INEQUALITIES FOR CONVEX AND s–CONVEX FUNCTIONS
ON $\Delta = [a, b] \times [c, d]$

M. EMIN ÖZDEMİR♦, HAVVA KAVURMACI♦♦, AHMET OCAK AKDEMİR♦, AND MERVE AVCİ♦

Abstract. In this paper, two new lemmas are proved and inequalities are established for co-ordinated convex functions and co-ordinated s–convex functions.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a < b$. The following double inequality:

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}$$

is well known in the literature as Hadamard’s inequality. Both inequalities hold in the reversed direction if $f$ is concave.

In [8], Orlicz defined $s$–convex functions as following:

**Definition 1.** A function $f : \mathbb{R}^{+} \to \mathbb{R}$, where $\mathbb{R}^{+} = [0, \infty)$, is said to be $s$–convex in the first sense if

$$f(\alpha x + \beta y) \leq \alpha^{s}f(x) + \beta^{s}f(y)$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha^{s} + \beta^{s} = 1$ and for some fixed $s \in (0, 1]$. We denote by $K_{1}^{s}$ the class of all $s$–convex functions.

**Definition 2.** A function $f : \mathbb{R}^{+} \to \mathbb{R}$, where $\mathbb{R}^{+} = [0, \infty)$, is said to be $s$–convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^{s}f(x) + \beta^{s}f(y)$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. We denote by $K_{2}^{s}$ the class of all $s$–convex functions.

Obviously, one can see that if we choose $s = 1$, both definitions reduced to ordinary concept of convexity.

For several results related to above definitions we refer readers to [6], [9], [13].

In [2], Dragomir defined convex functions on the co-ordinates as following:
Definition 3. Let us consider the bidimensional interval \( \Delta = [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \), \( c < d \). A function \( f : \Delta \rightarrow \mathbb{R} \) will be called convex on the co-ordinates if the partial mappings \( f_y : [a, b] \rightarrow \mathbb{R}, f_y(u) = f(u, y) \) and \( f_x : [c, d] \rightarrow \mathbb{R}, f_x(v) = f(x, v) \) are convex where defined for all \( y \in [c, d] \) and \( x \in [a, b] \). Recall that the mapping \( f : \Delta \rightarrow \mathbb{R} \) is convex on \( \Delta \) if the following inequality holds,

\[
f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \lambda f(x, y) + (1 - \lambda)f(z, w)
\]

for all \((x, y), (z, w) \in \Delta\) and \( \lambda \in [0, 1] \).

In [2], Dragomir established the following inequalities of Hadamard’s type for co-ordinated convex functions on a rectangle from the plane \( \mathbb{R}^2 \).

Theorem 1. Suppose that \( f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R} \) is convex on the co-ordinates on \( \Delta \). Then one has the inequalities;

\[
(1.1) \quad f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) \leq \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b f(x, \frac{c + d}{2}) dx + \frac{1}{d - c} \int_c^d f\left(\frac{a + b}{2}, y\right) dy \right]
\]

\[
\leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dx dy
\]

\[
\leq \frac{1}{4} \left[ \frac{1}{b - a} \int_a^b f(x, c) dx + \frac{1}{(b - a)} \int_a^b f(x, d) dx
\]

\[
+ \frac{1}{(d - c)} \int_c^d f(a, y) dy + \frac{1}{(d - c)} \int_c^d f(b, y) dy \right]\n\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\]

The above inequalities are sharp.

Similar results can be found in [5], [7], [10] and [11].

In [10], Alomari and Darus defined co-ordinated \( s \)-convex functions and proved some inequalities based on this definition.

Definition 4. Consider the bidimensional interval \( \Delta = [a, b] \times [c, d] \) in \([0, \infty)^2\) with \( a < b \) and \( c < d \). The mapping \( f : \Delta \rightarrow \mathbb{R} \) is \( s \)-convex on \( \Delta \) if

\[
f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \lambda^s f(x, y) + (1 - \lambda)^s f(z, w)
\]

holds for all \((x, y), (z, w) \in \Delta\) with \( \lambda \in [0, 1] \) and for some fixed \( s \in (0, 1] \).

In [12], Sarkanay et al. proved some Hadamard’s type inequalities for co-ordinated convex functions as followings:

Theorem 2. Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( f(x, y) \in C^2 \) is a convex function on the
co-ordinates on $\Delta$, then one has the inequalities:

\[
\begin{align*}
(1.2) & \quad \left| \frac{1}{4} f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| \\
& \leq \frac{(b - a)(d - c)}{16} \left( \frac{\partial^2 f}{\partial t \partial s} \left| a, c \right| + \frac{\partial^2 f}{\partial t \partial s} \left| a, d \right| + \frac{\partial^2 f}{\partial t \partial s} \left| b, c \right| + \frac{\partial^2 f}{\partial t \partial s} \left| b, d \right| \right)
\end{align*}
\]

where

\[
A = \frac{1}{2} \left[ \frac{1}{(b - a)} \int_a^b [f(x, c) + f(x, d)] dx + \frac{1}{(d - c)} \int_c^d \left[ f(a, y) dy + f(b, y) \right] dy \right].
\]

**Theorem 3.** Let $f: \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial t \partial s} \left| q \right|$, $q > 1$, is a convex function on the co-ordinates on $\Delta$, then one has the inequalities:

\[
\begin{align*}
(1.3) & \quad \left| \frac{1}{4} f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| \\
& \leq \frac{(b - a)(d - c)}{4 (p + 1)^{\frac{q}{2}}} \left( \frac{\partial^2 f}{\partial t \partial s} \left| q \right| a, c \right| + \frac{\partial^2 f}{\partial t \partial s} \left| q \right| a, d \right| + \frac{\partial^2 f}{\partial t \partial s} \left| q \right| b, c \right| + \frac{\partial^2 f}{\partial t \partial s} \left| q \right| b, d \right| \right)^{\frac{1}{q} - \frac{1}{p}}
\end{align*}
\]

where

\[
A = \frac{1}{2} \left[ \frac{1}{(b - a)} \int_a^b [f(x, c) + f(x, d)] dx + \frac{1}{(d - c)} \int_c^d \left[ f(a, y) dy + f(b, y) \right] dy \right]
\]

and $\frac{1}{p} + \frac{1}{q} = 1$.

**Theorem 4.** Let $f: \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial t \partial s} \left| q \right|$, $q \geq 1$, is a convex function on the co-ordinates on $\Delta$, then one has the inequalities:

\[
\begin{align*}
(1.4) & \quad \left| \frac{1}{4} f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| \\
& \leq \frac{(b - a)(d - c)}{16} \left( \frac{\partial^2 f}{\partial t \partial s} \left| q \right| a, c \right| + \frac{\partial^2 f}{\partial t \partial s} \left| q \right| a, d \right| + \frac{\partial^2 f}{\partial t \partial s} \left| q \right| b, c \right| + \frac{\partial^2 f}{\partial t \partial s} \left| q \right| b, d \right| \right)\frac{1}{q}
\end{align*}
\]

where

\[
A = \frac{1}{2} \left[ \frac{1}{(b - a)} \int_a^b [f(x, c) + f(x, d)] dx + \frac{1}{(d - c)} \int_c^d \left[ f(a, y) dy + f(b, y) \right] dy \right].
\]
In [1], Barnett and Dragomir proved an Ostrowski-type inequality for double integrals as following:

**Theorem 5.** Let \( f: [a, b] \times [c, d] \to \mathbb{R} \) be continuous on \([a, b] \times [c, d]\), \( f''_{xy} = \frac{\partial^2 f}{\partial x \partial y} \) exists on \((a, b) \times (c, d)\) and is bounded, that is

\[
\|f''_{xy}\|_\infty = \sup_{(x, y) \in (a, b) \times (c, d)} \left| \frac{\partial^2 f(x, y)}{\partial x \partial y} \right| < \infty,
\]

then we have the inequality:

\[
\left( 1.5 \right) \quad \left| \int_a^b \int_c^d f(s, t) dt ds - (b - a) \int_c^d f(x, t) dt \right|
\]

\[
- (d - c) \left| \int_a^b f(s, y) ds - (b - a)(d - c) f(x, y) \right|
\]

\[
\leq \left[ \frac{(b - a)^2}{4} + \left( x - \frac{a + b}{2} \right)^2 \right] \left( \frac{(d - c)^2}{4} + \left( y - \frac{c + d}{2} \right)^2 \right) \|f''_{xy}\|_\infty
\]

for all \((x, y) \in [a, b] \times [c, d]\).

