Linear complementary dual, maximum distance separable codes

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Abstract

Linear complementary dual (LCD) maximum distance separable (MDS) codes are constructed to given specifications. For given $n$ and $r < n$, with $n$ or $r$ (or both) odd, MDS LCD $(n, r)$ codes are constructed over finite fields whose characteristic does not divide $n$. Series of LCD MDS codes are constructed to required rate and required error-correcting capability. Given the field $GF(q)$ and $n/(q - 1)$, LCD MDS codes of length $n$ and dimension $r$ are explicitly constructed over $GF(q)$ for all $r < n$ when $n$ is odd and for all odd $r < n$ when $n$ is even. For given dimension and given error-correcting capability LCD MDS codes are constructed to these specifications with smallest possible length. Series of asymptotically good LCD MDS codes are explicitly constructed. Efficient encoding and decoding algorithms exist for all the constructed codes.

Linear complementary dual codes have importance in data storage, communications' systems and security.

1 Introduction

A Linear complementary dual, LCD, code is a linear code $C$ such that $C \cap C^\perp = 0$ where $C^\perp$ denotes the dual of $C$.

LCD codes have been studied extensively in the literature. For background, history and general theory consult the nice articles [2, 3, 4, 14] by Carlet, Mesnager, Tang and Qi. LCD codes were originally introduced by Massey in [11, 12]. These codes have been studied for improving the security of information on sensitive devices against side-channel attacks (SCA) and fault non-invasive attacks, see [5], and have found use in data storage and communications’ systems.

The necessary background on coding theory and field theory may be found in [1] or [13]. The finite field of order $q$ is denoted by $GF(q)$ and of necessity $q$ is a power of a prime.

Series of LCD MDS (maximum distance separable) codes are constructed to given requirements. For given odd $n$ and any $r < n$ and for given even $n$ and any odd $r < n$, MDS LCD $(n, r)$ codes are constructed over finite fields whose characteristic does not divide $n$. For a given rate and given error-correcting capability, MDS LCD codes are constructed to these specifications. For a given field $GF(q)$, and for each $n/(q - 1)$, LCD MDS codes of length $n$ and dimension $r$ are explicitly constructed for any given $r$ ($< n$) when $n$ is odd and for any given odd $r$ ($< n$) when $n$ is even.

Explicit series of asymptotically good LCD MDS codes are constructed. For given rate $R, 0 < R < 1$, infinite series of LCD MDS codes are constructed in which the ratio of the distance by the length approaches $(1 - R)$; the codes have efficient encoding and decoding algorithms.

Carlet and Guilley [6] investigate the application of binary LCD codes against side-channel attacks (SCA) and fault tolerant injection attacks (FIA). For the applications, LCD codes with specific dimension and specific error-correcting capability with length as small as possible are required. LCD MDS codes to required dimension and required error-correcting capability to smallest possible length for such a code are constructed here.

Carlet and Guilley [6] also showed that non-binary LCD codes in characteristic 2 can be transformed into binary LCD codes by expansion. Thus characteristic 2 LCD codes are of particular importance. The results here include infinite series of characteristic 2 LCD MDS codes. For characteristic 2 and given rate $R = \frac{r}{n}$ (reduced fraction, $0 < R < 1$) infinite series of LCD MDS codes over finite fields of characteristic 2 are constructed in which the limit of the ratio of the distance by the length approaches $(1 - R)$.

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Infinite series of LCD MDS codes are constructed over fields of prime characteristic in which the ratio of distance by length is \( (1 - R) \) for given \( R, 1 < R < 1 \); in these cases the arithmetic is modular arithmetic with complexity of operations \( \max\{O(n \log n), O(l^2)\} \).

The codes constructed are explicit and have efficient encoding and decoding algorithms. Efficient encoding and decoding algorithms for the constructed codes follow from the algorithms in \[7\]. The complexity of encoding and decoding is \( \max\{O(\log n), O(l^2)\} \) where \( n \) is the length and \( t = \lceil \frac{d}{2} \rceil \), \( (d \text{ the distance}) \), is the error-correcting capability of the code.

The codes are generated by rows in sequence of Fourier matrices over finite fields following the methods developed in \[7\] [8].

Carlet, Mesneger, Tang and Qi [3] also construct some LCD MDS codes. Mesneger, Tang and Qi [14] construct LCD codes using algebraic geometry methods. The codes here are different.

An LCD Hermitian code is defined as a code \( C \) such that \( C \cap C_H^\perp = 0 \) where \( C_H^\perp \) denotes the dual of \( C \) relative to the Hermitian product. Infinite series of Hermitian LCD MDS codes may also be constructed over fields of the form \( GF(q^2) \) using similar methods; the specifics are omitted.

### 1.1 Layout

The general construction is given within Section 2. The necessary background from \[7\] is summarised in Subsection 2.1 and Subsection 2.2 presents the general constructions. The complexity of encoding and decoding is discussed in Subsection 2.3. Section 2.4 constructs the codes over fields of characteristic 2. Subsection 2.5 presents specific simulations and samples with which to demonstrate the main constructions. A method for constructing LCD MDS codes to given rate and given error-correcting capability is derived in section 2.6 and samples are given. Section 2.7 gives the best length construction of LCD MDS codes for given dimension and required error-capability. Section 3 deals with deriving asymptotic constructions: for given rate \( R \) series of MDS LCD codes are constructed with this rate and in which the limit of the distance by the length approaches \( (1 - R) \). Subsection 3.1 derives such constructions over prime fields; Subsection 3.2 derives asymptotic such constructions over fields of particular characteristic 2.

