On the statistics of quantum transfer of non-interacting fermions in multi-terminal junctions

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Similarly to the recently obtained result for two-terminal systems, we show that there are constraints on the full counting statistics for non-interacting fermions in multi-terminal contacts. In contrast to the two-terminal result, however, there is no factorization property in the multi-terminal case.

Introduction. The problem of full counting statistics (FCS) of electronic charge transfer has been addressed since long time\(^1\) and the particular model of non-interacting fermions has been studied in detail in various setups. The FCS for transfer of non-interacting fermions is given by the Levitov–Lesovik determinant formula\(^2,3\) valid at arbitrary temperature and for an arbitrary time evolution of the scatterer. Recently, some properties of this result have been elucidated. First, in the particular case of charge transfer driven by a time-dependent bias voltage at zero temperature, the resulting FCS enjoys certain symmetries.\(^4,5\) Second, in the more general case of an arbitrary time-dependent scatterer and at arbitrary temperature, it has been shown that the FCS is factorizable into independent single-particle events.\(^6,7\)

In the present work, we generalize the result of Refs.\(^8,9\) to a multi-terminal setup. As in those works, we address the problem of determining which multi-channel charge transfers are possible and which are not in an arbitrary quantum pump, in the model of non-interacting fermions. In the two-terminal case, the constraint derived in Refs.\(^8,9\) is exact. In the multiterminal case, however, the problem is more complicated, and we have only partially solved it: we have formulated a necessary constraint (a “convexity condition”) on the charge-transfer statistics, without a proof (or a counterexample) that this constraint is sufficient. Also, there is no obvious physical interpretation of this constraint: we show that, unlike in the two-terminal case, our constraint cannot be interpreted as a factorization property of the charge-transfer statistics.

This work is partly based on the results reported in Ref.\(^10\).

Determinant formula. We first introduce notation and review the Levitov–Lesovik determinant formula\(^2,3\) for charge transfer of non-interacting fermions in application to a multi-lead setup. The notation and argument is fully parallel to that in Ref.\(^9\) where the two-lead case was considered.

We consider a contact with \(L\) leads, connected by an arbitrary time-dependent scatterer (see Fig. 1). To each lead (numbered \(i = 1, \ldots, L\)) we associate a “counting field”\(^1\) \(\lambda_i\) and a projector operator \(P_i\) acting in the single-particle Hilbert space. The leads are defined in such a way that

\[
\sum_{i=1}^{L} P_i = 1. \tag{1}
\]

Then the probabilities of the multi-lead charge transfers can be determined from the generating function

\[
\chi(\lambda_1, \ldots, \lambda_L) = \text{Tr} \left( \hat{\rho}_0 \hat{U}^\dagger \hat{e}^{i\lambda \hat{P}} \hat{U} e^{-i\lambda \hat{P}} \right) / \text{Tr} \hat{\rho}_0. \tag{2}
\]

Here the trace is taken in the multi-particle Fock space, \(\hat{\rho}_0\) is the initial density matrix, \(\hat{U}\) is the multi-particle evolution operator. We also use the shorthand notation \(\lambda \hat{P} = \sum_i \lambda_i \hat{P}_i\), where \(\hat{P}_i\) is the multi-particle operator (a fermionic bilinear\(^2\)) constructed from the projector \(P_i\) (it counts the particles in the lead \(i\)). As in the two-lead problem\(^2\) under the assumption that \(\hat{\rho}_0\) commutes with \(P_i\) (the absence of entanglement in the initial state), the Fourier components of the generating function \(\chi\) give the charge-transfer probabilities \(P_{q_1, \ldots, q_L}\),

\[
\chi(\lambda_1, \ldots, \lambda_L) = \sum_{q_1, \ldots, q_L = -\infty}^{\infty} P_{q_1, \ldots, q_L} \exp \left( i \sum_{i=1}^{L} \lambda_i q_i \right). \tag{3}
\]

Those probabilities are only non-zero for charge-conserving transfers with \(\sum_i q_i = 0\). This charge conservation corresponds to the symmetry of the generating function with respect to a simultaneous shift of all variables,

\[
\chi(\lambda_1, \ldots, \lambda_L) = \chi(\lambda_1 + \delta \lambda, \ldots, \lambda_L + \delta \lambda). \tag{4}
\]

As in the two-lead case, we define the complex variables

\[
u_i = e^{i\lambda_i}, \quad i = 1, \ldots, L, \tag{5}
\]
and consider the generating function as a function of $u_i$.

