Coherent states with minimum Gini uncertainty for finite quantum systems

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Abstract – Uncertainty relations ∆(ρ) ≥ η_d in terms of the Gini index are studied. The ‘Gini uncertainty constant’ η_d is estimated numerically and compared to an upper bound ˜η_d ≥ η_d. It is shown that for large d we get ˜η_d ≈ η_d. States |g⟩ with minimum Gini uncertainty and displacement transformations are used to define coherent states |α, β⟩_g (where α, β ∈ Z_d) with minimum Gini uncertainty (∆[|α, β⟩_g]⟨α, β|) ≈ η_d). The |α, β⟩_g resolve the identity, and therefore an arbitrary state can be expanded in terms of them. This expansion is robust in the presence of noise.

1. Introduction. – Uncertainty relations are very important in quantum physics. For the harmonic oscillator we have the Heisenberg uncertainty relation (and the Robertson-Schrödinger uncertainty relation which is related to it). We also have entropic uncertainty relations (e.g., [1]).

For systems with finite-dimensional Hilbert space (e.g., [2]) that we consider in this paper, we have the entropic uncertainty relations [3–6], the Donoho-Stark uncertainty relations [7], uncertainty relations that use the Gini index [8], etc.

The Gini index (e.g., [9,10]) is an alternative to the standard deviation and is popular in Mathematical Economics. If ρ is a density matrix, uncertainty relations with the Gini index consider the probability distributions P_X(r|ρ) and P_P(r|ρ) related to dual ‘position’ and ‘momentum’ bases (defined explicitly in Eq.(6) below). Then we calculate the Gini indices G_X(ρ) and G_P(ρ) for these probability distributions.

It has been shown [8] that

\[ \Delta(\rho) \geq \eta_d \]  

where \( \Delta(\rho) \) is defined in terms of Gini indices in Eq.(25) below, and \( \eta_d \) is a constant (defined in Eq.(24) below) which we call ‘Gini uncertainty constant’ and which depends only on the dimension \( d \). \( \eta_d \) is the analogue of \( \frac{1}{2} \) in the Heisenberg uncertainty relation for the harmonic oscillator.

In this paper:

- Using a large number of pure states, we estimate numerically \( \eta_d \) as functions of \( d \) (in fig.2). We also calculate analytically an upper bound

\[ \bar{\eta}_d = \frac{d - 1}{d + 1} \cdot \frac{\sqrt{d}}{1 + \sqrt{d}}, \]

and we show numerically that for large \( d \) we get \( \bar{\eta}_d \approx \eta_d \) (in fig.3).
We find numerically states $|g\rangle$ that satisfy the Gini uncertainty inequality as equality $(\Delta(|g\rangle|g\rangle) = \eta_2$. We use these states and displacement transformations, to define coherent mixed states $|\alpha,\beta\rangle_g$ (where $\alpha, \beta \in \mathbb{Z}_d$) with minimum Gini uncertainty $(\Delta[|\alpha,\beta\rangle_g g\langle\alpha,\beta|] \approx \eta_2)$. We show that they are related to each other through displacement transformations, and also that they resolve the identity. The resolution of the identity is used for an expansion of an arbitrary state in terms of them (Eq. (39)). It is shown that this expansion is robust in the presence of noise.

In section 2 we study a family of states with coherence properties in the sense of proposition [1]. In section 3 we estimate numerically $\eta_2$ and calculate analytically an upper bound $\tilde{\eta}_2$. For large $d$, we get $\eta_2 \approx \eta_2$.

In section 4 we define coherent states $|\alpha,\beta\rangle_g$ with minimum Gini uncertainty. They resolve the identity, and we can expand an arbitrary state in terms of them. One of the merits of studying coherent states, is that expansions in terms of them are robust in the presence of noise. We show that this is the case in the present context.

In section 5 we show that states that minimise the ‘Gini uncertainty relation’ do not minimise the ‘entropic uncertainty relation’. Different uncertainty relations are different constraints for the quantum formalism.

