Bounds on the Number of Edges in Hypertrees

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Abstract

Let $\mathcal{H}$ be a $k$-uniform hypergraph. A chain in $\mathcal{H}$ is a sequence of its vertices such that every $k$ consecutive vertices form an edge. In 1999 Katona and Kierstead suggested to use chains in hypergraphs as the generalisation of paths. Although a number of results have been published on hamiltonian chains in recent years, the generalisation of trees with chains has still remained an open area.

We generalise the concept of trees for uniform hypergraphs. We say that a $k$-uniform hypergraph $\mathcal{F}$ is a hypertree if every two vertices of $\mathcal{F}$ are connected by a chain, and an appropriate kind of cycle-free property holds. An edge-minimal hypertree is a hypertree whose edge set is minimal with respect to inclusion.

After considering these definitions, we show that a $k$-uniform hypertree on $n$ vertices has at least $n-(k-1)$ edges up to a finite number of exceptions, and it has at most $\binom{n}{k-1}$ edges. The latter bound is asymptotically sharp in 3-uniform case.

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1. Introduction

1.1. Definitions of Trees

A graph is called a tree if it is connected and cycle-free (a cycle-graph) on vertices $v_1, v_2, \ldots, v_n$ is the graph with edges $e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, \ldots, e_{n-1} = \{v_{n-1}, v_n\}, e_n = \{v_n, v_1\}$. Trees play an important role in many areas of applied mathematics, such as theory of algorithms, cryptography, data structures and information theory. It is well known that the following are equivalent for a graph $F$ on $n$ vertices.

- $F$ is a tree,
- any pair of vertices of $F$ are connected by a unique path,
- $F$ is an edge-minimal connected graph,
- $F$ is an edge-maximal cycle-free graph,
- $F$ is connected and has $n - 1$ edges.

Let us note that here and in the following, edge-minimal and edge-maximal refers to minimality or maximality with respect to inclusion of the set of edges.

Our main goal is to generalize the tree property for hypergraphs. It turns out that the situation is much more complicated than the graph case, one can only give lower and upper bounds for the number of edges in hypertrees.

1.2. Early generalisations of paths and cycles

Some earlier concepts of paths and cycles in hypergraphs are recalled in this subsection. Amongst those, Berge-paths and Berge-cycles are treated in more details.

A hypergraph is a set system i.e. a pair $(V, \mathcal{E})$ where $V$ is a nonempty finite set (called the set of vertices) and $\mathcal{E}$ is a family of subsets of $V$ (called the set of edges). An isolated vertex in a hypergraph is a vertex which does not contained in any edges. A loop is an edge of cardinality one.

The path-concept, introduced by Berge is one of the very earliest ones \[1\]. We will only consider uniform hypergraphs, but this definition works for arbitrary hypergraphs, too.
Definition 1. A Berge-path of length $l$ in a hypergraph $\mathcal{H}$ is a sequence $(v_1, e_1, v_2, e_2, \ldots, v_l, e_l, v_{l+1})$ such that

- $v_1, v_2, \ldots, v_l, v_{l+1}$ are distinct vertices of $\mathcal{H}$;
- $e_1, e_2, \ldots, e_l$ are distinct edges of $\mathcal{H}$;
- $v_k, v_{k+1} \in e_k$, for all $1 \leq k \leq l$.

If $l > 1$ and $v_1 = v_{l+1}$, then this “path” is called a Berge-cycle of length $l$.

Remark: For graphs the Berge-path and Berge-cycle is the same as the ordinary path and cycle. If $\mathcal{H}$ consists of large edges, then we have a considerable freedom in constructing a path on a given sequence of vertices because every edge of the path has two fixed vertices (on the path), but the other vertices can be chosen freely. Our definition of chain will be more restrictive.

Figure 1: A Berge-path on 5 vertices

For $u, v \in V$ let $u \equiv v$ denote the fact that there exists a Berge-path in $\mathcal{H}$ with endpoints $u$ and $v$. It is easy to see that “$\equiv$” is an equivalence relation on $V$ (in contrast to our chain concept; see Section 2). We define the connected components of $\mathcal{H}$ to be the equivalence classes of this relation.

We mention two results on Berge-cycle-free hypergraphs. In these, one can notice analogies with trees and forests of usual graph theory.

Theorem 2 (Berge [1]). If $\mathcal{H}$ is a hypergraph with $n$ vertices, $p$ connected components and $\mathcal{E} = \{e_i\}_{i=1}^m$, then it contains no Berge-cycles if and only if

$$\sum_{i=1}^m (|e_i| - 1) = n - p.$$
For graphs this gives that \( m = n - p \), which is a well known feature of forests with \( p \) connected components.

**Theorem 3 (Lovász [1])**. Let \( \mathcal{H} \) be a hypergraph with \( n \) vertices \( m \) edges and \( p \) connected components, which contains no Berge-cycles of length at least 3 and no loops. If \(|e_i \cap e_j| \leq 2\) for all \( e_i \neq e_j \), then

\[
\sum_{i=1}^{m} (|e_i| - 2) < n - p.
\]

This theorem shows that if \( \mathcal{H} \) is a connected 3-uniform hypergraph which contains no Berge-cycles of length greater than 2, then \( m < n - 1 \).

One of the earliest definitions of Hamiltonian cycle in hypergraphs was given by Bermond et. al. in 1976. The authors gave extensions of known results about Hamiltonian cycles of graphs to the hypergraph setting [4].

**Definition 4**. A cyclic permutation \((v_1, v_2, \ldots, v_n)\) of the vertices of \( \mathcal{H} \) is called a hypergraph hamiltonian cycle if for every \( 1 \leq i \leq n \) there exists an edge \( e_i \in \mathcal{E} \) such that \( v_i, v_{i+1} \in e_i \).

This definition is not equivalent to that of the hamiltonian Berge-cycle because in this case there can be identical edges among the \( e_i \)s.

