EXPONENTIAL LOCALIZATION OF HYDROGEN-LIKE ATOMS IN RELATIVISTIC QUANTUM ELECTRODYNAMICS

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Abstract. We consider two different models of a hydrogenic atom in a quantized electromagnetic field that treat the electron relativistically. The first one is a no-pair model in the free picture, the second one is given by the semi-relativistic Pauli-Fierz Hamiltonian. We prove that the no-pair operator is semi-bounded below and that its spectral subspaces corresponding to energies below the ionization threshold are exponentially localized. Both results hold true, for arbitrary values of the fine-structure constant, \(e^2\), and the ultra-violet cut-off, \(\Lambda\), and for all nuclear charges less than the critical charge without radiation field, \(Z_c = e^{-2}/(2/\pi + \pi/2)\). We obtain similar results for the semi-relativistic Pauli-Fierz operator, again for all values of \(e^2\) and \(\Lambda\) and for nuclear charges less than \(e^{-2}/\pi\).

1. Introduction

The existence of ground states of atoms and molecules described in the framework of non-relativistic quantum electrodynamics (QED) has been intensively studied in the past ten years. The first existence proofs have been given in [6, 8] for small values of the involved physical parameters, namely the fine-structure constant, \(e^2\), and the ultra-violet cut-off, \(\Lambda\). In [17] the existence of ground states for the Pauli-Fierz Hamiltonian has been established for arbitrary values of \(e^2\) and \(\Lambda\) assuming a certain binding condition which has been verified later on in [10] for helium-like atoms and in [20] for an arbitrary number of electrons. Moreover, infra-red finite algorithms and renormalization group methods have been applied to various models of non-relativistic QED to study their ground state energies and projections [7, 6, 7, 4, 5, 9, 15]. A question which arises naturally in this context is whether these results still hold true when the electrons are described by a relativistic operator. The aim of the present paper is to take one step forward in this direction. We study two different models that seem to be natural candidates for a mathematical analysis: The first one is given by...
the following no-pair operator,

\[ P_A^+ \left( D_A - \frac{\gamma}{|x|} + H_f \right) P_A^+. \]

Here \( D_A \) is the free Dirac operator minimally coupled to the quantized, ultraviolet cut-off vector potential, \( A \). (The symbol \( \hat{A} \) includes the fine-structure constant \( e^2 \).) \( \gamma \geq 0 \) is a coupling constant, \( H_f \) is the radiation field energy, and \( P_A^+ \) the spectral projection onto the positive spectral subspace of \( D_A \). The latter choice of projection is referred to as the free picture. The no-pair operator is thus acting on a projected Hilbert space where the electron and photon degrees of freedom are always linked together. The mathematical analysis of the analogue of this operator for molecules has been initiated in [25] where the stability of the second kind is shown under certain restrictions on \( e^2, \Lambda, \) and the nuclear charges. Moreover, in [24] the (positive) binding energy is estimated from above. There are numerous mathematical contributions on no-pair models where magnetic fields are not taken into account or treated classically; see, e.g., [30] for a list of references and also for a different choice of the projections. We remark that it is essential that the vector potential is included in the projection determining the no-pair model. For if \( P_A^+ \) is replaced by \( P_0^+ \) then the analogue of (1.1) describing \( N \) interacting electrons becomes unstable as soon as \( N \geq 2 \) [18, 25, 28]. Moreover, the operator in (1.1) is formally gauge invariant and this would not hold true anymore with \( P_0^+ \) in place of \( P_A^+ \). Gauge invariance plays, however, an important role in the proof of the existence of ground states as it permits to derive bounds on the number of soft photons. In fact, employing a mild infra-red regularization it is possible to prove the existence of ground states for the operator in (1.1) with \( P_A^+ \) replaced by \( P_0^+ \) [29, 22]. It seems, however, unlikely that the infra-red regularization can be dropped in this case [22].

The second operator studied in this article, the semi-relativistic Pauli-Fierz operator, is given as

\[ \sqrt{\vec{\sigma} \cdot (-i \nabla + \mathbf{A})}^2 + \frac{\gamma}{|x|} + H_f, \]

where \( \vec{\sigma} \) is a vector containing the Pauli spin matrices. For \( \gamma = 0 \) the fiber decomposition with respect to different values of the total momentum of this operator has been studied recently in [32]. Furthermore, it is remarked in [32] that for \( \gamma > 0 \), all eigenvalues of the operator in (1.2) are at least doubly degenerate since it anti-commutes with the time-reversal operator.

Typically, proving the existence of ground states in QED requires some information on the localization of low-lying spectral subspaces or at least of certain approximate ground state eigenfunctions. Here localization is understood with respect to the electronic degrees of freedom. In this paper we establish this
prerequisite for both models mentioned above by proving that spectral projectors corresponding to energies below the ionization thresholds are still bounded when multiplied with suitable exponential weight functions acting on the electron coordinates. These results hold true for all values of the fine-structure constant $e^2$ and the ultra-violet cut-off $\Lambda$, and for all coupling constants $\gamma$ below the critical values without quantized fields. That is, for $\gamma \in (0, 2/(2/\pi + \pi/2))$ in the case of the no-pair operator [4], and for $\gamma \in (0, 2/\pi)$ in the case of the semi-relativistic Pauli-Fierz operator. The ionization thresholds are defined as the infima of the spectra of the operators with $\gamma = 0$. Of course, our localization estimates are non-trivial only if the infima of the spectra for $\gamma > 0$ lie strictly below the ionization thresholds. In the present paper we verify this binding condition for sufficiently small values of $e^2$ and/or $\Lambda$. In fact, this perturbative result is a straightforward consequence of some of our technical lemmata. We remark that up to now it has actually not been known that the quadratic forms of both operators treated here are semi-bounded below when $\gamma$ varies in the parameter ranges given above and $e^2$ and $\Lambda$ are arbitrary. The proof of this is our first main result. For the semi-relativistic Pauli-Fierz operator we prove the semi-boundedness also in the critical case $\gamma = 2/\pi$. Moreover, the relation which determines the exponential decay rates, $a > 0$, of the semi-relativistic Pauli-Fierz operator in terms of the ionization threshold does not depend on $e^2$ and $\Lambda$ either. We have, however, to content ourselves with suboptimal estimates on $a$ because of technical reasons. In the case of the no-pair operator we find a relation between $a$ and the ionization threshold which does depend on $e^2$ and $\Lambda$ and it seems to be difficult to avoid this. In fact, what complicates the analysis of both models is the non-locality of the corresponding Hamiltonians. In this respect the no-pair operator is harder to analyze since also the potential and radiation field energy become non-local. In order to deal with this we derive various estimates on commutators involving the spectral projection $P_{\Lambda}^+$, exponential weights, and cut-off functions. We already obtained similar bounds for spectral projections of Dirac operators in classical magnetic fields in [31, 30]. However, since we are now dealing with quantized fields we additionally have to study commutators involving the quantized field energy.

We remark that the ionization threshold is expected to coincide with the energy value separating exponentially localized spectral subspaces from non-localized ones which requires also an upper bound on the energy of localized states. In non-relativistic QED this picture has been established in [16] again for arbitrary values of $e^2$ and $\Lambda$.

We further remark that the existence of ground states in a relativistic model describing both the photons and the electrons and positrons by quantized fields has been studied in [11, 12]. To this end infra-red and ultra-violet cut-offs for
the momenta of all involved particles are imposed in the interaction part of the Hamiltonian considered in \[11, 12\].

Finally, we would like to announce that this work will be continued by M. Könenberg and the present authors in [23] where the existence of ground states is established for both models treated in the present article.

This article is organized as follows. In the subsequent section we introduce the no-pair and semi-relativistic Pauli-Fierz operators and state our main results precisely. Section 3 provides various technical ingredients, for instance commutator estimates that describe the non-local properties of $P^+_A$. In Section 4 we prove the semi-boundedness for both models and, finally, in Section 5 we prove the exponential localization. The main text is followed by an appendix where we derive simple perturbative estimates on the ionization thresholds and ground state energies for small $e^2$ and/or $\Lambda$.

## 2. Definition of the Models and Main Results

In order to introduce the models treated in this article more precisely we first fix our notation and recall some standard facts. The state space of the quantized photon field is the bosonic Fock space,

$$
\mathcal{F}_b[\mathcal{H}] := \bigoplus_{n=0}^{\infty} \mathcal{F}_b^{(n)}[\mathcal{H}] \ni \psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots).
$$

It is modeled over the one photon Hilbert space

$$
\mathcal{H} := L^2(A \times \mathbb{Z}_2, dk), \quad \int dk := \sum_{\lambda \in \mathbb{Z}_2} \int_A d^3k.
$$

We assume that $A$ is $\mathbb{R}^3$ or $\mathbb{R}^3$ with a ball about the origin removed since these are the examples we encounter in [23]. $k = (k, \lambda)$ denotes a tupel consisting of a photon wave vector, $k \in A$, and a polarization label, $\lambda \in \mathbb{Z}_2$. Moreover, $\mathcal{F}_b^{(0)}[\mathcal{H}] := \mathbb{C}$ and $\mathcal{F}_b^{(n)}[\mathcal{H}] := L^2_s((A \times \mathbb{Z}_2)^n)$ is the subspace of all complex-valued, square integrable functions on $(A \times \mathbb{Z}_2)^n$ that remain invariant under permutations of the $n \in \mathbb{N}$ wave vector/polarization tupels. As usual we denote the vacuum vector by $\Omega := (1, 0, 0, \ldots) \in \mathcal{F}_b[\mathcal{H}]$. Many calculations will be performed on the following dense subspace of $\mathcal{F}_b[\mathcal{H}]$,

$$
\mathcal{C}_0 := \mathbb{C} \oplus \bigoplus_{n \in \mathbb{N}} C_0((A \times \mathbb{Z}_2)^n) \cap L^2_s((A \times \mathbb{Z}_2)^n). \quad (Algebraic direct sum.)
$$
The free field energy of the photons is the self-adjoint operator given by
\[
\mathcal{D}(H_f) := \left\{ (\psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}_b[H] : \sum_{n=1}^{\infty} \int \left| \sum_{j=1}^{n} \omega(k_j) \psi^{(n)}(k_1, \ldots, k_n) \right|^2 dk_1 \ldots dk_n < \infty \right\},
\]
and, for \( \psi \in \mathcal{D}(H_f) \),
\[
(H_f \psi)^{(0)} = 0, \quad (H_f \psi)^{(n)}(k_1, \ldots, k_n) = \sum_{j=1}^{n} \omega(k_j) \psi^{(n)}(k_1, \ldots, k_n), \quad n \in \mathbb{N}.
\]
Here the dispersion relation \( \mathcal{A} \times \mathbb{Z}_2 \ni k \mapsto \omega(k) \), depends only on \( k \) and not on \( \lambda \in \mathbb{Z}_2 \). Its precise form is not important in this paper. It could be any positive, polynomially bounded, measurable function. For definiteness we assume that \( 0 \leq \omega(k) \leq |k|, k \in \mathcal{A} \times \mathbb{Z}_2 \), since this is sufficient to apply our results in [23].

By symmetry and Fubini’s theorem
\[
(2.1) \quad \langle H_f^{1/2} \phi \mid H_f^{1/2} \psi \rangle = \int \omega(k) \langle a(k) \phi \mid a(k) \psi \rangle \, dk, \quad \phi, \psi \in \mathcal{D}(H_f^{1/2}),
\]
where \( a(k) \) annihilates a photon with wave vector/polarization \( k \),
\[
(a(k) \psi)^{(n)}(k_1, \ldots, k_n) = (n + 1)^{1/2} \psi^{(n+1)}(k, k_1, \ldots, k_n), \quad n \in \mathbb{N}_0,
\]
almost everywhere, and \( a(k) \Omega = 0 \). We further recall that the creation and the annihilation operators of a photon state \( f \in \mathcal{H} \) are given by
\[
(a^\dagger(f) \psi)^{(n)}(k_1, \ldots, k_n) = n^{-1/2} \sum_{j=1}^{n} f(k_j) \psi^{(n-1)}(\ldots, k_{j-1}, k_{j+1}, \ldots), \quad n \in \mathbb{N},
\]
\[
(a(f) \psi)^{(n)}(k_1, \ldots, k_n) = (n + 1)^{1/2} \int \mathcal{F}(k) \psi^{(n+1)}(k, k_1, \ldots, k_n) \, dk, \quad n \in \mathbb{N}_0,
\]
and \( (a^\dagger(f) \psi)^{(0)} = 0, a(f) \Omega = 0 \). We define \( a^\dagger(f) \) and \( a(f) \) on their maximal domains. The following canonical commutation relations hold true on \( \mathcal{C}_0 \),
\[
[a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0, \quad [a(f), a^\dagger(g)] = \langle f \mid g \rangle \mathbb{1},
\]
where \( f, g \in \mathcal{H} \). Moreover, we have \( \langle a(f) \phi \mid \psi \rangle = \langle \phi \mid a^\dagger(f) \psi \rangle \), \( \phi, \psi \in \mathcal{C}_0 \), and, by definition, \( a(f) \phi = \int \mathcal{F}(k) a(k) \phi \, dk, \phi \in \mathcal{C}_0 \).

Next, we describe the interaction between four-spinors and the photon field. The full Hilbert space containing all electron/positron and photon degrees of freedom is
\[
\mathcal{H} := L^2(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathcal{F}_b[H].
\]
It contains the dense subspace,
\[
\mathcal{D}_0 := C_0^\infty(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathcal{C}_0. \quad (\text{Algebraic tensor product})
\]
We consider general form factors fulfilling the following condition:
Hypothesis 1. For every \( k \in (\mathcal{A} \setminus \{0\}) \times \mathbb{Z}_2 \) and \( j \in \{1, 2, 3\} \), \( G^{(j)}(k) \) is a bounded continuously differentiable function, \( \mathbb{R}^3 \ni x \mapsto G^{(j)}_x(k) \), such that

\[
\int_{\omega(k)^\ell} \|G(k)\|_\infty^2 \, dk \leq d_\ell^2, \quad \ell \in \{-1, 0, 1\},
\]

and

\[
\int_{\omega(k)^{-1}} \|\nabla G(k)\|_\infty^2 \, dk \leq d_1^2,
\]

for some \( d_{-1}, \ldots, d_2 \in (0, \infty) \). Here \( G_x(k) = (G^{(1)}_x(k), G^{(2)}_x(k), G^{(3)}_x(k)) \) and \( \|G(k)\|_\infty := \sup_x |G_x(k)| \).

Although we are interested in the specific physical situation described in the following example we work with the more general hypothesis above since in our future applications we shall encounter truncated and discretized versions of the vector potential and the field energy. It will then be necessary to know that the results of the present article hold uniformly in the truncation and discretization and Hypothesis 1 is a convenient way to handle this.

Example. In the physical models we are interested in we have

\[
G^{e^2, \Lambda}_x(k) := e^2 \frac{\mathbb{1}_{\{|k| \leq \Lambda\}}}{\pi \sqrt{2|k|}} e^{-ik \cdot x} \varepsilon(k),
\]

for \( (x, k) \in \mathbb{R}^3 \times (\mathbb{R}^3 \times \mathbb{Z}_2) \) with \( k \neq 0 \). Here energies are measured in units of \( mc^2 \), \( m \) denoting the rest mass of an electron and \( c \) the speed of light. Length, i.e. \( x \), are measured in units of \( \hbar/(mc) \), which is the Compton wave length divided by \( 2\pi \). \( \hbar \) is Planck’s constant divided by \( 2\pi \). The photon wave vectors \( k \) are measured in units of \( 2\pi \) times the inverse Compton wavelength, \( mc/\hbar \). The parameter \( \Lambda > 0 \) is an ultraviolet cut-off and \( e^2 \approx 1/137 \) denotes Sommerfeld’s fine-structure constant which equals the square of the elementary charge in our units. One could equally well impose a smooth ultra-violet cut-off. The polarization vectors, \( \varepsilon(k, \lambda), \lambda \in \mathbb{Z}_2 \), are homogeneous of degree zero in \( k \) such that \( \{k, \varepsilon(k, 0), \varepsilon(k, 1)\} \) is an orthonormal basis of \( \mathbb{R}^3 \), for every \( k \in S^2 \). This corresponds to the Coulomb gauge. (In particular, the vector fields \( S^2 \ni k \mapsto \varepsilon(k, \lambda) \) cannot be continuous; see [27] for more information on the choice of \( \varepsilon \)).

