EQUIVARIANT COHOMOLOGY OF MOMENT-ANGLE
COMPLEXES WITH RESPECT TO COORDINATE SUBTORI

TARAS PANOV AND INDIRA ZEINIKESHEVA

Abstract. We compute the equivariant cohomology $H^*_T(Z_K)$ of moment-
angle complexes $Z_K$ with respect to the action of coordinate subtori $T \subset T^m$. 
We give a criterion for the equivariant formality of $Z_K$ and obtain specifications for the cases of flag complexes and graphs.

1. Introduction

Let $K$ be a simplicial complex on an $m$-element set $V$, and let $Z_K$ be the corresponding moment-angle complex $Z_K$. We study the equivariant cohomology of $Z_K$ with respect to the action of coordinate subtori $T_I \subset T^m$, where $I = \{i_1, \ldots, i_k\} \subset V$.

We construct two commutative integral dga models for $H^*_T(Z_K)$. The first is given by

$$(\Lambda[u_i : i \notin I] \otimes \mathbb{Z}[K], d), \quad du_i = v_i, \quad dv_i = 0,$$

where $\Lambda[u_i : i \notin I]$ is the exterior algebra on degree-one generators $u_i$, $i \notin I$, and $\mathbb{Z}[K]$ is the face ring of $Z_K$. The second dga model $R_I(K)$ is given by the quotient of the first one by the ideal generated by $u_iv_i$ and $v_i^2$ with $i \notin I$. As a result we obtain

**Theorem 3.3.** There are isomorphisms of rings

$$H^*_T(Z_K) \cong H(\Lambda[u_i : i \notin I] \otimes \mathbb{Z}[K], d) \cong H^*(R_I(K))$$

$$\cong \text{Tor}_{\mathbb{Z}[v_1, \ldots, v_m]}(\mathbb{Z}[v_i : i \in I], \mathbb{Z}[K]),$$

where $\mathbb{Z}[v_i : i \in I]$ is the $\mathbb{Z}[v_1, \ldots, v_m]$-module via the homomorphism sending $v_i$ to 0 for $i \notin I$.

When $I = V$, the dga model above reduces to the face ring $\mathbb{Z}[K]$ with zero differential, and we recover the integral formality result of [11].

When $I = \emptyset$, Theorem 3.3 gives the description of the ordinary integral cohomology of $Z_K$ of [2] and [5].

The additive (or $\mathbb{Z}[v_1, \ldots, v_m]$-module) isomorphism

$$H^*_T(Z_K) \cong \text{Tor}_{\mathbb{Z}[v_1, \ldots, v_m]}(\mathbb{Z}[v_i : i \in I], \mathbb{Z}[K]) \cong \text{Tor}_{\mathbb{Z}[v_i : i \notin I]}(\mathbb{Z}, \mathbb{Z}[K])$$

follows from the result of [8].

Next, we study the equivariant formality of $Z_K$, that is, whether $H^*_T(Z_K)$ is a free module over the polynomial ring $H^*_T(pt) = H^*(BT) = \mathbb{Z}[v_i : i \in I]$. We prove

2020 Mathematics Subject Classification. Primary 57S12; Secondary 13F55, 16E45, 55N91, 55R91.

Key words and phrases. moment-angle complex, equivariant cohomology, equivariant formality, graded modules over polynomial rings.

Research of T. Panov was carried out at the Steklov Mathematical Institute and funded by the Russian Science Foundation, grant 20-11-19098.

Research of I. Zeinikesheva was carried out at the Steklov International Mathematical Center and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075-15-2022-265).
Theorem 4.8. Let $\mathcal{K}$ be a simplicial complex on a finite set $V$. The following conditions are equivalent:

(a) For any $I \in \mathcal{K}$, the equivariant cohomology $H^*_I(Z_{\mathcal{K}})$ is a free module over $H^*(BT)$.

(b) There is a partition $V = V_1 \sqcup \cdots \sqcup V_p \sqcup U$ such that
$$\mathcal{K} = \partial \Delta(V_1) \ast \cdots \ast \partial \Delta(V_p) \ast \Delta(U),$$
where $\Delta(U)$ denotes a full simplex on $U$, and $\partial \Delta(V_i)$ denotes the boundary of a simplex on $V_i$.

(c) The rational face ring $\mathbb{Q}[\mathcal{K}]$ is a complete intersection ring (the quotient of the polynomial ring by an ideal generated by a regular sequence).

In the case of flag complexes we have the following specification:

Theorem 4.9. Let $\mathcal{K}$ be a flag complex on $V$. Then the following conditions are equivalent:

(a) $H^*_I(Z_{\mathcal{K}})$ is a free module over $\mathbb{Z}[v_i]$ for all $i$.

(b) $\mathcal{K} = \partial \Delta(V_1) \ast \cdots \ast \partial \Delta(V_p) \ast \Delta(U)$ where $|V_k| = 2$ for $k = 1, \ldots, p$.

A similar criterion holds in the case when $\mathcal{K}$ is a simple graph:

Theorem 4.11. Let $\mathcal{K}$ be a one-dimensional complex (a simple graph). Then the following conditions are equivalent:

(a) $H^*_I(Z_{\mathcal{K}})$ is a free module over $\mathbb{Z}[v_i]$ for any $i$.

(b) $\mathcal{K}$ is the one of the following: $\partial \Delta^2, \partial \Delta^1 \ast \partial \Delta^1, \partial \Delta^1, \Delta^1, \partial \Delta^1 \ast \Delta^0, \Delta^0$.

Along the way we establish some additional properties of the equivariant cohomology of $Z_{\mathcal{K}}$ and give illustrative examples.

2. Preliminaries

Let $\mathcal{K}$ be a simplicial complex on a finite $m$-element set $V$, which we often identify with the index set $[m] = \{1, 2, \ldots, m\}$. We refer to a subset $I = \{i_1, \ldots, i_k\} \subset V$ that is contained in $\mathcal{K}$ as a simplex. We assume that $\varnothing \in \mathcal{K}$ and allow ghost vertices, that is, one-element subsets $\{i\} \subset V$ such that $\{i\} \notin \mathcal{K}$.

