On Landauer principle and bound for infinite systems

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Abstract

Landauer principle provides a link between Shannon information entropy and Clausius thermodynamical entropy. We set up here a basic formula for the incremental free energy of a quantum channel, possibly relative to infinite systems, naturally arising by an Operator Algebraic point of view. By the Tomita-Takesaki modular theory, we can indeed describe a canonical evolution associated with a quantum channel state transfer. Such evolution is implemented both by a modular Hamiltonian and a physical Hamiltonian, the latter being determined by its functoriality properties. This allows us to make an intrinsic analysis, extending our QFT index formula, but without any a priori given dynamics; the associated incremental free energy is related to the logarithm of the Jones index and is thus quantised. This leads to a general lower bound for the incremental free energy of an irreversible quantum channel which is half of the Landauer bound, and to further bounds corresponding to the discrete series of the Jones index. In the finite dimensional context, or in the case of DHR charges in QFT, where the dimension is a positive integer, our lower bound agrees with Landauer bound.

*Supported in by the ERC Advanced Grant 669240 QUEST “Quantum Algebraic Structures and Models”, MIUR FARE R16X5RB55W QUEST-NET, GNAMPA-INdAM and Alexander von Humboldt Foundation.
1 Introduction

Quantum information is an increasing lively subject. At the present time, most papers in the subject deal with finite quantum systems, i.e. multi-matrix algebras. This is justified from the quantum computation point of view inasmuch as only few qubits quantum computers have so far been built up. However, from the functional analytic and conceptual point of view, it is natural to look at infinite quantum systems too. On one hand big quantum systems, as quantum computers in perspective or quantum black holes, may be idealised as infinite systems, on the other hand operator algebras not of type I reveal a rich structure that is not directly visible within the finite dimensional context and may thus both offer a new insight and provide effective tools for the analysis. In particular, looking at the subject from the Quantum Field Theory point of view is expected to lead to a new perspective within this framework.

In this paper we shall see a dynamics that is naturally associated with a quantum channel. At the root of our investigation is our analysis in [33] dealing with the time evolution in black hole thermodynamics, that also led us to a QFT index formula [34]. The dynamics in this paper is intrinsic and originates from the modular structure associated with Connes bimodules. Now, this evolution is canonically implemented both by a modular Hamiltonian and by a physical Hamiltonian, the latter being characterised by its good functoriality properties with respect to the bimodule tensor categorical structure. We can compare these two Hamiltonians; in the simplest factor case one is simply obtained by rescaling the other. We then define an intrinsic incremental free energy associated with the quantum channel. The incremental free energy turns out to be proportional to the logarithm of the Jones index; so, in particular, we obtain a lower bound which is related to the Landauer bound, and agrees with that in the finite dimensional context. The structure associated with quantum systems with a non trivial classical part, namely non-factor von Neumann algebras, is more involved; we shall deal here with the case where the classical part is finite dimensional.

Before we proceed with more details, we state some general facts that are at the basis of our investigation.

1.1 Entropy

We begin by recalling the basic forms of entropy that are related to our work.

1.1.1 Thermodinamical entropy

The concept of entropy arose in thermodynamics and is due to Clausius. If only reversible processes take place in an isolated, homogeneous system, then the integral over a closed path of the form $\frac{dQ}{T}$ vanishes: $\oint \frac{dQ}{T} = 0$. Thus $\frac{dQ}{T}$ is an exact differential form. As heat (energy) $Q$ and temperature $T$ are respectively an extensive and an intensive quantity, there must so exist an extensive state function $S$ such that $dS = \frac{dQ}{T}$. Clausius named this new appearing quantity entropy from Greek “transformation content”.

The second principle of thermodynamics asserts that, for an isolated system, the entropy never decreases

$$dS \geq 0$$

and indeed $dS = 0$ just for reversible transformations.
1.1.2 Information entropy

The theory of quantum information promotes classical information theory to the quantum world. One may say that a classical system is described by an abelian operator algebra $\mathcal{M}$, while a quantum system by a non-commutative operator algebra $\mathcal{M}$ and one firstly aims to describe the information carried by a state of a system $\mathcal{N}$ when transferred to a state of a system $\mathcal{M}$ by a classical/quantum channel.

Let’s consider finite systems first. In the classical case, the information of a state $\varphi$ on a system $\mathbb{C}^n$ $(n$-point space) with probability distribution $\{p_1, \ldots, p_n\}$ is measured by the Shannon entropy

$$S(\varphi) = -k \sum_i p_i \log p_i ,$$

where $k$ is a proportionality constant that for the moment may be put equal to 1. The point is that the probability of independent events is multiplicative, while information is additive, thus the information is to be proportional to the logarithm of the probability. So (1) measure the average information carried by the state $\varphi$.

In the quantum case $\mathcal{M}$ is a matrix algebra, $\varphi$ is a normalised, positive linear functional on $\mathcal{M}$ with density matrix $\rho$, and the information entropy is given by von Neumann entropy

$$S(\varphi) = -\text{Tr}(\rho \log \rho).$$

If the finite system is to encode both classical and quantum information, $\mathcal{M}$ is an arbitrary finite dimensional $C^*$-algebra.

A quantum channel between the two finite systems $\mathcal{N}$ and $\mathcal{M}$ is a completely positive map $\alpha : \mathcal{N} \to \mathcal{M}$ (possibly trace preserving, unital). Already within this relatively simple, finite dimensional setting one can see remarkable quantum structures and conceptual aspects.

Now, in the classical case, the passage from finite to infinite systems does not provide a new conceptual insight. On the other hand, we will see in this paper new aspects that do emerge in this passage from finite to infinite systems in the quantum case.

An infinite quantum system, possibly with a classical part too, will be described by a von Neumann algebra $\mathcal{M}$; the von Neumann entropy a normal state $\varphi$ on $\mathcal{M}$ makes no sense in this case, unless $\mathcal{M}$ is of type I; however Araki relative entropy between two faithful normal states $\varphi$ and $\psi$ on $\mathcal{M}$ is defined in general by

$$S(\varphi|\psi) \equiv -(\eta, \log \Delta_{\xi,\eta} \xi) ,$$

where $\xi, \eta$ are the vector representatives of $\varphi, \psi$ on the natural cone $L^2_+(\mathcal{M})$ and $\Delta_{\xi,\eta}$ is the relative modular operator associated with $\xi, \eta$, see [4, 38]. Relative entropy is one of the key concepts used in the following.

In this paper a quantum channel will be a normal, unital, completely positive $\alpha$ map between von Neumann algebras with finite dimensional centers, where $\alpha$ has finite index and $\varphi$ and $\psi$ input and output states with respect to $\alpha$. The index will be the Jones index of a normal bimodule canonically associated with $\alpha$. Modular theory and subfactor theories are indeed at the root of our analysis, as we shall later explain.

1.1.3 Statistical mechanics entropy

According to Boltzmann, for an isolated system in thermal equilibrium, the entropy is associated with the logarithmic counting of the number of all possible states of the system,
namely \( S = k \log W \), where \( W \) is the number of all possible microstates compatible with the given macrostate of the system; here \( k \) is the Boltzmann constant.

We can view \( S \) as a measure of our lack of knowledge about our system, thus getting a link between thermodynamical entropy and information entropy.

Since the the equilibrium distribution at inverse temperature \( \beta = \frac{1}{kT} \) is given by the Gibbs distribution \( e^{-\beta H}/Z(\beta) \) with \( H \) the energy and \( Z \) the normalising partition function, we see that, say in the quantum case, \( S \) is given by formula (2) with density matrix \( \rho = e^{-\beta H}/Z(\beta) \).

1.1.4 Black hole entropy

In black hole thermodynamics the laws of thermodynamics are promoted to a quantum black hole framework. In this context, entropy has been subject to several different interpretations, both as statistical mechanics entropy and as information entropy. Fundamental aspects here are provided by the Hawking thermal radiation, and by Bekenstein area law giving entropy a geometrical interpretation as proportional to the area of the black hole event horizon. We refer to the literature for more on this subject, e.g. [47].

1.2 Maxwell’s demon. Landauer bound

Towards the end of the 19th century, Maxwell suggested a thought experiment to show how the Second Law of Thermodynamics might hypothetically be violated, the so called Maxwell’s demon experiment, that we recall.

A gas is in equilibrium in a box and a wall is put to divide the box in two halves \( A \) and \( B \). A little being, the demon, controls a tiny door on the wall between \( A \) and \( B \). At an individual gas molecule approaches the door in \( A \) or \( B \), the demon quickly opens and shuts the door in order to allow only the faster molecules to pass from \( A \) to \( B \), and only the slower molecules to pass from \( B \) to \( A \). The average molecule speed thus increases in \( B \), and decreases in \( A \). Since faster molecules give rise to higher temperature, as a result the temperature in \( B \) becomes higher than in \( A \), so the entropy of the system decreases, thus violating the Second Law of Thermodynamics. Indeed a thermodynamical engine could extract work from this temperature difference.

Maxwell demon experiment, and its subsequent more refined versions, have long been a matter of debate in the Physics community, see e.g. [28]. An important contribution came by Szilard with a further idealisation where the gas has only one molecule. Szilard pointed out that the act of Maxwell’s demon to measuring molecular speed would require an expenditure of energy. So one must consider the entropy of the total system including the demon. The expenditure of energy by the demon would produce an increase of the entropy of the demon, which would be larger than the decrease of the entropy of the gas.

Rolf Landauer [27] realised however that some measuring processes need not increase thermodynamic entropy as long as they were thermodynamically reversible. Landauer argued that information is physical and this principle was central to solving the paradox of Maxwell’s demon. Bennett [1] noted that the demon has to memorise the information he acquires about the gas molecules. He argued that after a full cycle of information the demon’s memory has to be reset to its initial state to allow for a new iteration. According to Landauer’s principle, the erasure process will always dissipate more entropy than the demon produces during one cycle, in full agreement with the second law of thermodynamics. Tak-
ing into account Shannon information entropy, a generalised second law of thermodynamics holds true by considering the total system and the total (thermodynamical + information) entropy.

Landauer principle states that “any logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information bearing degrees of freedom of the information processing apparatus or its environment” [1].

Another way of phrasing this principle is that if an observer loses information about a physical system, the observer loses the ability to extract work from that system.

Landauer’s principle sets a lower bound of energy consumption of computation or logical operation, also known as Landauer limit. For an environment at temperature $T$, energy $E = ST$ must be emitted into that environment if the amount of added entropy is $S$. For a computational operation in which 1 bit of logical information is lost, the amount of entropy generated is at least $k \log 2$ and so the energy that must eventually be emitted to the environment is

$$E \geq kT \log 2,$$

with $k$ the Boltzmann constant. If no information is erased, a thermodynamically reversible logical operation is theoretically possible with no release of energy $E$, therefore the above bound concerns irreversible transformations.

Recently, physical experiments have tested Landauer’s principle and confirmed its predictions, see [5].

### 1.3 Underlying mathematical and physical context

We now recall a few facts that play a particular role in our paper.

#### 1.3.1 Modular theory, the intrinsic dynamics

As is well known, a von Neumann algebra $\mathcal{M}$ generalises at the same time the notion of (multi-)matrix algebra and the one of measure space: if $\mathcal{M}$ is finite dimensional then $\mathcal{M}$ is a direct sum matrix algebras, if $\mathcal{M}$ is abelian then $\mathcal{M}$ can be identified with $L^\infty(X,\mu)$ for some measure space $(X,\mu)$, so $\mathcal{M}$ is sometimes called a “noncommutative measure space”.

