PHASE TRANSITIONS ON HECKE C*-ALGEBRAS
AND CLASS-FIELD THEORY OVER \( \mathbb{Q} \)

MARCELO LACA AND MACHIEL VAN FRANKENHUIJSEN

Abstract. We associate a canonical Hecke pair of semidirect product groups to the ring inclusion of the algebraic integers \( \mathcal{O} \) in a number field \( K \), and we construct a C*-dynamical system on the corresponding Hecke C*-algebra, analogous to the one constructed by Bost and Connes for the inclusion of the integers in the rational numbers. We describe the structure of the resulting Hecke C*-algebra as a semigroup crossed product and then, in the case of class number one, analyze the equilibrium (KMS) states of the dynamical system. The extreme KMS\(_\beta\) states at low-temperature exhibit a phase transition with symmetry breaking that strongly suggests a connection with class field theory. Indeed, for purely imaginary fields of class number one, the group of symmetries, which acts freely and transitively on the extreme KMS\(_\infty\) states, is isomorphic to the Galois group of the maximal abelian extension over the field. However, the Galois action on the restrictions of extreme KMS\(_\infty\) states to the (arithmetic) Hecke algebra over \( K \), as given by class-field theory, corresponds to the action of the symmetry group if and only if the number field \( K \) is \( \mathbb{Q} \).

Introduction

The C*-dynamical system based on a noncommutative Hecke C*-algebra constructed by Bost and Connes from the inclusion of the integers in the rationals has inspired several authors to construct Hecke C*-algebras associated to algebraic number fields and function fields, e.g. [HLe, ALR, Coh]. These constructions share many of the interesting features of the Bost-Connes construction, in particular they all have a phase transition with spontaneous symmetry breaking at low temperature, a partition function related to the zeta function of the number field, and a unique type III factor equilibrium state at high temperature. The Bost-Connes C*-dynamical system has a group of symmetries that acts freely and transitively on the extreme KMS\(_\beta\) states for \( \beta > 1 \), and which is isomorphic to the Galois group of the maximal abelian extension \( \mathbb{Q}^{ab} \) of \( \mathbb{Q} \). Moreover, the embeddings of \( \mathbb{Q}^{ab} \) in \( \mathbb{C} \) are given by the evaluation of the extreme KMS\(_\infty\) states on an ‘arithmetic’ Hecke algebra over \( \mathbb{Q} \), and thus their model also has an arithmetic symmetry, in which the Galois group acts by Galois automorphisms on the values of extreme KMS\(_\infty\) states. This feature of having concrete Galois groups as symmetries has not been generalized to number fields, although in some cases the broken symmetries do have an interpretation as abstract Galois symmetries, e.g. [HLe, Coh].

In this work we were initially motivated by the observation that the symmetry groups of [HLe] are isomorphic to Galois groups for the nine imaginary quadratic fields of class number 1, see Remark 4.6. We study the Hecke C*-algebra canonically associated to the inclusion of the ring of integers \( \mathcal{O} \) in an algebraic number field \( K \). Specifically, we consider the full ‘\( ax+b \)’ group \( \mathcal{O} \rtimes \mathcal{O}^* \) of the ring of algebraic integers in that of the field, \( K \rtimes \mathbb{C}^* \). By ‘canonical’ here we mean that no cross section is

Date: September 28, 2004.

2000 Mathematics Subject Classification. Primary 46L55; Secondary 11R18, 82B10, 82B26.

Key words and phrases. Hecke algebra, phase transition, symmetry breaking, class field theory. Supported by the National Science and Engineering Research Council of Canada.
chosen to deal with the presence of the units $O^*$ in $O$. Although the inclusion of the full group of units in the subgroup causes some technical difficulties with the Hecke inclusion at the onset, these are offset later by an eventual simplification in the computation of the equilibrium states. Indeed, the unit group, which when infinite presented an obstacle to the computation of KMS states on the Hecke algebras of [ALR], is ‘factored out’ from our Hecke algebra, and as a result, it is the semigroup of principal integral ideals that acts as our renormalization semigroup of endomorphisms.

