On some quantum Hall states with negative flux

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Some recently observed fractional quantum Hall states are not easily explained in standard hierarchy/composite fermion schemes. This paper gives a brief introduction to some wavefunctions involving non-Abelian Read-Rezayi states with negative flux that have been proposed as candidates for these new quantum Hall fractions.

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I. INTRODUCTION

The fractional quantum Hall effect (FQHE) is a remarkable state of electronic matter that occurs in two-dimensions and under a strong magnetic field. From a practical point of view one observes experimentally a wealth of new thermodynamic phases in this special regime. While there is a theory for the most prominent states (in a sense given below), recent experiments have given evidence for more complexity for which there is yet no simple and universal theory. This seminar gives an overview of the situation as of end of 2008 and focuses on some recent proposals for wavefunctions describing these new quantum Hall states.

For the sake of completeness let us first set the stage for the FQHE. It is known to occur for particles confined in two spatial dimensions and subjected to a perpendicular magnetic field. In this set-up the one-body spectrum is drastically affected by the magnetic field: indeed kinetic energy is frozen and there is a set of exactly degenerate energy levels called Landau levels. These levels are separated by the cyclotron energy $\hbar \omega_c$, $\omega_c = eB/m$. Their degeneracy is given by $eB/\hbar$ times the area $A$ of the sample. Increasing the $B$ field leads to more degenerate levels and also to a larger separation between Landau levels. Imagine now that we have some particles in this situation and we look for the ground state. If we have $N_e$ electrons and we increase $B$, at some point there will be enough states in the lowest Landau level (LLL) to accommodate everybody. This will happen first when $N_e/eB_1/A \approx 1$. Beyond that value of $B$ there will be more states available and the problem of putting the electrons in the one-body orbitals becomes exponentially degenerate since the number of configurations is given by a binomial coefficient. If the cyclotron gap between the levels is large enough it is the mutual interactions between electrons that will determine the structure of the ground state. There is no longer any interplay between kinetic energy and potential energy. The FQHE is a "pure" interaction effect. Typical two-dimensional electron gases have densities of the order of $10^{11}$ cm$^{-2}$. The field required to put everybody in the LLL is thus $B_1 \approx 4 \times n(10^{11})$ Tesla. the number of occupied orbitals in the LLL is called in what follows the filling factor $\nu$, the ratio of electrons divided by the number of one-body quantum states in the LLL $\nu = nh/eB$.

Let us guess what would have been the reasoning of a condensed matter physicist before the eighties confronted with this situation. The exactly degenerate Landau levels are of course an idealized situation and the disorder present in the real-world samples broadens the levels which retains nevertheless a high density of states. These broadened levels are partially filled for $B > B_1$ so, as a function of strength of interactions, there are two plausible guesses: the first one is a Fermi liquid in a partially filled band. Since the band is narrow one can also envision the relevance of a Mott insulating state, i.e. a crystalline state of electrons. This second possibility is in fact realized at very small filling factor, $\nu \lesssim 1/7$. This is the so-called Wigner crystal. However the Fermi liquid phase does not happen and is replaced by a new kind of liquid state supporting fractionalization of quantum numbers. In fact there is a whole series of such liquids as a function of the filling factor $\nu$. Historically the first state for which R. B. Laughlin gave a satisfactory theoretical description was $\nu = 1/3$. The so-called "Composite Fermion" scheme developed by J. K. Jain gave a description of many other FQHE states. We will briefly give an overview of these theoretical approaches in section II.

Finally note that we will not discuss the transport properties of these FQHE states. It is certainly true that the study of a FQHE state starts by a measurement of resistivity. However the description of transport is a subject per se and is described in several Les Houches lectures. For us we just need to know that the FQHE states are liquids without any obvious local symmetry breaking, they do possess a gapped excitation spectrum and amongst the excitations there are unconventional quasiparticles with fractional charge and statistics. Experiments are very often limited to gap measurements. The "most prominent" states are the ones with largest gaps.
II. CLASSICAL HIERARCHIES

We now describe briefly the existing microscopic theory of the most prominent FQHE states. If we use the symmetric gauge for the external applied magnetic field $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$, then a basis of the one-body eigenstates for the LLL is given by:

$$\phi_m(z) = \frac{1}{\sqrt{2\pi 2^m m!}} z^m e^{-|z|^2/4\ell^2}, \quad (1)$$

where $m$ is a positive integer, $\ell = \sqrt{\hbar/eB}$ is the magnetic length, $z = x + iy$ is the complex coordinate in the plane. In the absence of interactions and external potential all these states have exactly the same eigenenergy $\frac{1}{2}\hbar\omega_c$.

