NOTE ON A NON LINEAR PERTURBATION OF THE IDEAL BOSE GAS

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Abstract. In this work we show that the introduction of a $U(1)$ symmetry breaking field in the energy operator of the boson-free gas, is equivalent, in the thermodynamic limit, to the inclusion, in the Hamiltonian of the ideal gas, of a non-linear function of the number operator associated with the zero mode. In other words, the limit pressures coincide. Moreover, both models undergo non conventional Bose-Einstein condensation (BEC) for strictly negative values of the chemical potential $\mu$. Finally, the proof of equivalence of limit pressures is extended to a class of full-diagonal models.

1. Introduction

Until 2013 it was believed that, from the point of view of a physical experiment, to confine a homogeneous system of Bose atoms, and to make it pass, subsequently, to the thermodynamic limit, would be an impossible task to perform. Thus [3],

In the magnetic traps, not only is the number of particles quite small, compared to the usual case, but the “boundary,” formed by a quadratic potential well, extends literally throughout the whole system. In order to take the thermodynamic limit in such a system it is necessary to weaken the potential so that, as the number of particles increases, the average density remains constant. This is well-defined mathematically, but is of course physically unrealizable. On the other hand, taking the box size to infinity in the homogeneous case is also unrealized experimentally.

Moreover, trapped gases were, generally speaking, spatially inhomogeneous. In this framework, to overcome the difficulties in defining pressure and volume for a gas confined in an inhomogeneous trap, it has been necessary to define macroscopic parameters that behave like them.

However, in 2013, BEC in a quasi uniform three dimensional potential of an optical trap box (cylindrical optical box) BEC was observed [4] [5].
The authors of ref.[4] point out that:

We have observed the Bose-Einstein condensation of an atomic gas in the (quasi)uniform three-dimensional potential of an optical box trap. Condensation is seen in the bimodal momentum distribution and the anisotropic time-of-flight expansion of the condensate. The critical temperature agrees with the theoretical prediction for a uniform Bose gas. The momentum distribution of a noncondensed quantum-degenerate gas is also clearly distinct from the conventional case of a harmonically trapped sample and close to the expected distribution in a uniform system. We confirm the coherence of our condensate in a matter-wave interference experiment. Our experiments open many new possibilities for fundamental studies of many-body physics.

Later, in the same article, they indicate that,

The thermodynamics of our gas are therefore very close to the textbook case of a uniform system and very different from the case of a harmonically trapped sample.

In this sense, this seems to be a suitable experimental scenario to test the consistency of the Bogoliubov’s theory -based on the concept of quasianaverages [12]-, about the spontaneous rupture of the $U(1)$ symmetry and simultaneous emergence of Bose Einstein condensation in the case of an ideal Bose gas. In other words, experiments could be carried out, in this framework, to study the thermodynamic behavior of an Ideal Bose gas system when it is disturbed by a nonvanishing external field, that breaks the symmetry $U(1)$. The question is: what is, in this case, the nature of such a condensation?

2. Basic Notions

2.1. Grand canonical and canonical ensembles. Let $\hat{H}_l$ be a self-adjoint operator on the Hilbert space $\mathcal{F}_B$ (Fock space), representing the energy operator of a Bose particle system. Let $\beta = T^{-1}$ and $\mu \in \mathbb{R}$ be the inverse temperature and the so-called chemical potential, respectively.

$\hat{H}_l(\mu)$ is defined as $\hat{H}_l(\mu) := \hat{H}_l - \mu \hat{N}$, where $\hat{N}$ is the total number operator given by $\hat{N} = \sum_p \hat{n}_p$, being $\hat{n}_p = \hat{a}^\dagger_p \hat{a}_p$ the number operator associated to the $p-$ mode.
The operators $\hat{a}_p$, $\hat{a}_p^\dagger$ defined on the Fock space, well-known as creations and annihilation operators, respectively, satisfies the commutation rules:

$$[\hat{a}_p, \hat{a}_q^\dagger] = \delta_{pq} I,$$

being $\delta_{pq}$ the kronecker delta and $I$ the identity operator.

$\hat{H}_l$ can be decomposed as the following sum: $\hat{H}_l = \hat{H}^0_l + \hat{H}_l^I$, where $\hat{H}^0_l = \sum_p \lambda_l(p) \hat{a}_p^\dagger \hat{a}_p$ and $\hat{H}^I_l = \sum_{p,q,r,s} U_{p,q,r,s} \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_r \hat{a}_s$ are the second quantizations of the laplacian operator and of the interaction $U$, respectively, both defined on the region of confinement of particles $\Lambda \subset \mathbb{R}^d$, with $d \in \mathbb{N}$.

We shall assume in this work, periodic boundary conditions. In this case all the subscripts $p$ belong to the set $\Lambda^* (\text{dual of } \Lambda)$ defined as $\Lambda^*_l = \{ p = (p_1, \ldots, p_d) \in \mathbb{R}^d : p_\alpha = 2\pi n_\alpha/l, n_\alpha \in \mathbb{Z}, \alpha = 1, 2, \ldots, d \}$, and $\lambda_l(p) = p^2/2$.

