Finiteness of attractors and repellers on sectional hyperbolic sets

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Abstract

We obtain an upper bound for the number of attractors and repellers that can appear from small perturbations of a sectional hyperbolic set. This extends results from [7] and [9].

1 Introduction

A sectional hyperbolic set is a partially hyperbolic set whose singularities are hyperbolic and whose central subbundle is sectionally expanding.

The result [9] asserts that for every sectional hyperbolic transitive attracting set Λ of a vector field $X$ on a compact 3-manifold there are neighborhoods $U$ of $X$ and $U$ of $Λ$ such that the number of attractors in $U$ of a vector field in $U$ is less than one plus the number of equilibria of $X$. This result was extended later in [11] by allowing $Λ$ to be an attracting set contained in the nonwandering set (rather than transitive). An extension of [9] to higher dimensions was recently obtained in [7]. The present work removes both transitivity and nonwandering hypotheses in order to prove that for every sectional hyperbolic set $Λ$ of a vector field $X$ on a compact manifold there are neighborhoods $U$ of $X$, $U$ of $Λ$ and a positive integer $n_0$ such that the number of attractors in $U$ of a vector field in $U$ is less than $n_0$. Let us state our result in a precise way.

Consider a compact manifold $M$ of dimension $n \geq 3$ (a compact $n$-manifold for short) with a Riemannian structure $\| \cdot \|$. We denote by $\partial M$ the boundary of $M$. Let $\mathcal{X}^1(M)$ be the space of $C^1$ vector fields in $M$ endowed with the $C^1$ topology. Fix $X \in \mathcal{X}^1(M)$, inwardly transverse to the boundary $\partial M$ and denotes by $X_t$ the flow of $X$, $t \in \mathbb{R}$. The maximal invariant set of $X$ is defined by

$$M(X) = \bigcap_{t \geq 0} X_t(M).$$

*Key words and phrases: Attractor, Repeller, Maximal invariant, Sectional-Anosov flow.

This work is partially supported by CAPES, Brazil.
Notice that $M(X) = M$ in the boundaryless case $\partial M = \emptyset$. A subset $\Lambda$ is called invariant if $X_t(\Lambda) = \Lambda$ for every $t \in \mathbb{R}$. We denote by $m(L)$ the minimum norm of a linear operator $L$, i.e., $m(L) = \inf_{\|v\| 
eq 0} \frac{\|Lv\|}{\|v\|}$.

**Definition 1.1.** A compact invariant set $\Lambda$ of $X$ is partially hyperbolic if there is a continuous invariant splitting $T_\Lambda M = E^s \oplus E^c$ such that the following properties hold for some positive constants $C, \lambda$:

1. $E^s$ is contracting, i.e., $\|DX_t(x)\big|_{E^s}\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

2. $E^s$ dominates $E^c$, i.e., $\frac{\|DX_t(x)\big|_{E^c}\|}{m(DX_t(x)\big|_{E^s})} \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

We say the central subbundle $E^c_x$ of $\Lambda$ is sectionally expanding if $\dim(E^c_x) \geq 2$ and $|\det(DX_t(x)\big|_{L_x})| \geq C^{-1}e^\lambda$, $\forall x \in \Lambda$ and $t > 0$ for all two-dimensional subspace $L_x$ of $E^c_x$. Here $\det(DX_t(x)\big|_{L_x})$ denotes the jacobian of $DX_t(x)$ along $L_x$.

Recall that a singularity of a vector field is hyperbolic if the eigenvalues of its linear part have non zero real part.

**Definition 1.2.** A sectional hyperbolic set is a partially hyperbolic set whose singularities (if any) are hyperbolic and whose central subbundle is sectionally expanding.

