The Exact Solution of the Cauchy Problem for a generalized "linear" vectorial Fokker-Planck Equation - Algebraic Approach

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Abstract

The exact solution of the Cauchy problem for a generalized "linear" vectorial Fokker-Planck equation is found using the disentangling techniques of R. Feynman and algebraic (operational) methods. This approach may be considered as a generalization of the Masuo Suzuki’s method for solving the 1-dimensional linear Fokker-Planck equation.

1 Introduction

The Fokker-Planck equations (FPE), the one-dimensional FPE

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x} [a(t, x)W] + \frac{\partial^2}{\partial x^2} [D(t, x)W(t, x)], \quad t \geq 0, \ x \in \mathbb{R}, \ (1)$$

and the "vectorial" FPE

$$\frac{\partial w}{\partial t} = -\nabla \cdot [a(t, x)w] + \nabla \nabla : \{ \hat{D}(t, x)w(t, x) \}, \quad t \geq 0, \ x \in \mathbb{R}^n, \ (2)$$

where $a(t, x) = (a_1(t, x), a_2(t, x), \ldots, a_n(t, x))^T$ is the "drift vector", $\hat{D}(t, x)$ is a symmetric non-negative definite "diffusion" tensor field of II rank, and $\nabla \nabla : \hat{D} = \frac{\partial^2 D_{ij}}{\partial x_i \partial x_j}$ (Einstein summation convention accepted), are widely used \cite{1-19} as a tool in modelling various processes in many areas of the theoretical and mathematical physics, chemistry and biology as well as in the pure and applied mathematics and in engineering: the nonequilibrium statistical mechanics (in particular in the theory of Brownian motion and similar phenomena: random walks, the fluctuations of the liquid surfaces, the local density fluctuations in fluids and solids, the fluctuations of currents, etc); the metrology (Josephson voltage standards); the laser physics; the turbulence theory;
the cellular behaviour; the neurophysiology; the population genetics; the mathematical
theory and applications of the stochastic processes,— to mention only a few of them.

Because of its importance there have been many attempts to solve FPE exactly or
approximately (for a review see [4, 6 - 11, 14, 19]). Among the recent investigations
on this problem noteworthy for us is the method of M. Suzuki [18].

In this paper we find the exact solution of following Cauchy problem:

\[
\frac{\partial u}{\partial t} = a_1(t)u(t, x) + a_2(t) \cdot \nabla u + a_3(t)x \cdot \nabla u + \hat{a}_4(t) : \nabla \nabla u, \quad u(0, x) = \phi(x),
\]

where \( \hat{a}_4(t) \) is a symmetric non-negative definite tensor function of second rank of the
scalar parameter \( t \).

It is easy to see that the Eq.(3) is connected with the “linear” vectorial FPE (2)
with a linear in \( x \) “drift vector” \( a(t, x) = b_1 + b_2x \) and an independent of \( x \) diffusion
tensor \( \hat{D} \). (Here \( b_1, b_2 \) and \( \hat{D} \) are functions of \( t \).) Therefore the Eq. (3) is a slight
generalization of the “linear” vectorial FPE (2) with \( t \)-dependent coefficients.

In the paper [20] the “isotropic” problems

\[
\frac{\partial u}{\partial t} = a_1u(t, x) + a_2 \cdot \nabla u + a_3x \cdot \nabla u + a_4\Delta u, \quad u(0, x) = \phi(x)
\]

and

\[
\frac{\partial u}{\partial t} = a_1(t)u(t, x) + a_2(t) \cdot \nabla u + a_3(t)x \cdot \nabla u + a_4(t)\Delta u, \quad u(0, x) = \phi(x)
\]

have been exactly solved (here \( a_4 \) and \( a_4(t) \) are arbitrary non-negative constant and
function of \( t \) respectively).

