Low temperature shape relaxation of 2-d islands by edge diffusion

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We present a precise microscopic description of the limiting step for low temperature shape relaxation of two dimensional islands in which activated diffusion of particles along the boundary is the only mechanism of transport allowed. In particular, we are able to explain why the system is driven irreversibly towards equilibrium. Based on this description, we present a scheme for calculating the duration of the limiting step at each stage of the relaxation process. Finally, we calculate numerically the total relaxation time as predicted by our results and compare it with simulations of the relaxation process.

I. INTRODUCTION

The understanding, description and control of structures at the nanometer scales is a subject of interest from the fundamental and applied points of view. From the fundamental point of view, there is a large literature concerning the growth of crystals and their shape. Yet, while the description of the equilibrium shape is rather clear, the dynamic description of crystal growth is still not well understood. In particular, we lack a complete understanding of the time scales involved in the relaxation process, and the mechanisms which irreversibly conduct the island to its equilibrium shape.

In this work, we study the shape relaxation of two dimensional islands in which activated diffusion of particles along the boundary is the only mechanism of transport allowed. In particular, we are able to explain why the system is driven irreversibly towards equilibrium. Based on this description, we present a scheme for calculating the duration of the limiting step at each stage of the relaxation process. Finally, we calculate numerically the total relaxation time as predicted by our results and compare it with simulations of the relaxation process.

The specific model under consideration consists of 2D islands having a perfect triangular crystalline structure. A very simple energy landscape for activated atomic motion was chosen: the aim being to point out the basic mechanisms of relaxation, and not to fit the specific behavior of a particular material. The potential energy $E_p$ of an atom is assumed to be proportional to the number $i$ of neighbors, and the kinetic barrier $E_{act}$ for diffusion is also proportional to the number of initial neighbors before the jump, regardless of the final number of neighbors: $E_{act} = -E_p = i \times E$ where $E$ sets the energy scale ($E = 0.1$ eV throughout the paper). Therefore, the probability $p_i$ per unit time that an atom with $i$ neighbors moves is $p_i = v_0 \exp[-i \times E/k_B T]$, where $v_0 = 10^{13} s^{-1}$ is the Debye frequency, $k_B$ is the Boltzmann constant and $T$ the absolute temperature. Hence, the average time in which a particle with $i$ neighbors would move is given by:

$$\tau_i = v_0^{-1} \exp[i \times E/k_B T] \quad (1)$$

The complete description of the model and of the simulation algorithm can be found in [4], where it was studied using Standard Kinetic Monte Carlo simulations. This simple kinetic model has only one parameter, the ratio $E/k_B T$. The temperature was varied from $83K$ to $500K$, and the number of atoms in the islands from 90 up to 20000. The initial configurations of the islands were elongated (same initial aspect ratio of about 10), and the simulations were stopped when the islands were close to equilibrium, with an aspect ratio of 1.2. The time required for this to happen was defined as the relaxation time corresponding to that island size and temperature. Concerning the dependence of the relaxation time on the size of the island, two different behaviors depending on temperature were distinguished. At high temperature, the relaxation time scaled as the number of atoms to the power 2, but this exponent decreased when
temperature was decreased. A careful analysis showed that the exponent tends towards 1 at low temperature. The dependence of the relaxation time as a function of temperature also changes, the activation energy was 0.3 eV at high temperature and 0.4 eV at low temperature. In this context, it is important to define what we call a low temperature: following we denote by $L_c$ the average distance between kinks on an infinite facet: we define the low temperature regime as that in which $L_c \gg L$ where $L$ is the typical size of our island, large facets are then visible on the island. It was shown that $L_c = \frac{2}{a} \exp(\frac{E}{2k_BT})$ where $a$ is the lattice spacing.

The behavior of the relaxation time as a function of the temperature and $N$, the number of particles of the island, can be summed up with two equations corresponding to the high and low temperature regimes:

$$t_{relaxation}^{HT} \propto \exp[3E/k_BT]N^2 \text{ for } N \gg L_c^2$$

$$t_{relaxation}^{LT} \propto \exp[4E/k_BT]N \text{ for } N \ll L_c^2$$

Replacing the temperature dependent factors by a function of $N_c$, the crossover island size (where $N_c = L_c^2 \propto \exp(E/k_BT)$), these two laws can be expressed as a unique scaling function depending on the rescaled number of particles $N/N_c$:

$$t_{relaxation} \propto \begin{cases} N_c^5 \left( \frac{N}{N_c} \right)^2 & \text{for } N \gg N_c \\ N_c^5 \frac{N}{N_c} & \text{for } N \ll N_c \end{cases}$$

So that the relaxation time is a simple monotonous function of $N/N_c$, and the temperature dependence is contained in $N_c$.

We will now focus on the precise microscopic description of the limiting step for relaxation in the low temperature regime.

II. DESCRIPTION OF THE LIMITING PROCESS AT LOW TEMPERATURE

A. Qualitative description

During relaxation at low temperature, islands are mostly in fully faceted configurations. Let us, for instance, consider an island in a simple configuration given by fig. 1. When $L$ is larger than $l$, the island is not in its equilibrium shape (which should be more or less a regular hexagon). To reach the equilibrium shape, matter has to flow from the “tips” of the island (facets of length $l$ in this case) to the large facets $L$. In this low temperature regime there are very few mobile atoms at any given time, therefore this mass transfer must be done step by step: the initial step being the nucleation of a “germ” of two bound atoms on a facet of length $L$ and then, the growth of this germ up to a size $L - 1$ due to the arrival of particles emitted from the kinks and corners of the boundary of the island. Thus the germ grows, and eventually completes a new row on the facet.

![FIG. 1. Configuration of the island we will consider in the qualitative description as well as in the quantitative one.](image)

This simple picture still leaves a basic question unanswered: the relatively faster formation of a new row on a small facet would lead the island further away from its equilibrium shape, and yet, we observe that this never happens. Indeed, sometimes a germ appears on a small facet but it eventually disappears afterwards, whereas the appearance of a germ on a large facet frequently leads to the formation of a new row, taking the island closer to its equilibrium shape.

These observations are at the root of irreversible nature of the relaxation, germs only grow and become stable on the large facet, so the island can only evolve to a shape closer to equilibrium. Yet, there is clearly no local drive for growth on large facets nor any mechanism inhibiting growth on small ones. In order to explain how this irreversibility comes about, we propose the following detailed description of the mechanism of nucleation and of growth of a germ.