### 1.3. Hamiltonian cycles and Dirac-type theorems for hypergraphs

Although our paper presents a new interesting definition of hypertrees, it has been inspired by some earlier papers on Dirac-type theorems for hypergraphs.

The first article in this area is dated to 1999 co-authored by Katona and Kierstead. It remains an active field of research, and a number of results have been published on hamiltonian chains in recent years by E. Szemerédi, V. Rödl, A. Ruciński, D. Kühn, D. Osthus, R. Mycroft, H. Hán, M. Schacht and others. For more details see [2, 3, 4, 5, 7, 8].

### 2. Definition of hypertrees

In this section, we generalise the concept of trees for \( k \)-uniform hypergraphs, where chains play the role of paths. After introducing the basic definitions, we discuss lower and upper bounds for the edge number of hypertrees and show that these are sharp in particular cases.
2.1. Possibilities for defining hypertrees

First, one has to clarify the notions of cycle, semicycle and chain. The relation between a chain and a path is similar to the relation between a tight hamiltonian cycle (see [5]) and a usual hamiltonian cycle. In the following sections we assume that there are no multiple edges in a hypergraph.

Definition 5 (Cycle). The \( k \)-uniform hypergraph \( C = (V, E) \) is a cycle if there exists a cyclic sequence \( v_1, v_2, \ldots, v_l \) of its vertices such that every vertex appears at least once (possibly more times) and for all \( 1 \leq i \leq l \), \( \{v_i, v_{i+1}, \ldots, v_{i+k-1}\} \) are distinct edges of \( C \). The length of the cycle \( C \) is the number of its edges: \( l \).

Note that a cycle has at least \( k + 1 \) edges. To see this, notice that if it is defined by the sequence \( v_1, v_2, \ldots, v_l \), then it has exactly \( l \) (different) edges by definition. \( k \)-uniformity implies that \( l \) is at least \( k \). However, \( l = k \) means that the \( k \) edges of the definition coincide, each covers the whole vertex set.

Cycles are too special structures for our purpose, so we use a weaker concept. If a chain intersects itself, then it contains a subhypergraph called a semicycle. Hypergraphs without semicycles are in close resemblance with (ordinary) forests.

Definition 6 (Semicycle). The nonempty \( k \)-uniform hypergraph \( C = (V, E) \) is a semicycle if there exists a sequence \( v_1, v_2, \ldots, v_l \) of its vertices such that every vertex appears at least once (possibly more times), \( v_1 = v_l \) and for all \( 1 \leq i \leq l-k+1 \), \( \{v_i, v_{i+1}, \ldots, v_{i+k-1}\} \) are distinct edges of \( C \). The length of the semicycle \( C \) is the number of its edges: \( l - k + 1 \).

Notice that a semicycle must have at least 3 edges. Clearly, one edge cannot form a semicycle. If a semicycle has only two edges, then it is defined by a sequence \( v_1, v_2, \ldots, v_k, v_{k+1} \) of \( k + 1 \) vertices. However, \( v_1 = v_{k+1} \), so there are \( k \) different vertices for two different edges, which is not enough.

Simple chains will play the most important role in defining hypertrees because we intend to require a natural chain-connectedness property.

Definition 7 (Chain). The nonempty \( k \)-uniform hypergraph \( L = (V, E) \) is a chain if there exists a sequence \( v_1, v_2, \ldots, v_l \) of its vertices such that every vertex appears at least once (possibly more times), \( v_1 \neq v_l \) and for all \( 1 \leq i \leq l-k+1 \), \( \{v_i, v_{i+1}, \ldots, v_{i+k-1}\} \) are distinct edges of \( C \). The length of the chain \( L \) is the number of its edges: \( l - k + 1 \).
Figure 2: A non-self-intersecting 3-uniform semicycle of length 5

Figure 3: A self-intersecting 3-uniform chain of length 7

A chain (cycle) is self-intersecting if there is a vertex appearing at least twice in its defining sequence. Similarly, a semicycle is self-intersecting if there is a vertex appearing at least twice in its defining sequence, except for the condition that the first and the last vertices must be identical.

Figure 4: A non-self-intersecting 3-uniform chain of length 3

Remark: A non-self-intersecting chain (cycle) is also called a tight-path (tight-cycle) in other related papers.

Remark: The length of a non-self-intersecting chain on $n$ vertices is $n - (k - 1)$, matching the fact that the length of a (2-uniform) path on $n$ vertices is $n - 1$. 

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Similarly, the length of a non-self-intersecting semicycle on $n$ vertices is $n - k + 2$, and for graphs, the length of a (2-uniform) cycle on $n$ vertices is $n$.

**Definition 8.** A $k$-uniform hypergraph $H$ is

- chain-connected if every pair of its vertices is connected by a chain, i.e. there exists a subhypergraph which is a chain and contains both vertices;
- semicycle-free if it contains no semicycle as subhypergraph;
- edge minimal/maximal with respect to the property $\phi$ if it has property $\phi$, and its edge set is minimal/maximal with respect to inclusion.

The following claim shows that to verify that a hypergraph is semicycle-free it is enough to show that it does not contain any non-self-intersecting semicycles.

**Claim 9.** If a $k$-uniform hypergraph $H$ contains a semicycle, then it also contains a non-self-intersecting semicycle.

**Proof:** Let $C$ be the shortest semicycle in $H$, defined by the vertex sequence $(x_1, x_2, \ldots, x_l)$. We show that $C$ is non-self-intersecting. Assume on the contrary that $C$ intersects itself. Then there exist indices $1 \leq i < j \leq l$ such that $x_i = x_j$ and $i \neq 1$ or $j \neq l$. However, the sequence $x_i, x_{i+1}, \ldots, x_j$ defines a semicycle which is shorter than $C$, a contradiction. □

**Corollary 10.** If a uniform hypergraph does not contain any non-self-intersecting semicycles, then it is semicycle-free.

