Abstract. In this research article, the techniques for computing an analytical solution of 2D fuzzy wave equation with some affecting term of force has been provided. Such type of achievement for the aforesaid solution is obtained by applying the notions of a Caputo non-integer derivative in the vague or uncertainty form. At the first attempt the fuzzy natural transform is applied for obtaining the series solution. Secondly the homotopy perturbation (HPM) technique is used, for the analysis of the proposed result by comparing the co-efficient of homotopy parameter $q$ to get hierarchy of equation of different order for $q$. For this purpose, some new results about Natural transform of an arbitrary derivative under uncertainty are established, for the first time in the literature. The solution has been assumed in term of infinite series, which break the problem to a small number of equations, for the respective investigation. The required results are then determined in a series solution form which goes rapidly towards the analytical result. The solution has two parts or branches in fuzzy form, one is lower branch and the other is upper branch. To illustrate the ability of the considered approach, we have proved some test problems.

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1. Introduction. Different differential, difference and integral equation and system have been used for many years to represent some real-world problems. Among them are Heat, Wave, KDV, Burgers, Laplace, Schrödinger and Sine-Gordon equations representing mostly physical phenomenon. Some mathematical models like SI, SIR, SEIR, HIV, and logistic models can be used for epidemiological phenomenon and social sciences. These equations have been analyzed for series, analytical, semi-analytical, and numerical solution to check their convergency and stability by various mathematical techniques. Since from the beginning of the 21st century, the field of Modern calculus like arbitrary order differential and anti-differential calculus have attained much interest by the scientists and scholars. Fractional calculus has more precise and realistic results in many fields like physic, engineering, biological sciences; therefore, it plays a vital role to solve various problems of practical life. Arbitrary order derivative is a global derivative which gives a lot of numbers of choices. Due to vast applications of fractional calculus, this field explored well in many articles, monographs, chapters, etc. see in detail [41, 35, 28, 46, 48, 6, 18, 38, 25, 39].

After the 1750’s “Reimann and Liouvilli”, “Euler and Fourier” have found an analytical solution to integer-order of differential and integral calculus. Later on, the field of fractional calculus initiated, and the researchers modeled many real-world problems by using fractional-order differential and integral equations in a much better way than the integer-order DEs, PDEs and IDEs. Fractional calculus have reduced many of the errors, which occurred in classical calculus. Application of fractional calculus may be seen in [41, 35, 46, 47, 26, 54, 34, 57, 55, 60, 56]. The researcher gave more valuable time in studying of fractional calculus. Surely non-integer order derivative is an anti-derivative of a definite type (means that the summation of the whole quantity) which make it generalized and globalized. The scholars investigated the existence theory, uniqueness results, approximation, optimization and stability analysis of the mathematical models, and obtained some beneficial results in this regards. Therefore, an arbitrary-order differential operator may be formulated in a large number of ways. Also, definite integration has no kernel of a regular type so a different type of kernel is provided as cited in different references. One such type of formula having currently gained more interest is of “ABC” non-integer derivative defined by “Attangana-Baleanu” and “Caputo” [5] in 2016. This arbitrary order derivative changed the “singular kernel” by “non-singular kernel” and because of this, it is studied on high-level [51, 50]. To deal with such type problems for analytical results, various methods including perturbation and decomposition techniques have been used increasingly (see [23, 24]). Furthermore, for numerical results, RK4 methods and generalized Taylor series methods were used in the last few years. Similarly, ABC method has also been used for numerical results (see [43, 3]). The aforesaid techniques is quickly converging and stable [31, 14].

Fractional calculus has many applications in the fields where data are not precise like biological, natural and engineering sciences [9, 16, 32, 36]. Before studying such problems, let us know about the fuzzy set. Zadeh introduced the concept of fuzzy set in 1965 [59], that how to measure uncertainty present in certain phenomena. The theory of fuzzy set is, therefore, extended to many other branches like topology, algebra, analysis, fuzzy logic, automata and many more. They further, elaborate the concept, by defining fuzzy-function and control [15]. Based on these results, the researchers extended this concept by introducing elementary fuzzy calculus [17, 22, 32]. Recently, fuzzy fractional differential and integral equations abbreviated
as FFDEs and FFIEs respectively got much interest due to its applicability in formulating real-society phenomenon. Esmail et al. studied some fundamental results of numerical analysis in [20]. To model these types of problems “fuzzy fractional differential equations” (FFDEs) and “fuzzy fractional integral equations” (FFIEs) can be used. Much more scholars and scientists consider this to analyze both the integer and fractional “fuzzy differential equations”. Due to the diverse applications “fuzzy differential equation” to model uncertain phenomena in various fields, such as biological and physical sciences, business etc., see [9, 16, 32, 36].

To investigate FFDEs different techniques like Fourier integral transform, “Laplace”, “Sumudu”, and Natural transform etc. were suggested in the past. Laplace transforms [33, 11, 7, 1, 52, 58] are widely used to solve the differential and integral equations. Later on, the Sumudu transform was introduced in 1993 and then extended to two variables by the same author in 2002 [10]. These transforms also have many uses in the solutions of various types of FDEs. By introducing “Natural transform” abbreviated by NT in 2008, initially used to the evaluation of fluid flowing situations of “Maxwell’s equations”. NT also used in the solution of ODEs [48, 6, 18, 12, 37]. NT has the ability of fast converging than Laplace and Sumudu transforms only by varying the parameters. Some of the iterative and series solution methods like HPM and improved HPM, “Addomian” decomposition and “Laplace Adomian decomposition technique”, “Taylor’s series technique”, also used for dealing such problems.

