2-Modules and the Representations of 2-Rings

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Abstract: In this paper, we develop 2-dimensional algebraic theory which closely follows the classical theory of modules. The main results are giving definitions of 2-modules and the representations of 2-rings. Moreover, for a 2-ring \( R \), we prove that its modules form a 2-abelian category.

Keywords: 2-Modules; Representation; 2-Rings; 2-Abelian Category

1 Introduction

Ordinary algebra (which we call it 1-dimensional algebra) is the study of algebraic structures on sets, i.e. of sets equipped with certain operations satisfying certain equations. The work on 2-dimensional algebra is to study the algebraic structures on groupoids\[19\].

One of the goals of higher-dimensional algebra is to categorify mathematical concepts, a very example is (symmetric) 2-groups \([2, 3, 5, 6, 10, 11, 12, 17, 19, 21]\), which play the role similar as (abelian) groups in 1-dimensional algebra. The higher-dimensional algebra was studied and used in many fields of mathematics such as algebraic geometry \([21, 3]\), topological field theory \([13, 14]\), etc.

In \([8]\), M.Jibladze and T.Pirashvili introduced categorical rings(We call them 2-rings). As 1-dimensional algebra, it’s natural to define 2-modules and the repre-

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sentations of 2-rings. M. Dupont in his PhD. thesis [19], mentioned the 2-modules, as the additive Gpd-functors, also proved that the 2-category formed by these 2-modules is a 2-abelian Gpd-category.

In this paper, we give a definition of the 2-module following the 1-dimensional case. When we finish this paper, we find that V. Schmitt gave another definition similar as 2-module([20]). But our definition is much closer to the classical case. The equivalence between our definition and M. Dupont’s definition will be given in our coming paper.

Abeian category plays an important role in homology theory, localization theory, representation theory, etc[1, 4, 18]. So 2-abelian category should be an important tool in studying higher dimensional algebraic theory.

In our coming papers, we will prove that for any a 2-abelian category $A$, there exists a 2-ring $R$, and the embedding of $A$ into the Gpd-category $(R\text{-}2\text{-}Mod)$, extending from Freyd-Mitchell Embedding Theorem[4], and study the (co)homology theory in 2-abelian category similar as in 1-dimensional case.

This paper is organized as follows:

After recalling basic facts on (symmetric)2-groups, 2-rings, we give the definitions of $R\text{-}2\text{-}modules$, $R\text{-}homomorphism$ between them, and morphism between $R\text{-}homomorphisms$, where $R$ is a 2-ring. As an application of $R\text{-}2\text{-}modules$, we also give the definition of representation of 2-rings. For our next work, we concretely construct the 2-category structure of all $R\text{-}2\text{-}modules$. In section 3, we prove $(R\text{-}2\text{-}Mod)$ is an additive, and also a 2-abelian Gpd-category by using the similar methods as M. Dupont discussing the symmetric 2-groups([19]). This section is the main part of this paper.

2 Basic Results on 2-Modules

Our goal in this section is to give the definition of $R\text{-}2\text{-}modules$, and the representations of 2-rings. We will begin by reviewing some definitions about (symmetric)2-groups and 2-rings, also called categorical groups and categorical rings in [5, 6, 8, 11, 17].

Definition 1. [8] A 2-group $A$ is a groupoid equipped with a monoidal structure,
i.e. a bifunctor \( + : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \), an unit object \( 0 \in \mathcal{A} \), and natural isomorphisms:

\[
\langle a, b, c \rangle: (a + b) + c \to a + (b + c),
\]

\[
l_a : 0 + a \to a,
\]

\[
r_a : a + 0 \to a
\]
satisfying the Mac Lane coherence conditions, i.e. the following diagrams commute:

Moreover, for each object \( a \in \mathcal{A} \), there exists an object \( a^* \in \mathcal{A} \), and an isomorphism \( \eta_a : a^* + a \to 0 \).

**Definition 2.** [19] Symmetric 2-group is a 2-group \( \mathcal{A} \), together with the natural isomorphism \( c_{a,b} : a + b \to b + a \) satisfies \( c_{a,b} \circ c_{b,a} = id \). Also, \( c_{a,b} \) is compatible with \( \langle -, -, - \rangle \) of the monoidal structure \( + \).

**Remark 1.** In our paper, when we say \( \mathcal{A} \) is a symmetric 2-group, it means \( (\mathcal{A}, 0, +, \langle - , -, - \rangle, l_-, r_-, \eta_-, c_{-, -}) \).

For any \( a, b, c, d \in obj(\mathcal{A})(\mathbb{R}) \),

\[
\langle a, b, c, d \rangle : (a + b) + (c + d) \to (a + c) + (b + d)
\]

will be denoted the composite canonical isomorphism in the following commutative diagram:
Definition 3. [19] Let $\mathcal{A}, \mathcal{B}$ be 2-groups. A homomorphism $F = (F, F_+, F_0) : \mathcal{A} \to \mathcal{B}$ consists of a functor $F : \mathcal{A} \to \mathcal{B}$ and two natural morphisms:

$$F_+(a, b) : F(a + b) \to F(a) + F(b),$$

$$F_0 : F(0) \to 0$$

such that the following diagrams commute:

\[ \begin{array}{c}
F((a+b)+c) & \xrightarrow{F(<a,b,c>)} & F(a+(b+c)) \\
F(a+b) + F(c) & \xrightarrow{F_+(a,b,c)} & F(a) + F(b+c) \\
(F(a) + F(b)) + F(c) & \xrightarrow{<F_a,F_b,F_c>} & F(a) + (F(b) + F(c)) \\
\end{array} \]

\[ \begin{array}{c}
F(a+0) & \xrightarrow{F_a} & Fa \\
F_+(a,0) & \xrightarrow{F_+} & Fa + F_0 \\
Fa + F_0 & \xrightarrow{F_a+F_0} & Fa + 0 \\
\end{array} \]
Remark 2. (i) The homomorphism $F = (F, F_+, F_0)$ is called strict when isomorphisms $F_+, F_0$ are identities.

(ii) If $A, B$ are two symmetric 2-groups, the homomorphism of symmetric 2-groups is the homomorphism $F = (F, F_+, F_0) : A \to B$, together with the following commutative diagram:

(iii) Given homomorphisms $A \xrightarrow{F=(F,F_+,F_0)} B \xrightarrow{G=(G,G_+,G_0)} C$ of 2-groups, their composition is $H = (H, H_+, H_0) : A \to C$, where $H = G \circ F : A \to C$ is a composition of functors, and $H_+, H_0$ are the following compositions:

$$H_+(a, c) : H(a + c) = (GF)(a + c) = G(F(a + c))$$

$$H_0 : (GF)(0) = G(F(0)) \xrightarrow{G(F_0)} G(0) \xrightarrow{G_0} 0.$$

It is easy to check $H = (H, H_+, H_0)$ is a homomorphism from $A$ to $C$.

Notation. The homomorphism $F = (F, F_+, F_0)$ of 2-groups is in fact a functor satisfies some compatible conditions(Fig.4.-6.). So we will only write $F$ for abbreviation.

Definition 4. Given homomorphisms $F, G : A \to B$ of 2-groups, a morphism from $F$ to $G$ is a natural transformation $\varepsilon : F \Rightarrow G$ such that, for any objects $a, b \in A$, the following diagrams commute:
Remark 3. The morphism between two homomorphisms $F$, $G$ of 2-groups $\mathcal{A}, \mathcal{B}$ is a natural isomorphism. In fact, for any object $a \in \mathcal{A}$, $\varepsilon_a : F(a) \to G(a)$ is a morphism in groupoid $\mathcal{B}$, so $\varepsilon_a$ is invertible.

Proposition 1. There is a 2-category $(2\text{-Gp})$ with 2-groups as objects, homomorphisms of 2-groups as 1-morphisms, and morphisms of homomorphisms as 2-morphisms. If the 2-groups are symmetric 2-groups, we denote this 2-category by $(2\text{-SGp})$.

Definition 5. A 2-ring is a symmetric 2-group $\mathcal{R}$, together with a bifunctor $\cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ (denoted by multiplication), an object $1 \in \mathcal{R}$, and natural isomorphisms:

\[
[r, s, t] : (r \cdot s) \cdot t \to r \cdot (s \cdot t) \text{ (associativity)},
\]
\[
\lambda_r : 1 \cdot r \to r \text{ (left unitality)},
\]
\[
\rho_r : r \cdot 1 \to r \text{ (right unitality)},
\]
\[
[r s_0]^+ : r \cdot (s_0 + s_1) \to r \cdot s_0 + r \cdot s_1 \text{ (left distributivity)},
\]
\[
<r_0 s] : (r_0 + r_1) \cdot s \to r_0 \cdot s + r_1 \cdot s \text{ (right distributivity)}.
\]

It is required that the $[\cdot, \cdot, \cdot]$, together with $\lambda_-$ and $\rho_-$ constitute a monoidal structure (Fig.1.-2. commute). Moreover, the following diagrams commute for all possible objects of $\mathcal{R}$:

\[
\begin{align*}
F(a + b) \xrightarrow{F} F(a) + F(b) \\
G(a + b) \xrightarrow{G} G(a) + G(b)
\end{align*}
\]

\[
\begin{align*}
F(0) \xrightarrow{\varepsilon_0} G(0)
\end{align*}
\]
\[
(r(s_0 + s_1))t \xrightarrow{(r^0 + r_1) t} (rs_0 + rs_1) t \xrightarrow{(r^0 + r_1) t < (r^0 t)} (rs_0 t + (rs_1) t)
\]

\[
[r, s_0 + s_1, t] \xrightarrow{[r, s_0 + s_1, t]} \xrightarrow{[r, s_0 + s_1, t] + [r, s_1, t]}
\]

\[
r((s_0 + s_1))t \xrightarrow{r < (r^0 t)} r(s_0 t) + r(s_1) t \xrightarrow{r(s_0 t) + r(s_1) t} r(s_0 t + s_1) t
\]

Fig. 9.

\[
(r_0 s + r_1 s) t \xrightarrow{(r_0 s t) + (r_1 s t) t} \xrightarrow{(r_0 s t) + (r_1 s t) t < (r_0 s t) t < (r_1 s t) t}
\]

\[
[(r_0 + r_1) s t] \xrightarrow{[(r_0 + r_1) s t] + [(r_1 + r_1) s t]}
\]

\[
(r_0 + r_1) (s t) \xrightarrow{r_0 (s t) + r_1 (s t)} r_0 (s t) + r_1 (s t)
\]

Fig. 10.

\[
1(r_0 + r_1) \xrightarrow{(r_0 + r_1) l} r_0 + r_1
\]

\[
\lambda_{\omega q} \xrightarrow{\lambda_{\omega q} + \lambda_q} r_0 + r_1
\]

\[
\rho_{\omega 4} \xrightarrow{\rho_{\omega 4} + \rho_4}
\]

\[
(r_0 + r_1) l \xrightarrow{r_0 l + r_1 l}
\]

Fig. 11.

\[
r((s_{00} + s_{01}) + (s_{10} + s_{11})) \xrightarrow{r((s_{00} + s_{01}) + (s_{10} + s_{11}))} r(s_{00} + s_{01}) + r(s_{10} + s_{11}) \xrightarrow{r(s_{00} + s_{01}) + r(s_{10} + s_{11})}
\]

\[
\xrightarrow{r(s_{00} + s_{01}) + (r_{10} + r_{11})} (r_{10} + r_{11}) t (s_0 + s_1)
\]

\[
\xrightarrow{r(s_{00} + s_{01}) + (r_{10} + r_{11}) t (s_0 + s_1) + \xrightarrow{r(s_{00} + s_{01}) + (r_{10} + r_{11}) t (s_0 + s_1)}}
\]

\[
\xrightarrow{(r_0 s_0 + r_1 s_0) + (r_1 s_0 + r_1 s_1)} (r_0 s_0 + r_1 s_0) + (r_1 s_0 + r_1 s_1)
\]

Fig. 12.
\[(r_0 + r_1) + (r_0 + r_1)s <_{\eta_0 + \eta_1} \eta_1 \rightarrow (r_0 + r_0)s + (r_0 + r_1)s \xrightarrow{\eta_0 [r + \eta_1]} (r_0 + r_0)s + (r_0 + r_1)s \]

\[(r_0 + r_0) + (r_0 + r_1)s <_{\eta_0 + \eta_1} \eta_1 \rightarrow (r_0 + r_0)s + (r_0 + r_1)s \xrightarrow{\eta_0 [r + \eta_1]} (r_0 + r_0)s + (r_0 + r_1)s \]

Fig.14.

Remark 4. 1. N.T. Quang in [14] discussed the relations between these 2-rings and Ann-categories.

2. If the above natural isomorphisms are identities, \( R \) is called a strict 2-ring.

**Definition 6.** [[8]] A homomorphism of 2-rings is a quadruple \( F = (F, F_+, F_\cdot, F_0) \), where \( F \) is a functor from \( R_1 \) to \( R_2 \), \( F_+, F_\cdot \) are natural morphisms of the forms

\[ F_+(r_0, r_1) : F(r_0 + r_1) \rightarrow F(r_0) + F(r_1) \]

\[ F(r_0, r_1) : F(r_0 \cdot r_1) \rightarrow F(r_0) \cdot F(r_1) \]

and \( F_1 : F(1) \rightarrow 1 \), \( F_0 : F(0) \rightarrow 0 \) are morphisms, such that \( (F, F_+, F_0), (F, F_\cdot, F_1) \) are monoidal functors with respect to the monoidal structures corresponding to the + and \( \cdot \), respectively. Moreover the diagrams commute:

Fig.15.

**Notation.** The homomorphism \( F = (F, F_+, F_\cdot, F_0) \) of 2-rings is in fact a functor \( F \) satisfies some compatible conditions(Fig.4.,5.,15.,16.). So, we will only write \( F \) for abbreviation.

**Definition 7.** The morphism of homomorphisms \( F, G : R_1 \rightarrow R_2 \) of 2-rings is a natural transformation \( \varepsilon : F \Rightarrow G \), such that, for any objects \( r, s \in R_1 \), the following diagrams commute:

Fig.16.
**Proposition 2.** [19] There is a 2-category with 2-rings as objects, homomorphisms of 2-rings as 1-morphisms, and morphisms of homomorphisms as 2-morphisms.

**Definition 8.** Let $\mathcal{R}$ be a 2-ring. An $\mathcal{R}$-2-module is a symmetric 2-group $\mathcal{M}$ equipped with a bifunctor $\cdot : \mathcal{R} \times \mathcal{M} \to \mathcal{M}$ (called operation of $\mathcal{R}$ on $\mathcal{M}$) denoted by $(r, m) \mapsto r \cdot m$ and natural isomorphisms:

- $a_{m,n}^r : r \cdot (m + n) \to r \cdot m + r \cdot n,$
- $b_{m,s}^r : (r + s) \cdot m \to r \cdot m + r \cdot n,$
- $b_{r,s,m}^r : (rs) \cdot m \to r \cdot (s \cdot m),$  
- $i_m : I \cdot m \to m,$  
- $z_r : r \cdot 0 \to 0$

such that the following diagrams commute:

Fig. 17.

**Fig. 18.**
Fig. 19.

Fig. 20.

Fig. 21.

Fig. 22.

Fig. 23.
Fig. 24.

\[
((r_1 + r_2) \cdot m) \xrightarrow{b_{\gamma,2,r,m}} (r_1 r_2) \cdot m
\]

\[
(r_1 + r_2) \cdot m \xrightarrow{b_{r,1,r,m}} (r_1 r_2) \cdot m
\]

\[
(r_1 + r_2) \cdot m \xrightarrow{b_{r,1,r,m} + b_{\gamma,2,r,m}} r_1 \cdot (r \cdot m) + r_2 (r \cdot m)
\]

Fig. 25.

\[
((r_0 + r_{01}) + (r_{01} + r_1)) \cdot m \xrightarrow{a_{m}^{n_0 \cdot r_{01} \cdot m} + r_{r_1} \cdot m} (r_0 + r_{01}) \cdot m + (r_{01} + r_1) \cdot m
\]

\[
((r_0 + r_{01}) + (r_{01} + r_1)) \cdot m \xrightarrow{a_{m}^{n_0 \cdot r_{01} \cdot m} + r_{r_1} \cdot m} (r_0 \cdot m + r_{01} \cdot m) + (r_{01} \cdot m + r_1 \cdot m)
\]

\[
(r_0 + r_{01}) \cdot m + (r_{01} + r_1) \cdot m \xrightarrow{a_{m}^{n_0 \cdot r_{01} \cdot m} + r_{r_1} \cdot m} (r_0 \cdot m + r_{01} \cdot m) + (r_{01} \cdot m + r_1 \cdot m)
\]

Fig. 26.

\[
r \cdot ((m_0 + m_{01}) + (m_0 + m_{1})) \xrightarrow{b_{m_0 + m_{01}, m_0 + m_{1}}} r \cdot (m_0 + m_{01}) + r \cdot (m_0 + m_{1})
\]

\[
r \cdot ((m_0 + m_{01}) + (m_0 + m_{1})) \xrightarrow{b_{m_0 + m_{01}, m_0 + m_{1}}} r \cdot (m_0 + m_{01}) + r \cdot (m_0 + m_{1})
\]

\[
(r \cdot m_0 + r \cdot m_{10}) + (r \cdot m_0 + r \cdot m_{1}) \xrightarrow{b_{m_0 + m_{01}, m_0 + m_{1}}} (r \cdot m_0 + r \cdot m_{10}) + (r \cdot m_0 + r \cdot m_{1})
\]

Fig. 27.

\[
1 \cdot (m_1 + m_2) \xrightarrow{i_{1}, m_{02}} 1 \cdot m_1 + 1 \cdot m_2
\]

\[
(r_{1}) \cdot m \xrightarrow{b_{r,1,m}} r \cdot (1 \cdot m)
\]

Fig. 28.

Fig. 29.
Definition 9. For two \( \mathcal{R} \)-2-modules \( \mathcal{M}, \mathcal{N} \), the \( \mathcal{R} \)-homomorphism between them is a quadruple \( F = (F, F_+, F_0, F_2) \), where \( (F, F_+, F_0) : \mathcal{M} \to \mathcal{N} \) is a homomorphism of symmetric 2-groups \( \mathcal{M}, \mathcal{N} \), \( F_2 \) is a natural isomorphism of the form

\[
F_2(r, m) : F(r \cdot m) \to r \cdot F(m),
\]

for \( r \in \mathcal{R}, \, m \in \mathcal{M} \), such that the following diagrams commute for all possible objects of \( \mathcal{R} \) and \( \mathcal{M} \), respectively.

Fig. 30.

![Diagram](image1)

Fig. 31.

![Diagram](image2)

Fig. 32.

![Diagram](image3)

Fig. 33.

![Diagram](image4)

Fig. 34.
Definition 10. The morphism of two $\mathcal{R}$-homomorphisms $F = (F, F_+, F_0, F_2)$, $G = (G, G_+, G_0, G_2) : \mathcal{M} \to \mathcal{N}$ of $\mathcal{R}$-2-modules is a natural transformation $\tau : F \Rightarrow G$ such that $\tau$ is the morphism from $(F, F_+, F_0)$ to $(G, G_+, G_0)$ as homomorphisms of symmetric 2-groups, moreover, the following diagram commutes, for $r \in \mathcal{R}, m \in \mathcal{M}$:

\[
\begin{array}{ccc}
F(r \cdot m) & \xrightarrow{F_2(r \cdot m)} & r \cdot F(m) \\
\downarrow^\tau & & \downarrow^{r \cdot \tau}
\end{array}
\]

Fig.37.

Notation. A homomorphism $F = (F, F_+, F_0, F_2)$ of $\mathcal{R}$-2-modules is in fact a functor $F$ satisfying some compatible conditions(Fig.32.-36.). So, we always write $F$ for abbreviation.

Theorem 1. Let $\mathcal{R}$ be a 2-ring. The $\mathcal{R}$-2-modules is a 2-category with objects $\mathcal{R}$-2-modules, denoted by $(\mathcal{R}$-2-Mod$)$. Its 1-morphisms are $\mathcal{R}$-homomorphisms between $\mathcal{R}$-2-modules, 2-morphisms are morphisms of $\mathcal{R}$-homomorphisms.

Proof. To prove this theorem we need to check the homomorphisms satisfy the definition of a 2-category [2].

1) 1-morphisms can be composed as a category.

Let $F : \mathcal{M}_1 \to \mathcal{M}_2$, $G : \mathcal{M}_2 \to \mathcal{M}_3$ be $\mathcal{R}$-homomorphisms. The composition of them is a composition functor $H \triangleq G \circ F : \mathcal{M}_1 \to \mathcal{M}_3$, together with natural
isomorphisms:

\[ H_+(m_1, m_2) : H(m_1 + m_2) = G(F(m_1 + m_2)) \xrightarrow{G(F_+)} G(F(m_1) + F(m_2)) \]
\[ G_+ : G(F(m_1)) + G(F(m_2)) = H(m_1) + H(m_2). \]

\[ H_0 : H(0) = G(F(0)) \xrightarrow{G(F_0)} G(0) \xrightarrow{G_0} 0. \]

\[ H_2(r, m) : H(r \cdot m) = G(F(r \cdot m)) \xrightarrow{G(F_2)} G(r \cdot F(m)) \xrightarrow{G_2} r \cdot G(F(m)) = r \cdot H(m). \]

2) 2-morphisms can be composed in two distinct ways.

- Vertical composition of 2-morphisms.

Let \( F, F', F'' : \mathcal{M}_1 \to \mathcal{M}_2 \) be \( \mathcal{R} \)-homomorphisms of \( \mathcal{R} \)-2-modules. \( \tau : F \Rightarrow F' \), \( \sigma : F' \Rightarrow F'' \) are morphisms between them, the composition of \( \tau \) and \( \sigma \) is \( \varepsilon = \sigma \circ \tau : F \Rightarrow F'' \) given by the family of morphisms \( \{ \varepsilon_m \triangleq \sigma_m \circ \tau_m : F(m) \to F''(m) \ \forall m \in \mathcal{M}_1 \} \) and such that the following diagram commutes:

- Horizontal composition of 2-morphisms.

Let \( \alpha : F \Rightarrow F' : \mathcal{M}_1 \to \mathcal{M}_2 \), \( \beta : G \Rightarrow G' : \mathcal{M}_2 \to \mathcal{M}_3 \) be morphisms of \( \mathcal{R} \)-homomorphisms. The composition of \( \alpha \) and \( \beta \) is \( \tau = \beta \circ \alpha : (G \circ F) \Rightarrow (G' \circ F') \) given by the family of morphisms \( \{ \tau_m \triangleq \beta_{F'(m)} \circ G(\alpha_m) : (G \circ F)(m) \to (G' \circ F')(m), \ \forall m \in \mathcal{M}_1 \} \), such that the following diagrams commute:
The above 1), 2), satisfy
(i) Composition of 1-morphisms is associative, i.e. for \( R \)-homomorphisms \( \mathcal{M}_1 \xrightarrow{F} \mathcal{M}_2 \xrightarrow{G} \mathcal{M}_3 \xrightarrow{H} \mathcal{M}_4 \), we have
\[
(H \circ G) \circ F = H \circ (G \circ F).
\]
In fact, there is an identity morphism \( \varphi : (H \circ G) \circ F \Rightarrow H \circ (G \circ F) \), defined by,
\[
\varphi_m \triangleq 1_m : ((H \circ G) \circ F)(m) = (H \circ G)(F(m)) = H(G(F(m))) = (H \circ (G \circ F))(m),
\]
for any \( m \in \mathcal{M}_1 \), also \( \varphi \) such that the Fig.4-6. and Fig.37. commute.

Moreover, for any \( \mathcal{R} \)-2-module, there is a \( \mathcal{R} \)-homomorphism \( 1_M = (1_M, id, id) : \mathcal{M} \rightarrow \mathcal{M} \) given by \( 1_M(m) = m, 1_M(f) = f \), for any object \( m \) and morphism \( f \) in \( \mathcal{M} \), such that, for any \( F : \mathcal{M} \rightarrow \mathcal{N} \), and \( G : \mathcal{K} \rightarrow \mathcal{M} \), we have \( F \circ 1_M = F \) and \( 1_M \circ G = G \).