With these definitions, at finite volume, it is possible to introduce the grand canonical partition function $\Xi_l(\beta, \mu)$ and the pressure $p_l(\beta, \mu)$:

$$\Xi_l(\beta, \mu) := \text{Tr}_{\mathcal{F}_B} \exp \left(-\beta \hat{H}_l(\mu) \right), \quad p_l(\beta, \mu) := \frac{1}{\beta V_l} \ln \Xi_l(\beta, \mu);$$

the canonical partition function $Z_{N,l}(\beta, \rho)$ and the free energy $f_l(\beta, \rho_l)$, where $\rho_l = \frac{N}{V_l}$:

$$Z_{N,l}(\beta, \rho) := \text{Tr}_{\mathcal{H}^{(N)}_B} \exp (-\beta \hat{H}^{(N)}_l), \quad f_l(\beta, \rho_l) := -\frac{1}{\beta V_l} \ln Z_{N,l}(\beta, \rho),$$

and, finally, the Gibbs states in the grand canonical ensemble and in the canonical ensemble:

$$\langle \cdot \rangle_{\hat{H}(\mu)} = \Xi^{-1}(\beta, \mu) \text{Tr}_{\mathcal{F}_B} \cdot \exp (-\beta \hat{H}(\mu)), \quad \langle \cdot \rangle_{\hat{H}^{(N)}_l} = Z_{N,l}(\beta, \rho) \text{Tr}_{\mathcal{H}^{(N)}_B} \cdot \exp (-\beta \hat{H}^{(N)}_l),$$

respectively.

The limit free energy $f(\beta, \rho)$ and the limit pressure $p(\beta, \mu)$ are defined as:

$$f(\beta, \rho) := \lim_{N_l, V_l \to \infty} f_l(\beta, \rho_l), \quad \text{assuming that } \lim_{N_l, V_l \to \infty} \rho_l = \rho = \text{constant},$$

and

$$p(\beta, \mu) := \lim_{V_l \to \infty} p_l(\beta, \mu), \quad \text{with}$$

$$\lim_{V_l \to \infty} \left( \frac{\hat{N}}{V_l} \right)_{\hat{H}(\mu)} = \lim_{V_l \to \infty} \rho_l = \rho(\mu) = \text{constant}.$$
On the other hand, stable systems are defined as those for which there exists $\mu_* \in \mathbb{R}$ such that only for $\mu \in (-\infty, \mu_*]$, $p(\beta, \mu) < \infty$, while superstable systems satisfy $p(\beta, \mu) < \infty$ for all values of $\mu$. Finally, if the following inequality (in the sense of operators)

$$\hat{H}_l' \geq -\frac{C_2}{V_l} \hat{N} + \frac{C_1}{V_l} \hat{N}^2$$

holds, the system is superstable.

2.2. Types of BEC.

- Condensation of type I corresponds to a macroscopic occupation of a finite number of states. Thus, a macroscopic occupation of the ground state, or traditional Bose-Einstein condensation, is given by the fulfillment of the condition

$$\lim_{V_l \to \infty} \left\langle \frac{\hat{n}_0}{V_l} \right\rangle_{\hat{H}(\mu)} = \rho_0 > 0.$$  

For the latter, in the condensed-uncondensed phase transition the appropriate order parameter is $\rho_0 = \rho - \rho_c$, being $\rho_c$ a critical density.

- Condensation of type II holds when there exists an infinite number of states macroscopically occupied.

- Condensation of the type III holds when there are not macroscopically occupied states but the following condition holds:

$$\lim_{\delta \to 0^+} \lim_{V_l \to \infty} \frac{1}{V_l} \sum_{p \in \Lambda^* : \lambda(p) < \delta} \langle \hat{n}_p \rangle_{\hat{H}(\mu)} > 0.$$  

The third type of Bose condensation, denominated generalized BEC (GBEC), was introduced by M. Girardeau in 1960 [1].

GBEC is more robust that the other kinds of condensation in the sense that it is independent of the shape of the confining region. Indeed, in the case of the free Bose gas, it always occurs for particle density values larger than a critical one.

These kinds of critical phenomena are in agreement with the standard phase transitions theory that identifies critical points with the emergence of singularities in the thermodynamic functions in the thermodynamic limit.

However, there is a fourth type of condensation independent on temperature and, for this reason, called non conventional. It is in the study of this phenomenon that we are interested in this work.
3. BEC and spontaneous symmetry breaking (SSB)

The standard strategy devoted to associate symmetry breaking with certain phase transition consists in introducing a small term on the original energy operator, preserving its self-adjointness but eliminating the symmetry corresponding to some conservation law.

Thus, in the case of Bose systems, it is possible to break the global $U(1)$ symmetry by adding the extra term $-\sqrt{V} \nu (\hat{a}_0 e^{-i\varphi} + \hat{a}^\dagger_0 e^{i\varphi})$ to the original energy operator $\hat{H}_l(\mu) = \hat{H}_l - \mu \hat{N}$, being $\hat{N}$ the total number operator, obtaining the new Hamiltonian $\hat{H}_{l,\nu,\varphi}(\mu) = \hat{H}_l(\mu) - \sqrt{V} \nu (\hat{a}_0 e^{-i\varphi} + \hat{a}^\dagger_0 e^{i\varphi})$ for which $[\hat{H}_{l,\nu,\varphi}(\mu), \hat{N}] \neq 0$, being $\nu \in \mathbb{R}^+$, $\varphi \in [0, 2\pi)$. In this case, the typical selection rules (degeneracy of the thermal averages),

$$\left| \left\langle \hat{a}^\dagger_0 \right| \hat{H}_{l,\nu,\varphi}(\mu) \right| = \left| \left\langle \hat{a}_0 \right| \hat{H}_l(\mu) \right| = 0,$$

being $\langle - \rangle_{\hat{H}_l(\mu)}$ the thermal average associated to $\hat{H}_l(\mu)$, at finite volume $V = l^d$, $d \in \mathbb{N}$, $d \geq 3$, do not hold anymore, i.e.:

$$\left| \left\langle \hat{a}^\dagger_0 \right| \hat{H}_{l,\nu,\varphi}(\mu) \right| = \left| \left\langle \hat{a}_0 \right| \hat{H}_{l,\nu,\varphi}(\mu) \right| = \sqrt{V} \eta_{l,\nu,\varphi} \neq 0,$$

where $\langle - \rangle_{\hat{H}_{l,\nu,\varphi}(\mu)}$ is the thermal average corresponding to the perturbed operator $\hat{H}_{l,\nu,\varphi}(\mu)$.