Due to the diverse applications of the NT, we will evaluate the semi-analytical solution of the given two-dimensional fuzzy non-integer order wave equation under Caputo fractional derivative by repeated iterative numerical techniques applying NT along with homotopy perturbation techniques (NTHPM).

\[ 
\begin{align*}
D_\theta^\alpha \tilde{U}(x, y, t, r) &= D_x^2 \tilde{U}(x, y, t, r) + D_y^2 \tilde{U}(x, y, t, r) + \tilde{F}(x, y, t, r), \quad 1 < \theta \leq 2, \\
\tilde{U}(x, y, 0) &= \tilde{G}_1(x, y), \\
\tilde{U}_t(x, y, 0) &= \tilde{G}_2(x, y),
\end{align*}
\]

(1)

de here \(D\) represent Caputo fractional derivative and external source \(\tilde{F}\) belongs to the field of continuous fuzzy mapping, and \(\tilde{G}_1, \tilde{G}_2\) are continuous fuzzy-valued functions.

Different methods like Laplace, Fourier, Sumudu, and Natural transforms, etc. [61, 19, 42, 27, 44, 45, 30, 4, 29, 49, 40, 2, 21] have been used to investigate wave and various differential equation. Some of the scholars analyzed two-dimensional wave equation by “double Laplace transform” without external force or function \(\tilde{F}\). However, in this article, a very authentic and generalized technique of NTHPM is used to solve the two-dimensional wave equation with an external force or function \(\tilde{F}\). To illustrate the said technique, we provide some numerical examples along and their graphs to investigate the validity of the considered analysis.

2. Background materials. Now we provide some basic concepts of fractional analysis (see [41, 35, 32, 53, 52]).

**Definition 2.1.** The NT for derivative of a function is

\[ N[D_\theta^\alpha Z(x, t)] = \frac{s^\theta}{u^\theta} N(Z(x, t)) - \sum_{i=0}^{k-1} \frac{s^{k-i-1}}{u^{k-i}} Z^i(x, 0). \]
**Definition 2.2.** (see [41, 35, 32, 53, 52]) The function $E_\beta(p)$ for $\beta > 0$ can be expressed as

$$E_\beta(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(i\beta + 1)}.$$  

**Definition 2.3.** Let $\mathcal{K} : R \to [0, 1]$ is known as “fuzzy number” if it satisfies

(i) $\mathcal{K}$ is “upper semi-continuous”;
(ii) $\mathcal{K}$ is fuzzy convex.
(iii) $\exists y_0 \in R : \mathcal{K}(y_0) = 1$;
(iv) $d\{\mathcal{K}(y) > 0, \text{ for } y \in R\}$ is compact.

**Definition 2.4.** In the parametrization situation “Fuzzy number” is represented by $[\mathcal{K}(x), \mathcal{K}(x)] = [x - 1, 1 - x]$, where $0 \leq x \leq 1$, satisfies the following

(i) $\mathcal{K}(x)$ is decreasing and continuous function in $(0, 1]$, and right continuous at 0.
(ii) $\mathcal{K}(x)$ is increasing and continuous function in $(0, 1]$, and right continuous at 0.
(iii) $\mathcal{K}(x) \geq \mathcal{K}(x)$.

Let $\kappa_1 \geq 0$, then various operations are defined as follow:

(i) Addition: $(\mathcal{U}(x), \mathcal{V}(x)) + (\mathcal{W}(x), \mathcal{W}(x)) = (\mathcal{U}(x) + \mathcal{W}(x), \mathcal{V}(x) + \mathcal{W}(x))$.
(ii) Subtraction: $(\mathcal{U}(x), \mathcal{V}(x)) - (\mathcal{W}(x), \mathcal{W}(x)) = (\mathcal{U}(x) - \mathcal{W}(x), \mathcal{V}(x) - \mathcal{W}(x))$.
(iii) Scalar multiplication: $\kappa_1 \cdot (\mathcal{V}(x)) = \begin{cases} (\kappa_1 \mathcal{U}(x_0), \kappa_1 \mathcal{V}(x_0)) \kappa_1 \geq 0, \\ (\kappa_1 \mathcal{V}(x_0), \kappa_1 \mathcal{V}(x_0)) \kappa_1 < 0. \end{cases}$

**Definition 2.5.** Let $D_1 : R \times R \to R_+ \cup \{0\}$ be an operator, $V = (\mathcal{V}(x), \mathcal{V}(x))$ and $W = (\mathcal{W}(x), \mathcal{W}(x))$ be any of the two number in parametric form of fuzzy number, then the “Hansdoff distance” from $V$ to $W$ will be computed as

$$D_1(V, W) = \sup_{r \in [0, 1]} \max\{|\mathcal{V}(r) - \mathcal{W}(r)|, |\mathcal{V}(r) - \mathcal{W}(r)|\}$$

In $R$, the metric space $D_1$, will satisfies

(i) $D_1(V + V_0, W + V_0) = D_1(V, W)$ for all $V_0, V, W \in R$,
(ii) $D_1(\kappa_1 \cdot V, \kappa_1 \cdot W) = |\kappa_1|D_1(V, W)$ for all $\kappa_1 \in R, V, W \in R$,
(iii) $D_1(V + \nu, W + V) \leq D_1(V, W) + D_1(V, W)$ for all $V, W, \nu, V \in R$,
(iv) $(R, D_1)$ is a metric called “complete space”.

**Definition 2.6.** Let $a, b \in E$, then, the “Hukuhara-difference (H-difference)” of $a$ and $b$ is denoted by $a \ominus_H b$, if exists.

It is easy to check that $a \ominus_H b \neq a + (-b)$.

**Definition 2.7.** Let us suppose $a_1, a_2 \in E$, gh-difference of two given fuzzy numbers is given by:

$$a_1 \ominus_H a_2 = a_3 \Leftrightarrow \begin{cases} (i) a_1 = a_2 + a_3, \\
\text{or} \\
(ii) a_2 = a_1 + (-1)a_3, \end{cases}$$

in which $a_3 \in E$.