Let $(X, A) = \{(X_1, A_1), \ldots, (X_m, A_m)\}$ be a sequence of $m$ pairs of pointed CW-complexes, $A_i \subset X_i$. For each subset $I \subset V$, define
$$(X, A)^I := \{(x_1, \ldots, x_m) \in \prod_{j=1}^m X_j: x_j \in A_j \text{ for } j \notin I\}.$$

The polyhedral product of $(X, A)$ corresponding to $\mathcal{K}$ is
$$(X, A)^{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} (X, A)^I = \bigcup_{I \in \mathcal{K}} \left( \prod_{i \in I} X_i \times \prod_{i \notin I} A_i \right).$$

Using the categorical language, denote by $\text{cat}(\mathcal{K})$ the face category of $\mathcal{K}$, with objects $I \in \mathcal{K}$ and morphisms $I \subset J$. Define the $\text{cat}(\mathcal{K})$-diagram
$$D_{\mathcal{K}}(X, A): \text{cat}(\mathcal{K}) \longrightarrow \text{top}, \quad I \longmapsto (X, A)^I,$$
which maps the morphism $I \subset J$ of $\text{cat}(\mathcal{K})$ to the inclusion of spaces $(X, A)^I \subset (X, A)^J$. Then we have
$$(X, A)^{\mathcal{K}} = \text{colim}_{I \in \mathcal{K}} D_{\mathcal{K}}(X, A) = \text{colim}(X, A)^I.$$

In the case when all the pairs $(X_i, A_i)$ are the same, i.e. $X_i = X$ and $A_i = A$ for $i = 1, \ldots, m$, we use the notation $(X, A)^{\mathcal{K}}$ for $(X, A)^{\mathcal{K}}$.
The moment-angle complex $Z_K$ is the polyhedral product $(D^2, S^1)^K$. We refer to \cite[Chapter 4]{Equivariant Cohomology of Moment-Angle Complexes} for more details and examples.

The face ring of $K$ is the quotient ring $$\mathbb{Z}[K] := \mathbb{Z}[v_1, \ldots, v_m]/I_K,$$
where $I_K$ is the ideal generated by the square-free monomials $v_I = \prod_{i \in I} v_i$ for which $I \subset V$ is not a simplex of $K$.

### 3. Equivariant cohomology

For an action of a topological group $G$ on a space $X$, the Borel construction is $$EG \times_G X := EG \times X/(e \cdot g^{-1}, g \cdot x) \sim (e, x),$$
where $EG$ is the universal right $G$-space, $e \in EG$, $g \in G$, $x \in X$. There is the Borel fibration $EG \times_G X \to BG$ over the classifying space $BG = EG/G$ with fibre $X$.

The equivariant cohomology $I$ of $X$ is

$$H^*_G(X) := H^*(EG \times_G X).$$

The torus $T^m = (S^1)^m$ acts on $Z_K = (D^2, S^1)^K$ coordinatewise. The universal bundle $ES^1 \to BS^1$ is the infinite-dimensional Hopf bundle $S^\infty \to CP^\infty$.

We consider the equivariant cohomology of $Z_K$ with respect to the action of coordinate subtori $T_I = \{(t_1, \ldots, t_m) \in T^m : t_j = 1 \text{ for } j \notin I\}$, where $I = \{i_1, \ldots, i_k\} \subset V$.

**Proposition 3.1.** There is a homotopy equivalence

$$ET_I \times_{T_I} Z_K \cong (Y, B)^K,$$

where

$$Y_i = \begin{cases} \mathbb{C}P^\infty, & i \in I, \\ D^2, & i \notin I, \end{cases} \quad B_i = \begin{cases} pt, & i \in I, \\ S^1, & i \notin I. \end{cases}$$

**Proof.** We have $$ET_I \times_{T_I} Z_K = ET_I \times_{T_I} (D^2, S^1)^K = (X, A)^K,$$
where

$$X_i = \begin{cases} S^\infty \times_{S^1} D^2, & i \in I, \\ D^2, & i \notin I, \end{cases} \quad A_i = \begin{cases} S^\infty \times_{S^1} S^1, & i \in I, \\ S^1, & i \notin I. \end{cases}$$

The result follows from the homotopy equivalence of pairs

$$(S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1) \cong (\mathbb{C}P^\infty, pt),$$

as in \cite[Theorem 4.3.2]{Equivariant Cohomology of Moment-Angle Complexes} where the case $I = \{m\}$ is treated.

Next we introduce two commutative dga models for the equivariant cohomology $H^*_G(Z_K)$. First, consider the dga

$$(\Lambda[u_i : i \notin I] \otimes \mathbb{Z}[K], d), \quad du_i = v_i, \quad dv_i = 0,$$

where $\Lambda[u_i : i \notin I]$ is the exterior algebra on generators indexed by $V - I$. The grading is given by $\deg u_i = 1$, $\deg v_i = 2$.

Second, consider the quotient dga

$$R_I(K) := \Lambda[u_i : i \notin I] \otimes \mathbb{Z}[K]/(u_i v_i = v_i^2 = 0, i \notin I),$$

noting that the ideal generated by $u_i v_i$ and $v_i^2$ with $i \notin I$ is $d$-invariant.

We denote by $C_*(X)$ and $C^*(X)$ the normalised singular chain dg coalgebra and singular cochain dg algebra of a space $X$, respectively. (A singular cochain is...
normalised if it vanishes on degenerate singular simplices [9, §VIII.6]; passing to normalised cochains does not change the quasi-isomorphism type of $C^*(X).$

**Theorem 3.2.** The singular cochain algebra $C^*(ET_1 \times T_2, \mathcal{Z}_K)$ is quasi-isomorphic to $(\Lambda[u_i; i \notin I] \otimes \mathbb{Z}[K], d)$ and $R_I(K)$. The quasi-isomorphisms are natural with respect to inclusion of subcomplexes.

