KERNEL REPRESENTATION FORMULA FROM COMPLEX TO REAL WIENER-ITÔ INTEGRALS AND VICE VERSA

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Abstract. We clearly characterize the relation between real and complex Wiener-Itô integrals. Given a complex multiple Wiener-Itô integral, we get explicit expressions for two kernels of its real and imaginary parts. Conversely, consider a two-dimensional real Wiener-Itô integral, we obtain the representation formula by a finite sum of complex Wiener-Itô integrals. The main tools are a recursion technique and Malliavin derivative operators. We build a bridge between real and complex Wiener-Itô integrals.

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Contents

Notations 2
1. Introduction 2
2. Preliminaries 5
2.1. Real isonormal Gaussian process 5
2.2. Complex isonormal Gaussian process 6
2.3. Malliavin derivative operators 8
3. Kernel representation formula from complex to real Wiener-Itô integrals 9
3.1. Uniqueness theorem 10
3.2. Representation theorem 13
3.3. Back to Itô’s theory 16
3.4. Generalized Stroock’s formula 18
4. Kernel representation formula from real to complex Wiener-Itô integrals 22
4.1. Uniqueness theorem 22
4.2. Representation theorem 23
5. Proofs of main results 25
5.1. Proof of Lemma 3.1 25
5.2. Proofs of Lemma 3.4, Proposition 3.5 and Theorem 3.3 26
5.3. Proofs of Theorem 3.7 and Theorem 3.9 27
5.4. Proof of Theorem 4.1 31
References 32

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Notations

\( \mathcal{H} \) : a real separable Hilbert space

\( \mathcal{H} ⊕ \mathcal{H} \) : the Hilbert space direct sum

\( \mathcal{H}_C \) : the complexification of \( \mathcal{H} \)

\( \mathcal{H} ⊙ m, (\mathcal{H} ⊕ \mathcal{H}) ⊙ m, \mathcal{H}_C ⊙ m \) : the \( m \) times symmetric tensor product of \( \mathcal{H}, \mathcal{H} ⊕ \mathcal{H}, \mathcal{H}_C \)

\( X, Y \) : the real Gaussian isonormal process over \( \mathcal{H} \)

\( W \) : the real Gaussian isonormal process over \( \mathcal{H} ⊕ \mathcal{H} \)

\( X_C, Y_C \) : the complexification of \( X, Y \)

\( Z \) : the complex Gaussian isonormal process over \( \mathcal{H}_C \)

\( \mathcal{H}_n(X), \mathcal{H}_n(Y), \mathcal{H}_n(W) \) : the \( n \)-th Wiener-Itô chaos of \( X, Y, W \)

\( \mathcal{H}_n^C(X), \mathcal{H}_n^C(W) \) : the complexification of \( \mathcal{H}_n(X), \mathcal{H}_n(W) \)

\( \mathcal{H}_{m,n}(Z) \) : the \( (m, n) \)-th complex Wiener-Itô chaos of \( Z \)

\( \text{symm}(f ⊗ g), f ⊗ g \) : symmetric tensor product of \( f \) and \( g \)

\( \Lambda \) : the set of all sequences \( p = \{p_k\}_{k=1}^{∞} \) of non-negative integers with a finite sum

\( \text{Re} z, \text{Im} z \) : the real and imaginary parts of a complex number \( z \)

\( \{e_k = e_k^1 + i e_k^2\}_{k≥1} \) : complete and orthogonal elements with norm \( \sqrt{2} \) in \( \mathcal{H}_C \),

\( e_k^1 = \text{Re} e_k, e_k^2 = \text{Im} e_k \)

\( \{u_{1,0}(k), v_{1,0}(k)\}_{k≥1} \) : a complete and orthonormal basis of \( \mathcal{H} ⊕ \mathcal{H} \),

\( u_{1,0}(k) = \frac{1}{\sqrt{2}}(e_k^1, -e_k^2), v_{1,0}(k) = \frac{1}{\sqrt{2}}(e_k^2, e_k^1) \)

1. Introduction

In 1951, Itô published the seminal article [16] and defined multiple Wiener integral with respect to a normal random measure which was introduced first and termed polynomial chaos by Wiener in [31]. Itô showed that multiple Wiener integrals of different degrees are orthogonal to each other and closely related to Hermite polynomials. By making use of the relation between multiple Wiener integrals and Hermite polynomials, Itô developed the chaos decomposition theory which leads to an orthogonal expansion of any square integrable functional of the normal random measure. Shortly afterwards, Itô in [17] established the theory of complex multiple Wiener-Itô integrals with respect to a complex normal random measure in 1952. Itô firstly defined Hermite polynomials of complex variables, also first called Hermite-Laguerre-Itô polynomials in [9], and showed the close relation between Hermite polynomials of complex variables and complex multiple Wiener-Itô integrals. Further, Itô utilized the results obtained to generalize and derive the spectral structure and ergodicity of the shift transformation of normal screw lines such as complex Wiener processes and complex normal stationary processes. Thereafter, there has been a recent interest in theoretical studies on complex Gaussian fields, specifically complex Wiener-Itô integrals, and Hermite-Laguerre-Itô polynomials. For example, in [6, 8, 10], the authors showed the fourth moment theorem for complex multiple Wiener-Itô integrals. That is, a sequence of complex multiple Wiener-Itô integrals converges in distribution to the bivariate normal distribution if and only if the absolute moment up to fourth converges. The product formula, independence and asymptotic moment-independence for complex multiple Wiener-Itô integrals were obtained in [7]. In [11], the
author got some properties of complex multiple Wiener-Itô integrals, complex Ornstein-Uhlenbeck operators and semigroups. For Hermite-Laguerre-Itô polynomials, detailed analytic properties, relation with real Hermite polynomials and other related researches can be found in [2, 12, 15].

The researches of complex Gaussian fields are strongly motivated by a variety of applications. For instance, in probabilistic models of cosmic microwave background radiation, it is important to understand high-frequency behaviour, namely central limit theorems for the Fourier coefficients associated with complex Gaussian subordinated fields which are identically distributed to functionals of complex Wiener-Itô integrals, see [13, 20] for more details. The cubic complex Ginzburg-Landau equation is one of the most crucial nonlinear partial differential equations in applied mathematics and physics which describes various physical phenomena such as nonlinear waves, second-order phase transition, superconductivity and superfluidity, see [3]. The stochastic cubic complex Ginzburg-Landau equation with complex space-time white noise on the three dimensional torus can be studied by using theories of complex Wiener-Itô integrals, see [13] for instance. The Ornstein-Uhlenbeck process was introduced in [4] to model the Chandler wobble or variation of latitude concerning with the rotation of the earth, and later has been heavily used in finance and econophysics. Statistical inference for parameter estimators of the Ornstein-Uhlenbeck process such as consistency and asymptotic normality can be obtained by using techniques of complex Wiener-Itô integrals, see [8, 27] for example. Moreover, the eigenfunctions of an Ornstein-Uhlenbeck operator which are closely related to the Hermite-Laguerre-Itô polynomials have been found to be useful for applications such as simulating rare events and approximating solutions to the Fokker-Planck equation, see [9, 30, 32, 33]. In the field of communication and signal processing, the noise is often supposed to be complex Gaussian noise, see [1, 5, 21, 26] for more details.

In order to investigate problems concerned with practical models mentioned above, we have to develop theories of complex square integrable functionals of a complex Gaussian process and deeply understand the structure of complex Wiener-Itô integrals based on Itô’s theories in [17]. We aim to not only explore specific properties of complex Wiener-Itô integrals but also establish the relation between real and complex Wiener-Itô integrals to make use of abundant theories of real Wiener-Itô integrals in [19, 22, 23, 24, 25] and so on. Chen and Liu in [10, Theorem 3.3] showed that the real and imaginary parts of a complex $(p, q)$-th Wiener-Itô integral can be expressed as a real Wiener-Itô integral of order $p + q$, respectively. This result provides a natural and direct perspective that one-dimensional complex Wiener-Itô integral can be regarded as a two-dimensional real Wiener-Itô integral. In this paper, we focus on completely characterizing the expressions for kernels of one-dimensional complex and two-dimensional real Wiener-Itô integrals and thus build a bridge between them. One the one hand, the explicit and computable representations for kernels guarantee that we can solve theoretical and practical problems concerning with complex Wiener-Itô integrals by utilizing the theories of two-dimensional real Wiener-Itô integrals. For example, a sufficient and necessary condition for the existence of the density of a complex Wiener-Itô integral is derived by using [22, Theorem 3.1] or [25, Theorem 3], see Corollary [3,10] also, in practical models, some statistical properties of a
complex Wiener-Itô integral can be easily verified with the help of explicit expressions for kernels, see Example 3.15 and Example 4.5 for instance. On the other hand, we realize that, for a complex Wiener-Itô integral of a higher order, the representations for kernels of real and imaginary parts are rather complicated and seems difficult to directly apply to some practical models. This reminds us the necessity to further develop the theories of complex Wiener-Itô integral itself.

[10, Theorem 3.3] is not convenient to use sometimes since it only shows the existences of kernels, for which expressions depend on some redundant parameters, of real and imaginary parts of a complex Wiener-Itô integral. In Section 3 we show how to get explicit representations for kernels of real and complex parts of a complex multiple Wiener-Itô integral. In Section 3.1, we remove those redundant parameters in the proof of [10, Theorem 3.3] and prove the uniqueness of the kernels of the real and imaginary parts, see Theorem 3.3. In Section 3.2, based on a complete orthonormal system of complex Wiener-Itô chaos shown by Itô in [17, Theorem 14] and defined as (2.4), we obtain more explicit recursion formulae by an induction argument, which makes full use of recursion formulae concerning complex multiple Wiener integrals established by Itô in [17, Theorem 9], see (3.7), (3.8) and Theorem 3.9. We stress that recursion formulae actually offer an algorithm to derive the representation for kernels of real and complex parts of a complex multiple Wiener-Itô integral of a higher order. In Section 3.4, we combine the Stroock’s formula with the relation between the real and complex Malliavin derivative operators, and then get another computable expressions for the kernels of the real and imaginary parts of a complex multiple Wiener-Itô integral, see Corollary 3.14. In a word, we refine [10, Theorem 3.3] and clearly characterize the kernels of the real and imaginary parts of a complex multiple Wiener-Itô integral.

In Section 4, we conversely consider a complex random variable whose real and imaginary parts are two real multiple Wiener-Itô integrals. Essentially, we show how to represent a two-dimensional real Wiener-Itô integrals as a finite sum of one-dimensional complex Wiener-Itô integrals. We firstly prove the uniqueness of this representation in Section 4.1, see Theorem 4.1. In Section 4.2, we get computable expressions for kernels of complex Wiener-Itô integrals by using the complex Stroock’s formula and the relation between the real and complex Malliavin derivative operators, see Theorem 4.3. Furthermore, in Theorem 4.4, we derive an equivalent condition that a two-dimensional real and a one-dimensional complex Wiener-Itô integral can be represented by each other.

The paper is organized as follows. Section 2 introduces some elements of the real and complex Gaussian isonormal process and Malliavin calculus. In Section 3.1, Section 3.2 and Section 3.4, we clearly characterize the kernels of real and imaginary parts of a complex multiple Wiener-Itô integral. In section 3.3, we revisit the classical theory of Itô’s complex multiple integrals and realize the results in Section 3.2 to complex multiple integrals with respect to a complex Brownian motion. In Section 4.1 and Section 4.2, we prove that a complex random variable, whose real and imaginary parts are two real multiple Wiener-Itô integrals, can be uniquely expressed as a finite sum of complex Wiener-Itô integrals and expressions for kernels of these complex Wiener-Itô integrals are obtained. The proofs of the main theorems of this paper are presented in Section 5.
2. Preliminaries

In this section, we briefly introduce some basic theories of the real isonormal Gaussian process, the complex isonormal Gaussian process and Malliavin calculus. See [10, 17, 23, 24] for more details.

2.1. Real isonormal Gaussian process. Suppose that $\mathcal{H}$ is a real separable Hilbert space with an inner product denoted by $(\cdot, \cdot)_{\mathcal{H}}$. Let $\|h\|_{\mathcal{H}}$ denote the norm of $h \in \mathcal{H}$. Consider an isonormal Gaussian process $X = \{X(h) : h \in \mathcal{H}\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where the $\sigma$-algebra $\mathcal{F}$ is generated by $X$. That is, $X = \{X(h) : h \in \mathcal{H}\}$ is a Gaussian family of centered random variables such that $\mathbb{E}[X(h)X(g)] = (h, g)_{\mathcal{H}}$ for any $h, g \in \mathcal{H}$.

For $n \geq 0$, the $n$-th Wiener-Itô chaos $\mathcal{H}_n(X)$ of $X$ is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(X(h)) : h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where $H_n(x)$ is the Hermite polynomial of degree $n$ defined by the equality

$$\exp \left\{ tx - \frac{1}{2} t^2 \right\} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x).$$

We denote by $\Lambda$ the set of all sequences $a = \{a_k\}_{k=1}^{\infty}$ of non-negative integers with a finite sum. Set $a! = \prod_{k=1}^{\infty} a_k!$ and $|a| = \sum_{k=1}^{\infty} a_k$. For $a = \{a_k\}_{k=1}^{\infty}, b = \{b_k\}_{k=1}^{\infty} \in \Lambda$, we define $a + b = \{a_k + b_k\}_{k=1}^{\infty}$ and say $a \leq b$ if $a_k \leq b_k$ for all $k \geq 1$. Let $\mathcal{H}_n^{\otimes n}$ and $\mathcal{H}_n^{\otimes n}$ denote the $n$-th tensor product and the $n$-th symmetric tensor product of $\mathcal{H}$, respectively. Let $\{\eta_k, k \geq 1\}$ be a complete orthonormal system in $\mathcal{H}$. For a sequence $m = \{m_k\}_{k=1}^{\infty} \in \Lambda$, define the Fourier-Hermite polynomial as

$$H_m = \frac{1}{\sqrt{m!}} \prod_{k=1}^{\infty} H_{m_k}(X(\eta_k)).$$

Denote by $\text{symm}(f \otimes g)$ the symmetrization of $f \otimes g$, where $f, g \in \mathcal{H}$. Let $|m| = n$. For $n \geq 1$, the mapping

$$I_n \left( \text{symm} \left( \otimes_{k=1}^{\infty} \eta_k^{m_k} \right) \right) := \sqrt{m!} H_m$$

provides a linear isometry between the symmetric tensor product $\mathcal{H}_n^{\otimes n}$, equipped with the norm $\sqrt{n!} \cdot \|\cdot\|_{\mathcal{H}_n^{\otimes n}}$, and the $n$-th Wiener-Itô chaos $\mathcal{H}_n(X)$. For $n = 0$, $I_0$ is the identity map. Note that Itô firstly proved (2.1) in [16, Theorem 3.1]. For any $f \in \mathcal{H}_n^{\otimes n}$, the random variable $I_n(f)$ is called the real $n$-th Wiener-Itô integral of $f$ with respect to $X$. Wiener-Itô chaos decomposition of $L^2(\Omega, \sigma(X), P)$ implies that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_n(X)$. That is, any random variable $F \in L^2(\Omega, \sigma(X), P)$ admits a unique expansion of the form

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_0 = \mathbb{E}[F]$, and $f_n \in \mathcal{H}_n^{\otimes n}$ with $n \geq 1$ are uniquely determined by $F$. 


