THE EFFECT OF REBATE VALUE AND SELLING PRICE-DEPENDENT DEMAND FOR A FOUR-LEVEL PRODUCTION MANUFACTURING SYSTEM

Umakanta Mishra
Department of Mathematics, School of Advanced Sciences
Vellore Institute of Technology, Vellore, 632014, India

Abu Hashan Md Mashud 2,3
2Department of Mathematics, Hajee Mohammad Danesh Science and Technology University
Dinajpur-5200, Bangladesh
3School of Engineering and IT, University of New South Wales (UNSW), Canberra, Australia

Sankar Kumar Roy
Department of Applied Mathematics with Oceanology and Computer Programming
Vidyasagar University, Midnapore-721102, WB, India

Md Sharif Uddin 5,6
5Department of Industrial Engineering, Prince Sattam bin Abdulaziz University
Alkharij, KSA, 16273
6Department of Mathematics
Jahangirnagar University, Savar, Dhaka-1342, Bangladesh

(Communicated by Daniel Oron)

Abstract. Price rebate is only permitted when purchases made by the customer exceed a predefined limit and they later buy other items from the purchaser. There are various forms of rebate used by production companies. This study provides a deteriorating inventory model of four-level production rates and derives the rebate-value-based demand with the product selling price under shortages. This model gives preference to optimal replenishment time, ordering quantity, rebate value, and selling price while maximizing total profit. This model first explores and discusses the demand function, which discretely hinges on the selling price of rebate value, followed by discussions on demand based on the selling price. This study proposes a solution through unique propositions and the construction of two algorithms that are suitable for four-level production; this has not yet been explored in-depth in the literature. Illustrative examples and a sensitivity analysis demonstrate the applicability of the proposed algorithms; the customer decides to buy a product that is larger than the minimum suitable for a price rebate and the buyer can then deal with a higher price rebate. The benefit of rebate marketing helps production companies increases conversion rates and encourages customers to purchase goods. This model demonstrates that proposing rebates can consume substantial pricing and inventory inferences and can result in a substantial increase in profit.

2020 Mathematics Subject Classification. Primary: 90B05; Secondary: 90C26;

Key words and phrases. Dimension theory, Poincaré recurrences, multifractal analysis, discrete-time model, singular Hopf bifurcation.

*Corresponding author: Abu Hashan Md Mashud.
1. Introduction. An economic production quantity model determines the quantity a company or a retailer must order to minimize the total inventory cost by balancing the inventory holding cost and fixed ordering cost in staple manufacturing products. Marketing deals with producers’ mail-in rebates and provides an opportunity for firms to advance their profits, thus building a closer relationship between producers and retailers. While rebate marketing potentially allows customers to save money, there is no superior technique for enticing them than a tax free gift, allowing them to purchase approximately what they need? For example, a super shop provides $ 40 cashback for a specific item when the customer paid full amount of $ 250 for a watch. There are lots of techniques for sales promotion. Over 20 % of total goods are purchases through promotional offers by the customers [40]. But all the promotional efforts are not beneficial for the retailer [38]. Moreover, it needs a check and balance. Rebate is one of the medium of sales promotion. Rebates are recognised as beneficial for moving goods, enhancing perceptibility, and evolving brand reliability. In business-to-business presentations, rebates can entice new clients, persuade them to try producer products, and incentivize the latest accounts to enhance their orders (see Fig 1). In particular, the demand rate directly affects lot size. Hence, new models have emerged which include optimal lot size, optimal price, and market spending. However, a decision variable such as rebate value was maintained under inventory management control. The literature includes a plethora of studies, such as Muzaffar et al. [30] anticipated a model with rebate value and delay in payments system where the nature of the retailers is heterogeneous for an automobile case. They showed that the rebate depends on the participation of customers. If the customers are not participating in the campaign eagerly, then some products need to hold for a long time which may damage the quality of the products. This phenomenon is known as the deterioration of products.

Deterioration of products is another major concern in managing a retailer’s inventory. Wee’s [42] pricing policy under the deteriorating inventory model in a declining market is in this direction. In that study, some technique of pricing with the effect of deterioration has been discussed by the author. Abad [1] extended the work of Wee [42] by assuming a large size and price under perishability and backordering. However, several producers have also begun to provide storage rebates (Yang and Dong [45], Hu et al. [12], Cao et al. [6]).

Wee [43] developed a model for replenishment policy under demand, based on price-and time-dependent deterioration [43]. Production rates are usually N levels in an inventory system; for instance, production rate usually changes from one level to another, based on time. This is indeed a desirable situation, wherein the rate of production is maintained at a lower level, leading to the avoidance of a large stock of manufactured items at the first stage, thus potentially reducing holding cost. Several studies related to different levels of production, such as those by Sivashankari and Panayappan [35] and Mishra [20], present the inventory policy under different rates of production with shortages. Demand is delicately related to price in competitive markets (Mashud et al. [21], [26]). Notably, deviations in demand determine cost and revenue arrangements; hence, price fitting must study the resulting demand variations. This study discusses pricing policy with the benefits of rebate value based on demand for four-level production rates of the inventory model. At the same time, the proposed model was developed considering the life of goods that deteriorate over time (e.g., flowers, medicines, fruits, volatile liquids, and sea products).
This study indicates that only Khouja [15] followed an analytic process, analogous to that considered in the present study, wherein the inventory consumes a demand as a function of rebate value and selling price. However, the setting of Khouja [15] is dissimilar from that discussed here; the emphasis is on the assessment of unconventional mechanisms for the synchronisation of rebate value and selling price under an economic order quantity (EOQ) model. This study notes that Sivashankari and Panayappan [35] investigated a setting where demand is constant; however, they considered two rates of production, without setting any rebate value and selling price, and constant demand rates that do not depend on production rates.

Prior studies seek to determine the demand based on rebate value and selling price under EOQ and attempt to study rebate value and selling price under a four-level production EPQ model was scant. After studying the literature review, focusing only on EOQ or EPQ models, it was found that no one covered a production inventory model with four different rates with retailer price and rebate value for operations managers who are interested in determining an initially low rate of production, avoiding huge quantum stock of manufactured items in the first phase and thereby resulting in reduced holding costs under shortages. After studying the above issues, the research question is: how can this study formulate a rebate value and selling price of fewer than four productions economic production quantity (EPQ) problems, considering shortage concerns? To respond to this, a model is developed herein for solving a four-level production EPQ model to help reduce the holding cost under a shortage condition while considering environmental parameters.

Prior studies formulate inventory models by assuming several types of deterioration (allowed/not allowed), demand based on price, and rebate value (allowed/not allowed). However, studies on a general inventory model that incorporate all parameters (four-level production, rebate value, and selling price) are scarce. Besides some other contributions are:


(i) To determine the optimal replenishment time, ordering quantity, rebate value, and selling price while maximizing total profit.

(ii) Through this proposed model, we first explore and discuss the demand function, a discrete function of the selling price of rebate value, followed by discussions on-demand based on selling price.

(iii) The motivation is to develop a standard inventory model comprising (i) four-level production and (ii) demand based on rebate value and selling price through shortages.

(iv) Pricing strategies are explored for the retailers simultaneously with some technique of using rebate in retailing business especially for a stapler manufacturing company.

The residual segments are arranged as follows: Section 2 presents the related literature. In Section 3, the relevant notation and assumptions are presented. The mathematical formulation, solution procedure, and optimal solution are derived from algorithms 1 and 2 in Section 4. Section 5 discusses several numerical examples, sensitivity analysis, and implications for model verification. In Section 6, the conclusions and forthcoming research track are presented.

2. Literature review. Over the years, many studies have been conducted on price-sensitive demand in inventory management. For instance, Chang et al. [7] anticipated an economic order quantity (EOQ) joint pricing model with optimum replenishment number and shortages, as well as discussing the optimum replenishment time for a fixed price. On the other hand, Ghosh et al. [30] projected an EOQ model of decaying products for a price dependent demand with back-ordering and lost sale. This model was solved analytically and the optimal replenishment size with product price was confirmed. Lu et al. [18] established a model based on the price and stock-reliant demand in an inventory system. Owing to the nonlinear profit function, they solved profit maximization for the joint pricing and replenishment time using Pontryagin’s maximum principle while Mashud [28] proposed a price-sensitive inventory model with shortages, in which the model is solved via the classical optimisation technique. Arcelus et al. [4] developed an optimising EOQ model for the manufacturers’ discount strategy to the ending buyer with price-dependent demand. It discusses the uncertain nature of demand, where the discount and price are not inclusive for the retailer. Khouja [15] formulated an EOQ model and solved for the optimal price, optimal order quantity, and rebate face value for a price and rebate sensitive demand while Yang and Dong [45] developed a logical EOQ model for online retailers’ product pricing and rebate strategies with a two-period setting. They proved that the retailer’s rebate decisions and pricing decisions are very important in a manufacturing system. Additionally, an economic production quantity (EPQ) model explores the order required by a company for maximization of the total profit by harmonising the holding and ordering costs. Shafieezadeh and Sadegheih [34] projected a joined multi-item and multi-echelon supply chain to optimise the total cost. Enriching the work of Shafieezadeh and Sadegheih [34], later models have adopted additional parameters, such as inflation, shortage, and combination price and stock-dependent demand (Shaikh et al. [36]). The literature on an EPQ model with different production rates has been discussed by Sivashankari and Panayappan [35]. They discussed two different production rates of the EPQ model with deteriorating items. To avoid manufacturing bulk amounts of products, the system is started at a low rate, which incurs a low holding
cost, and is quick to switch to a higher production rate. Mishra [20] extended the work of Sivashankari and Panayappan [35] by considering three rates of production for further reduction in holding costs.

