THE CHIRAL PHASE TRANSITION, RANDOM MATRIX MODELS, AND LATTICE DATA

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Abstract
We present two pieces of evidence in support of the conjecture that the microscopic spectral density of the Dirac operator is a universal quantity. First, we compare lattice data to predictions from random matrix theory. Second, we show that the functional form of the microscopic spectral correlations remains unchanged in random matrix models which take account of finite temperature. Furthermore, we present a random matrix model for the chiral phase transition in which all Matsubara frequencies are included.

1 Introduction
Motivated by the Leutwyler-Smilga sum rules for the inverse powers of the eigenvalues of the Dirac operator in a finite volume \cite{1}, Shuryak and Verbaarschot \cite{2} introduced the so-called microscopic spectral density of the Dirac operator $i\hat{D}$. This quantity is defined as

$$\rho_s(z) = \lim_{V \to \infty} \frac{1}{V^3} \rho\left(\frac{z}{V^3}\right),$$

(1)

where $\rho(\lambda) = \sum_n \delta(\lambda - \lambda_n)$ is the spectral density of the Dirac operator, $V$ is the space-time volume, and $\Sigma$ is the chiral condensate. The microscopic spectral density carries information on how the thermodynamic limit is approached which, among other things, is very useful for the analysis of lattice data, in particular in the chiral limit. It was conjectured that $\rho_s$ should be a universal quantity, i.e., that it is insensitive to the details of the dynamics. As a universal quantity, $\rho_s$ should depend only on the symmetries of the problem and, therefore, be calculable in a much simpler theory than QCD, namely random
matrix theory (RMT). In RMT, the matrix of the Dirac operator (including a mass term) in a chiral basis in Euclidean space is replaced by a random matrix according to
\[
\begin{bmatrix}
im & i\bar{\Phi} \\
(i\bar{\Phi})^\dagger & \text{im}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
im & W \\
W^\dagger & \text{im}
\end{bmatrix},
\]
where \( W \) is a random matrix of dimension \( N \). The average over gauge field configurations is replaced by the average over random matrices, and the weighting function \( \exp(-S_{\text{glue}}) \) is replaced by a simple Gaussian distribution \( P(W) \sim \exp(-N\Sigma^2 \text{Tr}WW^\dagger) \) of the random matrices. Depending on the representation of the fermions and the number of colors, there are three different universality classes which have been classified in Ref. [3]. They are described by the three chiral Gaussian ensembles: the chiral Gaussian Orthogonal Ensemble (chGOE) where \( W \) is real, the chiral Gaussian Unitary Ensemble (chGUE) where \( W \) is complex, and the chiral Gaussian Symplectic Ensemble (chGSE) where \( W \) is quaternion real. The microscopic spectral density has been computed for all three ensembles in the framework of RMT [4, 5].

There were a number of findings which added support to the universality conjecture for \( \rho_s \). We refer the reader to a recent review by Verbaarschot [6] which also gives a comprehensive account of other applications of RMT in QCD. In this paper, we present direct evidence for the universality of \( \rho_s \) by comparing lattice data to RMT predictions. This is done in Sec. 2.1. In Sec. 2.2 we add another piece of evidence by showing that the functional form of \( \rho_s \) and of the microscopic spectral correlations in general does not change in random matrix models which take account of finite temperature. In Sec. 3 we discuss a model for the finite-temperature chiral phase transition in which all Matsubara frequencies are included. Conclusions are drawn in Sec. 4.

2 Universality of \( \rho_s \) and of the microscopic spectral correlations

2.1 Lattice data and predictions from random matrix theory

We analyze lattice data obtained by Berbenni-Bitsch and Meyer for an SU(2) gauge theory with staggered fermions in the quenched approximation with \( \beta = 2.0 \). To study finite-volume effects, data were obtained for four different lattice sizes: 4\(^4\), 6\(^4\), 8\(^4\), and 10\(^4\). For each lattice, the complete spectrum of the Dirac operator was obtained for a very large number of configurations (9979, 9953, 3896, and 477 configurations for the 4\(^4\), 6\(^4\), 8\(^4\), and 10\(^4\) lattice, respectively). This was necessary to obtain good statistics. The numerical
eigenvalues of the Dirac operator satisfy an analytical sum rule with excellent precision (relative accuracy $\sim 10^{-8}$).

