Compactness of the resolvent for the Witten Laplacian

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Abstract. In this paper we consider the Witten Laplacian on 0-forms and give sufficient conditions under which the Witten Laplacian admits a compact resolvent. These conditions are imposed on the potential itself, involving the control of high order derivatives by lower ones, as well as the control of the positive eigenvalues of the Hessian matrix. This compactness criterion for resolvent is inspired by the one for the Fokker-Planck operator. Our method relies on the nilpotent group techniques developed by Helffer-Nourrigat [Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs, 1985].

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1. Introduction and main results

The Witten Laplacians on forms were initially introduced by E. Witten on a compact manifold, where he considered a new complex associated with the distorted exterior

\[ d_V = e^{-V} \circ d \circ e^V. \]

Then the Witten Laplacians on forms are defined by

\[ \Delta_V^{(c)} = d_V \circ d_V^* + d_V^* \circ d_V. \]

In this paper we will consider only the Witten Laplacians on 0-forms in the real space \( \mathbb{R}^n \), and in this case it reads

\[ \Delta_V^{(0)} = -\Delta_x + |\partial_x V(x)|^2 - \Delta_x V(x), \]

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and has the form of a Schrödinger operator $-\Delta + \bar{V}$ with $\bar{V} = |\partial_x V(x)|^2 - \Delta_x V(x)$. If replacing $V$ and $d$ respectively by $V/h$ and $hd$ in the distorted exterior we then get the semi-classical Witten Laplacian

$$\Delta_{V,h}^{(0)} = -h^2 \Delta_x + |\partial_x V|^2 - h \Delta_x V.$$ 

It is of interest in itself to analyze the spectrum of the semi-classical Witten Laplacian as the parameter $h \to 0$, cf. [8, 10, 12, 19, 20, 24, 25] and the references listed therein. If we introduce another parameter by

$$\tau = h^{-1},$$

then the semi-classical Witten Laplacian can be rewritten as

$$h^{-2} \Delta_{V,h}^{(0)} = -\Delta_x + \tau^2 |\partial_x V|^2 - \tau \Delta_x V = \Delta_{V}^{(0)}.$$ 

The latter operator is also closely related to the microhypoellipticity problem for the system of complex vector fields

$$P_j = \partial_{x_j} - i (\partial_{x_j} V(x)) \partial_t, \quad j = 1, \cdots, n, \quad i = \sqrt{-1}, \quad (1.1)$$

where limit $\tau \to +\infty$ has to be considered.

Our main goal of this paper is to explore the criterion by which the Witten Laplacian has a compact resolvent and thus admits purely discrete spectrum. This issue is closely linked with the exponential trend to the equilibrium for the spatially inhomogeneous kinetic systems, such as the non-selfadjoint Fokker-Planck and Boltzmann equations, cf. [3, 14–17]. Similar problems occur in the theory of the $\bar{\partial}$-Neumann problem, and we refer to [1, 4, 5, 7] and the surveys given in [6, 31], which reveal that there is a close relationship between the Witten Laplacians and the weighted $\square_b$-operator of the $\bar{\partial}$-complex.

By one of the elementary results on Schrödinger operators we see the Witten Laplacian is with a compact resolvent if

$$|\partial_x V(x)|^2 - \Delta_x V \to +\infty, \quad \text{as} \quad |x| \to +\infty.$$ 

More generally (see [8, 11] for instance), it is still true if

$$t |\partial_x V(x)|^2 - \Delta_x V \to +\infty, \quad \text{as} \quad |x| \to +\infty$$

for some $t \in [0, 2]$. The subject of compact resolvent for Witten Laplacian has already been explored extensively by Helffer-Nier [11] based on the idea of nilpotent Lie groups. This idea was initiated by Rothschild-Stein [30] when studying the hypoellipticity property of the Hörmander’s operators and Rothschild-Stein lifting theorem says that one can obtain the sharp local regularity by lifting the vector fields to nilpotent Lie groups and then using the analysis for the corresponding left invariant operators defined on the groups. This kind of nilpotent Lie techniques were developed further by Nourrigat [27–29] and Helffer-Nourrigat [13] for systems of pseudo-differential operators, where the pseudo-differential operators are approximated by operators defined in Euclidean space with polynomial coefficients and the problem is then reduced to the analysis of the operators with polynomial coefficients. When applying the nilpotent techniques to study the maximal estimate for
the specific Witten Laplacian, the property can be deduced from the analysis of the “limiting polynomials” (see [11] or Subsection 2.1 below for the precise definition), and based on this idea Helffer-Nier [11] have obtained the compact criteria for the resolvent of Witten Laplacian with specific potentials such as polynomials, the homogeneous and polyhomogeneous functions and the analytic functions. We also refer to the work of Helffer-Mohamed [9] for the first application of the hypoellipticity techniques to the compactness problems in mathematical physics. In this work we will give a new criterion, involving the similar conditions related to the local minimum problem. We remark that these conditions are imposed on the potential $V$ itself rather than on the “limiting polynomials”, which is somehow easy to check and apply to concrete examples. The criterion is inspirited by the one for Fokker-Planck operator [22] and the Helffer-Nier’s conjecture. The Helffer-Nier’s conjecture says the Fokker-Planck operator has a compact resolvent if and only if the Witten Laplacian $\Delta^{(0)}_{V/2}$ has a compact resolvent, which shows the close link between the compact resolvent property for the Fokker-Planck operator and the same property for the corresponding Witten Laplacian. The necessity part, that is the Witten Laplacian $\Delta^{(0)}_{V/2}$ has a compact resolvent if the Fokker-Planck operator $P$ is with a compact resolvent, has already proven by Helffer and Nier (c.f. [11, Theorem 1.1]). The reverse implication still remains open up to now, except for some special potentials (cf. [11, 17, 23] ). Recently the author [22] obtained a compactness criteria for the resolvent of Fokker-Planck operator, involving the control of the positive eigenvalues of the Hessian matrix of the potential, and the main assumption on $V$ there is that

$$\forall \ x \in \mathbb{R}^n, \ \sum_{j \in I_x} \lambda_j(x) \leq C \left(1 + |\partial_x V(x)|^2\right)^{2/3}, \quad (1.2)$$

with $\lambda_\ell, 1 \leq \ell \leq n$, the eigenvalues of the Hessian matrix $(\partial_{x_i} x_j V)_{1 \leq i, j \leq n}$ and

$$I_x = \left\{1 \leq \ell \leq n; \ \lambda_\ell(x) > 0\right\}. \quad (1.3)$$

Under the assumption (1.2) the author [22] proved the Fokker-Planck operator admits a compact resolvent, provided for some $\alpha \geq 0$,

$$\lim_{|x| \to +\infty} \ |\partial_x V(x)| = +\infty, \ \text{or} \ \lim_{|x| \to +\infty} \left(\alpha |\partial_x V(x)|^2 - \Delta_x V(x)\right) = +\infty. \quad (1.4)$$

In view of the necessity part of the Helffer-Nier’s Conjecture [11, Theorem 1.1], we see Witten Laplacian has also a compact resolvent under the same assumptions (1.2) and (1.3). As far as the Witten Laplacian is only concerned, we can go further by improving the conditions (1.2) and (1.4). Precisely, the main assumption on $V$ can be stated as follows.

Throughout the paper we will let $k \geq 2$ be a given integer, and define $\tilde{f}$ by setting

$$\tilde{f}(x) \overset{\text{def}}{=} \sum_{1 \leq |\alpha| \leq k} |\partial^\alpha V(x)|^{1/|\alpha|}. \quad (1.5)$$
Remark 1.3. As a result the Witten Laplacian and the constant τ
Moreover for any
where 0 < δ₁ < 1 is an arbitrarily small number, and M is defined by

\[ M(x) = \sum_{j \notin I_x} (-\lambda_j(x)). \]

Recall λ_\ell are the eigenvalues of the Hessian matrix \((\partial_{x_i}x_j V)_{1 \leq i,j \leq n} \).

(ii) There exists an arbitrarily small number 0 < δ₂ < 1, such that

\[ \forall \, |\alpha| = k + 1, \forall \, x \in \mathbb{R}^n, \quad |\partial^\alpha V(x)| \leq C_\alpha \left( 1 + 3(x) \right)^{k+1-\delta_2}, \]

where C_\alpha are constants depending only on α.

(iii) We have \( \tilde{f}(x) \to +\infty \) as |x| \to +\infty.

The main results can be stated as follows.

**Theorem 1.2.** Under Assumption 1.1 we can find two positive constant C, τ₀ > 0, such that

\[ \forall \, \tau \geq \tau_0, \forall \, u \in C_0^\infty(\mathbb{R}^n), \quad \|\tilde{f}_\tau u\|_{L^2}^2 \leq C \left( \Delta_{\tau V}^{(0)} u, u \right)_{L^2} + C \|u\|_{L^2}^2, \]

where \( \tilde{f}_\tau \) is defined by

\[ \tilde{f}_\tau(x) = \sum_{1 \leq |\alpha| \leq k} \tau^{1/|\alpha|} |\partial^\alpha V(x)|^{1/|\alpha|}. \]

Moreover for any \( \tau \) with 0 < \( \tau \) < \( \tau_0 \) we can find a constant C_\tau, depending on \( \tau \) and the constant C in (1.8), such that

\[ \forall \, u \in C_0^\infty(\mathbb{R}^n), \quad \|\tilde{f}_\tau u\|_{L^2}^2 \leq C_\tau \left( \Delta_{\tau V}^{(0)} u, u \right)_{L^2} + C \|u\|_{L^2}^2. \]

As a result the Witten Laplacian \( \Delta_{\tau V}^{(0)} \) has a compact resolvent for any \( \tau > 0 \).

**Remark 1.3.** (i) In view of Helffer-Nier’s conjecture we may expect that the Fokker-Planck operator is also with a compact resolvent under Assumption 1.1 above.

