Darboux transformation for two-level system

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Abstract

We develop the Darboux procedure for the case of the two-level system. In particular, it is demonstrated that one can construct the Darboux intertwining operator that does not violate the specific structure of the equations of the two-level system, transforming only one real potential into another real potential. We apply the obtained Darboux transformation to known exact solutions of the two-level system. Thus, we find three classes of new solutions for the two-level system and the corresponding new potentials that allow such solutions.

1 Introduction

It is well-known that some complex quantum systems, with a discrete energy spectrum, are situated in some special dynamical configuration in which only two stationary states are important. To describe such systems one can use appropriate models with two-level energy spectra. In a number of important cases a model for a two-level system in a time-dependent background is based on the following Schrödinger equation in 0 + 1 dimensions for a two-component time dependent spinor $\Psi(t)$,

$$i\frac{d\Psi}{dt} = (\sigma F) \Psi, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$F = (\varepsilon, 0, f(t)), \quad (\sigma F) = \begin{pmatrix} f(t) & \varepsilon \\ \varepsilon & -f(t) \end{pmatrix}. \quad (1)$$

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Here $\varepsilon$ is a constant and $f(t)$ is a real function (in what follows, we call $f(t)$ the potential)\(^1\). The components of the spinor obey the set of equations
\[
 i\dot{\psi}_2 + f\psi_2 = \varepsilon\psi_1, \quad i\dot{\psi}_1 - f\psi_1 = \varepsilon\psi_2.
\]
These equations and their solutions are studied in the present article.

Solutions of the two-level system with different potentials possess a wide range of applications, e.g., in quantum optics and in the semi-classical theory of laser. The model can be helpful to describe the behavior of molecule beams that cross a cavity immerse in a time dependent magnetic, or electric, field, as well as the behavior of an atom under the action of the electric field of a laser (see, for example, [1]). It can be mentioned, additionally, that two-level models are used to describe resonance absorption and nuclear induction experiments [3]. The two-level system with periodic (quasi-periodic) potentials $f(t)$ has been studied by several authors. They have considered various approximation methods for finding solutions of the equations (1), e.g. perturbational expansions [4], the method of averaging [2], and the rotating wave approximation method. For a review of these, and other methods, see [5].

One ought to say that equations (1) can be related to Zakharov-Shabat equations [6, 11]
\[
 i\dot{\psi}_1 + \varphi\psi_2 = \varepsilon\psi_1, \quad i\dot{\psi}_2 - \varphi^*\psi_1 = -\varepsilon\psi_2, \quad \varphi = f_1 + if_2.
\]
Indeed, for
\[
 f_1(i\tau) = f(\tau), \quad f_2 = 0, \quad \varepsilon = -i\varepsilon, \quad t = i\tau, \quad (\tau \text{ is real}),
\]
we can write the Zakharov-Shabat equations as follows
\[
 i\frac{d\Psi}{d\tau} = (\sigma F_{ZS})\Psi, \quad F_{ZS} = (0, f(\tau), \varepsilon), \quad \Psi = \begin{pmatrix} \psi_1(\tau) \\ \psi_2(\tau) \end{pmatrix}.
\]
The latter equation can be transformed into the equation (1) by the unitary transformation,
\[
 \Psi \rightarrow U\Psi, \quad U = \frac{1}{2} \left[1 + i(\sigma e)\right], \quad e = (1, 1, 1), \quad U^+ = U^{-1}.
\]

So far a few cases have been known when the two-level system admits exact solutions, see the pioneer work of Rabi [7] where a spatially homogeneous and time-dependent external magnetic field is analyzed, and the work [8] where exact solutions of the two-level system were found for the following specific forms of potentials $f$:
\[
 f(t) = \frac{r_0}{\cosh \tau}, \quad \tau = \frac{t}{T},
\]
\[
 f(t) = \frac{r_0}{T} \tanh \tau + \frac{r_1}{T},
\]
\(^1\)Here and in what follows $\hbar = c = 1$, and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices.
where $r_0, r_1$ and $T$ are some constants. In what follows, we call the potentials $f$ that admit exact solutions for the two-level system equations, exact solvable potentials.