These states have a definite chirality since $m$ is positive. The density associated with such a state is nonzero in the neighborhood of a ring centered on the origin (due to the gauge choice) of radius $\ell \sqrt{2m}$. We now consider the many-body problem with electrons all residing in the LLL. The states we describe are also spin polarized i.e. the Zeeman energy is large enough. This is not true in general of course and there are many interesting FQHE states involving the spin degree of freedom but the states at very high field presumably do not involve the spin. Each electron is described by a complex variable and thus the many-body wavefunction is of the form:

$$\Psi(z_1, \ldots, z_N) = f(z_1, \ldots, z_N) e^{-\sum_i |z_i|^2/4\ell^2}. \quad (2)$$

In general, contemplation of the full many-body wavefunction is not a very useful way to understand a physical system but in the FQHE physics this has proven to be the best approach (so far). Let us first understand what happens when the LLL is full of electrons, $\nu = 1$ or $B = B_1$ as defined in the introduction. We imagine a cylindrical infinite wall of a given radius $R$. This has the effect to send to infinity all states in Eq. (1) with $m$ larger than $R^2/2\ell^2$. Only the states close to the boundary will have wavefunctions different from the formula Eq. (1). This is a negligible effect for large systems. The number of states in the LLL is thus finite and equal to $M = R^2/2\ell^2 = eB/h \times$ (area of cylinder). If we fill exactly all these one-body states $\propto e/3$ with now a density which is $1/3$ of the previous $\nu = 1$ case hence it is a state with $\nu = 1/3$:

$$\Psi_{\nu=1/3} = \prod_{i<j} (z_i - z_j)^3 e^{-\sum_i |z_i|^2/4\ell^2}. \quad (3)$$

Why is this considered as being close to the truth for interacting electrons at $\nu = 1/3$? Contrary to the case of $\nu = 1$ this is not an exact eigenstate of the Coulomb problem. We first need to make a détour by looking at the two-body problem in the LLL. If we consider the kinetic energies of two charged particles then there is the following identity:

$$\frac{1}{2m}(p_1 + eA_1)^2 + \frac{1}{2m}(p_2 + eA_2)^2 = \frac{1}{2M}(P_{cm} + 2eA_{cm})^2 + \frac{1}{2\mu}(p_r + eA_r)^2. \quad (5)$$

where we have separation of the center of mass $M = 2m$ and the relative particle motions. The relative particle also is living in Landau levels so its kinetic energy is frozen. To find the eigenenergies of the two-body problem is now trivial: we just have to take expectation values of the interaction potential $V(r_1 - r_2)$ in the eigenstates of the relative particles given by eq. (1) with $8e^2$ in the exponential instead of $4\ell^2$ since $e/2$ appears in the relative kinetic energy. These eigenenergies are thus $V_m = \langle \phi_m | V | \phi_m \rangle$ for any (rotationally invariant!) potential (forgetting the
cyclotron energy independent of \( m \)). The exponent \( m \) appearing in the relative particle wavefunction is positive, i.e. the relative angular momentum is always positive. So any two-body interaction in the LLL is parametrized fully by the \( V_m \) coefficients called the pseudopotentials after D. Haldane. This peculiarity of the two-body problem also means that one can write the interaction Hamiltonian in a very special way:

\[
\mathcal{H} = \sum_{i<j} \sum_{m} V_m P_{ij}^m,
\]

where \( P_{ij}^m \) projects the couple of particles \( i, j \) onto relative angular momentum \( m \) and the sum over \( m \) is restricted to positive odd integers for spinless fermions due to Pauli statistics. For the repulsive Coulomb potential these pseudopotentials decrease with increasing values of \( m \). There is a very fundamental property of the Laughlin wavefunction: it is the unique smallest degree homogeneous polynomial which is a zero-energy eigenvalue of the special model where only \( V_1 \) is non-zero. This allows to understand why we believe that the Laughlin wavefunction captures the correct physics. When we vary pseudopotentials between the hard-core model with only non-zero \( V_1 \) and the true Coulomb problem, there is clear numerical evidence that nothing dramatic happens, i.e. we do not cross any phase boundary. This has been shown by D. Haldane by exact diagonalization of small systems.