In Section 1 we carry out the analysis of the structure of our Hecke algebra for a general number field $K$. We give a presentation in terms of generators and relations and also a realization as the crossed product of a certain commutative $*$-algebra by an action of the semigroup of principal integral ideals of $K$. This commutative $*$-algebra can be described in three different ways: first as the fixed point algebra of the action of the units on the group algebra of $K/O$, then as the Hecke $*$-algebra of an intermediate Hecke pair, and finally, via the Fourier transform in Section 2, as an algebra of continuous functions on the dual of $K/O$, which we view as the ‘adelic global inverse different’. Similar descriptions of the corresponding $C^*$-algebra are also possible thanks to a natural profinite compactification of the group $O^*$ that arises from their action on $K/O$, which allows us to consider the appropriate compact orbit space, see Lemma 2.3.

The Hecke $C^*$-algebra has a natural dynamics and a natural group of ‘geometric’ symmetries. These geometric symmetries are induced from the symmetries of the space of orbits of the additive classes in $K/O$ under the multiplicative action of the unit group $O^*$. To describe the phase transition of KMS states, in Section 3 we restrict our attention to fields of class number one. The phase transition and the symmetries for fields of higher class numbers require techniques beyond the ones developed in [L1] for lattice semigroups, and will be considered elsewhere.

Our main result is Theorem 3.1, where we show that for fields of class number one there is a unique KMS$_\beta$ state for $0 \leq \beta \leq 1$, while for $\beta > 1$ the symmetry group acts freely and transitively on extreme KMS$_\beta$ states. In the absence of real embeddings, that is, for purely imaginary fields, this symmetry group is isomorphic to the Galois group of the maximal abelian extension of the field, but in general, the symmetry group is missing the $\{\pm 1\}$ factors in the Galois group corresponding to complex conjugation on each real embedding.

To explore the possibility of a Galois action of the symmetry group, that is, the possibility of arithmetic symmetries, we consider in Section 4 the natural candidate for an arithmetic Hecke algebra over the field. When the extreme KMS$_\infty$ states are evaluated on this algebra, the resulting values are algebraic numbers, but the Hecke algebra model is based on torsion alone, in the sense that these numbers always lie in the maximal cyclotomic extension, and hence are not enough to support a free transitive action of the Galois group of the maximal abelian extension of the field. Indeed, in Theorem 4.4 when we compare the action of an idele as a geometric symmetry to its Galois action (via the Artin map) as an arithmetic symmetry on KMS$_\infty$ states, we discover that they are intertwined only when $K$ is equal $Q$.

This leads us to the conclusion that the connection with class field theory in the Bost–Connes Hecke $C^*$-algebra model is an exceptional feature derived from the peculiarities of the base field $Q$; specifically, a consequence of the Kronecker-Weber Theorem, by which the maximal abelian extension of $Q$ is in fact the maximal cyclotomic extension. However, the results of [BC] are too suggestive and the possibility of a noncommutative model that supports both arithmetic and geometric symmetries for number fields other than $Q$ is too tempting to be abandoned without
further effort. Indeed, as this work was being finished, we learned of a new construction due to Connes and Marcolli \cite{CM} in which a new \textit{C*-}dynamical system is proposed that promises to generate enough algebraic values for KMS$_\infty$ states to support free transitive Galois actions.

Acknowledgments: This research was initiated during a visit of M.L. to IHES, and carried out through several visits, of M.v.F. to the University of Newcastle, Australia and to the SFB 478 at the University of Muenster, and of M.L. to Utah Valley State College. We acknowledge the support of those institutions and we would like to thank the respective mathematics departments for their hospitality.