### 2 Construction

#### 2.1 Background material

The codes constructed are based on the methods of \[7\] [8], using the unit-derived schemes of \[9\] applied to Vandermonde and Fourier matrices.

A primitive \( n^{th} \) root of unity \( \omega \) in a field \( \mathbb{F} \) is an element \( \omega \) satisfying \( \omega^n = 1_{\mathbb{F}} \) but \( \omega^i \neq 1_{\mathbb{F}}, 1 \leq i < n \). The multiplicative identity of a field \( \mathbb{F} \) will be denoted by \( 1 \) rather than \( 1_{\mathbb{F}} \) when the field in question is clear. The field \( GF(q) \) (where \( q \) is a power of a prime) contains a primitive \( (q - 1) \) root of unity, \[11\] [13], and such a root is referred to as a primitive element in the field \( GF(q) \). In addition then the field \( GF(q) \) contains primitive \( n^{th} \) roots of unity for any \( n/(q - 1) \).

An \( (n, r) \) linear code is a linear code of length \( n \) and dimension \( r \); the rate is \( \frac{r}{n} \). An \( (n, r, d) \) linear code is a code of length \( n \), dimension \( r \) and (minimum) distance \( d \). The code is an MDS code provided \( d = (n - r + 1) \), which is the maximum distance an \( (n, r) \) code can attain. The error-capability of \( (n, r, d) \) is \( t = \lceil \frac{d - 1}{2} \rceil \) which is the maximum number of errors the code can correct successfully.

Let \( \omega \) be a primitive \( n^{th} \) root of unity in a field \( \mathbb{F} \). The Fourier matrix \( F_n \), relative to \( \omega \) and \( \mathbb{F} \), is the \( n \times n \) matrix

\[
F_n = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\
1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)}
\end{pmatrix}
\]
Then
\[
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\
1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)}
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)} \\
1 & \omega^{n-2} & \omega^{2(n-2)} & \ldots & \omega^{(n-2)(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega & \omega^2 & \ldots & \omega^{(n-1)}
\end{pmatrix} = nI_n
\]

The second matrix on the left of the equation is denoted by \( F_n^* \); then \( F_n F_n^* = nI_n \). This \( F_n^* \) can be obtained from \( F_n \) by replacing \( \omega \) by \( \omega^{n-1} \) and is a Fourier matrix itself. An \( n^{th} \) root of unity can only exist in a field whose characteristic does not divide \( 2 \).

2.2 LCD MDS codes: General constructions

In particular we have:

**Theorem 2.1** \([7]\) Let \( C \) be a code generated by taking any \( r \) rows of \( F_n \) in arithmetic sequence with arithmetic difference \( k \) satisfying \( \gcd(n, k) = 1 \). Then \( C \) is an MDS (maximum distance separable) \((n,r,n-r+1)\) code.

For ‘rows in sequence’ it is permitted that rows may wrap around and then \( e_k \) is taken to mean \( e_k \mod n \). Thus for example Theorem 2.2 could be applied to a code generated by \( \{e_r, \ldots, e_{n-1}, e_0, e_1, \ldots, e_s\} \).

There are similar theorems, see \([7]\), involving the more general Vandermonde matrices but these are not used here.

Let \( \mathbb{F}_n^* \) denote the matrix with \( \mathbb{F}_n F_n^* = nI_{n \times n} \). The rows of \( \mathbb{F}_n \) in order are denoted by \( \{e_0, e_1, \ldots, e_{n-1}\} \) and denote the columns of \( \mathbb{F}_n^* \) in order by \( \{f_0, f_1, \ldots, f_{n-1}\} \). Then it is important to note that \( f_i = e_{n-i}^T, e_i = f_{n-i}^T \) with the convention that suffixes are taken modulo \( n \). Also note \( e_i f_j = n \) and \( e_i f_j = 0, i \neq j \).

Thus
\[
\begin{pmatrix}
e_0 \\
e_1 \\
\vdots \\
e_{n-1}
\end{pmatrix}
(f_0, f_1, f_2, \ldots, f_{n-1}) =
\begin{pmatrix}
e_0 \\
e_1 \\
\vdots \\
e_{n-1}
\end{pmatrix}
(e_0^T, e_{n-1}^T, e_{n-2}^T, \ldots, e_1^T) = nI_n
\]

2.2 LCD MDS codes: General constructions

It is required to construct \((n,r)\) MDS LCD codes over finite fields.

First construct a Fourier \( n \times n \) matrix \( F_n \). There is then some restriction, depending on \( n \), on the characteristic of the finite fields that may be used; this is made more precise in subsection 2.2.1 below.

Denote the rows in order of \( F_n \) by \( \{e_0, e_1, \ldots, e_{n-1}\} \) and the columns in order of \( F_n^* \) by \( \{f_0, f_1, \ldots, f_{n-1}\} \). Suppose a dimension size \((2r+1)\) is required for the code. Let \( C = \{e_0, e_1, \ldots, e_r, e_{n-r}, e_{n-r+1}, \ldots, e_{n-1}\} \).

The generators of \( C \) can be given in sequence as \( \{e_{n-r}, e_{n-r+1}, \ldots, e_{n-1}, e_0, e_1, \ldots, e_r\} \). Thus the code \( C \) satisfies the criteria of Theorem 2.2 which implies that \( C \) is an \((n, 2r+1, n-2r)\) MDS code. Now \( C \ast (f_{r+1}, f_{r+2}, \ldots, f_{n-r-1}) = 0_{(2r+1) \times (n-2r+1)} \) (where \( \ast \) denotes matrix multiplication).