(ii) We need only verify the estimate (1.6) for these points where \( \Delta V \) is positive, since it obviously holds for the points where \( \Delta V \leq 0 \).

(iii) As an immediate consequence of (1.8) we have the maximal microhypoellipticity in the direction \( \tau > 0 \) (see [11]) for the system (1.1), that is, a constant C exists such that for any \( \tau > 0 \) and for any \( u \in C_0^\infty(\Omega) \) we have

\[ \|\partial_x u\|_{L^2}^2 + \tau^2 \| (\partial_x V) u\|_{L^2}^2 \leq C \|P_\tau u\|_{L^2}^2 + C \|u\|_{L^2}^2, \]

where \( \Omega \) is a neighborhood of the point \( x_0 \) such that \( \partial_x V(x_0) = 0 \), and

\[ P_\tau = \partial_x + (\partial_x V) \tau. \]
Note that \( \|P_x u\|_{L^2}^2 = (\Delta_N^{(0)} u, u)_{L^2} \), and the constant \( C \) here may depend on \( \Omega \).

We end the introduction with two examples which has already been explored in [11] by virtue of the “limiting polynomials” (see Subsection 2.1 below for the precise definition).

**Example** (Example 10.4.3.1 of [11]). Consider the potentials
\[
V_\delta(x) = x_1^2 x_2^2 + \delta (x_1^2 + x_2^2),
\]
with \( \delta \neq 0 \) given real number.

Here we apply Theorem 1.2. It is clear the statements (ii)-(iii) with \( k = 4 \) in Assumption 1.1 are fulfilled for the above \( V_\delta \). It remains to check the estimate (1.6) in the first statement (i). Direct computation gives
\[
|\nabla V_\delta(x)|^2 = 4 x_1^2 (x_2^2 + \delta)^2 + 4 x_2^2 (x_1^2 + \delta)^2
\]
and
\[
\sum_{j \in I_x} \lambda_j(x) + \sum_{j \not\in I_x} \lambda_j(x) = \Delta V_\delta(x) = 2 x_1^2 + 2 x_2^2 + 4 \delta.
\]
If \( \delta > 0 \), then for any \( x \in \mathbb{R}^n \),
\[
\sum_{j \in I_x} \lambda_j(x) + \sum_{j \not\in I_x} \lambda_j(x) \leq 2 \delta^{-2} |\partial_x V_\delta(x)|^2 + 4 \delta.
\]
Then the property in (1.6) is satisfied for \( \delta > 0 \).

Now suppose \( \delta < 0 \). For each \( (x_1, x_2) \in \mathbb{R}^2 \) we have three cases.

(a) If \( |x_1^2 + \delta| \leq -\delta/2 \) and \( |x_2^2 + \delta| \leq -\delta/2 \) then
\[
\sum_{j \in I_x} \lambda_j(x) + \sum_{j \not\in I_x} \lambda_j(x) \leq -2 \delta,
\]
and thus (1.6) holds.

(b) If \( |x_1^2 + \delta| \geq -\delta/2 \) and \( |x_2^2 + \delta| \geq -\delta/2 \) then
\[
|\nabla V_\delta(x)|^2 \geq \delta^2 |x|^2 \geq \delta^2 \left( \sum_{j \in I_x} \lambda_j(x) + \sum_{j \not\in I_x} \lambda_j(x) - 4 \delta \right),
\]
which yields (1.6).

(c) One of the terms \( |x_j^2 + \delta|, j = 1, 2 \), is bigger than \( -\delta/2 \) and the another one is smaller than \( -\delta/2 \). We may suppose without loss of generality that \( |x_1^2 + \delta| \leq -\delta/2 \) and \( |x_2^2 + \delta| \geq -\delta/2 \). Then we have
\[
-\frac{\delta}{2} \leq x_1^2 \leq -\frac{3 \delta}{2} \quad \text{and} \quad x_2^2 \geq -\frac{3 \delta}{2}, \quad \text{or} \quad -\frac{\delta}{2} \leq x_1^2 \leq -\frac{3 \delta}{2} \quad \text{and} \quad x_2^2 \leq -\frac{\delta}{2}.
\]
Note that \( \Delta V_\delta \) is bounded from above by a constant for the latter case in (1.10) and thus (1.6) holds; meanwhile for the former case we have, observing \( |x_2^2 + \delta| = x_2^2 + \delta \geq -\delta/2 \),
\[
|\nabla V_\delta(x)|^2 \geq (2 \delta)(x_2^2 + \delta)^2 \geq \delta^2 (x_2^2 + \delta)
\]
and
\[ \sum_{j \in I_x} \lambda_j(x) + \sum_{j \notin I_x} \lambda_j(x) \leq 2x_2^2 + \delta, \]
which also yields (1.6).

We then conclude that \( \triangle^{(0)}_{V_\delta} \) has a compact resolvent whenever \( \delta \neq 0 \). Note that \( \triangle^{(0)}_{V_\delta} \) doesn’t have compact resolvent when \( \delta = 0 \).

**Example** (Example 10.4.3.2 of [11]). Consider the potential \( \Phi_\delta \) defined by
\[ \Phi_\delta = (x_1^2 - x_2)^2 + \delta x_2^2. \]

By Proposition 10.21 of [11] we see \( \triangle^{(0)}_{V_\delta} \) has a compact resolvent if (and only if) \( \delta \neq 0 \).

If using Theorem 1.2 instead we can conclude that it is true for \( \delta \in \mathbb{R} \setminus \{0, -1\} \), and so our results can’t apply to this example when \( \delta = -1 \). To see this we need only verify the condition (1.6). Direct calculation gives
\[ |\partial_x \Phi_\delta| \sim 2 |x_1^2 - x_2| \cdot |x_1| + |x_1^2 - (1 + \delta)x_2|, \]
and the Hessian matrix \( H_{\Phi_\delta} \) of \( \Phi_\delta \) reads
\[ H_{\Phi_\delta} = \begin{pmatrix} 12x_1^2 - 4x_2 & -4x_1 \\ -4x_1 & 2(1 + \delta)x_2 \end{pmatrix}. \]

Write \( \mathbb{R}^2 = A_1 \cup A_2 \) with
\[ A_1 = \left\{ x = (x_1, x_2); \ |x_1| \geq 1 \right\}, \quad A_2 = \left\{ x = (x_1, x_2); \ |x_1| \leq 1 \right\}. \]

**Consider the case of** \( x \in A_1 \). Then
\[ |\partial_x \Phi_\delta| \sim 2 |x_1^2 - x_2| \cdot |x_1| + |x_1^2 - (1 + \delta)x_2| \geq |x_1^2 - x_2| + |\delta| \cdot |x_2| \]
Then the modulus of each entry in the matrix \( H_{\Phi_\delta} \) is bounded from above by \( |\partial_x \Phi_\delta| + 1 \), provided \( \delta \neq 0 \). Thus the condition (1.6) holds.

**Consider the case of** \( x \in A_2 \). Then the modulus of each entry in \( H_{\Phi_\delta} \) is bounded by \( |x_2| + 1 \). Moreover observe
\[ |\partial_x \Phi_\delta| \sim 2 |x_1^2 - x_2| \cdot |x_1| + |x_1^2 - (1 + \delta)x_2| \geq |1 + \delta| \cdot |x_2| - 1. \]
Thus the condition (1.6) holds in this case provided \( \delta \neq -1 \). As a result, it follows from Theorem 1.2 that \( \triangle^{(0)}_{\Phi_\delta} \) has a compact resolvent whenever \( \delta \neq 0, -1 \).

Finally we remark that \( \Phi_\delta \) violates (1.6) for \( \delta = -1 \) at the points \( (0, x_2) \) with \( x_2 \to -\infty \). Nonetheless, \( \triangle^{(0)}_{\Phi_{-1}} \) has indeed compact resolvent (see [11, Proposition 10.21]). It remains interesting to improve the condition (1.6) as sharp as possible, such that it can be applied to the potential \( \Phi_{-1} = x_1^4 - 2x_1^2x_2 \).
2. Proof of the main result

The proof is strongly inspired by the Helffer and Nourrigat’s recursion argument related to Kirillov’s theory, cf. [11, 26] for the induction arguments for Witten Laplacian and [13, 27, 28] for more general problems. Here we will follow the argument in the Nier’s lectures [26] and proceed through the subsections as below.

2.1. Helffer and Nourrigat’s Criteria for maximal estimates

In this part we recall the criteria for the maximal hypoellipticity developed by Helffer and Nourrigat [13] and its application to Witten Laplacian (see [11]).

Denote by $E_r$ the set of polynomials with degree less than or equal to $r$. A subset $\mathcal{L}$ of $E_r$ is called canonical if it has the following properties:

(i) If $p \in \mathcal{L}$ and $y \in \mathbb{R}^n$, then the polynomial defined by

$$q(x) = p(x + y) - p(y)$$

also lies in $\mathcal{L}$.

(ii) If $p \in \mathcal{L}$ and $\delta > 0$, then polynomial $q(x) = p(\delta x)$ also lies in $\mathcal{L}$.

(iii) $\mathcal{L}$ is a closed subset of $E_r$.

