SEQUENCE-COVERING MAPS ON GENERALIZED METRIC SPACES

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Abstract. Let \( f : X \to Y \) be a map. \( f \) is a sequence-covering map\(^2\) if whenever \( \{y_n\} \) is a convergent sequence in \( Y \) there is a convergent sequence \( \{x_n\} \) in \( X \) with each \( x_n \in f^{-1}(y_n) \); \( f \) is an 1-sequence-covering map\(^1\) if for each \( y \in Y \) there is \( x \in f^{-1}(y) \) such that whenever \( \{y_n\} \) is a sequence converging to \( y \) in \( Y \) there is a sequence \( \{x_n\} \) converging to \( x \) in \( X \) with each \( x_n \in f^{-1}(y_n) \). In this paper, we mainly discuss the sequence-covering maps on generalized metric spaces, and give an affirmative answer for a question in \(^1\) and some related questions, which improve some results in \(^{13, 16, 28}\), respectively. Moreover, we also prove that open and closed maps preserve strongly monotonically monolithicity, and closed sequence-covering maps preserve spaces with a \( \sigma \)-point-discrete \( k \)-network. Some questions about sequence-covering maps on generalized metric spaces are posed.

1. Introduction

A study of images of topological spaces under certain sequence-covering maps is an important question in general topology \(^{9, 11, 12, 13, 15, 18, 19, 20, 28}\). S. Lin and P. F. Yan in \(^{15}\) proved that each sequence-covering and compact map on metric spaces is an 1-sequence-covering map. Recently, F. C. Lin and S. Lin in \(^{13}\) proved that each sequence-covering and boundary-compact map on metric spaces is an 1-sequence-covering map. Also, the authors posed the following question in \(^{13}\):

**Question 1.1.** \(^{13}\) Question 3.6 Let \( f : X \to Y \) be a sequence-covering and boundary-compact map. Is \( f \) an 1-sequence-covering map if \( X \) is a space with a point-countable base or a developable space?

In this paper, we shall give an affirmative answer for Question 1.1.

S. Lin in \(^{16}\) Theorem 2.2 proved that if \( X \) is a metrizable space and \( f \) is a sequence-quotient and compact map, then \( f \) is a pseudo-sequence-covering map. Recently, C. F. Lin and S. Lin in \(^{13}\) proved that if \( X \) is a metrizable space and \( f \) is a sequence-quotient and boundary-compact map, then \( f \) is a pseudo-sequence-covering map. Hence we have the following Question 1.2.

**Question 1.2.** Let \( f : X \to Y \) be a sequence-quotient and boundary-compact map. Is \( f \) a pseudo-sequence-covering map if \( X \) is a space with a point-countable base or a developable space?

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On the other hand, the authors in [28] proved that each closed sequence-covering map on metric spaces is an 1-sequence-covering map. Hence we have the following Question 1.3.

**Question 1.3.** Let $f : X \to Y$ be a closed sequence-covering map. Is $f$ an 1-sequence-covering map if $X$ is a regular space with a point-countable base or a developable space?

In this paper, we shall give an affirmative answer for Question 1.2, which improves some results in [13] and [16], respectively. Moreover, we give an affirmative answer for Question 1.3 when $X$ has a point-countable base or $X$ is $g$-metrizable.

In [27], V. V. Tkachuk introduced the strongly monotonically monolithic spaces. In this paper, we also prove that strongly monotonically monolithic spaces are preserved by open and closed maps, and spaces with a $\sigma$-point-discrete $k$-network are preserved by closed sequence-covering maps.

### 2. Definitions and terminology

Let $X$ be a space. For $P \subset X$, $P$ is a sequential neighborhood of $x$ in $X$ if every sequence converging to $x$ is eventually in $P$.

**Definition 2.1.** Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a cover of a space $X$ such that for each $x \in X$, (a) if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$; (b) $\mathcal{P}_x$ is a network of $x$ in $X$, i.e., $x \in \bigcap \mathcal{P}_x$, and if $x \in U$ with $U$ open in $X$, then $P \subset U$ for some $P \in \mathcal{P}_x$.

1. $\mathcal{P}$ is called an sn-network for $X$ if each element of $\mathcal{P}_x$ is a sequential neighborhood of $x$ in $X$ for each $x \in X$.

2. $\mathcal{P}$ is called a weak base [1] for $X$ if whenever $G \subset X$ satisfying for each $x \in X$ there is a $P \in \mathcal{P}_x$ with $P \subset G$, $G$ is open in $X$. $X$ is $g$-metrizable [26] if $X$ is regular and has a $\sigma$-locally finite weak base.

**Definition 2.2.** Let $f : X \to Y$ be a map.

1. $f$ is a compact (resp. separable) map if each $f^{-1}(y)$ is compact (separable) in $X$;

2. $f$ is a boundary-compact (resp. boundary-separable) map if each $\partial f^{-1}(y)$ is compact (separable) in $X$;

3. $f$ is a sequence-covering map [25] if whenever $\{y_n\}$ is a convergent sequence in $Y$ there is a convergent sequence $\{x_n\}$ in $X$ with each $x_n \in f^{-1}(y_n)$;

4. $f$ is an 1-sequence-covering map [14] if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to $y$ in $Y$ there is a sequence $\{x_n\}$ converging to $x$ in $X$ with each $x_n \in f^{-1}(y_n)$;

5. $f$ is a sequentially quotient map [5] if whenever $\{y_n\}$ is a convergent sequence in $Y$ there is a convergent sequence $\{x_k\}$ in $X$ with each $x_k \in f^{-1}(y_n)$;

6. $f$ is a pseudo-sequence-covering map [9, 10] if for each convergent sequence $L$ in $Y$ there is a compact subset $K$ in $X$ such that $f(K) = L$.

It is obvious that
Remind readers attention that the sequence-covering maps defined the above-mentioned are different from the sequence-covering maps defined in [9], which is called pseudo-sequence-covering maps in this paper.

