Ample D4-D2-D0 Decay

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Abstract: We study the wall-crossing behavior of the index of BPS states for $D4-D2-D0$ brane systems on a Calabi-Yau 3-fold at large radius and point out that not only is the “BPS index at large radius” chamber-dependent, but that the changes in the index can be large in the sense that they dominate single-centered black hole entropy. We discuss implications for the weak coupling OSV conjecture. We also analyze the near horizon limit of multico-centered solutions, introduced in arXiv:0802.2257, for these particular configurations and comment on a general criterion, conjectured in arXiv:0802.2257, which identifies those multico-centered solutions whose near horizon limit corresponds to a geometry with a single asymptotic $AdS_3 \times S^2$ boundary.
1. Introduction and Conclusion

Consider type IIA string theory compactified on a Calabi-Yau manifold $X$. The space of BPS states associated to D-branes wrapping internal cycles has been a subject of much interest for over 10 years and continues to yield surprises. In particular, for both mathematical and physical reasons the index of BPS states of charge $\Gamma$ has been the focus of much recent research. This index, the second helicity supertrace of the space of BPS states, will be denoted $\Omega(\Gamma; t)$ where $t = B + iJ$ denotes a complexified Kähler class.\footnote{We follow the notation and conventions of \cite{9}, to which we refer for further background and references.} Physically, we are interested in $\Omega(\Gamma; t)$ because of its role in the Strominger-Vafa program of accounting for black hole entropy in terms of D-brane microstates \cite{20}. Mathematically, we expect that $\Omega(\Gamma; t)$ will eventually be identified as something like “the Euler character of the moduli space of stable objects in the bounded derived category on $X$ with stability condition $t$.”

At present, there are very few direct computations of the BPS indices, and those which have been carried out are only valid in the “large radius regime.” That is, they assume that the Kähler classes of all effective curves are large. In such a regime one can use geometric models such as D-brane gauge theories and/or M-brane worldvolume theories. One can then reduce the computation of BPS indices to a computation in a suitable conformal field theory. This has proven to be quite successful in a number of examples \cite{15,20}. Therefore, let us focus on the large radius limit of the BPS indices. To define it we choose some vector $B + iJ$ in the complexified Kähler cone and consider the limit

$$\lim_{\Lambda \rightarrow \infty} \Omega(\Gamma; \Lambda(B + iJ)) \quad (1.1)$$
There are of course other ways of “going to infinity” (for example, a different kind of limit is taken in [1]) but we restrict attention to (1.1) in this paper. We expect - on physical grounds - that this limit exists: In the large radius limit the physics is described by some D-brane gauge theory, and there should be a well-defined and finite-dimensional space of BPS states \( \mathcal{H}(\Gamma; t) \). Somewhat surprisingly, it was pointed out in [11] that the limit (1.1) depends on the direction \( B + iJ \) chosen in the Kähler cone, even for the D4-D2-D0 system studied in [15], and hence the “large-radius limit” of the index of BPS states is not well-defined without specifying more data. This fact has recently played an important role in [5]. Our point in the present paper is that in fact the dependence of the index on the direction \( B + iJ \) can be large and this has significant implications, as explained in more detail below.

It turns out that \( \Omega(\Gamma; t) \) is only piecewise constant as a function of \( t \), and it can jump discontinuously across walls of marginal stability. While \( \Omega(\Gamma; t) \) is difficult to compute it turns out that there are fairly simple formulae for the change of \( \Omega(\Gamma; t) \) across walls of marginal stability. The only wall-crossing formula we will need in this paper is the primitive wall-crossing formula of [9] which states the following: Suppose \( \Gamma_1, \Gamma_2 \) are primitive charge vectors such that \( \Gamma = \Gamma_1 + \Gamma_2 \) then the quantity

\[
\langle \Gamma_1, \Gamma_2 \rangle \Im(Z(\Gamma_1; t)(Z(\Gamma_2; t))^* \tag{1.2}
\]

is positive in the stable region and negative in the unstable region. (“Stable” and “unstable” refer to the physical stability of the multi-centered solutions of [8]. We refer to this as “Denef stability.”) If we cross the marginal stability wall, where the phases of \( Z(\Gamma_1, t) \) and \( Z(\Gamma_2, t) \) are equal, from the positive side to the negative side across a generic point \( t_{ms} \) then

\[
\Delta \Omega = (-1)^{(\Gamma_1, \Gamma_2)} - 1 |\langle \Gamma_1, \Gamma_2 \rangle| \Omega(\Gamma_1; t_{ms}) \Omega(\Gamma_2; t_{ms}). \tag{1.3}
\]

In [8] it was pointed out that for D6-D4-D2-D0 systems there is nontrivial wall-crossing at infinite radius. In [11, 9] it was shown that even for the D4-D2-D0 system with ample D4 charge \( P \), there are walls of marginal stability going to infinity. (Such examples are only possible when the dimension of the Kähler cone is greater than one [9].) One should therefore ask how large the discontinuities in \( \Omega \) can be across walls at infinity. In this note we show that they can be large in the following sense: If we consider charges \( \Gamma \) which support regular attractor points (hence the single-centered attractor solutions of [12, 19]) then it is not consistent with wall-crossing to assume that the contribution of such states dominate the large radius limit of \( \Omega \). We show this by exhibiting an explicit example.

Our example consists of a charge which supports a regular attractor point (hence a single-centered black hole), but which also supports a 3-centered solution. The three-centered solution decays across a wall in the Kähler cone which extends to arbitrarily large radius. The contribution of the single centered solution of charge \( \Gamma \) is predicted from supergravity to be

\[
\log |\Omega| \sim S_{BH}(\Gamma) := 2\pi \sqrt{-\hat{q}_0 P^3} \tag{1.4}
\]
In our example \( \Gamma \) will support a boundstate of charge \( \Gamma_1 + \Gamma_2 \) where \( \Omega(\Gamma_1) \) has bounded entropy and \( \Gamma_2 \) itself supports a regular attractor point, but \( S_{BH}(\Gamma_2) > S_{BH}(\Gamma) \). Thus the discontinuities in the index are competitive with the single-centered entropy.

This effect of entropy dominance of multi centered configurations over single-centered ones is similar to the “entropy enigma” configurations of \([9, 10]\). In that case, if we first take large \( J_\infty \) then under charge rescaling \( \Gamma \rightarrow \Lambda \Gamma \) single centered entropy scales as \( S_{BH} \sim \Lambda^2 \) while the two-centered solutions contribute to entropy as \( S_{2c} \sim \Lambda^3 \). On the other hand, if one holds the moduli at infinity, \( J_\infty \), fixed and scales \( \Gamma \), then the configuration will eventually become unstable and leave the spectrum. In the example of the present note, we again first take large \( J_\infty \). Then we find that under rescaling D4 charge \( P \rightarrow \Lambda P \) (holding the remaining components of \( \Gamma \) fixed) the single centered entropy scales as \( S_{BH} \sim \Lambda^2 \) while the three-centered entropy scales as \( S_{3c} \sim c_{3c} \Lambda^{3/2} \), with \( c_{3c} > c_{BH} \). Thus here the entropy dominance of multicentered configuration arises from the prefactor and not from the scaling exponent. In contrast to the entropy enigma configuration, if we fix moduli at infinity \( J_\infty \) and then scale \( P \), the configuration does not leave the spectrum, as shown at the end of section 3 below.

Like the “entropy enigma” configurations, the boundstates of the present note threaten to invalidate the weak-coupling version of the OSV conjecture \([18]\) (or its refined version \([9]\)). However, as discussed at length in \([9]\), (see especially section 7.4.2), since \( \Omega \) is an index there are potential cancellations between these configurations leading to the desired scaling \( \log \Omega \sim \Lambda^2 \) for uniformly scaled charges. The point of the present note is even if we assume that there are such miraculous cancellations the index will nevertheless have large discontinuities across the MS walls, even at large radius, and hence the weak coupling OSV conjecture is at best valid in special chambers of the Kähler cone. It is notable that the phenomenon we discuss cannot happen when the Kähler cone is one-dimensional. Moreover, our example only exists in the regime of weak topological string coupling, where \( |\hat{q}_0| \) is not much larger than \( P^3 \). This regime is already known to be problematical for the OSV conjecture \([9]\).

Of course, given a charge \( \Gamma = P + Q + q_0 dV \), with \( P \) in the Kähler cone, there is a natural direction singled out, namely the \( P \) direction. It is therefore natural to suppose that the refined OSV formula of \([9]\) should apply to

\[
\lim_{\Lambda \to \infty} \Omega(\Gamma; \Lambda z P)
\]  

where \( z = x + iy \) is a complex number, and indeed, several of the arguments in \([9]\) assumed (for simplicity) that \( J \) and \( P \) are proportional.

