On a question of B.J. Baker and M. Laidacker concerning disjoint compacta in $\mathbb{R}^N$

Olga Frolkina

Chair of General Topology and Geometry
Faculty of Mechanics and Mathematics
M.V. Lomonosov Moscow State University
and
Moscow Center for Fundamental and Applied Mathematics
Leninskie Gory 1, GSP-1
Moscow 119991, Russia

Abstract

We describe wild embeddings of polyhedra into $\mathbb{R}^N$ which show that the answer to the question of B.J. Baker–M. Laidacker (1989) concerning uncountable families of pairwise disjoint compacta can be twofold. The central idea of our construction is the use of specific wild Cantor sets, namely, Antoine–Blankinship–Ivanov necklaces and Krushkal sticky sets. Our basic tools are Antoine’s methods and Shtan’ko demension theory.

Keywords: Euclidean space, embedding, equivalence of embeddings, disjoint embeddings, dimension, Cantor set, Menger compactum, tame embedding, wild embedding, demension (= dimension of embedding)

2000 MSC: Primary 57N35; Secondary 57N45, 57N12, 57N13, 57N15, 57M30.

1. Introduction

The central theme of this work is disjoint embeddings: we would like to find uncountably many copies of a given compactum in $\mathbb{R}^N$ simultaneously,
so that they are mutually exclusive. This kind of problems goes back to the result of R.L. Moore (1928): any family of pairwise non-intersecting triodes in the plane is at most countable. Continua that can be placed in $\mathbb{R}^2$ in an uncountable number of disjoint homeomorphic copies satisfy further restrictions on their topological structure, but many questions still remain open [43], [31].

Placing an uncountable family of disjoint homeomorphic copies of a given compactum in the same ambient space imposes certain restrictions not only on the compactum itself, but also on the behaviour of the embeddings. As an example: concentric spheres in $\mathbb{R}^3$ of arbitrary radii $r > 0$ form a family of cardinality continuum. In contrast to this, by results of R.H. Bing, it is impossible to place an uncountable collection of pairwise disjoint wild closed surfaces in $\mathbb{R}^3$ (Definition 4). A short sketch of Bing’s idea can be found in [15, Thm. 3.6.1]; see also [30] for detailed references. For $N \geq 5$ there is a similar impossibility theorem for wild $(N-1)$-spheres in $\mathbb{R}^N$ [16, Thm. 10.5], [24, p. 383, Thm. 3C.2], [13, Thm. 1, 2].

This paper concerns with those collections of pairwise disjoint continua which are not only homeomorphic, but ambiently homeomorphic:

**Definition 1.** Two subsets $X, X' \subset \mathbb{R}^N$ are *ambiently homeomorphic* (or equivalently embedded) if there exists a homeomorphism $h$ of $\mathbb{R}^N$ onto itself such that $h(X) = X'$. In this case, we will write $h : (\mathbb{R}^N, X) \cong (\mathbb{R}^N, X')$.

As an example, let us recall the result of J.H. Roberts: for a “nicely embedded” snake-like continuum $K \subset \mathbb{R}^2$ there is an embedding $F : K \times C \to \mathbb{R}^2$ such that each $F(K \times \{t\})$, where $t \in C$, is embedded equivalently to the given $K$ [45, Thm. 1]. (Here $C$ is the Cantor set.) E.K. van Douwen proved that if a separable completely metrizable space $Y$ contains uncountably many mutually disjoint homeomorphic copies of a given compactum $X$, then $Y$ contains the product $X \times C$ [26, Thm. 1]; see [4, Thm. 2.2], [52, Thm. 1] for different proofs and further improvements. In general, we can not assert that the resulting images of $X \times \{t\}$, for $t \in C$, are embedded equivalently to each other or to some given copy of $X$.

By the classical Lefschetz–Menger–Nöbeling–Pontryagin–Tolstowa Embedding Theorem, each $k$-dimensional compactum embeds into $\mathbb{R}^{2k+1}$. In the case of a $k$-dimensional polyhedron $X$, the product $X \times I$ embeds into $\mathbb{R}^{2k+1}$ [44, Thm. 1.5]; but this statement does not “see” how “individual” images of $X \times \{t\}$, $t \in I$, are embedded.
1.1. The question of B.J. Baker and M. Laidacker

B.J. Baker and M. Laidacker posed the question [3, p. 209]: Let \( X \subset \mathbb{R}^{2k+1} \) be a \( k \)-dimensional continuum; is it true that \( \mathbb{R}^{2k+1} \) contains a family of pairwise disjoint compacta \( \{X_\alpha \mid \alpha \in \mathcal{A}\} \), where \( |\mathcal{A}| = c \) and \( \left( \mathbb{R}^{2k+1}, X_\alpha \right) \cong \left( \mathbb{R}^{2k+1}, X \right) \) for each \( \alpha \in \mathcal{A} \)?

Corollary 2 of [3] gives a positive answer under an additional “niceness” assumption on the embedding of \( X \); in general, the question remained open. In Section 2, we present two series of examples showing that the answer can be twofold, depending on more subtle properties of the given embedding \( X \subset \mathbb{R}^{2k+1} \).

In order to clarify which part of the Baker–Laidacker question is open, we need further preparations.

1.2. On (ambiently) universal spaces

In 1916, W. Sierpiński described a curve which is now well-known as the Sierpiński carpet, and proved that each 1-dimensional planar compactum can be embedded in the carpet (by that reason, the term “the Sierpiński universal plane curve” is also used). In 1921, he proved that the Cantor set is a universal space for the class of all zero-dimensional metrizable compacta.

In 1926, K. Menger defined a \( k \)-dimensional generalization \( M^k_N \subset \mathbb{R}^N \) of both the Cantor set and the Sierpiński carpet. The construction is recalled in Section 1.3. Menger conjectured that any \( k \)-dimensional compactum embeds into \( \mathbb{R}^{2k+1} \), and proved it for \( k = 1 \). For arbitrary \( k \) this was proven in 1931 independently by G. Nöbeling; by L.S. Pontrjagin and G. Tolstowa; and by S. Lefschetz. Menger conjectured that \( M^k_N \) is a universal space for the class of all \( k \)-dimensional compact subsets of \( \mathbb{R}^N \), and proved it for two cases: \( k = 1, N = 3 \); and \( k = N - 1 \) (for references, see [22]; the pictures of the Menger universal curve \( M^1_3 \) also known as the Menger sponge, the Menger cube, the Sierpiński cube, or the Sierpiński sponge, can be found in [25, p. 131–132]).

Embeddability of any \( k \)-dimensional compact metric space into \( M^k_{2k+1} \) was shown by S. Lefschetz [38]. In its full generality, Menger’s conjecture was proven by M.A. Shtan’ko: each \( k \)-dimensional compactum \( X \) embeddable in \( \mathbb{R}^N \) can be embedded into \( M^k_N \) [49, Thm. 1]; see also [51] or [25, Cor. 5.5.4]. But if \( X \) is already embedded in \( \mathbb{R}^N \), we can not guarantee that \( X \) can be position-wise embedded in \( M^k_N \) in the following sense:

**Definition 2.** Let \( X, Z \subset \mathbb{R}^N \). We say that \( X \) can be *ambiently embedded in* \( Z \), or \( X \) can be *position-wise embedded in* \( Z \) [3], if there exists a homeomorphism \( h \) of \( \mathbb{R}^N \) onto itself such that \( h(X) \subset Z \).
For $(k, N) = (0, 1)$ and $(k, N) = (1, 2)$ the existence of an ambient homeomorphism $h : \mathbb{R}^N \cong \mathbb{R}^N$ that takes a given $k$-dimensional compactum $X$ to $h(X) \subset M_k^N$ follows from arguments of Sierpiński, and for $(k, N) = (0, 2)$ — from Antoine’s result [35, Cor. 2.3.2], [42, Thm. 13.7]. In general, the existence of such an $h$ is equivalent to the inequality $\text{dem} X \leq k$ [49, Thm. 2], [51], [25, Thm. 3.5.1] (partial results were also obtained in [12], [14]). Here “dem” denotes “dimension”. This word does not contain mistakes: it abbreviates “dimension of embedding” introduced by M.A. Shtan’ko. The inequality $\text{dem} X \leq k$ means that $X \subset \mathbb{R}^N$ behaves geometrically much like a polyhedron of dimension $\leq k$. As an example, $\text{dem} M_k^N = k$.

