Invariant measures for typical continuous maps on manifolds

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Abstract
We study the invariant measures of typical $C^0$ maps on compact connected manifolds with or without boundary, and also of typical homeomorphisms. We prove that the weak* closure of the set of ergodic measures coincides with the weak* closure of the set of measures supported on periodic orbits and also coincides with the set of pseudo-physical measures. Furthermore, we show that this set has empty interior in the set of invariant measures.

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1. Introduction

In this article we study the structure of the invariant measures for typical continuous maps of a $C^1$ compact, connected manifold $M$ of finite dimension $m \geq 1$, with or without boundary. This generalizes the work in our previous article [CT] where we studied the case when $M$ is an interval. The study of the invariant measures for typical maps was initiated recently by Abdennur and Andersson [AA]. Previously studies of the dynamics of typical maps have concentrated on the topological properties; see [AHK, AP, H1, H2, KMOP, O, OU, PP, Y] and the references therein.

Let $\mathcal{E}_f$ denote the set of ergodic, $f$-invariant Borel probability measures, $\text{Per}_f$ the set of invariant measures supported on a single periodic orbit, and $\mathcal{O}_f$ denote the set of pseudo-physical
measures for \( f \), (see section 2.1 for the definition). We always have \( \text{Per}_f \subset \mathcal{E}_f \). Let \( \mathcal{C}(M) \) denote the set of continuous maps of \( M \) to itself and \( \mathcal{H}(M) \) denote the set of homeomorphisms of \( M \) to itself. We endow these spaces with the \( C^0 \) topology and say that a family of maps in \( \mathcal{C}(M) \) (resp. in \( \mathcal{H}(M) \)) is typical if it contains a countable intersection of open and dense family. Our main result is the following theorem:

**Theorem 1.** If \( f \) is typical in \( \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)), then

\[
\mathcal{E}_f = \text{Per}_f = \mathcal{O}_f.
\]

The following questions arise from theorem 1. Are all the invariant measures in the closure of the ergodic measures? Are all the invariant measures pseudo-physical? We prove that the answer is negative for \( C^0 \)-typical maps:

**Theorem 2.** If \( f \) is typical in \( \mathcal{C}(M) \) or if \( f \) is typical in \( \mathcal{H}(M) \), then the set \( \mathcal{O}_f \) has empty interior in the set of all \( f \)-invariant measures.

In particular an open and dense set of \( f \)-invariant measures is not in the closure of the ergodic measures. Theorem 2 implies that the typical behaviour of homeomorphisms on manifolds widely differ from the typical behaviour of \( C^1 \) diffeomorphisms. In fact, Gelfert and Kwietniak proved that for \( C^1 \)-typical diffeomorphisms the set of ergodic measures is dense in the space of invariant measures [GK, theorem 8.1].

The proof of theorem 1 is split into several pieces; its main components are theorems 9, 21, 22 and corollary 15. The main components of the proof of theorem 2 are theorems 5 and 12, proposition 8, and lemma 13.

The article is organized as follows. In section 2 we give some background material and prove a technical lemma on the approximation of measures. In section 3 we define the notion of shrinking sets, and we prove an extension of a result of Abdenur and Andersson, as well as some consequences of this result. We prove theorem 2 in section 4. Finally, sections 5 and 6 are dedicated to the proof of theorem 1.

### 2. The set-up

Let \( M \) be a compact, connected, \( C^1 \) manifold of finite dimension \( m \geq 1 \), with or without boundary. Once a Riemannian structure is chosen on \( M \), it defines a volume measure which we will refer to as the Lebesgue measure. It also defines a distance, \( \text{dist}(\cdot, \cdot) \), between points. We denote the \( C^0 \) distance on \( \mathcal{C}(M) \) by

\[
\rho(f, g) := \max_{x \in M} \text{dist}(f(x), g(x)).
\]

This metric makes the set \( \mathcal{C}(M) \) a complete metric space and generates the \( C^0 \) topology on \( \mathcal{C}(M) \). Similarly, \( \mathcal{H}(M) \) is a complete metric space with the \( C^0 \) topology generated by the distance

\[
\rho_H := \max\{\rho(f, g), \rho(f^{-1}, g^{-1})\}.
\]

The results of theorems 1 and 2 hold independently of the choice of the Riemannian structure on \( M \). In fact, the topological and Borel-measurable properties of the system depend on the metrizable topology of the manifold \( M \), but not on the particular choice of its metric dist. The properties related to the pseudo-physical measures depend only on the set of pseudo-physical measures inside the space of all the Borel invariant measures. But this set is preserved if we substitute the previously chosen Lebesgue measure by any finite Borel measure.
equivalent to it (see definition 3). Hence, the results remain unchanged if we change the choice of the volume form.

2.1. Pseudo-physical measures

For any point \( x \in M \) and \( f \in C(M) \), let \( p_\omega(x) \) be the set of the Borel probability measures on \( M \) that are the limits in the weak\(^*\) topology of the convergent subsequences of the sequence

\[
\left\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x} \right\}_{n \in \mathbb{N}}
\]

where \( \delta_y \) is the Dirac probability measure supported in \( y \in M \).

A measure \( \mu \) is called physical if the set of those \( x \in M \) for which \( p_\omega(x) = \{ \mu \} \) has positive Lebesgue measure. Note that we do not require physical measures to be ergodic.

In this article we consider a generalization of the above definition, introduced in [CE1] and studied in the \( C^1 \) case in [CE2, CCE1, CCE2]. We fix a distance, \( d(\cdot, \cdot) \), in the space of probability measures that endows the weak\(^*\) topology. It is easy to check that the following definition does not depend on the choice of this distance (see [CE1]).

**Definition 3.** A probability measure \( \mu \) is called pseudo-physical if for all \( \varepsilon > 0 \) the set

\[
A_\varepsilon(\mu) := \{ x \in M : d(p_\omega(x), \mu) < \varepsilon \}
\]

has positive Lebesgue measure. We denote by \( O_f \) the set of pseudo-physical measures for \( f \).

Note that \( O_f \) is always closed and non-empty, and that any pseudo-physical measure is automatically \( f \)-invariant, and we do not require a pseudo-physical measure to be ergodic. (In [CE1, CE2, CCE1] pseudo-physical measures were called SRB-like.)

2.2. The simplexes

We consider an atlas \( \{ (U_\alpha, \phi_\alpha) \} \) of the manifold \( M \), where \( U_\alpha \) are open sets whose union covers \( M \), and \( \phi_\alpha : U_\alpha \to \mathbb{R}^m \) are \( C^1 \) diffeomorphisms.

An Euclidean \( k \)-simplex (\( 0 \leq k \leq m \)) is the convex hull of a finite set \( \{ x_0, x_1, \ldots, x_k \} \subset \mathbb{R}^m \) such that \( \{ x_j - x_0 : 1 \leq j \leq k \} \) are linearly independent.

A \( k \)-simplex of \( M \) is a nonempty compact set \( T \) contained in a chart \( (U_\alpha, \phi_\alpha) \) such that \( \phi_\alpha(T) \) is an Euclidean \( k \)-simplex of \( \mathbb{R}^m \). Throughout the article \( T \) will denote the nonempty interior of an \( m \)-simplex \( T \).

\( M \) is triangulable if there exists a finite family \( \{ T_1, \ldots, T_t \} \) of \( m \)-simplexes (called a triangulation) such that \( T_i \cap T_j = \emptyset \) if \( i \neq j \) and \( \cup_i T_i = M \).

It is well known that any compact \( C^1 \) manifold is triangulable [W, M]. For any \( \varepsilon > 0 \) there exists a triangulation of \( M \) such that the diameters of all the simplexes \( T_i \) are smaller than \( \varepsilon \). To prove this it suffices to notice that an Euclidean \( m \)-simplex can be decomposed into a finite number of \( m \)-simplexes of arbitrarily small diameter.

Consider a triangulation \( T := T_m := \{ T_1^m, \ldots, T_t^m \} \) of \( M \). Define \( \partial T \) to be the union of the topological boundaries \( \partial T_i \) of the simplexes \( T_i \) of \( T \).

Consider an \( m \)-simplex \( T \) of \( M \) and its associated chart \( (U_\alpha, \phi_\alpha) \). Let \( c \) be a point in the interior of \( T \), which we will call "centroid". Let \( \{ x_0, x_1, \ldots, x_m \} \subset \mathbb{R}^m \) be the vertices of \( \phi_\alpha(T) \). Up to translation of the coordinates, it is not restrictive to assume that \( \phi_\alpha(c) \) of \( \phi_\alpha(T) \) is located at the origin. Suppose \( \lambda > 0 \) is not too large. We denote \( \lambda T \subset U_\alpha \) the new simplex such that \( \phi_\alpha(\lambda T) \) is the Euclidean \( m \)-simplex with vertices \( \{ \lambda x_0, \lambda x_1, \ldots, \lambda x_m \} \subset \phi_\alpha(U_\alpha) \).
and the chosen centroid \( \subset \) is well defined for all \( \lambda \), then there exists \( \varepsilon/\lambda \) is drawn with a solid line, while the simplex \( \lambda \) is drawn with a dotted line. The solid dot is the chosen centroid of the triangle located at the point \( c \in T = \text{int}(T) \subset U_\alpha \).

