Nonselfdual solutions for gauge fields in Schwarzschild and deSitter backgrounds for dimension $d \geq 4$.

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Abstract

A particularly simple class of nonselfdual solutions are obtained for gauge fields in Schwarzschild and deSitter backgrounds. For Lorentz signature these have finite energy and finite action for Euclidean signature. In each case one obtains either real or a pair of complex conjugate solutions. The actions are easily computed for any dimension $d$. Numerical values are given for $d = 4, 6, 7, 8, 9, 10$. It is explained why $d = 5$ is a very special case. Possible continuations and generalizations of the results obtained are indicated. A particular solution for $AdS_4$ background is presented in the Appendix.
1 Introduction:

In a previous paper [1], almost twenty years ago, we presented very simple, nonselfdual solutions for gauge fields in Schwarzschild and deSitter backgrounds in four dimensions. They were further discussed in Refs. 2 and 3. Here they are generalized to all dimensions \( d \geq 4 \), with Lorentz or Euclidean signature and \((d - 1)\) spatial dimensions. A solution in anti-deSitter background, restricted to \( d = 4 \), is studied in the Appendix.

Let us briefly recapitulate our previous results. The necessary steps for higher dimensions, as will be seen in the next section, will then be straightforward.

The background metrics are, for \( d = 4 \), in standard notations,

\[
ds^2 = \mp N dt^2 + N^{-1} dr^2 + r^2 d\Omega^2,
\]

where respectively,

\[
N = \left(1 - \frac{2M}{r}\right) \quad (\text{Schwarzschild}),
\]

and

\[
N = \left(1 - \Lambda r^2\right) \quad (\text{deSitter})
\]

The \( SU(2) \) generators being, in terms of the Pauli matrices

\[
\sigma_{ij} = \epsilon_{ijk} \left(\frac{1}{2} \sigma_k\right)
\]

the gauge fields, for our static, spherically symmetric ansatz, are given by

\[
A_0 = 0 \\
A_i = r^{-2} \left(K(r) - 1\right) \sigma_{ij} x_j = r^{-2} \left(K(r) - 1\right) \epsilon_{ijk} \left(\frac{1}{2} \sigma_k\right) x_j
\]

Finite energy (action) is obtained for Lorentz (Euclidean) signature, corresponding to (1.2) and (1.3) respectively, for

\[
K = \frac{r + 2Ma}{r + 2Mb}, \quad (a = -2.366, b = 4.098)
\]

and

\[
K = \frac{1 + a\Lambda r^2}{1 + b\Lambda r^2}, \quad (a = \pm i1.732, , b = -0.857 \mp i0.742)
\]

The exact values of \( a \) and \( b \) can be found in Ref.1. (The simple equations determining such parameters are given in Sec.2.)
For (1.2), when evaluating the energy or the action, apart from the standard angular integration, giving a factor $4\pi$, the radial integral is over the domain $[2M, \infty]$. For Lorentz signature one obtains a finite energy for this outer region. For Euclidean signature time becomes periodic with a period $8\pi M$ and the domain $[2M, \infty]$ for $r$ corresponds (as explained after (1.14)) to the entire one covered by the Kruskal coordinates. In this context one obtains a finite action for (1.5). This action is less than $8\pi^2$. It is

$$S = 8\pi^2(0.959) \quad (1.7)$$

This inequality is emphasized for the following reason. Selfdual and antiselfdual solutions can be obtained directly from the spin connections [1]. Setting

$$A_0 = \pm \left(\frac{1}{2} \frac{dN}{dr}\right)\left(\frac{1}{2} \sigma_i x_i\right)$$
$$A_i = r^{-2} \left(N^{\frac{1}{2}} - 1\right) \left(\frac{1}{2} \epsilon_{ijk} \sigma_k\right)x_j \quad (1.8)$$

one obtains the fundamental selfdual and antiselfdual solutions for the upper and the lower sign respectively. The Euclidean action and the topological index are respectively

$$S = 8\pi^2$$

and

$$P = \pm 1 \quad (1.9)$$

Thus we have a nonselfdual solution with lower action than the (anti)instanton with the lowest nontrivial index. (See the relevant remarks in Sec.4.) Note that for $M = 0$ in (1.5), $K = 1$ and hence $A_\mu = 0$ in (1.4). Thus the flat space limit is trivial. The curved metric is intrinsically necessary for the existence of such a solution.

For the deSitter case the domain of $r$ is the inner region up to the horizon, namely,$$
\left[0, \Lambda^{-\frac{1}{2}}\right] \quad (1.10)$$

For Euclidean signature the time period is $2\pi \Lambda^{-\frac{1}{2}}$. The complex action is, for upper and lower sign in (1.6) respectively,

$$S = 8\pi^2\left(1.755 \mp i4.197\right) \quad (1.11)$$

Such a complex solution can be considered in the context of a complexified gauge group, $SL(2, C)$ for our case. See the remarks and the references in [2]. The possible role of complex saddle points was also discussed in [2].
Before introducing the ansatz for gaugefields in higher dimensions (Sec.2), let us reca-
pitulate briefly some known but directly relevant results concerning the chosen metrics and
the traces of the generators of the gauge group $SO(d - 1)$.

