COMBINATIONS OF QUANTUM OBSERVABLES AND INSTRUMENTS

Stan Gudder
Department of Mathematics
University of Denver
Denver, Colorado 80208
sgudder@du.edu

“You should conduct research of such a high quality that people remember your name.” —Author, Unknown

Abstract

This article points out that observables and instruments can be combined in many ways that have natural and physical interpretations. We shall mainly concentrate on the mathematical properties of these combinations. Section 1 reviews the basic definitions and observables are considered in Section 2. We study parts of observables, post-processing, generalized convex combinations, sequential products and tensor products. These combinations are extended to instruments in Section 3. We consider properties of observables measured by combinations of instruments. We introduce four special types of instruments, namely Kraus, Lüders, trivial and semitrivial instruments. We study when these types are closed under various combinations. In this work, we only consider finite-dimensional quantum systems. A few of the results presented here have appeared in the author’s previous articles. [6, 7, 8].

1
1 Basic Definitions

Let $\mathcal{L}(H)$ be the set of linear operators on a finite-dimensional complex Hilbert space $H$. For $S, T \in \mathcal{L}(H)$ we write $S \leq T$ if $\langle \phi, S\phi \rangle \leq \langle \phi, T\phi \rangle$ for all $\phi \in H$. We define the set of effects by

$$\mathcal{E}(H) = \{ a \in \mathcal{L}(H): 0 \leq a \leq I \}$$

where $0, I$ are the zero and identity operators, respectively. The effects correspond to yes-no experiments and $a \in \mathcal{E}(H)$ is said to occur when a measurement of $a$ results in the outcome yes. We call $\rho \in \mathcal{E}(H)$ a partial state if $\text{tr} (\rho) \leq 1$ and $\rho$ is a state if $\text{tr} (\rho) = 1$. We denote the set of partial states by $\mathcal{S}_p(H)$ and the set of states by $\mathcal{S}(H)$. If $\rho \in \mathcal{S}(H), a \in \mathcal{E}(H)$, we call $P_\rho(a) = \text{tr} (\rho a)$ the probability that $a$ occurs in the state $\rho$ \cite{1,9,12,13}.

We denote the unique positive square-root of $a \in \mathcal{E}(H)$ by $a^{1/2}$. For, $a, b \in \mathcal{E}(H)$, their sequential product is the effect $a \circ b = a^{1/2}ba^{1/2}$, where $a^{1/2}ba^{1/2}$ is the usual operator product \cite{3,4}. We interpret $a \circ b$ as the effect that results from first measuring $a$ and then measuring $b$. Let $\Omega_A$ be a finite set. A finite observable \cite{9,15} with outcome space $\Omega_A$ is a subset

$$A = \{ A_x : x \in \Omega_A \} \subseteq \mathcal{E}(H)$$

such that $\sum_{x \in \Omega_A} A_x = I$. We denote the set of finite observables on $H$ by $\mathcal{O}(H)$. In the sequel, an observable will always mean a finite-observable. We interpret $A \in \mathcal{O}(H)$ as a measurement with possible outcomes $x \in \Omega_A$ and $A_x$ is the effect that occurs when the measurement result is $x$. If $A \in \mathcal{O}(H)$, we define the effect-values measure $X \mapsto A_X$ from $2^{\Omega_A}$ to $\mathcal{E}(H)$ by $A_X = \sum_{x \in X} A_x$. The distribution of $A \in \mathcal{O}(H)$ in the state $\rho \in \mathcal{S}(H)$ is defined by $\Phi^A_\rho(x) = \text{tr} (\rho A_x)$ for all $x \in \Omega_A$. Then

$$\Phi^A_\rho(X) = \sum_{x \in X} \Phi^A_\rho(x)$$

gives the probability that $A$ has an outcome in $X \subseteq \Omega_A$ when the system is in the state $\rho$. Notice that $X \mapsto \Phi^A_\rho(X)$ is a probability measure on $\Omega_A$.

An operation on $H$ is a completely positive, trace-reducing, linear map $\mathcal{A}: \mathcal{L}(H) \to \mathcal{L}(H)$ \cite{1,9,12,15}. Trace-reducing implies that $\mathcal{A}: \mathcal{S}_p(H) \to \mathcal{S}_p(H)$. According to Kraus’ Theorem \cite{9,12,15} every operation $\mathcal{A}$ has the
form \( A(T) = \sum_{i=1}^{n} S_i T S_i^* \) where \( S_i \in \mathcal{L}(H) \) satisfy \( \sum_{i=1}^{n} S_i^* S_i \leq I \). An operation \( A \) is a channel if \( A(\rho) \in \mathcal{S}(H) \) for all \( \rho \in \mathcal{S}(H) \) \[9, 12\]. In this case, the Kraus operators \( S_i \) satisfy \( \sum_{i=1}^{n} S_i^* S_i = I \). We denote the set of channels on \( H \) by \( \mathcal{C}(H) \). For a finite set \( \Omega_I \), a finite instrument with outcome space \( \Omega_I \) is a set of operations \( I = \{I_x : x \in \Omega_I\} \) such that \( \sum_{x \in \Omega_I} I_x \in \mathcal{C}(H) \) \[1, 6, 7, 9, 13\].

Defining \( I_X \) for \( X \subseteq \Omega_I \) by \( I_X = \sum_{x \in X} I_x \), we see that \( X \mapsto I_X \) is an operation-valued measure on \( H \). We denote the set of finite instruments on \( H \) by \( \text{In}(H) \). The distribution of \( I \in \text{In}(H) \) in the state \( \rho \in \mathcal{S}(H) \) is defined by \( \Phi_I(\rho)(x) = \text{tr}[I_x(\rho)] \) for all \( x \in \Omega_I \). Then 
\[
\Phi_I^T(X) = \sum_{x \in X} \Phi_I^T(x)
\]
gives the probability that \( I \) has an outcome in \( X \) when the system is in the state \( \rho \). As with observables, \( X \mapsto \Phi_I^T(X) \) gives a probability measure on \( \Omega_I \). If \( A \in \mathcal{O}(H) \), we say that an instrument \( I \in \text{In}(H) \) measures \( A \) (or is compatible with \( A \)) if \( \Omega_I = \Omega_A \) and \( \Phi_I^T(x) = \Phi_A^T(x) \) for all \( x \in \Omega_A \), \( \rho \in \mathcal{S}(H) \) \[11/12/13\]. This condition is equivalent to \( \text{tr}(%\rho A_X) = \text{tr}[I_X(\rho)] \) for all \( X \subseteq \Omega_A \), \( \rho \in \mathcal{S}(H) \).

If \( I \in \text{In}(H) \), there exists a unique \( \hat{I} \in \mathcal{O}(H) \) such that \( I \) measures \( \hat{I} \) \[9\]. However, an observable has many instruments that measure it. We view \( I \in \text{In}(H) \) as an apparatus that can be employed to measure the observable \( \hat{I} \in \mathcal{O}(H) \). However, \( I \) gives more information than \( \hat{I} \) because \( I_x(\rho) \in \mathcal{S}_p(H) \) updates the state \( \rho \) when the outcome \( x \) is observed. There is no corresponding unambiguous updating for observables.

2 Observables

This section discusses functions of observables and various combinations of observables. If \( A \in \mathcal{O}(H) \) and \( f : \Omega_A \rightarrow \Omega \) is a surjection, we define \( f(A) \in \mathcal{O}(H) \) to have outcome space \( \Omega \) and for every \( y \in \Omega \)
\[
f(A)_y = A_{f^{-1}(y)} = \sum_{x} \{A_x : f(x) = y\}
\]
We say that the observable \( f(A) \) is part of the observable \( A \) \[2/8/10/11\]. As its name suggests, we think of \( f(A) \) as an observable that measures only
a part of $A$. Two observables $A, B \in \mathcal{O}(H)$ are said to coexist if there exists a $C \in \mathcal{O}(H)$ and surjections $f : \Omega_C \to \Omega_A$, $g : \Omega_C \to \Omega_B$ such that $A = f(C)$, $B = g(C)$. In this way $A$ and $B$ can be simultaneously measured by measuring a single observable $C$. We say that $A, B \in \mathcal{O}(H)$ are jointly measurable if for all $\rho \in S(H)$ there exist probability measures $\mu_\rho$ on $\Omega_A \times \Omega_B$ such that $\mu_\rho (\{x\} \times \Omega_B) = \Phi^A_\rho (x)$ and $\mu_\rho (\Omega_A \times \{y\}) = \Phi^B_\rho (y)$, for all $x \in \Omega_A$, $y \in \Omega_B$. We call $\mu_\rho$ the joint distribution of $A, B$ in the state $\rho$.