The $\omega$-limit set of $p \in M$ is the set $\omega_X(\Lambda)$ formed by those $q \in M$ such that $q = \lim_{n \to \infty} X_{t_n}(p)$ for some sequence $t_n \to \infty$. We say that $\Lambda \subset M$ is transitive if $\Lambda = \omega_X(\Lambda)$ for some $p \in \Lambda$. We say that $\Lambda$ is singular if it contains a singularity; and attracting if $\Lambda = \cap_{t > 0} X_t(U)$ for some compact neighborhood $U$ of $\Lambda$. This neighborhood is called isolating block of $\Lambda$. It is well known that the isolating block $U$ can be chosen to be positively invariant, namely $X_t(U) \subset U$ for all $t > 0$. An attractor is a transitive attracting set. A repelling is an attracting for the time reversed vector field $-X$ and a repeller is a transitive repelling set.

With these definitions we can state our main result.

**Theorem A.** For every sectional hyperbolic set $\Lambda$ of a vector field $X$ on a compact manifold there are neighborhoods $U$ of $X$, $U$ of $\Lambda$ and $n_0 \in \mathbb{N}$ such that

$$\#\{L \subset U : L \text{ is an attractor or repeller of } Y \in U\} \leq n_0.$$ 

To finish we state a direct corollary of our result. Recall that a sectional-Anosov flow is a vector field whose maximal invariant set is sectional hyperbolic [8].

**Corollary 1.3.** For every sectional-Anosov flow of a compact manifold there are a neighborhood $U$ and $n_0 \in \mathbb{N}$ such that

$$\#\{L \text{ is an attractor or repeller of } Y \in U\} \leq n_0.$$
2 Proof

An useful property of sectional hyperbolic sets is given below.

**Lemma 2.1.** Let $X$ be a $C^1$ vector field of a compact $n$-manifold $M$, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a sectional hyperbolic set of $X$. Then, there is a neighborhood $U \subset \mathcal{X}^1(M)$ of $X$ and a neighborhood $U \subset M$ of $\Lambda$ such that if $Y \in U$, every nonempty, compact, non singular, invariant set $H$ of $Y$ in $U$ is hyperbolic saddle-type (i.e. $E^s \neq 0$ and $E^u \neq 0$).

*Proof.* See ([10]).

This following theorem examining the sectional hyperbolic splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ of a sectional hyperbolic set $\Lambda$ of $X \in \mathcal{X}^1(M)$ appears in [3].

**Theorem 2.2.** Let $X$ be a $C^1$ vector field of a compact $n$-manifold $M$, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a sectional hyperbolic set of $X$. If $\sigma \in \text{Sing}(X) \cap \Lambda$, then $\Lambda \cap W^{ss}(\sigma) = \{\sigma\}$.

We use it to prove the following.

**Proposition 2.3.** Let $X$ be a $C^1$ vector field of a compact $n$-manifold $M$, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a sectional hyperbolic set of $X$. Let $\sigma$ be a singularity of $X$ in $\Lambda$. Then, for every isolating block $U$ of $\Lambda$, there is a neighborhood $V$ of $W^{ss}(\sigma) \setminus \{\sigma\}$ in $U$ such that

$$(\cap_{t>0} Y_t(U)) \cap V = \emptyset,$$

for every $C^1$ vector field $Y$ close to $X$.

*Proof.* The equality in Theorem 2.2 implies that the negative orbit of every point in $W^{ss}(\sigma) \setminus \{\sigma\}$ leaves $\Lambda$. Hence we can arrange neighborhood $V$ containing $W^{ss}(\sigma) \setminus \{\sigma\}$ and such that

$$\Lambda \cap V = \emptyset$$

Since $U$ is the isolating block of $\Lambda$ we can find $T > 0$ such that

$$X_T(U) \cap V = \emptyset.$$

Hence

$$Y_T(U) \cap V = \emptyset,$$

for all $C^r$ vector field close to $X$. The result follows since $\cap_{t>0} Y_t(U) \subset Y_T(U)$. 

Next we recall the standard definition of hyperbolic set.
Definition 2.4. A compact invariant set $\Lambda$ of $X$ is hyperbolic if there are a continuous tangent bundle invariant decomposition $T_\Lambda M = E^s \oplus E^x \oplus E^u$ and positive constants $C, \lambda$ such that

- $E^x$ is the vector field’s direction over $\Lambda$.
- $E^s$ is contracting, i.e., $\| DX_t(x) \|_{E^s} \leq C e^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.
- $E^u$ is expanding, i.e., $\| DX_{-t}(x) \|_{E^u} \leq C e^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

A closed orbit is hyperbolic if it is also hyperbolic, as a compact invariant set. An attractor is hyperbolic if it is also a hyperbolic set.

