Aspects of the $q$–deformed Fuzzy Sphere

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Abstract

These notes are a short review of the $q$–deformed fuzzy sphere $S^2_{q,N}$, which is a “finite” noncommutative 2–sphere covariant under the quantum group $U_q(su(2))$. We discuss its real structure, differential calculus and integration for both real $q$ and $|q| = 1$. The algebra of functions on $S^2_{q,N}$ is isomorphic to the matrix algebra $Mat(N+1, \mathbb{C})$, but it carries additional structure which distinguishes it from $S^2_N$, related to its rotation symmetry under $U_q(su(2))$. For real $q$, we recover precisely the “discrete series” of Podles spheres \cite{4}. We describe its structure in general, including a covariant differential calculus and integration, and show how actions of Yang–Mills and Chern–Simons type arise naturally on this space. A much more detailed study of $S^2_{q,N}$ has been given by in \cite{1}. These considerations were motivated mainly by the work \cite{2} of Alekseev, Recknagel and Schomerus, who study the boundary conformal field theory describing spherical D–branes in the $SU(2)$ WZW model at level $k$. These authors extract an “effective” algebra of functions on the D–branes from the OPE of the boundary vertex operators. This algebra is twist–equivalent \cite{1} to the space of functions on $S^2_{q,N}$, if $q$ is related to the level $k$ by the formula

$$q = \exp\left(\frac{i\pi}{k + 2}\right).$$

2 The space $S^2_q$

First, recall that an algebra $\mathcal{A}$ is a $U_q(su(2))$–module algebra if there exists a map

$$U_q(su(2)) \times \mathcal{A} \to \mathcal{A},$$

$$(u, a) \mapsto u \triangleright a$$

(2.1)

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which satisfies \( u \triangleright (ab) = (u_{(1)} \triangleright a)(u_{(2)} \triangleright b) \) for \( a, b \in \mathcal{A} \). Here \( \Delta(u) = u_{(1)} \otimes u_{(2)} \) is the Sweedler notation for the coproduct of \( u \in U_q(su(2)) \).

A particularly simple way to define the \( q \)-deformed fuzzy sphere is as follows: Consider the spin \( \frac{N}{2} \) representation of \( U_q(su(2)) \),

\[
\rho : U_q(su(2)) \to \text{Mat}(N + 1, \mathbb{C}),
\]

which acts on \( \mathbb{C}^{N+1} \). With this in mind, it is natural to consider the simple matrix algebra \( \text{Mat}(N + 1, \mathbb{C}) \) as a \( U_q(su(2)) \)-module algebra, by \( u \triangleright M = \rho(u_{(1)})M\rho(Su_{(2)}) \). This defines \( S^2_{q, N} \). It is easy to see that under this action of \( U_q(su(2)) \), it decomposes into the irreducible representations

\[
S^2_{q, N} := \text{Mat}(N + 1, \mathbb{C}) = (1) \oplus (3) \oplus \ldots \oplus (2N + 1) \tag{2.3}
\]

(if \( q \) is a root of unity \( \mathbb{I} \), this holds provided \( N \leq \frac{k}{2} \), which we will assume here).

Let \( \{x_i\}_{i=+,-,0} \) be the weight basis of the spin 1 components, so that \( u \triangleright x_i = x_j \pi^j_i(u) \) for \( u \in U_q(su(2)) \). One can then show that they satisfy the relations

\[
\varepsilon^{ij}_k x_i x_j = \Lambda_N x_k, \\
g^{ij} x_i x_j = R^2. \tag{2.4}
\]

Here

\[
\Lambda_N = R \frac{[2]_{q^{N+1}}}{\sqrt{[N]_q[N+2]_q}}. \tag{2.5}
\]

\([n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\), and \( \varepsilon^{ij}_k \) and \( g^{ij} \) are the \( q \)-deformed invariant tensors. For example, \( \varepsilon^{33}_3 = q^{-1} - q \), and \( g^{1-1} = -q^{-1} \), \( g^{00} = 1 \), \( g^{-11} = -q \). In \( \mathbb{II} \), these relations were derived using a Jordan–Wigner construction. For \( q = 1 \), the relations of \( S^2_N \) are recovered, and for real \( q \neq 1 \) we obtain \( \varepsilon^{ij}_k x_i x_j = R(q - q^{-1}) x_k \) in the limit \( N \to \infty \).