The Schwarzschild metric for $d \geq 4$ in spherical coordinates [4] is given by

$$ds^2 = \mp N dt^2 + N^{-1} dr^2 + r^2 d\Omega_{d-2},$$  \hspace{1cm} (1.12)

where,

$$N = \left(1 - \left(\frac{C}{r}\right)^{(d-3)}\right),$$  \hspace{1cm} (1.13)

and $d\Omega_{(d-2)}$ is the line element on the unit $(d - 2)$-sphere.

Considering directly the Euclidean signature, the Kruskal coordinates are defined by

$$e^{(2kr^*)} = \frac{1}{4}(\eta^2 + \zeta^2)$$  
$$e^{ikt} = \left(\frac{\eta - i\zeta}{\eta + i\zeta}\right)^{\frac{1}{2}}$$  \hspace{1cm} (1.14)

where

$$r^* = \int N^{-1} dr = r + C^{(d-3)} \int \left(r^{(d-3)} - C^{(d-3)}\right)^{-1} dr$$

Using

$$\frac{1}{(x^n - 1)} = \frac{1}{n} \left(\frac{1}{x - 1} - \frac{x^{n-2} + 2x^{n-3} + \ldots + (n - 2)x + (n - 1)}{x^{n-1} + x^{n-2} + \ldots + x + 1}\right)$$  \hspace{1cm} (1.15)

it is easily seen, without evaluating $r^*$ completely in terms of partial fractions, that

$$r^* = r + \frac{C}{(d - 3)} \int \frac{dr}{r - C} + h(r)$$  \hspace{1cm} (1.16)

where the function $h(r)$ plays no role concerning the singularity at the horizon at $r = C$.
This can be verified by constructing $h(r)$ explicitly.

Thus

$$ds^2 = N \left(4k^2 e^{2kr^*}\right)^{-1} (d\zeta^2 + d\eta^2) + r^2 d\Omega_{(d-2)},$$  \hspace{1cm} (1.17)

where

$$e^{-2kr^*} = e^{-2kr} \left(r - C\right)^{-\left(\frac{2kC}{d-3}\right)} e^{-2kh(r)}$$
hence, setting

\[ k = -\frac{(d-3)}{2C} \]  

(1.18)

desingularizes the horizon. The domain \([C, \infty]\) ensures the positive definiteness of \((\eta^2 + \zeta^2)\) in (1.14).

From (1.14) the period is now found to be

\[ P_{(d)} = 2\pi|k|^{-1} = 4\pi C/(d-3) \]  

(1.19)

For \(d = 4\) and \(C = 2M\) one gets back the well known result

\[ P_{(4)} = 8\pi M \]

For the deSitter case, for all \(d\),

\[ N = (1 - \Lambda r^2) \]

and the period remains

\[ \tilde{P} = 2\pi \Lambda^{-\frac{1}{2}} \]  

(1.20)

Another necessary ingredient in evaluating the actions of our solutions will be seen (in Sec.3) to be the traces of the generators of \(SO(d-1)\). This will, of course, depend on the representation chosen. We will not always specify it in the following. But let us note here briefly the results for spinorial constructions of these generators.

Let, for \(i \neq j\),

\[ L_{ij}^{(n)} = -\frac{i}{2} \gamma_i^{(n)}\gamma_j^{(n)} \]  

(1.21)

where \(\gamma_i^{(n)}\) are the \(\gamma\)-matrices for \(n\) spatial dimensions, satisfying the Clifford algebra. Then the \(L\)'s satisfy the \(SO(n)\) algebra with Hermitian convention. Starting with the Pauli matrices for \(n = 3\) and

\[ L_{ij}^{(3)} = -\frac{i}{2} \sigma_i\sigma_j \]  

(1.22)

one proceeds in alternate steps for even and odd dimensions as follows.

Set

\[ \gamma_i^{(2p)} = \sigma_2 \otimes \gamma_i^{(2p-1)} \quad (i = 1, \ldots, 2p - 1) \]  

(1.23)

and

\[ \gamma_{2p}^{(2p)} = \sigma_1 \otimes I_{(2p-2)} \]  

(1.24)
In the next step, one sets, for \( n = (2p + 1) \),

\[
\gamma^{(2p+1)}_i = \gamma^{(2p)}_i \quad (i = 1, \ldots, 2p)
\]

\[
\gamma^{(2p+1)}_{2p+1} = \gamma^{(2p)}_1 \gamma^{(2p)}_2 \cdots \gamma^{(2p)}_{2p} \equiv \Gamma^{(2p)}_{2p+1}
\]  

(1.25)

Here \( \Gamma^{(2p)}_{2p+1} \) is the generalization of the chiral matrix \( \gamma_5 \) for \( n = 4 \).

