3-MANIFOLDS ADMITTING LOCALLY LARGE DISTANCE 2
HEEGAARD SPLITTINGS

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Abstract. From the view of Heegaard splitting, it is known that if a closed
orientable 3-manifold admits a distance at least three Heegaard splitting, then
it is hyperbolic. However, for a closed orientable 3-manifold admitting only
distance at most two Heegaard splittings, there are examples showing that it
could be reducible, Seifert, toroidal or hyperbolic. According to Thurston’s
Geometrization conjecture, the most important piece in eight geometries is
hyperbolic. So for a 3-manifold admitting a distance two Heegaard splittings,
it is critical to determine the hyperbolicity of it in studying Heegaard splittings.

Inspired by the construction of hyperbolic 3-manifolds admitting distance
two Heegaard splittings in [Qiu, Zou and Guo, Pacific J. Math. 275 (2015), no.
1, 231-255], we introduce the definition of a locally large geodesic in curve com-
plex and also a locally large distance two Heegaard splitting. Then we prove
that if a 3-manifold admits a locally large distance two Heegaard splitting,
then it is a hyperbolic manifold or an amalgamation of a hyperbolic manifold
and a Seifert manifold along an incompressible torus, while the example in
Section 3 shows that there is a non hyperbolic 3-manifold in this case. After
examining those non hyperbolic cases, we give a sufficient and necessary con-
dition for a hyperbolic 3-manifold when it admits a locally large distance two
Heegaard splitting.

Keywords: Hyperbolic 3-Manifold, Heegaard Distance, Curve Complex, Lo-
cally Large Geodesic.

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1. Introduction

In 1898, Heegaard [7] introduced a Heegaard splitting for a closed, orientable,
triangulated 3-manifold, i.e., there is a closed, orientable surface cutting this man-
ifold into two handlebodies. Later, Moise [15] proved that every closed, orientable
3-manifold admits a triangulation. So each closed orientable 3-manifold admits a
Heegaard splitting. This makes studying 3-manifolds through Heegaard splittings
possible.

One astonishing result proved by Haken [3] is that if all Heegaard splitting of a
3-manifold are reducible, i.e., there is an essential simple closed curve in Heegaard
surface bounding essential disks on both sides, then this manifold is reducible.
Later, Casson and Gordon [1] defined a weakly reducible Heegaard splitting and
proved that if a 3-manifold has a weakly reducible and irreducible Heegaard split-
ting, then it contains an embedded closed incompressible surface, i.e., it is Haken.

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Both of these two phenomenons drive people to think how Heegaard splittings reflect 3-manifolds.

For classifying 3-manifolds, Thurston [27] introduced the Geometrization conjecture (Haken version proved by Thurston [27] and full version proved by Perelman [18, 19, 20]) as follows: for any closed, irreducible, orientable 3-manifold, there are finitely many disjoint, non isotopy essential tori so that after cutting the manifold along those tori, each piece is one of eight geometries. Among all of these eight geometries, one is hyperbolic, another one is solvable and the left six pieces are Seifert. In these eight geometries, it is known that Seifert 3-manifolds have been completely classified. Moreover all of their irreducible Heegaard splittings are either vertical or horizontal, see [16]. Cooper and Scharlemann [2] studied all irreducible Heegaard splittings of a solvmanifold. And there are series of works on Heegaard splittings of some typical 3-manifolds, such as Lens space, surface $\times S^1$ etc.

With the curve complex defined by Harvey [6], Hempel [8] introduced an index-Heegaard distance for studying Heegaard splitting. Basically, this index-Heegaard distance is defined to be the length of a shortest geodesic in curve complex which connects these two boundaries of essential disks from different sides. Then he proved that all Heegaard splittings of a Seifert 3-manifold have distance at most two; if a 3-manifold contains an essential torus, then all Heegaard splittings of it have distance at most two, where this result is also proved by Hartshorn [5] and Scharlemann [23].

Combined with the Geometrization conjecture, if a 3-manifold admits a Heegaard splitting with Heegaard distance at least three, then it is hyperbolic. So it seems that if we fully understand all of distance two Heegaard splittings, then we can fully answer the question that how Heegaard splittings reflect 3-manifolds. So this question is reduced to

**Question** 1.1. What dose a 3-manifold look like if it only admits distance at most two Heegaard splittings?

Since the hyperbolic 3-manifolds are the most concerned, given a distance two Heegaard splitting, it is interesting to know whether the corresponding manifold is hyperbolic or not.

By the definition of a distance two Heegaard splitting, there is an essential simple closed curve and a pair of essential disks from different handlebodies so that this curve is disjoint from those two essential disks’ boundaries. It seems that this Heegaard splitting is simple and hence whether the manifold is hyperbolic or not should not be hard to answer. However, things for distance two Heegaard splittings are complicated because there are examples showing that a 3-manifold admitting a distance two Heegaard splitting could be Seifert, hyperbolic or contains an essential torus, see [5, 26, 21, 22].

Thompson [26] studied all distance two genus two Heegaard splittings and found that even for genus two Heegaard splittings, those manifolds could be very complicated. Later, Rubinstein and Thompson [22] extended this result to genus at least three cases. But their results give no sufficient conditions to determine whether it is hyperbolic or not.

According to Geometrization conjecture, except small Seifert 3-manifolds and hyperbolic 3-manifolds, all of 3-manifolds contain essential tori. It is known that the Heegaard splittings of small Seifert 3-manifolds are well understood. So to answer the question [1.1] the first step is to understand possible essential tori in a 3-manifold.
In [21], the authors studied the curve complex and introduced the definition of a locally large geodesic. Then they constructed infinitely many arbitrary large distance Heegaard splitting. In the proof of Theorem 1.3 in [21], they found that the locally large property of geodesics forces any geodesic realizing Heegaard distance to share some vertex $\gamma$ in common. So if the resulted manifold contains an essential torus $T^2$, then $T^2$ intersects this Heegaard surface in some essential simple closed curves, which are all isotopic to $\gamma$. Thus $T^2$ intersects that Heegaard surface in fixed essential simple closed curves. Under this circumstance, it seems the corresponding 3-manifold is not hard to understand. For this reason, we introduce the definition of a locally large distance two Heegaard splitting.

A length two geodesic realizing Heegaard distance is $G = \{\alpha, \gamma, \beta\}$, where $\alpha$ and $\beta$ bound essential disks from two sides of the Heegaard surface and $\gamma$ is disjoint from both $\alpha$ and $\beta$. As we know, there is a length two geodesics contains a non separating essential simple closed curve as its middle vertex realizing Heegaard distance. So we assume that $\gamma$ is represented by a non separating essential simple closed curve. Let $S$ be this Heegaard surface. We say the geodesic $G$ is locally large if for the surface $S_\gamma = S - \gamma$, $d_{S_\gamma}(a, b) \geq 11$, for any pair of $a$ and $b$ disjoint from $\gamma$, where both $a$ and $b$ bound essential disks in different sides of $S$ respectively. Moreover, by the definition of a locally large geodesic, we say a Heegaard splitting is locally large if there is a locally large geodesic realizing Heegaard distance.

The main result is

**Theorem 1.1.** If a closed orientable manifold $M$ admits a locally large distance two Heegaard splitting $V \cup_S W$, then $M$ is a hyperbolic manifolds or an amalgamation of a hyperbolic manifold and a small Seifert manifolds along an incompressible torus. Moreover there is only one essential torus in the non-hyperbolic case up to isotopy.