First, to create a germ, 2 atoms emitted from the corners of the island have to encounter on a facet. The activation energy required for this event is obviously independent of whether it occurs on a large facet or on a small facet. Once there is a germ of 2 atoms on a facet, the total energy of the island does not change when a particle is transferred from a kink to the germ (3 bonds are broken, and 3 are created) see Fig. 2. Clearly the same is true if a particle from the germ is transferred to its site of emission or any other kink. Thus, germs can grow or decrease randomly without energy variations driving the process. An exception to this occurs if the particle that reaches the germ is the last one of a row on a facet; in that case, the energy of the system decreases by 1 binding energy $E$. The island is then in a configuration from which it is extremely improbable to return to the previous configuration. For this to occur, a new germ would have to nucleate (and grow) on the original facet. This event is almost impossible in the presence of the kinks of the first growing germ, which act as traps for mobile atoms. Thus, when a germ nucleates on a facet, it can grow or decrease without changing the energy of the island except if a complete row on a facet disappears, in which case it “stabilizes”.
The scenario above explains why no new rows appear on small facets: when a germ grows on a small facet, since atoms come either from a small or a large facet, no complete row of a facet can disappear during the germ’s growth, and thus, the island never decreases its energy. On the other hand, when the germ grows on a large facet, the germ might grow or decrease, but if the size of the germ reaches the size of the small facet, the energy of the system will decrease and the system has almost no chance to go back its previous shape. We believe that this is the microscopic origin of irreversibility in the relaxation of this system. It should be stressed that this scenario for the growth and stabilization of germs is different from usual nucleation theory, where the germ has to overcome a free energy barrier to become stable.

This microscopic description also shows that the limiting step for this “row by row” relaxation mechanism is actually the formation on a large facet, of a germ of the size of the small facet. This fact allows us to estimate the duration of the limiting step at each stage of the relaxation process.

B. A Quantitative description

Based on our description of the process, we propose a scheme for calculating the time required to form a stable germ, i.e. a germ of size \( l \), on a facet of size \( L \). As mentioned above, the appearance of this stable germ is the limiting step for the formation of a new row on that facet.

The idea is to describe the growth of the germ as a succession of different island states, and calculate the probability and the time to go from one state to another in terms of the actual diffusive processes occurring on the island surface. These states form a Markov chain, the future evolution of the system being essentially determined by the states of the system, independently of its previous behavior.

As a further simplification, we consider a simple fully faceted island in an elongated hexagonal shape whose facets are of length \( L \) and \( l \), see Fig. 1; moreover, we normalize every length by the lattice spacing \( a \).

The different states we consider are (See Fig. 2 also):

- state 1 : one particle is on one end of a facet \( L \).
- state 2 : 2 particles are on the facet \( L \); one of them is on one end of the facet, and the other one has diffused from an end.
- state 3 : 2 particles are on the facet \( L \) but they are bonded together.
- state 4 : 3 bonded particles are on the facet \( L \).

\[ \ldots \]
- state \( n : n - 1 \) bonded particles are on the facet \( L \).

The goal of this calculation is to estimate the time to go from state 0 to state \( l + 1 \). We treat the problem as a discrete time Markov chain. The unit time being \( \tau_2 \), the typical time for a particle with 2 neighbors to move. This time is in fact the smallest relevant time of the system so that the operation of discretization does not affect the results. In the following, the time \( \tau_i \) is the average time for a particle with \( i \) neighbors to move: \( \tau_i = \nu_0^{-1} e^{\frac{E}{kT}} \). For clarity, we will use the term time for the discrete time of the Markov chain, and the term real time for the time of physical process. The obvious relation between the two time scales is: \( time = \frac{real \ time}{\tau_2} \).

We define the parameter \( \rho = \frac{\tau_2}{\tau_3} \). In the limit of small temperature, \( \rho \) is a very small quantity. Moreover, one can easily check that \( \rho = \frac{\exp(-\frac{Lc}{kT})}{\frac{L}{kT}} \) where \( Lc \) is the average distance between kinks defined in the first section. So that the condition \( Lc \gg L \) (low temperature regime) could be written as: \( \sqrt{\rho} L \ll 1 \) or \( \rho L \ll 1/L < 1 \).

Denote by \( \alpha_i \) the probability for the system in state \( i \) to stay in the state \( i \) the following step, \( p_i \) the transition probability for system in state \( i \) to go to state \( i+1 \), and \( q_i \) the transition probability for the system to go into state \( i - 1 \).

We have now to evaluate the different quantities \( \alpha_i \), \( p_i \) and \( q_i \) in terms of the diffusive processes that take place on the island’s boundary.

We first evaluate the quantities \( p_1, q_1 \) and \( \alpha_1 \). Let us assume that the states 0 and 2 are absorbent; the aver-
age time \( n_1 \) needed to leave state 1, corresponds to the average real time a particle stays on a facet starting on one of its edges, which act as traps. Since the particle performs a random walk, this can be readily calculated to be \( L \tau_2 \). So we should have:

\[
n_1 = \frac{1}{1 - \alpha_1} = L \tag{4}
\]

Moreover, the probability to go to state 2 is the probability that a new particle leaves a kink and reaches the facet while the first particle is still on the facet, we calculate this probability in Appendix A where we find:

\[
P = 1 - \left[ \frac{2 \sinh(2\sqrt{\rho}(L-1))}{\sinh(2\sqrt{\rho}L)} - \frac{\sinh(2\sqrt{\rho}(L/2-1))}{\sinh(2\sqrt{\rho}L/2)} \right].
\]

We expand expression Eq. 5 for small \( \rho \) keeping the first term:

\[
P = 2\rho(L-1) + o(\rho) \tag{6}
\]

We could also calculate the probability \( b_1 \), that the system in state 1 eventually reaches state 2 as:

\[
b_1 = \alpha_1 b_1 + p_1. \tag{7}
\]

Thus \( b_1 = \frac{p_1}{1 - \alpha_1} \), and using \( p_1 + \alpha_1 + q_1 = 1 \) we find:

\[
p_1 = \frac{P}{L} \approx 2\rho \frac{L-1}{L} + o(\rho) \tag{8}
\]

\[
\alpha_1 = 1 - \frac{1}{L} \tag{9}
\]

\[
q_1 = \frac{1}{L} - \frac{P}{L} \approx \frac{1}{L} - 2\rho \frac{L-1}{L} + o(\rho) \tag{10}
\]

