Classification of Special Anosov Endomorphisms of Nil-manifolds

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Abstract In this paper we give a classification of special endomorphisms of nil-manifolds: Let $f : N/\Gamma \to N/\Gamma$ be a covering map of a nil-manifold and denote by $A : N/\Gamma \to N/\Gamma$ the nil-endomorphism which is homotopic to $f$. If $f$ is a special $TA$-map, then $A$ is a hyperbolic nil-endomorphism and $f$ is topologically conjugate to $A$.

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1 Introduction

Finding a universal model for Anosov diffeomorphisms has been an important problem in dynamical systems. In this general context, Franks and Manning proved that every Anosov diffeomorphism of an infra-nil-manifold is topologically conjugate to a hyperbolic infra-nil-automorphism [7, 8, 12, 13] (According to Dekimpe’s work [5], some of their results are incorrect). Based on this result, Aoki and Hiraide have studied the dynamics of covering maps of a torus [2]. The importance of infra-nil-manifolds comes from the following Conjecture 1.1 and Theorem 1.2:

The first non-toral example of an Anosov diffeomorphism was constructed by Smale in [16]. He conjectured that, up to topological conjugacy, the construction in Smale’s example gives every possible Anosov diffeomorphism on a closed manifold.

Conjecture 1.1 Every Anosov diffeomorphism of a closed manifold is topologically conjugate to a hyperbolic affine infra-nil-automorphism.

Theorem 1.2 (Gromov [9]) Every expanding map on a closed manifold is topologically conjugate to an expanding affine infra-nil-endomorphism.

The conjecture has been open for many years (see [6, p. 48]). An interesting problem is to consider the conjecture for endomorphisms of a closed manifold. Our main theorem is a partial answer to the conjecture.

In this paper we give a classification of special endomorphisms of nil-manifolds (for definitions see the next section). In fact, Aoki and Hiraide [2] in 1994 proposed two problems:

1) Corresponding author
Problem 1.3  Is every special Anosov differentiable map of a torus topologically conjugate to a hyperbolic toral endomorphism?

Problem 1.4  Is every special topological Anosov covering map of an arbitrary closed topological manifold topologically conjugate to a hyperbolic infra-nil-endomorphism of an infra-nil-manifold?

Aoki and Hiraide answered Problem 1.3 partially as follows:

Theorem 1.5 ([2, Theorem 6.8.1]) Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a $\mathrm{TA}$-covering map of an $n$-torus and denote by $A : \mathbb{T}^n \to \mathbb{T}^n$ the toral endomorphism homotopic to $f$. Then $A$ is hyperbolic. Furthermore the inverse limit system of $(\mathbb{T}^n, f)$ is topologically conjugate to the inverse limit system of $(\mathbb{T}^n, A)$.

Theorem 1.6 ([2, Theorem 6.8.2]) Let $f$ and $A$ be as Theorem 1.5. Suppose $f$ is special, then the following statements hold:

1. if $f$ is a $\mathrm{TA}$-homeomorphism, then $A$ is a hyperbolic toral automorphism and $f$ is topologically conjugate to $A$,
2. if $f$ is a topological expanding map, then $A$ is an expanding toral endomorphism and $f$ is topologically conjugate to $A$,
3. if $f$ is a strongly special $\mathrm{TA}$-map, then $A$ is a hyperbolic toral endomorphism and $f$ is topologically conjugate to $A$.

In [17], Sumi has altered the condition “strongly special” (part (3) of Theorem 1.6) to just “special” as follows:

Theorem 1.7 ([17]) Let $f$ and $A$ be as Theorem 1.5. If $f$ is a special $\mathrm{TA}$-map, then $A$ is a hyperbolic toral endomorphism and $f$ is topologically conjugate to $A$.

In [18], Sumi supposed that:

“Every self-covering map of an infra-nil-manifold is homotopic to an infra-nil-endomorphism” (*)

and generalized (incorrectly) Theorem 1.5 and parts (1) and (2) of Theorem 1.6 for infra-nil-manifolds. According to [5], Dekimpe showed that assumption (*) is not true in general and there exist (interesting) diffeomorphisms and self-covering maps of an infra-nil-manifold which are not homotopic to an infra-nil-endomorphism. Sumi’s theorems are as follows:

Theorem 1.8 ([18, Theorem 1]) Let $f : N/\Gamma \to N/\Gamma$ be a covering map of an infra-nil-manifold and denote by $A : N/\Gamma \to N/\Gamma$ the infra-nil-endomorphism homotopic to $f$. If $f$ is a $\mathrm{TA}$-map, then $A$ is hyperbolic and the inverse limit system of $(N/\Gamma, f)$ is topologically conjugate to the inverse limit system of $(N/\Gamma, A)$.

Theorem 1.9 ([18, Theorem 2]) Let $f$ and $A$ be as in Theorem 1.8. Then the following statements hold:

1. if $f$ is a $\mathrm{TA}$-homeomorphism, then $A$ is a hyperbolic infra-nil-automorphism and $f$ is topologically conjugate to $A$,
2. if $f$ is a topological expanding map, then $A$ is an expanding infra-nil-endomorphism and $f$ is topologically conjugate to $A$. 
Dekimpe [5] in §4, gave an expanding map not topologically conjugate to an infra-nil-endomorphism and in §5, he gave an Anosov diffeomorphism not topologically conjugate to an infra-nil-automorphism.

According to [5], or Lemma 2.5, for every self-covering map \( f \) of a nil-manifold there exists a unique nil-endomorphism homotopic to \( f \). If we repair the assumption (*) by substituting nil-manifolds instead of infra-nil-manifolds, all the proofs of [18] are true for nil-manifolds. Thus we will use some of its results in this paper for nil-manifolds.

In the paper, by using Theorem 1.7, we partially answer Problem 1.4 of Aoki and Hiraide as follows:

**Theorem 1.10** (Main Theorem) Let \( f : N/\Gamma \to N/\Gamma \) be a covering map of a nil-manifold and denote by \( A : N/\Gamma \to N/\Gamma \) the nil-endomorphism homotopic to \( f \) (according to [5], such a unique homotopy exists for nil-manifolds). If \( f \) is a special TA-map, then \( A \) is a hyperbolic nil-endomorphism and \( f \) is topologically conjugate to \( A \).

**Corollary 1.11** If \( f : N/\Gamma \to N/\Gamma \) is a special Anosov endomorphism of a nil-manifold then it is conjugate to a hyperbolic nil-endomorphism.

2 Preliminaries

Let \( X \) and \( Y \) be compact metric spaces and let \( f : X \to X \) and \( g : Y \to Y \) be continuous surjection. Then \( f \) is said to be topologically conjugate to \( g \) if there exists a homeomorphism \( \varphi : Y \to X \) such that \( f \circ \varphi = \varphi \circ g \).

Let \( X \) be a compact metric space with metric \( d \). For \( f : X \to X \) a continuous surjection, we let

\[
X_f = \{ \tilde{x} = (x_i) : x_i \in X \text{ and } f(x_i) = x_{i+1}, i \in \mathbb{Z} \},
\]

\[
\sigma_f((x_i)) = (f(x_i)).
\]

The map \( \sigma_f : X_f \to X_f \) is called the shift map determined by \( f \). We call \((X_f, \sigma_f)\) the inverse limit of \((X, f)\). A homeomorphism \( f : X \to X \) is called expansive if there is a constant \( e > 0 \) (called an expansive constant) such that if \( x \) and \( y \) are any two distinct points of \( X \) then \( d(f^i(x), f^i(y)) > e \) for some integer \( i \). A continuous surjection \( f : X \to X \) is called \( c \)-expansive if there is a constant \( e > 0 \) such that for \( \tilde{x}, \tilde{y} \in X_f \) if \( d(x_i, y_i) \leq e \) for all \( i \in \mathbb{Z} \) then \( \tilde{x} = \tilde{y} \). In particular, if there is a constant \( e > 0 \) such that for \( x, y \in X \) if \( d(f^i(x), f^i(y)) \leq e \) for all \( i \in \mathbb{N} \) then \( x = y \), we say that \( f \) is positively expansive. A sequence of points \( \{x_i : a < i < b\} \) of \( X \) is called a \( \delta \)-pseudo orbit of \( f \) if \( d(f(x_i), x_{i+1}) < \delta \) for \( i \in (a, b-1) \). Given \( \epsilon > 0 \) a \( \delta \)-pseudo orbit of \( \{x_i\} \) is called to be \( \epsilon \)-traced by a point \( x \in X \) if \( d(f^i(x), x_i) < \epsilon \) for every \( i \in (a, b-1) \). Here the symbols \( a \) and \( b \) are taken as \( -\infty \leq a < b \leq \infty \) if \( f \) is bijective and as \( -1 \leq a < b \leq \infty \) if \( f \) is not bijective. \( f \) has the pseudo orbit tracing property (abbrev. POTP) if for every \( \epsilon > 0 \) there is \( \delta > 0 \) such that every \( \delta \)-pseudo orbit of \( f \) can be \( \epsilon \)-traced by some point of \( X \).

We say that a homeomorphism \( f : X \to X \) is a topological Anosov map (abbrev. TA-map) if \( f \) is expansive and has POTP. Analogously, we say that a continuous surjection \( f : X \to X \) is a topological Anosov map if \( f \) is \( c \)-expansive and has POTP, and say that \( f \) is a topological expanding map if \( f \) is positively expansive and open. We can check that every topological expanding map is a TA-map (see [2, Remark 2.3.10]).
Let $X$ and $Y$ be metric spaces. A continuous surjection $f : X \to Y$ is called a covering map if for $y \in Y$ there exists an open neighborhood $V_y$ of $y$ in $Y$ such that
\[
f^{-1}(V_y) = \bigcup_i U_i \quad (i \neq i' \Rightarrow U_i \cap U_i' = \emptyset)
\]
where each of $U_i$ is open in $X$ and $f_i(U_i) : U_i \to V_y$ is a homeomorphism. A covering map $f : X \to Y$ is especially called a self-covering map if $X = Y$. We say that a continuous surjection $f : X \to Y$ is a local homeomorphism if for $x \in X$ there is an open neighborhood $U_x$, of $x$ in $X$ such that $f(U_x)$ is open in $Y$ and $f_i(U_x) : U_x \to f(U_x)$ is a homeomorphism. It is clear that every covering map is a local homeomorphism. Conversely, if $X$ is compact, then a local homeomorphism $f : X \to Y$ is a covering map (see [2, Theorem 2.1.1]).

