Finite direct sums of cyclic embeddings
and an application to invariant subspace varieties
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Dedicated to Professor Fred Richman

Abstract: In his 1951 book “Infinite Abelian Groups”, Kaplansky gives a
combinatorial characterization of the isomorphism types of embeddings of
a cyclic subgroup in a finite abelian group. In this paper we use partial
maps on Littlewood-Richardson tableaux to generalize this result to finite
direct sums of such embeddings. As an application to invariant subspaces of
nilpotent linear operators, we develop a criterion to decide if two irreducible
components in the representation space are in the boundary partial order.

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Richardson tableau; partial map; representation space; boundary condition.

1 Introduction

Let $\Lambda$ be a discrete valuation ring with maximal ideal generator $p$ and radical
factor field $k$. We call an embedding $(A \subset B)$ of a submodule in a finite length
$\Lambda$-module cyclic provided $A$ is a cyclic $\Lambda$-module.

The aim of this paper is twofold. First we study (finite) direct sums of cyclic
embeddings. They form an important class of algebraic objects, for a variety of
reasons:

- There is a simple combinatorial description of the isomorphism types of di-
  rect sums of cyclic embeddings: They are given in terms of partial maps on
  the Littlewood-Richardson tableau associated with the embedding (The-
  orem 5.1).

- Sometimes, every embedding is a direct sum of cyclic embeddings: In
  particular, this is the case if in an embedding $(A \subset B)$, the submodule
  $A$ is $p^2$-bounded [1]. Here, the condition that the ambient space $B$ be
  of finite length can be relaxed in two ways as it suffices that $B$ is either
  finitely generated or bounded (Corollary 5.3).

- Often, there exist direct sums of cyclic embeddings of a given type: In
  Corollary 5.4 we give a combinatorial description for which isomorphism
types of modules $A$, $B$, $C$, there exists an embedding $(A \subset B)$, with
cokernel isomorphic to $C$, which is a direct sum of cyclic embeddings.
In particular, such embeddings exist whenever $B$ and $C$ are given by
partitions $\beta$ and $\gamma$ such that $\beta \setminus \gamma$ is a horizontal strip.

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second named author).
Our second goal is to shed light on the geometry of the representation space of invariant subspace varieties. Suppose that \( \Lambda \) is the power series ring \( k[[T]] \) with coefficients in an algebraically closed field \( k \). Then an embedding \( (A \subset B) \) is just a nilpotent linear operator \( B \) together with a \( T \)-invariant subspace \( A \).

The embeddings corresponding to partition type \( (\alpha, \beta, \gamma) \) form a constructible set \( \mathbb{V}^{\beta}_{\alpha,\gamma} \) of which the irreducible components are indexed by the Littlewood-Richardson tableaux of shape \( (\alpha, \beta, \gamma) \).

We are interested in how the irreducible components are linked in the representation space \( \mathbb{V}^{\beta}_{\alpha,\gamma} \). We illustrate the main results of the second part of this paper in an example.

Example: For \( \alpha = (3, 2), \beta = (5, 4, 3, 2, 1) \) and \( \gamma = (4, 3, 2, 1) \), we consider embeddings \( (A \subset B) \) where \( A \sim N_{\alpha} = \Lambda / (p^3) \oplus \Lambda / (p^2) \), \( B = N_{\beta} \), and where the cokernel \( B/A \) is isomorphic to \( N_{\gamma} \). Associated with each embedding is an LR-tableau (see Section 3); we denote by \( \mathbb{V}_{\Gamma} \) the collection of all embeddings in the affine variety \( \text{Hom}_k(N_{\alpha}, N_{\beta}) \) which have the corresponding tableau \( \Gamma \).

There are five LR-tableaux of shape \( \beta \setminus \gamma \) and content \( \alpha \):

\[
\begin{array}{c|c|c|c|c}
& 1 & 2 & 3 & 1 \\
\hline
\Gamma_1 & 2 & 3 & 1 & 3 \\
\Gamma_2 & 1 & 3 & 2 & 3 \\
\Gamma_3 & 2 & 3 & 3 & 1 \\
\Gamma_3 & 3 & 1 & 2 & 3 \\
\Gamma_4 & 3 & 2 & 1 & 3 \\
\end{array}
\]

Hence the space of all embeddings corresponding to partition type \( (\alpha, \beta, \gamma) \),

\[
\mathbb{V}^{\beta}_{\alpha,\gamma} = \{ f : N_{\alpha} \rightarrow N_{\beta} : \text{Cok} f \cong N_{\gamma} \} \subset \text{Hom}_k(N_{\alpha}, N_{\beta}),
\]

has five irreducible components, namely the closures \( \overline{\mathbb{V}}_{\Gamma_x} \) where \( x \) is one of the symbols in \( \{1, 2, 3a, 3b, 4\} \).

We are interested in the boundary of the irreducible components. Write

\[ \Gamma \prec_{\text{boundary}} \Gamma^* \text{ if } \mathbb{V}_{\Gamma^*} \cap \mathbb{V}_{\Gamma} \neq \emptyset. \]

Note that \( \Gamma \prec_{\text{boundary}} \Gamma^* \) and \( \Gamma^* \prec_{\text{boundary}} \Gamma \) cannot both hold. Namely, the dominance partial ordering for partitions induces a partial ordering on the set of LR-tableaux of a given shape; it turns out that whenever \( \Gamma \prec_{\text{boundary}} \Gamma^* \), then \( \Gamma \) and \( \Gamma^* \) are in the dominance partial ordering for tableaux. Hence the \textit{boundary relation}, defined to be the reflexive and transitive closure \( \leq_{\text{boundary}} \) of \( \prec_{\text{boundary}} \), is a geometrically motivated partial order on the set of LR-tableaux of shape \( \beta \setminus \gamma \) and content \( \alpha \). We denote this partially ordered set by \( \mathcal{P}_{\text{boundary}} \).

In this paper we provide a tool to determine this poset.

We observe that an LR-tableau \( \Gamma \) can be realized as the tableau of a direct sum of cyclic embeddings if and only if the tableau is a union of columns with subsequent entries (Lemma 5.2). Let \( \Gamma \) and \( \Gamma^* \) be two LR-tableaux of the same shape, both unions of columns with subsequent entries, such that they differ in exactly two columns. If the smallest entry occurs in a higher position in \( \Gamma^* \) then we say that \( \Gamma \) is obtained from \( \Gamma^* \) by an increasing box move.
In this situation we have the following result.

**Theorem 1.1.** If $\tilde{\Gamma}$ is obtained from $\Gamma$ by an increasing box move, then $\Gamma \prec_{\text{boundary}} \tilde{\Gamma}$.

Note that in our example, the skew diagram $\beta \setminus \gamma$ is a horizontal strip, hence any tableau of this shape is a union of columns with subsequent entries. Consider the first two tableaux, $\Gamma_1$ and $\Gamma_2$, they differ in the second and third column. More precisely, $\Gamma_2$ is obtained from $\Gamma_1$ by exchanging the contents of those two columns in such a way that the smallest entry is in a higher position in $\Gamma_2$, this is, $\Gamma_2$ is obtained from $\Gamma_1$ by an increasing box move. Further box moves yield relations corresponding to the remaining edges of the Hasse diagram of the poset $P_{\text{boundary}}$.

\[
P_{\text{boundary}} : \quad \begin{array}{c}
\Gamma_4 \\
\Gamma_3a \\
\Gamma_2 \\
\Gamma_1 \\
\end{array} \\
\begin{array}{c}
\Gamma_3b \\
\Gamma_1 \\
\end{array}
\]

Note that this poset is complete because it agrees with the dominance partial order.

In general, the poset $P_{\text{boundary}}$ may be difficult to determine. If, however, the shape $\beta \setminus \gamma$ of the LR-tableaux is a horizontal and vertical strip then it turns out that any two tableaux in dominance partial order can be transformed into each other by using increasing box moves [7].

Hence we obtain:

**Corollary 1.2.** Suppose $\alpha, \beta, \gamma$ are partitions such that $\beta \setminus \gamma$ is a horizontal and vertical strip. The following statements are equivalent for LR-tableaux $\Gamma, \tilde{\Gamma}$ of shape $\beta \setminus \gamma$ and content $\alpha$.

1. $\tilde{\Gamma}$ is obtained from $\Gamma$ by a sequence of increasing box moves.
2. $\Gamma \preceq_{\text{boundary}} \tilde{\Gamma}$.
3. $\Gamma$ and $\tilde{\Gamma}$ are in the dominance partial order.

Organization of the paper:

1. The first part of the paper consists of Sections 2 through 5. In particular, Theorem 5.1 gives the combinatorial characterization of direct sums of cyclic embeddings. This is a generalization of a result by Kaplansky for poles.

   - In Section 2 we introduce cyclic embeddings and poles, and review how height sequences determine poles, see Proposition 2.1.
Tableaux and partial maps on tableaux are introduced in Section 3, where the two concepts are used to characterize cyclic embeddings (Corollary 3.2).

We recall in Section 4 that each endo-submodule of a Λ-module B is cyclic, hence given by the tableau of the embedding \((a) \subset B\) where \(a\) is an endo-generator (Corollary 4.1). The natural filtration on \(B\) given by an endo-submodule will be used in the second part.

2. The second part consists of Sections 6 through 9. We investigate the representation space of invariant subspaces of nilpotent linear operators in Section 9, where the two results stated above are shown. Sections 6 through 8 prepare Section 9.

- In Section 6 we define the box-relation \(\prec_{\text{box}}\) on the set of Littlewood-Richardson tableaux of the same shape. Moreover, in Proposition 6.1 we show how two tableaux in box relation, say \(\Gamma \prec_{\text{box}} \tilde{\Gamma}\), give rise to a simultaneous pole decomposition of corresponding embeddings.

- The two embeddings differ in two summands, say \(R\) and \(R'\) in one and \(\tilde{R}\) and \(\tilde{R}'\) in the other embedding. In Section 7 we define two monomorphisms \(R' \to \tilde{R}'\) and \(\tilde{R} \to R\) which have isomorphic cokernels (Proposition 7.1).