Throughout the paper, we assume that the gh-differentiability exist.
Definition 2.8. Take the fuzzy operator $F : R \rightarrow R$, $F$ is a continuous function for a constant $y_0 \in [\beta_1, \beta_2]$, if and only if for all $\epsilon > 0$, there will be $\delta > 0$ : If $|y - y_0| < \delta$. Then
\[ D_1(F(y), F(y_0)) < \epsilon. \]

Definition 2.9. A continuous function or operator in level wise form $F : [\beta_1, \beta_2] \subset R \rightarrow R$ is defined at $a \in [\beta_1, \beta_2]$ if the set value function $F_x(y) = [F(y)]^x$ is defined at $y = a$ w.r.t: the $F$-metric space $D_1 \forall x \in [0, 1]$. We recall some important definitions of the fuzzy fractional calculus. 

Definition 2.10. Take $\nu : (c, d) \rightarrow R$ and $x_0 \in (c, d)$, then $\nu$ is “$gH$-differentiable” at $x_0$ if $\exists \nu'(x_0) \in R$:

1. $\exists \nu(x_0 + p) \ominus (x_0), \nu(x_0) \ominus (x_0 - F)$ and limit (in metric $D_1$, and for all $p > 0$ very small):
\[
\lim_{p \searrow 0} \frac{\nu(x_0 + p) \ominus (x_0)}{p} = \lim_{p \searrow 0} \frac{\nu(x_0) \ominus (x_0 - p)}{p} = \nu'_g(x_0).
\]

or (2) $\exists \nu(x_0) \ominus (x_0 + p), \nu(x_0 - p) \ominus (x_0)$ and limit (in metric $D_1$, and for all $p > 0$ sufficiently small):
\[
\lim_{p \searrow 0} \frac{\nu(x_0) \ominus (x_0 + p)}{-p} = \lim_{p \searrow 0} \frac{\nu(x_0 - p) \ominus (x_0)}{-p} = \nu'_g(x_0).
\]

We say that $\nu$ is (1)-differentiable if $\nu$ satisfies in the Case (1), of Definition 10, similarly for (2)-differentiability which satisfy in the Case (2). We recall some important definitions of the fuzzy fractional calculus. $L^p_{gH}(c, d)$, $1 \leq p < \infty$ stands for the set of all “fuzzy-valued measurable functions” $p$ on $[c, d]$ where $||P||_p = (\int_0^1 (d(P(t), 0))pdt)^{\frac{1}{p}}$.

$C^R[c, d]$ represents a space of all “fuzzy-valued functions” continuous in $[c, d]$. 

$AC^R[c, d]$ is the set of all “fuzzy-valued functions” which implies continuity.

Definition 2.11. Let $P \in C^E[a, b] \cap L^E[a, b]$ The “Riemann-Liouville integral” of the fuzzy-valued function $P$ is:
\[
RL^{\alpha}P(x) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{P(t)dt}{x - t}^{1-\alpha}, \quad x > a, \quad 0 < \alpha \leq 1.
\]

Definition 2.12. Let $F \in C^E \cap L^E$, $x_0 \in J$ and
\[
\Psi(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{\frac{d^k}{dt^k}F(t) \ominus \sum_{k=0}^{l-1} \frac{t^k}{k!}r_0^{(k)}(x - t)^\alpha}{(x-t)^\alpha} dt
\]
Then, $F$ is fuzzy Caputo fractional differentiable of order $0 < \alpha \leq 1$ at $x_0$, either

1. $\left(\mathcal{D}^\alpha_0 F\right)(x_0) = \lim_{\Delta x \to 0^+} \frac{\Psi(x_0 + \Delta x) \ominus \Psi(x_0)}{\Delta x} = \frac{\Psi(x_0) \ominus \Psi(x_0 - \Delta x)}{\Delta x},$

or

2. $\left(\mathcal{D}^\alpha_0 F\right)(x_0) = \lim_{\Delta x \to 0^+} \frac{\Psi(x_0) \ominus \Psi(x_0 + \Delta x)}{-\Delta x} = \frac{\Psi(x_0 - \Delta x) \ominus \Psi(x_0)}{-\Delta x},$

Note that $F$ is $(1-\alpha)$-differentiable if it satisfies in the case (1), and $F$ is $(2-\alpha)$-differentiable if it satisfies in the Case (2).
Main work. In the main work initially, the natural transform is applied to (1), and then homotopy perturbation techniques (NTHPM) is used to get the hierarchy of equations.

Definition 2.13.

\[ A = \{ \nu(t); \exists M, k_1, k_2 > 0, \ d(\nu(t), 0) < Me^{-\nu t}, \text{ if } t \in (-1)^i \times [0, \infty), \ i = 1, 2 \} \]

Then, the NT of fuzzy-valued function \( \nu(x,t) \) for \( t = 0 \) is defined as:

\[ N[\nu(x, t, r)] = R(x, s, u, r) = \int_0^\infty e^{-st} \nu(x, ut, r) dt, \]

where \( s > 0 \) and \( u > 0 \) are transform parameters.

Theorem 2.14. Let \( \mathcal{F}(t, r), \overline{\mathcal{F}}(t, r) \), be a fuzzy-valued function where \( \mathcal{F} \in A \). Then

\[ N[\mathcal{F}'(t, r)] = \frac{s}{u} N[\mathcal{F}(t, r)] \oplus \frac{\mathcal{F}(0, r)}{u}, \]

where \( \mathcal{F} \) is (1)-differentiable and

\[ N[\mathcal{F}'(t, r)] = -\frac{\mathcal{F}(0, r)}{u} \oplus (-1)^s \frac{s}{u} N[\mathcal{F}(t, r)], \]

where \( \mathcal{F} \) is (2)-differentiable provided that H-differences exist.