For even \( n \) one can take the chiral projections

\[
\frac{1}{2} \left( 1 \pm \Gamma^{(2p)}_{2p+1} \right) L^{(2p)}_{ij}
\]

(1.26)

which reduces the dimension by a factor 2.

For such spinorial constructions,

\[
Tr L^{(n)}_{ij} L^{(n)}_{i'j'} = \lambda^{(n)} \delta_{ii'} \delta_{jj'}
\]

(1.27)

where (ordering indices, say, as \( i < j \)) for odd \( n \),

\[
\lambda^{(n)} = 2^{(n-5)/2}
\]

and for even \( n \),

\[
\lambda^{(n)} = 2^{(n-4)/2}
\]

Chiral projections give, for even \( n \),

\[
\lambda^{(n)} = 2^{(n-6)/2}
\]

(1.28)

2 Metrics, ansatz for gauge fields and a class of solutions for \( d \geq 4 \):

We consider spherically symmetric, static metrics for dimensions \( d \geq 4 \). For the ansatz to be introduced below, the Kerr - Schild form of the metrics turn out to be convenient, for computation, to start with. The complications due to the nondiagonal form will be amply compensated by other good properties. In this form one has

\[
g_{\mu\nu} = \eta_{\mu\nu} + l_\mu l_\nu, \quad g^{\mu\nu} = \eta^{\mu\nu} - l^\mu l^\nu
\]

(2.1)

where

\[
\eta_{00} = \eta^{00} = -1, \eta_{ij} = \eta^{ij} = \delta_{ij}
\]

and
\[ l^\mu l^\nu \eta_{\mu\nu} = l^\mu l^\nu g_{\mu\nu} = 0 \]  
\hspace{1cm} (2.2)

For static spherical symmetry \( l_0 \) is a function of \( r \) only, where
\[ r^2 = \sum_{i=1}^{d-1} x_i^2 \]
and
\[ l_i = l_0 \frac{x_i}{r} \quad (i = 1, 2, \ldots, d - 1) \]
satisfying
\[ -l_0^2 + \sum_i l_i^2 = 0 \]  
\hspace{1cm} (2.3)

For Schwarzschild metric in \( d \) dimensions
\[ l_0^2 = (C/r)^{d-3} \quad (C > 0) \]  
\hspace{1cm} (2.4)

and for deSitter metric, for all \( d \),
\[ l_0^2 = \Lambda r^2 \quad (\Lambda > 0) \]  
\hspace{1cm} (2.5)

In this paper, seeking simplicity, we present explicit solutions for these two cases only. The Reissner-Nordstrom case with
\[ l_0^2 = (C/r)^{d-3} - (D/r^2)^{d-3} \]  
\hspace{1cm} (2.6)
will be excluded. The standard form in spherical coordinates is given by
\[ ds^2 = -Ndt^2 + N^{-1}dr^2 + r^2d\Omega_{(d-2)}, \]  
\hspace{1cm} (2.7)

where \( d\Omega_{(d-2)} \) is the line element on the unit \((d - 2)\)-sphere and
\[ N = \left(1 - l_0^2\right) \]  
\hspace{1cm} (2.8)

(For \( d = 4 \), for example, \( C = 2M \) and \( D = P^2 + Q^2 \) give back the well-known forms.) The coordinate transformation relating (2.1) an (2.7), namely
\[ x_0 = t + \int \frac{dr}{N} - r \]  
\hspace{1cm} (2.9)
does not affect our particularly simple ansatz for the gauge potentials to follow (with \( A_t = A_r = 0 \)).

*After* constructing the solutions using (2.1) the passage to Euclidean signature is best considered (rather than introducing imaginary \( l_0 \)) by directly starting from (2.7), leading to

\[
ds^2 = N dt^2 + N^{-1} dr^2 + r^2 d\Omega_{(d-2)}
\]  

(2.10)

The ansatz for the gauge potentials is

\[
  A_0 = 0
  \]

\[
  A_i = r^{-2} \left( K(r) - 1 \right) L_{ij} x_j \quad (i = 1, 2, \ldots, d - 1)
\]  

(2.11)

Here \( L_{ij} = -L_{ji} \) are hermitian \( SO(d - 1) \) rotation matrices satisfying

\[
  [L_{ij}, L_{i'j'}] = i \left( \delta_{ii'} L_{jj'} + \delta_{jj'} L_{ii'} - \delta_{ij} L_{ij'} - \delta_{ji} L_{ji'} \right)
\]  

(2.12)

For constructing solutions the Lie algebra is sufficient. But for evaluating actions one must specify the chosen representation before computing traces. Of particular interest are the cases discussed in Sec.1. But one can implement other representations, if so desired.