- In Section 8 we compute the tableau of the pull-back \(Q\) of the two monomorphisms. The pull-back gives rise to a one-parameter family \(Q(\mu)\) of embeddings which is such that \(Q(0)\) has tableau \(\tilde{\Gamma}\) and \(Q(\mu)\) has tableau \(\Gamma\) for \(\mu \neq 0\).

- In the last Section 9 we introduce the poset \(P_{\text{boundary}}\) and use the one-parameter family to establish the two results stated in the introduction.

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Dedication: The authors wish to dedicate this paper to Professor Fred Richman. His talk in 1999 about subgroups of \(p^\infty\)-bounded abelian groups [11] has introduced the second named author to the Birkhoff Problem. It was the first of many delightful presentations of Professor Richman, which in turn have led to numerous inspiring discussions with him. Since several years, the Birkhoff Problem and its many variations have been a major theme in the research of both authors.

2 Cyclic embeddings and poles

By mod\(\Lambda\) we denote the category of all (finite length) \(\Lambda\)-modules, and by \(S(\Lambda)\) the category of all short exact sequences in mod\(\Lambda\) with morphisms given by
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commutative diagrams. With the componentwise exact structure, \( S(\Lambda) \) is an exact Krull-Remak-Schmidt category. We denote objects in \( S(\Lambda) \) either as short exact sequences \( 0 \to A \to B \to C \to 0 \) of \( \Lambda \)-modules, or as embeddings \( (A \subset B) \).

An embedding \( (A \subset B) \) is called cyclic if \( A \) is an indecomposable \( \Lambda \)-module or zero. A cyclic embedding \( (A \subset B) \) is a pole if \( A \) is an indecomposable \( \Lambda \)-module and if \( (A \subset B) \) is an indecomposable object in \( S(\Lambda) \). An embedding of the form \( (0 \subset B) \) is called empty. By \( E_n \) we denote the empty embedding \( (0 \subset B) \), where \( B \) is cyclic of length \( n \).

In this section, we use Kaplansky’s height sequences to classify cyclic embeddings and poles up to isomorphy.

**Definition:**
- A height sequence is a strictly increasing sequence in \( \mathbb{N}_0 \cup \{\infty\} \) which reaches \( \infty \) after finitely many steps.
- We say an element \( a \in B, B \in \text{mod}\Lambda \), has height \( m \) if \( a \in p^mB \setminus p^{m+1}B \). In this case we write \( h(a) = m \). By definition, \( h(0) = \infty \).
- The height sequence for \( a \) in \( B \) is \( H_B(a) = (h(p^i a))_{i \in \mathbb{N}_0} \). Sometimes, we do not write the trailing entries \( \infty \).

**Proposition 2.1.** There is a one-to-one correspondence

\[ \{\text{poles}\}/\sim \overset{1-1}{\leftrightarrow} \{\text{finite non-empty strictly increasing sequences in } \mathbb{N}_0\} \]

We first construct a cyclic embedding \( (A \subset B) \) for a given strictly increasing sequence \( (m_i)_{0 \leq i \leq n} \). We say that \( (m_i) \) has a gap after \( m_\ell \) if \( \ell = n \) or \( m_{\ell+1} > m_\ell + 1 \). Let \( i_1 > i_2 > \cdots > i_s \) be such that \( (m_i) \) has gaps exactly after the entries \( m_{i_1} > m_{i_2} > \cdots > m_{i_s} \). For \( 1 \leq j \leq s \) put \( \beta_j = m_{i_j} + 1 \) and \( \ell_j = m_{i_j} - i_j \), then \( \beta \) and \( \ell \) are strictly decreasing sequences of positive and nonnegative integers, respectively. Let

\[ B = N_\beta = \bigoplus_{i=1}^s \Lambda/(p^{\beta_i}) \]

be generated by elements \( b_{\beta_j} \) of order \( p^{\beta_j} \). Let \( a = \sum_{j=1}^s p^{\ell_j} \cdot b_{\beta_j} \) and put \( A = (a) \). This yields a cyclic embedding \( P((m_i)) = (A \subset B) \).

**Example:** The height sequence \( (1, 3, 4) \) has gaps after 1 and 4, and hence gives rise to the embedding \( P((1, 3, 4)) = ((p^2b_3 + pb_2) \subset N_{(5,2)}) \).

\[ P((1, 3, 4)) : \]

In the picture, the columns correspond to the indecomposable direct summands of \( B \); the \( i \)-th box from the top in a column of length \( r \) represents the element \( p^{i+r}b_r \). The columns are aligned vertically to make the submodule generator(s) homogeneous, if possible.
Lemma 2.2 (see [2, Lemma 22]). Suppose the height sequence \((m_i)\) of some element \(a\) in \(B\) has a gap after \(m_\ell\). Then \(N_{(m_\ell+1)}\) occurs as a direct summand of \(B\).

Proof of the lemma. By assumption, \(h(p' a) = m_\ell\) and \(h(p^{\ell+1} a) > m_\ell + 1\). Hence there is \(y \in B\) with \(py = p^{\ell+1} a\) and \(h(y) > m_\ell\). Put \(z = p' a - y\). Then \(0 \neq z\), \(pz = 0\), and \(h(z) = m_\ell\). It follows that \(B\) has a direct summand isomorphic to \(N_{(m_\ell+1)}\): Since \(B\) satisfies

\[
(*) \quad \soc B \cap \rad^{m_\ell} B \neq \soc B \cap \rad^{m_\ell+1} B,
\]

one of the indecomposable direct summands of \(B\) does, but such a summand must have length \(m_\ell + 1\).

Proof of the proposition. \(\Box\)

- Clearly, for a pole \(((a) \subset B)\), the finite part of the height sequence \(H_B(a)\) is a finite strictly increasing sequence in \(\mathbb{N}_0\).
- By construction, a height sequence \((m_i)\) yields a cyclic embedding \(((a) \subset B)\). Since the height sequence is nonempty, \(a\) is nonzero. By Lemma 2.2, the embedding is indecomposable in \(S(\Lambda)\), hence a pole.
- Starting from a height sequence, it is easy to verify that the submodule generator of the constructed pole has the given height sequence.
- Starting from a pole \(((a) \subset B)\), the constructed pole \(P(H_B(a))\) has ambient space \(D\) which is isomorphic to a direct summand of \(B\), by Lemma 2.2, say \(B \cong D \oplus D'\). Since the submodule generators for \(((a) \subset B)\) and \(P(H_B(a))\) have the same height sequence, the transitivity statement in [2, Theorem 24] yields an isomorphism \(((a) \subset B) \cong P(H_B(a)) \oplus (0 \subset D')\) in \(S(\Lambda)\). Since the left side is indecomposable, we obtain that the poles \(((a) \subset B), P(H_B(a))\) are isomorphic.

\(\Box\)

Proposition 2.3. Every cyclic embedding in \(S(\Lambda)\) has the form

\[ P((m_i)) \oplus (0 \subset N_\beta) \]

for a (possibly constant) height sequence \((m_i)\) and a partition \(\beta\). More precisely, there is a one-to-one correspondence between the set of cyclic embeddings up to isomorphy and the set of pairs consisting of a height sequence and a partition.

Proof. Since \(S(\Lambda)\) is a Krull-Remak-Schmidt category, we can decompose any embedding as a direct sum of indecomposables. As \(\Lambda\) is a local ring, and as the submodule is cyclic, at most one of the summands has a nonzero submodule and hence is a pole. (The case where the submodule is zero is included as \(P((\infty))\) denotes the zero object in \(S(\Lambda)\).) The result follows from the previous proposition. \(\Box\)
Proposition 2.3 applies in particular to extended poles. As embeddings, they are the direct sum of a pole and a certain indecomposable empty embedding. Extended poles arise when due to a “box move” some subsequent entries in the height sequence are increased or decreased in such a way that a gap disappears.

**Lemma 2.4.** Suppose that for a given height sequence \((m_i)\), the gaps are listed in the subsequence \((m_{ij})\), \(j = 1, \ldots, s\), as above, but there is one non-gap included in this subsequence, say at \(m_{iu}\), for some \(u \in \{1, \ldots, s\}\). Let \(\delta_j = m_{ij} + 1\), \(k_j = m_{ij} - i_j\), \(D = N_{\delta_j}\) with generators \(d_{\delta_j}\) of order \(p^{k_j}\) for \(j = 1, \ldots, s\), and \(c = \sum_{j=1}^{s} p^{k_j} d_{\delta_j}\), as above, and denote this embedding as \(P((m_i) \vee m_{iu}) = ((c) \subset D)\).

Then \(P((m_i) \vee m_{iu}) \cong P((m_i)) \oplus (0 \subset N(\delta'))\) where \(\delta' = \delta_{iu}\).

**Proof.** Write \(P((m_i)) = ((a) \subset B)\) for the pole constructed as below Proposition 2.1. The isomorphism in Lemma 2.4 is given explicitly by the map \(D \to B \oplus N(\delta'):\)

\[
d_{\delta_j} \mapsto \begin{cases} b_{\delta_j} & \text{if } j \neq i_u, i_v \\ d' & \text{if } j = i_u \\ b_{\delta_j} - d' & \text{if } j = i_v \end{cases}
\]

if \(i_v\) is the gap directly following \(i_u\) (so \(i_v < i_u\)) and \(N(\delta')\) is generated by \(d'\). \(\square\)

**Example:** The height sequence \((1, 3, 4)\) has no gap after 3, so we may consider the extended pole \(P((1, 3, 4) \vee 3) = ((p^2d_5 + p^2d_4 + pd_2) \subset N_{(5,4,2)}):\)

\[
P((1, 3, 4) \vee 3) : \quad \cong P((1, 3, 4)) \oplus (0 \subset N_{(4)}) : \quad \begin{array}{|c|c|c|}
\hline
& & \\
\hline
\end{array} \oplus \begin{array}{|c|c|c|}
\hline
& & \\
\hline
\end{array}
\]

### 3 Tableaux

Suppose that in an embedding \((A \subset B)\), the submodule \(A\) has Loewy length \(r\). Then the sequence of epimorphisms

\[
B = B/p^\alpha A \to B/p^{\alpha-1} A \to \cdots \to B/pA \to B/A
\]

gives rise to a sequence of inclusions of partitions

\[
\beta = \gamma(r) \supset \gamma(r-1) \supset \cdots \supset \gamma(1) \supset \gamma(0) = \gamma,
\]

where the \(i\)-th partition records the lengths of the indecomposable summands of \(B/p^i A\). The partitions define the tableau \(\Gamma = [\gamma(0), \gamma(1), \ldots, \gamma(r)]\) which we picture as the Young diagram \(\gamma^{(r)}\) in which each box in the skew diagram \(\gamma^{(e)} \setminus \gamma^{(e-1)}\) carries an entry \(\square\). The shape of the tableau is given by the skew diagram \(\beta \setminus \gamma\), the content by the partition \(\alpha\) which counts in row \(e\) the number of boxes in \(\gamma^{(e)} \setminus \gamma^{(e-1)}\), so \(\alpha'_e = \text{len } p^e A/p^e A\) holds, or, equivalently, \(A \cong N_\alpha\).
Tableaux of embeddings are known to satisfy the Littlewood-Richardson (LR) condition: In each row, the entries are weakly increasing; in each column the entries are strictly increasing; and the lattice permutation property is satisfied: On the right hand side of each column, there occur at least as many entries as there are entries $e + 1$ ($e = 1, 2, \ldots$), see for example [5].

