SOLITON SOLUTIONS OF INTEGRABLE HIERARCHIES
AND COULOMB PLASMAS

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Abstract

Some direct relations between soliton solutions of integrable hierarchies and thermodynamical quantities of the Coulomb plasmas on the plane are revealed. We find that certain soliton solutions of the Kadomtsev-Petviashvili (KP) and B-type KP (BKP) hierarchies describe two-dimensional one or two component plasmas at special boundary conditions and fixed temperatures. It is shown that different reductions of integrable hierarchies describe one (two) component plasmas or dipole gases on one-dimensional submanifolds embedded in the two-dimensional space. We demonstrate application of the methods of soliton theory to statistical mechanics of such systems.

Keywords: Coulomb plasmas; Integrable hierarchies; Tau-functions; Solitons; Fermion systems; Dipole gases

1. Introduction

Recently we have shown [1] that the grand partition functions of some one-dimensional lattice gas models or equivalent to them partition functions of some Ising chains, coincide with the $N$-soliton tau-functions of various hierarchies of integrable nonlinear evolution equations. The present paper is the third one in the series and it comprises a detailed comparison of exactly solvable Coulomb plasma models on lattices with integrable equations. In extension of our previous considerations we discuss statistical mechanics of the Coulomb (logarithmic interaction)
gases on intrinsic two-dimensional geometric figures and various one-dimensional submanifolds of the plane. In this way we do not merely reinterpret previously known results [2]-[8], but also reveal a number of new exactly solvable models. It is natural to expect that the connection with integrable equations gives a clue to the classification of solvable (at fixed temperatures) plasma models.

A classical Coulomb plasma is a system of charged particles interacting through the Coulomb potential. In the Euclidean space $R^n$ the Coulomb potential is defined as a solution to the Poisson equation

$$\Delta V(r, r') = -\omega_n \delta(r - r'), \quad \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad r \in R^n$$

with $V(r, r')$ obeying certain boundary conditions. For the two-dimensional plane without boundaries, equation (1) has the following solution

$$V(z, z') = -\ln |z - z'|,$$

where $z = x + iy$ and $\omega_2 = 2\pi$. A system of particles forms a stable plasma if the opposite valued charges do not recombine with each other forming a gas of neutral molecules.

In general the location of plasma is constrained to some domain in which case there is a nontrivial interaction with the boundaries of this domain. In particular, the normal component of the electric field

$$\mathcal{E} = -\nabla V$$

should vanish on the surface of an ideal dielectric

$$\mathcal{E}_n = 0,$$

while the tangent component of this field vanishes on the surface of an ideal conductor

$$\mathcal{E}_t = 0.$$ 

A useful way of solving the Poisson equation with such boundary conditions is provided by the method of images. In the present work we consider systems where every charge has either a finite number of images created by the boundaries or boundary conditions create periodic lattices of images. In the case of a finite number of images the solution to (1) is a finite sum of logarithmic potentials (2) created by a charge and its images.

The energy of such systems of $N$ particles has the following form

$$E_N = \sum_{1 \leq i < j \leq N} Q_i Q_j V(z_i, z_j) + \sum_{1 \leq i \leq N} Q_i^2 v(z_i) + \sum_{1 \leq i \leq N} Q_i \Phi(z_i)$$
where $z_i = x_i + 1 \cdot y_i$ and $Q_i$ denote the coordinate and the charge of the $i$-th particle on the plane.

The first term in (6) is the energy of interaction between different charges. The second term is the sum of self-energies. This term is constant at certain boundary conditions, e.g. when the plasma is homogeneous or if it is restricted to a line with transverse boundary conditions. In general it arises as the energy of interaction between the charge and its own images. The third term describes an interaction of charges with external fields.

The grand partition function of a system of particles of $s$ different species is

$$G = \sum_{n_1=1}^{N_1} \cdots \sum_{n_s=1}^{N_s} \frac{\zeta_1^{n_1} \cdots \zeta_s^{n_s}}{n_1! \cdots n_s!} Z_{n_1 \ldots n_s}$$

$$= \sum_{n_1=1}^{N_1} \cdots \sum_{n_s=1}^{N_s} \frac{1}{n_1! \cdots n_s!} \int e^{-\Gamma H_{n_1 \ldots n_s} + \mu_1 n_1 + \ldots + \mu_s n_s} d\Omega, \quad (7)$$

where $\mu_1, \ldots, \mu_s$ denote chemical potentials ($\zeta_s = e^{\mu_s}$ are the fugacities) and $\Omega$ is the integration measure over the configuration space occupied by particles. In (7) we have introduced the dimensionless inverse temperature

$$\Gamma = \beta Q^2, \quad \beta = 1/kT \quad (8)$$

and the dimensionless Hamiltonian

$$H_{n_1 \ldots n_s} = \frac{1}{2} \sum_{i \neq j} q_i q_j V(z_i, z_j) + \sum_i q_i^2 v(z_i) + \sum_i q_i \phi(z_i), \quad (9)$$

where $q_i = Q_i/Q$ and $\phi(z) = \Phi(z)/Q$. For one and two component plasmas it is convenient to choose $Q = |Q_i|$, so that the dimensionless charges $q_i = 1$ in the one component case and $q_i = \pm 1$ in the two component case.

The two-dimensional one and two component plasma models have been solved for a variety of boundary conditions at the special value of the inverse temperature $\Gamma = 2$ (see, e.g. [2]-[8] and references therein). In the case of two component plasma an exact solution exists due to the well known correspondence between this system and the free fermion point of the Thirring/Sine-Gordon model. In technical terms the solution is possible due to different determinant representations (the Cauchy determinant for the two component plasma and the Vandermonde determinant for the one component case). Such determinant representations allow one to solve models of log-gases on a line with the transverse boundary conditions (e.g., see [8]). Note that the main part of the cited works deal with the continuous space models.
for an account of the lattice models, see, e.g. [1, 2, 3]. The literature on the Coulomb gases is enormous, many statistical mechanics models have been related to them [4], there is a relation to conformal field theories, etc. Still, the identification of Coulomb plasmas on lattices with some boundaries and of the famous multi-soliton systems has been missed in the previous investigations.

2. Basic observations

Let us consider a lattice version of (7). We suppose that each type of particles can occupy only a discrete set of points in the complex plane. E.g., in the two component case the positive and negative charge particles occupy sublattices \( \{z_+\} \) and \( \{z_-\} \). We denote the union of all sublattices as \( \{z\} \). No more than one particle is allowed at each site. In this case, the integrals in (7) are replaced by discrete sums over the lattice points and the whole partition function can be rewritten in the following form [10]

\[
G = \sum_{\{\sigma\}} \exp \left( \frac{1}{2} \sum_{z \neq z'} W(z, z') \sigma(z) \sigma(z') + \sum_{\{z\}} w(z) \sigma(z) \right),
\]

(10)

where

\[
W(z, z') = -\Gamma q(z)q(z')V(z, z'), \quad w(z) = \mu(z) - \Gamma \left( q^2(z)v(z) + q(z)\phi(z) \right)
\]

(11)

and \( \sigma(z) = 0 \) or \( 1 \) is an occupation number of the site with the coordinate \( z \). The variables \( q(z), \mu(z) \) are some functions of the lattice coordinates. For example, \( q(z+) = \pm 1, \mu(z+) = \mu_+ \) for the two component plasma.

