Optimized quantum state transfer through an XY spin chain

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Quantum state transfer along a one-dimensional spin chain has become a fundamental ingredient for quantum communication between distant nodes in a quantum network. We study the average fidelity of quantum state transfer (QST) along a XY spin chain by adjusting the basis identification between the first spin and the last spin. In a proper choice of the basis identification, we find that the QST fidelity depends only on the average parity of the initial state linearly. We propose a simple scheme to adjusting the basis identification to optimize the average fidelity such that it depends linearly on the absolute value of the average parity. In the case that the absolute value of the average parity is 1 we prove that the fidelity takes the maximum at any time over arbitrary initial state and basis identification.

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Introduction. — A quantum wire that builds the communication channel between distant nodes is a fundamental ingredient in a quantum network. The studies of a spin chain as a quantum wire are pioneered by Bose 1, where Bose showed that the high fidelity of state transfer could be achieved through a long unmodulated spin chain. Along an unmodulated spin chain, the perfect state transfer is possible only when the length of the spin chain is less than 4. It is shown in Refs. 2–5 that the perfect state transfer along an arbitrary long spin chain can be achieved by modulating the coupling strengths. Along this direction, an experiment in the framework of photonic lattices is reported to simulate the modulating coupling in Ref. 6. The schemes that use the spin chain without modulated coupling parameters in the limit of very weak endpoint couplings are discussed in Refs. 7–15, and the optimization of one or two weak endpoint couplings is further investigated in Refs. 16–18. It is also shown that it is possible to achieve perfect state transfer in a modulated spin chain without initialization in Refs. 19, 20.

Recently Godsil et al 21 proves a beautiful result for QST along an XX spin chain in the single excitation condition: XX spin chains can permit QST with a fidelity arbitrarily closed to 1, if and only if the number of nodes is \( N = p - 1, 2p - 1 \), where \( p \) is a prime, or \( N = 2^m - 1 \). However, numerical results shows that the time to achieve a pretty good fidelity is very long if \( N \) is large.

All the above results motivate us to explore a more general problem: For any given XY spin chain, what is the maximal QST fidelity it can takes at a given time \( t \)? Notice that there exists two factors affect the fidelity, one is the initial state of the spins except the sender, the other is the basis identification between the first spin and the last spin 22, 23. In other words, the central task is how to optimally exploit an XY spin chain to QST at any given time \( t \).

The article is organized as follows. First, we will introduce the model and explain the problem to be solved. Then we will show the dynamics of the average fidelity of QST relates with only the dynamics of observables for the output spin in the Heisenberg picture. Next we will study the dynamics of these observables in the Heisenberg picture, where a closed form is found for a general XY spin chain. Form the closed forms, we obtain the general relation between the fidelity and the parity of the initial state. Then the Laplace method is used to solve the Heisenberg equation to obtain the dynamics of the fidelity. Finally we propose a simple scheme to optimize the fidelity for arbitrary initial states at any given time.

Model and problem.— We consider a spin chain consisted of \( N \) qubits, modeled by the XY Hamiltonian:

\[
H = \sum_{i=1}^{N-1} J_i X_i X_{i+1} + K_i Y_i Y_{i+1},
\]

where \( X_i, Y_i \) are the Pauli matrices of spin \( i \), \( J_i \) and \( K_i \) are the coupling strengths.

The process of quantum state transfer along the spin chain is as follows. First, an unknown quantum state is prepared in the spin labeled with \( 1 \), next we allow the unitary evolution controlled by the Hamiltonian \( H \) for a time period \( t \). Then we check whether the unknown state has been transferred to another spin labeled with \( N \).

The quantity to characterize the QST process is the average fidelity for an unknown state transferring from spin 1 to spin \( N \), which is defined as

\[
F(t) = \int d\mu(\phi) \langle \rho_N | \text{Tr} \left( S \rho_N^\phi S^\dagger U(t) \rho_1^\phi \otimes \rho_{2 \cdots N} U^\dagger(t) \right) \rangle,
\]

where \( S \) is a unitary transformation on spin \( N \), \( \rho^\phi = |\phi\rangle \langle \phi| \), \( \rho_{2 \cdots N} \) is the initial state of the spins \( 2, 3, \cdots, N \), and \( U(t) \) is the unitary evolution of the system. The introduction of \( S \) means that we allow different identifications of the basis vectors between spin 1 and spin \( N \).

