OPEN CORE AND SMALL GROUPS IN DENSE PAIRS OF TOPOLOGICAL STRUCTURES

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Abstract. Dense pairs of geometric topological fields have tame open core, that is, every definable open subset in the pair is already definable in the reduct. We fix a minor gap in the published version of van den Dries’s seminal work on dense pairs of o-minimal groups, and show that every definable unary function in a dense pair of geometric topological fields agrees with a definable function in the reduct, off a small definable subset, that is, a definable set internal to the predicate.

For certain dense pairs of geometric topological fields without the independence property, whenever the underlying set of a definable group is contained in the dense-codense predicate, the group law is locally definable in the reduct as a geometric topological field. If the reduct has elimination of imaginaries, we extend this result, up to interdefinability, to all groups internal to the predicate.

Introduction

Tame topological structures and expansions by a predicate have often been considered from a model-theoretical point of view. Both $p$-adically and real closed fields are naturally endowed with a definable topology such that the field operations are continuous. The close interaction between the topological and algebraic properties of such structures is crucial to determine their model-theoretic behaviour. Several frameworks have been suggested to treat simultaneously archimedean and non-archimedean normed fields: in this paper, we will follow the topological approach proposed by Mathews [16], which was later on adopted by Berenstein, Dolich and Oshuus [2] to study the theory of dense pairs.

Robinson [21] showed that the theory of a real closed field $M$ equipped with a dense proper real closed subfield $P(M)$ is complete. Subsequently Macintyre [14] proved the same result for dense pairs of $p$-adically closed fields. The model-theoretical properties of dense pairs of o-minimal expansions of ordered abelian groups were thoroughly studied by van den Dries [7], who gave an explicit description of definable unary sets and functions, up to small sets. A set is small in a pair $(M,P(M))$ if it is contained in the image of the $P(M)$-points by a semialgebraic function. Though the theory of the pair is no longer o-minimal, every definable open set in the pair is actually definable in the reduct of $M$ as an ordered field, so the theory of the pair has o-minimal open core [7, 17, 18]. A similar result on the open core of pairs of $p$-adically and real closed fields has been recently obtained by

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Point [20] (see as well as work of Tressl [25]) as a by-product of the study of the theory of differentially closed topological fields. In Section 1, we will use a criterium (cf. Proposition 1.14) of geometric nature in order to provide a new proof of the following result:

**Theorem A.** Let \((M, E)\) be a dense pair of models of a geometric theory \(T\) of topological rings. Assume that every \(\mathcal{L}^P\)-definably closed set \(A\) is special, that is,

\[
\dim(a_1, \ldots, a_n/E) = \dim(a_1, \ldots, a_n/A \cap E) \quad \text{for all } a_1, \ldots, a_n \in A,
\]

where \(\dim\) is the dimension as a geometric structure.

Every open \(\mathcal{L}^P\)-definable subset of \(M^n\) over a special set \(A\) is \(\mathcal{L}^A\)-definable. Hence, the open core of the theory \(T^P\) is tame, for it is definable in the reduct \(T\).

After private communication with van den Dries, we provide in the Appendix a fix for a minor gap in the published version of [6, Corollary 3.4], and show, in the broader context of geometric topological structures, that every definable unary function in the pair agrees off a small subset with a function definable in the predicate. Note that the same gap also affects [2, Theorem 4.9].

Understanding the nature of groups definable in a specific structure is a recurrent topic in model theory. In [12], it was shown that a group definable in a \(p\)-adically or real closed field \(M\) is locally isomorphic to the connected component of the \(M\)-points of an algebraic group defined over \(M\). In particular, the group law is locally given by an algebraic map. Furthermore, if the group is Nash affine in a real closed field, this local isomorphism extends to a global isomorphism [13], since the Nash topology on Nash affine groups is noetherian. A crucial aspect to define the local isomorphism and the corresponding algebraic group through a group configuration diagram [11] is the strong interaction between the semialgebraic dimension in \(M\) and the transcendence degree, computed in the field algebraic closure of \(M\). This interaction will be captured in Definition 3.1 in an abstract set-up.

A first description of groups definable in dense pairs \((M, P(M))\) of real closed fields appears in [10]: the group law is locally semialgebraic (off a wider class of certain sets, encompassing small sets). However, if the group is small, particularly if the underlying set is a subset of some cartesian product of \(P(M)\), the above description does not provide any relevant information. In this note, we tackle the remaining case and consider groups whose underlying set is a subset of some cartesian product of \(P(M)\), following the proof of Hrushovski and Pillay [12]. A recent result of Eleftheriou [9] on elimination of imaginaries on \(P(M)\) with the induced structure from the pair (see Section 4) allows us to extend our description to all small groups, whenever the topological geometric field has elimination of imaginaries.

The first obstacle we encountered is the lack of a sensible notion of dimension in the pairs. In an arbitrary pair \((M, P(M))\) of topological structures, there is a rudimentary notion of dimension, given by the small closure: the union of all small definable sets. Since we are only interested in small sets, the small closure is not useful for our purposes. In Section 3, we attach a dimension to definable subsets of \(P(M)^k\) in terms of their honest definition, as introduced in [3], when the topological geometric structure does not have the independence property. Using this dimension, we produce in Section 3 a suitable group configuration diagram and show the following:
Theorem B. Consider a dense pair \((M, E)\) of models of a geometric NIP theory \(T\) of topological rings, with \(E = P(M)\), in the language \(\mathcal{L}^P = \mathcal{L} \cup \{P\}\). Assume that \(T\) is stably controlled with respect to a strongly minimal \(\mathcal{L}^0\)-theory \(T^0\) (see Definition 3.1). If \((G, \ast)\) is an \(\mathcal{L}^P\)-definable group over a special set \(D \subseteq M\) such that \(G \subseteq E^k\) for some \(k\), then the group law is locally \(\mathcal{L}^0\)-definable.

Furthermore, if \(T\) has elimination of imaginaries, then the same holds for all small groups \((G, \ast)\).

In case of the reals or the \(p\)-adics, we conclude that the group law is locally an algebraic group law. We do not know whether the group law is globally semialgebraic for small groups in dense pairs of \(p\)-adically or real closed fields.

1. Dense pairs of topological fields

In this section, we will recall the basic results on topological geometric structures.

Definition 1.1. A structure \(M\) in the language \(\mathcal{L}\) is a topological structure if there is a formula \(\theta(x, \bar{y})\), where \(x\) is a single variable, such that the collection \(\{\theta(M, \bar{a})\}_{\bar{a} \in M^n}\) forms a basis of a proper topology, that is, a \(T_1\) topology with no isolated points.

Remark 1.2. If \(M\) is a topological structure, then so is every model of the theory of \(M\).

Since the formula \(\theta(x; \bar{y})\) has the order property, the theory of a topological structure as above is always unstable [19, Proposition 1.2].

Definition 1.3. A cell in a topological structure \((M, \theta)\) is a definable set \(X \subseteq M^n\) such that \(\pi(X)\) is open and homeomorphic to \(X\), for some projection \(\pi: M^n \to M^m\) with respect to a given choice of coordinates. A topological structure \((M, \theta)\) has cell decomposition if, whenever \(A \subseteq M\), the domain \(X\) of every definable function \(F\) over \(A\) can be partitioned into finitely many cells \(X_1, \ldots, X_m\), each definable over \(A\), such that \(F\mid_{X_i}\) is continuous for every \(1 \leq i \leq m\).

Taking as a definable function the characteristic function of a definable set, it follows that every definable set can be partitioned into finitely many cells. However, since for the union of cells may itself be a cell, note that the above cell decomposition is weaker than the corresponding versions for real or \(p\)-adically closed fields.

The following definition was introduced in [2] in order to generalize the results of [3] from the o-minimal context to the \(p\)-adically closed one. Notice that the last item in the definition below is slightly stronger than the corresponding one in [3].

Definition 1.4. ([3, Definition 4.6] & [22, Definition 6.1])

A complete theory \(T\) in a language \(\mathcal{L}\) is a geometric theory of topological rings if all of the following conditions hold:

- The theory \(T\) eliminates \(\exists^\infty\).
- Algebraic closure satisfies the exchange principle in every model \(M\) of \(T\), that is, given a subset \(A\) and elements \(b\) and \(c\) of \(M\) such that \(c\) is in \(\text{acl}(A, b) \setminus \text{acl}(A)\), then \(b\) is algebraic over \(A \cup \{c\}\).
- The theory \(T\) has Skolem functions (so definable and algebraic closure agree).
- Every model \(M\) is a topological structure, witnessed by the formula \(\theta(x, \bar{y})\), with cell decomposition.
• Every definable unary subset of a model of $T$ is either finite or has non-empty interior.
• The language $\mathcal{L}$ contains the ring symbols and $M$ is an integral domain, with the natural interpretations, such that the ring operations are continuous.
• We have the following density conditions (cf. \cite{22, Definition 6.1}): for every $\theta(M, \bar{a})$ and every element $b$ in $\theta(M, \bar{a})$, there exists an open neighbourhood $V$ of $\bar{a}$ (in $M|\bar{a}$) and a tuple $\bar{c}$ such that for all $\bar{d}$ in $V$,
  \[ b \in \theta(M, \bar{c}) \subseteq \theta(M, \bar{d}). \]
  Furthermore, the set of $\bar{a}'$ such that
  \[ b \in \theta(M, \bar{a}') \subseteq \theta(M, \bar{a}) \]
  has non-empty interior.

In particular, we will say that two subsets $A$ and $B$ of $M$ are independent over a common subset $C \subseteq A \cap B$, denoted by
  \[ A \mid_C B, \]
if the geometric dimension
  \[ \dim(a_1, \ldots, a_n|C) = \dim(a_1, \ldots, a_n|B) \]
for any $a_1, \ldots, a_n$ in $A$. It follows from \cite{16, Theorem 9.5} that the above dimension coincides with the topological dimension, which is definable \cite{14, Corollary}.

**Example.** Real and $p$-adically closed fields, as well as real closed rings (in the sense of Cherlin and Dickmann) are geometric topological rings in the ring language \cite{16, 12}.

Recall that a dense pair of models of a geometric theory $T$ topological rings is a model $M$ of $T$ together with a proper elementary substructure $E \preceq M$ which is dense in $M$ with respect to the definable topology.

**Notation.** Given a language $\mathcal{L}$ and a geometric $\mathcal{L}$-theory $T$ of topological rings, we denote from now on by $T^P$ the theory of dense pairs $(M, E)$ of models of $T$ in the language $\mathcal{L}^P = \mathcal{L} \cup \{P\}$, where $P$ is a unary symbol whose interpretation is the proper submodel $E$. Henceforth, the symbol $P$ will be used to refer to the theory $T^P$: in particular, the definable closure in the language $\mathcal{L}^P$ will be denoted by $dcl^P$, whereas $dcl$ denotes the definable closure in $\mathcal{L}$. Given a subset $A \subseteq M$ of parameters, by $\mathcal{L}_A$-definable, resp. $\mathcal{L}^P_A$-definable, we mean $\mathcal{L}$-definable, resp. $\mathcal{L}^P_A$-definable, over $A$.