1. The Hecke C*-algebra

We use the following notation: $\mathcal{K}$ will denote an algebraic number field with ring of integers $\mathcal{O}$. The invertible elements of $\mathcal{K}$ form a multiplicative group, which will be denoted by $\mathcal{K}^*$; since $\mathcal{K}$ is a field, these are simply the nonzero elements of $\mathcal{K}$. Similarly, the group of invertible elements (units) of $\mathcal{O}$ will be denoted by $\mathcal{O}^*$; notice that $\mathcal{O}^*$ is strictly smaller than the multiplicative semigroup $\mathcal{O}^\times$ of nonzero integers.

To the inclusion of $\mathcal{O}$ in $\mathcal{K}$ we associate the following inclusion of \textit{`ax + b' groups'}

$$P_\mathcal{O} := \begin{pmatrix} 1 & \mathcal{O} \\ 0 & \mathcal{O}^* \end{pmatrix} \subset P_\mathcal{K} := \begin{pmatrix} 1 & \mathcal{K} \\ 0 & \mathcal{K}^* \end{pmatrix}.$$  

We shall prove that this is a Hecke (or almost normal) inclusion; in other words, that every double coset contains finitely many right cosets. This is equivalent to saying that the action of $P_\mathcal{O}$ by right-multiplication on the right coset space $P_\mathcal{O} \backslash P_\mathcal{K}$ has finite orbits. Before we can prove this, we need to develop some basic notation and results concerning the multiplicative action of the units $\mathcal{O}^*$ on $\mathcal{K}$ modulo $\mathcal{O}$.

For each $r \in \mathcal{K}$, denote by $\mathcal{O}^*_r$ the subgroup of $\mathcal{O}^*$ that fixes $r$ modulo $\mathcal{O}$:

$$\mathcal{O}^*_r = \{ u \in \mathcal{O}^* : ru = r \text{ mod } \mathcal{O} \}.$$  

More generally, for each fractional ideal $a$ of $\mathcal{K}$, denote by $\mathcal{O}^*_a$ the subgroup of $\mathcal{O}^*$ that fixes each element of $a$ modulo $\mathcal{O}$:

$$\mathcal{O}^*_a := \{ u \in \mathcal{O}^* : ru = r \text{ mod } \mathcal{O} \text{ for each } r \in a \} = \bigcap_{r \in a} \mathcal{O}^*_r.$$  

In particular, note that $\mathcal{O}^* = \mathcal{O}^*_{\mathcal{O}}$, that $\mathcal{O}^*_r$ depends only on the class of $r$ in $\mathcal{K}/\mathcal{O}$, and that $\mathcal{O}^*_a$ depends only on the ideal $a + \mathcal{O}$. Also, if $a$ and $b$ are ideals of $\mathcal{K}$ with $a \subseteq b$, then $\mathcal{O}^*_a \supseteq \mathcal{O}^*_b$.

\begin{lemma}
For every fractional ideal $a$ in $\mathcal{K}$, the index of $\mathcal{O}^*_a$ in $\mathcal{O}^*$ is finite.
\end{lemma}

\begin{proof}
Let $b := (a + \mathcal{O})^{-1}$. Then $b$ is an integral ideal and the multiplicative group $(\mathcal{O}/b)^*$ is finite. Let $n$ be the exponent of $(\mathcal{O}/b)^*$, so that $u^n = 1$ mod $b$ for every $u \in \mathcal{O}^*$. It follows that $ru^n = r$ mod $\mathcal{O}$ for every $r \in a$, hence $\mathcal{O}^*/\mathcal{O}^*_a$ has exponent at most $n$. Since $\mathcal{O}^*$ is finitely generated, this shows that the quotient $\mathcal{O}^*/\mathcal{O}^*_a$ is finite. \hfill $\square$