A 3-manifold is almost hyperbolic if either it is hyperbolic or it is an amalgamation of a hyperbolic 3-manifold and a small non hyperbolic 3-manifold along an incompressible torus. Then the conclusion of Theorem 1.1 says that

**Corollary 1.2.** A 3-manifold admitting a locally large distance two Heegaard splitting is almost hyperbolic.

The Geometrization Conjecture indicates that

1. a Seifert 3-manifold does not admits a complete hyperbolic structure;
2. a solvmanifold does not admits a complete hyperbolic structure;
3. an amalgamation of a complete hyperbolic 3-manifold and a small Seifert 3-manifold along a torus is not one of those eight geometries, see [25].

Thus combined with the result of Theorem 1.1,

**Corollary 1.3.** Neither a solvmanifold nor a Seifert 3-manifold admits a locally large distance two Heegaard splitting.

According to Geometrization conjecture, the most important piece of those eight geometries is hyperbolic. Thus giving a sufficient condition for a hyperbolic 3-manifold is critical in studying Heegaard splittings. But the example in Section 3 shows that the manifold $M$ in theorem 1.1 could be a non hyperbolic manifold. So to give a sufficient condition for a hyperbolic 3-manifold, we need to eliminate the possible essential torus in Theorem 1.1. For this purpose, we introduce some definitions.
An essential simple closed curve in Heegaard surface is a co-core for a handlebody if there is an essential disk in this handlebody so that its boundary intersects this curve in one point. The definition of a domain for an essential simple closed curve is in Section 5. The proof of Theorem 1.1 implies that under the locally large condition of \( \gamma \), the non hyperbolic case happens if two copies of \( \gamma \) bounds essential tori in both sides of \( S \) and \( \gamma \) is not co-core on either side of \( S \). So,

**Theorem 1.4.** Suppose that a closed orientable manifold \( M \) has a locally large distance two Heegaard splitting \( V \cup_{\gamma} W \). Let \( \gamma \) be an essential simple closed curve disjoint from a pair of essential disks from different sides of \( S \). Then \( M \) is hyperbolic if and only if either all domains of \( \gamma \) have euler characteristic number less than \(-1\) or \( \gamma \) is co-core for one side of \( S \).

We introduce the definition of a geodesic of curve complex in Section 2, construct a non hyperbolic 3-manifold in Section 3 and prove theorem 1.1 and 1.4 in Section 4 and 5.

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2. Some needed Lemmas

Let \( S \) be a compact surface of genus at least 1, and \( \mathcal{C}(S) \) be the curve complex of \( S \). We call a simple closed curve \( c \) in \( S \) is essential if \( c \) bounds no disk in \( S \) and is not parallel to \( \partial S \). It is known that each vertex of \( \mathcal{C}(S) \) is represented by the isotopy class of an essential simple closed curve in \( S \). For simplicity, we do not distinguish the essential simple closed curve \( c \) and its isotopy class \( c \) without any further notation.

Harvey [6] defined the curve complex \( \mathcal{C}(S) \) as follows: The vertices of \( \mathcal{C}(S) \) are the isotopy classes of essential simple closed curves on \( S \), and \( k+1 \) distinct vertices \( x_0, x_1, \ldots, x_k \) determine a \( k \)-simplex of \( \mathcal{C}(S) \) if and only if they are represented by pairwise disjoint essential simple closed curves. For any two vertices \( x \) and \( y \) of \( \mathcal{C}(S) \), the distance of \( x \) and \( y \) denoted by \( d_{\mathcal{C}(S)}(x, y) \), is defined to be the minimal number of \( 1 \)-simplexes in a simplicial path joining \( x \) to \( y \). In other words, \( d_{\mathcal{C}(S)}(x, y) \) is the smallest integer \( n \geq 0 \) such that there is a sequence of vertices \( x_0 = x, \ldots, x_n = y \) such that \( x_{i-1} \) and \( x_i \) are represented by two disjoint essential simple closed curves on \( S \) for each \( 1 \leq i \leq n \). For any two sets of vertices in \( \mathcal{C}(S) \), say \( X \) and \( Y \), \( d_{\mathcal{C}(S)}(X, Y) \) is defined to be \( \min \{ d_{\mathcal{C}(S)}(x, y) \mid x \in X, y \in Y \} \). For the torus or once punctured torus case, Masur and Minsky [12] define \( \mathcal{C}(S) \) as follows: The vertices of \( \mathcal{C}(S) \) are the isotopy classes of essential simple closed curves on \( S \), and \( k+1 \) distinct vertices \( x_0, x_1, \ldots, x_k \) determine a \( k \)-simplex of \( \mathcal{C}(S) \) if and only if \( x_i \) and \( x_j \) are represented by two simple closed curves \( c_i \) and \( c_j \) on \( S \) such that \( c_i \) intersects \( c_j \) in just one point for each \( 0 \leq i \neq j \leq k \). The following lemma is well known, see [11] [12] [13].

**Lemma 2.1.** \( \mathcal{C}(S) \) is connected, and the diameter of \( \mathcal{C}(S) \) is infinite.

A collection \( \mathcal{G} = \{ a_0, a_1, \ldots, a_n \} \) is a geodesic in \( \mathcal{C}(S) \) if \( a_i \in \mathcal{C}^0(S) \) and

\[
d_{\mathcal{C}(S)}(a_i, a_j) = | i - j |,
\]

for any \( 0 \leq i, j \leq n \). And the length of \( \mathcal{G} \) denoted by \( \mathcal{L}(\mathcal{G}) \) is defined to be \( n \). By Lemma 2.1 there is a shortest path in \( \mathcal{C}^1(S) \) connecting any two vertices of \( \mathcal{C}(S) \).
Thus for any two distance \( n \) vertices \( \alpha \) and \( \beta \), a geodesic \( \mathcal{G} \) connects \( \alpha \) and \( \beta \) if \( \mathcal{G} = \{ a_0 = \alpha, \ldots, a_n = \beta \} \). Now for any two sub-simplicial complex \( X, Y \subset \mathcal{C}(S) \), a geodesic \( \mathcal{G} \) realizing the distance of \( X \) and \( Y \) if \( \mathcal{G} \) connects an element \( \alpha \in X \) and an element \( \beta \in Y \) so that \( \mathcal{L}(\mathcal{G}) = d_{\mathcal{C}(S)}(X,Y) \).

Let \( F \) be a compact surface of genus at least 1 with non-empty boundary. Similar to the definition of the curve complex \( \mathcal{C}(F) \), we can define the arc and curve complex \( \mathcal{AC}(F) \) as follows:

Each vertex of \( \mathcal{AC}(F) \) is the isotopy class of an essential simple closed curve or an essential properly embedded arc in \( F \), and a set of vertices form a simplex of \( \mathcal{AC}(F) \) if these vertices are represented by pairwise disjoint arcs or curves in \( F \). For any two disjoint vertices, we place an edge between them. All the vertices and edges form 1-skeleton of \( \mathcal{AC}(F) \), denoted by \( \mathcal{AC}^1(F) \). And for each edge, we assign it length 1. Thus for any two vertices \( \alpha \) and \( \beta \) in \( \mathcal{AC}^1(F) \), the distance \( d_{\mathcal{AC}(F)}(\alpha, \beta) \) is defined to be the minimal length of paths in \( \mathcal{AC}^1(F) \) connecting \( \alpha \) and \( \beta \). Similarly, we can define the geodesic in \( \mathcal{AC}(F) \).

When \( F \) is a subsurface of \( S \), we call \( F \) is essential in \( S \) if the induced map of the inclusion from \( \pi_1(F) \) to \( \pi_1(S) \) is injective. Furthermore, we call \( F \) is a proper essential subsurface of \( S \) if \( F \) is essential in \( S \) and at least one boundary component of \( F \) is essential in \( S \). For more details, see [13].

