Multi-Component Gas-Dynamics and Turbulence

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Abstract

In the article, correct method for the kinetic Boltzmann equation asymptotic solution is formulated, the Hilbert method and the Enskog error are considered. The equations system of multi-component nonequilibrium gas-dynamics is derived, that corresponds to the first order in the approximate method for solution of the kinetic Boltzmann equation within the Struminskii approach. It is shown, that velocity distribution functions, received by the proposed method and by the Enskog method within the Enskog approach to solving of kinetic Boltzmann equation for gas mixture, are equivalent to first infinitesimal order terms (inclusive; accordingly, systems of gas-dynamic equations of the second order coincide), but, generally speaking, differ in the next order. An interpretation of turbulent gas flow is proposed, as stratified to components gas flow, which is described by the derived equations system of multi-component nonequilibrium gas-dynamics.

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I. INTRODUCTION

In 1912 Hilbert in [1], Chapter XXII, as an example of integral equation, considered the kinetic Boltzmann equation for one-component gas and proposed a “recipe” for its approximate (asymptotic) solution. Enskog concretized the Hilbert “recipe”. But meanwhile Enskog had made a not obvious logical mistake (see below section III) and, as the result, had formulated untrue method for (asymptotic) solution of the kinetic Boltzmann equation, having proposed to use “null” conditions, conditions (42)-(44) below with zero right-hand side, for determination of five arbitrary functional parameters of the first and following approximations of the velocity distribution function. As result of paralogism of the successive approximations method, partial time derivatives vanish in the necessary and sufficient conditions of solutions existence of integral equations of higher orders, see equations (46) below, and with them terms of gas-dynamic equations, corresponding viscosity, heat conductivity . . . , vanish. Enskog “improved” the situation by the introduction (see, for example, [2], Chapter 7, § 1, Section 5) of the unreasonable expansion of partial time derivative.

Below, on the example of proposed by Struminskii in [3] method for approximate solution of the kinetic Boltzmann equation for multi-component gas, it is shown, how it is necessary to change the Enskog method. The Struminskii approach differs from the Enskog approach to approximate solving of the kinetic Boltzmann equation for gas mixture (see, for example, [2], Chapter 8, § 2) in the expansion of the Boltzmann equation – see (2) below. It should be noted, that approaches to the approximate solving of the kinetic Boltzmann equation, that are close to that suggested by Struminskii, were considered previously in the kinetic theory of plasmas, see, e.g., [4], § 7.5. Struminskii’s paper [3] and, hence, his following papers, referring to [3] (for example, [5]), involves also errors in calculation of collision integrals. This led Struminskii to improper conclusions of general nature. Proposed change of the Struminskii method does not clear up the principal demerit of the Struminskii’s method, i.e. the lack of the explicit physical small parameter, in terms of which the expansion of the successive approximations method is performed (but it is not very significant for the content of this article, as we shall see below in the section V).

Further in this article calculations of collision integrals of the general form and for a specific model of the rigid sphere potential for the case, where separate components have, generally speaking, different mean velocities and temperatures, are presented. The equations
system of multi-component nonequilibrium gas-dynamics is derived, that corresponds to the first order in the approximate method for solution of the kinetic Boltzmann equation (the approximation order is defined below) within the Struminskii approach. It is shown, that velocity distribution functions, received by the described method and by the Enskog method within the Enskog approach to solving of kinetic Boltzmann equation for gas mixture, are equivalent to first infinitesimal order terms (inclusive; accordingly, systems of gas-dynamic equations of the second order coincide), but, generally speaking, differ in the next order. This difference is the possible reason, that going to higher order in the Enskog method does not lead to any real improvement in the results.

At the end of this article an interpretation of turbulent flows is proposed within the multi-component gas-dynamics.

The below notation is close to the one in [2]; this allows an easy comparison of the described below theory to the Enskog-Chapman theory and replacement of the treatment of details, common to the theories, by references to the appropriate points in [2]. Also it may be useful a familiarity with basics of the asymptotic expansions theory, accurately stated, for example, in [6], Chapter V.

II. MODIFIED STRUMINSKI’S METHOD FOR KINETIC BOLTZMANN EQUATION SOLUTION

The basic idea of the Struminskii’s approach is as follows. In Boltzmann equation for mixture of rarefied monatomic gases [see [2], Chapter 8, (1.1); the derivation of the Boltzmann equation and its domain of applicability are discussed, for example, in [2], Chapter 3 and 18, [4], Chapter 7, § 1 and the Bogolyubov paper [8], which is included in [2] as an addition]:

\[
\frac{\partial f_i}{\partial t} + \mathbf{c}_i \cdot \frac{\partial f_i}{\partial \mathbf{r}} + \frac{\mathbf{X}_i}{m_i} \cdot \frac{\partial f_i}{\partial \mathbf{c}_i} = \sum_j \int \int \int (f'_i f'_j - f_i f_j) g_{ij} b \, db \, d\epsilon \, d\mathbf{c}_j \\
= \sum_j \int \int (f'_i f'_j - f_i f_j) k_{ij} \, d\mathbf{k} \, d\mathbf{c}_j,
\]  

(1)
the formal parameter $\theta$ (used simply as an "indicator of smallness", $\theta = 1$) is included in somewhat another manner, than in the Enskog approach:

$$D_i f_i = -\frac{1}{\theta} J_i (f_i, f) - \sum_{j \neq i} J_{ij} (f_i, f_j). \quad (2)$$

In the Enskog approach multiplier $1 / \theta$ refers to the whole right-hand side:

$$D_i f_i = -\frac{1}{\theta} \sum_{j} J_{ij} (f_i, f_j) \quad (3)$$

– see [2], Chapter 7, § 1, Section 5.

In (1)-(2) subscripts $i, j$ number components of the mixture; $X_i$ – the external force, acting on molecule of $i$-th grade; $m_i$ – the mass of molecule of $i$-th grade; $g_{ij} = c_i - c_j$ of colliding particles; $b$ – aiming distance, $\epsilon$ – azimuthal angle, $k$ is unit vector directed to the center of mass of colliding particles from the point of their closest approach to each other, see [2], Chapter 3, figure 3; the scalar function $k_{ij} (g_{ij}, k)$ is defined by the equality

$$g_{ij} b d\theta d\epsilon \overset{\text{def}}{=} k_{ij} d\mathbf{k}; \quad (4)$$

in (2)

$$J_i (f_i, f) = \int \int (f_i f - f_i' f') k_i d\mathbf{k} d\mathbf{c}, \quad (5)$$

$$J_{ij} (f_i, f_j) = \int \int (f_i f_j - f_i' f_j') k_{ij} d\mathbf{k} d\mathbf{c}_j, \quad (6)$$

to differentiate between velocities of colliding molecules of the same grade in (5), we will denote one velocity by $c_i$ and the other by $c$ (without any subscript) and omit the subscript in the relevant velocity distribution function $f$ – cf. with [2], Chapter 3, § 5, Section 1; the other notation is essentially the same as in [2], Chapter 3, § 5 and Chapter 8, § 2.

Formally introduce parameter $\theta$ to the series of successive approximations for the velocity distribution function $f_i$:

$$f_i = f_i^{(0)} + \theta f_i^{(1)} + \theta^2 f_i^{(2)} + \cdots. \quad (7)$$

Write the differential part of equation (1) as:

$$D_i f_i = \left( \frac{\partial}{\partial t} + c_i \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{X_i}{m_i} \cdot \frac{\partial}{\partial c_i} \right) \left( f_i^{(0)} + \theta f_i^{(1)} + \cdots \right)$$

$$= D_i^{(1)} + \theta D_i^{(2)} + \theta^2 D_i^{(3)} + \cdots, \quad (8)$$

$$D_i^{(r)} = \frac{\partial f_i^{(r-1)}}{\partial t} + c_i \cdot \frac{\partial f_i^{(r-1)}}{\partial \mathbf{r}} + \frac{X_i}{m_i} \frac{\partial f_i^{(r-1)}}{\partial c_i}, \quad (9)$$
− cf. with \[2\], Chapter 7, § 1, Sections 4, 5 and \[3\]. In (8)-(9) the partial time derivative expansion

\[
\frac{\partial}{\partial t} = \sum_{r=0}^{\infty} \theta^r \frac{\partial}{\partial t^r},
\]

(10)

used by Enskog and then by Struminskii, is not. As a result, the below-discussed method for the Boltzmann equation solution differs in essence from the Enskog’s and Struminskii’s methods.

Having substituted (7) and (8) into (2) and equated coefficients in like powers of \(\theta\), we arrive at the system of equations of the successive approximations method for finding of functions \(f_i^{(r)}\). This system of equations can be written as [cf. with \[2\], Chapter 8, (2.5)-(2.8)]:

\[
D_i^{(r)} + \sum_{j \neq i} J_{ij}^{(r)} + \hat{J}_i^{(r)} + \check{J}_i^{(r)} = 0,
\]

(11)

where

\[
D_i^{(0)} = 0,
\]

(12)

\[
J_{ij}^{(0)} = 0,
\]

(13)

\[
\hat{J}_i^{(0)} = 0,
\]

(14)

\[
\check{J}_i^{(0)} = J_i \left( f_{i}^{(0)} f_{j}^{(0)} \right) + J_i \left( f_{i}^{(0)} f_{j}^{(1)} \right),
\]

(15)

\[
\hat{J}_i^{(1)} = 0,
\]

(16)

\[
J_{ij}^{(r)} = J_{ij} \left( f_{i}^{(0)} f_{j}^{(r-1)} \right) + \cdots + J_{ij} \left( f_{i}^{(r-1)} f_{j}^{(0)} \right),
\]

(17)

\[
\hat{J}_i^{(r+1)} = J_i \left( f_{i}^{(1)} f_{i}^{(r)} \right) + \cdots + J_i \left( f_{i}^{(r)} f_{i}^{(1)} \right),
\]

(18)

\[
\check{J}_i^{(r)} = J_i \left( f_{i}^{(0)} f_{i}^{(r)} \right) + J_i \left( f_{i}^{(r)} f_{i}^{(0)} \right),
\]

(19)

\(r = 1, 2, \ldots\)

The assumption that \(D_i^{(r)}\) does not contain \(f_i^{(r)}\), reduces the problem of the integro-differential equation solution to a simpler problem of the integral equation solution.

