The 6D Standing Wave Braneworld with Real Scalar Field

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Abstract

We introduce new non-static 6D braneworld model involving real 6D scalar field as the source of gravity. The brane is generated by the standing waves of gravity and real scalar field oscillating in the bulk. The oscillations are out of phase in time, so that the oscillation energy transfers back and forth between the standing waves of gravity and real scalar field. Underlying geometry of 6D spacetime has no horizons, the brane has one internal (on-brane) extra dimension which is compact and sufficiently small to describe our universe (hybrid compactification), and there is also one external to brane (off-brane) extra dimension, which is large or even infinite for the observers on the brane. We show that there are two different physical limits realizing the trapping of classical particles and light and also of massless scalar fields on the brane. In the first one, the amplitude of standing waves must be sufficiently large, while in the second one the amplitude of standing waves can be small enough. We show that in both cases particles and light, as well as massless scalar fields, are dynamically trapped on the brane by the pressure of bulk standing waves of gravity and real scalar field.

Keywords Branе · Standing waves · Trapping of particles · Zero modes

1 Introduction

Braneworld models with large extra dimensions [1–6] are very useful in addressing several insoluble problems in high energy physics (the hierarchy problem, the nature of flavor, etc.) (for reviews, see [7–10]). Most of the braneworlds are realized as static configurations, but there are also dynamical models, appearing mainly in cosmological studies, which involve time-dependent metrics and source matter fields, as well as branes with tensions varying in time [11–14]. One of the 5D braneworld models with non-stationary metric was proposed in [15–18] and was subsequently generalized to 6D in [19, 20]. In these models, the brane is generated by standing gravitational waves coupled to a phantom-like bulk scalar field, and rapid oscillations of these waves provide universal gravitational trapping of zero modes of all kinds of matter fields on the brane [20–25]. Later in [26] it was introduced another non-stationary 5D anisotropic braneworld generated by standing waves of the gravitational and real scalar field, instead of the phantom-like scalar fields of [15–20]. The metric of this model has horizon in the bulk, and in the case of small amplitudes of standing waves matter is actually trapped on the brane of the width equal to the horizon size, while in the case of large amplitudes of standing waves the zero modes of all kinds of matter fields are trapped on the brane of the width much less than the horizon size [27, 28].

In this article, we introduce a new non-static 6D braneworld model generated by standing gravitational waves coupled to real scalar field. It may seem that this new model is a straightforward generalization of the 5D braneworld introduced in [26] to 6D spacetime, but actually this is not quite true. First of all, the underlying geometry of our new 6D braneworld model does not have any horizons. In addition, the brane in our model has one internal (on-brane) extra dimension which is assumed to be compact and sufficiently small in order to describe our universe (hybrid compactification), and also the external to the brane (off-brane) extra dimension is large or even infinite for the observers on the brane, while in [26] it is compact and bounded by horizon. And, finally, the underlying metric of our model is symmetric in on-brane spatial coordinates corresponding to the spatial coordinates of our universe. We want to note that our braneworld model also differs
in underlying metric from that considered in [29], where authors also study the 6D standing-wave braneworld with normal matter as source.

Then, in the article, we show that there are two different physical situations (limits) realizing the trapping of classical particles and light and also of massless scalar fields on the brane in this new introduced model. One of them requires the amplitude of standing waves to be sufficiently large, while the other one admits the amplitude of standing waves to be small enough. We show that in both cases particles and light, as well as massless scalar fields, are dynamically trapped on the brane by the pressure of bulk standing waves of gravity and real scalar field.

2 The 6D Braneworld Model

We consider 6D spacetime with a single 5D-brane and non self-interacting real scalar field, \( \phi \). The action of the model is:

\[
S = \int d^6x \sqrt{g} \left( \frac{M^4}{2} R + \frac{1}{2} g^{AB} \partial_A \phi \partial_B \phi + L_{Brane} \right), \tag{1}
\]

where \( L_{Brane} \) is the brane Lagrangian and \( M \) is the fundamental scale, which is related to the 6D Newton’s constant, \( G = 1/(8\pi M^4) \). Capital Latin indices numerate 6D spacetime coordinates \((t, x, y, z, r, \theta)\), the signature of the metric is \((+, -, ..., -)\), and we use the units where \( c = \hbar = 1 \).

Variation of the action (1) with respect to metric \( g_{AB} \) and scalar field \( \phi \) gives us the Einstein equations:

\[
R_{AB} - \frac{1}{2} g_{AB} R = \frac{1}{M^4} (\sigma_{AB} + T_{AB}), \tag{2}
\]

where \( \sigma_{AB} \) and \( T_{AB} \) denote energy-momentum tensors of the brane and the bulk real scalar field, respectively, and

\[
T_{AB} = \partial_A \phi \partial_B \phi - \frac{1}{2} g_{AB} \partial^C \phi \partial_C \phi, \tag{3}
\]

and the equation of motion for the scalar field \( \phi \):

\[
\frac{1}{\sqrt{-g}} \partial_A \left( \sqrt{-g} g^{AB} \partial_B \phi \right) = 0. \tag{4}
\]

In the model, for the metric, we will use the following ansatz:

\[
ds^2 = \frac{e^s}{(1 + a |r|)^{1/2}} \left( dt^2 - dr^2 \right) - (1 + a |r|)^{1/2} \left[ e^u \left( dx^2 + dy^2 + dz^2 \right) + e^{-3u} d\theta^2 \right], \tag{5}
\]

where length-scale parameter \( a \) is a positive constant, and authors also study the 6D standing-wave braneworld with normal matter as source.

