Randomized Bregman Coordinate Descent Methods for Non-Lipschitz Optimization

Tianxiang Gao, Student Member, IEEE, Songtao Lu, Member, IEEE, Jia Liu, Senior Member, IEEE, and Chris Chu, Fellow, IEEE

Abstract—We propose a new randomized Bregman (block) coordinate descent (RBCD) method for minimizing a composite problem, where the objective function could be either convex or nonconvex, and the smooth part are freed from the global Lipschitz-continuous (partial) gradient assumption. Under the notion of relative smoothness based on the Bregman distance, we prove that every limit point of the generated sequence is a stationary point. Further, we show that the iteration complexity of the proposed method is $O(n\varepsilon^{-2})$ to achieve $\varepsilon$-stationary point, where $n$ is the number of blocks of coordinates. If the objective is assumed to be convex, the iteration complexity is improved to $O(n\varepsilon^{-1})$. If, in addition, the objective is strongly convex (relative to the reference function), the global linear convergence rate is recovered. We also present the accelerated version of the RBCD method, which attains an $O(n\varepsilon^{-1/\gamma})$ iteration complexity for the convex case, where the scalar $\gamma \in [1, 2]$ is determined by the generalized translation variant of the Bregman distance. Convergence analysis without assuming the global Lipschitz-continuous (partial) gradient sets our results apart from the existing works in the composite problems.

Index Terms—Bregman distance, Non-Lipschitz, Coordinate Descent, Convex and Nonconvex Optimization

I. INTRODUCTION

In this paper, we consider a composite optimization problem in the following form

$$\min_{x} F(x) \equiv f(x) + r(x),$$

where $r$ has $n$ separated blocks. More specifically, we have

$$r(x) = \sum_{i=1}^{n} r_i(x_i),$$

where $x_i$ denotes a subvector of $x$ with dimension $N_i$ such that $\sum_{i=1}^{n} N_i = N$, and each $r_i$ is a (possibly nonsmooth) convex function.

Due to the block separable structure, Problem (1) can be solved by (block) coordinate descent (CD) methods and/or their variants, especially in the large scale optimization problems. Roughly speaking, these methods are based on the strategy of selecting one coordinate/block of variables at each iteration using some index selection procedure (e.g., cyclic, greedy, randomized). This often dramatically reduces

the computational complexity of the algorithms per iteration as well as memory storage, making these methods simple and salable. See for instance [1]–[5] and references therein and a short summary in Table I, as well as the recent comprehensive review paper [6] for the up-to-date materials.

A widely used assumption in showing the convergence of CD methods in the literature is that the (partial) gradient of $f$ is globally Lipschitz-continuous. However, this could be a restrictive assumption violated in diverse applications in practice, such as matrix factorization [26], tensor decomposition [27], matrix/tensor completion [28], Poisson likelihood models [29], etc. Although this assumption may be relaxed by adopting conventional line search methods, the efficiency and computational complexity of the first-order methods are unavoidably distorted, especially when the size of the problem is large. In fact, this longstanding issue also appears in the classical proximal gradient descent (PGD) method. Fortunately, this issue is solved in [19]–[21]. They develop a new framework called Bregman proximal gradient descent (BPG) method that adapts the geometry of $f$ by the Bregman distance. In such a way, the decrease of the objective value can be still quantified. As a result, they are able to characterize the convergence behavior of BPG for minimizing convex composite problems without assuming globally Lipschitz-continuous gradient of the objective function. Further, this framework has been extended to the case of nonconvex optimization in [23].

Despite the crucial issue is solved in PGD-type methods, there are only few results on CD-type methods. A cyclic Bregman coordinate descent (CBCD) method has been proposed in [30], [31], but no rates are given. In [24], the authors provide the convergence rate result using randomized (block) coordinate selection strategy in a special case where $F$ is smooth convex and $r \equiv 0$. To the best of our knowledge, how to deal with this crucial issue is still an open problem, when using CD methods to solve a nonsmooth and convex/nonconvex Problem (1). Furthermore, the accelerated version of the RBCD method has not been proposed yet, and its iteration complexity analysis is still open as well. In this paper, we bridge these gaps by proposing a randomized Bregman (block) coordinate descent (RBCD) method and its accelerated variant. The comprehensive convergence analyses are established. The main contributions are highlighted as follows.

1) We propose a randomized Bregman (block) coordinate descent (RBCD) method to solve the composite problem where the smooth part does not have the global
Lipschitz-continuous (partial) gradient property.

2) By adapting the relative smoothness framework, we establish a rigorous convergence rate analysis of the RBCD method, showing that the convergence rate to a stationary point is $O(n\varepsilon^{-2})$ if $F$ is nonconvex, where $k$ is the number of iterations.

3) If $F$ is convex, RBCD achieves the global sublinear convergence rate of $O(n\varepsilon^{-1})$. The global linear convergence rate is obtained if $f$ is (relative) strongly convex.

4) The RBCD method can also be accelerated in the relative smoothness setting. The iteration complexity of $O(n\varepsilon^{-1/\gamma})$ can be obtained through the notion of generalized translation variant (explained in the latter section) of the Bregman distance.

### II. Preliminaries

**Notation.** Throughout this paper, we use bold upper case letters to denote matrices (e.g., $\mathbf{X}$), bold lower case letters to denote vectors (e.g., $\mathbf{x}$), and calligraphic letters (e.g., $\mathcal{X}$) are used to denote sets. We use $\| \cdot \|$ to denote the Euclidean norm. $\delta_X(x)$ represents the indicator function: $\delta_X(x) = 0$ if $x \in X$; otherwise, $\delta_X(x) = \infty$. If $\mathcal{X} = \mathbb{R}^n_+$, the indicator function becomes $\delta_x(x)$. For a function $f$, $\nabla f(x)$ denotes its gradient, while $\nabla_i f(x)$ is the partial gradient with respect to the $i$-th block. Let $f_i(x_i)$ be the function with respect to the $i$-th block, while the rest of blocks are fixed. Clearly, we have $\nabla_i f(x) = \nabla f_i(x_i)$. If $f$ is not differentiable, $\partial f$ denotes the subdifferential of $f$.

Given a convex function $\phi$, the Bregman proximal mapping of $\phi$ at a point $x$ is defined as

$$T_\phi(x) = \arg\min_{u} \phi(u) + D_h(u, x),$$

where $D_h(u, x) = h(u) - h(x) - \langle \nabla h(x), u - x \rangle$ is the Bregman distance with the reference convex function $h$. This mapping is well-defined since the functions $\phi$ and $h$ are convex. The convexity of $h$ also implies $D_h(x, y) \geq 0, \forall x, y$.

If, in addition, $h$ is strictly convex, $D_h(x, y) = 0$ if and only if $x = y$. In the rest of this paper, we assume $h$ is strictly convex. Note that $D_h(x, y)$ is not symmetric in general. Therefore, we use symmetric coefficient $\theta(h)$, defined by

$$\theta(h) = \inf_{x \neq y} \{D_h(x, y)/D_h(y, x)\},$$

(4) to measure the symmetry. When $\phi = \delta_x$, the Bregman proximal mapping reduces to the Bregman projection

$$P_{\mathcal{X}}^h(x) = \arg\min_{u} \{D_h(u, x) : u \in \mathcal{X}\}.$$

(5)

**Problem Formulation.** Our goal is to solve the following composite optimization problem

$$\text{minimize } F(x) \equiv f(x) + r(x),$$

(6)

where the following assumptions are made throughout this paper.

**Assumption 1.**

(i) $f$ is continuously differentiable.

(ii) $r$ is convex, block separable, proper and loser semi-continuous.

(iii) $F^* = \inf_{x} F(x) > -\infty$.

An estimate $x$ is said to be a stationary point of $F$ if it satisfies

$$0 \in \partial F \equiv \nabla f(x) + \partial r(x).$$

(7)

Note that the objective function $F$ could be convex or nonconvex since we don’t make the convexity assumption of $f$, which is the case in [24]. In addition, the function $r$ could be an indicator function of a closed convex set, so that the problem formulation in (6) includes the case where minimizing a nonsmooth objective function over a closed convex set.

### III. Randomized Bregman Coordinate Descent

In this section, we introduce the randomized Bregman (block) coordinate descent (RBCD) method for solving...
Lemma 1. The pair of functions \((g, h)\) is relatively smooth if and only if for all \(x\) and \(y\), it holds that
\[
g(y) - g(x) - \langle \nabla g(x), y - x \rangle \leq LD_h(y, x). \tag{10}
\]

Remark 1. When \(h = \frac{1}{2}\| \cdot \|^2\), the classical descent lemma is recovered, i.e.,
\[
g(y) - g(x) - \langle \nabla g(x), y - x \rangle \leq \frac{1}{2}\| y - x \|^2.
\]

To use Lemma 1 we additionally make the following assumptions for the rest of this paper.

