Serious mathematical defect in the important kinematics theorem known in continuum mechanics as Convection (or Transport) Theorem is reported. We claim that the traditional demonstration does not take into account a special constraint on integrand functions given in Lagrangian representation. Thus, we put in doubt that the traditional procedure for the transition from integral formulations of physical laws of some classical field theories to their differential form is mathematically rigorous. Reconsidered formulation shows the way how the system of fundamental differential equations of continuum mechanics and some other field theories could be modified without any change of the original integral formulation. The continuity equation and the differential form of the equation of motion for continuous media are discussed as examples of modification.

I. INTRODUCTION

At present, a huge amount of experimental data has been accumulated on deformable bodies, fluids, gases and plasmas. Mathematical understanding of their behavior, internal relationship between different concepts, models and observed phenomena are expressed in comprehensive modern descriptions generally referred as hydrodynamics, elasticity theory, electromagnetism, magnetohydrodynamics, plasma physics (as regards to the latter, nowadays a special priority, attention and technical support have been given to the solution of problems of thermonuclear reaction as alternative source of energy). The most part of inner links between these autonomous branches of scientific knowledge is mainly due to the common mathematical apparatus of continuum mechanics which brings them together as a part of a more general scheme.

Having in mind this unity of mathematical basis for all above-mentioned subdivisions of physics and mechanics, we shall refer in this work only to a mechanical theory of motion of continuous media which constitutes a significant part of continuum mechanics. The development of the theory of ordinary and partial differential equations, integral equation, differential geometry etc had great influence on conceptual, logical and mathematical structure of continuum mechanics and vise versa. In historical retrospective, it is now commonly accepted that the consolidation of mathematical and conceptual fundamentals of the theory of continuum mechanics was achieved by the end of the 19th century and, in main terms, coincided with the rigorization of the analysis completed by Weierstrass [1].

However, very recent indications [2] on ill-founded analysis in mathematical hydrodynamics put in doubt the fact that the process of rigorization of the fundamentals of classical field theory had been brought to the end in a correct way. More precisely, a detailed insight towards Lagrangian and Eulerian types of analytical description conventionally accepted for kinematics of continuous media shows that no equal standards of rigor are implied in both approaches to time derivatives. A reconsidered account [2] provided a mathematically rigorous analytical approach to the treatment of total time derivative in properly Eulerian description. Another serious defect had been detected in the demonstration of an important kinematics theorem known also as Convection (or Reynolds’ Transport) Theorem [2]. Elimination of both defects provided a necessary cross-verification for a modified differential form of continuity equation [2].

In what follows, one of our immediate purposes will be to fix basic conceptions of mathematical foundations of continuum mechanics and then we shall proceed to show the failure in the logic of the traditional approach to the formulation of the Convection Theorem and how its reconsideration motivates a new form of differential equations of motion for continuous media.

II. BASIC DESCRIPTIONS IN CONTINUUM MECHANICS

The basic mathematical idea of kinematics is that a continuous medium can be consistently conceived as an abstract geometrical object (for instance, collection of spatially distributed points etc). Thus, any deformation and motion is immediately associated with appropriate geometrical transformation [3]-[4]. Since any motion is always determined with respect to some reference frame, let us introduce a fixed coordinate system. To simplify our approach in order to highlight the nature of the defects inherent in the conventional demonstration of the Convection Theorem, we begin with the simplest assumptions of continuous geometrical transformations.

Actual mathematical formalism implies two complementary general descriptions of flow field kinematics. One of them, called Eulerian description (or representation), identifies a fixed volume element or a fixed point of space in a chosen reference system. All medium properties are described only as a function of local position \( \mathbf{r} \) and time \( t \). These independent variables are frequently regarded as Eulerian variables.