Shtan’ko dimension theory was developed with the aim of extending the notion of tameness from polyhedra (Section 1.5) to arbitrary compacta, see [48], [51], [28], [25], 3.4, 3.5], [20], [41]. We will not set forth the foundations of this theory in the present paper, indicating the suitable source if necessary.

1.3. The construction of Menger compacta $M_k^N$

Let $0 \leq k \leq N$. Take the standard cube $I^N = [0, 1]^N \subset \mathbb{R}^N$. Let $\mathcal{T}_j$ be the collection of all $N$-cubes obtained by subdividing $I^N$ into $3^{jN}$ congruent $N$-cubes by hyperplanes drawn perpendicular to the edges of $I^N$ at points dividing the edges into $3^j$ equal segments. The compactum $M_k^N$ is constructed inductively. The family $\mathcal{F}_0$ consists of the only element $I^N$. The collection $\mathcal{F}_{j+1}$ consists of all $v \in \mathcal{T}_{j+1}$ with the property: $v$ is a subset of some element $w \in \mathcal{T}_j$ such that $v$ intersects some $k$-face of $w$. Now let $P_j$ be the union of all elements of $\mathcal{F}_j$. The $k$-dimensional Menger compactum constucted in $\mathbb{R}^N$ is defined by

$$M_k^N = \bigcap_{j=0}^{\infty} P_j.$$

Taking $k = 0$, we get $M_0^N = C^N \cong C$, where $\cong$ stands for a homeomorphism.

For $1 \leq k \leq N$, the compactum $M_k^N$ is connected, therefore it is also called the $k$-dimensional Menger continuum constucted in $\mathbb{R}^N$. We also say the standard $k$-dimensional Menger continuum in $\mathbb{R}^N$, especially when we consider other inequivalent embeddings $M_k^N \hookrightarrow \mathbb{R}^N$. Observe that

$$C^N = M_0^N \subset M_1^N \subset \ldots \subset M_{N-1}^N \subset M_N^N = I^N.$$

For $N \geq 2k + 1$ we have $M_k^N \cong M_{2k+1}^N$. 

4
Let us emphasize that $M^k_N \subset \mathbb{R}^N$ is thought as a specific subset rather than an abstract compactum; it should be distinguished from an arbitrary embedding $M^k_N \hookrightarrow \mathbb{R}^N$ (compare with Proposition 1).

1.4. The case considered by B.J. Baker and M. Laidacker

B.J. Baker and M. Laidacker constructed, for any $k \geq 0$, an embedding $\Xi : M^{k}_{2k+1} \times \mathcal{C} \rightarrow \mathbb{R}^{2k+1}$ such that all compacta $\Xi(M^{k}_{2k+1} \times \{t\})$, where $t \in \mathcal{C}$, are obtained from each other by parallel translations. Moreover, by [49, Prop. 2] (see also [51, Prop. 9] or [25, Prop. 3.5.2]) each of $\Xi(M^{k}_{2k+1} \times \{t\})$ is embedded equivalently to the standard Menger compactum $M^k_{2k+1} \subset \mathbb{R}^{2k+1}$ described in Section 1.3. Hence their question (Section 1.1) has a positive answer for those $X \subset \mathbb{R}^{2k+1}$ that satisfy $\text{dim} \ X \leq k$ (equivalently, for $X$ ambiently embeddable into $M^k_{2k+1}$ [3, Cor. 2]). Our paper is devoted to those $X$ that cannot be ambiently embedded into $M^k_{2k+1}$.

Remark 1. [3, Cor. 2] strengthens the classical Lefschetz–Menger–Nöbeling–Pontryagin–Tolstowa theorem since each $k$-dimensional compactum can be embedded into the standard Menger compactum $M^k_{2k+1} \subset \mathbb{R}^{2k+1}$ [38].

1.5. On wild embeddings

Embeddings in our examples are wild. Theory of wild embeddings appeared at the beginning of the 20th century, in attempts to rigorously prove the Schoenflies theorem [42, Thm. 9.6, 10.3] and to generalize it to higher dimensions. Let us recall

Definition 3. A zero-dimensional compact set $X \subset \mathbb{R}^N$ is called tame if there exists a homeomorphism $h$ of $\mathbb{R}^N$ onto itself such that $h(X)$ lies on a straight line; otherwise, $X$ is called wild.

Definition 4. A subset of $\mathbb{R}^N$ is called a polyhedron if it is the union of a finite collection of simplices. A compactum $X \subset \mathbb{R}^N$ homeomorphic to a polyhedron is called tame if there exists a homeomorphism $h$ of $\mathbb{R}^N$ onto itself such that $h(X)$ is a polyhedron in $\mathbb{R}^N$; otherwise, $X$ is wild.

In 1921, L. Antoine proved that each zero-dimensional compactum in $\mathbb{R}^2$ is tame. In 1920 he sketched and in 1921 explicitly constructed a first wild Cantor set in $\mathbb{R}^3$ now called Antoine’s necklace [42, Section 18]. His construction was extended to higher dimensions independently by A.A. Ivanov.
Besides widely known Antoine’s necklace, there is an essentially different construction of a wild Cantor set in $\mathbb{R}^3$ given by P.S. Urysohn in 1922–23; for references, see [30]. Now, examples of wild Cantor sets in $\mathbb{R}^N$ include the ones with such strong properties as simply-connectedness of the complement, slipperiness, stickiness (see Section 3.2). The behaviour of wild Cantor sets resembles that of polyhedra of codimension 2 [28, Thm. 1.4], [25, Thm. 3.4.11].

Examples of wild 2-spheres in $\mathbb{R}^3$ (L. Antoine 1921, J.W. Alexander 1924), wild arcs in $\mathbb{R}^3$ (L. Antoine 1921; R.H. Fox and E. Artin 1948) and even everywhere wild arcs in $\mathbb{R}^3$ (L. Antoine 1924) are well-known. For $N \geq 3$, each uncountable compact subset of $S^{N-1}$ can be embedded in $\mathbb{R}^N$ in uncountably many inequivalent wild ways (R.B. Sher 1968, D.G. Wright 1986). For further information on topological embeddings, refer to [35], [46], [42], [25], [18], [56], [15], [16], [24]. For embeddings of arbitrary compact sets, the concept of tameness was introduced and deeply studied by M.A. Shtan’ko [48]–[51]. See Section 1.2 and references therein.

Our examples are based on Antoine’s ideas and methods, and on results of Shtan’ko.

1.6. Notation and agreements

All spaces are supposed to be metrizable, all maps are continuous. As a rule, the metric is denoted by $d$, this does not lead to ambiguous understanding.

$\mathbb{R}^N$ denotes Euclidean $N$-dimensional space with the usual metric. $S^N$ is the standard unit sphere in $\mathbb{R}^{N+1}$ with the induced metric. $I = [0, 1]$.

The symbol $X$ is used for an abstract space, to distinguish it from its homeomorphic copies which are embedded into a space $Y$; such embedded copies of $X$ are denoted by $X$ or $X_\alpha$.

For a subset $A$ of a metric space $Y$, the symbol $\overline{A}$ denotes the closure, $\hat{A}$ the interior, and $O_\varepsilon(A) = \{y \in Y \mid d(y, A) < \varepsilon\}$ the $\varepsilon$-neighbourhood. For non-empty compact subsets $A, B \subset Y$, the distance $d(A, B)$ is the minimum of distances between $a \in A$ and $b \in B$.

The symbol $Y \cong Y$ denotes a homeomorphism of $Y$ onto itself. Similarly, $h : (Y, A) \cong (Y, A)$ is a homeomorphism of pairs, i.e. a homeomorphism $h : Y \cong Y$ such that $h(A) = A$.

A self-homeomorphism $h$ of a metric space $Y$ is called an $\varepsilon$-homeomorphism if $d(x, h(x)) \leq \varepsilon$ for each $x \in Y$. 

6
For a homeomorphism $h : Y \cong Y$, its support $\text{supp } h$ is the closure of the set $\{x \in Y \mid h(x) \neq x\}$.

As usual, $\text{id}$ is the identity map.

For a topological manifold $M$, its boundary is denoted by $\partial M$.