Sometimes, there exists the possibility to construct new exact solutions of differential equations (in particular, of eigenvalue problems) with the help of the Darboux transformation method \cite{9, 10}. The idea of the Darboux transformation method is to find an operator (an intertwining operator) that relates solutions which correspond to different potentials. Thus, if one knows solutions for a given potential, and a Darboux transformation can be found, there exists a possibility to construct solutions for another potential and at the same time to find this potential. The method was applied for the first time by Darboux to find solutions of the Sturm-Liouville problem. Applications of the Darboux transformations to Schrödinger-type equations can be found in the survey \cite{12}.

For the generalization of the method to sets of differential equations see e.g. \cite{13}.

In the present article, we adapt the Darboux procedure to the case of the two-level system (Sect.II). In this respect, one ought to say that the Darboux transformation for the general Zakharov-Shabat equations where studied in \cite{10}, see also \cite{14}. However, such transformations cannot be directly used in the case under consideration since, in the general case, they violate the structure of the two-level system equations. We demonstrate that one can construct the Darboux intertwining operator that does not violate the specific structure of the equations of the two-level system, only transforming potentials $f(t)$ given by real functions into other potentials of the same type. Then (Sect.III) we apply the obtained Darboux transformation to known exact solutions of the two-level system. Thus, we find three classes of new solutions for the two-level system and the corresponding new potentials that allow such solutions.

### 2 Darboux transformation for two-level system

In this section, we adopt the Darboux procedure to equations \cite{11, 12}. These equations can be also written as an eigenvalue problem as follows:

\[
\hat{h}\Psi_\varepsilon = \varepsilon \Psi_\varepsilon, \quad \hat{h} = i\sigma_1 \frac{d}{dt} + V(t), \quad V(t) = i\sigma_2 f(t).
\]  

(8)

Suppose we know solutions of the problem for any (complex) $\varepsilon$. And suppose we can construct an intertwining operator $\hat{L}$, such that

\[
\hat{L}\hat{h} = \hat{h}_1 \hat{L},
\]

(9)

\[
\hat{h}_1 = i\sigma_1 \frac{d}{dt} + V_1(t), \quad V_1(t) = i\sigma_2 f_1(t).
\]

(10)

Then the eigenvalue problem for the operator $\hat{h}_1$, has the following solutions

\[
\hat{h}_1 \Phi_\varepsilon = \varepsilon \Phi_\varepsilon, \quad \Phi_\varepsilon = \hat{L} \Psi_\varepsilon.
\]

(11)
If the intertwining operator $\hat{L}$ is chosen to be

$$\hat{L} = \frac{d}{dt} + B, \quad (12)$$

where $A(t)$ and $B(t)$ are some time dependent $n \times n$ matrices, then the transformation from $\Psi_\varepsilon$ to $\Phi_\varepsilon$ is called the Darboux transformation.

There is a general method of constructing the intertwining operators $\hat{L}$ (see for example [13] and references there) for a given eigenvalue problem (8). However, for our purposes the direct application of the general method could not be useful. The point is that by application of this method one can violate the specific structure of the initial potential $V_0$, that is the final potential $V_1$ will not have the specific structure (9) with a real function $f_1(t)$. Then, the final set of equations is not a set of two-level system equations.

Thus, the peculiarity of our problem is that the matrix potentials $V_0$ and $V_1$ must obey some algebraic restrictions and the Darboux transformation has to respect these restrictions. In other words, we are looking for the Darboux transformations that do not change the form of the equations of the two-level system. The existence of such transformations is a nontrivial fact which we are going to verify below.

