Geometry of Coherent States : Some Examples of Calculations of Chern–Characters

Kazuyuki FUJII *†

Department of Mathematical Sciences
Yokohama City University
Yokohama 236-0027
JAPAN

Abstract

First we make a brief review of coherent states and prove that the resolution of unity can be obtained by the 1–st Chern character of some bundle. Next we define a Grassmann manifold for a set of coherent states and construct the pull–back bundle making use of a projector from the parameter space to this Grassmann manifold. We study some geometric properties (Chern–characters mainly) of these bundles.

Although the calculations of Chern–characters are in general not easy, we can perform them for the special cases. In this paper we report our calculations and propose some interesting problems to be solved in the near future.

---

*E-mail address : fujii@math.yokohama-cu.ac.jp
†Home-page : http://fujii.sci.yokohama-cu.ac.jp
1 Introduction

Coherent states or generalized coherent states play very important role in quantum physics, in particular, quantum optics, see [1] or [2] and its references. They also play an important one in mathematical physics. See the book [3]. For example, they are very useful in performing stationary phase approximations to path integral, see [4], [5] and [6].

In this paper we study coherent states from the geometric point of view. Namely for a set of coherent states satisfying a certain condition we can define a projector from a manifold consisting of parameters to (infinite-dimensional) Grassmann manifold · · · a kind of classifying map in K-Theory · · · and construct a pull–back bundle on the manifold. Making use of this we can calculate several geometric quantities, see for example [7].

In this paper we mainly focus on Chern–characters because they play an very important role in global geometry. But their calculations are not so easy. Our calculations are given only for $m = 1, 2$ (see the section 3). Even the case $m = 2$ the calculations are complicated enough. We leave the case $m = 3$ to the readers.

But it seems to the author that our calculations for $m = 2$ suggest some deep relation to recent non–commutative differential geometry or non–commutative field theory (see the section 4). But this is beyond the scope of this paper. We need further study.

By the way the hidden aim of this paper is to apply the results in this paper to Quantum Computation (QC) and Quantum Information Theory (QIT) if possible. As for QC or QIT see [8], [9] and [10] for general introduction. We are in particular interested in Holonomic Quantum Computation, see [11]–[15]. We are also interested in Homodyne Tomography [16], [17] and Quantum Cryptgraphy [18], [19].

In the forthcoming paper we want to discuss the applications.

2 Coherent States and Grassmann Manifolds
2.1 Coherent States

We make a brief review of some basic properties of coherent operators within our necessity, [2] and [3].

Let \( a(a^\dagger) \) be the annihilation (creation) operator of the harmonic oscillator. If we set \( N \equiv a^\dagger a \) (: number operator), then

\[
\begin{align*}
[N, a^\dagger] &= a^\dagger, \\
[N, a] &= -a, \\
[a^\dagger, a] &= -1.
\end{align*}
\]  

(1)

Let \( \mathcal{H} \) be a Fock space generated by \( a \) and \( a^\dagger \), and \( \{|n\rangle | n \in \mathbb{N} \cup \{0\} \} \) be its basis. The actions of \( a \) and \( a^\dagger \) on \( \mathcal{H} \) are given by

\[
\begin{align*}
a|n\rangle &= \sqrt{n}|n-1\rangle, \\
a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle, \\
N|n\rangle &= n|n\rangle
\end{align*}
\]  

(2)

where \( |0\rangle \) is a normalized vacuum (\( a|0\rangle = 0 \) and \( \langle 0|0 \rangle = 1 \)). From (2) state \( |n\rangle \) for \( n \geq 1 \) are given by

\[
|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle.
\]  

(3)

These states satisfy the orthogonality and completeness conditions

\[
\langle m|n \rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1.
\]  

(4)

Let us state coherent states. For the normalized state \( |z\rangle \in \mathcal{H} \) for \( z \in \mathbb{C} \) the following three conditions are equivalent :

\[
\begin{align*}
(i) \quad a|z\rangle &= z|z\rangle \quad \text{and} \quad \langle z|z \rangle &= 1 \\
(ii) \quad |z\rangle &= e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle = e^{-|z|^2/2} e^{za^\dagger}|0\rangle \\
(iii) \quad |z\rangle &= e^{za^\dagger-\overline{z}a}|0\rangle.
\end{align*}
\]  

(5)

(6)

(7)

In the process from (6) to (7) we use the famous elementary Baker-Campbell-Hausdorff formula

\[
e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B
\]  

(8)

whenever \([A, [A, B]] = [B, [A, B]] = 0\), see [2] or [3]. This is the key formula.
Definition  The state $|z\rangle$ that satisfies one of (i) or (ii) or (iii) above is called the coherent state.

The important feature of coherent states is the following partition (resolution) of unity.

$$\int_C \frac{[d^2z]}{\pi} |z\rangle\langle z| = \sum_{n=0}^{\infty} |n\rangle\langle n| = 1,$$  \hspace{1cm} (9)

where we have put $[d^2z] = d(\text{Re}z)d(\text{Im}z)$ for simplicity. We note that

$$\langle z|w\rangle = e^{-\frac{1}{2}|z|^2-\frac{1}{2}|w|^2+\bar{z}w} \implies |\langle z|w\rangle| = e^{-\frac{1}{4}|z-w|^2}, \quad \langle w|z\rangle = \overline{\langle z|w\rangle},$$  \hspace{1cm} (10)

so $|\langle z|w\rangle| < 1$ if $z \neq w$ and $|\langle z|w\rangle| \ll 1$ if $z$ and $w$ are separated enough. We will use this fact in the following.

