Photon gas with hyperbolic dispersion relations

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Abstract
We investigate the density of states for a photon gas confined in a nonmagnetic metamaterial medium in which some components of the permittivity tensor are negative. We study the effect of the resulting hyperbolic dispersion relations on the black body spectral density. We show that for both of the possible wavevector space topologies, the spectral density vanishes at a certain frequency. We obtain the partition function and derive some thermodynamical quantities of the system. To leading order, the results resemble those of a one- or two-dimensional photon gas with an enhanced density of states.

Keywords: hyperbolic dispersion, metamaterials, photon gas

1. Introduction

The density of states is a basic factor governing the physical properties of many statistical systems. This can be seen, for instance, by considering the effect of the electronic density of states on the conductivity or other properties of condensed matter systems. Similarly, there are many properties of optical systems which depend on the photonic density of states. This fact has been used in designing photonic devices with desired properties by altering their photonic density of states through a variety of techniques [1–4], and more recently, by using metamaterials [5]. The latter method relies on the properties of metamaterials, which, roughly speaking, are media with negative indices of refraction in certain frequency ranges.

The propagation of electromagnetic waves in media with negative permittivity and/or permeability was studied decades ago by Veselago [6]. This results, depending on the signs of the permittivity and the permeability, in the emergence of so-called evanescent or backward-propagating electromagnetic waves (for a brief review, see e.g., [7] and for a more extensive review see [8]). Following the experimental realization of such media [9–13], the subject has attracted a lot of interest in both science and technology. One can mention in this regard, on the practical side, the construction of various microwave devices ranging from antennas to band-pass filters and lenses, see e.g., [14] and references therein. On the other hand, from the more formal point of view, metamaterials are considered as arenas in which one can mimic certain theoretical models such as dynamic space-time, string theory D-branes and noncommutativity [15], space-times with a metric signature transition [16], Schwarzschild (anti-) de Sitter black holes [17], and motion around celestial bodies [18].

The present work aims to study the thermodynamics of a photon gas confined within a metamaterial medium, which usually obeys hyperbolic dispersion relations [19]. This is partly motivated by the recent interest in relativistic gases with modified dispersion relations. Such modifications have been considered within several contexts, namely, in [20–24], in connection with Lorentz invariance violating models within the context of quantum field theory; in [25], in the context of quantum gravity; in [26], in the context of high-energy astrophysics; in [27, 28], in the framework of noncommutative field theories; in [29], in the context of models with minimal length; and in [30], in the framework of Hořava gravity. Another motivation behind this work is the interest in developing different probes into the properties of metamaterials. As an example, spontaneous emission near hyperbolic metamaterials has been considered in [31, 32].

The hyperbolic dispersion relation associated with nanostructured metamaterials and its application in engineering the photonic density of states has been discussed in [5]. Here,
we study the statistical mechanics of a photon gas with such dispersion relations for different wavevector space topologies and obtain several new results. In particular, we show that they exhibit interesting properties such as vanishing of the density of states at certain frequencies and behaving like lower-dimensional usual photon gases but with an enhanced density of states.

In the next sections, after a brief review of the dispersion relation for the propagation of electromagnetic fields in an anisotropic medium, we show how hyperbolic dispersion relations could arise as a result of negative permittivities. We then obtain an expression for the photonic density of states in a rather simple model. We apply this to study black body radiation and show that this results in novel properties such as exhibiting the behaviour of a photon gas in a lower-dimensional ordinary medium. We study the thermodynamics of such media by calculating the partition function in the grand canonical ensemble. We conclude by discussing the results.

2. Hyperbolic dispersion relations

The propagation of electromagnetic waves is governed by the well-known Maxwell equations

\[ \nabla \cdot \mathbf{D} = -\rho, \]  
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \]  
\[ \nabla \cdot \mathbf{B} = 0, \]  
\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \]

in which \( \mathbf{D} = \varepsilon \cdot \mathbf{E} \) and \( \mathbf{B} = \mu \cdot \mathbf{H} \). In general, neglecting medium losses, the permittivity \( \varepsilon \) and the permeability \( \mu \) are symmetric tensors. Here, we consider an anisotropic dielectric medium with

\[ \varepsilon = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_0 & 0 \\ 0 & 0 & \mu_0 \end{pmatrix} \]

where \( \varepsilon_{x,y,z} \) are all constant, and seek plane wave solutions of the above equations in the absence of the electric charge \( \rho \) and the current \( \mathbf{J} \). Thus, we set

\[ \mathbf{E}(\mathbf{r}, t) = E_0 e^{i k \cdot \mathbf{r} - i \omega t}, \]

where the real part is to be taken. Inserting this back into the Maxwell equations (2) and (4) and making use of equation (1), one can show that the following relation holds

