PROPERTIES OF THE COSMOLOGICAL DENSITY DISTRIBUTION FUNCTION

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ABSTRACT

The properties of the probability distribution function of the cosmological continuous density field are studied. We present further developments and compare dynamically motivated methods to derive the PDF. One of them is based on the Zel’dovich approximation (ZA). We extend this method for arbitrary initial conditions, regardless of whether they are Gaussian or not. The other approach is based on perturbation theory with Gaussian initial fluctuations. We include the smoothing effects in the PDFs.

We examine the relationships between the shapes of the PDFs and the moments. It is found that formally there are no moments in the ZA, but a way to resolve this issue is proposed, based on the regularization of integrals. A closed form for the generating function of the moments in the ZA is also presented, including the smoothing effects. We suggest the methods to build PDFs out of the whole series of the moments, or out of a limited number of moments – the Edgeworth expansion. The last approach gives us an alternative method to evaluate the skewness and kurtosis by measuring the PDF around its peak. We note a general connection between the generating function of moments for small r.m.s $\sigma$ and the non-linear evolution of the overdense spherical fluctuation in the dynamical models.

All these approaches have been applied in 1D case where the ZA is exact, and simple analytical results are obtained. It allows us to study in details how these methods are related to each other. The 3D case is analyzed in the same manner and we found a mutual agreement in the PDFs derived by different methods in the the quasi-linear regime. Numerical CDM simulation was used to validate the accuracy of considered approximations. We explain the successful log-normal fit of the PDF from that simulation at moderate $\sigma$ as mere fortune, but not as a universal form of density PDF in general.

Subject headings: cosmology: theory — dark matter — galaxies: clustering
1. INTRODUCTION

One of the goals of the study of the nonlinear gravitational dynamics in an expanding universe is the determination of the statistical properties of the various cosmic fields that can be used to describe the matter distribution and motion. In the linear regime, for Gaussian initial conditions, this description is quite easy since it reduces to the behavior of the two-point correlation function, or equivalently to the shape of the power spectrum. Once nonlinear effects are taken into account this is no longer the case and the mathematical description of the statistical properties is more complicated. One can basically distinguish two approaches to describe the statistics of the non-linearities.

The description of statistics in terms of the correlation functions has been investigated since ’70s. In principle, a full knowledge of the matter field can be obtained from the shape of the $p$-point correlation functions (see Peebles 1980 for references). These functions are the solution of a hierarchy of dynamical equations, the BBGKY hierarchy. The spatial correlation functions have been measured in galaxy catalogues, or in numerical simulations. Progress was made to find a few low order correlation functions, theoretically, numerically and observationally (see Peebles 1993 for references). However, dynamical BBGKY equations have never been solved in general.

Another approach which drew much attention recently is based on the probability distribution function (PDF) of a cosmic field at a given point or PDFs at various points, or joint PDFs of several cosmic fields. The one-point density PDF, $P(\rho)\,d\rho$, is the primary object of study in this approach. The discrete analogy of the one point PDF is given by counts in cell probabilities that is basically what is obtained after smoothing of the discrete field by a sharp filter. It is important to note that the PDFs contain more statistical information than a few lower order correlation functions. Actually the moments of the PDF are the spatial averages (with the same window functions) of the correlation functions.

In practice, galaxy PDFs have been measured in various catalogues by Hamilton (1985) and Alimi, Blanchard & Schaeffer (1990) in the CfA survey, Bouchet et al. (1993) Kofman et al. (1994) in the IRAS surveys, Maurogordato, Schaeffer & Da Costa (1992) in the SSRS survey, Gaztañaga & Yokoyama (1993) both in the SSRS and CfA survey. The density PDF manifests significant non-Gaussian features in non-linear and even in quasi-linear regimes. One central question theories have to address is to determine what part of this non-Gaussianity came from non-linear dynamics, what part came from possible non-Gaussian initial conditions, and what part came from the galaxy biasing.

To see what kind of density PDFs emerge from gravitational dynamics, investigations have been made in numerical simulations by Bouchet, Schaeffer & Davis (1991), Weinberg and Cole (1992), Bouchet & Hernquist (1993), Juszkiewicz et al. (1993), Kofman et al. (1994).

In this paper we concentrate on the theoretical basis for the derivation of the cosmic density PDF. There were phenomenological attempts in the literature to design $P(\rho)$ – see, for instance, Saslaw and Hamilton (1984); Coles and Jones (1991). Here we discuss the derivation of the density PDF from gravitational dynamics, and its comparison against numerical simulation. It is a difficult problem to derive $P(\rho)$ for the general case. However, it can be studied in different regimes and approximations. One can distinguish different successive stages of the non-linear gravitational dynamics: quasi-linear regime, non-linear regime, and highly non-linear regime. The quasi-linear (or mildly non-linear) regime when $\sigma < 1$ can be investigated by the mean of the perturbation theory, unlike other regimes. In the non-linear regime $\sigma > 1$ the complexity of the dynamics makes analytical studies virtually impossible. Highly non-linear regime when $\sigma \gg 1$ might take place when the hypothetic hierarchical ansatz for the correlation functions is working. Indeed, another feature of the gravitational dynamics – self-similar solution (Davis & Peebles 1977) – is expected to be reached in this regime. The statistics of this regime has been a subject of consideration in many papers. In particular Balian & Schaeffer (1989) give relations of crucial interest that relate the properties of one-point density PDF to the ones of the correlation functions.

In this paper, we rather concentrate on the statistics in the quasi-linear regime. Fortunately, a fair fraction of the observational data does correspond to such a regime when the galaxy surveys are smoothed with a large enough radius (say $\gtrsim 8h^{-1}\text{Mpc}$). In the quasi-linear regime the two-point correlation function
is mainly determined by the initial conditions. The manifestation of the nonlinear features will be seen in the higher–order correlation functions, or equivalently in the departure of the shape of the distribution functions from Gaussian distributions. We have now various tools at our disposal to study the mildly nonlinear regime in details, so that specific predictions can be made that are directly derived from first principles.

One of the most successful approximations to describe the early nonlinear dynamics is the Zel’dovich approximation (Zel’dovich 1970). In this approximation, the displacement of the particles is extrapolated from its behavior in the linear regime. The whole local statistical properties of the field at the position of a particle is then determined by the statistical properties of the initial displacement field. For Gaussian initial condition, it is possible, for instance, to compute the density PDF by a change of variable starting with the initial statistical properties of the displacement. Actually, as it will be recalled in this paper, a lot more statistical properties can be derived with such an approximation (Kofman 1991, Kofman et al. 1994).

The second method is based on the calculations of the large–scale cumulants of the cosmic field from perturbation theory. Indeed, when one assumes Gaussian initial conditions, it turns out that it is possible to derive the leading term (with respect to the rms density fluctuation) of each cumulant (Bernardeau 1992). These first order contributions are expected to dominate the exact value of the cumulants as long as the rms fluctuation is accurately given by the linear theory. The shape of the density PDF is then obtained by a reconstruction of this function from the generating function of the moments.

The mathematical forms of the density PDFs derived analytically in these two methods are quite different. We compare the forms of the PDFs derived in the quasi-linear regime earlier in our papers, and show its mutual agreement. As this subject is in rapid development, many questions and controversies on the density PDF and moments are accumulating in the literature. Therefore we try to clarify many points related to the PDFs, generating functions, moments and smoothing in the quasi-linear regime. Presumably, there is no simple universal formula for $P(\rho)$ in general. However, one of the practical outputs of our paper is to provide a justification for the use a log-normal distribution as a successful fit for the actual cosmic density PDF. This fit is only accurate for the cosmological models based on the Gaussian initial statistics and realistic power spectrum, but not in the general case. Another practical output is a new view on the measurement of the lowest moments - skewness and kurtosis. Common belief is that they are affected by the high density tail which is difficult to measure. We will demonstrate that these moments can be evaluated from the form of the PDF around its maximum.

We will use the concepts of moments, cumulants, PDFs and generating function throughout the paper. The formal definitions of all of them are collected in Appendix A. However, we must stress that the introduction of quantities such as the generating functions of moments and cumulants is not an extra mathematical exercise but an useful tool to find the measurable statistics of the cosmic fields. There is a deep connection between generating function and the non-linear evolution of the spherical overdense fluctuation in the dynamical model of the gravitational instability. Hence, in most cases the generating function can be derived directly from the dynamical equations. We will present the closed forms of the generating functions in different approximations in the quasi-linear regime.

Throughout this analysis we got a series of new results: the generating functions of cumulants and finite moments in the Zel’dovich approximation, both with and without final smoothing; two methods of reconstruction of cosmological density PDF from, correspondingly, the whole (Laplace transformation) and partial (Edgeworth decomposition) series of moments; systematic mutual comparison of all these methods and their comparison against the calculations from N-body simulation; the range of validity of their fitting by the log-normal distribution; the extension of the derivation of density PDF in the Zel’dovich approximation for an arbitrary initial condition.

Section 2 of the paper is devoted to the 1D dynamics, in which the mathematical content is simpler than for the 3D dynamics. We apply general methods and mathematical tools in this case. Section 3 is devoted to the statistical properties that can be derived from the use of the Zel’dovich approximation for the 3D dynamics. In Section 4, we give the properties of the exact dynamics derived from direct perturbative calculation in the single stream regime. Section 5 is devoted to comparisons with numerical simulations. In the conclusive Section 6 we summarize the approximations that have been made and the results in a detailed
2. AN ILLUSTRATIVE EXAMPLE: THE 1D DYNAMICS

In this part we aim to present the analytical tools developed in the mildly nonlinear regime in case of a simpler dynamics. We assume in this part that the fluctuations are only one-dimensional. The general methods suggested here will be used in the next sections in the 3D case.

2.1. The construction of the density PDF from the Zel’dovich approximation

It is convenient to use the equation of motion in the Lagrangian description. In such a case the dynamics is described by the displacement, $S(q, t)$, of each particle from its initial position $q$. Its current Eulerian comoving position, $x$, is then given by

$$x = q + S(q, t).$$

(1)

For 1D case the mildly non-linear dynamics is quite simple. The reason is that the force exerted by a density perturbation over a given particle is independent of its distance to the particle. Therefore, before any shell crossing, the displacement of each particle depends on its Lagrangian position $q$ only. Then the displacement field for the growing mode can be factorized

$$S(q, t) = \Psi(q) D(t).$$

(2)

The 3D generalization of this form of the displacement is the Zel’dovich approximation (ZA). In 1D case the ZA is then identical to the exact dynamics. In relation (2) the displacement factorizes in a spatial function, $\Psi(q)$, which depends on the initial conditions, and a universal time dependent function $D(t)$ which contains by definition the time dependence of the growing modes. Let us call $\lambda_0$ the derivative of $-\Psi(q)$ with respect to $q$. The local comoving density (in units of the mean density, $\varrho = \rho / \bar{\rho}$) is then, by the mass conservation equation, given by

$$\varrho(q) = \frac{1}{|1 - D\lambda_0|}.$$ 

(3)

The reconstruction of the density PDF then is based on the statistical properties of $\lambda_0$. For Gaussian initial conditions, $\lambda_0$ simply obeys Gaussian statistics. The derivation of the shape of the density PDF is then just a matter of change of variable, and as it can be seen in (3) the density contrast will not be normally distributed. This is general feature due to the nature of this quantity. The positive density contrast can reach any large value while the negative density contrast is restricted by $\varrho \geq 0$. Hence the probability function $P(\varrho, D(t))$, meaning the fraction of volume with a given value of density, is expected to be very non-Gaussian even in the quasilinear stage. However, note that a simple change of variable leads to the density PDF at a given point, in Lagrangian space. The distribution in Eulerian space should take into account the fact that a given amount of matter in a dense spot occupies a small volume. The density PDF in Eulerian space is then obtained by divided the density distribution in Lagrangian space by the density, and multiplied by the numerical factor which controls the normalization. This factor is related to the number of streams $\langle N_s \rangle$, in cases $\sigma < 1$ under consideration it is very close to unity, and we ignore it (e.g. Gurbatov et al. 1991).