Let us recall the description of definable sets in dense pairs given by van den Dries via the notion of a small set \cite{6}. There is a natural notion of smallness and largeness for arbitrary pairs, as introduced by Casanovas and Ziegler \cite{8} in terms of saturated extensions. Their definition agrees with van den Dries’s original definition when the underlying theory is strongly minimal (cf. \cite{8, Section 2}). We will use the latter one, which resonates with the notion of internality to the predicate from geometric stability theory.

**Definition 1.5.** A set $X \subseteq M^n$ is small in the pair $(M, E)$ over $A$ if there is an $\mathcal{L}_A$-definable function $h : M^m \to M^n$ such that $X \subseteq h(E^m)$. 
In order to track the parameters necessary to describe definable sets in such dense pairs, we introduce the following definition.

**Definition 1.6.** A subset \( A \) of \( M \) is special if \( A \downarrow_{E \cap A} E \).

The following trivial observation will be used all throughout this work.

**Remark 1.7.** Subsets of \( E \) are special. Whenever \( B \) is a subset of \( E \) and \( A \) is special, so is \( dcl(A \cup B) \), with \( E \cap dcl(A \cup B) = dcl(B, E \cap A) \).

Though neither van den Dries nor Berenstein-Dolich-Onshuu refer to special sets when studying the theory \( T_P \) in [7, 2], the next results follow immediately by appropriately choosing the back-and-forth system:

**Fact 1.8.** (see [7, 2]) Let \( T \) be a geometric theory of topological rings.

1. The theory \( T^P \) of dense pairs is complete.
2. The type of a special set is uniquely determined by its quantifier-free \( L_P \)-type.
3. Any \( L^P \)-definable subset \( Y \) of \( E^n \) over a special set \( A \) is the trace of an \( L \)-definable subset \( Z \subseteq M^n \) over \( A \):
   \[ Z \cap E^n = Y. \]
4. The \( L^P \)-definable closure of a special set \( A \) agrees with its \( L \)-definable closure, and \( E \cap dcl^P(A) = dcl(E \cap A) \).
5. Every \( L^P \)-definable function \( f : M \to M \) over a special set \( A \) agrees with an \( L_A \)-definable function off some small set (cf. Appendix 3).
6. For every \( L^P \)-definable unary set \( X \subseteq M \) over a special set \( A \), there are pairwise disjoint \( L_A \)-definable open sets \( U, V \) and \( W \) of \( M \) such that:
   - \( M \setminus (U \cup V \cup W) \) is finite.
   - \( U \subseteq X \) and \( X \cap V = \emptyset \).
   - \( X \cap W \) is both dense and codense in \( W \).
   Thus, if \( X \subseteq M \) is small and \( L^P \)-definable, then the corresponding open set \( U \) is empty.

**Remark 1.9.** To prove the above fact in the context of geometric theories [3], it is crucial that a number of results in [2] also hold in this context. For example, we also have the following two relevant properties:

1. If \( (M, E) \) is a saturated model of \( T^P \), notice that \( M \) is not \( L^P \)-saturated over the set of parameters \( E \), but the rank of \( M \) over \( E \) (with respect to the pregeometry in \( T \)) is at least \( \kappa \) (see [3, Lemma 1.5]).
2. Every small and \( L \)-definable unary subset of \( M \) is finite: The set \( M \setminus E \) is codense in \( M \) (see the Claim in [2, Thm. 4.9]) and therefore no open \( L \)-definable subset of \( M \) can be small [7, Lemma 4.1]. We finish by applying Fact 1.8(6).

If \( X \subseteq M \) is open and \( L^P_A \)-definable, then \( X \) is \( L_A \)-definable, by Fact 1.8(6), for the corresponding set \( W \) is empty. The above characterization of 1-dimensional \( L^P \)-definable open sets as \( L \)-definable sets has been shown for arbitrary finite cartesian products of the universe in various settings [3, 14, 13, 32, 24, 25], archimedean and non-archimedean. For the sake of completeness, we will provide a different proof of geometric nature (cf. Theorem 1.16), which is an immediate consequence of
a general criterion (cf. Proposition 1.14) to determine whether an \( L^P \)-definable correspondence is \( L \)-definable. We first need an auxiliary result.

Henceforth, we work inside a sufficiently saturated dense pair of models \( (M,E) \) of a geometric theory \( T \) of topological rings.

Lemma 1.10. Given an \( L^P \)-definable unary function \( h \) with domain an infinite definable subset \( X \) of \( M \) over a special subset \( A \), there exist a special superset \( D \supseteq A \) and an \( L_0 \)-definable curve \( \alpha: \theta(M,\bar{b}) \to M^2 \) such that the intersection of \( \text{Im}(\alpha) \) with the graph of \( h \) is infinite.

Proof: If \( X \) is not small, Fact 1.8 yields the desired result. Thus, suppose that \( X \) is small, so \( X \subseteq f(E^\ell) \), for some \( L_A \)-definable map \( f \). By choosing a suitable cell decomposition of \( M^\ell \), we can assume that \( X = f(E^\ell \cap C) \), where \( C \) is an \( L_A \)-definable open cell \( C \) of \( M^\ell \) and \( f|_C \) is continuous.

Consider the \( L_A \)-definable open subset
\[
C_0 = \{ x \in C : f \text{ is locally constant at } x \}
\]
of \( C \). For each \( x \) in \( C_0 \), the fiber \( f^{-1}(f(x)) \) is \( \ell \)-dimensional. An easy dimension computation yields that \( f(C_0) \) must be finite.

Since \( X \) is infinite, the set \( C \setminus C_0 \) is infinite as well. If the interior of \( C \setminus C_0 \) happens to be trivial, a suitable cell decomposition yields an \( L_A \)-definable subset \( W \) of \( C \setminus C_0 \) and a projection \( \pi: M^\ell \to M^r \) for some choice of \( r \) coordinates, with \( r < \ell \) such that the restriction of \( \pi \) to \( W \) produces a homeomorphism between \( W \) and its open image \( \pi(W) \subseteq M^r \). Since \( f((C \setminus C_0) \cap E^\ell) \) is infinite, we may assume that \( f(W \cap E^\ell) \) is infinite as well, and proceed by induction on \( r \).

Therefore, we need only consider the case when \( C \setminus C_0 \) has non-empty interior. Renaming its interior as \( C \) again, we may assume that \( f \) is nowhere locally constant on \( C \).

Let now \( e \) be in \( C \cap E^\ell \). By the density condition in Definition 1.4, there is a basic neighbourhood \( U \) defined over some \( D_0 \subseteq E \) such that \( e \in U \). In particular, the open set \( U \cap C \) is defined over the set \( D = A \cup D_0 \cup \{ e \} \), which is again special, by the Remark 1.7.

We will show that there is a line \( J \) defined over \( E \) such that \( f(J \cap U \cap E^\ell) \) is infinite. Choose some basic neighbourhood \( \mathcal{V} \) of \( \emptyset \) in \( M^\ell \). Again, we may enlarge \( D \) to assume that \( \mathcal{V} \) is defined over \( D \). For each \( v \) in \( \mathcal{V} \), let \( J_v \) be the line
\[
J_v = \{ e + t \cdot v \}_{t \in M}
\]
and set
\[
\mathcal{V} = \{ v \in \mathcal{V} : f(J_v \cap U) \text{ is finite} \}.
\]
If \( \mathcal{V} \cap E^\ell \subseteq V \), then choose some \( v \) in \( \mathcal{V} \cap E^\ell \) of dimension \( \ell \) over \( D \). The ring operations are continuous, so there is a basic neighbourhood \( \theta(M,\bar{a}_0) \) of \( \emptyset \) such that for all \( t \) in \( \theta(M,\bar{a}_0) \), we have that \( e + t \cdot v \) is contained in \( U \). Again, we may assume that \( \bar{a}_0 \) lies in \( D \), by the the density condition in Definition 1.4.

The set \( f(J_v \cap U) \) is finite and contains \( f(e) \). Since the function \( f \) is continuous, the preimage \( f^{-1}(f(e)) \cap J_v \cap U \) is a non-empty open \( L \)-definable subset of \( J_v \cap U \). Thus, so is the \( L \)-definable subset of \( M \)
\[
\mathcal{E} = \{ t \in \theta(M,\bar{a}_0) : f(e + t \cdot v) = f(e) \}.
\]
Pick some $t$ in $T$ generic over $D \cup \{v\}$. Since
$$f(e + t \cdot v) = f(e),$$
we have that
$$\ell + 1 = \dim(v, t/D) = \dim(e + t \cdot v, t/D) = \dim(e + t \cdot v/D) + \dim(t/D, e + t \cdot v) = \dim(e + t \cdot v/D) + 1.$$ 

As the element $e + t \cdot v$ lies in the $L_D$-definable subset $f^{-1}(f(e))$ of $M^\ell$, the set $f^{-1}(f(e))$ is $\ell$-dimensional, so it contains an open subset of $M^\ell$, which contradicts our assumption that $f$ was nowhere locally constant on $C$.

Hence, there is some $v$ in $V \cap E^\ell$ which does not lie in $V$, that is, the set $f(J, \cap \mathcal{U})$ is infinite. Projecting onto a suitable coordinate, we find a basic open neighbourhood $\theta(M, b)$ defined over $E$ and an $L_E$-definable function $g : \theta(M, b) \to M^\ell$ such that $g(\theta(M, b) \cap E) \subseteq C$ and $f(g(\theta(M, b) \cap E))$ is infinite. As before, we may assume that $g$ is $L_D$-definable, up to enlarging $D$.

For every element $x$ in $\theta(M, b) \cap E$, the set $\text{dcl}(D, x)$ is special and $L_D$-definably closed, by the Remark 1.11. Note that it contains $h(x) = h \circ f \circ g(x)$. By saturation, there are finitely many $L_D$-definable functions $h_1, \ldots, h_n : \theta(M, b) \to M$ such that, whenever $x$ lies in $\theta(M, b) \cap E$, then $h(x) = h_i(x)$ for some $1 \leq i \leq n$. Without loss of generality, the function $h$ coincides with $h_1$ infinitely often on $\theta(M, b) \cap E$. Set now
$$\alpha : \theta(M, b) \to M^2$$
$$t \mapsto (f \circ g(t), h_1(t))$$

By construction, the graph of the curve $\alpha$ has an infinite intersection with the image of $h$. 

\begin{remark}
In the proof of the previous lemma, it was only used that $L$ contains the language of rings in order to define the line $J$. We do not know whether there is a more general set-up in which Lemma 1.10 holds.
\end{remark}

\begin{definition}
A geometric theory $T$ of topological rings is \emph{tame for pairs} if every $L_D$-definably closed subset in a dense pair $(M, E)$ of models of $T$ is special.
\end{definition}

\begin{example}
Real and $p$-adically closed fields are tame for pairs (it will be shown in Lemma 3.3).
\end{example}

\begin{question}
Is the theory of real closed rings tame for pairs?
\end{question}

For the rest of this section, we will assume that the geometric theory $T$ of topological rings is tame for pairs.

\begin{definition}
A correspondence $R \subseteq M^n \times M$ is a \emph{multi-function} if the collection $R(\bar{x}) = \{y \in M \mid (\bar{x}, y) \in R\}$ is finite. We will use the notation $R : M^n \to M$ to denote that $R$ is a multi-function.
\end{definition}

\begin{proposition}
Let $R : M^n \to M$ be an $L_D$-definable multi-function over a special subset $A$. The multi-function $R$ is $L_A$-definable if and only if the multi-function $R \circ \alpha$ is $L_D$-definable, whenever $D \supseteq A$ is special and $\alpha$ is an $L_D$-definable curve with domain an open subset of $M$.
\end{proposition}
Proof: One implication is trivial, so we need only show that $R$ is $\mathcal{L}_A$-definable, assuming it satisfies the right-hand condition. We prove it by induction on $n$. For $n = 1$, the curve $\alpha(t) = t$ is clearly $\mathcal{L}_A$-definable, and thus so is $R = R \circ \alpha$ by hypothesis.

