Ghosts as Negative Spinors

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Abstract

We study the properties of a BRST ghost degree of freedom complementary to a two-state spinor.

We show that the ghost may be regarded as a unit carrier of negative entropy.

We construct an irreducible representation of the $su(2)$ Lie algebra with negative spin, equal to $-\frac{1}{2}$, on the ghost state space and discuss the representation of finite SU(2) group elements.

The Casimir operator $J^2$ of the combined spinor-ghost system is nilpotent and coincides with the BRST operator $Q$. Using this, we discuss the sense in which the positive and negative spin representations cancel in the product to give an effectively trivial representation. We compute an effective dimension, equal to $\frac{1}{2}$, and character for the ghost representation and argue that these are consistent with this cancellation.

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1 Introduction

The technique of using ghosts to compensate unphysical degrees of freedom in the covariant quantization of systems with constraints was originally introduced by Faddev and Popov [1]. Tyutin, Becchi, Rouet and Stora [2] discovered that the ghost-enhanced system is invariant under a nilpotent (BRST) symmetry. This symmetry can be used to identify physical states and operators using the language of cohomology [3].

Ghosts give mathematical realization to the concept of negative degrees of freedom, which carry negative entropy or information. Although originally introduced as a mathematical device, these negative degrees of freedom are interesting to study in their own right.

In this article we will concentrate on the most elementary negative degree of freedom – a BRST ghost complementary to a single two-state spinor.

Since we will show that the ghost entropy ranges between 0 and $-1$ and cancels the entropy of the spinor, the ghost may be regarded as the negative version of a quantum bit, carrying $-1$ bit of information.

We then construct an irreducible representation of the $su(2)$ Lie algebra with negative spin, equal to $-\frac{1}{2}$, on the ghost state space. We also discuss the representation of finite $SU(2)$ group elements. While we are able to write down the matrix representation of a restricted class of group elements, we find that the Lie algebra representation does not exponentiate to a representation of the full group.

The Casimir operator $J^2$ of the combined spinor-ghost system is nilpotent and coincides with the BRST operator $Q$. Using this, we discuss the sense in which the positive and negative spin representations cancel in the product to give an effectively trivial representation. We compute an effective dimension, equal to $\frac{1}{2}$, and character for the ghost representation and argue that these are consistent with this cancellation.

2 The ghost

The definition of the ghost complementary to a two-state spinor may be motivated as follows.
Consider a Hamiltonian on the spinor state space of the form

\[ H_s = \omega \left( \begin{array}{c} -\frac{1}{2} \\ \frac{1}{2} \end{array} \right) \]

in a basis which we denote by \( \{|\pm \frac{1}{2}\rangle_s, |\frac{1}{2}\rangle_s\} \). The spinor partition function at finite inverse temperature \( \beta \) is given by

\[ Z_s(\beta) = \text{Tr} e^{-\beta H_s} = q^{-\frac{1}{2}}(1 + q), \quad q \equiv e^{-\omega \beta}. \]

We would like to introduce a ghost degree of freedom that will cancel the contribution of the spinor to the partition function. Assuming that the spinor and the ghost do not interact, the combined partition function \( Z(\beta) \) will factorize

\[ Z(\beta) = Z_s(\beta) Z_g(\beta). \]

and the ghost partition function should be

\[ Z_g(\beta) = \frac{q^{\frac{1}{2}}}{1 + q}. \]

Expanding this as a geometric series

\[ Z_g = q^{\frac{1}{2}} \left( 1 - q + q^2 - q^3 + \cdots \right), \quad (1) \]

we can write the ghost partition function in the form

\[ Z_g = \text{Tr} \ \eta e^{-\beta H_g}, \quad (2) \]

where the matrix in the exponent

\[ H_g = \omega \text{diag} \left( \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \right), \quad (3) \]

defines a Hamiltonian on an infinite-dimensional vector space with ordered basis denoted by

\[ \{|\frac{1}{2}\rangle_g, |\frac{3}{2}\rangle_g, |\frac{5}{2}\rangle_g, \ldots \} \]

and

\[ \eta = \text{diag} \ (1, -1, 1, -1, \ldots). \]
The presence of $\eta$ in the partition function may be motivated as follows. Writing the partition function of the combined spinor-ghost system as 

$$1 = (1 + q) (1 - q + q^2 - \cdots) = 1 + q - q + q^2 - q^2 + \cdots,$$

it is clear that the minus signs provided by $\eta$ cause the contributions of excited states to cancel pairwise, leaving only the contribution of the ground state. In the next section we will see that the ground state is the only physical state in the combined system. Since the partition function should only count physical states, we conclude that $\eta$ is necessary.