Then the density PDF in 1D reads

$$P_{1D}(\varrho)d\varrho = \frac{1}{(2\pi\sigma^2)^{1/2}} \left[ \exp\left(-\frac{\lambda^2}{2\sigma^2}\right) + \exp\left(-\frac{\lambda'^2}{2\sigma^2}\right) \right] \frac{d\varrho}{\varrho^3},$$

(4)

$$\lambda \equiv D \lambda_0 = 1 - \frac{1}{\varrho}, \quad \lambda' \equiv D \lambda'_0 = 1 + \frac{1}{\varrho}.$$ 

Note that $\sigma = \sigma_0 D(t)$, where $\sigma_0$ is the variance of the initial linear density contrast. In the limit of small $\sigma$ the distribution (4) transits to the Gaussian form. The high density asymptota $\varrho^{-3}$ that is seen in equation (4) is induced by the caustics.
2.2. Density PDF and moments

As recalled in the introduction, the knowledge of the shape of the PDF and the moments are intimately related. The very example we are presenting now is here to point out that this is not completely true. Indeed, none of the moments of the previous distribution are finite, because of the $\rho^{-3}$ asymptotics! It can be easily checked that even for an arbitrary small $\sigma$ the density PDF behaves like $\sqrt{2/\pi/\sigma \exp(-1/2\sigma^2)}\rho^{-3}$ at high density. However, physical density in caustics is finite. We expect that a physical process will operate as sort of regularization of the high density tail of $P(\rho)$. For example, Zel’dovich & Shandarin (1982) showed how physical properties of the hot dark matter lead to the finite density in the caustics. Obviously we do not know what will be the exact form of this regularization in general case, so we simply describe the regularization process by a cut-off at large density (where the physical regularizing processes are thought to occur) that makes the moments finite. The shape of this cut-off is somewhat arbitrary, but we will see that, to some extent, a fair fraction of the properties of the moments do not depend on the procedure that has been adopted. We chose two toy models of distribution, with the sharp cutoff and with the exponential cutoff, in the following form:

\[
P_{\text{reg1}}(\rho) \, d\rho = \frac{1}{C_1} P_{1D}(\rho) \, d\rho \quad \text{if} \quad \rho < \rho_c,
\]

\[
P_{\text{reg1}}(\rho) \, d\rho = 0 \quad \text{if} \quad \rho > \rho_c,
\]

and

\[
P_{\text{reg2}}(\rho) \, d\rho = \frac{1}{C_2} P_{1D}(\rho) \left[ 1 + \frac{\exp(x)}{\exp(x) + \exp(-x)} (\exp(-\rho/\rho_c) - 1) \right] \, d\rho,
\]

where $x = \rho - \rho_c$.

The coefficients $C_1$ and $C_2$ are simply normalization factors. The adopted values of $\rho_c$ will be quite large (above 5) insuring that the shape of the density PDF is not changed in the domain of interest that is for $0 < \rho < \text{several}$. The coefficients $C_1$ and $C_2$ are very close to unity in these cases. In Fig. 1 we present the distribution (4) and regularized distributions (5).

Once such a regularization is made, the moments of the distribution functions $\langle \delta^p \rangle = \int_0^\infty d\rho \, P(\rho)(\rho-1)^p$ can be easily calculated numerically. We present the results in terms of $S_p$ parameters adopted in the literature. These parameters are defined through the cumulants (see Appendix A, Eq. [A2]):

\[
S_p = \langle \delta^p \rangle / \sigma^{2(p-1)}.
\]

This is a generic definition which is not based on any a priori assumption. The $S_p$ parameters would be constants in the hierarchical ansatz, but in general they can be seen as functions of $\sigma$.

The parameter $S_3$ multiplied by $\sigma$ is the skewness, and $S_4$ multiplied by $\sigma^2$ is the kurtosis. The numerical calculations of the parameters $S_3$ and $S_4$ for the regularized distributions (5) and (6) for the exact Zel’dovich solution is shown in Fig. 2. The measurements have been made by calculating the actual second, third, and fourth moment of the regularized density PDF, and by taking the appropriate ratios. We see that these coefficients are finite. The shapes of $S_3(\sigma)$ and $S_4(\sigma)$ are universal in the interval of $\sigma$ from 0 up to the value $\sim 0.2$. These shapes are independent of the form of the cutoff, as well as on the parameter of truncation. In the limit of $\sigma \to 0$ we found $S_3 \approx 6$, and $S_4 \approx 72$. In the next section we will confirm these figures and their universality by analytical calculations. We conclude that the found values of $S_3(\sigma)$ and $S_4(\sigma)$ for $\sigma \lesssim 0.2$ are the proper moments in the 1D case. There is also some universality of these coefficients for $\sigma \gtrsim 1.2$, in the sense that it does not depend on the parameter $\rho_c$ for each panel. However, it does depend on the form of regularization. We suggest that the found values of $S_3(\sigma)$ and $S_4(\sigma)$ for $\sigma \gtrsim 1.2$ do not reflect the proper moments, but rather the shape of the adopted cutoff and thus do not have any physical meaning.

2.3. Calculation of $S_p$ for small $\sigma$

It is possible to calculate the $S_p$ series analytically when the variance $\sigma$ of the distribution function is small. This calculation requires the introduction of the generating functions of moments and cumulants (see
Appendix A). The formal definition of the generating function of the moments $\langle \delta^p \rangle$ is

$$M(\mu) = 1 + \sum_{p=1}^{\infty} \frac{\langle \delta^p \rangle \mu^p}{p!}$$

(8)

and the generating function of the cumulants $\langle \delta^p \rangle_c$ is

$$C(\mu) = \sum_{p=2}^{\infty} \frac{\langle \delta^p \rangle_c \mu^p}{p!}$$

(9)

Here $\mu$ is an auxiliary parameter. Using its definition, one can relate $M(\mu)$ to the shape of the density PDF:

$$M(\mu) = \exp C(\mu) = \int_0^\infty d\varrho P(\varrho) \exp([\varrho - 1]\mu)$$

(10)

Since we know the density PDF in the quasi-linear regime in the 1D problem, in principle, we can obtain the generating function $C(\mu)$ substituting the formula (4) into integral (10). When $\sigma \ll 1$ we can integrate (10) in a closed form, using the steepest descent method. The saddle point of the exponent is given by the equation

$$-\frac{1}{\sigma^2} \frac{d}{d\varrho} \left( \frac{\lambda(\varrho)^2}{2} \right) + \mu = 0$$

(11)

with $\lambda(\varrho) = 1 - 1/\varrho$. Note that the solution of this equation, $\lambda_s$, is such that $D\lambda_s$ is the function of the combination $\mu\sigma^2$ only. In 1D case the solution of equation (11) can actually be obtained straightforwardly. We will treat, however, this equation in a different manner, which will be much more appropriate in 3D case. We introduce the function $G(\tau)$,

$$G(\tau) = \frac{1}{1 + \tau} - 1, \quad \text{so that} \quad \varrho = G(-\lambda) + 1.$$  

(12)

After substitution of formula (12) into equation (11), the solution of equation (11), $\lambda_s$, is then given by the implicit equation

$$D\lambda_s = -\mu\sigma^2 \frac{dG}{d\tau} \bigg|_{\tau = -D\lambda_s}.$$  

(13)

The generating function $C(\mu)$ of the cumulants is obtained by taking the logarithm of the integral (10). In the low $\sigma$ limit it leads to retain only the term that is under the exponential at the saddle point position. We then obtain from (10),

$$C(\mu) = \mu G(-\tau_s) - \frac{\tau_s^2}{2\sigma^2}, \quad \tau_s = -D\lambda_s$$

(14)

where the saddle point $\lambda_s$ (or equivalently $\tau_s$) is given by the equation (13).

The reason we introduced $G(\tau)$ is the following. Equations (13) and (14) have a structure of the Legendre transformation from $G(\tau)$ to $C(\mu)$ with the variable of the transformation equal to unity afterwards. It turns out that the function $G(\tau)$ is the central object derived from the basic dynamical equations in the perturbation theory calculations (Bernardeau 1992). In the perturbation theory $G(\tau)$ is defined as the generating function of the other averages – vertices (see Sec. 4). It is remarkable that it corresponds to the dynamics of the “spherical” collapse – one dimensional in this case: $G(-\lambda)$ gives the density contrast of a collapsing object of linear overdensity $\lambda_0$, $\lambda = D(t) \lambda_0$, as function of time. We will discuss this derivation in general case in Sec. 4. The reason the Legendre transformation emerges is that two generating functions $G(\tau)$ and $C(\mu)$ considered as the sum over the corresponding averages are connected through that transformation. Hence, note that in the 1D case expression (14) for $C(\mu)$ can be obtained from the direct perturbation series expansion of the basic equations as well.
The crucial observation from the equation (14) is that the combination \( C(\mu)/\mu \) is a function of the combination \( \mu \sigma^2 \) only. What does it imply? The term in \( \mu^p \), the coefficient of which is the \( p^{th} \) cumulant we are looking for, is then proportional to \( \sigma^2(p-1) \). It rigorously demonstrates a scaling relation between the cumulants in the small \( \sigma \) limit (i.e. at large scales). The expression of these coefficients can be obtained with an expansion of the function \( C(\mu) \) with \( \mu: C(\mu) = \mu^2\sigma^2/2 + \mu^3\sigma^4 + 3\mu^4\sigma^6 + \ldots \). Then, for the skewness and kurtosis, for instance, we get

\[
S_3(0) = 6, \tag{15a}
\]

and

\[
S_4(0) = 72. \tag{15b}
\]

These results can be extended by the further perturbative calculation that takes into account the next terms of the expansion in the steepest descent method. It is then possible to get the first \( \sigma \) corrections for the expression of the skewness and the kurtosis. We then get

\[
S_3(\sigma) = 6 + 24\sigma^2 + O(\sigma^4), \tag{16a}
\]

and

\[
S_4(\sigma) = 72 + 810\sigma^2 + O(\sigma^4). \tag{16b}
\]

These expansions are not affected by the shape and parameters of the cut-offs. Theoretical curves (16) derived for small \( \sigma \) are plotted on Fig. 2, and indeed osculate the universal numerical curves for small \( \sigma \).

One of the challenges of the study of the large-scale structure formation is to find the \( \sigma \) dependence of the \( S_p \) parameters. From the theoretical point of view, it is not clear whether the \( \sigma \) dependence can be accurately determined with perturbation theory in the single stream approximation. Multistreaming may change the behavior in an unknown way. But we present here the results in 1D case to stress that perturbation theory does not prove at all that the parameters \( S_p \) are constant when \( \sigma < 1 \). The important property is that, for Gaussian initial fluctuations, they have finite limits at small \( \sigma \). Note that the expansion of \( S_p \) as a function of \( \sigma \) similar to formula (16) has never been done for the 3D case.

2.4. The reconstruction method from the generating function of the cumulants

In the previous section we solved the problem how to find the moments from the known PDF. When the whole series of the cumulants is known it is possible to construct the density PDF from the generating function of the cumulants by inversion of the relation (10) (the inverse Laplace transformation):

\[
P(\varrho)d\varrho = \frac{d\varrho}{2\pi i} \int_{-\infty}^{+\infty} d\mu \exp \left[ -C(\mu) - (\varrho - 1)\mu \right], \tag{17}
\]

where the integration is made in the complex plane.

In practice we can find the moments and generating function in the limit of small \( \sigma \). For the further analysis let us define the function \( \varphi(y) \) by the relation

\[
\varphi(y, \sigma) = \sum_{p=1}^{\infty} S_p(\sigma) \frac{(-1)^{p-1}}{p!} y^p = y - \sigma^2 C(-y/\sigma^2), \tag{18}
\]

where we define \( S_1 = 1, S_2 = 1 \). The advantage of this new function is that it is finite for an arbitrary \( y \) in the limit of small \( \sigma \). This function is the generating function of the parameters \( S_p(\sigma) \). Then from (17) and (18) the density PDF is given by

\[
P(\varrho)d\varrho = \frac{d\varrho}{2\pi i \sigma^2} \int_{-\infty}^{+\infty} dy \exp \left[ -\frac{\varphi(y, \sigma)}{\sigma^2} - \frac{\varrho y}{\sigma^2} \right], \tag{19}
\]
Balian & Schaeffer (1989) used this form for the hierarchical ansatz, when there is no dependence on \( \sigma \) in \( \varphi(y, \sigma) \). In the small \( \sigma \) limit the \( \sigma \) dependence that may be contained in \( \varphi(y, \sigma) \) vanishes, \( \varphi(y, \sigma) \rightarrow \varphi(y) \), and the results of Balian and Schaeffer (1989) can then be used here, too.

Note that general formula (19) does not assume, a priori, that \( \sigma \) is small (as for the Edgeworth expansion in the next subsection). It supposes, however, that the function \( \varphi(y, \sigma) \) that is used is valid in the domain of application. The reconstruction formula is of obvious interest when the function \( \varphi(y, \sigma) \) can be derived from the first principles as it has been shown when \( \sigma = 0 \) in 1D case. We will see that it is also the case for small \( \sigma \) limit in 3D case. The reconstruction formula (19) is thus of general interest.