A second, related, implication of our example concerns the modularity of generating functions for BPS indices. In \([9]\) a microscopic formulation of the “large radius” BPS indices was investigated by characterizing the BPS states as coherent sheaves supported on cycles in the linear system \( |P| \). Put differently, a D4 brane wraps a cycle \( \Sigma \in |P| \). There is a prescribed flux \( F \in H^2(\Sigma; \mathbb{Z}) \) and the system is bound to \( N \) anti-D0 branes. If we set \( d(F, N) = (-1)^{\dim \mathcal{M}} \chi(\mathcal{M}) \) where \( \mathcal{M} \) is the moduli space of supersymmetric configurations of this type then, it was claimed, the large radius BPS indices are finite
suns of the $d(F,N)$. On the other hand, duality symmetries of string theory imply that a certain generating function of the indices $d(F,N)$, denoted $Z_{D4D2D0}$, exhibits good modular behavior. It follows from the chamber dependence of the large radius limit of $\Omega$ that there must be chamber dependence of the $d(F,N)$. The chamber dependence of $d(F,N)$ raises the question of compatibility with the modularity of the partition function $Z_{D4D2D0}$. This partition function is also closely related to the $(0, 4)$ elliptic genus of the MSW string $[13, 4]$, and hence similar remarks might apply to that elliptic genus. The statement of modularity of these partition functions follows from very basic duality symmetries in string theory and conformal field theory which, one might guess, should be valid in every chamber of the Kähler cone. One might therefore expect that the change in the partition function must also be modular. It might be easier to verify this than it is to verify the modularity of the full partition function. One might approach this using the results of $[16]$: One must compute the change of the polar polynomial across a chamber and show that the associated cusp form vanishes. This appears to be a challenging computation, but one well worth doing if possible.

In section 5 we check what happens to our boundstate configurations in the near horizon scaling limit recently introduced in $[3]$. This is important since our observations regarding the entropy have the potential to lead to a troublesome contradiction with the AdS/CFT conjecture. If our configurations corresponded to states in the Cardy region of the holographic dual to an asymptotically $AdS_3 \times S^2$ geometry then there would be such a contradiction. Fortunately, our example turns out to be quite similar to that discussed in $[3]$: The first split $D4 \rightarrow D4 + D4$ corresponds to two infinitely separated $(AdS_3 \times S^2)$-like geometries, so there is no contradiction. These curious limiting geometries, and especially their holographic dual interpretation, deserve to be understood much better. Indeed, the existence of these $D4 \rightarrow D4 + D4$ decays suggests that in general one cannot identify the partition function $Z_{D4D2D0}$ of $[3]$ with the M5 elliptic genus of $[13, 4]!$. They might nevertheless agree in certain chambers of the Kähler cone (e.g. at the “AdS point” described in $[3]$). Clearly, this issue deserves to be understood better.

Finally, as a by-product of our investigation, in section 6 we discuss the general criterion, proposed in $[3]$, for $D$-brane configurations to have single asymptotic $AdS_3 \times S^2$ geometry in the near horizon limit. We give an argument, based on the Split Attractor Flow Conjecture, in favor of this criterion.

2. Some general remarks on stability at large radius

A thorough analysis of the possible walls at infinity for the D4D2D0 system, and the existence of split states in those regions is far beyond the scope of this modest note. We would, however, like to make a few elementary general points.

Let us consider a D4-D2-D0 charge $\Gamma = P + Q + q_0 dV$ splitting into a pair of charges $\Gamma = \Gamma_1 + \Gamma_2$ with

$$\Gamma_i = r_i + P_i + Q_i + q_{0,i} dV$$

(2.1)
Then \( r_1 = -r_2 = r \) and \( I_{12} = (\Gamma_1, \Gamma_2) = P_1 \cdot Q_2 - P_2 \cdot Q_1 - r q_0 \). The Denef stability condition is governed by the sign of \( I_{12} \) times the sign of \( Z_{12} := \operatorname{Im} Z_{1, hol} Z_{2, hol}^* \).

We are interested in the existence of walls at infinity. Let us consider walls which asymptotically contain lines in the Kähler moduli space. Thus, we set \( t \to \Lambda t \) and take \( \Lambda \to \infty \). If the leading term in \( Z_{12} \) at large \( \Lambda \) can change sign as the “direction” \( t \) is changed, then there will be asymptotic walls at infinity.

If \( r \) is nonzero then any wall that persists at infinity is necessarily an anti-MS wall, where the phases of \( Z(\Gamma_1; t) \) and \( Z(\Gamma_2; t) \) anti-align. There is no wall-crossing associated with such walls and thus we set \( r = 0 \).

When \( r = 0 \) (2.2) simplifies to

\[
Z_{12} = \frac{1}{4} \operatorname{Im} P_1 \cdot t^2 P_2 \cdot \bar{t}^2 \\
- \frac{1}{2} \operatorname{Im} (P_1 \cdot t^2 Q_2 \cdot \bar{t} + P_2 \cdot \bar{t}^2 Q_1 \cdot t) \\
+ \operatorname{Im} \left( Q_1 \cdot t Q_2 \cdot \bar{t} + \frac{1}{2} q_{0,1} P_2 \cdot \bar{t}^2 + \frac{1}{2} q_{0,2} P_1 \cdot t^2 \right) \\
- \operatorname{Im} (q_{0,1} Q_2 \cdot \bar{t} + q_{0,2} Q_1 \cdot t)
\]

(2.3)

For the generic direction \( t \) the leading behavior for \( \Lambda \to \infty \) will be governed by the sign of

\[
\operatorname{Im} P_1 \cdot t^2 P_2 \cdot \bar{t}^2 = (P_1 \cdot B \cdot J) P_2 \cdot B^2 - (P_2 \cdot B \cdot J) P_1 \cdot B^2 - (P_2 \cdot J^2 P_1 \cdot B \cdot J - P_1 \cdot J^2 P_2 \cdot B \cdot J)
\]

(2.4)

This vanishes in the one-modulus case, but is generically nonzero in the higher dimensional cases. Moreover, it is odd in \( B \). Therefore, just by changing the sign of \( B \) we change from a region of Denef stability to instability, and hence there are definitely walls at infinity.

As an example we analyze (2.4) for two particular examples of Calabi-Yau manifolds with a 2-parameter moduli space. The first case is the elliptic fibration \( \pi : X \to \mathbb{P}^2 \). A basis of divisors is \( D_1 = \alpha_f, D_2 = h \) with intersection products given by \( \alpha_f^3 = 9, \alpha_f^2 h = 3, \alpha_f h^2 = 1 \) and \( h^3 = 0 \). The second example is a blow-up of a hypersurface in \( \mathbb{P}^{(1,1,2,2)}[8] \) \( B \). A basis of divisors is \( H \) and \( L \) with intersection products given by \( H^3 = 8, H^2 L = 4, \)

\[ H L^2 = 0, L^3 = 0. \]

It turns out that in the elliptic fibration case (2.4) takes the form (here, superscripts denote components w.r.t. the basis \( D_1, D_2 \) above):

\[
16((B^1)^2 + (J^1)^2)(P_1^2 P_2^1 - P_1^1 P_2^2) (B^2 J^1 - B^1 J^2)
\]

(2.5)

and thus vanishes whenever \( P_1 \) becomes parallel to \( P_2 \) or \( B \) becomes parallel to \( J \). Assuming \( P_1 \) not parallel to \( P_2 \) there is exactly one wall, going to infinity with \( B \propto J \). In the case of \( \mathbb{P}^{(1,1,2,2)}[8] \) (2.4) looks like:

\[
(3 B_1 B_2 + B_2^2 + 3 J_1 J_2 + J_2^2)(P_1^{(2)} P_2^{(1)} - P_1^{(1)} P_2^{(2)})(B_2 J_1 - B_1 J_2)
\]

(2.6)
Here in addition to $B \propto J$ wall there is another wall for $3B_1 B_2 + B_2^2 + 3J_1 J_2 + J_2^2 = 0$, provided that $9B_1^2 - 12J_1 J_2 - 4J_2^2 > 0$. It is easy to see that on the $B \propto J$ wall the phases of the central charges align and hence, this is an MS and not an anti-MS wall. For the additional wall, presented above, the same is true.

It would be interesting to investigate these stable regions more thoroughly. The stability condition is necessary, but far from sufficient for the existence of BPS bound states, so one cannot immediately conclude that there is nontrivial wall-crossing. For simplicity we will henceforth take $B = 0$ in this paper. In this case the asymptotic walls are governed by the next largest term and the stability condition at large $\Lambda$ is governed by the sign of

$$ (P_2 \cdot J^2 Q_1 \cdot J - P_1 \cdot J^2 Q_2 \cdot J) $$

Again, in the one-modulus case this expression has a definite sign in accord with the analysis in [9], however, in the higher dimensional case it is perfectly possible for this quantity to change sign as $J$ changes direction in the Kähler cone. This is the example we will focus on.

3. An example

We now give an explicit example of a split of a D4D2D0 charge, which supports a single centered black hole, but which admits marginal stability walls at infinity describing a splitting into a pair of D4D2D0 systems in which the change in index $\Delta \Omega$ is larger than the single-centered entropy.

In order to have a single-centered solution we must assume $P$ is in the Kähler cone and the discriminant is positive. Therefore,

$$ \hat{q}_0 < 0 \quad \hat{q}_0 := q_0 - \frac{1}{2} Q^2|_P $$

where we recall that $Q^2|_P := (D_{ABC}P^C)^{-1} Q_A Q_B$.

In some chambers this charge can also support a multicentered solution where the first split in the attractor flow tree is given by

$$ \Gamma \rightarrow \Gamma_1 + \Gamma_2 $$

$$ \Gamma_1 = P_1 + \frac{\chi(P_1)}{24} dV $$

$$ \Gamma_2 = P_2 + Q + q_{0,2} dV $$

Here, $\Gamma_1$ is a pure D4-brane and $\Gamma_2$ is a D4-brane charge supporting a single-centered black hole: We will consider only charge configurations so that $\hat{q}_{0,2} < 0$, and hence $\Gamma_2$ has a regular attractor point.