Defining

\[
  F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]
\]  

(2.13)

one obtains

\[
  F_{0i} = 0
  
  F_{ij} = -r^{-2} \left( K^2 - 1 \right) L_{ij} + r^{-4} \left( r \frac{dK}{dr} - (K^2 - 1) \right) \left( x_i (L_{jk} x_k) - x_j (L_{ik} x_k) \right)
\]  

(2.14)

Now using (2.1),

\[
  F^{0i} = g^{0\mu} g^{i\nu} F_{\mu\nu} = r^{-1} l_0^2 \frac{dK}{dr} (L_{ik} x_k)
\]

and

\[
  F^{ij} = g^{i\mu} g^{j\nu} F_{\mu\nu} = W_1 L_{ij} + W_2 \left( x_i (L_{jk} x_k) - x_j (L_{ik} x_k) \right)
\]  

(2.15)

where

\[
  W_1 = -r^{-2} \left( K^2 - 1 \right)
\]

and

\[
  W_2 = r^{-4} \left( (1 - l_0^2 r) \frac{dK}{dr} - (K^2 - 1) \right)
\]  

(2.16)

The Euler-Lagrange equations of motion are, since \( |g| = 1 \) for the KS metric,
\[ D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + i [A_\mu, F^{\mu\nu}] = 0 \] (2.17)

It can be shown that
\[ D_\mu F^{\mu0} = 0 \]

and
\[ D_\mu F^{\mu j} = r^{-4} (r^{-(d-6)} \frac{d}{dr} (N r^{(d-4)} \frac{dK}{dr}) - (d - 3) K (K^2 - 1)) (L_{jk} x_k) \] (2.18)

Hence the equations of motion reduce to a single constraint (with \( N = 1 - l_0^2 \)),
\[ \frac{d}{dr} (N r^{(d-4)} \frac{dK}{dr}) = (d - 3) r^{(d-6)} K (K^2 - 1) \] (2.19)

The factor \((d-3)\) on the right corresponds to the fact that for \(d = 3\) one has the Abelian case with only \( L_{12} \). We exclude this by setting \( d \geq 4 \). The result (2.19) is obtained less simply by using directly spherical coordinates and (2.7). The equation (2.19) does not satisfy the Painleve criterion [5]. Though we cannot provide a general analysis, surprisingly simple particular solutions have been found. Interesting solutions can be found, not for flat space with \( N = 1 \), but remarkably for the curved ones given by (2.4) and (2.5). The special features arising for \( d = 5 \) will be discussed below.

### 2.1 Case 1. Schwarzschild background:

For
\[ N = (1 - (C/r)^{d-3}) \]

set
\[ K = \frac{r^{(d-3)} + aC^{(d-3)}}{r^{(d-3)} + bC^{(d-3)}} \] (2.20)

Inserting in (2.19) one obtains
\[ 3a + b(2d - 7) + (d - 1) = 0 \]
\[ a(a + b) - (d - 5)b = 0 \] (2.21)

This involves essentially solving only a quadratic in \( a \) or \( b \). For \( d = 4 \) our old results [1] are reproduced. Of the two real solutions only the one with
\[ a = -2.366, b = 4.098 \] (2.22)
gives finite energy (action) for Lorentz (Euclidean) signature.
For $d = 5$ one has a very special case, as is evident from the second equation of (2.21). Now the solution

$$a = 0, b = -(4/3)$$

leads to a divergent action since the domain of $(r/C)$ is $[1, \infty]$ and this includes a zero of the denominator of $K$. But now one can also consider the flat limit as follows.

Setting, for $(d = 5),$

$$a = \hat{a}/C^2, b = \hat{b}/C^2$$

the set (2.21) reduces to

$$3(\hat{a} + \hat{b}) + 4C^2 = 0$$
$$\hat{a}(\hat{a} + \hat{b}) = 0$$

Hence as $C \to 0$, setting ($\delta$ being an arbitrary real number)

$$-\hat{a} = \hat{b} = \delta^2$$

one obtains

$$(K - 1) = -\frac{2\delta^2}{r^2 + \delta^2}$$

Substituting this in (2.11) it is seen that for the convention, say,

$$\epsilon_{1234} = 1, \quad L_{12} = -L_{34} \quad (cyclic)$$

for the chirally projected $2 \times 2$ $SO(4)$ generators one obtains the famous BPST selfdual solution in $d = 4$ as a static one in $d = 5$ via our limiting process. Another convention gives the antiselfdual form.

