Research Article

On the Stability of One-Dimensional Wave Equation

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Received 5 August 2013; Accepted 16 September 2013

Academic Editors: K. Ammari, I. Canak, and M. M. Cavalcanti

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We prove the generalized Hyers-Ulam stability of the one-dimensional wave equation, \( u_{tt} = c^2 u_{xx} \), in a class of twice continuously differentiable functions.

1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with the metric \( d(\cdot, \cdot) \). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if a function \( h : G_1 \rightarrow G_2 \) satisfies the inequality \( d(h(xy), h(x)h(y)) < \delta \), for all \( x, y \in G_1 \), then there exists a homomorphism \( H : G_1 \rightarrow G_2 \) with \( d(h(x), H(x)) < \varepsilon \), for all \( x \in G_1 \) ?

The case of approximately additive functions was solved by Hyers [2] under the assumption that \( G_1 \) and \( G_2 \) are Banach spaces. Indeed, he proved that each solution of the inequality \( \|f(x+y) - f(x) - f(y)\| \leq \varepsilon \), for all \( x \) and \( y \), can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, \( f(x+y) = f(x) + f(y) \), is said to have the Hyers-Ulam stability.

Rassias [3] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

\[ \|f(x+y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p) \]  

and proved Hyers’ theorem. That is, Rassias proved the generalized Hyers-Ulam stability (or Hyers-Ulam-Rassias stability) of the Cauchy additive functional equation. Since then, the stability of several functional equations has been extensively investigated [4–9].

The terminologies, the generalized Hyers-Ulam stability, and the Hyers-Ulam stability can also be applied to the case of other functional equations, differential equations, and various integral equations.

Given a real number \( c > 0 \), the partial differential equation

\[ u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 \]  

is called the (one-dimensional) wave equation, where \( u_{tt}(x,t) \) and \( u_{xx}(x,t) \) denote the second time derivative and the second space derivative of \( u(x,t) \), respectively.

Let \( \varphi : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty) \) be a function. If, for each twice continuously differentiable function \( u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} \) satisfying

\[ \|u_{tt}(x,t) - c^2 u_{xx}(x,t)\| \leq \varphi(x,t) \quad (x,t \in \mathbb{R}), \]  

there exist a solution \( u_0 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} \) of the (one-dimensional) wave equation (2) and a function \( \Phi : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty) \) such that

\[ \|u(x,t) - u_0(x,t)\| \leq \Phi(x,t) \quad (x,t \in \mathbb{R}), \]  

where \( \Phi(x,t) \) is independent of \( u(x,t) \) and \( u_0(x,t) \), then we say that the wave equation (2) has the generalized Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability).

In this paper, using an idea from [10], we prove the generalized Hyers-Ulam stability of the (one-dimensional) wave equation (2).
2. Generalized Hyers-Ulam Stability

In the following theorem, using the d’Alembert method (method of characteristic coordinates), we prove the generalized Hyers-Ulam stability of the (one-dimensional) wave equation (2).

**Theorem 1.** Let a function \( \varphi : \mathbb{R} \times \mathbb{R} \to [0, \infty) \) be given such that the double integral

\[
\int_0^b \int_0^a \varphi \left( \frac{\mu + \nu, \mu - \nu}{2c} \right) d\mu d\nu
\]

exists for all \( a, b \in \mathbb{R} \). If a twice continuously differentiable function \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) satisfies the inequality

\[
|u(x, t) - c^2 u_{xx}(x, t)| \leq \varphi(x, t)
\]

for all \( x, t \in \mathbb{R} \), then there exists a solution \( u_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) of the wave equation (2) which satisfies

\[
|u(x, t) - u_0(x, t)| \leq \frac{1}{4c^2} \int_0^b \int_0^a \varphi \left( \frac{\mu + \nu, \mu - \nu}{2c} \right) d\mu d\nu
\]

for all \( x, t \in \mathbb{R} \).

**Proof.** Let us define a function \( \nu : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) by

\[
\nu(w, z) := u \left( \frac{w + z, w - z}{2c} \right).
\]

If we set \( w = x + ct \) and \( z = x - ct \), then we have \( u(x, t) = \nu(w, z) \) and

