ISOMORPHISM OF COMPACTIFICATIONS OF MODULI OF VECTOR BUNDLES: NONREDUCED MODULI

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Abstract. A morphism of the moduli functor of admissible semistable pairs to the Gieseker–Maruyama moduli functor (of semistable coherent torsion-free sheaves) with the same Hilbert polynomial on the surface, is constructed. It is shown that these functors are isomorphic, and main components of moduli scheme for semistable admissible pairs \((\tilde{S}, \tilde{L}), \tilde{E}\) are isomorphic to main components of the Gieseker–Maruyama moduli scheme.

Keywords: moduli space, semistable coherent sheaves, semistable admissible pairs, moduli functor, vector bundles, algebraic surface.

To the blessed memory of my Mum

Introduction

In the present article we continue to investigate the compactification of moduli of stable vector bundles on a surface by locally free sheaves. Various aspects of its construction and basic properties were given in preceding papers of the author [1]–[8].

In the present article \(S\) is smooth irreducible projective algebraic surface over an algebraically closed field \(k\) of characteristic 0, \(\mathcal{O}_S\) its structure sheaf, \(E\) a coherent torsion-free \(\mathcal{O}_S\)-module, \(E' := \text{Hom}_{\mathcal{O}_S}(E, \mathcal{O}_S)\) its dual \(\mathcal{O}_S\)-module. \(E'\) is reflexive and hence locally free. Everywhere in this article a locally free sheaf and its corresponding vector bundle are identified and both terms are used as synonyms. Let \(L\) be very ample invertible sheaf on \(S\); it is fixed and is used as a polarization. The symbol \(\chi(\cdot)\) denotes Euler–Poincaré characteristic, \(c_i(\cdot)\) \(i\)-th Chern class.

Definition 1. [4, 5] Polarized algebraic scheme \((\tilde{S}, \tilde{L})\) is called admissible if it satisfies one of the following conditions

i) \((\tilde{S}, \tilde{L}) \cong (S, L)\),

ii) \(\tilde{S} \cong \text{Proj} \bigoplus_{s \geq 0} (I[t] + (t))^{s} / (t^{s+1})\) where \(I = \text{Fitt}^0 \mathcal{E}xt^2(\kappa, \mathcal{O}_S)\) for Artinian quotient sheaf \(q_0 : \bigoplus \mathcal{O}_S \rightarrow \kappa\) of length \(l(\kappa) \leq c_2\), and \(\tilde{L} = L \otimes (\sigma^{-1} I \cdot \mathcal{O}_\tilde{S})\) is very ample invertible sheaf on the scheme \(\tilde{S}\); this polarization \(\tilde{L}\) is called distinguished polarization.

Recall the definition of a sheaf of 0-th Fitting ideals known from commutative algebra and involved in the previous definition. Let \(X\) be a scheme, \(F\) \(\mathcal{O}_X\)-module of finite presentation \(F_1 \xrightarrow{\varphi} F_0 \rightarrow F\). Without loss of generality we assume that \(\text{rank} F_1 \geq \text{rank} F_0\).

Definition 2. The sheaf of 0-th Fitting ideals of \(\mathcal{O}_X\)-module \(F\) is defined as \(\text{Fitt}^0 F = \text{im}(\bigwedge^{\text{rank} F_0} F_1 \otimes \bigwedge^{\text{rank} F_0} F_0' \xrightarrow{\varphi'} \mathcal{O}_X)\), where \(\varphi'\) is a morphism of \(\mathcal{O}_X\)-modules induced by \(\varphi\).

Remark 1. In further considerations we replace \(L\) by its big enough tensor power, if necessary for \(\tilde{L}\) to be very ample. This power can be chosen constant and fixed, as shown in [5]. All Hilbert polynomials are compute according to new \(L\) and \(\tilde{L}\) respectively.
As shown in [1], if \( \widetilde{S} \) satisfies the condition (ii) in the definition [1] it is decomposed into the union of several components \( \widetilde{S} = \bigcup_{i \geq 0} \widetilde{S}_i \). It has a morphism \( \sigma : \widetilde{S} \to S \) which is induced by the structure of \( \mathcal{O}_S \)-algebra on the graded object \( \bigoplus_{i \geq 0}(I[t]^i + (t)) \).

**Definition 3.** \([3]\) A *S*-stable (respectively, semistable) pair \( ((\widetilde{S}, \widetilde{L}), \widetilde{E}) \) is the following data:

- \( \widetilde{S} = \bigcup_{i \geq 0} \widetilde{S}_i \) — an admissible scheme, \( \sigma : \widetilde{S} \to S \) morphism which is called canonical, \( \sigma_i : \widetilde{S}_i \to S \) its restrictions on components \( \widetilde{S}_i, i \geq 0 \);
- \( \widetilde{E} \) vector bundle on the scheme \( \widetilde{S} \);
- \( \widetilde{L} \in \text{Pic} \widetilde{S} \) distinguished polarization;

such that

- \( \chi(\widetilde{E} \otimes \widetilde{L}^n) = rp(n) \), the polynomial \( p(n) \) and the rank \( r \) of the sheaf \( \widetilde{E} \) are fixed;
- the sheaf \( \widetilde{E} \) on the scheme \( \widetilde{S} \) is stable (respectively, semistable) due to Gieseker, i.e. for any proper subsheaf \( \widetilde{F} \subset \widetilde{E} \) for \( n \gg 0 \)

\[
\frac{h^0(\widetilde{F} \otimes \widetilde{L}^n)}{\text{rank } \widetilde{F}} < \frac{h^0(\widetilde{E} \otimes \widetilde{L}^n)}{\text{rank } \widetilde{E}},
\]

(respectively, \( \frac{h^0(\widetilde{F} \otimes \widetilde{L}^n)}{\text{rank } \widetilde{F}} \leq \frac{h^0(\widetilde{E} \otimes \widetilde{L}^n)}{\text{rank } \widetilde{E}} \));

- on each of additional components \( \widetilde{S}_i, i > 0 \), the sheaf \( \widetilde{E}_i := \widetilde{E}|_{\widetilde{S}_i} \) is quasi-ideal, i.e. admits a description of the form

\[
\widetilde{E}_i = \sigma_i^* \ker q_0/tors_i.
\]

for some \( q_0 \in \bigcup_{i \leq 2} \text{Quot}^1 \bigoplus \mathcal{O}_S \).

The definition of the subsheaf \( tors_i \) will be given below.

Pairs \( ((\widetilde{S}, \widetilde{L}), \widetilde{E}) \) such that \( (\widetilde{S}, \widetilde{L}) \cong (S, L) \) will be called \textit{S}-pairs.

In the series of articles of the author [1]—[6] a projective algebraic scheme \( \tilde{M} \) is built up as reduced moduli scheme of \( S \)-semistable admissible pairs and in [6] it is constructed as possibly nonreduced moduli space.

The scheme \( \tilde{M} \) contains an open subscheme \( \tilde{M}_0 \) which is isomorphic to the subscheme \( M_0 \) of Gieseker-semistable vector bundles in the Gieseker—Maruyama moduli scheme \( \mathcal{M} \) of torsion-free semistable sheaves whose Hilbert polynomial is equal to \( \chi(E \otimes L^n) = rp(n) \).