Let $\pi : Y \to X$ be a covering map. A homeomorphism $\alpha : Y \to Y$ is called a covering transformation for $\pi$ if $\pi \circ \alpha = \pi$. We denote by $G(\pi)$ the set of all covering transformations for $\pi$. It is easy to see that $G(\pi)$ is a group, which is called the covering transformation group for $\pi$.

Let $M$ be a closed smooth manifold and let $C^1(M, M)$ be the set of all $C^1$ maps of $M$ endowed with the $C^1$ topology. A map $f \in C^1(M, M)$ is called an Anosov endomorphism if $f$ is a $C^1$ regular map and if there exist $C > 0$ and $0 < \lambda < 1$ such that for every $\tilde{x} = (x_i) \in M_f = \{\tilde{x} = (x_i) : x_i \in M$ and $f(x_i) = x_{i+1}, i \in \mathbb{Z}\}$ there is a splitting
\[
T_{x_i}M = E_{x_i}^s \oplus E_{x_i}^u, \quad i \in \mathbb{Z}
\]
(we show this by $T_{\tilde{x}}M = \bigcup_i (E_{x_i}^s \oplus E_{x_i}^u)$) so that for all $i \in \mathbb{Z}$:

1. $D_{x_i}f(E_{x_i}^\sigma) = E_{x_{i+1}}^\sigma$ where $\sigma = s, u$,

2. for all $n \geq 0$
\[
\|D_{x_i}f^n(v)\| \leq C\lambda^n\|v\| \quad \text{if } v \in E_{x_i}^s,
\]
\[
\|D_{x_i}f^n(v)\| \geq C^{-1}\lambda^{-n}\|v\| \quad \text{if } v \in E_{x_i}^u.
\]

If, in particular, $T_{\tilde{x}}M = \bigcup_i E_{x_i}^u$ for all $\tilde{x} = (x_i) \in M_f$, then $f$ is said to be expanding differentiable map, and if an Anosov endomorphism $f$ is injective then $f$ is called an Anosov diffeomorphism. We can check that every Anosov endomorphism is a TA-map, and that every expanding differentiable map is a topological expanding map (see [2, Theorem 1.2.1]).

A map $f \in C^1(M, M)$ is said to be $C^1$-structurally stable if there is an open neighborhood $\mathcal{N}(f)$ of $f$ in $C^1(M, M)$ such that $g \in \mathcal{N}(f)$ implies that $f$ and $g$ are topologically conjugate. Anosov [1] proved that every Anosov diffeomorphism is $C^1$-structurally stable, and Shub [15] showed the same result for expanding differentiable maps. However, Anosov endomorphisms which are not diffeomorphisms nor expanding, are not $C^1$-structurally stable [11, 14].

A map $f \in C^1(M, M)$ is said to be $C^1$-inverse limit stable if there is an open neighborhood $\mathcal{N}(f)$ of $f$ in $C^1(M, M)$ such that $g \in \mathcal{N}(f)$ implies that the inverse limit $(M_f, \sigma_f)$ of $(M, f)$ and the inverse limit $(M_g, \sigma_g)$ of $(M, g)$ are topologically conjugate, i.e., there exists a homeomorphism $\varphi : M_g \to M_f$ such that $\sigma_f \circ \varphi = \varphi \circ \sigma_g$. Mané and Pugh [11] proved that every Anosov endomorphism is $C^1$-inverse limit stable.

We define special TA-maps as follows. Let $f : X \to X$ be a continuous surjection of a
A discrete topology. A uniform contains exactly one element of $H$ group free if no element other than the identity is of finite order. A $\Gamma$ with $\Gammao$ be Anosov, $A$ an endomorphism when $X$. Let $\text{Lie}(\cdot)$ be the algebra of all vector fields on $N$ space. Moreover, it is closed under Lie bracket. Thus $\text{Lie}(N)$ is a Lie subalgebra of the Lie algebra of all vector fields on $N$ and is called the Lie algebra of $N$. A nilpotent Lie group is a Lie group which is connected and whose Lie algebra is a nilpotent Lie algebra. That is, its Lie algebra’s central series eventually vanishes.

A group $G$ is a torsion group if every element in $G$ is of finite order. $G$ is called torsion free if no element other than the identity is of finite order. A discrete subgroup of a topological group $G$ is a subgroup $H$ such that there is an open cover of $H$ in which every open subset contains exactly one element of $H$. In other words, the subspace topology of $H$ in $G$ is the discrete topology. A uniform subgroup $H$ of $G$ is a closed subgroup such that the quotient space $G/H$ is compact.

Here, we bring the definitions of nil-manifolds and infra-nil-manifolds from Dekimpe in [4] and [5].

Let $N$ be a Lie group and $\text{Aut}(N)$ be the set of all automorphisms of $N$. Assume that $\overline{A} \in \text{Aut}(N)$ is an automorphism of $N$, such that there exists a discrete and cocompact subgroup $\Gamma$ of $N$, with $\overline{A}(\Gamma) \subseteq \Gamma$. Then the space of left cosets $N/\Gamma$ is a closed manifold, and $\overline{A}$ induces an endomorphism $A : N/\Gamma \rightarrow N/\Gamma$, $g\Gamma \mapsto \overline{A}(g)\Gamma$. If we require that this endomorphism to be Anosov, $\overline{A}$ must be hyperbolic (i.e., has no eigenvalue with modulus 1). It is known that this can happen only when $N$ is nilpotent. So we restrict ourselves to that case, where the resulting manifold $N/\Gamma$ is said to be a nil-manifold. Such an endomorphism $A$ induced by an automorphism $\overline{A}$ is called a nil-endomorphism and is said to be a hyperbolic nil-automorphism, when $\overline{A}$ is hyperbolic. If in the above definition, $\overline{A}(\Gamma) = \Gamma$, the induced map is called a nil-automorphism.
All tori, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ are examples of nil-manifolds.

Let $X$ be a topological space and let $G$ be a group. We say that $G$ acts (continuously) on $X$ if to $(g, x) \in G \times X$ there corresponds a point $g \cdot x$ in $X$ and the following conditions are satisfied:

1. $e \cdot x = x$ for $x \in X$ where $e$ is the identity,
2. $g \cdot (g' \cdot x) = gg' \cdot x$ for $x \in X$ and $g, g' \in G$,
3. for each $g \in G$ a map $x \mapsto g \cdot x$ is a homeomorphism of $X$.

When $G$ acts on $X$, for $x, y \in X$ letting

$$x \sim y \iff y = g \cdot x \quad \text{for some } g \in G$$

an equivalence relation $\sim$ in $X$ is defined. Then the identifying space $X/\sim$, denoted as $X/G$, is called the orbit space by $G$ of $X$. It follows that for $x \in X$, $[x] = \{g \cdot x : g \in G\}$ is the equivalence class.

An action of $G$ on $X$ is said to be properly discontinuous if for each $x \in X$ there exists a neighborhood $U(x)$ of $x$ such that $U(x) \cap gU(x) = \emptyset$ for all $g \in G$ with $g \neq e_G$. Here $gU(x) = \{g \cdot y : y \in U(x)\}$.

Now we give an extended definition of nil-manifolds. Let $N$ be a connected and simply connected nilpotent Lie group and $\text{Aut}(N)$ be the group of continuous automorphisms of $N$. Then $\text{Aff}(N) = N \rtimes \text{Aut}(N)$ acts on $N$ in the following way:

$$\forall (n, \gamma) \in \text{Aff}(N), \quad \forall x \in N : (n, \gamma) \cdot x = n \gamma(x).$$

So an element of $\text{Aff}(N)$ consists of a translational part $n \in N$ and a linear part $\gamma \in \text{Aut}(N)$ (as a set $\text{Aff}(N)$ is just $N \rtimes \text{Aut}(N)$) and $\text{Aff}(N)$ acts on $N$ by first applying the linear part and then multiplying on the left by the translational part). In this way, $\text{Aff}(N)$ can also be seen as a subgroup of $\text{Diff}(N)$.

Now, let $C$ be a compact subgroup of $\text{Aut}(N)$ and consider any torsion free discrete subgroup $\Gamma$ of $N \rtimes C$, such that the orbit space $N/\Gamma$ is compact. Note that $\Gamma$ acts on $N$ as being also a subgroup of $\text{Aff}(N)$. The action of $\Gamma$ on $N$ will be free and properly discontinuous, so $N/\Gamma$ is a manifold, which is called an infra-nil-manifold.

Klein bottle is an example of infra-nil-manifolds.

In what follows, we will identify $N$ with the subgroup $N \times \{\text{id}\}$ of $N \rtimes \text{Aut}(N) = \text{Aff}(N)$, $F$ with the subgroup $\{\text{id}\} \times F$ and $\text{Aut}(N)$ with the subgroup $\{\text{id}\} \times \text{Aut}(N)$.