Since the operator

$$U(z) = e^{za^\dagger - \bar{z}a} \quad \text{for} \quad z \in \mathbb{C}$$  \hspace{1cm} (11)

is unitary, we call this a (unitary) coherent operator. For these operators the following properties are crucial. For $z, w \in \mathbb{C}$

$$U(z)U(w) = e^{z\bar{w} - \bar{z}w} \quad U(w)U(z),$$  \hspace{1cm} (12)

$$U(z + w) = e^{-\frac{1}{2}(z\bar{w} - \bar{z}w)} \quad U(z)U(w).$$  \hspace{1cm} (13)

Here we list some basic properties of this operator.

(a) **Glauber Formula**  Let $A$ be any observable. Then we have

$$A = \int_C \frac{[d^2z]}{\pi} \text{Tr}[AU^\dagger(z)]U(z)$$  \hspace{1cm} (14)

This formula plays an important role in the field of homodyne tomography, [16] and [17].

(b) **Projection on Coherent State**  The projection on coherent state $|z\rangle$ is given by $|z\rangle\langle z|$. But this projection has an interesting expression:

$$|z\rangle\langle z| =: e^{-(a-z)(a-z)} :$$  \hspace{1cm} (15)

where the notation $: :$ means normal ordering.
This formula has been used in the field of quantum cryptography, \[18\] and \[19\]. We note that
\[
|z\rangle\langle w| \neq e^{-(a-z)\dagger(a-w)}:
\]
for \(z, w \in \mathbb{C}\) with \(z \neq w\).

## 2.2 Infinite Grassmann Manifolds and Chern–Characters

Let \(\mathcal{H}\) be a separable Hilbert space over \(\mathbb{C}\). For \(m \in \mathbb{N}\), we set
\[
St_m(\mathcal{H}) \equiv \{V = (v_1, \cdots, v_m) \in \mathcal{H} \times \cdots \times \mathcal{H} \mid V^\dagger V \in GL(m; \mathbb{C})\}.
\] (16)

This is called a (universal) Stiefel manifold. Note that the general linear group \(GL(m) \equiv GL(m; \mathbb{C})\) acts on \(St_m(\mathcal{H})\) from the right:
\[
St_m(\mathcal{H}) \times GL(m) \longrightarrow St_m(\mathcal{H}) : (V, a) \mapsto Va.
\] (17)

Next we define a (universal) Grassmann manifold
\[
Gr_m(\mathcal{H}) \equiv \{X \in M(\mathcal{H}) \mid X^2 = X, X^\dagger = X \text{ and } \text{tr}X = m\},
\] (18)

where \(M(\mathcal{H})\) denotes a space of all bounded linear operators on \(\mathcal{H}\). Then we have a projection
\[
\pi : St_m(\mathcal{H}) \longrightarrow Gr_m(\mathcal{H}) , \quad \pi(V) \equiv V(V^\dagger V)^{-1}V^\dagger ,
\] (19)

compatible with the action (17) \(\pi(Va) = Va(a^\dagger V^\dagger Va)^{-1}(Va)^\dagger = \pi(V)\).

Now the set
\[
\{GL(m), St_m(\mathcal{H}), \pi, Gr_m(\mathcal{H})\},
\] (20)

is called a (universal) principal \(GL(m)\) bundle, see \[7\] and \[10\]. We set
\[
E_m(\mathcal{H}) \equiv \{(X, v) \in Gr_m(\mathcal{H}) \times \mathcal{H} \mid Xv = v\}.
\] (21)

Then we have also a projection
\[
\pi : E_m(\mathcal{H}) \longrightarrow Gr_m(\mathcal{H}) , \quad \pi((X, v)) \equiv X .
\] (22)
The set
\[ \{ C^m, E_m(\mathcal{H}), \pi, Gr_m(\mathcal{H}) \} , \]  
\[ (23) \]
is called a (universal) \( m \)-th vector bundle. This vector bundle is one associated with the principal \( GL(m) \) bundle \[ (20) \].

Next let \( \mathcal{M} \) be a finite or infinite dimensional differentiable manifold and the map
\[ P : \mathcal{M} \longrightarrow Gr_m(\mathcal{H}) \]  
\[ (24) \]
be given (called a projector). Using this \( P \) we can make the bundles \[ (20) \] and \[ (23) \] pullback over \( \mathcal{M} \):
\[ \{ GL(m), \tilde{St}, \pi, \tilde{St}, \mathcal{M} \} \equiv P^* \{ GL(m), St_m(\mathcal{H}), \pi, Gr_m(\mathcal{H}) \} , \]  
\[ (25) \]
\[ \{ C^m, \tilde{E}, \pi, \tilde{E}, \mathcal{M} \} \equiv P^* \{ C^m, E_m(\mathcal{H}), \pi, Gr_m(\mathcal{H}) \} , \]  
\[ (26) \]
\[ \begin{array}{ccc}
GL(m) & \downarrow & GL(m) \\
\downarrow & & \downarrow \\
\tilde{St} & \longrightarrow & St_m(\mathcal{H}) \\
\downarrow & & \downarrow \\
\mathcal{M} & \overset{P}{\longrightarrow} & Gr_m(\mathcal{H}) \\
\end{array} \quad \begin{array}{ccc}
C^m & \downarrow & C^m \\
\downarrow & & \downarrow \\
\tilde{E} & \longrightarrow & E_m(\mathcal{H}) \\
\downarrow & & \downarrow \\
\mathcal{M} & \overset{P}{\longrightarrow} & Gr_m(\mathcal{H}) \\
\end{array} \]
see \[ [7] \]. \[ (26) \] is of course a vector bundle associated with \[ (23) \].

