A dynamical transition and metastability in a size-dependent zero-range process

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Abstract

We study a zero-range process with system-size dependent jump rates, which is known to exhibit a discontinuous condensation transition. Metastable homogeneous phases and condensed phases co-exist in extended phase regions around the transition, which have been fully characterized in the context of the equivalence and non-equivalence of ensembles. In this paper we report rigorous results on the large deviation properties and the free energy landscape which determine the metastable dynamics of the system. Within the condensed phase region we identify a new dynamic transition line which separates two distinct mechanisms of motion of the condensate, and provide a complete discussion of all relevant timescales. Our results are directly related to recent interest in metastable dynamics of condensing particle systems and apply to more general condensing particle systems, which exhibit the dynamical transition as a finite size effect.

Keywords: metastability, large deviations, critical phenomena, condensation

(Some figures may appear in colour only in the online journal)

1. Introduction

The understanding of metastable dynamics associated to phase transitions in complex many-body systems is a classical problem in statistical mechanics. It is rather well understood on a heuristic level, characterizing metastable states as local minima of the free energy landscape and the transitions between them occurring along a path of least action, corresponding to the classical Arrhenius law of reaction kinetics [1]. Since the classical work by Freidlin and Wentzell on random perturbations of dynamical systems [2], there have been various rigorous approaches in the context of stochastic particle and spin systems summarized in [3, chapter 4]...
A mathematically rigorous treatment of metastability remains an intriguing question and is currently a very active field in applied probability and statistical mechanics [5–9]. Most recently, potential theoretic methods [10] have been combined with a martingale approach to establish a general theory of metastability for continuous-time Markov chains [8, 11, 12]. The dynamics of condensation in driven diffusive systems has recently become an area of major research interest in this context. There have also been recent results on the inclusion process [13, 14] and systems exhibiting explosive condensation [15, 16]; however the zero-range process (ZRP) remains one of the most studied systems.

ZRPs are stochastic lattice gases with conservative dynamics introduced in [17], and the condensation transition in a particular class of these models was established in [18–21]. Many variants of this class have been studied in recent years including a non-Markovian version with slinky condensate motion [22, 23], see also [24, 25] for recent reviews of the literature. If the particle density $\rho$ exceeds a critical value $\rho_c$, the system phase separates into a homogeneous fluid phase at density $\rho_c$ and a condensate, which concentrates on a single lattice site and contains all the excess mass. The dynamics and associated timescales of this transition have been described heuristically in [26]. For large but finite systems, due to ergodicity, the location of the condensate changes on a slow timescale and converges to a random walk on the lattice in the limit of diverging density [27]. Recent extensions of these rigorous results include a non-equilibrium version of the dynamics [9, 12], and a thermodynamic scaling limit with a fixed supercritical density $\rho > \rho_c$ [28].

While the motion of the condensate is the only metastable phenomenon in the above results, a slight generalization studied in [29–31] exhibits metastable fluid states at supercritical densities, which are a finite-size effect and do not persist in the thermodynamic limit. The model we study here was first introduced in [32] motivated by experiments in granular media, and is a ZRP where the jump rates scale with the system size. This leads to an effective long-range interaction, and it is well known that these can give rise to metastable states that are persistent in the thermodynamic limit [1]. The condensation transition in this ZRP is discontinuous with metastable fluid and condensed states above and below the transition density, respectively, and the model has a rich phase diagram.

As a first main contribution of this work we identify a new dynamic transition within the condensed phase region which separates two distinct mechanisms of motion of the condensate. Secondly, we provide a complete discussion of all relevant timescales using a comprehensive approach in the context of large deviation theory, which proves to be a powerful tool for the characterization and study of phase transitions for non-equilibrium systems [1, 5]. All results we report here are based on rigorous work which is presented in more detail in [33], and apply to a general class of condensing particle systems, as explained in more detail in the conclusion. To our knowledge, this constitutes a first example of a dynamic analysis of a condensing particle system that exhibits extended regions in phase space with co-existing metastable states, and is an important step to extend recent rigorous results on the condensate dynamics in such systems.