Assume thus that $n > 1$ and the statement holds for all $k < n$. Given $b$ in $M$, consider the multi-function

$$R(\cdot, b): M^{n-1} \rightarrow M$$

which is clearly $\mathcal{L}_{\text{def}}^P(A, b)$-definable. Since it satisfies the same condition as $R$, we deduce that $R(\cdot, b)$ is $\mathcal{L}_{\text{def}}^P(A, b)$-definable, by induction on $n$ (Note that the assumption that the theory is tame for pairs yields that $\mathcal{L}_P$-definably closed sets are special). Compactness yields an $\mathcal{L}_A$-definable multi-function

$$\overline{R}: M^{n-1+\ell} \rightarrow M,$$

and an $\mathcal{L}_A$-definable map

$$g = (g_1, \ldots, g_\ell): M \rightarrow M^\ell$$

such that $R(x, b) = \overline{R}(x, g(b))$ (as sets) for all $(x, b)$ in $M^n$. Since the union of small sets is again small, there is an $\mathcal{L}_A$-definable map $\overline{g}: M \rightarrow M^\ell$ such that $g$ and $\overline{g}$ agree off a small $\mathcal{L}_A$-definable subset of $M$, by Fact \ref{fact:compactness}. Consider the following $\mathcal{L}_A$-definable multi-function:

$$R_0(x, b) = \overline{R}(x, \overline{g}(b)).$$

As before, for each $x$ in $M^{n-1}$, the multi-function

$$R_x: M \rightarrow M$$

$$b \mapsto R(x, b)$$

is $\mathcal{L}_{\text{def}}^P(A, x)$-definable. Hence the set

$$Y_x := \{b \in M : R_x(b) \neq R_0(x, b) \text{ (as sets)}\}$$

is $\mathcal{L}_{\text{def}}^P(A, x)$-definable. Since $Y_x$ is small for every $x$ in $M^{n-1}$, it must be finite, by the Remark \ref{remark:finite}. Compactness yields that there is some natural number $k$ such that $\text{card}(Y_x) \leq k$, for every $x$ in $M^{n-1}$.

By compactness, there is an $\mathcal{L}_A$-definable function $\gamma: M^{n-1} \rightarrow M^\ell$ and a formula $\varphi(y, w)$ such that $Y_x = \varphi(M, \gamma(x))$ for every $x$ in $M^{n-1}$. Consider now the $\mathcal{L}_A$-definable set

$$X = \{a \in M^{n-1} : Y_a \neq \emptyset\}.$$

The (finite) Skolem functions for $\varphi$ produce finitely many $\mathcal{L}_A$-definable functions $h_1, \ldots, h_k: M^{n-1} \rightarrow M$ such that $Y_x = \{h_1(x), \ldots, h_k(x)\}$ (possibly with repetitions) for every $x$ in $X$. If $Y_x$ has cardinality $\ell \leq k$, then note that $Y_x := \{h_1(x), \ldots, h_\ell(x)\}$ and $h_j(x) = h_\ell(x)$ for all $j \geq \ell$.

If $k = 0$, then we are done. Otherwise, let $\ell \leq k$ be the least cardinal of a non-empty fiber $Y_x$, for some $x$ in $X$. We will first show that

$$X_\ell := \{x \in M^{n-1} : \text{card}(Y_x) = \ell\}$$

is $\mathcal{L}_A$-definable. Consider the characteristic function $1_{X_\ell}$ of $X_\ell$ with domain $M^{n-1}$. Since functions are multi-functions, we need only show, by induction, that $1_X \circ \beta$
is $\mathcal{L}_D$-definable, whenever we take an $\mathcal{L}_D$-definable curve $\beta : I \to M$ with domain an open subset $I$ of $M$, where $D$ is special and $D \supseteq A$.

By Fact \ref{fact:open}, there are $\mathcal{L}_D$-definable open sets $U$, $V$ and $W$ of $M$ such that $M \setminus (U \cup V \cup W)$ is finite, the set $U$ is contained in

$$Z = \{ t \in I \mid 1_{X_t} \circ \beta(t) = 1 \},$$

the set $V$ is disjoint from $Z$ and $Z \cap W$ is dense and codense in $W$. It suffices to show that $W$ is empty. For each $i \leq k$, Fact \ref{fact:open} yields that there is an $\mathcal{L}_D$-definable function $\gamma_i : I \to M^{n-1}$ which agrees with $h_i \circ \beta$ off a small set $S_i$.

Note that the set

$$\{ t \in I : R(\beta(t), \gamma_i(t)) = R_0(\beta(t), \gamma_i(t)) \}$$

is $\mathcal{L}_D$-definable and contained in the small subset $S_i$, so it must be finite, by the Remark \ref{remark:finite}. Without loss of generality, we may assume that $\gamma_i(t)$ belongs to $Y_{\beta(t)}$.

Likewise, the $\mathcal{L}_D$-definable set

$$\{ t \in I \mid \#\{ \gamma_1(t), \ldots, \gamma_\ell(t) \} < \ell \}$$

is finite, for it is contained in the small set $S_1 \cup \cdots \cup S_\ell$. Thus, we may assume that $\#\{ \gamma_1(t), \ldots, \gamma_\ell(t) \} = \ell$, for all $t$ in $I$.

Since $W \setminus Z$ is dense in $W$, there is some $1 \leq j \leq k$ such that the set

$$Z_j = \{ t \in I \mid h_j(t) \notin \{ \gamma_1(t), \ldots, \gamma_\ell(t) \} \}$$

is infinite (note that it could be $j = 1$). Lemma \ref{lemma:finite} yields a special set $D_1 \supseteq D$ and an $\mathcal{L}_{D_1}$-definable curve $\alpha = (\alpha_1, \alpha_2)$ such that the intersection of $\text{Im}(\alpha)$ with the graph of $(h_j \circ \beta)|_{Z_j}$ is infinite. Consider the $\mathcal{L}_{D_1}$-definable set

$$\{ t \in W \mid t = \alpha_1(u) \text{ for some } u \in \text{Dom}(\alpha) \& \alpha_2(u) \notin \{ \gamma_1(t), \ldots, \gamma_\ell(t) \} \}$$

$$\& R(\beta \circ \alpha_1(u), \alpha_2(u)) \neq R_0(\beta \circ \alpha_1(u), \alpha_2(u)), $$

which is clearly infinite because it contains the projection on the first coordinate of the intersection of $\text{Im}(\alpha)$ with the graph of $(h_j \circ \beta)|_{Z_j}$. Thus, it must have non-empty interior, so find an $\mathcal{L}_{D_1}$-definable open subset of $W$ contained in $W \setminus Z$, which contradicts the fact that $Z$ is dense in $W$. Hence, the set $X_\ell$ is $\mathcal{L}_A$-definable, as desired.

Now, consider the $\mathcal{L}_A^\ell$-definable multi-function

$$H_\ell : X_\ell \to M : x \mapsto \{ h_1(x), \ldots, h_\ell(x) \} = \{ h_1(x), \ldots, h_\ell(x) \},$$

and let us show that $H_\ell$ is $\mathcal{L}_A$-definable, by induction on $n$. Let $\beta : I \to X_\ell$ be an $\mathcal{L}_D$-definable curve, with $D \supseteq A$ special. For each $i \leq k$, there is an $\mathcal{L}_D$-definable function $\gamma_i : I \to M^{n-1}$ which agrees with $h_i \circ \beta$ off a small set $S_i$.

As before, for all $t \in I$ and $i = 1, \ldots, \ell$ we can assume that $\gamma_i(t) \in Y_{\beta(t)}$ and $\text{card}\{ \gamma_1(t), \ldots, \gamma_\ell(t) \} = \ell$. In particular,

$$H_\ell \circ \beta(t) = \{ \gamma_1(t), \ldots, \gamma_\ell(t) \}$$

is $\mathcal{L}_D$-definable. Thus, we deduce that $H_\ell$ is $\mathcal{L}_A$-definable, as required.

By (finite) Skolem functions for the graph of $H_\ell$, there are $\mathcal{L}_A$-definable functions $\tilde{h}_1, \ldots, \tilde{h}_\ell : X_\ell \to M$ such that $H_\ell(x) = \{ \tilde{h}_1(x), \ldots, \tilde{h}_\ell(x) \}$ for all $x \in X_\ell$. In particular, for each $1 \leq j \leq \ell$, the multi-function $R(x, \tilde{h}_j(x))$ with domain $X_\ell$ is $\mathcal{L}_A$-definable: Indeed, given an $\mathcal{L}_D$-definable curve $\beta : I \to X_\ell$ over a special set $D \supseteq A$, the curve $x \mapsto (\beta(x), \tilde{h}_j(\beta(x)))$ is $\mathcal{L}_D$-definable. By hypothesis, so
is the multi-function $x \mapsto R(\beta(x), \tilde{h}_j(\beta(x)))$. By induction on $n$, we deduce that $x \mapsto R(x, \tilde{h}_j(x))$ is $\mathcal{L}_A$-definable, as desired.

Using the following multi-function $\tilde{R}$ defined as

$$
\tilde{R} : M^n \to M
\quad (x, y) \mapsto \begin{cases} 
R(x, \tilde{h}_j(x)), & \text{if } x \in X_\ell \text{ and } y = \tilde{h}_j(x), \text{ for some } 1 \le j \le k, \\
R_0(x, y), & \text{otherwise.}
\end{cases}
$$

iterating the previous argument, we find a multi-function $\tilde{R}_0 : M^n \to M$ which is $\mathcal{L}$-definable over $A$ such that the corresponding set

$$
\tilde{Y}_x := \{ b \in M : R(x, b) \neq \tilde{R}_0(x, b) \}
$$

is again $\mathcal{L}_{\text{def}}(A, x)$-definable and finite for each $x$ in $M^{n-1}$. Moreover, we know that $\text{card}(\tilde{Y}_x) \le k$ for every $x \in M^{n-1}$. On the other hand, the least cardinality $\ell$ of a non-empty fiber $\tilde{Y}_x$ for $x$ in $M^{n-1}$ is strictly greater than $\ell$. This process must therefore terminate after finitely many iterations, so the multi-function $R$ is $\mathcal{L}_A$-definable, as required. \hfill\Box

We obtain an immediate consequence of the previous result, considering characteristic functions of $\mathcal{L}_P$-definable sets.

**Corollary 1.15.** An $\mathcal{L}_P$-definable subset $X$ of $M^n$ over a special set $A$ is $\mathcal{L}_A$-definable if and only if the set $\{ t \in I : \alpha(t) \in X \}$ is $\mathcal{L}_D$-definable, whenever $\alpha : I \to M^n$ is an $\mathcal{L}$-definable curve over a special set $D \supseteq A$.