For this bundle the (global) curvature (2–) form \( \Omega \) is given by
\[ \Omega = PdP \wedge dP \]  
\[ (27) \]
making use of \[ (24) \], where \( d \) is the differential form on \( \Omega \). For the bundles Chern–characters play an essential role in several geometric properties. In this case Chern–characters are defined (see \[ [7] \], Section 11) by making use of
\[ \Omega, \Omega^2, \cdots, \Omega^{m/2}, \]  
\[ (28) \]
where we have assumed that \( m = \dim \mathcal{M} \) is even. We note that \( \Omega^2 = \Omega \wedge \Omega \), etc.
In this paper we don’t take the trace of (28), so it may be better to call them pre-Chern characters. We want to calculate them directly.

To calculate these quantities in infinite-dimensional cases is not so easy. In the next section let us calculate them in the special cases.

We now define our projectors for the latter aim. For \(z_1, z_2, \cdots, z_m \in \mathbb{C}\) we set

\[
V_m(z) = (|z_1\rangle, |z_2\rangle, \cdots, |z_m\rangle) \equiv V_m
\]

where \(z = (z_1, z_2, \cdots, z_m)\). Since \(V_m^\dagger V_m = \langle z_i|z_j\rangle \in M(m, \mathbb{C})\), we define

\[
D_m \equiv \{z \in \mathbb{C}^m | \det(V_m^\dagger V_m) \neq 0\} = \{z \in \mathbb{C}^m | V_m^\dagger V_m \in GL(m)\}.
\]

For example \(V_1^\dagger V_1 = 1\) for \(m = 1\), and for \(m = 2\)

\[
det(V_2^\dagger V_2) = \begin{vmatrix} 1 & a \\ \bar{a} & 1 \end{vmatrix} = 1 - |a|^2 \geq 0
\]

where \(a = \langle z_1|z_2\rangle\). So from (11) we have

\[
D_1 = \{z \in \mathbb{C} | \text{no conditions}\} = \mathbb{C},
\]

\[
D_2 = \{(z_1, z_2) \in \mathbb{C}^2 | z_1 \neq z_2\}.
\]

For \(D_m (m \geq 3)\) it is not easy for us to give a simple condition like (32).

**Problem** For the case \(m = 3\) make the condition (30) clear like (32).

At any rate \(V_m \in St_m(H)\) for \(z \in D_m\). Now let us define our projector \(P\) as follows:

\[
P : D_m \rightarrow Gr_m(H), \quad P(z) = V_m(V_m^\dagger V_m)^{-1}V_m^\dagger.
\]

In the following we set \(V = V_m\) for simplicity. Let us calculate (27). Since

\[
dP = dV(V^\dagger V)^{-1}V^\dagger - V(V^\dagger V)^{-1}(dV^\dagger V + V^\dagger dV)(V^\dagger V)^{-1}V^\dagger + V(V^\dagger V)^{-1}dV^\dagger
\]

\[
= V(V^\dagger V)^{-1}dV^\dagger \{1 - V(V^\dagger V)^{-1}V^\dagger\} + \{1 - V(V^\dagger V)^{-1}V^\dagger\}dV^\dagger
\]

where \(d = \sum_{j=1}^m (dz_j \frac{\partial}{\partial z_j} + d\bar{z}_j \frac{\partial}{\partial \bar{z}_j})\), we have

\[
PdP = V(V^\dagger V)^{-1}dV^\dagger \{1 - V(V^\dagger V)^{-1}V^\dagger\}.
\]
after some calculation. Therefore we obtain

\[ P dP \wedge dP = V(V^\dagger V)^{-1}[dV^\dagger \{1 - V(V^\dagger V)^{-1} V^\dagger \} dV](V^\dagger V)^{-1} V^\dagger. \]  \hspace{1cm} (34)

Our main calculation is \( dV^\dagger \{1 - V(V^\dagger V)^{-1} V^\dagger \} dV \), which is rewritten as

\[ dV^\dagger \{1 - V(V^\dagger V)^{-1} V^\dagger \} dV = [(1 - V(V^\dagger V)^{-1} V^\dagger) dV]^\dagger [(1 - V(V^\dagger V)^{-1} V^\dagger) dV] \]  \hspace{1cm} (35)

since \( Q \equiv 1 - V(V^\dagger V)^{-1} V^\dagger \) is also a projector (\( Q^2 = Q \) and \( Q^\dagger = Q \)). Therefore the first step for us is to calculate the term

\[ \{1 - V(V^\dagger V)^{-1} V^\dagger \} dV. \]  \hspace{1cm} (36)

Let us summarize our process of calculations :

1–st step \( \{1 - V(V^\dagger V)^{-1} V^\dagger \} dV \),

2–nd step \( dV^\dagger \{1 - V(V^\dagger V)^{-1} V^\dagger \} dV \) · · · (35),

3–rd step \( V(V^\dagger V)^{-1}[dV^\dagger \{1 - V(V^\dagger V)^{-1} V^\dagger \} dV](V^\dagger V)^{-1} V^\dagger \) · · · .

