Torsional Regularization of Self-Energy and Bare Mass of Electron

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In the presence of spacetime torsion, the momentum components do not commute; therefore, in quantum field theory, summation over the momentum eigenvalues will replace integration over the momentum. In the Einstein–Cartan theory of gravity, in which torsion is coupled to spin, the separation between the eigenvalues increases with the magnitude of the momentum. Consequently, this replacement regularizes divergent integrals in Feynman diagrams with loops by turning them into convergent sums. In this article, we apply torsional regularization to the self-energy of a charged lepton in quantum electrodynamics. We show that this procedure eliminates the ultraviolet divergence. We also show that torsion gives a photon a small nonzero mass, which regularizes the infrared divergence. In the end, we calculate the finite bare masses of the electron, muon, and tau lepton: 0.4329 MeV, 90.95 MeV, and 1543 MeV, respectively. These values constitute about 85\% of the observed, re-normalized masses.

1. Introduction.

In quantum electrodynamics, a calculation of the probability amplitude for a physical process will involve higher-order perturbation corrections represented by Feynman diagrams with closed loops of virtual particles\cite{1,2}. At one-loop level, there exist three kinds of diagrams. The first is vacuum polarization (charge screening): a photon creates a virtual electron-positron pair which annihilates to another photon. The second is self-energy, which is when an electron emits and reabsorbs a virtual photon. The third is a vertex, which is when an electron emits a photon, emits a second photon, and then reabsorbs the first\cite{3}. At higher-order levels, all Feynman diagrams are composed from these three fundamental diagrams.

In the four-momentum space, following the principle of superposition, these corrections involve the integration of Feynman propagators. These propagators are integrated over unbounded energy and momentum, resulting in integrals whose gauge-invariant parts are logarithmically divergent\cite{1,2}. This nonphysical result, called the ultraviolet divergence, is treated by regularization: a mathematical method of modifying the integrals to become finite. The two most common methods to eliminate this divergence are the Pauli–Villars regularization\cite{4} and the ’t Hooft–Veltman dimensional regularization\cite{5}. When a regulator vanishes, the integral in a four-momentum loop tends to infinity according to a known asymptotic form in the physical limit. This form can be absorbed by redefining the original (bare) mass, charge, or wave function, leading to a finite (dressed) value that is physically measured in the experiment. Such redefinition is called re-normalization\cite{1,6}. A bare quantity is divergent, but it is not measurable.

The integrals representing the self-energy and the vertex also have an infrared divergence, which arises from not accounting for soft (low-energy) photons that may exist in a final state\cite{1,2}. When adding soft braking radiation, the infrared divergences in the total cross-section cancel out. A small mass is given to the photon and then taken to zero in the physical limit to demonstrate this cancellation.

Torsional regularization, based on spacetime torsion\cite{7}, provides a physical mechanism that eliminates ultraviolet-divergent integrals in quantum electrodynamics and leads to a finite value of the bare electric charge\cite{8}. The consistency of the conservation law for the total angular momentum of a free Dirac particle in curved spacetime with relativistic quantum mechanics requires torsion\cite{9}. The most straightforward and most natural theory of gravity with torsion is the Einstein–Cartan theory, where torsion is coupled to spin\cite{10}. In the presence of the torsion tensor, the four-momentum components do not commute\cite{8}. Such that the momentum components satisfy a commutation relation analogous to that for the angular momentum components, a coordinate frame can be chosen\cite{11}.

In non-commutative momentum space, generated by torsion, integration over the momentum in Feynman diagrams will be replaced with summation over the discrete momentum eigenvalues. Following the Einstein–Cartan equations, the separation between the momentum eigenvalues increases with the magnitude of the momentum. Consequently, logarithmically divergent integrals in loop diagrams are replaced with convergent sums. Re-normalization becomes a
finite procedure and gives finite forms of the gauge-invariant vacuum polarization tensor, the running coupling, and the bare electric charge of the electron (and other charged particles) [8].