Finally, we introduce the self-adjoint Dirac matrices \( \alpha_1, \alpha_2, \alpha_3, \) and \( \beta \) that act on the four spinor components of an element from \( \mathcal{H} \). They are given by

\[
\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j \in \{1, 2, 3\}, \quad \beta := \alpha_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
where $\sigma_1, \sigma_2, \sigma_3$ denote the standard Pauli matrices, and fulfill the Clifford algebra relations

\begin{equation}
\alpha_i \alpha_j + \alpha_j \alpha_i = 2 \delta_{ij} \mathbb{1}, \quad i, j \in \{0, 1, 2, 3\}.
\end{equation}

The interaction between the electron/positron and photon degrees of freedom is now given as

$$
\mathbf{\alpha} \cdot \mathbf{A} \equiv \mathbf{\alpha} \cdot \mathbf{A}(\mathbf{G}) := \mathbf{\alpha} \cdot \mathbf{a}^\dagger(\mathbf{G}) + \mathbf{\alpha} \cdot \mathbf{a}(\mathbf{G}), \quad \mathbf{\alpha} \cdot \mathbf{a}^\dagger(\mathbf{G}) := \sum_{j=1}^{3} \alpha_j \mathbf{a}^\dagger(G^{(j)}_x),
$$

where $\mathbf{a}^\dagger$ is $a$ or $a^\dagger$. The following relative bounds are well-known and show that $\mathbf{\alpha} \cdot \mathbf{A}$ is a symmetric operator on $\mathcal{D}(H_1^{1/2}/f)$. (Henceforth we identify $H_1^{1/2}/f \otimes H_1^{1/2}/f$ etc.) For every $\psi \in \mathcal{D}(H_1^{1/2}/f)$,

\begin{equation}
\| \mathbf{\alpha} \cdot \mathbf{a}(\mathbf{G}) \psi \|_2 \leq d_2 - 1 \| H_1^{1/2}/f \psi \|_2,
\end{equation}

\begin{equation}
\| \mathbf{\alpha} \cdot \mathbf{a}^\dagger(\mathbf{G}) \psi \|_2 \leq d_2 - 1 \| H_1^{1/2}/f \psi \|_2 + d_0 \| \psi \|_2.
\end{equation}

(Notice that the $C^*$-equality and (2.5) imply $\| \mathbf{\alpha} \cdot \mathbf{u} \| = |\mathbf{u}|$, for every $\mathbf{u} \in \mathbb{R}^3$, whence $\| \mathbf{\alpha} \cdot \mathbf{z} \|_2 \leq 2|\mathbf{z}|_2$, for every $\mathbf{z} \in \mathbb{C}^3$. This is why the factor 2 appears on the left sides of (2.2) and (2.3).)

In order to define the no-pair and semi-relativistic Pauli-Fierz operators we recall that the free Dirac operator minimally coupled to $\mathbf{A}$ is given as

\begin{equation}
D_\mathbf{A} := \mathbf{\alpha} \cdot (-i\nabla + \mathbf{A}) + \beta := \sum_{j=1}^{3} \alpha_j (-i\partial x_j + \mathbf{a}^\dagger(G^{(j)}_x) + \mathbf{a}(G^{(j)}_x)) + \beta.
\end{equation}

A straightforward application of Nelson’s commutator theorem shows that $D_\mathbf{A}$ is essentially self-adjoint on $\mathcal{D}_0[2, 25, 32]$. We denote its closure starting from $\mathcal{D}_0$ again by the same symbol. Supersymmetry arguments [33] show that its spectrum is contained in the union of two half-lines,

$$
\sigma(D_\mathbf{A}) \subset (-\infty, -1] \cup [1, \infty).
$$

The no-pair operator acts in the projected Hilbert space

$$
\mathcal{H}_\mathbf{A} := P_\mathbf{A}^+ \mathcal{H},
$$

where

\begin{equation}
P_\mathbf{A}^+ := \mathbb{1}_{[0,\infty)}(D_\mathbf{A}) = \frac{1}{2} \mathbb{1} + \frac{1}{2} \text{sgn}(D_\mathbf{A}), \quad P_{\mathbf{A}}^- := \mathbb{1} - P_\mathbf{A}^+.
\end{equation}

A-priori it is defined by

\begin{equation}
H^{np}_\gamma \varphi^+ \equiv H^{np}_{\gamma, \mathbf{A}} \varphi^+ := P_\mathbf{A}^+ (D_\mathbf{A} - \frac{\gamma}{|\mathbf{x}|} + H_f) \varphi^+, \quad \varphi^+ \in P_\mathbf{A}^+ \mathcal{D}_0.
\end{equation}

We remark that $H^{np}_\gamma$ is actually well-defined since $P_\mathbf{A}^+$ maps $\mathcal{D}_0$ into $\mathcal{D}(|\mathbf{x}|^{-1}) \cap \mathcal{D}(H_f)$ by Lemmata [3,4] and [3,5(ii)] below. Our first main result shows that the
The critical coupling constant for the (electronic) Brown-Ravenhall operator, 
\[ B_\gamma^\alpha = P_0^+ (D_0 - \frac{\gamma}{|x|}) P_0^+ , \]
which is defined as a Friedrichs extension starting from \( P_0^+ C_0^\infty (\mathbb{R}^3, \mathbb{C}^4) \). The critical coupling constant for \( B_\gamma^\alpha \) has been determined in [14] and its value is
\[ \gamma_{cp} := 2/(2/\pi + \pi/2). \]
In [34] it is shown that the Brown-Ravenhall operator is strictly positive, \( B_\gamma^\alpha \geq (1 - \gamma) P_0^+ \), \( \gamma \in [0, \gamma_{cp}^\alpha] \).

**Theorem 2.1.** Assume that \( G \) fulfills Hypothesis [1]. Then there is a constant, \( c \in (0, \infty) \), such that, for all \( \gamma \in [0, \gamma_{cp}^\alpha] \), \( \delta > 0 \), \( \rho \in (0, 1 - \gamma/\gamma_{cp}^\alpha) \), and \( \varphi^+ \in P_A^+ \mathcal{D}_0 \), \( \|\varphi^+\| = 1 \),
\[ \langle \varphi^+ | (D_A - \frac{\gamma}{|x|} + \delta H_f) \varphi^+ \rangle \geq \frac{1}{1 + \rho} \langle \varphi^+ | B_{(1 + \rho)\gamma}^{\alpha} \varphi^+ \rangle + \langle \varphi^+ | P_0^- |D_0| \varphi^+ \rangle - c(\delta + \delta^{-2}) (d_1^2 + d_0^2 + d_{-1}^2 + (d_0^2 + d_{-1}^2)^2)/\rho^3 \).
\]
In particular, by the KLMN-theorem \( H_{\gamma}^{np} \) has a distinguished self-adjoint extension – henceforth denoted by the same symbol – such that \( Q(H_{\gamma}^{np}) = Q(H_0^{np}) \). Moreover, \( P_A^+ \mathcal{D}_0 \) is a form core for \( H_{\gamma}^{np} \).

**Proof.** This theorem is proved in Subsection 4.1. \( \square \)

On account of Tix’ inequality (2.12) and Theorem 2.1, we know that, for all \( \gamma \in (0, \gamma_{cp}^\alpha) \) and \( \delta \in (0, 1) \), we find constants \( c(\gamma), C(\gamma, \delta, d_{-1}, d_0, d_1) \in (0, \infty) \) such that
\[ P_A^+ (D_A - \frac{\gamma}{|x|} + \delta H_f) P_A^+ \geq c(\gamma) |D_0| - C(\gamma, \delta, d_{-1}, d_0, d_1), \]
in the sense of quadratic forms on \( Q(H_{\gamma}^{np}) \). We shall employ this bound in [23].

In the sequel we denote the ionization threshold of \( H_{\gamma}^{np} \) by
\[ \Sigma_{np} \equiv \Sigma_{np}(G) := \inf \{ \langle \varphi^+ | H_{\gamma}^{np} \varphi^+ \rangle : \varphi^+ \in P_A^+ \mathcal{D}_0, \|\varphi^+\| = 1 \}. \]
We denote the length of an interval \( I \subset \mathbb{R} \) by \( |I| \).

**Theorem 2.2.** There exist constants, \( k_1, k_2, k_3, k_4 \in (0, \infty) \) and, for all \( G \) fulfilling Hypothesis [1] and all \( \gamma \in (0, \gamma_{cp}^\alpha) \), we find some \( E \in (0, \infty) \), \( E \equiv E(\gamma, d_{-1}, d_0, d_1) \to 0 \), \( d_i \to 0 \), \( i \in \{-1, 0, 1\} \), such that the following holds true: Let \( I \subset (-\infty, \Sigma_{np}) \) be some compact interval and let \( a > 0 \) satisfy \( a \leq k_1(\gamma_{cp}^\alpha - \gamma)/(\gamma_{cp}^\alpha + \gamma) \) and \( \varepsilon := 1 - \max I/(\Sigma_{np} + E) - k_2a^2 > 0 \). Then \( \text{Ran}(1_I(H_{\gamma}^{np})) \subset \mathcal{D}(e^{a|\cdot|}) \) and
\[ \| e^{a|\cdot|} 1_I(H_{\gamma}^{np}) \|_{L(\mathcal{H}_A^+, \mathcal{H})} \leq (k_3/\varepsilon^2) (1 + |I|) e^{k_4a(\Sigma_{np} + E)/\varepsilon}. \]
Proof. This theorem is proved at the end of Subsection 5.2. □

Notice that the exponential decay rates \( a \) in Theorem 2.2 depend on the numbers \( d_i \) and \( \Sigma_{np} \) but not on the particular shape of the form factor \( G \). This information on \( a \) is sufficient in order to prove the existence of ground states.

We remark that in [30] the present authors prove that an eigenfunction for an eigenvalue \( \lambda < 1 \) of a one-particle no-pair operator in a classical magnetic field decays with an exponential rate \( a < \sqrt{1 - \lambda^2} \), for \( \lambda \in [0, 1) \), and \( a < 1 \), for \( \lambda < 0 \). This is the behaviour known from the square-root, or, Chandrasekhar operator. The idea used there to provide better decay rates does, however, not apply when the quantized field energy is present.

The simple perturbative estimates of the following remark ensure that the statement of Theorem 2.2 is non-trivial, i.e. that \( \inf \sigma(H_{\text{np}}^\gamma) < \Sigma_{np} \), at least for small values of \( d_{-1}, d_0, d_1 \). (Recall from [14] that \( \inf \sigma(B_{\gamma}^{\text{cl}}) < 1, \) for \( \gamma \in (0, \gamma_c^{np}) \).)

**Remark 2.3.** There is a constant, \( C_\gamma \in (0, \infty) \), depending only on \( \gamma \in (0, \gamma_c^{np}) \), such that, for all \( G \) fulfilling Hypothesis H with \( d_{-1}, d_0, d_1 \leq 1 \),

\[
0 \leq \Sigma_{np} - 1 \leq C_\gamma (d_{-1} + d_0 + d_1)
\]

and

\[
| \inf \sigma(H_{\text{np}}^\gamma) - \inf \sigma(B_{\gamma}^{\text{cl}}) | \leq C_\gamma (d_{-1} + d_0 + d_1).
\]

These bounds are derived in Appendix A. □

Next, we define the second operator studied in this article, the semi-relativistic Pauli-Fierz operator. It acts in the whole space \( \mathcal{H} \) and is à-priori given as

\[
H_{\gamma}^{\text{PF}} \varphi \equiv H_{\gamma, A}^{\text{PF}} \varphi := (|D_A| - \frac{\gamma}{|x|} + H_f) \varphi, \quad \varphi \in \mathcal{D}_0.
\]

In fact, the operator defined in (2.19) is a two-fold copy of the one given in (1.2) since

\[
|D_A| = \begin{pmatrix} T_A & 0 \\ 0 & T_A \end{pmatrix}, \quad T_A := \sqrt{\sigma \cdot (-i \nabla + A)}^2 + 1.
\]

We prefer, however, to consider the operator defined by (2.19) in order to have a unified notation. The critical constant for the semi-relativistic Pauli-Fierz operator is given by Kato’s constant,

\[
\gamma_{c}^{\text{PF}} := \frac{2}{\pi}.
\]

**Theorem 2.4.** There is some \( k \in (0, \infty) \) such that, for all \( \delta > 0 \) and \( G \) fulfilling Hypothesis H

\[
\frac{1}{4} \| |x|^{-1} \varphi \|^2 \leq \| (|D_A| + \delta H_f + (\delta^{-1} + \delta k^2) d_i^2) \varphi \|^2,
\]

Next, we define the second operator studied in this article, the semi-relativistic Pauli-Fierz operator. It acts in the whole space \( \mathcal{H} \) and is à-priori given as

\[
H_{\gamma}^{\text{PF}} \varphi \equiv H_{\gamma, A}^{\text{PF}} \varphi := (|D_A| - \frac{\gamma}{|x|} + H_f) \varphi, \quad \varphi \in \mathcal{D}_0.
\]

In fact, the operator defined in (2.19) is a two-fold copy of the one given in (1.2) since

\[
|D_A| = \begin{pmatrix} T_A & 0 \\ 0 & T_A \end{pmatrix}, \quad T_A := \sqrt{\sigma \cdot (-i \nabla + A)}^2 + 1.
\]

We prefer, however, to consider the operator defined by (2.19) in order to have a unified notation. The critical constant for the semi-relativistic Pauli-Fierz operator is given by Kato’s constant,

\[
\gamma_{c}^{\text{PF}} := \frac{2}{\pi}.
\]

**Theorem 2.4.** There is some \( k \in (0, \infty) \) such that, for all \( \delta > 0 \) and \( G \) fulfilling Hypothesis H

\[
\frac{1}{4} \| |x|^{-1} \varphi \|^2 \leq \| (|D_A| + \delta H_f + (\delta^{-1} + \delta k^2) d_i^2) \varphi \|^2,
\]
for all $\varphi \in \mathcal{D}_0$, and

\begin{equation}
\frac{2}{\pi} \frac{1}{|x|} \leq |D_A| + \delta H_f + (\delta^{-1} + \delta k^2) d_1^2,
\end{equation}

in the sense of quadratic forms on $\mathcal{D}_0$. In particular, for all $\gamma \in [0, \gamma_{\text{c}}^{\text{PF}}]$, $H_{\gamma}^{\text{PF}}$ has a self-adjoint Friedrichs extension – henceforth again denoted by the same symbol. For $\gamma \in [0, \gamma_{\text{c}}^{\text{PF}})$, we know that $\mathcal{Q}(H_{\gamma}^{\text{PF}}) = \mathcal{Q}(H_{0}^{\text{PF}})$ and $\mathcal{D}_0$ is a form core for $H_{\gamma}^{\text{PF}}$.

**Proof.** This theorem is proved in Subsection 4.2. □

Due to \cite{32} Proposition 1.2 we know that $|D_A| + H_f$ is essentially self-adjoint on $\mathcal{D}_0$, provided $d_{-1}, d_0, d_1$ are sufficiently small. Together with (2.21) and the Kato-Rellich theorem this shows that $H_{\gamma}^{\text{PF}}$ is essentially self-adjoint on $\mathcal{D}_0$ as long as $\gamma \in (0, 1/2)$ and $d_{-1}, d_0, d_1$ are small. In \cite{23} we shall extend this result to all values of $d_{-1}, d_0, d_1$.

We denote the ionization threshold of $H_{\gamma}^{\text{np}}$ by

\begin{equation}
\Sigma_{\text{PF}} \equiv \Sigma_{\text{PF}}(G) := \inf \left\{ \langle \varphi | H_{0}^{\text{PF}} \varphi \rangle : \varphi \in \mathcal{D}_0, \|\varphi\| = 1 \right\}.
\end{equation}

**Theorem 2.5.** There are constants, $k_1, k_2 \in (0, \infty)$, such that, for all $G$ fulfilling Hypothesis \cite{1} and $\gamma \in (0, \gamma_{\text{c}}^{\text{PF}})$, the following holds true: Let $I \subset (-\infty, \Sigma_{\text{PF}})$ be some compact interval and assume that $a \in (0, 1)$ satisfies $\varepsilon := \Sigma_{\text{PF}} - \max I - \frac{9a^2}{(1 - a^2)^2} > 0$. Then $\text{Ran}(\mathbb{I}_I(H_{\gamma}^{\text{PF}})) \subset \mathcal{D}(e^{\varepsilon|x|})$ and

\begin{equation}
\| e^{\varepsilon|x|} \mathbb{I}_I(H_{\gamma}^{\text{PF}}) \| \leq (k_1/\varepsilon^2) (1 + |I|) (\Sigma_{\text{PF}} + k_2 d_1^2) e^{c(\gamma) a (\Sigma_{\text{PF}} + k_2 d_1^2)/\varepsilon}.
\end{equation}

Here $c(\gamma) \in (0, \infty)$ depends only on $\gamma$.