By the metric (5), we want to describe the geometry of the 5D-brane placed at the origin of the large spacelike extra dimension \( r \) orthogonal to the brane. Among four spatial coordinates of the brane, three of them, \( x, y, \) and \( z \), denote the ordinary dimensions of our universe, while the remaining one, \( \theta \), is assumed to be compact, curled up to the unobservable sizes for the present energies (hybrid compactification). Analogous metric was considered in [26–28] for the anisotropic braneworld model in 5D spacetime. In those articles, the length-scale parameter \( a \) enters the 5D metric with negative sign, and so the metric has horizon at \( |r| = 1/|a| \) in the bulk. Contrary to that, our metric (5) does not have any horizon in the bulk, and also is symmetric in the brane coordinates \( x, y, \) and \( z \), which correspond to the coordinates of our universe.

We assume that the bulk scalar field \( \phi \), the source of gravity, depends only on time \( t \) and absolute value of \( r \) coordinate:

\[
\phi = \phi (r, |r|), \tag{7}
\]

and so, using (5) and (6), its equation of motion (4) reduces to:

\[
\phi'' + \left[ \frac{a}{1 + a |r|} + 2 \delta (r) \right] \phi' - \phi = 0. \tag{8}
\]

Here, in (8), and in what follows, overdots and primes always denote the derivatives with respect to \( t \) and \( |r| \), respectively.

Now, using (3), we rewrite Einstein (2) into the form:

\[
R_{AB} = \frac{1}{M^4} \left( \sigma_{AB} - \frac{1}{4} g_{AB} \sigma + \partial_A \phi \partial_B \phi \right), \tag{9}
\]

where \( \sigma = g^{AB} \sigma_{AB} \), and using the metric (5), split (9) into the system of equations for metric functions \( u, s \), and bulk scalar field:

\[
-3u'' + \frac{1}{2} s'' + \frac{a}{2 (1 + a |r|)} s' = \frac{1}{M^4} \phi^2, \tag{10}
\]

\[
-3u' u + \frac{a}{1 + a |r|} u' = 0, \tag{11}
\]

\[
-3u' - \frac{1}{2} s'' + \frac{a}{2 (1 + a |r|)} a = \frac{1}{M^4} \phi^2, \tag{12}
\]
and for brane tensions:

\[
\left( s' - \frac{3a}{2} \right) \delta (r) = \frac{1}{M^4} \left( \sigma_{tt} - \frac{1}{4} \left( 1 + a |r| \right)^2 e^\omega \right), \\
e^{-s^+} \left( -u' - \frac{a}{2} \right) \delta (r) = \frac{1}{M^4} \left( \sigma_{xx} + \frac{1}{4} \left( 1 + a |r| \right)^2 e^\omega \right), \\
e^{-s^+} \left( -u' - \frac{a}{2} \right) \delta (r) = \frac{1}{M^4} \left( \sigma_{yy} + \frac{1}{4} \left( 1 + a |r| \right)^2 e^\omega \right), \\
e^{-s^+} \left( -u' - \frac{a}{2} \right) \delta (r) = \frac{1}{M^4} \left( \sigma_{zz} + \frac{1}{4} \left( 1 + a |r| \right)^2 e^\omega \right), \\
\left( -s' - \frac{5a}{2} \right) \delta (r) = \frac{1}{M^4} \left( \sigma_{rr} + \frac{1}{4} \left( 1 + a |r| \right)^2 \right), \\
e^{-s^+} \left( 3u' - \frac{a}{2} \right) \delta (r) = \frac{1}{M^4} \left( \sigma_{\theta\theta} + \frac{1}{4} \left( 1 + a |r| \right)^2 e^\omega \right).
\]

The solution to the (8), (11) is:

\[
u(t, |r|) = A \sin \left( \omega t \right) J_0 \left( \frac{\omega}{a} (1 + a |r|) \right),
\]

\[\phi(t, |r|) = \sqrt{3} M^2 A \cos \left( \omega t \right) J_0 \left( \frac{\omega}{a} (1 + a |r|) \right),
\]

\[s(|r|) = \frac{3A^2(\omega a)^2}{(1 + a |r|)^{3/2}} \left[ J_0^2 \left( \frac{\omega}{a} (1 + a |r|) \right) + J_1^2 \left( \frac{\omega}{a} (1 + a |r|) \right) \right] - \left( \frac{\omega a}{1 + a |r|} \right) J_0 \left( \frac{\omega}{a} (1 + a |r|) \right) J_1 \left( \frac{\omega}{a} (1 + a |r|) \right) + D
\]

where \(A, D, \) and \(\omega\) are some constants, and \(J_0\) and \(J_1\) are Bessel functions of the first kind. It is easy to see that \(AJ_0 \left( \frac{\omega}{a} (1 + a |r|) \right)\) and \(\omega\) correspondingly define amplitude and frequency of standing waves.

