A note on the distribution of Skew Brownian motion with dry friction

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Abstract

This note concerns distributions of Skew Brownian motion with dry friction and its occupation time. These distributions were obtained in [2] by using the Laplace transform and joint characteristic functions. We provide an alternative approach to deriving these distributions. Our approach is based on using the results for Skew Brownian motion obtained in [3].

Keywords: Skew Brownian motion, Caughey-Dienes process, local time, occupation time

1 Introduction

Let $X_t = (X_t, t \geq 0)$ be a continuous time stochastic process defined as the solution of the following stochastic differential equation (SDE)

$$X_t = X_0 + \int_0^t m(X_s) ds + (2p - 1)L_t^{(0)} + W_t,$$

where $p \in (0, 1)$,

$$m(x) = m_1 1_{(x \geq 0)} + m_2 1_{(x < 0)}, x \in \mathbb{R},$$

for some constants $m_1$ and $m_2$,

$$L_t^{(0)} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{(x-\varepsilon \leq X_u \leq x+\varepsilon)} du$$

is the local time of the process at zero, and $(W_t, t \geq 0)$ is standard Brownian motion. The process $X_t$ is well known and widely used in many applications. If $m_1 = m_2 = 0$, then it is a driftless Skew Brownian motion (SBM) with parameter $p$ (e.g. see [1], [4] and references therein). The process $X_t$ naturally appears in [3] in the study of the two-valued local volatility model in finance. In the case $m_2 = -m_1 = m$ the process $X_t$ is known as the skew Caughey-Dienes process, or SBM with dry friction (e.g., see [2] and references therein). In [2] they derived

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the probability density function (pdf) of the skew Caughey-Dienes process and its occupation
time on the non-negative half-line. Their approach is based on using the Laplace transform
and joint characteristic functions, which require rather heavy computations. In this note, we
show how these distributions can be alternatively obtained by using the results of [3] for SBM.

2 Results

The distribution of the skew Caughey-Dienes process In the special case $m_2 = -m_1 = m$
equation (1) is as follows
\[ dX_t = -m \cdot \text{sgn}(X_t) + (2p - 1)dL_t^{(0)} + dW_t. \] (4)

Theorem 1 below concerns the distribution of the solution $X_t = (X_t, t \geq 0)$ of equation (4).

**Theorem 1.** Let $X_t = (X_t, t \geq 0)$ be the solution of equation (4). If $X_0 = 0$, then given $T > 0$
the pdf of $X_T$ is
\[ \phi_T(x) = \begin{cases} 2p \left( \frac{e^{-\frac{(mT + x)^2}{2\pi T}}}{\sqrt{2\pi T}} + \frac{m}{2} e^{-2mx} \left[ 1 + \text{Erf} \left( \frac{mT - x}{\sqrt{2T}} \right) \right] \right), & \text{if } x \geq 0, \\ 2q \left( \frac{e^{-\frac{(-mT + x)^2}{2\pi T}}}{\sqrt{2\pi T}} + \frac{m}{2} e^{2mx} \left[ 1 + \text{Erf} \left( \frac{mT + x}{\sqrt{2T}} \right) \right] \right), & \text{if } x < 0, \end{cases} \] (5)
where $q = 1 - p$ and Erf$(z)$ is the standard error function.

**Proof.** Fix $T > 0$ and define the following quantities
\[ \tau_0 = \min \{ t : X_t = 0 \}, \]
\[ \tau = \max \{ t \in (0, T] : X_t = 0 \}, \]
\[ V = \int_{\tau_0}^{\tau} 1_{\{X_t \geq 0\}} dt, \]
and the function
\[ \psi_{p,T}(t, v, x, l) = \begin{cases} 2p \cdot h(v, lp)h(t - v, lq)h(T - t, x), & \text{if } x \geq 0, \\ 2q \cdot h(v, lp)h(t - v, lq)h(T - t, x), & \text{if } x < 0, \end{cases} \] (9)
for $0 \leq v \leq t \leq T, l \geq 0$, where
\[ h(s, y) = \frac{|y|}{\sqrt{2\pi s^3}} e^{-\frac{y^2}{2s}}, \quad y \in \mathbb{R}, s \in \mathbb{R}_+, \] (10)
is the probability density function of the first passage time to zero of the standard BM starting
at $y$. It was shown in [3, Theorem 2] that, if $X_0 = 0$, then the joint density of $(\tau, V, X_T, L_T^{(0)})$
is given by
\[ \phi_T(t, v, x, l) = \psi_{p,T}(t, v, x, l) e^{-\frac{m_1 v + m_2 (mT + x)}{2} - l(m_1 p - qm_2 - m)x}, \] (11)
where \( \psi_{p,T}(t, v, z, l) \) is defined by equation (9). If \( m_1 = -m_2 = -m \), then the function (11) simplifies as follows

\[
\phi_T(t, v, x, l) = \psi_{p,T}(t, v, x, l) e^{-\frac{m^2}{2(t-v)} - m x \cdot \text{sgn}(x) + l m}
\]