**Proof.** This theorem is proved at the end of Subsection 5.3. □

The following remark again ensures that $\inf \sigma(H_{\gamma}^{\text{PF}}) < \Sigma_{\text{PF}}$, at least for small values of $d_{-1}, d_0, d_1$.

**Remark 2.6.** There is a constant, $C_\gamma \in (0, \infty)$, depending only on $\gamma \in (0, \gamma_{\text{c}}^{\text{np}})$, such that, for all $G$ fulfilling Hypothesis \cite{1} with $d_{-1}, d_0, d_1 \leq 1$,

\begin{equation}
0 \leq \Sigma_{\text{PF}} - 1 \leq C_\gamma (d_{-1} + d_0 + d_1)
\end{equation}

and

\begin{equation}
\left| \inf \sigma(H_{\gamma}^{\text{PF}}) - \inf \sigma(|D_0| - \frac{2}{|x|}) \right| \leq C_\gamma (d_{-1} + d_0 + d_1).
\end{equation}

These bounds are also derived in Appendix A. □


3. Commutator estimates

In order to study the non-local no-pair and semi-relativistic Pauli-Fierz operators we need some control on various commutators and error terms that typically appear in their analysis. They involve resolvents and spectral projections of \( D_A \), and multiplication operators, in particular, exponential weights or cut-off functions. Since we are dealing with quantized fields we also have to study commutators of the resolvents and spectral projections with the radiation field energy. The aim of this section is to provide appropriate bounds on the corresponding operator norms. Our estimates on the error terms involving the field energy are based on the next lemma. The following quantity appears in its statement and in various estimates below,

\[
\delta_\nu^2 \equiv \delta_\nu(E)^2 := 8 \int \frac{w_\nu(k,E)^2}{\omega(k)} \| G(k) \|_\infty^2 \, dk, \quad E, \nu > 0,
\]

where

\[
w_\nu(k,E) := E^{1/2-\nu} ((E + \omega(k))^{\nu+1/2} - E^{\nu} (E + \omega(k))^{1/2}).
\]

We observe that \( w_{1/2}(k,E) \leq \omega(k) \) and, hence,

\[
\delta_{1/2}(E) \leq 2d_1, \quad E > 0.
\]

Moreover,

\[
\delta_\nu(E) \leq \delta_\nu(1), \quad E \geq 1.
\]

Lemma 3.1. Let \( \nu, E > 0 \), and set

\[
\tilde{H}_f := H_f + E.
\]

Then

\[
\| [\alpha \cdot A, \tilde{H}_f^{-\nu} \tilde{H}_f^\nu] \psi \| \leq \delta_\nu(E) / E^{1/2}.
\]

Proof. We pick \( \phi, \psi \in \mathcal{D}_0 \) and write

\[
\langle \phi | [\alpha \cdot A, \tilde{H}_f^{-\nu}] \tilde{H}_f^\nu \psi \rangle
\]

\[
= \langle \phi | [\alpha \cdot a(G), \tilde{H}_f^{-\nu}] \tilde{H}_f^\nu \psi \rangle - \langle [\alpha \cdot a(G), \tilde{H}_f^{-\nu}] \phi | \tilde{H}_f^\nu \psi \rangle.
\]

By definition of \( a(k) \) and \( H_f \) we have the pull-through formula \( a(k) \theta(H_f) \psi = \theta(H_f + \omega(k)) a(k) \psi \), for almost every \( k \) and every Borel function \( \theta \) on \( \mathbb{R} \), which leads to

\[
[ a(k), \tilde{H}_f^{-\nu} \tilde{H}_f^\nu \psi
\]

\[
= \{ ((\tilde{H}_f + \omega(k))^{-\nu} - \tilde{H}_f^{-\nu}) (\tilde{H}_f + \omega(k))^{\nu+1/2} \} a(k) \tilde{H}_f^{-1/2} \psi.
\]
We denote the operator \{\cdots\} by \(F(k)\). Then \(F(k)\) is bounded and

\[
\|F(k)\| \leq \int_0^1 \sup_{t \geq 0} \left| \frac{d}{ds} \frac{(t + E + \omega(k))^{\nu + 1/2}}{(t + E + s \omega(k))^{\nu}} \right| ds \\
= - \int_0^1 \frac{d}{ds} \frac{(E + \omega(k))^{\nu + 1/2}}{(E + s \omega(k))^{\nu}} ds = w_\nu(k, E)/E^{1/2}.
\]

Using these remarks together with the Cauchy-Schwarz inequality and (2.1), we obtain

\[
\|\langle \phi | [\alpha \cdot a(G), \tilde{H}_f^{\nu}] \tilde{H}_f^{\nu} \psi \rangle\|
\leq \int \|\phi\| \|\alpha \cdot G(k)\| \|F(k)\| \|a(k) \tilde{H}_f^{-1/2} \psi\| dk
\leq \|\phi\| \left(2 \int \frac{\|F(k)\|^2}{\omega(k)} \|G(k)\|^2 \, dk\right)^{1/2} \left(\int \omega(k) \|a(k) \tilde{H}_f^{-1/2} \psi\|^2 \, dk\right)^{1/2}
\leq \tilde{\delta}_\nu(E) \|\phi\| \|H_f^{1/2} \tilde{H}_f^{-1/2} \psi\|.
\]

A similar argument applied to the second term in (3.5) yields

\[
\|\langle [\alpha \cdot a(G), \tilde{H}_f^{\nu}] \phi \mid \tilde{H}_f^{\nu} \psi \rangle\| \leq \tilde{\delta}_\nu(E) \|H_f^{1/2} \tilde{H}_f^{-1/2} \phi\| \|\psi\|,
\]

where \(\tilde{\delta}_\nu(E)\) is defined by (3.1) with \(w_\nu(k, E)\) replaced by

\[
\tilde{w}_\nu(k, E) := E^{1/2-\nu}(E^{\nu}(E + \omega(k))^{1/2} - E^{2\nu}(E + \omega)^{1/2-\nu}).
\]

Evidently, \(\tilde{w}_\nu \leq w_\nu\), thus \(\tilde{\delta}_\nu \leq \delta_\nu\), which concludes the proof. \(\square\)

It is a trivial but very useful observation that, by choosing \(E\) large enough, we can make to norm appearing in (3.4) as small as we please. For instance, this is exploited to ensure that certain Neumann series converge in the proof of the next corollary, where various commutation relations are established that are used many times in the sequel. In the whole paper it turns out to be convenient to replace \(H_f\) by \(\tilde{H}_f = H_f + E\) in order to deal with commutators involving the radiation field energy. Thanks to Lemma 3.1 commutators with inverse powers of \(\tilde{H}_f = H_f + E\) can always be treated as small error terms.

**Corollary 3.2.** Let \(z \in \mathbb{C}\) and \(L \in \mathcal{L}(L^2(\mathbb{R}^3_x, \mathbb{C}^4))\) be such that \(z \in \mathfrak{g}(DA) \cap \mathfrak{g}(DA + L)\) (where \(L \equiv L \otimes \mathbb{1}\)) and set

(3.6) \(R_{A,L}(z) := (DA + L - z)^{-1}, \quad R_A(z) := R_{A,0}(z)\).
Assume that $\nu, E > 0$ satisfy $\delta_\nu/E^{1/2} < 1/\|R_{A,L}(z)\|$, and introduce the following operators (recall \[3.3\] & \[3.4\]),

$$T_\nu := [\overline{H_f^{-\nu} \cdot \alpha \cdot A}, \overline{\nu}]_f.$$ \[3.7\]

$$\Xi_{\nu,L}(z) := \sum_{j=0}^{\infty} \{-R_{A,L}(z) T_\nu\}^j, \quad \Upsilon_{\nu,L}(z) := \sum_{j=0}^{\infty} \{-T_\nu^* R_{A,L}(z)\}^j.$$ \[3.8\]

Then

$$\|T_\nu\| \leq \delta_\nu/E^{1/2}, \quad \|\Xi_{\nu,L}(z)\|, \|\Upsilon_{\nu,L}(z)\| \leq (1 - \delta_\nu \|R_{A,L}(z)\|/E^{1/2})^{-1},$$ \[3.9\]

and

\begin{align*}
[R_{A,L}(z), \overline{H_f^{-\nu}}] &= R_{A,L}(z) \overline{[H_f^{-\nu}, \alpha \cdot A]} R_{A,L}(z) \quad \text{\[3.10\]} \\
\overline{H_f^{-\nu} R_{A,L}(z)} &= \Xi_{\nu,L}(z) R_{A,L}(z) \overline{H_f^{-\nu}}, \quad \text{\[3.11\]} \\
R_{A,L}(z) \overline{H_f^{-\nu}} &= \overline{H_f^{-\nu} R_{A,L}(z) \Upsilon_{\nu,L}(z)} \quad \text{\[3.12\]} \\
[R_{A,L}(z), \overline{H_f^{-\nu}}] \overline{H_f^{-\nu}} &= R_{A,L}(z) T_\nu \Xi_{\nu,L}(z) R_{A,L}(z). \quad \text{\[3.13\]}
\end{align*}

In particular, $R_{A,L}(z)$ maps $\mathcal{D}(1 \otimes H_f^1)$ into itself.

Proof. First, we remark that since $(D_A - z) \mathcal{D}_0$ is dense in $\mathcal{H}$ and since $z \in \partial(D_A + L)$ we also know that $(D_A + L - z) \mathcal{D}_0$ is dense. Next, we observe that, for every $\psi \in \mathcal{D}_0$, we have $\overline{H_f^{-\nu} \psi} \in \mathcal{D}(H_f^1) \subset \mathcal{D}(\alpha \cdot A)$, whence

\begin{align*}
[R_{A,L}(z), \overline{H_f^{-\nu}}] (D_A + L - z) \psi &= R_{A,L}(z) \overline{[H_f^{-\nu}, D_A]} \psi \\
&= R_{A,L}(z) \overline{[H_f^{-\nu}, \alpha \cdot A]} R_{A,L}(z) (D_A + L - z) \psi \\
&= R_{A,L}(z) T_\nu \overline{H_f^{-\nu} R_{A,L}(z) (D_A + L - z) \psi}.
\end{align*}

Since $(D_A + L - z) \mathcal{D}_0$ is dense and since $T_\nu$ and $[\overline{H_f^{-\nu}, \alpha \cdot A}]$ are bounded due to Lemma \[3.1\] this implies \[3.9\] and \[3.10\]. Then \[3.11\] follows from \[3.10\] and some elementary manipulations and \[3.13\] follows from \[3.10\] and \[3.11\]. Finally, the last assertion follows from \[3.12\] (which is just the adjoint of \[3.11\]) with $z$ and $L$ replaced by $\overline{\tau}$ and $L^*$ since $\Upsilon_{\mu,L}(z) = \Xi_{\nu,L^*}(z^*)$. \qed

We continue by stating some simple facts which are used in the proofs of the lemmata below: First, we have the following representation of the sign function of $D_A$ \[21\] Lemma VI.5.6],

\begin{align*}
\text{sgn}(D_A) \varphi &= \lim_{\tau \to \infty} \int_{-\tau}^{\tau} R_A(iy) \varphi \frac{dy}{\pi}, \quad \varphi \in \mathcal{H}.
\end{align*} \[3.14\]
Furthermore, since \((-1, 1) \subset \sigma(D_A)\) the spectral calculus yields, for all \(y \in \mathbb{R}\) and \(\kappa \in [0, 1)\),

\[
\| |D_A|^{\kappa} R_A(iy) \| \leq \frac{\mathbb{1}_{|y| < b(\kappa)}}{\sqrt{1 + y^2}} + \frac{c(\kappa) \mathbb{1}_{|y| \geq b(\kappa)}}{|y|^{1-\kappa}} =: \zeta_\kappa(y),
\]

where \(b(\kappa) := \kappa^{-1/2}(1 - \kappa)^{1/2} (1/0 := \infty)\), \(c(\kappa) := \kappa^{\kappa/2}(1 - \kappa)^{(1-\kappa)/2}\), and

\[
K(\kappa) := \int_{\mathbb{R}} \frac{\zeta_\kappa(y)}{\sqrt{1 + y^2}} \frac{dy}{2\pi} < \infty, \quad K(0) = \frac{1}{2}.
\]

Finally, to study their non-local properties we shall conjugate or commute various operators with exponential weight functions, \(e^F \equiv e^F \otimes 1\), acting on the electron coordinates, where

\[
F \in C^\infty(\mathbb{R}_x^3, \mathbb{R}) \cap L^\infty(\mathbb{R}_x^3, \mathbb{R}), \quad F(0) = 0, \quad F \geq 0 \text{ or } F \leq 0, \quad |\nabla F| \leq a,
\]

for some \(a \in [0, 1)\). The next lemma shows that the resolvent of \(D_A\) stays bounded after conjugation with \(e^F\). Its statement is actually well-known for classical magnetic fields; see, e.g. \([13]\). The proof presented in \([30\), Lemma 3.1\] applies, however, also to quantized fields without any change and we refrain from repeating it here.

\textbf{Lemma 3.3.} Let \(y \in \mathbb{R}, \ a \in [0, 1)\), and let \(F \in C^\infty(\mathbb{R}_x^3, \mathbb{R})\) have a fixed sign and satisfy \(|\nabla F| \leq a\). Then \(iy \in \sigma(D_A + i\alpha \cdot \nabla F)\),

\[
e^F R_A(iy) e^{-F} = (D_A + i\alpha \cdot \nabla F + iy)^{-1}|_{D(e^{-F})},
\]

and

\[
\| e^F R_A(iy) e^{-F} \| \leq \frac{\sqrt{6}}{\sqrt{1 + y^2}} \cdot \frac{1}{1 - a^2}.
\]

We define \(J : [0, 1) \to \mathbb{R}\) by

\[
J(0) := 1, \quad J(0) := \frac{\sqrt{6}}{1 - a^2}, \quad a \in (0, 1).
\]

\textbf{Lemma 3.4.} Let \(a, \kappa \in [0, 1)\) and let \(F\) satisfy \((3.17)\). For all \(\nu, E > 0\) with \(\delta_\nu J(a)/E^{1/2} < 1\), we define

\[
S_\nu^F := e^F \left[ \text{sgn}(D_A), \tilde{H}_{\nu}^{-\nu} \right] \tilde{H}_{\nu} e^{-F}
\]

on \(\mathcal{D}(\tilde{H}_{\nu}^F)\). Then

\[
\| |D_A|^\kappa S_\nu^F \| \leq (1 + a J(a)) \frac{K(\kappa) \delta_\nu J(a)/E^{1/2}}{1 - \delta_\nu J(a)/E^{1/2}}.
\]

14
In particular, $\text{sgn}(D_A)$ maps $\mathcal{D}(\mathbb{1} \otimes H_\nu^\nu)$ into itself and, if $E^{1/2} > \delta_\nu$, then the following identities hold true on $\mathcal{D}(\mathbb{1} \otimes H_\nu^\nu)$,

(3.21) $\tilde{H}_f^\nu \text{sgn}(D_A) = \text{sgn}(D_A) \tilde{H}_f^\nu + S_\nu \tilde{H}_f^\nu$,

(3.22) $\text{sgn}(D_A) \tilde{H}_f^\nu = \tilde{H}_f^\nu \text{sgn}(D_A) + \tilde{H}_f^\nu S_\nu$,

where $S_\nu := (S_0^\nu)^* \in \mathcal{L}(\mathcal{H})$.