The intertwining relation (8) with the operator $\hat{L}$ in the form (12) and the potential $V_1$ in the form (10) (such a choice is always possible [13]) leads to the following relations

$$\sigma_1 B - B \sigma_1 + \sigma_2 (f_1 - f) = 0, \quad (13)$$
$$\sigma_1 \dot{B} + \sigma_2 B f_1 - \sigma_2 \dot{f} - B \sigma_2 f = 0. \quad (14)$$

Let us choose

$$B = \alpha + i (f - \beta) \sigma_3, \quad (15)$$

where $\alpha(t)$ and $\beta(t)$ are some real functions. Then, we obtain for the function $f_1$,

$$f_1 = 2\beta - f, \quad (16)$$

and the equations for the functions $\alpha$ and $\beta$,

$$\dot{\alpha} - 2\beta (f - \beta) = 0, \quad \dot{\beta} + 2\alpha (f - \beta) = 0. \quad (17)$$

One can easily see that there is a first integral of the equation (17)

$$\alpha^2 + \beta^2 = R^2, \quad (18)$$

where $R$ is a real constant. Note that (18) is satisfied if we choose

$$\alpha = R \cos \mu, \quad \beta = f + R \sin \mu, \quad (19)$$

with $\mu(t)$ a real function. Substituting (19) into (17), we obtain for the function $\mu$ a transcendental differential equation

$$\dot{\mu} = 2 (R \sin \mu - f). \quad (20)$$
In what follows, we are going to find the functions $\alpha$ and $\beta$ independently, without the need to solve the equation (20). Thus, at the same time, we find in an indirect way solutions for the latter equation.

The time derivative in (12) can be taken from equation (8). Then we obtain, with account taken of (15),

$$
\Phi_\varepsilon = \left[ \alpha - i (\varepsilon \sigma_1 + \beta \sigma_3) \right] \Psi_\varepsilon.
$$

(21)

Thus, we see that the Darboux transformation (for equations (1, 2)) that respects the restriction (10) does exist. It has the algebraic form (21) and is determined by solutions of equations (17), or by equations (20). To finish the construction, one has to be able to represent solutions of the set (17) with the help of solutions of the initial equations (1, 2). Such a possibility exists and is described below.

Let us introduce the spinor $\bar{\Psi}$,

$$
\bar{\Psi} = -i \sigma_2 \Psi^* = \begin{pmatrix} -\psi_2^* \\ \psi_1^* \end{pmatrix}, \quad (\bar{\Psi}, \Psi) = 0,
$$

(22)

where $*$ denotes complex conjugation. As follows from (1), this spinor obeys the equation

$$
i \bar{\Psi} = (\sigma F^*) \bar{\Psi}, \quad F^* = (\varepsilon^*, 0, f).
$$

(23)

In addition, we introduce a complex vector $p(t)$,

$$
p = (\bar{\Psi}, \sigma \Psi).
$$

(24)

With account taken of (22), we can see that for any spinor $\Psi$, the relation

$$
p^2 = p_1^2 + p_2^2 + p_3^2 = (\bar{\Psi}, \Psi)^2 = 0
$$

holds true. An equation for the vector $p$ follows from (11) and (23),

$$
\dot{p} = 2 [F \times p].
$$

(26)

Let us introduce the functions $\alpha$ and $\beta$ as

$$
\alpha = -\varepsilon \frac{p_2}{p_3}, \quad \beta = -\varepsilon \frac{p_1}{p_3}.
$$

(27)

In virtue of (20), these functions are related as

$$
\alpha^2 + \beta^2 = -\varepsilon^2.
$$

(28)

It is easy to verify (using (20) and (28)) that the functions $\alpha$ and $\beta$ obey the set of equations (17). Setting $\varepsilon = -i R$ in (22)–(28), we can conclude that (28) coincides with (18). In addition, it follows from (20) that there exist such solutions of this equation such that $p_1$ and $p_2$ are real, whereas $p_3$ is imaginary and can be determined from (28). This provides the reality of the functions $\alpha$ and $\beta$. 