The right coset of $\gamma := \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix}$ is

$$P_{\mathcal{O}} \gamma = \left\{ \begin{pmatrix} 1 & y + xa \\ 0 & xu \end{pmatrix} : a \in \mathcal{O}, u \in \mathcal{O}^* \right\} = \begin{pmatrix} 1 & y + x\mathcal{O} \\ 0 & x\mathcal{O}^* \end{pmatrix},$$

with the obvious notation for a set of matrices. The image of $P_{\mathcal{O}} \gamma$ under the action of $\begin{pmatrix} 1 & a \\ 0 & u \end{pmatrix} \in P_{\mathcal{O}}$ by right-multiplication is simply $P_{\mathcal{O}} \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & u \end{pmatrix} = \begin{pmatrix} 1 & a + yu + x\mathcal{O} \\ 0 & x\mathcal{O}^* \end{pmatrix}$ and the corresponding right-orbit is

$$\left\{ \begin{pmatrix} 1 & a + yu + x\mathcal{O} \\ 0 & x\mathcal{O}^* \end{pmatrix} : a \in \mathcal{O}, u \in \mathcal{O}^* \right\}. $$
Next we compute explicitly the number $R(\gamma)$ of right cosets in the double coset $P_\mathcal{O}\gamma P_\mathcal{O}$ for each $\gamma \in \mathcal{P}_\mathcal{O}$.

**Lemma 1.2.** Suppose $\gamma = \begin{pmatrix} 1 & y \\ 0 & z \end{pmatrix} \in \mathcal{P}_K$, and let $a := (\mathcal{O} + x\mathcal{O})^{-1}$ be the inverse of the fractional ideal $\mathcal{O} + x\mathcal{O}$. Then

$$R(\gamma) = [O^* : O^*_{ga}] \cdot [O : (\mathcal{O} \cap x\mathcal{O})].$$

In particular, the pair $(P_\mathcal{O}, P_\mathcal{O})$ is a Hecke pair.

**Proof.** Suppose $u$ and $v$ are in $O^*$. It is easy to see that $u$ and $v$ are in the same class modulo $O_{y/x}$, if and only if, for any $a \in \mathcal{O}$, we have that $a + yu = a + yv \mod x\mathcal{O}$.

Hence, fixing $a$, we see that

$$\begin{pmatrix} 1 & a + yu + x\mathcal{O} \\ 0 & xO^* \end{pmatrix} \neq \begin{pmatrix} 1 & a + yv + x\mathcal{O} \\ 0 & xO^* \end{pmatrix}$$

if and only if $uO^*_{y/x} \neq vO^*_{y/x}$. On the other hand, fixing $u \in O^*/O^*_x$ we see that

$$\begin{pmatrix} 1 & a + yu + x\mathcal{O} \\ 0 & xO^* \end{pmatrix} \neq \begin{pmatrix} 1 & b + yu + x\mathcal{O} \\ 0 & xO^* \end{pmatrix}$$

only if $a - b \notin x\mathcal{O}$, that is, only if $a \neq b \mod (\mathcal{O} \cap x\mathcal{O})$. It follows that every right coset in the right-orbit of $P_\mathcal{O}\gamma$ arises from a group element $(\begin{pmatrix} 1 & a \\ 0 & u \end{pmatrix})$ with $u$ ranging over a set of representatives of $O^*/O^*_x$ and $a$ ranging over a set of representatives of $O/(O \cap x\mathcal{O})$. Hence $R(\gamma) \leq |O^*/O^*_x| |O/(O \cap x\mathcal{O})| < \infty$, proving that $(P_\mathcal{O}, P_\mathcal{O})$ is a Hecke pair.