In this article, we apply torsion regularization to the self-energy of a charged lepton. We review this procedure [8] in Section 2. Next, in Section 3, we review the self-energy of the electron in quantum electrodynamics [2]. We show how torsion eliminates the ultraviolet divergence of the self-energy in Section 4, which constitutes the main part of this work. Then, in Section 5, we show how torsion gives the photon a small mass and thus eliminates an infrared divergence. Lastly, we determine the bare masses of the charged leptons in Section 6.

Torsion may therefore physically eliminate the infinity problem in quantum field theory. This elimination would be a remarkable feature of the torsion tensor because torsion may also solve the singularity problem in general relativity [12]. Also, torsion may provide fermions with effective spatial extension [13] and explain the observed dynamics of the early Universe [14].

2. Torsion and Non-commutative Momentum.

This section reviews torsional regularization, following [8]. In the presence of torsion, the components \( p_i \) of the four-dimensional momentum operator satisfy a commutation relation

\[
[p_i, p_j] = 2i\hbar S^k_{ij}p_k,
\]

where \( S^k_{ij} \) is the torsion tensor [8]. For the Dirac fields, the spin tensor and thus the torsion tensor are completely antisymmetric [10]. In a coordinate frame, in which the torsion pseudovector has only the temporal component \( S^0 = -Q/(2\hbar) \), this commutation relation becomes

\[
[p_x, p_y] = iQp_z, \quad [p_y, p_z] = iQp_x, \quad [p_z, p_x] = iQp_y.
\]

We use \( c = 1 \). According to the Cartan equations, \( Q \) depends on the four-momentum and is proportional to \( p^3 \), where \( p^2 = (p^0)^2 - \mathbf{p}^2 \), where \( p^0 \) is the energy and \( \mathbf{p} \) is the momentum:

\[
Q = Up^3.
\]

The constant \( U \), which can be taken positive without loss of generality, is of the order of the squared inverse of the Planck mass \( M_P \). The cyclic commutation relations for the momentum can be written as

\[
[n_x, n_y] = in_z, \quad [n_y, n_z] = in_x, \quad [n_z, n_x] = in_y
\]

for the spatial components \( n_i \) of a vector

\[
\mathbf{n} = \frac{\mathbf{p}}{Q}.
\]

The relations (11) are analogous to those for the angular momentum. Consequently, the eigenvalues of \( \mathbf{n} \) are given by \( n = |\mathbf{n}| = \sqrt{j(j + 1)} \) and \( n_z = m \), where an integer \( j \geq 0 \) is the orbital quantum number and an integer \( m \in [-j, j] \) is the magnetic quantum number [11].

Torsional regularization [8] replaces integration over the momentum in Feynman diagrams with summation over the momentum eigenvalues related to the eigenvalues of \( \mathbf{n} \):

\[
\int \int \int d^n_xdn_ydn_z f(n) \rightarrow 4\pi \sum_{\text{eigenstates}} f(n)|n_z| = 4\pi \sum_{j=1}^{\infty} \sum_{m=-j}^{j} f(n)|m| = 4\pi \sum_{j=1}^{\infty} f(n)j(j+1) = 4\pi \sum_{j=1}^{\infty} f(n) n^2,
\]

where \( f(n) \) is an arbitrary scalar function of \( n \). According to this prescription, such a function is multiplied by the absolute value of the commutator of two integration variables, \( |[n_x, n_y]| = |n_z| \), and integration over continuous variables \( n_x, n_y, n_z \) is replaced with summation over the eigenvalues of \( \mathbf{n} \). For \( j = 0, m = 0 \) and thus \( n_z = 0 \), which does not contribute to the sum in (3).

\[\]
A Feynman diagram with a loop involves calculating an integral in the four-momentum space that has a form \( \int d^4l/(l^2 - \Delta + ie)^s \), where \( \Delta > 0 \) does not depend on the four-momentum \( l^i \), \( s \) is an integer, and \( e \to 0^+ \) \([1][2]\). Applying the Wick rotation, in which the temporal component of the four-momentum \( l^0 \) is replaced with \( il^0_E \) and thus \( l^2 = (l^0_E)^2 + l^2 \), turns the integration in spacetime with the Lorentzian metric signature to the integration in the four-dimensional Euclidean space. The integral becomes \( i(-1)^s \int d^4l_E/(l^2_E + \Delta)^s = i(-1)^s \int dl_E \sum l^2 = (l^0)^2 + l^2 = (l^0)^2 + U^2 n^2 l^6. \) (4)