5
Thus, we have expressed solutions of the set (17) via solutions of the initial equations (2) at $\varepsilon = -iR$, where $R$ is real. Substituting (27) into (21), one can find the final form of both the Darboux transformation for equations (11, 2) and the function $f_1(t)$ (which determines the new potential, see (10)),

$$
\Phi = q^{-1} \sigma^2 (\sigma \tilde{p}) \Psi, \quad \tilde{p} = (p_1, p_2, \varepsilon q/R),
$$

$$
f_1 = -2 \frac{p_1}{q} - f, \quad q = q(t) = \sqrt{p_1^2 + p_2^2}.
$$

Here the components $p_1$ and $p_2$ are constructed with the help of the equations (24) via solutions of equations (1, 2) at $\varepsilon = -iR$ with a real $R$. We stress that the constructed transformation preserves the form invariance of equations (1, 2). All the components of the transformation can be constructed in an algebraic manner via solutions of the initial equation.

### 3 New exact solutions for two-level systems

In the following, we apply the above consideration to obtain new solutions of equations (12).

#### 3.1 The first case

Let $f(t) = c_0 = \text{const}$ in equations (12). The corresponding solutions have the form

$$
\psi_1 = i (c_0 - \omega) p_0 \exp(i \omega t) - \varepsilon q_0 \exp(-i \omega t),
$$

$$
\psi_2 = i \varepsilon q_0 \exp(i \omega t) + (c_0 - \omega) q_0 \exp(-i \omega t),
$$

$$
\omega^2 = \sqrt{c_0^2 + \varepsilon^2}, \quad (30)
$$

where $q_0$ and $p_0$ are arbitrary complex constants. Using these solutions at $\varepsilon = -iR_0$ we find (using the above procedure) the functions $\alpha = \alpha_0(t)$ and $\beta = \beta_0(t)$ as well as the corresponding potential $f_1(t)$:

$$
\alpha_0(t) = -\frac{Q_0}{2 (Q_0 + c_0)}, \quad \beta_0(t) = c_0 + \frac{R_0^2 - c_0^2}{Q_0 + c_0},
$$

$$
Q_0 = Q_0(t) = \begin{cases} R_0 \cosh \varphi_0, & R_0^2 > c_0^2 \\ R_0 \cos \varphi_0, & R_0^2 < c_0^2 \end{cases},
$$

$$
\varphi_0 = 2 (\omega_0 t + \gamma_0), \quad \omega_0 = \sqrt{|R_0^2 - c_0^2|}.
$$

(31)

Here $\gamma_0$ is an arbitrary real constant.

The corresponding potential $f_1(t)$ reads:

$$
f_1(t) = 2\beta_0(t) - c_0 = c_0 + 2 \left( \frac{R_0^2 - c_0^2}{Q_0 + c_0} \right).
$$

(32)
For $c_0 \neq 0$ the function $f_1 (t)$ is not a particular case of the potential (6) and is a new solvable potential for equations (1, 2).

The spinor $\Phi_\epsilon$ can be easily constructed according to formula (21). We do not exhibit here its explicit form which is too cumbersome.

### 3.2 The second case

The Darboux transformation can be applied again to the exact solution obtained in the previous section. We represent here only the functions $\alpha_1 (t)$ and $\beta_1 (t)$ of such a transformation with $\varepsilon = -i R_1$. They are:

$$
\begin{align*}
\alpha_1 (t) &= R_1 S \left[2 \alpha_0 \left( c_0 \beta_0 - R_1^2 \right) Q_1 + \left( 2 \beta_0^2 - R_0^2 - R_1^2 \right) \dot{Q}_1 / 2 + 2 R_1 \alpha_0 \left( \beta_0 - c_0 \right) \right], \\
\beta_1 (t) &= -R_1 S \left\{ \left[ c_0 \left( 2 \beta_0^2 - R_0^2 + R_1^2 \right) - 2 \beta_0 R_1^2 \right] Q_1 - \alpha_0 \beta_0 \dot{Q}_1 \\
&\quad + R_1 \left[ R_1^2 - R_0^2 + 2 \beta_0 \left( \beta_0 - c_0 \right) \right] \right\},
\end{align*}
$$

(33)

where

$$
\begin{align*}
S^{-1} &= R_1 \left( R_0^2 + R_1^2 - 2 \beta_0 c_0 \right) Q_1 + \alpha_0 R_1 \dot{Q}_1 + \left( R_0^2 + R_1^2 \right) c_0 - 2 \beta_0 R_1^2, \\
Q_1 &= \left\{ \begin{array}{ll}
R_1 \cosh \phi_1, & R_1^2 > c_0^2, \\
R_1 \cos \phi_1, & R_1^2 < c_0^2,
\end{array} \right.