According to Ref. [3], staggered fermions in SU(2) should be described by the chGSE. Analytical results exist for the distribution of the smallest eigenvalue (only for the quenched approximation) [7],

$$P(\lambda_{\text{min}}) = \frac{\sqrt{\pi}}{2} c(\lambda_{\text{min}})^{3/2} I_{3/2}(c\lambda_{\text{min}}) e^{-\frac{1}{2}(c\lambda_{\text{min}})^2},$$

(3)

and for the microscopic spectral density (also for the unquenched case) [3],

$$\rho_s(z) = 2\pi(cz)^2 \int_0^1 du \int_0^1 dv \ [J_{a-1}(2uv)J_a(2ucz) - v J_{a-1}(2uc)J_a(2uv)z].$$

(4)

Hence, a direct comparison with lattice data is possible. Here, $a = N_f + 2\nu + 1$, where $N_f$ is the number of massless dynamical quarks and $\nu$ is the topological charge. The parameter $c$ sets the scale on which the eigenvalue is measured. It should be pointed out that $c$ is not a free parameter but determined by the lattice volume and the chiral condensate according to Eq. (1).

In Figure 1 we have plotted the data and the predictions from RMT for both $P(\lambda_{\text{min}})$ and $\rho_s(z)$ for all available lattice sizes except for the $10^4$ lattice where the number of configurations was too small to obtain sufficiently good statistics. The parameter $a$ in (4) is 1 since $N_f = 0$ and since $\nu_{\text{eff}} = 0$ on the lattice [3]. The data agree almost perfectly with the RMT predictions for all but the smallest lattice. The range over which $\rho_s$ agrees with the RMT prediction increases with the lattice size which is evident by the definition of $\rho_s$ in Eq. (4). We conclude that a lattice size of $6^4$ is sufficient to identify the universal microscopic spectral density of the Dirac operator.

Unquenched calculations and their analysis in terms of RMT are in progress. It will be very interesting to see the effect of a small dynamical quark mass in $\rho_s$.

### 2.2 The microscopic spectral density at finite temperature

Various random matrix models have been constructed for the spectrum of the Dirac operator at finite temperature to describe generic features of the chiral phase transition. We shall discuss these models in more detail in Sec. 3. The common feature of all these models is that a temperature-dependent matrix $Y$ is added to the random matrix $W$ in Eq. (2). The specific form of $Y$ depends on the choice of basis. However, as we shall see in Sec. 3 the matrix $Y$ can always be chosen real and diagonal. This defines the generic problem.
Figure 1: Distribution of the smallest eigenvalue $P(\lambda_{\text{min}})$ and microscopic spectral density $\rho_s(z)$ for three different lattice sizes (see text). Note that there are no free parameters in the RMT prediction.
The matrix of the Dirac operator (for \(m = 0\)) thus has the form
\[
\begin{bmatrix}
0 & W + Y \\
W^\dagger + Y & 0
\end{bmatrix}
\]
with \(Y = \text{diag}(Y_1, \ldots, Y_N)\). The \(Y_n\) are real and depend on \(T\). The dimension \(N\) of the matrix \(W\) can be identified with the space-time volume \(V\). For the moment, let us concentrate on the chGUE and on the quenched approximation. In this case, we can compute the chiral condensate which is given by
\[
\Xi(T) = \Sigma^2 \bar{x},
\]
where \(\Sigma\) is the chiral condensate at \(T = 0\) and \(\bar{x}\) is the only real and positive solution of
\[
\Sigma^2 = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{\bar{x}^2 + Y_n^2}
\]
or zero if no such solution exists [9]. The computation of the microscopic spectral density and the corresponding microscopic spectral correlations is more difficult since the standard methods involving orthogonal polynomials cannot be applied here due to the presence of the additional matrix \(Y\). We have used the graded eigenvalue method [10] and extended it to the present case with the additional chiral symmetry of the Dirac operator [11, 12]. A similar calculation has been done simultaneously and independently by the authors of Ref. [13].

