Electrostatic structures associated with dusty electronegative magnetoplasmas

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Abstract. By using the hydrodynamic equations of positive and negative ions, the Boltzmann electron density distribution and the Poisson equation with stationary dust, a three-dimensional (3D) Zakharov–Kuznetsov (ZK) equation is derived for small but finite amplitude ion-acoustic waves. However, the ZK equation is not appropriate to describe the system either at critical plasma compositions or in the vicinity of the critical plasma compositions. Therefore, the modified ZK (MZK) and extended ZK (EZK) equations are derived. The generalized expansion method is used to analytically solve the ZK, MZK and EZK equations. A new class of solutions that admits a train of well-separated bell-shaped periodic pulses is obtained. In certain conditions, the latter degenerates to either solitary or shock wave solutions. The effects of the physical parameters on the nonlinear structures are examined in many plasma environments having different negative ion species, such as D- and F-regions of the Earth’s ionosphere, as well as in laboratory plasma experiments.

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Numerical analysis of the solutions revealed that the profile of the nonlinear pulses suffers amplitude and width modifications due to enhancement of the dust practices, negative ions, positive-to-negative ion mass ratio and positive/negative ion cyclotron frequency. Furthermore, the necessary conditions for both solitons and shocks propagation as well as their polarity are examined.

Contents

1. Introduction
2. Basic equations and formulation of the problem
   2.1. Derivation of the ZK equation
   2.2. Derivation of the MZK and EZK equations
   2.3. Analytical solutions of the ZK, MZK and EZK equations
3. Numerical analyses and discussion
4. Summary
Appendix. Solutions of equations (30), (47) and (51) using the generalized expansion method
References

1. Introduction

It is well known that dust particles are common in the universe and they represent much of the solid matter in it. Dust particles often contaminate fully ionized or partially ionized gases and form the so-called ‘dusty plasma’, which occurs frequently in nature. In astrophysics, in the early 1930s, dust was shown to be present in the interstellar clouds where it appears as a selective absorption of stellar radiation interstellar reddening. Dust particles play a very important role in the solar system, in cometary tails, in planetary rings and also in the evolution of the solar system from its solar nebula to its present form. They are also found in environments such as production processes, flames, rocket exhausts, etching experiments and experiments on dust plasma crystals [1]. The dust particles are of micrometer or submicrometer size and their mass is large compared to the masses of the ion species. Due to the presence of such heavy particles, the plasma normal mode could be modified. For example, the ion-acoustic wave is one of the modified normal modes, which are called the dust ion-acoustic waves (DIAWs). Shukla and Silin [2] were the first to report theoretically the existence of DIAWs in unmagnetized dusty plasmas. Later, the DIAWs were observed experimentally in laboratory experiments [3, 4]. Furthermore, many efforts have been made to understand the properties of linear and nonlinear DIAWs in dusty plasmas ([5]–[7]; [8] and references therein; [9, 10]).

Negative ion plasma is a plasma containing both negative ion species and positive ion species in addition to electrons. This type of plasma is of great importance for various fields of plasma science and technology. The existence of a considerable number of negative ions in the Earth’s ionosphere [11] and cometary comae [12] is well known. Positive–negative ion plasmas are found in plasma processing reactors [13], in neutral beam sources [14] and in low-temperature laboratory experiments [15, 16]. Moreover, negative ions have been found to outperform positive ions in plasma etching. Therefore the importance of negative ion plasmas for the field of plasma physics is growing. To treat such plasmas skillfully, basic study and development of diagnostic techniques on negative ion plasmas are indispensable. Recently, the...
Cassini spacecraft has conclusively demonstrated the presence of heavy negative ions in the upper region of Titan’s atmosphere [17]. These particles may act as organic building blocks for even more complicated molecules. In a negative ion plasma, the number of electrons decreases according to the charge neutrality, i.e. \( n_e = n_e^- - n_e^+ \), where \( n_e^- \), \( n_e^+ \) and \( n_e^- \) are the electron, positive ion and negative ion densities, respectively. The resulting decrease in the shielding effect produced by electrons, which is one of the main effects governing the behavior of plasmas, characterizes the specific phenomena of negative ion plasmas. From this perspective, it seems to follow that the negative ions only influence plasmas secondarily. Although this is a fact, most of the phenomena are actually affected by the negative ions themselves as well as by the lack of electrons [16].

Recently, it was found that the presence of negative ions in dusty plasma could change the plasma composition and plasma transport properties [18], as well as the dust charges [19, 20]. For example, Kim and Merlino [19] reported the conditions under which dust grains could be positively charged in an electron-ion plasma with both positive and negative ions. Later, positively charged nanoparticles in the night time polar mesosphere were observed by Rapp et al [21]. Rosenberg and Merlino [22] investigated the effect of positive and negative dust grains on the ion-acoustic wave instability in a plasma with negative and positive ions. Due to the presence of a magnetic field, the behavior of the ion-acoustic waves in the presence of negatively charged ions and dust particles can drastically change. On the other hand, the presence of negative ions, as well as either positive or negative dust particles, can produce various nonlinear structures, such as solitons and shocks. There remains a wealth of other sources that point towards the existence of dust particles in space observation (cf [19, 22]). Mamun et al [23] have investigated the properties of solitary waves and double layers in an electronegative dusty plasma in a planar geometry, and compared the results with experiment [24]. It should be mentioned here that the origin and mechanism for the generation of dust particles in space plasma is still an important problem. Therefore, it is of practical interest to examine the effect of dust particles on the properties of the ion-acoustic excitations in dusty electronegative magnetoplasmas.

This paper is organized as follows. In section 2, we present the governing equations for the nonlinear DIAWs in positive–negative ion plasma. The reductive perturbation method is employed to derive the Zakharov–Kuznetsov (ZK) equation describing the system. Furthermore, we shall obtain appropriate equations, including higher-order nonlinearity, describing the evolution of the nonlinear pulses at critical plasma compositions and in the vicinity of the critical plasma compositions. The generalized expansion method is used to solve analytically the evolution equations, and obtain a train of well-separated bell-shaped periodic pulses that can change to solitary excitations as well as shock pulses. Section 3 contains the numerical results and discussion. Finally, the results are summarized in section 4.