From $d = 6$ onwards the solutions become complex. The corresponding finite complex action, or energy, will be obtained in the following section. The exact values can be obtained, for any $d$, immediately from (2.21). Some numerical values, giving a direct idea of variation with $d$, are given below. Both upper or both lower signs are to be taken. For

$$d = 6: \quad a = 0.500 \pm i 1.500, \quad b = -1.300 \mp i 0.900$$
$$d = 7: \quad a = 0 \pm i 1.732, \quad b = 0.857 \mp i 0.742$$
$$d = 8: \quad a = -0.167 \pm i 1.863, \quad b = -0.722 \mp i 0.621$$
$$d = 9: \quad a = -0.250 \pm i 1.984, \quad b = -0.659 \mp i 0.541$$
$$d = 10: \quad a = -0.300 \pm i 2.100, \quad b = -0.623 \mp i 0.485$$

2.2 Case 2. deSitter background:

For

$$N = (1 - \Lambda r^2)$$
one can satisfy (2.19) by setting

\[ K = \frac{1 + a\Lambda r^2}{1 + b\Lambda r^2} \]  (2.29)

with

\[ a(a + b) + 2(d - 5)b = 0 \]
\[ 3(d - 3)a - (d - 11)b + 2(d - 1) = 0 \]  (2.30)

Exact solutions can again be obtained by solving a quadratic in \( a \) or \( b \). Approximate numerical values are presented below.

For \( d = 4 \) our old results [1,2] are reproduced with

\[ a = \pm i1.732, \quad b = -0.857 \mp i0.742 \]  (2.31)

For the special case \( d = 5 \) the equations reduce to

\[ a(a + b) = 0 \]
\[ 3(a + b) + 4 = 0 \]  (2.32)

The one consistent solution

\[ a = 0, \quad b = -(4/3) \]  (2.33)

leads to divergence in \( K \) since (see Sec.3) one considers the domain \( 0 \leq \Lambda r^2 \leq 1 \). A flat space limit can again be considered as for the Schwarzschild case. This needs no repetition.

From \( d = 6 \) onwards (exhibiting a behavior complementary to the Schwarzschild case) the solutions become real. From \( d = 6 \) to \( d = 10 \), as will be seen in Sec.3, one has two real solutions, both giving finite action (or energy). for \( a \) and \( b \) one obtains the following results. For \( d = 6 \):

1. \( a = 0.807, \quad b = -0.547 \)
2. \( a = -6.193, \quad b = 9.147 \)

For \( d = 7 \):

1. \( a = -0.910, \quad b = -0.268 \)
2. \( a = -6.589, \quad b = 16.768 \)

For \( d = 8 \):

1. \( a = -0.901, \quad b = -0.159 \)
2. \( a = -7.765, \quad b = 34.159 \)  (2.34)
For $d = 9$:

(1) $a = -0.877$, $b = -0.108$

(2) $a = -9.123$, $b = 74.108$

For $d = 10$:

(1) $a = -0.853$, $b = -0.080$

(2) $a = -10.547$, $b = 203.480$

In the set (1) the negative value of $b$ satisfy $|b| \leq 1$ and in the set (2) $b > 0$. Thus in both cases divergence will be seen to be avoided (Sec.3). One may notice that variations (with $d$) are relatively small for the solutions (1) as compared to the solutions (2). It will be seen however (Sec.3) that the values of the action (or energy) for the two sets remain quite close for each $d$.

For $d = 11$ there is a sudden change. There is only one real solution with

$$a = -0.833, \quad b = 0.062$$

(2.35)

If the numerical values of $(a, b)$ for some particular $d$ are are found to be of interest in some context, they can be immediately obtained from (2.30).

### 3 Action (energy) for Euclidean (Lorentz) signature:

Since $|g| = 1$ for (2.1) and $F_{i0} = 0$ for our ansatz, for both signatures one computes, to start with,

$$\frac{1}{2} Tr \int dV_{(d-1)} (F_{ij} F^{ij})$$

(3.1)

where $dV_{(d-1)}$ is the volume element for the spatial dimensions. For our static, spherically symmetric ansatz the angular integrations merely give a factor equal to the surface area of the unit $(d-2)$-sphere, namely,

$$\Sigma_{(d-2)} = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}$$

(3.2)

The radial integration corresponds, for Schwarzschild and deSitter backgrounds respectively, to the domains

$$[C, \infty]$$

(3.3)

and

$$[0, \Lambda^{1\over 2}]$$

(3.4)

For Lorentz signature (3.3) corresponds to the outer region down to the horizon at $r = C$. For Euclidean signature it corresponds to the full domain of reality of the Kruskal-type
coordinates (Sec.1). For both signatures (3.4) corresponds to the inner region bounded by
the horizon at \( r = \Lambda^{\frac{1}{2}} \).