We conclude in section 6 with a discussion of our results.

2. States with coherence properties in finite quantum systems. – We consider a quantum system with variables in $\mathbb{Z}_d$, the ring of integers modulo $d$. $H_d$ is the $d$-dimensional Hilbert space describing these systems. Some of the results in these systems are different for odd or even $d$, and in this paper $d$ is an odd integer.

$|X; r\rangle$ where $r = -\frac{d-1}{2}, \ldots, \frac{d-1}{2}$, is an orthonormal basis which we call position basis (the $X$ in this notation is not a variable, but it simply indicates position basis). Through a Fourier transform we get another orthonormal basis that we call momentum basis:

$|P; r\rangle := F|X; r\rangle; \quad F = \frac{1}{\sqrt{d}} \sum_{r,s} \omega^{rs}|X; r\rangle\langle X; s|$;

$\omega = \exp\left(\frac{2\pi i}{d}\right); \quad r, s = -\frac{d-1}{2}, \ldots, \frac{d-1}{2}$. (3)

The displacement operators $Z^\alpha, X^\beta$ in the phase space $\mathbb{Z}_d \times \mathbb{Z}_d$, are given by

$$Z^\alpha := \sum_m \omega^{\alpha m}|X; m\rangle\langle X; m|$$

$$= \sum_m |P; m + \alpha\rangle\langle P; m|$$

$$X^\beta := \sum_m |X; m + \beta\rangle\langle X; m|$$

$$= \sum_m \omega^{-\beta m}|P; m\rangle\langle P; m|. \quad (4)$$

where $\alpha, \beta \in \mathbb{Z}_d$. General displacement operators are the unitary operators

$$D(\alpha, \beta) := Z^\alpha X^\beta \omega^{-2^{-1}\alpha \beta}$$

$$[D(\alpha, \beta)]^\dagger = D(-\alpha, -\beta); \quad \alpha, \beta \in \mathbb{Z}_d \quad (5)$$

The $2^{-1} = \frac{d+1}{2}$ exists in $\mathbb{Z}_d$ because $d$ is an odd integer. They act on position states as follows:

$$D(\alpha, \beta)|X; \kappa\rangle = \omega^{-2^{-1}\alpha \beta + \alpha \kappa}|X; \kappa + \beta\rangle \quad (6)$$

The $D(\alpha, \beta)\omega^\gamma$ with $\alpha, \beta, \gamma \in \mathbb{Z}_d$, form a representation of the Heisenberg-Weyl group with multiplication rule:

$$[D(\alpha_1, \beta_1)\omega^{\gamma_1}][D(\alpha_2, \beta_2)\omega^{\gamma_2}]$$

$$= D(\alpha, \beta)\omega^{\gamma} \quad (7)$$

where

$$\alpha = \alpha_1 + \alpha_2; \quad \beta = \beta_1 + \beta_2;$$

$$\gamma = \gamma_1 + \gamma_2 + 2^{-1}(\alpha_1 \beta_2 - \alpha_2 \beta_1) \quad (8)$$

If $|f\rangle$ is a state (‘fiducial state’), let $\Sigma(|f\rangle)$ be the set of $d^2$ density matrices

$$|\alpha, \beta\rangle_f := D(\alpha, \beta)|f\rangle; \quad \alpha, \beta \in \mathbb{Z}_d, \quad (9)$$

The fiducial vector should be a ‘generic’ vector (other than position or momentum states). They have ‘coherence properties’ described in the following proposition.

**Proposition 1.**

1. The displacement transformations $D(\kappa, \lambda)\exp(\imath \mu)$ leave invariant the set $\Sigma(|f\rangle)$, in the sense that they transform states in this set into other states in the same set:

$$[D(\kappa, \lambda)\omega^\nu]|\alpha, \beta\rangle_f = \omega^\nu|\alpha + \kappa, \beta + \lambda\rangle_f$$

$$\nu = \mu + 2^{-1}(\kappa \beta - \lambda \alpha) \quad (10)$$
Proof.