In order to consider the solution (12) as describing the brane at \(r = 0\), we need to require that the functions \(u, \phi\) and \(s\) vanish at \(r = 0\):

\[u(t, |r|)|_{r=0} = 0, \quad \phi(t, |r|)|_{r=0} = 0, \quad s(|r|)|_{r=0} = 0.
\]

All these conditions can be fulfilled by fixing \(\omega\) and \(D\) in the following way:

\[\omega = a Z_n ,
\]

\[D = -3A^2 Z_n^2 J_1^2 (Z_n) ,
\]

where \(Z_n\) denotes the \(n\)-th zero of the Bessel function \(J_0\). Consequently, the frequency \(\omega\) is quantized, it can take only discrete values in accordance with \(Z_n\), and in what follows for it we will use notation \(\omega_n\) with sub-index \(n\) reminding its connection with \(Z_n\):

\[\omega_n = a Z_n .
\]

Our solution (12) describes the brane at \(r = 0\) which is generated by the standing waves of bulk scalar field and gravity. And according to (12), the metric function \(u(t, |r|)\) and the bulk scalar field \(\phi(t, |r|)\), while having the same dependence on \(r\), are oscillating \(\pi/2\) out of phase in time, which means that oscillation energy is transferred back and forth between standing waves of gravity and scalar field.

There are only three free parameters in the model. These are:

\[a, \ A, \ Z_n .
\]

The first two parameters, \(a\) and \(A\), are continuous parameters, while the third parameter, \(Z_n\), is discrete because \(n\) can be only a natural number. In these free parameters, the solution (12) reads as follows:

\[u(t, |r|) = A \sin \left( a Z_n t \right) J_0 \left( Z_n (1 + a |r|) \right),
\]

\[\phi(t, |r|) = \sqrt{3} M^2 A \cos \left( a Z_n t \right) J_0 \left( Z_n (1 + a |r|) \right),
\]

\[s(|r|) = \frac{3A^2 Z_n^2}{(1 + a |r|)^{3/2}} \left[ J_0^2 \left( Z_n (1 + a |r|) \right) + J_1^2 \left( Z_n (1 + a |r|) \right) \right] - \frac{Z_n^{-1}}{(1 + a |r|)} J_0 \left( Z_n (1 + a |r|) \right) J_1 \left( Z_n (1 + a |r|) \right) - 3A^2 Z_n^2 J_1^2 (Z_n).
\]

Now, for any chosen values for these free parameters (17), it is easy to solve the (11) and find the brane energy-momentum tensor:

\[\sigma_R^A = M^4 \delta (r) diag \left[ \tau_r, \tau_x, \tau_y, \tau_z, \tau_\theta \right] ,
\]

where the brane tensions are:

\[\tau_r = -2a ,
\]

\[\tau_x = \tau_y = \tau_z = 0 ,
\]

\[\tau_\theta = -a \left( 3 + 4 A Z_n \sin (a \omega_n t) J_1 (Z_n) + 3A^2 Z_n^2 J_1^2 (Z_n) \right) ,
\]

\[3 \text{ Taylor Series and Time Averages of Metric Functions}
\]

In this section, we represent some useful expressions that will be used in the subsequent section to investigate trapping of matter on the brane by studying motion of classical particles and light in the background metric (5). The oscillating metric function \(u(t, |r|)\) enters the equations of motion in various forms, for example in the form of exponent in \(e^{\alpha u(t, |r|)}\), with \(\alpha\) being some real constant. In the limiting case, that realizes matter trapping, the standing wave frequency \(\omega_n\) is much larger than the frequencies corresponding to the energies \(E\) of particles trapped on the brane:

\[\omega_n = a Z_n \gg E .
\]
and so in the equations, we perform time averaging of the oscillating terms containing metric function \( u(t,r) \). Using the results of our previous papers \([21–23]\), we write the following expressions for time averages that will be useful in our calculations (time averages are denoted by angle brackets):

\[
\langle u f(r) \rangle = \langle \dot{u} f(r) \rangle = \langle u^t e^{a u} f(r) \rangle = 0,
\]

\[
\langle e^{a u} \rangle = \left\{ e^{a A} \text{sin}(a Z_n t) J_0(Z_n (1 + a |r|)) \right\} = I_0 (a A J_0 (Z_n (1 + a |r|))),
\]

where \( \alpha \) denotes any real constant, \( f(r) \) is any regular function depending only on \( r \), and \( I_0 \) is the zeroth-order modified Bessel function of the first kind.

We will also use the following Taylor series near the brane and asymptotic expansions far from the brane:

\[
\left. e^{a u} \right|_{a|r| \to 0} = 1 + \frac{1}{4} a^2 B (1 - (a |r|)) (a |r|)^2 + O \left( (a |r|)^4 \right),
\]

\[
\left. e^{a u} \right|_{a|r| \to 0} = 1 - \frac{1}{2} a^2 |r| + \left( \frac{B}{4} + \frac{3}{8} \right) (a |r|)^2 + O \left( (a |r|)^3 \right),
\]

\[
e^i \left|_{1 + a |r|} \right| = 1 + \left( 3B - \frac{3}{4} \right) a |r| + O \left( (a |r|)^2 \right),
\]

\[
e^{-i} \left|_{1 + a |r|} \right| = 1 - \left( 3B - \frac{3}{4} \right) a |r| + O \left( (a |r|)^2 \right),
\]

where \( \alpha \) still denotes any real constant, and the constant \( B \) is defined by free parameters \( A \) and \( Z_n \):

\[
B = A^2 Z_n^2 J_1^2 (Z_n).
\]