Given \( f \in \mathcal{H}^{\otimes p} \), \( g \in \mathcal{H}^{\otimes q} \), for \( r = 0, \ldots, p \wedge q \), the \( r \)-th contraction of \( f \) and \( g \) is an element of \( \mathcal{H}^{\otimes (p+q-2r)} \) defined by

\[
f \otimes_r g = \sum_{i_1, \ldots, i_r = 1}^{\infty} \langle f, \eta_{i_1} \otimes \cdots \otimes \eta_{i_r} \rangle_{\mathcal{H}^{\otimes r}} \otimes \langle g, \eta_{i_1} \otimes \cdots \otimes \eta_{i_r} \rangle_{\mathcal{H}^{\otimes r}}.
\]

Notice that \( f \otimes_r g \) is not necessarily symmetric, we denote by \( f \otimes g \) its symmetrization. \( [23] \) Proposition 2.7.10] provides the product formula for real multiple Wiener-Itô integrals as follows. For \( f \in \mathcal{H}^{\otimes p} \) and \( g \in \mathcal{H}^{\otimes q} \) with \( p, q \geq 0 \),

\[
\tag{2.2}
I_p(f)I_q(g) = \sum_{r=0}^{p+q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes g).
\]

2.2. Complex isonormal Gaussian process. Next, we introduce the complex isonormal Gaussian process. We complexify \( \mathcal{H} \), \( L^2(\Omega) \) in the usual way and denote by \( \mathcal{H}_C \), \( L^2_C(\Omega) \) respectively. Suppose \( h = f + ig \in \mathcal{H}_C \) with \( f, g \in \mathcal{H} \), we write

\[
X_C(h) := X(f) + iX(g),
\]

which satisfies \( \mathbb{E} \left[ X_C(h) X_C(h') \right] = \langle h, h' \rangle_{\mathcal{H}_C} \) with \( h' \in \mathcal{H}_C \). Let \( Y = \{ Y(h) : h \in \mathcal{H} \} \) is an independent copy of the isonormal Gaussian process \( X \) over \( \mathcal{H} \). Define \( Y_C(h) \) same as above. Let

\[
\tag{2.3}
Z(h) := \frac{X_C(h) + iY_C(h)}{\sqrt{2}}, \quad h \in \mathcal{H}_C,
\]

and we call \( Z = \{ Z(h) : h \in \mathcal{H}_C \} \) a complex isonormal Gaussian process over \( \mathcal{H}_C \), which is a centered symmetric complex Gaussian family satisfying

\[
\mathbb{E}[Z(h)]^2 = 0, \quad \mathbb{E}[Z(h)Z(\overline{h'})] = \langle h, h' \rangle_{\mathcal{H}_C}, \quad \forall h, h' \in \mathcal{H}_C.
\]

For each \( p, q \geq 0 \), let \( \mathcal{H}_{p,q}(Z) \) be the closed linear subspace of \( L^2_C(\Omega) \) generated by the random variables \( \{ J_{p,q}(Z(h)) : h \in \mathcal{H}_C, \| h \|_{\mathcal{H}_C} = \sqrt{2} \} \), where \( J_{p,q}(z) \) is the complex Hermite polynomial, or Hermite-Laguerre-Itô polynomial, given by

\[
\exp \left\{ \lambda \bar{z} + \bar{\lambda} z - 2|\lambda|^2 \right\} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\lambda^p \lambda^q}{p! q!} J_{p,q}(z), \quad \lambda \in \mathbb{C}.
\]

The space \( \mathcal{H}_{p,q}(Z) \) is called the \( (p, q) \)-th Wiener-Itô chaos of \( Z \).

Take a complete orthonormal system \( \{ \xi_k, k \geq 1 \} \) in \( \mathcal{H}_C \). For two sequences \( p = \{ p_k \}_{k=1}^{\infty}, q = \{ q_k \}_{k=1}^{\infty} \in \Lambda \), define a complex Fourier-Hermite polynomial or Fourier-Hermite-Laguerre-Itô polynomial as

\[
\tag{2.4}
J_{p,q} := \prod_{k=1}^{\infty} \frac{1}{\sqrt{2^{p_k+q_k} p_k! q_k!}} J_{p_k,q_k} \left( \sqrt{2} Z(\xi_k) \right).
\]

Then for any \( p, q \geq 0 \), the random variables

\[
\{ J_{p,q} : |p| = p, |q| = q \},
\]
form a complete orthonormal system in $\mathcal{H}_{p,q}(Z)$. As a consequence, the linear mapping

\begin{equation}
I_{p,q}\left(\text{symm}\left(\bigotimes_{k=1}^{\infty} \xi_k^{p_k}\right) \otimes \text{symm}\left(\bigotimes_{k=1}^{\infty} \bar{\xi}_k^{q_k}\right)\right) := \sqrt{p!q!}J_{p,q},
\end{equation}

provides an isometry from the tensor product $\mathcal{S}_C^{p} \otimes \mathcal{S}_C^{q}$, equipped with the norm $\sqrt{p!q!} \cdot \|\cdot\|_{\mathcal{S}_C^{p+q}}$, onto the $(p, q)$-th Wiener-Itô chaos $\mathcal{H}_{p,q}(Z)$. Note that (2.5) was proved by Itô in \cite{Ito1951} Theorem 13.2. For any $f \in \mathcal{S}_C^{p} \otimes \mathcal{S}_C^{q}$, $I_{p,q}(f)$ is called complex $(p, q)$-th Wiener-Itô integral of $f$ with respect to $Z$. Complex Wiener-Itô chaos decomposition of $L^2_C(\Omega, \sigma(Z), P)$ implies that $L^2_C(\Omega, \sigma(Z), P)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_{p,q}(Z)$. That is, any random variable $F \in L^2_C(\Omega, \sigma(Z), P)$ admits a unique expansion of the form

\begin{equation}
F = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} I_{p,q}\left(f_{p,q}\right),
\end{equation}

where $f_{0,0} = \mathbb{E}[F]$, and $f_{p,q} \in \mathcal{S}_C^{p} \otimes \mathcal{S}_C^{q}$ with $p + q \geq 1$, are uniquely determined by $F$.

Given $f \in \mathcal{S}_C^{a} \otimes \mathcal{S}_C^{b}$ and $g \in \mathcal{S}_C^{c} \otimes \mathcal{S}_C^{d}$, for $i = 0, \ldots, a \land d$, $j = 0, \ldots, b \land c$, the $(i, j)$-th contraction of $f$ and $g$ is an element of $\mathcal{S}_C^{(a+i-j)} \otimes \mathcal{S}_C^{(b+d-i-j)}$ defined by

\[
 f \otimes_{i,j} g = \sum_{l_1, \ldots, l_{i+j}=1}^{\infty} \langle f, \xi_{l_1} \otimes \cdots \otimes \xi_{l_i} \otimes \bar{\xi}_{l_{i+1}} \otimes \cdots \otimes \bar{\xi}_{l_{i+j}} \rangle \otimes \langle g, \xi_{l_{i+1}} \otimes \cdots \otimes \xi_{l_{i+j}} \otimes \bar{\xi}_{l_1} \otimes \cdots \otimes \bar{\xi}_{l_i} \rangle,
\]

and by convention, $f \otimes_{0,0} g = f \otimes g$ denotes the tensor product of $f$ and $g$. \cite[Theorem 2.1]{Ito1951} and \cite[Theorem A.1]{Ito1951} establish the product formula for complex Wiener-Itô integrals. For $f \in \mathcal{S}_C^{a} \otimes \mathcal{S}_C^{b}$ and $g \in \mathcal{S}_C^{c} \otimes \mathcal{S}_C^{d}$ with $a, b, c, d \geq 0$,

\[
 I_{a,b}(f)I_{c,d}(g) = \sum_{i=0}^{a \land d} \sum_{j=0}^{b \land c} \binom{a}{i} \binom{d}{i} \binom{b}{j} \binom{c}{j} i! j! I_{a+c-i-j, b+d-i-j}(f \otimes_{i,j} g).
\]

Here, we introduce some properties of real and complex polynomials that will be used in Section 3.1. Let $z = x + iy$ with $x, y \in \mathbb{R}$. \cite[Corollary 2.8]{Ito1951} shows that the real and complex Hermite polynomials satisfy

\begin{equation}
J_{p,q}(z) = \sum_{j=0}^{p+q} \binom{p+q-j}{r+s} \binom{p}{r} \binom{q}{s} (-1)^{q-s} H_j(x) H_{p+q-j}(y),
\end{equation}

and

\begin{equation}
H_m(x)H_n(y) = \frac{i^n}{2^{m+n}} \sum_{j=0}^m \sum_{r+s=j} \binom{m}{r} \binom{n}{s} (-1)^s J_{m+n-j}(z).
\end{equation}
According to [14] Proposition 3.11 or [28], for \( \theta \in [0, 2\pi) \), the real Hermite polynomials satisfy the invariant property
\[
H_n(x \cos \theta + y \sin \theta) = \sum_{j=0}^{n} \binom{n}{j} (\cos \theta)^j (\sin \theta)^{n-j} H_j(x) H_{n-j}(y).
\]

2.3. Malliavin derivative operators. Finally, we show some fundamental elements of Malliavin calculus. Let \( S \) denote the class of smooth random variables of the form \( F = f(X(h_1), \ldots, X(h_n)) \), where \( h_1, \ldots, h_n \in \mathcal{H} \), \( n \geq 1 \) and \( f \in C_p^{\infty}(\mathbb{R}^n) \), the set of all infinitely continuously differentiable real-valued functions such that all its partial derivatives have polynomial growth. Given \( F \in S \), the Malliavin derivative \( DF \) is a \( \mathcal{H} \)-valued random element given by
\[
DF = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} (X(h_1), \ldots, X(h_n)) h_i.
\]
The derivative operator \( D \) is a closable and unbounded operator from \( L^p(\Omega) \) to \( L^p(\Omega; \mathcal{H}) \) for any \( p \geq 1 \). By iteration, for \( k \geq 2 \), one can define \( k \)-th derivative \( D^k F \in L^p(\Omega; \mathcal{H}^{\otimes k}) \).

For any \( p \geq 1 \) and \( k \geq 0 \), let \( \mathbb{D}^{k,p} \) denote the closure of \( S \) with respect to the norm \( \| \cdot \|_{k,p} \) given by
\[
\| F \|_{k,p}^p = \sum_{i=0}^{k} \mathbb{E} \left( \| D^i F \|_{\mathcal{H}^{\otimes i}}^p \right).
\]
For any \( p \geq 1 \) and \( k \geq 0 \), we set \( \mathbb{D}^{\infty,p} = \bigcap_{k \geq 0} \mathbb{D}^{k,p} \), \( \mathbb{D}^{\infty} = \bigcap_{k \geq 0} \mathbb{D}^{k,\infty} \) and \( \mathbb{D}^{\infty} = \bigcap_{k \geq 0} \mathbb{D}^{k,\infty} \). If \( F = I_p(f) \) with \( f \in \mathcal{H}^{\otimes p} \), then \( I_p(f) \in \mathbb{D}^{\infty} \) and for any \( k \geq 0 \),
\[
D^k I_p(f) = \begin{cases} \frac{p!}{(p-k)!} I_{p-k}(f), & k \leq p, \\ 0, & k > p. \end{cases}
\]
In [29], the Stroock’s formula was established that for any random variable \( F \in L^2(\Omega, \sigma(X), P) \) expanded as \( F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n) \), if \( F \in \mathbb{D}^{n,2} \) for some \( n \geq 1 \), then
\[
f_p = \frac{1}{p!} \mathbb{E}[D^p F]
\]
for all \( p \leq n \).

In [11], Chen and Liu defined the complex Malliavin derivative operators \( \mathcal{D} \) and \( \overline{\mathcal{D}} \) as follows. Let \( S_Z \) denote the set of all smooth random variables of the form
\[
G = g(Z(h_1), \ldots, Z(h_m)),
\]
where \( h_1, \ldots, h_m \in \mathcal{H}_C \), \( m \geq 1 \) and \( g \in C_p^{\infty}(\mathbb{C}^m) \), the set of all infinitely continuously differentiable complex-valued functions such that all its partial derivatives have polynomial growth. If \( G \in S_Z \) with the form \( 2.11 \), then the complex Malliavin derivatives of \( G \) are the elements of \( L^2_C(\Omega; \mathcal{H}_C) \) defined by
\[
\mathcal{D} G = \sum_{j=1}^{m} \partial_j g(Z(h_1), \ldots, Z(h_m)) h_j,
\]
\[ \mathcal{D}G = \sum_{j=1}^{m} \tilde{\partial}_j g (Z(h_1), \ldots, Z(h_m)) \overline{h}_j, \]

where for complex numbers \( z_j = x_j + iy_j \) with \( x_j, y_j \in \mathbb{R} \) and \( j = 1, \ldots, m, \)

\[
\partial_j g = \frac{\partial}{\partial z_j} g(z_1, \ldots, z_m) = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) g(x_1, y_1, \ldots, x_m, y_m),
\]

\[
\bar{\partial}_j g = \frac{\partial}{\partial \overline{z}_j} g(z_1, \ldots, z_m) = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) g(x_1, y_1, \ldots, x_m, y_m),
\]

are the Wirtinger derivatives. One can define the iteration of the operator \( \mathcal{D} \) and \( \bar{\mathcal{D}} \) in such a way that \( \mathcal{D}^p \mathcal{D}^q G \) is a random variable with values in \( \mathcal{H}_C^{p+q} \otimes \mathcal{H}_C^{p+q} \) for any \( G \in \mathcal{S}_Z \).