(see figure 1). Clearly, there exists a \( \lambda_0 > 1 \) (depending on \( T \) and the chosen centroid \( c \)) such that \( \lambda T \) is well defined for all \( \lambda \in (0, \lambda_0) \).

2.3. A technical lemma

In the metrizable space of probability measures endowed with the weak* topology, consider the following distance between probabilities \( \mu, \nu \):

\[
d(\mu, \nu) := \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int_{\Omega} \Psi_i d\mu - \int_{\Omega} \Psi_i d\nu \right|
\]

where \( \{\Psi_i\}_{i=1}^{\infty} \) is a countable dense family in the space \( C^0(\Omega, [0, 1]) \).

We will use the following lemma on the approximation of measures.

**Lemma 4.** For any \( \varepsilon > 0 \), there exists \( q \geq 1 \) such that, if \( \mu \) and \( \nu \) are probability measures of \( M \) satisfying \( \text{supp}(\nu) \cup \text{supp}(\mu) \subset \bigcup_{j=1}^{m} I_j \) for some pairwise disjoint connected closed sets, if \( \text{diam}(I_j) \leq 1/q \), and if \( |\nu(I_j) - \mu(I_j)| \leq 1/\alpha qm \) for each \( j \), then \( d(\mu, \nu) < \varepsilon \).

**Proof.** Fix \( n \geq 1 \) such that \( \sum_{i=n}^{\infty} 2^{-i} < \varepsilon/2 \). Then, for any pair of probability measures \( \mu, \nu \) we obtain

\[
d(\mu, \nu) < \frac{\varepsilon}{2} + \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int_{\Omega} \Psi_i d\mu - \int_{\Omega} \Psi_i d\nu \right|
\]

The uniform continuity of the finite family of functions \( \{\Psi_i\}_{i=1}^{\infty} \) implies that there exists \( \delta > 0 \) such that, if \( \text{dist}(x_1, x_2) < \delta \), then \( \text{dist}(\Psi_i(x_1), \Psi_i(x_2)) < \varepsilon/2 \) for all \( 1 \leq i \leq n \). Fix \( q \in \mathbb{N}^+ \) such that \( 1/q < \min(\delta, \varepsilon/4) \).

The mean value theorem for integrals, yields for all \( 1 \leq i \leq n \)

\[
\int_{\bigcup_{j=1}^{m} I_j} \Psi_i d\mu = \sum_{j=1}^{m} \int_{I_j} \Psi_i d\mu = \sum_{j=1}^{m} \Psi_i(x_j) \mu(I_j) \text{ for some } x_j \in I_j,
\]

\[
\int_{\bigcup_{j=1}^{m} I_j} \Psi_i d\nu = \sum_{j=1}^{m} \int_{I_j} \Psi_i d\nu = \sum_{j=1}^{m} \Psi_i(x'_j) \nu(I_j) \text{ for some } x'_j \in I_j.
\]

From this we deduce

\[
\int_{\bigcup_{j=1}^{m} I_j} \Psi_i d\mu - \int_{\bigcup_{j=1}^{m} I_j} \Psi_i d\nu = \sum_{j=1}^{m} \left( \Psi_i(x_j) \mu(I_j) - \Psi_i(x'_j) \nu(I_j) \right)
\]
Since $|\nu(I_j) - \mu(I_j)| \leq 1/qm$ for each $j$, we conclude

$$\left| \int_M \Psi_i \, d\mu - \int_M \Psi_i \, dv \right| \leq \sum_{j=1}^m \frac{1}{qm} + \frac{\varepsilon}{4} \sum_{j=1}^m \nu(I_j) < \varepsilon/2 \quad \forall 1 \leq i \leq n.$$ 

Substituting this inequality in (1), finishes the proof of lemma 4. \hfill \square

### 3. Shrinking sets

A periodic shrinking set of period $p \geq 1$, is a nonempty open set $I$ whose closure $\overline{I}$ is an $m$-simplex, such that $\{f^j(I)\}_{0 \leq j \leq p-1}$ is a family of pairwise disjoint sets, $f^p(I) \subset I$, and $\text{diam}(f^j(I)) < \text{diam}(I)$ for all $1 \leq j \leq p - 1$.

An nonempty open set $J$, whose closure $\overline{J}$ is an $m$-simplex, is eventually periodic shrinking, if there exists a periodic shrinking set $I$ and a transience time $n \in \mathbb{N}^+$ such that $f^n(J) \subset I$, and $\text{diam}(f^j(J)) < \text{diam}(J)$ for all $1 \leq j \leq n - 1$.

Note that the same periodic or eventually periodic shrinking set for some $f \in \mathcal{C}(M)$ (resp. $\mathcal{H}(M)$) is also a periodic or eventually periodic shrinking set for all $g$ in a small neighborhood of $f$ in $\mathcal{C}(M)$ (resp. $\mathcal{H}(M)$).

If $I$ is a periodic or eventually periodic shrinking set with period $p$, then, by the Brouwer fixed point theorem, the map $P^p$ has a fixed point $x_0 \in I$; hence, the point $x_0$ is periodic for $f$ with period $p$.

The next theorem is a reformulation of [DOT, lemma 5.1] in the case of boundaryless manifolds, it is also a simplified version of the Shredding lemma of [AA]. The proof in [DOT, lemma 5.1] also works for manifolds with boundary, for completeness we will give a brief sketch of this proof.

**Theorem 5.** For a typical map $f \in \mathcal{C}(M)$ (resp. $f \in \mathcal{H}(M)$), Lebesgue a.e. point $x \in M$ belongs to a sequence $\{I_q\}_{q \in \mathbb{N}^+}$ of eventually periodic shrinking sets $I_q$ such that $\text{diam}(I_q) < 1/q$.

**Proof.** For $q, k \in \mathbb{N}^+$, denote by $\mathcal{S}_{q,k}$ the set of maps in $\mathcal{C}(M)$ (resp. $\mathcal{H}(M)$), for which there exists a finite family of nonempty open sets, which we denote by $\{I_1, \ldots, I_l\}$, such that:

(i) $\text{diam}(I_i) < 1/q$ for $i = 1, 2, \ldots, l$,

(ii) $I_i$ is a periodic or eventually periodic shrinking set,

(iii) $\text{Leb}(M \setminus \bigcup_{i=1}^l I_i) < 1/k$.

The set $\mathcal{S}_{q,k}$ is open in $\mathcal{C}(M)$ (resp. $\mathcal{H}(M)$) since, for each $f \in \mathcal{S}_{q,k}$, the same family of such shrinking sets of $f$, is also a family of shrinking sets satisfying conditions (i)--(iii) for any other map $g$ in a sufficiently small neighborhood of $f$.

Besides, in [DOT, lemma 5.1] it is proved that $\mathcal{S}_{q,k}$ is dense in $\mathcal{C}(M)$ (resp. $\mathcal{H}(M)$). This completes the proof of theorem 5, since the set

$$\mathcal{S} := \bigcap_{q \geq 1} \bigcap_{k \geq 1} \mathcal{S}_{q,k}$$

is a dense $G_\delta$-set. \hfill \square
When $M$ is a manifold without boundary, the following result was proven by Abdenur and Andersson in [AA], and it is also a consequence of [DOT, lemma 5.1]. Our proof works also when $M$ is a manifold with boundary.

**Corollary 6.** Let $f$ be a typical map in $C(M)$ (resp. $H(M)$). Then, for Lebesgue almost every point $x \in M$, the sequence

$$\left\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \right\}_{n \in \mathbb{N}^+}$$

of empirical probabilities is convergent.

**Proof.** Consider the families $S_{q,k}$ for $q, k \in \mathbb{N}^+$, defined at the beginning of the proof of theorem 5. We proved that each $S_{q,k}$ is open and dense, so typical $f$ belong to $S := \bigcap_{q,k \geq 1} S_{q,k}$. Thus, it is enough that the assertion of this corollary holds for all $f \in S$.

Consider a continuous function $\phi : M \to \mathbb{R}$. Since $M$ is compact, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that if $\text{dist}(x, y) < \delta$ then $|\phi(x) - \phi(y)| < \varepsilon$. Suppose $f \in S_{q,k}$ with $q$ such that $1/q < \delta$. Due to theorem 5, Lebesgue a.e. belongs to an eventually periodic shrinking set of diameter smaller than $1/q$. It is enough to prove the convergence of the corollary for any iterate of $x$ instead of $x$; thus we can assume that $x \in I_i$, where $I_i$ is a periodic shrinking set. Denote the period of $I_i$ by $p$. Then for $n > p$ we write $n = \ell p + r$ with $0 \leq r < p$. Since $I_i$ is a periodic shrinking set with diameter smaller than $1/q$ we have $\text{dist}(f^n x, f^r x) \leq \text{diam}(f^n I_i) \leq \text{diam}(I_i) < 1/q < \delta$.

