ON THE FREQUENCY OF VANISHING OF QUADRATIC TWISTS OF MODULAR L-FUNCTIONS

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Abstract. We present theoretical and numerical evidence for a random matrix theoretical approach to a conjecture about vanishings of quadratic twists of certain L-functions.

In this paper we present some evidence that methods from random matrix theory can give insight into the frequency of vanishing for quadratic twists of modular L-functions. The central question is the following: given a holomorphic newform \( f \) with integral coefficients and associated L-function \( L_f(s) \), for how many fundamental discriminants \( d \) with \( |d| \leq x \), does \( L_f(s, \chi_d) \), the L-function twisted by the real, primitive, Dirichlet character associated with the discriminant \( d \), vanish at the center of the critical strip to order at least 2?

This question is of particular interest in the case that the L-function is associated with an elliptic curve, in light of the conjecture of Birch and Swinnerton-Dyer. This case corresponds to weight \( k = 2 \). We will focus on this case for most of the paper, though we do make some remarks about higher weights (see (26) and below).

Suppose that \( E/Q \) is an elliptic curve with associated L-function

\[
L_E(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}
\]

for \( \Re s > 1 \). Then, as a consequence of the Taniyama-Shimura conjecture, recently solved by Wiles, Taylor, ([W], [TW]), and Breuil, Conrad, and Diamond, \( L_E \) is entire and satisfies a functional equation

\[
\left( \frac{2\pi}{\sqrt{N}} \right)^{-s} \Gamma(s)L_E(s) = \Phi_E(s) = w_E \Phi(1-s)
\]

where \( N \) is the conductor of \( E \) and \( w_E = \pm 1 \) is called the sign of the functional equation. Note that we have normalized the coefficients \( a_n \) so that the functional equation of \( L_E \) relates the values at \( s \) and \( 1-s \). The numbers \( a_n \) satisfy \( |a_n| \leq d(n) \), where \( d(n) \) is the number of positive divisors of \( n \), and \( a_n \) is an integer (related to the numbers of points on \( E \) mod \( p \) for primes \( p \) which divide \( n \)). Let

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$d$ represent a fundamental discriminant and let $\chi_d$ be the associated quadratic character (i.e. $\chi_d(n) = \left(\frac{d}{n}\right)$, the Kronecker symbol). We assume, for simplicity, that $(d, N) = 1$. Then the twisted $L$-function is

$$L_E(s, \chi_d) = \sum_{n=1}^{\infty} \frac{a_n^{*} \chi_d(n)}{n^s}.$$  

This $L$-function has functional equation

$$\left(\frac{2\pi}{|d|\sqrt{N}}\right)^{-s} \Gamma(s) L_E(s, \chi_d) = \Phi_E(s, \chi_d) = w_E(\chi_d(-N)) \Phi(1 - s, \chi_d);$$

it is actually the $L$-function of another elliptic curve, namely the quadratic twist of $E$ by $d$.

The conjecture of Birch and Swinnerton-Dyer predicts that the order of vanishing of the $L$-function of an elliptic curve at the central critical point $s = 1/2$ is the same as the rank of the Mordell-Weil group of the elliptic curve. Thus it would be of interest to find an asymptotic formula for

$$V_E(x) := \sum_{|d| \leq x} 1 \quad \text{where} \quad w_E(\chi_d(-N)) = 1, \quad L_E(1/2, \chi_d) = 0.$$  

since this would potentially be counting how often the twists of a given elliptic curve have rank at least 2. (Note that we have restricted the sum to twists for which the sign of the functional equation is +1; these $L$-functions vanish to order at least 2 because of the symmetry implied by the functional equation.) Goldfeld [G] has predicted that $V_E(x) = o(x)$. More specifically, he predicts that asymptotically, $1/2$ of all twists will have rank 0 and $1/2$ of all twists will have rank 1; consequently ranks 2 and higher should be infrequent. We will give a more precise conjecture about the frequency of twists with ranks at least 2.