2. Definitions and notation

We consider a ZRP on a one-dimensional lattice of $L$ sites with periodic boundary conditions. Particle configurations are denoted by vectors of the form $\eta = (\eta_x)_{x=1}^L$, where $\eta_x$ is the number of particles on site $x$, which can take any value in $\{0, 1, 2, \ldots\}$. In the ZRP particles jump off a site $x$ with a rate that depends only on the number of particles on the departure site, and then move to another site $y$ according to a random walk probability $p(x, y)$, which we take to be of
finite range, irreducible, and translation invariant. All results on the equivalence of ensembles and stationary large deviations are independent of the choice of \( \rho(x, y) \); however the choice can affect the metastable dynamics as discussed later.

The rate at which a particle exits a site is denoted by \( g(x, y) \), where the system size dependence of the jump rates is indicated by the subscript. We consider simple jump rates introduced in [32] of the form

\[
g_L(n) := \begin{cases} \frac{c}{n} & \text{if } n \leq aL, \\ 1 & \text{if } n > aL, \end{cases}
\]

for some \( c > 1 \) and \( a > 0 \). So sites containing more than \( aL \) particles lose particles more slowly, and \( a \) parameterizes the site occupation threshold between fast and slow jump rates.

It is well known that ZRPs, including size-dependent models, exhibit stationary distributions which factorize over lattice sites, see for example [25, 32, 34]. It is convenient to introduce a prior distribution (or reference measure) which is stationary, that will be used to characterize the canonical and grand-canonical distributions after proper renormalization. The prior probability distribution is also size-dependent and given by

\[
\mathbb{P}_L(\eta) := \frac{1}{Z_L} \prod_{i=1}^{L} w_L(\eta_i) e^{-\eta_i} \quad \text{with} \quad w_L(n) = \prod_{i=1}^{n} \frac{1}{g_L(i)} = \begin{cases} e^{-\eta_i} & \text{for } n \leq aL, \\ e^{-[aL]} & \text{for } n > aL, \end{cases}
\]

where the empty product is taken to be unity and the normalization factor is \( Z_L = \left( \sum_{n \geq 0} w_L(n) e^{-\eta} \right)^L \). The above weights \( w_L \) are themselves stationary for the ZRP, but not normalized, and the additional factor \( e^{-\eta} \) is a convenient choice so that they can be normalized, which allows the interpretation of free energies as large deviation rate functions (see [1]).

Since the dynamics are irreducible and conserve the total particle number, on a fixed lattice starting from any initial condition with a fixed number \( N \) of particles the system is ergodic. In the long time limit the distribution will converge to the corresponding canonical distribution \( \mathbb{P}_{L,N} := \mathbb{P}_L(\sum \eta_i = N) \), which is given by a conditional version of the reference measure

\[
\mathbb{P}_{L,N}(\eta) = \begin{cases} \frac{1}{Z_{L,N}} \mathbb{P}_L(\eta) & \text{if } \sum_{x=1}^{L} \eta_x = N, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( Z_{L,N} = \mathbb{P}_L(\sum_{x=1}^{L} \eta_x = N) \).

3. Results

The large scale behaviour and condensation transition can be characterized as usual by the canonical free energy, which is defined as

\[
f(\rho) := -\lim_{L \to \infty} \frac{1}{N/L} \log Z_{L,N}.
\]

Note that, since \( \mathbb{P}_L \) is a probability distribution, this is the large deviation rate function for the total number of particles under the reference measure \( \mathbb{P}_L \) (see also [1]) and is also the relative entropy of the canonical measures with respect to the reference distribution. Explicit computation can be done using the grand-canonical and a restricted grand-canonical ensemble which we outline in the appendix. In the following we simply report the main results.
There exists a transition density $\rho_{\text{trans}}$ characterized in (8), below which the system is typically in a fluid state with all particles distributed homogeneously. For $\rho > \rho_{\text{trans}}$, the system is in a condensed state, and phase separates into a single condensate site containing of order $(\rho - \rho_{c})L$ particles and a fluid background at density $\rho_{c}$. As usual, this is characterized by the free energy decomposing into a contribution $f_{\text{fluid}}$ from the fluid and $f_{\text{cond}}$ from the condensed phase. It is given by

$$f(\rho) = \begin{cases} f_{\text{fluid}}(\rho) & \text{if } \rho \leq \rho_{\text{trans}} \quad \text{(fluid state)}, \\ f_{\text{fluid}}(\rho_{c}) + f_{\text{cond}}(\rho - \rho_{c}) & \text{if } \rho > \rho_{\text{trans}} \quad \text{(condensed state)}. \end{cases}$$

