Research Article

On the System of Coupled Nondegenerate Kirchhoff Equations with Distributed Delay: Global Existence and Exponential Decay

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This paper studies the system of coupled nondegenerate viscoelastic Kirchhoff equations with a distributed delay. By using the energy method and Faedo-Galerkin method, we prove the global existence of solutions. Furthermore, we prove the exponential stability result.

1. Introduction

Let \( \mathcal{H} = \Omega \times (\tau_1, \tau_2) \times (0, \infty) \), in this work, we consider

\[
\begin{align*}
\rho_1 \Delta u - M(|u|^2) \Delta u &+ \int_0^\tau \left[ \varphi_1(t-s) \Delta u(x,t-s, \rho_1) - \int_0^\tau \left[ \varphi_1(t-s) \Delta u(x,t-s, \rho_1) ds \right] \right] dt + f_1(u, v) = 0, \\
\rho_2 \Delta v - M(|v|^2) \Delta v &+ \int_0^\tau \left[ \varphi_2(t-s) \Delta v(x,t-s, \rho_2) - \int_0^\tau \left[ \varphi_2(t-s) \Delta v(x,t-s, \rho_2) ds \right] \right] dt + f_2(u, v) = 0,
\end{align*}
\]

where \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), \( l > 0 \) and \( \Delta \) is the Laplacian operator, and the functions \( \mu_1, \mu_2 : [\tau_1, \tau_2] \rightarrow \mathbb{R} \) are bounded, with \( 0 \leq \tau_1 < \tau_2 \), and the relaxation functions are denoted by \( \varphi_1, \varphi_2 \). The function \( M \) is given by

\[
M : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad r \mapsto M(r) = a + br^\gamma,
\]

where \( a, b > 0 \), and \( \gamma \geq 1 \), and the functions \( f_1, f_2 \) will be defined later.

(1)
In 1976, Kirchhoff developed an equation describing the vibrations produced by a fixed series at its end, since it is considered a generalization of the d’Alembert equation, and it belongs to the wave equation models. Over time, many researchers and authors addressed these issues and problems with their continuous and rapid development, for example, see [1–4].

As for viscoelasticity, it is possible to delve into the following works for further clarification [3–10].

Also, the time or delay recorded in many natural and physical phenomena, especially problems resulting from vibrations, is an important factor for stability in general. And it has been studied extensively by many authors, including [5–7, 11–21]. Recently, in the presence of the varying delay, Mezouar and Boularrass studied system (1); for more information, see [22]. Based on these works, we in this work expand the results in [22] by adding the term of distributed delay.

We, under appropriate conditions, obtained the global existence of solutions, and we proved the exponential stability result of the system.

And we divided the paper into the following: in the second part, we set out the necessary hypotheses and the main result; in the third part, we prove the global existence of solutions, while in the fourth part, we present our result for exponential stability.

2. Preliminaries

In this section, we set the necessary hypotheses for proving the main result.

We need the following assumptions:

(A1) \( g_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \), \( i = 1, 2 \) are \( C^1 \) functions satisfying

\[
g(0) > 0, a - \int_0^\infty g_i(s)ds \geq k > 0, i = 1, 2. \tag{5}\]

(A2) \( \exists \xi_i > 0 \) satisfying

\[
g_i(t) \leq -\xi_i g_i(t), i = 1, 2, t \geq 0. \tag{6}\]

(A3) The number \( l \) satisfying \( 0 < l \leq \gamma \) and

\[
\begin{cases}
\leq \frac{2}{n-2} & \text{if } n > 2, \\
\gamma & \text{if } n \leq 2. 
\end{cases} \tag{7}
\]

(A4)

\[
\begin{cases}
f_1(u, v) = a_1 v + b_1 |v|^{q+1} |v|^{p-1} u, \\
f_2(u, v) = a_1 v + b_2 |v|^{q+1} |v|^{p-1} v,
\end{cases} \tag{8}
\]

where \( a_1 > 0, b_1 = (p + 1)(p + q), b_2 = (q + 1)(p + q) \) such that \( p \) and \( q \) are conjugate \( ((1/p) + (1/q) = 1) \), \( p, q < \gamma - (1/2) \) and satisfy

\[
2 \leq p, q \leq \sqrt{\frac{n}{2(n-2)}} \quad \text{if } n > 2, \\
\infty \quad \text{if } n \leq 2. \tag{9}
\]

We set the notations

\[
(g \circ \Psi)(t) := \int_0^t g(t-s)||\Psi(t) - \Psi(s)||^2 ds. \tag{10}
\]

As in [17], we introduce the new variables

\[
\begin{align*}
u_i(x, t - \varphi) &= \mathcal{I}_i(x, \rho, \varphi, t), \\
v_i(x, t - \varphi) &= \mathcal{V}_i(x, \rho, \varphi, t).
\end{align*} \tag{11}
\]

We have

\[
\begin{align*}
\rho \mathcal{I}_i(x, \rho, \varphi, t) + \mathcal{I}_i(x, \rho, \varphi, t) &= 0, \\
u_i(x, t) &= \mathcal{I}_i(x, 0, \varphi, t), \\
\rho \mathcal{V}_i(x, \rho, \varphi, t) + \mathcal{V}_i(x, \rho, \varphi, t) &= 0, \\
v_i(x, t) &= \mathcal{V}_i(x, 0, \varphi, t).
\end{align*} \tag{12}
\]

Consequently, problem (1) is equivalent to

\[
\begin{align*}
|M| u_{tt} - M(|M| u|^2) \Delta u - \Delta u_{tt} + \int_0^t g_1(t-s) \Delta u(s) ds - \int_{r_1}^{r_2} |\mu_1(q)| |\Delta \mathcal{I}_i(x, 1, q, t)| ds + f_1(u, v) &= 0, \\
|M| v_{tt} - M(|M| v|^2) \Delta v - \Delta v_{tt} + \int_0^t g_2(t-s) \Delta v(s) ds - \int_{r_1}^{r_2} |\mu_2(q)| |\Delta \mathcal{V}_i(x, 1, q, t)| ds + f_2(u, v) &= 0,
\end{align*} \tag{13}
\]

\[
\begin{align*}
\rho \mathcal{I}_i(x, \rho, \varphi, t) + \mathcal{I}_i(x, \rho, \varphi, t) &= 0, \\
\rho \mathcal{V}_i(x, \rho, \varphi, t) + \mathcal{V}_i(x, \rho, \varphi, t) &= 0,
\end{align*}
\]
where

\[(x, \rho, s, t) \in \Omega \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty)\]  

with the initial and boundary conditions

\[
\begin{aligned}
(u(x,0), v(x,0)) &= (u_0(x), v_0(x)), \text{ in } \Omega, \\
(u_t(x,0), v_t(x,0)) &= (u_1(x), v_1(x)), \text{ in } \Omega, \\
(u(x,-t), v_t(x,-t)) &= (f_0(x,t), g_0(x,t)), \text{ in } \Omega \times (0, \tau_2), \\
\varphi(x,\rho,0) &= f_0(x,\rho_0), \text{ in } \Omega \times (0, 1) \times (0, \tau_2), \\
\varphi_t(x,\rho,0) &= g_0(x,\rho_0).
\end{aligned}
\]

\[(14)\]

We need the following lemma.