So if \( F \) is an essential subsurface of \( S \), there is some connection between the \( \mathcal{AC}(F) \) and \( \mathcal{C}(S) \). For any \( \alpha \in \mathcal{C}^0(S) \), there is a representative essential simple closed curve \( \alpha_{geo} \) such that the intersection number \( i(\alpha_{geo}, \partial F) \) is minimal. Hence each component of \( \alpha_{geo} \cap F \) is essential in \( F \) or \( S - F \). Now for \( \alpha \in \mathcal{C}(S) \), let \( \kappa_F(\alpha) \) be isotopy classes of the essential components of \( \alpha_{geo} \cap F \).

For any \( \gamma \in \mathcal{C}(F) \), \( \gamma' \in \mathcal{C}^1(F) \) if and only if \( \gamma' \) is the essential boundary component of a closed regular neighborhood of \( \gamma \cup \partial F \). Now let \( \pi_F = \sigma_F \circ \kappa_F \). Then the map \( \pi_F \) is the subsurface projection defined in [13].

We say \( \alpha \in \mathcal{C}^0(S) \) cuts \( F \) if \( \pi_F(\alpha) \neq \emptyset \). If \( \alpha, \beta \in \mathcal{C}^0(S) \) both cut \( F \), we write \( d_{\mathcal{C}(F)}(\alpha, \beta) = \text{diam}_{\mathcal{C}(F)}(\pi_F(\alpha), \pi_F(\beta)) \). And if \( d_{\mathcal{C}(S)}(\alpha, \beta) = 1 \), then \( d_{\mathcal{AC}(F)}(\alpha, \beta) \leq 1 \) and \( d_{\mathcal{C}(F)}(\alpha, \beta) \leq 2 \).

The following is immediately followed from the above observation.

**Lemma 2.2.** Let \( F \) and \( S \) be as above, \( \mathcal{G} = \{ \alpha_0, \ldots, \alpha_k \} \) be a geodesic of \( \mathcal{C}(S) \) such that \( \alpha_j \) cuts \( F \) for each \( 0 \leq i \leq k \). Then \( d_{\mathcal{C}(F)}(\alpha_0, \alpha_k) \leq 2k \).

For essential curves \( \alpha, \beta \) in \( S \), let \( | \alpha \cap \beta | \) be the minimal geometric intersection number up to isotopy. We call \( \alpha \) and \( \beta \) intersect efficiently if the number of \( \alpha \cap \beta \) is equal to \( | \alpha \cap \beta | \).

One tool for studying the intersection between essential simple closed curves and arcs in \( S \) is the bigon criterion.

**Lemma 2.3.** [3] Let surface \( S \) be as above. Then for any two essential curves \( \alpha, \beta \) in \( S \), \( \alpha \) and \( \beta \) intersects efficiently if and only if \( \alpha \cup \beta \cup \partial S \) bounds no bigon or half-bigon in \( S \).

Assume that \( V \) is a non-trivial compression body, i.e., not the product I-bundle of a closed surface. Then there is an essential simple closed curve in \( \partial_+ V \) bounding an essential disk in \( V \). Let \( S \) be an essential subsurface in \( \partial_+ V \). We call \( S \) is a
hole for $V$ if for any essential disk $D \subset V$, $\pi_S(\partial D) \neq \emptyset$. Furthermore, we call an essential subsurface $S \subset \partial_+ V$ an incompressible hole for $V$ if $S$ is a hole for $V$ and is incompressible in $V$. Otherwise, $S$ is a compressible hole for $V$. Masur and Schleimer\cite{masur1995} studied the subsurface projection of an essential disk, and proved that:

**Lemma 2.4.** Let $V$ be a non-trivial compression body and $S$ be a compressible hole for $V$. Then for an essential disk $D$ in $V$, there are essential disks $D^1$ and $D^2$ satisfying:

- for $\partial D$, $\partial D^1$ and $\partial S$, they intersect efficiently;
- $\partial D^2 \subset S$;
- there is one component of $\partial D \cap S$ is disjoint from an component of $\partial D^1 \cap S$ and $\partial D^1 \cap \partial D^2 = \emptyset$. Furthermore, $d_{AC(S)}(\pi_S(\partial D), \partial D^2) \leq 3$.

**Proof.** See the proof of Lemma 11.5 and Lemma 11.7 \cite{masur1995}. \hfill $\Box$

Let $\{x_1, x_2, ..., x_n\}$ be a collection of some different points in $S$. For the manifold $S \times S^1$, $SC = \{x_i \times S^1, i = 1, ..., n\}$ is a collection of essential simple closed curves. A closed orientable 3-manifold is Seifert if it is obtained by doing Dehn surgeries along $SC$ as follows. We remove a regular neighborhood of $x_i \times S^1$ and glue back a solid torus where the meridian curve coincides with some $\beta_i/\alpha_i$ slope. If $\beta_i/\alpha_i \neq 0$, then this fiber is called exceptional. As we know, if all of these exceptional fiber are removed, then $M - \cup N(\beta_i/\alpha_i)$ is $F \times S^1$. Somehow, $M$ is represented as follows: $M = \{S, \beta_1/\alpha_1, ..., \beta_n/\alpha_n\}$, where $S$ is called the base surface. It is known that this representation is unique with some permutations in order.

In studying irreducible Heegaard splittings of a Seifert manifold $M$, there are two standard ones named as vertical and horizontal Heegaard splittings. To be clear, for a Seifert manifold $M = \{S, \beta_1/\alpha_1, ..., \beta_n/\alpha_n\}$ with projection $f : M \to S$, let $S = D \cup E \cup F$ be a cell decomposition where each component of $D$ or $F$ contains at most one singular point in its interior and each component of $E$ is a square with one pair of opposite edges in $D$ and the other one in $F$, where both $D \cup E$ and $E \cup F$ are connected. Then the union of $H_1 = f^{-1}(D) \cup E \times [0, \frac{1}{2}]$ is a handlebody and $H_2$, the complement of $H_1$ in $M$, is also a handlebody which is homeomorphic to $f^{-1}(F) \cup E \times [\frac{1}{2}, 1]$, where $S^1 = [0, 1]/\sim$. So $H_1 \cup \partial H_2$ is a Heegaard splitting of $M$, called a vertical Heegaard splitting. The construction of a vertical Heegaard splitting shows that for each Seifert manifold, it admits a vertical Heegaard splitting. Knowing that fact, people wonder that whether there is another type Heegaard splitting for a Seifert manifold in general or not. However, there is no other type of Heegaard splitting for a Seifert manifold in general. In \cite{MO}, Moriah and Schultens proved that except some special cases, almost all of Seifert 3-manifolds admit only vertical Heegaard splittings. They \cite{MO} also showed that there are horizontal Heegaard splittings for some Seifert 3-manifolds as follows. Taking a surface bundle $M_1 = F \times I/(x, 0) \sim (\psi(x), 1)$, where $\chi(F) \leq 0$ with one boundary component and $\psi : F \times \{1\} \to F \times \{0\}$ is a periodic homeomorphism and fixes $\partial F$ point by point. Let $M$ be a Dehn filling of $M_1 \cup D \times S^1$, where the longitude goes to $\partial F$. Then $\partial F \times \{0, \frac{1}{2}\}$ bounds an annulus $A$ in $D \times S^1$ which cuts out an I-bundle of $A$. It is not hard to see that $F \times \{0, \frac{1}{2}\} \cup A$ cuts $M$ into two handlebodies, where both of these two handlebodies are compact surfaces product I-bundles. So it gives a Heegaard splitting of $M$, called a horizontal Heegaard splitting. A result in \cite{MO} said that a Seifert manifold admits a horizontal Heegaard splitting if and only if
its eulerAngles is zero. Moreover, they [10] proved that all irreducible Heegaard splittings of a Seifert manifold is either vertical or horizontal.