Given $p \in E_r$, we denote by $\mathcal{L}_{p,0}$ the canonical set which contains all the polynomials $q$ of degree less than or equal to $r$ vanishing at 0 and such that there exists a sequence $y_j \in \mathbb{R}^n$ with $y_j \to 0$ and sequences $\tau_j$ and $h_j$ of positive numbers with $\tau_j \to +\infty$ and $h_j \to 0$, such that

$$q(x) = \lim_{j \to \infty} \tau_j h_j^{\alpha} \partial_x^\alpha (px) - p(y_j).$$

Remark 2.1. If $q \in \mathcal{L}_{p,0}$ then there exists a sequence $y_j \in \mathbb{R}^n$ with $y_j \to 0$ and sequences $\tau_j$ and $h_j$ of positive numbers with $\tau_j \to +\infty$ and $h_j \to 0$, such that

$$q(x) = \lim_{j \to +\infty} \tau_j \left[ p(y_j + h_j x) - p(y_j) \right].$$

Applying the results of Helffer-Nourrigat [11] to the system (1.1) gives the following

**Theorem 2.2 (Helffer and Nourrigat [11]).** Given $p \in E_r$. Assume that any nonzero $q \in \mathcal{L}_{p,0} \cap E_{r-1}$ has no local minimum in $\mathbb{R}^n$. Then there exists a constant $C > 0$ and a neighborhood $\Omega$ of 0, such that the following estimate

$$\|\partial_x u\|_{L^2}^2 + \tau^2 \| (\partial_x p) u \|_{L^2}^2 \leq C \left( \Delta^{(0)}_{\tau p} u, u \right)_{L^2} + C \|u\|_{L^2}^2$$

holds for all $\tau > 0$ and for all $u \in C_0^\infty(\Omega)$. 

2.2. Stability
In this part we will show that a stronger form of the estimate (1.6) is stable for the canonical set introduced above.

We first introduce some notations to be used throughout the paper. Given a function \( \rho \in C^2(\mathbb{R}^n) \), we denote by \( \lambda_{\rho, \ell} \), \( 1 \leq \ell \leq n \) the eigenvalues of the Hessian matrix \( (\partial_{x_i} \partial_{x_j} \rho)_{1 \leq i, j \leq n} \). And define \( I_{x, \rho} \) and \( M_\rho(x) \) by setting

\[
I_{x, \rho} = \{ 1 \leq \ell \leq n; \lambda_{\rho, \ell}(x) > 0 \} \tag{2.1}
\]

and

\[
M_\rho(x) = \sum_{j \notin I_{x, \rho}} (-\lambda_{\rho, j}(x)) = \sum_{j \notin I_{x, \rho}} |\lambda_{\rho, j}(x)| = \sum_{j=1}^n \max \{ -\lambda_{\rho, j}(x), 0 \}. \tag{2.2}
\]

We denote by \( B_\sigma \) the ball centered at 0 with radius \( \sigma \), i.e.,

\[
B_\sigma = \{ x \in \mathbb{R}^n; |x| < \sigma \}. \tag{2.3}
\]

The main result of this subsection can be stated as follows.

**Lemma 2.3.** Let \( p \in E_r \) satisfy that

\[
\forall x \in B_\sigma, \sum_{j \in I_{x, p}} \lambda_{p, j}(x) \leq C \left( M_p(x) + |\partial_x p(x)|^2 \right) \tag{2.4}
\]

for some \( \sigma > 0 \) and for some constant \( C > 0 \), where we use the notations given in (2.1)–(2.3). Then there exists a constant \( \tilde{C} \geq 1 \), depending only on the constant \( C \) above and the dimension \( n \), such that for any \( q \in \mathcal{L}_{p, 0} \) we have

\[
\forall x \in \mathbb{R}^n, \sum_{j \in I_{x, q}} \lambda_{q, j}(x) \leq \tilde{C} \left( M_q(x) + |\partial_x q(x)|^2 \right). \tag{2.5}
\]

As a result any \( q \in \mathcal{L}_{p, 0} \setminus \{0\} \) can not have any local minimum in \( \mathbb{R}^n \).

**Proof.** We begin with the proof of the first property (2.5). For any \( q \in \mathcal{L}_{p, 0} \), by Remark 2.1 we can find a sequence \( y_j \in \mathbb{R}^n \) with \( y_j \to 0 \) and sequences \( \tau_j \) and \( h_j \) of positive numbers with \( \tau_j \to +\infty \) and \( h_j \to 0 \), such that

\[
q(x) = \lim_{j \to +\infty} \tau_j \left[ p(y_j + h_j x) - p(y_j) \right] = \sum_{1 \leq |\beta| \leq r} \left( \lim_{j \to +\infty} \tau_j h_j^{|eta|} \partial^\beta p(y_j) \right) \frac{x^\beta}{\beta!}. \tag{2.6}
\]

This implies

\[
\forall |\alpha| \geq 1, \quad \partial^\alpha_x q(x) = \sum_{1 \leq |\beta| \leq r} \left( \lim_{j \to +\infty} \tau_j h_j^{|eta|} \partial^\beta p(y_j) \right) \frac{\partial^\alpha_x (x^\beta)}{\beta!}. \tag{2.6}
\]

On the other hand, using the Taylor expansion

\[
\tau_j \left[ p(y_j + h_j x) - p(y_j) \right] = \sum_{1 \leq |\beta| \leq r} \tau_j h_j^{|eta|} \partial^\beta p(y_j) \frac{x^\beta}{\beta!},
\]

we get
we have, for any $|\alpha| \geq 1$,
\[
\tau_j h_j^{\alpha} (\partial^\alpha p) (y_j + h_j x) = \tau_j \partial_x^\alpha \left( p(y_j + h_j x) - p(y_j) \right) = \sum_{1 \leq |\beta| \leq r} \frac{\tau_j h_j^{|\beta|} \partial^\beta p(y_j)}{\beta!} \partial_x^\alpha (x^\beta),
\]
which along with (2.6) yields
\[
\forall |\alpha| \geq 1, \quad \lim_{j \to +\infty} \tau_j h_j^{\alpha} (\partial^\alpha p) (y_j + h_j x) = \partial^\alpha q(x).
\]
In particular,
\[
\lim_{j \to +\infty} \tau_j h_j |(\partial_x p) (y_j + h_j x)| = |\partial_x q(x)| \quad (2.7)
\]
and
\[
\forall 1 \leq \ell \leq n, \quad \lim_{j \to +\infty} \tau_j h_j^2 \lambda_{p,\ell} (y_j + h_j x) = \lambda_{q,\ell} (x). \quad (2.8)
\]
Moreover observing $\mathcal{M}_q(x) = \sum_{1 \leq \ell \leq n} \max \{-\lambda_{q,\ell} (x), 0\}$ we have
\[
\lim_{j \to +\infty} \tau_j h_j^2 \mathcal{M}_p(y_j + h_j x) = \mathcal{M}_q(x) \quad (2.9)
\]
because of (2.8). For any $\ell \in I_{x,q}$ with $x \in \mathbb{R}^n$ given, we see $\lambda_{q,\ell} (x) > 0$. Then using (2.8) gives
\[
\lambda_{p,\ell} (y_j + h_j x) > 0
\]
for all $j$ large enough, since $\tau_j$ and $h_j$ are positive. Furthermore note $y_j \to 0, h_j \to 0$, and thus for any $x \in \mathbb{R}^n$ we have $y_j + h_j x \in B_\sigma$ for all $j$ large enough. Consequently it follows from (2.4) that, for all $j$ large enough,
\[
\lambda_{p,\ell} (y_j + h_j x) \leq C \left( \mathcal{M}_p(y_j + h_j x) + |\partial_x p(y_j + h_j x)|^2 \right).
\]
Combining the above estimate and (2.7)-(2.9) we obtain
\[
\lambda_{q,\ell} (x) = \lim_{j \to +\infty} \tau_j h_j^2 \lambda_{p,\ell} (y_j + h_j x) \\
\leq C \lim_{j \to +\infty} \left( \tau_j h_j^2 \mathcal{M}_p(y_j + h_j x) + \tau_j h_j^2 |\partial_x p(y_j + h_j x)|^2 \right) \\
= C \left( \mathcal{M}_q(x) + |\partial_x q(x)|^2 \right),
\]
which holds for any $\ell \in I_{x,q}$. This gives the first statement (2.5) as desired.

Next we prove the second statement. Let $q \in \mathcal{L}_{p,0}$ satisfy (2.9). For the symmetric Hessian matrix $(\partial_{x,i} q(x))_{1 \leq i,j \leq n}$, we can find a $n \times n$ orthogonal matrix $Q(x) = (q_{ij}(x))_{1 \leq i,j \leq n}$ such that
\[
Q^T \begin{pmatrix} 
\lambda_{q,1} & \lambda_{q,2} \\
& \ddots \\
& & \lambda_{q,n}
\end{pmatrix} Q = (\partial_{x,i} q)_{1 \leq i,j \leq n}, \quad (2.10)
\]
Define $a_{ij}, b_{ij}, c_{ij}, 1 \leq i, j \leq n$, as follows. $b_{ij} = 0$ if $i \neq j$, and

$$b_{jj}(x) = \begin{cases} \sqrt{C}, & \text{if } \lambda_{q,j}(x) \leq 0, \\ 1, & \text{if } \lambda_{q,j}(x) > 0, \end{cases}$$

(2.11)

with $\tilde{C} \geq 1$ the constant in (2.5). And

$$a_{ij} = \sum_{1 \leq k \leq n} (b_{kk}q_{ki}) (b_{kk}q_{kj}).$$

(2.12)