In particular, we conclude that the open core of the theory $T_P$ is tame, for it is definable in the reduct $T$ (cf. [1, 3, 20, 25]).

**Theorem 1.16.** Let $(M, E)$ be a dense pair of models of a geometric theory $T$ of topological rings such that $T$ is dense for pairs, that is, every $\mathcal{L}_P$-definably closed set is special. Every open $\mathcal{L}_P$-definable subset of $M^n$ over a special set $A$ is $\mathcal{L}_A$-definable. In particular, the topological closure $\text{cl}_M(X)$ of an $\mathcal{L}_A$-definable set $X$ of $M^n$ is $\mathcal{L}_A$-definable.

**Proof.** Let $U$ be an open subset of $M^n$ which is $\mathcal{L}_P$-definable over a special set $A$. By Corollary 1.15, it suffices to show that, given an $\mathcal{L}$-definable curve $\alpha$ over a special set $D \supseteq A$, the set $\{ t \in \text{Dom}(\alpha) \mid \alpha(t) \in U \}$ is $\mathcal{L}_D$-definable. Without loss of generality, we may assume that $\alpha$ is continuous. In particular, the preimage

$$
\alpha^{-1}(U) = \{ t \in \text{Dom}(\alpha) \mid \alpha(t) \in U \}
$$

is an open $\mathcal{L}_D^P$-definable subset of $M$, and thus $\mathcal{L}_D$-definable, by Fact 1.8. \hfill\Box

2. Honesty and Dimension

By Fact 1.8, any $\mathcal{L}_P$-definable subset $Y$ of $E^n$ is externally $\mathcal{L}$-definable: there is some $\mathcal{L}$-definable subset $Z$ in $M^n$, possibly with parameters from $M$, such that $Y = Z \cap M$. The possible external definitions for a given subset may be very different, as the following example shows:

**Example.** In the dense pair $(\mathbb{R}, \mathbb{R}_{\text{alg}})$, consider the $\mathcal{L}_P$-definable set

$$
Y = \{(x, y) \in \mathbb{R}^2_{\text{alg}} : x = y = 0 \}.
$$
Set \( Z = \{(x, y) \in \mathbb{R}^2 : x = y = 0\} \), which has dimension 0. Clearly \( Y = Z \cap \mathbb{R}_{\text{alg}}^2 \).

However, for \( Z_1 = \{(x, y) \in \mathbb{R}^2 : x = e \cdot y\} \), which has dimension 1, we still have that \( Y = Z_1 \cap \mathbb{R}_{\text{alg}}^2 \).

In a stable theory, externally definable subsets of any predicate are actually definable using parameters from the predicate, since \( \varphi \)-types are definable. For a general NIP theory, we can find external definitions using parameters in some elementary extension of the pair \( \mathbb{E} \).

**Fact 2.1.** [4] Corollary 4.10] If the geometric theory \( T \) of topological rings has NIP, then so does the theory \( T^P \) of dense pairs.

Henceforth, we work inside a sufficiently saturated dense pair of models \((M, E)\) of a geometric NIP theory \( T \) of topological rings.

**Fact 2.2.** [4] Theorem 3.13 & Proposition 3.23] Let \( M \) be a model of \( T^P \) with \( E = P(M) \), and \( \varphi(E, b) \subseteq E^n \) be an externally definable set, with \( \varphi(x, y) \) an \( \mathcal{L} \)-formula and \( b \) in \( M \). In some elementary extension \( M \preceq^P N \), there is an honest definition \( \chi(x) \) of \( \varphi(E, b) \), that is, an \( \mathcal{L} \)-formula \( \chi(x) \) with parameters in \( P(N) \) such that

\[
\varphi(E, b) \subseteq \chi(P(N)) \subseteq \varphi(P(N), b).
\]

We say that the set \( \chi(P(N)) \) is an honest definition of \( \varphi(E, b) \).

**Remark 2.3.** Notice that \( \chi(E) = \varphi(E, b) \), so \( \chi(x) \) is indeed an external definition. Moreover,

\[
\chi(P(N)) \subseteq \psi(P(N)), \quad \text{for any } \mathcal{L}_E\text{-formula } \psi(x) \text{ with } \varphi(E, b) \subseteq \psi(x).
\]

The honest definition does not depend on the formula \( \varphi(x, y) \) but solely on the set \( \varphi(E, b) \). Furthermore, if \( Z \) is an honest definition of the externally definable subset \( X \) of \( E^{n+m} \) and \( \pi : E^{n+m} \to E^n \) is projection onto \( m \) many coordinates, then \( \pi(Z) \) is an honest definition of \( \pi(X) \).

We will define the dimension of an externally definable subset of \( E^n \) in terms of the dimension of some honest definition. For that, we need the following auxiliary lemma. Given a subset \( C \subseteq M \), we denote by a \( C\text{-n-ball} \) the set

\[
B^C(\bar{a}) = \theta(C, \bar{a}_1) \times \cdots \times \theta(C, \bar{a}_n), \quad \text{for some tuple } \bar{a} = (\bar{a}_1, \ldots, \bar{a}_n) \text{ in } C.
\]

**Lemma 2.4.** Let \( N \) be an \( \mathcal{L}\)-elementary extension of a model \( M \) of \( T \) and \( X \subseteq N^n \) be an \( \mathcal{L}_X \)-definable set containing some \( M\)-n-ball \( B^M(\bar{a}) \). The topological dimension \( \dim(X) \) of \( X \) equals \( n \).

**Proof.** Since \( \dim(X) \leq n = \dim(N^n) \), we need only show \( \dim(X) \geq n \), by induction on \( n \). For \( n = 1 \), if \( \dim(X) \) were 0, then \( X \) would be finite, which contradicts the fact that, for a basic open set defined over \( M \), the set \( \theta(M, \bar{a}) \) is infinite.

Assume now that \( X \) is an \( \mathcal{L}_D \)-definable subset of \( N^{n+1} \), for some \( D \subseteq N \), and let \( \pi : N^{n+1} \to N^n \) be the projection onto the first \( n \) coordinates. Consider the definable set

\[
Y := \{(b_1, \ldots, b_n) \in \pi(X) : \pi^{-1}(b_1, \ldots, b_n) \cap X \text{ is infinite}\}.
\]

Denoting by \( \bar{a} = (\bar{a}_1, \ldots, \bar{a}_{n+1}) \) and \( \bar{a}' = (\bar{a}_1, \ldots, \bar{a}_n) \), we have that \( B^M(\bar{a}') = \pi(B^M(\bar{a}')) \subseteq Y \). By induction \( \dim(Y) = n \), so there is \( (b_1, \ldots, b_n) \) in \( Y \) such that \( \dim(b_1, \ldots, b_n|D) = n \). Since \( \pi^{-1}(b_1, \ldots, b_n) \cap X \) is infinite, there is \( b_{n+1} \) in
Thus, the model $M$ contained in $\phi$ that $\bar{Y}$ with $b$ in $M$ and $d_i$ in $P(N_i)$, for $i = 1, 2$.

Definition 2.6. The topological dimension of any two honest definitions of an $\mathcal{L}^P$-definable set $Y \subseteq E^n$ coincide.

Proof. Choose two $\mathcal{L}^P$-elementary extensions $N_1$ and $N_2$ of $M$ such that $Z_1 = \chi_1(P(N_1), d_1)$ and $Z_2 = \chi_2(P(N_2), d_2)$ are both honest definitions of $Y = \varphi(E, b)$, with $b$ in $M$ and $d_i$ in $P(N_i)$, for $i = 1, 2$.

By Remark 2.3, we may assume that $\dim(Z_1) = n$. There exists a $P(N_1)$-n-ball $\theta(P(N_1), \bar{a}_1) \times \cdots \times \theta(P(N_1), \bar{a}_n)$ contained in $\chi_1(P(N_1), d_1) \subseteq \varphi(P(N_1), b)$. The density condition in Definition 1.4 implies that we may assume that the tuples $\bar{a}_1, \ldots, \bar{a}_n$ belong to $P(N_1)$. Hence,

$$N_1 \models \exists \bar{z}_1 \in P \ldots \exists \bar{z}_n \in P \forall x_1 \in P \ldots \forall x_n \in P \left( \bigwedge_{i=1}^n \theta(x_i, \bar{z}_i) \implies \varphi(\bar{x}, b) \right).$$

Thus, the model $M$ must also satisfy the above formula. Choose a $P(M)$-n-ball contained in $\varphi(E, b) \subseteq \chi_2(P(N_2), d_2)$. Since $E = P(M) \preceq P(N_2)$, we conclude that $\dim(Z_2) = n$, by Lemma 2.4.

Definition 2.6. The dimension $\dim(Y)$ of an $\mathcal{L}^P$-definable subset $Y$ of $E^n$ is the topological dimension of some (or equivalently, any) honest definition of $Y$.

If $Y$ is $\mathcal{L}^P$-definable over $A$, a point $b$ in $Y$ is generic over $A$ if its geometric dimension $\dim(b/A)$ equals $\dim(Y)$.

For an $\mathcal{L}^P$-definable subset $Y$ of $E^n$, notice that $Y \times Y$ has dimension $2\dim(Y)$, since $Z \times Z$ is an honest definition of $Y \times Y$, whenever $Z$ is an honest definition of $Y$. If $Y \subseteq E^n$ is $\mathcal{L}$-definable over $E$, then $Y$ is an honest definition of itself, so its dimension as defined above agrees with its topological dimension.

We now give some equivalent (geometric) characterizations of the introduced dimension. We first compare it with the weakly o-minimal dimension from 1.4.

Lemma 2.7. Let $Y$ be an $\mathcal{L}^P$-definable subset of $E^n$. Then $\dim(Y)$ is the largest integer $k$ such that the subset $\pi(Y)$ of $E^k$ has non-empty interior, for some projection $\pi: E^n \rightarrow E^k$ onto $k$ coordinates.

Proof. In some $\mathcal{L}^P$-elementary extension $N$ of $M$, let $Z = \chi(P(N), d)$ be an honest definition of $Y = \varphi(E, b)$, with $b$ in $M$ and $d$ in $P(N)$. Let $k \leq n$ be the largest integer such that for some projection $\pi: E^n \rightarrow E^k$, the set $\pi(Y)$ has non-empty interior in $E^k$. By Fact 2.2, the set $\pi(Z)$ is an honest definition of $\pi(Y)$. The latter contains an $E$-$k$-ball $B^E(\bar{a})$, for some tuple $\bar{a}$, which we may assume lies in $E$, by the density condition in 1.4. Since $E \preceq P(N)$, Lemma 2.4 gives that $k = \dim \pi(Z) \leq \dim(Z) = \dim(Y)$.

In order to show that $\dim(Y) \leq k$, choose $\dim(Y) = \dim(Z)$ many coordinates such that, for the corresponding projection $\pi: P(N)^n \rightarrow P(N)^{\dim(Y)}$, the set $\pi(Z)$ has non-empty interior in $P(N)^{\dim(Y)}$. In particular, the set $\pi(\varphi(P(N), b))$ has non-empty interior. Since $M \preceq P$, we deduce that $\pi(\varphi(E, b))$ has non-empty interior in $E^{\dim(Y)}$, as desired.