From the definition of a Seifert 3-manifold, if $M$ has a genus at least 1 base surface $S$ or $S^2$ but with at least 4 exceptional fibers, $M$ contains an essential torus $T^2$. Then for any strongly irreducible Heegaard splitting $H_1 \cup \partial H_1, H_2$, there are two essential annuli $A_1 \subset H_1$ and $A_2 \subset H_2$ with $\partial A_1 \cap \partial A_2 = \emptyset$. If $M$ has $S^2$ as its base surface with at most three exceptional fibers, then

**Lemma 2.5.** (1) for a vertical Heegaard splitting, there are essential disks $D_1$ and $D_2$ from two sides of Heegaard surface so that their boundaries intersects in at most two points;

(2) for a horizontal Heegaard splitting, there is an essential simple closed curve $C$ and two essential annuli $A_1 = C \times [0, \frac{1}{2}]$ in $H_1$ and $A_2 = C \times [\frac{1}{2}, 1]$ in $H_2$ so that $\partial A_1 \cap \partial A_2$ contains at most one point.

**Proof.** The proof of second part is contained in the proof of Theorem 3.5 in [8]. So all we need to prove is the first part. Since a weakly reducible Heegaard splitting satisfies the conclusion, we only consider all strongly irreducible vertical Heegaard splittings of it. If this vertical Heegaard splitting has genus at least 3, Corollary 3.3 in [8] says that it has distance at most 1. Hence there are two essential disks satisfying the conclusion of Lemma 2.5. If this vertical Heegaard splitting has genus 2, by the definition, one handlebody $H_1$ is the union of two closed neighborhood of exceptional fibers and a rectangle $\times [0, \frac{1}{2}]$ and the other one $H_2$ is homeomorphic to an I bundle of one-holed torus with a non trivial Dehn surgery. It is not hard to see that removing two exceptional fibers reduces $M$ into a torus $\times I$ with a non trivial Dehn surgery, where we do the Dehn surgery along an simple closed curve $C$ and the union of a longitude and $C$ bounds an embedded annulus. The rectangle $\times [0, \frac{1}{2}]$ is isotopic to the closed neighborhood of an properly embedded unknotted arc which connects these two boundaries. After removing a rectangle $\times [0, \frac{1}{2}]$, it is changed into the handlebody $H_2$.

Let $a$ be a properly embedded arc in this rectangle where it connects a pair of opposite edges. Then $a \times [0, \frac{1}{2}]$ bounds an essential disk $D_1$ in $H_2$. Let $b$ be an properly embedded essential arc in once punctured torus which intersects the longitude empty. Then $b \times [\frac{1}{2}, 1]$ bounds an essential disk $D_2$ in $H_2$. It is not hard to see that $\partial D_1$ intersects $\partial D_2$ in two points. 

Hempel [8] showed that for a vertical Heegaard splitting, its genus equal to the sum of the number of rectangles and 1. So it means that only a Seifert 3-manifold with base surface $S^2$ with at most three exceptional fibers admits a genus 2 vertical Heegaard splitting. By the proof of Lemma 2.5 for a strongly irreducible vertical Heegaard splitting of genus 2.

**Corollary 2.6.** there are two essential disks $D_1$ and $D_2$ of two sides so that there are two non isotopy essential simple closed curve $C_1$ and $C_2$ in Heegaard surface disjoint from both of them.

**Proof.** Let $D_1$ and $D_2$ be as in Lemma 2.5. It is known that $H_2$ is an once punctured torus I-bundle with a non trivial Dehn surgery. Let $C_1$ be a longitude in upper boundary and $C_2$ be a longitude in lower boundary. Then these two curves satisfy the conclusion. 

\[\square\]
3. A toroidal manifold with a distance 2 Heegaard splitting

Let $M$ be a compression body with genus $2g - 1$, where $g \geq 2$, and $\partial_-M$ be a torus. Then there are two non-separating spanning annuli $A_1$ and $A_2$, i.e., the boundary of an essential annulus lies in different components of $\partial M$, such that $M - A_1 \cup A_2$ are two handlebodies $V_1$ and $V_2$ with same genera, see Figure 1.

![Figure 1. Annuli in M](image)

From Figure 1, $\partial V_1$ (resp. $\partial V_2$) consists of $S_1$ and an annulus $A_1^1$ (resp. $S_2$ and an annulus $A_2^1$). Since $S_1$ and $S_2$ are homeomorphic, there is a orientation reversing homeomorphism $f : S_1 \to S_2$ such that $f(\partial A_1^1 \cap S_1) = \partial A_2^1 \cap S_2$.

Since $S_1$ is a genus $g - 1 \geq 1$ surface with two boundary, the Projective Measured Lamination Space of $S_1$

$$PML(S_1) \cong S^{6g-9}$$

is not empty. It is known that the isotopy class of the boundary $C \subset S_1$ of an essential disk in $V_1$ is an element of $PML(S_1)$. Then the collection of all essential simple closed curves bounding disks is a subset of $PML(S_1)$. It is known that the intersection function on $ML(S)$ defined a weak*-topology on $ML(S)$, see [17]. Then there is a topology defined on $PML(S)$ induced by the projection $P : ML(S) \to PML(S)$. Under this topology, let $\mathcal{DS}_1 \subset PML(S_1)$ be the closure of all essential simple closed curves in $S_1$ which bound disks in $V_1$. So is $\mathcal{DS}_2$. By the symmetry of these two handlebodies $V$ and $V_2$, there is an automorphism of $h : S_1 \to S_1$ such that $h \circ f(\mathcal{DS}_2) \subset \mathcal{DS}_1$.

**Fact 3.1.** $\mathcal{DS}_1$ is nowhere dense.

**Note 3.1.** The proof is based on and contained in the proof of Theorem 1.2 [10]. For the integrity of this paper, we use the theory of Measured Lamination Space and rewrite it here.

Before proving Fact 3.1 we introduce a definition as follows. For any essential simple closed curve $\alpha \subset S_1$ bounding an essential disk in $V_1$, there is an disk system $\Gamma$ in $S_1$ such that

1. one of its vertices is $\alpha$;
2. all of its vertices are the isotopy classes of the boundaries of pairwise disjoint non-isotopic essential disks in $V_1$;
3. it splits $S_1$ into a collection of pairs of pants.
Proof. All we need to prove is $DS_1$ contains no open set in $PML(S_1)$.

Choosing an element $\alpha \in DS_1$ represented by an essential non-separating simple closed curve in $S_1$, by above argument, there is a disk system $\Gamma$ in $S_1$. For any element $\beta \in DS_1$ represented by an essential simple closed curve in $S_1$, by Lemma 2.20 we can isotope $\beta$ such that the intersection number $| \beta \cap \Gamma |$ is minimal. If $\beta$ intersects $\Gamma$ nonempty, then there is a wave $w$ corresponding to the outermost disk component in the complement of $\Gamma$ in $S_1$. Since the boundary of $\partial S_1$ bounds no essential disk in $V_1$, the wave $w$ is contained in a pair of pants bounded by the boundaries of essential disks. If $\beta$ intersects $\Gamma$ empty, then $\beta \in \Gamma$.