Definition 2.3. [23] Let \( A \) be a subset of a space \( X \). We call an open family \( \mathcal{N} \) of subsets of \( X \) an external base of \( A \) in \( X \) if for any \( x \in A \) and open subset \( U \) with \( x \in U \) there is a \( V \in \mathcal{N} \) such that \( x \in V \subset U \).

Similarly, we can define an externally weak base for a subset \( A \) of a space \( X \).

Throughout this paper all spaces are assumed to be Hausdorff, all maps are continuous and onto. The letter \( \mathbb{N} \) will denote the set of positive integer numbers. Readers may refer to [6, 8, 15] for unstated definitions and terminology.

3. SEQUENCE-COVERING AND BOUNDARY-COMPACT MAPS

Let \( \Omega \) be the sets of all topological spaces such that, for each compact subset \( K \subset X \in \Omega \), \( K \) is metrizable and also has a countably neighborhood base in \( X \). In fact, E. A. Michael and K. Nagami in [23] has proved that \( X \in \Omega \) if and only if \( X \) is the image of some metric space under an open and compact-covering map. It is easy to see that if a space \( X \) is developable or has a point-countable base, then \( X \in \Omega \) (see [4] and [27], respectively).

In this paper, when we say an \( snf \)-countable space \( Y \), it is always assume that \( Y \) has an \( sn \)-network \( \mathcal{P} = \bigcup \{ \mathcal{P}_y : y \in Y \} \) such that \( \mathcal{P}_y \) is countable and closed under finite intersections for each point \( y \in Y \).

Lemma 3.1. Let \( f : X \to Y \) be a sequence-covering and boundary-compact map, where \( Y \) is \( snf \)-countable. For each non-isolated point \( y \in Y \), there exists a point \( x_y \in \partial f^{-1}(y) \) such that whenever \( U \) is an open subset with \( x_y \in U \), there exists a \( P \in \mathcal{P}_y \) satisfying \( P \subset f(U) \)

Proof. Suppose not, there exists a non-isolated point \( y \in Y \) such that for every point \( x \in \partial f^{-1}(y) \), there is an open neighborhood \( U_x \) of \( x \) such that \( P \not\subset f(U_x) \) for every \( P \in \mathcal{P}_y \). Then \( \partial f^{-1}(y) \subset \bigcup \{ U_x : x \in \partial f^{-1}(y) \} \). Since \( \partial f^{-1}(y) \) is compact, there exists a finite subfamily of \( U \subset \bigcup \{ U_x : x \in \partial f^{-1}(y) \} \) such that \( \partial f^{-1}(y) \subset \bigcup U \).

We denote \( U \) by \( \{ U_i : 1 \leq i \leq n_0 \} \). Assume that \( \mathcal{P}_y = \{ P_n : n \in \mathbb{N} \} \) and \( \mathcal{W}_y = \{ F_n = \bigcap_{i=1}^{n_0} P_i : n \in \mathbb{N} \} \). It is obvious that \( \mathcal{W}_y \subset \mathcal{P}_y \) and \( F_{n+1} \subset F_n \) for every \( n \in \mathbb{N} \). For each \( 1 \leq n \leq n_0, n \in \mathbb{N} \), it follows that there exists \( x_{n,m} \in F_n \setminus f(U_m) \). Then denote \( y_k = x_{n,m} \), where \( k = (n-1)n_0 + m \). Since \( \mathcal{P}_y \) is a network at point \( y \) and \( F_{n+1} \subset F_n \) for every \( n \in \mathbb{N} \), \( \{ y_k \} \) is a sequence converging to \( y \) in \( Y \).

Because \( f \) is a sequence-covering map, \( \{ y_k \} \) is an image of some sequence \( \{ x_k \} \) converging to \( x \in \partial f^{-1}(y) \) in \( X \). From \( x \in \partial f^{-1}(y) \subset \bigcup U \) it follows that there

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1. Let \( f : X \to Y \) be a map. \( f \) is called a compact-covering map [23] if in case \( L \) is compact in \( Y \) there is a compact subset \( K \) of \( X \) such that \( f(K) = L \).
exists \(1 \leq m_0 \leq n_0\) such that \(x \in U_{m_0}\). Therefore, \(\{x\} \cup \{x_k : k \geq k_0\} \subseteq U_{m_0}\) for some \(k_0 \in \mathbb{N}\). Hence \(\{y\} \cup \{y_k : k \geq k_0\} \subseteq f(U_{m_0})\). However, we can choose an \(n > k_0\) such that \(k = (n - 1)n_0 + m_0 \geq k_0\) and \(y_k = x_{n,m_0}\), which implies that \(x_{n,m_0} \in f(U_{m_0})\). This contradicts to \(x_{n,m_0} \in F_n \setminus f(U_{m_0})\). \(\Box\)

The next lemma is obvious.

**Lemma 3.2.** Let \(f : X \rightarrow Y\) be 1-sequence-covering, where \(X\) is snf-countable. Then \(Y\) is snf-countable.

**Theorem 3.3.** Let \(f : X \rightarrow Y\) be a sequence-covering and boundary-compact map, where \(X\) is first-countable. Then \(Y\) is snf-countable if and only if \(f\) is an 1-sequence-covering map.

**Proof.** Necessity. Let \(y\) be a non-isolated point in \(Y\). Since \(Y\) is snf-countable, it follows from Lemma 3.1 that there exists a point \(x_y \in \partial f^{-1}(y)\) such that whenever \(U\) is an open neighborhood of \(x_y\), there is a \(P \in \mathcal{P}_y\) satisfying \(P \subseteq f(U)\). Let \(\{B_n : n \in \mathbb{N}\}\) be a countably neighborhood base at point \(x_y\) such that \(B_{n+1} \subseteq B_n\) for each \(n \in \mathbb{N}\). Suppose that \(\{y_n\}\) is a sequence in \(Y\), which converges to \(y\). Next, we take a sequence \(\{x_n\}\) in \(X\) as follows.