For \( p + q \geq 1, \mathcal{D}^p \mathcal{D}^q \) are closable from \( L^2_C(\Omega) \) to \( L^2_C(\Omega, \mathcal{H}_C^{p+q} \otimes \mathcal{H}_C^{p+q}) \) for every \( r \geq 1 \). Denote by \( \mathcal{D}^{p,r} \cap \mathcal{D}^{q,r} \) the closure of \( \mathcal{S}_Z \) with respect to the Sobolev seminorm \( \| \cdot \|_{p,q,r} \) given by

\[
\|G\|_{p,q,r}^r = \sum_{i=0}^{p-q} \sum_{j=0}^{p} \mathbb{E} \left( \left\| \mathcal{D}^i \mathcal{D}^j G \right\|_{\mathcal{H}_C^{p+q+i+j}} \right).
\]

In \([11, \text{Theorem 2.8}]\), the complex Stroock's formula was showed that for every random variable \( G \in L^2_C(\Omega, \sigma(Z), P) \) expressed as \( G = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} I_{p,q} (g_{p,q}) \), with \( g_{p,q} \in \mathcal{H}_C^{p} \otimes \mathcal{H}_C^{q} \), if \( G \in \mathcal{D}^{m,2} \cap \mathcal{D}^{n,2} \), then

\[(2.12) \quad g_{p,q} = \frac{1}{p!q!} \mathbb{E} \left[ \mathcal{D}^p \mathcal{D}^q G \right], \quad \forall \ p \leq m, q \leq n.\]

3. Kernel representation formula from complex to real Wiener-Itô integrals

We firstly define the real isonormal Gaussian process \( W \) over \( \mathfrak{H} \oplus \mathfrak{H} \). Let \( h, f \in \mathfrak{H} \), denote by \( (h, f) \in \mathfrak{H} \oplus \mathfrak{H} \) the Cartesian product of \( \mathfrak{H} \) and \( \mathfrak{H} \). With respect to the natural inner product, that is, for any \( h_1, h_2, f_1, f_2 \in \mathfrak{H} \),

\[
(h_1, f_1, h_2, f_2)_{\mathfrak{H} \oplus \mathfrak{H}} = (h_1, h_2)_{\mathfrak{H}} + (f_1, f_2)_{\mathfrak{H}},
\]

\( \mathfrak{H} \oplus \mathfrak{H} \) is a Hilbert space. \([10, \text{Proposition 4.2}]\) offers a realization of the isonormal Gaussian process \( W \) over the Hilbert space \( \mathfrak{H} \oplus \mathfrak{H} \) as

\[(3.1) \quad W = \{ W(h, f) = X(h) + Y(f) : h, f \in \mathfrak{H} \},\]

where \( X \) and \( Y \) are two real independent identically distributed isonormal Gaussian processes over \( \mathfrak{H} \). We denote \( \mathcal{H}_n(W) \) as the \( n \)-th Wiener-Itô chaos of \( W \).

The following lemma shows that we can obtain a complete orthonormal basis of \( \mathfrak{H} \oplus \mathfrak{H} \) by using the complete and orthogonal elements \( \{e_k\}_{k \geq 1} \) in \( \mathfrak{H}_C \) with \( \|e_k\|_{\mathfrak{H}_C} = \sqrt{2} \).

**Lemma 3.1.** Let \( \{e_k = e_k^1 + ie_k^2\}_{k \geq 1} \) be complete and orthogonal in \( \mathfrak{H}_C \). Suppose \( \|e_k\|^2_{\mathfrak{H}_C} = \|e_k^1\|^2_{\mathfrak{H}} + \|e_k^2\|^2_{\mathfrak{H}} = 2 \). Define \( u_{1,0}(k) = \frac{1}{\sqrt{2}} (e_k^1, -e_k^2) \) and \( v_{1,0}(k) = \frac{1}{\sqrt{2}} (e_k^2, e_k^1) \). Then \( \{u_{1,0}(k), v_{1,0}(k)\}_{k \geq 1} \) is a complete orthonormal basis of \( \mathfrak{H} \oplus \mathfrak{H} \).

The proof of Lemma 3.1 is presented in Section 5.1.
Remark 3.2. Actually, the definition of complex isonormal Gaussian process $Z$ over $\mathcal{H}_C$, see [2.3], leads us to construct the basis of $\mathcal{H} \oplus \mathcal{H}$ as above in Lemma 3.1. Specifically, calculating $Z(e_k)$ directly, we get

$$Z(e_k) = \frac{X_C(e_k) + iY_C(e_k)}{\sqrt{2}} = \frac{X(e_k^1) + iX(e_k^2) + iY(e_k^1) + iY(e_k^2)}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} [X(e_k^1) - Y(e_k^2)] + \frac{i}{\sqrt{2}} [X(e_k^2) + Y(e_k^1)]$$

$$= W \left( \frac{1}{\sqrt{2}} (e_k^1, -e_k^2) \right) + iW \left( \frac{1}{\sqrt{2}} (e_k^2, e_k^1) \right)$$

$$= I_1 \left( \frac{1}{\sqrt{2}} (e_k^1, -e_k^2) \right) + iI_1 \left( \frac{1}{\sqrt{2}} (e_k^2, e_k^1) \right)$$

$$= I_1 (u_{1,0}(k)) + iI_1 (v_{1,0}(k)),$$

where $I_1(\cdot)$ is the real 1-th Wiener-Itô integral with respect to $W$. Then,

$$(3.2) \quad I_{1,0}(e_k) = I_{1,0}(Z(e_k)) = Z(e_k) = I_1 (u_{1,0}(k)) + iI_1 (v_{1,0}(k)).$$

This is why we construct $\{u_{1,0}(k), v_{1,0}(k)\}_{k \geq 1}$ as the orthonormal basis of $\mathcal{H} \oplus \mathcal{H}$.

3.1. Uniqueness theorem. Let $p = \{p_k\}_{k=1}^{\infty}, q = \{q_k\}_{k=1}^{\infty} \in \Lambda$ with $|p| = p$ and $|q| = q$, respectively. For simplicity of presentation, define

$$a_{k,j} := i^{p_k+q_j-j} \sum_{r+s=j} \binom{p_k}{r} \binom{q_j}{s} (-1)^{q_j-s}.$$

For $j = \{j_k\}_{k=1}^{\infty} \in \Lambda$ with $j \leq p+q$, $\prod_{k=1}^{\infty} a_{k,j_k}$ is a complex number. Define $u(p, q), v(p, q) \in (\mathcal{H} \oplus \mathcal{H})^{\otimes (p+q)}$ as

$$u(p, q) = \sum_{j \leq p+q} \text{Re} \left( \prod_{l=1}^{\infty} a_{l,j_l} \right) \text{symm} \left( \otimes_{k=1}^{\infty} \left( u_{1,0}(k)^{\otimes j_k} \otimes v_{1,0}(k)^{\otimes (p_k+q_j-j_k)} \right) \right),$$

$$v(p, q) = \sum_{j \leq p+q} \text{Im} \left( \prod_{l=1}^{\infty} a_{l,j_l} \right) \text{symm} \left( \otimes_{k=1}^{\infty} \left( u_{1,0}(k)^{\otimes j_k} \otimes v_{1,0}(k)^{\otimes (p_k+q_j-j_k)} \right) \right).$$

Let $\{e_k\}_{k \geq 1}$ be as in the statement of Lemma 3.1, that is, $\{e_k\}_{k \geq 1}$ are complete and orthogonal with $\|e_k\|_{\mathcal{H}_C} = \sqrt{2}$ in $\mathcal{H}_C$. For $f \in \mathcal{S}_C^{\otimes p} \otimes \mathcal{S}_C^{\otimes q}$, there exists a unique sequence

$$\{c(p; q) : p, q \in \Lambda, |p| = p, |q| = q \} \subseteq \mathbb{C}$$

satisfying

$$\sum_{p, q \in \Lambda, |p|=p, |q|=q} |c(p; q)|^2 < \infty$$

such that

$$(3.3) \quad f = \sum_{p, q \in \Lambda, |p|=p, |q|=q} c(p; q) \text{symm} \left( \otimes_{k=1}^{\infty} e_k^{\otimes p_k} \right) \otimes \text{symm} \left( \otimes_{k=1}^{\infty} \varepsilon_k^{\otimes q_k} \right).$$

Now, we state the uniqueness theorem as follows.
Theorem 3.3 (Uniqueness Theorem). Suppose \( f \in \mathcal{S}_C^{op} \otimes \mathcal{S}_C^{eq} \) with the expansion given by (3.3). Then \( \mathcal{I}_{p,q}(f) \) admits the unique representation

\[
\mathcal{I}_{p,q}(f) = \sum_{p,q \in \Lambda, |p| = p, |q| = q} I_{p+q} \left( \text{Re} \left( c(p, q) \right) u(p, q) - \text{Im} \left( c(p, q) \right) v(p, q) \right) + i \sum_{p,q \in \Lambda, |p| = p, |q| = q} I_{p+q} \left( \text{Im} \left( c(p, q) \right) u(p, q) + \text{Re} \left( c(p, q) \right) v(p, q) \right),
\]

where \( I_{p+q}(\cdot) \) is the real \((p + q)\)-th Wiener-Itô integral with respect to \( W \).

Note that,

\[
\begin{align*}
\left\{ 2^{-\frac{p+q}{2}} e_k^{op} \otimes \bar{e}_k^{eq} : k \geq 1 \right\} \\
\subseteq \left\{ \left( \otimes_{k=1}^{\infty} e_k^{op} \right) \otimes \left( \otimes_{k=1}^{\infty} \bar{e}_k^{eq} \right) : p = \{p_k\}_{k=1}^{\infty}, q = \{q_k\}_{k=1}^{\infty} \in \Lambda, |p| = p, |q| = q \right\},
\end{align*}
\]

and the latter is a complete orthonormal basis of \( \mathcal{S}_C^{op} \otimes \mathcal{S}_C^{eq} \). Therefore, the proof of uniqueness theorem (Theorem 3.3) is divided into following three steps:

- Unique representation for \( \mathcal{I}_{p,q}(e_k^{op} \otimes \bar{e}_k^{eq}) \), namely, Lemma 3.4.
- Unique representation for \( \mathcal{I}_{p,q}(\text{symm}(\otimes_{k=1}^{\infty} e_k^{op}) \otimes \text{symm}(\otimes_{k=1}^{\infty} \bar{e}_k^{eq})) \), namely, Proposition 3.5.
- Unique representation for \( \mathcal{I}_{p,q}(f) \) with \( f \in \mathcal{S}_C^{op} \otimes \mathcal{S}_C^{eq} \).

Lemma 3.4. For \( e_1 = e_1^1 + i e_1^2 \in \mathcal{S}_C \) with \( \|e_1\|_{\mathcal{S}_C} = \sqrt{2} \),

\[
(3.4) \quad \mathcal{I}_{p,q}(e_1^{op} \otimes \bar{e}_1^{eq}) = \sum_{j=0}^{p+q} \sum_{r+s=j} \binom{p}{r} \binom{q}{s} (-1)^{q-s} I_{p+q}(u_{1,0}(1)^{\otimes j} \otimes v_{1,0}(1)^{\otimes (p+q-j)}),
\]

where

\[
u_{1,0}(1) = \frac{1}{\sqrt{2}} (e_1^1, -e_1^2), \quad v_{1,0}(1) = \frac{1}{\sqrt{2}} (e_1^2, e_1^1),
\]

and \( I_{p+q}(\cdot) \) is the real \((p + q)\)-th Wiener-Itô integral with respect to \( W \).

Proposition 3.5. For \( p = \{p_k\}_{k=1}^{\infty}, q = \{q_k\}_{k=1}^{\infty} \in \Lambda \) with \(|p| = p\) and \(|q| = q\),

\[
\mathcal{I}_{p,q}(\text{symm}(\otimes_{k=1}^{\infty} e_k^{op}) \otimes \text{symm}(\otimes_{k=1}^{\infty} \bar{e}_k^{eq})) = I_{p+q}(u(p, q)) + i I_{p+q}(v(p, q)),
\]

where \( I_{p+q}(\cdot) \) is the real \((p + q)\)-th Wiener-Itô integral with respect to \( W \).

Proofs of Lemma 3.4, Proposition 3.5 and Theorem 3.3 are presented in Section 5.2.

Remark 3.6. Chen and Liu in [11] Theorem 3.3 showed that for \( f \in \mathcal{S}_C^{op} \otimes \mathcal{S}_C^{eq} \), there exist \( u, v \in (\mathcal{S} \oplus \mathcal{S})^{(p+q)} \) such that

\[
\text{Re} \mathcal{I}_{p,q}(f) = I_{p+q}(u), \quad \text{Im} \mathcal{I}_{p,q}(f) = I_{p+q}(v),
\]
where \( I_{p,q}(\cdot) \) is the real \((p+q)\)-th Wiener-Itô integral with respect to \( W \). In order to prove [10, Theorem 3.3], Chen and Liu introduced \( n + 1 \) parameters \( 0 < \theta_n < \cdots < \theta_0 < \pi \) and defined the invertible matrix \( M \) as

\[
M = M (\theta_0, \ldots, \theta_n) = (M_{ij})_{0 \leq i,j \leq n} \]

\[
= \begin{bmatrix}
(sin \theta_0)^n (\frac{n}{1}) (sin \theta_0)^{n-1} \cos \theta_0 & \cdots & (\frac{n}{n-1}) \sin \theta_0 (cos \theta_0)^{n-1} (cos \theta_0)^n \\
(sin \theta_1)^n (\frac{n}{1}) (sin \theta_1)^{n-1} \cos \theta_1 & \cdots & (\frac{n}{n-1}) \sin \theta_1 (cos \theta_1)^{n-1} (cos \theta_1)^n \\
\vdots & \ddots & \vdots \\
(sin \theta_n)^n (\frac{n}{1}) (sin \theta_n)^{n-1} \cos \theta_n & \cdots & (\frac{n}{n-1}) \sin \theta_n (cos \theta_n)^{n-1} (cos \theta_n)^n
\end{bmatrix}.
\]

They used the properties of Hermite polynomials (2.7) and (2.9), solved \( n + 1 \) linear equations and expressed \( J_{k,n-k} (e_1^{\otimes k} \otimes \overline{e}_1^{\otimes (n-k)}) \) (see [10, Equation (21)]) as

\[
J_{k,n-k} (Z(e_1)) = \sum_{j=0}^{n} c_j H_j (Re Z(e_1)) H_{n-j} (Im Z(e_1))
\]

\[
= \sum_{j=0}^{n} c_j \sum_{l=0}^{n} (M^{-1})_{jl} H_n (W (cos \theta_l u_{1,0}(1) + sin \theta_l v_{1,0}(1)))
\]

where \( c_j = i^{n-j} \sum_{r+s=j} \binom{k}{r} \binom{n-k}{s} (-1)^{n-k-s} \).