Using the summary of split attractor flows in the appendix, we see that a necessary condition for the existence of the split realization is that the flow crosses a wall of marginal stability for $\Gamma_1$ and $\Gamma_2$, at a positive value of the flow parameter $s$. Using notations from Appendix A the flow parameter is given by:
\[ s_{ms}^{12} = 2 - (Q \cdot J - P \cdot B \cdot J) \left( \frac{1}{2} P_1 \cdot (J^2 - B^2) + \frac{\chi(P_1)}{24} \right) - \frac{1}{2} P \cdot (J^2 - B^2) + Q \cdot B - q_0 \right) \left. \right|_{\infty} \]

Here \(|_{\infty}\) means that complexified Kähler moduli \( t = B + iJ \) are evaluated at spatial infinity. The vanishing locus of \( s_{ms} \) is the wall of marginal stability. This is a rather complicated expression, but it simplifies if the starting point is chosen to have zero \( B \)-field. In that case the parameter along the flow \( s_{ms}^{12} \), for which the wall is crossed is

\[ s_{ms}^{12} = 2 - \left. \frac{Q \cdot J \left( \frac{1}{2} P_1 \cdot J^2 + \frac{\chi(P_1)}{24} \right)}{\left( \frac{1}{3} J^3 \right)^{\frac{1}{2}} P \cdot (J^2 - B^2) + Q \cdot B - q_0 + iQ \cdot J - iP \cdot B \cdot J \right|_{\infty} \cdot P_1 \cdot Q} \quad (3.3) \]

which further simplifies in the large \( J \) limit to

\[ s_{ms}^{12} = -2 \left. \frac{Q \cdot J \cdot P_1 \cdot J^2}{\left( \frac{1}{3} J^3 \right)^{\frac{1}{2}} P \cdot J^2} \right|_{\infty} \quad (3.4) \]

The condition \( s_{ms}^{12} > 0 \) (which is equivalent to the Denef stability condition) imposes a restriction on \( Q \), because we must have \((Q J_{\infty})(P_1 Q) < 0\) while both \( P_1 \) and \( J_{\infty} \) are in Kähler cone. There are plenty of charges that satisfy this condition and we’ll give a numerical example below.

We are not quite done constructing the split attractor flow tree because \( \Gamma_1 \) is a polar charge, and must itself be realized as a mult centered solution.

As discussed in appendix A, for an attractor tree to exist all its edges must exist and moreover all its terminal charges must support BPS states. The charge \( \Gamma_2 \) supports a regular black hole. Meanwhile, \( \Gamma_1 \) is realized as a flow, splitting into \( D6 \) and \( \overline{D6} \) as in [9]:

\[ \Gamma_1 \rightarrow \Gamma_3 + \Gamma_4 \]
\[ \Gamma_3 = e^{P_1/2} \]
\[ \Gamma_4 = -e^{-P_1/2} \quad (3.6) \]

So for the whole tree to exist we need

- \( s_{ms}^{12} > 0 \) for the split \( \Gamma \rightarrow \Gamma_1 + \Gamma_2 \) to exist
- \( s_{ms}^{34} > 0 \) for the split \( \Gamma_1 \rightarrow \Gamma_3 + \Gamma_4 \) to exist
- \( s_{0}^{34} > s_{ms}^{34} \) where \( s_{0}^{34} \) is the value when the flow reaches zero of the charge \( Z(\Gamma_1) \)

These conditions are sufficient because the charges \( \Gamma_3 \) and \( \Gamma_4 \) exist everywhere in moduli space and \( \Gamma \) and \( \Gamma_2 \) support black holes. It is also easy to see that both walls are MS and not anti-MS walls. It turns out that above conditions are always satisfied if
• $J_\infty$ is on stable side of the wall, corresponding to $s_{ms}^{12} > 0$

• $P_1 \ll J_\infty$ component-wise in a basis of Kähler cone

To see this we estimate $s_{ms}^{34}$ and $s_0^{34}$ in the large $J_\infty$ limit. Recall from appendix A that

$$s_{ms}^{34} = \frac{\langle \Gamma_3, \Delta H \rangle - \langle \Gamma_3, \Gamma \rangle s_{ms}^{12}}{\langle \Gamma_3, \Gamma_4 \rangle}.$$  (3.7)

Now plugging the expression for $\Delta H$ from (A.6) we can estimate $\langle \Gamma_3, \Delta H \rangle \sim \frac{J_\infty^3}{3\sqrt{4/3}J_\infty^2}$. Using $\langle \Gamma_3, \Gamma_4 \rangle = -\frac{P_1^3}{6}$ and the fact that $s_{ms}^{12} \sim O(\frac{1}{J_\infty^{1/2}})$ is small we get

$$s_{ms}^{34} \sim \frac{2J_\infty^3}{\sqrt{4/3}J_\infty^2 P_1^3}. \quad (3.8)$$

To find $s_0^{34}$ we equate the central charge to zero $Z(\Gamma_1; t) = 0$ to get the vanishing locus:

$$-\frac{\chi(P_1)}{24} - \frac{1}{2} P_1 \cdot B^2 + \frac{1}{2} P_1 \cdot J^2 = 0, \quad P_1 \cdot B \cdot J = 0 \quad (3.9)$$

Moduli along the flow of charge $\Gamma_1$ are determined again by (A.6) with $\Gamma(s) = s\Gamma_1 + s_{ms}^{12}\Gamma - \Delta H$. Recalling that $s_{ms}^{12} \sim O(\frac{1}{J_\infty^{1/2}})$ this can be written as

$$\Gamma(s) = \left( O(\frac{1}{J_\infty^{5/2}}), s P_1 + O(\frac{1}{J_\infty^{1/2}}), O(\frac{1}{J_\infty^{1/2}}), s \frac{\chi(P_1)}{24} - \frac{J_\infty^3}{2\sqrt{4/3}J_\infty^{3/2}} \right) \quad (3.10)$$

Plugging this $\Gamma(s)$ into (A.6) and taking into account that $s_0^{34} \sim O(J_\infty^{3/2})$, as we will see below, we find that

$$J(s_0^{34})^a \sim P_1^a \sqrt{-\frac{6}{P_1^3} \left( \frac{\chi(P_1)}{24} - \frac{J_\infty^3}{2s_0^{34} \sqrt{4/3}J_\infty^{3/2}} \right)} \quad (3.11)$$

and $B^a(s_0^{34})$ is small. Now we can solve (3.10) for $s_0^{34}$ to find:

$$s_0^{34} \sim \frac{6J_\infty^3}{\sqrt{4/3}J_\infty^{3/2} (P_1)^3} \quad (3.12)$$

Thus we see from (3.8) and (3.12) that the existence conditions are indeed satisfied: $s_0^{34} > s_{ms}^{34}$.

We conclude with a numerical example, checking explicitly that such split solutions exist. We consider again the elliptic fibration example and $\mathbb{P}^{(1,1,2,2,2)}[8]$ of $\mathfrak{g}$.

The initial charge is of the form $\Gamma = P + Q + q_0 dV$, where $P = (50, 50)$, $Q = (-1, 3)$, $q_0 = -10$. The starting point of the flow is $J_\infty = (500, 100)$, which indeed lies on stable side of MS wall in (3.4). The pure $D4$ has charge $P_1 = (1, 2)$. All the existence conditions are found to be satisfied for both Calabi-Yau manifolds. As we'll discuss in the next section, the entropy of this three-centered configuration is expected to be larger than the one from

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the single-centered realization of the same total charge. The numerical examples confirm
this claim in both cases.

Now we will justify the remark made in the introduction about the existence of the
split state for \( P \to \infty \). We take \( B_\infty = 0 \) and evaluate (3.3). Evaluating (3.4) in the limit
\( P \to \infty \) and with fixed \( J_\infty \) produces an expression almost identical to (3.3). In particular,
it remains positive, but does go to zero. The second split \( \Gamma_1 \to \Gamma_3 + \Gamma_4 \) will therefore
happen very close to starting point in moduli space and hence \( J_\infty \gg P_1 \) will guarantee
that the second split exists. This proves that our example exists in the \( P \to \infty \) limit if it
existed in \( J_\infty \to \infty \) limit.

4. Comparison of the entropies

Now let us compare the discontinuity \( \Delta \Omega \) of the BPS index with the contribution of the
single-centered (black hole) solutions to the “large radius” index \( \Omega(\Gamma; J_\infty) \).

We first assume that the dominant contribution to the large radius entropy is that
of the single-centered solutions, if they exist. We will then show that this assumption is
inconsistent with the wall-crossing phenomena.

The black hole contribution to \( \Omega \) can be approximated using the equation from the
attractor mechanism

\[
\Omega_{BH}(\Gamma) := \exp S_{BH}(\Gamma) = \exp \left[ 2\pi \sqrt{-\hat{q}_0 P^3/6} \right], \tag{4.1}
\]

The discontinuity of the index across the wall \( \Gamma \to \Gamma_1 + \Gamma_2 \) is given by

\[
\Delta_{12} \Omega(\Gamma; t_{ms}) = (-1)^{(\Gamma_1, \Gamma_2)-1} |(\Gamma_1, \Gamma_2)| \Omega(\Gamma_1; t_{ms}^{12}) \Omega(\Gamma_2; t_{ms}^{12}) \tag{4.2}
\]

Here the indices of \( \Gamma_1 \) and \( \Gamma_2 \) are evaluated on the MS wall. As we have said, the state with
charge \( \Gamma_1 \) is realized as a split attractor flow splitting into pure \( D_6 \) and \( \overline{D_6} \) with fluxes.
The index of \( \Gamma_1 \) is polynomial in charges and is given by \( \Omega(\Gamma_1) = (-1)^{I(P_1)-1} \Gamma(P_1) \) where
\( I(P) := \frac{P^3}{6} + \frac{\alpha(X) \cdot P}{12} \). Again using our assumption we would estimate that the index of \( \Gamma_2 \)
can again be approximated by the black hole contribution:

\[
\Omega(\Gamma_2, J_\infty) \sim \Omega_{BH}(\Gamma_2) = \exp \left[ 2\pi \sqrt{-\hat{q}_{0,2} P_2^3/6} \right] \tag{4.3}
\]

since \( \Gamma_2 \) supports a single-centered black hole.