In Fig. 3 the density PDF in 1D case is calculated numerically with the reconstruction formula (19), where \( \varphi(y) \) is calculated from (14) and (18) in the small \( \sigma \) approximation, i.e. ignoring \( \sigma \)-dependence in \( \varphi(y, \sigma) \). It is then compared to the original shape of the density PDF (Eq. [4]). We find that the reconstruction method based on the generating function works very well for the density interval \( 0 < y < 2 \), slightly worsening as \( \sigma \) increases. The shape of the high-density tail at \( y > 2 \) of the PDF from the reconstruction method differs from that of the actual PDF. For the form (19), the density PDF has an exponential cutoff as \( \exp[-y/(x_s \sigma^2)] \) with \( x_s = 27/4 = 6.75 \) (see Balian & Schaeffer 1992), which is very different from the \( y^{-3} \) cutoff of formula (4). This discrepancy is just due to the fact that we ignore the \( \sigma \)-dependence in the parameters \( S_p \). Therefore the discrepancy is increasing with \( \sigma \). It is quite interesting that for moderate \( \sigma \), the \( \sigma \)-dependence affects the high-density tail of PDF, but does not affect its shape around the maximum, nor the low-density tail (see the Table in Sec. 6).

2.5. Reconstruction of PDF from a few cumulants: The Edgeworth expansion

In the previous section it has been shown how it is possible to derive the shape of the density PDF out of the generating function. In this section we report a method that can be used to recover the shape of the PDF when only a limited number of cumulants is known. In practice we may have only a few lowest cumulants, such as the skewness and the kurtosis. In the case of the weakly non-linear dynamics, when slight departure from the initial Gaussian distribution is expected, one can use the general decomposition series around the Gaussian PDF induced by the first non-zero cumulants. This decomposition is known as the Gram-Charlier series (Kendall & Stuart, 1958). Longuet-Higgins (1963) applied the Edgeworth form of the Gram-Charlier series to the statistics of the weakly non-linearities in the theory of 2D sea waves. Inspired by this paper, we suggest to use the Edgeworth’s decomposition for the 1D and 3D cosmological density PDF (as reported in Kofman 1993). We understand that similar ideas were independently suggested by Juszkiewicz et al. (1993).

The Edgeworth expansion can be derived from the form (19) of the density PDF. Assuming that the density contrast \( \delta \) is of the order of \( \sigma \) and small, the relevant values of \( y \) are also of the order of \( \sigma \) and are thus assumed to be small. It is then legitimate to expand the function \( \varphi(y) \)

\[
\varphi(y, \sigma) \approx y - \frac{1}{2} y^2 + \frac{S_1}{3!} y^3 - \frac{S_4}{4!} y^4 + \frac{S_5}{5!} y^5 \pm \ldots, \tag{20}
\]

where \( S_p = S_p(\sigma) \). To calculate the density PDF, we substitute the expansion (20) into the the integral (19). Then we make a further expansion of the non-Gaussian part of the factor \( \exp[-\varphi(y)/\sigma^2] \) with respect to both \( y \) and \( \sigma \) assuming they are of the same order, see Appendix B for details.

Finally we obtain the so-called Edgeworth form of the Gram-Charlier series for density PDF

\[
P(\delta) d\delta = \frac{1}{(2\pi \sigma^2)^{1/2}} \exp \left( -\frac{\nu^2}{2} / 2 \right) \left[ 1 + \sigma \frac{S_3}{6} H_3(\nu) + \sigma^2 \left( \frac{S_4}{24} H_4(\nu) + \frac{S_5^2}{72} H_6(\nu) \right) \right. \\
+ \sigma^3 \left( \frac{S_6}{120} H_5(\nu) + \frac{S_4 S_5}{144} H_7(\nu) + \frac{S_5^3}{1296} H_9(\nu) \right) + \ldots \bigg] d\delta, \tag{21}
\]

where \( H_n(\nu) \) are the Hermite polynomials (see Appendix B), \( \nu = \delta/\sigma \). This is a universal form for any slightly non-Gaussian dynamical models, i.e. when \( \sigma \) is small and \( S_p \) are finite. The actual forms of the parameters \( S_p = S_p(\sigma) \) which depend on particular dynamics, affect the expansion (21) with respect to \( \sigma \).
Longuet-Higgins (1963), Juszkiewicz et al. (1993) and Kofman (1993) gave it up to \( \sigma^2 \)-correction. We give the Edgeworth expansion up to \( \sigma^3 \)-correction, for which the \( \sigma \)-dependence of \( S_3 \) (see, for instance, eq.[16a]) has to be taken into account. The resulting approximate form (21) is a Gaussian distribution multiplied by a corrective function – in the square brackets – expanded with respect to \( \sigma \). The zero order of this expansion gives simply a Gaussian distribution; the first order corrects it by taking into account the skewness; the second order by taking into account both the skewness and the kurtosis, etc.

Thus, it is possible to get an approximate form of the density PDF from a few lowest known cumulants. This method is irreplaceable when only a few cumulants have been derived from the first principles. However, it is important to note that this expansion has been made possible only for slightly non-Gaussian regime. The validity domain of the form (21) is limited to finite values of \( \delta/\sigma \).

Let us now apply the Edgeworth expansion (21) to the 1D gravitational dynamics. We have to take into account the \( \sigma \) dependence of \( S_3 \) given in equation (16a), and then obtain,

\[
P(\delta)d\delta = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp(-\nu^2/2) \left[ 1 + \sigma H_3(\nu) + \sigma^2 \left( 3H_4(\nu) + \frac{1}{2}H_6(\nu) \right) \right.
\]

\[
\left. + \sigma^3 \left( 11H_5(\nu) + 3H_7(\nu) + \frac{1}{6}H_9(\nu) + 4H_3(\nu) \right) + ... \right] d\delta.
\]

We plot the density PDF in 1D case reconstructed from the Edgeworth expansion (22) in Fig. 4, and compare it with the actual PDF (4), for \( \sigma = 0.1 \), and \( \sigma = 0.3 \). We can see that a few iterations of the expansion (22) reproduce the peak of \( P(\delta) \) in the interval \( |\delta| \lesssim 0.5 \) around it for small \( \sigma \) relatively well. It reproduces well the shift of the maximum towards the low density. It completely fails to reproduce \( P(\delta) \) outside of this interval where spurious wiggles appear. For a given value of \( \sigma \), each next \( \sigma \) iteration quite slowly improves the approximation. Unfortunately, the method is rapidly worsening as \( \sigma \) increases, and in practice is useless for \( \sigma \gtrsim 0.5 \).

The usual measurements of the lowest cumulants are significantly affected by the high density tail of the PDF, i.e. the rare events. It is interesting to note, that in the context of the reconstruction methods, the lowest cumulants alone are responsible for the shift of the peak of \( P(\delta) \). Therefore the measurement of the shape of the PDF maximum, which statistically is more robust, can provide an alternative method of evaluation of the lowest cumulants.

3. THE 3D DYNAMICS WITH THE ZEL’DOVICH APPROXIMATION

We consider here the statistics of the cosmic density field in 3D Zel’doovich approximation in a similar manner as it was done in 1D case. The most important change is that the Zel’dovich solution no longer reproduces the exact dynamics but is an approximation. It is, however, thought to be a good description, so that it is worth investigating the statistical properties that can be obtained in this approximation. We report here a different derivation of what have been done by Kofman (1991) and Kofman et al. (1994) for Gaussian initial fluctuations. The new method (Kofman 1994) also allows to extend the results to non-Gaussian adiabatic initial fluctuations.

3.1. Zel’dovich approximation for filtered initial fluctuations

For the sake of simplicity we assume that the universe is filled by the pressureless matter. The growing mode of adiabatic perturbations is \( D(t) \). Let \( \mathbf{x}, \mathbf{v} = a \mathbf{dx}/dt, \rho(t, \mathbf{x}) \) and \( \phi(t, \mathbf{x}) \) be, respectively, the comoving coordinates, peculiar velocity, density of dark matter and peculiar Newtonian gravitational potential. It is convenient to introduce a new time variable, \( \tilde{D}(t) \), then a comoving velocity is \( \mathbf{u} = d\mathbf{x}/d\tilde{D} \). The growing mode is non-rotational, so that the velocity field is potential. Let \( \Phi \) be the velocity potential so that \( \mathbf{u} = \nabla_{\mathbf{x}} \Phi \).

The gravitational dynamics of the cosmological system is complicated and requires the N-body simulations. For interesting cosmological models such as the CDM scenario, the structure formation looks
like complicated hierarchical pancaking and clustering from very small to large cosmological scales (e.g., see Shandarin & Zel’dovich 1989, Kofman et al. 1992). However the gravitational clustering at sufficiently large scales $R$ can be considered in the quasilinear theory in a single stream regime ignoring small scale details. For this goal we use the Zel’dovich approximation but apply it for the filtered initial gravitational potential. This approach, sometimes called the truncated Zel’dovich approximation, was used for different purposes in papers Bond and Couchman (1986), Kofman (1991), Shandarin (1992), Kofman et al. (1992), Coles et al. (1993), Kofman et al. (1994). From a mathematical point of view the truncated Zel’dovich approximation means just that the Zel’dovich approximation is applied to the truncated initial potential to ensure being in the single stream quasilinear regime.

To describe the motion of particles we can introduce the tensor of the velocity derivatives $S_{ij}(x,t) = -\nabla_{x_i}v_j$ in the Eulerian space; for the potential motion it is reduced to $S_{ij} = -\nabla_{x_i}\nabla_{x_j}\Phi$. Let $\lambda_i$ be its eigenvalues. The field of the $S_{ij}(t)$-tensor evolves in time, its initial value (in the Lagrangian space) coincides with the Lagrangian deformation tensor $D_{ij} = -\nabla_{\lambda_i}\nabla_{\lambda_j}\Phi_0$. From dynamical equations one can easily show that the eigenvalues $\lambda_i(t)$ are related to their initial values $\lambda_{0i}$, by $\lambda_i(t) = \lambda_{0i}/(1 + D(t)\lambda_{0i})$. The local density can be obtained by the inverse of the Jacobian of the transformation between $q$ and $x$, so that

$$\theta = \frac{\theta_0}{|(1 - D\lambda_{01})(1 - D\lambda_{02})(1 - D\lambda_{03})|}.$$  

(23)

In the ZA the statistics of the evolved field can then entirely be obtained by the statistics of the initial local density $\theta_0$ and the initial eigenvalues $\lambda_{0i}$s. For adiabatical perturbations $\theta_0 = 1$.

3.2. Joint PDF in the ZA for arbitrary initial statistics

In the last section 3.1 all relationships were obtained without making any assumption on the initial statistics. The cosmic density PDF can then be obtained from the initial joint PDF of all involved cosmic fields: $W_0(\theta_0, \lambda_{01}, \lambda_{02}, \lambda_{03}, \bar{u}_0, \Phi_0) \, d\theta_0 \, d\lambda_{01} \, d\lambda_{02} \, d\lambda_{03} \, d^3u_0 \, d\Phi_0$. That the statistics can be completely made from the statistical behavior of all variables is entirely due to the use of the Zel’dovich approximation. If this approximation were released it would no longer be true.

The density PDF can be obtained in general case of arbitrary initial condition by integrating the combination $\delta(\theta - 1) \times W_0(\theta_0, \lambda_{01}, \lambda_{02}, \lambda_{03}, \bar{u}_0, \Phi_0)$ over all involved variables except density. This is one of the new results of our paper. Density PDF for non-Gaussian initial fluctuations will be considered in a separate paper (Kofman 1994).