### 3 BRST cohomology

In BRST theories [3], the analysis of physical states and operators is carried out in terms of an operator $Q$ that is hermitian and nilpotent. In other words,

$$Q^\dagger = Q, \quad Q^2 = 0.$$

Physical states satisfy 

$$Q |\phi\rangle = 0,$$

and are regarded as equivalent if they differ by a $Q$-exact state. In other words,

$$|\phi\rangle \sim |\phi\rangle + Q |\chi\rangle,$$

where $|\chi\rangle$ is an arbitrary state. More formally, physical states are elements of the cohomology of $Q$, defined as the quotient vector space

$$\text{ker} \, Q / \text{im} \, Q.$$

The inner product on this quotient space may be defined in terms of the original inner product by noting that all elements of $\text{im} \, Q$ are orthogonal to all elements of $\text{ker} \, Q$, so that the induced inner product defined on equivalence classes $[|\phi\rangle] = |\phi\rangle + \text{im} \, Q$ in the cohomology by

$$\langle \phi + \text{im} \, Q |\phi' + \text{im} \, Q \rangle \equiv \langle \phi |\phi' \rangle, \quad \phi, \phi' \in \text{ker} \, Q$$

is well defined.
A hermitian operator $A$ is regarded as physical if $[A, Q] = 0$. This ensures that $A$ leaves $\text{im} \, Q$ invariant, so that the reduced operator $[A]$ defined on the cohomology classes by

$$[A] (|\phi\rangle + \text{im} \, Q) = A |\phi\rangle + \text{im} \, Q$$

is well-defined. In particular, the hamiltonian is required to be physical

$$[Q, H] = 0.$$ 

In order to define a BRST cohomology in the system at hand, we need to postulate a BRST operator $Q$. $Q$ should commute with the combined hamiltonian which, in the ordered basis

$$\{|-\frac{1}{2}\rangle_s \otimes |\frac{1}{2}\rangle_g, |\frac{1}{2}\rangle_s \otimes |\frac{1}{2}\rangle_g, |-\frac{1}{2}\rangle_s \otimes |\frac{3}{2}\rangle_g, |\frac{1}{2}\rangle_s \otimes |\frac{3}{2}\rangle_g, \ldots\},$$

has the form

$$H = H_s + H_g = \begin{pmatrix} 0 & 1 & \ldots \\ 1 & 2 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$ 

Thus, $Q$ takes the two-dimensional subspace formed by each pair of excited states with the same energy eigenvalue onto itself. Since the combined partition function $Z = 1$, the cohomology of $Q$ cannot contain excited states. Therefore, $Q$ cannot be zero in any of the excited subspaces. Since we need to have $Q^2 = 0$, we see that in each of these subspaces there has to be pair of linearly independent vectors $|v_1\rangle$ and $|v_2\rangle$ such that $Q |v_1\rangle = |v_2\rangle$ and $Q |v_2\rangle = 0$. This is precisely what one needs to eliminate these excited states from the cohomology.