(ii) Vertical composition is associative, i.e. for any 2-morphisms as in the diagram

\[
\begin{align*}
\begin{array}{c}
\xymatrix{
\mathcal{M}_1 
\ar[r]^-F & \mathcal{M}_2 
\ar[r]^-G & \mathcal{M}_3 
\ar[r]^-H & \mathcal{M}_4
}
\end{array}
\end{align*}
\]

From the composition of morphisms in \( \mathcal{M}_2 \), we have
\[
((\varepsilon \circ \tau) \circ \sigma)_m = (\varepsilon_m \circ \tau_m) \circ \sigma_m = \varepsilon_m \circ (\tau_m \circ \sigma_m) = \varepsilon_m \circ (\tau \circ \sigma)_m.
\]
Moreover, for any \( F : \mathcal{M}_1 \rightarrow \mathcal{M}_2 \), there exists \( 1_F : \mathcal{F} \Rightarrow F \) given by \( 1_F \triangleq 1_F \),
where \( (1_F)_m = 1_{Fm} \), such that, for any \( \sigma : F \Rightarrow G \), we have \( \sigma \circ 1_F = \sigma \), since
\[
(\sigma \circ 1_F)_m = \sigma_m \circ (1_F)_m = \sigma_m \circ 1_{Fm} = \sigma_m,
\]
for any object \( m \in \mathcal{M}_1 \).

(iii) Horizontal composition is associative, i.e. for any 2-morphisms as in the following diagram:
We need to check

\[(\gamma \ast \beta) \ast \alpha = \gamma \ast (\beta \ast \alpha).\]

In fact, for any object \(m \in \mathcal{M}_1\), from the definition of the horizontal composition and natural transformation, the following commutative ensures the associativity.

(iv) Vertical composition and horizontal composition of 2-morphisms satisfy the exchange law.

For any 1-morphisms and 2-morphisms as in the following diagram

\[
\begin{array}{c}
\mathcal{M}_1 \\ \beta_1 \downarrow \\
\mathcal{M}_2 \\ \alpha_1 \downarrow \\
\mathcal{M}_3 \\ \beta_2 \downarrow \\
\end{array}
\]

We need to check

\[(\beta' \circ \beta) \ast (\alpha' \circ \alpha) = (\beta \ast \alpha) \circ (\beta' \ast \alpha').\]

In fact, for any \(m \in \mathcal{M}_1\), we have the following commutative diagram
Note that, the above diagram commutes, since the small diagrams I and II commute from the properties of natural transformations $\beta, \beta'$.

The above ingredients satisfy the definition of 2-category, so all $\mathcal{R}$-2-modules form a 2-category. $\square$

**Proposition 3.** Let $\mathcal{A}$ be a symmetric 2-group, $\mathcal{E}_{nd}\mathcal{A} = \{F : \mathcal{A} \to \mathcal{A} \text{ is an endomorphism of } \mathcal{A}\}$. Then $\mathcal{E}_{nd}\mathcal{A}$ is a 2-ring.

**Proof.** Step 1. $\mathcal{E}_{nd}\mathcal{A}$ is a category consists of the following data:

- Object is an endomorphism $F : \mathcal{A} \to \mathcal{A}$.

- Morpshism is a morphism between two $\mathcal{R}$-endomorphisms. Composition of morphisms:

$$
\text{Hom}(F, G) \times \text{Hom}(G, H) \longrightarrow \text{Hom}(F, H)
$$

$$(\tau, \sigma) \mapsto \sigma \circ \tau
$$

is defined by, $(\sigma \circ \tau)_A \triangleq \sigma_A \circ \tau_A$ as a composition of morphisms in $\mathcal{A}$, for any $A \in \mathcal{A}$, which is a morphism from $F$ to $H$, for $\tau, \sigma$ such that Fig.7. commutes.

The above ingredients are subject to the following axioms:

(1) For $F \in \text{obj}(\mathcal{E}_{nd}\mathcal{A})$, there is an identity $1_F \in \text{Hom}(F, F)$, defined by $(1_F)_A \triangleq 1_{F(A)}$, for any $A \in \mathcal{A}$, such that for any $\tau : F \Rightarrow G$, $\tau \circ 1_F = \tau$, since $(\tau \circ 1_F)_A = \tau_A \circ 1_{FA} = \tau_A$.

(2) Given morphisms $\tau : F \Rightarrow G$, $\sigma : G \Rightarrow H$, and $\epsilon : H \Rightarrow J$, from the associativity of composition of morphisms in $\mathcal{A}$, we have

$$(\epsilon \circ \sigma) \circ \tau = \epsilon \circ (\sigma \circ \tau).$$
Step 2. $\mathcal{E}nd\mathcal{A}$ is a groupoid, i.e. for any morphism $\tau : F \Rightarrow G : \mathcal{A} \to \mathcal{A}$ has an inverse. For any object $a \in \mathcal{A}$, $\tau_a : F(a) \to G(a)$ ia a morphism in $\mathcal{A}$, and $\mathcal{A}$ is a groupoid, $\tau_a$ has an inverse $(\tau_a)^*$, so the inverse of $\tau$ is defined by $(\tau^*)_a \triangleq (\tau_a)^*$.

Step 3. $\mathcal{E}nd\mathcal{A}$ is a monoidal category.

- There is a monoidal structure on $\mathcal{E}nd\mathcal{A}$, i.e. there is a bifunctor

  $$ + : \mathcal{A} \times \mathcal{A} \to \mathcal{A} $$

  $$ (F, G) \mapsto F + G, $$

  $$ (\tau : F \Rightarrow G, \tau' : F' \Rightarrow G') \mapsto \tau + \tau' : F + F' \Rightarrow G + G' $$

  where $F + G$ and $\tau + \tau'$ are given as follows:

  $$(F + G)(a) \triangleq F(a) + G(a), \ (\tau + \tau')_a \triangleq \tau_a + \tau'_a$$ under the monoidal structure of $\mathcal{A}$.

  $$(F + G)(a + b) \to (F + G)(a) + (F + G)(b)$$ is the composition

  $$(F + G)(a + b) = F(a + b) + G(a + b) \xrightarrow{F(a, b) + G(a, b)} (Fa + Fb) + (Ga + Gb) \xrightarrow{(Fa + Fb) + (Ga + Gb)} (Fa + Ga) + (Fb + Gb) = (F + G)(a) + (F + G)(b).$$

  $$(F + G)(0) : (F + G)(0) \to 0$$ is the composition

  $$(F + G)(0) = F0 + G0 \xrightarrow{F0 + G0} 0 + 0 \xrightarrow{l_0} 0.$$  

Since $F$ and $G$ are endomorphisms of symmetric 2-group $\mathcal{A}$, and $\tau$, $\tau'$ are morphisms of homomorphisms, so $F + G \in \mathcal{E}nd\mathcal{A}$, and $\tau + \tau'$ is morphism of homomorphism.

- There is a unit object $0 = (0, 1, 0) : \mathcal{A} \to \mathcal{A}$, given by $0(a) \triangleq 0$, $\forall a \in \mathcal{A}$, where $0$ is the unit object of $\mathcal{A}$.

- There are natural isomorphisms:

  $$< F, G, H > : (F + G) + H \Rightarrow F + (G + H), $$

  $$l_F : 0 + F \to F, $$

  $$r_F : F + 0 \to F$$
given by:

\[
<F, G, H>_a \triangleq <Fa, Ga, Ha> : ((F + G) + H)(a) = (Fa + Ga) + Ha \\
\rightarrow Fa + (Ga + Ha) = (F + (G + H))(a),
\]

\[
(l_F)_a : (0 + F)(a) = 0(a) + Fa = 0 + Fa \xrightarrow{t_{Fa}} Fa,
\]

\[
(r_F)_a : (F + 0)(a) = Fa + 0(a) = Fa + 0 \xrightarrow{r_{Fa}} Fa,
\]

such that Fig.1.-2. commute.

**Step 3.** \(\mathcal{E}nd\mathcal{A}\) is a symmetric 2-group.

(i) Every object of \(\mathcal{E}nd\mathcal{A}\) is invertible, i.e. \(\forall F \in \mathcal{E}nd\mathcal{A}, \exists F^* \in \mathcal{E}nd\mathcal{A}\), and natural isomorphism \(\eta_F : F^* + F \rightarrow 0\). In fact, let \(F^*(a) \triangleq (F(a))^*\), \((\eta_F)_a \triangleq \eta_{Fa}\), where \((F(a))^*\) is the inverse of \(F(a)\), and \(\eta_{Fa} : (F(a))^* + F(a) \rightarrow 0\) is a natural isomorphism in \(\mathcal{A}\).

(ii) For any \(F, G \in \mathcal{E}nd\mathcal{A}\), there is a natural isomorphism \(c_{F,G} : F+G \rightarrow G+F\), given by \((c_{F,G})_a \triangleq c_{Fa,Ga} : Fa + Ga \rightarrow Ga + Fa\), and since \(\mathcal{A}\) is symmetric, so \(c_{Fa,Ga}\) is an isomorphism, such that \(c_{Fa,Ga} \circ c_{Ga,Fa} = id\), then \(c_{F,G} \circ c_{G,F} = id\).

**Step 4.** \(\mathcal{E}nd\mathcal{A}\) is a 2-ring.

We need to give another monoidal structure on \(\mathcal{E}nd\mathcal{A}\), satisfy some compatibilities.

(i) There are a bifunctor \(\cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}\), by \(F \cdot G \triangleq F \circ G\), \(\tau \cdot \tau' \triangleq \tau' \ast \tau\), \(F \circ G\) is the composition of homomorphisms, see Remark 2.(ii), and \(\tau' \ast \tau\) is the horizontal composition of 2-morphisms in the 2-category \((2\text{-SGp})\).

Using the same method in proof of Theorem 1, we know that \(F \cdot G \in \mathcal{E}nd\mathcal{A}\), \(\tau \cdot \tau'\) is the morphism in \(\mathcal{E}nd\mathcal{A}\).

(ii) The unit object of \(\mathcal{E}nd\mathcal{A}\) under the monoidal structure \(\cdot\) is \((I, id, id)\), which is for any object \(a\) and morphism \(f\) in \(\mathcal{A}\), \(I(a) \equiv a\), \(I(f) \equiv f\).
(iii) There are natural isomorphisms:

\[ \begin{align*}
[F, G, H] : (F \cdot G) \cdot H & \Rightarrow F \cdot (G \cdot H), \\
\lambda_F : I \cdot F & \Rightarrow F, \\
\rho_F : F \cdot I & \Rightarrow F, \\
[F_G^G] : F \cdot (G + G') & \Rightarrow F \cdot G + F \cdot G', \\
<_{F'} G : (F + F') \cdot G & \Rightarrow F \cdot G + F' \cdot G
\end{align*} \]

given by:

\[ \begin{align*}
[F, G, H]_a = \text{id} : ((F \cdot G) \cdot H)(a) = (F \cdot (G \cdot H))(a), \\
(\lambda_F)_a = \text{id} : (I \cdot F)(a) = (I \circ F)(a) = I(Fa) = Fa, \\
(\rho_F)_a = \text{id} : (F \cdot I)(a) = (F \circ I)a = F(I(a)) = Fa, \\
([F_G^G])_a = \text{id} : F \cdot (G + G')(a) = F((G + G')(a)) = F(Ga + G'a) \xrightarrow{F \cdot} (F \cdot G + F \cdot G')(a), \\
(<_{F'} G])_a = \text{id} : ((F + F') \cdot G)(a) = (F + F')G(a) = F(Ga) + F'(Ga) = (F \cdot G)(a) + (F' \cdot G)(a).
\end{align*} \]

The above ingredients satisfy the following conditions:

1) \((I, [\_, \_, \_], \lambda_\_, \rho_\_)\) constitute a monoidal structure (obviously, since all of them are identities.)

2) Fig.8-14. commute for all possible objects of \(\mathcal{E}nd\mathcal{A}\), from the definition of homomorphisms.

\(\square\)

Remark 5. For an Ann-category \(\mathcal{A}\), the category \(\mathcal{E}nd(\mathcal{A})\) is also an Ann-category([15]), then is a 2-ring([14]).

Similarly, as the 1-dimensional representation of rings[16], we give

**Definition 11.** A representation of a 2-ring \(\mathcal{R}\) is a 2-ring homomorphism \(F : \mathcal{R} \rightarrow \mathcal{E}nd\mathcal{A}\) of 2-rings, where \(\mathcal{A}\) is a symmetric 2-group.

**Proposition 4.** Let \(\mathcal{M}\) be a symmetric 2-group. Every representation \(F : \mathcal{R} \rightarrow \mathcal{E}nd\mathcal{M}\) equips \(\mathcal{M}\) with the structure of a \(\mathcal{R}\)-2-module. Conversely, every \(\mathcal{R}\)-2-module \(\mathcal{M}\) determines a representation \(F : \mathcal{R} \rightarrow \mathcal{E}nd\mathcal{M}\).

**Proof.** For a representation \(F : \mathcal{R} \rightarrow \mathcal{E}nd\mathcal{M}\), we will prove \(\mathcal{M}\) is a \(\mathcal{R}\)-2-module. There are a bifunctor \(\cdot : \mathcal{R} \times \mathcal{M} \rightarrow \mathcal{M}\), defined by \(r \cdot m \triangleq F(r)(m)\). and natural
isomorphisms:

\[ a_{m,n}^r : r \cdot (m + n) = F(r)(m + n) \xrightarrow{F(r)_+} F(r)(m) + F(r)(n) = r \cdot m + r \cdot n, \]
\[ b_{m}^{r,s} : (r + s) \cdot m = F(r + s)(m) \xrightarrow{F_+} (F(r) + F(s))(m) = F(r)m + F(s)m = r \cdot m + s \cdot m, \]
\[ b_{r,s,m} : (rs) \cdot m = F(rs)(m) \xrightarrow{F} (F(r)F(s))(m) = F(r)(F(s)(m)) = r \cdot (s \cdot m), \]
\[ i_m : I \cdot m = F(I)(m) \xrightarrow{F_1} I(m) = m, \]
\[ z_r : r \cdot 0 = F(r)(0) \xrightarrow{F(0)} 0. \]

It is easy to check Figs.18-31. commute, since F is homomorphism of 2-rings.

Conversely, if \( \mathcal{M} \) is a \( \mathcal{R} \)-2-module, by Proposition 1, we know that \( \text{End} \mathcal{M} \) is a 2-ring. Now we give a homomorphism

\[ F : \mathcal{R} \longrightarrow \text{End} \mathcal{M} \]
\[ r \mapsto F(r)(m) \triangleq r \cdot m, \]

where \( r \cdot m \) is the operation of \( \mathcal{R} \) on \( \mathcal{M} \).

Next we will prove F is a homomorphism of 2-rings.

- \( F : \mathcal{R} \rightarrow \text{End} \mathcal{M} \) is a functor.

  (i) \( F(r) \in \text{End} \mathcal{M} \), i.e. \( F(r) \) is an endomorphism of symmetric 2-group \( \mathcal{M} \).

  In fact, \( F(r) \triangleq r \cdot \) is a functor, for fixed \( r \in \mathcal{R} \) from \( \cdot \) being a bifunctor.

  Moreover, there are natural isomorphisms:

\[ F(r)_+(m_1, m_2) \triangleq a_{m_1,m_2}^r : F(r)(m_1 + m_2) = r \cdot (m_1 + m_2) \rightarrow r \cdot m_1 + r \cdot m_2 = F(r)(m_1) + F(r)(m_2), \]
\[ F(r)_0 \triangleq z_r : F(r)(0) = r \cdot 0 \rightarrow 0. \]

such that the following diagrams commute.
(ii) For any morphism \( f : r_1 \to r_2 \) in \( \mathcal{R} \), \( F(f) : F(r_1) \to F(r_2) \) is a morphism in \( \text{End} \mathcal{M} \).

(iii) For any morphisms \( r_1 \xrightarrow{f_1} r_2 \xrightarrow{f_2} r_3 \), we have \( F(f_2 \circ f_1) = F(f_2) \circ F(f_1) \), and also for any object \( r \in \mathcal{R} \), \( F(1_r) = 1_{F(r)} \).

In fact, by the properties of bifunctor \( \cdot : \mathcal{R} \times \mathcal{M} \to \mathcal{M} \), it is easy to prove (ii), (iii).

- There are natural isomorphisms:
  \[
  F_+(r_0, r_1) : F(r_0 + r_1) \to F(r_0) + F(r_1),
  \]
  \[
  F(r_0, r_1) : F(r_0 r_1) \to F(r_0) F(r_1),
  \]
  \[
  F_1 : F(1) \to 1
  \]
  given by the following ways: for any object \( m \in \mathcal{M} \),
  \[
  F_+(r_0, r_1)_m \triangleq b_{r_0, r_1}^m : F(r_0 + r_1)(m) = (r_0 + r_1) \cdot m \to r_0 \cdot m + r_1 \cdot m = (F(r_0) + F(r_1))(m),
  \]
  \[
  F(r_0, r_1)_m \triangleq b_{r_0 r_1, m} : F(r_0 r_1)(m) = (r_0 r_1) \cdot m \to r_0 \cdot (r_1 \cdot m) = F(r_0) F(r_1),
  \]
  \[
  (F_1)_m \triangleq i_m : F(1)(m) = 1 \cdot m \to m = 1(m).
  \]
• The above ingredients satisfy the following conditions:

(i) \((F, F_+, F_0), (F, F, F_1)\) are monoidal functors with respect to the monoidal structures on 2-rings \(\mathcal{R}\) and \(\mathcal{E}nd\mathcal{M}\).

(ii) The following diagrams commute:

\[
\begin{align*}
F((r_1 + r_2)r)(m) & \longrightarrow F(r_1r + r_2r)(m) \\
(F(r_1 + r_2)r)(m) & \quad \quad (F(r_1r) + F(r_2r))(m) \\
((Fr_1 + Fr_2)r)(m) & \longrightarrow (Fr_1Fr + Fr_2Fr)(m)
\end{align*}
\]

Fig.24.

\[
\begin{align*}
F(r(r_1 + r_2))(m) & \longrightarrow F(rr_1 + rr_2)(m) \\
(F(r)F(r_1 + r_2))(m) & \quad \quad F(rr_1 + rr_2)(m) \\
(F(r)(Fr_1 + Fr_2))(m) & \longrightarrow (Frr_1 + Frr_2)(m)
\end{align*}
\]

Fig.25.

\(\square\)

3 \((\mathcal{R}-2\text{-Mod})\) is a 2-Abelian \(Gpd\)-category

In this section, we will show that \((\mathcal{R}-2\text{-Mod})\) is a 2-abelian \(Gpd\)-category under the the definition of 2-abelian \(Gpd\)-category given in \[19\]. First we will give several definitions similar as \[5, 6, 7, 11, 19\].

**Definition 12.** \[17\] Let \(\mathcal{M}, \mathcal{N}\) be two \(\mathcal{R}\)-2-modules, where \(\mathcal{R}\) is an 2-ring. The functor \(0 : \mathcal{M} \rightarrow \mathcal{N}\) which sends each morphism to the identity of the unit object of \(\mathcal{N}\), is a \(\mathcal{R}\)-homomorphism, called the zero-morphism.

**Definition 13.** Let \(F : \mathcal{A} \rightarrow \mathcal{B}\) be a 1-morphism in \((\mathcal{R}-2\text{-Mod})\). The kernel of \(F\) is a triple \((KerF, e_F, \varepsilon_F)\), where \(KerF\) is an \(\mathcal{R}\)-2-module, \(e_F : KerF \rightarrow \mathcal{A}\) is a
1-morphism, and $\varepsilon_F : F \circ e_F \Rightarrow 0$ is a 2-morphism, satisfies the universal property in the following sense:

For given $K \in \text{obj}(\mathcal{R}\text{-}2\text{-Mod})$, a 1-morphism $G : K \to A$, and a 2-morphism $\varphi : F \circ G \Rightarrow 0$, there exist a 1-morphism $G' : K \to \text{Ker}F$ and a 2-morphism $\varphi' : e_F \circ G' \Rightarrow G$, such that $\varphi'$ is compatible with $\varphi$ and $\varepsilon_F$, i.e. the following diagram commutes:

\[
\begin{array}{c}
F \circ G \ar[d]_{F \varphi} \ar[r]^{e_F \circ G} & 0 \ar[d]^	ext{can} \\
F \circ G' \ar[r]_\varphi & 0
\end{array}
\]

Moreover, if $G''$ and $\varphi''$ satisfy the same conditions as $G'$ and $\varphi'$, then there exists a unique 2-morphism $\psi : G'' \Rightarrow G'$, such that

\[
\begin{array}{c}
e_F \circ G' \ar[d]_\psi \ar[r]^{e_F \circ G} & e_F \circ G' \\
G' \ar[rr]_\varphi & & G'
\end{array}
\]

**Theorem 2.** For any 1-morphism $F : A \to B$ in $(\mathcal{R}\text{-}2\text{-Mod})$, the kernel of $F$ exists.

**Proof.** We will construct the kernel of $F$ as follows:

- There is a category $\text{Ker}F$ consists of:
  - Object is a pair $(A, a)$, where $A \in \text{obj}(A), a : F(A) \to 0$ is a morphism in $B$.
  - Morphism $f : (A, a) \to (A', a')$ is a morphism $f : A \to A'$ in $\mathcal{A}$, such that $a' \circ F(f) = a$.
  - Composition of morphisms. Given morphisms $(A, a) \xrightarrow{f'} (A', a') \xrightarrow{f} (A'', a'')$, their composition $f' \circ f : A \to A''$ is just the composition of morphisms in $\mathcal{A}$, such that $a'' \circ F(f' \circ f) = a'' \circ (F(f') \circ F(f) = a' \circ F(f) = a$.

The above ingredients satisfy the following axioms:
(1) For any \((A,a) \in \text{obj}(KerF)\), there exists \(1_{(A,a)} \triangleq 1_A : (A,a) \to (A,a)\) such that any \(f : (A,a) \to (A',a')\), \(f \circ 1_{(A,a)} = f\).

(2) Composition is associative: Given morphisms

\[
A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4.
\]

We have \((f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)\) in \(A\), so it is true in \(KerF\).

- \(KerF\) is a groupoid with zero object \((0,F_0)\), where \(0\) is unit object of \(A\), 
  \(F_0 : F(0) \to 0\) is a morphism in \(A\).
  For any morphism \(f : (A,a) \to (A_1,a_1)\) of \(KerF\), is a morphism \(f : A \to A_1\) in \(A\) such that \(a_1 \circ Ff = a\). \(A\) is an \(R\)-2-module, as a morphism in \(A\), 
  \(f : A \to A_1\) has inverse \(g : A_1 \to A\) such that \(g \circ f = 1_A\) and \(a \circ Fg = a_1\), so is the inverse of \(f\) in \(KerF\).

- \(KerF\) is a monoidal category.
  There is a bifunctor

\[
+ : KerF \times KerF \to KerF
\]

\[
((A_1, a_1), (A_2, a_2)) \mapsto (A, a) \triangleq (A_1, a_1) + (A_2, a_2)
\]

\[
((A_1, a_1) \xrightarrow{f_1} (A_1', a_1'), (A_2, a_2) \xrightarrow{f_2} (A_2', a_2')) \mapsto (A_1, a_1) + (A_1', a_1') \xrightarrow{f_1 + f_2} (A_2, a_2) + (A_2', a_2')
\]

where \(A \triangleq A_1 + A_2\), \(a\) is the composition 

\[
F(A) = F(A_1 + A_2) \xrightarrow{F_+} FA_1 + FA_2 \xrightarrow{a_1 + a_2} 0 + 0 \xrightarrow{l_0} 0,
\]

\(f_1 + f_2 : A_1 + A_1' \to A_2 + A_2'\) is an addition of morphisms in \(A\) under monoidal structure of \(A\), such that the following diagram commutes:
Then $f_1 + f_2$ is a morphism in $\text{Ker} F$.