From Penner and Harer [17], there is always a birecurrent maximal train track $\tau$ in $S_1$ such that it intersects all the wave like $w$ for the disk system $\Gamma$. Then there is a minimal measured lamination $\mathcal{L}$ carried by $\tau$ intersecting all the wave like $w$ such that the complement of it in $S_1$ is a disk or a one-holed disk with a finite points removed from its boundary, where the one holed disk contains one boundary of $S_1$. Then $\mathcal{L}$ is not in $DS_1$ because it intersects each element in $DS_1$ non empty.

It is known that the collection of essential simple closed curves in $S_1$ is dense in $PML(S_1)$. Then there is a sequence $\{c_1, ..., c_n, ...\}$ converging to $\mathcal{L}$ in $PML(S_1)$, where $c_i$ is represented by an essential simple closed curve. Hence there is a number $N$ such that $c_{N+1}$ intersects all the waves like $w$ for the disk system $\Gamma$. So there is a neighborhood $U$ of $c_{N+1}$ in $PML(S_1)$ disjoint from $DS_1$ in $PML(S_1)$.

Now suppose that there is an open set $U' \subset DS_1$. Then there is an automorphism $f : S_1 \to S_1$, where $f(DS_1) = DS_1$, and a non separating essential curve $\alpha_1 \in U'$ bounding disk in $V_1$ such that $f(\alpha_1) = \alpha$ and $f(U') \subset DS_1$ is an open neighborhood of $\alpha$ in $PML(S_1)$.

For each essential simple closed curve $c \subset S_1$ which intersects $\alpha$ nonempty, let $\tau_\alpha$ be the Dehn twist along $\alpha$ in $S_1$. It is known that $\tau_\alpha^n(c)$ is closed to $c$ in $PML(S_1)$. Then $\tau_\alpha^n(c_{N+1}) \subset f(U')$ for $n$ large enough. Hence there is an open subset $U_1 \subset U$ such that $\tau_\alpha^n(U_1) \subset f(U')$. It means that $f^{-1} \circ \tau_\alpha^n(U_1) \subset U'$. Then $\tau_\alpha^{-n} \circ f(DS_1) \neq DS_1$. But since $\alpha$ bounds an essential disk in $V_1$, both of these two maps $\tau_\alpha$ and $\tau_\alpha^{-1}$ map $DS_1$ into $DS_1$. Hence $\tau_\alpha^{-n} \circ f$ maps $DS_1$ into $DS_1$. A contradiction.

Since the collection of those stable and unstable laminations of all pseudo anosov automorphisms in $S_1$ is dense in $PML(S_1)$, there is a pseudo anosov map $g$ in $S_1$ such that the stable lamination are not in $DS_1$. By the proof of Theorem 2.7 [8], if $n$ is large enough, then $d_{C(S_1)}(g^n(\alpha), h \circ f(\beta)) \geq 11$ for any $\alpha$ and $\beta$ bounding essential disks in $V_1$ and $V_2$ respectively.

For constructing a non hyperbolic 3-manifold, we set $M_1$ be $V_2 \cup g^n \circ h \circ f V_1$ along $S_2$ and $S = \partial V_1$ in $M_1$. After pushing $S$ a little into the interior of $M$, $S$ splits $M_1$ into a handlebody $V$ and a compression body $W$, see Figure 2.

Note: $S$ is colored in green and $S_1$ is colored in red, where $S$ is parallel to the union of $S_1$ and an annulus $A \subset \partial M_1$.

A Heegaard splitting is weakly reducible if there are a pair of essential disks from different sides of the Heegaard surface so that their boundaries intersect empty. Otherwise, the Heegaard splitting is strongly irreducible.

Fact 3.2. The Heegaard splitting $V \cup S W$ is strongly irreducible.
Proof. Suppose not. Then the Heegaard splitting is weakly reducible. So there are a pair of essential disks $D \subset V$ and $E \subset W$ so that $\partial D \cap \partial E = \emptyset$. From the construction of $M_1$, $A$ is incompressible in $M_1$. Let $S_{1,1} \subset S_1$ be $S_1 - N(\partial S_1)$, where $N(\partial S_1)$ is a regular neighborhood of $\partial S_1$ in $S_1$. After pushing the closure of $A \cup N(\partial S_1)$ a little into $M_1$ so that it is disjoint from $S_{1,1}$, $A \cup N(\partial S_1)$ is turned into an embedded annulus $A_{1,1}$. Then $S$ is isotopic to $S_{1,1} \cup A_{1,1}$. It is not hard to see that every essential disk of $V$ (resp. $W$) has the property that its boundary cuts $S_{1,1}$. It means that $S_{1,1}$ is a hole for both of $V$ and $W$. By the construction of $V \cup W$, there is an essential disk in $V$ (resp. $W$) with its boundary in $S_{1,1}$. Then $S_{1,1}$ is a compressible hole for both of $V$ and $W$.

By Lemma 2.4, for the essential disk $D$, there is an essential disk $D_1 \subset V$ such that

1. $\partial D_1 \subset S_{1,1}$;
2. there is one component $a \subset \partial D \cap S_{1,1}$ such that $d_{c(S_{1,1})}(\pi_{S_{1,1}}(a), \partial D_1) \leq 3$;
3. Similarly for the essential disk $E$, there is an essential disk $E_1 \subset W$ such that
   1. $\partial E_1 \subset S_{1,1}$;
   2. there is one component $b \subset \partial E \cap S_1$ such that $d_{c(S_{1,1})}(\pi_{S_{1,1}}(b), \partial E_1) \leq 3$;
Since $\partial D \cap \partial E = \emptyset$, by Lemma 2.2 $d_{c(S_{1,1})}(\pi_{S_{1,1}}(a), \pi_{S_{1,1}}(b)) \leq 2$. By triangle inequality,

$$d_{c(S_{1,1})}(\partial D_1, \partial E_1) \leq 8.$$  

Since $S_{1,1}$ is an essential subsurface of $S_1$, every essential simple closed curve in $S_{1,1}$ is an essential simple closed curve in $S_1$. Then

$$d_{c(S_1)}(\partial D_1, \partial E_1) \leq 8.$$  

Since $S_1 \subset \partial V_1$, it is not hard to see that $D_1$ is also an essential disk in $V_1$. So is the disk $E_1$. Then the inequality above implies that

$$d_{c(S_1)}(g^n(\alpha), h \circ f(\beta)) \leq 8,$$

for some pair of $\alpha$ and $\beta$ bounding essential disks in $V_1$ and $V_2$ respectively.

It contradicts the assumption of $M_1$.  

It is known that every Heegaard splitting of a boundary reducible 3-manifold is weakly reducible. Then the torus boundary $T^2_1$ of $M_1 = V_1 \cup f_{\partial V_2}$, $V_2$ is incompressible.

Let $ST_1$ and $ST_2$ be two solid torus. Let $A^2_1 \subset \partial ST_1$ be an incompressible annulus so that the core circle of $A^2_1$ intersects the meridian circle in at least two points up to isotopy. Similarly, choose an annulus $A^2_2$ in the boundary $ST_2$. After
gluing \( ST_1 \) and \( ST_2 \) together along a homeomorphism between \( A_1^2 \) and \( A_2^2 \), the resulted manifold \( M_2 \) is a small Seifert 3-manifold with only one torus boundary \( T_2^3 \), where \( T_2^3 \) is incompressible.

Let \( h_1 : T_1^2 \to T_2^2 \) be a homeomorphism such that \( h_1(\partial S_1) = \partial A_1^2 \). Then \( M^* = M_1 \cup h_1 M_2 \) is closed and \( T_2^3 \) is incompressible in \( M^* \).