In [1], Sarıkaya proved an Ostrowski-type inequality for double integrals and gave a corollary as following:

**Theorem 6.** Let \( f: [a, b] \times [c, d] \to \mathbb{R} \) be an absolutely continuous functions such that the partial derivative of order 2 exist and is bounded, i.e.,

\[
\left| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right| = \sup_{(x, y) \in (a, b) \times (c, d)} \left| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right| < \infty
\]

for all \((t, s) \in [a, b] \times [c, d]\). Then we have,

\[
(1.6) - \int_a^b [(\alpha_2 - c) f(t, c) + (d - \beta_2) f(t, d)] dt
\]

\[
- \int_c^d [((\alpha_1 - a) f(a, s) + (b - \beta_1) f(b, s))] ds + \int_a^b \int_c^d f(t, s) ds dt
\]

\[
\leq \left[ \frac{(\alpha_1 - a)^2 + (b - \beta_1)^2}{2} + \frac{(a + b - 2\alpha_1)^2 + (a + b - 2\beta_1)^2}{2} \right]
\]

\[
\times \left[ \frac{(\alpha_2 - c)^2 + (d - \beta_2)^2}{2} + \frac{(c + d - 2\alpha_2)^2 + (c + d - 2\beta_2)^2}{2} \right] \left| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right|_\infty
\]

for all \((\alpha_1, \alpha_2), (\beta_1, \beta_2) \in [a, b] \times [c, d]\) with \(\alpha_1 < \beta_1, \alpha_2 < \beta_2\) where

\[
H(\alpha_1, \alpha_2, \beta_1, \beta_2) = (\alpha_1 - a) [(\alpha_2 - c) f(a, c) + (d - \beta_2) f(a, d)]
\]

\[
+ (b - \beta_1) [(\alpha_2 - c) f(b, c) + (d - \beta_2) f(b, d)]
\]
and
\[
G (\alpha_1, \alpha_2, \beta_1, \beta_2) = \begin{pmatrix}
(\beta_1 - \alpha_1) \left[ (\alpha_2 - c) f \left( \frac{a + b}{2}, c \right) + (d - \beta_2) f \left( \frac{a + b}{2}, d \right) \right] \\
+ (\beta_2 - \alpha_2) \left[ (\alpha_1 - a) f \left( a, \frac{c + d}{2} \right) + (b - \beta_1) f \left( b, \frac{c + d}{2} \right) \right].
\end{pmatrix}
\]

Corollary 1. Under the assumptions of Theorem 6, we have
\[
\left| (b - a) (d - c) f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + \int_a^b \int_c^d f(t, s) ds dt - (d - c) \int_a^b f \left( t, \frac{c + d}{2} \right) dt - (b - a) \int_c^d f \left( \frac{a + b}{2}, s \right) ds \right| \leq \frac{1}{16} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} (b - a)^2 (d - c)^2.
\]

In [3], Pachpatte established a new Ostrowski type inequality similar to inequality (1.5) by using elementary analysis.

The main purpose of this paper is to establish inequalities of Ostrowski-type for co-ordinated convex functions by using Lemma 2 and to establish some new Hadamard’s type inequalities for co-ordinated s–convex functions.

2. INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

To prove our main result, we need the following lemma which contains kernels similar to Barnett and Dragomir’s kernels in [1], [see the paper [1], proof of Theorem 2.1].

Lemma 1. Let \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta = [a, b] \times [c, d] \). If \( \frac{\partial^2 f}{\partial t \partial s} \in L(\Delta) \), then the following equality holds:
\[
f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) - \frac{1}{(d - c)} \int_c^d f \left( \frac{a + b}{2}, y \right) dy - \frac{1}{(b - a)} \int_a^b f \left( x, \frac{c + d}{2} \right) dx + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx =\]
\[
\int_a^b \int_c^d p(x, t) q(y, s) \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b - t}{b - a} + \frac{t - a}{b - a} \frac{d - s}{d - c} + \frac{s - c}{d - c} \right) ds dt
\]
where
\[
p(x, t) = \begin{cases}
(t - a), & t \in \left[ a, \frac{a + b}{2} \right] \\
(t - b), & t \in \left( \frac{a + b}{2}, b \right)
\end{cases}
\]
and
\[
q(y, s) = \begin{cases}
(s - c), & s \in \left[ c, \frac{c + d}{2} \right] \\
(s - d), & s \in \left( \frac{c + d}{2}, d \right)
\end{cases}.
\]
for each \( x \in [a, b] \) and \( y \in [c, d] \).
Proof. Integration by parts, we can write

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(x,t) q(y,s) \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b-t}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{b-a} c + \frac{s-c}{d-c} d \right) dsdt
\]

\[
= \int_c^d q(y,s) \left[ \int_a^b \frac{\partial f}{\partial s} \left( b-t \frac{b-a}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{b-a} c + \frac{s-c}{d-c} d \right) dt \right. \\
\left. + \int_a^b \frac{\partial^2 f}{\partial s^2} \left( b-t \frac{b-a}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{b-a} c + \frac{s-c}{d-c} d \right) dt \right] ds \\
\]

\[
= \int_c^d q(y,s) \left\{ \left[ (t-a) \frac{\partial f}{\partial s} \left( b-t \frac{b-a}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{b-a} c + \frac{s-c}{d-c} d \right) \right]_a^b \\
- \int_a^b \frac{\partial f}{\partial s} \left( b-t \frac{b-a}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{b-a} c + \frac{s-c}{d-c} d \right) dt \right. \\
\left. + \left[ (t-b) \frac{\partial f}{\partial s} \left( b-t \frac{b-a}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{b-a} c + \frac{s-c}{d-c} d \right) \right]_a^b \\
- \int_a^b \frac{\partial^2 f}{\partial s^2} \left( b-t \frac{b-a}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{b-a} c + \frac{s-c}{d-c} d \right) dt \right\} ds.
\]

We obtain

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(x,t) q(y,s) \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b-t}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{b-a} c + \frac{s-c}{d-c} d \right) dsdt
\]

\[
= (b-a) \int_c^d q(y,s) \left\{ \frac{\partial f}{\partial s} \left( a+b, \frac{d-s}{2}, \frac{d-s}{d-c} c + \frac{s-c}{d-c} d \right) \\
- \int_a^b \frac{\partial f}{\partial s} \left( \frac{a+b}{2}, \frac{d-s}{2}, \frac{d-s}{d-c} c + \frac{s-c}{d-c} d \right) dt \right\} ds.
\]