Assumption 2. The functions \((f_i, h_i)\) are relatively smooth with constants \(L_i > 0, \forall i\).

With the relative smoothness between \((f_i, h_i)\), the following result shows the basic descent property of the proposed method.

Lemma 2. For any \(x\), and any \(i \in \{1, 2, \cdots, n\}\), let \(x^+\) be defined as in Eq. \(8\). Then we have
\[
F(x^+) \leq F(x) - \frac{1 + \theta_i - L_i}{\alpha} D_h(T_i(x), x_i), \tag{11}
\]
where \(\theta_i = \theta(h_i)\). In particular, with \(0 < \alpha < \frac{1 + \theta_i}{L_i}\), a sufficient descent in the objective value of \(F\) is guaranteed.

Maximizing the function \(g(\alpha) = (1 + \theta_i - L_i)\alpha\) with respect to \(\alpha\) yields the stepsize \(\alpha^* = \frac{1 + \theta_i}{L_i}\). Substituting the obtained stepsize into (11) yields the following result.

Corollary 1. For any \(x\), let \(x^+\) to be defined as in Eq. \(8\). With stepsize \(\alpha = \frac{1 + \theta_i}{2L_i}\), we have
\[
F(x^+) \leq F(x) - L_i D_h(T_i(x), x_i). \tag{12}
\]

With the stepsize \(\alpha = \frac{1 + \theta_i}{2L_i}\), Corollary 1 quantifies the descent in the objective value. Therefore, the stepsize \(\alpha^* = \frac{1 + \theta_i}{L_i}\) is an appropriate choice for Algorithm 1.

Since only one block is selected and updated per iteration, the quantity \(D_h(x^+, x)\) introduced in [20, 21] cannot be used to measure the optimality of the RBCD method. Given an estimate \(x\), we introduce the reference function \(H\) and the corresponding Bregman mapping as follows:
\[
H(x) = \sum_{i=1}^{n} L_i h_i(x_i), \tag{13}
\]
\[
D_H(y, x) = \sum_{i=1}^{n} L_i D_h(y_i, x_i) \tag{14}
\]
\[
T(x) = \argmin_{u} \langle \nabla f(x), u - x \rangle + D_H(u, x) + r(u). \tag{15}
\]

Based on this mapping, the following result shows that the quantity \(D_H(T(x), x)\) can be used to measure the optimality of \(F\).

Lemma 3. A vector \(x\) is a stationary point of \(F\) if and only if \(D_H(T(x), x) = 0\).

Clearly, when \(F\) is convex, then the current estimate \(x\) is a global minimum if \(D_H(T(x), x) = 0\).

A. Convex and strongly convex case

In this subsection, we provide the convergence analysis for the case where \(F\) is convex. Since \(r\) is convex, we have \(f\) is also convex. We use \(E_i\) (or \(E_{i_0}\)) to denote the expectation with respect to a single random variable \(i\) (or \(i_0\)). We use \(E\) to denote the expectation with respect to all random variables \(\{i_0, i_1, \cdots\}\).
Lemma 4. prove the convergence results of the RBCD method.

\[ g(y) \geq g(x) + \langle \nabla g(x), y - x \rangle + \mu D_h(y, x). \]  

(16)

Note that if \( \mu = 0 \), the classical convexity for a smooth function \( g \) is recovered. Moreover, when \( h = \frac{1}{\theta} \| \cdot \| \), the classical strongly convexity is recovered. In the rest of this subsection, we assume \( f \) is strongly convex relative to \( H \).

Assumption 3. \( f \) is \( \mu \)-strongly convex relative to \( H \), i.e., there exists a scalar \( \mu \geq 0 \) such that

\[ f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \mu D_H(y, x). \]  

(17)

Since \( r \) is assumed to be convex, the function \( F \) is also \( \mu \)-strongly convex relative to \( H \), i.e.,

\[ F(y) \geq F(x) + \langle \nabla f(x), y - x \rangle + \mu D_H(y, x). \]  

(18)

for some \( v \in \partial F(x) \). Moreover, by Assumption 2 we have

\[ f(T_i(x)) \leq f(x) + \langle \nabla f(x), T_i(x) - x \rangle + L_i D_h(T_i(x), x). \]  

(19)

Substituting \( y = T_i(x) \) in Eq. (17) and combining it with the inequality (19), we immediately obtain that \( \mu \leq 1 \).

The following lemma provides the key inequalities used to prove the convergence results of the RBCD method.

Lemma 4. For any vector \( x \), let \( x^+ \) to be defined as in E.q. (8) by picking up \( i \in \{1, 2, \ldots, n\} \) uniformly at random. Set stepsize \( \alpha = \frac{1 + \theta}{2L} \). For any vector \( u \), the expectation of \( F(x^+) \) satisfies

\[ E_i[F(x^+)] \leq \frac{1}{n} \left[ (n - 1)F(x) + F(u) + (1 - \mu)D_H(u, x) - D_H(u, T_i(x)) \right]. \]  

(20)

and the expectation of \( D_H(x^+, x) \) satisfies

\[ E_i[D_H(u, x^+)] = \frac{n - 1}{n} D_H(u, x) + \frac{1}{n} D_H(u, T_i(x)). \]  

(21)

By applying Lemma 4 the main convergence results are established in Theorem 1. Note that this result generalizes Theorem 1 through replacing the proximal mapping by the classical strongly convexity is recovered.

Theorem 1. Let \( \{x_k\} \) be the sequence generated by Algorithm 1. Then for any \( k \geq 0 \), the iterates \( x_k \) satisfies

\[ E[F(x^k) - F(x^\star)] \leq \frac{n}{n + k} \left( F(x^\star) - F(x^0) + D_H(x^\star, x_0) \right). \]  

(22)

Further, if \( f \) is \( \mu \)-strongly convex relative to \( H \), then

\[ E[F(x^k) - F(x^\star)] \leq \left( 1 - \frac{(1 + \theta)\mu}{n(1 + \theta\mu)} \right)^k \left( F(x^0) - F(x^\star) + D_H(x^\star, x^0) \right). \]  

(23)

where \( \theta = \min \{\theta_i\} \).

Therefore, if \( F \) is convex, the sequence \( \{x_k\} \) needs at most \( \mathcal{O}(n\varepsilon^{-1}) \) to converge to an \( \varepsilon \)-solution. Further, the classical linear convergence rate is obtained if \( f \) is strongly convex (relative to \( H \)).

B. Nonconvex case

In this subsection, we establish the convergence results for the case where \( F \) is nonconvex. Since \( r \) is convex, \( f \) is nonconvex. Due to the nonconvexity, it is of interest to find a stationary point. Lemma 3 implies that \( D_H(T(x), x) \) can be used to measure the optimality. The following result shows the descent property of the proposed method in terms of the optimality gap \( D_H(T(x), x) \).

Lemma 5. For any \( x \), let \( x^+ \) to be defined as in E.q. (8) by picking up the index \( i \) uniformly at random. Let \( \alpha = \frac{1 + \theta}{2L} \). Then the following inequality holds:

\[ E[F(x^+)] \leq F(x) - \frac{1}{n} D_H(T(x), x). \]  

(24)

Using Lemma 3, we can establish the convergence results of the RBCD method for nonconvex \( F \).

Theorem 2. Let \( \{x_k\} \) to be the sequence generated by Algorithm 1. Let stepsize \( \alpha = \frac{1 + \theta}{2L} \), then

(i) The sequence \( \{F(x^k)\} \) is non-increasing.

(ii) \( \sum_{k=0}^{\infty} E[D_H(T(x^k), x^k)] < \infty \), and hence the sequence \( \{E[D_H(T(x^k), x^k)]\} \) converges to zero.

(iii) \( \forall k \geq 0, \) we obtain

\[ \min_{0 \leq i \leq k} E[D_H(T(x^i), x^k)] \leq \frac{n}{k + 1} (F(x^0) - F^*), \]  

(25)

where \( F^* = \inf F(x) \rightarrow -\infty \).

(iv) Every limit point of \( \{x_k\} \) is a stationary point.