2. Basic equations and formulation of the problem

We consider 2D, magnetized and collisionless four-component plasmas consisting of positive ions, negative ions, electrons and stationary dust particles. The external magnetic field is directed in the \( x \) axis, i.e. \( \mathbf{B} = B_0 \hat{x} \), where \( \hat{x} \) is the unit vector along the \( x \) axis. The propagation of the nonlinear electrostatic excitations is governed by a system of fluid equations for the positive and negative ion fluids, distinguished by using the index ‘+’ and ‘−’, respectively. The
dynamics are governed by the continuity equation
\[ \frac{\partial n_{+,-}}{\partial t} + \nabla \cdot (n_{+,-}u_{+,-}) = 0, \]  
\[ (1) \]
and the momentum equations
\[ m_+ \left( \frac{\partial}{\partial t} + u_+ \cdot \nabla \right) u_+ = -e\nabla \phi + \frac{e}{c} (u_+ \times B_0\hat{\lambda}), \]
\[ m_- \left( \frac{\partial}{\partial t} + u_- \cdot \nabla \right) u_- = e\nabla \phi - \frac{e}{c} (u_- \times B_0\hat{\lambda}). \]
\[ (2) \]
\[ (3) \]
The Poisson equation reads
\[ \nabla^2 \phi = 4\pi e (n_{-} - n_{+} + n_e - \delta Z_d n_d). \]
\[ (4) \]
In equations (1)–(4), \( n_{+,-} \) is the positive (negative) ion number density, while \( n_d \) and \( n_e \) \((= n_{e0} \exp(e\phi/k_B T_e))\) are the dust and electron densities, respectively. Furthermore, \( u_{+,-} \) is the positive (negative) ion fluid velocity, \( \phi \) is the electrostatic wave potential, \( e(Z_d) \) is the magnitude of electron (dust) charge, \( m_{+,-} \) is the mass of positive (negative) ion, \( B_0 \) is the magnitude of the ambient magnetic field and \( c \) is the speed of light. \( \delta = +1 \) for positive dust particles and \( \delta = -1 \) for negative dust particles.

Equations (1)–(4) may be cast in a reduced (non-dimensional) form, for convenience of manipulation. For positive ions,
\[ \frac{\partial \bar{n}_+}{\partial \bar{t}} + \frac{\partial \bar{n}_+ \bar{u}_{+x}}{\partial \bar{x}} + \frac{\partial \bar{n}_+ \bar{u}_{+y}}{\partial \bar{y}} = 0, \]
\[ (5) \]
\[ \frac{\partial \bar{n}_{+x}}{\partial \bar{t}} + \left( \bar{n}_{+x} \frac{\partial}{\partial \bar{x}} + \bar{n}_{+y} \frac{\partial}{\partial \bar{y}} \right) \bar{u}_{+x} + \frac{\partial \bar{\phi}}{\partial \bar{x}} = 0, \]
\[ (6) \]
\[ \frac{\partial \bar{n}_{+y}}{\partial \bar{t}} + \left( \bar{n}_{+x} \frac{\partial}{\partial \bar{x}} + \bar{n}_{+y} \frac{\partial}{\partial \bar{y}} \right) \bar{u}_{+y} - \omega_e \bar{u}_{+z} = 0, \]
\[ (7) \]
\[ \frac{\partial \bar{n}_{+z}}{\partial \bar{t}} + \left( \bar{n}_{+x} \frac{\partial}{\partial \bar{x}} + \bar{n}_{+y} \frac{\partial}{\partial \bar{y}} \right) \bar{u}_{+z} + \omega_e \bar{u}_{+y} = 0, \]
\[ (8) \]
for negative ions,
\[ \frac{\partial \bar{n}_-}{\partial \bar{t}} + \frac{\partial \bar{n}_- \bar{u}_{-x}}{\partial \bar{x}} + \frac{\partial \bar{n}_- \bar{u}_{-y}}{\partial \bar{y}} = 0, \]
\[ (9) \]
\[ \frac{\partial \bar{n}_{-x}}{\partial \bar{t}} + \left( \bar{n}_{-x} \frac{\partial}{\partial \bar{x}} + \bar{n}_{-y} \frac{\partial}{\partial \bar{y}} \right) \bar{u}_{-x} - Q \frac{\partial \bar{\phi}}{\partial \bar{x}} = 0, \]
\[ (10) \]
\[ \frac{\partial \bar{n}_{-y}}{\partial \bar{t}} + \left( \bar{n}_{-x} \frac{\partial}{\partial \bar{x}} + \bar{n}_{-y} \frac{\partial}{\partial \bar{y}} \right) \bar{u}_{-y} - Q \frac{\partial \bar{\phi}}{\partial \bar{y}} - \omega_e \bar{u}_{-z} = 0, \]
\[ (11) \]
\[ \frac{\partial \bar{n}_{-z}}{\partial \bar{t}} + \left( \bar{n}_{-x} \frac{\partial}{\partial \bar{x}} + \bar{n}_{-y} \frac{\partial}{\partial \bar{y}} \right) \bar{u}_{-z} - \omega_e \bar{u}_{-y} = 0, \]
\[ (12) \]
and for electrons,
\[ \bar{n}_e = \alpha \exp \bar{\phi}. \]
\[ (13) \]
Finally, the system is closed by the Poisson equation as
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \bar{\phi} = \bar{n}_- - \bar{n}_+ + \bar{n}_e - \delta \gamma. \tag{14}
\]
Adopting the neutrality hypothesis, the heavy plasma species density \( n_d \) (including positive and negative dust as an ensemble) will be taken to be fixed and continuous, thus providing a globally neutralizing background. The variables appearing in equations (5)–(14) have been scaled by appropriate quantities. Thus, the density \( n_j \) (for \( j = +, -, d \) and \( e \)) is normalized by the unperturbed positive ion density \( n_{e0} \), \( u_{e, -} \) is scaled by the positive ion sound speed \( C_{s+} = (k_B T_e/m_+)^{1/2} \), the potential \( \phi \) is normalized by \( k_B T_e/e \) and the positive- (negative-) ion cyclotron frequency \( \omega_{c\pm} = eB_0/(m_\pm c) \) is normalized by the ion plasma period \( \omega_{p+} = (4\pi e^2 n_{e0}/m_+)^{-1/2} \). The space and time variables are in units of the characteristic Debye length \( \lambda_{D+} = (k_B T_e/4\pi e^2 n_{e0})^{1/2} \) and the ion plasma period \( \omega_{p+} \), respectively. We define the mass ratio \( Q = m_+/m_- \). The neutrality condition implies
\[
\alpha + \beta = 1 + \delta \gamma, \tag{15}
\]
where \( \alpha = n_{e0}/n_{+0} \), \( \beta = n_{-0}/n_{+0} \) and \( \gamma = Z_Q n_{d0}/n_{e0} \) (the index ‘0’ denotes the unperturbed density states). The upper bar in equations (5)–(14) will be omitted henceforth.

