LOGARITHMIC BLOCH SPACE AND ITS PREDUAL

Miroslav Pavlović

Abstract. We consider the space $\mathcal{B}_1^{\log \alpha}$, of analytic functions on the unit disk $D$, defined by the requirement $\int_D |f'(z)|\phi(|z|)dA(z) < \infty$, where $\phi(r) = \log^\alpha(1/(1-r))$ and show that it is a predual of the “$\log^\alpha$-Bloch” space and the dual of the corresponding little Bloch space. We prove that a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n \downarrow 0$ is in $\mathcal{B}_1^{\log \alpha}$ iff $\sum_{n=0}^{\infty} \log^\alpha(n+2)/(n+1) < \infty$ and apply this to obtain a criterion for membership of the Libera transform of a function with positive coefficients in $\mathcal{B}_1^{\log \alpha}$. Some properties of the Cesàro and the Libera operator are considered as well.

1. Introduction and some results

Let $H(D)$ denote the space of all functions analytic in the unit disk $D$ of the complex plane. Endowed with the topology of uniform convergence on compact subsets of $D$, the class $H(D)$ becomes a complete locally convex space. In this paper we are concerned with the predual of the space $\mathcal{B}_1^{\log \alpha}$, $\alpha \in \mathbb{R}$,

\[ \mathcal{B}_1^{\log \alpha} = \left\{ f \in H(D) : |f'(z)| = O\left((1-|z|)^{-1} \log^\alpha \frac{2}{1-|z|}\right) \right\}. \]

The norm in $\mathcal{B}_1^{\log \alpha}$ is defined by

\[ \|f\|_{\mathcal{B}_1^{\log \alpha}} = |f(0)| + \sup_{z \in D} |f'(z)|(1-|z|)\log^{-\alpha} \frac{1}{1-|z|}. \]

The subspace, $b_1^{\log \alpha}$, of $\mathcal{B}_1^{\log \alpha}$ is defined by replacing “$O$” with “$o$”. It will be proved:

(A) The dual of $b_1^{\log \alpha}$ is isomorphic to $\mathcal{B}_1^{\log \alpha}$,

\[ \mathcal{B}_1^{\log \alpha} = \left\{ f : \|f\|_{\mathcal{B}_1^{\log \alpha}} = |f(0)| + \int_D |f'(z)| \log^\alpha \frac{2}{1-|z|}dA(z) < \infty \right\}. \]

2010 Mathematics Subject Classification: 30D55.

Key words and phrases: Libera operator, Cesàro operator, Hardy spaces, logarithmic Bloch type spaces, predual.

Supported by MNTR Serbia, Project ON174017.

Communicated by Gradimir Milovanović.
and the dual of $\mathcal{B}_{\log^\alpha}^1$ is isomorphic to $\mathcal{B}_{\log^\alpha}$, in both cases with respect to the bilinear form

\begin{equation}
\langle f, g \rangle = \lim_{r \uparrow 1} \sum_{n=0}^{\infty} \hat{f}(n)\hat{g}(n)r^{2n}.
\end{equation}

(In (1.1) $dA$ stands for the normalized Lebesgue measure on $\mathbb{D}$.) This extends the well-known result on the Bloch space and the little Bloch space $\mathcal{B} := \mathcal{B}_{\log^0}$.

These spaces are Banach spaces, and the space $\mathcal{B}_{\log^\alpha}$ coincides with the closure in $\mathcal{B}_{\log^\alpha}$ of the set of all polynomials. The space $\mathcal{B}_{\log^1} := \mathcal{B}_{\log^0}$ occurs naturally in the study of pointwise multipliers on the usual Bloch space $\mathcal{B} := \mathcal{B}_{\log^0}$ (see [3]).

One of interesting properties of $\mathcal{B}_{\log^\alpha}$ is described in the following theorem:

**Theorem 1.1.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $\{a_n\}$ is a nonincreasing sequence, of real numbers, tending to zero. Let $\alpha \geq -1$. Then $f$ belongs to $\mathcal{B}_{\log^\alpha}$ if and only if

\begin{equation}
S_{\alpha}(f) := \sum_{n=0}^{\infty} \frac{a_n \log(n+2)}{n+1} < \infty.
\end{equation}

Moreover, there is a constant $C$ independent of $\{a_n\}$ such that $S_{\alpha}(f)/C \leq \|f\|_{\mathcal{B}_{\log^\alpha}} \leq CS_{\alpha}(f)$.

**Proof.** See Section 4. □

In the case $\alpha = 0$, this assertion is proved in [17]. We can take $a_n$ to be the coefficients of the Libera transform of a function with positive coefficients. Namely, if $g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n$ and

\begin{equation}
\sum_{n=0}^{\infty} \frac{|\hat{g}(n)|}{n+1} < \infty,
\end{equation}

then the Libera transform $\mathcal{L}g$ of $g$ is well defined as

\begin{equation}
\mathcal{L}g(z) = \frac{1}{1-z} \int_{1}^{1} f(\zeta) d\zeta = \sum_{n=0}^{\infty} z^n \sum_{k=n}^{\infty} \frac{\hat{g}(k)}{k+1}
\end{equation}

(see, e.g., [12]). If $\hat{g} \geq 0$, then condition (1.4) is also necessary for the existence of the integral in (1.5): take $z = 0$ to conclude that (1.5) implies the convergence of the integral

$$
\int_{0}^{1} g(t) dt = \sum_{n=0}^{\infty} \frac{\hat{g}(n)}{n+1}.
$$

Then, as an application of Theorem 1.1 we get

**Theorem 1.2.** Let $\alpha > -1$, let $g \in H(\mathbb{D})$, and $\hat{g} \geq 0$. Then $\mathcal{L}g$ is in $\mathcal{B}_{\log^\alpha}$ if and only if

$$
K_{\alpha}(g) := \sum_{n=0}^{\infty} \frac{\hat{g}(n)\log(n+2)}{n+1} < \infty.
$$

We have $K_{\alpha}(g)/C \leq \|\mathcal{L}g\|_{\mathcal{B}_{\log^\alpha}} \leq CK_{\alpha}(g)$. 

Proof. See Section 4.

In the general case, the integral in (1.5) need not exist, but it certainly exists if \( g \in H(D) \), which means that \( g \) is analytic in a neighborhood of the closed disk. By using Theorem 1.1 we shall prove that \( \overline{L} \) cannot be extended to a bounded operator from \( \mathcal{B}_{\log^{\alpha}}^{1} \) to \( H(D) \), if \( \alpha < 0 \). In the case \( \alpha \geq 0 \), every function \( g \in \mathcal{B}_{\log^{\alpha}}^{1} \) satisfies (1.4), whence \( L \) is well defined, and we will show that \( L \) maps this space into \( \mathcal{B}_{\log^{\alpha-1}}^{1} \), when \( \alpha > 0 \). If \( \alpha = 0 \) we need a sort of “iterated” logarithmic space.

