GRAVITATIONAL DRESSING OF $N = 2$ SIGMA–MODELS BEYOND LEADING ORDER

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Abstract

We study the $\beta$–function of the $N = 2$ $\sigma$–model coupled to $N = 2$ induced supergravity. We compute corrections to first order in the semiclassical limit, $c \to -\infty$, beyond one–loop in the matter fields. As compared to the corresponding bosonic, metric $\sigma$–model calculation, we find new types of contributions arising from the dilaton coupling automatically accounted for, once the Kähler potential is coupled to $N = 2$ supergravity.
1 Introduction

Matter systems which are not conformally invariant, when coupled to two–dimensional gravity, induce propagating modes of the latter due to the appearance of the one–loop trace anomaly \([1]\). Thus it is of interest to study the gravitational back reaction and analyze how physical quantities are affected by the presence of this dynamically generated gravitational field. At one–loop order in the matter fields exact results (i.e. to all orders in the gravitational coupling) have been obtained for the \(\beta\)–functions of bosonic \([2]\) as well as supersymmetric \(N = 1, 2\) \(\sigma\)–models \([3]\). More precisely it has been shown that the effect of induced gravity for the \(N = 0, 1\) theories is simply to multiplicatively renormalize the \(\beta\)–functions as follows:

\[
N = 0 \quad \beta_{G}^{(1)} = \frac{\kappa + 2}{\kappa + 1} \beta_{0}^{(1)} \quad \kappa + 2 = \frac{1}{12} [c - 13 - \sqrt{(1 - c)(25 - c)}]
\]

\[
N = 1 \quad \beta_{G}^{(1)} = \frac{\kappa + 3}{\kappa + 1} \beta_{0}^{(1)} \quad \kappa + \frac{3}{2} = \frac{1}{8} [c - 5 - \sqrt{(1 - c)(9 - c)}]
\]

where \(\kappa\) is the central charge of the gravitational \(SL(2R)\) Kac–Moody algebra. For the \(N = 2\) \(\sigma\)–model the coupling to supergravity does not produce any new divergent contribution so that the one–loop \(\beta\)–function does not receive a gravitational dressing. These results indicate that to first order in perturbation theory for the matter fields, even if the renormalization group trajectories might be affected (e.g. for the \(N = 0, 1\) theories), the critical points of the various theories are the same as in the absence of gravity, since the \(\beta\)–function is at most rescaled by a multiplicative factor.

The expectation that this result may be universal, valid to all orders in the matter loops, fails however to be fulfilled at least in the case of the bosonic \(\sigma\)–model. Explicit calculations \([4, 5]\) at two–loop order in the matter and to leading order in the semiclassical limit, \(c \to -\infty\), have shown that new structures are produced so that the multiplicative renormalization of the \(\beta\)–function is not maintained. Even though one would always prefer a simple answer, this result does not come as a surprise and it indicates that new physics is produced by the coupling to a curved quantum spacetime.

It becomes of interest to address the corresponding question for the supersymmetric theories. The model with \(N = 1\) supersymmetry is for several aspects very similar
to the bosonic one. On a flat two-dimensional worldsheet it has been studied at the perturbative level using the quantum–background field expansion in normal coordinates [3] and the $\beta$–function has been computed up to high orders in the loop expansion. The leading contribution, proportional to the Ricci tensor, is at one loop, while the next to the leading nonvanishing correction is at four loops [7]. As mentioned above the coupling to $N = 1$ induced supergravity affects multiplicatively the one–loop $\beta$–function. We have not performed the explicit calculation at higher–loop level but, in a way completely parallel to the bosonic example, we expect that due to the interaction with the gravitational field new structures will arise modifying the flat fixed points of the theory. Although it might be of interest to know the exact expression of the gravitational modifications, in a sense no novelties are expected.

The $\sigma$–model with $N = 2$ supersymmetry is singled out already at the one–loop level being its $\beta$–function completely unaffected by the coupling to supergravity. In this case the interest in studying the situation at higher perturbative orders in the matter fields is at least two folded: first there is the question whether this absence of gravitational renormalization will persist beyond the leading correction. The other issue that can be studied in the $N = 2$ context is the influence on the metric $\beta$–function of the dilaton coupling. Indeed in the $N = 2$ model the dilaton term is automatically accounted for through the coupling of the Kähler potential to the supergravity fields [8]: although the classical dilaton coupling vanishes, it reappears and becomes relevant at the quantum level. We have found that its presence introduces significant modifications to the renormalization group trajectories of the $N = 2$ $\sigma$–model coupled to quantum dynamical $N = 2$ supergravity.