(3)

As derived in (A.4) in the appendix

$$f_{\text{fluid}}(\rho) = \rho \log \rho - (1 + \rho) \log (1 + \rho) + \rho \log (ce) - \log \left(1 - \frac{1}{ce}\right),$$

(4)

which is the relative entropy of a geometric distribution with density $\rho$ with respect to the reference measure. This geometric distribution can be interpreted as the fluid phase as is discussed in the appendix. The critical background density in the condensed phase is given in (A.2) as $\rho_{c} = 1/(c - 1)$. The condensate contribution

$$f_{\text{cond}}(m) = \lim_{L \to \infty} \frac{1}{L} \log P_{L}(\eta_{1} = M) = \begin{cases} m + m \log c & \text{if } m < a, \\ m + a \log c & \text{if } m \geq a, \end{cases}$$

(5)

is determined by the reference probability of a single site containing $mL + o(L)$ particles, since the condensate has no associated entropy. Note that even though fluid and condensate coexist in the condensed state, there is no free energy contribution from the interface since the stationary distributions factorize, and the combinatorial factor of $L$ possible positions for the condensate location only contributes on a sub-exponential scale. This lack of surface tension also implies that the condensed phase consists of a single site, in contrast to other systems with non-product stationary distributions [35, 36].

The behaviour of the free energy is dominated by typical stationary configurations, which are illustrated in figure 1 along with the phase diagram of the model. As observed in [32], since the background density $\rho_{l} < \rho_{\text{trans}}$ is strictly smaller than the transition density, the phase transition is discontinuous and local observables such as the bulk density and the current change discontinuously at the transition density. This is in contrast to the condensation transition in ZRPs without size-dependent rates, where observables are continuous functions at the transition point in the thermodynamic limit. Such systems exhibit discontinuous behaviour only in special scaling limits to study finite-size effects around the critical point [29].

**Metastable states.** The phase diagram in figure 1 also contains information about metastable states. They can be identified as local minima of the large deviation rate function $I_{\rho}(m)$ for the maximum occupation number $M_{L}(\eta) = \max_{\eta} \eta_{1}$, as is shown in figure 2. This characterizes the exponential rate of decay of the canonical probability to observe a maximum of size $mL + o(L)$, i.e.

$$P_{L,N}(M_{L} = M) \approx e^{-I_{\rho}(m)L} \quad \text{as } L \to \infty, \ M/L \to m.$$  

In order to calculate $I_{\rho}(m)$, we first find joint large deviations of the maximum and the density under the prior distribution described by a rate function $I(\rho, m)$. Precisely, we can show that the following limit exists for $\rho > 0$ and $m \in [0, \rho]$
The limit is independent of details of the sequences $N/L$ and $M/L$ so long as $m \in (0, \rho]$; if $M/L \to 0$ we require that $M$ is not too small (essentially, $M$ must be larger than the normal fluctuations in the bulk of order $\sqrt{L}$).

We find that for each $\rho > 0$ and $m \in [0, \rho]$ the joint rate function for the density and maximum is specified exactly by the recursion

$$I(\rho, m) := -\lim_{L \to \infty} \frac{1}{L} \log \mathbb{P} \left( \sum_{x=1}^{L} \eta_{x} = N, \mathcal{M}_{L} = M \right),$$

where $N/L \to \rho, M/L \to m$. 

*Figure 1.* Phase diagram with the following phase regions: (I) for $\rho < \rho_{c} + a$ there is a unique fluid state and particles are distributed homogeneously. (II) For $\rho_{c} + a < \rho < \rho_{\text{trans}}$ an additional metastable condensed state exists. (III) For $\rho > \rho_{\text{trans}}$ the condensed state becomes stable, and the fluid state remains metastable for all densities. The new transition density $\rho_{\text{dyn}}$ (12) characterizes a change in the mechanism for condensate motion, which is explained in figure 3. Typical stationary configurations for fluid and condensed states are shown on the right.