As \( (f_{r+1}, f_{r+2}, \ldots, f_{n-r-1}) = (e_{n-r-1}^T, e_{n-r-2}^T, \ldots, e_{r+1}^T) \) this gives \( C^\perp = \langle e_{r+1}, e_{r+2}, \ldots, e_{n-r-1} \rangle \) and so \( C \cap C^\perp = 0 \). Thus \( C \) is an \((n, 2r+1, n-2r)\) MDS LCD code.

When choosing the rows from the Fourier matrix \( F_n \) for the generating matrix, it is noted that if \( e_i \) is chosen then also \( e_{n-i} \) must be chosen when an LCD code is required. For an MDS code to be constructed the rows chosen must also be in arithmetic sequence with arithmetic difference \( k \) satisfying \( \gcd(k, n) = 1 \) in order to satisfy the conditions of Theorem 2.1.

In general when choosing rows from the Fourier matrix to form the generator matrix if the row \( e_i \) is chosen then also the row \( e_{n-i} \) must also be chosen when an LCD code is required; to get an MDS code
the rows should finish up in sequence so that Theorem 2.1 may be applied. When \( n \) is even it is not possible by this method to get an even dimension \( r \) and rows in sequence to satisfy Theorem 2.1 as the arithmetic differences \( k \) that can be obtained are also even and then \( \gcd(n, k) \geq 2 \).

The above argument allows the construction of \((n, 2r+1)\) LCD MDS codes whether \( n \) is even or odd and for any (odd) \((2r+1)\). If \( n \) is odd this also allows the construction of \((n, r)\) LCD MDS for any \( r \) since if \( \mathcal{C} \) is an LCD MDS code then so is \( \mathcal{C}^\perp \).

It is worthwhile to construct directly \((n, r)\) LCD MDS codes for odd \( n \) and any \( r \). If \( r \) is odd then proceed as before. Suppose \( r = 2t \) is even and \( n \) is odd. Define \( \mathcal{C} = \langle e_1, e_3, \ldots, e_{r-1}, e_{n-r+1}, e_{n-r+3}, \ldots, e_{n-1} \rangle \).

Then \( \mathcal{C} \) can be given in sequence \( \langle e_{n-r+1}, e_{n-r+3}, \ldots, e_{n-1}, e_1, e_3, \ldots, e_{r-1} \rangle \) with arithmetic difference \( 2 \); now \( \gcd(n, 2) = 1 \) and hence by Theorem 2.1 \( \mathcal{C} \) is an MDS code. It is also seen to be an LCD code since if \( e_i \in \mathcal{C} \) then also \( e_{n-i} \in \mathcal{C} \). Thus \( \mathcal{C} \) is and LCD MDS \((n, r)\) code.

From a Fourier \( n \times n \) matrix other LCD MDS \((n, r)\) codes are obtained for given \( r \) by varying the allowed arithmetic difference as per Theorem 2.1 \( r \) must be odd if \( n \) is even. Let \( k < n, \gcd(k, n) = 1 \) and suppose that \( r \) is odd; when \( r \) is even and \( n \) is odd the constructions are similar. Then define \( \mathcal{C} = \langle e_0, e_k, \ldots, e_{rk}, e_{n-rk}, e_{n-rk+k}, \ldots, e_{n-k} \rangle \). Here the suffices \( rk \) mean \( r \times k \) and all suffices are taken mod \( n \). The generators of \( \mathcal{C} \) can be given in arithmetic sequence as \( \langle e_{n-rk}, e_{n-rk+k}, \ldots, e_{n-k}, e_0, e_k, \ldots, e_{rk} \rangle \).

Thus \( \mathcal{C} \) as constructed is an \((n, 2r+1, n-2r)\) MDS code. As with the case \( k = 1 \) above it is seen that \( \mathcal{C} \) is also an LCD code.

2.3 Complexity

Efficient encoding and decoding algorithms exist for these codes by the methods/algorithms developed in [7]. In general the complexity is \( \max \{ O(n \log n), O(t^2) \} \) where \( n \) is the length and \( t \) is the error-correcting capability, that is, \( t = \lceil \frac{d-1}{2} \rceil \) where \( d \) is the distance. See the algorithms in [7] for details.

2.4 Comment on direct sum

Suppose \( \mathcal{C} \oplus \mathcal{C}^\perp = \mathbb{F}_n \) for an LCD MDS code \( \mathcal{C} \) as described herein. Then it is relatively easy to write any element \( w \in \mathbb{F}_n \) as a unique sum \( w = w_1 + w_2 \) with \( w_1 \in \mathcal{C}, w_2 \in \mathcal{C}^\perp \). This is because \( \mathcal{C} \) is formed from certain rows of a Fourier matrix and \( \mathcal{C}^\perp \) is simply the rest of the rows of the matrix. It is then a matter of calculating \( F_n^{-1} w \). \( F_n^{-1} \) is also a Fourier matrix divided by \( n \) and this calculation is easy. Then choose the coefficients of \( w \) corresponding to the rows chosen for \( \mathcal{C} \) and \( \mathcal{C}^\perp \). This would suggest a nearest neighbour decoding method by [11], see more detail in [10]; the method would require in addition finding and implementing a nearest word function from \( \mathcal{C}^\perp \) to \( \mathcal{C} \). This is not better nor quicker than the direct decoding algorithmic method developed in [7] for the codes.

2.5 Characteristic 2

Carlet and Guilley show [6] that non-binary LCD codes in characteristic \( 2 \) can be transformed into binary LCD codes by expansion. Thus we look at characteristic \( 2 \) in particular. Let \( GF(2^k) \) be a field of characteristic \( 2 \). Then there exists a primitive \((2^k - 1)\) root of unity \( \omega \) in \( GF(2^k) \). Let \( n = (2^k - 1) \) and \( F_n \) be the Fourier \( n \times n \) formed using \( \omega \). Build LCD MDS codes of length \( 2^k - 1 \) and required rate as follows.