For the price-dependent demand scope, Ouyang et al. [33] developed an optimal joint pricing inventory model with an ordering problem under an order-size-dependent trade credit environment. Ho [11] anticipated a new model, extending the effort of Ouyang et al. [33] by allowing for price-and-credit-linked demand hinges on a two-level trade credit strategy. Conversely, Soni and Patel [37] discovered an inventory model for the retailer’s joint pricing non-instantaneous deteriorating items using a credibility constraint and shows the benefits of joint pricing problem significantly while Hasan et al. [10] further anticipated an inventory model for pricing strategies for non-instantaneously deteriorating agricultural items where they have showed two different discount policy for two different types of products and explores the strong impact of pricing on inventory model. Mashud et al. [29] developed an inventory model for the newsboy problem, considering deteriorating items in which they show that just-in-time delivery can optimise retailer profit. Önal et al. [31] studied a mathematical deteriorating inventory model using price and stock-reliant demand. The profit of retailers has been derived via a metaheuristics algorithm (Tabu Search), using self-space and backroom storage capacity, while Mishra et al. [27] projected a sustainable inventory model for deteriorating items considering carbon emissions; they considered flowers as deteriorating items and provided some managerial insights into how one can preserve flowers at controllable temperatures at minimum expense. Later, Jadidi et al. [13] modified the study of Önal et al. [31] by bearing in mind stochastic demand with a price discount inventory model to determine optimum values of price and discount. In that paper, the extensive facilities of using a discount in inventory models have been explored meticulously.

Mashud et al. [24] developed a single-level production rate for imperfect items with a discount facility. They show a controllable carbon emission can benefit the retailer in multi ways. Also, they provided some pricing strategies, and a suitable preservation technology can efficiently limit the deterioration of the products. However, Manna et al. [19] projected a two-level production rate with defective items and advertisement costs. The paper critically evaluates the impacts of two level production rate and the benefits from advertisement policy. Liuxin et al. [17] established a model for counting stock, time, and price-reliant demand in a finite planning horizon. They considered a multi-period to determine the maximum average profit by giving preference to proper pricing and replenishment policy. To identify the exact pricing and replenishment policy an algorithm is proposed in this paper which also helps to obtain the optimum average profit of the model. Another study derives optimisation results for price and cycle time, showing how all results apply for particular demand and deterioration functions through a profit-optimising deterministic inventory model under linear cost (O’Neill and Sanni [32]). Thereafter, an inventory model including joint optimisation and dynamic pricing with a replenishment cycle and deterioration rate where demand was developed to find an optimal dynamic pricing via Pontryagin’s maximum principle (Tashakkor et al. [39]).

In this paper, a rebate value and selling price-dependent demand in which four different levels of production are derived, and it is possible that production started at one rate, after some time, it may be switched over to another rate, after that it may be switched over to another rate, and after some time, it may be switched over
to another rate. More production rate situation is desirable in the sense that by starting at a low rate of production, a large quantum stock of staple manufacturing product at the initial stage is avoided and leading to a reduction in the holding cost. We summarise and compare various forms of EOQ/EPQ models in Table 1 and state the potential research contributions that we intend to make.

**Table 1:** Prior studies and Assessments among the research:

| Reference                        | Inventory type | Production Level | Demand based on rebate value and selling price | Backorder | Profit |
|----------------------------------|----------------|------------------|-----------------------------------------------|-----------|--------|
| Khouja [15]                      | Integrated     | x                | x                                             | ✓         | ✓      |
| khouja et al. [16]               | Integrated     | x                | x                                             | ✓         | ✓      |
| Arcelus, et al. [4]              | Integrated     | x                | x                                             | ✓         | ✓      |
| Wong et al. [44]                 | Integrated     | x                | x                                             | ✓         | ✓      |
| Caliskan-Demirag et al. [5]      | Integrated     | x                | x                                             | ✓         | ✓      |
| Ho [11]                          | Integrated     | x                | x                                             | x         | ✓      |
| Sivashankari and Panayappan [35] | EPQ            | ✓                | x                                             | x         | ✓      |
| Mishra [20]                      | EPQ            | ✓                | x                                             | x         | ✓      |
| Lu et al. [18]                   | EOQ            | x                | x                                             | x         | ✓      |
| Mishra et al. [23]               | EOQ            | x                | x                                             | ✓         | x      |
| Manna et al. [19]                | EPQ            | x                | x                                             | x         | ✓      |
| Jadidi et al. [13]               | Integrated     | x                | x                                             | ✓         | ✓      |
| Chen [8]                         | Integrated     | x                | x                                             | ✓         | ✓      |
| Mishra [22]                      | EPQ            | ✓                | x                                             | x         | ✓      |
| Yang and Dong [45]               | Integrated     | x                | ✓                                             | x         | ✓      |
| Hu et al. [12]                   | Integrated     | x                | ✓                                             | x         | ✓      |
| Liuxin et al. [17]               | Integrated     | x                | x                                             | ✓         | ✓      |
| Chernonog and Avinadav [9]       | Integrated     | x                | x                                             | x         | ✓      |
| Zhan et al. [46]                 | Integrated     | x                | ✓                                             | ✓         | ✓      |
| Cao et al. [6]                   | Integrated     | x                | ✓                                             | x         | ✓      |
| Mishra et al. [25]               | EOQ            | x                | x                                             | x         | ✓      |
| Khakzad and Gholamian [14]       | Integrated     | x                | x                                             | x         | x      |

Prior studies focus on different types of inventory models; however, the contributions are different in assisting production managers who are involved in four production levels (complex). In reality, any production system can face three different situations concerning the EPQ issue: rates of production, demand, and shortages. In this research, we covered all possible cases by developing four rates of production inventory models with retailer price and rebate value under shortages. These models may be useful for operations managers who are interested in determining
an initially low rate of production, avoiding large quantum stock of manufactured items at the first stage and thereby reducing the holding cost.

3. Notations and assumptions. Here, we describe the notations and assumptions of the projected model.

3.1. Notation.

| Parameters | Description                                      | Units          |
|------------|--------------------------------------------------|----------------|
| $t_7$      | Total cycle time (decision variable)             | Week           |
| $P$        | Sales price (decision variable)                  | $/unit         |
| $r$        | Face value of the rebate (decision variable)     | $/unit         |
| $P$        | Production rates at time                         | Units          |
| $D$        | Demand rates at time                             | Units          |
| $\omega$  | The serving /processing cost per rebate redeemed | $/unit/time unit |
| $c_h$      | per unit holding cost                            | $/unit/time unit |
| $c_p$      | per unit production cost                         | $/unit/time unit |
| $A$        | Per cycle setup cost                             | $/cycle/run    |
| $SP$       | Total sales revenue of the product               | $/week         |
| $PC$       | Total Production cost                            | $/week         |
| $c_s$      | Per unit shortage cost                           | $/unit/time unit |
| $OC$       | Setup cost                                       | $/week         |
| $HC$       | Total Holding cost                               | $/week         |
| $DC$       | Deteriorating cost                               | $/week         |
| $SC$       | Total shortage cost                              | $/week         |
| $RC$       | Rebate redemption cost is a reduction on a sale price | $/week   |
| $g(t)$     | Level of inventory of the system at any time     | Units          |
| $\Pi = \Pi(t_7, r, P)$ | Per unit total profit | $/week         |
| $\Pi' = \Pi(t_7', p')$ | Per unit optimum total profit | $/week         |

3.2. Assumptions. The assumptions are used in the intended model are:

I The production rate is always higher than the demand rate (i.e. $P = D(p, r)\mu > D(p, r) = D$), where the demand rate$L = D(p, r)$ is a function of price and rebate face value.

II The linearly increasing demand function in terms of price and rebate value is $D = D(p, r) = a - bp + dr$ where $a>0, b>0$ and $d>0$ (motivated by Khouja [15]). Here, $D$ rises as soon as $d$ units for every single unit cost increase in the face value of the rebate $r$ and $D$ reduces once $b$ units for every unit cost decline in price. This relationship $\frac{d}{b}$ is recognised as rebate efficiency. The proportion of rebates kept hinges on the face value relative to the reference price $\omega = r/P_1$ ($P_1$ denotes the value of the rebate amount, at which 100% of rebate agreements are saved). The serving/handling cost per rebate saved is $\omega$ which is attained in addition to the face value.

III In this study, we consider expensive products that encourage more rebates than cheap products. Therefore, based on this theme, the demand function is constructed in a relation between rebate and selling price as $D = D(p, r) = a \sqrt{r} e^{-bd}$ where $a>0, b>0$ and $d>0$.

IV Constant deterioration rate is $\theta$.

V In a given cycle, no replacement of deteriorated items occurs.

VI A combination of four different production rates is considered with a single item.

VII Shortages are acceptable and it is completely backlogged.

VIII A continuous production system is considered.
4. Mathematical framework and solution procedure. In this section, two cases are discussed, based on the rebate value. Case 1 addresses the demand hinging on selling price and rebate value; in Case 2, the discussion is limited to selling price-dependent demand.