Proof. Let \( \mathcal{F} \) is (1)-differentiable, then

\[ \mathcal{F}'(t, r) = [\mathcal{F}(t, r), \overline{\mathcal{F}}(t, r)], \]

and

\[ \begin{cases} N[\mathcal{F}'(t, r)] = \frac{s}{u} N[\mathcal{F}(t, r)] - \frac{\mathcal{F}(0, r)}{u}, \\ N[\overline{\mathcal{F}}(t, r)] = \frac{s}{u} N[\mathcal{F}(t, r)] - \frac{\mathcal{F}(0, r)}{u}, \end{cases} \]

Then

\[ N[\mathcal{F}'(t, r)] = N[\mathcal{F}(t, r), \overline{\mathcal{F}}(t, r)] = \left[ N\mathcal{F}(t, r), N\overline{\mathcal{F}}(t, r) \right] \]

\[ = \left[ \frac{s}{u} N\mathcal{F}(t, r) - \frac{\mathcal{F}(0, r)}{u}, \frac{s}{u} N\overline{\mathcal{F}}(t, r) - \frac{\mathcal{F}(0, r)}{u} \right], \]

\[ = \frac{s}{u} \left[ N\mathcal{F}(t, r), N\overline{\mathcal{F}}(t, r) \right] \oplus \frac{\mathcal{F}(0, r)}{u}, \]

Indeed, under (2)-differentiability the proof is similar to Case (1). Hence, we only write it.

Now, we extend the previous results to the fractional case when the order of differentiability is \( \theta \), where \( 0 < \theta \leq 1 \), in the Caputo sense, to obtain natural transform of fractional derivative of fuzzy-valued function \( \mathcal{F} \).

Theorem 2.15. Let \( \mathcal{F} \in A \) be a fuzzy valued function and \( 0 < \theta \leq 1 \), then

\[ N[D^\theta \mathcal{F}(t, r)] = \frac{s^\theta}{u^\theta} N[\mathcal{F}(t, r)] \oplus \frac{s^{\theta-1}}{u^\theta} \mathcal{F}(0, r), \]

when \( \mathcal{F} \) is (1, \( \theta \))-differentiable and

\[ N[D^\theta \mathcal{F}(t, r)] = \frac{s^{\theta-1}}{u^\theta} \mathcal{F}(0, r) \oplus \frac{s^\theta}{u^\theta} N[\mathcal{F}(t, r)], \]
when $\mathcal{F}$ is $(2, \theta)$-differentiable.

**Proof.** Let $\mathcal{F}$ is $(1, \theta)$-differentiable, then

$$N[D_\theta^\alpha \mathcal{F}(t, r)] = \frac{s^r}{u^r} N[D_\theta^\alpha \mathcal{F}(t, r)] - \frac{s^r-1}{u^r} \mathcal{F}(0, r),$$

$$N[D_t^\theta \mathcal{F}(t, r)] = \frac{s^r}{u^r} N[D_t^\theta \mathcal{F}(t, r)] - \frac{s^r-1}{u^r} \mathcal{F}(0, r),$$

then, we obtain

$$N[D_\theta^\alpha \mathcal{F}(t, r)] = N[D_\theta^\alpha \mathcal{F}(t, r), D_\theta^\alpha \mathcal{F}(t, r)],$$

$$= \left[ N[D_\theta^\alpha \mathcal{F}(t, r)], N[D_t^\theta \mathcal{F}(t, r)] \right],$$

$$= \left[ \frac{s^r}{u^r} N[D_\theta^\alpha \mathcal{F}(t, r)] - \frac{s^r-1}{u^r} \mathcal{F}(0, r), \frac{s^r}{u^r} N[D_t^\theta \mathcal{F}(t, r)] - \frac{s^r-1}{u^r} \mathcal{F}(0, r) \right],$$

$$= \frac{s^r}{u^r} [N[D_\theta^\alpha \mathcal{F}(t, r)], N[D_t^\theta \mathcal{F}(t, r)]] \ominus \frac{s^r-1}{u^r} [\mathcal{F}(0, r), \mathcal{F}(0, r)],$$

$$= \frac{s^r}{u^r} [N[D_\theta^\alpha \mathcal{F}(t, r)] \ominus \frac{s^r-1}{u^r} [\mathcal{F}(0, r)],$$

provided that the H-difference exists. The proof for $\mathcal{F}$ when it is $(2, \theta)$-differentiable is completely similar. \hfill $\square$

**Theorem 2.16.** Let $\mathcal{F} \in A, 1 + \alpha = \theta, \quad 0 < \alpha \leq 1, \quad \text{then} \quad 1 < \theta \leq 2$ and the natural transform of fractional derivative of order $\theta$ for fuzzy-valued function $\mathcal{F}$ is given by:

$$N[D_t^\theta \mathcal{F}(t, r)] = \frac{s^r}{u^r} [N[\mathcal{F}(t, r)] \ominus \frac{s^r-1}{u^r} [\mathcal{F}(0, r) \ominus \frac{s^r-2}{u^r-1} [\mathcal{F}'(0, r)],$$

where $\mathcal{F}$ is $(1)$-differentiable and $D_t^\alpha [\mathcal{F}(t, r)] = (1, \alpha)$-differentiable and

$$N[D_t^{\theta-1} \mathcal{F}(t, r)] = -\frac{s^{\theta-1}}{u^{\theta-1}} \mathcal{F}(0, r) \ominus (-1) \frac{s^\theta}{u^\theta} [N[\mathcal{F}(t, r)] \ominus \frac{s^\theta-2}{u^\theta-1} [\mathcal{F}'(0, r)],$$

where $\mathcal{F}$ is $(1)$-differentiable and $D_t^{\alpha-1} [\mathcal{F}(t, r)] = (2, \alpha)$-differentiable and