The parameter \( B \) contains \( Z_n \) in the form \( Z_n^2 J_1^2 (Z_n) \). Using the following well-known asymptotic expansions for Bessel functions:

\[
J_0(x) \mid_{x \to +\infty} = -\sqrt{\frac{2}{\pi}} \sin \left( x + \frac{\pi}{4} \right) x^{-\frac{1}{2}} + O \left( x^{-\frac{3}{2}} \right),
\]

\[
x^2 J^2_1(x) \mid_{x \to +\infty} = \frac{2}{\pi} \cos^2 \left( x + \frac{\pi}{4} \right) x - \frac{3}{4\pi} \cos (2x)
+ O \left( x^{-1} \right),
\]

it is easy to show that

\[
Z_n^2 J_1^2 (Z_n) = \frac{2}{\pi} \frac{Z_n}{Z_n + O \left( Z_n^{-1} \right)}.
\]

Using (37) and the fact that \( Z_n \) increases from \( Z_1 = 2.40482 \) to infinity when the natural number \( n \) runs from 1 to infinity, for parameter \( B \), we can safely use the following expression:

\[
B = \frac{2}{\pi} A^2 Z_n.
\]

And finally, using (38), the expressions (31) and (32) can be written as:

\[
u \mid_{a |r| \to \infty} = \sin (a Z_n t) \frac{\sin (Z_n + Z_n a |r| + \frac{\pi}{4})}{\sqrt{\frac{\pi}{2} Z_n^2 (a |r|)^2}} + O \left( (a |r|)^{-\frac{3}{2}} \right),
\]

\[
s \mid_{a |r| \to \infty} = 3Ba |r| + O \left( (a |r|)^{-1} \right),
\]

\[
u \mid_{a |r| \to \infty} = \sin (a Z_n t) \frac{\sin (Z_n + Z_n a |r| + \frac{\pi}{4})}{\sqrt{\frac{\pi}{2} Z_n^2 (a |r|)^2}} + O \left( (a |r|)^{-\frac{3}{2}} \right),
\]

\[
s \mid_{a |r| \to \infty} = 3Ba |r| + O \left( (a |r|)^{-1} \right).
\]

4 Trapping of Classical Particles and Light on the Brane

In this section, we consider motion of classical particles and photons in our 6D model with background metric (5) and show that they are trapped on the brane. Classical particles move according to the geodesic equation:

\[
\frac{d^2 x^A}{d\lambda^2} + \Gamma^A_{BC} \frac{dx^B}{d\lambda} \frac{dx^C}{d\lambda} = 0,
\]

where \( \Gamma^A_{BC} \) are the Christoffel symbols.
where $\lambda$ is a trajectory parameter. In our model, the nonvanishing Christoffel symbols are:

\[
\Gamma_{xx}^t = \Gamma_{yy}^t = \Gamma_{zz}^t = \frac{\dot{u}}{2}(1 + a \ |r|)^{\frac{3}{2}}e^{-s},
\]

\[
\Gamma_{ii}^t = \text{sgn} (r) \left( \frac{s'}{2} - \frac{3a}{8(1 + a \ |r|)} \right),
\]

\[
\Gamma_{\theta\phi}^t = -\frac{3\dot{u}}{2}(1 + a \ |r|)^{\frac{3}{2}}e^{-3u-s},
\]

\[
\Gamma_{xx}^r = \Gamma_{yy}^r = \Gamma_{zz}^r = \text{sgn} (r) \left( \frac{d}{2} + \frac{a}{4(1 + a \ |r|)} \right),
\]

\[
\Gamma_{xx}^r = \Gamma_{yy}^r = \Gamma_{zz}^r = \frac{\dot{u}}{2},
\]

\[
\Gamma_{xx}^r = \Gamma_{yy}^r = \Gamma_{zz}^r = -\text{sgn} (r) \left( \frac{d}{2} + \frac{a}{4(1 + a \ |r|)} \right),
\]

\[
\Gamma_{\theta\phi}^r = \text{sgn} (r) \left( \frac{3d}{2} - \frac{a}{4(1 + a \ |r|)} \right)(1 + a \ |r|)^{\frac{1}{2}}e^{-3u-s},
\]

\[
\Gamma_{\theta\phi}^r = -\text{sgn} (r) \left( \frac{3d}{2} - \frac{a}{4(1 + a \ |r|)} \right),
\]

\[
\Gamma_{\theta\phi}^r = -\frac{3\dot{u}}{2}.
\]

(42)

For reasons of simplicity, let us consider particle moving in the $xy$-plane. For such motion:

\[
dy = dz = d\theta = 0,
\]

(43)

and from (41), we get the following system of equations:

\[
\frac{d^2t}{d\lambda^2} + \Gamma_{xx}^t \left( \frac{dx}{d\lambda} \right)^2 + 2\Gamma_{xt}^t \frac{dt}{d\lambda} \frac{dr}{d\lambda} = 0,
\]

(44)

\[
\frac{d^2x}{d\lambda^2} + 2\Gamma_{xx}^t \frac{dx}{d\lambda} \frac{dt}{d\lambda} + 2\Gamma_{xt}^x \frac{dx}{d\lambda} \frac{dr}{d\lambda} = 0,
\]