The Laughlin wavefunction can only describe liquids with filling factors \( 1/m \), \( m \) odd while there are many more FQHE states in the real world. We start to trying to rewrite the Laughlin state as a determinant. We like determinant since Slater determinants are the simplest way to get an efficient description of atoms, molecules, nuclei and solids. With Slater determinants we can make particle-hole excitations and hence construct excited states on top of the ground state. This is a very desirable theoretical tool. We do the following manipulation:

\[
\Psi_{\nu=1/3} = \prod_{i<j} (z_i - z_j) \times \prod_{i<j} (z_i - z_j),
\]

where the ubiquitous exponential factor is not written for clarity. The last factor is the Vandermonde determinant. We note that:

\[
\prod_{i\neq j} (z_i - z_j) = \prod_{j\neq 1} (z_1 - z_j) \cdots \prod_{j\neq N} (z_N - z_j) \equiv J_1 \cdots J_N,
\]

where we have defined the so-called Jastrow factors \( J_i \). These factors can be distributed along the columns of the Vandermonde determinant:

\[
\Psi_{\nu=1/3} = J_1 \cdots J_N \times \prod_{i<j} (z_i - z_j) = \begin{vmatrix}
J_1 & \ldots & J_N \\
J_1 & \ldots & J_N \\
\vdots & \vdots & \vdots \\
J_1 & \ldots & J_N \\
\end{vmatrix}.
\]

So this is a Slater determinant provided we change the rules in the following way: instead of using orbitals \( z^m \) which are bona fide one-body orbitals we now use pseudo-orbitals \( z^m J \) where \( J \) effectively repels all the other particles. This is not really a one-body object but within this construct we can play the same usual Slater-like construction of excited states and so on. The first appearance of the Vandermonde determinant in Eq.(7) is suggestive of a flux reduction effect of correlations. Indeed the Vandermonde determinant is the ground state at \( \nu = 1 \). It is as if the correlation factors \( J_1 \cdots J_N \) reduce the magnetic field from \( B \) to \( B_{eff} = B - 2nh/e \) so that \( \nu = 1/3 \) becomes \( \nu = 1 \). If we have:

\[
\Phi_\nu = \prod_{i<j} (z_i - z_j)^2 \Phi_\nu^*,
\]

then the two filling factors are related by \( 1/\nu = 2 + 1/\nu^* \). This is intuitively reasonable: the Jastrow factor repels the particles and flattens the pancake i.e. the charge distribution of the wavefunction. Let us now reason in terms of the effective filling factor \( \nu^* \). If \( \nu \) is greater than \( 1/3 \) it implies that the effective \( \nu^* \) is now larger than one. We thus have to occupy Landau levels higher than the LLL in the Slater determinant. There is nothing wrong with that, provided we project back to the LLL. There will be special filling when there is filling of an integer number \( p \) of LLs. It is natural but not immediately obvious to expect that such wavefunctions will have to do with incompressible FQHE states. The candidate “composite fermion” (CF) states are thus:

\[
\Phi_{\nu=p/(2p+1)} = \mathcal{P}_{LLL} \prod_{i<j} (z_i - z_j)^2 \Phi_{\nu^*=p}
\]
For example the case \( p = 2 \) involves the second LL which is spanned by the one-body orbitals \( z^* z^m \) with \( m \geq 0 \). We can make a Slater determinant by putting half of the electrons in the pseudo LLL and the other half in the second pseudo-LL:

\[
\Phi_{\nu=2} = \begin{vmatrix}
1 & \ldots & 1 \\
z_1 & \ldots & z_N \\
\vdots & \vdots & \vdots \\
z_1^{N/2-1} & \ldots & z_N^{N/2-1} \\
z_1^* & \ldots & z_N^* \\
z_1^* z_1 & \ldots & z_N^* z_N \\
\vdots & \vdots & \vdots \\
z_1^{N/2-1} z_1 & \ldots & z_N^{N/2-1} z_N \\
\end{vmatrix}.
\]  

(12)

This is essentially the original Jain proposal - it has proven extremely successful. However manipulation of this wavefunction is inconvenient in practice due to the projection that one has to perform after multiplication by the Jastrow factor. Jain and Kamilla have shown that this scheme may be slightly altered to be much more tractable while retaining all its good quantitative properties. The idea is again to distribute all Jastrow factor. Jain and Kamilla have shown that this scheme may be slightly altered to be much more tractable.