The other approach, called Lagrangian description, identifies an individual bit of continuous medium (charac-
characterized by an initial position-vector \( \mathbf{a} \) at some chosen time instant and gives account of the medium properties along their trajectory. This approach associates non-zero motion with a non-zero continuous geometrical transformation \( H_t \). Thus, the set \( H_t \Omega_0 \) (or a position-vector \( H_t \mathbf{a} \)) represents the same individual bit of continuous medium (or point-particle) at time \( t \). Continuity requirement on \( H_t \) involves a natural limitation: the bounding surface of the closure \( H_t \Omega_t \Omega_0 \) always consists of the same medium elements, i.e. \( \partial \Omega_t = H_t \partial \Omega_0 \) and there is no flux of medium particles through the boundary at any instants of time. Thus, under this conditions the transformation \( H_t \mathbf{a} \) is always the point transformation and the function \( H_t \mathbf{a} = \mathbf{r}(t, \mathbf{a}) \) describes a law of motion of a point. The position-vector \( \mathbf{a} \) denotes the initial position of an individual particle and, therefore, can be used as a label for constant identification of the particle at any instant of time. An initial set of identified particles is equivalent to the set of labels \( \{ \mathbf{a} \} \) regarded sometimes as Lagrangian parameters. The assumption on continuity of geometrical transformation \( H_t \) is equivalent to the requirement that the function \( \mathbf{r}(t, \mathbf{a}) \), which describes the law of motion, possesses continuous partial derivatives with respect to all variables, i.e. \( t \) and \( \mathbf{a} \).

In considering the motion of continuous medium as a set of individual mutually interacting point-particles (or volume bits), Lagrangian approach is indispensable as the first step, implying by it the individualization of particles by a set of labels \( \{ \mathbf{a} \} \). Thus, it should be emphasized again that the geometrical transformation \( H_t \mathbf{a} = \mathbf{r}(\mathbf{a}, t) \) gives the complete picture of a motional history for every individual particle from the set \( \{ \mathbf{a} \} \). The detailed description of the law of motion implies an introduction of certain additional concepts such as the velocity and acceleration of particles of a continuous medium. In Lagrangian description velocity and acceleration are defined as the first and the second order partial time derivative with respect to \( \mathbf{r} \), respectively:

\[
\mathbf{v} = \frac{\partial}{\partial t} \mathbf{r}(t, \mathbf{a}); \quad \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial^2}{\partial t^2} \mathbf{r}(t, \mathbf{a}) \quad (2.1)
\]

In the context of Eulerian description, \textit{a priori} there is no identification and hence no explicit consideration of the function \( \mathbf{r} = \mathbf{r}(t, \mathbf{a}) \). The primary notion is the velocity field as a function of position \( \mathbf{r} \) in space and time \( t \) on some domain of a continuous medium:

\[
\frac{d\mathbf{r}}{dt} = \mathbf{v}(t, \mathbf{r}) \quad (2.2)
\]

where variables \( \mathbf{r} \) and \( t \) are independent.

Picking up some initial point \( \mathbf{a} = \mathbf{r}(t_0) \), one selects from a congruence (a set of integral curves of (2.2)) a unique solution. Thus, a formulation of the initial Cauchy problem

\[
\frac{d\mathbf{r}}{dt} = \mathbf{v}(t, \mathbf{r}); \quad \mathbf{r}(t_0) = \mathbf{a} \quad (2.3)
\]

is mathematically equivalent to an act of identification, allowing any solution of (2.3) to be represented as in Lagrangian description \( \mathbf{r} = \mathbf{r}(t, \mathbf{a}) \). There is a general consensus that this procedure can be taken as a rule for translating from one to the other description. Thus, if some medium quantity \( f \) is defined in Eulerian representation as \( f(t, \mathbf{r}) \), then there is an obvious translation rule to its Lagrangian representation [4]:

\[
g(t, \mathbf{a}) = f(t, \mathbf{r}(t, \mathbf{a})) \quad (2.4)
\]

Convective (or Euler’s material) derivative is introduced as:

\[
\frac{\partial}{\partial t} g(t, \mathbf{a}) = \frac{d}{dt} f(t, \mathbf{r}(t, \mathbf{a})) = \frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f \quad (2.5)
\]