By integrating again, we get

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(x,t) q(y,s) \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b-t}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{b-a} c + \frac{s-c}{d-c} d \right) dsdt
\]

\[
= (b-a) \left\{ \int_c^d (s-c) \frac{\partial f}{\partial s} \left( a+b, \frac{d-s}{2}, \frac{d-s}{d-c} c + \frac{s-c}{d-c} d \right) ds \\
+ \int_a^b (s-d) \frac{\partial f}{\partial s} \left( a+b, \frac{d-s}{2}, \frac{d-s}{d-c} c + \frac{s-c}{d-c} d \right) ds \\
- \int_a^b (s-c) \frac{\partial f}{\partial s} \left( b-t \frac{b-a}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{b-a} c + \frac{s-c}{d-c} d \right) ds \\
+ \int_a^b (s-d) \frac{\partial f}{\partial s} \left( b-t \frac{b-a}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{b-a} c + \frac{s-c}{d-c} d \right) dt \right\}.
\]
By calculating the above integrals, we have

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(x,t) q(y,s) \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b-t}{b-a} a + \frac{t-a}{b-a} b, \frac{d-s}{d-c} c + \frac{s-c}{d-c} d \right) \, ds \, dt
\]

\[
= (b-a)(d-c) f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)
- (b-a) \int_c^d f \left( \frac{a+b}{2}, \frac{d-s}{d-c} c + \frac{s-c}{d-c} d \right) \, ds
- (d-c) \int_a^b f \left( \frac{b-t}{b-a} b + \frac{t-a}{b-a} a, \frac{c+d}{2} \right) \, dt
\]

\[
\int_a^b \int_c^d f \left( \frac{b-t}{b-a} b + \frac{t-a}{b-a} a, \frac{d-s}{d-c} c + \frac{s-c}{d-c} d \right) \, ds \, dt.
\]

Using the change of the variable \( x = \frac{b-a}{b-a} a + \frac{d-c}{d-c} b \) and \( y = \frac{b-a}{b-a} c + \frac{b-a}{b-a} d \), then dividing both sides with \( (b-a) \times (d-c) \), this completes the proof. \( \square \)

**Theorem 7.**  Let \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta = [a, b] \times [c, d] \). If \( \left| \frac{\partial^2 f}{\partial t \partial s} \right| \) is a convex function on the co-ordinates on \( \Delta \), then the following inequality holds:

\[
(2.1) \quad \left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \right|
- \frac{1}{(d-c)} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy
- \frac{1}{(b-a)} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx
\]

\[
\leq \frac{(b-a)(d-c)}{64} \left[ \left| \frac{\partial^2 f}{\partial t \partial s} (a,c) \right| + \left| \frac{\partial^2 f}{\partial t \partial s} (b,c) \right| + \left| \frac{\partial^2 f}{\partial t \partial s} (a,d) \right| + \left| \frac{\partial^2 f}{\partial t \partial s} (b,d) \right| \right].
\]

**Proof.** From Lemma 2 and using the property of modulus, we have

\[
\left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \right|
- \frac{1}{(d-c)} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy
- \frac{1}{(b-a)} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx
\]

\[
\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d |p(x,t) q(y,s)| \left| \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b-t}{b-a} a + \frac{t-a}{b-a} b, \frac{d-s}{d-c} c + \frac{s-c}{d-c} d \right) \right| \, ds \, dt
\]
Since $|\frac{\partial^2 f}{\partial t \partial s}|$ is co-ordinated convex, we can write

$$
\begin{align*}
&\left| f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
&- \frac{1}{(d - c)} \int_c^d f \left( \frac{a + b}{2}, y \right) dy - \frac{1}{(b - a)} \int_a^b f \left( x, \frac{c + d}{2} \right) dx \\
&+ \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f (x, y) dy dx \\
&\leq \frac{1}{(b - a)(d - c)} \\
&\times \int_c^d |q(y, s)| \left\{ \int_a^{\frac{a + b}{2}} (t - a) \left[ \frac{b - t}{b - a} \frac{\partial^2 f}{\partial t \partial s} \left( a, \frac{d - s}{d - c} + \frac{s - c}{d - c} \right) \right] dt \\
&+ \int_a^{\frac{b}{2}} (b - t) \left[ \frac{t - a}{b - a} \frac{\partial^2 f}{\partial t \partial s} \left( a, \frac{d - s}{d - c} + \frac{s - c}{d - c} \right) \right] dt \\
&+ \int_{\frac{b}{2}}^{\frac{b - t}{b - a}} \left[ \frac{b - t}{b - a} \frac{\partial^2 f}{\partial t \partial s} \left( b, \frac{d - s}{d - c} + \frac{s - c}{d - c} \right) \right] dt \} ds.
\end{align*}
$$

By computing these integrals, we obtain

$$
\begin{align*}
&\left| f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
&- \frac{1}{(d - c)} \int_c^d f \left( \frac{a + b}{2}, y \right) dy - \frac{1}{(b - a)} \int_a^b f \left( x, \frac{c + d}{2} \right) dx \\
&+ \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f (x, y) dy dx \\
&\leq \frac{(b - a)}{8(d - c)} \left\{ \int_c^d |q(y, s)| \left| \frac{\partial^2 f}{\partial t \partial s} \left( a, \frac{d - s}{d - c} + \frac{s - c}{d - c} \right) \right| \right. \\
&\left. + \int_c^d |q(y, s)| \left| \frac{\partial^2 f}{\partial t \partial s} \left( b, \frac{d - s}{d - c} + \frac{s - c}{d - c} \right) \right| \right\} ds.
\end{align*}
$$
Using co-ordinated convexity of \( \frac{\partial^2 f}{\partial t \partial s} \), again, we get

\[
\left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) - \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) dy - \frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) dx \right|
\]

\[
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dydx \leq \frac{(b-a)}{8(d-c)} \left\{ \int_c^d (s-c) \left[ \frac{d-s}{d-c} \frac{\partial^2 f}{\partial t \partial s}(a,c) \right] ds + \int_c^d (d-s) \left[ \frac{s-c}{d-c} \frac{\partial^2 f}{\partial t \partial s}(a,d) \right] ds \right. \\
+ \left. \int_c^{c+d} (s-c) \left[ \frac{d-s}{d-c} \frac{\partial^2 f}{\partial t \partial s}(b,c) \right] ds + \int_c^{c+d} (d-s) \left[ \frac{s-c}{d-c} \frac{\partial^2 f}{\partial t \partial s}(b,d) \right] ds \right\}.
\]

By a simple computation, we get the required result.

\[
\square
\]

**Remark 1.** Suppose that all the assumptions of Theorem 7 are satisfied. If we choose \( \frac{\partial^2 f}{\partial t \partial s} \) is bounded, i.e.,

\[
\left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty = \sup_{(t,s) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right| < \infty,
\]

we get

\[
(2.2)
\left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) - \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) dy - \frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) dx \right|
\]

\[
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dydx \leq \frac{(b-a)}{16} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty
\]

which is the inequality \((1.7)\).