Our paper is organized as follows: in the next section we define the model in $N = 2$ superspace and we briefly summarize the relevant steps which lead to the identification of the dilaton coupling. In section 3 we present the quantization of the matter–supergravity coupled system. We use a formulation of the theory in $d = 2 - 2\epsilon$ dimensions and follow closely the perturbative approach in Refs. [9, 5, 10] to which the reader should refer to for details on the quantization of the gravitational fields and of the $N = 2$ $\sigma$–model respectively. Our results are contained in section 4. We conclude with some final comments. Notations and useful formulae for Kähler manifolds are in Appendix A, details of an explicit calculation in Appendix B.

## 2 N=2 matter–supergravity model

We study the $N = 2$ supersymmetric $\sigma$–model coupled to $N = 2$ supergravity using a superfield formulation. The complete action can be written in $d = 2 - 2\epsilon$ dimensions by
dimensional reduction from the $N = 1$ theory in four dimensions [11]

\[ S = S_{SG} + S_{M} \]
\[ = -2 \frac{d-1}{d-2} \kappa^2_0 \int d^d x d^2 \theta d^2 \bar{\theta} \ E^{-1} + \int d^d x d^2 \theta d^2 \bar{\theta} \ E^{-1} K(\Phi^\mu, \bar{\Phi}^{\bar{\mu}}) \quad (2.1) \]

where $E$ is the $N = 2$ vielbein superdeterminant, and $K$ is the Kähler potential of the $n$–dimensional complex manifold described by $n$ covariantly chiral superfields $\Phi^\mu$, $\nabla_\alpha \Phi^\mu = 0$, $\mu = 1, \cdots, n$. (One could add to this action a superpotential, a chiral integral term, but due to the $N = 2$ nonrenormalization theorem it would play no role in the analysis of the renormalization properties of the theory).

In two dimensions and in its minimal formulation, the constraints of $N = 2$ supergravity are solved in terms of two prepotentials, a real vector superfield $H_a$ and a chiral scalar compensator $\sigma$ [13]. In conformal gauge only the compensator is relevant since $H_a$ contains pure gauge modes. Away from two dimensions $H_a$ cannot be gauged away completely. However we expect the perturbative calculation of the $\beta$–function, which is evaluated in the limit $\epsilon \to 0$, to be independent of $H_a$. Indeed the validity of this result has been checked explicitly in the bosonic case up to two loops in the matter fields [3]. Dropping the dependence on $H_a$, with a natural definition of conformal gauge in $d$ dimensions [3], we write

\[ E = e^{\frac{d-2}{d-2} (\sigma + \bar{\sigma})} \]
\[ (2.2) \]

so that the action for the matter system becomes

\[ S_M = \int d^d x d^2 \theta d^2 \bar{\theta} \ e^{(\sigma + \bar{\sigma})} K(\Phi^\mu, \bar{\Phi}^{\bar{\mu}}) \]
\[ (2.3) \]

Note that defining

\[ \tilde{K}(\Phi^\mu, \bar{\Phi}^{\bar{\mu}}, \sigma, \bar{\sigma}) \equiv e^{(\sigma + \bar{\sigma})} K(\Phi^\mu, \bar{\Phi}^{\bar{\mu}}) \]
\[ (2.4) \]

the action in eq. (2.3) describes a $\sigma$–model on a $(n + 1)$–dimensional complex manifold with Kähler potential $\tilde{K}$.

In order to clarify the role played by the $N = 2$ supergravity $\sigma$ superfield, it is convenient to reexpress the theory in $N = 1$ superspace. To this end we introduce coordinates $\theta_1^\alpha = \theta^\alpha + \bar{\theta}^\alpha$, $\theta_2^\alpha = \theta^\alpha - \bar{\theta}^\alpha$ and corresponding derivatives $D_{1\alpha} = D_{\alpha} + \bar{D}_{\alpha}$, $D_{2\alpha} = D_{\alpha} - \bar{D}_{\alpha}$, and define the $N = 1$ projections of the superfields as

\[ \Psi^\mu(\theta_1) \equiv \Phi^\mu(\theta, \bar{\theta}) \Big|_{\theta_2=0} \quad \sigma_1(\theta_1) \equiv \sigma(\theta, \bar{\theta}) \Big|_{\theta_2=0} \]
\[ (2.5) \]