It follows from the stable manifold theory [6] that if $p$ belongs to a hyperbolic set $\Lambda$, then the following sets

$$W^s_X(p) = \{ x : d(X_t(x), X_t(p)) \to 0, t \to \infty \},$$
$$W^u_X(p) = \{ x : d(X_t(x), X_t(p)) \to 0, t \to -\infty \},$$

are $C^1$ immersed submanifolds of $M$ which are tangent at $p$ to the subspaces $E^s_p$ and $E^u_p$ of $T_p M$ respectively. Similarly,

$$W^s_X(p) = \bigcup_{t \in \mathbb{R}} W^s_{X}(t(p)),$$
$$W^u_X(p) = \bigcup_{t \in \mathbb{R}} W^u_{X}(t(p)).$$

are also $C^1$ immersed submanifolds tangent to $E^s_p \oplus E^u_p$ and $E^s_p \oplus E^u_p$ at $p$ respectively. Moreover, for every $\epsilon > 0$ we have that

$$W^s_X(p, \epsilon) = \{ x : d(X_t(x), X_t(p)) \leq \epsilon, \forall t \geq 0 \},$$
$$W^u_X(p, \epsilon) = \{ x : d(X_t(x), X_t(p)) \leq \epsilon, \forall t \leq 0 \}$$

are closed neighborhoods of $p$ in $W^s_X(p)$ and $W^u_X(p)$ respectively.

Let $O = \{ X_t(x) : t \in \mathbb{R} \}$ be the orbit of $X$ through $x$, then the stable and unstable manifolds of $O$ defined by

$$W^s(O) = \bigcup_{x \in O} W^s(x),$$
$$W^u(O) = \bigcup_{x \in O} W^u(x)$$

are $C^1$ submanifolds tangent to the subbundles $E^s_\Lambda \oplus E^u_\Lambda$ and $E^u_\Lambda \oplus E^u_\Lambda$ respectively.

A homoclinic orbit of a hyperbolic periodic orbit $O$ is an orbit in $\gamma \subset W^s(O) \cap W^u(O)$. If additionally $T_q M = T_q W^s(O) + T_q W^u(O)$ for some (and hence all) point $q \in \gamma$, then we say that $\gamma$ is a transverse homoclinic orbit of $O$.

Definition 2.5. The homoclinic class $H(O)$ of a hyperbolic periodic orbit $O$ is the closure of the union of the transverse homoclinic orbits of $O$. We say that an invariant set $L$ is a homoclinic class if $L = H(O)$ for some hyperbolic periodic orbit $O$. 

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We denote by:
\( \text{Sing}(X) \) the set of singularities of \( X \).
\( \text{Cl}(A) \) the closure of \( A, A \subset M \).
If \( \delta > 0 \), \( B_\delta(A) = \{ x \in M : d(x,A) < \delta \} \), where \( d(\cdot, \cdot) \) is the metric in \( M \).

**Lemma 2.6.** Let \( X \) be a \( C^1 \) vector field of a compact \( n \)-manifold \( M, X \in \mathcal{X}^1(M) \).
Let \( \Lambda \in M \) be a hyperbolic set of \( X \). Then, there is a neighborhood \( U \subset \mathcal{X}^1(M) \) of \( X \), a neighborhood \( U \subset M \) of \( \Lambda \) and \( n_0 \in \mathbb{N} \) such that
\[ \# \{ L \subset U : L \text{ is homoclinic class of } Y \in U \} \leq n_0 \]
for every vector field \( Y \in U \).