Let \((2r+1)\) be the dimension for the required rate. Then let \( \mathcal{C} = \langle e_0, e_1, \ldots, e_r, e_{n-r}, \ldots, e_{n-1} \rangle \) and \( \mathcal{C} \) is an LCD MDS \((n, 2r+1)\) code by the general construction in section 2.2 above.

Let \( 2r \) be the dimension for the required rate. Note that \( n \) is odd.

Let \( \mathcal{C} = \langle e_1, e_3, e_5, \ldots, e_{2r+1}, e_{n-2r+1}, e_{n-2r+3}, \ldots, e_{n-1} \rangle \).

This is in sequence \( \langle e_{n-2r+1}, e_{n-2r+3}, \ldots, e_{n-1}, e_1, e_3, \ldots, e_{2r-1} \rangle \) with arithmetic difference \( 2 \) and \( \gcd(n, 2) = 1 \) and so satisfies the criteria of Theorem 2.1. Then \( \mathcal{C} \) is an LCD MDS \((n, 2r)\) code.
There are as noted before other ways to construct the codes using \( k \) with \( \gcd(n,k) = 1 \) giving \( \phi(n) \) different codes of required form. The size of the field must increase as the length increases. These are near maximum length codes for the field \( GF(2^k) \).

Similarly for given \( GF(2^k) \) LCD MDS codes of the form \((m,r)\) may be constructed for any \( m/(2^k-1) \) over this field. Note that \( m \) is odd.

### 2.6 The fields

Suppose \( n \) is given and it is required to find the the finite fields over which a Fourier \( n \times n \) matrix exists. The following argument is essentially taken from [7]. It is included for clarity and completeness and is necessary for deciding on the relevant fields.

Note first of all that the field must have characteristic which does not divide \( n \) in order for the Fourier \( n \times n \) matrix to exist over the field.

**Proposition 2.1** There exists a finite field of characteristic \( p \) containing an \( n^{th} \) root of unity for given \( n \) if and only if \( p \nmid n \).

**Proof:** Let \( p \) be a prime which does not divide \( n \). Hence \( p^{\phi(n)} \equiv 1 \mod n \) by Euler’s theorem where \( \phi \) denotes the Euler \( \phi \) function. More specifically let \( \beta \) be the least positive integer such that \( p^n \equiv 1 \mod n \). Consider \( GF(p^n) \). Let \( \delta \) be a primitive element in \( GF(p^n) \). Then \( \delta \) has order \( (p^n-1) \) in \( GF(p^n) \) and \( (p^n-1) = sn \) for some \( s \). Thus \( \omega = \delta^s \) has order \( n \) in \( GF(p^n) \).

On the other hand if \( p \nmid n \) then \( n = 0 \) in a field of characteristic \( p \) and so no \( n^{th} \) root of unity can exist in the field. \( \square \)

In fact the proof constructs the smallest field of characteristic \( p \) with an \( n^{th} \) root of unity when \( p \nmid n \). The Fourier \( n \times n \) matrix may then be constructed over this field. If \( p \) is a prime not dividing \( n \) find the least positive power (it exists by Euler’s Theorem) such that \( p^n \equiv 1 \mod n \) and then the field \( GF(p^n) \) contains an \( n^{th} \) root of unity from which the Fourier \( n \times n \) over \( GF(p^n) \) may be constructed.

#### 2.6.1 Required fields

Suppose \( n = 52 \). The prime divisors of \( n \) are 2, 13 so take any other prime \( p \) and then there is a field of characteristic \( p \) which contains a \( 52^{nd} \) root of unity. For example take \( p = 3 \). Know \( 3^{\phi(52)} \equiv 1 \mod 52 \) and \( \phi(52) = 24 \) but indeed \( 3^6 \equiv 1 \mod 52 \). Thus the field \( GF(3^6) \) contains a primitive \( 52^{nd} \) root of unity and the Fourier \( 52 \times 52 \) matrix exists in \( GF(3^6) \). Also \( 5^4 \equiv 1 \mod 52 \), and so \( GF(5^4) \) can be used. Now \( 5^4 = 625 < 729 = 3^6 \) so \( GF(5^4) \) is a smaller field with which to work.

Even better though is \( GF(53) = \mathbb{Z}_{53} \) which is a prime field. This has an element of order 52 from which the Fourier \( 52 \times 52 \) matrix can be formed. Now \( \omega = (2 \mod 53) \) is an element of order 52 in \( GF(53) \). Work and codes with the resulting Fourier \( 52 \times 52 \) matrix can then be done in modular arithmetic, within \( \mathbb{Z}_{53} \), using powers of \((2 \mod 53)\).

#### 2.7 LCD MDS sample cases.

The best way to understand the general constructions is by looking at suitable relatively small samples and prototypes; but in general there is no restriction on the length or on the dimension for which MDS LCD codes can be constructed by the method.

**Length 7.** Consider a Fourier \( 7 \times 7 \) matrix \( F_7 \) over a field \( F \). Suitable fields need to be of characteristic not dividing 7 and containing an element of order 7. Thus \( GF(2^3), GF(3^3), GF(5^3), GF(11^2), GF(13^2), \ldots \) are suitable fields over which LCD MDS \((7,r)\) codes may be constructed by the methods. Now \( GF(2^3) \) is of characteristic 2, which is good and may be required, and this also is the smallest field with a primitive \( 7^{th} \) root of unity.