\[ \frac{k^2}{\varepsilon_x} + \frac{k^2}{\varepsilon_y} + \frac{k^2}{\varepsilon_z} = \frac{1}{\varepsilon_0 c^2} \]

in which \( \varepsilon_i = \frac{\varepsilon_i}{\varepsilon_0} \) (\( i = x, y, z \)), and \( c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \) is the speed of light in vacuum. If we confine ourselves to the special case of a uniaxial medium with \( \varepsilon_x = \varepsilon_y \), we can obtain the following simpler relation

\[ \frac{k^2}{\varepsilon_z} + \frac{k^2}{\varepsilon_x} = \frac{\omega^2}{c^2}. \]

For ordinary media, \( \varepsilon_x \) and \( \varepsilon_z \) are positive and can be constant. However, for metamaterials, either of these can have negative values. Also, in general, metamaterials are always highly dispersive media, i.e., they exhibit both temporal (frequency dependence) and spatial (wavevector dependence) dispersions [33]. Thus, the explicit forms of the permittivity components depend on the structure of the medium under consideration. Since the spatial dispersion effects are usually much weaker than the temporal ones, here, for convenience, we consider a dissipationless medium and ignore the spatial dispersion. In the presence of dissipation, which is the case for the metal–dielectric metamaterials, the permittivity components are complex and the above simple hyperbolic dispersion get replaced by hyperbolic-like dispersion relations [34]. However, we expect the results obtained here to be qualitatively valid for low-loss media as well.

We consider a medium with

\[ \varepsilon_z = 1 \]
\[ \varepsilon_x = 1 - \frac{\Omega^2}{\omega^2} \]

in which \( \Omega \) is some cutoff frequency. With this choice, we obtain negative values for \( \varepsilon_x \) for frequencies under the cutoff value. This characterizes a nonmagnetic metamaterial with negative \( xx \) and \( yy \) components of the permittivity. Thus, equation (8) reduces to

\[ \frac{k^2}{\varepsilon_x} + \frac{k^2}{\varepsilon_z} = \frac{\omega^2}{c^2}. \]

An example of such dispersion relations has been used in [35] (see also [36]) to study the physics of a vacuum in a strong magnetic field in which the permittivity components are

\[ \varepsilon_{xx} = \varepsilon_{yy} = \frac{1 + \alpha}{1 - \alpha} \]

and

\[ \varepsilon_{zz} = \alpha \left( 1 - \frac{\omega_0^2}{\omega^2} \right) \]

where \( \omega_0 \) is a constant frequency proportional to the inverse London penetration depth and \( 0 \leq \alpha \leq 1 \) is also a constant.

In this work, we confine ourselves to the case of nonmagnetic metamaterials. To take metamaterials with a magnetic response into account, one would be faced with a more complicated picture, in which a variety of mode-dependent dispersion relations (classified in [37]) should be considered.

3. The density of states

To study the statistical mechanics of the system, we first need to calculate the density of states. This can be obtained, as usual, from the wavevector space volume of the shell enclosed by surfaces defined by \( \omega \) and \( \omega + d\omega \). Obviously, this volume is infinite, giving rise to a divergent density of states. However, in practice, there is a bound on the values of \( k \) which is imposed by the physical properties of the medium. An
example of this is discussed in [5], in which the metamaterial under study is fabricated via a special nanopatterning with a characteristic patterning scale \( a \). This puts an upper cutoff value on \( k \) which is of the order of \( \frac{a}{c} \). Thus, we need to calculate the above volume, taking the constraint \( k < k_c \) into account, \( k_c \) being the cutoff value. The volume inside the hyperbolic surface described by equation (11) (depicted in figure 1) and the planes \( k_x = \pm k_0 \), where \( k_0 \equiv \frac{\Omega^2 - \omega^2}{c a^2} \), equals

\[
V = \frac{2\pi}{3c^3\Omega^3} \omega^2(\Omega^2 - \omega^2)(4\Omega^2 - \omega^2) \]

where we have used \( k_c = \Omega \). From this, we obtain

\[
\frac{dV}{d\omega} = \frac{4\pi}{3c^3\Omega^3} \omega^4(4\Omega^4 - 10\Omega^2\omega^2 + 3\omega^4). \tag{12}
\]

Basically, this gives the density of states in the interval \( [\omega, \omega + d\omega] \). However, there are some exotic properties associated with this relation, namely, the appearance of negative values. This is because of the presence of a fixed cutoff value for \( k \). In certain regions, by increasing \( \omega \), the value of \( k_0 \) decreases so the net effect is a decrease in the enclosed volume. The behaviour of \( \frac{dV}{d\omega} \) is depicted in figure 2.