The formal parameter \(\theta\) is not introduced to the Boltzmann equation in the Enskog approach (\[2\], Chapter 7, § 1, Section 5), but the \(\theta\) is introduced to the series of successive approximations for the velocity distribution functions in another manner than in (7):

\[
f_i = \frac{1}{\theta} f_i^{(0)} + f_i^{(1)} + \theta f_i^{(2)} + \cdots.
\]

(20)
The result is the same, if expansion (7) is used for the velocity distribution function, but multiplier \( 1 / \theta \) referring to the whole right-hand side is introduced to equation (3). The successive approximations \( f_i^{(0)}, f_i^{(1)}, f_i^{(2)} \ldots \) calculated within the Enskog approach prove ordered in (inverse) mixture molecule number density \( n \): \( f_i^{(0)} \) is proportional to \( n \), \( f_i^{(1)} \) is independent of \( n \), etc., – see [2], Chapter 7, § 1, Section 5 and [2], Chapter 7, § 2. Ipso facto, a physical substantiation of the using of the successive approximations method for finding of a solution of the kinetic Boltzmann equation appears. For Struminskii’s approach it is inconveniently to define explicitly a small physical parameter, in terms of which the expansion proceeds.

Below, when discussing the approximation order, we will consider the approximation order equal to the value of the superscript \( r \) in (11). In the zeroth approximation we obtain the following integral equation for determination of function \( f_i^{(0)} \):

\[
J_i \left( f_i^{(0)}, f^{(0)} \right) = 0. \tag{21}
\]

Having multiplied equation (21) by \( \ln f_i \), integrated over all \( c_i \), and transformed the integrals with taking into account

\[
\iiint \phi_i f_i' f_j' k_{ij} \, dk \, dc_i \, dc_j = \iiint \phi_i f_i' f_j k_{ij} \, dk \, dc_i \, dc_j, \tag{22}
\]

with equality (22) being independent of the form of functions \( \phi, f \) (cf. with [2], Chapter 3, § 3, Section 3), similarly to [2], Chapter 4, § 3, we find:

\[
\frac{1}{4} \iiint \ln \left( \frac{f_i^{(0)} f^{(0)}}{f_i^{(0)} f^{(0)} } \right) \left( f_i^{(0)} f^{(0)} - f_i^{(0)} f^{(0)} \right) k_i \, dk \, dc \, dc_i = 0. \tag{23}
\]

The integrand in (23) cannot be greater than zero, therefore the integral in (23) can be equal to zero only when the integrand (it is supposed, that all considered functions are continuous in each point of the range of their definition) vanishes for all values of the variables, i.e.

\[
f_i^{(0)} f^{(0)} = f_i^{(0)} f^{(0)} \tag{24}
\]

or

\[
\ln f_i^{(0)} + \ln f^{(0)} - \ln f_i^{(0)} \ln f^{(0)} = 0. \tag{25}
\]

Hence, \( \ln f_i^{(0)} \) should be expressed linearly in terms of additive collision invariants \( \psi_i^{(1)} = m_i, \psi_i^{(2)} = m_i c_i, \psi_i^{(3)} = \frac{1}{2} m_i c_i^2 \).
For the collision of the \(i\)-th molecule with the \(j\)-th molecule the conservation of invariant \(\psi_i^{(l)}\) is expressed by equality:

\[
\psi_i^{(l)'} + \psi_j^{(l)'} - \psi_i^{(l)} - \psi_j^{(l)} = 0 \quad (l = 1, 2, 3).
\] (26)

As molecule velocities \(c_i', c_j'\) upon the collision are determined completely in terms of the molecule velocities \(c_i, c_j\) before the collision by two parameters specifying the collision, for example, by aiming distance \(b\) and azimuthal angle \(\epsilon\) (see above), from four scalar equations (26), corresponding to conservation of energy and three components of momentum, like in [2], Chapter 3, § 2 we conclude, that no additive collision invariants linearly independent of \(\psi_i^{(l)}\) exist. However, this conclusion seems not quite stringent logically.

Thus,

\[
\ln f_i^{(0)} = \alpha_{i}^{(1,0)} + \alpha_{i}^{(2,0)} \cdot m_i c_i - \alpha_{i}^{(3,0)} \frac{1}{2} m_i c_i^2,
\] (27)

where \(\alpha_{i}^{(1,0)}\) and \(\alpha_{i}^{(3,0)}\) are some scalar functions of \(r\) and \(t\), independent of \(c_i\), and \(\alpha_{i}^{(2,0)}\) is a vector function of \(r\) and \(t\). Or

\[
\ln f_i^{(0)} = \ln \alpha_{i}^{(0,0)} - \alpha_{i}^{(3,0)} \frac{1}{2} m_i \left( c_i - \frac{\alpha_{i}^{(2,0)}}{\alpha_{i}^{(3,0)}} \right)^2,
\] (28)

where \(\alpha_{i}^{(0,0)}\) is a new scalar function of \(r\) and \(t\). I.e. the general solution of equation (21) can be written in the form of Maxwell function:

\[
f_i^{(0)} = \beta_i^{(1)} \left( \frac{m_i}{2\pi k \beta_i^{(3)}} \right)^\frac{3}{2} e^{-\frac{m_i}{2k} (c_i - \beta_i^{(2)})^2 / 2k \beta_i^{(3)}},
\] (29)

where

\[
\beta_i^{(1)} = \alpha_{i}^{(0,0)} \left( \frac{2\pi}{m_i \alpha_{i}^{(3,0)}} \right)^\frac{3}{2},
\] (30)

\[
\beta_i^{(2)} = \frac{\alpha_{i}^{(2,0)}}{\alpha_{i}^{(3,0)}},
\] (31)

\[
\beta_i^{(3)} = \frac{1}{k \alpha_{i}^{(3,0)}}.
\] (32)

By definition particle number density, mean velocity and temperature of the \(i\)-th compo-
nent are introduced:

\[ n_i \overset{\text{def}}{=} \int f_i \, dc_i, \]  
\[ n_i m_i u_i \overset{\text{def}}{=} \int m_i c_i f_i \, dc_i, \]  
\[ \frac{3}{2} n_i kT_i \overset{\text{def}}{=} \int \frac{1}{2} m_i (c_i - u_i)^2 f_i \, dc_i. \]

From (33)-(35) we obtain equality:

\[ \frac{3}{2} n_i kT_i + \frac{1}{2} n_i m_i u_i^2 = \int \frac{1}{2} m_i c_i^2 f_i \, dc_i, \]

which is convenient for using further instead of definition (35).

Together with asymptotic expansion (7), according to definitions (33), (34), (36), it is necessary to define asymptotic expansion for particle number density \( n_i \) of the \( i \)-th component

\[ n_i = n_i^{(0)} + \theta n_i^{(1)} + \theta^2 n_i^{(2)} + \cdots, \]

mean velocity \( u_i \) of the \( i \)-th component

\[ u_i = u_i^{(0)} + \theta u_i^{(1)} + \theta^2 u_i^{(2)} + \cdots \]

and temperature \( T_i \) of the \( i \)-th component

\[ T_i = T_i^{(0)} + \theta T_i^{(1)} + \theta^2 T_i^{(2)} + \cdots. \]

Having substituted (7) and (37)-(39) into (33), (34), (36) and equated zeroth infinitesimal order terms, arbitrary functions \( \beta_i^{(1)}(r, t), \beta_i^{(2)}(r, t) \) and \( \beta_i^{(3)}(r, t) \), that appear in (29), can be juxtaposed with the zeroth approximations to local values of particle number density, mean velocity, and temperature of the \( i \)-th component:

\[ \beta_i^{(1)}(r, t) = n_i^{(0)}(r, t), \]
\[ \beta_i^{(2)}(r, t) = u_i^{(0)}(r, t), \]
\[ \beta_i^{(3)}(r, t) = T_i^{(0)}(r, t). \]

Functions (40) are found from the first-order gas-dynamic equations system (below we will see, why from this equations system):

\[ \int \psi_i^{(l)} \left( D_i^{(1)} + \sum_{j \neq i} J_{ij}^{(1)} \right) \, dc_i = 0 \quad (l = 1, 2, 3). \]
Similarly, functions \( f_i^{(r)} \) \((r = 1, 2 \ldots)\), found at the \( r \)-th step of the successive approximations method, prove, see below, parametrically dependent on 5 arbitrary scalar functions of \( r, t \). Having equated terms of the same infinitesimal order in (33), (34), (36):

\[
\int f_i^{(r)} \, dc_i = n_i^{(r)}, \tag{42}
\]

\[
\int m_i c_i f_i^{(r)} \, dc_i = m_i \left( n_i u_i \right)^{(r)} = m_i \sum_{s=0}^{r} n_i^{(r-s)} u_i^{(s)}, \tag{43}
\]

\[
\int \frac{1}{2} m_i c_i^2 f_i^{(r)} \, dc_i = \frac{3}{2} k \left( n_i T_i \right)^{(r)} + \frac{1}{2} m_i \left( n_i u_i^2 \right)^{(r)} = \frac{3}{2} k \sum_{s=0}^{r} n_i^{(r-s)} T_i^{(s)} + \frac{1}{2} m_i \sum_{s=0}^{r} \sum_{q=0}^{s} n_i^{(r-s)} u_i^{(s-q)} \cdot u_i^{(q)}, \tag{44}
\]

we obtain 5 conditions for each subscript \( i \), that can be used to express 5 arbitrary functional parameters in \( f_i^{(r)} \) as functions of

\[
n_i^{(r)} = n_i^{(r)}(r, t), \tag{45a}
\]

\[
u_i^{(r)} = u_i^{(r)}(r, t), \tag{45b}
\]

\[
T_i^{(r)} = T_i^{(r)}(r, t), \tag{45c}
\]

Functions (45) can be found from the \((r + 1)\)-th order equations system:

\[
\int \psi_i^{(l)} \left( D_i^{(r+1)} + \sum_{j \neq i} J_{ij}^{(r+1)} + \dot{J}_i^{(r+1)} \right) \, dc_i = 0 \quad (l = 1, 2, 3). \tag{46}
\]