It follows from Theorem 1 of Auslander in [3], that $\Gamma \cap N$ is a uniform lattice of $N$ and that $\Gamma/(\Gamma \cap N)$ is a finite group. This shows that the fundamental group of an infra-nil-manifold $N/\Gamma$ is virtually nilpotent (i.e., has a nilpotent normal subgroup of finite index). In fact $\Gamma \cap N$ is a maximal nilpotent subgroup of $\Gamma$ and it is the only normal subgroup of $\Gamma$ with this property. (This also follows from [3].)

If we denote by $p : N \times C \to C$ the natural projection on the second factor, then $p(\Gamma) = \Gamma \cap N$ is a uniform lattice of $N$ and that $\Gamma/(\Gamma \cap N)$ is a finite group. Let $F$ denote this finite group $p(\Gamma)$, then we will refer to $F$ as being the holonomy group of $\Gamma$ (or of the infra-nil-manifold $N/\Gamma$). It follows that $\Gamma \subseteq N \rtimes F$. In case $F = \{\text{id}\}$, so $\Gamma \subseteq N$, the manifold $N/\Gamma$ is a nil-manifold. Hence, any infra-nil-manifold $N/\Gamma$ is finitely covered by a nil-manifold $N/(\Gamma \cap N)$. This also explains the prefix “infra”.

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**References:**

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[2] Auslander M. (1960) Actions of groups on nilmanifolds.

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**Note:** The document discusses the concept of nil-manifolds, infra-nil-manifolds, and the actions of groups on manifolds. It provides a clear explanation of the definitions and properties of these mathematical objects, including the orbit space, fundamental group, and the role of the holonomy group.
Fix an infra-nil-manifold $N/\Gamma$, so $N$ is a connected and simply connected nilpotent Lie group and $\Gamma$ is a torsion free, uniform discrete subgroup of $N \rtimes F$, where $F$ is a finite subgroup of $\text{Aut}(N)$. We will assume that $F$ is the holonomy group of $\Gamma$ (so for any $\mu \in F$, there exists an $n \in N$ such that $(n, \mu) \in \Gamma$).

We can say that an element of $\Gamma$ is of the form $n\mu$ for some $n \in N$ and some $\mu \in F$. Also, any element of $\text{Aff}(N)$ can uniquely be written as a product $n\psi$, where $n \in N$ and $\psi \in \text{Aut}(N)$. The product in $\text{Aff}(N)$ is then given as

$$\forall n_1, n_2 \in N, \ \forall \psi_1, \psi_2 \in \text{Aut}(N): n_1\psi_1 n_2\psi_2 = n_1\psi_1(n_2)\psi_1\psi_2.$$ 

Now we can define infra-nil-endomorphisms as follows:

Let $N$ be a connected, simply connected nilpotent Lie group and $F \subseteq \text{Aut}(N)$ a finite group. Assume that $\Gamma$ is a torsion free, discrete and uniform subgroup of $N \rtimes F$. Let $\overline{A}: N \rtimes F \to N \rtimes F$ be an automorphism, such that $\overline{A}(F) = F$ and $\overline{A}(\Gamma) \subseteq \Gamma$, then, the map

$$A: N/\Gamma \to N/\Gamma, \quad \Gamma \cdot n \mapsto \Gamma \cdot \overline{A}(n)$$

is the infra-nil-endomorphism induced by $\overline{A}$. In case $\overline{A}(\Gamma) = \Gamma$, we call $A$ an infra-nil-automorphism.

In the definition above, $\Gamma \cdot n$ denotes the orbit of $n$ under the action of $\Gamma$. The computation above shows that $A$ is well defined. Note that infra-nil-automorphisms are diffeomorphisms, while in general an infra-nil-endomorphism is a self-covering map.

The following theorem shows that the only maps of an infra-nil-manifold, that lift to an automorphism of the corresponding nilpotent Lie group are exactly the infra-nil-endomorphisms defined above.

**Theorem 2.2** ([5, Theorem 3.4]) Let $N$ be a connected and simply connected nilpotent Lie group, $F \subseteq \text{Aut}(N)$ a finite group and $\Gamma$ a torsion free discrete and uniform subgroup of $N \rtimes F$ and assume that the holonomy group of $\Gamma$ is $F$. If $\overline{A}: N \to N$ is an automorphism for which the map

$$A: N/\Gamma \to N/\Gamma, \quad \Gamma \cdot n \mapsto \Gamma \cdot \overline{A}(n)$$

is well defined (meaning that $\Gamma \cdot \overline{A}(n) = \Gamma \cdot \overline{A}(\gamma \cdot n)$ for all $\gamma \in \Gamma$), then

$$\overline{A}: N \rtimes F \to N \rtimes F : x \mapsto \phi x \phi^{-1} \quad \text{(conjugation in Aff}(N))$$

is an automorphism of $N \rtimes F$, with $\overline{A}(F) = F$ and $\overline{A}(\Gamma) \subseteq \Gamma$. Hence, $A$ is an infra-nil-endomorphism.

Let $X$ be a topological space. We write $\Omega(X; x_0)$ the family of all closed paths from $x_0$ to $x_0$. Let $\Omega(X; x_0)/\sim$ be the identifying space with respect to the equivalence relation $\sim$ by homotopy. We write this set

$$\pi_1(X; x_0) = \Omega(X; x_0)/\sim.$$ 

The group $\pi_1(X; x_0)$ is called the fundamental group at a base point $x_0$ of $X$. If, in particular, $\pi_1(X; x_0)$ is a group consisting of the identity, then $X$ is said to be simply connected with respect to a base point $x_0$.

Let $x_0$ and $x_1$ be points in $X$. If there exists a path $w$ joining $x_0$ and $x_1$, then we can define a map

$$w_2 : \Omega(X; x_1) \to \Omega(X; x_0), \quad \text{by} \ w_2(u) = (w \cdot u) \cdot \overline{w},$$
where \( u \in \Omega(X;x_1), (w \cdot u) \) is the concatenation of \( w \) and \( u \) and \( \overline{w} \) is \( w \) in reverse direction. For \( u,v \in \Omega(X;x_1) \) suppose \( u \sim v \). Then \( w_2(u) \sim w_2(v) \) and thus \( w_2 \) induces a map
\[
w_\ast : \pi_1(X;x_1) \to \pi_1(X;x_0), \quad \text{by } w_\ast([u]) = [w_2(u)],
\]
this map is an isomorphism (see \cite[Lemma 6.1.4]{2}).

**Remark 2.3** If \( X \) is a path connected space then we can remove the base point and write \( \pi_1(X;x_0) = \pi_1(X) \).

Let \( f,g : X \to Y \) be homotopic and \( F \) a homotopy from \( f \) to \( g \) (\( f \sim g \, (F) \)). Then for \( x_0 \in X \) we can define a path \( w \in \Omega(Y; f(x_0), g(x_0)) \) by
\[
w(t) = F(x_0, t), \quad t \in [0,1],
\]
and the relation between homomorphisms \( f_\ast : \pi_1(X;x_0) \to \pi_1(Y;f(x_0)) \) and \( g_\ast : \pi_1(X;x_0) \to \pi_1(Y;g(x_0)) \) is: \( g_\ast = \overline{w} \circ f_\ast \) (see \cite[Lemma 6.1.9]{2}).

Let \( X \) and \( Y \) be topological spaces and \( f : X \to Y \) a continuous map. Take \( x_0 \in X \) and let \( y_0 = f(x_0) \). It is clear that \( fu = f \circ u \in \Omega(X,y_0) \) for \( u \in \Omega(X;x_0) \). Thus we can find a map
\[
f_2 : \Omega(X;x_0) \to \Omega(Y;y_0), \quad \text{by } f_2(u) = fu,
\]
where \( u \in \Omega(X;x_0) \). If \( u \sim v \, (F) \) for \( u,v \in \Omega(X;x_0) \), then we have \( fu \sim fv \, (f \circ F) \), from which the following map will be induced:
\[
f_\ast : \pi_1(X;x_0) \to \pi_1(Y;y_0), \quad \text{by } f_\ast([u]) = [f_2(u)] = [fu].
\]
It is easy to check that \( f_\ast \) is a homomorphism. We say that \( f_\ast \) is a *homomorphism induced from a continuous map \( f : X \to Y \).

Let \( G \) be a group. We say that a path connected topological space \( X \) is of type \( K(G,1) \) if \( \pi_1(X) = G \) and \( \pi_k(X) = 0 \) for \( k \neq 1 \). \( \pi_k(X) \) is the \( k \)th homotopy group of \( X \).

**Proposition 2.4** (\cite[Proposition 6.7.8]{2}) Let \( N \) be a topological space of type \( K(G,1) \) and let \( M \) be a compact connected topological manifold. Let \( x_0 \in M \) and \( y_0 \in N \). Then, given a homomorphism \( \varphi : \pi_1(M,x_0) \to \pi_1(N,y_0) \), there exists a continuous map \( f : M \to N \) with \( f(x_0) = y_0 \) such that \( f_\ast = \varphi \). Conversely, if \( f,g : M \to N \) are continuous maps with \( f(x_0) = g(x_0) = y_0 \) and if \( f_\ast, g_\ast : \pi_1(M,x_0) \to \pi_1(N,y_0) \) satisfies \( f_\ast(\alpha) = \rho g_\ast(\alpha) \rho^{-1}, \forall \alpha \in \pi_1(M,x_0) \) for some \( \rho \in \pi_1(N,y_0) \), then \( f \) and \( g \) are homotopic.

**Lemma 2.5** (\cite[Remark 6.7.9]{2}) Let \( f,g : N/\Gamma \to N/\Gamma \) be continuous maps of a nil-manifold and let \( f(x_0) = g(x_0) \) for some \( x_0 \in N/\Gamma \). Then \( f \) and \( g \) are homotopic if and only if \( f_\ast = g_\ast : \pi_1(N/\Gamma, x_0) \to \pi_1(N/\Gamma, f(x_0)) \). From this fact we have that if \( f : N/\Gamma \to N/\Gamma \) is a continuous map, then there is a unique nil-endomorphism \( A : N/\Gamma \to N/\Gamma \) homotopic to \( f \).