Proof. We set $L := i\alpha \cdot \nabla F$ so that $e^F R_A(iy) e^{-F} = R_{A,L}(iy), \; y \in \mathbb{R}$, and $\|L\| \leq a$. Combining (3.13) with (3.14) we obtain, for all $\phi \in \mathcal{D}(|D_A|^\kappa)$ and $\psi \in \mathcal{D}(H_\nu^\nu)$,

$$\left| \left| \left| D_A \right| \left| \kappa \phi \right| S_\nu^F \psi \right| \right| \leq \int_R \left| \left| \left| D_A \right| \left| \kappa \phi \right| R_{A,L}(iy) T_\nu \Xi_{\nu,L}(iy) R_{A,L}(iy) \psi \right| \right| dy \xi$$

$$\leq \|T_\nu\| \int_R \|D_A\|^{\kappa} R_{A,L}(iy) \phi \| \Xi_{\nu,L}(iy)\| \|R_{A,L}(iy)\| \frac{dy}{\pi} \|\phi\| \|\psi\|.$$ 

Here we estimate $\|T_\nu\|$ by means of (3.4) and we write $|D_A|^\kappa R_{A,L}(iy) = |D_A|^\kappa R_A(iy) (1 - L R_{A,L}(iy))$ in order to apply (3.15). Moreover, (3.8) and Lemma 3.3 show that $\|\Xi_{\nu,L}(iy)\| \leq (1 - \delta_\nu J(a)/E^{1/2})^{-1}$, for all $y \in \mathbb{R}$. Altogether these remarks yield the asserted estimate. Now, the following identity in $\mathcal{L}(\mathcal{H})$,

$$\text{sgn}(D_A) \tilde{H}_f^\nu = \tilde{H}_f^\nu \text{sgn}(D_A) - \tilde{H}_f^\nu (S_\nu^0)^*,$$

shows that $\text{sgn}(D_A)$ maps the domain of $H_\nu^\nu$ into itself and that (3.21) is valid. Taking the adjoint of (3.21) and using $[\tilde{H}_f^\nu \text{sgn}(D_A)]^* = \text{sgn}(D_A) \tilde{H}_f^\nu$ (which is true since $\tilde{H}_f^\nu \text{sgn}(D_A)$ is densely defined and $\text{sgn}(D_A) = \text{sgn}(D_A)^{-1} \in \mathcal{L}(\mathcal{H})$) we also obtain (3.22). \hfill $\square$

Lemma 3.5. (i) For all $E > 0$, $\nu \geq 0$, with $\delta_{\nu+1/2}/E^{1/2} < 1$, and $\chi \in C^\infty(\mathbb{R}^3, [0, 1])$, the following resolvent formulas are valid,

$$R_0(iy) \chi - \chi R_A(iy) \right) \tilde{H}_f^{\nu-1/2} =$$

(3.23) $R_0(iy) \tilde{H}_f^{\nu} \left\{ \alpha \cdot (i\nabla \chi + \chi A) + T_\nu^* \chi \right\} \tilde{H}_f^{\nu-1/2} R_A(iy) \Xi_{\nu+1/2,0}(-iy)^*,$

and

(3.24) $\tilde{H}_f^{\nu-1/2} \left( R_0(iy) \chi - \chi R_A(iy) \right) = R_0(iy) \tilde{H}_f^{\nu-1/2} \alpha \cdot (A \chi - i\nabla \chi) R_A(iy) .$

(ii) If $\delta_\nu(1) < \infty$, for some $\nu \geq 1$, then $P_\lambda^+$ maps the subspace $\mathcal{D}(D_0 \otimes H_\nu^\nu)$ into itself.
Proof. (i): A short computation using (3.11) yields, for every \( \varphi \in \mathcal{D}_0 \),
\[
\tilde{H}_f^{-\nu-1/2} \left( R_A(-iy) \chi - \chi R_0(-iy) \right) (D_0 + iy) \varphi \\
= -\Xi_{\nu + 1/2,0}(-iy) R_A(-iy) \tilde{H}_f^{-\nu-1/2} \alpha \cdot (\chi A + i \nabla \chi) \varphi \\
= -\Xi_{\nu + 1/2,0}(-iy) R_A(-iy) \tilde{H}_f^{-1/2} \left\{ \alpha \cdot (\chi A + i \nabla \chi) + \chi T_0 \right\} \times \\
\times \tilde{H}_f^{-\nu} R_0(-iy) (D_0 + iy) \varphi .
\]
Now, \( (D_0 - iy) \mathcal{D}_0 \) is dense in \( \mathcal{H} \) and \( \tilde{H}_f^{-1/2} \alpha \cdot A \) is bounded due to (2.6) and (2.7). Therefore, the previous computation implies an operator identity in \( \mathcal{L}(\mathcal{H}) \) whose adjoint is (3.23), and (3.24) is derived in a similar fashion.

(ii): For \( \nu \geq 1 \), (3.23) shows that the range of \( R_A(iy) \tilde{H}_f^{1/2-\nu} \) is contained in \( \text{Ran}(R_0(iy) \otimes \tilde{H}_f^{-\nu+1}) \subset \mathcal{D}(D_0 \otimes H_f^{1/2-\nu}) \), for every \( y \in \mathbb{R} \). Moreover, we know from (2.9) and Lemma 3.3 that \( \tilde{H}_f^{1/2-\nu} P_A \tilde{H}_f^{1/2-\nu} \in \mathcal{L}(\mathcal{H}) \). Now, let \( \varphi \in \mathcal{D}(D_0 \otimes H_f^{1/2}) \). Then \( D_A \varphi \in \mathcal{D}(\tilde{H}_f^{-\nu+1/2}) \) and it follows that
\[
P_A^+ \varphi = R_A(0) \tilde{H}_f^{1/2-\nu} (\tilde{H}_f^{\nu-1/2} P_A \tilde{H}_f^{1/2-\nu}) \tilde{H}_f^{\nu-1/2} D_A \varphi \in \mathcal{D}(D_0 \otimes H_f^{-1/2}).
\]
Furthermore, we know that \( P_A^+ \varphi \in \mathcal{D}(1 \otimes H_f^\nu) \) by Lemma 3.4. \( \square \)

In our study of the exponential localization we shall often encounter error terms involving the operator
\[
(3.25) \quad \mathcal{K}_{\chi,F} := [P_A^+, \chi e^F] e^{-F},
\]
where \( \chi \in C^\infty(\mathbb{R}_x^3, [0,1]) \) and \( F : \mathbb{R}_x^3 \to \mathbb{R} \) are functions of the electron coordinates and \( F \) satisfies (3.17). The operator norm bounds derived in Lemma 3.6 below provide the necessary control on \( \mathcal{K}_{\chi,F} \).

**Lemma 3.6.** Let \( a, \kappa \in [0,1) \) and let \( F \) satisfy (3.17). Then we have, for all \( \nu, E > 0 \) with \( E^{1/2} > \delta \nu \), \( J(a) \), and \( \chi \in C^\infty(\mathbb{R}_x^3, [0,1]) \),
\[
(3.26) \quad \| D_A \|^{\kappa} \mathcal{K}_{\chi,F} \| \leq K(\kappa) J(a) (a + \| \nabla \chi \|) ,
\]
\[
(3.27) \quad \| D_A \|^{\kappa} \tilde{H}_f^{\nu} \mathcal{K}_{\chi,F} \tilde{H}_f^{\nu} \| \leq K(\kappa) J(a) \frac{a + \| \nabla \chi \| \infty}{(1 - \delta \nu J(a)/E^{1/2})^2} .
\]
In particular,
\[
(3.28) \quad \| e^F P_A^+ e^{-F} \| \leq 1 + a J(a)/2 .
\]
Let \( k \) be the universal constant appearing in Theorem 2.4. There is another universal constant, \( C \in (0, \infty) \), such that, for all \( E > \max\{4 J(a)^2, (1 + k^2)\} d_1' \),
\[
(3.29) \quad \| |x|^{-1/2} \mathcal{K}_{\chi,F} \tilde{H}_f^{-1/2} \| \leq C J(a) \frac{a + \| \nabla \chi \| \infty}{(1 - 2 d_1 J(a)/E^{1/2})^2} .
\]
Proof. Using the notation introduced in (3.6) and Lemma 3.3, we have
\[
(3.30) \quad \left[ R_A(iy), \chi e^F \right] e^{-F} = R_A(iy) M R_{A,L}(iy),
\]
where
\[
(3.31) \quad M := i\alpha \cdot (\nabla \chi + \chi \nabla F) \quad \text{and} \quad L := i\alpha \cdot \nabla F.
\]
By means of (2.5) we find \(\|M\| \leq (a + \|\nabla \chi\|_\infty)\). Moreover, \(\|R_{A,L}(iy)\| \leq J(a) (1 + y^2)^{-1/2}\) by Lemma 3.3 and the operator \(\Xi_{\nu,L}(iy)\) given by (3.1) satisfies
\[
\|\Xi_{\nu,L}(iy)\| \leq (1 - \delta_\nu J(a)/E^{1/2})^{-1}.
\]
We further set \(\tilde{\Xi}_{\nu}(iy) := \sum_{l=0}^{\infty} \{ -T_l R_A(iy) \}^l\) so that \(\Xi_{\nu,L}(iy) = R_A(iy) \tilde{\Xi}_{\nu}(iy)\) and \(\|\tilde{\Xi}_{\nu}(iy)\| \leq (1 - \delta_\nu/E^{1/2})^{-1}\). On account of (3.11) we now obtain, for all \(\phi, \psi \in \mathcal{D}_0\),
\[
(3.32) \quad \left| \left\langle |D_A|^\kappa \phi \left| \tilde{H}_f^{-\nu} K_{\chi,F} \tilde{H}_f^\nu \psi \right| \right\rangle \right| \leq \frac{1}{2\pi} \int_{\mathbb{R}} \left| \left\langle |D_A|^\kappa \phi \left| \tilde{H}_f^{-\nu} R_A(iy) M R_{A,L}(iy) \tilde{H}_f^\nu \psi \right| \right\rangle \right| \, dy.
\]
In the second step we used (3.31) twice and \([\tilde{H}_f^{-\nu}, M] = 0\). Applying the various norm bounds mentioned above together with (3.15) we see that (3.27) holds true for the first choice of the signs \(\pm\). To obtain (3.27) with the second choice of signs we proceed analogously applying (3.12) instead of (3.11). (Notice that, by Lemma 3.5, \(K_{\chi,F}\) maps \(\mathcal{D}_0\) into \(\mathcal{D}(H_f^\nu)\).) Also (3.26) is proved in the same way. (Just ignore \(\tilde{H}_f\).) (3.29) follows from (3.27) and the inequality (2.22) from Theorem 2.4 which is proved independently. We also use (3.2) to derive (3.29). \(\square\)

**Lemma 3.7.** Let \(a \in [0, 1)\) and let \(F\) satisfy (3.11). Let \(\nu, E > 0\) such that \(\delta_\nu J(a)/E^{1/2} \leq 1/2\). Then
\[
(3.33) \quad \| |D_A| \left[ \chi_1 e^F, [P_A^+, \chi_2 e^{-F}] \right] \| \leq J(a) \prod_{i=1,2} (a + \|\nabla \chi_i\|_\infty),
\]
\[
(3.34) \quad \| \tilde{H}_f^{-\nu} \left[ \chi_1 e^F, [P_A^+, \chi_2 e^{-F}] \right] \tilde{H}_f^{-\nu} \| \leq 8 J(a) \prod_{i=1,2} (a + \|\nabla \chi_i\|_\infty),
\]
\[
(3.35) \quad \| \frac{1}{|x|} \left[ \chi_1 e^F, [P_A^+, \chi_2 e^{-F}] \right] \tilde{H}_f^{-1/2} \| \leq 8^{3/2} J(a) \prod_{i=1,2} (a + \|\nabla \chi_i\|_\infty).
\]
In (3.35) we assume that \(E \geq (4d_1 J(a))^2\).

**Proof.** Let \(\phi, \psi \in \mathcal{D}_0\), \(\|\phi\| = \|\psi\| = 1\). First, we derive a bound on
\[
I_{\phi,\psi} := \int_{\mathbb{R}} \left| \left\langle |D_A| \phi \left| \tilde{H}_f^{-\nu} \left[ \chi_1 e^F, [R_A(iy), \chi_2 e^{-F}] \right] \tilde{H}_f^{-\nu} \psi \right| \right\rangle \right| \, dy.
\]
Expanding the double commutator we get
\[ [\chi_1 e^F, [R_A(iy), \chi_2 e^{-F}]] = \eta(\chi_1, \chi_2, F; y) + \eta(\chi_2, \chi_1, -F; y), \]
where
\[ \eta(\chi_1, \chi_2, F; y) := R_A(iy) \alpha \cdot (\nabla \chi_1 + \chi_1 \nabla F) e^F R_A(iy) e^{-F} \alpha \cdot (\nabla \chi_2 - \chi_2 \nabla F) R_A(iy). \]

Writing \( L := i\alpha \cdot \nabla F \), we obtain
\[
\int_R \left| \left| D_A \phi \right| \left| \tilde{H}_f^\nu \eta(\chi_1, \chi_2, F; y) \tilde{H}_f^{-\nu} \psi \right| \right| dy \frac{dy}{2\pi} 
\leq \int_R \left| \left| \phi \right| \left| D_A R_A(iy) \Upsilon_{\nu,0}(iy) \alpha \cdot (\nabla \chi_1 + \chi_1 \nabla F) \times R_A, L(iy) \Upsilon_{\nu,0}(iy) \alpha \cdot (\nabla \chi_2 - \chi_2 \nabla F) R_A(iy) \Upsilon_{\nu,0}(iy) \psi \right| \right| dy \frac{dy}{2\pi}
\]
(3.36) \leq \frac{(a + \|\nabla \chi_1\|)(a + \|\nabla \chi_2\|)}{(1 - \delta_x/ E^3/2)^2} \cdot \frac{J(a)}{1 - \delta_x J(a)/ E^3/2} \int_R \frac{dy}{2\pi (1 + y^2)}.

A bound analogous to (3.36) holds true when the roles of \( \chi_1 \) and \( \chi_2 \) are interchanged and \( F \) is replaced by \( -F \). Consequently, \( I_{\phi, \psi} \) is bounded by two times the right hand side of (3.36). Altogether this shows that (3.33) and (3.34) hold true. (Just ignore \( |D_A| \) or \( \tilde{H}_f \), respectively, in the above argument.) (3.35) follows from (3.33) and (3.34) and the inequality
\[ \| x^{-1} \varphi \|^2 \leq 4 \| D_A \varphi \|^2 + 4 \| \tilde{H}_f^{1/2} \varphi \|^2, \quad \varphi \in \mathcal{D}_0, \]
which is true for \( E \geq d_1^2 \) and derived independently in the proof of Theorem 2.4 given below; see (4.7).

In what follows we set, for every \( F \) satisfying (3.17),
(3.37) \[ P_A^F := e^F P_A^+ e^{-F}, \]
so that
(3.38) \[ \chi P_A^F = P_A^+ \chi - K_{\chi,F}, \quad P_A^F \chi = \chi P_A^+ - K_{\chi,-F}. \]

Corollary 3.8. Let \( O \) be \( D_A, \frac{1}{|x|}, \tilde{H}_f \), or any element of \( \mathcal{L}(\mathcal{H}) \) with \( \| O \| \leq 1 \). Then there exists some universal constant \( K \in (0, \infty) \) such that, for all \( E \geq (4d_1 J(a))^2, a \in (0, 1), F \) satisfying (3.17), \( \chi \in C^\infty(\mathbb{R}^3, [0, 1]), \epsilon > 0 \), and \( \varphi \in \mathcal{D}_0 \cup P_A^+ \mathcal{D}_0 \),
\[
\left| \langle \varphi \mid P_A^F \chi O \chi P_A^F \varphi \rangle - \langle \varphi \mid \chi P_A^+ O P_A^+ \chi \varphi \rangle \right|
\leq \epsilon \langle \varphi \mid \chi P_A^+ |O| P_A^+ \chi \varphi \rangle + (1 + \epsilon^{-1}) K(a + \|\nabla \chi\|_\infty)^2 J(a)^2 \| \tilde{H}_f^{1/2} \varphi \|^2.
\]
Moreover, if $\mathcal{O}$ is self-adjoint, then

$$
\left| \text{Re} \left[ \langle \varphi | P_\mathcal{A}^F \mathcal{O} P_\mathcal{A}^F \varphi \rangle - \langle \varphi | P_\mathcal{A}^+ \mathcal{O} P_\mathcal{A}^+ \varphi \rangle \right] \right| 
\leq K a^2 J(a)^2 \left( \left\| \widetilde{H}_f^{1/2} \varphi \right\|^2 + \left\| \widetilde{H}_f^{1/2} P_\mathcal{A}^+ \varphi \right\|^2 \right).
$$

(3.40)

If $\mathcal{O} = D_\mathcal{A}$ or $\mathcal{O} \in \mathcal{L}(\mathcal{H})$ then we can replace the norms $\left\| \widetilde{H}_f^{1/2} \varphi \right\|^2$ and $\left\| \widetilde{H}_f^{1/2} P_\mathcal{A}^+ \varphi \right\|^2$ on the right sides of (3.39) and (3.40) by $\| \varphi \|^2$ and $2 \| \varphi \|^2$, respectively.