**Proof.** By the stability of hyperbolic sets we can fix a neighborhood \( U \subset M \) of \( \Lambda \), a neighborhood \( U \subset \mathcal{X}^1(M) \) of \( X \) and \( \epsilon > 0 \) such that every hyperbolic set \( H \subset U \) of every \( Y \in U \) satisfies that
\[ W^{ss}_Y(x, \epsilon), W^{uu}_Y(x, \epsilon) \text{ have uniform size } \epsilon \text{ for all } x \in H \] (1)

By contradiction, we suppose that there exists a sequence of vector fields \( X^n \in U \) converging to \( X \) such that
\[ \# \{ L \subset U : L \text{ is homoclinic class of } X^n \} \geq n \]

It is well known [5] that the periodic orbits are dense in \( L^n \subset \Lambda^n = \Lambda_{X^n} \), for all \( n \in \mathbb{N} \). Moreover, these homoclinic classes are pairwise disjoint.

Let \( \epsilon > 0 \) be the uniform size by (1), and let \( \eta > 0 \) be such that \( 0 < \eta < \frac{\epsilon}{2} \).
Since \( U \) is neighborhood of \( \Lambda \), \( \text{Cl}(U) \) is compact neighborhood of \( \Lambda \), then we can cover \( \text{Cl}(U) \) with a finite number of balls with radius \( \frac{\eta}{2} \). We denote this finite number by \( n_0 \).

Thus, if two periodic points \( p_1, p_2 \in L \) satisfies \( d(p_1, p_2) < \eta \), then
\[ W^{ss}_X(p_1, \epsilon) \cap W^{uu}_X(p_2, \epsilon) \neq \emptyset \] (2)

Therefor, for every vector field \( X^N \) with \( N > n_0 \), we have that there are homoclinic classes \( L^i, L^j \) of \( X^N \) in \( \text{Cl}(U) \) contained in the same ball with radius \( \frac{\eta}{2}, 1 \geq i < j \geq N \).

Since \( L^i \) and \( L^j \) are homoclinic classes, there are periodic points \( p^i \) and \( p^j \) of \( L^i \) and \( L^j \) respectively satisfying (2), then \( p^i \) and \( p^j \) belongs to the same homoclinic class and this imply \( L^i = L^j \). Thus, the sequence \( (L^n)_{n \in \mathbb{N}} \), is constant for \( n \) enough large. This is a contradiction and the proof follows. \( \square \)

**Lemma 2.7.** Let \( X \) be a \( C^1 \) vector field of a compact \( n \)-manifold \( M, n \geq 3, X \in \mathcal{X}^1(M) \). Let \( \Lambda \in M \) be a sectional hyperbolic set of \( X \). Let \( Y^n \) be a sequence of vector fields converging to \( X \) in the \( C^1 \) topology. There is a neighborhood
$U \subset M$ of $\Lambda$, such that if $R^n$ is a repeller of $Y^n$, $R^n \subset \cap_{t>0}Y^n_t(U)$ for each $n \in \mathbb{N}$, then the sequence $(R^n)_{n \in \mathbb{N}}$ of repellers do not accumulate on the singularities of $X$, i.e.,

$$Sing(X) \cap Cl(\cup_{n \in \mathbb{N}} R^n) = \emptyset$$

Proof. Let $\sigma \in Sing(X)$ and we denote $\Lambda_Y = \cap_{t>0}Y^n_t(U)$. Fix the neighborhood $U$ of $\Lambda$ as in Lemma 2.1 and thus we can assume that $U$ is an isolating block of $\Lambda$. Assume by contradiction that

$$Sing(X) \cap Cl(\cup_{n \in \mathbb{N}} R^n) \neq \emptyset.$$ 

Then, exist a sequence $(x_n)_{n \in \mathbb{N}}$, with $x_n \in R^n \subset \Lambda_Y$, for all $n \in \mathbb{N}$, and such that

$$x_n \rightarrow \sigma.$$ 

Since $\Lambda$ is sectional hyperbolic set, we have (by Theorem 2.2) that $\Lambda \cap W^{ss}_X(\sigma) = \{\sigma\}$, and as $Y^n \rightarrow X$ (by Proposition 2.3), there is a neighborhood $V$ of $W^{ss}(\sigma) \setminus \{\sigma\}$ in $M$ such that $\Lambda_Y \cap V = \emptyset$, for $n \in \mathbb{N}$ large enough.