Suppose \( H \) is \( \sigma \)-strongly convex with respect to the Euclidean norm \( \| \cdot \| \). Then we have \( D_H(y, x) \geq \frac{\sigma}{2} \| y - x \|^2 \). Combining the strongly convexity of \( H \) with Theorem 2 we immediately obtain the following convergence rate result

\[ \min_{0 \leq i \leq k} E[T(x^i) - x^k] \leq 2 \frac{n}{\sigma(k + 1)} (F(x^0) - F^*). \]  

(26)

Therefore, the sequence \( \{x_k\} \) converges to a stationary point at the rate of \( \mathcal{O}(\sqrt{n}/\varepsilon) \). In another word, to obtain an \( \varepsilon \)-stationary point, i.e., \( \|T(x) - x\| \leq \varepsilon \), the RBCD method needs to run \( \mathcal{O}(n\varepsilon^{-2}) \) iterations.

V. ACCELERATED RANDOMIZED BREGMAN COORDINATE DESCENT

In this section, we restrict ourselves to the unconstrained smooth minimization problem as follows

\[ \min_{x \in \mathcal{X}} f(x), \]  

(27)

where \( f \) is convex and satisfies Assumption 1. The closed convex set \( \mathcal{X} \) satisfies \( \mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \), such that \( x_i \in \mathcal{X}_i \), \( \forall i \). It is equivalent to consider \( r_i \) as an indicator function of the closed convex set \( \mathcal{X}_i \).
The accelerated randomized Bregman coordinate descent (ARBCD) method is given as Algorithm 2. At the k-th iteration, the ARBCD method selects a coordinate $i_k$ uniformly at random, and generates the three vectors $y^k$, $x^{k+1}$, and $z^{k+1}$, where the vectors $y^k$ and $x^{k+1}$ are the affine combinations of $x^k$ and $z^k$, and $y^k$, $z^k$, and $z^{k+1}$, respectively, and the vector $z^{k+1}$ is obtained as follows:

$$z^{k+1} = \arg\min_{u \in \mathcal{X}} (\nabla f(y^k), u_{i_k} - y^k_{i_k}) + (n\beta_k)^{-1} D_H(u, z^k).$$

(28)

Note that Step 1 and 3 of Algorithm 2 need $O(N)$ operations, while $O(1)$ operations are usually expected in a general coordinate descent method. In the latter section, we will show an efficient implementation of the ARBCD method so that the ARBCD method only needs $O(1)$ operations at each iteration.

**Algorithm 2:** Accelerated Randomized Bregman (Block) Coordinate Descent (ARBCD).

**Input:** initial $x_0$ and $\gamma$

Initialize: $z^0 = x^0$ and $\beta_0 = 1$

for $k = 1, 2 \cdots$ do

1) $y^k = (1 - \beta_k)x^k + \beta_k z^k$

2) Choose $i_k \in \{1, 2, \cdots, n\}$ uniformly at random

Compute $z^{k+1}$ by E.q. (28)

3) $x^{k+1} = y^k + n\beta_k (z^{k+1} - z^k)$

4) Choose $\beta_{k+1} \in (0, 1]$ such that $

\frac{1 - \beta_k}{\beta_{k+1}} \leq \frac{1}{\beta_k}$

end

VI. CONVERGENCE ANALYSIS OF ARBCD

To better understand the proposed method, we make the following definitions and observations. First, we define the vector $z^{k+1}$ as follows:

$$z^{k+1} = \arg\min_{u \in \mathcal{X}} (\nabla f(y^k), u - y^k) + (n\beta_k)^{-1} D_H(u, z^k),$$

(29)

which is the full-dimensional update version of $z_{i_k}^{k+1}$ in E.q. (28). Therefore, the vector $z^{k+1}$ can be computed by:

$$z_{i_k}^{k+1} = \begin{cases} \hat{z}_{i_k}^{k+1}, & \text{if } i = i_k, \\ z_k, & \text{if } i \neq i_k. \end{cases}$$

(30)

It follows from the definition of $x^{k+1}$ in Step 3 of Algorithm 2 that we have:

$$x_{i_k}^{k+1} = \begin{cases} y_{i_k} + n\beta_k (z_{i_k}^{k+1} - z_k^k), & \text{if } i = i_k, \\ y_k, & \text{if } i \neq i_k. \end{cases}$$

(31)

Clearly, the vector $x^{k+1}$ and $y^k$ are only one coordinate part from each other, which satisfies the relative smoothness property in Assumption 2.

One of the challenges to establish the convergence results is from the nature of Bregman distances. Since a Bregman distance is in general not a norm, it does not hold the homogeneous translation invariant, i.e.,

$$\|u + \theta(v - w)\| = |\theta| \|v - w\|, \quad \forall \alpha, u, v, w.$$  

(32)

To handle this issue, [22] introduces the notion of triangle scaling property (TSP).

**Definition 3.** [22] Definition 2] The Bregman distance defined with a convex reference function $h$ has the triangle scaling property if there exists some scalar $\gamma > 0$ such that for all $u, v, w$:

$$D_h((1 - \theta)u + \theta v, (1 - \theta)u + \theta w) \leq \theta \gamma D_h(v, w), \forall \theta \in [0, 1].$$

(33)

In contrast, we introduce the more general notion of the generalized translation invariant (GTI) in the following definition, and show it is equivalent to triangle scaling property, when restricting $\theta \in [0, 1]$.

**Definition 4.** [Generalized Translation Invariant] The Bregman distance defined with a convex reference function $h$ has the generalized translation invariant property if there exists some scalar $\gamma \geq 0$ such that for all $u, v, w$:

$$D_h(u + \theta(v - w), u) \leq |\theta| \gamma D_h(v, w). \quad \forall \theta \in \mathbb{R}.$$  

(34)

**Lemma 6.** The Bregman distance has the generalized translation invariant with $\theta \in [0, 1]$ if and only if it holds the triangle scaling property.

**Remark 2.** Here we gives three examples to show the existences of GNI in some Bregman divergences, while the proof is included in Appendix.

(i) The norms. Let $\| \cdot \|_A$ be a norm, $A$ be a positive define matrix, $h(x) = (1/2)||x||_A^2$, and $D_h(x, y) = (1/2)||x - y||_A^2 = (1/2)x^TAy$. It is easy to see that $\gamma = 2$.

(ii) The Kullback-Leibler (KL) divergence. Let $h$ be the negative Boltzmann-Shannon entropy:

$$h(x) = \sum_{i=1}^N x_i \log x_i$$

defined over $\mathbb{R}_+^N$. The Bregman distance is given by:

$$D_{KL}(x, y) = \sum_{i=1}^N \left( x_i \log \left( \frac{x_i}{y_i} \right) - x_i + y_i \right).$$  

(35)

It can be shown that $\gamma = 1$.

(iii) The Itakura-Saito (IS) distance. Let $h$ be the Burg’s entropy: $h(x) = -\sum_{i=1}^N \log x_i$ on $\mathbb{R}_+^N$. The Bregman distance associated with $h$ is given by:

$$D_{IS}(x, y) = \sum_{i=1}^M \left( -\log \left( \frac{x_i}{y_i} \right) + \frac{x_i}{y_i} - 1 \right).$$  

(36)

To satisfy the definition of GNI, we must have $\gamma = 0$. Similar to TSP, however, $\gamma = 0$ is the uniform value for $D_{IS}$, and the intrinsic $\gamma$ value can be $2$ if the three points are close to each other [22, Theorem 1].

Note that the GTI is more general since TSP needs $\theta \in [0, 1]$, but GTI holds for all $\theta \in \mathbb{R}$.

To use the notion of GTI, we make the following assumption.

**Assumption 4.** The Bregman distances $D_h(\cdot, \cdot)$ have the generalized translation invariant with the constant $\gamma > 0$, $\forall \theta$.

Using the notion of GTI, we will show that the ARBCD method converges with a sublinear rate of $O(n\epsilon^{-1/\gamma})$. We
start with recalling the critical lemma \[33\] Lemma 3.2 for a Bregman proximal mapping.

**Lemma 7.** \[33\] **Lemma 3.2** For a convex function \( \phi \) and a vector \( x \), if the Bregman proximal mapping is defined as
\[
x^+ = \argmin \phi(u) + D_h(u, x),
\]
and then
\[
\phi(u) + D_h(u, x) \geq \phi(x^+) + D_h(x^+, x) + D_h(u, x^+), \forall u.
\]

The key relationship between two consecutive iterates in Algorithm 3 is established in the following lemma.

**Lemma 8.** Suppose Assumptions 1, 2, and 4 holds. For any vector \( u \), the sequences generated by Algorithm 3 satisfy, for all \( k \geq 0 \),
\[
E_{i_k} \left[ 1 - \frac{\beta_{k+1}}{\beta_k} \left( f(x^{k+1}) - f(u) \right) + n^\gamma D_H(u, z^{k+1}) \right] \\
\leq 1 - \frac{\beta_k}{\beta_{k+1}} (f(x^k) - f(u)) + n^\gamma D_H(u, z^k).
\]

The following lemma introduces a sequence \( \{\beta_k\} \) that satisfies the condition in Step 4 of Algorithm 3.