**Theorem 8.** Let \( f : \Delta = [a,b] \times [c,d] \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta = [a,b] \times [c,d] \). If \( \left| \frac{\partial^2 f}{\partial t \partial s} \right|^q \), \( q > 1 \), is a convex function on the co-ordinates on \( \Delta \),
then the following inequality holds:

\[
\left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) + \frac{1}{(b-a) (d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx \right| \\
- \frac{1}{(d-c)} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy - \frac{1}{(b-a)} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx \leq \frac{1}{(b-a) (d-c)} \int_a^b \int_c^d |p(x,t) q(y,s)| \left| \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b-t}{b-a} + \frac{a-b}{d-c} + \frac{d-s}{d-c} + \frac{s-c}{d-c} \right) \right|^q \, ds \, dt
\]

By applying the well-known Hölder inequality for double integrals, then one has

\[
\left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) + \frac{1}{(b-a) (d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx \right| \\
- \frac{1}{(d-c)} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy - \frac{1}{(b-a)} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx \leq \frac{1}{(b-a) (d-c)} \int_a^b \int_c^d |p(x,t) q(y,s)| \left| \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b-t}{b-a} + \frac{a-b}{d-c} + \frac{d-s}{d-c} + \frac{s-c}{d-c} \right) \right|^q \, ds \, dt
\]

Since \( |\frac{\partial^2 f}{\partial t \partial s}|^q \) is a co-ordinated convex function on \( \Delta \), we can write for all \((t, s) \in [a, b] \times [c, d]\)

\[
\left| \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b-t}{b-a} + \frac{a-b}{d-c} + \frac{d-s}{d-c} + \frac{s-c}{d-c} \right) \right|^q \leq \frac{b-t}{b-a} \left| \frac{\partial^2 f}{\partial t \partial s} \left( \frac{a}{d-c} + \frac{s-c}{d-c} \right) \right|^q \\
+ \frac{t-a}{b-a} \left| \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b}{d-c} + \frac{s-c}{d-c} \right) \right|^q
\]
Using inequality of (2.5) in (2.4), we get

\[
\left( \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b - t}{b - a} + \frac{t - a}{b - a} + \frac{d - s}{d - c} + \frac{s - c}{d - c} \right) \right)^q \leq \left( \frac{b - t}{b - a} \right) \left( \frac{d - s}{d - c} \right) \left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q + \left( \frac{b - t}{b - a} \right) \left( \frac{s - c}{d - c} \right) \left| \frac{\partial^2 f}{\partial t \partial s} (a, d) \right|^q + \left( \frac{a - t}{b - a} \right) \left( \frac{d - s}{d - c} \right) \left| \frac{\partial^2 f}{\partial t \partial s} (b, c) \right|^q + \left( \frac{a - t}{b - a} \right) \left( \frac{s - c}{d - c} \right) \left| \frac{\partial^2 f}{\partial t \partial s} (b, d) \right|^q.
\]

Using inequality of (2.5) in (2.4), we get

\[
\left| f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + \frac{1}{b - a} \int_a^b f (x, y) \, dy \right| - \frac{1}{(b - a)(d - c)} \int_c^d f \left( \frac{a + b}{2}, y \right) \, dy - \frac{1}{(b - a)} \int_a^b f \left( x, \frac{c + d}{2} \right) \, dx \leq \frac{(b - a)(d - c)}{4 (p + 1)^q} \left( \left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (a, d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (b, d) \right|^q \right)^{\frac{1}{q}}.
\]

where we have used the fact that

\[
\left( \int_a^b \left| p(x, t) q(y, s) \right|^p \, dt \, ds \right)^{\frac{1}{p}} = \frac{[(b - a)(d - c)]^{1 + \frac{1}{p}}}{4 (p + 1)^{\frac{q}{p}}}.
\]

This completes the proof. \(\square\)

**Remark 2.** Suppose that all the assumptions of Theorem 8 are satisfied. If we choose \(\frac{\partial^2 f}{\partial t \partial s}\) is bounded, i.e.,

\[
\left\| \frac{\partial^2 f (t, s)}{\partial t \partial s} \right\|_{\infty} = \sup_{(t, s) \in (a, b) \times (c, d)} \left| \frac{\partial^2 f (t, s)}{\partial t \partial s} \right| < \infty,
\]

we get

\[
\left( \frac{a + b}{2}, \frac{c + d}{2} \right) - \frac{1}{(b - a)} \int_a^b f \left( \frac{a + b}{2}, y \right) \, dy - \frac{1}{(b - a)} \int_a^b f \left( x, \frac{c + d}{2} \right) \, dx + \frac{1}{(b - a)(d - c)} \int_c^d f \left( x, \frac{c + d}{2} \right) \, dx \leq \frac{(b - a)(d - c)}{4 (p + 1)^{\frac{q}{p}}} \left\| \frac{\partial^2 f (t, s)}{\partial t \partial s} \right\|_{\infty}.
\]
Theorem 9. Let \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta = [a, b] \times [c, d] \). If \( \left| \frac{\partial^2 f}{\partial t \partial s} \right|^q, q \geq 1 \), is a convex function on the co-ordinates on \( \Delta \), then the following inequality holds:

\[
\left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \right. \\
- \frac{1}{(d-c)} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy - \frac{1}{(b-a)} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx \leq \frac{1}{16} \\
\times \left( \int_a^b \int_c^d |p(x, t) q(y, s)| \left| \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b-t}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{d-c} c + \frac{s-c}{d-c} d \right) \right| \, ds \, dt \right)^{\frac{q}{4}}.
\]

Proof. From Lemma 2, we have

\[
\left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \\
- \frac{1}{(d-c)} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy - \frac{1}{(b-a)} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx \right| \leq \frac{1}{(b-a)(d-c)} \\
\int_a^b \int_c^d |p(x, t) q(y, s)| \left| \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b-t}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{d-c} c + \frac{s-c}{d-c} d \right) \right| \, ds \, dt
\]

By applying the well-known Power mean inequality for double integrals, then one has

\[
\left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \\
- \frac{1}{(d-c)} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy - \frac{1}{(b-a)} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx \right| \leq \frac{1}{(b-a)(d-c)} \\
\times \left( \int_a^b \int_c^d |p(x, t) q(y, s)| \, ds \, dt \right)^{1-\frac{q}{4}} \\
\left( \int_a^b \int_c^d |p(x, t) q(y, s)| \left| \frac{\partial^2 f}{\partial t \partial s} \left( \frac{b-t}{b-a} + \frac{t-a}{b-a} b, \frac{d-s}{d-c} c + \frac{s-c}{d-c} d \right) \right| ^q \, ds \, dt \right)^{\frac{1}{q}}.
\]
Since $|\frac{\partial^2 f}{\partial t \partial s}|^q$ is a co-ordinated convex function on $\Delta$, we can write for all $(t, s) \in [a, b] \times [c, d]$

\begin{equation}
\frac{\partial^2 f}{\partial t \partial s} \left( \begin{array}{c}
\frac{b-t}{b-a} + \frac{t-a}{b-a} + \frac{d-s}{d-c} + \frac{s-c}{d-c}
\end{array} \right)^q
\end{equation}

\begin{align}
&\leq \left( \frac{b-t}{b-a} \left( \frac{d-s}{d-c} \right) \left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q \\
&+ \left( \frac{b-t}{b-a} \left( \frac{s-c}{d-c} \right) \left| \frac{\partial^2 f}{\partial t \partial s} (a, d) \right|^q \\
&+ \left( \frac{t-a}{b-a} \left( \frac{d-s}{d-c} \right) \left| \frac{\partial^2 f}{\partial t \partial s} (b, c) \right|^q \\
&+ \left( \frac{t-a}{b-a} \left( \frac{s-c}{d-c} \right) \left| \frac{\partial^2 f}{\partial t \partial s} (b, d) \right|^q \right) .
\end{align}