In (41) and (46), to simplify the further transformations according to the definitions of the pressure tensor of the \( i \)-th component

\[
p_i \overset{\text{def}}{=} \int m_i \left( c_i - u_i \right) \left( c_i - u_i \right)_i f_i \, dc_i = \int m_i c_i f_i \, dc_i - n_i m_i u_i u_i, \tag{47}
\]

the vector of heat flow density of the \( i \)-th component

\[
q_i \overset{\text{def}}{=} \int \frac{1}{2} m_i \left( c_i - u_i \right)^2 \left( c_i - u_i \right)_i f_i \, dc_i = \int \frac{1}{2} m_i c_i^2 f_i \, dc_i - p_i \cdot u_i - \frac{3}{2} n_i k T_i u_i, \tag{48}
\]

and the temperature of the \( i \)-th component \((35), \Psi_i^{(1)} = m_i, \Psi_i^{(2)} = m_i C_i, \Psi_i^{(3)} = \frac{1}{2} m_i C_i^2, \) where \( C_i = (c_i - u_i) \), can be used instead of \( \psi_i^{(l)} \).
Functions $f_i^{(r)}$, $r = 1, 2 \ldots$ are solutions to integral equations (11), which can be rewritten as

$$D_i^{(r)} + \sum_{j \neq i} J_{ij}^{(r)} + \tilde{J}_i^{(r)} = -\check{J}_i^{(r)}. \quad (49)$$

The left-hand side of (49) includes only functions known from the previous step of the successive approximations method. Unknown function $f_i^{(r)}$ appears linearly only in the right-hand side of equation (49). Therefore, the general solution of equation (49) is $\Xi_i^{(r)} + \xi_i^{(r)}$, where $\xi_i^{(r)}$ is the general solution of homogeneous integral equation

$$\check{J}_i^{(r)} = J_i^{(0)} f_i^{(0)} + J_i^{(r)} f_i^{(r)} = 0, \quad (50)$$

and $\Xi_i^{(r)}$ is some partial solution of inhomogeneous equation (49).

Solutions $\Xi_i^{(r)}$ and $\xi_i^{(r)}$ will be sought in the form of $\Xi_i^{(r)} = f_i^{(0)} \phi_i^{(r)}$ and $\xi_i^{(r)} = f_i^{(0)} \phi_i^{(r)}$, where $\phi_i^{(r)}$ and $\phi_i^{(r)}$ are new unknown functions. In view of (24),

$$J_i^{(0)} f_i^{(0)} \phi_i^{(r)} + J_i^{(r)} f_i^{(r)} \phi_i^{(0)} = \iint f_i^{(0)} f_i^{(0)} \left( \phi_i^{(r)} + \phi_i^{(r)} - \phi_i^{(r)} - \phi_i^{(r)} \right) k_i \, dk \, dc$$

$$\iint f_i^{(0)} f_i^{(0)} \left( \phi_i^{(r)} + \phi_i^{(r)} - \phi_i^{(r)} - \phi_i^{(r)} \right) k_i \, dk \, dc \equiv n_i^2 I_i \left( \phi^{(r)} \right) = 0. \quad (51)$$

Having multiplied equation (51) by $\phi_i^{(r)} \, dc_i$, integrated over all $c_i$, and transformed the integrals with account for (22), we arrive at:

$$\frac{1}{4} \iint f_i^{(0)} f_i^{(0)} \left( \phi_i^{(r)} - \phi_i^{(r)} - \phi_i^{(r)} \right)^2 k_i \, dk \, dc \, dc_i = 0. \quad (52)$$

From (52) we conclude, cf. with (23) and (27), that $\phi_i^{(r)}$ is a linear combination of additive collision invariants $\psi_i^{(l)}$:

$$\phi_i^{(r)} = \alpha_i^{(1,r)} \psi_i^{(1)} + \alpha_i^{(2,r)} \psi_i^{(2)} + \alpha_i^{(3,r)} \psi_i^{(3)}, \quad (53)$$

where $\alpha_i^{(1,r)}$ and $\alpha_i^{(3,r)}$ are arbitrary scalar functions of $r$ and $t$, and $\alpha_i^{(2,r)}$ is an arbitrary vector function of $r$ and $t$. In place of the additive invariants $\psi_i^{(l)}$, functions $\Psi_i^{(l)}$ can also be used, which are additive invariants of collision of particles of the same grade:

$$\phi_i^{(r)} = \alpha_i^{(1,r)} \Psi_i^{(1)} + \alpha_i^{(2,r)} \Psi_i^{(2)} + \alpha_i^{(3,r)} \Psi_i^{(3)}, \quad (54)$$
where \( \alpha_i^{(1,r)′} \) and \( \alpha_i^{(3,r)′} \) are new arbitrary scalar functions of \( r \) and \( t \), and \( \alpha_i^{(2,r)′} \) is a new arbitrary vector function of \( r \) and \( t \). Thus,

\[
\xi_i^{(r)} = f_i^{(0)} \left( \alpha_i^{(1,r)′} + \alpha_i^{(2,r)′} \cdot m_i c_i + \alpha_i^{(3,r)′} \frac{1}{2} m_i c_i^2, \right).
\] (55)

To make use the results of the integral equation theory, transform equation (49) to the standard form. The right-hand side of integral equation (49), i.e. \( n_i^2 I_i (\Phi(r)) \), which is a function of \( c_i \) (and, naturally, of \( r \) and \( t \), to simplify the notation, the evident dependencies are not specified) can be represented as

\[
n_i^2 I_i (\Phi(r)) = K_0 (c_i) \Phi_i^{(r)} (c_i) + \int K (c_i, c) \Phi^{(r)} (c) \, dc,
\] (56)

where

\[
K_0 (c_i) = \iint f_i^{(0)} f_i^{(0)} k_i \, dk \, dc
= f_i^{(0)} \iint |c_i - c| b \, db \, dc,
\] (57)

and \( K (c_i, c) \) is a symmetric function of \( c_i, c \) (see [1], Chapter XXII and [2], Chapter 7, § 6). Hence, equation (49) can be rewritten as:

\[
F_i^{(r)} (c_i) = K_0 (c_i) \Phi_i^{(r)} (c_i)
+ \int K (c_i, c) \Phi^{(r)} (c) \, dc,
\] (58)

\( F_i^{(r)} (c_i) \) in (58) denotes the left-hand side of integral equation (49). Linear integral equation (58) is reduced by transformation

\[
\Phi_i^{(r)} (c_i) = \tilde{\Phi}_i^{(r)} (c_i)
\sqrt{K_0 (c_i)}
\] (59a)

\[
F_i^{(r)} (c_i) = \tilde{F}_i^{(r)} (c_i) \sqrt{K_0 (c_i)},
\] (59b)

\[
K (c_i, c) = \tilde{K} (c_i, c) \sqrt{K_0 (c_i) K_0 (c)}
\] (59c)

maintaining the kernel symmetry to the linear integral equation of the second kind with symmetric kernel

\[
\tilde{F}_i^{(r)} (c_i) = \tilde{\Phi}_i^{(r)} (c_i) + \int \tilde{K} (c_i, c) \tilde{\Phi}^{(r)} (c) \, dc.
\] (60)
As homogeneous integral equation (51) has nonzero solutions (53), corresponding to equation (60), homogeneous integral equation

\[ 0 = \tilde{\phi}^{(r)}(c_i) + \int \tilde{K}(c_i, c) \tilde{\phi}^{(r)}(c) \, dc. \]  

has nonzero solutions

\[ \tilde{\phi}^{(r)}(c_i) = \phi^{(r)}(c_i) \sqrt{K_0(c_i)} \]

\[ = \left( \alpha^{(1,r)} \psi^{(1)} + \alpha^{(2,r)} \cdot \psi^{(2)} + \alpha^{(3,r)} \psi^{(3)} \right) \]

\[ \times \sqrt{K_0(c_i)}. \]  

Therefore, according to the second Fredholm alternative ([9], Chapter III, §§ 2, 3 or [10], Section 15.3-7), if \( \tilde{K}(c_i, c) \) is piecewise-continuous and normalizable and \( \tilde{F}^{(r)}(c_i) \) is continuous and square-integrable (these conditions are assumed met), the necessary and sufficient condition of solution existence of equation (60), with taking into account symmetry of kernel \( \tilde{K}(c_i, c) \), is the orthogonality of \( \tilde{\phi}^{(r)}(c_i) \) (with weight function 1) to each solution \( \phi^{(r)}(c_i) \) of equation (61).

Thus, the necessary and sufficient condition of the existence of a solution to equation (49) is the orthogonality of the left-hand side of the equation to functions \( \psi^{(l)}(c_i) \) [or, which is the same, to functions \( \Psi^{(l)}(c_i) \)], i.e. satisfaction of equalities (41), (46), which, on the other hand, can be considered as equations for determination of \( n^{(r)}_i, u^{(r)}_i, T^{(r)}_i, r = 0, 1, 2 \ldots \).

The partial solution of inhomogeneous integral equation (49) \( \Xi^{(r)}_i = f^{(l)}(c_i) \) can be constructed, for example, using expansion \( \Phi^{(r)}(c_i) \) in series in terms of Sonin polynomials with expansion coefficients, depending on \( r \) and \( t \), as this is done in [2], Chapter 7 and 8.

### III. SOME REMARKS

By the successive approximations method we receive, generally speaking, asymptotic solution of the task. Upper statement of the method for the kinetic Boltzmann equation solution is close to [2], Chapter 7 and 8. Somewhat more accurately the method for the kinetic Boltzmann equation solution can be described within the theory of asymptotic expansions with variable coefficients [6], Chapter V, § 2, Section 5. Additional conditions (for example, initial or boundary conditions for a differential equation), if there are ones, it is necessary to expand into asymptotic series, and to solve, equating coefficients at same terms.
of unified scale of comparison (§ 2, Chapter V, § 2, Section 1 (usually, the scale of comparison is the set of functions $\theta^r$, defined on the neighborhood filter of the point $\theta = 0$), the resulting system of equations. If this system of equations can be resolved, we have asymptotic solution of the starting equation, satisfying supplementary conditions. Sometimes the found asymptotic solution is regular ([11], Chapter 1), i.e. analytically depending on $\theta$, solution of the task. For example, the equation (see [12], Chapter V, § 2 or [13], Chapter IV, § 7.1)

$$\theta \frac{df}{dt} + f = 0, \quad f(0) = 0$$

(63)

with zero initial condition have with regard to comparison scale of function $\theta^r$, asymptotic solution $f^{(r)}(t) \equiv 0, (r = 0, 1, 2 \ldots)$, that is also the exact solution of the task. However, the task

$$\theta \frac{df}{dt} + f = 0, \quad f(0) = 1$$

(64)

has not analogous asymptotic solution, as the zero-order asymptotic solution of the differential equation $f^{(0)}(t) \equiv 0$ contradicts to the initial condition $f^{(0)}(0) = 1$. This is not serious weakness of the successive approximations method. In the task (64) one can introduce new function $g(t)$:

$$f(t) = \exp(-t/\theta) g(t)$$

(65)

(cf. with [11]; though discussed by Lomov expansions are not asymptotic expansions with variable coefficients, and accordingly Lomov’s approach as a whole seems not quite correct, in the monography [11] is actually shown, that similar replacements of functions allow to obtain asymptotic solutions for a wide class of problems) and by the successive approximations method receive asymptotic solution of the task $g^{(0)}(t) \equiv \text{const} = 1, g^{(r)}(t) \equiv \text{const} = 0 (r = 1, 2 \ldots)$, that is again the exact solution.