(1) Eq. (10) is proved using Eq. (7).

(2) We need to prove that

$$\frac{1}{d} \sum_{\alpha, \beta} \langle \alpha, \beta \rangle f (\alpha, \beta) = 1.$$  \hspace{1cm} (11)

This reduces to proving that

$$\frac{1}{d} \sum_{\alpha, \beta} \langle X; \kappa | \alpha, \beta \rangle f (\alpha, \beta) | X; \lambda \rangle = \delta(\kappa, \lambda).$$  \hspace{1cm} (12)

where $\delta(\kappa, \lambda)$ is the Kronecker delta. Using Eq. (6) this reduces to proving that

$$\frac{1}{d} \sum_{\alpha, \beta} \omega^{\alpha(\kappa - \lambda)} \langle X; \kappa - \beta | f \rangle \langle f | X; \lambda - \beta \rangle = \delta(\kappa, \lambda).$$  \hspace{1cm} (13)

But

$$\frac{1}{d} \sum_{\alpha} \omega^{\alpha} = \delta(\kappa, \lambda),$$  \hspace{1cm} (14)

and this reduces to

$$\delta(\kappa, \lambda) \sum_{\beta} |\langle f | X; \kappa - \beta \rangle|^2 = \delta(\kappa, \lambda),$$  \hspace{1cm} (15)

which is true.

Using the resolution of the identity in Eq. (11) we expand an arbitrary state $|s\rangle$ as

$$|s\rangle = \sum_{\alpha, \beta} s_{\alpha, \beta} |\alpha, \beta\rangle f; \quad s_{\alpha, \beta} = \frac{1}{d} f (\alpha, \beta | s\rangle.$$  \hspace{1cm} (16)

The states $|\alpha, \beta\rangle f$ in Eq. (9) have the coherence properties described in proposition 1, and have been studied using the language of analytic functions (and in particular Theta functions) in refs. [11][12]. These states are not (in general) minimum uncertainty states. In this paper we identify among them, which states have minimum quantum uncertainty (defined through Gini uncertainties).

### 3. Gini uncertainties for finite quantum systems.

If $\rho$ is a density matrix, the probability distributions related to the position and momentum basis, are

$$P_X(r|\rho) = \langle X; r | \rho | X \rangle; \quad \sum_{r=0}^{d-1} P_X(r|\rho) = 1; \quad r \in \mathbb{Z}_d$$

$$P_P(r|\rho) = \langle P; r | \rho | P \rangle; \quad \sum_{r=0}^{d-1} P_P(r|\rho) = 1. \hspace{1cm} (17)$$

They are both measurable quantities.

**Definition 2.** The Gini index with respect to the position basis is given by one of the following equivalent to each other relations (e.g., proposition 6.1 in ref [13]):

1. We use a permutation $\pi_X$ to order the probabilities $P_X(r|\rho)$ in ascending order:

$$P_X(\pi_X(0)|\rho) \leq P_X(\pi_X(1)|\rho) \leq \ldots \leq P_X(\pi_X(d-1)|\rho)$$  \hspace{1cm} (18)

Then

$$G_X(\rho) = 1 - \frac{2}{d+1} \sum_{r=0}^{d-1} |P_X(\pi_X(r)|\rho) - P_X(\pi_X(r+1)|\rho)|$$  \hspace{1cm} (19)

2. 

$$G_X(\rho) = \frac{1}{2(d+1)} \sum_{r,s} |P_X(r|\rho) - P_X(s|\rho)| \hspace{1cm} (20)$$

Roughly speaking the standard deviation is sum of differences squared, while the Gini index is sum of the absolute values of the differences. The $G_X(\rho)$ indicates the Gini uncertainty in $P_X(r|\rho)$. $G_X(\rho) = \frac{d-1}{d+1}$ indicates a certain outcome (one $P_X(r|\rho)$ equal to 1 and the others equal to zero), while $G_X(\rho) = 0$ indicates the most uncertain outcome ($P_X(r|\rho) = \frac{1}{d}$). We note that the probabilities $P_X(r|\rho)$, and therefore $G_X(\rho)$, are measurable quantities.

