. 7(a) we see that for smaller fixed \(r\) the correlation \(C\) approaches a constant value independent of \(r\) as \(L\) increases. For larger \(r\), the correlation function \(C\) decays exhibiting finite size effects. E.g., the value of \(r\) at which \(C\) drops to, say, 0.6, doubles every time \(L\) is doubled. This is confirmed by the overlapping graphs in Fig. 7(b), where \(C\) is displayed as a function of the scaled variable \(r/L\).

Since the absolute average value of the slope of each graph in Fig. 7 first increases for small \(r\) and then decreases when \(r\) approaches \(L\), there is an intermediate regime on the logarithmic scale (less than 1/3 of a decade) where the slope is almost constant, leading to an apparent power law corresponding to co-dimension \(d_{co} = 0.35\) (indicated by the dashed line). This behavior, although with a co-dimension \(d_{co} = 0.25\) instead, prompted the authors of 30 to suggest that the spanning cluster is a fractal with dimension \(d_f = 1.75\) 31. Their data corresponds to \(L = 96\), which is the second left-most graph in Fig. 7(a).

However, in the fractal regime, we would expect the \(C(r)\) graphs corresponding to ever increasing \(Ls\) to be linear continuations (on the logarithmic scale) of each other with their cutoffs ever increasing with \(L\). Instead we see in Fig. 7(a) graphs that keep shifting to the right, clearly exhibiting a finite size (\(L\)-dependent) effect. We examined this for both the fixed \(u\) and the “almost disconnected cluster” ensembles, and found no appreciable difference between the overall behavior of the correlation function. (It should be noted that when Ref. 30 was written, the absence of a critical point of 2DVSP was not clearly recognized.) These observations convince us that in \(d = 2\) the spanning cluster is indeed compact and its linear size is proportional to \(L\), while its mass is proportional to \(L^2\), consistently with the visual inspection of Fig. 6.

The linear size of a cluster can be quantified by its radius of gyration \(R_g\), defined as 1, 30

\[
R_g^2 = \frac{1}{s} \sum_{i=1}^{s} (\vec{r}_i - \vec{r}_{cm})^2,
\]

where \(s\) is the number of sites in the cluster, \(\vec{r}_i\) are the cluster sites, and \(\vec{r}_{cm}\) is the position of the center of mass of the cluster. It should be noted that due to the periodic boundary conditions in the horizontal direction both the positions of steps \(\{\vec{r}_i\}\) and the position of center of mass \(\vec{r}_{cm}\) are not always uniquely defined. The proper choices are made to minimize the resulting \(R_g\). In regular percolation problems, the mean \(R_g\) of the clusters, as well as \(R_g\) of typical large cluster scales as the correlation length \(\xi\).

We examined the relation between \(R_g\) of various clusters and their mass for clusters in the entire range of sizes \(s\). The particular values of linear extent \(R_g\) of various clusters with given specific mass \(s\) are broadly scattered but the average values of \(R_g^2(s)\) are proportional to \(s\) leading to conclusion that the clusters are not fractal, similarly to the largest cluster. As we mentioned before, the largest cluster has mass proportional to \(L^2\). It is natural to expect that their linear size is proportional to \(L\). Figure 8 depicts \(R_g\) of the largest cluster as a function of \(L\), for \(L\) ranging from 4 to 512 and for \(u = 0.4, 0.7, ..., 1.9\). We see that \(R_g\) clearly displays the expected linear scaling with \(L\), although the slope of linear curves in Fig. 8 slowly decreases with increasing \(u\).

The jagged boundary of the clusters seen in Fig. 6 is formed by segments of a RW. We define the cluster hull perimeter \(H\) as the total mass of cluster sites bordering the sites visited by the random walk. We considered the hull perimeter only for the largest cluster. For fixed \(L\) we may expect the numerical value of the perimeter to decrease as \(u \to 0\) since the spanning cluster will essentially have no boundaries, and on the other hand for \(u \gg 1\) the perimeter will again be small due to decrease in the typical size of the largest cluster. Somewhere at the intermediate values of \(u\), possibly around \(u^*\), we will see large hull perimeters. Unlike Bernoulli percolation, our clusters do not have internal boundaries due to holes inside a cluster, and therefore their entire perimeter belongs to the hull. For Bernoulli site percolation at \(d = 2\) and \(p = p_c\), both the mass and the hull of the spanning cluster are fractal, and the hull scales as \(H \sim L^{D_H}\) with fractal dimension \(D_H = 1.74\) 48, 49.