Let $\varphi$ be a faithful normal state of $\mathcal{M}$ (noncommutative integral). The Tomita-Takesaki modular theory provides a canonical, intrinsic dynamics associated with $\varphi$: a one-parameter group of automorphism $\sigma^\varphi$ of $\mathcal{M}$,

$$t \in \mathbb{R} \mapsto \sigma^\varphi_t \in \text{Aut}(\mathcal{M}),$$

called the modular group of $\varphi$. Among its remarkable properties, we mention here the following:

- $\sigma^\varphi$ is a purely noncommutative object: $\sigma^\varphi$ acts identically if and only if $\varphi$ is tracial, therefore $\sigma^\varphi$ is not visible within the classical abelian case.
- $\sigma^\varphi$ does not depend on $\varphi$ up to inner automorphisms by Connes Radon-Nikodym theorem [7]; in particular, $\sigma^\varphi$ can be an outer action only in the infinite dimensional case, indeed in the type $\text{III}$ case.
- $\sigma^\varphi$ is characterised by the KMS thermal equilibrium condition at inverse temperature $\beta = -1$ with respect to the state $\varphi$. The KMS condition appears in Quantum Statistical Mechanics, see Section 1.3.3; thus modular theory is directly connected with Physics.
By the last point, if we have a physical evolution satisfying the KMS condition at inverse temperature $\beta > 0$ w.r.t. a state $\varphi$, then we may identify this evolution with the rescaled modular group $t \mapsto \sigma^{\varphi}_{-t/\beta}$, yet the modular evolution exists independently of any underlying physical setting.

### 1.3.2 Jones index

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of factors. The Jones index $[\mathcal{M} : \mathcal{N}]$ measures the relative size of $\mathcal{N}$ in $\mathcal{M}$. It was originally defined for factors with a tracial state [22], then extended to arbitrary inclusions of factors with a normal conditional expectation $\varepsilon : \mathcal{M} \to \mathcal{N}$ by Kosaki [25], and then in [30]. One of the main properties of the index is that its values are quantised

$$[\mathcal{M} : \mathcal{N}] = 4 \cos^2 \frac{\pi}{k}, \quad k = 3, 4, \ldots \quad \text{if} \quad [\mathcal{M} : \mathcal{N}] < 4,$$

by Jones’ theorem [22].

In general, the well behaved index to be considered is the one with respect to the minimal expectation, the minimum over all possible indices. The multiplicativity of the minimal index is shown in [26, 32] and refs therein.

Jones’ index appears in many contexts in Mathematics and in Physics. In [30] we showed the index-statistics theorem

$$\text{DHR dimension} = \sqrt{\text{Jones index}},$$

a relation between the Doplicher-Haag-Roberts statistical dimension of a superselection sector in Quantum Field Theory and the index of a localised endomorphism that represents the sector. Furthermore, our formula in QFT [33] provides an interpretation of the logarithm of the index from the entropy viewpoint, see also Sections 1.3.4 and 4 here.

In this paper, we are going to consider possibly infinite physical systems with a finite classical part, so von Neumann algebras $\mathcal{N}, \mathcal{M}$ with finite dimensional centers. If $\mathcal{N} \subset \mathcal{N}$, the minimal expectation and the minimal index can then be defined, and the latter is a scalar subject to the above restriction (3) if the centers of $\mathcal{N}$ and $\mathcal{M}$ intersect trivially, see [18, 13] and refs therein.

In a forthcoming paper [15], we shall see how to define a non-scalar dimension that is multiplicative in this general framework. This allows us to define a “functorial” Hamiltonian. Here, in order to focus more on the quantum information side, we do not dwell on non-scalar dimension issues. In order to delving further into the general case, we refer to [15].

### 1.3.3 Quantum relativistic statistical mechanics

Let $\mathcal{M}$ be a finite purely quantum system, namely $\mathcal{M}$ is the algebra on $n \times n$ complex matrices. The time evolution $\tau$ is a one-parameter automorphism group of $\mathcal{M}$ implemented by a one-parameter group $U$ of unitaries in $\mathcal{M}$, namely $\tau_t(X) = U(t)XU(-t)$, $X \in \mathcal{M}$. We have $U(t) = e^{itH}$ where $H \in \mathcal{M}$ is a positive selfadjoint operator, the Hamiltonian.

As is well known, a state $\varphi$ of $\mathcal{M}$ in thermal equilibrium at inverse temperature $\beta > 0$ is characterised by the Gibbs condition, namely

$$\varphi(X) = \text{Tr}(\rho X),$$

where $\rho$ is the density matrix $\rho = e^{-\beta H} / \text{Tr}(e^{-\beta H})$. 

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At infinite volume, $\mathcal{M}$ becomes an infinite dimensional operator algebra, in general no trace $\text{Tr}$ exists any longer on $\mathcal{M}$ and the evolution $\tau$ is not inner. A state $\varphi$ in thermal equilibrium at inverse temperature $\beta > 0$ is now characterised by the KMS condition:

For every $X,Y \in \mathcal{M}$, there is a function $F_{XY} \in A(S_\beta)$ such that

\begin{align}
(a) \quad F_{XY}(t) &= \omega(X\tau_t(Y)) , \\
(b) \quad F_{XY}(t + i\beta) &= \omega(\tau_t(Y)X) ,
\end{align}

(4)

where $A(S_\beta)$ is the algebra of functions analytic in the strip $S_\beta = \{0 < \Im z < \beta\}$, bounded and continuous on the closure $\overline{S_\beta}$.

The KMS condition generalises the Gibbs to infinite systems, see [17, 45, 4].

As said, if $\mathcal{M}$ is a von Neumann algebra and $\varphi$ a faithful normal state of $\mathcal{M}$, then $\tau$ can be identified with the rescaled modular group $\sigma^\varphi$ of $\varphi$. For an evolution on a $C^*$-algebra $\mathfrak{A}$ with KMS state $\varphi$, one gets into the von Neumann algebra framework by considering the weak closure $\mathcal{M}$ of $\mathfrak{A}$ in the GNS representation associated with $\varphi$, the KMS condition holds then on $\mathcal{M}$ too.

In quantum relativistic statistical mechanics, locality and relativistic invariance have both to hold. In essence, as far as we are concerned here, quantum relativistic statistical mechanics is the study of KMS states in Quantum Field Theory.

### 1.3.4 The analog of the Kac-Wakimoto formula and a QFT index theorem

A local conformal net $\mathcal{A}$ on $S^1$ is the operator algebraic framework to study chiral Conformal Quantum Field Theory, see [23]. If $L_0$ is the conformal Hamiltonian (generator of the rotation one parameter unitary group) in the vacuum representation, and $L_\rho$ is the conformal Hamiltonian in any representation $\rho$ of $\mathcal{A}$, one expects the following formula to hold:

\[ \lim_{t \to 0^+} \frac{\text{Tr}(e^{-tL_\rho})}{\text{Tr}(e^{-tL_0})} = d(\rho) , \]

(5)

here $d(\rho)$ is the dimension of $\rho$ (see Section 1.3.2 and [35]).

Although formula (5) has been checked for most models, see [46, 24], it stands up as an important unproven conjecture.

In [33], we have however proven an analog of formula (5) that holds true in full generality, where the rotation flow is replaced by a geometric KMS flow in QFT, for instance the dilation flow in CFT or the boost flows in general Quantum Field Theory [3, 19]. To stay in a specific context, let $H_0$ and $H_\rho$ be the Hamiltonian for a uniformly accelerated observer in the Minkowski spacetime with acceleration $a > 0$, equivalently $H_0$ and $H_\rho$ are the generators the geodesic flow evolution in the Rindler spacetime, respectively in the vacuum representation and in the representation $\rho$; then

\[ (\Omega, e^{-tH_\rho}\Omega)|_{t=\beta} = d(\rho) , \]

(6)

with $\Omega$ the vacuum vector and $\beta = \frac{2\pi}{a}$ the inverse Hawking-Unruh temperature, see also Section 4.

The proof of formula (6) is based on a tensor categorical analysis and has been subsequently extended as a ground for a QFT index theorem [34].
In this paper, we generalise formula (6) without any reference to a given KMS physical flow, we indeed rely on an intrinsic modular evolution so to set up a universal formula. Of course, this formula gives back (6) and other formulas when applied to specific physical settings. Because of its generality, it can now be applied also to the Quantum Information setting, the context we are mainly going to consider in the following.

2 Bimodules and quantum channels

In order to simplify our exposition, we assume all von Neumann algebras in this paper to have separable predual, namely to be representable on a separable Hilbert space. And all Hilbert spaces will be separable.

2.1 Connes bimodules

We start by recalling the basic facts on normal bimodules over von Neumann algebras (see [9, 42, 41, 31, 12]) and develop further material.

Given von Neumann algebras \( \mathcal{N} \) and \( \mathcal{M} \), by a \( \mathcal{N} - \mathcal{M} \)-bimodule \( \mathcal{H} \) we mean a Hilbert space \( \mathcal{H} \) with a normal left action of \( \mathcal{N} \) and a right action of \( \mathcal{M} \). Namely we have a normal representation \( \ell \) of \( \mathcal{N} \) on \( \mathcal{H} \) and a normal anti-representation \( r \) of \( \mathcal{M} \) on \( \mathcal{H} \) (thus \( r(m_1 m_2) = r(m_2) r(m_1) \)) such that \( \ell(N) \) and \( r(M) \) mutually commute. Possibly, the natural notation

\[ n \xi m \equiv \ell(n) r(m) \xi, \quad n \in \mathcal{N}, \ m \in \mathcal{M}, \ \xi \in \mathcal{H}, \]

will be used.

Let \( \mathcal{M}^o \) be the von Neumann algebra opposite to \( \mathcal{M} \), with \( m \mapsto m^o \) the natural anti-isomorphism of \( \mathcal{M} \) with \( \mathcal{M}^o \). Since an anti-representation \( r \) of \( \mathcal{M} \) corresponds to a representation \( r^o \) of \( \mathcal{M}^o \), i.e. \( r^o(m^o) = r(m) \), a \( \mathcal{N} - \mathcal{M} \)-bimodule \( \mathcal{H} \) corresponds to a binormal representation \( \pi \) of \( \mathcal{N} \otimes \mathcal{M}^o \), the algebraic tensor product of \( \mathcal{N} \) and \( \mathcal{M}^o \), namely \( \pi \) is a representation of \( \mathcal{N} \otimes \mathcal{M}^o \) on \( \mathcal{H} \) whose restriction both on \( \mathcal{N} \otimes 1 \) and \( 1 \otimes \mathcal{M}^o \) is normal, indeed \( \pi(n \otimes m^0) = \ell(n) r(m) \). The representation \( \pi \) extends and corresponds uniquely to a representation of the maximal tensor product \( C^* \)-algebra \( \mathcal{N} \otimes_{\max} \mathcal{M}^o \), the completion of \( \mathcal{N} \otimes \mathcal{M}^o \) with respect to the maximal \( C^* \)-norm.

So there are natural notions of direct sum of bimodules and intertwiner between bimodules: they are the ones that appear when we view a \( \mathcal{N} - \mathcal{M} \) bimodule as a representation of \( \mathcal{N} \otimes_{\max} \mathcal{M}^o \), indeed bimodules form a \( C^* \)-category.

We shall denote by \( L^2(\mathcal{M}) \) the identity \( \mathcal{N} - \mathcal{M} \) bimodule, which is unique (up to unitary equivalence). If \( \mathcal{M} \) acts on a Hilbert space \( \mathcal{H} \) with cyclic and separating vector \( \xi \), then \( \mathcal{H} = L^2(\mathcal{M}) \) with actions \( \ell(m) \eta = m \eta, \ r(m) \eta = J m^* J \eta, \ \eta \in \mathcal{H} \), where \( J \) is the modular conjugation of \( \mathcal{M} \) associated with \( \xi \), indeed with any cyclic (thus separating) vector in the natural positive cone \( L^2(\mathcal{M})_+ \) given by \( \xi \).