The integration over \( l^0 \) can be replaced with the integration over \( l \):

\[
dl^0 = d\frac{dl^0}{dl} = dl \frac{1 - 3U^2 n^2 l^4}{(1 - U^2 n^2 l^4)^{1/2}}.
\]

The integration over \( l \) can be replaced with the integration over \( n = 1/Q \) and then changed to the summation over \( j \) in the non-commutative momentum space:

\[
\int \iint dl_x dl_y dl_z f(l^2) \to \int \iint dn_x dn_y dn_z J f(Q^2 n^2) \to 4\pi \sum_{j=1}^{\infty} J f(Q^2 n^2) n^2,
\] (5)

where \( J = \partial(l_x l_y l_z)/\partial(n_x, n_y, n_z) \) is the Jacobian of the transformation from \( l \) to \( n \). Differentiating \( \Box \) with respect to \( n_x \), using \( n^2 = n_x^2 + n_y^2 + n_z^2 \), gives \( 2\Box(\partial l/\partial n_x) = 6U^2 n^2 l^2 (\partial l/\partial n_x) + 2U^2 l^6 n_x \), which is equivalent to

\[
\frac{\partial l}{\partial n_x} = \frac{U^2 l^5 n_x}{1 - 3U^2 n^2 l^4}.
\]

The transformation derivatives are thus

\[
\begin{align*}
\frac{\partial l_x}{\partial n_x} &= \frac{\partial(Q n_x)}{\partial n_x} = Q + 3U n_x l^2 \frac{\partial l}{\partial n_x} = \frac{Q}{1 - 3U^2 n^2 l^4} [1 - 3U^2 l^4 (n_x^2 + n_z^2)], \\
\frac{\partial l_x}{\partial n_y} &= \frac{\partial(Q n_x)}{\partial n_y} = 3U n_x l^2 \frac{\partial l}{\partial n_y} = \frac{Q}{1 - 3U^2 n^2 l^4} (3U^2 l^4 n_x n_y),
\end{align*}
\]

and similarly for other components. They give the Jacobian:

\[
J = \det \begin{pmatrix}
\frac{\partial l_x}{\partial n_x} & \frac{\partial l_x}{\partial n_y} & \frac{\partial l_x}{\partial n_z} \\
\frac{\partial l_y}{\partial n_x} & \frac{\partial l_y}{\partial n_y} & \frac{\partial l_y}{\partial n_z} \\
\frac{\partial l_z}{\partial n_x} & \frac{\partial l_z}{\partial n_y} & \frac{\partial l_z}{\partial n_z}
\end{pmatrix} = \frac{Q^3}{1 - 3U^2 n^2 l^4}.
\]

Consequently, integrating over the Euclidean four-momentum \( l^i \) and using the prescription \( \Box \) gives

\[
\int dl^0 dl = \int dl \frac{dl^0}{dl} J dn = \int dl \frac{Q^3}{(1 - U^2 n^2 l^4)^{1/2}} = 2 \int_0^{1/(U n)} dl \frac{U^3 l^9}{(1 - U^2 n^2 l^4)^{1/2}}
\]

\[
\to 8\pi \sum_{j=1}^{\infty} \int_0^{1/(U n)} dl \frac{U^3 l^9}{(1 - U^2 n^2 l^4)^{1/2}} n^2,
\]

with \( n = \sqrt{j + 1} \). The integral \( \int dl^0 dl/(l^2 + \Delta)^s \) in the presence of the Einstein–Cartan torsion is therefore \( \Box \)