2.1. Derivation of the ZK equation

To investigate the nonlinear propagation of the electrostatic DIAWs, we shall employ the reductive perturbation method [25]. According to this method, the independent variables can be stretched as
\[
X = \varepsilon^{1/2}(x - \lambda t), \quad Y = \varepsilon^{1/2}y \quad \text{and} \quad \tau = \varepsilon^{3/2}t, \tag{16}
\]
where \( \varepsilon \) is a small (real) parameter and \( \lambda \) is the wave propagation speed. The dependent variables are expanded as
\[
\Psi = \Psi^{(0)} + \sum_{n=1}^{\infty} \varepsilon^n \Psi^{(n)}, \tag{17}
\]
where
\[
\Psi = [n_+, n_-, n_e, u_{+, x}, u_{-, x}, \phi]^T \tag{18}
\]
and
\[
\Psi^{(0)} = [1, \beta, \alpha, 0, 0, 0]^T. \tag{19}
\]
The transverse velocity \( (y \text{ and } z \text{ components}) \) of the positive/negative ion fluid is expanded as
\[
u_{+, y} = \varepsilon^{3/2} \tilde{u}_{+, y}^{(1)} + \varepsilon^2 \tilde{u}_{+, y}^{(2)} + \varepsilon^{5/2} \tilde{u}_{+, y}^{(3)} + \cdots, \tag{20}
\]
and so forth, i.e. similar expressions hold upon switching sign \( + \rightarrow - \) and/or axis \( y \rightarrow z \) in the index notation.

Employing the variable stretching (16) and the expansions (17)–(20) into equations (5)–(14), we may now isolate distinct orders in \( \varepsilon \) and derive the corresponding variable contributions. The lowest-order equations in \( \varepsilon \) read
\[
r_{+}^{(1)} = \frac{1}{\lambda} \phi_{+}^{(1)}, \quad u_{+}^{(1)} = \frac{1}{\lambda} \phi_{+}^{(1)}, \quad u_{+, x}^{(1)} = \frac{1}{\omega_{c+}} \frac{\partial \phi_{+}^{(1)}}{\partial Y}, \tag{21}
\]
The Poisson equation provides the compatibility condition
\[ \alpha - \frac{1}{\lambda^2} - \frac{\beta Q}{\lambda^2} = 0. \]  

(23)

The next order in \( \varepsilon \) yields
\[ \frac{\partial n_+^{(2)}}{\partial X} = \left[ \frac{1}{\lambda^2} \frac{\partial \phi^{(1)}}{\partial X} + \frac{2}{\lambda^3} \frac{\partial \phi^{(1)}}{\partial \tau} + \frac{3}{2\lambda^4} \frac{\partial \phi^{(1)2}}{\partial X} + \frac{1}{\lambda} \frac{\partial u_+^{(2)}}{\partial Y} \right], \]

(24)

\[ \frac{\partial n_-^{(2)}}{\partial X} = \left[ -\frac{\beta Q}{\lambda^2} \frac{\partial \phi^{(1)}}{\partial X} - \frac{2\beta Q}{\lambda^3} \frac{\partial \phi^{(1)}}{\partial \tau} + \frac{3\beta Q}{2\lambda^4} \frac{\partial \phi^{(1)2}}{\partial X} + \frac{\beta}{\lambda} \frac{\partial u_-^{(2)}}{\partial Y} \right], \]

(26)

\[ \frac{\partial n_+^{(2)}}{\partial X} = \left[ \lambda \frac{\partial^2 \phi^{(1)}}{\partial X\partial Y} \right], \]

(25)

\[ \frac{\partial n_-^{(2)}}{\partial X} = \left[ \lambda \frac{Q}{\omega_{c-}^2} \frac{\partial^2 \phi^{(1)}}{\partial X\partial Y} \right], \]

(27)

and
\[ \frac{\partial \phi^{(1)}}{\partial X^2} + \frac{\partial \phi^{(1)}}{\partial Y^2} = n_e^{(2)} + n_-^{(2)} - n_+^{(2)}. \]

(29)

From the second terms in \( \varepsilon \) together with (21), (22) and (23), we obtain the desired ZK equation,
\[ \frac{\partial \phi^{(1)}}{\partial \tau} + A \frac{\partial \phi^{(1)}}{\partial X} + B \frac{\partial^3 \phi^{(1)}}{\partial X^3} + D \frac{\partial^3 \phi^{(1)}}{\partial X \partial Y^2} = 0. \]

(30)

The nonlinearity \( A \) and the dispersion coefficients \( B \) and \( D \) are given by
\[ A = B \left[ \frac{3}{\lambda^4} - \frac{3\beta Q^2}{\lambda^4} - \alpha \right], \]

(31)

\[ B = \left[ \frac{\lambda^3}{\omega_{c-}^2} \right], \]

(32)

and
\[ D = B \left[ 1 + \frac{\beta Q}{\omega_{c-}^2} + \frac{1}{\omega_{c-}^2} \right]. \]

(33)

2.2. Derivation of the MZK and EZK equations

It is obvious that, at some critical compositions, e.g. for critical negative ion density \( \beta = (3 - \alpha \lambda^4)/3Q^2 \), the quadratic nonlinearity coefficient \( A \) acquires negligible value. Therefore, an analogous equation (or simple ad-hoc stretching (16)) can be formulated and we may require
a different perturbative scaling by adjusting our parameter scaling appropriately. Let us now introduce new stretched space–time coordinates [7]:

\[ X = \varepsilon(x - \lambda t), \quad Y = \varepsilon y \quad \text{and} \quad \tau = \varepsilon^3 t. \]  

We shall use expansions (17)–(19) but the transverse velocities will be given as

\[ u_{+,y} = \varepsilon^2 u_{+,y}^{(1)} + \varepsilon^3 u_{+,y}^{(2)} + \varepsilon^4 u_{+,y}^{(3)} + \cdots. \]  