**Real structure.** In order to define a real noncommutative space, one must specify a star structure on the algebra of functions. Since \( S^2_{q, N} \) should decompose into unitary representations of \( U_q(su(2)) \), we restrict to the cases \( q \in \mathbb{R} \) and \( |q| = 1 \). In either case, the star structure on \( S^2_{q, N} = \text{Mat}(N + 1, \mathbb{R}) \) is defined to be the usual hermitean adjoint of matrices. In terms of the generators \( x_i \), this becomes

\[
x^*_i = g^{ij} x_j, \quad \text{if } q \in \mathbb{R} \tag{2.6}
\]

and

\[
x^*_i = -\omega x_i \omega^{-1} = x_j \rho(L^{-j}_k)q^{-2}g^{ki}, \quad \text{if } |q| = 1. \tag{2.7}
\]

Here \( \omega \in \hat{U}_q(su(2)) \) generates the quantum Weyl reflection \( \mathbb{III} \),

\[
\Delta(\omega) = \mathcal{R}^{-1} \omega \otimes \omega, \quad \omega^2 = v\epsilon, \quad v = S\mathcal{R}_2\mathcal{R}_1 q^{-H}, \tag{2.8}
\]
and $L^{-i}_j = \pi^i_j (R^{-1}_1 R^{-1}_2) \in U_q(su(2))$. Using the map $\rho$ (2.4), this amounts to the star structure $H^* = H, (X^\pm)^* = X^\mp$ defining the compact form $U_q(su(2))$ for both $q \in \mathbb{R}$ and $|q| = 1$. In the case $q \in \mathbb{R}$, we have recovered precisely the discrete series of Podles spheres [4].

To summarize, $S^2_{q,N}$ is same algebra $\text{Mat}(N + 1, \mathbb{C})$ as $S^2_N \equiv S^2_{q=1,N}$, but its symmetry $U_q(su(2))$ acts on it in a way which is inequivalent to the undeformed case. It admits additional structure compatible with this symmetry, such as a differential calculus and an integral. This will be discussed next.

3 Differential calculus and integration

In order to write down Lagrangians, it is convenient to use the notion of an (exterior) differential calculus. A covariant differential calculus over $S_{q,N}$ is a graded bimodule $\Omega^* = \bigoplus_n \Omega^n$ over $S_{q,N}$ which is a $U_q(su(2))$–module algebra, together with an exterior derivative $d$ which satisfies $d^2 = 0$ and the graded Leibnitz rule.

The structure of the calculus is determined by requiring covariance, and a systematic way to derive it is given in [1]. Here we will simply quote the most important features. First, the modules $\Omega^n$ turn out to be free over $S_{q,N}$ both as left and right modules with $\dim \Omega^n = (1, 3, 3, 1)$ for $n = (0, 1, 2, 3)$, and vanish for higher $n$. In particular, it is not possible to have a calculus with only “tangential” forms; this means that vector fields over $S_{q,N}$ will in general also contain “radial” components. As suggested by the dimensions, there exists a canonical map

$$^*H : \Omega^n \rightarrow \Omega^{3-n}, \quad (3.1)$$

which satisfies $(^*H)^2 = id$, and respects the $U_q(su(2))$– and $S_{q,N}$–module structures. It satisfies furthermore

$$\alpha(^*H \beta) = (^*H \alpha) \beta$$

for any $\alpha, \beta \in \Omega^*$. Moreover, there exists a special one–form

$$\Theta \in \Omega^1_{q,N}$$

which is a singlet under $U_q(su(2))$, and generates the calculus as follows:

$$df = [\Theta, f], \quad d\alpha^{(1)} = [\Theta, \alpha^{(1)}]_+ - ^*H(\alpha^{(1)}), \quad d\alpha^{(2)} = [\Theta, \alpha^{(2)}]$$

for any $f \in S_{q,N}$ and $\alpha^{(i)} \in \Omega^i$. One can verify that

$$d\Theta = \Theta^2 = ^*H(\Theta), \quad \text{but not as bimodules}$$
and \([f, \Theta^3] = 0\) with \(\Theta^3 \neq 0\). There also exists a star structure \([\text{I}]\) on \(\Omega^*\) for both \(q \in \mathbb{R}\) and \(|q| = 1\), which makes it a covariant * calculus.