Since \(B_n\) is an open neighborhood of \(x_y\), it follows from the Lemma 3.1 that there exists a \(P_n \in \mathcal{P}_y\) such that \(P_n \subseteq f(B_n)\) for each \(n \in \mathbb{N}\). Because every \(P \in \mathcal{P}_y\) is a sequential neighborhood, it is easy to see that for each \(n \in \mathbb{N}\), \(f(B_n)\) is a sequential neighborhood of \(y\) in \(Y\). Therefore, for each \(n \in \mathbb{N}\), there is an \(i_n \in \mathbb{N}\) such that \(y_i \in f(B_n)\) for every \(i \geq i_n\). Suppose that \(1 < i_n < i_{n+1}\) for every \(n \in \mathbb{N}\). Hence, for each \(j \in \mathbb{N}\), we take

\[
x_j \in \begin{cases} f^{-1}(y_j), & \text{if } j < i_1, \\ f^{-1}(y_j) \cap B_n, & \text{if } i_n \leq j < i_{n+1}. \end{cases}
\]

We denote \(S = \{x_j : j \in \mathbb{N}\}\). It is easy to see that \(S\) converges to \(x_y\) in \(X\) and \(f(S) = \{y_n\}\). Therefore, \(f\) is an 1-sequence-covering map.

Sufficiency. It easy to see that \(Y\) is snf-countable by Lemma 3.2 \(\Box\)

We don’t know whether, in Theorem 3.3, \(f\) is an 1-sequence-covering map when \(X\) is only first-countable. However, we have the following Theorem 3.6 which gives an affirmative answer for Question 1.1. Firstly, we give some technique lemmas.

**Lemma 3.4.** \([23]\) If \(X \in \Omega\), then every compact subset of \(X\) has a countably external base.

**Lemma 3.5.** Let \(f : X \rightarrow Y\) be a sequence-covering and boundary-compact map. If \(X \in \Omega\), then \(Y\) is snf-countable.

**Proof.** Let \(y\) be a non-isolated point for \(Y\). Then \(\partial f^{-1}(y)\) is non-empty and compact for \(X\). Therefore, \(\partial f^{-1}(y)\) has a countably external base \(\mathcal{U}\) in \(X\) by Lemma 3.4.

Let

\[
\mathcal{V} = \{\bigcup \mathcal{F} : \text{There is a finite subfamily } \mathcal{F} \subseteq \mathcal{U} \text{ with } \partial f^{-1}(y) \subseteq \bigcup \mathcal{F} \}.
\]

Obviously, \(\mathcal{V}\) is countable. We now prove that \(f(\mathcal{V})\) is a countable sn-network at point \(y\).

1) \(f(\mathcal{V})\) is a network at \(y\).

Let \(y \in U\). Obviously, \(\partial f^{-1}(y) \subset f^{-1}(U)\). For each \(x \in \partial f^{-1}(y)\), there exist an \(U_x \in \mathcal{U}\) such that \(x \in U_x \subset f^{-1}(U)\). Therefore, \(\partial f^{-1}(y) \subset \bigcup \{U_x : x \in \partial f^{-1}(y)\}\).
\( \partial f^{-1}(y) \). Since \( \partial f^{-1}(y) \) is compact, it follows that there exists a finite subfamily \( F \subset \{ U_x : x \in \partial f^{-1}(y) \} \) such that \( \partial f^{-1}(y) \subset \bigcup F \subset f^{-1}(U) \). It is easy to see that \( F \in \mathcal{V} \) and \( y \in \bigcup f(F) \subset U \).

(2) For any \( P_1, P_2 \in f(\mathcal{V}) \), there exists a \( P_3 \in f(\mathcal{V}) \) such that \( P_3 \subset P_1 \cap P_2 \).

It is obvious that there exist \( V_1, V_2 \in \mathcal{V} \) such that \( f(V_1) = P_1, f(V_2) = P_2 \), respectively. Since \( \partial f^{-1}(y) \subset V_1 \cap V_2 \), it follows from the similar proof of (1) that there exists a \( V_3 \in \mathcal{V} \) such that \( \partial f^{-1}(y) \subset V_3 \subset V_1 \cap V_2 \). Let \( P_3 = f(V_3) \). Hence \( \partial f^{-1}(y) \subset V_3 \subset \bigcup f(V) \subset f^{-1}(U) \). Therefore, \( \{ x_n \} \) is eventually in \( V \), and this is implied that \( \{ y_n \} \) is eventually in \( P \).

Therefore, \( f(\mathcal{V}) \) is a countable sn-network at point \( y \). \( \square \)

**Theorem 3.6.** Let \( f : X \to Y \) be a sequence-covering and boundary-compact map. If \( X \in \Omega \), then \( f \) is an 1-sequence-covering map.

**Proof.** From Lemma 3.5 it follows that \( Y \) is snf-countable. Therefore, \( f \) is an 1-sequence-covering map by Theorem 3.3. \( \square \)

By Theorem 3.6, it easily follows the following Corollary 3.7 which gives an affirmative answer for Question 1.1.

**Corollary 3.7.** Let \( f : X \to Y \) be a sequence-covering and boundary-compact map. Suppose also that at least one of the following conditions holds:

1. \( X \) has a point-countable base;
2. \( X \) is a developable space.

Then \( f \) is an 1-sequence-covering map.

**Lemma 3.8.** Let \( f : X \to Y \) be a sequence-covering map, where \( Y \) is snf-countable and \( \partial f^{-1}(y) \) has a countably external base for each point \( y \in Y \). Then, for each non-isolated point \( y \in Y \), there exists a point \( x_y \in \partial f^{-1}(y) \) such that whenever \( U \) is an open subset with \( x_y \in U \), there exists a \( P \in \mathcal{P}_y \) satisfying \( P \subset f(U) \).