The operator $Q$ can only be hermitian if the inner product is not positive definite. Indeed, assuming hermiticity, the inner product of $|v_2\rangle$ with itself is

$$\langle v_2 | v_2 \rangle = \langle Q | v_1 \rangle | Q | v_1 \rangle = \langle v_1 | Q^2 | v_1 \rangle = 0.$$
In other words, the vector $v_2$ is null (has zero norm). A suitable indefinite inner product is obtained by defining the inner product on the ghost state space as

$$
\eta_{mn} \equiv \langle m|n \rangle_g = (-)^{m-\frac{1}{2}} \delta_{mn}, \quad m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots,
$$

or

$$
\eta = \text{diag} (1, -1, 1, -1, \ldots).
$$

In the product space, the inner product is then

$$
G \equiv 1_s \otimes \eta = \begin{pmatrix}
1 & 1 & & \\
1 & & 1 & \\
& 1 & & \\
& & & -1 \\
& & & -1 \\
& & & 1 \\
& & & 1 \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}.
$$

Given this inner product, we will show in later sections that one can construct a hermitian representation of the $su(2)$ Lie algebra on the state space. In this representation, the Casimir operator $J^2$ is nonzero but satisfies $(J^2)^2 = 0$. It is therefore a very natural candidate for $Q$, and we will indeed choose $Q = J^2$. This will have the added benefit that the $su(2)$ generators will be physical operators in the sense discussed above.

In the above basis, the operator $Q = J^2$ has the form (see section 9)

$$
Q = \begin{pmatrix}
0 & & & \\
& -1 & i & \\
& i & 1 & \\
& & & -2 \\
& & & 2i \\
& & & 2i \\
& & & 2 \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}.
$$

It is easily checked that $Q^2 = 0$ and that $Q$ is hermitian with respect to the inner product $G$, as follows from

$$
Q = Q^\dagger = GQ^+G,
$$
where the dagger denotes hermitian conjugation with respect to the indefinite inner product $\langle \cdot | \cdot \rangle$ on the state space and the $+$ sign denotes the usual matrix adjoint (11).

With respect to the basis consisting of the ground state and the null excited states

$$|n\rangle \equiv \frac{1}{\sqrt{2}} \left( i \left| -\frac{1}{2} \rightangle_s \otimes |n + \frac{1}{2}\rangle_g + \left| \frac{1}{2} \rightangle_s \otimes |n - \frac{1}{2}\rangle_g \right),$$

$$|\tilde{n}\rangle \equiv \frac{1}{\sqrt{2}} \left( -i \left| -\frac{1}{2} \rightangle_s \otimes |n + \frac{1}{2}\rangle_g + \left| \frac{1}{2} \rightangle_s \otimes |n - \frac{1}{2}\rangle_g \right),$$

$$(8)$$

$G$ has the form

$$G = \begin{pmatrix}
1 & 1 \\
1 & -1 \\
-1 & \ddots
\end{pmatrix},$$

and $Q$ is

$$Q = \begin{pmatrix}
0 & 0 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
& & \ddots
\end{pmatrix}.$$  

$$(9)$$

By construction, the cohomology of $Q$ consists of the single zero-energy state

$$\left| -\frac{1}{2} \rightangle_s \otimes \left| \frac{1}{2} \rightangle_g + \text{im} \ Q.$$

4 Indefinite inner product spaces

We review a few facts regarding operators on indefinite inner product spaces such as (5), also called Krein spaces in the literature [4].
Unitary and hermitian operators on an indefinite product space are often called pseudo-unitary and pseudo-hermitian to emphasize that the inner product is indefinite.

It is important to be aware that not all results that are valid for positive definite spaces are valid when the inner product is not positive definite. For example, not all pseudo-hermitian operators are diagonalizable. A good counterexample is precisely the operator \( Q = J^2 \) above.

In a basis where the metric tensor is represented by the matrix \( \eta \), the matrix representation of a unitary transformation \( U \) satisfies

\[
U^{-1} = U^\dagger = \eta U^+ \eta, \tag{10}
\]

where the dagger denotes hermitian conjugation with respect to the indefinite inner product and the plus sign denotes the ordinary matrix adjoint.

A hermitian transformation satisfies

\[
A = A^\dagger = \eta A^+ \eta. \tag{11}
\]

Taking \( U = e^{i\epsilon A} \), it follows that the infinitesimal version of the pseudo-unitarity condition (10) is just the pseudo-hermiticity condition (11).