The following definition of Gieseker-semistability is used.

**Definition 4.** \([6]\) The coherent \( \mathcal{O}_S \)-sheaf \( E \) is stable (respectively, semistable) if for any proper subsheaf \( F \subset E \) of rank \( r' = \text{rank } F \) for \( n \gg 0 \)

\[
\frac{\chi(E \otimes L^n)}{r} > \frac{\chi(F \otimes L^n)}{r'}, \quad \left( \text{respectively, } \frac{\chi(E \otimes L^n)}{r} \geq \frac{\chi(F \otimes L^n)}{r'} \right).
\]

Let \( E \) be a semistable locally free sheaf. Then, obviously, the sheaf \( I = \mathcal{F}itt^0 \mathcal{E}xt^1(E, \mathcal{O}_S) \) is trivial and \( \widetilde{S} \cong S \). In this case \( ((\widetilde{S}, \widetilde{L}), \widetilde{E}) \cong ((S, L), E) \) and we have a bijective correspondence \( \tilde{M}_0 \cong M_0 \).

Let \( E \) be a semistable nonlocally free coherent sheaf; then the scheme \( \tilde{S} \) contains reduced irreducible component \( \tilde{S}_0 \) such that the morphism \( \sigma_0 := \sigma|_{\tilde{S}_0} : \tilde{S}_0 \to S \) is a morphism of blowing up of the scheme \( S \) in the sheaf of ideals \( I = \mathcal{F}itt^0 \mathcal{E}xt^1(E, \mathcal{O}_S) \).

Formation of a sheaf \( I \) is an approach to the characterization singularities if the sheaf \( E \) i.e. its difference from a locally free sheaf. Indeed, the quotient sheaf \( \chi := E^{\text{crys}}/E \) is Artinian of length not greater then \( c_2(E) \), and \( \mathcal{E}xt^1(E, \mathcal{O}_S) \cong \mathcal{E}xt^2(\chi, \mathcal{O}_S) \). Then \( \mathcal{F}itt^0 \mathcal{E}xt^2(\chi, \mathcal{O}_S) \) is a sheaf of ideals of (in general case nonreduced) subscheme \( Z \) of bounded length \([6]\) supported at finite set of points on the surface \( S \). As it is shown in [4], others irreducible components \( \tilde{S}_i, i > 0 \) of the scheme \( \tilde{S} \) in general case carry nonreduced scheme structure.

Each semistable coherent torsion-free sheaf \( E \) corresponds to a pair \( ((\widetilde{S}, \widetilde{L}), \widetilde{E}) \) where \( (\widetilde{S}, \widetilde{L}) \) defined as described.
Now we describe the construction of the subsheaf tors in \([0,1]\). Let \(U\) be Zariski-open subset in one of components \(\widetilde{S}_i, i \geq 0\), and \(\sigma^*E|\widetilde{S}_i(U)\) corresponding group of sections. This group is \(O_{\widetilde{S}_i}(U)\)-module. Sections \(s \in \sigma^*E|\widetilde{S}_i(U)\) annihilated by prime ideals of positive codimensions in \(O_{\widetilde{S}_i}(U)\), form a submodule in \(\sigma^*E|\widetilde{S}_i(U)\). This submodule is denoted as \(\text{tors}_i(U)\). The correspondence \(U \mapsto \text{tors}_i(U)\) defines a subsheaf \(\text{tors}_i \subset \sigma^*E|\widetilde{S}_i\). Note that associated primes of positive codimensions which annihilate sections \(s \in \sigma^*E|\widetilde{S}_i(U)\), correspond to subsheaves supported in the preimage \(\sigma^{-1}(\text{Supp } x) = \bigcup_{i \geq 0} \widetilde{S}_i\). Since by the construction the scheme \(\widetilde{S} = \bigcup_{i \geq 0} \widetilde{S}_i\) is connected [3], subsheaves \(\text{tors}_i, i \geq 0\), allow to construct a subsheaf \(\text{tors} \subset \sigma^*E\). The former subsheaf is defined as follows. A section \(s \in \sigma^*E|\widetilde{S}_i(U)\) satisfies the condition \(s \in \text{tors}|\widetilde{S}_i(U)\) if and only if

- there exist a section \(y \in O_{\widetilde{S}_i}(U)\) such that \(ys = 0\),
- at least one of the following two conditions is satisfied: either \(y \in p\), where \(p\) is prime ideal of positive codimension; or there exist Zariski-open subset \(V \subset \widetilde{S}\) and a section \(s' \in \sigma^*E(V)\) such that \(V \supset U\), \(s'|U = s\), and \(s'|_{V \cap \widetilde{S}_0} \in \text{tors}(\sigma^*E|\widetilde{S}_0)(V \cap \widetilde{S}_0)\). In the former expression the torsion subsheaf \(\sigma^*E|\widetilde{S}_0\) is understood in usual sense.

The role of the subsheaf \(\text{tors} \subset \sigma^*E\) in our construction is analogous to the role of torsion subsheaf in the case of reduced and irreducible base scheme. Since no confusion occur, the symbol \(\text{tors}\) is understood everywhere in described sense. The subsheaf \(\text{tors}\) is called a torsion subsheaf.

In [5] it is proven that sheaves \(\sigma^*E/\text{tors}\) are locally free. The sheaf \(\widetilde{E}\) include in the pair \(((\widetilde{S}, \widetilde{L}), \widetilde{E})\) is defined by the formula \(\widetilde{E} = \sigma^*E/\text{tors}\). In this circumstance there is an isomorphism \(H^n(\widetilde{S}, \widetilde{E} \otimes L) \cong H^n(S, E \otimes L)\).

In the same article it was proven that the restriction of the sheaf \(\widetilde{E}\) to each of components \(\widetilde{S}_i, i > 0\), is given by the quasi-ideality relation \([0,1]\) where \(q_0 : O_{\widetilde{S}_i}^n \rightarrow \varpi\) is an epimorphism defined by the exact triple \(0 \rightarrow E \rightarrow E^{\vee \vee} \rightarrow \varpi \rightarrow 0\) in view of local freeness of the sheaf \(E^{\vee \vee}\).

Resolution of singularities of a semistable sheaf \(E\) can be globalized in a flat family by means of the construction developed in various versions in [2, 3, 5]. Let \(T\) be a reduced irreducible quasi-projective scheme, \(\Xi\) a sheaf of \(O_{T \times S}\)-modules, \(L\) invertible \(O_{T \times S}\)-sheaf very ample relatively \(T\) and such that \(L_{|T \times S} = L\), and \(\chi(\Xi \otimes L^n_{|T \times S}) = r(p(n)\) for all closed points \(t \in T\). We also assume that \(T\) contains nonempty open subset \(T_0\) such that \(\Xi|T_0 \times S\) is locally free \(O_{T_0 \times S}\)-module. Then following objects are defined:

- \(\widetilde{T}\) integral normal scheme obtained as a blowing up \(\phi : \widetilde{T} \rightarrow T\) of the scheme \(T\),
- \(\pi : \Sigma \rightarrow \widetilde{T}\) flat family of admissible schemes with invertible \(O_{\Sigma}\)-module \(\widetilde{L}\) such that \(\widetilde{L}_{|T \times S}\) distinguished polarization of the scheme \(\pi^{-1}(t)\),
- \(\widetilde{E}\) locally free \(O_{\Sigma}\)-module and \((\pi^{-1}(t), \widetilde{L}_{|\pi^{-1}(t)}, \widetilde{E}_{|\pi^{-1}(t)})\) is \(S\)-semistable admissible pair.