Thus

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j x) - \frac{1}{p} \sum_{j=0}^{p-1} \phi(f^j x) \right| < \varepsilon. \quad (2)$$

Then

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j x) - \frac{1}{p} \sum_{j=0}^{p-1} \phi(f^j x) \right| \leq \left| \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j x) - \frac{1}{\ell p} \sum_{j=0}^{\ell p-1} \phi(f^j x) \right| + \left| \frac{1}{\ell p} \sum_{j=0}^{\ell p-1} \phi(f^j x) - \frac{1}{p} \sum_{j=0}^{p-1} \phi(f^j x) \right|. \quad (3)$$

Using (2) we see that second term is bounded by $\varepsilon/p < \varepsilon$. Using the triangle inequality once more we can bound the first term by

$$\left| \frac{1}{n} \sum_{j=\ell p}^{n-1} \phi(f^j x) \right| + \left| \frac{r}{n \ell p} \sum_{j=0}^{\ell p-1} \phi(f^j x) \right|. \quad (3)$$

Since $r$ is bounded, for $n$ sufficiently large this is bounded by $\varepsilon$, and the difference in (3) is bounded by $2\varepsilon$.

We deduce that...
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) - \liminf_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) < 4\varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, the result follows.

A map \( f \in C(M) \) is \textit{Lebesgue-a.e. strongly non positively expansive}, if for any real number \( \alpha > 0 \), and for Lebesgue a.e. \( x \in M \),

\[ \text{Leb}\left( \{ y \in M : \text{dist}(f^n(x), f^n(y)) < \alpha \ \forall \ n \in \mathbb{N} \} \right) > 0. \]

In the terminology of [AM2, definition 2.1], this property is referred to as \textit{the Lebesgue measure not being positively expansive} (see also [AM1]).

Moreover, using the notation of [AM2, section 2]:

\[ \Phi_{\alpha}(x) := \{ y \in M : \text{dist}(f^n(y), f^n(x)) < \alpha \}. \]

Namely, the positive expansivity of the Lebesgue measure fails not only in some points, but Lebesgue a.e..

**Corollary 7.** Typical maps in \( C(M) \) (resp. \( f \in H(M) \)) are Lebesgue-a.e. strongly non positively expansive.

**Proof.** Take any periodic or eventually periodic set \( I \) with diameter smaller than \( \alpha \). Then, \( \text{diam}(f^j(I)) < \alpha \) for all \( j \geq 0 \). Any two points \( x, y \in I \) satisfy \( \text{dist}(f^n(x), f^n(y)) < \alpha \) for all \( j \geq 0 \). Thus for any point \( x \in I \) we have

\[ \text{Leb}\left( \{ y \in M : \text{dist}(f^n(x), f^n(y)) < \alpha \ \forall \ n \in \mathbb{N} \} \right) \geq \text{Leb}(I) > 0. \]

Thus the result follows directly from theorem 5.

An \( f \in C(M) \) is called \textit{positively expansive} if there exists a constant \( \alpha > 0 \), called the \textit{expansivity constant}, such that, for any two points \( x, y \in M \), if \( \text{dist}(f^n(x), f^n(y)) \leq \alpha \) for all \( n \in \mathbb{N} \), then \( x = y \). Clearly a map which is Lebesgue a.e. strongly non positively expansive is not positively expansive. For compact connected manifolds with boundary it is known that every map is not positively expansive (theorem 2.2.19, [AH]). On the other hand, if \( M \) has no boundary, then we have examples of positively expansive maps: toral endomorphisms all of whose eigenvalues have absolute value strictly greater than one.

### 3.1. The sets \( AA \) and \( AA_1 \)

For any \( x \in M \) we define

\[ \mu_x := \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \quad (4) \]

when this limit exists; namely, when \( p_\omega(x) = \{ \mu_x \} \). Let

\[ AA := \{ x \in M : p_\omega(x) = \{ \mu_x \} \} \] and \( AA_1 := \{ x \in AA : \mu_x \in C_f \} \quad (5) \]
For typical continuous maps or homeomorphisms, corollary 6 states
\[ \text{Leb}(AA) = \text{Leb}(M), \] (6)
while [CE1, theorem 1.5] implies that
\[ \text{Leb}(AA_1) = \text{Leb}(M). \] (7)

**Proposition 8.** If \( f \) is typical in \( \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)) then
\[ O_f = \{ \mu_x : x \in AA_1 \}. \]
The set \( \{ x \in AA_1 : d(\mu_x, \mu_{x_0}) < \varepsilon \} \) has positive Lebesgue measure for every \( \varepsilon > 0 \) and for every \( x_0 \in AA_1 \).

**Proof.** By definition of the set \( AA_1 \) we have \( \{ \mu_x : x \in AA_1 \} \subset O_f \). Besides, since \( O_f \) is closed in the weak* topology (see [CE1, theorem 1.3]), we have \( \{ \mu_x : x \in AA_1 \} \subset O_f \).

We turn to the other inclusion. Suppose \( \mu \in O_f \), i.e. \( \text{Leb}\{ x \in M : d(p_\omega(x), \mu) < \varepsilon \} > 0 \) for all \( \varepsilon > 0 \).

Since \( \text{Leb}(AA_1) = 1 \) and \( p_\omega(x) = \mu_x \) for \( x \in AA_1 \), this is equivalent to
\[ \text{Leb}\{ x \in AA_1 : d(\mu_x, \mu) < \varepsilon \} > 0 \] for all \( \varepsilon > 0 \).

Thus \( \mu \in \{ \mu_x : x \in AA_1 \} \). Since \( \mu \in O_f \) is arbitrary we conclude \( O_f \subset \{ \mu_x : x \in AA_1 \} \). The first assertion of proposition 8 is proven.

To prove the second assertion recall the definition of pseudo-physical measure, \( \mu \in O_f \) if the set \( \{ x \in M : d(p_\omega(x), \mu) < \varepsilon \} \) has positive Lebesgue measure. Combining this with equality (7), we deduce that \( \{ x \in AA_1 : d(\mu_x, \mu) < \varepsilon \} \) has positive Lebesgue measure, for any \( \mu \in O_f \), in particular for \( \mu_{x_0} \) for any \( x_0 \in AA_1 \). □

### 3.2. Approximation of pseudo-physical measures by periodic measures

For a typical map, each pseudo-physical measure can be arbitrarily approximated by atomic measures supported on periodic points, more precisely

**Theorem 9.** Typical maps \( f \in \mathcal{C}(M) \) (resp. \( f \in \mathcal{H}(M) \)), satisfy \( O_f \subset \text{Per}_f \).

To prove this theorem we need the following lemma.

**Lemma 10.** Let \( f \in \mathcal{C}(M) \) (resp. \( f \in \mathcal{H}(M) \)) and \( \mu \) be an \( f \)-invariant measure. Assume that \( I \subset M \) is a periodic shrinking set of period \( p \), and denote \( K := \bigcup_{j=0}^{p-1} f^j(I) \). Then
\[ \mu(f^j(T)) = \mu(T) = \frac{\mu(K)}{p} \quad \forall j \geq 0. \] (8)

If \( \mu \) is additionally ergodic, then \( \mu(K) \) is either zero or one.

**Proof.** For any \( j \geq 0 \) let \( j = kp - i \) with \( k \in \mathbb{N}^+ \) and \( 1 \leq i \leq p \). Using that \( \mu \) is \( f \)-invariant, that \( f^p(T) \subset I \) and that \( f^{-j}(f^j(T)) \supset T \) for all \( j \geq 0 \), we obtain:
\[ \mu(f^j(T)) = \mu(f^{-j}(f^j(T))) \geq \mu(T) \geq \mu(f^k\mu(T)) = \mu(f^{-j}(f^k\mu(T))) \geq \mu(f^j(T)). \]

Hence, all the inequalities above are equalities; and thus \( \mu(f^j(T)) = \mu(T) \) for all \( j \geq 0 \). Assertion (8) follows, since \( \mu(K) = \sum_{j=0}^{p-1} \mu(f^j(T)) \).

Finally, since \( f(K) \subset K \), we have \( K \subset f^{-1}(f(K)) \subset f^{-1}(K) \). Since \( \mu \) is ergodic, the set \( K \) must have \( \mu \)-measure equal to zero or one. \( \square \)

**Proof of theorem 9.** Fix a typical \( f \in C(M) \) (or \( \mathcal{H}(M) \)). Choose \( q_n \geq 1 \) such that any two measures satisfying the \( q_n \)-approach conditions of lemma 4, are mutually at distance smaller than \( 1/n \). Let \( \mu \in O_f \).

