Ultraviolet fixed points in conformal gravity
and general quadratic theories

Nobuyoshi Ohta\textsuperscript{1,4} and Roberto Percacci\textsuperscript{2,3}

\textsuperscript{1} Department of Physics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan
\textsuperscript{2} International School for Advanced Studies, via Bonomea 265, 34136 Trieste, Italy
\textsuperscript{3} INFN, Sezione di Trieste, Italy

E-mail: ohtan@phys.kindai.ac.jp and percacci@sissa.it

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Abstract
We study the beta functions for four-dimensional conformal gravity using two different parametrizations of metric fluctuation, linear split and exponential parametrization. We find that after imposing the traceless conditions, the beta functions are the same in four dimensions though the dependence on the dimensions are quite different. This indicates the universality of these results. We also examine the beta functions in general quadratic theory with the Einstein and cosmological terms for exponential parametrization, and find that it leads to results for beta functions of dimensionful couplings different from linear split, though the fact that there exists a nontrivial fixed point remains the same and the fixed points also remain the same.

Keywords: asymptotic safety, functional renormalization group, conformal gravity, quadratic gravity, fixed point

(Some figures may appear in colour only in the online journal)

1. Introduction

It has long been known that Einstein gravity theory with quadratic curvature terms are renormalizable [1]. Early attempts at computing beta functions are made in [2, 3], and the correct beta functions for the dimensionless couplings have been obtained by Avramidi and Barvinsky [4]. The theories are shown to be asymptotically free [3]. However it is known that this class of theories suffers from the problem of ghosts and cannot give healthy physical theories. For this reason, the theories are not taken seriously as a candidate of quantum gravity.

\textsuperscript{4} Author to whom any correspondence should be addressed.
Recently, however, it has been shown that a special class of such theories can be unitary in three dimensions. These are called new massive gravity \cite{5}. The classification of all possible unitary theories in three dimensions with quadratic curvature terms is given in \cite{6}. With quadratic curvature terms, it is expected that these theories may give unitary and renormalizable quantum gravity. If the unitary theories are also renormalizable, this would be the first example of the complete theory of quantum gravity although in three dimensions. Unfortunately it turns out that the unitarity and renormalizability are just incompatible with each other in this class of three-dimensional theories \cite{7–9}. Given this result, it seems that the only possible way to make sense of these theories is to resort to the notion of asymptotic safety \cite{10–13}. These are those theories which have nontrivial ultraviolet fixed points and could provide consistent UV completion of the theories within Wilson’s formulation of renormalization. Asymptotic safety may be as predictive as asymptotic freedom.

In our previous papers \cite{14, 15} we have studied this class of theories in diverse dimensions and derived beta functions for the coupling constants in the theories. We have found that there are certainly nontrivial fixed points as well as Gaussian fixed points in all dimensions studied, thus confirming and extending earlier results in four dimensions \cite{4, 16–23}. We have also studied the theories near four dimensions, and found that there is an additional fixed point slightly away from four dimensions, but this fixed point disappears just on dimension four. We suggested that this additional fixed point may correspond to a Weyl-invariant theory because at that point the coefficient of the Weyl-non-invariant term vanishes, and the reason that this fixed point disappears in $D = 4$ is attributed to the fact that our approach did not take into account the Weyl invariance of the theory at that fixed point; we did not impose that the trace of the metric fluctuation would vanish. This is all right outside $D = 4$ since there is no such invariance, but becomes a problem in $D = 4$. However the fixed point value was slightly different from the one computed for conformal gravity in an earlier paper \cite{17}. It is one of the purposes of this paper to further study the fixed points in the conformal theories.

Another related issue is the gauge- and parametrization-dependence of the fixed points. In all attempts at the quantum treatment of gauge theories, we have to fix the gauge, and it is an important issue to know if and how the physical result depends on the gauge. In the gravity theory, there is also a problem how to parametrize the fluctuation around backgrounds. Here we also study how our result may change if we use different parametrization of the metric. Namely we calculate the beta functions in $D = 4$ theory with Weyl curvature squared together with Gauss-Bonnet terms, by imposing the traceless condition. The calculation is done in the standard linear split of the metric fluctuation:

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}. \quad (1.1)$$

We then find that the result agrees with the earlier one. We also evaluate beta functions in a new parametrization of a nonlinear exponential type \cite{24}:

$$g_{\mu\nu} = \delta_{\mu\nu} (e^h)^\nu_\mu. \quad (1.2)$$

in order to check if this is a universal result.

There are some advantages of using the latter parametrization:

1. The metric in (1.2) is always positive for large fluctuation, whereas this is not true for the linear split (1.1).
2. It helps to separate the gauge dependence of the obtained results \cite{25–28}.
3. It turns out that the unphysical singularities which often appear in the flow equations can be avoided for the nonlinear parametrization (1.2) \cite{28}. 

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It is thus interesting to also check the result in the new parametrization. We find that the result does not change for dimensionless couplings, thus confirming the universality of the beta functions in this case. However, we find that the beta functions of the dimensionful couplings, the Newton constant and cosmological terms, do change. We discuss physical implications of these results.

This paper is organized as follows. In section 2, we present the theory we consider, and summarize our conventions. In section 3, we derive the quadratic part of our action for the linear split (1.1) and make gauge fixing. In section 4, we show how the result in section 3 is modified for the nonlinear parametrization (1.2). We also discuss the case of Einstein theory with the cosmological constant and quadratic terms in the curvature in the nonlinear parametrization (1.2). This serves to check how the result in our previous paper [15] changes for different parametrization. In section 6, we briefly summarize the functional renormalization group equations (FRGE), and then derive the beta functions in the above cases in section 7. Section 8 is devoted to our conclusions.

2. The action

We consider a theory invariant under the Weyl transformation in four dimensions. The action is given by

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\lambda} C^2 - \frac{1}{\rho} E \right], \]  

(2.1)

containing two dimensionless couplings \( \lambda \) and \( \rho \). Here

\[ C^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{3} R^2, \]  

(2.2)

is the square of the Weyl tensor and

\[ E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2, \]  

(2.3)

is the Gauss–Bonnet combination, which is topological in four dimensions. Nevertheless, we should keep this term in discussing renormalization of the theory.

The action (2.1) is invariant under the Weyl rescaling

\[ g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}, \]  

(2.4)

with \( \Omega \) a function of spacetime.

The action (2.1) has the alternative form

\[ S = \int d^4x \sqrt{-\tilde{g}} \left[ \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma R_{\mu\nu\alpha\beta}^2 \right], \]  

(2.5)

where

\[ \alpha = -\frac{1}{\rho} + \frac{1}{6\lambda}, \quad \beta = \frac{4}{\rho} - \frac{1}{\lambda}, \quad \gamma = -\frac{1}{\rho} + \frac{1}{2\lambda}. \]  

(2.6)

We use the background field method and split the metric into general backgrounds and quantum parts, as given in (1.1) and (1.2). The cutoff is constructed in such a way that the flow equation is invariant under background transformations, both diffeomorphisms and Weyl transformations. Then we have to fix the gauge; we choose the gauge for the quantum Weyl transformations by imposing the trace of the metric fluctuation as zero. The final result should not depend on the gauge due to the Weyl invariance.
3. Quadratic expansion of the action in the linear split

In order to derive the effective action at the one-loop level, or to calculate the one-loop beta functions, we need the expansion of the action to second order in $h_{\mu\nu}$. This calculation is discussed in detail in [15] for the linear split (1.1). In this and the next three sections, the result is presented for any dimensions $D$, though our main concern in the present paper is $D = 4$.