**Lemma 1.** The energy functional \(E\), given by

\[
E(t) = \frac{1}{1+2} \left( \|u_t\|_{L^2}^2 + \|v_t\|_{L^2}^2 \right) + \frac{b}{2(y+2)} \\
\cdot \left( \|\nabla u\|_{L^{2(y+2)}}^2 + \|\nabla v\|_{L^{2(y+2)}}^2 \right) + \frac{1}{2} \left( a - \int_0^t g_1(s)ds \right) \|\nabla u\|^2 \\
+ \frac{1}{2} \left( g_1^a(t) + \|v_t\|^2 + \frac{1}{2} \left( g_2^a(t) + \|v_t\|^2 \right) \right) \\
+ \frac{1}{2} \left( g_1^b(t) + \|v_t\|^2 + \frac{1}{2} \left( g_2^b(t) + \|v_t\|^2 \right) \right) \\
\cdot \|\nabla \varphi\|^2 + \|\mu_2(\rho)\| \|\nabla \varphi\| dx + a \int_\Omega uv dx \\
+ (p + q) \int_\Omega |u|^p |v|^q dx,
\]

satisfies

\[
E'(t) \leq -\beta \int_{\Omega} \left( |\mu_1(\rho)| \|\nabla \varphi(x,1,q,t)\|^2 + |\mu_2(\rho)| \right) \\
\cdot \|\nabla \varphi(x,1,q,t)\|^2 d\rho + \lambda \left( \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right) \\
+ \frac{1}{2} \left( g_1^c(t) + \|v_t\|^2 + \frac{1}{2} \left( g_2^c(t) + \|v_t\|^2 \right) \right) \\
\cdot \|v_t\|^2 - \frac{1}{2} g_2(t) \|v_t\|^2 dx,
\]

\[
(17)
\]

\[
where \ \beta = \left( (1 - \delta_t)/2 \right > 0, \\
and \ \lambda = \max \left\{ \lambda_1 = \left( (\delta_t + 1)/2 \right \} \int_{\Omega} |\mu_1(\rho)| d\rho, \lambda_2 = \left( (\delta_t + 1)/2 \right \} \int_{\Omega} |\mu_2(\rho)| d\rho \right\}, \delta_t < 1.
\]

**Proof.** Multiplying equation (13) by \(u_t, v_t\), and we use (15), one gets

\[
d \frac{dt}{2} \left\{ \frac{1}{1+2} \|u_t\|_{L^2}^2 + b \frac{2}{2(y+1)} \|\nabla u_t\|_{L^{2(y+1)}}^2 + \frac{1}{2} \left( a - \int_0^t g_1(s)ds \right) \|\nabla u\|^2 \\
+ \frac{1}{2} g_1(t) \|\nabla u(t)\|^2 + \int_\Omega u_t \mu_1(\rho) \|\Delta \varphi(x,1,q,t)\| d\rho dx \\
+ \frac{1}{2} g_2(t) \|\nabla v(t)\|^2 + \int_\Omega v_t \mu_2(\rho) \|\Delta \varphi(x,1,q,t)\| d\rho dx \\
+ \frac{1}{2} \int_{\Omega} v_t v_{1}^q dx + b_1 \int_\Omega u_t |u_t|^{p-1} |v_{1}^q| dx + \frac{d \|v_t\|_{L^2}}{dt} \left( \|v_t\|^2 \right) \\
+ \frac{1}{2} \|v_t\|^2 + \frac{1}{2} (g_2^d(t) \|v_t\|^2) - \frac{1}{2} (g_2^f(t) \|v_t\|^2) \\
+ \frac{1}{2} (g_2^g(t) \|v_t\|^2 + \frac{1}{2} (g_2^h(t) \|v_t\|^2)) \right\} \|v_t\|^2 \\
+ \frac{1}{2} \int_\Omega v_t v_{1}^q dx + b_2 \int_\Omega v_t |v_{1}^q| dx.
\]

\[
(18)
\]

And multiplying equation (13) by \(\Delta \varphi |\mu_1(\rho)|\), and integrating the result over \(\Omega \times (0,1) \times (\tau_1, \tau_2)\), one gets

\[
d \frac{dt}{2} \left\{ \int_{\Omega} \int_0^{r_1} \frac{\int_\varphi \mu_1(\rho) \|\nabla \varphi\|^2 dx dq dx}{d\rho} \\
+ \frac{1}{2} \int_\varphi \mu_1(\rho) \|\nabla \varphi\|^2 dx dq dx \\
+ \frac{1}{2} \int_\varphi \mu_2(\rho) \|\nabla \varphi\|^2 dx dq dx \\
+ \frac{1}{2} \int_\varphi \mu_1(\rho) \|\nabla \varphi\|^2 dx dq dx \\
+ \frac{1}{2} \int_\varphi \mu_2(\rho) \|\nabla \varphi\|^2 dx dq dx \\
+ \frac{1}{2} \int_\varphi \mu_1(\rho) \|\nabla \varphi\|^2 dx dq dx \\
+ \frac{1}{2} \int_\varphi \mu_2(\rho) \|\nabla \varphi\|^2 dx dq dx
\]

\[
(19)
\]

Similarly, multiplying equation (13) by \(\Delta \varphi |\mu_2(\rho)|\), we find

\[
d \frac{dt}{2} \left\{ \int_{\Omega} \int_0^{r_1} \frac{\int_\varphi \mu_2(\rho) \|\nabla \varphi\|^2 d\rho} d\rho \\
+ \frac{1}{2} \int_\varphi \mu_2(\rho) \|\nabla \varphi\|^2 d\rho \\
+ \frac{1}{2} \int_\varphi \mu_1(\rho) \|\nabla \varphi\|^2 d\rho \\
+ \frac{1}{2} \int_\varphi \mu_2(\rho) \|\nabla \varphi\|^2 d\rho \\
+ \frac{1}{2} \int_\varphi \mu_1(\rho) \|\nabla \varphi\|^2 d\rho \\
+ \frac{1}{2} \int_\varphi \mu_2(\rho) \|\nabla \varphi\|^2 d\rho \\
+ \frac{1}{2} \int_\varphi \mu_1(\rho) \|\nabla \varphi\|^2 d\rho
\]

\[
(20)
\]
by using the inequalities of Young and Cauchy-Schwartz for $\delta_1 > 0$, we have

$$
\int_\Omega \nabla u_j \cdot \nabla \varphi(x, 1, q, t) \, dq \, dx \\
\leq \frac{\delta_1}{2} \left( \int_{\tau_1}^{\tau_2} |\mu_1(q)| \, dq \right) \|\nabla u_j\|^2 \quad (21)
$$

Similarly, we get

$$
\int_\Omega \nabla v_j \cdot \nabla \varphi(x, 1, q, t) \, dq \, dx \\
\leq \frac{\delta_1}{2} \left( \int_{\tau_1}^{\tau_2} |\mu_2(q)| \, dq \right) \|\nabla v_j\|^2 + \frac{\delta_1}{2} \int_{\tau_1}^{\tau_2} \langle |\mu_2(q)| \rangle \|\nabla \varphi(x, 1, q, t)\|^2 \, dq. \quad (22)
$$

By summing (18)–(20) and using (21) and (22), and choosing $\delta_1$ such that $\delta_1 < 1$, we find (16) and (17). This completes the proof.