Then we can verify that, for any $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$,

$$\sum_{1 \leq i, j \leq n} a_{ij} \eta_i \eta_j = \sum_{1 \leq k \leq n} b_{kk} \sum_{1 \leq i \leq n} q_{ki} \eta_i \left( \sum_{1 \leq j \leq n} q_{kj} \eta_j \right)$$

$$\geq \sum_{1 \leq k \leq n} \left( \sum_{1 \leq i \leq n} q_{ki} \eta_i \right) \left( \sum_{1 \leq j \leq n} q_{kj} \eta_j \right)$$

$$\geq \sum_{1 \leq i, j \leq n} \left( \sum_{1 \leq k \leq n} q_{ki} q_{kj} \right) \eta_i \eta_j$$

$$= |\eta|^2,$$

the last line using the fact that $Q(x)$ is an orthogonal. Thus $(a_{ij})_{n \times n}$ is a positive-definite matrix. Similarly we use the relations (2.10) and (2.12) to compute, letting $\delta_{k\ell}$ be the the Kronecker delta function,

$$\sum_{1 \leq i, j \leq n} a_{ij}(x) \partial_{x_i} \partial_{x_j} q(x)$$

$$= \sum_{1 \leq i, j \leq n} \left( \sum_{k=1}^{n} (b_{kk}(x)q_{ki}(x)) (b_{kk}(x)q_{kj}(x)) \right) \left( \sum_{\ell=1}^{n} q_{\ell i}(x)q_{\ell j}(x)\lambda_{q,\ell}(x) \right)$$

$$= \sum_{1 \leq k, \ell \leq n} b_{kk}(x)^2 \lambda_{q,\ell}(x) \left( \sum_{i=1}^{n} q_{ki}(x)q_{\ell i}(x) \right) \left( \sum_{j=1}^{n} q_{kj}(x)q_{\ell j}(x) \right)$$

$$= \sum_{1 \leq k, \ell \leq n} b_{kk}(x)^2 \lambda_{q,\ell}(x) \delta_{k\ell} = \sum_{1 \leq k \leq n} b_{kk}(x)^2 \lambda_{q,k}(x)$$

$$= \sum_{i \in I_{x,q}} \lambda_{q,i}(x) + \tilde{C} \sum_{i \notin I_{x,q}} \lambda_{q,i}(x),$$

the last equality following from (2.11). Now suppose $q$ satisfies (2.5). Then it follows from the above equalities that for any $x \in \mathbb{R}^n$ we have, observing $M_q = - \sum_{i \notin I_{x,q}} \lambda_{q,i}$,

$$\sum_{1 \leq i, j \leq n} a_{ij}(x) \partial_{x_i} \partial_{x_j} q(x) \leq \tilde{C} \left( M_q(x) + |\partial_{x} q(x)|^2 \right) = \tilde{C} M_q(x) = \tilde{C} |\partial_{x} q(x)|^2.$$

As a result, by maximum principle for elliptic equations we conclude that $q$ can not have any local minimum in $\mathbb{R}^n$ unless it is a constant. Observe $q$ vanishes at 0. Thus any $q \in \mathcal{L}_{p,0} \setminus \{0\}$ can not have any local minimum in $\mathbb{R}^n$. The proof of Lemma 2.3 is thus complete. \[\square\]
As an immediate consequence of Theorem 2.2 and Lemma 2.3, we have

**Corollary 2.4.** Let \( p \in E_r \). Suppose that there are two constants \( C, \sigma > 0 \) such that

\[
\forall \ x \in B_\sigma, \quad \sum_{j \in I_{x,p}} \lambda_{p,j}(x) \leq C \left( M_p(x) + |\partial_x p(x)|^2 \right),
\]

where we use the notations given in (2.1) - (2.3). Then we can find a constants \( c_0 > 0 \) depending only on \( p \) such that, decreasing \( \sigma \) if necessary,

\[
\|\partial_x u\|_{L^2}^2 + \tau^2 \| (\partial_x p) u \|_{L^2}^2 \leq c_0 \left( \| (\partial_x + \tau (\partial_x p)) u \|_{L^2}^2 + c_0 \| u \|_{L^2}^2 \right)
\]

holds for any \( u \in C_0^\infty(B_\sigma) \) and for any \( \tau > 0 \). By density arguments we see the above estimate still holds for any \( u \in H_0^1(B_\sigma) \) with \( H_0^1(B_\sigma) \) the classical Sobolev space. Note that \( \| (\partial_x + \tau (\partial_x p)) u \|_{L^2}^2 = (\Delta_{\tau p}^0 u, u)_{L^2} \) if \( u \in C_0^\infty(B_\sigma) \).

**2.3. Localization**

Here we introduce some partitions of unity related to a slowly varying metric. Recall a metric \( g \) is slowly varying if we can find two constant \( C \geq 1 \) and \( r > 0 \) such that

\[
\forall \ x, y, T \in \mathbb{R}^n, \quad g_x(y - x) \leq r^2 \implies \forall \ T \in \mathbb{R}^n, \quad C^{-1} \leq \frac{g_x(T)}{g_y(T)} \leq C. \quad (2.13)
\]

And we refer to [18, 21] for more details on the metrics and the symbol space related to a metric.

**Remark 2.5.** In order to prove a metric \( g \) is slowly varying we ask only that

\[
\exists \ C \geq 1, \quad \exists \ r > 0, \quad \forall \ x, y, T \in \mathbb{R}^n, \quad g_x(y - x) \leq r^2 \implies g_y(T) \leq C g_x(T),
\]

which is sufficient to give the previous property (2.13), see [21, Remark 2.2.2] for instance.

Now we define \( f \) by setting

\[
f(x) = \sum_{1 \leq |\alpha| \leq k} \left( 1 + |\partial^\alpha V(x)|^2 \right)^{\frac{1}{2|\alpha|}}, \quad (2.14)
\]

which is a regularization of the function \( \tilde{f} \) in (1.5). Observe \( \tilde{f} \leq f \leq C_k \left( 1 + \tilde{f} \right) \) for some constant \( C_k \) depending only on \( k \). Let \( V \) be the potential satisfying (1.7) in Assumption 1.1. Then for any multi-index \( \alpha \) with \( |\alpha| = k + 1 \) we can find a constant \( C_\alpha \) depending on \( \alpha \) such that

\[
\forall \ x \in \mathbb{R}^n, \quad |\partial^\alpha V(x)| \leq C_\alpha f(x)^{k+1-\delta_2}, \quad (2.15)
\]

with \( \delta_2 > 0 \) the arbitrarily small number given in (1.7). Moreover letting \( \varepsilon > 0 \) be a small number to be determined further, we define the metric \( g_{x,\varepsilon} \) as follows.

\[
g_{x,\varepsilon} = \varepsilon f(x)^2 |dx|^2. \quad (2.16)
\]

Next we will show that the metric defined above is slowly varying.
Lemma 2.6. Let $V$ be the potential satisfying the condition (2.15). Then the metric defined by (2.16) is slowly varying, i.e., we can find two constants $C_*, r > 0$, depending only the constants in (2.15) but independent of $\varepsilon$, such that if $g_{x,\varepsilon}(y - x) \leq \varepsilon r^2$ then

$$C_*^{-2} \leq \frac{g_{x,\varepsilon}}{g_{y,\varepsilon}} \leq C_*^2.$$  

In order to prove the above lemma we need the bootstrap principle. Here we refer to [32, Proposition 1.21].

Proposition 2.7. For each $x \in \mathbb{R}^n$ we have two statements, a “hypothesis” $H(x)$ and a “conclusion” $C(x)$, with the following assertions listed subsequently fulfilled.

(i) If $C(x)$ is true for some $x_0$ then $H(x)$ holds for all $x$ in a neighborhood of $x_0$.

(ii) If $x_j, j \geq 1$, is a sequence in $\mathbb{R}^n$ which converges to some $\tilde{x}$, and if $C(x_j)$ is true for all $j \geq 1$, then $C(\tilde{x})$ is true.

(iii) $H(x)$ is true for at least one $x \in \mathbb{R}^n$.

(iv) If $H(x)$ is true for some $x \in \mathbb{R}^n$ then so is $C(x)$ for the same $x$.

Then $C(x)$ is true for all $x \in \mathbb{R}^n$.

The proof of the proposition above is just the same as that in [32, Proposition 1.21], with the time interval $I$ therein replaced by $\mathbb{R}^n$.

Proof of Lemma 2.6. Note that

$$g_{x,\varepsilon}(y - x) \leq \varepsilon r^2 \iff |y - x| \leq f(x)^{-1}r.$$  

Then in view of Remark 2.5 we only need show that

$$\exists \, C_* > 0, \ \forall \, x, y \in \mathbb{R}^n, \ |y - x| \leq rf(x)^{-1} \implies f(y) \leq C_* f(x).$$  

(2.17)

To do so we use bootstrap arguments stated in Proposition 2.7. Let $0 < r < 1$ to be determined later. We define a continuous function $\psi_r(x)$ by setting

$$\psi_r(x) = \max_{z \in \{ |z - x| \leq rf(x)^{-1} \}} \frac{f(z)}{f(x)}.$$  

Let $C_* > \psi_1(0) + 1$ be a parameter to be chosen later, and let $H(x)$ denote the statement that

$$\psi_r(x) \leq 2C_*$$  

and let $C(x)$ denote the statement that

$$\psi_r(x) \leq C_*.$$  

The continuity of $\psi_r$ gives the following assertions:

(i) If $C(x)$ is true for some $x_0$ then $H(x)$ holds for all $x$ in a neighborhood of $x_0$.

(ii) If $x_j, j \geq 1$, is a sequence in $\mathbb{R}^n$ which converges to some $\tilde{x}$, and if $C(x_j)$ is true for all $j \geq 1$, then $C(\tilde{x})$ is true.