**Theorem 2.6** (\cite[Theorem 6.3.4]{2}) If \( \pi : Y \to X \) is the universal covering, then for each \( b \in Y \)
\begin{enumerate}
  \item the map \( \alpha \mapsto \alpha(b) \) is a bijection from \( G(\pi) \) onto \( \pi^{-1}(\pi(b)) \),
  \item the map \( \psi_b : G(\pi) \to \pi_1(X,\pi(b)) \) by \( \alpha \mapsto [\pi \circ u_{\alpha(b)}] \) is an isomorphism where \( u_{\alpha(b)} \) is a path from \( b \) to \( \alpha(b) \).
\end{enumerate}
Furthermore, the action of \( G(\pi) \) on \( Y \) is properly discontinuous and \( Y/G(\pi) \) is homeomorphic to \( X \).
Theorem 2.7 ([2, Theorem 6.3.7]) Let $G$ be a group and $X$ a topological space. Suppose that $G$ acts on $X$ and the action is properly discontinuous. Then

1. the natural projection $\pi : X \rightarrow X/G$ is a covering map,
2. if $X$ is simply connected, then the fundamental group $\pi_1(X/G)$ is isomorphic to $G$.

Corollary 2.8 Let $N/\Gamma$ be an infra-nil-manifold and $\pi : N \rightarrow N/\Gamma$ be the natural projection. Then

$$\Gamma \cong \pi_1(N/\Gamma) \cong G(\pi).$$

Proof Since $\Gamma$ acts on $N$ properly discontinuous, the natural projection $\pi : N \rightarrow N/\Gamma$ is a covering map. Since $N$ is simply connected, by Theorem 2.7 we have $\Gamma \cong \pi_1(N/\Gamma)$.

On the other hand, since $N$ is simply connected and $\Gamma$ acts on $N$ properly discontinuous the natural projection $\pi : N \rightarrow N/\Gamma$ is the universal covering map. So by Theorem 2.6 we have $\Gamma \cong G(\pi)$. \qed

From now on we only consider $N/\Gamma$ as a nil-manifold.

Lemma 2.9 Let $f : N/\Gamma \rightarrow N/\Gamma$ be a continuous map of a nil-manifold, and $A : N/\Gamma \rightarrow N/\Gamma$ be the unique nil-endomorphism homotopic to $f$, then $\overline{f}_* = \overline{A}_* : \Gamma \rightarrow \Gamma$.

Proof By Corollary 2.8, $\overline{f}_*$ and $\overline{A}_*$ are two maps on $\Gamma$. For $[e] = \{x \in N : \gamma(x) = \gamma \cdot x = e$ for some $\gamma \in \Gamma\}$, we have

$$f([e]) = f \circ \pi(e) = \pi \circ \overline{f}(e) = \pi(\overline{f}(e)) = \pi(e) = [e] = A([e]).$$

So according to Lemma 2.5, $\overline{f}_* = \overline{A}_*$. \qed

Lemma 2.10 ([18, Lemma 1.3]) Let $f : N/\Gamma \rightarrow N/\Gamma$ be a self-covering map of a nil-manifold and $A : N/\Gamma \rightarrow N/\Gamma$ denote the nil-endomorphism homotopic to $f$. If $f$ is a TA-covering map, then $A$ is hyperbolic.

Lemma 2.11 ([18, Lemma 1.5]) Let $f : N/\Gamma \rightarrow N/\Gamma$ be a self-covering map and let $\overline{f} : N \rightarrow N$ be a lift of $f$ by the natural projection $\pi : N \rightarrow N/\Gamma$. If $f$ is a TA-covering map then $\overline{f}$ has exactly one fixed point.

For continuous maps $f$ and $g$ of $N$ we define $D(f, g) = \sup\{D(f(x), g(x)) : x \in N\}$ where $D$ denotes a left invariant, $\Gamma$-invariant Riemannian distance for $N$. Notice that $D(f, g)$ is not necessary finite.

Suppose that $f : N/\Gamma \rightarrow N/\Gamma$ is a TA-covering map. Let $A : N/\Gamma \rightarrow N/\Gamma$ be the nil-endomorphism homotopic to $f$, and let $\overline{A} : N \rightarrow N$ be the automorphism which is a lift of $A$ by the natural projection $\pi$. Since $D_{\overline{A}}$ is hyperbolic by Lemma 2.10, the Lie algebra $\text{Lie}(N)$ of $N$ splits into the direct sum $\text{Lie}(N) = E_s^e \oplus E_u^e$ of subspaces $E_s^e$ and $E_u^e$ such that $D_{\overline{A}}(E_s^e) = E_s^e$, $D_{\overline{A}}(E_u^e) = E_s^u$ and there are $c > 1, 0 < \lambda < 1$ so that for all $n \geq 0$

$$\|D_{\overline{A}}^n(v)\| \leq c\lambda^n\|v\| \quad (v \in E_s^e),$$
$$\|D_{\overline{A}}^{-n}(v)\| \leq c\lambda^n\|v\| \quad (v \in E_u^e),$$

where $\|\cdot\|$ is the Riemannian metric. Let $\overline{L}^\sigma(v) = \exp(E_{\sigma}^e)$ ($\sigma = s, u$) and let $\overline{L}^\sigma(x) = x \cdot \overline{L}^\sigma(\sigma = s, u)$ for $x \in N$. Since left translations are isometries under the metric $D$, it follows that for all $x \in N$

$$\overline{L}^\sigma(x) = \{y \in N : D(\overline{A}(x), \overline{A}(y)) \rightarrow 0 \ (i \rightarrow \infty)\},$$
Lemma 2.12 ([10, Lemma 3.2]) For \( x, y \in N \), \( L^{s}(x) \cap L^{u}(y) \) consists of exactly one point.

For \( x, y \in N \) denote by \( \beta(x, y) \) the point in \( L^{s}(x) \cap L^{u}(y) \).

Lemma 2.13 ([10, Lemma 3.2])

1. For \( L > 0 \) and \( \epsilon > 0 \) there exists \( J > 0 \) such that for \( x, y \in N \) if \( D(\overline{A}(x), \overline{A}(y)) \leq L \) for all \( i \) with \( i \leq J \), then \( D(x, y) \leq \epsilon \).

2. For given \( L > 0 \), if \( D(\overline{A}(x), \overline{A}(y)) \leq L \) for all \( i \in \mathbb{Z} \), then \( x = y \) (\( x, y \in N \)).

Lemma 2.14 ([18, Lemma 2.3]) Under the assumptions and notations as above, there is a unique map \( \overline{h} : N \rightarrow N \) such that

1. \( \overline{A} \circ \overline{h} = \overline{h} \circ f \),

2. \( D(\overline{h}, id_{N}) \) is finite,

where \( id_{N} : N \rightarrow N \) is the identity map of \( N \). Furthermore \( \overline{h} \) is surjective and uniformly continuous under \( D \).

In addition, if \( f \) is not an expanding map then \( \overline{h} \) is a homeomorphism, i.e., \( \overline{h} \) is \( D \)-biuniformly continuous. (See [2, Proposition 8.4.2].)

Lemma 2.15 ([18, Lemma 2.4]) For the semiconjugacy \( \overline{h} \) of Lemma 2.14, we have the following properties:

1. There exists \( K > 0 \) such that \( D(\overline{h}(x\gamma), \overline{h}(x)\gamma) < K \) for \( x \in N \) and \( \gamma \in \Gamma \).

2. For any \( \lambda > 0 \), there exists \( L \in \mathbb{N} \) such that \( D(\overline{h}(x\gamma), \overline{h}(x)\gamma) < \lambda \) for \( x \in N \) and \( \gamma \in \overline{A}_{L}(\Gamma) \).

3. For \( x \in N \) and \( \gamma \in \bigcap_{i=0}^{\infty} \overline{A}_{i}(\Gamma) \), we have \( \overline{h}(x\gamma) = \overline{h}(x)\gamma \).

4. For \( x \in N \) and \( \gamma \in \Gamma \), we have \( \overline{h}(x\gamma) \in \overline{L}^{u}(\overline{h}(x)) \).

Remark 2.16 By part (2) of Theorem 2.14, there is a \( \delta_{K} > 0 \) such that \( D(\overline{h}(x), x) < \delta_{K} \) for \( x \in N \), we have (see [2, p. 270 (8.5)])

\[
\overline{W}^{s}(x) \subset U_{\delta_{K}}(\overline{L}^{s}(\overline{h}(x))) \quad \text{and} \quad \overline{W}^{u}(x; e) \subset U_{\delta_{K}}(\overline{L}^{u}(\overline{h}(x))).
\]

By Lemma 2.11 if \( f : N/\Gamma \rightarrow N/\Gamma \) is a TA-map and \( \overline{f} : N \rightarrow N \) a lift of it, then there exists a unique fixed point say \( b \) such that \( \overline{f}(b) = b \). For simplicity we can suppose that \( b = e \). Indeed, we can choose a homeomorphism \( \varphi \) of \( N \) such that \( \varphi(\pi(b)) = e \). Then \( \varphi \circ f \circ \varphi^{-1} \) is a TA-covering map such that \( \varphi \circ f \circ \varphi^{-1}(e) = e \).

Let \( x \in N \), we define the stable set and unstable sets of \( x \) for \( f \) and \( A \) as follow (for more details see [2]):

\[
\overline{W}^{s}(x) = \left\{ y \in N : \lim_{i \to -\infty} D(\overline{f}^{i}(x), \overline{f}^{i}(y)) = 0 \right\},
\]

\[
\overline{W}^{u}(x; e) = \left\{ y \in N : \lim_{i \to -\infty} D(\overline{f}^{i}(x), \overline{f}^{i}(y)) = 0 \right\},
\]

where \( e = (\ldots, e, e, e, \ldots) \).