Sarnak has predicted that $V_E(x)$ should be about $x^{3/4}$. His reasoning has to do with the formulas of Waldspurger [Wa], Shimura [Sh], and Kohnen and Zagier [KZ] which relate the value of $L_E(1/2, \chi_d)$ to the Fourier coefficient of a half-integral weight modular form. Roughly,

$$L_E(1/2, \chi_d) = \kappa_E c_E(|d|) \sqrt{d}$$  

where $\kappa_E$ depends only on $E$ and where the integers $c_E(|d|)$ are the Fourier coefficients of a half-integral weight form. The Ramanujan conjecture for these coefficients predicts that $c_E(|d|) \ll |d|^{1/4+\epsilon}$ for every $\epsilon > 0$. If $c_E(d)$ takes on each integer value up to $|d|^{1/4}$ about the same number of times for $|d| \leq x$, then it should take the value 0 about $x^{3/4}$ times.

Using random matrix theory, we would like to give a conjecture of the form

$$V_E(x) \sim b_E x^{3/4} (\log x)^{\epsilon_E}$$

for certain constants $b_E$ and $\epsilon_E$. The basic idea is to regard the family

$$F_{E^+} = \{ L_E(s, \chi_d) : w_E \chi_d(-N) = +1 \}$$

as an example of a family of $L$-functions associated to a random $E$.
as an orthogonal family, in the sense of the families introduced by Katz and Sarnak ([KS1], [KS2]). More specifically, this family conjecturally has symmetry type $O^+$. Thus, for example, we believe that the statistics of the low lying zeros of the $L$-functions in this family will match the statistics of eigenvalues near 1 of the matrices in $SO(2N)$.

The point of departure for our conjectures is the work of Keating and Snaith [KeSn1] and [KeSn2] (see also [BH] and [CF]) which indicates that the moments
\[ M_E(T, k) = \frac{1}{T^*} \sum_{|d| \leq T} L_E(1/2, \chi_d)^k \]
(with $T^* = \sum_{L_E(s, \chi_d) \in F_{E+}}|d|$) apparently behave like the moments of the characteristic polynomials of matrices in $SO(2N)$ where $N$ is of size $\log T$. Precisely, they conjecture that
\[ M_E(T, k) \sim g_k(O^+)a_k(E) (\log T)^{(k-1)/2} \]
where
\[ g_k(O^+) = \prod_{p} \left(1 - \frac{1}{p}\right)^{k(k-1)/2} \left(\frac{1 - \frac{a_p}{p} + \frac{1}{p}}{2} + \left(1 + \frac{a_p}{p} + \frac{1}{p}\right)^{-k}\right) \left(\frac{p}{p+1} + \frac{1}{p+1}\right) \]

This conjecture arises from arithmetical considerations together with the fact from random matrix theory that the moments of the characteristic polynomials of matrices in $SO(2N)$, evaluated at the point 1, averaged over the group can be explicitly evaluated. Thus,
\[ M_O(N, s) = \int_{SO(2N)} |\det(U - I)|^s \, dU \]
\[ = 2^{2Ns} \prod_{j=1}^{N} \frac{\Gamma(N + j - 1)\Gamma(s + j - 1/2)}{\Gamma(j - 1/2)\Gamma(s + j + N - 1)} \]

where $dU$ is the Haar measure for $SO(2N)$. The connection with $g_k$ is that
\[ M_O(N, k) \sim g_k(O^+)N^{k(k-1)/2} \]
as $N \to \infty$. Note that the formula for $g_k(O^+)$ can be extended to all real $k$ by
\[ g_k(O^+) = 2^{k^2/2} \frac{G(1+k)\sqrt{\Gamma(1+2k)}}{\sqrt{G(1+2k)\Gamma(1+k)}} \]
where $G$ is Barnes’ Double Gamma function.

Continuing to follow Keating and Snaith ([KeSn2], equations (74) - (81)), we observe that knowledge of all the complex moments of characteristic polynomials, evaluated at 1, in $SO(2N)$ gives complete information about the density function for the distribution of values of the characteristic polynomials at this point. Specifically, the latter is the Mellin transform of the former:

$$(14) \quad P_O(N, x) = \frac{1}{2\pi i x} \int_{(c)} M_O(N, s)x^{-s}ds$$

where $(c)$ denotes the vertical line path from $c - i\infty$ to $c + i\infty$; this formula is valid for all real $x$. Note that $P_O(N, x) dx$ gives the probability that $\det(U - I) = x$ for an element $U$ of $SO(2N)$.