For a Bose system undergoing BEC, in the thermodynamic limit, we have

$$\lim_{\nu \to 0} \lim_{V \to \infty} \eta^2_l = \left\{ \begin{array}{ll} \rho_0 \neq 0 & \text{if } \rho > \rho_c \\ 0 & \text{if } \rho \leq \rho_c \end{array} \right.$$

being $\rho_c$ a critical density of particles.

From a mathematical point of view, for $\rho \leq \rho_c$, in the uncondensed phase, it is possible to make a limit exchange, obtaining:

$$\lim_{\nu \to 0} \lim_{V \to \infty} \eta_{l,\nu,\varphi} = \lim_{V \to \infty} \lim_{\nu \to 0} \eta_{l,\nu,\varphi} = 0.$$

However, for $\rho > \rho_c$,

$$\lim_{\nu \to 0} \lim_{V \to \infty} \eta_{l,\nu,\varphi} \neq \lim_{V \to \infty} \lim_{\nu \to 0} \eta_{l,\nu,\varphi}$$

In this context the limit thermal averages defined as

$$\ll \cdot, \cdot \gg := \lim_{h \to 0} \lim_{V \to \infty} \langle - \rangle_{\hat{H}_{l,\nu,\varphi}(\mu)}$$

have been denominated Bogoliubov’s quasiaverages or anomalous averages. In fact, this notion was introduced for the first time by N.N.Bogoliubov [12].

Thus, the degeneracy of regular averages, produced by the presence of additive conservation laws (or equivalently, by the invariance of the
Hamiltonian with respect to certain groups of transformations) is reflected by the dependence of quasi averages on the extra infinitesimal term. In this sense Bogoliubov claimed that the latter are more “physical” than the regular averages \[12\]. However this procedure, in some cases, has been applied without having necessarily a clear physical meaning.

We are assuming that other types of degeneracy do not exist and, thus, the introduction of the term [...], is sufficient for the removal of the degeneracy. (N. N. Bogoliubov \[12\])

Let \(\hat{\rho}_{0,l} = V^{-1} \hat{a}_{0}^\dagger \hat{a}_{0}, \hat{\eta}_{l} = V^{-\frac{1}{2}} \hat{b}_{0}\). In the case of the free Bose gas, for

\[
\hat{H}_{0,l,\nu,\varphi} = \sum_{\mathbf{p}} \lambda_{l}(\mathbf{p}) \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} - \sqrt{V} \nu \left( \hat{a}_{0} e^{-i\varphi} + \hat{a}_{0}^\dagger e^{i\varphi} \right),
\]

the following limits

\[
\lim_{\nu \to 0^+} \lim_{V \to \infty} \left\langle \hat{\rho}_{0,l} \right\rangle_{\hat{H}_{0,l,\nu,\varphi}(\mu)} = \rho_{0}, \quad \lim_{\nu \to 0^+} \lim_{V \to \infty} \left\langle \hat{\eta}_{l} \right\rangle_{\hat{H}_{0,l,\nu,\varphi}(\mu)} = \sqrt{\rho_{0} e^{i\varphi}}
\]

hold \[12\]. In other words:

\[
\lim_{\nu \to 0^+} \lim_{V \to \infty} \left\langle \hat{\rho}_{0,l} \right\rangle_{\hat{H}_{0,l,\nu,\varphi}(\mu)} = \lim_{\mu \to 0^+} \lim_{\nu \to \infty} | \langle \hat{\eta}_{l} \rangle_{\hat{H}_{0,l,\nu,\varphi}(\mu)} |^2 = \rho_{0}.
\]

In what follows, in order to simplify the notation, we will omit the angle \(\varphi\) in the subscripts of any mathematical expression (thermal averages, Hamiltonians, etc.).

Following step by step the strategy developed by Bogoliubov, let

\[
\hat{a}_{0} = -\frac{\nu}{\mu} e^{i\varphi} \sqrt{V} + \hat{b}_{0}, \quad \hat{a}_{0}^\dagger = -\frac{\nu}{\mu} e^{-i\varphi} \sqrt{V} + \hat{b}_{0}^\dagger.
\]

Substituting these operators in the original energy operator, we obtain:

\[
\hat{H}_{0,l,\nu}(\mu) = -\mu \hat{b}_{0}^\dagger \hat{b}_{0} + \sum_{\mathbf{p} \in \Lambda_{l} \backslash \{0\}} \left( \frac{\mathbf{p}^2}{2} - \mu \right) \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{\nu^2 V}{\mu}.
\]

In that follows, it will be assumed that,

\[
\mu = \mu_{*} = -\frac{\nu}{\sqrt{\rho_{0}}},
\]

where \(\rho_{0}\) is a strictly positive real constant.

Clearly,

\[
\left\langle \hat{\eta}_{0} \right\rangle_{\hat{H}_{0,l,\nu}(\mu_{*})} = \left\langle \hat{b}_{0} \right\rangle_{\hat{H}_{0,l,\nu}(\mu_{*})} = 0.
\]

This implies that:

\[
\left\langle \hat{a}_{0}^\dagger \right\rangle_{\hat{H}_{0,l,\nu}(\mu_{*})} = e^{-i\varphi} \sqrt{V \rho_{0}}, \quad \left\langle \hat{a}_{0} \right\rangle_{\hat{H}_{0,l,\nu}(\mu_{*})} = e^{i\varphi} \sqrt{V \rho_{0}}
\]
Besides,

\[ \langle \hat{b}_0^\dagger \hat{b}_0 \rangle_{\tilde{H}_{l,\nu}^0(\mu_*)} = (\exp \beta (-\mu_*) - 1)^{-1}, \quad \langle \hat{n}_p \rangle_{\tilde{H}_{l,\nu}^0(\mu_*)} = \left( \exp \beta \left( \frac{p^2}{2} - \mu_* \right) - 1 \right)^{-1}. \]