In eq. (2.3) now we integrate on the $\theta_2$ variables following Ref. [3] and obtain

\[ S_M = \frac{1}{4} \int d^d x d^2 \theta_1 (D_2)^2 \tilde{K} \Big|_{\theta_2=0} \]

\[ (2.6) \]
the action (2.6) using real coordinates $\Psi \pm \bar{\Psi}$ where

$$\chi$$

where we have defined $J^i$. At this stage, in terms of the complex structure

$$N$$

integration by parts in (2.7) leads to the

$$N$$

It is easy to identify the terms which correspond to dilaton–type couplings. Indeed,

$$N$$

where $G_{ij}$ is the Kähler metric. We note that in a $(n + 1)$–dimensional complex manifold

the model takes the familiar form

$$N$$

where we have defined $\chi \equiv (\Psi^1, \ldots, \Psi^{2n}, \sigma_1 + \bar{\sigma}_1, \sigma_1 - \bar{\sigma}_1)$ and

$$N$$

It is easy to identify the terms which correspond to dilaton–type couplings. Indeed,

integration by parts in (2.7) leads to the $N = 1$ dilaton action

$$N$$

where $E_1 = e^{\frac{d}{2}J^i(\sigma_1 + \bar{\sigma}_1)}$ is the $N = 1$ vielbein superdeterminant and the $N = 1$ scalar

curvature in $d = 2 - 2\epsilon$ dimensions is given by

$$N$$

The remaining terms which contain $\sigma_1 - \bar{\sigma}_1$, can be identified as the $N = 2$ supersymmetric

partners of the dilaton couplings. In section 4 we will show that these vertices give rise to divergent corrections to the metric $G_{ij}$, not expressible as geometric objects on the manifold.
3 Quantization in superspace

We study now the quantization of the system described by the action in (2.1) directly in $N = 2$ superspace. For the bosonic case a covariant procedure of quantization for the Hilbert–Einstein action away from two dimensions has been proposed in Ref. [9]. Within this approach it has been shown that the coupling of nonconformal ($c \neq 0$) matter to gravity leads to a one–loop renormalization of the gravitational coupling constant $\kappa_0^2$. As a consistency check of the procedure, the exact results of two–dimensional quantum gravity [12] have been reobtained for $\epsilon \to 0$, in the strong coupling regime ($\kappa^2 \gg |\epsilon|$). This analysis has been extended to the $N = 1$ supersymmetric system [14].

In the $N = 2$ case one can use a parallel formulation. The renormalization of the gravitational coupling constant is given by

$$\frac{1}{\kappa_0^2} = \mu^{-2\epsilon} \left( \frac{1}{\kappa^2} - \frac{1}{2} \frac{c - 1}{\epsilon} \right)$$

(3.1)

Therefore, in complete analogy with the bosonic example (see Ref. [5], eq. (2.13)), we take as asymptotic behavior of $\kappa_0^2$ in the limits $c \to -\infty$, $\epsilon \to 0$, and in the strong coupling regime

$$\kappa_0^2 \sim -\frac{2\epsilon}{c}$$

(3.2)

Alternatively this can be viewed as a definition of induced $d$-dimensional gravity, to leading order in the anomaly coefficient $c$.

Our aim is to compute radiative gravitational corrections to the $\beta$–functions of the $N = 2$ $\sigma$–model to leading order in the semiclassical limit $c \to -\infty$. Therefore we consider contributions with only one $\kappa_0^2$ factor (see eq. (3.2)). Since the dependence on the $H_\sigma$ supergravity vector field has been dropped, we only need compute the $\sigma$ chiral compensator propagator and its couplings to the $\Phi$ matter fields. Following a standard procedure, we use the quantum–background field method and perform a linear splitting

$$\Phi^\mu \to \Phi^\mu + \Phi_0^\mu \quad \bar{\Phi}^\mu \to \bar{\Phi}^\mu + \bar{\Phi}_0^\mu$$

(3.3)

in the action (2.1). With a rescaling $\epsilon\sigma \to \sigma$, we obtain

$$S = \int d^d x d^2 \theta d^2 \bar{\theta} \left[ 1 + (\sigma + \bar{\sigma}) + \frac{1}{2}(\sigma + \bar{\sigma})^2 + \cdots \right]$$

$$\left[ -2 \frac{d-1}{d-2} \kappa_0^{-2} + K(\Phi_0, \bar{\Phi}_0) + K_\mu(\Phi_0, \bar{\Phi}_0) \Phi^\mu + K_{\bar{\mu}}(\Phi_0, \bar{\Phi}_0) \bar{\Phi}^{\bar{\mu}} + 
+ \frac{1}{2} K_{\mu\nu}(\Phi_0, \bar{\Phi}_0) \Phi^\mu \Phi^\nu + \frac{1}{2} K_{\bar{\mu}\bar{\nu}}(\Phi_0, \bar{\Phi}_0) \bar{\Phi}^{\bar{\mu}} \bar{\Phi}^{\bar{\nu}} + K_{\mu\bar{\nu}}(\Phi_0, \bar{\Phi}_0) \Phi^\mu \bar{\Phi}^{\bar{\nu}} + \cdots \right]$$