The conjugate \( \mathcal{H} \) of the \( \mathcal{N} - \mathcal{M} \) bimodule \( \mathcal{H} \) is the \( \mathcal{M} - \mathcal{N} \) bimodule over the conjugate Hilbert space \( \bar{\mathcal{H}} \) with actions \( m \cdot \xi \cdot n = n^* \xi m^* \), namely

\[ \bar{\ell}(m) \xi = r(m^*) \xi, \quad \bar{r}(n) \xi = \ell(n^*) \xi. \]  

Moreover, there exists an (internal) tensor product of bimodules and of intertwiners ([9, 42]): if \( \mathcal{H} \) is a \( \mathcal{M}_1 - \mathcal{M}_2 \) bimodule and \( \mathcal{K} \) is a \( \mathcal{M}_2 - \mathcal{M}_3 \) bimodule, then \( \mathcal{H} \otimes \mathcal{K} \) is a \( \mathcal{M}_1 - \mathcal{M}_3 \) bimodule, whose representation is given by \( \ell(n_1) \bar{\ell}(m_2) \xi \bar{r}(n_3) \eta \) on the Hilbert space \( \mathcal{H} \otimes \mathcal{K} \).
bimodule. We shall give the definition of the tensor product, up to unitary equivalence, later below.

Note that we have also the operation of external tensor product. Let \( \mathcal{H}_k \) be a \( N_k - M_k \) bimodule, \( k = 1, 2 \). The external tensor product \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) is the obvious \( N_1 \otimes N_2 - M_1 \otimes M_2 \) bimodule on the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). We shall use the symbol \( \otimes \) for the tensor product, and the larger \( \bigotimes \) for the usual external tensor product.

Let now \( \mathcal{N} \) and \( \mathcal{M} \) be von Neumann algebras with finite dimensional centers and \( \mathcal{H} \) a \( \mathcal{N} - \mathcal{M} \) bimodule. We shall say that \( \mathcal{H} \) has finite index if the inclusion \( \ell(\mathcal{N}) \subset r(\mathcal{M})' \) has finite index ([25, 18, 13]). In this case, if the centers are finite dimensional, the index of \( \mathcal{H} \) is defined by

\[
\text{Ind}(\mathcal{H}) \equiv [r(\mathcal{M})' : \ell(\mathcal{N})] = [\ell(\mathcal{N})' : r(\mathcal{M})],
\]

here the square brackets denote the minimal index of an inclusion of von Neumann algebras with finite dimensional centers (see [31, 13] and references therein), we refer to [15] for more on the index structure we need here.

We shall say that \( \mathcal{H} \) is connected if

\[
\ell(\mathcal{N}) \cap r(\mathcal{M}) = \ell(Z(\mathcal{N})) \cap r(Z(\mathcal{M})) = \mathbb{C}.
\]

Here \( Z(\mathcal{\cdot}) \) denotes the center. If \( \mathcal{H} \) has finite index, then \( \mathcal{H} \) is the direct sum of finitely many connected bimodules. For all results in this paper, we may deal with connected bimodules only, the general case being an immediate consequence by considering the direct sum along the atoms of \( \ell(Z(\mathcal{N})) \cap r(Z(\mathcal{M})) \).

Note that \( \text{Ind}(\mathcal{H}) \) is a element is the \( \ell(\mathcal{N}) \cap r(\mathcal{M}) \) and \( \text{Ind}(\mathcal{H}) \geq 1 \), in particular the index is a scalar if \( \mathcal{H} \) is connected.

Let \( \text{Ind}(\mathcal{H}) < \infty \) and \( \mathcal{H} \) be connected as above. The (scalar) dimension of \( \mathcal{H} \) is the square root the index

\[
d_{\mathcal{H}} \equiv \sqrt{\text{Ind}(\mathcal{H})}.
\]

Given faithful normal states \( \varphi, \psi \) on \( \mathcal{N} \) and \( \mathcal{M} \), we define the modular operator \( \Delta_\mathcal{H}(\varphi|\psi) \) of \( \mathcal{H} \) with respect to \( \varphi, \psi \) as

\[
\Delta_\mathcal{H}(\varphi|\psi) \equiv d(\varphi \cdot \ell^{-1} \cdot \epsilon) / d(\psi \cdot r^{-1}), \tag{8}
\]

where the right hand side is the Connes’ spatial derivative [8] for the pair \( r(\mathcal{M})', r(\mathcal{M}) \) w.r.t. the states \( \varphi \cdot \ell^{-1} \cdot \epsilon \) and \( \psi \cdot r^{-1} \) and \( \epsilon : r(\mathcal{M})' \to \ell(\mathcal{N}) \) is the minimal conditional expectation, see also [44, 48].

For simplicity, in this paper we may sometime assume that the left and right actions are faithful, i.e. \( \ell \) and \( r \) are injective maps, which is automatic if \( \mathcal{N} \) and \( \mathcal{M} \) are factors. This assumption can be easily removed: let \( p \) the central projection of \( \ell \) (i.e. \( 1 - p \) is the largest projection in \( Z(\mathcal{N}) \) in the kernel of \( \ell \)) and similarly \( q \) the central support of \( r \). Then \( \ell_p \equiv \ell|_{\mathcal{N}_p} \) and \( r_q \equiv r|_{\mathcal{M}_q} \) are injective maps, and we may consider \( \mathcal{H} \) as a \( \mathcal{N}_p - \mathcal{M}_q \) bimodule with left and right actions \( \ell_p \) and \( r_q \). In (8) \( \varphi \) and \( \psi \) are then replaced by the positive linear functionals \( \varphi|_{\mathcal{N}_p} \) and \( \psi|_{\mathcal{M}_q} \).

The following important lemma follows by combining the properties of the spatial derivative [8] and Takesaki’s theorem on conditional expectations [45].

**Lemma 2.1.** With \( \Delta \equiv \Delta_\mathcal{H}(\varphi|\psi) \), for every \( t \in \mathbb{R} \) we have

\[
\Delta^{it} \ell(n) \Delta^{-it} = \ell(\sigma_t^\varphi(n)), \quad \Delta^{it} r(m) \Delta^{-it} = r(\sigma_t^\psi(m)), \tag{9}
\]

\( n \in \mathcal{N}, \ m \in \mathcal{M} \).
Proof. \((d(\varphi \cdot \ell^{-1} \cdot \varepsilon)/d(\psi \cdot r^{-1}))^it\) implements \(\sigma^\varphi_{t^it-1,\varepsilon}\) on \(r(M)'\) and \(\psi \cdot r^{-1}\) on \(r(M)\). By Takesaki’s theorem \(\sigma^\varphi_{t^it-1,\varepsilon}\) restricts to \(\sigma^\varphi_{t^it-1}\) on \(\ell(N)'\). Since \(\sigma^\varphi_{t^it-1}(\ell(n)) = \ell(\sigma^\varphi_t(n))\), we have have the first equality in (9). The second equality is also clear now.

For every \(t \in \mathbb{R}\), \(\Delta_H^it(\varphi, \psi)\) is then a unitary intertwiner between the bimodule \(\mathcal{H}\) and the twisted bimodule \(\mathcal{H}_t\) where \(\ell, r\) are replaced by \(\ell \cdot \sigma^\varphi_t\) and \(r \cdot \sigma^\psi_t\). So, in particular, the notions of internal tensor product and conjugation are defined for the modular unitaries.

Note also that

\[
\Delta^itX\Delta^{-it} = X, \quad X \in (\ell(N) \vee r(M))',
\]

because the minimal conditional expectation is tracial on the relative commutant.

The definition of \(\Delta_H(\varphi|\psi)\) is not symmetric. If we define the “right” modular operator \(\Delta^H_{\text{right}}(\varphi|\psi) = d(\varphi \cdot \ell^{-1})/d(\psi \cdot r^{-1} \cdot \varepsilon)\), with \(\varepsilon : \ell(N)' \to r(M)\) the minimal conditional expectation, we have by Kosaki’s formula [25]

\[
\Delta^H_{\text{right}}(\varphi|\psi) = \text{Ind}(\mathcal{H}) \cdot \Delta_H(\varphi|\psi).
\]

We call \(\log \Delta_H(\varphi|\psi)\) the modular Hamiltonian of \(\mathcal{H}\) with respect to the states \(\varphi\) and \(\psi\).

Let \(\mathcal{N}, \mathcal{M}\) be von Neumann algebras and \(\mathcal{H}\) a \(\mathcal{N} - \mathcal{M}\) bimodule. Let \(\mathcal{F}\) be a type I\(_\infty\) factor, namely \(\mathcal{F}\) is isomorphic to \(B(K)\) with \(K\) a separable, infinite-dimensional Hilbert space, and consider the identity \(\mathcal{F} - \mathcal{F}\) bimodule \(L^2(\mathcal{F})\) (\(B(K)\) acting on the Hilbert-Schmidt operators by left and right multiplication). Then the external tensor product \(\mathcal{H} \otimes L^2(\mathcal{F})\) is naturally a \(\mathcal{N} \otimes \mathcal{F} - \mathcal{M} \otimes \mathcal{F}\) bimodule, the “ampliation” of \(\mathcal{H}\). As \(\mathcal{N} \otimes \mathcal{F}, \mathcal{M} \otimes \mathcal{F}\) are properly infinite, by considering the ampliation, most proofs can be easily reduced to the case of bimodules over properly infinite von Neumann algebras.

We begin to analyse the functoriality properties of the modular Hamiltonian in the factorial case.

The internal tensor product \(\mathcal{H} \otimes \mathcal{K}\) of bimodules \(\mathcal{H}\) and \(\mathcal{K}\) was defined in [9, 42], we shall explain it in the next section.

Theorem 2.2. Let \(\mathcal{M}_k\) be factors, \(\varphi_k\) a faithful normal state of \(\mathcal{M}_k, k = 1, 2, 3\). With \(\mathcal{H}, \mathcal{H}'\) finite index \(\mathcal{M}_1 - \mathcal{M}_2\) bimodules and \(\mathcal{K}\) a \(\mathcal{M}_2 - \mathcal{M}_3\) finite index bimodule, we have

\(a\) \(\Delta^it_H(\varphi_1|\varphi_2) \otimes \Delta^it_K(\varphi_2|\varphi_3) = \Delta^it_{\mathcal{H} \otimes \mathcal{K}}(\varphi_1|\varphi_3)\)

\(b\) \(\Delta^it_H(\varphi_2|\varphi_1) = \text{Ind}(\mathcal{H})^it \cdot \Delta^{-it}_{\mathcal{H}}(\varphi_1|\varphi_2)\).

If \(T : \mathcal{H} \to \mathcal{H}'\) is a unitary intertwiner

\(c\) \(T\Delta^it_H(\varphi_1|\varphi_2) = \Delta^it_{\mathcal{H}'}(\varphi'_k|\varphi'_k)T, \) where \(\varphi'_k\) is the state obtained by \(\varphi_k\) the adjoint action of \(T\), \(k = 1, 2\).

Here, in (a), the tensor product is taken w.r.t. the state \(\varphi_2\).

Proof. We sketch the proof of \(a\), more details can be found in [15], see also the comment after Proposition 2.8 below.

With the identification \(\mathcal{H} \simeq L^2(\mathcal{M}_2)\), we have that \(t \mapsto u_t \equiv \Delta^it_H(\varphi_1|\varphi_2)\Delta^{-it}_{\varphi_2}\) is the Connes Radon-Nikodym cocycle \(u_t^{(1)} = (\Phi_1 : \Phi_1 : \Phi_2)_t\), with \(\Phi_1\) the (minimal) left inverse.
of \( \rho_1 \). Similarly, with the identification \( K \simeq L^2_\rho(\mathcal{M}_2) \), we have that \( u^{(2)}_t = \Delta^K_{it}(\varphi_2|\varphi_3)\Delta^{-it}_{\varphi} = (D\varphi_2 \cdot \Phi_2 : D\varphi_3)_{it} \), with \( \Phi_2 \) the left inverse of \( \rho_2 \).

We may now identify \( L^2_\rho(\mathcal{M}_2) \) with \( \rho_2 L^2(\mathcal{M}_2) \); then the tensoriality of the Connes cocycle (extending the arguments is [33]) implies that

\[
\Delta^{it}_H(\varphi_1|\varphi_2) \otimes \Delta^{it}_k(\varphi_2|\varphi_3) = \Delta^{it}_{H\otimes K}(\varphi_1|\varphi_3) ,
\]
hence (a).