Now we are in a position to consider an important kinematics theorem which concerns the time rate of change of \( f \)-content of any volume integral (i.e. not only infinitesimal volume elements). Its formulation can be found in any basic text on fluid dynamics or elasticity theory. For our convenience, we shall use the exposition and symbolic notations implemented in [4]. As the first step, let us define \( f \)-content of some deformable moving volume domain \( \Omega_t \) and its time derivative in Eulerian and Lagrangian representations, respectively:

\[
F(t) = \int_{\Omega_t} f(t, \mathbf{r}) dV; \quad \frac{dF}{dt} = \frac{D}{Dt} \int_{\Omega_t} f(t, \mathbf{r}) dV \quad (2.6)
\]

and

\[
F(t) = \int_{\Omega_t} f(t, \mathbf{r}(t, \mathbf{a})) dV = \int_{\Omega_t} g(t, \mathbf{a}) dV;
\]

\[
\frac{dF}{dt} = \frac{\partial}{\partial t} \int_{\Omega_t} g(t, \mathbf{a}) dV \quad (2.7)
\]

where \( \Omega_t = H_t \Omega_0 \).

The geometrical transformation \( H_t \) is algebraically represented by the Jacobian determinant \( J = \det \left| \frac{\partial H_t \mathbf{a}}{\partial \mathbf{a}} \right| \) which has the following partial time derivative [4]:

\[
\frac{\partial}{\partial t} J = \frac{\partial}{\partial t} \det \left| \frac{\partial H_t \mathbf{a}}{\partial \mathbf{a}} \right| = (\nabla \cdot \mathbf{v}) J \quad (2.8)
\]

where \((\nabla \cdot \mathbf{v}) = \text{div} \mathbf{v}\).

Thus, in the framework of Lagrangian representation the evolution of a medium \( f \)-content can be written in
Theorem 1: Let \( \mathbf{v} \) be a vector field generating a fluid flow through a fixed 3-dimensional domain \( V \) and if \( f(\mathbf{r}, t) \in C^1(\mathcal{V}) \), then

\[
\frac{D}{Dt} \int_{\mathcal{V}} f(\mathbf{r}) \, d\mathbf{r} = \int_{\mathcal{V}} \left( \frac{\partial f}{\partial t} + f(\nabla \cdot \mathbf{v}) \right) \, d\mathbf{r} \quad (3.1)
\]

where \( \rho \) is the mass density. The equation (3.2) can be regarded as a modified continuity equation which was already independently obtained due to reconsidered approach to total time derivatives in properly Eulerian description [2]. Mathematical soundness and applicability of the equation (3.2) was also analytically verified on a simple one-dimension example of an ideal flow (see Appendix B in [2]).

In continuum mechanics, there are many physical quantities such as linear and angular momentums, energy as well as some other scalar, vector or tensor characteristics which undergo time variations during the motion of any given volume of a medium. If this quantities are continuous functions of the coordinates everywhere inside the spatial domain \( V_t \), then the mathematical procedure used for the demonstration of the Convection Theorem (3.2) also remains valid. Thus, for any moving individual macroscopic volume \( V_t \), the equation for time variation of linear momentum will take the following form for each spatial component \( v^i \) of the velocity vector \( \mathbf{v} \):

\[
\frac{D}{Dt} \int_{V_t} \rho v^i \, d\mathbf{v} = \int_{V_t} \left[ \frac{\partial \rho v^i}{\partial t} + \rho v^i \, div(\mathbf{v}) \right] \, d\mathbf{v} \quad (3.3)
\]
The concept of force is introduced in continuum mechanics phenomenologically by analogy with classical mechanics. In other words, different forces which act on the volume \( V_t \) are responsible for the time variation of momentum:

\[
\frac{D}{Dt} \int_{V_t} \rho v dV = \int_{V_t} \left[ \frac{\partial \rho v}{\partial t} + \rho v \text{div}(v) \right] dV + \int_{\partial V_t} \rho f dS \tag{3.4}
\]

where \( f \) is a density of all external mass forces and \( P \) is a surface stress force represented by a stress tensor \( P^{ik} \).