As in Ref. 9, we assume, in addition to the absence of entanglement of the initial state, that both $\hat{\rho}_0$ and $\hat{U}$ are exponentials of fermionic bilinears (which reflects our assumption of non-interacting fermions). Under those assumptions, we repeat the calculation of Ref. 9 and arrive at the resulting determinant formula

$$\chi(\lambda_1, \ldots, \lambda_L) = \det \left[ 1 + n_F(U^† e^{i\lambda P} U e^{-i\lambda P}) \right], \quad (6)$$

which involves only operators in the single-particle Hilbert space with the occupation-number operator

$$n_F = \frac{\rho_0}{\rho_0 + 1}. \quad (7)$$

Convexity condition. Similarly to the trick employed in Ref. 9, we can rewrite the determinant formula by defining the hermitian “effective-transparency operators”

$$\hat{X}(i) = (1 - n_F) P_i + n_F^{1/2} U^† P_i U n_F^{1/2}. \quad (8)$$

After simple algebra [using the completeness relation (1)], one can re-express the generating function (6) as

$$\chi(u_1, \ldots, u_L) = \det \left[ e^{-i\lambda P} \sum_{i=1}^L u_i \hat{X}(i) \right]. \quad (9)$$

The eigenvalues of the operators $\hat{X}(i)$ are bounded between 0 and 1, which allows us to prove a certain constraint on the zeroes (roots) of the generating function $\chi$. An elegant form of this constraint can be formulated in terms of the convex envelope (convex hull) $H_c(X)$ of a given set of complex numbers $X$: a minimal convex set containing $X$ (see Fig. 2a). The constraint may now be cast in the form of two conditions that need to be satisfied:

1. For any root of the characteristic function $\chi(u_1, \ldots, u_L) = 0$, the convex envelope $H_c(\{u_1, \ldots, u_L\})$ contains zero.

2. If $\chi(u_1, \ldots, u_L) = 0$ and if zero belongs to the boundary of $H_c(\{u_1, \ldots, u_L\})$, then those of the points $\{u_1, \ldots, u_L\}$ that do not lie on the straight segment of the boundary of $H_c(\{u_1, \ldots, u_L\})$ containing zero, can be arbitrarily changed while still satisfying the equation $\chi(u_1, \ldots, u_L) = 0$ (Fig. 2b).

The proof of Condition 1 is easy: if $|\Psi\rangle$ is a zero mode of the operator in the determinant (9), then

$$\sum_{i=1}^L u_i \langle \Psi|\hat{X}(i)|\Psi\rangle = 0. \quad (10)$$

Since all the coefficients $\langle \Psi|\hat{X}(i)|\Psi\rangle$ are non-negative real numbers (whose sum equals one), zero belongs to the convex envelope of $u_1, \ldots, u_L$.

FIG. 2: (a): Illustration of the definition of the convex envelope (convex hull). The shaded region shows the convex envelope of the points $u_1, \ldots, u_6$ in the complex plane. If the points $u_1, \ldots, u_6$ correspond to a root of the generating function, then Condition 1 of the constraint claims that zero must belong to the shaded region. (b): Illustration of Condition 2 of the constraint. In this figure (with the points $u_1, \ldots, u_6$ corresponding to a root of the generating function), the points $u_2, u_3,$ and $u_6$ can be changed arbitrarily, and the new set of points will still give a root of the generating function.

To prove Condition 2, consider again a root $(u_1, \ldots, u_L)$ of the generating function and the corresponding zero mode $|\Psi\rangle$. If zero lies at the boundary of the convex envelope $H_c(\{u_1, \ldots, u_L\})$, then the linear combination (10) contains nonvanishing coefficients $\langle \Psi|\hat{X}(i)|\Psi\rangle$ only for variables $u_i$ which belong to the same straight segment of the boundary containing zero. All the other coefficients necessarily vanish, which, by virtue of the non-negativity of $\hat{X}(i)$, implies $\hat{X}(i)|\Psi\rangle = 0$. Therefore all those variables $u_i$ may be changed arbitrarily while $|\Psi\rangle$ will remain a zero mode. This completes the proof of Condition 2.