In the paper [21] we have found the exact solutions of the following Cauchy prob-
lems:

\[
\frac{\partial u}{\partial t} = a_1u(t, x) + a_2 \cdot \nabla u + a_3x \cdot \nabla u + \hat{a}_4 : \nabla \nabla u, \quad u(0, x) = \phi(x)
\]

and

\[
\frac{\partial u}{\partial t} = a_1(t)u(t, x) + a_2(t) \cdot \nabla u + a_3(t)x \cdot \nabla u + a_4(t)\hat{a} : \nabla \nabla u, \quad u(0, x) = \phi(x)
\]

where \( \hat{a}_4 \) and \( \hat{a} \) are symmetric non-negative definite tensors of second rank and \( a_4(t) \)
is a scalar function; \( a_4(t) > 0 \). (It is obvious that the problem (3) is more general
than the problem (6); \( \hat{a}_4(t) \) is arbitrary symmetric non-negative definite tensor
function of second rank, while in (7) \( \hat{a}_4(t) \) has a special form: \( \hat{a}_4(t) = a_4(t)\hat{a} \).

Our method may be regarded as a combination of the disentangling techniques of
R. Feynman [22] with the operational methods developed in the functional analysis
and in particular in the theory of pseudodifferential equations with partial derivatives
[23] - [27]. As we have emphasized in [20] and [21] this approach is an extension and
generalization of the M. Suzuki’s method [18] for solving the one-dimensional linear
FPE (1).
2 Exact Solution of the Cauchy Problem (3)

In view of the t-dependence of the coefficients in the Eq. (3), formally we have for the solution of the initial value problem (3) an ordered exponential

\[ u(t, x) = \left( \exp_+ \int_0^t [a_1(s) + a_2(s) \cdot \nabla + a_3(s)x \cdot \nabla + \hat{a}_4(s) : \nabla \nabla] \, ds \right) \phi(x), \tag{8} \]

where

\[ \exp_+ \int_0^t \hat{C}(s) \, ds \equiv T - \exp \int_0^t \hat{C}(s) \, ds \]

\[ = \hat{I} + \lim_{k \to \infty} \sum_{n=1}^k \int_0^t \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_{n-1}} \int_0^{t_n} \hat{C}(t_1) \hat{C}(t_2) \ldots \hat{C}(t_n). \tag{9} \]

If we introduce the operators

\[ \hat{A}(t) = a_2(t) \cdot \nabla + a_3(t)x \cdot \nabla \quad \text{and} \quad \hat{B}(t) = \hat{a}_4(t) : \nabla \nabla, \tag{10} \]

we may write (8) in the form

\[ u(t, x) = e^{\int_0^t a_1(s) \, ds} \left( \exp_+ \int_0^t [\hat{A}(s) + \hat{B}(s)] \, ds \right) \phi(x), \tag{11} \]

as the first term in the exponent commutes with all others.

To proceed with the pseudodifferential operator in Eq. (11) we shall use the theorem of M.Suzuki [18] :

If

\[ [\hat{A}(t), \hat{B}(t)] = \alpha(t, s) \hat{B}(s), \]

then

\[ \exp_+ \int_0^t [\hat{A}(s) + \hat{B}(s)] \, ds = \left( \exp_+ \int_0^t \hat{A}(s) \, ds \right) \left( \exp_+ \int_0^t \hat{B}(s) e^{-\int_0^s \alpha(u, s) \, du} \, ds \right). \tag{12} \]

In our case we have

\[ [\hat{A}(s), \hat{B}(s')] \equiv [a_2(s) \cdot \nabla + a_3(s)x \cdot \nabla, \hat{a}_4(s') : \nabla \nabla] = -2a_3(s)\hat{a}_4(s') : \nabla \nabla \equiv -2a_3(s)\hat{B}(s'). \tag{13} \]

Therefore from (12) we obtain

\[ \exp_+ \int_0^t [\hat{A}(s) + \hat{B}(s)] \, ds = \left( \exp_+ \int_0^t \hat{A}(s) \, ds \right) \left( \exp_+ \int_0^t \hat{B}(s) e^{\int_0^s \alpha(u, s) \, du} \, ds \right). \tag{14} \]

The linearity of the integral and the explicit form of \( \hat{A} \) (see Eq. (10)) permit to write the first factor in (14) in terms of usual, not ordered, operator valued exponent

\[ \exp_+ \int_0^t \hat{A}(s) \, ds \equiv \exp_+ \int_0^t [a_2(s) \cdot \nabla + a_3(s)x \cdot \nabla] \, ds = e^{2a_2(t) \cdot \nabla + a_3(t)x \cdot \nabla}. \tag{15} \]
For convenience we introduce the following notations:
\[
\alpha_1(t) = \int_0^t a_1(s)ds, \quad \vec{\alpha}_2(t) = \int_0^t a_2(s)ds, \quad \alpha_3(t) = \int_0^t a_3(s)ds.
\] (16)