Recall that similar expressions hold upon switching the sign + \rightarrow - and/or the axis \( y \rightarrow z \) in the index notation of equation (35). We substitute the stretching (34) and the expansions (17)–(19) and (35) into the basic equations (5)–(14). Of course, the lowest order of \( \varepsilon \) recovers the linearized solutions (21) and (22) as well as the compatibility condition (23). For the next order of \( \varepsilon \), we obtain

\begin{align*}
    n_+^{(2)} & = \frac{1}{\lambda^2} \phi_+^{(2)} + \frac{3}{2\lambda^4} \phi_+^{(1)2}, \\
    \nu_+^{(2)} & = \frac{1}{\lambda} \phi_+^{(2)} + \frac{1}{2\lambda^3} \phi_+^{(1)2}, \\
    u_+^{(2)} & = \frac{\lambda}{\omega_{c+}^2} \frac{\partial^2 \phi_+^{(1)}}{\partial X \partial Y}, \\
    n_-^{(2)} & = -\frac{\beta Q}{\lambda^2} \phi_-^{(2)} + \frac{3\beta Q^2}{2\lambda^4} \phi_-^{(1)2}, \\
    \nu_-^{(2)} & = \frac{-Q}{\lambda} \phi_-^{(2)} + \frac{Q^2}{2\lambda^3} \phi_-^{(1)2}, \\
    u_-^{(2)} & = \frac{-\lambda Q}{\omega_{c-}^2} \frac{\partial^2 \phi_-^{(1)}}{\partial X \partial Y},
\end{align*}

while Poisson’s equation gives

\[ \left( \alpha - \frac{1}{\lambda^2} - \frac{\beta Q}{\lambda^2} \right) \phi_+^{(2)} + \left( \frac{3}{\lambda^4} - \frac{3\beta Q^2}{\lambda^4} - \alpha \right) \phi_+^{(1)2} = 0. \]  

The coefficient of \( \phi_+^{(2)} \) is identically zero, due to the compatibility condition (23), while the coefficient of \( \phi_+^{(1)} \) is precisely \( A/B \), vanishing in the case at hand. Thus, Poisson’s equation is automatically satisfied. The next order in \( \varepsilon \) gives the following equations:

\begin{align*}
    \frac{\partial n_+^{(3)}}{\partial X} & = \left[ \frac{15}{2\lambda^6} \phi_+^{(1)2} \frac{\partial \phi_+^{(1)}}{\partial X} + \frac{3}{\lambda^4} \frac{\partial \phi_+^{(1)}}{\partial X} \frac{\partial \phi_+^{(1)}}{\partial X} + \frac{2}{\lambda^3} \frac{\partial \phi_+^{(1)}}{\partial \tau} + \frac{1}{\omega_{c+}^2} \frac{\partial \phi_+^{(1)}}{\partial X} \frac{\partial \phi_+^{(1)}}{\partial Y} + \frac{1}{\lambda^2} \frac{\partial \phi_+^{(3)}}{\partial X} \right], \\
    \frac{\partial n_-^{(3)}}{\partial X} & = \left[ -\frac{15\beta Q}{2\lambda^6} \phi_-^{(1)2} \frac{\partial \phi_-^{(1)}}{\partial X} + \frac{3\beta Q^2}{\lambda^4} \frac{\partial \phi_-^{(1)}}{\partial X} \frac{\partial \phi_-^{(1)}}{\partial X} - \frac{2\beta Q}{\lambda^3} \frac{\partial \phi_-^{(1)}}{\partial \tau} - \frac{\beta Q}{\omega_{c-}^2} \frac{\partial \phi_-^{(1)}}{\partial X} \frac{\partial \phi_-^{(1)}}{\partial Y} - \frac{\beta Q}{\lambda^2} \frac{\partial \phi_-^{(3)}}{\partial X} \right], \\
    \frac{\partial n_\varepsilon^{(3)}}{\partial X} & = \alpha \frac{\partial \phi_-^{(3)}}{\partial X} + \alpha \frac{\partial \phi_-^{(1)2}}{\partial X} + \alpha \frac{\partial \phi_-^{(1)3}}{6} \frac{\partial \phi_-^{(1)}}{\partial X}.
\end{align*}
and
\[ \frac{\partial^2 \phi^{(1)}}{\partial X^2} + \frac{\partial^2 \phi^{(1)}}{\partial Y^2} = n_e^{(3)} + n_n^{(3)} - n_+^{(3)}. \]  
(46)

Eliminating the third-order variables from equations (43)–(46), we finally obtain a modified ZK (MZK) equation as
\[ \frac{\partial \phi^{(1)}}{\partial \tau} + C \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial X} + B \frac{\partial^3 \phi^{(1)}}{\partial X^3} + D \frac{\partial^3 \phi^{(1)}}{\partial X \partial Y^2} = 0, \]
(47)
where \( B \) and \( D \) are given by (32) and (33), respectively, and \( C \) is given by
\[ C = \frac{1}{2} B \left[ \frac{15(1 + \beta Q^3)}{\lambda^6} - \alpha \right]. \]  
(48a)

Recall that equation (47) was derived at \( A = 0 \), i.e. it is correct only at the vanishing of the nonlinear coefficient \( A \). So, we have to ensure that \( A = 0 \) in equation (48a). Therefore, from equation (31), we obtain \( \beta Q^2 = 1 - (\alpha \lambda^4/3) \), which can be used in equation (48a) to obtain
\[ C \equiv C' = \frac{1}{2} B \left[ \frac{15(1 + (1 - (\alpha \lambda^4/3)) Q)}{\lambda^6} - \alpha \right]. \]  
(48b)

Summarizing the above analysis, we have developed a theory for nonlinear excitations in multicomponent plasmas, in fact reducing the system of plasma–fluid equations to the ZK equation. At critical composition of negative ion concentration, i.e. \( \beta \equiv \beta_c = (3 - \alpha \lambda^4)/3 Q^2 \), the nonlinear coefficient \( A \) vanishes. Since the pulses evolve when there is a balance between the effects of dispersion and nonlinearity, the required balance is missing at \( \beta \equiv \beta_c \). Equation (47) describes the pulse excitations in this case. However, an important question arises here regarding the strength of the nonlinearity \( A \). Specifically, one may wonder whether the coefficient \( A \) may acquire small values and what happens then. Therefore, we may anticipate that higher-order nonlinearity enters into play, if \( A \) is of the order of, say, \( \varepsilon \). To answer this question, one should obtain an appropriate equation that describes the evolution of the system in this case. Now, we shall therefore explicitly assume that \( A \sim \varepsilon \), in order to examine the behavior of the nonlinear waves in multicomponent plasma in the vicinity of the critical composition of negative ion concentration \( \beta_c \). We have shown that the right-hand side of Poisson’s equation is given to \( \text{O}(\varepsilon^2) \) by
\[ n_e^{(2)} + n_n^{(2)} - n_+^{(2)} = \left( \frac{3}{\lambda^4} - \frac{3 \beta Q^2}{\lambda^4} - \alpha \right) \phi^{(1)2}. \]  
(49)