Then, we define

\[ \rho_{c,I}(\beta, \mu_*) = \frac{1}{V} \sum_{\mathbf{p} \in \Lambda^* \setminus \{0\}} \left( \exp \beta \left( \frac{p^2}{2} - \mu_* \right) - 1 \right)^{-1}. \]

Passing to the thermodynamic limit, we get:

\[ \rho_c(\beta, \mu_*) = \frac{1}{(2\pi)^d} \int \left( \exp \beta \left( \frac{p^2}{2} - \mu_* \right) - 1 \right)^{-1} d^3 \mathbf{p}. \]

On the other hand,

\[ \langle \hat{b}_0^\dagger \hat{b}_0 \rangle_{\tilde{H}_{l,\nu}^0(\mu_*)} = \left\langle \left( \frac{\hat{a}_0^\dagger}{\sqrt{V}} - \sqrt{\rho_0} e^{-i\phi} \right) \left( \frac{\hat{a}_0}{\sqrt{V}} - \sqrt{\rho_0} e^{i\phi} \right) \right\rangle_{\tilde{H}_{l,\nu}^0(\mu_*)} \]

\[ = \lim_{V \to \infty} \frac{1}{V} \left( \exp \beta (-\mu_*) - 1 \right)^{-1} = 0. \]

This leads to,

\[ \lim_{V \to \infty} \frac{1}{V} \left( \langle \hat{a}_0^\dagger \hat{a}_0 \rangle_{\tilde{H}_{l,\nu}^0(\mu_*)} - \sqrt{\rho_0} e^{-i\phi} \langle \hat{a}_0^\dagger \rangle_{\tilde{H}_{l,\nu}^0(\mu_*)} \right) \]

\[ - \sqrt{\rho_0} e^{i\phi} \langle \hat{a}_0 \rangle_{\tilde{H}_{l,\nu}^0(\mu_*)} + \rho_0 = 0. \]

\[ \lim_{V \to \infty} \frac{1}{V} \langle \hat{a}_0^\dagger \hat{a}_0 \rangle_{\tilde{H}_{l,\nu}^0(\mu_*)} = \lim_{V \to \infty} \left| \left( \frac{\hat{a}_0}{\sqrt{V}} \right)_{\tilde{H}_{l,\nu}^0(\mu_*)} \right|^2 = \rho_0. \]

The role of the coupled external source, which is very unique, should not be exaggerated from a physical point of view. It is rather the superposition or transition from the ground state to a coherent state that best reflects, in that sense, the spontaneous break of symmetry. Such superposition disappears when the thermodynamic limit is reached. There, both states (with the same energy) the fundamental and the coherent - translation of the first one- become orthogonal between them. Mathematically speaking, unlike the finite systems, in the thermodynamic limit we have infinite inequivalent representations of the Bose commutation rules or, in other words, infinite representations of the broken symmetry. In this sense, the Bogoliubov’s approach consists in explicitly fixing one of them.
4. Non linear perturbation of the Ideal Bose Gas

Our purpose, in this work, is to replace in eq.(1) the external source $-\sqrt{V} \nu (\hat{a}_0 e^{-i\varphi} + \hat{a}_0^\dagger e^{i\varphi})$ by $-2\nu \sqrt{V} \sqrt{\hat{n}_0 + 1}$. In this case, the latter expression is expandable in powers series of the operator $\hat{n}_0$ (spectral theorem). This substitution is motivated by the fact that:

$$\hat{a}_0 \varphi_n = \sqrt{n_0} \varphi_{n-1}, \quad \hat{a}_0^\dagger \varphi_n = \sqrt{n_0 + 1} \varphi_{n+1},$$

where $\{\varphi_n\}$ is a set of orthonormal eigenfunctions of $\hat{n}_0$.

Thus, in this section we consider a model of a Bose gas whose energy operator corresponds to the sum of the Hamiltonian of the free Bose gas with a nonlinear perturbation represented by the square root of the number operator associated to the zero mode.

$$\hat{H}^{app}_{\mu,\nu} = \sum_{p \in \Lambda^*} \lambda(p) \hat{a}_p^\dagger \hat{a}_p - 2\nu \sqrt{V} \sqrt{\hat{n}_0 + 1} - \mu \hat{N}, \quad \nu > 0,$$  

The Hamiltonian given by eq.(2) represents a stable model defined in the domain $\mathcal{D} = \{ (\beta, \mu) : \beta > 0, \mu < 0 \}$.

On the other hand, let $\hat{H}^{0}_{\mu,\nu}(\mu)$ is defined as:

$$\hat{H}^{0}_{\mu,\nu}(\mu) = \hat{H}^{0}_l(\mu) - \nu \sqrt{V} (e^{i\varphi} \hat{a}_0^\dagger + e^{-i\varphi} \hat{a}_0), \quad \nu > 0.$$  

Note that $[\hat{H}^{app}_{\mu,\nu}(\mu), \hat{N}] = 0$, i.e. the energy operator given by eq.(2) preserves the $U(1)$ symmetry. However, $[\hat{H}^{0}_{\mu,\nu}(\mu), \hat{N}] \neq 0$, i.e., the continuous gauge symmetry associated with the $U(1)$ group is broken by the external field $-\nu \sqrt{V} (e^{i\varphi} \hat{a}_0^\dagger + e^{-i\varphi} \hat{a}_0)$.

In the next section a strong connection between the critical behaviour of both models, in the thermodynamic limit, will be established.

The main purpose of this work is to determine explicit expressions for the limit pressures of the model given by eq.(2) in the framework of the so called Laplace principle (see Appendix) and the Large Deviations Method based in two theorems proved by S. R. S. Varadhan [2]. Moreover, it shall be proven the existence of a phase characterizes by the emergence of non conventional Bose-Einstein condensation, i.e., the existence of an independent on temperature condensate.
4.1. Limit pressure and nonconventional condensation.