Figure 9 depicts the hull perimeter \(H\) of the largest cluster of our problem as a function of \(L\), for \(L\) ranging from 4 to 512 and for \(u = 0.4, 0.7, ..., 1.9\) on a logarithmic scale. For fixed large \(L\) the hull perimeter \(H\) slightly depends on \(u\) reaching maximum close to \(u = 1\). With increasing \(L\) all of graphs approach straight lines with slopes corresponding to power \(D_H = 1.33 \pm 0.01\), where the size of the error provides a subjective estimate of uncertainty in extrapolation as well as slight differences between different values of \(u\). The dashed line in Fig. 9 has slope 4/3 and provides guide to the eye. (The statistical errors are negligible.) The measured exponent 4/3 coincides with the well-known theoretical result proven by Lawler et al. 50 (and conjectured by Mandelbrot 52) that the fractal dimension of the frontier of a Brownian motion in \(d = 2\) is 4/3.
V. CLUSTER STATISTICS

The mean number of clusters of size $s$ per lattice site $n_s$ as defined in Sec. III is one of the most revealing features of a percolating system. Despite the differences between 2DVSVP and the usual Bernoulli percolation we will attempt to follow a similar logic while pointing out important differences between the systems. If a system lacks a length or mass scale, then we expect the functions characterizing the system to be power laws. In particular, one might expect $n_s = As^{-\tau}$, where $\tau$ is called a Fisher exponent [1], while $A$ is a constant, possibly dependent on some microscopic properties and details. In Bernoulli percolation such dependence is valid on scales $r$ much larger than the lattice constant $a$ but smaller than the correlation length $\xi$, i.e., for cluster masses satisfying $1 \ll s \ll s_c$, where $s_c$ is a typical mass of a cluster of linear size $\xi$. (Typically, there is a power law dependence between $s_c$ and $\xi$.) At length scales $r \sim \xi$ the power law is corrected by some cutoff function $F_c$, which is $\approx 1$ for $s \ll s_c$ and drops to 0 as $s_c$ is exceeded. The overall shape of the
dependence is
\[ n_s = A s^{-\gamma} F_c. \]  

It is frequently assumed in Bernoulli percolation that the cutoff function depends only on the ratio \( s/s_c \), although away from the threshold a more complicated dependence on \( p \) might appear \[51\]. At the percolation threshold (\( \xi = \infty \)) the cutoff is absent, and therefore Eq. (6) dictates that an infinite sum \( \sum s n_s \approx \sum s A s^{1-\gamma} \) converges, and therefore \( \tau > 2 \). Indeed for two-dimensional Bernoulli percolation \( \tau = 197/81 \approx 2.05 \) \[1\].

In the 2DVSP problem function \( n_s \) plays a somewhat different role. In percolation problems with a threshold, there is some correlation length \( \xi \) and the therefore a very large system of linear size \( L \) can be treated as a collection \( (L/\xi)^d \) independent systems. Consequently, even a single very large realization of the system assures that most cluster sizes \( s \) will be present and \( n_s \) can be naturally treated as a continuous function representing frequency of clusters of size \( s \). In the 2DVSP, there is no correlation length and \( L \) is the only large length scale. In a single sample there are only few large clusters of size \( s \approx L^2 \) and, consequently, if we avoid averaging over samples for most large \( s \) we will have vanishing \( n_s \) and only few particular values of \( s \) will produce \( n_s = 1/L^2 \). An increase of \( L \) will not improve that situation. Only the averaging over ensemble will produce a continuous function \( n_s \) of \( s \). Nevertheless, we expect to have a large range of scale-free behavior, and, as in the case of regular percolation, we hope that the ensemble averaged \( n_s \) has a shape given by Eq. (8).