**Lemma 9.** \[32\] **Lemma 3.1** The sequence \( \beta_k = \frac{\gamma}{k+\gamma} \) satisfies
\[
\frac{\beta_{k+1} - 1}{\beta_{k+1}} \leq \frac{1}{\beta_k}, \quad \forall k \geq 0.
\]

Combining Lemma 8 with Lemma 9 the main convergence results for the ARBCD are established in the following theorem.

**Theorem 3.** Suppose Assumptions 1, 2, and 4 hold. If \( \beta_k = \frac{\gamma}{k+\gamma} \) for all \( k \geq 0 \), then the following inequality holds, for any vector \( u \),
\[
E \left[ f(x^{k+1}) - f(u) \right] \leq \left( \frac{n^\gamma}{k+\gamma} \right)^{\gamma} D_H(u, x^0), \quad \forall k \geq 0.
\]

Note that due to the affine combinations in Step 1 and 3 of Algorithm 2 the current implementation requires \( O(N) \) operations. In the next section, we introduce an efficient implementation so that only \( O(1) \) operations are needed at each iteration.

### VII. Efficient Implementation

In order to avoid full-dimensional vector operations, the previous works \[12\], \[34\] propose a strategy that changes the variables for the accelerated coordinated descent methods in the global Lipschitz-continuous (partial) gradient setting. Here we show this scheme can be adapted so that the full-dimensional operations can be avoided in the relative smoothness setting, which is given as Algorithm 3. Instead of computing the vector \( z^{k+1} \), a search direction \( d_{i_k}^k \) is computed in Algorithm 3 as follows
\[
d_{i_k}^k = \argmin_{\gamma_k, \delta_k} + d \in X \left\langle \nabla f_i, \gamma_i \right\rangle (\gamma_k, \delta_k, u^k + v^k), d \rangle + (n\beta_k)^{\gamma_k} L_{i_k} D_h(v_{i_k}^k + d, v_{i_k}^k).
\]

**Algorithm 3: Efficient implementation of ARBCD.**

**Input:** initial \( x_0 \) and \( \gamma \)

Initialize: \( v^0 = x^0, u^0 = 0 \) and \( \beta_0 = 1 \)

for \( k = 1, 2 \cdots \) do
1. Choose \( i_k \in \{1, 2, \cdots, n\} \) uniformly at random
2. \( v_{i_k}^{k+1} = v_{i_k}^k + d_{i_k}^k \)
3. \( u_{i_k}^{k+1} = u_{i_k}^k - \frac{1}{\beta_k} d_{i_k}^k \)
4. Compute \( \beta_{k+1} \) from \( \frac{1 - \beta_{k+1}}{\beta_{k+1}} = \lambda \)
end

return \( \beta_{k+1} u_{k+1} + v_{k+1} \)

**Proposition 1.** The sequences \( \{x^k, y^k, z^k\} \) and \( \{u^k, v^k\} \) generated from Algorithm 2 and 3 respectively, satisfy
\[
z^k = v^k
\]
\[
x^k = \beta_{k+1} u^k + v^k
\]
\[
y^k = \beta_{k+1} u^k + v^k
\]
for all \( k \geq 1 \). That is, these two algorithms are equivalent.

Note that in Algorithm 3 only a single block coordinate of the vectors \( u^k \) and \( v^k \) are updated at each iteration, which cost \( O(N_i) \) operations. Although computing the partial gradient in Eq. (42) may still cost full-dimensional operations in general, the previous works \[11\], \[12\], \[34\] introduce a number of optimization problems where the partial gradient can be computed cheaply without actually forming \( y^k \).

### VIII. Numerical Experiments

To showcase the strength of the proposed methods, we consider two applications of relatively smooth convex optimization: Poisson inverse problem, and relative-entropy nonnegative regression.

#### A. Poisson linear inverse problem

A large number of problems in nuclear medicine, night vision, astronomy and hyperspectral imaging can be described as inverse problems where data measurements are collected according to a Poisson process whose underlying intensity function is indirectly related to an object of interest through a linear system. This class of problems have been studied intensively in the literature. See for instance \[35\]–\[37\] and references therein, as well as a more recent comprehensive review \[38\] for the up-to-date references.

Formally, in a Poisson inversion problem we are given a nonnegative observation matrix \( A \in \mathbb{R}_{+}^{M \times N} \), a noisy measurement vector \( b \in \mathbb{R}_{+}^{M} \), and the goal is to recover the signal or image of interest \( x \in \mathbb{R}_{+}^{N} \). Under the Poisson assumption, we can rewrite the observation model as follows
\[
b \sim \text{Poisson}(Ax).
\]

Therefore, a natural and widely used measure of proximity of two nonnegative vectors is based on the KL divergence.
Particularly, minimizing the KL-divergence $D_{KL}(b, Ax)$ is equivalent to maximize the Poisson log-likelihood function. The optimization problem can be formulated as follows

$$\min_{x \geq 0} f(x) \equiv D_{KL}(b, Ax). \quad (47)$$

To apply the RBCD and ARBCD methods, we need to identify a series of adequate reference functions $h_i$. Here we use Burg’s entropy and the corresponding Bregman distance, i.e., the IS distance.

**Lemma 10.** Let $f_i(x_i) = D_{KL}(b, Ax)$ and $h_i(x_i)$ be defined as

$$h_i(x_i) = -\log x_i. \quad (48)$$

Then the functions $(f_i, h_i)$ are relatively smooth with any scalar $L_i$ satisfying

$$L_i \geq \|b\|_1 = \sum_{i=1}^{M} b_i. \quad (49)$$

Equipped with Lemma 10, Theorem 1 is applicable and warrants the convergence. Since $\theta(h_i) = 0$, we can take the stepsize $\alpha_k = 1/\|b\|_1 \forall k \geq 0$. To solve Poisson inverse problems, the E.q. (28) can be written as

$$T_i(x) = \arg\min_{u_i \geq 0} (\nabla f_i(x_i), u_i) + 2\|b\|_1 D_{IS}(u_i, x_i). \quad (50)$$

It follows from [22] Theorem 1 that the intrinsic TSE of a Bregman distance is 2, even the uniform TSE is not. In addition, [22] numerically shows the convergence and efficiency of the Accelerated Bregman Proximal method (ABPG) with $\gamma = 2$. Thus, we here also use $\gamma = 2$ for the ARBCD method. As a result, E.q. (28) becomes

$$z_{ik}^{k+1} = \arg\min_{u_{ik} \geq 0} (\nabla f_i(x_i), u_{ik}) + 2n\beta_k \|b\|_1 D_{IS}(u_{ik}, z_{ik}^k). \quad (51)$$

We compare the proposed algorithms RBCD and ARBCD with two state-of-the-art algorithms: Bregman Proximal Gradient (BPG) method [20] and accelerated Bregman Proximal Gradient (ABPG) [22] method. All algorithms are implemented in Matlab code.

Figure 1 shows the computational results for a randomly generated dataset with $M = 500$ and $N = 500$. The entries in $A$ and $b$ are generated randomly from a uniform distribution over the interval $[0, 1]$. Each algorithm starts with the same initial values. Note that the CD-type methods has a inner loop of $N$ iterations as their computational complexity is $N$ times cheaper than the gradient-based methods. As a result, the computational complexity in each iteration is identical.

In Figure 1 we can see the RBCD method is only slightly better than the BPG method, because the RBCD method uses the most updated coordinate to update, and BPG and RBCD methods use the same stepsize $\alpha_k = 1/\|b\|_1$. Figure 1 also shows that the accelerated methods ABPG and ARBCD are both faster than their non-accelerated variants. We can also conclude that the ARBCD method is faster than the other methods. It is well-known that the accelerated (proximal) gradient method does not guarantee the descent in the objective values at each iteration. Instead, the number of ripples are on the traces of the objective values. This criteria can be found on the ABPG method as well in Figure 1. On the other hand, we do not find such ripples or bumps from the ARBCD method. Particularly, Figure 1 shows that the ARBCD method provides consistent descent in the objective values.

