On State Counting and Characters

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Abstract

We outline the relationship between the thermodynamic densities and quasi-particle descriptions of spectra of RSOS models with an underlying Bethe equation. We use this to prove completeness of states in some cases and then give an algorithm for the construction of branching functions of their emergent conformal field theories. Starting from the Bethe equations of $D_n$ type, we discuss some aspects of the $Z_n$ lattice models.
1. Introduction

A number of questions of interest regarding interacting many-body quantum systems are posed within the framework of the limit of the number of particles going to infinity. Two situations in which will be relevant to the discussion in this paper are the thermodynamics at non-zero temperatures and the low-energy eigenvalue spectrum of the hamiltonian. To set up the thermodynamic formalism, it is imperative that one has a control over counting the states of the system, in order to compute the entropy. On the other hand, the low-energy spectrum is, generically, of a form that is interpreted as that describing “quasi-particles,” which are then said to populate the allowed energy levels subject to composition rules. In order to determine these rules, it is again necessary to classify and count the states.

The main point of this paper is twofold – first, we point out a direct (and innocent-looking) connection between the thermodynamic formalism and the quasi-particle description in a wide class of one-dimensional quantum spin systems whose spectral information is expressed in terms of solutions to the Bethe equations. This will then enable us to carry out a detailed counting of the states on a finite lattice, and consequently to construct the branching functions of the emergent conformal field theories.

A quick word about where to place these results: The correspondence between 2-dimensional lattice models of statistical mechanics, their 1-dimensional spin-chain counterparts and their field theory limits – both conformal (massless) and their integrable off-critical (massive) counterparts, have long been explored in the context of the particular models we consider. In [3], the study of the critical generalized RSOS models was undertaken using ”Bethe ansatz techniques,” that had earlier been developed for excitation spectra [2] and for thermodynamics [2], (see also [7] [8]). In addition, the authors of [3] computed the central charges by taking the $T \to 0$ limit of the thermodynamic calculation of the entropy, following [4]. These methods have been further extended for the critical models of higher rank algebras ([5], [13]) and for the off-critical models whose masses have been determined and S-matrices computed in [12]. The key ingredient in all of these studies is the densities of the roots of the Bethe equations, and to quote the authors of [12], ”the set of densities ... determines all macroscopical observables in the model (pg. 307).”

In [15], [16], the composition rules ([10]) and the quasi-particle spectrum ([14]) were used to construct the partition function of the low-temperature quantum 3-state Potts spin chain, which were shown to to give $q$-series expansions of known modular branching
functions of the associated conformal field theories. The work of [21] used the method of [20] to extract the central charge by taking the $q \to 1^-$ limit of the $q$-series, uncovering, in the process, the form of the dilogarithm identities very similar to those that had featured in the central charge computations via the thermodynamics of these Bethe ansatz solvable models. This similarity was exploited to formulate new $q$-series identities for branching functions in [17], [18], [22] and [23]. The work of [24] served as a bridge between the thermodynamic calculations of [5], [6] and [13] and the quasi-particle approach of [15] and [16], thus “explaining” away the origin of these identities into the structure of the Bethe equations themselves. The key step that allowed this connection is the innocuous equation (4.4). The proofs of some of these forms were already known ([28]), but since then, proofs of some more of these identities have appeared ([29], [30]).

A brief outline of this paper: Section 2 explicitly shows the counting procedure that was followed in [10] to get the composition rules for the simplest series of RSOS models. Section 3 outlines the procedure for those models based on the simply laced algebras. Section 4 sets up the correspondence between the thermodynamic and the quasi-particle descriptions. Section 5 outlines the algorithm for setting up the $q$-series that count the low-lying states which are (conjectured, in general, to be) branching functions in the vacuum sector of the coset conformal field theory. Section 6 talks about the $Z_n$ models of [31] as an example, and the Appendix works out a useful combinatoric identity to count the states in the ground state sector of arbitrary spin $su(2)$ models.

2. Counting states in the ABF models

The RSOS models that were introduced in [4]. In [4], the eigenvalues of the transfer matrix of these models were expressed in terms of the roots of the following equation (in their notation, $l = r - 2$):

$$
\left[ \frac{\sinh \left( \frac{\pi}{2(l+2)}(\lambda_j + i) \right)}{\sinh \left( \frac{\pi}{2(l+2)}(\lambda_j - i) \right)} \right]^N = \Omega_j \prod_{k=1}^{N/2} \frac{\sinh \left( \frac{\pi}{2(l+2)}(\lambda_j - \lambda_k + 2i) \right)}{\sinh \left( \frac{\pi}{2(l+2)}(\lambda_j - \lambda_k - 2i) \right)}.
$$

(2.1)

$\Omega_j$ is a phase factor, $N$ the size of the chain and $l$ is an integer.

The solutions to these equations are assumed to be of the “string” form [1] and following [4], we write them in the form

$$
\lambda_j = \lambda + i(j + 1 - 2j_1), \quad 1 \leq j_1 \leq j, \quad 1 \leq j \leq l, \quad \Im \lambda = 0.
$$

(2.2)
These assumptions were made in the bulk in [3], (with a number of caveats referring to the appearance of solutions not of the above form) but here we shall impose these on finite-size lattices. (Of course, the same thermodynamic limit can be recovered.) On multiplying out the equations for the components of each string, [7] we get a set of equations for the real parts, \( \lambda \), of the strings. We then take the logarithm of (2.1) so that the integer branches are distinct:

\[
Nt_j(\lambda^j_\mu) = 2\pi i I_{j\mu} + \sum_{k=1}^l \sum_{\nu=1}^{M_k} \Theta_{jk}(\lambda^j_\mu - \lambda^k_\nu),
\]

(2.3)

where \( \lambda^j_\mu \) labels the center of the \( \mu \)-th string of length \( j \) which is a root of eq. (2.1). The functions \( t_j \) and \( \Theta_{jk} \) are defined below.

\[
t_j(\lambda) = f(\lambda; |j - 1| + 1),
\]

(2.4)

\[
\Theta_{jk}(\lambda) = \left( f(\lambda; |j - k|) + 2 \sum_{i=1}^{\min(j,k)-1} f(\lambda; |j - k| + 2i) + f(\lambda; j + k) \right)
\]

(2.5)

\[
f(\lambda; n) = \frac{1}{2\pi i} \ln \left( \frac{\sinh \frac{1}{2}(in - \lambda)}{\sinh \frac{1}{2}(in + \lambda)} \right),
\]

(2.6)

for integer values of \( n/(l + 2) \), and is 0 otherwise.