**Theorem 4.1.** For \((\beta, \mu) \in \mathcal{D}, \nu > 0,\) in the thermodynamic limit,

\[
p^{\text{app}}(\beta, \mu, \nu) = -\frac{\nu^2}{\mu} + p^{\text{id}}(\beta, \mu),
\]

where \(p^{\text{app}}(\beta, \mu, \nu), p^{\text{id}}(\beta, \mu)\) are the the limit pressures of the system whose Hamiltonian is given by the energy operator of eq. (2) and the energy operator given by eq. (1), but excluding the mode 0, respectively.

**Proof.** Let,

\[
\hat{H}_{\text{app}}^{\text{app}} = \sum_{\mathbf{p} \in \Lambda^*} \lambda(\mathbf{p}) \hat{n}_\mathbf{p} - 2\sqrt{V} \nu \sqrt{\hat{n}_0 + 1}.
\]

Note that the function \(h(x) = ax + b\sqrt{x+c}, a, b \in \mathbb{R}, x \in [0, \infty), c > 0\) is either an infinitely increasing or an infinitely decreasing mapping on \([0, \infty)\) except the case \(a < 0, b > 0\).

Let \(\{g_l\}\) be a sequence of functions defined on \([0, \infty)\) given as,

\[
g_l(x) = (\mu - \lambda(0))x + 2\nu \sqrt{x + \frac{1}{V}}, x \in [0, \infty),
\]

whose first and second derivatives are,

\[
g'_l(x) = (\mu - \lambda(0)) + \nu \left(x + \frac{1}{V}\right)^{-1/2},
\]

\[
g''_l(x) = -\frac{\nu}{2} \left(x + \frac{1}{V}\right)^{-3/2} < 0,
\]

respectively.

From these facts, it follows that \(g_l(x)\) is a concave function attaining its global maximum at \(x^*_l = \left(\frac{\nu}{\lambda((0) - \mu)}\right)^2 - \frac{1}{V}\), for a large enough value of \(V\) such that \(x^*_l \geq 0\) and

\[
\lim_{V \to \infty} \sup_{x \in [0, \infty)} g_l(x) = \lim_{V \to \infty} \sup_{x \in [0, \infty)} g_l(x^*_l) = -\frac{\nu^2}{\mu},
\]

being \(\mu < 0, \lambda(0) = 0\).

Use will be made of the so-called large deviations method, based on the Laplace principle, for obtaining a closed analytical expression for \(p^{\text{app}}(\beta, \mu, \nu)\). Since \(\hat{H}_{\text{app}}^{\text{app}}(\mu)\) is a diagonal operator with respect to the number operators, the finite pressure can be written as,
\[ p^\text{app}_t(\beta, \mu, \nu) = \frac{1}{\beta V} \ln \text{Tr}_{F_B} \exp\{-\beta \hat{H}^\text{app}_{t,\mu}(\mu)\} \]

\[ p^\text{app}_t(\beta, \mu, \nu) = \frac{1}{\beta V} \ln \sum_{n_0=0}^{\infty} \exp \beta \{(\mu - \lambda(0))n_0 + 2\sqrt{V} \nu \sqrt{n_0 + 1}\} \]

\[ + \frac{1}{\beta V} \ln \sum_{p \in \lambda^\ast \setminus \{0\}, n_p} \exp \beta \{(\mu - \lambda(p))n_p\}. \]

Noting, that

\[ p^\text{app}_0(\beta, \mu, \nu) = \frac{1}{\beta V} \ln \sum_{n_0=0}^{\infty} \exp \beta \{(\mu - \lambda(0))n_0 + 2\sqrt{V} \nu \sqrt{n_0 + 1}\} \]

\[ = \frac{1}{\beta V} \ln \sum_{n_0=0}^{\infty} \exp \beta V \left\{ (\mu - \lambda(0))\frac{n_0}{V} + 2\nu \sqrt{\frac{n_0}{V} + \frac{1}{V}} \right\} \]

\[ = \frac{1}{\beta V} \ln \sum_{n_0=0}^{\infty} \exp \left\{ \beta V g_t \left( \frac{n_0}{V} \right) \right\}. \]

It is not hard to see that \( \{p^\text{app}_t(\beta, \mu, \nu)\} \) is a sequence of Darboux sums, then, in the thermodynamic limit the Laplace principle leads to the following expression,

\[ p^\text{app}(\beta, \mu, \nu) = -\frac{\nu^2}{\mu} + p^\text{id}'(\beta, \mu). \]

\[ \square \]

**Theorem 4.2.** For \((\beta, \mu) \in \mathcal{D}, \nu > 0\), in the thermodynamic limit, the Bose Gas with Hamiltonian given by eq.(2) undergoes non conventional condensation if and only if, the ideal gas whose energy operator is given by eq.(1) also displays independent on temperature condensation. Moreover,

\[ p^\text{id}(\beta, \mu, \nu) = p^\text{app}(\beta, \mu, \nu), \]

and the amount of condensate satisfies:

\[ \rho^\text{app}_0(\mu, \nu) = p^\text{id}_0(\mu, \nu) = \frac{\nu^2}{\mu^2} \]
Proof. Note that:

\[ p_{id}^l(\beta, \mu, \nu) = \frac{1}{\beta V} \ln \left( 1 - e^{\beta \mu} \right) - \frac{\nu^2}{\mu} + p_l^{id'}(\beta, \mu). \]  

(7)

From this it follows that:

\[ p_{id}^l(\beta, \mu, \nu) - p_{app}^l(\beta, \mu, \nu) = \frac{1}{\beta V} \ln \left( 1 - e^{\beta \mu} \right) + p_l^{id'}(\beta, \mu) - p_l^{id'}(\beta, \mu). \]

Thus, in the thermodynamic limit, for fixed values of \( \beta \) and \( \mu < 0 \),

\[ p_{id}^l(\beta, \mu, \nu) = p_{app}^l(\beta, \mu, \nu). \]

