INCREASING STABILITY OF THE INVERSE BOUNDARY VALUE PROBLEM FOR THE SCHRÖDINGER EQUATION

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ABSTRACT. In this work we study the phenomenon of increasing stability in the inverse boundary value problem for the Schrödinger equation. This problem was previously considered by Isakov in which he discussed the phenomenon in different ranges of the wave number (or energy). The main contribution of this work is to provide a unified and easier approach to the same problem based on the complex geometrical optics solutions.

1. INTRODUCTION

Most of inverse problems are known to be severely ill-posed. This weakness makes it extremely difficult to design reliable reconstruction algorithms in practice. However, in some cases, it has been observed numerically that the stability increases with respect to some parameter such as the wave number (or energy) (see, for example, [4] for the inverse obstacle scattering problem). Several rigorous justifications of the increasing stability phenomena in different settings were obtained by Isakov et al [7, 9, 10, 1, 2]. In particular, in [10], Isakov considered the Helmholtz equation with a potential

\[(\Delta + k^2 + q(x))u(x) = 0\] in \(\Omega \subset \mathbb{R}^n\)

with \(n \geq 3\). He obtained stability estimates of determining \(q\) by the Dirichlet-to-Neumann map for different ranges of \(k\), which demonstrate the increasing stability phenomena in \(k\). The purpose of this work is to provide a more straightforward way to derive a similar estimate for the inverse boundary value for (1.1). In [10], Isakov used real geometrical optics solutions for the large wave number \(k\). In this work, by more careful choice of an additional large parameter and a priori constraints we are able to use complex geometrical optics (CGO) solutions introduced by Calderón [3] and Sylvester-Uhlmann [12] for all \(k \geq 1\). This will simplify the proof in [10]. Recently similar results were obtained by Isaev and Novikov [8] by using less explicit and more complicated methods of scattering theory.

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In this work, instead of considering the Dirichlet-to-Neumann map, we define the boundary measurements to be the Cauchy data corresponding to \((1.1)\)

\[
C_q = \left\{ \left. \left( u|_{\partial \Omega}, \frac{\partial u}{\partial \nu} \right) \right| u \text{ is a solution to } (1.1) \right\}.
\]

Hereafter, \(\nu\) is the unit outer normal vector of \(\partial \Omega\). Assume that \(C_{q_1}\) and \(C_{q_2}\) are two Cauchy data associated with refraction indices \(q_1\) and \(q_2\), respectively. To measure the distance between two Cauchy data, we define

\[
\text{dist}(C_{q_1}, C_{q_2}) = \max \left\{ \frac{\| (f,g) - (\tilde{f}, \tilde{g}) \|_{H^{1/2} \oplus H^{-1/2}}}{\| (f,g) \|_{H^{1/2} \oplus H^{-1/2}}}, \frac{\| (f,g) - (\tilde{f}, \tilde{g}) \|_{H^{1/2} \oplus H^{-1/2}}}{\| (f,g) \|_{H^{1/2} \oplus H^{-1/2}}}, \right. \left. \frac{\| (f,g) - (\tilde{f}, \tilde{g}) \|_{H^{1/2} \oplus H^{-1/2}}}{\| (f,g) \|_{H^{1/2} \oplus H^{-1/2}}} \right\},
\]

where

\[
\| (f,g) \|_{H^{1/2} \oplus H^{-1/2}} = \left( \|f\|^2_{H^{1/2}(\partial \Omega)} + \|g\|^2_{H^{-1/2}(\partial \Omega)} \right)^{1/2}.
\]

Our main theorem is stated as follows.

**Theorem 1.1.** Let \(n \geq 3\). Assume \(C_{q_1}\) and \(C_{q_2}\) are Cauchy data corresponding to \(q_1(x)\) and \(q_2(x)\), respectively. Let \(s > n/2\) and \(M > 0\). Assume \(\|q\|_H^{(\Omega)} \leq M\) and \(\supp (q_1 - q_2) \subset \Omega\). Denote \(\tilde{q}\) the zero extension of \(q_1 - q_2\). Then for \(k \geq 1\) and \(\text{dist}(C_{q_1}, C_{q_2}) \leq 1/e\) we have the following stability estimate:

\[
\| \tilde{q} \|_{H^{-s}(\mathbb{R}^n)} \leq Ck^{4} \text{dist}(C_{q_1}, C_{q_2}) + C \left( k + \log \frac{1}{\text{dist}(C_{q_1}, C_{q_2})} \right)^{-(2s-n)},
\]

where \(C > 0\) depends only on \(n, s, \Omega, M\) and \(\supp (q_1 - q_2)\).