In this situation there is a blowup morphism \(\Phi : \Sigma \rightarrow \widetilde{T} \times S\) and

\[(\Phi_*\widetilde{E})^{\vee \vee} = (\phi, Id_S)^*\Xi;\]
what follows from the coincidence of reflexive sheaves at right hand side and at left hand side, on the open subset apart the subset of codimension 3. It is important that the scheme \(\widetilde{T} \times S\) is integral and normal.

The described mechanism was called in [5] a standard resolution.

In [8] the procedure of standard resolution is generalized to the case of families with nonreduced base. It is shown that transformation of the family of torsion-free coherent sheaves \(E\) can be done in such a way that we get a family of admissible semistable pairs \((\pi : \Sigma \rightarrow T, \widetilde{L}), \Xi\) with the same base \(T\), i.e. base scheme does not undergo a birational transformation and \(\phi\) is identity isomorphism.
Remark 2. In [8] we did not prove the relation analogous to (0.2). Then when speaking of standard resolution of the family with nonreduced base we do not mention such a relation.

In section 1 we remind the definition of the functor $f_{GM}$ of moduli of coherent torsion-free sheaves ("Gieseker – Maruyama functor") (1.3, 1.4) and improve the definition of the functor $f$ of moduli of admissible semistable pairs (1.2, 1.1). The rank $r$ and polynomial $p(n)$ are fixed and equal for both moduli functors. After that we give the description of the transformation of a family of semistable admissible pairs $((\pi : \tilde{\Sigma} \to T, \tilde{L}), \tilde{E})$ with (possibly, nonreduced) base scheme $T$ to a family $E$ of coherent torsion-free semistable sheaves with the same base $T$. The transformation provides a morphism of the functor of admissible semistable pairs $f$ to Gieseker – Maruyama functor $f_{GM}$.

In section 2 we show that the morphism of functors we constructed is an inverse for the morphism $\kappa : f_{GM} \to f$ built up in [8]. In this way the functors of interest (namely, their subfunctors corresponding to families containing locally free sheaves and $S$-pairs respectively) are isomorphic.

In the present article we prove following results.

**Theorem 1.** There is a natural transformation $\tau : f \to f_{GM}$ of every maximal closed irreducible subfunctor of the moduli functor of admissible semistable pairs containing $S$-pairs to the corresponding maximal closed irreducible subfunctor of Gieseker – Maruyama moduli functor which contains locally free sheaves with same rank and Hilbert polynomial. This natural transformation is inverse to the natural transformation $\kappa : f_{GM} \to f$ constructed in [8] and induced by the procedure of standard resolution developed in the same article. Hence both morphisms of nonreduced moduli functors $\kappa : f_{GM} \to f$ and $\tau : f \to f_{GM}$ are isomorphisms.

**Corollary 1.** The union of main components of nonreduced moduli scheme $\tilde{M}$ for $f$ is isomorphic to the union of main components of nonreduced Gieseker – Maruyama scheme $\tilde{M}$ for sheaves with same rank and Hilbert polynomial.

1. Morphism of moduli functors

Following [10] ch. 2, sect. 2.2, we recall some definitions. Let $C$ be a category, $C'$ its dual, $C' = \text{Funct}(C^\circ, \text{Sets})$ category of functors to the category of sets. By Yoneda’s lemma, the functor $C \to C' : F \mapsto (\mathcal{F} : X \mapsto \text{Hom}_C(X, F))$) includes $C$ into $C'$ as full subcategory.

**Definition 5.** [10] ch. 2, definition 2.2.1] The functor $f \in \text{Ob} C'$ is corepresented by the object $M \in \text{Ob} C$, if there exist a $C'$-morphism $\psi : f \to M$ such that any morphism $\psi' : f \to \mathcal{F}'$ factors through the unique morphism $\omega : M \to \mathcal{F}'$.

**Definition 6.** The scheme $\tilde{M}$ is a coarse moduli space for the functor $f$ if $f$ is corepresented by the scheme $\tilde{M}$.

Let $T, S$ be schemes over a field $k$, $\pi : \tilde{\Sigma} \to T$ a morphism of $k$-schemes. We introduce the following

**Definition 7.** The family of schemes $\pi : \tilde{\Sigma} \to T$ is birationally $S$-trivial if there exist isomorphic open subschemes $\tilde{\Sigma}_0 \subset \tilde{\Sigma}$ and $\Sigma_0 \subset T \times S$ and there is a scheme equality $\pi(\tilde{\Sigma}_0) = T$.

The former equality means that all fibres of the morphism $\pi$ have nonempty intersections with the open subscheme $\tilde{\Sigma}_0$.

In particular, if $T = \text{Spec} k$ then $\pi$ is a constant morphism and $\tilde{\Sigma}_0 \cong \Sigma_0$ is open subscheme in $S$.

Since in the present paper we consider only $S$-birationally trivial families, they will be referred to as birationally trivial families.
We consider sets of families of semistable pairs
\[ \left\{ \begin{array}{l}
\pi : \Sigma \to T \text{ birationally } S\text{-trivial,} \\
\mathcal{L}_t \in \text{Pic} \Sigma \text{ flat over } T,
\end{array} \right. \]
for \( m \gg 0 \) very ample relatively \( T \),
\[ \forall t \in T \mathcal{L}_t = \mathcal{L}|_{H_{-1}(t)} \text{ ample;} \]
\[ (\pi^{-1}(t), \mathcal{L}_t) \text{ admissible scheme with distinguished polarization;} \]
\[ \chi(L^p_t) \text{ does not depend on } t, \]
\[ \mathcal{E} \text{ locally free } \mathcal{O}_\Sigma \text{ sheaf flat over } T; \]
\[ \chi((\mathcal{E} \otimes \mathcal{L}^n)|_{H_{-1}(t)}) = rp(n); \]
\[ ((\pi^{-1}(t), \mathcal{L}_t), \mathcal{E}|_{H_{-1}(t)}) \text{ - semistable pair} \]
and a functor
\[ (\mathfrak{F}_T = (\text{Sets})^o \to (\text{Sets}) \]
from the category of \( k \)-schemes to the category of sets. It attaches to any scheme \( T \) the set of equivalence classes of families of the form \((\mathfrak{F}_T / \sim)\).

The equivalence relation \( \sim \) is defined as follows. Families \((\pi : \Sigma \to T, \mathcal{L}, \mathcal{E})\) and \((\pi' : \Sigma' \to T, \mathcal{L}', \mathcal{E}')\) from the class \( \mathfrak{F}_T \) are said to be equivalent (notation: \((\pi : \Sigma \to T, \mathcal{L}, \mathcal{E}) \sim (\pi' : \Sigma' \to T, \mathcal{L}', \mathcal{E}')\) if
1) there exist an isomorphism \( \iota : \Sigma \to \Sigma' \) such that the diagram
\[ \begin{array}{ccc}
\Sigma & \sim & \Sigma' \\
\downarrow \pi & & \downarrow \pi' \\
T & & T
\end{array} \]
commutes.
2) There exist line bundles \( L', L'' \) on the scheme \( T \) such that \( \iota^* \mathcal{E}' = \mathcal{E} \otimes \pi^* L', \iota^* \mathcal{L}' = \mathcal{L} \otimes \pi^* L'' \).