We can do the same with state 2, knowing that the probability for two particles to stick (state 3) is \( \lambda/L \), where \( \lambda = (\cosh + \sqrt{\pi}) \pi \) (the calculation of this probability is carried out in appendix B). The time to leave state 2 is \( \kappa L \), where \( \kappa \) is a numerical constant given by \( \kappa = \pi \sum_{k=1}^{\infty} \frac{1}{2k+1} \) (see appendix B). Thus, we obtain:

\[
p_2 = \frac{\lambda}{\kappa L^2} \tag{11}
\]

\[
\alpha_2 = 1 - \frac{1}{\kappa L} \tag{12}
\]

\[
q_2 = \frac{1}{\kappa L} - \frac{\lambda}{\kappa L^2} \tag{13}
\]

In order to obtain a chain which can be treated analytically, we assume that the probability to go from state 3 to state 4, is the same as the probability to go from state 4 to state 5 and, in general, that the probabilities to go from state \( i \) to state \( i + 1 \) are \( p_i = p_{i+1} = p \) for \( i \geq 3 \). Similarly, we assume that \( \alpha_i = \alpha_{i+1} = \alpha \) for \( i \geq 3 \) and \( q_i = q_{i+1} = q \) for \( i > 3 \).

To calculate the probabilities \( p, q, \alpha \), in appendix B we have calculated the average real time \( t_{p,q,\alpha} \) to go from state \( i \) to state \( i + 1 \) assuming the average distance between kinks and the germ is \( L/2 \): \( t_{p,q,\alpha} = L/4 * \tau_3 + L(L - 2)/4 * \tau_2 \). Moreover, since \( p = q \), we can calculate \( p, q \) and \( \alpha \):

\[
p = 2\rho/L - \frac{2(L-2)}{L} \rho^2 + o(\rho^2) \tag{14}
\]

\[
\alpha = 1 - \frac{4\rho}{L} + \frac{4(L-2)}{L} \rho^2 + o(\rho^2) \tag{15}
\]

\[
q = 2\rho/L - \frac{2(L-2)}{L} \rho^2 + o(\rho^2) \tag{16}
\]

When 2 particles are bonded on the facet (state 3), the probability to go to state 2 should practically be equal to \( q \), we will assume this to be the case.

So far, we have omitted the possibility that the germ can also nucleate on a small facet \( l \). To take this into account, we have to consider new states:

- state \(-1\): one particle is on the facet \( l \) on one of its edges.
- state \(-2\): 2 particles are on the facet \( l \) one of them is on the edge of the facet, and the other has diffused from an edge.
- state \(-3\): 2 particles are on the facet \( l \) but they are bonded.
- state \(-4\): 3 bonded particles are on the facet \( l \).
- \ldots
- state \(-l\): \( l - 1 \) bonded particles are on the facet \( l \).

As discussed above, if the system arrives to state \(-l\), a row on a small row is completed, which is not an absorbing state, and since this row cannot grow any further the system can only go back to state \(-l + 1\) or stay in state \(-l\).

The different probabilities of transition from one state to another in this branch of the chain are the same as the ones calculated before, replacing \( L \) by \( l \). Thus we have \( q_{-i} = p_i(L \Rightarrow l) \), \( p_{-i} = q_i(L \Rightarrow l) \) and \( \alpha_{-i} = \alpha_i(L \Rightarrow l) \) for \( i \geq 1 \) except for the state \(-l \), where we always have \( \alpha_{-i} = \alpha(L \Rightarrow l) \), but, \( p_{-1} = 2 * q(L \Rightarrow l) \). In the following, we will use the notation: \( p_i^* = p_i(L \Rightarrow l) \), the values of \( p_i \) where we have replace \( L \) by \( l \).

To complete the calculation, we now have to determine \( p_0, q_0 \) and \( \alpha_0 \). The average real time the system stays in state 0, assuming states 1 and \(-1\) are absorbing, is almost \( \tau_3/2 \) if we take into account that there are two kinks, one at each end of the facet. The probability that the germ nucleates on the facet \( L \) is simply \( \frac{L}{2\tau_3} \). From this we deduce:

\[
p_0 = 2\frac{L}{L + 2l} \tag{17}
\]

\[
\alpha_0 = 1 - 2\rho \tag{18}
\]

\[
q_0 = 2\frac{L}{L + 2l} \tag{19}
\]

The diagram of the entire Markov chain is then given by Fig. 4. And we will calculate the time to go from state
0 to state \( l+1 \).

![Diagram of the entire Markov chain with two branches: the upper one for the germs growing on a small facet and the other one for the germs growing on the large one, state \( l+1 \) is the only absorbing state.](image)

The state \( l+1 \) is absorbent (as discussed above, when the size of the germ reaches \( l \) on a large facet, the system cannot go back to the initial state). Let us call \( n_i \) the average time to go from state \( i \) to state \( l+1 \). We can write:

\[
\begin{align*}
n_{-l} &= 1 + 2q^*n_{-l-1} + \alpha^*n_{-l} \\ n_{-k} &= 1 + q^*n_{-k-1} + \alpha^*n_{-k} + p^*n_{-k+1} \\
& \quad \text{with } 3 \leq k \leq l \\
& \quad \text{...} \\
n_{-2} &= 1 + q_2^*n_{-1} + \alpha_2^*n_{-2} + p_2^*n_{-3} \\ n_{-1} &= 1 + q_1^*n_0 + \alpha_1^*n_{-1} + p_1^*n_{-2} \\ n_0 &= 1 + q_0^*n_{-1} + \alpha_0^*n_0 + p_0^*n_1 \\ n_1 &= 1 + q_1^*n_0 + \alpha_1^*n_1 + p_1^*n_2 \\ n_2 &= 1 + q_2^*n_1 + \alpha_2^*n_2 + p_2^*n_3 \\
& \quad \text{...} \\
n_k &= 1 + q_nk_{-1} + \alpha nk + pn_{k+1} \\
& \quad \text{with } 3 \leq k \leq l - 1
\end{align*}
\]