Let \( S^* = S_1 \cup A_1^2 \). Then \( S^* \) splits \( M^* \) into two 3-manifolds, denoted by \( V^* \) and \( W^* \) respectively. Moreover it is unknown whether all the \( C \)

Then it is a geodesic in \( \mathcal{C}(S) \) and realizes the Heegaard distance.

For the Heegaard splitting \( V^* \cup S_2 \), \( W^* \), there maybe many geodesics in \( \mathcal{C}(S^*) \) realizing the Heegaard distance. Moreover it is unknown that whether all the geodesics have a common vertex or not. But for the non hyperbolic example \( M^* = \)
$V^* \cup_S W^*$ in Section 3 there is an essential non-separating simple closed curve $\gamma^*$ such that for any pair of essential simple closed curves $\alpha$ and $\beta$ disjoint from $\gamma^*$ bounding essential disks in $V^*$ and $W^*$ respectively, $d_{C(S^*)}(\alpha, \beta) \geq 11$.

**Fact 4.1.** Every geodesic realizing the distance of $V^* \cup_S W^*$ has $\gamma^*$ as one of its vertices.

**Proof.** Suppose not. Then there is one geodesic

$$\mathcal{G}_1 = \{\alpha_1, \gamma_1, \beta_1\}$$

so that

1. it realizes the Heegaard distance;
2. $\gamma_1$ is not isotopic to $\gamma^*$.

Let $S_{\gamma^*}$ be the closure of the complement of $\gamma^*$ in $S^*$. Since $\gamma^*$ bounds essential disks in neither $V^*$ nor $W^*$ and is non-separating, $S_{\gamma^*}$ is a compressible hole for both of these two disk complexes of $V^*$ and $W^*$. By Lemma 2.4 for $\alpha_1$ (resp. $\beta_1$), there is an essential disk $D$ (resp. $E$) so that

1. $\partial D$ (resp. $\partial E$) is disjoint from $\gamma^*$;
2. there is an essential disk $D_1$ (resp. $E_1$) is disjoint from $D$ (resp. $E$);
3. there is one component of $\partial D_1 \cap S_{\gamma^*}$ (resp. $\partial E_1 \cap S_{\gamma^*}$) disjoint from one component of $\partial D_1 \cap S_{\gamma^*}$ (resp. $\partial E_1 \cap S_{\gamma^*}$).

Then by Lemma 2.2

$$d_{C(S^*)}(\pi_{S^*}(a), \partial D) \leq 3 \text{ and } d_{C(S^*)}(\pi_{S^*}(b), \partial E) \leq 3.$$ 

Since each component of $\gamma_1 \cap S_{\gamma^*}$ is disjoint from $a$ and $b$ and not isotopic to $\gamma^*$, by Lemma 2.2

$$d_{C(S^*)}(\pi_{S^*}(a), \pi_{S^*}(b)) \leq 4.$$ 

Then by triangle inequality, $d_{C(S^*)}(\partial D, \partial E) \leq 10$. It contradicts the assumption of $\gamma^*$. \qed

For the closed orientable irreducible 3-manifold $M^*$, Geometrization Conjecture says that there are finitely many essential tori so that after cutting $M^*$ along these tori, each piece is either hyperbolic, Seifert or Solvable. Thus to understand the geometry of $M^*$, the first thing is to check the possible embedded essential tori in it. It is known that for any possible essential torus $T^2$ in $M^*$, by Schultens’ Lemma 24, they can be isotoped to a general position that $T^2 \cap S^*$ consists of essential simple closed curves in both $S^*$ and $T^2$. After pushing the possible boundary parallel annulus to the other side, we assume that each component of $T^2 \cap V^*$ (resp. $T^2 \cap W^*$) is an essential annulus in $V^*$ (resp. $W^*$). On one side, a boundary compression on an essential annulus produces an essential disk. So for each component of $\gamma \subset T^2 \cap S^*$, there is a geodesic containing it as its one vertex, which realizes Heegaard distance. On the other side, by Fact 4.1 each geodesic realizing distance of Heegaard splitting $V^* \cup_S W^*$ shares the same vertex $\gamma^*$. Hence each component of $T^2 \cap S^*$ is isotopic to $\gamma^*$.

After doing a boundary compression on one annulus component of $T^2 \cap V^*$, there is an essential separating disk $D$ in $V^*$ so that

1. $D$ cuts out a solid torus $ST$ in $V^*$;
2. each component of $T^2 \cap V^*$ lies in $ST$.

The reason for case (2) happening is that we choose a boundary compression disk for $T^2 \cap V^*$ so that its interior intersects them empty. Then after doing boundary
compression along this disk, the resulted disk $D$ is disjoint from all components of $T^2 \cap V^*$. Since these two boundaries of this annulus are isotopic, $D$ cuts out a solid torus from $V^*$. Then all components of $T^* \cap V^*$ are contained in this solid torus after isotopy.

As all components of $T^2 \cap V^*$ are pairwise disjoint, all these components of $T^2 \cap V^*$ are parallel, i.e., any two components of $T^2 \cap V^*$ cuts out an I-bundle of annulus. So are $T^2 \cap W^*$. Since the union of all these annuli is $T^2$,

**Fact 4.2.** $T^2$ intersects $V^*$ in only one essential annulus.

**Proof.** Suppose not. Then there are at least two essential annulus in $V^*$. And there is an essential disk $D \subset V^*$ such that $D$ cuts out a solid torus containing $T^2 \cap V^*$. For $T^2 \cap W^*$, there is also an essential disk $E \subset W^*$ such that $E$ cuts out a solid torus containing $T^2 \cap W^*$, see Figure 3.

![Figure 3. Parallel Annuli](image)

Since the distance of Heegaard splitting $V^* \cup S \cup W^*$ is 2, $\partial D \cap \partial E \neq \emptyset$. It means that the red circles coincide with the blue circles in Figure 3. Then the essential annulus bounded by the red circles in $V^*$ and the essential annulus bounded by the blue circles in $W^*$ are patched together in $T^2$. And the resulted manifold is a torus or a kleinian bottle. But $T^2$ contains no Kleinian bottle as its subset. So the resulted manifold is a torus which is the $T^2$, where it intersects $S^*$ in only two simple closed curves. A contradiction.

Moreover, the proof of Fact 4.2 indicates that

**Fact 4.3.** if $M^*$ is toroidal, there is only one essential separating torus in $M^*$ up to isotopy.

We begin to prove Theorem 4.1 which is rewritten as follows.

**Theorem 4.1.** For a manifold $M$ admitting a distance 2, genus at least 2 Heegaard splitting $V \cup S \cup W$, if there is an essential non-separating simple closed curve $\gamma$ in $S$ so that

1. $\gamma$ bounds no essential disk in $V$ or $W$;
2. there is a geodesic realizing Heegaard distance of $V \cup S \cup W$ with $\gamma$ as one of its vertices;


(3) for any pair of essential simple closed curves $\alpha$ and $\beta$ bounding disks in $V$ and $W$ respectively, if they are disjoint from $\gamma$, then $d_{c(S_i)}(\alpha, \beta) \geq 11$, then $M$ is either a hyperbolic 3-manifold or an amalgamation of a hyperbolic 3-manifold and a small Seifert 3-manifold along an incompressible torus.

**Proof.** Since $M$ admits a distance 2 Heegaard splitting, by Haken’s Lemma, $M$ is irreducible. It is known that every irreducible closed orientable 3-manifold $M$ contains an essential torus or not. In the later case, by Geometrization conjecture, $M$ is either a small Seifert 3-manifold or a hyperbolic 3-manifold.