Concerning (b), we may assume \( \mathcal{H} = L^2_\rho(\mathcal{M}) \), so \( \mathcal{H} = \rho L^2(\mathcal{M}) \). Then by Kosaki’s formula

\[
\Delta(\varphi_1|\varphi_2) = d\varphi_1 \Phi_1/d\varphi_2 = (d\varphi_2/d\varphi_1 \cdot \Phi_1)^{-1} = \text{Ind}(H)(d\varphi_2 \cdot \Phi_2 / d\varphi_1)^{-1} = \text{Ind}(H)\Delta(\varphi_2|\varphi_1)^{-1} .
\]

(c) follows by the canonicity of the spatial derivative and of the minimal expectation. \( \square \)

Theorem 2.2 does not extends directly to the non factor case. For a finite-index inclusion of von Neumann algebras with finite dimensional centers, there exists an associated minimal conditional expectation; however the composition of minimal expectations may fail to be minimal. In other words, the multiplicativity of the minimal index does not always hold in the non factor case.

As will be explained in [15], for a finite-index \( \mathcal{N} - \mathcal{M} \)-bimodule \( \mathcal{H} \) with finite dimensional \( Z(\mathcal{N}) \) and \( Z(\mathcal{M}) \), we have to consider the (matrix) dimension \( D_H \). This gives a strictly positive linear map \( D_H : \mathcal{H} \to \mathcal{H} \) (still denoted by \( D_H \)) and \( D_H \) belongs to \( l(\mathcal{N})' \cap r(\mathcal{M})' \). In particular \( D_H^{it} \), \( t \in \mathbb{R} \), is a unitary intertwiner of the bimodule \( \mathcal{H} \). The scalar dimension is then given by

\[
d_H = ||D_H||
\]

(Hilbert linear operator norm of the matrix \( D_H \)), if \( \mathcal{H} \) is connected.

By [15], (a) in Theorem 2.2 can then be written also as

\[
\Delta^{it}_H(\varphi_1|\varphi_2)D^{it}_H \otimes \Delta^{it}_\mathcal{K}(\varphi_2|\varphi_3)D^{it}_\mathcal{K} = \Delta^{it}_{\mathcal{H}\otimes \mathcal{K}}(\varphi_1|\varphi_3)D^{it}_{\mathcal{H}\otimes \mathcal{K}} ;
\]

notice here that \( D^{it}_H \) is an intertwiner \( \mathcal{H} \to \mathcal{H} \) and \( \Delta^{it}_H(\varphi_1|\varphi_2) \) is an intertwiner \( \mathcal{H} \to \mathcal{H}_t \), so \( \Delta^{it}_\mathcal{K}(\varphi_2|\varphi_3)D^{it}_\mathcal{K} \) is an intertwiner \( \mathcal{H}_t \to \mathcal{H}_t \), and similarly \( \Delta^{it}_\mathcal{K}(\varphi_2|\varphi_3)D^{it}_\mathcal{K} \) is an intertwiner \( \mathcal{K} \to \mathcal{K}_t \), therefore the internal tensor product in (11) is defined.

By the above theorem and comments, we define the (physical) Hamiltonian \( K_H(\varphi, \psi) \) of \( \mathcal{H} \) with respect to \( \varphi \) and \( \psi \) by rescaling the modular Hamiltonian in (8)

\[
K_H(\varphi|\psi) = \log \Delta_H(\varphi|\psi) + \log D_H .
\]

Alternatively, the definition of the physical Hamiltonian can be given as follows. If \( \mathcal{H} \) is a \( \mathcal{N} - \mathcal{M} \) bimodule with \( \mathcal{N}, \mathcal{M} \) factors we put

\[
K_H(\varphi|\psi) = \log \Delta_H(\varphi|\psi) + \log d_H .
\]

If \( \mathcal{H} \) is a \( \mathcal{N} - \mathcal{M} \) bimodule with \( Z(\mathcal{N}), Z(\mathcal{M}) \) finite dimensional, then \( \mathcal{N} = \bigoplus_i \mathcal{N}_i \) and \( \mathcal{M} = \bigoplus_j \mathcal{M}_j \) (direct sum of algebras corresponding to the atoms \( p_i \) of \( Z(\mathcal{N}) \) and \( q_j \) of \( Z(\mathcal{M}) \)). Here \( \mathcal{N}_i = \mathcal{N}_{p_i} \) and \( \mathcal{M}_j = \mathcal{M}_{q_j} \) are factors. We then have the Hilbert space decomposition \( \mathcal{H} = \bigoplus_{i,j} \mathcal{H}_{ij} \) corresponding to the projections \( p_i q_j \) providing a representation of the direct sum of algebras \( \mathcal{N}_i \otimes \mathcal{M}_j \). (We may say that \( \mathcal{H} \) is the external direct sum of the \( \mathcal{H}_{ij} \).)
Then $\mathcal{H}_{ij}$ is a $\mathcal{N}_i - \mathcal{M}_j$ bimodule with $\mathcal{N}_i, \mathcal{M}_j$ factors, and we put

$$K_\mathcal{H}(\varphi|\psi) = \bigoplus_{i,j} K_{\mathcal{H}_{ij}}(\varphi_i|\psi_i).$$

Here

$$K_{\mathcal{H}_{ij}}(\varphi_i|\psi_i) = \log \Delta_\mathcal{H}(\varphi|\psi)_{|\mathcal{H}_{ij}} + d_{\mathcal{H}_{ij}} = \log \Delta_\mathcal{H}(\varphi_i|\psi_j) + d_{\mathcal{H}_{ij}},$$

and $\varphi_i = \varphi|\mathcal{N}_{p_i}, \psi_j = \psi|\mathcal{M}_{q_j}$.

The one parameter unitary group

$$U_t^K(\varphi|\psi) = \Delta_{\mathcal{H}}(\varphi|\psi) D_{\mathcal{H}}^{it}$$

generated by $e^{itK_\mathcal{H}(\varphi|\psi)}$ naturally transforms under the tensor categorical operations, in particular the equation $U_t^K(\varphi|\psi) = \bar{U}_t^K(\varphi|\psi)$ fixes the rescaling of the modular Hamiltonian in the factor case. We state its main properties in the following theorem.

**Theorem 2.3.** Let $\mathcal{M}_k$ be a von Neumann algebra with finite-dimensional center, $\varphi_k$ a faithful normal state of $\mathcal{M}_k$, $k = 1, 2, 3$. With $\mathcal{H}, \mathcal{H}'$ finite index $\mathcal{M}_1 - \mathcal{M}_2$ bimodules and $\mathcal{K}$ a $\mathcal{M}_2 - \mathcal{M}_3$ finite index bimodule, $U_t^K(\varphi,\psi)$ is the unique one parameter unitary group of automorphisms of $\mathcal{H}$ that depends naturally on $\mathcal{H}, \varphi, \psi$. Namely

$$U_t^K(\varphi|\psi)\ell(n)U_{-t}^{-1}(\varphi|\psi) = \ell(\sigma^K(\varphi|\psi)n), \quad U_t^K(\varphi|\psi)r(m)U_{-t}^{-1}(\varphi|\psi) = r(\sigma^K(\varphi|\psi)m) \quad (12)$$

and

(a) $U_t^H\otimes^K(\varphi_1|\psi_3) = U_t^K(\varphi_1|\varphi_2)\otimes U_t^K(\varphi_2|\varphi_3)$;

(b) $U_t^H(\varphi_2|\varphi_1) = \bar{U}_t^{-1}(\varphi_2|\varphi_1)$;

If $T : \mathcal{H} \to \mathcal{H}'$ is a unitary intertwiner

(c) $TU_t^H(\varphi|\psi) = U_t^{H'}(\varphi'|\varphi_2)T$, where $\varphi_k$ and $\varphi_k'$ are related by the adjoint action of $T$, $k = 1, 2$.

**Proof.** The first statement (12) follows directly from (9).

Concerning (a), we decompose $\mathcal{H} = \bigoplus_{i,k} \mathcal{H}_{ik}$ canonically into external direct sum of factorial bimodules $\mathcal{H}_{ik}$, and similarly for $\mathcal{K} = \bigoplus_{k,j} \mathcal{K}_{kj}$. Then

$$\mathcal{H} \otimes \mathcal{K} = \bigoplus_{i,j,k} \mathcal{H}_{ik} \otimes \mathcal{K}_{kj}$$

and we apply (a) of Theorem 2.2.

Our statement (b) here is satisfied because the statement (b) in Theorem 2.2 holds true in more generality in the case of bimodules over von Neumann algebras with finite dimensional centers.

Similarly, (c) here follows by (c) in Theorem 2.2, which is satisfied in the finite dimensional center case too.

Of course, eq. (12) is equivalent to the requirement that $\text{Ad}U_t^{H}(\psi,\varphi)$ satisfies the KMS condition on $\ell(\mathcal{N})$ and $r(\mathcal{M})$.

We shall further examine the modular and the physical Hamiltonians in the sequel. In a first instance, the reader might prefer to use only the scalar dimension and deal here just with the factorial case.
2.2 Completely positive maps

Let \( \mathcal{N} \) and \( \mathcal{M} \) be von Neumann algebras and \( \alpha : \mathcal{N} \to \mathcal{M} \) a completely positive, normal, unital map. Thus \( \alpha \) is a linear map from \( \mathcal{N} \) to \( \mathcal{M} \) which is normal (equivalently, continuous in the ultra-weak topology), preserving the unity, such that the natural map \( \alpha \otimes \text{id}_k : \mathcal{N} \otimes \text{Mat}_k(\mathbb{C}) \to \mathcal{M} \otimes \text{Mat}_k(\mathbb{C}) \) is positive for every \( k \in \mathbb{N} \). It follows that ||\( \alpha || = 1 \) and \( \alpha(n^*n) \geq \alpha(n)^*\alpha(n) \), \( n \in \mathcal{N} \).

**Lemma 2.4.** Let \( \alpha : \mathcal{N} \to \mathcal{M} \) be a completely positive, normal, unital map as above and \( \mathcal{F} \) a von Neumann algebra. The map \( \alpha \otimes \text{id} : \mathcal{N} \odot \mathcal{F} \to \mathcal{M} \odot \mathcal{F} \) is positive.

**Proof.** Since \( \mathcal{F} \) is contained in \( B(\mathcal{K}) \) with \( \mathcal{K} \) the underlying Hilbert space of \( \mathcal{F} \), we may assume that \( \mathcal{F} = B(\mathcal{K}) \). We shall show that \( \alpha \otimes \text{id} \) extends to a normal, completely positive map of \( \mathcal{N} \otimes \mathcal{F} \) to \( \mathcal{M} \otimes \mathcal{F} \) (tensor product of von Neumann algebras).

With \( e_n \in \mathcal{F} \) a sequence of finite rank projections increasing to 1, clearly \( \alpha \otimes \text{id} \) is positive on \( \bigcup_n (\mathcal{N} \otimes e_n \mathcal{F} e_n) \) because \( \alpha \) is completely positive. So, since \( \bigcup_n e_n \mathcal{F} e_n \) is weakly dense in \( \mathcal{F} \), it suffices to show that \( \alpha \otimes \text{id} \) is normal. Now, if \( \varphi \) and \( \psi \) belongs to the predual of \( \mathcal{N} \) and \( \mathcal{F} \), the linear functional \( \varphi \otimes \psi \cdot \alpha \otimes \text{id} = \varphi \cdot \alpha \otimes \psi \) belongs to the predual of \( \mathcal{M} \otimes \mathcal{F} \). Thus the transpose of \( \alpha \otimes \text{id} \) (as linear bounded operator) maps a total set of normal linear functionals into normal linear functionals, so it is normal. \( \square \)

Let \( \mathcal{M} \) be a von Neumann algebra and \( \varphi \) a normal faithful state of \( \mathcal{M} \). Consider the identity \( \mathcal{M} - \mathcal{M} \) bimodule \( L^2(\mathcal{M}) \) and \( \xi_\varphi \in L^2(\mathcal{M})_+ \) the vector representative of \( \varphi \). The sesquilinear form on \( \mathcal{M} \) associated with \( \varphi \) is \( \langle m_1, m_2 \rangle_\varphi \equiv (\xi_\varphi, m_1 J_m_2 J_\xi_\varphi) \), with \( J \) the modular conjugation in \( L^2(\mathcal{M}) \).