If we use (2.9) in (2.8), we get

\begin{align}
&\left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\
&- \frac{1}{(d-c)} \int_c^d f \left( \frac{a+b}{2}, y \right) dy - \frac{1}{(b-a)} \int_a^b f \left( x, \frac{c+d}{2} \right) dx \right|
\end{align}

\begin{align}
&\leq \frac{1}{(b-a)(d-c)} \left\{ \left( \int_a^b \int_c^d |p(x, t) q(y, s)| ds dt \right)^{1-\frac{1}{q}} \\
&\times \left( \int_a^b \int_c^d |p(x, t) q(y, s)| \left| \frac{\partial^2 f}{\partial t \partial s} \left( \begin{array}{c}
\frac{b-t}{b-a} a + \frac{t-a}{b-a} b + \frac{d-s}{d-c} c + \frac{s-c}{d-c} d
\end{array} \right) \right|^q ds dt \right)^{\frac{1}{q}} \\
&\leq \left( \int_a^b \int_c^d |p(x, t) q(y, s)| ds dt \right)^{1-\frac{1}{q}} \\
&\times \left( \int_a^b \int_c^d |p(x, t) q(y, s)| \left[ \left( \frac{b-t}{b-a} \left( \frac{d-s}{d-c} \right) \left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q + \left( \frac{b-t}{b-a} \left( \frac{s-c}{d-c} \right) \left| \frac{\partial^2 f}{\partial t \partial s} (a, d) \right|^q \\
&+ \left( \frac{t-a}{b-a} \left( \frac{d-s}{d-c} \right) \left| \frac{\partial^2 f}{\partial t \partial s} (b, c) \right|^q + \left( \frac{t-a}{b-a} \left( \frac{s-c}{d-c} \right) \left| \frac{\partial^2 f}{\partial t \partial s} (b, d) \right|^q \right) \right]^{1-\frac{1}{q}}.
\end{align}

Computing the above integrals and using the fact that

\begin{align}
\left( \int_a^b \int_c^d |p(x, t) q(y, s)| ds dt \right)^{1-\frac{1}{q}} = \left( \frac{(b-a)^2 (d-c)^2}{16} \right)^{1-\frac{1}{q}}.
\end{align}
This completes the proof. □

3. INEQUALITIES FOR CO-ORDINATED s–CONVEX FUNCTIONS

To prove our main results we need the following lemma:

Lemma 2. Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) be an absolutely continuous function on \( \Delta \) where \( a < b, \ c < d \) and \( t, \lambda \in [0, 1] \), if \( \frac{\partial^2 f}{\partial t \partial \lambda} \in L(\Delta) \), then the following equality holds:

\[
\frac{f(a,c) + r_2 f(a,d) + r_1 f(b,c) + r_1 r_2 f(b,d)}{(r_1 + 1)(r_2 + 1)} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dx dy \\
- \left( \frac{r_2}{r_2+1} \right) \frac{1}{d-c} \int_c^d f(b,y) dy - \left( \frac{1}{r_1 + 1} \right) \frac{1}{d-c} \int_c^d f(a,y) dy \\
- \left( \frac{r_2}{r_2+1} \right) \frac{1}{b-a} \int_a^b f(x,d) dx - \left( \frac{1}{r_2 + 1} \right) \frac{1}{b-a} \int_a^b f(x,c) dx \\
= \frac{(b-a)(d-c)}{(r_1 + 1)(r_2 + 1)} \\
\times \int_0^1 \int_0^1 ((r_1 + 1) t - 1) ((r_2 + 1) \lambda - 1) \frac{\partial^2 f}{\partial t \partial \lambda} (tb + (1-t) a, \lambda d + (1-\lambda) c) dt d\lambda
\]

for some fixed \( r_1, r_2 \in [0, 1] \).

Proof. Integration by parts, we get

\[
\int_0^1 \int_0^1 ((r_1 + 1) t - 1) ((r_2 + 1) \lambda - 1) \frac{\partial^2 f}{\partial t \partial \lambda} (tb + (1-t) a, \lambda d + (1-\lambda) c) dt d\lambda \\
= \int_0^1 ((r_2 + 1) \lambda - 1) \left[ \int_0^1 ((r_1 + 1) t - 1) \frac{\partial^2 f}{\partial t \partial \lambda} (tb + (1-t) a, \lambda d + (1-\lambda) c) dt d\lambda \right] \\
= \int_0^1 ((r_2 + 1) \lambda - 1) \left[ \frac{((r_1 + 1) t - 1) \frac{\partial f}{\partial \lambda} (tb + (1-t) a, \lambda d + (1-\lambda) c)}{b-a} \right]_0^1 \\
- \frac{r_1 + 1}{b-a} \int_0^1 \frac{\partial f}{\partial \lambda} (tb + (1-t) a, \lambda d + (1-\lambda) c) dt d\lambda
\]

\[
= \int_0^1 \int_0^1 ((r_1 + 1) t - 1) ((r_2 + 1) \lambda - 1) \frac{\partial^2 f}{\partial t \partial \lambda} (tb + (1-t) a, \lambda d + (1-\lambda) c) dt d\lambda \\
= \int_0^1 ((r_2 + 1) \lambda - 1) \left[ \frac{r_1}{b-a} \frac{\partial f}{\partial \lambda} (b, \lambda d + (1-\lambda) c) + \frac{1}{b-a} \frac{\partial f}{\partial \lambda} (a, \lambda d + (1-\lambda) c) \\
- \frac{r_1 + 1}{b-a} \int_0^1 \frac{\partial f}{\partial \lambda} (tb + (1-t) a, \lambda d + (1-\lambda) c) dt d\lambda
\]
Again by integration by parts, we have

\[
\int_0^1 ((r_2 + 1) \lambda - 1) \left[ \frac{r_1}{b-a} \frac{\partial f}{\partial \lambda}(b, \lambda d + (1 - \lambda) c) + \frac{1}{b-a} \frac{\partial f}{\partial \lambda}(a, \lambda d + (1 - \lambda) c) \right] \, d\lambda \\
- \frac{r_1 + 1}{b-a} \int_0^1 \frac{\partial f}{\partial \lambda} (tb + (1-t) a, \lambda d + (1 - \lambda) c) \, dt \, d\lambda \\
= \frac{r_1}{b-a} \frac{(r_2 + 1) \lambda - 1}{d-c} f(b, \lambda d + (1 - \lambda) c) \bigg|_0^1 - \frac{r_1 (r_2 + 1)}{(b-a)(d-c)} \int_0^1 f(b, \lambda d + (1 - \lambda) c) \, d\lambda \\
+ \frac{1}{b-a} \frac{(r_2 + 1) \lambda - 1}{d-c} f(a, \lambda d + (1 - \lambda) c) \bigg|_0^1 - \frac{r_2 (r_2 + 1)}{(b-a)(d-c)} \int_0^1 f(a, \lambda d + (1 - \lambda) c) \, d\lambda \\
- \frac{r_1 + 1}{b-a} \int_0^1 ((r_2 + 1) \lambda - 1) \frac{\partial f}{\partial \lambda} (tb + (1-t) a, \lambda d + (1 - \lambda) c) \, d\lambda \bigg|_0^1 dt.
\]

Computing these integrals and by using the results, we obtain

\[
\int_0^1 \int_0^1 ((r_2 + 1) t - 1) ((r_2 + 1) \lambda - 1) \frac{\partial^2 f}{\partial t \partial \lambda} (tb + (1-t) a, \lambda d + (1 - \lambda) c) \, dt \, d\lambda \\
= \frac{1}{(b-a)(d-c)} \left[ f(a, c) + r_2 f(a, d) + r_1 f(b, c) + r_1 r_2 f(b, d) \\
- r_1 (r_2 + 1) \int_0^1 f(b, \lambda d + (1 - \lambda) c) \, d\lambda - (r_2 + 1) \int_0^1 f(a, \lambda d + (1 - \lambda) c) \, d\lambda \\
- r_2 (r_1 + 1) \int_0^1 f(tb + (1-t) a, d) \, dt - (r_2 + 1) \int_0^1 f(tb + (1-t) a, c) \, dt \\
+ (r_1 + 1) (r_2 + 1) \int_0^1 \int_0^1 f(tb + (1-t) a, \lambda d + (1 - \lambda) c) \, dtd\lambda \right].
\]