\[
\begin{align*}
\int \frac{dl^0}{(l^2 + \Delta)^s} &\to 8\pi \sum_{j=1}^{\infty} \int_0^{1/(U n)} dl \frac{U^3 l^9}{(1 - U^2 n^2 l^4)^{1/2}} n^2 \frac{1}{(l^2 + \Delta)^s} = 8\pi \sum_{j=1}^{\infty} \int_0^{1} d\xi \frac{U^3 \xi^9 n^2 (U n)^5}{(1 - \xi^4)^{1/2}(\xi^2 + U \Delta n)^s} \\
&= \sum_{j=1}^{\infty} \int_0^{1} d\xi \frac{8\pi U^2 \xi^9 n^3 (U n)^5}{(1 - \xi^4)^{1/2}(\xi^2 + U \Delta n)^s} = \sum_{j=1}^{\infty} \int_0^{1} d\xi \frac{4\pi U^2 \xi^9 n^3 (U n)^5}{(1 - \xi^4)^{1/2}(\xi^2 + U \Delta n)^s} = 4\pi \sum_{j=1}^{\infty} \int_0^{\pi/2} d\phi \frac{U^2 \xi^9 n^3 (U n)^5}{(\sin \phi + U \Delta n)^s},
\end{align*}
\] (6)
using a series of substitutions: \( U n d^2 = \zeta^2 = \zeta = \sin \phi \). For \( U < 0 \), \( U \) in the sum-integral \( f \) is replaced with \(|U|\). At large values of \( j \), the sum-integral \( f \) behaves as \( \sim \sum_{j=1}^{\infty} j^{-3} \) for any \( s \), so it converges, regularizing the original integral.

In the limit of continuous momentum space, \( U \to 0 \), the separation between adjacent values of \( j \) has no effect on the integral, so the summation over \( j \) can be replaced with the integration over \( y \), where \( y = \sqrt{j(j+1)} \). For \( s = 3 \), this limit is equal to the original, finite value of the integral. For \( s = 2 \), the integral in this limit diverges as \( \sim \ln(U) \), equivalently to the Pauli–Villars regularization.

### 3. Electron Self-energy.

This section reviews the self-energy of the electron in quantum electrodynamics, following [2]. The Feynman propagator for an electron with bare mass \( m_0 \) and four-momentum \( p_\mu \) is given by

\[
G = \frac{i(\not{p} + m_0)}{(p^2 - m_0^2 + i\epsilon)},
\]

where \( \not{p} = \gamma^\mu p_\mu \) and \( \gamma^\mu \) are the Dirac matrices. We use \( \hbar = 1 \). The propagator in the second-order in \( e \) is given by

\[
G\Sigma G = \frac{i(\not{p} + m_0)}{(p^2 - m_0^2 + i\epsilon)}[-i\Sigma_2(p)] \frac{i(\not{p} + m_0)}{(p^2 - m_0^2 + i\epsilon)},
\]

where \( \Sigma_2 \) represents the electron self-energy at one-loop level:

\[
-i\Sigma_2(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(k + m_0)}{k^2 - m_0^2 + i\epsilon} \gamma^\nu \frac{-i}{(p - k)^2 - \mu^2 + i\epsilon}.
\]

Infrared divergence is regularized by adding a small photon mass \( \mu \), which will be determined in Section 5. This propagator describes an electron that emits a photon and reabsorbs it. Introducing a Feynman parameter \( x \) combines two denominators into one:

\[
\frac{1}{k^2 - m_0^2 + i\epsilon (p - k)^2 - \mu^2 + i\epsilon} = \int_0^1 \frac{1}{(k^2 - 2xk \cdot p + xp^2 - x\mu^2 - (1-x)m_0^2 + i\epsilon)^2} dx
\]

Consequently, one obtains

\[
i\Sigma_2(p) = -e^2 \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{-2x \not{l} + 4m_0}{|l^2 - \Delta + i\epsilon|},
\]

where

\[
\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2.
\]

Applying the Wick rotation gives an integral in the four-dimensional Euclidean space:

\[
\Sigma_2(p) = e^2 \int_0^1 dx \int \frac{d^4l_E}{(2\pi)^4} \frac{-2x \not{l} + 4m_0}{|l^2_E + \Delta|^2}.
\]

The full propagator of an electron is equal to the sum of terms that involve powers of one-particle-irreducible diagrams:

\[
G + G\Sigma G + G\Sigma G\Sigma G + \cdots = \frac{i}{\not{p} - m_0 - \Sigma(\not{p})}.
\]