**Proof.** Suppose not, there exists a non-isolated point \( y \in Y \) such that for every point \( x \in \partial f^{-1}(y) \), there is an open neighborhood \( U_x \) of \( x \) such that \( P \not\subset f(U_x) \) for every \( P \in \mathcal{P}_y \). Let \( B \) be a countably external base for \( \partial f^{-1}(y) \). Therefore, for each \( x \in \partial f^{-1}(y) \), there exists a \( B_x \in B \) such that \( x \in B_x \subset U_x \). For each \( x \in \partial f^{-1}(y) \), it follows that \( P \not\subset f(B_x) \) whenever \( P \in \mathcal{P}_y \). Assume that \( \mathcal{P}_y = \{ P_n : n \in \mathbb{N} \} \) and \( \mathcal{W}_y = \{ F_n = \bigcap_{i=1}^{n} P_i : n \in \mathbb{N} \} \). We denote \( \{ B_x \in B : x \in \partial f^{-1}(y) \} \) by \( \{ B_m : m \in \mathbb{N} \} \). For each \( n, m \in \mathbb{N} \), it follows that there exists \( x_{n,m} \in F_n \setminus f(B_m) \). For \( n \geq m \), we denote \( y_k = x_{n,m} \) for \( k = m + n(n-1)/2 \).

Since \( \mathcal{P}_y \) is a network at point \( y \) and \( F_{n+1} \subset F_n \) for every \( n \in \mathbb{N} \), \( \{ y_k \} \) is a sequence converging to \( y \) in \( Y \). Because \( f \) is a sequence-covering map, \( \{ y_k \} \) is an image of some sequence \( \{ x_k \} \) converging to \( x \in \partial f^{-1}(y) \) in \( X \). From \( x \in \partial f^{-1}(y) \subset \bigcup \{ B_m : m \in \mathbb{N} \} \) it follows that there exists a \( B_m \in \mathbb{N} \) such that \( B_m \) is an open neighborhood at \( x \). Therefore, \( \{ x \} \cup \{ x_k : k \geq k_0 \} \subset f(B_m) \). However, we can choose a \( k \geq k_0 \) and an \( n \geq m_0 \) such that \( y_k = x_{n,m_0} \), which implies that \( x_{n,m_0} \in f(B_m) \). This contradictions to \( x_{n,m_0} \in F_n \setminus f(B_m) \). \( \square \)
Theorem 3.9. Let $f : X \to Y$ be a sequence-covering and boundary-separable map. If $X$ has a point-countable base and $Y$ is snf-countable, then $f$ is an 1-sequence-covering map.

Proof. Obviously, $\partial f^{-1}(y)$ has a countably external base for each point $y \in Y$. Therefore, it is easy to see by Lemma 3.8 and the proof of Theorem 3.3. \qed

Remark. We can’t omit the condition “$Y$ is snf-countable” in Theorem 3.9. Indeed, the sequence fan $S_{\omega}$ is the image of metric spaces under the sequence-covering $s$-maps by [15, Corollary 2.4.4]. However, $S_{\omega}$ is not snf-countable, and therefore, $S_{\omega}$ is not the image of metric spaces under an 1-sequence-covering map.

In this section, we finally give an affirmative answer for Question 1.2.

Lemma 3.10. [5] Let $f : X \to Y$ be a map. If $X$ is a Fréchet space, then $f$ is a sequentially quotient map if and only if $Y$ is a Fréchet space and $f$ is a sequentially quotient map.

Theorem 3.11. Let $f : X \to Y$ be a boundary-compact map. If $X \in \Omega$, then $f$ is a sequentially quotient map if and only if it is a pseudo-sequence-covering map.

Proof. First, suppose that $f$ is sequentially quotient. If $\{y_n\}$ is a non-trivial sequence converging to $y_0$ in $Y$, put $S_1 = \{y_0\} \cup \{y_n : n \in \mathbb{N}\}$, $X_1 = f^{-1}(S_1)$ and $g = f|_{X_1}$. Thus $g$ is a sequentially quotient, boundary compact map. So $g$ is a pseudo-open map by Lemma 3.10. Since $X \in \Omega$, let $\{U_n\}_{n \in \mathbb{N}}$ be a decreasingly neighborhood base of compact subset $\partial g^{-1}(y_0)$ in $X_1$. Thus $\{U_n \cup \text{Int}(g^{-1}(y_0))\}_{n \in \mathbb{N}}$ is a decreasingly neighborhood base of $g^{-1}(y_0)$ in $X_1$. Let $V_n = U_n \cup \text{Int}(g^{-1}(y_0))$ for each $n \in \mathbb{N}$. Then $y_0 \in \text{Int}(g(V_n))$, thus there exists an $i_n \in \mathbb{N}$ such that $y_i \in g(V_n)$ for each $i \geq i_n$, so $g^{-1}(y_i) \cap V_n \neq \emptyset$. We can suppose that $1 < i_n < i_{n+1}$. For each $j \in \mathbb{N}$, we take

$$x_j \in \begin{cases} f^{-1}(y_j), & \text{if } j < i_1, \\ f^{-1}(y_j) \cap V_n, & \text{if } i_n \leq j < i_{n+1}. \end{cases}$$

Let $K = \partial g^{-1}(y_0) \cup \{x_j : j \in \mathbb{N}\}$. Clearly, $K$ is a compact subset in $X_1$ and $g(K) = S_1$. Thus $f(K) = S_1$. Therefore, $f$ is a pseudo-sequence-covering map.

Conversely, suppose that $f$ is a pseudo-sequence-covering map. If $\{y_n\}$ is a convergent sequence in $Y$, then there is a compact subset $K$ in $X$ such that $f(K) = \{y_n\}$. For each $n \in \mathbb{N}$, take a point $x_n \in f^{-1}(y_n) \cap K$. Since $K$ is compact and metrizable, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. So $f$ is sequentially quotient. \qed

Corollary 3.12. Let $f : X \to Y$ be a boundary-compact map. Suppose also that at least one of the following conditions holds:

1. $X$ has a point-countable base;
2. $X$ is a developable space.