$C$ is the Cantor set.

$| \cdot |$ is the cardinality of a set; $c = |\mathbb{R}|$ is the cardinality of the continuum. $\mathcal{A}$ is usually an index set.

2. Statements

2.1. Negative examples

In our first “negative” result, the dimension of the ambient space is at least 4. The 3-dimensional case is treated below, in Proposition 1.

**Theorem 1.** For any $N \geq 4$ and any uncountable compactum $\mathfrak{X} \subset S^{N-1}$ there exists an embedding $f : \mathfrak{X} \to \mathbb{R}^N$ such that
1) the image $X := f(\mathfrak{X})$ can not be position-wise embedded in $M^{N-3}_N$, and
2) it is impossible to place in $\mathbb{R}^N$ uncountably many pairwise disjoint compacta ambiently homeomorphic to $X$.

Theorem 1 is proved in Section 3.3

As a consequence, we get “negative” examples for the Baker–Laidacker problem, assuming that the dimension of the ambient space is at least 5:

**Corollary 1.** For each $k \geq 2$ there exists an embedding $f : I^k \to \mathbb{R}^{2k+1}$ such that
1) the $k$-cell $Q := f(I^k)$ can not be position-wise embedded into $M^{k}_{2k+1}$ (and even into $M^{2k-2}_{2k+1}$), and
2) it is impossible to place in $\mathbb{R}^{2k+1}$ uncountably many pairwise disjoint $k$-cells embedded equivalently to $Q$.

Corollary 1 can not be “word-to-word” extended to the case of $2k+1 = 3$, since each arc in $\mathbb{R}^3$ can be position-wise embedded in $M^1_3$ [12, Satz 4]. That is, desired examples should have a more complicated structure.

**Proposition 1.** There exist an embedding $f : M^1_3 \to \mathbb{R}^3$ and a closed broken line $L \subset \mathbb{R}^3$ such that the compactum $X = L \cup f(M^1_3)$ has the following properties:
1) $X$ is connected,
2) \( \dim X = 1 \),
3) \( X \) can not be position-wise embedded into \( M^1_3 \),
4) it is impossible to place in \( \mathbb{R}^3 \) uncountably many pairwise disjoint compacta embedded equivalently to \( X \).

**Proof.** The construction of the standard Menger continuum \( M^1_3 \subset \mathbb{R}^3 \) can be thought of as starting from \( I^3 \) and drilling through rectangular channels. If we drill knotted channels instead, taking care of their size and position, the resulting compactum \( T \subset \mathbb{R}^3 \) will be homeomorphic to \( M^1_3 \) by the Anderson Characterization Theorem \([1, \text{Thm. XII}]\); this is the image of the desired embedding: \( T = f(M^1_3) \). And it will be impossible to remove some unknotted closed broken line \( L \) from \( T \) by a small push. With details and proof, such an embedding is presented in \([11]\); with less details, a similar example is described in \([40]\). Without proof, a similar construction was given in \([29, \text{p. 788–789}]\) in connection with the problem of coincidence of Brouwer–Menger–Urysohn and Alexandroff dimensions. Thereby, we will refer to the paper \([11]\) containing maximum details. The desired properties follow from Bothe’s results. Indeed, 1) is by construction. 2) is proved in \([11]\). The set \( T \) can not be position-wise embedded in \( M^1_3 \) by \([11]\), this implies 3). Finally, let us prove 4). Suppose the contrary; by Corollary \([4]\) for any \( \varepsilon > 0 \) there is a homeomorphism \( g : \mathbb{R}^3 \cong \mathbb{R}^3 \) such that \( X \cap g(X) = \emptyset \) and \( d(x, g(x)) \leq \varepsilon \) for each \( x \in X \). Hence \( T \cap g(L) = \emptyset \); this is contradicted by \([11, \text{p. 255}]\).

**Remark 2.** In Proposition \([11]\) the compactum \( T = f(M^1_3) \) itself already has properties 1)–3). We conjecture that \( T \) also has property 4); verification of this would require additional reasoning which, in our opinion, can be carried out similarly to \([11]\).

### 2.2. Affirmative examples

**Theorem 2.** Suppose that \( N \geq 4, 1 \leq k \leq N - 3, \) and \( 2k + 1 \leq N \). Let \( P \subset \mathbb{R}^N \) be a \( k \)-dimensional polyhedron such that for some point \( a \in P \), its neighbourhood in \( P \) is homeomorphic to the interval \((0, 1)\). Then there exists an embedding \( F : P \times C \to \mathbb{R}^N \) with the properties:
1) all compacta \( F(P \times \{t\}) \), where \( t \in C \), are embedded into \( \mathbb{R}^N \) equivalently to each other, and
2) no one of \( F(P \times \{t\}) \), where \( t \in C \), can be position-wise embedded into \( M^{N-3}_N \) (hence also into \( M^k_N \)).
For the proof, see Section 3.4. As a consequence, we get a series of “affirmative” examples for the Baker–Laidacker question:

**Corollary 2.** Let \( k \geq 2 \). Let \( P \) be a \( k \)-dimensional compact polyhedron \( P \subset \mathbb{R}^{2k+1} \) such that for some point \( a \in P \), its neighbourhood in \( P \) is homeomorphic to the interval \((0,1)\). Then there exists an embedding \( F : P \times \mathcal{C} \to \mathbb{R}^{2k+1} \) such that the compacta \( F(P \times \{ t \}), t \in \mathcal{C}, \) are pairwise ambiently homeomorphic, and no one of them can be position-wise embedded into \( M_{2k+1}^k \) (and even into \( M_{2k+1}^{2k-2} \)).

**Remark 3.** In [30], for each \( N \geq 3 \) and each \( 1 \leq k \leq N - 1 \) we described an embedding \( F : I^k \times \mathcal{C} \to \mathbb{R}^N \) such that all \( k \)-cells \( F(I^k \times \{ t \}), t \in \mathcal{C}, \) are inequivalently embedded in \( \mathbb{R}^N \); no one of these cells can be position-wise embedded in \( M_{N-3}^N \). (For \( N = 2 \), the situation is different: any Cantor fence \( I \times \mathcal{C} \hookrightarrow \mathbb{R}^2 \) is ambiently equivalent to the standard one [3, Thm. 2], [33].) We believe that both Theorem 2 and Corollary 2 hold true for \( P = I^k \); this probably can be obtained using ideas from [30].

Recall that the question of Baker and Laidacker concerns connected spaces (continua). But it also makes sense for non-connected spaces. In this case, another construction is possible:

**Theorem 3.** For each \( N \geq 3 \) and each \( 0 \leq k \leq N - 3 \) there exists an embedding \( F : (I^k \times \mathcal{C}) \times \mathcal{C} \to \mathbb{R}^N \) such that
1) all compacta \( F((I^k \times \mathcal{C}) \times \{ t \}), t \in \mathcal{C}, \) are embedded in \( \mathbb{R}^N \) equivalently to each other, and
2) no one of these compacta can be position-wise embedded into \( M_{N-3}^N \) (hence into \( M_N^N \) also).

Taking \( k \geq 2 \) and \( N = 2k + 1 \) in Theorem 3, we get new affirmative examples for the Baker–Laidacker problem, except for the connectivity requirement. The proof of Theorem 3 is given in Section 3.5.

3. Proofs

3.1. Main observations regarding disjoint embeddings

In our proofs we rely on the following assertion. It is probably well known (at least partially), but we did not find it in the literature. The proof does not contain fundamentally new ideas; however, we present it for completeness.
Condition (iii) is rather strong, as is shown in [17] and in a recent arxiv preprint “An answer to a question of J.W. Cannon and S.G. Wayment” by the author.

**Theorem 4.** Let \( \mathcal{X}, Y \) be compact spaces. Let \( e : \mathcal{X} \to Y \) be an embedding; denote \( X = e(\mathcal{X}) \). The following are equivalent:

(i) there exists an embedding \( \Psi : \mathcal{X} \times \mathcal{C} \to Y \) such that \( (Y, \Psi(\mathcal{X} \times \{t\})) \cong (Y, X) \) for each \( t \in \mathcal{C} \),

(ii) there exists a collection \( \{X_\alpha \mid \alpha \in \mathcal{A}\} \) of \( |\mathcal{A}| = \mathfrak{c} \) pairwise disjoint subsets of \( Y \) such that \( (Y, X_\alpha) \cong (Y, X) \) for any \( \alpha \in \mathcal{A} \),

(iii) for any \( \varepsilon > 0 \) there exists a homeomorphism \( h : Y \cong Y \) such that \( X \cap h(X) = \emptyset \) and \( d(x, h(x)) \leq \varepsilon \) for each \( x \in Y \).