Using the change of the variable \( x = tb + (1-t) a \) and \( y = \lambda d + (1 - \lambda) c \) for \( t, \lambda \in [0,1] \) and multiplying the both sides by \( \frac{(b-a)(d-c)}{(r_1+1)(r_2+1)} \), we get the required result. \( \square \)

**Theorem 10.** Let \( f : \Delta = [a, b] \times [c, d] \subset [0, \infty)^2 \to \mathbb{R} \) be an absolutely continuous function on \( \Delta \). If \( \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right| \) is \( s \)-convex function on the co-ordinates on
Proof. From Lemma 3, we can write:

\[
\Delta, \text{ then one has the inequality:}
\]

\[
\frac{f(a,c) + r_2f(a,d) + r_1f(b,c) + r_1r_2f(b,d)}{(r_1 + 1)(r_2 + 1)} - \frac{1}{d - c} \left[ \left( \frac{r_2}{r_2 + 1} \right) \int_c^d f(b,y)dy + \left( \frac{1}{r_1 + 1} \right) \int_c^d f(a,y)dy \right] + \frac{1}{b - a} \left[ \left( \frac{r_2}{r_2 + 1} \right) \int_a^b f(x,d)dx + \left( \frac{1}{r_2 + 1} \right) \int_a^b f(x,c)dx \right] + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x,y)dxdy \leq \frac{1}{(r_1 + 1)(r_2 + 1)(s + 1)^2(s + 2)^2} \times \left( MN \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right| (a,c) + LN \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right| (a,d) + KM \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right| (b,c) + KL \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right| (b,d) \right)
\]

where

\[
M = \left( s + 1 + 2(r_1 + 1) \left( \frac{r_1}{r_1 + 1} \right)^{s+2} - r_1 \right)
\]

\[
N = \left( s + 1 + 2(r_2 + 1) \left( \frac{r_2}{r_2 + 1} \right)^{s+2} - r_2 \right)
\]

\[
L = \left( r_2(s + 1) + 2 \left( \frac{1}{r_2 + 1} \right)^{s+1} - 1 \right)
\]

\[
K = \left( r_1(s + 1) + 2 \left( \frac{1}{r_1 + 1} \right)^{s+1} - 1 \right)
\]
By using co-ordinated $s-$convexity of $f$, we have

\[
\left. \frac{f(a,c) + r_2 f(a,d) + r_1 f(b,c) + r_1 r_2 f(b,d)}{(r_1 + 1)(r_2 + 1)} \right| \\
\left. - \left( \frac{r_2}{r_2 + 1} \right) \frac{1}{d-c} \int_c^d f(b,y)dy - \left( \frac{1}{r_1 + 1} \right) \frac{1}{d-c} \int_c^d f(a,y)dy \right| \\
\left. - \left( \frac{r_2}{r_2 + 1} \right) \frac{1}{b-a} \int_a^b f(x,d)dx - \left( \frac{1}{r_2 + 1} \right) \frac{1}{b-a} \int_a^b f(x,c)dx \right| \\
\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dxdy \right| \\
= \frac{(b-a)(d-c)}{(r_1 + 1)(r_2 + 1)} \\
\times \int_0^1 \int_0^1 \left| ((r_1 + 1) t - 1) ((r_2 + 1) \lambda - 1) \right| \\
\left\{ t^s \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (b, \lambda d + (1 - \lambda) c) \right| + (1-t)^s \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (a, \lambda d + (1 - \lambda) c) \right| \right\} dt \ d\lambda.
\]

By calculating the above integrals, we get

\[
(3.2) \quad \int_0^1 \left| ((r_1 + 1) t - 1) \right| \left\{ t^s \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (b, \lambda d + (1 - \lambda) c) \right| \\
+ (1-t)^s \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (a, \lambda d + (1 - \lambda) c) \right| \right\} dt \\
= \int_0^{1/r_1+1} \left( (1 - (r_1 + 1) t) \right) \left\{ t^s \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (b, \lambda d + (1 - \lambda) c) \right| \\
+ (1-t)^s \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (a, \lambda d + (1 - \lambda) c) \right| \right\} dt \\
+ \int_{1/r_1+1}^1 ((r_1 + 1) t - 1) \left\{ t^s \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (b, \lambda d + (1 - \lambda) c) \right| \\
+ (1-t)^s \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (a, \lambda d + (1 - \lambda) c) \right| \right\} dt \\
= \frac{1}{(s+1)(s+2)} \left[ \left( r_1 \left( s + 1 \right) + 2 \left( \frac{1}{r_1 + 1} \right)^{s+1} - 1 \right) \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (b, \lambda d + (1 - \lambda) c) \right| \\
+ \left( s + 1 + 2 (r_1 + 1) \left( \frac{1}{r_1 + 1} \right)^{s+2} - r_1 \right) \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (a, \lambda d + (1 - \lambda) c) \right| \right].
\]
By a similar argument for other integrals, by using co-ordinated $s$–convexity of $f$, we get

$$
\int_0^1 \left[(r_2 + 1) t - 1\right] \left\{ \frac{\partial^2 f}{\partial t \partial \lambda} (b, \lambda d + (1 - \lambda) c) \right\} d\lambda = \\
\int_0^{r_2 + 1} \left(1 - (r_2 + 1) t\right) \left\{ \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| + (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| \right\} d\lambda \\
+ \int_0^1 \left((r_2 + 1) t - 1\right) \left\{ \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| + (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right| \right\} d\lambda \\
= \frac{1}{(s + 1)(s + 2)} \left[ r_2 (s + 1) + 2 \left( \frac{1}{r_2 + 1} \right)^{s+1} - 1 \right] \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| \\
+ \frac{1}{(s + 1)(s + 2)} \left[ r_2 (s + 1) + 2 \left( \frac{1}{r_2 + 1} \right)^{s+1} - 1 \right] \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| \\
+ \frac{1}{(s + 1)(s + 2)} \left[ s + 1 + 2 (r_2 + 1) \left( \frac{r_2}{r_2 + 1} \right)^{s+2} - r_1 \right] \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| \\
+ \frac{1}{(s + 1)(s + 2)} \left[ s + 1 + 2 (r_2 + 1) \left( \frac{r_2}{r_2 + 1} \right)^{s+2} - r_1 \right] \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|.
$$

By using these in (3.2), we obtain the inequality (3.1). \hfill \square

**Corollary 2.** (1) If we choose $r_1 = r_2 = 1$ in (3.1), we have

$$
\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| \leq \frac{1}{(s + 1)^2 (s + 2)^2} \left( \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| \right)
$$

(2) If we choose $r_1 = r_2 = 0$ in (3.1), we have

$$
\left| f(a, c) - \frac{1}{d - c} \int_c^d f(a, y) dy - \frac{1}{b - a} \int_a^b f(x, c) dx \right| \\
+ \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dxdy \leq \frac{(b - a)(d - c)}{(s + 1)^2 (s + 2)^2} \left( \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) + (s + 1) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) + (s + 1) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| \right| \right)$$
Remark 3. If we choose $s = 1$ in \([3,3]\), we get an improvement for the inequality \([1,2]\).