3.3. Density PDF for Gaussian initial fluctuations

In this Section we study the statistics of the continuous cosmological fields evolving from the initial Gaussian fluctuations, which is the general frame for most cosmological models. For the Gaussian initial conditions we can omit $\bar{u}_0$ and $\Phi_0$ in the initial joint PDF and write it as

$$W_0(\theta_0, \lambda_{01}, \lambda_{02}, \lambda_{03}) \, d\theta_0 \, d\lambda_{01} \, d\lambda_{02} \, d\lambda_{03} = \delta(\theta_0 - 1) \, d\theta_0 \, M_0(\lambda_{01}, \lambda_{02}, \lambda_{03}) \, d\lambda_{01} \, d\lambda_{02} \, d\lambda_{03}. \quad (24)$$

The first factor – the Dirac $\delta$-function – is the initial density distribution function, which corresponds to the perfectly homogeneous density distribution $\theta_0 = 1$. This is just the formal limit of the Gaussian density distribution with $\sigma \to 0$. This factorization is expected to take place for cosmological models with small adiabatical initial fluctuations. The second factor is the joint distribution function of the eigenvalues of the initial deformation tensor for an initial Gaussian displacement field (Doroshkevich 1970)

$$d\lambda_{01} \, d\lambda_{02} \, d\lambda_{03} \, M_0(\lambda_{01}, \lambda_{02}, \lambda_{03}) = d\lambda_{01} \, d\lambda_{02} \, d\lambda_{03} \times \frac{55/2 \pi^{5/2}}{27 \sigma_0^6} \, (\lambda_{01} - \lambda_{02})(\lambda_{01} - \lambda_{03})(\lambda_{02} - \lambda_{03}) \, \exp \left[ -\frac{1}{\sigma_0^2} \left( 3J_{01}^2 - \frac{15}{2}J_{02} \right) \right], \quad (25)$$

where $J_{01} = \lambda_{01} + \lambda_{02} + \lambda_{03}$ and $J_{02} = \lambda_{01}\lambda_{02} + \lambda_{01}\lambda_{03} + \lambda_{02}\lambda_{03}$.
The shape of the density PDF can then be obtained by the change of variable \( \varrho_0 \) to \( \varrho \) and by integrating over \( \lambda_0 \)s in (24). We then have

\[
P(\varrho, D) d\varrho = d\varrho \int d\lambda_{01} d\lambda_{02} d\lambda_{03} \delta[\varrho(1 - D\lambda_{01})(1 - D\lambda_{02})(1 - D\lambda_{03})] - 1] \\
\times M_0(\lambda_{01}, \lambda_{02}, \lambda_{03}).
\]

We get the formal expression for the Eulerian density PDF in the ZA.

Substituting the expression (25) into the integral (26), after some tedious algebra, we can reduce the integral (26) to a simpler one-dimensional integral which has to be evaluated numerically,

\[
P(\varrho, D) d\varrho = \frac{9 \cdot 5^{3/2}}{4\pi N_s \varrho^3 \sigma^4} \int_0^{\infty} ds e^{-(s-3)/2\sigma^2} \\
\times \left(1 + e^{-6s/\sigma^2}\right)^2 \left(e^{-\beta_1^2/2\sigma^2} + e^{-\beta_2^2/2\sigma^2} - e^{-\beta_3^2/2\sigma^2}\right),
\]

where the parameter \( \sigma(t) = D(t)\varrho_0 \) is the standard deviation of the density fluctuations \( \varrho/\bar{\varrho} \) in the linear theory, and \( N_s \) is the mean number of streams, \( N_s = 1 \) in the single stream regime. The expression (27) was derived by a different method earlier on by Kofman (1991), Kofman et al. (1994).

This 3D formula corresponds to the formula (4) derived in 1D. The analytical expression is obviously different, but the method to build it follows the same scheme: the density PDF is obtained from the joint PDF of the eigenvalues \( \lambda_0 \) of the initial local deformation tensor. In the limit of very small \( \sigma \) the formula (27) is reduced to the Gaussian distribution. We plot the PDF calculated numerically from formula (27) in Fig. 5. For moderate values of \( \sigma \) density PDF calculated in the ZA is in good agreement with the PDF from N-body CDM simulations, see Kofman et al. (1994), and also Fig. 7. One can expect formula (27) works even better for those models, for which the pancaking is more pronounced.

### 3.4. Calculation of \( S_p \) for small \( \sigma \) in the ZA

The 3D density PDF in the ZA also has the caustic-induced \( \varrho^{-3} \)-asymptota. As a result, the moments cannot be formally defined in the ZA for any given value of \( \sigma \). However, we can regularize the PDF in the Zel’’dovich approximation by cutting off the high density tail as we did in Sec. 2.2. for 1D case. Then we are able to define the cumulants and derive their large scale properties. We use the same two regularizations, the sharp cutoff and the exponential cutoff as for the 1D case, see equation (4). The parameters \( S_3 \) and \( S_4 \), defined as in relation (7) and calculated with numerical integrations of the moments, are plotted in Fig. 6 as functions of \( \sigma \). They exhibit a very similar qualitative behavior than for the 1D case. Numerical shapes of \( S_p(\sigma) \) are universal for small \( \sigma \approx 0.3 \), i.e. independent of the form of the cutoff and on the parameter of truncation. However, their quantities in the small \( \sigma \) limit are changed compared with 1D case. In 3D case we found \( S_3 \approx 4 \) and \( S_4 \approx 30 \). For \( \sigma \approx 0.3 \), \( S_3 \) and \( S_4 \) depend on the shape of the regularization functions, therefore the moments in the ZA for moderate and large \( \sigma \) are poor-defined.

In the small \( \sigma \) limit more advanced analytical progress can be done for the Gaussian initial fluctuation. Grinstein & Wise (1987), Munshi & Starobinsky (1993) calculated \( S_3(0) = 4 \); Bernardeau et al. (1994), Catelan & Moscardini (1993) calculated \( S_3(0) \) and \( S_4(0) = 272/9 \), all from the perturbation theory around the ZA. The new result we present in the rest of this section is the analytical derivation of the generating function of the cumulants, and consequently, in principle, \( all \) the cumulants themselves, in the ZA.

The most straightforward method of the calculation of the generating function \( C(\mu) \) and cumulants is the same as for the 1D case. Let us use the integral (10) for \( C(\mu) \), where for the density PDF in the ZA we use the integral (26), which includes the joint PDF of the eigenvalues \( \lambda_{0i} \) and the \( \delta \)-function of the density.
Integrating the $\delta$-function over the density, we get the integral over $\lambda_{0i}$-s

$$
\exp C(\mu) = \frac{5^{5/2} 27}{8 \pi \sigma_0^6} \int \frac{d\lambda_{01} \, d\lambda_{02} \, d\lambda_{03}}{|(1 - D\lambda_{01})(1 - D\lambda_{02})(1 - D\lambda_{03})|} \left( \lambda_{01} - \lambda_{02} \right) \left( \lambda_{01} - \lambda_{03} \right) \left( \lambda_{02} - \lambda_{03} \right)
\times \exp \left[ -\frac{1}{\sigma_0^2} \left( 3 J_{01}^2 - \frac{15}{2} J_{02}^2 \right) + \mu \left( \frac{1}{|1 - D\lambda_{01}|(1 - D\lambda_{02})(1 - D\lambda_{03})|} - 1 \right) \right].
$$

(28)

In the limit of small $\sigma$ we can apply the steepest descent method to evaluate this integral. This is a triple integral that, however, can be calculated since $\lambda_{0i}$-s are involved in the integrand in symmetric combinations only. There are three equations for the saddle point $(\lambda_{s1}, \lambda_{s2}, \lambda_{s3})$ in the 3D $\lambda_i$-space, which admit a simple symmetric solution

$$
\lambda_{01} = \lambda_{02} = \lambda_{03} = \lambda_s,
\quad 3D\lambda_s = \frac{\sigma^2 \mu}{(1 - D\lambda_s)^4}.
$$

(29)

Note that $D\lambda_s$ is a function of the combination $\sigma^2 \mu$ only, as it was in 1D case. This fifth-order equation cannot be solved analytically. Despite that, the introduction of the machinery of $G(\tau)$-function described in Sec. 2.3 helps to overcome the problem.

The saddle point equation (29) can be rewritten in terms of $G^Z(\tau)$-function. Indeed, the equation

$$
\tau_s = \mu \sigma^2 \frac{dG^Z}{d\tau}(\tau_s), \quad \tau_s \equiv -3D\lambda_s,
$$

(30)

is reduced to the algebraic equation (29), if we choose

$$
G^Z(\tau) = \frac{1}{(1 + \tau/3)^3} - 1.
$$

(31)

Then the leading factor of the integral (28) for small $\sigma$, emerging from the steepest descent method, gives us the expression for $C(\mu)$:

$$
C^Z(\mu) = \mu G^Z(\tau_s) - \frac{\tau_s^2}{2 \sigma^2},
$$

(32a)

or equivalently

$$
\varphi^Z(y) = y + y G^Z(\tau_s) + \frac{\tau_s^2}{2}, \quad \tau_s = -\frac{y}{2 \sigma^2} \frac{dG^Z(\tau)}{d\tau}.
$$

(32b)

The formulae (31), (32) are the 3D counterparts of (12) and (14) for the 1D case. Again, equations (32) have the structure of the Legendre transformation from $G^Z(\tau)$ to $C^Z(\mu)$ with the variable of the transformation equal to unity afterwards. It is remarkable, that the formulae (31) and (32) can be derived independently from the perturbation series based on the dynamical equations of ZA, similar to the method of Bernardeau (1992) based on the perturbation series of cosmological equations (see also Sec. 4), and what we discussed in Sec. 2. With this approach we obtain form (32) for $C^Z(\mu)$ where the function $G^Z(\tau)$ describes the collapse of the density contrast of a symmetric perturbation of linear overdensity $-\tau$ in the equations corresponding to the three dimensional ZA. The solution of this problem is given by formula (31). Note, that the form (31) has been derived here by a totally different method!

Using the expression (32), we can derive analytically the full series of the $S_p(0)$ parameters in the ZA. For instance we have

$$
S_3^Z(0) = 4,
$$

(33a)

$$
S_4^Z(0) = \frac{272}{9} \approx 30,
$$

(33b)

$$
S_5^Z(0) = \frac{3080}{9} \approx 342.
$$

(33c)
The first two figures are represented by circles in Fig. 6, and give the small \( \sigma \) limit of the numerical curves \( S_3(\sigma) \) and \( S_1(\sigma) \). The next terms of the asymptotic series of the integral (28) would allow, in principle, to derive analytically the \( \sigma \) expansion of the parameters \( S_p(\sigma) \), similar to 1D decompositions (16). We leave this exercise out of the scope of the paper. It is interesting to note that the skewness, kurtosis, etc., in the 3D case are systematically smaller then corresponding numbers of the 1D case, c.f. (33) and (15). As numerical curves of Figs. 2 and 6 show, the \( \sigma \) dependence (where it is well-defined) is also weaker in 3D case than in 1D case. In the ZA the departure from the Gaussian distribution is than faster in 1D case then in 3D case.

3.5 Effects of the final smoothing in the ZA for small \( \sigma \)

In the previous section we found the parameters \( S_p \) and the generating function in the limit of small \( \sigma \) in the framework of the ZA. Using the machinery of the reconstruction, described in Sec. 2., we can obtain the density PDF in this limit from the generating function (19), or from the Edgeworth asymptotic expansion (21). However, it has little sense since we know the PDF (27) for an arbitrary \( \sigma \) in the ZA. What is of greater interest is to use the reconstruction techinics when the final smoothing effects in the generating function and \( S_p \) are taken into account. It can be done for small \( \sigma \) only, and allows, in principle, to reconstruct the density PDF in the ZA with final smoothing.

To get the PDF for the continuous field for different \( \sigma \), we need to filter the Eulerian density field, either in observational surveys, N-body simulations, or analytical approximations. The truncated Zel’dovich approximation, introduced in Sec. 3.1, is based on the smoothing of the initial fluctuations. We recall, that the initial smoothing is inevitable anyway, otherwise we get the multiple streaming regime, for which the ZA is not applicable at all. The fact that the order of the smoothing and the dynamical evolution not commute was noticed by Kofman et al. (1994). It is quite complicated to incorporate analytical approaches given in that paper or in Sec. 3, with the final smoothing. The reason is that the smoothed density does not only depends on the local eigenvalues of the deformation tensor at one point, but on the behavior of the deformation tensor in a whole smoothing large area.

When the filtering is taken into account, the PDF is expected to slightly depend on the shape of the power spectrum, since it is known after Goroff et al. (1986) that the parameters \( S_p \) depend on it. On the other hand, the PDF (27) does not have any dependence with the power spectrum and is only defined by the value of the rms density fluctuation \( \sigma \), showing some limitations for the practical use of this result. For the sake of simplicity, we will assume in the following, that the dependence on the power spectrum can be reduced to the dependence on the effective index \( n \) at the scale at which the field is filtered. The index \( n \) is the logarithmic derivative of \( \sigma \) with the scale \( R \).

There was an attempt by Padmanabhan and Subramanian (1993) to take into account the additional, final smoothing. They made an additional approximation within the ZA, assuming that the typical collapse is spherical. Unfortunately this assumption is valid for a small fraction of the Lagrangian volume, and the resulting distribution poorly fit the numerical PDF. Juszkiewicz et al. (1993) suggested an interesting phenomenological conjecture in order to take into account the final smoothing effects. They pointed out that the \( \sigma \) and \( n \) dependence of the density PDF (for small \( \sigma \)) is mainly contained in the \( S_3 \sigma \) combination. In the context of the ZA this property leads to the substitution \( \sigma \rightarrow \sigma S_3(n)/4 \) in the density PDF (27).