3 Examples of Calculations of Chern–Characters

In this section we calculate the Chern–characters only for the cases \( m = 1 \) and \( m = 2 \). Even for \( m = 2 \) the calculation is complicated enough. For \( m \geq 3 \) calculations become miserable.

3.1 \( M=1 \)

In this case \( \langle z | z \rangle = 1 \), so our projector is very simple to be

\[ P(z) = |z \rangle \langle z |. \]  \hspace{1cm} (37)

In this case the calculation of curvature is relatively simple. From (34) we have

\[ P dP \wedge dP = |z \rangle \{d\langle z | (1 - |z \rangle \langle z|) d|z \rangle \} \langle z | = |z \rangle \langle z | \{d\langle z | (1 - |z \rangle \langle z|) d|z \rangle \}. \]  \hspace{1cm} (38)
Since $|z\rangle = \exp(-\frac{1}{2}|z|^2)\exp(za^\dagger)|0\rangle$ by (4),
\[
d|z\rangle = \left\{ \left( a^\dagger - \frac{z}{2}\right) dz - \frac{z}{2}d\bar{z} \right\} |z\rangle = \left\{ a^\dagger dz - \frac{1}{2}(\bar{z}dz + zd\bar{z}) \right\} |z\rangle = \left\{ a^\dagger dz - \frac{1}{2}d(|z|^2) \right\} |z\rangle,
\]
so that
\[
(1 - |z\rangle\langle z|)d|z\rangle = (1 - |z\rangle\langle z|)a^\dagger |z\rangle dz = (a^\dagger - \langle z|a^\dagger|z\rangle) |z\rangle dz = (a - z)^\dagger dz|z\rangle
\]
because $(1 - |z\rangle\langle z|)|z\rangle = 0$. Similarly $d\langle z|(1 - |z\rangle\langle z|) = \langle z|(a - z)d\bar{z}$.

Let us summarize:
\[
(1 - |z\rangle\langle z|)d|z\rangle = (a - z)^\dagger dz|z\rangle, \quad d\langle z|(1 - |z\rangle\langle z|) = \langle z|(a - z)d\bar{z}. \tag{39}
\]
Now we are in a position to determine the curvature form (38).
\[
d\langle z|(1 - |z\rangle\langle z|)d|z\rangle = \langle z|(a - z)(a - z)^\dagger |z\rangle d\bar{z} \wedge dz = d\bar{z} \wedge dz
\]
after some algebra. Therefore
\[
\Omega = PdP \wedge dP = |z\rangle\langle z|d\bar{z} \wedge dz. \tag{40}
\]
From this result we know
\[
\frac{\Omega}{2\pi i} = |z\rangle\langle z|dz \wedge \frac{dy}{\pi}
\]
when $z = x + iy$. This just gives the resolution of unity in (3).

### 3.2 $M=2$ ... Main Result

First of all let us determine the projector. Since $V = (|z_1\rangle, |z_2\rangle)$ we have easily
\[
P(z_1, z_2) = (|z_1\rangle, |z_2\rangle) \left( \begin{array}{cc} 1 & \langle z_1|z_2\rangle \\ \langle z_2|z_1\rangle & 1 \end{array} \right)^{-1} \left( \begin{array}{c} \langle z_1| \\ \langle z_2| \end{array} \right)
\]
\[
= \frac{1}{1 - |\langle z_1|z_2\rangle|^2} (|z_1\rangle, |z_2\rangle) \left( \begin{array}{cc} 1 & -\langle z_1|z_2\rangle \\ -\langle z_2|z_1\rangle & 1 \end{array} \right) \left( \begin{array}{c} \langle z_1| \\ \langle z_2| \end{array} \right)
\]
\[
= \frac{1}{1 - |\langle z_1|z_2\rangle|^2} (|z_1\rangle\langle z_1| - \langle z_2|z_1\rangle|z_2\rangle\langle z_1| - \langle z_1|z_2\rangle|z_1\rangle\langle z_2| + |z_2\rangle\langle z_2|). \tag{41}
\]
Let us calculate (36) : Since
\[ dV = (d|z_1), d|z_2) = \left( \left\{ a^* dz_1 - \frac{1}{2} d(|z_1|^2) \right\} |z_1), \left\{ a^* dz_2 - \frac{1}{2} d(|z_2|^2) \right\} |z_2) \]
the straightforward calculation leads
\[ \{1 - V(V^*V)^{-1}V^*\}dV = (K_1, K_2) \] (42)
where
\[ K_1 = \left( \left( a - z_2 \right)^* - \frac{\bar{z}_1 - \bar{z}_2}{1 - |\langle z_1|z_2 \rangle|^2} \right) dz_1|z_1) + \frac{\langle z_2|z_1 \rangle (\bar{z}_1 - \bar{z}_2)}{1 - |\langle z_1|z_2 \rangle|^2} d|z_1|z_2), \]
\[ K_2 = \frac{\langle z_1|z_2 \rangle (\bar{z}_2 - \bar{z}_1)}{1 - |\langle z_1|z_2 \rangle|^2} dz_2|z_1) + \left( \left( a - z_1 \right)^* - \frac{\bar{z}_2 - \bar{z}_1}{1 - |\langle z_1|z_2 \rangle|^2} \right) dz_2|z_2). \] (43)
Therefore by (35)
\[ dV^*\left(1 - V(V^*V)^{-1}V^*\right)dV = \left( \begin{array}{cc} F_{11} & F_{12} \\ F_{21} & F_{22} \end{array} \right) \] (44)
where
\[ F_{11} = \left( 1 - \frac{|z_1 - z_2|^2 |\langle z_1|z_2 \rangle|^2}{1 - |\langle z_1|z_2 \rangle|^2} \right) dz_1 \land dz_1 , \quad F_{12} = \langle z_1|z_2 \rangle \left( 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1|z_2 \rangle|^2} \right) dz_1 \land dz_2 , \]
\[ F_{21} = \langle z_2|z_1 \rangle \left( 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1|z_2 \rangle|^2} \right) dz_2 \land dz_1 , \quad F_{22} = \left( 1 - \frac{|z_1 - z_2|^2 |\langle z_1|z_2 \rangle|^2}{1 - |\langle z_1|z_2 \rangle|^2} \right) dz_2 \land dz_2 . \] (45)
Now we are in a position to determine the curvature form (34). Since
\[ V(V^*V)^{-1} = \frac{1}{1 - |\langle z_1|z_2 \rangle|^2} (|z_1) - \langle z_2|z_1 \rangle |z_2), |z_2) - \langle z_1|z_2 \rangle |z_1) , \]
\[ (V^*V)^{-1}V^* = \frac{1}{1 - |\langle z_1|z_2 \rangle|^2} \left( \frac{\langle z_1| - \langle z_2|z_1 \rangle \langle z_2 \rangle}{\langle z_2| - \langle z_1|z_2 \rangle \langle z_1 \rangle} \right) \]
we obtain
\[ \Omega = PdP \land dP \]
\[ = \frac{1}{(1 - |\langle z_1|z_2 \rangle|^2)^2} \left( |z_1) \langle z_1|L_1 - \langle z_2|z_1 \rangle |z_2) \langle z_1|L_2 - \langle z_1|z_2 \rangle |z_1) \langle z_2|L_3 + |z_2) \langle z_2|L_4 \right) , \] (46)
where