For pedagogical reasons, we shall carry out in some detail, the counting procedure which is central to all further physical elaborations. We define

\[
Z_j(\lambda) \equiv t_j(\lambda) - \frac{1}{N} \sum_{k=1}^l \sum_{\nu=1}^{M_k} \Theta_{jk}(\lambda, \lambda^k_\nu),
\]

(2.7)

so that the (half-) integers \( I_{j\mu} \) satisfy

\[
Z_j(\lambda^j_\mu) = 2\pi i \frac{I_{j\mu}}{N}.
\]

(2.8)

If we assume that \( Z_j(\lambda) \) is monotonic, then the range of the integers, \( \Delta I_j \equiv I_{j,max} - I_{j,min} \) is set by taking the difference of the limiting values, \( Z_j(\pm\infty) \). A result used repeatedly is the following:

\[
d_n \equiv \frac{1}{2\pi i} [f(+\infty, n) - f(-\infty, n)] = \frac{l + 2 - n}{l + 2}, \quad n \leq l + 1.
\]

(2.9)

Therefore,

\[
\Delta I_j = N d_j - \sum_{k=1}^l M_k \left\{ (1 - \delta_{jk}) d_{j-k} + d_{j+k} + 2 \sum_{i=1}^{\min(j,k)-1} d_{j-k+2i} \right\},
\]

(2.10)
where $\min(i, j)$ picks out the smaller of the two values $i$ and $j$. Substitution (2.9) into (2.10) and performing the sum over the variable $i$ takes the form

\[
(l + 2)\Delta I_j = N(l + 2 - j) - \sum_{k=1}^{l} M_k \left\{ \left( (l + 2) - |j - k| \right) - \delta_{jk} \left( (l + 2) - |j - k| \right) \right.
\]

\[
\left. + \left( (l + 2) - (j + k) \right) + 2 \left( (l + 2) - |j - k| \right) (\min(j, k) - 1) \right. 
\]

\[
- 2\min(j, k) (\min(j, k) - 1) \right\} . \tag{2.11}
\]

Splitting up the sum over $k$ to get past the $\min(j, k)$ function, we get

\[
(l + 2)\Delta I_j = N((l + 2) - j) - \sum_{k=1}^{j-1} M_k \left\{ \left( (l + 2) - j + k \right) + \left( (l + 2) - j - k \right) \right.
\]

\[
\left. + 2 \left( (l + 2) - j + k \right) (k - 1) - 2k(k - 1) \right\} 
\]

\[
- M_j \left\{ \left( (l + 2) - 2j \right) - 2j(j - 1) + 2(l + 2)(j - 1) \right\} 
\]

\[
- \sum_{k=j+1}^{l} M_k \left\{ \left( (l + 2) - k + j \right) + \left( (l + 2) - j - k \right) \right. 
\]

\[
\left. + 2 \left( (l + 2) - k + j \right) (j - 1) - 2j(j - 1) \right\} 
\]

\[
= N((l + 2) - j) - \left( (l + 2) - j \right) \sum_{k=1}^{j-1} kM_k + 2j^2 M_j + (l + 2)M_j 
\]

\[
- 2(l + 2)jM_j - 2j \sum_{k=j+1}^{l} \left( (l + 2) - k \right) M_k . \tag{2.12}
\]

Consider the special case of the strings of length $l$: setting $j = l$ above,

\[
(l + 2)\Delta I_l = N(l + 2 - l) - 2l(l + 2) \sum_{k=1}^{l-1} kM_k + (2l^2 + (l + 2) - 2(l + 2)l)M_l 
\]

\[
= 2N - 4 \sum_{k=1}^{l-1} kM_k + M_l(2 - 3l) , \tag{2.13}
\]

\[
= 4 \left( \frac{N}{2} - \sum_{k=1}^{l} kM_k \right) + 4lM_l + M_l(2 - 3l) , 
\]

\[
= (l + 2)M_l ,
\]

where we have used the sum rule on the total number of roots in the last step. We thus notice the remarkable feature that is enforced by the string hypothesis and the monotonicity assumptions on $Z(\lambda)$ that $\Delta I_l = M_l$, i.e., all the allowed values for the integers that
specify the locations of the real parts of strings of length \( l \) are always occupied – there are “no holes” in this sector [5]. Going back to (2.12), we replace the \( M_l \) occurring in the last sum by \( \frac{N}{2\pi} \sum_{k=1}^{l-1} k M_k \), and after a few simplifications and dividing by \( (l + 2) \), we get

\[
\Delta I_j = N(1 - \frac{j}{l}) + 2\sum_{l=1}^{j} k M_k(j - l) + 2\sum_{k=j+1}^{l-1} M_k(k - l) + M_j, \\
= N(1 - \frac{j}{l}) + 2\sum_{k=1}^{l-1} \left\{ \min(j, k) - \frac{jk}{l} \right\} M_k \\
= N \left\{ \min(j, 1) - \frac{j \cdot 1}{l} \right\} + 2\sum_{k=1}^{l-1} \left\{ \min(j, k) - \frac{jk}{l} \right\} M_k.
\]

(2.14)

Note that the coefficients in the sum in the last line in (2.14) are the \( j, k \)th matrix elements of the inverse Cartan matrix of the \( A_{l-1} \) root system! We write the coefficient of \( N \) in a suggestive way for comparison with later results.

These integer specifications can be used to label the eigenvalues of the hamiltonian. Does this give a complete classification? We shall evaluate the number of ways these integers can be ascribed to the \( \lambda \)s that determine the eigenvalues, i.e., perform the following combinatorial sum:

\[
S = \sum_{\{M_j\}} \prod_j \left( \frac{\Delta I_j}{M_j} \right).
\]

(2.15)

To do this we shall follow [7]. First of all, we reinstate \( M_l \) into (2.15) by replacing \( M_1 \) by the sum rule, re-expressing it in the variables \( M_2, \ldots, M_l \)

\[
S = \sum_{\{M_j\}} \left( \frac{N/2 + \sum_{k=2}^{l} (k - 2) M_k}{N/2 - \sum_{k=2}^{l} k M_k} \right) \prod_{j=2}^{l-1} \left( M_j + 2\sum_{k=j+1}^{l} (k - j) M_k \right),
\]

(2.16)

where we notice that \( M_2 \) now occurs in only two places, and the sum over \( M_2 \) is of the form

\[
\sum_{j \geq 0} \binom{C}{A - 2j} \binom{B + j}{j}
\]

(2.17)

which can be evaluated as the coefficient of \( x^A \) in \((1 + x)^C(1 - x^2)^{-B-1}\).

We are now left with the following:

\[
\sum_{\{M_k\}, k \geq 3} \frac{1}{2\pi} \oint dx \left( \frac{1 + x}{x} \right)^{N/2} \frac{1}{(1 - x^2)} \prod_{k=3}^{l} \left( \frac{x^k(1 + x)^{k-2}}{(1 - x^2)^{2k-2}} \right) M_k \left( \frac{\Delta I_k}{M_k} \right),
\]

(2.18)
where the contour is around the origin. The sum over $M_3$ is now of the form $\sum_{j \geq 0} \binom{B+j}{j} y^j$, which is just $(1 - y)^{-B-1}$. We can carry on doing this for successive values of $k$. In [7], this iterated sum has been encoded in the following form:

$$S = \frac{1}{2\pi} \oint \frac{dx}{x} \left( \frac{1 + x}{x} \right)^{N/2} \prod_{k=3}^{l} \frac{1}{1 - u_j^{-1}},$$

(2.19)

where

$$(u_j - 1)^2 = u_{j-1} u_{j+1}, \quad u_3 = \frac{x^3}{(1 + x)(1 - x^2)}, \quad u_2 = x^{-2}. \quad (2.20)$$