On the other hand, using the Griffiths Lemma \[13\]

\[ \frac{\partial}{\partial \mu} p_{id}^l(\beta, \mu, \nu) - \frac{\partial}{\partial \mu} p_{app}^l(\beta, \mu, \nu) = \frac{1}{V} \left( \frac{1}{e^{-\beta \mu} - 1} \right). \]

In this case,

\[ p_{app}^l(\beta, \mu, \nu) = \frac{\nu^2}{\mu^2} + p_c(\beta, \mu), \]

(8)

\[ p_{id}^l(\beta, \mu, \nu) = \frac{\nu^2}{\mu^2} + \frac{1}{V} \left( \frac{1}{e^{-\beta \mu} - 1} \right) + p_{c,l}(\beta, \mu). \]

(9)

From eqs.(8) and (9) we get:

\[ \rho_{0}^{app}(\beta, \nu, \nu) = \rho_{app}^l(\beta, \mu, \nu) - p_c(\beta, \mu), \]

(10)

\[ \rho_{0}^{id}(\beta, \mu, \nu) = \rho_{id}^l(\beta, \mu, \nu) - p_{c,l}(\beta, \mu), \]

(11)

being, as before:

\[ p_{c,l}(\beta, \mu) = \frac{1}{V} \sum_{\mathbf{p} \in \Lambda^* \setminus \{0\}} \left( \exp \beta \left( \frac{\mathbf{p}^2}{2} - \mu \right) - 1 \right)^{-1}, \]

\[ p_c(\beta, \mu) = \frac{1}{(2\pi)^d} \int \left( \exp \beta \left( \frac{\mathbf{p}^2}{2} - \mu \right) - 1 \right)^{-1} d^3 \mathbf{p}. \]

Since,
\[
\lim_{V \to \infty} \frac{1}{\beta V} \left( \frac{1}{e^{-\beta \mu} - 1} \right) = 0,
\]

for the fixed parameters \((\beta, \mu) \in D, \nu > 0\), and from the expressions in eqs. (10) and (11), we have that both systems, simultaneously, undergo non conventional condensation. Moreover, the amount of condensate is given as:

\[
\rho_{0}^{\text{app}}(\mu, \nu) = \rho_{0}^{\text{id}}(\mu, \nu) = \frac{\nu^{2}}{\mu^{2}}.
\]

The Bogoliubov’s approach considers a chemical potential \(\mu_{*} \) such that \(\mu_{*} = -\nu \sqrt{\rho_{0}}\), being \(\rho_{0}\) a real and strictly positive constant.

On the other hand, unlike the system given by the Hamiltonian in eq. (1), the system whose energy operator is represented by eq. (2) preserves the \(U(1)\) symmetry.

If \(\rho_{0}^{\text{id}}(\beta, \mu, \nu) = \rho_{0} = \text{constant} \neq 0, \rho_{0} > 0, \mu_{l} < 0\), we have that:

\[
\rho_{0} \sim \frac{1}{V (-\mu_{l} + \mu_{l}^{2}/2)} + \frac{\nu^{2}}{\mu_{l}^{2}}.
\]

Thus, for values of \(\mu_{l}\) in a small neighborhood of zero,

\[
\beta V \rho_{0} \sim -\frac{1}{\mu_{l}} + \frac{\beta V \nu^{2}}{\mu_{l}^{2}}.
\]

By solving the second order equation in \(\mu_{l}\), we obtain:

\[
\mu_{l} \sim -\frac{1}{2\beta V \rho_{0}} \left( 1 + \sqrt{1 + (2\beta V \nu)^{2} \rho_{0}} \right).
\]

Finally, taking the thermodynamic limit:

\[
\lim_{V \to \infty} \mu_{l} = \mu^{*} = -\frac{\nu}{\sqrt{\rho_{0}}}.
\]

For the free Bose gas, this result means that in the domain \(D\), in spite of that the chemical potential \(\mu_{l}\) depends on the inverse temperature \(\beta\) at finite volume, in the thermodynamic limit \(\mu\) depends only on the fixed parameters \(\rho_{0}, \nu\).
4.2. **Full diagonal models.** Let \( \hat{H}^{FD}_l(\mu) \) be the energy operator defined as:

\[
\hat{H}^{FD}_l(\mu) = \hat{H}^0_l + \frac{a}{2V} \left( \hat{N}^2 - \hat{N} \right) + \frac{1}{2V} \sum_{p,p'} v(p - p') \hat{n}_p \hat{n}_{p'}.
\]

(12)

\( \hat{H}^{FD}_l(\mu) \) belongs to a class of energy operators so-called **full diagonal Bose Hamiltonians.** Clearly, \( \hat{H}^{FD}_l(\mu) \) satisfies the commutation rule

\[
[\hat{H}^{FD}_l(\mu), \hat{N}] = 0.
\]

For example, \( \hat{H}^0_l \) is a full diagonal mode. If \( a > 0 \), and 
\( v(p - p') \geq 0 \) these are superstable systems, i.e., their limit pressures exist for all real value of \( \mu \).

Let \( \hat{H}^{FD}_{l,\nu}(\mu) \), \( \hat{H}^{FD,app}_{l,\nu}(\mu) \) be the following operators:

\[
\hat{H}^{FD}_{l,\nu}(\mu) = \hat{H}^{FD}_l(\mu) - \nu \sqrt{V} (\hat{a}^\dagger_0 + \hat{a}_0), \quad \nu > 0,
\]

(13)

\[
\hat{H}^{FD,app}_{l,\nu}(\mu) = \hat{H}^{FD}_l(\mu) - 2\nu \sqrt{V} \sqrt{\hat{n}_0 + 1}, \quad \nu > 0.
\]

(14)

In this case, \([\hat{H}^{FD,app}_{l,\nu}(\mu), \hat{N}] = 0\), and \([\hat{H}^{FD}_{l,\nu}(\mu), \hat{N}] \neq 0\).