\[
L_1 = \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_1 \wedge dz_1 - |\langle z_1 | z_2 \rangle|^2 \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_2 \wedge dz_1 \\
- |\langle z_1 | z_2 \rangle|^2 \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_1 \wedge dz_2 + |\langle z_1 | z_2 \rangle|^2 \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_2 \wedge dz_2
\]

\[
L_2 = \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_1 \wedge dz_1 - \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_2 \wedge dz_1 \\
- |\langle z_1 | z_2 \rangle|^2 \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_1 \wedge dz_2 + \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_2 \wedge dz_2 ,
\]

\[
L_3 = \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_1 \wedge dz_1 - |\langle z_1 | z_2 \rangle|^2 \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_2 \wedge dz_1 \\
- \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_1 \wedge dz_2 + \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_2 \wedge dz_2 ,
\]

\[
L_4 = |\langle z_1 | z_2 \rangle|^2 \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_1 \wedge dz_1 - |\langle z_1 | z_2 \rangle|^2 \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_2 \wedge dz_1 \\
- |\langle z_1 | z_2 \rangle|^2 \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_1 \wedge dz_2 + \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} \, d\bar{z}_2 \wedge dz_2 .
\]

(47)

This is our main result. Next let us calculate \( \Omega^2 \) (\( \Omega^k = 0 \) for \( k \geq 3 \)) : From \( 46 \) we obtain after long calculation

\[
\Omega^2 = \frac{1}{(1 - |\langle z_1 | z_2 \rangle|^2)^4} \left( |\langle z_1 | z_1 \rangle M_1 - \langle z_2 | z_1 \rangle |z_2\rangle \langle z_1 | M_2 - \langle z_1 | z_2 \rangle |z_1\rangle \langle z_2 | M_3 + |z_2\rangle \langle z_2 | M_4 \right) \\
\times d\bar{z}_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 ,
\]

(48)

where

\[
M_1 = |\langle z_1 | z_2 \rangle|^2 (1 - |\langle z_1 | z_2 \rangle|^2)^2 + 2|\langle z_1 | z_2 \rangle|^2 (1 - |\langle z_1 | z_2 \rangle|^2) |z_1 - z_2|^2 \\
- |\langle z_1 | z_2 \rangle|^2 (1 + 2|\langle z_1 | z_2 \rangle|^2) |z_1 - z_2|^4 ,
\]

\[
M_2 = (1 - |\langle z_1 | z_2 \rangle|^2)^2 + 2|\langle z_1 | z_2 \rangle|^2 (1 - |\langle z_1 | z_2 \rangle|^2) |z_1 - z_2|^2
\]