Actually, the right hand side of (3.5) does not depend on these redundant parameters \( \theta_0, \ldots, \theta_n \). From the perspective on Wiener-Itô integrals, utilizing the linearity, we get that

\[
\sum_{l=0}^{n} (M^{-1})_{jl} H_n (W (cos \theta_l u_{1,0}(1) + sin \theta_l v_{1,0}(1)))
\]

\[
= \sum_{l=0}^{n} (M^{-1})_{jl} I_n ((cos \theta_l u_{1,0}(1) + sin \theta_l v_{1,0}(1))^{\otimes n})
\]

\[
= \sum_{l=0}^{n} (M^{-1})_{jl} I_n \left( \sum_{r=0}^{n} \binom{n}{r} (cos \theta_l)^r (sin \theta_l)^{n-r} u_{1,0}(1)^{\otimes r} \otimes v_{1,0}(1)^{\otimes (n-r)} \right)
\]

\[
= \sum_{r=0}^{n} \sum_{l=0}^{n} (M^{-1})_{jl} M_{lr} I_n (u_{1,0}(1)^{\otimes r} \otimes v_{1,0}(1)^{\otimes (n-r)})
\]

\[
= \sum_{r=0}^{n} (M^{-1} M)_{jr} I_n (u_{1,0}(1)^{\otimes r} \otimes v_{1,0}(1)^{\otimes (n-r)})
\]

\[
= I_n (u_{1,0}(1)^{\otimes j} \otimes v_{1,0}(1)^{\otimes (n-j)})
\]

That is,

\[
I_{k,n-k} (e_1^{\otimes k} \otimes \overline{e}_1^{\otimes (n-k)}) = J_{k,n-k} (Z(e_1))
\]
expression for $\theta$ (3.5) which is exactly what we state in Lemma 3.4 and implies that the right hand side of (3.2).

Representation theorem. In this section, we directly get more explicit recursion formulae for kernels of the real (Theorem 3.3). This leads to somewhat complicated representation for $\theta$ (3.9)

$$\sum_{j=0}^{n} \binom{n-j}{r} \binom{n-k}{s} (-1)^{n-k-s} I_n \left( u_{1,0}(1)^\otimes j \otimes v_{1,0}(1)^\otimes (n-j) \right),$$

which is exactly what we state in Lemma 3.4 and implies that the right hand side of (3.5) does not depend on these redundant parameters $\theta_0, \ldots, \theta_n$. As a by-product, we get an expression for $\text{symm} \left( u_{1,0}(1)^\otimes j \otimes v_{1,0}(1)^\otimes (n-j) \right)$ as

$$\sum_{j=0}^{n} (M^{-1})_{jj} (\cos \theta_{1} u_{1,0}(1) + \sin \theta_{1} v_{1,0}(1))^\otimes n, \quad \forall 0 < \theta_n < \ldots < \theta_0 < \pi.$$

3.2. Representation theorem. Note that

$$\left\{ 2^{-\frac{p+q}{2}} (\otimes_{k=1}^{\infty} e_k^{\otimes p}) \otimes (\otimes_{k=1}^{\infty} \bar{e}_k^{\otimes q}) \right\} : \mathcal{P} = \{ p_k \}_{k=1}^{\infty}, q = \{ q_k \}_{k=1}^{\infty} \in \Lambda, \left| p \right| = p, \left| q \right| = q \}

is a complete orthonormal basis of $\mathcal{Q}_C^{\otimes p} \otimes \mathcal{Q}_C^{\otimes q}$. In Section 3.1 based on the representation for $\mathcal{I}_{p,q} (e_k^{\otimes p} \otimes \bar{e}_k^{\otimes q})$ with $k \geq 1$ (see Lemma 3.4), we prove the uniqueness theorem (Theorem 3.3). This leads to somewhat complicated representation for

$$\mathcal{I}_{p,q} \left( \text{symm} \left( \otimes_{k=1}^{\infty} e_k^{\otimes p} \right) \otimes \text{symm} \left( \otimes_{k=1}^{\infty} \bar{e}_k^{\otimes q} \right) \right).$$

In this section, we directly get more explicit recursion formulae for kernels of the real and imaginary parts of $\mathcal{I}_{p,q} \left( \text{symm} \left( \otimes_{k=1}^{\infty} e_k^{\otimes p} \right) \otimes \text{symm} \left( \otimes_{k=1}^{\infty} \bar{e}_k^{\otimes q} \right) \right)$ by an induction argument.

Before illustrating the representation theorem, we introduce some notations. Define $k = (k_1, k_2, \ldots)$ and $j = (j_1, j_2, \ldots)$. For ease of notations, we write

$$\mathcal{I}_{p,q}(k, j) : = \mathcal{I}_{p,q}(\text{symm}(e_{k_1} \otimes \cdots \otimes e_{k_p}) \otimes \text{symm}(\bar{e}_{j_1} \otimes \cdots \otimes \bar{e}_{j_q}))$$

$$= \mathcal{I}_{p,q}(e_{k_1} \otimes \cdots \otimes e_{k_p} \otimes \bar{e}_{j_1} \otimes \cdots \otimes \bar{e}_{j_q}),$$

and

$$u_{p,q}(k, j) := u_{p,q}(k_1, \ldots, k_p; j_1, \ldots, j_q),$$

$$v_{p,q}(k, j) := v_{p,q}(k_1, \ldots, k_p; j_1, \ldots, j_q),$$

where $u_{p,q}(k_1, \ldots, k_p; j_1, \ldots, j_q), v_{p,q}(k_1, \ldots, k_p; j_1, \ldots, j_q) \in (\mathcal{Q} \oplus \mathcal{Q})^{\otimes (p+q)}$ are recursively defined by

$$u_{0,1}(j) = u_{1,0}(j), \quad v_{0,1}(j) = -v_{1,0}(j),$$

$$u_{p,q}(k, j) = u_{p-1,q}(k, j) \otimes u_{1,0}(k_p) - v_{p-1,q}(k, j) \otimes v_{1,0}(k_p)$$

$$- u_{p,q-1}(k, j) \otimes u_{0,1}(j_q),$$

$$v_{p,q}(k, j) = u_{p-1,q}(k, j) \otimes v_{1,0}(k_p) + v_{p-1,q}(k, j) \otimes u_{1,0}(k_p)$$

$$+ u_{p,q-1}(k, j) \otimes v_{0,1}(j_q) + v_{p,q-1}(k, j) \otimes u_{0,1}(j_q).$$

Now we explain that $u_{p,q}(k, j)$ and $v_{p,q}(k, j)$ are well defined. Since $\mathcal{I}_{p,q}(j, k) = \mathcal{I}_{p,q}(k, j)$, we have

$$u_{p,q}(k, j) = u_{q,p}(j, k) \quad \text{and} \quad v_{p,q}(k, j) = -v_{q,p}(j, k).$$
Theorem 3.9. The recursion representation given by (3.12) where, after a tedious calculation, \( f \) is well defined. Similarly, one can show that \( v_{p,q}(k,j) \) is well defined.

**Theorem 3.7.** For \( p \geq 0, q \geq 0, p + q > 0 \), we have
\[
\mathcal{I}_{p,q}(k,j) = I_{p+q}(u_{p,q}(k,j)) + i I_{p+q}(v_{p,q}(k,j)),
\]
where \( I_{p+q}(-) \) is the real \((p+q)\)-th Wiener-Itô integral with respect to \( W \).

**Remark 3.8.** Take \( k = (1, \ldots, 1, 0, \ldots) \), \( j = (1, \ldots, 1, 0, \ldots) \) and consider \( \mathcal{I}_{p,q}(e_1^{\otimes p} \otimes e_1^{\otimes q}) \).

Using the recursion formulae of \( u_{p,q}(k,j) \) and \( v_{p,q}(k,j) \) repeatedly, we further get
\[
(3.10) \quad u_{p,q}(k,j) + iv_{p,q}(k,j) = \text{symm} \left( \sum_{j=0}^{p} \sum_{l=0}^{q-1} \binom{p}{j} \binom{q-1}{l} (f_{p,q}(j,l) + ig_{p,q}(j,l)) \right),
\]
where, after a tedious calculation, \( f_{p,q}(j,l) \) and \( g_{p,q}(j,l) \) are obtained as
\[
(3.11) \quad f_{p,q}(j,l) + ig_{p,q}(j,l) = i^{j+l}(-1)^j u_{1,0}(1)^{\otimes (p+q-(j+l+1))} \otimes v_{1,0}(1)^{\otimes (j+l)} \otimes (u_{1,0}(1) - iv_{1,0}(1)).
\]

One can verify that (3.10) is equal to the kernel expression in (3.4) in the sense of symmetry.

Let \( \{e_k\}_{k \geq 1} \) with \( \|e_k\|_{\mathcal{H}_C} = \sqrt{2} \) be complete and orthogonal elements in \( \mathcal{H}_C \). For \( f \in \mathcal{H}_C^{\otimes p} \otimes \mathcal{H}_C^{\otimes q} \), there exists a unique sequence
\[
\{b_{p,q}(k,j) = b(k_1, \ldots, k_p; j_1, \ldots, j_q), k_1, \ldots, k_p, j_1, \ldots, j_q \geq 1 \} \subseteq \mathbb{C}
\]
such that
\[
(3.12) \quad f = \sum_{k_1, \ldots, k_p, j_1, \ldots, j_q = 1}^{\infty} b_{p,q}(k,j) \text{symm}(e_{k_1} \otimes \cdots \otimes e_{k_p}) \otimes \text{symm}(\overline{e_{j_1}} \otimes \cdots \otimes \overline{e_{j_q}}).
\]

Now we are able to restate the uniqueness theorem (Theorem 3.3) as a version of recursion representation.

**Theorem 3.9 (Representation Theorem).** Suppose \( f \in \mathcal{H}_C^{\otimes p} \otimes \mathcal{H}_C^{\otimes q} \) with the expansion given by (3.12). Then \( \mathcal{I}_{p,q}(f) \) admits the unique representation
\[
\mathcal{I}_{p,q}(f) = \sum_{k_1, \ldots, k_p, j_1, \ldots, j_q = 1}^{\infty} I_{p+q} \left( \text{Re} \left( b_{p,q}(k,j) u_{p,q}(k,j) \right) - \text{Im} \left( b_{p,q}(k,j) v_{p,q}(k,j) \right) \right) + i \sum_{k_1, \ldots, k_p, j_1, \ldots, j_q = 1}^{\infty} I_{p+q} \left( \text{Im} \left( b_{p,q}(k,j) u_{p,q}(k,j) \right) + \text{Re} \left( b_{p,q}(k,j) v_{p,q}(k,j) \right) \right),
\]
where $I_{p+q}(\cdot)$ is the real $(p + q)$-th Wiener-Itô integral with respect to $W$.

We prove Theorem 3.7 by induction and Theorem 3.9 by an approximation argument in Section 5.3.

From now on, we assume that $D$ is the real Malliavin derivative operator with respect to the isonormal Gaussian process $W$ over $\mathfrak{F} \oplus \mathfrak{F}$. As a corollary of Theorem 3.7 we obtain a sufficient and necessary condition of the existence of density of the complex Wiener-Itô integral $\mathcal{I}_{p,q}(k,j)$.

**Corollary 3.10.** Consider a complex random variable

$$\mathcal{I}_{p,q}(e_{k_1} \otimes \cdots \otimes e_{k_p} \otimes \overline{e_{j_1}} \otimes \cdots \otimes \overline{e_{j_q}}) := F_1 + iF_2.$$  

Without loss of generality, we assume that $1 \leq k_1 \leq \ldots \leq k_p, 1 \leq j_1 \leq \ldots \leq j_q$. Then the law of the two-dimensional random vector $(F_1, F_2)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$ if and only if $p \neq q$ or if $p = q$ and there exists $1 \leq l \leq p$ such that $k_l \neq j_l$.

**Proof.** From Theorem 3.7

$$F_1 = I_{p+q}(u_{p,q}(k,j)), \quad F_2 = I_{p+q}(v_{p,q}(k,j)).$$

By (3.7) and the definition of symmetric tensor product,

$$u_{p,q}(k,j) = \frac{1}{p} + \frac{1}{q} \left[ \sum_{l=1}^{p} (u_{p-1,q}(\hat{k}_l,j) \otimes u_{1,0}(k_l)) - v_{p-1,q}(\hat{k}_l,j) \otimes v_{1,0}(k_l) \right] + \sum_{r=1}^{1} (u_{p,q-1}(k,j_r) \otimes u_{0,1}(j_r) - v_{p,q-1}(k,j_r) \otimes v_{0,1}(j_r)), $$

where the notations of $u_{p-1,q}(\hat{k}_l,j), v_{p-1,q}(\hat{k}_l,j), u_{p,q-1}(k,j_r)$ and $v_{p,q-1}(k,j_r)$ can be found in (5.3). Then,

\begin{equation}(3.13)\quad DF_1 = I_{p+q-1} \left( \sum_{l=1}^{p} u_{p-1,q}(\hat{k}_l,j) \right) u_{1,0}(k_l) - I_{p+q-1} \left( \sum_{l=1}^{p} v_{p-1,q}(\hat{k}_l,j) \right) v_{1,0}(k_l) \\
+ I_{p+q-1} \left( \sum_{r=1}^{1} u_{p,q-1}(k,j_r) \right) u_{0,1}(j_r) - I_{p+q-1} \left( \sum_{r=1}^{1} v_{p,q-1}(k,j_r) \right) v_{0,1}(j_r)
\end{equation}

\begin{equation}(3.14)\quad DF_2 = \sum_{l=1}^{p} I_{p+q-1}(u_{p-1,q}(\hat{k}_l,j))u_{1,0}(k_l) - \sum_{l=1}^{p} I_{p+q-1}(v_{p-1,q}(\hat{k}_l,j))v_{1,0}(k_l) \\
+ \sum_{r=1}^{1} I_{p+q-1}(u_{p,q-1}(k,j_r))u_{0,1}(j_r) + \sum_{r=1}^{1} I_{p+q-1}(v_{p,q-1}(k,j_r))v_{0,1}(j_r).\end{equation}
The law of \((F_1, F_2)\) is absolutely continuous with respect to Lebesgue measure if and only if \(DF_1(\omega)\) and \(DF_2(\omega)\) are linearly independent in \(\mathcal{F}_1 \otimes \mathcal{F}_2\) for a.s. \(\omega\), see [22, Theorem 3.1]. (The absolute continuity of the law is also equivalent to the linear independence of \(u_{p,q}(k,j)\) and \(v_{p,q}(k,j)\) in \((\mathcal{F}_1 \otimes \mathcal{F}_2) \odot (p+q)\) by [25, Theorem 3]. However, it is easier to verify the linearly independence in \(\mathcal{F}_1 \otimes \mathcal{F}_2\) than in \((\mathcal{F}_1 \otimes \mathcal{F}_2) \odot (p+q)\).) Therefore, it suffices to show that \(DF_1(\omega)\) and \(DF_2(\omega)\) are linearly independent in \(\mathcal{F}_1 \otimes \mathcal{F}_2\) for a.s. \(\omega\) if and only if \(p \neq q\) or if \(p = q\) and there exists \(1 \leq l \leq p\) such that \(k_l \neq j_l\).