It is easy to check numerically that $D_{IS}$ does not hold GTI or TSP property for any scalar $\gamma > 0.5$. We conduct another experiment to explore the impact of the parameter $\gamma$. Figure 2 shows the convergence behaviors of the ABP and ARBCD methods with $\gamma = 0.1, 1.0$ and $2.0$. The larger $\gamma$ is, the more acceleration the ABPG method obtains. However, it seems the ARBCD method holds the opposite relationship with the $\gamma$ values. The ARBCD method achieves the maximum acceleration when the $\gamma$ is minimum.

**B. Relative-entropy nonnegative regression**

Anther formulation to solve the nonnegative linear inverse problem introduced in Section VII-A is to minimize $D_{KL}(Ax, b)$, i.e.,

$$\min_{x \geq 0} f(x) \equiv D_{KL}(Ax, b). \quad (52)$$

Fig. 1: Poisson inverse problem: synthetic dataset with $M = 500$ and $N = 500$.

Fig. 2: Poisson inverse problem: synthetic dataset with varying $\gamma$ values.
the ARBCD method is faster than the rest methods. As the faster than the BPG and ABPG methods, respectively, and Figure 1, where the RBCD and ARBCD methods are slightly shows the almost identical convergence behaviors as in Figure 4 shows improved convergence for the generated dataset with a = 500 where the second inequality is due to D_h(x_i, T_i(x)) \geq \theta D_h(T_i(x), x_i). Since x^*_j = x_j \forall i \neq j, we obtain

\[ F(x^+) \leq F(x) - \left( \frac{1 + \theta}{\alpha} - L_i \right) D_h(T_i(x), x_i). \]

causes the divergence of the ARBCD method. Therefore, the choice of the hyperparameter \( \gamma \) has significant influence on the performance of the ARBCD method.

IX. CONCLUSION

In this paper, we propose a randomized Bregman (block) coordinate descent (RBCD) method and its accelerated variant ARBCD method for minimizing a composite problem, where the smooth part of the objective function does not satisfies the global Lipschitz-continuous (partial) gradient property. By using the relative smoothness, we establish the iteration complexity of \( O(n \varepsilon^{-2}) \) to obtain an \( \varepsilon \)-stationary point in the case where \( F \) is nonconvex. Besides, the iteration complexity is improved to \( O(n \varepsilon^{-1}) \) if \( f \) is convex, and the global linear convergence rate can be achieved by RBCD if \( f \) is strongly convex. We introduce the notion of generalized translation invariant. Thanks to this notion, we are able to establish the convergence result for the ARBCD method which uses the acceleration technique. Thus, the iteration complexity is further improved to \( O(n \varepsilon^{-1/\gamma}) \) by the ARBCD method.

APPENDIX

A. Proof of Lemma 2

Proof: From the relative smoothness, we obtain

\[ f(x^+) \leq f(x) + \langle \nabla_i f(x), T_i (x) - x_i \rangle + L_i D_h(T_i(x), x_i). \]  

(56)

From the optimality of \( T_i(x) \) in (9), we have

\[ \nabla_i f(x) + \frac{1}{\alpha} (\nabla_h_i(T_i(x)) - \nabla_h_i(x_i)) + v_i^+ = 0, \]

for some \( v_i^+ \in \partial r_i(T_i(x)) \). The convexity of \( r_i \) implies

\[ r_i(x_i) - r_i(T_i(x)) \geq \langle v_i^+, x_i - T_i(x) \rangle \]

\[ = - \langle \nabla_i f(x) + \frac{1}{\alpha} (\nabla h_i(T_i(x)) - \nabla h_i(x_i)), x_i - T_i(x) \rangle \]

\[ = - \langle \nabla_i f(x), x_i - T_i(x) \rangle + \frac{1}{\alpha} (D_h(x_i, T_i(x)) + D_h(T_i(x), x_i)) \]

(57)

Combining E.q. (56) and (57) yields

\[ f(x^+) + r_i(T_i(x)) \leq f(x) + r_i(x_i) + L_i D_h(T_i(x), x_i) \]

\[ - \frac{1}{\alpha} (D_h(x_i, T_i(x)) + D_h(T_i(x), x_i)) \]

\[ \leq f(x) + r_i(x_i) - \left( \frac{1 + \theta}{\alpha} - L_i \right) D_h(T_i(x), x_i), \]

\[ F(x^+) \leq F(x) - \left( \frac{1 + \theta}{\alpha} - L_i \right) D_h(T_i(x), x_i). \]
B. Proof of Lemma \[3\]

Proof: (\(\implies\)). Suppose \(x\) is a stationary point. Then we have
\[
\nabla f(x) + v = 0,
\]
for some \(v \in \partial r(x)\). From the convexity of \(r\), it follows that for any vector \(u\)
\[
r(u) \geq r(x) - \langle \nabla f(x), u - x \rangle.
\]
(58)

By the optimality of \((15)\), we obtain
\[
\nabla f(x) + \nabla H(T(x)) - \nabla H(x) + v^+ = 0,
\]
for some \(v^+ \in \partial r(T(x))\). It follows that
\[
r(x) - r(T(x)) \geq -\langle \nabla f(x), x - T(x) \rangle
- \langle \nabla H(T(x)) - \nabla H(x), x - T(x) \rangle.
\]
(60)

Let \(u = T(x)\) and combine the equations (58) and (60). Then we obtain
\[
0 \geq D_H(T(x)) + D_H(T(x), x).
\]
Since \(D_H(x, T(x)), D_H(T(x), x) \geq 0\), we obtain
\(D_H(T(x), x) = 0\).

\((\impliedby)\). Suppose \(D_H(T(x), x) = 0\). The (strict) convexity of \(H\) implies \(T(x) = x\). From (59), we obtain
\[
0 \in \nabla f(x) + \partial r(x),
\]
which indicates \(x\) is a stationary point.

C. Proof of Lemma \[4\]

Proof: Since each block \(i\) is selected uniformly at random, we have
\[
E_i[F(x^+)] = \sum_{i=1}^{n} \frac{1}{n} F(x^+)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} f(x^+) + r(x^+)
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} f(x) + \langle \nabla_i f(x), T_i(x) - x_i \rangle
+ L_i D_H(T_i(x), x_i) + r(T_i(x)) + \sum_{j \neq i} r_j(x_j)
\]
\[
(i) \leq \frac{1}{n} [n f(x) + \nabla f(x), T(x) - x]
+ D_H(T(x), x) + r(T(x)) + (n - 1)r(x)]
= \frac{1}{n} [(n - 1) F(x) + f(x) + \nabla f(x), T(x) - x]
+ D_H(T(x), x) + r(T(x))
\]
\[
(ii) \leq \frac{1}{n} [(n - 1) F(x) + f(u) - \mu D_H(u, x)
+ D_H(T(u), x) + r(u) + \langle \nabla H(T(x)) - \nabla H(x), u - T(x) \rangle)]
\]
\[
= \frac{1}{n} [(n - 1) F(x) + F(u) + (1 - \mu) D_H(u, x) - D_H(u, T(x)) + \frac{k + 1}{n} \{ F(x) - F(x^*) \}]
\]
where (i) follows from the relative smoothness of \((f_i, h_i)\); (ii) uses the fact of \(T_i(x) = T(x)\); (iii) is based on the convexity of \(f\) and \(r\); (iv) uses the fact of \((\nabla h(z) - \nabla h(x), y - z) = D_h(y, x) - D_h(y, z) - D_h(z, x)\).

For any vector \(u\), we have
\[
D_H(u, x^+)
= \sum_{i} L_i D_H(u_i, T_i(x)) + \sum_{j \neq i} L_j D_H(u_j, x_j)
= \sum_{i} L_i D_H(u_i, T_i(x)) - L_i D_H(u_i, x_i) + D_H(u, x) \quad (61)
\]
Taking the expectation of Eq. (61) with respect to \(i\) yields
\[
E_i[D_H(u, x^+)]
= \sum_{i} \frac{1}{n} [D_H(u, x) - L_i D_H(u_i, x_i) + L_i D_H(u_i, T_i(x))]
= \frac{1}{n} \{ n D_H(u, x) - D_H(u, x) + D_H(u, T(x)) \}
= D_H(u, x) - \frac{1}{n} [D_H(u, x) - D_H(u, T(x))]
\]

D. Proof of Theorem \[7\]

Proof: Combining (21) with (20), let \(u = x^*\), and we have
\[
E_i[F(x^+)] \leq E[F(x)] + D_H(x^*, x) - D_H(x^*, x^+)
- \frac{1}{n} [F(x) - F(x^*)]
\]
(62)

Taking the expectation of (63) with respect to \(\{i_0, i_1, \cdots\}\) yields
\[
E[F(x^+)] \leq E[F(x)] + D_H(x^*, x) - D_H(x^*, x^+)
- \frac{1}{n} [F(x) - F(x^*)]
\]
(63)