Moreover, there are natural isomorphisms:

(i) $< (A_1, a_1), (A_2, a_2), (A_3, a_3) > \cong < A_1, A_2, A_3 >: ((A_1, a_1) + (A_2, a_2)) + (A_3, a_3) = ((A_1 + A_2) + A_3, a) \to (A_1 + (A_2 + A_3), a') = (A_1, a_1) + ((A_2, a_2) + (A_3, a_3))$, such that the following diagram commutes:

(ii) $l_{(A,a)} \cong l_A : (0, F_0) + (A, a) = (0 + A, a') \to (A, a)$, such that

(iii) $r_{(A,a)} \cong r_A : (A, a) + (0, F_0) = (A + 0, a') \to (A, a)$, such that
commutes.

From the definition of morphism in $Ker F$ and $\mathcal{A}$ being a monoidal category, the above isomorphisms satisfy the Mac Lane coherence conditions.

• $Ker F$ is a symmetric 2-group.

  - For any $(A, a) \in obj(Ker F)$, $A \in obj(\mathcal{A})$, $a : F(A) \to 0$. Since $\mathcal{A}$ is a symmetric 2-group, so $A$ has an inverse $A^*$, and natural isomorphism $\eta_A : A^* + A \to 0$. Let $a^*$ be a composition $F(A^*) \cong (F(A))^* \to 0^* \cong 0$. We have $(A, a)^* \triangleq (A^*, a^*)$, and natural isomorphism $\eta_{(A,a)} \triangleq \eta_A : (A, a)^* + (A, a) \to (0, F_0)$. It is easy to check $\eta_{(A,a)}$ is a morphism in $Ker F$.

  - For any objects $(A_1, a_1), (A_2, a_2) \in Ker F$, there is a natural isomorphism
    \[ c_{(A_1,a_1),(A_2,a_2)} : (A_1, a_1) + (A_2, a_2) \to (A_2, a_2) + (A_1, a_1), \]
given by
    \[ c_{A_1, A_2} : A_1 + A_2 \to A_2 + A_1, \]
and since $c_{A_1, A_2} \circ c_{A_2, A_1} = id$, so $c_{A_1, A_2} \circ c_{A_2, A_1} = id$.

• $Ker F$ is an $\mathcal{R}$-2-module.

There is a bifunctor
\[ \cdot : \mathcal{R} \times Ker F \longrightarrow Ker F \]
\[ (r, (A, a)) \mapsto r \cdot (A, a) \triangleq (r \cdot A, a') \]
\[ (r_1 \xrightarrow{\varphi} r_2, (A_1, a_1) \xrightarrow{f} (A_2, a_2)) \mapsto r_1 \cdot (A_1, a_1) \xrightarrow{\varphi \cdot f} r_2 \cdot (A_2, a_2) \]
where $a' : F(r \cdot A) \xrightarrow{F_2} r \cdot F(A) \xrightarrow{r \cdot a} r \cdot 0 \xrightarrow{z} 0$, $\varphi \cdot f$ is defined under the operation of $\mathcal{R}$ on $\mathcal{A}$, and such that the following diagram commutes:
Moreover, there are natural isomorphisms:

(1) \( a^r_{(A_1,a_1),(A_2,a_2)} : r \cdot ((A_1, a_1) + (A_2, a_2)) \to r \cdot (A_1, a_1) + r \cdot (A_2, a_2) \), as follows:

\[ r \cdot ((A_1, a_1) + (A_2, a_2)) \cong (r \cdot (A_1 + A_2), a') , \ r \cdot (A_1, a_1) + r \cdot (A_2, a_2) \cong (r \cdot A_1 + r \cdot A_2, a''), \]

where \( a' \) and \( a'' \) are the compositions as in the following diagrams, such that the diagram commutes:

(2) \( b^{r_1,r_2}_{(A,a)} : (r_1 + r_2) \cdot (A, a) \to r_1 \cdot (A, a) + r_2 \cdot (A, a) \), as follows:

\[ (r_1 + r_2) \cdot (A, a) \cong ((r_1 + r_2) \cdot A, a') , \ r_1 \cdot (A, a) + r_2 \cdot (A, a) \cong (r_1 \cdot A + r_2 \cdot A, a''), \]

\( b^{r_1,r_2}_{(A,a)} \cong b_A^{r_1,r_2} \), where \( a' \) and \( a'' \) are the compositions as in the following diagrams, such that the diagram commutes:
(3) \( b_{r_1,r_2,(A,a)} : (r_1 r_2) \cdot (A, a) \rightarrow r_1 \cdot (r_2 \cdot (A, a)) \), as follows:
\[
(r_1 r_2) \cdot (A, a) \triangleq ((r_1 r_2) \cdot A, a'), \quad (r_1 \cdot (r_2 \cdot (A, a))) \triangleq (r_1 \cdot (r_2 \cdot A), a''), \quad b_{r_1,r_2,(A,a)} \triangleq b_{r_1,r_2,A},
\]
where \( a' \) and \( a'' \) are the compositions as in the following diagrams, such that the diagram commutes:

(4) \( i_{(A,a)} : 1 \cdot (A, a) \rightarrow (A, a) \), as follows:
\[
1 \cdot (A, a) \triangleq (1 \cdot A, a'), \quad i_{(A,a)} \triangleq 1_A,
\]
where \( a' \) is the composition as in the following diagrams, such that the diagram commutes:
(5) $z_r : r \cdot (0, F_0) \to (0, F_0)$, as follows: 
$r \cdot (0, F_0) \triangleq (r \cdot 0, F'_0)$, $z_r$ is just the natural isomorphism in $A$, such that the following diagram commutes:

The above natural isomorphisms, together with the basic facts about $\mathcal{R}$-2-module $A$ satisfy Fig.18.-31. commute.

- $Ker F$ is a kernel of $F : A \to B$.

(i) There is a functor 

$$e_F : Ker F \to A$$

$$(A, a) \mapsto A$$

$$(A_1, a_1) \xrightarrow{f} (A_2, a_2) \mapsto f : A_1 \to A_2$$

such that, for $(A_1, a_1) \xrightarrow{f_1} (A_2, a_2) \xrightarrow{f_2} (A_3, a_3)$, we have $e_F(f_2 \circ f_1) = f_2 \circ f_1 = e_F(f_2) \circ e_F(f_1)$.

And also there are natural isomorphisms:

$$(e_F)_+ = id : e_F((A_1, a_1) + (A_2, a_2)) = A_1 + A_2 = e_F((A_1, a_1) + e_F((A_2, a_2)),$$

$$(e_F)_0 = id : e_F(0, F_0) = 0,$$

$$(e_F)_2 = id : e_F(r \cdot (A, a)) = r \cdot A = r \cdot e_F(A, a).$$

30
Obviously, Fig.32.-36. commute.

(ii) There is a 2-morphism:

$$\varepsilon_F : F \circ e_F \Rightarrow 0$$

given by, $(\varepsilon_F)_{(A,a)} \triangleq a : (F \circ e_F)(A,a) = F(A) \rightarrow 0$, for any $(A,a) \in \text{obj}(\text{Ker}F)$.

(iii) $(\text{Ker}F, e_F, \varepsilon_F)$ satisfies the universal property.

For any $K \in \text{obj}(\mathcal{R}\text{-}2\text{-Mod})$, 1-morphism $G : K \rightarrow A$, and 2-morphism $\varphi : F \circ G \Rightarrow 0$. There is a 1-morphism:

$$G' : K \rightarrow \text{Ker}F$$

$$A \mapsto (G(A), \varphi_A)$$

$$A_1 \xrightarrow{f} A_2 \mapsto (G(A_1), \varphi_{A_1}) \xrightarrow{G(f)} (G(A_2), \varphi_{A_2})$$

It is easy to see $G'$ is well-defined and $G'$ is an $\mathcal{R}$-homomorphism.

Define a 2-morphism

$$\varphi' : e_F \circ G' \Rightarrow G$$

by

$$\varphi'_A \triangleq 1_{GA} : (e_F \circ G')(A) = e_F(G(A), \varphi_A) = GA \rightarrow GA,$$

for any $A \in \text{obj}(K)$.

$\varphi'$ is compatible with $\varphi$ and $\varepsilon_F$, i.e. Fig.38. commutes.

If $G'', \varphi''$ satisfy the same conditions as $G', \varphi'$.

To define $\psi : G'' \Rightarrow G'$, we need to define, for any $A \in \text{obj}(K)$, a morphism $\psi_A : G''(A) \rightarrow G'(A)$, which is defined by $\psi_A \triangleq \varphi''_A$ from the commutative diagram Fig.39..

Obviously, $\psi''$ is unique for $\varphi''$ is.

\[\square\]

**Definition 14.** Let $F : A \rightarrow \mathcal{B}$ be a 1-morphism in ($\mathcal{R}\text{-}2\text{-Mod}$). The cokernel of $F$ is the triple $(\text{Coker}F, p_F, \pi_F)$, where $\text{Coker}F$ is an $\mathcal{R}$-2-module, $p_F : \mathcal{B} \rightarrow \text{Coker}F$ is a 1-morphism, and $\pi_F : p_F \circ F \Rightarrow 0$ is a 2-morphism, satisfies the universal property in the following sense:

Given an $\mathcal{R}$-2-module $\mathcal{K}$, a 1-morphism $G : \mathcal{B} \rightarrow \mathcal{K}$, and a 2-morphism $\varphi : G \circ F \Rightarrow 0$, there exist a 1-morphism $G' : \text{Coker}F \rightarrow \mathcal{K}$ and a 2-morphism $\varphi' : G' \circ p_F \Rightarrow G$, such that $\varphi'$ is compatible with $\varphi$ and $\pi_F$, i.e. the following diagram commutes.
Moreover, if $G''$ and $\varphi''$ satisfy the same conditions as $G'$ and $\varphi'$, then there exists a unique 2-morphism $\psi : G'' \Rightarrow G'$, such that

$$
\require{AMScd}
\begin{align*}
\vcenter{\xymatrix{ G' \circ p_F & G' \circ 0 \\
\downarrow \varphi' \ar[rr]_-{1_{G'}} & & G' \circ 0 \\
G' \circ F & 0 \\
\downarrow \psi \ar[uu]_-{1_{G'F}} & \\
G' \circ F & 0 }}
\end{align*}
$$

Fig. 40.

commutes.

**Theorem 3.** For any 1-morphism $F : A \to B$ in $(\mathcal{R}\text{-}2\text{-}Mod)$, the cokernel of $F$ exists.

**Proof.** We need to construct an the cokernel of $F$ as follows:

- Let $\text{Coker}F$ be a category consisting of:
  - Objects are those of $B$.
  - Morphism of $B_1 \to B_2$ is the equivalence class of $(f, A)$, denoted by $[f, A]$, where $A \in \text{obj}(A)$, $f : B_1 \to B_2 + F(A)$, and for two morphisms $(f, A)$, $(f', A') : B_1 \to B_2$ are equivalent if and only if there exists an isomorphism $\alpha : A \to A'$ in $A$, such that $(1_{B_2} + F(\alpha)) \circ f = f'$.

- Composition of morphisms in $\text{Coker}F$.

Let $B_1 \xrightarrow{[f_1, A_1]} B_2 \xrightarrow{[f_2, A_2]} B_3$ be morphisms in $\text{Coker}F$. Then $[f_2, A_2] \circ [f_1, A_1] \equiv [f, A]$, where $A = A_1 + A_2$, $f$ is the composition

$$
B_1 \xrightarrow{f_1} B_2 + FA_1 \xrightarrow{f_1 + 1_{FA_1}} (B_3 + FA_2) + FA_1 \xrightarrow{(B_3 + FA_2, FA_1)} B_3 + (FA_2 + FA_1) \xrightarrow{1_{B_3} + FA_1} B_3 + (FA_1 + FA_2) \xrightarrow{1_{B_3} + F^{-1}} B_3 + F(A_1 + A_2).
$$

The above composition of morphisms in $\text{Coker}F$ is well-defined, since, if $(f_1, A_1)$ and $(f'_1, A'_1)$ are equivalent, i.e. $\exists \alpha : A_1 \to A'_1$, such that $(1_{B_2} +
\(F(\alpha) \circ f_1 = f'_1\). Then there exists \(\beta \triangleq \alpha + 1_{A_2} : A_1 + A_2 \to A'_1 + A_2\), such that the following diagram commutes:

![Diagram](image)

The above objects and morphisms, together with composition of morphisms satisfy the following axioms:

(1) For any \(B \in \mathcal{B}\), there exists identity morphism \([\widetilde{1}_B, 0] : B \to B\) in \(\text{Coker} F\), where 0 is unit object of \(\mathcal{A}\), \(\widetilde{1}_B : B \xrightarrow{r_B^{-1}} B + 0 \xrightarrow{1_B + F_0^{-1}} B + F0\), such that for any morphism \([f, A] : B \to B_1\), \([f, A] \circ [\widetilde{1}_B, 0] = [f, A]\).

In fact, \([f, A] \circ [\widetilde{1}_B, 0] = [f', A']\), where \(A' = A + 0\), \(f'\) is the composition \(B \xrightarrow{r_B^{-1}} B + 0 \xrightarrow{1_B + F_0^{-1}} B + F0 \xrightarrow{f + 1_{F0}} (B_1 + FA) + F0 \xrightarrow{<B_1, FA,F0>} B_1 + (FA + F0) \xrightarrow{1_{B_1} + c_{FA,F0}} B_1 + (F0 + FA) \xrightarrow{1_{B_1} + F_{0}^{-1}} B_1 + F(0 + A)\). There exists an isomorphism \(l_A : 0 + A \to A\), together with the properties of \(F\), such that \((f', A')\) is equivalent to \((f, A)\), so they are same morphism in \(\text{Coker} F\).

(ii) Associativity of compositions. Given morphisms

\[
B_1 \xrightarrow{[f_1,A_1]} B_2 \xrightarrow{[f_2,A_2]} B_3 \xrightarrow{[f_3,A_3]} B_4.
\]

There exists an isomorphism \(<A_1, A_2, A_3> : (A_1 + A_2) + A_3 \to A_1 + (A_2 + A_3)\), such that

\[
[f_3, A_3] \circ ([f_2, A_2] \circ [f_1, A_1]) = ([f_3, A_3] \circ [f_2, A_2]) \circ [f_1, A_1]
\]

up to equality in \(\text{Coker} F\).
• \(\text{Coker} F\) is a groupoid with zero object 0, where 0 is zero object of \(B\).

For any morphism \([f, A] : B_1 \to B_2\) in \(\text{Coker} F\), since \(f : B_1 \to B_2 + FA\) is a morphism in \(B\), and \(B\) is a groupoid, there exists \(f' : B_2 + FA \to B_1\), such that \(f' \circ f = \text{id}\). For any \(A \in \text{obj}(A)\), since \(A\) is 2-group, there exists \(A^* \in \text{obj}(A)\), and a natural isomorphism \(\eta_A : A + A^* \to 0\).

Let \(f^*\) be the composition
\[
\begin{align*}
B_2 & \to B_2 + 0 \to B_2 + F0 \to B_2 + F(A + A^*) \\
B_2 + (FA + F(A^*)) & \to (B_2 + FA) + F(A^*) \to B_1 + F(A^*).
\end{align*}
\]

From \(\eta_A\), the properties of 2-groups and \(F\), we get
\[
\[(f^*, A^*) \circ (f, A)] = [1_{B_1}, 0].
\]

• \(\text{Coker} F\) is a monoidal category.

There is a bifunctor

\[
+: \text{Coker} F \times \text{Coker} F \to \text{Coker} F
\]

\[
(B_1, B_2) \mapsto B_1 + B_2,
\]

\[
(B_1 \xrightarrow{[f_1, A_1]} B_2, B_3 \xrightarrow{[f_2, A_2]} B_4) \mapsto B_1 + B_3 \xrightarrow{[f, A]} B_2 + B_4
\]

where \(B_1 + B_2\) is the addition under the monoidal structure of \(B\), \(A \triangleq A_1 + A_2\), \(f\) is the following composition

\[
\begin{align*}
B_1 + B_3 & \xrightarrow{f_1 + f_2} (B_2 + FA_1) + (B_4 + FA_2) \\
& \xrightarrow{\beta_{B_2}^{B_1} FA_1} (B_1 + B_4) + (FA_1 + FA_2) \\
& \xrightarrow{1+F^{-1}} (B_1 + B_4) + F(A_1 + A_2).
\end{align*}
\]

From the definition of +, the natural isomorphisms are those of them in \(B\), then the Mac Lane coherence conditions hold.

• \(\text{Coker} F\) is a symmetric 2-group.

For any object \(B \in \text{Coker} F\), \(B \in B\), since \(B\) is a 2-group, there exist \(B^* \in B\), and \(\eta_B : B^* + B \to 0\). Let \(\eta_B^* : B^* + B \to B^* + B + 0 \to B^* + B + F0\), then \((\eta_B^*, 0)\) is an isomorphism in \(\text{Coker} F\).

For any two objects \(B_1, B_2 \in \text{Coker} F\), \(B_1, B_2 \in B\), \(B\) is symmetric monoidal
There exists $c_{B_1, B_2} : B_1 + B_2 \to B_2 + B_1$, such that $c_{B_1, B_2} \circ c_{B_2, B_1} = 1_{B_1 + B_2}$. Let $c'_{B_1, B_2}$ be the composition

$$B_1 + B_2 \xrightarrow{c_{B_1, B_2}} B_2 + B_1 \xrightarrow{r_{B_2} + B_1} (B_2 + B_1) + 0 \xrightarrow{1_{B_2 + B_1} + F_0} (B_2 + B_1) + F_0.$$ 

Obviously, $c'_{B_1, B_2} \circ c'_{B_2, B_1} = 1_{B_1 + B_2}$.

- **Coker $F$** is an $R$-2-module.

  There is a bifunctor

  \[ \cdot : \mathcal{R} \times \text{Coker} F \to \text{Coker} F \]

  \[ (r, B) \mapsto r \cdot B, \]

  \[ (r \xrightarrow{\varphi} r_2, B_1 \xrightarrow{[f, A]} B_2 \mapsto r_1 \cdot B_1 \xrightarrow{[f', A']} r_2 \cdot B_2 \]

  where $r \cdot B \in \mathcal{B}$ under the operation of $\mathcal{R}$ on $\mathcal{B}$, $A' \triangleq r_2 \cdot A$, $f'$ is the composition

  \[ r_1 \cdot B_1 \to r_2 \cdot B_1 \to r_2 \cdot (B_2 + FA) \to r_2 \cdot B_2 + r_2 \cdot FA \to r_2 \cdot B_2 + F(r_2 \cdot A). \]

  The natural isomorphisms are all induced by the the natural isomorphisms in $\mathcal{B}$, so they satisfy Fig.18–31. commute.

- There is a functor:

  \[ p_F : \mathcal{B} \to \text{Coker} F \]

  \[ B \mapsto B, \]

  \[ B_1 \xrightarrow{f} B_2 \mapsto B_1 \xrightarrow{[f', 0]} B_2 \]

  where $f'$ is the composition

  \[ f' : B_1 \xrightarrow{f} B_2 \xrightarrow{r_{B_2}} B_2 + 0 \xrightarrow{1_{B_2 + F_0}^{-1}} B_2 + F_0. \]

  - $p_F$ is a functor.

  Let $B_1 \xrightarrow{f_1} B_2 \xrightarrow{f_2} B_3$ be morphisms in $\mathcal{B}$, we need to prove $p_F(f_2 \circ f_1) = p_F(f_2) \circ p_F(f_1)$. In fact,

  \[ p_F(f_1) = [f_1', 0] : B_1 \to B_2 \to B_2 + 0 \to B_2 + F_0, \]

  \[ p_F(f_2) = [f_2', 0] : B_2 \to B_3 \to B_3 + 0 \to B_3 + F_0, \]
\[ p_F(f_1 \circ f_2) = [f, 0] : B_1 \rightarrow B_3 \rightarrow B_3 + 0 \rightarrow B_3 + F0, \]

\[ p_F(f_2) \circ p_F(f_1) : B_1 \rightarrow B_2 + F0 \rightarrow (B_3 + F0) + F0 \rightarrow B_3 + (F0 + F0) \rightarrow B_3 + F(0 + 0) \rightarrow B_3 + F0. \]

For any identity morphism \(1_B : B \rightarrow B\) in \(B\), we have \(p_F(1_B) = (\sim_1, 0) : B \rightarrow B\).

· There are natural isomorphisms:

\[ (p_F)_+ = id : p_F(B_1 + B_2) = B_1 + B_2 = p_F(B_1) + p_F(B_2), \]

\[ (p_F)_0 = id : p_F(0) = 0, \]

\[ (p_F)_2 = id : p_F(r \cdot B) = r \cdot B = r \cdot p_F(B). \]

Obviously, \(p_F\) is an \(R\)-homomorphism.

Define a 2-morphism

\[ \pi_F : p_F \circ F \Rightarrow 0 \]

given by, for any \(A \in \text{obj}(A)\),

\[ (\pi_F)_A = [(\pi_F)'_A, A] : (p_F \circ F)(A) = p_F(FA) = FA \rightarrow 0 \]

where \((\pi_F)'_A : FA \rightarrow FA + 0 \rightarrow FA + F0\).

Next, we will show \(p_F, \pi_F\) satisfy the universal property. For any \(K \in \text{obj}(R-2-\text{Mod})\), 1-morphism \(G : B \rightarrow K\), and \(\varphi : G \circ F \Rightarrow 0\) in \((R-2-\text{Mod})\), define a homomorphism

\[ G' : \text{Coker} F \rightarrow K \]

\[ B \mapsto G(B), \]

\[ B_1 \xrightarrow{[f, A]} B_2 \mapsto G'(B_1) \xrightarrow{G'([f, A])} G(B_2) \]

Since \(G\) is an \(R\)-homomorphism, so \(G'\) is.

Define a 2-morphism \(\varphi' : G' \circ p_F \Rightarrow G\), by

\[ \varphi'_B \triangleq 1_{GB} : (G' \circ p_F)(B) = G'(B) = G(B), \]

for any \(B \in \text{obj}(B)\), such that Fig.40. commutes.

If \(G''\) and \(\varphi''\) satisfy the same conditions as \(G'\) and \(\varphi'\), there is a 2-morphism

\[ \psi : G'' \Rightarrow G' \]

defined by, \(\psi_B \triangleq \varphi''_B : G''(B) \rightarrow G'(B)\), for each \(B \in \text{obj}(B)\),

such that the diagram Fig.41. commutes.

Obviously, \(\psi''\) is the unique 2-morphism for \(\varphi''\) is.
A groupoid enriched category (for short, a Gpd-category) is in fact a 2-category, satisfies some special properties, which plays an important role in 2-abelian category (more details see [19]). Next we will give some results about \((\mathcal{R}-2\text{-Mod})\).

**Lemma 1.** The 2-category \((\mathcal{R}-2\text{-Mod})\) is a Gpd-category.

**Proof.** \((\mathcal{R}-2\text{-Mod})\) contains the following ingredients:

1. For any \(A, B \in \text{obj}(\mathcal{R}-2\text{-Mod})\), \(\text{Hom}(A, B)\) is a groupoid, with \(\mathcal{R}\)-homomorphisms from \(A\) to \(B\) as its objects and the morphisms of two \(\mathcal{R}\)-homomorphisms as its morphisms.

Composition of morphisms. Let \(\tau : F \Rightarrow G, \sigma : G \Rightarrow H\) be morphisms in \(\text{Hom}(A, B)\). \(\sigma \circ \tau : F \Rightarrow H\) is given by

\[
(\sigma \circ \tau)_A \triangleq \sigma_A \circ \tau_A : FA \xrightarrow{\tau_A} GA \xrightarrow{\sigma_A} HA.
\]

It is easy to check \(\sigma \circ \tau\) is a morphism of \(\mathcal{R}\)-homomorphisms from \(F\) to \(H\) (see Theorem 1).

The above objects and morphisms satisfy the following axioms:

(i) For any \(F \in \text{Hom}(A, B)\), \(\exists 1_F : F \Rightarrow F\), defined by \(1_F)_A \triangleq 1_{FA}, \forall A \in A\), such that for any \(\tau : F \Rightarrow G, \sigma : H \Rightarrow F\), we have \(\tau \circ 1_F = \tau, 1_F \circ \sigma = \sigma\), since \((\tau \circ 1_F)(A) = \tau_A \circ 1_{FA} = \tau_A \circ 1_{FA} = \tau_A, (1_F \circ \sigma)(A) = (1_{FA} \circ \sigma_A) = 1_{FA} \circ \sigma_A = \sigma_A\), for any \(A \in A\).