**Lemma 2.1.** (i) For every $A \in \mathcal{O}(H)$, $\rho \in S(H)$, $y \in f(\Omega_A)$ we have that

$$\Phi^{f(A)}_\rho (y) = \Phi^A_\rho [f^{-1}(y)] = \sum_x \{ \Phi^A_\rho (x) : f(x) = y \}$$

(ii) If $A$ and $B$ coexist, then $A$ and $B$ are jointly measurable.

**Proof.** (i) For every $y \in \Omega_{f(A)}$ we obtain

$$\Phi^{f(A)}_\rho (y) = \text{tr} [\rho f(A)_y] = \text{tr} [\rho A_{f^{-1}(y)}] = \Phi^A_\rho [f^{-1}(y)]$$

$$= \sum_x \{ \Phi^A_\rho (x) : f(x) = y \}$$

(ii) Since $A$ and $B$ coexist, there exists a $C \in \mathcal{O}(H)$ such that $A = f(C)$, $B = g(C)$. For $\rho \in S(H)$, define the probability measure $\mu_\rho$ on $\Omega_A \times \Omega_B$ by

$$\mu_\rho (x,y) = \text{tr} [\rho C_{f^{-1}(x)}g^{-1}(y)]$$

We then obtain

$$\mu_\rho (\{x\} \times \Omega_B) = \text{tr} [\rho C_{f^{-1}(x)}] = \text{tr} [\rho f(C)_x] = \text{tr} (\rho A_x) = \Phi^A_\rho (x)$$

and in a similar way, $\mu_\rho (\Omega_A \times \{y\}) = \Phi^B_\rho (y)$.

We do not know whether the converse of Lemma 2.1(ii) holds.

Let $\Omega_A$ be the outcome space for $A \in \mathcal{O}(H)$ and let $\Omega$ be another finite set. Suppose $\mu : \Omega_A \times \Omega \to [0,1]$ satisfies $\sum_{y \in \Omega} \mu_{xy} = 1$ for every $x \in \Omega_A$. We call $\mu$ a transition probability from $\Omega_A$ to $\Omega$. The condition $\sum_{y \in \Omega} \mu_{xy} = 1$ says that $x$ transitions into some $y \in \Omega$ with probability 1. A post-processing of $A$ is an observable $B = \mu \cdot A \in \mathcal{O}(H)$ with outcome space $\Omega$ defined by
\[ B_y = \sum_{x \in \Omega} \mu_{xy} A_x \] Notice that \( B \) is indeed an observable because \( B_y \geq 0 \) for all \( y \in \Omega \) and
\[
\sum_{y \in \Omega} B_y = \sum_{x \in \Omega} \sum_{y \in \Omega} \mu_{xy} A_x = \sum_{x \in \Omega} A_x = I
\]
We interpret \( B = \mu \cdot A \) as first measuring \( A \) and then processing the result with transitions to the outcome space \( \Omega_B = \Omega \). One way of post-processing \( A \) is by employing another observable \( B \) and a collection of states \( \alpha_x, x \in \Omega_A \). We then define
\[
\mu_{xy} = \text{tr} (\alpha_x B_y) = \Phi^B_{\alpha_x}(y)
\]
and write \( \mu \cdot A = \text{Post}_{(\alpha, B)}(A) \). We call \( \text{Post}_{(\alpha, B)}(A) \) the \textit{post-processing} of \( A \) relative to \( (\alpha, B) \). We can also post-process a probability measure \( \nu \) on \( \Omega_A \) to a probability measure \( \mu \cdot \nu \) on \( \Omega \) by defining \( (\mu \cdot \nu)_y = \sum_{x \in \Omega_A} \mu_{xy} \nu_x \).

\textbf{Lemma 2.2.} (i) \( \Phi^\mu_{\cdot A} = \mu \cdot \Phi^A \). (ii) \( \Phi^\rho_{\text{Post}_{(\alpha, B)}(A)}(y) = \sum_{x \in \Omega_A} \Phi^B_{\alpha_x}(y) \Phi^A_{\rho}(x) \).

\textbf{Proof.} (i) For all \( y \in \Omega \) we have that
\[
\Phi^\rho_{\mu \cdot A}(y) = \text{tr} (\rho (\mu \cdot A)_y) = \text{tr} \left[ \rho \sum_{x \in \Omega_A} \mu_{xy} A_x \right] = \sum_{x \in \Omega_A} \mu_{xy} \text{tr} (\rho A_x) = \sum_{x \in \Omega_A} \mu_{xy} \Phi^A_{\rho}(x) = \mu \cdot \Phi^A_{\rho}(y)
\]
The result follows. (ii) Applying (i) gives
\[
\Phi^\rho_{\text{Post}_{(\alpha, B)}(A)}(y) = \sum_{x \in \Omega_A} \mu_{xy} \Phi^A_{\rho}(x) = \sum_{x \in \Omega_A} \Phi^B_{\alpha_x}(y) \Phi^A_{\rho}(x)
\]

Let \( A^i \in \mathcal{O}(H) \) with outcome spaces \( \Omega^i, i = 1, 2, \ldots, n \) and let \( \lambda_i \in (0, 1) \) with \( \sum \lambda_i = 1 \). A \textit{generalized convex combination} of \( A^i \) has outcome space \( \Omega = \bigcup_{i=1}^n \Omega^i \) and is the observable define by
\[
\left( \bigvee_{i=1}^n \lambda_i A^i \right)_x = \sum_{i=1}^n \{ \lambda_i A^i_x : x \in \Omega^i \}
\]
for all \( x \in \Omega \). The two extreme cases of a generalized convex combination are when \( \Omega_i = \Omega, i = 1, 2, \ldots, n \) and when \( \Omega^i \cap \Omega^j = \emptyset, i \neq j, i, j = 1, 2, \ldots, n \).
The first case is called a convex combination and is denoted by \( \sum_{i=1}^{n} \lambda_i A^i \). The second case is called a convex union and is denoted by \( \bigcup_{i=1}^{n} \lambda_i A^i \). We have that \( \left( \sum_{i=1}^{n} \lambda_i A^i \right)_x = \sum_{i=1}^{n} \lambda_i A^i_x \) for all \( x \in \Omega \) and \( \left( \bigcup_{i=1}^{n} \lambda_i A^i \right)_x = \lambda_j A^j_x \) where \( x \in \Omega' \). When \( A = \bigvee_{i=1}^{n} \lambda_i A^i \) we obtain

\[
\Phi^A_\rho(x) = \sum_{i=1}^{n} \{ \lambda_i \text{tr} (\rho A^i_x): x \in \Omega^i \} = \sum_{i=1}^{n} \{ \lambda_i \Phi^A_\rho(x): x \in \Omega^i \}
\]