In this Section we present the generalization (in the limit of small \( \sigma \)) of the generating function (32) in the ZA that takes the final smoothing into account. The top-hat filtering is the simplest model of smoothing for which a complete analytical study can be done. We report the results of a general method developed by Bernardeau (1994) and applied here for the ZA. The basic idea is that the non-linear contributions to the \( S_p \) parameters are not affected by the filtering at a given mass \( M_0 \) scale (top-hat filtering in the Lagrangian space). This property is valid for the ZA as well as for the exact single stream cosmological dynamics, but may not be true for other dynamical approximations. The filtering effects at a given radius \( R_0 \) can then be calculated from a transformation of the density PDF from Lagrangian space to Eulerian space: \( \int_{R_0}^{\infty} g P_e(\rho, R_0) \rho d\rho = \int_{R_0}^{\infty} P_L(\rho, M_0) \rho d\rho \), where \( M_0 = 4\pi R_0^3/3 \) (see Bernardeau 1994 for details). It is striking to see that the result in terms of a \( \mathcal{G} \)-function can be derived from that through (32) and the reconstruction formula (19). For the ZA with final smoothing this function will be denoted as \( \mathcal{G}^{ZS} \). The mathematical
transformation to obtain $G^{ZS}$ from $G^Z$ (corresponding to the case without smoothing) is then given by the formula,

$$G^{ZS}(\tau) = G^Z \left\{ \tau \frac{\sigma \left[ (1 + G^Z(S_\tau))^1/3 R_0 \right]}{\sigma(R_0)} \right\},$$  \hspace{1cm} (34)

where $R_0$ is the filtering scale and $\sigma(R_0)$ is the rms density fluctuation at this scale. The generating function of the cumulants is then given by

$$\varphi^{ZS}(y) = y + yG^{ZS}(\tau) + \frac{1}{2} \tau^2, \quad \tau = -y \frac{d}{d\tau}G^{ZS}(\tau).$$  \hspace{1cm} (35)

For instance the coefficients $S^{ZS}_3(0)$ and $S^{ZS}_4(0)$, for a power law spectrum of index $n$, are

$$S^{ZS}_3(0) = 4 - (n + 3),$$

$$S^{ZS}_4(0) = \frac{272}{9} - \frac{50}{3} (n + 3) + \frac{7}{3} (n + 3)^2.$$  \hspace{1cm} (36)

The smoothing correction given by formulae (34), (35) are quite complicated, but final formulae (36) are rather readable.

Fortunately, for one of the most interesting cases $n = -1$, corresponding to the CDM power spectrum on the galaxy clustering scale, there is a simple solution of equation (34)

$$G^{ZS}(\tau) = (1 - \tau/3)^3 - 1,$$  \hspace{1cm} (37)

note difference with eq. (31) for $G^Z(\tau)$. Using this simple form, it is possible then in case $n = -1$ to build the PDF from the reconstruction formula (19). We present the result for PDF for this interesting particular case in Fig. 5 and compare it to the shape of the density PDF in the absence of smoothing obtained both with the formula (27) and with the reconstruction formula (19). As in 1D, the reconstruction method (that neglects the $\sigma$-dependence of the $S_p$ parameters) leads to a less extended high density (exponential) tail, but does not change very much the low density behavior. When the filtering effects are taken into account the density PDF, however, tends to have weaker non-Gaussian features both at the high- and low-density tail. The recipe to build density PDF in ZA with smoothing for arbitrary $n$ is consistent in substituting eqs.(34), (31) into the general reconstruction formula (19), but valid for small $\sigma$ only.

4. SUMMARIZING PERTURBATION SERIES FROM EXACT DYNAMICS

In the previous section we calculated the statistics using the Zel’dovich approximation. Beyond the Zel’dovich approximation, that is when one assumes only the single stream regime, the general form of the density PDF, that would be the counterpart of (27) in ZA, has never been obtained. However, using perturbation theory for Gaussian initial conditions, it is possible to derive cumulants of the density PDF. The approximation which is admitted to apply the perturbation theory, is that the gravitational clustering at sufficiently large scales can be considered in the single stream regime ignoring small scale details. This approach to derive cumulants in quasi-linear dynamics has been intensively used in the literature (Peebles 1980, Fry 1984, Goroff et al. 1986, Bouchet et al. 1992, Juszkiewicz et al. 1993, etc.). Within this approximation to the complicated actual dynamics, the closed form for the generating function of the cumulants is rigorously derived in the limit of small $\sigma$ (Bernardeau 1992, 1994). Since the cumulants very slowly increase with $\sigma$ while $\sigma < 1$, at least for $n \approx -1$, these results can be extrapolated across the whole quasilinear regime, which makes them very useful. In this section we first recall the results of calculation of $S_p(0)$ and the PDF based on the perturbation theory without final smoothing, as it was derived in Bernardeau (1992). Then we report the corresponding results when final smoothing is included (Bernardeau 1994), which are the most important for practical applications. Then we show how the general method of the Edgeworth expansion can be used with the cosmological dynamical perturbation theory.

4.1. Reconstruction of PDF through $S_p$ in the small $\sigma$ limit from the exact dynamics
In 1D, the large–scale limit (e.g. [15]) of the parameters \( S_p \) is exact. In 3D the Zel’dovich solution is really an approximation, at any stage of the dynamics, so that the large–scale limit of the parameters \( S_p(0) \) from the ZA is also an approximation to the exact \( S_p(0) \). However, it is possible to calculate the values of \( S_p(0) \) from the exact single stream cosmological dynamics, without having to know the shape of the density PDF a priori.

The principle of the calculation has been given by Bernardeau (1992) and is summarized in Appendix C. In this section we recall the results that have been obtained to compare them with those obtained from other methods. One can use the basic single stream cosmological equation of gravitational instability, and seek the solution in the form of perturbation series, e.g. \( \delta = \sum_{\nu=1}^{\infty} \delta^{(\nu)} \), see Appendix C. Then there is a set of basic equations in each order \( p \) for \( \delta^{(p)} \). Let us define the following connected (normalized) averages: \( \nu_p \equiv \langle \delta^{(p)}(\delta^{(1)})^p \rangle_c/\sigma^{2p} \). Next, we construct the generating function of these averages

\[
\mathcal{G}(\tau) = \sum_{p=1}^{\infty} \frac{\nu_p}{p!} (-\tau)^p.
\]

The choice of that particular combination is based on the observation that there is a closed equation for \( \mathcal{G}(\tau) \). Indeed, multiplying equations for \( \delta^{(p)} \) by \( (\delta^{(1)})^p \) and averaging the result, and then summarizing the hierarchy of equations, one can derive the single equation for \( \mathcal{G}(\tau) \) (eq. [C7]). It is remarkable that this equation does not contain space derivatives, and has exactly the same form as the equation of the spherically symmetric collapse of the “overdense” \( \mathcal{G}(\tau) \) with a “scalar factor” \( \tau \). A good approximate analytical solution of the “spherical collapse” is given by

\[
\mathcal{G}(\tau) \approx \frac{1}{(1 + \tau/1.5)^{1.5}} - 1. \quad (38)
\]

The function \( \mathcal{G}(\tau) \), in principle, depends on the cosmological parameters, but weakly. The closed form (38) is actually a fitting formula that turns out to be extremely accurate for any cosmological parameters.

The generating function \( \varphi(y) \) of the \( S_p(0) \) parameters (18) can then be built with the generating function (38) of \( \nu_p \) parameters. In Field Theory the averages like \( \langle \delta^{(p)}(\delta^{(1)})^p \rangle_c \) are known as vertices, the averages like the cumulants \( \langle \delta^{(p)} \rangle_c \) then are trees. They can be represented by diagrams where factor \( \sigma \) corresponds to the lines, and factor \( \nu_p \) to the vertices (for a vertex connecting \( p \) lines, see Appendix C for details). There is a very useful and deep general result which links the generating functions of vertices and cumulants: the generating function of cumulants is connected to the generating function of vertices through the Legendre transformation (see Bernardeau & Schaeffer 1992 for references). In the cosmological context the Legendre transformation reads as

\[
\varphi(y) = y + y \mathcal{G}(\tau) + \frac{1}{2} \tau^2, \quad \tau = -y \mathcal{G}'(\tau). \quad (39)
\]

Now using (39) and (20), one can derive the whole series of the \( S_p(0) \) parameters. For instance, we have

\[
S_3(0) = \frac{34}{7} \approx 4.9,
\]
\[
S_4(0) = \frac{60712}{1323} \approx 45.9. \quad (40)
\]

This approach allows to derive the parameters \( S_p \) for \( \sigma = 0 \). Direct perturbative calculations also allow to obtain the parameters \( S_3(0) \), \( S_4(0) \) (Peebles 1980, Fry 1984) and, additionally, should admit derivation of their further \( \sigma \) dependence. Unfortunately, no \( \sigma \) corrections of \( S_p(\sigma) \) were obtained from the perturbation series so far, but analysis of the numerical simulations indicates that \( \sigma \) dependence of \( S_p(\sigma) \) is rather weak in the quasilinear regime (Juszkiewicz et al 1993, and hereafter §5), at least for the interesting cases \( n = 0, -1 \). Outside of the quasilinear regime, \( S_p(\sigma) \) dependence might be noticeable, as some numerical simulations indicate (Lucchin et al. 1994). Note, however, possible impacts of the numerical effects on these calculations (Colombi et al. 1993).
We can reconstruct the density PDF by substituting (39) in the general formula (19). The resulting shape of the density PDF is presented in Fig. 7. Note that this PDF depends on $\sigma$ only and not on the power spectrum, as it was in the ZA without smoothing.

4.2. The final smoothing in $S_p$ and PDF for small $\sigma$

It is also possible to get the full series of the $S_p^S(0)$ parameters when the final smoothing, with a top-hat window function, is taken into account. We can do it via the same method we used earlier for the ZA in Sec. 3.6. The generating function of the cumulants is defined via a $G^S$-function given by the relationship,

$$G^S(\tau) = G\left(\frac{\tau (1 + G^S(\tau))^{1/3} R_0}{\sigma(R_0)}\right).$$

(41)

where $R_0$ is the filtering scale and $\sigma(R_0)$ is the rms density fluctuation at this scale. The generating function of the cumulants is then given by

$$\varphi^S(y) = y + yG^S(\tau) + \frac{1}{2} \tau^2, \quad \tau = -\frac{d}{d\tau} G^S(\tau).$$

(42)

For instance the coefficients $S_3^S(0)$ and $S_4^S(0)$, for a power law spectrum of index $n$, are

$$S_3^S(0) = \frac{34}{7} - (n + 3),$$

$$S_4^S(0) = \frac{60712}{1323} - \frac{62}{3} (n + 3) + \frac{7}{3} (n + 3)^2.$$  (43)

We derive (43) from the generating function (41). The same coefficients $S_3^S(0)$ and $S_4^S(0)$ as function of $n$ were derived independently directly from the perturbation theory (Juszkiewicz et al. 1993, Bernardeau 1993), which confirms the validity of the general formula (41).

Substituting (42) into (19), we can reconstruct the density PDF under all three assumptions. Now it also depends on the power spectrum. The properties of the resulting density PDF are given in details by Bernardeau (1994), and are recalled in the Table of Sec. 6 below.

Despite the fact that the $\sigma$ dependence on the $S_p$ parameters has been neglected, it seems that the formula (19), with the generating function from (42), is the most reliable analytical expression for the density PDF in the mildly non-linear regime (see Sec. 5) for $n \gtrsim -2$.

4.3. The Edgeworth expansion in 3D case

The accuracy of the Edgeworth decomposition in a realistic case of 3D exact single stream dynamics with final smoothing is worth examining. Using the expressions of $S_3$ and $S_4$ (eq. [43]) we can use the decomposition (22) up to the second order in $\sigma$. Note that the use of this decomposition up to the third order would require the determination of the first $\sigma$-correction $\sim \sigma^2$ in $S_3$ which we do not know yet.

In Fig. 8 we plot the density PDF in 3D case reconstructed from the Edgeworth expansion (22) with $S_p(0)$ from (43) for $n = -1$, and compare it with the PDF from (42) and (19), for $\sigma = 0.3$ and $\sigma = 0.5$. We can see that a couple of iterations of the expansion (22) reproduce the peak of $P(\delta)$ in the interval $|\delta| \lesssim 0.5$ around it for small $\sigma$ relatively well. It reproduces well the shift of the maximum towards the low density. It fails to reproduce $P(\delta)$ outside of this interval. For a given value of $\sigma$, each next $\sigma$ iteration improves the approximation quite slowly. The reconstruction is rapidly worsening as $\sigma$ increases, and in practice is useless for $\sigma \gtrsim 0.5$.