We first summarize the final form for the linear split (1.1), where we have dropped terms with linear derivatives acting on the fluctuation and terms with two derivatives acting on a background curvature (omitting indices, these are of the form $h(\nabla R)\nabla h$ and $h(\nabla \nabla R)h$; such terms do not contribute to the final results, see for example [4, 18]). The terms proportional to $\alpha$, $\beta$ and $\gamma$ can be written in the form [15, 29]

$$
\alpha h^{\mu\nu} \left[ \nabla_\mu \nabla_\nu \nabla_\beta - \bar{g}_{\mu\nu} \nabla_\beta \nabla_\alpha - \bar{g}_{\alpha\beta} \nabla_\mu \nabla_\beta + \bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} \Box^2 - \bar{R} \bar{g}_{\mu\nu} \nabla_\alpha \nabla_\beta \right] + 2\bar{R} \bar{g}_{\mu\nu} \bar{g}_{\beta\nu} + \bar{R}_{\mu\nu} \bar{R}_{\nu\beta} - \frac{1}{4} J_{\mu\nu\alpha\beta} \bar{R}^2 \right] h^{\alpha\beta},
$$

$$
\beta h^{\mu\nu} \left[ \frac{1}{2} \nabla_\mu \nabla_\nu \nabla_\alpha \nabla_\beta - \bar{g}_{\mu\alpha} \nabla_\alpha \nabla_\beta - \bar{g}_{\nu\beta} \nabla_\mu \nabla_\alpha \nabla_\beta + \frac{1}{4} \left( \bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} + \bar{g}_{\mu\beta} \bar{g}_{\nu\alpha} \right) \Box^2 + \frac{1}{2} \bar{R}_{\mu\nu} \nabla_\alpha \nabla_\beta \right] + 2\bar{R}_{\mu} \bar{g}_{\nu\beta} \nabla_\nu \nabla_\alpha + \frac{3}{2} \bar{g}_{\beta\alpha} \bar{R}_{\mu\nu} \nabla^\nu \nabla_\alpha + \bar{R}_{\mu\nu} \nabla_\nu \nabla_\alpha \nabla^\beta \right] + \frac{1}{4} \left( 2\bar{g}_{\mu\nu} \bar{g}_{\nu\beta} - \bar{g}_{\mu\nu} \bar{g}_{\nu\beta} \right) \bar{R}^{\rho\lambda} \nabla_\rho \nabla_\lambda \nabla_\alpha - \frac{3}{2} \bar{R}_{\mu} \bar{R}_{\nu\alpha\beta} + \frac{1}{2} \bar{g}_{\nu\alpha} \bar{R}_{\mu\rho} \bar{R}_\rho - \frac{1}{2} \bar{g}_{\nu\beta} \bar{R}_{\mu\rho} \bar{R}_\rho - \frac{1}{4} J_{\mu\nu\alpha\beta} \bar{R}^2 \right] h^{\alpha\beta},
$$

$$
\gamma h^{\mu\nu} \left[ \nabla_\mu \nabla_\nu \nabla_\beta + \bar{g}_{\mu\nu} \bar{g}_{\beta\alpha} \Box^2 - 2\bar{g}_{\mu\nu} \bar{g}_{\beta\alpha} \nabla_\mu \nabla_\alpha - 2\bar{g}_{\mu\nu} \bar{R}_{\mu\rho\lambda\sigma} \nabla^\rho \nabla^\lambda + 3\bar{R}_{\mu\nu\alpha\beta} \Box \right] - 4\bar{R}_{\mu\nu} \nabla^\alpha \nabla_\beta - 4\bar{g}_{\mu\nu} \bar{R}_{\mu\rho} \nabla^\rho \nabla_\alpha - 2\bar{g}_{\mu\nu} \bar{R}_{\mu\rho} \nabla^\rho \nabla_\alpha + \bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} \bar{R}_{\rho\lambda\sigma} \nabla^\rho \nabla^\lambda

- 2\bar{g}_{\mu\nu} \bar{R}_{\mu\alpha\sigma\beta} \nabla^\sigma \nabla^\beta + 4\bar{g}_{\mu\nu} \nabla_\alpha \nabla_\beta + 2\bar{g}_{\mu\nu} \bar{R}_{\mu\rho\lambda\sigma} \nabla^\rho \nabla^\lambda - 2\bar{g}_{\mu\nu} \bar{R}^{\rho\lambda} \bar{R}_{\mu\rho\lambda\sigma} - \bar{R}_{\mu\nu} \nabla_\alpha \nabla_\beta + 3\bar{R}_{\mu\nu} \bar{R}^{\rho\lambda} \bar{R}_{\rho\lambda\sigma} + \bar{R}_{\mu\nu} \nabla_\alpha \nabla_\beta - 3\bar{R}_{\mu\nu} \bar{R}^{\rho\lambda} \bar{R}_{\rho\lambda\sigma} + \frac{1}{4} J_{\mu\nu\alpha\beta} \bar{R}^2 \right] h^{\alpha\beta}.
$$

(3.1)

Here and in what follows, a bar indicates that the quantity is evaluated on the general background; the indices are raised, lowered and contracted by the background metric $\bar{g}$, the covariant derivative $\nabla$ is constructed with the background metric. The tensor $J$ is defined by

$$
J_{\mu\nu\alpha\beta} = \delta_{\mu\nu,\alpha\beta} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta},
$$

(3.2)

where

$$
\delta_{\mu\nu,\alpha\beta} = \frac{1}{2} \left( \bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} + \bar{g}_{\mu\beta} \bar{g}_{\nu\alpha} \right) \equiv \hat{1},
$$

(3.3)
is the identity in the space of symmetric tensors. We should note that due to the presence of the external factors of $h$, the expression in the square bracket is automatically symmetrized under the interchanges $\mu \leftrightarrow \nu$, $\alpha \leftrightarrow \beta$ and $(\mu, \nu) \leftrightarrow (\alpha, \beta)$.

The BRST transformation for the fields is found to be

$$
\delta_B g_{\mu
u} = -\delta \lambda \left[ g_{\mu
u} \nabla_\rho e^\rho + g_{\nu\mu} \nabla_\xi e^\xi \right] \equiv -\delta \lambda D_{\mu
u,\rho} e^\rho,
$$

$$
\delta_B e^\mu = -\delta \lambda e^\nu \nabla_\rho e^\mu, \quad \delta_B e_\mu = i\delta B_\mu, \quad \delta_B B_\mu = 0,
$$

(3.4)

which is nilpotent. Here $e^\rho$, $e_\mu$ and $B_\mu$ are the Faddeev–Popov ghost, anti-ghost and an auxiliary field, respectively, and $\delta \lambda$ is an anticommuting parameter. The gauge fixing term and the Faddeev–Popov ghost terms are concisely written as

$$
\mathcal{L}_{GF+FP} / \sqrt{-g} = i\delta_B \left[ \bar{e}_\mu Y^{\mu\nu} \left( \chi_\nu - \frac{a}{2} B_\nu \right) \right] / \delta \lambda
$$

$$
= -B_\nu Y^{\mu\nu} \chi_\nu + i\bar{e}_\nu Y^{\mu\nu} \left( \nabla_\lambda D_{\lambda,\rho} + b \nabla_\rho D_{\lambda,\nu} \right) e^\rho + \frac{d}{2} B_\nu Y^{\mu\nu} B_\nu
$$

$$
\simeq -\frac{1}{2a} Y^{\mu\nu} \chi_\nu + i\bar{e}_\nu Y^{\mu\nu} \left[ g_{\rho\nu} \square + 2b \nabla_\rho \nabla_\nu + R_{\rho\nu} \right] e^\rho,
$$

(3.5)

where the auxiliary field $B_\mu$ is integrated out in the last line. Here

$$
\chi_\nu \equiv \nabla^\nu h_{\mu\nu} + b \nabla_\nu h,
$$

$$
Y^{\mu\nu} \equiv \bar{g}_{\rho\nu} \square + c \nabla_\rho \nabla_\nu - d \nabla_\nu \nabla_\mu.
$$

(3.6)

where $a$, $b$, $c$ and $d$ are gauge parameters. We choose them such that the non-minimal four derivative terms $\nabla_\mu \nabla_\nu \nabla_\lambda \nabla_\beta \bar{g}_{\rho\nu} \square$ and $\bar{g}_{\rho\nu} \nabla_\mu \nabla_\nu \nabla_\lambda \nabla_\beta$ cancel. This leads to the choice [18]

$$
a = \frac{1}{\beta + 4\gamma}, \quad b = \frac{4\alpha + \beta}{4(\gamma - \alpha)}, \quad c - d = \frac{2(\gamma - \alpha)}{\beta + 4\gamma} - 1.
$$

(3.7)

In order to simplify the gauge-fixing term, we will further choose $d = 1$. With these choices, the ghost operator is

$$
\Delta_{gh} = \epsilon^{\mu\nu} \square + (1 + 2b) \nabla_\mu \nabla_\nu + R_{\mu\nu}.
$$

(3.8)

In addition, we also impose the traceless condition on $h_{\mu\nu}$:

$$
h_{\mu\nu} = 0.
$$

(3.9)

This does not introduce any ghost contribution, so we can simply impose this.

