Preconditioning Based on LU Factorization in Iterative Method for Solving Systems of Linear Algebraic Equations with Sparse Matrices

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Abstract. A new approach to preconditioning based on LU - decomposition for solving systems of linear algebraic equations by iterative methods is proposed. The approach makes it possible to effectively to find an acceptable solution with a minimum filling of sparse matrices.

1. Introduction

The solution of linear algebraic equations systems is one of the main numerical linear algebra problems. All methods for solving linear algebraic systems can be divided into two classes: direct and iterative (approximate). Each method has advantages and disadvantages. Choice of the method depends on the structure of the equations systems matrix: size, conditionality, symmetry, filling (the creation of new nonzeros), the specifics of the nonzero location in the matrix, etc. At the present time, iterative are more interesting to solve linear algebraic equations systems with sparse matrices of large dimensions [7].

Sparse matrices sources are mentioned in solving various practical problems: structural analysis, the theory of electrical networks and energy distribution systems, numerical solution of differential equations, graph theory, as well as genetics, social science, behavioral sciences and others. Due to the fact that the dimension of the matrices in most cases is very large, it is necessary to organize the storage of sparse objects (matrices, vectors) in such a way that only non-zero elements are in the computer's memory. This will reduce the computation time of the algorithm, as well as save memory [2].

An interesting case is linear algebraic equations systems with nonsymmetric ill-conditioned sparse matrices of large dimensions. The projection methods are the most effective and stable among the iterative methods for solving linear systems. They are related to the projection onto Krylov subspaces. They do not require finding optimal iteration parameters and a priori information about the spectrum of the initial matrix, and allow work with different types preconditioners [2, 6].

Iterative methods preconditioners can be based on classical iterative methods (Jacobi, Gauss-Seidel, etc.) or different modifications of the initial matrix factorization (the Cholesky factorization, LU factorization, etc.). In this paper is proposed a preconditioner applied to linear algebraic equations systems when it is solved by iterative methods of the Krylov subspace and based on the LU factorization of the matrix. It provides a smaller matrix filling, significantly accelerate convergence and accuracy of the solution compared to other preconditioners.

2. Statement of the Problem

Consider the system linear algebraic equations
\[ Ax = b, \]  

where \( A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n, \det A \neq 0. \)

This system can be solved by direct and iterative methods. Iterative methods are considered to reduce filling for sparse linear systems. Iterative methods always give an approximate solution. In order to obtain an exact solution, an infinite number of iterations is necessary, so the criteria for stopping the method are introduced. The convergence rate of the method is determined by the number of approximations that are constructed by the method until the stopping criterion is satisfied. Due to the accumulation of computational error, a method that converges in theory can diverge in practice. In order to avoid this, the system preconditioning is used. It is transition to linear systems with the same solution and matrix with the best qualities by multiplying the system with a matrix of a special kind. Among the iterative methods projection methods are the most stable, effective, and also allowing for solution and matrix with the best qualities by multiplying the system with a matrix of a special kind.

order to avoid this, the system preconditioning is used. It is transition to linear systems with the same solution and matrix with the best qualities by multiplying the system with a matrix of a special kind. Among the iterative methods projection methods are the most stable, effective, and also allowing for work with preconditioners of different types, and especially methods associating with the projection onto Krylov subspaces.

Instead of the original system (1), consider a new system

\[ \tilde{A} \tilde{x} = \tilde{b} \]  

where \( \tilde{b} = M_1^{-1}b, \tilde{A} = M_1^{-1}AM_2^{-1}, \tilde{x} = M_2x, M_1, M_2 - \) nonsingular matrices of dimension \( n. \)

The process of transition from (1) to (2) in order to improve the characteristics of the matrix to accelerate the convergence to the solution is called preconditioning, the matrix \( M^{-1} \) by the preconditioner matrix, and the system (2) is called preconditioned [2, 6]. Let's consider the left preconditioning, which is most often used in practice. In the case of left preconditioning, system (1) takes the form

\[ M^{-1}Ax = M^{-1}b, \]  

which due to the non-singularity of \( M \) has the same exact \( \tilde{x}. \) Denote by \( \tilde{A} = M^{-1}A, \tilde{b} = M^{-1}b, \) and write (3) in the form

\[ \tilde{A} \tilde{x} = \tilde{b}. \]  

3. A Preconditioner Based on LU Factorization with a Partial Pivoting and the Filling Threshold

In this paper we propose a preconditioner applied to the linear systems (1) when it is solved by iterative methods of the Krylov subspace and based on the LU factorization of the matrix \( A. \) We construct the preconditioner matrix \( M = LU \) for the transition from system (1) to system (4). The matrix \( M \) obvious satisfies all the requirements for the preconditioner matrix. It is close to the matrix \( A; \) it is easily computable by the formulas of LU-decomposition [1]; finally, it is easily invertible, since it is the product of two triangular matrices. Thus, choice is a good enough way of preconditioning.

The choice of the preconditioner matrix \( M = LU \) for the system linear algebraic equations with sparse matrices of large dimensions implies filling the portrait of the matrices \( L \) and \( U, \) i.e. the appearance in these matrices of nonzeros in those positions \( a_{ij} = 0. \) This increases the amount of memory that is required to store the preconditioner matrix. It is can be avoided if the \( LU \) factorization of the matrix \( A \) is used as a preconditioner without filling. In this case, the matrix is given in the form \( M = \tilde{L} \tilde{U}, \) where \( \tilde{L} \) and \( \tilde{U} \) are lower- and upper-triangular matrices, respectively. The matrices \( \tilde{L} \) and \( \tilde{U} \) are computed in the same way as the matrices \( L \) and \( U, \) using the \( LU \) factorization of the matrix [1], only with the difference that if \( a_{ij} = 0 \) then we set \( \tilde{l}_{ij} = 0 \) and \( \tilde{u}_{ij} = 0 [2, 6]. \)

Denote this preconditioner \( LU (0) \). He quite effectively copes with the solution of problems, if the matrix of the system with a diagonal dominance, but for nonsymmetric matrices give an increasing of solution relative error.