As the $S_3$ and $S_4$ parameters are lower than in 1D, the accuracy of the decomposition in 3D case is better. Correspondingly, in 3D case with the final smoothing, the accuracy of the Edgeworth decomposition is also better for smaller $S_3$ and $S_4$, i.e. for larger index $n$. Additionally, from the comparison of 1D and
For the spherical geometry of cells. Fig. 10 shows the Eulerian density PDFs at the top-hat filtering of particle distribution at different time steps.

5. COMPARISON WITH PDF AND \(S_p\) FROM N-BODY SIMULATIONS

We have derived analytically two forms (27) and (19) of density PDF in two reasonable dynamical approximations, the ZA and the perturbation theory in a single stream regime. Apparently, there is no universal formula for the PDF in general. Now we compare our analytical results with those from cosmological N-body simulations. The distribution function (27) has already been tested in a previous work, Kofman et al. (1994) with a density field that had been filtered with a Gaussian window function. From the theoretical point of view, the effect of filtering is better known for a top–hat window function. We therefore run a new series of tests with the top-hat smoothing.

We used a large numerical simulation kindly provided by Couchman (Couchman 1991). The simulation has been made in a box of \(200 \, h^{-1}\text{Mpc}\) size with periodic boundary conditions, and contains \(2.1 \times 10^6\) particles. It used an adaptive \(P^3M\) code and the initial conditions correspond to a CDM power spectrum with \(\Omega = 1\), \(H_0 = 50\text{km s}^{-1}\text{Mpc}^{-1}\) and the bias parameter \(b \approx 1.0\). We made three filterings with a top–hat window function at two different time steps, at redshifts \(z = 0.6\) and \(z = 0\). The three different filtering radii we choose were, \(5 \, h^{-1}\text{Mpc}\), \(10 \, h^{-1}\text{Mpc}\), and \(15 \, h^{-1}\text{Mpc}\). The errors for all the measures that have been made have been determined by dividing the simulation box in eight equal subsamples and by making eight different measurements.

5.1. Calculation of \(S_p\) from N-body simulations

The first test is the determination of the parameters \(S_3\) and \(S_4\) compared to the theoretical predictions. We calculated them from the counts of particles in the ensemble of \(50^3\) spheres disposed on a grid. Thus, it corresponds to a spherical top-hat filtering. As the number of particles is quite significant, the shot noise effects are negligible and have been neglected to compute the moments of the measured distributions. The resulting values of \(S_3\) and \(S_4\) are plotted in Fig. 9 as a function of \(\sigma\). For each filtering radius, we calculated the initial effective index \(n\) of the power spectrum to derive the expected value of of \(S_p(0)\) coefficients from formula (43). For the three filtering radii \(5 \, h^{-1}\text{Mpc}\), \(10 \, h^{-1}\text{Mpc}\), and \(15 \, h^{-1}\text{Mpc}\) we get correspondingly \(n \approx -1.3, -1.0, -0.7\). Correspondingly, three values \(S_3(0) \approx 3.2, 2.9, 2.6\) and three values \(S_4(0) \approx 17.7, 13.3, 10.6\) at \(\sigma = 0\) plotted in Fig. 9 without error bars are these theoretical predictions.

Each curve is related to a different filtering radius: circles, squares and triangles correspond respectively to the smoothing with \(5 \, h^{-1}\text{Mpc}\), \(10 \, h^{-1}\text{Mpc}\), and \(15 \, h^{-1}\text{Mpc}\). First point without error bars on each curve is the theoretical prediction at \(\sigma = 0\) we just discussed, which also can be interpreted as the initial time step at redshift \(z \to \infty\). Two other points on each curve correspond to two other time steps, at the redshift \(z = 0.6\) and at present \(z = 0\). It can be seen that the theoretical prediction (43) – how \(S_p(0, n)\) depend on the initial index \(n – \) is well reproduced by the extrapolation of the numerical curves backward to \(z \to \infty\), i.e. to \(\sigma \to 0\). Moreover, for a given smoothing radius, the \(S_p\) parameters do not exhibit any \(\sigma\) dependence within the error bars. This result makes the theoretical derivation of the \(S_p(0)\) series particularly attractive. It is also obvious that error bars are bigger for bigger filtering scale. Other numerical results by Bouchet & Hernquist (1992), and Lucchin et al. (1993) indicate, however, that a variation with \(\sigma\) may be significative, especially in the nonlinear regime, and for low values of \(n\).

5.2. Calculation of PDF from N-body simulations

The second numerical test is the construction of the density PDF from the N-body simulation. We used the top-hat filtering of particle distribution at different time steps \(z = 0.6\) and \(z = 0\) for different filtering radii \(5 \, h^{-1}\text{Mpc}\) and \(15 \, h^{-1}\text{Mpc}\). Top-hat smoothing allows us to construct the PDF as count-in-cell statistics for the spherical geometry of cells. Fig. 10 shows the Eulerian density PDFs at \(z = 0.6\) for two smoothing radii \(5 \, h^{-1}\text{Mpc}\) and \(15 \, h^{-1}\text{Mpc}\), which corresponds to \(\sigma = 0.92\) and \(\sigma = 0.29\); as well as the PDFs at \(z = 0\) for the same smoothing but with \(\sigma = 1.52\) and \(\sigma = 0.47\). This choice of parameters covers a broad range
of non-linear stages, \(0.3 \lesssim \sigma \lesssim 1.5\). The error bars are the standard deviation of the mean in eight equal sub-samples. We plot \(P(\varrho)\) over wide range of density up to \(\varrho = 10\).

We compare the numerical PDF with the two most advanced analytical predictions. First, we plot the density PDF from (27) derived in ZA without smoothing, for four corresponding values of \(\sigma\). Formula (27) approximates numerical PDF with top-hat smoothing very well up to \(\sigma \lesssim 0.5\) in the range \(0 < \varrho \lesssim \) 3, then starts to overestimate the high-density tail. This is the range of validity of the underlying assumptions of the Zel’dovich approximation without final smoothing. For larger \(\sigma\) the approximation (27) is slightly worsening, and is out of applicability in the multiple stream regime \(\sigma > 1\). Our conclusion confirms of that of Kofman et al. (1994), based on the Gaussian filtering.

We also plot the density PDF from the exact perturbation theory in the single stream regime and with the final smoothing. Assuming that \(S_p\) parameters are constant in quasilinear regime, we substitute the generating function \(\varphi^S(y)\) from equation (42) into the reconstruction formula (19), and calculate \(P(\varrho)\) for corresponding values of \(\sigma\) and \(n\). The results are presented in Fig. 10, and show a remarkable agreement with the numerical density PDF over the whole range of \(\varrho\). We extrapolate the theoretical PDF from the perturbation theory for non-small \(\sigma\) beyond the range of its validity. However, the agreement with the numerical PDF is striking up to the maximal used values \(\sigma \approx 1.5\) and \(\varrho \approx 10\), and so far there are no signs of deviation even for higher \(\sigma\)!

Plausible explanation of why formulae (19), (42) work so well is that \(S_p(\sigma, n)\) parameters depends very weakly on \(\sigma\) up to moderate \(\sigma\), at least for \(n \lesssim -2\). It has been explicitly checked for \(S_2\) and \(S_3\), but it should also be true for any of them otherwise the low- and high-density tails would not have been reproduced so accurately. Then it is not too surprising that the density PDF (27) from ZA for which the \(S_p\) parameters are not constant (see Fig. 6) does not fit well the low- and high-density tails for \(n \lesssim -2\). For \(n \lesssim -2\), however, the \(S_p(\sigma)\) parameters may have a stronger dependence on \(\sigma\), and the ZA could then provide a more reliable PDF. In any case it would be interesting to check these theoretical predictions against the PDF from N-body simulations for higher \(\sigma\).

5.3. Fitting by the log-normal distribution

As it was noted a long time ago by Hubble (1934), the galaxy count distribution in the plane cells on the sky might be well described by the log-normal distribution. The log-normal distribution fits the observed galaxy PDF from 3D surveys as well (Hamilton 1985, Bouchet et al. 1993, Kofman et al. 1994). The log-normal density distribution reads as

\[
P_{\log}(\varrho)d\varrho = \frac{1}{(2\pi \sigma_1^2)^{1/2}} \exp \left[ -\frac{\left( \ln \varrho + \sigma_1^2/2 \right)^2}{2\sigma_1^2} \right] \frac{d\varrho}{\varrho}, \quad \sigma_1^2 = \ln(1 + \sigma^2).
\]

Kofman et al. (1994) found that the log-normal distribution is an excellent approximation to the density PDF from N-body CDM simulation for moderate values of \(\varrho\) in the used range \(\varrho \leq 5\). In Fig. 10 we also compared the density PDF from N-body CDM simulation with the log-normal distribution. We also found a striking agreement between the log-normal PDF and that from N-body CDM simulation for the tested values of \(0.3 < \sigma < 1.5\) in the tested range of \(\varrho \leq 10\!\). Such a remarkable fitting inspires the thought that there might be a strong dynamical reason to manifest the log-normal features of the density PDF. For instance, Coles & Jones (1991) argued for the log-normal mapping of the linear density field to describe its non-linear evolution. Their log-normal model is universal for any spectral index \(n\). Unfortunately the log-normal mapping does not work (Coles et al. 1993).

Why does the log-normal density PDF work so well?

We argue that the log-normal successful fit can be seen as a mere coincidence due to the shape of the CDM power spectrum at moderate \(\sigma\). The log-normal PDF is not a universal form of the cosmic density PDF due to the non-linear dynamics, but is rather a convenient fit for the particular region in the plane of \((\sigma, n)\)-parameters. This region of \((\sigma, n)\)-parameters includes the CDM model at moderate \(\sigma\). It explains why the log-normal PDF fits the results of N-body CDM simulations. Consequently, the “log-normalish” features of the observed density PDF mean that the realistic cosmological model corresponds to that \((\sigma, n)\)-region, i.e. close to the CDM model in this respect.
Let us consider the properties of the cumulants of the log-normal distribution. The first two $S_p$ parameters of this distribution as a function of $\sigma$ are

$$S_3^{\log}(\sigma) = 3 + \sigma^2,$$  \hspace{1cm} (45a)  

$$S_4^{\log}(\sigma) = 16 + 15\sigma^2 + 6\sigma^4 + \sigma^6,$$  \hspace{1cm} (45b)  

for an arbitrary $\sigma$. For the sake of completeness note that the log-normal distribution has $S_p^{\log}(0) = p^{p-2}$, and its $G^{\log}(\tau)$-function is simply $\exp(-\tau)$.

Parameters $S_3^{\log}(\sigma)$ and $S_4^{\log}(\sigma)$ are plotted on Fig. 11 as dashed lines. We compare curves (45) with the values of $S_3$ and $S_4$ from N-body CDM simulations for four values of $\sigma = 0.29, 0.47, 0.92, 1.52$. The log-normal curves (45) are shown to match the CDM parameters for moderate values of $\sigma$ reasonably well. Consequently, we predict the agreement worsening as far as $S_p$-s are departing from each other. The accuracy of the log-normal distribution is good essentially due to the particular slope of the CDM power spectrum at corresponding smoothing scales at moderate $\sigma$. A less steep power spectrum, for instance, would not have led to the same level of agreement. The formula (42) with (19) shows what the expected dependence of the density PDF with $n$ is (see Bernardeau 1994 for more details). The log-normal distribution is expected to fail for the higher values of $\sigma$ where the density PDFs from N-body simulations exhibit a power law behavior (see for instance, Bouchet & Hernquist 1993), which is not the case for the form (44). The low- and high-density tails of the log-normal distribution are very different from those given by the reconstruction formula (19), as it is shown in the Table below.

In practice, however, for an observed power spectrum $n \approx -1$ at moderate $\sigma$, the log-normal distribution is a very effective and simple fit for the density PDF in the mildly non-linear regime. In accompanying paper (Kofman & Bernardeau, 1994) we address the domain of validity of the log-normal fit more specifically.