By the definition of pseudo-physical measure, the set \( A_n \subset M \) of points \( y \) such that \( d(p\omega(y), \mu) < 1/n \), has positive Lebesgue measure. Since \( f \) is typical in \( C(M) \) or \( \mathcal{H}(M) \), we have Lebesgue a.e. \( y_n \in A_n \) satisfy \( p\omega(y_n) = \{ \mu_{y_n} \} \) (corollary 6) and is contained in an arbitrarily small eventually periodic shrinking set (theorem 5). Therefore,

\[ d(\mu_{y_n}, \mu) < 1/n, \]

and \( \mu_{y_n} \) is supported on \( K_n := \bigcup_{i=1}^{p_n} f^i(T_n) \), where \( T_n \) is a periodic shrinking set of period \( p_n \), such that \( \text{diam}(T_n) \leq 1/q_n \). Applying lemma 10 and the definition of shrinking set yields \( \mu_{y_n}(f(T_n)) = 1/p_n \).

Since the set \( T_n \) is shrinking periodic with period \( p_n \), there exists at least one periodic orbit of period \( p_n \) in \( K_n \). Consider the periodic invariant measure \( \nu_n \) supported on this periodic orbit. It also satisfies \( \nu_n(f(T)) = 1/p_n \). So, applying lemma 4 we deduce that

\[ \exists \nu_n \in \text{Per}_f: d(\mu_{y_n}, \nu_n) \leq 1/n. \]

Combining inequalities (9) and (10), we deduce that there exists sequence of periodic atomic measures \( \nu_n \) such that \( d(\nu_n, \mu) < 2/n \), finishing the proof of theorem 9. \( \square \)

**4. Most invariant measures are not pseudo-physical**

In this section we will prove theorem 2. We start with an extension of theorem of Abdenur and Andersson in [AA, theorem 3.6]:

**Theorem 11 (Extension of Abdenur–Andersson theorem).** For typical maps in \( C(M) \) there does not exist physical measures.

**Proof.** In [AA, theorem 3.6], Abdenur and Andersson prove this assertion in the case that \( M \) is a compact connected \( C^1 \) manifold without boundary. Now, let us prove that it also holds when \( M \) has boundary. Given a manifold with boundary \( M \), take two disjoint copies \( M_1, M_2 \) and identify each boundary point in \( M_1 \) with the same point in \( M_2 \). We obtain the double cover \( D(M) \) of \( M \) which is a \( C^1 \) manifold without boundary with the same dimension as \( M \). See for example 9.32 in [L] for details of this construction.
Suppose that \( M \) has boundary, and denote by \( V \) the family of all the continuous map \( f \in C(M) \) such that \( f(M) \subset M \setminus \partial M \). By construction \( V \) is open and dense in \( C(M) \). For each \( f \in V \), construct the nonempty set \( S_f \subset C(D(M)) \) composed by all the continuous maps \( g : D(M) \mapsto D(M) \) such that \( g|_{M_1} = f \).

We claim that for any open set \( U \subset V \) the set
\[
U' := \bigcup_{f \in U} S_f = \left\{ g \in C(D(M)) : g|_{M_1} = f \in U \right\}
\]
is open in \( C(D(M)) \); in particular \( V' \) is open in \( C(D(M)) \). This claim follows from that fact that the restriction that sends each map \( g \in C(D(M)) \) to the map \( g|_{M_1} : M_1 \mapsto g(M_1) \subset M \) is a continuous operator. Therefore, the preimage \( U' \) by that restriction, of the open family \( U \subset V \subset C(M) \subset C(M, D(M)) \), is open.

Next we claim that a converse-like statement is true: for any open set \( U' \subset V' \subset C(D(M)) \) the set
\[
U := \left\{ f \in V : \exists g \in U' \subset C(D(M)) \text{ such that } g|_{M_1} = f \right\}
\]
is open in \( C(M) \). In fact, since \( U' \subset V' \), the restriction \( g|_{M_1} \) for each \( g \in U' \), belongs to \( V' \). Besides, the restriction that sends each map \( g \in V' \) to the map \( f := g|_{M_1} \in V \) is an open operator. Therefore, the image \( U \) by that restriction, of the open family \( U' \), is open.

Since theorem 11 holds for typical maps \( g \in C(D(M)) \), and \( V' \) is nonempty and open in \( C(D(M)) \), it also holds for typical maps \( g \in V' \). Namely, there exists a sequence \( \{U'_n\}_{n \geq 1} \) of open and dense subsets of \( V' \) such that assertions (a) and (b) hold for all \( g \in \bigcap_{n \geq 1} U'_n \subset V' \).

Consider the preimages \( U_n \subset V \) defined in (11), of the sets \( U_n \). Since \( U_n \) is open and dense in \( V' \), the above assertions imply that \( U_n \) is open and dense in \( V \); hence also in \( C(M) \).

If theorem 11 holds for a map \( g \in V' \subset C(D(M)) \), then it also holds for the restriction \( g|_{M_1} \in V \subset C(M) \). We conclude that it holds for the countable intersection \( \bigcap_{n \geq 1} U_n \) of open and dense sets in \( C(M) \). In other words, theorem 11 holds for typical maps in \( C(M) \).

As pointed out by an anonymous referee, the Shredding lemma in [AA] implies the density of the property of having at least two shrinking sets (or two disjoint trapping regions, as they are called in [AA]). The proof of the Shredding lemma in [AA], even if stated for boundaryless manifolds, works as well for manifolds with boundary. Thus we have

**Remark 12.** There is an open and dense set of maps in \( C(M) \) (resp. \( H(M) \)) which are not uniquely ergodic.

Now, we state the main lemma to be used to prove theorem 2.

**Lemma 13.** Suppose that \( f \) is a typical map in \( C(M) \) (resp. \( H(M) \)), \( \mu_1 \) is \( f \)-invariant and supported on \( K = \bigcup_{i=0}^{p-1} f^i(I) \), where \( I \) is a periodic shrinking set of period \( p \), and that \( \mu_2 \) is \( f \)-invariant such that \( \mu_2(K) = 0 \). Then, no convex combination \( \nu = \lambda\mu_1 + (1 - \lambda)\mu_2 \) with \( 0 < \lambda < 1 \) is pseudo-physical.

**Proof.** Since \( I \) is a shrinking periodic set with period \( p \), it is is open, \( I \) is an \( m \)-simplex, and \( f^p(I) \subset I \). Consider the continuous real function \( \psi : M \to [0, 1] \) defined by
\[ \psi(x) := \frac{\text{dist}(x, M \setminus I)}{\text{dist}(x, f^p(T)) + \text{dist}(x, M \setminus I)}. \]  

(12)

It satisfies \( \psi|_{f^n(T)} = 1, 0 < \psi(x) < 1 \) if \( x \in I \setminus f^p(T) \), and \( \psi(x) = 0 \) if \( x \notin I \).

Since \( \mu_1 \) is supported on \( K = \bigcup_{j=0}^{p-1} f^j(I) \), we can apply lemma 10 to deduce that

\[ \mu_1(T) = \mu_1(f^p(T)) = \frac{1}{p}; \quad \text{hence} \mu_1(I \setminus f^p(T)) = 0. \]

We obtain \( \int \psi \, d\mu_1 = \mu_1(f^p(T)) = 1/p \) and \( \int \psi \, d\mu_2 = \mu_2(I) = 0 \). Therefore

\[ 0 < \int \psi \, d\nu = \lambda/p < 1/p. \]

Choose \( \varepsilon > 0 \) such that for any measure \( \mu \), if \( d(\nu, \mu) < \varepsilon \), then \( \int \psi \, d\mu > 0 \) and \( \int \psi \, d\mu < 1/p \).

Consider the set

\[ A_\varepsilon^\prime(\nu) := \{ x \in AA : \text{dist} (\mu_x, \nu) < \varepsilon \}, \]

where the set \( AA \) is defined in equality (5). We claim that the set \( A_\varepsilon^\prime(\nu) \) is empty. Arguing by contradiction, assume that there exists \( x \in A_\varepsilon^\prime(\nu) \). Then, from the choice of \( \varepsilon \), we have

\[ 0 < \int \psi \, d\mu_x < 1/p, \]

and from (4) we deduce that there exists \( n_0 \geq 1 \) such that

\[ \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x)) = \int \psi \left( \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \right) > 0 \quad \forall \ n \geq n_0. \]

Thus, there exists \( n_1 \geq 1 \) such that \( \psi(f^{n_1}(x)) > 0 \); hence \( f^{n_1}(x) \in I \). This implies that the future orbit of \( f^n x \) is contained in \( K \). Hence, \( \mu_x \) is supported on \( K \). Since \( \mu_x \) is \( f \)-invariant, we apply lemma 10 to deduce that \( \mu_x(f^p(T)) = 1/p \) and \( \mu_x(I \setminus f^p(T)) = 0 \). Therefore,

\[ \int \psi \, d\mu_x = \mu_x(f^p(T)) = \frac{1}{p}, \]

contradicting (13). We have proved that \( A_\varepsilon^\prime(\nu) \) is empty.