Now, let us write out the \( \tau \)-function of \( N \)-soliton solutions of some integrable hierarchy in the Hirota form [11]

\[
\tau_N = \sum_{\sigma=0,1} \exp \left( \frac{1}{2} \sum_{z \neq z'} A_{zz'} \sigma(z) \sigma(z') + \sum_{\{z\}} \theta(z) \sigma(z) \right),
\]

(12)

where the variable \( z \) takes \( N \) discrete values describing spectral characteristics of solitons. The function \( \theta(z) \) parameterizes phases of solitons with the index \( z \) and \( A_{zz'} \) is the phase shift acquired as a result of the scattering of solitons with the indices \( z \) and \( z' \) on each other.

Evidently, the expressions (10) and (12) have the same form. One just needs to make proper identifications between the phase shifts \( A_{zz'} \) and the interaction potentials \( W(z, z') \), and between the phases \( \theta(z) \) and the function \( w(z) \).
Concluding this section we present brief explanations on the structure of the expression (12). One can track appearance of the \( \tau \)-function on the simplest example of the Korteweg-de Vries (KdV) equation which emerges as a compatibility condition of two linear problems:

\[
L\psi(x) \equiv -\psi_{xx}(x) + u(x)\psi(x) = \lambda\psi(x), \quad (13)
\]

\[
\psi_t(x,t) = B\psi(x,t), \quad B \equiv -4\partial_x^3 + 6u(x,t)\partial_x + 3u_x(x,t). \quad (14)
\]

Eigenvalues of the Schrödinger operator (13) do not depend on “time” \( t \): \( \partial\lambda/\partial t = 0 \). As a result, the compatibility condition of (13) and (14) takes the following operator form:

\[
L_t = [B,L],
\]

and yields the KdV equation

\[
u_t + uu_{xx} - 6uu_x = 0,
\]

describing the motion of shallow water. Stable traveling wave solutions of this equation are called solitons. The inverse scattering method allowed one to find an explicit expression for the general \( N \)-soliton solution

\[
u(x,t) = -2\partial_x^2 \ln \tau_N(x,t), \quad (15)
\]

where \( \tau_N \) is the determinant of some \( N \times N \) matrix called the tau-function. It can be written in the form

\[
\tau_N = \sum_{\mu_i=0,1} \exp \left( \sum_{1 \leq i < j \leq N} A_{ij}\mu_i\mu_j + \sum_{1 \leq i \leq N} \theta_i\mu_i \right),
\]

(16)

\[
\theta_i = k_i x - k_i^3 t + \theta_i^{(0)}, \quad i, j = 1, 2, \ldots, N.
\]

(17)

The variable \( k_i \) is related to the \( i \)-th eigenvalue of the operator \( L \) in the standard quantum mechanical meaning, \( \lambda_i = -k_i^2/4 \), and it parameterizes the amplitude of the \( i \)-th soliton which is proportional to \( k_i^2 \). The scattering process is described by the time evolution from \( t \to -\infty \) to \( t \to +\infty \). In this picture \( \theta_i^{(0)}/k_i \) are the zero time phases of solitons and \( k_i^2 \) are equal to their velocities.

The phase shifts \( A_{ij} \) describe relative retardation of \( i \)-th soliton after the interaction with the \( j \)-th one. It is determined by the formula

\[
e^{A_{ij}} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2},
\]

(18)

In the standard approach \( k_i \) are real. However, formally one can take them as complex variables as well. Then it is seen from (14) that the plane coordinates \( z_i \) of some particular configuration of the Coulomb particles (see below) play the role of \( k_i \) and correspond thus to complex amplitudes and velocities of solitons. This leads to complexification of the function
$u(x,t)$ which requires somewhat different physical interpretation of the KdV equation. However, for complex $k_i$ there is no complete coincidence of (10) and (12) — the signs of modules in the Coulomb potential make $G$ (10) real and there are no such signs in (18) and, so, $\tau_N$ is complex. On the one hand, one may place the plasma onto the $x$ or $y$ axis in order to reach the coincidence and this one-dimensional situation was considered in [1]. On the other hand, one may try to identify KdV solitons with complex $k_i$ with a special system of interacting electric and magnetic charges [9]. We shall not consider the latter situation here, but rather concentrate upon the purely electric systems on the plane with real energy of interaction.

There are generalizations of the expressions (17)-(18) such that the corresponding $u(x,t,\ldots)$ satisfy higher order members of the KdV-hierarchy, sin-Gordon, Kadomtsev–Petviashvili (KP), Toda, and some other integrable equations [11]. Note that all these hierarchies can be derived within the free fermion formalism which is outlined in the concluding section of the present article. In the next section we consider in detail an identification of the KP hierarchy solitons and plasma particles.

3. KP hierarchy

The KP hierarchy which starts from the KP equation

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right)$$

is obtained through a generalization of the KdV formalism. The KdV equation is an equation which describes the eigenvalue preserving deformations of a second order differential operator, while the KP equation describes that of a first order pseudo-differential operator:

$$L(t, \partial) \psi(t, k) = k \psi(t, k),$$

$$\frac{\partial}{\partial t_n} \psi(t, k) = B_n(t, \partial) \psi(t, k),$$

where $t = (t_1 = x, t_2 = y, t_3 = t, \ldots), \partial = \frac{\partial}{\partial t_1}$, and

$$L(t, \partial) = \partial + u_1(t) \partial^{-1} + u_3(t) \partial^{-2} + \ldots,$$

$$B_n(t, \partial) = \partial^n + \sum_{j=0}^{n-1} b_{nj}(t) \partial^j.$$  \hspace{1cm} (19)

The KP hierarchy is a system of nonlinear differential equations for $u_j(t)$ and $b_{nj}(t)$, resulting from the Zakharov-Shabat type compatibility conditions of linear equations given above

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad \frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} = [B_n, B_m].$$
As in the KdV case, the potential \( u = u_1 \) is expressed in terms of the tau function \((15)\) and \(N\)-soliton solution is presentable in the Hirota form \((12)\) with the following parameterization of the soliton phases and phase shifts \[(12)\]:

\[
A_{zz'} = \ln \left( \frac{(a_z - a_{z'}) (b_{z} - b_{z'})}{(a_z + b_{z'}) (b_{z} + a_{z'})} \right),
\]

\[
\theta(z) = \theta^{(0)}(z) + \sum_{p=1}^{\infty} (a_z^p - (-b_z)^p) t_p,
\]

where \( t_p \) is the \( p \)-th KP “time” and \( a_z, b_z \) are some arbitrary functions of \( z \).