Now discuss what is the "size" of the maximal under inclusion of those open subschemes \( \Sigma_0 \) in a family of admissible schemes \( \Sigma \), which are isomorphic to appropriate open subschemes in \( T \times S \) in the definition \( \mathfrak{F}_T \). The set \( F = \Sigma \setminus \Sigma_0 \) is closed. If \( T_0 \) is open subscheme in \( T \) whose points carry fibres isomorphic to \( S \), then \( \Sigma_0 \not\supseteq \pi^{-1} T_0 \) (inequality is true because \( \pi(\Sigma_0) \supseteq T \) in the definition \( \mathfrak{F}_T \)). The subscheme \( \Sigma_0 \) which is open in \( T \times S \) and isomorphic to \( \Sigma_0 \), is such that \( \Sigma_0 \not\supseteq T_0 \times S \). If \( \pi : \Sigma \to T \) is family of admissible schemes then \( \Sigma_0 \cong \Sigma \setminus F \), and \( F \) is (set-theoretically) the union of additional components of fibres which are non-isomorphic to \( S \).

The Gieseker–Maruyama functor
\[ \mathfrak{F}^GM : (\text{Sets})^o \to \text{Sets}, \]
attaches to any scheme \( T \) the set of equivalence classes of families of the following form \( \mathfrak{F}^GM_T / \sim \), where
\[ \mathfrak{F}^GM_T = \left\{ \begin{array}{l}
\mathcal{E} \text{ sheaf of } \mathcal{O}_{T \times S} \text{ - modules flat over } T; \\
\mathcal{L} \text{ invertible sheaf of } \mathcal{O}_{T \times S} \text{ - modules,}
\end{array} \right. \]
ample relatively to \( T \)
\[ \text{and such that } L_t := \mathcal{L}|_{\times S} \cong L \text{ for any point } t \in T; \]
\[ E_t := \mathcal{E}|_{\times S} \text{ torsion-free and Gieseker-semistable; } \]
\[ \chi(E_t \otimes L^p_t) = rp(n). \]

Families \( \mathcal{E}, \mathcal{L} \) and \( \mathcal{E}', \mathcal{L}' \) from the class \( \mathfrak{F}^GM_T \) are said to be equivalent (notation: \((\mathcal{E}, \mathcal{L}) \sim (\mathcal{E}', \mathcal{L}')\)), if there exist linebundles \( L', L'' \) on the scheme \( T \) such that \( \mathcal{E}' = \mathcal{E} \otimes p^* L', \mathcal{L}' = \mathcal{L} \otimes p^* L'' \) where \( p : T \times S \to T \) is projection onto the factor.
Remark 3. Since Pic $(T \times S) = \text{Pic} T \times \text{Pic} S$, our definition of the moduli functor $\mathcal{F}^G$ is equivalent to the standard definition which can be found, for example, in [10]; the difference in choice of polarizations $L$ and $\mathbb{L}'$ having isomorphic restrictions on fibres over the base $T$, is avoided by the equivalence which is induced by tensoring by inverse image of an invertible sheaf $L''$ from the base $T$.

The morphism of functors $\xi : \mathcal{F}^G \to \mathcal{F}$ is defined by commutative diagrams

\[
\begin{array}{ccc}
T & \xrightarrow{\iota} & \mathcal{F}^G / \sim \\
\downarrow & & \downarrow \\
\mathcal{F} / \sim
\end{array}
\]

where $T \in \text{ObSch}_k$, $\xi(T) : (\mathcal{F}_T^G / \sim) \to (\mathcal{F}_T / \sim)$ is a morphism in the category of sets (mapping).

Remark 4. We consider the subfunctors in $\mathcal{F}^G$ (resp. in $\mathcal{F}$) which correspond to unions of maximal irreducible substacks containing locally free sheaves (resp. $S$-pairs). Then any family $(L, E)$ (resp. $(\pi : \Sigma \to T, \tilde{L}, \tilde{E})$) with base $T$ can be included into the family $(\mathbb{L}', \mathbb{E}')$ (resp. $(\pi' : \Sigma' \to T', \mathbb{L}', \mathbb{E}')$) with some connected and, possibly, nonreduced base $T'$ and containing locally free sheaves (resp. $S$-pairs), according to fibred diagrams

\[
\begin{array}{ccc}
T \times S & \xrightarrow{\iota, \iota} & \Sigma' \\
\downarrow & & \downarrow \\
T' \times S & \xrightarrow{\iota'} & \tilde{\Sigma}'
\end{array}
\]

Namely, $E = (i, id_2)^* \mathbb{E}'$ (resp., $\Sigma = \Sigma' \times_{T'} T$, $\tilde{\iota} : \Sigma \hookrightarrow \Sigma'$ is induced morphism of immersion, $\tilde{\mathbb{E}} = \tilde{\iota}^* \mathbb{E}'$, $\tilde{\mathbb{L}} = \tilde{\iota}^* \mathbb{L}'$). We can assume that $T'$ is such that its reduction $T'_{\text{red}}$ is an irreducible scheme. This means that we consider admissible semistable pairs which are deformation equivalent to $S$-pairs [3].

We mean under the Gieseker - Maruyama scheme $\overline{M}$ the union of those components of nonreduced moduli scheme for semistable coherent torsion-free sheaves, which contain locally free sheaves, and under the moduli scheme $\overline{M} \cap S$ the union of its components containing $S$-pairs.

Further we show that there is a morphism of the nonreduced moduli functor of admissible semistable pairs to the nonreduced Gieseker – Maruyama moduli functor. Namely, for any scheme $T$ we build up a correspondence $((\pi : \Sigma \to T, \mathbb{L}, \mathbb{E})) \mapsto (L, E)$. It defines a set mapping $\{((\pi : \Sigma \to T, \mathbb{L}, \mathbb{E})) / \sim\} \to \{\{L, E\} / \sim\}$. This means that the family of semistable coherent torsion-free sheaves $E$ with the same base $T$ can be constructed by any family $((\pi : \Sigma \to T, \mathbb{L}), \mathbb{E})$ of admissible semistable pairs which is birationally trivial and flat over $T$. Let $((\Sigma, \mathbb{L}, \mathbb{E}))$ be the maximal open subscheme of $\overline{M}$ which is isomorphic to an open subscheme of the product $T \times S$. Choose $m \gg 0$ such that the morphism of $\mathcal{O}_\Sigma$-modules $\pi^* \pi_* (\mathbb{E} \otimes \mathbb{L}^m) \to \mathbb{E} \otimes \mathbb{L}^m$ is surjective. After tensoring $\mathbb{E}$ by an appropriate invertible sheaf from the base $T$ if necessary, for locally free $\mathcal{O}_T$-sheaf $V := \pi_* (\mathcal{E} \otimes \mathbb{L}^m)$ we have $\pi^* V \otimes \mathbb{L}^{-m}|_{\Sigma_0} \cong V \otimes \mathbb{L}^{-m}|_{\Sigma_0}$ in view of the isomorphism $\Sigma_0 \cong \Sigma_0$, and there is an
Consider relative Grothendieck's scheme Quot \( r^p(n)(V \boxtimes L^{-m}) \rightarrow T \). It carries a universal \( \mathcal{O}_{\text{Quot}^{r^p(n)}(V \boxtimes L^{-m})} \times S \)-quotient sheaf

\[
\mathcal{E} = V \boxtimes r^p(n)(V \boxtimes L^{-m}) \rightarrow E_{\text{Quot}}.
\]

The morphism \( \pi^* V \rightarrow \tilde{E} \boxtimes \tilde{L}^m \) induces a morphism of \( T \)-schemes

\[
\Sigma_0 \rightarrow \text{Quot}^{r^p(n)}(V \boxtimes L^{-m}) \times S,
\]

which is locally closed immersion.