The physical (observed) mass \( m \) is located at a pole of the above denominator that is a solution of

\[
[\not{p} - m_0 - \Sigma(\not{p})]|_{\not{p} = m} = 0.
\]

The full propagator has a simple pole shifted from \( m_0 \) by the self-energy \( \Sigma(\not{p}) \). Near the pole, the denominator is equal to \( (\not{p} - m)(1 - d\Sigma/d\not{p})|_{\not{p} = m} \), which gives the re-normalization constant of the electron wave function: \( Z_2^{-1} = (1 - d\Sigma/d\not{p})|_{\not{p} = m} \). Therefore, the physical mass is shifted from the bare mass in the second order in \( e \) by

\[
\delta m = m - m_0 = \Sigma_2(\not{p} = m) \approx \Sigma_2(\not{p} = m_0).
\]
Consequently, one obtains
\[ \delta m = 2m_0 e^2 \int_0^1 dx \int \frac{d^4l_E}{(2\pi)^4} \frac{2 - x}{[l_E^2 + \Delta]^2}, \]
where
\[ \Delta = (1 - x)^2 m_0^2 + x\mu^2. \]

4. Torsional Regularization of Ultraviolet Divergence.

Now, we proceed to the main calculation of this work. In the presence of torsion, integration over the momentum is replaced with summation over the momentum eigenvalues according to (6):
\[ \int \frac{d^4l_E}{(l_E^2 + \Delta)^2} \rightarrow 4\pi U^{s-2} \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4 \phi [l(l+1)]^{(s-3)/2}}{[\sin \phi + U\Delta \sqrt{l(l+1)}]^s}. \]

For \( s = 2 \), this replacement gives
\[ \int \frac{d^4l_E}{(l_E^2 + \Delta)^2} \rightarrow 4\pi \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4 \phi [l(l+1)]^{-1/2}}{[\sin \phi + U\Delta \sqrt{l(l+1)}]^2}. \]

With the prescription (9), the mass shift (7) becomes
\[ \delta m = \frac{8\pi m_0 e^2}{(2\pi)^4} \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \int_0^1 dx \frac{\sin^4 \phi (2 - x)[(l(l+1)]^{-1/2}}{[\sin \phi + U\Delta \sqrt{l(l+1)}]^2}. \]

Since \( U \sim M_p^{-2}, \) \( U\Delta \) is on the order of \((m_0/M_p)^2 \ll 1\). In this limit, the separation between adjacent values of \( l \) does not affect significantly the integrand and thus the summation over \( l \) in (10) can be replaced with the integration over \( n \), where \( n = \sqrt{l(l+1)} \). Consequently, we obtain
\[ \delta m = \frac{8\pi m_0 e^2}{(2\pi)^4} \int_0^{\pi/2} d\phi \int_0^1 dx \int_{\sqrt{2}}^{\infty} dn \frac{\sin^4 \phi (2 - x)n^{-1}}{[\sin \phi + U\Delta n]^2}. \]

Substituting from \( n \) to a new variable \( w = U\Delta n \) gives
\[ \delta m = \frac{8\pi m_0 e^2}{(2\pi)^4} \int_0^{\pi/2} d\phi \int_0^1 dx \int_{\sqrt{2}}^{\infty} dw \frac{dw}{\sin \phi + \sqrt{2U\Delta} w \sin \phi + w^2} \]
\[ = \frac{2m_0\alpha}{\pi^2} \int_0^1 dx (2 - x) \int_0^{\pi/2} d\phi \left[ -\frac{\sin^3 \phi}{\sin \phi + \sqrt{2U\Delta}} + \sin^2 \phi \ln \left( \frac{\sin \phi + \sqrt{2U\Delta}}{\sqrt{2U\Delta}} + 1 \right) \right], \]
where \( \alpha = e^2/4\pi \) is the fine structure constant. This integral is convergent as long as \( U \neq 0 \), that is, in the presence of torsion, and as long as \( \Delta \neq 0 \).