The number of states in the interval between \( \omega \) and \( \omega + d\omega \) may be obtained by multiplying a factor of \( \frac{2\pi}{3c^3\Omega^3} \) by the absolute value of \( \frac{dV}{d\omega} \). Here, \( V \) is the volume of the medium and the factor \( \frac{2}{3} \) is inserted to account for two polarization states of each photon, and to include the positive octant only. Thus, the density of states \( \rho(\omega) \) is given by

\[
\rho(\omega) \, d\omega = \frac{V}{3\pi^2c^3\Omega^3} |\omega|4\Omega^4 - 10\Omega^2\omega^2 + 3\omega^4| \, d\omega. \tag{12}
\]

For \( \omega > \Omega \), the medium obeys the usual spherical dispersion relation, and we have \( \rho(\omega) \, d\omega = \frac{V}{3\pi^2c^3\Omega^3} \omega^2 \, d\omega \).

An immediate effect of the above modified relation for the density of states can be seen by studying black body radiation. To see this, we consider a metamaterial medium of the type described above and a gas of photons in equilibrium with the medium at a temperature \( T = \frac{1}{k_B} \) (\( k_B \) being the Boltzmann constant). The energy density of the radiation emitted inside the medium at this temperature is given by the following expression

\[
u(x) \, d\omega = \frac{1}{V} \frac{\hbar \rho(\omega) \, d\omega}{e^{\beta\hbar\omega} - 1} \tag{13}
\]

where \( \rho(\omega) \) is the spectral density. For the usual spherical dispersion relation, this leads to the well-known Stefan–Boltzmann law. However, using the hyperbolic density of states, equation (12), the result would be different. In practice, we need not use the whole expression equation (12) here, but rather a simplified version would be sufficient. To see this, we first rewrite the above expression in the following form

\[
u(x) = \frac{1}{3\pi^2c^3\hbar^3\beta^4} \frac{|4X^4 - 10X^2\omega^2 + 3\omega^4|}{X^3} \frac{x^2}{e^x - 1}. \tag{14}
\]

where \( x = \beta \hbar \omega \) and \( X = \beta \hbar \Omega \). Now, we note that for a medium with nanoscale characteristic patterning length \( a \sim 10^{-8} \) m, for room temperatures, we have \( X \sim 10^7 \), which means that we can confine ourselves practically to the region \( x \ll X \). Thus, we obtain

\[
u(x) = \frac{4X}{3\pi^2c^3\hbar^3\beta^4} \frac{x^2}{e^x - 1}. \tag{15}
\]

The functional form of this expression resembles its counterpart for a usual two-dimensional medium. This might be explained from the dispersion relation equation (11), where in the left-hand side we have something like the norm of a vector computed with respect to a metric with two space-like components and a time-like one. Also, compared with a two-dimensional photon gas, here the spectral density contains an extra \( \frac{4X}{3\pi^2c^3\hbar^3\beta^4} \) factor, which is of the order of the inverse characteristic length \( a^{-1} \), leading to a huge enhancement of the density. The above relation is plotted in figure 3, which shows radical enhancement of the spectral density compared with a usual photon gas in 3 + 1 Minkowskian space.

In spite of the above arguments, one may still be interested in studying the behaviour of the spectral density for
we can write the grand partition function as follows

Considering a grand canonical ensemble for the photon gas, be derived from the partition function in the appropriate limit.

4. The partition function

Qualitatively, this looks like the previous case.

Inserting equation (12) into the above expression, we get

\[ \ln Q = \frac{-V\Omega}{3\pi^2 c^3 \beta^3 h^2} \int_0^{\frac{c}{\omega}} f(x) \, dx \]  (18)

where \( \alpha \approx 0.67 \) is the root of \( 4 - 10x^2 + 3x^4 = 0 \). Now, using our previous argument on the largeness of \( X = \beta h\Omega \) for a nanopatterned metamaterial, we can ignore the last two integrals in the above expression. This leaves us with

\[ \ln Q = \frac{-V\Omega}{3\pi^2 c^3 \beta^3 h^2} \int_0^{\frac{c}{\omega}} f(x) \, dx \]

in which

\[ f(x) = x \left( 4 - 10\frac{x^2}{X^2} + 3\frac{x^4}{X^4} \right) \ln(1 - e^{-x}). \]

In view of the largeness of \( X \), this can be approximated as

\[ \ln Q \approx \frac{V\Omega}{6\pi^2 c^3 \beta^3 h^2} \int_0^{\infty} 4x^2 - \frac{5x^4}{X^2} + \frac{x^6}{X^4} \, dx. \]  (19)

Inserting the values of the Bose–Einstein integrals above, this reduces to

\[ \ln Q \approx \frac{4V\Omega}{3\pi^2 c^3 \beta^3 h^2} \left( \zeta(3) - \frac{15}{X^2} \zeta(5) + \frac{90}{X^4} \zeta(7) \right) \]  (20)

where \( \zeta \) is the Riemann zeta function.