It has been shown [8] that

$$0 \leq G_X(\rho) \leq \frac{d-1}{d+1}; \quad G_X(\rho|\rho(\alpha, \beta)) = G_X(\rho), \hspace{1cm} (21)$$

where

$$\rho(\alpha, \beta) = D(\alpha, \beta) |\alpha, \beta\rangle f D(\alpha, \beta)|.$$  \hspace{1cm} (22)
In a similar way to \( \mathcal{G}_X(\rho) \), we define the Gini index \( \mathcal{G}_P(\rho) \) in terms of the \( P_P(r|\rho) \) with respect to the momentum basis. We also define the sum
\[
\mathcal{G}_{XP}(\rho) := \mathcal{G}_X(\rho) + \mathcal{G}_P(\rho).
\]
Both \( \mathcal{G}_X(\rho) \) and \( \mathcal{G}_P(\rho) \) are measurable quantities using two different ensembles describing the density matrix \( \rho \) (because the projectors \( |X;r\rangle\langle X;r| \) and \( |P;s\rangle\langle P;s| \) do not commute). Therefore \( \mathcal{G}_{XP}(\rho) \) is a measurable quantity.

Ref. [8] has shown that \( \mathcal{G}_{XP}(\rho) \) cannot take values which are arbitrarily close to \( 2\frac{d+1}{d+2} \). Therefore
\[
\eta_d := 2\frac{d-1}{d+1} - \sup_{\rho} \mathcal{G}_{XP}(\rho) > 0.
\]
We refer to this as the ‘Gini uncertainty constant’, and we get the uncertainty relation
\[
\Delta(\rho) := 2\frac{d-1}{d+1} - \mathcal{G}_{XP}(\rho) \geq \eta_d.
\]
Using Eq. (21) we show that
\[
\Delta[\rho(\alpha, \beta)] = \Delta(\rho).
\]
Also we note that
\[
\mathcal{G}_{XP}\left(\frac{1}{d^2}\right) = 0; \quad \Delta\left(\frac{1}{d^2}\right) = 2\frac{d-1}{d+1}.
\]
If \( \rho, \sigma \) are two density matrices, and \( p \) a probability then
\[
\mathcal{G}_X[p\rho + (1-p)\sigma] \leq p\mathcal{G}_X(\rho) + (1-p)\mathcal{G}_X(\sigma).
\]
Let \( \rho \) be a density matrix and \( \lambda_i, |e_i\rangle \) its eigenvalues and eigenvectors:
\[
\rho = \sum_i \lambda_i |e_i\rangle\langle e_i|; \quad \sum_i \lambda_i = 1.
\]
Then Eq. (28) shows that
\[
\mathcal{G}_X(\rho) \leq \sum_i \lambda_i \mathcal{G}_X(|e_i\rangle\langle e_i|)
\]
\[
\mathcal{G}_P(\rho) \leq \sum_i \lambda_i \mathcal{G}_P(|e_i\rangle\langle e_i|)
\]
and therefore
\[
\mathcal{G}_{XP}(\rho) \leq \mathcal{G}_{XP}(|e_{\text{max}}\rangle\langle e_{\text{max}}|)
\]
where \( |e_{\text{max}}\rangle \) is the eigenstate with maximum \( \mathcal{G}_{XP}(|e_i\rangle\langle e_i|) \).

**Numerical estimate of the Gini uncertainty constant \( \eta_d \).** We consider a large set \( R(d) \) of random pure states. Then
\[
\sup_{\rho} \mathcal{G}_{XP}(\rho) > \max_{\rho \in R(d)} \mathcal{G}_{XP}(\rho)
\]
Eq. (31) shows that for the supremum we only need to consider pure states. The random pure states were produced using Qiskit [14].