For Lorentz signature one thus obtains a finite energy for our solutions. For Euclidean
signature the time becomes periodic, for the Schwarzschild and the deSitter metrics re-
spectively, with a period (Sec.1),

\[
P_{(d)} = \frac{4\pi C}{(d - 3)}
\]  
\[
\tilde{P} = \frac{2\pi}{\Lambda^{\frac{1}{2}}}
\]  

Hence by multiplying the integral (3.1) by \( P_{(d)} \) and \( \tilde{P} \) respectively one obtains
now the total action. The radial integral will be real or complex according to the
background and the dimension considered. For complex solutions one can obtain a real
action by considering a doubled block-diagonal form of the rotation matrices \( L_{ij} \) and thus
 treating the complex conjugate solutions together. Keeping various possibilities in mind,
at this stage, let us set (using the ordering \( i < j \) to avoid ambiguities)

\[
\text{Tr}(L_{ij}L_{i'j'}) = \lambda_{(d)} \delta_{ii'}\delta_{jj'}
\]  

(See Sec.1 for evaluation of \( \lambda_{(d)} \) corresponding to spinorial constructions, setting
\( d = (n+1). \)) To evaluate (3.1) we need the traces, with \( (i, j = 1, \ldots, d - 1) \),

\[
T_1 = \text{Tr}(L_{ij}L_{ij})
\]
\[
T_2 = -\lambda_{(d)}(d - 2)
\]
\[
T_3 = \lambda_{(d)}(d - 2)
\]  

Using (2.14) and (3.9) one obtains after grouping terms,

\[
\frac{1}{2} \text{Tr}F_{ij}F^{ij} = \frac{\lambda_{(d)}}{2}(d - 2)(d - 2)r^{-4}\left(2N\left(r\frac{dK}{dr}\right)^2 + (d - 3)(K^2 - 1)^2\right)
\]  

3.1 The radial integral:

(1) For the Schwarzschild case the radial integral is
\[ \int_C drr^{(d-6)} \left( 2N \left( r \frac{dK}{dr} \right)^2 + (d - 3) \left( K^2 - 1 \right)^2 \right) \]

\[ C^{(d-5)} \int_1^\infty dxx^{(d-6)} \left( 2N \left( x \frac{dK}{dx} \right)^2 + (d - 3) \left( K^2 - 1 \right)^2 \right) \equiv C^{(d-5)} I_{(d)} \]

where now

\[ N = \left( 1 - x^{-(d-3)} \right), \quad K = \frac{x^{(d-3)} + a}{x^{(d-3)} + b} \]

The total Euclidean action, for example, is

\[ \frac{1}{2} P_{(d)} \Sigma_{(d-2)} \lambda_{(d)} (d - 2) C^{(d-5)} I_{(d)} \]

For \( d = 4 \)

\[ P_{(4)} = 4\pi C = 8\pi M, \quad \Sigma_{(2)} = 4\pi, \quad \lambda_{(4)} = (1/2) \]

leading to a total action

\[ 8\pi^2 I_{(4)} = 8\pi^2 (0.959) \]

which reproduces exactly our old result [1].

Leaving aside the exceptional case \( d = 5 \), we give below the values of \( I_{(d)} \) for the solutions \((2.28)\). \( I_{(d)} \) can be expressed in terms of standard integrals for the general case. Direct numerical evaluation gives, corresponding to the upper and the lower sign in \((2.28)\) respectively,

\[ I_{(6)} = 7.605 \mp i9.141 \]
\[ I_{(7)} = 7.021 \mp i16.788 \]
\[ I_{(8)} = 5.217 \mp i25.378 \]
\[ I_{(9)} = 2.109 \mp i34.906 \]
\[ I_{(10)} = -2.320 \mp i45.312 \]

Note that the real part becomes negative for \( d = 10 \), indicating dominant contribution of terms bilinear in the imaginary parts.

(2) For the deSitter case the radial integral is

\[ \int_0^\Lambda^{-1/2} drr^{(d-6)} \left( 2N \left( r \frac{dK}{dr} \right)^2 + (d - 3) \left( K^2 - 1 \right)^2 \right) \]
\[ = \Lambda^{-(d-5)/2} \int_0^1 dxx^{(d-6)} \left( 2N \left( x \frac{dK}{dx} \right)^2 + (d - 3) \left( K^2 - 1 \right)^2 \right) \equiv \Lambda^{-(d-5)/2} \tilde{I}_{(d)} \]
where now

\[ N = (1 - x^2), \quad K = \frac{1 + ax^2}{1 + bx^2} \]

The total Euclidean action, for example, is

\[
(1/2) \tilde{P} \Sigma_{(d-2)} \lambda_{(d)} (d - 2) \Lambda^{-(d-5)/2} \tilde{I}_{(d)}\]

For \( d = 4 \), with \( \tilde{P} = 2\pi \Lambda^{-1/2} \), one obtains the total action as

\[
4\pi^2 \tilde{I}_{(4)} = 4\pi^2 (3.510 \mp i8.394) = 8\pi^2 (1.755 \mp i4.197)
\]

This is again our old result [1, 2].