The criticism of the successive approximations method ([12], Chapter V, § 2 or [13], Chapter IV, § 7.1, possibly, reflects a dissatisfaction of authors with the unreasonable expansion of partial time derivative in the Enskog method.

Hilbert, having marked in [1], Chapter XXII, that the expansion

$$F = \frac{\Phi}{\lambda} + \Psi + X\lambda + \cdots,$$

(66)
analogous to (20) upper, (Hilbert considered only one-component gas; we maintain here Hilbert notation, however clear enough from a context) is the power series in (small parameter) \( \lambda \), satisfying to the Boltzmann equation and such, that expressions [cf. with (33)-(36) and (37)-(39)]

\[
\int \psi^{(i)} F \, d\omega = \frac{1}{\lambda} \int \psi^{(i)} \Phi \, d\omega + \int \psi^{(i)} \Psi \, d\omega + \lambda \int \psi^{(i)} X \, d\omega + \cdots \quad (i = 1, 2, 3, 4, 5)
\]

for \( t = t_0 \) pass into power series

\[
\Lambda^{(i)} = \frac{f^{(i)}}{\lambda} + g^{(i)} + \lambda h^{(i)} + \cdots \quad (i = 1, 2, 3, 4, 5),
\]

in the theorem, closing his work, has formulated ”recipe” for (asymptotic) solution of the kinetic Boltzmann equation, in which he has proposed five arbitrary functional parameters of functions \( \Phi, \Psi, X \ldots \) to define ”from five partial differential equations”, analogous (41), (46), ”at that for \( t = t_0 \)” to preset

\[
\int \psi^{(i)} \Phi \, d\omega = \lambda \Lambda^{(i)} \quad (i = 1, 2, 3, 4, 5),
\]

\[
\int \psi^{(i)} \Psi \, d\omega = 0 \quad (i = 1, 2, 3, 4, 5),
\]

\[
\int \psi^{(i)} X \, d\omega = 0 \quad (i = 1, 2, 3, 4, 5).
\]

In notation from (33)-(36), (37)-(39), (41), (46) upper, Hilbert proposed simply to use special initial condition

\[
n (r, t_0) = n^{(0)} (r, t_0),
\]

\[
u (r, t_0) = u^{(0)} (r, t_0),
\]

\[
T (r, t_0) = T^{(0)} (r, t_0)
\]

or

\[
\int \psi^{(l)} f^{(r)} \, d\mathbf{c} \bigg|_{t=t_0} \equiv 0 \quad (l = 1, 2, 3)
\]

for \( r = 1, 2 \ldots \). It allowed him, as corollary of the theorem, to formulate ”fundamental result for the theory of gases: the state of stable gas at any \( t \) is uniquely determinated, if for it at \( t = t_0 \) density, temperature and velocity are known as function of a point of space”.  

14
"For the further substantiation of the gas theory" it would be necessary to supplement Hilbert’s theorem with explicit definition of five arbitrary functional parameters of functions $f_i^{(r)}$, found on the $r$-th step ($r = 0, 1, 2 \ldots$) of the successive approximations method, through gas physical parameters (33), (34), (36), (42) - (44), but Hilbert had not made it.

Enskog formulated up the Hilbert ”recipe” for concrete calculations. However meantime Enskog had made a logical mistake. He used ”null” conditions (75) identically, at any $t$, not just at $t = t_0$ (see [2], Chapter 7, § 1, Section 1):

$$\int \psi^{(l)} f^{(r)} \, dc \equiv 0 \quad (l = 1, 2, 3)$$

for $r = 1, 2 \ldots$ From the point of the successive approximations method view Enskog instead of (37)-(39) had supposed

$$n (r, t, \theta) = \theta^0 n (r, t, \theta) + \theta^1 0 + \theta^2 0 + \cdots, \quad (77)$$

$$u (r, t, \theta) = \theta^0 u (r, t, \theta) + \theta^1 0 + \theta^2 0 + \cdots, \quad (78)$$

$$T (r, t, \theta) = \theta^0 T (r, t, \theta) + \theta^1 0 + \theta^2 0 + \cdots. \quad (79)$$

If $n$, $u$ and $T$ did not depend from $r$ and $t$, it would mean, that Enskog used simultaneously different scales of comparison \{ $n (\theta), \theta^1, \theta^2 \ldots$ \}, \{ $u (\theta), \theta^1, \theta^2 \ldots$ \}, \{ $T (\theta), \theta^1, \theta^2 \ldots$ \} in the successive approximations method, that is already wrong. In a general case, when $n$, $u$ and $T$ depend from $r$ and $t$, the sums (77)-(79) cannot even be considered as asymptotic expansions with variable coefficients.

Infringement of logic of the successive approximations method is immediately appeared in that from the equations, analogous (46) ($r = 1, 2 \ldots$), in compliance with (76) partial time derivatives vanish

$$\int \psi^{(l)} \frac{\partial f^{(r)}}{\partial t} \, dc = \frac{\partial}{\partial t} \int \psi^{(l)} f^{(r)} \, dc = 0 \quad (l = 1, 2, 3), \quad (80)$$

and with them terms of gas-dynamic equations, corresponding viscosity, heat conductivity \ldots, vanish. Somehow to correct the situation, Enskog has been forced to enter unreasonable expansion of partial time derivative (10).

IV. CALCULATION OF DEFINITE MULTIDIMENSIONAL INTEGRALS

In this section we are dealing with calculation of definite multidimensional integrals

$$\iiint \int \int \int \int \psi_i^{(l)} \left( f_i^{(0)} f_j^{(0)} - f_i^{(0)} f_j^{(0)} \right) g_{ij} b \, db \, dc_i \, dc_j. \quad (81)$$
In (81) \( \Psi_i^{(1)} = m_i, \Psi_i^{(2)} = m_i C_i, \Psi_i^{(3)} = \frac{1}{2} m_i C_i^2, C_i = c_i - u_i; \)

\[
f_i^{(0)} = n_i \left( \frac{m_i}{2\pi kT_i} \right)^{\frac{3}{2}} e^{-m_i(c_i - u_i)^2/2kT_i},
\]

is the Maxwell function of distribution of velocities of the \( i \)-th component particles, the prime in the distribution function means, that the distribution of the particle velocities \( c_i' \) after the collision is considered. The other notation is specified above.

According to (22), integral (81) can be transformed as follows:

\[
\int \int \int \Psi_i^{(l)} (f_i^{(0)} f_j^{(0)} - f_i^{(0)} f_j^{(0)}) g_{ij} db d\epsilon dc_i dc_j
\]

\[
= \int \int \int \Psi_i^{(l)} f_i^{(0)} f_j^{(0)} g_{ij} b' db' d\epsilon' dc_i' dc_j
\]

\[
- \int \int \int \Psi_i^{(l)} f_i^{(0)} f_j^{(0)} g_{ij} b db dc_i dc_j
\]

\[
= \int \int \int (\Psi_i^{(l') - \Psi_i^{(l)}}) f_i^{(0)} f_j^{(0)} g_{ij} b db dc_i dc_j.
\]

(83)

As the particle mass is conserved in the collision, for \( \Psi_i^{(1)} = m_i \) integral (83) vanishes. In the two other instances, generally speaking, this is not the case because there is no summation over the components, cf. with \( \text{[7]} \), Chapter 7, (2.33).

Hereafter statements of the two following simple propositions are used several times.

**Proposition 1.** \( f \) is assumed to be a ruled function on \( \mathbb{R} \) with values in \( \mathbb{R} \), \( w \in \mathbb{R}^3 \) be a fixed nonzero vector, \( n \in \mathbb{R}^3 \) be a unit vector. In this case

\[
\int_{\Omega_n} f (w \cdot n) n d\Omega_n = \frac{2\pi w}{w} \int_0^\pi f (w \cos (\theta)) \cos (\theta) \sin (\theta) d\theta.
\]

(84)

In the left-hand side of (84) the integral is taken over all directions of vector \( n, w \cdot n \) is the scalar product of vectors \( w \) and \( n \).

**Remark.** If \( w \) is a zero vector, then the right-hand side of (84) is set equal to 0.

**Proof.** Select the system of spherical coordinates, such that the polar axis direction be the same as the direction of the vector \( w \). Resolve the vector \( n \) into two components: parallel \( (n_\parallel) \) and perpendicular \( (n_\perp) \) to the vector \( w \):

\[
n = n_\parallel + n_\perp = \frac{(w \cdot n) w}{w^2} + n_\perp.
\]

(85)
Having substituted expression (85) for the vector $n$ into the left-hand side of (84) and integrated over the azimuthal angle, we obtain the required equality (85), as in the integration over the azimuthal angle the $n_\bot$ containing term vanishes.

**Proposition 2.** $E$ and $F$ is assumed to be two complete normalized spaces over field $\mathbb{R}$, $u$ be a continuous linear map of $E$ into $F$. In this case, if $f$ is a ruled function on interval $I \subset \mathbb{R}$ with its values in $E$, then $u \circ f$ is the ruled function on $I$ with its values in $F$ and

$$\int_a^b u(f(t)) \, dt = u \left( \int_a^b f(t) \, dt \right).$$

(86)

**Proof.** Equality (86) follows immediately from the expression for the derivative of composite function $u \circ f$; the details of the proof can be found in [6], Chapter II, § 1, Section 5.

In these propositions ruled functions can be replaced by better known continuous functions.