Remark 2.17 By Lemma 2.14, since \( \overline{h} \) is \( D \)-uniformly continuous then \( \overline{h}^{\ast}(x) = \overline{L}^{s}(\overline{h}(x)) \) and \( \overline{h}(\overline{W}^{u}(x; e)) = \overline{L}^{u}(\overline{h}(x)) \).

Lemma 2.18 The following statements hold:

1. \( \overline{W}^{s}(x)\gamma = \overline{W}^{s}(x\gamma) \) for \( \gamma \in \Gamma \) and \( x \in N \),

2. \( \overline{W}^{u}(x; e)\gamma = \overline{W}^{u}(x\gamma; e) \) for \( \gamma \in \Gamma \) and \( x \in N \).
Proof  It is an easy corollary of [2, Lemma 6.6.11]. According to Corollary 2.8, in the mentioned lemma put \( \Gamma \) instead of \( G(\pi) \) and \( N \) instead of \( \overline{X} \).

Lemma 2.19  The following statements hold:

1. \( \overline{L}^u(x) \gamma = \overline{L}^u(x \gamma) \) for \( \gamma \in \Gamma \) and \( x \in N \),
2. \( \overline{L}^s(x) \gamma = \overline{L}^s(x \gamma) \) for \( \gamma \in \Gamma \) and \( x \in N \).

Proof  Proof is the same as in Lemma 2.18.

Lemma 2.20 ([18, Lemma 5.4])  Let \( N/\Gamma \) be a nil-manifold. If \( f : N/\Gamma \to N/\Gamma \) is a TA-covering map, then the nonwandering set \( \Omega(f) \) coincides with the entire space \( N/\Gamma \).

Lemma 2.21 ([2, Lemma 8.6.2])  For \( \epsilon > 0 \) there is \( \delta > 0 \) such that if \( D(x,y) < \delta \), \( x, y \in N \) then \( \overline{W}^s(x) \subset U(\overline{W}^s(y)) \) and \( \overline{W}^a(x; e) \subset U(\overline{W}^a(y; e)) \), where for a set \( S \), \( U_\epsilon(S) = \{ y \in N : D(y, S) < \epsilon \} \).

3  Proof of Main Theorem

In this section we suppose that \( f : N/\Gamma \to N/\Gamma \) is a special TA-covering map of a nil-manifold which is not injective or expanding, and \( A : N/\Gamma \to N/\Gamma \) is the unique nil-endomorphism homotopic to \( f \).

Sketch of Proof  By Lemma 2.14, there is a unique semiconjugacy \( \overline{h} : N \to N \) between \( \overline{f} \) and \( \overline{A} \), such that by Proposition 3.2 (3), \( \overline{h}(v \gamma) = \overline{h}(v) \gamma \), for each \( \gamma \in \overline{W}^u(e; e) \cap \Gamma \) and \( v \in \overline{W}^u(e; e) \). Through Proposition 3.3 to Proposition 3.13 we show that for all \( \gamma \in \Gamma \) and \( x \in N \), \( \overline{h}(x \gamma) = \overline{h}(x) \gamma \). Based on this result, \( \overline{h} \) induces a homeomorphism \( h : N/\Gamma \to N/\Gamma \) which is the conjugacy between \( f \) and \( A \).

To prove the main theorem we need some consequential lemmas and propositions.

Lemma 3.1  The following statements hold:

1. Let \( D \) be the metric of \( N \) as above. For each \( x \in N \), \( D(x^{-1}, e) = D(x, e) \).
2. If \( x \in \overline{W}^u(e; e) \), then \( \overline{W}^u(x; e) = \overline{W}^u(e; e) \).
3. If \( x \in \overline{L}^u(e) \), then \( \overline{L}^u(x) = \overline{L}^u(e) \).

Proof  (1)

\[
D(x^{-1}, e) = D(x^{-1}e, x^{-1}x) = D(e, x) \quad (D \text{ is left invariant}) = D(x, e)
\]

(2) Since \( x \in \overline{W}^u(e; e) \), we have \( D(\overline{f}^i(x), \overline{f}^i(e)) \to 0 \) as \( i \to -\infty \). Let \( y \in \overline{W}^u(e; e) \), then \( D(\overline{f}^i(y), \overline{f}^i(e)) \to 0 \) as \( i \to -\infty \). We have,

\[
D(\overline{f}^i(y), \overline{f}^i(x)) < D(\overline{f}^i(y), \overline{f}^i(e)) + D(\overline{f}^i(e), \overline{f}^i(x)) \to 0 \quad \text{as} \quad i \to -\infty.
\]

So, \( y \in \overline{W}^u(x; e) \), i.e., \( \overline{W}^u(e; e) \subset \overline{W}^u(x; e) \). Conversely, if \( y \in \overline{W}^u(x; e) \) then \( D(\overline{f}^i(y), \overline{f}^i(x)) \to 0 \) as \( i \to -\infty \), and

\[
D(\overline{f}^i(y), \overline{f}^i(e)) < D(\overline{f}^i(y), \overline{f}^i(x)) + D(\overline{f}^i(x), \overline{f}^i(e)) \to 0 \quad \text{as} \quad i \to -\infty.
\]

So, \( y \in \overline{W}^u(e; e) \), i.e., \( \overline{W}^u(x; e) \subset \overline{W}^u(e; e) \).

(3) Its proof is the same as part (2).
For simplicity, let $\Gamma_{\overline{\mathcal{T}}} = \overline{W}^u(e; e) \cap \Gamma$ and $\Gamma_{\overline{T}} = \overline{T}^u(e) \cap \Gamma$.

**Proposition 3.2** The following statements hold:

1. $\Gamma_{\overline{T}}$ and $\Gamma_{\overline{\mathcal{T}}}$ are subgroups of $\Gamma$.
2. $\overline{T}^u \subset \Gamma_{\overline{\mathcal{T}}}$.
3. $\overline{h}(v\gamma) = \overline{h}(v)\gamma$, for each $\gamma \in \Gamma_{\overline{\mathcal{T}}}$ and $v \in \overline{W}^u(e; e)$.
4. If $\overline{W}^u(\gamma_1; e) = \overline{W}^u(\gamma_2; e)$, for some $\gamma_1, \gamma_2 \in \Gamma$, then we have $\overline{h}(x\gamma_1^{-1})\gamma_1 = \overline{h}(x\gamma_2^{-1})\gamma_2$, for $x \in \overline{W}^u(\gamma_1; e)$.

**Proof**

1. Let $\gamma_1, \gamma_2 \in \Gamma_{\overline{\mathcal{T}}}$. Since $\Gamma$ is a group we have $\gamma_1\gamma_2^{-1} \in \Gamma$. Now consider that $\gamma_1, \gamma_2 \in \overline{L}^u(e)$, since $A^i(e) = e$, for all $i$, then by definition,

$$\lim_{i \to -\infty} D(A^i(\gamma_1), e) = \lim_{i \to -\infty} D(A^i(\gamma_1), A^i(e)) = 0,$$

$$\lim_{i \to -\infty} D(A^i(\gamma_2), e) = \lim_{i \to -\infty} D(A^i(\gamma_2), A^i(e)) = 0. \quad (3.1)$$

As $D$ is left invariant we have

$$0 \leq \lim_{i \to -\infty} D(A^i(\gamma_1\gamma_2^{-1}), A^i(e))$$

$$= \lim_{i \to -\infty} D(A^i(\gamma_1)A^i(\gamma_2^{-1}), e)$$

$$= \lim_{i \to -\infty} D(A^i(\gamma_1)A^{-i}(\gamma_2), A^i(\gamma_1)A^{-i}(\gamma_1))$$

$$= \lim_{i \to -\infty} D(A^{-i}(\gamma_2), A^{-i}(\gamma_1)) \quad (D \text{ is left invariant})$$

$$\leq \lim_{i \to -\infty} D(A^{-i}(\gamma_2), e) + D(e, A^{-i}(\gamma_1))$$

$$= \lim_{i \to -\infty} D(A^i(\gamma_2), e) + D(A^i(\gamma_1), e) = 0 \quad \text{(Lemma 3.1 (1) and (3.1))}.$$ 

Thus $\gamma_1\gamma_2^{-1} \in \overline{L}^u(e)$ and $\overline{L}^u(e)$ is a subgroup of $N$. So $\overline{L}^u(e) \cap \Gamma$ is a subgroup of $\Gamma$.

For the second part, Let $\gamma_1, \gamma_2 \in \Gamma_{\overline{\mathcal{T}}}$. Since $\Gamma$ is a group we have $\gamma_1\gamma_2^{-1} \in \Gamma$. Now consider that $\gamma_1, \gamma_2 \in \overline{W}^u(e; e)$. Then,

$$\overline{W}^u(e; e)\gamma_1 = \overline{W}^u(e; e) \quad \text{(Lemma 2.18)}$$

$$= \overline{W}^u(\gamma_1; e)$$

$$= \overline{W}^u(e; e) \quad \text{(Lemma 3.1 (2)).}$$

Similarly, $\overline{W}^u(e; e)\gamma_2 = \overline{W}^u(e; e)$. So we have $\overline{W}^u(e; e)\gamma_1 = \overline{W}^u(e; e)\gamma_2$ and then $\gamma_1\gamma_2^{-1} \in \overline{W}^u(e; e)$, and we have the result.