Also for the state
\[
\rho = |s\rangle\langle s|; \quad |s\rangle = \sqrt{\frac{\sqrt{d}}{2\sqrt{d} + 2}}(|X;0\rangle + |P;0\rangle)
\]
where \( a, b \in \mathbb{Z}_d \), we find
\[
\mathcal{G}_{XP}(\rho) = \frac{d-1}{d+1} \left(1 + \frac{1}{1 + \sqrt{d}}\right).
\]
Therefore
\[
\sup_{\rho} \mathcal{G}_{XP}(\rho) \geq \frac{d-1}{d+1} \left(1 + \frac{1}{1 + \sqrt{d}}\right),
\]
and
\[
\eta_d \leq \bar{\eta}_d; \quad \bar{\eta}_d = \frac{d-1}{d+1} \cdot \frac{\sqrt{d}}{1 + \sqrt{d}}.
\]

Fig. 1: \( \mathcal{G}_{XP}(\rho) \) for 400 random pure states (\( d = 7 \))
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get a very similar (but not exactly the same) value for
\[ \max_{\rho \in \mathbb{R}^{|d|}} [G_{XP}(\rho)]. \]
We also note that the result obeys the inequality in Eq. (35).

Using 400 random pure states for each value of the odd dimension \( d \), we calculated
\[ \max_{\rho \in \mathbb{R}^{|d|}} [G_{XP}(\rho)] \]
(which is a lower bound for \( \sup_{\rho} [G_{XP}(\rho)] \)), and in turn a numerical estimate of \( \eta_d \), for odd values of \( d \). The results are shown in fig. 2 (the ‘wiggles’ are an artefact of the numerical work).

For large \( d \) (e.g. \( d > 41 \)) fig. 3 shows that
\[ \delta := \eta_d - \tilde{\eta}_d \approx 0. \quad (37) \]

According to Eq. (26), the \( d^2 \) states
\[ |\alpha, \beta\rangle = D(\alpha, \beta)|g\rangle \quad (38) \]
also have minimum Gini uncertainty
\( \Delta[|\alpha, \beta\rangle_g \langle \alpha, \beta|] \approx \eta_d \). In addition to that these states also obey proposition 1, i.e., they are related to each other through displacement transformations, and they resolve the identity. Consequently we can expand an arbitrary state in terms of these states (as in Eq. (16)):
\[ |s\rangle = \sum_{\alpha, \beta} s_{\alpha, \beta} |\alpha, \beta\rangle_g; \quad s_{\alpha, \beta} = \frac{1}{d} g(\alpha, \beta|s). \quad (39) \]

As an example we take \( d = 3 \) and
\[ |s\rangle = \begin{pmatrix} 0.5040 - 0.1526i \\ 0.3283 + 0.1757i \\ 0.8324 + 0.0231i \end{pmatrix} \quad (40) \]

The 9 component vectors are given in table 2.

There is redundancy in coherent states (in the present context we have \( d^2 \) coherent states in a \( d \)-dimensional Hilbert space). This redundancy makes

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**Fig. 2:** \( \eta_d \) as a function of (odd) \( d \).

**Fig. 3:** \( \delta = \eta_d - \tilde{\eta}_d \) as a function of (odd) \( d \).
the expansion in terms of coherent states, robust in the
presence of noise. We show this for the above example.

There are many ways of adding noise to the exact
result. Here we multiplied each of the components
sα,β in table 2 with 1 + λα,β where λα,β is a random number
uniformly distributed in the interval (−ε, ε):

\[ |\bar{s}\rangle = \sum_{\alpha,\beta} (1 + \lambda_{\alpha,\beta}) s_{\alpha,\beta}|\alpha,\beta\rangle \]

We then calculated the ‘error vector’

\[ |e\rangle := |\bar{s}\rangle - |s\rangle = \sum_{\alpha,\beta} \lambda_{\alpha,\beta} s_{\alpha,\beta}|\alpha,\beta\rangle \]

and its magnitude

\[ |||e||| = \sqrt{\sum_{i=1}^{d} |e_i|^2} \]

We repeated that numerical experiment 10 times and
found an average value of |||e|||av = 0.01116 for ε = 0.3,
and |||e|||av = 0.02095 for ε = 0.5. It is seen that the
expansion is robust in the presence of noise.

\[ E_S(\rho) := E_X(\rho) + E_P(\rho) - \ln d \geq 0. \] (45)

We use logarithms with base e.