**Example 1.** Let \( \{a_1,a_2,a_3\}, \{b_1,b_2,b_3\} \in \mathcal{O}(H) \). Define \( A^1, A^2 \in \mathcal{O}(H) \) by \( \Omega_{A^i} = \{x_1,x_2,x_3\}, i = 1,2, A^1_{x_1} = a_1, A^2_{x_1} = b_1, \) \( j = 1,2,3 \). Then for the convex combination \( A = \frac{1}{2} A^1 + \frac{1}{2} A^2 \) we have that \( \Omega_A = \{x_1,x_2,x_3\} \) and \( A_{x_1} = \frac{1}{2} (a_1 + b_1), i = 1,2,3 \). Now define \( B^1, B^2 \in \mathcal{O}(H) \) by \( \Omega_{B^i} = \{x_1,x_2,x_3\}, \Omega_{B^2} = \{y_1,y_2,y_3\} \) where \( \Omega_{B^1} \cap \Omega_{B^2} = \emptyset \) and \( B^1_{x_1} = a_i, B^2_{y_i} = b_i, i = 1,2,3 \). Then for the convex union \( B = \frac{1}{2} B^1 \cup \frac{1}{2} B^2 \) we have

\[
\Omega_B = \Omega_{B^1} \cup \Omega_{B^2} = \{x_1,x_2,x_3,y_1,y_2,y_3\}
\]

and \( B_{x_1} = \frac{1}{i} a_i, i = 1,2,3, B_{y_i} = \frac{1}{i} b_i, i = 1,2,3 \). For another example, define \( C^1, C^2 \in \mathcal{O}(H) \) by \( \Omega_{C^1} = \{x_1,x_2,x_3\}, \Omega_{C^2} = \{x_1,y_2,y_3\} \) where \( \{x_2,x_3\} \cap \{y_2,y_3\} = \emptyset \) and \( C^1_{x_1} = a_i, i = 1,2,3, C^2_{x_1} = b_1, C^2_{y_i} = b_i, i = 2,3 \). Then for the generalized convex combination \( C = \frac{1}{2} C^1 \lor \frac{1}{2} C^2 \) we have

\[
\Omega_C = \Omega_{C^1} \cup \Omega_{C^2} = \{x_1,x_2,x_3,y_2,y_3\}
\]

and \( C_{x_1} = \frac{1}{2} (a_1 + b_2), C_{x_2} = \frac{1}{2} a_2, C_{x_2} = \frac{1}{2} a_2, C_{y_2} = \frac{1}{2} b_2, C_{y_3} = \frac{1}{2} b_3 \). To illustrate the large number of possibilities even in this simple case, define \( D^1, D^2 \in \mathcal{O}(H) \) by \( \Omega_{D^1} = \{x_1,x_2,x_3\}, \Omega_{D^2} = \{x_1,y_2,y_3\} \) where \( x_3 \neq y_3 \) and \( D^1_{x_1} = a_i, D^2_{x_1} = b_i, D^2_{y_3} = b_3 \). For the generalized convex combination \( D = \frac{1}{2} D^1 \lor \frac{1}{2} D^2 \) we have \( \Omega_D = \{x_1,x_2,x_3,y_3\} \) and \( D_{x_1} = \frac{1}{2} (a_1 + b_1), D_{x_2} = \frac{1}{2} (a_2 + b_2), D_{x_3} = \frac{1}{2} a_3, D_{y_3} = \frac{1}{2} b_3 \).

**Theorem 2.3.** (i) \( f \left( \bigvee_{i=1}^{n} \lambda_i A^i \right) = \sum_{i,x} \{ \lambda_i A^i_x: x \in \Omega_i, f(x) = y \} \)

(ii) \( f \left( \sum_{i=1}^{n} \lambda_i A^i \right) = \sum_{i=1}^{n} \lambda_i f(A^i) \).

(iii) \( f \left( \bigcup_{i=1}^{n} \lambda_i A^i \right) = \sum_{i=1}^{n} \lambda_i f|_{\Omega_i}(A^i) \).
Proof. (i) Letting \( f: \cup \Omega_i \rightarrow \Omega \) be a surjection, if \( y \in \Omega \) we have that

\[
 f \left( \bigvee_{i=1}^{n} \lambda_i A^i \right)_y = \left( \bigvee_{i=1}^{n} \lambda_i A^i \right)_{f^{-1}(y)} = \sum_x \left\{ \left( \bigvee_{i=1}^{n} \lambda_i A^i \right)_x : f(x) = y \right\} = \sum \{ \lambda_i A^i_x : x \in \Omega_i, f(x) = y \}
\]

(ii) In this case \( \Omega_i = \Omega_j \) for all \( i, j = 1, \ldots, n \) so \( \cup \Omega_i = \Omega_j \) for all \( j = 1, 2, \ldots, n \). Hence, by (i) we obtain

\[
 f \left( \bigvee_{i=1}^{n} \lambda_i A^i \right)_y = \sum \{ \lambda_i A^i_x : f(x) = y \} = \sum \lambda_i A^i_{f^{-1}(y)} = \sum \lambda_i f(A^i)_y
\]

The result follows. (iii) In this case \( \Omega_i \cap \Omega_j = \emptyset \) for all \( i \neq j \). Hence, if \( x \in \cup \Omega_i \), then \( x \in \Omega_i \) for a unique \( i \). Then \( \left( \bigcup_{i=1}^{n} \lambda_i A^i \right)_x = \lambda_i A^i_x \) where \( x \in \Omega_j \) and by (i) we have that

\[
 f \left( \bigcup_{i=1}^{n} \lambda_i A^i \right)_y = \sum \sum_x (\lambda_i A^i : f|_{\Omega_i}(x) = y) = \sum \lambda_i (f|_{\Omega_i})^{-1}(y) = \sum \lambda_i f|_{\Omega_i}(A^i)_y
\]

The result follows. \(\square\)

Notice that \( \mu \cdot (\sum \lambda_i A^i) = \sum \lambda_i (\mu \cdot A^i) \) because

\[
 \left[ \mu \cdot (\sum \lambda_i A^i) \right]_y = \sum_x \mu_x \sum_i \lambda_i A^i_x = \sum_i \lambda_i \sum_x \mu_x A^i = \sum_i \lambda_i (\mu \cdot A^i)_y
\]

In general, \( \mu \cdot (\bigvee \lambda_i A^i) \neq \bigvee \lambda_i (\mu \cdot A^i) \) because the \( A^i \) can have different outcome spaces so \( \mu \cdot A^i \) is not defined.

We call the observables \( F^j \) with outcome space \( \Omega = \{x_1, x_2, \ldots, x_n\} \) defined by \( F^j x_i = \delta_{ij} \) *identity observables.* If \( A \in \mathcal{O}(H) \) with \( \Omega_A = \Omega \) and \( \lambda \in (0, 1] \), we call \( B \in \mathcal{O}(H) \) given by \( B = (1 - \lambda)F^j + \lambda A \) the observable \( A \text{ with noise factor } (1 - \lambda)/\lambda \) \cite{9, 10}.

**Theorem 2.4.** If \( A^i \in \mathcal{O}(H) \) and \( \lambda_i \in [0, 1] \) with \( \sum \lambda_i = 1 \), \( i = 1, 2, \ldots, n \), then there exists \( A \in \mathcal{O}(H) \) and surjections \( f_j: \Omega_A \rightarrow \Omega_{A^j} \) such that

\[
 f_j(A) = (1 - \lambda_j)F^j + \lambda_j A^j
\]

for some \( x_j \in \Omega_{A^j}, j = 1, 2, \ldots, n \).
Proof. We can assume, without loss of generality, that $\Omega_{A_i} \cap \Omega_{A_j} = \emptyset$ for $i \neq j$. Letting $A = \bigcup_{i=1}^{n} \lambda_i A_i$ we have that $\Omega_A = \bigcup_{i=1}^{n} \Omega_{A_i}$. Define the functions $f_j : \Omega_A \rightarrow \Omega_{A_j}$ by $f_j(x) = x$ for all $x \in \Omega_{A_j}$ and $f_j(x) = x_j \in \Omega_{A_j}$ for $x \not\in \Omega_{A_j}$. Then we obtain

$$f_j \left( \bigcup_{i=1}^{n} \lambda_i A_i^i \right)_{x} = \left( \bigcup_{i=1}^{n} \lambda_i A_i^i \right)_{f_j^{-1}(x)} = \sum_{i} \lambda_i A_i^{i} f_j^{-1}(x) \cap \Omega_i$$

$$= \sum_{i \neq j} \lambda_i A_i^i + \lambda_j A_j^i = \sum_{i \neq j} \lambda_i I_x^i + \lambda_j A_j^i$$

and for $x = x_j$ we have that

$$f_j \left( \bigcup_{i=1}^{n} \lambda_i A_i^i \right)_{x} = \left( \bigcup_{i=1}^{n} \lambda_i A_i^i \right)_{f_j^{-1}(x)} = \left( \bigcup_{i} \lambda_i A_i^i \right)_{x} = \lambda_j A_j^i$$

Hence, if $A = \bigcup_{i=1}^{n} \lambda_i A_i^i$ then

$$f_j(A) = (1 - \lambda) I_x^j + \lambda_j A_j^i$$

Since the $f_j(A)$ in Theorem 2.3 are all parts of the same observable $A$, we see that the $f_j(A) = (1 - \lambda_j) I_x^j + \lambda_j A_j^i$ mutually coexist, $j = 1, 2, \ldots, n$. We conclude that any set of observables $A^i$, $j = 1, 2, \ldots, n$, “almost coexist” in the sense that a noisy version of $A_j^i$ is a part of an observable $A$, $j = 1, 2, \ldots, n$.