The boundary condition for this process is \( n_{l+1} = 0 \). The calculation of \( n_0 \) is quite straightforward but tedious, it is carried out in the appendix E. We find that in the limit of small temperatures, the typical real time needed to nucleate a germ of size \( l \) on a facet of size \( L \) is given by:

\[
\tau(L, l) \approx \frac{\tau^2_L}{\tau^2_L (L + 2l)} \left( \frac{L}{\alpha} - 1 \right)(l - 1) + 1
\]

III. ESTIMATION OF THE RELAXATION TIME

Scaling laws

In this section we calculate the typical time required for an island to relax from an initial out of equilibrium shape. We assume that at all times, the instantaneous shape of the island can be characterized by the lengths \( L \) and \( l \) of its long and short facets respectively. Then, following the discussion in the previous sections, a new row of particles will appear on a long facet after a time \( \tau(L, l) \). Thus, calling \( v(L) \) the normal speed of the large facet and taking the particle size as our unit distance, we have:

\[
v(L) \approx \frac{1}{\tau(L, l)} \approx \frac{4\tau_2(L-1)}{1/(L-1)(l-1) + 1}
\]

The scaling properties of the relaxation time can be deduced by noticing that the length scales involved scale as \( N^{1/2} \), where \( N \) is the number of atoms of the island. Thus we renormalize the lengths by \( x \rightarrow x' = N^{-1/2}x \). Then, to scale out the size dependence as \( N \) grows, one must rescale time by \( t \rightarrow t' = tN^{-1} \). This is the result obtained in [8]: at low temperature, the relaxation time is proportional to the number of the atoms of the island. But, as we expect our results for the time required to complete a row at each stage, as given in Eq. E10, to be relatively accurate, we can go beyond the scaling properties and use it to calculate numerically the time required for the complete relaxation process, including the corrections arising from the lower order terms.

In what follows, we establish the differential equations which permit the calculation of the full relaxation time of an island. As mentioned above, we still consider the simple island of Fig. [E] where \( v(L) \) is the normal speed of the facet \( L \), and \( v(l) \) that of the facet \( l \). We now consider \( L \) and \( l \) as continuous variables, which considerably simplifies the calculation.

Conservation of the matter imposes the relation:

\[
Lv(L) + 2lv(l) = 0 \tag{32}
\]

Moreover, we can find geometric relations between \( L, l, v(L), v(l) \):

\[
v(L) = \sqrt{3}/2 \frac{dl}{dt} \tag{33}
\]

\[
v(l) = \sqrt{3}/4 \frac{dl}{dt} + \frac{dL}{dt} \tag{34}
\]

So finally we find:

\[
\frac{dl}{dt} = \frac{2}{\sqrt{3}} \frac{1}{\tau(L,l)} \tag{35}
\]

\[
\frac{dL}{dt} = -\frac{2}{\sqrt{3}} \left( \frac{L}{l} + 1 \right) \frac{1}{\tau(L,l)} \tag{36}
\]
To integrate these equations numerically: we use Eq. E10, and for \( p1,q1 \) and \( \alpha_1 \) the exact estimation using Eq. A9, as well as the explicit values of the quantities \( q_i, \alpha_i \) and \( p_i \). We start the integration from an island of aspect ratio of \( R = 10 \), and stop it when the aspect ratio is \( R = 1.2 \). Aspect ratios are explicitly calculated as:

\[
R = \frac{r_x}{r_y} \\
r_x^2 = \frac{1}{S} \iint_{\text{Island Surface}} (x - x_G)^2 \, dx \, dy \\
r_y^2 = \frac{1}{S} a \iint_{\text{Island Surface}} (y - y_G)^2 \, dx \, dy \\
S = \iint_{\text{Island Surface}} dx \, dy
\]

(37) (38) (39) (40)

Where \( x_G \) and \( y_G \) give the position of the center of gravity of the island. We report in Fig. 5 the relaxation time as a function of temperature and island size. Our results have a rather good quantitative agreement with those obtained from direct simulations of the system. At higher temperatures, multiple nucleation processes, the presence of many mobile particles on the island’s boundary and the failure of our hypothesis \( L_c \gg L \) (Sect. II B) invalidate our picture and the relaxation becomes driven by the coarsed grained curvature of the boundary, which leads to Mullins classical theory.