Using the density matrices |g⟩ defined earlier, we calculated the

\[ \mathcal{E}_d := \mathcal{E}[|0,0\rangle_g g|0,0\rangle] \]

In fig. 4 we plot the \( \mathcal{E}_d \) as a function of d. It is seen that the states |0,0⟩g which are states with minimum
Gini uncertainty (∆[0,0]g = 0, 0] ≈ ηd), are not states
with minimum entropic uncertainties.

Different uncertainty relations are different constraints for the quantum formalism.

6. Discussion. – We studied uncertainty relations
for systems with finite-dimensional Hilbert space, using the Gini index (Eq.(25)). They involve the quantity \( \mathcal{G}_{XP} \) in Eq.(23), which as we explained earlier is a measurable quantity. We estimated numerically \( \eta_d \), as a function of d. We also gave the analytical upper bound \( \eta_d \) and shown that for large d, we get \( \eta_d \approx \eta_d \).

We found numerically states |g⟩ that satisfy the Gini uncertainty inequality, as an equality (∆|g⟩= 0, 0] ≈ ηd). We used these states to define the coherent states |α, β⟩g (Eq.(38)) that have minimum Gini uncertainty. They resolve the identity, and an expansion of an arbitrary state in terms of them is given in Eq.(39). It has been shown with numerical examples, that this expansion is robust in the presence of noise.

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| \(s_{-1,-1}\) | 1, -1, -1) \(g\) | \(0.2527 - 0.0295i\) |
|----------------|-----------------|-----------------|
|                 |                 | 0.0844 + 0.0181i |
|                 |                 | 0.1074 - 0.0148i |
| \(s_{-1,0}\) | -1, 0) \(g\) | \(0.0893 + 0.0016i\) |
|                 |                 | 0.2093 + 0.0081i |
|                 |                 | 0.0664 + 0.0255i |
| \(s_{-1,1}\) | -1, 1) \(g\) | \(0.0923 + 0.0310i\) |
|                 |                 | 0.1222 - 0.0029i |
|                 |                 | 0.2869 - 0.0008i |
| \(s_{0,-1}\) | 0, -1) \(g\) | \(-0.0672 - 0.0952i\) |
|                 |                 | -0.0361 - 0.0161i |
|                 |                 | 0.0217 + 0.0447i |
| \(s_{0,0}\) | 0, 0) \(g\) | \(-0.0338 - 0.0055i\) |
|                 |                 | 0.0270 + 0.0757i |
|                 |                 | 0.0234 - 0.0140i |
| \(s_{0,1}\) | 0, 1) \(g\) | \(-0.0108 - 0.0733i\) |
|                 |                 | -0.0488 + 0.0791i |
|                 |                 | 0.2180 + 0.0109i |
| \(s_{1,-1}\) | 1, -1) \(g\) | \(0.0688 + 0.0071i\) |
|                 |                 | -0.0191 + 0.0136i |
|                 |                 | -0.0126 - 0.0266i |
| \(s_{1,0}\) | 1, 0) \(g\) | \(0.0151 - 0.0175i\) |
|                 |                 | 0.0169 + 0.0517i |
|                 |                 | -0.0159 - 0.0094i |
| \(s_{1,1}\) | 1, 1) \(g\) | \(-0.0367 + 0.0287i\) |
|                 |                 | -0.0274 - 0.0517i |
|                 |                 | 0.1370 + 0.0078i |

Table 2: The vectors in the expansion in Eq.(39) for the state in Eq.(40)