This argument only gives an upper bound for $R(\gamma)$ because the same right coset may arise from different combinations of representatives of $O^*/O^*_x$ and of $O/(O \cap x\mathcal{O})$. To account for the redundancy, and to compute $R(\gamma)$, let $u_1, u_2 \in O^*/O^*_x$ and $a_1, a_2 \in O/(O \cap x\mathcal{O})$. The right cosets corresponding to $(\begin{pmatrix} 1 & a \\ 0 & u \end{pmatrix})$ coincide if and only if $a_1 + yu_1 + x\mathcal{O} = a_2 + yu_2 + x\mathcal{O}$, equivalently, if and only if $a + y(u_1 - u_2) = x\mathcal{O}$, where we have put $u := u_2/u_1$ and $a := (a_2 - a_1)/u_1$. Note that for each $u$, this is possible for at most one value of $a$ in $O/(O \cap x\mathcal{O})$. Furthermore,

$$a + y(u - 1) \in x\mathcal{O} \quad \text{for some } a$$

$$\iff y(u - 1) \in \mathcal{O} + x\mathcal{O}$$

$$\iff y(u - 1) \in \mathcal{O} \quad \text{for every } z \in a$$

$$\iff (zy)u = zy \mod \mathcal{O} \quad \text{for every } z \in a$$

$$\iff u \in O_{ga}^*.$$ 

Thus for each fixed pair of representatives $(u_1, a_1)$ of $(O^*/O^*_x) \times (O/(O \cap x\mathcal{O}))$ there are exactly $|O_{ga}^*/O^*_x|$ pairs $(u_2, a_2)$, with $u_2 = uu_1$ and $a_2 = a_1 + au_1$ (where $a$ is the unique element modulo $O \cap x\mathcal{O}$ such that $a + y(u - 1) \in x\mathcal{O}$), such that $(\begin{pmatrix} 1 & a \\ 0 & u \end{pmatrix})$ is in the right coset of $(\begin{pmatrix} 1 & a_1 \\ 0 & u_1 \end{pmatrix})$. Since $[O^* : O^*_x]/[O_{ga}^* : O^*_x] = [O^* : O_{ga}^*]$, the lemma follows.

The **Hecke algebra** $\mathcal{H}(P_\mathcal{O}, P_\mathcal{O})$ is, by definition, the convolution $*$-algebra of complex–valued bi-invariant functions on $P_\mathcal{O}$ that are supported on finitely many double cosets. The convolution product is defined by

$$f * g(\gamma) = \sum_{\gamma_1 \in P_\mathcal{O} \setminus P_\mathcal{O}} f(\gamma \gamma_1^{-1})g(\gamma_1),$$

which is a finite sum because each double coset contains finitely many right cosets; the involution is given by $f^*(\gamma) = \overline{f(\gamma^{-1})}$, and the identity is the characteristic
function of $P\mathcal{O}$. The same convolution formula, with $g$ replaced by a square integrable function $\xi$ on the space $P\mathcal{O}\setminus P\mathcal{K}$ defines an involutive representation $\lambda$ of $\mathcal{H}(P\mathcal{K}, P\mathcal{O})$ on the Hilbert space $\ell^2(P\mathcal{O}\setminus P\mathcal{K})$:

$$\lambda(f)\xi = f * \xi \quad \text{for } f \in \mathcal{H} \text{ and } \xi \in \ell^2(P\mathcal{O}\setminus P\mathcal{K}).$$

(1.4) We shall refer to $\lambda$ as the Hecke representation of $\mathcal{H}(P\mathcal{K}, P\mathcal{O})$; the norm closure of its image $\lambda(\mathcal{H}(P\mathcal{K}, P\mathcal{O}))$ is, by definition, the Hecke $C^*$-algebra, and is denoted by $C^*_\mathcal{H}(P\mathcal{K}, P\mathcal{O})$. Both $\mathcal{H}(P\mathcal{K}, P\mathcal{O})$ and $C^*_\mathcal{H}(P\mathcal{K}, P\mathcal{O})$ come with a canonical time evolution by automorphisms $\{\sigma_t : t \in \mathbb{R}\}$, given by

$$\sigma_t(f)(\gamma) = (\frac{R(\gamma)}{L(\gamma)})^{it}f(\gamma), \quad \gamma \in P\mathcal{K}, \quad t \in \mathbb{R},$$

where $L(\gamma)$ denotes the number of left cosets in the double coset of $\gamma$, and is equal to $R(\gamma^{-1})$. A routine calculation shows that this action is spatially implemented on $\ell^2(P\mathcal{O}\setminus P\mathcal{K})$, by the unitary group defined by $(U_t\xi)(\gamma) = (\frac{R(\gamma)}{L(\gamma)})^{it}\xi(\gamma)$ for $\xi \in \ell^2(P\mathcal{O}\setminus P\mathcal{K})$. See [K, Bi, BC] for details.