For small positive $x$, the pole of $M_O(N, s)$ at $s = -1/2$ determines the dominating behavior of $P_O(N, x)$. In fact, we see that

$$(15) \quad P_O(N, x) \sim x^{-1/2}2^{-N} \Gamma(N)^{-1} \prod_{j=1}^{N} \frac{\Gamma(N + j - 1)\Gamma(j)}{\Gamma(j - 1/2)\Gamma(j + N - 3/2)}$$

$$:= x^{-1/2}h(N)$$

as $x \to 0^+$. As $N \to \infty$,

$$(16) \quad h(N) \sim 2^{-7/8}G(1/2)\pi^{-1/4}N^{3/8}.$$

An interpretation of the above is that the probability that an element of $SO(2N)$ has a characteristic polynomial whose value at 1 is $X$ or smaller is

$$(17) \quad \sim \int_0^X x^{-1/2}h(N) \, dx = 2X^{1/2}h(N).$$

We apply this reasoning to the values of $L_E(1/2, \chi_d)$. In particular, by (6) the fact that the $c_E(|d|)$ are integers implies that these values are discretized. If, for example, it is known that

$$(18) \quad L_E(1/2, \chi_d) < \kappa_E/\sqrt{|d|},$$

then it follows that $L_E(1/2, \chi_d) = 0$. Similarly, if

$$(19) \quad \kappa_E/\sqrt{|d|} \leq L_E(1/2, \chi_d) < 4\kappa_E/\sqrt{|d|},$$

then it must be the case that $L_E(1/2, \chi_d) = \kappa_E/\sqrt{|d|}$.

We assume now that the distribution of values of $L_E(1/2, \chi_d)$ will behave like the values of the determinants of random orthogonal matrices with some suitable restrictions and use this assumption to conjecture results about the frequency of vanishing of $L_E(1/2, \chi_d)$. The restrictions we have in mind are of an arithmetical nature. First of all, we want to include the arithmetical factor $a_k(E)$. We expect that

$$(20) \quad M_E(T, s) \sim a_s(E)M_O(N, s)$$
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with $N \sim \log T$. Thus, using

$$P_E(T, x) = \frac{1}{2\pi i} \int_{(c)} M_E(T, s)x^{-s}ds$$

(21)

together with (20), we reiterate the conjecture of [KeSn2] (equation (81)) that

$$P_E(T, x) \sim a_{-1/2}x^{-1/2}h(N)$$

(22)

should approximate for small $x$ the probability density function for values of $L_E(1/2, \chi_d)$. Of course, this formula cannot be too accurate as we have already remarked that the values of $L_E(1/2, \chi_d)$ are discretized. The precise nature of this discretization is somewhat involved; it involves the constant $\kappa_E$ for which we have explicit formulas, but it also involves the coefficients $c_E(|d|)$ whose arithmetic nature is difficult to describe. Simplistically, we would like to use the integral of (22), as in (17), and (18) to predict that

$$\#\{|d| \leq T : L_E(1/2, \chi_d) = 0, L_E(s, \chi_d) \in \mathcal{F}_E^+\} \sim \frac{8}{\sqrt[4]{T\log T^{3/2}}}h(N)$$

(23)

with $N \sim \log T$. However, the $c(|d|)$ are divisible by some predetermined powers of 2 which change this discretization. For example, in the case of the congruent number curve $E_{32}: y^2 = x^3 - x$, the number $c(|d|)$ is divisible by $\tau(d)$ for squarefree $d$ where $\tau(d)$ is the number of divisors of $d$. Thus, if

$$L_E(1/2, \chi_d) < \kappa_{E_{32}}\tau(d)^2 / |d|^{1/2},$$

(24)

then $L_{E_{32}}(1/2, \chi_d) = 0$. If we introduce the factor $\tau(d)$ into (23) it will raise the expected frequency of vanishing by a factor of about $\log T$ giving a total order of magnitude $T^{3/4}(\log T)^{11/8}$ for the frequency of vanishing. We expect this to be the correct order of magnitude but are not able to say yet what constant we expect.