(45)

\[
\frac{d^2r}{d\lambda^2} + \Gamma_{xx}^r \left( \frac{dr}{d\lambda} \right)^2 + \Gamma_{rr}^r \left( \frac{dt}{d\lambda} \right)^2 = 0.
\]

(46)

The explicit forms of these equations are:

\[
\frac{d^2t}{d\lambda^2} + \frac{\dot{u}}{2}(1 + a \ |r|)^{\frac{3}{2}}e^{-s-u} \left( \frac{dx}{d\lambda} \right)^2 + \text{sgn} (r) \left( \frac{s'}{2} - \frac{3a}{8(1 + a \ |r|)} \right) \frac{dt}{d\lambda} \frac{dr}{d\lambda} = 0,
\]

(47)

\[
\frac{d^2x}{d\lambda^2} + \frac{\dot{u}}{2}(1 + a \ |r|)^{\frac{3}{2}}e^{-s-u} \left( \frac{dx}{d\lambda} \right)^2 + \text{sgn} (r) \left( \frac{d}{2} + \frac{a}{4(1 + a \ |r|)} \right) \frac{dx}{d\lambda} \frac{dr}{d\lambda} = 0,
\]

(48)

\[
\frac{d^2r}{d\lambda^2} + \frac{\dot{u}}{2}(1 + a \ |r|)^{\frac{3}{2}}e^{-s-u} \left( \frac{dr}{d\lambda} \right)^2 + \text{sgn} (r) \left( \frac{d}{2} + \frac{a}{4(1 + a \ |r|)} \right) \frac{dr}{d\lambda} \frac{dt}{d\lambda} = 0.
\]

(49)

Dividing by $\frac{dx}{d\lambda}$, (48) can be rewritten as:

\[
\frac{d}{d\lambda} \left[ \ln \left( \frac{dx}{d\lambda} \right) + u + \frac{1}{2} \ln (1 + a \ |r|) \right] = 0,
\]

(50)

which immediately gives us the integral

\[
\frac{dx}{d\lambda} = \xi e^{-u} (1 + a \ |r|)^{-\frac{1}{2}},
\]

(51)

where the integration constant $\xi$ can be associated with the velocity of particle along the brane.

Now, using (51), we rewrite (47) in the form:

\[
\frac{d^2t}{d\lambda^2} + \xi^2 \frac{\dot{u}}{2}(1 + a \ |r|)^{\frac{3}{2}}e^{-s-u} + \text{sgn} (r) \left( \frac{s'}{2} - \frac{3a}{8(1 + a \ |r|)} \right) \frac{dt}{d\lambda} \frac{dr}{d\lambda} = 0.
\]

(52)

In this equation, we can neglect the second term containing $\dot{u}$ because the time average of this term is equal to 0 according to (21) and (22):

\[
\left[ \xi^2 \frac{\dot{u}}{2}(1 + a \ |r|)^{\frac{3}{2}}e^{-s-u} \right] = 0,
\]

(53)

and as we will see soon (see formula (55)) on the brane ($r = 0$) the geodesic parameter $\lambda$ will coincide with the coordinate time $t$. What remains can be rewritten in the form:

\[
\frac{d}{d\lambda} \left[ \ln \left( \frac{dt}{d\lambda} \right) + s - \frac{3}{4} \ln (1 + a \ |r|) \right] = 0,
\]

(54)

which, in analogy with (50) and (51), gives us

\[
\frac{dt}{d\lambda} = e^{-s}(1 + a \ |r|)^{\frac{3}{2}},
\]

(55)

where the integration constant we have chosen is equal to 1, so that on the brane, i.e., at $r = 0$, the geodesic parameter $\lambda$ will coincide with the coordinate time $t$.

Now, inserting (51) and (55) into the (49), and multiplying it by $(1 + a \ |r|)^{-\frac{3}{2}}e^{\frac{3}{2} \frac{dt}{d\lambda}}$, we get the equation:

\[
\frac{d}{d\lambda} \left[ e^{\frac{3}{2} \frac{dt}{d\lambda}} \right] + \xi^2 (1 + a \ |r|)^{-\frac{1}{2}}e^{-u} -(1 + a \ |r|)^{\frac{3}{2}}e^{-s} = 0.
\]

(56)

Its first integral is:

\[
\frac{e^{\frac{3}{2} \frac{dt}{d\lambda}}}{(1 + a \ |r|)^{\frac{3}{2}}} + \xi^2 (1 + a \ |r|)^{-\frac{1}{2}}e^{-u} -(1 + a \ |r|)^{\frac{3}{2}}e^{-s} = C,
\]

(57)

with $C$ being the integration constant.