The counterpart of the previous theorem among graphs states that if a graph contains a closed walk, then it also contains a cycle. Among hypergraphs, this theorem becomes false if we use cycle instead of semicycle. So, considering the semicycle as the generalisation of the graph-theoretical cycle seems to be a fruitful idea.

**Claim 11.** A semicycle-free $k$-uniform hypergraph does not contain a self-intersecting chain.
Proof: Let $L = (x_1, \ldots, x_l)$ be a chain in $\mathcal{H}$, and assume indirectly that it is self-intersecting. Then there exists $1 \leq i < j \leq l$ such that $x_i = x_j$ ($i \neq 1$ or $j \neq l$ since $x_1 \neq x_l$). Now the part $\mathcal{C} = (x_i, \ldots, x_j)$ of the chain induces a semicycle, which is a contradiction. □

At this point, we can define hypertrees the same way we defined trees in Section [1]. Some of these definitions are not compatible with the concept of chain, while others may be too general. One has to take into consideration that chain-connectedness is not a transitive property, and that there are two different possibilities to generalize cycles.

Let $\mathcal{F}$ be a $k$-uniform hypergraph. Let us explore the various possible generalisations of the equivalent tree definitions.

1. $\mathcal{F}$ is connected and cycle-free: here there are two possibilities: $\mathcal{F}$ is chain-connected and cycle-free, or $\mathcal{F}$ is chain-connected and semicycle-free.
   We require the stronger semicycle-freeness condition instead of cycle-freeness for hypertrees. Although cycle-freeness is a weaker condition, later it turns out that it is not that much weaker, but there will be a proof where the absence of semicycles will play an important role.

2. Every pair of vertices of $\mathcal{F}$ is connected by exactly one path: this is not a useful definition in this context, because in this case there could be no chains in $\mathcal{F}$ with length more than one: If two edges $e, f \in \mathcal{E}$ intersect each other in 2 vertices, for example, $e \cap f = \{u, v\}$, then $u$ and $v$ are connected by two chains ($e$ and $f$) of length one, contradicting the definition.
   So, $\mathcal{F}$ can only be a kind of block-design, which are already well investigated, therefore we rejected this idea.

3. $\mathcal{F}$ is an edge-minimal connected graph: the generalisation of this definition is that $\mathcal{F}$ is an edge-minimal chain-connected hypergraph. This is not a subcase of the first definition unless we require the semicycle-free property. The example below points to following counterexample proves this observation.
   Let $n \geq 5$, $V = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n\}$ and $\mathcal{C}$ be the 3-uniform cycle determined by the vertex-sequence $x_1, x_2, \ldots, x_n$. Furthermore, let
   $$\mathcal{E}_y = \{\{x_{i-1}, x_i, y_i\}: 1 \leq i \leq n\}, \quad \mathcal{E}_z = \{\{z_i, x_i, x_{i+1}\}: 1 \leq i \leq n\}$$
(the indices are understood cyclicly). Then \( \mathcal{F} = (V, E(C) \cup E_y \cup E_z) \) is edge-minimal chain-connected hypergraph, but it is not semicycle-free, moreover it is not cycle-free because \( C \) is its subhypergraph.

4. \( F \) is edge-maximal cycle-free graph: the generalisation of this definition is that \( F \) is an edge-maximal semicycle-free hypergraph. Again, this is not a subcase of the first definition unless we require chain-connectedness. Instead of semicycle-freeness one could also require cycle-freeness here.

To sum these ideas, we would like to give definitions which are strong enough, but yield us not only the trivial structures.

**Definition 12 (Hypertree).** The \( k \)-uniform hypergraph \( \mathcal{F} \) is a hypertree if it is chain-connected and semicycle-free.

![Figure 5: A 3-uniform hypertree of order 5](image)

**Definition 13 (Edge-minimal hypertree).** \( \mathcal{F} \) is an edge-minimal hypertree if it is a hypertree, and deleting any of its edge \( e \), \( \mathcal{F}\{e\} \) is not a hypertree any more (i.e. chain-connectedness does not hold).

**Definition 14 (Edge-maximal hypertree).** \( \mathcal{F} \) is an edge-maximal hypertree if it is a hypertree, and adding any new edge \( e \) to it, \( \mathcal{F} \cup \{e\} \) is not a hypertree any more (i.e. semicycle-freeness does not hold).

In this way, the edge maximal/minimal hypertrees are also hypertrees, but they are the extreme cases among hypertrees.
The main reason to use semicycle-free property is that Claim 9 implies that every chain is non-self-intersecting in a hypertree. Without this property we should face with substantially more complicated case-analysis.

We mentioned above that being “connected by chain” is not a transitive property, thus it is not an equivalence relation. This fact is the main difficulty in this topic. It also means that it is usually hard to mimic the graph theoretic proofs, we need to invent new ideas.

**Definition 15 (l-hypertree).** *A hypertree is called an l-hypertree if all chains in it have length at most l.*

In Section 5 it will be shown that there are better bounds for the edge number of an l-hypertree if \( l < k \).
To every hypergraph we assign a graph, which more or less preserves the structure of chains. This will be a useful tool in some proofs.

**Definition 16 (Tight line graph).** Let $\mathcal{H} = (V, E)$ be a $k$-uniform hypergraph. The tight line graph of $\mathcal{H}$ is the graph $L_\mathcal{H} = (\mathcal{E}, E)$ where $E = \{\{e, f\}: e, f \in \mathcal{E}, |e \cap f| = k - 1\}$.

This definition is not equivalent to the definition of the usual line graph, where all pairs of intersecting edges are adjacent. In our case we are only interested in the substantial intersections (of size $(k - 1)$) since the edges of chains are joined in this way.