Let \( F_7 \) be a Fourier matrix constructed from a primitive \( 7^{th} \) root of unity \( \omega \). Denote the rows of \( F_7 \) in order by \( \langle e_0,e_1,\ldots,e_6 \rangle \) and the columns of \( F_7^t \) in order by \( \langle f_0,f_1,\ldots,f_6 \rangle \). Then \( e_i f_j = 7 \delta_{ij} \) and \( f_i = e_{7-i}^{-1} \) with the convention that \( e_7 = e_0 \).

Construct \((7,3,5)\) and \((7,5,3)\) MDS LCD codes as follows. Let \( C = \langle e_0,e_1,e_6 \rangle \). Notice that the rows of \( C \) are in sequence \( \{e_6,e_0,e_1\} \) and so satisfy the conditions of Theorem 2.2. Thus \( C \) generates an
(7, 3, 5) MDS code. Check that $C$ generates an LCD code. Now $C \ast (f_2, f_3, f_4, f_5) = 0_{3 \times 4}, (f_2, f_3, f_4, f_5) = (e_5, e_6, e_3, e_2^T)$ (where $\ast$ is matrix multiplication) and hence $C^\perp = \langle e_2, e_3, e_4, e_5 \rangle$. Thus $C \cap C^\perp = 0$ as required. Hence $C$ is an (7, 3, 5) MDS LCD code.

Now let $C = \langle e_0, e_1, e_2, e_5, e_6 \rangle$. Notice that the rows of $C$ are in sequence $\{e_5, e_6, e_0, e_1, e_2\}$ and so satisfy the conditions of Theorem 2.2. Thus $C$ is an (7, 5, 3) MDS code. It is now necessary to show that $C$ generates an LCD code. Now $C \ast (f_3, f_4) = 0_{5 \times 2}, (f_3, f_4) = (e_4, e_5)$ and hence $C^\perp = \langle e_3, e_4 \rangle$. Thus $C \cap C^\perp = 0$ as required.

Other LCD MDS codes may be constructed as follows. Note gcd(3, 7) = 1. Thus let $C = \langle e_0, e_1, e_3, e_6, e_7 - 6, e_7 - 3 \rangle = \langle e_1, e_4, e_6, e_9 \rangle$. Thus $C$ satisfies the conditions of Theorem 2.1 and so is an LCD code. It is also an LCD code as is easily checked since $e_7$ is included so is $e_7 - 1$. Thus this $C$ is an LCD MDS (7, 5, 3) code.

The method allows the construction of $\phi(7) = 6$ MDS LCD codes of forms (7, 5, 3).

Consider now constructing an LCD MDS code $(7, r)$ where $r$ is even. Consider $r = 4$. Let $C = \langle e_1, e_3, e_4, e_6 \rangle$. This is given in sequence $\langle e_1, e_4, e_6, e_9 \rangle$ with arithmetic difference 2 and gcd(7, 2) = 1. Hence by Theorem 2.1 $C$ is an LCD code. It is also an LCD code and is therefore an MDS LCD (7, 4) code.

Is there a prime field with an element of order 7? Yes $GF(29)$ has an element of order 7. Then $\omega = (7 \mod 29)$ has order 7 in $GF(29) = \mathbb{Z}_{29}$ and may be used to form a Fourier $7 \times 7$ matrix in $GF(29)$. The calculations are then done in modular arithmetic – in $\mathbb{Z}_{29} = GF(29)$.

**Length 11.** Suppose a length 11 LCD mds code is required. Let $F_{11}$ be a Fourier $11 \times 11$ matrix over some field $F$. Appropriate fields include $GF(2^{10}), GF(3^3), GF(23)$ since an element of order 11 exists in these fields. Note that $GF(23) = \mathbb{Z}_{23}$ is a prime field and the arithmetic is modular arithmetic; for example $2 \mod 23$ has order 11 in $GF(23)$ and this (modular) element could be used to generate the Fourier matrix.

Now $(11, 9, 3), (11, 7, 5), (11, 5, 7), (11, 3, 9)$ MDS LCD codes can be generated codes with $F_{11}$.

Denote the rows in order of $F_{11}$ by $\{e_0, e_1, \ldots, e_{10}\}$ and the columns in order of $F_{11}^T$ by $\{f_0, f_1, \ldots, f_{10}\}$.

As noted $e_i f_j = 11 \delta_{ij}, e_i^T = f_{11 - i}$.

Now let $C = \langle e_0, e_1, e_2, e_3, e_4, e_7, e_8, e_9 \rangle$. Then $C$ can be given in sequence $\{e_7, e_8, e_9, e_{10}, e_0, e_1, e_2, e_3, e_4\}$ and hence satisfies the criteria of Theorem 2.1 and thus is a (11, 9, 3) MDS code. Then $C \ast (f_5, f_6) = 0_{9 \times 2}$. Now $(f_5, f_6) = (e_6, e_5^T)$ and hence $C^\perp = \langle e_5, e_6 \rangle$ and so $C \cap C^\perp = 0$. Thus $C$ is an (11, 9, 3) MDS LCD code.

The $(11, 7, 5), (11, 5, 7), (11, 3, 9)$ MDS LCD required codes are constructed similarly.

The methods as noted allows the construction of $\phi(11) = 10$ LCD MDS codes of each type.

By methods/algorithms of [2] the codes have efficient encoding and decoding algorithms.

Also $(11, r)$ MDS LCD codes for $r$ even are produced using $\langle e_1, e_3, \ldots, e_8, e_{10} \rangle$ similar to the methods in the case of length 7.