Then, the quadratic terms in the action can be written in the form $h_{\mu\nu} K_{\mu\nu,\alpha\beta} h^{\alpha\beta}$, where

$$
K = K \square^2 + D_{\mu\lambda} \nabla_\nu \nabla_\lambda + W.
$$

(3.10)

The explicit forms of the coefficients are [15]

$$
(K)_{\mu\nu,\alpha\beta} = \frac{\beta + 4\gamma}{4} \left( \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} + \frac{4\alpha + \beta}{4(\gamma - \alpha)} \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} \right),
$$

(3.11)

$$
(D_{\mu\lambda})_{\mu\nu,\alpha\beta} = -2\gamma \bar{g}_{\mu\nu} R_{\alpha\beta\lambda} + 4\gamma \bar{g}_{\mu\nu} R_{\alpha\beta\lambda} + (\beta + 3\gamma) \bar{g}_{\mu\lambda} R_{\nu\alpha\beta} - (2\beta + 4\gamma) \bar{g}_{\mu\lambda} R_{\nu\alpha\beta} - 2\gamma \bar{g}_{\mu\nu} R_{\alpha\beta\lambda} - 2\alpha \bar{g}_{\mu\nu} R_{\alpha\beta\lambda} + 2\alpha \bar{g}_{\mu\lambda} R_{\alpha\beta\lambda} + 2\gamma \bar{g}_{\mu\lambda} R_{\alpha\beta\lambda}.
$$
\[
+ \frac{\alpha}{2} \bar{R} \left( g_{\mu\nu} \bar{g}_{\alpha\beta} - g_{\mu\nu} R_{\alpha\beta} - 2 g_{\mu\nu} \bar{g}_{\alpha\beta} + 2 g_{\mu\nu} \bar{g}_{\alpha\beta} \right) \\
+ 2 \gamma g_{\mu\nu} \bar{g}_{\alpha\beta} \bar{R}_{\mu\nu} + \left( \frac{\beta}{2} + \gamma \right) \bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} \bar{R}_{\mu\nu} - \frac{\beta}{4} g_{\mu\nu} \bar{g}_{\alpha\beta} \bar{R}_{\mu\nu},
\]

(3.12)

\[
(W)_{\mu\nu,\alpha\beta} = \frac{3}{2} \gamma \bar{g}_{\mu\nu} \bar{R}_{\mu} \bar{R}_{\nu} + 4 \gamma \bar{R}_{\mu\nu} \bar{R}_{\nu} + \bar{R}_{\mu\nu} \bar{R}_{\nu} \bar{R}_{\mu} \\
- \gamma \bar{R}_{\mu\nu} \bar{R}_{\mu} \bar{R}_{\nu} + (\beta + 5 \gamma) \bar{R}_{\mu\nu} \bar{R}_{\mu} \bar{R}_{\nu} \\
+ 6 \gamma \bar{R}_{\mu} \bar{R}_{\mu} \bar{R}_{\nu} + \left( \frac{\beta}{2} + \gamma \right) \bar{R}_{\mu} \bar{R}_{\nu} \\
+ \alpha \bar{R} \left( \frac{1}{2} \bar{R}_{\mu\nu} + \frac{3}{2} \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} - \bar{g}_{\alpha\beta} \bar{R}_{\mu\nu} \right) \\
+ \alpha \bar{R} \bar{R}_{\mu\nu} + \frac{1}{8} \left( \bar{R}^2 + \beta \bar{R}^2 + \gamma \bar{R}^2 \right) \\
\times \left( \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} - 2 \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} \right) + \left( \frac{5}{2} \beta + 4 \gamma \right) \bar{g}_{\mu\nu} \bar{R}_{\nu} \bar{R}_{\mu} \\
- \gamma \bar{g}_{\mu\nu} \bar{R}_{\mu\nu} \bar{R}_{\nu} \bar{R}_{\mu} \bar{R}_{\nu} - \beta \bar{g}_{\mu\nu} \bar{R}_{\mu} \bar{R}_{\nu} \bar{R}_{\mu} \bar{R}_{\nu} - (\beta + 4 \gamma) \bar{g}_{\mu\nu} \bar{R}_{\mu\nu} \bar{R}_{\nu} \bar{R}_{\nu},
\]

(3.13)

where we have dropped terms with two derivatives acting on a background curvature, and the symmetrization $\alpha \leftrightarrow \beta$ and $\mu \leftrightarrow \nu$ and $(\mu, \nu) \leftrightarrow (\alpha, \beta)$ should be understood.

To impose the gauge condition (3.9), we should multiply the projection operator

\[
P_{\mu\nu,\alpha\beta} = \delta_{\mu\nu,\alpha\beta} - \frac{1}{D} \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta}.
\]

(3.14)

After this, the above kinetic operators are transformed into

\[
\left( \tilde{K} \right)_{\mu\nu,\alpha\beta} = (KP)_{\mu\nu,\alpha\beta} = \frac{\beta + 4 \gamma}{4} P_{\mu\nu,\alpha\beta}, \\
\left( \tilde{D}_{\nu,\alpha\beta} \right)_{\mu\nu,\alpha\beta} = (PD_{\nu,\alpha\beta} P_{\mu\nu,\alpha\beta} \\
\left( \tilde{W} \right)_{\mu\nu,\alpha\beta} = (PWP)_{\mu\nu,\alpha\beta}.
\]

(3.15)

As a final step, we factorize the tensor $\tilde{K}$ in the operator $\mathcal{K}$. Since the projector $P$ in (3.14) plays the role of the identity matrix, we find

\[
\mathcal{K} = \tilde{K} \mathcal{H}; \quad \mathcal{H} = \Box^2 + V_{\nu\lambda} \nabla^\nu \nabla^\lambda + U,
\]

(3.16)

where

\[
V_{\nu\lambda} = \frac{4}{\beta + 4 \gamma} \bar{D}_{\nu\lambda}, \quad U = \frac{4}{\beta + 4 \gamma} \bar{W}.
\]

(3.17)

The form of the coefficients $V_{\nu\lambda}$ and $U$ and their traces are reported in appendix A.1.

4. Quadratic expansion of the action for conformal theory in the nonlinear exponential parametrization

If we use the exponential type parametrization for the fluctuation (1.2), we find that the quadratic terms are slightly different. We can easily derive the difference by noting that (1.2) gives
Thus if we restrict our attention only to the second order terms, they are generated from the linear term in the expansion of the action, which is

\[ \mathcal{L}^{(1)} = \alpha \left( \frac{1}{2} R_{\mu\nu}^2 h - 2 \bar{R} R_{\mu\nu} h^{\mu\nu} \right) + \beta \left( \frac{1}{2} R_{\mu\nu}^2 h - 2 \bar{R} R_{\mu\nu} R_{\rho\sigma} h^{\rho\sigma} \right) + \gamma \left( \frac{1}{2} R_{\mu\nu\rho\lambda}^2 h - 2 \bar{R} R_{\mu\nu\rho\lambda} \bar{R}_{\rho\gamma\lambda} h^{\rho\gamma} h^{\mu\nu} \right), \]  

up to terms which do not contribute to our results. These terms generate second order terms by the replacement

\[ h_{\mu\nu} \rightarrow \frac{1}{2} h_{\mu\lambda} h_{\nu}^{\lambda}. \]  

This does not affect \( K \) in (3.11) and \( D_{\rho\lambda} \) in (3.12), but modifies \( W \) in (3.13) to

Note that the additional terms are proportional to the field equation because the variation of the linear term gives precisely the field equation. After imposing the gauge condition (3.9), by multiplying the projector (3.14), we get modified \( U \) which is given in appendix A.2.

5. Quadratic expansion of the action with Einstein and cosmological terms in the nonlinear exponential parametrization

Here let us consider the action

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{\kappa^2} (R - 2\Lambda) + \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma R_{\mu\nu\rho\lambda}^2 \right]. \]  

We use the exponential parametrization (1.2) in order to see if the result changes from when we use the linear split (1.1) [15]. In this case, our \( K \) remains the same as (3.11), but we do not have projection. This gives the same \( D_{\rho\lambda} \) as in our previous paper [15] and modifies \( W \) in (3.13) to
We get modified $U$ which is given in appendix A.3. We see that the result becomes simpler than the other parametrization [15].