The union of two tableaux is taken row-wise, so if $E = [\varepsilon(i)]_{0 \leq i \leq s}$ and $F = [\zeta(i)]_{0 \leq i \leq t}$ are tableaux then the partitions $\gamma(i)$ for $\Gamma = E \cup F$ are given by taking the union (of ordered multi-sets): $\gamma(i) = \varepsilon(i) \cup \zeta(i)$ where $\varepsilon(i) = \varepsilon(s)$ for $i \geq s$ and $\zeta(i) = \zeta(t)$ for $i \geq t$.

It is easy to see that if embeddings $(A \subset B), (C \subset D)$ have tableaux $E, F$, respectively, then the direct sum $(A \subset B) \oplus (C \subset D)$ has tableau $\Gamma = E \cup F$.

We can also decompose some tableaux into columns:

**Column tableaux** (which we call *columns* if the context is clear) are tableaux which have only one column, and where the entries in this column are subsequent natural numbers (not necessarily starting at 1). Each column is one of the following: For $1 \leq e \leq f$, we denote by $C(e, f)_n$ the 1-column tableau of height $n$ with entries $e, \ldots, f$. Formally, $C(e, f)_n = [\gamma(0), \ldots, \gamma(f)]$ where $\gamma(0) = \cdots = \gamma(e - 1) = (n - f + e - 1), \gamma(e) = (n - f + e), \ldots, \gamma(f) = (n)$. By $C(1, 0)_n$ we denote the empty column of length $n$. Note that the lattice permutation property is satisfied if and only if $e = 1$.

**Definition:** A partial map $g$ on an LR-tableau $\Gamma$ assigns to each box $e$ with $e > 1$ a box $d$ with entry $d = e - 1$ such that

1. $g$ is one-to-one,

2. for each box $b$, the row of $g(b)$ is above the row of $b$.

**Remark:** Given an LR-tableau $\Gamma$, the existence of at least one partial map follows from the lattice permutation property.

Given a partial map $g$ on an LR-tableau $\Gamma$, a *jump* in row $r$ is a box $b$ in this row with the property that either $b$ has entry $\blacksquare$ or that $g(b)$ occurs in row $s$ with $s < r - 1$. We say a partial map $g$ on $\Gamma$ has the *empty box property (EBP)* if for each $r$, there are at least as many columns in $\Gamma$ of exactly $r - 1$ empty boxes as there are jumps in row $r$.

**Example:** Consider the following LR-tableau

\[
\begin{array}{c|c|c}
1 & 2 & 3 \\
2 & 1 & 2 \\
3 & & \\
\end{array}
\]

\[
\begin{array}{c|c|c}
1 & & \\
2 & & \\
3 & 1 & 2 \\
\end{array}
\]

**Definition:** A partial map $g$ on an LR-tableau $\Gamma$ assigns to each box $e$ with $e > 1$ a box $d$ with entry $d = e - 1$ such that

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**Example:** Consider the following LR-tableau

\[
\begin{array}{c|c|c}
1 & 2 & 3 \\
2 & 1 & 2 \\
3 & 1 & 2 \\
\end{array}
\]
To define a partial map, we have to assign to each box $\square$ a corresponding box $\blacksquare$ and to each box $\blacksquare$ a corresponding box $\bigcirc$. Note that we can do this in four different ways. To specify the maps, we distinguish boxes of the same entry by indicating the row on the right; for the two boxes in the first row, we use the letters L and R. Consider the partial map $g$ defined as follows:

$$g : \square \mapsto \blacksquare, \quad \blacksquare \mapsto \bigcirc, \quad \bigcirc \mapsto \square$$

Note that $g$ satisfies (EBP) since there is a column of 2 empty boxes corresponding to the jump $\square \mapsto \blacksquare$. However, the partial map $g' : \square \mapsto \blacksquare, \quad \blacksquare \mapsto \bigcirc, \quad \bigcirc \mapsto \square$ does not satisfy (EBP), because there is no column of 3 empty boxes corresponding to the jump at $\square \mapsto \blacksquare$.

We collect some properties of tableaux for cyclic embeddings.

**Proposition 3.1.** Suppose the embedding $(A \subset B)$ is cyclic with $A$ a module of Loewy length $r$.

1. In the tableau, each entry $1, \ldots, r$ occurs exactly once.
2. The height sequence $(m_i)$ of the submodule generator $a$ determines the rows in $\Gamma$ in which the entries occur, and conversely. More precisely, the entry $e$ occurs in row $m_e - 1 + 1$.
3. The LR-tableau $\Gamma$ of a cyclic embedding is a union of columns with disjoint entries.
4. There is a unique partial map on $\Gamma$; it satisfies the (EBP).

**Proof.** 1. For the first statement note that for each $0 < e \leq r$, the module $B/p^{e-1}A$ is a factor module of $B/p^eA$ of colength one.
2. For any embedding, the number of boxes in the first $m$ rows of $\gamma^{(e)}$ is the length of $(B/p^eA)/(p^mB + p^eA)/p^eA) = B/(p^eA + p^mB)$. Here, the modules $B/(p^{e-1}A + p^mB)$ and $B/(p^eA + p^mB)$ have the same length if $m \leq m_{e-1}$ and different lengths otherwise since $p^{e-1}a \in p^{m_{e-1}}B \setminus p^{m_{e-1}+1}B$.
3. By construction of $P((m_i))$, each gap in the height sequence gives rise to a new summand of $B$ and hence to a new column in the tableau. It follows from 2. that $\Gamma$ is a union of columns.
4. There is a unique partial map on $\Gamma$ given by assigning to a box $\square$ where $e > 1$ the unique box with entry $e - 1$. Since $\Gamma$ is a union of columns, the (EBP) is satisfied.

**Corollary 3.2.** 1. There is a one-to-one correspondence between cyclic embeddings up to isomorphy and LR-tableaux in which each entry occurs at most once.
2. Under this correspondence, a pole corresponds to a tableau in which the number of columns equals the number of gaps in the height sequence.

Proof. 1. By the Green-Klein theorem, any LR-tableau \( \Gamma \) can be realized as the tableau of an embedding \( (A \subset B) \). If \( \Gamma \) has at most one box \( 1 \), then \( A \) is cyclic since \( \dim A/pA = \dim B/pA - \dim B/A = \#(\text{boxes } 1) \). The height sequence of the generator of \( A \) can be read off using Lemma 3.1. 2. The uniqueness of the isomorphism type follows from Proposition 2.3.

2. Regarding poles, we have seen in the proof of Proposition 2.1 that the summands of the ambient space of a pole correspond to the gaps in the height sequence of the submodule generator. The result follows from Proposition 2.3. \( \square \)

4 Endo-submodules

Given a \( \Lambda \)-module \( B \), we obtain a precise description for the submodules of \( B \) when considered as a module over its endomorphism ring. We use these endo-submodules to define a natural filtration for a pole or extended pole.

We recall from \cite{2} Theorem 24] that a height sequence \( (m_i) \) defines an \( \text{End}(B) \) submodule consisting of all elements of \( B \) with height sequence \( \geq (m_i) \) (this is the “fully transitive” property). The partial ordering on height sequences is given by \( (m_i) \geq (q_i) \) if \( m_i \geq q_i \) holds for all \( i \in \mathbb{N} \).

It is shown in \cite{2} Theorem 25] that every \( \text{End}(B) \)-submodule of \( B \) is cyclic, and hence is determined uniquely, up to equality, by the height sequence of a generator.

Suppose \( a \in B \) has height sequence \( (m_i) \) with gaps exactly after \( m_{i_1} > m_{i_2} > \cdots > m_{i_s} \). We can specify the corresponding \( \text{End}(B) \)-submodule \( X \) explicitly. Namely, with \( a = \sum_j p^{\ell_j} b_{\beta_j} \) as above, also the summands \( p^{\ell_j} b_{\beta_j} \) are in \( X \), and so are their images under endomorphisms of \( B \). Put \( k_j = \beta_j - \ell_j \) for \( 1 \leq j \leq s \), then

\[
\text{End}(B) \cdot a = \sum_{j=1}^s (\text{rad}^{k_j} B \cap \text{soc}^{k_j} B).
\]

With this notation, the endo-submodule \( \text{End}(B) \cdot a \) can also be written as an intersection of sums:

\[
\text{End}(B) \cdot a = \text{rad}^{k_1} B \cap (\text{soc}^{k_1} B + \text{rad}^{k_2} B) \cap \cdots \cap (\text{soc}^{k_{s-1}} B + \text{rad}^{k_s} B) \cap \text{soc}^{k_s} B
\]

To verify this, use induction and the modular law.

The following result leads to a quick proof for the formula for the number of \( \text{End}(B) \)-submodules of a finite length \( \Lambda \)-module \( B \).

Corollary 4.1. Suppose the \( \Lambda \)-module \( B \) is given by a partition \( \beta \). Any two of the following sets are in one-to-one correspondence.
1. The set of $\text{End}(B)$-submodules of $B$.