Let \( T' \subset \text{Quot}^{r^p(n)}(V \boxtimes L^{-m}) \) be (possibly nonreduced) subscheme formed by all quotient sheaves of the form \( q_t : V \otimes L_{-m} \rightarrow E_t \) such that \( q_t |_{\Sigma_0 \cap (t \times S)} = (\Sigma_0 \cap (t \times S)) \otimes \mathcal{E} \). The symbol \( V \) denotes a \( k \)-vector space \( V \cong H^0(\tilde{S}_t = \pi^{-1}(t), \tilde{E}_t \boxtimes \tilde{L}_t^m) \) of dimension \( rp(m) \) which is isomorphic to the fibre of the vector bundle \( V \) at a point \( t \in T \). Equivalently, \( T' \) is a scheme-theoretic image of the subscheme \( \Sigma_0 \) in \( \text{Quot}^{r^p(n)}(V \boxtimes L^{-m}) \). We have the following commutative diagram of \( T \)-schemes with fibred square

\[
\begin{array}{ccc}
\Sigma_0 & \rightarrow & T' \times S \\
\tau & \downarrow & \downarrow \tau \\
T' & \rightarrow & \text{Quot}^{r^p(n)}(V \boxtimes L^{-m})
\end{array}
\]

The double arrow in the diagram of schemes means that the scheme-theoretic image of the morphism \( \tau \) coincides with its target, i.e. the image of the scheme \( T' \) under the morphism \( \tau \) has same scheme structure as \( T \). It it true because the image of the subscheme \( \Sigma_0 \) under the projection on \( T \) coincides with \( T \).

We claim that \( \tau \) is an isomorphism. For the proof consider at first a closed point \( m \in T \), the image of the fibre \( \Sigma_0 \cap \pi^{-1}(m) \) in \( \text{Quot}^{r^p(n)}(V \boxtimes L^{-m}) \times S \) (which is denoted also as \( \Sigma_0 \cap \pi^{-1}(m) \)) and an epimorphism corresponding to the point \( m \)

\[
V \boxtimes L_{-m} |_{\Sigma_0 \cap \pi^{-1}(m)} \rightarrow E|_{\Sigma_0 \cap \pi^{-1}(m)}.
\]

Note that in the situation of the remark 4 the image of the subset \( \pi^{-1}(m) \setminus (\Sigma_0 \cap \pi^{-1}(m)) \) in \( S \) is a finite collection of points on the surface \( S = m \times S \subset \text{Quot}^{r^p(n)}(V \boxtimes L^{-m}) \times S \). Indeed, this is a "limit" of closed immersions of surfaces isomorphic to \( S \) with semistable locally free sheaves on them (in the sense of Gieseker – Maruyama functor). It is a surface isomorphic to \( S \) again, with semistable coherent torsion-free sheaf on it.

The subscheme \( U := (\Sigma_0 \cap \pi^{-1}(m)) \subset m \times S \) is non-affine and strictly greater then any proper affine subscheme in \( S \) which does not contain the subset \( S \setminus U \).

We will show that the morphism \[ \text{ker} (V \boxtimes L_{-m} |_{\Sigma_0 \cap \pi^{-1}(m)} \rightarrow E|_{\Sigma_0 \cap \pi^{-1}(m)}) \] has a unique continuation to the whole of the subscheme \( m \times S \). This means that the submodule

\[
E := E|_{\Sigma_0 \cap \pi^{-1}(m)}
\]

has a unique continuation to the whole of the subscheme \( m \times S \). In the sequel we use the notation \( E := E|_{m \times S} \).

For any open \( U' \subset S \) such that \( U' \cap (S \setminus U) \neq \emptyset \), the element \( f \in (V \boxtimes L_{-m})(U') = V \boxtimes L_{-m}|_{m \times S}(U') \) vanishing in \( E(U' \cap U) \), maps to \( 0 \in E(U') \) on the whole of \( U' \). Then in the fibre over the closed point \( m \in T \) there is a unique continuation of the epimorphism \( V \otimes L_{-m} |_U \rightarrow E|_U \) to the homomorphism \( V \otimes L_{-m} \rightarrow E \). Since the morphism \( \text{Ker}_\text{red} : \tilde{M} \rightarrow \tilde{M} \)
\(\tilde{\mathcal{M}}\) is bijective [5] and any semistable pair \((\tilde{S}, \tilde{L}), \tilde{E}\) corresponds to a coherent semistable torsion-free sheaf \(E\), and for \(m \gg 0\) the homomorphism we built \(V \otimes L^{-m} \to E\) defines a point in \(Q \subset \text{Quot}^{\text{red}}(V \otimes L^{-m})\), then this continuing homomorphism is an epimorphism.

Now turn to the continuation of the epimorphism \(V \boxtimes L^{-m}|_{\tilde{\mathcal{M}}_0} \to \text{Quot}|_{\tilde{\mathcal{M}}_0}\). Let \(U' \subset T' \times S\) be an open subscheme such that \(U' \cap \tilde{\Sigma}_0 \neq \emptyset\). Assume that there exist an element \(f \in (V \boxtimes L^{-m})(U')\) vanishing in \(\text{Quot}|_{T' \times S}(U' \cap \tilde{\Sigma}_0)\) but not in \(\text{Quot}|_{T' \times S}(U')\). This means that \(f|_{\tilde{\Sigma}}(\tilde{T}' \times S, \tilde{\Sigma}_0) \neq 0\) what leads to the decomposition of irreducible topological space \(T' \times S\) into the disjoint union of two open subsets. This contradiction proves unique continuation of the epimorphism \(V \boxtimes T \circ \text{Quot}^{\text{red}}(\mathbb{V} \boxtimes L^{-m}) \boxtimes L^{-m}|_{\tilde{\mathcal{M}}_0} \to \text{Quot}|_{\tilde{\mathcal{M}}_0}\) to the epimorphism \(V \boxtimes T \circ \text{Quot}^{\text{red}}(\mathbb{V} \boxtimes L^{-m}) \boxtimes L^{-m}|_{T' \times S} \to E\) where \(E := \text{Quot}|_{T' \times S}\).

Note that the correspondence we built \(T \to T'\) is functorial and yields in a morphism of functors \(\text{Mor}_{\text{Sch}}(\mathcal{F}, \mathcal{F}') \to \text{Mor}_{\text{Sch}}(\mathcal{F}, \mathcal{F}')\) and hence in natural transformation of functors of points for schemes \(T\) and \(T'\). This means [11, lecture 3, Proposition] that there is a morphism of schemes \(\tau^{-1}: T \to T'\) which is inverse to \(\tau\).