The description of the simple model considered in this work is certainly not exact and there are other effects that could be taken into account. Perhaps the most important effect we have overstepped at low temperatures, is related to our assumption that after the germ stabilizes a single new row is formed on the large facet. The differential equations for the evolution of the island were derived from this assumption. It is clear that this is not correct: our estimation of the time required to stabilize a germ starts from a fully faceted configuration, and once a complete row on a small facet disappears the germ becomes stable and a full row on the large facet can be formed. Once this row is finished, it is very unlikely that the island will be in a fully faceted configuration again, leaving at least an extra kink on the boundary. This gives rise to extra sources and traps for mobile particles, which might have affected the relaxation rate. Another issue is the assumption of the faceted island with only two facets sizes: a more detailed characterization may be relevant especially in the early stages of relaxation. It is clear that more accurate models for specific systems can also be constructed. These could take into account the dependence of the edge-diffusion coefficients on the orientation of the facet, as well as the dependence of emission rates on the local geometry. Such dependences have been studied, for example, by \( \alpha_i \). In terms of the elements of description we use, inclusion of these effects would be achieved by changing the values of \( \tau_2 \) (diffusion time) and \( \tau_3 \) (emission time from kinks and corners) depending on the orientation of the facets involved in each event. Thus, the nucleation time would depend on the facet upon which it happens. Such a dependence of the nucleation time may drive the island toward a non regular hexagon equilibrium shape \( \alpha_i \) and would reproduce the phenomenology of a larger variety of materials. These changes will affect the temperature dependent prefactors in our results, as these depend on the temperature through \( \tau_2 \) and \( \tau_3 \).
However, the size dependence of the nucleation time and of the relaxation time, which is where the departure from Mullin’s theory is evidenced, would stay the same. From a more general point of view, only the diffusion of particles along the perimeter of islands has been taken into account in this work. In real systems, other mechanisms can contribute to the transport of the matter which leads to relaxation: volume diffusion and transport through the two dimensional gas of particle surrounding the island. Volume diffusion is usually a much slower process than the other two, and can usually be neglected safely. On the other hand, it is well known that edge diffusion is more efficient for short trips whereas transport through the 2D gas is faster on long distances. Following Pimpinelli and Villain in[4], a characteristic length \( r_1 \) beyond which edge diffusion is less efficient than transport through the gas can be evaluated: 
\[
r_1 \approx \sqrt{D_s \tau_v}
\]
where \( D_s \) is the edge diffusion coefficient and \( 1/\tau_v \) the probability per unit time a given particle leaves the island. So that our assumptions should be valid for islands with a number \( N \) of particles such that: 
\[
N \ll D_s \tau_v.
\]
Since the activation energy for edge diffusion is smaller than the activation energy for evaporation, \( r_1 \) is a decreasing quantity with temperature, and we expect \( r_1 \) to be very large at low temperature. Thus, this mechanism is essentially irrelevant in the description of the evolution of nanometer structures at low temperatures. Moreover, recent experimental results have shown that supported Ag two dimensional islands relax via atomic diffusion on the island perimeter, without significant contribution from exchange with the two dimensional gas. Finally, our results are to be compared with a recent theoretical study concerning the relaxation of three dimensional crystallites. This study also points out two relaxation regimes as a function of temperature. At high temperature the relaxation scales in accordance with the results derived from Mullins’ theory, whereas at low temperature the relaxation time becomes an exponential function of the size of the crystallites. So that the effects of lowering the temperature are qualitatively different for two dimensional and three dimensional crystallites: in two dimensions, lowering the temperature decreases the strength of the dependence of the relaxation time as a function of the size of the crystallites (as it crosses over from a \( N^2 \) dependence to a \( N \) dependence), whereas it increases this the strength in three dimensions. In both cases, the limiting step is the nucleation of a germ on a facet: a unidimensional germ in two dimensions, and a two dimensional germ in three dimensions. The difference stems from the fact that in the two dimensional case, the activation energy for the creation of the germ does not depend on the size of the island, it is always constant: \( 4E \), and it stabilizes when a row on a small facet has been removed. In the three dimensional case, this activation energy depend on the size of the crystallite. The transfer of a particle from a tip of the crystallite to the germ has a gain in volume energy (depending on the size of the islands) and a loss in edge energy of the germ (depending on the size of the germ). Summing these two terms, an energy barrier proportional to the size of the crystallite appears for the creation of a stable germ. The exponential behavior of the relaxation time as a function of \( N \) is a consequence of this energy barrier dependence. Finally, we believe that the overall picture presented here, while still oversimplified, seems to be complete enough to provide a general picture of the processes leading to the shape relaxation of two dimensional islands at low temperatures.

We acknowledge useful discussions with P. Jensen, J. Wittmer and F. Niceas. We are grateful with an anonymous referee for his many useful comments. H.L. also acknowledges partial financial support from CONACYT and DGAPA-UNAM, and is thankful with Univ. Claude Bernard, Lyon 1, for the invitation during which part of this work was done.

**APPENDIX A: CALCULATION OF THE PROBABILITY TO HAVE 2 PARTICLES ON THE FACET**

We calculate the probability \( P \) of having 2 particles on a facet with absorbing boundaries, knowing that at time \( t = 0 \), one particle is on one edge of the facet (abscissa 1), and that the other particle can appear on the facet with a probability per unit time \( 1/\tau \). Relatively to our Markov chain, this probability is the probability the system in state 1 eventually reaches state 2.

We denote by \( S(x,t) \) the probability that a particle on the facet at position \( x \) at time \( t = 0 \) is still on the facet at time \( t \). Then, \( S(x,t) \) satisfies the usual diffusion equation:

\[
\frac{\partial S(x,t)}{\partial t} = D \frac{\partial^2 S(x,t)}{\partial x^2}
\]

(A1)

This equation is to be solved with the conditions: 
\( S(x,0) = 1 \) for every \( x \in [0,L] \) (i.e. we are sure to find the particle on the facet at time \( t = 0 \)), \( S(0,t) = S(L,t) = 0 \) for every \( t \) (i.e. the boundaries sides of the facet are absorbent). The solution of Equation [A1] is

\[
S(x,t) = \sum_{n=0}^{+\infty} \frac{4}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi x}{L}\right) e^{-\frac{D(2n+1)^2t}{L^2}}.
\]

(A2)

To take into account the appearance of particles on the facet, we assume the process to be Poissonian so that the probability to have a particle appearing at time \( t \) is \( \frac{1}{\tau} e^{-\frac{t}{\tau}} \). Thus the probability \( P \) that two particles are on the facet is:

\[
P = \int_0^\infty \frac{1}{\tau} e^{-\frac{t}{\tau}} S(x,t) dt
\]

(A3)
To take into account that particles can appear on the facet on both ends, we take \( \tau = \tau_3/2 \), which holds at low temperatures. This leads to the expression:

\[
P = \frac{4 \rho^2}{\pi} \frac{1}{\delta^2 + (2n+1)^2} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\chi)}{2n+1}
\]

where:

\[
\delta^2 = \frac{2L^2}{D\pi^2 \tau_3}
\]

\[
\chi = \frac{\pi x}{L}
\]

Using formula\[A5\]:

\[
\zeta(\delta, \chi) = \frac{\sum_{k=1}^{\infty} \cos(k\chi)}{\delta^2 + k^2}
\]

we have:

\[
\sum_{n=0}^{\infty} \frac{\sin((2n+1)\chi)}{2n+1 \delta^2 + (2n+1)^2} = -\frac{1}{2\delta^2} \left[ \int_0^\chi \zeta(\delta, \chi') - \frac{1}{4} \zeta(\delta/4, 2\chi') \right] d\chi'
\]

So that, using Eqs.\[A3\],\[A6\] we finally find the following expression for \( P \):

\[
P = 1 - \left[ \frac{2\sinh(2\sqrt{\rho}(L-1))}{\sinh(2\sqrt{\rho}L)} - \frac{\sinh(2\sqrt{\rho}(L/2-1))}{\sinh(2\sqrt{\rho}L/2)} \right]
\]

\[
(P = 2\rho(L-1) + o(\rho))\]