**Proof.** Let \( d \) be any fixed metric on \( Y \), and endow the space \( C(Y, Y) \) of continuous maps with the distance \( \rho(f, g) = \sup\{d(f(x), g(x)) \mid x \in Y\} \).

\((iii) \Rightarrow (i)\) is proved by the standard Cantor tree type argument, similarly to [45], [9]; compare with [25, Exercise 4.8.1 and p. 180, Remark], [27].

By compactness of \( Y \), property \((iii)\) remains valid after replacing \( X \) by any \( X' \) such that \( (Y, X') \cong (Y, X) \).

Let \( \{\varepsilon_m\} \) be a sequence of positive numbers. (In fact, each of its elements will be selected based on what has already been done in the previous step; the details will become clear later.)

**Step 1.** Apply \((iii)\) to \( X \). Take homeomorphisms \( h_0, h_1 : Y \cong Y \) such that \( \rho(h_0, id_Y) \leq \varepsilon_1 \), \( \rho(h_1, id_Y) \leq \varepsilon_1 \), \( h_0(X) \cap h_1(X) = \emptyset \).

**Step 2.** Apply \((iii)\) to \( h_0(X) \) and to \( h_1(X) \). Take homeomorphisms \( h_{00}, h_{01}, h_{10}, h_{11} : Y \cong Y \) such that

\[
\max\{\rho(h_{00}, id_Y), \rho(h_{01}, id_Y), \rho(h_{10}, id_Y), \rho(h_{11}, id_Y)\} \leq \varepsilon_2,
\]

and

\[
h_{00}h_0(X), \ h_{01}h_0(X), \ h_{10}h_1(X), \ h_{11}h_1(X)
\]

are pairwise disjoint.

We continue in the same way. On **Step m** we apply \((iii)\) to \( 2^{m-1} \) compacta constructed on **Step** \( (m-1) \), and obtain a collection of homeomorphisms \( \{h_{a_1a_2...a_m} \mid a_1, \ldots, a_m \in \{0;1\}\} \) such that

\[
\max\{\rho(h_{a_1a_2...a_m}, id_Y) \mid a_1, \ldots, a_m \in \{0;1\}\} \leq \varepsilon_m
\]
and all $2^m$ compacta 

$$h_{a_1a_2...a_m}h_{a_1a_2...a_{m-1}}...h_{a_1a_2h_{a_1}(X)} \text{ for } a_1, \ldots, a_m \in \{0; 1\}$$

are pairwise disjoint. Assume that the sequence $\{\varepsilon_m\}$ is rapidly decreasing, so that we have: for each $\sigma = (a_1, a_2, a_3, \ldots) \in \{0, 1\}^\mathbb{N}$ the sequence

$$h_{a_1}, h_{a_1a_2}, h_{a_1a_2a_3}h_{a_1a_2}h_{a_1}, \ldots$$

converges to a homeomorphism $h_\sigma : Y \cong Y$ \cite{[4], [8], Thm. 7}, \cite{[25], Prop. 2.2.2}.

Fix any homeomorphism $\xi : C \cong \{0; 1\}^\mathbb{N}$. The desired embedding $\Psi$ is defined by

$$X \times C \xrightarrow{e \times \xi} X \times \{0; 1\}^\mathbb{N} \to Y, \quad (x, t) \mapsto (e(x), \xi(t)) \mapsto h_{\xi(t)}(e(x)).$$

For $t, t' \in C$ the desired homeomorphism $(Y, \Psi(X \times \{t\})) \cong (Y, \Psi(X \times \{t'\}))$ can be defined as the composition $h_{\xi(t')}(h_{\xi(t)})^{-1}$.

$(i) \Rightarrow (ii)$ is evident.

$(ii) \Rightarrow (iii)$. For each $\alpha \in A$ fix a homeomorphism $g_\alpha : (Y, X) \cong (Y, X_\alpha)$. The space of continuous maps $C(Y, Y)$ is complete by compactness of $Y$. Hence the uncountable set $\{g_\alpha \mid \alpha \in A\}$ contains a countable subset of pairwise distinct elements $f_*, f_1, f_2, \ldots$ such that $\lim_{m \to \infty} \rho(f_m, f_*) = 0$. Let $h_m := (f_*)^{-1}f_m : Y \cong Y$. We have

$$X \cap h_m(X) = (f_*)^{-1}(f_*(X) \cap f_m(X)) = (f_*)^{-1}(\emptyset) = \emptyset.$$ 

It is easy to verify that $\lim_{m \to \infty} \rho(\text{id}_Y, h_m) = 0$, hence $X$ satisfies $(iii)$.

Remark 4. Analyzing the proof of $(ii) \Rightarrow (iii)$ of Theorem 4 we easily see that in the case of $Y = S^N$ (or any oriented closed manifold), $h$ can be taken to be orientation-preserving.

Remark 5. In the case of $Y = \mathbb{R}^N$, conditions $(i)$ and $(ii)$ of Theorem 4 are equivalent. This can be shown following the same proof, applied to the one-point compactification $S^N$; all maps $g_\alpha$ and $h_{a_1...a_m}$ should be taken to preserve the added compactifying point.
Corollary 3. Let $\mathfrak{X}$ be a compact space. Let $e : \mathfrak{X} \to \mathbb{R}^N$ be an embedding, and denote $X = e(\mathfrak{X})$. Suppose that for any $\varepsilon > 0$ there exists a homeomorphism $h : \mathbb{R}^N \cong \mathbb{R}^N$ with the properties:
1) $X \cap h(X) = \emptyset$,
2) $\text{supp } h \subset O_\varepsilon(X)$, and
3) $d(x, h(x)) \leq \varepsilon$ for each $x \in \mathbb{R}^N$.

Then there exists an embedding $\psi : \mathfrak{X} \times C \to \mathbb{R}^N$ such that $\left(\mathbb{R}^N, \psi(\mathfrak{X} \times \{t\})\right) \cong \left(\mathbb{R}^N, X\right)$ for each $t \in C$.

Proof. Let $Y$ be a closed ball in $\mathbb{R}^N$ that contains $X$ in its interior. By assumptions, for each $\varepsilon > 0$ there exists an $\varepsilon$-homeomorphism $f : Y \cong Y$ such that $X \cap f(X) = \emptyset$ and $f|_{\partial Y} = \text{id}$. By Theorem 4 we get an embedding $g : \mathfrak{X} \times C \to Y$ such that $\left(Y, g(\mathfrak{X} \times \{t\})\right) \cong \left(Y, X\right)$ for each $t \in C$. The desired embedding $\psi$ can be obtained as the composition of $g$ and the inclusion $Y \subset \mathbb{R}^N$.

Corollary 4. Let $X \subset \mathbb{R}^N$ be a compact set. Suppose that there exists an uncountable collection $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ of subsets of $\mathbb{R}^N$ such that $\left(\mathbb{R}^N, X_\alpha\right) \cong \left(\mathbb{R}^N, X\right)$ for each $\alpha \in \mathcal{A}$.

Then for each bounded open subset $U \subset \mathbb{R}^N$ containing $X$ and for each $\varepsilon > 0$ there exists an orientation-preserving homeomorphism $g : \mathbb{R}^N \cong \mathbb{R}^N$ such that
1) $X \cap g(X) = \emptyset$, and
2) $d(x, g(x)) \leq \varepsilon$ for any $x \in U$.

Observe that we do not require $g(U)$ to be a subset of $U$.

Proof. Let $q \in S^N$ be the north pole. Let $p : S^N \setminus \{q\} \to \mathbb{R}^N$ be the stereographic projection map, and $i = p^{-1} : \mathbb{R}^N \to S^N$. In this proof, $d$ denotes the distance in $\mathbb{R}^N$; the standard distance in $S^N$ induced from $\mathbb{R}^{N+1}$ is denoted by $D$.