4.1. Case 1 (demand depends on selling price and rebate value). This section discusses a production-deteriorating inventory model with four different levels of production, price, and rebate-values-dependent demand. Initially, the production starts at a certain rate but may thereafter be switched to another rate, maintaining the production rate “variable” at all times. For such a variable production rate, a huge quantum stock of manufactured items can be avoided in the first stage, resulting in reduced holding costs. The detailed mathematical derivation and solution procedures are as follows:

During $0 \leq t \leq t_1$ the rate of production is $D(p, r)\mu$ and the rate of demand is $D(p, r)$ while in time $t_1 \leq t \leq t_2$, the production rate is $\nu D(p, r)\mu$ and the demand rate is $\nu D(p, r)$, where “$\nu$” is constant. The inventory is gathered at a rate of $\nu D(p, r)(\mu - 1)$. In the interval $t_2 \leq t \leq t_3$, the production rate is $\xi D(p, r)\mu$ and the demand rate is $\xi D(p, r)$, where “$\xi$” is a constant. The inventory accrues at a rate of $\xi D(p, r)(\mu - 1)$. The inventory level diminishes because $D(p, r)$ and $\theta$ at time $t_4 \leq t \leq t_5$. During the time between $t_5 \leq t \leq t_6$ shortages gather at a rate of $D(p, r)$. Thus, at time, $t_5 \leq t \leq t_6$ it is a prerequisite to hype $S$ units of items. Again, the production process starts at time $t_6 \leq t \leq t_7$ at a proportion of $D(p, r)\mu$ to recuperate both the older shortages in time $t_5 \leq t \leq t_6$. Time $t_7$ is required for all units $q = D(p, r)t_7$. Figure 2 presents the differential equation for labeling the system in the time interval $0 \leq t \leq t_7$, where, at any time $t$, $q(t)$ refers to the stock level of the system.

\[ q'(t) + \theta q(t) = D(p, r)(\mu - 1), \quad 0 \leq t \leq t_1 \]  

**Figure 2.** Representing inventory vs time
Therefore, ignoring the second and higher degrees of \( \theta \)

Maximum inventory \( q \)

\[
q' (t) + \theta q(t) = \nu D(p,r)(\mu - 1), \quad t_1 \leq t \leq t_2
\]  
(2)

\[
q' (t) + \theta q(t) = \xi D(p,r)(\mu - 1), \quad t_2 \leq t \leq t_4
\]  
(3)

\[
q' (t) + \theta q(t) = \rho D(p,r)(\mu - 1), \quad t_3 \leq t \leq t_4
\]  
(4)

\[
q' (t) + \theta q(t) = -D(p,r), \quad t_4 \leq t \leq t_5
\]  
(5)

\[
q' (t) = -D(p,r), \quad t_5 \leq t \leq t_6
\]  
(6)

\[
q' (t) = D(p,r)(\mu - 1), \quad t_6 \leq t \leq t_7
\]  
(7)

Satisfying the boundary situations are

\[
q(0) = 0, \quad q(t_1) = q_1, \quad q(t_2) = q_2, \quad q(t_3) = q_3, \quad q(t_4) = q_4,
\]

\[
q(t_5) = 0 \quad \text{and} \quad q(t_6) = -S \quad \text{and} \quad q(t_7) = 0.
\]  
(8)

Resolving the equations (1)-(7) produces,

\[
q(t) = \frac{D(p,r)(\mu - 1)}{\theta} [1 - e^{-\theta t}], \quad 0 \leq t \leq t_1
\]  
(9)

\[
q(t) = \frac{\nu D(p,r)(\mu - 1)}{\theta} [1 - e^{-\theta t}], \quad t_1 \leq t \leq t_2
\]  
(10)

\[
q(t) = \frac{\xi D(p,r)(\mu - 1)}{\theta} [1 - e^{-\theta t}], \quad t_2 \leq t \leq t_3
\]  
(11)

\[
q(t) = \frac{\rho D(p,r)(\mu - 1)}{\theta} [1 - e^{-\theta t}], \quad t_3 \leq t \leq t_4
\]  
(12)

\[
q(t) = \frac{D(p,r)}{\theta} [e^{\theta (t_5 - t)} - 1], \quad t_4 \leq t \leq t_5
\]  
(13)

\[
q(t) = -D(p,r)[t_5 - t], \quad t_5 \leq t \leq t_6
\]  
(14)

\[
q(t) = D(p,r)(\mu - 1)[t_7 - t], \quad t_6 \leq t \leq t_7
\]  
(15)

Maximum inventory \( q_1 \): From equations [1] and [8], using the Taylor series and ignoring the second and higher degrees of \( \theta (0 < \theta < 1) \).

Therefore,

\[
q_1 = D(p,r)(\mu - 1)t_1
\]  
(16)

Maximum inventory \( q_2 \): From equations (2) and (8), using the Taylor series and ignoring the second and higher degrees of \( \theta (0 < \theta < 1) \).

Therefore,

\[
q_2 = \nu D(p,r)(\mu - 1)t_2
\]  
(17)

Maximum inventory \( q_3 \): From equations (3) and (8), using the Taylor series and ignoring the second and higher degrees of \( \theta (0 < \theta < 1) \).

Therefore,

\[
q_3 = \xi D(p,r)(\mu - 1)t_3
\]  
(18)

Maximum inventory \( q_4 \): From equations (4) and (8), using the Taylor series and ignoring the second and higher degrees of \( \theta (0 < \theta < 1) \).

Therefore,

\[
q_4 = \rho D(p,r)(\mu - 1)t_4
\]  
(19)

From equations [14] and [8], the shortage level is \( S \) obtained as follows:

\[
q(t_6) = -S \Rightarrow D(p,r)(t_6 - t_5) = -S.
\]

From equations [15] and [8], one can obtain the level of shortage \( S \) as:
Now, the total profit comprises of numerous cost components which are:

(i) Sales revenue per unit time

\[ SP = D(p, r)p \]  \hspace{1cm} (21)

(ii) Per unit time production cost

\[ PC = D(p, r)c_p \]  \hspace{1cm} (22)

(iii) Set up cost per unit time

\[ OC = \frac{A}{t_7} \]  \hspace{1cm} (23)

(iv) The per unit time holding cost is:

\[
HC = \frac{c_h}{t_7} \left[ \int_0^{t_1} q(t)dt + \int_{t_2}^{t_3} q(t)dt + \int_{t_4}^{t_5} q(t)dt \right] + \frac{\nu D(p, r)(\mu - 1)}{\theta} \int_{t_2}^{t_3} [1 - e^{-\theta t}] dt + \frac{\rho D(p, r)(\mu - 1)}{\theta} \int_{t_3}^{t_4} [1 - e^{-\theta t}] dt
\]

\[
+ \int_{t_4}^{t_5} D(p, r)\left[ e^{\theta(t_5-t)} - 1 \right] dt
\]

\[
= \frac{c_h}{t_7} \left[ \int_0^{t_1} D(p, r)(\mu - 1) \left[ t + \frac{e^{-\theta t}}{\theta} \right] dt + \nu D(p, r)(\mu - 1) \left[ t + \frac{e^{-\theta t}}{\theta} \right] dt \right]
\]

\[
+ \frac{\xi D(p, r)(\mu - 1)}{\theta} \left[ t + \frac{e^{-\theta t}}{\theta} \right] dt + \frac{\rho D(p, r)(\mu - 1)}{\theta} \left[ e^{\theta(t_4-t)} - 1 \right] dt - \int_{t_4}^{t_5} [1 - e^{-\theta t}] dt
\]

\[
\Rightarrow HC
\]

\[
= \frac{c_h}{2t_7} \left[ D(p, r)(\mu - 1)t_1^2 + \nu D(p, r)(\mu - 1)(t_2^2 - t_1^2) + \xi D(p, r)(\mu - 1)(t_3^2 - t_2^2)
\]

\[
+ \rho D(p, r)(\mu - 1)(t_4^2 - t_3^2) + D(p, r)(t_5 - t_4)^2 \right]
\]  \hspace{1cm} (25)

(v) Deteriorating cost per unit time

\[
DC = \frac{\theta c_p}{t_7} \left[ \int_0^{t_1} q(t)dt + \int_{t_2}^{t_3} q(t)dt + \int_{t_4}^{t_5} q(t)dt \right] + \frac{\nu D(p, r)(\mu - 1)}{\theta} \int_{t_2}^{t_3} [1 - e^{-\theta t}] dt + \frac{\xi D(p, r)(\mu - 1)}{\theta} \int_{t_2}^{t_4} [1 - e^{-\theta t}] dt + \frac{\rho D(p, r)(\mu - 1)}{\theta} \int_{t_3}^{t_4} [1 - e^{-\theta t}] dt
\]
Hence, the per unit time total profit is:

\[ \Pi(t_1, t_2, t_3, t_4, t_5, t_6, t_7, r, p) = SP - PC - OC - HC - DC - SC - RC \]

\[ \Rightarrow \Pi = D(p, r)p - D(p, r)\gamma - A \frac{c_h + \theta c_p}{t_7} \left[ D(p, r)(\mu - 1)\gamma^2 \right] \\
+ \nu D(p, r)(\mu - 1)(\gamma^2 - \rho^2) \gamma^2 + \xi D(p, r)(\mu - 1)(\gamma^2 - \rho^2) \gamma^2 \\
+ \rho D(p, r)(\mu - 1)(\gamma^2 - \rho^2) \gamma^2 + D(p, r)(\mu - 1)\gamma^2 \]

\[ \Rightarrow \Pi = \frac{D(p, r)c_n D(p, r)(\mu - 1)}{P_1} \left[ t_7 - t_5 \right]^2 - r(r + w)D(p, r) \]