If we take

\[
a_z = z = x + iy, \quad b_z = -z^* = -x + iy, \quad y \geq 0
\]

then

\[
A_{zz'} = W(z, z') = -2V(z, z'),
\]

where

\[
V(z, z') = -\ln |z - z'| + \ln |z^* - z'|
\]

is the potential at the point \( z \) created by a positive unit charge particle placed at the point \( z' \) over the conducting surface occupying the \( y \leq 0 \) region (since \( V(z, z') \) solves the Poisson equation with the tangent boundary condition \((4)\) at \( y = 0 \)). Using the method of images one may say that this is an effective potential created by a positive charge at the point \( z' \), \( \Im z' > 0 \), and its image of opposite charge located at \( (z')^* \). Comparing \((21)\) with the original definition \((11)\) one finds that the temperature \((8)\) is fixed and equals to

\[
\Gamma = 2.
\]

Thus the reduction of KP \(N\)-soliton solution described above corresponds to the Coulomb plasma in the upper half plane \( \Im z > 0 \) with metallic boundary along the \( x \)-axis. This situation is depicted in the Fig. 1.

Note that the choice \( a_z = z, b_z = z^* \) gives a similar situation — a plasma in the right half plane with the metallic boundary along the \( y \) axis (evidently this configuration is reached by the rotation of the coordinates \( z \to iz \)).

Let us shift \( z \to z + ia \), \( a \) real and take the limit \( a \to \infty \), i.e. take the plasma far away from the boundary. This leads to some divergences in the energy which can be removed by addition of an appropriate diverging constant to the initial Hamiltonian. As a result one gets the pure plasma system at the inverse temperature \( \Gamma = 2 \). This temperature corresponds to the
Figure 1: KP equation: one component two-dimensional plasma above an ideal conductor. Positive charges are shown as white squares while their negative images are shown as black squares. Interactions between different charges are shown by dashed lines, while the interactions between charges and their own images are shown by the solid lines.

The identification \( w(z) = \theta(z) \) allows one to write the following expression for the zero-time phases \( \theta^{(0)}(z) \) in (20):

\[
\theta^{(0)}(z) = \mu - \Gamma (\ln |z^* - z| + \phi(z)), \quad \Gamma = 2,
\]

(23)

where the second term corresponds to the "charge-image" self-interaction energy, and the last term describes the potential of the field created by the neutralizing background of the density \( \rho(z) \):

\[
\Delta \phi(z) = -2\pi \rho(z), \quad \frac{\partial \phi}{\partial x}\bigg|_{y=0} = 0.
\]

(24)

Contributions from the KP “times”

\[
\sum_{p=0}^{\infty} (z^p - (z^*)^p) t_p = -\Gamma \phi_{ext}(z)
\]

(25)

correspond to an external electric field. One can see that (25) is a sum of the harmonic polynomials, \( p \)-th polynomial being a fundamental antisymmetric polynomial of the group of reflections (the Coxeter group) of \( 2p \)-gon. Since the Laplacian of this part is zero the

standard normalization in the random matrix theory \([3]\). Note that the normalization of the temperature in our previous papers \([1]\) was chosen as \( \Gamma = 1 \) since there we were discussing Ising chains without detailed comparison with the Coulomb systems which is a goal of the present work.

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corresponding density of charges is zero, i.e. this field is generated by external distant charges. For instance, the contribution of the first time \((z - z^*)t_1\) corresponds to the homogeneous electric field perpendicular to the boundary. For reality of the potential the time \(t_1\) has to be purely imaginary.

The system of distant charges \(g_i\) located at the points \(w_i\) above the conductor create the following electrostatic potential at the point \(z\)

\[
\phi_{ext}(z) = -\sum_i g_i \ln \frac{|z - w_i|}{|z - w_i^*|}.
\]

Since the external charges are far from the origin \(|z| \ll |w_i|\), we can expand the potential \(\phi_{ext}(z)\) in the Taylor series and get as KP times

\[
t_p \equiv \frac{\Gamma}{2p} \sum_i g_i \left( \frac{1}{(w_i^*)^p} - \frac{1}{w_i^p} \right).
\]

In this picture the KP times take imaginary values automatically.

One may conclude that the general imaginary times evolution of a special system of KP hierarchy solitons describes electrostatics of a plasma in a varying external electric field.

It is worth of mentioning that sometimes the continuous limit is not physical below some temperature. Indeed, let us consider the partition function of \(N\) particle gas at the inverse temperature \(\Gamma\) in a square domain \((0 \leq x = \Re z \leq L) \times (0 \leq y = \Im z \leq L)\)

\[
Z_N = \int_0^L dx_1 \int_0^L dy_1 \cdots \int_0^L dx_N \int_0^L dy_N \exp \left( \Gamma \sum_{i<j} \ln \frac{|z_i - z_j|}{|z_i - z_j^*|} - \Gamma \sum_i \ln |z_i - z_i^*| \right).
\]

Introducing new dimensionless variables \(z_i/L\) one gets the following expression for the thermodynamic potential or the pressure \(P\) in appropriate units

\[
P = \frac{\ln Z_N}{N \Gamma} = \frac{(2 - \Gamma) \ln L}{\Gamma} + \text{terms independent on } L.
\]

We see that at the inverse temperature \(\Gamma = 2\) the average pressure changes the sign. The system is inhomogeneous and the local pressure is different from the average one. It is clear that the local pressure changes the sign at the boundary and at the inverse temperatures \(\Gamma \geq 2\) the particles stick to the surface of the conductor.

As a result, the continuous space version of the model in a domain with the conducting boundary does not make physical sense if the temperature is low enough. However, the situation is different if plasma is confined to a domain which does not touch the boundary (as in the situation depicted in the Fig. 1). E.g., one can avoid such an unphysical behavior in the
discrete (lattice) version of the model. In the latter case the lattice spacing plays the role of the radius of a hard core repulsive interaction which prevents collapse of the system. Note that the identification with the KP soliton systems takes place exactly at the critical temperature $\Gamma = 2$.

Closing this section let us note that the equation (1) and the boundary conditions (4), (5) hold for an appropriate conformal change of the variable $z \rightarrow f(z)$. For instance, choosing soliton parameters in (20) as follows $a_z = z^2, b_z = -(z^*)^2$, we obtain

$$W(z, z') = 2 \ln \left| \frac{z^2 - (z')^2}{(z^*)^2 - (z')^2} \right|$$

corresponding to the interaction of two charged particles in the rectangular corner with metallic walls along the $x$ and $y$ axes. Higher degree monomial maps $z \rightarrow z^n$ put the plasma into the corner with the $\pi/n$ angle between the conducting walls.

The exponential map

$$a_z = \exp \left( \frac{\pi z}{L} \right), \quad b_z = -\exp \left( \frac{\pi z^*}{L} \right),$$

(26)
generates the $W$-potential

$$W(z, z') = 2 \ln \left| \frac{\sinh \frac{\pi}{2L} (z - z')}{\sinh \frac{\pi}{2L} (z^* - z')} \right|,$$

(27)

which describes the plasma in the strip $\Im z = (0, L)$ between two parallel conductors.