3. Global Existence

**Theorem 2.** Suppose that (5)–(8) hold. Then, given $(u_0, v_0) \in (H^2(\Omega) \cap H_0^1(\Omega))^2$, $(u_1, v_1) \in (H_0^1(\Omega))^2$, and $(f_0, g_0) \in (H^1((\Omega, (0, 1), (\tau_1, \tau_2)))^2$, there exists a weak solution $(u, v, \mathcal{X}, \mathcal{Y})$ of problem (13)–(15) such that

$$(u, v, \mathcal{X}, \mathcal{Y}) \in L^{\infty}(\mathbb{R}_+, \mathcal{H}_j), u_0, v_1$$

$$(u_{n+1}, v_{n+1}) \in L^2(\mathbb{R}_+, H_0^1(\Omega)), u_{n+1}, v_{n+1}$$

where

$$\mathcal{H}_j = (H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H^1((\Omega, (0, 1), (\tau_1, \tau_2)))^2).$$

**Proof.** Let the Galerkin basis $u_j, v_j, \mathcal{X}_j, \mathcal{Y}_j$ for $n \geq 1$, we set

$$W_n = \text{span}\{u_1, u_2, \ldots, u_n\},$$

$$K_n = \text{span}\{v_1, v_2, \ldots, v_n\}.$$  

The sequences $\mathcal{X}_j(x, r, p), \mathcal{Y}_j(x, r, p)$ are defined for $1 \leq j \leq n$ by

$$\mathcal{X}_j(x, 0, p) = u_j(x), \mathcal{Y}_j(x, 0, p) = v_j(x).$$

Then, taking $\mathcal{X}_j(x, 0, p), \mathcal{Y}_j(x, 0, p)$ by over $L^2((0, 1) \times (r_1, r_2))$ and denoting

$$Z_n = \text{span}\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n\},$$

$$Y_n = \text{span}\{\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n\}.\quad (27)$$

Given initial data $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1, v_1 \in H_0^1(\Omega)$, and $f_0, g_0 \in L^2((\Omega \times (0, 1) \times (\tau_1, \tau_2)))$, we define the approximations

$$u_m = \sum_{j=1}^{n} g_{jm}(t) u_j(x),$$

$$v_m = \sum_{j=1}^{n} h_{jm}(t) v_j(x),$$

$$\mathcal{X}_m = \sum_{j=1}^{n} k_{jm}(t) \mathcal{X}_j(x, r, p),$$

$$\mathcal{Y}_m = \sum_{j=1}^{n} k_{jm}(t) \mathcal{Y}_j(x, r, p).\quad (28)$$

It investigates the following problem:

$$\begin{align*}
\left[ u_{mt}, u_{mtt}, v_{mj}, v_{mj} \right] + M \left[ \|\nabla u_m(t)\| \right] \left[ \nabla u_m, \nabla v_m \right] \\
+ \left[ \nabla u_{mtt}, \nabla u_j \right] + \left[ f_1(u_m, v_m), u_j \right] \\
- \int_0^t \left[ g_1(t-s) \right] \left[ \nabla u_{ms} \right], (s), \nabla u_j \right] ds \\
+ \int_{\tau_1}^{\tau_2} \left[ |\mu_1(q)| \right] \left[ \nabla \mathcal{X}_{m}(x, 1, q, r), \nabla u_j \right] \, dq = 0, \\
\left[ v_{mj}, v_{mj} \right] + M \left[ \|\nabla v_m(t)\| \right] \left[ \nabla v_m, \nabla v_j \right] \\
+ \left[ \nabla v_{mtt}, \nabla v_j \right] + \left[ f_2(u_m, v_m), v_j \right] \\
- \int_0^t \left[ g_2(t-s) \right] \left[ \nabla v_{ms} \right], (s), \nabla v_j \right] ds \\
+ \int_{\tau_1}^{\tau_2} \left[ |\mu_2(q)| \right] \left[ \nabla \mathcal{Y}_{m}(x, 1, q, r), \nabla v_j \right] \, dq = 0,
\end{align*}\quad (29)$$

with initial conditions

$$u_m(0) = u_0^m, \ u_{m0}(0) = u_1^m,$$

$$v_m(0) = v_0^m, \ v_{m0}(0) = v_1^m,$$

$$\mathcal{X}_m(0) = \mathcal{X}_0^m, \ \mathcal{Y}_m(0) = \mathcal{Y}_0^m.\quad (30)$$
which satisfies

\[ u_0^{m} \rightarrow u_0, \text{in } H^2(\Omega) \cap H^1_0(\Omega), \]
\[ u_1^{m} \rightarrow u_1, \text{in } H^1_0(\Omega), \]
\[ v_0^{m} \rightarrow v_0, \text{in } H^2(\Omega) \cap H^1_0(\Omega), \]
\[ v_1^{m} \rightarrow v_1, \text{in } H^1_0(\Omega), \]
\[ G_0^{m} \rightarrow G_0, \text{in } L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2)), \]
\[ Y_0^{m} \rightarrow Y_0, \text{in } L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2)). \]  

(31)

Noting that \((l(2(l + 1)) + 1/(2(l + 1)) + 1/2 = 1, by using Hölder's inequality, we get

\[ \left( |u_m^{l_1}| u_{m, l} u_{j} \right) = \int_{\Omega} |u_m^{l_1}| u_{m, l} u_{j} \, dx \]
\[ \leq \left( \int_{\Omega} |u_m^{l_1}|^{2(l+1)} \, dx \right)^{1/(2(l+1))} \cdot \|u_{m, l}\|_{2(l+1)} \|u_j\|. \]  

(32)

As (8) holds, using the embedding of Sobolev, the terms \(|u_m^{l_1}| u_{m, l} u_{j}\) and \(|v_m^{l_1}| v_{m, l} v_{j}\) in (29) make sense (see [22]).

First estimate.