The internal energy is given by

\[ U = -\frac{\partial}{\partial \beta} \ln Q \]

\[ \approx \frac{8V\Omega}{3\pi^2 c^3 \beta^3 h^2} \left( \zeta(3) - \frac{30}{X^2} \zeta(5) + \frac{270}{X^4} \zeta(7) \right) \]  (21)

which resembles the energy of a usual photon gas in two dimensions together with correction terms of the form \( X^{-2} \) and \( X^{-4} \) which are very small for a metamaterial medium with nanoscale patterning length. Comparing the last two equations, one reaches the following relation

\[ P = \frac{1}{2} \frac{U}{V} \left( 1 - O(X^{-2}) \right) \]  (22)

in which higher order correction terms are neglected. Other thermodynamical quantities such as entropy can also be obtained from the above partition function. The result of these calculations is consistent with the result obtained in the previous section, according to which the photon gas under consideration behaves like an ordinary photon gas in 2 + 1 dimensions with some small corrections.

It would also be interesting to consider a metamaterial with \( \varepsilon_x = \varepsilon_y > 0 \) and \( \varepsilon_z < 0 \). This can be modelled by interchanging \( \varepsilon_x \) and \( \varepsilon_z \) in equations (9) and (10). The relevant dispersion relation is then

\[ k_x^2 + k_y^2 = \left( \frac{\Omega^2}{\omega^2} - 1 \right) k_z^2 = -\frac{\Omega^2 - \omega^2}{c^2}. \]  (23)

In this case, the wavevector space configuration is again a hyperboloid, but with a different topology, namely, a two-sheeted hyperboloid depicted in figure 5.

In this case, we are interested in the volume between the surfaces spanned by the relation equation (23) and the cutoff planes \( k_z = \pm k_0 = \frac{\omega}{\varepsilon_z} \sqrt{\Omega^2 - \omega^2} \). Again, this volume can decrease or increase with increasing values of \( \omega \). The
under the assumption that the dielectric coefficients of the metric signatures. The relevant calculations are performed of physical systems in space-times with non-Lorentzian topology, and the density of states can decrease or increase the wavevector space is still hyperbolic, but with a different negative, the surface spanned by the dispersion relation in some interesting features such as vanishing at certain frequencies. For the case where only the density of states, which in the case where the $\omega$ component is

$$\rho(\omega) = \frac{V}{3\pi^2 c^3} \left( \frac{\sqrt{2}}{\sqrt{2} - \frac{\omega}{\Omega}} \right) \frac{\omega^2}{2}.$$

This shows a constant density of states corrected with a $\omega^2$ term. One may interpret this as the density of states of a photon gas in 1 + 1 dimensions for large values of $\Omega$.

5. Discussion and conclusions

We have considered a photonic gas with a modified dispersion relation propagating in a nonmagnetic metamaterial medium with some negative components of the permittivity tensor. From the hyperbolic dispersion relations, we obtained the density of states, which in the case where the $xx$ and $yy$ components of the permittivity tensor are negative shows some interesting features such as vanishing at certain frequencies. For the case where only the $zz$ component is negative, the surface spanned by the dispersion relation in the wavevector space is still hyperbolic, but with a different topology, and the density of states can decrease or increase with frequency. We studied the thermodynamics of the system and showed that it resembles a photonic gas in an ordinary one- or two-dimensional medium, but with enhanced density of states. The results are of interest in studying the behaviour of physical systems in space-times with non-Lorentzian metric signatures. The relevant calculations are performed under the assumption that the dielectric coefficients of the medium have a plasma-like frequency dependence but are independent of the wavevector. It would be interesting to consider more general situations, where these are functions of both the frequency and the wavevector.

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![Figure 5](https://via.placeholder.com/150)

Figure 5. The surface spanned by the dispersion relation $k_0^2 + k_y^2 - (\frac{\omega^2}{c^2} - 1)k_z^2 = -\frac{\omega^2}{c^2}$. 

![Figure 6](https://via.placeholder.com/150)

Figure 6. The function $\nu' = \frac{\nu}{\nu_0}$ (in units of $\frac{4\pi\omega^2}{\epsilon_0}$) in terms of $\omega$ (in units of $\Omega$) for the surface $k_0^2 + k_y^2 - (\frac{\omega^2}{c^2} - 1)k_z^2 = -\frac{\omega^2}{c^2}$.
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