As $x_n \rightarrow \sigma$, for $\epsilon > 0$ uniform size, $W^{ss}_Y(x_n, \epsilon) \rightarrow W^{ss}_X(\sigma, \epsilon)$ in the sense $C^1$ manifolds [11].

Then, for $n \in \mathbb{N}$ enough large, $W^{ss}_Y(x_n, \epsilon) \cap V \neq \emptyset$. Note that $W^{ss}_Y(x_n, \epsilon) \subset W^{ss}_Y(x^n) \subset R^n$, since is repeller of $Y^n$. Hence $R^n \cap V \neq \emptyset$, then $\Lambda_Y \cap V \neq \emptyset$. This is a contradiction. \thickhline

Let $M$ be a compact $n$-manifold, $n \geq 3$. Fix $X \in \chi^1(M)$, inwardly transverse to the boundary $\partial M$. We denotes by $X_t$ the flow of $X$, $t \in \mathbb{R}$.

There is also a stable manifold theorem in the case when $\Lambda$ is sectional hyperbolic set. Indeed, denoting by $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ the corresponding the sectional hyperbolic splitting over $\Lambda$ we have from [8] that the contracting subbundle $E^s_\Lambda$ can be extended to a contracting subbundle $E^s_U$ in $M$. Moreover, such an extension is tangent to a continuous foliation denoted by $W^{ss}$ (or $W^{ss}_X$ to indicate dependence on $X$). By adding the flow direction to $W^{ss}$ we obtain a continuous foliation $W^s$ (or $W^s_X$) now tangent to $E^u_U \oplus E^c_U$. Unlike the Anosov case $W^s$ may have singularities, all of which being the leaves $W^{ss}(\sigma)$ passing through the singularities $\sigma$ of $X$. Note that $W^s$ is transverse to $\partial M$ because it contains the flow direction (which is transverse to $\partial M$ by definition).

It turns out that every singularity $\sigma$ of a sectional hyperbolic set $\Lambda$ satisfies $W^{ss}_X(\sigma) \subset W^s_X(\sigma)$. Furthermore, there are two possibilities for such a singularity, namely, either $\dim(W^{ss}_X(\sigma)) = \dim(W^s_X(\sigma))$ (and so $W^{ss}_X(\sigma) = W^s_X(\sigma)$) or $\dim(W^s_X(\sigma)) = \dim(W^{ss}_X(\sigma)) + 1$. In the later case we call it Lorenz-like according to the following definition.

**Definition 2.8.** We say that a singularity $\sigma$ of a sectional-Anosov flow $X$ is Lorenz-like if $\dim(W^s(\sigma)) = \dim(W^{ss}(\sigma)) + 1$. 

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Let $\sigma$ be a singularity Lorenz-like of a sectional hyperbolic set $\Lambda$. We will denote $\dim(W^s(X)(\sigma)) = s$ and $\dim(W^s(X)(\sigma)) = u$, therefore $\sigma$ has a $(s + 1)$-dimensional local stable manifold $W^s(X)(\sigma)$. Moreover $W^s(X)(\sigma)$ separates $W^s_{\text{loc}}(\sigma)$ in two connected components denoted by $W^s_{\text{loc}}(\sigma)$ and $W^s_{\text{loc}}(\sigma)$ respectively.

**Definition 2.9.** A singular-cross section of a Lorenz-like singularity $\sigma$ will be a pair of submanifolds $\Sigma^t, \Sigma^b$, where $\Sigma^t, \Sigma^b$ are cross sections and:

\[ \Sigma^t \text{ is transversal to } W^s_{\text{loc}}(\sigma). \]
\[ \Sigma^b \text{ is transversal to } W^s_{\text{loc}}(\sigma). \]

Note that every singular-cross section contains a pair singular submanifolds $l^t, l^b$ defined as the intersection of the local stable manifold of $\sigma$ with $\Sigma^t, \Sigma^b$ respectively.