As the sequences \(u_0^{m}, v_0^{m}, u_1^{m}, v_1^{m}, G_0^{m}, (\cdots, 0)\) and \(Y_0^{m}, (\cdots, 0)\) converge and from (17) and Gronwall's lemma, we get \(C_1 > 0\) independent of \(m\) such that

\[ E_m(t) + \int_{\tau_1}^{\tau_2} \left( |\mu_1(q)| \|\nabla m_i(x, 1, q, t)\|^2 + |\mu_2(q)| \|\nabla Y_m(x, 1, q, t)\|^2 \right) \, dq \leq C_1, \]  

(33)

where

\[ E_m(t) = \frac{1}{b} \left( \frac{\|u_m\|^{2_{[1]} + \|v_m\|^{2_{[1]}}}}{2} + \frac{b}{2(\gamma + 2)} \right) \frac{\|\nabla u_m\|^{2_{(\gamma+2)}}}{2} \]
\[ + \|\nabla v_m\|^{2_{(\gamma+2)}} \right) + \frac{1}{2} \left( a - \int_{0}^{t} g_1(s) \, ds \right) \|\nabla u_m\|^2 \]
\[ + \frac{1}{2} \left( a - \int_{0}^{t} g_1(s) \, ds \right) \|\nabla v_m\|^2 + \frac{1}{2} \left( \|\nabla u_m\|^2 + \|\nabla v_m\|^2 \right) \]
\[ + \frac{1}{2} \left( g_1 \nabla v_m \right)(t) + \frac{1}{2} \left( g_1 \nabla v_m \right)(t) \]
\[ + \frac{1}{2} \int_{\tau_1}^{\tau_2} \left( |\mu_1(q)| \|\nabla m_i(x, 1, q, t)\|^2 + |\mu_2(q)| \|\nabla Y_m(x, 1, q, t)\|^2 \right) \, dq \]
\[ + a \int_{\Omega} u_m v_m \, dx + (p + q) \int_{\Omega} |u_m| \|v_m\|^{q+1} \, dx, \]  

(34)

using (33) and (8), one gets

\[ u_m, v_m \text{ are bounded in } L^\infty_{loc}(\mathbb{R}_+, H^1_0(\Omega)), \]
\[ u_{m, l}, v_{m, l} \text{ are bounded in } L^\infty_{loc}(\mathbb{R}_+, H^1_0(\Omega)), \]
\[ \mathcal{X}_m(x, \rho, q, t), \mathcal{Y}_m(x, \rho, q, t) \text{ are bounded in } L^\infty_{loc}(\mathbb{R}_+, H^1_0(\Omega)) \times (0, 1) \times (\tau_1, \tau_2). \]  

(35)

The second estimate.

We multiply equation (29)_{1,2} by \(g_{j} \nu, h_{j} \nu\); by summing \(j\) from 1 to \(n\), one gets

\[ \int_{\Omega} \left| u_{m, l} \right|^2 \, dx + \int_{\Omega} M(\|u_{m, l}(t)\|) \|u_{m, l}\| \, dx \]
\[ + \int_{\Omega} \|\nabla u_{m, l}\|^2 \, dx + \int_{\Omega} f_1(u_m, v_m) u_{m, l} \, dx \]
\[ - \int_{\tau_1}^{\tau_2} g_1(t-s) \|u_m(s)\| \nu_{m, l} \, ds \]
\[ + \int_{\tau_1}^{\tau_2} |\mu_1(\rho)| \|\nabla m_i(x, 1, \rho, t)\| \nu_{m, l} \, dx = 0, \]  

(36)

By differentiating (29)_{3,4}, we get

\[ (q \mathcal{X}_{m, l}(x, \rho, q, t) + \mathcal{X}_{m, l}(x, \rho, q, t), \mathcal{X}_{l}) = 0, \]
\[ (q \mathcal{Y}_{m, l}(x, \rho, q, t) + \mathcal{Y}_{m, l}(x, \rho, q, t), \mathcal{Y}_{l}) = 0. \]  

(37)

And we multiply (37), by \(\mathcal{X}_{j, l}\) and (37)_{2} by \(\mathcal{Y}_{j, l}\); by summing \(j\) from 1 to \(n\), we have

\[ \frac{1}{2} \int_{\Omega} \|\mathcal{X}_{m, l}\|^2 + \frac{1}{2} \int_{\Omega} \|\mathcal{Y}_{m, l}\|^2 = 0, \]  

(38)

Integrating the result (38) over \((0, 1)\) with respect to \(\rho\), we obtain

\[ \frac{1}{2} \int_{\Omega} \|\mathcal{X}_{m, l}\|^2 \, d\rho + \frac{1}{2} \|\mathcal{X}_m(x, 1, q, t)\|^2 - \frac{1}{2} \|u_{m, l}(x, t)\|^2 = 0, \]
\[ \frac{1}{2} \int_{\Omega} \|\mathcal{Y}_{m, l}\|^2 \, d\rho + \frac{1}{2} \|\mathcal{Y}_m(x, 1, q, t)\|^2 - \frac{1}{2} \|v_{m, l}(x, t)\|^2 = 0. \]  

(39)
Summing (36) and (39) and using $M(r) \geq a$, we get

\[
\begin{align*}
\int_{\Omega} |u_{m,t}|^2 |u_{m,t}|^2 dx + & \|\nabla u_{m,t}\|^2 \\
+ \frac{1}{2} \frac{d}{dt} \int_{0}^{t} \|\mathcal{X}_{m}(x, 1, \rho, t)\|^2 \\
\leq & \frac{1}{2} \|u_{m,t}\|^2 - \int_{\Omega} a \nabla u_{m} \nabla u_{m,t} dx - \int_{\Omega} f_{1}(u_{m}, \nu_{m}) u_{m,t} dx \\
& + \int_{\Omega} \int_{0}^{t} g_{1}(t-s) \nabla u_{m}(s) \nabla u_{m,t} ds dx \\
& - \int_{\Omega} \int_{0}^{t} |\mu_{1}(q)| \nabla \mathcal{X}_{m}(x, 1, q, t) \nabla u_{m,t} d\rho dx,
\end{align*}
\]

and, by using the inequality of Young, we get

\[
\begin{align*}
\int_{\Omega} a \nabla u_{m} \nabla u_{m,t} dx \\ \\
\leq \eta \|\nabla u_{m,t}\|^2 + \frac{a^2}{4\eta} \|\nabla u_{m}\|^2,
\end{align*}
\]

we have

\[
\begin{align*}
\int_{\Omega} \int_{0}^{t} g_{1}(t-s) \nabla u_{m}(s) \nabla u_{m,t} ds dx \\ \\
\leq \eta \|\nabla u_{m}\|^2 + \frac{(a-k)g_{1}(0)}{4\eta} \int_{0}^{t} \|\nabla u_{m}(s)\|^2 ds,
\end{align*}
\]

Similarly, we get

\[
\begin{align*}
\int_{\Omega} \int_{t_{1}}^{t_{2}} |\mu_{1}(q)| \nabla \mathcal{X}_{m}(x, 1, q, t) \nabla u_{m,t} d\rho dx \\ \\
\leq \eta \lambda_{1} \|\nabla u_{m}\|^2 + \frac{1}{4\eta} \int_{t_{1}}^{t_{2}} |\mu_{1}(q)| \|\nabla \mathcal{X}_{m}(x, 1, q, t)\|^2 d\rho,
\end{align*}
\]

substituting (41)–(45) into (40), and using (17), one gets

\[
\begin{align*}
\int_{\Omega} |u_{m,t}|^2 |u_{m,t}|^2 dx + & \left(1 - \left(\eta(\lambda_{1} + 2) + \frac{(1 + b_{1})C_{s}^{2}}{2}\right)\right) \\
\cdot \|\nabla u_{m,t}\|^2 + & \frac{1}{2} \frac{d}{dt} \int_{0}^{t} \|\mathcal{X}_{m}(x, 1, q, t)\|^2 \\
\leq C_{2} + & \frac{1}{4\eta} (a-k)g_{1}(0)C_{1} T,
\end{align*}
\]

where $C_{2} > 0$ depends on $\eta, a, \alpha, C_{s}, b_{1}, b_{2}, p, q, C_{1}$. 