We are now ready to discuss the beta functions in the renormalization group (RG) equation.

6. Derivation of beta functions from the functional renormalization group equation

In the Wilsonian RG, we consider the effective action $\Gamma_k$ describing physical phenomena at momentum scale $k$, which can be regarded as the lower limit of the functional integration and the infrared cutoff. The dependence of the effective action on $k$ gives the RG flow, which can be written as an FRGE [11] having on the rhs a trace of functions of the kinetic operators.

Up to this point, we have considered the action in Minkowski space. In the following derivation of beta functions, we make Euclideanization.

In our quadratic action, we have the three operators: $\mathcal{H}$ acting on the graviton $h_{\mu\nu}$, the ghost operator $\Delta_{gh}$ and the third ghost operator $Y_{\nu\mu}$. Let us choose cutoffs for the graviton, ghost and third ghost to be functions of these operators, respectively: $KR_k(\mathcal{H})$ for the graviton, $R_k(\Delta_{gh})$ for the ghosts and $R_k(Y)$ for the third ghost. The FRGE says that

$$\left(D^\alpha_{\mu\nu}\right)_{\mu\nu\alpha\beta} = -2\alpha \bar{R}_{\mu\nu\alpha\beta} + 4\beta g_{\mu\nu} \bar{R}_{\alpha\beta} + (\beta + 3\gamma) \bar{R}_{\mu\nu\alpha\beta}$$

$$= (2\beta + 4\gamma) \bar{R}_{\mu\nu\alpha\beta} + 2\alpha g_{\mu\nu} \bar{R}_{\alpha\beta} + 2\beta g_{\mu\nu} \bar{R}_{\alpha\beta}$$

$$+ \left(\frac{\alpha}{2} \bar{R} + \frac{1}{4\alpha^2}\right)\left(\bar{R}_{\mu\nu\alpha\beta} - \bar{R}_{\nu\mu\alpha\beta} - 2\bar{R}_{\mu\nu\alpha\beta} - 2\bar{R}_{\nu\mu\alpha\beta} + 2\bar{R}_{\alpha\beta}\right)$$

$$+ 2\gamma g_{\mu\nu} \bar{R}_{\alpha\beta} + \left(\frac{\alpha}{2} + \gamma\right) \bar{R}_{\mu\nu\alpha\beta} - \beta \bar{R}_{\mu\nu\alpha\beta}.$$

(5.2)

$$\left(W_{\mu\nu,\alpha\beta}\right)_{\mu\nu\alpha\beta} = \frac{1}{2} \gamma \bar{R}_{\mu\nu} \rho_{\alpha\beta} + 4\gamma \bar{R}_{\mu\nu\alpha\beta} \bar{R}_{\rho_{\alpha\beta}}$$

$$- \gamma \bar{R}_{\mu\nu\alpha\beta} \rho_{\alpha\beta} + (\beta + 5\gamma) \bar{R}_{\rho_{\alpha\beta}} \bar{R}_{\rho_{\alpha\beta}} + \left(\frac{\alpha}{2} + \gamma\right) \bar{R}_{\rho_{\alpha\beta}}$$

$$+ \left(\frac{\alpha}{2} + \gamma\right) \bar{R}_{\rho_{\alpha\beta}} + \left(\frac{\alpha}{2} + \gamma\right) \bar{R}_{\rho_{\alpha\beta}} + \frac{1}{8} \alpha R^2 + \beta \bar{R}^2 + \gamma \bar{R}^2 + \frac{1}{\alpha^2} (\bar{R}^2 - 2\Lambda)$$

$$\times \bar{R}_{\rho_{\alpha\beta}}$$

$$+ \left(\frac{5}{2} \beta + 4\gamma\right) \bar{R}_{\rho_{\alpha\beta}} + \frac{1}{8} \alpha R^2 + \beta \bar{R}^2 + \gamma \bar{R}^2 - \frac{1}{\alpha^2} (\bar{R}^2 - 2\Lambda).$$

(5.3)

We get modified $U$ which is given in appendix A.3. We see that the result becomes simpler than the other parametrization [15].
\[ \hat{I}_k = \frac{1}{2} \text{Tr} \frac{R_k(H)}{P_k(H)} - \text{Tr} \frac{\dot{R}_k(\Delta_{gh})}{P_k(\Delta_{gh})} - \frac{1}{2} \text{Tr} \frac{\dot{R}_k(Y)}{P_k(Y)}, \]  

(6.1)

where we define \( P_k(z) = z + R_k(z) \), and the dot represents the derivative with respect to \( \ln k \).

One can obtain the beta functions of \( \alpha, \beta, \gamma \) by calculating the terms in the rhs proportional to \( \int dx \sqrt{g} R \), \( \int dx \sqrt{g} R_{\mu\nu}R^{\mu\nu} \) and \( \int dx \sqrt{g} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \).

We can compute the rhs using the following general formulas for the trace of a function of an operator. Calling \( W_\tilde{\ } \) the Laplace transform of \( W \), we have for a differential operator of order \( p \) in \( D \) dimensions:

\[ \text{Tr}[W(\Delta)] = \sum_n \sum_\lambda W(\lambda_n) = \sum_n \int_0^\infty dx e^{-\lambda_n s} W(s) = \int_0^\infty ds W(s) \text{Tr} e^{-s\Delta} \]

\[ = \sum_{n=0}^{\infty} B_{2n}(\Delta) \int_0^\infty ds W(s)s^{-\frac{D}{2} + \frac{2n}{p}}, \]  

(6.2)

where \( B_{2n} \) are the coefficients appearing in the expansion of the heat kernel of the operator. The \( Q \)-functionals are given (for \( m > 0 \)) by

\[ Q_m(W) = \int_0^\infty ds W(s)s^{-m} = \frac{1}{\Gamma(m)} \int_0^\infty dz z^{m-1} W(z). \]  

(6.3)

The last form is the more useful one. (It is obtained more easily going from right to left. Insert the Laplace expansion of \( W \) in the rhs, exchange the order of the integrations over \( s \) and \( z \), and then use the integral representation of the Gamma function.) For \( m=0 \), one has \( Q_0(W) = W(0) \).

With this formula the FRGE, expanded up to terms quadratic in curvature, is

\[ \hat{I}_k = \frac{1}{2} B_4(H) Q(4, (D - 4)/4) - B_4(\Delta_{gh}) Q(2, (D - 4)/2) \]

\[ - \frac{1}{2} B_4(Y) Q(2, (D - 4)/2). \]  

(6.5)

We have to calculate the \( Q \)-functionals \( Q(p, m) = Q_m\left(\frac{R_k}{P_k}\right) \), for an operator of order \( p \). For convenience, we choose the cutoff profile \( R_k(z) = (k^p - z)\theta(k^p - z) = k^p(1 - y)\theta(1 - y), \)

(6.6)

\[ \dot{R}_k(z) = pk^p\theta(k^p - z) = pk^p\theta(1 - y), \]  

(6.7)

\[ P_k(z) = z + R_k(z) = k^p \quad \text{for} \quad z < k^p, \]  

(6.8)

\[ \frac{\dot{R}_k}{P_k} = p \theta(1 - y). \]  

(6.9)

For \( m \geq 0 \), we find

\[ Q(p, m) = \frac{1}{\Gamma(m)} \int_0^\infty dy y^{m-1} \frac{\dot{R}_k(z)}{P_k(z)} = \frac{k^p}{\Gamma(m)} \int_0^\infty dy y^{m-1} p\theta(1 - y) \]

\[ = \frac{pk^p}{\Gamma(m)} \int_0^1 dy y^{m-1} = \frac{pk^p}{\Gamma(m + 1)}. \]  