10
This is a second main result in this paper.

We want to calculate them up to this case. Therefore let us give an explicit form to the projector:

\[
P(z_1, z_2, z_3) = (|z_1\rangle, |z_2\rangle, |z_3\rangle) \begin{pmatrix}
1 & \langle z_1 | z_2 \rangle & \langle z_1 | z_3 \rangle \\
\langle z_2 | z_1 \rangle & 1 & \langle z_2 | z_3 \rangle \\
\langle z_3 | z_1 \rangle & \langle z_3 | z_2 \rangle & 1
\end{pmatrix}^{-1} \begin{pmatrix}
\langle z_1 | \\
\langle z_2 | \\
\langle z_3 |
\end{pmatrix}
\]

\[
P(z_1, z_2, z_3) = \frac{1}{\det M} \left[ \{1 - |\langle z_2 | z_3 \rangle|^2\}|z_1\rangle\langle z_1 | - \{(z_1|z_2) - \langle z_1 | z_3 \rangle\langle z_3 | z_2 \rangle\}|z_1\rangle\langle z_2 | - \{(z_1|z_2) - \langle z_1 | z_3 \rangle\langle z_3 | z_2 \rangle\}|z_2\rangle\langle z_1 | + \{1 - |\langle z_1 | z_3 \rangle|^2\}|z_2\rangle\langle z_2 | - \{(z_2|z_3) - \langle z_2 | z_1 \rangle\langle z_1 | z_3 \rangle\}|z_2\rangle\langle z_3 | - \{(z_2|z_3) - \langle z_2 | z_1 \rangle\langle z_1 | z_3 \rangle\}|z_3\rangle\langle z_2 | + \{1 - |\langle z_1 | z_2 \rangle|^2\}|z_3\rangle\langle z_3 | \right],
\]

This is a second main result in this paper.

We have calculated the Chern–characters for \( m = 2 \). Since our results are in a certain sense “raw” (remember that we don’t take the trace), one can freely “cook” them. We leave it to the readers.

### 3.3 Problems

Before concluding this section let us propose problems:

**Problem 3** For the case of \( m = 3 \) perform the similar calculations!

We want to calculate them up to this case. Therefore let us give an explicit form to the projector:

\[
M_3 = (1 - |\langle z_1 | z_2 \rangle|^2)^2 + 2|\langle z_1 | z_2 \rangle|^2(1 - |\langle z_1 | z_2 \rangle|^2)|z_1 - z_2|^2
\]

\[
- |\langle z_1 | z_2 \rangle|^2(2 + |\langle z_1 | z_2 \rangle|^2)|z_1 - z_2|^4,
\]

\[
M_4 = |\langle z_1 | z_2 \rangle|^2(1 - |\langle z_1 | z_2 \rangle|^2)^2 + 2|\langle z_1 | z_2 \rangle|^2(1 - |\langle z_1 | z_2 \rangle|^2)|z_1 - z_2|^2
\]

\[
- |\langle z_1 | z_2 \rangle|^2(1 + 2|\langle z_1 | z_2 \rangle|^2)|z_1 - z_2|^4.
\]

(49)

This is a second main result in this paper.

We have calculated the Chern–characters for \( m = 2 \). Since our results are in a certain sense “raw” (remember that we don’t take the trace), one can freely “cook” them. We leave it to the readers.
where
\[ \det M = 1 - |\langle z_1 | z_2 \rangle|^2 - |\langle z_2 | z_3 \rangle|^2 - |\langle z_3 | z_1 \rangle|^2 + \langle z_1 | z_2 \rangle \langle z_2 | z_3 \rangle \langle z_3 | z_1 \rangle + \langle z_1 | z_3 \rangle \langle z_3 | z_2 \rangle \langle z_2 | z_1 \rangle. \]

Perform the calculations of \( \Omega, \Omega^2 \) and \( \Omega^3 \). Moreover

**Problem** ∞ For the general case perform the similar calculations (if possible).

It seems to the author that the calculations in the general case are very hard.

4 A Supplement

In this section we make a brief supplement to the discussion in the preceding section.

For \( V = V_m \) in (29) we set
\[ \tilde{V} = V (V^\dagger V)^{-1/2} , \]
then \( \tilde{V} \) satisfies the equation \( \tilde{V}^\dagger \tilde{V} = 1_m \). Then the canonical connection form \( \mathcal{A} \) is given by
\[ \mathcal{A} \equiv \tilde{V}^\dagger d\tilde{V} = (V^\dagger V)^{-1/2} V^\dagger dV (V^\dagger V)^{-1/2} + (V^\dagger V)^{1/2} d(V^\dagger V)^{-1/2} \\
= (V^\dagger V)^{1/2} \{(V^\dagger V)^{-1} V^\dagger dV\} (V^\dagger V)^{-1/2} + (V^\dagger V)^{1/2} d(V^\dagger V)^{-1/2} , \]
so the (local) curvature form \( \mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \) becomes
\[ \mathcal{F} = (V^\dagger V)^{-1/2} [dV^\dagger \{1 - V(V^\dagger V)^{-1} V^\dagger\} dV] (V^\dagger V)^{-1/2} \]
because of
\[ d\{(V^\dagger V)^{-1} V^\dagger dV\} + \{(V^\dagger V)^{-1} V^\dagger dV\} \wedge \{(V^\dagger V)^{-1} V^\dagger dV\} \\
= (V^\dagger V)^{-1} V^\dagger dV \{1 - V(V^\dagger V)^{-1} V^\dagger\} dV . \]