From estimate (1.2), it is obvious that the stability behaves more like Lipschitz type when \(k\) is large. We would like to point out that unlike in the acoustic case where the constant associated with the Lipschitz estimate grows exponentially in \(k\) [11], the constant here grows only polynomially in \(k\). Similarly, the corresponding constant obtained in [10] (see estimate (8) there) also grows polynomially in \(k\).

The paper is organized as follows. In Section 2, we will collect some known results about the CGO solutions and an estimate for the difference of potentials, which are essential tools in the proof. In Section 3, we present a detailed proof of Theorem 1.1.

## 2. Preliminaries

To begin, we state the existence of CGO solutions for \((1.1)\). These special solutions are first constructed by Sylvester and Uhlmann [12]. Another construction based on the Fourier series is given by Hähner [6].

**Lemma 2.1.** Let \(s > n/2\). Assume that \(\zeta = \eta + i\xi \ (\eta, \xi \in \mathbb{R}^n)\) satisfies

\[
|\eta|^2 = k^2 + |\xi|^2 \quad \text{and} \quad \eta \cdot \xi = 0,
\]

i.e., \(\zeta \cdot \zeta = k^2\). Then there exist constants \(C_*\) and \(C > 0\), which are independent of \(k\), such that if \(|\xi| > C_* \|q\|_H^{(\Omega)}\) then there exists a solution \(u\) to the equation \((1.1)\) of the form

\[
u(x) = e^{i\zeta \cdot x} (1 + \psi(x)),
\]
where \( \psi \) has the estimate
\[
\|\psi\|_{H^s(\Omega)} \leq \frac{C}{|\xi|}\|q\|_{H^s(\Omega)}.
\]

**Remark 2.2.** Note that the correction term \( \psi \) decays in \( \text{Im}\zeta \). This property is crucial in obtaining that the constant associated with the Lipschitz estimate grows only polynomially in \( k \).

Next inequality is an easy consequence of Alessandrini’s identity. We refer to [5] for the proof.

**Proposition 2.3.** Let \( u_l \) and \( C_q \) be solution and Cauchy data to the equation (1.1) with \( q = q_l \), respectively \((l = 1, 2)\). Then the following estimate holds:
\[
\left| \int_{\Omega} \left( q_2 - q_1 \right) u_1 u_2 \ dx \right| \\
\leq \left\| \left( u_1, \frac{\partial u_1}{\partial \nu} \right) \right\|_{H^{1/2} \oplus H^{-1/2}} \left\| \left( u_2, \frac{\partial u_2}{\partial \nu} \right) \right\|_{H^{1/2} \oplus H^{-1/2}} \text{dist}(C_{q_1}, C_{q_2}).
\]

## 3. Proof of main theorem

To prove Theorem 1.1, we first derive two lemmas.

**Lemma 3.1.** Under the assumptions in Theorem 1.1,
\[
|\mathcal{F}q(r\omega)| \leq C k^4 e^{C_a} \text{dist}(C_{q_1}, C_{q_2}) + \frac{C}{a} \|q\|_{H^{-s}(\mathbb{R}^n)}
\]
holds for \( k \geq 1, r \geq 0, \omega \in \mathbb{R}^n \) with \( |\omega| = 1 \) and \( a > C_* M \) with \( k^2 + a^2 > r^2/4 \), where \( C > 0 \) depends only on \( n, s, M, \Omega \) and \( \text{supp}(q_1 - q_2) \) and \( C_* \) is the constant given in Lemma 2.1.