We can make several comments on the obtained result. First, in the particular case of two leads $(L = 2)$, this constraint is equivalent to that found in Ref. 3 (the variable $u$ in that work corresponds to the ratio $u_1/u_2$ in our present notation). Second, while our constraint is a necessary condition for realizability of a given statistics in a non-interacting fermionic system, we could not determine if it is also a sufficient one. Moreover, we do not have any algorithm which would determine if a given charge-transfer statistics is realizable (or design a suitable quantum evolution if it is). Those interesting ques-
tions are left for future studies. Third, our criterion is technically difficult to check in its full formulation for all roots \((u_1, \ldots, u_L)\). However, for practical applications, one may test the constraint on suitably chosen families of roots (e.g., one-parametric families), either analytically or numerically.

**Non-factorizability.** In the two-terminal case, the “convexity condition” derived above implies a factorizability of the charge transfer statistics: the probabilities of a given charge transfer are the same as in a superposition of some single-electron transfer processes (whose transfer probabilities depend in a non-trivial way on the evolution of the quantum system). One can see that it is not the case in the multi-terminal \((L > 2)\) case.

This can be most easily demonstrated with a counter-example involving only a finite number of electrons (in the wave packet formalism of Ref. [11]), to which our result is also applicable. Consider two fermions sent into a stationary multi-terminal contact along two terminals (labeled 1 and 2) with exactly the same shape of wave packets (Fig. 3). Then, due to the Fermi statistics of particles, the probabilities to have both fermions scattered to the same lead vanish. The resulting generating function will therefore have the form

\[
\chi(u_1, \ldots, u_L) = \frac{1}{u_1u_2} \sum_{i<j} \alpha_{ij} u_i u_j, \tag{11}
\]

where \(\alpha_{ij} = |s_{1i}s_{2j} - s_{1j}s_{1i}|^2\) are the probabilities of various two-particle transfer events constructed out of the single-particle scattering amplitudes \(s_{ij}\) (which are assumed to be time and energy independent). On the other hand, the factorizability of the charge transfer would imply

\[
\chi(u_1, \ldots, u_L) = \frac{1}{u_1u_2} \left( \sum_i p_i u_i \right) \left( \sum_i p'_i u_i \right) \tag{12}
\]

for some probabilities \(p_i\) and \(p'_i\). One can verify that if one considers a statistics [11] with all \(\alpha_{ij}\) nonzero (which is possible), then such a statistics is not factorizable in the form [12].

**Conclusion.** To summarize, we have considered the problem of possible full counting statistics for non-interacting fermions in coherent multi-terminal systems. We have obtained a necessary condition for a full counting statistics to be realizable. Like in the two-terminal case, this condition may be used to prove impossibility of certain sets of charge-transfer probabilities (one can easily construct examples of such impossible statistics).

At the same time, the problem of designing an actual “quantum pump” for a given charge-transfer statistics (or even merely proving its possibility) appears much more difficult in the multi-terminal case than in the two-terminal one. While in the two-terminal case, the full counting statistics of non-interacting fermions is conveniently parameterized by the spectral density of “effective transparencies”, we are not aware of a similar parameterization in the multi-terminal case. In the formulation with a finite number of particles, even the question of the dimensionality of the space of all possible full counting statistics remains open. All those interesting questions deserve further study, in particular in the context of using quantum contacts for generating entangled states [12].

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Many-Body entanglement: a new application of the full counting statistics.

A good practical test of the constraint is obtained by restricting $u_i$ to a one-parameter family $u_i = a_i z + b_i$ for some fixed sets of real numbers $a_i$ and $b_i$ with the condition that all $a_i \geq 0$ (nothing is assumed about $b_i$). Then the equation $\chi(z) = 0$ must either have only real roots $z$ or be identically satisfied for all $z$. This form of the constraint is convenient for numerical tests by using a large number of randomly chosen vectors $a_i$ and $b_i$. 