\[
\phi_T(t, v, x, l) = \begin{cases} 
2p \cdot h(v, lp) h(t - v, lq) h(T - t, x) e^{-\frac{m^2}{2} - m x + l m}, & \text{if } x \geq 0, \\
2q \cdot h(v, lp) h(t - v, lq) h(T - t, x) e^{-\frac{m^2}{2} + m x + l m}, & \text{if } x < 0,
\end{cases}
\]

for \( 0 \leq v \leq t \leq T, l \geq 0 \). Using the convolution property of the hitting times of Brownian motion and the fact that \( p + q = 1 \), we have that

\[
\int_0^t h(v, lp) h(t - v, lq) dv = h(t, l),
\]

which gives the joint density of \( (\tau, X_T, L_T^{(0)}) \), namely,

\[
\phi_T(t, x, l) = \int_0^t \phi_T(t, v, x, l) dv = \begin{cases} 
2p \cdot h(t, l) h(T - t, x) e^{-\frac{m^2}{2} - m x + l m}, & \text{if } x \geq 0, \\
2q \cdot h(t, l) h(T - t, x) e^{-\frac{m^2}{2} + m x + l m}, & \text{if } x \geq 0,
\end{cases}
\]

Using the convolution property of the hitting times one more time gives that

\[
\int_0^T h(t, l) h(T - t, x) dt = h(T, l + |x|).
\]

Therefore, the joint density of \( X_T \) and \( L_T^{(0)} \) is

\[
\phi_T(x, l) = \begin{cases} 
2p \cdot h(T, l + x) e^{-\frac{m^2}{2} - m x + l m}, & \text{if } x \geq 0, \\
2q \cdot h(T, l - x) e^{-\frac{m^2}{2} + m x + l m}, & \text{if } x < 0,
\end{cases}
\]

It is left to integrate out the local time in order to obtain the pdf of \( X_T \). Integration gives that

\[
\int_0^\infty \phi_T(x, l) dl = \phi_T(x),
\]

where \( \phi_T(x) \) is the function defined in (5), as claimed.

**The distribution of the occupation time** Let \( X_t = (X_t, t \geq 0) \) be the solution of equation (1). Given \( T > 0 \) define

\[
U = \int_0^T 1_{\{X_t \geq 0\}} dt,
\]

i.e. \( U \) is the occupation time of the non-negative half-line during the time period \( [0, T] \) (the occupation time). In [2] the pdf of the occupation time is expressed in term of a double integral of a rather complicated function. We show that this pdf can be obtained as an integral of a function of one variable, which is explicitly expressed in terms of the complementary error function \( \text{Erfc}(z) = 1 - \text{Erf}(z) \).
It is easy to see that if $X_0 = 0$, then $U = V + T - \tau$, if $X_T \geq 0$, and $U = V$, if $X_T < 0$, where quantities $\tau$ and $V$ are defined in (7) and (8) respectively. This immediately gives \[\text{equation (15)}\] for the joint density $\varphi_T(t, u, x, l)$ of $(\tau, U, X_T, L_T(0))$, which is as follows

$$\varphi_T(t, u, x, l) = \begin{cases} 
2p \cdot h(u + t - T, lp)h(T - u, lq)h(T - t, x)e^{-\frac{m^2\tau}{2} - x^2 + lm}, 
& \text{if } x \geq 0, \ l > 0, \ \text{and } t \leq T, \ T - t \leq u \leq T, \\
2q \cdot h(u, lp)h(t - u, lq)h(T - t, x)e^{-\frac{m^2\tau}{2} + x^2 + lm}, 
& \text{if } x < 0, \ l > 0, \ \text{and } 0 \leq u \leq t \leq T.
\end{cases} \tag{16}$$

Using that

$$\int_{u}^{T} h(u - T + t, lp)h(T - t, x)dt = h(u, lp + x) \text{ for } x \geq 0,$$
$$\int_{u}^{T} h(t - u, lq)h(T - t, x)dt = h(T - u, lq + |x|) \text{ for } x < 0,$$

we obtain the joint density $\varphi_T(u, x, l)$ of $(U, X_T, L_T(0))$. Integrating over the variable $x$ gives the joint density of the occupation time $U$ and the local time $L_T(0)$, that is

$$\varphi_T(u, l) = 2e^{-\frac{m^2\tau}{2} + lm} \left( pF(u, l, p) + qF(T - u, l, q) \right), \tag{17}$$

where

$$F(y, l, c) = e^{-\frac{y^2}{2}} \frac{m}{\sqrt{2\pi y}} \text{Erfc} \left[ \frac{l - my + \sqrt{2y}}{\sqrt{2y}} \right] e^{lmc + \frac{m^2}{2}y}. \tag{18}$$

Thus, the pdf of the occupation time is given by the integral $\varphi_T(u) = \int_{0}^{\infty} \varphi_T(u, l)dl$, where the function $\varphi_T(u, l)$ is explicitly expressed in terms of the complementary error function, as claimed.

**References**

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