The major difficulties in the calculation of integral (83) are associated with the fact that parameters of the Maxwell functions for the $i$-th and the $j$-th components are not equal:

$$u_i \neq u_j, \quad T_i \neq T_j.$$  

(87)

As a result, it is not easy to get rid of the scalar products of vectors in the exponent (it is desirable that the expression for the exponent be as simple as possible).

As the scattering angle depends on the module of relative velocity of colliding particles [see, for example, [2], Chapter 3, § 4, Section 2 or [7], Chapter 1, (5.26)], it is natural to transfer in (83) to new variables – center-of-mass velocity $G_{ij}$ and relative colliding particle velocity $g_{ij}$, which are related with the particle velocities $c_i$ and $c_j$ as:

$$c_i = G_{ij} + \frac{m_j}{m_i + m_j} g_{ij},$$

(88)

$$c_j = G_{ij} - \frac{m_i}{m_i + m_j} g_{ij},$$

(89)

– cf. with [2], Chapter 9, § 2. For further simplification of the exponent vector $G_{ij}$ can be replaced by vector $\tilde{G}_{ij}$ resulting from $G_{ij}$ in an arbitrary affine transformation, for example, the one, which is a composition of shift, homothety (multiplication by a scalar), and rotation. The rotation arbitrariness is reduced to the freedom in choosing of direction of the polar
axis in the transition to the spherical coordinate system. Similarly, the vector \( g_{ij} \) can be replaced by the vector \( \tilde{g}_{ij} \), resulting from \( g_{ij} \) in composition of arbitrary homothety and arbitrary rotation. The shift of the origin of the vector \( g_{ij} \) would lead to a parametric dependence of the final integral on vectors \( u_i \) and \( u_j \) (cf. with [14], Chapter 3), which is undesirable, as integral (83) is supposed to be reduced to Chapman-Cowling integral \( \Omega_{ij}^{(l,s)} \) [see [2], Chapter 9, § 3, (3.29) and [7], Chapter 7, (4.34)].

In view of the aforesaid, make the following substitution of variables \( G_{ij} \) and \( g_{ij} \):

\[
\begin{align*}
g_{ij} &= z_1 \tilde{g}_{ij}, \\
G_{ij} &= z_2 \tilde{G}_{ij} + z_3 \tilde{g}_{ij} + \frac{u_i + u_j}{2}.
\end{align*}
\]

In (90)-(91) the scalar factors \( z_1, z_2, \) and \( z_3 \) are selected from the condition that the coefficients of \( \tilde{g}_{ij}^2 \) and \( \tilde{G}_{ij}^2 \) in the exponent be equal to 1 and the coefficient of the scalar product \( \tilde{g}_{ij} \cdot \tilde{G}_{ij} \) be equal to 0 (compare to the method of variable separation):

\[
\begin{align*}
z_1 &= \sqrt{\frac{2(m_i T_j + m_j T_i)}{m_i m_j}}, \\
z_2 &= \sqrt{\frac{2T_i T_j}{m_i T_j + m_j T_i}}, \\
z_3 &= \frac{2(T_i - T_j)}{m_i + m_j} \sqrt{\frac{m_i m_j}{2(m_i T_j + m_j T_i)}}.
\end{align*}
\]

Analogous substitutions of variables can be used in more complicated situations, for example, discussed in [14], Chapter 3.

With new variables the exponent can be written in the following form:

\[
- \left[ \tilde{g}_{ij}^2 + \tilde{G}_{ij}^2 + a_0 w^2 + a_1 \tilde{g}_{ij} \cdot w + a_2 \tilde{G}_{ij} \cdot w \right],
\]

where

\[
\begin{align*}
w &= \frac{u_i - u_j}{2}, \\
a_0 &= \frac{m_i}{2T_i} + \frac{m_j}{2T_j}, \\
a_1 &= -2 \sqrt{\frac{2m_i m_j}{m_i T_j + m_j T_i}}, \\
a_2 &= \left( \frac{m_j}{T_j} - \frac{m_i}{T_i} \right) \sqrt{\frac{2T_i T_j}{m_i T_j + m_j T_i}}.
\end{align*}
\]
Determine Jacobian of transformation of variables \((c_i, c_j) \rightarrow (\tilde{g}_{ij}, \tilde{G}_{ij})\) [see (90)-(91)]:

\[
\frac{\partial (c_i, c_j)}{\partial (\tilde{g}_{ij}, \tilde{G}_{ij})} = \frac{\partial (c_i, c_j)}{\partial (g_{ij}, G_{ij})} \frac{\partial (g_{ij}, G_{ij})}{\partial (\tilde{g}_{ij}, \tilde{G}_{ij})} = \frac{z_1^3 z_2^3}{\partial (c_i, c_j)} = \frac{z_1^3 z_2^3}{\partial (c_i - c_j, c_j)} = z_1^3 z_2^3. 
\]

Now consider the case, where \(\Psi_i = \Psi_i = m_i (c_i - u_i)\). In view of (95), (100), (90)-(91) and the equality, following from the definition of \(k\) upper,

\[
m_i (c_i^l - c_i) = \frac{m_i m_j}{m_i + m_j} (g_{ij} - g_{ij}) = -2 \frac{m_i m_j}{m_i + m_j} (g_{ij} \cdot k) k
\]

integral (S3) can be rewritten as:

\[
\int \int \int m_i (c_i^l - c_i) f_j^{(0)} g_{ij} b \, db \, d\epsilon c_i \, dc_j
\]

\[
= -2 \frac{m_i m_j}{m_i + m_j} \frac{z_i^3 z_2^3}{n_i} \left( \frac{m_i}{2 \pi k T_i} \right) \left( \frac{m_j}{2 \pi k T_j} \right) \frac{z_i^3 z_2^3}{n_j} \int \int \int (\tilde{g}_{ij} \cdot k) k
\]

\[
\times \exp \left( - \left[ \tilde{g}_{ij} + \tilde{G}_{ij} + a_0 w^2 + a_1 \tilde{g}_{ij} \cdot w + a_2 \tilde{G}_{ij} \cdot w \right] \right)
\]

\[
x \tilde{g}_{ij} b \, db \, d\tilde{G}_{ij} \, d\tilde{g}_{ij} \, db. 
\]

Integrating with respect to \(\epsilon\) in (102) (with fixed \(\tilde{g}_{ij}\) and \(\tilde{G}_{ij}\)), resolve vector \(k\) into two components: the ones parallel and perpendicular to vector \(\tilde{g}_{ij}\) - cf. with the proof of Proposition I:

\[
\int (\tilde{g}_{ij} \cdot k) \, k \, db = 2 \pi \cos^2 \left( \frac{\pi - \chi}{2} \right) \tilde{g}_{ij} = \pi (1 - \cos \chi) \tilde{g}_{ij}.
\]

When integrating over \(\tilde{G}_{ij}\) and directions of vector \(\tilde{g}_{ij}\), use Proposition I. As a result we arrive at

\[
\int \int m_i (c_i^l - c_i) f_j^{(0)} g_{ij} b \, db \, d\epsilon c_i \, dc_j
\]

\[
= 16 n_i n_j m_i T_j + m_i T_j \frac{w \sqrt{\pi}}{(m_i + m_j) w} \frac{e^{-\frac{2m_i m_j w^2}{m_i T_j + m_j T_j}}}{\xi^2}
\]

\[
\times \int \int e^{-\tilde{g}_{ij}^2} [\tilde{g}_{ij} \xi \cosh (\tilde{g}_{ij} \xi) - \sinh (\tilde{g}_{ij} \xi)] \tilde{g}_{ij}^2 (1 - \cos \chi) b \, db \, d\tilde{g}_{ij}. 
\]
In $(104)$

$$\xi = a_1 w,$$  \hspace{1cm} (105)

factor $a_1$ is determined by formula $(98)$. It is easy to check that the singularity at $\xi = 0$, which is possible when $w = 0$, is actually absent in the right-hand side. Expression $(104)$ differs from Sruminskii’s expression, i.e. $(8)$.

The case, where $\Psi_i^{(3)} = \Psi_i^{(3)} = \frac{1}{2} m_i (c_i - u_i)^2$, differs from the just considered one in the factor of the exponent in the right-hand side of $(102)$. Transform difference $\Psi_i^{(l') - \Psi_i^{(l)}}$ according to $(90)$, $(91)$ and $(2)$, Chapter 3, (4.9) and taking into account that only the relative particle velocity direction changes during the collision ($g_{ij} = g'_{ij}$):

$$\Psi_i^{(3)} = \Psi_i^{(3)} = \frac{m_i}{2} [(c'_{i} - u_i)^2 - (c_i - u_i)^2]$$

$$= \frac{m_i}{2} (c'_{i} - c_i) (c'_{i} + c_i - 2u_i)$$

$$= \frac{m_i m_j}{m_i + m_j} \left( \{ g'_{ij} - g_{ij} \} \cdot \{ G_{ij} - u_i \} \right)$$

$$= -2 z_1 \frac{m_i m_j}{m_i + m_j} (g_{ij} \cdot k)$$

$$\times \left( k \cdot \left\{ z_2 \tilde{G}_{ij} + z_3 \tilde{g}_{ij} - \frac{u_i - u_j}{2} \right\} \right).$$  \hspace{1cm} (106)

With respect to its arguments the scalar product is a bilinear continuous function, therefore Proposition $(2)$ can be applied. On the integration with respect to $\epsilon$, similarly to $(103)$, we arrive at:

$$-2 z_1 \frac{m_i m_j}{m_i + m_j} \int (g_{ij} \cdot k) \left( k \cdot \left\{ z_2 \tilde{G}_{ij} + z_3 \tilde{g}_{ij} - \frac{u_i - u_j}{2} \right\} \right) d\epsilon$$

$$= -2 \pi z_1 \frac{m_i m_j}{m_i + m_j} (1 - \cos \chi) \left( \tilde{g}_{ij} \cdot \left\{ z_2 \tilde{G}_{ij} + z_3 \tilde{g}_{ij} - \frac{u_i - u_j}{2} \right\} \right).$$  \hspace{1cm} (107)

Perform the integration over $\tilde{G}_{ij}$ and directions of vector $\tilde{g}_{ij}$ using Proposition $(1)$

$$\int \int \frac{m_i}{2} \left[ (c'_{i} - u_i)^2 - (c_i - u_i)^2 \right] f_i^{(0)} f_j^{(0)} g_{ij} b \, db \, d\epsilon \, dc_i \, dc_j$$

$$= 16 n_i n_j \frac{\sqrt{w}}{\xi} e^{-\frac{2m_i m_j w^2}{\xi (r_j^2 + m_j^2)}} \int \int e^{-\tilde{g}_{ij}^2}$$

$$\times \left\{ - D_{1,ij} \frac{w}{\xi} [\tilde{g}_{ij} \cosh (\tilde{g}_{ij} \xi) - \sinh (\tilde{g}_{ij} \xi)] - 2 D_{2,ij} \tilde{g}_{ij}^2 \sinh (\tilde{g}_{ij} \xi) \right\}$$

$$\times \tilde{g}_{ij}^2 (1 - \cos \chi) b \, db \, d\tilde{g}_{ij}.$$  \hspace{1cm} (108)
In (108):

\[ D_{1,ij} = \frac{2m_jT_i}{m_i + m_j}, \]  

\[ D_{2,ij} = \frac{m_im_j(T_i - T_j)}{2(m_i + m_j)^2} \sqrt{\frac{2T_i}{m_i} + \frac{2T_j}{m_j}}. \]  

The other notations are the same as in (104).