**Proof.** We will use CGO solutions (2.1) with appropriately chosen parameter \( \zeta \). Let us denote \( \zeta_l = \eta_l + i\xi_l, l = 1, 2. \) We can choose \( \omega, \tilde{\omega} \in \mathbb{R}^n \) satisfying
\[
\omega \cdot \omega^\perp = \omega \cdot \tilde{\omega}^\perp = \omega^\perp \cdot \tilde{\omega}^\perp = 0 \quad \text{and} \quad |\omega^\perp| = |\tilde{\omega}^\perp| = 1.
\]
Now we set
\[
\xi_1 = a\omega^\perp, \quad \eta_1 = -\frac{r}{2}\omega + \sqrt{k^2 + a^2 - \frac{r^2}{4}}\tilde{\omega}^\perp, \\
\xi_2 = -\xi_1 \quad \text{and} \quad \eta_2 = -r\omega - \eta_1,
\]
and thus
\[
\xi_l \cdot \eta_l = 0, \quad |\eta_l|^2 = k^2 + |\xi_l|^2
\]
and \( |\xi_l| = a \geq C_* M \geq C_* \|q_l\|_{H^s(\Omega)}. \) From Lemma 2.1, there exist CGO solutions \( u_l(x) = e^{i\xi_l \cdot x}(1 + \psi_l(x)) \) to equation (1.1) with \( q = q_l \), where \( \psi_l \) satisfies
\[
\|\psi_l\|_{H^s(\Omega)} \leq \frac{C}{|\xi_l|}\|q_l\|_{H^s(\Omega)}.
\]
Note that \( \psi_l \) also satisfies the estimate
\[
(3.2) \quad \|\psi_l\|_{H^s(\Omega)} \leq \frac{C}{|\xi_l|}\|q_l\|_{H^s(\Omega)} \leq \frac{CM}{a} < \frac{CM}{C_* M} = \frac{C}{C_*}.
\]
Now, by Proposition 2.3 and using the relation \(-r\omega = \zeta_1 + \zeta_2\), we have that

\[
\int_{\Omega} \tilde{q}(x)e^{-ir\omega\cdot x}(1 + \psi_1)(1 + \psi_2) \, dx = \int_{\Omega} (q_2 - q_1)u_1u_2 \, dx
\]

\[
\leq \left\| \left( u_1, \frac{\partial u_1}{\partial \nu} \right) \right\|_{H^{1/2} \cap H^{-1/2}} \left\| \left( u_2, \frac{\partial u_2}{\partial \nu} \right) \right\|_{H^{1/2} \cap H^{-1/2}} \text{dist}(C_{q_1}, C_{q_1}).
\]

Subsequently, we obtain

\[
(3.3) \quad |F\tilde{q}(r\omega)| = \left| \int_{\Omega} \tilde{q}(x)e^{-ir\omega\cdot x} \, dx \right|
\]

\[
\leq \left| \int_{\Omega} \tilde{q}(x)e^{-ir\omega\cdot x}(1 + \psi_1)(1 + \psi_2) \, dx \right|
\]

\[
+ \int_{\Omega} \tilde{q}(x)e^{-ir\omega\cdot x}(\psi_1 + \psi_2 + \psi_1 \psi_2) \, dx
\]

\[
\leq \left\| \left( u_1, \frac{\partial u_1}{\partial \nu} \right) \right\|_{H^{1/2} \cap H^{-1/2}} \left\| \left( u_2, \frac{\partial u_2}{\partial \nu} \right) \right\|_{H^{1/2} \cap H^{-1/2}} \text{dist}(C_{q_1}, C_{q_1})
\]

\[
+ \int_{\Omega} \tilde{q}(x)e^{-ir\omega\cdot x}(\psi_1 + \psi_2 + \psi_1 \psi_2) \, dx.
\]