5 The oscillator representation

Now that we have determined the spectrum and the inner product on the ghost state space, we can write the partition function (2) in terms of a bosonic oscillator satisfying

\[
[a, a^\dagger] = -1 \tag{12}
\]

as

\[
Z_g = \text{Tr} \eta e^{-\beta H_g},
\]

where

\[
H_g = -\omega a^\dagger a, \quad \eta = (-)^N, \quad N = -a^\dagger a.
\]

The minus sign on the right hand side of (12) ensures that the inner product will have the indefinite form (5). We take the following normalization convention for the basis used in (2)

\[
|m\rangle_g \equiv (-)^{m - \frac{1}{2}} \frac{1}{\sqrt{(m - \frac{1}{2})!}} (a^\dagger)^{m - \frac{1}{2}} |1\rangle_g, \quad m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \tag{13}
\]
6 Entropy

From the expression (2), we see that the finite temperature Boltzmann density matrix has the form

\[
\rho_g(\beta) \equiv \frac{1}{Z_g} e^{-\beta H_g} \eta, \\
= (1 + q) \text{diag}(1, -q, q^2, -q^3, \ldots), \quad q = e^{-\beta}.
\] (14)

The density matrix has unit trace, as it should. However, it has both positive and negative diagonal entries, so that the usual probabilistic interpretation is not valid if we regard the ghost in isolation. However, it is possible to formally define an entropy for the ghost.

Due to the negative eigenvalues in the ghost density matrix, we have to modify the conventional definition \( S(\rho) \equiv -\text{Tr} \rho \log \rho \) of the von Neumann entropy [5]. A suitable expression that is invariant under pseudo-unitary transformations \( \rho \to U\rho U^{-1} \) is

\[
S(\rho) \equiv -\frac{1}{2} \text{Tr} \rho \log (\rho)^2.
\] (15)

This expression reduces to the von Neumann definition when the eigenvalues are positive.

It is straightforward to verify that, with this definition, the entropy is additive for spinor-ghost states of the factorizable form \( \rho = \rho_s \otimes \rho_g \). In other words

\[
S(\rho_s \otimes \rho_g) = S_s(\rho_s) + S_g(\rho_g).
\]

Calculating the ghost entropy for the finite temperature density matrix (14), we find

\[
S_g = -\log(1 + q) - \frac{q}{1+q} \log q = -S_s.
\]

We therefore see that the ghost carries negative entropy that exactly compensates that of the original spinor. The ghost entropy is always between zero and \(-1\), attaining the former in the pure state at zero temperature and the latter in the maximally mixed state at infinite temperature.

We can also express the entropy in terms of the partition function by noting that for a finite temperature density matrix the definition (15) gives

\[
S(\beta) = \left(1 - \beta \frac{\partial}{\partial \beta}\right) \log Z.
\] (16)
Cancellation of entropy between the spinor and ghost then follows trivially from combined with the relation $1 = Z_s Z_g$.

7 $su(2)$ and negative spin

In this section we will construct an irreducible representation of the $su(2)$ Lie algebra on the ghost state space. This representation has spin equal to $-\frac{1}{2}$, and is pseudo-hermitian with respect to the indefinite inner product in the ghost state space.

For later reference, let us write down the spin-$\frac{1}{2}$ representation

$$J_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
J_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
J_z = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

On the ghost state space, a set of generators hermitian with respect to the indefinite inner product is given by

$$J_x \equiv \frac{i}{2} \left( \sqrt{N+1} a - a^\dagger \sqrt{N+1} \right),
J_y \equiv -\frac{i}{2} \left( \sqrt{N+1} a + a^\dagger \sqrt{N+1} \right),
J_z \equiv N + \frac{1}{2}.
$$

It is easy to check that these generators satisfy the $su(2)$ algebra

$$[J_x, J_y] = i J_z, \quad [J_y, J_z] = i J_x, \quad [J_z, J_x] = i J_y.$$

We can define raising and lowering operators

$$J_+ \equiv J_x + i J_y = -i a^\dagger \sqrt{N+1},
J_- \equiv J_x - i J_y = i \sqrt{N+1} a,$$

satisfying $J_\pm^2 = J_\mp$ and

$$[J_+, J_-] = 2 J_z.$$
It is straightforward to calculate
\[ J^2 = \frac{1}{2} (J_+ J_- + J_- J_+) + J_z^2 = -\frac{1}{4}. \]
Since \( J^2 = j(j + 1) \), where \( j \) denotes the spin, we see that
\[ j = -\frac{1}{2}. \]
In other words, the representation that we have constructed has negative spin.