**Proof.** We combine the arguments of [11], [3, §4.5, §8.1] and [6].

The acyclicity of the ideal generated by $v_i^2$ and $u_i u_i$ for $i \notin I$ is treated in the same way as [3, Lemma 3.2.6], where the case $I = \emptyset$ is treated. This gives a quasi-isomorphism $\Lambda[u_i; i \notin I] \otimes \mathbb{Z}[K] \xrightarrow{\sim} R_I(K)$. For the remaining quasi-isomorphism $R_I(K) \simeq C^*(ET_1 \times T_2, \mathcal{Z}_K)$, we use the homotopy equivalent polyhedral product model $(Y, B)^K$ of Proposition 3.1.

Throughout the proof, we use the following zig-zag of quasi-isomorphisms of dgas [10, §7.2]:

\[ \begin{align*}
C^*(X) \otimes C^*(Y) &\xrightarrow{\sim} \text{Hom}(C_*(X) \otimes C_*(Y), \mathbb{Z}) \simeq C^*(X \times Y),
\end{align*} \]

where the arrow on the right is the dual of the Eilenberg–Zilber map $\Lambda[\bigotimes_{i \in I} Z[i]] \rightarrow C_*(X \times Y)$, which is a quasi-isomorphism of dg coalgebras [4, (17.6)].

First consider the case $K = \emptyset$ with $m$ ghost vertices. Then $\mathcal{Z}_K = T^m$ and

\[ ET_1 \times T_2, \mathcal{Z}_K \simeq T^m / T_1 = (Y, B)^K = \prod_{i \in I} S^1, \]

whereas $R_I(K) = \Lambda[u_i; i \notin I]$. There is a quasi-isomorphism $\Lambda[u] = H^*(S^1) \rightarrow C^*(S^1)$ mapping $u$ to its representing singular 1-cocycle (here it is important that we work with normalised cochains). Applying (3.1) we obtain the required quasi-isomorphism $\Lambda[u_i; i \notin I] \simeq C^*(\prod_{i \in I} S^1)$.

Now consider the case $m = 1$ and $K = \Delta^0$, a 0-simplex. If $I = \emptyset$, then $(Y, B)^K = D^2$ and $R_I(K) = \Lambda[u] \otimes \mathbb{Z}[v]/(uv = v^2 = 0)$. Let $\varphi : [01] \rightarrow D^2$ be the standard parametrisation of the boundary circle $S^1$, viewed as a singular 1-simplex. Let $\psi : [012] \rightarrow D^2$ be a singular 2-simplex such that $\psi|_{[12]} = \varphi$ and $\psi|_{[02]} = \psi|_{[01]}$ are constant maps to the basepoint $1 \in S^1$. Then $\partial \varphi = 0$ and $\partial \psi = \varphi$, as we work with the normalised chains. Now if $\alpha \in C^1(D^2)$ is the 1-cocycle dual to $\varphi$ and $\beta \in C^2(D^2)$ is dual to $\psi$, then $\Lambda[u] \otimes \mathbb{Z}[v]/(uv = v^2 = 0) \rightarrow C^*(D^2)$ mapping $u$ to $\alpha$ and $v$ to $\beta$ is a quasi-isomorphism. If $I = \{1\}$, then $(Y, B)^K = \mathbb{C}P^\infty$ and $R_I(K) = \mathbb{Z}[v]$. There is a quasi-isomorphism $\mathbb{C}P^\infty = H^*(\mathbb{C}P^\infty) \rightarrow C^*(\mathbb{C}P^\infty)$ mapping $v$ to its representing singular 2-cocycle.

Next consider the case $K = \Delta^{m-1} = \Delta[m]$, the full simplex on $[m]$. Applying (3.1) and the K"unneth theorem, we obtain a zig-zag of quasi-isomorphisms

\[ \begin{align*}
R_I(\Delta[m]) &= \Lambda[u_i; i \notin I] \otimes \mathbb{Z}[v_1, \ldots, v_m] / (u_i v_i, v_i^2; i \notin I) \\
\Rightarrow \quad \bigotimes_{i \in I} \mathbb{Z}[v_i] &\otimes \bigotimes_{i \notin I} \left( \Lambda[u_i] \otimes \mathbb{Z}[v_i] / (u_i v_i, v_i^2) \right) \xrightarrow{\sim} \bigotimes_{i \in I} C^*(\mathbb{C}P^\infty) \otimes \bigotimes_{i \notin I} C^*(D^2) \\
\Rightarrow \quad \cdots &\rightarrow C^* \left( \prod_{i \in I} \mathbb{C}P^\infty \times \prod_{i \notin I} D^2 \right) = C^*((Y, B)^{\Delta[m]}),
\end{align*} \]

which completes the proof for the case $K = \Delta[m]$.

The general case is proved by induction on the number of simplices in $K$ using the naturality with respect of inclusion of subcomplexes and the Mayer–Vietoris sequence, as in [6, Theorem 1]. Namely, we add simplices one by one to the empty simplicial complex on $[m]$ and use the zig-zag of dga maps between the two short exact sequences for any two simplicial complexes $K_1$ and $K_2$ on $[m]$:  
\[ 0 \rightarrow R_I(K_1 \cup K_2) \rightarrow R_I(K_1) \oplus R_I(K_2) \rightarrow R_I(K_1 \cap K_2) \rightarrow 0 \]
and
\[ 0 \to C^\ast((Y, B)^{K_1 \cup K_2}) \to C^\ast((Y, B)^{K_1}) \oplus C^\ast((Y, B)^{K_2}) \to C^\ast((Y, B)^{K_1 \cap K_2}) \to 0 \]

The zig-zags between the middle and right nonzero terms are quasi-isomorphisms by induction. Then the zig-zag on the left is also a quasi-isomorphism by the cohomology long exact sequence and five lemma.