\phi_1 = 2 \left( \omega_1 t + \gamma_1 \right), \quad \omega_1 = \sqrt{|R_1^2 - c_0^2|}.
\end{align*}
$$

Here the functions $\alpha_0 (t)$ and $\beta_0 (t)$ are defined by equations (31) and $\gamma_1$ is a real constant.

The corresponding solvable potential $f'_1 (t)$ has the form:

$$
f'_1 (t) = 2 \beta_1 (t) + f_1 (t) = 2 \left[ \beta_1 (t) + \beta_0 (t) \right] - c_0.
$$

(34)

We stress that for nonzero $R_0$ and $R_1$ this potential is not a particular case of any known exact solvable potentials.

As before, the spinor $\Phi_2$ can be easily constructed according to formula (21) and is not exhibited here.

### 3.3 The third case

Now we assume that the function $f$ has the form (7). In this case, solutions of (1, 2) can be written as (see (5))

$$
\begin{align*}
\psi_1 &= (1 - z)^\nu E \left[ c_1 z^\mu F (a + 1; b; c; z) + c_2 z^{-\mu} F (\bar{a} + 1; \bar{b}; \bar{c}; z) \right], \\
\psi_2 &= (1 - z)^\nu \left[ (r_0 - r_1 + 2 i \mu) c_1 z^\mu F (a, b + 1; c; z) + (r_0 - r_1 - 2 i \mu_0) c_2 z^{-\mu} F (\bar{a}, \bar{b} + 1; \bar{c}; z) \right],
\end{align*}
$$

(35)
where

\[ z = \frac{1}{2} (1 + \tanh r) , \quad a = \mu + \nu + ir_0 , \quad b = \mu + \nu - ir_0 , \quad \bar{a} = -\mu + \nu + ir_0 , \]

\[ \bar{b} = -\mu + \nu - ir_0 , \quad c = 1 + 2\mu , \quad \bar{c} = 1 - 2\mu , \quad E = \varepsilon T , \]

and \( c_1 \) and \( c_2 \) are complex constants. If the following relations are satisfied

\[ 4\mu^2 + E^2 + (r_0 - r_1)^2 = 0 , \]
\[ 4\nu^2 + E^2 + (r_0 + r_1)^2 = 0 , \]

we can identify \( F(a, b; c; z) \) with the hyper-geometrical function.

We are going to construct the operator \( \hat{L} \) in the case where \( \mu \) and \( \nu \) are real. Therefore, setting \( E = -iR \) in (36), the reality condition will be satisfied if

\[ R^2 > \max (r_0 \pm r_1)^2 . \]

In this case, we can write

\[ \mu_0 = \frac{1}{2} \sqrt{R^2 - (r_0 - r_1)^2} , \quad \nu_0 = \frac{1}{2} \sqrt{R^2 - (r_0 + r_1)^2} , \]

and the expressions (35) become

\[ \psi_1^{(0)} = -iR (1 - z)^{\nu_0} (c_1 z^{\mu_0} F_0 + c_2 z^{-\mu_0} F_1) , \]
\[ \psi_2^{(0)} = (1 - z)^{\nu_0} [(r_0 - r_1 + 2i\mu_0) c_1 z^{\mu_0} F_0 + (r_0 - r_1 - 2i\mu_0) c_2 z^{-\mu_0} F_1^*] , \]
\[ F_0 = F(a_0 + 1, a_0^*; 1 + 2\mu_0; z) , \quad F_1 = F(\bar{a}_0 + 1, \bar{a}_0^*; 1 - 2\mu_0; z) , \]
\[ a_0 = \mu_0 + \nu_0 + ir_0 , \quad \bar{a} = -\mu_0 + \nu_0 + ir_0 . \]