The choice

$$a_z = \exp \left( -\frac{\pi x}{L} \right), \quad b_z = -\exp \left( -\frac{\pi (x + \alpha)}{L} \right),$$

results in

$$W(x, x') = \ln \frac{\sinh^2 \frac{\pi (x - x')}{2L}}{\sinh \frac{\pi}{2L} (x - x' - \alpha) \sinh \frac{\pi}{2L} (x - x' + \alpha)},$$

(28)

Solution of the Poisson equation with periodic boundary condition along the $y$-axis with the period $2L$ is given by the potential

$$V(z, z') = -\ln \left| \sinh \frac{\pi}{2L} (z - z') \right|.$$ 

Therefore one can interpret (28) as the interaction energy of two neutral dipoles in the periodic background with the distance between charges in the molecule equal to $\alpha$ (the internal energy of dipoles is neglected). These dipoles all lie on the $x$-axis and have identical orientation. Since $W = -\Gamma V$, we see that the effective inverse temperature is twice smaller than in the previous cases, i.e. $\Gamma = 1$. 

10
Let us choose parameters in (20) as specific double periodic functions

\[ a_z = \text{sn}^2 z, \quad b_z = -\text{sn}^2 z^*, \]

where \( \text{sn} z \) is the Jacobian elliptic function with the periods \( L_x, L_y \). Then one gets the \( W \)-potential

\[
W(z, z') = 2 \ln \left| \frac{\text{sn}^2 z - \text{sn}^2 z'}{\text{sn}^2 z^* - \text{sn}^2 z'} \right| = 2 \ln \left| \frac{\theta_1(z - z')\theta_1(z + z')}{\theta_1(z^* - z')\theta_1(z^* + z')} \right|,
\]

where \( \theta_1(z) \) is the Jacobi theta-function

\[
\theta_1(z) = 2 \sum_{n=0}^{\infty} (-1)^n \exp \left( -\pi(n + 1/2)^2 \frac{L_y}{L_x} \right) \sin \frac{\pi(2n + 1)z}{L_x}.
\]

It vanishes when \( z \) lies on the boundary of a \( L_x \times L_y \) rectangle, i.e. we have the plasma in a box with the conducting walls (\( \Gamma = 2 \)). For small \( z, z' \) (or for the large size box \( L_x, L_y \to \infty \)) one recovers plasma in the rectangular corner.

4. KP hierarchy: the two component plasma case

Let us consider the situation when lattice points \( \{z\} \) consist of two subsets \( \{z_\pm\}, \{z\} = \{z_-\} \cup \{z_+\} \). Choosing the following identification of parameters in (20)

\[
a_{z_-} = z_-, \quad b_{z_-} = -z_-^*, \quad a_{z_+} = z_+^*, \quad b_{z_+} = -z_+,
\]

we get a model of the two component plasma (the plasma consisting of two species of opposite charges) above an ideal conductor at the inverse temperature \( \Gamma = 2 \):

\[
W(z_\pm, z'_\pm) = 2 \ln |z_\pm - z'_\pm| - 2 \ln |z_\pm^* - z'_\pm|,
\]

\[
W(z_\pm, z'_\mp) = 2 \ln |z_\pm^* - z'_\mp| - 2 \ln |z_\pm - z'_\mp|,
\]

\[
\theta(z_\pm) = \mu_\pm - 2 \ln |z_\pm - z_\pm^*| \mp 2\phi(z_\pm).
\]

After the conformal transformations \( z \to z^2, e^z \), etc one gets the two-component plasma in the metallic rectangular corner, a strip, etc.

Note that the bulk properties of the two component plasma are different from the ones of the one component systems. Actually, statistical mechanics of the one component plasma is not stable without metallic boundary or neutralizing background, since its particles tend toward boundaries repelling each other with long range (logarithmic) forces. In the two component case
screening is possible if the system is neutral and the temperature is high enough. Simple scaling
analysis shows that the pure two component plasma undergoes a transition in the bulk at $\Gamma = 2$
\cite{2}. At this temperature an association of opposite charges into molecules (neutralization of
plasma) takes place. Again, there is no such a transition in the lattice case because of the
hard-core repulsion at the lattice spacing distance.

It is well known that the two component homogeneous plasma without boundaries is a
specific representation of the quantum Sine-Gordon or Thirring model \cite{13}. The inverse tem-
perature (22) corresponds to the free fermion point, but the model is (in principle) integrable
for any $\Gamma$. The Sine-Gordon model requires renormalization if the coupling constant exceeds a
critical value. The lattice spacing (in the case of the lattice plasma) is equivalent to the intro-
duction of a cutoff in the corresponding field theory. Thus it is natural to associate the case
of two component lattice plasma above the metallic boundary with some discretized boundary
Sine-Gordon or Thirring model.

5. KP hierarchy: some reductions

In this section we consider a number of one-dimensional reductions of the KP hierarchy and
some self-similar soliton solutions. We describe here only the most popular integrable systems
and do not cover all possible cases and their Coulomb gas interpretations.

1) We begin with the reduction considered earlier in the literature \cite{2}. In this case, plasma
is restricted to the line $y = Y$. The $W$-potential is translationally invariant and equals to

$$W(x, x') = \ln \frac{(x - x')^2}{Y^2 + (x - x')^2}.$$  

Self-interaction energies of charges with their own images are constant and may be neglected.

2) Reduction to the KdV hierarchy. In this case particles move along the vertical line $x = 0$
and the $W$-potential is

$$W(y, y') = \ln \frac{(y - y')^2}{(y + y')^2}.$$  \hspace{1cm}(31)

Now a non-trivial self-interaction term $\propto \ln |2y|$ enters the definition of soliton phases.

3) Discrete KdV hierarchy. The phase shifts have the form \cite{14}

$$A_{z z'} = \ln \left( \frac{\sinh \frac{1}{2}(z - z')}{\sinh \frac{1}{2}(z + z')} \right)^2.$$  

Taking $z$ to be purely imaginary $z = i\alpha$ we get the following result

$$W(\alpha, \alpha') = \ln \left( \frac{\sin \frac{1}{2}(\alpha - \alpha')}{\sin \frac{1}{2}(\alpha + \alpha')} \right)^2,$$
which corresponds to the plasma restricted to an arc with the center of the corresponding circle at the conductor’s surface. The self-interaction energy is $-\ln |2 \sin \alpha|$. It is sufficient to put $z = e^{i\alpha}$ in (21) in order to get this system from the KP soliton solutions.

4) The reductions admitting both left and right moving solitons correspond to the plasma restricted to domains of disjoint parts. E.g., the Toda lattice case [11]

$$A_{zz'} = \ln \left( \frac{\epsilon(z)z - \epsilon(z')z'}{1 - \epsilon(z)\epsilon(z')zz'} \right)^2, \quad \epsilon(z) = \pm 1$$

corresponds for $z = e^{i\alpha}$ to the two component plasma on the half circle with the center lying upon the conductor surface. The positive charge particles occupy the subsector $\alpha \in [0, \pi/2]$ and the negative charge particles are located in the subsector $\alpha \in [\pi/2, \pi]$.