**Proof.** In view of (3.38) and Lemma 3.5(ii) we have the following operator identity on $\mathcal{D}_0 \cup P_\mathcal{A}^+ \mathcal{D}_0$,

$$
P_\mathcal{A}^F \chi \mathcal{O} \chi P_\mathcal{A}^F - \chi P_\mathcal{A}^+ \mathcal{O} P_\mathcal{A}^+ \chi
= -K^{*}_{\chi,-F} \mathcal{O} P_\mathcal{A}^+ \chi - \chi P_\mathcal{A}^+ \mathcal{O} \mathcal{K}_{\chi,F} - \mathcal{K}^{*}_{\chi,-F} \mathcal{O} \mathcal{K}_{\chi,F}.
$$

(3.41)

Consequently, the term on the left side of (3.39) is less than or equal to

$$
\left\| \mathcal{O}^{1/2} P_\mathcal{A}^+ \chi \varphi \right\| \left\{ \sum_{\bar{\nu} = \pm} \left\| \mathcal{O}^{1/2} \mathcal{K}_{\chi,\bar{\nu}F} \varphi \right\| \right\} + \prod_{\bar{\nu} = \pm} \left\| \mathcal{O}^{1/2} \mathcal{K}_{\chi,\bar{\nu}F} \varphi \right\|.
$$

Therefore, (3.39) follows from Lemma 3.6.

In order to derive (3.40) we write $\mathcal{K}_F := \mathcal{K}_{1,F}$ and infer from (3.41) that

$$
\text{Re} \left[ \langle \varphi | P_\mathcal{A}^F \mathcal{O} P_\mathcal{A}^F \varphi \rangle - \langle \varphi | P_\mathcal{A}^+ \mathcal{O} P_\mathcal{A}^+ \varphi \rangle \right]
= -\text{Re} \left[ \langle \varphi | P_\mathcal{A}^+ \mathcal{O} \mathcal{K}_F + \mathcal{K}_{-F} \rangle \varphi \rangle \right] - \text{Re} \left[ \langle \varphi | \mathcal{K}^{*}_{\chi,-F} \mathcal{O} \mathcal{K}_{\chi,F} \varphi \rangle \right],
$$

(3.42)

where

$$
\mathcal{K}_F + \mathcal{K}_{-F} = \left[ \left[ P_\mathcal{A}^+, e^F \right], e^{-F} \right].
$$

Therefore, (3.40) follows from Lemma 3.7 applied to the first term in (3.42) and Lemma 3.6 applied to the second term in (3.42). (In the case $\mathcal{O} = \widetilde{H}_f$ we apply (3.34) with $\nu = 1/2$.) $\square$

**Lemma 3.9.** For all $\kappa \in [0, 1)$, $E > (2d_1)^2$, and $\chi \in C^\infty(\mathbb{R}^3, [0, 1])$,

$$
\left\| D_0 \partial^\kappa (P_\mathcal{A}^+ \chi - \chi P_\mathcal{A}^+) \widetilde{H}_f^{-1/2} \right\| \leq K(\kappa) \left( \| \nabla \chi \|_\infty + (d_0^2 + 2d_1^2)^{1/2} \right),
$$

(3.43)

$$
\left\| D_\mathcal{A} \partial^\kappa (P_\mathcal{A}^+ \chi - \chi P_\mathcal{A}^+) \widetilde{H}_f^{-1/2} \right\| \leq K(\kappa) \left( \| \nabla \chi \|_\infty + (d_0^2 + 2d_1^2)^{1/2} \right).
$$

(3.44)
Proof. Combining (3.23) with (3.14) we find, for \( \phi, \psi \in \mathcal{D}_0 \),

\[
\left| \langle |D_0|^\kappa \phi \mid (P_0^+ \chi - \chi P_A^+) \tilde{H}_f^{-1/2} \psi \rangle \right| = \int_r \left| \langle |D_0|^\kappa \phi \mid R_0(iy) \alpha \cdot (i\nabla \chi - \chi A) \tilde{H}_f^{-1/2} R_A(iy) \mathcal{Y}_{1/2,0}(iy) \psi \rangle \right| \frac{dy}{2\pi} \\
\leq \int_r \left| |D_0|^\kappa R_0(iy) \right| \|\phi\| \left( \|\nabla \chi\|_\infty + \|\alpha \cdot A \tilde{H}_f^{-1/2}\| \right) \cdot \|R_A(iy)\| \|\mathcal{Y}_{1/2,0}(iy)\| \|\psi\| \frac{dy}{2\pi}.
\]

On account of (3.15) and \( \|\mathcal{Y}_{1/2,0}(iy)\| \leq (1 - 2d_1/E^{1/2})^{-1} \) this implies (3.43). The bound (3.44) is proved analogously by interchanging the roles of \( D_0 \) and \( D_A \) and using the adjoint of (3.24). \( \square \)

Corollary 3.10. For all \( \varepsilon > 0 \), \( \chi \in C_0^\infty(\mathbb{R}^3, [0,1]) \), and \( \varphi^+ \in P^+_A \mathcal{D}_0 \),

\[
|||D_0|^{1/2} P_0^- \chi \varphi^+|| \leq \varepsilon \left( H_f^{1/2} \chi \varphi^+ \right)^2 + \frac{c^4}{4\varepsilon^2} \left| P_0^- \chi \varphi^+ \right|^2,
\]

where \( c \) denotes the right hand side of (3.43) with \( \kappa = 3/4 \). Moreover, we have, for \( \varepsilon, \tau > 0 \),

\[
\left| \langle P_0^+ \chi \varphi^+ \mid \frac{1}{|x|} P_0^- \chi \varphi^+ \rangle \right| \\
\leq \tau \left\| |D_0|^{1/2} P_0^+ \chi \varphi^+ \right\|^2 + \varepsilon \left\| H_f^{1/2} \varphi^+ \right\|^2 + \frac{c^4 \pi^6}{2^{11}\varepsilon^2 \tau^3} \left\| P_0^- \chi \varphi^+ \right\|^2.
\]

Proof. Using (3.43) (which is certainly valid also with \( P_A^+ \) replaced by \( P_A^- \)), we first observe that

\[
P_0^- \chi \varphi^+ = (P_0^- \chi - \chi P_A^-) \varphi^+ \in \mathcal{D}(|D_0|^{3/4}).
\]

This permits to get

\[
\left\| |D_0|^{1/2} P_0^- \chi \varphi^+ \right\|^2 \\
\leq \left\| |D_0|^{1/4} P_0^- \chi \varphi^+ \right\|^2 \left\| |D_0|^{3/4} (P_0^- \chi - \chi P_A^+) \varphi^+ \right\| \\
\leq \left\| |D_0|^{1/4} P_0^- \chi \varphi^+ \right\|^2 \left| H_f^{1/2} \varphi^+ \right\| \\
\leq \frac{c^2}{2\varepsilon} \left\| P_0^- \chi \varphi^+ \right\|^2 + \varepsilon \left\| H_f^{1/2} \varphi^+ \right\|^2 \\
\leq \frac{1}{2} \left\| |D_0|^{1/2} P_0^- \chi \varphi^+ \right\|^2 + \frac{c^4}{8\varepsilon^2} \left\| P_0^- \chi \varphi^+ \right\|^2 + \frac{\varepsilon}{2} \left\| H_f^{1/2} \varphi^+ \right\|^2,
\]

which implies (3.45). The bound (3.46) follows from (3.45) and Kato’s inequality, \( |x|^{-1} \leq (\pi/2)|D_0| \). \( \square \)
4. Semi-boundedness

In the following two subsections we prove Theorems 2.1 and 2.4 which state that the no-pair and relativistic Pauli-Fierz operators are semi-bounded provided the coupling constant in front of the Coulomb potential stays below the critical values $\gamma_c^{np} = 2/(2/\pi + \pi/2)$ and $\gamma_c^{PF} = 2/\pi$, respectively.

4.1. The no-pair operator: Semi-boundedness.

Proof of Theorem 2.1. We pick some $\rho \in (0, 1 - \gamma/\gamma_c^{np})$ and set $\gamma_\rho := (1 + \rho) \gamma$. By virtue of Lemma 3.5(ii) we have $P_0^+ \varphi^+ \in D(D_0 \otimes H_f^{1/2})$, whence

$$
\langle \varphi^+ | (D_A - \frac{\omega}{|x|}) \varphi^+ \rangle = \frac{\gamma}{\gamma_\rho} \langle \varphi^+ | P_0^+ (D_0 - \frac{\omega}{|x|}) P_0^+ \varphi^+ \rangle + (1 - \gamma/\gamma_\rho) \langle \varphi^+ | P_0^+ D_0 \varphi^+ \rangle + \langle \varphi^+ | \alpha \cdot A \varphi^+ \rangle
$$

(4.1)

$$
+ \langle \varphi^+ | P_0^- (D_0 - \frac{\omega}{|x|}) P_0^- \varphi^+ \rangle + 2\gamma \Re \langle \varphi^+ | P_0^- \frac{1}{|x|} P_0^- \varphi^+ \rangle
$$

(4.2)

$$
- 2\gamma \Re \langle \varphi^+ | P_0^- | D_0 | \varphi^+ \rangle - \| |D_0|^{1/2} P_0^- \varphi^+ \|^2.
$$

(4.3)

(4.4)

We employ (3.45) with $\varepsilon = \delta/4$ to estimate the second term in (4.3) from below by $-(\delta/4)\langle \varphi^+ | \bar{H}_f \varphi^+ \rangle - (4c^4/\delta^2) \| \varphi^+ \|^2$. Here $\bar{H}_f = H_f + E$ and we choose $E = 16d_1^2$. Then $c^4$ is proportional to $(d_0^2 + 2d_{-1}^2)^2$. The term in (4.3) can be estimated from below by means of (3.46), where we choose $E = 16d_1^2$, $\varepsilon = \delta/(8\gamma)$, and $\tau = (1 - \gamma/\gamma_\rho)/(2\gamma) = \rho/(2\gamma[1 + \rho])$. With this choice of $\tau$ the portion of the kinetic energy in (4.1) compensates for the contribution coming from the first term on the right side in (3.46). By Kato’s inequality the term in (4.2) is bounded from below by $-(1 + \pi\gamma/2) \| |D_0|^{1/2} P_0^- \varphi^+ \|^2$, which we estimate further by means of (3.45) with $E = 16d_1^2$ and $\varepsilon = \delta/(4 + 2\pi\gamma)$. Combining these remarks with $\alpha \cdot A \geq -\delta/4$ we arrive at (2.13). □

4.2. The semi-relativistic Pauli-Fierz operator: Semi-boundedness.

Lemma 4.1. There is a constant $k \in (0, \infty)$ such that, for all $E \geq k^2 d_1^2$ and all $\phi \in \mathcal{D}_0$,

$$
\text{Re} \langle |D_A| \phi | \bar{H}_f \phi \rangle \geq (1 - k d_1 E^{-1/2}) \| |D_A|^{1/2} \bar{H}_f^{1/2} \phi \|^2.
$$

(4.5)
Proof. Let $\phi \in \mathcal{D}_0$ and set $\psi := \tilde{H}_f^{1/2} \phi$. Using (3.7) and (3.21), we have
\[
\text{Re} \left< D_A \tilde{H}_f^{-1/2} \psi \mid \text{sgn}(D_A) \tilde{H}_f^{1/2} \psi \right>
= \text{Re} \left< (D_A - T_{1/2}) \psi \mid \tilde{H}_f^{-1/2} \text{sgn}(D_A) \tilde{H}_f^{1/2} \psi \right>
= \text{Re} \left< (D_A - T_{1/2}) \psi \mid (\text{sgn}(D_A) - S_{1/2}) \psi \right>
\geq \langle |D_A| \psi \mid \psi \rangle - \| |D_A|^{1/2} \psi \| \| |D_A|^{1/2} S_{1/2} \psi \| - \|T_{1/2}\| (1 + \|S_{1/2}\|) \|\psi\|^2.
\]
Together with (3.2), (3.8), and (3.20) this gives the asserted estimate. \qed

Proof of Theorem 2.4. We pick some $\delta > 0$ and choose $E = (\delta^{-2} + k^2) d_1^2$, where $k$ is the constant appearing in Lemma 4.1. To start with we recall that
\[
(4.6) \quad D_A \phi = (-i\nabla + A)^2 \phi + S \cdot B \phi + \phi, \quad \phi \in \mathcal{D}_0,
\]
where the entries of the formal vector $S$ are $S_j = \sigma_j \otimes 1_2$ and $B$ is the magnetic field, i.e.
\[
S \cdot B = S \cdot a^\dagger (\nabla_x \wedge G) + S \cdot a (\nabla_x \wedge G).
\]
A standard estimate using Hypothesis II shows that, for every $\varphi \in \mathcal{D}_0$,
\[
|\langle \varphi \mid S \cdot B \varphi \rangle| \leq 2 d_1 \|\varphi\| \|H_f^{1/2} \varphi\| \leq \delta \langle \varphi \mid H_f \varphi \rangle + d_1^2 \delta^{-1} \|\varphi\|^2.
\]
Using $E \geq k^2 d_1^2$ in the third step, we thus obtain, for all $\phi \in \mathcal{D}_0$,
\[
\frac{1}{4} \langle \phi \mid x^{-2} \phi \rangle \leq \langle \phi \mid (-i\nabla + A)^2 \phi \rangle
\leq \langle D_A \phi \mid D_A \phi \rangle + \delta \langle \phi \mid (H_f + \delta^{-2} d_1^2) \phi \rangle - \|\phi\|^2
= \langle D_A \phi \mid D_A \phi \rangle + \delta \langle \phi \mid \tilde{H}_f \phi \rangle - (\delta k^2 d_1^2 + 1) \|\phi\|^2
\leq \langle D_A \phi \mid D_A \phi \rangle + \langle \phi \mid \delta^2 \tilde{H}_f^2 \phi \rangle + 2\text{Re} \langle |D_A| \phi \mid \delta \tilde{H}_f \phi \rangle
- (\delta k^2 d_1^2 + 3/4) \|\phi\|^2
= \|(|D_A| + \delta \tilde{H}_f) \phi\|^2 - (\delta k^2 d_1^2 + 3/4) \|\phi\|^2.
\]
Here we also used a diamagnetic inequality in the first step. The diamagnetic inequalities used here and in the first step of (4.8) below are well-known at least for classical magnetic fields. They hold true, however, also for quantized fields due to an argument by J. Fröhlich which is presented in [3] and [24]; see also [19] [20]. (The basic underlying observation is that all components $A_i(x)$ and $A_j(y)$, $i, j \in \{1, 2, 3\}$, $x, y \in \mathbb{R}^3$, of the vector potential commute and can hence be diagonalized simultaneously. In this way the problem is reduced to the
classical situation.) Since the square root is operator monotone it follows that, for all \( \phi \in \mathcal{D}_0 \),

\[
\frac{2}{\pi} \langle \phi \mid |x|^{-1} \phi \rangle \leq \langle \phi \mid -i\nabla + A \mid \phi \rangle \leq \langle \phi \mid (|D_A| + \delta \tilde{H}_f) \phi \rangle.
\]

\[\square\]

5. Exponential localization

5.1. Outline of the proof. Our next aim is to prove the main Theorems 2.2 and 2.5 which assert that low-lying spectral subspaces of the no-pair and semi-relativistic Pauli-Fierz operators are exponentially localized. We recall the general strategy of the proofs in this subsection and apply the results to the no-pair and semi-relativistic Pauli-Fierz operators in Subsections 5.2 and 5.3 respectively. The basic idea underlying the proofs is essentially due to \cite{6} and described briefly in Lemma 5.1. The technical Lemma 5.2 summarizes (and simplifies) a part of a proof from \cite{31}. Occasionally, we will also benefit from some observations made in \cite{16}.