Cesàro operator. The dual of \( H(D) \) is equal to \( H(D) \), where “\( g \in H(D) \)” means that \( g \) is holomorphic in a neighborhood of \( D \) (depending on \( g \)). The duality pairing is given

\[
\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\hat{g}(n),
\]

where \( f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in H(D) \) and \( g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n \in H(D) \), and the series is absolutely convergent (see, e.g., [8]). The Cesàro operator is defined on \( H(D) \) as

\[
Cf(z) = \sum_{n=0}^{\infty} z^n \frac{1}{n+1} \sum_{k=0}^{n} a_k, \quad f \in H(D).
\]

It is easy to verify that the adjoint of \( C : H(D) \mapsto H(D) \) is equal to \( \overline{C} : H(D) \mapsto H(D) \), under the pairing (1.6), and vice versa (see, e.g., [12]).

The operators \( C \) and \( \overline{L} \) acting on \( H^p \) spaces were first studied by Siskakis in 1987. In [21] he proved that \( C \) is bounded on \( H^p \) for \( 1 < p < \infty \), and that \( \overline{L} \) can be extended to a bounded operator on \( H^p \), \( 1 < p < \infty \), and obtained some results on their spectra and norms. A few years later he proved the boundedness of the Cesàro operator on \( H^1 \) ([22]), while Miao proved its boundedness on \( H^p \) for \( 0 < p < 1 \) ([10]). A short proof of the boundedness of \( C \) on \( H^p \), \( 0 < p < \infty \), as well as a stronger result, can be also found in Nowak [11]. However, \( H^\infty \) is not mapped into itself by \( C \) (see [4]). If we write (1.7) as

\[
zCf(z) = \int_{0}^{z} \frac{f(\zeta)}{1 - \zeta} d\zeta,
\]

and hence

\[
(zCf(z))' = \frac{f(z)}{1 - z},
\]

we conclude that \( C \) maps \( H^\infty \) into the Bloch space (see [4]).

On the other hand, by using the inequality

\[
|f(z)| = O\left( \log \frac{2}{1 - |z|} \right), \quad f \in \mathcal{B},
\]

and the analogous inequality for \( f \in b \) (replace “\( O \)” with “\( o \)”), we get:

(B) The operator \( C \) maps the space \( \mathcal{B} \) into \( \mathcal{B}_{\log}^{1} \), and \( b \) into \( b_{\log}^{1} \).
One of our aims is to generalize this assertion to some other values of \( \alpha \) and then use assertion (A) together with the duality between \( \mathcal{C} \) and \( \overline{\mathcal{C}} \) to obtain an alternative proof of some results on the action of \( \mathcal{L} \) from \( \mathfrak{B}_{\alpha+1} \) to \( \mathfrak{B}_\alpha \), where

\[
\mathcal{L} f(z) = \int_0^1 f(t + (1-t)z) \, dt.
\]

In particular we have:

(C) The operator \( \mathcal{L} \) is well defined on \( \mathfrak{B}_{\log} \) and maps it into \( \mathfrak{B} \).

It should be noted that: (a) \( \mathfrak{B} \subset H^1 \); (b) \( \mathcal{L} \) does not map \( \mathfrak{B} \) into \( H^1 \) (see [17]); and (c) \( \mathcal{L} \) maps \( \mathfrak{B} \) into BMOA ([12], which improves an earlier result, namely that \( \mathcal{L} \) maps \( \mathfrak{B} \) into \( B_{\log} \), which improves an earlier result, namely that \( \mathcal{L} \) maps \( \mathfrak{B} \) into \( B_{\log} \).

The formula (1.8) is obtained from (1.5) by integrating over the straight line joining \( z \) and 1. A sufficient (not necessary [18]) condition for the possibility of such integration is (1.4) \( (g = f) \).

In proving some of our results, in particular assertions (A) and (B), we use a sequence of polynomials constructed in [6] (see also [7] and [16]) to decompose the space into a sum which resembles a sum of finite-dimensional spaces (see Section 3).

2. Some more results

Some elementary facts concerning the cases when \( \mathcal{L} f \) is well defined are collected in the following theorem, where

\[
\ell_{1-1} = \left\{ g \in H(D) : \|g\|_{\ell_{1-1}} = \sum_{n=0}^{\infty} \frac{|\hat{g}(n)|}{n+1} < \infty \right\}.
\]

**Theorem 2.1.** Let \( \alpha \in \mathbb{R} \). Then:

(a) \( \mathfrak{B}_{\log} \subset \ell_{1-1} \) for all \( \alpha \);

(b) \( \mathfrak{B}_{\log} \subset \ell_{1-1} \) if and only if \( \alpha \geq 0 \);

(c) if \( \alpha < 0 \), then \( \overline{\mathcal{C}} \) cannot be extended to a continuous operator from \( \mathfrak{B}_{\log} \) to \( H(D) \).

**Proof.** See Section 4. \( \square \)

**Remark 2.1.** The inclusions in (a) and (b) are continuous. Assertion (c) says much more than simply that \( \mathfrak{B}_{\log} \subset \ell_{1-1} \).

In the context of the action of \( \mathcal{C} \) and \( \mathcal{L} \) some new spaces occur: the space \( \mathfrak{B}_{\log} \) is defined by the requirement

\[
|f'(z)| = \mathcal{O}\left( \log \log \frac{4}{1-|z|} \right),
\]

the space \( \mathfrak{b}_{\log} \) defined by replacing “\( \mathcal{O} \)” with “\( o \)”, and the space \( \mathfrak{B}_{\log} \) defined by

\[
\int_D |f'(z)| \log \log \frac{4}{1-|z|} \, dA(z) < \infty.
\]

Our next result is
Theorem 2.2. (a) If \( \alpha > -1 \), then \( C \) maps the space \( B^{\log \alpha} \), resp. \( b^{\log \alpha} \), into \( B^{\log \alpha+1} \), resp. \( b^{\log \alpha+1} \).

(b) \( C \) maps the space \( B^{\log -1} \), resp. \( b^{\log -1} \), into \( B^{\log} \), resp. \( b^{\log} \).

Proof. See Section 5. \( \square \)

Remark 2.2. If \( f \in B^{\log \alpha} \) and \( \alpha < -1 \), then, as it can easily be shown, \( f \in A(D) \), where \( A(D) \) is the disk-algebra, i.e., the subset of \( H^\infty \) consisting of those \( f \) which have a continuous extension to the closed disk. Moreover, the modulus of continuity of the boundary function \( f(\zeta), \zeta \in \partial D \), satisfies the condition

\[
\omega(f, t) = O(t \left( \log 2 t \right)^{\alpha+1}), \quad t \downarrow 0.
\]

This follows from the inequality

\[
\omega(f, t) \leq C \int_{1-t}^1 M_\infty(r, f') dr,
\]

see \([15, \text{Theorem 2.2}]\). It should be noted that the modulus of continuity of \( f_* \) is “proportional” to that of \( f(z), z \in D \), see \([23, 19]\).