**Frame.** The most convenient basis to work with is the “frame” generated by one–forms \(\theta^a \in \Omega^1\) for \(a = +, -, 0\), which satisfy

\[
[\theta^a, f] = 0, \\
\theta^a \theta^b = -q^2 \hat{R}^{ab}_{cd} \theta^d \theta^c, \\
*H \theta^a = -\frac{1}{q^2 |2| q^2} \varepsilon^{ac} \theta^c \theta^b, \\
\theta^a \theta^b \theta^c = -\Lambda^2 \frac{q^6}{R^2} \varepsilon^{cba} \Theta^3.  
\]

(3.3) (3.4) (3.5)

Such \(\theta^a\) exist and are essentially unique. The disadvantage is that they have a somewhat complicated transformation law

\[
u \triangleright \theta^a = u_1 S u_3 \pi^a_b (S u_2) \theta^b.
\]

One can alternatively use a basis of one–forms \(\xi_i\) which transform as a vector under \(U_q(su(2))\), but then the commutation relations are more complicated \([\text{I}]\).

**Integration.** The unique invariant integral of a function \(f \in S^2_{q,N}\) is given by its quantum trace as an element of \(Mat(N + 1, \mathbb{C})\),

\[
\int f := \frac{4\pi R^2}{[N + 1]_q} \text{Tr}_q(f) = \frac{4\pi R^2}{[N + 1]_q} \text{Tr}(f q^{-H}),
\]

normalized such that \(\int 1 = 4\pi R^2\). Invariance means that \(\int u \triangleright f = \varepsilon(u) \int f\). It is useful to define also the integral of forms, by declaring \(\Theta^3\) to be the “volume form”. Writing any 3–form as \(\alpha^{(3)} = f \Theta^3\), we define \(\int \alpha^{(3)} = \int f \Theta^3 := \int f\). Using the correct cyclic property of this integral, one can then verify Stokes theorem

\[
\int d\alpha^{(2)} = \int [\Theta, \alpha^{(2)}] = 0.
\]

(3.6)

4 Gauge Fields

Actions for gauge theories arise in a very natural way on \(S^2_{q,N}\). We shall describe (abelian) gauge fields as one–forms

\[
B = \sum B_a \theta^a \in \Omega^1,
\]
expanded in terms of the frames $\theta^a$. They have 3 independent components, which means that $B$ has also a radial component. However, the latter cannot be disentangled from the other components, since it is not possible to construct a covariant calculus with 2 tangential components only.

We propose that Lagrangians for gauge fields should contain no explicit derivatives in the $B$ fields. The kinetic terms will then arise naturally upon a shift of the field $B$,

$$B = \Theta + A.$$ 

The only Lagrangian of order $\leq 3$ in $B$ which after this shift contains no linear terms in $A$ is the “Chern-Simons” action

$$S_{CS} := \frac{1}{3} \int B^3 - \frac{1}{2} \int B \ast_H B = -\text{const} + \frac{1}{2} \int AdA + \frac{2}{3} A^3. \quad (4.1)$$

Going to order 4 in $B$, we define the curvature as

$$F := B^2 - *_H B = dA + A^2,$$

using (3.2). Then a “Yang–Mills” action is naturally obtained as

$$S_{YM} := \int F *_H F = \int (dA + A^2) *_H (dA + A^2). \quad (4.2)$$

These are precisely the kind of actions that have been found in the string-induced low-energy effective action on $D$–branes in the $SU(2)_k$ WZW model, in the leading approximation.

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