Similarly, we get

\[
\begin{align*}
\int_{\Omega} |v_{m,t}|^2 |v_{m,t}|^2 dx + & \left(1 - \left(\eta(\lambda_{2} + 2) + \frac{(1 + b_{2})C_{s}^{2}}{2}\right)\right) \\
\cdot \|\nabla v_{m,t}\|^2 + & \frac{1}{2} \frac{d}{dt} \int_{0}^{t} \|\mathcal{Y}_{m}(x, 1, q, t)\|^2 \\
\leq C_{2} + & \frac{1}{4\eta} (a-k)g_{2}(0)C_{1} T,
\end{align*}
\]
Integrating (41) over \((0, t)\), we get

\[
\int_0^t \int_\Omega \left( \left| \nabla u_{m(t)}(\sigma) \right| \left| \nabla v_{m(t)}(\sigma) \right| \right)^2 dxd\sigma \\
+ \left( 1 - \left\{ \eta(\lambda_1 + 2) + \frac{(1 + b_1)C_2^2}{2} \right\} \right) \\
\cdot \int_0^t \left( \left| \nabla u_{m(t)}(\sigma) \right| \right)^2 d\sigma + \frac{1}{2} \int_0^t \left| \nabla u_{m(t)} \right|^2 dQ \\
+ \frac{1}{2} \int_0^t \left| \nabla u_{m(t)}(x, 1, q, \sigma) \right|^2 d\sigma \\
\leq \left( C_2 + \frac{1}{4\eta} (a - k) g_1(0) C_1 T \right) T,
\]

(47)

At this stage, choosing \(\eta > 0\) such that

\[
1 - \left\{ \eta(\lambda_1 + 2) + \frac{(1 + b_1)C_2^2}{2} \right\} > 0, \text{ for } i = 1, 2,
\]

we find

\[
\int_0^t \left( \left| \nabla u_{m(t)}(\sigma) \right| \right)^2 d\sigma \\
+ \frac{1}{2} \int_0^t \left( \left| \nabla u_{m(t)} \right|^2 \right) dQ \leq C_3.
\]

(49)

We have from (17) and (49) that there exist subsequences \((u_k)\) of \((u_m)\) and \((v_k)\) of \((v_m)\) such that

\[
(u_k, v_k) \rightharpoonup (u, v) \text{ weakly star in } L^{\infty}(0, T, H_0^1(\Omega)),
\]

\[
(u_k, v_k) \rightharpoonup (u, v) \text{ weakly star in } L^{\infty}(0, T, H_0^1(\Omega)),
\]

\[
(u_{m(t)}, v_{m(t)}) \rightharpoonup (u_t, v_t) \text{ weakly star in } L^2(0, T, H_0^1(\Omega)),
\]

\[
(\mathcal{X}, \mathcal{Y}_k) \rightharpoonup (\mathcal{X}, \mathcal{Y}) \text{ weakly star in } L^\infty(0, T, L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))),
\]

\[
(\mathcal{X}_k, \mathcal{Y}_k) \rightharpoonup (\mathcal{X}_*, \mathcal{Y}_*) \text{ weakly star in } L^\infty(0, T, L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))).
\]

(50)

We work now with the nonlinear term. From (17), we find

\[
\left\| u_{k(t)} \right\|_{L^2(0, T; L^2(\Omega))}^2 = \int_0^T \left\| u_{k(t)} \right\|_{L^2(\Omega)}^{2(1+1)} dt \\
\leq C_*^{2(1+1)} \int_0^T \left\| u_{k(t)} \right\|_{L^2(\Omega)}^{2(1+1)} dt \leq C_4,
\]

(51)

where \(C_4\) depends only on \(C_*\), \(C_1\), \(T, l\).

And from the theorem of Aubin-Lions (see Lions [23]), we deduce that there exists a subsequence of \((u_k)\), given by \((u_{k(t)})\), such that

\[
u_{k(t)} \rightharpoonup u_t \text{ strongly in } L^2(0, T, L^2(\Omega)),
\]

(52)

we get

\[
u_{k(t)} \to u_t \text{ almost everywhere in } \Omega \times \mathbb{R}_+.
\]

(53)

Hence,

\[
u_{k(t)} \to |u_t|^{|u_t|} u_t \text{ almost everywhere in } \Omega \times \mathbb{R}_+.
\]

(54)

Thus, using (46) and (48) and the Lions lemma, we derive

\[
u_{k(t)} \to |u_t|^{|u_t|} u_t \text{ weakly in } L^2(0, T, L^2(\Omega)).
\]

(55)

Similarly,

\[
u_{k(t)} v_{k(t)} \to |v_t|^{|v_t|} v_t \text{ weakly in } L^2(0, T, L^2(\Omega)),
\]

(56)

\[
(\mathcal{X}_k, \mathcal{Y}_k) \rightharpoonup (\mathcal{X}, \mathcal{Y}) \text{ strongly in } L^2(0, T, L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))),
\]

which implies

\[
(\mathcal{X}_*, \mathcal{Y}_*) \rightharpoonup (z, y) \text{ almost everywhere in } \Omega \times (0, 1) \times (\tau_1, \tau_2).
\]

(58)

The sequences \((u_k)\) and \((v_k)\) satisfy

\[
f_1(u_k, v_k) \to f_1(u, v) \text{ strongly in } L^2(0, T, L^2(\Omega)),
\]

(59)

We have

\[
\left\| f_1(u_k, v_k) - f_1(u, v) \right\|^2 = \int_\Omega \left| |v_m|^{|v_m|} u_m - |v|^{|v|} u \right|^2 dx.
\]

(60)

Noting that \((l/2p) + (1/2q) + (1/2) = 1\), by applying the generalized Hölder’s and Young’s inequalities, and (8), we get

\[
\left\| f_1(u_k, v_k) - f_1(u, v) \right\|^2 \leq C \left[ \left\| \nabla (u_m - u) \right\|^2 + \left\| \nabla (v_m - v) \right\|^2 \right].
\]

(61)
As \((u_k)\) and \((v_k)\) are Cauchy sequences in \(L^\infty(0, T, H^1_0(\Omega))\) (prove it as in [1]), then we get \((59)_1\). Similarly, we get the convergence \((59)_2\).