It may be more illuminating to realise the dgas in question as the limits of dgas corresponding to simplices \( I \in \mathcal{K} \). Namely, given a subset \( J \subset V \), let \( \Delta(J) \) denote a simplex on \( J \), viewed as a simplicial complex on \( V \) (with ghost vertices \( V - J \)). Then
\[
R_I(\Delta(J)) = \Lambda[u_i : i \notin I] \otimes \mathbb{Z}[v_j : j \in J]/(u_j v_j = v_j^2 = 0, j \in J - I),
\]
Consider the \( \text{cat}^{\text{op}}(\mathcal{K}) \)-diagram
\[
\mathcal{R}_{I,K} : \text{cat}^{\text{op}}(\mathcal{K}) \to \text{dga}, \quad J \mapsto R_I(\Delta(J)),
\]
sending a morphism \( J_1 \subset J_2 \) of \( \text{cat}^{\text{op}}(\mathcal{K}) \) to the surjection of dgas \( R_I(\Delta(J_2)) \to R_I(\Delta(J_1)) \). Then
\[
R_I(\mathcal{K}) = \lim_{J \in \mathcal{K}} \mathcal{R}_{I,K} = \lim_{J \in \mathcal{K}} R_I(\Delta(J))
\]
Similarly, we have a \( \text{cat}^{\text{op}}(\mathcal{K}) \)-diagram
\[
\mathcal{C}_{I,K} : \text{cat}^{\text{op}}(\mathcal{K}) \to \text{dga}, \quad J \mapsto C^\ast((Y, B)^J).
\]
The zig-zag of quasi-isomorphisms (3.2) induces an objectwise weak equivalence of diagrams \( \mathcal{R}_{I,K} \simeq \mathcal{C}_{I,K} \). The canonical maps \( \mathcal{R}_{I,K}(J) \to \lim \mathcal{C}_{I,K}|_{\text{cat}^{\text{op}}(\partial \Delta(J))} \) and \( \mathcal{C}_{I,K}(J) \to \lim \mathcal{C}_{I,K}|_{\text{cat}^{\text{op}}(\partial \Delta(J))} \) are fibrations (surjections of dgas). Therefore, both diagrams \( \mathcal{R}_{I,K} \) and \( \mathcal{C}_{I,K} \) are Reedy fibrant (see [3, Appendix C.1]). Their limits are therefore quasi-isomorphic. Thus, we obtain the required zig-zag of quasi-isomorphisms of dgas
\[
R_I(\mathcal{K}) = \lim_{J \in \mathcal{K}} R_I(\Delta(J)) \simeq \lim_{J \in \mathcal{K}} C^\ast((Y, B)^J) \leftarrow C^\ast(\text{colim}_{J \in \mathcal{K}}(Y, B)^J) = C^\ast((Y, B)\mathcal{K}),
\]
where the second-to-last map is a quasi-isomorphism by excision (or by Mayer–Vietoris).

For the equivariant cohomology, we obtain

**Theorem 3.3.** There are isomorphisms of rings
\[
H_T^\ast(Z_{\mathcal{K}}) \cong H^\ast(\Lambda[u_i : i \notin I] \otimes \mathbb{Z}[\mathcal{K}], d) \cong H^\ast(R_I(\mathcal{K}), d)
\]
\[
\cong \text{Tor}_{\mathbb{Z}[v_1, \ldots, v_m]}(Z[v_i : i \in I], \mathbb{Z}[\mathcal{K}]),
\]
where \( Z[v_i : i \in I] \) is the \( \mathbb{Z}[v_1, \ldots, v_m] \)-module via the homomorphism sending \( v_1 \) to 0 for \( i \notin I \).

**Proof.** The first two isomorphisms follow from Theorem 3.2. For the last one, consider the Koszul resolution \( \Lambda[u_i : i \notin I] \otimes \mathbb{Z}[v_i : i \notin I] \to \mathbb{Z} \) of the augmentation \( \mathbb{Z}[v_i : i \notin I] \)-module \( \mathbb{Z} \). Tensoring it with \( \mathbb{Z}[v_i : i \in I] \) we obtain a free resolution of the \( \mathbb{Z}[v_1, \ldots, v_m] \)-module \( \mathbb{Z}[v_i : i \in I] \):
\[
\Lambda[u_i : i \notin I] \otimes \mathbb{Z}[v_1, \ldots, v_m] \to \mathbb{Z}[v_i : i \in I].
\]
Then \( \text{Tor}_{\mathbb{Z}[v_1, \ldots, v_m]}(\mathbb{Z}[v_i : i \in I], \mathbb{Z}[\mathcal{K}]) \) is the cohomology of the complex obtained by applying \( \otimes_{\mathbb{Z}[v_1, \ldots, v_m]}\mathbb{Z}[\mathcal{K}] \) to the resolution above, which gives \( \Lambda[u_i : i \notin I] \otimes \mathbb{Z}[\mathcal{K}] \). \qed
When \( I = [m] \), we obtain that the singular cochain algebra of \( ET^m \times_{T^m} \mathbb{Z}_K \simeq (CP^\infty, pt)^K \) is quasi-isomorphic to \( \mathbb{Z}[K] \) with zero differential, which is the integral formality result of [11].

When \( I = \emptyset \), we obtain the description of the ordinary integral cohomology of \( \mathbb{Z}_K \) of [2] and [5].

4. Equivariant formality

A \( T^K \)-space \( X \) is called *equivariantly formal* if \( H^*_T(X) \) is free as a module over \( H^*_T(pt) = H^*(BT^K) \). The latter condition implies that the spectral sequence of the bundle \( ET \times_{T^m} X \rightarrow BT^K \) collapses at the \( E_2 \) page.

Using the results of the previous section, we obtain that \( \mathbb{Z}_K \) is equivariantly formal with respect to the action of \( T_I \) if \( \text{Tor}_{\mathbb{Z}[v_1, \ldots, v_m]}(\mathbb{Z}[v_1 : i \in I], \mathbb{Z}[K]) \) is free as a module over \( H^*(BT_I) = \mathbb{Z}[v_1 : i \in I] \).

**Lemma 4.1.** Let \( K = \Delta[m] \). Then \( H^*_T(\mathbb{Z}_K) \) is free as a \( H^*(BT_I) \)-module, for any \( I \subset [m] \).

Proof. For \( K = \Delta[m] \), we have \( \mathbb{Z}_K \cong D^m \) is \( T_I \)-equivariantly contractible. Hence, \( H^*_T(\mathbb{Z}_K) \cong H^*_T(pt) = H^*(BT_I) \) is a free \( H^*(BT_I) \)-module.