5) The Boussinesq equation [13] corresponds to a one component plasma on disjoint halves of the hyperbola $3x^2 = y^2 + 1$ situated above the conductor occupying the $y \leq 0$ half plane:

$$A_{yy'} = \ln \left( \frac{(\epsilon(y)x(y) - \epsilon(y')x(y'))^2 + (y - y')^2}{(\epsilon(y)x(y) - \epsilon(y')x(y'))^2 + (y + y')^2} \right), \quad x(y) = \sqrt{\frac{y^2 + 1}{3}}, \quad \epsilon(y) = \pm 1.$$

It would be interesting to find the equation whose soliton phase shifts are given by the elliptic functions (29).

Consider some specific lattice plasma configurations associated with self-similar soliton solutions of integrable equations.

Let us take the KP hierarchy with parameters (26) and restrict the corresponding plasma to the line parallel to the conductor surfaces

$$\Im z = Y.$$

Applying this constraint, we get from (27)

$$W(x - x') = -\ln \left( \sin^2 \frac{\pi Y}{L} \coth^2 \frac{\pi (x - x')}{2L} + \cos^2 \frac{\pi Y}{L} \right).$$

The simplest self-similar reduction corresponds to the homogeneous one-dimensional lattice parallel to the $x$-axis. Then the $n$-th charge has the coordinates $z_n = X + hn + iY$, where $X$ is some fixed constant and $h$ is the lattice spacing. In this case soliton momenta $a_z$ and $b_z$ form one geometric progression (26). The simplest KdV self-similar reduction corresponds to the case when the line is located at equal distances between parallel conductors $Y = L/2$:

$$W(x - x') = \ln \tanh \frac{\pi (x - x')}{2L}.$$
\(M\)-periodic reduction corresponds to the situation when soliton momenta are composed from \(M\) distinct geometric progressions. In this case the lattice \(\{z\}\) consists of \(M\) homogeneous sublattices. In the plasma language it corresponds to the model where plasma moves on \(M\) distinct parallel lines between two conductors with the coordinates \(\{z\} = \{iY_p + X_p + nh, p = 1, \ldots, M, n = 0, 1, \ldots\}\). If one sets \(Y_p = L/2\) then all these sublattices are situated upon the middle line and this case corresponds to the general self-similar KdV soliton potentials of [16].

In a similar way one can place plasma upon \(M\) parallel lines with fixed \(x\)-coordinates, \(x = X_p\), which are situated above the \(y \leq 0\) conductor. In this case self-similar lattices are described by restriction of \(y\) to the union of \(M\) geometric progressions \(\{z\} = \{X_p + iY_p q^n, q < 1\}\). However, such configurations are not safe from the collapse of particles on the walls. The KP self-similar systems are richer than the ones of the KdV or BKP (to be considered below) cases due to the presence of non-trivial translational parts \(X_p\) (or \(Y_p\)) in the parameterization of the corresponding soliton spectral data \(a_z, b_z\).

6. Correspondence with Ising lattices

It is well known that lattice gas models are related to the Ising models [10]. Indeed, substituting

\[\sigma(z) = (s(z) + 1)/2, \quad s(z) = \pm 1,\]

into (10), we get a formula for the partition function of an Ising model up to some constant multiplicative factor

\[G = \sum_{\{s\}} \exp \left( \frac{1}{2} \sum_{z \neq z'} J(z, z') s(z)s(z') + \sum_{\{z\}} H(z)s(z) \right),\]  

(33)

where

\[J(z, z') = \frac{1}{4} W(z, z')\]

are the exchange constants and

\[H(z) = \frac{1}{2} w(z) + \frac{1}{4} \sum_{z', z' \neq z} W(z, z')\]  

(34)

denotes the external magnetic field. Here we have absorbed the temperature variable \(\Gamma\) into the definition of \(J(z, z')\) and \(H(z)\).

Note that the Ising models emerging in this way look natural only in the reduced one-dimensional case, when there are no constraints upon the values of spins at different points.
In the two-dimensional picture with boundaries one has to assume that the configuration of spins satisfies some geometric constraints which do not have natural meaning similar to the one existing in the electrostatics. This leads to the exchange which depends not only on the distance between the spins but on the distance to the boundaries as well.

Technically, it appears that the Ising representation of the lattice plasma grand partition function is more convenient for writing it in the determinant form.

7. BKP hierarchy

Consider the BKP equation:
\[
\frac{\partial}{\partial t_1} \left( 9 \frac{\partial u}{\partial t_5} - 5 \frac{\partial^3 u}{\partial t_3 \partial t_1^2} + \frac{\partial^5 u}{\partial t_5^2} - 30 \frac{\partial u}{\partial t_3 \partial t_1} + 30 \frac{\partial^3 u}{\partial t_1^2} + 60 \left( \frac{\partial u}{\partial t_1} \right)^3 \right) - 5 \frac{\partial^2 u}{\partial t_3^2} = 0.
\]

This equation involves three independent variables \( t_1, t_3, t_5 \). It is the first equation of the B-type KP (BKP) hierarchy which is a reduction of the general KP hierarchy \([12, 17]\). This reduction is achieved by setting \( b_{n0} = 0 \) in (19), which assumes, in turn, that even “times” evolution is absent, \( t_{2p} = 0 \). The BKP tau-function is defined via the relation
\[
u(t_1, t_3, \ldots) = \frac{\partial}{\partial t_1} \ln \tau_{BKP}.
\]

\( N \)-soliton solution of the BKP hierarchy has the same form (12) with the following phase shifts and soliton phases \([12]\):
\[
A_{zz'} = \ln \left( \frac{(a_z - a_{z'}) (b_z - b_{z'}) (a_z - b_{z'}) (b_z - a_{z'})}{(a_z + a_{z'}) (b_z + b_{z'}) (a_z + b_{z'}) (b_z + a_{z'})} \right),
\]
\[
\theta(z) = \theta^{(0)}(z) + \sum_{p=1}^{\infty} (a_z^{2p-1} + b_z^{2p-1}) t_{2p-1},
\]
where \( t_{2p-1} \) are the BKP “times”.

If we take one component plasma and fix
\[
a_z = z = x + iy, \quad b_z = z^* = x - iy, \quad x > 0, \quad y \geq 0,
\]
then \( A_{zz'} = W(z, z') = -2V(z, z') \), where
\[
V(z, z') = -\ln |z - z'| - \ln |z^* - z'| + \ln |z + z'| + \ln |z^* + z'|
\]
is the total interaction energy between unit charge particles at the points \( z, \Im z > 0, \Re z > 0 \), and \( z', \Im z' > 0, \Re z' > 0 \), in a domain of the upper right quarter of the plane with an ideal dielectric boundary (cf. (4)) along the \( x \)-axis and an ideal conductor wall along the \( y \)-axis (cf.
Figure 2: BKP equation: a one component two-dimensional plasma in the corner between an ideal dielectric (the horizontal axis) and an ideal conductor (the vertical axis). Positive charges are shown as white squares while their negative charge images are shown as the black squares. Interactions between different charges are shown by dashed lines, while the self-interactions between charges and their own images are shown by the solid lines.
This situation is depicted in the Fig. 2. According to (3) the inverse temperature is fixed and equals to $\Gamma = 2$.