It is interesting to note that for \( u_i = u_j \) integral (104) and the first term in (108) vanish and the second term in (108) is:

\[ \sim \text{const} (T_j - T_i), \]  

that corresponds to energy transfer from the "hot" components to the "cold", see the gas-dynamic equations system below. In view of the sign of \( a_1 \) (98) and definition of \( \xi \) (105), the first term leads to temperature increase with \( w \neq 0 \).

V. FIRST-ORDER EQUATIONS SYSTEM OF MULTI-COMPONENT NONEQUILIBRIUM GAS-DYNAMICS

Above the gas-dynamic equations system has been derived, in a sense, as a "by-product" during the Boltzmann equation solution. More generally, the gas-dynamic equations system can be written in the form of the transport equations, cf. with [2], Chapter 3, (1.12) and [7], Chapter 7, (2.31).

Having multiplied the Boltzmann equation for the \( i \)-th component (1) by \( \Psi_i(l) \) and integrated over all values of \( c_i \) (it is assumed that all the integrals obtained below converge and products like \( \Psi_i(l)X_i f_i \) tend to zero, when \( c_i \) tends to infinity), we arrive at:

\[ \int \Psi_i(l) \left( \frac{\partial f_i}{\partial t} + c_i \cdot \frac{\partial f_i}{\partial x} + \frac{X_i}{m_i} \cdot \frac{\partial f_i}{\partial c_i} \right) \, dc_i = \sum_j \int \int \Psi_i(l) \left( f_i^j f_j^i - f_i f_j \right) g_{ij} b \, db \, de \, dc_i \, dc_j. \]  

(112)
The terms in the left-hand side of this equation can be transformed:

\[
\int \Psi_i(\mathbf{l}) \frac{\partial f_i}{\partial t} d\mathbf{c}_i = \frac{\partial}{\partial t} \int \Psi_i(\mathbf{l}) f_i d\mathbf{c}_i - \int \frac{\partial \Psi_i(\mathbf{l})}{\partial t} f_i d\mathbf{c}_i = \frac{\partial}{\partial t} \left( n_i \overline{\Psi_i} \right) - n_i \overline{\frac{\partial \Psi_i}{\partial t}}, \tag{113}
\]

\[
\int \Psi_i(\mathbf{l}) \mathbf{c}_i \cdot \frac{\partial f_i}{\partial \mathbf{r}} d\mathbf{c}_i = \frac{\partial}{\partial \mathbf{r}} \cdot \int \Psi_i(\mathbf{l}) f_i d\mathbf{c}_i - \int \mathbf{c}_i \cdot \frac{\partial \Psi_i(\mathbf{l})}{\partial \mathbf{r}} f_i d\mathbf{c}_i = \frac{\partial}{\partial \mathbf{r}} \cdot n_i \overline{\Psi_i} \mathbf{c}_i - n_i \overline{\mathbf{c}_i} \cdot \frac{\partial \Psi_i}{\partial \mathbf{r}}, \tag{114}
\]

\[
\int \Psi_i(\mathbf{l}) \mathbf{X}_i \cdot \frac{\partial f_i}{\partial \mathbf{c}_i} d\mathbf{c}_i = - \int \left( \frac{\partial}{\partial \mathbf{c}_i} \cdot \overline{\Psi_i(\mathbf{l})} \mathbf{X}_i \right) f_i d\mathbf{c}_i = -n_i \frac{\partial}{\partial \mathbf{c}_i} \cdot \overline{\Psi_i(\mathbf{l})} \mathbf{X}_i. \tag{115}
\]

In (113)-(115), the bar, as usually, denotes the average of the quantity

\[
\overline{V}_i = \frac{1}{n_i} \int V_i f_i d\mathbf{c}_i; \tag{116}
\]

\( \mathbf{r} \) and \( \mathbf{c}_i \) are considered as independent variables. In view of (113)-(115), from (112) we obtain

\[
\frac{\partial}{\partial t} \left( n_i \overline{\Psi_i} \right) + \frac{\partial}{\partial \mathbf{r}} \cdot n_i \overline{\Psi_i} \mathbf{c}_i - n_i \left( \frac{\partial \overline{\Psi_i}}{\partial t} + \mathbf{c}_i \cdot \frac{\partial \overline{\Psi_i}}{\partial \mathbf{r}} + \frac{\partial}{\partial \mathbf{c}_i} \cdot \overline{\Psi_i(\mathbf{l})} \mathbf{X}_i \right) = \sum_j \int \int \int \Psi_i(\mathbf{l}) \left( f_i^j f_j^j - f_i f_j \right) g_{ij} b d\mathbf{b} d\mathbf{c}_i d\mathbf{c}_j, \tag{117}
\]

– the transfer equation for the \( \Psi_i(\mathbf{l}) \), that refers to particles of the \( i \)-th grade.

To derive the equations of mass, momentum and energy transport for the \( i \)-th component from (117), sequentially substitute \( m_i, m_i (\mathbf{c}_i - \mathbf{u}_i), \frac{1}{2} m_i (\mathbf{c}_i - \mathbf{u}_i)^2 \) for \( \Psi_i(\mathbf{l}) \) into (117).

In the Enskog-Chapman theory, in view of the additional summation over \( i \), the right-hand side of (117) vanishes always. However, if the velocity distribution functions for some components are Maxwell functions (82) with different parameters of mean velocity and temperature \( (\mathbf{u}_i \neq \mathbf{u}_j, T_i \neq T_j) \), for example, due to some external effects (see below), then nonzero terms remain in the right-hand side of (117). In this case equations (117) \( (l = 1, 2, 3) \) are the same as equations (41). Thus, on straightforward transformations we arrive at the following gas-dynamic equations system [cf. with 7, Chapter 7, (2.42), (2.45),
\[ n_i m_i \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial r} \cdot n_i^{(0)} p_i = n_i X_i - n_i m_i u_i \cdot \frac{\partial}{\partial r} u_i, \quad (118) \]

\[ \frac{\partial n_i}{\partial t} = -\frac{\partial}{\partial r} \cdot n_i u_i, \quad (119) \]

\[ \frac{\partial \hat{E}_i}{\partial t} + \frac{\partial}{\partial r} \cdot q_i^{(0)} + p_i^{(0)} \cdot \frac{\partial u_i}{\partial r} - \sum_{j \neq i} I_{e, ij}^{(0)} = -\frac{\partial}{\partial r} \cdot \hat{E}_i u_i. \quad (120) \]

In (118)-(120):

\[ p_i^{(0)} = n_i m_i (c_i - u_i) (c_i - u_i)^{(0)} = n_i k T_i U = p_i^{(0)} U \quad (121) \]

is the \( i \)-th component pressure tensor, \( p_i^{(0)} \) is the hydrostatic pressure, \( U \) is the unit tensor, double product of two second rank tensors \( w \) and \( w' \) ([2], Chapter 1, § 3) is the scalar 
\[ w : w' = \sum_{\alpha} \sum_{\beta} w_{\alpha \beta} w'_{\beta \alpha} = w' : w, \]

\[ q_i^{(0)} = \frac{1}{2} n_i m_i (c_i - u_i)^2 (c_i - u_i)^{(0)} = 0 \quad (122) \]

is the \( i \)-th component heat flux density vector,

\[ \hat{E}_i = \frac{1}{2} n_i m_i (c_i - u_i)^2 \quad (123) \]

is the internal energy of particles of the \( i \)-th component per unit volume, which is equal, in this case, to energy of their translational motion, however, the energy transfer equation, written in form (120), apparently, can be used in more general cases as well (cf. with [7], Chapter 7, § 6); in (121)-(123) superscript \( (0) \) denotes averaging (116) with Maxwell function \( f_i^{(0)} \) from (82).

In (119)-(120) \( I_{p, ij}^{(0)} \), \( I_{e, ij}^{(0)} \) denote integrals (104) and (108), respectively. When averaging the last term in the left-hand side of (117), external force \( X_i \), acting on the particle of the \( i \)-th grade, is assumed independent of the particle velocity.

VI. VALUES OF KINETIC INTEGRALS FOR INTERACTION POTENTIAL OF RIGID SPHERES

Integral terms \( I_{p, ij}^{(0)} \), \( I_{e, ij}^{(0)} \), appearing in multi-component gas-dynamics equations system (118)-(120), are quite complex functions of mean velocities and temperatures of separate
components, mainly, because of a complex dependence of deflection angle $\chi$ on relative velocity of colliding particles [cf. with \textsuperscript{[2]}, Chapter 1, (5.26)]:

$$\chi (b, g) = \pi - 2b \int_{r_m}^{\infty} \frac{dr}{r^2} \frac{r^2}{1 - \frac{v^2}{w^2}} \sqrt{1 - \frac{\phi(r)}{w^2}}.$$  \hfill (124)

The component temperatures appear in the resultant expressions as a making non-dimensional factor.