Necessity: Otherwise, we have \(p = q\) and for any \(1 \leq l \leq p\), \(k_l = j_l\). Fix \(\omega\),

\[
DF_2(\omega) = \sum_{l=1}^{p} I_{2p-1} \left( v_{p-1,p}(\mathbf{k}_l, k) + v_{p,p-1}(k, \mathbf{k}_l) \right) u_{1,0}(k_l)
\]

\[
+ \sum_{l=1}^{p} I_{2p-1} \left( u_{p-1,p}(\mathbf{k}_l, k) - u_{p,p-1}(k, \mathbf{k}_l) \right) v_{1,0}(k_l)
\]

\[= 0,
\]

which contradicts the assumption that for a.s. \(\omega\), \(DF_1(\omega)\) and \(DF_2(\omega)\) are linearly independent in \(\mathcal{F}_1 \otimes \mathcal{F}_2\).

Sufficiency: Observing whether the coefficients of orthogonal elements

\[
\{u_{1,0}(k_l), u_{1,0}(j_r), v_{1,0}(k_l), v_{1,0}(j_r)\}_{1 \leq l \leq p, 1 \leq r \leq q} \subseteq \mathcal{F}_1 \otimes \mathcal{F}_2
\]

in (3.13) and (3.14) can cancel out, we have that \(DF_1(\omega) \neq 0\) for a.s. \(\omega\). Moreover, \(DF_2(\omega) \neq 0\) for a.s. \(\omega\) if \(p \neq q\) or if \(p = q\) and there exists \(1 \leq l \leq p\) such that \(k_l \neq j_l\). Fix \(\omega\), combining the orthogonality of \(\{u_{1,0}(k_l), u_{1,0}(j_r), v_{1,0}(k_l), v_{1,0}(j_r)\}_{1 \leq l \leq p, 1 \leq r \leq q}\) and its coefficient vectors are orthogonal in \(\mathbb{R}^{2(p+q)}\), we get the conclusion.

\[\square\]

3.3. **Back to Itô’s theory.** Assume that the underlying Hilbert space \(\mathcal{F} = L^2(T, \mathcal{B}, \mu)\), where \((T, \mathcal{B})\) is a measurable space and \(\mu\) is a \(\sigma\)-finite measure without atoms. Then the linear mapping \(\mathcal{I}_{p,q}\) coincides with a multiple Wiener-Itô integrals defined by Itô in [17]. According to [17, Theorem 3.1], there exists a continuous complex normal random measure \(\mathcal{M} = \{M(B) : B \in \mathcal{B}, \mu(B) < \infty\}\) on \((T, \mathcal{B})\), such that, for every \(B, C \in \mathcal{B}\) with finite measure,

\[\mathbb{E}[M(B)\bar{M}(C)] = \mu(B \cap C).
\]

For the off-diagonal simple function \(f \in \mathcal{F}_C^{\otimes p} \otimes \mathcal{F}_C^{\otimes q}\) of the form

\[f(t_1, \ldots, t_p, s_1, \ldots, s_q) = \sum a_{i_1 \ldots i_p j_1 \ldots j_q} 1_{B_{i_1} \times \cdots \times B_{i_p} \times B_{j_1} \times \cdots \times B_{j_q}},\]
where $1_A$ is the indicator function of a set $A$, Itô defined the multiple integral $I_{p,q}(f)$ as

$$I_{p,q}(f) = \sum_{a_{i_1\cdots i_d}M(B_{i_1})\cdots M(B_{i_d})}.$$ 

Then by density argument, Itô extended the multiple integrals to any $f \in \mathcal{S}_C^{\otimes p} \otimes \mathcal{S}_C^{\otimes q}$ as follows,

$$I_{p,q}(f) = \int \cdots \int f(t_1, \ldots, t_p; s_1, \ldots, s_q) \, dM(t_1) \cdots dM(t_p) \, \overline{dM(s_1)} \cdots \overline{dM(s_q)}.$$ 

Specifically, let $\mathcal{H} = L^2(\mathbb{R}^+) \otimes (B_1(t), B_2(t))_{t \geq 0}$ be a two-dimensional Brownian motion. $\zeta_t := B_1(t) + B_2(t)$ is called complex Brownian motion. We can extend $(1_{[0,t]}, 1_{[0,t]}) \mapsto B_1(t_1) + B_2(t_2)$ with $t_1, t_2 \geq 0$ to a real isonormal Gaussian process denoted by $\widehat{M}$ over $\mathcal{H} \otimes \mathcal{H}$, namely the stochastic integral with respect to two-dimensional Brownian motion.

For $0 \leq r \leq p + q$, by the definition of Itô’s iterated integral, we know that

$$(3.15) \quad I_{d_{p+q-r}} := I_{p,q} \left( (a_{i_1,0}(1) \otimes f_{p+q-r}) \otimes a_{i_1,0}(1) \right)$$

$$= \frac{(p + q)!}{2^{p+q}r!} \sum_{i_1, \ldots, i_p + q = 1} (-1)^{p+q-r} \int_0^\infty \int_0^{t_p} \cdots \int_0^{t_1} e_{i_1}^{p+q-r} (t_{p+q-r}) \, dB_1 (t_1)$$

$$\cdots e_{i_p}^{p+q-r} (t_{p+q-r}) \, dB_3^{p+q+r+1} (t_{p+q-r+1}) \cdots dB_3^{p+q+r+1} (t_{p+q-r+1}),$$

where $I_{p+q} (\cdot)$ is $(p+q)$-th Wiener-Itô integral with respect to $\widehat{M}$ and $d_{p+q-r}$ denotes the number of $j \in \{1, \ldots, p+q-r\}$ such that $i_j = 2$ and $d_0 = 0$ for $r = p + q$.

Then using (3.10), (3.11) and (3.15), we can express $I_{p,q} (e_{i_1}^{\otimes p} \otimes e_{i_2}^{\otimes q})$ as Itô’s iterated integral

$$I_{p,q} (e_{i_1}^{\otimes p} \otimes e_{i_2}^{\otimes q})$$

$$= \sum_{j=0}^{p} \sum_{l=0}^{q-1} \binom{p}{j} \binom{q-1}{l} \left( I_{p+q} (f_{p,q}(j,l)) + iI_{p+q} (g_{p,q}(j,l)) \right)$$

$$= \sum_{j=0}^{p} \sum_{l=0}^{q-1} \binom{p}{j} \binom{q-1}{l} i^{j+l-1} (-1)^j (I_{d_{p+q-(j+1)}} - iI_{d_{p+q-(j+i+1)}}).$$

For example, take $p = q = 1$, we get that

$$I_{1,1} (e_{i_1} \otimes e_{i_1}) = I_{d_2} + I_{d_0} + i (-I_{d_4} + I_{d_4})$$

$$= \frac{2!}{2} \sum_{i_1, i_2 = 1} (-1)^{i_2} \int_0^\infty \int_0^{t_2} e_{i_1}^{i_1} (t_1) e_{i_1}^{i_2} (t_2) \, dB_{i_1} (t_1) \cdots dB_{i_2} (t_2)$$

$$+ \frac{2!}{2} \sum_{i_1, i_2 = 1} (-1)^{i_2} \int_0^\infty \int_0^{t_2} e_{i_1}^{i_1} (t_1) e_{i_1}^{i_2} (t_2) \, dB_{3-i_1} (t_1) \cdots dB_{3-i_2} (t_2)$$

$$= \int_0^\infty \int_0^{t_2} e_{i_1}^{i_1} (t_1) e_{i_1}^{i_2} (t_2) \, dB_{i_1} (t_1) \cdots dB_{i_2} (t_2) + \int_0^\infty \int_0^{t_2} e_{i_1}^{i_1} (t_1) e_{i_1}^{i_2} (t_2) \, dB_{2} (t_1) \cdots dB_{2} (t_2).$$
expressed as

\[- \int_0^\infty \int_0^{t_2} e_1^1(t_1) e_1^2(t_2) \, dB_1(t_1) \, dB_2(t_2) - \int_0^\infty \int_0^{t_2} e_1^2(t_1) e_1^1(t_2) \, dB_2(t_1) \, dB_1(t_2) + \int_0^\infty \int_0^{t_2} e_1^2(t_1) e_1^1(t_2) \, dB_1(t_1) \, dB_2(t_2) + \int_0^\infty \int_0^{t_2} e_1^1(t_1) e_1^2(t_2) \, dB_1(t_1) \, dB_2(t_2) \, .\]

Remark 3.2. For a smooth random variable

\[G, \tag{3.16}\]

Lemma 3.11. For any operator \((\mathcal{D})\) as

\[A \in \mathbb{D}^p(\Omega; \mathcal{F} \oplus \bar{\mathcal{F}}) \text{ where } (\mathcal{D}) \in \mathbb{D}^1.\]

Note that for \(D F \in L^p(\Omega; \mathcal{F} \oplus \bar{\mathcal{F}})\) for \(F \in \mathbb{D}^{1-p}\), we denote \(DF\) by \((D_1F, D_2F)\). In particular, for

\[F = f(W(h_1, f_1), \ldots, W(h_n, f_n)) \, ,\]

where \((h_1, f_1), \ldots, (h_n, f_n) \in \mathcal{F} \oplus \bar{\mathcal{F}}, n \geq 1\) and \(f \in C_\infty^p(\mathbb{R}^n)\), \(D_1F\) and \(D_2F\) are \(\mathcal{F}\)-valued random elements given by

\[
D_1F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (W(h_1, f_1), \ldots, W(h_n, f_n)) \, h_i, \\
D_2F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (W(h_1, f_1), \ldots, W(h_n, f_n)) \, f_i.
\]

For any operator \((A_1, A_2)\) and \((A_3, A_4)\), we define the tensor product of \((A_1, A_2)\) and \((A_3, A_4)\) as

\[(A_1, A_2) \otimes (A_3, A_4) = (A_1 A_3) \otimes (A_2 A_4)^T,\]

where \((\cdot)^T\) denotes the transposition of a matrix or a vector. Note that for \(k \geq 2\) and \(F \in \mathbb{D}^{k,p}\), \(D^k F = (D_1, D_2)^{\otimes k} F \in L^p(\Omega; (\mathcal{F} \oplus \bar{\mathcal{F}})^{\otimes k})\).

The following Lemma establishes the relation between the real Malliavin derivative operator \(D = (D_1, D_2)\) and the complex Malliavin derivative operators \(\mathcal{D}, \bar{\mathcal{D}}\).

Lemma 3.11.

\[\mathcal{D} = \frac{D_1 - iD_2}{\sqrt{2}}, \quad \bar{\mathcal{D}} = \frac{D_1 + iD_2}{\sqrt{2}}.\]

Proof. By Remark 3.2 for a smooth random variable \(G\) with the form \((2.11)\), \(G\) can be expressed as

\[G = g(Z(h_1), \ldots, Z(h_m)) \, ,\]

\[= g_1 \left( \frac{W(h_1, -h_1^2)}{\sqrt{2}}, \frac{W(h_1^2, h_1^1)}{\sqrt{2}}, \ldots, \frac{W(h_m, -h_m^2)}{\sqrt{2}}, \frac{W(h_m^2, h_m^1)}{\sqrt{2}} \right) + ig_2 \left( \frac{W(h_1, -h_1^2)}{\sqrt{2}}, \frac{W(h_1^2, h_1^1)}{\sqrt{2}}, \ldots, \frac{W(h_m, -h_m^2)}{\sqrt{2}}, \frac{W(h_m^2, h_m^1)}{\sqrt{2}} \right),\]
where \( h_j = h_1^j + ih_2^j \in \mathcal{H}_C \) with \( 1 \leq j \leq m \) and \( g_1, g_2 \in C^\infty_p(\mathbb{R}^{2m}) \). Then by the definitions of the complex Malliavin derivative operators \( \mathcal{D}, \mathcal{D}^\ast \) and real Malliavin derivative operator \( D = (D_1, D_2) \), we get the conclusion.

\[ \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \]

Remark 3.12. \((3.16)\) is entirely analogous to the definition of Wirtinger derivatives

\[ \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \]

for a complex number \( z = x + iy \) with \( x, y \in \mathbb{R} \).

For \( F \in L_\mathbb{C}^2(\Omega, \sigma(Z), P) \), by chaos decomposition \((2.6)\) and Theorem 3.9, we have

\[
F = \sum_{p=0}^\infty \sum_{q=0}^\infty I_{p,q}(f_{p,q}) = \sum_{p=0}^\infty \sum_{q=0}^\infty I_{p+q}(u_{p,q}) + i \sum_{p=0}^\infty \sum_{q=0}^\infty I_{p+q}(v_{p,q}),
\]

where \( I_{p+q}(\cdot) \) is the real \((p + q)\)-th Wiener-Itô integral with respect to \( W \) and \( u_{p,q}, v_{p,q} \in (\mathcal{H} \oplus \bar{\mathcal{H}})^{c(p+q)} \) are the kernels of the real and imaginary parts of \( I_{p,q}(f_{p,q}) \). Thus, \( \text{Re} F \) and \( \text{Im} F \) can be uniquely expanded into series of \( I_n(\cdot) \) as follows

\[ \text{Re} F = \sum_{n=0}^\infty I_n(f_n), \quad f_n = \sum_{p=0}^n u_{p,n-p}, \]

\[ \text{Im} F = \sum_{n=0}^\infty I_n(g_n), \quad g_n = \sum_{p=0}^n v_{p,n-p}. \]

Combining \((3.16)\) and Stroock’s formula \((2.10)\) for real Wiener-Itô integrals, we obtain the computable expressions of \( f_n \) and \( g_n \), which can be considered as a generalized Stroock’s formula.