**Claim 4.4.** $M$ is not a small Seifert 3-manifold.

**Proof.** Suppose not. Then $M$ is a small Seifert 3-manifold. Hence it has $S^2$ as its base surface with at most three exceptional fibers. If $M$ has only one or two exceptional fibers, then $M$ is a Lens space. But all genus at least 2 Heegaard splitting of a Lens space is stabilized, reducible, i.e., they all have distance 0. So $M$ contains three exceptional fibers.

Moriah and Schultens [10] proved that each irreducible Heegaard splitting of $M$ is either vertical or horizontal. For the Heegaard splitting $V \cup_S W$, if it is vertical, then it has genus 2. By Corollary 2.4, there are two essential disks $D_1$ and $D_2$ from two sides of $S$ and two non isotopy disjoint essential simple closed curves $C_1$ and $C_2$ so that both $C_1$ and $C_2$ are disjoint from $\partial D_1$ and $\partial D_2$. But under the condition that $d_{c(S_i)}(\alpha, \beta) \geq 11$, by the proof of Fact 4.1, $C_1$ is isotopic to $C_2$. so it is impossible. Hence it is a horizontal Heegaard splitting.

Recall that for a horizontal Heegaard splitting, $M_1 = F \times I/(x, 0) \sim (\psi(x), 1)$, where $\partial F$ is connected and $\psi | \partial F \times I = Id$, and $M = M_1 \cup B^2 \times S^1$. And $V = F \times [0, \frac{1}{2}]$ (resp. $W$ is homeomorphic to $F \times [\frac{1}{2}, 1]$). By Lemma 2.5, there is an essential simple closed curve $C \in F$ so that $A_1 = C \times [0, \frac{1}{2}]$ and $A_2 = C \times [\frac{1}{2}, 1]$ so that $\partial A_1$ intersects $\partial A_2$ in at most one point.

It is not hard to see that there are a pair of essential disks of two sides of $S$ so that their boundary disjoint from $C \times \{\frac{1}{2}\}$. By the proof of Fact 4.1, $C \times \{\frac{1}{2}\}$ is isotopic to $\gamma$. Let $a$ be an arc in $F$ disjoint from $C$. Then there is an essential disk $D_1 = a \times [0, \frac{1}{2}]$ (resp. $D_2 = a \times [\frac{1}{2}, 1]$) disjoint from $C \times \{\frac{1}{2}\}$. Thus $D_1 \cap A_1 = \emptyset$ (resp. $D_2 \cap A_2 = \emptyset$). Hence

$$d_{c(S_i)}(\partial D_1, \partial D_2) \leq diam_{c(S_i)}(\partial D_1, \partial A_1) + + diam_{c(S_i)}(\partial A_1, \partial A_2) + + diam_{c(S_i)}(\partial D_2, \partial A_2) \leq 1 + 2 + 1 = 4.$$

It contradict the choice of $\gamma$. 

So $M$ is hyperbolic or toroidal. If $M$ is a hyperbolic manifold, then the proof ends. So we assume that $M$ contains an essential torus $T^2$. By Fact 4.1, 4.2 and 4.3

(1) it contains only one essential torus $T^2$ up to isotopy, where it is separating;
(2) each component of $T^2 \cap S$ is isotopic to $\gamma$;
(3) $T^2 \cap V$ (resp. $T^2 \cap W$) splits $V$ (resp.$W$) into a solid torus and a handlebody.
Let $A$ be an annulus bounds by $T^2 \cap S$ in $S$ and $S_A = S - A = S_\gamma$. Let $M_1$ be the amalgamation of these two solid tori along $A$. It is not hard to see that $M_1$ is a small Seifert manifold with a disk as its base surface.

Let $M_2 = M - M_1$. In the manifold $M_2$, $\partial S_A$ consists of two isotopic essential simple closed curves in $\partial M_2 = T^2$. And $S_A$ cuts $M_2$ into two handlebodies. Let $S_2$ be the union of $S_A$ and an annulus $A^*$ bounded by $\partial S_A$ in $\partial M_2$, see Figure 4.

![Figure 4. Heegaard surface $S_2$](image)

After pushing $S_2$ a little into the interior of $M_2$, $S_2$ cuts $M_2$ into a handlebody and a compression body. Then there is a Heegaard splitting $V_2 \cup S_2 W_2$ for $M_2$. Similar to the proof of Fact 3.2, $S_2$ is a strongly irreducible Heegaard surface.

Remember that $S_2$ is also contained in $M$. So

**Fact 4.5.** $S_2$ share the essential subsurface $S_A$ with $S$ in common.

*Proof.* See Figure 4. □

From Figure 4 every essential disk in $V_2$ or $W_2$ with its boundary disjoint from $\partial S_A$ is a compression disks of $S_A$ in $V$ or $W$ respectively.

**Claim 4.6.** $M_2$ is irreducible, boundary irreducible, atoroidal and anannular.

*Proof.* Since $M$ is irreducible and $T^2$ is incompressible, $M_2$ is irreducible and boundary irreducible. By Fact 4.3, $M$ contains only one essential torus $T^2$ up to isotopy. Then $M_2$ is atoroidal. Now suppose $M_2$ contains an essential annulus $A_1$. By Schultens’ Lemma [23], $A_1 \cap S_2$ are all essential simple closed curves in both $A_1$ and $S_2$. After pushing all the boundary parallel annuli to the different side of $S_2$, $A_1 \cap V_2$ (resp. $A_1 \cap W_2$) are essential annuli. We say at least one component $\gamma_1 \subset A_1 \cap S_2$ is not isotopic to $\gamma$. For if not, then there is an I-bundle of $\partial M_2 = T^2$ containing $A_1$ after some isotopy, which means that $A_1$ is inessential. Then there is an essential disk $D_1 \subset V_2$ (resp. $E_1 \subset W_2$) disjoint from $\gamma_1$.

Since $S_2$ cuts $M_2$ into a handlebody and a compression body, let $V_2$ be the handlebody. From Figure 4 $S_2$ is the union of $S_A$ and annulus $A^*$, where $V_2$ is a disk sum of a handlebody and I-bundle of the annulus $A^*$. Then the boundary of each essential disk in $V_2$ intersects $S_A$ nonempty. So $S_A$ is a compressible hole. By a similar argument, $S_A$ is also a compressible hole for $W_2$. Then by the proof of Fact 4.4, there is a pair of essential disks $D \subset V_2$ for $D_1$ and $E \subset W_2$ for $E_1$ so that $\partial D$ and $\partial E$ are both disjoint from $\partial S_A$ and

$$d_C(S_A)(\partial D, \partial E) \leq 10.$$  

Remember that each essential disk in $V_2$ or $W_2$ disjoint from $\partial S_A$ is still an essential disk in $V$ or $W$ respectively and $S_A = S_\gamma$. Then it contradicts the choice of Heegaard splitting $V \cup_S W$. 
By Thurston’s hyperbolic theorem of Haken manifolds, $M_2$ is hyperbolic.