**Definition 17.** The hypergraph $\mathcal{H}$ is line-graph-connected if its tight line graph is connected.

This will be used in the proof of the lower bound for the edge number of hypertrees. The proof will be easier for line-graph-connected hypertrees, while the general case can be traced back to the tight line graph components.

![Figure 8: The tight line graph of the hypertree of figure 5, which is not line-graph-connected.](image)

2.2. Basic types of edge-minimal hypertrees

In the following subsection we are going to review some special types of edge-minimal hypertrees. We will sometimes refer back to these constructions later.

**Definition 18 (Tight star).** The $k$-uniform hypergraph $\mathcal{S}_n$ of order $n$ is a tight star if $n \geq k$ and there exist $u_1, u_2, \ldots, u_{k-1} \in V(\mathcal{S}_n)$ such that

$$\mathcal{E}(\mathcal{S}_n) = \{\{u_1, u_2, \ldots, u_{k-1}, w\}: w \in V(\mathcal{S}_n), w \neq u_i, \text{ for } 1 \leq i \leq k-1\}.$$
Claim 19. Every tight star is an edge-minimal hypertree.

Note, that the two most simple class of hypertrees are the tight stars and the non-self-intersecting chains (tight paths). From now on, we will write star instead of tight star.

Non-self-intersecting chains and semicycles will be denoted by roman capital letters, usually by $L$ and $C$.

Definition 20 ($l$-Flower). The hypergraph $V_n$ of order $n$ is an $l$-flower if $n > k > l \geq 1$, $V(V_n) = \{v_1, \ldots, v_{n-l}, u_1, \ldots, u_l\}$ and

$$E(V_n) = \{\{v_i, v_{i+1}, \ldots, v_{i+k-l-1}, u_1, \ldots, u_l\} : 1 \leq i \leq n-l\},$$

where indices are understood cyclicly. A 1-flower is simply called flower.

Note, that a $(k-1)$-flower is the same as a star.
Claim 21. Let $2 < k < n$. The $k$-uniform flower $\mathcal{V}_n$ is a hypertree if and only if $2k - 1 \leq n \leq 4(k - 1)$.

It is an edge-minimal hypertree if and only if $3(k - 1) \leq n \leq 4(k - 1)$.

The proof is quite simple and left to the reader. The key is the fact, that $v_i$ and $v_j$ are connected by a chain in $\mathcal{V}_n$ if and only if $\min\{|j-i|, n-1-|j-i|\} < 2k - 2$. $v_i$ and $v_j$ are connected by two edge-disjoint chains (or $v_i = v_j$ if it is the first vertex of a semicycle) if and only if $\max\{|j-i|, n-1-|j-i|\} < 2k - 2$.

Definition 22 (Focus-vertex). Let $\mathcal{F} = (V, E)$ be an edge-minimal hypertree. Then $v \in V$ is a focus-vertex of $\mathcal{F}$ if it is contained in every edge of $\mathcal{F}$.

Obviously, an edge-minimal hypertree may have at most $k$ focus-vertices, and only the trivial hypertree (which consists of 1 edge) has exactly $k$ of them. Hypertrees with $k - 1$ focus-vertices are stars, and flowers have one focus-vertex. It is an interesting question in itself that what is the maximum number of edges in a $3$-uniform hypertree on $n$ vertices with exactly 1 focus-vertex. Removing this focus-vertex leads to a graph-theoretical problem.

Definition 23 (Geometric hypertree). A hypertree $\mathcal{F}$ is $l$-geometric if every $l$-element set of its vertices is contained in exactly one edge.

For example, the coincidence-hypergraph of the Fano-plane and other Steiner-systems are geometric hypertrees. $l$-geometric hypertrees are exactly the $l-(n, k, 1)$ block-designs. Therefore, every geometric hypertree with the same parameters has an equal number of edges that can be computed from the parameters. (We will see that this is not true for all hypertrees.)

- Every $k$-uniform $2$-geometric hypertree is edge-minimal.
- A $k$-uniform $l$-geometric hypertree has exactly $(\frac{n}{l})\binom{k}{l}$ edges which is well known from the theory of block-designs.

Definition 24 (Recursive hypertree). An edge-minimal hypertree $\mathcal{F}$ is recursive if it can be obtained by the following recursive construction:

1. One edge consisting of $k$ vertices is a recursive hypertree.
2. If we add a new vertex \( v \) and some new edges all containing \( v \) to a recursive hypertree such that the resulting hypergraph is still an edge-minimal hypertree, then this is also a recursive hypertree.

Can we build up all edge-minimal hypertrees with recursive construction? The answer is negative: deleting any vertex \( v_i \) and the two edges containing it from the 3-uniform flower \( \mathcal{V}_7 \), we get a non-chain-connected hypergraph (see Definition 20) since \( v_{i-1} \) and \( v_{i+1} \) become chain-disconnected. If we delete the focus-vertex \( u_1 \) in turn, then we have to erase all edges, so the remaining hypergraph obviously not chain-connected. This shows that \( \mathcal{V}_7 \) is not a recursive hypertree.

3. Lower bound for the edge number of hypertrees

A tree on \( n \) vertices has exactly \( n - 1 \) edges. In case of hypertrees the situation is more complicated. Chains and stars have \( n - (k - 1) \) edges, however, \((k - 1)\)-geometric hypertrees have \( \frac{1}{k} \binom{n}{k-1} \) edges. Anyway, it seems like \( n - (k - 1) \) is the tight lower bound for the number of edges, however, it turns out that it is true only if \( n > n_0(k) \).

**Theorem 25.** Let \( \mathcal{F} \) be a \( k \)-uniform chain-connected hypergraph with \( n \) vertices and \( m \) edges. If \( n \geq (k - 1)^2 \), then \( m \geq n - (k - 1) \).