These are essentially Chebyshev polynomials of the second kind, i.e. $u_j(x) = U_j^2(1/2, 1+x)$, where

$$U_n(\cos \phi) = \frac{\sin((n+1)\phi)}{\sin \phi}. \quad (2.21)$$

By looking at the poles and residues of $U_l(z)/U_{l+1}(z)$, we notice that

$$\frac{U_l(z)}{U_{l+1}(z)} = \frac{1}{l+2} \sum_{j=1}^{l+1} \frac{\sin^2(\frac{\pi j}{l+2})}{z - \cos(\frac{\pi j}{l+2})}. \quad (2.22)$$

Therefore we can express the sum as

$$S = \frac{4}{l+2} \sum_{j=1}^{[(l+1)/2]} \frac{1}{2\pi} \oint \frac{dx}{x} \left( \frac{1 + x}{x} \right)^{N/2} \sin^2(\frac{\pi j}{l+2}) \frac{\sin^2(\frac{\pi j}{l+2})}{1 - (4 \cos^2(\frac{\pi j}{l+2}) - 1)x}. \quad (2.23)$$

(notice that the square roots cancel out). By deforming the contour to pick up all the other poles except at the origin gives us

$$S = \frac{2}{l+2} \sum_{j=1}^{l+1} \sin^2\left(\frac{\pi j}{l+2}\right)(2 \cos\left(\frac{\pi j}{l+2}\right))^N, \quad (2.24)$$

which is precisely the multiplicity of the singlet in the $N$-fold tensor product of the fundamental representation of $U_q(su(2))$ at $q = \exp(\frac{2\pi i}{l+2})$. This is the expected number because of the equivalence [38] of the construction of the model state space to truncated tensor product representations of quantum groups at roots of unity. (This method of performing the sum, which is lifted from [4], where it was used for the $l \to \infty$ case, was deployed for the $l = 4$ case in [11]. See also [39].)
3. For the simply-laced algebras.

A bit of history: In [5], higher spin representations of $su(2)$ were studied. From the $su(2)$ models, we can generalize even further, by considering the models of [11] as solved in [6], by starting with the corresponding Bethe equations. In fact, the authors of [6] made the observation that the structure of the Bethe equations could be cast in a way that could be generalized to the case of all simply laced algebras even though the models whose spectrum these equations would parametrize were not known. This game was further extended to the elliptic case in [12] and by Kuniba [13] who set up the thermodynamics of (hypothetical) systems whose energy eigenvalues were parametrised by solutions to the (trigonometric) Bethe equations associated with all (i.e. not just simply-laced) untwisted affine algebras.

We shall briefly outline the counting procedure for the simply-laced cases which closely follows section 1. This will highlight the key feature characteristic of the connection between the quasi-particle description (which is a recasting of the map between the integers and momenta in a suggestive language that posits this physical paradigm) and the thermodynamic formalism.

For the simply-laced algebras, the Bethe equations are of the form [3]:

\[
\left[ \frac{\sinh\left(\frac{\pi}{2L}(\lambda_j^{(a)} + is\delta_{ap})\right)}{\sinh\left(\frac{\pi}{2L}(\lambda_j^{(a)} - is\delta_{ap})\right)} \right]^N = \Omega_j^{(a)} \prod_{b=1}^r \prod_{k=1}^{N_b} \frac{\sinh\left(\frac{\pi}{2L}(\lambda_j^{(a)} - \lambda_k^{(b)} + iC_{ab})\right)}{\sinh\left(\frac{\pi}{2L}(\lambda_j^{(a)} - \lambda_k^{(a)} - iC_{ab})\right)}. \tag{3.1}
\]

\[
N_a = Ns[C^{-1}]_{ap}, \tag{3.2}
\]

where $N$ is the size of the lattice $C$ is the Cartan matrix, $L = l + g$, for integer $l$, $g$ is the dual Coxeter number and $s$ characterizes the type of fusion. In what follows, the string hypothesis (as in [3]) can be written as:

\[
\lambda_j^{(a)} = \lambda + i(j + 1 - 2j_1), \quad 1 \leq j_1 \leq j, \tag{3.3}
\]

$\lambda$ is real and denotes the center of the string and $j$ denotes its length, which is further assumed to satisfy $1 \leq j \leq l$. We shall therefore impose $\sum_{j=1}^l jM_j^{(a)} = N_a$, where $M_j^{(a)}$ denotes the number of strings of colour $(a)$, length $j$ and $N_a$ is given by (3.2). As before, we multiply out the Bethe equations for the components of each string, and end up with equations for the real parts of the roots. We then take the logarithm of the multiplied out (3.1) so that the integer branches are distinct:

\[
Nt_{j,s}^{(a)}(\lambda_{j\mu}^{(a)}) = 2\pi i I_{j\mu}^{(a)} + \sum_{b=1}^r \sum_{k=1}^{M_k^{(b)}} \sum_{\nu=1}^{M_{\nu}^{(b)}} \Theta_{jk}^{(ab)} (\lambda_{\mu}^{(a)} - \lambda_{\nu}^{(b)}), \tag{3.4}
\]
where \( \lambda^j_{\mu}(a) \) labels the center of the \( \mu \)-th string of length \( j \) and color \((a)\) which is a root of eq. (3.1). The functions \( t_{j,s}^{(a)} \) and \( \Theta_{jk}^{(ab)} \) are defined below.

\[
t_{j,s}^{(a)}(\lambda) = \delta_{ap} \sum_{k=1}^{\min(j,s)} f(\lambda; |j-s| + 2k - 1), \quad (3.5)
\]

\[
\Theta_{jk}^{(ab)}(\lambda) = \delta_{ab} \left( f(\lambda; |j-k|) + 2 \sum_{i=1}^{\min(j,k)-1} f(\lambda; |j-k| + 2i) + f(\lambda; j+k) \right) - I_{ab} \left( \sum_{i=1}^{\min(j,k)} f(\lambda; |j-k| + 2i - 1) \right), \quad (3.6)
\]

\[
f(\lambda; n) = \frac{1}{2\pi i} \ln \left( \frac{\sinh \frac{1}{2\pi} (in - \lambda)}{\sinh \frac{1}{2\pi} (in + \lambda)} \right), \quad (3.7)
\]

for integer values of \( n/L \), and is 0 otherwise. \( I_{ab} \) is the incidence matrix of the respective Dynkin diagrams. As in section 2, we define the corresponding \( Z(\lambda) \), this time with an extra colour index. Assuming monotonicity, we find that for strings of length \( l \), the range of integers \( \Delta I_j^{(a)} \) coincides with the total number of strings of that length, \( M_l^{(a)} \), for all colour labels \((a)\). There are no holes in the distribution of integers for these strings. Since the \( l \)-strings do not contribute to the counting of states, we can eliminate \( M_l^{(a)} \) using the sum rule (3.2). The ranges of integers associated with the other strings are:

\[
\Delta I_j^{(a)} = N \delta_{ap} \left[ C^{-1}_{A_{l-1}} \right]_{js} + M_j^{(a)} - \sum_{b=1}^{r} \sum_{k=1}^{l-1} (C_{\mathcal{G}})_{ab} \left[ C^{-1}_{A_{l-1}} \right]_{jk} M_k^{(b)} , \quad j < l, \quad (3.8)
\]

where \( C_{\mathcal{G}} \) is the Cartan matrix of the (simply-laced) Lie algebra \( \mathcal{G} \) and (as before) \( C^{-1}_{A_{l-1}} \) is the inverse Cartan matrix of \( A_{l-1} \). Note that this specializes to the case of \( su(2) \) considered earlier, for \( s = 1 \). This gives a combinatorial count of the number of states of the form described above,

\[
\sum_{\{M_j^{(a)}\}} \prod_{j,a} (\Delta I_j^{(a)} / M_j^{(a)}). \quad (3.9)
\]