(3.4)
where $\Phi^\mu$, $\bar{\Phi}^\bar{\mu}$, $\sigma$, $\bar{\sigma}$ are the quantum fields. From (3.4) we have the supergravity propagator in the usual form for chiral superfields (in our conventions $\Box = \frac{1}{2} \partial^a \partial_a$)

$$\langle \sigma(x, \theta, \bar{\theta}) \bar{\sigma}(x', \theta', \bar{\theta}') \rangle = -\frac{\kappa_0^2}{d-1} \epsilon^{-1} \Box^{-1} \delta^{(2)}(x-x') \delta^{(2)}(\theta-\theta') \delta^{(2)}(\bar{\theta}-\bar{\theta}')$$  \hspace{1cm} (3.5)

Since we have performed a linear splitting the various terms in the expansion (3.4) are not manifestly covariant under reparametrization of the manifold. However, in the final result the local divergent contributions to the effective action, and correspondingly the counterterms, will be in covariant form [10].

In Ref. [10] where the flat (i.e. in the absence of supergravity) $N=2$ $\sigma$–model was studied, a general procedure for the perturbative evaluation of the ultraviolet divergences was outlined, which on the basis of dimensional considerations and $N=2$ supersymmetry, lead to simplified Feynman rules. In particular the relevant contributions to the tree–level two–point function $\langle \Phi^\mu \bar{\Phi}^{\bar{\mu}} \rangle$ were computed to all orders in the background, and an effective matter propagator was obtained

$$\langle \Phi^\mu(x, \theta, \bar{\theta}) \bar{\Phi}^{\bar{\mu}}(x', \theta', \bar{\theta}') \rangle = -K^{\mu\bar{\nu}...} \Box^{-1} \delta^{(2)}(x-x') \delta^{(2)}(\theta-\theta') \delta^{(2)}(\bar{\theta}-\bar{\theta}')$$ \hspace{1cm} (3.6)

where $K^{\mu\bar{\nu}}$ is the inverse of the Kahler metric. In addition a simplified set of effective vertices was introduced: indeed it was shown that Feynman diagrams containing vertices of the form $K_{\mu\nu\rho...}$ and/or $K_{\bar{\mu}\bar{\nu}\bar{\rho}...}$ with only unbarred or only barred indices do not contribute to divergent quantum corrections. Therefore, in the case of flat $N=2$ superspace the counterterms are always in terms of derivatives of the Kahler metric and expressible as geometric objects (products of the Riemann tensor and its derivatives) in the final result.

In the presence of propagating gravity fields, the same dimensional arguments do apply and exactly the same conclusions can be drawn as far as the matter effective propagator and the matter effective self–interactions are concerned. On the other hand, whenever matter–supergravity couplings are involved some care is needed in order to identify correctly the relevant ones. In this case one has to keep vertices where at least one quantum field has opposite chirality with respect to the others. Therefore, we cannot discard vertices with only unbarred or barred indices on $K$ if a quantum gravity line of opposite chirality is present, i.e. vertices like $K_{\mu\nu...\rho} \Phi^\mu \Phi^\nu... \bar{\Phi}^{\bar{\rho}}$ are relevant.

With this set of rules in mind we proceed as follows: first, since we study supergravity effects in the semiclassical limit $c \rightarrow -\infty$, at any loop order we draw all the diagrams which have only one gravity line (the leading order in $1/c$). Then on each diagram we perform the $D$–algebra as explained in Ref. [10] and reduce the corresponding expressions.
to standard momentum integrals which we evaluate using supersymmetric dimensional regularization and minimal subtraction. In order to extract the overall divergence we subtract the ultraviolet subdivergences corresponding to lower–order renormalizations and remove infrared infinities by using the procedure described in Refs. [15, 16, 5].