It rests to confirm ourselves that the subscheme \(T' \subset \text{Quot}^{\text{red}}(V \boxtimes L^{-m})\) in whole lies in the subscheme \(Q\) corresponding to semistable coherent torsion-free sheaves. For this purpose we assume that \(T' = \text{Spec} \ A\) for \(A\) being a local \(k\)-algebra of finite type with a maximal ideal \(\mathfrak{m}\). The closed point \(\mathfrak{m} \in T'\) corresponding to admissible semistable pair \((\tilde{S}, \tilde{L}), \tilde{E}\), is taken in our construction to a coherent \(L\)-semistable torsion-free sheaf \(E_{\mathfrak{m}}\). Then, passing to localizations of the ring \(A\) in each of its prime ideals \(p \in \text{Spec} \ A = T'\), we can conclude that all semistable admissible pairs which correspond to points of the scheme \(T'\), are taken to semistable torsion-free coherent sheaves. Then \(T_{\text{red}}\) belongs to the subscheme \(Q\) of semistable coherent torsion-free sheaves. Properties of torsion-freeness and of Gieseker-semistability are open in flat families of coherent sheaves. Then if \(T'_{\text{red}}\) belongs to the subscheme of semistable torsion-free sheaves the same is true for \(T'\).

Indeed, assume that \(T'\) does not belong to the subscheme of torsion-free semistable sheaves. Since \(E_{\mathfrak{m}}\) is semistable and torsion-free, then there exists a nonempty closed subscheme in \(T'\) such that it contains a closed point corresponding to non-semistable sheaf or to a sheaf with torsion. This is impossible because the only closed point of the scheme \(T'\) corresponds to semistable torsion-free sheaf.

We have proven that there is a natural transformation \(\zeta: \mathcal{I} \to \mathcal{I}^{GM}\) of the functor of admissible semistable pairs to the functor of semistable coherent torsion-free sheaves. This natural transformation is defined by the series if commutative diagrams

\[
\begin{array}{ccc}
T & \xrightarrow{\mathcal{I}} & \mathcal{M}/\sim \\
\downarrow & & \downarrow \\
\mathcal{I}^{GM} & \xrightarrow{\zeta^{GM}} & \mathcal{M}/\sim 
\end{array}
\]

and leads to the morphism of moduli schemes \(\tau: \tilde{M} \to \tilde{M}\) by well-known procedure. In particular, the deduction of a morphism of moduli schemes from the morphism of functors can be found in [5].

Remark 5 In [5] the natural transformation \(\kappa: \mathcal{I}^{GM} \to \mathcal{I}\) and the corresponding morphism of moduli schemes \(\kappa: \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}\) are built up.

2. ISOMORPHISM OF MODULI FUNCTORS

The construction of the previous section establishes the morphism of functors \(\zeta: \mathcal{I} \to \mathcal{I}^{GM}\). In the paper [5] the morphism in opposite direction \(\kappa\) is constructed. It is necessary to prove that these two morphisms are mutually inverse and hence provide an isomorphism of functors.
First we show that \( \tau \circ \xi = id_{GM} \). For this purpose take a family of semistable torsion-free coherent sheaves \( E \) and a family of polarizations \( L \). Tensoring these sheaves if necessary by appropriate invertible \( \mathcal{O}_T \)-sheaves, assume that locally free sheaves \( p_*(E \otimes L^m) \) and \( p_*L^m \) have 1st Chern class equal to 0. Apply the procedure of standard resolution from \( S \) to the family chosen. This leads to a family of admissible semistable pairs \( (\tau : \Sigma \to T, \tilde{\xi}) \). Now perform a transformation from sect.1 of the present paper. We get a family of coherent torsion-free semistable sheaves \( E' \) again. Now, tensoring both \( \mathcal{O}_T \)-modules by appropriate invertible sheaves form the base \( T \) and getting \( \mathcal{O}_T \)-modules \( E' \) and \( L' \) respectively, we consider locally free \( \mathcal{O}_T \)-sheaves \( p_*(E' \otimes L'^m) \) and \( p_*L'^m \) as having 1st Chern class equal to 0. Families \( (L, E) \) and \( (L', E') \) coincide along the subscheme \( \Sigma \) in the notation of sect.1. According to the reasoning in sect. 1, they coincide on the whole of the product \( \Sigma = T \times S \). From this we conclude that \( (L, E) \sim (L', E') \). The proof done means that the natural transformation \( \xi : f^{GM} \to f \) is a section of the natural transformation \( \tau \) and hence the morphism of moduli spaces \( \kappa : M \to \tilde{M} \) is a section of the morphism \( \tau : M \to \tilde{M} \). Then \( \kappa : M \to \tilde{M} \) is a closed immersion. This follows from the simple lemma.

Lemma 1. Let \( f : X \to Y \) be a scheme morphism and \( s : Y \to X \) its section. Then \( s \) is closed immersion.

Proof. Since \( f \circ s = id_Y \), the morphism \( s \) maps the scheme \( Y \) isomorphically to its image in \( X \). Pass to affine subschemes and assume that \( X = \text{Spec } A \), \( Y = \text{Spec } B \), where \( A, B \) are commutative rings, \( f^2 : B \to A \), \( s^2 : A \to B \) their homomorphisms inducing corresponding scheme morphisms. Now \( s^2 \circ f^2 = id_B \). Then \( f^2 \) maps \( B \) isomorphically to its image in \( A \) and \( s^2 \) maps a subring \( f^2(B) \subset A \) to \( B \). Hence \( s^2 \) is surjective homomorphism and \( B \) is isomorphic to a quotient ring of \( A \). Now conclude that \( s : \text{Spec } B \to \text{Spec } A \) is closed immersion. 

We have proven that any admissible semistable pair \( ((\tilde{S}, \tilde{L}), \tilde{E}) \) corresponds to a coherent semistable torsion-free sheaf \( E \), and there is a closed immersion \( \tilde{M} \hookrightarrow M \).

Now confirm that \( \xi \circ \tilde{\xi} = id_{\tilde{L}} \). Let \( T_0 \) be maximal nonempty open subscheme in \( T \) whose closed points correspond to \( S \)-pairs. Restriction of the family \( (\tilde{\Sigma} \to T, \tilde{\xi}) \) to \( T_0 \) induces a locally closed immersion \( \mu_0 : T_0 \hookrightarrow \text{Quot}_{op}(V \boxtimes L^{-m}) \). This immersion in the composite with the structure projection to the base \( T \) provides an isomorphism \( \mu_0(T_0) \cong T_0 \). In this circumstance \( \mu_0(E_{\text{quot}}) = E_{\boxtimes l} \big|_{T_0} \).