The definition of pseudo-physical measure \( \mu \) asserts that

\[ \text{Leb}(A_{\varepsilon}(\mu)) > 0 \quad \forall \ \varepsilon > 0. \]

But (6) implies that \( \text{Leb}(A_{\varepsilon}^\prime(\nu)) = \text{Leb}(A_{\varepsilon}(\nu)) \), which equals zero because \( A_{\varepsilon}^\prime(\nu) = \emptyset \). We conclude that \( \nu \) is not pseudo-physical. \( \square \)

4.1. End of the proof of theorem 2

Proof. From [CE1, theorem 1.3] the set \( O_f \) is closed. Let us prove that its interior in \( \mathcal{M}_f \) is empty. Fix \( \mu \in O_f \). Since a typical map is not uniquely ergodic (recall remark 12), there exists an ergodic measure \( \nu \neq \mu \). Denote by \( \varepsilon := d(\mu_x, \nu) > 0 \). Consider an arbitrary \( \delta \in (0, \varepsilon/2) \). 

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From proposition 8, we can find \( x_0 \in \mathcal{A}_1 \) such that \( d(\mu_n, \mu) < \delta \). So, to prove that \( \mu \) does not belong to the interior of \( \mathcal{O}_f \) in \( \mathcal{M}_f \), it is enough to prove that the \( \delta \)-neighborhood of \( \mu_n \) in \( \mathcal{M}_f \) is not contained in \( \mathcal{O}_f \). To do that, it is enough to find a sequence \( \{\mu_n\}_n \subset \mathcal{M}_f \) converging to \( \mu_n \) and such that \( \mu_n \not\in \mathcal{O}_f \).

Applying theorem 5 and equality (7), and taking into account the second part of proposition 8, the point \( x_0 \) could be chosen to belong to arbitrarily small eventually periodic shrinking sets. Therefore, for any fixed \( q \in \mathbb{N}^+ \) (that will be chosen later), the probability measure \( \mu_n \) is supported on the orbit of a periodic shrinking set \( I_q \) of diameter smaller than \( 1/q \). Denote by \( p \) the period of \( I_q \), and denote

\[
K = \bigcup_{j=0}^{p-1} f^j(I_q).
\]

We have \( \mu_n(K) = 1 \). So, applying lemma 10 we deduce

\[
\mu_n(f^j(I)) = 1/p, \quad 0 \leq j \leq p - 1.
\]

Applying lemma 4, construct \( q \geq 1 \) such that if \( \mu \) is a probability measure satisfying \( \mu(f^j(I)) = \mu_n(f^j(I)) = 1/p \) with \( \text{diam}(f^j(I)) < 1/q \) for all \( 0 \leq j \leq p - 1 \), then \( d(\mu, \mu_n) < \delta \). Since \( d(\mu, \mu_n) < \epsilon > \delta \), we deduce that \( \nu(f^j(I)) \neq 1/p \) for some \( j \). But applying lemma 10 \( \nu(f^j(I)) = \nu(K)/p \) for all \( \nu(K) \in \{0, 1\} \). Since \( \nu \) is ergodic, we conclude that \( \nu(K) = 0 \). Therefore, applying lemma 13, \( \mu_n := \lambda_n \mu_x + (1 - \lambda_n)\nu \in \mathcal{M}_f \setminus \mathcal{O}_f \) for all \( 0 < \lambda_n < 1 \). Taking \( \lambda_n \to 1^- \), we obtain \( \mu_n \to \mu_x \), with \( \mu_n \in \mathcal{M}_f \setminus \mathcal{O}_f \), as wanted.

\[\square\]

5. Pseudo-physical, ergodic and periodic measures

A \( \delta \)-pseudo-orbit of \( f \) is a sequence \( \{y_n\}_{n \in \mathbb{N}} \subset \mathcal{M} \) such that

\[
\text{dist}(f(y_n), y_{n+1}) < \delta \quad \forall \ n \in \mathbb{N}.
\]

A \( \delta \)-pseudo-orbit \( \{y_n\}_{n \in \mathbb{N}} \) is periodic with period \( p \geq 1 \), if

\[
y_{n+p} = y_n \quad \forall \ n \in \mathbb{N}.
\]

A map \( f \in C(\mathcal{M}) \) has the periodic shadowing property if for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that, if \( \{y_n\}_{n \in \mathbb{N}} \) is any periodic \( \delta \)-pseudo-orbit, then, at least one periodic orbit \( \{f^n(y)\}_{n \in \mathbb{N}} \) satisfies

\[
\text{dist}(f^n(x), y_n) < \epsilon \quad \forall \ n \in \mathbb{N}.
\]

We will also use the following result, which is the accumulation of many authors’ work.

**Theorem 14.** Typical maps \( f \in C(\mathcal{M}) \) (resp. \( \mathcal{H}(\mathcal{M}) \)) have the periodic shadowing property.

**Proof.** The statement follows from [KMOP, theorem 1.2] and [PP, theorem 1]. Earlier special cases where treated in [Y] and [O]. \[\square\]

Recall that \( \mathcal{E}_f \) denotes the set of ergodic measures, and \( \text{Per}_f \) denotes the set of invariant measures supported on periodic orbits of \( f \). As a consequence of theorem 14:

**Corollary 15.** For a typical map \( f \in C(\mathcal{M}) \) (resp. \( f \in \mathcal{H}(\mathcal{M}) \)) we have \( \mathcal{E}_f \subset \overline{\text{Per}_f} \).
Remark 16. The closing lemma and the closeability properties (see for instance [CS, definition 2.1] and [GK, definitions 4.1 and 4.5]), imply that for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that, for a finite piece of orbit \( \{f^i(y)\}_{0 \leq i \leq p} \) verifying \( \text{dist}(f^p(y), y) < \delta \), there exists a periodic point \( x \), such that \( \text{dist}(f^i(y), f^i(x)) < \epsilon \) for all \( 0 \leq j \leq p \). For typical \( f \), the latter condition is a consequence of the periodic shadowing property stated in theorem 14, and it implies (using well known arguments) that \( E_f \subset F_{\text{Per}_f} \). This is the proof of corollary 15. For a seek of completeness we include here the details:

Proof of corollary 15. From the definition of distance in the weak* topology of the space of probability measures it is standard to check that for all \( \epsilon_0 > 0 \), there exists \( \epsilon > 0 \), such that, for any two points \( x_1, x_2 \in M \),

\[
\text{dist}(x_1, x_2) < \epsilon \; \Rightarrow \; d(\delta_{x_1}, \delta_{x_2}) < \epsilon_0.
\]

Fix any \( \mu \in E_f \). Since \( \mu \) is ergodic, we have \( p\omega(x) = \{\mu\} \) for \( \mu \)-a.e. \( x \in M \). Fix such a point \( x \); then there exists \( n_0 \geq 1 \) such that

\[
d \left( \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}, \mu \right) < \epsilon_0 \quad \forall \; n \geq n_0. \tag{14}
\]

Given \( \epsilon \), choose \( \delta > 0 \) given by theorem 14. By Poincaré Recurrence lemma, the point \( x \) can be chosen to be recurrent. Thus

\[
d(f^{p-1}(x), x) < \delta \text{ for some } p \geq n_0.
\]

Construct the periodic \( \delta \)-pseudo-orbit \( \{y_n\}_{n \in \mathbb{N}} \) of period \( p \) defined by \( y_n = f^n(x) \) for all \( 0 \leq n < p \), \( y_{n+p} = y_n \) for all \( n \geq 0 \). Applying theorem 14, there exists a periodic orbit \( \{f^n(z)\}_{n \geq 0} \) such that \( \text{dist}(f^n(z), y_n) < \epsilon \) \( \forall \; n \geq 0 \). By construction, if \( ip \leq n < (i+1)p \) and \( i \geq 0 \) then \( \text{dist}(f^n(z), f^{n-yp}(x)) < \epsilon \). Thus, from the choice of \( \epsilon \), we obtain \( d(\delta_{f^n(z)}, \delta_{f^{n-yp}(x)}) < \epsilon_0 \). Denote by \( q \) the period of \( z \). Taking into account that balls are convex in the weak*-distance in the space of probabilities, we deduce

\[
d \left( \frac{1}{qp} \sum_{j=0}^{qp-1} \delta_{f^{jz}(z)}, \frac{1}{qp} \sum_{j=0}^{qp-1} \delta_{f^{jz}(z)} \right) < \epsilon_0.
\]

For the atomic invariant measure \( \nu \) supported on the periodic orbit of \( z \), we have

\[
\nu = \frac{1}{q} \sum_{j=0}^{q-1} \delta_{f^{jz}(z)} = \frac{1}{qp} \sum_{j=0}^{qp-1} \delta_{f^{jz}(z)}.
\]

Thus,

\[
d \left( \nu, \frac{1}{p} \sum_{j=0}^{p-1} \delta_{f^{jz}(z)} \right) < \epsilon_0.
\]

Together with (14), this implies that for any \( \epsilon_0 \) the given ergodic measure \( \mu \) is \( 2\epsilon_0 \)-approxi-
mated by some measure \( \nu \in \text{Per}_f \), finishing the proof of corollary 15.