2. The set of embeddings of the form $((a) \subset B)$, up to isomorphy.

3. The set of LR-tableaux with outer shape $\beta$ in which each entry occurs at most once.

As a consequence, each set has exactly $\prod_i (1 + \beta_i - \beta_{i+1})$ many elements.

Proof. In each of the sets, the elements are given by height sequences. The correspondence follows from Corollary [3,2] and Proposition [3,1] 2., together with the isomorphy between the lattices of $\text{End}(B)$-submodules of $B$ and of height sequences of elements in $B$ in [2, Theorem 25]. For the formula we count the third set. Note that tableau properties require that each entry be the rightmost entry in its row since there are no multiple entries. Hence in the $i$-th column, there may be between 0 and $\beta_i - \beta_{i+1}$ many entries. The expression is the product of the number of choices for the number of entries in each column. \qed

Definition: An element $a \in B$ defines a filtration for $B$, as follows. Let $F_0 = \text{End}(B) \cdot a$ and, for $i \in \mathbb{Z} \setminus \{0\}$, put $F_i = p^{-i} F_0$. We call $(F_i)_i$ the filtration of $B$ centered at $a$.

Clearly, the filtration $\cdots \subset F_{-1} \subset F_0 \subset F_1 \subset \cdots$ has factors which are semisimple $\Lambda$-modules, and $a \in F_i$ for $i \geq 0$.

Lemma 4.2. Suppose $a = \sum p^{\ell_j} b_{\beta_j}$ is the submodule generator of a pole or extended pole.

1. The elements $p^{\ell_j} b_{\beta_j}$ form a minimal generating set for the $\Lambda$-module $F_0$.

2. For $\alpha \in \mathbb{Z}$, the $k$-vector space $F_\alpha / F_{\alpha - 1}$ has basis given by the residue classes of the elements $p^{\ell_j - \alpha} b_{\beta_j}$ where $j$ is such that $\ell_j - \beta_j < \alpha \leq \ell_j$.

3. The residue class of the submodule generator $a$ is homogeneous in $F_0 / F_{-1}$ in the sense that it is the sum of the basic elements in 2.

4. Similarly, for $\alpha > 0$, the residue class of $p^\alpha a$ is a homogeneous element in $F_{-\alpha} / F_{-\alpha - 1}$.

Proof. 1. For a pole $((a) \subset B)$ with $a = \sum p^{\ell_j} b_{\beta_j}$, the sequences $\ell_j$ and $k_j = \beta_j - \ell_j$ are strictly decreasing, hence a term $p^{\ell_v} b_{\beta_v}$ occurs in exactly one summand in $\text{End}(B) \cdot a = \sum (\text{rad}^{k_j} B \cap \text{soc}^{k_j} B)$. If the pole is extended, then the sequence $k_j$ is still strictly decreasing, but $\ell_j$ is stationary at one place, say $\ell_v = \ell_{v-1}$. The corresponding summand $p^{\ell_v} b_{\beta_v}$ occurs in exactly two terms in the expression $\text{End}(B) \cdot a = \sum (\text{rad}^{k_j} B \cap \text{soc}^{k_j} B)$, in each case, it is not in the radical.

2. The second statement follows from the first.

3. and 4. The last two statements follow from the first two. \qed
5  Direct sums of cyclic embeddings

We can now give a combinatorial description for finite direct sums of cyclic embeddings, up to isomorphy.

Definition:  
- Two partial maps $g$, $g'$ on an LR-tableau $\Gamma$ are equivalent if $g' = \pi^{-1} \circ g \circ \pi$ for some permutation $\pi$ on the set of non-empty boxes in $\Gamma$ which preserves the entry and the row.
- Two pairs $(\Gamma, g)$, $(\Delta, h)$, each consisting of an LR-tableau and a partial map on the tableau, are equivalent if $\Gamma = \Delta$ and if the partial maps $g, h$ are equivalent.

Example: Consider the LR-tableau from the example in Section 3.

\[
\begin{array}{ccc}
1 & 1 & 2 \\
& 2 & 3
\end{array}
\]

The maps $g$ and $g'$ are not equivalent. But $g$ is equivalent to the partial map

\[ h : 1 \mapsto 2^*, 2^* \mapsto 1^*, 2^* \mapsto 1^*. \]

(The there is a fourth partial map on $\Gamma$, it is equivalent to $g'$.)

Clearly, given two equivalent partial maps on an LR-tableau, then one satisfies (EBP) if and only if the other does.

Theorem 5.1. There is a one-to-one correspondence between the isomorphism types of direct sums of cyclic embeddings, and the equivalence classes of pairs $(\Gamma, g)$ where $\Gamma$ is a Littlewood-Richardson tableau and $g$ a partial map on $\Gamma$ which satisfies (EBP).

Proof. Given a direct sum of cyclic embeddings, each summand gives rise to a tableau with a uniquely determined partial map that satisfies (EBP). This map has at most one orbit, it records the isomorphism type of a pole (Proposition 3.1). The tableau of the direct sum is given by taking the row-wise union of the tableaux of the summands; it admits a partial map which is (boxwise) defined by the partial maps for the summands; this map satisfies (EBP) because the restricted maps do. Given two isomorphic direct sum decompositions, the two unordered list of height sequences of the poles involved are equal, hence the two associated partial maps differ by conjugation by a permutation of the boxes which preserves rows and entries.

Conversely, suppose $g$ is a partial map on $\Gamma$ with (EBP). For each orbit $\mathcal{O}$ of $g$, the boxes in $\mathcal{O}$ together with the empty parts of columns which correspond to the jumps (recall the definition of the (EBP)), constitute the tableau of a pole. The empty parts of columns in $\Gamma$ not used by the (EBP) define the tableau of an empty embedding. Since all boxes are accounted for, $\Gamma$ is the union of all the tableaux. Hence the sum of the poles and the empty embedding has $\Gamma$ as its
If \( g' \) is an equivalent partial map on \( \Gamma \), then the boxes in the orbits may differ, but not the rows in which they occur. Thus the corresponding unordered list of height sequences of poles is equal, and so is the partition which records those empty parts of columns which are not used up by the jumps in the height sequences. Hence the associated embeddings are isomorphic.

The two assignments are inverse to each other since this is true for indecomposable embeddings.

There is a different way to read off from a given Littlewood-Richardson tableau if it is the tableau of a direct sum of cyclic embeddings.

**Lemma 5.2.** A Littlewood-Richardson tableau \( \Gamma \) is the tableau of a direct sum of cyclic embeddings if and only if \( \Gamma \) is a union of columns.

**Proof.** In Proposition 3.1 we have seen that the tableau of a cyclic embedding is a union of columns, and this property is preserved under taking direct sums. Conversely, if the tableau is a union of columns, then the lattice permutation property allows for the construction of a partial map with (EBP). The result follows from Theorem 5.1.

We conclude this section with examples and remarks.

**Example:** In general, the embedding corresponding to a given LR-tableau is not determined uniquely, up to isomorphy, by the tableau alone. In each case, the following embeddings will have the same given LR-tableau.

1. Two nonisomorphic sums of poles, given by nonequivalent partial maps.

\[
\begin{align*}
\Gamma : & \quad P((1,3)) \oplus P((0)) : \\
& \quad P((0,3)) \oplus P((1)) : \\
& \quad P((0,2)) \oplus P((1)) : \\
& \quad P((1,2)) \oplus P((0)) : \\
& \quad \oplus (0 \subset N(2)) : \\
& \quad E \oplus (0 \subset N(3)) : \\
& \quad \oplus P((0,1)) :
\end{align*}
\]

2. A sum of two poles vs. a sum of two poles and an empty embedding.

\[
\begin{align*}
\Gamma : & \quad P((0,2)) \oplus P((1)) : \\
& \quad P((1,2)) \oplus P((0)) : \\
& \quad \oplus (0 \subset N(2)) : \\
& \quad \oplus P((0,1)) :
\end{align*}
\]

3. The sum of poles is determined uniquely by the partial map; but there is also an embedding which is not a sum of poles.

\[
\begin{align*}
\Gamma : & \quad P((0,1,3)) \oplus P((2)) : \\
& \quad E \oplus (0 \subset N(3)) :
\end{align*}
\]

4. Two partial maps, up to equivalence, only one has the (EBP).

\[
\begin{align*}
\Gamma : & \quad P((0,1,3)) \oplus P((2)) : \\
& \quad \oplus P((0,2)) :
\end{align*}
\]
(5) Two partial maps with (EBP), up to equivalence, but only one Klein tableau (see below).

\[
\begin{array}{c}
\Gamma : \\
\begin{array}{c}
\begin{array}{c}
1 \\
3 \\
2
\end{array}
\end{array}
\end{array}
\oplus P((0, 2, 3)) \\
\oplus P((1, 2)) \\
\oplus (0 \subset N(3)) : \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2 \\
1 \\
3
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\oplus P((1, 2, 3)) \\
\oplus P((0, 2)) \\
\oplus (0 \subset N(3)) :
\end{array}
\]

Remark: A Klein tableau consists of a Littlewood-Richardson tableau $\Gamma$ together with an equivalence class of a partial map $g$ under a relation (K). The additional condition (iii) in [4, Page 64] requires that whenever the box above $b = (r_0, d)$ is of the form $b' = (r_0 - d)$ with $d = e - 1$, then $g(b) = b'$. One can verify that a partial map with (EBP) is always equivalent to one which satisfies this condition (iii). Here, two partial maps $g, g'$ on $\Gamma$ are in relation (K) if for each box $b$ with entry $e > 1$, the boxes $g(b), g'(b)$ are in the same row. Often, Klein tableaux are presented by adding to each entry $e > 1$ the row of $g(e)$ as a subscript. Example (5) above shows that relation (K) in general is coarser than the above equivalence relation for partial maps.

Remark: The proof of Theorem 5.1 shows how to read off the direct sum decomposition from a given pair $(\Gamma, g)$. Each orbit of $g$ gives rise to a pole, its height sequence is given by Proposition 3.1. To each jump in the height sequence, there is a corresponding empty part of a column in $\Gamma$. The empty boxes in those columns which do not correspond to any pole determine the remaining empty embedding.