If we restricted to prime twists ($|d| = p$) then the extra powers of 2 are not so significant. This leads to

**Conjecture 1.** Let $E$ be an elliptic curve defined over $Q$. Then there is a constant $c_E > 0$ such that

$$\sum_{p \leq T \atop L_E(1/2, \chi_p) = 0 \atop L_E(s, \chi_p) \in \mathcal{F}_E^+} 1 \sim c_E T^{3/4}(\log T)^{-5/8}.$$  

(25)

We will return to a discussion of the value of $c_E$ in another paper.

We remark that our arguments apply equally well to newforms $f$ of weight 4 with integral Fourier coefficients and even functional equation. Here we have a different discretization and expect that

$$\sum_{p \leq T \atop L_f(1/2, \chi_p) = 0} 1 \sim c_f T^{1/4}(\log T)^{-5/8}.$$  

(26)

Note in particular that the exponent on $T$ is now $1/4$. For a newform of weight 6 or higher, we expect that there will be at most a finite number of twists that vanish. We have some numerical evidence to support these conjectures.

For the remainder of this paper we would like to discuss a numerical experiment which allows us to skirt the delicate issue of the arithmetic nature of the $c(|d|)$. For a prime $p$ we consider the ratios

$$R_p(T) = \left( \sum_{|d| \leq T} 1 / \sum_{|d| \leq T} \frac{L_E(1/2, \chi_d)}{L_E(s, \chi_d) \in F_E^+} \right) / \left( \sum_{|d| \leq T} 1 / \sum_{|d| \leq T} \frac{L_E(1/2, \chi_d)}{L_E(s, \chi_d) \in F_E^+} \right).$$

By considering this ratio, the powers of $T$, of log $T$, and the constants intrinsic to the curve $E$ should all cancel out.

More generally, let

$$Q_p(k) = \lim_{T \to \infty} \frac{\sum_{|d| \leq T, \chi_d(p) = \pm 1} L_E(1/2, \chi_d)^k}{\sum_{|d| \leq T, \chi_d(p) = \pm 1} L_E(s, \chi_d) \in F_E^+}.$$

assuming that this limit exists. What is a reasonable conjecture for $Q_p(k)$?

Using standard techniques from analytic number theory (see [I]), we can evaluate $Q_p(1)$. Based on the analysis involved in such an evaluation, we expect that

$$Q_p(k) = \frac{(p + 1 + a_p)^k}{(p + 1 - a_p)^k},$$

where $a_p$ is the $p$-th Fourier coefficient of the modular form associated with $E$. The heuristics are as follows. Consider either sum that appears in (28)

$$\sum_{|d| \leq T, \chi_d(p) = \pm 1} L_E(1/2, \chi_d)^k = \sum_{|d| \leq T, \chi_d(p) = \pm 1} \left( \sum_{n=1}^{\infty} \frac{a_n \chi_d(n)}{n} \right)^k = \sum_{n=1}^{\infty} \frac{b_n \chi_d(n)}{n}$$

where

$$b_n = \sum_{n=n_1 \cdots n_k} a_{n_1} a_{n_2} \cdots a_{n_k},$$

the sum being over all ways of writing $n$ as a product of $k$ factors. Summing over $d$, the main contribution to (30) comes from those $n$’s that are of the form $p^r m^2$, i.e. a power of $p$ times a perfect square (since, unless $(d, n) > 1$, these always have $\chi_d(n) = \chi_d(p)^r$ while, for other $n$’s, we get cancellation as we sum over $d$). So, the main contribution to (30) is roughly

$$\sum_{p^r m^2} \frac{b_p r m^2 \chi_d(p)^r}{p^r m^2}.$$
But, $b_{uv} = b_u b_v$ when $(u, v) = 1$, so the inner sum above equals

$$\sum_{(m, p) = 1} \frac{b_m}{m^2} \sum_{r=0}^{\infty} \frac{b_{pr} \chi_d^r(p)}{p^r}.$$ 

Now,

$$\sum_{r=0}^{\infty} \frac{b_{pr} \chi_d^r(p)}{p^r} = \left( \sum_{r=0}^{\infty} \frac{a_{pr} \chi_d^r(p)}{p^r} \right)^k.$$ 