The spectra of both the no-pair and the semi-relativistic Pauli-Fierz operators will certainly be continuous up to their minima, at least for the physically interesting choice of the form factor. In particular, we cannot employ eigenvalue equations to derive exponential decay estimates. (Of course, this would be possible if were only interested in the exponential localization of ground state eigenfunctions.) According to \cite{6} a possibility to handle this is to smooth out the spectral projection and to apply a suitable integral representation of the smoothed projection. We shall employ the following formula due to Amrein et al. \cite[Theorem 6.1.4(b)]{1} which holds for every \( f \in C_0^\infty(\mathbb{R}), \nu \in \mathbb{N}, \) and every self-adjoint operator, \( X, \) in some Hilbert space,

\[
f(X) = \sum_{\kappa=0}^{\nu-1} \frac{1}{\pi^{\kappa!}} \int_{\mathbb{R}} f^{(\kappa)}(\lambda) \Im \left[ i^{\kappa} (X - \lambda - i)^{-1} \right] d\lambda
\]

\[+ \int_0^1 \frac{t^{\nu-1}}{\pi(\nu - 1)!} \int_{\mathbb{R}} f^{(\nu)}(\lambda) \Im \left[ i^{\nu} (X - \lambda - it)^{-1} \right] d\lambda dt.\]

The following lemma is essentially due to \cite{6}.

**Lemma 5.1.** Let \( X \) and \( Y \) be self-adjoint operators in \( \mathcal{H} \) with a common domain. Let \( a > 0 \) and \( I \subset \mathbb{R} \) be a compact interval such that \( I \subset \sigma(Y) \). Assume that there exist \( C, C' \in (0, \infty) \) and another compact interval \( J \subset \sigma(Y) \) such that \( J \supset I \) and that, for all \( F \) satisfying (3.17),

\[
\left\| e^F (X - Y) \right\| \leq C, \quad \sup_{(\lambda,t) \in J \times (0,1]} \left\| e^F (Y - \lambda \pm it)^{-1} e^{-F} \right\| \leq C'.
\]

\[23\]
Then \( \text{Ran}(1_I(X)) \subset \mathcal{D}(e^{a|x|}) \) and

\[
\| e^{a|x|} 1_I(X) \| \leq c(I, J) C C',
\]

where

\[
(5.3) \quad c(I, J) = k \left( 1 + |J| + \text{dist}(I, J')^{-1} \right),
\]

for some universal constant \( k \in (0, \infty) \).

**Proof.** We find some \( f \in C_0^\infty(\mathbb{R}, [0, 1]) \) such that \( f \equiv 1 \) on \( I \) and \( \text{supp}(f) \subset J \).
Then \( e^F 1_I(X) = e^F (f(X) - f(Y)) 1_I(X) \), since \( J \subset g(Y) \). Here we can rewrite \( f(X) - f(Y) \) by means of \((5.1)\). On account of \((5.2)\) and the second resolvent identity we have, for every \( \lambda \in J \) and \( t \in (0, 1] \),

\[
(5.4) \quad \left\| e^F \left\{ (X - \lambda \pm it)^{-1} - (Y - \lambda \pm it)^{-1} \right\} \right\| \leq C C'/t.
\]

Now, we observe that the factor \( t^{\nu-1} \) in \((5.1)\) compensates for the \( 1/t \) singularity in \((5.4)\) if we choose \( \nu = 2 \). Using these remarks we readily find some \( c(I, J) \in (0, \infty) \) such that, for all \( F \) satisfying \((5.17)\), we have \( \|e^F 1_I(X)\| \leq c(I, J) C C' \).
By an appropriate choice of \( f \) we can ensure that \( c(I, J) \) has the form given in \((5.3)\). But then \( \|e^{a|x|} 1_I(X)\| \leq c(I, J) C C' \) holds true also as a consequence of the monotone convergence theorem applied to a suitable increasing sequence of weights \( F_1, F_2, \ldots \), where each \( F_j \) satisfies \((3.17)\). \( \square \)

To verify the second condition in \((5.2)\) the following lemma is helpful.

**Lemma 5.2.** Let \( Y \) be a positive operator in \( \mathcal{H} \) which admits \( \mathcal{D}_0 \) as a form core. Set \( b := \inf \sigma(Y) \) and let \( J \subset (-\infty, b) \) be some compact interval. Let \( a \in (0, 1) \) and assume that, for all \( F \) satisfying \((3.17)\), we have \( e^{\pm F} \mathcal{Q}(Y) \subset \mathcal{Q}(Y) \).
\( \text{(Notice that } e^{\pm F} \text{ maps } \mathcal{D}_0 \text{ into itself.) Assume further that there exist constants } c(a), f(a), g(a), h(a) \in [0, \infty) \text{ such that } c(a) < 1/2 \text{ and } b g(a) + h(a) < b - \max J \text{ and, for all } F \) satisfying \((3.17)\) and \( \varphi \in \mathcal{D}_0 \),

\[
(5.5) \quad \left\| e^{aF} e^{aF} (Y - \varphi) \right\| \leq c(a) \left\| e^{aF} (Y - \varphi) + f(a) \left\| \varphi \right\| \right\|,
\]

\[
(5.6) \quad \text{Re} \left\langle \varphi \left| e^{aF} e^{aF} \varphi \right\rangle \geq (1 - g(a)) \left\| e^{aF} \right\| \varphi \rangle - h(a) \left\| \varphi \right\|^2.
\]

Then we have, for all \( F \) satisfying \((3.17)\),

\[
(5.7) \quad \sup_{(\lambda, t) \in J \times (0, 1]} \left\| e^{aF} (Y - \lambda \pm it)^{-1} e^{aF} \right\| \leq (b - \max J - h(a) - b g(a))^{-1}.
\]

**Proof.** Since \( e^{aF} \) is an isomorphism on \( \mathcal{H} \) the densely defined operators \( e^{aF} Y e^{aF} \) and \( Y \) have the same resolvent set and

\[
(5.8) \quad \mathcal{R}_F(z) := e^{aF} (Y - z)^{-1} e^{aF} = (e^{aF} Y e^{aF} - z)^{-1}, \quad \text{for } z \in g(Y).
\]

24
In particular, \( e^F Y e^{-F} \) is closed because its resolvent set is not empty. Since \( e^{-F} \) is a self-adjoint isomorphism we further know that
\[
(5.9) \quad (e^F Y e^{-F})^* = e^{-F} (e^F Y)^* = e^{-F} Y e^F.
\]
By assumption we have
\[
(5.10) \quad \mathcal{D}(e^F Y e^{-F}) = e^F \mathcal{D}(Y) \subset e^F \mathcal{Q}(Y) \subset \mathcal{Q}(Y).
\]
Condition (5.5) and \( c(a) < 1/2 \) imply that \( (e^F Y e^{-F})|_{\mathcal{Q}_0} \) has a distinguished closed and sectorial extension which we denote by \( Y_F \). This extension is the only closed extension having the properties \( \mathcal{D}(Y_F) \subset \mathcal{Q}(Y), \mathcal{D}(Y^+_F) \subset \mathcal{Q}(Y) \), and \( it \in \mathcal{Q}(Y_F) \), for all \( t \in \mathbb{R} \) such that \( |t| \) is larger than some positive constant; see \[21\]. Thanks to (5.8), (5.9), and (5.10), we know that \( e^F Y e^{-F} \) is a closed extension enjoying all these properties, whence \( Y_F = e^F Y e^{-F} \). By virtue of (5.6) we have, with \( \delta := b - \max J - b g(a) - h(a) > 0 \) and for all \( \lambda \in J \), \( t \in (0, 1] \), \( \varphi \in \mathcal{D}_0 \),
\[
(5.11) \quad \Re \langle \varphi \mid Y_F - \lambda \pm it \rangle \varphi \geq \{(1 - g(a)) b - \lambda - h(a)\} \|\varphi\|^2 \geq \delta \|\varphi\|^2.
\]
Therefore, the numerical range of \( Y_F - \lambda \pm it \) is contained in the half space \( \{\zeta \in \mathbb{C} : \Re \zeta \geq \delta\} \) \[21\]. Theorem VI.1.18 and Corollary VI.2.3]. Moreover, by (5.8) the deficiency of \( Y_F - \lambda \pm it \) is zero, and we may hence estimate the norm of \( (Y_F - \lambda \pm it)^{-1} \) by the inverse distance of \( \lambda \pm it \) to the numerical range of \( Y_F \) \[21\] Theorem V.3.2]. We thus obtain the estimate \( \| (Y_F - \lambda \pm it)^{-1} \| \leq \delta^{-1} \), for all \( \lambda \in J \) and \( t \in (0, 1] \), which together with (5.8) proves the lemma. \( \square \)

5.2. **The no-pair operator: Localization.** To begin with we introduce a scaled partition of unity. Namely, we pick some \( \tilde{\mu} \in C^\infty(\mathbb{R}, [0, 1]) \) such that \( \tilde{\mu} \equiv 1 \) on \( \{|x| \leq 1\} \) and \( \tilde{\mu} \equiv 0 \) on \( \{|x| \geq 2\} \) and observe that \( \theta := \tilde{\mu}^2 + (1 - \tilde{\mu})^2 \geq 1/2 \). Then we set, for \( R \geq 1 \) and \( x \in \mathbb{R}^3 \), \( \mu_{1,R}(x) := \tilde{\mu}(x/R)/\theta^{1/2}(x/R) \), and \( \mu_{2,R}(x) := (1 - \tilde{\mu}(x/R))/\theta^{1/2}(x/R) \), so that \( \mu_{1,R}^2 + \mu_{2,R}^2 = 1 \). We define
\[
(5.12) \quad e(\gamma) := \inf \sigma(H^\gamma + E P^+_A), \quad \gamma \in [0, \gamma^\gamma_c),
\]
where \( H^\gamma \) is considered as an operator acting in \( \mathcal{H}_A^+ \). The parameter \( E > 0 \) is chosen sufficiently large later on. We shall apply Lemmata 5.1 and 5.2 with
\[
(5.13) \quad X^\gamma = H^\gamma + E P^+_A + P^-_A H_f P^-_A + e(0) P^-_A,
\]
\[
(5.14) \quad Y^\gamma = X^\gamma + (e(0) - e(\gamma_R)) P^+_A \mu_{1,R}^2 P^+_A,
\]
where \( H^\gamma \) is now considered as an operator acting in \( \mathcal{H} \) and
\[
(5.15) \quad \gamma_R := (1 + 1/R) \gamma/(1 - c/R), \quad R > c.
\]
Here \( c \geq 1 \) is the constant appearing in Lemma 5.3 \( X^\gamma \) and \( Y^\gamma \) are self-adjoint on the same domain \( \mathcal{D}(X^\gamma) = \mathcal{D}(Y^\gamma) = \mathcal{D}(H^\gamma) \cap \mathcal{D}(P^-_A H_f P^-_A) \) and both operators admit \( \mathcal{D}_0 \) as a form core.
The idea to define the comparison operator $Y_{\gamma}^{\text{np}}$ by essentially adding only a cut-off function located in a ball about the origin to $X_{\gamma}^{\text{np}}$ is borrowed from [10]. An obvious consequence of this choice (and the bound (3.28)) is that the first condition in (5.2) is fulfilled.

**Lemma 5.3.** Let $\gamma \in [0, \gamma_c^{\text{np}})$ and $E > \max\{1, (2d)_1^2\}$. Then there exist universal constants $c, c' \in [1, \infty)$ such that, for all $R > \max\{c, (\gamma + c_c^{\text{np}})/(\gamma_c^{\text{np}} - \gamma)\}$ and all $\varphi \in \mathcal{D}_0$, $\varphi^+ := P_{\tilde{A}}^+ \varphi$,

$$\langle \varphi | Y_{\gamma}^{\text{np}} \varphi \rangle \geq c(0) \|\varphi\|^2 - (c'/R) (c(0) + |c(\gamma)|) \|\varphi^+\|^2.$$

**Proof.** Let $\varphi \in \mathcal{D}_0$ and set $\varphi^+ := P_{\tilde{A}}^+ \varphi$. Since $\nabla = \sum_{i=1,2} \mu_{i,R} \nabla \mu_{i,R}$ we have

$$\langle \varphi | (H_{\gamma}^{\text{np}} + E P_{\tilde{A}}^+) \varphi \rangle = \sum_{i=1,2} \langle \varphi^+ | P_{\tilde{A}}^+ \mu_{i,R} (D_{\tilde{A}} - \gamma/|x| + \tilde{H}_f) \mu_{i,R} P_{\tilde{A}}^+ \varphi^+ \rangle,$$

where $\tilde{H}_f = H_f + E$. On account of Corollary 3.8 with $\varepsilon = 1/R$ we thus have, for all $R \geq 1$,

$$\langle \varphi | (H_{\gamma}^{\text{np}} + E P_{\tilde{A}}^+) \varphi \rangle \geq (1 - 1/R) \sum_{i=1,2} \langle \varphi^+ | \mu_{i,R} P_{\tilde{A}}^+ (D_{\tilde{A}} + \tilde{H}_f) P_{\tilde{A}}^+ \mu_{i,R} \varphi^+ \rangle$$

$$- (1 + 1/R) \langle \varphi^+ | \mu_{1,R} P_{\tilde{A}}^+ \gamma/|x| P_{\tilde{A}}^+ \mu_{1,R} \varphi^+ \rangle - \|\mu_{2,R}^2 \gamma/|x|\|_\infty \|\varphi^+\|^2$$

$$- \sum_{i=1,2} \frac{3K(1 + R) \|\nabla \mu_{i,1}\|^2}{R^2} \|\tilde{H}_f^{1/2} \varphi^+\|^2.$$  \hspace{1cm} (5.16)

($K$ is the constant appearing in Corollary 3.8) We set $C_{\mu} := \sum_{i=1,2} \|\mu_{i,1}\|_\infty$ and apply Corollary 3.8 once more to obtain

$$\|\tilde{H}_f^{1/2} \varphi^+\|^2 = \sum_{i=1,2} \|\tilde{H}_f^{1/2} \mu_{i,R} P_{\tilde{A}}^+ \varphi^+\|^2$$

$$\leq 2 \sum_{i=1,2} \langle \varphi^+ | \mu_{i,R} P_{\tilde{A}}^+ \tilde{H}_f P_{\tilde{A}}^+ \mu_{i,R} \varphi^+ \rangle + \frac{2K C_{\mu}}{R} \|\tilde{H}_f^{1/2} \varphi^+\|^2.$$

Here we also estimated $1 + 1/R \leq 2$. This implies, for $R \geq 4K C_{\mu}$,

$$\|\tilde{H}_f^{1/2} \varphi^+\|^2 \leq 4 \sum_{i=1,2} \langle \varphi^+ | \mu_{i,R} P_{\tilde{A}}^+ \tilde{H}_f P_{\tilde{A}}^+ \mu_{i,R} \varphi^+ \rangle.$$
Combining the previous estimate with (5.16) and setting \( c := 1 + 24 K C_{\mu} \) we arrive at

\[
\langle \varphi \mid (H_{\gamma}^{np} + EP_{A}^{+}) \varphi \rangle \\
\geq (1 - c/R) \langle \mu_{1,R} \varphi^{+} \mid (H_{\gamma}^{np} + EP_{A}^{+}) \mu_{1,R} \varphi^{+} \rangle \\
= (1 - c/R) \langle \mu_{2,R} \varphi^{+} \mid (H_{\gamma}^{np} + EP_{A}^{+}) \mu_{2,R} \varphi^{+} \rangle - (\gamma/R) \| \varphi^{+} \|^2
\]

Next, we show that the conditions (5.5) and (5.6) required in Lemma 5.2 are satisfied. We abbreviate

\[
\Delta(\gamma_R) := e(0) - e(\gamma_R) = \Sigma_{np} - \inf \sigma(H_{\gamma}^{np}).
\]

**Lemma 5.4.** There is some constant \( k_1 \in (0, \infty) \) such that, for all \( \gamma \in (0, \gamma_{np}^{\mu}) \), \( \mathbf{G} \) fulfilling Hypothesis 7, \( a \in (0, 1/2], \) all \( F \) satisfying (3.17), all sufficiently large \( E > 0 \) (depending only on \( d_{-1}, d_0, d_1, \) and \( \gamma \)), and all \( \varphi \in \mathcal{D}_0 \),

(5.17) \[
|\text{Re} \left[ \langle \varphi \mid (e^F Y_{\gamma}^{np} e^{-F} - Y_{\gamma}^{np}) \varphi \rangle \right] | \leq k_1 a^2 \langle \varphi \mid (Y_{\gamma}^{np} + \Delta(\gamma_R) + \Sigma_{np}) \varphi \rangle,
\]

and

(5.18) \[
|\langle \varphi \mid (e^F Y_{\gamma}^{np} e^{-F} - Y_{\gamma}^{np}) \varphi \rangle | \leq k_1 a \langle \varphi \mid (c(\gamma) Y_{\gamma}^{np} + \Delta(\gamma_R) + \Sigma_{np}) \varphi \rangle,
\]

where \( c(\gamma) = (\gamma_{np}^{\mu} + \gamma)/(\gamma_{np}^{\mu} - \gamma) \).