**APPENDIX B: PROBABILITY THAT TWO PARTICLES STICK ON THE FACE**

In this part we evaluate the probability \( P_\triangle \) that two particles stick on a facet with absorbing boundaries, knowing that at time \( t = 0 \), one particle is on one end of the facet, and the other is at a position \( x_0 \) on the face. This problem can be mapped to a 2-d problem in which, at time \( t \), the first particle is at position \( y \), and the second particle is at position \( x \). This virtual particle moves diffusively in a square of side \( L \), starting from position \( (x_0, a) \) (\( a \) is the lattice spacing). The quantity we are looking for is the probability for this virtual particle to reach the diagonal \( y = x \) of the square. Thus we can consider the motion of the virtual particle in the triangle \( 0 \leq x \leq y \leq L \). We call \( D(x, y) \) the probability that a virtual particle starting at time \( t = 0 \) from \( (x, y) \) leaves the triangle by the diagonal, \( V(x, y) \), the probability that this particle leaves the triangle by its vertical side, and \( H(x, y) \) the probability that this particle leaves the triangle by its horizontal side. We use here a continuous description: the discrete problem being far too difficult. It can be easily seen that \( D, V \) and \( H \) satisfy Laplacian equations:

\[
\Delta D(x, y) = 0 \quad \Delta V(x, y) = 0 \quad \Delta H(x, y) = 0
\]

With the conditions:

\[
D(x, x) = 1 \quad D(x, 0) = 0 \quad D(0, y) = 0
\]

\[
V(x, x) = 0 \quad V(x, 0) = 0 \quad V(L, y) = 1
\]

\[
H(x, x) = 0 \quad H(x, 0) = 1 \quad H(0, y) = 0
\]

for \( \forall \, x \in [0, L] \) and \( \forall \, y \in [0, L] \)

And moreover we should have:

\[
D(x, y) + V(x, y) + H(x, y) = 1
\]

which states that the particle is sure to leave the triangle since all sides are absorbent. Instead of calculating directly \( D(x, y) \), we will calculate \( V(x, y) \) and \( H(x, y) \).

We first calculate \( V_\square(x, y) \) and \( H_\square(x, y) \), which are the probability that one brownian particle in a square with absorbing sides, respectively leaves the square by the vertical side \( x = L \) and by the horizontal side \( y = 0 \). So we have:

\[
\Delta V_\square(x, y) = 0 \quad \Delta H_\square(x, y) = 0
\]

with boundary conditions:

\[
V_\square(x, L) = 0 \quad V_\square(x, 0) = 0 \quad V_\square(0, y) = 0 \quad V_\square(L, y) = 1
\]

\[
H_\square(x, x) = 0 \quad H_\square(x, 0) = 1 \quad H_\square(0, y) = 0 \quad H_\square(x, L) = 0
\]

for \( \forall \, x \in [0, L] \) and \( \forall \, y \in [0, L] \)

The solution of equations Eq.\[B6\], with these conditions is:

\[
H_\square(x, y) = \frac{4}{\pi} \sum_{m=0}^{\infty} \sin\left(\frac{(2m+1)\pi x}{2m+1}\right) \sin\left(\frac{(2m+1)\pi (L-y)}{2m+1}\right)
\]

\[
V_\square(x, y) = \frac{4}{\pi} \sum_{m=0}^{\infty} \sin\left(\frac{(2m+1)\pi x}{L}\right) \sin\left(\frac{(2m+1)\pi y}{L}\right)
\]

One now can deduce the values of \( V(x, y) \) and \( H(x, y) \) of our initial problem with a superposition of solutions to impose the condition: \( V(x, x) = 0 \) and \( H(x, x) = 0 \):

\[
V(x, y) = V_\square(x, y) - V_\square(y, x)
\]

\[
H(x, y) = H_\square(x, y) - H_\square(y, x)
\]
Using Eq. B3, B13, B14, B11, B12 we finally find an expression for \( D(x_0, a) \):

\[
D(x_0, a) = 1 - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin \left( \frac{(2m+1)\pi a}{L} \right)}{2m+1} \left[ \frac{\sinh \left( \frac{(2m+1)\pi x_0}{L} \right)}{\sinh(2m+1)\pi} - \frac{\sin \left( \frac{(2m+1)(L-x_0)}{L} \right)}{\sin(2m+1)\pi} \right]
\]

We now have to calculate the probability that the second particle is at position \( x_0 \) when the other one appears on the facet. We denote this probability \( P(x_0) \). Then the probability we are looking for is simply:

\[
P_{\Delta} = \int_0^L D(x_0, a) P(x_0) dx_0 \quad \text{(B15)}
\]

We are actually able to calculate the exact probability \( P(x_0) \), but then we are not able to find a simple expression for \( P_{\Delta} \), so we prefer to make the following approximation: in the limit of small temperature, the typical time needed for a particle to appear on the facet is about \( \tau_\Delta \), which is long compared to the typical time a particle lasts on the facet (about \( L\tau_\delta \) for a particle that starts near the edge). Thus, to a good approximation, the probability \( P(x_0) \) has reached its stationary value. Taking into account only the first term of the series, we have:

\[
P(x_0) = \frac{\pi}{2L} \sin \frac{\pi x_0}{L} \quad \text{(B16)}
\]

With this expression Eq. B13 becomes easy to calculate and we find:

\[
P_{\Delta} = 1 - \frac{\sinh \frac{\pi(L-a)}{L}}{\sinh \pi} - \frac{\sin \frac{\pi a}{L}}{\sin \frac{\pi}{L}} = \left( \frac{\cosh \pi + 1}{\sinh \pi} \right) \frac{\pi a}{L} - \frac{\pi^2 a^2}{2L^2} + o(1/L^2) \quad \text{(B17)}
\]

So that we find that the leading term of \( P_{\Delta} \) is proportional to \( 1/L \). We write this as:

\[
P_{\Delta} = \frac{1}{L} + o(1/L^2) \quad \text{(B19)}
\]

with \( \lambda = \left( \frac{\cosh \pi + 1}{\sinh \pi} \right) \pi \quad \text{(B20)}

**APPENDIX C: CALCULATION OF THE TIME TO LEAVE STATE 2**

We calculate in this part the average time \( < \tau > \) for two particles on the facet either to bond, or either one of them to reach the boundary. This time corresponds to the time the system stays in state 2 in the Markov chain. Using the description of appendix 3 with the virtual particle in the triangle, we are looking for the average time this particle needs to leave the triangle.