**Proposition 2.5.** Let \( \alpha : \mathcal{N} \to \mathcal{M} \) be a normal, unital, completely positive map as above and \( \varphi \) a faithful normal state of \( \mathcal{M} \). There exists a \( \mathcal{N} - \mathcal{M} \) bimodule \( H_\alpha \), a unit vector \( \xi \in H_\alpha \), such that
\[
\langle m_1, m_2 \rangle_\varphi \equiv (\xi_\varphi, m_1 J_m_2 J_\xi_\varphi) = \langle \alpha(n), m \rangle_\varphi,
\]
with \( \xi \) is cyclic for \( \ell(\mathcal{N}) \vee r(\mathcal{M}) \) (with \( n_1 m = \ell(n) r(m) \xi \) and \( \ell, r \) the left and right actions on \( H_\alpha \)).

Let \( \beta : \mathcal{N} \to \mathcal{M} \) be a normal, unital, completely positive map and \( \mathcal{K} \) a \( \mathcal{N} - \mathcal{M} \) bimodule with cyclic vector \( \eta \in \mathcal{K} \). If \( \langle \alpha(n), m \rangle_\varphi = \langle \eta, \beta(n) \eta m \rangle \), there exists a \( \mathcal{N} - \mathcal{M} \) bimodule unitary equivalence \( U : \mathcal{K} \to H_\alpha \) such that \( U^* \alpha(\cdot) U = \beta \) and \( U \eta = \xi \).

**Proof.** The map
\[
n, m^o \in \mathcal{N} \times \mathcal{M}^o \mapsto \langle \alpha(n), m^o \rangle_\varphi
\]
is bilinear, so it gives a linear map \( \tilde{\varphi} \) on \( \mathcal{N} \odot \mathcal{M}^o \)
\[
\tilde{\varphi}(n \otimes m^o) = \langle \alpha(n), m^o \rangle_\varphi.
\]
Now
\[ \tilde{\varphi} = \varphi_\xi \cdot \pi \cdot \alpha \otimes \text{id} \]
on \mathcal{N} \odot \mathcal{M} \), where \( \pi \) is the representation of \( \mathcal{M} \odot \mathcal{M}' \) giving the identity \( \mathcal{M} \rightarrow \mathcal{M} \) bimodule \( L^2(\mathcal{M}) \) and \( \varphi_\xi = (\xi, \xi) \) with \( \xi \) the vector representative of \( \varphi \) in \( L^2_+ (\mathcal{M}) \); thus \( \tilde{\varphi} \) is positive being the composition of positive maps by Lemma 2.4.

Let \( \mathcal{H}_\alpha \) be the \( \mathcal{N} \rightarrow \mathcal{M} \) bimodule given by the GNS representation of \( \tilde{\varphi} \). Clearly eq. (13) holds with \( \xi \) the GNS vector of \( \tilde{\varphi} \).

The uniqueness of \( \mathcal{H}_\alpha \) follows easily by the uniqueness of the GNS representation. \( \square \)

Let \( \mathcal{N} \) and \( \mathcal{M} \) be von Neumann algebras and \( \rho : \mathcal{N} \rightarrow \mathcal{M} \) be a normal homomorphism. The \( \mathcal{N} \rightarrow \mathcal{M} \) bimodule \( L^2_\rho (\mathcal{M}) \) is defined as follows: \( L^2_\rho (\mathcal{M}) = L^2 (\mathcal{M}) \) as a Hilbert spaces, the right actions on \( L^2_\rho (\mathcal{M}) \) and \( L^2 (\mathcal{M}) \) are the same, while the left action of \( \mathcal{N} \) on \( L^2_\rho (\mathcal{M}) \) is twisted by \( \rho \), namely
\[ \ell_\rho (n) r_\rho (m) \eta = \rho (n) \eta m, \]
where \( \ell_\rho \) and \( r_\rho \) are the actions on \( L^2_\rho (\mathcal{M}) \) and the right hand side carries the actions on \( L^2 (\mathcal{M}) \).

It can be easily seen that \( L^2_\rho_1 (\mathcal{M}) \) is unitarily equivalent to \( L^2_\rho (\mathcal{M}) \) iff there exists a unitary \( u \in \mathcal{M} \) (acting on the left on \( L^2 (\mathcal{M}) \)) such that \( \rho_2 (n) = u \rho_1 (n) u^* \).

Similarly, if \( \theta : \mathcal{N} \rightarrow \mathcal{M} \) is a normal homomorphism, we can define the \( \mathcal{M} \rightarrow \mathcal{N} \) bimodule \( \theta L^2 (\mathcal{M}) \) by twisting the right \( \mathcal{M} \) action on \( L^2 (\mathcal{M}) \) by \( \theta \), and also \( \theta L^2_\rho (\mathcal{M}) \) is the \( \mathcal{N} \rightarrow \mathcal{M} \) bimodule where both left and the right actions on \( L^2 (\mathcal{M}) \) are twisted.

**Corollary 2.6.** Let \( \mathcal{N}, \mathcal{M} \) be properly infinite von Neumann algebras, \( \mathcal{H} \) a \( \mathcal{N} \rightarrow \mathcal{M} \) bimodule. Then \( \mathcal{H} \) is unitarily equivalent to \( L^2_\rho (\mathcal{M}) \) for some homomorphism \( \rho : \mathcal{N} \rightarrow \mathcal{M} \).

If we fix a faithful normal state \( \varphi \) of \( \mathcal{M} \), there is a canonical choice for \( \rho \).

Similarly, \( \mathcal{H} \) is unitarily equivalent to \( \theta L^2 (\mathcal{N}) \) for some homomorphism \( \theta : \mathcal{M} \rightarrow \mathcal{N} \); and \( \theta \) can be chosen canonically if we fix a normal faithful state of \( \mathcal{N} \).

**Proof.** Since \( r(\mathcal{M}) \) is properly infinite with properly infinite commutant \( r(\mathcal{M})' \supset \ell(\mathcal{N}) \), we may identify (up unitary equivalence) \( \mathcal{H} \) with \( L^2 (\mathcal{M}) \) as a right \( \mathcal{M} \)-module. Then the left action \( \ell \) of \( \mathcal{N} \) is our homomorphism \( \rho : \mathcal{N} \rightarrow \mathcal{M} \).

Now, if \( \varphi \) is given, we may construct \( L^2 (\mathcal{M}) \) by the GNS representation of \( \varphi \), with cyclic vector \( \xi_\varphi \). The the above identification \( r(\mathcal{M})' \simeq \mathcal{M} \) is given by composing the adjoint action of the modular conjugation of \( r(\mathcal{M}), \xi_\varphi \) with the *-operation, so the construction of \( \rho \) is canonical.

The last part is similarly proven. \( \square \)

With \( \rho : \mathcal{N} \rightarrow \mathcal{M} \) a homomorphism and \( L^2_\rho (\mathcal{M}) \) as above, the conjugate \( L^2_\bar{\rho} (\mathcal{M}) \) is then a \( \mathcal{M} \rightarrow \mathcal{N} \) bimodule that is unitarily equivalent to \( L^2_\rho (\mathcal{N}) \) for some homomorphism \( \bar{\rho} : \mathcal{M} \rightarrow \mathcal{N} \) by Corollary 2.6. \( \bar{\rho} \) is called the conjugate homomorphism of \( \rho \).

**Proposition 2.7.** \( L^2_\rho (\mathcal{M}) \simeq \rho L^2 (\mathcal{N}) \). More generally, we have
\[ \rho_1 L^2_\rho_2 (\mathcal{M}) \simeq L^2_{\rho_1} (\mathcal{N}) \rho_2 , \]
where \( \rho_1 : \mathcal{M}_1 \rightarrow \mathcal{N} \), \( \rho_2 : \mathcal{M}_2 \rightarrow \mathcal{M} \) and \( \rho : \mathcal{N} \rightarrow \mathcal{M} \) are homomorphisms.
Proof. The proof is essentially given in [31], it follows from the formula \( \bar{\rho} = \rho^{-1} \cdot \gamma_{\rho} \), where \( \gamma_{\rho} : M \to \rho(N) \) is the canonical endomorphism. \( \quad \Box \)

We define now the tensor product \([9, 42]\). Let \( M_1, M_2, M_3 \) be properly infinite von Neumann algebras, \( H \) a \( M_1 - M_2 \) bimodule, \( K \) a \( M_2 - M_3 \) bimodule. Then \( H \otimes K \) will be a \( M_1 - M_3 \) bimodule. We consider indeed the relative tensor product: we choose a faithful normal state \( \varphi \) on \( M_2 \) and define the tensor product \( H \otimes_{\varphi} K \) w.r.t. \( \varphi \), yet \( H \otimes_{\varphi} K \) does not depend on \( \varphi \) up to unitary equivalence. So \( H \otimes K \) is well defined up to unitary equivalence and \( H \otimes_{\varphi} K \) is a canonical representative for \( H \otimes K \).

We may identify \( H \) with \( L^2(M_2) \) as a right \( M_2 \)-module (canonically by the GNS representation of \( \varphi \) and \( K \) with \( L^2(M_2) \) as a left \( M_2 \)-module (again canonically by the GNS representation of \( \varphi \)). Then \( H \otimes K \) is \( L^2(M_2) \) where the \( M_1 \) action comes from the left module action of \( M_1 \) on \( H \) and the right \( M_3 \) action comes from the left action of \( M_3 \) on \( K \).

In other words, we make the bimodule identification \( H = L^2_{\rho_1}(M_2) \) and \( K = \rho_2 L^2(M_2) \) with homomorphisms \( \rho_1 : M_1 \to M_2 \) and \( \rho_2 : M_3 \to M_2 \), and then
\[
L^2_{\rho_1}(M_2) \otimes_{\rho_2} L^2(M_2) = \rho_2 L^2_{\rho_1}(M_2).
\]

It follows by Proposition 2.7 that
\[
L^2_{\rho_1}(M_2) \otimes L^2_{\rho_2}(M_3) \simeq L^2_{\rho_1 \rho_2}(M_3).
\]

We now define the tensor product of intertwiners. Let \( H \) and \( H_1 \) be \( N - M \) bimodules with actions \( \ell, r \) and \( \ell_1, r_1 \). Then \( T \in \text{Hom}(H, K) \) if \( T \in B(H, K) \) and \( T \) intertwines \( \ell \) with \( \ell_1 \) and \( r \) with \( r_1 \). Thus, with \( H = L^2_{\rho}(M) \), \( H = L^2_{\rho_1}(M) \),
\[
T \in \text{Hom}(L^2_{\rho}(M), L^2_{\rho_1}(M)) \iff T \in B(L^2(M)), T \in M & T \rho(n) = \rho_1(n)T, n \in N,
\]
where \( T \in M \) means that \( T \) belongs to \( M \) acting on the left on \( L^2(M) \). Similarly, with \( K = \theta L^2(M) \), \( K_1 = \theta_1 L^2(M) \) \( M - L \) bimodules and \( \theta : L \to M \) a homomorphism, we have
\[
S \in \text{Hom}(\theta L^2(M), \theta_1 L^2(M)) \iff S \in B(L^2(M)), S \in M' & S \theta(z) = \theta_1(z)S, z \in L,
\]
with \( M \) as above (i.e. \( S \) belongs to \( M \) acting on the right on \( L^2(M) \)).

Then \( T \otimes S \in \text{Hom}(H \otimes K, H_1 \otimes K_1) \) is given by
\[
(T \otimes S) \xi = T \xi S, \xi \in L^2(M),
\]
(left and right action on \( L^2(M) \), with the Hilbert space bimodule identifications \( H \otimes K = \theta L^2(M)_\rho \) and \( H_1 \otimes K_1 = \theta_1 L^2(M)_{\rho_1} \) (thus \( H \otimes K = H_1 \otimes K_1 = L^2(M) \) as Hilbert spaces).

The following proposition shows that the tensor product of intertwiners is a direct generalisation of the Doplicher-Haag-Roberts notion \([11]\), see \([31]\).