We shall use the equality of binomial coefficients

\[
\binom{A}{B} = \binom{A}{A-B} \quad (3.10)
\]

to define a new variable

\[
N_j^{(a)} = \Delta I_j^{(a)} - M_j^{(a)}, \quad (3.11)
\]
which counts the number of holes. In these variables, the allowed range of integers takes the form:

\[
\Delta I^{(a)}_j = N\delta_{sj} [C^{-1}_G]_{ap} + N^{(a)}_j - \sum_{b=1}^{r} \sum_{k=1}^{l-1} (C^{-1}_G)_{ab} [C_{A_{-1}}]_{jk} N^{(b)}_k. \tag{3.12}
\]

The observation that \(\Delta I^{(a)}_j = M^{(a)}_l\) had been first made in the context of thermodynamic densities studied in [3], [4]. The striking difference is, of course, that in the thermodynamic formalism, \(N\) is a large number, and only a single order in \(N\) is retained, and there was no reason to expect this to be true at any finite \(N\). (That the authors of [3] notice deviations from the string hypothesis (3.3) makes it all the more surprising.) We shall exploit this coincidence in order to do away with the complication of choosing branches as in (3.4).

4. The thermodynamic formalism.

The procedure of setting up the thermodynamics has now become standard, going back to [3]. We shall reproduce the equations in [13] where the thermodynamics of systems based on the Bethe equations corresponding to all simply-laced algebras were studied.

\[
\left[ \frac{\sinh \left( \frac{\pi}{2L} (\lambda^{(a)}_j + i(s\omega_p|\alpha_a)) \right)}{\sinh \left( \frac{\pi}{2L} (\lambda^{(a)}_j - i(s\omega_p|\alpha_a)) \right)} \right]^N = \Omega^{(a)}_j \prod_{b=1}^{r} \prod_{k=1}^{N_b} \frac{\sinh \left( \frac{\pi}{2L} (\lambda^{(a)}_j - \lambda^{(b)}_k + i(\alpha_a|\alpha_b)) \right)}{\sinh \left( \frac{\pi}{2L} (\lambda^{(a)}_j - \lambda^{(a)}_k - i(\alpha_a|\alpha_b)) \right)}, \tag{4.1}
\]

\[
N_a = Ns [C^{-1}]_{ap}, \tag{4.2}
\]

where \(N\) is the size of the lattice \(\alpha_a\) are the simple roots, \(\omega_p\) are the fundamental weights, \((\cdot|\cdot)\) is the canonical bilinear form on the dual to the Cartan subalgebra, \(C\) is the Cartan matrix, \(L = l + g\), for integer \(l\), \(g\) is the dual Coxeter number and \(s\) characterizes the type of fusion.

The string hypothesis for the solutions of (4.1) is

\[
\lambda^{(a)}_j = \lambda + i t^{-1}_a (j + 1 - 2j_1), \quad 1 \leq j_1 \leq j \tag{4.3}
\]

with \(t^{-1}_a = (\alpha_a|\alpha_a)/2\). \(\lambda\) is real and denotes the center of the string and \(j\) denotes its length, which is further assumed to satisfy \(1 \leq j \leq t_al\). We shall therefore impose \(\sum_{j=1}^{t_al} j M^{(a)}_j = N_a\), where \(M^{(a)}_j\) denotes the number of strings of colour \((a)\), length \(j\) and \(N_a\) is given by (4.2).
To make the idea of upgrading bulk quantities to finite lattices concrete, we define densities for strings and holes [3] by requiring their integrals over all $\lambda$ to be exactly equal to the number of strings and holes divided by $N$, the size of the system. This is the precise statement that rules the interpolation between the bulk and finite lattices. In the notation of [13],

$$\hat{\rho}_j^{(a)}(0) = M_j^{(a)}/N, \quad \hat{\sigma}_j^{(a)}(0) = N_j^{(a)}/N,$$

(4.4)

where $\hat{f}(0)$ denotes the Fourier transform of $f$ for zero argument:

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(\lambda)e^{-i\lambda x}d\lambda, \quad \hat{f}(0) = \int_{-\infty}^{\infty} f(\lambda)d\lambda.$$

(4.5)

One would expect this correspondence to be only true modulo correction terms vanishingly small compared to $N$. In the thermodynamic limit, because solving the integral equations involves taking derivatives on the densities, one can choose the branches of the logarithms (of the Bethe equations) freely. What we see is that there exists a choice of branch on the finite lattice that makes the translation of statements true in the bulk to finite lattices possible.

The equations governing the densities that determine the thermodynamics of these systems as studied in [13] are

$$\delta_{pa} \hat{A}_{pa}^{(l)sm} = \hat{\sigma}_{m}^{(a)} + \sum_{b=1}^{r} \sum_{k=1}^{t_b l - 1} \hat{M}_{ab} \hat{A}_{ab}^{(l)mk} \rho_k^{(b)},$$

(4.6)

where

$$\hat{A}_{ab}^{(J)mk} = \hat{A}_{ba}^{(J)km} = \frac{\sinh\left(\min(m/t_a, k/t_b)x\right) \sinh((J - \max(m/t_a, k/t_b))x)}{\sinh(x/t_{ab}) \sinh(Jx)},$$

$$\hat{M}_{ab} = \hat{M}_{ba} = \frac{t_b}{t_{ab}} C_{ab} + 2\delta_{ab} \left(\cosh(x/t_a) - 1\right),$$

(4.7)

and $t_{ab} = \max(t_a, t_b)$.

Setting $x$ (the Fourier transform variable) to zero, and applying the definition (4.4) to (4.6), we get

$$N_j^{(a)} = N\delta_{ap}[C^{-1}_{\lambda_{\nu+1}}]js - \sum_{b=1}^{r} \sum_{k=1}^{t_b l - 1} K_{ab}^{jk} M_k^{(b)},$$

(4.8)

where

$$K_{ab}^{jk} \equiv (\alpha_a|\alpha_b)\{\min(t_b j, t_a k) - \frac{jk}{l}\}.$$

(4.9)
We can also solve for $M_j^{(a)}$ in terms of $N_j^{(a)}$ to get

$$M_j^{(a)} = N \sum_{k=1}^{t_p l - 1} (K^{-1})_{ab}^{jk} (C^{-1}_{t_p l - 1})_{ks} \sum_{b=1}^{r} \sum_{k=1}^{t_p l - 1} (K^{-1})_{ab}^{jk} N_k^{(b)}, \quad (4.10)$$

where $K^{-1}$ is defined by

$$\sum_{c=1}^{r} \sum_{m=1}^{t_a l - 1} (K^{-1})_{ac}^{jm} K_{cb}^{rmk} = \delta_{ab} \delta_{jk}. \quad (4.11)$$

In the above, the index $t_a l$ never shows up because there are never any holes in this sector, and the sum rule (4.2) is used to eliminate the dependence on $M_{t_p l}^{(a)}$. Note that for the simply-laced cases, $K$ factorizes into a “level” and a “rank” piece, and (4.8) reduces to (3.8) using (3.11). The range of integers for the finite system, under this correspondence, is $\Delta I_j^{(a)} = M_j^{(a)} + N_j^{(a)}$.