Let us call \( S(x_0, y_0, t) \) the survival probability: i.e. the probability a particle starting at \( (x_0, y_0) \) at time \( t = 0 \) is still in the triangle at time \( t \). The average time \( < \tau(x_0, y_0) > \) the particle stays in the triangle is given by:

\[
< \tau(x_0, y_0) > = \int_0^\infty S(x_0, y_0, t) dt \quad \text{(C1)}
\]

Moreover calling \( P_{\Delta}(x, y, x_0, y_0, t) \) the probability that one particle starting at \( (x_0, y_0) \) at time \( t = 0 \) is at position \( (x, y) \) at time \( t \), we have:

\[
S(x_0, y_0, t) = \int_\Delta P_{\Delta}(x, y, x_0, y_0, t) dxdy \quad \text{(C2)}
\]

Finally, we are only interested in \( < \tau(x_0, 1) > \) since we know that our virtual particle starts at \( (x_0, a) \), so we have to average the time \( < \tau(x_0, 1) > \) over all values \( x_0 \). To do this we use the approximate distribution given in Eq. B11:

\[
< \tau > = \int_0^L \int_0^\infty \int_\Delta P_{\Delta}(x, y, x_0, 1, t) dxdy dt d\tau \quad \text{(C4)}
\]

We now have to calculate \( P_{\Delta}(x, y, x_0, 1, t) \). As before, we find the solution in a square, and then by superposition, we deduce the solution in the triangle:

\[
P_{\Delta}(x, y, x_0, 1, t) = P_{\xi}(x, y, x_0, 1, t) - P_{\xi}(x_0, 1, x, y, t) = \frac{4}{L^2} \sum_{m,n=1}^{\infty} \sin \frac{m\pi x_0}{L} \sin \frac{n\pi y}{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{L} e^{-\frac{m^2+n^2}{L^2} x_0} \quad \text{(C5)}
\]

We find \( < \tau > \) integrating Eq. C4 using Eq. C5, C6 over \( x_0 \) first, then over \( x \) and \( y \), and finally over \( t \), we find:

\[
< \tau > = \frac{L^2}{D\pi^4} \sum_{k=1}^{\infty} \frac{1}{1 + 4k^2} \sin \frac{2k\pi}{L} \left[ \frac{2}{k} - \frac{1}{k(2k+1)} \right] \quad \text{(C7)}
\]

We interested in the leading term as \( L \to \infty \), let us define a function \( f(u) \) by:

\[
f(u) = \sum_{k=1}^{\infty} \frac{1}{1 + 4k^2} \sin(ku) \left[ \frac{2}{k} - \frac{1}{k(2k+1)} \right] \quad \text{(C8)}
\]

\( f(u) \) is a normally convergent series, so that:

\[
\frac{df(u)}{du} = \sum_{k=1}^{\infty} \frac{1}{1 + 4k^2} \cos(ku) \left[ \frac{2}{k} - \frac{1}{k(2k+1)} \right] \quad \text{(C9)}
\]

\[
\sum_{k=1}^{\infty} \frac{1}{1 + 4k^2} \left[ 2 - \frac{1}{(2k+1)} \right] \quad \text{(C10)}
\]
which is also convergent. The second term of the development is of order \( u^2 \) or smaller, so that integrating Eq. (10), and using it in Eq. (11), we find:

\[
< \tau > = \frac{2L}{D\pi^2} \sum_{k=1}^{\infty} \frac{1}{1 + 4k^2} \left[ 2 - \frac{1}{(2k+1)} \right] + O(1/L)
\]

(C11)

Thus \( < \tau > \) is proportional to \( L \), and we write:

\[
< \tau > = \kappa L \tau_2 + O(1/L)
\]

(C12)

with \( \kappa = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{1 + 4k^2} \left[ 2 - \frac{1}{(2k+1)} \right] \)

(C13)

APPENDIX D: CALCULATION OF THE TIME A PARTICLE NEEDS TO GO TO CENTER OF THE FACET

We now calculate the average real time \( t_{p,q,\alpha} \) to go from state \( i \) to state \( i+1 \) assuming the average distance between the kinks or corners from which particles are emitted and the germ is \( L/2 \). This again can be posed as a Markov chain where:

- state 0 : no particle is on the facet;
- state 1 : one particle coming from a kink is in position 1 on the facet;
- state 2 : the particle is at position 2;
- state \( k \) : the particle is at position \( k \); and state \( L/2 \) is absorbent.

Again we use \( \tau_2 \) as the unit time. Using the same definition for the quantities \( p_i, q_i \) and \( \alpha_i \) as in the main text, we have:

\[
p_i = 1/2
\]

(D1)

\[
\alpha_i = 0
\]

(D2)

\[
q_i = 1/2
\]

(D3)

with \( i \geq 1 \)

To calculate \( p_0 \) and \( \alpha_0 \) we know that the real time an atom needs to leave a kink is \( \tau_3 \), and as there are two kinks at the ends of the facet, \( \tau_3/2 \) is, to a good approximation, the average time to leave state 0. Thus that we find:

\[
p_0 = 2\rho
\]

(D4)

\[
\alpha_0 = 1 - 2\rho
\]

(D5)

Calling \( n_i \) the average time to go from state \( i \) to the absorbent state \( L/2 \), we have the equations:

\[
n_0 = 1 + \alpha_0 n_0 + p_0 n_1
\]

(D6)

\[
n_k = 1 + \frac{n_{k-1}}{2} + \frac{n_{k+1}}{2}
\]

(D7)

with \( k \geq 1 \)

\[
n_{L/2} = 0
\]

(D8)

and \( n_{L/2} = 0 \). Summing equations (D7) from \( k = 1 \) to \( j \), and then from \( j = 1 \) to \( L/2 - 1 \), and using Eq. (D8) one finds:

\[
n_1 = \frac{L - 2}{L} n_0 + \frac{L - 2}{2}
\]

(D9)

And Eq. (D6) permits to obtain \( n_0 \):

\[
n_0 = \frac{L}{2p_0} + \frac{L(L - 2)}{4}
\]

(D10)

Going back to real time, one finds the average time \( \tau \) a particle needs to leave a kink and reach the center of the facet is:

\[
t_{p,q,\alpha} = \frac{L}{4} \tau_3 + \frac{L(L - 2)}{4} \tau_2
\]

(D11)

APPENDIX E: CALCULATION OF \( N_0 \)