$$N[D_t^{\theta-2} \mathcal{F}(t, r)] = -\frac{s^{\theta-2}}{u^{\theta-2}} \mathcal{F}(0, r) \ominus (-1) \frac{s^\theta}{u^\theta} [N[\mathcal{F}(t, r)] \ominus \frac{s^\theta-1}{u^\theta-1} [\mathcal{F}(0, r)],$$

where $\mathcal{F}$ is $(2)$-differentiable and $D_t^{\alpha-2} [\mathcal{F}(t, r)] = (1, \alpha)$-differentiable and

$$N[D_t^{\theta-1} \mathcal{F}(t, r)] = -\frac{s^{\theta-1}}{u^{\theta-1}} \mathcal{F}(0, r) \ominus (-1) \frac{s^\theta}{u^\theta} [N[\mathcal{F}(t, r)] \ominus \frac{s^\theta-1}{u^\theta-1} [\mathcal{F}(0, r)],$$

where $\mathcal{F}$ is $(2)$-differentiable and $D_t^{\alpha-1} [\mathcal{F}(t, r)] = (2, \alpha)$-differentiable.

**Proof.** We only prove the case that $\mathcal{F}$ is $(2)$-differentiable and $D_t^\alpha \mathcal{F}(t, r)$ is $(2, \alpha)$-differentiable and the other cases are completely similar to the demonstration of
the proof of this case. Under the stated assumptions, we get:

\[
N[D^\theta_0 \mathcal{F}(t,r)] = N[D^\theta_0 \mathcal{F}(t,r)],
\]

\[
= -\frac{s^\theta}{u^\theta} \mathcal{F}(0,r) \ominus (-1)^{\frac{s^\theta}{u^\theta}} N[\mathcal{F}(t,r)],
\]

\[
= -\frac{s^\theta}{u^\theta} \mathcal{F}(0,r) \ominus (-1)^{\frac{s^\theta}{u^\theta}} \left( \frac{\mathcal{F}(0,r)}{u} \ominus (-1)^{\frac{s^\theta}{u^\theta}} N[\mathcal{F}(t,r)] \right),
\]

\[
= -\frac{s^\theta}{u^\theta} \mathcal{F}(0,r) \ominus \frac{s^\theta}{u^\theta} \mathcal{F}(0,r) + \frac{s^\theta+1}{u^\theta+1} N[\mathcal{F}(t,r)],
\]

\[
= -\frac{s^\theta-2}{u^\theta-1} \mathcal{F}(0,r) \ominus \frac{s^\theta-1}{u^\theta-1} \mathcal{F}(0,r) + \frac{s^\theta}{u^\theta} N[\mathcal{F}(t,r)],
\]

provided that H-difference exists.

In this part, we demonstrate our work for the case that the given unknown fuzzy-valued function \( \tilde{u}(x,y,t) \) is (1)-differentiable and \( \tilde{u}'(x,y,t) \) is \((1,\alpha)\)-differentiable w.r.t. \( t \) and also \( \tilde{u} \) is twice differentiable w.r.t. \( x \) and \( y \).

\[
N\left[D^\theta_x \tilde{u}(x,y,t,r)\right] = N\left[D^\theta_x \tilde{u}(x,y,t,r) + D^\theta_y \tilde{u}(x,y,t,r) + \mathcal{F}(x,y,t,r)\right], \quad 1 < \theta \leq 2,
\]

\[
\tilde{u}(x,y,0,r) = \tilde{g}_1(x,y,r),
\]

\[
\tilde{u}_t(x,y,0,r) = \tilde{g}_2(x,y,r),
\]

due to given initial conditions, we have:

\[
\frac{s^\theta}{u^\theta} N[\tilde{u}(x,y,t,r)] \ominus \frac{s^\theta-1}{u^\theta} \tilde{G}_1(x,y,r) \ominus \frac{s^\theta-2}{u^\theta-1} \tilde{G}_2(x,y,r)
\]

\[
= N \left[D^\theta_x \tilde{u}(x,y,t,r) + D^\theta_y \tilde{u}(x,y,t,r) + \mathcal{F}(x,y,t,r)\right],
\]

(5)

or

\[
N[\tilde{U}(x,y,t,r)] = \frac{\tilde{g}_1(x,y,r)}{s} + \frac{u}{s^2} \tilde{g}_2(x,y,r)
\]

\[
+ \frac{u^\theta}{s^\theta} N \left[D^2_x \tilde{u}(x,y,t,r) + D^2_y \tilde{u}(x,y,t,r) + \mathcal{F}(x,y,t,r)\right].
\]

Now, implementing natural transform on both sides of above estimates, we can write

\[
\tilde{U}(x,y,t,r) = N^{-1} \left[ \frac{\tilde{g}_1(x,y,r)}{s} + \frac{u}{s^2} \tilde{g}_2(x,y,r) \right]
\]

\[
+ N^{-1} \left[ \frac{u^\theta}{s^\theta} N \left[D^2_x \tilde{u}(x,y,t,r) + D^2_y \tilde{u}(x,y,t,r) + \mathcal{F}(x,y,t,r)\right] \right].
\]

As there is no non-linear term so we can define formal perturbation technique in the form of power series in the parameter \( q \) as

\[
\tilde{u}(x,y,t,r) = \sum_{k=0}^{\infty} q^k \tilde{z}_k(x,y,t,r),
\]
Then, we obtain the solution in the series form as

\[
\sum_{k=0}^{\infty} \tilde{Z}(x, y, t, r) = \tilde{S}_1(x, y, r) + t\tilde{S}_2(x, y, r) + qN^{-1}\left[\frac{\theta^\theta}{\Gamma(\theta + 1)} \left[\int_0^1 \left(\sum_{k=0}^{\infty} q^k \tilde{Z}_k(x, y, t, r) + \mathcal{F}(x, y, t, r)\right)\right]\right].
\]