Further, one can write an explicit formula for $a_{pr}$ in terms of $a_p$ and $p$. Assume that $E$ has good reduction mod $p$. Writing the corresponding Euler factor $(1 - a_p p^{-s} + p^{1-2s}) = (1 - ap^{-s})(1 - \beta p^{-s})$, with $\alpha + \beta = a_p$ and $\alpha \beta = p$, we have, using partial fractions, $a_{pr} = (\alpha^r + 1 - \beta^{r+1})/(\alpha - \beta)$. Substituting this into (31), and summing the geometric series we get

$$\left( \sum_{r=0}^{\infty} \frac{a_{pr} \chi_d^r(p)}{p^r} \right)^k = \frac{p^{2k}}{(p + 1 - \chi_d(p) a_p)^k}.$$ 

Hence, the only difference in the numerator and denominator of (28), asymptotically, is a factor of

$$\frac{(p + 1 + a_p)^k}{(p + 1 - a_p)^k}.$$ 

Thus, by the random matrix theory considerations above (as in (15)), taking $k = -1/2$ leads to

**Conjecture 2.** With $R_p(T)$ defined as above, and $E$ having good reduction mod $p$,

$$R_p = \lim_{T \to \infty} R_p(T) = \sqrt{\frac{p + 1 - a_p}{p + 1 + a_p}}.$$ 

Note that the number $N_p$ of points on the elliptic curve $E$ over the finite field $F_p$ of $p$ elements can be computed as

$$N_p = p + 1 - a_p$$

so that the above ratio is the square-root of the ratio of the number of points on $E/F_p$ to the number of points on $E^* / F_p$ for any character $\chi$ with $\chi(p) = -1$.

We conclude with some numerical evidence to support these conjectures. We consider three elliptic curves which we call $E_{11}$, $E_{19}$, and $E_{32}$. These are associated to the unique newforms of weight 2 and levels 11, 19, and 32 respectively. The values of $L(s, \chi_d)$ were evaluated by computing the $c(|d|)$'s of the corresponding weight 3/2 forms. For the level 11 and 19 curves, we only computed these for $d < 0$ and $d$ odd. For the level 32 curve, we computed these for all odd $d$. The relevant form for the level 32 case is described in [Ko]. The forms for the level 11
Table 1. A table in support of Conjecture 2, comparing $R_p$ v.s. $R_p(T)$ for the three elliptic curves $E_{11}$, $E_{19}$, $E_{32}$ ($T$ equal to $333605031$, $263273979$, $930584451$ respectively). More of this data, for $p < 2000$, is depicted in the figures below. The 0 entries for $p = 11$ and $p = 19$ are explained by the fact that we are restricting ourselves to twists with even functional equation, $w_{E\chi}(N) = 1$. Hence for $E_{11}$ and $E_{19}$, we are only looking at twists with $\chi_d(11) = \chi_d(19) = -1$.

and 19 cases can be computed according to [Gr] and were given to us by Fernando Rodriguez-Villegas.
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Figure 1. Pictures depicting $R_p/R_p(T)$, for $p < 2000, T$ as in Table 1.

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Figure 2. Figures in support of Conjecture 1. These depict the l.h.s. of (25) divided by $T^{3/4}(\log T)^{-5/8}$. For the level 11 and 19 curves, we only looked at twists with $d < 0$, $d$ prime, even functional equation. We also depict the l.h.s of (23) divided by $T^{3/4}(\log T)^{11/8}$ for the level 32 curve and odd $d$. While the pictures are reasonably flat, $\log(T)$ is almost constant for most of the interval in question. The flatness we are observing reflects the main dependence on $T^{3/4}$.

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Figure 3. The first picture depicts the value distribution of $L_{E_{11}}(1/2, \chi_d)$, for prime $|d|, -788299808 < d < 0$, even functional equation, compared to $P_O(N, x)$, with $N = 20$. For contrast, we depict, in the second picture, the value distribution of $L(1/2, \chi_d)$ (Dirichlet L-functions) for all fundamental $800000 < |d| < 1000000$. Here, the Katz-Sarnak philosophy predicts a Unitary Symplectic family, and so we compare the data against $P_{USp}(N, x)$, $N = 5$.

In these pictures, we have renormalized the $L$ values so as to have the same means as $P_O(20, x)$, and $P_{USp}(5, x)$ respectively, and have not incorporated the $a_k$ values into the pictures.

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