Now, using the Taylor’s expansion theorem in (26) and ignoring the second- and higher-order terms of \( \theta \) for a small value of \( \theta \):

\[ \Rightarrow DC \]

\[ \Rightarrow DC = \frac{c_h}{2t_7} \left[ D(p, r)(\mu - 1)\gamma^2 + \nu D(p, r)(\mu - 1)(\gamma^2 - \rho^2) \gamma^2 + \xi D(p, r)(\mu - 1)(\gamma^2 - \rho^2) \gamma^2 \right] \\
+ \rho D(p, r)(\mu - 1)(\gamma^2 - \rho^2) \gamma^2 + D(p, r)(\mu - 1)\gamma^2 \]

\[ \Rightarrow SC = \frac{c_s}{t_7} \left[ \int_{t_5}^{t_6} q(t)dt + \int_{t_6}^{t_7} q(t)dt \right] \\
= \frac{c_s}{t_7} \left[ \int_{t_5}^{t_6} D(p, r)[t - t_5]dt + D(p, r) \int_{t_5}^{t_7} D(\nu)(\mu - 1)[t - t_7] - t]dt \right] \\
= \frac{D(p, r)c_s D(p, r)(\mu - 1)}{D(p, r)\gamma^2} \left[ t_7 - t_5 \right]^2 \]

\[ \Rightarrow RC = \frac{r(r + w)D(p, r)}{P_1} \]

Hence, the per unit time total profit is:

\[ \Pi(t_1, t_2, t_3, t_4, t_5, t_6, t_7, r, p) = SP - PC - OC - HC - DC - SC - RC \]

Let us assume \( t_1 = \alpha t_5, t_2 = \beta t_5, t_3 = \gamma t_5 \) and \( t_4 = \lambda t_5 \). Therefore, per unit time, the total profit can be presented as:

\[ \Rightarrow \Pi(t_5, t_7, r, p) = D(p, r)p - D(p, r)\gamma - A \frac{c_h + \theta c_p}{t_7} \left[ D(p, r)(\mu - 1)\gamma^2 \right] \\
+ \nu D(p, r)(\mu - 1)(\gamma^2 - \rho^2) \gamma^2 + \xi D(p, r)(\mu - 1)(\gamma^2 - \rho^2) \gamma^2 \\
+ \rho D(p, r)(\mu - 1)(\gamma^2 - \rho^2) \gamma^2 + D(p, r)(\mu - 1)\gamma^2 \]
\[
\Pi(t_5, t_7, r, p) = \left( D(p, r) - \frac{r(r + w)D(p, r)}{P_1} \right) \cdot t_5.
\]

### Solution Procedure.

**Proposition 1.** If \(r, p\) and the cycle time \(t_7\) is fixed, then the total profit per unit time \(\Pi(t_5, t_7, r, p)\) is concave in cycle time \(t_5\).

**Proof.** The first and second partial derivatives of the function \(\Pi(t_5, t_7, r, p)\) with respect to \(t_5\) are as follows:

\[
\frac{\partial \Pi(t_5, t_7, r, p)}{\partial t_5} = \frac{c_h + \theta c_p}{t_7} \left[ D(p, r)(\mu - 1)\alpha^2 t_5 + \nu D(p, r)(\mu - 1)(\beta^2 - \alpha^2)t_5 \\
+ \xi D(p, r)(\mu - 1)(\gamma^2 - \beta^2)t_5 + \rho D(p, r)(\mu - 1)(\lambda^2 - \beta^2)t_5 \\
+ D(p, r)t_5(1 - \lambda)^2 \right] + \frac{2D(p, r)c_s D(p, r)(\mu - 1)}{D(p, r)(\mu - 1)t_7} \left| t_7 - t_5 \right|
\]

and

\[
\frac{\partial^2 \Pi(t_5, t_7, r, p)}{\partial t_5^2} = \frac{c_h + \theta c_p}{t_7} \left[ D(p, r)(\mu - 1)\alpha^2 + \nu D(p, r)(\mu - 1)(\beta^2 - \alpha^2) \\
+ \xi D(p, r)(\mu - 1)(\gamma^2 - \beta^2) \\
+ \rho D(p, r)(\mu - 1)(\lambda^2 - \beta^2) + D(p, r)(1 - \lambda)^2 \right] - \frac{2D(p, r)c_s D(p, r)(\mu - 1)}{D(p, r)(\mu - 1)} < 0.
\]

Therefore

\[
\frac{2\Pi(t_5, t_7, r, p)}{\partial t_5} = 0 \Rightarrow t_5 = R t_7
\]

where

\[
R = \frac{2D(p, r)c_s D(p, r)(\mu - 1)}{D(p, r)(\mu - 1)c_h + \theta c_p} \left[ D(p, r)(\mu - 1)\alpha^2 \\
+ \nu D(p, r)(\mu - 1)(\beta^2 - \alpha^2) + \xi D(p, r)(\mu - 1)(\gamma^2 - \beta^2) \\
+ \rho D(p, r)(\mu - 1)(\lambda^2 - \beta^2) + D(p, r)(1 - \lambda)^2 \right] + \frac{2D(p, r)c_s D(p, r)(\mu - 1)}{D(p, r)(\mu - 1)}
\]

Placing the values of \(R t_7\) in the position of \(t_5\) in Equation (31), we can obtain the new profit function, that is:

\[
\Pi(t_7, r, p) = D(p, r)p - D(p, r)c_p - \frac{A}{t_7} \left[ c_h + \theta c_p \right] \\
\left[ D(p, r)(\mu - 1)\alpha^2 R^2 t_7 + \nu D(p, r)(\mu - 1)(\beta^2 - \alpha^2)R^2 t_7 \\
+ \xi D(p, r)(\mu - 1)(\gamma^2 - \beta^2)R^2 t_7 + \rho D(p, r)(\mu - 1)(\lambda^2 - \beta^2)R^2 t_7 \\
+ D(p, r)(1 - \lambda)^2 R^2 t_7 \right] \\
- \frac{D(p, r)c_s D(p, r)(\mu - 1)(t_7 - R t_7)^2}{D(p, r)\mu t_7} - \frac{r(r + w)D(p, r)}{P_1}.
\]
Proposition 2. If \( r \) and \( p \) are stable, then equation (36) provides the optimum value in cycle time \( t_7 \).

Proof. The first-and second-order derivatives of the function \( \Pi(t_7, p, r) \) regarding \( t_7 \) are:

\[
\Rightarrow \frac{\partial [\Pi(t_7, r, p)]}{\partial t_7} = -\frac{A}{t_7^2} \left[ c_h + \theta c_p \right]
\]

\[
= \left[ D(p, r)(\mu - 1)\alpha^2 R^2 + \nu D(p, r)(\mu - 1)(\beta^2 - \alpha^2)R^2 \\
+ \xi D(p, r)(\mu - 1)(\gamma^2 - \beta^2)R^2 + \rho D(p, r)(\mu - 1)(\lambda^2 - \beta^2)R^2 \\
+ D(p, r)(1 - \lambda)^2 R^2 \right] - \frac{D(p, r)c_s D(p, r)(\mu - 1)(1 - R)^2}{D(p, r)\mu}
\]

\[
\Rightarrow \frac{\partial^2 [\Pi(t_7, r, p)]}{\partial t_7^2} = -\frac{2A}{t_7^4} < 0
\]

For the values of \( t_7 \), then \( \frac{\partial [\Pi(t_7, r, p)]}{\partial t_7} = 0 \)

\[
\Rightarrow -\frac{1}{2}D(p, r)R^2 \left[ (\lambda - 1)^2 + \alpha^2(\mu - 1) + (\beta^2 - \alpha^2)(\mu - 1)\nu + (\gamma^2 - \beta^2)(\mu - 1)\xi \\
+ (\lambda^2 - \beta^2)(\mu - 1)\rho \right] (c_h + \theta c_p)
\]

\[
- \frac{D(p, r)(R - 1)^2(\mu - 1)c_s}{\mu} + \frac{A}{t_7^4} = 0.
\]

\]

Proposition 3. If \( p \) and \( t_7 \) are fixed, then equation (36) provides the optimum value in \( r \).

Proof. The first-and second order derivatives of function \( \Pi(t_7, p, r) \) with respect to \( r \) are:

\[
\Rightarrow \frac{\partial [\Pi(t_7, r, p)]}{\partial r} = [D(p, r)p']' - D'(p, r)c_p - \frac{A}{t_7^2} \left[ c_h + \theta c_p \right]
\]

\[
\Rightarrow \frac{\partial^2 [\Pi(t_7, r, p)]}{\partial r^2} = [D(p, r)p'']'' - D''(p, r)c_p - \frac{A}{t_7^2} \left[ c_h + \theta c_p \right]
\]

\[
\left[ (\mu - 1)\alpha^2 R^2 t_7 + \nu(\mu - 1)(\beta^2 - \alpha^2)R^2 t_7 \\
+ \xi(\mu - 1)(\gamma^2 - \beta^2)R^2 t_7 + \rho(\mu - 1)(\lambda^2 - \beta^2)R^2 t_7 + (1 - \lambda)^2 R^2 t_7 \right]
\]

\[
- \frac{D'(p, r)c_s(\mu - 1)(t_7 - R t_7)^2}{\mu t_7} \frac{r(r + w)D'(p, r) + (2r + w)D(p, r)}{P_1}' < 0
\]
for finding the values of \( p \), then 

\[
\frac{\partial [\Pi(t_r, r, p)]}{\partial p} = 0
\]

\[
\Rightarrow [D(p, r)p']' - D'(p, r)c_p - A \frac{D'(p, r)[c_h + \theta c_p]}{t_7}
\]

\[
- \left[ (\mu - 1)\alpha^2 R^2 t_7 + \nu(\mu - 1)(\beta^2 - \alpha^2) R^2 t_7 + \xi(\mu - 1)(\gamma^2 - \beta^2) R^2 t_7 + \rho(\mu - 1)(\lambda^2 - \beta^2) R^2 t_7 + (1 - \lambda)^2 R^2 t_7 \right]
\]

\[
= - \frac{D'(p, r)c_s(\mu - 1)(t_7 - R t_7)^2}{\mu t_7} = \frac{r(r + w)D'(p, r)}{P_1} + (2r + w)D(p, r) = 0. \tag{41}
\]

Proposition 4. If \( r \) and \( t_7 \) are fixed, then equation (36) provides the optimum value in \( p \).