Form a Grassmannian bundle \( \text{Grass}(V, r) \to T \) of \( r \)-quotient spaces of fibres of the vector bundle \( V \). The fibre of the bundle \( \text{Grass}(V, r) \to T \) at the point \( t \in T \) is usual Grassman variety \( G(V, r) \). Since all vector spaces \( V_t \cong V \) are isomorphic as having equal dimensions, all fibres of the Grassmannian bundle are also isomorphic: \( G(V, r) \cong G(V, r) \). For an admissible semistable pair \( ((\tilde{S}, \tilde{L}), \tilde{E}) \) for \( m \gg 0 \) there is a closed immersion \( j : \tilde{S} \hookrightarrow G(V, r) \) which is defined by the epimorphism of locally free sheaves \( H^0(\tilde{S}, \tilde{E} \otimes L^m) \boxtimes L^{-m} = \tilde{E} \). Let \( \mathcal{O}_{G(V, r)}(1) \) be a positive generator of the group \( \text{Pic } G(V, r) \), then \( P(n) := \chi(j^*\mathcal{O}_{G(V, r)}(n)) \) is the Hilbert polynomial of the closed subscheme \( j(\tilde{S}) \). Fix the polynomial \( P(n) \). Consider the Hilbert scheme \( \text{Hilb}^{P(n)} \text{Grass}(V, r) \) of subschemes in \( \text{Grass}(V, r) \) having Hilbert polynomial equal to \( P(n) \), and its universal subscheme \( \text{Univ}^{P(n)} \text{Grass}(V, r) \to \text{Hilb}^{P(n)} \text{Grass}(V, r) \).
The family of admissible semistable pairs \( ((\pi : \hat{\Sigma} \to T, \hat{\Sigma}), \hat{E}) \) induces the following diagram with fibred square and immersions with "relative" schemes

\[
\begin{array}{ccc}
\hat{\Sigma} & \xrightarrow{\hat{p}} & \text{Univ}^P \text{Grass}(V, r) \\
\pi & \downarrow & \downarrow \phi \\
T & \xrightarrow{\mu} & \text{Hilb}^P \text{Grass}(V, r)
\end{array}
\]

The family of interest contains \( S \)-pairs; take a maximal open \( \hat{\Sigma}_0 \subset \hat{\Sigma} \) which is isomorphic to an open subset in the product \( T \times S \). Consider the relative Grothendieck’s scheme \( \text{Quot}^r \text{Grass}(V \boxtimes L^{-m}) \) and the diagonal immersion

\[
\delta_0 : \hat{\Sigma}_0 \hookrightarrow \text{Univ}^P \text{Grass}(V, r) \times_T \text{Quot}^r \text{Grass}(V \boxtimes L^{-m}).
\]

Let a morphism \( \varphi \) be defined as a composite of \( T \)-morphisms

\[
\begin{array}{ccc}
\hat{\Sigma}_0 & \xrightarrow{\delta_0} & \text{Univ}^P \text{Grass}(V, r) \times_T \text{Quot}^r \text{Grass}(V \boxtimes L^{-m}) \\
\varphi & \downarrow \varphi' & \downarrow \varphi' \circ \mu \\
\text{Quot}^r \text{Grass}(V \boxtimes L^{-m}) & \end{array}
\]

Due to the reasoning on the natural transformation \( t \) in the previous section, the immersion \( \delta_0 \) is continued to the immersion

\[
\delta : \hat{\Sigma} \hookrightarrow \text{Univ}^P \text{Grass}(V, r) \times_T \text{Quot}^r \text{Grass}(V \boxtimes L^{-m}).
\]

The morphism \( \varphi \) has the scheme \( T' \) as its image. It is isomorphic to \( T \). The isomorphism is provided by the projection on the base \( T \) of relative schemes.

The immersion \( \mu' : \varphi(\hat{\Sigma}_0) \cong T \hookrightarrow \text{Quot}^r \text{Grass}(V \boxtimes L^{-m}) \) gives rise to a family of coherent semistable torsion-free sheaves \( E = \mu'^* \text{Quot} \). The family of polarizations \( L \) is given, for example, by the formula \( L = O_T \boxtimes L \).

Now perform a standard resolution of the family \( E \) as described in [8]. We are interested in such a version of standard resolution which does not change the base \( T \). The standard resolution suggested in [8] involves a blowing up \( \sigma : \hat{\Sigma} \to \Sigma \) of the product \( \Sigma = T \times S \) in the sheaf of Fitting ideals \( I = \text{Fit}^b \text{Ext}^1(E, O_E) \). In the same article we proved that the composite \( f := p_1 \circ \sigma : \hat{\Sigma} \to T \) is flat morphism. We obtain a family of admissible semistable pairs \( ((\pi' : \Sigma' \to T, \Sigma'), \Sigma') \), where \( \Sigma' = \hat{\Sigma}, \pi' = f, \Sigma' = \sigma* E \boxtimes \sigma^{-1} I \cdot O_{\Sigma'}, \Sigma' \) is such as described in [8]. It gives rise to a locally free sheaf \( \pi'_* (\hat{E}' \boxtimes (\hat{\Sigma}')^m) \). We need the following proposition which will be proven below.

**Proposition 1.** After tensoring by appropriate invertible \( O_T \)-sheaves following locally free sheaves are isomorphic: \( V = \pi_* (E \boxtimes \Omega^m), V_0 = p_* (E \boxtimes \Omega^m), \) and \( V' = \pi'_* (\hat{E}' \boxtimes (\hat{\Sigma}')^m) \).

By the proposition [1] there are following immersions of \( T \)-schemes

\[
\delta' : \hat{\Sigma}' \hookrightarrow \text{Univ}^P \text{Grass}(V, r) \times_T \text{Quot}^r \text{Grass}(V \boxtimes L^{-m})
\]

and

\[
\mu' : \hat{\Sigma}' \hookrightarrow \text{Univ}^P \text{Grass}(V, r).
\]

The scheme \( \hat{\Sigma}' \) contains an open subscheme \( \hat{\Sigma}'_0 \) which is isomorphic to the subscheme \( \hat{\Sigma}_0 \). By this isomorphism and by the construction of standard resolution there are isomorphisms \( \hat{\Sigma}'|\hat{\Sigma}'_0 \cong \hat{\Sigma}|\hat{\Sigma}_0 \) and \( \hat{E}'|\hat{\Sigma}'_0 \cong \hat{E}|\hat{\Sigma}_0 \). Besides, fibres of families \( \hat{\Sigma} \) and \( \hat{\Sigma}' \) in closed points coincide.
Remark 6. The coincidence along reductions follows from the uniqueness of the scheme closure for $\mu(\Sigma_{\text{red}}) = \tilde{\mu}'(\Sigma'_{\text{red}})$ in $\text{Univ}^{P(n)}\text{Grass}(V, r) \times_T T_{\text{red}}$. This implies the isomorphism of reduced moduli functors $f_{\text{red}}^* G_M \cong f_{\text{red}}^*$ and hence the isomorphism of reduced moduli schemes $M_{\text{red}} \cong M_{\text{red}}$.

Now note that under the projection
\[ \pi : \text{Univ}^{P(n)}\text{Grass}(V, r) \to \text{Hilb}^{P(n)}\text{Grass}(V, r) \]
we have
\[ \pi(\mu(\Sigma_0)) = \mu(T) = \mu'(T) = \pi(\tilde{\mu}'(\Sigma'_0)). \]
This subscheme is mapped by the structure projection $\text{Hilb}^{P(n)}\text{Grass}(V, r) \to T$ to the base $T$ isomorphically.