We now assume that the deviation of the system from the state of critical density is \( \text{O}(\varepsilon) \), i.e. the coefficient of \( \phi^{(1)2} \) is \( \text{O}(\varepsilon^2) \), and thus the right-hand side of (49) is \( \text{O}(\varepsilon^3) \). So, in Poisson’s equation to \( \text{O}(\varepsilon^3) \), we have to take this quantity into account [26]. Then (46) will be replaced by
\[ \frac{\partial^2 \phi^{(1)}}{\partial X^2} + \frac{\partial^2 \phi^{(1)}}{\partial Y^2} = n_e^{(3)} + n_n^{(3)} - n_+^{(3)} + n_e^{(2)} + n_n^{(2)} - n_+^{(2)}, \]  
(50)
where \( n_e^{(2)} \), \( n_+^{(2)} \) and \( n_n^{(2)} \) are still given by (28), (36) and (39), respectively. If we omit the rather lengthy and cumbersome calculations, we obtain the EZK equation:
\[ \frac{\partial \phi^{(1)}}{\partial \tau} + (A \phi^{(1)} + C \phi^{(1)2}) \frac{\partial \phi^{(1)}}{\partial X} + B \frac{\partial^3 \phi^{(1)}}{\partial X^3} + D \frac{\partial^3 \phi^{(1)}}{\partial X \partial Y^2} = 0, \]
(51)
which is the evolution equation in the vicinity of the critical negative ion density. It consists of the nonlinear terms of the ZK equation (30) and the MZK equation (47). Thus the evolution
equation (51) can be studied for the particular cases of the ZK or the MZK equation and serves especially as the transitive link between the various ZK equations. Note that the nonlinear and dispersion coefficients are still given by (31), (32), (33) and (48a).

2.3. Analytical solutions of the ZK, MZK and EZK equations

The analytical solutions of equations (30), (47) and (51) can be obtained using the generalized expansion method [27]. The mathematical details are given in the appendix. For simplicity, we shall use the notation \( \phi = \phi^{(1)} \) to summarize the final solutions of the evolution equations.

(i) Equation (30) has a solitary-wave solution in the form

\[
\phi = \phi_0 \text{sech}^2 \left( \frac{L_X X + L_Y Y - \vartheta \tau}{W} \right),
\]

where \( \phi_0 = 3\vartheta / (AL_X) \) is the maximum (potential perturbation) amplitude, \( W = \sqrt{4\Gamma/\vartheta} \) is the pulse width, \( L_X \) and \( L_Y \) are the direction cosines (i.e. \( L_X^2 + L_Y^2 = 1 \)), \( \Gamma = L_X(BL_X^2 + DL_Y^2) \) and \( \vartheta \) is the soliton speed to be determined later.

(ii) Equation (47) has a localized-pulse solution as

\[
\phi = \pm \sqrt{6\vartheta \Gamma / L_X} \text{sech} \left( \sqrt{2(L_X X + L_Y Y - \vartheta \tau)} \right),
\]

(iii) Equation (51) admits both solitary and shock wave solutions, which are given as, respectively,

\[
\phi = - \frac{2c_2}{c_3 \pm \sqrt{\Delta}} \cosh \left( \sqrt{c_3(L_X X + L_Y Y - \vartheta \tau)} \right),
\]

where \( A, B, D, C \) and \( C' \) are given by (31), (32), (33), (48a) and (48b), respectively.

3. Numerical analyses and discussion

There are many types of plasmas present in space and used in laboratory experiments. For example, \((\text{H}^+, \text{O}^-_2)\) and \((\text{H}^+, \text{H}^-)\) plasmas have been found in the D- and F-regions of the Earth’s ionosphere, but \((\text{Ar}^+, \text{F}^-)\) plasma was used to study the ion-acoustic wave propagation [28] in laboratory experiments. It is clear that the three types of plasmas have different components, which result in different mass ratios \( Q (= m_+/m_-) \). As expected, the mass ratio plays a role in the properties of the wave propagation. Therefore, part of our interest is in investigating the effect of the mass ratio \( Q \) on the existence of the pulse excitations. The mass ratios for \((\text{H}^+, \text{O}^-_2)\), \((\text{H}^+, \text{H}^-)\) and \((\text{Ar}^+, \text{F}^-)\) plasmas are 0.03, 1 and 2.1, respectively. Firstly, we shall investigate the properties of the solitary excitations represented by equation (30) (which has a solution (52)). Note that the sign of the nonlinear coefficient \( A \) affects the sign of \( \phi_0 \), which in turn determines
the polarity of the potential perturbation \( \phi^{(1)} \equiv \varphi \). Therefore, both positive and negative pulses may possibly be obtained, for different parameter values, respectively, representing a potential hump or a dip (an electrostatic potential hole). Furthermore, note that the soliton amplitude \( \phi_0 \) is proportional to the soliton speed \( \vartheta \) and inversely proportional to the soliton width \( W \). Thus, faster solitons will be taller and narrower, while slower ones will be shorter and wider. The qualitative features of the soliton properties are thus recovered [29].