In the simplest case of particles, interacting according to the law of rigid spheres, the following analytical expressions for $I_{p,ij}^{(0)}$ and $I_{e,ij}^{(0)}$ have been derived:

$$I_{p,ij}^{(0)} = n_i n_j \frac{m_i T_j + m_j T_i}{m_i + m_j} \frac{w \sqrt{\pi}}{2 \sigma_{ij}^2} \exp \left( -\frac{\xi^2}{4} \right) \left[ 2 \xi (\xi^2 + 2) + \sqrt{\pi} (\xi^4 + 4 \xi^2 - 4) \text{erf} \left( \frac{\xi}{2} \right) \right], \hfill (125)$$

$$I_{e,ij}^{(0)} = -n_i n_j \frac{\sqrt{\pi}}{2 \sigma_{ij}^2} \exp \left( -\frac{\xi^2}{4} \right) \left[ 2D_{1,ij} w \xi (\xi^2 + 2) + 2D_{2,ij} \xi^2 (\xi^2 + 10) \right]$$

$$-n_i n_j \frac{\pi}{2 \sigma_{ij}^2}$$

$$\times \left[ D_{1,ij} w \left( \xi^4 + 4 \xi^2 - 4 \right) + D_{2,ij} \xi \left( \xi^4 + 12 \xi^2 + 12 \right) \right] \text{erf} \left( \frac{\xi}{2} \right). \hfill (126)$$

In (125)-(126) notations from (96), (98), (105), (109)-(110) are used.

**VII. PROPOSED METHOD AND ENSKOG-CHAPMAN THEORY**

Let’s consider, what do nonzero conditions (42)-(44) and expansions (37)-(39) for one-component gas lead to; for one-component gas the Enskog approach and the Struminskii approach coincide.

Differential equations (41), from which functions $n^{(0)}(r, t), u^{(0)}(r, t), T^{(0)}(r, t)$ are found, for one-component gas can be written in the form [cf. with (118)-(123)]:

$$\frac{Dn^{(0)}}{Dt} = -n^{(0)} \frac{\partial}{\partial r} \cdot u^{(0)}, \hfill (127)$$

$$\frac{Dn^{(0)} m u^{(0)}}{Dt} = n^{(0)} \mathbf{X} - \frac{\partial p^{(0)}}{\partial r}, \hfill (128)$$

$$\frac{D T^{(0)}}{Dt} = -\frac{2p^{(0)}}{3n^{(0)} k} \frac{\partial}{\partial r} \cdot u^{(0)} = -\frac{2T^{(0)}}{3} \frac{\partial}{\partial r} \cdot u^{(0)}. \hfill (129)$$
In (127)-(129)
\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}(0) \cdot \frac{\partial}{\partial \mathbf{r}},
\]
\[
p^{(0)} = n^{(0)} k T^{(0)}.
\]

From (127), (129) we receive, that in the first order gas-dynamic flow is adiabatic:
\[
\frac{D}{Dt} \left[ n^{(0)} \left( T^{(0)} \right)^{-\frac{3}{2}} \right] = 0.
\]

For one-component gas integral equation (49), \( r = 1 \), from which \( f^{(1)} = f^{(0)} \Phi^{(1)} \) is found, with taking into account (29), (56) and (127)-(132), can be written as (cf. with [2], Chapter 7, § 3):
\[
- n^2 I \left( \Phi^{(1)} \right) = \frac{\partial f^{(0)}}{\partial t} + \mathbf{c} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{r}} + \frac{\mathbf{X}}{m} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{c}} = f^{(0)} \left[ \left( C^2 - \frac{5}{2} \right) \mathbf{C} \cdot \frac{\partial \ln T^{(0)}}{\partial \mathbf{r}} + 2 \mathbf{C} : \frac{\partial}{\partial \mathbf{r}} \mathbf{u}^{(0)} \right].
\]

In (133) \( C = \mathbf{c} - \mathbf{u}^{(0)} \),
\[
\mathbf{C} = \left( \frac{m}{2kT^{(0)}} \right)^{\frac{1}{2}} \mathbf{C},
\]
\( \mathbf{C} \) is the module of vector \( \mathbf{C} \); for arbitrary second-rank tensor \( \mathbf{w} \)
\[
\overset{\circ}{\mathbf{w}} = \mathbf{w} - \frac{1}{3} \mathbf{U} (\mathbf{U} : \mathbf{w})
\]
- tensor with zero trace.

The general (scalar) solution of equation (133), being the sum of some partial solution of equation (133) and general solution of homogeneous equation \( I \left( \phi^{(1)} \right) = 0 \), cf. with (54), we can seek in the form of:
\[
f^{(1)} = f^{(0)} \left[ - \frac{1}{n^{(0)}} \left( \frac{2kT^{(0)}}{m} \right)^{\frac{1}{2}} \mathbf{A} \cdot \frac{\partial \ln T^{(0)}}{\partial \mathbf{r}} - \frac{1}{n^{(0)}} \mathbf{B} : \frac{\partial}{\partial \mathbf{r}} \mathbf{u}^{(0)} \right] + f^{(0)} \left( \alpha^{(1,1)} + \alpha^{(2,1)} \cdot m \mathbf{C} + \alpha^{(3,1)} \frac{1}{2} m \mathbf{C}^2 \right),
\]
where the vector function \( \mathbf{A} \) is partial solution of the equation
\[
n I \left( \mathbf{A} \right) = f^{(0)} \left( C^2 - \frac{5}{2} \right) \mathbf{C},
\]
and the tensor function \( \mathbf{B} \) is partial solution of the equation
\[
n I \left( \mathbf{B} \right) = 2 f^{(0)} \overset{\circ}{\mathbf{C}} \mathbf{C}.
\]
Solvability conditions of equations (137) and (138) are satisfied, i.e. (see the section II upper):

\[
\int \Psi^{(l)} f^{(0)} \left( C^2 - \frac{5}{2} \right) C \, dc = 0 \quad (l = 1, 2, 3),
\]

\[
\int \Psi^{(l)} f^{(0)} \mathring{C} C \, dc = 0 \quad (l = 1, 2, 3).
\]

In (139) and (140) \( \Psi^{(1)} = m, \Psi^{(2)} = mC, \Psi^{(3)} = \frac{1}{2} mC^2 \).

Because \( r, t \) and \( u^{(0)} \) do not explicitly occur in equations (137) and (138), and right-hand side of equation (138) is symmetric tensor with zero trace, solutions \( A \) and \( B \) can be sought in the form of:

\[
A = A \left( n^{(0)}, C, T^{(0)} \right) C,
\]

\[
B = B \left( n^{(0)}, C, T^{(0)} \right) \mathring{C} C,
\]

where \( A \left( n^{(0)}, C, T^{(0)} \right) \) and \( B \left( n^{(0)}, C, T^{(0)} \right) \) – scalar functions of \( n^{(0)}, C \) and \( T^{(0)} \). It is possible to impose an additional condition on the solution \( A \) (cf. with [2], Chapter 7, § 3, Section 1):

\[
\int C^2 f^{(0)} A \left( n^{(0)}, C, T^{(0)} \right) \, dc = 0.
\]

Having substituted (136) in (42)-(44) with taking into account (29), (40) and (143), we arrive at:

\[
n^{(1)} = \int f^{(0)} \phi^{(1)} \, dc
\]

\[
= n^{(0)} \alpha^{(1,1)} + mn^{(0)} u^{(0)} \cdot \alpha^{(2,1)}
\]

\[
+ \frac{1}{2} n^{(0)} \left[ 3kT^{(0)} + m(u^{(0)})^2 \right] \alpha^{(3,1)},
\]

\[
m(nu)^{(1)} = \int mcf^{(0)} \phi^{(1)} \, dc
\]

\[
= mn^{(0)} u^{(0)} \left( \alpha^{(1,1)} + nu^{(0)} \cdot \alpha^{(2,1)} \right) + mn^{(0)} kT^{(0)} \alpha^{(2,1)}
\]

\[
+ \frac{1}{2} mn^{(0)} u^{(0)} \left[ 5kT^{(0)} + m(u^{(0)})^2 \right] \alpha^{(3,1)},
\]

\[
\frac{3}{2} k (nT)^{(1)} + \frac{1}{2} m(nu^2)^{(1)} = \int \frac{1}{2} mc^2 f^{(0)} \phi^{(1)} \, dc
\]

\[
= \frac{1}{2} n^{(0)} \left[ 3kT^{(0)} + m(u^{(0)})^2 \right] \alpha^{(1,1)}
\]

\[
+ \frac{1}{2} mn^{(0)} \left[ 5kT^{(0)} + m(u^{(0)})^2 \right] u^{(0)} \cdot \alpha^{(2,1)}
\]

\[
+ \frac{5}{4} n^{(0)} kT^{(0)} \left[ 3kT^{(0)} + 2m(u^{(0)})^2 \right] \alpha^{(3,1)}
\]

\[
+ \frac{1}{4} n^{(0)} m^2 u^{(0)}^4 \alpha^{(3,1)}.
\]
In (144)-(146) vanishing integrals (cf. with [2], Chapter 7, § 3, Section 1) are neglected. From (144)-(146) we have:

\[ \alpha^{(1,1)} = \frac{n^{(1)}}{n^{(0)}} - \frac{3}{2} \frac{T^{(1)}}{T^{(0)}}, \]

\[ \alpha^{(2,1)} = \frac{u^{(1)}}{kT^{(0)}}, \]

\[ \alpha^{(3,1)} = \frac{1}{kT^{(0)}} \frac{T^{(1)}}{T^{(0)}}. \]  

To first infinitesimal order terms (see. [6], Chapter V, § 2, definition 2) expression

\[ f^{(0)} + f^{(0)} \left( \frac{n^{(1)}}{n^{(0)}} - \frac{3}{2} \frac{T^{(1)}}{T^{(0)}} + \frac{u^{(1)}}{kT^{(0)}} \cdot mC + \frac{1}{kT^{(0)}} \frac{T^{(1)}}{T^{(0)}} \frac{1}{2} mC^2 \right) \]  

coincides with the asymptotic expansion of the solution \( \tilde{f}^{(0)} \) in the Enskog-Chapman theory

\[ \tilde{f}^{(0)} = n^{[1]} \left( \frac{m}{2\pi kT^{[1]}} \right)^{\frac{3}{2}} e^{-\frac{m}{kT^{[1]}}} \]

where \( n^{[1]} = n^{(0)} + n^{(1)}, u^{[1]} = u^{(0)} + u^{(1)} \) and \( T^{[1]} = T^{(0)} + T^{(1)} \), cf. with Taylor expansion of function \( \tilde{f}^{(0)} \) about the point \( (n^{(0)}, u^{(0)}, T^{(0)}) \). This assertion can be written in the form of:

\[ \tilde{f}^{(0)} \sim f^{(0)} + f^{(0)} \left( \frac{n^{(1)}}{n^{(0)}} - \frac{3}{2} \frac{T^{(1)}}{T^{(0)}} + \frac{u^{(1)}}{kT^{(0)}} \cdot mC + \frac{1}{kT^{(0)}} \frac{T^{(1)}}{T^{(0)}} \frac{1}{2} mC^2 \right). \]