**Lemma 4.2.** Let \( K = \partial \Delta[m] \), the boundary of a simplex on \([m]\). Then \( \mathbb{Z}_K \cong S^{2m-1} \) and \( H^*_T(\mathbb{Z}_K) \) is free as a \( H^*(BT_I) \)-module, for any \( I \subset [m] \).

Proof. Consider the spectral sequence of the bundle \( ET_I \times_{T_I} \mathbb{Z}_K \rightarrow BT_I \) with fibre \( \mathbb{Z}_K \cong S^{2m-1} \). We claim that the homomorphism \( H^*(ET_I \times_{T_I} \mathbb{Z}_K) \rightarrow H^*(\mathbb{Z}_K) \) induced by the inclusion of the fibre is surjective. Indeed, by the construction of the previous section, \( H^*(ET_I \times_{T_I} \mathbb{Z}_K) \rightarrow H^*(\mathbb{Z}_K) \) is the cohomology homomorphism induced by the dga map

\[
(\Lambda[u_i : i \not\in I] \otimes \mathbb{Z}[K], d) \rightarrow (\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[K], d).
\]

We have \( H^*(\mathbb{Z}_K) = \mathbb{Z}[1, u_1, v_1 \cdot \cdots \cdot v_m], \) where \([u_1, v_1, \ldots, v_m] \in H^{2m-1}(\mathbb{Z}_K)\) denotes the cohomology class of the cocycle \( u_1v_1 \cdot \cdots \cdot v_m \) with \( v_i \) omitted (note that \( \mathbb{Z}[K] = \mathbb{Z}[v_1, \ldots, v_m]/(v_1 \cdot \cdots \cdot v_m) \)). Choosing \( i \not\in I \) we get that \([u_1, v_1, \ldots, v_m] \) also represents a nontrivial cohomology class in \( H^*(ET_I \times_{T_I} \mathbb{Z}_K) \) (here we use the fact that \( I \neq [m] \)). Hence, \( H^*(ET_I \times_{T_I} \mathbb{Z}_K) \rightarrow H^*(\mathbb{Z}_K) \) is surjective.

Now \( H^q(ET_I \times_{T_I} \mathbb{Z}_K) \rightarrow H^q(\mathbb{Z}_K) \) is the edge homomorphism

\[
H^*(ET_I \times_{T_I} \mathbb{Z}_K) \rightarrow E^{0,q}_* \rightarrow E^{0,q}_* = H^q(\mathbb{Z}_K)
\]

of the spectral sequence. Its surjectivity implies \( E^{0,q}_2 = E^{0,q}_\infty \), that is, all the differentials from the first column are trivial. By the multiplicative structure in the spectral sequence, all other differentials are also trivial. We obtain \( H^*(ET_I \times_{T_I} \mathbb{Z}_K) \cong E_\infty = E_2 \cong H^*(BT_I) \otimes H^*(\mathbb{Z}_K) \), a free \( H^*(BT_I) \)-module.

Let \( K_1 \) and \( K_2 \) be simplicial complexes on the sets \( V_1 \) and \( V_2 \), respectively. Their *join* is the simplicial complex on \( V_1 \sqcup V_2 \) given by

\[
K_1 \star K_2 = \{I_1 \sqcup I_2 \subset V_1 \sqcup V_2 : I_1 \subset K_1, \ I_2 \subset K_2\}.
\]

**Lemma 4.3.** Let \( I_1 \subset V_1, \ I_2 \subset V_2, \ V = V_1 \sqcup V_2, \ I = I_1 \sqcup I_2 \) and \( K = K_1 \star K_2 \). Suppose that \( H^*_T(\mathbb{Z}_{K_1}) \) is free as a \( H^*(BT_{I_1}) \)-module, and \( H^*_T(\mathbb{Z}_{K_2}) \) is free as a \( H^*(BT_{I_2}) \)-module. Then \( H^*_T(\mathbb{Z}_K) \) is free as a \( H^*(BT_I) \)-module.

Proof. We have \( \mathbb{Z}_K \cong \mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2} \) by [3, Proposition 4.1.3]. Then

\[
H^*_T(\mathbb{Z}_K) \cong H^*_T(\mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2}) = H^*(ET_I \times_{T_I} (\mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2})) \cong \big( (ET_{I_1} \times_{T_{I_1}} \mathbb{Z}_{K_1}) \times (ET_{I_2} \times_{T_{I_2}} \mathbb{Z}_{K_2}) \big) \\
\cong H^*(ET_{I_1} \times_{T_{I_1}} \mathbb{Z}_{K_1}) \otimes H^*(ET_{I_2} \times_{T_{I_2}} \mathbb{Z}_{K_2}) = H^*_T(\mathbb{Z}_{K_1}) \otimes H^*_T(\mathbb{Z}_{K_2}).
\]
The second-to-last isomorphism above follows by the Künneth formula, because $H^*(BT_1 \times T_1, \mathbb{Z}_K)$ is a free $\mathbb{Z}$-module (as it is a free $\mathbb{Z}[v_i : i \in I_1]$-module). The claim follows, since $H^*(BT_1) = H^*(BT_1) \otimes H^*(BT_1)$.

Lemma 4.4. Let $I \notin K$. Then $H^*_T(\mathbb{Z}_K)$ is not free as a module over $H^*(BT_1)$.

Proof. Take $v_I = \prod_{i \in I} v_i \in H^*(BT_1)$. Then $v_I \cdot 1 = [v_I] = 0$, because $v_I$ represents zero in $H^*_T(\mathbb{Z}_K) = H(\Lambda[u_i : i \notin I] \otimes \mathbb{Z}[K])$. Hence, $1$ is a $H^*(BT_1)$-torsion element, and $H^*_T(\mathbb{Z}_K)$ is not free as a $H^*(BT_1)$-module.

Let $K$ be a simplicial complex on $V$ and $V' \subset V$. The subcomplex $K' = \{ I \in K : I \subset V' \}$ is called a full subcomplex (an induced subcomplex on $V'$). Equivalently, $K' \subset K$ is a full subcomplex if any missing face of $K'$ is a missing face of $K$.