**Remark 4.1.** The main result (Theorem 1.1) of Johnson, Minsky and Moriah’s paper \cite{9} says that for a Heegaard splitting $V \cup_S W$, if there is an essential subsurface $F \subset S$ such that the distance of these two projections of disk complexes $D(V)$ and $D(W)$ into $F$, denoted by $d_F(S)$, satisfies that $d_F(S) > 2g(S) + 2$, then up to an ambient isotopy, any Heegaard splitting of $M$ with genus less than or equal to $g(S)$ has the subsurface $F$ in common. For the Heegaard splitting in Theorem 4.1, if condition (3) is updated into $d_S((\alpha, \beta)) \geq \max\{2g(S) + 3, 11\}$, then any Heegaard splitting $S'$ of it with genus less than or equal to $g(S)$ has $S_1$ in common up to an ambient isotopy. Since $\partial S_1$ bounds no disk in $M$, $S_1 \subset S'$ is essential. By the calculation of the euler characteristic number, $\partial S_1$ bounds an annulus $A$ in $S'$. The proof of Theorem 1.1 implies that $A$ is parallel to an annulus in $S$. So the Heegaard $S$ is the unique minimal Heegaard surface up to isotopy.

5. **Proof of Theorem 1.4**

Let $M, V \cup_S W$ and $\gamma$ be the same as in Theorem 4.1. On one side, since $\gamma$ is the middle vertex of a geodesic realizing Heegaard distance, there are two essential compression disks for $S$ disjoint from $\gamma$ from two sides of $S$. On the other side, as $\gamma$ is incompressible on both of these two sides of $S$, there is an essential subsurface $F \subset S$ containing $\gamma$ so that each essential simple closed curve in $F$ bounds no essential disk on either side of $S$. Then there is a maximal (defined later) essential surface $F$ containing $\gamma$ so that there is no essential, i.e., incompressible and non-peripheral, simple closed curve in $F$ so that it bounds an essential disk on either side of $S$.

It is possible that there are many essential surfaces satisfying the property above. Thus we shall introduce some definitions for distinguishing all those surfaces. We call two subsurface $F_1$ and $F_2$ are same if $F_1$ is isotopic to $F_2$ in $S$. For a collection of different subsurfaces, we define a partial order as follows. For any two essential subsurface $F_1$ and $F_2$ of $S$, $F_1 < F_2$ if $F_1$ can be isotoped into $F_2$ and $-\chi(F_1) < -\chi(F_2)$. Since there is a lowest bound for all Euler characteristic numbers of those subsurface, there is a maximal essential subsurface for any sequence of subsurfaces in order. For convenience, for each one of these maximal essential subsurfaces, we call it a domain of $\gamma$.

Throughout the proof of Theorem 4.1, the case that $M$ contains an essential torus means that (1) two copies of $\gamma$ bounds an essential annulus in both $V$ and $W$, namely, one domain of $\gamma$ is an once punctured torus in $S$; (2) $\gamma$ is not a co-core in either of these two sides of $S$. So to eliminate the possible essential tori in $M$, it is sufficient to add some conditions related to these two cases (1) and (2).

We assemble the above argument as the following proposition.

**Proposition 5.1.** Let $M, V \cup_S W$ and $\gamma$ be the same as in Theorem 4.1. If either each domain of $\gamma$ has the Euler characteristic number less than -1 or $\gamma$ is a co-core for one side of $S$, then $M$ is hyperbolic.

**Proof.** Suppose not. Then $M$ is not hyperbolic. Since $M$ admits a locally large distance two Heegaard splitting, by Theorem 4.1, $M$ contains an essential annulus $T^2$. 

The proof of Theorem 4.1 suggests that $T^2$ intersects $S$ in two copies of $\gamma$. It means that two copies of $\gamma$ bounds an essential annulus $A_1$ (resp. $A_2$) in $V$ (resp. $W$). Then there are two one hole tori of $S$ containing $\gamma$ in its interior from two sides of $S$. Thus for either side of $S$, there is one domain of $\gamma$ with Euler characteristic number equal to -1.

**Claim 5.1.** $\gamma$ is not a co-core for either side of $S$.

**Proof.** Suppose not. Without loss of genericity, $\gamma$ is a co-core of the handlebody $V$. Then there is an essential disk $D$ so that $\partial D \cap \gamma$ in one point. Then $\partial N(\partial D \cup \gamma)$ bounds an essential disk $D_1$, which cuts $V$ into a solid torus $ST$ and a small genus handlebody. Since the annulus $A_1$ is essential in $V$ and $V$ is irreducible, by standard innermost disk surgery, $A \cap D_1 = \emptyset$. Then $A_1 \subset ST$.

As $\gamma$ is a co-core, the disk $D$ intersects $A_1$ in one essential arc. Then there is a boundary compression disk $D_0 \subset D$ for $A_1$ in $ST$ so that after doing a boundary compression along $D_0$, $A_1$ is changed into a trivial disk in $V$. A contradiction. □

Thus these two conclusions contradict the assumption of $\gamma$. □

Moreover, the Proposition 5.1 can be updated into the following theorem, which is the Theorem 1.4.

**Theorem 5.2.** Let $M, V \cup_S W$ and $\gamma$ be the same as in Proposition 5.1. Then $M$ is hyperbolic if and only if either each domain of $\gamma$ has the Euler characteristic number less than -1 or $\gamma$ is a co-core for one side of $S$.

**Proof.** For the forward direction. Suppose that (1) there are two domains $F_1$ and $F_2$ of $\gamma$, where both $F_1$ and $F_2$ are one hole disks and $\partial F_1$ (resp. $\partial F_2$) bounds an essential disk in $V$ (resp. $W$), and (2) $\gamma$ is not a co-core for both sides of $S$. Then $\partial F_1$ (resp. $\partial F_2$) cuts out a solid torus in $V$ (resp. $W$) containing $\gamma$. Let $A$ be closed regular neighborhood of $\gamma$. Since $\gamma$ is not a co-core for either side of $S$, by the standard combinatorial techniques, $\partial A$ bounds two essential annuli $A_1$ and $A_2$ in both of $V$ and $W$ respectively, see Figure 5.

![Figure 5. The Essential Annulus $A_1$](image)

Then $A_1 \cup A_2$ is a torus $T^2$ or a Kleinian bottle $K$. Since $A_1 \cup A_2$ is separating in $M$, it is a torus $T^2$.

**Claim 5.2.** $T^2$ is essential in $M$. 
Proof. By Figure 5, $A_1$ (resp. $A_2$) cuts out a solid torus $ST_1$ (resp. $ST_2$), where both of these two solid tori have the annulus $A$ as their common boundaries surface. Since $\gamma$, the core curve of $A$, is not a co-core of either of these two handlebodies $V$ and $W$, $M_2 = ST_1 \cup_A ST_2$ is a small Seifert space. Since $A$ is incompressible in both $ST_1$ and $ST_2$, $A$ is incompressible in $M_2$. For if $T^2 = \partial M_2$ is compressible in $M_2$, then the compression disk $D$ intersects $A$ nonempty up to isotopy. Otherwise, either $A_1$ or $A_2$ is compressible in $ST_1$ or $ST_2$ respectively. Then there is an outermost disk $D_0 \subset D$ for $A$. Without loss of genericity, we assume that $D_0 \subset ST_1$ in $V$. After doing boundary compression on $A_1$ along $D_0$, $A_1$ is changed into a trivial disk in $V$, which is impossible.

Let $M_1 = M - M_2$. The proof of Fact 3.2 suggests that $T^2$ is incompressible in $M$. So $T^2$ is incompressible in $M$. □

So $M$ contains an essential torus $T^2$. It contradicts the assumption that $M$ is hyperbolic.

For the backward direction. The proof is contained in proof of Proposition 5.1. □

Remark 5.1. The conclusion of Theorem 1.4 says that although these 3-manifolds which admit distance two Heegaard splittings are complicated, we can still get some kind of classification of them suggested by Geometrization Conjecture as we consider all those locally large distance two Heegaard splittings.

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