Corollary 2.8. Let $Y_1 \subseteq Y_2$ be $\mathcal{L}^P$-definable subsets of $E^n$. Then $\dim(Y_1) \leq \dim(Y_2)$. 
Lemma 2.9. Given an $L^p$-definable subset $Y$ of $E^n$,
\[ \dim(Y) = \max\{\dim(Z_0) : Z_0 \subseteq Y \text{ and } Z_0 \text{ is } L\text{-definable over } E\}. \]

Proof. By Corollary 2.8, the value $\dim(Y)$ is at least the maximum of the values on the right-hand side of the equation. We need only prove the other inequality. Choose an honest definition $Z = \chi(P(N),d)$ of $Y = \varphi(E,b)$ in some $L^p$-elementary extension $N$ of $M$, with $b$ in $M$ and $d$ in $P(N)$. Since $M \preceq^p N$ and the topological dimension is definable, there is some $d$ in $E$ such that $\chi(E,d) \subseteq \varphi(E,b) = Y$ with $\dim(\chi(M,d)) = \dim(Y)$, as desired. \hfill \Box

Density of the predicate $P$ implies the existence of (non-standard) elements of the appropriate dimension for any honest definition of an $L^p$-definable subset of $E^n$.

Lemma 2.10. Let $Y \subseteq E^n$ be an $L^p$-definable subset with honest definition $Z \subseteq P(N)^n$ having parameters over $P(N)$, for some $\kappa$-saturated elementary extension $M \preceq^p N$. Whenever $Z$ is definable over a subset $B$ of $N$ of size strictly less than $\kappa$, there is a generic element $a$ in $Z$, that is, with $\dim(a/B) = \dim(Z)$.

Proof. By Remark 2.3, we may assume that $\dim(Z)$ equals $n$. Write $Z = \chi(P(N),d)$, for some tuple $d$ in $P(N)$.

We need only show that the collection of formulae
\[ \Sigma(x) = \{\chi(x,d)\} \cup \{x \in P\} \cup \{-\psi(x,b) : \dim(\psi(x,b)) < n\} \] is finitely satisfiable. Indeed, given $L_B$-definable subsets $X_1,\ldots,X_k$ of $N^n$ with $\dim(X_i) < n$, the $L$-definable set
\[ \chi(N,d) \setminus \bigcup_{i=1}^k X_i \]
has dimension $n$, so it contains some non-empty open subset of $N^n$. By density, it contains an element $a$ in $P(N)^n$, as desired. \hfill \Box

Corollary 2.11. If $Y$ is $L^p$-definable subset of $E^n$ and $X$ is an $L$-definable subset of $M^n$ such that $Y \subseteq X$, then $\dim(Y) \leq \dim(X)$.

Proof. There is some finite subset $B \subseteq M$ such that $Y = \varphi(E)$ and $X = \psi(M)$, for some $L_B$-formulae $\varphi$ and $\psi$. In a sufficiently saturated elementary extension $M \preceq^p N$, choose an honest definition $Z \subseteq P(N)^n$ of $Y = \varphi(E)$. By Lemma 2.10, there is a generic element $a$ in $Z$ over $B$. As $Z \subseteq \varphi(P(N)) \subseteq \psi(N)$, we obtain $\dim(Y) = \dim(a/B) \leq \dim(X)$, as desired. \hfill \Box

For the rest of this section, we assume that our geometric NIP theory $T$ of topological rings is tame for pairs (cf. Definition 1.12).

By Theorem 1.16, the topological closure $cl_{M^n}(Y)$ of an $L^p$-definable subset $Y$ of $E^n$ is $L_E$-definable. We will now show that the dimension of the $E$-trace of the boundary $(cl_{M^n}(Y) \setminus Y) \cap E^n$ is strictly less than the dimension of $Y$.

Proposition 2.12. Let $Y$ be an $L^p$-definable subset of $E^n$.
\begin{enumerate}
\item If $cl_{M^n}(Y)$ denotes the topological closure, then $\dim(Y) = \dim(cl_{M^n}(Y))$;
\item $\dim(Y) = \min\{\dim(X) : X \subseteq M^n \text{ is } L\text{-definable and } Y \subseteq X\}$;
\item $\dim((cl_{M^n}(Y) \setminus Y) \cap E^n) < \dim(Y)$.
\end{enumerate}
Proof. We will prove (1), by induction on \( n \). By Corollary 2.11, we need only prove that \( \dim(Y) \geq \dim(\text{cl}_{M^n}(Y)) \). For \( n = 1 \), the result is obvious, since an externally definable subset of \( E \) of dimension 0 is finite, and thus so is its topological closure.

Assume first that \( \dim(\text{cl}_{M^n}(Y)) = n \) and write \( Y = X \cap E^n \), for some \( \mathcal{L} \)-definable subset \( X \) of \( M^n \). Note that \( \dim(X) = n \), for otherwise the set \( \text{cl}_{M^n}(Y) \setminus X \) has non-empty interior, so it contains a basic non-empty open subset \( U = \theta(M, \bar{a}_1) \times \cdots \times \theta(M, \bar{a}_n) \), but

\[
\emptyset \neq U \cap Y = U \cap X \cap E^n \subseteq U \cap X = \emptyset.
\]

The density condition in Definition 1.4 implies that the set \( X \) contains a basic non-empty open subset

\[
\theta(M, \bar{b}_1) \times \cdots \times \theta(M, \bar{b}_n),
\]

for some tuples \( \bar{b}_1, \ldots, \bar{b}_n \) in \( E \). Since

\[
\theta(E, \bar{b}_1) \times \cdots \times \theta(E, \bar{b}_n) \subseteq X \cap E^n = Y,
\]

we deduce that \( \dim(Y) \geq n \), by Corollary 2.13.

In case \( \dim(\text{cl}_{M^n}(Y)) < n \), choose some projection \( \pi : M^n \to M^k \), with \( k = \dim(\text{cl}_{M^n}(Y)) < n \), such that \( \pi(\text{cl}_{M^n}(Y)) \) has non-empty interior in \( M^k \). By induction, we have that \( \dim(\pi(Y)) = \dim(\text{cl}_{M^k}(\pi(Y))) \). If \( \dim(\text{cl}_{M^k}(\pi(Y))) = k \) then \( k = \dim(\text{cl}_{M^k}(\pi(Y))) = \dim(\pi(Y)) \) and thus \( \dim(Y) \geq k \), by Lemma 2.7.

Hence, we need only show that \( \dim(\text{cl}_{M^k}(\pi(Y))) \) cannot be strictly less than \( k \). Otherwise, the set \( \pi(\text{cl}_{M^n}(Y)) \setminus \text{cl}_{M^k}(\pi(Y)) \) has non-empty interior, so choose an element \( a \) in \( \pi(\text{cl}_{M^n}(Y)) \) and a basic non-empty neighbourhood \( U \) in \( M^k \) such that \( a \) belongs to \( U \subseteq \pi(\text{cl}_{M^n}(Y)) \setminus \text{cl}_{M^k}(\pi(Y)) \). In particular, there is some tuple \( \bar{a} \) in \( \text{cl}_{M^n}(Y) \) such that \( \pi(\bar{a}) = a \), so the open set \( \pi^{-1}(U) \) contains the accumulation point \( \bar{a} \). Hence the set

\[
Z = \pi^{-1}(U) \cap Y
\]

cannot be empty. However, the set \( \pi(Z) \) is contained in \( U \cap \pi(Y) = \emptyset \), which is a contradiction.

In order to prove (2), observe that \( \dim(Y) \) is less than or equal to the right-hand side, by Corollary 2.11. Let \( X \) be an \( \mathcal{L} \)-definable subset of \( M^n \) containing \( Y \) of least possible dimension. Notice that \( X \cap \text{cl}_{M^n}(Y) \) is again \( \mathcal{L} \)-definable, by Theorem 1.16, and contains \( Y \). In particular,

\[
\dim(Y) \leq \dim(X) \leq \dim(X \cap \text{cl}_{M^n}(Y)) \leq \dim(\text{cl}_{M^n}(Y)) \leq \dim(Y).
\]

We conclude now by showing (3). Let \( X \) be an \( \mathcal{L} \)-definable subset of \( M^n \) of least possible dimension containing \( Y \). If \( Y = X_1 \cap E^n \), for some \( \mathcal{L} \)-definable subset \( X_1 \) of \( M^n \), notice \( X \cap X_1 \) is again \( \mathcal{L} \)-definable and \( Y = X \cap X_1 \cap E^n \), so

\[
\dim(X) \leq \dim(Y) \leq \dim(X \cap X_1) \leq \dim(X).
\]

We may therefore assume that \( Y = X \cap E^n \). We have that

\[
(\text{cl}_{M^n}(Y) \setminus Y) \cap E^n \subseteq (\text{cl}_{M^n}(Y) \cap E^n) \setminus Y \subseteq (\text{cl}_{M^n}(X) \cap E^n) \setminus Y \subseteq (\text{cl}_{M^n}(X) \setminus X) \cap E^n.
\]

Corollaries 2.8 and 2.11 yield that

\[
\dim((\text{cl}_{M^n}(Y) \setminus Y) \cap E^n) \leq \dim((\text{cl}_{M^n}(X) \setminus X) \cap E^n) \leq \dim(\text{cl}_{M^n}(X) \setminus X) < \dim(X) \leq \dim(Y),
\]

and thus \( \dim(Y) \leq \dim(X) \leq \dim(X \cap X_1) \leq \dim(X) \).

\[
\dim(\text{cl}_{M^n}(Y) \setminus Y) \leq \dim(\text{cl}_{M^n}(X) \setminus X) \leq \dim(X) \leq \dim(Y).
\]
as desired.

3. Small groups

Every definable group $G$ in a geometric structure $M$ becomes naturally a topological group with respect to a definable topology on $G$ (which need not coincide with the induced topology) [18, Theorem 7.1.10]. The study of groups in [12] rely on the close relation between a real or $p$-adically closed field and its algebraic closure. For a similar analysis of groups for geometric topological fields, we introduce the following notion.

Definition 3.1. A geometric theory $T$ is stably controlled if there is a strongly minimal theory $T^0$ which eliminates quantifiers and imaginaries in a sublanguage $L^0 \subseteq L$ such that:

- The theory $T$ contains $T^0$, that is, every model $M$ of $T$ embeds as an $L^0$-substructure of some model $N$ of $T^0$. In particular, the $L^0$-definable closures of $M$ in the various models of $T^0$ are definably isomorphic (since $T^0$ eliminates quantifiers), and analogously for the $L^0$-algebraic closures.
- The underlying $L^0$-structure of a model $M$ of $T$ is $L^0$-definably closed in some (equivalently, every) model $N$ of $T^0$ containing $M$.
- In every model $M$ of $T$, the dimension given by the algebraic closure with respect to the language $L$ coincides with the $L^0$-Morley rank, computed inside the $L^0$-algebraic closure, which is strongly minimal.

Example. Real and $p$-adically closed fields in the ring language are stably controlled by the strongly minimal theory $\mathbb{A}CF_0$ [12].

Lemma 3.2. If a geometric theory of topological rings $T$ is stably controlled, then it is tame for pairs, that is, in a dense pair $(M, E)$ of models of $T$, every $L^P$-definably closed set is special.