By equating coefficient of \(q\) on both sides, we get

\[
\begin{align*}
\mathcal{O}(q^0) : \tilde{Z}_0(x, y, t, r) &= \tilde{S}_1(x, y, r) + t\tilde{S}_2(x, y, r), \\
\mathcal{O}(q^1) : \tilde{Z}_1(x, y, t, r) &= N^{-1}\left[\frac{\theta^\theta}{\Gamma(\theta + 1)} \left[\int_0^1 \left(\sum_{k=0}^{\infty} q^k \tilde{Z}_k(x, y, t, r) + \mathcal{F}(x, y, t, r)\right)\right]\right], \\
\mathcal{O}(q^2) : \tilde{Z}_2(x, y, t, r) &= N^{-1}\left[\frac{\theta^\theta}{\Gamma(\theta + 1)} \left[\int_0^1 \left(\sum_{k=0}^{\infty} q^k \tilde{Z}_k(x, y, t, r) + \mathcal{F}(x, y, t, r)\right)\right]\right], \\
\mathcal{O}(q^3) : \tilde{Z}_3(x, y, t, r) &= N^{-1}\left[\frac{\theta^\theta}{\Gamma(\theta + 1)} \left[\int_0^1 \left(\sum_{k=0}^{\infty} q^k \tilde{Z}_k(x, y, t, r) + \mathcal{F}(x, y, t, r)\right)\right]\right], \\
&\vdots \\
\mathcal{O}(q^{n+1}) : \tilde{Z}_{n+1}(x, y, t, r) &= N^{-1}\left[\frac{\theta^\theta}{\Gamma(\theta + 1)} \left[\int_0^1 \left(\sum_{k=0}^{\infty} q^k \tilde{Z}_k(x, y, t, r) + \mathcal{F}(x, y, t, r)\right)\right]\right].
\end{align*}
\]

Then, we obtain the solution in the series form as

\[
\tilde{U}(x, y, t, r) = \lim_{q \to 1} \sum_{n=0}^{\infty} q^n \tilde{Z}(x, y, t, r),
\]

or \(\tilde{U}(x, y, t, r) = \tilde{Z}_0(x, y, t, r) + \tilde{Z}_1(x, y, t, r) + \tilde{Z}_2(x, y, t, r) + \ldots\), which is the required series whose convergence for each lower and upper approximations has already proved in [8].

**Illustrative problems.** Here, we take some illustration of two dimension fuzzy wave non-integer order differential equations, then the NTHPM is used to take the numerical simulation.

**Example 1.** Take the following fuzzy wave model as

\[
\begin{align*}
\mathcal{D}_x^\theta \tilde{U}(x, y, t, r) &= \mathcal{D}_y^\theta \tilde{U}(x, y, t, r) + \mathcal{D}_t^\theta \tilde{U}(x, y, t, r) + y + x, \quad 1 < \theta \leq 2, 1 < t < 0 \\
\tilde{U}(x, y, r, 0) &= \sin(y + x)\tilde{K}(r), \\
\tilde{U}(x, y, r, 0) &= \tilde{K}(r)\cos(y + x),
\end{align*}
\]

here \(\tilde{K}(r) = [\tilde{K}(r), \tilde{K}(r)] = [r - 1, 1 - r]\) Taking the aforesaid techniques as in (5), we obtain the first three terms of the semi-analytical solution as under

\[
\begin{align*}
\tilde{Z}_0(x, y, t; r) &= \sin(y + x)\tilde{K}(r) + t\tilde{K}(r)\cos(x + y), \\
\tilde{Z}_0(x, y, t; r) &= \tilde{K}(r)\sin(x + y) + t\tilde{K}(r)\cos(x + y), \\
\tilde{Z}_1(x, y, t; r) &= -[\sin(y + x)\tilde{K}(r) - (x + y)]\frac{\theta^\theta}{\Gamma(\theta + 1)} - \tilde{K}(r)[\cos(y + x)]\frac{\theta^{\theta+1}}{\Gamma(\theta + 2)}, \\
\tilde{Z}_1(x, y, t; r) &= -[\tilde{K}(r)\sin(y + x) - y - x]\frac{\theta^\theta}{\Gamma(\theta + 1)} - \tilde{K}(r)[\cos(x + y)]\frac{\theta^{\theta+1}}{\Gamma(\theta + 2)}.
\end{align*}
\]
Using (6), one may evaluate the result in the form as

\[ \tilde{u}(x, y; t; r) = \tilde{z}_0(x, y; t; r) + \tilde{z}_1(x, y; t; r) + \tilde{z}_2(x, y; t; r) \ldots, \]

such that

\[ \tilde{u}(x, y; t; r) = \tilde{z}_0(x, y; t; r) + \tilde{z}_1(x, y; t; r) + \tilde{z}_2(x, y; t; r) \ldots, \]

\[ \overline{u}(x, y; t; r) = \overline{z}_0(x, y; t; r) + \overline{z}_1(x, y; t; r) + \overline{z}_2(x, y; t; r) \ldots. \]

Where source term in (8) is \( \overline{F}(x, y; t; r) = x + y \). If there is no source term, then we have:

\[ \tilde{u}(x, y; t; r) = \sum_{i=0}^{\infty} \left( \frac{(-1)^i i^i \theta}{\Gamma(i + 1)} \sin(y + x) \overline{K}(r) + \frac{(-1)^i i^i \theta + 1}{\Gamma(i + 2)} \overline{K}(r) \cos(x + y) \right), \]

\[ \overline{u}(x, y; t; r) = \sum_{i=0}^{\infty} \left( \frac{(-1)^i i^i \theta}{\Gamma(i + 1)} \sin(y + x) \overline{K}(r) + \frac{(-1)^i i^i \theta + 1}{\Gamma(i + 2)} \cos(x + y) \overline{K}(r) \right). \]

Next we have provided simulation of Example 1 at various non-integer order \((1 < \theta \leq 2)\) for upper and lower branches or parts of fuzzy solution are given in figures 1-4 in 3D and 2D form respectively. The figures 5 and 6 shows the comparison of series solution of example 1 by LADM and Natural transform with Homotopy perturbation method (NTHPM) in 3D and 2D form. The two similar color legends in each and every graph shows upper and lower portion of fuzzy solution respectively.