Multiplying \((29)\) by \(\Psi(t) \in \mathcal{D}(0, T)\) and integrating the result over \((0, T)\), we get

\[
-\frac{1}{I+1} \int_0^T \left( |u_m|^i |u_{mm}, u_i| \right) \Psi^i(t) dt \\
+ \int_0^T \mathcal{M}(\|\nabla u_m(t)\|) (\nabla u_m, \nabla u_i) \Psi^i(t) dt \\
+ \int_0^T (\nabla u_m, \nabla u_i) \Psi^i(t) dt + \int_0^T \left( f_i(u_m, v_m), u_i \right) \Psi^i(t) dt \\
- \int_0^T \int_{\tau_i} g_i(t-s) (\nabla u_m(s), \nabla u_i) \Psi^i(t) ds dt \\
+ \int_0^T \int_{\tau_i} \mu_i(t) (\nabla \mathcal{L}_m(x, 1, \rho, t), \nabla u_i) \Psi^i(t) dq dt = 0,
\]

and

\[
\frac{1}{I+1} \int_0^T \left( |v_m|^i |v_{mm}, v_i| \right) \Psi^i(t) dt \\
+ \int_0^T \mathcal{M}(\|\nabla v_m(t)\|) (\nabla v_m, \nabla v_i) \Psi^i(t) dt \\
+ \int_0^T (\nabla v_m, \nabla v_i) \Psi^i(t) dt + \int_0^T \left( f_i(u_m, v_m), v_i \right) \Psi^i(t) dt \\
- \int_0^T \int_{\tau_i} g_i(t-s) (\nabla v_m(s), \nabla v_i) \Psi^i(t) ds dt \\
+ \int_0^T \int_{\tau_i} \mu_i(t) (\nabla \mathcal{G}_m(x, 1, \rho, t), \nabla v_i) \Psi^i(t) dq dt = 0,
\]

\[
\int_0^T \left( \mathcal{Q} \mathcal{L}_m(x, \rho, t) + \mathcal{Z}_m(x, \rho, t), \mathcal{L}_m \right) \Psi^i(t) dt = 0,
\]

\[
\int_0^T \left( \mathcal{Q} \mathcal{G}_m(x, \rho, t) + \mathcal{Z}_m(x, \rho, t), \mathcal{G}_m \right) \Psi^i(t) dt = 0,
\]

\(\forall j = 1, \ldots, m\).  \(\tag{62}\)

We obtain \((62)\) by the convergence of \((50), (54), (56), \) and \((59)\). This completes the proof.

4. Exponential Decay

In this section, the stability result of the system \((13)-(15)\) is proved.

We need the following lemmas.

Lemma 3. The functional

\[
F_i(t) = \frac{1}{I+1} \int_\Omega \left( |u_i|^i |u_i| + |v_i|^i |v_i| \right) \text{d}x \\
+ \int_\Omega (\nabla u_i \nabla v_i, \nabla v_i) \text{d}x,
\]

satisfies

\[
F_i(t) \leq \frac{1}{I+1} \left( \|u_i\|^{i+1}_{I+1} + \|v_i\|^{i+1}_{I+1} \right) \\
+ \left( \frac{I+1}{I+2} \right)^{i+1} \left( \|\nabla u_i\|^2 + \|\nabla v_i\|^2 \right)
\]

\[
+ \left( \frac{I+1}{I+2} \right)^{i+1} \left( C_{i+1} + \frac{c}{2} \right) \left( \|\nabla u_i\|^2 + \|\nabla v_i\|^2 \right), \tag{64}\]

and

\[
F'_i(t) \leq \frac{1}{I+1} \left( \|u_i\|^{i+1}_{I+1} + \|v_i\|^{i+1}_{I+1} \right) \\
+ \left( \frac{I+1}{I+2} \right)^{i+1} \left( \|\nabla u_i\|^2 + \|\nabla v_i\|^2 \right)
\]

\[
+ \left( \frac{I+1}{I+2} \right)^{i+1} \left( C_{i+1} + \frac{c}{2} \right) \left( \|\nabla u_i\|^2 + \|\nabla v_i\|^2 \right) + \left( \frac{I+1}{I+2} \right)^{i+1} \left( \frac{\epsilon}{C_0} + \frac{2}{C_0} \right) \left( |u_i|^{i+1}_{I+1} + |v_i|^{i+1}_{I+1} \right)
\]

\[
\cdot \left( \|\nabla u_i\|^2 + \|\nabla v_i\|^2 \right) + \frac{C_{i+1}}{4C_0} \left( |\nabla u_i|^{i+1}_{I+1} + |\nabla v_i|^{i+1}_{I+1} \right)^2
\]

\[
+ \frac{C_{i+1}}{4C_0} \left( |\nabla u_i|^{i+1}_{I+1} + |\nabla v_i|^{i+1}_{I+1} \right)^2 \tag{65}\]

Proof.

(1) By applying the inequalities of Young and Poincare', we find

\[
|F_i(t)| \leq \frac{1}{I+1} \left( |u_i|^{i+1}_{I+1} + \left( \frac{I+1}{I+2} \right)^{i+1} \left( \|u_i\|^2 + \|v_i\|^2 \right) \right)
\]

\[
+ \left( \frac{I+1}{I+2} \right)^{i+1} \left( \|\nabla u_i\|^2 + \|\nabla v_i\|^2 \right)
\]

\[
+ \left( \frac{I+1}{I+2} \right)^{i+1} \left( C_{i+1} + \frac{c}{2} \right) \left( \|\nabla u_i\|^2 + \|\nabla v_i\|^2 \right), \tag{66}\]

and

(2) Direct computation using integration by parts, we get

\[
F_i(t) = \int_\Omega \left( |u_i|^i |u_i| \right) u dx + \frac{1}{I+1} \left( |u_i|^{i+1}_{I+1} + \int_\Omega \left( |v_i|^i |v_i| \right) v dx \\
+ \frac{1}{I+1} \left( \|u_i\|^{i+1}_{I+1} \right) + \|u_i\|^2 \\
- \int_\Omega \left( |\nabla u_i|^{i+1}_{I+1} + \|\nabla u_i\|^2 \right) - \int_\Omega \left( \|\nabla v_i\|^2 \right) \|\nabla v_i\|^2 \\
\cdot \left( \|\nabla u_i\|^2 + \|\nabla v_i\|^2 \right) + \left( \frac{I+1}{I+2} \right)^{i+1} \left( \frac{C_{i+1}}{4C_0} + \frac{c}{2} \right) \right)
\]

\[
\cdot \left( \|\nabla u_i\|^2 + \|\nabla v_i\|^2 \right)
\]

\[
\cdot \left( \|\nabla u_i\|^2 + \|\nabla v_i\|^2 \right) + \left( \frac{I+1}{I+2} \right)^{i+1} \left( \frac{\epsilon}{C_0} + \frac{2}{C_0} \right) \left( |u_i|^{i+1}_{I+1} + |v_i|^{i+1}_{I+1} \right)
\]

\[
\cdot \left( \|\nabla u_i\|^2 + \|\nabla v_i\|^2 \right) + \frac{C_{i+1}}{4C_0} \left( |\nabla u_i|^{i+1}_{I+1} + |\nabla v_i|^{i+1}_{I+1} \right)^2
\]

\[
+ \frac{C_{i+1}}{4C_0} \left( |\nabla u_i|^{i+1}_{I+1} + |\nabla v_i|^{i+1}_{I+1} \right)^2 \tag{65}\]
Lemma 4. The functional

\[
F_2(t) = \int_{\Omega} \left( \Delta u - \frac{1}{l+1} |u|^l u \right) g_1(t-s)(u(t) - u(s)) \, ds \, dx \\
+ \int_{\Omega} \left( \Delta v - \frac{1}{l+1} |v|^l v \right) g_2(t-s)(v(t) - v(s)) \, ds \, dx,
\]