(iii) $H(0)$ is true, recalling $C_* > \psi_1(0) \geq \psi_r(0)$ for $0 < r < 1$. 
Then by Proposition 2.7 we see $C(x)$ will be true for all $x \in \mathbb{R}^n$ if we can show that

$H(x)$ is true for some $x \in \mathbb{R}^n \implies C(x)$ is also true for the same $x$. (2.18)

In the following arguments we will prove the property (2.18) under the hypothesis in (2.17). We will use $C_j \geq 1$, $j \geq 1$, to denote different constants which depend only on the integer $k$ and the constants given in (2.15). For any $\alpha$ with $1 \leq |\alpha| \leq k$ we have the Taylor expansion for $\partial^\alpha V$:

$$
\partial^\alpha V(y) = \sum_{|\beta| \leq k-|\alpha|} \frac{\partial^{\beta+\alpha} V(x)}{\beta!} (y - x)^\beta
$$

Moreover for the last term in (2.19), we use (2.15) to compute, supposing $H(x)$.

$$
\left| \sum_{|\beta| = k+1 - |\alpha|} \frac{\beta |\beta|}{\beta!} (y - x)^\beta \int_0^1 (1 - \theta)^{|\beta| - 1} \partial^{\beta+\alpha} V(x + \theta(y - x)) d\theta \right|
$$

(2.20)

From the definition of $f$ and the fact $0 < r < 1$ it follows that if $|y - x| \leq r f(x)^{-1}$ then

$$
\left| \sum_{|\beta| \leq k - |\alpha|} \frac{\partial^{\beta+\alpha} V(x)}{\beta!} (y - x)^\beta \right| \leq C_1 f(x)^{|\alpha|}.
$$

Moreover for the last term in (2.19), we use (2.15) to compute, supposing $|y - x| \leq r f(x)^{-1},$

$$
\left| \sum_{|\beta| = k+1 - |\alpha|} \frac{\beta |\beta|}{\beta!} (y - x)^\beta \int_0^1 (1 - \theta)^{|\beta| - 1} \partial^{\beta+\alpha} V(x + \theta(y - x)) d\theta \right|
$$

$$
\leq C_2 \sum_{|\beta| = k+1 - |\alpha|} r^{|eta|} f(x)^{-|\beta|} \int_0^1 \left( f(x + \theta(y - x)) \right)^{k+1-\delta_2} d\theta
$$

$$
\leq r C_2 f(x)^{|\alpha| - \delta_2} \sum_{|\beta| = k+1 - |\alpha|} \int_0^1 \left( \frac{f(x + \theta(y - x))}{f(x)} \right)^{k+1-\delta_2} d\theta
$$

$$
\leq r C_3 f(x)^{|\alpha|} \psi_r(x)^{k+1-\delta_2},
$$

the second inequality using the fact that $0 < r < 1$ and the last inequality following from the definition of $\psi_r$. As a result the validity of $H(x)$ gives that

$$
\left| \sum_{|\beta| = k+1 - |\alpha|} \frac{\beta |\beta|}{\beta!} (y - x)^\beta \int_0^1 (1 - \theta)^{|\beta| - 1} \partial^{\beta+\alpha} V(x + \theta(y - x)) d\theta \right|
$$

$$
\leq r C_3 f(x)^{|\alpha|} (2 C_*)^{k+1-\delta_2},
$$

which along with (2.19) - (2.20) yields that for any $\alpha$ with $1 \leq |\alpha| \leq k$ and for any $y \in \mathbb{R}^n$ with $|y - x| \leq r f(x)^{-1}$, we have

$$
|\partial^\alpha V(y)| \leq C_1 f(x)^{|\alpha|} + r C_3 f(x)^{|\alpha|} (2 C_*)^{k+1-\delta_2}
$$
and thus, observing $C_1, C_3, C_\ast \geq 1$,
\[
|\partial^\alpha V(y)|^{1/|\alpha|} \leq C_1 f(x) + r^{1/|\alpha|} C_3 f(x)(2C_\ast)^{(k+1-\delta_2)/|\alpha|} \\
\leq C_1 f(x) + r^{1/k} C_3 f(x)(2C_\ast)^{k+1}.
\]
This implies, in view of the definition of $f$,
\[
f(y) \leq C_4 f(x) + r^{1/k} C_4 f(x)(2C_\ast)^{k+1},
\]
that is,
\[
\frac{f(y)}{f(x)} \leq C_4 + r^{1/k} C_4 (2C_\ast)^{k+1}.
\]
Observe the above inequality holds for all $y$ such that $|y - x| \leq rf(x)^{-1}$ and thus
\[
\psi_r(x) \leq C_4 + r^{1/k} C_4 (2C_\ast)^{k+1}.
\]
Now we choose $C_\ast$ such that
\[
C_\ast = 2C_4 + \psi_1(0) + 1
\]
and choose such a small $r$ that
\[
r^{1/k}(2C_\ast)^{k+1} \leq 1.
\]
Then (2.21) gives $\psi_r(x) < 2C_4 < C_\ast$ and thus $C(x)$ holds, completing the proof of the property (2.18). As a result we use Proposition 2.7 to conclude that
\[
\psi_r(x) \leq C_\ast
\]
for all $x \in \mathbb{R}^n$, with the constants $C_\ast$ and $r$ chosen above. This yields the assertion (2.17) as desired, completing the proof of Lemma 2.6.

Let $g_\varepsilon$ be the metric given by (2.16). We denote by $S(1, g_\varepsilon)$ the class of smooth real-valued functions $a(x)$ satisfying the following condition:
\[
\forall \gamma \in \mathbb{Z}^n_+, \forall x \in \mathbb{R}^n, \quad |\partial^\gamma a(x)| \leq C_\gamma \left(\varepsilon^{1/2} f(x)\right)^{|\gamma|},
\]
with $C_\gamma$ the constants depending only on $\gamma$, but independent of $\varepsilon$. The space $S(1, g_\varepsilon)$ endowed with the seminorms
\[
|a|_{\ell, S(1, g_\varepsilon)} = \sup_{x \in \mathbb{R}^n, |\gamma| = \ell} \left(\varepsilon^{1/2} f(x)\right)^{-|\gamma|} |\partial^\gamma a(x)|, \quad \ell \geq 0,
\]
becomes a Fréchet space.

The main feature of a slowly varying metric is that it enables us to introduce some partitions of unity related to the metric. We can apply [18, Lemma 1.4.9 and Theorem 1.4.10] to $\|y\|_x = \left(g_{x, \varepsilon}(y) / (\varepsilon r^2)\right)^{1/2}$ with $r$ the number given in Lemma 2.6; this gives the following lemma (see also [18, Lemma 18.4.4] with $c$ therein replaced by $\varepsilon r^2$).
Lemma 2.8 (Partition of unity). Let \( g_{\varepsilon} \) be the metric given by (2.16) and let \( r, C_* \) be the constants given in Lemma 2.6. We can find a sequence \( x_{\mu} \in \mathbb{R}^n, \mu \geq 1, \) such that the union of the balls
\[
\Omega_{\mu, \varepsilon, r} = \left\{ x \in \mathbb{R}^n; \ g_{x_{\mu}, \varepsilon}(x - x_{\mu}) < \frac{\varepsilon r^2}{2} \right\}
\]
covers the whole space \( \mathbb{R}^n \). Moreover there exists a positive integer \( N \), depending only on \( C_* \) and the dimension \( n \) but independent of \( \varepsilon \), such that the intersection of more than \( N \) balls is always empty. One can choose a family of nonnegative functions \( \{ \varphi_{\mu, \varepsilon} \}_{\mu \geq 1} \) uniformly bounded in \( S(1, g_{\varepsilon}) \) with respect to \( \mu \), such that
\[
supp \varphi_{\mu, \varepsilon} \subset \Omega_{\mu, \varepsilon, r}, \quad \sum_{\mu \geq 1} \varphi_{\mu, \varepsilon}^2 = 1 \quad \text{and} \quad \sup_{\mu \geq 1} |\partial_x \varphi_{\mu, \varepsilon}(x)| \leq C\varepsilon^{1/2} f(x),
\]
where \( C \) is a constant independent of \( \varepsilon \). Here by uniformly bounded in \( S(1, g_{\varepsilon}) \) with respect to \( \mu \), we mean
\[
\sup_{\mu} |\varphi_{\mu, \varepsilon}|_{\ell, S(1, g_{\varepsilon})} \leq C_\ell, \quad \ell \geq 0,
\]
with \( C_\ell \) constants depending only on \( \ell \).

Remark 2.9. It follows from Lemma 2.6 that for any \( \mu \geq 1 \) one has
\[
\forall \ x, y \in \Omega_{\mu, \varepsilon, r}, \quad C_*^{-1} f(y) \leq f(x) \leq C_* f(y),
\]
where \( C_* \) is the constant given in Lemma 2.6.

2.4. Proof of Theorem 1.2

This part is devoted to proving Theorem 1.2 and we only need to prove the estimates (1.8) and (1.9), and the compactness of the resolvent for Witten Laplacian will follow immediately from these estimates due to (iii) in Assumption 1.1. In the proof we let \( \tilde{f}, f \) be the functions introduced respectively in (1.5) and (2.14), satisfying that
\[
\tilde{f} \leq f \leq C_k \left( 1 + \tilde{f} \right)
\]
for some constant \( C_k \) depending only on \( k \). Recall \( E_k \) is the set of polynomials with degree less than or equal to \( k \), and \( B_\sigma \) denotes the ball centered at 0 with radius \( \sigma \).