**Proof.** Let \( \varphi \in \mathcal{D}_0 \) and let \( Y_{\gamma}^{np,F} \) denote the operator obtained by replacing the projections \( P_{A}^{+} \) and \( P_{A}^{-} \) in \( Y_{\gamma}^{np} \) by \( P_{A}^{F} \) and \( P_{A}^{-F} := e^F P_{A}^{-} e^{-F} \), respectively, i.e.

\[
Y_{\gamma}^{np,F} := P_{A}^{F} (D_{A} - \frac{\gamma}{|x|} + \tilde{H}_f + \Delta(\gamma_R) \mu_{2,R}^2) P_{A}^{F} + P_{A}^{-F} \tilde{H}_f P_{A}^{-F} + \Sigma_{np} P_{A}^{-F},
\]

where \( \Delta(\gamma_R) = e(0) - e(\gamma_R) \). Then \( e^{-F} D_{A} e^{F} = D_{A} - i\alpha \cdot \nabla F \) implies

\[
\langle \varphi \mid e^F Y_{\gamma}^{np} e^{-F} \varphi \rangle = \langle \varphi \mid Y_{\gamma}^{np,F} \varphi \rangle = i \langle \varphi \mid P_{A}^{F} \alpha \cdot \nabla F P_{A}^{F} \varphi \rangle
\]

\[
= i \langle \varphi \mid P_{A}^{+} \alpha \cdot \nabla F P_{A}^{+} \varphi \rangle + i \langle \varphi \mid K_{F} \alpha \cdot \nabla F K_{F} \varphi \rangle
\]

\[
- i \langle \varphi \mid K_{F} \alpha \cdot \nabla F P_{A}^{+} \varphi \rangle - i \langle \varphi \mid P_{A}^{+} \alpha \cdot \nabla F K_{F} \varphi \rangle.
\]

Since \( \|\alpha \cdot \nabla F\| \leq a \) and \( \|K_{F}\| \leq a J(a)/2 \) we thus obtain

(5.19) \[
|\langle \varphi \mid e^F Y_{\gamma}^{np} e^{-F} \varphi \rangle - \langle \varphi \mid Y_{\gamma}^{np,F} \varphi \rangle | \leq a (1 + a J(a) + a^2 J(a)^2/4) \|\varphi\|^2.
\]
Since \( \text{Re} [i \langle \varphi \mid P_A^+ \mathbf{a} \cdot \nabla F P_A^- \varphi \rangle] = 0 \) we further have
\[
\text{Re} \left[ \langle \varphi \mid e^F Y_{\gamma^p} e^{-F} \varphi \rangle - \langle \varphi \mid Y_{\gamma}^{np,F} \varphi \rangle \right] \leq a^2 (J(a) + a J(a)^2/4) \| \varphi \|^2.
\]

Assuming \( E \geq (4d_1 J(a))^2 \) we next apply Corollary 3.8 (Estimate (3.40) and its obvious analogue for \( P_A^- \)) to each of the six terms in \( \text{Re} [Y_{\gamma}^{np,F} - Y_{\gamma^p}^{np}] \) (involving the operators \( D_A, \tilde{\Delta}_f, P_A^+ \tilde{\Delta}_f P_A^+, \Delta(\gamma_R) \mu_{1,R}^2, \) and \( \Sigma_{np} \), respectively). As a result we find some universal constant, \( k_2 \in (0, \infty) \), such that, for all \( \varphi \in \mathcal{D}_0 
abla (\Delta(\gamma_R) + \Sigma_{np}) \| \varphi \|^2.
\]

In order to derive (5.18) we apply Corollary 3.6 (Estimate (3.39) with \( \varepsilon = a \) and its obvious analogue for \( P_A^- \)) to each of the six terms in \( Y_{\gamma}^{np,F} - Y_{\gamma^p}^{np} \). Proceeding in this way we find some universal constant, \( k_3 \in (0, \infty) \), such that, for every \( \varphi \in \mathcal{D}_0 
abla (\Delta(\gamma_R) + \Sigma_{np}) \| \varphi \|^2.
\]

As above we argue that \( Y_{\gamma^p}^{np} + \tilde{\Delta}_f \leq k_4 Y_{\gamma^p}^{np} = k_4 (Y_{\gamma^p}^{np} + P_A^+ \gamma P_A^-) \) and it follows from Theorem 2.1 that \( P_A^+ \gamma P_A^- \leq 2\gamma (\gamma_{\gamma^p} - \gamma)^{-1} Y_{\gamma}^{np} \), provided \( E > 0 \) is sufficiently large depending on \( d_{-1}, d_0, d_1 \), and \( \gamma \). Combining these remarks with (5.19) we arrive at (5.18). □

In the following lemma we verify another assumption made in Lemma 5.2.
Lemma 5.5. There exist constants $c_1, c_2 \in (0, \infty)$ such that, for all $F : \mathbb{R}^3 \to \mathbb{R}$ satisfying (5.17) and all $\varphi \in \mathcal{D}_0$,

\begin{equation}
\langle e^F \varphi \mid Y_0^{np} e^F \varphi \rangle \leq c_1 \|e^F\|^2 \langle \varphi \mid Y_0^{np} \varphi \rangle + c_2 \|e^F\| \|\varphi\|^2.
\end{equation}

In particular, $e^F \mathcal{Q}(Y_0^{np}) \subset \mathcal{Q}(Y_0^{np})$, for every $\gamma \in [0, \gamma_c^{np})$.

Proof. It is clear that we only have to comment on the unbounded terms in $Y_0^{np}$. In [31] Equation (4.24) and the succeeding paragraphs we proved that

\begin{equation}
\langle \varphi \mid e^F P_\Lambda^\pm (\pm D_\Lambda) e^F \varphi \rangle \leq c_3 \|e^F\|^2 \langle \varphi \mid P_\Lambda^\pm (\pm D_\Lambda) \varphi \rangle + c_4 \|e^F\| \|\varphi\|^2,
\end{equation}

for every $\varphi \in \mathcal{D}_0$. We derived this bound in [31] for classical vector potentials. The proof works, however, also for the quantized vector potential without any change. Moreover, we only treated the choice of the plus sign in (5.24). But again an obvious modification of the proof in [31] shows that (5.24) is still valid when we choose the minus sign. (This will actually be necessary only in the next subsection where we treat the semi-relativistic Pauli-Fierz operator.) On account of (5.21) it thus remains to show that $\|\widetilde{H}_f^{1/2} P_\Lambda^\pm e^F \varphi\| \leq c_5 \|e^F\| \|\widetilde{H}_f^{1/2} \varphi\|$. This follows, however, immediately from (3.21) which implies $\|H_f^{1/2} P_\Lambda^\pm e^F \varphi\| \leq (1 + \|S_{1/2}\|/2) \|e^F \widetilde{H}_f^{1/2} \varphi\|$. From these remarks we readily derive the asserted estimate which shows that $e^F \mathcal{Q}(Y_0^{np}) \subset \mathcal{Q}(Y_0^{np})$ holds true. But from Theorem 2.1 we know that $\mathcal{Q}(Y_0^{np}) = \mathcal{Q}(Y_0^{np})$, for every $\gamma \in [0, \gamma_c^{np})$. \hfill \Box

Proof of Theorem 2.2. Assume that $\gamma < \gamma_c^{np}$ and let $I \subset \mathbb{R}$ be a compact interval with max $I < \Sigma_{np}$. We fix some $E \in [1, \infty)$ and set $I_E := I + E$. In the following we assume that $E$ is so large that Lemmata 5.3 and 5.4 are applicable. (Then $E$ depends on $d_\Lambda, d_0, d_1$, and $\gamma$.) Let $k_1$ be the constant appearing in the statement of Lemma 5.4, $\lambda := \max I_E$, and $e(0) = \Sigma_{np} + E$. We assume that $a \in (0, 1/2)$ is so small that $k_1 a (\gamma_c^{np} + \gamma)(\gamma_c^{np} - \gamma)^{-1} < 1/2$ and $\varepsilon := \{1 - (\lambda/e(0)) - 5k_1 a^2\}/4 > 0$. On account of Lemma 5.3 we may fix some $R \geq 1$ such that $b := \inf \sigma(Y_0^{np}) \geq e(0) - \varepsilon$, which implies $1/b < (1/e(0))(1 + 2\varepsilon)$. (We can choose $R = c_1 e(0)/\varepsilon$, for some universal constant $c_1$.) By virtue of Lemmata 5.4 and 5.5 we can then apply Lemma 5.2 with $g(a) := k_1 a^2$, $h(a) = k_1 a^2 (\Delta(\gamma_R) + \Sigma_{np})$, and $J := I_E + [-b \varepsilon, b \varepsilon]$. In view of Theorem 2.1 we can further assume that $\Delta(\gamma_R) \leq \Sigma_{np} + E = e(0)$. From these remarks we infer that $b - \max J - b g(a) - h(a) \geq b \varepsilon$. This ensures that the second condition in (5.2) is fulfilled with $C' \leq 1/(b \varepsilon)$. The first bound in (5.2) is also valid since $X_\gamma^{np} - Y_\gamma^{np} = (\Delta(\gamma_R) P_\Lambda^+ \mu_1^{1/2} R P_\Lambda^+) \quad \text{and} \quad \|e^F P_\Lambda^+ \mu_1^{1/2} R P_\Lambda^+\| \leq \|P_\Lambda^+\| \|e^F \mu_1^{1/2} R\| \leq \text{const} \cdot e^{2aR}$ Then Lemma 5.1 with $\text{dist}(I, J^c) = b \varepsilon$ and $|J| = |I| + 2b \varepsilon$ implies
that
\[ \| e^{i|x|} 1_{I_E}(X_{\gamma}^p) P_A^+ \| \leq \text{const} \cdot \{ 1 + (1 + |I|)/(b\varepsilon) + 1/(b\varepsilon)^2 \} e(0) e^{2\alpha R}. \]

Since \(1/b \leq 2/e(0) \leq 2\) and \(P^+_A X_{\gamma}^p = X_{\gamma}^p P^+_A = (H_{\gamma}^p + E P^+_A) \oplus 0\) this proves Theorem 2.2. (Keeping track of all conditions imposed on \(\mu_1, \mu_2, \mu_3, \mu_4\)) we see that we can choose \(-E\) proportional to the term in the second line in (2.13), whence \(E \to 0, d_i \to 0, i \in \{-1, 0, 1\}\.) \(\square\)

5.3. The semi-relativistic Pauli-Fierz operator: Localization. Again we employ the partition of unity \(\mu_1, \mu_2, \mu_3, \mu_4 = 1\) constructed in the first paragraph of Subsection 5.2. We set \(\gamma \in [0, \gamma_c]^{\text{PF}}, \) so that \(\epsilon(0) = \Sigma_{\gamma, \text{PF}}\) and apply Lemma 5.1 with \(X_{\gamma}^{\text{PF}} = H_{\gamma}^{\text{PF}}, \) \(Y_{\gamma}^{\text{PF}} = H_{\gamma}^{\text{PF}} + (\epsilon(0) - \epsilon(\gamma_R)) \mu_1, R \geq 1.\)

Of course, \(X_{\gamma}^{\text{PF}}\) and \(Y_{\gamma}^{\text{PF}}\) are self-adjoint on the same domain and both admit \(D_0\) as a form core. The remaining conditions of Lemma 5.1 are easier to verify than in the previous subsection since only the kinetic energy term in the semi-relativistic Pauli-Fierz operator is non-local.

Lemma 5.6. There is some \(C \in (0, \infty)\) such that, for all \(\gamma \in (0, \gamma_c]^{\text{PF}}, R \geq \gamma_c^{\text{PF}} / (\gamma_c^{\text{PF}} - \gamma), G\) fulfilling Hypothesis 7 and \(\varphi \in D_0,\)

\[ \langle \varphi | H_{\gamma}^{\text{PF}} \varphi \rangle \geq \epsilon(\gamma_R) \| \mu_1 R \varphi \|^2 + \epsilon(0) \| \mu_2 R \varphi \|^2 - \frac{\epsilon(0) + \epsilon(\gamma_R) + C}{R} \| \varphi \|^2. \]

Proof. Let \(\varphi \in D_0.\) We write \(|D_A| = P^+_A D_A P^+_A - P^-_A D_A P^-_A\) and obtain by means of Corollary 3.8 (and its obvious analogue for \(P^-_A\))

\[ \langle \varphi | |D_A| \varphi \rangle = \sum_{\zeta = \pm} \sum_{i=1,2} \langle \varphi | P^\zeta_A \mu_i (\zeta 1) D_A \mu_i P^\zeta_A \varphi \rangle \]

\[ \geq (1 - 1/R) \sum_{i=1,2} \langle \varphi | \mu_i |D_A| \mu_i \varphi \rangle - \frac{2C\mu K(1 + R)}{R^2} \| \varphi \|^2, \]

where \(C\) is \(\| \nabla \mu_1 \|_\infty + \| \nabla \mu_2 \|_\infty.\) The remaining term, \(-\gamma/|x| + \tilde{H}_f,\) in \(H_{\gamma}^{\text{PF}}\) commutes with \(\mu_1\) and \(\mu_2,\) so the assertion becomes evident. \(\square\)

In the next lemma we verify the conditions (5.5) and (5.6) of Lemma 5.2. In contrast to the previous subsection we can now choose \(g = 0\) in (5.6). This
Lemma 5.7. For all \( a \in (0, 1) \), \( F \) satisfying (3.17), \( \gamma \in (0, \gamma_c^{PF}) \), \( G \) fulfilling Hypothesis \( \mathcal{H} \) and \( \varphi \in \mathcal{D}_0 \),

\[
\operatorname{Re} \left[ \langle \varphi | (e^F H_\gamma^{PF} e^{-F} - H_\gamma^{PF}) \varphi \rangle \right] \leq (3/2) a^2 J(a)^2 \| \varphi \|^2.
\]

Moreover, for every \( \varepsilon > 0 \), there is some constant, \( C(a, \gamma, \varepsilon) \in (0, \infty) \), such that

\[
\operatorname{Re} \left[ \langle \varphi | (e^F H_\gamma^{PF} e^{-F} - H_\gamma^{PF}) \varphi \rangle \right] \leq \varepsilon \langle \varphi | H_\gamma^{PF} \varphi \rangle + C(a, \gamma, \varepsilon) \| \varphi \|^2.
\]

Proof. On \( \mathcal{D}_0 \) the operator \( \operatorname{Re} [e^F H_\gamma^{PF} e^{-F} - H_\gamma^{PF}] \) appearing on the left side of (5.26) equals

\[
\operatorname{Re} \left[ e^F |D_\varphi| e^{-F} - |D_\varphi| \right] = \frac{1}{2} \left[ e^{-F}, [ |D_\varphi|, e^F ] \right] = \frac{1}{2} \left[ e^{-F}, D_\varphi [\text{sgn}(D_\varphi), e^F] - i \alpha \cdot (\nabla F) e^F \text{sgn}(D_\varphi) \right] = \frac{1}{2} D_\varphi [e^{-F}, [\text{sgn}(D_\varphi), e^F]] - i \alpha \cdot \nabla F (K_{0,-} + K_{0,F}),
\]

where we use the notation (3.25). On account of Lemmata 3.6 and 3.7 this implies (5.26). Moreover, since

\[
e^F |D_\varphi| e^{-F} - |D_\varphi| = -2 D_\varphi K_F + i \alpha \cdot (\nabla F) e^F \text{sgn}(D_\varphi) e^{-F}
\]

holds true on \( \mathcal{D}_0 \), the left hand side of (5.27) is less than or equal to

\[
\varepsilon_1 \langle \varphi | |D_\varphi| \varphi \rangle + \varepsilon_1^{-1} \| |D_\varphi|^{1/2} K_F \| ^2 \| \varphi \|^2 + a \| e^F \text{sgn}(D_\varphi) e^{-F} \| ^2 \| \varphi \|^2 \leq \varepsilon_1 \text{const}(\gamma) \langle \varphi | H_\gamma^{PF} \varphi \rangle + \text{const}(a, \varepsilon_1) \| \varphi \|^2,
\]

for every \( \varepsilon_1 > 0 \). This proves (5.27). \( \square \)

Lemma 5.8. There exist constants, \( c_1, c_2 \in (0, \infty) \), such that, for all \( a \in (0, 1) \) and \( F \) satisfying (3.17),

\[
\langle e^F Y_0^{PF} e^F \varphi \rangle \leq c_1 \| e^F \| ^2 \langle \varphi | Y_0^{PF} \varphi \rangle + c_2 \| e^F \| ^2 \| \varphi \|^2, \quad \varphi \in \mathcal{D}_0.
\]

In particular, \( e^F Q(Y_\gamma^{PF}) \subset Q(Y_\gamma^{PF}) \), for every \( \gamma \in (0, \gamma_c^{PF}) \).