Concerning the Libera operator we shall prove, besides Theorem 2.1(c), the following facts.

Theorem 2.3. (a) If \( \alpha > 0 \), then \( L \) is well defined on \( B^{1, \log \alpha} \) and maps this space to \( B^{1, \log \alpha-1} \).

(b) \( L \) is well defined on \( B^{1, \log} \) and maps this space into \( B^{1, \log} \).

(c) \( L \) is well defined on \( B^{1} \) and maps it into \( B^{1, \alpha} \) for all \( \alpha < -1 \).

Proof. See Section 5. \( \square \)

Theorem 2.4. Let \( \alpha \in \mathbb{R} \). Then the dual of \( b^{\log \alpha} \), resp. \( B^{1, \log \alpha} \), is isomorphic to \( B^{1, \log \alpha-1} \), resp. \( B^{1, \log \alpha} \), under the pairing \((1.2)\). Similarly, the dual of \( b^{\log} \), resp. \( B^{1, \log} \), is isomorphic to \( B^{1, \log} \), resp. \( B^{1, \log} \), under the same pairing.

Proof. See Section 6. \( \square \)

Remark 2.3. The phrase “the dual of \( X \) is isomorphic to \( Y \) under the pairing \((1.2)\)” means that if \( f \in X \) and \( g \in Y \), then the limit in \((1.2)\) exists and the functional \( \Phi(f) = \langle f, g \rangle \) is bounded on \( X \); and on the other hand, if \( \Phi \in X^* \), then there exists \( g \in Y \) such that \( \Phi(f) = \langle f, g \rangle \), and moreover, there exists a constant \( C \) independent of \( g \) such that \( \|g\|_Y/C \leq \|\Phi\| \leq C\|g\|_Y \).

As an application of Theorems 2.2, 2.3, and 2.4, one can prove the following fact.

Theorem 2.5. Let \( \alpha > 0 \). Then the adjoint (with respect to \((1.2)\)) of the operator \( L : B^{1, \log \alpha} \rightarrow B^{1, \log \alpha-1} \) is equal to \( C : B^{1, \log \alpha-1} \rightarrow B^{1, \log \alpha} \). The adjoint of the operator \( C : b^{\log \alpha-1} \rightarrow b^{\log \alpha} \) is equal to \( L : b^{\log \alpha} \rightarrow b^{1, \log \alpha-1} \). The analogous assertions hold in the case when \( \alpha = 0 \).
3. Decompositions

In [6], a sequence \( \{V_n\}_{n=0}^{\infty} \) was constructed in the following way.
Let \( \omega \) be a \( C^\infty \)-function on \( \mathbb{R} \) such that

1. \( \omega(t) = 1 \) for \( t \leq 1 \),
2. \( \omega(t) = 0 \) for \( t \geq 2 \),
3. \( \omega \) is decreasing and positive on the interval \((1, 2)\).

Let \( \varphi(t) = \omega(t/2) - \omega(t) \), and let \( V_0(z) = 1 + z \), and, for \( n \geq 1 \),

\[
V_n(z) = \sum_{k=0}^{\infty} \varphi(k/2^n-1)\zeta^k = \sum_{k=2^n-1}^{2^{n+1}-1} \varphi(k/2^n-1)\zeta^k.
\]

The polynomials \( V_n \) have the following properties:

\begin{align*}
(3.1) & \quad g(z) = \sum_{n=0}^{\infty} V_n \ast g(z), \text{ for } g \in H(D); \\
(3.2) & \quad \|V_n \ast g\|_p \leq C\|g\|_p, \text{ for } g \in H^p, \ p > 0; \\
(3.3) & \quad \|V_n\|_p \leq 2^{n(1-1/p)}, \text{ for all } p > 0,
\end{align*}

where \( \ast \) denotes the Hadamard product. Here \( \|h\|_p \) denotes the norm in the \( p \)-Hardy space \( H^p \),

\[
\|h\|_p = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})| \, d\theta \right)^{1/p} = \sup_{0 < r < 1} M_p(r, g).
\]

We need additional properties.

**Lemma 3.1.** Let \( P(z) = \sum_{k=m}^{j} a_k z^k \), \( m < j \). Then

\[
r^m \|P\|_p \leq M_p(r, P) \leq r^m \|P\|_p, \quad 0 < r < 1.
\]

When applied to the polynomial \( P = V_n \ast g' \), this gives:

\[
2^{n+1-1} \|V_n \ast g'\|_p \leq M_p(r, V_n \ast g') \leq 2^{n-1-1} \|V_n \ast g'\|_p \text{ for } n \geq 1.
\]

Another inequality will be used (see [16 Exercise 7.3.5]):

\[
2^{n-1} \|V_n \ast g'\|_p / C \leq \|V_n \ast g'\|_p \leq C2^{n+1} \|V_n \ast g'\|_p \text{ for } n \geq 1,
\]

where \( C \) is a constant independent of \( n \) and \( g \).

**Theorem 3.1.** Let \( \alpha \in \mathbb{R} \), and \( f \in H(D) \). Then:

(i) \( f \in B_{ \log^\alpha } \) if and only if \( \sup_{n \geq 0} (n+1)^{-\alpha} \|V_n \ast f\|_\infty < \infty \).
(ii) \( f \in B_{ \log^{\alpha-1} } \) if and only if \( \lim_{n \to \infty} (n+1)^{1-\alpha} \|V_n \ast f\|_\infty = 0 \).
(iii) \( f \in B_{ \log^{\alpha} } \) if and only if \( \sum_{n=0}^{\infty} (n+1)^{\alpha} \|V_n \ast f\|_1 < \infty \).

Moreover, the inequality

\[
C^{-1} \|f\|_{B_{ \log^\alpha } } \leq \sup_{n \geq 0} (n+1)^{-\alpha} \|V_n \ast g\|_\infty \leq C^{-1} \|f\|_{B_{ \log^\alpha } }
\]

holds, where \( C \) is independent of \( f \). The analogous inequality holds in the case of

(iii) as well.
For the proof we need the following reformulation of [9, Proposition 4.1].