Remark: An algebraic description of finite direct sums of cyclic embeddings is given in [1, in particular Lemma 1 and Theorem 2]. Here, the multiplicity of a pole $P$ as a direct summand of a given embedding $M$ is the dimension of the $k$-vector space $F_P(M)$, for a suitable functor $F_P : S(\Lambda) \to \text{mod}k$.

It remains to verify the two remaining claims in the Introduction.

Corollary 5.3. Suppose that $(A \subset B)$ is an embedding of $\Lambda$-modules where $A$ is $p^2$-bounded and where $B$ is either finitely generated or bounded. Then the embedding is a direct sum of cyclic embeddings.

Proof. The case where $B$ is finitely generated is covered in [10, Corollary 5.4]. There is exactly one indecomposable embedding where $B$ is not bounded, namely $(0 \subset \Lambda)$.

For the bounded case, note that an embedding $(A \subset B)$ with $p^2A = 0 = p^nB$ can be considered a module over the ring

\[
R = \begin{pmatrix}
\Lambda/(p^2) & \Lambda/(p^2) \\
0 & \Lambda/(p^n)
\end{pmatrix},
\]

so [11, Theorem 1] yields the claim. □

Let $\alpha, \beta, \gamma$ be partitions. Using Lemma 5.2 it is possible to decide whether there exists an embedding $N_\alpha \to N_\beta$ with cokernel $N_\gamma$ which is a direct sum of
cyclic embeddings. Recall from [3] that there exists a short exact sequence
\[ \mathcal{E} : 0 \to N_\alpha \to N_\beta \to N_\gamma \to 0 \]
if and only if there exists an LR-tableau of shape $\beta \setminus \gamma$ and content $\alpha$. As a consequence, we obtain

**Corollary 5.4.** Given partitions $\alpha, \beta, \gamma$, there exists an embedding $N_\alpha \to N_\beta$ with cokernel $N_\gamma$ which is a direct sum of cyclic embeddings if and only if there is an LR-tableau of shape $\beta \setminus \gamma$ and content $\alpha$ which is a union of columns. \(\square\)

We are particularly interested in the following situation.

**Corollary 5.5.** Suppose $\Gamma$ is a Littlewood-Richardson tableau of shape $(\alpha, \beta, \gamma)$ such that $\beta \setminus \gamma$ is a horizontal strip. Then there exists an embedding $(A \subset B)$ which is a direct sum of cyclic embeddings and which has tableau $\Gamma$.

**Proof.** Since $\Gamma$ is a Littlewood-Richardson tableau, there exists a partial map $g$ on $\Gamma$. Since $\beta \setminus \gamma$ is a horizontal strip, each column in the tableau $\Gamma$ has at most one entry, so the (EBP) is trivially satisfied. The embedding corresponding to the partial map $g$ is a direct sum of cyclic embeddings, and it has the tableau $\Gamma$. \(\square\)

### 6 Box moves

In the second part of this paper we are interested in the “transition” between the collections of embeddings which are given by two different but “similar” LR-tableaux. In this section we show that whenever the two tableaux differ by a so called box move, then they can be realized by embeddings which have “compatible” decompositions as direct sums of cyclic embeddings.

**Definition:** Suppose that $\Gamma, \bar{\Gamma}$ are LR-tableaux of the same shape and content, and that both are unions of columns in such a way that they differ in exactly two of those columns. We say $\bar{\Gamma}$ is obtained from $\Gamma$ by an increasing (decreasing) box move if the smallest entry in those two columns in $\bar{\Gamma}$ is in a higher (lower) position than in $\Gamma$. We write $\Gamma \leq _{\text{box}} \bar{\Gamma}$ where $\leq _{\text{box}}$ denotes the reflexive and transitive closure of the relation given by increasing box moves.

**Remark:** Suppose $\bar{\Gamma}$ is obtained from $\Gamma$ by an increasing box move. Since $\Gamma, \bar{\Gamma}$ have the same shape, the two columns in which $\Gamma$ and $\bar{\Gamma}$ differ have the same number of entries, more precisely, they are columns of the form $C(e, f)_n$, $C(e', f')_{n'}$ in $\Gamma$ and $C(e', f')_n$, $C(e, f)_{n'}$ in $\bar{\Gamma}$, for suitable numbers $n > n'$, $e < e'$, $f < f'$ such that $f - e = f' - e'$.

**Example:** 1. The name box move originates from dealing with LR-tableaux which are horizontal and vertical strips; the box move is simply given by exchanging two boxes in the tableau. In the example in the introduction, $\Gamma_2$ is...
obtained from $\Gamma_1$ by an increasing box move.

\[
\begin{array}{|c|c|c|c|}
\hline
& 1 & & \\
\hline
& 2 & & \\
\hline
& 3 & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
 & 1 & & \\
\hline
 & 2 & & \\
\hline
 & 3 & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
\text{box} & 1 & & \\
\hline
 & 2 & & \\
\hline
 & 3 & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & & \\
\hline
3 & 1 & & \\
\hline
2 & 3 & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & & \\
\hline
3 & 1 & & \\
\hline
2 & 3 & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
\text{box} & 1 & & \\
\hline
 & 2 & & \\
\hline
 & 3 & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & & \\
\hline
3 & 1 & & \\
\hline
2 & 3 & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & & \\
\hline
3 & 1 & & \\
\hline
2 & 3 & & \\
\hline
\end{array}
\]

(We exchanged entries 1 and 3 in columns 2 and 3.)

2. Both tableaux $\Gamma = C(1, 2) \cup C(3, 4) \cup C(1, 2) \cup C(1, 2)$ and $\tilde{\Gamma} = C(3, 4) \cup C(1, 2) \cup C(1, 2)$ are unions of columns (the first in two different ways). $\Gamma$ is obtained from $\Gamma$ by an increasing box move: $\Gamma \lessdot_{\text{box}} \tilde{\Gamma}$.

\[
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\]

Our aim is to show:

**Proposition 6.1.** Suppose $\Gamma, \tilde{\Gamma}$ are tableaux of the same shape and content such that $\tilde{\Gamma}$ is obtained from $\Gamma$ by an increasing box move. Then each of the tableaux $\Gamma, \tilde{\Gamma}$ is the tableau of a sum of poles and an empty embedding. These direct sum decompositions differ, up to isomorphy and reordering of the summands, in exactly two poles and perhaps in the empty embeddings.

**Proof.** We construct partial maps $g, \tilde{g}$ for $\Gamma, \tilde{\Gamma}$, respectively, which both satisfy the (EBP) and which differ in exactly two orbits. The claim follows from Theorem 5.1 and its proof, see the Remark on the correspondence between partial maps and the pole decomposition.

Suppose the tableaux $\Gamma, \tilde{\Gamma}$ differ in the columns $C(e, f)_n, C(e', f')_{n'}$ for $\Gamma$ and $C(e', f')_{n'}, C(e, f)_{n'}$ for $\tilde{\Gamma}$, with $n > n'$ and $e < e'$, as in the remark above. We identify the complements $\Gamma \setminus (C(e, f)_n \cup C(e', f')_{n'})$ and $\tilde{\Gamma} \setminus (C(e', f')_{n'} \cup C(e, f)_{n'})$.

The two orbits of $g$ and $\tilde{g}$ which differ will be as follows. The orbits of $g$ which contain the non-empty boxes in $C(e, f)_n$ and in $C(e', f')_{n'}$, correspond to the orbits of $\tilde{g}$ which contain the non-empty boxes in $C(e', f')_{n'}$ and in $C(e', f')_{n}$, respectively.

More precisely, here are the partial maps, which we define entry by entry on the boxes in the columns.

For a partial map $g$ and $x \geq 2$, denote by $g_x$ the restriction of the domain of $g$ to boxes of the form $b_x$. Clearly, $g$ is given by the sequence of its restrictions $(g_x)_{x \geq 2}$. Depending on the subscript $x$, we use either the lattice permutation property of $\Gamma$ to define the map $g_x$, and then $\tilde{g}_x$, or the lattice permutation property of $\tilde{\Gamma}$ to define $\tilde{g}_x$, and then $g_x$.

For an entry $x$ different from $e, e', f+1, f'+1$, define $g_x$ as usual, that is, as in the proof of Theorem 5.1. For boxes $b$ with entry $x$, put

$$
g_x(b) = \begin{cases}
\text{the box above } b & \text{if } b \in C(e', f')_{n'} \cup C(e, f)_{n'} \\
g_x(b) & \text{otherwise.}
\end{cases}
$$
If $e > 1$, define $\tilde{g}_e$ as usual. Let $b'$ be the box $e$ in $C(e, f)_{n'}$. For boxes $b$ with entry $e$ put

$$g_e(b) = \begin{cases} 
\tilde{g}_e(b') & \text{if } b \text{ is the box } e \text{ in } C(e, f)_n \\
\tilde{g}_e(b) & \text{otherwise}.
\end{cases}$$

(Since $n > n'$, the box $b$ with entry $e$ in $C(e, f)_n$ is lower in $\Gamma$ than the box $b'$ in $\tilde{\Gamma}$. Hence we pick $\tilde{g}_e$ first and define $g_e(b) = \tilde{g}_e(b')$.)