The quantum counterterms we want to compute correspond to local corrections to the Kähler potential. In dimensional regularization they have the form

$$K \to K + \sum_{n=1}^{\infty} \sum_{l=n}^{\infty} \frac{1}{\epsilon^n} K^{(n,l)}$$

so that the renormalization of the Kähler metric is given by

$$G_{\mu \bar{\nu}}^B = G_{\mu \bar{\nu}}^R + \sum_{n=1}^{\infty} \sum_{l=n}^{\infty} \frac{1}{\epsilon^n} T^{(n,l)}_{\mu \bar{\nu}}$$

Correspondingly the $\beta$–function is

$$\beta_{\mu \bar{\nu}}(G^R) = 2\epsilon G_{\mu \bar{\nu}}^R + 2 \left( 1 + \lambda \frac{\partial}{\partial \lambda} \right) \sum_{l=1}^{\infty} T^{(1,l)}_{\mu \bar{\nu}} (\lambda^{-1} G^R)|_{\lambda=1}$$

Therefore in order to evaluate the perturbative corrections to the $\beta$–functions we concentrate on the first order pole contributions in the $\epsilon$–expansion. This allows to discard Feynman diagrams which reduce to tadpole–like diagrams when in the process of performing the $D$-algebra, matter propagators are cancelled by momentum factors. Indeed it is easy to show that after subtraction of subdivergences they give rise to higher order $1/\epsilon$ poles. On the contrary, a careful analysis of the subtraction of subdivergences shows that in general we cannot drop corresponding diagrams in which a gravity propagator would be cancelled. We illustrate this rather subtle point in Appendix B with a specific example.

As it has been observed in Ref. [3], for the $N = 2$ theories the cosmological term is given by a chiral integral and as such it is not renormalized. Consequently the Liouville field does not acquire anomalous dimensions and the physical scale is not modified with respect to the standard renormalization scale. This implies that the renormalization group $\beta$–functions in the presence of $N = 2$ dynamical gravity are not rescaled by a multiplicative factor. The only possible corrections, if present at all, must be given by new, nontrivial contributions from gravity propagating inside the loops. At one loop no modification has been found [3].

In the next section we present the explicit calculation of the gravitational dressing of the $\beta$–function at two loops in the matter fields.
4 Gravitational dressing at two–loop order

Now we consider quantum corrections at two loops in the presence of gravity. (Note that whenever we say at \( n \)-loop order in the matter fields we are considering \( n \)-loop diagrams with a gravity propagator inserted, so that we are effectively at \( n + 1 \) loops.) Since the \( \sigma \) propagator is \( O(\varepsilon^2) \), the first divergent, gravitational–induced contribution can only arise at two loops in the matter fields.

In the absence of supergravity the second order correction to the Kähler potential has been computed in Ref. [10] and it is given by \( K^{(2,2)} = R \). Therefore, as it is well known, in the flat case the \( \beta \)-function does not receive any contribution at two loops.

In order to evaluate the gravitational dressing we need consider two-loop matter graphs with one \( \sigma \) propagator inserted. Keeping in mind the set of effective Feynman rules discussed in the previous section, one selects the relevant diagrams which in the end will give contributions proportional to \( 1/\varepsilon \). They are drawn in Figs. 1, 2, 3 where the structure of the \( D \)-derivatives is explicitly indicated. We have not drawn diagrams containing two–point matter–gravity vertices that by integration by parts of \( D^2 \) factors reduce immediately to the structures shown in the figures. Their dependence on the background contributes to covariantize the matter couplings according to the formulae (A.8–A.10) (see Appendix A for more details).

We perform the \( D \) algebra in such a way to reduce all the diagrams in Figs. 1, 2, 3 to the diagrams 1a, 2a and 3a respectively. Again this can be achieved by partial integration: at the vertices which contain only one \( D^2 \) (\( \bar{D}^2 \)) and a number of \( \bar{D}^2 \) (\( D^2 \)) we integrate by parts one \( \bar{D}^2 \) (\( D^2 \)) and use the relations \( \bar{D}^2 D^2 \bar{D}^2 = \square \bar{D}^2 \), \( D^2 \bar{D}^2 D^2 = \square D^2 \) to cancel the propagator of the corresponding line. Then the \( \bar{D}^2 \) (\( D^2 \)) factor is integrated back to the original line. Applying this procedure a number of times one obtains the above stated result. The dependence on the background fields can be reconstructed diagram by diagram looking at the structure of the vertices, so that including combinatoric factors and making use of the relations (A.6, A.8–A.10) we obtain the following result: from the diagrams in Fig.1

\[ \frac{1}{6} D_\mu D_\nu D_\rho K D^{\mu} D^{\nu} D^{\rho} K I_1 \]  