Example 2. Let us take another example of fuzzy wave equation of non-integer order with ICs as:

\[ D_t^\theta \tilde{u}(x, y; t; r) = D_x^2 \tilde{u}(x, y; t; r) + D_y^2 \tilde{u}(x, y; t; r) + y + (x + t^2), \quad 1 < \theta \leq 2, \]

\[ \tilde{u}(x, y; r, 0) = e^{(x)} \overline{K}(r), \]

\[ \tilde{u}_t(x, y; r, 0) = e^{(y)} \overline{K}(r), \quad 0 < x, y < 1 \]

(10)

here \( \overline{K}(r) = [\overline{K}(r), \overline{K}(r)] = [r - 1, 1 - r] \), applying NTHPM in this case, to get

\[ \tilde{z}_0(x, y; t; r) = [e^x + x e^y] \overline{K}(r), \]

\[ \tilde{z}_0(x, y; t; r) = \overline{K}(r)[e^x + x e^y] \]

\[ \tilde{z}_1(x, y; t; r) = \left[ e^x + (y + x) \overline{K}(r) \right] \frac{t^\theta}{\Gamma(\theta + 1)} + \overline{K}(r) e^y \frac{t^{\theta + 1}}{\Gamma(\theta + 2)} + \frac{t^{\theta + 2}}{\Gamma(\theta + 3)}. \]
Figure 1. Representation of three dimensional (3D) graph of fuzzy solution at four different fractional order of $\theta$ with $y = 0.5, t = 0.5$, of example 1. The two similar color legends represents upper and lower portion of fuzzy solution respectively.

Figure 2. Representation of three dimensional (3D) graph of fuzzy solution at four other different fractional order of $\theta$ with $y = 0.5, t = 0.5$ of example 1. The two similar color legends represents upper and lower portion of fuzzy solution respectively.
Figure 3. Representation of two dimensional (2D) graph of fuzzy solution at four different fractional order of $\theta$ with $x = 0.5, y = 0.5, t = 0.5$ of example 1. The two similar color legends represents upper and lower portion of fuzzy solution respectively.

Figure 4. Representation of two dimensional (2D) graph of fuzzy solution at four other different fractional order of $\theta$ with $x = 0.5, y = 0.5, t = 0.5$ of example 1. The two similar color legends represents upper and lower portion of fuzzy solution respectively.
Figure 5. Representation of comparison of fuzzy solution for upper and lower branches of example 1 by LADM and NTHPM in three dimension (3D). The two similar color legends represents upper and lower portion of fuzzy solution respectively.

Figure 6. Representation of comparison of fuzzy solution for upper and lower branches of example 1 by LADM and NTHPM in two dimension (2D). The two similar color legends represents upper and lower portion of fuzzy solution respectively.
and similarly higher terms can be calculated. In view (6), fuzzy solution can be written in the following fashion:

\[ \tilde{u}(x, y, t; \tau) = \tilde{z}_0(x, y, t; \tau) + \tilde{z}_1(x, y, t; \tau) + \tilde{z}_2(x, y, t; \tau) \ldots, \]

which implies

\[ \underline{u}(x, y, t; \tau) = \underline{z}_0(x, y, t; \tau) + \underline{z}_1(x, y, t; \tau) + \underline{z}_2(x, y, t; \tau) \ldots, \]
\[ \overline{u}(x, y, t; \tau) = \overline{z}_0(x, y, t; \tau) + \overline{z}_1(x, y, t; \tau) + \overline{z}_2(x, y, t; \tau) \ldots. \]

When no external force term or source term the solution of (10), is provided as:

\[ \underline{u}(x, y, t; \tau) = \sum_{i=0}^{\infty} \left[ \frac{t^{i\theta}}{\Gamma(i\theta + 1)} e^{x} \tilde{\kappa}(x) + \frac{t^{i\theta + 1}}{\Gamma(i\theta + 2)} e^{y} \tilde{\kappa}(x) \right], \]
\[ \overline{u}(x, y, t; \tau) = \sum_{i=0}^{\infty} \left[ \frac{t^{i\theta}}{\Gamma(i\theta + 1)} e^{x} \tilde{\kappa}(x) + \frac{t^{i\theta + 1}}{\Gamma(i\theta + 2)} e^{y} \tilde{\kappa}(x) \right]. \]

Further we have given simulation of Example 2 at various non-integer order (1 < \theta \leq 2) for upper and lower branches or parts of fuzzy solution are given in figures 7-10 in 3D and 2D form respectively. The two similar color legends represents upper and lower portion of fuzzy solution respectively. The figures 11 and 12 shows the comparison of series solution of example 2 by Laplace Adomian decomposition method(LADM) and Natural transform with Homotopy perturbation method(NTHPM) in 3D and 2D form.