**Lemma 3.2.** Let $\varphi$ be a continuous function on the interval $(0, 1]$ such that $\varphi(x)/x^\gamma$ $(0 < x < 1)$ is nonincreasing, and $\varphi(x)/x^\beta$ $(0 < x < 1)$ is nondecreasing, where $\beta$ and $\gamma$ are positive constants independent of $x$. Let

$$F_1(r) = (1 - r)^{-1/q} \varphi(1 - r) \sup_{n \geq 1} \lambda_n r^{2n+1-1},$$

$$F_2(r) = (1 - r)^{1/q} \varphi(1 - r) \sum_{n=0}^{\infty} \lambda_n r^{2n-1-1},$$

where $\lambda_n \geq 0$, $0 < q \leq \infty$. If $F = F_1$ or $F = F_2$, then

$$C^{-1} \|F\|_{L^q(0, 1)} \leq \|\{\varphi(2^{-n})\lambda_n\}\|_{c^0} \leq C \|F\|_{L^q(0, 1)}.$$

**Proof of Theorem 3.1.** Case (i). Let $\varphi(x) = x \log x$, and $q = \infty$. That $\varphi$ is normal follows from the relation

$$\lim_{x \downarrow 0} \frac{x \varphi'(x)}{\varphi(x)} = 1.$$

Let $\lambda_n = 2^n \|V_n \ast f\|_\infty$. By (3.1), (3.2), (3.4), and (3.5), we have

$$C^{-1} |\hat{f}(1)| + C^{-1} \sup_{n \geq 1} \lambda_n r^{2n+1-1} \leq M_\infty(r, f') \leq C|\hat{f}(1)| + C \sum_{n=1}^{\infty} \lambda_n r^{2n-1-1}.$$

Hence, by Lemma 3.2 we obtain the desired result.

Case (ii). In this case we can proceed in two ways:

1° Modify the proof of Lemma 3.2 to get the inequalities

$$C^{-1} \|F\|_{C^0[0, 1]} \leq \|\{\varphi(2^{-n})\lambda_n\}\|_{c^0} \leq C \|F\|_{C^0[0, 1]},$$

where $C^0[0, 1] = \{u \in C[0, 1] : u(1) = 0\}$ and $c^0$ is the set of the sequences tending to zero.

2° Consider the spaces $b_{\log} \subseteq B_{\log}$ and $X = \{f : \|V_n \ast f\| = o((n + 1)^\alpha)\}$, which is, by assertion (i) and its proof, a subspace of a space $Y$ isomorphic to $B_{\log}$. It is not hard to show that the polynomials are dense in both $b_{\log}$ and $X$. This proves (ii).

Case (iii). In this case we use the function $\varphi(x) = x \log x$, let $q = 1$, and then proceed as in the proof of (i). The details are omitted. This concludes the proof of the theorem.

**Remark 3.1.** By choosing $\phi(x) = x \log \log \frac{1}{x}$, then we can conclude that Theorem 3.1 remains true if $\log^\alpha$, resp. $(n + 1)^\alpha$, are replaced with $\log \log$, resp. $\log(n + 2)$.

---

1 Following Shields and Williams [20], we call such a function normal.
4. Functions with decreasing coefficients

Proof of Theorem 1.1 Assuming that (1.3) holds, we want to prove that

\[ \|f\|_{B^1_{\log}} \leq C a_0 + C \sum_{n=1}^{\infty} a_{2^n-1} (n+1)^\alpha. \]

According to Theorem 3.1 and its proof, we have

\[ C^{-1} \cdot \|f\|_{B^1_{\log}} \leq a_0 + \sum_{n=1}^{\infty} (n+1)^\alpha \|V_n * f\|_1 \leq C \|f\|_{B^1_{\log}}. \]

Let \( n \geq 1, m = 2^{n-1}, \) and \( Q_k = \sum_{j=m}^{\infty} \varphi(j/m) e_j. \) Since \( Q_{4m-1} = V_n, \) we have

\[ V_n * f = \sum_{k=m}^{4m-1} \varphi(k/m) a_k e_k = \sum_{k=m}^{4m-1} (a_k - a_{k+1}) Q_k + a_{4m} Q_{4m-1} = \sum_{k=m}^{4m-1} (a_k - a_{k+1}) Q_k + a_{4m} V_n. \]

On the other hand, \( Q_k = V_n * \Delta_{n,k}, \) where

\[ \Delta_{n,k} = \sum_{j=2^{n-1}}^{k} z^j, \quad 2^{n-1} \leq k \leq 2^{n+1}. \]

By (3.2), with \( g = \Delta_{n,k}, \) we have

\[ \|Q_k\|_1 \leq C \|\Delta_{n,k}\|_1 \leq C \log(k + 1 - 2^{n-1}) \leq C(n + 1). \]

Combining these inequalities we get

\[ \|V_n * f\|_1 (n+1)\alpha \leq C \sum_{k=m}^{4m-1} (a_k - a_{k+1}) (n+1)^{\alpha+1} + C a_{4m} \|V_n\|_1 (n+1)\alpha \]

\[ \leq C(n+1)^{\alpha+1} (a_m - a_{4m}) + C a_{4m} (n+1)^\alpha \]

\[ = C(n+1)^{\alpha+1} (a_{2n-1} - a_{2n+1}) + C(n+1)^\alpha a_{2n+1}. \]

Here we have used the relation \( \|V_n\|_1 \leq C \) (see (5.3)!) Thus

\[ (n+1)^\alpha \|V_n * f\|_1 \leq C(n+1)^{\alpha+1} (a_{2n-1} - a_{2n}) \]

\[ + C(n+1)^{\alpha+1} (a_{2n} - a_{2n+1}) + C(n+1)^\alpha a_{2n+1}, \]

and therefore it remains to estimate the sums

\[ S_1 = \sum_{n=1}^{\infty} (n+1)^{\alpha+1} (a_{2n-1} - a_{2n}) \quad \text{and} \quad S_2 = \sum_{n=1}^{\infty} (n+1)^{\alpha+1} (a_{2n} - a_{2n+1}). \]

If \( \alpha > -1, \) then

\[ (n+1)^{\alpha+1} \leq C \sum_{k=1}^{n} (k+1)^\alpha, \]
and hence
\[ S_1 \leq C \sum_{n=1}^{\infty} (a_{2n-1} - a_{2n}) \sum_{k=1}^{n} (k+1)^{\alpha} = C \sum_{k=1}^{\infty} (k+1)^{\alpha} \sum_{n=k}^{\infty} (a_{2n-1} - a_{2n}) \]
\[ = C \sum_{k=1}^{\infty} (k+1)^{\alpha} a_{2k-1}. \]

In the case of \( S_2 \) we get
\[ S_2 \leq C \sum_{k=1}^{\infty} (k+1)^{\alpha} a_{2k}, \]
which completes the proof of “if” part of the theorem in the case \( \alpha > -1 \). If \( \alpha = -1 \), then
\[ \| V_n \ast f \|_1 \sum_{k=1}^{\infty} (k+1)^{\alpha} a_{2k} \]
\[ \leq C \sum_{k=1}^{\infty} (k+1)^{\alpha} a_{2k+1}, \]
from which we get the desired result in the case \( \alpha = -1 \).