Now, let us investigate this (57) in the region near the brane where $a \ |r| < 1$. It is obvious that the width of this region along the extra dimension $r$ is of order of $a^{-1}$, and when $a \ll 1$, it can be sufficiently large. In this region, we
can use Taylor expansions from the previous section. And so, using (25), (27), and (28), (57) can be presented as:

\[
1 + \left( B - \frac{3}{4} \right)a \left| r \right| \left( \frac{dr}{d\lambda} \right)^2 - \frac{1}{2} \xi^2 a \left| r \right| + \left( B - \frac{3}{4} \right)a \left| r \right| = C,
\]

(58)

where all constant values coming from Taylor expansions we have included into the integration constant \( C \), and the parameter \( B \) is defined by (34). Now, let us show that in the case when the parameter \( B \) is large:

\[ B = A^2 Z_n^2 J_1^2 (Z_n) \gg 1, \quad (59) \]

particles and light are trapped on the brane. It must be mentioned that this condition can be realized in two different physical situations. In the first one, the amplitude \( A \) of the standing waves is large, i.e., we have:

\[ A \gg 1. \quad (60) \]

It is obvious that in this situation the requirement (59) is fulfilled because for all values of \( n \) (see also (38)):

\[ Z_n \geq Z_1 = 2.40482 > 1. \quad (61) \]

In the second physical situation, the amplitude of the standing waves is small, i.e., we have:

\[ A \ll 1, \quad (62) \]

and simultaneously the frequency of the standing waves is sufficiently large, i.e., the zero \( Z_n \) of the Bessel function \( J_0 \) is sufficiently large, so that:

\[ Z_n \gg \frac{1}{A}. \quad (63) \]

It is easy to see that in this situation the requirement (59) is also fulfilled.

So, in both physical situations, parameter \( B \) is large (see (59)), and the (58) can be rewritten in the form:

\[ (1 + a B \left| r \right|) \left( \frac{dr}{d\lambda} \right)^2 + a B \left| r \right| = C = \rho^2, \quad (64) \]

where, taking into account, the fact that the left-hand side of this equation is always positive, we have renamed the integration constant \( C \) by \( \rho^2 \) to emphasize that it must be positive. From the equation, it is easy to see that \( B a \left| r \right| \) plays the role of trapping gravitational potential.

Now, inserting into the line element (5) expressions (43), (51), (55), and (64), and using formulas (25), (27), and (28), we express the interval in terms of the parameter \( \lambda \):

\[
d s^2 = \frac{e^{\xi} (d t^2 - d r^2)}{(1 + a \left| r \right|)^{\frac{3}{2}}} - \frac{e^{\rho^2} d x^2}{(1 + a \left| r \right|)^{-\frac{1}{2}}} = \left[ 1 - \left( \xi^2 + \rho^2 \right) - 2 \left( 1 + \rho^2 \right)a B \left| r \right| \right] d \lambda^2. \quad (65)
\]

Taking this expression on the brane, \( r = 0 \), it is easy to see that for light \( \xi^2 + \rho^2 = 1 \) and for massive particles \( \xi^2 + \rho^2 < 1 \).

It is obvious from (64) that motions of particles and light along the extra dimension \( r \) are confined, and they are trapped on the brane in the region:

\[ |r| \leq \frac{\rho^2}{a B}. \quad (66) \]

so that the width of the localization area \( \Delta \) is of the order of \( (a B)^{-1} \)

\[ \Delta \sim \frac{1}{a B}. \quad (67) \]

5 Localization of Scalar Fields

In this section, we consider the localization of a real massless scalar field \( \Phi (x^4) \) on the brane in the case (59), i.e., when \( B \gg 1 \). The Klein-Gordon equation for 6D scalar field \( \Phi (x^4) \), obtained from the 6D action:

\[ S = \frac{1}{2} \int d^6 x \sqrt{g} g^{AB} \partial_A \Phi \partial_B \Phi, \quad (68) \]

has form

\[ \frac{1}{\sqrt{g}} \partial_A \left( \sqrt{g} g^{AB} \partial_B \Phi \right) = 0. \quad (69) \]

Using (5) and (6), this equation can be written as:

\[
\left[ \partial_t^2 - \frac{e^{\xi}}{(1 + a \left| r \right|)^{\frac{3}{2}}} \left( \partial_x^2 + \partial_y^2 + \partial_z^2 + e^{\rho^2} \partial_\rho^2 \right) \right] \Phi
= \frac{\partial_r \left[ (1 + a \left| r \right|) \partial_r \Phi \right]}{(1 + a \left| r \right|)}. \quad (70)
\]

Now, in our case \( B \gg 1 \), it is easy to see that the metric functions \( u \) and \( s \) obey the relation:

\[
\left| \frac{u (t, \left| r \right|)}{s (\left| r \right|)} \right| \ll 1, \quad (71)
\]

which holds for any \( t \) and \( r \). For the regions close to the brane \( a \left| r \right| \to 0 \) and far from the brane \( a \left| r \right| \to \infty \), this relation straightforwardly follows from expressions (29), (30), (39), and (40). So, in the scalar field (70), we can drop the function \( u \) in the exponents and rewrite it as:

\[
\left[ \partial_t^2 - \frac{e^{\xi}}{(1 + a \left| r \right|)^{\frac{3}{2}}} \left( \partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_\rho^2 \right) \right] \Phi
= \frac{\partial_r \left[ (1 + a \left| r \right|) \partial_r \Phi \right]}{(1 + a \left| r \right|)}. \quad (72)
\]

We look for the solution to this equation in the form:

\[
\Phi (t, x, y, z, r, \theta) = \phi (x^4) \chi (\left| r \right|) \Theta (\theta)
= \phi (x^4) \sum_{l, m} \chi_m (\left| r \right|) e^{il \frac{\theta}{\Theta}}, \quad (73)
\]