**Proposition 3.13** (Generalized Stroock’s formula). If \( F \in D^{m,2} \cap \bar{D}^{m,2} \) for some \( m \geq n \) with the expansions of \( \text{Re} F = \sum_{n=0}^\infty I_n(f_n) \) and \( \text{Im} F = \sum_{n=0}^\infty I_n(g_n) \), where \( f_n, g_n \in (\mathcal{H} \oplus \bar{\mathcal{H}})^{c(n)} \) and \( I_n(\cdot) \) is the real \( n \)-th Wiener-Itô integral with respect to \( W \). Then \( f_n \) and \( g_n \) are uniquely defined as

\[ f_n + ig_n = \frac{1}{n!} 2^{-\frac{n}{2}} \mathbb{E} \left[ (D + \bar{D}, i(D - \bar{D}))^{\otimes n} F \right]. \]

**Proof.** By the Stroock’s formula \((2.10)\) and \((3.16)\), for \( n \leq m \), we get

\[
f_n = \frac{1}{n!} \mathbb{E} [D^n \text{Re} F] = \frac{1}{n!} \mathbb{E} \left[ (D^1, D^2)^{\otimes n} \text{Re} F \right] = \frac{1}{n!} 2^{-\frac{n}{2}} \mathbb{E} \left[ (D + \bar{D}, i(D - \bar{D}))^{\otimes n} \text{Re} F \right].
\]

By a similar argument, we obtain that

\[
g_n = \frac{1}{n!} 2^{-\frac{n}{2}} \mathbb{E} \left[ (D + \bar{D}, i(D - \bar{D}))^{\otimes n} \text{Im} F \right].
\]

Then we get the conclusion. \( \square \)
In Theorem 3.9, we establish a method to get the kernels of the real and imaginary parts of a complex Wiener-Itô integral by given complete and orthogonal elements in $H_C$. However, it is difficult to get explicit expressions for the kernels when the basis of $H_C$ cannot be determined. With Proposition 3.13, we overcome this difficulty and obtain another expressions of the kernels in terms of complex Malliavin derivative operators $D$ and $\bar{D}$.

**Corollary 3.14.** $F = I_{p,q}(f)$ with $f \in \mathcal{H}^{\otimes p}_C \otimes \mathcal{H}^{\otimes q}_C$ admits the unique representation

$$I_{p,q}(f) = I_{p+q}(u) + iI_{p+q}(v),$$

where $u, v \in (\mathcal{H} \oplus \mathcal{H})^{\otimes (p+q)}$ are defined as

$$u + iv = \frac{1}{(p + q)!} 2^{\frac{p+q}{2}} (D + \bar{D}, i(D - \bar{D}))^{\otimes (p+q)} F,$$

and $I_{p+q}(\cdot)$ is the real $(p + q)$-th Wiener-Itô integral with respect to $W$.

By Corollary 3.14, given a complex Wiener-Itô integral, we can consider it as a two-dimensional random vector whose components are real Wiener-Itô integrals and expressions for the kernels are explicit. Then the asymptotic normality of it can be proved by utilizing the multidimensional version of fourth moment theorem (see [23, Theorem 5.2.7, Theorem 6.2.3]). (With [10, Theorem 3.3], we can not implement this method since the kernels of real and imaginary parts are unclear.)

**Example 3.15.** In [8], Chen, Hu and Wang considered the least squares estimator $\hat{\gamma}_T$ of the drift coefficient $\gamma$ for the complex-valued Ornstein-Uhlenbeck processes disturbed by fractional noise, and get the strong consistency and the asymptotic normality of $\hat{\gamma}_T$. The numerator $F_T$ of the statistic $\sqrt{T}(\hat{\gamma}_T - \gamma)$ is a $(1,1)$-th complex Wiener-Itô integral with respect to a complex fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, \frac{3}{4})$. Namely, assume that $\gamma \in C$ is unknown,

$$F_T = I_{1,1}(\psi_T(t,s)), \quad \psi_T(t,s) = \frac{1}{\sqrt{T}} e^{-\gamma(t-s)} 1_{\{0 \leq s \leq t \leq T\}},$$

$$\bar{F}_T = I_{1,1}(\phi_T(t,s)), \quad \phi_T(t,s) = \frac{1}{\sqrt{T}} e^{-\gamma(s-t)} 1_{\{0 \leq t \leq s \leq T\}}.$$

The underlying Hilbert space is defined as

$$\mathcal{H} := \left\{ f | f: \mathbb{R}_+ \to \mathbb{R}, \|f\|_\mathcal{H} := \int_0^\infty \int_0^\infty f(s)f(t)\varphi(s,t)dsdt < \infty \right\},$$

with $\alpha_H = H(2H - 1)$, $\varphi(s,t) = \alpha_H |s - t|^{2H-2}$ and inner product

$$\langle f, g \rangle_\mathcal{H} = \int_0^\infty \int_0^\infty f(s)g(t)\varphi(s,t)dsdt.$$

We complexify $\mathcal{H}$ in the usual way and denote by $\mathcal{H}_C$. In [8], in order to show the asymptotic normality of $F_T$, the authors firstly established some equivalent conditions
for the complex fourth moment theorem (see [8, Theorem 1.3]), and then made use of results obtained, calculated accurately
\[
\lim_{T \to \infty} \mathbb{E} \left[ |F_T|^2 \right], \quad \lim_{T \to \infty} \mathbb{E} \left[ F_T^2 \right],
\]
and verified the contraction condition
\[
\lim_{T \to \infty} \|\psi_T \otimes_{0,1} \phi_T\|_{\mathbb{C}^2} = 0.
\]
Now, by Corollary 3.14, we get that
\[
F_T = F_{1,T} + iF_{2,T} = I_2(u_T + iv_T),
\]
where \( u_T + iv_T \) are defined as (3.17) below. This means that we can regard \( F_T \) as two-dimensional random vector \((F_{1,T}, F_{2,T})\) and utilize the real fourth moment theorem (see [23, Theorem 5.2.7, Theorem 6.2.3]) to prove the asymptotic normality of \( F_T \). In this way, we need to calculate
\[
\lim_{T \to \infty} \mathbb{E} \left[ F_{i,T}F_{j,T} \right], \quad i, j = 1, 2,
\]
and show that
\[
\lim_{T \to \infty} \|u_T \otimes_1 u_T\|_{\mathbb{C}^2} = 0, \quad \lim_{T \to \infty} \|v_T \otimes_1 v_T\|_{\mathbb{C}^2} = 0.
\]
Next, we derive the expression of the kernel \( u_T + iv_T \) by Corollary 3.14. Calculating directly, we have
\[
\left(D + \bar{D}, i(D - \bar{D})\right)^{\otimes 2} = \left( \begin{array}{cc} D + \bar{D} \\ i(D - \bar{D}) \end{array} \right) \otimes \left( \begin{array}{cc} D + \bar{D} \\ i(D - \bar{D}) \end{array} \right) = \left( \begin{array}{cc} D_2 + D\bar{D} + \bar{D}D + D^2 \\ i(D_2 + D\bar{D} - \bar{D}D - D^2) \\ i(D_2 - D\bar{D} + \bar{D}D - D^2) \\ -D_2 + \bar{D}D + \bar{D}D - D^2 \end{array} \right).
\]
Then according to Corollary 3.14 and the fact that \( D_2 F = D^2 F = 0 \), we get
\[
(3.17) \quad u_T(t, s) + iv_T(t, s) = \frac{1}{4} \left(D + \bar{D}, i(D - \bar{D})\right)^{\otimes 2} F
\]
\[
= \frac{1}{4} \left( \begin{array}{cc} \mathcal{D}\bar{D} + \bar{D}\mathcal{D} \\ i(D\bar{D} - \bar{D}D) \\ i(-D\bar{D} + \bar{D}\mathcal{D}) \\ D\bar{D} + \bar{D}\mathcal{D} \end{array} \right) F = \frac{1}{4} \left( \begin{array}{cc} \psi(t, s) + \psi(s, t) \\ i(\psi(t, s) - \psi(s, t)) \\ -i(\psi(t, s) - \psi(s, t)) \\ \psi(t, s) + \psi(s, t) \end{array} \right)
\]
\[
= \frac{1}{4\sqrt{T}} \left( \begin{array}{cc} -ie^{-\gamma|t-s|} \left(1_{\{0 \leq t \leq s \leq T\}} - 1_{\{0 \leq s \leq t \leq T\}} \right) \\ ie^{-\gamma|t-s|} \left(1_{\{0 \leq t \leq s \leq T\}} - 1_{\{0 \leq s \leq t \leq T\}} \right) \\ e^{-\gamma|t-s|} \left(1_{\{0 \leq t \leq s \leq T\}} - 1_{\{0 \leq s \leq t \leq T\}} \right) \end{array} \right).
\]
It is worth noting that the representation theorem (Theorem 3.9 or Corollary 3.14) we established does provide a new method to solve problems concerning complex Wiener-Itô integrals. The explicit expressions for kernels of real and imaginary parts enable us to
do accurate calculations, although the difficulties to overcome and techniques used are similar in the above two methods.

4. Kernel representation formula from real to complex Wiener-Itô integrals

4.1. Uniqueness theorem. In Section 3, we show how to get the kernels of real and complex parts of a complex multiple Wiener-Itô integral. Conversely, in this section, we consider a complex random variable, whose real and imaginary parts are two real multiple Wiener-Itô integrals of the same order, and prove that it can be uniquely expressed as a finite sum of complex Wiener-Itô integrals.

Let \( m = \{m_k\}_{k=1}^{\infty}, n = \{n_k\}_{k=1}^{\infty} \in \Lambda \) with \(|m| + |n| = p\). Let \( p = \{p_k\}_{k=1}^{\infty} := m + n\), then \(|p| = p\). For simplicity of presentation, define

\[
\tilde{a}_{k,j} := \sum_{r+s=j} \binom{m_k}{r} \binom{n_k}{s} (-1)^s.
\]

For \( j = \{j_k\}_{k=1}^{\infty} \in \Lambda \) with \( j \leq p\), \( \prod_{k=1}^{\infty} a_{k,j_k} \) is a complex number. Define \( g_{l,p-l}(m,n) \in \hat{F}_{C} \otimes \hat{F}_{C}(p-l) \) with \( 0 \leq l \leq p \) as

\[
g_{l,p-l}(m,n) = \sum_{j \leq l, j_1 = j} \left( \prod_{i=1}^{\infty} \tilde{a}_{i,j_i} \right) \text{symm} \left( \otimes_{k=1}^{\infty} e_k^{\otimes j_k} \right) \otimes \text{symm} \left( \otimes_{k=1}^{\infty} \tilde{e}_k^{\otimes (p_k-j_k)} \right).
\]

Suppose \( g_1, g_2 \in (\hat{F} \oplus \hat{F})^{op} \), then there exists a unique sequence

\[
\{\tilde{c}(m,n) : m, n \in \Lambda, |m| + |n| = p\} \subset \mathbb{C}
\]

satisfying \( \sum_{m,n \in \Lambda, |m| + |n| = p} |\tilde{c}(m,n)|^2 < \infty \) such that

\[
(4.1) \qquad g_1 + ig_2 = \sum_{m,n \in \Lambda, |m| + |n| = p} \tilde{c}(m,n) \text{symm} \left( \otimes_{k=1}^{\infty} \left( u_{1,0}(k)^{\otimes m_k} \otimes v_{1,0}(k)^{\otimes n_k} \right) \right).
\]

**Theorem 4.1.** Suppose \( g_1, g_2 \in (\hat{F} \oplus \hat{F})^{op} \) and \( g_1 + ig_2 \) is given by (4.1). Then \( I_p(g_1) + iI_p(g_2) \) admits the representation

\[
I_p(g_1) + iI_p(g_2) = \sum_{l=0}^{p} I_{l,p-l}(\tilde{g}_{l,p-l}),
\]

where \( \tilde{g}_{l,p-l} \in \hat{F}_{C} \otimes \hat{F}_{C}(p-l) \), \( 0 \leq l \leq p \) are defined as

\[
\tilde{g}_{l,p-l} := \sum_{m,n \in \Lambda, |m| + |n| = p} \tilde{c}(m,n)g_{l,p-l}(m,n).
\]

We prove Theorem 4.1 in Section 5. Moreover, by an induction argument and utilizing recursion formula concerning real multiple Wiener-Itô integral given by Itô (see [16, Equation 3.4]), we can offer a recursion representation version of Theorem 4.1. The proof of this result is quite similar to that of Theorem 3.9 and so is omitted.
4.2. Representation theorem. For an integer $1 \leq j \leq 2^p$, $j - 1$ can be uniquely expressed as a binary number

$$j - 1 = \sum_{l=1}^{p} a_{jl}2^{l-1}, \quad a_{jl} \in \{0, 1\}.$$  

Combining the real Stroock’s formula (2.10) with the fact that $(D_1, D_2)^{\otimes p}$ is a $2^p$-dimensional column vector defined as

$$(D_1, D_2)^{\otimes p} = (D_{a_{j1}+1}D_{a_{j2}+1} \cdots D_{a_{jp}+1})_{1 \leq j \leq 2^p},$$

we get the following lemma.

**Lemma 4.2.** Suppose that $I_p(g)$ with $g = (g_j)_{1 \leq j \leq 2^p} \in (\mathcal{F} \oplus \mathcal{F})^{\otimes p}$ is a real $p$-th Wiener-Itô integral with respect to $W$, then

$$g_j = \frac{1}{p!} D_{a_{j1}+1}D_{a_{j2}+1} \cdots D_{a_{jp}+1} I_p(g).$$

For $1 \leq j \leq 2^p$ and $0 \leq k \leq p$, let $b_{kj}$ denote the number of 1 in $\{a_{j1}, \ldots, a_{jk}\}$, and $c_{kj}$ denote the number of 1 in $\{a_{j+1}, \ldots, a_{jp}\}$. Set $b_{0j} = 0$, $1 \leq j \leq 2^p$. Define column vectors

$$V_k = (V_{kj})_{1 \leq j \leq 2^p} = \left((1 + i)^{b_{kj}} i^{c_{kj}}\right)_{1 \leq j \leq 2^p}, \quad 0 \leq k \leq p.$$

**Theorem 4.3.** For any $g_1 = (g_{1j})_{1 \leq j \leq 2^p}, g_2 = (g_{2j})_{1 \leq j \leq 2^p} \in (\mathcal{F} \oplus \mathcal{F})^{\otimes p}$, consider the complex random variable $I_p(g_1) + iI_p(g_2)$, where $I_p(\cdot)$ is the $p$-th real multiple Wiener-Itô integral with respect to $W$. Then

$$I_p(g_1) + iI_p(g_2) = \sum_{k=0}^{p} \mathcal{I}_{k,p-k}(g_{k,p-k}),$$

where $g_{k,p-k} \in \mathcal{F}_{C}^{\otimes k} \otimes \mathcal{F}_{C}^{\otimes (p-k)}, 0 \leq k \leq p$, are defined as

$$g_{k,p-k} = \frac{2^{-p/2}p!}{k!(p-k)!} \sum_{j=1}^{2^p} V_{kj}(g_{1j} + ig_{2j}).$$

**Proof.** According to Theorem 4.1, there uniquely exist $g_{k,p-k} \in \mathcal{F}_{C}^{\otimes k} \otimes \mathcal{F}_{C}^{\otimes (p-k)}$ with $k = 0, \ldots, p$ such that

$$I_p(g_1) + iI_p(g_2) = \sum_{k=0}^{p} \mathcal{I}_{k,p-k}(g_{k,p-k}).$$

Combining complex Stroock’s formula (2.12) and Equation (3.16), we have

$$g_{k,p-k} = \frac{1}{k!(p-k)!} D^k D^{p-k} (I_p(g_1) + iI_p(g_2))$$

$$= \frac{2^{-p/2}}{k!(p-k)!} (D_1 - iD_2)^k (D_1 + iD_2)^{p-k} (I_p(g_1) + iI_p(g_2))$$
\[
= \frac{2^{-p/2}p!}{k!(p-k)!} \sum_{j=1}^{2p} V_{kj}(g_{1j} + ig_{2j}),
\]
where the last equality follows from Lemma 4.2 and the fact that
\[
(D_1 - iD_2)^k(D_1 + iD_2)^{p-k} = \sum_{j=1}^{2p} V_{kj}D_{a_{j1}+1}D_{a_{j2}+1} \cdots D_{a_{jp}+1}.
\]

By Theorem 4.3, we get the following theorem. This theorem shows that if the kernels \(g_1 = (g_{1j})_{1 \leq j \leq 2p}, g_2 = (g_{2j})_{1 \leq j \leq 2p} \in (\mathfrak{H} \oplus \mathfrak{F})^{\otimes p}\) satisfy condition (4.3), then the two-dimensional random vector \(I_p(g_1), I_p(g_2)\) can be regarded as a complex multiple Wiener-Itô integral. This means that we can utilize the theory of complex multiple Wiener-Itô integrals to solve the problems concerning two-dimensional random vectors whose components are real multiple Wiener-Itô integrals of the same order.