Obviously for a given Hamiltonian, the fidelity \( F(t) \) depends on the choices of \( S \) and \( \rho_{2 \cdots N} \). The aim of this
Letter is to analyze how the choices of $S$ and $\rho_{2\ldots N}$ affect the fidelity, and how to design the proper choices such that the fidelity is optimized. Most importantly, what is the maximal fidelity among all the choices? Is there a simple scheme to attain the maximal fidelity?

**Fidelity of QST in Heisenberg picture.**— We start with the analysis of what need to be calculated to determine the fidelity $F(t)$.

If we introduce a quantum channel

$$\mathcal{E}_t(\rho_1) = \text{Tr}_{2\ldots N}(U(t)\rho_1 \otimes \rho_{2\ldots N}U^\dagger(t)),$$

then the fidelity $F(t)$ can be regarded as the fidelity between the unitary transformation $S$ and the quantum operation $\mathcal{E}_t$, which was simplified in Refs. [23, 24] as

$$F(t) = \frac{1}{2} + \frac{1}{12} \sum_{\alpha \in \{X,Y,Z\}} \text{Tr}(S\alpha NS^\dagger \mathcal{E}_t(\alpha_1)).$$

The meaning of the above equation is that the average over all one-spin states can be reduced to the average over six states, namely all the eigen-states of $X$, $Y$ and $Z$. Hence it makes the average fidelity becomes observable in experiments.

Note that $S\alpha NS^\dagger = \sum_{\beta} R_{\alpha\beta} \beta_S$ with $R$ being a rotation matrix. Therefore we obtain

$$F(t) = \frac{1}{2} + \frac{1}{12} \sum_{\alpha,\beta} R_{\alpha\beta} \text{Tr}(\beta(t)\alpha_1 \rho_{2\ldots N}),$$

where $\beta_N(t)$ is the Pauli matrix $\beta_N$ in the Heisenberg picture. Eq. (1) directly relates the fidelity $F(t)$ with the Pauli matrices of spin $N$ in the Heisenberg picture. Thus to obtain the fidelity $F(t)$ we only need to calculate the dynamics of the Pauli matrices of spin $N$.

Because the time-dependent state in the Schrodinger picture contains the dynamics of all the system’s observables, the above result implies that it possibly simplifies the study of the QST fidelity if we adopt the Heisenberg picture other than the Schrodinger picture. We will show it is indeed the case for the XY spin chain in the following.

**Fidelity and parity.**— We start to analyze the dynamics of $X_N(t)$, which satisfies the Heisenberg equation

$$\frac{dX_N}{dt} = i[H,X_N].$$

To solve the above equation, we first find the set of operators including $X_N$, which is closed under the action $[H,\cdot]$. We adopt the method given in Ref. [20], which is demonstrated in a graph shown in Fig. 1. Every node in the graph represents an operator. If we investigate the evolution of the operator $\hat{O}$, for example $\hat{X}_N$ as we analyze in this section, then put it in the first node. The node adjacent to the first node is got by commuting first node with Hamiltonian $H$. Other nodes are got by the same way until we get all the elements of the closed operators set. An outgoing (incoming) edge corresponds to a $+\,(-)$ sign.