In view of (3.3), we want to estimate \(\left\| (u_1, \partial u_1/\partial \nu) \right\|_{H^{1/2} \cap H^{-1/2}}\). Recall that \(u_1\) solves (1.1) with \(q = q_1\). Using assumptions \(\|q\|_{H^s(\Omega)} \leq M\), and \(s > n/2\), and \(k \geq 1\), we have that

\[
\left\| \frac{\partial u_1}{\partial \nu} \right\|_{H^{-1/2}(\partial \Omega)} \leq Ck^2 \left\| u_1 \right\|_{L^2(\Omega)} + C \left\| \nabla u_1 \right\|_{L^2(\Omega)}
\]

and thus

\[
\left\| \left( u_1, \frac{\partial u_1}{\partial \nu} \right) \right\|_{H^{1/2} \cap H^{-1/2}} \leq Ck^2 \left\| u_1 \right\|_{L^2(\Omega)} + C \left\| \nabla u_1 \right\|_{L^2(\Omega)}.
\]

We now choose \(R_0 > 0\) large enough such that \(\Omega \subset B_{R_0}(0)\). Then we have

\[
\left| u_1(x) \right| \leq e^{-\xi_1 \cdot x} (1 + \left| \psi_1(x) \right|) \leq Ce^{\xi_1 |R_0|} = Ce^{\alpha R_0}
\]

since

\[
\left| \psi_1(x) \right| \leq \|\psi_1\|_{L^\infty(\Omega)} \leq C\|\psi_1\|_{H^s(\Omega)} \leq C
\]

by \(s > n/2\) and (3.2). It follows that

\[
\left\| u_1 \right\|_{L^2(\Omega)} \leq Ce^{\alpha R_0}.
\]

On the other hand, in view of \(\left\| \nabla \psi_1 \right\|_{L^2(\Omega)} \leq \|\psi_1\|_{H^s(\Omega)} \leq C\) \((s > n/2 \geq 3/2 > 1)\) and (3.1), we can estimate

\[
\left\| \nabla u_1 \right\|_{L^2(\Omega)} = \left| iu_1 \zeta_1 + e^{i \bar{\zeta}_1} \cdot \nabla \psi_1 \right|_{L^2(\Omega)} \leq \left| \zeta_1 \right| \left\| u_1 \right\|_{L^2(\Omega)} + e^{\xi_1 |R_0|} \left\| \nabla \psi_1 \right\|_{L^2(\Omega)}
\]

\[
\leq C(k + |\zeta_1|)e^{\alpha R_0} + Ce^{\xi_1 |R_0|} = C(k + a)e^{\alpha R_0} + Ce^{\alpha R_0} \leq Cke^{Ca}.
\]

Summing up, we obtain

\[
(3.4) \quad \left\| \left( u_1, \frac{\partial u_1}{\partial \nu} \right) \right\|_{H^{1/2} \cap H^{-1/2}} \leq Ck^2 \left\| u_1 \right\|_{L^2(\Omega)} + C \left\| \nabla u_1 \right\|_{L^2(\Omega)}
\]

\[
\leq Ck^2 e^{Ca} + Cke^{Ca} \leq Ck^2 e^{Ca}.
\]

Note that here \(C\) depends on \(n, s, M,\) and the diameter of \(\Omega\).
Let \( \chi \in C_0^\infty(\Omega) \) be a cut-off function satisfying \( \chi \equiv 1 \) near \( \text{supp} (q_1 - q_2) \), then we have

\[
\left| \int_\Omega \tilde{q}(x)e^{-ir\omega \cdot x}(\psi_1 + \psi_2 + \psi_1 \psi_2) \, dx \right| \\
= \left| \int_\Omega \tilde{q}(x)\chi(x)e^{-ir\omega \cdot x}(\psi_1 + \psi_2 + \psi_1 \psi_2) \, dx \right| \\
\leq \int_\Omega |\tilde{q}(x)||\chi(\psi_1 + \psi_2 + \psi_1 \psi_2)| \, dx \\
\leq \|\tilde{q}\|_{H^{-s}(\Omega)}\|\chi(\psi_1 + \psi_2 + \psi_1 \psi_2)\|_{H^s(\Omega)}.
\]

Since \( s > n/2 \) and (3.2), we can estimate

\[
\|\chi(\psi_1 + \psi_2 + \psi_1 \psi_2)\|_{H^s(\Omega)} \\
\leq \|\chi\|_{H^s(\Omega)}(|\psi_1|_{H^s(\Omega)} + |\psi_2|_{H^s(\Omega)} + |\psi_1\psi_2|_{H^s(\Omega)}) \\
\leq \|\chi\|_{H^s(\Omega)}\left(\frac{CM}{a} + \frac{CM}{C_*} + \frac{CM}{C_*} \right) \leq \frac{C}{a}.
\]

Finally, (3.1) follows from (3.3), (3.4), (3.5), and (3.6). \( \square \)

The following lemma is an easy corollary of Lemma 3.1.