**Proof:** Let \( L \) be the tight line graph of \( \mathcal{F} \), and let \( \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_l \) denote the vertex sets of the connected components of \( L \). Furthermore, let \( V_i = \bigcup\{e: e \in \mathcal{E}_i\}, i = 1, \ldots, l \) be the projections of the components to the vertex set \( V \), called the classes of \( \mathcal{F} \). Then it can be easily shown that the following hold:

- \( \bigcup_{i=1}^l \mathcal{E}_i = \mathcal{E}, \mathcal{E}_i \cap \mathcal{E}_j = \emptyset \) if \( i \neq j \).
- \( \bigcup_{i=1}^l V_i = V \) since every vertex of \( \mathcal{F} \) is contained in some edges.
- \( \forall u, v \in V, \exists i \) such that \( u, v \in V_i \).

The last point follows from the fact that \( \mathcal{F} \) is chain-connected and the edges of a chain form the vertex set of a path in the tight line graph. Therefore, the edges of the chain that connects \( u \) and \( v \), are in the same tight line graph component \( \mathcal{E}_i \) of \( L \), whose projection \( V_i \) contains the vertices of the chain, in particular both \( u \) and \( v \).

Now we are going to prove a lemma which says that the theorem is true for line-graph-connected hypertrees, so it is true for every class.
Lemma 26. If $\mathcal{H} = (V', \mathcal{E}')$ is a line-graph-connected $k$-uniform hypergraph that contains no isolated vertex, then $|\mathcal{E}'| \geq |V'| - (k - 1)$.

Proof: Let $e \in \mathcal{E}'$ be an arbitrary edge and $\mathcal{X}_1 = (e, \{e\})$ be a subhypergraph. We will define a subhypergraph-sequence $\mathcal{X}_1 \subset \mathcal{X}_2 \subset \ldots \subset \mathcal{X}_m$, for which $\mathcal{X}_m = \mathcal{H}$ and for every $i$, $i = |\mathcal{E}(\mathcal{X}_i)| \geq |V(\mathcal{X}_i)| - (k - 1)$. This implies the lemma.

Let $m = |\mathcal{E}'|$. Assume that we have already defined $\mathcal{X}_{i-1}$ for some $i - 1 < m$, and let $L$ denote the tight line graph of $\mathcal{H}$. Since we are not in the $m$th step and $L$ is connected, there must be an edge in $L$ between $\mathcal{E}(\mathcal{X}_{i-1})$ and $\mathcal{E}(\mathcal{X}_{i-1})$. Let $f_{i-1} \in \mathcal{E}(\mathcal{X}_{i-1})$ and $e_i \notin \mathcal{E}(\mathcal{X}_{i-1})$ denote the two endpoints of this edge. Then $|f_{i-1} \cap e_i| = k - 1$, so $|e_i \cap V(\mathcal{X}_{i-1})| \geq k - 1$. Let $\mathcal{X}_i = \mathcal{X}_{i-1} \cup \{e_i\} = (V(\mathcal{X}_{i-1}) \cup e_i, \mathcal{E}(\mathcal{X}_{i-1}) \cup \{e_i\})$.

Thus starting from $\mathcal{X}_1$, in every step we add a new edge, and we continue this process until all the edges of $\mathcal{H}$ are added. At the last step $\mathcal{X}_m = \mathcal{H}$ will hold since there is no isolated vertex.

By induction, it can be shown that for every index $i$, $|\mathcal{E}(\mathcal{X}_i)| \geq |V(\mathcal{X}_i)| - (k - 1)$. If $i = 1$, then the claim is trivially true. Assume that the claim is true for $i - 1$, namely, $|\mathcal{E}(\mathcal{X}_{i-1})| \geq |V(\mathcal{X}_{i-1})| - (k - 1)$. In the $i^{th}$ step we add one new edge and at most one new vertex to $\mathcal{X}_{i-1}$ because $e_i$ intersects $V(\mathcal{X}_{i-1})$ in at least $k - 1$ vertices. Thus $|\mathcal{E}(\mathcal{X}_i)| \geq |V(\mathcal{X}_i)| - (k - 1)$ also holds.

Continuing the proof of the theorem, all of the hypergraphs $(V_i, \mathcal{E}_i)$ meet the conditions of the lemma, so $|\mathcal{E}_i| \geq |V_i| - (k - 1)$ for $i = 1, 2, \ldots, l$. Therefore,

$$|\mathcal{E}| = \sum_{i=1}^{l} |\mathcal{E}_i| \geq \sum_{i=1}^{l} (|V_i| - (k - 1)) = \left(\sum_{i=1}^{l} |V_i|\right) - l(k - 1).$$

Let $\sigma = \sum_{i=1}^{l} |V_i|$. For proving $m \geq n - (k - 1)$ it is enough to show that $\sigma - l(k - 1) \geq n - (k - 1)$, or equivalently

$$\sigma \geq n + (l - 1)(k - 1).$$

Notice that $\sigma \geq n$ certainly holds since every vertex of $\mathcal{F}$ is covered by one of the classes.

We may assume that $|V_i| < n$ for every $i$, otherwise there would be a line-graph component $(V_j, \mathcal{E}_j)$ such that $|V_j| = n$ and $n - (k - 1) = |V_j| - (k - 1) \leq |\mathcal{E}_j| \leq |\mathcal{E}|$, which would finish the proof.
We introduce a parameter \( r = \min_{x \in V} |\{i : x \in V_i\}| \), which helps us in splitting the problem into two parts. It is clear that \( r \geq 2 \) always holds since if \( r = 1 \), then there would be a vertex \( x \), which is covered by only one class, say \( V_j \). However, for every \( y \in V \) the pair \( \{x, y\} \) of vertices has to be covered by a class, which can only be \( V_j \). Hence \( V_j = V \), which contradicts the previous assumption.

To complete the proof, we only need two inequalities.

**Lemma 27.**

1. \( \sigma \geq rn \),
2. \( \sigma \geq n + r - 1 + (l - r)k \).