**Theorem 4.4.** Given a two-dimensional random vector \((I_p(g_1), I_p(g_2))\) with \(g_1 = (g_{1j})_{1 \leq j \leq 2p}, g_2 = (g_{2j})_{1 \leq j \leq 2p} \in (\mathfrak{H} \oplus \mathfrak{F})^{\otimes p}\), whose components are real multiple Wiener-Itô integrals with respect to \(W\). If there exists a unique \(0 \leq k \leq p\) such that

\[
\begin{align*}
\sum_{j=1}^{2p} V_{kj}(g_{1j} + ig_{2j}) & \neq 0, \\
\sum_{j=1}^{2p} V_{ij}(g_{1j} + ig_{2j}) & = 0, \quad l \neq k, 0 \leq l \leq p,
\end{align*}
\]

then

\[
I_p(g_1) + iI_p(g_2) = \mathcal{I}_{k,p-k}(g_{k,p-k}),
\]

where \(g_{k,p-k}\) is defined as (4.2).

**Example 4.5.** Back to Example 3.15, \(u_T(t, s) + iv_T(t, s) \in (\mathfrak{H} \oplus \mathfrak{F})^{\otimes 2}\) is given by (3.17). For \(p = 2\),

\[
V_0 = (1, i, i, -1)^T, \quad V_1 = (1, -i, i, 1)^T, \quad V_2 = (1, -i, -i, -1)^T.
\]

According to Theorem 4.3, \(I_2(u_T) + iI_2(v_T)\) can be uniquely expressed as

\[
I_2(u_T) + iI_2(v_T) = \mathcal{I}_{0,2}(g_{0,2}) + \mathcal{I}_{1,1}(g_{1,1}) + \mathcal{I}_{2,0}(g_{2,0}),
\]

where

\[
\begin{align*}
g_{0,2} & = \frac{e^{-\gamma|t-s|}}{8\sqrt{T}} (1 - 1_{0 \leq s \leq t \leq T} + 1_{0 \leq t \leq s \leq T} - 1_{0 \leq s \leq t \leq T}) = 0, \\
g_{1,1} & = \frac{e^{-\gamma|t-s|}}{2\sqrt{T}} (1 + 1_{0 \leq s \leq t \leq T} - 1_{0 \leq t \leq s \leq T}) = \frac{1}{\sqrt{T}} e^{-\gamma (t-s)} 1_{0 \leq s \leq t \leq T} = \psi_T(t, s), \\
g_{2,0} & = \frac{e^{-\gamma|t-s|}}{8\sqrt{T}} (1 + 1_{0 \leq s \leq t \leq T} - 1_{0 \leq t \leq s \leq T} + 1_{0 \leq t \leq s \leq T} - 1_{0 \leq s \leq t \leq T} - 1) = 0.
\end{align*}
\]

That is to say, the condition (4.3) is satisfied. Therefore,

\[
I_2(u_T) + iI_2(v_T) = \mathcal{I}_{1,1}(g_{1,1}) = \mathcal{I}_{1,1}(\psi_T).
\]
Remark 4.6. Combining the main results we established in Section 3 and Section 4, we derive that

\[
L^2_C(\Omega, \sigma(Z), P) = \bigoplus_{p=0}^{\infty} \bigoplus_{q=0}^{\infty} \mathcal{H}_{p,q}(Z) = \bigoplus_{p=0}^{\infty} \bigoplus_{k+l=p} \mathcal{H}_{k,l}(Z) = \bigoplus_{p=0}^{\infty} \left( \mathcal{H}_p(W) + i \mathcal{H}_p(W) \right).
\]

(4.4) was proved by an existence proof in [10]. We further clearly characterize the kernels of real and complex Wiener-Itô integrals in this paper. We can understand chaos derived that

Combining the main results we established in Section 3 and Section 4, Remark 4.6.

Proof of Lemma 3.1. Since \(\{e_k\}_{k \geq 1}\) are orthogonal in \(\mathcal{F}_C\) and \(\|e_k\|^2_{\mathcal{H}_C} = 2\), we have

\[
2\delta_{kj} = \langle e_k, e_j \rangle_{\mathcal{H}_C} = \langle e_k^1 + ie_k^2, e_j^1 + ie_j^2 \rangle_{\mathcal{H}_C} = \left( \langle e_k^1, e_j^1 \rangle_{\mathcal{H}} + \langle e_k^2, e_j^2 \rangle_{\mathcal{H}} \right) + i \left( \langle e_k^1, e_j^2 \rangle_{\mathcal{H}} - \langle e_k^2, e_j^1 \rangle_{\mathcal{H}} \right),
\]

where \(\delta_{kj} = 1_{\{k=j\}}\). It implies that for any \(k\) and \(j\),

\[
\langle e_k^1, e_j^1 \rangle_{\mathcal{H}} + \langle e_k^2, e_j^2 \rangle_{\mathcal{H}} = 2\delta_{kj},
\]

\[
\langle e_k^2, e_j^1 \rangle_{\mathcal{H}} - \langle e_k^1, e_j^2 \rangle_{\mathcal{H}} = 0.
\]

Then,

\[
\langle u_{1,0}(k), u_{1,0}(j) \rangle_{\mathcal{H}} = \frac{1}{2} \left( \langle e_k^1, e_j^1 \rangle_{\mathcal{H}} + \langle e_k^2, e_j^2 \rangle_{\mathcal{H}} \right) = \delta_{kj},
\]

\[
\langle v_{1,0}(k), v_{1,0}(j) \rangle_{\mathcal{H}} = \frac{1}{2} \left( \langle e_k^2, e_j^2 \rangle_{\mathcal{H}} + \langle e_k^1, e_j^1 \rangle_{\mathcal{H}} \right) = \delta_{kj},
\]

\[
\langle u_{1,0}(k), v_{1,0}(j) \rangle_{\mathcal{H}} = \frac{1}{2} \left( \langle e_k^1, e_j^2 \rangle_{\mathcal{H}} - \langle e_k^2, e_j^1 \rangle_{\mathcal{H}} \right) = 0,
\]

which shows that \(\{u_{1,0}(k), v_{1,0}(k)\}_{k \geq 1}\) are orthonormal in \(\mathcal{F} \oplus \bar{\mathcal{F}}\).

5. Proofs of Main Results

5.1. Proof of Lemma 3.1.
Next, we show the completeness of \(\{u_{1,0}(k), v_{1,0}(k)\}_{k \geq 1}\). Suppose that \(w = (w_1, w_2) \in \mathcal{H} \oplus \mathcal{H}\) satisfies \(\langle w, u_{1,0}(k) \rangle_{\mathcal{H} \oplus \mathcal{H}} = 0\) and \(\langle w, v_{1,0}(k) \rangle_{\mathcal{H} \oplus \mathcal{H}} = 0\) for any \(k \geq 1\). That is,
\[
\langle w, u_{1,0}(k) \rangle_{\mathcal{H} \oplus \mathcal{H}} = \frac{1}{\sqrt{2}} \left( \langle w_1, e_{1,k}^1 \rangle_{\mathcal{H}} - \langle w_2, e_{2,k}^1 \rangle_{\mathcal{H}} \right) = 0,
\]
\[
\langle w, v_{1,0}(k) \rangle_{\mathcal{H} \oplus \mathcal{H}} = \frac{1}{\sqrt{2}} \left( \langle w_1, e_{1,k}^2 \rangle_{\mathcal{H}} + \langle w_2, e_{2,k}^1 \rangle_{\mathcal{H}} \right) = 0.
\]
Let \(\tilde{w} = w_2 + iw_1\), then
\[
\langle \tilde{w}, e_k \rangle_{\mathcal{H}_C} = \langle w_2 + iw_1, e_k^1 + ie_k^2 \rangle_{\mathcal{H}_C} = \left( \langle w_2, e_k^1 \rangle_{\mathcal{H}} + \langle w_1, e_k^2 \rangle_{\mathcal{H}} \right) + i \left( \langle w_1, e_k^1 \rangle_{\mathcal{H}} - \langle w_2, e_k^2 \rangle_{\mathcal{H}} \right) = 0,
\]
which indicates that \(\tilde{w} = 0\) by the completeness of \(\{e_k\}_{k \geq 1}\) in \(\mathcal{H}_C\). Therefore, \(w = (w_1, w_2) = 0\) in \(\mathcal{H} \oplus \mathcal{H}\).

5.2. **Proofs of Lemma 3.4, Proposition 3.5 and Theorem 3.3.** From now on, for the real multiple Wiener-Itô integral \(I_p(\cdot)\) with respect to \(W\), if the kernel is complex with the form \(f + ig\), where \(f, g \in (\mathcal{H} \oplus \mathcal{H})^{\otimes p}\), we set
\[
I_p(f + ig) = I_p(f) + iI_p(g).
\]

**Proof of Lemma 3.4.** By the relation between real and complex Hermite polynomials (2.7), we have
\[
I_{p,q} \left( e_1^{\otimes p} \otimes \bar{e}_1^{\otimes q} \right) = J_{p,q} \left( Z(e_1) \right)
\]
\[
= \sum_{j=0}^{p+q} \sum_{r+s=j}^{p+q-j} \binom{p}{r} \binom{q}{s} (-1)^{q-s} H_j \left( \text{Re}Z(e_1) \right) H_{p+q-j} \left( \text{Im}Z(e_1) \right)
\]
\[
= \sum_{j=0}^{p+q} \sum_{r+s=j} \binom{p}{r} \binom{q}{s} (-1)^{q-s} H_j \left( W(u_{1,0}(1)) \right) H_{p+q-j} \left( W(v_{1,0}(1)) \right)
\]
\[
= \sum_{j=0}^{p+q} \sum_{r+s=j} \binom{p}{r} \binom{q}{s} (-1)^{q-s} I_j \left( u_{1,0}(1)^{\otimes j} \right) I_{p+q-j} \left( v_{1,0}(1)^{\otimes (p+q-j)} \right).
\]
According to the definition of real Wiener-Itô integral (2.1),
\[
I_j \left( u_{1,0}(1)^{\otimes j} \right) I_{p+q-j} \left( v_{1,0}(1)^{\otimes (p+q-j)} \right) = I_{p+q} \left( u_{1,0}(1)^{\otimes j} \otimes v_{1,0}(1)^{\otimes (p+q-j)} \right).
\]
Then the proof is completed.

**Proof of Proposition 3.5.** Using the definitions of real and complex Wiener-Itô integrals, (2.1) and (2.5), respectively, we have
\[
I_{p,q} \left( \text{symm} \left( \bigotimes_{k=1}^\infty e_k^{\otimes pk} \right) \otimes \text{symm} \left( \bigotimes_{k=1}^\infty \bar{e}_k^{\otimes qk} \right) \right)
\]
\[
= \prod_{k=1}^\infty I_{p_k,q_k} \left( e_k^{\otimes pk} \otimes \bar{e}_k^{\otimes qk} \right)
\]
Proof of Theorem 3.3. For $f \in \mathcal{H}_C^{\otimes p} \otimes \mathcal{H}_C^{\otimes q}$ with the expansion given by \eqref{eq:expansion}, \( I_{p,q} (f) = \sum_{p,q \in \Lambda, |p|=p, |q|=q} c(p,q) I_{p,q} (\text{symm} (\otimes_{k=1}^{\infty} e_k^{\otimes p_k}) \otimes \text{symm} (\otimes_{k=1}^{\infty} e_k^{\otimes q_k})) \)

\[ = \sum_{p,q \in \Lambda, |p|=p, |q|=q} c(p,q) I_{p+q} (u(p,q) + iv(p,q)) \]

We complete the proof. \( \square \)

5.3. Proofs of Theorem 3.7 and Theorem 3.9. For $1 \leq r \leq q + 1$, we write \( I_{p,q}(k,j_r) := I_{p,q}(e_{k_1} \otimes \cdots \otimes e_{k_p} \otimes e_{j_1} \otimes \cdots \otimes e_{j_{r-1}} \otimes e_{j_{r+1}} \otimes \cdots \otimes e_{j_{q+1}}) \).

For $1 \leq l \leq p + 1$, we also define \( I_{p,q}(\hat{k}_l,j) \) in the same way. The following useful lemma is rephrased from \cite[Theorem 9]{17}.

Lemma 5.1. For $p \geq 0$, $q \geq 0$, we have

\begin{align}
I_{p+1,q}(k,j) &= I_{p,q}(k,j) I_{0,0} (e_{k_{p+1}}) - 1_{\{q > 0\}} \sum_{r=1}^{q} 2\delta_{j_r,k_{p+1}} I_{p,q-1}(k,j_r), \\
I_{p,q+1}(k,j) &= I_{p,q}(k,j) I_{0,1} (e_{j_{p+1}}) - 1_{\{p > 0\}} \sum_{l=1}^{p} 2\delta_{k_l,j_{q+1}} I_{p-1,q}(\hat{k}_l,j),
\end{align}

where $1_A$ is the indicator function of a set $A$ and $\delta_{k,j} = 1_{\{k = j\}}$.