5. Torsional Regularization of Infrared Divergence.

According to the Einstein–Cartan theory of gravity, a fermion field is a source of torsion. When a photon is coupled to a lepton, it is also minimally coupled to the torsion tensor:
\[ \mathcal{L} \sim F^2 \sim (\partial A + SA)^2, \]
where \( \mathcal{L} \) is the Lagrangian density, \( F \) symbolizes the electromagnetic field tensor, \( A \) symbolizes the electromagnetic potential, and \( S \) symbolizes the torsion tensor. Therefore, the term \( S^2A^2 \) arising from this coupling generates a mass
term $\mu^2 A^2$, so the mass of a photon is on the order of magnitude of the torsion tensor, which is on the order of $U m^3$.

Using $p = m_0$, as in (7), gives

$$\mu \sim m_0^3 U, \quad U \sim \frac{1}{M^6} \quad (12)$$

For the entire range of $x \in [0, 1]$, $\Delta$ in (8) is thus different from zero. Consequently, torsion removes infrared divergence and makes a lepton self-energy. For a Feynman diagram that converges in the limit $\mu \to 0$, the small nonzero mass $\mu$ does not significantly affect its amplitude.

6. Calculation and Discussion.

Since $U \Delta \ll 1$, the integral (11) can be approximated as

$$\delta m = \frac{2m_0 \alpha}{\pi^2} \int_0^1 dx (2-x) \int_0^{\pi/2} d\phi \left[ -\sin^3 \phi + \sin^2 \phi \ln \left( \frac{\sin \phi}{\sqrt{2U\Delta}} \right) \right]$$

$$= \frac{2m_0 \alpha}{\pi^2} \int_0^1 dx (2-x) \int_0^{\pi/2} d\phi \left[ -\sin^2 \phi - \sin^2 \phi \ln \left( \sqrt{2U} m_0^2 \Delta \right) + \sin^2 \phi \ln(\sin \phi) \right]$$

$$= \frac{2m_0 \alpha}{\pi^2} \int_0^1 dx (2-x) \left[ \int_0^{\pi/2} d\phi \sin^2 \phi \left[ -1 - \ln(\sqrt{2U} m_0^2) - \ln \left( \frac{\Delta}{m_0^2} \right) \right] + \int_0^{\pi/2} d\phi \sin^2 \phi \ln(\sin \phi) \right],$$

where $\Delta$ is given by (8). After integrating over $\phi$, we obtain

$$\delta m = \frac{2m_0 \alpha}{\pi^2} \int_0^1 dx (2-x) \left[ \frac{\pi}{4} \left( -1 - \ln(\sqrt{2U} m_0^2) - \ln \left( \frac{\Delta}{m_0^2} \right) \right) + N \right],$$

where

$$N = \int_0^{\pi/2} \sin^2 \phi \ln(\sin \phi) d\phi \approx -0.151697.$$  

Consequently, the mass shift is equal to

$$\delta m = \frac{2m_0 \alpha}{\pi^2} \left[ \frac{\pi}{4} \left( -1 - \ln(\sqrt{2U} m_0^2) \right) + N \right] - \frac{2m_0 \alpha}{\pi^2} \int_0^1 dx (2-x) \frac{\pi}{4} \ln \left( \frac{\Delta}{m_0^2} \right)$$

$$= \frac{3m_0 \alpha}{4\pi} \left( 4N - 1 - \ln(\sqrt{2U} m_0^2) \right) - \frac{m_0 \alpha}{2\pi} \int_0^1 dx (2-x) \ln \left( \frac{(1-x)^2 + \mu^2}{m_0^2} \right), \quad (13)$$

With the substitution of $U$ and $\mu$ (12), we use (13) to determine the observed mass $m = m_0 + \delta m$ as a function of the bare mass $m_0$. Conversely, we obtain the bare mass from the observed mass. For the electron, $m = 0.510999$ MeV gives $m_0 = 0.432928$ MeV. For the muon, $m = 105.658$ MeV gives $m_0 = 90.9514$ MeV. For the tau lepton, $m = 1776.86$ MeV gives $m_0 = 1542.63$ MeV.