**Lemma 3.2.** Suppose that the assumptions in Theorem 1.1 hold. Let \( R > C_*M \) with \( C_* \) being the constant given in Lemma 2.1. Then for \( k \geq 1, r \geq 0 \) and \( \omega \in \mathbb{R}^n \) with \( |\omega| = 1 \), the following estimates hold true: if \( 0 \leq r \leq k + R \) then

\[
|\mathcal{F}\tilde{q}(r\omega)| \leq CK^4e^{CR}\text{dist}(C_{q_1}, C_{q_2}) + \frac{C}{R}\|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)};
\]

if \( r \geq k + R \) then

\[
|\mathcal{F}\tilde{q}(r\omega)| \leq CK^4e^{Cr}\text{dist}(C_{q_1}, C_{q_2}) + \frac{C}{r}\|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}.
\]

**Proof.** It is enough to take \( a = R \) when \( 0 \leq r \leq k + R \), and take \( a = r \) when \( r \geq k + R \) in Lemma 3.1. \( \square \)

Now we prove our main theorem.

**Proof of Theorem 1.1.** Written in polar coordinates, we have that

\[
\|\tilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} = C \int_0^\infty \int_{|\omega|=1} |\mathcal{F}\tilde{q}(r\omega)|^2(1 + r^2)^{-s}r^{n-1} \, d\omega dr \\
= C \left( \int_0^{k+R} \int_{|\omega|=1} |\mathcal{F}\tilde{q}(r\omega)|^2(1 + r^2)^{-s}r^{n-1} \, d\omega dr + \int_{k+R}^T \int_{|\omega|=1} |\mathcal{F}\tilde{q}(r\omega)|^2(1 + r^2)^{-s}r^{n-1} \, d\omega dr + \int_T^\infty \int_{|\omega|=1} |\mathcal{F}\tilde{q}(r\omega)|^2(1 + r^2)^{-s}r^{n-1} \, d\omega dr \right) \\
=: C(I_1 + I_2 + I_3),
\]

where \( R > C_*M \) and \( T \geq k + R \) are parameters which will be chosen later.
In the same way, using estimate (3.8), we have

\begin{align*}
I_3 & \leq C \int_T^\infty \|q_1 - q_2\|^2_{L^2(\Omega)} (1 + r^2)^{-s} r^{n-1} \, dr \\
& \leq C T^{-m} \left( \varepsilon \|q_1 - q_2\|^2_{H^{-s}(\Omega)} + \frac{1}{\varepsilon} \|q_1 - q_2\|^2_{H^s(\Omega)} \right) \\
& \leq C T^{-m} \left( \|\overline{q}\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{1}{\varepsilon} \right)
\end{align*}

for \( \varepsilon > 0 \), where \( m := 2s - n \).

On the other hand, by estimate (3.7), we can obtain

\begin{align*}
I_1 & \leq \int_0^{k+R} \left( C k^4 e^{CR} \text{dist}(C_{q_1}, C_{q_2}) + \frac{C}{r} \|\overline{q}\|_{H^{-s}(\mathbb{R}^n)} \right)^2 (1 + r^2)^{-s} r^{n-1} \, dr \\
& \leq C \left( k^8 e^{CR} \text{dist}(C_{q_1}, C_{q_2})^2 + \frac{1}{R^2} \|\overline{q}\|^2_{H^{-s}(\mathbb{R}^n)} \right) \int_0^{k+R} (1 + r^2)^{-s} r^{n-1} \, dr \\
& = C \left( k^8 e^{CR} \text{dist}(C_{q_1}, C_{q_2})^2 + \frac{1}{R^2} \|\overline{q}\|^2_{H^{-s}(\mathbb{R}^n)} \right).
\end{align*}