We now consider a limit of large charges. We hold \( P_1 \) fixed and take \( P \to \infty \) along
some direction in the Kähler cone. Then from Eqs. (4.1), (4.3) the indices of \( \Gamma \) and \( \Gamma_2 \) will
be exponentially large for large \( P \) while \( \Omega(\Gamma_1) \) is a known, bounded function of \( P_1 \). This
means that to compare the contributions (4.1) and (4.2) we need to compare the exponents:

\[
-\hat{q}_0 P^3 \quad \text{vs} \quad -\hat{q}_{0,2} P_2^3 \tag{4.4}
\]

In this limit we can write

\[
P_2^3 = P^3 - 3P^2 \cdot P_1 + ... = P^3 \left( 1 - \frac{3P^2 \cdot P_1}{P^3} + O(1/|P|^2) \right). \tag{4.5}
\]
Moreover, since $q_0$ is conserved at the vertex

$$\hat{q}_{0,2} = \hat{q}_0 + \frac{1}{2} Q^2 |_P - \frac{\chi(P_1)}{24} - \frac{1}{2} Q^2 |_{P_2} \quad (4.6)$$

In taking our charge limit we can make $q_{0,2}$ sufficiently negative that $\hat{q}_{0,2}$ and $\hat{q}_0$ are both negative. Now we can write

$$-\hat{q}_{0,2} P_2^3 = -\hat{q}_0 P^3 \left(1 - \frac{\chi(P_1)}{24 \hat{q}_0} - \frac{1}{2 \hat{q}_0} (Q^2 |_{P_2} - Q^2 |_P) - \frac{3 P^2 \cdot P_1}{P^3} + O(1/|P|^2) \right) \quad (4.7)$$

Since $\hat{q}_0$ is negative we see from (4.7) that the contribution of the $\Gamma \rightarrow \Gamma_1 + \Gamma_2 \rightarrow (\Gamma_3 + \Gamma_4) + \Gamma_2$ split attractor flow will be greater in the $P \rightarrow \infty$ limit provided that

$$\frac{\chi(P_1)}{24 \hat{q}_0} + \frac{1}{2 \hat{q}_0} (Q^2 |_{P_2} - Q^2 |_P) - \frac{3 P^2 \cdot P_1}{P^3} > 0 \quad (4.8)$$

The first term of (4.8) is always positive, while the second term can have both signs. The third term is always negative. However, for parametrically large $P$ and fixed $Q$ the second and third terms are suppressed, so the expression is positive. Thus we find that in the limit described above, the split flow configuration has greater entropy than the black hole contribution:

$$\Omega_{BH}(\Gamma) \ll \Delta_{12} \Omega(\Gamma; t_{ms}) . \quad (4.9)$$

Thus, as explained in the introduction, not only does the value of the index $\Omega$ depend on the direction in which $J$ is taken to infinity, but this dependence can be very strong, and even dominate single-centered black hole entropy.

One might worry that there are other split flow realizations of the charge $\Gamma$, with the same wall of marginal stability as the one we are studying, which produce a cancellation in $\Delta \Omega$. For example, the charge $\Gamma_2$ might well support multi-centered solutions. However, by our hypothesis, the single-centered entropy dominates the multi-centered ones, so such a cancellation cannot occur. Then (4.9) leads to a contradiction and hence we conclude that it cannot be that single-centered entropy dominates the entropy at infinity in all chambers.

Remarks

1. In the context of topological string theory the topological string coupling $g_{top} \sim \sqrt{-q_0/P^3}$ [18]. The effect we are discussing does not appear in the strong coupling regime, in harmony with the arguments in [3]. However, it does appear in the problematic weak coupling regime.

2. Interestingly, this phenomenon will not occur with splits into two single-centered attractors. If $q_{0,i} < 0$ for both $i = 1, 2$ and $P_1, P_2$ are in the Kähler cone then (taking $Q_i = 0$ for simplicity) one can show that

$$S_{BH}(\Gamma) > S_{BH}(\Gamma_1) + S_{BH}(\Gamma_2) \quad (4.10)$$

as expected. We do not know of a proof of the analogous statement for $Q_i \neq 0$. 

- 10 -
3. In principle the example we have given can be extended by replacing $\Gamma_1$ by an arbitrary extreme polar state in the sense of [9]. Following [9], the charges $\Gamma_1 \longrightarrow \Gamma_3 + \Gamma_4$ can be parametrized as

\[
\begin{align*}
\Gamma_3 &= re^{S_1}(1 - \beta_1 + n_1w) \\
\Gamma_4 &= -re^{S_2}(1 - \beta_2 + n_2w) \\
\Gamma_1 &= r \left( 0, \frac{\hat{P}}{2} + \Delta \beta, \frac{\hat{P}\beta^2}{24} + \frac{\hat{P}\beta}{2} + \frac{S\Delta \beta}{2} - \Delta nw \right) \quad (4.11)
\end{align*}
\]

where $\hat{P} = S_1 - S_2$, $S = S_1 + S_2$, $\beta = \beta_1 + \beta_2$, $\Delta \beta = \beta_2 - \beta_1$, $\Delta n = n_2 - n_1$. For sufficiently small $\beta_i$ and $n_i$ and $S_1 \equiv P_1/2$, $S_2 \equiv -P_1/2$, the charge $\Gamma_1$ is very close to a pure $D4$-brane and all existence conditions are still satisfied. The 3-centered entropy dominance also continues to hold.

5. M-theory lift and its near-horizon limit

In this section we discuss the M-theory lift of the 3-centered configuration of interest and analyze its near horizon limit following the procedure of [5]. Our motivation here is to relate our configurations to the MSW conformal field theory, and to check that there is no contradiction with AdS/CFT.

The solution to the attractor equations in the effective 4d $\mathcal{N} = 2$ SUGRA for a general multicentered configuration can be written (in the regime of large Kähler classes) in terms of harmonic functions ([1], eq. (2.8)):

\[
\begin{align*}
&ds^2_{4d} = -\frac{1}{\Sigma} (dx_0 + \sqrt{G_4} \omega)^2 + \Sigma (d\vec{x})^2, \\
&A^0 = \frac{\partial \log \Sigma}{\partial H_0} \left( \frac{dx_0}{\sqrt{G_4}} + \omega \right) + \omega_0, \\
&A^A = \frac{\partial \log \Sigma}{\partial H_A} \left( \frac{dx_0}{\sqrt{G_4}} + \omega \right) + A^A_d, \\
&t^A = \frac{H^A}{H^0} + \frac{y^A}{Q^2} \left( i\Sigma - \frac{L}{H^0} \right), \quad (5.1)
\end{align*}
\]

where
\[ *d\omega = \frac{1}{\sqrt{G_4}} (dH, H), \quad d\omega_0 = \frac{1}{\sqrt{G_4}} * dH^0 \]
\[ dA_d^A = \frac{1}{\sqrt{G_4}} * dH^A, \quad \Sigma = \sqrt{Q^3 - L^2 / (H^0)^2} \]

\[ L = H_0(H^0)^2 + \frac{1}{3} D_{ABC} H^A H^B H^C - H^A H_A H^0, \quad (5.2) \]
\[ Q^3 = \left( \frac{1}{3} D_{ABC} y^A y^B y^C \right)^2, \quad D_{ABC} y^A y^B = -2H_C H^0 + D_{ABC} H^A H^B \]
\[ H \equiv (H^0, H^A, H_A, H_0) := \sum_a \frac{\Gamma_a \sqrt{G_4}}{|x - \bar{x}_a|} - 2\text{Im} \left( e^{-i\alpha} \Omega \right)|_{\bar{x}=\infty}, \quad (5.3) \]

\[ A = 1, \ldots, h^{1,1}(X) \] are components relative to a basis \( D_A \) for \( H^2(X, \mathbb{Z}) \), \( * \) is the Hodge star with respect to the Euclidean metric \( d\bar{x}^2 \) on \( \mathbb{R}^3 \), and we choose a solution \( y^A \) of the quadratic equations such that \( y^A D_A \) is in the Kähler cone. The Calabi-Yau volume in string units is given by

\[ \tilde{V}_{IIA} = \frac{D_{ABC}}{6} J^A J^B J^C = \frac{1}{2} \Sigma^3 Q^3 \]

and \( G_4 \) is the 4-dimensional Plank constant, determined in terms of the string length \( l_s \) and string coupling \( g_s \) by

\[ G_4 = \frac{l_s^2 g_s^2}{32 \pi^2 \tilde{V}_{IIA,\infty}}. \quad (5.4) \]

The above equations assume \( H^0(\bar{x}) \) is nonzero, but they have a smooth limit as \( H^0 \rightarrow 0 \). (See [17] eq. (9.21) for the relevant expansions.)