Figure 10 depicts a logarithmic plot of the cluster number per site \( n_s \) for \( L \) ranging from 4 to 4096 and \( u = 1.3 \). All the graphs consist of a relatively straight region for \( s \ll L^2 \) and a cutoff around \( 0.3L^2 \). Close to the cutoff, the curves exhibit somewhat unusual behavior described in the next paragraph. We excluded the area close to the cutoff from our analysis of scaling behavior. While we used rather large statistical samples, the frequencies of finding a cluster of some particular (large) \( s \) for large \( L \)s become very low and the resulting curves are very “noisy.” The curves presented in Fig. 10 are “smoothed” by averaging the results for a particular \( s \) over a range \( \sim \sqrt{s} \). This procedure has almost no effect for moderate \( L \)s, but distorts the “tails” of the curves on large lattices: For \( L \geq 512 \), when \( n_s \) drops below a certain (small) value, where in the entire ensemble of \( 7 \cdot 10^5 \) samples there is about one cluster of each size \( s \), the a averaging procedure distorts the curve, because it is just an averaging \( 0s \) and 1s. We therefore truncated the curves when this value is reached. For \( L = 512 \) the truncation appears when the cutoff is reached, while for larger \( L \)s the graphs are truncated even before reaching the cutoff.

All the curves for \( L < 512 \) in Fig. 10 exhibit a sharp cutoff and the position of that cutoff increases by a factor of 4 every time \( L \) doubles. Such \( L \)-dependence is consistent with the results that we had in Sec. 11 Therefore, the cutoff function depends of \( s/s_c \), with \( s_c \sim L^2 \) with some dependence on \( u \). Near the cutoff position the function \( F_c \) has some structure: In Fig. 10 instead of being just a simple monotonic drop from 1 to 0, it actually increases above 1 before the drop thus moderating the decay of \( n_s \) close to \( s_c \). The behavior close to the cutoff depends on \( u \): For smaller us the “bump” in \( F_c \) becomes even more pronounced. Such non-trivial shape of \( F_c \) apparently reflects the fact that large-\( s \) part of \( n_s \) attempts to depict few very large clusters by using a smooth function of \( s \).

Before the cutoff, the curves in Fig. 10 are fairly straight and approximately follow a slope that very slowly increases and reaches the value of \( \tau \approx -1.83 \) (or slightly larger) for the largest \( L \)s. The dashed line in Fig. 10 indicates slope 1.85. Such behavior is a significant deviation from the expectation that \( \tau \) should exceed 2. If this represents an asymptotic trend, then the requirement for the \( \sum s n_s \) to be finite for ever increasing \( L \), i.e., with the power-law cutoff increasing as \( L^2 \) would require the prefactor of the power law in Eq. (8) to decrease \( A \sim L^{-2} \). While the vertical position of the curves in Fig. 10 slightly decreases with increasing \( L \) it is extremely weak, possibly dependent on \( 1/\ln L \). Strong \( L \)-dependence of \( A \) would also cast doubt on our assumption of scale-independence of the results on the intermediate scales. The results described in this paragraph, namely \( \tau < 2 \) and \( A \) almost independent of \( L \), are not mutually consistent.

Special properties of Gaussian RWs in \( d = 2 \) can be used to advance a theoretical heuristic argument that \( \tau = \tau_{th} = 2 \). Long RWs on a lattice can be treated as Gaussian RWs in continuum. As has been mentioned in the before-last paragraph of Sec. 11 a Gaussian RW can be coarse-grained or fine-grained exactly: The increase of lattice size from \( L \) to \( \lambda L \) and number of steps
N from $uL^2$ to $u\lambda^2L^2$ is equivalent to keeping the system size unchanged, while increasing $N$ by a factor $\lambda^2$ and decreasing the step size $\ell$ by factor $\lambda$. Positions of every $\lambda^2$rd step of this new fine-grained configuration will be distributed exactly as the positions of the original Gaussian RW before it was fine-grained. Thus, the process of fine-graining replaces each step of the RW by $\lambda^2$ smaller steps but does not change the paths of RWs on larger scales. We saw in Sec. II that the clusters have compact interiors, and therefore their volume will not change, except for being measured in smaller units, i.e., a cluster of $s$ sites will become a cluster of $\lambda^2s$ sites, of the same shape, although with more jagged boundaries. (This argument assumes that the slight changes in the fine-grained boundaries created by the RW do not break up fragile clusters that have “bottleneck” or join clusters which were “almost connected” in the original geometry, or at least, such changes are very rare.) Thus by assuming, that nothing changed in the overall geometry, or at least, such changes are very rare.) Thus the final Gaussian RW before it was fine-grained. Thus, the result is valid for arbitrary $\lambda$ and, in particular, by taking $\lambda = 1/L$ we observe that $L^4n_s(L)$ becomes a function of only of $s/L^2$. Figure 11 depicts the scaled number of clusters as a function of scaled cluster mass, and the graphs for various $L$s almost (but not completely) collapse. Furthermore, the scaled cutoffs are very similar and appear at about $s_c/L^2 \approx 0.3$, yet again confirming relation $s_c \approx L^2$.