5. Quasi-particles and $q$-series.

Now for some physics. The first thing to remember is that the hamiltonians are connected to transfer matrices of two dimensional statistical mechanical systems with positive-definite Boltzmann weights, and for either of the $\pm$ signs the ground state has to be unique by the Perron-Frobenius theorem. Our classification (assumed complete) via the partition of the roots of (4.1) into stringy clusters for the states in this sector (where the ground state is expected to lie) indicates that the only states that meet this criterion are given by the partition for which the summand in binomial coefficients gives 1. Using the variable variable $N_j^{(a)}$ which counts the number of holes in the allowed distribution of integers, and look for solutions of $N_j^{(a)} = 0$. The observation that the range of allowed integers for strings of length $t_a l$ was equal to the number $M_{t_a l}$ itself indicates that the state in which all the roots are of this type is one such candidate for a ground state. The only other state is then the state which has a distribution of strings given by

$$M_j^{(a)} = N \sum_{k=1}^{t_p l - 1} (K^{-1})_{ab}^{jk} (C^{-1}_{t_p l - 1})_{ks}. \quad (5.1)$$

We thus have the two ground states for the two hamiltonians that differ by an overall sign. Any other partition of the roots then corresponds to an excited state in either model.
\[
\Delta I_j^{(a)} = N \delta_{ap}[C_{A_{pl-1}}^{-1}]_{js} + M_j^{(a)} - \sum_{b=1}^{r} \sum_{k=1}^{t_{bl-1}} K^{jk}_{ab} M_k^{(b)}, \quad (5.2)
\]

and

\[
\Delta I_j^{(a)} = N \sum_{k=1}^{t_{pl-1}} (K^{-1})_{ajp}[C_{A_{pl-1}}^{-1}]_{ks} + N_j^{(a)} - \sum_{b=1}^{r} \sum_{k=1}^{t_{bl-1}} (K^{-1})_{ab} N_k^{(b)} \quad (5.3)
\]

encode the taxonomy of all the states of the system except the ground state which has been “subtracted out.” We can then regard (5.2) and (5.3) as the fundamental equations that determine the excitation spectra of the two models, with the numbers \(M_j^{(a)}\) and \(N_j^{(a)}\) counting the number of these excitations.

A generically striking property that the spectrum of a many-body quantum system (such as the ones described here) possesses is one described as a quasi-particle form. This refers to the additive decomposition of the eigenvalues thus:

\[
\lim_{N \to \infty} E - E_{GS} = \sum_{\alpha, \text{rules}} \sum_{j} n_{\alpha} e^\alpha(p_j^\alpha), \quad (5.4)
\]

where \(E\) refers to the eigenvalues of the Hamiltonian \((E_{GS}\) being the smallest eigenvalue) and \(N\) refers to the size of the system. The functions in the summand are the dispersion curves of what are defined to be quasi-particles of different species \(\alpha\), and the index \(j\) runs over their number \(n_{\alpha}\). The momentum of a particular eigenstate is

\[
P - P_{GS} = \sum_{\alpha, \text{rules}} \sum_{j} p_j^\alpha. \quad (5.5)
\]

The “rules” determine the composition of the eigenstate in question, and in particular, could depend on the statistics of the excitations.

The \(n_{\alpha}\) quasi-particles of (5.4) and (5.5) are the excitations which number \(M_j^{(a)}\) and \(N_j^{(a)}\) above the two vacua in the thermodynamic limit. The completeness argument on the finite lattice (which we have demonstrated only for a subset of the cases under consideration) indicates the completeness of the particle picture of the (separable) Hilbert space of the emergent field theory.

The momentum eigenvalue (5.5) is written as a sum of the logarithm of the term on the left hand sides of (4.1). Thus, the sum of the integers for any set of roots gives the total momentum (with the appropriate \(2\pi \frac{2}{N}\) taken into account) of the state that these roots correspond to. Therefore we see that for every term in \(\Delta I_j^{(a)} = M_j^{(a)} + N_j^{(a)}\) for which
the term proportional to $N$ does not have support, the contribution of these strings (or holes) to the momentum of the eigenstate at order one ($N^0$) must necessarily vanish. For a system with no mass gap the energy contribution must consequently be zero to order one. However, they could (and do) contribute to the spectrum at order $1/N$. These have been termed “ghost” excitations in [19]. The strings (and holes) for which $\Delta I_j^{(a)} \sim N$ have an extensive single-excitation Hilbert space dimension and constitute the order one excitation spectrum, and the coefficient of $N$ encodes the Brillouin zone scheme of these particles. (The relationship of these spectra to their fractional statistics [25] has been mentioned in [15] and in [26], where the relationship of the statistical exclusion coefficient to the central charge of the conformal field theory has been shown.)

It is seen in numerical studies (which provided the background for ref. [10]), that within each class of states with a particular root content, the sum of the absolute values of the integers gives a good estimate of the (approximate) degeneracy of the levels, i.e., those states with the same value for the sum of the absolute value of the integers had energy eigenvalues that were almost equal. This is reminiscent of the conformal field theory definitions of energy and momentum being the sum and differences of the eigenvalues of $L_0$ and $\bar{L}_0$. One could then presume that this degeneracy would become exact in the thermodynamic limit, and that these integers keep track of, in a rather robust fashion and on a finite lattice, a classification of states that take on significance (in terms of dynamical symmetries) only in the thermodynamic limit. We can then hope to count degenerate states in the $N \to \infty$ limit by keeping track of these integers. (For more discussion on quasi-particles and state counting in the thermodynamic limit, see [19].)

In other words, the binomial counting may be refined by introducing a grading, i.e., by introducing a variable $q$, whose powers are the integers. Consider, for example, any 3 of the 5 integers in the set $\{ \pm 2, \pm 1, 0 \}$ can be chosen such that their sum takes values in the set $\{ \pm 3, \pm 2, \pm 1, 0 \}$. The multiplicity of their occurrence can be observed from the coefficient of the appropriate power of $q$ in

$$\binom{5}{3}_q = q^{-3} + q^{-2} + 2q^{-1} + 2 + 2q + q^2 + q^3,$$

(5.6)

where

$$\binom{A + B}{B} q^{\frac{1}{2} AB} \left[ \begin{array}{c} A + B \\ B \end{array} \right]$$

(5.7)
are defined in terms of $q$-binomial coefficients,

$$
\begin{bmatrix} A \\ B \end{bmatrix} = \frac{(q; q)_A}{(q; q)_{A-B}(q; q)_B},
$$

(5.8)

where

$$(q; q)_A = \prod_{j=1}^{A} (1 - q^j),$$

(5.9)

(non-zero only for integers, $A$ and $B$, with $0 \leq A \leq B$).