Next, define $\tilde{g}_{f'+1}$ as usual. There is a unique box $b'$ such that $\tilde{g}_{f'+1}(b')$ is the box $e$ in $C(e', f')_{n'}$. For a box $b$ with entry $f' + 1$ define

$$g_{f'+1}(b) = \begin{cases} 
\text{the box } e & \text{if } b = b' \\
\tilde{g}_{f'+1}(b) & \text{otherwise}.
\end{cases}$$

Assume first that $e' \neq f + 1$. Define $g_{e'}$ as usual. Let $b'$ be the box $e$ in $C(e', f')_{n'}$. For boxes $b$ with entry $e'$ put

$$\tilde{g}_{e'}(b) = \begin{cases} 
g_{e'}(b') & \text{if } b \text{ is the box } e \text{ in } C(e', f')_n \\
\text{the box above } b & \text{if } e < e' < f \text{ and } b \in C(e, f)_{n'} \\
g_{e'}(b) & \text{otherwise}.
\end{cases}$$

Define $g_{f+1}$ as usual. There is a unique box $b'$ such that $g_{f+1}(b')$ is the box $e$ in $C(e, f)_n$. For boxes $b$ with entry $f + 1$ define

$$\tilde{g}_{f+1}(b) = \begin{cases} 
\text{the box } e & \text{if } b = b' \\
\text{the box above } b & \text{if } b \in C(e', f') \setminus e \\
g_{f+1}(b) & \text{otherwise}.
\end{cases}$$

It remains to deal with the case $e' = f + 1$, define $g_{e'}$ as usual. There is a unique box $b'$ such that $g_{e'}(b')$ is the box $e$ in $C(e, f)_n$. Let $b'$ be the box $e$ in $C(e', f')_{n'}$. For a box $b$ with entry $e'$ define

$$\tilde{g}_{e'}(b) = \begin{cases} 
\text{the box } e & \text{if } b = b' \\
g_{e'}(b) & \text{if } b \text{ is the box } e \text{ in } C(e', f')_n \\
g_{e'}(b) & \text{otherwise}.
\end{cases}$$

The (EBP) is satisfied for the maps in the proof of Theorem 5.1 by construction, the property also holds for the modified maps.

**Example:** In the above example, the poles are given by the orbits of the maps $g$ and $\tilde{g}$. For $\Gamma$, the orbit $\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \to \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$ yields the pole $P = P((3, 4))$, the orbit $\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \to \begin{array}{c} 2 \\ 3 \\ 4 \\ 1 \end{array} \to \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \to \begin{array}{c} 4 \\ 1 \\ 2 \\ 3 \end{array}$ yields $P' = P((0, 1, 2, 3))$. $\Gamma$ is the tableau for the direct sum of those two poles and the empty embedding $E_{(2)}$.

For $\tilde{\Gamma}$, the orbit $\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \to \begin{array}{c} 2 \\ 3 \\ 1 \end{array}$ yields $\tilde{P} = P((2, 3))$, while the long orbit gives rise to $\tilde{P}' = P((0, 1, 3, 4))$. The tableau for the direct sum of these two poles is $\tilde{\Gamma}$.

Note that there are monomorphisms $\tilde{P} \to P$ and $P' \oplus E_{(2)} \to \tilde{P}'$ which both have cokernel $E_{(1)}$. \qed
In $\Gamma$: $C = C(e,f)_n$; $C' = C(e',f')_{n'}$; in $\tilde{\Gamma}$: $\tilde{C} = C(e,f)_{n'}$, $\tilde{C}' = C(e',f')_n$

Suppose that the columns are parts of the LR-tableaux of (possibly extended) poles $R$, $R'$, $\tilde{R}$, $\tilde{R}'$, respectively. We show in this section that there are monomorphisms $\tilde{R} \to R$, $R' \to \tilde{R}'$ which both have cokernel $E_{(n-n')}$.

More precisely, in the proof of Proposition 6.1 we have constructed partial maps $g, \tilde{g}$ which differ in exactly the two orbits which correspond to those columns. Hence there are poles with the following properties:

$$
P = P((m_i)), \quad m_{e-1} = n - f + e, \ldots, m_{f-1} = n
$$

$$
P' = P((m'_i)), \quad m'_{e'-1} = n' - f + e, \ldots, m'_{f'-1} = n'
$$

$$
\tilde{P} = P((\tilde{m}_i)), \quad \tilde{m}_{e-1} = n' - f + e, \ldots, \tilde{m}_{f-1} = n'
$$

and $\tilde{m}_i = m_i$ for $i \notin \{e-1, \ldots, f-1\}$

$$
\tilde{P}' = P((\tilde{m}'_i)), \quad \tilde{m}'_{e'-1} = n - f + e, \ldots, \tilde{m}'_{f'-1} = n
$$

and $\tilde{m}'_i = m'_i$ for $i \notin \{e'-1, \ldots, f'-1\}$

Comparing the height sequences $(m_i)$, $(\tilde{m}_i)$, we see that $\tilde{m}_i = m_i - n + n'$ for $i \in \{e-1, \ldots, f-1\}$ but $\tilde{m}_i = m_i$ otherwise. As $n > n'$ then, whenever $e > 2$, the sequence $(m_i)$ has a gap after $m_{e-2}$ while the sequence $(\tilde{m}_i)$ has a gap only if $\tilde{m}_{e-1} \neq \tilde{m}_{e-2} + 1$. Hence, if $e > 2$ and $\tilde{m}_{e-1} = \tilde{m}_{e-2} + 1$, the ambient space for $P$ contains a summand $N_{(\tilde{m}_{e-1})}$ which is missing in $\tilde{P}$. In this case, we add this missing summand back in: $\tilde{R} = \tilde{P} \oplus E_{(\tilde{m}_{e-1})}$. By Lemma 2.4, $\tilde{R} \cong P((\tilde{m}_i) \vee \tilde{m}_{e-2})$. Similarly, there may be missing gaps after $m_{f-1}$ for the sequence $(m_i)$ when compared with $(\tilde{m}_i)$; after $m'_{f'-1}$ for the sequence $(\tilde{m}'_i)$
when compared to \((m'_i)\); and after \(m'_{e-2}\) for the sequence \((m'_i)\) when compared to \(\check{m}'_{e-1}\). Formally we define:

\[
R = \begin{cases} 
P \oplus E_{(m_{f-1})} \cong P((m_i) \vee m_{f-1}) & \text{if } m_f = m_{f-1} + 1 \\
P & \text{otherwise} 
\end{cases}
\]

\[
R' = \begin{cases} 
P' \oplus E_{(m'_{e-2})} \cong P((m'_i) \vee m'_{e-2}) & \text{if } e' \geq 2 \text{ and } m_{e-1} = m_{e-2} + 1 \\
P' & \text{otherwise} 
\end{cases}
\]

\[
\check{R} = \begin{cases} 
P \oplus E_{(\check{m}_{e-2})} \cong P((\check{m}_i) \vee \check{m}_{e-2}) & \text{if } e \geq 2 \text{ and } \check{m}_{e-1} = \check{m}_{e-2} + 1 \\
P & \text{otherwise} 
\end{cases}
\]

\[
\check{R}' = \begin{cases} 
P' \oplus E_{(\check{m}'_{e-1})} \cong P((\check{m}'_i) \vee \check{m}'_{e-1}) & \text{if } \check{m}'_{e-1} = \check{m}'_{e-2} + 1 \\
P' & \text{otherwise} 
\end{cases}
\]

The monomorphisms are as follows. Write \(R = ((a) \subset B)\) where \(B = N_{\beta}\) and \(a = \sum p^{i} b_{\beta_i}\). Then \(\beta\) has a unique part \(n\), say \(\beta_u = n\), and \(\ell_u = n - f\) (since \(f\) is minimal with respect to the property that \(p^{i} : p^{e+n} b_{n} = 0\)). Then \(\check{R} = ((a) \subset \check{B})\) is the submodule of \(R\) the ambient space of which is generated by \(b_{\beta_1}, \ldots, b_{\beta_{u-1}}, p^{n-n'} b_{\beta_u}, b_{\beta_{u+1}}, \ldots\). Thus, \(\check{B} = N_{\check{\beta}}\), where \(\check{\beta}\) is obtained from \(\beta\) replacing \(\beta_u = n\) by \(\check{\beta}_u = n'\). Hence \(a = \sum p^{i} b_{\check{\beta}_i}\) where \(\check{\ell}_i = \ell_i\) for \(i \neq u\) and \(\check{\ell}_u = n' - f\). Clearly, \(R/\check{R} \cong E_{(n-n')}\).

Similarly, write \(\check{R}' = ((c) \subset \check{D})\). If \(\check{D} = N_{\check{\delta}}\) then \(\check{\delta}\) contains a unique part \(n\), say \(\check{\delta}_v = n\). Write \(c = \sum p^{i} d_{\check{\delta}_i}\), then \(\check{k}_v = n - f'\), as above.

The submodule \(\check{R}'\) of \(\check{R}\) has ambient space \(D\) generated by \(d_{\check{\delta}_1}, \ldots, d_{\check{\delta}_{v-1}}, p^{n-n'} d_{\check{\delta}_v}, d_{\check{\delta}_{v+1}}, \ldots\), so \(D \cong N_{\delta}\) where the partition \(\delta\) is obtained from \(\check{\delta}\) by replacing \(\check{\delta}_v = n\) by \(\delta_v = n'\). We write \(c = \sum p^{i} d_{\delta_i}\) where \(k_v = n' - f'\) and \(k_i = k_i\) for \(i \neq v\). Clearly, \(\check{R}'/\check{R}' \cong E_{(n-n')}\).

We summarize:

\textbf{Proposition 7.1.} Suppose the LR-tableau \(\Gamma\) is obtained from \(\Gamma\) by an increasing box move.

1. Then there are cyclic embeddings \(R, \check{R}, R', \check{R}'\) and an embedding \(S\), such that \(S \oplus R \oplus \check{R}\) has tableau \(\Gamma\) and \(S \oplus R' \oplus \check{R}'\) has tableau \(\check{\Gamma}\).

2. There are monomorphisms of embeddings \(\check{R} \rightarrow R\) and \(R' \rightarrow \check{R}'\) such that both maps have cokernel \(E_{(n-n')},\) where \(n\) and \(n'\) are the lengths of the columns in which the entries are exchanged.

\(\square\)

\section{An extension of two cyclic embeddings}

In the set-up from the previous section, we show that there is an embedding \(Q\) and two exact sequences of embeddings with \(Q\) as middle term and for which
the sum of the end terms have tableau $\Gamma$ and $\tilde{\Gamma}$, respectively. The main result is that $Q$ has LR-tableau $\Gamma$.