Also note that $\dim(l^*) = \dim(W^{ss}(\sigma))$.

If $* = t, b$ then $\Sigma^*$ is a hypercube of dimension $(n - 1)$, i.e., diffeomorphic to $B^u[0, 1] \times B^{ss}[0, 1]$, with $B^u[0, 1] \approx I^u, B^{ss}[0, 1] \approx I^s, I^k = [-1, 1]^k, k \in \mathbb{Z}$ and where:

- $B^{ss}[0, 1]$ is a ball centered at zero and radius 1 contained in $\mathbb{R}^{\dim(W^{ss}(\sigma))} = \mathbb{R}^s$.
- $B^u[0, 1]$ is a ball centered at zero and radius 1 contained in $\mathbb{R}^{\dim(W^u(\sigma))} = \mathbb{R}^{n-s-1}$.

Let $f : B^u[0, 1] \times B^{ss}[0, 1] \to \Sigma^*$ be the diffeomorphism, where
\[ f(\{0\} \times B^{ss}[0, 1]) = l^* \]
and $\{0\} = 0 \in \mathbb{R}^u$. Hence, we denoted the boundary of $\Sigma^*$ for $\partial\Sigma^*$, and $\partial\Sigma^* = \partial^h\Sigma^* \cup \partial^v\Sigma^*$ such that
\[ \partial^h\Sigma^* = \{ \text{the union of the boundary submanifolds which are transverse to } l^* \} \]
\[ \partial^v\Sigma^* = \{ \text{the union of the boundary submanifolds which are parallel to } l^* \}. \]

Moreover,
\[ \partial^h\Sigma^* = (I^u \times [\bigcup_{j=0}^{-1}(I^j \times \{-1\} \times I^{s-j-1})) \bigcup (I^u \times [\bigcup_{j=0}^{-1}(I^j \times \{1\} \times I^{s-j-1})]) \]
\[ \partial^v\Sigma^* = ([\bigcup_{j=0}^{-1}(I^j \times \{-1\} \times I^{u-j-1})) \times I^s) \bigcup ([\bigcup_{j=0}^{-1}(I^j \times \{1\} \times I^{u-j-1})] \times I^s) \]
and where $I^0 \times I = I$.

Hereafter we denote $\Sigma^* = B^u[0, 1] \times B^{ss}[0, 1]$.

**Proposition 2.10.** Let $X$ be a $C^1$ vector field of a compact $n$-manifold $M$, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a sectional hyperbolic set of $X$. Then, there are neighborhoods $U$ of $X$, $U$ of $\Lambda$ and $n_0 \in \mathbb{N}$ such that
\[ \# \{ A \subset U : A \text{ is an attractor of } Y \in U \} \leq n_0. \]

**Proof.** The proof is by contradiction, i.e., suppose that for \( n \in \mathbb{N} \), we have that for all neighborhood \( U \) of \( X \), exists \( Y \in U \) such that
\[ \# \{ A \subset U : A \text{ is an attractor of } Y \in U \} \geq n. \]

Then, there is a sequence of vectors fields \((X^n)_n \in \mathbb{N}\), such that \( X^n \xrightarrow{C^1} X \), and a sequence \((A^n)_n \in \mathbb{N}\) where \( A^n \) is an attractor of vector field \( X^n \), for all \( n \).

By compactness we can suppose that the attractors are non-singular, since the singularities are isolated. Fix the neighborhood \( U \) of \( \Lambda \) as in Lemma 2.1 and thus we can assume that \( U \) is an isolating block of \( \Lambda \).

We claim that the sequence \((A^n)_n \in \mathbb{N}\) of attractors accumulate on the singularities of \( X \), otherwise \( Sing(X) \cap Cl(\cup_{n \in \mathbb{N}} A^n) = \emptyset \), then there is \( \delta > 0 \), such that \( B_\delta(Sing(X)) \cap (\cup_{n \in \mathbb{N}} A^n) = \emptyset \).