**Proposition 2.8.** Let \( \rho, \rho' : M_1 \to M_2 \), \( \theta, \theta' : M_2 \to M_3 \) be normal homomorphisms and
\[
T \in \text{Hom}(L^2_{\rho}(M_2), L^2_{\rho'}(M_2)), S \in \text{Hom}(L^2_{\theta}(M_3), L^2_{\theta'}(M_3)) \text{ intertwiners. Then } T \in M_2,
\]
\[
S \in M_3 	ext{ and }
\]
\[
T \otimes S = \theta(T)S = S \theta'(T)
\]
as elements of \( M_3 \).
Proof. This follows easily by using Proposition 2.7. □

We go back to Theorem 2.2 for a moment. With the notation therein, we may make an identification of the form \( \mathcal{H} = L^2_p(M_2) \), \( \mathcal{K} = _gL^2(M_2) \), thus \( \mathcal{H} \otimes \mathcal{K} = _gL^2_p(M_2) \). Since \( \Delta^{it} \equiv \Delta^{it}_H(\varphi_1|\varphi_2) \otimes \Delta^{it}_K(\varphi_2|\varphi_3) \) is a tensor product of intertwiners, it follows by property (9) that

\[
\Delta^{it} \rho(m_1) \Delta^{-it} = \rho(\sigma^{\varphi_1}_t(m_1)) , \quad \Delta^{it} \theta(m_3) \Delta^{-it} = \theta(\sigma^{\varphi_3}_t(m_3)) .
\]

This is a main implication of (a) in Theorem 2.2.

The above definition of tensor product is given for bimodules over properly infinite von Neumann algebras. The general case may be defined, up to unitary equivalence, by considering the ampliation of the bimodules (tensoring with a type \( I_\infty \) factor and reducing by minimal projections).

We now give a generalisation of Stinespring dilation theorem, that allows to dilate completely positive, normal maps between von Neumann algebras to a normal homomorphism, and show the uniqueness of the minimal dilation, cf. [9], see also [2].

Theorem 2.9. Let \( \alpha : N \to M \) be a normal, completely positive unital map between the properly infinite von Neumann algebras \( N, M \).

There exist an isometry \( v \in M \) and a homomorphism \( \rho : N \to M \) such that

\[
\alpha(x) = v^* \rho(x)v , \quad x \in N .
\]

Moreover the dilation pair \((\rho, v)\) can be chosen to be minimal: the support of \( \rho(N)v \) in \( \rho(N)vM \) is equal to 1 (i.e. \( \rho(N)v\mathcal{H} = \mathcal{H} \) with \( \mathcal{H} \) the underlying Hilbert space).

If \((\rho', v')\) is another minimal dilation pair for \( \alpha \), there exists a unique unitary \( u \in M \) such that

\[
\rho'(x) = u\rho(x)u^* , \quad v' = uv . \tag{15}
\]

Proof. Let \( \varphi \) be a faithful normal state of \( M \) and \( \mathcal{H}_\alpha \) the \( N - M \) bimodule associated with \( \alpha \) with cyclic vector \( \xi \in \mathcal{H}_\alpha \) by Proposition 2.5. By Proposition 2.6 we may identify \( \mathcal{H}_\alpha \) with \( L^2_p(M) \) with \( \rho : N \to M \) a homomorphism. Then eq. (13) reads as an equation in \( L^2(M) \):

\[
(\xi, \alpha(n)\xi m) = (\eta, \rho(n)\eta m)
\]

for some vector \( \eta \in L^2(M) \) cyclic for \( \rho(N)vM \).

The map \( v : \xi m \mapsto \eta m, m \in M, \) is isometric and its closure is an isometry \( v : L^2(M) \to L^2(M) \) with final projection the orthogonal projection \( p \) onto \( \eta M \). As \( v \) commutes with the right action of \( M \) on \( L^2(M) \), we have \( v \in M \) (acting on the left on \( L^2(M) \)) and

\[
(\xi, v^* \rho(n)v\xi m) = (\eta, \rho(n)\eta m) = (\xi, \alpha(n)\xi m),
\]

thus \( \alpha = v^* \rho(\cdot)v \) by Proposition 2.5.

Now, the left support of \( v \) is the left support of \( \rho(N)p \), namely the projection onto the closure of \( \rho(N)\eta M \), which is equal to 1 by the cyclicity of \( \eta \); thus the pair \((\rho, v)\) is minimal.

The uniqueness of the minimal pair \((\rho, v)\) stated in (15) follows by Proposition 2.5. To show that the choice of the unitary \( u \in M \) in (15) is unique, note that the equation

\[
u\rho(n)v = \rho'(n)uv = \rho'(n)v'
\]

uniquely determines \( u \) by the minimality assumption. □
Corollary 2.10. Let $\mathfrak{A}$ be a unital $C^*$-algebra and $\Phi : \mathfrak{A} \to \mathcal{M}$ a completely positive, unital map from $\mathfrak{A}$ into a properly infinite von Neumann algebras $\mathcal{M} \subset B(\mathcal{H})$.

There exist an isometry $v \in \mathcal{M}$ and a representation $\rho$ of $\mathfrak{A}$ on $\mathcal{H}$ with $\rho(\mathfrak{A}) \subset \mathcal{M}$ such that

$$\Phi(x) = v^* \rho(x)v, \quad x \in \mathfrak{A}.$$ 

Proof. Let $\psi$ be a faithful normal state of $\mathcal{M}$ and $\varphi \equiv \psi \cdot \Phi$ its pullback to a state of $\mathfrak{A}$. Then $\Phi$ factors through the GNS representation of $\mathfrak{A}$ given by $\varphi$:

$$\begin{array}{cc}
\mathfrak{A} & \overset{\Phi}{\longrightarrow} \mathcal{M} \\
\pi_{\varphi} \downarrow & \downarrow \Phi_0 \\
\mathcal{N} & \longrightarrow \mathcal{M}
\end{array}$$

with $\mathcal{N} \equiv \pi_{\varphi}(\mathfrak{A})'$ and $\Phi_0 : \mathcal{N} \to \mathcal{M}$ a completely positive map. Indeed if $a \in \mathfrak{A}$ we have

$$\pi_{\varphi}(a) = 0 \implies \varphi(a^*a) = 0 \implies \psi \cdot \Phi(a^*a) = 0 \implies \Phi(a^*a) = 0 \implies \Phi(a) = 0$$

since $\Phi(a^*)\Phi(a) \leq \Phi(a^*a)$. As $\psi \cdot \Phi_0$ is normal on $\mathcal{N}$, it follows easily that $\Phi_0$ is normal.

We now apply Theorem 2.9 to $\Phi_0$. If $\mathcal{N}$ is properly infinite we get immediately our statement. In general, we may consider $\mathcal{N} \otimes \mathcal{F}$, with $\mathcal{F}$ a type $I_{\infty}$ factor, and a faithful normal conditional expectation $\varepsilon : \mathcal{N} \otimes \mathcal{F} \to \mathcal{N}$, apply Theorem 2.9 to $\Phi_0 \cdot \varepsilon$ and read the formula for $\Phi_0 = \Phi_0 \cdot \varepsilon|_{\mathcal{N}}$. \qed

The following corollary extends to the infinite-dimensional case the known construction of Kraus operators.

Corollary 2.11. Let $\alpha : \mathcal{M} \to \mathcal{M}$ be a completely positive, normal, unital map with $\mathcal{M}$ a type $I_{\infty}$ factor (i.e. $\mathcal{M}$ is a von Neumann algebra isomorphic to $B(\mathcal{H})$, $\dim \mathcal{H} = \infty$). There exists a sequence of elements $T_i \in \mathcal{M}$ with $\sum_i T_i T_i^* = 1$ (Kraus operators) such that

$$\alpha(m) = \sum_i T_i m T_i^*. \quad m \in \mathcal{M}.$$ 

Proof. Write $\alpha = v^* \rho(\cdot)v$ by Theorem 2.9, with $\rho$ an endomorphism of $\mathcal{M}$ and $v \in \mathcal{M}$ an isometry. As shown in [29], every endomorphism of a type $I$ factor is inner, namely there exists a sequence of isometries $v_i \in \mathcal{M}$ with $\sum_i v_i v_i^* = 1$ such that

$$\rho(m) = \sum_i v_i m v_i^*, \quad m \in \mathcal{M}.$$ 

Thus

$$\alpha(m) = v \rho(m) v^* = \sum_i v^* v_i m v_i^* v = \sum_i T_i m T_i^*, \quad m \in \mathcal{M},$$

with $T_i = v^* v_i$. \qed

The homomorphism $\rho : \mathcal{N} \to \mathcal{M}$ associated with $\alpha$ in Theorem 2.9 is called the (minimal) dilation of $\alpha$ (defined up to inner automorphisms) and $(\rho, v)$ a minimal dilation pair for $\alpha$. 

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Corollary 2.12. Let $\alpha : N \to M$ be a normal, completely positive, unital map, with $M$ properly infinite. With $\varphi$ a normal faithful state of $M$, the bimodule $H_\alpha$ associated with $\alpha$ is unitarily equivalent to $L^2_\rho(M)$, with $\rho$ the minimal dilation of $\alpha$. In particular, $H_\alpha$ does not depend on $\varphi$ up to unitary equivalence.

Proof. Immediate by Theorem 2.9 and its proof. □

Let $\alpha : M_1 \to M_2$ and $\beta : M_2 \to M_3$ be completely positive, normal, unital maps of properly infinite von Neumann algebras and $\rho, \theta$ the corresponding minimal dilations. Then $\theta \rho$ is a dilation homomorphism of $\beta \alpha$. Indeed, write $\alpha = v^* \rho(\cdot) v$ and $\beta = w^* \theta(\cdot) w$. Then $\beta \alpha = w^* \theta(v^*) \theta(\cdot) \theta(v) w$, so $\theta \rho$ is a dilation homomorphism of $\beta \alpha$. In general, $\beta \alpha \theta(v) w$ could fail to be minimal.

We now define the transpose of a completely positive map, see [38].

Proposition 2.13. Let $N, M$ be von Neumann algebras and $\alpha : N \to M$ a normal, completely positive, unital, faithful map. Let $\varphi$ a faithful normal state of $M$ and consider the state $\psi = \varphi \cdot \alpha$ on $N$.

There exists a unique normal, completely positive, unital map $\alpha' : M \to N$ such that

$$\langle \alpha(n), m \rangle_\varphi = \langle \alpha'(m), n \rangle_\psi.$$  \hspace{1cm} (16)

$\alpha'$ is called the transpose of $\alpha$.

Proof. For a fixed positive $m \in M$, the linear functional $n \mapsto \langle \alpha(n), m \rangle_\varphi = (\xi_\varphi, \alpha(n) \xi_\varphi) = \langle m, \alpha'(m) \rangle_\psi$, so there is a positive element $\alpha'(m) \in N$, with $\|\alpha'(m)\| \leq \|m\|$, such that $\langle \alpha(n), m \rangle_\varphi = \langle \alpha'(m), n \rangle_\psi$; then $\alpha'$ is defined by linearity and satisfies eq. (16), so it is normal.

To see that $\alpha'$ is completely positive, note that the transpose of $\alpha \otimes \text{id} : N \otimes \text{Mat}_k(\mathbb{C}) \to M \otimes \text{Mat}_k(\mathbb{C})$ with respect to the state $\varphi \otimes \tau$ ($\tau$ the normalised trace) is of $\alpha' \otimes \text{id}$, which is therefore positive. □

Proposition 2.14. With $\alpha$ and $\alpha'$ as above, we have $H_{\alpha'} = \overline{H}_{\alpha}$, namely the bimodule associate with $\alpha'$ is unitarily equivalent to the conjugate of the bimodule associated with $\alpha$.

In particular, if $N, M$ are properly infinite, the dilation $\overline{\rho}$ of $\alpha'$ is conjugate to the dilation of $\alpha$.