Proof. The first-and second order differentiations of \( \Pi(t_r, r, p) \) with respect to \( p \):

\[
\Rightarrow \frac{\partial^2 [\Pi(t_r, r, p)]}{\partial p^2} = [D(p, r)p]' - D'(p, r)c_p - A \frac{D'(p, r)[c_h + \theta c_p]}{t_7}
\]

\[
= \frac{D'(p, r)[c_h + \theta c_p]}{2} \left[ (\mu - 1)\alpha^2 R^2 t_7 + \nu(\mu - 1)(\beta^2 - \alpha^2) R^2 t_7 + \xi(\mu - 1)(\gamma^2 - \beta^2) R^2 t_7 + \rho(\mu - 1)(\lambda^2 - \beta^2) R^2 t_7 + (1 - \lambda)^2 R^2 t_7 \right]
\]

\[
= - \frac{D'(p, r)c_s(\mu - 1)(t_7 - R t_7)^2}{\mu t_7} - \frac{r(r + w)D'(p, r)}{P_1} < 0
\]

For finding the values of \( p \), then 

\[
\frac{\partial [\Pi(t_r, r, p)]}{\partial p} = 0
\]

\[
\Rightarrow [D(p, r)p]' - D'(p, r)c_p - A \frac{D'(p, r)[c_h + \theta c_p]}{t_7}
\]

\[
= \frac{D'(p, r)[c_h + \theta c_p]}{2} \left[ (\mu - 1)\alpha^2 R^2 t_7 + \nu(\mu - 1)(\beta^2 - \alpha^2) R^2 t_7 + \xi(\mu - 1)(\gamma^2 - \beta^2) R^2 t_7 + \rho(\mu - 1)(\lambda^2 - \beta^2) R^2 t_7 + (1 - \lambda)^2 R^2 t_7 \right]
\]

\[
= - \frac{D'(p, r)c_s(\mu - 1)(t_7 - R t_7)^2}{\mu t_7} - \frac{r(r + w)D'(p, r)}{P_1} = 0. \tag{42}
\]
Proposition 5. The profit function \( \Pi(t_7, r, p) \) provides optimum profit in \( t_7, r \) and \( p \) and hence entails unique optimum values.

Proof. The profit function \( \Pi(t_7, r, p) \) provides optimum profit in \( t_7, r \) and \( p \) if \( \Pi(t_7, r, p) \) is negative definite, i.e., if each eigenvalue of the Hessian matrix

\[
H = \begin{pmatrix}
\frac{\partial^2 \Pi(t_7, r, p)}{\partial t^2} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial t \partial p} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial t \partial r} \\
\frac{\partial^2 \Pi(t_7, r, p)}{\partial r \partial t} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial r^2} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial r \partial p} \\
\frac{\partial^2 \Pi(t_7, r, p)}{\partial p \partial t} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial p \partial r} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial p^2}
\end{pmatrix}
\]

is negative.

It is almost impossible to solve the above Hessian matrix via classical techniques so a search algorithm to solve the Hessian matrix is proposed in Algorithm 1, which allows one to obtain the optimal solutions of \( t_7, r, p \).

Algorithm 1.

Stage 1 Starting with \( i = 0 \) and the initial trial value of \( p_0 \) and \( r_0 \).

Stage 2 For given \( r_i \) and \( p_i \), find optimal \( R \) and \( t_7 \).

Stage 3 Consumption of the outcome achieved in Stage 2, determining the optimal values of \( r_{i+1} \) and \( p_{i+1} \).

Stage 4 If the variance among two successful iterations is satisfactorily small, we set \( r^* = r_{i+1}, p^* = p_{i+1} \) and \( t_7^* = t_7 \).

Stage 5 Then \( (t_7^*, r^*, p^*) \) and \( \Pi^* = \Pi(t_7^*, r^*, p^*) \) are the optimal solution and the algorithm is stopped.

Stage 6 Otherwise, set \( i = i + 1 \).

4.2. Case 2 (demand depends on selling price). In this section, we discuss a situation where demands are selling price reliant while assumptions and model descriptions are analogous to Sections 2 and Section 3.1, with \( D = D(p) \) and without rebate redemption costs.

4.2.1. Solution procedure. However, to avoid redundancy, theoretical derivations are overlooked in this section. The solution procedure proposed in Algorithm 2 can be performed to obtain the optimal solutions of \( t_7, p \) and \( \Pi(t_7, p) \).

Algorithm 2.

Step 1 : Starting with \( i = 0 \) and the initial trial value of \( p_0 \).

Step 2 : From any particular \( p_i \), obtain optimum \( R \) and \( t_7 \).

Step 3 : To define the optimal value of \( p_{i+1} \), the results obtained in Step 2 are used.

Step 4 : If the variation of two successive iterations tends to zero, then set \( p^* = p_{i+1} \) and \( t_7^* = t_7 \).
Step 5: Then \((t^*_7, p^*)\) and \(\Pi^* = \Pi(t^*_7, p^*)\) are the optimal solution and the algorithm is stopped.

5. Numerical examples and sensitivity analysis.

5.1. Case study. This section presents a scheme to solve dissimilar inventory problems with the selling price and rebate value under a shortage. All models proposed in the previous sections to obtain the optimal solutions in two sub-cases (selling price with and without rebate value). The case instances demonstrate the applicability of the different sub-cases. The inventory model is utilized to regulate the time, find a better selling price with/without rebate value, and optimize profit. For a better example of using the optimal solution policies (with shortage) and to see how the solution process is suggested for this work, we present several numerical examples. This section aims to solve four-rate production inventory problems with the selling price and rebate value under shortage cases using different models. To explain all numerical examples more precisely and to evaluate the evidence of the numerical example (applicability), we conducted a case study based on the staple manufacturing product (Figure 3). This company’s production inventory system is very similar to the proposed model, as they study different types of demand. However, in this section, the data used are illustrative.

Here, considering the demand function is non-linear because the demand to go up whenever we drop the price. However, with a linear process, we get the same increase in demand by cutting the price in half or giving the product away for free from half price. Here we used the demand function as the non-linear function was decreasing the price by fixed percentage increases consumption.

Figure 3. Staple manufacturing product

5.2. Numerical Illustration.

Example 1.

For instance, via the previous theory, one can use the following example: demand parameters \(D(p, r) = 80 - 4p + 2r\). Let us consider the following parametric values:\(A = $40 per set up, \mu = $20 units, c_h = $0.06/units, c_s = $0.04/units, c_p =\)
$0.03 \text{ units}, \theta = 0.2, \nu = 2, \xi = 3, \rho = 4, \alpha = 0.2, \beta = 0.4, \gamma = 0.6, \lambda = 0.8, P_1 = 3 \text{ and } w = 0.3$. Using Mathematica-9.0 and Algorithm 1, the values in Table 2 are obtained by starting with $p = 5$ and $r = 4$, from which one can easily conclude that the optimum values of the decision are $R^* = 0.0216282, t^*_7 = 5.22363$ week, $r^* = 0.600007, p^* = 10.4472$ and $\Pi(t^*_7, r^*, p^*) = 380.657/\text{week}$. In addition, we can obtain $t^*_1 = 0.0225956 \text{ week}, t^*_2 = 0.0451912 \text{ week}, t^*_3 = 0.0677868 \text{ week}, t^*_4 = 0.112978, t^*_5 = 4.9681 \text{ week}, q^*_1 = 16.9199 \text{ units}, q^*_2 = 67.6795 \text{ units}, q^*_3 = 152.279 \text{ units}, q^*_4 = 270.718 \text{ units}, q^*_5 = 205.87 \text{ units} \text{ and } S^* = 191.346 \text{ units}$. The above outcomes are optimum as the eigenvalues of the Hessian matrix

$$H = \begin{pmatrix} \frac{\partial^2 \Pi(t_7, r, p)}{\partial t^2} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial t \partial r} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial t \partial p} \\ \frac{\partial^2 \Pi(t_7, r, p)}{\partial r \partial t} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial r^2} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial r \partial p} \\ \frac{\partial^2 \Pi(t_7, r, p)}{\partial p \partial t} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial p \partial r} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial p^2} \end{pmatrix}$$

are $-0.56127, -8, \text{ and } -28.2742$. Thus, the objective functions are concave and unimodal. Expanding the data specified in Example 1, one can graphically illustrate that the concavity of $\Pi(t^*_7, r^*, p^*)$, as shown in Figure 5, concerning $t_7$ when $r$ and $p$ are constant. The concavity of $\Pi(t^*_7, r^*, p^*)$ is shown in Figure 4 with respect to $p$ when $t_7$ and $r$ are fixed. The concavity of $\Pi(t^*_7, r^*, p^*)$ is shown in Figure 8 with respect to $r$ when $t_7$ and $p$ are fixed. Hence $t_7, r$ and $p$ is a unique solution.