To prove “only if” part we use Hardy’s inequality in the form
\[ \pi M_1(r, g) \geq \sum_{n=0}^{\infty} |\hat{g}(n)|^{\frac{1}{\alpha}} n^{\alpha}. \]

It follows that
\[ \int_{D} |f'(z)| \log^{\alpha} \frac{2}{1-|z|} dA(z) = 2 \int_{0}^{1} M_1(r, f') \log^{\alpha} \frac{2}{1-r} dr \]
\[ \geq 2 \sum_{n=1}^{\infty} a_{n} \frac{1}{n+1} \int_{0}^{1} \log^{\alpha} \frac{2}{1-r} r^{n} dr \]
Now the desired result follows from the inequality
\[ \int_{0}^{1} \varphi(1-r) r^{n} dr \geq c \varphi \left( \frac{1}{n+1} \right) \] (\( c = \text{const.} > 0 \)),
valid for any normal function \( \varphi \) (see [9, Lemma 4.1]).

Before proving Theorem 1.2 some remarks are in order. Let \( f(z) = \sum_{n=0}^{\infty} a_{n} z^{n} \), \( a_{n} \geq 0 \). In order that \( \mathcal{L}f \) be well defined by (1.8) it is necessary and sufficient that
\[ \sum_{n=0}^{\infty} \frac{a_{n}}{n+1} < \infty. \]

We already have mentioned in Introduction that this condition implies the existence of the integral in (1.8). In fact, this integral converges uniformly on compact subsets of \( \mathbb{D} \), which means that the limit
\[ \lim_{x \uparrow 1} \int_{0}^{x} f(t + (1-t)z) dt \]
exists and is uniform in \( |z| < \rho \), for every \( \rho < 1 \). This guarantees that \( \mathcal{L}f \) is analytic. On the other hand, if the integral in (1.8) exists, then we take \( z = 0 \) to conclude that (4.1) holds.
Proof of Theorem 1.2. The Taylor coefficients of $Lf$ are

$$b_n = \sum_{k=n}^{\infty} \frac{a_n}{n+1}.$$  

The sequence $\{b_n\}$ is nonincreasing so we can apply Theorem 1.1 to conclude that $Lf \in \mathfrak{B}_{\log^\alpha}$ if and only if

$$\sum_{n=0}^{\infty} \frac{\log^\alpha(n+2)}{n+1} \sum_{k=n}^{\infty} \frac{a_k}{k+1} = \sum_{k=0}^{\infty} \frac{a_k}{k+1} \sum_{n=0}^{k} \frac{\log^\alpha(n+2)}{n+1} < \infty.$$  

Now the desired result follows from the estimate

$$C^{-1} \log^{\alpha+1}(k+2) \leq \sum_{n=0}^{k} \frac{\log^\alpha(n+2)}{n+1} \leq C \log^{\alpha+1}(k+2),$$  

which holds because $\alpha > -1$. \hfill \Box

Remark 4.1. The above proof shows that $Lf$ belongs to $\mathfrak{B}_{\log^{-1}}$ if and only if

$$\sum_{n=0}^{\infty} \frac{a_n}{\log \log(n+4)} < \infty.$$  

Now we pass to the proof of Theorem 2.1.

Proof of Theorem 2.1 (a). Let $g \in \mathfrak{B}_{\log^\alpha}$. Then

$$M_2(r,g') \leq C(1-r)^{-1} \log \frac{2}{1-r}.$$  

Proof of Theorem 2.1 (c). Since $\mathfrak{B}_{\log^\alpha} \subset \mathfrak{B}_{\log^\beta}$ for $\beta < \alpha$, we may assume that $-1 < \alpha < 0$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \log^{-\varepsilon - \alpha}(n+2),$$  

where $\varepsilon > 1$. Condition (4.1) holds because $\varepsilon > 1$. For every $r \in (0,1)$ the function $f_r(z) = f(rz)$ belongs to $H(D)$ and, by Theorem 1.1 and its proof, the set $\{f_r : 0 < r < 1\}$ is bounded in $\mathfrak{B}_{\log^\alpha}$. On the other hand,

$$\overline{L}(f_r)(0) = \sum_{k=0}^{\infty} \frac{r^k}{(k+1) \log(k+2)}.$$  

Now choose $\varepsilon = 1 - \alpha > 1$ (because $\alpha < 0$) to get

$$\overline{L}(f_r)(0) = \sum_{k=0}^{\infty} \frac{r^k}{(k+1) \log(k+2)} \rightarrow \infty \quad (r \uparrow 1).$$  

This contradicts the fact that if a set $X \subset \mathfrak{B}_{\log^\alpha}$ is bounded and $\overline{L}$ is bounded on $\mathfrak{B}_{\log^\alpha}$, then the set $\{\overline{L}(f) : f \in X\}$ is bounded because the functional $h \mapsto h(0)$ is continuous on $H(D)$. This completes the proof. \hfill \Box

Proof of Theorem 2.1 (a). Let $g \in \mathfrak{B}_{\log^\alpha}$. Then

$$M_2(r,g') \leq C(1-r)^{-1} \log \frac{2}{1-r}.$$
It follows that
\[ 2^n \left( \sum_{k=2^n}^{2^n+1-1} |\hat{g}(k)|^2 \right)^{1/2} r^{2^n+1} \leq C(1-r)^{-1} \log \alpha \frac{2}{1-r}. \]
Taking \( r = 1 - 2^{-n}, n \geq 1 \), we get
\[ \left( \sum_{k=2^n}^{2^n+1-1} |\hat{g}(k)|^2 \right)^{1/2} \leq C \log \alpha (n + 1). \]
Hence,
\[ 2^{-n} \sum_{k=2^n}^{2^n+1-1} |\hat{g}(k)| \leq \left( 2^{-n} \sum_{k=2^n}^{2^n+1-1} |\hat{g}(k)|^2 \right)^{1/2} \leq 2^{-n/2} \log \alpha (n + 1). \]
This gives the result. \( \square \)

**Proof of Theorem 2.1 (b).** In this case we use Hardy's inequality as in the proof of Theorem 1.1 to get
\[ \|g\|_{B^1_{\log \alpha}} \geq c \sum_{n=0}^{\infty} \frac{\hat{g}(n) \log \alpha (n + 2)}{n + 1}. \]
This proves the result because \( \alpha > 0 \). \( \square \)

5. **Proofs of Theorem 2.2 and 2.3**

Define the operator \( \mathcal{R} : H(\mathbb{D}) \mapsto H(\mathbb{D}) \) by
\[ \mathcal{R}f(z) = \sum_{n=0}^{\infty} (n+1)f(n)z^n = \frac{d}{dz} (zf(z)). \]
By using Theorem 3.1 and the relation
\[ (5.1) \quad C^{-1} 2^n \|V_n * f\|_p \leq \|V_n * \mathcal{R}f\|_p \leq C 2^n \|V_n * f\|_p \quad (n \geq 0), \]
on one proves that the norm in \( \mathfrak{B}_{\log \alpha} \) is equivalent to
\[ \sup_{z \in \mathbb{D}} (1 - |z|) \log^{-\alpha} \frac{2}{1 - |z|} |\mathcal{R}f(z)|. \]