Hence we observed that the set of $N$ operators

$$\{X_{N+2-2m}Z^N_{N+3-2m}, Y_{N+1-2m}Z^N_{N+2-2m}\}$$

with $m \in \{1,2,\ldots,[N]/2\}$ are closed under the action $[H,\cdot]$, where $Z^m = \prod_{i=m} Z_i$. So $X_N(t)$ can be expanded as

$$X_N(t) = \sum_m (a_{2m-1}X_{N+2-2m}Z_{N+3-2m}^N + a_{2m}Y_{N+1-2m}Z_{N+2-2m}^N).$$

Notice that the system is invariant under the transformation

$$X_n \rightarrow Y_n, Y_n \rightarrow X_n, Z_n \rightarrow -Z_n, J_i \leftrightarrow K_i.$$  (4)

Thus we have

$$Y_N(t) = \sum_m (b_{2m-1}Y_{N+2-2m}Z_{N+3-2m}^N + b_{2m}X_{N+1-2m}Z_{N+2-2m}^N),$$

where

$$b_{2m-1}(t) = a_{2m-1}^{J_i+K_i}(t),$$

$$b_{2m}(t) = -a_{2m}^{J_i+K_i}(t).$$

Then $Z_n(t)$ can be obtained by $Z_n(t) = -iX_N(t)Y_N(t)$.

From Eq. (4), the parts of $X_N(t)$, $Y_N(t)$, and $Z_n(t)$ that contribute to the fidelity $F(t)$ must contain $X_1$, $Y_1$, or $Z_1$. So we can write $X_N$ and $Y_N$ into two parts, one part that contributes to $F(t)$, and the other that does not. When $N$ is odd, we get

$$X_N(t) = a_NX_1Z_2^N + \bar{X}_N, \quad Y_N(t) = b_NY_1Z_2^N + \bar{Y}_N.$$  

When $N$ is even, we have

$$X_N(t) = a_NY_1Z_2^N + \bar{X}_N, \quad Y_N(t) = b_NX_1Z_2^N + \bar{Y}_N.$$  

![FIG. 1: The sets of operators including $X_N$ or $Y_N$ that are closed under the action of $[H,\cdot]$. (a) The case when $N$ is odd. (b) The case when $N$ is even.](image-url)
According to the above results, it is reasonable to take
\( S = I (R_{\alpha \beta} = \delta_{\alpha \beta}) \) when \( N \) is odd, and \( S = \exp(i\frac{\pi}{2}Z_N) \)
\( (R_{XY} = R_{XZ} = R_{ZZ} = 1 \) and all other elements of \( R \)
are 0) when \( N \) is even.

Therefore we obtain the fidelity
\[
F(t) = \frac{1}{2} + \frac{(a_N + a_N^{\dagger + K}) \langle Z_N^2 \rangle + a_N a_N^{\dagger + K}}{6}. \tag{8}
\]

This implies that the fidelity of quantum state transfer is determined by the average value of the parity for
the initial state \( \rho_{2\ldots N} \) and the coefficient \( a_N(t) \). For the initial states \( \rho_{2\ldots N} \) that have the same average value of parity, the fidelity \( F(t) \) will be the same.

Structure of the solution of \( a_N(t) \).— Now we come
to the solution of \( a_N(t) \), which is determined from the
Heisenberg equation for \( X_N \):
\[
\frac{da}{dt} = Ga, \tag{9}
\]
where \( a = (a_1(t), a_2(t), \ldots, a_N(t))^T \) with \( T \) being the
transpose operation, \( G \) is a tri-diagonal matrix
\[
G = \begin{bmatrix}
0 & -2J_{N-1} & 0 & 0 & \cdots \\
2K_{N-1} & 0 & 2J_{N-2} & 0 & \cdots \\
0 & -2J_{N-2} & 0 & -2K_{N-3} & \cdots \\
0 & 0 & 2K_{N-3} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}. \tag{10}
\]
The initial condition is \( a(0) = (1, 0, 0, \ldots, 0)^T \). Note
that we have the normalization condition \( \sum_i a_i^2 = 1 \) arising from \( \text{Tr}(X_N^2(t)) = 2 \). Since the matrix \( G \) satisfies
\( G^\dagger = -G \), Eq. \( \text{(9)} \) can be imagined as a Schrödinger equation
with the Hamiltonian being \( iG \) in \( N \)-dimensional
Hilbert space, which greatly reduces the computational complexity for our problem.