2. Take $\gamma \in \Gamma_{\overline{T}}$ such that $\gamma \notin \Gamma_{\overline{\mathcal{T}}}$. So, $\gamma \notin \overline{L}^u(e)$ and for each $n \in \mathbb{Z}$, $n \neq 0$, $\gamma^n \notin \overline{L}^u(e)$. On the other hand, by part (1), Remark 2.16 and the fact that $\overline{h}(e) = e$, for all $n \in \mathbb{Z}$, we have $\gamma^n \in \overline{W}^u(e; e) \subset U_{\delta_k}(\overline{L}^u(e))$, which is impossible.

3. Let $\gamma \in \Gamma_{\overline{T}}$ and $v \in \overline{W}^u(e; e)$. We have

$$v\gamma \in \overline{W}^u(e; e)\gamma$$

$$= \overline{W}^u(e; e) \quad \text{(Lemma 2.18)}$$

$$= \overline{W}^u(\gamma; e)$$

$$= \overline{W}^u(e; e), \quad \text{(Lemma 3.1 (2))}$$
On the other hand, by part (4) of Lemma 2.15, 
\[ \bigcup \gamma \in \Gamma, \quad \bar{h}(v) \gamma \in \bar{h}(W^u(e; e)) \gamma \]

By part (2), \( \gamma \in \Gamma, \) Thus
\[ \bar{h}(v) \gamma \in \bar{h}(W^u(e; e)) \gamma = \bigcup \gamma \in \Gamma, \quad \bar{h}(v) \gamma = \bar{h}(v) \gamma \] (Remark 2.17)
\[ \bar{h}(v) \gamma = \bar{h}(v) \gamma = \bigcup \gamma \in \Gamma, \quad \bar{h}(v) \gamma = \bar{h}(v) \gamma = \bigcup \gamma \in \Gamma, \] (Lemma 2.19)
\[ \bar{h}(v) \gamma = \bar{h}(v) \gamma = \bigcup \gamma \in \Gamma, \quad \bar{h}(v) \gamma = \bar{h}(v) \gamma = \bigcup \gamma \in \Gamma, \] (Lemma 3.1 (3)).

Again by Lemma 3.1 (3) and last part of the above relation, \( \bar{h}^\prime(\bar{h}(v) \gamma) = \bar{h}^\prime(e) \), and
\[ \bar{h}(v) \gamma \in \bar{T}^\prime(e) = \bar{T}^\prime(\bar{h}(v) \gamma). \]

On the other hand, by part (4) of Lemma 2.15, \( \bar{h}(v) \gamma \in \bar{T}^\prime(\bar{h}(v) \gamma). \) Since \( \bar{T}^\prime(\bar{h}(v) \gamma) \cap \bar{T}^\prime(\bar{h}(v) \gamma) = \{ \bar{h}(v) \gamma \} \) (see [18, Lemma 2.1]), then \( \bar{h}(v) \gamma = \bar{h}(v) \gamma. \)

(4) Let \( x \in W^u(\gamma_1; e) = W^u(\gamma_2; e). \) For \( \gamma_1, \gamma_2 \in \Gamma, \) we have \( \gamma_2 \in W^u(\gamma_2; e) = W^u(\gamma_1; e) = W^u(e; e) \gamma_1. \) Thus, \( \gamma_2 \gamma_1^{-1} \in W^u(e; e), \) and then \( \gamma_2 \gamma_1^{-1} \in \Gamma. \) Similarly, \( x \gamma_1^{-1}, x \gamma_2^{-1} \in W^u(e; e). \)

Now, by part (3),
\[ \bar{h}(x \gamma_1^{-1}) \gamma_1 = \bar{h}(x \gamma_2^{-1} \gamma_2 \gamma_1^{-1}) \gamma_1 \]
\[ = \bar{h}(x \gamma_2^{-1}) \gamma_2 \gamma_1^{-1} \gamma_1 \]
\[ = \bar{h}(x \gamma_2^{-1}) \gamma_2. \]

According to part (4) of Proposition 3.2, we can define a map \( \bar{h}^:\cup_{\gamma \in \Gamma} W^u(\gamma; e) \to \cup_{\gamma \in \Gamma} \bar{T}^\prime(\gamma), \) by
\[ \bar{h}^\prime(x) = \bar{h}(x \gamma^{-1}) \gamma, \quad x \in W^u(\gamma; e) \quad (\gamma \in \Gamma). \]

The next proposition shows some properties of \( \bar{h}^\prime: \)

**Proposition 3.3** The following statements hold:

1. \( \bar{A} \circ \bar{h} = \bar{h} \circ \bar{f} \) on \( \bigcup_{\gamma \in \Gamma} W^u(\gamma; e), \)
2. \( D(\bar{h}^\prime, \text{id}_{|_{\bigcup_{\gamma \in \Gamma} W^u(\gamma; e)}}) < \infty, \)
3. \( \bar{h}^\prime(\gamma) = \gamma \) for \( \gamma \in \Gamma, \)
4. if \( x \in W^u(\gamma; e) \) (\( \gamma \in \Gamma \)), then \( \bar{h}^\prime(x) \in \bar{T}^\prime(\gamma) \) and \( \bar{h}^\prime(x) \in \bar{T}^\prime(\bar{h}(x)). \)
5. if \( y \in W^s(x) \) for \( x, y \in \bigcup_{\gamma \in \Gamma} W^u(\gamma; e), \) then \( \bar{h}^\prime(y) \in \bar{T}^\prime(\bar{h}(x)). \)

**Proof**

1. Suppose that \( x \in W^u(\gamma; e) = W^u(e; e) \gamma, \) for some \( \gamma \in \Gamma. \) Then
\[ x \gamma^{-1} \in W^u(e; e). \] (3.2)

By [2, p. 205] we have \( \bar{f}(W^u(\gamma; e)) = W^u(\bar{f}(\gamma); e). \) Here \( \bar{f}(\gamma) \) means \( \bar{f}_\gamma(\gamma) \) which by Lemma 2.9 is equal to \( \bar{A}_\gamma(\gamma) \) and \( \bar{A}_\gamma(\gamma) \in \Gamma. \) Therefore,
\[ \bar{f}(x) \in \bar{f}(W^u(\gamma; e)) = W^u(\bar{A}_\gamma(\gamma); e) = W^u(e; e) \bar{A}_\gamma(\gamma), \]
so,
\[ (\overline{f}(x))(\overline{A}_s(\gamma))^{-1} \in \overline{W}'(e; e). \] (3.3)

Thus we have
\[
\overline{A} \circ \overline{h}(x) = \overline{A}(\overline{h}(x\gamma^{-1})\gamma) \\
= \overline{A}(\overline{h}(x\gamma^{-1})) \quad ((3.2) \text{ and Proposition 3.2 (3)}) \\
= \overline{A} \circ \overline{h}(x) \\
= \overline{h} \circ \overline{f}(x) \quad \text{(Lemma 2.14)} \\
= \overline{h}(\overline{f}(x))(\overline{A}_s(\gamma))^{-1}(\overline{A}_s(\gamma)) \\
= \overline{h}(\overline{f}(x))(\overline{A}_s(\gamma))^{-1}(\overline{A}_s(\gamma)) \quad ((3.3) \text{ and Proposition 3.2 (3)}) \\
= \overline{h}(\overline{f}(x)) \\
= \overline{h} \circ \overline{f}(x).
\]

(2) Let \( x \in \overline{W}'(\gamma; e) \), for some \( \gamma \in \Gamma \), and let \( \delta_K > 0 \) be satisfying \( D(\overline{h}, \text{id}_N) < \delta_K \). Then
\[
D(\overline{h}(x), x) = D(\overline{h}(x\gamma^{-1})\gamma, x) \\
= D(\overline{h}(x\gamma^{-1})\gamma, x\gamma^{-1}) \\
= D(\overline{h}(x\gamma^{-1}), x\gamma^{-1}) \quad (D \text{ is } \Gamma\text{-invariant}) \\
< \delta_K.
\]

(3) For any \( \gamma \in \Gamma \), by definition we have
\[
\overline{h}(\gamma) = \overline{h}(\gamma\gamma^{-1})\gamma = \overline{h}(e)\gamma = e\gamma = \gamma.
\]

(4) Let \( x \in \overline{W}'(\gamma; e) \), for some \( \gamma \in \Gamma \). We have
\[
\overline{h}(x) = \overline{h}(x\gamma^{-1})\gamma \in \overline{h}(\overline{W}'(\gamma; e)\gamma^{-1})_\gamma \\
= \overline{h}(\overline{W}'(e; e)\gamma^{-1})_\gamma \quad \text{(Lemma 2.18)} \\
= \overline{h}(\overline{W}'(e; e))_\gamma \\
= \overline{L}'(e)\gamma \quad \text{(Remark 2.17)} \\
= \overline{L}'(\gamma), \quad \text{(Lemma 2.19)}
\]
and
\[
\overline{h}(x) = \overline{h}(x\gamma^{-1}) \gamma \\
\in \overline{L}'(\overline{h}(x)\gamma^{-1})_\gamma \quad \text{(Lemma 2.15 (4))} \\
= \overline{L}'(\overline{h}(x))\gamma^{-1}_\gamma \quad \text{(Lemma 2.19 (1))} \\
= \overline{L}'(\overline{h}(x)).
\]

(5) By the second part of proof of (4), we have
\[
\overline{L}'(\overline{h}(y)) = \overline{L}'(\overline{h}(y)) = \overline{h}(\overline{W}'(y)) = \overline{h}(\overline{W}'(x)) = \overline{L}'(\overline{h}(x)) = \overline{L}'(\overline{h}(x)),
\]
so, \( \overline{h}(y) \in \overline{L}'(\overline{h}(x)) \).

\[\Box\]

**Lemma 3.4** ([18, Lemma 7.6]) \hspace{1em} For each \( u, v \in N \), \( \overline{W}'(u; e) \cap \overline{W}'(v) \) is the set of one point.
According to the above lemma, define \( \tau(u, v) = \overline{W}^u(u; e) \cap \overline{W}^s(v) \).