Leaving aside again the case \( d = 5 \), we give the values of \( \tilde{I}_{(d)} \) for the solutions (2.34). For each value of \( d \) (from 6 to 10) one now has two real solutions ((1) and (2) in (2.34)). Considering them in order one obtains the following values:

\[
(1) \quad \tilde{I}_{(6)} = 0.495, \quad (2) \quad \tilde{I}_{(6)} = 2.447
\]
\[
(1) \quad \tilde{I}_{(7)} = 1.112, \quad (2) \quad \tilde{I}_{(7)} = 1.824
\]
\[
(1) \quad \tilde{I}_{(8)} = 1.134, \quad (2) \quad \tilde{I}_{(8)} = 1.589
\]
\[
(1) \quad \tilde{I}_{(9)} = 1.096, \quad (2) \quad \tilde{I}_{(9)} = 1.473
\]
\[
(1) \quad \tilde{I}_{(10)} = 1.056, \quad (2) \quad \tilde{I}_{(10)} = 1.395
\]

Finite values are obtained in each case since \((1 + b\Lambda r^2)\) has no zero in the domain \([0, 1]\) of \( \Lambda r^2 \). For the anti-deSitter case there is no horizon and our solutions, even apart from the question of the values of \( b \), lead to divergent actions due to asymptotic properties for \( d \geq 6 \).

4 Remarks:

Various aspects of nonselfdual solutions have been studied by a number of authors. An incomplete list of references is provided [6,7,8,9,10,11]. Here we have shown how the simplest static, spherically symmetric curved spaces provide surprising new possibilities. Our results are not limited to existence theorems. We provide explicit solutions and hence complete information concerning the gauge potentials at every space-time point. The simplicity of our solutions should permit a relatively easy study of normal modes and (un)stability properties. Maintaining the spherical symmetry one can proceed as for sphalerons [12,13,14,15,16], the best known class of nonselfdual solutions. The starting point, without Higgs fields and with our explicit solutions for the gauge potentials, should be even simpler. Such a study would yet involve considerable numerical computations. It is deferred to another paper. But we add some relevant comments.
We started by pointing out the striking fact that the action (1.7) of our nonselfdual solution (Schwarzschild, \( d = 4 \)) is lower than that for the lowest-action, non-trivial, selfdual solution (\( P = 1 \)). But our solution is not topologically stabilized. As an evident consequence of \( F_{0i} = 0 \) it has zero index (\( P = 0 \)). Considering only \( F_{ij} \), it is seen that as \( r \to \infty \) the magnitude falls faster (\( \approx r^{-3} \)) than that for a monopole. Presumably our solution provides a minimal saddle-point between two topologically distinct vacua. But a precise statement needs further study. Apart from the common factors (\( 4\pi \) from angular integrations and \( 8\pi M \) for the period) one can compare the radial contributions. The action densities of the selfdual and the nons elfdual cases are respectively

\[
\frac{6M^2}{r^4}
\]

and (with \( a = -2.366, b = 4.098 \))

\[
\frac{4M^2(a-b)^2}{r^2(r+2Mb)^3}(3r^2 + 2M(2(a+b) - 1)r + 2M^2(a+b)^2)
\]

As \( r \to \infty \) both fall as \( r_4 \) but for (4.2) with a larger numerical coefficient. On the other hand (4.2) starts from a lower value at \( r = 2M \). The total effects are given by (1.9) and (1.7) respectively. One further point should be noted. For Euclidean signature selfdual gauge field configurations, having zero energy-momentum tensor, do not perturb the metric. Thus one obtains, effectively, a solution for the combined gravitation-gauge field system. For a nons elfdual system this is no longer the case. Hence there a ”background approximation” (ignoring the back reaction of the gauge field on the metric) is involved. But again, this is true not only for curved but also for a flat background, the latter being held fixed to be flat even in presence of other fields.

For deSitter background (\( d = 4 \)) we obtained a pair of complex solutions. The conformal properties of the metric and of gauge fields in four dimensions permitted us to reinterpret our static solutions as time-dependent complex ones in flat space-time [2]. This is in sharp contrast to the Schwarzschild case where the solution vanishes with \( M \).

After the very special case of \( d = 5 \), discussed separately, from \( d = 6 \) onwards the complementary behaviours for these metrics continue with a crossover. Schwarzschild solutions become complex while the deSitter ones become real. It is remarkable that both real solutions (\( d = 6, 7, 8, 9, 10 \)) give finite action, the ratio of the two actions changing slowly with \( d \). The prospect of mapping out the action landscape, with ridges and valleys, in the neighbourhoods of these solutions is intriguing. The special nature of \( d = 11 \) has been pointed in (2.35).