Figure 1 presents the contour plot of the wave amplitude \( \phi_0 \) with dust density/concentration \( \gamma \) and negative ion density/concentration \( \beta \) for positive dust particles at different values of mass ratio \( Q \). Firstly, the light-colored regions correspond to higher (lower) values of the positive (negative) wave amplitude. For \((\text{H}^+, \text{O}_2^-)\) plasma in the Earth’s ionosphere, the mass ratio \( Q = 0.03 \) is low; it is seen that the positive potential has a wider region than the negative potential (cf figure 1(a)). The behavior of the wave amplitude \( \phi_0 \) for the other plasma types is presented in figures 1(b) and (c) for the Earth’s ionosphere (\(\text{H}^+, \text{H}^-\)) and laboratory experiment (\(\text{Ar}^+, \text{F}^-\)) with larger mass ratios \( Q = 1 \) and 2.1, respectively. It is seen that, for larger mass ratio, the negative potential becomes much more dominant than the positive potential. Increasing dust density \( \gamma \) leads to a decrease (increase) in the positive (negative) wave amplitude. However, increasing negative ion density \( \beta \) enhances (decreases) the positive (negative) pulse amplitude. The contour plot of the wave amplitude \( \phi_0 \) with dust concentration \( \gamma \) and negative ion concentration \( \beta \) for negative dust particles and different values of mass ratio \( Q \) is depicted in figure 2. It is obvious that increasing the dust density \( \gamma \) and negative ion density \( \beta \) increases (decreases) the positive (negative) pulse amplitude. From equation (52), it is clear that the width \( W \) of the solitary pulses depends on dust density \( \gamma \), negative ion density \( \beta \) and positive/negative ion cyclotron frequency \( \omega_\gamma \). The effects of the dust density \( \gamma \) and the negative ion density \( \beta \) on the solitary width \( W \) are depicted in figure 3 for \((\text{H}^+, \text{H}^-)\) plasma in the Earth’s ionosphere, where the mass ratio \( Q = 1 \). It is seen that, for positive dust particles (cf Figure 3(a)), the increase in dust concentration \( \gamma \) (negative ion concentration \( \beta \)) shrinks (en-hances) the pulse width. For negative dust particles (cf Figure 3(b)), increasing both dust density \( \gamma \) and negative ion density \( \beta \) makes the solitary pulses much wider. For different mass ratios of the Earth’s ionosphere plasma (\(\text{H}^+, \text{O}_2^-\)) and laboratory plasma (\(\text{Ar}^+, \text{F}^-\)), we obtain the same qualitative behavior as in figure 3. The effects of positive/negative ion cyclotron frequency \( \omega_{\xi+,\text{...}} \) on the solitary waves are depicted in figure 4. Here, we shall consider the positive dust case since the negative dust case gives the same qualitative behavior as positive dust. It is seen that increasing the positive ion cyclotron frequency \( \omega_\gamma \) has an important role, i.e. it decreases the solitary pulse width for (\(\text{H}^+, \text{O}_2^-\)) plasma with low mass ratio \( Q = 0.03 \), but increasing the negative ion cyclotron frequency \( \omega_\gamma \) has no significant effect. For (\(\text{H}^+, \text{H}^-\)) and (\(\text{Ar}^+, \text{F}^-\)) plasmas with higher mass ratios \( Q = 1 \) and 2.1, respectively (cf Figures 4(b) and (c)), both positive and negative ion cyclotron frequencies \( \omega_{\xi+,\text{...}} \) make the width narrower (i.e. more spiky).

Recall that, at a critical composition of negative ion concentration (i.e. \( \beta \equiv \beta_c = (3 - \alpha \lambda^3)/3Q^2 \)), the nonlinear coefficient \( A \) in equation (30) acquires negligible value. Since the solitons evolve when there is a balance between the effects of dispersion and nonlinearity, the required balance is missing at \( \beta_c \). Equation (47) describes the pulse excitations in this case, which have a solitary pulse solution given by equation (53). Now, it is important to examine the properties of the electrostatic excitations for equation (53). It is clear that the width has the same behavior as equation (52) but the amplitude depends now on a new nonlinear coefficient \( C' \) (given by equation (48b)). It is obvious that, to have real amplitude, the nonlinear coefficient \( C' \) must be positive, which we shall examine now. We have plotted the contour plot of \( C' \) for
Figure 1. The contour plot of the amplitude (given by equation (52)), with $\gamma$ and $\beta$ for positive dust particles, where (a) $Q = 0.03$, (b) $Q = 1$ and $Q = 2.1$. Here, we have used $L_x = 0.8$. The light-colored regions correspond to higher (lower) values of the positive (negative) wave amplitude.
Figure 2. The contour plot of the amplitude (given by equation (52)), with $\gamma$ and $\beta$ for negative dust particles, where (a) $Q = 0.03$, (b) $Q = 1$ and $Q = 2.1$. Here, we have used $L_x = 0.8$. The light-colored regions correspond to higher (lower) values of the positive (negative) wave amplitude.
positive and negative dust particles in figures 5 and 6, respectively, for the Earth’s ionosphere plasma ($H^+, H^-)$, where the mass ratio $Q = 1$. Figures 5(a) and 6(a) show the regions for positive and negative $C'$. For negative $C'$, the solitary pulses cannot exist due to the imaginary amplitude, but for positive $C'$ the solitary excitations propagate. The behavior of the solitary pulses given by equation (53) is depicted in figures 7 and 8 for positive and negative dust particles, respectively. It is seen that, for dust polarity, the increase in negative ion concentration $\beta$ enhances the soliton amplitude. For positive (negative) dust particles, the amplitude shrinks (increases) with an increase in the dust concentration $\gamma$.

Now, it is interesting to examine the behavior of the nonlinear excitations in the vicinity of the critical plasma composition described by equation (51). Equation (51) admits both solitary and shock solutions represented by equations (54) and (55), respectively. From equation (54), it is obvious that the solitary pulses can only propagate for positive $\Delta$. The latter is depicted

Figure 3. The contour plot of the width (given by equation (52)), with $\gamma$ and $\beta$ for (a) positive dust particles and (b) negative dust particles, where $Q = 1$, $L_x = 0.8$, $\omega_{c+} = 0.03$ and $\omega_{c-} = 0.05$. 

New Journal of Physics 12 (2010) 073010 (http://www.njp.org/)
Figure 4. The contour plot of the width (given by equation (52)), with $\omega_{c+}$ and $\omega_{c-}$ for positive dust particles, where (a) $Q = 0.03$, (b) $Q = 1$ and $Q = 2.1$. Here, we have used $\gamma = \beta = 0.2$ and $L_x = 0.8$. 
Figure 5. The contour plot of (a) the nonlinear coefficients $C$ (given by equation (48a)) and $C'$ (given by equation (48b)). (b) The determinant $\Delta$ and (c) the nonlinear coefficient $A$ are depicted against $\gamma$ and $\beta$, for positive dust particles, where $Q = 1$ and $L_x = 0.8$. Note that the upper (lower) regions of the lines $C = 0$ and $C' = 0$ are negative (positive).
Figure 6. The contour plot of (a) the nonlinear coefficient $C$ (given by equation (48a)) and $C'$ (given by equation (48b)). (b) The determinant $\Delta$ and (c) the nonlinear coefficient $A$ are depicted against $\gamma$ and $\beta$, for negative dust particles, where $Q = 1$ and $L_x = 0.8$. Note that the upper (lower) regions of the lines $C = 0$ and $C' = 0$ are negative (positive).
Figure 7. The contour plot of the amplitude (given by equation (53)), with γ and β for positive dust particles, where (a) \( Q = 0.03 \), (b) \( Q = 1 \) and \( Q = 2.1 \). Here, we have used \( L_x = 0.8 \). The light-colored regions correspond to higher (lower) values of the positive (negative) wave amplitude.
Figure 8. The contour plot of the amplitude (given by equation (53)), with $\gamma$ and $\beta$ for negative dust particles, where (a) $Q = 0.03$, (b) $Q = 1$ and $Q = 2.1$. Here, we have used $L_x = 0.8$. The light-colored regions correspond to higher (lower) values of the positive (negative) wave amplitude.
Figure 9. The solitary potential profile $\varphi$ (given by equation (54)) is depicted against $\xi$ (a) positive dust particles, for positive potential (thin curves) $\gamma = 0.4$ and $\beta = 0.2$ (black-solid line), $\gamma = 0.4$ and $\beta = 0.3$ (red-dashed line) and $\gamma = 0.5$ and $\beta = 0.2$ (blue-dotted line). For negative potential (thick curves) $\gamma = 0.2$ and $\beta = 0.4$ (black-solid line), $\gamma = 0.2$ and $\beta = 0.5$ (red-dashed line) and $\gamma = 0.3$ and $\beta = 0.4$ (blue-dotted line). (b) Negative dust particles, for positive potential (thin curves) $\gamma = 0.2$ and $\beta = 0.1$ (black-solid line), $\gamma = 0.2$ and $\beta = 0.15$ (red-dashed line) and $\gamma = 0.25$ and $\beta = 0.1$ (blue-dotted line). For negative potential (thick curves) $\gamma = 0.3$ and $\beta = 0.4$ (black-solid line), $\gamma = 0.3$ and $\beta = 0.45$ (red-dashed line) and $\gamma = 0.35$ and $\beta = 0.4$ (blue-dotted line). Here, $Q = 1$, $L_x = 0.8$, $\omega_{e+} = 0.3$ and $\omega_{e-} = 0.5$.