Construct the following:

$$
\begin{align*}
    f_p &= \sum_{\{M_j^{(a)}\}} \prod_{j,(a)} q^{-\frac{1}{2} M_j^{(a)} N_j^{(a)} (M_j^{(a)})} \left[ M_j^{(a)} + N_j^{(a)} (M_j^{(a)}) \right], \\
    f_h &= \sum_{\{N_j^{(a)}\}} \prod_{j,(a)} q^{-\frac{1}{2} N_j^{(a)} M_j^{(a)} (N_j^{(a)})} \left[ N_j^{(a)} + M_j^{(a)} (N_j^{(a)}) \right],
\end{align*}
$$

(5.10)

where the parantheses are used to indicate that we write $f_p$ only in the $M_j^{(a)}$ variables and $f_h$ only in the $N_j^{(a)}$ variables. Since large values of $\lambda$ correspond to small momentum contributions and therefore, low energy states, and the integers have a monotonic dependence on $\lambda$, the largest integers correspond to the lowest energy. We therefore assign zero momentum to the term in $\Delta I_j^{(a)}$ proportional to $N$ (which is the edge of the Brillouin zone) in the power of $q$ in the factor pre-multiplying the $q$-binomial coefficients.

We can now count the low energy states in the thermodynamic limit, $N \to \infty$. For the massless case, this is equivalent to computing the partition function of the theory in the particular sector, with $q = \exp(-\frac{2\pi \nu}{N T})$, in the limit $N \to \infty$, $T \to 0$, $NT$ finite, $\nu$ being
the speed of sound. In the $N \to \infty$ limit, the $f_p$ and $f_h$ are re-defined as follows

\[
\chi_p = \lim_{N \to \infty} f_p = \lim_{N \to \infty} \sum_{\{M_j^{(a)}\}} q^{-\frac{1}{2} M_j^a K^a M_j^\times} \times \prod_{j,a} \left[ \frac{N \delta_{ap} [G_{A_{p+j-1}}]_j + M_j^{(a)} + \sum_{b=1}^r \sum_{k=1}^{t_{b+j-1}} K_{ab}^{jk} M_k^{(b)}}{M_j^{(a)}} \right],
\]

\[
\chi_h = \lim_{N \to \infty} f_h = \lim_{N \to \infty} \sum_{\{N_j^{(a)}\}} q^{-\frac{1}{2} N_j^a (K^{-1}) \cdot N} \times \prod_{j,a} \left[ N \sum_{k=1}^{t_{b+j-1}} (K^{-1})_{ap} [G_{A_{p+j-1}}]_k + \sum_{b=1}^r \sum_{k=1}^{t_{b+j-1}} (K^{-1})_{ap} N_k^{(b)} + N_j^{(a)} \right],
\]

(5.11)

with the condition that

\[
\lim_{N \to \infty} \left[ N \right]_m = \frac{1}{(q; q)_m}.
\]

These are conjectured to be the branching functions (in the vacuum sector) corresponding to the conformal field theories constructed as cosets (a la GKO) [27]. Some of these have been conjectured earlier (see [19] for references). The branching functions these $q$-series expansions correspond to are listed in [13]. The central charges which can be obtained by taking the $q \to 1$ limit as in [20], [21] and [18] are also listed in [13]. There exists a proof of the $su(2)$ formulas in [28] for the $M$-type variables ($\chi_p$) and in [30] for the $N$-type variables ($\chi_h$).

6. The $Z_n$ models and some “Snake-Oil”.

In [18] it was pointed out that the $q$-series corresponding to what in our notation would be $G = D_n$, $s = 1$ and $p = 1$ and in the $M$ variable coincided with the vacuum character of $Z_n$ parafermions (with the colour index $(a)$ taking values up to the rank, $n$).

We have obtained [24] this $q$-series from a Bethe equation, without any reference to a physical model. The Fateev-Zamolodchikov model [31] in its scaling limit is known to give rise to the parafermionic conformal field theory. The $Z_3$ case was studied in [10] [14] using the formulation of [33], and we saw that the formulas in section 1 reproduced (modulo an orbifolding which we shall indicate later) those results (for the $Q = 0$ sector of the
3-state Potts model). To confirm whether this counting procedure in terms of the roots of (3.1) gives the total number of \( Q = 0 \) states, we need to execute a corresponding binomial sum.

We shall number the nodes of the Dynkin diagram of \( D_n \) such that the point of bifurcation is at 3 and the ends of the fork are at 1 and 2. The expression we are going to evaluate is

\[
S_M = \sum_{\{m_j\}} \binom{M + m_3}{2m_1} \binom{m_3}{2m_2} \binom{m_1 + m_2 + m_4}{2m_3} \binom{m_3 + m_5}{2m_4} \cdots \binom{m_{n-2} + m_n}{2m_{n-1}} \binom{m_{n-1}}{2m_n}.
\]

Recall that for the critical 3-state Potts case, which is the \( n = 3 \) case of the \( Z_n \) models of Fateev and Zamolodchikov, the number of genuine \( O(1) \) quasi-particle excitations were both even and odd in number unlike the corresponding orbifold-related RSOS model which only had an even number in keeping with the \( Z_2 \) invariance. The even sector only picks up the states with the \( Z_2 \) positive charge states. We therefore expect that to recover the complete \( Z_n \) charge zero sector we will have to sum over \( m_1 \) half integral as well, which will necessarily make the associated \( m_2 \) to be half-integral too.

Following the elegant and extremely useful method advocated by H. Wilf called the “Snake Oil” method [32], we define

\[
f(x) = \sum_M x^M S_M,
\]

interchange the order of summation and perform the sum over \( M \) first. We shall follow the convention that the binomial coefficient \( \binom{A}{B} \) vanishes if \( B < 0 \) or if \( A \) is a positive integer less than \( B \). Using the formula (valid for \(|x| < 1\),

\[
\sum_{k \geq 0} \binom{k}{B} x^k = \frac{1}{(1-x)(x)} B,
\]

we get

\[
\sum_{M \geq 0} x^M \binom{M + m_3}{2m_1 + \alpha} = \frac{x^{2m_1 + \alpha - m_3}}{(1-x)^{2m_1 + \alpha + 1}}.
\]

(Here \( \alpha = 0, 1 \) which we have introduced in order to keep track of the odd/evenness of the number of excitations.) Plugging this result into the sum over \( m_1 \), we then need to perform

\[
\frac{x^{-m_3 + \alpha}}{(1-x)^{\alpha + 1}} \sum_{m_1} \left( \frac{x}{1-x} \right)^{m_1} \binom{m_1 + m_2 + m_4 + \alpha}{2m_3}.
\]
which is again of the form (6.2). After some simplification, the sum over \( m_2 \) then takes the form
\[
\left( \frac{1-x}{x} \right)^{2m_4} \cdot \frac{1-x}{1-2x} \cdot \frac{x^{3m_3}}{(1-2x)^{2m_3}} \sum_{m_2} \left( \frac{1-x}{x} \right)^{2m_2+\alpha} \left( \frac{m_3}{2m_2 + \alpha} \right) \cdot (6.3)
\]