**Lemma 3.5** For \( \epsilon > 0 \), there is \( \delta > 0 \) such that

\[
D(u, v) < \delta \Rightarrow \max\{D(\tau(u, v), u), D(\tau(u, v), v)\} < \epsilon \quad (u, v \in N).
\]

**Proof** Let \( \epsilon > 0 \) be given. Since \( \overline{h} \) is \( D \)-biuniformly continuous there exists \( \epsilon' > 0 \) such that

\[
D(x, y) < \epsilon' \Rightarrow D(\overline{h}^{-1}(x), \overline{h}^{-1}(y)) < \epsilon \quad (x, y \in N).
\]

By [2, Theorem 6.6.5] or [18, Lemma 7.2] since \( N \) is simply connected, \( \overline{h} \) has local product structure (for definition and details, see [2]), and then for \( \epsilon > 0 \) there exists \( \delta' > 0 \) such that

\[
D(u, v) < \delta' \Rightarrow \max\{D(\beta(u, v), u), D(\beta(u, v), v)\} < \epsilon' \quad (u, v \in N).
\]

Again since \( \overline{h} \) is \( D \)-biuniformly continuous, there exists \( \delta > 0 \) such that

\[
D(u, v) < \delta \Rightarrow D(\overline{h}(u), \overline{h}(v)) < \delta' \quad (u, v \in N).
\]

We know that \( \overline{h}(\tau(u, v)) = \beta(\overline{h}(u), \overline{h}(v)) \) therefore

\[
D(u, v) < \delta \Rightarrow D(\overline{h}(u), \overline{h}(v)) < \delta'
\]

\[
\Rightarrow \max\{D(\beta(\overline{h}(u), \overline{h}(v)), \overline{h}(u)), D(\beta(\overline{h}(u), \overline{h}(v)), \overline{h}(v))\} < \epsilon'
\]

\[
\Rightarrow \max\{D(\overline{h}(\tau(u, v)), \overline{h}(u)), D(\overline{h}(\tau(u, v)), \overline{h}(v))\} < \epsilon'
\]

\[
\Rightarrow \max\{D(\overline{h}(u, v), u), D(\overline{h}(u, v), v)\} < \epsilon.
\]

\( \square \)

**Proposition 3.6** \( \overline{h} \) is \( D \)-uniformly continuous.

**Proof** Suppose that the statement is false. So there is \( \epsilon_0 > 0 \), for all \( \delta > 0 \), there are \( x, y \in \bigcup_{\gamma \in \Gamma} \overline{W}^u(\gamma; e) \) such that

\[
D(x, y) < \delta \quad \text{and} \quad D(\overline{h}(x), \overline{h}(y)) > \epsilon_0.
\] (3.4)

By definition of \( \overline{L}^u(x) \) (\( x \in N, \sigma = s, u \)), for \( w \in \overline{L}^s(v) \) there is \( \epsilon_1 > 0 \) such that

\[
D(v, w) < \epsilon_0/2 \Rightarrow D(\overline{L}^u(w), \overline{L}^u(w)) > \epsilon_1.
\] (3.5)

Take \( n > 0 \) and \( \delta_1 > 0 \) such that \( \epsilon^n_1 \geq 2\delta_K \) and \( \delta_1^n \leq 2\delta_K \).

By Lemma 2.21, there exists \( \delta_2 > 0 \) such that

\[
D(v, w) < \delta_2 \Rightarrow \overline{W}^u(w, e) \subset U_{\delta_1}(\overline{W}^u(v, e)).
\] (3.6)

Since \( \overline{h} \) is continuous, take \( \delta_3 > 0 \) such that

\[
D(u, v) < \delta_3 \Rightarrow D(\overline{h}(u), \overline{h}(v)) < \epsilon_0/2.
\] (3.7)

By Lemma 3.5, there is \( 0 < \delta < \delta_2 \) such that

\[
D(x, y) < \delta \Rightarrow D(y_{\gamma_{y}}^{-1}, \gamma(\gamma_{y}^{-1}) = D(y, \gamma(\gamma_{y}) < \delta_3 \quad (x, y \in N).
\]

(3.8)

Now consider \( x, y \in \bigcup_{\gamma \in \Gamma} \overline{W}^u(\gamma; e) \) satisfy (3.4). There exist \( \gamma_{x}, \gamma_{y} \in \Gamma \) such that \( x \in \overline{W}^u(\gamma_{x}, e) \) and \( y \in \overline{W}^u(\gamma_{y}, e) \). We have

\[
D(\overline{h}(x), \overline{h}(\gamma_{y}^{-1}x)) \geq D(\overline{h}(x), \overline{h}(y)) - D(\overline{h}(y), \overline{h}(\gamma_{y}^{-1}x)) \geq \epsilon_0 - D(\overline{h}(y_{\gamma_{y}}^{-1}), \gamma_{y}^{-1}x, \overline{h}(\gamma_{y}^{-1}x) \gamma_{y}) \quad (by \ (3.4))
\]
On the other hand,

\[ D(\overline{u}^x(\gamma_x,e), \overline{u}^y(\gamma_y,e)) > \frac{\epsilon_0}{2} = \epsilon_0/2 \quad \text{by (3.7) and (3.8)}. \] (3.9)

By Proposition 3.3 (4)

\[ x \in \overline{W}^u(\gamma_x,e) \Rightarrow \overline{u}(x) \in \overline{L}^u(\gamma_x), \]
\[ \tau(y,x) \in \overline{W}^u(\gamma_y,e) \Rightarrow \overline{u}(\tau(y,x)) \in \overline{L}^u(\gamma_y). \]

Thus by Proposition 3.3 (5), (3.9) and (3.5) we have

\[ D(\overline{L}^u(\gamma_x), \overline{L}^u(\gamma_y)) > \epsilon_1. \]

Suppose \( \gamma = \gamma_y \gamma_x^{-1} \). We have

\[ \gamma \gamma_x = \gamma_y \not\subset U_{\epsilon_1}(\overline{L}^u(\gamma_x)) \Rightarrow \gamma \not\subset U_{\epsilon_1}(\overline{L}^u(\gamma_x \gamma_x^{-1})) = U_{\epsilon_1}(\overline{L}^u(e)) \]
\[ \Rightarrow \gamma^n(e) \not\subset U_{\epsilon_1}(\overline{L}^u(e)) \supset U_{2 \delta_K}(\overline{L}^u(e)). \] (3.10)

On the other hand,

\[ \overline{W}^u(\gamma \gamma_x; e) = \overline{W}^u(\gamma_y; e) \]
\[ = \overline{W}^u(y; e) \quad (y \in \overline{W}^u(\gamma_y; e)) \]
\[ \subset U_{\delta_1}(\overline{W}^u(x; e)) \quad \text{by (3.6)} \]
\[ = U_{\delta_1}(\overline{W}^u(\gamma_x; e)) \quad (x \in \overline{W}^u(\gamma_x; e)) \]
\[ = U_{\delta_1}(\overline{W}^u(e; e) \gamma_x). \]

Now we have

\[ \overline{W}^u(\gamma \gamma_x; e) \gamma_x^{-1} \subset U_{\delta_1}(\overline{W}^u(e; e)) \Rightarrow \overline{W}^u(\gamma \gamma_x \gamma_x^{-1}; e) \subset U_{\delta_1}(\overline{W}^u(e; e)) \]
\[ \Rightarrow \overline{W}^u(\gamma; e) \subset U_{\delta_1}(\overline{W}^u(e; e)) \]
\[ \Rightarrow \overline{W}^u(\gamma^2; e) \subset U_{\delta_1}(\overline{W}^u(e; e)) \]
\[ \Rightarrow \overline{W}^u(\gamma^n; e) \subset U_{\delta_1}(\overline{W}^u(e; e)) \quad \text{by induction} \]
\[ \Rightarrow \gamma^n(e) \subset U_{\delta_1}(\overline{W}^u(e; e)) \subset U_{2 \delta_K}(\overline{L}^u(e)) \]
\[ \Rightarrow \gamma^n(e) \subset U_{2 \delta_K}(\overline{L}^u(e)). \] (3.11)

Finally, (3.10) and (3.11) make a contradiction.

Let \( \bar{u} = (u_i) \in N_{\bar{T}} \) and for each \( i \in \mathbb{Z}, \bar{T}_{u_{i-1},u_{i+1}} \) be the lift of \( f \) by \( \pi \) such that \( \bar{T}(u_i) = u_{i+1} \) and define

\[ \bar{T}_{\bar{u}} = \begin{cases} \bar{T}_{u_{i-1},u_{i}} \circ \cdots \circ \bar{T}_{u_0,u_1} & \text{for } i > 0, \\
(\bar{T}_{u_{i-1},u_{i+1}})^{-1} \circ \cdots \circ (\bar{T}_{u_{-1},u_{0}})^{-1} & \text{for } i < 0, \\
nid & \text{for } i = 0. \end{cases} \]

We define a map \( \tau_{\bar{u}} = \tau_{\bar{u}}^f : N \to (N/\Gamma)_f \) by

\[ \tau_{\bar{u}}(x) = (\pi \circ \bar{T}_{\bar{u}})(x)_{\underbrace{\infty}_{i=-\infty}} \quad (x \in N). \]

Since \( \bar{T}(e) = e \), then \( \tau_e(e) = \tau_x(e) = (\pi(e))_{\underbrace{\infty}_{i=-\infty}}. \)

**Lemma 3.7** (\([2, \text{Lemma } 6.6.8 \ (1)]\)) If \( x \in X \) and \( \bar{u} \in N_{\bar{T}} \) then \( \pi(\overline{W}^u(x; \bar{u})) = W^u(\tau_{\bar{u}}(x)). \)
Let $X$ be a compact metric set and $f : X \to X$ a continuous surjection. A point $x \in X$ is said to be a nonwandering point if for any neighborhood $U$ of $x$ there is an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. The set $\Omega(f)$ of all nonwandering points is called the nonwandering set. Clearly $\Omega(f)$ is closed in $X$ and invariant under $f$.