(ii) Associativity of the composition. Given morphisms

\[
F_1 \xrightarrow{\tau_1} F_2 \xrightarrow{\tau_2} F_3 \xrightarrow{\tau_3} F_4
\]

in \(\text{Hom}(A, B)\). Then \((\tau_3 \circ \tau_2) \circ \tau_1 = \tau_3 \circ (\tau_2 \circ \tau_1)\), since

\[
((\tau_3 \circ \tau_2) \circ \tau_1)_A = ((\tau_3)_A \circ (\tau_2)_A) \circ (\tau_1)_A = (\tau_3)_A \circ ((\tau_2)_A \circ (\tau_1)_A) = (\tau_3 \circ (\tau_2 \circ \tau_1))_A,
\]

for \(\forall A \in \text{obj}(A)\).

(iii) For any morphism \(\tau : F \Rightarrow G : A \rightarrow B, \exists \tau^* : G \Rightarrow F\), such that \(\tau^* \circ \tau \triangleright 1_F\). In fact, \(\forall A \in A, \tau_A : FA \rightarrow GA\) is a morphism in \(B\), and \(B\) is a
groupoid, so \( \exists (\tau_A)^* : GA \to FA \), such that \((\tau_A)^* \circ \tau_A \preceq 1_F A\). Define \( \tau^* : G \Rightarrow F \) by \( (\tau^*)_A \triangleq (\tau_A)^* \), such that \((\tau^* \circ \tau)_A = (\tau^*)_A \circ \tau_A \preceq 1_F A = (1_F)_A\).

(2). For any \( A \in \text{obj}(\mathcal{R}\text{-}2\text{-Mod}) \), there is an \( \mathcal{R} \)-homomorphism \( 1_A : A \to A \), defined by \( 1_A(A) = A, \ \forall A \in A \).

(3). For any \( A, B, C \in \text{obj}(\mathcal{R}\text{-}2\text{-Mod}) \), there is a functor composition

\[
\begin{align*}
\text{comp} : \text{Hom}(A, B) \times \text{Hom}(B, C) & \longrightarrow \text{Hom}(A, C) \\
(F, G) & \mapsto \text{comp}(F, G) \triangleq G \circ F,
\end{align*}
\]

where \( \beta \circ \alpha \) is the horizontal composition in Theorem 1.

From Theorem 1, we also have

(4). For all \( A, B, C, D \in \text{obj}(\mathcal{R}\text{-}2\text{-Mod}) \), and \( F \in \text{Hom}(A, B) \), \( G \in \text{Hom}(B, C) \), \( H \in \text{Hom}(C, D) \), there is a natural transformation

\[
\alpha_{H,G,F} = \text{id} : (H \circ G) \Rightarrow H \circ (G \circ F).
\]

(5). For all \( A, B \in \text{obj}(\mathcal{R}\text{-}2\text{-Mod}) \), \( F \in \text{Hom}(A, B) \), there are natural transformations

\[
\begin{align*}
\rho_F : F \circ 1_A & \Rightarrow F, \\
\lambda_F : 1_B \circ F & \Rightarrow F
\end{align*}
\]

given by, for any \( A \in \text{obj}(\mathcal{A}) \), \( f : A_1 \to A_2 \) in \( \mathcal{A} \), \( (\rho_F)_A = \text{id} \), \( (\lambda_F)_A = \text{id} \), \( (\rho_F)_f = F(f) \), \( (\lambda_F)_f = F(f) \). Obviously, \( \rho_F \), \( \lambda_F \) are morphisms in \( \text{Hom}(A, B) \).

Given \( \mathcal{R}\text{-}2\text{-module homomorphisms} \mathcal{A} \overset{F}{\to} \mathcal{B} \overset{G}{\to} \mathcal{C} \overset{H}{\to} \mathcal{D} \overset{K}{\to} \mathcal{E} \), since \( \alpha_{-, -} =, \rho_-, \lambda_- \) are identities, so they satisfy the following diagrams commute:

\[
\begin{array}{c}
\xymatrix{
((K \circ H) \circ G) \circ F \ar[d]^{a_{K,H,G,F}} \ar[r]_{a_{K,H,G,F}} & (K \circ H) \circ (G \circ F) \ar[d]^{a_{K,H,G,F}} \\
(K \circ (H \circ G)) \circ F & K \circ (H \circ (G \circ F))
}
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{
(G \circ 1_B) \circ F \ar[d]^{a_{G,F}} \ar[r]_{a_{G,F}} & G \circ (1_B \circ F) \ar[d]^{a_{G,F}} \\
G \circ F & G \circ F
}
\end{array}
\]
Corollary 1. \((\mathcal{R}2\text{-Mod})\) is a \(\text{Gpd}^*\)-category, where \(\text{Gpd}^*\) means the pointed groupoid appearing in [19].

Proof. For any \(A, B \in \text{obj}(\mathcal{R}2\text{-Mod})\), \(\text{Hom}(A, B)\) is a pointed groupoid where the point is just the zero \(\mathcal{R}\)-homomorphism from \(A\) to \(B\). Using the similar methods in the proof of Lemma 1, it is easy to prove it.

Lemma 2. For any \(A, B \in \text{obj}(\mathcal{R}2\text{-Mod})\), \(\text{Hom}(A, B)\) is a symmetric 2-group.

Proof. From Lemma 1, \(\text{Hom}(A, B)\) is a groupoid, so we need to give the monoidal structure on it, and prove it is symmetric under this monoidal structure.

There is a bifunctor:

\[ + : \text{Hom}(A, B) \times \text{Hom}(A, B) \rightarrow \text{Hom}(A, B) \]

\[(F, G) \mapsto F + G, \]

\[(\tau : F \Rightarrow F', \sigma : G \Rightarrow G') \mapsto \tau + \sigma : F + G \Rightarrow F' + G' \]

given by, \((F + G)(A) \triangleq FA + GA\), \((\tau + \sigma)_A \triangleq \tau_A + \sigma_A\) under the monoidal addition in \(\mathcal{B}\), for any \(A \in \mathcal{A}\).

The unit object \(0 \in \text{Hom}(A, B)\) is just the zero \(\mathcal{R}\)-homomorphism of \(\mathcal{R}\)-2-modules.

Moreover, there are natural isomorphisms:

\[ < F, G, H > : (F + G) + H \Rightarrow F + (G + H), \]

\[ l_F : 0 + F \Rightarrow F, \]

\[ r_F : F + 0 \Rightarrow F \]

defined by, \(< F, G, H >_A \triangleq < FA, GA, HA >, (l_F)_A = l_{FA}, (r_F)_A = r_{FA}, \forall A \in \mathcal{A}\).

Since \(\mathcal{B}\) is a monoidal category, the Mac Lane coherence conditions hold, i.e. Fig.1-2. commute.

For any \(F, G \in \text{obj}(\text{Hom}(\mathcal{A}, \mathcal{B}))\), \((c_{F,G})_A \triangleq c_{FA,GA}, \exists c_{F,G} : F + G \Rightarrow G + F\), such that \(c_{FA,GA} \circ c_{GA,FA} \cong 1_{GA+FA}\). Then \(c_{F,G} \circ c_{G,F} = 1_{G+F}\).

For any \(F \in \text{obj}(\text{Hom}(\mathcal{A}, \mathcal{B}))\). Define \(F^* : \mathcal{B} \rightarrow \mathcal{A}\) by \(F^*(A) = (FA)^*, \forall A \in \mathcal{A}\), where \((FA)^*\) is the inverse of \(FA\) in \(\mathcal{B}\), with natural isomorphism \(\eta_{FA} : (FA)^* + FA \Rightarrow 0\). So there is a natural isomorphism \(\eta_F : F^* + F \Rightarrow 0\), given by \((\eta_F)_A \triangleq \eta_{FA}^*\).
Lemma 3. \((\mathcal{R} \text{-} 2\text{-Mod})\) is a presemiadditive \textit{Gpd}-category refer to the Definition 218 in [19].

\textbf{Proof.} For any \(\mathcal{A}, \mathcal{B} \in \text{obj}(\mathcal{R} \text{-} 2\text{-Mod})\), \(\text{Hom}(\mathcal{A}, \mathcal{B})\) is a symmetric monoid groupoid (Lemma 2) with transformations natural in each variables:

\[\varphi^H_{G_1, G_2} : H \circ (G_1 + G_2) \Rightarrow H \circ G_1 + H \circ G_2,\]
\[\psi^{H_1, H_2}_G : (H_1 + H_2) \circ G \Rightarrow H_1 \circ G + H_2 \circ G,\]
\[\varphi^H_0 : 0 \Rightarrow H \circ 0,\]
\[\psi^0_G : 0 \Rightarrow 0 \circ G,\]

defined by, \(\forall A \in \mathcal{A}\),

\[(\varphi^H_{G_1, G_2})_A \triangleq (H_+)(G_1, G_2)_A : (H \circ (G_1 + G_2))(A) = H(G_1 A + G_2 A) \Rightarrow H(G_1 A) + H(G_2 A),\]
\[(\psi^{H_1, H_2}_G)_A \triangleq \text{id} : ((H_1 + H_2) \circ G)(A) = H_1(GA) + H_2(GA),\]
\[(\varphi^H_0)_A \triangleq H_0 : 0(A) = 0 \Rightarrow H(0A) = H(0),\]
\[(\psi^0_G)_A \triangleq \text{id} : 0(A) = 0 \Rightarrow 0(GA) = 0.\]

The above natural transformations satisfy the following conditions:

\textbf{(a)} For \(\mathcal{A} \in \text{obj}(\mathcal{R} \text{-} 2\text{-Mod})\), \(H \in \text{Hom}(\mathcal{B}, \mathcal{C})\), the functor \(H \circ - : \text{Hom}(\mathcal{A}, \mathcal{B}) \Rightarrow \text{Hom}(\mathcal{A}, \mathcal{C})\), with \(\varphi^H_{G_1, G_2}\) and \(\varphi^H_0\), is symmetric monoidal.

In fact, \((G_1 + G_2)(A) \triangleq G_1 A + G_2 A, \ \forall A \in \mathcal{A}\). Moreover, \(F\) satisfies the following commutative diagrams:
So \( H \circ - \) is a symmetric monoidal functor.

(b) For \( G \in \text{Hom}(\mathcal{A}, \mathcal{B}), \ C \in \text{obj}(\mathcal{R}-2\text{-Mod}) \), the functor \(- \circ G : \text{Hom}(\mathcal{B}, \mathcal{C}) \to \text{Hom}(\mathcal{A}, \mathcal{C})\), with \( \psi^G_{H_1,H_2} \) and \( \psi^0_G \) symmetric monoidal in the same methods as in (a).

(c) For all \( H_1, H_2 \in \text{Hom}(\mathcal{B}, \mathcal{C}) \), the transformations \( \psi^H_{H_1,H_2} \) and \( \psi^0 \) are monoidal functors. Since \( H_1, H_2 \) are \( \mathcal{R} \)-homomorphisms.

(d) For \( G : \mathcal{B} \to \mathcal{C}, \ H : \mathcal{C} \to \mathcal{D} \), and \( F, F' \in \text{Hom}(\mathcal{A}, \mathcal{B}), \ \alpha_{H,G,-} \) is a monoidal natural identity, i.e. the following diagrams commute:

\[
\begin{array}{c}
\xymatrix{ 
(H \circ G) \circ F + (H \circ G) \circ F' 
\ar[rr]^{\alpha_{H,G,F} + \alpha_{H,G,F'}} 
& & 
H \circ (G \circ F) + H \circ (G \circ F') 
\ar[ll]_{\psi^G_{F,F'}} 
\ar[u]_{\psi^H_{F,F'}} 
\ar[d]_{\alpha_{H,G,F,F'}} 
\ar[r]_H 
\ar[u]_{H \circ (G + F)} 
\ar[d]_{H \circ (G \circ F)} 
& 
H \circ (G \circ (F + F')) 
\ar[l]_{\alpha_{H,G,F,F'}} 
\ar[u]_{H \circ G} 
\ar[d]_{H \circ (G \circ F)} 
& 
H \circ 0 
\ar[l]_{\alpha_{H,G,-}} 
\ar[u]_{H \circ G} 
\ar[d]_{H \circ (G \circ F)} 
\ar[r]^H 
\ar[u]_{H \circ 0} 
\ar[d]_{H \circ (G \circ F)} 
& 
H \circ (G \circ 0) 
\ar[l]_{\alpha_{H,G,-}} 
\ar[u]_{H \circ G} 
\ar[d]_{H \circ (G \circ F)} 
\end{array}
\]

Similarly, the following conditions hold.

(d) For \( F \in \text{Hom}(\mathcal{A}, \mathcal{B}), \ H \in \text{Hom}(\mathcal{C}, \mathcal{D}), \ \alpha_{H,-,F} \) is a monoidal natural transformation.

(e) For \( F \in \text{Hom}(\mathcal{A}, \mathcal{B}), \ G \in \text{Hom}(\mathcal{B}, \mathcal{C}), \ \alpha_{-,G,F} \) is a monoidal natural transformation.

(f) For all \( \mathcal{A}, \mathcal{B} \in \text{obj}(\mathcal{R}-2\text{-Mod}) \), since \( \lambda_\_ \) and \( \rho_\_ \) are identities, so the following unit natural transformations are monoidal:
i.e. the following diagrams commute:

\[ \begin{array}{c}
1_B \circ (F_1 + F_2) \\
\downarrow \quad \downarrow \\
F_1 + F_2 \\
\downarrow \quad \downarrow \\
(F_1 + F_2) \circ 1_A
\end{array} \]

\[ \begin{array}{c}
1_B \circ 1_B \\
\downarrow \quad \downarrow \\
F_1 + F_2 \\
\downarrow \quad \downarrow \\
F_1 \circ 1_A + F_2 \circ 1_A
\end{array} \]

Proposition 5. \((\mathcal{R}\text{-2-Mod})\) has all finite biproducts.

Proof. For any two objects \(A, B\) in \((\mathcal{R}\text{-2-Mod})\), there is a new object \(A \times B\) in \((\mathcal{R}\text{-2-Mod})\).

- \(A \times B\) is a category consisting of the following data:
  - Objects are pairs \((A, B)\), where \(A \in \text{obj}(\mathcal{A})\), \(B \in \text{obj}(\mathcal{B})\).
  - A morphism between \((A_1, B_1)\) and \((A_2, B_2)\) is a pair \((f, g)\), where \(f : A_1 \to A_2\) is a morphism in \(\mathcal{A}\), \(g : B_1 \to B_2\) is a morphism in \(\mathcal{B}\).
  - Composition of morphisms. Given morphisms \((A_1, B_1) \xrightarrow{(f_1, g_1)} (A_2, B_2) \xrightarrow{(f_2, g_2)} (A_3, B_3)\), \((f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)\) is also a morphism in \(\mathcal{A} \times \mathcal{B}\).

The above ingredients satisfy the following axioms:

1. For any \((A, B) \in \text{obj}(\mathcal{A} \times \mathcal{B})\), there exists a morphism \(1_{(A,B)} \triangleq (1_A, 1_B) : (A, B) \to (A, B)\), such that for any morphism \((f, g) : (A, B) \to (A_1, B_1)\), \((f, g) \circ 1_{(A,B)} = (f, g)\).

2. Associativity of compositions. Given morphisms

\[ \begin{array}{c}
(A_1, B_1) \xrightarrow{(f_1, g_1)} (A_2, B_2) \xrightarrow{(f_2, g_2)} (A_3, B_3) \xrightarrow{(f_3, g_3)} (A_4, B_4),
\end{array} \]
for \((f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)\) and \((g_3 \circ g_2) \circ g_1 = g_3 \circ (g_2 \circ g_1)\), we have

\[
((f_3, g_3) \circ (f_2, g_2)) \circ (f_1, g_1) = (f_3, g_3) \circ ((f_2, g_2) \circ (f_1, g_1)).
\]

- \(\mathcal{A} \times \mathcal{B}\) is a symmetric 2-group.

For any morphism \((f, g) : (A_1, B_1) \to (A_2, B_2)\) in \(\mathcal{A} \times \mathcal{B}\), and \(\mathcal{A}, \mathcal{B}\) are groupoids, there exist \(f^* : A_2 \to A_1\), \(g^* : B_2 \to B_1\), such that \(f^* \circ f = 1_{A_1}, g^* \circ g = 1_{B_1}\). So there exists \((f, g)^* \triangleq (f^*, g^*) : (A_2, B_2) \to (A_1, B_1)\), such that \((f, g)^* \circ (f, g) = 1_{(A_1, B_1)}\).

There is an unit object \(0 = (0_A, 0_B)\) in \(\mathcal{A} \times \mathcal{B}\), where \(0_A\) and \(0_B\) are unit objects of \(\mathcal{A}\) and \(\mathcal{B}\), respectively.

There are a bifunctor

\[
+: (\mathcal{A} \times \mathcal{B}) \times (\mathcal{A} \times \mathcal{B}) \to (\mathcal{A} \times \mathcal{B})
\]

\[
((A_1, B_1), (A_2, B_2)) \mapsto (A_1, B_1) + (A_2, B_2) \triangleq (A_1 + A_2, B_1 + B_2),
\]

\[
((f_1, g_1), (f_2, g_2)) \mapsto (f_1 + g_1) + (f_2 + g_2) \triangleq (f_1 + f_2, g_1 + g_2)
\]

and natural isomorphisms:

\[
< (A_1, B_1), (A_2, B_2), (A_3, B_3) > : ((A_1, B_1) + (A_2, B_2)) + (A_3, B_3)
\]

\[
\to (A_1, B_1) + ((A_2, B_2) + (A_3, B_3)),
\]

\[
l_{(A,B)} : 0 + (A, B) \to (A, B),
\]

\[
r_{(A,B)} : (A, B) + 0 \to (A, B)
\]

given by \(< (A_1, B_1), (A_2, B_2), (A_3, B_3) > \triangleq (< A_1, A_3 >, < B_1, B_2, B_3 >),
\]

\[
l_{(A,B)} \triangleq (l_A, l_B), \ r_{(A,B)} \triangleq (r_A, r_B).\] Obviously, the Mac Lane coherence conditions hold.

For any \((A, B) \in \text{obj}(\mathcal{A} \times \mathcal{B})\), \(A \in \mathcal{A}, B \in \mathcal{B}\), there exist \(A^* \in \mathcal{A}, B^* \in \mathcal{B}\), and natural isomorphisms \(\eta_A : A^* + A \to 0, \ \eta_B : B^* + B \to 0\). Then there exists \((A, B)^* \triangleq (A^*, B^*),\) and natural isomorphism \(\eta_{(A,B)} \triangleq (\eta_A, \eta_B) : (A, B)^* + (A, B) \to 0\).

For any two objects \((A_1, B_1), (A_2, B_2) \in (\mathcal{A} \times \mathcal{B})\), since there are natural isomorphisms \(c_{A_1, A_2} : A_1 + A_2 \to A_1 + A_1, \ c_{B_1, B_2} : B_1 + B_2 \to B_2 + B_1\), with \(c_{A_1, A_2} \circ c_{A_2, A_1} = id, \ c_{B_1, B_2} \circ c_{B_2, B_1} = id\). Then we get a natural isomorphism

\[
c_{(A_1, B_1),(A_2, B_2)} \triangleq (c_{A_1, A_2}, c_{B_1, B_2}) : (A_1, B_1) + (A_2, B_2) \to (A_2, B_2) + (A_1, B_1),
\]

with \(c_{(A_1, B_1),(A_2, B_2)} \circ c_{(A_2, B_2),(A_1, B_1)} = id\).
• $A \times B$ is an $\mathcal{R}$-2-module.

There is a bifunctor

$$\cdot : \mathcal{R} \times (A \times B) \rightarrow (A \times B)$$

$$(r, (A, B)) \mapsto r \cdot (A, B) \triangleq (r \cdot A, r \cdot B),$$

$$(r_1 \circ r_2, (A_1, B_1)) \mapsto (r_1 \cdot A_1, r_2 \cdot A_2) \mapsto (r_1 \cdot B_1, r_2 \cdot B_2)$$

since $A$, $B$ are $\mathcal{R}$-2-modules.

Also, there are natural isomorphisms:

$$a_{(A_1, B_1), (A_2, B_2)} \triangleq (a_{A_1, A_2}, a_{B_1, B_2}) : r \cdot ((A_1, B_1) + (A_2, B_2)) \rightarrow (r \cdot (A_1 + A_2), r \cdot (B_1 + B_2)),$$

$$b_{(A,B)} \triangleq (b_{A}, b_{B}) : (r_1 + r_2) \cdot (A, B) \rightarrow r_1 \cdot (A, B) + r_2 \cdot (A, B),$$

$$b_{(A,B)} \triangleq (b_{r_1, r_2, A}, b_{r_1, r_2, B}) : (r_1 r_2) \cdot (A, B) \rightarrow r_1 \cdot (r_2 \cdot (A, B)),$$

$$i_{(A,B)} \triangleq (i_A, i_B) : I \cdot (A, B) \rightarrow (I \cdot A, I \cdot B),$$

$$z_r \triangleq (z_r, z_r) : r \cdot 0 = r \cdot (0, 0) \rightarrow 0.$$

Since $A$, $B$ are $\mathcal{R}$-2-modules, their natural isomorphisms make Fig.18.-Fig.31. commute, so do $a_{(A_1, B_1), (A_2, B_2)}$, $b_{(A,B)}$, $b_{(A,B)}$, $i_{(A,B)}$ and $z_r$.

• $A \times B$ is the biproduct of $A$ and $B$.

We need to prove $A \times B$ is not only the product but also the coproduct of $A$ and $B$.

There are $\mathcal{R}$-homomorphisms:

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B$$

$$A \leftarrow (A, B) \rightarrow B,$$

$$A_1 \xleftarrow{f} A_2 \leftarrow ((A_1, B_1) \xrightarrow{(f, g)} (A_2, B_2)) \rightarrow B_1 \xrightarrow{g} B_2$$

and the faithful $\mathcal{R}$-homomorphisms:

$$A \xrightarrow{i_1} A \times B \xleftarrow{i_2} B$$

$$A \rightarrow (A, 0),$$

$$(0, B) \leftarrow B$$

Next, we will show that $(A \times B, i_1, i_2)$ satisfies the universal property of coproduct [5, 19].
For any \( R \)-2-module \( K \) and \( R \)-homomorphisms \( F_1 : \mathcal{A} \to K \), \( F_2 : \mathcal{B} \to K \), there is an \( R \)-homomorphism \( G : \mathcal{A} \times \mathcal{B} \to K \), with
\[
G(A, B) = F_1 A + F_2 B,
\]
and isomorphisms \( l_1 : G \circ i_1 \Rightarrow F_1 \), \( l_2 : G \circ i_2 \Rightarrow F_2 \), given by
\[
(l_1)_A : (G \circ i_1)(A) = G(A, 0) = F_1 A + F_2 0 \to F_1 A + 0 \to F_1 A,
\]
\[
(l_2)_B : (G \circ i_2)(B) = G(0, B) = F_1 0 + F_2 B \to 0 + F_2 B \to F_2 B.
\]
So \( \mathcal{A} \times \mathcal{B} \) is a coproduct.

Using the Proposition 225 in [19], \( (\mathcal{A} \times \mathcal{B}, p_1, p_2) \) is a product, such that
\[
p_2 \circ i_1 = 0, \quad p_1 \circ i_1 = 1_{\mathcal{A}}, \quad p_1 \circ i_2 = 0, \quad p_2 \circ i_2 = 1_{\mathcal{B}}.
\]
So, \( \mathcal{A} \times \mathcal{B} \) is a biproduct.

\[\square\]

**Definition 15.** ([19] Definition 242) Let \( \mathcal{C} \) be a Gpd-category.

1. We say that \( \mathcal{C} \) is semiadditive if it is presemiadditive and has all finite biproducts.

2. We say that \( \mathcal{C} \) is additive if it is preadditive and has all finite biproducts.

**Corollary 2.** \((\mathcal{R}-2-\text{Mod})\) is an additive Gpd-category.

Next, we use the definitions of pips(copips) and roots(coroots) given in [8, 19] to give the next definitions in \((\mathcal{R}-2-\text{Mod})\).