Consequently (from now on “′” means \[\frac{d}{dt}\])
\[
\alpha_1'(t) = a_1(t), \quad \vec{\alpha}_2'(t) = a_2(t), \quad \alpha_3'(t) = a_3(t),
\]
\[
\alpha_1(0) = 0, \quad \vec{\alpha}_2(0) = 0, \quad \alpha_3(0) = 0.
\] (17)

Thus we obtain from the Eq. (11)
\[
u(t, x) = e^{\alpha_1(t)e^{[\vec{\alpha}_2(t)+\alpha_3(t)x]}} \nabla \left( \exp_{+} \int_{0}^{t} \hat{a}_4(s)e^{2\alpha_3(s)} : \nabla \nabla ds \right) \phi(x).
\] (18)

Finally using the formulae (see [20] and [21])
\[
\left[ \exp_{+} \left( \int_{0}^{t} \hat{\Psi}(s) : \nabla \nabla ds \right) \right] \phi(x)
\]
\[
= \frac{1}{\sqrt{\det(4\pi \hat{\tau}(t))}} \int_{\mathbb{R}^n} \left\{ \exp \left[ -(x - y) \cdot \frac{\hat{\tau}^{-1}(t)}{4} \cdot (x - y) \right] \right\} \phi(y)dy,
\] (19)

where
\[
dy = dy_1dy_2 \ldots dy_n, \quad \hat{\tau}(t) = \int_{0}^{t} \hat{\Psi}(s)ds
\]
and
\[
e^{\vec{\alpha}_2(t) \cdot \nabla + \alpha_3(t)x \cdot \nabla} g(x) = g \left( xe^{\alpha_3(t)} + \int_{0}^{t} a_2(s)e^{\alpha_3(s)}ds \right) \equiv g(z),
\] (20)

we find from the Eq. (18) the following expression for the exact solution of the Cauchy problem (3) \((\hat{\Psi}(s) = \hat{a}_4(s) \exp[2a_3(s)])\) :
\[
u(t, x) = \frac{e^{\alpha_1(t)}}{\sqrt{\det(4\pi \hat{\tau}(t))}} \int_{\mathbb{R}^n} \left\{ \exp \left[ -(z - y) \cdot \frac{\hat{\tau}^{-1}(t)}{4} \cdot (z - y) \right] \right\} \phi(y)dy,
\] (21)

where
\[
\hat{\tau}(t) = \int_{0}^{t} \hat{a}_4(s)e^{2\alpha_3(s)}ds
\]
is a symmetric non-negative definite second rank tensor function of \(t\), \(dy = dy_1 \ldots dy_n\) and \(z\) is defined in (20).

Substituting the expression (21) in the Eq. (3) we see immediately that the function \(\nu(t, x)\) is a solution of the problem (3), and, according to the Cauchy theorem, it is the only classical solution of this problem.
3 Concluding remarks

- The exact solutions of the Cauchy problem (3) is obtained using the algebraic method we have described.

- When $\dot{a}_4(t)$ is scalar: $\dot{a}_4(t) = a_4(t)\dot{1}$ (in this case $\dot{a}_4 : \nabla \nabla = a_4 \Delta$) the “anisotropic” problem (3) turns to the “isotropic” one, with the exact solution found in [20]. It is easy to check that the solution (21) turns to the solution obtained in [20] (there is an error in [20]: the sign before $a_2$ in the Eqs. (17) and (34) there, should be (+)).

- In the case $\dot{a}_4(t) = a_4(t)\dot{a}$ the Cauchy problem (3) reduces to the problem (7) treated in [21]. In this case the solution (21) turns to the solution obtained in [21].

- For different choices of the coefficients $a_j$ and $a_2$ the Eq. (3) may be regarded also as a set of different diffusion equations. Therefore from the formula (21) we obtain the exact solutions of the Cauchy problems for this set of diffusion equations.

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