Lemma 4.5. If $H^*_T(\mathbb{Z}_K)$ is a free $H^*(BT_1)$-module and $K'$ is a full subcomplex of $K$ such that $I \notin K'$, then $H^*_T(\mathbb{Z}_{K'})$ is also a free $H^*(BT_1)$-module.

Proof. Since $K'$ is a full subcomplex, there is a retraction $\mathbb{Z}_K \to \mathbb{Z}_K \to \mathbb{Z}_{K'}$ (see [13, Proposition 2.2] or [12, Lemma 4.2]), which is $T_I$-equivariant for any $I \subset V'$. It follows that $H^*_T(\mathbb{Z}_{K'})$ is a direct summand in the free $H^*(BT_1)$-module $H^*_T(\mathbb{Z}_K)$. Hence, $H^*_T(\mathbb{Z}_{K'})$ is also free.

The equivariant cohomology $H^*_T(\mathbb{Z}_K)$ may fail to be free as a $H^*(BT_1)$-module even when $I$ is a simplex of $K$.

Example 4.6. Let $K$ be an $m$-cycle (the boundary of an $m$-gon), with vertices numbered counter-clockwise. Let $I = \{ i \}$, so that $T_I$ is the $i$th coordinate circle $S^1_i$. When $m = 3$ or $m = 4$, $H^*_T(\mathbb{Z}_K)$ is free over $\mathbb{Z}[v_i]$ for all $i$ by Lemma 4.2 and Lemma 4.3. Suppose that $m \geq 5$. Then the nonzero cohomology class in $H^*_T(\mathbb{Z}_K)$ represented by the cocycle $u_1 v_3 \in \Lambda[u_1, \ldots, u_{m-1}] \otimes \mathbb{Z}[K]$ is a $\mathbb{Z}[v_m]$-torsion element. Indeed, $v_m \cdot [u_1 v_3] = [u_1 v_3 v_m] = 0$, since $v_3 v_m = 0$ in $\mathbb{Z}[K]$ for $m \geq 5$. Hence, $H^*_T(\mathbb{Z}_K)$ is not free as a $\mathbb{Z}[v_m]$-module.

Recall that a missing face (a minimal non-face) of a simplicial complex $K$ on $V$ is a subset $I \subset V$ such that $I \notin K$ but every proper subset of $I$ is in $K$. In other words, $I$ is a missing face if $\partial \Delta(I)$ is a subcomplex of $K$, but $\Delta(I)$ is not. We denote by MF($K$) the set of missing faces of $K$.

Generalising Example 4.6, we have

Lemma 4.7. Let $I_1$ and $I_2$ be missing faces of $K$, and suppose that $I = I_1 - I_2$ is nonempty. Then $H^*_T(\mathbb{Z}_K)$ is not free as a module over $H^*(BT_1)$.

Proof. Since $I_1$ and $I_2$ are distinct missing faces, we have $I_2 \subsetneq I_1$. Take $j \in I_2 - I_1$. Then $j \notin I$. The cocycle $u_j v_{I_2 - j}$ represents a nontrivial cohomology class in $H^*_T(\mathbb{Z}_K) = H(\Lambda[u_i : i \notin I] \otimes \mathbb{Z}[K])$.

We claim that the cohomology class $[u_j v_{I_2 - j}] \in H^*_T(\mathbb{Z}_K)$ is a $H^*(BT_1)$-torsion element. Indeed, take $v_I = \prod_{i \in I} v_i \in H^*(BT_1)$. Then

$$v_I \cdot [u_j v_{I_2 - j}] = [u_j v_I v_{I_2 - j}] = [u_j v_{I_2 - I_2}] = 0,$$

since $v_{I_1} = 0$ in $\mathbb{Z}[K]$. Hence, $H^*_T(\mathbb{Z}_K)$ is not free as a $H^*(BT_1)$-module.

Theorem 4.8. Let $K$ be a simplicial complex on a finite set $V$. The following conditions are equivalent:

(a) For any $I \in K$, the equivariant cohomology $H^*_T(\mathbb{Z}_K)$ is a free module over $H^*(BT_1)$.
(b) There is a partition \( V = V_1 \sqcup \cdots \sqcup V_p \sqcup U \) such that
\[
\mathcal{K} = \partial \Delta(V_1) * \cdots * \partial \Delta(V_p) * \Delta(U),
\]
where \( \Delta(U) \) denotes a full simplex on \( U \), and \( \partial \Delta(V_i) \) denotes the boundary of a simplex on \( V_i \).

(c) The rational face ring \( \mathbb{Q}[\mathcal{K}] \) is a complete intersection ring (the quotient of the polynomial ring by an ideal generated by a regular sequence).

Proof. (a) \( \Rightarrow \) (b) We have \( \mathbb{Q}[\mathcal{K}] = \mathbb{Q}[v_1, \ldots, v_m]/(t_1, \ldots, t_p) \) where \( t_k = \prod_{i \in V_k} v_i \) is a square-free monomial and \( V_k \) is a missing face of \( \mathcal{K} \), for \( k = 1, \ldots, p \). Suppose some of these missing faces intersect nontrivially, say, \( V_1 \cap V_2 \neq \emptyset \). Then \( I = V_1 - V_2 \) is nonempty, and \( H_{\mathcal{I}}(\mathbb{Z}[\mathcal{K}]) \) is not a free \( H^*(BT) \)-module by Lemma 4.7. A contradiction. Hence, \( V_1, \ldots, V_p \) are pairwise non-intersecting, so \( \mathcal{K} \) is as described in (b).

(b) \( \Rightarrow \) (a) Write \( I = I_1 \sqcup \cdots \sqcup I_p \sqcup J \), where \( I_k \subseteq V_k \), \( J \subseteq U \). Then \( H_{\mathcal{I}}(\mathbb{Z}[\mathcal{K}]) \) is a free \( H^*(BT) \)-module by Lemmas 4.1, 4.2 and 4.3.