This solution of 4d supergravity can be lifted to 5d supergravity. To do this we use the standard relation between \( M \)-theory and IIA geometries

\[ ds_{5d}^2 = \frac{R^2}{4} e^{2\phi} (d\psi + A^0)^2 + e^{2\phi} ds_{4d}^2, \]
\[ Y^A = \tilde{V}_{IIA}^{-1/3} J^A, \quad A_{5d}^A = A^A + B^A (d\psi + A^0) \quad (5.5) \]

Here \( R \) is the M-theory circle radius, \( \psi \sim \psi + 4\pi \), \( Y^A \) are 5d SUGRA moduli, and \( \phi(\bar{x}) \) is the 10d dilaton field, normalized as \( \phi(\infty) = 0 \). Note that the Calabi-Yau volume in 11d Planck units is

\[ \tilde{V}_M = e^{-2\phi} \tilde{V}_{IIA} / g_s^2. \quad (5.6) \]

The near horizon limit of the \( M \)-theory solution, introduced in [9], may be described as follows. Beginning with a solution (5.1) we introduce a family of BPS solutions of the 4d supergravity equations, parametrized by \( \lambda \in [1, \infty) \). The expressions that get modified
under this deformation are given by

\[ ds_{4d, \lambda}^2 = \frac{1}{\Sigma^\lambda} (dx_0 + \lambda^{-3/2} \sqrt{G_4} \omega^\lambda)^2 + \lambda^{-6} \Sigma^\lambda (d\bar{x})^2, \]

\[ \mathcal{A}^0_\lambda = \frac{\partial \log \Sigma^\lambda}{\partial H^\lambda_0} \left( \lambda^{3/2} \frac{dx_0}{\sqrt{G_4}} + \omega^\lambda \right), \]

\[ \star d\omega^\lambda = \frac{\lambda^{-3/2}}{\sqrt{G_4}} \langle dH^\lambda, H^\lambda \rangle, \quad d\omega^0_\lambda = \frac{\lambda^{-3/2}}{\sqrt{G_4}} \star dH^0_\lambda, \]

\[ H^\lambda := \lambda^{3/2} \sum_a \Gamma_a \sqrt{G_4} \frac{1}{|\bar{x} - \bar{x}^\lambda_a|} - 2\text{Im}(e^{-i\alpha_\infty} \Omega)|_{B_\infty + i\lambda J_\infty} \]

(5.7)

Here, \( \Omega = -\frac{1}{\sqrt{4/3} J^3} e^{B+iJ} \) and for brevity we omit the corresponding formulae for \( A^A_\lambda \) and \( A^A_{d, \lambda} \). The vectors \( \bar{x}^\lambda_a \) used to define \( H^\lambda \) can be taken to be any solution of the integrability constraints

\[ \sum_{b \neq a} \left\langle \Gamma_a, \Gamma_b \right\rangle \bar{x}^\lambda_{ab} = -\lambda^{-3} \sqrt{\frac{3}{G_4 J^3_\infty}} \text{Im} \left( e^{-i\alpha_\infty, \lambda} \int \Gamma_b e^{-(B_\infty + i\lambda J_\infty)} \right) \quad \forall b. \]  

(5.8)

where \( x^\lambda_{ab} := |\bar{x}^\lambda_a - \bar{x}^\lambda_b| \) and \( e^{i\alpha_\infty, \lambda} \) is the phase of the total central charge at \( B_\infty + i\lambda J_\infty \). We choose \( \bar{x}^\lambda_a \) to coincide with our original solution at \( \lambda = 1 \), and let them depend continuously on \( \lambda \). Clearly there is some degree of arbitrariness at this stage.

The above family of solutions can be obtained from original ones by scaling (5.2)

\[ \bar{x} \rightarrow \lambda^{-3} \bar{x}, \quad l_s \rightarrow \lambda^{-3/2} l_s, \quad g^2_s \rightarrow \lambda^3 g^2_s, \quad G_4 \rightarrow \lambda^{-3} G_4, \quad J_\infty \rightarrow \lambda J_\infty, \quad B_\infty \rightarrow B_\infty \]

(5.9)

but we prefer to keep \( \bar{x}, l_s, G_4 \) fixed and change the solution according to (5.7). The constant \( G_4 \), and the coordinate system, in these equations is \( \lambda \)-independent.

Now consider the corresponding \( \lambda \)-deformed 5d geometries. Since the moduli \( t^A(\bar{x}; \lambda) \) determined by (5.1) scale as \( \lambda^0 \) for \( \lambda \rightarrow \infty \) (at least when \( H^0(\bar{x}) \neq 0 \)) it is clear that if the \( \bar{x}^\lambda_a \) have a well-defined limit then there are well-defined limiting moduli \( \tau^A(\bar{x}) := \lim_{\lambda \rightarrow \infty} t^A(\bar{x}; \lambda) \). One must be careful because the limits \( \bar{x} \rightarrow \infty \) and \( \lambda \rightarrow \infty \) do not commute. Indeed \( t^A(\bar{x}; \lambda) \rightarrow B^A_\infty + i\lambda J^A_\infty \) as \( \bar{x} \rightarrow \infty \) for any fixed \( \lambda \) while \( \tau^A(\bar{x}) \) has asymptotics for large \( x = |\bar{x}| \):

\[ \tau^A = D^{AB} Q_B + \mathcal{O}(1/x) + i \sqrt{\frac{3|x|}{P^3}} (J^3_\infty / 3)^{1/4} P^A (1 + \mathcal{O}(1/x)) \]

(5.10)

2In principle some components of the moduli space of solutions to (5.8) might be obstructed by the positivity of the discriminant.

\[ \quad \]

\[ \quad \]
This implies that the 5d SUGRA moduli $Y^A(\vec{x})$ have well-behaved large $\vec{x}$ asymptotics

$$Y^A(\vec{x}) = \frac{P^A}{(P^3/6)^{1/3}} + \mathcal{O}(1/|\vec{x}|). \quad (5.11)$$

Moreover, since the 10d dilaton scales according to (5.6) as $e^{2\phi^A(\vec{x})} = \frac{\tilde{V}_M}{\lambda^2 V_M}$ ($\tilde{V}_M$ is $\lambda$ independent), $e^{2\phi^A(\vec{x})}$ for fixed $\vec{x}$ scales as $\lambda^{-3}$. Note, however, that in the other order of limits $\phi^A(\infty) = 0$. The corresponding 5d metric for the deformed solution $\lambda^2 ds^2_{5d,\lambda}$ has a well-defined limit. Reference [5] shows that this limiting solution defines a geometry which is asymptotically $AdS_3 \times S^2$, where there is a nontrivial connection on the (trivial) $S^2$ bundle over the asymptotic $AdS_3$ region.

The upshot is that if we can choose the centers $\vec{x}_a^\lambda$, constrained by (5.8), so that the $\vec{x}_a^\lambda$ have a well-defined finite limit as $\lambda \to \infty$ then, by AdS/CFT, the BPS states corresponding to the multicentered solution at $\lambda = 1$ should correspond to BPS states in the MSW conformal field theory. However, it can happen that as $\lambda \to \infty$ the distances between the centers $\vec{x}_a^\lambda$ cannot remain bounded. In this case the behavior of the limiting geometry is more complicated, and might involve, for example, “several $AdS_3 \times S^2$ geometries at infinite separation.” In particular, note that if the total $D6$ charge vanishes then $\alpha_{\infty,\lambda} \to 0$ and hence those integrability equations (5.8) with $\Gamma^0_b = 0$ have a zero on the RHS. This might force some centers to move to infinity.

In view of the above results we next turn to our 3-centered configuration and examine the integrability conditions on the positions of the three centers. For the set of charges described in section 3 we have two independent equations:

$$-\langle \Gamma_2, \Gamma_3 \rangle_{x_{23}^\lambda} + \langle \Gamma_3, \Gamma_4 \rangle_{x_{34}^\lambda} = \theta_3^\lambda \quad (5.12)$$

$$-\langle \Gamma_2, \Gamma_4 \rangle_{x_{24}^\lambda} - \langle \Gamma_3, \Gamma_4 \rangle_{x_{34}^\lambda} = \theta_4^\lambda$$

where $\theta_b^\lambda$ denote (minus) the right-hand-sides of (5.8). The intersections of charges take the form:

$$\langle \Gamma_3, \Gamma_4 \rangle = -\frac{P^3}{6} := c$$

$$\langle \Gamma_2, \Gamma_3 \rangle = \left(\frac{P \cdot P_2^2}{8} - \frac{P^3}{8} + q_{0.2}\right) - \frac{Q \cdot P_1}{2} := a - b$$

$$\langle \Gamma_2, \Gamma_4 \rangle = -\left(\frac{P \cdot P_2^2}{8} - \frac{P^3}{8} + q_{0.2}\right) - \frac{Q \cdot P_1}{2} := -a - b \quad (5.13)$$

Using the charges of section 3 and the limit $P \to \infty$ holding $P_1$ fixed, we have $a \gg b, c$ and $c < 0$. As for the sign of $b$ we first choose $b > 0$ and explain the case $b < 0$ later. Equations (5.12) determine $x_{23}^\lambda$ and $x_{24}^\lambda$ in terms of $x_{34}^\lambda$. As discussed above, there is still freedom in choosing the dependence of $x_{34}^\lambda$ on $\lambda$. One way to fix this freedom is to choose
$x_{34}^{\lambda}$ independent of $\lambda$. The relations between $x_{ab}^{\lambda}$, following from (5.12) are subject to the triangle inequalities. The moduli space of solutions will generically consist of several intervals on the $x_{34}^{\lambda}$ line. The relation between these intervals and topologies of attractor flow trees is the essence of the Split Attractor Flow Conjecture (SAFC) [6], which we recall in Appendix A for convenience.