Table 2: Computational results

| Number of iterations | $R$    | $t_7$   | $r$    | $p$    | $\Pi(t_7, r, p)$ |
|----------------------|--------|---------|--------|--------|-----------------|
| 1                    | 0.216284 | 3.97675 | 1.31944 | 14.028 | 334.61          |
| 2                    | 0.216279 | 6.36706 | 0.665818| 10.9352| 377.988         |
| 3                    | 0.0216281| 5.34862 | 0.604776| 10.4854| 380.469         |
| 4                    | 0.0216282| 5.23315 | 0.600339| 10.4499| 380.643         |
| 5                    | 0.0216282| 5.2243  | 0.60018 | 10.4473| 380.656         |
| 6                    | 0.0216282| 5.22366 | 0.600007| 10.4472| 380.657         |
| 7                    | 0.0216282| 5.22363 | 0.600007| 10.4472| 380.657         |
| 8                    | 0.0216282| 5.22363 | 0.600007| 10.4472| 380.657         |

Figure 4. The concave nature of $\Pi$ with respect to $P$ Example 1 and 2.
Example 2.
For instance, in the previous theory, we consider the following example: the demand parameter \(D(p, r) = 80 - 4p\), i.e., the demand is influenced by the selling price only. Suppose that the parametric values are \(A = \$40\) per set up, \(\mu = \$20\) units, \(c_h = \$0.06/units, c_s = \$0.04/units, c_p = \$0.03\) units, \(\theta = 0.2, \nu = 2, \xi = 3, \rho = 4, \alpha = 0.2, \beta = 0.4, \gamma = 0.6, \lambda = 0.8\). Using Mathematica-9.0 and Algorithm 1, the values in Table 3 are obtained by starting with \(p = 5\) which clearly shows that the optimum values are \(R^* = 0.0216282, t^*_7 = 4.23357\) week, \(p^* = 10.1707\) and \(\Pi(t^*_7, p^*) = \$377.01/week\). In addition, we can obtain \(t^*_1 = 0.0183129\) week, \(t^*_2 = 0.03662581\) week, \(t^*_3 = 0.0549387\) week, \(t^*_4 = 0.0732516\) week, \(t^*_5 = 0.0915645, t^*_6 = 4.02647\) week, \(q^*_1 = 13.6802\) units, \(q^*_2 = 54.7209\) units, \(q^*_3 = 123.122\) units, \(q^*_4 = 218.884\) units, \(q^*_5 = 166.452\) units and \(S^* = 154.709\) units. The above outcomes are optimum as the eigenvalues of the Hessian matrix

\[
H = \begin{bmatrix}
\frac{\partial^2\Pi(t^*_7, p)}{\partial t^*_7 \partial t^*_7} & \frac{\partial^2\Pi(t^*_7, p)}{\partial t^*_7 \partial p} \\
\frac{\partial^2\Pi(t^*_7, p)}{\partial p \partial t^*_7} & \frac{\partial^2\Pi(t^*_7, p)}{\partial p \partial p}
\end{bmatrix}
\]

are -4.89947, -66.8339 and -464.372. Thus, the objective function is concave and unimodal. By consuming the data set in Example 3, we can graphically display that the concavity of as shown in Figure 5, regarding when and are constant. The concavity of is shown in Figure 7 concerning when and are constant. The concavity of is shown in Figure 8 with respect to when and are constant. Hence \(t^*_7\) and \(p^*\) is a unique solution.

Table 3 : Computational results:

| Number of iterations | \(R\)    | \(t^*_7\) | \(p\)  | \(\Pi(t^*_7, p)\) |
|----------------------|----------|-----------|--------|-------------------|
| 1                    | 0.0216284| 4.23357   | 10.1707| 377.01            |
| 2                    | 0.0216282| 4.23357   | 10.1707| 377.01            |
| 3                    | 0.0216282| 4.23357   | 10.1707| 377.01            |

Figure 5. The concave nature of \(\Pi\) with respect to \(t^*_7\) for Example 1 and 2.

Example 3.
To demonstrate the previous theory, one can take the demand parameter as \(D(p, r) = 200r^2e^{-0.4p}\). Suppose that the parametric values are \(A = \$40\) per set up, \(\mu = \$20\) units, \(c_h = \$0.06/units, c_s = \$0.04/units, c_p = \$0.03\) units, \(\theta =\)
0.2, \nu = 2, \xi = 3, p = 4, \alpha = 0.2, \beta = 0.4, \gamma = 0.6, \lambda = 0.8, P_1 = 3 and w = 0.3.
Using Mathematica -9.0 and Algorithm 1, the values in Table 4 is obtained by
starting with \( p = 5 \) and \( r = 4 \), which clearly shows that the optimum values are
\( R^* = 0.0216282, t_7^* = 2.536966 \) week, \( r^* = 2.66465, p^* = 5.34994 \) and \( \Pi(t_7^*, r^*, p^*) = \$401.946/week \). we can also obtain \( t_1^* = 0.0109742 \) week, \( t_2^* = 0.0219484 \) week,
\( t_3^* = 0.0329225 \) week, \( t_4^* = 0.0438967 \) week, \( t_5^* = 0.0548709 \) week, \( t_6^* = 2.41286 \) week,
\( q_1^* = 34.8389 \) units, \( q_2^* = 139.357 \) units, \( q_3^* = 313.549 \) units, \( q_4^* = 557.421 \) units,
\( q^* = 423.888 \) units and \( S^* = 393.985 \) units. The mentioned outcomes are optimum
as the eigenvalues of the Hessian matrix
\[
H = \begin{pmatrix}
\frac{\partial^2 \Pi(t_7, r, p)}{\partial t^2} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial t \partial r} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial t \partial p} \\
\frac{\partial^2 \Pi(t_7, r, p)}{\partial r \partial t} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial r^2} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial r \partial p} \\
\frac{\partial^2 \Pi(t_7, r, p)}{\partial p \partial r} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial p \partial t} & \frac{\partial^2 \Pi(t_7, r, p)}{\partial p^2}
\end{pmatrix}
\]
are \(-4.89947, -66.8339\) and \(-464.372\).

Thus, the objective function is concave and unimodal. By consuming the data
set in Example 3, we can graphically display that the concavity of \( \Pi(t_7^*, r^*, p^*) \)
as shown in Figure 5, regarding \( t_7 \) when \( r \) and \( p \) are constant. The concavity of
\( \Pi(t_7^*, r^*, p^*) \) is shown in Figure 7 concerning \( p \) when \( t_7 \) and \( r \) are constant.
The concavity of \( \Pi(t_7^*, r^*, p^*) \) is shown in Figure 8 with respect to \( r \) when \( t_7 \) and \( p \)
is constant. Hence, \( t_7, r \) and \( p \) is a unique solution.

**Table 4**: Computational results

| Number of iterations | \( R \) | \( t_7 \) | \( r \) | \( p \) | \( \Pi(t_7, r, p) \) |
|----------------------|-------|-------|-----|-----|------------------|
| 1                    | 0.0216287 | 1.5758 | 3.40356 | 8.37926 | 301.769 |
| 2                    | 0.0216285 | 3.64039 | 3.7101 | 8.37926 | 290.055 |
| 3                    | 0.0216285 | 3.33961 | 3.23438 | 7.74399 | 334.006 |
| 4                    | 0.0216285 | 3.35767 | 2.9662 | 6.58827 | 276.844 |
| 5                    | 0.0216286 | 2.91956 | 2.82123 | 5.97419 | 393.994 |
| 6                    | 0.0216286 | 2.71482 | 2.74503 | 5.66496 | 399.362 |
| 7                    | 0.0216286 | 2.62285 | 2.70566 | 5.50926 | 401.031 |
| 8                    | 0.0216286 | 2.57330 | 2.6855 | 5.43053 | 401.587 |
| 9                    | 0.0216286 | 2.55819 | 2.67523 | 5.39073 | 401.791 |
| 10                   | 0.0216286 | 2.54765 | 2.67001 | 5.36975 | 401.875 |
| 11                   | 0.0216286 | 2.54236 | 2.66736 | 5.36036 | 401.912 |
| 12                   | 0.0216286 | 2.53690 | 2.66602 | 5.35518 | 401.929 |
| 13                   | 0.0216286 | 2.53834 | 2.66534 | 5.35257 | 401.938 |
| 14                   | 0.0216286 | 2.53766 | 2.66499 | 5.35124 | 401.942 |
| 15                   | 0.0216286 | 2.53732 | 2.66482 | 5.35056 | 401.944 |
| 16                   | 0.0216286 | 2.53713 | 2.66473 | 5.35023 | 401.945 |
| 17                   | 0.0216286 | 2.53705 | 2.66468 | 5.35005 | 401.946 |
| 18                   | 0.0216286 | 2.53701 | 2.66466 | 5.34996 | 401.946 |
| 19                   | 0.0216286 | 2.53698 | 2.66465 | 5.34992 | 401.946 |
| 20                   | 0.0216286 | 2.63697 | 2.66465 | 5.34999 | 401.946 |
| 21                   | 0.0216286 | 2.53696 | 2.66465 | 5.34999 | 401.946 |
| 22                   | 0.0216286 | 2.53696 | 2.66465 | 5.34999 | 401.946 |
Figure 6. The concave nature of $\Pi$ with respect to $t_7$ for Example 3 and 4.