(68)

satisfies

\[
F_2(t) \leq \frac{1}{l+2} \left( \| u \|_{l+2}^2 + \| v \|_{l+2}^2 \right) + \frac{1}{2} \left( \| \nabla u \|^2 + \| \nabla v \|^2 \right)
\]

\[
+ \left( \frac{(l+1)^{-1}}{l+2} (a-k)^{l+2} \right) \| \nabla u \|^{l+1} + \| \nabla v \|^{l+1} \right) + \frac{1}{2} (a-k)
\]

\[
\cdot \left\{ 1 + \frac{(l+1)^{-1}}{l+2} (a-k)^{l+2} \right\} (g_1 \circ \nabla u + g_2 \circ \nabla v),
\]

(69)

and for any \( \varepsilon_2 > 0 \),

\[
F_2'(t) \leq \frac{1}{l+1} \left[ \left( 1 - \int_0^t g_1(s) \, ds \right) \| u \|_{l+1}^2 + \left( 1 - \int_0^t g_2(s) \, ds \right) \| v \|_{l+1}^2 \right]
\]

\[
\cdot \| u \|_{l+1}^2 + \left( 2 \varepsilon_2 (a-k)^2 + \frac{\alpha C_2^4}{2} \right) \left( \| u \|^2 + \| v \|^2 \right)
\]

\[
+ \left( a-k \right) \left( \frac{(l+1)^{-1}}{l+2} (g_1(0))^{l+2} \right) \| u \|^2 + \| v \|^2
\]

\[
+ \frac{C_2^4}{2} \left( g_1(0)^{l+2} \right) \| u \|^2 + \| v \|^2
\]

\[
\cdot \left\{ 1 - \frac{(l+1)^{-1}}{l+2} (a-k)^{l+2} \right\} (g_1 \circ \nabla u + g_2 \circ \nabla v),
\]

(70)

Proof.

(1) By using Young’s inequality and the conjugate exponents \( p' = (l+2)/(l+1), \ q' = l+2, \) and Hölder’s inequality, we obtain

\[
\left| \int_{\Omega} \frac{1}{l+1} |u|^l u \int_0^t g_1(t-s)(u(t) - u(s)) \, ds \, dx \right|
\]

\[
\leq \frac{1}{l+2} \left| u \right|_{l+2}^2 + \left( \frac{(l+1)^{-1}}{l+2} \right) \left[ (a-k)^{l+2} \right]
\]

\[
\cdot \left( 2^{l+1} (a-k) \| u \|^{l+1} + \frac{1}{2} (g_1 \circ \nabla u) \right),
\]

(71)

Similarly, we get

\[
\left| \int_{\Omega} \frac{1}{l+1} |v|^l v \int_0^t g_2(t-s)(v(t) - v(s)) \, ds \, dx \right|
\]

\[
\leq \frac{1}{l+2} \left| v \right|_{l+2}^2 + \left( \frac{(l+1)^{-1}}{l+2} \right) \left[ (a-k)^{l+2} \right]
\]

\[
\cdot \left( 2^{l+1} (a-k) \| v \|^{l+1} + \frac{1}{2} (g_2 \circ \nabla v) \right),
\]

(72)
Using Young’s, Cauchy-Schwarz, Hölder’s, and Poincaré’s inequalities, and   \( k \leq 1 \), we obtain (70).

At this point, let us introduce the functional given by

\[
F_2(t) = \int_{\Omega} \left( \frac{1}{2} \|\nabla v_t\|^2 + \frac{1}{2} (a - k)(g_2 \circ \nabla v_t) \right) dx dt
\]

\[
F_3(t) = \int_{\Omega} \left( \rho e^{-\phi} \left( |\mu_1(\rho)| \|\mathcal{X}\|^2 + |\mu_2(\rho)| \|\mathcal{Y}\|^2 \right) \right) dx dt
\]

\[
F_3(t) = -\eta_1 \int_{\tau_1}^{T} \rho \left( \|\mu_1(\rho)\| \|\mathcal{X}\|^2 + |\mu_2(\rho)| \|\mathcal{Y}\|^2 \right) dx dt
\]

\[
F_3(t) = -\eta_1 \int_{\tau_1}^{T} \rho \left( \|\mu_1(\rho)\| \|\mathcal{X}\|^2 + |\mu_2(\rho)| \|\mathcal{Y}\|^2 \right) dx dt
\]
As $-e^\varphi$ is an increasing function, we have $-e^\varphi \leq -e^{-\varepsilon_1}$, for any $\varphi \in [\tau_1, \tau_2]$.

Then, setting $\eta_1 = e^{-\varepsilon_1}$, we find (78).

**Theorem 6.** Assume (5)–(8) hold, then $\exists \varepsilon_1, \xi_2 > 0$ such that the energy functional (16) satisfies

$$E(t) \leq \xi_2 e^{-\varepsilon_1 t}, \forall t \geq t_0.$$  \hspace{1cm} (81)

**Proof.** We define the functional of Lyapunov

$$\mathscr{L}(t) = NE(t) + F_1(t) + N_2 F_2(t) + F_3(t),$$  \hspace{1cm} (82)

where $N, N_2 > 0$.

First, if we let

$$\mathcal{K}(t) = F_1(t) + N_2 F_2(t) + F_3(t),$$  \hspace{1cm} (83)

then, by (64), (69), and (77), we get

$$|\mathcal{K}(t)| \leq c E(t).$$  \hspace{1cm} (84)

Consequently,

$$|\mathcal{K}(t)| = |\mathscr{L}(t) - NE(t)| \leq c E(t),$$  \hspace{1cm} (85)

which yields

$$(N - c) E(t) \leq \mathcal{K}(t) \leq (N + c) E(t).$$  \hspace{1cm} (86)