**Proof:**

(1) We are going to compute \( \sigma \) in two ways. We can sum up the size of the classes or we can sum up for all vertices the number of classes containing them. Thus \( \sigma = \sum_{x \in V} |\{i : x \in V_i\}| \geq rn \), which verifies the first inequality.

(2) For every vertex \( x \) the union of the classes containing \( x \) must be \( V \), since they have to cover every pair \( \{x, y\} \) of vertices. Choose such a vertex \( x \) that is contained in exactly \( r \) classes. Now \( \sum_{i : x \in V_i} |V_i| \geq n + r - 1 \) since we count every vertex at least once and \( x \) exactly \( r \) times. On the other hand, \( \sum_{i : x \notin V_i} |V_i| \geq (l - r)k \) because the size of every class is at least \( k \).

Combining these inequalities we obtain \( \sigma = \sum_{i : x \in V_i} |V_i| + \sum_{i : x \notin V_i} |V_i| \geq n + r - 1 + (l - r)k \), which proves the second part of the lemma. \( \square \)

Using the lemma, it is enough to show that either \( rn \geq n + (l - 1)(k - 1) \) or \( n + r - 1 + (l - r)k \geq n + (l - 1)(k - 1) \) holds, since these imply \( \sigma \geq n + (l - 1)(k - 1) \) and therefore \( m \geq n - (k - 1) \) will hold.

It is easy to see that obtain

\[
rn \geq n + (l - 1)(k - 1) \Leftrightarrow l - 1 \leq \frac{(r - 1)n}{k - 1},
\]

and

\[
n + r - 1 + (l - r)k \geq n + (l - 1)(k - 1) \Leftrightarrow l - 1 \geq (r - 1)(k - 1).
\]

Therefore our claim, \( m \geq n - (k - 1) \), does not hold only if \( \frac{(r - 1)n}{k - 1} < l - 1 < (r - 1)(k - 1) \). This implies the condition \( n < (k - 1)^2 \) (we have seen above
that \( r \neq 1 \), so \( n \geq (k - 1)^2 \) implies \( m \geq n - (k - 1) \), which completes the proof. □

The lower bound of Theorem 25 is sharp since chains and stars on \( n \) vertices have exactly \( n - (k - 1) \) edges.

It is an interesting question if it is possible to eliminate the condition \( n \geq (k - 1)^2 \). Or otherwise, what are the sharp lower and upper bounds for sizes of \( k \)-uniform counterexamples?

We have partial answers for these questions. Let us call a hypertree a counterexample if it does not fulfil \( m \geq n - (k - 1) \).

Claim 28.

• For \( k = 3, 4, 5 \), there are no \( k \)-uniform counterexamples, so Theorem 25 holds without any condition for \( n \).

• For every \( k \geq 6 \), there is a \( k \)-uniform counterexample.

The proof of the first part is a fairly straightforward case analysis using combinatorial arguments and relations between the parameters \( n, m, k, r \) and \( l \) (see \( r \) and \( l \) in the proof of Theorem 25), so it is left to the reader. The second part is obtained from the following simple construction.

Let \( V' = \{v_1, v_2, \ldots, v_{k-6}\} \), \( V_x = \{x_1, x_2, x_3\} \), \( V_y = \{y_1, y_2, y_3\} \), \( V_z = \{z_1, z_2, z_3\} \), \( V = V_x \cup V_y \cup V_z \cup V' \) and \( E = \{V_x \cup V_y \cup V_z, V_x \cup V' \cup V_z, V_y \cup V' \cup V_z\} \). Then the hypergraph \( F = (V, E) \) is an edge-minimal hypertree that has \( k + 3 \) vertices and 3 edges. Hence \( m = 3 < 4 = n - (k - 1) \), so \( F \) is a counterexample if \( V' \) exists, i.e. \( k \geq 6 \).

Let us see, how many vertices a counterexample may have for a fixed \( k \).

Claim 29.

• If \( k \geq 6 \), then there exist a \( k \)-uniform counterexample of order \( k + 3 \);

• if \( k \geq 6 \) even, then there exist a \( k \)-uniform counterexample of order \( \frac{k(k-2)}{2} \);

• if \( k \geq 6 \) odd, then there exist a \( k \)-uniform counterexample of order at least \( \frac{(k-1)(k-4)}{2} + 1 \);
The previous construction proves the first part. It can be easily seen that it is a sharp lower bound since every hypertree with 2 edges must have \( k + 1 \) vertices, thus a counterexample with \( k + 2 \) vertices cannot exist.

For the second statement, we use \( c \) clusters of vertices, each of size \( \frac{k}{2} \), where \( c \) depends on \( k \), and each pair of clusters form an edge.

Now, we have \( n = \frac{ck}{2} \) vertices and \( m = \binom{c}{2} \) edges. The condition being a counterexample can be formulated as

\[
\binom{c}{2} < \frac{ck}{2} - (k - 1).
\]

To maximize \( n \) for a fixed \( k \), we must maximize the integer \( c \) subject to the previous constraint. \( c \) attains its maximal value at \( k - 2 \), and the hypergraph (which is obviously chain-connected) obtained in this way has \( \frac{k(k-2)}{2} \) edges.

The third claim can be proven with a minor modification of the previous proof. Take the construction above with \( c \) clusters, each of size \( \frac{k-1}{2} \). Now add an extra vertex to each edge, and maximize \( c \) subject to the related, modified constraint.

4. Upper bounds for the edge number of hypertrees

In this section we prove an upper bound for the edge number. It is also shown that the given bound is asymptotically sharp in 3-uniform case.

It is obvious that a \( k \)-uniform hypergraph has at most \( \binom{n}{k} \) edges. In the case of hypertrees, the order of magnitude is one less.

**Theorem 30.** If \( \mathcal{F} = (V, \mathcal{E}) \) is a semicycle-free, \( k \)-uniform hypergraph with \( n \) vertices and \( m \) edges, then \( m \leq \binom{n}{k-1} \).