**Definition 16.** Let \( F : \mathcal{A} \to \mathcal{B} \) be \( \mathcal{R} \)-homomorphism.

- The pip of \( F \) is given by an \( \mathcal{R} \)-2-module \( \text{Pip}F \), two zero \( \mathcal{R} \)-2-module homomorphisms \( 0 : \text{Pip}F \to \mathcal{A} \), and morphism \( \sigma : 0 \Rightarrow 0 \) of \( \mathcal{R} \)-homomorphisms as in the following diagram:
such that \( F \ast \sigma = 1_0 : 0 \Rightarrow 0 : PipF \to B \), and for any other \( \mathcal{D} \) as in the following diagram

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\sigma} & A \\
\downarrow & & \downarrow f \\
0 & & B
\end{array}
\]

with \( F \ast \alpha = 1_0 \). There is an \( \mathcal{R} \)-homomorphism \( G : \mathcal{D} \to PipF \), such that \( \sigma \ast G = \alpha \). \( G \) is unique up to an invertible morphism of \( \mathcal{R} \)-homomorphisms, i.e. if there is a \( G' : \mathcal{D} \to PipF \), with \( \sigma \ast G' = \alpha \), there exists a unique isomorphism \( \tau : G' \Rightarrow G \), such that

\[
\begin{array}{ccc}
\sigma \ast G' & \xrightarrow{\alpha} & \sigma \ast G \\
\downarrow & \swarrow & \downarrow \\
\alpha & & \alpha
\end{array}
\]

commutes.

• The copip of \( F \) is given by an \( \mathcal{R} \)-2-module \( CopipF \), two zero \( \mathcal{R} \)-homomorphisms \( 0 : B \to CopipF \), and morphism of \( \mathcal{R} \)-homomorphisms \( \sigma : 0 \Rightarrow 0 : B \to CopipF \), as in the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & \searrow & \downarrow 0 \\
& CopipF & \\
\downarrow & \swarrow & \\
& \mathcal{D} &
\end{array}
\]

such that \( \sigma \ast F = 1_0 : 0 \Rightarrow 0 : A \to CopipF \), and for any other

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & \searrow & \downarrow 0 \\
& \mathcal{D} &
\end{array}
\]

with \( \alpha \ast F = 1_0 \). There is an \( \mathcal{R} \)-homomorphism \( G : CopipF \to \mathcal{D} \), such that \( G \ast \sigma = \alpha \). \( G \) is unique up to an invertible morphism of \( \mathcal{R} \)-homomorphisms, i.e. if there is a \( G' : CopipF \to \mathcal{D} \), with \( G' \ast \sigma = \alpha \), there exists a unique isomorphism \( \tau : G' \Rightarrow G \).
Definition 17. Let $\alpha : 0 \Rightarrow 0 : \mathcal{A} \to \mathcal{B}$ be 2-morphism in $(\mathcal{R}\text{-}2\text{-Mod})$.

- The root of $\alpha$ is an $\mathcal{R}\text{-}2$-module $\text{Root}\alpha$ and an $\mathcal{R}$-homomorphism $F : \text{Root}\alpha \to \mathcal{A}$ as in the following diagram

\[
\begin{array}{ccc}
\text{Root}\alpha & \xrightarrow{F} & \mathcal{A} & \xrightarrow{\alpha} & \mathcal{B} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
\]

such that $\alpha \ast F = 1_0 : 0 \Rightarrow 0 : \mathcal{D} \to \mathcal{B}$, for any other as in

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{G} & \mathcal{A} & \xrightarrow{\alpha} & \mathcal{B} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
\]

with $\alpha \ast G = 1_0$, there exist an $\mathcal{R}$-homomorphism $G' : \mathcal{D} \to \text{Root}\alpha$ and an invertible morphism $\varphi : F \circ G' \Rightarrow G$. The pair $(G', \varphi)$ is unique up to an invertible morphism, i.e. If $(G'', \varphi')$ satisfies the same conditions as $(G', \varphi)$, there exists a unique $\tau : G'' \Rightarrow G'$, such that

\[
\begin{array}{ccc}
F \circ G' & \xrightarrow{\varphi} & F \circ G' \\
\downarrow & & \downarrow \\
G & \xrightarrow{G' \ast \varphi' \ast G'' \ast F} & G
\end{array}
\]

commutes.

- The coroot of $\alpha$ is an $\mathcal{R}\text{-}2$-module $\text{Coroot}\alpha$, and an $\mathcal{R}$-homomorphism $F : \mathcal{B} \to \text{Coroot}\alpha$ with $F \ast \alpha = 1_0 : 0 \Rightarrow 0 : \mathcal{A} \to \text{Coroot}\alpha$. For any other $G : \mathcal{B} \to \mathcal{D}$, with $G \ast \alpha = 1_0$, there exist an $\mathcal{R}$-homomorphism $G' : \text{Coroot}\alpha \to \mathcal{D}$, and an invertible 2-morphism $\varphi : G' \circ F \Rightarrow G$ as in the following diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\alpha} & \mathcal{B} & \xrightarrow{F} & \text{Coroot}\alpha \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{D} & \xrightarrow{G} & \mathcal{B} & \xrightarrow{\varphi} & \text{Coroot}\alpha
\end{array}
\]
The pair \((G', \varphi)\) is unique up to an invertible 2-morphism, i.e. if \((G'', \varphi')\) satisfies the same conditions as \((G', \varphi)\), there exists a unique \(\tau : G'' \Rightarrow G'\), such that

\[
\begin{array}{ccc}
G' \circ F & \overset{F\tau}{\Rightarrow} & G' \circ F \\
\downarrow \varphi & & \downarrow \varphi \\
G & \downarrow \varphi & \downarrow \varphi \\
\end{array}
\]

commutes.

Next, we will use definitions 16-17 to give the existence of (Co)Pip and (Co)Root, based on the results of symmetric 2-groups in [8, 19].

- Existence of pip of F.
  (i) There is a category \(\text{Pip}F\) consisting the following data:
    - Object is a morphism \(a : 0 \to 0\) in \(A\), such that \(Fa = 1_{F0}\), where 0 is the unit object of \(A\).
    - Morphism of \(\varphi : a \Rightarrow b : 0 \to 0\) is just the identity if and only if \(a = b\).
    - Composition of morphisms is a composition of identity morphisms in \(A\) which is also identity.

  Obviously, the above data satisfy the necessary conditions in the definition of category, and \(\text{Pip}F\) is also a groupoid.

  (ii) \(\text{Pip}F\) is a symmetric 2-group with monoidal structure as follows:

  \(\text{Pip}F\) has a unit object \(1 : 0 \to 0\), which is the identity of 0.

  There is a bifunctor

  \[
  + : \text{Pip}F \times \text{Pip}F \rightarrow \text{Pip}F
  \]

  \[
  (a : 0 \rightarrow 0, b : 0 \rightarrow 0) \mapsto a + b \triangleq a \circ b : 0 \rightarrow 0
  \]

  Also, \(+\) maps identity morphism to identity morphism in \(\text{Pip}F\). Moreover, there are natural identities:

  \[
  (a + b) + c = (a \circ b) \circ c = a \circ (b \circ c) = a + (b + c),
  \]

  \[
  1 + a = 1 \circ a = a,
  \]

  \[
  a + 1 = a \circ 1 = a
  \]
satisfy the Mac Lane coherence conditions (Fig. 1-2.).

For any object \( a : 0 \to 0 \) in \( \text{Pip}F \), which is a morphism in \( \mathcal{A} \), and since \( \mathcal{A} \) is a groupoid, \( a \) is invertible.

For any two objects \( a, b : 0 \to 0 \) in \( \text{Pip}F \), there exists identity \( a + b = b + a \), since \( a, b \) are endomorphisms of 0.

(iii) \( \text{Pip}F \) is an \( \mathcal{R}-2 \)-module.

There is a bifunctor:

\[
\star : \mathcal{R} \times \text{Pip}F \longrightarrow \text{Pip}F
\]

\[
(r, 0 \stackrel{a}{\to} 0) \mapsto r \star a
\]

\[
(r_1 \stackrel{\varphi}{\to} r_2, a \stackrel{id}{\to} a) \mapsto 0 \stackrel{\varphi \ast id}{\to} 0
\]

where \( r \star a = z_r \circ r \cdot a \cdot z_r^{-1} \) is the composition

\[
0 \xrightarrow{z_r^{-1}} r \cdot 0 \xrightarrow{r \cdot a} r \cdot 0 \xrightarrow{z_r} 0,
\]

such that \( F(r \star a) = 1_{F_0} \). \( \varphi \ast id = z_{r_2} \ast (\varphi \cdot id) \ast z_{r_1}^{-1} = id. \)

In fact, using Fig. 37. and \( Fa = 1_{F_0} \),

\[
F(r \star a) = F(z_r \circ r \cdot a \cdot z_r^{-1}) = F(z_r) \circ F(r \cdot a) \circ F(z_r^{-1}) = F(z_r) \circ F_2^{-1} \circ r \cdot F(a) \circ F_2 \circ F(z_r^{-1}) = 1_{F_0}.
\]

Moreover, there are natural isomorphisms:

(1) \( (r_1 r_2) \ast a = r_1 \ast (r_2 \ast a) \).

In fact,

\[
r \ast (a_1 + a_2) = r \ast (a_1 + a_2) \triangleq z_r \circ (r \cdot (a_1 \circ a_2)) \circ z_r^{-1} = z_r \circ (r \cdot a_1 \circ a_2) \circ z_r^{-1} = z_r \circ r \cdot a_1 \circ r \cdot a_2 \circ z_r^{-1} = r \ast a_1 \circ r \ast a_2 = r \ast a_1 + r \ast a_2,
\]

\[
(r_1 r_2) \ast a \triangleq z_{r_1 r_2} \circ ((r_1 r_2) \cdot a) \circ z_{r_1 r_2}^{-1} = z_r \circ (r_1 \cdot z_{r_2}) \circ b_{r_1 r_2} \ast 0 \circ ((r_1 r_2) \cdot a) \circ b_{r_1 r_2}^{-1} \circ (r_1 \cdot z_{r_2}^{-1}) \circ z_{r_1 r_2}^{-1},
\]

\[
r_2 \ast a \triangleq z_{r_2} \circ (r_2 \cdot a) \circ z_{r_2}^{-1},
\]

\[
r_1 \ast (r_2 \ast a) \triangleq z_{r_1} \circ r_1 \cdot (r_2 \ast a) \circ z_{r_1}^{-1} = z_{r_1} \circ r_1 \cdot z_{r_2} \circ r_1 \cdot (r_2 \cdot a) \circ r_1 \cdot z_{r_2}^{-1} \circ z_{r_1}^{-1}.
\]

Next, we will check \( b_{r_1 r_2} \circ ((r_1 r_2) \cdot a) \circ b_{r_1 r_2}^{-1} = r_1 \cdot (r_2 \cdot a) \).

Let \( F, G : \mathcal{R} \times \mathcal{R} \times \mathcal{A} \to \mathcal{A} \) be functors, given by,

\[
F(r_1, r_2, 0) \triangleq (r_1 r_2) \cdot 0,
\]

\[
G(r_1, r_2, 0) \triangleq r_1 \cdot (r_2 \cdot 0).
\]
for $r_1, r_2 \in \text{obj}(\mathcal{R})$. $b_{r_1, r_2}$ is a natural transformation from $F$ to $G$, so for $a : 0 \to 0$ in $\mathcal{A}$, we have the following commutative diagram:

$$
\begin{array}{ccc}
(r_1, r_2) \cdot 0 & \xrightarrow{(r_1, r_2) \cdot a} & (r_1, r_2) \cdot 0 \\
\downarrow b_{r_1, r_2} & & \downarrow b_{r_1, r_2} \\
(r_1 + r_2) \cdot 0 & \xrightarrow{(r_1 + r_2) \cdot a} & (r_1 + r_2) \cdot 0
\end{array}
$$

(2) $(r_1 + r_2) \ast a = r_1 \ast a + r_2 \ast a$.

In fact,

$$(r_1 + r_2) \ast a \triangleq z_{r_1+r_2} \circ (r_1 + r_2) \cdot a \circ z_{r_1+r_2}^{-1} = (z_{r_1} + z_{r_2}) \circ b_0^{r_1,r_2} \circ (r_1 + r_2) \cdot a \circ (b_0^{r_1,r_2})^{-1} \circ (z_{r_1}^{-1} + z_{r_2}^{-1}).$$

Next, we will check $b_{r_1,r_2}^{-1} \circ (r_1 + r_2) \cdot a \circ (b_0^{r_1,r_2})^{-1} = r_1 \cdot a + r_2 \cdot a$.

Let $F, G : \mathcal{R} \times \mathcal{R} \times \mathcal{A} \to \mathcal{A}$ be functors, given by, for $r_1, r_2 \in \text{obj}(\mathcal{R})$,

$$F(r_1, r_2, 0) \triangleq (r_1 + r_2) \cdot 0,$$

$$G(r_1, r_2, 0) \triangleq r_1 \cdot 0 + r_2 \cdot 0.$$ $b_{r_1,r_2}$ is a natural transformation from $F$ to $G$, so for $a : 0 \to 0$ in $\mathcal{A}$, we have the following commutative diagram:

$$
\begin{array}{ccc}
(r_1 + r_2) \cdot 0 & \xrightarrow{(r_1 + r_2) \cdot a} & (r_1 + r_2) \cdot 0 \\
\downarrow b_{r_1, r_2} & & \downarrow b_{r_1, r_2} \\
(r_1 + r_2) \cdot 0 & \xrightarrow{(r_1 + r_2) \cdot a} & (r_1 + r_2) \cdot 0
\end{array}
$$

Similarly, we have (3) $1 \ast a = a$, (4) $r \ast 1 = 1$.

The above natural isomorphisms are identities, so they make Fig.18.-31. commute, then $\text{Pip} F \in \text{obj} (\mathcal{R}-2\text{-Mod})$.

(iv) $\text{Pip} F$ is the pip of $F$.

There are zero homomorphisms:

$$0 : \text{Pip} F \to \mathcal{A}$$

$$0 \xrightarrow{a} 0 \mapsto 0.$$
Remark 6. For one object \( a \in \text{Pip}F \), it is a morphism in \( A \), so we can consider the above zero morphisms as one maps to the source of \( a \), another maps to the target of \( a \).

A 2-morphism

\[
\sigma : 0 \Rightarrow 0
\]

\[(\sigma)_a \triangleq a\]

such that, \((1_F \ast \sigma)_a = (1_F)_0(a) \circ F((\sigma)_a) = 1_{F0}\), then \( F \ast \sigma = 1_0 \).

If \( D \in \text{obj}(R\text{-2-Mod}) \), and \( \alpha : 0 \Rightarrow 0 : D \to A \), with \( F \ast \alpha = 1_0 \). There exists an \( R \)-homomorphism \( G : D \to \text{Pip}F \), by \( G(d) \equiv \alpha_d : 0 \to 0 \), and since \( F \ast \alpha = 1_0 \), so \( F(\alpha_d) = 1_{F0} \). Also for any \( d \in D \), \((\sigma \ast G)_d = \sigma_{Gd} = G(d) = \alpha_d\), i.e. \( \sigma \ast G = \alpha \).

From the given \( R \)-homomorphism \( G \), it is easy to see that \( G \) is unique up to an invertible 2-morphism.

• Existence of the copip of \( F \).

(i) There is a category consisting of the following data:

· A unique object is denoted by \( * \).

· Morphism from \( * \) to \( * \) is the object \( B \in B \). Two morphisms \( B_1, B_2 : * \to * \) are equal, if there exist \( A \in \text{obj}(A) \), and \( b : B_1 \to FA + B_2 \). Denote the equivalence class of morphisms by \([B]\).

· Composition of morphisms:

Let \( * \xrightarrow{[B_1]} * \xrightarrow{[B_2]} * \) be morphisms in \( \text{Copip}F \). We have \([B_2] \circ [B_1] \triangleq [B_1 + B_2]\), which is well-defined. In fact, if \( B_1, B_1' \) are equal, i.e. \( \exists A_1 \in \text{obj}(A) \), and \( b_1 : B_1 \to FA_1 + B_1' \). There exist \( A_1 \in \text{obj}(A) \), and \( b : B_1 + B_2 \to (FA_1 + B_1') + B_2 \to FA_1 + (B_1' + B_2) \), so \( B_1 + B_2 \), \( B_1' + B_2 \) are equal.

The above data satisfy the following axioms:

(1) For the unique object \( * \), there exists an identity morphism \( 1 : * \to * \), which in fact is the unit object \( 0 \) of \( B \), such that for any morphism \([B]\), there are \([B] \circ 1 = [0 + B] = [B]\), \( 1 \circ [B] = [B + 0] = [B]\). We will to show the first equality. In fact, there exist \( 0 \in \text{obj}(A) \), and \( b : 0 + B \xrightarrow{F_0^{-1} + 1_B} F0 + B \).

(2) Associativity of composition.
Given morphisms \( [B_1] \rightarrow [B_2] \rightarrow [B_3] \), \( [B_3] \circ ([B_2] \circ [B_1]) \triangleq (B_1 + B_2) + B_3 \) is equal to \( ([B_3] \circ [B_2]) \circ [B_1] \triangleq B_1 + (B_2 + B_3) \), since there exist \( 0 \in \text{obj}(A) \), and morphism \( b : (B_1 + B_2) + B_3 \xrightarrow{\text{cop}_1(B_1,B_2,B_3)} B_1 + (B_2 + B_3) \xrightarrow{1_{B_1+ (B_2 + B_3)}} 0 + B_1 + (B_2 + B_3) \).

For any morphism \([B]\) in \( \text{Cop}_F \), \( B \in \text{obj}(B) \), and \( B \) is a 2-group, so \( B \) is invertible, then \([B]\) is invertible, i.e. \( \text{Cop}_F \) is a groupoid.

(ii) \( \text{Cop}_F \) is a symmetric 2-group.

There is a bifunctor

\[
+ : \text{Cop}_F \times \text{Cop}_F \rightarrow \text{Cop}_F
\]

\[
(\ast, \ast) \mapsto \ast
\]

\[
(\ast \xrightarrow{[B_1]} \ast, \ast \xrightarrow{[B_2]} \ast) \mapsto \ast \xrightarrow{[B_1 + B_2]} \ast
\]

Moreover, the natural isomorphisms are identities, so they satisfy the Mac Lane coherence conditions. The inverse of an object is just itself.

(iii) \( \text{Cop}_F \) is an \( \mathcal{R} \)-2-module.

We can give the trivial bifunctor

\[
\star : \mathcal{R} \times \text{Cop}_F \rightarrow \text{Cop}_F
\]

\[
(r, \ast) \mapsto \ast
\]

(iv) \( \text{Cop}_F \) is the copip of \( F \).

There are two zero \( \mathcal{R} \)-morphisms

\[
0 : B \rightarrow \text{Cop}_F
\]

\[
B \mapsto \ast
\]

Remark 7. For one object \( B \in \mathcal{B} \), it is a morphism in \( \text{Cop}_F \), so we can consider the above zero morphisms as one maps to the source of \( B \), another maps to the target of \( B \).

There is a morphism between the above two zero \( \mathcal{R} \)-homomorphisms

\[
\sigma : 0 \Rightarrow 0
\]

\[
\sigma_B \triangleq B : \ast \rightarrow \ast
\]
such that $\sigma \ast F = 1_0$.

If $\mathcal{D} \in \text{obj}(\mathcal{R}\text{-2-Mod})$, and $\alpha : 0 \Rightarrow 0 : \mathcal{B} \rightarrow \mathcal{D}$, with $\alpha \ast F = 1_0$. There exists an $\mathcal{R}$-homomorphism

$$G : \text{Copip}F \longrightarrow \mathcal{D}$$

$$\ast \mapsto G(\ast) \equiv 0,$$

$$[B] : \ast \mapsto \ast \mapsto G([B]) \triangleq \alpha_B : 0 \rightarrow 0$$

where $0$ is the unit object of $\mathcal{D}$. For any $B \in \text{obj} \mathcal{B}$, $(G \ast \sigma)_B = (1_G)_0(B) \circ G(\sigma_B) = (1_G)_* \circ G(B) = G(B) = \alpha_B$, i.e. $G \ast \sigma = \alpha$.

From the definition of $G$, $G$ is unique up to an invertible 2-morphism.

**Definition 18.** ([19], Proposition 179.) Let $\mathcal{C}$ be a Gpd*-category with zero object and all the kernels and cokernels. We say that $\mathcal{C}$ is 2-Puppe-exact if the following property holds.

For every morphism $f : A \rightarrow B$ in $\mathcal{C}$, $\omega_f$ and $\omega_f$ are equivalent in the following diagrams:

\[\begin{array}{ccc}
\text{Ker} f & \xrightarrow{e_f} & A & \xrightarrow{f} & B \\
\downarrow & & \downarrow & & \downarrow \\
\text{Coker} e_f & \xrightarrow{\pi_f} & \text{Root} \sigma & \xrightarrow{\varphi} & \text{Ker} q_f
\end{array}\]

\[\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
\text{Pip} f & \xrightarrow{\pi_f} & \text{Ker} q f
\end{array}\]

\[\begin{array}{ccc}
\text{Copip}\ f & \xrightarrow{\varphi} & \text{Ker} f \\
\downarrow & & \downarrow \\
\text{Pip} f & \xrightarrow{\pi_f} & \text{Ker} q f
\end{array}\]

**Definition 19.** ([19], Definition 183.) A 2-abelian Gpd-category is a 2-Puppe-exact Gpd*-category which has all the finite products and coproducts.

The following four Propositions are the factorization systems in 2-category ($\mathcal{R}\text{-2Mod}$) in the sense of [7, 11, 19].

**Proposition 6.** Every 1-morphism $F : \mathcal{A} \rightarrow \mathcal{B}$ in ($\mathcal{R}\text{-2-Mod}$) factors as the following composite, where $\widehat{E}_F$ is surjective, $\widehat{\Omega}_F$ is an equivalence, and $\widehat{M}_F$ is full and faithful.

$$\mathcal{A} \xrightarrow{\widehat{E}_F} \text{Im}^1_{pl} F \xrightarrow{\widehat{\Omega}_F} \text{Im}^2_{pl} F \xrightarrow{\widehat{M}_F} \mathcal{B}.$$  

**Proof.** Step 1. The $\mathcal{R}$-module $\text{Im}^1_{pl} F$ is described in the following way:
• Category $Im_{pl}^1 F$ consists of:
  
  - Objects are those of $\mathcal{A}$.
  
  - Morphism $f : A \to A'$ in $Im_{pl}^1 F$ is a morphism $F(f) : FA \to FA'$ in $\mathcal{B}$. The composition of morphisms are those of $\mathcal{B}$, i.e. morphisms $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ in $Im_{pl}^1 F$ are $FA_1 \xrightarrow{F(f_1)} FA_2 \xrightarrow{F(f_2)} FA_3$ in $\mathcal{B}$, so the composition $f_2 \circ f_1 : A_1 \to A_3$ is $F(f_2 \circ f_1) = F(f_2) \circ F(f_1)$ in $\mathcal{B}$.

The above ingredients satisfying the following axioms:

1. For any object $A \in Im_{pl}^1 F$, there exist an identity morphism $1_A : A \to A$ given by $F(1_A) \triangleq 1_{FA} : FA \to FA$ in $\mathcal{B}$, such that for any morphism $f : A_1 \to A_2$, we have $f \circ 1_{A_1} = f$, $1_{A_2} \circ f = f$, since $F(f \circ 1_{A_1}) = F(f) \circ F(1_{A_1}) = F(f) \circ 1_{FA_1} = F(f)$, $F(1_{A_2} \circ f) = F(1_{A_2}) \circ F(f) = 1_{FA_2} \circ F(f) = F(f)$.