An invariant measure \( \mu \) is called infinitely shrinked if there exists a sequence \( \{I_q\}_{q \geq 0} \) of periodic shrinking intervals \( I_q \) of periods \( p_q \) such that \( \text{diam}(I_q) < 1/q \) and \( \mu(\bigcup_{j=1}^{p_q} f^j(I_q)) = 1 \) for all \( q \gg 1 \). We denote by \( \text{Shr}_f \subset \mathcal{M}_f \) the set of infinitely shrinked invariant measures.

From theorem 5, Lebesgue-a.e. \( x \in M \) belongs to a sequence of eventually periodic or periodic shrinkage sets \( J_q \) with \( \text{diam}(J_q) < 1/q \). Every eventually periodic shrinkage set \( J_q \) wavers under \( f \) until it drops into a periodic shrinking set \( I_q \) with \( \text{diam}(I_q) < \text{diam}(J_q) < 1/q \). By the definition of periodic shrinking set, every point of \( I_q \) has all the measures of \( p_\omega(x) \) supported on the compact set
\[
K_{\omega q} := \bigcup_{j=0}^{p_q-1} f^j(\bar{I}_q).
\]

In particular for Lebesgue almost all \( x \in AA_1 \), the limit measure \( \mu_x \) defined by \( (4) \), is supported on \( K_{\omega q} \). Thus, for a.e. \( x \in AA_1 \) we have \( \mu_x \in \text{Shr}_f \). Taking into account \( (7) \), this implies that the set
\[
AA_2 := \{ x \in AA_1 : \mu_x \in \text{Shr}_f \}
\]
has full Lebesgue measure.

**Proposition 17.** For a typical map \( f \in \mathcal{C}(M) \) (resp. \( f \in \mathcal{H}(M) \)),
\[
O_f = \{ \mu_x : x \in AA_2 \} = \text{Shr}_f.
\]

**Proof.** By construction, \( AA_2 \subset AA_1 \). So, applying proposition 8, we obtain:
\[
\{ \mu_x : x \in AA_2 \} \subset \{ \mu_x : x \in AA_1 \} = O_f.
\]

To obtain the opposite inclusion, we apply [CE1, theorem 1.5]): \( O_f \) is the minimal weak\(^*\)-compact set of probability measures, that contains \( p_\omega(x) \) for Lebesgue a.e. \( x \). Since \( \{ \mu_x : x \in AA_2 \} \) is weak\(^*\)-compact and contains \( p_\omega(x) \) for Lebesgue almost all \( x \) (because \( \text{Leb}(AA_2) = \text{Leb}(M) \)), we conclude that
\[
\{ \mu_x : x \in AA_2 \} \supset O_f.
\]

The inclusion \( \{ \mu_x : x \in AA_2 \} \subset \text{Shr}_f \) follows trivially from the definition of the set \( AA_2 \). Now, let us prove the opposite inclusion. We will prove that every shrinking measure is pseudo-physical. Let \( \mu \in \text{Shr}_f \). For any \( \varepsilon > 0 \), choose \( q \gg 1 \) as in lemma 4. By the definition of shrinking measure, there exists a periodic shrinking set \( I_\varepsilon \) such that \( \mu \) is supported on
\[
K_{\mu} := \bigcup_{j=0}^{p_\varepsilon-1} f^j(\overline{I_\varepsilon})
\]
for the periodic shrinking set \( I_\varepsilon \) of period \( p_\varepsilon \), such that \( \text{diam}(\overline{I_\varepsilon}) < 1/q' \); hence \( \text{diam}(f^j(\overline{I_\varepsilon})) < 1/q' \implies 1 \leq j \leq p_\varepsilon, \mu(K_{\mu}) = 1 \).

Besides, for any point \( x \in I_\varepsilon \), any measure in \( p_\omega(x) \) is also supported on \( K_{\mu} \). If additionally \( x \in AA_1 \), then \( p_\omega(x) = \{ \mu_x \} \), so \( \mu_x(K_{\mu}) = 1 \). Finally, applying lemmas 4 and 10, we deduce that the measures \( \mu \in \text{Shr}_f \) and \( \mu_x \) given above satisfy
\[ d(\mu, \mu_x) < \varepsilon \] for any \( x \in I_q \cap AA_1 \).

Since \( \text{Leb}(I'_q \cap AA_1) = \text{Leb}(I'_q) > 0 \), the basin \( A_c(\mu) \) has positive Lebesgue measure; namely \( \mu \) is pseudo-physical.

We have shown that every shrinking measure is pseudo-physical. Since the set \( O_f \) of pseudo-physical measures is closed, we conclude
\[ \text{Shr}_f \subset O_f, \]
finishing the proof of proposition 17.

**Theorem 18.** For any map \( f \in C(M) \) (resp. \( f \in H(M) \)), if \( \mu \in \text{Shr}_f \), then it is ergodic.

Before proving theorem 18 let us deduce its main consequence:

**Corollary 19.** For a typical map \( f \in C(M) \) (resp. \( f \in H(M) \)):
\[ O_f = \text{Shr}_f \subset \bigcap \overline{\text{Per}}_f. \]

The corollary immediately follows by combining corollary 15 with proposition 17 and theorem 18. At the end of the next section we will prove that for typical maps these sets are all equal.

**Proof of theorem 18.** Fix \( f \in C(M) \). Suppose \( \mu \in \text{Shr}_f \), and \( \mu_1, \mu_2 \in M_f \) such that
\[ \mu = \lambda \mu_1 + (1 - \lambda) \mu_2, \quad \text{with} \quad 0 < \lambda < 1. \]
(15)

We shall prove that \( \mu_1 = \mu_2 = \mu \); namely \( \mu \) is extremal in the convex compact set of invariant measures; hence ergodic.

Take arbitrary \( \varepsilon > 0 \) and fix \( q \geq 1 \) as in lemma 4. By the definition of infinitely shrinking measures, there exists a periodic shrinking set \( I_q \), with \( \text{diam}(I_q) < 1/q \), and period \( p_q \), whose \( f \)-orbit \( K_q \) supports \( \mu \). The definition of periodic shrinking set and lemma 10 tell us:
\[ \mu(f^j(I_q)) = \frac{1}{p_q}, \quad \text{diam}(f^j(I_q)) < 1/q \quad \forall 1 \leq j \leq p_q. \]

Since \( \mu(K_q) = 1 \), from (15) we deduce \( \mu_1(K_q) = \mu_2(K_q) = 1 \). Applying lemma 10 we obtain
\[ \mu_1(f^j(I_q)) = \mu_2(f^j(I_q)) = \frac{1}{p_q} \quad \forall 1 \leq j \leq p_q. \]

So, lemma 4 implies \( d(\mu_1, \mu) < \varepsilon \), and \( d(\mu_2, \mu) < \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, we conclude that \( \mu = \mu_1 = \mu_2 \); hence \( \mu \) is ergodic.

**6. All ergodic measures are pseudo-physical**

**Definition 20.** Let \( q \geq 1 \) and \( x_0 \) be a periodic point with period \( r \geq 1 \). We call the (invariant) measure
\[ \nu = \frac{1}{r} \sum_{j=0}^{r-1} \delta_{f^j(x_0)} \in \text{Per}_f \]

a $q$-shrunk periodic measure, if there exists some periodic shrinking set $I$, with diameter smaller than $1/q$, with period $p \geq 1$ such that $\nu$ is supported on $K := \bigcup_{j=0}^{p-1} f(I)$. From the definition of periodic shrinking set, the period $p$ must divide $r$. We denote by $\text{Shr}_q\text{Per}_f$ the set of $q$-shrunk periodic measures.

We say that an invariant measure $\mu$, is $\varepsilon$-approached by $q$-shrunk periodic measures if there exists $\nu \in \text{Shr}_q\text{Per}_f$ such that $d(\mu, \nu) < \varepsilon$. We denote by $A\text{Shr}_{\varepsilon,q}\text{Per}_f$ the set of measures that are $\varepsilon$-approached by $q$-shrunk periodic measures.

**Theorem 21.** For any map $f \in C(M)$ (resp. $f \in \mathcal{H}(M)$)

\[ \bigcap_{\varepsilon > 0} \bigcap_{q \geq 1} A\text{Shr}_{\varepsilon,q}\text{Per}_f \subset \mathcal{O}_f. \]

**Proof.** Fix $\varepsilon > 0$, and choose $q \geq 1$ as in lemma 4, such that $1/q < \varepsilon$. For any $\mu_q \in A\text{Shr}_{1/q}\text{Per}_f$, denote by $\nu_q$ a measure in $\text{Shr}_q\text{Per}_f$ such that $d(\mu_q, \nu_q) < 1/q < \varepsilon$.