Note that, since the entire state space is generated by applying the raising operator \( J_+ \) to the vacuum state, the representation is irreducible. Also, in contrast to the positive spin irreducible representations, it is infinite-dimensional.

However, a more useful definition of the dimension of a representation is given by the character of the identity \( \chi(1) \). In a later section, we shall see that for the ghost representation this is equal to \( \frac{1}{2} \). We will therefore argue that the dimension is effectively finite.

The lowest weight state is the ghost ground state \( |\frac{1}{2}\rangle_g \), which has \( J_z \) eigenvalue \( m = \frac{1}{2} \). As usual for a lowest weight state, it satisfies the relation \( m = -j \).

A peculiar feature of the representation is its asymmetry with respect to interchange of \( J_- \) and \( J_+ \). In particular, there is no highest weight state \( |\frac{1}{2}\rangle_g \) satisfying \( J_+ |\frac{1}{2}\rangle_g = 0 \).

It is straightforward to check that, in terms of the basis vectors (13) we have
\[ J_+ |m\rangle = i \left( m + \frac{1}{2} \right) |m + \frac{1}{2}\rangle, \quad m = \frac{1}{2}, \frac{3}{2}, \ldots \]
\[ J_- |m + 1\rangle = i \left( m + \frac{1}{2} \right) |m\rangle. \quad (18) \]
The factor \( m + \frac{1}{2} \) is the continuation to negative spin \( j = -\frac{1}{2} \) of the standard expression \( [(j + m + 1)(j - m)]^{\frac{1}{2}} \) from the representation theory of \( su(2) \).

### 8 Finite \( SU(2) \) Transformations

In this section we will discuss the representation of finite \( SU(2) \) transformations on the ghost state space. Rather than exponentiating the \( su(2) \)
generators by brute force, we will present a simple guess for the finite form of the transformations, prove that these indeed represent SU(2) transformations, and then relate them to the Lie algebra generators.

Our first observation is that states in the combined spinor-ghost space of the special form

\[
\begin{pmatrix}
1 \\
z \\
z^2 \\
\vdots
\end{pmatrix} \otimes \begin{pmatrix}
1 \\
z \\
z^2 \\
\vdots
\end{pmatrix}
\]

have unit normalization with respect to the indefinite inner product; in particular \((1 + zz^*)(1 - zz^* + (zz^*)^2 - \ldots) = 1\).

Now consider the effect of the SU(2) transformation \[6\]

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
-\bar{\beta} & \bar{\alpha}
\end{pmatrix}.
\]

The spinor state becomes

\[
(a + bz) \begin{pmatrix}
1 \\
z'
\end{pmatrix},
\]

where \(z'\) is given by the \(CP^1\) conformal transformation

\[
z' = \frac{c + dz}{a + bz}.
\]

Since the representation has to preserve the inner product, our ansatz is that the state after the rotation will be of the normalized form

\[
\begin{pmatrix}
1 \\
z'
\end{pmatrix} \otimes \begin{pmatrix}
1 \\
z'^2 \\
\vdots
\end{pmatrix}.
\]

If true, we see from (20) that the transformed ghost state has to be

\[
\frac{1}{a + bz} \begin{pmatrix}
1 \\
z'
\end{pmatrix} = \frac{1}{a + bz} \begin{pmatrix}
1 \\
\frac{c + dz}{a + bz} \\
\frac{c + dz}{a + bz}^2 \\
\vdots
\end{pmatrix}.
\]
This will give us the information we need to determine the ghost SU(2) transformation matrices in terms of the parameters $a, b, c$ and $d$. Indeed, we will first show that this transformation can be written in terms of a linear operator on the ghost state space. We will then verify that these linear operators represent SU(2) under composition.