Then $f$ is a sequentially quotient map if and only if it is a pseudo-sequence-covering map.

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2$S_{\omega}$ is the space obtained from the topological sum of $\omega$ many copies of the convergent sequence by identifying all the limit points to a point.

3$X$ is said to be a Fréchet space if $x \in \overline{P} \subset X$, there is a sequence in $P$ converging to $x$ in $X$.

4$f$ is a pseudo-open map if whenever $f^{-1}(y) \subset U$ with $U$ open in $X$, then $y \in \text{Int}(f(U))$. 
Question 3.13. Let $f : X \to Y$ be a sequence-covering and boundary-compact (or compact) map. Is $f$ an 1-sequence-covering map if one of the following conditions is satisfied?

1. Every compact subset of $X$ is metrizable;
2. Every compact subset of $X$ has countable character.

Remark If $X$ satisfies the conditions (1) and (2) in Question 3.13, then $f$ is an 1-sequence-covering map by Theorem 3.6.

4. Sequence-covering maps on $g$-metrizable spaces

In this section, we mainly discuss sequence-covering maps on spaces with a specially weak base.

Lemma 4.1. Let $f : X \to Y$ be a sequence-covering and boundary-compact map. For each non-isolated point $y \in Y$, there exist a point $x \in \partial f^{-1}(y)$ and a decreasingly weak neighborhood base $\{B_n\}_n$ at $x$ such that for each $n \in \mathbb{N}$, there are a $P \in \mathcal{P}_y$ and $i \in \mathbb{N}$ with $P \subset f(B_n)$ if $X$ and $Y$ satisfy the following (1) and (2):

1. $Y$ is snf-countable;
2. Every compact subset of $X$ has a countably externally weak base in $X$.

Proof. Suppose not, there exists a non-isolated point $y \in Y$ such that for every point $x \in \partial f^{-1}(y)$ and every decreasingly weak neighborhood base $\{B_n\}_n$ of $x$, there is an $n \in \mathbb{N}$ such that $P \subset f(B_n)$ for every $P \in \mathcal{P}_y$. Since $\partial f^{-1}(y)$ is compact, it follows that $\partial f^{-1}(y)$ has a countably externally weak base $\mathcal{B}$ of $X$. Without loss of generality, we can assume that $\mathcal{B}$ is closed under finite intersections. Therefore, for each $x \in \partial f^{-1}(y)$, there exists a $B_x \in \mathcal{B}$ such that $P \subset f(B_x)$ for every $P \in \mathcal{P}_y$. Next, using the argument from the proof of Lemma 3.8, this leads to a contradiction. □

The following Lemma 4.2 is easily to check, and hence we omit it.

Lemma 4.2. Let $X$ have a compact-countable weak base. Then every compact subset of $X$ has a countably externally weak base in $X$.

Theorem 4.3. Let $f : X \to Y$ be a sequence-covering and boundary-compact map, where $X$ has a compact-countable weak base. Then $Y$ is snf-countable if and only if $f$ is an 1-sequence-covering map.

Proof. Necessity. Let $y$ be a non-isolated point in $Y$. Since $X$ has a compact-countable weak base, it follows from Lemmas 4.1 and 4.2 that there exists a point $x_y \in \partial f^{-1}(y)$ and a decreasingly countably weak base $\{B_n : n \in \mathbb{N}\}$ at point $x_y$ such that for each $n \in \mathbb{N}$, there is a $P \in \mathcal{P}_y$ satisfying $P \subset f(B_n)$. Suppose that $\{y_n\}$ is a sequence in $Y$, which converges to $y$. Then we can take a sequence $\{x_n\}$ in $X$ by the similar argument from the proof of Theorem 3.3. Therefore, $f$ is an 1-sequence-covering map.

Sufficiency. By Lemma 3.2 $Y$ is snf-countable. □

We don’t know whether the condition “compact-countable weak base” on $X$ can be replaced by “point-countable weak base” in Theorem 4.3.

Corollary 4.4. Let $f : X \to Y$ be a sequence-covering and boundary-compact map, where $X$ is $g$-metrizable. Then $Y$ is snf-countable if and only if $f$ is an 1-sequence-covering map.
Each closed sequence-covering map on metric spaces is 1-sequence-covering [28]. Now, we improve the result in the following theorem.

**Theorem 4.5.** Let $f : X \to Y$ be a closed sequence-covering map, where $X$ is $g$-metrizable. Then $f$ is an 1-sequence-covering map.

**Proof.** Since $X$ is $g$-metrizable and $f$ is a closed sequence-covering map, $Y$ is $g$-metrizable [21, Theorem 3.3]. Therefore, $f$ is a boundary-compact map by [21, Corollary 2.2]. Hence $f$ is an 1-sequence-covering map by Corollary 4.6. □

**Question 4.6.** Let $f : X \to Y$ be a sequence-covering and boundary-compact map. If $X$ is $g$-metrizable, then is $f$ an 1-sequence-covering map?

### 5. Closed sequence-covering maps

Say that a Tychonoff space $X$ is strongly monotonically monolithic [27] if, for any $A \subset X$ we can assign an external base $\mathcal{O}(A)$ to the set $\overline{A}$ in such a way that the following conditions are satisfied:

(a) $|\mathcal{O}(A)| \leq \max\{|A|, \omega\}$;
(b) if $A \subset B \subset X$ then $\mathcal{O}(A) \subset \mathcal{O}(B)$;
(c) if $\alpha$ is an ordinal and we have a family $\{A_\beta : \beta < \alpha\}$ of subsets of $X$ such that $\beta < \beta' < \alpha$ implies $A_\beta \subset A_{\beta'}$, then $\mathcal{O}(\bigcup_{\beta < \alpha} A_\beta) = \bigcup_{\beta < \alpha} \mathcal{O}(A_\beta)$.