The solutions of the standard Yang-Mills Lagrangian are no longer independent of conformal factors in the metric for \( d > 4 \). (In particular, the lack of scale invariance is signalled by the persistence of the parameters \( C \) and \( \Lambda \) in the actions.) So a direct passage to flat space is not possible, preventing a comparison of the actions of our real deSitter solutions with those of possible flat-space solutions in higher dimensions – the fundamental octonionic
instanton [17,18] in eight dimensions, for example. For a Lagrangian quartic in $F_{\mu\nu}$ conformal invariance is restored in 8 dimensions. (See Ref.19 and the references cited there for the general case of 4$p$ dimensions.) But our class of nonselfdual solutions have not been constructed in the framework of these generalized Lagrangians.

We have left various possible generalizations unexplored. To start with one can consider more general metrics as backgrounds. Solving (2.19) with $N$ given by the Reissner-Nordstrom metric is the most immediate possibility. (But one encounters here new problems.) For both $d = 4$ and $d = 8$ the Kerr metric can be related to the Schwarzschild one through an imaginary translation [20]. One can implement a corresponding translation in the gauge potentials to see whether it can be adapted, or not, to obtain solutions for Kerr backgrounds. The $AdS$ background, briefly introduced in the Appendix, evidently deserves a more thorough study. We hope to explore such possibilities elsewhere.

One reason for presenting our solutions, restricted as they are, is the pleasant simplicity attained. We show that curvature, in some cases, can open doors rather than erect barriers. Another reason is the current broad interest, in the context of strings and branes, in solutions for higher dimensions. Without citing references, let us state that nonselfdual, non-BPS, solutions deserve scrutiny.

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Appendix: A nonselfdual solution for $AdS_4$ background.

For $d = 4$ a divergent solution in deSitter background was shown [1,3] to be related through conformal transformations to meron-type solutions in flat space-time. This, particularly simple, solution corresponds (with $\Lambda > 0$) to

$$N = (1 - \Lambda r^2), \quad K = N^{-\frac{1}{2}} = (1 - \Lambda r^2)^{-\frac{1}{2}} \quad (A.1)$$

Here we note that changing the sign before $\Lambda$ one obtains the anti-deSitter case ($AdS_4$) and

$$N = (1 + \Lambda r^2), \quad K = N^{-\frac{1}{2}} = (1 + \Lambda r^2)^{-\frac{1}{2}} \quad (A.2)$$

This provides a solution of (2.19) with $d = 4$ namely of

$$\frac{d}{dr} \left( N \frac{dK}{dr} \right) = r^{-2} K \left( K^2 - 1 \right) \quad (A.3)$$

where now $K$ is no longer singular. Henceforth, in the Appendix, we set $\Lambda = 1$ for simplicity. The radial integral (3.16), in absence of the horizon, should now be replaced by

$$\tilde{I}(4) = \int_0^\infty dx x^{-2} \left( 2N \left( x \frac{dK}{dx} \right)^2 + \left( K^2 - 1 \right)^2 \right) \quad (A.4)$$

where

$$N = (1 + x^2), \quad K = (1 + x^2)^{-\frac{1}{2}}$$

One obtains

$$\tilde{I}(4) = 3 \int_0^\infty \frac{x^2}{(1 + x^2)^2} dx = \frac{3\pi}{4} \quad (A.5)$$

Thus, corresponding to (3.10), one obtains a finite spatial integral

$$\frac{3\pi^2}{2} \quad (A.6)$$

The factor from the time integration depends on the chosen context. Now there is no horizon to be desingularized and the discussion of Sec.1 is not directly relevant. But one can start by considering the hypersurface

$$- t_1^2 - t_2^2 + x_1^2 + x_2^2 + x_3^2 = -1 \quad (A.7)$$

In terms of the spherical coordinates

$$\begin{align*}
(x_1, x_2, x_3) & \rightarrow (r, \theta, \phi) \\
(t_1, t_2) & \rightarrow (T, \psi) \quad (A.8)
\end{align*}$$
the metric on the hypersurface

\[ r^2 - T^2 = -1 \]

is

\[ ds^2 = -(1 + r^2)d\psi^2 + (1 + r^2)^{-1}dr^2 + r^2d\Omega_2 \]  \hspace{1cm} (A.9)

In this context the \( \psi \)-integration gives a factor \( 2\pi \) and one obtains a total action

\[ 3\pi^3 \]  \hspace{1cm} (A.10)

But often it is preferable to consider the covering space (CAdS) replacing \( \psi \in S^1 \) by \( t \in \mathbb{R} \). Then the action is evidently divergent.

The solution of (A.1) with the square root involved for \( K \) seems to be specific to \( d = 4 \). But it would be interesting to search for suitable generalizations, related to this class, for higher dimensions.
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