For $A, B \in \mathcal{O}(H)$, we define their sequential product $A \circ B \in \mathcal{O}(H)$ by $\Omega_{A \circ B} = \Omega_A \times \Omega_B$ and

$$(A \circ B)_{(x, y)} = A_x \circ B_y = A_x^{1/2} B_y A_y^{1/2}$$

If $X \subseteq \Omega_A \times \Omega_B$, we have that

$$(A \circ B)_X = \sum_{(x, y) \in X} (A \circ B)_{(x, y)} = \sum_{(x, y) \in X} A_x \circ B_y$$

It follows that $(A \circ B)_{(x, y)} = A_x \circ B_y$ but $(A \circ B)_{X \times \{y\}} \neq A_X \circ B_y$, in general. Moreover,

$$\Phi^{A \circ B}_{\rho}(x, y) = \text{tr} \left[ \rho (A \circ B)_{(x, y)} \right] = \text{tr} (\rho A_x \circ B_y)$$
and if $X \subseteq \Omega_A \times \Omega_B$ then
\[ \Phi_{\rho}^{A \circ B}(X) = \sum_{(x,y) \in X} \text{tr} (\rho A_x \circ B_y) \]

We also define the observable $(B \mid A)$ with $\Omega_{(B \mid A)} = \Omega_B$ and $(B \mid A)_y = \sum_{x \in \Omega_A} (A_x \circ B_y)$. We call $(B \mid A)$ the observable $B$ conditioned on $A$ [5, 6, 8]. We then have that
\[ \Phi_{\rho}^{(B \mid A)}(y) = \text{tr} [\rho (B \mid A)_y] = \text{tr} \left[ \rho \sum_{x \in \Omega_A} (A_x \circ B_y) \right] = \sum_{x \in \Omega_A} \text{tr} (\rho A_x \circ B_y) \]
for all $Y \subseteq \Omega_B$ we obtain
\[ \Phi_{\rho}^{(B \mid A)}(Y) = \sum_{y \in Y} \sum_{x \in \Omega_A} \text{tr} (\rho A_x \circ B_y) = \sum_{x \in \Omega_A} \text{tr} (\rho A_x \circ B_Y) \]

Defining the functions $f : \Omega_A \times \Omega_B \rightarrow \Omega_B$, $g : \Omega_A \times \Omega_B \rightarrow \Omega_A$ by $f(x, y) = y$ for all $x \in \Omega_A$ and $g(x, y) = x$ for all $y \in \Omega_B$ we see that
\[ f(A \circ B)_y = (A \circ B)_f^{-1}(y) = \sum_x \{(A \circ B)_{(x,y)} : f(x, y) = y\} \]
\[ = \sum_{x \in \Omega_A} (A_x \circ B_y) = (B \mid A)_y \]

and
\[ g(A \circ B)_x = (A \circ B)_{g^{-1}(x)} = \sum_y \{(A \circ B)_{(x,y)} : g(x, y) = x\} \]
\[ = \sum_{y \in \Omega_B} (A_x \circ B_y) = A_x \]

Hence, $(B \mid A) = f(A \circ B)$ and $A = g(A \circ B)$. We conclude that $(B \mid A)$ and $A$ coexist. In general, $(B \mid A)$ and $B$ need not coexist. Also $(B \mid A)$ and $(C \mid A)$ need not coexist even though they both coexist with $A$.

**Theorem 2.5.** (i) $A \circ \left( \bigvee_{i=1}^{n} \lambda_i B^i \right) = \bigvee_{i=1}^{n} \lambda_i A \circ B^i$.

(ii) $(\bigvee_{i=1}^{n} \lambda_i B^i \mid A) = \bigvee_{i=1}^{n} \lambda_i (B^i \mid A)$.
Proof. (i) For all \( x \in \Omega_A, y \in \bigcup \Omega_i \) with \( \Omega_i = \Omega_{B_i} \) we have that

\[
\left( A \circ \bigvee_{i=1}^n \lambda_i B^i \right)_{(x,y)} = A_x \circ \left( \bigvee \lambda_i B^i \right)_y = A_x \circ \sum_i \{ \lambda_i B^i_y : y \in \Omega_i \}
\]

\[
= \sum_i \{ \lambda_i A_x \circ B^i_y : y \in \Omega_i \}
\]

\[
= \left\{ \bigvee \lambda_i (A \circ B^i)(x,y) : y \in \Omega_i \right\}
\]

\[
= \left( \bigvee_{i=1}^n \lambda_i A \circ B^i \right)_{(x,y)}
\]

The result follows. (ii) For all \( x \in \Omega_A, y \in \bigcup \Omega_i \), it follows from (i) that

\[
\left( \bigvee_{i=1}^n \lambda_i B^i \mid A \right)_{(x,y)} = \sum_{x \in \Omega_A} A_x \circ \left( \bigvee \lambda_i B^i \right)_y = \sum_{x \in \Omega_A} \bigvee \lambda_i A_x \circ B^i_y
\]

\[
= \sum_{x \in \Omega_A} \sum_i \{ \lambda_i A_x \circ B^i_y : y \in \Omega_i \} = \sum_{y \in \Omega_i} \sum_{x \in \Omega_A} A_x \circ B^i_y
\]

\[
= \bigvee \left( \lambda_i \sum_{x \in \Omega_A} A_x \circ B^i_y \right) = \left[ \bigvee \lambda_i (B^i \mid A) \right]_{(x,y)}
\]

The result follows. \( \square \)

In general, \( \bigvee \lambda_i B^i \circ A \neq \bigvee \lambda_i (B^i \circ A) \) and \( (A \mid \bigvee \lambda_i B^i) \neq \bigvee \lambda_i (A \mid B^i) \).