To simplify the notation, we often write products in the Hecke algebra simply as $fg$ instead of $f * g$, provided there is no risk of confusion, and we use square brackets to indicate the characteristic function of a subset of $P\mathcal{K}$. Thus, the Hecke $*$-algebra $\mathcal{H}(P\mathcal{K}, P\mathcal{O})$ is the linear span of the characteristic functions $[P\mathcal{O}\gamma P\mathcal{O}]$ of double cosets of elements $\gamma$ in $P\mathcal{K}$.

Next we consider two maps $\mu : \mathcal{O}^* \to \mathcal{H}(P\mathcal{K}, P\mathcal{O})$ and $\theta : \mathcal{K} \to \mathcal{H}(P\mathcal{K}, P\mathcal{O})$ defined as follows. Let $N_a = |O/\mathcal{O}a\mathcal{O}|$ be the absolute norm of $a \in \mathcal{O}^*$, and let

$$\mu_a := \frac{1}{\sqrt{N_a}}\left[P\mathcal{O}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} P\mathcal{O}\right];$$

(1.5) for each $r \in \mathcal{K}$ let

$$\theta_r := \frac{1}{R(r)}\left[P\mathcal{O}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} P\mathcal{O}\right],$$

(1.6) where we use the shorthand notation $R(r)$ for $R\left(\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}\right)$, the number of right cosets in the double coset $P\mathcal{O}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} P\mathcal{O}$. Recall that $R(r) = [\mathcal{O}^* : \mathcal{O}r\mathcal{O}]$ by Lemma [12].

Remark 1.3. Since $P\mathcal{O}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} P\mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} P\mathcal{O}$ and $N_{ua} = N_u$ for every $u \in \mathcal{O}^*$, it is clear that the map $\mu$ factors through the quotient $\mathcal{O}^* \to \mathcal{O}^*/\mathcal{O}^*$, and hence can be viewed as an injective map from the semigroup of principal integral ideals into $\mathcal{H}(P\mathcal{K}, P\mathcal{O})$. Similarly, the double coset of $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ is

$$P\mathcal{O}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} P\mathcal{O} = \begin{pmatrix} 1 & \mathcal{O} + 3 \mathcal{O}^* \\ 0 & \mathcal{O}^* \end{pmatrix} = \begin{pmatrix} 1 & (\mathcal{O} + 3) \mathcal{O}^* \\ 0 & \mathcal{O}^* \end{pmatrix},$$

where no link is implied between the two occurrences of $\mathcal{O}^*$ in the last expression. Hence, $\theta$ factors through the quotient $K \to K/\mathcal{O}$. In fact, $\theta_r$ depends on $r$ only through the orbit of $r + \mathcal{O} \in K/\mathcal{O}$ under the multiplicative action of $\mathcal{O}^*$ on $K/\mathcal{O}$ We shall denote the set of such orbits by $(K/\mathcal{O})\mathcal{O}^*$. It is clear that different orbits give different $\theta_r$’s, because the corresponding supports are disjoint.