6. CONCLUSION

6.1. Theoretical framework

Let us recall the assumptions that have been made throughout the paper to derive the various presented results. As we noted in Section 3.1, the actual gravitational dynamics for the realistic cosmological scenarios is quite complicated and includes the superposition of hierarchical pancaking and clustering across the vast range of cosmological scales. In the non-linear regime, the basic equations in terms of the continuous cosmic fields are multiple stream. However, at large scales, ignoring small scale substructures, we expect that the gravitational clustering is simpler, and might be approximated by the single stream regime. Unfortunately, this transition to the large–scale single stream description (i.e. the demonstration that, indeed, the small scale details can be ignored for the large–scale dynamics) has never been done, even in the linear regime. The common (but formally unjustified yet) belief is that after the large–scale filtering the perturbation theory, or truncated ZA can be applied. It relates to a lesser extent to the N-body simulations with inevitable truncation of the genuine power spectrum. We are also working within this assumption, basically because the results we derive are in good agreement with the N-body simulations. However, we clearly understand the nature of the approximation and the need to justify it.

The usual practice is to filter the initial fluctuations in order to model the large scale dynamics. Thus, our first assumption is that

1a) the initial fluctuations are smoothed to ensure being in the single stream regime, the particular filtering scale $R$ is controlled by the rms density contrast, $\sigma = \sigma(R)$.

We used two theoretical approaches (in particular, to derive the density PDF). One is to assume that

2a) the truncated Zel’dovich approximation can be used to describe the large–scale dynamics.

The results which might be derived under these two assumptions are given in §3. Note that the ZA can be used whatever the nature of the initial conditions is, Gaussian or not.
The other approach is the perturbation theory for the cosmological equation of continuity, the Euler and the Poisson equations. This method intrinsically relies on the hypothesis of Gaussian initial conditions. But in any case, even in the single stream approximation, the equations are fully non-linear and it is difficult, if not impossible, to find an exact solution. For instance, the general form of the \( S_p(\sigma) \) parameters as functions of \( \sigma \) are beyond our current skills of analytic calculations. We are then led to derive them in their leading order in \( \sigma \), that is

2b) the cumulants are derived in the limit of small \( \sigma \).

This calculation can be done rigorously from the summarized perturbation series. Note, that in the paper Bernardeau (1992) the PDF was derived under the assumption 1a) and 2b), contrary to how this paper is sometimes quoted.

So, within the assumption 1a) and 2a) or 1a) and 2b) it is possible to make interesting theoretical predictions, for instance, get the dynamical derivations for the one-point density PDF and moments. However, these calculations do not take into account the final filtering that, in practice, cannot be avoided. This is a crucial step since it alters the statistical properties of the cosmic fields. If the scale of the final filtering is larger than the scale of the initial filtering, then the initial filtering is irrelevant, and we can replace the assumption 1a) by the assumption that

1b) this is the final smoothing only, which assures that the large–scale dynamics can be accurately described by the single stream approximation.

Even within the simplified dynamics given by the ZA it is no longer possible to derive the density PDF under this last assumption. The only known approach that allows to do this comes from perturbative calculations, since it makes it possible to derive the leading order of the cumulants of the final density field. It is then a considerable improvement compared to previous results. The assumptions 1b) and 2b) then lead to a density PDF in very good agreement with the numerical results.

6.2. Results

We derived the density PDF, cumulants in forms of \( S_p(\sigma) \)-parameters and their generating functions in case of 1D gravitational dynamics, in the 3D Zel’dovich approximation with and without final smoothing, in the perturbation theory extrapolated over mildly non-linear regime, with and without final smoothing, for the log-normal distribution, and from cosmological N-body CDM simulation. We summarize the quantitative results in the Table.

We also present new qualitative results stemming from our study. As one can see from the Table, the values \( S_p(0) \) are constants which characterize the particular dynamical model. For example, these values are different for 1D gravitational instability, 3D Zel’dovich approximation, single stream 3D gravitational instability, sea waves dynamics, etc. The common trend is that \( S_p \) is rapidly increasing with \( p \), but at a different rate. The \( \sigma \)-dependence of the \( S_p(\sigma) \) parameters also characterizes the particular dynamics, and as we illustrated for different models, might also be quite different. The smoothing effects introduces an extra dependence on the power spectrum index \( n \), \( S_p(\sigma, n) \). We can expect \( S_p(\sigma, n) \) from different models can coincide in some regions of parameters (\( \sigma, n \)). It allows to construct simple fitting formula, as we found it for the log-normal distribution.

\( S_p(0) \)-coefficients are related to the particular non-Gaussian statistics which emerges from the non-linear dynamics. The usual practice of their direct measurements from observations is significantly affected by the final sample volume. We, however, learned from the Edgeworth expansion that \( S_p(0) \) coefficients are related to the shape of the PDF peak. It gives us an alternative method to evaluate the skewness and kurtosis by measuring the PDF around its maximum, which is statistically more robust. This approach might be interesting in other contexts, such as to constrain the skewness of the cosmological \( \Delta T/T \) fluctuations, or skewness and kurtosis of the cosmological velocity field and its divergence.

There is a deep connection between the generating function of \( S_p(0) \) parameters and \( G(\tau) \)-function – the generating function of vertices, and the non-linear dynamics of the spherical overdense fluctuation in the particular dynamical model. The solution of the last problem in the particular dynamical model
gives the $G(\tau)$-function, which allows to reconstruct the density PDF in the limit of small $\sigma$. This is a remarkably simple and general prescription. From the Table note a simple generalization of the $G(\tau)$ for the models without final smoothing: 

$$G(\tau) = (1 + \tau/\alpha)^{-\alpha} - 1.$$  

For the Zel’’dovich approximation in the space of $N$-dimensions, $\alpha = N$. For the 3D perturbation theory $\alpha \approx 1.5$. In the limit $N \gg 1$ we get $G(\tau) = \exp(-\tau) - 1$, which coincides with that of the log-normal distribution.

We have derived the cosmological density PDF in two different approximations. The one based on ZA gives the density PDF in the early non-linear regime, $\sigma \lesssim 0.5$, and is expected to improve for the models with $n \lesssim -2$, for which pancaking is more pronounced. This approach remains irreplaceable for models with non-Gaussian initial density fluctuations. However, we found that the best theoretical model for the density PDF evolving from Gaussian distribution, in the case of $n \gtrsim -2$, is based on the perturbation theory when the final smoothing is included. This PDF works remarkably well for significant range of $\sigma$ in the mildly non-linear regime. Both approaches provide us with an efficient machinery to deal with one-point statistics for a broad range of models. In the Table we summarize the properties of the various approximations that have been presented and used in this paper. It shows, in particular, that the low-density tail is not affected by the reconstruction method. On the other hand, the shape of the high-density tail is dramatically modified. Both tails are also slightly modified, where the smoothing effects are taken into account.

It was noted earlier and further confirmed that the log-normal distribution is an excellent fit for the density PDF from CDM non-linear dynamics. We found an explanation of this mystery, based on the properties of cumulants. The log-normal distribution fits well in the particular range of the parameter space $(n, \sigma)$ around $n \sim -1$, $\sigma \sim 1/2$, and worsening outside of this region. By chance, the popular CDM model at moderate $\sigma$ corresponds to this region. Thus, some “log-normal” features of the observed density PDF would mean that realistic cosmological model is close to that range of parameters.

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APPENDIX A: Statistical technics

In this appendix, we give technical definitions of interest for statistical studies, such as the moments of the density distribution function, its cumulants, the generating functions of the moments and of the cumulants. Relationship between those quantities are also given. The moments \( \langle \delta^p \rangle \) of the distribution function, \( P(\varrho) \), are given by the integrals,

\[
\langle \delta^p \rangle = \int_0^\infty d\varrho \; P(\varrho)(\varrho - 1)^p.
\]  

(A1)

The cumulants can then be obtained from the moments. The \( p^{th} \) cumulants, \( \langle \delta^p \rangle_c \), is defined recursively from the \( p^{th} \) moment following the rules,

\[
\langle \delta \rangle_c = 0,
\]

\[
\langle \delta^2 \rangle_c = \langle \delta^2 \rangle \equiv \sigma^2,
\]

\[
\langle \delta^3 \rangle_c = \langle \delta^3 \rangle,
\]

\[
\langle \delta^4 \rangle_c = \langle \delta^4 \rangle - 3 \langle \delta^2 \rangle_c^2,
\]

\[
\langle \delta^5 \rangle_c = \langle \delta^5 \rangle - 10 \langle \delta^2 \rangle_c \langle \delta^3 \rangle_c,
\]

\[
\ldots
\]

(A2)

In general, to obtain the \( p^{th} \) cumulant, one should consider all the decompositions of a set of \( p \) points in its subsets (but the one being only the set itself); multiply, for each decomposition, the cumulants corresponding to each subset and subtract the results of all these products obtained in that way from the \( p^{th} \) moment.

Each cumulant, in a given order, actually contains a piece of information about the shape of the PDF that cannot be derived from the lower order cumulants. For instance, the second cumulant gives the width of the distribution, the third cumulant measures its asymmetry, the fourth – its flatness, etc. The Gaussian distribution is characterized by only one non-zero cumulant (whereas all even moments are nonzero): the second one, which gives its variance. However, in general, a distribution function will be characterized by the whole series of cumulants. Therefore, to treat more easily the series of the cumulants, we are led to define other mathematical tools of great practical interest. There are the generating functions of the moments and the cumulants. The generating function of the moments is defined by,

\[
\mathcal{M}(\mu) = 1 + \sum_{p=1}^{\infty} \frac{\langle \delta^p \rangle \mu^p}{p!},
\]

(A3)

The generating function of the cumulants, \( \mathcal{C}(\lambda) \), is defined in a similar way,

\[
\mathcal{C}(\mu) = \sum_{p=2}^{\infty} \frac{\langle \delta^p \rangle_c \mu^p}{p!}.
\]

(A4)

One fundamental result of the statistics is that \( \mathcal{M}(\mu) \) and \( \mathcal{C}(\mu) \) are connected in a simple way. Indeed, we have

\[
\mathcal{M}(\mu) = \exp[\mathcal{C}(\mu)].
\]

(A5)

We omit here the rigorous demonstration of this property (see, e.g. Balian & Schaeffer 1989). By expanding \( \exp[\mathcal{C}(\mu)] \) with respect to \( \mu \), one can easily verify that the first few moments are given correctly.

Using this property, it is then possible to relate the generating function of the moments, or of the cumulants, to the shape of the PDF. Indeed, we have

\[
\mathcal{M}(\mu) \exp[\mathcal{C}(\mu)] = \sum_{p=0}^{\infty} \int_0^\infty d\varrho \; P(\varrho) \frac{[(\varrho - 1)\mu]^p}{p!}
\]

\[
= \int_0^\infty d\varrho \; P(\varrho) \exp[(\varrho - 1)\mu].
\]

(A6)
APPENDIX B: Derivation of the Edgeworth expansion

In this appendix we present the derivation of the Edgeworth expansion. It is based on the reconstruction (19) for the density PDF. The generating function \( \varphi(y) \) can be expanded with respect to \( y \) (eq. [20]). We then have to expand the non-Gaussian part of the exponent in (19),

\[
\exp \left[ -\frac{\varphi(y)}{\sigma^2} \right] \approx \exp \left[ -\frac{-y + y^2}{2\sigma^2} \right] \times \left[ 1 - \frac{S_3}{3! \sigma^2} y^3 + \frac{S_4}{4! \sigma^2} y^4 + \frac{S_5^2}{2(3!)^2 \sigma^4} y^6 - \frac{S_5 S_4}{5! \sigma^2} y^5 - \frac{S_3}{3! 4! \sigma^4} y^7 - \frac{S_3^3 y^9}{(3!)^4 \sigma^6} + \ldots \right].
\]

This expansion should be made with respect to both \( y \) and \( \sigma \), assuming they are of the same order. The resulting value of the density PDF requires the determination of integrals of the form

\[
I_n = \int_{-i\infty}^{+i\infty} \frac{dy}{2\pi i \sigma^2} \exp \left[ \frac{y^2}{2\sigma^2} + \frac{\nu - 1}{\sigma^2} y \right] y^n,
\]

for \( n \geq 3 \) that gives

\[
I_n = \frac{(-1)^n}{(2\pi \sigma^2)^{1/2}} \exp \left( -\nu^2/2 \right) \sigma^n H_n(\nu), \quad \nu = \delta/\sigma
\]

where \( H_n(\nu) \) are the Hermite polynomials,

\[
H_n(\nu) \equiv (-1)^n \exp(\nu^2/2) \frac{d^n}{d\nu^n} \exp(-\nu^2/2) = \nu^n - \frac{n(n-1)}{1!} \frac{\nu^{n-2}}{2} + \frac{n(n-1)(n-2)(n-3)}{2!} \frac{\nu^{n-4}}{2^2} - \ldots
\]