In particular, the intersection of $L$-definably closed special subsets of $M$ is again special, by Fact [13].

Proof. Given an $L^P$-definably closed subset $A$ of $M$, we need to show that $A \downarrow_{A \cap E} E$. Choose a model $K$ of the strongly minimal $L^0$-theory $T^0$ controlling $T$ such that $M$ embeds into $K$ as an $L^0$-substructure. Denote by $dcl^0$ and $acl^0$ the definable and algebraic closures in $K$.

Given a tuple $a$ in $A$, elimination of imaginaries yields that the $L^0$-type $q$ of $a$ over $acl^0(E)$ is stationary. Its canonical base $cb(q)$ lies in $dcl^0(E, a) \subseteq M$. Thus $cb(q)$ lies in $acl^0(E) \cap M \subseteq dcl(E) = E$, for $E$ is an elementary $L$-substructure of $M$. Any $L^P$-automorphism of $M$ fixing $A$ pointwise must permute $E$ as a set, so $cb(q)$ lies in $dcl^P(A) = A$. Hence, the restriction of $q$ to $E \cap A$ is stationary, and

$$\dim(a/E) = RM^0(a/E) = RM^0(a/E \cap A) = \dim(a/E \cap A),$$

as desired. □

If the theory $T$ of $M$ is stably controlled with respect to the strongly minimal theory $T^0$, a verbatim adaptation of [12] Theorem A] yields that there are:

- a connected group $(H, \circ)$ which is $L^0$-definable in $T^0$ with parameters from $M$;
- open $L_M$-definable subsets $U$, $V$ and $W$ of $G$;
such that

\[ f(x) \circ g(y) = h(x \ast y) \quad \text{for every } (x, y) \text{ in } U \times V. \]

In particular, the group law on \( G \) is locally \( \mathcal{L}_0 \)-definable. In the case of \( p \)-adically or real closed fields, we conclude that the group law on \( G \) is locally algebraic.

We will prove an analogon for certain small groups definable in dense pairs.

**Theorem 3.3.** Let \( T \) be a geometric NIP theory of topological rings, which is stably controlled with respect to the strongly minimal \( \mathcal{L}^0 \)-theory \( T^0 \). Consider a dense pair \((M, E)\) of models of \( T \), with \( E = P(M) \), in the language \( \mathcal{L}^P = \mathcal{L} \cup \{P\} \). If \((G, \ast)\) is an \( \mathcal{L}^P \)-definable group over a special set \( D \subseteq M \) such that \( G \subseteq E^k \) for some \( k \), then there are:

- an \( \mathcal{L}_E \)-definable subset \( Z \) of \( G \) of dimension \( \dim(G) \);
- an \( \mathcal{L}_D \)-definable connected group \((H, \circ)\) over \( E \);
- relatively open \( \mathcal{L}_E \)-definable subsets \( U, V \) and \( W \) of \( Z \) with \( U \ast V \subseteq W \);
- relatively open \( \mathcal{L}_E \)-definable subsets \( U', V' \) and \( W' \) of \( H(E) \) with \( U' \circ V' \subseteq W' \);
- \( \mathcal{L}_E \)-definable homeomorphisms

\[ f : U \to U', \quad g : V \to V' \quad \text{and} \quad h : W \to W'; \]

such that

\[ f(x) \circ g(y) = h(x \ast y) \quad \text{for every } (x, y) \text{ in } U \times V. \]

**Proof.** The proof is an adaptation of the analogous result [12, Theorem A], so we will only remark the relevant steps, to avoid repetitions. We may assume that the special set \( D \) is finite-dimensional and \( \mathcal{L}^P \)-definably closed, by Fact 2.2. Observe that \( A_0 \) is special in \( N \).

Let

\[ Y = \{(a, b, a \ast b) : a, b \in G\} \subseteq G^3, \]

be the graph of the group operation \( \ast \). The set \( Y \) is \( \mathcal{L}^P \)-definable over \( D \), so choose an honest definition \( Z_Y \) of \( Y \) defined over a finite set \( A_0 \) of \( P(N) \), where \( N \) is a sufficiently saturated \( \mathcal{L}^P \)-elementary extension \( N \) of \( M \), by Fact 2.2. Observe that \( A_0 \) is special in \( N \).

Fact 2.2 yields that the image \( Z_i \) of the \( i \)-th-projection \( \pi_i : Z_Y \to G(P(N)^k) \) is an honest definition of \( G \), so

\[ \dim(Z_1) = \dim(Z_2) = \dim(Z_3) = \dim(G), \]

by Corollary 2.3. Since \( Z_Y \subseteq \{(a, b, a \ast b) : a, b \in G(P(N)^k)\} \), we conclude that for each pair \((a, b)\) in

\[ Z_{1,2} := (\pi_1 \times \pi_2)(Z_Y) \subseteq G(P(N)^k) \times G(P(N)^k), \]

there is a unique \( c \) in \( Z_3(P(N)^k) \) such that \((a, b, c)\) belongs to \( Z_Y \). Hence, the set \( Z_Y(P(N)^{3k}) \) is the graph of an \( \mathcal{L} \)-definable function \( \circ \) over \( A_0 \):

\[ Z_{1,2}(P(N)^{2k}) \to Z_3(P(N)^k) \]

\[ (a, b) \to a \circ b. \]

Note that \( Z_{1,2}(P(N)^{2k}) \) is an honest definition of \( G \times G \), so \( \dim(Z_{1,2}) = 2 \dim(G) \).
Since $M \preceq^P N$, the set $M$ is special in $N$. In particular, the set $D$ remains special in $N$, and so is $D_1 = \dcl(D, A_0)$, by the Remark 4.7. Clearly $D_1$ is finite-dimensional. Observe that $Z_V$, the projections $Z_1$, $Z_2$ and $Z_3$, as well as the map $\circ$ are all $\mathcal{L}$-definable over $D_1$.

By Lemma 2.10 choose a generic point $(a, b)$ in $Z_{1, 2}(P(N)^{2k})$ with $c = a \circ b$ in $Z_3$, so

$$\dim(a/b, D_1) = 2 \dim(G).$$

The pair $(a, b)$ is the starting point of the group configuration theorem, working in the $\mathcal{L}_0$-algebraic closure $\acl^0(\mathcal{P}(N))$ with respect to the theory $T^0$.

Given any two elements $x$ and $y$ of $G(\mathcal{P}(N)^k)$, the set $D_1 \cup \{x, y\}$ is again special in $N$, by the Remark 4.7. Let $P(D_1)$ denote $D_1 \cap P(N)$. By Fact 4.8, we have that $P(dcl^0(D_1 \cup \{x, y\})) = dcl(P(X_1), x, y)$, so the element $x \cdot y$ lies in $dcl(P(D_1), x, y)$. Moreover, given any special set $D' \supseteq D$, if $x$ and $y$ are both generic in $G(\mathcal{P}(N)^k)$ and independent over $D'$ then so is $x \cdot y$, which is independent from each of the factors over $D'$.

Because of this, the same arguments as in [12, Proposition 3.1] carry on to obtain a finite-dimensional subset $D_2$ of $P(N)$ containing $P(D_1)$, a connected group $H \subseteq \acl^0(\mathcal{P}(N))^k$ which is $\mathcal{L}_0$-definable over $D_2$ in the theory $T^0$, and points $a'$, $b'$ and $c'$ of $H(\mathcal{P}(N)^k)$, such that

- $a'$ and $b'$ are $D_2$-generic independent elements of $H(\mathcal{P}(N)^k)$;
- $a' \cdot b' = c'$;
- $\dcl(aD_2) = \dcl(a'D_2)$, $\dcl(bD_2) = \dcl(b'D_2)$ and $\dcl(cD_2) = \dcl(c'D_2)$.

In particular, there is an $\mathcal{L}_{D_2}$-definable bijection $f$ from a subset of $G(\mathcal{P}(N)^k)$ to a subset of $H(\mathcal{P}(N)^k)$, with $f(a) = a'$. We now proceed as in [12, Lemma 4.8]. Denote by $Z$ the $\mathcal{L}$-definable set $Z_1 \cup Z_2 \cup Z_3$. Note that $Z(\mathcal{P}(N)^{k}) \subseteq G(\mathcal{P}(N)^k)$, and all three elements $a$, $b$ and $c = a \cdot b$ lie in $Z$. Since $a$ in $Z$ and $a'$ in $H(\mathcal{P}(N)^k)$ are each generic over $D_2$, there are relatively open $\mathcal{L}_{D_2}$-definable neighbourhoods $U$ of $a$ in $Z$ and $U'$ of $a'$ in $H(\mathcal{P}(N))$, such that $f : U \to U'$ is a homeomorphism. Similarly, there exists $\mathcal{L}_{D_2}$-definable homeomorphisms $g : V \to V'$ and $h : W \to W'$ with $g(b) = b'$ and $h(c) = c'$, for some relatively open $\mathcal{L}_{D_2}$-definable open subsets $V$ and $W$ of $Z$, resp. $V'$ and $W'$ of $H(\mathcal{P}(N))$.

Now, the function $\circ$ is $\mathcal{L}_{A_0}$-definable and $A_0 \subseteq P(D_1) \subseteq D_2$. The $\mathcal{L}_{D_2}$-definable set

$$\{(x, y) \in Z_{1, 2} \cap (U \times V) \cap \circ^{-1}(W) : f(x) \circ g(y) = h(x \circ y)\} \subseteq Z \times Z,$$

contains the point $(a, b)$, which has dimension $2 \dim(Z)$ over $D_2$. There are open $\mathcal{L}_{\mathcal{P}(N)}$-definable neighbourhoods $U_1$ of $a$, resp. $V_1$ of $b$, in $Z$ such that $U_1 \times V_1$ is contained in the above set. Hence, we have that

$$U_1 \subseteq U, \ V_1 \subseteq V \quad \text{and} \quad U_1 \ast V_1 = U_1 \circ V_1 \subseteq W$$

with $f(x) \circ g(y) = h(x \circ y) = h(x \cdot y)$ for every $(x, y)$ in $U_1 \times V_1$. Finally, rename $U_1$ and $V_1$ by $U$ and $V$, respectively. Since $M \preceq^P N$, we can obtain the corresponding definable sets and homeomorphisms in our pair $(M, E)$, all definable over $E$, with $Z \subseteq G$ of dimension $\dim(G)$.

If the theory $T$ is an o-minimal expansion of a real closed field, we can provide an intrinsic definition of the local isomorphism of Theorem 3.3.

Recall that (the theory of) a weakly minimal expansion of an ordered abelian group $M$ is non-valuational if there are no proper definable subgroups of $M$ (cf. 26).
Lemma 1.5]). Every o-minimal theory is non-valuational, since they are definably complete, so definable cuts are realized. A typical example of a valuational theory is the theory of a non-archimedian real closed field with a distinguished predicate for the convex hull of the integers.