(c) \( \Rightarrow \) (b) Suppose \( \mathbb{Q}[\mathcal{K}] = \mathbb{Q}[v_1, \ldots, v_m]/(t_1, \ldots, t_p) \) where \( (t_1, \ldots, t_p) \) is a regular sequence. We can assume that \( t_k = \prod_{i \in V_k} v_i \) where \( V_k \) is a missing face of \( \mathcal{K} \), for \( k = 1, \ldots, p \). Suppose some of these missing faces intersect nontrivially, say, \( V_1 \cap V_2 \neq \emptyset \). Then \( t_2 \cdot \prod_{i \in V_1 - V_2} v_i = t_1 \cdot \prod_{j \in V_2 - V_1} v_j \), so \( t_2 \) is a zero divisor in \( \mathbb{Q}[v_1, \ldots, v_m]/(t_1) \). A contradiction. Hence, \( V_1, \ldots, V_p \) are pairwise non-intersecting, so \( \mathcal{K} \) is as described in (b).

The equivalence (b) \( \Rightarrow \) (c) of Theorem 4.8 was noted in [11, §5].

Recall that \( \mathcal{K} \) is called a flag complex if each of its missing faces has two vertices. A simplicial complex \( \mathcal{K} \) is flag if and only if it has no ghost vertices and any set of vertices of \( \mathcal{K} \) which are pairwise connected by edges spans a simplex. In the case of flag complexes we have the following specification of the criterion in Theorem 4.8.

**Theorem 4.9.** Let \( \mathcal{K} \) be a flag complex on \( V \). Then the following conditions are equivalent:

(a) \( H^*_{\mathcal{I}}(\mathbb{Z}[\mathcal{K}]) \) is a free module over \( \mathbb{Z}[v_i] \) for all \( i \).

(b) \( \mathcal{K} = \partial \Delta(V_1) * \cdots * \partial \Delta(V_p) * \Delta(U) \) where \( |V_k| = 2 \) for \( k = 1, \ldots, p \).

Proof. Implication (b) \( \Rightarrow \) (a) follows from Theorem 4.8, so we only need to prove (a) \( \Rightarrow \) (b). Let \( V_1, V_2 \) be missing faces. Then \( |V_1| = |V_2| = 2 \). If \( V_1 \cap V_2 \neq \emptyset \), then \( V_1 - V_2 = \{i\} \) for some \( i \in V \). Then \( H^*_{\mathcal{I}}(\mathbb{Z}[\mathcal{K}]) \) is not free as a \( \mathbb{Z}[v_i] \)-module by Lemma 4.7. This contradiction shows that all missing faces of \( \mathcal{K} \) are pairwise non-intersecting, so \( \mathcal{K} \) is as in (b).

Example 4.10. Let \( \mathcal{K} \) be the simplicial complex on 5 vertices with \( \text{MF}(\mathcal{K}) = \{I_1, I_2\} \), where \( I_1 = \{1, 2, 3\} \) and \( I_2 = \{3, 4, 5\} \). Then \( H^*_{\mathcal{I}}(\mathbb{Z}[\mathcal{K}]) \) is not free as a \( H^*(BT) \)-module for \( I = \{1, 2\} \) (or for \( I = \{4, 5\} \)) due to Lemma 4.7. However, \( H^*_{\mathcal{I}}(\mathbb{Z}[\mathcal{K}]) \) is a free \( H^*(BS^1) \)-module for all \( I \). Indeed, it can be shown by the
methods of \cite{7, 8} or \cite{1} that $Z_K \cong S^5 \vee S^5 \vee S^8$. The ordinary cohomology is generated by the following monomials in the Koszul algebra $\Lambda[u_1, \ldots, u_5] \otimes \mathbb{Z}[K]$:

$$H^*(Z_K) = \mathbb{Z}[1, [uv_{i_1-k}], [uv_{i_2-j}], [uv_{i_1}uv_{i_2}v_{i_3-i_4}]],$$

where $k \in I_1, j \in I_2, [uv_{i_1-k}], [uv_{i_2-j}] \in H^5(Z_K), i_1 \in I_1 - I_2, i_2 \in I_2 - I_1$ and $[uv_{i_1}uv_{i_2}v_{i_3-i_4}] \in H^3(Z_K)$.

Since both $I_1 - I_2$ and $I_2 - I_1$ contain two elements, we can choose $k, j, i_1, i_2$ such that $i \notin \{k, j, i_1, i_2\}$. Then the monomials $uv_{i_1-k}, uv_{i_2-j}, u_{i_1}u_{i_2}v_{i_3-i_4}$ represent nontrivial cohomology classes in $H^5(Z_K)$. This implies that the homomorphism $H^5_{S_1}(Z_K) \to H^*(Z_K)$ is surjective. Therefore, the spectral sequence of the bundle $ES_1^1 \times_S Z_K \to BS_1^1$ collapses at the $E_2$ page, as in the proof of Lemma 4.2. It follows that $H^5_{S_1}(Z_K) \cong H^*(BS_1^1) \otimes H^*(Z_K)$, a free $H^*(BS_1^1)$-module.

The equivalence similar to that of Theorem 4.9 also holds when $K$ is one-dimensional.

**Theorem 4.11.** Let $K$ be a one-dimensional complex (a simple graph). Then the following conditions are equivalent:

(a) $H^*_S(Z_K)$ is a free module over $\mathbb{Z}[v_i]$ for any $i$.

(b) $K$ is the one of the following: $\partial \Delta^2, \partial \Delta^1 \ast \partial \Delta^1, \partial \Delta^3, \partial \Delta^1 \ast \Delta^0, \Delta^0$.

**Proof.** Implication (b)⇒(a) follows from Theorem 4.8, so we only need to prove (a)⇒(b). We consider several cases.

Case 1: $K$ is a tree. If it has no more than three vertices, then $K$ is $\Delta^1, \partial \Delta^1 \ast \Delta^0$ or $\Delta^0$. In each of these cases $H^*_S(Z_K)$ is a free $\mathbb{Z}[v_i]$-module by Theorem 4.8.