(4.1)

where \( I_1 \) is the contribution corresponding to diagram 1a. Diagrams in Fig. 2 sum up to

\[ \frac{1}{2} D_\mu D_\nu D_\rho K D^{\mu} D^{\nu} D^{\rho} K I_2 \]  

(4.2)

where \( I_2 \) corresponds to the diagram 2a, whereas from the diagrams in Fig.3 we obtain

\[ \frac{1}{4} R_{\mu\bar{\nu}\nu\bar{\rho}} D^{\mu} D^{\nu} K D^{\bar{\mu}} D^{\bar{\rho}} K I_3 \] 

(4.3)
\[ I_3 \text{ being the contribution of diagram } 3a. \text{ To evaluate } I_1, I_2 \text{ and } I_3 \text{ we first complete the } D-\text{algebra as indicated in Figs. 4, 5, 6. Whenever we produce a } \Box \text{ factor on a matter line we drop the corresponding tadpole contribution. Moreover we integrate } \partial-\text{derivatives by parts in order to reduce all the contributions to products of tadpoles that, as explained in the previous section and in Appendix B, we keep only if the cancelled propagator is a gravity line. In dimensional regularization and in our IR subtraction scheme } [13, 5], \text{ the elementary tadpole integral } I = \int \frac{d^d \rho}{(2\pi)^d} \frac{1}{\rho^2} \text{ is computed by shifting the propagator } \frac{1}{\rho^2} \to \frac{1}{\rho^2} + \frac{\epsilon}{\epsilon} \delta^{(2)}(p^2), \text{ so that one has} \]

\[ I \equiv \int \frac{d^d \rho}{(2\pi)^d} \frac{1}{\rho^2} \to \frac{1}{4\pi} \frac{1}{\epsilon} \tag{4.4} \]

In this fashion the overall divergence of the diagrams in Figs. 4, 5, 6 can be easily determined

\[ I_1 = -\kappa_0 \epsilon I^3 = \frac{1}{(4\pi)^3} \frac{2}{c} \frac{1}{\epsilon} \]

\[ I_2 = \kappa_0 \epsilon (-I^3 + \frac{2}{3} I^3) = \frac{1}{(4\pi)^3} \frac{2}{3c} \frac{1}{\epsilon} \]

\[ I_3 = \frac{\kappa_0 \epsilon}{3} \frac{4}{3} I^3 = -\frac{1}{(4\pi)^3} \frac{8}{3c} \frac{1}{\epsilon} \tag{4.5} \]

Collecting the results from eqs. (4.1–4.3) and (1.3) we obtain the final expression for the \(1/c\)-correction to the Kähler potential at two–loop order

\[ K^{(1,2)} = \frac{1}{(4\pi)^3} \frac{1}{3c} \left[ D_\mu D_\nu D_\rho K D^\mu D^\nu D^\rho K + D_\mu D_\nu D_\rho K D^\mu D^\nu D^\rho K \right. \]

\[ \left. -2R_{\mu\nu\rho\sigma} D^\mu D^\nu K D^\rho D^\sigma K \right] \tag{4.6} \]

The result can be expressed in terms of real coordinates as

\[ K^{(1,2)} = \frac{1}{(4\pi)^3} \frac{1}{3c} \left[ 2D_i D_j D_k K D^i D^j D^k K - R_{ijklm} D^i D^j D^k D^m K - 2R \right] \tag{4.7} \]

From the expression (4.6) we immediately obtain the \(1/\epsilon\)-correction to the Kähler metric

\[ T^{(1,2)}_{\mu\nu} = \frac{\partial K^{(1,2)}}{\partial \Phi^\mu \partial \Phi^\nu} \tag{4.8} \]

We conclude that in the presence of propagating gravity the matter \(\beta\)-function is corrected at two loops

\[ \beta^{(2)}_{\mu\nu} = \frac{1}{(4\pi)^3} \frac{2}{3c} \left[ D_\mu D_\nu \left[ D_\rho D_\delta D_\tau K D^\rho D^\delta D^\tau K + D_\rho D_\delta D_\tau K D^\rho D^\delta D^\tau K \right. \right. \]

\[ \left. \left. -2R_{\rho\delta\sigma\tau} D^\rho D^\delta K D^\sigma D^\tau K \right] \right] \tag{4.9} \]
In order to reexpress the above result in terms of real coordinates we need introduce the complex structure $J^i_j$. We have

$$\beta^{(2)}_{ij} = (D_i D_j + J^m_i J^m_j D_mD_n) K^{(1,2)} \tag{4.10}$$

where $K^{(1,2)}$ is given in Eq. (4.7).