Suppose that $U$ and $\varepsilon$ are given. Take a bounded open subset $V \subset \mathbb{R}^N$ such that $\overline{U} \subset V$.

For brevity, let $\hat{X} := i(X)$, $\hat{X}_\alpha := i(X_\alpha)$, $\hat{U} := i(U)$, $\hat{V} := i(V)$. Observe that $\left(S^N, \hat{X}_\alpha\right) \cong \left(S^N, \hat{X}\right)$ for each $\alpha \in \mathcal{A}$.

The closure of $\hat{V}$ in $S^N$ is a compact set which does not contain $q$. Hence the restriction map $p|_{\hat{V}} : \hat{V} \to V$ is well-defined and uniformly continuous.
Choose a $\gamma > 0$ such that for any $x, x' \in \hat{V}$ with $D(x, x') \leq \gamma$, we have $d(p(x), p(x')) \leq \varepsilon$. We may moreover assume that

$$O_\gamma(\hat{U}) \subset \hat{V} \quad \text{and} \quad O_\gamma(\hat{U}) \cap O_\gamma(q) = \emptyset.$$  

Using Remark 4, take an orientation-preserving homeomorphism $h : S^N \cong S^N$ such that $\hat{X} \cap h(\hat{X}) = \emptyset$ and $D(x, h(x)) \leq \frac{\gamma}{2}$ for each $x \in S^N$. In particular, we have $D(q, h(q)) \leq \frac{\gamma}{2}$. Take an orientation-preserving homeomorphism $f : S^N \cong S^N$ such that $f h(\hat{X}) = \hat{X}$ and $D(q, f(x)) \leq \frac{\gamma}{2}$ for each $x \in S^N$. Consequently, $\hat{x}$ and $h(\hat{x})$ lie in $O_\gamma(\hat{U}) \subset \hat{V}$. Together with the choice of $\gamma$, this implies

$$d(x, g(x)) = d(p(\hat{x}), ph'(\hat{x})) \leq \varepsilon.$$  

Corollary 4 is proved.

3.2. Pushing Cantor sets off themselves

R.J. Daverman conjectured [22, Conj. 1] that for any two Cantor sets $X$ and $X'$ in $\mathbb{R}^N$ and any $\varepsilon > 0$ there is an $\varepsilon$-homeomorphism $h : \mathbb{R}^N \cong \mathbb{R}^N$ such that $X \cap h(X') = \emptyset$. This is known to be true for $N \leq 3$ [47, Thm. 1].

However, for each $N \geq 4$ V. Krushkal constructed a sticky (wild) Cantor set in $\mathbb{R}^N$ [37, Thm. 1.1]: it cannot be isotoped off of itself by any sufficiently small ambient isotopy. (Compare [55].)

Modifying Alexander’s idea, J. Kister observed that each pair of $\varepsilon$-close homeomorphisms of $\mathbb{R}^N$ is connected by an ambient $\varepsilon$-isotopy [36, Thm. 1] (for general manifolds, see [19], [21]). Hence it is impossible to take the Krushkal set off itself by any sufficiently small ambient homeomorphism. This can also be derived directly from Krushkal’s arguments. In fact, Krushkal
constructed a Cantor set $K \subset \mathbb{R}^N$ together with a bounded open neighbourhood $U \subset \mathbb{R}^N$ and an $\varepsilon > 0$ so that the following is satisfied: if $h : \mathbb{R}^N \cong \mathbb{R}^N$ is an orientation-preserving homeomorphism satisfying $d(x, h(x)) \leq \varepsilon$ for each $x \in U$, then $K \cap h(K) \neq \emptyset$. Corollary 4 now implies:

**Corollary 5.** Let $K \subset \mathbb{R}^N$ be a Krushkal Cantor set, $N \geq 4$. Suppose that $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ is a family of pairwise disjoint Cantor sets in $\mathbb{R}^N$ such that each $X_\alpha$ is ambiently homeomorphic to $K$. Then $\mathcal{A}$ is no more than countable.

This Corollary will be used in the proof of Theorem 1.

**Remark 6.** The property mentioned immediately before Corollary 5 implies: if $K \subset \mathbb{R}^N$ is the Cantor set constructed by Krushkal and $g : \mathbb{R}^N \cong \mathbb{R}^N$ is a homeomorphism, then $g(K)$ is also a sticky Cantor set. For this reason, by the Krushkal Cantor set we will mean any Cantor set embedded equivalently to the concrete one described by Krushkal.

In contrast to Kruskal Cantor sets, it is possible to place in $\mathbb{R}^N$ a family of pairwise disjoint equivalently embedded Antoine–Blankinship–Ivanov Cantor sets (see Section 1.5). This will be used in the proof of Theorem 2.

The dimension of each wild Cantor set in $\mathbb{R}^N$ equals $N - 2$ (see [28, Thm. 1.4], [25, Thm. 3.4.11]; in the case of $N = 4$, refer to [50] or [7, p. 5]). The polyhedra constructed in our paper can not be position-wise embedded into $M_N^{N-3}$ since they contain wild Cantor sets.

### 3.3. Proof of Theorem 1

It is well-known that each uncountable compactum contains a Cantor set. Moreover, for any uncountable compactum $\mathcal{X} \subset S^{N-1}$ we can choose a Cantor set $\mathcal{X}_0 \subset \mathcal{X}$ such that $\mathcal{X}_0$ is cellularly separated in $S^{N-1}$: $\mathcal{X}_0 = \bigcap_{m=1}^{\infty} \bigcup_{\ell=1}^{\ell_m} V_{m,\ell}$, where each $V_{m,\ell} \subset S^{N-1}$ is homeomorphic to $\mathbb{R}^{N-1}$, the closure $\overline{V_{m,\ell}}$ of each $V_{m,\ell}$ in $S^{N-1}$ is a topological $(N - 1)$-cell, and $\overline{V_{m,\ell}} \cap \overline{V_{m,n}} = \emptyset$ for each $m \geq 1$ and each $1 \leq \ell \neq n \leq \ell_m$ [35, I.3, I.4].

Take a Krushkal Cantor set $K \subset \mathbb{R}^N$. Fix any homeomorphism $f_0 : \mathcal{X}_0 \cong K$. Let $F : S^{N-1} \to \mathbb{R}^N$ be any embedding that extends $f_0$. (The existence of such extension, even with the additional property of being piecewise-linear on $S^{N-1} \setminus \mathcal{X}_0$, is proved using “horn pulling method” which is introduced
in the works of L. Antoine and J. Alexander; see [30, Stat. 4] for detailed references. The idea can be found in [42, Thm. 18.6, 18.7], [25, Example 2.7.1].) Let $f := F|_X$ and $X := f(\mathcal{X})$. We have
\[ \text{dem } X = \text{dem } f(\mathcal{X}) \geq \text{dem } f(\mathcal{X}_0) = \text{dem } K = N - 2 > N - 3 = \text{dem } M_N^{N-3}, \]
which implies 1). Assertion 2) follows from Corollary 3.

3.4. Proof of Theorem 2

We will construct an embedding $f : P \to \mathbb{R}^N$ which satisfies assumptions of Corollary 3. There exist a point $a \in P$ and a number $\rho > 0$ such that $O_\rho(a) \cap P \cong (0,1)$. We may moreover assume that $O_\rho(a) \cap P$ is a straight line segment; denote it $J$ for brevity.

Take a Blankinship–Ivanov Cantor set $A \subset \mathbb{R}^N$ (this is a wild Cantor set constructed for $N \geq 4$ using solid tori $I^2 \times (S^1)^{N-2}$, analogously to Antoine’s Necklaces [10], [33], [34], [25, Example 4.7.1]). We may assume that $A \subset O_\rho(a)$.

As in the proof of Theorem 1, construct an embedding $f_0 : J \to O_\rho(a)$ such that $f_0|_{\partial J} = \text{id}$, $f_0(J) \supset A$, and the restriction of $f_0$ to $J \setminus (f_0)^{-1}(A)$ is piecewise-linear. Define an embedding $f : P \to \mathbb{R}^N$ by
\[ f(x) = \begin{cases} 
  x & \text{for } x \in P \setminus J, \\
  f_0(x) & \text{for } x \in J.
\end{cases} \]

For brevity, denote $\Sigma := f(J)$ and $\Pi := \overline{P \setminus J}$. We have $\Pi \cup \Sigma = f(P)$ and $\Pi \cap \Sigma = \partial \Sigma$. Replacing the segment $J$ with a smaller one if necessary, we may assume that $\dim \Pi = k$.