Proof. By eq. (16) the natural state $\overline{\varphi}$ and $\overline{\psi}$ on $N \otimes M^o$ and $M \otimes N^o$ as in eq. (14) are related by $\overline{\varphi}(n \otimes m^o) = \overline{\psi}(m \otimes n^o)$. The statement follows by eq. (7). □

Let $\alpha : N \to M$ be a normal, completely positive unital map. We shall say that $\alpha$ is covariant if there exist faithful normal positive functional $\varphi$ on $N$ and $\psi$ on $M$ such that

$$\alpha(\sigma_i^\varphi(n)) = \sigma_i^\psi(\alpha(n)), \quad n \in N.$$  \hspace{1cm} (17)

Next immediate proposition shows that the covariance condition naturally extends the trace preserving condition to infinite systems.
Proposition 2.15. Let \( \mathcal{N}, \mathcal{M} \) be finite dimensional and \( \alpha : \mathcal{N} \to \mathcal{M} \) a completely positive, unital map and assume that \( \mathcal{M} \) is the algebra generated by \( \alpha(\mathcal{N}) \).

Then \( \alpha \) is covariant if it is trace preserving, namely \( \varphi = \psi \cdot \alpha \) with \( \varphi \) and \( \psi \) tracial states of \( \mathcal{N} \) and \( \mathcal{M} \).

Proof. The modular group \( \sigma^\varphi \) is the identity iff \( \varphi \) is a trace. Thus \( \alpha \) is covariant if \( \alpha \) is trace preserving as (17) trivially holds if \( \varphi \) and \( \psi \) are tracial.

\[ \square \]

Lemma 2.16. Let \( \mathcal{M} \) be a von Neumann algebra, \( \psi \) a faithful normal positive functional of \( \mathcal{M} \) and \( v \in \mathcal{M} \) an isometry. With \( \psi' \) the positive linear functional on \( \mathcal{M} \) given by

\[ \psi'(m) = \psi(v^*mv) + \psi((1-p)m(1-p)), \]

we have

\[ \sigma_t^\psi(v) = u_t^* v, \quad t \in \mathbb{R}, \]

where \( u_t = (D\psi : D\psi)_t \). Moreover \( \psi'_v = \psi \) with \( \psi'_v(m) \equiv \psi'(vmv^*) \).

Proof. Let \( \psi' \) be the normal positive linear functional on \( \mathcal{M} \) given (18). Let \( \theta : \mathcal{M} \to \mathcal{M}_\rho \) the isomorphism \( m \mapsto vmv^* \), then \( \psi = \psi'_v = \psi'|_{\mathcal{M}_\rho} \cdot \theta \), so

\[ \sigma^\psi = \theta^{-1} \cdot \sigma^{\psi'} \cdot \theta, \]

As \( \sigma^{\psi'}|_{\mathcal{M}_\rho} = \sigma^{\psi}|_{\mathcal{M}_\rho} \), we have

\[ \sigma^\psi = \theta^{-1} \cdot \sigma^{\psi'} \cdot \theta, \]

namely

\[ \sigma_t^\psi(m) = v^*\sigma_t^{\psi'}(v)\sigma_t^\psi(m)\sigma_t^{\psi'}(v^*)v = u_t^*\sigma_t^{\psi'}(m)u_t, \]

where \( u_t = \sigma_t^{\psi'}(v^*)v = (D\psi : D\psi)_t \). We thus have \( 1 = \sigma_t^{\psi}(v^*)u_t^* v \), namely (19) holds.

\[ \square \]

Proposition 2.17. Let \( \alpha : \mathcal{N} \to \mathcal{M} \) be as above and \( \alpha = v^* \rho(\cdot)v \) its dilation. If \( \alpha \) is covariant then \( \rho \) is covariant.

Proof. Suppose \( \alpha \) is covariant and let \( \varphi \) and \( \psi \) be as in eq. (17). We have:

\[ v^* \rho(\sigma_t^\psi(n))v = \sigma_t^\psi(v^*\rho(n)v) = \sigma_t^\psi(v^*)\sigma_t^\psi(\rho(n))\sigma_t^\psi(v), \]

thus

\[ v^* \rho(n)v = \sigma_t^\psi(v^*)\rho_t(n)\sigma_t^\psi(v) \]

with \( \rho_t \equiv \sigma_t^\psi \rho \sigma_t^\psi \).

Let \( \psi' \) be the positive linear functional given by (19). Then \( \sigma_t^\psi(v) = u_t^* v \) with \( u_t = (D\psi : D\psi)_t \). We have

\[ v^* \rho(n)v = \sigma_t^\psi(v^*)\rho_t(n)\sigma_t^\psi(v) = v^* u_t \rho_t(n) u_t^* v = v^* \sigma_t^{\psi'} \rho \sigma_t^{\psi'}(n)v, \]

thus \( \rho = \sigma_t^{\psi'} \rho \sigma_t^{\psi'} \), namely \( \rho \) is covariant.

\[ \square \]
Corollary 2.18. Let $\mathcal{N}, \mathcal{M}$ be von Neumann algebras with finite dimensional centers, $\alpha : \mathcal{N} \to \mathcal{M}$ a normal, completely positive, unital map and $\mathcal{H}_\alpha$ the associated bimodule. The following are equivalent:

(a) $\alpha$ has finite index,

(b) $\text{Hom}(\mathcal{H}_\alpha, \mathcal{H}_\alpha)$ is finite dimensional and both $\alpha$ and $\alpha'$ have covariant dilation.

Proof. With $\rho$ and $\rho'$ the dilation homomorphisms of $\alpha$ and $\alpha'$, we have that $\rho'$ is conjugate to $\rho$. By Takesaki theorem $\rho$ (resp. $\rho'$) is covariant iff there exists a normal faithful conditional expectation from $\mathcal{M}$ onto $\rho(\mathcal{N})$ (resp. from $\mathcal{N}$ onto $\rho'(\mathcal{M})$). The result then follows as in [31].

3 Quantum channels

Let $\mathcal{N}, \mathcal{M}$ be von Neumann algebras with finite dimensional centers $Z(\mathcal{N}), Z(\mathcal{M})$.

The index of a completely positive, normal, unital map $\alpha : \mathcal{N} \to \mathcal{M}$ is now defined by

$$\text{Ind}(\alpha) \equiv \text{Ind}(\mathcal{H}_\alpha),$$

with $\mathcal{H}_\alpha$ the bimodule associated with $\alpha$.

Therefore, if $\mathcal{N}$ and $\mathcal{M}$ are properly infinite,

$$\text{Ind}(\alpha) = \text{Ind}(\rho),$$

where $\rho : \mathcal{N} \to \mathcal{M}$ is the dilation homomorphism of $\alpha$ and $\text{Ind}(\rho) = [\mathcal{M} : \rho(\mathcal{M})].$

By a quantum channel $\alpha : \mathcal{N} \to \mathcal{M}$ we shall mean a completely positive, normal, unital map $\alpha$ with finite index. Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a quantum channel and $\varphi_{\text{in}}$ a faithful normal state of $\mathcal{M}$. We consider $\varphi_{\text{in}}$ as an input state for $\alpha$. The output state on $\mathcal{N}$ is then defined by

$$\varphi_{\text{out}} = \varphi_{\text{in}} \cdot \alpha.$$

With $\mathcal{H}_\alpha$ the $\mathcal{N} - \mathcal{M}$ bimodule associated with $\alpha$ and $\xi \in \mathcal{H}_\alpha$ the cyclic vector associated with $\varphi_{\text{in}}$ as in Prop. 2.5, we have

$$(\xi, m) = \varphi_{\text{in}}(m), \quad (\xi, n\xi) = \varphi_{\text{out}}(n),$$

namely

$$\varphi_{\text{in}} = \varphi_{\xi} \cdot r, \quad \varphi_{\text{out}} = \varphi_{\xi} \cdot \ell,$$

where $\varphi_{\xi}$ is the vector state associated with $\xi$ and $r, \ell$ the left and right actions of $\mathcal{N}, \mathcal{M}$ on $\mathcal{H}_\alpha$ that we may assume to be faithful.

Clearly the transpose map $\alpha'$ interchanges $\varphi_{\text{in}}$ with $\varphi_{\text{out}}$:

$$\varphi_{\text{in}} = \varphi_{\text{out}} \cdot \alpha'.$$

Let $\varepsilon : r(\mathcal{M})' \to \ell(\mathcal{N})$ be the minimal expectation. Then

$$\Phi = \ell^{-1} \cdot \varepsilon : r(\mathcal{M})' \to \mathcal{N}$$
is a quantum channel from \(r(\mathcal{M})' \simeq \mathcal{M}\) to \(\mathcal{N}\) called the left inverse of \(\alpha\). Notice that \(\Phi \cdot \ell\) is the identity on \(\mathcal{N}\).

We define the modular operator \(\Delta_{\alpha,\varphi_{\text{in}}}\) of \(\alpha\) with respect to the initial state \(\varphi_{\text{in}}\) as the modular operator of the associated \(\mathcal{N} - \mathcal{M}\) bimodule \(\mathcal{H}_\alpha\) with respect to the initial and the final state:

\[
\Delta_{\alpha,\varphi_{\text{in}}} = \Delta_{\mathcal{H}_\alpha}(\varphi_{\text{in}}|\varphi_{\text{out}}) = d\varphi_{\text{out}} \cdot \ell^{-1} \cdot \varepsilon / d\varphi_{\text{in}} \cdot r^{-1},
\]

thus \(\Delta_{\alpha,\varphi_{\text{in}}} = d(\varphi_\xi \cdot \varepsilon |_{r(\mathcal{M})'}) / d(\varphi_\xi |_{r(\mathcal{M})})\). Then \(\Delta_{\alpha,\varphi_{\text{in}}}\) is a positive, non-singular selfadjoint operator on \(\mathcal{H}_\alpha\) and we have

\[
\Delta_{\alpha,\varphi_{\text{in}}}^\ell(n)\Delta_{\alpha,\varphi_{\text{in}}}^{-\ell}(n) = \ell(\sigma^\text{out}_t(n)) , \quad \Delta_{\alpha,\varphi_{\text{in}}}^\ell(m)\Delta_{\alpha,\varphi_{\text{in}}}^{-\ell}(m) = r(\sigma^\text{in}_t(m)),
\]

where \(\sigma^\text{in/out}\) is the modular group of \(\mathcal{N}/\mathcal{M}\) w.r.t. \(\varphi^\text{in/out}\).

The entropy of a quantum channel \(\alpha\) in the initial state \(\varphi_{\text{in}}\) is now defined as

\[
S_{\alpha,\varphi_{\text{in}}} \equiv - (\xi, \log \Delta_{\alpha,\varphi_{\text{in}}} \xi).
\]

Let’s write the above formulas in the homorphism case. Assume that \(\mathcal{N}, \mathcal{M}\) are properly infinite and identify the bimodule \(\mathcal{H}_\alpha\) with \(L^2(\mathcal{M})\), with \(\rho: \mathcal{N} \to \mathcal{M}\) the dilation of \(\alpha\). So now \(\ell = \rho\), the \(\rho\)-twisted left action of \(\mathcal{N}\) on \(L^2(\mathcal{M})\), and \(r\) is the right action of \(\mathcal{M}\) on \(L^2(\mathcal{M})\). We consider the case \(\alpha = \rho\). We choose a faithful normal state \(\varphi_{\text{in}} = \omega\) on \(\mathcal{M}\); then \(\varphi_{\text{out}} = \omega \cdot \rho\) is the output state on \(\mathcal{N}\). With \(\varepsilon: \mathcal{M} \to \rho(\mathcal{N})\) the minimal expectation, \(\Phi = \rho^{-1} \cdot \varepsilon: \mathcal{M} \to \mathcal{N}\) be the left inverse of \(\rho\) and we have

\[
\Delta_{\alpha,\varphi_{\text{in}}} = \Delta(\varphi_{\text{out}}\Phi|\omega) = \Delta(\omega \cdot \varepsilon|\omega) = \Delta_{\hat{\xi},\hat{\xi}}\]

where \(\xi, \hat{\xi}\) are the vector representatives in \(L^2(\mathcal{M})_+\) of the states \(\omega\) and \(\omega \cdot \varepsilon\) on \(\mathcal{M}\) and \(\Delta_{\hat{\xi},\hat{\xi}}\) is the relative modular operator of \(\hat{\xi}, \hat{\xi}\). Thus

\[
S_{\rho,\omega} = - (\xi, \log \Delta_{\hat{\xi},\hat{\xi}}\xi)
\]

is indeed Araki’s relative entropy. In particular

\[
S_{\alpha,\varphi_{\text{in}}} \geq 0.
\]

This positivity of the entropy holds for any quantum channel \(\alpha\) by similar arguments.