$f$ is said to be topologically transitive (here $X$ may be not necessarily compact), if there is $x_0 \in X$ such that the orbit $O^+(x_0) = \{f^i(x_0) : i \in \mathbb{Z}^{\geq 0}\}$ is dense in $X$. It is easy to check that if $X$ is compact, a continuous surjection $f : X \to X$ is topologically transitive if and only if for any $U, V$ nonempty open sets there is $n > 0$ such that $f^n(U) \cap V \neq \emptyset$.

A continuous surjection $f : X \to X$ of a metric space is topologically mixing if for nonempty open sets $U, V$ there exists $N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$. Topological mixing implies topological transitivity.

For continuing, we need the next theorem for which proof one can see [2, Theorem 3.4.4].

**Theorem 3.8** (Topological decomposition theorem) Let $f : X \to X$ be a continuous surjection of a compact metric space. If $f : X \to X$ is a $TA$-map, then the following properties hold:

1. (Spectral decomposition theorem due to Smale) The nonwandering set, $\Omega(f)$, contains a finite sequence $B_i (1 \leq i \leq l)$ of $f$-invariant closed subsets such that
   - $\Omega(f) = \bigcup_{i=1}^{l} B_i$ (disjoint union),
   - $f|_{B_i} : B_i \to B_i$ is topologically transitive.

   Such the subsets $B_i$ are called basic sets.

2. (Decomposition theorem due to Bowen) For $B$ a basic set there exist $a > 0$ and a finite sequence $C_i (0 \leq i \leq a - 1)$ of closed subsets such that
   - $C_i \cap C_j = \emptyset (i \neq j)$, $f(C_i) = C_{i+1}$ and $f^a(C_i) = C_i$,
   - $B = \bigcup_{i=0}^{a-1} C_i$,
   - $f^a|_{C_i} : C_i \to C_i$ is topologically mixing,

Such the subsets $C_i$ are called elementary sets.

**Lemma 3.9** ([18, Lemma 5.4]) $\Omega(f) = N/\Gamma$.

**Lemma 3.10** $N/\Gamma$ is indeed an elementary set.

**Proof** By Lemma 2.11, let $\overline{f} : N \to N$ be the lift of $f$ such that $\overline{f}(e) = e$. By the commuting diagram:

$$
\begin{array}{ccc}
N & \xrightarrow{\overline{f}} & N \\
\downarrow{\pi} & & \downarrow{\pi} \\
N/\Gamma & \xrightarrow{f} & N/\Gamma
\end{array}
$$

we have

$$f([e]) = f(\pi(e)) = \pi(\overline{f}(e)) = \pi(e) = [e].$$

Therefore, $[e]$ is a fixed point of $f$.

By Lemma 3.9, $\Omega(f) = N/\Gamma$. As $N$ is connected and $\pi$ is a continuous surjection then $N/\Gamma$ is connected. In the proof of part (1) of spectral decomposition theorem, they prove that basic sets are close and open. Hence by connectedness of $\Omega(f) = N/\Gamma$, it consists of only one basic set, say $B$. On the other hand, by part (2) of spectral decomposition theorem, $N/\Gamma = B$ is the
union of elementary sets. There is an elementary set, say \( C \), such that \( [e] \subset C \). Since elementary sets are disjoint, by condition \( f(C_i) = C_{i+1} \), \( N/\Gamma = B \) consists of only one elementary set. \( \square \)

**Lemma 3.11** ([2, Remark 5.3.2 (2)]) Let \( f : X \to X \) be a TA-map of a compact metric space and let \( C \) be an elementary set of \( f \). If \( \bar{x} = (x_i) \in N_f \) and \( x_i \in C \) for all \( i \in \mathbb{Z} \) then \( W^u(\bar{x}) \cap C \) is dense in \( C \).

**Lemma 3.12** \( \bigcup_{\gamma \in \Gamma} W^u(\gamma; e) \) is dense in \( N \).

**Proof** By Lemmas 2.18 and 3.7 we have

\[
\bigcup_{\gamma \in \Gamma} W^u(\gamma; e) = \bigcup_{\gamma \in \Gamma} (W^u(e; e))_\gamma = \pi^{-1}(W^u(\tau(e))). \tag{3.12}
\]

We have \( \tau(e) = (\pi(e))_{\infty}^{\infty} \in (N/\Gamma)f \). On the other hand, Since by Lemma 3.10, \( \Omega(f) = N/\Gamma \) is an elementary set, say \( C \), and for \( (\pi(e))_{\infty}^{\infty} \) we have \( \pi(e) \in N/\Gamma = C \) for all \( i \in \mathbb{Z} \), by Lemma 3.11 we have

\[
W^u(\tau(e)) = W^u(\tau(e)) \cap (N/\Gamma) = W^u(\tau(e)) \cap C
\]

is dense in \( C = N/\Gamma \). By relation (3.12), we have the desired result. \( \square \)

By Proposition 3.6, \( \bar{h}^u \) is extended to a continuous map \( \bar{h} : N \to N \). From Proposition 3.3 (1), (2) and (3), and Lemma 2.14, we have \( \bar{h} = \tilde{h} \) and \( \bar{h}(\gamma) = \gamma \) for all \( \gamma \in \Gamma \).

**Proposition 3.13** For all \( \gamma \in \Gamma \) and \( x \in N \), \( \bar{h}(x\gamma) = \bar{h}(x)\gamma \).

**Proof** According to Lemma 2.15 (4), we have

\[
\bar{h}(x\gamma) \in \bar{L}^u(\bar{h}(x)\gamma). \tag{3.13}
\]

Suppose that \( x \in \bigcup_{\gamma \in \Gamma} W^u(\gamma; e) \). Then there is \( \gamma_x \in \Gamma \) such that \( x \in W^u(\gamma_x; e) \). For each \( \gamma \in \Gamma \) we have

\[
x\gamma \in W^u(\gamma_x; e) \gamma = W^u(\gamma_x\gamma; e).
\]

Thus

\[
\bar{h}(x\gamma) \in \overline{h(W^u(\gamma_x\gamma; e))} = \bar{L}^u(\bar{h}(\gamma_x\gamma)) \quad \text{(by Remark 2.17)} \nonumber
\]

\[= \bar{L}^u(\gamma_x\gamma). \tag{3.14}\]

On the other hand,

\[
\bar{h}(x)\gamma \in \overline{h(W^u(\gamma_x; e))}\gamma = (\overline{L}^u(\bar{h}(\gamma_x)))\gamma \quad \text{(by Remark 2.17)} \nonumber
\]

\[= \overline{L}^u(\gamma_x)\gamma = \overline{L}^u(\gamma_x\gamma) \quad \text{(by Lemma 2.19)}. \tag{3.15}\]

By (3.15), we have \( \overline{L}^u(\gamma_x\gamma) = \overline{L}^u(\bar{h}(x)\gamma) \). Therefore, by (3.14) we have

\[
\bar{h}(x\gamma) \in \overline{L}^u(\bar{h}(x)\gamma). \tag{3.16}
\]

By (3.13) and (3.16) we have

\[
\bar{h}(x\gamma) \in \overline{L}^u(\bar{h}(x)\gamma) \cap \overline{L}^u(\bar{h}(x)\gamma) = \{\bar{h}(x)\gamma\}. \tag{3.17}
\]
Thus for each \( x \in \bigcup_{\gamma \in \Gamma} W^u(\gamma; e) \) we have \( \overline{h}(x\gamma) = \overline{h}(x)\gamma \). Since \( \overline{h} \) is continuous and \( \bigcup_{\gamma \in \Gamma} W^u(\gamma; e) \) is dense in \( N \), we have the desired result. \( \square \)

**The End of Main Theorem’s Proof**  According to Proposition 3.13, \( \overline{h} \) induces a homeomorphism \( h : N/\Gamma \to N/\Gamma \) such that \( h \circ \pi = \pi \circ \overline{h} \), i.e., the following diagram commutes:

\[
\begin{array}{ccc}
N & \xrightarrow{\overline{h}} & N \\
\downarrow{\pi} & & \downarrow{\pi} \\
N/\Gamma & \xrightarrow{h} & N/\Gamma
\end{array}
\]

\( h \) is the conjugacy between \( f \) and \( A \). For if \( x \in N/\Gamma \) then there is \( y \in N \) such that \( x = \pi(y) \) and

\[
h \circ f(x) = h \circ f(\pi(y)) = h(f \circ \pi(y)) = h(\pi \circ \overline{f}(y)) = h(\pi \circ \overline{f}(y)) = h(\pi \circ \overline{f}(y)) = \pi(A \circ \overline{h}(y)) = \pi \circ \overline{f}(y) = A(\pi \circ \overline{f}(y)) = A(h \circ \pi(y)) = A \circ h(\pi(y)) = A \circ h(x).
\]

So the Main Theorem is proved. \( \square \)

**Proof of Corollary 1.11**  As mentioned in Section 2, every endomorphism of a compact metric space is a covering map. Every Anosov endomorphism is a TA-map (see [2, Theorem 1.2.1]). Every diffeomorphism is special (since it is injective). For every diffeomorphism or special expanding map of a nil-manifold, by (repaired for nil-manifolds) Theorem 1.9, it is conjugate to a hyperbolic nil-automorphism or an expanding nil-endomorphism, respectively, which are hyperbolic nil-endomorphisms. In Theorem 1.10, we prove the case that \( f \) is not injective or expanding. So in this case \( f \) is conjugate to a hyperbolic nil-endomorphism too. \( \square \)

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