The differential form of the relation (3.4) written in components is often regarded as equation of motion of a continuous medium in Eulerian coordinates of the fixed reference system:

\[
\rho \frac{\partial v^i}{\partial t} + \rho v^j \frac{\partial v^i}{\partial x^j} = \rho f^i + \frac{\partial P^{ik}}{\partial x^k} \tag{3.5}
\]

where \( v = \{v^i\} \) and \( r = \{x^i\} \). Taking into account that the modified continuity equation (3.2) can be rewritten in components as:

\[
\frac{\partial \rho v^i}{\partial t} + \rho \frac{\partial v^i}{\partial x^k} = 0 \tag{3.6}
\]

the equation of motion (3.5) takes a more simple form:

\[
\rho \frac{\partial v^i}{\partial t} = f^i \rho + \frac{\partial P^{ik}}{\partial x^k} \tag{3.7}
\]

Importantly to emphasize that no changes were assumed for the analytical representation of forces in the right-hand side of (3.4). The modification is concerned only the left-hand side of (3.4) which refers to the time variation of momentum. Thus, in the traditional approach the partial time derivative \( \frac{\partial}{\partial t} \) is replaced by Euler’s derivative \( \frac{D}{Dt} \).

Nevertheless, we shall limit our consideration here only by examples of mass and linear momentum time variations since they suffice to show the way how the system of fundamental differential equation of continuum mechanics and other field theories could be modified. To conclude the Section, we would like to stress another important feature of the given approach: reconsidered fundamental differential equations of continuum mechanics can be derived without any change in their original integral formulation.

\[
\begin{align*}
\text{IV. CONCLUSIONS} \\
\text{Rational examination of mathematical foundations of} \\
\text{continuum mechanics as being of central importance and} \\
\text{wide appeal in physical field theories, shows serious} \\
\text{defect at the very basic level. To be more specific, we claim} \\
\text{to have found that the traditional demonstration of the} \\
\text{important kinematics theorem of continuum mechanics} \\
\text{known as Convection (or Transport) Theorem does not} \\
\text{take into account a special constraint on integrand func-} \\
\text{tions given in Lagrangian representation and, as a con-} \\
\text{sequence, it is not already based on a mathematically} \\
\text{rigorous approach.}
\end{align*}
\]

Any modification of the conventional procedure would imply undeniable changes in the set of basic differential equations of continuum mechanics as well as some other autonomous branches of physical science such as electromagnetism, magnetohydrodynamics, plasma physics etc. Moreover, in this work we show that these modifications would not be accompanied by any change in corresponding integral formulations, leaving them untouched. The latter fact is important from the practical meaning, since the prevailing amount of experimental data in physical field theories was basically classified in form of integral laws. Thus, the major point that emerges from the above considerations is that the traditional transition from original integral formulations of physical laws of classical field theories to their differential form may come in conflict with the mathematical rigor.

\[
\begin{align*}
\text{REFERENCES} \\
1. & \ M. \ Kline, \ Mathematical \ Thought \ from \ Ancient \ to \ Modern \ Times, \ Vol. \ 2 \ (Oxford \ University \ Press, \ New \ York, \ 1972) \\
2. & \ R. \ Smirnov-Rueda, \ Found. \ Phys., \ 35(10) \ (2005) \\
3. & \ B. \ Dubrovin, \ S. \ Novikov \ and \ A. \ Fomenko, \ Modern \ Geometry, \ Vol. \ 1 \ (Ed. \ Mir, \ Moscow, \ 1982) \\
4. & \ R.E. \ Meyer, \ Introduction \ to \ Mathematical \ Fluid \ Dynamics \ (Wiley, \ 1972)
\end{align*}
\]