In this paper we propose a preconditioner applied to linear systems (1) when it is solved by iterative methods of the Krylov subspace and based on the \( LU \) factorization of the matrix \( A. \) It provides a smaller filling of the system matrix, as well as a higher convergence rate and accuracy of the solution in comparison with the preconditioner \( LU (0) \). We consider the system (1), the nonsingular matrix \( A \) in the general case can contain zero elements on the main diagonal, then not one of the considered preconditions can be applied due to division by zero. In the case of ill-conditioned matrices, the nonoccurrence of a preconditioner lead to the impossibility of finding a system solution.
We propose a method of $LU$ ($t$) factorization with pivoting. It solves the problem of division by zero, which allows us to apply it to nonsingular matrices $A$ with an arbitrary portrait. Using the same matrix filling threshold [7, 8] reduces the amount of memory required to store the preconditioner matrix. In practice, a partial pivoting for constructing such a preconditioner is quite sufficient.

Since the complete factorization leads to the filling of the matrix portrait, without any constraints, we introduce a certain level of filling. In the $LU$ factorisation algorithm with a partial pivoting [7] we introduce the parameter $t$, called the filling threshold and define as:

$$t = \frac{\|A\|_2}{\varepsilon_m},$$  

(5)

where $\varepsilon_m$ – is a machine epsilon, and $\|A\|_2$ - is the Euclidean norm of the matrix.

Such a method of excluding nonzeros is not good in the case of unscaled matrices, the large value of $\|A\|_2$, the filling criterion given by (5) becomes too high and can not be applied. Then the filling threshold is given by $t = \varepsilon$, where $\varepsilon$ is some given accuracy.

The constructed algorithms research was carried out on real problems, as matrices from Harwell-Boeing collection were chosen as test matrices [5]. Table 1 lists the matrices considered together with their characteristics: dimensions, number of nonzeros elements and application area.

**Table 1. The tested matrices characteristics.**

| HB matrix | Size | Nonzeros | Application area          |
|-----------|------|----------|---------------------------|
| MCCA      | 180  | 2659     | Astrophysics              |
| MCFE      | 765  | 24382    | Astrophysics              |
| ARC 130   | 130  | 1037     | Different area            |
| FS 183 1  | 183  | 998      | Chemical kinetics         |
| FS 541 4  | 541  | 4285     | Different area            |
| WATT 1    | 1856 | 11360    | Petroleum kinetics        |

The results of the matrices filling as a result of the linear systems solution using a preconditioner based on the $LU$ factorization with the pivot and the parameter are presented in table 2.

**Table 2. The matrix filling.**

| Matrix | LU factorization, filling | LU($t$) factorization with pivoting, filling |
|--------|----------------------------|---------------------------------------------|
| ARC 130| 8183                       | 1150                                       |
| FS 183 1| 12904                      | 6128                                       |
| FS 541 4| 111871                     | 39923                                      |
| MCCA   | 1603                       | 3735                                       |
| MCFE   | 56073                      | 62949                                      |
| WATT 1 | 206542                     | 187387                                     |

In Figure 1, we see the filling that was obtained after solving linear systems with one of the matrices [5] FS 183 1.

It is seen the combination of the pivot choice and the parameter in the preconditioner based on the $LU$ factorisation makes it possible to reduce the filling.

We present the results of the linear systems solution by iterative methods for one of the matrices on the table 1.

In accordance with tables 2 and 3, in comparison with the preconditioner based on $LU$ factorization, the preconditioner with the $LU$ ($t$) factorization with the pivoting and the filling threshold is as good as the solution accuracy and the convergence rate, but reduces the memory cost by several time. The methods with a preconditioner based on the $LU$ factorization are superior in accuracy and speed of convergence to methods with Jacobi preconditioner.
Figure 1. Portraits of the matrix FS 183 1 after LU factorization (a, b), after LU (t) factorization with pivoting and the filling threshold (c, d).

Figure 2. Portraits of the matrix FS 541 4 after LU factorization (a, b), after LU (t) factorization with pivoting and the filling threshold (c, d).

Table 3. The result of the quasi-minimal residuals method for the matrix FS 541 4.

| Method                                      | Preconditioner | Iteration count | Relative error |
|---------------------------------------------|----------------|-----------------|----------------|
| Quasi-minimal residuals method              | --             | 1076            | 0.45e-5        |
|                                             | Jacobi         | 209             | 0.85e-5        |
| LU factorization                            | 2              | 0.11e-10        |
| LU(0) factorization                         | 9              | 0.11e-5         |
| LU(t) factorization with pivoting           | 2              | 0.64e-11        |
4. Conclusion
A new approach to preconditioners based on LU-decomposition and Jacobi for solving ill-posed problems is considered. On the basis of the preconditioner with LU factorization a modification has been developed. It makes possible to reduce the matrix filling without significant loss of the solution accuracy, which is confirmed experimentally, especially when using it in the quasi-minimal residual method and the biconjugate gradient method. In addition, the developed preconditioner with the LU (t) factorization with the pivoting is applicable to matrices with an arbitrary portrait. This system linear algebraic equations in most cases without the preconditioner is not possible to solve.

5. References
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