2. Given morphisms
   
   $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4$,
   
   for $F((f_3 \circ f_2) \circ f_1) = F(f_3) \circ F(f_2) \circ F(f_1) = F(f_3 \circ (f_2 \circ f_1))$, so the associativity of composition is true.

• $Im_{pl}^1 F$ is a symmetric monoidal groupoid.

The unit object of $Im_{pl}^1 F$ is just the unit object of $\mathcal{A}$.

There is a bifunctor

$$+ : Im_{pl}^1 F \times Im_{pl}^1 F \longrightarrow Im_{pl}^1 F$$

$$(A, A') \mapsto A + A'$$

$$(A_1 \xrightarrow{f} A_2, A'_1 \xrightarrow{f'} A'_2) \mapsto A_1 + A'_1 \xrightarrow{f + f'} A_2 + A'_2$$

where $A + A'$ is an addition of objects of $\mathcal{A}$, $f + f'$ is a composition morphism $F(f + f') : F(A_1 + A'_1) \xrightarrow{f + f'} FA_1 + FA'_1 \xrightarrow{F + F + F} FA_2 + FA'_2 \xrightarrow{F + F + F} F(A_2 + A'_2)$.

Moreover, there are natural isomorphisms:

$$< A_1, A_2, A_3 > : (A_1 + A_2) + A_3 \to A_1 + (A_2 + A_3),$$

$$l_A : 0 + A \to A,$$

$$r_A : A + 0 \to A,$$

$$c_{A_1,A_2} : A_1 + A_2 \to A_2 + A_1$$

defined by the images of the natural isomorphisms in $\mathcal{B}$ under $F$. As $\mathcal{B}$ is a symmetric monoidal groupoid, so they satisfy the Mac Lane coherence conditions, and $Im_{pl}^1 F$ is a symmetric monoidal groupoid.
• $\text{Im}_{pl}^1 F$ is a symmetric 2-group.

We need to show every object of $\text{Im}_{pl}^1 F$ is invertible. In fact, for any object $A \in \text{Im}_{pl}^1 F$, $A \in \text{obj}(A)$, and $A$ is 2-group, there exist $A^* \in \text{obj}(A)$ and natural isomorphism $\eta_A : A^* + A \to 0$, so there are $A^* \in \text{obj}(\text{Im}_{pl}^1 F)$, and composition isomorphism $\eta_{FA} : FA^* + FA \xrightarrow{F_+^{-1}} F(A^* + A) \xrightarrow{F^{(\eta_A)}} F0 \xrightarrow{F_0} 0$ in $\mathcal{B}$, which gives a natural isomorphism in $\text{Im}_{pl}^1 F$.

• $\text{Im}_{pl}^1 F$ is an $\mathcal{R}$-2-module.

There is a bifunctor

$$\mathcal{R} \times \text{Im}_{pl}^1 F \longrightarrow \text{Im}_{pl}^1 F$$

$$(r, A) \mapsto r \cdot A,$$

$$(r_1 \xrightarrow{\varphi} r_2, A_1 \xrightarrow{f} A_2) \mapsto r_1 \cdot A_1 \xrightarrow{\varphi \cdot f} r_2 \cdot A_2$$

where $r \cdot A$ is the operation of $\mathcal{R}$ on $A$, $\varphi \cdot f : r_1 \cdot A_1 \to r_2 \cdot A_2$ is a composition $F(\varphi \cdot f) : F(r_1 \cdot A_1) \xrightarrow{F_2} r_1 \cdot FA_1 \xrightarrow{\varphi \cdot F(f)} r_2 \cdot FA_2 \xrightarrow{F_2^{-1}} F(r_2 \cdot A_2)$ in $\mathcal{B}$.

Moreover, there are natural isomorphisms:

$$F(a_{A_1, A_2}^r) : r \cdot (A_1 + A_2) \to r \cdot A_1 + r \cdot A_2,$$

$$F(b_A^{r_1, r_2}) : (r_1 + r_2) \cdot A \to r_1 \cdot A + r_2 \cdot A,$$

$$F(b_{r_1, r_2, A}) : (r_1 r_2) \cdot A \to r_1 \cdot (r_2 \cdot A),$$

$$F(i_A) : I \cdot A \to A,$$

$$F(z_r) : r \cdot 0 \to 0$$

defined by the images of natural isomorphisms in $\mathcal{A}$ under $F$. Since $\mathcal{A}$ is an $\mathcal{R}$-2-module and $F$ is a functor, so Fig.18.-31. commute.

**Step 2.** The $\mathcal{R}$-2-module $\text{Im}_{pl}^2 F$ is described in the following way:

• Category $\text{Im}_{pl}^2 F$ consists of the following data:

  • Objects are the triples $(A, \varphi, B)$, where $A \in \text{obj}(A)$, $B \in \text{obj}(B)$, $\varphi : FA \to B$.

  • Morphism of $(A_1, \varphi_1, B_1) \to (A_2, \varphi_2, B_2)$ is the morphism $g : B_1 \to B_2$ in $\mathcal{B}$. Composition of morphisms and identity morphism are those of $\mathcal{B}$, and associativity of composition naturally holds.
• $\text{Im}_{pl}^2F$ is a symmetric monoidal groupoid.

The unit object is $(0, F_0, 0)$, where the first 0 is the unit object of $A$, the last one is the unit object of $B$, $F_0 : F0 \to 0$.

There is a bifunctor

$$+ : \text{Im}_{pl}^2F \times \text{Im}_{pl}^2F \longrightarrow \text{Im}_{pl}^2F$$

$$(A_1, \varphi_1, B_1), (A_2, \varphi_2, B_2) \mapsto (A_1, \varphi_1, B_1) + (A_2, \varphi_2, B_2) \triangleq (A, \varphi, B),$$

$$(g, g') \mapsto g + g'$$

where $A \triangleq A_1 + A_2$, $B \triangleq B_1 + B_2$, $\varphi$ is a composition $F(A_1 + A_2) \xrightarrow{F_*} F A_1 + FA_2 \xrightarrow{\varphi_1 + \varphi_2} B_1 + B_2$, $g + g'$ is a morphism of $g$, $g'$ under the monoidal structure of $B$.

Moreover, there are natural isomorphisms:

$$< (A_1, \varphi_1, B_1), (A_2, \varphi_2, B_2), (A_3, \varphi_3, B_3) > \triangleq < B_1, B_2, B_3 > : ((A_1, \varphi_1, B_1) + (A_2, \varphi_2, B_2)) + (A_3, \varphi_3, B_3) \rightarrow (A_1, \varphi_1, B_1) + ((A_2, \varphi_2, B_2) + (A_3, \varphi_3, B_3)),$$

$$l_{(A, \varphi, B)} \triangleq l_B : (0, F_0, 0) + (A, \varphi, B) \rightarrow (A, \varphi, B),$$

$$r_{A, \varphi, B} \triangleq r_B : (A, \varphi, B) + (0, F_0, 0) \rightarrow (A, \varphi, B),$$

$$c_{(A_1, \varphi_1, B_1), (A_2, \varphi_2, B_2)} \triangleq c_{B_1, B_2} : (A_1, \varphi_1, B_1) + (A_2, \varphi_2, B_2) \rightarrow (A_2, \varphi_2, B_2) + (A_1, \varphi_1, B_1)$$

where $< B_1, B_2, B_3 >$, $l_B$, $r_B$, $c_{B_1, B_2}$ are natural isomorphisms in $B$, satisfy the Mac Lane coherence conditions, and also $F$ is a functor, then $\text{Im}_{pl}^2F$ is a symmetric monoidal category.

For any morphism $g : (A_1, \varphi_1, B_1) \rightarrow (A_2, \varphi_2, B_2)$ in $\text{Im}_{pl}^2F$ is a morphism $g : B_1 \rightarrow B_2$ in $B$ and $B$ is a groupoid, there exists $g^* : B_2 \rightarrow B_1$, such that $g^* \circ g = 1_{B_1}$, so there is $g^* : (A_2, \varphi_2, B_2) \rightarrow (A_1, \varphi_1, B_1)$, such that $g^* \circ g = 1_{(A_1, \varphi_1, B_1)}$ in $\text{Im}_{pl}^2F$.

• $\text{Im}_{pl}^2F$ is a symmetric 2-group.

For any object $(A, \varphi, B)$ in $\text{Im}_{pl}^2F$, $A \in \text{obj}(A)$, $B \in \text{obj}(B)$, $\varphi : FA \rightarrow B$, for $A$, $B$ are 2-groups, there exist $A^* \in \text{obj}(A)$, $B^* \in \text{obj}(B)$, and natural isomorphisms $\eta_A : A^* + A \rightarrow 0$, $\eta_B : B^* + B \rightarrow 0$. So there exist $(A, \varphi, B)^* \triangleq (A^*, \varphi^*, B^*)$, and natural isomorphism $\eta_{(A, \varphi, B)} \triangleq \eta_B$.

• $\text{Im}_{pl}^2F$ is an $\mathcal{R}$-2-module.
There is a bifunctor
\[ \star : \mathcal{R} \times \text{Im}_2^{\text{pl}} F \longrightarrow \text{Im}_2^{\text{pl}} F \]
\[ (r, (A, \varphi, B)) \mapsto r \star (A, \varphi, B) \triangleq (r \cdot A, \tilde{r} \cdot \varphi, r \cdot B) \]
\[ (r_1 \xrightarrow{\pi} r_2, (A_1, \varphi_1, B_1) \xrightarrow{g} (A_2, \varphi_2, B_2)) \mapsto \pi \cdot g \]
where \( r \cdot A, r \cdot B \) are operations of \( \mathcal{R} \) on \( A, B \), respectively, \( \tilde{r} \cdot \varphi \) is a composition \( F(r \cdot A) \xrightarrow{F_2} r \cdot FA \xrightarrow{r \cdot \varphi} r \cdot B \), \( \pi \cdot g \) is under the operation of \( \mathcal{R} \) on \( B \).

Moreover, there are natural isomorphisms given by the natural isomorphisms in \( \mathcal{B} \) from its \( \mathcal{R} \)-2-module structure.

**Step 3.** \( \text{Im}_1^{\text{pl}} F, \text{Im}_2^{\text{pl}} F \) are equivalent \( \mathcal{R} \)-2-modules.

Define a functor
\[ \widehat{\Omega}_F : \text{Im}_1^{\text{pl}} F \longrightarrow \text{Im}_2^{\text{pl}} F \]
\[ A \mapsto (A, 1_{FA}, FA), \]
\[ A_1 \xrightarrow{g} A_2 \mapsto FA_1 \xrightarrow{g} FA_2 \]
From the definition of \( \widehat{\Omega}_F \), we see that \( \widehat{\Omega}_F \) restricts on morphisms of \( \text{Im}_1^{\text{pl}} F \) to be identity, so \( \widehat{\Omega}_F \) is a functor.

Also, there are natural morphisms:
\[ (\widehat{\Omega}_F)_+ \triangleq F_+ : \widehat{\Omega}_F(A_1 + A_2) = (A_1 + A_2, 1_{F(A_1 + A_2)}, F(A_1 + A_2)) \rightarrow \widehat{\Omega}_F(A_1) + \widehat{\Omega}_F(A_2) = (A_1, 1_{FA_1}, FA_1) + (A_2, 1_{FA_2}, FA_2) = (A_1 + A_2, -, FA_1 + FA_2), \]
\[ (\widehat{\Omega}_F)_0 \triangleq F_0 : \widehat{\Omega}_F(0) = (0, 1_{F0}, F0) \rightarrow (0, F0, 0), \]
\[ (\widehat{\Omega}_F)_2 \triangleq F_2 : \widehat{\Omega}_F(r \cdot A) = (r \cdot A, 1_{F(r \cdot A)}, F(r \cdot A)) \rightarrow r \star \widehat{\Omega}_F(A) = r \star (A, 1_{FA}, FA) = (r \cdot A, -, r \cdot FA) \]
such that \( (\widehat{\Omega}_F, (\widehat{\Omega}_F)_+, (\widehat{\Omega}_F)_0, (\widehat{\Omega}_F)_2) \) is an \( \mathcal{R} \)-homomorphism, since \( F \) is.

Define a functor
\[ \widehat{\Omega}_F^{-1} : \text{Im}_1^{\text{pl}} F \longrightarrow \text{Im}_2^{\text{pl}} F \]
\[ (A, \varphi, B) \mapsto A, \]
\[ (A_1, \varphi_1, B_1) \xrightarrow{g} (A_2, \varphi_2, B_2) \mapsto \widehat{\Omega}_F^{-1}(g) \]
where \( \hat{\Omega}^{-1}_F(g) = \varphi_2^{-1} \circ g \circ \varphi_1 \) is the composition \( FA_1 \xrightarrow{\varphi_1} B_1 \xrightarrow{g} B_2 \xrightarrow{\varphi_2^{-1}} FA_2 \).

For any identity morphism \( 1_B : (A, \varphi, B) \to (A, \varphi, B) \) in \( Im^{2}_{pl}F \), we have \( \hat{\Omega}^{-1}_F(1_B) = \varphi^{-1} \circ 1_B \circ \varphi = 1_{FA} \). Given morphisms \( (A_1, \varphi_1, B_1) \xrightarrow{g_1} (A_2, \varphi_2, B_2) \xrightarrow{g_2} (A_3, \varphi_3, B_3) \), we have \( \hat{\Omega}^{-1}_F(g_2 \circ g_1) = \varphi_3^{-1} \circ (g_2 \circ g_1) \circ \varphi_1 \), \( \hat{\Omega}^{-1}_F(g_2) = \varphi_3^{-1} \circ g_2 \circ \varphi_2 \), \( \hat{\Omega}^{-1}_F(g_1) = \varphi_2^{-1} \circ g_1 \circ \varphi_1 \), so \( \hat{\Omega}^{-1}_F(g_2 \circ g_1) = \hat{\Omega}^{-1}_F(g_2) \circ \hat{\Omega}^{-1}_F(g_1) \). Then \( \hat{\Omega}^{-1}_F \) is a functor.

Also, there are natural morphisms:

\[
(\hat{\Omega}^{-1}_F)_{+} = id : \hat{\Omega}^{-1}_F((A_1, \varphi_1, B_1) + (A_2, \varphi_2, B_2)) = \hat{\Omega}^{-1}_F(A_1 + A_2, -, B_1 + B_2) = A_1 + A_2 \\
(\hat{\Omega}^{-1}_F)_{0} = id : \hat{\Omega}^{-1}_F(0, 1_{F0}, 0) = 0 \to 0, \\
(\hat{\Omega}^{-1}_F)_2 = id : \hat{\Omega}^{-1}_F(r \ast (A, \varphi, B)) = \hat{\Omega}^{-1}_F(r \cdot A, (r \cdot \varphi), r \cdot B) = r \cdot A \to r \cdot \hat{\Omega}^{-1}_F(A, \varphi, B) = r \cdot A.
\]

Obviously, \((\hat{\Omega}^{-1}_F, (\hat{\Omega}^{-1}_F)_{+}, (\hat{\Omega}^{-1}_F)_0, (\hat{\Omega}^{-1}_F)_2)\) is an \( \mathcal{R} \)-homomorphism.

Next, we will check \( \hat{\Omega}^{-1}_F \circ \hat{\Omega}_F = 1, \hat{\Omega}_F \circ \hat{\Omega}^{-1}_F \Rightarrow 1 \).

\[
\begin{align*}
(\hat{\Omega}^{-1}_F \circ \hat{\Omega}_F)(A) &= \hat{\Omega}^{-1}_F(A, 1_{FA}, FA) = A, \ \forall A \in \text{obj}(Im^{1}_{pl}F), \\
(\hat{\Omega}^{-1}_F \circ \hat{\Omega}_F)(g) &= (\hat{\Omega}^{-1}_F)(g) = g, \ \forall g \in \text{Mor}(Im^{1}_{pl}F).
\end{align*}
\]

There is a morphism of \( \mathcal{R} \)-homomorphisms

\[
\tau : \hat{\Omega}_F \circ \hat{\Omega}^{-1}_F \Rightarrow 1 : Im^{2}_{pl}F \longrightarrow Im^{2}_{pl}F,
\]

given by

\[
\tau_{(A, \varphi, B)} \triangleq \varphi : (\hat{\Omega}_F \circ \hat{\Omega}^{-1}_F)(A, \varphi, B) = \hat{\Omega}_F(A) = (A, 1_{FA}, FA) \to (A, \varphi, B).
\]

For a morphism \( g : (A_1, \varphi_1, B_1) \to (A_2, \varphi_2, B_2) \), \( (\hat{\Omega}_F \circ \hat{\Omega}^{-1}_F)(g) = \hat{\Omega}_F(\varphi_2^{-1} \circ g \circ \varphi_1) = \varphi_2^{-1} \circ g \circ \varphi_1 \), i.e. \( \tau \) is a natural transformation.

\[
\begin{align*}
(\hat{\Omega}_F \circ \hat{\Omega}^{-1}_F)((A_1, \varphi_1, B_1) + (A_2, \varphi_2, B_2)) &= (A_1 + A_2, 1_{FA_1 + FA_2}, F(A_1 + A_2), \\
(\hat{\Omega}_F \circ \hat{\Omega}^{-1}_F)(A_1, \varphi_1, B_1) + (\hat{\Omega}_F \circ \hat{\Omega}^{-1}_F)(A_2, \varphi_2, B_2) &= (A_1 + A_2, -, FA_1 + FA_2), \\
\tau_{(A_1, \varphi_1, B_1) + (A_2, \varphi_2, B_2)} &= (\varphi_1 + \varphi_2) \circ F_, \\
\tau_{(A_i, \varphi_i, B_i)} &= \varphi_i, \ \text{for } i = 1, 2.
\end{align*}
\]

Thus Fig.7 commutes. We can also get commutative diagram Fig.37. in the similar ways.

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Since \( \varphi : FA \to B \) is a morphism in \( \mathcal{B} \), and \( \mathcal{B} \) is a groupoid, so \( \tau_{(A, \varphi, B)} \) is an isomorphism.

**Step 4.** There is a surjective \( \mathcal{R} \)-homomorphism.

Define a functor

\[
\hat{E}_F : \mathcal{A} \to \text{Im}_{pl}^1 F
\]

\[
A \mapsto A,
\]

\[
f : A_1 \to A_2 \mapsto Ff : FA_1 \to FA_2
\]

For any \( 1_A : A \to A \) in \( \mathcal{A} \),

\[
\hat{E}_F(1_A) = F(1_A) = 1_{FA}.
\]

Given morphisms \( A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \) in \( \mathcal{A} \),

\[
\hat{E}_F(f_2 \circ f_1) = F(f_2 \circ f_1) = Ff_2 \circ Ff_1 = \hat{E}_F(f_2) \circ \hat{E}_F(f_1).
\]

There are natural isomorphisms:

\[
(\hat{E}_F)_+ = id : (\hat{E}_F)(A_1 + A_2) = A_1 + A_2 = \hat{E}_F(A_1) + \hat{E}_F(A_2),
\]

\[
(\hat{E}_F)_0 = id : \hat{E}_F(0) = 0,
\]

\[
(\hat{E}_F)_2 = id : \hat{E}_F(r \cdot A) = r \cdot A = r \cdot \hat{E}_F(A).
\]

Obviously, \( (\hat{E}_F, (\hat{E}_F)_+, (\hat{E}_F)_0, (\hat{E}_F)_2) \) is an \( \mathcal{R} \)-homomorphism.

For every \( C \in \text{obj}(\mathcal{R}-2\text{-Mod}) \), there is a functor

\[
\Phi : \text{Hom}(\text{Im}_{pl}^1 F, C) \to \text{Hom}(\mathcal{A}, C)
\]

\[
G \mapsto G \circ \hat{E}_F,
\]

\[
\alpha : G_1 \Rightarrow G_2 \mapsto \alpha \ast \hat{E}_F \triangleq \alpha \ast 1_{\hat{E}_F}
\]

where \( \alpha \ast 1_{\hat{E}_F} \) is the horizontal composition of 2-morphisms. It is easy to check \( \Phi \) is an \( \mathcal{R} \)-homomorphism.

If \( \alpha, \beta : G_1 \Rightarrow G_2 : \text{Im}_{pl}^1 F \to C \), such that \( \alpha \ast \hat{E}_F = \beta \ast \hat{E}_F \), i.e. \( \forall A \in \text{obj}(\mathcal{A}) \),

\[
(\alpha \ast \hat{E}_F)(A) = G_2((1_{\hat{E}_F})_A) \circ \alpha_{\hat{E}_F(A)} = 1_{G_2A} \circ \alpha_A = \alpha_A.
\]

\[
(\beta \ast \hat{E}_F)(A) = G_2((1_{\hat{E}_F})_A) \circ \beta_{\hat{E}_F(A)} = 1_{G_2A} \circ \beta_A = \beta_A.
\]
Then \( \alpha = \beta \). So we proved \( \Phi \) is a faithful functor.

From above results, we get a surjective \( \mathcal{R} \)-homomorphism \( \hat{E}_F : A \to \text{Im}_{pl}^1 F \).

**Step 5.** There is a fully faithful \( \mathcal{R} \)-homomorphism.

Define a functor

\[
\hat{M}_F : \text{Im}_{pl}^2 F \to \mathcal{B}
\]

\[
(A, \varphi, B) \mapsto B,
\]

\[
g : (A_1, \varphi_1, B_1) \to (A_2, \varphi_2, B_2) \mapsto g : B_1 \to B_2.
\]

For any identity morphism \( 1_{(A, \varphi, B)} = 1_B : (A, \varphi, B) \to (A, \varphi, B) \),

\[
\hat{M}_F(1_{(A, \varphi, B)}) = \hat{M}_F(1_B) = 1_B = 1_{(A, \varphi, B)}.
\]

For morphisms \( (A_1, \varphi_1, B_1) \xrightarrow{g_1} (A_2, \varphi_2, B_2) \xrightarrow{g_2} (A_3, \varphi_3, B_3) \),

\[
\hat{M}_F(g_2 \circ g_1) = g_2 \circ g_1 = \hat{M}_F(g_2) \circ \hat{M}_F(g_1).
\]

There are natural morphisms:

\[
(\hat{M}_F)_+ = \text{id} : \hat{M}_F((A_1, \varphi_1, B_1) + (A_2, \varphi_2, B_2)) = \hat{M}_F(A_1 + A_2, -, B_1 + B_2) = B_1 + B_2
\]

\[
\to \hat{M}_F(A_1, \varphi_1, B_1) + \hat{M}_F(A_2, \varphi_2, B_2) = B_1 + B_2,
\]

\[
(\hat{M}_F)_0 = \text{id} : \hat{M}_F(0, F_0, 0) = 0,
\]

\[
(\hat{M}_F)_1 = \text{id} : \hat{M}_F(\Omega \otimes (A, \varphi, B)) = \hat{M}_F(\Omega \otimes A, -, \Omega \otimes B) = \Omega \otimes B = \Omega \cdot \hat{M}_F(A, \varphi, B).
\]

\[\hat{M}_F, (\hat{M}_F)_+, (\hat{M}_F)_0, (\hat{M}_F)_2\) is an \( \mathcal{R} \)-homomorphism.

For any pairs of objects \( (A_1, \varphi_1, B_1), (A_2, \varphi_2, B_2) \) in \( \text{Im}_{pl}^2 F \) with \( g : B_1 \to B_2 \) in \( \mathcal{B} \), there exists \( g : (A_1, \varphi_1, B_1) \to (A_2, \varphi_2, B_2) \) in \( \text{Im}_{pl}^2 F \), such that \( \hat{M}_F(g) = g \). Thus \( \hat{M}_F \) is full.

For given morphisms \( g_1, g_2 : (A_1, \varphi_1, B_1) \to (A_2, \varphi_2, B_2) \) in \( \text{Im}_{pl}^2 F \) such that \( \hat{M}_F(g_1) = \hat{M}_F(g_2) : B_1 \to B_2 \), since \( \hat{M}_F(g_i) = g_i, \ i = 1, 2 \). So \( g_1 = g_2 \). Thus \( \hat{M}_F \) is faithful.