Consider the periodic shrinking set $I = I(\nu_q)$ and the compact set $K$ for $\nu_q$ from definition 20. From the definition of periodic shrinking set, any point $x \in I$ satisfies $f^n(x) \subset K = \bigcup_{j=0}^{p-1} f^j(I)$ for all $n \geq 0$. So, any measure $\nu \in p\omega(x)$ is supported on $K$. Also $\nu_q$ is supported on $K$. Thus, applying lemma 10, we deduce that

\[ \nu(f^j(I)) = \nu_q(f^j(I)) = \frac{1}{p} \quad \forall \ 1 \leq j \leq p; \quad \text{diam}(f^j(I)) < \frac{1}{q}. \]

Now, from lemma 4, we obtain $d(\nu_q, \nu) < \varepsilon$ for all $\nu \in p\omega(x)$, for all $x \in I$. Thus, for any $x \in I$

\[ d(p\omega(x), \nu_q) < \varepsilon, \quad \text{hence} \quad d(p\omega(x), \mu_q) < 2\varepsilon. \quad (16) \]

Note that when we vary the value of $\varepsilon > 0$, the value of $q$, and thus also the measures $\nu_q$ and $\mu_q$ and the set $I$, may change. So, from the above inequality we can not deduce that each $\mu_q$ is pseudo-physical. Nevertheless, we have proved that for any fixed value of $\varepsilon > 0$ there exists $q \geq 1$ such that inequality (16) holds for all $\mu_q \in A\text{Shr}_{1/q}\text{Per}_f$.

Now, consider any measure $\mu'_q \in A\text{Shr}_{1/q}\text{Per}_f$. Thus, there exists $\mu_q \in A\text{Shr}_{1/q}\text{Per}_f$ such that

\[ d(\mu'_q, \mu_q) < \varepsilon. \]

Combining this with (16) we deduce that, for all $\varepsilon > 0$ there exists $q \geq 1$ such that, for any measure $\mu'_q \in A\text{Shr}_{1/q}\text{Per}_f$ there exists an open set $I$ (the periodic shrinking set $I(\nu_q)$ for the measure $\nu_q$ associated to $\mu'_q$) such that

\[ d(p\omega(x), \mu'_q) < 3\varepsilon \quad \forall \ x \in I. \quad (17) \]
So, if \( \mu \in \bigcap_{\varepsilon > 0} \bigcap_{q \geq 1} \overline{\text{AShr}_{\varepsilon q, r} Per_f} \), then, for all \( \varepsilon > 0 \) there exists an open set \( I \) satisfying assertion (17). Thus \( \text{Leb}(A_{1\varepsilon}(\mu)) \geq \text{Leb}(I) > 0 \) for all \( \varepsilon > 0 \); hence \( \mu \in \mathcal{O}_f \), as wanted. \( \square \)

The last ingredient of the proof of theorem 1 is the following theorem.

**Theorem 22.** For a typical map \( f \in \mathcal{C}(M) \) (resp. \( f \in \mathcal{H}(M) \)),

\[
\text{Per}_f \subseteq \bigcap_{q \geq 1} \overline{\text{AShr}_{\varepsilon q, r} Per_f} \quad \forall \varepsilon > 0.
\] (18)

Before proving theorem 22, let us introduce the following definition:

**Definition 23.** Fix \( q, r \in \mathbb{N}^+ \). A good \( q, r \)-covering \( \mathcal{U}_{qr} \) for \( f \in \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)), is a finite family of open simplexes (i.e. the interiors of simplexes) such that

1. \( \text{diam}(U_i) < 1/q \) for any \( U_i \in \mathcal{U}_{qr} \).
2. There exists a periodic shrinking set \( I_i \), with period \( p_i \leq r \), with \( p_i \) that divides \( r \), such that \( I_i \subseteq U_j \).
3. \( \mathcal{U}_{qr} \) covers the compact set \( \text{Per}(f, r) := \{ x \in M : f^r x = x \} \).

We call a map \( f \in \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)) a good \( q, r \)-covered map, if there exists a good \( q, r \)-covering \( \mathcal{U}_{qr} \) for \( f \). We denote by \( \mathcal{P}_{qr} \subseteq \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)) the set of all good \( q, r \)-covered maps.

**Proof of theorem 22.** We claim that, for fixed \( q, r \geq 1 \), the set \( \mathcal{P}_{qr} \) is open in \( \mathcal{C}(M) \). Fix \( f \in \mathcal{P}_{qr} \), and denote its good \( q, r \)-covering by \( \mathcal{U}_{qr} = \{ U_1, U_2, \ldots, U_h \} \). The compact set \( K = M \setminus \bigcup_{i=1}^h U_i \) does not intersect the compact set \( \{ f^i(x) = x \} \). Let us prove that for all \( g \in \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)) close enough to \( f \), the same compact set \( K \) (defined for the same covering \( \mathcal{U}_{qr} \)) does not intersect \( \{ g^i(x) = x \} \). In fact, the real function \( \phi(g) := \text{dist}(f^i(\cdot), \cdot) \) depends continuously on \( f \). Since \( \min_{x \in K} \phi_f(x) > 0 \), we deduce \( \min_{x \in K} \phi_g(x) > 0 \) for all \( g \in \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)) close enough to \( f \). In other words, \( \mathcal{U}_{qr} \) also covers the fixed points of \( g \). Thus, the good \( q, r \)-covering of \( f \), is also a covering satisfying conditions (1) and (2) of definition 23, for any \( g \in \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)) close enough to \( f \). Now, let us prove that condition (3) for \( g \) is satisfied by the same covering \( \mathcal{U}_{qr} \), provided that \( g \) is close enough to \( f \). Consider a \( f \)-shrinking periodic set \( I_i \subseteq U_j \in \mathcal{U}_{qr} \), of period \( p_i \). Now \( I_i \) is a periodic shrinking set with the same period \( p_i \) for all \( g \) sufficiently close to \( f \). Since the family \( \{ I_i \}_{1 \leq i \leq n} \) of shrinking periodic sets to be preserved is finite, we conclude that (3) is also satisfied for any \( g \) sufficiently close to \( f \) and thus \( \mathcal{P}_{qr} \) is open in \( \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)).

In lemma 24, we will prove that \( \mathcal{P}_{qr} \) is dense in \( \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)). Let us conclude the proof of theorem 22 assuming that lemma 24 is proven. Observe that, for fixed \( q, r \geq 1 \), any \( f \in \mathcal{P}_{qr} \) has the following property: any point \( x_0 \) fixed by \( f \) (in particular any periodic point \( x_0 \) of periodic \( r \)) is \((1/q)\)-near all the points of a periodic shrinking set \( I_0 \) with diameter smaller than \( 1/q \), and with period \( p_0 \leq r \) dividing \( r \).

Besides, any periodic shrunk set of period \( p_0 \) has at least one periodic point \( y_0 \), fixed by \( f^{p_0} \). We deduce that \( I_0 \), whose diameter is smaller than \( 1/q \), contains a periodic point \( y_0 \). Using the definition of the set of measures \( \text{Shr}_r \text{Per}_f \), we summarize this assertion as follows:

\[
\text{for all } x_0 \text{ with period } r, \text{ there exists } (19)
\]
\[ \nu_0 := \frac{1}{q} \sum_{j=0}^{q-1} \delta_{f^j(x_0)} \in \text{Shr}_q \text{Per}_f, \text{ with } \text{dist}(y_0, x_0) < 1/q. \]

For \( r \geq 1 \) fixed, consider \( f \in \bigcap_{q \geq 1} \mathcal{P}_{q,r} \). Let \( \mu_0 := \frac{1}{q} \sum_{j=0}^{q-1} \delta_{f^j(x_0)} \). Fix \( \varepsilon > 0 \) and choose \( q' \geq 1 \) as in lemma 4. Since \( f' \) is continuous for each \( j \), we can find a \( q > q' \) so that if \( \text{dist}(x,y) < 1/q \), then \( \text{dist}(\delta_{f^j(x)}, \delta_{f^j(y)}) < 1/q' \) for all \( j \in \{0, 1, \ldots, r\} \). Then lemma 4 implies that \( d(\mu_0, \nu_0) < \varepsilon \).

We have shown that for any given periodic orbit \( \{f^j(x_0)\}_{0 \leq j < r} \) of period \( r \), the distance between the periodic measure supported on it, and some measure \( \nu_q \in \text{Shr}_q \text{Per}_f \), for all \( q \) large enough, is smaller than \( \varepsilon \). In other words, any periodic measure supported on a periodic orbit of period \( r \), belongs to \( \bigcap_{q \geq 1} \overline{\text{AShr}_{x_0} \text{Per}_f} \) for all \( \varepsilon > 0 \).