We use the cokernel maps from Proposition 7.1 to construct $Q$ as a pullback in the category of homomorphisms of $\Lambda$-modules.

$$
\begin{array}{c}
0 \\
\downarrow \\
\tilde{R} \\
\downarrow \\
0 \\
\end{array}
\begin{array}{c}
\rightarrow \\

\begin{array}{c}
0 \\
\downarrow \\
R' \\
\downarrow \\
0 \\
\end{array}
\begin{array}{c}
\rightarrow \\

\begin{array}{c}
Q \\
\rightarrow \\
R \\
\rightarrow \\
0 \\
\end{array}
\begin{array}{c}
0 \\
\downarrow \\
\tilde{R}' \\
\downarrow \\
E_{(n-n')} \\
\rightarrow \\
0 \\
\end{array}
\end{array}
\end{array}
$$

In this example, the middle row is not split exact:

**Example:** Consider the following two tableaux from the example in the Introduction.

\[
\Gamma = \Gamma_2 : \begin{array}{|c|c|}
\hline
1 & 2 \\
\hline
1 & 3 \\
\hline
2 &   \\
\hline
\end{array}
\quad \tilde{\Gamma} = \Gamma_{3a} : \begin{array}{|c|c|}
\hline
1 & 2 \\
\hline
2 & 3 \\
\hline
3 &   \\
\hline
\end{array}
\]

$\tilde{\Gamma}$ is obtained from $\Gamma$ by an increasing box move. It gives rise to the following pole decomposition.

For $\Gamma$, $R = P((2,4))$: \begin{array}{|c|c|}
\hline
1 &   \\
\hline
2 & 3 \\
\hline
3 &   \\
\hline
\end{array} \quad R' = P((0,1,3) \lor 0): \begin{array}{|c|c|}
\hline
1 & 2 \\
\hline
3 &   \\
\hline
   &   \\
\hline
\end{array} \cong \begin{array}{|c|c|}
\hline
1 & 2 \\
\hline
3 &   \\
\hline
   &   \\
\hline
\end{array}

For $\tilde{\Gamma}$, $\tilde{R} = P((1,4))$: \begin{array}{|c|c|}
\hline
1 &   \\
\hline
2 & 3 \\
\hline
3 &   \\
\hline
\end{array} \quad \tilde{R}' = P((0,2,3) \lor 2): \begin{array}{|c|c|}
\hline
1 & 2 \\
\hline
3 &   \\
\hline
   &   \\
\hline
\end{array} \cong \begin{array}{|c|c|}
\hline
1 & 2 \\
\hline
3 &   \\
\hline
   &   \\
\hline
\end{array}

Note that the embeddings $\tilde{R} \rightarrow R$ and $R' \rightarrow \tilde{R}'$ both have cokernel $E_{(1)}$.

The pullback $Q$ of the diagram has the following direct sum
Let $F \subseteq \mathcal{Q}$ of the above equation, is just the length of the module of boxes which are empty or have entry at most $\ell$ and $H/p$. We denote by $H$ the notation from above Proposition 7.1, in particular, $\mathcal{R} = ((a) \subset B)$ where $a = \sum p^i b_\alpha$ and $\mathcal{R}' = ((c) \subset D)$ where $c = \sum p^i d_\beta$.

Put $Q = ((r, s) \subset B \oplus D)$ where $r = (a, p^{n'-f} d_{n'})$ and $s = (0, c)$.

Clearly, $\mathcal{R}'$ embeds into the second component in $Q$; the cokernel of this embedding is $R$.

The module $\mathcal{R}$ embeds into the first component and the summand of the second component generated by $d_{n'}$; for $i \neq u$, $\tilde{b}_\beta_i$ is mapped to $b_\beta_i$ while $\tilde{b}_\beta_u = b_{n'}$ is mapped to $(p^{n-n'} b_n, d_{n'})$. In particular, $p^{f} b_{u} = p^{n'-f} b_{n'}$ maps to $(p^{n-f} b_n, p^{n'-f} d_{n'}) = (p^{f} b_n, p^{n'-f} d_{n'})$, so $a \in \tilde{B}$ is sent to $(a, p^{n'-f} d_{n'}) = r \in Q$. The cokernel of this map is $Q \rightarrow \tilde{R}'$; here the second component $\mathcal{D}$ of the ambient space is included into $\mathcal{D}$, as in the embedding $\mathcal{R}' \rightarrow \tilde{R}'$ above, and the map $B \rightarrow \tilde{D}$ is given by $b_\beta_i \mapsto 0$ if $i \neq u$ and $b_\beta_u = b_{n_1} \mapsto -d_{n_1}$.

With these modules and maps, all the squares in the pullback diagram are commutative.

**Proposition 8.1.** The embeddings $Q, \mathcal{R} \oplus \mathcal{R}'$ have the same tableau.

**Proof.** Suppose $\mathcal{R} \oplus \mathcal{R}'$ has tableau $\Gamma = (\gamma^{(\ell)})_\ell$ and $Q$ has tableau $\Delta = (\delta^{(\ell)})_\ell$. The two tableaux are equal if we can show for each pair of natural numbers $\ell$, $m$ that the tableaux $\Gamma, \Delta$ have the same number of boxes which are empty or have entry at most $\ell$ in the first $m$ rows:

$$(\gamma^{(\ell)})_1 + \cdots + (\gamma^{(\ell)})_m = (\delta^{(\ell)})_1 + \cdots (\delta^{(\ell)})_m.$$  

We denote by $H$ the common ambient space of $\mathcal{R} \oplus \mathcal{R}' = ((a, c) \subset B \oplus D)$ and of $Q = ((r, s) \subset B \oplus D)$, and denote the two subspaces by $U$ and $W$, respectively. If the embedding $\{U \subset H\}$ has tableau $\Gamma$, the embedding $\{(U + p^m H)/p^m H \subset H/p^m H\}$ has the tableau which consists of the first $\ell$ rows of $\Gamma$. The number of boxes which are empty or have entry at most $\ell$, given by the left hand side of the above equation, is just the length of the module

$$(H/p^m H)/(p^f U + p^m H)/p^m H) \cong H/(p^f U + p^m H).$$  

Let $(\mathcal{F}_a)$ and $(\mathcal{G}_a)$ be the filtrations of $B$ and $D$ centered at the elements $a$ and $c$, respectively, see Lemma 4.2. Consider the submodule generators for $Q$, decomposition.

\[
\begin{array}{cccc}
\begin{array}{|c|c|c|c|}
\hline
b_2 & b_3 & b_4 & b_5 \\
\hline
b_1 & b_2 & b_3 & b_4 \\
\hline
b_0 & & & \\
\hline
\end{array} & \mathcal{R} & \mathcal{R} & \mathcal{R} \\
\hline
\end{array}
\]

Each isomorphism is given by a change of the generators of the ambient space: $b_2' = b_2 + b_1; b_2'' = b_2 + pb_3, b_4' = -b_3; b_4'' = b_4 + b_5'$. The last sum contains a summand which is an indecomposable embedding which is not cyclic, see [12] (6.5)
\[ r = (a, p^{n'-f}d_n') \text{ and } s = (0, c). \] We define a filtration \((\mathcal{H}_\alpha)\) for \(H = B \oplus D\) by adjusting the filtrations \((\mathcal{F}_\alpha)\) and \((\mathcal{G}_\alpha)\) for \(B\) and \(D\) such that the elements \(r\) and \(s\) are homogeneous in degrees \(f\) and \(f'\), respectively:

\[ \mathcal{H}_\alpha = \mathcal{F}_{\alpha-f} \oplus \mathcal{G}_{\alpha-f'}. \]

We have seen above that we need to show that for all \(\ell, m \in \mathbb{N}\), the \(\Lambda\)-modules \(p^\ell U + p^m H\) and \(p^\ell W + p^m H\) have the same length. For this, it suffices to show that for each \(\alpha \in \mathbb{Z}, \ell, m \in \mathbb{N}\), the subspaces \(U_{\alpha,\ell,m}\) and \(W_{\alpha,\ell,m}\) of \(V_{\alpha} = \mathcal{H}_\alpha/\mathcal{H}_{\alpha-1}\) have the same dimension:

\[ U_{\alpha,\ell,m} = \frac{(p^\ell U + p^m H) \cap \mathcal{H}_\alpha + \mathcal{H}_{\alpha-1}}{\mathcal{H}_{\alpha-1}}; \quad W_{\alpha,\ell,m} = \frac{(p^\ell W + p^m H) \cap \mathcal{H}_\alpha + \mathcal{H}_{\alpha-1}}{\mathcal{H}_{\alpha-1}}. \]

This statement holds for \(\alpha > f - \ell\) and for \(\alpha \leq 0\) since \(U_{\alpha,\ell,m} = W_{\alpha,\ell,m}\).

Two cases remain:

Assume \(1 \leq \alpha \leq \max\{f - \ell, 1 + f - e\}\). Then \(V_{\alpha}\) has dimension \(a + v\), a basis is induced by the generators of \(B\) and \(D\): \(b_{\beta_1}, \ldots, b_{\beta_n} = b_n; d_{\delta_1}, \ldots d_{\delta_v} = d_{n'}\). Here is a sketch of the basis for \(V_{\alpha}\) together with the subpace generators for \(U_{\alpha,0,0}\) and \(W_{\alpha,0,0}\):

\[
\begin{align*}
U_{\alpha,0,0} : & \quad \begin{array}{cccc}
\text{I} & \text{II} & \cdots & \text{V} \\
\beta_1 & \beta_2 & \cdots & \delta_1 \\
b_n & d_{\delta_v - 1} & \delta_1 \\
d_{n'} & & & \\
\end{array} \\
W_{\alpha,0,0} : & \quad \begin{array}{cccc}
\text{I} & \text{II} & \cdots & \text{V} \\
\beta_1 & \beta_2 & \cdots & \delta_1 \\
b_n & d_{\delta_v - 1} & \delta_1 \\
d_{n'} & & & \\
\end{array}
\end{align*}
\]

If \(\alpha < n' - m\) then both \(U_{\alpha,\ell,m}\) and \(W_{\alpha,\ell,m}\) have dimension two, as there is no contribution from the term \(p^m H\). Otherwise, the basic element corresponding to \(d_{n'}\) is contained in the space generated by \(p^m H\), and hence \(U_{\alpha,\ell,m} = W_{\alpha,\ell,m}\).