Figure 7. The concave nature of $\Pi$ with respect to $p$ for Example 3 and 4.

Figure 8. The concave nature of $\Pi$ with respect to $r$ for Example 1 and 2.
Example 4.
For instance, from the previous theory, one can set the demand parameter \(D(p, r) = 200r^2e^{-0.4p} \), i.e., the demand only hinges on the selling price. Suppose that the parametric values are \(A = \$40 \) per set up, \(\mu = \$20 \) units, \(c_h = \$0.06/\text{units} \), \(c_s = \$0.04/\text{units} \), \(c_p = \$0.03 \) units, \(\theta = 0.2, \nu = 2, \xi = 3, \rho = 4, \alpha = 0.2, \beta = 0.4, \gamma = 0.6, \lambda = 0.8 \). Using Mathematica-9.0 and Algorithm 2, the values in Table 5 are obtained by initially setting \(p = 5 \) which clearly shows that the optimum values are \(R^* = 0.0216282 \), \(t_7^* = 4.08432 \text{ week} \), \(p^* = 2.83048 \) and \(\Pi(t_7^*, p^*) = \$151.369/\text{week} \).

In addition, we can obtain \(t_1^* = 0.0176675 \text{ week} \), \(t_2^* = 0.0353349 \text{ week} \), \(t_3^* = 0.0530215 \text{ week} \), \(t_4^* = 0.0706698 \text{ week} \), \(t_5^* = 0.0883373 \text{ week} \), \(t_6^* = 3.88452 \text{ week} \), \(q_1^* = 21.6398 \) units, \(q_2^* = 86.5591 \) units, \(q_3^* = 194.758 \) units, \(q_4^* = 346.236 \) units, \(q^* = 263.296 \) units and \(S^* = 244.722 \) units. The above outcomes are optimum as the eigenvalues of the Hessian matrix

\[
H = \begin{pmatrix}
\frac{\partial^2 \Pi(t_7, p)}{\partial t_7^2} & \frac{\partial^2 \Pi(t_7, p)}{\partial t_7 \partial p} \\
\frac{\partial^2 \Pi(t_7, p)}{\partial p \partial t_7} & \frac{\partial^2 \Pi(t_7, p)}{\partial p^2}
\end{pmatrix}
\]

are -1.17417 and -25.7861.

Thus, the objective function is concave and unimodal. Expanding the data assumed in Example 4, one can graphically illustrate that the concavity of \(\Pi(t_7^*, p^*)\) is revealed in Figure 5 with respect to \(t_7\) when \(p\) is constant. The concavity of \(\Pi(t_7^*, p^*)\) is provided in Fig 7 regarding \(p\) when \(t_7\) is constant. Hence, \(t_7\) and \(p\) is a unique solution.

Table 5: Computational results

| Number of iterations | \(R\)     | \(t_7\)     | \(p\)     | \(\Pi(t_7, p)\) |
|----------------------|-----------|-------------|-----------|-----------------|
| 1                    | 0.0216279 | 6.30321     | 2.99373   | 144.63          |
| 2                    | 0.0216284 | 4.21988     | 2.84046   | 151.042         |
| 3                    | 0.0216284 | 4.09248     | 2.83108   | 151.35          |
| 4                    | 0.0216284 | 4.08481     | 2.83052   | 151.368         |
| 5                    | 0.0216284 | 4.08435     | 2.83049   | 151.369         |
| 6                    | 0.0216284 | 4.08433     | 2.83048   | 151.369         |
| 7                    | 0.0216284 | 4.08432     | 2.83048   | 151.369         |
| 8                    | 0.0216284 | 4.08432     | 2.83048   | 151.369         |

5.3. Effect of demand on profit, order quantity and shortages. To show the actual effect of demand function on profit of the chain, order quantity and shortages a graph for different values is presented at Figure 9.
Figure 9. Comparison of profit, ordering quantity and shortages concerning altered demand.

From Figure 9, the profit is maximized when the demand is a function of price and rebate value. From Figure 9, it is clear that for all the stated three cases, the profit is maximum when the demand depends on the selling price and rebate value exponentially. However, the graph shows that in the case of order quantity and shortages, the second-highest value of profit occurs when the demand of the products depends on exponential selling price, while the exponential selling price and rebate value have a significant effect on order quantity. It is also clear that the exponential selling price-dependent demand has less sensitiveness among the model’s profit, ordering quantity, and shortages.

5.4. Sensitivity analysis and managerial insights for industry.

This study examines the consequences of changes in the major system parameter, scaling demand parameter, price sensitivity, production cost, shortage cost, reference price sensitive parameter, rebate effectiveness, and rebate value at which 100% of rebate offers are saved by altering the parameters, allowing for one parameter at a time and keeping the rest untouched.

Figs. 10 – 17 show the variation in total profit per unit time for variables $A, a, b, c_p, c_s, d, w$ and $P_1$.

This study develops the following managerial insights based on the sensitivity analysis results:

- If the producer jointly optimizes the rebate and selling price, then the profit increases. Therefore, producers should consider rebate and selling price-dependent demand. From Figure 9, the profit is higher when the demand is a function of price and rebate value.
- When the scaling demand parameter increases (Figure 10), the total profit increases. In a real sense, this usually occurs when the buyer’s demand increases, which results in additional profit for the retailer.
- The price elasticity demand parameter with a decreasing rebate effect has been depicted in Figure 11. The rebate value is absent in terms of the item price, which is the linear demand case. The question of which case is more
Figure 10. The consequence of $\Pi$ concerning $a$ for Example 1, 2, 3 and 4

Figure 11. The consequence of $\Pi$ concerning $b$ for Example 1, 2, 3 and 4

Figure 12. The consequence of $\Pi$ concerning $c_p$ for Example 1, 2, 3 and 4
suitable depends upon the price of the items. The linear demand model can be fit for low-priced items.

- An increase in the production cost per unit time (Figure 12) results in a decline in the total profit. From an economic standpoint, if the supplier
Figure 16. The consequence of Π concerning $P_1$ for Example 1,2,3 and 4

provides a higher production cost per unit, then the producers will order a smaller quantity.

- With the escalation of shortage costs (Figure 13), the total profit will increase. From an economic perspective, producers will negate shortages when the shortage cost is very high.
- Producers do not have the face value of the rebate (Figure 14) and have less total profit than those with the rebate face value. Producers invest the rebate’s face value to sell their products in a shorter duration, reducing many real-world complications. Therefore, the demand function minimizes total profit with the least quantity and lowest setup cost.
- The above sensitivity analysis (Figures 14 to 16) shows that the total profit consuming rebates growth forms a practical rebate function. This is noticed because of the intensification in demand owing to the rebate. Marketing produces several encouragements for mounting rebate effectiveness, which can also contribute to amplified salvation. Based on numerical examples of rebate salvation, it appears that salvation will not be inclined to the point when the cost should balance the rise in profits.
- The above sensitivity analysis (Figures 14 to 16) shows that the rise in profit from expending rebates is a function of the reference price. This indicates that the customer’s salvation decision is associated with the time and effort required to complete the salvation necessity proportionate to the assessment they identify on their time. However, this suggestion is not irrefutable.

6. Conclusions. This study constructed an inventory model with four-level production based on rebate value and selling price. The current models do not provide adequate attention to parameters, such as rebate value and selling price, in multiple production-level environments. Additionally, this research sought to preference ideal unique replenishment time, ordering quantity, and rebate value while simultaneously maximizing the total profit of the manufacturer. This study also presented a four-level production rate algorithm for a deteriorating inventory model for the proposed research. We first discuss the demand function based on the selling price of the rebate value, followed by the price on the next. The model demonstrates a unique optimum solution with several propositions and suitable algorithms. Numerical examples justify the applicability of the model and the elasticity of numerous key parameters is presented in terms of sensitivity. The key findings are:
Suppose the manufacturer invests in rebate value to sell their products in a shorter duration. The demand function shortens the time duration to maximize the profit with minimum quantity minus the setup cost. The maximum reference price, the upper increase in profit contributed by rebates. However, the upper proportion of the rebate values offers should not be saved.

This suggests that mail-in rebates are often useless for large ticket items. For that product, significant rebates affecting demand could also be quite substantial and within the range of minimum dollars. Hence, rebate values are often saved, making the rebate plan ineffective. Actually, they will explain the deficiency of mail-in rebates value for giant ticket-like staple companies (frequently rebates for products are instant at buying and hence act as a sort of price drop).

The profit is higher when the demand is a function of price and rebate value. Therefore, the manufacturer invests in rebate value to sell their products.

In addition to these contributions, it is possible to modify the model by implementing numerous rebate-value programs. This limitation and weakness of this study is the consequence of streamlining rebate salvation on its strength and salvation rates. If shortening the salvation procedure has a more significant effect on rebate than on salvation rates, the reduction will increase profits; if not, the net effect can be a fall in profit. Lastly, future scopes are necessary to detect the reference prices of buyer groups and are related to non-refundable income. In addition, this study may be extended to include stochastic demand patterns with functions of time and price with five production levels.

Conflicts of Interest: The authors declare that, there is no conflict of interest.