When applying the Laplace transformation on Eq. \( \text{(9)}, \) we get
\[
A\tilde{a}(p) = a(0), \tag{11}
\]
where \( A = p - G \).

According to Cramer’s rule in the basic matrix theory, we have
\[
\tilde{a}_N = \frac{\det A^{(N)}_N}{\det A_N}, \tag{12}
\]
where \( A^{(N)}_N \) is the matrix \( A \) whose \( N \)-th column vector replaced by \( a(0) \).

Notice that
\[
\det A^{(N)}_N = \begin{cases}
(-1)^{M+1} \prod_{i=1}^{M} (2K_{2i-1}) \prod_{i=2}^{M} (2J_{2i-2}) & \text{if } N = 2M, \\
(-1)^M \prod_{i=1}^{M} (2K_{2i}) (2J_{2i-1}) & \text{if } N = 2M + 1.
\end{cases} \tag{13}
\]
The aim is to find \( \det A_N \), denoted as \( F_N \). For the tri-
diagonal matrix \( A \), we have the iterative relation for its
determinant
\[
F_{2m} = pF_{2m-1} + 4K_{N-2m+1}^2 F_{2m-2}, \tag{14a}
\]
\[
F_{2m-1} = pF_{2m-2} + 4J_{N-2m+2}^2 F_{2m-3}. \tag{14b}
\]
The initial condition is \( F_1 = 0 \) and \( F_0 = 1 \).

Then we can prove that
\[
F_N = \begin{cases}
\prod_{i=1}^{M} (p^2 + q_i^2) & \text{if } N = 2M, \\
p \prod_{i=1}^{M} (p^2 + s_i^2) & \text{if } N = 2M + 1.
\end{cases} \tag{15}
\]

In general, we assume that \( q_i \neq q_j \) and \( s_i \neq s_j \) for any
\( i \neq j \). So the inverse Laplace transformation of \( \tilde{a}_N \) is
\[
\begin{aligned}
a_N(t) &= \sum_{i=1}^{M} \frac{\sin(q_i t)}{q_i} \prod_{i \neq j} (q_i^2 - q_j^2), \quad \text{for } N = 2M, \\
&= \sum_{i=0}^{M} \frac{\sin(s_i t)}{s_i} \prod_{i \neq j} (s_j^2 - s_i^2), \quad \text{for } N = 2M + 1,
\end{aligned} \tag{16}
\]
where \( s_0 = 0 \).

Let us consider the special case where \( J_i = \frac{1}{2} \); \( K_i = \frac{K}{2} \).
Then we obtain \( q_k = \sqrt{1 + 2K \cos \varphi + K^2} \), with \( \varphi \) is the
roots of the equation
\[
\csc(\varphi)(K \sin(\varphi M) - \sin(M \varphi)) = 0,
\]
and \( s_k = 0 \), \( s_k = \sqrt{1 + 2K \cos \frac{k\pi}{M} + K^2}, \ k = 1, 2, \ldots, M - 1 \). When \( K = 1 \), that is XX model, \( q \) and \( s \)
are reduced to
\[
q_k = s_k = 2 \cos\left(\frac{k\pi}{N+1}\right), k = 1, 2, \ldots, M.
\]

The detailed derivation of the above solution is given in
the appendix.

Now we demonstrate our results numerically. First, we
show how the parity affects the QST fidelity in Fig. \( \text{2a} \) for
the XX model with \( J = K = 1 \) and \( N = 50 \). Before the
time \( t \simeq 14 \), \( F(t) \simeq \frac{1}{2} \), which implies that the signal of
the input state propagates along the chain with the Lieb-
Robinson velocity \( [27–29] \). Obviously the fidelity \( F(t) \)
oscillates with time for different parities, which reflects
the signal of the input state propagates in the spin chain.
In general, the fidelity for arbitrary initial state at any
time is between the one for the parity of +1 and that for
the parity of −1. Particularly, even the average parity is zero, i.e. for a maximal mixed state, the fidelity might
be larger than 1/2.