That is, \( (V^\dagger V)^{-1} V^\dagger dV \) is a main term of \( \mathcal{A} \). By the way the relation between (34) and \( \mathcal{F} \) above is given by
\[ PdP \wedge dP = \tilde{V} \mathcal{F} \tilde{V}^\dagger . \]
For the case of \( m = 2 \) let us calculate the connection form \((52)\). But since the calculation is very complicated, we only calculate the main term in \((52)\), which is essential to calculate the curvature form as shown above. Since
\[
V^\dagger dV = \begin{pmatrix}
\bar{z}_1 dz_1 - \frac{1}{2}d(|z_1|^2) & \langle z_1 | z_2 \rangle \{ \bar{z}_1 d\bar{z}_2 - \frac{1}{2}d(|z_2|^2) \} \\
\langle z_2 | z_1 \rangle \{ \bar{z}_2 d\bar{z}_1 - \frac{1}{2}d(|z_1|^2) \} & \bar{z}_2 d\bar{z}_2 - \frac{1}{2}d(|z_2|^2)
\end{pmatrix}
\]
we have
\[
(V^\dagger V)^{-1} V^\dagger dV = \frac{1}{1 - |\langle z_1 | z_2 \rangle|^2} \begin{pmatrix} N_{11} & N_{12} \\
N_{21} & N_{22} \end{pmatrix}
\]
where
\[
N_{11} = (\bar{z}_1 - |\langle z_1 | z_2 \rangle|^2 \bar{z}_2) dz_1 - \frac{1}{2} \left( 1 - |\langle z_1 | z_2 \rangle|^2 \right) d(|z_1|^2), \quad N_{12} = \langle z_1 | z_2 \rangle (\zbar_1 - \zbar_2) d\zbar_2,
\]
\[
N_{21} = \langle z_2 | z_1 \rangle (\zbar_2 - \zbar_1) d\zbar_1, \quad N_{22} = (\zbar_2 - |\langle z_1 | z_2 \rangle|^2 \zbar_1) d\zbar_2 - \frac{1}{2} \left( 1 - |\langle z_1 | z_2 \rangle|^2 \right) d(|z_2|^2).
\]

Here we note that
\[
(V^\dagger V)^{1/2} = \begin{pmatrix} s & \langle z_1 | z_2 \rangle \langle z_1 | z_2 \rangle t \\
\langle z_1 | z_2 \rangle \langle z_1 | z_2 \rangle t & s \end{pmatrix},
\]
\[
(V^\dagger V)^{-1/2} = \frac{1}{\sqrt{1 - |\langle z_1 | z_2 \rangle|^2}} \begin{pmatrix} s & -\langle z_1 | z_2 \rangle \langle z_1 | z_2 \rangle t \\
-\langle z_1 | z_2 \rangle \langle z_1 | z_2 \rangle t & s \end{pmatrix},
\]
where
\[
s = \frac{1}{2} \{ (1 + |\langle z_1 | z_2 \rangle|)^{1/2} + (1 - |\langle z_1 | z_2 \rangle|)^{1/2} \},
\]
\[
t = \frac{1}{2} \{ (1 + |\langle z_1 | z_2 \rangle|)^{1/2} - (1 - |\langle z_1 | z_2 \rangle|)^{1/2} \}.
\]

From \((52)\) and the formulas above we can obtain the explicit form of \( A \). We leave it to the readers.

Last let us give the explicit form to \( F \) in \((53)\) making use of \((54)\) and \((59)\) :
\[
F = \frac{1}{1 - |\langle z_1 | z_2 \rangle|^2} \left( \begin{array}{cc} s & -\langle z_1 | z_2 \rangle \langle z_1 | z_2 \rangle t \\
-\langle z_1 | z_2 \rangle \langle z_1 | z_2 \rangle t & s \end{array} \right) \left( \begin{array}{cc} F_{11} & F_{12} \\
F_{21} & F_{22} \end{array} \right) \left( \begin{array}{cc} s & -\langle z_1 | z_2 \rangle \langle z_1 | z_2 \rangle t \\
-\langle z_1 | z_2 \rangle \langle z_1 | z_2 \rangle t & s \end{array} \right)
\]
\[
= \frac{1}{1 - |\langle z_1 | z_2 \rangle|^2} \left( \begin{array}{cc} F_{11} & F_{12} \\
F_{21} & F_{22} \end{array} \right)
\]
where

\[ F_{11} = \frac{1 + \sqrt{1 - |\langle z_1 | z_2 \rangle|^2}}{2} \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_1 \wedge dz_1 + \frac{|\langle z_1 | z_2 \rangle|^2}{2} \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_2 \wedge dz_1 \]

\[ - \frac{|\langle z_1 | z_2 \rangle|^2}{2} \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_2 \wedge dz_1 \]

\[ + \frac{|\langle z_1 | z_2 \rangle|^2}{2} \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_1 \wedge dz_2 \]

\[ + \frac{1 - \sqrt{1 - |\langle z_1 | z_2 \rangle|^2}}{2} \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_2 \wedge dz_2 , \]

\[ F_{12} = -\frac{\langle z_1 | z_2 \rangle}{2} \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_1 \wedge dz_1 \]

\[ + \frac{\langle z_1 | z_2 \rangle}{2} \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_2 \wedge dz_1 \]

\[ + \frac{\langle z_1 | z_2 \rangle}{2} \left\{ 1 + \frac{1 - |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_2 \wedge dz_1 \]

\[ - \frac{\langle z_1 | z_2 \rangle}{2} \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_1 \wedge dz_2 \]

\[ - \frac{\langle z_1 | z_2 \rangle}{2} \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_2 \wedge dz_2 , \]