**Exercises:** Over what finite fields do $11 \times 11$ Fourier matrices exist? What is the smallest non-prime field over which an $11 \times 11$ Fourier matrix exists? (See Section 2.2)

**Relatively large sample with modular arithmetic.** Consider $GF(257) = \mathbb{Z}_{257}$ and 257 is prime. Construct the Fourier matrix $F_{256}$ with a primitive 256th root of unity $\omega$ in $GF(257)$. Since the order of 3 mod 257 is 256 then a choice for $\omega$ is $\omega = (3 \mod 257)$. Denote the rows of $F_{256}$ in order by $\{e_0, e_1, \ldots, e_{255}\}$.

Suppose a dimension $r = 2t + 1$ is required $t \geq 0$. Choose $C = \langle e_0, e_1, \ldots, e_t, e_{256-t}, e_{256-t-1}, \ldots, e_{255} \rangle$.

Then $C$ generates an MDS LCD $(256, r)$ code. The arithmetic is modular arithmetic, $mod 257$, and is dealing with powers of $\mod 257$.

Note for example that $(5 \mod 257)$ or $(7 \mod 257)$ could also be used to generate the Fourier $256 \times 256$ matrix.

The method allows the construction of $\phi(256) = 128$ such LCD MDS $(256, 2r + 1)$ codes. For larger primes the number that can be construction is substantial and cryptographica methods can be devised. For example for the prime $p = 2^{11} - 1$ the Fourier $(p - 1) \times (p - 1)$ Fourier matrix exists over $GF(p)$ and $\phi(p - 2) = 534600000$. 

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2.8 To required dimension and error-correcting capability

Let a dimension \( k \) be given and it is required to construct LCD MDS codes with this dimension and required error-correcting capability as per [6].

Let \( k \) be the required dimension and LCD MDS codes of the form \((n, r, d)\) are required where \( \lceil \frac{d}{n} \rceil \) is the required error-correcting capability.

By the bound for codes it follows that \( d \leq n - k + 1 \) from which \( n \geq k + d - 1 \). So \( n = k + d - 1 \) is the smallest the length \( n \) can be.

Consider then \( n = k + d - 1 \). Construct a Fourier \( n \times n \) matrix over a finite field. Such matrices may be constructed over a field of characteristic not dividing \( n \) and of large enough order so that the field contains an element of order \( n \), see section 2.6. If \( k \) is odd or if \( n \) is odd then by the general method of Section 2.2 LCD MDS codes may be constructed of the form \((n, k)\). The distance of the code is \( n - k + 1 = d \) as required. If both \( n \) and \( k \) are even then \( n = k + d - 1 \) implies that \( d \) is odd. Replace \( d \) by \( d + 1 \) which is even but also \( \lceil \frac{d}{n} \rceil = \left\lfloor \frac{d}{r} \right\rfloor \) for odd \( d \) require a length \( n = k + d \) which is odd. Now construct LCD MDS \((n, k)\) codes by the methods of Section 2.2 general which has distance \( n - k + 1 = d + 1 \) which has error-correcting capability \( \left\lfloor \frac{d}{n} \right\rfloor \) as required.

2.8.1 Sample

Consider requiring a dimension \( 7 \) which can correct \( 3 \) errors. Then a \((n, 7, 7)\) code requires \( n \geq 13 \). Let \( n = 13 \). Now construct a \( 13 \times 13 \) Fourier matrix over a finite field. Then by method of subsection 2.2 construct the LCD MDS code \((13, 7, 7)\). The dimension is \( 7 \) as required and the error-correcting capability is \( \left\lfloor \frac{13}{7} \right\rfloor = 3 \) as required.

Over which fields may the Fourier \( 13 \times 13 \) matrix be constructed? See section 2.6. For characteristic \( 2 \) the field required is \( GF(2^{12}) \) as the order of \( 2 \) mod \( 13 \) is \( 12 \). The field is large. Now the order of \( 3 \) mod \( 13 \) is \( 3 \) and so the field \( GF(3^2) \) could be used which is nice.

By methods of [4] (Proposition 3) this \((13, 7, 7)\) LCD MDS code over \( GF(3^2) \) could be expanded to a \((39, 21, \geq 7)\) LCD code over \( GF(3) \). However LCD MDS \((39, 21, 19)\) codes may be obtained over other fields, for example over \( GF(2^{12}) \) or \( GF(5^4) \), by the general methods of 2.2.

Now \( GF(53) \) contains an element of order \( 13 \) so this prime field \( GF(53) = \mathbb{Z}_{53} \) could be used with modular arithmetic. Then \( \omega = (10 \mod 53) \) has order \( 13 \) and so this (modular) element could be used to generate the Fourier \( 13 \times 13 \) matrix over \( GF(53) \). There are other (modular) elements of order \( 13 \) in \( GF(53) \) from which the Fourier \( 13 \times 13 \) matrix can be constructed.

2.8.2 Samples 2

Suppose a dimension \( 227 \) is required which can correct \( 14 \) errors. Thus a \((n, 227, 29)\) code is required. Now \( n \geq 227 + 29 - 1 = 255 \). Let \( n = 255 \). Construct a Fourier \( 255 \times 255 \) matrix over a finite field and then by general method of Subsection 2.2 construct the LCD MDS \((255, 227)\) code over this field; this has distance \( 29 \) and error-correcting capability of \( 14 \).

Over what fields may the Fourier \( 255 \times 255 \) matrix be constructed? Now the order of \( 2 \) mod \( 255 \) is \( 8 \) so the field \( GF(2^8) \) may be used and has an element of order \( 255 \) which in this case is a primitive element in \( GF(2^8) \).

In the prime field \( GF(257) \) the Fourier \( 256 \times 256 \) exists. By method of Subsection 2.2 construct the \((256, 227)\) LCD MDS code over \( GF(257) \). This has distance \( 30 \) and so has error-correcting capability of \( \left\lfloor \frac{256}{227} \right\rfloor = 14 \) also. Then \( \omega = (3 \mod 257) \) has order \( 256 \) in \( GF(257) \) and may be used to generate the Fourier \( 256 \times 256 \) matrix over \( GF(257) = \mathbb{Z}_{257} \).