**Step 6.** \( F = \hat{M}_F \circ \hat{\Omega}_F \circ \hat{E}_F \).

For any \( A \in \text{obj}(\mathcal{A}) \),

\[
(\hat{M}_F \circ \hat{\Omega}_F \circ \hat{E}_F)(A) = (\hat{M}_F \circ \hat{\Omega}_F)(A) = \hat{M}_F(A, 1_{FA}, FA) = FA.
\]
For any morphism $f : A_1 \to A_2$ in $\mathcal{A}$,

$$(\widehat{M}_F \circ \widehat{\Omega}_F \circ \widehat{E}_F)(f) = (\widehat{M}_F \circ \widehat{\Omega}_F)(Ff) = \widehat{M}_F(Ff) = Ff.$$ 

\[ \square \]

**Proposition 7.** For each $\mathcal{R}$-homomorphism $F : \mathcal{A} \to \mathcal{B}$ in $(\mathcal{R}, 2$-Mod), $\widehat{E}_F : \mathcal{A} \to Im_{\mathcal{R}}F$ is the cokernel of the kernel of $F$.

**Proof.** We know that the kernel of $F$ is $(\text{Ker} F, e_F, \varepsilon_F)$, where $\text{Ker} F$ is a category with objects are pairs $(A, a)$, where $A \in \text{obj}(\mathcal{A})$, $a : FA \to 0$, $e_F(A, a) = A$, $(\varepsilon_F)_a = a$.

- Let us describe the $\text{Coker} e_F$ in the following ways:
  - Objects are the objects of $\mathcal{A}$.
  - Morphism from $A_1$ to $A_2$ is the equivalence class of the triple $(N, a, f)$, denote by $[N, a, f]$, where $N \in \text{obj}(\mathcal{A})$, $a : FN \to 0$ is a morphism in $\mathcal{B}$, $f : A_1 \to N + A_2$ in $\mathcal{A}$, and for two morphisms $(N_1, a_1, f_1), (N_2, a_2, f_2) : A_1 \to A_2$ are equal if there exists a morphism $n : N_1 \to N_2$ in $\mathcal{A}$, such that the following diagrams commute:

  ![Diagram 42](image)

  ![Diagram 43](image)

  - Composition of morphisms. Let $A_1 \xrightarrow{[N_1, a_1, f_1]} A_2 \xrightarrow{[N_2, a_2, f_2]} A_3$ be morphisms in $\text{Coker} e_F$, $[N, a, f] \triangleq [N_2, a_2, f_2] \circ [N_1, a_1, f_1]$, where $N \triangleq N_1 + N_2$, $a$ is the composition $F(N_1 + N_2) \xrightarrow{Fk} FN_1 + FN_2 \xrightarrow{a_1 + a_2} 0 + 0 = 0$, $f$ is the composition $A_1 \xrightarrow{f_1} N_1 + A_2 \xrightarrow{1_{N_1} + f_2} N_1 + (N_2 + A_3) \xrightarrow{\langle N_1, N_2, A_3 >^{-1}} (N_1 + N_2) + A_3 = N + A_3$.

  This composition is well-defined, since $(N_1, a_1, f_1), (N_1', a_1', f_1')$ are equal, i.e. $\exists n_1 : N_1 \to N_1'$, such that $(n_1 + 1_{A_2}) \circ f_1 = f_1'$, $a_1' \circ Fn_1 = a_1$. There exists $n \triangleq n_1 + 1_{N_2} : N_1 + N_2 \to N_1' + N_2$, such that Fig.42.-43. commute, then $(N_2, a_2, f_2) \circ (N_1, a_1, f_1)$ is equal to $(N_2, a_2, f_2) \circ (N_1', a_1', f_1')$.

- There is a morphism of $\mathcal{R}$-homomorphisms

  $$\delta : \widehat{E}_F \circ e_F \Rightarrow 0$$
given by
\[
\delta_{(A,a)} \triangleq a : (\hat{E}_F \circ e_F)(A, a) = (\hat{E}_F)(A) = A \to 0(A, a) = 0,
\]
for any \((A, a) \in \text{obj}(\text{Ker}F)\), which is a morphism in \(\text{Im}_{\text{pl}}F\).

For any morphism \(f : (A_1, a_1) \to (A_2, a_2)\), we have
\[
(\hat{E}_F \circ e_F)(f) = f : (\hat{E}_F \circ e_F)(A_1, a_1) = A_1 \to (\hat{E}_F \circ e_F)(A_2, a_2) = A_2
\]
in \(\text{Im}_{\text{pl}}F\) is the morphism \(Ff : FA_1 \to FA_2\) in \(\mathcal{B}\), \(\delta_{(A_i, a_i)} = a_i, \ i = 1, 2\).

From \(a_2 \circ Ff = a_1\), \(\delta\) is a natural transformation.

Using the properties of \(F\), it is easy to check \(\delta\) is a morphism of \(\mathcal{R}\)-homomorphisms.

By the universal property of the cokernel, there is an \(\mathcal{R}\)-homomorphism
\[
\Phi : \text{Cokere}_F \longrightarrow \text{Im}_{\text{pl}}F
\]
\[
A \mapsto A,
\]
\[
[N, a, f] : A_1 \to A_2 \mapsto \Phi(N, a, f) : A_1 \to A_2
\]
where \((N, a, f)\) is the representation element of equivalence class of \([N, a, f]\), \(\Phi(N, a, f)\) is the following composition of morphisms in \(\mathcal{B}\),
\[
FA_1 \xrightarrow{Ff} F(N + A_2) \xrightarrow{F} FN + FA_2 \xrightarrow{a + 1_{FA_2}} 0 + FA_2 \xrightarrow{I_{FA_2}} FA_2.
\]

\(\Phi\) is well-defined, if \((N_1, a_1, f_1), (N_2, a_2, f_2)\) are equal, i.e. \(\exists n : N_1 \to N_2\), such that
\[
(n + 1_{A_2}) \circ f_1 = f_2, \ a_2 \circ Fn = a_1.
\]

We get \(\Phi(N_1, a_1, f_1) = \Phi(N_2, a_2, f_2)\) from the following commutative diagrams

For any identity morphism \([0, F_0, l^{-1}_a] : A \to A\) in \(\text{Cokere}_F\), we have
\[
\Phi(0, F_0, l^{-1}_a) = l_{FA} \circ (F + 1_{FA}) \circ F_+ \circ F(l^{-1}_a) = 1_{FA}.
\]
For given morphisms $A_1 \xrightarrow{[N_1, a_1, f_1]} A_2 \xrightarrow{[N_2, a_2, f_2]} A_3$ in $\text{Coker}_F$, from the basic properties of morphisms in $\text{Coker}_F$ and $F$, we have
\[
\Phi((N_2, a_2, f_2) \circ (N_1, a_1, f_1)) = \Phi(N_2, a_2, f_2) \circ \Phi(N_1, a_1, f_1).
\]
Moreover, there are natural morphisms:
\[
\Phi_+ = id : \Phi(A_1 + A_2) = A_1 + A_2 = \Phi A_1 + \Phi A_2,
\]
\[
\Phi_0 = id : \Phi(0) = 0,
\]
\[
\Phi_2 = id : \Phi(r \cdot A) = r \cdot A = r \cdot \Phi A.
\]
\Phi is an $R$-homomorphism.

- $\Phi$ is an equivalent $R$-homomorphism.

For every $C \in \text{obj}(\mathcal{R}2\text{-Mod})$, there is a functor
\[
\Psi : \text{Hom}(\text{Im}_{pl}^1 F, C) \longrightarrow \text{Hom}(\text{Coker}_F, C)
\]
\[
G \mapsto G \circ \Phi,
\]
\[
\alpha : G_1 \Rightarrow G_2 \mapsto \alpha \ast \Phi
\]
where $\alpha \ast 1_\Phi$ is the horizontal composition of 2-morphisms.

If $\alpha, \beta : G_1 \Rightarrow G_2 : \text{Im}_{pl}^1 F \rightarrow C$, such that $\alpha \ast \Phi = \beta \ast \Phi$, i.e. $\forall A \in \text{obj}(\text{Coker}_F)$,
\[
(\alpha \ast \Phi)(A) = G_2((1_\Phi)A) \circ \alpha_{\Phi(A)} = 1_{G_2 A} \circ \alpha_A = \alpha_A,
\]
\[
(\beta \ast \Phi)(A) = G_2((1_\Phi)A) \circ \beta_{\Phi(A)} = 1_{G_2 A} \circ \beta_A = \beta_A.
\]
Then $\alpha = \beta$. So $\Psi$ is a faithful functor, and $\Phi$ is surjective.

For any two objects $A_1$, $A_2$ in $\text{Coker}_F$, and morphism $g : FA_1 \rightarrow FA_2$ in $\text{Im}_{pl}^1 F$. Set $N = A_1 + A_2^*$, where $A_2^*$ is an inverse of $A_2$, together with natural isomorphism $\eta_{A_2} : A_2^* + A_2 \rightarrow 0$, denote by $\eta'_{A_2} = \eta_{A_2} \circ c_{A_2, A_2^*} : A_2 + A_2^* \rightarrow 0$, $a$ is the composition
\[
A_1 \xrightarrow{r_{A_1}} A_1 + 0 \xrightarrow{1_{A_1} + \eta_{A_2}^{-1}} A_1 + (A_2^* + A_2) \xrightarrow{<A_1, A_2^*, A_2>^{-1}} (A_1 + A_2^*) + A_2 = N + A_2.
\]

After calculations, we have $\Phi(N, b, f) = g$, then $\Phi$ is full.

For two morphisms $[N_1, a_1, f_1], [N_2, a_2, f_2] : A_1 \to A_2$ in $\text{Cokere}_F$, such that $\Phi(N_1, a_1, f_1) = \Phi(N_2, a_2, f_2)$.

Let $n$ be the composition

$$N_1 \xrightarrow{r_{N_1}} N_1 + 0 \xrightarrow{1 + \eta_{A_2}} N_1 + (A_2 + A_2^*) \xrightarrow{\langle N_1, A_2, A_2^* \rangle^{-1}} (N_1 + A_2) + A_2^* \xrightarrow{f_1^{-1} + 1 + A_2^*} A_1 + A_2^* \xrightarrow{f_2 + 1 + A_2^*} (N_2 + A_2) + A_2^* \xrightarrow{\langle N_2, A_2, A_2^* \rangle} N_2 + (A_2 + A_2^*) \xrightarrow{1 \eta_{N_2} + \eta_{A_2}'} N_2 + 0 \xrightarrow{l_{N_2}} N_2,$$

then $[N_1, a_1, f_1], [N_2, a_2, f_2] : A_1 \to A_2$ are equal in $\text{Cokere}_F$, thus $\Phi$ is faithful.

$\Phi : \text{Cokere}_F \to \text{Im}_1^{\text{pl}} F$ is an equivalent $\mathcal{R}$-homomorphism.

\[ \square \]

**Proposition 8.** For each $\mathcal{R}$-homomorphism $F : A \to B$ in $(\mathcal{R}-2\text{-Mod})$, $\hat{M}_F : \text{Im}_1^{\text{pl}} F \to B$ is the root of the copip of $F$.

**Proof.** The copip of $F$, is the $\mathcal{R}$-2-module $\text{Copip}_F$, together with 2-morphism $\sigma : 0 \Rightarrow 0 : \mathcal{B} \to \text{Copip}_F$. As a category, $\text{Copip}_F$ has the unique object $*$ and morphisms are objects of $\mathcal{B}$, and $\sigma(B) = B : * \to *$.

**Step 1.** Let us describe the root of $\sigma : 0 \Rightarrow 0 : \mathcal{B} \to \text{Copip}_F$.

- The category $\text{Root}_\sigma$ consists of the following data:
  - Objects are $B \in \text{obj}(\mathcal{B})$, such that $\sigma_B = B = 0$ in $\text{Copip}_F$, i.e. there exist $A \in \text{obj}(\mathcal{A})$ and $g : B \to FA + 0$.
  - Morphisms are morphisms in $\mathcal{B}$.
  - Composition of morphisms and the unit object are just the composition of morphisms in $\mathcal{B}$, and the unit object $0$ in $\mathcal{B}$, respectively.

  From the $\mathcal{R}$-2-module structure of $\mathcal{B}$, we can give $\text{Root}_\sigma$ a $\mathcal{R}$-2-module structure.

- The $\mathcal{R}$-2-module $\text{Root}_\sigma$ is the root of $\sigma$.
There is an $\mathcal{R}$-homomorphism

$$R : \text{Root}\sigma \longrightarrow \mathcal{B}$$

$$B \mapsto B,$$

$$g : B_1 \rightarrow B_2 \mapsto g : B_1 \rightarrow B_2.$$

Since $\text{Root}\sigma$ and $\mathcal{B}$ have the same $\mathcal{R}$-2-module structure, then $R$ is an $\mathcal{R}$-homomorphism.

Also, for any $B \in \text{obj}(\text{Root}\sigma)$, $(\sigma \ast R)(B) = \sigma(B) = B = 0 = 1_0(B)$, i.e. $\sigma \ast R = 1_0$.

For $\mathcal{D} \in \text{obj}(\mathcal{R}\text{-2-Mod})$ and an $\mathcal{R}$-homomorphism $G : \mathcal{D} \rightarrow \mathcal{B}$, such that $\sigma \ast G = 1_0$, there exist an $\mathcal{R}$-homomorphism

$$G' : \mathcal{D} \longrightarrow \text{Root}\sigma$$

$$D \mapsto G(D),$$

$$d : D_1 \rightarrow D_2 \mapsto d : D_1 \rightarrow D_2$$

and a 2-morphism $\alpha : G \Rightarrow R \circ G'$ given by $\alpha_D \triangleq 1_{G(D)} : G(D) \rightarrow (R \circ G'(D)) = R(G(D)) = G(D)$.

For any $\mathcal{C} \in \text{obj}(\mathcal{R}\text{-2-Mod})$, there is a functor

$$\Psi : \text{Hom}(\mathcal{C}, \text{Root}\sigma) \longrightarrow \text{Hom}(\mathcal{C}, \mathcal{B})$$

$$H \mapsto R \circ H,$$

$$\tau : H_1 \Rightarrow H_2 \mapsto R \ast \tau$$

where $R \ast \tau = 1_R \ast \tau$ is the horizontal composition of 2-morphisms. For all $H_1, H_2 : \mathcal{C} \rightarrow \text{Root}\sigma$, and $\beta : R \circ H_1 \Rightarrow R \circ H_2$, there exists a unique $\chi : H_1 \Rightarrow H_2$ given by $\chi_C \triangleq \beta_C : H_1 C \rightarrow H_2 C$, so $R$ is fully faithful.

**Step 2.** $\text{Root}\sigma$ and $\text{Im}_{pl}^2 F$ are equivalent in $(\mathcal{R}\text{-2-Mod})$.

Recall

$$\widehat{M}_F : \text{Im}_{pl}^2 F \longrightarrow \mathcal{B}$$

$$(A, \varphi, B) \mapsto B,$$

$$g : (A_1, \varphi_1, B_1) \rightarrow (A_2, \varphi_2, B_2) \mapsto g : B_1 \rightarrow B_2.$$
We have \((\sigma \ast \hat{M}_F)_{(A,\varphi,B)} = \sigma_B = B, \forall (A,\varphi,B) \in \text{obj}(\text{Im}_2^2 F)\), so \(\sigma \ast \hat{M}_F = 1_0\), and under the definition of root, there is an \(\mathcal{R}\)-homomorphism

\[
\Phi : \text{Im}_2^2 F \longrightarrow \text{Root}_\sigma
\]

\[(A,\varphi,B) \mapsto B,\]

\[g : (A_1,\varphi_1,B_1) \mapsto (A_2,\varphi_2,B_2) \mapsto g : B_1 \rightarrow B_2.\]

\(\Phi\) is well-defined, i.e. \(\Phi(A,\varphi,B) = B\) is an object of \(\text{Root}_\sigma\), since there exist \(A \in \text{obj}(A)\), and \(\bar{\varphi} : B \rightarrow FA + 0\) under the composition

\[
B \xrightarrow{\varphi^{-1}} FA \xrightarrow{r_{FA}^{-1}} FA + 0,
\]

i.e. \(\bar{\varphi} = r_{FA}^{-1} \circ \varphi^{-1} = (\varphi \circ r_{FA})^{-1}\).

There is an \(\mathcal{R}\)-homomorphism

\[
\Phi^{-1} : \text{Root}_\sigma \longrightarrow \text{Im}_2^2 F
\]

\[B \mapsto (A,\varphi,B),\]

\[g : B_1 \rightarrow B_2 \mapsto g : B_1 \rightarrow B_2\]

where \((A,\varphi,B) \in \text{obj}(\text{Im}_2^2 F)\), given by the following way:

For \(B \in \text{obj}(\text{Root}_\sigma)\), there exist \(A' \in \text{obj}(A)\) and \(g' : B \rightarrow FA' + 0\), so \(A\) is the just \(A'\) of \(A\), and \(\varphi\) is the composition \(FA \xrightarrow{r_{FA}^{-1}} FA + 0 \xrightarrow{(g')^{-1}} B\). Since, \(\Phi^{-1}\) restricting on the morphisms is identity, then \(\Phi\) is an \(\mathcal{R}\)-homomorphism.

For any \(B \in \text{obj}(\text{Root}_\sigma), g \in \text{Mor}(\text{Root}_\sigma)\), we have

\[
(\Phi \circ \Phi^{-1})(B) = \Phi(A,\varphi,B) = B,
\]

\[
(\Phi \circ \Phi^{-1})(g) = g.
\]

Also, for any \((A,\varphi,B) \in \text{obj}(\text{Im}_2^2 F)\), and \(g \in \text{Mor}(\text{Im}_2^2 F)\), we have

\[
(\Phi^{-1} \circ \Phi)(A,\varphi,B) = \Phi^{-1}(B) = B,
\]

\[
(\Phi^{-1} \circ \Phi)(g) = g.
\]

Then \(\Phi \circ \Phi^{-1} = 1_{\text{Root}_\sigma}, \Phi^{-1} \circ \Phi = 1_{\text{Im}_2^2 F}\). \(\square\)

**Proposition 9.** Every 1-morphism \(F : A \rightarrow B\) in \((\mathcal{R} \text{-} 2\text{-Mod})\) factors as the following composite, where \(E_F\) is full and surjective, \(\Omega_F\) is an equivalence, and \(M_F\) is faithful.

\[
A \xrightarrow{E_F} \text{Im}^1 F \xrightarrow{\Omega_F} \text{Im}^2 F \xrightarrow{M_F} B.
\]
Proof. Step 1. The \(R\)-2-module \(Im^1 F\) is described in the following way:

- Category \(Im^1 F\) consists of:
  - Objects are those of \(\mathcal{A}\).
  - Morphisms are the equivalent classes of morphisms in \(\mathcal{A}\), for two morphisms \(f_1, f_2 : A_1 \to A_2\) are equal in \(Im^1 F\), if \(F f_1 = F f_2\) in \(\mathcal{B}\), denote by \([f_1]\). The composition and identities are those of \(\mathcal{A}\) up to equivalence.

- \(Im^1 F\) is a symmetric 2-group.

The unit object is just the unit object \(0\) of \(\mathcal{A}\).

There is a bifunctor

\[
+ : Im^1 F \times Im^1 F \longrightarrow Im^1 F
\]

\[
(A_1, A_2) \mapsto A_1 + A_2,
\]

\[
(A_1 \xrightarrow{[f_1]} A'_1, A_2 \xrightarrow{[f_2]} A'_2) \mapsto A_1 + A_2 \xrightarrow{[f_1 + f_2]} A'_1 + A'_2
\]

where \(A_1 + A_2\), \([f_1 + f_2]\) are given under the monoidal structure of \(\mathcal{A}\). If \(f_1, f'_1 : A_1 \to A_2\) are equal in \(Im^1 F\), i.e. \(F f_1 = F f'_1\). From the following commutative diagram

\[
\begin{array}{ccc}
F(A_1 + A_2) & \xrightarrow{F(f_1 + f_2)} & F(A'_1 + A'_2) \\
\downarrow F & & \downarrow F \\
FA_1 + FA_2 & \xrightarrow{f_1 + f_2} & FA'_1 + FA'_2
\end{array}
\]

we have \(F([f_1 + f_2]) = F([f'_1 + f_2])\), then \(f_1 + f_2\), \(f'_1 + f_2\) are equal in \(Im^1 F\).

Moreover, there are natural isomorphisms:

\[
<A_1, A_2, A_3> : A_1 + A_2 + A_3 \to A_1 + A_2 + A_3,
\]

\[
l_A : 0 + A \to A,
\]

\[
r_A : A + 0 \to A,
\]

\[
c_{A_1, A_2} : A_1 + A_2 \to A_2 + A_1,
\]

given by the natural isomorphisms in \(\mathcal{A}\). Since \(\mathcal{A}\) is a symmetric monoidal category, so is \(Im^1 F\).
Since any morphism \([f] : A \rightarrow A'\) in \(Im'F\) is in fact a morphism in \(\mathcal{A}\) up to equivalence, and \(\mathcal{A}\) is a groupoid, there exists \(f^* : A' \rightarrow A\) in \(\mathcal{A}\), such that 
\([f^*] \circ [f] = [(f^* \circ f)] = 1\) in \(Im'F\).

For any object \(A \in \text{obj}(Im^1F)\), \(A \in \mathcal{A}\), there exist \(A^* \in \text{obj}(A)\), and 
\(\eta_A : A^* + A \rightarrow 0\), so there are \(A^* \in \text{obj}(Im^1F)\), and \(\eta_A : A^* + A \rightarrow 0\).

- \(Im^1F\) is an \(\mathcal{R}\)-2-module.

The \(\mathcal{R}\)-2-module structure is induced from the \(\mathcal{R}\)-2-module structure of \(\mathcal{A}\).

**Step 2.** The \(\mathcal{R}\)-2-module \(Im^2F\) is given by following data:

- The category \(Im^2F\) consists of:
  - Objects are the triple \((A, \varphi, B)\), where \(A \in \text{obj}(\mathcal{A})\), \(B \in \text{obj}(\mathcal{B})\), \(\varphi : FA \rightarrow B\) in \(\mathcal{B}\).
  - Morphism from \((A_1, \varphi_1, B_1)\) to \((A_2, \varphi_2, B_2)\) is the equivalent class of a pair \((f, g)\), denote by \([f, g]\), where \(f : A_1 \rightarrow A_2, g : B_1 \rightarrow B_2\), such that \(g \circ \varphi_1 = \varphi_2 \circ Ff\), and for two morphisms \((f, g), (f', g') : (A_1, \varphi_1, B_1) \rightarrow (A_2, \varphi_2, B_2)\) are equal if \(g = g'\), or \(Ff = Ff'\).
  - Composition of morphisms \((A_1, \varphi_1, B_1) \xrightarrow{[f_1, g_1]} (A_2, \varphi_2, B_2) \xrightarrow{[f_2, g_2]} (A_3, \varphi_3, B_3)\) is given by \([f, g] = [f_2, g_2] \circ [f_1, g_1]\), where \(f = f_2 \circ f_1, g = g_2 \circ g_1\), such that 

\[
\varphi_3 \circ F(f_2 \circ f_1) = \varphi_3 \circ F(f_2) \circ F(f_1) = g_2 \circ \varphi_2 \circ F(f_1) = g_2 \circ g_1 \circ \varphi_1 = (g_2 \circ g_1) \circ \varphi_1.
\]

Also the above composition is well-defined, if \((f_1, g_1), (f'_1, g'_1)\) are equal, then 
\((f_2, g_2) \circ (f_1, g_1), (f_2, g_2) \circ (f'_1, g'_1)\) are equal, because of \(g_2 \circ g_1 = g_2 \circ g'_1\).

- \(Im^2F\) is a symmetric 2-group.

The unit object is \((0, F_0, 0)\), where the first 0 is the unit object of \(\mathcal{A}\), the last 0 is the unit object of \(\mathcal{B}\) and \(F_0 : F0 \rightarrow 0\).