The resulting form of the density PDF is

\[
P(\delta)d\delta = \frac{1}{(2\pi \sigma^2)^{1/2}} \exp \left( -\nu^2/2 \right) \left[ 1 + \sigma \frac{S_3}{6} H_3(\nu) + \sigma^2 \left( \frac{S_4}{24} H_4(\nu) + \frac{S_5^2}{72} H_6(\nu) \right) + \sigma^3 \left( \frac{S_5}{120} H_5(\nu) + \frac{S_4 S_3}{144} H_7(\nu) + \frac{S_3^3}{1296} H_9(\nu) \right) + \ldots \right] d\delta.
\]
APPENDIX C: \(S_p(0)\) from the Perturbation Theory

In this appendix we give a brief sketch of the derivation of some technical details used Sec. 4.1. Cosmological gravitational instability of the perfect fluid without pressure in the single stream regime is described by the continuity equation

\[
\frac{\partial \rho}{\partial t} + 3H \rho + \frac{1}{a} \nabla \cdot (\rho \mathbf{v}) = 0, \tag{C1a}
\]

the Euler equation

\[
\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a}(\mathbf{v} \cdot \nabla) \mathbf{v} + H \mathbf{v} = -\frac{1}{a} \nabla \phi, \tag{C1b}
\]

and the Poisson equation

\[
\nabla^2 \phi = 4\pi G a^2 (\rho - \bar{\rho}). \tag{C1c}
\]

It is convenient to use \(a(t)\) as a new time variable, and the velocity potential \(\Phi\) for the potential flow under consideration, see §3.1 for definition. Let us consider the case of the Einstein-de Sitter Universe. Then three equations (C1) can be rewritten in form of two equations:

\[
a \frac{\partial}{\partial a} \delta(x, a) + (1 + \delta(x, a)) \Delta \Phi(x, a) + \nabla \delta(x, a) \cdot \nabla \Phi = 0 \tag{C2a}
\]

and

\[
a \frac{\partial}{\partial a} \Delta \Phi(x, a) + \frac{1}{2} \Delta \Phi(x, a) + \nabla \Phi(x, a) \cdot \nabla (\Delta \Phi(x, a))
\]

\[
\quad + \sum_{\alpha, \beta = 1}^{3} \Phi_{\alpha \beta}(x, a) \Phi_{\alpha \beta}(x, a) + \frac{3}{2} \delta(x, a) = 0, \tag{C2b}
\]

with \(\Phi_{\alpha \beta}(x, a) = (1/aH)^2 \partial^2 \Phi(x, a)/\partial x_\alpha \partial x_\beta\) where \(x_\alpha\) is the component \(\alpha\) of \(x\).

Let us seek the solution of the equations (C2) in the form of perturbation series, e.g. \(\delta = \sum_{p=1}^{\infty} \delta^{(p)}\), \(\nabla^2 \Phi = \sum_{p=1}^{\infty} (\nabla^2 \Phi)^{(p)}\), etc. Then we obtain the hierarchy of equations for values \(\delta^{(p)}\), \((\nabla^2 \Phi)^{(p)}\) etc., in each order \(p\) of the perturbation series. Then, at the lowest order of \(\sigma\), the cumulant \(\langle \delta^p \rangle_c\) is given by a term in the form

\[
\langle \delta^p \rangle_c = \sum_{\text{combinations}, p(i)} \langle \prod_{i=1}^{p} \delta^{(p(i))} \rangle_c \tag{C3a}
\]

where the sum is taken over all the possible combinations \(p(i), i = 1 \ldots p\), for which

\[
p(i) \geq 1, \quad \sum_{i=1}^{p} p(i) = 2p - 2. \tag{C3b}
\]

This particular result is correct due to the hypothesis of the Gaussian initial conditions. The terms in (C3b) are all products of the rms density fluctuation at the power \(2p - 2\) by some combination of the vertices,

\[
\nu_p \equiv \langle \delta^{(p)} \delta^{(1)} \rangle_c^{1/\sigma^2}. \tag{C4}
\]

(and respectively \(\mu_p\) for \(\nabla^2 \Phi\), etc.) The parameters \(S_p\) (eq. [7]) for \(\sigma = 0\) are then given by a combination of the vertices, \(\nu_p\), that technically corresponds to a tree summation. The relationship between the \(S_p\) parameters and the vertices can be written in a closed form at the level of their corresponding generating functions. Let us define the function \(G(\tau)\) by

\[
G(\tau) = \sum_{p=1}^{\infty} \frac{\nu_p}{p!} (-\tau)^p. \tag{C5}
\]
(a similar one can be defined for other vertices). Then we have

$$\varphi(y, 0) = y + y \bar{G}(\tau) + \frac{\tau^2}{2}, \quad \tau = -y \frac{d}{d\tau} \bar{G}(\tau),$$  \hspace{1cm} (C6a)

where $\varphi(y, \sigma)$ is defined in (20).

The problem then is to derive the function $\bar{G}(\tau)$ from the equations (C2). F.B. 1992 derived useful properties of the generating functions of vertices. Multiplying equations (C2a) by $(\delta^{(1)})^P$ and averaging the result, and repeating similar operation with eq. (C2b), and then using properties of the generating functions of vertices, one can finally get the single equation for $\bar{G}(\tau)$:

$$- (1 + \bar{G}) \tau^2 \frac{d^2}{d\tau^2} \bar{G} + \frac{4}{3} \left( \tau \frac{d}{d\tau} \bar{G} \right)^2 - \frac{3}{2} (1 + \bar{G}) \tau \frac{d}{d\tau} \bar{G} + \frac{3}{2} \bar{G}(1 + \bar{G})^2 = 0. \quad \hspace{1cm} (C7)$$

with $\bar{G}(\tau) \sim -\tau$ when $\tau \to 0$.

It is remarkable, that this equation does not contain any space derivatives, and has exactly the same form as the equation of the spherically symmetric collapse of the “overdense” $\bar{G}(\tau)$ with a “scalar factor” $\tau$. The analytical solution of the equation (C7) is well known. When $\tau < 0$

$$\bar{G} = \frac{9}{2} \frac{(\theta - \sin \theta)^2}{(1 - \cos \theta)^3} - 1, \quad \tau = -\frac{3}{5} \left( \frac{3}{4} (\theta - \sin \theta) \right)^{2/3} \quad \hspace{1cm} (C8a)$$

and when $\tau > 0$

$$\bar{G} = \frac{9}{2} \frac{(\sinh \theta - \theta)^2}{(\cosh \theta - 1)^3} - 1, \quad \tau = \frac{3}{5} \left( \frac{3}{4} (\sinh \theta - \theta) \right)^{2/3}. \quad \hspace{1cm} (C8b)$$

It turns out that the form (38) for $\bar{G}(\tau) \approx 1/(1 + \tau/1.5)^{1.5} - 1$ is a very good fit to the exact solution (C8).

In general case of arbitrary background cosmology the function $\bar{G}(\tau)$ depends on the cosmological parameters. Fortunately, this dependence is weak, and the form (38) can be accurately used in general case.
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Table and Figure captions

Table 1. Properties of the density PDF obtained with the various approximations described in the text. The first two columns give the values of the $S_3$ and $S_4$ parameters. In general case they depend on $\sigma$. A couple of first terms of the $\sigma$-expansion are given in cases they are known. The third column gives the shape of the function $G(\tau)$ which is used to derive the function $\varphi(y)$, involved in the reconstruction formula (19). The last two columns give the high- and low-density asymptotes of the resulting distributions.

Fig 1. Shape of the density PDF for 1D dynamics and for $\sigma = 0.5$ (eq. [4]) (solid line). The dotted, dashed, long dashed and dotted dashed lines show the shape of the density PDF defined in Eq. (6) for respectively $\rho_c=5, 7.5, 10, 12.5$.

Fig 2. Coefficients after regularization for the 1D dynamics. The filled circles correspond to the theoretical limits for $\sigma \to 0$ (Eqs. [15]), and the thick solid lines correspond to the theoretical curves (Eqs. [16]). The dotted, dashed, long dashed and dotted dashed lines are for a sharp cutoff (Eq. [5], left panel) or an exponential cutoff (Eq. [6], right panel) for respectively $\rho_c =5, 7.5, 10, 12.5$.

Fig 3. The shape of the density PDF as obtained from eq. (4), thin solid lines, and by the reconstruction method [eq.(19)], thick solid lines, for $\sigma = 0.3$ (left panel) and $\sigma = 0.5$ (right panel).

Fig 4. The Edgeworth expansion (Eq. [22]) compared to the form (4) for $\sigma = 0.1$ (left panel) and $\sigma = 0.3$ (right panel). The dashed line corresponds to the case when the skewness only is taken into account (up to $\sigma$ correction), the long dashed lines when both the skewness and the kurtosis are taken into account, up to $\sigma^2$ corrections, and the dotted line when the expansion is made up to the $\sigma^3$ order (Eq. [22]).

Fig 5. Shape of the density PDF for $\sigma = 0.3$ (left panel) and $\sigma = 0.5$ (right panel) for various methods based on the Zel’dovich approximation. The solid line is the shape obtained by the Zel’dovich approximation, (Eq. [27]); the other curves have been obtained by the reconstruction formula (19) by assuming that the ratios $\langle \delta^p \rangle / \langle \delta^2 \rangle^{p-1} = S_p^Z$ equal their low-$\sigma$ limit in the ZA without taking into account the smoothing effects (dashed lines), and taking into account the smoothing effects for $n = -1$ (long dashed lines).

Fig 6. Parameters after regularization for the 3D dynamics. The dotted, dashed, long dashed and dotted dashed lines are for the sharp cutoff (left panel) or the exponential cutoff (right panel) for respectively $\rho_c =5, 7.5, 10, 12.5$ in the regularized expression (27). The filled circles correspond to their theoretical limit at small $\sigma$ and the squares to their theoretical limits when the filtering effects are taken into account, for $n = -1$.

Fig 7. Shape of the density PDF for $\sigma = 0.3$ (left panel) and $\sigma = 0.5$ (right panel) from the reconstruction formula (19) using the value of the cumulants from (38) for the exact dynamics in the $\sigma \to 0$ limit. The solid line is the density PDF obtained by assuming that the values $S_p$ equal their low $\sigma$ limit without taking into account the smoothing effects. The dashed line is obtained when the smoothing effects are taken into account assuming that the initial power spectrum is $P(k) \propto k^{-1}$. The long dashed line is the log-normal distribution.

Fig 8. The Edgeworth expansion (Eq. [21]) compared to the density PDF obtained with (42) and the reconstruction formula (19) for $n = -1$, $\sigma = 0.3$ (left panel) and $\sigma = 0.5$ (right panel). The dashed line corresponds to the case where the skewness only is taken into account (up to $\sigma$ correction), and the long dashed lines to the case where both the skewness and the kurtosis are taken into account, up to $\sigma^2$ corrections. The long dashed line is the lognormal distribution.
Fig 9. The coefficients $S_3(\sigma)$ and $S_4(\sigma)$ as functions of $\sigma$ in a CDM numerical simulation. The circles, squares and triangles correspond respectively to the smoothing radii of 15, 10, 5 $h^{-1}$Mpc. Three points at each curve corresponds to three different time steps, at $z = \infty$, 0.6, 0. The values at $\sigma = 0$ (or $z \to \infty$) are the theoretical predictions from (43) taking into account the filtering effects.

Fig 10. The density PDF for CDM initial conditions. The points are measured in a numerical simulation at two different time steps corresponding to $a/a_0 = 0.6$ (upper panel) and $a/a_0 = 1$ (lower panel) and at two different smoothing radii $R_0 = 5h^{-1}$Mpc and $R_0 = 15h^{-1}$Mpc. The rms density fluctuation are then respectively $\sigma = 0.92$ and $\sigma = 0.29$ in the upper panel and $\sigma = 1.52$ and $\sigma = 0.47$ in the lower panel. The error bars have been obtained by dividing the sample into eight subsamples. The solid line is the prediction given by (19) when the smoothing effects are taken into account (with Eq. [42]). The dashed line is the prediction (27) from the ZA and the long dashed line is the lognormal distribution (44).

Fig 11. The measured values of the $S_3(\sigma)$ and $S_4(\sigma)$ parameters in the CDM simulation as functions of $\sigma$. The solid lines correspond to the values of these coefficients from formula (43) for $n = -1$. The thick dashed lines correspond to the log-normal distribution, eq. (45). The $S_3$ and $S_4$ values in CDM and log-normal models overlap for moderate $\sigma \sim 0.5$ only.