*Figure 2.* Left: the rate function $I_{\rho}(m)$ (7) for $a = 0.2, c = 3.5$ and various values of $\rho$, showing a single well for $\rho < \rho_{c} + a$ and a double well for higher densities. The change of global minimum happens at $\rho = \rho_{\text{trans}}$. Right: the exponential rates of the lifetimes and condensate motion time, given by (11), for $a = 0.2, c = 3.5$, characterizing the transition densities $\rho_{\text{trans}}$ (8) and $\rho_{\text{dyn}}$ (12).
\[ I(\rho, m) = \begin{cases} f_{\text{cond}}(m) + f_{\text{fluid}}(\rho - m) & \text{if } m < a \text{ or } m > (\rho - \rho_c), \\ f_{\text{cond}}(m) + \inf_{m \in [0, m]} I(\rho - m, m_2) & \text{otherwise}. \end{cases} \]  

(6)

The iteration in the second case closes after finitely many steps for each \( \rho < \infty \) and \( m \in [0, \rho] \). The first term \( f_{\text{cond}}(m) \) is the contribution of the maximum to the rate function and the second term is the contribution due to the bulk of the system. The infimum in the second line of (6) arises since a large deviation outside the range \( m < a \text{ or } m > (\rho - \rho_c) \), which is always atypical and never locally stable, may be realized by configurations with more than one macroscopically occupied site.

Given \( I(\rho, m) \), the canonical free energy and large deviations of the maximum under the canonical distributions are straightforward to compute

\[
f(\rho) = \inf_{m \in [0, \rho]} I(\rho, m) \quad \text{and} \quad I_{\rho}(m) := -\lim_{L \to \infty} \frac{1}{L} \log P_L(N(M_L = M) = I(\rho, m) - f(\rho),
\]

where again \( N/L \to \rho \) and \( M/L \to m \). Note that \( f(\rho) \) is simply a contraction over the most likely value for \( M_L \) and gives the normalization of the rate function \( f_{\rho} \).

Below \( \rho_c + a \) there is a unique minimum of \( I_{\rho}(m) \) at \( m = 0 \) which corresponds to the fluid phase. Above \( \rho_c + a \) there is another local minimum at \( m = \rho - \rho_c \) which corresponds to the condensed state. The fluid state exists for all densities \( \rho \) and parameter values \( a \geq 0, \ c \geq 1 \), and is stable for \( \rho < \rho_{\text{trans}} \) and metastable above (cf figure 2). The transition density \( \rho_{\text{trans}} \) is then characterized by both local minima of \( I_{\rho}(m) \) being of equal depth, i.e.

\[ I_{\rho_{\text{trans}}}(0) = I_{\rho_{\text{trans}}}(\rho - \rho_c) = 0. \]

(8)

With the above results, this is equivalent to

\[
f_{\text{fluid}}(\rho_{\text{trans}}) = f_{\text{cond}}(\rho_{\text{trans}} - \rho_c) + f_{\text{fluid}}(\rho_c) \quad \text{with} \quad \rho_c = \frac{1}{c - 1}.
\]

(9)

We stress again that, although metastable fluid states can exist in the standard ZRP close to the critical point in a suitable scaling limit, these do not persist in the thermodynamic limit above the critical point [29].

The dynamic transition. Above \( \rho_{\text{trans}} \), a typical stationary configuration is phase separated with the condensate on a single site. Analogous to previous results [27], due to translation invariance and ergodicity on large finite systems, the condensate will change location due to fluctuations. We show in this section that there is a dynamical phase transition at a density \( \rho_{\text{dyn}} > \rho_{\text{trans}} \), where the mechanism of condensate relocation changes abruptly. For large densities \( \rho > \rho_{\text{dyn}} \) the typical mechanism for this relocation is to stay phase separated and grow a second condensate (see figure 3 (IIIb)), which is the same mechanism as identified in other super-critical ZRPs, see for example [27, 28]. This mechanism exhibits an interesting spatial dependence on the underlying random walk probabilities \( p(x, y) \), which can lead to a non-uniform motion of the condensate, for example if \( p(x, y) \) is nearest-neighbour symmetric. For densities \( \rho_{\text{trans}} < \rho < \rho_{\text{dyn}} \) the typical mechanism is to dissolve the condensate and enter an intermediate metastable fluid state (see figure 3 (IIIa)). Since the system relaxes to a translation invariant metastable fluid state before the condensate reforms, the condensate reforms at a site uniformly at random, independently of the geometry of the lattice. This is very different from mechanism (IIIb) where the intermediate state is a saddle point with two condensates of equal height (cf figure 4). In both cases, the life time of intermediate states is...
negligible compared to the timescale on which the condensate moves and on this timescale the transition happens instantaneously.