To exhibit the linear operator realizing the transformation (23), we observe that, as long as $a \neq 0$ and $|z| < |a/b|$, we can expand the expression for the transformed ghost state in terms of positive powers of $z$. Doing this explicitly, we see that the expression (23) is equivalent to the linear transformation

$$
\frac{1}{a} \cdot \begin{pmatrix}
1 & -\frac{b}{a} & \left(\frac{b}{a}\right)^2 & \cdots \\
\frac{c}{a} & \left(\frac{d - 2bc}{a^2}\right) & \cdots & \\
\left(\frac{c}{a}\right)^2 & \cdots & \\
\vdots & \vdots & \\
\end{pmatrix}
\begin{pmatrix}
1 \\
z \\
z^2 \\
\vdots
\end{pmatrix}.
$$

The expression for the general element of this candidate SU(2) matrix is easily calculated to be

$$
U_{ij} = \sum_{k=0}^{\min(i,j)} (-)^{j-k} \binom{i + j - k}{i} \binom{i}{k} a^{-i-j+k-1} b^{j-k} c^{i-k} d^k,
$$

for $i, j = 0, 1, 2, \ldots$

To show that these linear operators $U$ represent SU(2) transformations under composition, it is enough to show that this is true on states of the special form

$$
\begin{pmatrix}
1 \\
z \\
z^2 \\
\vdots
\end{pmatrix},
$$

because these states linearly span the whole ghost state space. For these states, the effect of $U$ is by construction given by the expression (24). It is readily verified that the conformal transformations $z \to \left(\frac{z + dz}{a + bz}\right)$ in (23) form a realization of SU(2) under composition. For the prefactor in (23), it is easy to check that, under composition of transformations parametrized
by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \), the prefactor becomes by linearity

\[
\frac{1}{(a + bz)} \frac{1}{(a' + b'(z + dz/(a+dz)))} = \frac{1}{(a'a + b'c) + (a'b + b'd)z},
\]

as required. This confirms that matrices of the form (25) represent SU(2) transformations.

So far we have not been very careful with respect to the domains of the transformations \( U_{ij} \). In fact, the domain of the linear operator defined by (25) is not the entire state space. This can be seen from the growth condition \(|z| < |a/b|\) required for equivalence of (23) and (25). Because the composition of two transformations of the form (25) only makes sense if the range of the first overlaps with the domain of the second, matrices of the form (25) cannot be arbitrarily composed. In other words, the formal sums appearing in the matrix multiplication may not converge. Although one may be able to make sense of these formal sums by analytic continuation, doing this will be outside the scope of the present article.

Also note that the representation matrix becomes singular when \( a \to 0 \), which happens for rotations by \( \pi \) that take the unit vector \( z \) to \(-z\). This is a consequence of the fact, noted in the previous section, that the representation has no highest weight state, since these rotations would normally exchange lowest and highest weight states.

The conclusion is that the Lie algebra representation of the previous section does not exponentiate to a give a representation of the full group.

We can prove that the representation matrix \( U \) is pseudo-unitary on its domain by showing that it conserves the norm of all states of the form

\[
\begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots 
\end{pmatrix},
\]

whose linear span is the entire state space. In particular, applying (23), the norm of the transformed state becomes

\[
\left| \frac{1}{a + bz} \right|^2 \left( 1 - \left| \frac{c + dz}{a + bz} \right|^2 + \left| \frac{c + d}{a + bz} \right|^4 + \ldots \right)
\]

14
\[
\begin{align*}
&= \frac{1}{|a+bz|^2 + |c+dz|^2} \\
&= \frac{1}{1 + \bar{z}z},
\end{align*}
\]

where in the last line we have used the fact that the SU(2) matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is unitary, so that its columns are orthonormal. This is just the norm of the original state, which completes the proof.

Note that pseudo-unitarity with respect to the indefinite inner product

\[
U^{-1} = \eta U^\dagger \eta.
\]
implies that the rows or columns of the matrix \( U \) are orthonormal with respect to the inner product \( \eta \).