### 3.1 The physical Hamiltonian

The modular group \(t \mapsto \sigma^\omega_t\) of a von Neumann algebra \(\mathcal{M}\) is an intrinsic dynamics associated with a faithful normal state \(\omega\) of \(\mathcal{M}\). However the evolution parameter \(t\) is canonical up to a scaling in the sense it is uniquely determined when we fix the inverse temperature \(\beta = 1/kT\), namely \(t \mapsto \tau_t = \sigma^\omega_{-\beta t}\) is the unique one parameter automorphism group of \(\mathcal{M}\) that satisfies the \(\beta\)-KMS condition with respect to \(\omega\) as stated in (4) of Section 1.3.3.

The modular unitary group \(t \mapsto \Delta^\omega_t\) is a canonical implementation of \(\sigma^\omega\) on \(L^2(\mathcal{M})\) (with respect to the vector representative of \(\omega\) in \(L^2(\mathcal{M})_+\)):

\[
\Delta^\omega_t m \Delta^{-\omega}_{-t} = \sigma^\omega_t (m), \quad m \in \mathcal{M}.
\]
Analogously, the modular group of a quantum channel $\alpha : \mathcal{N} \to \mathcal{M}$ is an intrinsic dynamics associated with $\alpha$ and an initial (faithful, normal) state $\varphi_{\text{in}}$ of $\mathcal{M}$; and the choice of the inverse temperature $\beta$ uniquely fixes the scaling parameter.

The modular unitary group $\Delta(\varphi_{\text{in}}|\varphi_{\text{out}})_{it}$ of $\alpha$ is a canonical implementation of the modular automorphism group $\sigma_{t}^{\text{in}} \odot \sigma_{t}^{\text{out}}$ on $\mathcal{H}_{\alpha}$, namely

$$\Delta(\varphi_{\text{in}}|\varphi_{\text{out}})_{it} \equiv \ell^{it}(n)\Delta(\varphi_{\text{in}}|\varphi_{\text{out}})^{-it} \equiv \ell(\sigma_{t}^{\text{out}}(n)),$$  \hspace{1cm} (23)

$$\Delta(\varphi_{\text{in}}|\varphi_{\text{out}})_{it} \equiv r^{it}(m)\Delta(\varphi_{\text{in}}|\varphi_{\text{out}})^{-it} \equiv r(\sigma_{t}^{\text{in}}(m)),$$  \hspace{1cm} (24)

where $\varphi_{\text{out}} = \varphi_{\text{in}} \cdot \alpha$.

Now, there exists another canonical implementation of $\sigma_{t}^{\text{in}} \odot \sigma_{t}^{\text{out}}$ on $\mathcal{H}_{\alpha}$. This is determined through the tensor categorical structure provided by the family of finite index bimodules. It is defined by

$$U_{t}^{\alpha}(\varphi_{\text{in}}|\varphi_{\text{out}}) \equiv U_{t}^{\mathcal{H}_{\alpha}}(\varphi_{\text{in}}|\varphi_{\text{out}})$$

and is the unique one-parameter unitary group on $\mathcal{H}_{\alpha}$ that implements $\sigma_{t}^{\text{in}} \odot \sigma_{t}^{\text{out}}$ and naturally behaves under the tensor categorical operations; in particular

$$U_{t}^{\alpha}(\varphi_{\text{out}}|\varphi_{\text{in}}) = \overline{U_{t}^{\alpha}(\varphi_{\text{in}}|\varphi_{\text{out}})}.$$

By Theorem 2.2 we have

$$U_{t}^{\alpha}(\varphi_{\text{in}}|\varphi_{\text{out}}) = \Delta^{it}_{\alpha,\varphi_{\text{in}}}D(\alpha)^{it},$$

where $D(\alpha)$ is the matrix dimension of $\mathcal{H}_{\alpha}$ [15].

We call $U_{\alpha}$ the physical unitary evolution associated with $\alpha$.

More generally, when we consider the $\beta$-rescaling of the modular parameter, we get the physical unitary evolution $t \mapsto U_{\alpha}(-\beta^{-1}t)$ at inverse temperature $\beta$. Its self-adjoint generator on $\mathcal{H}_{\alpha}$ is the physical Hamiltonian $H_{\alpha,\varphi_{\text{in}}}$ associated with $\alpha$ and the state $\varphi_{\text{in}}$ at inverse temperature $\beta$.

By the above discussion, we have

$$\beta H_{\alpha} = -\log \Delta_{\alpha,\varphi_{\text{in}}} - \log D(\alpha).$$  \hspace{1cm} (25)

The mean energy $E_{\alpha,\varphi_{\text{in}}}$ of $\alpha$ is defined by

$$E_{\alpha,\varphi_{\text{in}}} \equiv \langle \xi, H_{\alpha,\varphi_{\text{in}}} \xi \rangle,$$

with $\xi$ the cyclic vector in $\mathcal{H}_{\alpha}$ associated with $\varphi_{\text{in}}$.

Then the (incremental) free energy $F_{\alpha,\varphi_{\text{in}}}$ is defined by the thermodynamical relation

$$F_{\alpha,\varphi_{\text{in}}} = E_{\alpha,\varphi_{\text{in}}} - \beta^{-1}S_{\alpha,\varphi_{\text{in}}},$$  \hspace{1cm} (26)

where the entropy $S_{\alpha,\varphi_{\text{in}}}$ of $\alpha$ in the initial state $\varphi_{\text{in}}$ is defined as above as a relative entropy:

$$S_{\alpha,\varphi_{\text{in}}} = -\langle \xi, \log \Delta_{\alpha,\varphi_{\text{in}}} \xi \rangle.$$  

It can be shown that the free energy is also expressed by the relative partition formula:

$$F_{\alpha,\varphi_{\text{in}}} \equiv -\beta^{-1}\log \langle \xi, e^{-\beta H_{\alpha,\varphi_{\text{in}}}} \xi \rangle,$$

(cf. [33]).
Theorem 3.1. We have:

\[ F_{\alpha,\varphi_{\text{in}}} = -\beta^{-1}(\xi, \log D(\alpha)\xi) \]  

(27)

Proof. By evaluating both sides of equation (25) on the vector state associated with \( \xi \) we have

\[ \beta(\xi, H_{\alpha}\xi) = -(\xi, \log \Delta_{\alpha,\varphi_{\text{in}}}\xi) - (\xi, \log D(\alpha)\xi) \]

namely

\[ E_{\alpha,\varphi_{\text{in}}} - \beta^{-1}S_{\alpha,\varphi_{\text{in}}} = -\beta^{-1}(\xi, \log D(\alpha)\xi) \]

thus eq. (27) follows by the thermodynamical relation (26). \( \square \)

Since \( D(\alpha) \geq 1 \), as a consequence of Theorem 3.1 we have the negativity of the incremental free energy

\[ -F_{\alpha,\varphi_{\text{in}}} \geq 0 \]

Notice that, if \( D(\alpha) \) is a scalar, we have

\[ F_{\alpha,\varphi_{\text{in}}} = -\beta^{-1}\log d(\alpha) \]

which is independent of \( \varphi_{\text{in}} \); this holds, in particular, in the factor case.

In general, we define the free energy of a quantum channel \( \alpha \) as

\[ F_{\alpha} \equiv \inf F_{\alpha,\varphi_{\text{in}}} \]

infimum over all (faithful, normal) initial states \( \varphi_{\text{in}} \). Possibly, \( -F_{\alpha} \) could be named free energy (or absolute free energy).

Corollary 3.2. We have

\[ -F_{\alpha} = \beta^{-1}\log D_{\text{max}}(\alpha) \]

where \( D_{\text{max}}(\alpha) \) is the largest entry of the matrix dimension \( D(\alpha) \).

Proof. \( D(\alpha) \) acts on \( \mathcal{H}_{\alpha} \) as a diagonal operator on \( \mathcal{H}_{\alpha} \) with eigenvectors \( \{d_{ij}\} \), the dimension of the factorial components of \( \mathcal{H}_{\alpha} \). Thus the supremum of \( (\xi, \log D(\alpha)\xi) \) over all unit vectors \( \xi \) is equal to the largest entry \( D_{\text{max}}(\alpha) \) of \( D(\alpha) \). \( \square \)

By Jones’ theorem [22], the dimension (i.e. square root of the index) \( d \) of a subfactor is quantised: if \( d < \infty \) then

\[ d = 2\cos(\pi/n) \quad n = 3, 4, \ldots \quad \text{or} \quad d \geq 2 \]

in particular

\[ d \neq 1 \implies d \geq \sqrt{2} \]

so we have the following corollary.

Corollary 3.3. If the free energy \( F_{\alpha} \) is non-zero, then

\[ -F_{\alpha} \geq \frac{1}{2}kT \log 2 \]

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The above corollary gives a general lower bound for the opposite of the free energy \(-F_\alpha\). It is half of the Landauer bound derived in [27].

If \(\alpha\) is a quantum channel between finite dimensional quantum systems \(\mathcal{N}, \mathcal{M}\) or, more generally, if \(\mathcal{N}, \mathcal{M}\) are type I von Neumann algebras, then \(D_{\text{max}}(\alpha)\) is a positive integer thus (unless \(\alpha\) is reversible and \(D(\alpha) = 1\))

\[-F_\alpha \geq kT \log 2,\]  

which is indeed the lower bound derived by Landauer in a finite dimensional context.

In Quantum Field Theory on the \(n + 1\) Minkowski spacetime, \(n \geq 2\), a DHR charge \(\alpha\) has Fermi-Bose statistics and the dimension of \(\alpha\) is a positive integer [11]. So, also in this case, the lower bound (28) holds, if \(\alpha\) is irreversible.

4 Final comments

We end up with a few comments.

The vacuum geometric modular action in QFT is related with the Hawking-Unruh effect, see [43] for the Schwarzschild black hole case [43]. The evolution parameter of the modular group is proportional to the proper time of the geodesic observer. Our work here gives, in particular, a further viewpoint concerning this evolution in a charged state [33, 34].

One may read our present paper also in relation with the intrinsic, modular interpretation of time proposed by Connes and Rovelli in [10].

Landauer principle has been recently considered within the \(C^*-\)algebraic context in [21].

As mentioned, Jones’ index is related to entropy [40, 33]; indeed it appears in different quantum information contexts, see [37, 14].

One may wonder about possible relations with other forms of entropy in Quantum Field Theory, entanglement entropy in particular. We refer to [39, 20] for recent Operator Algebraic analyses on entanglement entropy in QFT.

5 Outlook

Our work is going to be naturally supplemented by two forthcoming papers.

The paper [15] concerns the mathematical methods underlying our analysis, pointing to a clarification about the notion of dimension for bimodules over von Neumann algebras with non-trivial, finite dimensional centers (for related analysis in this context, see [16, 13] and refs. therein). In particular, it will discuss the functoriality properties of the matrix dimension. The study of the non trivial center case is motivated in order to deal with a general quantum systems with a non trivial classical component too.

The paper [36] will discuss, among other things, the Bekenstein bound in the context of black hole information theory. The discussion made in [33], and the results therein, together with a due interpretation, naturally give us a rigorous derivation of this bound. Our formula for the incremental free energy can be read in this framework. A quite similar discussion can be found in a more recent paper [6], which is however heuristic (as local von Neumann algebras are of type III).

Acknowledgements. We would like to thank the Isaac Newton Institute for Mathematical Sciences in Cambridge and the Simons Foundation for the hospitality during the program.
“Operator algebras: subfactors and their applications” in January-February and May-June 2017, where part of this work was carried on and presented at the June workshop. We also thank Luca Giorgetti for several comments.

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