Theorem 11. Let $f : \Delta = [a, b] \times [c, d] \subset [0, \infty)^2 \rightarrow [0, \infty)$ be an absolutely continuous function on $\Delta$. If $\frac{\partial^2 f}{\partial \sigma \lambda}$ is $s$-convex function on the co-ordinates on $\Delta$, for some fixed $s \in (0, 1]$ and $p > 1$, then one has the inequality:

\[
\begin{align*}
&f(a, c) + r_2 f(a, d) + r_1 f(b, c) + r_1 r_2 f(b, d) \\
&- \left( \frac{r_2}{r_2 + 1} \right) \frac{1}{d - c} \int_c^d f(b, y)dy - \left( \frac{1}{r_1 + 1} \right) \frac{1}{d - c} \int_c^d f(a, y)dy \\
&- \left( \frac{r_2}{r_2 + 1} \right) \frac{1}{b - a} \int_a^b f(x, d)dx - \left( \frac{1}{r_2 + 1} \right) \frac{1}{b - a} \int_a^b f(x, c)dx \\
&+ \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y)dydx \\
&= \left( \frac{(b-a)(d-c)}{(r_1+1)(r_2+1)} \right) \left( \frac{1 + r_1^{\frac{s}{p} - 1}}{(r_1+1)^{\frac{s}{p}} (r_2+1)^{\frac{s}{p}} (p+1)^{\frac{s}{p}}} \right) \\
&\times \left( \frac{\partial^2 f}{\partial \sigma \lambda} \right)^q (a, c) + \frac{\partial^2 f}{\partial \sigma \lambda} \right|^q (a, d) + \frac{\partial^2 f}{\partial \sigma \lambda} \right|^q (b, c) + \frac{\partial^2 f}{\partial \sigma \lambda} \right|^q (b, d) \right)^{\frac{1}{q}}
\end{align*}
\]

for some fixed $r_1, r_2 \in [0, 1]$, where $q = \frac{p}{p - 1}$.

Proof. Let $p > 1$. From Lemma 3 and using the Hölder inequality for double integrals, we can write

\[
\begin{align*}
&f(a, c) + r_2 f(a, d) + r_1 f(b, c) + r_1 r_2 f(b, d) \\
&- \left( \frac{r_2}{r_2 + 1} \right) \frac{1}{d - c} \int_c^d f(b, y)dy - \left( \frac{1}{r_1 + 1} \right) \frac{1}{d - c} \int_c^d f(a, y)dy \\
&- \left( \frac{r_2}{r_2 + 1} \right) \frac{1}{b - a} \int_a^b f(x, d)dx - \left( \frac{1}{r_2 + 1} \right) \frac{1}{b - a} \int_a^b f(x, c)dx \\
&+ \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y)dydx \\
&= \left( \frac{(b-a)(d-c)}{(r_1+1)(r_2+1)} \right) \left( \int_0^1 \int_0^1 \left| ((r_1+1)t - 1)((r_2+1)\lambda - 1)^p dt d\lambda \right| \right)^{\frac{1}{p}} \\
&\times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial \sigma \lambda} (tb + (1-t)a, \lambda d + (1-\lambda)c) \right|^q dt d\lambda \right)^{\frac{1}{q}}.
\end{align*}
\]
Since \( \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q \) is \( s \)-convex function on the co-ordinates on \( \Delta \), we can write for \( t, \lambda \in [0, 1] \)

\[
\left| \frac{\partial^2 f}{\partial t \partial \lambda} (tb + (1 - t) a, \lambda d + (1 - \lambda) c) \right|^q \leq t^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, \lambda d + (1 - \lambda) c) \right|^q + (1 - t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, \lambda d + (1 - \lambda) c) \right|^q
\]

and

\[
\left| \frac{\partial^2 f}{\partial t \partial \lambda} (tb + (1 - t) a, \lambda d + (1 - \lambda) c) \right|^q \leq t^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right|^q + t^s (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right|^q + \lambda^s (1 - t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right|^q + (1 - \lambda)^s (1 - t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|^q
\]

thus, we obtain

\[
\left| f(a, c) + r_2 f(a, d) + r_1 f(b, c) + r_1 r_2 f(b, d) \right| \frac{1}{(r_1 + 1) (r_2 + 1)} + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dxdy \]

\[
- \left( \frac{r_2}{r_2 + 1} \right) \frac{1}{d - c} \int_c^d f(b, y) dy - \left( \frac{1}{r_1 + 1} \right) \frac{1}{d - c} \int_c^d f(a, y) dy \]

\[
- \left( \frac{r_2}{r_2 + 1} \right) \frac{1}{b - a} \int_a^b f(x, d) dx - \left( \frac{1}{r_1 + 1} \right) \frac{1}{b - a} \int_a^b f(x, c) dx
\]

\[
= \frac{(b - a)(d - c)}{(r_1 + 1) (r_2 + 1)} \left( \frac{1 + r_1^s}{r_1 + 1} \right) \left( \frac{1 + r_2^s}{r_2 + 1} \right) (s + 1) \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) + \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) + \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) + \frac{\partial^2 f}{\partial t \partial \lambda} (b, d)
\]

Which completes the proof of the inequality (3.4). \( \square \)
Corollary 3. (1) Under the assumptions of Theorem 12, if we choose $r_1 = r_2 = 1$ in (3.4), we have

\begin{equation}
\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| \leq \frac{1}{4} \left[ \frac{1}{d-c} \int_c^d [f(b, y) + f(a, y)] dy + \frac{1}{b-a} \int_a^b [f(x, d) + f(x, c)] dx \right] \\
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)dxdy \right| = \frac{(b-a)(d-c)}{4 (p+1)^{\frac{s}{2}}}
\times \left( \frac{\frac{\partial^2 f}{\partial t \partial \lambda}}{q} (a, c) + \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) + \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) + \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right)^{\frac{1}{s}}.
\end{equation}

(2) Under the assumptions of Theorem 12, if we choose $r_1 = r_2 = 0$ in (3.4), we have

\begin{equation}
\left| f(a, c) - \frac{1}{d-c} \int_c^d f(a, y)dy - \frac{1}{b-a} \int_a^b f(x, c)dx \right| + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)dxdy \right| = \frac{(b-a)(d-c)}{4 (p+1)^{\frac{s}{2}}}
\times \left( \frac{\frac{\partial^2 f}{\partial t \partial \lambda}}{q} (a, c) + \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) + \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) + \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right)^{\frac{1}{s}}.
\end{equation}

Remark 4. If we choose $s = 1$ in (3.5), we obtain an improvement for the inequality (3.6).

Theorem 12. Let $f : \Delta = [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}$ be an absolutely continuous function on $\Delta$. If $\left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q$ is $s$-convex function on the co-ordinates on
for some fixed \( s \in (0, 1) \) and \( q \geq 1 \), then one has the inequality:

\[
\left| \frac{f(a, c) + r_2 f(a, d) + r_1 f(b, c) + r_1 r_2 f(b, d)}{(r_1 + 1)(r_2 + 1)} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dxdy \right|
\]

\[
- \left( \frac{r_2}{r_2 + 1} \right) \frac{1}{d-c} \int_c^d f(b, y) dy - \left( \frac{1}{r_1 + 1} \right) \frac{1}{d-c} \int_c^d f(a, y) dy
\]

\[
- \left( \frac{r_2}{r_2 + 1} \right) \frac{1}{b-a} \int_a^b f(x, d) dx - \left( \frac{1}{r_2 + 1} \right) \frac{1}{b-a} \int_a^b f(x, c) dx
\]

\[
\leq \frac{(b-a)(d-c)}{(r_1 + 1)(r_2 + 1)} \left( \frac{1 + r_1^2}{4 (r_1 + 1)(r_2 + 1)} \right)^{1 - \frac{1}{q}} MN \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (a, c) + LN \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (a, d) + KM \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (b, c) + KL \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (b, d) \right)^{\frac{1}{q}}
\]

for some fixed \( r_1, r_2 \in [0, 1] \).