The left hand side of Eq. (9) is independent of $\lambda$, and the equation must be valid for an arbitrary $\lambda$ in the power-law regime that is only possible, when $n_s = As^{-2}$ leading to the “theoretical” value of Fisher exponent $\tau_{th} = 2$. A slope of -2 is indicated by the dotted line in Fig. 10 and seems to be slightly larger than the approximate slope of -1.83 seen in the graph. Later we will explore the possibility that our results did not yet converge to their asymptotic values. If the Fisher exponent is 2, then the coefficient $A$ cannot be constant: To maintain a finite $\sum_s s n_s$, it must decrease as $1/\ln L$. Indeed, $A$ slowly decreases with increasing $L$. Our heuristic argument is not accurate enough to determine the logarithmic terms either in the prefactor, or even in the s-dependence of $n_s$.

In Sec. III we discussed the possibility of extremely slow convergence of the numerical results when the $L$-dependence may be as slow as $1/\ln L$. We attempted to study the apparent discrepancy between the heuristic result $\tau_{th} = 2$ and the measured $\tau \approx 1.83$. We measured the weak dependence on $L$ of the effective exponent $\tau$. The exponent has been extracted from a linear fit on a logarithmic scale in the range $1 \leq s \leq 0.01L^2$, which avoids the very peculiar behavior of the curves near the cutoff. The data points for $\tau$ in Fig. 12 are arithmetic means of the slopes in the two halves of the range, $1 < s < 0.1L$ and $0.1L < s < 0.01L^2$. (For $L = 4096$ the slope in the entire range is 1.86, which is slightly larger than this mean.) Figure 12 depicts $\tau$ as a function of $1/\ln L$ for $L = 64, 128, 256, 512$ and for $u = 0.7, 1.0, \ldots, 2.8$. For $u = 1.3$ the simulations have been extended to $L = 1024, 2048, 4096$. Slightly nonlinear behavior (on the logarithmic scale) of the graphs in Fig. 10 introduces possible systematic errors as large as 0.05. Such errors, combined with extremely slow $L$-dependence prevent a reliable extrapolation to $L \to \infty$. Nevertheless, Fig. 12 demonstrates the plausibility of the asymptotic value $\tau_{th} = 2$ that is indicated by an arrow.

Usage of the continuous (ensemble averaged) function $n_s$ obscures the fact that there are only few large clusters, and therefore in each sample there are no clusters for most large values of $s$. It is therefore, beneficial to examine these statistics from a different point of view. We denote $P_k$ as the mass (volume) of the $k$th largest cluster, divided by the lattice volume $L^2$. In particular, the largest cluster strength $P$ equals $P_1$, and the identity $\sum_k P_k = p$ is trivial. By construction $P_k$ is monotonous and should exhibit a more “continuous” behavior than $n_s$ because for large clusters (small $k$), $P_k$ should converge in the $L \to \infty$ limit just as $P$ does, and for smaller clusters the differences between cluster sizes are small and there are many of them.

Figure 13 depicts $P_k$ as a function of the cluster mass.
FIG. 12. Plot of the effective exponents \( \tau \) defined in Eq. (9), as a function of \( 1/\ln L \), for \( u = 0.7, 1.0, ..., 2.8 \) (top to bottom) and \( L = 64, 128, 256, 512 \). For \( u = 1.3 \) the range of \( L \) was extended to \( L = 1024, 2048, 4096 \). The exponents are extracted from a linear fits (on logarithmic scale) in the range \( 1 \leq s \leq 0.01L^2 \) (see text). Estimated systematic errors are as large as 0.05. Arrow near the vertical axis indicates the expected theoretical value \( \tau_h = 2 \).