By derivation (82) and applying (17), (65), (70), (78), and (6), one gets

$$\mathcal{L}'(t) \leq \frac{1}{l+1} \left\{ (1 - h_0) + N_1 \right\} \left\{ \|u_l\|^{\alpha_2}_{\mathcal{L}^2} + \|v_l\|^{\alpha_2}_{\mathcal{L}^2} \right\}$$

$$\left\{ + \left\{ \lambda (1 + N) + N_1 + \varepsilon_2 - h_0 \right\} \left\{ \|\nabla u_l\|^2 + \|\nabla v_l\|^2 \right\} \right.$$

$$\left\{ + \varepsilon_2 M_0 \left\{ (a - k) + \frac{(l + 1)^{-1}}{l + 2} (h_2 C_*)^{\varepsilon_2} 2^{(l+1)} + R_1 \right\} \right.$$

$$\left\{ + N_1 \left\{ \varepsilon_1 (a - k + \lambda - k) + \frac{b_1 + b_2}{2} \right\} C_*^2 \right\}$$

$$\left\{ + \left\{ \frac{2\varepsilon_2 (a - k)^2 + \frac{\alpha C_*^2}{2}}{2} \right\} \left\{ \left\{ \frac{\|\nabla u_l\|^2 + \|\nabla v_l\|^2}{2} \right\}$

$$\left\{ + \left\{ \frac{-1}{\xi^2} \left\{ \frac{M_0}{4\varepsilon_2} \right\} + \left\{ \frac{2\varepsilon_2 + \lambda}{4\varepsilon_2} \right\} \left\{ \frac{\alpha C_*^2}{2} \right\} (a - k) + \frac{N_1}{4\varepsilon_1} \right\} \right.$$

$$\left\{ + \frac{N_1}{2} \left\{ \frac{h_1}{4\varepsilon_2} \left\{ \frac{(l + 1)^{-1}}{l + 2} (h_1)^2 C_*^{\alpha_2} \right\} \right\} \right.$$}

$$\cdot \left\{ \left( g_{1*}^c \nabla u + \left( g_{2*}^c \nabla v \right) \right) - \eta_1 \int_{\tau_1}^{\tau_2} \rho \left\{ \mu_1 (\rho) \right\} \|\nabla X\|^2 \right\}$$

$$\left. + \left\{ \mu_2 (\rho) \right\} \|\nabla Y\|^2 \left\{ dp \right\} - \left\{ \eta_1 + N_1 + \varepsilon_2 \right\} \frac{N_1}{4\varepsilon_1} \right\} \right.$$

$$\left\{ \int_{\tau_1}^{\tau_2} \left\{ \left( \mu_1 (\rho) \right) \|\nabla X(x, \rho, t)\|^2 \right\} dp \right\}$$

$$\left. + \left\{ \mu_2 (\rho) \right\} \|\nabla Y(x, \rho, t)\|^2 \left\{ dp \right\} \right\}.$$

where $h_0 = \min \left\{ \int_0^1 g_1(s) ds \right\}, h_2 = \min \left\{ \int_0^1 g_2(s) ds \right\}, M_0 = \max \left\{ (\|\nabla u_l\|^2), M(\|\nabla v_l\|^2) \right\}, h_1 = \min \left\{ \int_0^1 g_1(0, s) ds \right\}, h_2 = \max \left\{ \int_0^1 g_1(1, s) ds \right\}, h_c = \max \left\{ \xi_1, \xi_2 \right\},$ and $R_1 = \min \left\{ b_1 (C_*^2 (\eta_1 + \varepsilon_2)) + b_2 (C_*^2 (\eta_1 + \varepsilon_2)), C_*^2 (\eta_1 + \varepsilon_2) \right\}$.

At this stage, choosing two fixed numbers $N, N_1, \varepsilon_2 > 0, \lambda > 0, \alpha_1 > 0,$ and

$$h_1 - \lambda (1 + N) - N_1 \geq 0,$$

we choose $\varepsilon_2$ small enough such that

$$\alpha_2 = \eta_1 - 1 + N_1 > 0.$$  \hspace{1cm} (87)

we choose $\varepsilon_2$ small enough that

$$\alpha_3 = N \beta - \varepsilon_2 - \frac{N_1}{4\varepsilon_1} > 0,$$

$$\alpha_4 = \left\{ \frac{-\varepsilon_2 M_0 \left\{ (a - k) + \frac{(l + 1)^{-1}}{l + 2} (h_2 C_*)^{\varepsilon_2} 2^{(l+1)} + R_1 \right\} + N_1 \left\{ k - \varepsilon_1 (a - k + \lambda) - \frac{b_1 + b_2}{2} + \alpha \right\} \frac{C_*^2}{2} \right\} \right.$$

$$\left. - \left\{ \frac{2\varepsilon_2 (a - k)^2 + \frac{\alpha C_*^2}{2}}{2} \right\} \right\} > 0,$$

$$\alpha_5 = \left\{ \frac{\frac{1}{\xi^2} \left\{ \frac{M_0}{4\varepsilon_2} \right\} + \left\{ \frac{2\varepsilon_2 + \lambda}{4\varepsilon_2} \right\} \left\{ \frac{\alpha C_*^2}{2} \right\} (a - k) - \frac{N_1}{4\varepsilon_1} \right\} \right.$$

$$\left. - \left\{ \frac{h_1}{4\varepsilon_2} \left\{ \frac{(l + 1)^{-1}}{l + 2} (h_1)^2 C_*^{\alpha_2} \right\} \right\} \right\} > 0.$$  \hspace{1cm} (89)

Thus, we get

$$\mathcal{L}'(t) \leq \frac{-1}{l+1} \left\{ \varepsilon_1 \right\} \left\{ \frac{\|\nabla u_l\|^{\alpha_2}_{\mathcal{L}^2} + \|v_l\|^{\alpha_2}_{\mathcal{L}^2}}{2} \right\} - \alpha_3 \left\{ \|\nabla u_l\|^2 + \|\nabla v_l\|^2 \right\}$$

$$- \alpha_4 \left\{ \|\nabla u_l\|^2 + \|\nabla v_l\|^2 \right\} - \alpha_5 \left\{ \left( g_{1*}^c \nabla u + \left( g_{2*}^c \nabla v \right) \right) \right\}.$$
for some additional help they provided to him during his studies. Moreover, he thanks them for the hours they spent with him.

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\[
- \eta \int_{\Omega} \rho (|\mu_1(\rho)||\mathcal{L}|^2 + |\mu_2(\rho)||\mathcal{Y}|^2) d\rho d\theta \\
+ \alpha \int_{\Omega} (|\mu_1(\rho)||\mathcal{L}(x, 1, \rho, t)||^2 + |\mu_2(\rho)||\mathcal{Y}(x, 1, \rho, t)||^2) d\rho d\theta,
\]

(91)

\[
c_1 E(t) \leq \mathcal{L}(t) \leq c_2 E(t), \forall t \geq 0,
\]

(92)

using (16), estimates (91) and (86), respectively, we get

\[
\mathcal{L}'(t) \leq -k_1 E(t) - k_2 E'(t), \forall t \geq t_0,
\]

(93)

for some \(k_1, k_2, c_1, c_2 > 0\).

By the combination of (93) with (92), we obtain

\[
\mathcal{R}'(t) \leq -\lambda \mathcal{R}(t),
\]

(94)

where

\[
\mathcal{R}(t) = \mathcal{L}(t) + k_2 E(t) \sim E(t).
\]

(95)

Integrating the result (94) over \((t_0, t)\), we find

\[
\mathcal{R}(t) \leq \mathcal{R}(t_0) e^{-\lambda (t-t_0)}, \forall t_0 \geq t.
\]

(96)

It follows from (95) that (81) holds. This completes the proof.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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