To derive this transition, with the same approach as above we can calculate the canonical large deviations of the maximum and the second most occupied site $M_L^{(2)}$:

$$I_{\rho}^{(2)}(m_1, m_2) = -\lim_{L \to \infty} \frac{1}{L} \log P_{L} \left[ M_L = M_1, M_L^{(2)} = M_2 \right]$$

$$= f_{\text{cond}}(m_1) + I(\rho - m_1, m_2) - f(\rho),$$

(10)

where $N/L \to \rho, M_1/L \to m_1$ and $M_2/L \to m_2$. This rate function essentially gives rise to a free energy landscape for the maximum and second most occupied site.

In order for the condensate to move, the system must go via a state in which the maximum and second most occupied site differ in occupation by at most a single particle. In order to reach the diagonal $m_1 = m_2$ from a condensed state with $m_1 > 0$ and $m_2 = 0$ there are two relevant paths as shown in figure 4. The first one is along the axis $m_2 = 0$ toward the metastable fluid state with $m_1 = m_2 = 0$ following the black line (mechanism IIIa). The second one is along the red line with $m_1 + m_2 = \rho - \rho_L$, growing a second condensate reaching the diagonal at the the local minimum of the blue curve, which is a saddle point in the full landscape (mechanism IIIb). The associated heights of the saddle points are given by

$$\Delta_1^{\text{cond}}(\rho) := I_{\rho}(a) - I_{\rho}(\rho - \rho_L),$$

$$\Delta_2^{\text{cond}}(\rho) := I_{\rho}^{(2)}(\rho - \rho_L - a, a) - I_{\rho}(\rho - \rho_L) = a \log c.$$ (11)

Plugging (4) into (7) it is easy to see that $\Delta_1^{\text{cond}}(\rho)$ is increasing from 0 for $\rho \geq \rho_L + a$ (see figure 2 right). Since $\Delta_2^{\text{cond}}(\rho)$ is constant, this implies that there is a dynamic transition at a density $\rho_{\text{dyn}}$ characterized by
In this formalism we can also include

\[ \Delta_{\text{cond}}(\rho) \] for \( \rho < \rho_{\text{dyn}} \) (mechanism (IIIa)),

\[ \Delta_{\text{cond}}(\rho) \] for \( \rho > \rho_{\text{dyn}} \) (mechanism (IIIb)).

(12)

Figure 4. Mechanisms for condensate motion from the free energy landscape, \( a = 0.1, c = 4 \). The top row shows mechanism (IIIa) with \( \rho_{\text{trans}} < \rho = 0.8 < \rho_{\text{dyn}} \), the bottom row mechanism (IIIb) with \( \rho = 1 > \rho_{\text{dyn}} \). Left: surface plot of \( I_\rho^{(2)}(m_1, m_2) \) (10), the arrowed line indicates the minimal action path to the diagonal which has to be reached for condensate motion. Right: the dashed blue line shows the landscape along the diagonal \( I_\rho^{(2)}(\rho, \rho) \). The path along the \( x \)-axis \( I_\rho^{(2)}(x, 0) \) is shown as a full black line, and is chosen in mechanism (IIIa). The dashed red line shows the path to the diagonal by growing a second condensate with constant bulk density \( \rho_c \), chosen in mechanism (IIIb). \( \Delta_{\text{cond}}^1, \Delta_{\text{cond}}^2 \) denote the respective exponential costs for the paths (11).

\[ \Delta_{\text{fluid}}(\rho) := I_\rho(\alpha) - I_\rho(0) \]