Equations (137) and (138) differ from analogous equations [2], Chapter 7, (3.9) and (3.10) in the Enskog-Chapman theory only in use \( n^{(0)}, u^{(0)} \) and \( T^{(0)} \) instead of \( n, u \) and \( T \) (i.e. \( n^{[1]}, u^{[1]} \), and \( T^{[1]} \)). In first and second terms in the right-hand side of (138), in (141) and (142) functions \( n^{(0)}, u^{(0)} \) and \( T^{(0)} \), as upper in (44), can be, respectively, replaced by functions \( n^{[1]}, u^{[1]} \) and \( T^{[1]} \). Therefore to first infinitesimal order terms expression

\[ f^{(0)} \left[ - \frac{1}{n^{(0)}} \left( \frac{2kT^{(0)}}{m} \right) A \cdot \frac{\partial \ln T^{(0)}}{\partial r} - \frac{1}{n^{(0)}} B \cdot \frac{\partial}{\partial r} u^{(0)} \right] \]  

coincides with the solution \( \tilde{f}^{(1)} \) in the Enskog-Chapman theory

\[ \tilde{f}^{(1)} = \tilde{f}^{(0)} \left[ - \frac{1}{n^{[1]}} \left( \frac{2kT^{[1]}}{m} \right) \tilde{A} \cdot \frac{\partial \ln T^{[1]}}{\partial r} - \frac{1}{n^{[1]}} \tilde{B} \cdot \frac{\partial}{\partial r} u^{[1]} \right]. \]

Consequently, to first infinitesimal order terms the solution \( f^{[1]} = f^{(0)} + f^{(1)} \) coincides with the solution \( \tilde{f}^{[1]} = \tilde{f}^{(0)} + \tilde{f}^{(1)} \), received in the Enskog-Chapman theory:

\[ \tilde{f}^{[1]} \sim f^{[1]} \]
As a result, with the same exactness expressions for heat flux density vector (2, Chapter 7, § 4)

\[ \tilde{q}^{[1]} = \tilde{q}^{(1)} = q^{(1)} = \int \frac{1}{2} m (c - u^{[1]})^2 (c - u^{[1]}) f^{[1]} dc \]

\[ \tilde{q}^{[1]} \sim q^{[1]} = q^{[1]} = \int \frac{1}{2} m (c - u^{[1]})^2 (c - u^{[1]}) f^{[1]} dc \]

\[ \sim - \frac{2k^2 T^{(0)} \partial T^{(0)}}{3m} \int \mathbf{A} \cdot \mathbf{I} (\mathbf{A}) dc \]

\[ \sim - \frac{2k^2 T^{[1]} \partial T^{[1]}}{3m} \int \tilde{\mathbf{A}} \cdot \mathbf{I} (\tilde{\mathbf{A}}) dc \]

\[ = - \lambda \frac{\partial T^{[1]}}{\partial r} \]  

(156)

and pressure tensor

\[ \tilde{p}^{[1]} \sim p^{[1]} = \int m (c - u^{[1]}) (c - u^{[1]}) f^{[1]} dc \]

\[ \sim \left( n^{(0)} k T^{(0)} + n^{(1)} k T^{(0)} + n^{(0)} k T^{(1)} \right) U \]

\[ \sim - \frac{1}{5} k T^{(0)} \frac{\partial}{\partial r} u^{(0)} \int \mathbf{B} : \mathbf{I} (\mathbf{B}) dc \]

\[ \sim n^{[1]} k T^{[1]} U - \frac{1}{5} k T^{[1]} \frac{\partial}{\partial r} u^{[1]} \int \tilde{\mathbf{B}} : \mathbf{I} (\tilde{\mathbf{B}}) dc \]

\[ = n^{[1]} k T^{[1]} U - 2\mu \frac{\partial}{\partial r} u^{[1]}, \]  

(157)

coincide, cf. with the contrary assertion, for example, in [15], [12], [13]. In (157) notation is used: for arbitrary second-rank tensor \( w \)

\[ (\overline{w})_{\alpha\beta} = \frac{1}{2} (w_{\alpha\beta} + w_{\beta\alpha}) \]  

(158)

– corresponding symmetric tensor.

Having solutions \( f^{(0)} \) and \( f^{(1)} \), we can choose: one can solve separately gas-dynamic equations systems of the first and the second orders (127)-(129) and (49), \( r = 1 \), and separately find \( n^{(0)}, u^{(0)}, T^{(0)} \) and \( n^{(1)}, u^{(1)}, T^{(1)} \), or, substituting function \( f^{[1]} = f^{(0)} + f^{(1)} \) in the system of transfer equations (or summing systems of equations (127)-(129) and (49), \( r = 1 \)), immediately seek solutions \( n^{[1]}, u^{[1]}, T^{[1]} \) of the system (117) of, generally speaking, singularly perturbed differential equations.

Analogous results can be received for multi-component gas within the Enskog approach. These questions together with third order gas-dynamic equations system within the Enskog approach will be considered in a next article.
VIII. TURBULENCE AS MULTI-COMPONENT GAS DYNAMICS

As is well known, laminar flow becomes the turbulent flow, when some parameter characterizing the flow, namely, Reynolds number

\[ R = \frac{\rho uL}{\mu} > 1. \]  \hspace{1cm} (159)

In (159), \( \rho \) is the density of gas, \( u \) and \( L \) are some characteristic macroscopic velocity and linear size of the flow, \( \mu \) is the coefficient of viscosity. Having rewritten (159) as

\[ R = \frac{\rho u^2}{\mu L}, \]  \hspace{1cm} (160)

– cf. with the expression for viscosity tensor in (157), the Reynolds number can be treated as the ratio of the macroscopic momentum flux, proportional to \( f(0) \), to the viscosity-induced microscopic momentum flux, proportional to \( \tilde{f}^{(1)} \). Roughly speaking, viscosity ”aligning” the gas molecules according to a Maxwellian distribution at the same mean velocity and temperature can ”process” the microscopic momentum flux alone. However, if the macroscopic flux outperforms the microscopic, the gas flow, necessarily, comes to be stratified to components. The flow stratification to components can be also caused by external factors.

If

\[ R \sim \frac{f^{(0)}}{\tilde{f}^{(1)}} \rightarrow \infty, \]  \hspace{1cm} (161)

then that turbulent flow must be described by the gas-dynamic equations system, corresponding to the first approximation order in the approximate method for solution of kinetic Boltzmann equation, i.e. without viscosity and heat conductivity. But the gas-dynamic equations system of the first approximation order in the Enskog-Chapman theory can not describe turbulent flow with the entropy increase, see (132). Using the gas-dynamic equations system (118)-(120) resolves this paradox.

If gas-dynamic equations do not describe turbulent gas flows, then either something has been missed during the transition from the exact solution of the kinetic Boltzmann equation to its approximate solution (by the Enskog method) and then to the gas-dynamic equations, or the kinetic Boltzmann equation does not describe turbulent gas flows and requires replacement. However the last, i.e. necessity of replacement of the kinetic Boltzmann equation on another kinetic equation at transition from gas laminar flow to turbulent gas flow, seems ill-founded.
The gas dynamics of the components with the velocity distribution functions, close to the Maxwell functions of different mean velocities and temperatures, should be described by equations (118)-(120). From this point of view, the observed stochasticity of the turbulent flow is similar to the stochasticity of the Brownian motion. They differ in scale: in the Brownian motion that particle moves stochastically, whose mass is comparable to the mass of separate gas molecules, while in the turbulent flow that body moves stochastically, whose mass is comparable to the mass of separate gas components. In (119)-(120) the integral terms (proportional to $n_i$, $n_j$) can be huge, it explains unexpected power of turbulent effects.

[1] David Hilbert, *Grundzüge einer Allgemeinen Theorie der Linearen Integralgleichungen* (Teubner, Leipzig and Berlin, 1912) [in German].

[2] Sydney Chapman and T. G. Cowling, *The Mathematical Theory of Non-uniform Gases* (Cambridge University Press, Cambridge, 1952).

[3] V. V. Struminskii, *Prikladnaya Mathematica i Mechnica* 38, 203 (1974) [Applied Mathematics and Mechanics 38, 203 (1974)].

[4] K. P. Gurov, *Foundation of Kinetic Theory* (Nauka, Moscow, 1966) [in Russian].

[5] V. V. Struminskii and V. Ju. Velikodniy, Soviet Physics-Doklady 266, 28 (1982) [Sov. Phys. Dokl. 266, 28 (1982)].

[6] N. Bourbaki, *Functions of a Real Variable*, Elements of Mathematics, Book IV (Springer, Berlin, 2004).

[7] J. O. Hirschfelder, Ch. F. Curtiss, and R. B. Bird, *Molecular Theory of Gases and Liquids* (Wiley, New York, 1954).

[8] N. N. Bogolyubov, *Problems of the Dynamic Theory in Statistical Physics* (Gostechizdat, Moscow-Leningrad, 1946) [Russian Edition].

[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1 (Wiley, New York, 1989).

[10] Granino A. Korn and Theresa M. Korn, *Mathematical Handbook*, 2nd ed. (McGraw-Hill, New York, 1968).

[11] S. A. Lomov, *Introduction in the General Theory of Singular Pertubances* (Nauka, Moscow, 1965) [in Russian].

[12] Carlo Cercignani, *Theory and Application of the Boltzmann Equation* (Scottish Academic
[13] P. Résibois and M. De Leener, *Classical Kinetic Theory of Fluids* (Wiley, New York, 1977).

[14] V. N. Oraevskiy, J. V. Konikov, and G. V. Khazanov, *Transport Processes in Anisotropic Near-Earth Plasma* (Nauka, Moscow, 1985) [in Russian].

[15] V. V. Struminskii, Soviet Physics-Doklady 158, 70 (1964) [Sov. Phys. Dokl. 158, 70 (1964)].