(6.10)
Next we list the necessary heat kernel coefficients. From \([30]\), we have
\[
B_4(H) = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(D/4)}{2\Gamma((D - 2)/2)} \times \int d^{D}x \text{tr} \left[ \frac{\hat{I}}{90} \hat{R}_{\alpha\beta\gamma\delta}^2 - \frac{\hat{I}}{90} \hat{R}_{\alpha\beta}^2 + \frac{1}{36} \hat{R}^2 + \frac{1}{6} \hat{R}_{\alpha\beta} \hat{R}^\alpha \hat{R}^\beta \right] \\
- \frac{2}{D - 2} U - \frac{1}{6(D - 2)} \left( 2\hat{R}_{\alpha\beta} V^\alpha \right) \\
+ \frac{1}{4(D^2 - 4)} \left( V^\rho V^\lambda + 2V^\rho V^\lambda \right) \right],
\]
where \(\hat{I}\) is the identity defined in \((3.3)\) and \(\hat{R}_{\alpha\beta}\) is the commutator of the covariant derivatives acting on the tensor \(h^{\alpha\beta}: \hat{R}_{\alpha\beta} = [\nabla_\alpha, \nabla_\beta].\) Collecting, we find
\[
B_4(H) = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(D/4)}{2\Gamma((D - 2)/2)} \times \int d^{D}x \left[ \hat{R}_{\mu\nu\rho\lambda}^2 \left( \frac{(D - 1)(D + 2)}{180} - \frac{D + 2}{6} + \frac{2A_1}{D - 2} + \frac{12D_1}{D^2 - 4} \right) \\
- \hat{R}_{\mu\nu}^2 \left( \frac{(D - 1)(D + 2)}{180} + \frac{2A_2}{D - 2} + \frac{C_1}{3(D - 2)} - \frac{12D_2}{D^2 - 4} \right) \\
+ \hat{R}^2 \left( \frac{(D - 1)(D + 2)}{72} - \frac{2A_3}{D - 2} + \frac{B_1}{6(D - 2)} - \frac{C_2}{3(D - 2)} + \frac{12D_3}{D^2 - 4} \right) \right],
\]
where the constants \(A_1, B_1, C_1\) and \(D_1\) are defined in the appendix. Note that because we take the trace over the traceless symmetric tensor space, we have to use \(\text{tr} (\hat{I}) = \frac{(D - 1)(D + 2)}{2}\), which is one of the differences from the case in \([15]\).

From \([31]\), we have for the Euclidean operator
\[
Y_{\mu\nu} = -\tilde{g}_{\mu\nu} \Box + \sigma_Y \nabla_\mu \nabla_\nu + \hat{R}_{\mu\nu},
\]
with \(\sigma_Y = 1 - 2 \frac{\gamma - \alpha}{\beta + 4\gamma}\).

B_4(Y) = \frac{1}{(4\pi)^{D/2}} \int d^{D}x \sqrt{-\tilde{g}} \left[ \frac{D - 16 + (1 - \sigma_Y)^{\frac{4-D}{2}} \hat{R}_{\mu\nu\rho\lambda}^2}{180} \\
- \frac{D - 91 + (1 - \sigma_Y)^{\frac{4-D}{2}} \hat{R}_{\mu\nu}^2}{180} + \frac{D - 13 + (1 - \sigma_Y)^{\frac{4-D}{2}} \hat{R}^2}{72} \right],
\]
whereas for the Euclidean ghost operator
\[
\Delta_{gh\mu\nu} = -\tilde{g}_{\mu\nu} \Box + \sigma_g \nabla_\mu \nabla_\nu - \hat{R}_{\mu\nu},
\]
with \(\sigma_g = -(1 + 2b) = \left( 1 + 2 \frac{\beta + 4\alpha}{4(\gamma - \alpha)} \right).\)
The $B_4$ agree with the formulas in [18] for $D = 4$, but the dependence on $D$ is more complicated than appears there.

Substituting the heat kernel coefficients in equation (6.5), and extracting the coefficients of $\bar{R}^2$, $\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}$, and $\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}$, we obtain the beta functions of $\alpha$, $\beta$ and $\gamma$. It turns out that the scalar curvature squared $\bar{R}^2$ is absent, so our result is consistent with conformal invariance.

7. Beta functions in four dimensions

In this section, we summarize our results for beta functions in four dimensions.

7.1. Conformal gravity in the linear split

In four dimensions, the couplings $\alpha$ and $\beta$ (or equivalently $\lambda$ and $\rho$) are all dimensionless. We define

$$\theta \equiv \frac{\lambda}{\rho},$$

and consider beta functions for the couplings. For $D = 4$ and linear parametrization in section 3, we have the traces in (A.1). Using these results in the formulae in section 4, we find

$$\beta_\lambda = -\frac{1}{(4\pi)^2} \frac{199}{15} \lambda^2,$$

$$\beta_\rho = \frac{1}{(4\pi)^2} \frac{87}{20} \rho^2,$$

in agreement with [17]. We also find

$$\beta_\theta = \frac{1}{(4\pi)^2} \frac{261}{60} - 796\theta \lambda.$$

The fixed point of $\theta$ is $\theta_\text{FP} = \frac{261}{796} = 0.32789$. The fixed point value that we found in our previous paper [15] as a candidate for four-dimensional conformal theory is $\theta = 0.325296$. As we will see, this value itself is closer to the fixed point in gravity theory without conformal
invariance. It seems that the main difference comes from the difference in the contribution of the identity operator noted below equation (6.12). This in turn means the difference in the contribution of the trace part of \( h_{\mu \nu} \).

### 7.2. Conformal theory in the exponential parametrization

On the other hand, it is interesting to check how the results change if we use the nonlinear exponential parametrization in section 4. The trace of \( U \) drastically simplifies as given in appendix A.2. We find beta functions for these coefficients for \( D = 4 \):

\[
\begin{align*}
\beta_\lambda &= -\frac{1}{(4\pi)^2} \frac{199}{15} \lambda^2, \\
\beta_\rho &= -\frac{1}{(4\pi)^2} \frac{87}{20} \rho^2, \\
\beta_\theta &= \frac{1}{(4\pi)^2} \frac{261 - 796\theta}{60} \lambda.
\end{align*}
\tag{7.4}
\]

Somewhat to our surprise, we find that these results agree with the above beta functions with the separate parametrization even though the traces of \( V \) and \( U \) are so different.

### 7.3. Quadratic theory with the Einstein and cosmological terms

We have also examined the beta functions in the presence of Einstein and cosmological terms in the exponential parametrization. We use the couplings \( \Lambda = \Lambda/k^2 \), \( \tilde{G} = G k^2 \). The results for the traces are summarized in appendix A.3. In this case, we find that if we set the scalar curvature squared \( R^2 \) to zero from the outset in the action (2.1), we get singular beta functions. So we first set the coefficients in the action (2.5) as

\[
\alpha = -\frac{1}{\rho} + \frac{1}{\xi} + \frac{1}{6\lambda}, \quad \beta = \frac{4}{\rho} - \frac{1}{\lambda}, \quad \gamma = -\frac{1}{\rho} + \frac{1}{2\lambda}.
\tag{7.5}
\]

We also define

\[
\xi = -(D - 1) \frac{\Lambda}{\omega}.
\tag{7.6}
\]

We obtain results that should be compared with those in our previous paper [15].

The beta functions are found to be

\[
\begin{align*}
\beta_\lambda &= -\frac{1}{(4\pi)^2} \frac{133}{10} \lambda^2, \\
\beta_\rho &= -\frac{1}{(4\pi)^2} \frac{196}{45} \rho^2, \\
\beta_\xi &= -\frac{1}{(4\pi)^2} \frac{5(1 + 12\omega + 8\omega^2)}{4\omega^2} \lambda^2, \\
\beta_\omega &= -\frac{1}{(4\pi)^2} \frac{25 + 1098\omega + 200\omega^2}{60} \lambda, \\
\beta_\theta &= \frac{1}{(4\pi)^2} \frac{7(56 - 171\theta)}{90} \lambda.
\end{align*}
\tag{7.7}
\]

We find that these agree with the results in [15]. The fixed points of \( \lambda \) and \( \theta \) are \( \lambda_\ast = 0 \) and \( \theta_\ast = \frac{56}{171} = 0.3275 \), respectively.
We also have the beta functions for the Newton and cosmological constants. For simplicity we omit tildes on these.

\[ \beta_\Lambda = -2\Lambda + \frac{3 + 34\omega + 40\omega^2}{12(4\pi)^2\omega}\lambda\Lambda - \frac{171 + 298\omega + 152\omega^2 + 16\omega^3}{36\pi(1 + \omega)}G\Lambda \]

\[ + \frac{283 + 664\omega + 204\omega^2 - 128\omega^3 - 32\omega^4}{144\pi(1 + \omega)^2}G - \frac{1 + 10\omega}{4(4\pi)^2\omega}\lambda + \frac{1 + 20\omega^2}{64(4\pi)^3\omega^2G}\lambda^2, \]

\[ \beta_G = 2G - \frac{5 - 26\omega - 40\omega^2}{12(4\pi)^2\omega}\lambda G - \frac{171 + 298\omega + 152\omega^2 + 16\omega^3}{36\pi(1 + \omega)}G^2. \]  