(13)

as the depth of the fluid minimum, which provides another characterization of \( \rho_{\text{trans}} \) as

\[ \Delta_{\text{fluid}}(\rho) = \Delta_{\text{cond}}^1(\rho) \] as illustrated in figure 2 on the right. After a straightforward computation this leads to

\[ \rho_{\text{dyn}} = \rho_{\text{trans}} - a \quad \text{and therefore} \quad \rho_{\text{dyn}} > \rho_c + 2a. \]

(14)

Note that the saddle point at \( m_1 = (\rho - \rho_c - a), m_2 = a \) corresponding to \( \Delta_{\text{cond}}^2(\rho) \) only exists if \( \rho > \rho_c + 2a \) and the system can sustain two macroscopically occupied sites. So while mechanism IIIa exists for all densities \( \rho > \rho_c + a \) and therefore for \( \rho > \rho_{\text{trans}} \), mechanism IIIb exists only for \( \rho > \rho_c + 2a \) (see figure 2 right). This is larger than \( \rho_{\text{trans}} \) for \( a \) large enough, but always below \( \rho_{\text{dyn}} \) when mechanism IIIb becomes typical.
The exponential timescales associated with the corresponding activation times of the metastable motion are directly related to the saddle point heights (as predicted by the Arrhenius law). Also, the dynamics are expected to concentrate in the thermodynamic limit on the least action path in the free energy landscape (10) (see [2, 5]). The expected lifetime of the fluid state, condensed state, and time to observe condensate motion are defined by

\[ T_{\text{fluid}}(\rho, L) = \mathbb{E}^\rho \left[ \inf \left\{ t > 0 \mid M_L > a \right\} \right], \]
\[ T_{\text{cond}}(\rho, L) = \mathbb{E}^\rho \left[ \inf \left\{ t > 0 \mid M_L < a \right\} \right], \]
\[ T_{\text{move}}(\rho, L) = \mathbb{E}^\rho \left[ \inf \left\{ t > 0 \mid M_L = M^{(2)}_L \right\} \right]. \]

Here the expectations \( \mathbb{E}^\rho \), \( \mathbb{E}_{\rho}^{\text{cd}} \) are with respect to the dynamics with system size \( L \) and density \( \rho \) started from a configuration in the fluid and condensed states, respectively. The exponential growth of the life times with the system size is then related to the saddle point structure as follows

\[ \lim_{L \to \infty} \frac{1}{L} \log T_{\text{fluid}} = \Delta_1^{\text{fluid}}(\rho), \quad \lim_{L \to \infty} \frac{1}{L} \log T_{\text{cond}} = \Delta_1^{\text{cond}}(\rho), \]
\[ \lim_{L \to \infty} \frac{1}{L} \log T_{\text{move}} = \min \left\{ \Delta_1^{\text{cond}}(\rho), \Delta_2^{\text{cond}}(\rho) \right\}. \]

This behaviour and the dynamic transition are confirmed in simulations shown in figure 5 for symmetric nearest-neighbour dynamics in one-dimension.

On the basis of our results, using the techniques of [8], convergence to the predicted limit process for the condensate motion can be established mathematically rigorously for reversible dynamics (symmetric \( p(x, y) \)). This still requires a significant effort and technical estimates to bound finite-size errors, which remains an open mathematical problem. Our results on the mechanisms of condensate motion are independent of \( p(x, y) \) and reversibility, but for non-reversible ergodic dynamics the remaining technical difficulties to show convergence to the expected limit process are even harder, and additional restrictions may apply. First rigorous results on non-reversible condensate motion have just recently been achieved in [9, 12].
4. Conclusion

In this paper we have completely characterized the relevant time scales and the candidates for transition paths of metastable condensate motion in a ZRP with size-dependent rates (1), and thereby established a dynamical phase transition which has not been observed before. Our results are based on mathematically rigorous work presented in detail in [33]. Based on previous studies of finite-size effects in more classical zero-range condensation models without size-dependent rates, the results on the dynamic phase transition also apply in these models in a scaling limit around the critical point and are relevant in related applications [29–31].