**Proof.** From Lemma 3 and using the well-known Power-mean inequality, we can write

\[
\left| \frac{f(a, c) + r_2 f(a, d) + r_1 f(b, c) + r_1 r_2 f(b, d)}{(r_1 + 1)(r_2 + 1)} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dxdy \right|
\]

\[
- \left( \frac{r_2}{r_2 + 1} \right) \frac{1}{d-c} \int_c^d f(b, y) dy - \left( \frac{1}{r_1 + 1} \right) \frac{1}{d-c} \int_c^d f(a, y) dy
\]

\[
- \left( \frac{r_2}{r_2 + 1} \right) \frac{1}{b-a} \int_a^b f(x, d) dx - \left( \frac{1}{r_2 + 1} \right) \frac{1}{b-a} \int_a^b f(x, c) dx
\]

\[
= \frac{(b-a)(d-c)}{(r_1 + 1)(r_2 + 1)} \left( \int_0^1 \int_0^1 \left| ((r_1 + 1) t - 1) ((r_2 + 1) \lambda - 1) \right| dtd\lambda \right)^{1 - \frac{1}{q}}
\]

\[
\times \left( \int_0^1 \int_0^1 \left| ((r_1 + 1) t - 1) ((r_2 + 1) \lambda - 1) \right| \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (tb + (1-t) a, \lambda d + (1 - \lambda) c) dtd\lambda \right)^{\frac{1}{q}}
\]

Since \( \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q \) is \( s \)-convex function on the co-ordinates on \( \Delta \), we can write for \( t, \lambda \in [0, 1] \)

\[
\left| \frac{\partial^2 f}{\partial t \partial \lambda} (tb + (1-t) a, \lambda d + (1 - \lambda) c) \right|^q
\]

\[
\leq t^q \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, \lambda d + (1 - \lambda) c) \right|^q + (1-t)^q \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, \lambda d + (1 - \lambda) c) \right|^q
\]
and

\[
\left| \frac{\partial^2 f}{\partial t \partial \lambda} (tb + (1-t) a, \lambda d + (1-\lambda) c) \right|^q \leq t^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (b, d) + t^s (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (b, c) + \lambda^s (1-t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (a, d) + (1-\lambda)^s (1-t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (a, c)
\]

hence, it follows that

\[
\left| \frac{f(a, c) + r_2 f(a, d)}{(r_1 + 1) (r_2 + 1)} \right|^q - \left( \frac{r_2}{r_2 + 1} \right)^\frac{1}{4} \int_a^b f(x, d) dx - \left( \frac{1}{r_1 + 1} \right)^\frac{1}{4} \int_a^d f(x, y) dy \leq \frac{(b-a)(d-c)}{(r_1 + 1) (r_2 + 1)} \left[ \frac{1}{4} \right]^\frac{1}{4} \times \left( \int_0^1 \int_0^1 \left| ((r_1 + 1) t - 1) ((r_2 + 1) \lambda - 1) \right| \left\{ t^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (b, d) + t^s (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (b, c) + \lambda^s (1-t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (a, d) + (1-\lambda)^s (1-t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (a, c) \right\} dtd\lambda \right]^\frac{1}{q}
\]

By a simple computation, one can see that

\[
\left( \int_0^1 \int_0^1 \left| ((r_1 + 1) t - 1) ((r_2 + 1) \lambda - 1) \right| \left\{ t^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (b, d) + t^s (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (b, c) + \lambda^s (1-t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (a, d) + (1-\lambda)^s (1-t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (a, c) \right\} dtd\lambda \right]^\frac{1}{q} \leq \left( M N \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (a, c) + LN \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (a, d) + KM \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (b, c) + KL \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q (b, d) \right)^\frac{1}{q}
\]

where \( K, L, M \) and \( N \) as in Theorem 11. By substituting these in (3.6) and simplifying we obtain the required result. \( \square \)
Corollary 4. (1) Under the assumptions of Theorem 13, if we choose \( r_1 = r_2 = 1 \), we have
\[
\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) - \frac{1}{4} \left[ \frac{1}{d-c} \int_c^d [f(b, y) + f(a, y)] dy + \frac{1}{b-a} \int_a^b [f(x, d) + f(x, c)] dx \right] \right|
\approx \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx
\leq \frac{(b-a)(d-c)}{4} \left( \frac{1}{4} \right)^{1 - \frac{q}{4}}
\times \left( MN \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (a, c) + LN \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (a, d) + KM \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (b, c) + KL \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (b, d) \right) \frac{1}{(s+1)^2 (s+2)^2}
\]

(2) Under the assumptions of Theorem 13, if we choose \( r_1 = r_2 = 0 \), we have
\[
\left| f(a, c) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx - \frac{1}{d-c} \int_c^d f(a, y) dy - \frac{1}{b-a} \int_a^b f(x, c) dx \right|
\leq \frac{(b-a)(d-c)}{4} \left( \frac{1}{4} \right)^{1 - \frac{q}{4}}
\times \left( MN \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (a, c) + LN \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (a, d) + KM \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (b, c) + KL \left| \frac{\partial^2 f}{\partial x \partial y} \right|^q (b, d) \right) \frac{1}{(s+1)^2 (s+2)^2}
\]

Remark 5. Under the assumptions of Theorem 13, if we choose \( r_1 = r_2 = 1 \) and \( s = 1 \), we have an improvement for the inequality (1.4).

References

[1] N.S. Barnett and S.S. Dragomir, An Ostrowski type inequality for double integrals and applications for cubature formulae, Soochow J. Math., 27 (1) (2001), 1-10.
[2] S.S. Dragomir, On Hadamard’s inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese Journal of Math., 5, 2001, 775-788.
[3] B.G. Pachpatte, A new Ostrowski type inequality for double integrals, Soochow J. Math., 32 (2) (2006), 317-322.
[4] M. Z. Sarikaya, On the Ostrowski type integral inequality for double integral, http://arxiv.org/abs/1005.0454v1
[5] M. Alomari and M. Darus, On the Hadamard’s inequality for log – convex functions on the co-ordinates, Journal of Inequalities and Appl., 2009, article ID 283147.
[6] H. Hudzik and L. Maligranda, Some remarks on \( s \)-convex functions, Aequationes Math., 48 (1994), 100-111.
[7] M.K. Bakula and J. Pečarić, On the Jensen’s inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese Journal of Math., 5, 2006, 1271-1292.
[8] W. Orlicz, A note on modular spaces , I, Bull. Acad. Polon. Sci. Math. Astronom. Phys., 9 (1961), 157-162.
[9] S.S. Dragomir and S. Fitzpatrick, The Hadamard’s inequality for $s$–convex functions in the second sense, Demonstratio Math., 32 (4) (1999), 687-696.
[10] M. Alomari and M. Darus, The Hadamard’s inequality for $s$–convex functions of 2-variables, Int. Journal of Math. Analysis, 2(13), 2008, 629-638.
[11] M.E. Özdemir, E. Set, M.Z. Sarıkaya, Some new Hadamard’s type inequalities for co-ordinated $m$–convex and $(\alpha, m)$–convex functions, Accepted.
[12] M.Z. Sarıkaya, E. Set, M. Emin Özdemir and S.S. Dragomir, New some Hadamard’s type inequalities for co-ordinated convex functions, Accepted.
[13] U.S. Kirmaci, M.K. Bakula, M.E. Özdemir and J. Pečarić, Hadamard-type inequalities for $s$–convex functions, Applied Mathematics and Computation, 193 (2007), 26-35.

*Atatürk University, K.K. Education Faculty, Department of Mathematics, 25240, Erzurum, Turkey
E-mail address: emos@atauni.edu.tr
E-mail address: hkavurmaci@atauni.edu.tr

♣ Ağrı İbrahim Çeçen University, Faculty of Science and Arts, Department of Mathematics, 04100, Ağrı, Turkey
E-mail address: ahmetakdemir@agri.edu.tr
E-mail address: merveavci@gmail.com