\[ F_{21} = -\frac{\langle z_2 | z_1 \rangle}{2} \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_1 \wedge dz_1 \]

\[ + \frac{\langle z_2 | z_1 \rangle}{2} \left\{ 1 + \frac{1 - |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_2 \wedge dz_1 \]

\[ + \frac{\langle z_2 | z_1 \rangle}{2} \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_2 \wedge dz_1 \]

\[ - \frac{\langle z_2 | z_1 \rangle}{2} \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_2 \wedge dz_2 , \]

\[ F_{22} = \frac{1 - \sqrt{1 - |\langle z_1 | z_2 \rangle|^2}}{2} \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_1 \wedge dz_1 \]

\[ - \frac{|\langle z_1 | z_2 \rangle|^2}{2} \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_2 \wedge dz_1 \]

\[ - \frac{|\langle z_1 | z_2 \rangle|^2}{2} \left\{ 1 - \frac{|z_1 - z_2|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_1 \wedge dz_2 \]

\[ + \frac{1 + \sqrt{1 - |\langle z_1 | z_2 \rangle|^2}}{2} \left\{ 1 - \frac{|z_1 - z_2|^2 |\langle z_1 | z_2 \rangle|^2}{1 - |\langle z_1 | z_2 \rangle|^2} \right\} d\bar{z}_2 \wedge dz_2 . \]
We have calculated the curvature form $\mathcal{F}$. This one has an interesting structure (compare $\mathcal{F}_{11}$ with $\mathcal{F}_{22}$ and $\mathcal{F}_{12}$ with $\mathcal{F}_{21}$). The author believes strongly that our $\mathcal{F}$ gives a solution of field equations of some non–commutative field theory.

5 Discussion

We have calculated Chern–characters for pull–back bundles on $\mathcal{D}_1$ and $\mathcal{D}_2$, and suggested a relation with some non–commutative field theory (or non–commutative differential geometry) for the case $m = 2$. For the case $m = 3$ we have left the calculations to the readers as a problem.

Our paper is based on coherent states, but we believe that one can trace the same process for generalized coherent states (namely geometry of generalized coherent states). See [20] and [21] for generalized coherent states. In the forthcoming paper [22] we will report this.

When the author was performing the calculations in section 3, the paper [23] appeared. It seems that our paper has some deep relation to it.

References

[1] L. Mandel and E. Wolf : Optical Coherence and Quantum Optics, Cambridge University Press, 1995.

[2] J. R. Klauder and Bo-S. Skagerstam (Eds) : Coherent States, World Scientific, Singapore, 1985.

[3] A. Perelomov : Generalized Coherent States and Their Applications, Springer–Verlag, 1986.
[4] K. Funahashi, T. Kashiwa, S. Sakoda and K. Fujii : Coherent states, path integral, and semiclassical approximation, J. Math. Phys., 36(1995), 3232.

[5] K. Funahashi, T. Kashiwa, S. Sakoda and K. Fujii : Exactness in the Wentzel-Kramers-Brillouin approximation for some homogeneous spaces, J. Math. Phys., 36(1995), 4590.

[6] K. Fujii, T. Kashiwa, S. Sakoda : Coherent states over Grassmann manifolds and the WKB exactness in path integral, J. Math. Phys., 37(1996), 567.

[7] M. Nakahara : Geometry, Topology and Physics, IOP Publishing Ltd, 1990.

[8] H-K. Lo, S. Popescu and T. Spiller (eds) : Introduction to Quantum Computation and Information, 1998, World Scientific.

[9] A. Hosoya : Lectures on Quantum Computation (in Japanese), 1999, Science Company (in Japan).

[10] K. Fujii : Introduction to Grassmann Manifolds and Quantum Computation, quant-ph/0103011.

[11] K. Fujii : Note on Coherent States and Adiabatic Connections, Curvatures, J. Math. Phys., 41(2000), 4406, quant-ph/9910069.

[12] K. Fujii : Mathematical Foundations of Holonomic Quantum Computer, to appear in Rept. Math. Phys., quant-ph/0004102.

[13] K. Fujii : More on Optical Holonomic Quantum Computer, quant-ph/0005129.

[14] K. Fujii : Mathematical Foundations of Holonomic Quantum Computer II, quant-ph/0101102.

[15] K. Fujii : From Geometry to Quantum Computation, quant-ph/0107128.

[16] G. M. D’Ariano, L. Maccone and M. G. A. Paris : Quorum of observables for universal quantum estimation, quant-ph/0006006.
[17] M. G. A. Paris : Entanglement and visibility at the output of a Mach–Zehnder interferometer, quant-ph/9811078.

[18] K. Banaszek : Optical receiver for quantum cryptography with two coherent states, quant-ph/9901067.

[19] K. Banaszek and K. Wodkiewicz : Direct Probing of Quantum Phase Space by Photon Counting, atom-ph/9603003.

[20] K. Fujii : Basic Properties of Coherent and Generalized Coherent Operators Revisited, Mod. Phys. Lett. A, 16(2001), 1277, quant-ph/0009012.

[21] K. Fujii : Note on Extended Coherent Operators and Some Basic Properties, quant-ph/0009116.

[22] K. Fujii : Geometry of Generalized Coherent States (tentative), in progress.

[23] M. Spradlin and A. Volovich : Noncommutative solitons on Kahler manifolds, hep-th/0106180.