**Proof:** We give an injective function \( \varphi: \mathcal{E} \rightarrow \binom{V}{k-1} \) below.

Let us construct a maximal chain in \( \mathcal{F} \) (i.e. it cannot be continued with more edges). Such a chain exists because \( \mathcal{F} \) is semicycle-free. Starting with any edge, we construct the chain edge by edge in greedy way by appending a new edge to the end of the chain. The semicycle-free property implies that every chain is non-self-intersecting, hence the chain expands with a new vertex (which is not used so far) in every step. Sooner or later the vertices of \( \mathcal{F} \) run out, and we cannot expand our chain anymore.

Let us take the last edge of a maximal chain, and assign the set consisting of the last \( k - 1 \) vertices of the chain to this edge. Obviously, this \( (k - 1) \)-set
is contained only in the edge to which we assigned it, otherwise the chain
could be continued.

Now, let us delete this edge from \( F \), and repeat the same process to
the remaining hypergraph. The subhypergraphs inherit the semicycle-free
property, so we can execute the assignment again. We continue deleting
edges until the edges of \( F \) run out, i.e. \( \varphi \) is defined everywhere.

It remains to prove that \( \varphi \) is injective. Assume indirectly that we assign
the same vertex set \( T \) to two different edges \( e_1 \) and \( e_2 \). Without loss of
generality, we may assume that during the assigning process we have deleted
\( e_1 \) earlier than \( e_2 \). So, \( e_2 \) still exists at the moment of deleting \( e_1 \), and both
edges contains the set \( T \) because of the definition of \( \varphi \). This is a contradiction,
thus \( \varphi \) must be injective.

Due to the injectivity we have \( |\varphi(\mathcal{E})| = m \), combining this with \( |\varphi(\mathcal{E})| \leq \binom{n}{k-1} \) we obtain the theorem. \( \square \)

Remark: Theorem 30 remains true even if we only require \( F \) to be cycle-free
because we can also find a maximal chain in that case (otherwise there would
be a \( k-1 \)-tuple of vertices, which appears at least twice as \( k-1 \) consecutive
vertices of the chain). The same proof also works in this case.

In particular, the statement is true also for hypertrees (since they are
semicycle-free).

The following construction proves the asymptotic sharpness of the upper
bound of Theorem 30 in 3-uniform case.

Let \( V = \{v_i\}_{i=1}^n \) be a set of vertices and \( F = (V, \mathcal{E}) \) be an arbitrary 3-
uniform hypertree. Now let \( B(F) = (V \cup V', \mathcal{E} \cup \mathcal{E}') \) denote the 3-uniform
hypergraph where \( V' = \{0, 1\}^n \),

\[
\mathcal{E}' = \left\{ \{v_i, u, w\} : v_i \in V, \ u, w \in V' \text{ and the } i\text{th bit is the first bit where } u \text{ and } w \text{ differ} \right\}.
\]

**Theorem 31.** If \( F \) is a hypertree, then \( B(F) \) is also a hypertree.

**Proof:**

(1) Chain-connectedness:

Any two vertices from \( V \) are connected by a chain because \( F \) is a hyper-
tree, and all of its edges belong to \( B(F) \), too.

For \( u, w \in V' \) let \( i \) denote the position of the first bit where they differ.
Then by definition \( \{v_i, u, w\} \in \mathcal{E}' \), thus \( u \) and \( w \) are connected by a chain of
length one in \( B(F) \).
In case of \( u \in V' \), \( v_i \in V \) consider a vertex \( w \in V' \) whose first \( i-1 \) bits are the same as the first \( i-1 \) bits of \( u \), but its \( i \)th bit differs from the \( i \)th bit of \( u \). Such a \( w \) certainly exists since the preceding condition can be satisfied due to \( 1 \leq i \leq n \). Then, by definition, \( \{v_i, u, w\} \in E' \), so \( u \) and \( v_i \) are connected with a chain of length one.

(2) Semicycle-freeness:

Assume indirectly that \( B(F) \) contains a semicycle \( C \). If the first edge of \( C \) belongs to \( E \), then \( F \) entirely contains \( C \), otherwise the first edge of \( C \) that intersects \( V' \), would include exactly two vertices from \( V \), but such an edge does not exist. However, \( F \) cannot contain \( C \) since \( F \) is semicycle-free, which leads to contradiction.

Hence, we have the first edge of \( C \) in \( E' \). Then the second edge contains at least one vertex from \( V' \), so this edge also belongs to \( E' \). The same argument shows that every edge of \( C \) is in \( E' \). Let us notice that only one edge can contain a pair \( \{u, w\} \subseteq V' \) since the first place where the bits of \( u \) and \( v \) differ, uniquely determines the third vertex of the edge containing \( \{u, w\} \), which is in \( V \). So, if two adjacent vertices of the semicycle are in \( V' \), then these vertices are the last two vertices of the semicycle since only one edge of \( C \) can contain them.

Let \( \{v_{i_1}, u_1, u_2\} \) be the first edge of \( C \). If we write down the vertices of the semicycle in a sequence, denoting the vertices from \( V \) by \( v_{i_k} \) and those from \( V' \) by \( u_k \), there are 3 possible sequences:

- \( v_{i_1}u_1u_2 \cdot \) this cannot be a semicycle because it would have less than 5 vertices,
- \( u_1v_{i_1}u_2u_3 \cdot \) the same argument works in this case,
- \( u_1u_2v_{i_1}u_3u_4 \cdot \) this cannot be a semicycle since \( u_1 \) and \( u_2 \), \( u_2 \) and \( u_3 \), \( u_3 \) and \( u_4 \) differ in the \( i \)th bit, so \( u_1 \) and \( u_4 \) differ in the \( i \)th bit, but they have to be identical in the semicycle.