**Proof of Theorem 2.2 (a).** Let \( \alpha > -1 \) and \( f \in \mathfrak{B}_{\log \alpha} \). Then, by integration,
\[ |f(z)| \leq \log^{\alpha+1} \frac{1}{1 - |z|}. \]
Since
\[ \mathcal{R}f(z) = \frac{f(z)}{1 - z}, \]
we see that
\[ |\mathcal{R}f(z)| \leq C (1 - |z|)^{-1} \log^{\alpha+1} \frac{1}{1 - |z|}. \]
The result follows. \( \square \)
Proof of Theorem 2.2 (b). The function \( \varphi(x) = x \log \log(4/x) \) is normal because \( \lim_{x \to 0} x \varphi'(x)/\varphi(x) = 1 \). Hence, arguing as in the proof of Theorem 3.1 we conclude that \( f \in B_{\log \log} \) if and only if \( \sup_{n \geq 0} \| V_n \ast f \|_{\infty} / \log(n + 2) < \infty \). Then using (5.1) we find that \( g \in B_{\log \log} \) if and only if

\[
|Rg(z)| \leq C(1 - |z|)^{-1} \log \log \frac{4}{1 - |z|}.
\]

The rest of the proof is the same as in the case of (a).

Remark 5.1. In the case of the small spaces the proofs are similar and therefore omitted.

For the proof of Theorem 2.3 we need the following lemma [18]:

Lemma 5.1. If \( f \in \ell^1 \), then \( Lf \) is well defined by (1.8) and the inequality

\[
(5.2) \quad rM_1(r, (Lf)'') \leq 2(1 - r)^{-1} \int_r^1 M_1(s, f') \, ds, \quad 0 < r < 1,
\]

holds.

Before passing to the proof observe that \( \mathcal{B}_{1, \log}^{a} \subset \mathcal{B}^{1} \) and \( \mathcal{B}^{1}_{\log} \subset \mathcal{B}^{1} \), and, since \( \mathcal{B}^{1} \subset H^{1} \), we see that in all cases of Theorem 2.3 the operator \( L \) is well defined.

Proof of Theorem 2.3 (a). We have, by (5.2),

\[
\int_{D} |(Lf)'(z)| \log^{\alpha - 1} \frac{2}{1 - |z|} \, dA(z) = 2 \int_0^1 M_1(r, (Lf)') \log^{\alpha - 1} \frac{2}{1 - r} \, dr
\]

\[
\leq 4 \int_0^1 (1 - r)^{-1} \log^{\alpha - 1} \frac{2}{1 - r} \, dr \int_r^1 M_1(s, f') \, ds
\]

\[
= 4 \int_0^1 M_1(s, f') \, ds \int_0^s (1 - r)^{-1} \log^{\alpha - 1} \frac{2}{1 - r} \, dr
\]

\[
\leq C \int_0^1 M_1(s, f') \log^{\alpha} \frac{2}{1 - s} \, ds.
\]

A standard application of the maximum modulus principle shows that the inequality remains valid if we replace \( ds \) with \( s \, ds \). This gives the result.

The proofs of Theorem 2.3 (b) and (c), are similar and we omit them.

6. Proof of Theorem 2.4

We consider a more general situation. Let \( X \subset H(D) \) (with continuous inclusion) be a Banach space such that the functions \( f_w(z) = f(wz), |w| \leq 1, \) belong to \( X \) whenever \( f \in X \), and \( \sup_{|w| \leq 1} \| f_w \|_X \leq \| f \|_X \). Such a space is said to be homogeneous (see [2]). A homogeneous space satisfies the condition

\[
(6.1) \quad \| V_n \ast f \|_X \leq C \| f \|_X, \quad f \in X,
\]

where \( C \) is independent of \( n \) and \( f \).
If in addition
\[
\lim_{r \to 1} \| f - f_r \| X = 0, \quad f \in X,
\]
then the dual of $X$ can be identified with the space, $X'$, of those $g \in H(\mathbb{D})$ for which limit (6.2) exists for all $f \in X$ (see [11, 2]). Also, the dual of a homogeneous space $X$ satisfying (6.2) can be realized as the space of coefficient multipliers, $(X, A(\mathbb{D}))$, from $X$ to $A(\mathbb{D})$; in this case we have $(X, A(\mathbb{D})) = (X, H^\infty) =: X^*$ (see [2]). The norm in $X^*$ is introduced as
\[
\| g \|_{X^*} = \sup \| f \ast g \|_X : f \in X, \quad \| f \|_X \leq 1,
\]
and, if $X$ is homogeneous and satisfies (6.2), it is equal to
\[
\| g \|_{X^*} = \sup \{ \| f \ast g \|_X : f \in X, \quad \| f \|_X \leq 1 \}.
\]
There is another way to express $(f, r)$, when $f \in X$, $X$ satisfies (6.2), and $g \in X'$; namely, in this case, the function $f \ast g$ belongs to $A(\mathbb{D})$, and we have $(f, g) = (f \ast g)(1)$ (see [13, 2]).

We fix a sequence $\lambda = \{ \lambda_n \}_{n=0}^\infty$ of positive real numbers such that
\[
0 < \inf_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_n}, \quad \sup_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_n} < \infty.
\]

It is clear that the spaces $H^p (0 < p \leq \infty)$, $A(\mathbb{D})$, $\mathfrak{B}_{\log^a}$, $\mathfrak{B}_{\log^\infty}$, and $\mathfrak{B}_{\log^a}$ are homogeneous. Among them only $H^\infty$ and $\mathfrak{B}_{\log^a}$ do not satisfy condition (6.2).

Consider the following three spaces of sequences $\{ f_n \}_{n=0}^\infty$, $f_n \in H(\mathbb{D})$:

(a) $v_0(\lambda, X) = \{ f_n : \lim_{n \to \infty} \| f_n \| X / \lambda_n = 0 \}$;

(b) $\ell^\infty(\lambda, X) = \{ f_n : \sup_{n \geq 0} \lambda_n \| f_n \| X < \infty \}$;

(c) $\ell^1(\lambda, X) = \{ f_n : \sum_{n=0}^\infty \| f_n \| X / \lambda_n < \infty \}$.

We also define the spaces $v_0(\lambda, X)$, $V^\infty(\lambda, X)$, and $V^1(\lambda, X)$ (as subsets of $H(\mathbb{D})$) by replacing $f_n$ with $V_n \ast f$ in (a), (b), and (c), respectively. The proof of the following lemma is rather easy, and is therefore left to the reader.