FIG. 13. Logarithmic plot of \( P_k \), averaged over \( 7 \times 10^5 \) samples, as a function of the cluster mass index \( k \), for \( u = 1.3 \) and for \( L = 4, 8, ..., 4096 \) (left to right). The dashed line indicates slope \(-1.20\).

Index \( k \), for \( L \) ranging from 4 to 512 and \( u = 1.3 \). The graphs are obtained from the average of \( P_k \) over many configurations, in the same manner as the graphs of \( n_s \) in Fig. 10. For \( k > 10 \) the graphs appear to roughly converge to a straight line representing a power law \( Bk^{-q} \), up to a sharp cutoff \( k_c \), which increases with increasing \( L \). The effective exponent \( q \) slightly decreases with increasing \( L \) and reaches \( q = 1.20 \) for the largest sample. (The estimated systematic errors of the effective exponents are smaller than 0.05.) The prefactor \( B \) is almost independent of \( L \). We note that most of the mass of the vacant clusters is contained in three or four largest clusters, while the remainder contains a small fraction of vacant sites. Since large \( k \) corresponds to small clusters, the value of \( k_c \) is not evident. Clearly, the total number of clusters is significantly smaller than the number of sites, and therefore \( k_c \lesssim L^2 \). Even a tighter bound can be obtained by demanding the cutoff appears when the cluster sizes reach single site, i.e., \( k_c \approx L^{2/3}B^{1/3} \). The changing shape of the cutoff does not permit exact evaluation of its power dependence on \( L \) but it seems to increase slightly slower than \( L^2 \).

Both \( P_k \) and \( n_s \) describe the same cluster statistics from slightly different points of view. Assuming that they both are power laws, it should be possible to relate them. The cluster size \( s \) for a specific fixed index \( k \) fluctuates, but it is possible to treat the mean of \( s \) as a function of \( k \). Similarly, for a specific \( s \) we can define the mean value of \( k \). Both of these \( s-k \) relations are expected to be the same power laws. As long as the fluctuations of \( s \) for fixed \( k \), or, alternatively, the fluctuations of \( k \) for fixed \( s \) are small we can proceed with our derivation by assuming an approximate deterministic relation \( s(k) \). (For power law distributions such relation can be used even when the fluctuations are large: One gets correct relations between the exponents.) We will extract this relation from the mass of the \( j \)th cluster \( L^2P_k = L^2Bk^{-q} = s \). If there are \( N_s \) clusters of size \( s \), the change of an \( s \) cluster’s mass by a unit advances its cluster index \( k \) by \( N_s \). Therefore, we expect \( \frac{ds}{dk} = -N_s \), or

\[
\frac{dk}{d(L^2Bk^{-q})} = -L^2As^{-\tau} = -L^2A(L^2Bk^{-q})^{-\tau}, \quad (10)
\]

which relates the exponents \( q \) and \( \tau \) by

\[
q = \frac{1}{\tau - 1}. \quad (11)
\]

The slope of the graphs in Fig. 13 indicates that \( q \approx 1.20 \), and this corresponds due to Eq. (11) to \( \tau \approx 1.83 \). The latter value is the same as the (approximate) measured value of \( \tau \) in the graphs of Fig. 10. It is interesting to note that that our heuristic estimate \( \tau_h = 2 \) corresponds to the exponent \( q = q_{th} = 1 \). By examining the dependence of the effective exponent \( q \) on \( 1/\ln L \) we note that 1 is the likely asymptotic value of \( q \) for \( L \to \infty \), although large error bars prevent exact extrapolation. Further examination of the coefficients in Eq. (10) shows that the product of the prefactors \( AB^{1-\tau} \approx L^{2(\tau - 2)} \). Since neither \( A \) nor \( B \) show significant \( L \)-dependence, the anticipated asymptotic value \( \tau = \tau_h = 2 \) seems to be consistent with our results.