The compactum $f(P)$ can not be position-wise embedded into $M_N^{N-3}$ since
\[ \text{dem } f(P) \geq \text{dem } A = N - 2 > N - 3 = \text{dem } M_N^{N-3}. \]

In the rest of the proof, we show that $f$ satisfies assumptions of Corollary 3.

Step 1. Pushing a “singularity” $A$ off itself.

Take a positive number $\alpha$ such that $\alpha < d(\Pi, A)$. Recall that the set $A$ has a special structure: it is an intersection of sets which are the unions of disjoint solid tori $I^2 \times (S^1)^{N-2}$. By [25, p. 171, Remark] there exists an $\alpha$-homeomorphism $g_1 : \mathbb{R}^N \cong \mathbb{R}^N$ such that
\[ A \cap g_1(A) = \emptyset \quad \text{and} \quad \text{supp } g_1 \subset O_\rho(A). \]
By the choice of $\alpha$, we have

$$\Pi \cap g_1(A) = \emptyset \quad \text{and} \quad g_1|_{\Pi} = id.$$ 

**Step 2. Pushing a “singularity” off the arc $\Sigma$.**

Step 2 can be skipped putting $g_2 := id$ if $\Sigma \cap g_1(A) = \emptyset$. If not, do as follows. Take a positive number $\beta$ such that

$$\beta < d(\Pi \cup A, g_1(A)) \quad \text{and} \quad \alpha + \beta < d(\Pi, A).$$

Let us construct a $\beta$-homeomorphism $g_2 : \mathbb{R}^N \cong \mathbb{R}^N$ such that

$$\Sigma \cap g_2 g_1(A) = \emptyset \quad \text{and} \quad supp g_2 \subset O_\beta(\Sigma \cap g_1(A)).$$

To prove that the desired $g_2$ exists, consider $L := \Sigma \setminus O_\beta(A)$. By construction, $L$ is a finite 1-dimensional polyhedron. By the choice of $\beta$, we have

$$\Sigma \cap g_1(A) \cap O_\beta(A) \subset g_1(A) \cap O_\beta(A) = \emptyset,$$

therefore

$$\Sigma \cap g_1(A) = L \cap g_1(A).$$

Recall that $\text{dem} g_1(A) = \text{dem} A = N - 2$, and apply the General Position Theorem [51, Thm. 10], [23, Cor. 3.4.7] to $L$ and $g_1(A)$ in the $\beta$-neighbourhood of their intersection. We thus get a desired $g_2$.

Observe that

$$(\Pi \cup \Sigma) \cap g_2 g_1(A) = \emptyset.$$ 

**Step 3. Pushing the rest part of $\Sigma$ off the initial compactum $\Pi \cup \Sigma$.**

At Step 3, we will construct a special self-homeomorphism $g_3$ of $\mathbb{R}^N$ whose properties include the equality

$$(\Pi \cup \Sigma) \cap g_3 g_2 g_1(\Sigma) = \emptyset.$$ 

Take a positive number $\gamma$ such that

$$\gamma < d(\Pi \cup \Sigma, g_2 g_1(A)) \quad \text{and} \quad \alpha + \beta + \gamma < d(\Pi, A).$$

Take an open neighbourhood $W$ of $A$ in $\mathbb{R}^N$ such that $O_\gamma(g_2 g_1(A)) = g_2 g_1(W)$. Denote $L' := \Sigma \setminus W$. By construction, $L'$ is a finite polyhedron with

$$\dim L' \leq 1,$$

and

$$(\Pi \cup \Sigma) \cap g_2 g_1(\Sigma \setminus L') \subset (\Pi \cup \Sigma) \cap g_2 g_1(W) = (\Pi \cup \Sigma) \cap O_\gamma(g_2 g_1(A)) = \emptyset.$$
Therefore
\[(\Pi \cup \Sigma) \cap g_2g_1(\Sigma) = (\Pi \cup \Sigma) \cap g_2g_1(L').\]

Recall that
\[\text{dem}(\Pi \cup \Sigma) = N - 2 \quad \text{and} \quad \text{dem } g_2g_1(L') = \text{dim } L' = \text{dim } L' \leq 1.\]

By the General Position Theorem [51, Thm. 10], [25, Cor. 3.4.7], there exists a \(\gamma\)-homeomorphism \(g_3 : \mathbb{R}^N \cong \mathbb{R}^N\) such that
\[(\Pi \cup \Sigma) \cap g_3g_2g_1(L') = \emptyset \quad \text{and} \quad \text{supp } g_3 \subset O_{\gamma}((\Pi \cup \Sigma) \cap g_2g_1(L')) \tag{1}\]

By construction, \(g_3\) restricted on \(O_{\gamma}(g_2g_1(A)) = g_2g_1(W)\) is the identity map, hence
\[(\Pi \cup \Sigma) \cap g_3g_2g_1(\Sigma \cap W) = (\Pi \cup \Sigma) \cap g_2g_1(W) = \emptyset \tag{2}\]

From (1) and (2) we get the desired property:
\[(\Pi \cup \Sigma) \cap g_3g_2g_1(\Sigma) = \emptyset.\]

In addition, the inequality \(\alpha + \beta + \gamma < d(\Pi, A)\) implies
\[A \cap g_3g_2g_1(\Pi) = \emptyset.\]

**Step 4. Pushing the rest part of the compactum \(\Pi \cup \Sigma\) off itself.**

Take a positive number \(\delta\) such that
\[\delta < \min\{d(\Pi \cup \Sigma, g_3g_2g_1(\Sigma)); \quad d(A, g_3g_2g_1(\Pi))\}.\]

Denote \(L'' := \Sigma \setminus O_{\delta}(A)\). Again, \(L''\) is a finite polyhedron with \(\text{dim } L'' \leq 1\).

Observe that
\[\text{dem}(\Pi \cup L'') = k \quad \text{and} \quad \text{dem } g_3g_2g_1(\Pi) = \text{dim } \Pi = k.\]

By assumption, \(2k + 1 \leq N\). Applying the General Position Theorem to \(\Pi \cup L''\) and \(g_3g_2g_1(\Pi)\), we find a \(\delta\)-homeomorphism \(g_4 : \mathbb{R}^N \cong \mathbb{R}^N\) such that
\[(\Pi \cup L'') \cap g_4g_3g_2g_1(\Pi) = \emptyset \quad \text{and} \quad \text{supp } g_4 \subset O_{\delta}((\Pi \cup L'') \cap g_3g_2g_1(\Pi)).\]

By construction, \(O_{\delta}(A) \cap g_4g_3g_2g_1(\Pi) = \emptyset\). Hence
\[(\Pi \cup \Sigma) \cap g_4g_3g_2g_1(\Pi) = \emptyset \tag{3}\]
Observe that $g_4$ restricted on $O_\delta(g_3g_2g_1(\Sigma))$ is the identity map, consequently
\[(\Pi \cup \Sigma) \cap g_4g_3g_2g_1(\Sigma) = (\Pi \cup \Sigma) \cap g_3g_2g_1(\Sigma) = \emptyset \quad (4)\]
From (3) and (4) we finally obtain
\[(\Pi \cup \Sigma) \cap g_4g_3g_2g_1(\Pi \cup \Sigma) = \emptyset.

The composition $h := g_4g_3g_2g_1$ is the desired homeomorphism. Indeed:
\[\text{supp } h \subset O_{\alpha+\beta+\gamma+\delta}(f(P)), \text{ and } h \text{ moves each point of } \mathbb{R}^N \text{ no further than } \alpha + \beta + \gamma + \delta \text{ (which can be made smaller than an arbitrary given positive number } \varepsilon).\]

3.5. Proof of Theorem [X]

By assumption, $N - k \geq 3$. Take an embedding $f : C \times C \to \mathbb{R}^{N-k}$ such that all $A_t := f(C \times \{t\})$, where $t \in C$, are wild Cantor sets embedded in $\mathbb{R}^{N-k}$ equivalently to each other. For example, we may take Antoine–Blankingship–Ivanov sets [25, Remark on p. 171 and Exercise 4.8.1]. Applying the idea from [13, p. 479], define a new embedding $F$ by multiplying with $I^k$:
\[I^k \times (C \times C) \xrightarrow{id \times f} I^k \times \mathbb{R}^{N-k} \subset \mathbb{R}^k \times \mathbb{R}^{N-k} = \mathbb{R}^N.\]
For each $t \in C$, denote $X_t := F(I^k \times C \times \{t\}) = I^k \times A_t \subset \mathbb{R}^N$. All $X_t$’s are embedded into $\mathbb{R}^N$ equivalently to each other.