Second, we demonstrate how the rate \( J/K \) affects the
fidelity \( F(t) \) for a XY model in a spin chain with \( N = 50 \)
in Fig. \( \text{2b} \). Note that the excitation number \( \sum_i Z_i \) is
not conserved in the case \( J \neq K \). When \( J = 0 \), the fidelity
is 1/2 at any time, which implies that the signal of the
input state can never be transferred in this case. When
\( J/K \) is near 1, the behavior of the fidelity is similar to
that of the XX model. However, when \( J/K \) is far from 1,
it shows a different oscillation behavior: the delay start
time and the increasing oscillation amplitude.
Fidelity optimization by adjusting basis identification.— In Figs. 2 and 3 the fidelity \( F(t) < \frac{1}{2} \) for some \( t \). As we know, the fidelity can reach \( \frac{1}{2} \) without any connection. Therefore, we can always make the fidelity not less than \( \frac{1}{2} \) by adjusting basis identification. Here we emphasize that the basis identification needs not any real operation, but only an agreement about the basis map between the first spin and the last one.

Here we propose a simple scheme to adjust the unitary gate \( S \). We gives four unitary operations \( i^{ab}Y_N^aX_N^b \) with \( a, b \in \{0, 1\} \). Among the four choices, the optimized fidelity is

\[
F(t) = \frac{1}{2} + \left( |a_N| + |a_N^{J_1+K_1}| \right) \left( Z_2^N \right) + |a_Na_N^{J_1+K_1}| \right). 
\]

(17)

From Eq. (17), if we increase the amplitude of the parity, we can improve the fidelity \( F(t) \). However, since the parity of spin 2 to N is a collective observable, we have no idea to increase its absolute value to improve the fidelity by manipulating only the last spin. In addition, Eq. (17) implies that the optimized fidelity is not less than \( \frac{1}{2} \) if we make a proper choice of \( a, b \).

Obviously, our optimized scheme can get the maximal fidelity when \( \langle Z_2^N \rangle = 1 \), namely,

\[
F_{op}(t) = \frac{1}{2} + \left( |a_N| + |a_N^{J_1+K_1}| \right) + |a_Na_N^{J_1+K_1}| \right). 
\]

(18)

Note that a similar result is given in Ref. [22] provided that \( \langle Z_2^N \rangle = 1 \). In fact, it is also the maximal fidelity for arbitrary initial state and arbitrary unitary transformation \( S \), i.e., for any XY spin chain, we have

\[
F_{op}(t) = \max_{S, \rho \rightarrow \rho_{op}} F(t). 
\]

(19)

The detail of the proof is given in the appendix.

We illustrate our optimized scheme of the QST fidelity in Fig. 4 for the XY model with \( N = 50, J/K = 0.8 \), and \( \langle Z_2^N \rangle = 1 \). The optimized QST fidelity is a piecewise function, where different unitary gates \( S \) are taken in different pieces.
Discussion and conclusion.— Our result shows that the QST fidelity linearly depends on the parity of the initial state. Then the maximal fidelity can be taken at the parity \( \langle a_N^2 \rangle = \pm 1 \). To achieve the perfect (or pretty good) QST, it is sufficient to consider the case with the maximal parity. In other words, if the perfect (or pretty good) QST cannot arrive at the initial state with the maximal parity, it also can not arrived for arbitrary initial state. Notice that only the parity of the initial state is relative, so when the initial state is in the eigen-subspace of the parity, the QST fidelity will take the maximum.

In the Schrödinger picture, when the initial state is complex or the Hamiltonian does not conserve the excitation number, the QST will not be equivalent to the propagation of the single excitation. Eq. (9) implies that, in terms of our optimizing scheme the maximal average fidelity can be arrived when the operations propagation occurs in the Heisenberg picture for any initial states.