Next, we show that our Hecke algebra is universal for its relations, see Theorem [16]. We need two lemmas to understand the algebra generated by $\theta_r, r \in K/\mathcal{O}$. Denote by $e_r$ the indicator function

$$e_r := \left[P\mathcal{O}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\right]$$
of the right coset of \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), and denote by \( f_r \) the indicator function

\[
f_r := \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} P_\O
\]

of the left coset of \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

**Lemma 1.4.** For any subgroup \( \S \) of \( \O_r^* \) of finite index in \( \O^* \) we have

\[
\theta_r = \frac{1}{|\O^* : \S|} \sum_{u \in \O/\S} e_{ur} = \frac{1}{|\O^* : \S|} \sum_{u \in \O/\S} f_{ur}.
\]

**Proof.** The right orbit of \( P_\O \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) consists of the right cosets \( \begin{pmatrix} 1 & ru + \S \\ 0 & u \end{pmatrix} \) for \( u \in \O^* \), and it is clear that the union of these right cosets is the common support of both \( \theta_r \) and \( \sum_{u \in \O/\S} e_{ur} \). Further, for \( \gamma \) in this support, there is a unique \( u \in \O^* \) modulo \( \O_r^* \) such that \( P_\O \gamma = P_\O \begin{pmatrix} 1 & ru \\ 0 & u \end{pmatrix} \), and, modulo \( \S \), there are \( [\O^* : \S]/[\O^* : \O_r^*] \) such \( u \)'s. Hence the value of the right-hand side of (1.7) at \( \gamma \) is \( [\O^* : \S]/[\O^* : \O_r^*] = 1/[\O^* : \O_r^*] \), which, by definition, is the value of \( \theta_r \) at \( \gamma \). The proof of the second equality is entirely analogous. \( \square \)

The functions \( e_r \) and \( f_r \) are not bi-invariant, so they are not in \( \H \), but \( e_r \) is left–invariant, and \( f_r \) is right–invariant, and it still makes sense to compute the convolution product \( f_r * e_s \) using formula (1.6). We notice however that this product may fail to be left or right–invariant.

**Lemma 1.5.** For \( r, s \in \K \) the convolution \( f_r * e_s \) is the indicator function

\[
f_r * e_s = \left[ \bigcup_{u \in \O^*} \begin{pmatrix} 1 & ru + s + \O \\ 0 & u \end{pmatrix} \right] = \sum_{u \in \O^*} \begin{pmatrix} 1 & ru + s + \O \\ 0 & u \end{pmatrix}.
\]

**Proof.** Since \( e_s(\gamma_1) = 1 \) if and only if the right coset \( \gamma_1 \) is precisely \( P_\O \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), we have \( f_r * e_s(\gamma) = 1 \) or 0 according to whether \( f_r(\gamma \gamma_1^{-1}) = 1 \) or 0 for \( \gamma_1 = P_\O \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Now \( f_r(\gamma \gamma_1^{-1}) = 1 \) if and only if \( \gamma \gamma_1^{-1} = \begin{pmatrix} 1 & ru + s + a \\ 0 & u \end{pmatrix} \) as a left coset, which means \( \gamma = \begin{pmatrix} 1 & ru + s + a \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Finally, the characteristic function of the union can be replaced by a pointwise sum because the sets are mutually disjoint, so that only one term is nonzero at each point. \( \square \)

**Proposition 1.6.**

1. The \( \ast \)-subalgebra \( \mathfrak{A} \) of \( \H(P_K, P_\O) \) generated by the set \( \{ \theta_r : r \in (\K/\O)\O_r^* \} \) is universal for the relations:

\[
\theta_0 = 1,
\theta_{ur} = \theta_r = \theta_r^*, \quad r \in \K/\O, w \in \O^*,
\theta_r \theta_s = \frac{1}{R(r)} \frac{1}{R(s)} \sum_{u \in \O^*} \sum_{v \in \O^*} \delta_{ur+vs}, \quad r, s \in \K/\O.
\]

Moreover, the generating set \( \{ \theta_r \} \) is a linear basis for \( \mathfrak{A} \), and \( \mathfrak{A} \) is commutative.