Finally, we can explicitly relate these finite transformations to the ghost Lie algebra generators we wrote down before. Taking the one-parameter subgroup

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix},
\]
we can differentiate the corresponding ghost representation matrix in (25) to get the Lie algebra element

\[
\frac{d}{d\theta_{\theta=0}} U(\theta) = \frac{1}{2} \begin{pmatrix} -1 & -1 & -2 & -3 & \cdots \\ -1 & -2 & -3 & \cdots \\ -2 & -3 & \cdots \\ -3 & \cdots \\ \vdots & \vdots & \ddots \\
\end{pmatrix}
\]

This is just our ghost \textit{su}(2) generator \( iJ_x = -\frac{1}{2} \left( \sqrt{N+1}a + a^\dagger \sqrt{N+1} \right) \).

Similarly, the generators \( iJ_y \) and \( iJ_z \) are the Lie algebra generators corresponding respectively to

\[
\begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}
\]

and

\[
\begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}.
\]
9 Combining representations

In this section we study the product representation in the spinor-ghost system. We will discuss the sense in which the positive and negative spin representations cancel to give an effectively trivial representation.

First, notice that there is a lowest weight state in the spinor-ghost product state space given by

\[ |0\rangle \equiv \left| -\frac{1}{2}\right>_s \otimes \left| \frac{1}{2}\right>_g \]

By definition, it is annihilated by \( J^- \equiv J_-^s + J_-^g \). Using the relation

\[ J^2 = J_+ J_- + J_z^2 - J_z \]

it trivially follows that

\[ J^2 |0\rangle = 0. \]

We are therefore dealing with an \( su(2) \) representation of spin 0 in the product space. What is unusual about this representation, however, is that \( J_+ |0\rangle \neq 0 \). However, this state has zero norm. In fact, this is true for all higher states in the ladder since

\[ J_+^n |0\rangle = i^n n! \sqrt{2} |n\rangle , \]

where \( |n\rangle \) is defined in (8) and is null. In general, we would like to truncate the ladder of states generated by \( J_+ \) as soon as we reach a null state.

In section (3), we chose the BRST operator \( Q \) equal to \( J^2 \). In terms of the basis (8), it is not hard to calculate

\[ J^2 |\tilde{n}\rangle = 2n |n\rangle , \quad J^2 |n\rangle = 0, \]

leading to the matrix representations (7) and (9). Since \( J^2 \) is the Casimir operator, the generators \( J_i \) all commute with \( Q \). In other words, they are physical operators in the sense of section 3.

We can therefore consistently reduce these operators to the cohomology of \( Q \), and use (4) to define the induced operators \([J_+], [J_-] \) and \([J_z] \) on the quotient space by

\[ [J_i] (|\phi\rangle + \text{im} \ Q) = J_i |\phi\rangle + \text{im} \ Q. \]

The cohomology consists of the single class \(|0\rangle + \text{im} \ Q \), and is a one-dimensional, positive definite Hilbert space. The induced generators \{\([J_+], [J_-], [J_z] \)\} are zero, corresponding to the trivial representation of \( su(2) \).
10 Character and dimension formulae

The characters of the positive spin representations of $SU(2)$ are given by

\[ \chi^{(j)}(\theta) = \frac{\sin\left(j + \frac{1}{2}\right) \theta}{\sin \frac{1}{2}\theta}, \]

and satisfy the orthogonality relation

\[ \frac{1}{2\pi} \int_0^{2\pi} d\theta \chi^{j_1}(\theta) \chi^{j_2}(\theta) (1 - \cos \theta) = \delta_{j_1 j_2}. \tag{26} \]

We remind the reader that a character is a function on the conjugacy classes of a group. For $SU(2)$, all rotations through the same angle $\theta$ are in the same class, irrespective of the direction of their axes.

We will now attempt to define a meaningful character for the spin $(-\frac{1}{2})$ representation.

In accordance with the discussion of the previous section, we would like the character of the product representation to be equal to that of the trivial representation. The expression for the character should therefore be invariant under the BRST reduction used to eliminate the null states in the product representation. An expression that ignores the discarded states is given by

\[ \chi^{(-\frac{1}{2}) \otimes (\frac{1}{2})}(\theta) = \text{Tr} \left( 1_s \otimes \eta \right) e^{i\theta J_z}, \]

where $\eta = (-)^N_g$ and $e^{i\theta J_z}$ is the representative of the conjugacy class corresponding to angle $\theta$. Here $J_z = J_s^z \otimes 1_g + 1_s \otimes J_g^z$.