To carry out the calculation, we can first calculate the \( n_k \) for \( k \geq 1 \). Noting that \( p = q \), we have from Eq. (28):

\[
(n_{k+1} - n_k) = (n_k - n_{k-1}) - 1/p,
\]

(E1)

so that summing from \( k=4 \) to \( j \), and then from \( j=4 \) to \( l \), and using \( n_{l+1} = 0 \), we find:

\[
(l - 1)n_3 = (l - 2)n_2 + 1/p\left(\frac{(l - 2)(l - 1)}{2}\right)
\]

(E2)

Inserting this result in Eq. (27) we find \( n_2 \) as a function of \( n_1 \), in the same way, using Eq. (26), one can deduce the equation for \( n_1 \):

\[
n_1 (1 - \alpha_1 - p_1 B) = 1 + p_1 A + q_1 n_0
\]

(E3)

where :

\[
A = \frac{1 + \rho l - 2}{1 - \alpha_2 - p_2 \frac{l - 2}{2}}
\]

(E4)

\[
B = \frac{q_2}{1 - \alpha_2 - p_2 \frac{l - 2}{2}}
\]

(E5)

We now calculate the case \( k < 0 \). Calling \( m_j = n_{-j} \) for every \( j \), and summing Eq. (21) from \( j \) to \( z - 1 \), one finds:

\[
m_z - m_{z-1} = m_j - m_{j-1} - \frac{z - j}{p^*},
\]

(E7)

and Eq. (20) yields: \( m_j - m_{j-1} = \frac{1}{2p^*} \). Using this in Eq. (E7) and taking \( j = 3 \), one finds:

\[
m_3 - m_2 = \frac{l - 3}{p^*} + \frac{1}{2p^*}
\]

(E8)
Using Eqs. 23, 24, one finds:

\[ n_{-1} = \frac{1}{q_1} + \frac{p_1}{q_1^2 q_2} + \frac{p_1^2 p_2^2}{q_1^3 q_2^2} (l - 5/2) + n_0. \]  

(E9)

One can know obtain the value of \( n_0 \) from Eq. 23, 25 and 23:

\[ p_0 (1 - \frac{q_1}{1 - \alpha_1 - p_1 B}) n_0 = 1 + p_0 \frac{1 + p_1 A}{1 - \alpha_1 - p_1 B} + q_0 \left( \frac{1}{q_1} + \frac{p_1}{q_1^2 q_2} + \frac{p_1^2 p_2^2}{q_1^3 q_2^2} (l - 5/2) \right) \]  

(E10)

Then, using Eq. 23, 25 we can calculate A and B:

\[ A = \frac{L}{4 \rho} \frac{(L - 1)(L - 2)}{(L - 1)(L - 1) + 1} + \frac{\lambda}{\rho} L^2 (l - 1) + \frac{L(L-2)(l-2)(l-1)}{4} + o(1) \]

\[ B = 1 - \frac{1}{(\lambda - 1)(L - 1) + 1} \]

And finally, the expression of \( n_0 \) is:

\[ n_0 = \frac{1}{\rho^2} \frac{(L + 2)}{4L(L - 1)} \left[ -1 \right. \right. \]

\[ + O(1/\rho) \]  

(E11)

We report here only the leading term: the expressions of the quantities \( p_i, q_i \) and \( \alpha_i \) permit us to calculate \( n_0 \) up to order 1, but these terms are truly ugly and it does not seem relevant to give them here.

Going back to real time, we find that the time \( \tau(L, l) \) to nucleate a new row on a facet is given by:

\[ \tau(L, l) = \frac{\tau_3^2}{\tau_3} \frac{(L + 2)}{4L(L - 1)} \left[ -1 \right. \right. \]

\[ + O(\tau_3) \]  

(E12)

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1. M. Lagally, Physics Today 46(11), 24 (1993) and references therein; H. Gleiter, Nanostructured Materials 1 1 (1992); Z. Zhang and M. G. Lagally, Science 276, 377 (1997)
2. P. Jensen, Rev. Mod. Phys. 71, 1695 (1999) and references therein
3. P. Nozières, in : Solids far from equilibrium, Ed. C. Godrèche (Cambridge Univ. Press, Cambridge, 1992)
4. A. Pimpinelli and J. Villain Physics of Crystal Growth (Cambridge University Press, 1998)
5. P. Jensen, N. Combe, H. Larralde, J. L. Barrat, C. Misbah and A. Pimpinelli, Eur. Phys. J. B 11, 497-504 (1999)
6. C. Herring, Physics of Powder Metallurgy (McGraw-Hill Book New York, Company, Inc. 1951) Ed. W. E. Kingston; C. Herring, Phys Rev 82, 87 (1951); W.W. Mullins, J. Appl. Phys. 28, 333 (1957) and 30, 77 (1959); F.A. Nichols and W.W. Mullins, J. Appl. Phys. 36, 1826 (1965); F.A. Nichols, J. Appl. Phys. 37, 2805 (1966)
7. This formulation of the relaxation time as a function of \( N/N_c \) permits to resolve the apparent paradox mentioned in 5. Namely, that extrapolating the low temperatures regime to higher temperatures, one could infer that large islands would relax faster at high temperature than at low temperature. The curve \( \log(t_{relaxation}) \) as a function of \( \log(N) \) at a given temperature \( T_1 \) is obtained by translation of the curve at temperature \( T_0 \) by a vector \( \frac{N_0(T_1)}{N_0(T_0)} \left[ \frac{1}{5} \right] \).
8. see for example : J. Langer in : Solids far from equilibrium, Ed. C. Godrèche (Cambridge Univ. Press, Cambridge, 1992)
9. R. Ferrando and G. Tréglia, Phys. Rev. B 50, 12104 (1994).
10. T. Michély, M. Hohage, M. Bott and G. Comsa, Phys. Rev. Lett. 70, 3943 (1993).
11. C. R. Stoldt, A. M. Cadilhe, C. J. Jenks, J. -M. Wen and, J. W. Evans and P. A. Thiel, Phys. Rev. Lett. 81, 2950 (1998).
12. N. Combe, P. Jensen and A. Pimpinelli, Phys. Rev. Lett. 85, 110 (2000).
13. I.S. Gradshteyn and I.M. Ryzhik Table of Integrals, series, and Products(Academic Press San Diego) Ed. Alan Jeffrey p. 47