REFERENCES

[1] P. L. Abad, Optimal pricing and lot sizing under conditions of perishability and partial backordering, Management Science, 42 (1996), 1093–1104.
[2] P. L. Abad and C. K. Jaggi, A joint approach for setting unit price and the length of the credit period for a seller when end demand is price sensitive, International Journal of Production Economics, 83 (2003), 115–122.
[3] A. Abdul, A. Jolson Marin and Y. Darmon Reney, A model for optimizing the refund value in rebate promotions, Journal of Business Research, 29 (1994), 239–245.
[4] F. A. Arcelus, S. Kumar and G. Srinivasan, Pricing and rebate policies in the two-echelon supply chain with asymmetric information under-price-dependent stochastic demand, International Journal of Production Economics, 113 (2008), 598–618.
[5] O. Caliskan-Demirag, Y. (Frank) Chen and J. Li, Customer and retailer rebates under risk aversion, International Journal of Production Economics, 133 (2011), 736–750.
[6] K. Cao, G. Han, B. Xu and J. Wang, Gift card payment or cash payment: Which payment is suitable for trade-in rebate?; Transportation Research Part E: Logistics and Transportation Review, 134 (2020), 101857.
[7] H. J. Chang, J. T. Teng, L. Y. Ouyang and C. Y. Dye, Retailer’s optimal pricing and lot-sizing policies for deteriorating items with partial backlogging, European J. Oper. Res., 168 (2005), 51–64.
[8] T. H. Chen, Optimizing pricing replenishment and rework decision for imperfect and deteriorating items in a manufacturer-retailer channel, International Journal of Production Economics, 183 (2017), 539–550.
[9] T. Chernonog and T. Avinadav, Pricing and advertising in a supply chain of perishable products under asymmetric information, International Journal of Production Economics, 209 (2019), 249–264.
[10] R. Hasan, A. H. M. Mashud, Y. Daryanto and H. M. Wee, A non-instantaneous inventory model of agricultural products considering deteriorating impacts and pricing policies, Kybernetes, 50 (2020).
C. H. Ho, The optimal integrated inventory policy with price-and-credit-linked demand under two-level trade credit, *Computers and Industrial Engineering*, 60 (2011), 117–126.

S. Hu, Z.-J. Ma and J. B. Sheu, Optimal prices and trade-in rebates for successive-generation products with strategic consumers and limited trade-in duration, *Transportation Research Part E: Logistics and Transportation Review*, 124 (2019), 92–107.

O. M. Jadidi, Y. Jaber and S. Zolfaghari, Joint pricing and inventory problem with price dependent stochastic demand and price discounts, *Computers and Industrial Engineering*, 114 (2017), 45–53.

A. Khakzad and M. R. Gholamian, The effect of inspection on deterioration rate: An inventory model for deteriorating items with advanced payment, *Journal of Cleaner Production*, 2541 (2020), 120117.

M. Khouja, A joint optimal pricing, rebate value, and lot-sizing model, *European Journal of Operational Research*, 174 (2006), 706–723.

M. Khouja, M. Hadzikadic and M. A. Zaffar, An agent based modeling approach for determining optimal price-rebate schemes, *Simulation Modelling Practice and Theory*, 16 (2008), 111–126.

C. Liuxin, C. Xian, M. F. Keblis and L. Gen, Optimal pricing and replenishment policy for deteriorating inventory under stock-level-dependent, time-varying and price-dependent demand, *Computers and Industrial Engineering*, 135 (2019), 1294–1299.

L. Lu, J. Zhang and W. Tang, Optimal dynamic pricing and replenishment policy for perishable items with inventory-level-dependent demand, *Internat. J. Systems Sci.*, 47 (2016), 1480–1494.

A. K. Manna, J. K. Dey and S. K. Mondal, Imperfect production inventory model with production rate dependent defective rate and advertisement dependent demand, *Computers and Industrial Engineering*, 104 (2017), 9–22.

U. Mishra, An inventory model for controllable probabilistic deterioration rate under shortages, *Evolving Systems*, 7 (2016), 287–307.

A. H. M. Mashud, D. Roy, Y. Daryanto, R. K. Chakrabortty and M.-L. Tseng, A sustainable inventory model with controllable carbon emissions, deterioration and advance payments, *Journal of Cleaner Production*, 296 (2021), 126608.

U. Mishra, Optimizing three-rates-of-production inventory model under market selling price and advertisement cost with deteriorating items, *International Journal of Management Science and Engineering Management*, 13 (2018), 295–305.

U. Mishra, L. E. Cardenas-Barron, S. Tiwari, A. A. Shaikh and G. Treviño-Garza, An inventory model under price and stock dependent for controllable deterioration rate with shortages and preservation technology investment, *Ann. Oper. Res.*, 254 (2017), 165–190.

A. H. M. Mashud, D. Roy, Y. Daryanto and M. H. Ali, A Sustainable inventory model with imperfect products, deterioration, and controllable emissions, *Mathematics*, 8 (2020), 2049.

U. Mishra, J.-Z. Wu and M.-L. Tseng, Effects of a hybrid-price-stock dependent demand on the optimal solutions of a deteriorating inventory system and trade credit policy on remanufactured product, *Journal of Cleaner Production*, 241 (2019), 118282.

A. H. M. Mashud, M. Parvin, U. Mishra, M.-L. Tseng and M. K. Lim, A sustainable inventory model with controllable carbon emissions for green-warehouse farms, *Journal of Cleaner Production*, 298 (2021), 126777.

U. Mishra, A. H. M. Mashud, M. L. Tseng and J.-Z. Wu, Optimizing a sustainable supply chain inventory model for controllable deterioration and emission rates in a greenhouse farm, *Mathematics*, 9 (2021), 495.

A. H. M. Mashud, A deteriorating inventory model with different types of demand and fully backlogged shortages, *International Journal of Logistics systems and Management*, 36 (2020), 16–45.

A. H. M. Mashud, H. M. Wee, C. V. Huang and J.-Z. Wu, Optimal replenishment policy for deteriorating products in a newsboy problem with multiple just-in-time deliveries, *Mathematics*, 8 (2020), 1981.

A. Muzaffar, M. N. Malik and S. Deng, Efficacy of retailer rebates and delayed incentives under customer heterogeneity, *RAIRO Oper. Res.*, 55 (2021), 1695–1713.

M. Önal, A. Yenipazarli and O. E. Kundakcioglu, A mathematical model for perishable products with price-and-displayed-stock-dependent demand, *Computers and Industrial Engineering*, 102 (2016), 246–258.
[32] B. O’Neill and S. Sanni, Profit optimisation for deterministic inventory systems with linear cost, Computers and Industrial Engineering, 122 (2018), 303–317.

[33] L. Y. Ouyang, C. H. Ho and C. H. Su, An optimization approach for joint pricing and ordering problem in an integrated inventory system with order-size dependent trade credit, Computers and Industrial Engineering, 57 (2009), 920–930.

[34] M. Shafieezadeh and A. Sadegheih, Developing an integrated inventory management model for multi-item multi-echelon supply chain, International Journal of Advanced Manufacturing Technology, 72 (2014), 1099–1119.

[35] C. K. Sivashankari and S. Panayappan, Production inventory model for two-level production with deteriorative items and shortages, International Journal of Advanced Manufacturing Technology, 76 (2015), 2003–2014.

[36] A. A. Shaikh, A. H. M. Mashud, M. S. Uddin and M. A. A. Khan, Non-instantaneous deterioration inventory model with price and stock dependent demand for fully backlogged shortages under inflation, International Journal of Business Forecasting and Marketing Intelligence, 3 (2017), 152–164.

[37] H. N. Soni and K. A. Patel, Joint pricing and replenishment policies for non-instantaneous deteriorating items with imprecise deterioration free time and credibility constraint, Computers and Industrial Engineering, 66 (2013), 944–951.

[38] S. Srinivasan, K. Pawels, D. M. Hanssens and M. Dekimpe, Do promotions benefit manufacturer, retailers or both?, Management Science, 50 (2004), 617–629.

[39] N. Tashakkor, S. H. Mirmohammadi and M. Iranpoor, Joint optimization of dynamic pricing and replenishment cycle considering variable non-instantaneous deterioration and stock-dependent demand, Computers and Industrial Engineering, 123 (2018), 232–241.

[40] H. Teunter, Analysis of Sales Promotion Effects on Household Purchasing Behavior, ERIM Ph. D. research series in management, Erasmus University, Rotterdam, 2002.

[41] C. Wang and R. Huang, Pricing for seasonal deteriorating products with price- and ramp-type time-dependent demand, Computers and Industrial Engineering, 77 (2014), 29–34.

[42] H. M. Wee, Joint pricing and replenishment policy for deteriorating inventory with declining market, International Journal of Production Economics, 40 (1995), 163–171.

[43] H. M. Wee, A replenishment policy for items with a price-dependent demand and a varying rate of deterioration, Production Planning and Control, 8 (1997), 494–499.

[44] W. K. Wong, J. Qi and S. Y. S. Leung, Coordinating supply chains with sales rebate contracts and vendor-managed inventory, International Journal of Production Economics, 120 (2009), 151–161.

[45] L. Yang and S. Dong, Rebate strategy to stimulate online customer reviews, International Journal of Production Economics, 204 (2018), 99–107.

[46] J. Zhan, X. Chen and Q. Hu, The value of trade credit with rebate contract in a capital-constrained supply chain, International Journal of Production Research, 57 (2019), 379–396.