Summing over \(l = 0, 1, \cdots, k - 1\) yields
\[
E[F(x^k)] \leq F(x^0) + D_H(x^*, x^0) - E[D_H(x^*, x^k)]
- \frac{1}{n} \sum_{l=0}^{k} E[F(x^l) - F(x^*)]
\]
\[
\leq F(x^0) + D_H(x^*, x^0) - \frac{1}{n} \sum_{l=0}^{k} E[F(x^l) - F(x^*)]
\]
\[
\leq F(x^0) + D_H(x^*, x^0) - \frac{k}{n} E[F(x^{k+1}) - F(x^*)],
\]
where the last inequality is because \(\{F(x^l)\}\) is a descent sequence. Subtracting \(F(x^*)\) on both sides and rearrange yields
\[
E[F(x^k)] \leq F(x^0) - F(x^*) + D_H(x^*, x^0) + \frac{k}{n} E[F(x^{k+1}) - F(x^*)] \leq F(x^0) - F(x^*) + D_H(x^*, x^0).
\]
Dividing both sides by \(\frac{n+k}{n}\) yields the desired result.
If $f$ is $\mu$-strongly convex relative to $H$, we have

$$
\mathbb{E}[F(x^+)] = M \leq \frac{n-1}{n} F(x) + \frac{1}{n} F(x^+) + \frac{1}{n} L_i H(T_i(x), x_i).
$$

Subtracting $F(x^+)$ on the both sides and rearrange yields

$$
\mathbb{E}[F(x^+) - F(x^+)] 
\leq F(x) - F(x^+) + H(T_i(x), x_i).
$$

The relative strongly convexity of $F$ implies

$$
F(x) - F(x^+) + \mu H(x^+, x) 
\geq \mu H(x^+, x^+),
$$

Define

$$
\beta = \frac{(1 + \theta)\mu}{1 + \theta},
$$

Clearly, we have $\beta \leq 1$ since $\mu \leq 1$. Then

$$
F(x) - F(x^+) + \mu H(x^+, x) 
\geq \beta F(x) - F(x^+) + \mu H(x^+, x).
$$

Combining the inequality above with (64) yields

$$
\mathbb{E}[F(x^+) - F(x^+) + H(x^+, x^+)] 
\leq \left(1 - \frac{\beta}{n}\right) (F(x) - F(x^+) + H(x^+, x)).
$$

Taking the expectation with respect to $\{i_0, i_1, \cdots\}$ on the both sides of the relation above, we have

$$
\mathbb{E}[F(x^k) - F(x^+) + H(x^+, x^k)] 
\leq \left(1 - \frac{\beta}{n}\right)^k (F(x^0) - F(x^+) + H(x^+, x^0)).
$$

Dropping $H(x^+, x^k)$ on the left hand yields the desired result.

**E. Proof of Lemma 5**

**Proof:** Taking the expectation of (12) with respect to $i$ yields

$$
\mathbb{E}_i[F(x^+)] 
\leq F(x) - M_i D_i(T_i(x), x_i).
$$

$$
= F(x) - \sum_{i=1}^n \frac{1}{n} L_i D_i(T_i(x), x_i)
= F(x) - \frac{1}{n} \sum_{i=1}^n L_i D_i(T_i(x), x_i)
= F(x) - \frac{1}{n} D_i(T(x), x),
$$

where $(i)$ is because $T_i(x) = T(x)_i$.

**F. Proof of Theorem 2**

**Proof:** (i). The result is directly obtained from Lemma 5 (ii). Taking the expectation of (24) with respect to all variables and rearranging yields

$$
\mathbb{E}[H(T(x^i), x^i)] 
\leq n \mathbb{E}(F(x^i) - F(x^{i+1})).
$$

Taking the telescopic sum of the above inequality for $l = 0, 1, \cdots, k$ gives us

$$
\sum_{l=1}^k \mathbb{E}[H(T(x^i), x^i)] 
\leq n (F(x^0) - \mathbb{E}[F(x^k)])
\leq n (F(x^0) - F^*).
$$

Since $F$ is lower bounded, taking the limit $k \to \infty$ yields the desired result.

(iii). The inequality (66) further implies that

$$
(k + 1) \min_{0 \leq i \leq k} \mathbb{E}[H(T(x^i), x^i)] 
\leq n (F(x^0) - F^*)
\leq n (F(x^0) - F^*).
$$

Dividing $k + 1$ on both sides gives us the desired result.

(iv). Let $x^*$ be a limit point of $\{x^k\}$ and there exists a subsequence $\{x^{k_p}\}$ such that $x^{k_p} \to x^*$ as $p \to \infty$.

Since the functions $r_i$ are lower semi-continuous, we have for all $i$,

$$
\liminf_{p \to \infty} r_i(x^{k_p}) \geq r_i(x^*).
$$

At the $k$-th iteration, suppose the index $i$ is selected, then the convexity of $r_i$ implies that

$$
r_i(x^{k+1}_i) - r_i(x^*_i)
\leq \langle \nabla f(x^k) + \nabla h_i(x^{k+1}_i), x^{k+1}_i - x^*_i \rangle
$$

Let $\{x^{k_i}\}$ be the subsequence of $\{x^{k_p}\}$ such that the index $i$ is selected. Choosing $k = k_i - 1$ in the above inequality, and letting $q \to y$ yields

$$
\limsup_{q \to \infty} r_i(x^{k_q}) \leq r_i(x^*_i),
$$

where we use the facts $x^{k_q} \to x^*$ as $q \to \infty$. Thus, combining (68) with (67), we have

$$
\lim_{q \to \infty} r_i(x^{k_q}) = r_i(x^*_i).
$$

Since $i$ is selected arbitrarily, we have

$$
\lim_{p \to \infty} r_i(x^{k_p}) = r_i(x^*_i), \quad \forall i.
$$

Furthermore, by the continuity of $f$, we obtain

$$
\lim_{p \to \infty} F(x^{k_p}) = \lim_{p \to \infty} \left\{ f(x^{k_p}) + \sum_{i=1}^n r_i(x^{k_p}) \right\}
= f(x^*) + \sum_{i=1}^n r_i(x^*_i) = F(x^*).
$$

From (ii) and Lemma 5 it follows that $x^*$ is a stationary point of $F$. 

G. Proof of Lemma 2

Proof: Suppose the Bregman distance $D_h(\cdot, \cdot)$ holds the generalized translation variant, and let $u = (1 - \theta)x + \theta w$ for any $x$. Then we have

$$D_h((1 - \theta)x + \theta w, (1 - \theta)x + \theta w) \leq |\theta|^\gamma D_h(v, w), \quad \forall \theta \in \mathbb{R}.$$ 

Since the above inequality holds for all $\theta$, it must hold for $\theta \in [0, 1]$.

\[
D_h(y + \theta(v - w), y) \leq \theta D_h(v, w), \quad \forall \theta \in [0, 1]. \tag{69} 
\]

Therefore, the generalized translation invariant holds for $\theta \in [0, 1]$.

H. Proof of Remark 2

(i) It is easy to verify that

\[
\frac{1}{2}||u + \theta(v - w) - u||_A^2 = \frac{1}{2}\theta^2||v - w||_A^2.
\]

(ii) Without loss the generality, we assume $N = 1$. Using the log sum inequality, we obtain

\[
D_{KL}(u + \theta(v - w), u) \\
= (u + \theta(v - w)) \log \left( \frac{u + \theta(v - w)}{u} \right) + \theta(v - w) \\
= (u + \theta(v - w)) \log \left( \frac{u + \theta(v - w)}{u} \right) + \theta(v - w) \\
+ (\theta v - u - \theta(v - w)) \log \left( \frac{\theta v - u - \theta(v - w)}{\theta v - u} \right) \\
- (\theta v - u - \theta(v - w)) \log \left( \frac{\theta v - u - \theta(v - w)}{\theta v - u} \right) \\
\leq \theta v \log \left( \frac{v}{w} \right) + \theta(v - w) \\
= \theta D_{KL}(v, w).
\]

(iii) Without loss generality, we assume $N = 1$. As the GNI property in Definition 4 is defined for all $u, v, w$, we consider a special case of $u = \theta w$. Then, we have

\[
D_{BS}(u + \theta(v - w), u) = D_{BS}(\theta v, \theta w) \\
= -\log \left( \frac{\theta v}{\theta w} + \frac{\theta v}{\theta w} - 1 \right) \\
= D_{BS}(v, w).
\]

To obtain $D_{BS}(\theta v, \theta w) \leq |\theta|^\gamma D_{BS}(v, w)$ for all $\theta \in \mathbb{R}$, we must have $\gamma = 0$, otherwise $\theta > 1$ for all $\theta \in (0, 1)$.