From the properties of the trace, it immediately follows that

\[ \chi^{(-\frac{1}{2}) \otimes (\frac{1}{2})}(\theta) = \chi^{(-\frac{1}{2})}(\theta) \chi^{(\frac{1}{2})}(\theta), \]

where

\[ \chi^{(-\frac{1}{2})}(\theta) = \text{Tr} \eta e^{i\theta J_s^z}, \]
\[ \chi^{(\frac{1}{2})}(\theta) = \text{Tr} e^{i\theta J_z}. \]

Furthermore, we have

\[ \chi^{(-\frac{1}{2})}(\theta) = \text{Tr} \eta e^{i\theta J_s} = e^{i\theta/2} \left( 1 - e^{i\theta} + e^{2i\theta} - \ldots \right) = \frac{1}{2 \cos \frac{1}{2} \theta}, \tag{27} \]
which is just the inverse of the character

\[ \chi\left(\frac{1}{2}\right)(\theta) = \frac{\sin \theta}{\sin \frac{\theta}{2}} = 2 \cos \frac{1}{2}\theta \]

of the spin \( \frac{1}{2} \) representation. This means that

\[ \chi\left(-\frac{1}{2}\right)\chi\left(\frac{1}{2}\right) = 1 = \chi^{(0)}. \]

This is consistent with the fact that the product of the spinor and the ghost gives an effectively trivial representation.

The character \( \chi\left(-\frac{1}{2}\right) \) is not normalizable with respect to the measure in (26). Indeed, because of the singularity at \( \theta = \pi \), it does not live in the same function space. However, we can take its inner product with positive spin characters, which means that we may regard it as a distribution.

Note that the singularity at \( \theta = \pi \) is to be expected. Indeed, we have seen already that certain elements in the conjugacy class of rotations by \( \pi \) are ill-defined (notably the finite rotations in section 8 where \( a \to 0 \)). The singularity in the character reflects this fact.

The dimension of a representation is conventionally related to the character as \( d = \chi(1) \). For the ghost, this gives a fractional dimension

\[ d = \frac{1}{2}. \]

This is consistent with the argument that the combined spinor-ghost system effectively transforms in the trivial representation of \( \text{SU}(2) \), which has dimension 1. Indeed, the dimension of a product representation is the product of the dimensions, so that the combined system will have effective dimension \( \frac{1}{2} \cdot 2 = 1 \), which is indeed correct.

The dimension can also be related to the entropy as follows. Conventionally, the entropy of a maximally mixed state is related to the dimension of the state space \( \mathcal{H} \) by

\[ S_{\text{extremal}} = \log_2 \dim (\mathcal{H}). \]

For the ghost, we have seen that \( S_{\text{extremal}} = -1 \), consistent with a dimension \( \dim (\mathcal{H}) = \frac{1}{2} \).
11 Conclusion

In this article, we studied the properties of a BRST ghost degree of freedom complementary to a two-state spinor. We showed that the ghost entropy ranges between 0 and $-1$ and cancels the entropy of the spinor.

We then constructed an irreducible representation of the $su(2)$ Lie algebra of negative spin, equal to $-\frac{1}{2}$, on the ghost state space. We were also able to exhibit representation matrices for finite $SU(2)$ transformations, although we found that these become singular for certain $SU(2)$ elements, and cannot be arbitrarily composed because of domain issues. In other words, the Lie algebra representation does not exponentiate to a give a representation of the whole group.

Since we chose the BRST operator $Q$ to be equal to the Casimir operator $J^2$ of the product representation, the generators of the product representation all commute with $Q$. Using this, we discussed the sense in which the positive and negative spin representations combine to give an effectively trivial representation in the product space and showed that a character can be defined for the negative representation in a way that is consistent with this cancellation. We argued that a fractional effective dimension of $\frac{1}{2}$ can be assigned to the ghost representation.

The methods of this article can be expanded to the study of arbitrary negative spin representations of $su(2)$. Work in this direction is in progress [7].

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