Let \((M, E)\) now be a dense pair of models of \(T\) and fix a special subset \(D \subseteq M\). The (weak) Shelah’s expansion \([23]\) \(E_{\text{ext}}\) of \(E\) is the structure in the language \(\mathcal{L}^{\text{ext}}\) consisting of relational symbols \(R_X\) for each \(\mathcal{L}^D\)-definable subset \(X\) of \(E^n\), as \(n\) varies, whose universe is \(E\) and the interpretation in \(E_{\text{ext}}\) of each \(R_X\) is \(X\). Since the projection of an \(\mathcal{L}^D\)-definable subset of \(E^n\) is again \(\mathcal{L}^D\)-definable, the theory \(\text{Th}(E_{\text{ext}})\) eliminates quantifiers. Thus, definable sets in \(E_{\text{ext}}\) are exactly the \(\mathcal{L}^D\)-definable sets, which are exactly the traces on \(E\) of \(\mathcal{L}_D\)-definable subsets of \(M\), by Fact [18][1]. In particular, the structure \(E_{\text{ext}}\) is a weakly o-minimal \([1]\).

Furthermore, it is non-valuational: given a subgroup \(H\) of \((E, +, \leq)\) definable in \(E_{\text{ext}}\), the set \(\text{cl}_M(H)\) is a subgroup of \((M, +)\) and it is \(\mathcal{L}\)-definable, by Theorem [1][10]. Since \(M\) is o-minimal, it follows that \(\text{cl}_M(H)\) is trivial or \(\text{cl}_M(H) = M\). As the characteristic is 0, the only finite additive subgroup is the trivial one. Assume hence that \(\text{cl}_M(H) = M\), so \(\text{dim}(E \setminus H) < \text{dim}(H) = 1\), by Proposition [2][12]. In particular, the set \(E \setminus H\) has dimension 0, so it is finite, which yields that \(E = H\), since \(E\) is divisible.

Hence, the (honest) dimension defined in Definition 2.6 for the non-valuational weakly o-minimal structure \(E_{\text{ext}}\) is a dimension in the sense of [23], for it coincides with the topological dimension, by Lemma 2.7. Theorem [24], Theorem 4.6] yields that every \(\mathcal{L}^D\)-definable group \(G \subseteq E^k\) (which are \(\mathcal{L}^{\text{ext}}\)-definable) has a topology \(\tau\) which turns it into a topological group. Specifically, there is an \(\mathcal{L}^D\)-definable \(\tau\)-open subset \(L\) of \(G\) such that the topology \(\tau\) restricted to \(L\) coincides with the topology induced from \(E^k\) and finitely many translates of \(L\) cover \(G\), with \(\text{dim}(G \setminus L) < \text{dim}(G)\).

We can now conclude the following result:

**Corollary 3.4.** Let \(T\) be an o-minimal expansion of a real closed field, which is stably controlled by the strongly minimal \(\mathcal{L}^0\)-theory \(T^0\). Consider a dense pair \((M, E)\) of models of \(T\), with \(E = P(M)\), in the language \(\mathcal{L}^P = \mathcal{L} \cup \{P\}\). If \((G, \ast)\) is an \(\mathcal{L}^P\)-definable group over a special set \(D \subseteq M\) such that \(G \subseteq E^k\) for some \(k\), then there is an \(\mathcal{L}^P\)-definable local isomorphism between a neighbourhood of the identity in \(G\) and a neighbourhood of the identity in \(H(E)\), for some connected group \(H\) which is \(\mathcal{L}^0\)-definable over \(E\).

**Proof.** Enlarging \(D\), we may assume that all the sets and maps in Theorem 3.3 are definable over the finite-dimensional special set \(D\). Let now \(\tau\) be the topology of \(G\) described above and \(L \subseteq G\) be the \(\mathcal{L}^D\)-definable \(\tau\)-open subset of \(G\) such that \(\tau\) restricted to \(L\) coincides with the topology induced from \(E^k\) and finitely many translates of \(L\) cover \(G\), with \(\text{dim}(G \setminus L) < \text{dim}(G)\).

We first show that the subset \(U \cap L\) has non-empty interior in the set \(U\) obtained in Theorem 3.3. Since

\[
U = (U \cap L) \cup (U \setminus L),
\]

we have that \(\text{dim}(U \cap L) = \text{dim}(U) = \text{dim}(G)\), for \(\text{dim}(G \setminus L) < \text{dim}(G)\). By Proposition [2][12][3], the dimension of the set \(\text{cl}_{M^\ast}(U \setminus L)\) equals \(\text{dim}(U \setminus L) < \text{dim}(G)\). Now,

\[
(U \cap L) \setminus \text{int}_U(U \cap L) \subseteq \text{cl}_{M^\ast}(U \setminus L),
\]
shown in [12, Corollary 4.9], the set $U$ is a homeomorphism and a local isomorphism, as required.

Require that:

- operations are continuous with respect to the definable topology.
- Furthermore, we provide a short proof of Eleftheriou’s result, which becomes somewhat easier in our particular setting.

Recall that the language $L$ expands the language of rings and that the ring operations are continuous with respect to the definable topology. Furthermore, we require that:

- the geometric NIP theory $T$ eliminates imaginaries;
- the intersection of two definably closed special sets is special: this is the case, whenever $T$ is stably controlled with respect to a strongly minimal theory $T^0$, by Lemma 3.2.

**Corollary 4.1.** [3, Lemma 3.1]

Let $A$ and $A'$ be two definably closed special subsets of $M$. Given a subset $Y$ of $E^n$, which is both $L^P$-definable over both $A$ and $A'$, the set $Y$ is $L^P$-definable over $A \cap A'$.

**Proof.** The proof goes by induction on $\dim(Y)$. For $\dim(Y) = 0$, it is obvious, since $Y$ is finite. Otherwise, if $cl_{M^n}(Y)$ denotes the topological closure of $Y$, note that $cl_{M^n}(Y) \cap E^n$ is the disjoint union of $Y$ and the set

$$cl_{M^n}(Y) \cap E^n \setminus Y = (cl_{M^n}(Y) \setminus Y) \cap E^n.$$

Therefore, it suffices to show that both $cl_{M^n}(Y) \cap E^n$ and $(cl_{M^n}(Y) \cap E^n) \setminus Y$ are $L^P$-definable over $A \cap A'$. By Theorem 1.10, the set $cl_{M^n}(Y)$ is $L$-definable over $A$ and $A'$, so it is $L$-definable over $A \cap A'$, by elimination of imaginaries of the theory $T$. 

\[ \square \]
Finally, the dimension of the $\mathcal{L}^P$-definable set $(\text{cl}_{M^n}(Y) \setminus Y) \cap E^n$ is strictly less than $\dim(Y)$, by Proposition 2.12[d]. Since $(\text{cl}_{M^n}(Y) \setminus Y) \cap E^n$ is both $\mathcal{L}^P$-definable over $A$ and $A'$, it is definable over $A \cap A'$, by induction. \hfill $\square$

We say that a definable set $Z$ in a fixed complete theory has a field of definition if there is a smallest definably closed subset $B$ over which $Z$ is definable. Clearly, a theory with elimination of imaginaries has fields of definitions for all definable sets. If a theory has fields of definitions for all definable sets, then it has weak elimination of imaginaries. Indeed, given an imaginary $e = a/E$, let $B$ be a field of definition of the definable class $E(x,a)$. By minimality, the set $B$ equals $\text{dcl}(b)$, for some finite tuple $b$ such that $e$ is definable over $b$. We need only show that the type $\text{tp}(b/e)$ is algebraic. Observe that the collection of realisations of $\text{tp}(b/e)$ lies in the set $B$, so its cardinality is bounded. Compactness yields that this set must be finite, so $b$ is algebraic over $e$, as desired.

**Fact 4.2.** [Theorem C and Corollary 1.4]

For every $\mathcal{L}^P$-definable small subset $X \subseteq M^n$ over a special set $A$, there is an $\mathcal{L}^P$-definable injection $X$ into $E^l$ over $A$, for some natural number $l$.

**Proof.** By Fact 1.8, we may assume that $A$ is $\mathcal{L}^P$-definably closed. By smallness of $X$ over $A$, there is an $\mathcal{L}_A$-definable function $h : M^m \to M^n$, with $X \subseteq h(E^m)$. For tuples of $E^m$, set $e_1 \sim e_2$ if $h(e_1) = h(e_2)$.

Let us first show that, for each tuple $e$ in $E^m$, its $\sim$-class has a field of definition in $T^P$ working over the parameter set $A$. Clearly, the $\sim$-class $Y$ of $e$ is $\mathcal{L}^P$-definable over $\text{dcl}(Ae)$, since the latter is special and $\mathcal{L}^P$-definably closed, by the Remark 1.3 and Fact 1.8[d]. Choose a tuple $d$ in $E$ of minimal length such that the set $Y$ is definable over the special set $\text{dcl}(Ad)$. Suppose that $\text{dcl}(Ad)$ is not a field of definition of $Y$. Hence, there is an $\mathcal{L}^P$-definably closed set $B \supseteq A$ (hence special) over which $Y$ is defined, but

$$A \subseteq A' = \text{dcl}(Ad) \cap B \subset \text{dcl}(Ad).$$

Our assumptions yield that the set $A'$ is special, since both $B$ and $\text{dcl}(Ad)$ are. By Corollary 4.1, the set $Y$ is definable over $A'$. Choose an $A$-independent tuple $u$ in $M$ with $A' = \text{dcl}(Au)$. Though $u$ need not be in $E$, it lies in $\text{dcl}(Ad) \subseteq \text{dcl}(AE)$. The independence

$$A, u \not\overset{E' \cap \text{dcl}(Au)}{\perp} E,$$

implies that $\dim(u/A, E \cap \text{dcl}(Au)) = 0$. Choose a tuple $f$ of $E$ of length $|u|$ which is $\mathcal{L}$-interdefinable with $u$ over $A$. By the exchange principle, we have that $|f| < |d|$, contradicting our choice of $d$. Thus, the set $Y$ has $\text{dcl}(Ad)$ as a field of definition (working over $A$).

Since $T$ has finite Skolem functions, the set $Y$ has a canonical parameter in $E$, working over $A$. By compactness, there exists an $\mathcal{L}_A^P$-definable injection which sends the $\sim$-class of $e$ to some tuple in $E^l$, for some fixed $l$. Every point of $X$ is of the form $h(e)$, for some $e$ in $E^m$. Any two representatives lie in the same $\sim$-class, so composing we obtain an $\mathcal{L}^P$-definable injection $X \to E^l$ over $A$, as desired. \hfill $\square$
Fact 4.2 yields that every $L^P$-definable small group of $(M, E)$ is, up to $L^P$-definability, a subset of some cartesian product $E^k$, so we conclude the following:

**Corollary 4.3.** If the tame topological geometric theory $T$ with NIP eliminates imaginaries, is stably controlled by a theory $T^0$ and expands a topological field, then every small group $(G, \ast)$ has locally, up to $L^P$-interdefinability, an $L^0$-definable group law with parameters from $E$.

**Question.** Is the group law of every small group in a dense pair of real closed fields semialgebraic, up to $L^P$-interdefinability?

### 5. Appendix: reproving Fact 1.8

In [7, Corollary 3.4], it was stated that $L^P$-definable function $F : M \rightarrow M$ agrees off some small subset of $M$ with a function $\hat{F} : M \rightarrow M$ that is $L$-definable. However, the proof only yields that $F$ agrees off some small subset of $M$ with one of finitely many $L$-definable functions $\hat{F}_1, \ldots, \hat{F}_\ell : M \rightarrow M$. Note that the same gap also affects [2, Theorem 4.9].