**Theorem 4.3.**

\[
p_{FD}(\beta, \mu, \nu) = p_{FD,app}(\beta, \mu, \nu).
\]

(15)

**Proof.** For this kind of models in ref. [11] it has been proved that:

\[
\lim_{V \to \infty} \left\langle \frac{\hat{a}^\dagger_0}{\sqrt{V}} \right\rangle_{\hat{H}^{FD}_{l,\nu}(\mu)} = \lim_{V \to \infty} \left\langle \frac{\hat{a}_0}{\sqrt{V}} \right\rangle_{\hat{H}^{FD}_{l,\nu}(\mu)}
\]

\[
= \text{sgn} \nu \lim_{V \to \infty} \sqrt{V^{-1}} \left\langle \hat{a}^\dagger_0 \hat{a}_0 \right\rangle_{\hat{H}^{FD}_{l,\nu}(\mu)}.
\]

(16)

In our case \( \text{sgn} \nu = + \).

Let define \( \delta_{p_l}, \) and \( \delta_H \) as

\[
\delta_{p_l} = p_{FD}(\beta, \mu, \nu) - p_{FD,app}(\beta, \mu, \nu),
\]

\[
\delta_H = \hat{H}^{FD,app}_{l,\nu}(\mu) - \hat{H}^{FD}_{l,\nu}(\mu) = \nu \sqrt{V} \left( 2\sqrt{\hat{n}_0 + 1} - (\hat{a}^\dagger_0 + \hat{a}_0) \right),
\]

respectively.
Note that $\hat{H}_{FD,app}(\mu)$ preserves the $U(1)$ symmetry. This fact and the left hand side Bogoliubov’s inequality (see the Appendix) lead to:

$$\delta p_l \geq 2\nu \left\langle \sqrt{\hat{\rho}_{0,l} + \frac{1}{V}} \right\rangle_{\hat{H}_{l,\nu}^{FD,app}(\mu)} \geq 0.$$ 

Moreover, in the thermodynamic limit we get,

$$\lim_{V \to \infty} \delta p_l = \geq 2\nu \lim_{V \to \infty} \left\langle \sqrt{\hat{\rho}_{0,l}} \right\rangle_{\hat{H}_{l,\nu}^{FD,app}(\mu)} \geq 0. \quad (17)$$

From the right hand Bogoliubov’s inequality and the Jensen inequality (see Appendix) we obtain:

$$\delta p_l \leq \frac{\nu}{\sqrt{V}} \left\langle 2\sqrt{\hat{n}_0 + 1} - (\hat{a}_0^\dagger + \hat{a}_0) \right\rangle_{\hat{H}_{l,\nu}^{FD}(\mu)} \leq \nu \left( 2\sqrt{\hat{\rho}_{0,l} + \frac{1}{V}} - \left\langle \frac{\hat{a}_0^\dagger}{\sqrt{V}} \right\rangle_{\hat{H}_{l,\nu}^{FD}(\mu)} - \left\langle \frac{\hat{a}_0}{\sqrt{V}} \right\rangle_{\hat{H}_{l,\nu}^{FD}(\mu)} \right). \quad (18)$$

Finally, taking the limit $V \to \infty$ and using the expressions in eq. (16) and the inequalities (17) and (18), we obtain: $0 \leq \lim_{V \to \infty} \delta p_l \leq 0$. Hence

$$p^{FD}(\beta, \mu, \nu) = p^{FD,app}(\beta, \mu, \nu).$$

\[\square\]

A well-known example of a full diagonal Hamiltonian is associated to the mean field model, whose energy operator with an additional term broken the $U(1)$ symmetry, is given by the expression:

$$\hat{H}_{l,\nu}^{MF} = \hat{H}^0 + \frac{a}{2V} \left( \hat{N}^2 - \hat{N} \right) - \nu \sqrt{V} (\hat{a}_0^\dagger + \hat{a}_0),$$

where $a > 0$, $V$ is the volume of the region enclosing the particle system and $\nu \in \mathbb{R}$.

In this case, the operator $\hat{H}_{l,\nu}^{MF,app}$ has the following form:

$$\hat{H}_{l,\nu}^{MF,app} = \hat{H}^0 + \frac{a}{2V} \left( \hat{N}^2 - \hat{N} \right) - \nu \sqrt{V} \sqrt{\hat{n}_0 + 1}.$$ 

As before, $\nu > 0$. 

4.3. Conclusions.

a. For fixed parameters $\mu < 0$, $\nu > 0$, the pressures and the density of particles in the condensates of the systems whose operators are given by eqs. (1) and (2), in the thermodynamic limit, coincide. Thus,

$$p_{\text{app}}(\beta, \mu, \nu) = p_{\text{id}}(\beta, \mu, \nu) = -\frac{\nu^2}{\mu} + p_{\text{id}}'(\beta, \mu),$$

$$\rho_0^{\text{id}}(\mu, \nu) = \rho_0^{\text{app}}(\mu, \nu) = \frac{\nu^2}{\mu^2},$$

i.e., both models are equivalent in a thermodynamic sense and they undergo, simultaneously, non conventional BEC in $\mathcal{D}$.

b. The full diagonal models, with coupled external sources given in eqs. (13) and (14), are thermodynamically equivalent.

c. Despite what has been said in a) and b), the external source $-2\nu\sqrt{V}\sqrt{n_0} + 1$ does not remove the degeneracy of the regular averages.