For simplicity, we write

\begin{align}
&u_{p,q}(k,j_r) := u_{p,q}(k_1, \ldots, k_p; j_1, \ldots, j_{r-1}, j_{r+1}, \cdots, j_{q+1}), \quad \text{for } 1 \leq r \leq q + 1, \\
u_{p,q}(\hat{k}_l,j) := u_{p,q}(k_1, \ldots, k_{l-1}, k_{l+1}, \ldots, k_{p+1}; j_1, \cdots, j_q), \quad \text{for } 1 \leq l \leq p + 1, \\
v_{p,q}(k,j_r) := v_{p,q}(k_1, \ldots, k_p; j_1, \ldots, j_{r-1}, j_{r+1}, \cdots, j_{q+1}), \quad \text{for } 1 \leq r \leq q + 1, \\
v_{p,q}(\hat{k}_l,j) := v_{p,q}(k_1, \ldots, k_{l-1}, k_{l+1}, \ldots, k_{p+1}; j_1, \cdots, j_q), \quad \text{for } 1 \leq l \leq p + 1.
\end{align}
By (3.7) and (3.8), we get that
\[
w_{p,q}(k,j) := u_{p,q}(k,j) + iv_{p,q}(k,j)
= (u_{p-1,q}(k,j) + iv_{p-1,q}(k,j)) \otimes (u_{1,0}(k_p) + iv_{1,0}(k_p))
= (u_{p,q-1}(k,j) + iv_{p,q-1}(k,j)) \otimes (u_{0,1}(j_q) + iv_{0,1}(j_q)).
\]

Define \( w_{1,0}(j) := u_{1,0}(j) + iv_{1,0}(j) \) and \( w_{0,1}(j) := u_{0,1}(j) + iv_{0,1}(j) \).

**Proof of Theorem 3.7.** The proof is by induction. We use \( A(p,q) \) to denote the equation
\[
I_{p,q}(k,j) = I_{p+q}(u_{p,q}(k,j) + iv_{p,q}(k,j)) = I_{p+q}(w_{p,q}(k,j)).
\]

For \( p \geq 0, q \geq 0, p + q > 0 \), let \( B(p,q) \) denote the equation
\[
(p + q)I_{p+q-1}(w_{p,q}(k,j) \otimes w_{1,0}(k_{p+1})) = 1_{\{q \geq 0\}} \sum_{r=1}^{q} 2\delta_{j_r,k_{p+1}}I_{p,q-1}(k,\hat{j}_r).
\]

We will prove \( A(p,q) \) and \( B(p,q) \) hold by induction with respect to \( n := p + q \) in three steps. See [Figure 1] for complete ideas of proof and [Figure 2] for details.

![Figure 1](image-url)

**Figure 1.** The black part represents that conditions \( A \) and \( B \) hold for \((p,q)\) with \( p + q \leq m \). The blue part represents conditions \( A \) and \( B \) for \((p,q)\) with \( p + q \geq m + 1 \). We prove the blue part by induction with respect to \( p + q \).

**Step 1** Suppose \( n = p + q = 1 \). For \((p,q) = (1,0)\), (3.2) implies \( A(1,0) \). By (3.6),
\[
I_{0,1}(\eta_j) = \overline{I_{1,0}(\eta_j)} = I_1(u_{1,0}(j)) - iI_1(v_{1,0}(j)) = I_1(u_{0,1}(j) + iv_{0,1}(j)).
\]

It follows that \( A(0,1) \) holds.

A direct calculation shows that
\[
I_0(w_{0,1}(j_1) \otimes w_{1,0}(k_1))
\]
Therefore, we have \( B \). This implies \( B \), where the last equality follows from (1.1) and (2.2), we have \( \delta \). 

In Step 2, we prove \( \delta \) with \( \delta \) and \( \delta \) hold. In Step 3, we prove \( \delta \) with \( \delta \) and \( \delta \) hold.

\[
= (u_{1,0}(j_1) - iv_{1,0}(j_1)) \tilde{\otimes}_1 (u_{1,0}(k_1 + iv_{1,0}(k_1))
= u_{1,0}(j_1) \tilde{\otimes}_1 u_{1,0}(k_1) + v_{1,0}(j_1) \tilde{\otimes}_1 v_{1,0}(k_1) + i(u_{1,0}(j_1) \tilde{\otimes}_1 v_{1,0}(k_1) - v_{1,0}(j_1) \tilde{\otimes}_1 u_{1,0}(k_1))
= 2\delta_{j_1,k_1}.
\]

Therefore, we have \( B(0,1) \). Similarly, we get that

\[
I_0 (w_{1,0}(k_1) \tilde{\otimes}_1 w_{1,0}(k_2)) = 0,
\]

which implies \( B(1,0) \).

**Step 2** Suppose \( A(p, q) \) and \( B(p, q) \) hold for \( n = p + q \leq m \) with \( m \geq 1 \). In this step, we will prove that \( A(p, q) \) holds for \( n = m + 1 \).

It suffices to prove \( A(p + 1, q) \) and \( A(p, q + 1) \) hold for \( p + q = m \). By (5.1), \( A(p, q) \), \( A(1,0) \) and (2.2), we have

\[
I_{p+1,q}(k,j)
= I_{p,q}(k,j)I_{1,0}(e_{k_{p+1}}) - 1_{\{q>0\}} \sum_{r=1}^{q} 2\delta_{j_r,k_{p+1}} I_{p,q-1}(k, j_r)
= I_{p+q}(w_{p,q}(k,j))I_1(w_{1,0}(k_{p+1})) - 1_{\{q>0\}} \sum_{r=1}^{q} 2\delta_{j_r,k_{p+1}} I_{p,q-1}(k, j_r)
= \sum_{r=0}^{1} r! \binom{p+q}{r} \binom{1}{r} I_{p+q+1-2r}(w_{p,q}(k,j) \tilde{\otimes}_r w_{1,0}(k_{p+1}))
- 1_{\{q>0\}} \sum_{r=1}^{q} 2\delta_{j_r,k_{p+1}} I_{p,q-1}(k, j_r)
= I_{p+q+1}(w_{p,q}(k,j) \tilde{\otimes}_1 w_{1,0}(k_{p+1})) + (p+q) I_{p+q+1-2}(w_{p,q}(k,j) \tilde{\otimes}_1 w_{1,0}(k_{p+1}))
- 1_{\{q>0\}} \sum_{r=1}^{q} 2\delta_{j_r,k_{p+1}} I_{p,q-1}(k, j_r)
= I_{p+q+1}(w_{p+1,q}(k,j)),
\]

where the last equality follows from \( B(p, q) \) and (5.4). Therefore, \( A(p + 1, q) \) holds.
By (3.9) and $B(q, p)$, we have
\[
(p + q)I_{p+q-1}(w_{p,q}(k,j)\hat{\otimes}_1 w_0, 1(j_{q+1})) = (p + q)I_{p+q-1}(w_{q,p}(j,k)\hat{\otimes}_1 w_{1,0}(j_{q+1}))
\]
\[
= 1_{(p > 0)} \sum_{i=1}^{p} 2\delta_{i,j_{q+1}}I_{q,p-1}(j,k) = 1_{(p > 0)} \sum_{i=1}^{p} 2\delta_{i,j_{q+1}}I_{p-1,q}(k,j).
\]
Combining this equation and (5.2), we obtain $A(p, q + 1)$ by a similar argument of $A(p + 1, q)$.

**Step 3** Suppose $A(p, q)$ holds for $n = p + q \leq m + 1$ and $B(p, q)$ holds for $p + q \leq m$. In this step, we will prove $B(p, q)$ holds for $n = m + 1$.

It suffices to show that $B(p + 1, q)$ and $B(p, q + 1)$ hold for $p + q = m$, where the equation $B(p + 1, q)$ is
\[
(p + q + 1)I_{p+q}(w_{p+1,q}(k,j)\hat{\otimes}_1 w_{1,0}(k_{p+2})) = 1_{(q > 0)} \sum_{r=1}^{q} 2\delta_{j_{r},k_{p+2}}I_{p+1,q-1}(k,j_r).
\]
By $A(p + 1, q - 1)$, we have
\[
I_{p+q}(w_{p+1,q-1}(k,j_r)) = I_{p+q}(w_{p+1,q-1}(k,j_r)).
\]
To prove $B(p + 1, q)$, it suffices to show that
\[
(5.5) \quad (p + q + 1) (w_{p+1,q}(k,j)\hat{\otimes}_1 w_{1,0}(k_{p+2})) = 1_{(q > 0)} \sum_{r=1}^{q} 2\delta_{j_{r},k_{p+2}}w_{p+1,q-1}(k,j_r).
\]
By (5.4), we have
\[
w_{p+1,q}(k,j)\hat{\otimes}_1 w_{1,0}(k_{p+2}) = [w_{p,q}(k,j)\hat{\otimes}_1 w_{1,0}(k_{p+1})] \otimes w_{1,0}(k_{p+2})
\]
\[
= \frac{p + q}{p + q + 1} [w_{p,q}(k,j)\hat{\otimes}_1 w_{1,0}(k_{p+1})] \otimes w_{1,0}(k_{p+1})
\]
\[
= \frac{1}{p + q + 1} 1\{q > 0\} \sum_{r=1}^{q} 2\delta_{j_{r},k_{p+2}}w_{p,q-1}(k,j_r)\hat{\otimes}_1 w_{1,0}(k_{p+1})
\]
\[
= \frac{1}{p + q + 1} 1\{q > 0\} \sum_{r=1}^{q} 2\delta_{j_{r},k_{p+2}}w_{p+1,q-1}(k,j_r),
\]
where the second to last equality follows from $B(p,q)$ and the last equality follows from (5.4). This means (5.5) and thus $B(p + 1, q)$ holds. Using the similar argument as above, we can get $B(p,q + 1)$.

Hence by the inductive method, $A(p, q)$ and $B(p, q)$ hold for $p, q \geq 0$ and $p + q > 0$. The proof is completed.

**Proof of Theorem 3.9** For $f \in S_p^{\otimes p} \otimes S_q^{\otimes q}$ with the expansion given by (3.12), by the linearity of mappings $I_{p,q}(\cdot)$ and $I_p(\cdot)$, we have
\[
I_{p,q}(f) = \sum_{k_1,\ldots,k_{p+1},j_{p+1}=1}^{\infty} b_{p,q}(k,j)I_{p,q}(k,j)
\]
\[ \sum_{k_1, \ldots, k_p, j_1, \ldots, j_q=1}^\infty b_{p,q}(k,j) I_{p+q} (u_{p,q}(k,j) + i v_{p,q}(k,j)) \]

\[ \sum_{k_1, \ldots, k_p, j_1, \ldots, j_q=1}^\infty I_{p+q} (b_{p,q}(k,j) (u_{p,q}(k,j) + i v_{p,q}(k,j))) . \]

\[ \square \]

5.4. Proof of Theorem [4.1]

**Proof of Theorem [4.1]** Step 1 For \( m + n = p \), we show the unique representation for

\[ I_p (u_{1,0}(1)^\otimes m \otimes v_{1,0}(1)^\otimes n) . \]

Combining the definition of real Wiener-Itô integral [2.1] and the relation between real and complex Hermite polynomials [2.8], we have

\[ I_m (u_{1,0}(1)^\otimes m) I_n (v_{1,0}(1)^\otimes n) \]

\[ = H_m (W (u_{1,0}(1))) H_n (W (v_{1,0}(1))) \]

\[ = \frac{i}{2^p} \sum_{j=0}^{p} \sum_{r+s=j} \left( \begin{array}{c} m \\ r \end{array} \right) \left( \begin{array}{c} n \\ s \end{array} \right) (-1)^s J_{j,p-j} (Z (e_1)) \]

\[ = \frac{i}{2^p} \sum_{j=0}^{p} \sum_{r+s=j} \left( \begin{array}{c} m \\ r \end{array} \right) \left( \begin{array}{c} n \\ s \end{array} \right) (-1)^s I_{j,p-j} (e_1^\otimes j \otimes \overline{e}_1^\otimes (p-j)). \]

**Step 2** For \( m = \{m_k\}_{k=1}^\infty, n = \{n_k\}_{k=1}^\infty \in \Lambda \) with \( |m| + |n| = p \), we give the representation for

\[ I_p (\text{symm} (\otimes_{k=1}^\infty (u_{1,0}(k)^\otimes m_k \otimes v_{1,0}(k)^\otimes n_k))) . \]

Let \( p = \{p_k\}_{k=1}^\infty := m + n \). By the definitions of real and complex Wiener-Itô integrals, [2.1] and [2.5], respectively, we have

\[ I_p (\text{symm} (\otimes_{k=1}^\infty (u_{1,0}(k)^\otimes m_k \otimes v_{1,0}(k)^\otimes n_k))) \]

\[ = \prod_{k=1}^\infty I_{p_k} (u_{1,0}(k)^\otimes m_k \otimes v_{1,0}(k)^\otimes n_k) \]

\[ = \prod_{k=1}^\infty \left( \sum_{j=0}^{p_k} \hat{a}_{k,j} \mathcal{I}_{j,p_k-j} (e_k^\otimes j \otimes \overline{e}_k^\otimes (p_k-j)) \right) . \]

\[ = \sum_{j \leq p} \left( \prod_{k=1}^\infty \hat{a}_{i,j} \right) \mathcal{I}_{\sum_{k=1}^\infty j_k, p-\sum_{k=1}^\infty j_k} (\otimes_{k=1}^\infty (e_k^\otimes j_k \otimes \overline{e}_k^\otimes (p_k-j_k))) \]

\[ = \prod_{l=0}^p \mathcal{I}_{d,p-l} \left( \sum_{j \leq p, |j|=l} \left( \prod_{k=1}^\infty \hat{a}_{i,j} \right) \otimes_{k=1}^\infty (e_k^\otimes j_k \otimes \overline{e}_k^\otimes (p_k-j_k)) \right) . \]
Step 3 For $g_1, g_2 \in (\mathfrak{g} \oplus \mathfrak{g})^\otimes p$ and $g_1 + ig_2$ is given by \[41\],
\[
I_p(g_1 + ig_2) = \sum_{m,n \in \Lambda, |m|+|n|=p} \tilde{c}(m;n) \left( \text{symm} \left( \otimes_{k=1}^{\infty} (u_{1,0}(k)^{\otimes m_k} \otimes v_{1,0}(k)^{\otimes n_k}) \right) \right)
\]
\[
= \sum_{m,n \in \Lambda, |m|+|n|=p} \tilde{c}(m;n) \sum_{l=0}^{p} I_{l,p-l}(g_{l,p-l}(m,n))
\]
\[
= \sum_{l=0}^{p} I_{l,p-l} \left( \sum_{m,n \in \Lambda, |m|+|n|=p} \tilde{c}(m;n)g_{l,p-l}(m,n) \right).
\]

□

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