(7.8)

The beta functions for \( G \) and \( \Lambda \) in the linear split (1.1) were found to be [15]

\[ \beta_\Lambda = -2\Lambda + \frac{1 + 86\omega + 40\omega^2}{12(4\pi)^2\omega}\lambda\Lambda - \frac{171 + 298\omega + 152\omega^2 + 16\omega^3}{36\pi(1 + \omega)}G\Lambda \]

\[ + \frac{283 + 664\omega + 204\omega^2 - 128\omega^3 - 32\omega^4}{144\pi(1 + \omega)^2}G - \frac{1 + 10\omega}{4(4\pi)^2\omega}\lambda + \frac{1 + 20\omega^2}{64(4\pi)^3\omega^2G}\lambda^2, \]

\[ \beta_G = 2G - \frac{3 + 26\omega - 40\omega^2}{12(4\pi)^2\omega}\lambda G - \frac{171 + 298\omega + 152\omega^2 + 16\omega^3}{36\pi(1 + \omega)}G^2. \]  

(7.9)

(7.10)

Thus we find that the beta functions for dimensionful couplings are different though those for the dimensionless couplings remain the same. This suggests the universality of the latter beta functions. We can also see that only the second terms of both the beta functions for \( G \) and \( \Lambda \) are different, but these terms vanish at the fixed point of \( \lambda_0 = 0 \), so the fixed points are not affected by the difference. This suggests that our results are parameterization independent. For a picture of the flow in the \( \Lambda-G \) plane, see [32].

8. Conclusions

In this paper, we have studied the beta functions for conformal gravity in the linear split and exponential parametrization of the metric fluctuation and found that they agree with each other. This is an indication that these results are universal. This study was partly motivated by our study of fixed points in Einstein theory with cosmological and quadratic curvature terms [15] where we find an extra fixed point in addition to those already known. We conjectured that this might correspond to conformal gravity in four dimensions because the coefficient of \( \xi \) for the scalar curvature squared vanishes at the fixed point. However, our analysis indicates that this does not seem to be the case since the fixed point values are different for the conformal gravity independent of the parametrization. The contribution from the trace modes of the metric seems to make the crucial difference.

To check the parametrization dependence of this approach, we have also examined the beta functions for the Einstein theory with the cosmological constant and quadratic terms. We have found that the beta functions for dimensionless couplings are the same, but those for dimensionful couplings are slightly different. This suggests again that the beta functions for dimensionless couplings are universal, but those for dimensionful couplings are not. However, we find that the fixed points are the same because the difference disappears at the fixed point. These are physically reasonable results.
As mentioned in the introduction, there is an issue of gauge-dependence in these results. It would be very interesting to study the gauge and parametrization independence and other issues, to which we hope to return in a separate publication [32].

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Appendix. \( U, V \) and their traces

A.1. Linear split

The tensors \( U \) and \( V \) are given in (3.17) for the case of linear split (1.1) after the traceless projector (3.14) is multiplied. The explicit form of \( U \) is given by

\[
U = \frac{4}{\beta + 4\gamma}
\left[ \frac{1}{4} \left( \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma R_{\mu\nu\rho\sigma}^2 \right) \left( \frac{D + 12}{D^2} \frac{\partial}{\partial \gamma} g_{\mu\nu} \delta_{\alpha\beta} - \delta_{\mu\nu} g_{\alpha\beta} \right) + \frac{3}{2D} \alpha \left( D \frac{\partial}{\partial \gamma} R_{\mu\nu} \right) - 2g_{\alpha\beta} R_{\mu\nu} - 2g_{\mu\nu} R_{\alpha\beta} \right] \left( \frac{D + 3\beta + \gamma}{D^2} \frac{\partial}{\partial \gamma} g_{\alpha\beta} + \alpha g_{\alpha\beta} \frac{\partial}{\partial \gamma} \right) - 2\frac{3\beta + 2\gamma}{D} g_{\alpha\beta} R_{\mu\nu} R_{\rho\sigma} + \frac{4}{D} \frac{\partial}{\partial \gamma} g_{\alpha\beta} R_{\mu\nu} R_{\rho\sigma} + \frac{3}{2D} \frac{\partial}{\partial \gamma} \left( \frac{D}{2} \frac{\partial}{\partial \gamma} R_{\mu\nu} \right) - \frac{3}{2D} \frac{\partial}{\partial \gamma} \left( \frac{D}{2} \frac{\partial}{\partial \gamma} R_{\mu\nu} \right) - \frac{3}{2D} \frac{\partial}{\partial \gamma} \left( \frac{D}{2} \frac{\partial}{\partial \gamma} R_{\mu\nu} \right)
\]

The expression for \( V^{\rho\lambda} \) is \( V^{\rho\lambda} = (V^{\rho\lambda})_{\mu,\alpha,\beta} \)

\[
V^{\rho\lambda} = \frac{4}{\beta + 4\gamma} \sum_{i=1}^{20} b_i k_i
\]

where

\[
\begin{align*}
k_1 &= g^{\rho\lambda} g_{\alpha\beta} R_{\mu\nu,\alpha} \\
k_2 &= \delta_{\mu,\nu,\alpha} g^{\rho\lambda} \\
k_3 &= g^{\rho\lambda} R_{\mu\nu,\beta} \\
k_4 &= \delta_{\rho,\lambda} R_{\mu\nu,\alpha} \\
k_5 &= \delta_{\rho,\mu} R_{\nu,\beta} \\
k_6 &= \delta_{\rho,\nu} (\beta R_{\mu,\lambda}) \\
k_7 &= \frac{1}{2} \delta_{\rho,\mu} \delta_{\lambda,\nu} R_{\mu,\lambda} \\
k_8 &= \delta_{\rho,\mu} R_{\lambda,\beta} \\
k_9 &= g_{\mu,\nu} (\delta_{\rho,\lambda} R_{\gamma,\beta}) \\
k_{10} &= \frac{1}{2} \delta_{\rho,\mu} \delta_{\lambda,\nu} R_{\mu,\lambda} \\
k_{11} &= \frac{1}{2} \delta_{\rho,\mu} \delta_{\lambda,\nu} R_{\mu,\lambda} \\
k_{12} &= \frac{1}{2} \delta_{\rho,\mu} \delta_{\lambda,\nu} R_{\mu,\lambda} \\
k_{13} &= \frac{1}{2} \delta_{\rho,\mu} \delta_{\lambda,\nu} R_{\mu,\lambda} \\
k_{14} &= \frac{1}{2} \delta_{\rho,\mu} \delta_{\lambda,\nu} R_{\mu,\lambda} \\
k_{15} &= \frac{1}{2} \delta_{\rho,\mu} \delta_{\lambda,\nu} R_{\mu,\lambda} \\
k_{16} &= \frac{1}{2} \delta_{\rho,\mu} \delta_{\lambda,\nu} R_{\mu,\lambda} \\
k_{17} &= \frac{1}{2} \delta_{\rho,\mu} \delta_{\lambda,\nu} R_{\mu,\lambda} \\
k_{18} &= \frac{1}{2} \delta_{\rho,\mu} \delta_{\lambda,\nu} R_{\mu,\lambda} \\
k_{19} &= \frac{1}{2} \delta_{\rho,\mu} \delta_{\lambda,\nu} R_{\mu,\lambda} \\
k_{20} &= \frac{1}{2} \delta_{\rho,\mu} \delta_{\lambda,\nu} R_{\mu,\lambda}
\end{align*}
\]
and
\[ b_1 = -2\gamma, \quad b_2 = \frac{\alpha}{2} \bar{R}, \quad b_3 = \beta + 3\gamma, \]
\[ b_4 = 2\gamma, \quad b_5 = -\alpha \bar{R}, \quad b_6 = \frac{\beta}{2} + \gamma, \]
\[ b_7 = -4\gamma, \quad b_8 = -2\beta - 4\gamma, \quad b_9 = -2\gamma, \]
\[ b_{10} = -2\alpha, \quad b_{11} = \frac{4}{D} \gamma, \quad b_{12} = \frac{4}{D} \gamma, \]
\[ b_{13} = \frac{\alpha - \beta - \gamma}{D}, \quad b_{14} = \frac{\alpha - \beta - \gamma}{D}, \]
\[ b_{15} = \frac{2\beta}{D}, \quad b_{16} = \frac{2\beta}{D}, \quad b_{17} = b_{18} = \frac{2\alpha}{D} \bar{R}, \]
\[ b_{19} = \frac{(D + 6) \alpha + 2(\beta + \gamma)}{2D^2} \bar{R}, \quad b_{20} = -\frac{(D + 4)}{2D^2} \beta - \frac{D - 4}{D^2} \gamma. \]  

(A.4)

In deriving this result, one has to pay special attention to the symmetry \((\mu, \nu) \leftrightarrow (\alpha, \beta)\). The following results are obtained using software Math Tensor run on Mathematica. It is important to realize that the indices are symmetrized. For example, the indices \(\rho\) and \(\lambda\) on \(V\) must be symmetrized in making products.