VI. CONCLUSIONS AND DISCUSSION

The study of RWs in \( d = 2 \) is a very old and well explored subject. We concentrated on percolation aspects
of sites not visited by the random walk on a periodic lattice both because this is the lower critical dimension of slightly more conventional percolation problems of vacant sites in $d \geq 3$, and because this problem exhibits features that are absent in “typical” lower critical dimension problems. As far as it was possible, we used the tools of percolation theory to analyze the problem, although certain features were very different from regular percolation, and even different from the behavior of Bernoulli percolation at its lower critical dimension.

We have demonstrated that contrary to older results the cluster interiors are not fractal, although their boundaries are. The only macroscopic scale of the 2DVSP problem is the lattice size $L$ and at shorter distances the behavior is scale-free and can be described using power laws. The 2DVSP problem converges very slowly to “large $L$ limit.” Our results indicate that the approach to asymptotic behavior is as slow as the decay to zero of $1/\ln L$. Consequently, all measurements even at large $L$ correspond to intermediate effective behavior. We suggested a heuristic argument setting the values of the exponents $\tau$ and $q$ describing the cluster size distribution. Measured values of the exponents are distinct but seem to move towards the “theoretical” values with increasing $L$.

For $u \ll 1$ the problem of cluster size distribution can be described on an infinite lattice without the need to introduce periodic boundary conditions. For $u \gg 1$ the spanning clusters are virtually non-existent and one needs to understand clusters created by $\mathcal{N}_{cr} \gg 1$ almost independent pieces of the RW. Most of our measurements were performed for $u = O(1)$, where the situation is most diverse, and might be different from the behavior at very small or very large $u$. Our study, does not resolve this problem. Moreover, the approach to the limit of $p \to 1$ is much slower for larger $u$ values, and therefore various parts of the curves, such as shown in Figs. 3 to 5 converge at different rates to their asymptotic values. Therefore the shape of the curves may keep changing with increasing $L$.

The graphs in Fig. 13 have rather distinct behavior for $k \lesssim 4$ as opposed to larger $k$s. Such separation into “large” clusters (small $k$), and ”small” clusters (larger $k$) provides possible clues into the $u$-dependence of the features. By examining similar graphs for different values of $u$ we observed that increasing $u$ decreases the sizes of the “large” clusters, but increases the sizes of the clusters with larger $k$ towards the larger cluster sizes. The $u$-dependence of all the properties requires a more systematic study.

Our theoretical arguments did not go beyond an approximate “heuristic” approach. However, RW is a rather well understood object, and it is conceivable that more accurate predictions can be made analytically.

ACKNOWLEDGMENTS

Y.K. thanks M. Kardar for stimulating discussions. This work was supported by the Israel Science Foundation Grant No. 453/17.