If the plasma is taken far away from the corner by appropriate translations, then one gets the pure plasma system at the effective temperature $\Gamma = 2$. Sliding along the $y$-axis to infinity one comes to the previously considered plasma associated with the KP equation. Sliding along the $x$-axis requires a renormalization of the zero energy level after which one gets a plasma above the surface of a dielectric.

If we place charges upon the $y = 0$ axis, then the dielectric boundary condition disappears and we have

$$W(x, x') = \ln \left( \frac{x - x'}{x + x'} \right)^4,$$

which corresponds to the plasma model induced by the KdV equation at the inverse temperature $\Gamma = 4$, which is twice higher than in the appropriate KP reduction case.

Identification of the initial phase $\theta^{(0)}(z)$ in (20) is as follows

$$\theta^{(0)}(z) = 2 \ln |z^* - z| - 2 \ln |z^* + z| - 2 \ln |2z| + \mu - 2\phi(z),$$

where the meaning of all terms is similar to that of (23), (24). Contribution of the BKP “times”

$$\sum_{p=1}^{\infty} (z^{2p-1} + (z^*)^{2p-1}) t_{2p-1}$$

(36)
corresponds to electric field created by external charges. The potential (36) satisfies all the boundary conditions (4), (5). The $p$-th polynomial in the sum (36) is the fundamental antisymmetric polynomial of the reflection group of the $2(2p - 1)$-gon. Again, the evolution of solitons under the BKP flow corresponds to the evolution of plasma under a motion of distant external charges.

In complete parallel with the KP case one can consider conformal transformations $z \rightarrow z^n, e^z$, etc and map plasma to various geometric configurations. Considering systems with two sublattices like (30) one arrives at the model of two component plasma in the metal-dielectric corner or other bounded regions. This leads again to a boundary Sine-Gordon model at the free fermion point corresponding to the temperature $\Gamma = 2$. 

17
8. BKP hierarchy: a dipole gas on a line between two ideal conductors

In this section we consider real self-similar reductions of the BKP hierarchy corresponding to dipole gases on a line between two conductors.

Let us write out once more the $N$-soliton $\tau$-function of the BKP hierarchy in the slightly different notations

$$
\tau_N = \sum_{\sigma=0,1} \exp \left( \sum_{1 \leq i < j \leq N} A_{ij} \sigma_i \sigma_j + \sum_{i=1}^N \theta_i \sigma_i \right),
$$

$$
e^{A_{ij}} = \frac{(a_i - a_j)(b_i - b_j)(a_i - b_j)(b_i - a_j)}{(a_i + a_j)(b_i + b_j)(a_i + b_j)(b_i + a_j)},
$$

where $i, j$ are the numbers of solitons (they are equivalent in meaning to the coordinate $z$ of plasma particles). Choosing

$$
a_i = \exp(-\pi hi/L), \quad b_i = -\exp(-\pi (hi + \alpha)/L), \quad i = 1, \ldots, N,
$$

we get an expression for the grand partition function of the form (10) for a homogeneous lattice gas. The particles of the gas interact via the following $W$-potential

$$
W_d(i - j) = W(h(i - j)) - \frac{1}{2} W(h(i - j) - \alpha) - \frac{1}{2} W(h(i - j) + \alpha)
$$

with $W$ given by (32). It is seen immediately that this is the potential of two dipole molecules consisting of two opposite charges situated at the distance $\alpha$ from each other without the part describing interaction of charges inside the dipoles (which is constant). The distance between molecules at $i$-th and $j$-th site is $h(i - j)$, where $h$ denotes the lattice spacing. The dipoles move along the line situated at equal distances $1/2L$ from two parallel conductors (see the Fig. 3). All dipoles are oriented in one direction. Due to the dipole gas interpretation one has the effective temperature $\Gamma = 1$ — this is a demonstration of some discrete temperature renormalization effect.

If one takes $a_i$ as in (37) but changes the sign of $b_i$ (i.e. shifts $\alpha \to \alpha + 1L$), then the signs of the second and third terms in (38) are changed too. This situation corresponds to a dipole gas on the middle line, the dipoles being charged molecules of total charge +2 with the same distance between charges $\alpha$. These molecules are positioned similar to the previous case.

Another lattice gas model of charged dipoles appears if one replaces $\alpha$ in (38) by $\alpha + 1L$, $0 < \alpha < L$. In this case charged dipoles are positioned vertically and symmetrically with respect to the middle line $y = L/2$. The case when $\alpha$ in (38) is replaced by $\alpha L$, $0 < \alpha < L$, corresponds to
the neutral dipoles gas in the strip between dielectric walls. The dipoles are perpendicular to the middle line \( y = L/2 \), similar to the previous case, and have an identical orientation.

The \( M \)-periodic self-similar reductions \([1, 16]\), when \( a_{i+M} = qa_i, b_{i+M} = qb_i \), describe the gas consisting of \( M \) different types of dipoles. In some particular cases this leads to dipole gases with different orientations of neutral or charged molecules in a strip between the conducting walls.

For instance, for the choice

\[
a_{2i} = e^{-\frac{\pi}{L}(2ih-\alpha/2)}, \quad b_{2i} = -e^{-\frac{\pi}{L}(2ih+\alpha/2)},
\]

\[
a_{2i+1} = e^{-\frac{\pi}{L}((2i+1)h+\alpha/2)}, \quad b_{2i+1} = -e^{-\frac{\pi}{L}((2i+1)h-\alpha/2)}
\]

neutral dipoles situated on the even and odd sites have opposite directions. (Here one may note that the formal substitution \( \sigma_i \rightarrow (-1)^i \sigma_i + 1 - (-1)^i \) in the grand partition function converts the interaction energy between molecules to the previous form \([18]\). However, this transformation changes the form of interaction with external fields.)

The \( 2M \)-periodic reduction

\[
a_{2j,M-M} = e^{-\frac{\pi}{L}(2jh-\alpha_1)} \\
\ldots \\
a_{2j,M-1} = e^{-\frac{\pi}{L}(2jh-\alpha_1)} \\
a_{2j,M} = e^{-\frac{\pi}{L}((2j+1)h+\alpha_1)} \\
\ldots \\
a_{2j,M+M-2} = e^{-\frac{\pi}{L}((2j+1)h+\alpha_{M-1})} \\
a_{2j,M+M-1} = e^{-\frac{\pi}{L}((2j+1)h+\alpha_{M})}
\]

\[
b_{2j,M-M} = -e^{-\frac{\pi}{L}(2jh+\alpha_1)} \\
\ldots \\
b_{2j,M-1} = -e^{-\frac{\pi}{L}(2jh+\alpha_1)} \\
b_{2j,M} = -e^{-\frac{\pi}{L}((2j+1)h-\alpha_1)} \\
\ldots \\
b_{2j,M+M-2} = -e^{-\frac{\pi}{L}((2j+1)h-\alpha_{M-1})} \\
b_{2j,M+M-1} = -e^{-\frac{\pi}{L}((2j+1)h-\alpha_{M})}
\]

describes a gas of neutral dipoles lying on the middle line between two ideal dielectrics. Dipoles are pointed normally to the boundaries. They can switch their orientations (“up” and “down”) and internal degrees of freedom characterized by \( M \) different dipole moments \( \alpha_j \). In general, we have to introduce \( M \) different chemical potentials describing “internal energy” of the dipole molecule.