The trace of \(U\) is given as
\[
\text{tr} \ U = \delta_{\mu\nu,\alpha\beta} U_{\mu\nu,\alpha\beta} = A_1 \bar{R}_{\alpha\beta}^2 + A_2 \bar{R}_{\mu\nu}^2 + A_3 \bar{R}^2, \tag{A.5}
\]
where
\[
A_1 = \frac{-24\gamma + 2(3\beta + 13\gamma)D + 5\gamma D^2 - \gamma D^3}{2D(\beta + 4\gamma)},
\]
\[
A_2 = \frac{-24\beta + 2(4\alpha + 5\beta + 12\gamma)D + 5\beta D^2 - \beta D^3}{2D(\beta + 4\gamma)},
\]
\[
A_3 = \frac{-24\alpha + 2(3\alpha + \beta + 2\gamma)D + 5\alpha D^2 - \alpha D^3}{2D(\beta + 4\gamma)}. \tag{A.6}
\]

For \(D = 4\), these reduce to
\[
A_1 = 3, \quad A_2 = \frac{4(\alpha + \beta + 3\gamma)}{\beta + 4\gamma}, \quad A_3 = \frac{2\alpha + \beta + 2\gamma}{\beta + 4\gamma}. \tag{A.7}
\]

Next,
\[
\text{tr} \left( V_{\rho}^\mu \bar{R} \right) = B_1 \bar{R}, \tag{A.8}
\]

with
\[
B_1 = \frac{(D + 2) \left\{ D(D - 1)(D - 2)\alpha + (D^2 - 7D + 4)\beta - 2(D^2 + 2D + 4)\gamma \right\}}{D(\beta + 4\gamma)}. \tag{A.9}
\]

For \(D = 4\),
\[
B_1 = \frac{12(3\alpha - \beta - 7\gamma)}{\beta + 4\gamma}. \tag{A.10}
\]

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Next,
\[
\text{tr} \left( V_{\mu}^\lambda \tilde{R}_{\lambda\rho} \right) = C_1 \tilde{R}_{\mu\nu}^2 + C_2 \tilde{R}^2,
\]  
with
\[
C_1 = \frac{8(\beta - 2\gamma) - 2(4\alpha + 3\beta + 6\gamma)D - (3\beta + 2\gamma)D^2 + (\beta + 2\gamma)D^3}{D(\beta + 4\gamma)},
\]
\[
C_2 = \frac{4(3\alpha - \beta - \gamma) - 2(2\alpha + \beta + 3\gamma)D - (\alpha + 4\gamma)D^2 + \alpha D^3}{D(\beta + 4\gamma)}.
\]  
(A.12)

For \( D = 4 \),
\[
C_1 = \frac{8(-\alpha + \gamma)}{\beta + 4\gamma}, \quad C_2 = \frac{11\alpha - 3\beta - 23\gamma}{\beta + 4\gamma}.
\]  
(A.13)

Finally
\[
\frac{1}{48} \text{tr} \left( V_{\mu}^\lambda V_{\lambda}^{\nu} \right) + \frac{1}{24} \text{tr} \left( V_{\rho}^\lambda V_{\lambda}^{\rho} \right) = D_1 \tilde{R}_{\mu\nu\rho\lambda}^2 + D_2 \tilde{R}_{\mu\nu}^2 + D_3 \tilde{R}^2,
\]  
where
\[
D_1 = \frac{D^2(D + 2)\beta^2 + 2D(D + 2)(3D + 4)\beta\gamma + (9D^3 + 48D^2 + 64D - 64)\gamma^2}{4D(\beta + 4\gamma)^2},
\]
\[
D_2 = \frac{1}{12D^2(\beta + 4\gamma)^2} \left[ 8D^3(D + 1)\alpha^2 + 32D(-D^2 + D + 4)\alpha\gamma 
+ 16(D^2 + D - 4)D\alpha\beta 
+ 4(D + 2)(7D^3 + 5D^2 - 16)\beta\gamma 
+ (3D^4 + 5D^3 + 30D^2 + 16D + 32)\beta^2 
+ 4(D^5 + 14D^4 + 30D^3 + 2D^2 - 80D + 32)\gamma^2 \right],
\]
\[
D_3 = \frac{1}{24D^2(\beta + 4\gamma)^2} \left[ D \left( D^5 + D^4 - 2D^3 - 24D^2 - 24D + 16 \right) \alpha^2 
+ 2 \left( D^5 - D^4 - 2D^3 - 20D^2 - 24D + 96 \right) \alpha\beta 
- 4 \left( D^5 + 2D^4 - 8D^3 - 32D^2 + 32D + 96 \right) \alpha\gamma 
+ \left\{ D(D - 8)(D^2 + D + 6) - 32 \right\} \beta^2 
- 4(D + 2)(D^3 + 8D^2 - 6D + 8)\beta\gamma 
- 4(D^4 + 17D^3 + 28D^2 + 4D - 64)\gamma^2 \right].
\]  
(A.15)

For \( D = 4 \),
\[
D_1 = 6, \quad D_2 = \frac{2(20\alpha^2 + 8\alpha\beta + 13\beta^2 - 8\alpha\gamma + 96\beta\gamma + 196\gamma^2)}{3(\beta + 4\gamma)^2},
\]
\[
D_3 = \frac{43\alpha^2 + 10\alpha\beta - 7\beta^2 - 46\alpha\gamma - 66\beta\gamma - 109\gamma^2}{6(\beta + 4\gamma)^2}.
\]  
(A.16)

These results for quadratic curvature terms agree with those in [17].
A.2. Nonlinear exponential parametrization

When we use the exponential parametrization (1.2), the tensor \( V \) is the same as the linear split, but the tensor \( W \) changes to (4.5). The tensor \( U \), after the projection by (3.14), is then given by

\[
U = \frac{4}{\beta + 4\gamma} \left[ \frac{2}{D^2} \left( \alpha \tilde{R}^2 + \beta \tilde{R}_{p}^2 + \gamma \tilde{R}_{p\sigma\tau} \right) \tilde{g}_{\mu\nu} \tilde{g}_{\alpha\beta} + \frac{\alpha}{2D} \left( D \tilde{g}_{\mu\nu} \tilde{R}_{\mu\nu} - 4 \tilde{g}_{\alpha\beta} \tilde{R}_{\mu\nu} \right) \right. \\
- 4 \tilde{g}_{\mu\nu} \tilde{R}_{\alpha\beta} \tilde{R}_{\mu\nu} \tilde{R}_{\alpha\beta} \tilde{R}_{\gamma\delta} \tilde{R}_{\gamma\delta} \tilde{R}_{\mu\nu} \tilde{R}_{\alpha\beta} \tilde{R}_{\gamma\delta} \tilde{R}_{\gamma\delta} \tilde{R}_{\mu\nu} \tilde{R}_{\alpha\beta} \tilde{R}_{\gamma\delta} \tilde{R}_{\gamma\delta} \\
\left. + \left( \frac{5}{2} \beta + 4\gamma \right) \tilde{g}_{\mu\nu} \tilde{R}_{\alpha\beta} \tilde{R}_{\mu\nu} \tilde{R}_{\alpha\beta} \tilde{R}_{\gamma\delta} \tilde{R}_{\gamma\delta} \tilde{R}_{\mu\nu} \tilde{R}_{\alpha\beta} \tilde{R}_{\gamma\delta} \tilde{R}_{\gamma\delta} \tilde{R}_{\mu\nu} \tilde{R}_{\alpha\beta} \tilde{R}_{\gamma\delta} \tilde{R}_{\gamma\delta} \right].
\]