The presence of the \( \alpha \)s alongside the summed (dummy) variable indicates that the sum is performed over all (even and odd) the integers, and thus we have
\[
f(x) = \sum_{m_n} \ldots \sum_{m_3} \frac{1-x}{1-2x} \left( \frac{x}{1-2x} \right)^{2m_3} \left( \frac{x}{x} \right)^{2m_4} \left( \frac{m_3 + m_5}{2m_4} \right) \times \left( \frac{m_4 + m_6}{2m_5} \right) \ldots \left( \frac{m_{n-2} + m_n}{2m_{n-1}} \right) \left( \frac{m_{n-1}}{2m_n} \right) \cdot (6.4)
\]
Definition \( f_3 \equiv \frac{(x/(1-2x))^2}{2} \), so that upon performing the sum over \( m_3 \),
\[
f(x) = \frac{1-x}{1-2x} \frac{1}{1-f_3} \sum_{m_n, \ldots, m_4} f_{3-m_5} \left( \frac{1-x}{x} \frac{f_3}{1-f_3} \right)^{2m_4} \left( \frac{m_4 + m_6}{2m_5} \right) \times \left( \frac{m_5 + m_7}{2m_6} \right) \ldots \left( \frac{m_{n-2} + m_n}{2m_{n-1}} \right) \left( \frac{m_{n-1}}{2m_n} \right) \cdot (6.5)
\]
Defining
\[
f_4 \equiv \left( \frac{1-x}{x} \frac{f_3}{1-f_3} \right)^2 = \left( \frac{x}{1-3x} \right)^2, \quad (6.6)
\]
and iterating this process, we end up with
\[
f(x) = \frac{1-x}{1-2x} \prod_{j=3}^n \frac{1}{1-f_j}, \quad (6.7)
\]
with
\[
(f_j^{-1} - 1)^2 = f_{j-1}^{-1} f_{j+1}^{-1}, \quad (6.8)
\]
and the initial two values of \( f_j \) for \( j = 3, 4 \) are defined above. It can be checked that a solution to these recurrence relations with the given initial data is given by
\[
f_j = \left( \frac{x}{1 - (j-1)x} \right)^2, \quad (6.9)
\]
Plugging it all into (6.4), we get
\[
f(x) = \frac{1-(n-1)x}{1-nx} = \sum n^{M-1} x^M, \quad (6.10)
\]
which precisely counts the number of states in the \((Z_n\) charge) \( Q = 0 \) sector of the Fateev-Zamolodchikov spin chain.
Going back to the characters, we now consider those in the $N$ variables, which we now expect are those that describe the conformal field theory of the anti-ferromagnetic chain. In \[18\] and \[22\], it was conjectured (based on expansions of the formulas \[6.11\] as $q$-series, using Mathematica\textsuperscript{TM}) that these characters are those at radii $\sqrt{\frac{n}{2}}$ on the Gaussian line of the moduli space of $c = 1$ theories. This expectation has been borne out in \[34\] for odd $n$.

7. Discussion.

In order to describe the physics that the solutions to the Bethe equations encoded, we invoked the positivity of the Boltzmann weights of non-existent models. For the anti-ferromagnetic $D_n$ models, the classification scheme seemed to describe the states of the $Z_n$ models of \[31\] whose Boltzmann weights were, however, not positive for the anti-ferromagnetic regime (of the associated spin chain). This condition of “confinement of holes” served to rearrange the classification of states in a useful way giving correct results. (In passing, let us note that in \[35\], the $E_8$ structure of the Bethe equations emerges owing to a similar occurrence of a “‘frozen’ Dirac sea.”)

The more interesting questions that remain to be answered involve cases where the dependence of the energy functional on the roots of the Bethe equations is non-trivial, as in the case of the superintegrable chiral Potts model \[36\], where similar completeness studies have been made \[37\].

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References

[1] H. A. Bethe, Z. Phys. 71 (1931), 205.
[2] E. Lieb, Phys. Rev. 130 (1963), 1616.
[3] C. N. Yang and C. P. Yang, J. Math. Phys. 10 (1969), 1115.
[4] G. E. Andrews, R. J. Baxter and P. J. Forrester, J. Stat. Phys. 35 (1984) 193.
[5] V. V. Bazhanov and N. Yu. Reshetikhin, Int. J. Mod. Phys. A4 (1989) 115.
[6] V. V. Bazhanov and N. Yu. Reshetikhin, J. Phys. A23 (1990), 1477.
[7] M. Takahashi, Prog. Theo. Phy. 46 (1971), 401.
[8] M. Takahashi and M. Suzuki, 48 (1972), 2187, M. Gaudin, Phys. Rev. Lett., 26 (1971), 1301.
[9] H. M. Babujian, Nucl. Phys. B 215, (1983), 317.
[10] G. Albertini, S. Dasmahapatra and B. M. McCoy, Int. J. Mod. Phys. A7, Suppl. 1A (1992) 1.
[11] M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Comm. Math. Phys., 119 (1988), 543.
[12] V. V. Bazhanov and N. Yu. Reshetikhin, Prog. Theor. Phys. Suppl. 102 (1990) 301.
[13] A. Kuniba, Nucl. Phys. B389 (1993) 209.
[14] G. Albertini, S. Dasmahapatra and B. M. McCoy, Phys. Lett. A 170 (1992) 397.
[15] R. Kedem and B. M. McCoy, J. Stat. Phys. 71, (1993), 865.
[16] S. Dasmahapatra, R. Kedem, B. M. McCoy and E. Melzer, J. Stat. Phys. (in press), hep-th/9304150.
[17] R. Kedem, T. R. Klassen, B. M. McCoy, and E. Melzer, Phys. Lett. B 304, 263.
[18] R. Kedem, T. R. Klassen, B. M. McCoy, and E. Melzer, Phys. Lett. B 307, (1993), 68.
[19] S. Dasmahapatra, R. Kedem, T. R. Klassen, B. M. McCoy and E. Melzer in Yang-Baxter Equations in Paris, World Sci., also in Int. J. Mod. Phys. B, Vol. 7, (1993), 3617.
[20] B. Richmond and G. Szekeres, J. Austral. Soc. (Series A) 31 (1981), 362.
[21] W. Nahm, A. Recknagel, M. Terhoeven, Mod. Phys. Lett. A, 8 (1993), 1835.
[22] M. Terhoeven, Bonn preprint, hep-th.
[23] A. Kuniba, T. Nakanishi, J. Suzuki, Harvard preprint HUPT-92/A069.
[24] S. Dasmahapatra, ICTP preprint IC/93/91, hep-th/9305024.
[25] F. D. M. Haldane, Phys. Rev. Lett. 61, 937 (1991).
[26] S. Dasmahapatra, to be published in the proceedings of the summer school and workshop on string theory held in Trieste, 1993.
[27] P. Goddard, A. Kent and D. Olive, Comm. Math. Phys., 103 (1986), 105.
[28] J. Lepowsky and M. Primc, Structure of standard modules for the affine Lie algebra $A_{1}^{(1)}$, Cont. Math., Vol. 46 (AMS, Providence, 1985).
[29] B. Feigin and A. Stoyanovsky, preprint, RIMS-942, 1993.
[30] A. Berkovich, preprint, BONN-HE-94-04, hep-th/9403073.
[31] V. A. Fateev and A. B. Zamolodchikov, Phys. Lett. A 92 (1982), 37.
[32] H. Wilf in “Surveys in Combinatorics, 1989,” ed. J. Siemons, LMS Lec. Not. Ser. 141, Cambridge, 1989.
[33] G. Albertini, J. Phys. A 25 (1991), 1799.
[34] G. Albertini, preprint, SB-ITP-93-64, [hep-th/9310133](http://arxiv.org/abs/hep-th/9310133) (to appear in Int. J. Mod. Phys. A).
[35] V. V. Bazhanov, B. Nienhuis and S. O. Warnaar, Phys. Lett. B 322 (1994), 198.
[36] G. Albertini, B. M. McCoy, J. H. H. Perk, Eigenvalue spectrum of the superintegrable chiral Potts model, Adv. in Pure Math. 19 (1989) 1.
[37] S. Dasmahapatra, R. Kedem and B. M. McCoy, Spectrum and completeness of the three-state superintegrable chiral Potts model, Nucl. Phys. B396 (1993) 506.
[38] F. M. Goodman and T. Nakanishi, Phys. Lett. B262 (1991), 259.
[39] A. N. Kirillov, preprint, [hep-th/9312084](http://arxiv.org/abs/hep-th/9312084).
[40] A. N. Kirillov, J. Sov. Math., 36 (1987).
[41] S. Kerov, A. N. Kirillov and N. Yu. Reshetikhin, J. Sov. Math. 41 (1988), 916, A. N. Kirillov and N. Yu. Reshetikhin, J. Sov. Math. 41 (1988), 925, A. N. Kirillov and N. Yu. Reshetikhin, Lett. Math. Phys. 12, (1986), 500.
8. Appendix