If \( A \in \mathcal{O}(H_1) B \in \mathcal{O}(H_2) \), we define the tensor product \( A \otimes B \in \mathcal{O}(H_1 \otimes H_2) \) by \( \Omega_A \otimes B = \Omega_A \times \Omega_B \) and \( (A \otimes B)(x,y) = A_x \times B_y \). If \( \mu_{xy} \) and \( \nu_{uv} \) are transition probabilities, we define the transition probability

\[
\mu \bullet \nu_{(x,u),(y,v)} = \mu_{xy} \nu_{uv}
\]

We see that \( \mu \bullet \nu \) is indeed a transition probability because

\[
\sum_{(y,v)} \mu \bullet \nu_{(x,u),(y,v)} = \sum_{y,v} \mu_{xy} \nu_{uv} = \sum_y \mu_{xy} \sum_v \nu_{uv} = 1
\]

If \( f: \Omega_A \to \Omega_1, g: \Omega_B \to \Omega_2 \), we define the function \( f \times g: \Omega_A \times \Omega_B \to \Omega_1 \times \Omega_2 \) by

\[
f \times g(x, y) = (f(x), g(y))
\]

The next result summarizes combinations with \( A \otimes B \).
Theorem 2.6. (i) If $A \in \mathcal{O}(H_1)$, $B \in \mathcal{O}(H_2)$, then $(\mu \cdot A) \otimes (\nu \cdot B) = (\mu \cdot \nu) \cdot (A \otimes B)$. (ii) If $A \in \mathcal{O}(H_1)$, $B \in \mathcal{O}(H_2)$, then $f(A) \otimes g(B) = f \times g(A \otimes B)$. (iii) $A \otimes (\bigvee \lambda_i B_i) = \bigvee \lambda_i A \otimes B^i$ and $(\bigvee \lambda_i B^i) \otimes A = \bigvee \lambda_i B^i \otimes A$. (iv) If $A, C \in \mathcal{O}(H_1)$ and $B, D \in \mathcal{O}(H_2)$, then
\[
[(A \otimes B) \circ (C \otimes D)]((x,y),(u,v)) = [(A \circ C) \otimes (B \circ D)]((x,u),(y,v))
\]

Proof. (i) For all applicable $x, y, u, v$ we have that
\[
[(\mu \cdot A) \otimes (\nu \cdot B)](y,z) = \sum_x \mu_{xy} A_x \otimes \sum_u \nu_{u_z} B_u
\]
\[
= \sum_{x,u} \mu_{xy} \nu_{u_z} (A \otimes B)(x,u)
\]
\[
= \sum_{x,u} \mu \cdot \nu((x,u),(y,z)) (A \circ B)(x,u)
\]
\[
= [(\mu \cdot \nu) \cdot (A \otimes B)](y,z)
\]
The result follows. (ii) Letting $h = f \times g$ we obtain
\[
[f(A) \otimes g(B)](u,v) = f(A)u \otimes g(B)v = A_{f^{-1}(u)} \otimes B_{g^{-1}(v)}
\]
\[
= \sum_x \{A_x : f(x) = u\} \otimes \sum_y \{B_y : g(y) = v\}
\]
\[
= \sum_{x,y} \{A_x \otimes B_y : f(x) = u, g(y) = v\}
\]
\[
= \sum_{x,y} \{A_x \otimes B_y : h(x,y) = (u,v)\}
\]
\[
= \sum_{x,y} \{(A \otimes B)_{(x,y)} : h(x,y) = (u,v)\}
\]
\[
= (A \otimes B)_{h^{-1}(u,v)} = [h(A \otimes B)](u,v)
\]
The result follows. (iii) For all applicable $x, y$ we have that
\[
\left[ A \otimes \left( \bigvee \lambda_i B_i \right) \right]_{(x,y)} = A_x \otimes \left( \bigvee \lambda_i B_i \right)_y = A_x \otimes \sum_i \{\lambda_i B^i_y : y \in \Omega_i\}
\]
\[
= \sum_i \{\lambda_i A_y \otimes B^i_y : y \in \Omega_i\}
\]
\[
= \sum_i \{\lambda_i A_x \otimes B^i_y : y \in \Omega_i\}
\]
\[ \left( \bigvee \lambda_i A \otimes B^i \right)_{(x,y)} \]

The result follows. (iv) For all applicable \( x, y, u, v \) we obtain

\[
\begin{align*}
[(A \otimes B) \circ (C \otimes D)]_{((x,y),(u,v))} &= (A \otimes B)_{(x,y)} \circ (C \otimes D)_{(u,v)} = (A_x \otimes B_y) \circ (C_u \otimes D_v) \\
&= (A_x \otimes B_y)^{1/2} (C_u \otimes D_v) (A_x \otimes B_y)^{1/2} \\
&= (A_x^{1/2} \otimes B_y^{1/2}) (C_u \otimes D_v) (A_x^{1/2} \otimes B_y^{1/2}) \\
&= A_x^{1/2} C_u A_x^{1/2} \otimes B_y^{1/2} D_v B_y^{1/2} = A_x \circ C_u \otimes B_y \circ D_v \\
&= (A \circ C)_{(x,u)} \otimes (B \circ D)_{(y,v)} = [(A \circ C) \otimes (B \circ D)]_{((x,u),(y,v))}
\end{align*}
\]

We see from Theorem 2.6(iv) that \((A \otimes B) \circ (C \otimes D) \neq (A \circ C) \otimes (B \circ D)\), in general. It also follows from Theorem 2.6(ii) that if \( A, B \in \mathcal{O}(H_1) \) coexist and \( C, D \in \mathcal{O}(H_2) \) coexist, then \( A \otimes C \) and \( B \otimes D \) coexist. Indeed, we have observables \( E \in \mathcal{O}(H_1) \), \( F \in \mathcal{O}(H_2) \) and functions \( f_1, g_2, f_2, g_2 \) such that \( A = f_1(E), B = g_1(E), C = f_2(F), D = g_2(F) \). Applying Theorem 2.6(ii) gives

\[
\begin{align*}
A \otimes C &= f_1(E) \otimes f_2(F) = f_1 \times f_2(E \otimes F) \\
B \otimes D &= g_1(E) \otimes g_2(F) = g_1 \times g_2(E \otimes F)
\end{align*}
\]

Hence, \( A \otimes C \) and \( B \otimes D \) coexist.

We have gone from \( \mathcal{O}(H_1), \mathcal{O}(H_2) \) to obtain observables in \( \mathcal{O}(H_1 \otimes H_2) \). We can also go the other way to reduce observables in \( \mathcal{O}(H_1 \otimes H_2) \) to elements of \( \mathcal{O}(H_1) \) and \( \mathcal{O}(H_2) \). If \( A \in \mathcal{O}(H_1 \otimes H_2) \), we define the reduced observables \( A^1 \in \mathcal{O}(H_1), A^2 \in \mathcal{O}(H_2) \) by \( A^1_x = \frac{1}{n_2} \text{tr}_{H_2} A_x \) for all \( x \in \Omega_A \) where \( n_2 = \text{dim} \ H_2 \) and \( \text{tr}_{H_2} \) is the partial trace with respect to \( H_2 \) [5, 8, 9] and similarly \( A^2_x = \frac{1}{n_1} \text{tr}_{H_1} A_x \). To check that \( A^1 \) is indeed an observable, we see that \( A^1_x \geq 0 \) and

\[
\sum_{x \in \Omega_A} A^1_x = \frac{1}{n_2} \sum_{x \in \Omega_A} \text{tr}_{H_2} (A_x) = \frac{1}{n_2} \text{tr}_{H_2} \left( \sum_{x \in \Omega_A} A_x \right) = \frac{1}{n_2} \text{tr}_{H_2} (I)
\]

\[
= \frac{1}{n_2} \text{tr}_{H_2} (I_1 \otimes I_2) = \frac{1}{n_2} (I_2) I_1 = I_1
\]

where \( I_1, I_2 \) are the identity operators on \( H_1, H_2 \), respectively.
If $A \in \mathcal{O}(H_1)$, $B \in \mathcal{O}(H_2)$, we have the observable $C = A \otimes B \in \mathcal{O}(H_1 \otimes H_2)$. It is interesting to note that

$$(A \otimes B)^1_{(x,y)} = \frac{1}{n_2} \operatorname{tr}_H(A \otimes B)_{(x,y)} = \frac{1}{n_2} \operatorname{tr}_H(A_x \otimes B_y) = \frac{1}{n_2} (\operatorname{tr}_B y) A_x$$

Hence, $(A \otimes B)^1_{\{x\} \times \Omega_B} = A_x$ and $(A \otimes B)^1_{\Omega_A \times \{y\}} = \frac{1}{n_2} (\operatorname{tr}_B y) I_1$. In a similar way, $(A \otimes B)^2_{(x,y)} = \frac{1}{n_1}(\operatorname{tr}_A x) B_y$. For $A, B \in \mathcal{O}(H_1 \otimes H_2)$ we obtain

$$(A \circ B)^1_{(x,y)} = \frac{1}{n_2} \operatorname{tr}_H(A \circ B)_{(x,y)} = \frac{1}{n_2} \operatorname{tr}_H A_x \circ B_y = \frac{1}{n_2} \operatorname{tr}_H A_x^{1/2} B_y A_x^{1/2}$$

On the other hand, we have that

$$(A^1 \circ B^1)_{(x,y)} = A_x^1 \circ B_y^1 = \frac{1}{(n_2)^2} (\operatorname{tr}_H A_x^1) \circ (\operatorname{tr}_H B_y^1) = \frac{1}{(n_2)^2} (\operatorname{tr}_H A_x^1)^{1/2} (\operatorname{tr}_H B_y^1) (\operatorname{tr}_H A_x^1)^{1/2}$$

It follows that $(A \circ B)^1 \neq A^1 \circ B^1$, in general, so sequential products need not be preserved under reduction. The next result shows that the other combinations we considered are preserved.