in figures 5(b) and 6(b) for positive and negative dust particles, respectively, for the Earth’s ionosphere plasma ($H^+$, $H^-$), as an example, where the mass ratio $Q = 1$. It is seen that $\Delta$ is usually positive, except for a certain region (represented by the white part). It is obvious that the charge neutrality condition is not satisfied in that region. In figure 9, we have numerically analyzed the solitary pulse solution (54) and investigated how the negative ions and dust particles change the profile of the electrostatic excitation. It turns out that both positive and negative potentials can propagate, depending on the concentration values of dust particles $\gamma$ and negative ions $\beta$. On the other hand, for the positive dust case, both positive and negative potentials exist. For high values of negative ions $\beta$, the pulse is more spiky, but the pulse
Figure 10. The profile $\varphi$ (given by equation (A.6)) is depicted against $\xi$ for $\gamma = 0.4$, $\beta = 0.2$, $Q = 1$, $L_x = 0.8$, $\omega_{+e} = 0.3$, $\omega_{-e} = 0.5$, $\vartheta = 0.01$, $c_2 = -1$ and $c_4 = 1$. Here, $m = 0.3$ (red-dotted line), $m = 0.4$ (blue-dashed line) and $m = 1$ (black-solid line).

becomes wider with an increase in positive dust particles $\gamma$. The effect of the negative dust particles has different scenarios, as depicted in figure 9(b). Increasing the negative dust particle density $\gamma$ and the negative ion density $\beta$ causes the propagation of taller and narrower positive pulses but shorter and wider negative pulses.

Now, we shall discuss the existence of the double layers (shocks). As we saw above in equation (55), $\Gamma$ is always positive, therefore the existence of the double layer requires $C < 0$ (given by equation (48a)). The Earth’s ionosphere plasma ($\text{H}^+$, $\text{H}^-$) will be used as an example to investigate numerically the nonlinear coefficient $C$. The numerical analyses in figures 5(a) and 6(a) show that, for low negative ion concentration $\beta$, the dominant situation corresponds to $C > 0$, while for high negative ion density $\beta$, the nonlinear coefficient $C < 0$. However, it is clear from figures 5(a) and (b) and 6(a) and (b) that the region of negative $C$ usually overlaps with the region of positive $\Delta$. Physically, double layers propagate when the nonlinear-dispersion balance is missing and cannot then survive the solitary pulses propagating. On the other hand, when the nonlinear-dispersion balance is achieved, the double layers cannot exist. Thus, even for $C < 0$, the solitary pulse still exists (due to $\Delta > 0$) and therefore the double layers cannot propagate in the present plasma system. It is expected that, for non-Maxwellian electron distributions (such as nonthermal distribution, non-isothermal distribution), the double layers could exist, which will be considered in the future.

First of all, in view of the analysis and interpretation of our results, we should point out that solutions (A.5)–(A.7) admit a train of well-separated bell-shaped periodic pulses. The latter can degenerate to solitary pulses at certain values of $m$ (where $m$ is the modulus of the three Jacobi elliptic functions $\text{cn}$, $\text{dn}$ and $\text{sn}$). On the other hand, when $m \to 0$, the Jacobi elliptic functions degenerate to the triangular functions, i.e. $\text{sn} \to \sin$, $\text{cn} \to \cos$ (which support periodic solutions). However, for $m \to 1$, the Jacobi elliptic functions degenerate to the hyperbolic functions, i.e. $\text{sn} \to \tanh$, $\text{cn} \to \text{sech}$ (which support solitary solutions). In figure (10), we have numerically analyzed solution (A.6) ‘as an example’ for different values of $m$. It is clear that, for small values of $m$, solution (A.6) admits periodic pulses, whereas increasing $m \to 1$
leads to propagation of the solitary pulses. This result could not have been obtained using the travelling wave directly (see, e.g., [8]), but it is obvious that the generalized expansion method supports periodic, solitary and double layer solutions, depending on the physical parameters of the system.

4. Summary

In this paper, we have studied the nonlinear propagation of ion-acoustic waves in dusty electronegative magnetoplasmas, where a background of stationary dust was considered. We have derived the ZK equation, the MZK equation (at critical plasma composition) and the EZK equation (in the vicinity of the critical plasma composition). Using the generalized expansion method, a new class of solutions of the evolution equations that admits a train of well-separated bell-shaped periodic pulses is obtained. In certain conditions, these solutions degenerate to solitary and shock wave solutions. We have used the present model to investigate the behavior of the nonlinear structures in different plasma environments with different negative ion species, such as D- and F-regions of the Earth’s ionosphere (H+, O−2) and (H+, H−), as well as in laboratory plasma experiment, (Ar+, F−). Numerical analysis of the solutions revealed that the profile of the nonlinear pulses suffers amplitude and width modifications due to enhancement of the dust particle density γ, negative ion density β, positive-to-negative ion mass ratio Q and positive/negative ion cyclotron frequency ωc+−. Furthermore, the necessary conditions for the propagation of both solitary pulses and shock waves, as well as their polarity, are examined.