\[
\Phi_{\nu=2/5} = \prod_{i<j} (z_i - z_j)^2 \begin{vmatrix}
1 & \ldots & 1 \\
z_1 & \ldots & z_N \\
\vdots & \vdots & \vdots \\
z_1^{N/2-1} & \ldots & z_N^{N/2-1} \\
z_1 \Sigma_1 & \ldots & \Sigma_N \\
z_1 \Sigma_1 \Sigma_1 & \ldots & z_N \Sigma_N \Sigma_N \\
\vdots & \vdots & \vdots \\
z_1^{N/2-1} \Sigma_1 & \ldots & z_N^{N/2-1} \Sigma_N \\
\end{vmatrix},
\]  

(13)

where \( \Sigma_i = \sum_{j \neq i} \frac{1}{z_i - z_j} \). This is a typical example of the CF scheme. Note that one can change the partitioning between the two LLs involved by putting say \( N_1 \) electrons in the LLL and \( N_2 \) in the second LL with \( N_1 + N_2 = N \). All these states are observed in exact diagonalizations in the unbounded disk geometry. The reason why we believe in the CF is slightly different wrt the Laughlin state. Here there is no simple Hamiltonian for which such states are unique exact ground states. But the spectroscopy of low-lying levels is correctly reproduced: we find the good quantum numbers and ordering of levels in the CF scheme when compared to exact diagonalization results. This CF scheme gives a reasonable description of FQHE states at \( \nu = p/(2p+1) \). By multiplying by extra powers of the Vandermonde it is easy to generate candidates for \( p/(4p+1), p/(6p+1) \) ... without new ideas. In the CF folklore we say that CF are electrons dressed by two flux tubes which means that there is a Jastrow factor squared that appears in the trial wavefunction.

Note now that for \( B \) less than \( 2 nh/e \) the effective field is now negative. This happens for fractions between \( \nu = 1 \) and the limiting case \( \nu = 1/2 \) (which is a compressible state). This is easily included in the CF scheme since changing the sign of \( B \) amounts to complex conjugation. We just have to use \( p \) filled CF levels with negative fluxes in Eq. (11). While there are more derivative operators, it also leads to satisfactory wavefunctions describing fractions now at \( p/(4p-1) \), \( p/(4p-1) \) and so on. This CF construction also explains naturally the occurrence of a gapless compressible state at \( \nu = 1/2 \) : this is the value for which the net effective flux is zero, suggestive of freely moving CF particles. For many years it has been known that one fraction with \( \nu = 5/2 \) does not fit into the CF scheme. This FQHE state is in the second LL and is in fact a state with partial filling \( 1/2 \) of the second LL. It is reasonable to expect that the filled LLL with the two spin values i.e. \( \nu = 2 \) plays the role of an inert dielectric medium renormalizing the interactions hence we should find a replica of the FQHE phenomenon in the second LL if the interactions are not dramatically altered.

In fact there is a big difference which is the appearance of a FQHE at \( \nu = 5/2 \) with an odd denominator, forbidden from CF/hierarchical constructions. The most successful candidate to describe this state is the Moore-Read Pfaffian state, which is a state with pairing between the effective CF particles. This state has fractionally charged excitations with charge \( e/4 \) for which there is some experimental evidence. It is described below.
Recent experiments have uncovered states displaying FQHE at filling factors \( \nu = 4/11, 5/13, 4/13, 6/17, \) and \( 5/17 \) that do not belong to the primary FQHE sequences. In addition, there is also evidence for two new even-denominator fractions \( \nu = 3/10, \) and \( 3/8. \) This is very unusual since the only previously known example of an even-denominator fraction is the elusive \( \nu = 5/2 \) state. The state \( 3/8 \) has been also observed in the N=1 LL at total filling factor \( \nu = 2 + 3/8. \) The new odd-denominator fractions can be explained by hierarchical reasoning in the spirit of the original Halperin-Haldane hierarchy. For example, at \( \nu = 4/11, \) the CFs have an effective filling factor \( \nu_{CF} = 1 + 1/3. \) If the interactions between the CFs have a repulsive short-range core then it is plausible that they will themselves form a standard Laughlin liquid at filling factor \( 1/3 \) within the second CF Landau level. It should be pointed out that this construction of “second generation” of composite fermions is part of the standard lore of the hierarchical view of the FQHE states since the CF construction and the older Halperin-Haldane hierarchy can be related by a change of basis in the lattice of quantum numbers. 

Since the even-denominator fractions requires clustering they do not fit naturally in this picture. There is no natural “Grand Unification” of all these new fractions in the hierarchical constructs.