[1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, 2nd ed. (Taylor and Francis, London, UK, 1991).
[2] G. Grimmett, *Percolation*, 2nd ed. (Springer, Berlin, 1999).
[3] P. J. Flory, J. Am. Chem. Soc. 63, 3083 (1941).
[4] W. H. Stockmayer, J. Chem. Phys. 12, 125 (1944).
[5] S. R. Broadbent and J. M. Hammersley, Math. Proc. Cambridge Philos. Soc. 53, 629 (1957).
[6] A. A. Sabert, Phys. Rev. Lett. 110, 178501 (2013).
[7] P. Grassberger, Math. Biosci. 63, 157 (1983).
[8] J. T. Matamalas, A. Arenas, and S. Gómez, Sci. Adv. 4, eaau4212 (2018).
[9] S. G. Ankelal, P. Balukudumbi, and M. A. Bevan, Phys. Rev. E 73, 020403 (2006).
[10] A. Coniglio, H. E. Stanley, and W. Klein, Phys. Rev. Lett. 42, 518 (1979).
[11] H. Tsurusawa, M. Leocmach, J. Russo, and H. Tanaka, Sci. Adv. 5, eaau6090 (2019).
[12] M. Adam, M. Delsanti, J. Munch, and D. Durand, Physica A 163, 85 (1990).
[13] F. Taubert, R. Fischer, J. Groeneveld, S. Lehmann, M. S. Müller, E. Rödig, T. Wiegand, and A. Huth, Nature 554, 519 (2018).
[14] H. A. Makse, S. Havlin, and H. E. Stanley, Nature 377, 608 (1995).
[15] S. Havlin, H. E. Stanley, A. Bashan, J. Gao, and D. Y. Kenett, Chaos Solit. Fract. 72, 4 (2015).
[16] T. Kalisky and R. Cohen, Phys. Rev. E 73, 035101 (2006).
[17] I. Derényi, G. Palla, and T. Vicsek, Phys. Rev. Lett. 94, 160202 (2005).
[18] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. 85, 5468 (2000).
[19] H. Berry, J. Pelta, D. Lairez, and V. Larreta-Garde, Biochimica et Biophysica Acta 1524, 110 (2000).
[20] G. C. Fadda, D. Lairez, B. Arrio, J.-P. Carton, and V. Larreta-Garde, Biophys. J. 85, 2808 (2003).
[21] G. Toulouse, Il Nuovo Cimento B (1971-1996) 23, 234 (1974).
[22] W. T. Harris, J. Phys. C 7, 1671 (1974).
[23] A. Weinrib, Phys. Rev. B 29, 387 (1984).
[24] A. Coniglio and A. Fierro, in *Encyclopedia of Complexity and Systems Science*, part 3 (Springer, New York, 2009) pp. 1596–1615.
[25] G. Gori, M. Michelangeli, N. Defenu, and A. Tombeloni, Phys. Rev. E 96, 012108 (2017).
[26] O. Riordan and L. Warna, Science 333, 322 (2011).
[27] R. M. D’Souza and J. Nagler, Nat. Phys. 11, 531 (2015).
[28] Y. Kantor, Phys. Rev. B 33, 3522 (1986).
[29] Y. Kantor and M. Kardar, Phys. Rev. E 100, 022215 (2019).
[30] J. R. Banavar, M. Mulhikumar, and J. F. Willemes, J. Phys. A: Math. Gen. 18, 61 (1985).
[31] T. Abete, A. de Candia, D. Lairez, and A. Coniglio, Phys. Rev. Lett. 93, 228301 (2004).
[32] B. B. Mandelbrot, The Fractal Geometry of Nature (Freeman, New York, 1982).
[33] A. Dvoretzky and P. Erdős, in Proc. 2nd Berkeley Symposium. Math. Stat. and Prob. (University of California Press, Berkeley, 1951) p. 33.
[34] G. H. Vineyard, J. Math. Phys. 4, 1191 (1963).
[35] R. J. Rubin and G. H. Weiss, J. Math. Phys. 23, 250 (1982).
[36] G. Pólya, Mathematische Annalen 84, 149 (1921).
[37] B. D. Hughes, Random Walks and Random Environments, Vol. 1 (Clarendon Press, Oxford, 1995).
[38] M. J. A. M. Brummelhuis and H. J. Hilhorst, Physica A 185, 35 (1992).
[39] D. J. Aldous, Z. Wahrscheinlichkeitstheorie verw. Gebiete 62, 361 (1983).
[40] M. J. A. M. Brummelhuis and H. J. Hilhorst, Physica A 176, 387 (1991).
[41] A.-S. Sznitman, The Annals of Probability 36, 1 (2008).
[42] V. Sidoravicius and A.-S. Sznitman, Commun. Pure Appl. Math. 62, 0831 (2009).
[43] A.-S. Sznitman, Ann. Math. 171, 2039 (2010).
[44] R. Balázs, Electron. Commun. Probab. 20, 1 (2015).
[45] J. Hoshen and R. Kopelman, Phys. Rev. B 14, 3438 (1976).
[46] Y. M. Strelniker, S. Havlin, and A. Bunde, in Encyclopedia of Complexity and Systems Science edited by R. A. Meyers (Springer, New York, 2009) pp. 3847–3858.
[47] A. Kapitulnik, A. Aharony, G. Deutscher, and D. Stauffer, J. Phys. A: Math Gen. 16, L269 (1984).
[48] T. Grossman and A. Aharony, J. Phys. A: Math. Gen. 19, L745 (1986).
[49] R. F. Voss, J. Phys. A: Math. Gen. 17, L373 (1984).
[50] G. F. Lawler, O. Schramm, and W. Werner, Math. Res. Lett. 8, 401 (2001).
[51] B. Ding, C. Li, M. Zhang, G. Lu, and F. Ji, Eur. Phys. J. B 87, 79 (2014).