I. Proof of Lemma 8

Proof: With simple algebra operations, we have

\[
x^{k+1} - y^k = n \left[ \beta_k(z^{k+1} - y^k) + (1 - \beta_k)(x^k - y^k) \right].
\]

Based on the relation in E.q. (31), we know $x^{k+1}$ and $y^k$ satisfy the relative smoothness property since they are only one coordinate different from each other. Therefore, we obtain

\[
f(x^{k+1}) \leq f(y^k) + (\nabla y_k f(y^k), x^{k+1} - y^k) + L_{ik} D_h(x^{k+1}_i, y^k_i) \\
= f(y^k) + (\nabla y_k f(y^k), x^{k+1}_i - y^k_i) \\
+ L_{ik} D_h(y^k_i + n\beta_k(z^{k+1}_i - z^k_i), y^k_i) \\
\leq f(y^k) + (\nabla y_k f(y^k), x^{k+1}_i - y^k_i) \\
+ (n\beta_k)^\gamma L_{ik} D_h(z^{k+1}_i, z^k_i) \\
= f(y^k) + n\beta_k(\nabla y_k f(y^k), z^{k+1}_i - y^k_i) \\
+ n(1 - \beta_k)(\nabla y_k f(y^k), x^{k+1}_i - y^k_i) \\
+ (n\beta_k)^\gamma L_{ik} D_h(z^{k+1}_i, z^k_i) \\
\leq \beta_k \left[ f(y^k) + n(\nabla y_k f(y^k), z^{k+1}_i - y^k_i) \right] \\
+ (1 - \beta_k) \left[ f(y^k) + n(\nabla y_k f(y^k), x^{k+1}_i - y^k_i) \right] \\
+ (n\beta_k)^\gamma L_{ik} D_h(z^{k+1}_i, z^k_i),
\]

where (i) is using the generalized translation invariant, (ii) is due to E.q. (70), and (iii) is due to E.q. (30). Taking the expectation with respect to $i_k$ on both sides yields for all $u$

\[
E_{i_k} f(x^{k+1}) \\
\leq \beta_k \left[ f(y^k) + nE_{i_k}(\nabla y_k f(y^k), z^{k+1}_i - y^k_i) \right] \\
+ (1 - \beta_k) \left[ f(y^k) + nE_{i_k}(\nabla y_k f(y^k), x^{k+1}_i - y^k_i) \right] \\
+ (n\beta_k)^\gamma E_{i_k} \left[ L_{i_k} D_h(z^{k+1}_i, z^k_i) \right] \\
\leq \beta_k \left[ f(y^k) + \langle \nabla f(y^k), z^{k+1}_i - y^k_i \rangle \right] \\
+ (1 - \beta_k) \left[ f(y^k) + \langle \nabla f(y^k), x^{k+1}_i - y^k_i \rangle \right] \\
+ n(1 - \beta_k) \left[ \beta_k D_H(z^{k+1}_i, z^k_i) \right] \\
\leq (1 - \beta_k) f(x^k) \\
+ \beta_k \left[ f(y^k) + \langle \nabla f(y^k), u - y^k \rangle \right] \\
+ (n\beta_k)^\gamma - D_H(u, z^{k+1}_i) \\
\leq (1 - \beta_k) f(x^k) \\
+ \beta_k \left[ f(u) + (n\beta_k)^\gamma D_H(u, z^{k+1}_i) \right] \\
- (n\beta_k)^\gamma D_H(u, z^{k+1}_i),
\]

where (i) is because the $i_k$-th coordinate is selected uniformly at random, (ii) is due to the convexity of $f$, (iii) is due to the definition of $z^{k+1}_i$ in E.q. (29) and applying Lemma 7 with $\phi(u) = (n\beta_k)^{-1 - \gamma} \left[ f(y^k) + \langle \nabla f(y^k), u - y^k \rangle + \delta x(u) \right]$, and (iv) is due to the convexity of $f$. Subtracting $f(u)$ on both sides gives us

\[
E_{i_k} f(x^{k+1}) - f(u) \leq (1 - \beta_k)(f(x^k) - f(u)) \\
+ n(1 - \beta_k)^\gamma D_H(u, z^{k+1}_i) \\
- (n\beta_k)^{-1 - \gamma} D_H(u, z^{k+1}_i),
\]

(70)
Dividing $\beta_k^n$ on both sides, we have
\[
\frac{1}{\beta_k^n} \mathbb{E}_{i_k} [f(x^{k+1}) - f(u)] \leq \frac{1 - \beta_k^n}{\beta_k^n} (f(x^k) - f(u)) + n^{-1} D_H(u, z^k) - n^{-1} D_H(u, z^{k+1}).
\]
Taking the expectation of $D_H(u, z^{k+1})$ with respect to $\mathbb{E}_{i_k}$ yields
\[
\mathbb{E}_{i_k} [D_H(u, z^{k+1})] = \mathbb{E}_{i_k} [D_H(u, z^k) - L_{i_k} D_h(u_{i_k}, z_{i_k}) + L_{i_k} D_h(u_{i_k}, z_{i_k}^{k+1})]
= \sum_{i_k=1}^{n} \frac{1}{n} [D_H(u, z^k) - L_{i_k} D_h(u_{i_k}, z_{i_k}^k) + L_{i_k} D_h(u_{i_k}, z_{i_k}^{k+1})]
= \frac{1}{n} [n D_H(u, z^k) - D_H(u, z^k) + D_H(u, z^{k+1})]
= D_H(u, z^k) - \frac{1}{n} [D_H(u, z^k) - D_H(u, z^{k+1})].
\]
Multiplying both sides by $n^\gamma$, we obtain
\[
n^\gamma \mathbb{E}_{i_k} [D_H(u, z^{k+1})] = n^\gamma D_H(u, z^k) - n^{\gamma-1} [D_H(u, z^k) - D_H(u, z^{k+1})]
\]
Combining (72) with (71), we have
\[
\mathbb{E}_{i_k} \left[ \frac{1 - \beta_k}{\beta_k^n} (f(x^{k+1}) - f(u)) + n^\gamma D_H(u, z^{k+1}) \right] \leq \frac{1 - \beta_k}{\beta_k^n} (f(x^k) - f(u)) + n^\gamma D_H(u, z^k).
\]
Finally applying the condition in Step 4 of Algorithm 2 yields the desired result.

J. Proof of Theorem 3

Proof: Taking the expectation with respect to \{i_0, i_1, \ldots, i_k\} yields
\[
\mathbb{E} \left[ \frac{1 - \beta_k^{k+1}}{\beta_k^{k+1}} (f(x^{k+1}) - f(u)) + n^\gamma D_H(u, z^{k+1}) \right] \leq \mathbb{E} \left[ \frac{1 - \beta_k}{\beta_k^n} (f(x^k) - f(u)) + n^\gamma D_H(u, z^k) \right].
\]
The direct consequence of Eq. (74) is, for any $u$,
\[
\mathbb{E} \left[ \frac{1 - \beta_k^{k+1}}{\beta_k^{k+1}} (f(x^{k+1}) - f(u)) + n^\gamma D_H(u, z^{k+1}) \right] \leq \left( 1 - \beta_k \right) (f(x^0) - f(u)) + n^\gamma D_H(u, z^0).
\]
Using $D_H(u, z^{k+1}) \geq 0$, and the initialization $\beta_0 = 1$ and $z^0 = x^0$, we obtain
\[
\mathbb{E} \left[ \frac{1 - \beta_k^{k+1}}{\beta_k^{k+1}} (f(x^{k+1}) - f(u)) \right] \leq n^\gamma D_H(u, x^0),
\]
which implies
\[
\mathbb{E} \left[ f(x^{k+1}) - f(u) \right] \leq n^\gamma \beta_k^n D_H(u, x^0) = \left( \frac{n^\gamma}{k + \gamma} \right)^{\gamma} D_H(u, x^0).
\]