Here, we have taken into consideration that every edge of \( C \) comes from \( E' \), and such an edge contains two vertices from \( V' \). The point at the end of the sequences indicates the fact that the given sequence cannot be continued.

We get a contradiction again, so there is no semicycle in \( B(F) \), implying that \( B(F) \) is a hypertree. □

Let us count the number of vertices and edges of the hypertree obtained in this way. \( |V \cup V'| = n + 2^n \), \( |E \cup E'| \geq |E'| = \binom{n}{2} \) since exactly one edge
of \(E'\) belongs to every pair of vertices of \(V'\). So,
\[
\frac{|E(B(F))|}{\binom{|V(F)|}{2}} \geq \frac{(2n)}{(n+2n)} \rightarrow 1 \text{ if } n \rightarrow \infty.
\]
This matches the bound in Theorem 30.

Thus, if \(F_1, F_2, \ldots\) is a hypertree sequence for which \(\lim_{n \to \infty}|V(F_n)| = \infty\) is true, then \(|E(B(F_n))| \sim (\binom{|V(F)|}{2})\).

If we want to get a hypertree with relatively few vertices and high number of edges, then we can apply the operator \(B()\) more times one after the other.

This result verifies that the bound of Theorem 30 is asymptotically sharp in 3-uniform case. In \(k\)-uniform case, it is an open question to find a similar construction.

**Remark:** Theorem 31 implies that from the viewpoint of the edge number, requiring cycle-free or semicycle-free property does not mean a big difference in 3-uniform case since the cycle-free property guarantees that the number of edges is at most \(\binom{n}{2}\), and nor the stronger semicycle-freeness can lower this bound.

5. Upper bound for \(l\)-hypertrees

We have seen the upper bound for the edge number of a hypertree. One can get better bound under the assumption \(F\) contains no long chains.

**Theorem 32.** If \(1 \leq l \leq k\) and \(F = (V, E)\) is a \(k\)-uniform \(l\)-hypertree with \(n\) vertices and \(m\) edges, then \(m \leq \frac{1}{k-l+1}\binom{n}{k-l-1}\).

**Proof:** The proof is similar to the proof of Theorem 30. We give \(k-l+1\) injective functions, \(\varphi_1, \varphi_2, \ldots, \varphi_{k-l+1}: E \rightarrow \binom{V}{k-l-1}\) for which \(\text{Im} \varphi_i \cap \text{Im} \varphi_j = \emptyset\), for all \(1 \leq i < j \leq k-l+1\).

Let \(L\) be a maximal chain in \(F\). Such a chain exists because \(F\) is semicycle-free. Let \(e\) be the last edge of \(L\). The length of this chain is at most \(l \leq k\), therefore there are at least \(k-l+1\) vertices contained in the intersection of all edges of \(L\). Let us denote the set of these vertices by \(U\). If \(u \in U\) and some edge \(f \in E, f \neq e\) covers \(e \setminus \{u\}\), then \(L\) could be continued by \(f\) (because we have a freedom in ordering the elements of \(U\)), which is a contradiction.

Let \(u_1, \ldots, u_{k-l+1} \in U\) be distinct vertices and \(\varphi_i(e) = e \setminus \{u_i\}\), for all \(i = 1, \ldots, k-l+1\). Then delete \(e\) and repeat the whole process for the
remaining hypergraph. In the end, \( \varphi_i \)'s will be defined everywhere. The previous observation shows that the conditions required for \( \varphi_i \)'s are satisfied.

So, \((k-l+1)|E| = |\text{Im}\varphi_1| + \ldots + |\text{Im}\varphi_{k-1}| = |\text{Im}\varphi_1 \cup \ldots \cup \text{Im}\varphi_{k-1}| \leq \binom{n}{k-1}\), hence \(|E| \leq \frac{1}{k-l+1} \binom{n}{k-1}\). \(\square\)

**Remark:** Semicycle-freeness is a crucial premise of this proof, and there are no trivial extension to the case when we exchange semicycle freeness by cycle freeness because a maximal chain of length \(l\) could be extended with an edge to form a semicycle of length \(l+1\).

The most important case is \(l = 2\), when the number of edges is at most \(\frac{1}{k-1} \binom{n}{k-1}\).

### 6. Open problems

There are many interesting open questions related to hypertrees such that: “What is the maximal number of edges in a \(k\)-uniform edge-minimal hypertree of order \(n\)?” or “What is the minimal number of edges in a \(k\)-uniform edge-maximal hypertree of order \(n\)?”.

Based on our research, we propose the following two conjectures:

**Conjecture 33.** For every \(k\)-uniform edge-minimal hypertree \(F = (V, E)\) on \(n\) vertices \(|E| \leq \frac{1}{k-1} \binom{n}{2}\) holds.

**Conjecture 34.** Every \(3\)-uniform edge-maximal hypertree on \(n\) vertices has at least \(\frac{1}{2} \binom{n}{2} - O(n)\) edges.

It remained an open question whether or not the upper bound of the edge number of \(l\)-hypertrees stated in Theorem 32 is asymptotically sharp.

However, we can also modify the definition of edge-minimal hypertrees a bit. Instead of edge-minimal hypertrees, it is interesting to study edge-minimal chain-connected hypergraphs or local hypertrees, which can contain long semicycles, but no short ones. Similarly, we can study edge-maximal semicycle-free hypergraphs instead of edge-maximal hypertrees.

What if we allow a chain to use an edge more times? Then our theorems and definitions would alter more or less. In this case, one can show forbidden substructures in edge-minimal hypertrees such as the complete hypergraph \(K_{k+2}^{(k)}\).

A special kind of \(3\)-uniform hypertrees is simultaneously edge-minimal and edge-maximal \(3\)-uniform hypertrees. This is a small subclass, and its elements have nearly \(\frac{1}{2} \binom{n}{2}\) edges by our conjectures.
Another open problem is to give tight lower and upper bounds for the number of edges in recursive hypertrees.

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