From [27, Proposition 2.5] it follows that a Tychonoff space with a point-countable base is strongly monotonically monolithic. Moreover, if $X$ is a strongly monotonically monolithic space, then it is easy to see that $X \in \Omega$ by [27, Theorem 2.7].

**Lemma 5.1.** Let $f : X \to Y$ be a closed sequence-covering map, where $X$ is a strongly monotonically monolithic space. Then $Y$ contains no closed copy of $S_\omega$.

**Proof.** Suppose that $Y$ contains a closed copy of $S_\omega$, and that $\{y\} \cup \{y_i(n) : i, n \in \mathbb{N}\}$ is a closed copy of $S_\omega$ in $Y$, here $y_i(n) \to y$ as $i \to \infty$. For every $k \in \mathbb{N}$, put $L_k = \cup \{y_i(n) : i \in \mathbb{N}, n \leq k\}$. Hence $L_k$ is a sequence converging to $y$. Let $M_k$ be a sequence of $X$ converging to $u_k \in f^{-1}(y)$ such that $f(M_k) = L_k$. We rewrite $M_k = \cup \{x_i(n, k) : i \in \mathbb{N}, n \leq k\}$ with each $f(x_i(n, k)) = y_i(n)$.

Case 1: $\{u_k : k \in \mathbb{N}\}$ is finite.

There are a $k_0 \in \mathbb{N}$ and an infinite subset $\mathbb{N}_1 \subset \mathbb{N}$ such that $M_k \to u_{k_0}$ for every $k \in \mathbb{N}_1$, then $X$ contains a closed copy of $S_\omega$. Hence $X$ is not first countable. This is a contradiction.

Case 2: $\{u_k : k \in \mathbb{N}\}$ has a non-trivial convergent sequence in $X$.

Without loss of generality, we suppose that $u_k \to u$ as $k \to \infty$. Since $X$ is first-countable, let $\{U_m\}$ be a decreasingly and open neighborhood base of $X$ at point $u$ with $\bigcup_{m \in \mathbb{N}} U_m = \{u\}$. Fix $n$, pick $x_{i_m}(n, k_m) \in U_{i_m} \cap \{x_i(n, k_m)\}$. We can suppose that $i_m < i_{m+1}$, then $\{f(x_{i_m}(n, k_m))\}$ is a subsequence of $\{y_i(n)\}$. Since $f$ is closed, $\{x_{i_m}(n, k_m)\}_{i_m}$ is not discrete in $X$. Then there is a subsequence of $\{x_{i_m}(n, k_m)\}_{i_m}$ converging to a point $b \in X$ because $X$ is a first-countable space. It is easy to see that $b = u$ by $x_{i_m}(n, k_m) \in U_m$ for every $m \in \mathbb{N}$. Hence $x_{i_m}(n, k_m) \to u$ as $m \to \infty$. Then $\{u\} \cup \{x_{i_m}(n, k_m) : n, m \in \mathbb{N}\}$ is a closed copy of $S_\omega$ in $X$. Thus, $X$ is not first countable. This is a contradiction.

Case 3: $\{u_k : k \in \mathbb{N}\}$ is discrete in $X$. 

Let $B = \{u_k : k \in \mathbb{N}\} \cup \{M_k : k \in \mathbb{N}\}$. Since $X$ is strongly monotonically monolithic, $\overline{B}$ is metrizable. Hence there exists a discrete family $\{V_k\}_{k \in \mathbb{N}}$ consisting of open subsets of $\overline{B}$ with $u_k \in V_k$ for each $k \in \mathbb{N}$. Pick $x_{ik}(1,k) \in V_k \cap \{x_i(1,k))_i$ such that $\{f(x_{ik}(1,k))\}_k$ is a subsequence of $\{y_i(n)\}$. Since $\{x_{ik}(1,k)\}_k$ is discrete in $\overline{B}$, $\{f(x_{ik}(1,k))\}_k$ is discrete in $Y$. This is a contradiction.

In a word, $Y$ contains no closed copy of $S_\omega$. \hfill \Box

**Lemma 5.2.** Let $f : X \to Y$ be a closed sequence-covering map, where $X$ is a strongly monotonically monolithic space. Then $\partial f^{-1}(y)$ is compact for each point $y \in Y$.

**Proof.** From Lemma 5.1 it follows that $Y$ contains no closed copy $S_\omega$. Since $X$ is a strongly monotonically monolithic space, every closed separable subset of $X$ is metrizable, and hence is normal. Therefore, $\partial f^{-1}(y)$ is countable compact for each point $y \in Y$ by [21] Theorem 2.6. From [27] Theorem 2.7 it easily follows that every countable compact subset of $X$ is compact. \hfill \Box

**Theorem 5.3.** Let $f : X \to Y$ be a closed sequence-covering map, where $X$ is a strongly monotonically monolithic space. Then $f$ is an 1-sequence-covering map.

**Proof.** It is easy to see by Lemma 5.2 and Theorem 3.6. \hfill \Box

**Corollary 5.4.** Let $f : X \to Y$ be a closed sequence-covering map, where $X$ is a Tychonoff space with a point-countable base. Then $f$ is an 1-sequence-covering map.

In fact, we can replace “Tychonoff” by “regular” in Corollary 5.4, and hence we have the following result.

**Corollary 5.5.** Let $f : X \to Y$ be a closed sequence-covering map, where $X$ is a regular space with a point-countable base. Then $f$ is an 1-sequence-covering map.

**Proof.** Since $X$ has a point-countable base and $f$ is a closed sequence-covering map, $Y$ has a point-countable base by [21] Theorem 3.1. Therefore, $f$ is a boundary-compact map by [22] Lemma 3.2. Hence $f$ is an 1-sequence-covering map by Corollary 3.7. \hfill \Box

We don’t know whether, in Corollary 5.5, the condition “$X$ has a point-countable base” can be replaced by “$X \in \Omega$”. So we have the following question.