**Theorem 2.7.** (i) If $A \in \mathcal{O}(H_1 \otimes H_2)$, then $f(A)^i = f(A^i)$, $i = 1, 2$. (ii) If $A, B \in \mathcal{O}(H_1 \otimes H_2)$ coexist, then $A^i$ and $B^i$ coexist, $i = 1, 2$. (iii) If $A \in \mathcal{O}(H_1 \otimes H_2)$, then $(\mu \cdot A)^i = \mu \cdot A^i$, $i = 1, 2$. (iv) If $A^j \in \mathcal{O}(H_1 \otimes H_2)$, then $(\bigvee \lambda_j A^j)^i = \bigvee \lambda_j (A^j)^i$, $i = 1, 2$.

**Proof.** We prove these results for $i = 1$ and the proofs for $i = 2$ are similar.

(i) For all $y \in \Omega_{f(A)}$ we obtain

$$f(A)^1_y = \frac{1}{n_2} \operatorname{tr}_H f(A)_y = \frac{1}{n_2} \operatorname{tr}_H A f^{-1}(y) = \frac{1}{n_2} \operatorname{tr}_H \left( \sum_x \{ A_x : f(x) = y \} \right)$$

$$= \sum_x \left\{ \frac{1}{n_2} \operatorname{tr}_H A_x : f(x) = y \right\} = \sum_x \{ A^1_x : f(x) = y \} = f(A^1)_y$$

The result follows. (ii) If $A, B$ coexist, there exist $C \in \mathcal{O}(H_1 \otimes H_2)$ and functions $f, g$ such that $A = f(C)$, $B = g(C)$. Applying (i) gives $A^1 = f(C)^1 = f(C^1)$ and $B^1 = g(C)^1 = g(C)^1$. Hence, $A^1$ and $B^1$ coexist.

(iii) For all applicable $y$ we have that

$$(\mu \cdot A)^1_y = \frac{1}{n_2} \operatorname{tr}_H (\mu \cdot A)_y = \frac{1}{n_2} \operatorname{tr}_H \left( \sum_x \mu_{xy} A_x \right) = \sum_x \mu_{xy} \frac{1}{n_2} \operatorname{tr}_H A_x$$
= \sum_x \mu_{xy} A^1_x = (\mu \cdot A^1)_y

The result follows. (iv) For all applicable \( x \) we obtain

\[
\left( \bigvee \lambda_i A^i \right)^1_x = \frac{1}{n_2} \text{tr}_{H_2} \left( \bigvee \lambda_i A^i \right)_x = \frac{1}{n_2} \text{tr}_{H_2} \left[ \sum_i \{ \lambda_i A^i_x : x \in \Omega_i \} \right]
\]

\[
= \sum_i \{ \lambda_i \text{tr}_{H_2} A^i_x : x \in \Omega_i \} = \sum_i \{ \lambda_i (A^i)_x : x \in \Omega_i \}
\]

\[
= \left[ \bigvee \lambda_i (A^i)^1 \right]_x
\]

This proves the result.

Although we have not found a counterexample, we conjecture that the converse of Theorem 2.7(ii) does not hold.

3 Instruments

For instruments, we define post-processing \( \mu \cdot \mathcal{I} \), parts \( f(\mathcal{I}) \), coexistence and generalized convex combinations \( \bigvee \lambda_i \mathcal{I}^i \) as we did for observables. The next theorem shows that these definitions are consistent.

**Theorem 3.1.** (i) \( f(\mathcal{I})^\wedge = f(\hat{\mathcal{I}}) \) (ii) \( (\bigvee \lambda_i \mathcal{I}^i)^\wedge = \bigvee \lambda_i \mathcal{I}^i \wedge \). (iii) \( (\mu \cdot \mathcal{I})^\wedge = \mu \cdot \hat{\mathcal{I}} \).

**Proof.** (i) For all \( x \in \Omega_\mathcal{I} \) and \( \rho \in S(H) \) we have that

\[
\text{tr} \left[ \rho f(\hat{\mathcal{I}})_x \right] = \text{tr} \left[ \rho \hat{\mathcal{I}} f^{-1}(x) \right] = \text{tr} \left[ \mathcal{I} f^{-1}(x) \right] = \text{tr} \left[ f(\mathcal{I})_x (\rho) \right] = \text{tr} \left[ \rho f(\mathcal{I})^\wedge_x \right]
\]

Hence, \( f(\mathcal{I})^\wedge = f(\hat{\mathcal{I}})_x \) for all \( x \in \Omega_\mathcal{I} \) and the result follows. (ii) For all \( x \in \bigcup_i \Omega_i \) where \( \Omega_i = \Omega_\mathcal{I}^i \) we obtain

\[
\text{tr} \left[ \rho \left( \bigvee \lambda_i \mathcal{I}^i \wedge \right)_x \right] = \text{tr} \left[ \rho \sum_x \{ \lambda_i \mathcal{I}^i_x \wedge : x \in \Omega_i \} \right] = \sum_x \{ \lambda_i \text{tr}(\rho \mathcal{I}^i_x) \wedge : x \in \Omega_i \}
\]

\[
= \sum_x \{ \lambda_i \text{tr} \left[ \mathcal{I}^i_x (\rho) \right] : x \in \Omega_i \}
\]

\[
= \text{tr} \left[ \sum_x \{ \lambda_i \mathcal{I}^i_x (\rho) : x \in \Omega_i \} \right]
\]

14
\[
= \text{tr} \left[ \left( \bigvee x \lambda_i \mathcal{I}^i \right)_x (\rho) \right] = \text{tr} \left[ \rho \left( \bigvee x \lambda_i \mathcal{I}^i \right)_x \right]
\]

Hence, \( \bigvee x \lambda_i \mathcal{I}^i_x = \left( \bigvee x \lambda_i \mathcal{I}^i \right)_x \) for all \( x \in \bigcup \Omega_i \) and this gives the result.

(iii) For all \( y \in \Omega_{\mu, \mu} \) we have that

\[
\text{tr} \left[ \rho (\mu \cdot \tilde{I})_y \right] = \text{tr} \left( \rho \sum_x \mu_{xy} \tilde{I}_x \right) = \sum_x \mu_{xy} \text{tr} \left( \rho \tilde{I}_x \right) = \sum_x \mu_{xy} \text{tr} \left[ \mathcal{I}_x (\rho) \right] \\
= \text{tr} \left[ \sum_x \mu_{xy} \mathcal{I}_x (\rho) \right] = \text{tr} \left[ (\mu \cdot \mathcal{I})_y (\rho) \right] = \text{tr} \left[ \rho (\mu \cdot \mathcal{I})_y \right]
\]

We conclude that \( (\mu \cdot \mathcal{I})_y = (\mu \cdot \tilde{I})_y \) for all \( y \in \Omega_{\mu, \mu} \) and this proves the result. \( \square \)

Applying Theorem 3.1(i) we obtain the following.

**Corollary 3.2.** If \( \mathcal{I}, \mathcal{J} \in \text{In} (\mathcal{H}) \) coexist, then \( \tilde{\mathcal{I}}, \tilde{\mathcal{J}} \) coexist.