![Figure 1: The two contributing topologies of attractor trees. The left tree is the main example of this paper. The right tree also exists, for our charges, in certain regions of moduli space.](image)

In the present case the two possible attractor flow tree topologies are shown in Figure 1. To identify the region corresponding to the left tree, we tune the moduli at infinity to be close to the $D4 \rightarrow D4 + D4$ MS wall. This means choosing $\theta^\lambda_2 = - (\theta^\lambda_3 + \theta^\lambda_4)$ close to zero. We can then write the triangle inequalities as follows:

\[
\frac{a-b}{c-\theta^\lambda_4 x_{34}} + \frac{a+b}{c+\theta^\lambda_4 x_{34}} \geq 1 \\
\frac{a-b}{c-\theta^\lambda_3 x_{34}} + 1 \geq \frac{a+b}{c+\theta^\lambda_3 x_{34}} \\
1 + \frac{a+b}{c+\theta^\lambda_4 x_{34}} \geq \frac{a-b}{c-\theta^\lambda_3 x_{34}}
\]

(5.14)

Close to the MS wall $\theta^\lambda_2 = 0$, we can write $\theta^\lambda_3 = - \theta^\lambda_3 - \theta^\lambda_4$, solve inequalities (5.14) and expand the solution to first order in $\theta^\lambda_2$. Using in addition the relations between the magnitudes of $a, b, c$, we get the following solutions to (5.14):

\[
-\frac{c}{\theta^\lambda_4} + \frac{c}{2(\theta^\lambda_4)^2} \theta^\lambda_2 \leq x_{34} \leq \frac{2a}{\theta^\lambda_4} - \frac{a}{(\theta^\lambda_4)^2} \theta^\lambda_2 \\
x_{34} \leq -\frac{2b-c}{\theta^\lambda_4} - \frac{a(2b+c)}{2b(\theta^\lambda_4)^2} \theta^\lambda_2 \\
x_{34} \leq -\frac{c}{\theta^\lambda_4} + \frac{ac}{2b(\theta^\lambda_4)^2} \theta^\lambda_2
\]

(5.15)

It is easy to see from these inequalities that for $\theta^\lambda_2 < 0$ the solution consists of a point and an interval:
\[ x_{34} \in \left\{ -\frac{c}{\theta_4^2} + \frac{ac}{2b(\theta_4^2)\theta_2^2} \right\} \bigcup \left\{ \frac{2b - c}{\theta_4^2} + \frac{a(2b - c)}{2b(\theta_4^2)^2} \theta_2^4, \frac{2a}{\theta_4^2} - \frac{a}{(\theta_4^2)^2} \theta_2^2 \right\}. \]  

(5.16)

\[ D4 \rightarrow D4D4 \quad D4 \rightarrow D6\bar{D}6 \]

Figure 2: The two intervals, corresponding to topologies of Figure 1.

On the other hand for \( \theta_2^4 > 0 \) the point disappears, and the solution is just an interval. Thus, under the SAFC correspondence, the attractor tree topology of our main example is identified with the component of the moduli of solutions to (5.12), given by the point on the \( x_{34}^4 \) line. In the above we have chosen a definite sign of \( b \), but it is easy to check that choosing \( b < 0 \) would lead to the existence of a point for \( \theta_2^4 > 0 \), and absence of it for \( \theta_2^4 < 0 \). This can also be seen from the stability condition for the \( D4 \rightarrow D4D4 \) split, \( -\frac{\theta_2^4}{(\theta_1^4)^2} > 0 \), taking into account \( \langle \Gamma_1, \Gamma_2 \rangle = 2b \).

Having identified the intervals with the corresponding topologies we can investigate what happens to each interval as we change \( \lambda \) from 1 to \( \infty \). From the functional form of \( \theta_4^4 \) it is easy to see that \( \theta_2^4 = O(\lambda^{-2}) \) and \( \theta_4^4 = O(1) \) as \( \lambda \rightarrow \infty \). Thus in the near horizon limit the point on the \( |\vec{x}_{34}^4| \) line corresponding to the topology of interest goes to \( |\vec{x}_{34}^4| = -\frac{\theta_4^4}{\theta_1^4} \). This means that \( \vec{x}_{23}^4, \vec{x}_{24}^4 \rightarrow \infty \) as \( \lambda \rightarrow \infty \) and we get an infinite separation between charges \( \Gamma_2 \) and \( \Gamma_3 + \Gamma_4 \).

The conclusion is that our 3-centered configuration does not correspond to a single smooth geometry with \( AdS_3 \times S^2 \) asymptotics in the near horizon limit of [5]. This is just as well, as pointed out in the introduction.

6. Some general remarks on holographic duals of \( D4D4 \) boundstates.

As a byproduct of our investigation of the previous section we would like to make some more general remarks concerning the relation between the split attractor flows and the existence of a near horizon geometry with a single \( AdS_3 \times S^2 \) boundary. In [5] it is stated that configurations with the first split of the type \( D4 \rightarrow D4 + D4 \) do not correspond to geometries with a single \( AdS_3 \times S^2 \) boundary. In this section we will refine this statement.

We begin with the integrability conditions:

\[ \sum_{b \neq a} \frac{\langle \Gamma_a, \Gamma_b \rangle}{x_{ab}} = \theta_a \quad \theta_a := 2\text{Im}(e^{-ia}Z(\Gamma_a))_\infty \]  

(6.1)

and denote by \( M(\theta) \) the moduli space of solutions in \( \vec{x}_a \) to (6.1). The decomposition of the charges in the first split defines a disjoint decomposition of the charges into two sets \( A \bigcup B \). Then, summing (6.1) over all charges in one cluster we get:
\[
\sum_{a \in A, b \in B} \frac{\langle \Gamma_a, \Gamma_b \rangle}{x_{ab}} = \theta_A := 2\text{Im}(e^{-i\alpha} Z(\Gamma_A))_\infty
\] (6.2)

**Conjecture 1:** The component of \(M(\theta)\) that corresponds to a topology with the first split \(D4 \rightarrow D4 + D4\) according to \(A \sqcup B\) under the SAFC, has the property: if \(\sum_{a \in A} \theta_a \rightarrow 0\), then \(x_{ab} \rightarrow \infty\) for \( \forall a \in A, b \in B\).

We do not know the proof of this statement but our previous 3-centered example can serve as an illustration of it. A suggestive argument here is the following: Tune the moduli at infinity \(t_\infty\) close to the MS wall of the first split. Then, according to the SAFC, for the \(D4 \rightarrow D4D4\) component of moduli space the \(D4\) clusters will become separated, and denoting the maximum size of these clusters by \(d\), we can write (6.2) as

\[
\frac{\langle \Gamma_A, \Gamma_B \rangle}{r_{AB}} \left(1 + O\left(\frac{d}{r_{AB}}\right)\right) = \theta_A.
\] (6.3)

If one could argue, that as \(\theta_A \rightarrow 0\) the sizes of clusters will remain much smaller than the separation between them \(d \ll r_{AB}\), then we necessarily have \(r_{AB} \rightarrow \infty\) and Conjecture 1 follows. Unfortunately, in general the sizes of clusters can grow as we change \(\theta_a\)'s, so this argument does not always apply and one needs a more detailed knowledge of the moduli space of solutions to (6.1).

A related issue that we wish to address is a conjecture of [5], relating multicentered solutions with single \(AdS_3 \times S^2\) near horizon geometry and attractor flow trees at the “AdS point.” The “AdS point” is given by

\[
t_{AdS} = D^{AB} Q_B + i\infty P_A
\] (6.4)

This is a point on the boundary of moduli space given by \(\lim_{u \rightarrow \infty} D^{AB} Q_B + iu P_A\) and we are considering limits of attractor flows with \(D^{AB} Q_B + iu P_A\) as an initial point. Note that it is naturally selected by the near horizon limit (5.10). Note that the component of moduli space with first split \(D4 \rightarrow D4 + D4\), does not correspond to a single \(AdS_3 \times S^2\), and this component also does not exist at the AdS point. This motivated [5] to suggest:

**Conjecture 2:** There is a one to one correspondence between (i) components of the moduli space of lifted multicentered solutions with a single \(AdS_3 \times S^2\) asymptotic geometry and (ii) attractor flow trees starting at the AdS point.

We now give an argument in favor of this conjecture. As discussed in Appendix A, the attractor tree is specified by the \(H\)-functions:

\[
H(s^{(a)}) = \Gamma^{(a)} s^{(a)} - \Delta H^{(a)},
\] (6.5)

where \(s^{(a)}\) is the parameter along the flow on the \(a\)-th edge. The rescaling in (5.7) leading to the near horizon limit of [4] results in changing the \(H\)-functions to

\[
H(s^{(a)}) \rightarrow H^{\lambda}(s^{(a)}) = \lambda^{3/2} \Gamma^{(a)} s^{(a)} - \Delta H^{(a)}\lambda.
\] (6.6)
According to (3.3), $\Delta H_\lambda^{(a)}$ depend linearly on and are completely determined in terms of $\Delta H_\lambda$, and $\Delta H_\lambda = 2\text{Im}(e^{-i\Omega}t^\perp_\infty)$, where $t^\perp_\infty := B_\infty + i\lambda J_\infty$. As the solution for the moduli are homogeneous of degree zero in $H$, we can replace these $H^\lambda$-functions with:

$$H^\lambda(s^{(a)}) \longrightarrow \tilde{H}^\lambda(s^{(a)}) = \Gamma^{(a)} s^{(a)} - \Delta \tilde{H}^{(a)}_0,$$

$$\Delta \tilde{H}^\lambda = \lambda^{-3/2}2\text{Im}(e^{-i\Omega})|_{t^\perp_\infty}.$$

We will refer to the split flow defined by (6.7) as a $\lambda$-deformed flow. Note that for $\lambda$-deformed flows the values of MS wall crossings parameters $s^{(a)}_{ms} \lambda$ in (3.3) will depend on $\lambda$. Our argument will be based on two assumptions:

**Assumption 1:** There is a $\lambda$-deformed version of the SAFC. That is, the components of the moduli space of $\lambda$-deformed solutions (5.4) are in one to one correspondence with $\lambda$-deformed attractor flow trees.