It is possible to make a construction based on the idea of composite bosons that carry now an odd number of flux quanta i.e. a Jastrow factor to an odd power in the trial wavefunction. This has to be contrasted with the previous CF construction where we used only even powers of the Jastrow factor. These fluxes may be positive or negative. One can then exploit the possibility of clustering of bosons in the lowest LL (LLL). Indeed it has been suggested that incompressible liquids of Bose particle may form at fillings \( \nu = k/2 \) with integer \( k. \) We will now write down spin-polarized FQHE wavefunctions on the disk and spherical geometry. By construction they reside entirely in the LLL and have filling factor \( \nu = k/(3k \pm 2). \) While the positive flux series already appeared in the work of Read and Rezayi, the negative flux series is new. These series produce candidate wavefunctions for all the states observed by Pan et al. beyond the main CF sequences, thus unifying even and odd denominator fractions. For the fraction \( 3/7, \) the negative flux candidate wavefunction has an excellent overlap with the Coulomb ground state obtained by exact diagonalization on the sphere for \( N=6 \) electrons.

The first observation is that some of the new fractions of ref. \( 2 \) are of the form \( p/(3p \pm 1). \) This would be natural for the FQHE of bosons where one expects the formation of composite fermions with an odd number of flux tubes, i.e. \( 1 \) CF and \( 3 \) CF. The \( 1 \) CF lead to a series of Bose fractions at \( \nu = p/(p+1) \) which has nothing to do with the present problem. But if the \( 3 \) CFs fill an integer number of pseudo-Landau levels then this leads to magic filling factors \( p/(3p \pm 1). \) Indeed there is evidence from theoretical studies of bosons in the LLL with dipolar interactions that \( 1 \) CF do appear. This suggests that composite bosons may form in the electronic system, three flux tubes bound to one electron. \( 3 \) CBs, the attachment may be with statistical flux along or against the applied magnetic field. If \( \nu \) stands for the electron filling factor and \( \nu^* \) the \( 3 \) CB filling factor, they are related by \( 1/\nu = 3 + 1/\nu^*. \) The relationship between the wanted electronic trial wavefunction and the CB wavefunction is:

\[
\Psi^{\text{Fermi}}_\nu(z_i) = \mathcal{P}_{\text{LLL}} \prod_{i<j} (z_i - z_j)^3 \Phi^{\text{Bose}}_{\nu^*}(\{z_i, z_i^*\}).
\]  

(14)

The Laughlin-Jastrow factor \( \prod_{i<j} (z_i - z_j)^3 \) transforms bosons into fermions and adequately takes into account the Coulomb repulsion. The next step is to find candidates for the trial state \( \Phi^{\text{Bose}}_{\nu^*}. \) It has been suggested that bosons in the LLL may form incompressible states for \( \nu^* = k/2. \) There is some evidence that they are described by the Read-Rezayi parafermionic states with clustering of \( k \) particles:

\[
\Phi^{\text{RR}}_{\nu^* = k/2} = \mathcal{S} \left[ \prod_{i_1 < j_1} (z_{i_1} - z_{j_1})^3 \cdots \prod_{i_k < j_k} (z_{i_k} - z_{j_k})^3 \right].
\]  

(15)

In this equation, the \( \mathcal{S} \) symbol means non-symmetrization of the product of Laughlin-Jastrow factors over all partition of \( N \) particles in subsets of \( N/k \) particles (\( N \) being divisible by \( k \)) (the ubiquitous exponential factor appearing in all LLL states has been omitted for clarity). While the relevance of such states to bosons with contact interactions is not clear, it has been shown that longer-range interactions like dipolar interaction may help stabilize these states. Since the CBs are composite objects it is likely that their mutual interaction has also some long-range character. It is thus natural to try the ansatz \( \Phi^{\text{Bose}}_{\nu^*} = \Phi^{\text{RR}}_{\nu^* = k/2} \) in Eq. (14). This leads to a series of states with electron filling factor \( \nu = k/(3k + 2) \) which is in fact the \( M = 3 \) case of the generalized \( (k, M) \) states constructed by Read and Rezayi. In this construction, the flux attached to the boson is positive. It is also natural to construct wavefunctions with negative flux attached to the CBs. Now the Bose function depends only upon the antiholomorphic coordinates:

\[
\Phi^{\text{Bose}}_{\nu^*}(\{z_i^*\}) = (\Phi^{\text{RR}}_{\nu^* = k/2}(\{z_i\}))^*.
\]  