Recall that a wild Cantor set in $\mathbb{R}^N$ can not be locally 1-co-connected; see [32] or [3, Thm. 5.1] for $N = 3$, [39] for $N \geq 5$, and [50] or [7, p. 5] for $N = 4$. (A closed subset $X$ of $\mathbb{R}^d$ is called locally 1-co-connected, or briefly 1-LCC if for each $x \in X$, any neighbourhood $U$ of $x$ in $\mathbb{R}^d$ contains a smaller neighbourhood $V$ of $x$ such that every map $\gamma : S^1 = \partial(I^2) \to V \setminus X$ can be extended to a map $\Gamma : I^2 \to U \setminus X$. Refer to [48], [28], [25, 3.4], [20] for details.)

Hence for any $t \in C$, the set $A_t$ is not 1-LCC in $\mathbb{R}^{N-k}$. Consequently, $X_t$ is not 1-LCC in $\mathbb{R}^N$. By [14, proof of Thm. 2] or [48] (one may also refer to [28, Thm. 1.4] or [25, Thm. 3.4.11]), $X_t$ can not be position-wise embedded into $M_{N-3}^N$.

4. Appendix

The construction of Baker and Laidacker includes careful consideration of metric properties, providing $c$ congruent compacta. In this section, we
give a short topological proof for a weaker statement (Corollary 6). We also state several consequences, including a generalization of results obtained by R.B. Sher and E.R. Apodaca.

**Corollary 6.** For each $k \geq 0$, there exists an embedding $\psi : M^k_{2k+1} \times C \to \mathbb{R}^{2k+1}$ such that for each $t \in C$ the image $\psi(M^k_{2k+1} \times \{t\})$ is ambiently homeomorphic to the standard Menger compactum $M^k_{2k+1} \subset \mathbb{R}^{2k+1}$.

**Proof.** The standard Menger compactum $M^k_{2k+1}$ can be taken off itself by a small self-homeomorphism of $\mathbb{R}^{2k+1}$ with support in an arbitrarily small neighbourhood of $M^k_{2k+1}$ [49, Prop. 1], [51, Prop. 8, Thm. 10], [25, Cor. 3.4.7, Thm. 3.5.1]. The conditions of Corollary 3 are thus satisfied.

The Baker–Laidacker theorem implies the following. Let $\mathcal{H}$ be a family of compacta whose dimensions do not exceed $k$, and $|\mathcal{H}| \leq c$; then it is possible to embed all members of $\mathcal{H}$ in $\mathbb{R}^{2k+1}$ simultaneously, so that the images are pairwise disjoint [3, Cor. 1]. Observing that there are exactly $c$ compact subsets of $\mathbb{R}^{2k+1}$, we immediately get:

**Corollary 7.** For each $k \geq 0$, there exists a family $\mathcal{H}$ of pairwise disjoint compacta in $\mathbb{R}^{2k+1}$ such that for each compactum $X$ with $\dim X \leq k$ the family $\mathcal{H}$ contains $c$ elements homeomorphic to $X$.

Assuming the Continuum Hypothesis, R.B. Sher proved [47, Thm. 3]: there is a collection $\mathcal{H}$ of pairwise disjoint arcs in $\mathbb{R}^3$ such that if $A \subset \mathbb{R}^3$ is an arc whose wild set is a compact 0-dimensional set, then there is an $A' \in \mathcal{H}$ which is embedded in $\mathbb{R}^3$ equivalently to $A$.

Actually, without the Continuum Hypothesis we show that $\mathbb{R}^3$ is “a universal storage” of all possible knots and arcs (this also implies [2, Thm. 4, 5]):

**Corollary 8.** There exists a family $\mathcal{H}$ of pairwise disjoint compacta in $\mathbb{R}^3$ with the property: for any one-dimensional compactum $X \subset \mathbb{R}^3$ that can be represented as the union of no more than countably many ANR-sets, we have

$$|\{Y \in \mathcal{H} \mid (\mathbb{R}^3, Y) \cong (\mathbb{R}^3, X)\}| = c.$$
Proof of Corollary 8. Any $X$ which satisfies our assumptions can be position-wise embedded into $M_3^1$ [12, Satz 4]. The set of all compact subsets of $\mathbb{R}^3$ has cardinality $\mathfrak{c}$, hence the desired statement follows from [3, Thm. 1] or from Corollary 6.

Similarly to Corollary 8, we get

**Corollary 9.** There exists a family $\mathcal{H}$ of pairwise disjoint compacta in $\mathbb{R}^3$ such that for any zero-dimensional compactum $X \subset \mathbb{R}^3$ there is an $Y \in \mathcal{H}$ which is embedded in $\mathbb{R}^3$ equivalently to $X$.

**Proof.** Each zero-dimensional compactum $X \subset \mathbb{R}^3$ can be position-wise embedded into $M_3^1$; this can be derived from [12, Thm. 1] using the idea from [12, Proof of Satz 4]. Alternatively, we may directly apply [12, Satz 4] if we recall the Denjoy–Riesz theorem: for each $N \geq 1$ and each zero-dimensional compactum $X \subset \mathbb{R}^N$ there exists an arc $\hat{X} \subset \mathbb{R}^N$ with $X \subset \hat{X}$.

**Final remarks**

**Remark 7.** It remains an open question whether there exists a 1-dimensional compactum $X \subset \mathbb{R}^3$ such that: $X$ can not be position-wise embedded into $M_3^1$, and $\mathbb{R}^3$ contains an uncountable family of pairwise disjoint compacta embedded equivalently to $X$.

**Remark 8.** For each $N \geq 4$, D.G. Wright described cellular arcs $L_1, L_2$ in $\mathbb{R}^N$ such that $L_1$ cannot be slipped off $L_2$, and consequently $X = L_1 \cup L_2$ is sticky [54]. This could possibly be used to construct new “negative” examples for the question of B.J. Baker and M. Laidacker.

**References**

[1] R.D. Anderson, One-dimensional continuous curves and a homogeneity theorem, Ann. Math. (2) 68 (1958) 1–16.

[2] E.R. Apodaca, On the simultaneous embedding of uncountably many distinct wild arcs with one wild endpoint in $E^3$, a geometric approach, Fundam. Math. 113 (1981) 175–186.
[3] D.J. Baker, M. Laidacker, Embedding uncountably many mutually exclusive continua into Euclidean space, Can. Math. Bull. 32 (1989) 2, 207–214.

[4] H. Becker, F. van Engelen, J. van Mill, Disjoint embeddings of compacta, Mathematika 41 (2) (1994) 221–232.

[5] A.S. Besicovitch, On homoeomorphism of perfect plane sets, Proc. Camb. Philos. Soc. 54 (1958) 168–186.

[6] M. Bestvina, Characterizing $k$-dimensional universal Menger compacta. Mem. Am. Math. Soc. 71, No. 380, 110 p. (1988).

[7] M. Bestvina, R.J. Daverman, G.A. Venema, J.J. Walsh, A 4-dimensional 1-LCC shrinking theorem, Topol. Appl. 110 (2001) 3–20.

[8] R.H. Bing, Tame Cantor sets in $E^3$, Pacif. J. Math. 11 (2) (1961) 435–446.

[9] R.H. Bing, Each disk in $E^3$ contains a tame arc, Am. J. Math. 84 (1962) 583–590.

[10] W.A. Blankinship, Generalization of a construction of Antoine, Ann. Math. (2) 53 (1951) 276–297.

[11] H.G. Bothe, Ein eindimensionales Kompaktum im $E^3$, das sich nicht lagetreu in die Mengersche Universalkurve einbetten läßt, Fundam. Math. 54 (1964) 251–258.