Suppose $K$ has more than three vertices. Then $K$ has a connected induced subgraph $K_i$ on 4 vertices, which has the form $\begin{array}{ccc}
0 & 1 & 2 \\
1 & 2 & 3
\end{array}$ or $\begin{array}{ccc}
0 & 1 & 2 \\
1 & 2 & 0
\end{array}$. In both cases, there are $I_1, I_2 \in MF(K)$ such that $I_1 - I_2 = \{i\}$ for some $i$. Then $H^*_S(Z_{K_i})$ is not free over $\mathbb{Z}[v_i]$ by Lemma 4.7, and $H^*_S(Z_K)$ is also not free by Lemma 4.5. A contradiction.

Case 2: $K$ is a disjoint union of trees. If $K$ has two vertices, then $K = \partial \Delta^1$.

Suppose $K$ has more than two vertices. Write $K = K_1 \sqcup \cdots \sqcup K_n$ where each $K_j$ is a tree. Then each $K_j$ has at most three vertices by Case 1. Take $I_1 = \{i, j\}$, $I_2 = \{k, j\}$, where $i \in K_1, j \in K_2$ and $\{k, j\} \notin K$. Then $I_1, I_2 \in MF(K)$ and $I_1 - I_2 = \{i\}$. Hence, $H^*_S(Z_K)$ is not a free $\mathbb{Z}[v_i]$-module by Lemma 4.7. A contradiction.

Case 3: $K$ has a 3-cycle. If $K$ is a 3-cycle, then $K = \partial \Delta^2$.

Suppose $K$ has at least 4 vertices. Consider the induced subgraph on 4 vertices containing a 3-cycle. There are four cases:

![Graphs showing different cases](image)

In the first two cases, take $I_1 = \{3, 4\}, I_2 = \{2, 4\} \in MF(K)$. In the last two cases, take $I_1 = \{1, 2, 3\}$ and $I_2 = \{1, 2, 4\} \in MF(K)$. Then $I_1 - I_2 = \{3\}$ and $H^*_S(Z_K)$ is not a free $\mathbb{Z}[v_3]$-module by Lemma 4.7. A contradiction.

Case 4: $K$ has no 3-cycles and has a 4-cycle. If $K$ is a 4-cycle, then $K = \partial \Delta^1 \ast \partial \Delta^1$.
Suppose $K$ has more than 4 vertices. Consider the induced subgraph on 5 vertices containing a 4-cycle. Since there are no 3-cycles, there are three cases:

In all cases take $I_1 = \{2, 4\}$, $I_2 = \{4, 5\}$ in $\text{MF}(K)$, then $I_1 - I_2 = \{2\}$ and $H^*_S(\mathbb{Z}_K)$ is not a free $\mathbb{Z}[v_2]$-module by Lemma 4.7. A contradiction again.

Case 5: each minimal cycle in $K$ has length at least 5. Then $K$ has an induced subgraph $K_1$ which is an $m$-cycle with $m \geq 5$. As in Example 4.6, we have that $H^*_S(\mathbb{Z}_{K_1})$ is not free as a $\mathbb{Z}[v_m]$-module. So $H^*_S(\mathbb{Z}_K)$ is also not free by Lemma 4.5. A contradiction. □

References

[1] Abramyan, Semyon; Panov, Taras. Higher Whitehead products in moment-angle complexes and substitution of simplicial complexes. Tr. Mat. Inst. Steklova 305 (2019), 7–28 (Russian); Proc. Steklov Inst. Math. 305 (2019), no. 1, 1–21 (English translation).

[2] Baskakov, Ilia; Buchstaber, Victor; Panov, Taras. Cellular cochain algebras and torus actions. Uspekhi Mat. Nauk 59 (2004), no. 3, 159–160 (Russian); Russian Math. Surveys 59 (2004), no. 3, 562–563 (English translation).

[3] Buchstaber, Victor; Panov, Taras. Toric topology. Mathematical Surveys and Monographs, vol. 204, American Mathematical Society, Providence, RI, 2015.

[4] Eilenberg, Samuel; Moore, John C. Homology and fibrations. I. Coalgebras, cotensor product and its derived functors. Comment. Math. Helv. 40 (1966), 199–236.

[5] Franz, Matthias. On the integral cohomology of smooth toric varieties. Proc. Steklov Inst. Math. 252 (2006), 53–62.

[6] Franz, Matthias. Dga models for moment-angle complexes. Preprint (2020); arXiv:2006.01571.

[7] Gribić, Jelena; Theriault, Stephen. Homotopy theory in toric topology. Uspekhi Mat. Nauk 71 (2016), no. 2, 3–80; Russian Math. Surveys 71 (2016), no. 2, 185–251 (English).

[8] Luo, Shisen; Matsumura, Tomoo; Moore, W. Frank. Moment angle complexes and big Cohen–Macaulayness. Algebr. Geom. Topol. 14 (2014), no. 1, 379–406.

[9] Mac Lane, Saunders. Homology. Springer, Berlin, 1963.

[10] McCleary, John. A user’s guide to spectral sequences. Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001.

[11] Notbohm, Dietrich; Ray, Nigel. On Davis–Januszkiewicz homotopy types. I. Formality and rationalisation. Algebr. Geom. Topol. 5 (2005), 31–51.

[12] Panov, Taras; Theriault, Stephen. The homotopy theory of polyhedral products associated with flag complexes. Compositio Math. 155 (2019), no. 1, 206–228.

[13] Panov, Taras; Veryovkin, Yakov. Polyhedral products and commutator subgroups of right-angled Artin and Coxeter groups. Mat. Sbornik 207 (2016), no. 11, 105–126 (Russian); Sbornik Math. 207 (2016), no. 11, 1582–1600 (English translation).

Faculty of Mathematics and Mechanics, Lomonosov Moscow State University, Russia;
Faculty of Computer Science, National Research University Higher School of Economics, Moscow, Russia;
Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia;
Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia
Email address: tpanov@mech.math.msu.su

Faculty of Mathematics and Mechanics, Lomonosov Moscow State University, Russia;
Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia
Email address: znikzk@gmail.com