Our results depend on the orders of magnitude of the constants $U$ and $\mu$ (12), but they are not significantly affected by changing their exact values. For the electron, increasing $U$ by a factor of 10 gives $m_0 = 0.434408$ MeV and decreasing it by the same factor gives $m_0 = 0.431457$ MeV. Increasing or decreasing $\mu$ by a factor of 10 gives $m_0 = 0.432928$ MeV. Accordingly, the value of $\mu$ is less significant than the value of $U$ in the shift from the bare mass to the re-normalized mass.

We conclude that the bare masses of charged leptons constitute about 85% of their observed, re-normalized masses. To compare, the bare electric charge is about 22% higher than the observed, re-normalized charge. These numbers might change slightly when the contributions from the W and Z bosons are included.

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[1] R. P. Feynman, Phys. Rev. 76, 769 (1949); J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, 1965); C. Nash, *Relativistic Quantum Fields* (Academic Press, 1978); F. Mandl and G. Shaw, *Quantum Field Theory* (Wiley,
1993); W. Greiner and J. Reinhardt, Quantum Electrodynamics (Springer, 1994); M. Maggiore, A Modern Introduction to Quantum Field Theory (Oxford University Press, 2005); C. Itzykson and J.-B. Zuber, Quantum Field Theory (Dover, 2006); K. Huang, Quantum Field Theory: From Operators to Path Integrals (Wiley, 2010).

[2] M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory (Addison-Wesley, 1995).

[3] J. Schwinger, Phys. Rev. 75, 651 (1949); Phys. Rev. 82, 664 (1951).

[4] W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949).

[5] G. 't Hooft and M. Veltman, Nucl. Phys. B 44, 189 (1972).

[6] F. J. Dyson, Phys. Rev. 75, 486 (1949); M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

[7] E. Schrödinger, Space-time Structure (Cambridge University Press, 1954).

[8] N. Popławski, Found. Phys. 50, 900 (2020); arXiv:1807.07068.

[9] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. 48, 393 (1976).

[10] T. W. B. Kibble, J. Math. Phys. 2, 212 (1961); D. W. Sciama, in Recent Developments in General Relativity, p. 415 (Pergamon, 1962); Rev. Mod. Phys. 36, 463 (1964); Rev. Mod. Phys. 36, 1103 (1964); F. W. Hehl and B. K. Datta, J. Math. Phys. 12, 1334 (1971); E. A. Lord, Tensors, Relativity and Cosmology (McGraw-Hill, 1976); V. de Sabbata and M. Gasperini, Introduction to Gravitation (World Scientific, 1985); K. Nomura, T. Shirafuji, and K. Hayashi, Prog. Theor. Phys. 86, 1239 (1991); V. de Sabbata and C. Sivaram, Spin and Torsion in Gravitation (World Scientific, 1994); N. Popławski, Classical Physics: Spacetime and Fields, arXiv:0911.0334.

[11] P. A. M. Dirac, The Principles of Quantum Mechanics (Oxford University Press, 1930); L. D. Landau and E. M. Lifshitz, Quantum Mechanics: Non-Relativistic Theory (Pergamon, 1977); J. J. Sakurai, Modern Quantum Mechanics (Addison-Wesley, 1994).

[12] F. W. Hehl, P. von der Heyde, and G. D. Kerlick, Phys. Rev. D 10, 1066 (1974); B. Kuchowicz, Gen. Relativ. Gravit. 9, 511 (1978); M. Gasperini, Phys. Rev. Lett. 56, 2873 (1986); N. J. Popławski, Phys. Lett. B 694, 181 (2010); Phys. Lett. B 701, 672 (2011); Gen. Relativ. Gravit. 44, 1007 (2012); N. Popławski, Phys. Rev. D 85, 107502 (2012); G. Unger and N. Popławski, Astrophys. J. 870, 78 (2019); J. L. Cubero and N. J. Popławski, Class. Quantum Grav. 37, 025011 (2020); N. Popławski, Zh. Eksp. Teor. Fiz. 159, 448 (2021); J. Exp. Theor. Phys. 132, 374 (2021).

[13] N. J. Popławski, Phys. Lett. B 690, 73 (2010); Phys. Lett. B 727, 575 (2013).

[14] N. Popławski, Astrophys. J. 832, 96 (2016); S. Desai and N. J. Popławski, Phys. Lett. B 755, 183 (2016).