In this appendix, we shall evaluate the sum over the products of binomial coefficients for the case of $su(2)$, arbitrary spin $s$, by the “Snake-Oil” method. We shall warm up to it by first re-doing the sum for the $s = 1$ case. Let us write the combinatorial summand in terms of the “hole” variables (we shall change notation, replacing $N_j$ by $m_j$, and $N/2 \equiv M$):

$$S_M = \sum_{m_1, m_2, \ldots, m_n} \left( \begin{array}{c} M + m_2 \\ 2m_1 \end{array} \right) \left( \begin{array}{c} m_1 + m_3 \\ 2m_2 \end{array} \right) \left( \begin{array}{c} m_2 + m_4 \\ 2m_3 \end{array} \right) \ldots \left( \begin{array}{c} m_{n-2} + m_n \\ 2m_{n-1} \end{array} \right) \left( \begin{array}{c} m_{n-1} \\ 2m_n \end{array} \right). \quad (8.1)$$

Next, define

$$f(x) = \sum_M x^M S_M, \quad (8.2)$$

interchange the order of summation and perform the sum over $M$ first:

$$\sum_{M \geq 0} x^M \left( \begin{array}{c} M + m_2 \\ 2m_1 \end{array} \right) = \sum_{M \geq 0} x^{-m_2} x^{M+m_2} \left( \begin{array}{c} M + m_2 \\ 2m_1 \end{array} \right).$$

We shall follow the convention that the binomial coefficient $\left( \begin{array}{c} A \\ B \end{array} \right)$ vanishes if $B < 0$ or if $A$ is a positive integer less than $B$. Using the formula (valid for $|x| < 1$,

$$\sum_{k \geq 0} \left( \begin{array}{c} k \\ B \end{array} \right) x^k = \frac{1}{(1 - x)^B}, \quad (8.3)$$

we get

$$f(x) = \sum_{m_1, m_2, \ldots, m_n} \frac{x^{-m_2}}{1 - x} y_1^{m_1} \left( \begin{array}{c} m_1 + m_3 \\ 2m_2 \end{array} \right) \left( \begin{array}{c} m_2 + m_4 \\ 2m_3 \end{array} \right) \ldots \left( \begin{array}{c} m_{n-2} + m_n \\ 2m_{n-1} \end{array} \right) \left( \begin{array}{c} m_{n-1} \\ 2m_n \end{array} \right),$$

where we have defined $y_1 \equiv (x/(1 - x))^2$. In the same way, we can now perform the sum over $m_1$ and be left with

$$f(x) = \sum_{m_2, \ldots, m_n} \frac{x^{-m_2}}{1 - x} \frac{y_1^{-m_3}}{1 - y_1} \left( \frac{y_1}{1 - y_1} \right)^2 \left( \begin{array}{c} m_2 + m_4 \\ 2m_3 \end{array} \right) \ldots \left( \begin{array}{c} m_{n-2} + m_n \\ 2m_{n-1} \end{array} \right) \left( \begin{array}{c} m_{n-1} \\ 2m_n \end{array} \right) \quad (8.4)$$

$$= \sum_{m_2, \ldots, m_n} \frac{1}{1 - x} \frac{y_1^{-m_3}}{1 - y_1} y_2^{m_2} \left( \begin{array}{c} m_2 + m_4 \\ 2m_3 \end{array} \right) \ldots \left( \begin{array}{c} m_{n-2} + m_n \\ 2m_{n-1} \end{array} \right) \left( \begin{array}{c} m_{n-1} \\ 2m_n \end{array} \right),$$

where

$$y_2 \equiv \left( \frac{1}{\sqrt{x}} \frac{y_1}{1 - y_1} \right)^2. \quad (8.5)$$
The pattern is thus evident and the process is repeated until we end up with

\[ f(x) = \prod_{j=0}^{n-1} \frac{1}{(1 - y_j)}, \quad (8.6) \]

where

\[ y_{-1} \equiv 1, y_0 = x, \quad \text{and} \quad (y_j^{-1} - 1)^2 = y_{j-1}^{-1}y_{j+1}^{-1}. \quad (8.7) \]

These are the same recurrence relations as before, only this time,

\[ y_j^{-1/2} = U_{j+1}(\frac{1}{2\sqrt{x}}) \quad (8.8) \]

At this point it is easy to see that if we were to consider any other representation, not necessarily \( s = 1 \), there would be a term involving \( M \) in the binomial coefficient at a different place, the \( s^{th} \) place. That is there would be a factor of the form

\[ \left( \frac{m_{s-1} + m_{s+1} + N}{2m_s} \right), \]

(\( \text{where} \ N \text{ is really equal to } N/2 \)), which would give rise to a power \( (y_{s-1})^{-N} \) picked up while redifining the summation variable so that the required sum is simply the constant term (coefficient of \( x^0 \)) in

\[ f(x) = U_1(\frac{1}{2\sqrt{x}})U_s^{2N}(\frac{1}{2\sqrt{x}})U_{n+1}(\frac{1}{2\sqrt{x}}) \]

Once again, using (2.22) and closing the contours over the zeroes of the denominator we arrive at

\[ S_N = \frac{2}{l+2} \sum_{j=1}^{l+1} \sin^2\left(\frac{\pi j}{l+2}\right)U_s^N\left(\cos\left(\frac{\pi j}{l+2}\right)\right), \quad (8.9) \]

with \( l = n + 1 \). This specializes correctly to the \( s = 1 \) case evaluated above.

As is immediately obvious, this method can be used for evaluating the sums for cases where the quadratic form \( K_{ij}^{\theta^{ab}} \) involves only the \( i, j \) or the \( a, b \) indices. The other cases require the introduction of multiple variable generating functions and multidimensional residues are required. For more results and techniques associated with counting solutions of Bethe equations, see [40] and [41].