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where Greek indices numerate 4D coordinates \( t, x, y, \) and \( z, \) and \( \theta_0 \) is some constant normalizing the curled up coordinate \( \theta \) as
\[
0 \leq \frac{\theta}{\theta_0} \leq 2\pi, \tag{74}
\]
and 4D factor \( \varphi (x^\mu) \) of the scalar field wavefunction obeys the equation:
\[
\eta^{\rho \beta} \partial_\rho \partial_\beta \varphi (x^\mu) = -m_0^2 \varphi (x^\mu). \tag{75}
\]

In what follows, we assume:
\[
\varphi (x^\mu) = e^{-i(Et - p_x x - p_y y - p_z z)}. \tag{76}
\]

Then, extra dimension factor \( \chi_m (|r|) \) of the 6D scalar field wavefunction obeys the equation:
\[
\chi_m'' + \frac{a}{1 + a |r|} \chi_m' - E^2 \left( \frac{e^s}{1 + a |r|} \right)^{\frac{3}{2}} - 1 \chi_m = 0, \tag{77}
\]
with the boundary conditions:
\[
\begin{align*}
\chi_m' &|_{|r|=0} = 0, \tag{78} \\
\chi_m &|_{|r|\to \infty} = 0. \tag{79}
\end{align*}
\]

In the case of massless 4D scalar field \((m_0 = 0)\) the extra dimension factor \( \chi_0 \) corresponding to s-wave \((l = 0)\) bulk zero mode obeys the equation:
\[
\chi_0'' + \frac{a \text{sgn}(r)}{1 + a |r|} \chi_0' - E^2 \left( \frac{e^s}{1 + a |r|} \right)^{\frac{3}{2}} - 1 \chi_0 = 0. \tag{80}
\]

To show that this equation has the solution localized on the brane, we investigate it in two limiting regions: close to the brane \((a|r| \to 0)\) and far from the brane \((a|r| \to \infty)\).

In our case, i.e., when \( B \gg 1 \), in the limiting region \(a|r| \to 0\), (80) has the following asymptotic form (see expression (30)):
\[
\chi_0'' + a \text{sgn}(r) \chi_0' - 3E^2 B a |r| \chi_0 = 0, \tag{81}
\]
which has the general solution
\[
\chi_0 = e^{-\frac{a|z|}{2}} \left[ C_1 Ai \left( \frac{1}{4} k^{-\frac{3}{2}} + \frac{1}{4} a |r| \right) \right. \\
\left. + C_2 Bi \left( \frac{1}{4} k^{-\frac{3}{2}} + \frac{1}{4} a |r| \right) \right], \tag{82}
\]
where \( C_1 \) and \( C_2 \) are integration constants, \( Ai \) and \( Bi \) are Airy functions, and the parameter \( k \) is
\[
k = \frac{3BE^2}{a^2}. \tag{83}
\]

To fulfill the condition (78), the constants \( C_1 \) and \( C_2 \) must obey the relation:
\[
C_2 = \frac{-2k^4 Ai' \left( \frac{1}{4} k^{-\frac{3}{2}} \right) + Ai \left( \frac{1}{4} k^{-\frac{3}{2}} \right)}{2k^4 Bi' \left( \frac{1}{4} k^{-\frac{3}{2}} \right) - Bi \left( \frac{1}{4} k^{-\frac{3}{2}} \right)} C_1, \tag{84}
\]
where \( Ai' \) and \( Bi' \) denote the first derivatives of Airy functions. Then, at \( r = 0 \), i.e., on the brane, the function \( \chi_0 (|r|) \) will have the following series expansion:
\[
\chi_0 (|r|)_{|r|=0} = C \left( 1 + \frac{k}{6} a^3 |r|^3 \right) + O (a^4 |r|^4). \tag{85}
\]

where the constant \( C \) is defined as:
\[
C = \frac{Ai \left( \frac{1}{4} k^{-\frac{3}{2}} \right) Bi' \left( \frac{1}{4} k^{-\frac{3}{2}} \right) - Ai' \left( \frac{1}{4} k^{-\frac{3}{2}} \right) Bi \left( \frac{1}{4} k^{-\frac{3}{2}} \right)}{Bi' \left( \frac{1}{4} k^{-\frac{3}{2}} \right) - \frac{1}{2} k^{-\frac{1}{2}} Bi \left( \frac{1}{4} k^{-\frac{3}{2}} \right) C_1. \tag{86}
\]

In the second limiting region \( a|r| \to \infty \), (80) has the following asymptotic form (see the expression (40)):
\[
\chi_0'' + \frac{a \text{sgn}(r)}{1 + a |r|} \chi_0' - E^2 \left( \frac{e^s}{1 + a |r|} \right)^{\frac{3}{2}} - 1 \chi_0 = 0, \tag{87}
\]
and in our case \( B \gg 1 \), making change to variable
\[
z = 1 + a|r|, \tag{88}
\]
in the region \( z \to \infty \) this equation gets the form
\[
\frac{d^2 \chi_0 (z)}{dz^2} + \frac{1}{z} \frac{d\chi_0 (z)}{dz} - \frac{E^2}{a^2} e^{3Bz} \chi_0 (z) = 0. \tag{89}
\]