Proof of Theorem 1.2 (Maximal estimate). We begin with the first assertion (1.8) and will prove it by contradiction. To do so suppose that, contrary to (1.8), for any \( \ell \geq 1 \) and for any \( \tau > 0 \), there exists a function \( u_\ell = u_{\ell, \tau} \in C_0^\infty(\mathbb{R}^n) \) with \( u_\ell \neq 0 \), such that
\[
\| \tilde{f} u_\ell \|_{L^2}^2 > \ell \left( \Delta_{\tau V}^{(0)} u_\ell, u_\ell \right)_{L^2} + \ell \| u_\ell \|_{L^2}^2.
\]
Here and throughout the proof we will write \( u_\ell \) instead of \( u_{\ell, \tau} \), omitting the dependence of \( \tau \), to simplify the notation. For given \( 0 < \varepsilon < 1 \) to be
determined further, we let \( \{ \varphi_{\mu, \varepsilon} \}_{\mu \geq 1} \) be the partition of unity given in Lemma 2.8, which satisfies that
\[
\text{supp } \varphi_{\mu, \varepsilon} \subset \Omega_{\mu, \varepsilon, r} = \left\{ x \in \mathbb{R}^n; \ |x - x_\mu| < \frac{r}{\sqrt{2}f(x_\mu)} \right\}
\]
with \( r > 0 \) the number given in Lemma 2.6 and that
\[
|\partial_x \varphi_{\mu, \varepsilon}| \leq \tilde{C}_* \varepsilon^{1/2} f \tag{2.25}
\]
with \( \tilde{C}_* \) a constant independent of \( \varepsilon \) and \( \mu \). To simplify the notations we will use \( C_j, j \geq 1 \), in the following discussions to denote the suitable constants which depend on \( \tilde{C}_* \) above but are independent of \( \varepsilon, \tau, \mu \) and \( \ell \). By the IMS localization formula (cf. [2, Theorem 3.2]) we obtain
\[
\left( \Delta_{\tau V}^{(0)} \varphi_{u, \ell} \right)_L^2 = \sum_{\mu \geq 1} \left( \Delta_{\tau V}^{(0)} (\varphi_{\mu, \varepsilon} u_{\ell}) , \varphi_{\mu, \varepsilon} u_{\ell} \right)_L^2 - \sum_{\mu \geq 1} \| (\partial_x \varphi_{\mu, \varepsilon}) u_{\ell} \|_L^2
\]
\[
\geq \sum_{\mu \geq 1} \left( \Delta_{\tau V}^{(0)} (\varphi_{\mu, \varepsilon} u_{\ell}) , \varphi_{\mu, \varepsilon} u_{\ell} \right)_L^2 - C_1 \varepsilon \sum_{\mu \geq 1} \| f(x) \varphi_{\mu, \varepsilon} u_{\ell} \|_L^2,
\]
where the last inequality follows from (2.25) and the fact that the intersection of more than \( N \) balls \( \Omega_{\mu, \varepsilon, r} \) is always empty with \( N \) a fixed integer given in Lemma 2.8 independent of \( \varepsilon \). As a result we combine (2.24) and the above estimate to conclude
\[
\sum_{\mu \geq 1} \| \tilde{f} \varphi_{\mu, \varepsilon} u_{\ell} \|_L^2
\]
\[
> \sum_{\mu \geq 1} \ell \left[ \left( \Delta_{\tau V}^{(0)} (\varphi_{\mu, \varepsilon} u_{\ell}) , \varphi_{\mu, \varepsilon} u_{\ell} \right)_L^2 + \| \varphi_{\mu, \varepsilon} u_{\ell} \|_L^2 - C_1 \varepsilon \| f(x) \varphi_{\mu, \varepsilon} u_{\ell} \|_L^2 \right].
\]
Thus for any \( \ell \), there exists a positive integer \( \mu_\ell \), depending only on \( \ell \), such that
\[
\| \tilde{f} \varphi_{\mu_\ell, \varepsilon} u_{\ell} \|_L^2
\]
\[
> \ell \left[ \left( \Delta_{\tau V}^{(0)} (\varphi_{\mu_\ell, \varepsilon} u_{\ell}) , \varphi_{\mu_\ell, \varepsilon} u_{\ell} \right)_L^2 + \| \varphi_{\mu_\ell, \varepsilon} u_{\ell} \|_L^2 - C_1 \varepsilon \| f(x) \varphi_{\mu_\ell, \varepsilon} u_{\ell} \|_L^2 \right].
\]
As a result we use (2.23) to conclude that, for all \( \ell \) large enough such that \( \ell > 1/\varepsilon \),
\[
0 > \ell \left[ \left( \Delta_{\tau V}^{(0)} (\varphi_{\mu_\ell, \varepsilon} u_{\ell}) , \varphi_{\mu_\ell, \varepsilon} u_{\ell} \right)_L^2 + (1 - \varepsilon C_2) \| \varphi_{\mu_\ell, \varepsilon} u_{\ell} \|_L^2
\]
\[
- \varepsilon C_2 \| \tilde{f}(x) \varphi_{\mu_\ell, \varepsilon} u_{\ell} \|_L^2 \right] \tag{2.26}
\]
with
\[
\text{supp } (\varphi_{\mu_\ell, \varepsilon} u_{\ell}) \subset \Omega_{\mu_\ell, \varepsilon, r} = \left\{ x \in \mathbb{R}^n; \ |x - x_{\mu_\ell}| < \frac{r}{\sqrt{2}f(x_{\mu_\ell})} \right\}
\]
We claim that there exists a subsequence \( \{ x_{\mu\ell_j} \} \) of \( x_{\mu\ell} \) such that
\[
\lim_{j \to +\infty} |x_{\mu\ell_j}| = +\infty. \tag{2.27}
\]
Otherwise we can find a constant \( R > 0 \) such that
\[
\forall \ell \geq 1, \quad |x_{\mu\ell}| \leq R,
\]
which yields, using the notation \( M_R \) defined as \( \max_{|x| \leq R} f(x) \) and observing \( f \geq 1 \),
\[
\left\{ x; \; |x - x_{\mu\ell}| < \frac{r}{\sqrt{2M_R}} \right\} \subseteq \Omega_{\mu\ell,\varepsilon,r} = \left\{ x; \; |x - x_{\mu\ell}| < \frac{r}{\sqrt{2f(x_{\mu\ell})}} \right\} \tag{2.28}
\]
and
\[
\bigcup_{\ell \geq 1} \Omega_{\mu\ell,\varepsilon,r} \subseteq \left\{ x \in \mathbb{R}^n; \; |x| < R + 2^{-1/2}r \right\}. \]
We then have a contradiction, since the Lebesgue measure of the set on the right hand side is finite and independent of \( \ell \), meanwhile the Lebesgue measure of
\[
\bigcup_{\ell \geq 1} \Omega_{\mu\ell,\varepsilon,r}
\]
is \( +\infty \) due to (2.28) and the fact that the intersection of more than \( N \) balls \( \Omega_{\mu\ell,\varepsilon,r} \) is always empty. The contradiction implies the conclusion (2.27) and thus
\[
\lim_{j \to +\infty} f(x_{\mu\ell_j}) = +\infty \tag{2.29}
\]
because of the statement (iii) in Assumption 1.1.

Now we denote
\[
v_j = \varphi_{\mu\ell_j,\varepsilon} u_{\ell_j}, \quad j \geq 1.
\]
Then
\[
supp v_j \subset \left\{ x \in \mathbb{R}^n; \; |x - x_{\mu\ell_j}| < \frac{r}{\sqrt{2f(x_{\mu\ell_j})}} \right\},
\]
and furthermore, in view of (2.26),
\[
0 > \left( \triangle^{(0)}_{\tau V} v_j, \; v_j \right)_{L^2} + (1 - \varepsilon C_2) \| v_j \|_{L^2}^2 - \varepsilon C_2 \| f(x)v_j \|_{L^2}^2. \tag{2.30}
\]
In the following discussion we will derive a contradiction through several steps, starting from the estimate (2.30).

**Step 1.** We define
\[
w_j(x) = v_j \left( x_{\mu\ell_j} + f(x_{\mu\ell_j})^{-1}x \right).
\]
Then \( w_j \in C_0^\infty(\mathbb{R}^n) \) with
\[
supp w_j \subset \left\{ x \in \mathbb{R}^n; \; |x| < r/\sqrt{2} \right\}, \quad j \geq 1.
\]
Using the changes of variable $x = x_{\mu_{j}} + f(x_{\mu_{j}})^{-1}y$ in (2.30) for the $L^2$ integration, we obtain

$$0 > \left( \Delta_{q_j}^{(0)} w_j, w_j \right)_{L^2} + \frac{1 - \varepsilon C_2}{f(x_{\mu_{j}})^2} \| w_j \|^2_{L^2} - \varepsilon C_2 \| \tilde{f}_{q_j} w_j \|^2_{L^2},$$

where

$$q_j(x) = V(x_{\mu_{j}} + f(x_{\mu_{j}})^{-1}x) - V(x_{\mu_{j}})$$

and

$$\tilde{f}_{q_j}(x) = \sum_{1 \leq |\alpha| \leq k} \tau^{1/|\alpha|} |\partial^{\alpha} q_j(x)|^{1/|\alpha|}.$$  

The inequality (2.31) implies

$$\| w_j \|_{L^2} + \| \partial_x w_j \|_{L^2} > 0$$

and thus we can define

$$\zeta_j = \frac{w_j}{\left( \| w_j \|^2_{L^2} + \| \partial_x w_j \|^2_{L^2} \right)^{1/2}}.$$ 

Then we have, recalling $B_\sigma = \{ x \in \mathbb{R}^n; \ |x| < \sigma \}$,

$$\zeta_j \in C_0^\infty (B_{r/\sqrt{2}}), \quad \| \zeta_j \|^2_{L^2} + \| \partial_x \zeta_j \|^2_{L^2} = 1,$$

and, dividing both sides of (2.31) by the factor $\| w_j \|^2_{L^2} + \| \partial_x w_j \|^2_{L^2}$,

$$0 > \left( \Delta_{q_j}^{(0)} \zeta_j, \zeta_j \right)_{L^2} + \frac{1 - \varepsilon C_2}{f(x_{\mu_{j}})^2} \| \zeta_j \|^2_{L^2} - \varepsilon C_2 \| \tilde{f}_{q_j} \zeta_j \|^2_{L^2}.$$ 

Thus

$$\liminf_{j \to +\infty} \left[ \left( \Delta_{q_j}^{(0)} \zeta_j, \zeta_j \right)_{L^2} - \varepsilon C_2 \| \tilde{f}_{q_j} \zeta_j \|^2_{L^2} \right] \leq 0,$$

since it follows from (2.29) that

$$\frac{1 - \varepsilon C_2}{f(x_{\mu_{j}})^2} \| \zeta_j \|^2_{L^2} \to 0 \text{ as } \to +\infty.$$