5. Appendix

5.1. Laplace principle.

**Proposition 5.1.** Let $G : I \to \mathbb{R}$ be a continuous function defined on the interval $I$, and bounded above by the constant $M$ for all $x \in I$. It is assumed that there exists $\alpha > 0$ such that for $|x|$ large enough,

$$G(x) < \alpha |x|.$$  

Then,

$$\lim_{N \to \infty} \frac{1}{N} \ln \left( \int_I e^{NG(x)} dx \right) = \sup_{x \in I} \{G(x)\}. \quad (19)$$

5.2. Bogoliubov’s Inequalities. Let $\hat{H}_{a,l}$ and $\hat{H}_{b,l}$ be selfadjoint operators defined on $\mathcal{D} \subset \mathcal{F}_B$. $p_{a,l}(\beta, \mu)$, $p_{b,l}(\beta, \mu)$ represent the grand canonical pressures and the free canonical energies corresponding to the operators $\hat{H}_{a,l}$, $\hat{H}_{b,l}$. In this case the following well known Bogoliubov inequalities,

$$\langle \frac{\hat{H}_{a,l}(\mu) - \hat{H}_{b,l}(\mu)}{V} \rangle_{\hat{H}_{a,l}(\mu)} \leq p_{b,l}(\beta, \mu) - p_{a,l}(\beta, \mu) \leq \langle \frac{\hat{H}_{a,l}(\mu) - \hat{H}_{b,l}(\mu)}{V} \rangle_{\hat{H}_{b,l}(\mu)}, \quad (20)$$
hold, where \( \langle - \rangle_{\hat{H}_{a,l}(\mu)} \), \( \langle - \rangle_{\hat{H}_{b,l}(\mu)} \) are the Gibbs states in the grand canonical ensemble associated to the Hamiltonians \( \hat{H}_{a,l}, \hat{H}_{b,l} \), respectively.

5.3. **Jensen’s Inequality.** Let \( \hat{H}_l \) be a self-adjoint operator, diagonal with respect to the number operators. Since the spectrum of \( \hat{H}_l \), coincides with the set of non negative integers, this model can be classically understood by using non negative random variables defined on a suitable probability space \( \Omega_l \).

Let \( \Omega_l \) be the countable set of sequences \( \omega = \{ \omega(p) \in \mathbb{N} : p \in \Lambda_l^* \} \subset \mathbb{N} \cup \{0\} \) satisfying

\[
\sum_{p \in \Lambda_l^*} \omega(p) < \infty.
\]

The basic random variables are the occupation numbers \( \{ n_p : j = 1, 2, ..., \} \). They are defined as the functions \( n_p : \Omega_l \to \mathbb{N} \) given as \( n_p(\omega) = \omega(p) \) for any \( \omega \in \Omega_l \). The total number of particles in the configuration \( \omega \) is denoted as \( N(\omega) \). Then the total number, excluded the zero mode is denoted as \( N'(\omega) \).

In this framework, the Gibbs state can be written by replacing \( \hat{H}_l \), by a function \( H_l : \Omega_l \to \mathbb{R} \), representing the projection of the energy operator on the occupation-number basis of the Bose Fock space.

Let \( \mathbb{P} \) be a probability defined for any \( \omega \in \Omega_l \) as

\[
\mathbb{P}[\omega] = \left[ \sum_{\omega \in \Omega_l} \exp(-\beta[H_l(\mu)](\omega)) \right]^{-1} \exp(-\beta[H_l(\mu)](\omega)).
\]

For arbitrary \( S \subset \Omega \) this implies that

\[
\mathbb{P}[S \subset \Omega_l] = \left[ \sum_{\omega \in \Omega_l} \exp(-\beta[H_l(\mu)](\omega)) \right]^{-1} \sum_{\omega \in S} \exp(-\beta[H_l(\mu)](\omega)).
\]

In this case, \( \langle \hat{X} \rangle_{\hat{H}_l(\mu)} \equiv \mathbb{E}[X] \), being \( X : \Omega_l \to \mathbb{R} \) the function corresponding to the projection of the operator \( \hat{X} \) on the occupation-number basis.

Thus, the expectation of \( X \) respect to \( \mathbb{P} \) is defined as:

\[
\mathbb{E}[X] = \sum_{\omega \in \Omega_l} X(\omega) \mathbb{P}[\omega].
\]

If \( X : \mathbb{R} \to \mathbb{R} \) is a concave function, the following Jensen’s inequality:

\[
\mathbb{E}[f(X)] \leq f(\mathbb{E}[X]),
\]
5.4. Griffiths Lemma.

**Lemma 5.2.** (Griffiths [13]) Let \( \{g_n : I \to \mathbb{R}, I \equiv (a, b) \subset \mathbb{R}\}_{n \in \mathbb{N}} \) be a sequence of convex functions on \( I \) with a pointwise limit \( g(x) \), which, of course is convex. Let \( G^+_n(x) \) [resp. \( G^-_n(x) \)] be the right (resp. left) derivatives of \( g_n(x) \), and similarly for \( G^+(x) \), \( G^-(x) \). Then, for all \( x \in I \),

\[
\lim_{n \to \infty} \sup_{x} G^+_n(x) \leq G^+(x), \quad \lim_{n \to \infty} \sup_{x} G^-_n(x) \geq G^-(x). \tag{25}
\]

In particular, if all the \( g_n \) and \( g \) are differentiable at some point \( x \in I \), then

\[
\lim_{n \to \infty} \frac{dg_n(x)}{dx} = \frac{dg(x)}{dx}. \tag{26}
\]

**Proof.** Fix \( x \in I \) and \( x \pm y \in I \),

\[
g_n(x + y) \geq g_n(x) + yG^+_n(x),
\]

\[
g_n(x - y) \geq g_n(x) - yG^-_n(x).
\]

Fix \( y \) and take the limit \( n \to \infty \). Then,

\[
\lim_{n \to \infty} \sup_{x} G^+_n(x) \leq y^{-1}[g(x + y) - g(y)]
\]

and similarly for \( \lim_{n \to \infty} \inf_{x} G^-_n(x) \). Now let \( y \downarrow 0 \).

\[\square\]

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