The constants \( c_1 \) and \( c_2 \) will be chosen such that the relation

\[ \frac{c_1}{c_2} = p^{2\mu_0} (r_0 - r_1 - 2i\mu_0) R^{-1} = p^{2\mu_0} e^{-2i\varphi_0} \]

is satisfied, where \( p \) is a new real constant and \( \varphi_0 \) is a constant phase defined, in agreement with (37), by the expression

\[ (r_0 - r_1 + 2i\mu_0) R^{-1} = e^{2i\varphi_0} . \]

For such a choice of the constants \( c_1 \) and \( c_2 \), the solutions (35) assume the form

\[ \psi_1^{(0)} = -iR (1 - z)^{\nu_0} \sqrt{c_1 c_2} A , \quad \psi_2^{(0)} = R (1 - z)^{\nu_0} \sqrt{c_1 c_2} A^* , \]
\[ A = (pz)^{\mu_0} e^{-i\varphi_0} F_0 + (pz)^{-\mu_0} e^{i\varphi_0} F_1 . \]

Using the above solutions (35) and (36) and the expression (24) and (27), the constants \( \alpha \) and \( \beta \) are seen to be real, and they can be written as

\[ \alpha = \frac{iR (A^2 - A^2)}{2T AA^*} , \quad \beta = f + \frac{R (A^2 + A^2)}{2T AA^*} , \]

\[ 8 \]
with $f$ defined by (7). In this case, the Darboux transformation provides exact solutions of equations (1, 2) with a potential given by

$$f_1 = \frac{R (A^* + A)}{|A|^2} - r_0 \tanh \frac{\tau}{T} - r_1.$$  

(43)

We remark that the new potential, as well as the corresponding solutions, are only expressed via the hyper-geometric functions.

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**References**

[1] Nussenzveig H.M., *Introduction to Quantum Optics*, (Gordon and Breach, New York 1973)

[2] Barata J.C., Wreszinski W.F., *Strong-Coupling Theory of Two-Level Atoms in Periodic Fields*, Phys. Rev. Lett. 84, 2112 (2000)

[3] Rabi I.I., Ramsey N.F., Schwinger J., *Use of Rotating Coordinates in Magnetic Resonance Problems*, Rev. Mod. Phys. 26, 167 (1945)

[4] Barata J.C., Cortez D.A., *Time evolution of two-level systems driven by periodic fields*, Phys Lett. A 301, 350 (2002)

[5] Grifoni M., Hanggi P., *Driven quantum tunneling*, Phys. Rep. 304, 229 (1998)

[6] Novikov S., et al, *Theory of Solitons*, (Consultants Bureau, New York, London 1984)

[7] Rabi I.I., *Space Quantization in a Gyrating Magnetic Field*, Phys. Rev. 51, 652 (1937)

[8] Bagrov V.G., Barata J.C.A., Gitman D.M., Wreszinski W.F., *Aspects of two-level systems under external time-dependent fields*, J. Phys A34, 10869 (2001)

[9] Darboux G., *Sur une proposition relative aux équations linéaires*, C.R. Acad. Sci. 94, 1456 (1882)

[10] Matveev V.B., Salle M.A., *Darboux transformations and solutions*, (Springer-Verlag, Berlin 1991)
[11] McLaughlin D.W., *Whiskered Tori for NLS Equations*, in "Important Developments in Soliton Theory", Ed. A.S. Fokas, V.E. Zakharov pp 537-558 (Springer-Verlag, 1993)

[12] Bagrov V.G., Samsonov B.F., *Darboux transformations of the Schrödinger equation*. Physics of Particles and Nuclei, *28*(4), 374 (1997)

[13] Nieto I.M., Pecheritsin A.A., Samsonov B.F., *Intertwining technique for the one-dimensional stationary Dirac equation*, Ann. Phys. *305*, No 2, 151 (2003)

[14] Rogers C., Schief W.K., *Backlund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory* (Cambridge Univ. Press 2002)