In the same way, using estimate (3.8), we have

\begin{align*}
I_2 & \leq C \int_{k+R}^T \left( C k^4 e^{Cr} \text{dist}(C_{q_1}, C_{q_2}) + \frac{C}{r} \|\overline{q}\|_{H^{-s}(\mathbb{R}^n)} \right)^2 (1 + r^2)^{-s} r^{n-1} \, dr \\
& \leq C k^8 \text{dist}(C_{q_1}, C_{q_2})^2 \int_{k+R}^T e^{Cr} (1 + r^2)^{-s} r^{n-1} \, dr \\
& \quad + C \|\overline{q}\|^2_{H^{-s}(\mathbb{R}^n)} \int_{k+R}^T (1 + r^2)^{-s} r^{n-1} \, dr \\
& \leq C \left( k^8 e^{CT} \text{dist}(C_{q_1}, C_{q_2})^2 + \frac{1}{R^2} \|\overline{q}\|^2_{H^{-s}(\mathbb{R}^n)} \right),
\end{align*}

where we have used

\begin{align*}
\int_{k+R}^T e^{Cr} (1 + r^2)^{-s} r^{n-1} \, dr & \leq e^{CT} \int_{k+R}^T (1 + r^2)^{-s} r^{n-1} \, dr \\
& \leq e^{CT} \int_0^\infty (1 + r^2)^{-s} r^{n-1} \, dr = Ce^{CT},
\end{align*}

\begin{align*}
\int_{k+R}^T (1 + r^2)^{-s} r^{n-1} \, dr & \leq \int_{k+R}^T r^{-2s+n-1} \, dr \\
& \leq \frac{1}{2s - n + 2} \frac{1}{(k+R)^{2s-n+2}} \leq \frac{C}{(k+R)^2} \leq \frac{C}{R^2},
\end{align*}
and $s > n/2$, $k \geq 1$. Combining (3.9)–(3.12) gives
\begin{equation}
\|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}^2 \leq C(I_1 + I_2 + I_3)
\end{equation}
\begin{align*}
\leq & \, C \left( k^8 e^{CR} \operatorname{dist}(C_{q_1}, C_{q_2})^2 + \frac{1}{R^2} \|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}^2 \right) \\
& \quad + C \left( k^8 e^{CT} \operatorname{dist}(C_{q_1}, C_{q_2})^2 + \frac{1}{R^2} \|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}^2 \right) \\
& \quad + C T^{-m} \left( \varepsilon \|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}^2 + \frac{1}{\varepsilon} \right) \\
\leq & \, C \left( \frac{2}{R^2} + \varepsilon T^{-m} \right) \|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}^2 + C k^8 e^{CR} \operatorname{dist}(C_{q_1}, C_{q_2})^2 \\
& \quad + C k^8 e^{CT} \operatorname{dist}(C_{q_1}, C_{q_2})^2 + \frac{C T^{-m}}{\varepsilon}.
\end{align*}

To continue, we consider the following two cases:
\begin{enumerate}[(i)]
\item $k + R \leq p \log \frac{1}{A}$ and
\item $k + R \geq p \log \frac{1}{A},$
\end{enumerate}
where $R > C_* M$ and $p > 0$ are constants which will be determined later. We begin with the first case (i). Taking
\begin{equation}
R > 2\sqrt{C}
\end{equation}
and $\varepsilon = e^{Tm}$ ($c \ll 1$), we deduce that
\begin{equation}
\|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}^2 \leq C k^8 A + C k^8 e^{CT} A + C T^{-2m}
\end{equation}
for any $T \geq k + R$ by (3.13), where $A = \operatorname{dist}(C_{q_1}, C_{q_2})^2$.