K. Proof of Proposition 2

Proof: It is straightforward to see that $x^0 = y^0 = z^0 = \hat{v}^0$. Suppose the recursive hypotheses hold for the $k$-th iteration. From the optimality of Eq. (42), we have
\[
\langle \nabla_i f(\beta_k u^k + v^k), d_i^k \rangle + (n \beta_k) \gamma^{-1} L_{i_k} D_h(v_{i_k}^k + d_i^k, v_{i_k}^k) \leq \langle \nabla_i f(y_i^k), z_{i_k}^{k+1} - z_{i_k}^k \rangle + (n \beta_k) \gamma^{-1} L_{i_k} D_h(z_{i_k}^{k+1}, v_{i_k}^k),
\]
where (i) is due to the optimality, and (ii) is due to the recursive hypotheses. Similarly, from the optimality of Eq. (28), we obtain
\[
\langle \nabla_i f(y_i^k), z_{i_k}^{k+1} - z_{i_k}^k \rangle + (n \beta_k) \gamma^{-1} L_{i_k} D_h(z_{i_k}^{k+1}, z_{i_k}^k) \leq \langle \nabla_i f(y_i^k), d_i^k \rangle + (n \beta_k) \gamma^{-1} L_{i_k} D_h(v_{i_k}^k + d_i^k, v_{i_k}^k),
\]
where (i) is due to the optimality, and (ii) is due to the recursive hypotheses. Combining (75) and (76) yields
\[
z_{i_k}^{k+1} = z_{i_k}^k + d_{i_k}^k = v_{i_k}^k + d_{i_k}^k = v_{i_k}^{k+1},
\]
or equivalently
\[
z_{i_k}^{k+1} = v_{i_k}^{k+1}.
\]
From Step 3 of Algorithm 3 we have
\[
u_{i_k}^{k+1} = u^k - \frac{1 - n \beta_k}{\beta_k} (v_{i_k}^{k+1} - v^k).
\]
Then, we have
\[
\beta_k^{k+1} u_{i_k}^{k+1} + v_{i_k}^{k+1} \overset{(i)}{=} \beta_k^{k+1} \left( u^k - \frac{1 - n \beta_k}{\beta_k} (v_{i_k}^{k+1} - v^k) \right) + v_{i_k}^{k+1}
= \beta_k^{k+1} u^k - (1 - n \beta_k) (v_{i_k}^{k+1} - v^k) + v_{i_k}^{k+1}
= \beta_k^{k+1} u^k + v^k + n \beta_k (v_{i_k}^{k+1} - v^k)
\overset{(ii)}{=} y^k + n \beta_k (z_{i_k}^{k+1} - z^k)
= x_{i_k}^{k+1},
\]
where (i) is due to Eq. (77) and (ii) is due to the recursive hypotheses. Finally, we have
\[
\beta_k^{K+1} u_{i_k}^{K+1} + v_{i_k}^{K+1} \overset{(i)}{=} \beta_k^{K+1} (x_{i_k}^{K+1} - v_{i_k}^{K+1}) + v_{i_k}^{K+1}
\overset{(ii)}{=} (1 - \beta_k^{K+1}) (x_{i_k}^{K+1} - v_{i_k}^{K+1}) + v_{i_k}^{K+1}
= (1 - \beta_k^{K+1}) x_{i_k}^{K+1} + \beta_k^{K+1} v_{i_k}^{K+1}
\overset{(iii)}{=} (1 - \beta_k^{K+1}) x_{i_k}^{K+1} + \beta_k^{K+1} z_{i_k}^{K+1}
= y_{i_k}^{K+1},
\]
where (i) and (iii) is due to recursive hypotheses, and (ii) is due to Step 4 of Algorithm 3.
L. Proof of Lemma \[10\]

**Proof:** Let us fix coordinate \( j \). Then we have

\[
f_j(x_j) = \sum_{i=1}^{M} \left( b_i \log \left( \frac{b_i}{a_{ij}} \right) + \langle a_i, x \rangle - b_i \right),
\]

where \( a_i \) is the \( i \)-th row of \( A \). The first- and second-order derivatives of \( f_j \) are given by

\[
f_j'(x_j) = \sum_{i=1}^{M} \left( 1 - \frac{b_i}{a_{ij}} \right) a_{ij}, \]
\[
f_j''(x_j) = \sum_{i=1}^{M} \frac{b_i a_{ij}^2}{(a_i, x)^2}.
\]

It follows from the nonnegativity of \( A \) and \( x \) that we have

\[
\frac{a_{ij}^2}{(a_i, x)^2} \leq \frac{1}{x_j^2}.
\]

Applying the inequality above yields

\[
f_j''(x_j) = \sum_{i=1}^{M} \frac{b_i a_{ij}^2}{(a_i, x)^2} \leq \left( \sum_{i=1}^{M} b_i \right) \frac{1}{x_j^2} = \left( \sum_{i=1}^{M} b_i \right) h_j''(x_j).
\]

M. Proof of Lemma \[11\]

**Proof:** Fixing the \( j \)-th coordinate of \( x \), define \( f_j(x_j) \) as follows

\[
f_j(x_j) = \sum_{i=1}^{M} \left( (a_i, x) \log \left( \frac{(a_i, x)}{b_i} \right) + b_i - (a_i, x) \right)
\]

Then the first- and second-derivatives of \( f_j \) are given by

\[
f_j'(x_j) = \sum_{i=1}^{M} a_{ij} \left( \log \left( \frac{a_i, x}{b_i} \right) \right),
\]
\[
f_j''(x_j) = \sum_{i=1}^{M} \frac{a_{ij}^2}{(a_i, x)}.
\]

Using the nonnegativity of \( A \) and \( x \), we obtain \( a_{ij}x_j \leq (a_i, x) \), which further implies

\[
\frac{a_{ij}^2}{(a_i, x)} \leq \frac{a_{ij}}{x_j}.
\]

Invoking the inequality above, we obtain the desired result

\[
f_j''(x_j) = \sum_{i=1}^{M} \frac{a_{ij}^2}{(a_i, x)} \leq \sum_{i=1}^{M} \frac{a_{ij}}{x_j} \leq \left( \sum_{i=1}^{M} a_{ij} \right) h_j''(x_j).
\]
[25] T. Gao, S. Lu, J. Liu, and C. Chu, “Leveraging two reference functions in block bregman proximal gradient descent for non-convex and non-lipschitz problems,” *arXiv preprint arXiv:1912.07527*, 2019.

[26] D. D. Lee and H. S. Seung, “Learning the parts of objects by non-negative matrix factorization,” *Nature*, vol. 401, no. 6755, pp. 788, 1999.

[27] Y.-D. Kim and S. Choi, “Nonnegative tucker decomposition,” in *Proceedings of IEEE Conference on Computer Vision and Pattern Recognition*, pp. 1–8, 2007.

[28] Y. Xu, W. Yin, Z. Wen, and Y. Zhang, “An alternating direction algorithm for matrix completion with nonnegative factors,” *Frontiers of Mathematics in China*, vol. 7, no. 2, pp. 365–384, 2012.

[29] N. He, Z. Harchaoui, Y. Wang, and L. Song, “Fast and simple optimization for poisson likelihood models,” *arXiv preprint arXiv:1608.01264*, 2016.

[30] M. Ahookhosh, L. T. K. Hien, N. Gillis, and P. Patrinos, “Multi-block bregman proximal alternating linearized minimization and its application to sparse orthogonal nonnegative matrix factorization,” *arXiv preprint arXiv:1908.01402*, 2019.

[31] X. Wang, X. Yuan, S. Zeng, J. Zhang, and J. Zhou, “Block coordinate proximal gradient method for nonconvex optimization problems: Convergence analysis,” 2018.

[32] S. Bonettini, M. Prato, and S. Rebegoldi, “A cyclic block coordinate descent method with generalized gradient projections,” *Applied Mathematics and Computation*, vol. 286, pp. 288–300, 2016.

[33] G. Chen and M. Teboulle, “Convergence analysis of a proximal-like minimization algorithm using bregman functions,” *SIAM Journal on Optimization*, vol. 3, no. 3, pp. 538–543, 1993.

[34] Y. T. Lee and A. Sidford, “Efficient accelerated coordinate descent methods and faster algorithms for solving linear systems,” in *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*. IEEE, pp. 147–156, 2013.

[35] I. Csiszar et al., “Why least squares and maximum entropy? an axiomatic approach to inference for linear inverse problems,” *The annals of statistics*, vol. 19, no. 4, pp. 2032–2066, 1991.

[36] A. P. Dempster, N. M. Laird, and D. B. Rubin, “Maximum likelihood from incomplete data via the em algorithm,” *Journal of the Royal Statistical Society: Series B (Methodological)*, vol. 39, no. 1, pp. 1–22, 1977.

[37] H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of inverse problems*, vol. 375, Springer Science & Business Media, 1996.

[38] M. Bertero, P. Boccacci, G. Desiderà, and G. Vicipi, “Image deblurring with poisson data: from cells to galaxies,” *Inverse Problems*, vol. 25, no. 12, pp. 123006, 2009.