Thus, in the same way as in \([7]\), we define
\[ H = \cap_{t \in \mathbb{R}} X_t \left( U \setminus B_{\delta/2}(Sing(X)) \right) \tag{3} \]

By definition \( Sing(X) \cap H = \emptyset \), \( H \) is compact since \( \Lambda \) is, and \( H \) is a nonempty compact set \([7]\), which is clearly invariant for \( X \). It follows that \( H \) is hyperbolic by Lemma 2.1 and by Lemma 2.6 there is \( n_0 \in \mathbb{N} \) such that the sequence of attractors is bounded by \( n_0 \), that is a contradiction.

Then, the sequence \((A^n)_n \in \mathbb{N}\) of attractors accumulate on the singularities of \( X \), i.e., \( Sing(X) \cap Cl(\cup_{n \in \mathbb{N}} A^n) \neq \emptyset \). Thus, exists \( \sigma \in U \) such that \( \sigma \in Sing(X) \cap Cl(\cup_{n \in \mathbb{N}} A^n) \).

The subbundle \( E^s \) of \( \Lambda \) extends to a contracting invariant subbundle on the whole \( U \) and we take a continuous (not necessarily invariant) extension of \( E^c \) in \( U \). We have that this extension persists by small perturbations of \( X \) \([6]\) and we denote the splitting by \( E^{s,n} \oplus E^{c,n} \), where \( E^{s,0} \oplus E^{c,0} = E^s \oplus E^c \). We can assume that \( \sigma(X^n) = \sigma \) and \( l^t \cup l^b \subset W^s_{X^n}(\sigma) \) for all \( n \).

As before we fix a coordinate system \((x,y) = (x^*,y^*) \) in \( \Sigma^s \) with \( * = t,b \) and such that \( \Sigma^* = B^u[0,1] \times B^{ss}[0,1] \) and \( l^* = \{0\} \times B^{ss}[0,1] \) with respect to \((x,y)\).

Denote by \( \Pi^* : \Sigma^s \to B^u[0,1] \) the projection, where \( \Pi^*(x,y) = x \) and for \( \Delta > 0 \) we define \( \Sigma^s,\Delta = B^u[0,\Delta] \times B^{ss}[0,1] \).

Then, by Theorem 2.2 we have that \( \Lambda \cap W^s_{X^n}(\sigma) = \{\sigma\} \) and by Lemma 2.1 \( A^n \) is a hyperbolic attractor of type saddle of \( X^n \) for all \( n \). Then by \([7]\) for every isolating block \( U \) of \( \Lambda \) we can choose \( \Sigma^t, \Sigma^b \), singular-cross section for \( \sigma \) in \( U \) such that
\[ (\cap_{t > 0} X^n_t(U)) \cap (\partial^b \Sigma^t \cup \partial^h \Sigma^b) = \emptyset \tag{4} \]
and we have that there is \( n_1 \) such that \( A^{n_1} \cap \text{int}(\Sigma^s,\Delta_0) \neq \emptyset \).
We shall assume that $A^{n_1} \cap \text{int}(\Sigma^t, \Delta_0) \neq \emptyset$ (Analogous proof for the case $* = b$). By (4) we have $A^{n_1} \cap \partial^h \Sigma^t, \Delta_0 = \emptyset$, and by compactness we have that there is $p \in \Sigma^t, \Delta_0 \cap A^{n_1}$ such that

$$\text{dist}(\Pi^t(\Sigma^t, \Delta_0 \cap A^{n_1}), 0) = \text{dist}(\Pi^t(p), 0),$$

where $\text{dist}$ denotes the distance in $B^n[0, \Delta_0]$. Note that $\text{dist}(\Pi^t(p), 0)$ is the minimum distance of $\Pi^t(\Sigma^t, \Delta_0 \cap A^{n_1})$ to 0 in $B^n[0, \Delta_0]$.

As $W^u_{X^{n_0}}(p) \subset A^{n_1}$, since $A^{n_1}$ is attractor, we have that $W^u_{X^{n_1}}(p) \cap \Sigma^t, \Delta_0$ contains some compact manifold $K^{n_1}$.