**Example 3.** Take the following fuzzy wave equation of fractional order with ICs as

\[ D_{t}^{\alpha} \tilde{u}(x, y, t; \tau) = D_{t}^{\gamma} \tilde{u}(x, y, t; \tau) + D_{t}^{\gamma} \tilde{u}(x, y, t; \tau) + (\sin(y + x) + t), \quad 1 < \theta \leq 2, 0 < \tau < t < 1, \]
\[ \tilde{u}(x, y, 0) = e^{\theta x} \tilde{\kappa}(x), \]
\[ \tilde{u}(x, y, 0) = \tilde{\kappa}(x)e^{(x-y)}, \quad 0 < x, y < 1. \]  

we aim to obtain the series solutions by the same procedure as above

\[ \underline{z}_0(x, y, t; \tau) = [e^{\theta x} \tilde{\kappa}(x) + t e^{-(y+\gamma)}], \]
\[ \overline{z}_0(x, y, t; \tau) = \tilde{\kappa}(x)[e^{\theta x} + t e^{-(x+y)}], \]
**Figure 7.** Representation of three dimensional (3D) graph of fuzzy solution at four different fractional order of $\theta$ with $y = 0.5$, $t = 0.5$ of example 2. The two similar color legends represents upper and lower portion of fuzzy solution respectively.

**Figure 8.** Representation of three dimensional (3D) graph of fuzzy solution at four other different fractional order of $\theta$ with $y = 0.5$, $t = 0.5$ of example 2. The two similar color legends represent upper and lower portion of fuzzy solution respectively.
Figure 9. Representation of two dimensional (2D) graph of fuzzy solution at four different fractional order of $\theta$ with $x = 0.5$, $y = 0.5$, $t = 0.5$ of example 2. The two similar color legends represents upper and lower portion of fuzzy solution respectively.

Figure 10. Representation of two dimensional (2D) graph of fuzzy solution at four other different fractional order of $\theta$ with $x = 0.5$, $y = 0.5$, $t = 0.5$ of example 2. The two similar color legends represents upper and lower portion of fuzzy solution respectively.
Figure 11. Representation of comparison of fuzzy solution for upper and lower branches of example 2 by LADM and NTHPM in three dimension(3D). The two similar color legends represents upper and lower portion of fuzzy solution respectively.

Figure 12. Representation of comparison of fuzzy solution for upper and lower branches of example 2 by LADM and NTHPM in two dimension(2D). The two similar color legends represents upper and lower portion of fuzzy solution respectively.
In this way all the other terms can be computed easily. Applying (6), fuzzy solution can be written:

\[ U(x, y, t, r) = \tilde{Z}_0(x, y, t, r) + \tilde{Z}_1(x, y, t, r) + \tilde{Z}_2(x, y, t, r), \]

which implies

\[ U(x, y, t, r) = \tilde{Z}_0(x, y, t, r) + \tilde{Z}_1(x, y, t, r) + \tilde{Z}_2(x, y, t, r), \]

when no external force term or source term then the solution of (12), can be computed as

\[ U(x, y, t, r) = \sum_{i=0}^{\infty} \left[ \frac{t^{i\theta}}{\Gamma(i\theta + 1)} \left( \tilde{Z}(r) 2^i e^{s+y} + (-1)^i \sin(x + y) \right) + \frac{t^{i\theta+1}}{\Gamma(i\theta + 2)} \tilde{Z}(r) e^{-(s+y)} \right]. \]

Further we have given simulation of Example 3 at various non-integer order (1 < \theta \leq 2) for upper and lower branches or parts of fuzzy solution are given in figures 13-16 in 3D and 2D form respectively while figures 17 and 18 shows the comparison of series solution of example 3 by LADM and Natural transform with Homotopy perturbation method (NTHPM) in 3D and 2D form. The two similar color legends in each graph represents upper and lower portion of fuzzy solution respectively.

**Conclusion and discussion.** In this article, an analytical solutions of two dimensions fuzzy wave equation along with external force or external source function have been successfully established. The series or semi-analytical solution of the considered equation in sense of fractional Caputo derivative has been developed by Natural transform method along-with Homotopy perturbation method (NTHPM). By multiplying the fuzzy number we have obtained the upper and lower parts of the required solution. The comparison of NTHPM with LADM have also been simulated for each example. The obtained results have been testified by providing some relevant examples. Graphs of the numerical results at different non-integer orders of the said problems are also provided i.e for 1 < \theta \leq 2. Moreover, The techniques adopted in this article to solve fuzzy fractional wave equation has not established before and can be applied to all those areas of physical and natural sciences where uncertainty lies in different phenomenon like quantum mechanics and different theories of orbits or orbital around the nucleus. Therefore, this work will open new doors in the area of fuzzy calculus as well in fuzzy fractional calculus. For future work,
Figure 13. Representation of three dimensional (3D) graph of fuzzy solution at four different fractional order of \( \theta \) with \( y = 0.5, t = 0.5 \), of example 3. The two similar color legends represents upper and lower portion of fuzzy solution respectively.

Figure 14. Representation of three dimensional (3D) graph of fuzzy solution at four other different fractional order of \( \theta \) with \( y = 0.5, t = 0.5 \) of example 3. The two similar color legends represents upper and lower portion of fuzzy solution respectively.
Figure 15. Representation of two dimensional (2D) graph of fuzzy solution at four different fractional order of $\theta$ with $x = 0.5, y = 0.5, t = 0.5$ of example 3. The two similar color legends represents upper and lower portion of fuzzy solution respectively.

Figure 16. Representation of two dimensional (2D) graph of fuzzy solution at four other different fractional order of $\theta$ with $x = 0.5, y = 0.5, t = 0.5$ of example 3. The two similar color legends represents upper and lower portion of fuzzy solution respectively.
Figure 17. Representation of comparison of fuzzy solution for upper and lower branches of example 3 by LADM and NTHPM in three dimension(3D). The two similar color legends represents upper and lower portion of fuzzy solution respectively.

Figure 18. Representation of comparison of fuzzy solution for upper and lower branches of example 3 by LADM and NTHPM in two dimension(2D). The two similar color legends represents upper and lower portion of fuzzy solution respectively.
we aim to apply the aforementioned approach under “Atangana-Baleanu Caputo” (ABC) and Caputo-Fabrizio (CF) fractional derivative for fuzzy-valued functions based on both types of fuzzy differentiability.

Conflict of interest. The authors declare that they have no conflict of interest.

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