2.9 To given rate and error-correcting capability

Suppose an LCD MDS code of rate \( R = \frac{n}{n} \) is required which can correct \( \geq t \) errors. In reduced fraction form \( n \) or \( r \) is odd; both could be odd. Define \( n_i + i \times n_r + i \times r \) for odd \( i \). Let \( i \) be the least such that \( t \leq \left\lfloor \frac{2n - 1}{2} \right\rfloor \). Construct the Fourier \( n_i \times n_r \) matrix over a suitable finite field and from this construct by the general method the MDS LCD \((n_i, r_i)\) code. The rate is \( R \) and the error correcting capability is \( \left\lfloor \frac{n_i - r_i}{2} \right\rfloor \geq t \). Other more direct ways may be applied in particular instances.
2.9.1 Sample: Rate $\frac{8}{7}$ required.

Suppose a rate $R = \frac{8}{7}$ is required which can correct 25 errors. Thus an MDS LCD code $(n, r, \geq 51)$ is required where $\frac{n}{n-r+1} = \frac{8}{7}$. This gives $n = n-r+1 = n(1-R)+1$ thus requiring $50 = n(\frac{8}{7}) \geq 50$. This requires $n = 175$ and $r = 125$. So construct a $175 \times 175$ Fourier matrix and define $C = (e_0, e_1, \ldots, e_{173}, e_{174}, \ldots, e_{174})$. The rows are in sequence and hence $C$ is an $(175, 125, 51)$ MDS code by Theorem 2.2. That it is also an LCD code follows from the general method.

Exercise: Over which finite fields can a Fourier $175 \times 175$ be constructed? What is the smallest finite field over which a Fourier $175 \times 175$ can be constructed?

2.9.2 Sample: Rate $\frac{5}{7}$ required.

Suppose a rate $R = \frac{5}{7}$ is required which can correct 25 errors. Require a code $(n, r, \geq 51)$ such that $\frac{n}{n-r+1} = \frac{5}{7}$. Then require $n-r+1 = n(1-R)+1 = n(1-\frac{5}{7})+1 = n(\frac{2}{7})+1 \geq 51$. Thus $n = 400$, $r = 350$. Hence a $(400, 350, 51)$ code is required or better. Now the general method requires the dimension to be odd. Thus require a $(400, 349, 52)$ code which has slightly less rate than $R$, which is okay for a system which can transmit at rate $R$, and this can correct 25 errors.

Now 401 is prime so the field $GF(401)$ has an element of order 400 from which the Fourier $400 \times 400$ matrix may be defined over the prime field $GF(401)$. A suitable element of order 400 in $GF(401)$ is (3 mod 401).

A code with characteristic 2 may be required so then consider constructing a $(401, 351, 51)$ code which has rate slightly greater than $\frac{5}{7}$ or a $(401, 349, 53)$ code which has rate slightly less than $\frac{5}{7}$. Here though the order of 2 mod 401 is 200 so would require a field $2^{200}$ to get an element of order 401. However the order of 2 mod 399 is 18 requiring a field $GF(2^{18})$ so try to be satisfied with length 399!

Form the Fourier $399 \times 399$ matrix in $GF(2^{18})$ and use the general method; then deduce an MDS LCD $(399, 349, 51)$. This has rate $0.87468..$ and can correct 25 errors.

Using the general method as described requires a $(8i, 7i)$ MDS LCD code such that $8i - 7i \geq 50$. This requires $i \geq 50$ but requiring odd $i$ gives $i = 51$ as the least such. Thus construct a $(408, 357, 51)$ LCD MDS code of rate $\frac{51}{50}$ and can correct 25 errors. Now 409 is prime so the field $GF(409)$ may be used in which there exists a 408 root of unity. A solution is $\omega = (21 \mod 409)$ and the order of this $\omega$ is 408 in $GF(409)$ and may be used to construct the $408 \times 408$ Fourier matrix over $GF(409)$ from which the MDS LCD $(408, 357)$ code may be derived.

2.9.3 Rate $R = \frac{4}{7}$ in prescribed field type.

Suppose a rate of $R = \frac{4}{7}$ is required in a field of characteristic 2 for an MDS LCD code which can correct at least 25 errors. An $(n, r, n-r+1)$ MDS LCD code is required and the $n$ must be odd in order to find characteristic 2 field with an element of order $n$. Then $d = n-r+1 \geq 51$ in order to correct 25 errors. Thus $n(1-\frac{4}{7}) \geq 50$ which means $n \geq 250$. Take $n = 255$ and then the field $GF(2^8)$ has an element of order 255. Now $255 \times \frac{4}{7} = 204$. But we need the dimension to be odd in order for the general method to work to get an LCD code. Take $r = 203$ and then the $(255, 203, 53)$ code has rate $\frac{203}{255} = 0.7960..$ which is slightly less than $\frac{4}{7}$, which is fine as the given system can transmit at rate $\frac{4}{7}$.

Thus construct the Fourier $255 \times 255$ code over $GF(2^8)$ using the primitive 255 root of unity. Then construct the MDS LCD $(255, 203)$ code by the general method above:

$C = (e_0, e_1, \ldots, e_{101}, e_{154}, e_{155}, \ldots, e_{254})$. Then $C$ is an MDS LCD $(255, 203)$ code over $GF(2^8)$.