The presence of the complex structure in the metric $\beta$–function is consistent with the fact that in the $N = 1$ formalism the gravity–matter coupling in the bare action (2.7) is indeed proportional to $J^i_j$.

## 5 Conclusions

The main result of our work is contained in eq. (4.9): the $\beta$–function of the $N = 2$ supersymmetric $\sigma$–model receives its first gravitational correction at two loops in the matter fields. The dressing, absent at lower orders, is given by structures which are not geometric objects of the Kähler manifold. This new type of divergences can be interpreted as dilatonic contributions. As mentioned earlier, once the interaction between the Kähler potential and $N = 2$ supergravity is switched on, supergravity–dilaton vertices are automatically included: they vanish in the classical limit, being $O(\epsilon)$, but at the quantum level a dilaton term is induced in the effective action. At two loops in the matter system with one insertion of $N = 2$ supergravity, it is the dilaton which gives a nonvanishing correction to the metric $\beta$–function.

We observe that our calculation is not affected by scheme dependence ambiguities. In fact the first, nonvanishing, gravitational correction is the one in eq. (4.9). Therefore no $O(1/c)$ conventional subtraction ambiguities can be produced from finite subtractions proportional to lower–loop counterterms. Moreover, even if one were to be perverse and take into account finite subtractions proportional to the one–loop matter counterterm, the ambiguities would be given by geometric structures and they would never mix with the terms in eq. (4.9).

In order to fully understand the role played by the dilaton field, it would be quite interesting to consider the $N = 1$ supersymmetric $\sigma$–model with both metric and dilaton couplings to dynamical supergravity. For this system one could compute the corrections to the metric $\beta$–function induced by the supergravity–dilaton interactions and interpret then the equations $\beta_{ij} = 0$ as equations of motion for the noncritical superstring.

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A Kähler manifolds

A Kähler manifold is a complex manifold with vanishing torsion. It is endowed with a complex structure \( J^i_j \) which satisfies \( J^2 = -1 \) and is an isometry of the metric

\[
J^i_j J^k_l G_{ik} = G_{jl} \tag{A.1}
\]

Moreover it is covariantly constant as a consequence of the vanishing of the torsion. Therefore it is always possible to choose a suitable set of complex coordinates on the manifold so that the complex structure has the standard form

\[
J^\mu_{\bar{\nu}} = i \delta^\mu_{\nu} \quad J^{\bar{\mu}}_{\bar{\nu}} = -i \delta^{\bar{\mu}}_{\bar{\nu}} \tag{A.2}
\]

In this coordinate system the Kähler metric satisfies

\[
\partial_\rho G^\mu_{\bar{\nu}} = \partial_\mu G^\rho_{\bar{\nu}} \quad \partial_\beta G^\mu_{\bar{\nu}} = \partial_\beta G^\mu_{\bar{\nu}} \quad G_{\mu\nu} = 0 \tag{A.3}
\]

and locally it can be expressed in terms of the Kähler potential \( K \) as

\[
G^\mu_{\bar{\nu}} = \frac{\partial K}{\partial \Phi^\mu \partial \bar{\Phi}^{\bar{\nu}}} \tag{A.4}
\]

In general we use the following notation

\[
K_{\mu_1 \cdots \mu_p \bar{\nu}_1 \cdots \bar{\nu}_q} \equiv \frac{\partial^\mu}{\partial \Phi^\mu_1 \cdots \partial \Phi^\mu_p} \frac{\partial^{\bar{\nu}}}{\partial \bar{\Phi}^{\bar{\nu}_1} \cdots \partial \bar{\Phi}^{\bar{\nu}_q}} K \tag{A.5}
\]

so that \( G^{\mu_{\bar{\nu}}} = K_{\mu_{\bar{\nu}}} \).