We have that $K^{n_1}$ is transverse to $\Pi^t$ (i.e. $K^{n_1}$ is transverse to the curves $(\Pi^t)^{-1}(c)$, for every $c \in B^n[0, \Delta_0]$). First we denote $\Pi^t(K^{n_1}) = K^{n_1}$, the image of $K^{n_0}$ by the projection $\Pi^t$ in $B^n[0, \Delta_0]$. Note that $K^{n_1} \subset B^n[0, \Delta_0]$ and $\Pi^t(p) \in \text{int}(K^{n_1})$.

Since $\dim(K^{n_1}) = \dim(B^n[0, \Delta_0]) = (n - s - 1)$, there is $z_0 \in K^{n_0}$ such that

$$\text{dist}(\Pi^t(z_0), 0) < \text{dist}(\Pi^t(p), 0).$$

As $A^{n_1} \cap \partial^h \Sigma^t, \Delta_0 = \emptyset$, $K^{n_1} \subset W^u_{X^{n_1}}(p)$ and $\dim(K^{n_1}) = \dim(B^n[0, \Delta_0])$, we conclude that $\text{dist}(\Pi^t(\Sigma^t, \Delta_0 \cap A^{n_1}), 0) = 0$, and this last equality implies that

$$A^{n_1} \cap l^t \neq \emptyset.$$

Since $l^t \subset W^s_{X^{n_1}}(\sigma)$ and $A^{n_1}$ is closed invariant set for $X^{n_1}$ we conclude that $\sigma \in A^{n_1}$. This is a contradiction, since by hypotheses we have that $A^n$ is non-singular for all $n \in \mathbb{N}$ and the proof follows.

\[\square\]

**Proof of Theorem [1]** We prove the theorem by contradiction. Let $X$ be a $C^1$ vector field of a compact $n$-manifold $M$, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \in M$ be a sectional hyperbolic set of $X$. Then, we suppose that there is a sequence $(X^n)_{n \in \mathbb{N}} \subset \mathcal{X}^1(M)$, $X^n \xrightarrow{C^1} X$ such that every $X^n$ exhibits $n$ attractors or repellers, with $n > n_0$. By Proposition 2.10 there is a neighborhood $U \subset \mathcal{X}^1(M)$ of $X$ and a neighborhood $U \subset M$ of $\Lambda$ such that the attractors in $U$ are finite for all $Y \in U$. Thus, we are left to prove only for the repeller case. We denote by $R^n$ a repeller of $X^n$ in $\cap_{t \geq 0} X^n(U) = \Lambda_{X^n}$. Since $\Lambda_{X^n}$ arbitrarily close to $\Lambda$ and since $R^n \in \Lambda_{X^n}$, $R^n$ also is arbitrarily close to $\Lambda$, we can assume that $L^n$ belongs to $\Lambda$ for all $n$.

Let $(R^n)_{n \in \mathbb{N}}$ be the sequence of repellers contained in $\Lambda$. By the Lemma 2.7 we have that

$$\text{Sing}(X) \cap \text{Cl}(\cup_{n \in \mathbb{N}} R^n) = \emptyset$$
Then, we have that there is $\delta > 0$, such that $B_\delta(Sing(X)) \cap (\cup_{n \in \mathbb{N}} R^n) = \emptyset$.

As in (3) we define $H = \bigcap_{t \in \mathbb{R}} X_t (U \setminus B_{\delta/2}(Sing(X)))$. It follows that $H$ is hyperbolic by Lemma 2.1 and by the Lemma 2.6 we have that there is a neighborhood $U \subset X^1(M)$ of $X$, a neighborhood $U \subset M$ of $H$, and $n_1 \in \mathbb{N}$ such that

$$\#\{R \subset U : R \text{ is a repeller of } Y \in U\} \leq n_1 \leq n_0$$

for every vector field $Y \in U$. This is a contradiction, since by hypotheses we have that

$$\#\{R \subset H : R \text{ is a repeller of } Y \in U\} \geq n > n_0.$$
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