In summary, the analytical result on the QST fidelity along an XY spin chain is given in the Heisenberg picture. It shows that the QST fidelity only depends on the average value of the parity of the initial states. We also propose a simple scheme to optimize the QST fidelity by adjusting the basis identification with one of the operations \( I, X, Y, Z \) on the final spin, which ensures the QST fidelity is not less than \( \frac{1}{2} \) at any time for any initial states. We prove that in terms of our optimizing scheme the maximal average fidelity can be arrived when the amplitude of the average parity is 1. Therefore we give a scheme on how to optimally use a XY spin chain to transfer an unknown quantum state at any given time.

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SOLUTIONS OF \( F_N = 0 \) IN THE XY MODEL

Consider the special case of XY model, where \( J_i = \frac{K_i}{2} \). Then we can rewrite Eq. (14)

\[
\frac{F_{2m}}{F_{2m-1}} = \left\{ \begin{array}{cl} \frac{p^2 + K^2}{p} & \text{if } F_{2m-2} > 0 \\ 1 & \text{if } F_{2m-2} < 0 \end{array} \right.
\]

with

\[
\left[ \begin{array}{c} F_0 \\ F_{-1} \end{array} \right] = M \left[ \begin{array}{c} 1 \\ 0 \end{array} \right],
\]

So

\[
\left[ \begin{array}{c} F_{2M} \\ F_{2M-1} \end{array} \right] = M^M \left[ \begin{array}{c} 1 \\ 0 \end{array} \right],
\]

where

\[
M = \left[ \begin{array}{cc} p^2 + K^2 & p \\ p & 1 \end{array} \right].
\]
The eigenvalues of the matrix $\mathcal{M}$ are
\[
\frac{1}{2} \left( -\sqrt{(-K^2 - p^2 - 1)^2 - 4K^2} + K^2 + p^2 + 1 \right)
\]
and
\[
\frac{1}{2} \left( \sqrt{(-K^2 - p^2 - 1)^2 - 4K^2} + K^2 + p^2 + 1 \right).
\]
Its eigenvectors are
\[
\begin{bmatrix}
-\sqrt{K^2 + 2K^2 p^2 - 2K^2 + p^2 + 1 - p^2 + 1}

\sqrt{K^2 + 2K^2 p^2 - 2K^2 + p^2 + 1 - p^2 + 1}
\end{bmatrix}
\]
\[
\begin{bmatrix}
-\sqrt{K^2 + 2K^2 p^2 - 2K^2 + p^2 + 1 - p^2 + 1}

\sqrt{K^2 + 2K^2 p^2 - 2K^2 + p^2 + 1 - p^2 + 1}
\end{bmatrix}
\].

Note that the eigenvectors are not normalized. After directly compute and let $p^2 = -1 - 2K \cos \varphi - K^2$, we get
\[
\begin{bmatrix}
F_{2M} \\
F_{2M-1}
\end{bmatrix} = \begin{bmatrix}
(-1)^M K^{M-1} \csc(\varphi)(K \sin(M+1) \varphi + \sin(M \varphi)) \\
(-1)^M K^{M-1} \csc(\varphi) p \sin(M \varphi)
\end{bmatrix}.
\]

When $N = 2M - 1$, the roots of the equation $F_N = 0$ are $p = \pm i \sqrt{1 + 2K \cos \varphi + K^2}$ and $p = 0$, where \( \varphi = \frac{k \pi}{2M-1} \), $k = 1, 2, \cdots, M - 1$.

When $N = 2M$, the roots of the equation $F_N = 0$ are $p = \pm i \sqrt{1 + 2K \cos \varphi + K^2}$, with $\varphi$ is the roots of the equation
\[
\csc(\varphi)(K \sin(M \varphi + \varphi) - \sin(M \varphi)) = 0.
\]

For the XX model, that is $K = 1$, the roots of the equation $F_N = 0$ are reduced to
\[
p = \pm i 2 \cos \frac{\varphi}{2},
\]
with $\varphi = \frac{2k \pi}{N+1}$, $k = 1, 2, \cdots, N$

**PROOF OF THE OPTIMIZED FIDELITY**

Here we will prove that for any XY spin chain
\[
F_{op}(t) = \max_{S,\rho_{2N}} F(t),
\]
where
\[
F_{op}(t) = \frac{1}{2} + \left( |a_N| + |a_N J^{++K_i}| \right) / 6.
\]