**Question 5.6.** Let $f : X \to Y$ be a closed sequence-covering map. If $X \in \Omega$ (and $X$ is regular), then is $f$ an 1-sequence-covering map?

**Corollary 5.7.** Let $f : X \to Y$ be a closed sequence-covering map, where $X$ is a strongly monotonically monolithic space. Then $f$ is an almost-open map.\footnote{\textit{f is an almost-open map}\textsuperscript{2} if there exists a point $x_y \in f^{-1}(y)$ for each $y \in Y$ such that for each open neighborhood $U$ of $x_y$, $f(U)$ is a neighborhood of $y$ in $Y$.}

**Proof.** $f$ is an 1-sequence-covering map by Theorem 5.3. For each point $y \in Y$, there exists a point $x_y \in f^{-1}(y)$ satisfying the Definition 2.2(4). Let $U$ be an open neighborhood of $x_y$. Then $f(U)$ is a sequential neighborhood of $y$. Indeed, for each sequence $\{y_n\} \subset Y$ converging to $y$, there exists a sequence $\{x_n\} \subset X$ such that $\{x_n\}$ converges to $x_y$ and $x_n \in f^{-1}(y_n)$ for each $n \in \mathbb{N}$. Obviously, $\{x_n\}$ is eventually in $U$, and therefore, $\{y_n\}$ is eventually in $f(U)$. Hence $f(U)$
is a sequential neighborhood of \( y \). Since \( X \) is first-countable, \( Y \) is a Fréchet space. Then \( f(U) \) is a neighborhood of \( y \). Otherwise, suppose \( y \in Y \setminus \text{int}(f(U)) \), and therefore, \( y \in Y \setminus f(U) \). Since \( Y \) is Fréchet, there exists a sequence \( \{y_n\} \subset Y \setminus f(U) \) converging to \( y \). This is a contradiction with \( f(U) \) is a sequential neighborhood of \( y \). Therefore, \( f \) is an almost-open map.

\[ \square \]

**Remark** In [27], V. V. Tkachuk has proved that closed maps don’t preserve strongly monotonically monolithic spaces. However, if perfect maps\(^6\) preserve strongly monotonically monolithic spaces, then it is easy to see that closed sequence-covering maps preserve strongly monotonically monolithity by Lemma 5.2. So we have the following two questions.

**Question 5.8.** Do closed sequence-covering maps (or an almost open and closed maps) preserve strongly monotonically monolithity?

**Question 5.9.** Do perfect maps preserve strongly monotonically monolithity?

In [27], V. V. Tkachuk has also proved that open and separable maps preserve strongly monotonically monolithity. However, we have the following result.

**Theorem 5.10.** Let \( f : X \rightarrow Y \) be an open and closed map, where \( X \) is a strongly monotonically monolithic space. Then \( Y \) is a strongly monotonically monolithic space.

**Proof.** From [21] Theorem 3.4 it follows that \( f \) is a sequence-covering map. Therefore, \( \partial f^{-1}(y) \) is compact for each point \( y \in Y \) by Lemma 5.2. Then \( \partial f^{-1}(y) \) is metrizable by [27] Theorem 2.7, and hence it is separable, for each point \( y \in Y \). For each point \( y \in Y \), if \( y \) is a non-isolated point, let \( A_y \) be a countable dense set in the subspace \( \partial f^{-1}(y) \); if \( y \) is an isolated point, then we choose a point \( x_y \in f^{-1}(y) \) and let \( A_y = \{x_y\} \).

Let \( B \subset Y \). Put \( A_B = \bigcup \{A_y : y \in B\} \) and \( \mathcal{N}(B) = \{f(W) : W \in \mathcal{O}(A_B)\} \). It is easy to see that \( \mathcal{N}(B) \) satisfies the conditions (a)-(c) of the definition of strongly monotonically monolithicity. Therefore, we only need to prove that \( \mathcal{N}(B) \) is an external base for \( \overline{B} \). For each point \( y \in \overline{B} \), let \( U \) be open subset in \( Y \) with \( y \in U \).

Case 1: \( y \) is a non-isolated point in \( Y \).

Since \( f \) is an open map, \( \emptyset \neq f^{-1}(y) \subset \overline{f^{-1}(B)} \), and hence \( \partial f^{-1}(y) \subset \overline{f^{-1}(B)} \). Take any point \( x \in \partial f^{-1}(y) \). Then \( x \in \overline{A_B} \). Therefore, there exists a \( V \in \mathcal{O}(A_B) \) such that \( x \in V \subset f^{-1}(U) \). So \( W = f(V) \in \mathcal{N}(B) \) and \( y \in W \subset U \).

Case 2: \( y \) is an isolated point in \( Y \).

It is easy to see that \( \{y\} \in \mathcal{N}(B) \), and therefore, \( y \in \{y\} \subset U \).

In a word, \( \mathcal{N}(B) \) is an external base for \( \overline{B} \). \( \square \)

Let \( \mathcal{B} = \{B_\alpha : \alpha \in H\} \) be a family of subsets of a space \( X \). \( \mathcal{B} \) is point-discrete (or weakly hereditarily closure-preserving) if \( \{x_\alpha : \alpha \in H\} \) is closed discrete in \( X \), whenever \( x_\alpha \in B_\alpha \) for each \( \alpha \in H \).

It is well-known that metrizability, \( g \)-metrizability, \( \aleph \)-spaces, and spaces with a point-countable base are preserved by closed sequence-covering maps, see [21], [28]. Next, we shall consider spaces with a \( \sigma \)-point-discrete \( k \)-network, and shall prove that spaces with \( \sigma \)-point-discrete \( k \)-network are preserved by closed sequence-covering maps. Firstly, we give some technique lemmas.

\(^6\)A map \( f \) is called perfect if \( f \) is a closed and compact map
Lemma 5.11. Let $X$ be an $\mathfrak{N}_1$-compact space\footnote{A space $X$ is called \emph{$\mathfrak{N}_1$-compact} if each subset of $X$ with a cardinality of $\mathfrak{N}_1$ has a cluster point.} with a $\sigma$-point-discrete network. Then $X$ has a countable network.