The result of the calculation is very simple: The functional form of \(\rho_s\) and of the microscopic spectral correlations remains unchanged provided that we take into account the temperature dependence of the chiral condensate [14]. Specifically, we find that the \(k\)-point functions can be written as
\[
R_k(x_1, \ldots, x_k) = \det[C(x_p, x_q)]_{p,q=1,\ldots,k},
\]
where in the microscopic limit the function \(C\) is given by
\[
C(x_p, x_q) = 2N\Xi u_p J_1(u_p)J_0(u_q) - u_q J_0(u_p)J_1(u_q)
\]
with \(u_p = 2N\Xi x_p\) and \(u_q = 2N\Xi x_q\). Hence, the only change with respect to the zero-temperature case is that the microscopic variables are rescaled by \(\Xi\), the temperature-dependent generalization of \(\Sigma\). This result adds further support to the conjecture that \(\rho_s\) (and the microscopic spectral correlations) are universal.
So far, the calculation has been done for the quenched approximation only. The unquenched case gives rise to some technical complications, but we have no doubt that the same result will be obtained: The functional form of the microscopic spectral density and the microscopic spectral correlations will not change provided that the arguments are rescaled by the temperature-dependent chiral condensate. We make the same conjecture for the chGOE and the chGSE although it is not clear at the moment how the calculation should be done in these two cases since the graded eigenvalue method requires the computation of an Itzykson-Zuber-like integral which is not yet known for the chGOE and the chGSE. Since the microscopic spectral correlations are known at zero temperature for all three chiral ensembles (even in the presence of massless dynamical quarks) and since the temperature dependence of the chiral condensate can be computed easily for each ensemble, the finite-temperature results thus follow immediately.

3 A random matrix model for the chiral phase transition with all Matsubara frequencies included

As we have already mentioned in the previous section, at finite temperature a temperature-dependent matrix $Y$ has to be added to the random matrix $W$ representing the matrix elements of the Dirac operator. Why this is so will become obvious below. Since $P(W)$ is invariant under a transformation of the basis, we can always make $Y$ diagonal by a suitable basis transformation. While at zero temperature the choice of basis was completely irrelevant we now have to specify the basis with respect to which the matrix elements of the Dirac operator are evaluated. A different choice of basis will be reflected in a different form of the matrix $Y$. In Refs. [15, 16] a plane wave basis was chosen whereas the basis of Ref. [9] consisted of (anti-) instanton zero modes. In [15], only the lowest Matsubara frequency was included in $Y$ since it is sufficient to determine the properties of the chiral phase transition at $T = T_c$. We now present a model which includes all Matsubara frequencies, thereby extending the validity of the model to lower temperatures.\footnote{After completion of this work, we were informed by M. A. Nowak that an equivalent model, derived from the NJL model and additionally including a chemical potential, was discussed recently in Ref. [17]. One can show that the two models yield identical results.} (For a comparison with the model presented in Ref. [16] see below.)

If one chooses a plane-wave basis $\phi_k(x)e^{i\omega_k \tau}$ at finite temperature, the $\partial_0$-term in $i\hat{D}$ gives rise to a term proportional to $\omega_k = (2k + 1)\pi T$ ($k$ integer) in the matrix of the Dirac operator. These terms are included in $Y$. The basis ...
states \( \phi_k(x) \) corresponding to the same Matsubara frequency \( \omega_k \) are coupled by a random matrix. In the same way, basis states belonging to different Matsubara frequency should be coupled by random matrices. The matrix of the Dirac operator at finite temperature can thus be written in the form

\[
\begin{bmatrix}
im & W + \Omega \\
W^\dagger + \Omega & \nim
\end{bmatrix}
\]

(10)

with

\[
\Omega = \text{diag}(\pi T, \ldots, \pi T, -\pi T, \ldots, -\pi T, -3\pi T, \ldots, -3\pi T, \ldots).
\]

(11)

The dimension of a submatrix of \( \Omega \) belonging to one single Matsubara frequency is equal to \( K \) which should be identified with the three-volume \( V_3 \). The four-volume is \( V_4 = V_3/T = K/T \). The elements of the random matrix \( W \) are distributed according to \( P(W) \sim \exp(\text{Tr} WW^\dagger) \). The model presented in Ref. [16] also took into account all Matsubara frequencies. However, basis states corresponding to different Matsubara frequencies were coupled by the same random matrix. There is no reason why this should be the case, and the resulting temperature dependence of the chiral condensate shows non-analytic features which are physically unrealistic.