(A.17)

The trace is given as

\[
\text{tr } U = \delta^{\mu\nu,\alpha\beta} U_{\mu\nu,\alpha\beta} = A_1 \tilde{R}_{\mu\nu}^2 + A_2 \tilde{R}_{\mu\nu}^2 + A_3 \tilde{R}_{\mu\nu}^2,
\]

(A.18)

where

\[
A_1 = \frac{-8\gamma + (3\beta + 10\gamma)D + \gamma D^2}{D(\beta + 4\gamma)},
\]

\[
A_2 = \frac{-8\beta + 2(2\alpha + \beta + 6\gamma)D + \beta D^2}{D(\beta + 4\gamma)},
\]

\[
A_3 = \frac{-8\alpha + (\beta + 2\gamma)D + \alpha D^2}{D(\beta + 4\gamma)}.
\]

(A.19)

Note that these are quite different from the corresponding ones (A.6) in the linear split where they contain cubic terms in dimension \( D \). Nevertheless, it turns out that they agree with those in the linear split in four dimensions:

\[
A_1 = 3, \quad A_2 = \frac{4(\alpha + \beta + 3\gamma)}{\beta + 4\gamma}, \quad A_3 = \frac{2\alpha + \beta + 2\gamma}{\beta + 4\gamma}.
\]

(A.20)

The rest of the traces of \( V \) are the same as in the previous section A.1.

A.3. Nonlinear exponential parametrization in Einstein-quadratic theory

Here we consider the theory with quadratic curvature terms and the Einstein and cosmological terms, as considered in our previous paper [15]. We use the exponential parametrization (1.2). In this case, we do not impose the traceless condition, and we simply use

\[
\left( K^{-1} \right)_{\mu\nu}^{\alpha3} = \frac{4}{\beta + 4\gamma} \left( \delta_{\mu\nu}^{\alpha3} - \Omega g_{\mu\nu} g^{\alpha3} \right),
\]

(A.21)

with

\[
\Omega \equiv \frac{4\alpha + \beta}{\Sigma}, \quad \Sigma \equiv 4(\gamma - \alpha) + D(4\alpha + \beta).
\]

(A.22)
After some work we find

\[
(U)_{\mu\nu,\alpha\beta} = \frac{4}{\beta + 4\gamma} \left[ \frac{1}{2} \gamma \bar{g}_{\alpha\beta} \tilde{R}_{\nu}{}^{\rho\lambda\sigma} \bar{R}_{\mu}{}^{\rho\lambda\sigma} - \gamma \bar{R}_{\nu}{}^{\rho\lambda\sigma} \bar{R}_{\mu}{}_{\rho\lambda\sigma} + 4 \gamma \bar{R}_{\rho\lambda\sigma} \bar{R}_{\mu}{}_{\rho\lambda\sigma} \right] \\
- 3 \gamma \left( \bar{R}_{\nu}{}_{\rho\lambda\sigma} + \bar{R}_{\mu}{}^{\rho\lambda\sigma} \right) + \left( \frac{\beta}{2} + \gamma \right) \bar{R}_{\nu}{}_{\rho\lambda\sigma} \bar{R}_{\mu}{}_{\rho\lambda\sigma} - \frac{\gamma}{2} \bar{g}_{\alpha\beta} \bar{R}_{\mu}{}_{\rho\lambda\sigma} \bar{R}_{\nu}{}_{\rho\lambda\sigma} \\
+ \frac{\Omega}{2} S^2 \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} + \left( \frac{\alpha \bar{R}}{2} + \frac{1}{4 \kappa^2} \right) \left( \bar{R}_{\nu}{}_{\rho\lambda\sigma} + \bar{g}_{\alpha\beta} \bar{R}_{\mu}{}_{\rho\lambda\sigma} - \bar{g}_{\mu\nu} \bar{R}_{\alpha\beta} \right) \\
+ \left( \frac{\beta}{2} + 4 \gamma \right) \bar{g}_{\alpha\beta} \bar{R}_{\mu}{}_{\rho\lambda\sigma} \bar{R}_{\nu}{}_{\rho\lambda\sigma} + \left( \beta + 5 \gamma \right) \bar{R}_{\rho\lambda\sigma} \bar{R}_{\mu}{}_{\rho\lambda\sigma} \bar{R}_{\nu}{}_{\rho\lambda\sigma} - \frac{\beta}{2} \bar{g}_{\alpha\beta} \bar{R}_{\mu}{}_{\rho\lambda\sigma} \bar{R}_{\nu}{}_{\rho\lambda\sigma} \\
- 2 \Omega \left( \gamma \bar{g}_{\mu\nu} \bar{R}_{\rho\lambda\sigma} \bar{R}_{\alpha \beta} + \beta \bar{g}_{\mu\nu} \bar{R}_{\rho\lambda\sigma} \bar{R}_{\beta}^{\alpha} \right) + \alpha \bar{R}_{\mu}{}_{\rho\lambda\sigma} \bar{R}_{\nu}{}_{\rho\lambda\sigma} - \frac{\Omega_1 - 2 \Omega}{2 \kappa^2} \left( \bar{R} - 4 \Lambda \right) \Omega \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} - 2 \left( \beta + 2 \gamma \right) \bar{g}_{\mu\nu} \bar{R}_{\alpha \beta}^{\rho\lambda} \bar{R}_{\rho\lambda\sigma} \bar{R}_{\mu\nu} \bar{R}_{\alpha \beta} \right] \text{,} \tag{A.23}
\]

where we have defined

\[
S^2 = \alpha \bar{R}^2 + \beta \bar{R}_{\mu\nu}^2 + \gamma \bar{R}_{\mu\nu\rho\lambda\sigma}^2 + \frac{1}{\kappa^2} \left( \bar{R} - 2 \Lambda \right), \tag{A.24}
\]

and

\[
\Omega_1 = \frac{10 \alpha + 3 \beta + 2 \gamma}{\Sigma}, \quad \Omega_3 = \frac{3 \alpha + \beta + \gamma}{\Sigma}, \tag{A.25}
\]

with \( \Sigma \) given in (A.22).

The trace of \( U \) is given as

\[
\text{tr } U = \delta_{\mu\nu,\alpha\beta} U_{\mu\nu,\alpha\beta} = A_1 \bar{R}_{\mu\nu\rho\lambda\sigma}^2 + A_2 \bar{R}_{\mu\nu}^2 + A_3 \bar{R}^2 + A_4 \bar{R}_{\kappa^2} + A_5 \frac{\Lambda}{\kappa^2}, \tag{A.26}
\]

where

\[
A_1 = \frac{1}{(\beta + 4 \gamma) \Sigma} \left[ 12 (D - 1) \alpha \beta + 2 (2 D^2 + 17 D - 28) \alpha \gamma + 3 D \beta^2 \right. \\
+ (D^2 + 10 D + 4) \beta \gamma \\
+ \left. (6 D + 4) \gamma^2 \right],
\]

\[
A_2 = \frac{1}{(\beta + 4 \gamma) \Sigma} \left[ 16 (D - 1) \alpha^2 + 2 (2 D^2 + 3 D - 12) \alpha \beta + 16 (3 D - 2) \alpha \gamma \right. \\
+ (D^2 + 2 D - 8) \beta^2 + 2 (9 D - 4) \beta \gamma + 48 \gamma \right],
\]

\[
A_3 = \frac{1}{(\beta + 4 \gamma) \Sigma} \left[ 2 (2 D^2 - 3 D - 8) \alpha^2 + (D^2 + 4 D - 12) \alpha \beta + 2 (7 D - 12) \alpha \gamma \right. \\
+ (\beta + 2 \gamma) (2 D + 4 \gamma) \right],
\]

\[
A_4 = \frac{1}{(\beta + 4 \gamma) \Sigma} \left[ D - 2 \right] + 4 D \alpha + (D + 2) \beta + 8 \gamma],
\]

\[
A_5 = \frac{1}{(\beta + 4 \gamma) \Sigma} 4 D (\alpha - \gamma). \tag{A.27}
\]

For the traces of \( V \), we should use those in our previous paper [15].
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