**Lemma 6.1.** If $X$ is a homogeneous space, then so are $v(\lambda, X)$, $V^\infty(\lambda, X)$, and $V^1(\lambda, X)$. The spaces $v_0(\lambda, X)$ and $V^1(\lambda, X)$ satisfy (6.2). The space $v_0(\lambda, X)$ is equal to the closure in $V^\infty(\lambda, X)$ of the sets of all polynomials.

Theorem 2.4 will be deduced from Theorem 3.1 and the following.

**Proposition 6.1.** If $X$ is a homogeneous space satisfying (6.2), then the dual of $v_0(\lambda, X)$, resp. $V^1(\lambda, X)$, is isomorphic to $V^1(\lambda, X')$, resp. $V^\infty(\lambda, X')$, with respect to (1.2).

In proving we use ideas from [13, 14, 7]. For the proof we need the following lemma.

**Lemma 6.2.** The operator $T(\{ f_n \}) = \sum_{n=0}^\infty V_n \ast f_n$ acts as a bounded operator from $Y$ to $Z$, where $Y$ is one of the spaces $v_0(\lambda, X)$, $\ell^\infty(\lambda, X)$, and $\ell^1(\lambda, X)$, while $Z$ is $v_0(\lambda, X)$, $V^\infty(\lambda, X)$, and $V^1(\lambda, X)$, respectively.
Then where, by definition, \( w_j = f_j = 0 \) for \( j < 0 \). It follows that
\[
\|V_n \ast T(\{f_j\})\|_X \leq C \sum_{j=n}^{n+1} \|f_j\|_X,
\]
where we have used (6.1). Now the proof is easily completed by using (6.3). □

**Lemma 6.3.** Let \( g \in (v_0(\lambda, X))' \), resp. \( g \in (V^1(\lambda, X))' \), and define the operator \( S \) on \( \mathcal{C}_0(\lambda, X) \), resp. \( \mathcal{L}^1(\lambda, X) \), by
\[
S(\{f_n\}) = T(\{f_n\}) \ast g = \sum_{k=0}^{\infty} f_k \ast V_k \ast g.
\]
Then \( S \) maps \( \mathcal{C}_0(\lambda, X) \), resp. \( \mathcal{L}^1(\lambda, X) \), into \( H^\infty \) and \( \|S\| \leq C\|g\|_{(v_0(\lambda, X))'} \), resp. \( \|S\| \leq C\|g\|_{(V^1(\lambda, X))'} \).

**Proof.** By the preceding lemma, we have
\[
\|S(\{f_n\})\|_\infty \leq \|T(\{f_n\})\|_{(v_0(\lambda, X))'} \|g\|_{(v_0(\lambda, X))'} \leq C\|\{f_n\}\|_{\mathcal{C}_0(\lambda, X)} \|g\|_{(v_0(\lambda, X))'}.\]
This proves the result in one case. In the other case the proof is the same. □

**Proof of Proposition 6.1.** Define the polynomials \( P_n \ (n \geq 0) \) by
\[
P_n = V_{n-1} + V_n + V_{n+1}.
\]
Hence
\[
V_n = \sum_{j=0}^{\infty} V_j \ast V_n = (V_{n-1} + V_n + V_{n+1}) \ast V_n = P_n \ast V_n.
\]
Let \( f \in v_0(\lambda, X) \) and \( g \in V^1(\lambda, X)' \). It is easily verified that, when \( 0 < r < 1 \),
\[
(f \ast g)(z) = \sum_{n=0}^{\infty} (f \ast V_n \ast g)(z) = \sum_{n=0}^{\infty} (P_n \ast f \ast V_n \ast g)(z), \quad z \in \mathbb{D},
\]
the series being absolutely convergent. Since
\[
\|P_n \ast f \ast V_n \ast g\|_\infty \leq \|P_n \ast f\|_X \|V_n \ast g\|_{X'},
\]
we have
\[
\|f \ast g\|_\infty \leq \sum_{n=0}^{\infty} \|P_n \ast f\|_X \|V_n \ast g\|_{X'} \leq C \sum_{n=0}^{\infty} \|P_n \ast f\|_X \|V_n \ast g\|_{X'}
\]
\[
= C \sum_{n=0}^{\infty} (\|P_n \ast f\|_X/\lambda_n) (\lambda_n\|V_n \ast g\|_{X'}) \leq C\|f\|_{v_0(\lambda, X)} \|g\|_{V^1(\lambda, X')},
\]
This proves the inclusion \( V^1(\lambda, X') \subset (v_0(\lambda, X))' \).
To prove the converse, let \( g \in (v_0(\lambda,X))^* \). Let \( S \) denote the operator defined in Lemma 6.3. By Lemma 6.3, \( S \) acts as a bounded operator from \( v_0(\lambda,X) \) into \( H^\infty \) and we have \( \|S\| \leq C \|g\|v_0(\lambda,X)^* \). Now it suffices to prove that
\[
\|S\| \geq (1/2)\|\{g_n\}\|_{\ell^1(\lambda,X)} = (1/2)\|g\|_{V^1(\lambda,X)}.
\]
For each \( n \geq 0 \) choose \( f_n \in X \) so that \( \|f_n\|_X = 1 \) and \( (f_n,g_n) \) is a real number such that \( (f_n,g_n) \geq (1/2)\|g_n\|_{X^*} \). If \( \{a_n\} \) is a finite sequence of non-negative real numbers, then
\[
S(\{a_nf_n\}) = \sum_{n=0}^{\infty} a_n(f_n,g_n) \geq (1/2)\sum_{n=0}^{\infty} a_n\|g_n\|_{X^*} = (1/2)\sum_{n=0}^{\infty} (a_n/\lambda_n)\lambda_n\|g_n\|_{X^*}.
\]
Hence, by taking the supremum over all \( \{a_n\} \) such that \( 0(\lambda_n) \), we get \( S(\lambda_nf_n) \geq (1/2)\sum_{n=0}^{\infty} \lambda_n\|g_n\|_{X^*} \). Since \( \|\{a_nf_n\}\|_{\ell^1(\lambda,X)} \leq 1 \), where \( a_n = \lambda_n \) for \( 0 \leq n \leq N \) (\( N \in \mathbb{N} \)) and \( a_n = 0 \) for \( n > N \) we see that \( \|S\| \geq (1/2)\|g\|_{V^1(\lambda,X^*)} \), as desired. This completes the proof that \( v_0(\lambda,X') = V^1(\lambda,X') \). In a similar way one proves that \( V^1(\lambda,X') = V^\infty(\lambda,X') \), which is all what has to be proved.