Proof. Let $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ be a $\sigma$-point-discrete network for $X$, where each $\mathcal{P}_n$ is a point-discrete family for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let

$$B_n = \{ x \in X : |\mathcal{P}_n|_x > \omega \}.$$

Claim 1: $\{ P \setminus B_n : P \in \mathcal{P}_n \}$ is countable.

Suppose not, there exist an uncountable subset $\{ P_\alpha : \alpha < \omega_1 \} \subset \mathcal{P}_n$ and $\{ x_\alpha : \alpha < \omega_1 \} \subset X$ such that $x_\alpha \in P_\alpha \setminus B_n$. Since $\mathcal{P}_n$ is a point-discrete family and $X$ is $\mathfrak{N}_1$-compact, $\{ x_\alpha : \alpha < \omega_1 \}$ is countable. Without loss of generality, we can assume that there exists $x \in X \setminus B_n$ such that each $x_\alpha = x$. Therefore, $x \in B_n$, a contradiction.

Claim 2: For each $n \in \mathbb{N}$, $B_n$ is a countable and closed discrete subspace for $X$.

For each $Z \subset B_n$ with $|Z| \leq \omega_1$, let $Z = \{ x_\alpha : \alpha \in \Lambda \}$. By the definition of $B_n$ and Well-ordering Theorem, it is easy to obtain by transfinite induction that $\{ P_\alpha : \alpha \in \Lambda \} \subset \mathcal{P}_n$ such that $x_\beta \in P_\alpha$ and $P_\alpha \neq P_\beta$ for each $\alpha \neq \beta$. Therefore, $Z$ is a countable and closed discrete subspace for $X$. Hence $B_n$ is a countable and closed discrete subspace.

For each $n \in \mathbb{N}$, let $\mathcal{P}'_n = \{ P \setminus B_n : P \in \mathcal{P}_n \} \cup \{ \{ x \} : x \in B_n \}$. Then $\mathcal{P}'_n$ is a countable family.

Obviously, $\bigcup_{n \in \mathbb{N}} \mathcal{P}'_n$ is a countable network for $X$. \hfill $\Box$

The proof of the following lemma is an easy exercise.

Lemma 5.12. Let $\{ F_\alpha \}_{\alpha \in A}$ be a point-discrete family for $X$ and countably compact subset $K \subset \bigcup_{\alpha \in A} F_\alpha$. Then there exists a finite family $\mathcal{F} \subset \{ F_\alpha \}_{\alpha \in A}$ such that $K \subset \bigcup \mathcal{F}$.

Lemma 5.13. Let $\mathcal{P}$ be a family of subsets of a space $X$. Then $\mathcal{P}$ is a $\sigma$-point-discrete $wcs^*$-network\footnote{A family $\mathcal{P}$ of $X$ is called a $wcs^*$-network if, whenever a sequence $\{ x_n \}$ converges to $x \in U$ with $U$ open in $X$, there are a $P \in \mathcal{P}$ and a subsequence $\{ x_{n_i} \}$ of $\{ x_n \}$ such that $x_{n_i} \in P \subset U$ for each $n_i \in \mathbb{N}$.} for $X$ if and only if $\mathcal{P}$ is a $\sigma$-point-discrete $k$-network\footnote{A family $\mathcal{P}$ of $X$ is called a $k$-network if whenever $K$ is a compact subset of $X$ and $K \subset U$ with $U$ open in $X$, there is a finite subfamily $\mathcal{P}' \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{P}' \subset U$.} for $X$.

Proof. Sufficiency. It is obvious. Hence we only need to prove the necessity.

Necessity. Let $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ be a $\sigma$-point-discrete $wcs^*$-network, where each $\mathcal{P}_n$ is a point-discrete family for each $n \in \mathbb{N}$. Suppose that $K$ is compact and $K \subset U$ with $U$ open in $X$. For each $n \in \mathbb{N}$, let

$$\mathcal{P}'_n = \{ P \in \mathcal{P}_n : P \subset U \}, \quad F_n = \bigcup \mathcal{P}'_n.$$

Then there exists $m \in \mathbb{N}$ such that $K \subset \bigcup_{k \leq m} F_k$. Suppose not, there is a sequence $\{ x_n \} \subset K$ with $x_n \in K \setminus \bigcup_{i \leq n} F_i$. By Lemma 5.11 it is easy to see that $K$ is metrizable. Therefore, $K$ is sequentially compact. It follows that there exists a convergent subsequence of $\{ x_n \}$. Without loss of generality, we assume that $x_n \to x$. Since $\mathcal{P}$ is a $wcs^*$-network, there exist a $P \in \mathcal{P}$, and a subsequence $\{ x_{n_i} \}$ of $\{ x_n \}$ such that $\{ x_{n_i} \} : i \in \mathbb{N} \subset P \subset U$. Therefore, there exists $l \in \mathbb{N}$
such that $P \in P'$. Choose $i > l$, since $P \subset F_l$, $x_n \in F_l$, a contradiction. Hence there exists $m \in \mathbb{N}$ such that $K \subset \bigcup_{k \leq m} F_k$. By Lemma 5.12 there is a finite family $P'' \subset \bigcup_{i \leq m} P'_i$ such that $K \subset \bigcup P'' \subset U$. Therefore, $P$ is a $k$-network. □

**Theorem 5.14.** Closed sequence-covering maps preserve spaces with a $\sigma$-point-discrete $k$-network.

**Proof.** It is easy to see that closed sequence-covering maps preserve spaces with a $\sigma$-point-discrete $\omega cs^*$-network. Hence closed sequence-covering maps preserve spaces with a $\sigma$-point-discrete $k$-network by Lemma 5.13. □

**Question 5.15.** Do closed maps preserve spaces with a $\sigma$-point-discrete $k$-network?

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