Unlike the other concepts, we must define sequential products of instruments differently from that of observables. If \( \mathcal{I}, \mathcal{J} \in \text{In} (\mathcal{H}) \), then their sequential product \( \mathcal{I} \circ \mathcal{J} \in \text{In} (\mathcal{H}) \) is defined by \( \Omega_{\mathcal{I} \circ \mathcal{J}} = \Omega_{\mathcal{I}} \times \Omega_{\mathcal{J}} \) and \( (\mathcal{I} \circ \mathcal{J})_{x,y}(\rho) = \mathcal{J}_y [\mathcal{I}_x (\rho)] \) for all \( \rho \in \mathcal{S} (\mathcal{H}) \). We define the conditional instrument \( (\mathcal{J} | \mathcal{I}) \in \text{In} (\mathcal{H}) \) by \( \Omega_{(\mathcal{J} | \mathcal{I})} = \Omega_{\mathcal{J}} \) and

\[
(\mathcal{J} | \mathcal{I})_y (\rho) = \sum_{x \in \Omega_{\mathcal{I}}} (\mathcal{I} \circ \mathcal{J})_{(x,y)} (\rho) = \sum_{x \in \Omega_{\mathcal{I}}} \mathcal{J}_y (\mathcal{I}_x (\rho)) = \mathcal{J}_y [\Omega_{\mathcal{I}} (\rho)]
\]

Of course, \( \mathcal{I}_{\Omega_{\mathcal{I}}} \) is the channel given by \( \mathcal{I} \). Unlike for observables, the next theorem has a second part.

**Theorem 3.3.** (i) \( \mathcal{I} \circ (\bigvee x \lambda_i \mathcal{J}^i) = (\bigvee x \lambda_i \mathcal{I} \circ \mathcal{J}^i) \). (ii) \( (\bigvee x \lambda_i \mathcal{J}^i) \circ \mathcal{I} = (\bigvee x \lambda_i \mathcal{J}^i \circ \mathcal{I}) \)

**Proof.** We let \( x \in \Omega_{\mathcal{I}}, y \in \bigcup \Omega_i \) where \( \Omega_i = \Omega_{\mathcal{J}_i} \) and \( \rho \in \mathcal{S} (\mathcal{H}) \) be arbitrary elements. (i) The following steps hold:

\[
(\mathcal{I} \circ \bigvee x \lambda_i \mathcal{J}^i)_{(x,y)} (\rho) = \left( \bigvee x \lambda_i \mathcal{J}^i \right)_y (\mathcal{I}_x (\rho)) = \sum_y \left\{ \lambda_i \mathcal{J}^i_y [\mathcal{I}_x (\rho)] : y \in \Omega_i \right\} \\
= \sum_y \left\{ \lambda_i (\mathcal{I} \circ \mathcal{J}^i)_{(x,y)} (\rho) : y \in \Omega_i \right\} \\
= (\bigvee x \lambda_i \mathcal{I} \circ \mathcal{J}^i)_{(x,y)} (\rho)
\]
The result now follows. (ii) The following steps hold:

\[
\bigvee \lambda_i J^i \circ I (x,y) (\rho) = I_y \left( \bigvee \lambda_i J^i_x (\rho) \right) = I_y \left[ \sum_x \left\{ \lambda_i J^i_x (\rho) : x \in \Omega_i \right\} \right]
\]

\[
= \sum_x \left\{ \lambda_i I_y (J^i_x (\rho) ) : x \in \Omega_i \right\}
\]

\[
= \sum_x \left\{ \lambda_i (J^i \circ I)_{(x,y)} (\rho) : x \in \Omega_i \right\}
\]

\[
= \bigvee \left( \lambda_i J^i \circ I \right)_{(x,y)} (\rho)
\]

The result follows. \(\Box\)

Most of the theorems in Section 2 concerning observables hold for instruments and the proofs are similar so we shall not repeat them. We will mainly concentrate on various types of instruments that we now define. We say that an instrument \( I \in \text{In} (H) \) is:

- **Kraus** if it has the form \( I_x (\rho) = S_x \rho S_x^* \) where \( S_x \in \mathcal{L} (H) \) with \( \sum S_x^* S_x = I \),

- **Lüders** if \( I_x (\rho) = \mathcal{L}_x^A (\rho) = A^{1/2}_x \rho A^{1/2}_x \) where \( A \in \mathcal{O} (H) \),

- **Trivial** if \( I_x (\rho) = \text{tr} (\rho A_x) \alpha \) where \( A \in \mathcal{O} (H) \), \( \alpha \in \mathcal{S} (H) \),

- **Semitrivial** if \( I_x (\rho) = \text{tr} (\rho A_x) \alpha_x \) where \( A \in \mathcal{O} (H) \), \( \alpha_x \in \mathcal{S} (H) \).

Notice that a Lüders instrument is a special case of a Kraus instrument and a trivial instrument is a special case of a semitrivial instrument. An interesting example of a semitrivial instrument is

\[
I_x (\rho) = \frac{\text{tr} (\rho A_x)}{\text{tr} (A_x)} A_x
\]

It is easy to check that the observable measured by the Kraus instrument is \( \hat{I}_x = S_x^* S_x \) and the other three types of instruments measure the observable \( A \). This also shows that an observable is measured by many different instruments. We call \( S_x \) the operators for the Kraus instrument \( I_x (\rho) = S_x \rho S_x^* \).

We say that two observables \( A, B \in \mathcal{O} (H) \) commute if \( A_x B_y = B_y A_x \) for all \( x \in \Omega_A, y \in \Omega_B \).
Theorem 3.4. (i) \((L^A \circ L^B)^\wedge = (L^A)^\wedge \circ (L^B)^\wedge = A \circ B\). (ii) \(L^A \circ L^B\) is a Lüders instrument if and only if \(A\) and \(B\) commute and if in this case \(L^A \circ L^B = L^{A \circ B}\). (iii) If \(I\) and \(J\) are Kraus instruments with operators \(S_x, T_y\), respectively, then \(I \circ J\) is a Kraus instrument with operators \(T_y S_x\). (iv) If \(I, J\) are simitrivial with observables \(A, B\) and states \(\alpha_x, \beta_y\), respectively, then \(I \circ J\) is semitrivial with observable \(C_{(x,y)} = \text{tr} \left( \alpha_x B_y A_x \right)\) and states \(\beta_y\). Moreover, \(J | I\) is semitrivial with observable \(\text{Post}_{(a,B)}(\tilde{I})\) and states \(\beta_y\).

Proof. (i) For all \(x \in \Omega_A, y \in \Omega_B\) and \(\rho \in \mathcal{S}(H)\) we have that

\[
\text{tr} \left[ \rho(L^A \circ L^B)^\wedge_{(x,y)} \right] = \text{tr} \left[ (L^A \circ L^B)_{(x,y)}(\rho) \right] = \text{tr} \left[ L^B_y(L^A_x(\rho)) \right] = \text{tr} \left( B_y^{1/2} A_x^{1/2} \rho A_x^{1/2} B_y^{1/2} \right) = \text{tr} \left( \rho A_x^{1/2} B_y A_x^{1/2} B_y^{1/2} \right)
\]

It follows that \((L^A \circ L^B)^\wedge = A \circ B = (L^A)^\wedge \circ (L^B)^\wedge\). (ii) As in (i) we have that

\[
(L^A \circ L^B)_{(x,y)}(\rho) = B_y^{1/2} A_x^{1/2} \rho A_x^{1/2} B_y^{1/2}
\]

On the other hand

\[
L^{A \circ B}_{(x,y)}(\rho) = (A \circ B)^{1/2}_{(x,y)} \rho(A \circ B)^{1/2}_{(x,y)} = (A_x^{1/2} B_y A_x^{1/2})^{1/2} \rho(A_x^{1/2} B_y A_x^{1/2})^{1/2}
\]