**Proof of Theorem 2.4.** First we prove that \((b_{\log^a})' = B_1^{\log^a}\). By Theorem 3.1, we have \( b_{\log^a} = v_0(\lambda,A(D)) \), where \( \lambda_n = (n+1)^a \). Hence, by Proposition 6.1, the dual of \( b_{\log^a} \) is isomorphic to \( V^1(\lambda,A(D))' \). In order to estimate \( \|V_n \ast g\|_{A(\lambda)}' \) first observe that \( H^1 \subset A(\lambda) \) and moreover \( \|V_n \ast g\|_{A(\lambda)} \leq \|V_n \ast g\|_1 \). On the other hand, let \( \Phi \) be a bounded linear functional on \( A(D) \), let \( \Phi_0 \) be the Hahn/Banach extension of \( \Phi \) to \( hC(D) \), and choose \( g \in A(D)^a \) so that \( \Phi(f) = (f,g) \) for all \( f \in A(D) \). By the Riesz representation theorem, we have
\[
\Phi_0(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{-i\theta}) \, d\mu(e^{i\theta})
\]
\[
= \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{-i\theta})g(re^{i\theta}) \, d\theta = \lim_{r \to 1^-} \sum_{n=0}^{\infty} f(n)g(n)r^{2n},
\]
and \( \|\mu\| = \|\Phi\| = \|\Phi_0\| \). In particular, taking \( f(w) = (1 - zw)^{-1} \), where \( z \in D \) is fixed, we get \( \int_0^{2\pi} f(1 - e^{i\theta}z)^{-1} \mu(e^{i\theta}) = g(z) \). Hence
\[
R^1g(z) = \frac{1}{2\pi} \int_0^{2\pi} (1 - e^{-i\theta}z)^{-2} \mu(e^{i\theta}),
\]
and hence, by integration, \( M_1(r,R^1f) \leq \|\mu\|(1 - r^2)^{-1} = \|g\|_{A(\lambda)}(1 - r^2)^{-1} \). Now we proceed as in the proof of Theorem 3.1 to conclude that \( \|V_n \ast g\|_1 \leq C\|V_n \ast g\|_{A(\lambda)} \). It follows that \( g \in (b_{\log^a})' \) if and only if \( g \in V^1(\lambda,H^1) \), i.e., by Theorem 3.1, \( g \in B_1^{\log^a} \).

In proving that \((B_1^{\log^a})' \) is isomorphic to \( B_1^{\log^a} \), we use the inclusions \( H^\infty \subset (H^1)' \subset B_1^{\log^a} \), and then proceed as above.

**Remark 6.1.** The above proof of Theorem 2.4 certainly is not the simplest one. However, it can be applied to prove some general duality and multipliers theorems (see 13, 14, 7). For instance, the dual of \( b_{\log^a} \) is isomorphic to \( B_1^{\log^a} \), and the dual of \( B_1^{\log^a} \) is isomorphic to \( B_1^{\log^a} \).
References

1. A. Aleman, M. Carlsson, A.-M. Persson, Preduals of $Q_p$-spaces, Complex Var. Elliptic Equ. 52(7) (2007), 605–628.
2. O. Blasco, M. Pavlović, Coefficient multipliers on Banach spaces of analytic functions, Rev. Mat. Iberoam. 27(2) (2011), 415–447.
3. L. Brown, A. L. Shields, Multipliers and cyclic vectors in the Bloch space, Michigan Math. J. 38 (1991), 141–146.
4. N. Danikas, A. G. Siskakis, The Cesàro operator on bounded analytic functions, Analysis 13(3) (1993), 295–299.
5. N. Danikas, S. Ruscheweyh, A. Siskakis, Metrical and topological properties of a generalized Libera transform, Arch. Math. 63(6) (1994), 517–524.
6. M. Jevtić, M. Pavlović, On multipliers from $H^p$ to $ℓ^q$ $(0 < q < p < 1)$ Arch. Math. 56 (1991), 174–180.
7. ______, Coefficient multipliers on spaces of analytic functions, Acta Sci. Math. 64 (1998), 531–545.
8. D. H. Luecking, L. A. Rubel, Complex Analysis, A Functional Analysis Approach, Springer-Verlag, New York, 1984.
9. M. Mateljević, M. Pavlović, $L^p$ behaviour of the integral means of analytic functions, Studia Math. 77 (1984), 219–237.
10. J. Xiao, The Cesàro operator is bounded on $H^p$ for $0 < p < 1$, Proc. Am. Math. Soc. 116 (1992), 1077–1079.
11. M. Nowak, Another proof of boundedness of the Cesàro operator on $H^p$, Ann. Univ. Mariae Curie-Skłodowska, Sect. A 54 (2000), 75–78.
12. M. Nowak, M. Pavlović, On the Libera operator, J. Math. Anal. Appl. 370(2) (2010), 588–599.
13. M. Pavlović, Mixed norm spaces of analytic and harmonic functions. I, Publ. Inst. Math., Nouv. Sér. 40(54) (1986), 117–141.
14. ______, Mixed norm spaces of analytic and harmonic functions. II, Publ. Inst. Math., Nouv. Sér. 41(55) (1987), 97–110.
15. ______, On the moduli of continuity of $H^p$ functions with $0 < p < 1$, Proc. Edinburgh Math. Soc. 35 (1992), 89–100.
16. ______, Introduction to Function Spaces on the Disk, Posebna Izd., Mat. Inst., Beograd 20, 2004.
17. ______, Analytic functions with decreasing coefficients and Hardy spaces, Proc. Edinburgh Math. Soc. 56 (2013), 623–625.
18. ______, Definition and properties of the Libera operator on mixed norm spaces, Sci. World J. (2014), Article ID 590656, 15 pages.
19. L. A. Rubel, A. L. Shields, B. A. Taylor, Mergelyan sets and the modulus of continuity of analytic functions, J. Approximation Theory 15(1) (1975), 23–40.
20. A. L. Shields, D. L. Williams, Bounded projections, duality, and multipliers in spaces of analytic functions, Trans. Am. Math. Soc. 162 (1971), 287–302.
21. A. G. Siskakis, Composition semigroups and the Cesàro operator on $H^p$, J. London Math. 36(2) (1987), 153–164.
22. ______, The Cesàro operator is bounded on $H^1$, Proc. Am. Math. Soc. 110 (1990), 461–462.
23. P. M. Tamrazov, Contour and solid structural properties of holomorphic functions of a complex variable, (in Russian), Uspehi Mat. Nauk 28 (1973), 131–161. English translation in Russian Math. Surveys 28 (1973), 141–173.
24. Jie Xiao, Cesàro-type operators on Hardy, BMOA and Bloch spaces, Arch. Math. 68(5) (1997), 398–406.

Faculty of Mathematics, University of Belgrade (Received 24 12 2015)
Belgrade, Serbia
pavlovic@matf.bg.ac.rs