**Assumption 2:** The $\lambda$-deformed solution “survives” the near horizon limit, i.e. it corresponds to an asymptotically $AdS_3 \times S^2$ geometry, iff the corresponding $\lambda$-deformed attractor flow tree has all its flow parameters $s^{(a)}_{ms} \lambda$ nonzero (and positive) in the limit $\lambda \longrightarrow \infty$. The attractor flow tree exists at the $AdS$ point iff all it’s flow parameters $s^{(a)}_{ms}$ stay nonzero (and positive) as it’s starting point approaches $AdS$ point.

The second assumption is of course closely related to Conjecture 1 above, because for the first split $D4 \longrightarrow D4 + D4$ we have $s_{ms} = \sum_{A,B} a_{AB} s^{(a)}_{AB}$. Given the above assumptions we want to prove that there is a one to one correspondence between $\lambda$-deformed attractor flow trees, that “survive” the near horizon limit in the sense of Assumption 2, and regular (not $\lambda$-deformed) attractor flow trees, that start at the $AdS$ point (i.e. that have initial point approaching this boundary point as $\lambda \longrightarrow \infty$).

First, we note that the first split of a $\lambda$-deformed flow that “survives” the limit must be $D4 \longrightarrow D6 + \overline{D6}$. To see this we use (3.6), to estimate the $\lambda$ dependence of $\Delta \tilde{H}^\lambda$:

$$\tilde{H}^\lambda = (\Delta \tilde{H}^0, \Delta \tilde{H}^A, \Delta \tilde{H}_A, \Delta \tilde{H}_0) \sim (\lambda^{-4}, \lambda^{-2}, \lambda^{-2}, \lambda^0).$$

(6.8)

From this we find that for $D4 \longrightarrow D6 + \overline{D6}$ the flow parameter of the first split is $s^{\lambda}_{ms} \sim \lambda^0$, while for $D4 \longrightarrow D4 + D4$ it is $s^{\lambda}_{ms} \sim \lambda^{-2}$. This means that only $D4 \longrightarrow D6 + \overline{D6}$ is a valid split in the limit $\lambda \longrightarrow \infty$.

For the chosen attractor trees we next look at the first edge of the flow tree in the moduli space. Using formula A.6 from Appendix A, the complexified Kähler moduli are:

$$B^\lambda_A(s) = D^{AB} \left( sP^C - \Delta \tilde{H}^C_X \right) (sQ_B - \Delta \tilde{H}^{\lambda}_B)$$

$$J^\lambda_A(s) = (sP^A - \Delta \tilde{H}^A_X) \sqrt{-6(sq_0 - \Delta \tilde{H}^0_0 - 1/2Q^2(s))/(sP - \Delta \tilde{H}^{\lambda}_0)}$$

(6.9)

Figure 3 shows that the flow starts at $t^\perp_\infty$; but for the flow parameter $s \sim \frac{1}{\lambda^2}$ the first term in $(sP^A - \Delta \tilde{H}^A_X)$ becomes comparable with second term and then starts to dominate, so that the flow will go along the $P$ direction. The transition from $J_\infty$ asymptotics to $P$
Figure 3: The behavior of the flow for the first edge of the tree.

asymptotics occurs around $s \sim \frac{1}{\lambda^2}$. Also note that the first split $D6\bar{D}6$ occurs long after this region at $s_{ms}^{(a)\lambda} \sim \lambda^0$.

Now choose a value $\tilde{s}^{\lambda}$ of the flow parameter that goes to zero more slowly than $\frac{1}{\lambda^2}$, e.g. $\tilde{s}^{\lambda} \sim \frac{1}{\lambda^{2-\epsilon}}$, with small $\epsilon > 0$. From (6.9), it follows that $J^{A}(\tilde{s}^{\lambda})$ will approach the $P$ direction as $\lambda \to \infty$, and grow as $\lambda^{1-\epsilon/2}$, i.e.

$$t^{A}(\tilde{s}^{\lambda}) \sim D^{AB}(P)Q_{B}(1 + O(\lambda^{-\epsilon})) + i\lambda^{1-\epsilon/2}P^{A} \text{const} \left(1 + O\left(\frac{1}{\lambda}\right)\right). \tag{6.10}$$

We can think of the part of the attractor flow tree that starts at $J^{A}(\tilde{s}^{\lambda})$ as a tree on its own. It is again constructed in terms of $H$-functions, but now the $\Delta \tilde{H}^{\lambda}$ function will look like:

$$\Delta \tilde{H}^{\lambda} = \lambda^{-3/2}2\text{Im}(e^{-i\alpha}\Omega)|_{t(\tilde{s}^{\lambda})}. \tag{6.11}$$

The only difference of this $\Delta \tilde{H}^{\lambda}$ with the $\Delta H$ of the $\lambda$-undeformed flow with starting point given by (5.10), is the overall factor $\lambda^{-3/2}$. Denoting the flow parameters for all edges of the tree collectively by $s$, we can introduce new parameters $s' = \lambda^{3/2} s$, in terms of which the $H$-functions will look like the ones for the $\lambda$-undeformed flow with starting point given by (6.10). It follows from Appendix A that the existence conditions, written in terms of parameters $s'$, are the same as those written in terms of $s$, and furthermore the non-zero $s_{ms}^{(a)\lambda}$ will correspond to non-zero $s_{ms}^{(a)\lambda}$ since $s_{ms}^{(a)\lambda} = \lambda^{3/2} s_{ms}^{(a)\lambda}$. By virtue of Assumption 2, the $\lambda$-deformed flow tree that "survives" the near horizon limit has all its flow parameters $s_{ms}^{(a)\lambda}$ non-zero, and the corresponding $\lambda$-undeformed flow tree with starting point (6.10) exists at the AdS point.

In order to prove Conjecture 2 in the other direction consider a family of attractor flow trees whose initial point approaches the AdS point. Note that only the trees with the first split $D4 \to D6\bar{D}6$ exist in this limit, as shown in [5], eq.(3.64). Without loss of generality, for sufficiently large $\lambda$ we can choose the initial points to be given by the right-hand side of (6.10) for some $t_{\infty}$. Now, due to Assumption 2, the existence of the attractor flow tree at the AdS point means that in the limit $\lambda \to \infty$ all the flow parameters of these trees, $s_{ms}^{(a)\lambda}$, stay non-zero. The dependence on $\lambda$ in $s_{ms}^{(a)\lambda}$ originates from the dependence in
the starting point (6.10). We can use the discussion above to argue that there exists a corresponding \( \lambda \)-deformed flow tree, starting at \( t_\infty \) and passing through the point (6.10) at some parameter \( \hat{s}^3 \). For this \( \lambda \)-deformed flow tree to "survive" the limit \( \lambda \rightarrow \infty \) we must have all \( s_{ms}^{(a)} \) non-zero and positive, due to Assumption 2. As the relation between the \( \lambda \)-scaling and \( \theta \)-scaling (i.e. those with nonzero \( D \) coefficient of the \( \lambda \) we expect does not vanish generically. For example for figure 4, intersection products of the charges, which does not vanish in these examples and hence it is a combination of the form:

\[
\theta(\Gamma) := 2\text{Im}(e^{-i\alpha} Z(\Gamma))\_{\infty}
\]

(6.12)

According to (A.4), for each edge \( a \) the flow parameter \( s_{ms}^{(a)} \) is given by a linear combination, with rational coefficients, of \( \theta(\Gamma_i) \), where \( i \) runs over all the intermediate charges occurring in the path from the root of the tree to the edge \( a \). For the \( \lambda \)-deformed flow these \( \theta(\Gamma_i) \) have a definite scaling under \( \lambda \)-scaling. For instance, since the first split is always \( \Gamma(D4) \rightarrow \Gamma_1 + \Gamma_2 \) where \( \Gamma_1 \) and \( \Gamma_2 \) have nonzero (and opposite) \( D6 \) charge, we have \( \theta(\Gamma_1) = -\theta(\Gamma_2) \sim \lambda^0 \) and \( \theta(\Gamma_1) \) will enter the expressions for all \( s_{ms}^{(a)} \). Other \( \theta(\Gamma_i) \) will in general have \( \mathcal{O}(\lambda^0) \) scaling (i.e. those with nonzero \( D6 \) charge) but, examining examples, we find that the coefficient of the \( \lambda^0 \) term will be some complicated nonlinear expression in terms of the intersection products of the charges, which does not vanish in these examples and hence we expect does not vanish generically. For example for figure 4, \( s_{ms}^{(4)} \) for the edge with \( \Gamma_4 \), it is a combination of the form:

\[
s_{ms}^{(4)} = \frac{\theta_5 - \frac{\langle \Gamma_3 \Gamma_6 \rangle}{\langle \Gamma_3 \Gamma_4 \rangle} \theta_3 + \frac{\langle \Gamma_3 \Gamma_5 \rangle}{\langle \Gamma_3 \Gamma_4 \rangle} \theta_1 - \frac{\langle \Gamma_5 \Gamma_6 \rangle}{\langle \Gamma_5 \Gamma_6 \rangle} \theta_1}{\langle \Gamma_5 , \Gamma_6 \rangle}
\]