**Step 2.** Let $q_j$ be given in (2.32) with $V$ satisfying Assumption 1.1. We will prove here that there exists a subsequence of $q_j$, still denoted by $q_j$, and a polynomial $p \in E_k \setminus \{ 0 \}$, such that

$$\forall \ 0 \leq |\beta| \leq k, \ \forall \ x \in B_{r/\sqrt{2}}, \ \lim_{j \to +\infty} \partial^{\beta} q_j(x) = \partial^{\beta} p(x),$$

and that, using the notations given in (2.1) and (2.2),

$$\forall \ x \in B_{r/\sqrt{2}}, \ \sum_{j \in I_{x,p}} \lambda_{p,j}(x) \leq C_3 \left( M_p(x) + \| \partial_x p(x) \|^2 \right).$$
We begin with the proof of (2.35). To do so we use Taylor’s expansion to write

\[
q_j(x) = \sum_{1 \leq |\alpha| \leq k} \frac{f(x_{\mu_{\ell_j}})^{-|\alpha|} \partial^\alpha V(x_{\mu_{\ell_j}})}{\alpha!} x^\alpha + f(x_{\mu_{\ell_j}})^{-(k+1)} \sum_{|\alpha|=k+1} \frac{|\alpha| x^\alpha}{\alpha!} \int_0^1 (1 - \theta)^k \partial^\alpha V(x_{\mu_{\ell_j}} + \theta x f(x_{\mu_{\ell_j}})^{-1}) d\theta.
\]

(2.37)

Moreover observe \( f(x_{\mu_{\ell_j}})^{-|\alpha|} \partial^\alpha V(x_{\mu_{\ell_j}}), 1 \leq |\alpha| \leq k, \) is an uniformly bounded sequence with respect to \( j \) and thus we can find a subsequence, still denoted by \( f(x_{\mu_{\ell_j}})^{-|\alpha|} \partial^\alpha V(x_{\mu_{\ell_j}}) \), such that

\[
\forall 1 \leq |\alpha| \leq k, \lim_{j \to +\infty} f(x_{\mu_{\ell_j}})^{-|\alpha|} \partial^\alpha V(x_{\mu_{\ell_j}}) = A_\alpha.
\]

(2.38)

This gives

\[
\sum_{1 \leq |\alpha| \leq k} |A_\alpha| \frac{1}{k!} = \lim_{j \to +\infty} f(x_{\mu_{\ell_j}})^{-1} \sum_{1 \leq |\alpha| \leq k} \left| \partial^\alpha V(x_{\mu_{\ell_j}}) \right| \frac{1}{|\alpha|} = \lim_{j \to +\infty} \frac{\tilde{f}(x_{\mu_{\ell_j}})}{f(x_{\mu_{\ell_j}})} > 0,
\]

the last inequality using (2.23) and the fact that \( \tilde{f}(x_{\mu_{\ell_j}}) \to +\infty \) as \( j \to +\infty \).

As a result, defining \( p \) by

\[
p = \sum_{1 \leq |\alpha| \leq k} \frac{A_\alpha}{\alpha!} x^\alpha,
\]

(2.39)

we see \( p \in E_k \setminus \{0\} \) and the first term on the right side of (2.37) converges to \( p(x) \). In order to treat the remainder term in (2.37) we use (1.7) and (2.22) to obtain that, for any \( \gamma \) with \( |\gamma| = k + 1 \), and for any \( |x| < r/\sqrt{2} \) and any \( \theta \in [0, 1] \),

\[
\left| \partial^\gamma V \left( x_{\mu_{\ell_j}} + \theta x f(x_{\mu_{\ell_j}})^{-1} \right) \right| \leq C_\gamma \left[ f(x_{\mu_{\ell_j}} + \theta x f(x_{\mu_{\ell_j}})^{-1}) \right]^{k+1-\delta_2} \leq \tilde{C}_\gamma f(x_{\mu_{\ell_j}})^{k+1-\delta_2},
\]

where \( \delta_2 > 0 \) is an arbitrary small number and \( C_\gamma, \tilde{C}_\gamma \) are two constants depending only on \( \gamma \). This implies, for any \( x \in B_{r/\sqrt{2}} \),

\[
f(x_{\mu_{\ell_j}})^{-(k+1)} \sum_{|\alpha|=k+1} \left| \frac{|\alpha| x^\alpha}{\alpha!} \int_0^1 (1 - \theta)^k \partial^\alpha V(x_{\mu_{\ell_j}} + \theta x f(x_{\mu_{\ell_j}})^{-1}) d\theta \right| \leq C_4 f(x_{\mu_{\ell_j}})^{-\delta_2} \to 0, \text{ as } j \to +\infty,
\]

the last line using (2.29). As a result we have

\[
\forall x \in B_{r/\sqrt{2}}, \lim_{j \to +\infty} q_j(x) = p(x).
\]
Similarly, for any \(1 \leq |\beta| \leq k\),
\[
\partial^\beta q_j(x) = f(x_{\mu_j})^{-|\beta|} (\partial^\beta V)(x_{\mu_j} + f(x_{\mu_j})^{-1}) x
\]
\[
= f(x_{\mu_j})^{-|\beta|} \sum_{0 \leq |\gamma| \leq k - |\beta|} \frac{f(x_{\mu_j})^{-|\gamma|} \partial^{\gamma+\beta} V(x_{\mu_j})}{\gamma!} x^\gamma
\]
\[
+ f(x_{\mu_j})^{-(k+1)} \sum_{|\gamma| = k+1 - |\beta|} \frac{|\gamma| x^\gamma}{\gamma!} \times \int_0^1 (1 - \theta)^{|\gamma|-1} \partial^{\gamma+\beta} V(x_{\mu_j} + \theta f(x_{\mu_j})^{-1}) d\theta,
\]
with the remainder term above trending to 0 as \(j \to +\infty\) for any \(x \in B_{r/\sqrt{2}}\). Meanwhile for the first term on the right hand side, we have
\[
f(x_{\mu_j})^{-|\beta|} \sum_{0 \leq |\gamma| \leq k - |\beta|} \frac{f(x_{\mu_j})^{-|\gamma|} \partial^{\gamma+\beta} V(x_{\mu_j})}{\gamma!} x^\gamma
\]
\[
= \sum_{\alpha \geq |\beta| \leq k} \frac{f(x_{\mu_j})^{-|\alpha|} \partial^{\alpha} V(x_{\mu_j})}{(\alpha - \beta)!} x^{\alpha-\beta} \to \sum_{\alpha \geq |\beta| \leq k} \frac{A_\alpha}{(\alpha - \beta)!} x^{\alpha-\beta} = \partial^\beta p(x)
\]
as \(j \to +\infty\), the last line using (2.38) and (2.39). Combining the above relations we obtain the first assertion (2.35).

It remains to show (2.36). Recall
\[
q_j(x) = V(x_{\mu_j} + f(x_{\mu_j})^{-1}) x - V(x_{\mu_j}).
\]
It then follows from (2.35) that for any \(1 \leq |\beta| \leq 2\) and for any \(x \in B_{r/\sqrt{2}}\) we have
\[
\partial^\beta p(x) = \lim_{j \to +\infty} \partial^\beta q_j(x) = \lim_{j \to +\infty} f(x_{\mu_j})^{-|\beta|} (\partial^\beta V)(x_{\mu_j} + f(x_{\mu_j})^{-1}) x.
\]
This implies for any \(x \in B_{r/\sqrt{2}}\) we have, using the notation (2.2)
\[
|\partial_x p(x)| = \lim_{j \to +\infty} \left| f(x_{\mu_j})^{-1} (\partial_x V)(x_{\mu_j} + f(x_{\mu_j})^{-1}) x \right|,
\]
\[
\lambda_{p,i}(x) = \lim_{j \to +\infty} f(x_{\mu_j})^{-2} \lambda_{V,i}(x_{\mu_j} + f(x_{\mu_j})^{-1}) x
\]
and
\[
\mathcal{M}_p(x) = \lim_{j \to +\infty} f(x_{\mu_j})^{-2} \mathcal{M}_V(x_{\mu_j} + f(x_{\mu_j})^{-1}) x.
\]
Now let \(x \in B_{r/\sqrt{2}}\) and let \(i \in I_{x,p}\). Then we have \(\lambda_{p,i}(x) > 0\), which along with (2.41) yields
\[
\lambda_{V,i}(x_{\mu_j} + f(x_{\mu_j})^{-1}) x > 0
\]
for all $j$ large enough. As a result it follows from (1.6) that
\[
\lambda_{V,i}(x_{\mu_{\ell_j}} + f(x_{\mu_{\ell_j}})^{-1}x) \\
\leq C_5 \left( M_V(x_{\mu_{\ell_j}} + f(x_{\mu_{\ell_j}})^{-1}x) + |\partial_x V(x_{\mu_{\ell_j}} + f(x_{\mu_{\ell_j}})^{-1}x)|^2 \right) \\
+ C_5 \left( \sum_{2 \leq |\alpha| \leq k} |\partial^\alpha V(x_{\mu_{\ell_j}} + f(x_{\mu_{\ell_j}})^{-1}x)|^{(2-\delta_1)/|\alpha|} + 1 \right),
\]
which holds for all $j$ large enough. Then using (2.40)-(2.42) yields, for any $i \in I_{x,p}$ with $x \in B_{r/\sqrt{2}}$,
\[
\lambda_{p,i}(x) = \lim_{j \to +\infty} f(x_{\mu_{\ell_j}})^{-2} \lambda_{V,i}(x_{\mu_{\ell_j}} + f(x_{\mu_{\ell_j}})^{-1}x) \\
\leq C_5 \lim_{j \to +\infty} \left[ f(x_{\mu_{\ell_j}})^{-2} M_V(x_{\mu_{\ell_j}} + f(x_{\mu_{\ell_j}})^{-1}x) \\
+ |f(x_{\mu_{\ell_j}})^{-1}\partial_x V(x_{\mu_{\ell_j}} + f(x_{\mu_{\ell_j}})^{-1}x)|^2 \\
+ f(x_{\mu_{\ell_j}})^{-\delta_1} \sum_{2 \leq |\alpha| \leq k} |f(x_{\mu_{\ell_j}})^{-|\alpha|}\partial^\alpha V(x_{\mu_{\ell_j}} + f(x_{\mu_{\ell_j}})^{-1}x)|^{2-\delta_1/|\alpha|} \\
+ f(x_{\mu_{\ell_j}})^{-2} \right] \\
= C_5 \left( M_p(x) + |\partial_x p(x)|^2 \right),
\]
the last line holding because $f(x_{\mu_{\ell_j}})^{-1} \to 0$ as $j \to +\infty$ and for any $2 \leq |\alpha| \leq k$ we have
\[
|f(x_{\mu_{\ell_j}})^{-|\alpha|}\partial^\alpha V(x_{\mu_{\ell_j}} + f(x_{\mu_{\ell_j}})^{-1}x)|^2 \\
\leq \left( f(x_{\mu_{\ell_j}})^{-1} f(x_{\mu_{\ell_j}} + f(x_{\mu_{\ell_j}})^{-1}x) \right)^{|\alpha|} \leq C_6
\]
due to (2.22). We have proven (2.36).