(16)
The projection onto the LLL in Eq. (14) means that the electronic wavefunction can be written as:

$$\Psi_{\nu}^{\text{Fermi}}(\{z_i\}) = \Phi_{\nu}^{RR}(\{\frac{\partial}{\partial z_i}\}) \prod_{i<j} (z_i - z_j)^3.$$  

(17)

The filling factor of this new series of states is now $\nu = k/(3k - 2)$. These states can be written in the spherical geometry with the help of the spinor components $u_i = \cos(\theta_i/2)e^{i\phi_i/2}$, $v_i = \sin(\theta_i/2)e^{-i\phi_i/2}$ ($\{\theta_i, \phi_i\}$ being standard polar coordinates) by making the following substitutions:

$$z_i - z_j \rightarrow u_i v_j - u_j v_i, \quad \partial_{z_i} - \partial_{z_j} \rightarrow \partial_{u_i} \partial_{v_j} - \partial_{v_i} \partial_{u_j}.$$  

(18)

This construction leads to wavefunctions that have zero total angular momentum $L = 0$ as expected for liquid states. On the sphere these two series of states have a definite relation between the number of flux quanta through the surface and the number of electrons. The positive flux series has $N_\phi = N/\nu - 5$ while the negative flux series has $N_\phi = N/\nu - 1$. Even when these states have the same filling factor as standard hierarchy/composite fermion states, the shift (the constant term in the $N_\phi - N$ relation) is in general different. The positive flux series starts with the Laughlin state for $\nu = 1/5$ at $k = 1$, the $k = 2$ state is the known Pfaffian state at $\nu = 1/4$, at $k = 3$ there is a state with $\nu = 3/11$ which competes with the $4\text{CF}$ state with negative flux, at $k = 4$ the competition is with the similar $\nu = 2/7$ $4\text{CF}$ state. This series also contains $5/17$ at $k = 5$, $3/8$ at $k = 6$, and $4/11$ at $k = 8$. The negative flux series starts with the filled Landau level at $k = 1$ and contains notably $5/13$ ($k = 5$), $3/8$ ($k = 6$), $4/11$ ($k = 8$), $6/17$ ($k = 12$). It is not likely that these states will compete favorably with the main sequence CF states in view of the remarkable stability of the latter. However the situation is open concerning the exotic even denominator and the unconventional odd-denominator states. Also the CF states may be destabilized by slightly tuning the interaction potential. It is known for example that there is a window of stability for a non-Abelian $\nu = 2/5$ state in the N=1 LLL which is obtained by slightly decreasing the $V_1$ pseudopotential component with respect to its Coulomb value.

A similar phenomenon seems to happen at $\nu = 3/7$ in the LLL. The conventional CF state at this filling factor is a member of the principal sequence of states. It is realized for $N = 9$ electrons at $N_\phi = 16$ in the spherical geometry. There is a singlet ground state and a well-defined branch of neutral excitations for $L = 2, 3, 4, 5$: see
Fig. (1b)). The negative-flux state Eq. (17) requires \( N_\phi = 20 \) for the same number of particles. At this flux for pure Coulomb interaction there is simply a set of nearly degenerate states without evidence for an incompressible state; see Fig. (1b). If the pseudopotential \( V_1 \) is decreased from its Coulomb LLL value, the CF state is quickly destroyed (Fig.(1c)) but there is appearance of a possibly incompressible state precisely at the special shift predicted above : Fig.(1d). There is a \( L = 0 \) ground state and a branch of excited states for \( L = 2, 3, 4 \). To check if this state has anything to do with the new negative flux state proposed above, the overlap between the candidate wavefunction for \( k = 3 \) in Eq.(17) and the numerically obtained ground state is displayed in Fig.(2) for \( N=6 \) electrons at \( N_\phi = 13 \). Even for the pure Coulomb interaction the squared overlap is 0.9641 and it rises up to 0.99054 for \( V_1 = 0.885V_{\text{Coulomb}} \).

More numerical evidences may be found in ref. (12) concerning some of the other fractions in these series like 3/8 and 3/11. It is difficult to study states with high-order \( k \)-clustering since they require at least 2\( k \) particles. It should be noted that so far these new series of states have not been derived from correlators of a conformal field theory. It is known that the Read-Rezayi states can all be derived from expectation values of fields of parafermionic CFTs. Proving that the new states we described above can be derived from a unitary CFT would be an indication that they describe incompressible candidate FQHE states.

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