Appendix. Solutions of equations (30), (47) and (51) using the generalized expansion method

To obtain the possible analytical solutions of equations (30), (47) and (51), we assume that

\[ \phi = \phi^{(1)}(\xi) \quad \text{and} \quad \xi = L_X X + L_Y Y - \vartheta \tau, \]  

where \( L_X \) and \( L_Y \) are the direction cosines (i.e. \( L_X^2 + L_Y^2 = 1 \)) and \( \vartheta \) is the acoustic speed to be determined later. Putting (A.1) into (51), we obtain

\[ -\vartheta \phi' + A_0\phi\phi' + B_0\phi^2\phi' + \gamma \phi''' = 0, \]

where \( A_0 = AL_X \), \( B_0 = CL_X \) and \( \Gamma = L_X(BL_X^2 + DL_Y^2) \). According to the generalized expansion method [27] and due to the mixed nonlinearity, the solution of equation (A.2) can be represented by

\[ \phi = \sum_{i=0}^{2} a_i \omega^i, \]

with

\[ \frac{d\omega}{d\xi} = k \sum_{i=0}^{4} c_i \omega^i \]

where \( a_0, a_1, a_2, c_0, c_1, c_2, c_3 \) and \( c_4 \) are arbitrary constants to be determined later and \( k = \pm 1 \). Substituting equation (A.3) into equation (A.2) and making use of equation (A.4), we obtain a
Hence, solution (A.10) reduces to

\[ \varphi = \phi_0 \text{sech}^2 \left( \frac{\xi}{W} \right), \]  

(A.10)
where \( \phi_0 = 3\vartheta/(AL_X) \) is the maximum (potential perturbation) amplitude and \( W = \sqrt{4L_X(BL_X^2 + DL_Y^2)/\vartheta} \) is the pulse width.

**Case B:** for \( B_0 \neq 0 \) (i.e. \( C \neq 0 \)), solutions of equations (47) and (51)

\[
\varphi = -\frac{A_0}{2B_0} + k \sqrt{\frac{6\Gamma c_2 m^2}{B_0 (2m^2 - 1)}} \, \text{cn} \left( \sqrt{\frac{c_2}{(2m^2 - 1)}} \, \xi \right), \tag{A.11}
\]

with \( c_0 = -\frac{c_2^2 m^2 (1 - m^2)}{c_4 (2m^2 - 1)^2}, \quad c_2 > 0, \quad c_4 < 0, \)

\[
\varphi = -\frac{A_0}{2B_0} + k \sqrt{\frac{6\Gamma c_2}{B_0 (2 - m^2)}} \, \text{dn} \left( \sqrt{\frac{c_2}{(2 - m^2)}} \, \xi \right), \tag{A.12}
\]

with \( c_0 = \frac{c_2^2 (1 - m^2)}{c_4 (2 - m^2)^2}, \quad c_2 > 0, \quad c_4 < 0, \)

and

\[
\varphi = -\frac{A_0}{2B_0} + k \sqrt{\frac{6\Gamma c_2 m^2}{B_0 (m^2 + 1)}} \, \text{sn} \left( \sqrt{\frac{-c_2}{(m^2 + 1)}} \, \xi \right), \tag{A.13}
\]

with \( c_0 = \frac{c_2^2 m^2}{c_4 (m^2 + 1)^2}, \quad c_2 < 0, \quad c_4 > 0, \)

where \( m \) is a modulus of the Jacobian elliptic function and \( c_1 = c_3 = 0 \). As \( m \to 1 \), the Jacobi doubly periodic solution (A.13) degenerates to the bell-shaped solitary wave:

\[
\varphi = -\frac{A_0}{2B_0} + k \sqrt{\frac{6\Gamma c_2}{B_0}} \sech \left( \sqrt{\frac{c_2}{2}} \, \xi \right), \quad \text{with} \ c_2 > 0, \quad c_4 < 0, \tag{A.14}
\]

where the arbitrary constant \( c_0 \) vanishes. Note that, for \( A_0 = 0 \) (i.e. \( A = 0 \)), the last equation provides us with the bell-shaped solitary wave solution of equation (47) as

\[
\varphi = k \sqrt{\frac{6\Gamma c_2}{B_0}} \sech \left( \sqrt{\frac{c_2}{2}} \, \xi \right), \tag{A.15}
\]

where \( c_2 = \vartheta/\Gamma \) and \( B_0 = C'L_X \). Recall that we should use \( C' \) (from equation (48b)) in the expression of \( B_0 \), since \( A = 0 \) in this case.

Again, as \( m \to 1 \), solution (A.13) can degenerate to the kink-type wave solution:

\[
\varphi = -\frac{A_0}{2B_0} + k \sqrt{\frac{3\Gamma c_2}{B_0}} \tanh \left( \sqrt{\frac{-c_2}{2}} \, \xi \right), \quad c_2 < 0, \quad c_4 > 0, \tag{A.16}
\]

where \( c_0 = c_2^2 / 4c_4 \) and \( c_2 = A_2^2 / 12\Gamma B_0 \). In solutions (A.11)–(A.16), the acoustic speed \( \vartheta = \frac{1}{2}(-A_0^2 / 2B_0 + 2\Gamma c_2) \) where \( c_2 \neq A_0^2 / 4\Gamma B_0 \). Furthermore, the generalized expansion method
provides us with further analytical solutions of the EZK equation (51) as

\[
\varphi = -\frac{2c_2}{c_3 + k\sqrt{\Delta} \cosh\left(\sqrt{c_2}\xi\right)} ,
\]

with \( \Delta = c_3^2 - 4c_2c_4 \), \( c_0 = c_1 = 0 \), \( c_2 = \frac{\vartheta}{\Gamma} \), \( c_3 = -\frac{A_0}{3\Gamma} \), \( c_4 = -\frac{B_0}{6\Gamma} \),

(A.17)

and

\[
\varphi = -\frac{A_0}{2B_0} \left[ 1 + k \coth\left(\frac{\sqrt{-A_0^2}}{24\Gamma B_0^\vartheta} \xi\right) \right] ,
\]

(A.18)

with \( \vartheta = -\frac{A_0^2}{6B_0} \) and \( B_0 < 0 \).

Recall that the obtained solutions (A.14) and (A.15) are valid only for the MZK equation (47), where \( A_0 = A L_X = 0 \) (so, in this case, we must use the expression of \( C' \) from (48b)), while solutions (A.13), (A.16), (A.17) and (A.18) can be used only for the EZK equation (51) where both \( A \) and \( C \) are not zero (so, in this case, we must use the expression of \( C' \) from (48a)).

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