**Step 3.** Let $\zeta_j$, $j \geq 1$, be given in Step 1. Observe $\zeta_j \in C_0^\infty(B_{r/\sqrt{2}})$ for all $j$. Then in view of the condition (2.33), we conclude that there exists a subsequence of $\zeta_j$, still denoted by $\zeta_j$, and a $\zeta \in H^1_0(B_{r/\sqrt{2}})$ such that
\[
\zeta_j \to \zeta \text{ weakly in } H^1_0(B_{r/\sqrt{2}}), \tag{2.43}
\]
and
\[
\zeta_j \to \zeta \text{ strongly in } L^2(B_{r/\sqrt{2}}) \tag{2.44}
\]
due to the compact injection of $H^1_0(B_{r/\sqrt{2}})$ into $L^2(B_{r/\sqrt{2}})$. The weak convergence (2.43) implies
\[
\|\zeta\|_{L^2}^2 + \|\partial_x \zeta\|_{L^2}^2 \leq \left( \lim_{j \to +\infty} \|\zeta_j\|_{H^1} \right)^2 \leq \lim \inf_{j \to +\infty} \|\zeta_j\|_{H^1} \\
= \lim \inf_{j \to +\infty} \|\partial_x \zeta_j\|_{L^2}^2 + \|\zeta\|_{L^2}^2,
\]
the last equality using (2.44). Thus
\[ \| \partial_x \zeta \|_{L^2}^2 \leq \lim \inf_{j \to +\infty} \| \partial_x \zeta_j \|_{L^2}^2. \] (2.45)
From (2.33), (2.35) and (2.44) it follows that, observing \( \text{supp} \zeta_j \subset B_{r/\sqrt{2}} \) for all \( j \geq 1 \),
\[ \lim_{j \to +\infty} \int_{\mathbb{R}^n} \left( \tau^2 |\partial_x q_j|^2 - \tau \Delta q_j \right) |\zeta_j|^2 \, dx = \int_{\mathbb{R}^n} \left( \tau^2 |\partial_x p|^2 - \tau \Delta p \right) |\zeta|^2 \, dx \] (2.46)
and
\[ \lim_{j \to +\infty} \| \tilde{f}_{\tau q_j} \zeta_j \|_{L^2}^2 = \lim_{j \to +\infty} \| \tilde{f}_{\tau p} \zeta \|_{L^2}^2. \] (2.47)
Consequently, observe
\[ \| (\partial_x + \tau (\partial_x p)) \zeta \|_{L^2}^2 = \| \partial_x \zeta \|_{L^2}^2 + \int_{\mathbb{R}^n} \left( \tau^2 |\partial_x p|^2 - \tau \Delta p \right) |\zeta|^2 \, dx, \]
and thus using (2.45)-(2.47) gives
\[ \| (\partial_x + \tau (\partial_x p)) \zeta \|_{L^2}^2 \leq \frac{\varepsilon C_2}{2} \| \tilde{f}_{\tau p} \zeta \|_{L^2}^2 \]
\[ \leq \lim \inf_{j \to +\infty} \left[ \left( \Delta_{\tau q_j} (\zeta_j) \right)_{L^2} - \varepsilon C_2 \| \tilde{f}_{\tau q_j} (x) \zeta_j \|_{L^2} \right] \leq 0, \] (2.48)
the last inequality following from (2.34). Moreover in view of (2.36) we can apply Corollary 2.4 to conclude that, decreasing \( r \) if necessary so that \( r/\sqrt{2} \leq \sigma \) with \( \sigma \) given in Corollary 2.4
\[ \| \partial_x \zeta \|_{L^2}^2 + \tau^2 \| (\partial_x p) \zeta \|_{L^2}^2 \leq C_7 \left( \| (\partial_x + \tau (\partial_x p)) \zeta \|_{L^2}^2 + \| \zeta \|_{L^2}^2 \right). \] (2.49)
Here the constant \( C_7 \) may depend on the polynomial \( p \), but is independent of \( \tau \). On the other hand, note \( p \in E_k \) and then we can use the Baker-Campbell-Hausdorff formula (see [26, Lemma 4.14] for instance) to obtain that
\[ \tau^{2/k} \| \zeta \|_{L^2}^2 \leq C_8 \| \tilde{f}_{\tau p} \zeta \|_{L^2}^2 \leq C_9 \left( \| (\partial_x + \tau (\partial_x p)) \zeta \|_{L^2}^2 + \tau^2 \| (\partial_x p) \zeta \|_{L^2}^2 \right). \]
This along with (2.49) implies that
\[ \| \tilde{f}_{\tau p} (x) \zeta \|_{L^2} \leq C_{10} \left( \| (\partial_x + \tau_0 (\partial_x p)) \zeta \|_{L^2} \right) \] (2.50)
for some \( \tau_0 \) large enough. Note (2.48) holds for arbitrary \( \tau \) and thus we combine the above estimate and (2.48) to get
\[ \| (\partial_x + \tau_0 (\partial_x p)) \zeta \|_{L^2} \leq \varepsilon C_2 \| \tilde{f}_{\tau p} \zeta \|_{L^2} \leq \varepsilon C_2 C_{10} \left( \| (\partial_x + \tau_0 (\partial_x p)) \zeta \|_{L^2} \right). \]
Thus letting \( \varepsilon = 1/(2C_2 C_{10}) \) we obtain \( \| (\partial_x + \tau_0 (\partial_x p)) \zeta \|_{L^2} = 0 \), and thus \( \| \zeta \|_{L^2} = 0 \) in view of (2.50). Furthermore using (2.49) for \( \tau = \tau_0 \) gives \( \| \partial_x \zeta \|_{L^2} = 0 \). This contradicts (2.43) and (2.44), since \( \| \zeta_j \|_{H^1_{\tau}} = 1 \) by (2.33).

The contradiction yields the first property (1.8) in Theorem 1.2. \( \square \)

Completeness of the proof of Theorem 1.2. In this part we will prove the second property (1.9) in Theorem 1.2. Recall we have already proven that
\[ \forall \tau \geq \tau_0, \forall u \in C^\infty_0 (\mathbb{R}^n), \quad \| \tilde{f}_{\tau} u \|_{L^2} \leq C \left( \Delta_{\tau} (u, u) \right)_{L^2} + C \| u \|_{L^2}^2, \] (2.51)
for some \( \tau_0 > 0 \) and \( C \geq 1 \). It remains to consider \( \tau \) with \( 0 < \tau < \tau_0 \).
Let $\tau_0$ and $C$ be the constants in (2.51). For any $\tau$ with $0 < \tau < \tau_0$ we take $m = m_\tau$ by
\[
m = \max \left\{ 1, \sqrt{(2C - 1)/2C\tau_0/\tau} \right\}.
\]
Then direct verification shows
\[
1 - \frac{1}{2C} \leq \left( \frac{m\tau}{\tau_0} \right)^2 \leq 1.
\] (2.52)

Note $m \geq 1$ and thus we have the comparison in the sense of quadratic forms on $C_0^\infty(\mathbb{R}^n)$:
\[
\triangle_{\tau_0}^{(0)} V \leq m \triangle_{\tau}^{(0)} + (\tau_0^2 - m^2) |\partial_x V|^2 - (\tau_0 - m\tau) \Delta V
\]
\[
\leq m \triangle_{\tau}^{(0)} + \left( 1 - (m\tau/\tau_0)^2 \right) \bar{f}_{\tau_0}^2
\]
\[
\leq m \triangle_{\tau}^{(0)} + \frac{1}{2} \triangle_{\tau_0}^{(0)} + \frac{1}{2},
\]
the last inequality holding because it follows from (2.52) and (2.51) that, for any $u \in C_0^\infty(\mathbb{R}^n)$,
\[
\left( 1 - (m\tau/\tau_0)^2 \right) \| \bar{f}_{\tau_0} u \|_{L^2}^2 \leq \frac{1}{2C} \| \bar{f}_{\tau_0} u \|_{L^2}^2 \leq \frac{1}{2} \left( \triangle_{\tau_0}^{(0)} u, u \right)_{L^2} + \frac{1}{2} \| u \|_{L^2}^2.
\]
Consequently we have
\[
0 \leq \triangle_{\tau_0}^{(0)} V \leq 2m \triangle_{\tau}^{(0)} + 1,
\]
which yields that, for any $u \in C_0^\infty(\mathbb{R}^n)$ and for any $0 < \tau < \tau_0$,
\[
\| \bar{f}_{\tau} u \|_{L^2}^2 \leq \| \bar{f}_{\tau_0} u \|_{L^2}^2 \leq C \left( \triangle_{\tau_0}^{(0)} u, u \right)_{L^2} + C\| u \|_{L^2}^2
\]
\[
\leq 2mC \left( \triangle_{\tau}^{(0)} u, u \right)_{L^2} + 2C\| u \|_{L^2}^2.
\]
This gives (1.9), completing the proof of Theorem 1.2. □

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