Going through the same steps as in [9], we obtain a saddle-point equation similar to (7) from which the chiral condensate follows as \( \langle \bar{q}q \rangle = \Sigma^2 \bar{x} \). This equation reads (in the chiral limit \( m = 0 \))

\[
\Sigma^2 = \frac{1}{V_4} \sum_{n=1}^{N} \frac{1}{\bar{x}^2 + \Omega_n^2} = \frac{2K}{K/T} \sum_{k=0}^{N/2K} \frac{1}{\bar{x}^2 + \omega_k^2},
\]

(12)

where the upper limit \( N/2K \) of the sum should go to infinity. We now perform the sum, choose units in which \( \langle \bar{q}q \rangle(T=0) = 1 \), and express the result in terms of \( \langle \bar{q}q \rangle \) to obtain

\[
\langle \bar{q}q \rangle = \tanh \left( \frac{\langle \bar{q}q \rangle}{T/T_c} \right).
\]

(13)

This equation can only be solved numerically, and the result is plotted in Figure 2. The low-temperature behavior of the chiral condensate can be obtained analytically in this model. We have

\[
\langle \bar{q}q \rangle \approx 1 - 2 \exp \left( -\frac{2}{T/T_c} \right) \quad \text{as} \quad T \to 0.
\]

(14)

We observe that the chiral condensate does not change significantly unless the temperature is close to \( T_c \). While this is in qualitative agreement with lattice
data [18] Eq. (14) does not agree with the well-established low-temperature expansion of chiral perturbation theory [19]. The reason for this discrepancy is that the random-matrix model in its present form is effectively zero-dimensional in space and, therefore, unable to describe the effect of the Goldstone bosons adequately. In order to account for the Goldstone bosons, additional non-random parts would have to be added to the random matrix.

At the critical temperature, we can compute the critical exponents $\beta$, $\gamma$, and $\delta$. These are defined using the reduced temperature $t = (T - T_c)/T_c$ by

\[
\langle \bar{q}q \rangle \sim |t|^{\beta}, \quad \chi \sim |t|^{-\gamma}
\]

with $\chi = \partial \langle \bar{q}q \rangle / \partial m |_{m=0}$, and $\langle \bar{q}q \rangle \sim m^{1/\delta}$ at $T = T_c$. We obtain $\beta = 1/2$, $\gamma = 1$, and $\delta = 3$ in agreement with [15]. This supports the position taken in [15] that only the lowest Matsubara frequency is responsible for the critical behavior at the phase transition.

4 Conclusions

We have presented very direct evidence for the universality of the microscopic spectral density of the Dirac operator by comparing lattice data obtained by Berbenni-Bitsch and Meyer to predictions from random matrix theory. Impressive agreement was found already for relatively small lattices. Clearly, in the microscopic limit the lattice data are described by random matrix theory. Thus, RMT is a very useful tool in the analysis of lattice data and may provide...
guidance for further lattice calculations. It remains to be seen whether RMT can also prove useful with respect to simplifying the generation of gauge field configurations. This would be enormously helpful, in particular close to the chiral limit.

The universality of the microscopic spectral correlations of the Dirac operator was further supported by a generic finite-temperature calculation in which it was found that the functional form of the microscopic spectral correlations does not change with temperature provided that the temperature dependence of the chiral condensate is taken into account in the microscopic scale. Although this calculation was only done for the chGUE in the quenched approximation, it is more than likely that analogous results will be obtained in the unquenched case and for the other two chiral ensembles.

A model was presented for the chiral phase transition at finite temperature. This model took account of all Matsubara frequencies. It reproduced the observation from lattice data that the chiral condensate remains almost constant up to temperatures close to $T_c$ and also confirmed the position of Ref. [15] that the lowest Matsubara frequency is responsible for the properties of the chiral phase transition. Reasons for discrepancies with the low-temperature expansion of chiral perturbation theory have been discussed.

A number of very interesting topics such as the effect of a chemical potential [20, 21] and the question of topological charge and the axial U(1) anomaly [22] have not even been mentioned in this article. Although the field of applying RMT to problems of QCD is a very recent one, many useful applications have already been found, and we are convinced that this will not cease to be the case in future work.

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