By the construction and by the universal property of the Hilbert scheme we have
\[ \tilde{\mu}(\Sigma) = \pi^{-1} \pi(\tilde{\mu}(\Sigma_0)) = \pi^{-1} \mu(T) = \pi^{-1} \mu'(T) = \pi^{-1} \pi(\tilde{\mu}'(\Sigma'_0)) = \tilde{\mu}'(\Sigma'). \]
Isomorphisms $\Sigma \cong \tilde{\mu}(\Sigma)$ and $\Sigma' \cong \tilde{\mu}'(\Sigma')$ complete the proof.

Proof of proposition 7. Consider an epimorphism of $\mathcal{O}_{T \times S}$-modules
\[ V \boxtimes L^{-m} \to \Sigma, \]
associated with the immersion $T \hookrightarrow \text{Quot}^r(V \boxtimes L^{-m})$. Tensoring by $\mathcal{O}_S$-sheaf $L^m$ and formation of a direct image $p_*$ lead to the morphism of locally free $\mathcal{O}_T$-modules $\psi : V \to p_*(\Sigma \boxtimes L^m)$. The sheaf on the right hand side differs from $V_0$ by tensoring by some invertible $\mathcal{O}_T$-module which proves the proposition for $\mathcal{O}_T$-modules $V$ and $V_0$.

Now turn to the pair $V_0$ and $V'$. Recall that sheaves $\Sigma$ and $\Sigma'$ are obtained by standard resolution of the family $E$. In this procedure one gets an epimorphism $\sigma^* \Sigma \to \Sigma'$ [8]. Twisting by $(\Sigma')^m$ and formation of the direct image $\sigma_*$ lead to the morphism of $\mathcal{O}_{T \times S}$-modules
\[ \sigma_* (\sigma^* \Sigma \boxtimes (\Sigma')^m) \to \sigma_* (\Sigma' \boxtimes (\Sigma')^m). \]
(2.1)
Now we need the following lemma which generalizes well-known projection formula.

Lemma 2. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces such that $f_* \mathcal{O}_X = \mathcal{O}_Y$, $E$ $\mathcal{O}_Y$-module of finite presentation, $\mathcal{F} \mathcal{O}_X$-module. Then there is a monomorphism $\mathcal{E} \otimes f_* \mathcal{F} \hookrightarrow f_* (\mathcal{E} \otimes \mathcal{F})$.

Proof of lemma 2. Fix any finite presentation for $\mathcal{E}$: $E_1 \to E_0 \to \mathcal{E} \to 0$, where $E_0, E_1$ are locally free $\mathcal{O}_Y$-modules. Formation of an inverse image $f^*$, tensoring by $\otimes_X \mathcal{F}$ followed by formation of a direct image $f_*$ lead to a complex
\[ \cdots \to f_* [f^* E_1 \otimes_X \mathcal{F}] \to f_* [f^* E_1 \otimes_X \mathcal{F}] \to f_* [f^* \mathcal{E} \otimes_X \mathcal{F}] \to \cdots \]
Due to the usual projection formula, first two terms equal $E_1 \otimes_Y f_* \mathcal{F}$ and $E_0 \otimes_Y f_* \mathcal{F}$ respectively. Then we have
\[ \mathcal{E} \otimes_Y f_* \mathcal{F} = \text{coker} (E_1 \otimes_Y f_* \mathcal{F} \to E_0 \otimes_Y f_* \mathcal{F}) \hookrightarrow f_* [f^* \mathcal{E} \otimes_X \mathcal{F}] . \]
This proves the lemma.

Applying the lemma we get the monomorphism
\[ (E \boxtimes L^{-m}) \otimes f_* (\sigma^{-1} \mathbb{1} \cdot \mathcal{O}_X)^m \hookrightarrow f_* (\sigma^* \Sigma \boxtimes (\Sigma')^m). \]
(2.2)
Formation of inverse image $p_*$ in both morphisms \((2.1)\) and \((2.2)\) and the equality $\pi = p \circ \sigma$ lead to the diagram

\[
\begin{array}{ccc}
E \otimes L^m & \xrightarrow{\sigma_* (\sigma^{-1} \cdot \mathcal{I}_\Sigma)^m} & p_* [E \otimes L^m] \\
\downarrow \pi_* \sigma E \otimes (L')^m & & \downarrow \pi_* [E' \otimes (L')^m] \\
\end{array}
\]

Upper horizontal arrow is induced by the inclusion $\sigma^{-1} \cdot \mathcal{I}_\Sigma \hookrightarrow \mathcal{O}_\Sigma$. Lower horizontal arrow is an epimorphism since $m \gg 0$ and $\widetilde{L}'$ is ample relatively to the projection $\pi$. Since the homomorphism $E$ is an automorphism. Let the reduction $\Phi : M \rightarrow M'$ be of use below.

In the diagram \((2.3)\) sheaves from right hand side are locally free of rank $rp(m)$. Tensoring $E$ (or $\widetilde{E}$) by an appropriate invertible sheaf $\mathcal{L}$ from the base $T$ we make sheaves from the right hand side in \((2.3)\) coincide on the open subset out of a subscheme of codimension $\geq 2$. Besides, it is known \([12]\) that for a sheaf of ideals $I$ on a scheme $\Sigma$ and for any invertible $\mathcal{O}_\Sigma$-sheaf $L'$ blowing ups of this scheme defined by sheaves $I$ and $I \otimes L'$, are isomorphic. Then tensoring if necessary the sheaf of ideals $I$ by an appropriate invertible $\mathcal{O}_T$-sheaf $L'$, we achieve that sheaves in the upper row of the diagram \((2.3)\) coincide on the open subset out of subscheme of codimension $\geq 2$. This transformation leads to tensoring the sheaf $L'$ by $\pi^* L'$.

After such a transformation the upper horizontal arrow in \((2.3)\) is a canonical morphism of the sheaf $p_* ([E \otimes L^m] \otimes \sigma_* (\sigma^{-1} \cdot \mathcal{I}_\Sigma)^m)$ to its reflexive hull. The skew arrow is the morphism of the same sheaf to a locally free sheaf which is obviously reflexive. Hence this morphism factors through the reflexive hull and gives rise to a (double dual to $\eta$) morphism $\eta^{\vee \vee} : p_* [E \otimes L^m] \rightarrow \pi_* [E' \otimes (L')^m]$ of locally free $\mathcal{O}_T$-sheaves. It is an isomorphism on open subset out of a subscheme of codimension not less than 2. Besides, the restrictions of both locally free sheaves to the reduction $T_{red}$ also coincide. The coincidence of sheaves on the whole of the scheme $T$ follows from the following simple algebraic lemma.

**Lemma 3.** Let $A$ be a commutative ring, $M$ finite $A$-module, $\Phi : M \rightarrow M$ its $A$-endomorphism. Let the reduction $\Phi_{red} : M_{red} \rightarrow M_{red}$ is $A_{red}$-automorphism. Then $\Phi$ is $A$-automorphism.

\[\Box\]

**Proof of lemma** Since the homomorphism $\Phi_{red}$ is surjective and the nilradical includes in the Jacobson radical then, due to \([13]\) ch. 2, exercise 10], $\Phi$ is also surjective. Then by \([14]\) theorem 2.4 $\Phi$ is an automorphism.  \[\Box\]

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