The only nonvanishing components of the connection are \( \Gamma^\mu_{\nu\rho} \) and \( \Gamma^\mu_{\bar{\nu}\bar{\rho}} \), and the Riemann and Ricci tensors are given by

\[
R^{\mu_{\bar{\nu}}}_{\mu\bar{\nu}} = K^{\mu_{\bar{\nu}}}_{\mu\bar{\nu}} - K^{\mu_{\bar{\nu}}}_{\mu\bar{\eta}} K^{\rho\bar{\rho}}_{\rho\bar{\eta}} \tag{A.6}
\]

\[
R^{\mu}_{\mu\bar{\nu}} \equiv R^\rho_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \log \det K_{\sigma\bar{\sigma}} \tag{A.7}
\]

We list some useful identities involving covariant derivatives of the Kähler potential which have been used in the calculation of the corrections to the metric \( \beta \)-function

\[
D_\mu D_{\nu} K = K^{\rho}_{\mu\nu} - K^{\rho}_{\mu\bar{\nu}} K^{\sigma}_{\rho} \tag{A.8}
\]

\[
D_{\bar{\mu}} D_{\bar{\nu}} D_\rho K = [-K^{\rho}_{\mu\nu} + K^{\rho}_{\mu\bar{\nu}} K^{\sigma}_{\nu}] K^{\sigma}_{\bar{\sigma}} K_{\sigma} \tag{A.9}
\]
\[ D_\mu D_\nu D_\rho K = K_{\mu\nu\rho} - K^{\sigma\rho} \left[ K_{\mu\sigma\rho} K_\sigma - 3 K_{\sigma(\mu\nu} K_{\rho)\sigma} + 3 K^{\rho\eta} K_{\sigma(\mu\nu} K_{\rho)\sigma\eta} K_\eta \right] \]  

(A.10)

At the perturbative level these covariant couplings come from resummation of different contributions involving mixed matter–gravity vertices. As an example we consider a vertex proportional to \( K_{\mu\nu} \) with two matter and one gravity lines as in Fig. 7a. As shown in Fig. 7b a graph with a mixed two–point vertex reduces to the one in Fig. 7a once the \( D^2 \) on the gravity line has been integrated by parts on the matter. Its background structure \( -K_{\mu\rho} K^{\rho\bar{\sigma}} K_{\bar{\sigma}} \) is indeed what one needs in order to covariantize \( K_{\mu\nu} \) as in Eq. (A.8).

B  An example

In this appendix we show on a simple example how one must carefully operate when momentum factors, which cancel the gravity propagator, are produced by \( D \)–algebra manipulations.

To be pedagogical let us consider first the graph in Fig. 8a where the \( D^2 \) factors are explicitly indicated at one vertex. By performing the \( D \)–algebra the matter propagator carrying the \( \bar{D}^2 \) factor can be immediately cancelled and we are left with the momentum structure shown in Fig. 8b. This graph is tadpole–like and does not contribute to the \( 1/\epsilon \) pole. In fact, in the evaluation of the corresponding integral we must subtract two one–loop subdivergences (corresponding to the \( A \) and \( B \) loops), and one two–loop subdivergence (corresponding to the subgraph \( A \cup B \)) and in so doing we obtain

\[ (\epsilon^2 I)^2 - 2 \frac{1}{\epsilon} (\epsilon^2 I) I - \left[ I^2 - 2 \frac{1}{\epsilon} I \right]_{div} (\epsilon^2 I) = 0 \]  

(B.1)

where \( I \) is the elementary tadpole integral (4.4) and we have neglected \( 1/4\pi \)-factors for notational convenience.

Let us consider now the graph in Fig. 8c. In this case by performing the \( D \)–algebra the gravity propagator is cancelled and the resulting momentum structure is shown in Fig. 8d. Again we are lead to evaluate a tadpole–like integral, but now none of the subgraphs is divergent because either it does not include the (cancelled) gravity line and then it is finite by itself, or else it includes the cancelled gravity and then it becomes finite once multiplied by the \( \epsilon^2 \) carried by the gravity propagator. Hence there are no subdivergences and the result is simply

\[ (\epsilon^2 I)^2 = \frac{1}{\epsilon} \]  

(B.2)
More generally, when the gravity propagator has been cancelled we can obtain contributions of order $1/\epsilon$ even if the graph corresponding to the momentum structure is tadpole-like. In these cases we cannot discard the graph.
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Figure 1: Two-loop divergent diagrams contributing to formula (4.1).

The dashed line denotes the gravity propagator.
Figure 2: Two-loop diagrams contributing to formula (4.2).
Figure 3: Two-loop diagram contributing to formula (4.3).
Figure 4: D-algebra for the diagram in Fig. 1a.
Figure 5: D-algebra for the diagram in Fig. 2a. The arrows on the propagators denote contracted momentum factors.
Figure 6: D-algebra for the diagram in Fig. 3a.
Figure 7: Covariantization of the vertex with two matter and one gravity lines.
Figure 8: D-algebra for the example in Appendix B.