Suppose a rate of at least $R = \frac{4}{7}$ is required in a prime field for an MDS LCD code which correct at least 25 errors. An $(n, r, n-r+1)$ MDS LCD code is required. Then $d = n-r+1 \geq 51$ in order to correct 25 errors. Thus $n(1-\frac{4}{7}) \geq 50$ which means $n \geq 250$. The next prime $\geq n+1 = 257$. This has an element of order 256. Now $\frac{4}{7} \times 256$ is required for the dimension which must be odd for the construction in section 2.2 and hence require $r = 205$. Then the code $(256, 205, 51)$ created over $GF(257)$ has rate $\frac{205}{256}$ which is 0.8007.. only slightly greater than the required rate.

3 Asymptotics

Construct an infinite series of LCD MDS codes $(n_i, r_i)$ codes where the rate $\frac{n_i}{r_i} = R = \frac{4}{7}$ (fixed) with $0 < R < 1$. Then for such a series it would follow that $\lim_{i \to \infty} = 1 - R$. 

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We can assume that either \( r \) or \( n \) in the fraction \( R = \frac{r}{n} \) is odd.

From a Fourier \( n \times n \) matrix construct an MDS LCD \((n, r)\) code; as either \( n \) or \( r \) is odd this can be done by the general method of section 2.2. Let \( n_1 = n, r_1 = r \) and define \( n_i = i \times n \) and \( r_i = i \times r \) for odd \( i \) that is for \( i = 1, 3, 5, \ldots \). Then either \( n_i \) or \( r_i \) is odd. Construct the \( n_i \times n_i \) Fourier matrix over some finite field. By the general method, from this Fourier matrix construct an \((n_i, r_i)\) LCD MDS code. The rate of the code is \( R = \frac{r_i}{n_i} \). The distance of the code is \((n_i - r_i + 1)\) and the ratio of the distance to the length is \( \frac{n_i - r_i + 1}{n_i} = 1 - R + \frac{1}{n_i} \). As \( i \to \infty \) this ratio of the distance to the length approaches \((1 - R)\).

Note that \( 0 < R < 1 \) if and only if \( 0 < (1 - R) < 1 \) so could obtain an infinite series of LCD MDS codes in which the ratio of the distance to the length is required to approach a specific \( R \) with \( 0 < R < 1 \).

The explicit codes produced have efficient encoding and decoding algorithms by the algorithmic methods in [7].

### 3.1 Modular arithmetic; prime fields

Let \( p \) be a prime and consider \( GF(p) \). Then The Fourier matrix of size \((p - 1) \times (p - 1)\) may be formed over \( GF(p) \). Suppose a rate \( R \) LCD MDS code is required over \( GF(p) \). Let \( r = \lfloor (p - 1) \ast R \rfloor + 1 \) or \( r = \lfloor (p - 1) \ast R \rfloor \) so that \( r \) turns out to be odd. Construct the \((p - 1, r)\) LCD MDS code by the general method.

Consider an infinite set of prime \( p_1, p_2, \ldots \). Then construct for each \( p_i \) an LCD MDS code \( C_i \) close to the given rate \( R \) over the field \( GF(p_i) \). Then the limit of these codes as \( i \to \infty \) is \((1 - R)\).

For example take all the primes of the form \( 4n + 1 \), namely 5, 13, 17, 29, \ldots and require a rate of \( R = \frac{3}{4} \). In \( GF(4n + 1) = \mathbb{Z}_{4n + 1} \) construct the Fourier \( 4n \times 4n \) matrix using a primitive \((4n)^{th}\) root of unity. Let \( r = 3n \) if \( n \) is odd and \( r = 3n - 1 \) when \( n \) is even. By the general method construct LCD MDS \((4n, r)\) codes. These are LCD MDS codes \((4, 3, 2), (12, 9, 4), (16, 11, 6), (28, 21, 8), (36, 27, 10), (40, 29, 12), (52, 39, 14) \ldots over the prime fields \( GF(5), GF(13), GF(17), GF(29), GF(37), GF(41), GF(53) \ldots \) respectively. If the exact rate \( \frac{3}{4} \) is required then only include primes of the form \( 4n + 1 \) where \( n \) is odd. The limit of the ratio of the distance to the length of this series of codes is \((1 - R) = \frac{1}{4} \).

### 3.2 Characteristic 2

Suppose characteristic 2 is required. Then it is required that the \( n_i \) are odd so that a Fourier \( n_i \times n_i \) may be constructed over a field of characteristic 2. Thus for \( R = \frac{r}{n} \) it is required that \( n \) be odd and then proceed as in section 3.1.

Here’s an example with characteristic 2. Let \( R = \frac{1}{2} \). An infinite series of LCD MDS with this ratio is required where the ratio of the distance to the length approaches \((1 - R)\). Let \( n_i = i \times 7 \) and \( r_i = i \times 5 \) for \( i = 1, 3, 5, \ldots \). Now \( n_i \) is odd so there exists a Fourier \( n_i \times n_i \) matrix in a field of characteristic 2. Construct the LCD MDS code \((n_i, r_i)\) over this field by the general method of section 2.2. The code has rate \( \frac{1}{4} \) for all \( i \). As \( i \to \infty \) the ratio of the distance to the length approaches \((1 - \frac{1}{4}) = \frac{3}{4} \). The codes are the LCD MDS codes \(\{(7, 5), (21, 15), (35, 25)\ldots\} \) over different fields of characteristic 2. The \((7, 5)\) code could be over \( GF(2^3) \), the \((21, 15)\) code could be over \( GF(2^6) \), and the \((35, 25)\) code could be over \( GF(2^{12}) \). The \((63, 35)\) LCD MDS code could be constructed over \( GF(2^6) \) as the order of 2 mod 63 is 6.

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