Without loss of generality, we prove the result in the case that $N$ is odd. Inserting Eq. (3) and Eq. (5) into Eq. (1), we get
\[
X_N(t) = a_N X_1 Z_N^N + \bar{X}_N,
\]
\[
Y_N(t) = b_N Y_1 Z_N^N + Y_N.
\]

Then
\[
Z_N(t) = a_N b_N Z_t - i a_N X_1 Z_N^N \bar{Y}_N
\]
\[
- i b_N Y_1 \bar{X}_N Z_N^N - i \bar{X}_N \bar{Y}_N.
\]

Therefore
\[
F(t) = \frac{1}{2} + \frac{1}{6} \left( R_{xx} a_N \langle Z_N^N \rangle + R_{yy} b_N \langle Z_N^N \rangle + R_{zz} a_N b_N
\]
\[
+ R_{xx} a_N \langle -i \bar{Z}_N \bar{Y}_N \rangle + R_{yy} b_N \langle -i \bar{X}_N \bar{Z}_N \rangle
\]
\[
= \frac{1}{2} + \frac{1}{6} \left( a_N (R_{xx} \langle Z_N^N \rangle + R_{yy} \langle -i \bar{Z}_N \bar{Y}_N \rangle)
\]
\[
+ b_N (R_{yy} \langle Z_N^N \rangle + R_{yy} \langle -i \bar{X}_N \bar{Z}_N \rangle) + R_{xx} a_N b_N \right)
\]
\[
\leq \frac{1}{2} + \frac{1}{6} \left( |a_N| \sqrt{R_{xx}^2 + R_{yy}^2} \langle Z_N^N \rangle^2 + \langle -i \bar{Z}_N \bar{Y}_N \rangle^2
\]
\[
+ |b_N| \sqrt{R_{yy}^2 + R_{xx}^2} \langle Z_N^N \rangle^2 + \langle -i \bar{X}_N \bar{X}_N \rangle^2
\]
\[
+ |a_N b_N| \langle R_{zz} \rangle \right)
\]
\[
\leq \frac{1}{2} + \frac{1}{6} (|a_N| + |b_N| + |a_N b_N|),
\]
where we used
\[
R_{xx}^2 + R_{xx}^2 \leq R_{xx}^2 + R_{xx}^2 + R_{yy}^2 = 1,
\]
\[
R_{yy}^2 + R_{yy}^2 \leq R_{yy}^2 + R_{yy}^2 + R_{yy}^2 = 1,
\]
\[
R_{zz}^2 \leq R_{xx}^2 + R_{yy}^2 + R_{zz}^2 = 1.
\]

In addition, we define
\[
Z_L = Z_N^N,
\]
\[
X_L = -i Z_N^N \bar{Y}_N
\]
\[
\sqrt{1 - b_N^2}.
\]

It is easy to check that
\[
Z_L = Z_L^1
\]
\[
X_L = X_L^1
\]
\[
Z_L^2 = X_L^2 = 1
\]
\[
Z_L X_L = -X_L Z_L.
\]

So $Z_L$ and $X_L$ can be regarded as the logical qubit Pauli operators, which implies that
\[
\sqrt{\langle Z_L^2 \rangle^2 + \langle -i Z_L \bar{Y}_N \rangle^2} \leq \sqrt{\langle Z_L \rangle^2 + \langle X_L \rangle^2} \leq 1.
\]

Similarly, we have
\[
\sqrt{\langle Z_L^2 \rangle^2 + \langle -i Z_L \bar{X}_N \rangle^2} \leq 1.
\]

The maximal fidelity is taken if and only if all the qualities are satisfied. First, we note that $R_{zz}^2 = 1$. Hence we get $R_{xx}^2 = R_{yy}^2 = 0$. So $R_{xx}^2 = R_{yy}^2 = 1$. Therefore $\langle Z_L^2 \rangle = 1$. 