The two mechanisms for condensate motion we identified depend purely on the stationary distribution of the process and the resulting free energy landscape. It is well known that a large class of condensing particle systems exhibits the same factorized stationary distributions as the above models, including also models beyond zero-range dynamics [16, 25]. Therefore, the dynamic phase transition is expected to be a generic phenomenon in a large class of models, and the two mechanisms identified are also expected to be of a general nature.

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Appendix

A standard approach for explicit computations of free energies is the use of grand-canonical distributions

\[ P_{L,\mu}(\eta) = \frac{e^{\mu \sum_{x=1}^L \eta_x}}{Z_{L,\mu}}, \quad \text{with} \quad Z_{L,\mu} = \left( e^{\mu \sum_{x=1}^L \eta_x} \right)_\mu = \left( \left( e^{\mu \eta} \right)_\mu \right)^L. \]

The mean particle density is fixed by the conjugate parameter \( \mu \in \mathbb{R} \), called the chemical potential. Note that the equality on the right-hand side follows since the reference measure factorizes over lattice sites and the marginals on each site are identical. The grand-canonical distributions are well defined for all \( \mu \in (-\infty, 1) \). For fixed \( L \), as \( \mu \to 1 \) the normalization \( Z_{L,\mu} \) and the average particle density \( \langle \eta \rangle_{\mu} \) diverge.

The grand-canonical pressure is given by the point-wise limit of the scaled cumulant generating function

\[ p(\mu) = \lim_{L \to \infty} \frac{1}{L} \log Z_{L,\mu} = \begin{cases} \log \frac{c - e^{-1}}{c - e^{-\mu}} & \text{if } \mu < 1, \\ \infty & \text{if } \mu \geq 1. \end{cases} \quad (A.1) \]

The density can be computed as \( R(\mu) = \partial_\mu p(\mu) \) and, as discussed in previous work [25, 32], the critical density is given by
\[ \rho_c := \lim_{\rho \to 1} R(\rho) = \lim_{\mu \to c} \frac{e^{\mu-1}}{e^{\mu}-1} = \frac{1}{c - 1} < \infty. \quad (A.2) \]

Although \( p(\mu) \) does not exist above \( \mu_c = 1 \) it can be extended analytically up to \( 1 + \log c \). It turns out that this extended pressure is exactly the one associated to the grand-canonical distributions conditioned on no site containing more than \( aL \) particles. These restricted grand-canonical distributions can be interpreted as metastable fluid states (see [33] for details), and their pressure is given by

\[ p_{\text{fluid}}(\mu) = \log \left( \frac{c - e^{-1}}{c - e^{\mu-1}} \right), \quad \text{where} \quad \mu < 1 + \log c. \quad (A.3) \]

The free energy of the fluid phase is then given by the Legendre–Fenchel transform of the pressure

\[ f_{\text{fluid}}(\rho) := \sup_{\mu \in \mathbb{R}} \left[ \mu \rho - p_{\text{fluid}}(\mu) \right], \quad (A.4) \]

which is explicitly given in (4). There are further interesting questions related to the equivalence of canonical and grand-canonical ensembles, which are discussed rigorously in [33].