[12] H.G. Bothe, Universalmengen bezüglich der Lage im $E^n$, Fundam. Math. 56 (1964) 203–212.

[13] J.L. Bryant, Concerning uncountable families of $n$-cells in $E^n$, Michigan Math. J. 15 (1968) 477–479.

[14] J.L. Bryant, On embedding of compacta in euclidean space, Proc. Am. Math. Soc. 23 (1969) 46–51.

[15] C.E. Burgess, J.W. Cannon, Embeddings of surfaces in $E^3$, Rocky Mountain J. of Math. 1 (2) (1971) 259–344.
[16] C.E. Burgess, Embeddings of surfaces in Euclidean three-space, Bull. Am. Math. Soc. 81 (1975) 5, 795–818.

[17] J.W. Cannon, S.G. Wayment, An imbedding problem, Proc. Am. Math. Soc. 25 (1970) 566–570.

[18] A.V. Chernavskii, Geometric Topology of Manifolds. (Russian) Itogi Nauki, Algebra, Topologiya 1962, 161–187 (1964).

[19] A.V. Chernavskii, Local contractibility of the group of homeomorphisms of a manifold, Math. USSR, Sb. 8:3 (1969) 287–333; transl. from: Mat. Sb., n. Ser. 79(121):3(7) (1969) 307–356.

[20] A.V. Chernavskii, On the work of L.V. Keldysh and her seminar, Russian Math. Surveys 60 (4) (2005) 589–614; transl. from: Usp. Mat. Nauk 60 (4) (2005) 11–36.

[21] A.V. Chernavskii, Local contractibility of the homeomorphism group of $\mathbb{R}^n$, Proc. Steklov Inst. Math. 263 (2008) 189–203; transl. from: Tr. Mat. Inst. Steklova 263 (2008) 201–215.

[22] A.Ch. Chigogidze. Inverse spectra. North-Holland Mathematical Library. 53. Amsterdam: Elsevier. x, 421 p. (1996).

[23] R.J. Daverman, On the absence of tame disks in certain wild cells, Geom. Topol., edited by L.C. Glaser and T.B. Rushing, Proc. Conf. Park City 1974, Lect. Notes Math. 438 (1975) 142–155.

[24] R.J. Daverman, Embeddings of $(n−1)$-spheres in Euclidean $n$-space, Bull. Am. Math. Soc. 84 (1978) 3, 377–405.

[25] R.J. Daverman, D.A. Venema. Embeddings in Manifolds. Graduate Studies in Mathematics 106. Providence, RI: American Mathematical Society. 2009.

[26] E.K. van Douwen, Uncountably many pairwise disjoint copies of one metrizable compactum in another, Topol. Appl. 51 (1993) 87–91.

[27] W.T. Eaton, A Generalization of the Dog Bone Space to $E^n$, Proc. Am. Math. Soc. 39 (2) (1973) 379–387.
[28] R.D. Edwards, Demension theory, I. Geom. Topol., edited by L.C. Glaser and T.B. Rushing, Proc. Conf. Park City 1974, Lect. Notes Math. 438 (1975) 195–211.

[29] F. Frankl, L. Pontrjagin, Ein Knotensatz mit Anwendung auf die Dimensionstheorie, Math. Ann. 102 (1930) 785–789.

[30] O. Frolkina, Wild high-dimensional Cantor fences in $\mathbb{R}^n$, Part I, Topol. Appl. 258 (2019) 451–464.

[31] L.C. Hoehn, An uncountable family of copies of a non-chainable tree-like continuum in the plane, Proc. Am. Math. Soc. 141 (2013) 7, 2543–2556.

[32] T. Homma, On tame imbedding of 0-dimensional compact sets in $E^3$, Yokohama Math. J. 7 (1959) 191–195.

[33] A.A. Ivanov, Isotopy of compacta in Euclidean spaces, Soviet Math. Dokl. (=Dokl. Akad. Nauk SSSR) 71 (6) (1950) 1021–1022 (in Russian). See Zbl 0037.26305.

[34] A.A. Ivanov, Isotopy of compacta in Euclidean spaces, Doctoral dissertation (Candidate of Sciences), St. Petersburg Department of V.A. Steklov Institute of Mathematics of the USSR Academy of Sciences, Moscow–Leningrad, 1950. 63 p. (In Russian.)

[35] L.V. Keldysh, Topological imbeddings in Euclidean space (English. Russian original), Proc. Steklov Inst. Math. 81 (1966), 203 p. Transl. from: Tr. Mat. Inst. Steklov. 81 (1966), 184 p.

[36] J. Kister, Small isotopies in euclidean spaces and 3-manifolds, Bull. Am. Math. Soc. 65 (1959) 371–373.

[37] V. Krushkal, Sticky Cantor sets in $\mathbb{R}^d$, J. Topol. Anal. 10 (2018) 2, 477–482.

[38] S. Lefschetz, On compact spaces, Ann. Math. (2) 32 (1931) 521–538.

[39] D.R. McMillan jr., Taming Cantor sets in $E^n$, Bull. Am. Math. Soc. 70 (1964) 706–708.

[40] D.R. McMillan jr., H. Row, Tangled embeddings of one-dimensional continua, Proc. Am. Math. Soc. 22 (1969) 378–385.
[41] S. Melikhov, Review of “Embeddings in manifolds” by R.J. Daverman and G.A. Venema, Mathematical Reviews, MR2561389 (2011g:57025).

[42] E.E. Moise. Geometric Topology in Dimensions 2 and 3. Springer-Verlag, 1977.

[43] D. Repovš, A.B. Skopenkov, E.V. Ščepin, On uncountable collections of continua and their span, Colloq. Math. 69 (1995) 2, 289–296.

[44] D. Repovš, A.B. Skopenkov, E.V. Ščepin, On embeddability of $X \times I$ into Euclidean space, Houston J. Math. 21 (1995) 1, 199–204.

[45] J.H. Roberts, Concerning atriodic continua, Monatshefte f. Math. 37 (1930) 223–230.

[46] T.B. Rushing. Topological embeddings. Pure and Applied Mathematics. Vol. 52, New York and London, Academic Press, 1973.

[47] R.B. Sher, Families of arcs in $E^3$, Trans. Am. Math. Soc. 143 (1969) 109–116.

[48] M.A. Shtan’ko, The embedding of compacta in Euclidean space, Math. USSR-Sb. 12:2 (1970) 234–254. (Transl. from: Mat. Sb. 83 (125) (1970) 2 (10) 234–255.) Announcement appeared in Dokl. Akad. Nauk SSSR 86 (1969) 1269–1272 (= Soviet Math. Dokl. 10 (1969) 758–761).

[49] M.A. Stahn’ko, Solution of Menger’s problem in the class of compacta, Sov. Math., Dokl. 12 (1971), 1846–1849. Transl. from: Dokl. Akad. Nauk SSSR 201 (1971), 1299–1302.

[50] M.A. Stan’ko, Embedding compacta in Euclidean space of dimension 5, 4 and 3, Math. USSR-Sb. 28:4 (1976), 563–569 (1978). Transl. from: Mat. Sbornik, n. Ser. 99 (141) (1976) 626–633.

[51] M.A. Shtan’ko, Dimension of embedding and approximation of compact sets. Solution of the Menger problem: abstract of the dissertation for the degree of Doctor of Physical and Mathematical Sciences: 01.01.04. Moscow, 1982. 113 p. (In Russian.)

[52] S. Todorčević, Embeddability of $K \times C$ into $X$, Bull. Cl. Sci. Math. Nat. Sci. Math. 114 (1997) 22, 27–35.
[53] E.D. Tymchatyn, R.B. Walker, Taming the Cantor fence, Topol. Appl. 83 (1998) 45–52.

[54] D.G. Wright, Sticky arcs in \( E^n \) (\( n \geq 4 \)), Proc. Am. Math. Soc. 66 (1977) 181–182.

[55] D.G. Wright, Pushing a Cantor set off itself, Houston J. Math. 2 (1976) 439–447.

[56] H. Zieschang, A.V. Chernavskii, Geometric topology of manifolds. (English. Russian original.) Progress in Mathematics Vol. 6: Topology and Geometry, 3–49 (1970); transl. from Itogi Nauki Tekh., Ser. Algebra, Topologiya, Geom. 1965, 219–261 (1967).