Now, introducing the new function \( g_0 (z) \) as
\[
\chi_0 (z) = z^{-\frac{1}{2}} g_0 (z), \tag{90}
\]
we get the equation:
\[
\frac{d^2 g_0 (z)}{dz^2} - \frac{E^2}{a^2} e^{3Bz} g_0 (z) = 0, \tag{91}
\]
which in terms of new variable
\[
y = e^{3Bz}, \tag{92}
\]
gets the form
\[
\frac{d^2 g_0 (y)}{dy^2} + \frac{d g_0 (y)}{dy} - \frac{E^2}{9a^2 B^2} g_0 (y) = 0. \tag{93}
\]
This equation has the following general solution:
\[
g_0 (y) = C_3 K_0 (\kappa \sqrt{y}) + C_4 I_0 (\kappa \sqrt{y}), \tag{94}
\]
where \( C_3 \) and \( C_4 \) are some integration constants, \( I_0 \) and \( K_0 \) are zero-order modified Bessel functions of the first and second kinds, respectively, and the positive constant \( \kappa \) is defined as

\[
\kappa = \frac{2E}{3aB}.
\]

Then, using (88), (90), and (92), for the \( \chi_0 \) we have:

\[
\chi_0 (|r|)|_{|a|r| \to \infty} = C_3 \frac{K_0 \left( \kappa e^2 Ba |r| \right)}{|r|} + C_4 \frac{I_0 \left( \kappa e^2 Ba |r| \right)}{|r|}.
\]

From (96) we get that in the limiting region \( a |r| \to \infty \) the zero mode has the following asymptotic form:

\[
\chi_0 (|r|)|_{|a|r| \to \infty} = C_3 \frac{e^{- \kappa \frac{1}{2} Ba |r|}}{|r|} \left[ 1 - \frac{1}{8} \kappa^{-1} e^{- \frac{1}{2} Ba |r|} + O \left( e^{-3 Ba |r|} \right) \right],
\]

where all the constant factors, that appeared before the square brackets, were included in the constant \( C_3 \). Taking into account (71), the action (68) for the 6D scalar field zero mode can be written in the form:

\[
S_0 = \pi \theta_0 \int d^4 x \int d r \left[ F_1 \partial_\nu \varphi^2 - F_2 \left( \partial_\nu \varphi^2 + \partial_\nu \varphi^2 + \partial_\nu \varphi^2 \right) - F_3 \varphi^2 \right],
\]

where integration over \( \theta \) has already been carried out, and \( F_i \) functions are defined as:

\[
F_1 \left( a |r| \right) = (1 + a |r|) \chi_0^2, \quad F_2 \left( a |r| \right) = (1 + a |r|)^{- \frac{3}{2}} \chi_0^2, \quad F_3 \left( a |r| \right) = (1 + a |r|) \chi_0^2.
\]

Using expressions (85) and (98), it is easy to find that these functions in the two limiting regions, i.e., near the brane ( \( a |r| \to 0 \) ) and far from the brane ( \( a |r| \to \infty \) ), have the following asymptotic expansions:

\[
F_1 \left( a |r| \to 0 \right) = \frac{C_2}{(1 + a |r|)} + \frac{O \left( a^3 |r|^3 \right)}{3},
\]

\[
F_2 \left( a |r| \to 0 \right) = \frac{C_2}{4} \left( 1 - \frac{3}{4} a |r| + \frac{5}{32} a^2 |r|^2 \right) + \frac{O \left( a^3 |r|^3 \right)}{3},
\]

\[
F_3 \left( a |r| \to 0 \right) = \frac{C_2 e^{- \frac{3}{2} a |r|}}{1 - \frac{1}{4} \kappa^{-1} e^{- \frac{1}{2} a |r|} + O \left( e^{-3 a |r|} \right)}.
\]

Now, it is easy to see that the integral over coordinate \( r \) in (99) is finite. Indeed, in the region near the brane, \( a |r| \to 0 \), the functions \( F_i \left( r \right) \) in the integrand behave in a regular way (see expressions (103), (104), and (105)), and in the region far from the brane, \( a |r| \to \infty \), the integral is convergent due to the fact that functions \( F_i \) at infinity sharply decrease as \( e^{- \frac{3}{2} a |r|} \), that is, as an exponential of an exponential (see expressions (106), (107), and (108)). This means that the scalar field zero mode wavefunction is localized on the brane. Also note that on the brane, \( r = 0 \), due to the expressions (103), (104), and (105), the Lagrangian in the action (99) gets the standard 4D form for the massless scalar field.

6 Conclusions

In the paper, we have introduced new non-static 6D braneworld model involving real 6D bulk scalar field as the source of gravity. The brane is generated by the standing waves of gravity and real scalar field oscillating in the bulk. The oscillations are out of phase in time, so that the oscillation energy transfers back and forth between the standing waves of gravity and real bulk scalar field. Underlying geometry of 6D spacetime has no horizons, the
brane has one internal (on-brane) extra dimension which is compact and sufficiently small to describe our universe (hybrid compactification), and there is also one external to brane (off-brane) extra dimension, which is large or even infinite for the observers on the brane. It is shown that in the case of large model parameter $B \gg 1$, the classical particles and light, as well as massless scalar fields, are localized on the brane. It is also shown that there are two different physical situations (limits) realizing the condition $B \gg 1$. In the first physical situation, the amplitude of standing waves must be sufficiently large, while in the second physical situation the amplitude of standing waves can be small enough. In both cases, particles and light, as well as massless scalar fields, are dynamically trapped on the brane by the pressure of standing waves of gravity.

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