Another possible generalization describes mixtures of the \(+2\) charge molecules with both parallel and perpendicular orientations of dipoles. Such models can describe polar plasmas where molecules can perform discrete rotations by \( \pi/2 \). More complicated types of mixtures of plasma particles are possible as well. Several physical variables can be calculated here. These
are the number density, polarization and the pressure. One particular model is solved in the next section.

9. Solution of a dipole gas model

It is known from the theory of solitons that the \( \tau \)-function (12) can be represented as a determinant of some matrix. In general it is necessary to apply the inverse scattering method for an auxiliary linear problem to write solutions in such a form.

Determinant representations are of great help for evaluations of the partition functions, since the corresponding matrices appear to have the Toeplitz form in some physically interesting cases and, as a result, they can be diagonalized by the discrete Fourier transformation. Here we consider only the BKP case, since some models related to the KP hierarchy happen to be considered already in the literature, see, e.g. [2].

As shown in [4], the \( N = 2p \) soliton BKP \( \tau \)-function, or the corresponding Ising model partition function (33) admits the following determinant form depending on the soliton parameters \( a(z), b(z) \) and the magnetic field (34):

\[
G_{2p} = \left( \prod_{z \neq z'} \frac{(a(z) + a(z'))(b(z) + b(z'))(a(z) - b(z'))(b(z) - a(z'))}{(a(z) - a(z'))(b(z) - b(z'))(a(z) + b(z'))(b(z) + a(z'))} \right)^{1/8} \sqrt{\det Gr},
\] (39)
where matrix elements of the matrix $\mathbf{Gr}$ are

$$
\mathbf{Gr}_{z,z'} = g(z)g(z') \frac{b(z) - b(z')}{b(z) + b(z')} e^{H(z) + H(z')} + g(z)g(z')^{-1} \frac{b(z) + a(z')}{b(z) - a(z')},
$$

and

$$
g(z) = \left( \prod_{z'} \frac{(a(z) - a(z'))(b(z) + b(z'))(b(z) - a(z'))(a(z) + b(z'))}{(a(z) + a(z'))(b(z) - b(z'))(b(z) + a(z'))(a(z) - b(z'))} \right)^{1/4}.
$$

We consider the situation when the external potential is constant, $w = \mu$, i.e. the dipoles interact only between themselves.

Since the potential (38) is of the short range type, the magnetic field in the corresponding Ising model (34) is homogeneous in the bulk. We can neglect the field inhomogeneities at the edges in the thermodynamic limit $N \to \infty$. From (34) it follows that

$$
H = \frac{1}{2}(\mu + C), \quad C = \frac{1}{2} \sum_{i=1}^{\infty} W_d(i).
$$

Consider the dipole gas model described in the Fig. 3. Substituting (37) into (11) we see that the matrix $\mathbf{Gr}_{ij}$ has the Toeplitz form, $\mathbf{Gr}_{ij} = \mathbf{Gr}_{i-j}$, in the thermodynamic limit $p \to \infty$. Therefore it is diagonalized by the discrete Fourier transformation. The final answer for the thermodynamic potential per site or the pressure in appropriate units $P(\mu) = \lim_{p \to \infty} (\ln G)/2p$ can be found from the results of calculations of the free energy per site of the corresponding Ising chain (11):

$$
P(\mu) = \frac{1}{4} \ln \frac{(q, q, bq, q/b, q)_{\infty}}{(-q, -q, -bq, -q/b, q)_{\infty}} + \frac{1}{4\pi} \int_{0}^{2\pi} d\nu \ln |2\rho(\nu)|,
$$

where $q = e^{-\pi h/L}$, $b = -e^{-\pi \alpha/L}$, and

$$
\rho(\nu) = \cosh(C + \mu) + \frac{(-q; q)_{\infty}^2}{(-e^{i\nu}, -qe^{i\nu}; q)_{\infty}} \left( \frac{(b^{-1}e^{i\nu}, qbe^{-i\nu}; q)_{\infty}}{(b^{-1}, qb; q)_{\infty}} + \frac{(be^{i\nu}, qbe^{-i\nu}; q)_{\infty}}{(b, qb^{-1}; q)_{\infty}} \right).
$$

The standard notations for the $q$-shifted factorials

$$
(a; q)_0 = 1, \quad (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}),
$$

$$
(a_1, a_2, \ldots, a_n; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_n; q)_n
$$

are used in these formulae.
Taking the derivative with respect to $\mu$ we find the number density of molecules $n(\mu)$:

$$n(\mu) = \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{1}{\pi} \int_0^\pi \frac{d\nu}{1 + d(\nu) \cosh(\mu + C)} \right) \tanh(C + \mu), \quad (42)$$

where

$$d(\nu) = \frac{(qb, q/b; q)_\infty (|b|^{-1/2} + |b|^{1/2}) \theta_2(\nu, q^{1/2})}{(-q; q)_{2\infty}^2 2 \text{Re} \theta_2(\nu - (1/2) \ln|b|; q^{1/2})}.$$  

Here $\theta_2(\nu, q^{1/2})$ is the Jacobi theta-function

$$\theta_2(\nu, q^{1/2}) = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2/2} \cos(2n + 1)\nu$$

$$= 2q^{1/8} \cos \nu (q; q)_\infty (-qe^{2i\nu}; q)_\infty (-qe^{-2i\nu}; q)_\infty.$$  

In a similar way one can analyze other dipole gas models, in particular, the general $M$-periodic reduction cases, mentioned in the previous section.

In conclusion of the discussion of the BKP hierarchy it should be mentioned that statistical mechanics interpretation of the KP reductions of the C and D types (CKP and DKP) \cite{17} is not known to the authors. One may try to find a Coulomb gas realization of soliton solutions of these and other known integrable equations which were not considered in this paper.

10. Conclusions

Constructions considered so far have one essential drawback from the point of view of statistical physics: the partition functions derived from the tau functions of classical integrable hierarchies correspond only to some fixed temperatures. A possible way of overcoming this obstacle is to look for appropriate quantum generalizations of classical hierarchies.

It is well known (e.g., see \cite{13}) that the grand partition function of two component plasma can be expressed in terms of the following equivalent field theories: Sine-Gordon, Thirring or sigma-model.

Lattice versions of the neutral Coulomb plasma correspond to some discretizations of the above models: lattice Sine-Gordon, scalar Hubbard or XXZ model. Since these three models are equivalent, we take the scalar Hubbard model as their representative.