By (3.1), we conclude that \(L^A \circ L^B\) is a Lüders instrument if and only if \(B_y^{1/2} A_x^{1/2} = A_x^{1/2} B_y^{1/2}\) which is equivalent to \(A_x B_y = B_y A_x\) for all \(x, y\). In this case \(L^A \circ L^B = L^{A \circ B}\). (iii) Since

\[
(I \circ J)_{(x,y)}(\rho) = J_y(I_x(\rho)) = T_y S_x \in \rho S_x^* T_y = (T_y S_x) \rho (T_y S_x)^*
\]

we conclude that \(I \circ J\) is a Kraus instrument with operators \(T_y S_x\). (iv) Since

\[
(I \circ J)_{(x,y)}(\rho) = \text{tr} \left( \rho A_x J_y(\alpha_x) \right) = \text{tr} \left( \rho A_x A_x \right) \text{tr} \left( \alpha_x B_y \right) \beta_y = \text{tr} \left( \rho C_{(x,y)} \right) \beta_y
\]

we conclude that \(I \circ J\) is semitrivial with observable \(C_{(x,y)}\) and states \(\beta_y\). The last statement follows from

\[
(J | I)_y(\rho) = \sum_x (I \circ J)_{(x,y)}(\rho) = \text{tr} \left( \rho \sum_x \text{tr} \left( \alpha_x B_y A_x \right) \right) \beta_y = \text{tr} \left( \rho \text{Post}_{(a,B)}(\tilde{I})_y \right) \beta_y
\]
Corollary 3.5. If $\mathcal{I}$, $\mathcal{J}$ are trivial with observables $A$, $B$ and states $\alpha$, $\beta$, respectively, then $\mathcal{I} \circ \mathcal{J}$ is trivial with observable $C_{(x,y)} = \text{tr} (\alpha B_y)A_x$ and state $\beta$. Moreover, $(\mathcal{J} | \mathcal{I})_y(p) = \text{tr} (\alpha B_y)\beta$ so $(\mathcal{J} | \mathcal{I})$ is trivial with observable $\text{tr} (\alpha B_y)I$ and state $\beta$.

Example 2. This example illustrates that $(\mathcal{I} \circ \mathcal{J})^\wedge \neq \hat{\mathcal{I}} \circ \hat{\mathcal{J}}$ except for Lüders instruments. If $\mathcal{I}$ and $\mathcal{J}$ are Kraus instruments with operators $S_x$, $T_y$, respectively, we have seen that $(\mathcal{I} \circ \mathcal{J})^\wedge_{(x,y)} = S_x^*T_y T_y S_x$. However,

$$(\hat{\mathcal{I}} \circ \hat{\mathcal{J}})_{(x,y)} = \hat{\mathcal{I}}_x \circ \hat{\mathcal{J}}_y = \hat{\mathcal{I}}_x^{1/2} \hat{\mathcal{J}}_y^{1/2} = (S_x^* S_x)^{1/2} T_y (S_x^* S_x)^{1/2}$$

Hence, $(\mathcal{I} \circ \mathcal{J})^\wedge \neq \hat{\mathcal{I}} \circ \hat{\mathcal{J}}$, in general. If $\mathcal{I}$, $\mathcal{J}$ are trivial instruments with operators $A$, $B$ and states $\alpha$, $\beta$, respectively, then

$$(\hat{\mathcal{I}} \circ \hat{\mathcal{J}})_{(x,y)} = \hat{\mathcal{I}}_x \circ \hat{\mathcal{J}}_y = A_x \circ B_y = A_x^{1/2} B_y A_x^{1/2}$$

However, we have seen that

$$(\mathcal{I} \circ \mathcal{J})^\wedge_{(x,y)} = \text{tr} (\alpha B_y)A_x$$

Hence, $(\mathcal{I} \circ \mathcal{J})^\wedge \neq \hat{\mathcal{I}} \circ \hat{\mathcal{J}}$, in general. □

Example 3. We first show that $f(\mathcal{L}^A)$ is not a Lüders instrument and $f(\mathcal{L}^A) \neq \mathcal{L}^{f(A)}$ in general. To show this, we have that

$$f(\mathcal{L}^A)_y(p) = \mathcal{L}^{A}_{f^{-1}(y)}(p) = \sum_x \left\{ A_x^{1/2} \rho A_x^{1/2} : f(x) = y \right\} \tag{3.2}$$

which is not a Lüders instrument, in general. However, $\mathcal{L}^{f(A)}$ is a Lüders instrument so $f(\mathcal{L}^A) \neq \mathcal{L}^{f(A)}$. To be explicit we obtain

$$\mathcal{L}^{f(A)}_y(p) = f(A)_y^{1/2} \rho f(A)_y^{1/2} = A^{1/2} f^{-1}_y(p)A^{1/2} f^{-1}_y$$

$$= \left( \sum_x \{ A_x : f(x) = y \} \right)^{1/2} \rho \left( \sum_x \{ A_x : f(x) = y \} \right)^{1/2}$$

which is different than $f(\mathcal{L}^A)$ in [322]. If $\mathcal{I}_x(p) = S_x \rho S_x^*$ is a Kraus instrument, then $f(\mathcal{I})$ need not be a Kraus instrument. Indeed,

$$f(\mathcal{I})_y(p) = \mathcal{I}_{f^{-1}(y)}(p) = \sum_x \{ \mathcal{I}_x(p) : f(x) = y \} = \sum_x \{ S_x \rho S_x^* : f(x) = y \}$$
which is not a Kraus instrument, in general. We leave it to the reader to show that if $\mathcal{I}$ is semitrivial, then $f(\mathcal{I})$ need not be semitrivial. However, if $\mathcal{I}(\rho) = \text{tr} (\rho A_x)\alpha$ is trivial, then $f(\mathcal{I})$ is trivial with observable $f(A)$ and state $\alpha$. Indeed,

$$
f(I)_y(\rho) = \sum_x \{I_x(\rho) : f(x) = y\} = \sum_x \{\text{tr} (\rho A_x)\alpha : f(x) = y\}
$$

$$
= \text{tr} \left[ \rho \sum_x \{A_x : f(x) = y\} \right] \alpha = \text{tr} [\rho f(A)_y] \alpha
$$

References

[1] P. Busch, M. Grabowski and P. Lahti, *Operational Quantum Physics*, Springer-Verlag, Berlin, 1995.

[2] S. Fillipov, T. Heinosaari and L. Leppäjärvi, Simulability of observables in general probabilistic theories, *Phys. Rev. A* 97, 062102 (2018).

[3] S. Gudder and R. Greechie, Sequential Products on effect algebras, *Rep. Math. Phys.* 49, 87–111 (2002).

[4] S. Gudder and G. Nagy, Sequential quantum measurements, *J. Math. Phys.* 42, 5212–5222 (2001).

[5] S. Gudder, Conditioned observables in quantum mechanics, arXiv:quant-ph 2005.04775 (2020).

[6] ———, Quantum instruments and conditioned observables, arXiv:quant-ph 2005.08117 (2020).

[7] ———, Finite quantum instruments, arXiv:quant-ph 2005.13642 (2020).

[8] ———, Parts and composites of quantum systems, arXiv:quant-ph 2009.07371 (2020).

[9] T. Heinosaari and M. Ziman, *The Mathematical Language of Quantum Theory*, Cambridge University Press, Cambridge, 2012.

[10] T. Heinosaari, D. Reitzner, R. Stano and M. Ziman, Coexistence of quantum operations, *J. Phys. A* 42, 365302 (2009).
[11] T. Heinosaari, T. Miyandera and D. Reitzner, Strongly incompatible quantum devices, *Found. Phys.* **44**, 34–57 (2014).

[12] K. Kraus, *States, Effects and Operations*, Springer-Verlag, Berlin, 1983.

[13] P. Lahti, Coexistence and joint measurability in quantum mechanics, *Int. J. Theor. Phys.* **42**, 893–906 (2003).

[14] G. Lüders, Über due Zustandsänderung durch den Messprozess, *Ann. Physik* **6**, 322–328 (1951).

[15] M. Nielsen and I. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, 2000.