Now we choose $T = p \log (1/A)$, which is greater than or equal to $k + R$ by the condition (i). Our current aim is to show that there exists $C_1 > 0$ such that
\begin{equation}
k^8 e^{CT} A \leq C_1 \left( k + \log \frac{1}{A} \right)^{-2m}
\end{equation}
and
\begin{equation}
T^{-2m} \leq C_1 \left( k + \log \frac{1}{A} \right)^{-2m}.
\end{equation}
Substituting (3.16) and (3.17) into (3.15) clearly implies (1.2). We remark that (3.17) is equivalent to
\begin{equation}
C_1^{-1/2m} \left( k + \log \frac{1}{A} \right) \leq p \log \frac{1}{A}.
\end{equation}

Since we have
\begin{equation}
k + \log \frac{1}{A} \leq (k + R) + \log \frac{1}{A} \leq (p + 1) \log \frac{1}{A}
\end{equation}
by (i), condition (3.18) (i.e. (3.17)) holds whenever
\begin{equation}
C_1^{-1/2m} \leq \frac{p}{p + 1}.
\end{equation}
On the other hand, condition (3.16) is equivalent to
\begin{equation}
8 \log k + (Cp - 1) \log \frac{1}{A} + 2m \log \left( k + \log \frac{1}{A} \right) \leq \log C_1.
\end{equation}
Using (i), we can bound the left-hand side of (3.20) by

\[ (\text{LHS of (3.20)}) \leq 8 \log p + 2m \log(p + 1) + (Cp - 1) \log \frac{1}{A} + 2(m + 4) \log \log \frac{1}{A}, \]

Choosing

\[ p \leq \frac{1}{2C}, \]

we can see that

\[ (\text{LHS of (3.20)}) \leq 8 \log \frac{1}{2C} + 2m \log \left( \frac{1}{2C} + 1 \right) - \frac{1}{2} \log \frac{1}{A} + 2(m + 4) \log \log \frac{1}{A}. \]

Therefore, condition (3.20) (i.e. (3.16)) is satisfied provided

\[ 8 \log \frac{1}{2C} + 2m \log \left( \frac{1}{2C} + 1 \right) + 2(m + 4) \log \log \frac{1}{A} \leq \log C. \]  

Next we consider case (ii). We choose \( T = k + R \) and observe that the term \( I_2 \) in (3.9) does not appear in this case. Hence, instead of (3.13), we have

\[ \| \tilde{q} \|_{L^2(\mathbb{R}^n)} \leq C \left( \frac{1}{R^2} + \varepsilon T^m \right) \| \tilde{q} \|_{H^{-s}(\mathbb{R}^n)} + Ck^8e^{CR} \text{dist}(Cq_1, Cq_2)^2 + \frac{CT^{-m}}{\varepsilon} \]

Setting \( \varepsilon = T^m/R^2 \) implies that

\[ \| \tilde{q} \|_{H^{-s}(\mathbb{R}^n)} \leq \frac{2C}{R^2} \| \tilde{q} \|_{H^{-s}(\mathbb{R}^n)} + Ck^8e^{CR}A + CR^2(k + R)^{-2m}. \]

Now we choose

\[ R > 2\sqrt{C} \]

and obtain that

\[ \| \tilde{q} \|_{H^{-s}(\mathbb{R}^n)} \leq Ck^8A + C(k + R)^{-2m}, \]

which implies the desired estimate (1.2) since from condition (ii) we have

\[ k + R \geq \frac{k}{2} + \frac{k + R}{2} \geq \frac{k}{2} + \frac{p}{2} \log \frac{1}{A} \geq \frac{\min(p, 1)}{2} \left( \frac{k + \log \frac{1}{A}}{2} \right). \]

As the last step, we choose appropriate \( R, p, \) and \( C_1 \) to complete the proof. We first pick \( R > C_1M \) sufficiently large satisfying (3.14) and (3.23) and then choose \( p \) small enough satisfying (3.21). Finally, we take \( C_1 \) large enough satisfying (3.19) and (3.22). \( \square \)
4. Conclusion

We think that increasing stability is an important feature of the inverse boundary problem for the Schrödinger potential which should lead to higher resolution of numerical algorithms. It is important to collect numerical evidence of this phenomenon. Our method is based on the CGO solutions constructed in [6] where the constants in Lemma 2.1 are explicit. So most likely one can give explicit constants in Theorem 1.1 at least for particular domains $\Omega$ like balls. Contrary to the acoustic case [11], the constants in the estimate (1.2) depend only polynomially on $k$. It is an important and challenging question to determine whether the exponential dependence on $k$ of the estimates in [11] is indeed generic if there are no assumptions on rays.

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