It remains to consider the case where \(\max\{f - \ell, 1 + f - e\} < \alpha \leq f - \ell\) (this case does not occur if \(e = 1\)). Here, the space \(V_{\alpha}\) contains at least two additional basic elements, given by \(b_{\beta_{n'}}\) and \(d_{\delta_v + 1}\). Again, if \(\alpha < n' - m\), then both \(U_{\alpha,\ell,m}\) and \(W_{\alpha,\ell,m}\) have the same dimension \(2 + \dim[(p^m H \cap \mathcal{H}_\alpha) + \mathcal{H}_{\alpha-1}]/\mathcal{H}_{\alpha-1}\); otherwise the basic element corresponding to \(d_{n'}\) is contained in the space generated by \(p^m H\), and then the subspaces are equal: \(U_{\alpha,\ell,m} = W_{\alpha,\ell,m}\).

We have seen that for all choices of \(\alpha, \ell\) and \(m\), the spaces \(U_{\alpha,\ell,m}\), \(W_{\alpha,\ell,m}\) have the same dimension; it follows that \(Q\) and \(R \oplus R'\) have the same tableau: \(\Delta = \Gamma\).

The following example is a minor modification of the previous and may help to follow the details in the proof.

\textit{Example:} Consider the LR-tableaux:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4
\end{array}
\quad
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4
\end{array}
\]
Finite direct sums of cyclic embeddings

$\widetilde{\Gamma}$ is obtained from $\Gamma$ by an increasing box move which exchanges the columns $C(2,3)_5$ and $C(3,4)_4$ in $\Gamma$ by $C(3,4)_5$ and $C(2,3)_4$. It gives rise to the following pole decomposition: $R = P((0,3,4))$, $R' = P((0,1,2,3,5) \lor 1)$, $\widetilde{R} = P((0,2,3))$, $\widetilde{R}' = P((0,1,3,4,5) \lor 4)$.

$R : \begin{array}{|c|c|c|}
\hline
\hline
\hline
\hline
\hline
\end{array}$

$R' : \begin{array}{|c|c|c|}
\hline
\hline
\hline
\hline
\hline
\end{array}$

$\widetilde{R} : \begin{array}{|c|c|c|}
\hline
\hline
\hline
\hline
\hline
\end{array}$

$\widetilde{R}' : \begin{array}{|c|c|c|}
\hline
\hline
\hline
\hline
\hline
\end{array}$

In the proof we show that $Q$ and $R \oplus R'$ have the same tableau. The columns in the diagrams are labeled by the generators of $H_\Lambda$; the boxes in a row labeled $\alpha$ correspond to a basis for $V_\alpha$. Note that the submodule generators $r$, $s$ of $Q$ and $(a,0)$ and $(0,c)$ of $R \oplus R'$ are homogeneous elements in degrees $f$ and $f'$, respectively.

$Q : \begin{array}{|c|c|c|}
\hline
\hline
\hline
\hline
\hline
\end{array}$

$R \oplus R' : \begin{array}{|c|c|c|}
\hline
\hline
\hline
\hline
\hline
\end{array}$

9 The boundary relation for invariant subspace varieties

In this section we assume that the field $k$ is algebraically closed and that the discrete valuation ring $\Lambda$ is a $k$-algebra, for example the power series ring $\Lambda = k[[T]]$ or the localization of the polynomial ring $\Lambda = k[T]_{(T)}$. In each case, a finite dimensional $\Lambda$-module $B$ is a nilpotent linear operator, determined up to isomorphy by the partition which lists the sizes of the Jordan blocks. A cyclic embedding $(A \subset B)$ consists also of a subspace $A \subset B$, invariant under the action of the operator, and such that the induced action on $A$ gives rise to at most one Jordan block.

Here is our main result:

Theorem 9.1. Suppose the LR-tableau $\widetilde{\Gamma}$ is obtained from $\Gamma$ by an increasing box move. There is a one-parameter family of embeddings $Q(\mu)$ such that $Q(\mu)$ has tableau $\Gamma$ for $\mu \neq 0$ and tableau $\widetilde{\Gamma}$ for $\mu = 0$.

Example: We continue the example from the beginning of Section 8. For $\mu \in k$, the module $Q(\mu)$ is defined as follows:

$Q(\mu) : \begin{array}{|c|c|c|}
\hline
\hline
\hline
\hline
\hline
\end{array}$

$Q : \begin{array}{|c|c|c|}
\hline
\hline
\hline
\hline
\hline
\end{array}$

$R \oplus R' : \begin{array}{|c|c|c|}
\hline
\hline
\hline
\hline
\hline
\end{array}$

$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\hline
\hline
\hline
\hline
\end{array}$
We have also pictured $Q$ and $\bar{R} \oplus \bar{R}'$. Clearly, for $\mu = 0$, $Q(\mu) \cong \bar{R} \oplus \bar{R}'$.

For $\mu \neq 0$, write $Q(\mu)$ as the embedding $((x, y) \subset B \oplus D)$ where $B \oplus D$, as the ambient space of $R \oplus R'$, is generated by $b_5, b_3, b_2, d_4, d_1$, as before, and $x = p^3b_5 + pd_2, y = pb_3 + \mu d_2 + pd_4 + d_1$. The successive substitutions in $Q(\mu)$:

$d_2' = pb_3 + \mu d_2$ (so $y = d_2' + pd_4 + d_1, x = p^3b_5 - p^2b_3/\mu + d_2'/\mu$) and $x'' = \mu x, b_5'' = \mu b_5, b_3'' = -b_3$ (so $x'' = p^3b_5'' + p^2b_3'' + pd_2'$) show that $Q(\mu) \cong Q$.

Proof of Theorem 9.1 The proof will follow the above example. We assume

For $LR$-tableaux $\Gamma$, $(\mu, s)$ embedding $((x, y) \subset B \oplus D)$ where $x = (a - p^{k_1}b_n, p^{n''-f}d_n'), y = (p^{k_1+n'-n''}b_n, c + (\mu - 1)p^{k_1}d_n')$.

For $\mu = 0$, there is a decompositions of $Q(0)$ given by the decomposition of the ambient space as

$$B \oplus D = \left( \bigoplus_{i \neq u} b_{\beta_i, \Lambda} \oplus d_{\delta_i, \Lambda} \right) \oplus \left( b_{\beta_u, \Lambda} \oplus \bigoplus_{i \neq v} d_{\delta_i, \Lambda} \right).$$

Note that $x$ is in the first summand, $\bar{B}$ say, and $\bar{R} \cong ((x) \subset \bar{B})$. Also $y$ is in the second summand, $\bar{D}$ say, and $\bar{R}' \cong ((y) \subset \bar{D})$.

Let $\mu \neq 0$, it remains to show that $Q(\mu) \cong Q$. Substitute $d_n' = p^{n-n'}, b_n + \mu d_n', D' = \bigoplus_{i \neq v} d_{\delta_i, \Lambda} \oplus d_n', \Lambda$, this yields $y = (0, (c - p^{k_1}d_n') + p^{k_1}d_n') \in B \oplus D'$, $x = (a - p^{k_1}b_n, p^{n'-f}/\mu b_n, p^{n''-f}/\mu d_n')$. Then substitute $x'' = x, b_5'' = \mu b_{\beta_i}$ for $i \neq u$ and $b_n'' = -b_n, a'' = \sum p^{k_1}b_{\beta_i}''$ to obtain $x'' = (a'', p^{n'-f}d_n')$. This shows

$$Q(\mu) = ((x'', y) \subset B \oplus D') \cong ((r, s) \subset B \oplus D) = Q.$$

We obtain the following consequence for the representation space of invariant subspaces of nilpotent linear operators. Let $\alpha, \beta, \gamma$ be partitions. Then the subset $\mathcal{V}_{\alpha, \beta}^\delta$ of the affine variety $\text{Hom}_k(N_\alpha, N_\beta)$ consisting of all $k[T]$-linear monomorphisms with cokernel isomorphic to $N_\gamma$ is constructible. If $\mathcal{V}_\Gamma$ denotes the subset of all embeddings with tableau $\Gamma$, then the irreducible components of $\mathcal{V}_{\alpha, \beta}^\delta$ are given by the closures $\mathcal{V}_\Gamma$, where $\Gamma$ runs over all LR-tableaux of shape $(\alpha, \beta, \gamma)$.

For LR-tableaux $\Gamma$, $\tilde{\Gamma}$ of the same shape we write $\Gamma \sim_{\text{boundary}} \tilde{\Gamma}$ if $\mathcal{V}_\Gamma \cap \mathcal{V}_{\tilde{\Gamma}} \neq \emptyset$. Theorem 9.1 yields the following combinatorial criterion for the boundary relation which we stated above as Theorem 1.1.
Corollary 9.2. Suppose \( \tilde{\Gamma} \) is obtained from \( \Gamma \) by an increasing box move. Then \( \Gamma \prec_{\text{boundary}} \tilde{\Gamma} \).

For tableaux \( \Gamma, \tilde{\Gamma} \) of the same shape \((\alpha, \beta, \gamma)\), we have seen in [8] that the boundary relation implies the dominance relation (which is given by the dominance relation for the defining partitions). As a consequence, the reflexive and transitive closure of the boundary relation \( \prec_{\text{boundary}} \) defines a partial order \( \leq_{\text{boundary}} \) on the set of LR-tableaux of the given shape \((\alpha, \beta, \gamma)\). This is the poset \( P_{\text{boundary}} \).

In case the skew diagram \( \beta \setminus \gamma \) is a horizontal and vertical strip, the dominance relation is generated by the increasing box moves ([7]). As a consequence we can describe the relation \( \leq_{\text{boundary}} \) in case \( \beta \setminus \gamma \) is a horizontal and vertical strip.

Corollary 9.3 (Corollary [12]). The reflexive and transitive closure of the boundary relation is a partial order on the set of all LR-tableaux of a given shape \((\alpha, \beta, \gamma)\). In case \( \beta \setminus \gamma \) is a horizontal and vertical strip, the dominance order, the closure of the box order, and the closure of the boundary relation agree.

The case where the induced operator on the subspace is \( T^2 \)-bounded has been studied in [5, 6]. Here, the boundary relation (and not only its closure) is equivalent to the dominance order ([6, Section 5.3]).

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