The two component plasma at the temperature \cite{22} is mapped onto the free fermion point of the one-dimensional scalar Hubbard model. It can also be mapped to free spin 1/2 fermions on the lattice \cite{7}. The Coulomb plasma at the temperatures different from \cite{22} corresponds to interacting fermions.
From our point of view, the relation of Coulomb plasmas to the theory of integrable hierarchies described here sheds some new light onto the relation between fermion models and different kinds of plasmas. Indeed, it is known [17] that the soliton equations can be derived in the framework of the free fermion formalism and they are equivalent to Plucker relations on the infinite dimensional Grassmanian manifold. We remind briefly basic points of this formalism.

Let us take fermions on the one-dimensional lattice. Their creation and annihilation operators \( \psi^*_i, \psi_i \) satisfy the relations:

\[
\{\psi^*_i, \psi_j\} = \delta_{ij}, \quad \{\psi_i, \psi_j\} = 0, \quad i, j \in \mathbb{Z}.
\]

The neutral vacuum is defined as the state where the sites with \( i < 0 \) are empty and other sites are filled by fermions (actually, it is a kink state of the XX model or the free fermion point of the scalar Hubbard model).

The action of the fermion operators on the vacuum is

\[
\psi_n \langle \text{vac} \rangle = 0, \quad n < 0, \quad \psi^*_n \langle \text{vac} \rangle = 0, \quad n \geq 0,
\]

\[
\langle \text{vac} | \psi_n = 0, \quad n \geq 0, \quad \langle \text{vac} | \psi^*_n = 0, \quad n < 0.
\]

The fermion field operators

\[
\psi(\lambda) = \sum_{k=-\infty}^{\infty} \psi_k \lambda^k, \quad \psi^*(\lambda) = \sum_{k=-\infty}^{\infty} \psi^*_k \lambda^{-k}
\]

evolve under the KP flow as follows

\[
e^{H(t)} \psi(\lambda) e^{-H(t)} = e^{\theta(t, \lambda)} \psi(\lambda), \quad e^{H(t)} \psi^*(\lambda) e^{-H(t)} = e^{-\theta(t, \lambda)} \psi^*(\lambda),
\]

\[
\theta(t, \lambda) = \sum_{n=1}^{\infty} t_n \lambda^n,
\]

where the generating function of KP Hamiltonians \( H(t) \) has the form

\[
H(t) = \sum_{n=1}^{\infty} H_n t_n, \quad H_n = -\sum_{k=-\infty}^{\infty} : \psi_k \psi^*_{k+n} :,
\]

where the colons mean the normal ordering with respect to the vacuum \( |\text{vac}\rangle \). The \( \tau \)-function is defined as follows

\[
\tau(t, g) = \langle \text{vac} | e^{H(t)} g | \text{vac} \rangle, \quad g \in G,
\]

where \( G \) is a subgroup of the fermion algebra preserving one-fermion states

\[
G = \left\{ g | \ g \mathcal{V} g^{-1} = \mathcal{V}, \ g \mathcal{V}^* g^{-1} = \mathcal{V}^* \right\}
\]
and $\mathcal{V} = \oplus_{i \in \mathbb{Z}} C\psi_i$, $\mathcal{V}^* = \oplus_{i \in \mathbb{Z}} C\psi_i^*$. Choosing the element $g$ of $G$ in the definition of $\tau$-function as 

$$g = \exp \sum_{i=1}^{N} \gamma_i \psi(a_i)\psi^*(-b_i),$$

where $\gamma_i$ are some arbitrary constants related to the zero time phases of solitons, one gets the KP $N$-soliton $\tau$-function (12), (20).

Thus, the two component plasma at the temperature (22) is described simultaneously by the KP hierarchy and the free fermions model. Variation of the temperature from this value, which in some models (e.g., in the one-dimensional Ising chains picture) is not qualitatively distinguished from the other ones, would correspond to a generalization of the free fermion formalism for integrable hierarchies to the case of interacting fermions. The KP hamiltonians which are of the form of hopping terms of the Hubbard model, suggest that such a generalization is possible, provided KP hamiltonians are replaced by conserved quantities of a generalization of the spinless Hubbard or XX model. For instance, in the XX limit of the XXZ model derivatives of the transfer matrix are in the algebra (43)

$$(\ln T(u))^{(n)} \propto H_{-n} + H_n.$$  

One may conjecture that the derivatives of general XXZ transfer matrix belong to a generalization of the algebra of free fermion Hamiltonians of the KP hierarchy.

Here we encounter one difficulty. Roughly speaking, the algebra of the KP Hamiltonians (43) has two times as many Hamiltonians as derivatives of the XXZ transfer matrix. Probably there are some extra integrals of motion which are reduced to $H_n - H_{-n}$ in the free fermion limit. As an example, we can mention that some extra nonlocal constants of motion have been derived for the Hubbard model in [18].

Another possible way of generalization is due to the approach described, e.g., in [19], where some generalization of the nonlinear Schrödinger equation is presented. Unfortunately the notion of $\tau$-function is clearly defined only in the limiting cases of the free fermion points (e.g., for the impenetrable Bose gas). It is not obvious that solutions of the corresponding integro-differential equations can be expressed in terms of a $\tau$-function at general couplings and that such a function would make sense in the plasma picture.

Concluding this article we discuss possible physical significance of the models considered in it going beyond the Coulomb gas picture. As was shown in [1] there are nice interpretations of the one-dimensional cases from the point of view of Ising magnets. However, various boundary conditions arising within the intrinsically two-dimensional Coulomb interaction systems look
somewhat artificial in the Ising picture. Still, a number of 2D Ising models with such non-local exchange can be formulated which are exactly solvable by the techniques due to Gaudin (see the third paper in [2]).

Another possible application concerns the fractional quantum Hall effect (FQHE). Remind that the two-dimensional one component Coulomb plasma in the uniform neutralizing background of the density $1/2m$ has the following partition function

$$Z = \int dz_1 \ldots dz_N \exp \left( -2mE \right), \quad E = -\sum_{i<j} \log |z_i - z_j| + \frac{1}{4m} \sum_i |z_i|^2.$$

It was shown by Laughlin that the $n$-point correlation functions in the appropriate state of the FQHE at the filling factor $1/m$ coincide with those of the one component Coulomb plasma. In particular, the normalization factor of the corresponding wave function is given by $Z$.

Our model is a bit different from the pure one component plasma, since we have non-trivial boundary conditions. However, by placing the domain of concentration of the charged particles far from the boundary we get the Laughlin plasma. It is also possible that the plasma with boundaries has some meaning in the fractional quantum Hall picture as well describing there some boundary effects. $\tau$-functions of the classical hierarchies correspond to boundary Laughlin states at the full filling $1/m = 1$. Nontrivial examples $m > 1$ could be described by $\tau$-functions of the generalized hierarchies discussed above.

Acknowledgments. The authors are indebted to F.D.M. Haldane, V.I. Inozemtsev, V.E. Korepin, S.P. Novikov and V.B. Priezzhev for some remarks and stimulating discussions. The work of I.L. is supported by a fellowship from NSERC (Canada), V.S. is supported in part by the RFBR (Russia) grant 97-01-01041 and the INTAS grant 96-0700.

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