There is a bifunctor
\[ + : Im^2F \times Im^2F \rightarrow Im^2F \]
\[
((A_1, \varphi_1, B_1), (A_2, \varphi_2, B_2)) \mapsto (A_1, \varphi_1, B_1) + (A_2, \varphi_2, B_2) = (A, \varphi, B),
\]
\[
(A_1, \varphi_1, B_1) \xrightarrow{[f, g]} (A_2, \varphi_2, B_2), (A'_1, \varphi'_1, B'_1) \xrightarrow{[f', g']} (A'_2, \varphi'_2, B'_2) \mapsto [f + f', g + g']
\]
where \(\varphi\) is the composition \(F(A_1 + A_2) \xrightarrow{F_+} FA_1 + FA_2 \xrightarrow{\varphi_1 + \varphi_2} B_1 + B_2\), 
\([f + f', g + g']\) makes the following diagram commutes:
Similar as the above steps, the definition of addition of morphisms is well-defined.

Moreover, there are natural isomorphisms:

\[ \langle (A_1, \varphi_1, B_1), (A_2, \varphi_2, B_2), (A_3, \varphi_3, B_3) \rangle \triangleq \langle A_1, A_2, A_3 \rangle, < B_1, B_2, B_3 > \],

\[ l_{(A,\varphi,B)} = (l_A, l_B), \]

\[ r_{(A,\varphi,B)} = (r_A, r_B), \]

\[ c_{(A_1,\varphi_1,B_1),(A_2,\varphi_2,B_2)} = (c_{A_1,A_2}, c_{B_1,B_2}). \]

Using the usual methods, we can check the above natural isomorphisms are well-defined, and satisfy the conditions of symmetric 2-group.

- \( \text{Im}^2F \) is an \( \mathcal{R} \)-2-module.

There is a bifunctor

\[ \cdot \colon \mathcal{R} \times \text{Im}^2F \to \text{Im}^2F \]

\[ (r, (A, \varphi, B)) \mapsto r \cdot (A, \varphi, B) \triangleq (r \cdot A, r \cdot \varphi, r \cdot B), \]

\[ (r_1 \alpha \mapsto r_2, (A_1, \varphi_1, B_1) ) \xrightarrow{[f,g]} (A_2, \varphi_2, B_2) \mapsto r \cdot (A_1, \varphi_1, B_1) \xrightarrow{r \cdot [f,g]} r \cdot (A_2, \varphi_2, B_2) \]

where \( r \cdot \varphi : F(r \cdot A) \xrightarrow{F_r} r \cdot FA \xrightarrow{r \cdot \varphi} r \cdot B \), from \( r \cdot \) is a functor and \( F \) is an \( \mathcal{R} \)-homomorphism, we have the following commutative diagram:
Also, if \((f, g), (f', g')\) are equal, then \(\varphi \cdot (f, g), \varphi \cdot (f', g')\) are equal too.

Moreover, there are natural isomorphisms:

\[
\begin{align*}
\alpha_r^{r(A_1, \varphi_1, B_1), (A_2, \varphi_2, B_2)} & \triangleq (\alpha_r^{r(A_1, A_2), A_r(B_1, B_2)}), \\
\beta_{r_1, r_2}^{r(A, \varphi, B)} & \triangleq (\beta_{r_1, r_2}^{r(A, B)}), \\
b_{r_1, r_2}^{r(A, \varphi, B)} & \triangleq (b_{r_1, r_2}^{A, r}, b_{r_1, r_2}^{B, r}), \\
i_{r(A, \varphi, B)} & \triangleq (i_A, i_B), \\
z_r & \triangleq (z_r, z_r).
\end{align*}
\]

By the usual methods, we can check the above natural isomorphisms are well-defined, and Fig.18.-31. commute.

**Step 3.** \(Im^1 F, \, Im^1 F\) are equivalent as \(\mathcal{R}\text{-2-modules.}\)

Define a functor

\[
\Omega_F : Im^1 F \longrightarrow Im^2 F
\]

\[
A \mapsto (A, 1_{FA}, FA), \quad [f] : A_1 \rightarrow A_2 \mapsto [f, Ff] : (A_1, 1_{FA_1}, FA_1) \rightarrow (A_2, 1_{FA_2}, FA_2)
\]

the above definition is well-defined. In fact, if \(f, f' : A_1 \rightarrow A_2\) are equal in \(Im^1 F\), i.e. \(Ff = Ff'\), then we have \((f, Ff), (f', Ff')\) are equal in \(Im^2 F\), i.e. \(\Omega_F(f), \, \Omega_F(f')\) are equal.

For any identity morphism \(1_A : A \rightarrow A\) in \(Im^1 F\), \(\Omega_F(1_A) = (1_A, F(1_A) = (1_A, 1_{FA}) : (A, 1_{FA}, FA) \rightarrow (A, 1_{FA}, FA)\) is the identity morphism in \(Im^2 F\).

For any morphisms \(A_1 \xrightarrow{[f_1]} A_2 \xrightarrow{[f_2]} A_3\) in \(Im^1 F\), we have \(\Omega_F([f_2 \circ f_1]) = \Omega([f_2 \circ f_1]) = [f_2 \circ f_1, F(f_2 \circ f_1)] = [f_2 \circ f_1, Ff_2 \circ Ff_1] = [f_2, Ff_2] \circ [f_1, Ff_1] = \Omega_F([f_2]) \circ \Omega_F([f_1]).\) Then \(\Omega_F\) is a functor.

There are natural isomorphisms:

\[
\begin{align*}
(\Omega_F)_+ & \triangleq (1_{A_1 + A_2}, F_+) : \Omega_F(A_1 + A_2) = (A + A, 1_{F(A_1 + A_2)}, F(A_1 + A_2)) \rightarrow \Omega_F(A_1) + \Omega_F(A_2) \\
& = (A_1, 1_{FA_1}, FA_1) + (A_2, 1_{FA_2}, FA_2) = (A + A, 1_{FA_1} + 1_{FA_2}) \circ F_+, FA_1 + FA_2), \\
(\Omega_F)_0 & \triangleq (1_0, 0) : \Omega_F(0) = (0, 1_{F0}, F0) \rightarrow (0, F0, 0), \\
(\Omega_F)_2 & \triangleq (1_{r \cdot A}, F_2) : \Omega_F(r \cdot A) = (r \cdot A, 1_{F(r \cdot A)}, F(r \cdot A)) \rightarrow r \cdot \Omega_F(A) = r \cdot (A, 1_{FA}, FA) \\
& = (r \cdot A, r \cdot 1_{FA} \circ F_2, r \cdot FA) = (r \cdot A, F_2, r \cdot FA).
\end{align*}
\]
After basic calculations, the above natural isomorphisms are well-defined, and 
\(((\Omega F)_+, (\Omega F)_0, (\Omega F)_2)\) is an \(R\)-homomorphism, i.e. Fig.32–36.commute.

Define a functor

\[
\Omega_F^{-1} : Im^2 F \longrightarrow Im^1 F
\]

\[
(A, \varphi, B) \mapsto A,
\]

\[
(A_1, \varphi_1, B_1) \xrightarrow{[f,g]} (A_2, \varphi_2, B_2) \mapsto A_1 \xrightarrow{[f]} A_2.
\]

If \((f,g), (f',g')\) are equal in \(Im^2 F\), i.e. \(Ff = Ff'\), then \(f, f'\) are equal in \(Im^1 F\).

From the definition of \(\Omega_F^{-1}\), we see that \(\Omega_F^{-1}\) maps morphisms of \(Im^2 F\) to the first part of them, so \(\Omega_F^{-1}\) is an \(R\)-homomorphism with \((\Omega_F^{-1})_+ = id, (\Omega_F^{-1})_0 = id, (\Omega_F^{-1})_2 = id\).

For any \(A \in obj(Im^1 F)\), and any representative morphism \(f : A_1 \to A_2\) in \(Im^1 F\), we have

\[
(\Omega_F^{-1} \circ \Omega_F)(A) = \Omega_F^{-1}(A, 1_{FA}, FA) = A,
\]

\[
(\Omega_F^{-1} \circ \Omega_F)(f) = \Omega_F^{-1}(f, Ff) = f.
\]

So \(\Omega_F^{-1} \circ \Omega_F = 1_{Im^1 F}\).

There is a morphism of \(R\)-homomorphisms:

\[
\tau : \Omega_F \circ \Omega_F^{-1} \to 1 : Im^2 F \longrightarrow Im^2 F,
\]

\[
\tau_{(A,\varphi,B)} \triangleq (1_A, \varphi) : (\Omega_F \circ \Omega_F^{-1})(A, \varphi, B) = \Omega_F(A) = (A, 1_{FA}, FA) \to (A, \varphi, B).
\]

For any morphism \((f,g) : (A_1, \varphi_1, B_1) \to (A_2, \varphi_2, B_2)\) in \(Im^2 F\), with \(g \circ \varphi_1 = \varphi_2 \circ Ff, (\Omega_F \circ \Omega_F^{-1})(f,g) = \Omega_F(f) = (f, Ff)\), we have the following commutative diagram:

\[
\begin{array}{c}
(A_1,1_{FA},FA_1) \xrightarrow{(f,g)} (A_2,1_{FA},FA_2) \\
\downarrow (1_{A_1},\eta_1) \quad \downarrow (1_{A_2},\eta_2) \\
(A,\varphi_1,B_1) \xrightarrow{(f,g)} (A_2,\varphi_2,B_2)
\end{array}
\]
So $\tau$ is a natural transformation.

\[(\Omega_F \circ \Omega_F^{-1})(A_1, \varphi_1, B_1) + (A_2, \varphi_2, B_2) = (A_1 + A_2, 1_{F(A_1 + A_2)}, F(A_1 + A_2)),\]
\[(\Omega_F \circ \Omega_F^{-1})(A_1, \varphi_1, B_1) + (\Omega_F \circ \Omega_F^{-1})(A_2, \varphi_2, B_2) = (A_1 + A_2, F_+, FA_1 + FA_2),\]
\[\tau(A_1, \varphi_1, B_1) + (A_2, \varphi_2, B_2) = \left(1_{A_1 + A_2}, (\varphi_1 + \varphi_2) \circ F_+ \right),\]
\[\tau(A_i, \varphi_i, B_i) = (1_{A_i}, \varphi_i), \text{ for } i = 1, 2.\]

Thus Fig.7 commutes. We can also get commutative diagrams Fig.37. in the similar way. Then $\tau$ is a morphisms of $R$-homomorphisms.

Since $\tau_{(A, \varphi, B)} = (1_A, \varphi)$ and $Im^2F$ is a groupoid, so $\tau$ is an isomorphism.

**Step 4.** There is a full and surjective $R$-homomorphism $E_F$.

Define a functor

\[
E_F : A \rightarrow Im^1F
\]
\[
A \mapsto A
\]
\[
f : A_1 \rightarrow A_2 \mapsto [f] : A_1 \rightarrow A_2
\]

Obviously, $E_F$ is an $R$-homomorphism.

**Step 5.** There is a faithful $R$-homomorphism $M_F$.

Define a functor

\[
M_F : Im^2F \rightarrow B
\]
\[
(A, \varphi, B) \mapsto B,
\]
\[
(A_1, \varphi_1, B_1) \xrightarrow{[f, g]} (A_2, \varphi_2, B_2) \mapsto B_1 \xrightarrow{g} B_2.
\]

Obviously, $M_F$ is a faithful $R$-homomorphism.

**Step 6.** $F = M_F \circ \Omega_F \circ E_F$.

For any $A \in obj(A)$ and any morphism $f \in Mor(A)$, we have

\[(M_F \circ \Omega_F \circ E_F)(A) = (M_F \circ \Omega_F)(A) = M_F(A, 1_{FA}, FA) = FA,\]
\[(M_F \circ \Omega_F \circ E_F)(f) = (M_F \circ \Omega_F)(f) = M_F(f, Ff) = Ff.\]
Proposition 10. For each \( \mathcal{R}\)-homomorphism \( F : \mathcal{A} \to \mathcal{B} \) in \((\mathcal{R} \text{-} 2\text{-}\text{Mod})\), \( M_F : \text{Im}^2 F \to \mathcal{B} \) is the kernel of the cokernel of \( F \).

Proof. The cokernel of \( F \) is \((\text{Coker} F, p_F, \pi_F)\), where \( \text{Coker} F \) is a category whose object is the object of \( \mathcal{B} \), morphism is \((f, A) : B_1 \to B_2\), with \( A \in \text{obj}(\mathcal{A}) \), \( f : B_1 \to B_2 + FA, p_F(B) = B\), \((\pi_F)_A = [(\pi_F)_A' A], \) where \((\pi_F)'_A : FA \to FA + 0 \to FA + 0\).

- Let us describe the \( \text{Kerp}_F \) in the following way:
  - Objects are the triple \((B, A, b)\), where \( B \in \text{obj}(\mathcal{B}) \), \( A \in \text{obj}(\mathcal{A}) \), \( b : B \to FA + 0\).
  - Morphism from \((B, A, b)\) to \((B', A', b')\) is the equivalence class of a pair \((g, f)\), denote by \([g, f]\), where \( g : B \to B'\), \( f : A \to A'\), such that \((Ff + 1_0) \circ b = b' \circ g\), and for two morphisms \((g, f), (g', f') : (B, A, b) \to (B', A', b')\) are equal, if \( g = g'\).
  - Composition of morphisms. Given morphisms \((B_1, A_1, b_1) \xrightarrow{[g_1,f_1]} (B_2, A_2, b_2)\), \((g_2, f_2) \circ (g_1, f_1) \equiv [g_2 \circ g_1, f_2 \circ f_1]\). If \((g_1, f_1), (g_1', f_1') : (B_1, A_1, b_1) \to (B_1, A_1, b_1)\) are equal, i.e. \( g_1 = g_1'\), then \((g_2, f_2) \circ (g_1, f_1), (g_2, f_2) \circ (g_1', f_1')\) are equal.

There is a bifunctor

\[
+ : \text{Kerp}_F \times \text{Kerp}_F \longrightarrow \text{Kerp}_F
\]

\[
((B_1, A_1, b_1), (B_2, A_2, b_2)) \mapsto (B_1, A_1, b_1) + (B_2, A_2, b_2) \equiv (B_1 + B_2, A_1 + A_2, b), \quad ((g_1, f_1), (g_2, f_2)) \mapsto (g, f) \equiv (g_1 + g_2, f_1 + f_2)
\]

Using the similar methods in Theorem 2, \( \text{Kerp}_F \) is an \( \mathcal{R} \)-2-module.

- There is a 2-morphism \( \epsilon : p_F \circ M_F \Rightarrow 0\), given by \( \epsilon_{(A, \varphi, B)} \equiv [l_{FA} \circ \varphi^{-1}, A] : (p_F \circ M_F)(A, \varphi, B) = p_F(B) = B \to 0\). By the universal property of the kernel, there is an \( \mathcal{R} \)-homomorphism

\[
\Theta : \text{Im}^2 F \longrightarrow \text{Kerp}_F
\]

\[
(A, \varphi, B) \mapsto (B, A, b), \quad [f, g] : (A, \varphi, B) \to (A', \varphi', B') \mapsto [g, f] : (B, A, b) \to (B', A', b')
\]

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where \( b, b' \) are the compositions \( B \xrightarrow{\varphi^{-1}} FA \xrightarrow{r_{FA}} FA + 0, \) \( B' \xrightarrow{(\varphi')^{-1}} FA' \xrightarrow{r_{FA'}} FA' + 0 \), respectively, such that

\[
(Ff + 1_0) \circ b = (Ff + 1_0) \circ r_{FA} \circ \varphi^{-1} = r_{FA'} \circ Ff \circ \varphi^{-1} = r_{FA'} \circ (\varphi')^{-1} \circ g = b' \circ g.
\]

If \((f, g), (f', g') : (A, \varphi, B) \rightarrow (A', \varphi', B')\) are equal, i.e. \( g = g' \), then \( \Theta(f, g), \Theta(f', g') \) are equal in \( Ker_{pF} \).

- There is an \( R \)-homomorphism

\[
\Theta^{-1} : Ker_{pF} \rightarrow Im^2 F
\]

\[
(B, A, b) \mapsto (A, B, \varphi),
\]

\[
[g, f] : (B, A, b) \rightarrow (B', A', b') \mapsto [f, g] : (A, B, \varphi) \rightarrow (A', B', \varphi')
\]

where \( \varphi = b^{-1} \circ r_{FA}^{-1} : FA \rightarrow FA + 0 \rightarrow B \), and \((f, g)\) satisfy

\[
g \circ \varphi = g \circ b^{-1} \circ r_{FA}^{-1} = (b')^{-1} \circ (Ff + 1_0) \circ r_{FA}^{-1} = (b')^{-1} \circ r_{FA'}^{-1} \circ Ff = \varphi' \circ Ff.
\]

Similarly, if \((g, f), (g', f')\) are equal in \( Im^2 F \), then \( \Theta^{-1}(g, f), \Theta^{-1}(g', f') \) are equal in \( Ker_{pF} \), too. Moreover, \( \Theta^{-1} \) is an \( R \)-homomorphism.

For any \((A, \varphi, B) \in obj(Im^2 F), (f, g) \in Mor(Im^2 F)\), we have

\[
(\Theta^{-1} \circ \Theta)(A, \varphi, B) = \Theta^{-1}(B, A, r_{FA}^{-1} \circ \varphi^{-1}) = (A, r_{FA}^{-1} \circ \varphi^{-1})^{-1} \circ r_{FA}^{-1}, B) = (A, \varphi, B),
\]

\[
(\Theta^{-1} \circ \Theta)(f, g) = \Theta^{-1}(g, f) = (f, g).
\]

For any \((B, A, b) \in obj(Ker_{pF}), (g, f) \in Mor(Ker_{pF})\), we have

\[
(\Theta \circ \Theta^{-1})(B, A, b) = \Theta(B, A, b^{-1} \circ r_{FA}^{-1}) = (B, A, r_{FA}^{-1} \circ (b^{-1} \circ r_{FA}^{-1})) = (B, A, b),
\]

\[
(\Theta \circ \Theta^{-1})(g, f) = \Theta(f, g) = (g, f).
\]

Thus \( \Theta : Im^2 F \rightarrow Ker_{pF} \) is an equivalent.

\[\square\]

**Proposition 11.** For each \( R \)-homomorphism \( F : A \rightarrow B \) in \((R-2-Mod), E_F : A \rightarrow Im^1 F \) is the coroot of the pip of \( F \).

**Proof.** The pip of \( F \) is an \( R \)-2-module \( PipF \), together with 2-morphism \( \pi_F : 0 \Rightarrow 0 : PipF \rightarrow A \), whose component at \( a \) is \( a \) itself, where \( a : 0 \rightarrow 0 \) is an object of \( PipF \).
Let us describe $\text{Coroot}_{\pi F}$ in the following way:

- Objects are the objects of $\mathcal{A}$.
- Morphisms are the morphisms of $\mathcal{A}$ up to equivalent relations, for two morphisms $f, f' : A_1 \to A_2$ which are equal in $\text{Coroot}_{\pi F}$, if there exists $a : 0 \to 0$ in $\mathcal{A}$, such that $Fa = 1_{F0}$ and
  \[ f \circ r_{A_1} = r_{A_2} \circ (f' + a). \]
- Composition of morphisms and identity morphisms are the composition and identity in $\mathcal{A}$ up to equivalence, and also well-defined. In fact, given morphism $f_2 : A_2 \to A_3$ and equal morphisms $f_1, f'_1 : A_1 \to A_2$ in $\text{Coroot}_{\pi F}$, for equivalence of $f_1, f'_1$, there exists $a : 0 \to 0$, such that $Fa = 1_{F0}$, $f_1 = r_{A_1} \circ (f'_1 + a) \circ r_{A_1}^{-1}$. Then there exists $a : 0 \to 0$, such that
  \[ f_2 \circ f_1 \circ r_{A_1} = f_2 \circ r_{A_2} \circ (f'_1 + a) = r_{A_3} \circ (f'_2 + 1_0) \circ (f'_1 + a) = r_{A_3} \circ (f'_2 \circ f'_1 + a). \]

Then $f_2 \circ f_1, f_2 \circ f'_1$ are equal in $\text{Coroot}_{\pi F}$.

$\text{Coroot}_{\pi F}$ is an $R$-2-module from the $R$-2-module structure of $\mathcal{A}$ up to equivalence.

There is an $R$-homomorphism

\[ R : \mathcal{A} \longrightarrow \text{Coroot}_{\pi F} \]

\[ A \mapsto A, \]

\[ f : A_1 \to A_2 \mapsto f : A_1 \to A_2 \]

such that for any $a : 0 \to 0$ in $\text{Pip}F$, $(R \ast \pi_F)_a = R(a) = a = (1_0)_a$, i.e. $R \ast \pi_F = 1_0$.

For $K \in \text{obj}(\mathcal{R}\text{-2-Mod})$, and an $R$-homomorphism $G : \mathcal{A} \to K$, such that $G \ast \pi_F = 1_0$, there exist an $R$-homomorphism

\[ G' : \text{Coroot}_{\pi_F} \longrightarrow K \]

\[ A \mapsto GA, \]

\[ f : A_1 \to A_2 \mapsto G(f) : G(A_1) \to G(A_2) \]

and a 2-morphism

\[ \alpha : G' \circ R \Rightarrow G \]

\[ A \mapsto \alpha_A \triangleq id_{GA} : (G \circ R)(A) = G'(A) = GA \to GA. \]
From the given $G$, we know that $G'$ is an $R$-homomorphism, $\alpha$ is a 2-morphism.

For every $C \in \text{obj}(R\text{-}2\text{-Mod})$, there is an $R$-homomorphism

$$- \circ R : \text{Hom}(\text{Coroot}\pi_F, C) \to \text{Hom}(A, C)$$

$$H \mapsto H \circ R,$$

$$\tau : H_1 \Rightarrow H_2 \mapsto \tau \circ R \triangleq \tau \circ 1_R : H_1 \circ R \Rightarrow H_2 \circ R$$

such that for any objects $H_1$, $H_2$ in $\text{Hom}(\text{Coroot}\pi_F, C)$, and a 2-morphism $\beta : H_1 \circ R \Rightarrow H_2 \circ R$, there is a 2-morphism $\tau : H_1 \Rightarrow H_2$ given by $\tau_A \triangleq \beta_A : H_1 A \to H_2 A$. Also, if $\tau_1$, $\tau_2 : H_1 \Rightarrow H_2 : \text{Coroot}\pi_F \to C$, such that $\tau_1 \circ R = \tau_2 \circ R : H_1 \circ R \Rightarrow H_2 \circ R$, then, for any $A \in \text{obj}(A)$, $(\tau_1)_A = (\tau_2)_A = (\tau_1 \circ R)_A = (\tau_2 \circ R)_A$, i.e. $\tau_1 = \tau_2$. So $R$ is full and surjective, and then $(\text{Coroot}\pi_F, R)$ is the coroot of $\pi_F$.

• There is an equivalence between $\text{Coroot}\pi_F$ and $\text{Im}^1 F$.

From the definition of $E_F : A \to \text{Im}^1 F$, we have $E_F \circ \pi_F = 1_0$, from the universal property of coroot of $\pi_F$, there is an $R$-homomorphism

$$\Theta : \text{Coroot}\pi_F \to \text{Im}^1 F$$

$$A \mapsto A$$

$$f : A_1 \to A_2 \mapsto f : A_1 \to A_2.$$

Also there is an $R$-homomorphism

$$\Theta^{-1} : \text{Im}^1 F \to \text{Coroot}\pi_F$$

$$A \mapsto A,$$

$$f : A_1 \to A_2 \mapsto f : A_1 \to A_2.$$

In [19], the author proved that $\Theta$ and $\Theta^{-1}$ are well-defined homomorphism of symmetric 2-groups. Since $\text{Coroot}\pi_f$, $\text{Im}^1 F$ have the same $R$-2-module structure and $\Theta \circ \Theta^{-1} = 1, \Theta^{-1} \circ \Theta = 1$, then $\text{Coroot}\pi_f$, $\text{Im}^1 F$ are equivalent $R$-2-modules.

\[ \Box \]

In the sense of 2-abelian $Gpd$-category, and from Propositions 6-11, we have

**Theorem 4.** The $Gpd$-category $(R\text{-}2\text{-Mod})$ is a 2-abelian $Gpd$-category.
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