After private communication with van den Dries, we will provide in this appendix a proof of [7, Corollary 3.4] in the broader context of geometric topological structures, fixing the gap in the published version.

**In this appendix, we work in a sufficiently saturated dense pair of models $(M, E)$ of a geometric theory $T$ of topological rings.**

As in [7, Theorem 2.5], a straightforward adaptation of the back-and-forth system yields items (1), (2) and (3) in Fact 1.8.

**Lemma 5.1.** Let $D$ be a special subset of $M$ and $(U_y : y \in Y)$ a uniformly $L_D$-definable family of open subsets of $M$, parametrized over the $L_D$-definable subset $Y \subseteq M^k$. The set

$$\bigcup_{y \in Y \cap E^k} U_y$$

is $L_D$-definable.

**Proof.** Naming the parameters in $D$, we may clearly assume that $D = \emptyset$. By cell decomposition, we may assume that $Y$ is a cell. We will prove it by induction on $k$, the initial case $k = 0$ being trivial.

**Claim.** We may assume that $\dim(Y) = k$.

**Proof of Claim.** Otherwise, there is some projection $\pi : M^k \rightarrow M^\ell$, with $\ell < k$, such that $\pi|_Y$ is a homeomorphism between $Y$ and the open subset $\pi(Y)$ of $M^\ell$. By Fact 1.8 (3), the $L^P$-definable subset $\pi(Y \cap E^k) \subseteq E^\ell$ is the trace of an $L$-definable set $Y_1$, that is, we have that $Y_1 \cap E^\ell = \pi(Y \cap E^k)$. Considering the intersection, we may assume that $Y_1 \subseteq \pi(Y)$, so denote by $\rho : Y_1 \rightarrow Y$ the inverse of $\pi|_Y$ restricted to $Y_1$. Reparametrizing the family $(U_y : y \in Y)$ as $(U_{\rho(y)} : y \in Y_1)$ and decomposing $Y_1$ into cells, we conclude the desired result by induction. $\square$ Claim

Hence, the cell $Y$ is open in $M^k$. For $x$ in the open $L$-definable subset

$$U = \bigcup_{y \in Y} U_y$$

$$U = \bigcup_{y \in Y} U_y$$
of $M$, consider the corresponding $L$-definable sets
\[ Y_x = \{ y \in Y \mid x \in U_y \}, \]
and
\[ B = \{ x \in U \mid \text{int}(Y_x) = \emptyset \}. \]

Claim. The set $B$ is finite.

Proof of Claim. If $B$ were infinite, its interior $\text{int}(B)$ is non-empty. Set
\[ Y_1 = \bigcup_{x \in \text{int}(B)} Y_x = \{ y \in Y \mid U_y \cap \text{int}(B) \neq \emptyset \}. \]

For each $y$ in $Y_1$, the open set $U_y \cap \text{int}(B)$ is not empty. By Skolem functions in the theory $T$, choose some $x = f(y)$ in $U_y \cap \text{int}(B)$, for some $L$-definable map $f : Y_1 \to M$. In particular, there is a basic open neighborhood of $f(y)$ contained in $U_y \cap \text{int}(B)$, that is,
\[ f(y) \in \theta(M, g(y)) \subseteq U_y \cap \text{int}(B), \]
for some $L$-definable Skolem function $g : Y_1 \to M^s$. We may assume, partitioning $Y_1$, that both $f$ and $g$ are continuous.

If $Y_1$ were not empty, choose some element $y$ in $Y_1$ and consider $g(y)$ in $M^s$. By the density condition, there is an open neighborhood $V$ of $g(y)$ and a tuple $\bar{c}$ such that for all $\bar{d}$ in $V$,
\[ f(y) \in \theta(M, \bar{c}) \subseteq \theta(M, \bar{d}). \]

Let us first show that the open neighborhood $g^{-1}(V)$ of $y$ is contained in $Y_{f(y)}$: given $z$ in $g^{-1}(V)$, we need to show that $f(y)$ belongs to $U_z$. Now, the element $g(z)$ belongs to $V$, so
\[ f(y) \in \theta(M, \bar{c}) \subseteq \theta(M, g(z)). \]
By construction, the set $\theta(M, g(z)) \subseteq U_z \cap \text{int}(B)$, so $f(y)$ lies in $U_z$ as desired. Hence $Y_{f(y)}$ has non-empty interior, which contradicts that $f(y)$ belongs to $B$. In particular, the set $Y_1$ is empty and thus so is $\text{int}(B)$, since $Y_x \neq \emptyset$ for every $x$ in $U$. Therefore, the set $B$ is finite, as desired. \hfill \Box Claim

Now, the finite subset
\[ B' = B \cap \left( \bigcup_{y \in Y \cap E^k} U_y \right) \]
is trivially $L$-definable. Note that
\[ \bigcup_{y \in Y \cap E^k} U_y = B' \cup \bigcup_{y \in Y \cap E^k} (U_y \setminus B). \]
We need only show that
\[ \bigcup_{y \in Y \cap E^k} (U_y \setminus B) = \bigcup_{y \in Y} (U_y \setminus B), \]
in order to conclude that $\bigcup_{y \in Y \cap E^k} U_y$ is $L$-definable.

Given $x$ in $(U_y \setminus B)$ for some $y$ in $Y$, we have that $\text{int}(Y_x) \neq \emptyset$, since $x$ does not lie in $B$. Hence, there is an open neighbourhood $V$ of $y$ contained in $Y_x$, so choose an element $y_0$ in $V \cap E^k$, by density. Thus $x$ lies in $U_{y_0} \subseteq \bigcup_{y \in Y \cap E^k} U_y$, as desired. \hfill \Box
Fact 5.2. ([11] Theorem 2.5]) Every $L^P$-formula $\phi(x_1, \ldots, x_n)$ is equivalent in $T^P$ to a boolean combination of formulae of the form
$$ \exists y_1 \cdots \exists y_m (P(y_1) \land \cdots \land P(y_m) \land \psi(x_1, \ldots, x_n, y_1, \ldots, y_m)) $$
where for some $L$-formula $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$.

We will now describe $L^P$-definable subsets of $M$, which is exactly [7, Corollary 3.5]. We will obtain such a description using Fact 5.2 (instead of deducing it from [7, Cor 3.4]).

Proposition 5.3. For every $L^P$-definable subset $S \subseteq M$ over an special set $D$, there are $L^P_D$-definable small sets $X_1$ and $X_2$, and an $L_D$-definable subset $S' \subseteq M$ such that
$$ S = (S' \setminus X_1) \cup X_2. $$

Proof. Let $F$ denote the family of subsets of $M$ of the form
$$(S' \setminus X_1) \cup X_2$$
with $S'$, $X_1$ and $X_2$ as in the statement. Since every subset of a small subset is again small, it is easy to check that $F$ is closed under boolean combinations. Thus, by Fact 5.2 we need only show that each unary definable set $S$ of the form
$$ \exists y_1 \cdots \exists y_m (P(y_1) \land \cdots \land P(y_m) \land \psi(x, d, y_1, \ldots, y_m)) $$
belongs to $F$, where $d$ is a tuple of parameters in $D$. Set $Y \subseteq M^n$ be the set $\exists x \exists y \exists z \cdots \exists w \exists P(y_1) \land \cdots \land P(y_m) \land \psi(x, d, y_1, \ldots, y_m)$. By uniform cell decomposition, for each $y$ in $Y$, the $L_D$-definable subset of $\psi(M, d, y)$ of $M$ is a union of a finite set $F_y$ (whose cardinality is bounded by some $r$ in $\mathbb{N}$ uniformly on $y$) and an open $L_D$-definable subset $U_y$ of $M$. Since $T$ has finite Skolem functions, there are $L_D$-definable functions $g_1, \ldots, g_r : Y \to M$ such that $F_y = \{ g_1(y), \ldots, g_r(y) \}$ for each $y$ in $Y$.

The $L$-definable set
$$ X_2 = \bigcup_{y \in Y \cap E^m} F_y = \bigcup_{i=1}^r g_i(E^k), $$
is small. By Lemma 5.1 the set
$$ S' = \bigcup_{y \in Y \cap E^m} U_y $$
is $L_D$-definable. Clearly, the set $S = S' \cup X_2$, as desired. \hfill \square

Remark 5.4. In particular, if a small subset $X \subseteq M$ is $L^P$-definable over some set of parameters $D$, then there is an $L_D$-definable map $h : M^m \to M$ such that $X \subseteq h(E^m)$.

The proof of [11] Theorem 4], and more generally of [7, Theorem 4.9], only need as main ingredient the Proposition 5.3, and not [7, Corollary 3.4]. Hence, we obtain verbatim a proof of the following result:

Proposition 5.5. (cf. [7, Theorem 4]) For every $L^P$-definable unary set $X \subseteq M$ over a special set $D$, there are pairwise disjoint $L_D$-definable open sets $U$, $V$, $W_1$ and $W_2$ of $M$ such that:

- $M \setminus (U \cup V \cup W_1 \cup W_2)$ is finite.
- $U \subseteq X$ and $X \cap V = \emptyset$. 

\begin{itemize}
\item $X \cap W_1$ is dense in $W_1$ and small.
\item $X \cap W_2$ is dense and codense in $W_2$ and co-small in $W_2$.
\end{itemize}

We have now all the ingredients in order to fill the gap of [7, Corollary 3.4].

**Theorem 5.6.** Any $\mathcal{L}^P$-definable function $F : M \to M$ definable over an special set $D$ agrees off some small subset of $M$ with an $\mathcal{L}_D$-definable function $\hat{F} : M \to M$.

**Proof.** As stated in the beginning of the Appendix, the proof of [6, Corollary 3.4] yields that there are finitely many $\mathcal{L}_D$-definable maps $\hat{G}_1, \ldots, \hat{G}_\ell : M \to M$ and a small $\mathcal{L}^P$-definable set $X$ such that for all $x$ in $M \setminus X$, we have that $F(x) = \hat{G}_i(x)$ for some $1 \leq i \leq \ell$.

Clearly, if $\ell = 1$, we are done. Otherwise, consider the set $X_1 = \{ x \in M \mid F(x) = \hat{G}_1(x) \}$, and let $U, V, W_1$ and $W_2$ be the corresponding open $\mathcal{L}_D$-definable subsets of $M$ for $X_1$, as in Proposition 5.3. Define $F_1 = M \setminus (U \cup V \cup W_1 \cup W_2)$ and

$$\hat{G}_2 : M \to M$$

$$x \mapsto \begin{cases} 
\hat{G}_1(x) & \text{if } x \in U \cup W_2 \\
\hat{G}_2(x) & \text{otherwise.}
\end{cases}$$

The subset $X' = X \cup (X_1 \cap W_1) \cup (W_2 \setminus X_1) \cup F_1$ is clearly small, and note that for $x$ in $M \setminus X'$, the value $F(x)$ equals $\hat{G}_2(x)$ or $\hat{G}_i(x)$, for some $3 \leq i \leq \ell$. We proceed now by induction on $\ell$ in order to produce an $\mathcal{L}_D$-definable function $\hat{F}$, which agrees with $F$ off a small subset of $M$. \hfill $\square$

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