Proof. Of course, \( \| \mu_{1,R} e^F \varphi \| ^2 \leq \| e^F \| ^2 \| \mu_{1,R} \varphi \| ^2 \) and, since \( H_f \) and \( e^F \) commute, \( \| H_f^{1/2} e^F \varphi \| ^2 \leq \| e^F \| ^2 \| H_f^{1/2} \varphi \| ^2 \). To conclude we write \( |D_\varphi| = P_\varphi^+ D_\varphi - P_\varphi^- D_\varphi \) and again employ the bound (5.24) derived in [31]. \( \square \)
Proof of Theorem 2.5. Let $\gamma \in (0, \gamma_{c_{PF}})$, let $I \subset (-\infty, \Sigma_{PF})$ be some compact interval, and let $a \in (0, 1)$ satisfy $\varepsilon := (\Sigma_{PF} - \max I - (3/2) a^2 J(a)^2)/3 > 0$. By virtue of Lemma 5.6 we may choose $R \geq \gamma_{c_{PF}}/(\gamma_{c_{PF}} - \gamma)$ so large that $Y_{PF}^\gamma \geq \varepsilon(0) - \varepsilon$. On account of Lemmata 5.7 and 5.8 we may apply Lemma 5.2 with $J = I + [-\varepsilon, \varepsilon]$ and $h(a) = (3/2) a^2 J(a)^2$, $g(a) = 0$, and $c(a) = 1/4$. It ensures that the second condition in (5.2) is fulfilled also, with $C \geq \varepsilon(0) - (\gamma_{PF} - \gamma)\varepsilon$. Moreover, $(\varepsilon F(X_{PF}^\gamma - Y_{PF}^\gamma)) = (\varepsilon(0) - \varepsilon(\gamma_R))\|\varepsilon F \mu_{1, R}^F\|$ and $\|\varepsilon F \mu_{1, R}^F\| \leq e^{2aR}$, so the first condition in (5.2) is fulfilled also, with $C = (\varepsilon(0) - (\gamma_{PF} - \gamma)\varepsilon) e^{2aR}$. Therefore, Theorem 2.5 is a consequence of Lemma 5.1 and (2.22), which implies that $|\varepsilon(\gamma_R)| \leq \Sigma_{PF} + \text{const} \cdot d_1^2$. □

Appendix A. The ground state energy and ionization threshold for small field strength

In this appendix we prove the perturbative estimates on the ground state energies and ionization thresholds of the no-pair and semi-relativistic Pauli-Fierz operators asserted in Remarks 2.3 and 2.6 respectively. In the whole appendix we always assume that

$$0 < d_1 \leq 1,$$
$$0 < d_2 \leq 1.$$

Moreover, we fix some value of $E$ such that

(A.1) \quad $2d_1/E^{1/2} \leq 1/2$.

We start with the semi-relativistic Pauli-Fierz operator.

Proof of Remark 2.6. For every $\gamma \in (0, \gamma_{c_{PF}})$, we let $E_{el}^C(\gamma)$ denote the (positive) ground state energy of Chandrasekhar’s operator, $|D_0| - \gamma/|x|$, and $\phi_{el}^C(\gamma)$ a corresponding ground state eigenfunction. For $\varepsilon > 0$, we set $\gamma_{\varepsilon} := 1/(1 + \varepsilon)\gamma$. Using the minimax principle and Kato’s inequality, which can be written as $1/|x| \leq (\gamma_{c_{PF}} - \gamma)^{-1}(|D_0| - \gamma/|x|)$, it is easy to see that

$$0 \leq E_{el}^C(\gamma_{\varepsilon}) - E_{el}^C(\gamma) \leq \frac{\varepsilon}{1 + \varepsilon} c(\gamma) E_{el}^C(\gamma), \quad c(\gamma) := \frac{\gamma}{\gamma_{c_{PF}} - \gamma}.$$

Next, let $\varphi \in H^{1/2}(\mathbb{R}^3) \otimes \mathcal{H}_0$. On account of Lemma 3.9 we have

$$|\langle \varphi | P_{A}^\pm (\pm D_0) P_{A}^\pm \varphi \rangle - \langle \varphi | P_{0}^\pm (\pm D_0) P_{0}^\pm \varphi \rangle|$$

(A.2) \quad $\leq \varepsilon |\langle \varphi | P_{0}^\pm (\pm D_0) P_{0}^\pm \varphi \rangle| + \frac{(1 + \varepsilon)}{(1 - 2d_1/E^{1/2})^2} \|\tilde{H}_{f}^{1/2} \varphi\|^2$.
Moreover, by virtue of Lemma 3.4 we find, for every $\delta > 0$,
\[
\left| \left< \varphi \mid P_A^+ \alpha \cdot A P_A^+ \varphi \right> \right| \\
\leq \| \varphi \| \| \alpha \cdot A \tilde{H}_f^{-1/2} P_A^+ \| \left( 1 + \frac{1}{2} \cdot \frac{d_1/E^{1/2}}{1 - 2d_1/E^{1/2}} \right) \| \tilde{H}_f^{1/2} \varphi \| \\
\text{(A.3)} \\
\leq C d_* \left( \delta \left< \varphi \mid \tilde{H}_f \varphi \right> + \delta^{-1} \| \varphi \|^2 \right),
\]
where $C \in (0, \infty)$ is some universal constant. Here we used (2.6), (2.7), $d_* := d_0^2 + 2d_1^2$, and (A.4). Since $|D_A| = P_A^+ D_A P_A^+ - P_A^- D_A P_A^-$ and $\tilde{H}_f \Omega = E \Omega$ the above estimates with $\varphi = \phi_{el}(\gamma) \otimes \Omega$ and $\delta = 1$ show that
\[
\left< \phi_{el}(\gamma) \otimes \Omega \mid |D_A| - \frac{\gamma}{|x|} + \tilde{H}_f \right> \phi_{el}(\gamma) \otimes \Omega \\
\leq (1 + \varepsilon) \left< \phi_{el}(\gamma) \mid |D_0| - \gamma / |x| \right> \phi_{el}(\gamma) + E \\
\text{(A.4)} \\
\leq (1 + \varepsilon)(1 + c(\gamma)) E_{el}(\gamma) + E + \left( \frac{1}{\varepsilon} O(d_*^2) + O(d_*) \right) (1 + E),
\]
provided (A.1) holds true. If we choose $\varepsilon = d_*$, then we find some $C_0' \in (0, \infty)$ such that
\[
\inf \sigma(H_0^{PF}) \leq E_{el}(\gamma) + C_0' (d_1 + d_*) .
\]
If we set $\gamma = 0$ in (A.1) and replace $\phi_{el}(\gamma)$ by some normalized $\chi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ satisfying $\left( \chi \mid |D_0| \chi \right) \leq 1 + \varepsilon$, then we obtain
\[
\left< \chi \otimes \Omega \mid (H_0^{PF} + E)\chi \otimes \Omega \right> \leq (1 + \varepsilon)^2 + E + \left( \frac{1}{\varepsilon} O(d_*^2) + O(d_*) \right) (1 + E).
\]
Choosing $\varepsilon = d_*$ as above we we find some $C_0' \in (0, \infty)$ such that
\[
1 \leq \Sigma_{PF}^{\Omega} \leq 1 + C_0' (d_1 + d_*) .
\]
It remains to derive the lower bound on $\inf \sigma(H_0^{PF})$. To this end we set $\tilde{\gamma} := \gamma / (1 - \varepsilon)$, for some $\varepsilon > 0$ such that $\tilde{\gamma} < \gamma^{PF}$. Moreover, we choose $\delta = 1/(2C d_*)$ in (A.3). Then (A.2) and (A.3) permit to get, for every $\varphi \in \mathcal{H}_0$,
\[
\left< \varphi \mid (H_0^{PF} + E) \varphi \right> \geq (1 - \varepsilon) \left< \varphi \mid |D_0| - \tilde{\gamma} / |x| \right> \varphi \\
- (1 - (1 + 1/\varepsilon) O(d_*^2) - 1/2) \left< \varphi \mid H_f + E \right> \varphi - 2 C d_*^2 \| \varphi \|^2 .
\]
Here we again made use of (A.1). So, choosing $\varepsilon = d_*$ and using
\[
\left< \varphi \mid (|D_0| - \tilde{\gamma} / |x|) \right> \varphi \geq E_{el}(\tilde{\gamma}) \geq \left( 1 - \frac{\varepsilon}{1 - \varepsilon} \cdot \frac{\gamma}{\gamma^{PF} - \gamma} \right) E_{el}(\gamma) ,
\]
which is a straightforward consequence of the minimax principle, we find some $C_0'' \in (0, \infty)$ such that
\[
\left< \varphi \mid H_0^{PF} \varphi \right> \geq (1 - C_0'' d_*) E_{el}(\gamma) - O(d_*^2) \| \varphi \|^2 ,
\]
for all sufficiently small values of $d_*$. 
\[\square\]
Proof of Remark 2.3. For $\gamma \in (0, \gamma^\text{np})$, we let $E_{\text{el}}^B(\gamma)$ and $\phi_{\text{el}}^B(\gamma)$ denote the ground state energy and a normalized ground state eigenfunction of the Brown-Ravenhall operator, that is,

$$P_0^+ (D_0 - \frac{\gamma}{|x|}) P_0^+ \phi_{\text{el}}^B(\gamma) = E_{\text{el}}^B(\gamma) \phi_{\text{el}}^B(\gamma).$$

It is known that $E_{\text{el}}^B(\gamma) \in [1 - \gamma_\varepsilon, 1)$ [34]. We set

$$\gamma_\varepsilon := (1 - \varepsilon)/\gamma/\gamma + \varepsilon, \quad \varepsilon \in (0, 1).$$

Then a standard argument based on the inequality [14]

$$\gamma^\text{np} \langle \phi | x^{-1} \phi \rangle \leq \langle P_0^+ \phi | D_0 P_0^+ \phi \rangle, \quad \phi \in H^{1/2}(\mathbb{R}^3, \mathbb{C})$$

and the minimax principle shows that

$$(A.5) \quad 0 \leq E_{\text{el}}^B(\gamma_\varepsilon) - E_{\text{el}}^B(\gamma) \leq \frac{\varepsilon}{1 + \varepsilon} \frac{\gamma}{\gamma^\text{np} - \gamma} E_{\text{el}}^B(\gamma).$$

Using (A.1), (A.2), (A.3) with $\delta = 1$, and

$$P_0^+ \phi_{\text{el}}^B(\gamma_\varepsilon) = \phi_{\text{el}}^B(\gamma_\varepsilon), \quad \|\phi_{\text{el}}^B(\gamma_\varepsilon)\| = 1, \quad \tilde{H}_f \phi_{\text{el}}^B(\gamma_\varepsilon) \otimes \Omega = E \phi_{\text{el}}^B(\gamma_\varepsilon) \otimes \Omega,$$

we deduce that

$$\langle \phi_{\text{el}}^B(\gamma_\varepsilon) \otimes \Omega | P_0^+ D_0^A P_0^+ \phi_{\text{el}}^B(\gamma_\varepsilon) \otimes \Omega \rangle \leq (1 + \varepsilon) \langle \phi_{\text{el}}^B(\gamma_\varepsilon) \otimes \Omega | D_0 \phi_{\text{el}}^B(\gamma_\varepsilon) \otimes \Omega \rangle$$

$$(A.6) \quad + (1 + 1/\varepsilon) O(d^2 E + C d (E + 1)).$$

Moreover, since $P_0^+ = (1/2) 1 + (1/2) \text{sgn}(D_0)$, Lemma 3.3 yields

$$\langle \phi_{\text{el}}^B(\gamma_\varepsilon) \otimes \Omega | P_0^+ \tilde{H}_f P_0^+ \phi_{\text{el}}^B(\gamma_\varepsilon) \otimes \Omega \rangle \leq (1 + \varepsilon') \| P_0^+ \tilde{H}_f^{1/2} \phi_{\text{el}}^B(\gamma_\varepsilon) \otimes \Omega \|^2$$

$$+ (1 + \frac{1}{\varepsilon'}) \| S_{1/2} \tilde{H}_f^{1/2} \phi_{\text{el}}^B(\gamma_\varepsilon) \otimes \Omega \|^2 \leq E \| P_0^+ \phi_{\text{el}}^B(\gamma_\varepsilon) \otimes \Omega \|^2 + (\varepsilon' E + (1 + 1/\varepsilon') O(d^2 E)) \|\phi_{\text{el}}^B(\gamma_\varepsilon) \otimes \Omega\|^2.$$
Putting the estimates above together we arrive at

\[
\langle \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega | P_{A}^{+} (D_{A} - \gamma/|x| + H_{f} + E) P_{A}^{+} \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega \rangle \\
\leq (1 + \varepsilon) \langle \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega | P_{0}^{+} (D_{0} - \gamma_{\varepsilon}/|x|) P_{0}^{+} \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega \rangle \\
+ E \left\| P_{A}^{+} \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega \right\|^{2} \\
+ \{ \varepsilon' E + (1 + 1/\varepsilon') \mathcal{O}(d_{1}^2) + (1 + 1/\varepsilon) E \mathcal{O}(d_{1}^2) \\
+ C_{d_{1}} (1 + E) \} \left\| \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega \right\|^{2}.
\]

(A.7)

On the other hand,

\[
\left\| \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega \right\|^{2} = \left\| P_{0}^{+} \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega \right\|^{2} \\
\leq (1 + \varepsilon) \left\| P_{A}^{+} \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega \right\|^{2} + (1 + \frac{1}{\varepsilon}) \left\| (P_{A}^{+} - P_{0}^{+}) \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega \right\|^{2} \\
\leq (1 + \varepsilon) \left\| P_{A}^{+} \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega \right\|^{2} + (1 + \frac{1}{\varepsilon}) \mathcal{O}(d_{1}^2) E \left\| \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega \right\|^{2}.
\]

(A.8)

We may assume that \( \langle \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega | H^{np}_{\gamma} \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega \rangle \) is positive. (For otherwise the upper bound on \( \inf \sigma(H^{np}_{\gamma}) \) holds true trivially.) Choosing \( \varepsilon = d_{*}, \varepsilon' = d_{1}, \) and using (A.5), (A.7), and (A.8), we find some \( C_{\gamma} \in (0, \infty) \) such that

\[
\langle \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega | P_{A}^{+} (D_{A} - \gamma/|x| + H_{f}) P_{A}^{+} \phi^{B}_{el}(\gamma_{\varepsilon}) \otimes \Omega \rangle \\
\leq \frac{(1 + \varepsilon) E_{d}(\gamma_{\varepsilon}) + \mathcal{O}(d_{1} + d_{*})}{(1 - \mathcal{O}(d_{*}))/ (1 + \varepsilon)} \leq E_{d}(\gamma) + C_{\gamma} (d_{1} + d_{*}),
\]

for all sufficiently small values of \( d_{*} \). Repeating the same argument with \( \gamma = 0 \) and with \( \phi^{B}_{el}(\gamma_{\varepsilon}) \) replaced by some normalized \( \chi \in C_{\infty}^{0}(\mathbb{R}^{3}, \mathbb{C}^{4}) \) with \( \langle \chi | P_{0}^{+} D_{0} P_{0}^{+} \chi \rangle \leq 1 + \varepsilon, \) we obtain the estimate

\[
1 \leq \Sigma_{np} \leq 1 + \mathcal{O}(d_{1} + d_{*}).
\]

Since the lower bound on \( \inf \sigma(H^{np}_{\gamma}) \) follows from Theorem 2.1 with \( \rho = d_{*} \) this concludes the proof. \( \square \)

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