\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) & = \nu_w(w, z) \frac{\partial w}{\partial t} + \nu_z(w, z) \frac{\partial z}{\partial t} \\
& = c \nu_w(w, z) - c \nu_z(w, z), \\
\frac{\partial u}{\partial x}(x, t) & = \nu_{ww}(w, z) \frac{\partial w}{\partial x} + \nu_{ww}(w, z) \frac{\partial z}{\partial x} \\
& + \nu_{wz}(w, z) \frac{\partial w}{\partial x} + \nu_{wz}(w, z) \frac{\partial z}{\partial x} \\
& = \nu_{ww}(w, z) + 2 \nu_{wz}(w, z) + \nu_{zz}(w, z),
\end{align*}
\]

for all \( w, z \in \mathbb{R} \). Hence, we have

\[
\frac{\partial u}{\partial t}(x, t) - c^2 \frac{\partial u}{\partial x}(x, t) = -4c^2 \nu_{ww}(w, z),
\]

for any \( x, t \in \mathbb{R} \). Thus, it follows from inequality (6) that

\[
|\nu_{ww}(w, z)| \leq \frac{1}{4c^2} \varphi \left( \frac{w + z, w - z}{2c} \right),
\]

for any \( w, z \in \mathbb{R} \). Therefore, we get

\[
-\frac{1}{4c^2} \int_0^b \int_0^a \varphi \left( \frac{\mu + \nu, \mu - \nu}{2c} \right) d\mu d\nu
\]

\[
\leq \int_0^b \int_0^a \nu_{ww}(w, z) d\mu d\nu
\]

\[
\leq \frac{1}{4c^2} \int_0^b \int_0^a \varphi \left( \frac{\mu + \nu, \mu - \nu}{2c} \right) d\mu d\nu
\]

or equivalently

\[
|\nu(w, z) - \nu(w, 0) - \nu(0, z) + \nu(0, 0)|
\]

\[
\leq \frac{1}{4c^2} \int_0^b \int_0^a \varphi \left( \frac{\mu + \nu, \mu - \nu}{2c} \right) d\mu d\nu,
\]

for all \( w, z \in \mathbb{R} \).

On account of (8), we get

\[
\nu(w, z) = u \left( \frac{w + z, w - z}{2c} \right), \quad \nu(w, 0) = u \left( \frac{w}{2c}, \frac{w}{2c} \right),
\]

\[
\nu(0, z) = u \left( \frac{z}{2}, \frac{z}{2} \right), \quad \nu(0, 0) = u(0, 0).
\]

Hence, it follows from (13) and the last equalities that

\[
\left| u \left( \frac{w + z}{2c}, \frac{w - z}{2c} \right) - u \left( \frac{w}{2c}, \frac{w}{2c} \right) - u \left( \frac{z}{2}, \frac{z}{2c} \right) \right| + u(0, 0)
\]

\[
\leq \frac{1}{4c^2} \int_0^b \int_0^a \varphi \left( \frac{\mu + \nu, \mu - \nu}{2c} \right) d\mu d\nu,
\]

for all \( w, z \in \mathbb{R} \).

If we set \( w = x + ct \) and \( z = x - ct \) in the last inequality, then we obtain

\[
\left| u(x, t) - u_0(x, t) \right|
\]

\[
\leq \frac{1}{4c^2} \int_0^b \int_0^a \varphi \left( \frac{\mu + \nu, \mu - \nu}{2c} \right) d\mu d\nu,
\]

for all \( x, t \in \mathbb{R} \), where we set

\[
u_0(x, t) := u \left( \frac{x}{2} + \frac{c}{2} t, \frac{x}{2c} \right)
\]

\[
+ \frac{4c^2}{2} \left( \frac{x}{2} - \frac{c}{2} t, \frac{x}{2c} + \frac{t}{2} \right) - u(0, 0).
\]
Corollary 2. Given a constant \( \alpha > 0 \), let a function \( \varphi : \mathbb{R} \times \mathbb{R} \to [0, \infty) \) be given as

\[
\varphi(x, t) = \alpha e^{-x^2 - t^2}.
\]  

If a twice continuously differentiable function \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) satisfies inequality (6), for all \( x, t \in \mathbb{R} \), then there exists a solution \( u_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) of the wave equation (2) which satisfies

\[
|u(x, t) - u_0(x, t)| \leq \frac{\alpha \pi}{8c^2} \left| \text{erf} \left( \frac{x - ct}{\sqrt{2}} \right) - \text{erf} \left( \frac{x + ct}{\sqrt{2}} \right) \right|,
\]

for all \( x, t \in \mathbb{R} \).

Proof. Since

\[
\left| \int_0^b \int_0^a \varphi \left( \frac{\mu + y}{2}, \frac{\mu - y}{2c} \right) d\mu d\nu \right| = \left| \int_0^b \int_0^a \alpha e^{-y^2/2} d\mu d\nu \right|
\]

\[
= \alpha \left| \int_0^b e^{-y^2/2} \left( \int_0^a \frac{e^{\mu \nu}}{\sqrt{\pi}} \right) d\nu \right|
\]

\[
= \alpha \left| \int_0^b \left( \frac{2}{\sqrt{\pi}} \right)^{b/2} e^{-y^2/2} \right| \left( \frac{2}{\sqrt{\pi}} \right)^{a/2} \left( \frac{1}{\sqrt{2}} \right) < \infty,
\]

for all \( a, b \in \mathbb{R} \), in view of Theorem 1, we conclude that the statement of this corollary is true.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (no. 2013R1A1A2005557).

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