Finally, if \( f \in \mathcal{P} := \bigcap_{q \geq 1} \bigcap_{r \geq 1} \mathcal{P}_{q,r} \), then all its periodic measures (supported on periodic orbits of any period \( r \)) will belong to \( \overline{\text{AShr}_{x_0} \text{Per}_f} \) for all \( q \geq 1 \) and for all \( \varepsilon > 0 \). In brief, if \( f \in \mathcal{P} \), then
\[ \bigcap_{q \geq 1} \text{Per}_f \subset \overline{\text{AShr}_{x_0} \text{Per}_f} \quad \forall \varepsilon > 0. \]

As \( \mathcal{P}_{q,r} \) is open and dense in \( \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)), the maps \( f \in \mathcal{P} \) are typical. This ends the proof of theorem 22, provided that lemma 24 is proven.

**Lemma 24.** For each \( q, r \geq 1 \), the set \( \mathcal{P}_{q,r} \) is dense in \( \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)).

**Proof.** We will use the notation that we defined in section 2.2. Fix \( f \in \mathcal{C}(M) \) (resp. \( \mathcal{H}(M) \)) and \( \varepsilon > 0 \). Consider \( 0 < \delta := \frac{1}{q} < \min \left\{ \frac{1}{q}, \frac{1}{2} \right\} \) such that \( \text{dist}(x,y) < \delta \) implies \( \text{dist}(f(x), f(y)) < \varepsilon \). We wish to \( \varepsilon \)-perturb \( f \) into new map \( g \in \mathcal{P}_{q,r} \). Consider a triangulation \( T := \{T_1, \ldots, T_n\} \) of \( M \) such that the diameters of all the simplexes \( T_i \) are at most \( \delta/3 \). By modifying \( T \) we can suppose that if \( T_i \cap \text{Per}(f,r) \neq \emptyset \) then \( T_i \cap \text{Per}(f,r) \neq \emptyset \), where \( T_i = \text{int}(T_i) \); and furthermore, for some \( x_i \in T_i \cap \text{Per}(f,r) \) its orbit \( \{x_i, f(x_i), \ldots, f^{r-1}(x_i)\} \cap \partial T = \emptyset \). Indeed, suppose some \( T_i \) does not satisfy these conditions. If no point of \( \{x_i, f(x_i), \ldots, f^{r-1}(x_i)\} \cap \partial T \) belongs to the interior \( T_i \), then we can modify the triangulation by first slightly moving one corner of \( T_i \) so that in the resulting triangulation some point \( x_i \in \text{Per}(f,r) \) belongs to the interior \( T_i \), and all the other simplexes of \( T \) which satisfied this condition before, still satisfy it.

If the orbit of \( x_i \) does not lay in the interior of the simplexes, let \( 1 \leq j \leq r - 1 \) be the first time that \( f^j(x_i) \in \partial T \). Again we modify the triangulation by slightly moving one corner of the triangulation so that in the modified triangulation \( f^j(x_i) \notin \partial T \), and all the other simplexes of \( T \) which satisfied this condition before, still satisfy it. Repeat this procedure a finite number of times to produce the desired triangulation. We can assume that the perturbations are so small that in the resulting triangulation the diameters of all the simplexes are at most \( \delta/2 \).

Let \( T^{(1)} := \{T^{(1)}_1, \ldots, T^{(1)}_l\} := \{T_i \in T : T_i \cap \text{Per}(f,r) \neq \emptyset\} \) and for each \( i \) let \( x_i \) be a point in \( T^{(1)}_i \cap \text{Per}(f,r) \). Using \( x_i \) as a centroid, consider a real number \( \lambda_{T^{(1)}_i} > 1 \) near enough 1, such that \( \lambda_{T^{(1)}_i} T^{(1)}_i \) is well defined. Let \( \lambda_T := \min \{\lambda_{T^{(1)}_i} : 1 \leq i \leq l\} \). For \( \lambda \in (1, \lambda_T) \) we have \( x_i \in T_i \subset T_i \subset \lambda T_i \), and \( \{\lambda T_1, \ldots, \lambda T_l\} \) is an open cover of \( \text{Per}(f,r) \). For the rest of the
proof we fix \( \lambda \in (1, \lambda_T) \), so close to 1 such that besides, each of the points \( x_i \) does not lie inside \( \lambda T_i \) for any \( j \neq i \), and furthermore such that the diameters of the \( \lambda T_i \) are at most \( \delta \).

Denote

\[
F := \{ x_i : 1 \leq i \leq l \}, \quad F' := \bigcup_{j=0}^{r-1} f^j(F).
\]

Hence \( F \) is consists of exactly \( l \) different points that are fixed by \( f' \), and \( F' \) consists of at most \( rl \) different points, also fixed by \( f' \).

We will define an homeomorphism \( h : M \mapsto M \). Consider a chart \( (U_\alpha, \phi_\alpha) \) such that \( \lambda T_i \subset U_\alpha \). Choose \( \eta > 0 \) so small such that for each \( x \in F' \) the solid ball \( B(\phi_\alpha(x), \eta) \subset \mathbb{R}^m \) does not intersect \( \phi_\alpha(U_\alpha \cap \partial(\lambda T)) \), such that the diameter of \( B(x, \eta) := \phi^{-1}(B(x, \eta)) \) is at most \( \delta \), and such that the finite family \( \{ B(x, \eta) : x \in F' \} \) is composed of pairwise disjoint sets. We define \( h(y) = y \) if \( y \notin \bigcup_{x \in F'} B(x, \eta) \). On the complement we define \( h \) as follows. Fix \( x \in F' \), and consider polar/spherical coordinates \( (s, \theta) : s \in [0, 1], \theta \in S^m \) to describe the ball \( B(x, \eta) \). Finally use \( \phi^{-1}_\alpha \) to pull back these coordinates to \( B(x, \eta) \).

Fix a certain \( k \geq 1 \) sufficiently large (that will be chosen later), and define \( h : B(x, \eta) \mapsto B(x, \eta) \) by setting \( h(s, \theta) := (s^k, \theta) \). This construction for each point \( x \in F' \) completes the definition of the homeomorphism \( h \). Thus if \( f \) is continuous then so is

\[
g := f \circ h,
\]

and if \( f \) is an homeomorphism, then \( g \) is also an homeomorphism.

Note that for any \( y \in M \) the map \( h \) satisfies

\[
dist(y, h(y)) < \delta \quad \text{and} \quad dist(y, h^{-1}(y)) < \delta.
\]

Since \( g = f \circ h \) and \( dist(h(y), y) < \delta \) we have

\[
\rho(f, g) < \varepsilon.
\]

If besides \( f \) is an homeomorphism, then \( g^{-1} = h^{-1} \circ f^{-1} \) and

\[
dist(g^{-1}(y), f^{-1}(y)) = dist(h^{-1}(z), z) < \delta < \varepsilon.
\]

Thus

\[
\rho(g^{-1}, f^{-1}) < \varepsilon.
\]

We have proved that \( g \) is an \( \varepsilon \)-perturbation of \( f \) in \( C(M) \) (resp. \( H(M) \)). Now, to end the proof of the lemma, it is enough to choose \( k \) such that the set \( B(x, \eta) \) contains a periodic shrinking set of period that divides \( r \) for each point \( x \) of the finite set \( F' \).

From the above construction, we have \( h(0, \theta) = 0 \). Therefore, \( g(x) = g(0, \theta) = f(0, \theta) = f(x) \) for all \( x \in F' \); thus each point \( x \) of \( F' \) will be fixed by the map \( g' \).

We claim that if we choose \( k \) large enough, then \( g \) has a shrinking set around the point \( x \in T_i \cap F' \) contained in \( B(x, \eta) \). We consider the \( (s, \theta) \) coordinates in the set \( B(x, \eta) \). Fix \( \eta_1 \in (0, \eta) \). For each \( j = 0, 1, \ldots, r - 1 \) choose a simplex \( T_j \) containing \( f(x) \) such that \( T_j \subset B(f^j(x), \eta_1) \). Next choose \( \eta_2 \in (0, \eta_1) \) small enough such that \( f(B(f^j(x), \eta_2)) \subset I_{j+1} \) for \( j = 0, 1, \ldots, r - 1 \). In these coordinates \( B(x, \eta) \) is given by \( \{(s, \theta) : s \in [0, \eta_1]\} \). Choose \( k > 1 \) so that \( \eta_1^k < \eta_2 \). For each \( j = 1, \ldots, r \) we have
Thus the simplexes $\tilde{I}_j$ are the desired shrinking sets. This finishes the proof of lemma 24. $\square$

6.1. End of the proof of theorem 1

**Proof.** In corollary 19 we have proved that $O_f \subset \overline{\gamma_f} = \overline{\text{Per}_f}$. Combining theorems 21 and 22, we deduce that

$$\text{Per}_f \subset \bigcap_{\varepsilon > 0} \bigcap_{q \geq 1} \text{AShr}_{\varepsilon,q} \text{Per}_f \subset O_f.$$

Since $O_f$ is closed, we conclude that $O_f = \overline{\gamma_f} = \overline{\text{Per}_f}$, as wanted. $\square$

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