References

[1] Touchette H 2009 The large deviation approach to statistical mechanics Phys. Rep. 478 1–69
[2] Freidlin M I and Wentzell A D 2012 Random Perturbations of Dynamical Systems 3rd edn (Berlin: Springer)
[3] Oliveira R I and Vares M E 2005 Large Deviations and Metastability (Cambridge: Cambridge University Press)
[4] Bovier A 2009 Metastability Methods of Contemporary Mathematical Statistical Physics 1970 edn (Berlin: Springer) 177–221
[5] Touchette H 2014 Equivalence and nonequivalence of ensembles: thermodynamic, macrostate, and measure levels arXiv:1403.6608v1 [cond-mat.stat-mech]
[6] Fernandez R and Manzo F 2014 Asymptotically exponential hitting times and metastability: a pathwise approach without reversibility arXiv:1406.2637v1 [math.PR]
[7] Cassandro M, Galves A, Olivieri E and Vares M E 1984 Metastable behavior of stochastic dynamics: a pathwise approach J. Stat. Phys. 35 603–34
[8] Beltrán J and Landim C 2013 A martingale approach to metastability Probab. Theory Relat. Fields (online first)
[9] Landim C 2014 Metastability for a Non-reversible dynamics: the evolution of the condensate in totally asymmetric zero range processes Commun. Math. Phys. 330 1–32
[10] Bovier A, Eckhoff M, Gayrard V and Klein M 2002 Metastability and low lying spectra in reversible Markov chains Commun. Math. Phys. 255 219–55
[11] Beltrán J and Landim C 2010 Tunneling and metastability of continuous time Markov chains J. Stat. Phys. 140 1–50
[12] Beltrán J and Landim C 2012 Tunneling and metastability of continuous time Markov chains II, the nonreversible case J. Stat. Phys. 149 598–618
[13] Cao J, Chleboun P and Grosskinsky S 2014 Dynamics of condensation in the totally asymmetric inclusion process J. Stat. Phys. 155 523–43
[14] Grosskinsky S, Redig F and Vafayi K 2013 Dynamics of condensation in the symmetric inclusion process Electron. J. Probab. 18 66
[15] Waclaw B and Evans M R 2012 Explosive condensation in a mass transport model Phys. Rev. Lett. 108 070601
[16] Waclaw B and Evans M R 2014 Condensation in stochastic mass transport models: beyond the zero-range process J. Phys. A: Math. Theor. 47 95001
[17] Spitzer F 1970 Interaction of Markov processes Adv. Math. 5 246–90
[18] Drouffe J-M, Godrèche C and Camia F 1998 A simple stochastic model for the dynamics of condensation J. Phys. A: Math. Gen. 31 L19
[19] Evans M R 2000 Phase transitions in one-dimensional nonequilibrium systems Braz. J. Phys. 30 42–57
[20] Grosskinsky S, Schütz G M and Spohn H 2003 Condensation in the zero range process: stationary and dynamical properties J. Stat. Phys. 113 389–410
[21] Godrèche C 2003 Dynamics of condensation in zero-range processes J. Phys. A: Math. Gen. 36 6313
[22] Hirschberg O, Mukamel D and Schütz G M 2012 Motion of condensates in non-Markovian zero-range dynamics J. Stat. Mech. Theory Exp. 2012 08014
[23] Hirschberg O, Mukamel D and Schütz G M 2009 Condensation in temporally correlated zero-range dynamics Phys. Rev. Lett. 103 90602
[24] Godrèche C and Luck J M 2012 Condensation in the inhomogeneous zero-range process: an interplay between interaction and diffusion disorder J. Stat. Mech. Theory Exp. 2012 12013
[25] Chleboun P and Grosskinsky S 2014 Condensation in stochastic particle systems with stationary product measures J. Stat. Phys. 154 432–65
[26] Godrèche C and Luck J M 2005 Dynamics of the condensate in zero-range processes J. Phys. A: Math. Gen. 38 7215
[27] Beltrán J and Landim C 2011 Metastability of reversible condensed zero range processes on a finite set Probab. Theory Relat. Fields 152 781–807
[28] Armendáriz I, Grosskinsky S and Loulakis M 2015 in preparation
[29] Chleboun P and Grosskinsky S 2010 Finite size effects and metastability in zero-range condensation J. Stat. Phys. 140 846–72
[30] Armendáriz I, Grosskinsky S and Loulakis M 2013 Zero-range condensation at criticality Stoch. Process. Appl. 123 3466–96
[31] Evans M R, Majumdar S N and Zia R K P 2006 Canonical analysis of condensation in factorized steady states J. Stat. Phys. 123 357–90
[32] Grosskinsky S and Schütz G M 2008 Discontinuous condensation transition and nonequivalence of ensembles in a zero-range process J. Stat. Phys. 132 77–108
[33] Chleboun P 2015 in preparation
[34] Evans M R and Hanney T 2005 Nonequilibrium statistical mechanics of the zero-range process and related models J. Phys. A. Math. Gen. 38 R195
[35] Evans M R, Hanney T and Majumdar S N 2006 Interaction driven real-space condensation Phys. Rev. Lett. 97 010602
[36] Waclaw B, Sopik J, Janke W and Meyer-Ortmanns H 2009 Mass condensation in one-dimension with pair-factorized steady states J. Stat. Mech. Theory Exp. 2009 P10021