(6.13)

Here \( \theta_5 \sim \lambda^{-2}, \theta_1 \sim \lambda^0, \theta_3 \sim \lambda^0 \). If we assume that all \( D6 \) branes have \( D6 \) charges \( \pm 1 \), then in the limit \( \lambda \rightarrow \infty \) \( \theta_1 = -\theta_3 \), the leading coefficient of \( s_{ms}^{(4)} \) is proportional to

\[
-\langle \Gamma_3 , \Gamma_6 \rangle \langle \Gamma_5 , \Gamma_1 \rangle + \langle \Gamma_3 , \Gamma_5 \rangle \langle \Gamma_5 , \Gamma_6 \rangle + \langle \Gamma_1 , \Gamma_6 \rangle \langle \Gamma_5 , \Gamma_6 \rangle + \langle \Gamma_1 , \Gamma_6 \rangle \langle \Gamma_5 , \Gamma_3 \rangle
\]

(6.14)

which has no reason to vanish. In this way we can argue that all \( s_{ms}^{(a)} \lambda \) will have an order \( \sim \lambda^0 \) contribution whose coefficient will not scale to zero as \( \lambda \rightarrow \infty \), at least not in general.

To summarize, we have shown that there is a one to one correspondence between \( \lambda \)-deformed attractor flow trees that “survive” the near horizon limit, and regular attractor flow trees, starting at AdS point. If one grants Assumptions 1 and 2 this would actually prove Conjecture 2, and hence the conjecture of [3].

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A. Attractor flow trees

In this appendix we summarize some facts about attractor flow trees.

Consider type IIA string theory on $M_4 \times X$, where $X$ is a Calabi-Yau of generic holonomy. BPS states in the theory are labeled by their electromagnetic charges $\Gamma = (p^0, p^a, q_a, q_0)$.

The low energy theory is $N = 2$ supergravity coupled to $n_V = h^{1,1}(X) + 1$ vector multiplets representing complexified Kähler moduli of $X$. In this low energy theory BPS states are realized as single or multicentered black hole solutions.

It was conjectured in [6, 7] that the existence of multicentered BPS solutions of supergravity can be analyzed in terms of the the existence of split attractor flow trees. Some attempts at making this conjecture more precise were made in [9, 5].

**Split Attractor Flow Conjecture (SAFC):**

a) The components of the moduli spaces (in $\vec{x}_i$) of the multicentered BPS solutions with constituent charges $\Gamma_i$ and background $t_\infty$, are in 1-1 correspondence with the attractor flow trees beginning at $t_\infty$ and terminating on attractor points for $\Gamma_i$.

b) For a fixed $t_\infty$ and total charge $\Gamma$ there are only a finite number of attractor flow trees.

A practical recipe of identifying the intervals with the corresponding tree topologies is the following: tune the moduli at infinity such that they approach the first MS wall of a given attractor flow tree. Then, as we change the moduli across that MS wall, the corresponding component of moduli space of solutions to (6.1) ceases to exist. In this paper we assume the truth of the split attractor flow conjecture and simply establish the existence of attractor flow trees.

We now give an explicit description of an attractor flow tree.

First, we introduce some notation. For a general tree we denote quantities, related to particular vertex, by $X^{(\vec{e})}$ for quantity $X$. Here $\vec{e}$ is a vector of $+$ and $-$ signs and the sequence of $+$ and $-$ corresponds to sequence of right and left turns that one needs to
make when going from the origin of the tree to that vertex (the origin itself will have no superscript).

The attractor equation for the edge starting at vertex \( (a) \), looks like:

\[
2e^{-U}Im(e^{-i\alpha^{(a)}(t)}\Omega(t)) = -H(s^{(a)}), \tag{A.1}
\]

where \( \Omega(t) = -\frac{1}{\sqrt{4/3J^3}}e^{B+iJ} \) (in IIA picture), \( e^U \) is the metric warp factor, \( \alpha^{(a)} \) is the phase of central charge \( Z(\Gamma^{(a)}) \), \( s^{(a)} \) is a parameter of the flow on this edge, and

\[
H(s^{(a)}) = \Gamma^{(a)}s^{(a)} - \Delta H^{(a)}. \tag{A.2}
\]

\( \Delta H^{(a)} \) depends only on the moduli at infinity and is determined recursively by summing contributions from the origin of the tree up to vertex \( (a) \):

\[
\Delta H = 2Im(e^{-i\alpha}\Omega)|_{t=\infty},
\]

\[
\Delta H^{(+)} = \Delta H^{(-)} = \Delta H - \Gamma s_{ms},
\]

\[
\Delta H^{(++)} = \Delta H^{(-+)} = \Delta H^{(+)} - \Gamma^{(+)}s^{(+)}_{ms},
\]

\[
\Delta H^{(-+)} = \Delta H^{(-)} = \Delta H^{(-)} - \Gamma^{(-)}s^{(-)}_{ms}, \ldots \tag{A.3}
\]

where \( s^{(a)}_{ms} \) are values of parameters along the flow, for which surfaces of marginal stability are crossed:

\[
s_{ms}^{(+)} = \frac{\langle \Gamma^{(+)}\Delta H^{(+)} \rangle}{\langle \Gamma^{(+)\Gamma^{(+)}} \rangle},
\]

\[
s_{ms}^{(-)} = \frac{\langle \Gamma^{(-+)}\Delta H^{(-)} \rangle}{\langle \Gamma^{(-+)\Gamma^{(-)}} \rangle}, \ldots \tag{A.4}
\]

The solution to the attractor equations (A.1), that is, the image of the flow in moduli space, can be written in closed form in terms of the entropy function \( S(p,q) \) [2]:

\[
t^A(s^{(a)}) = \frac{\partial S}{\partial \delta_\lambda^A} + \pi ip^A \left|_{(p,q)=H(s^{(a)})} \right., \tag{A.5}
\]

Here, the parameter \( s^{(a)} \) varies as: \( s^{(a)} \in (0,\infty) \) for the terminal edge, and \( s^{(a)} \in (0,s^{(a)}_{ms}) \) for an inner edge.

For a given attractor tree to exist, all its edges have to exist. Terminal edges exist if the discriminants of terminal charges are positive, or if the terminal charge is pure electric or magnetic, which corresponds to the flow going to the boundary of moduli space. Inner edges exist if:

1. The flow reaches the MS wall at a positive flow parameter \( s^{(a)}_{ms} > 0 \)
2. And, an MS wall (not an anti-MS wall) is crossed, i.e. $rac{Z(\Gamma^{(a+)})}{Z(\Gamma^{(a-)})}|_{s_{ms}} > 0$

3. And, the MS wall is crossed before the flow hits a zero of the central charge (if present):

$$s_{ms}^{(a)} \leq s_0^{(a)} \quad \text{or} \quad s_0^{(a)} \leq 0$$

where $s_0^{(a)}$ is the value where the flow crashes on a zero.

For a D4-D2-D0 charge we give explicit formulae for attractor flow in moduli space:

$$t^a(s) = D(P(s))^{ab}Q_b(s) + iP^a(s)\sqrt{-6q_0(s)/P^3(s)}$$

$$\Gamma(s) = p^0(s) + P(s) + Q(s) + q_0(s)dV = s\Gamma - \Delta H$$

$$\Delta H = \frac{2Im(Z\Omega)}{|Z|}\bigg|_\infty = \frac{2}{3J^3} \left(2\frac{Q \cdot J + P \cdot B \cdot J}{P \cdot J^2} - J + J^2\frac{Q \cdot J + P \cdot B \cdot J}{P \cdot J^2} + \frac{J^3}{6}\right)\bigg|_\infty$$

(A.6)

In the formula for $\Delta H$ we used the large $J_\infty$ approximation and dropped relative corrections of order $O(J^{-2})$. The expression for $t^a(s)$ was found from (A.5) putting $p^0(s) = 0$. Strictly speaking, this is not true because already $\Delta H$ contains non-zero contribution to $p^0(s)$. To estimate the error that we make, take the expression for the moduli for a 1-parameter moduli space and expand it around $p^0(s) = 0$. The first correction looks like:

$$\delta_1 t^a(s) = \left[\frac{2Q(s)^2 - 3P(s)q_0(s)}{P(s)^3} + i\frac{\sqrt{3}P(s)Q(s)(2Q(s)^2 - 3P(s)q_0(s))}{3P(s)^3\sqrt{P(s)^2Q(s)^2 - 2P(s)q_0(s)}}\right] p^0(s)$$

(A.7)

Focusing on $J_\infty$ dependence, $\Gamma(s)$ in (A.6) can be written as

$$\Gamma(s) = \left(O(J^{-5/2}), sP + O(J^{-1/2}), sQ + O(J^{-1/2}), sq_0 + O(J^{3/2})\right)$$

(A.8)

This means that, for instance, for $s$ of order $s \sim J^{-1/2+\epsilon}$ with $0 \leq \epsilon \leq 2$ (which covers all the cases of interest in this paper) the correction in (A.7) is of order

$$\delta_1 t^a(s) \sim O(J^{-2\epsilon}) + iO(J^{-1-3/2\epsilon})$$

(A.9)

and can be neglected in large $J_\infty$ limit.

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