2. For each \( d \in \O^* \), the subset \( \{ \theta_r : rd \in \O \} \) spans a finite dimensional \( \ast \)-subalgebra \( \mathfrak{A}(d) \) of \( \mathfrak{A} = \bigcup_{d \in \O^*} \mathfrak{A}(d) \).

3. The algebra \( \mathfrak{A} \) embeds in the Hecke \( \ast \)-algebra, where it generates a \( \ast \)-subalgebra \( A_\theta \) that is universal for the above relations (interpreted now in the category of \( \ast \)-algebras).

4. The map

\[
\theta_r \mapsto \frac{1}{R(r)} \sum_{u \in \O^*} \delta_{ur} \in C^*(\K/\O).
\]
determines an embedding of $\mathcal{A}$ as a $*$-subalgebra of $\mathbb{C}[\mathcal{K}/\mathcal{O}]$ and of $A_0$ as a $C^*$-subalgebra of $C^*(\mathcal{K}/\mathcal{O})$.

Proof. The first two relations are immediate to verify. In order to prove the third one, let $O_{r,s}^*$ be the subgroup of units that fix both $r$ and $s$ modulo $\mathcal{O}$. Clearly $O_{r,s}^* = O_{r+s}^*$, so it is of finite index in $O^*$ and by Lemma 1.4

$$\theta_r \ast \theta_s = \frac{1}{[O^* : O_{r,s}^*]_2} \sum_{u,v \in O^*/O_{r,s}^*} f_{ur} \ast e_{uv}. \tag{1.8}$$

Using Lemma 1.3 we write the indicator function $f_{ur} \ast e_{uv}$ as a sum of indicator functions,

$$f_{ur} \ast e_{uv} = \sum_{w \in O^*} \left[ \begin{pmatrix} 1 & w & \mathcal{O} \\ 0 & w \end{pmatrix} \right].$$

In the triple summation resulting in (1.8), we first sum over $u$ and $v$ replacing $u$ by $u/w$ when summing over $w$, to obtain

$$\theta_r \ast \theta_s = \frac{1}{[O^* : O_{r,s}^*]_2} \sum_{w \in O^*/O_{r,s}^*} \sum_{u,v \in O^*/O_{r,s}^*} \left[ \begin{pmatrix} 1 & \mathcal{O} \\ 0 & w \end{pmatrix} \right].$$

Fixing now the values of $r, s, u, v$, the sum over $w$ gives the indicator function $e_{ur+vs}$, so that

$$\theta_r \ast \theta_s = \frac{1}{[O^* : O_{r,s}^*]_2} \sum_{u,v \in O^*/O_{r,s}^*} e_{ur+vs} = \frac{1}{[O^* : O_{r,s}^*]_2} \sum_{w \in O^*/O_{r,s}^*} e_{w(ur+vs)},$$

where we have introduced an extra sum over $w$. Now, in the sum over $u$ and $v$, we replace $u$ by $wu$ and $v$ by $vw$ to obtain

$$\theta_r \ast \theta_s = \frac{1}{[O^* : O_{r,s}^*]_2} \sum_{w \in O^*/O_{r,s}^*} \sum_{u,v \in O^*/O_{r,s}^*} e_{w(ur+vs)}.$$ Finally, we apply Lemma 1.3 to the sum over $w$ to obtain

$$\theta_r \ast \theta_s = \frac{1}{[O^* : O_{r,s}^*]_2} \sum_{u,v \in O^*/O_{r,s}^*} \theta_{ur+vs}.$$ It is clear from this that the $\theta_r$ commute with each other, and that their linear span is multiplicatively closed. We have already seen that $\theta_r = \theta_1$ if and only if $r$ and $s$ are in the same $O^*$-orbit in $\mathcal{K}/\mathcal{O}$, so as a set, $\{\theta_r : r \in (\mathcal{K}/\mathcal{O})O^*\}$ is linearly independent, because the supports of any two distinct elements are disjoint. It follows that this set is a linear basis for $\mathcal{A}$. Conversely, if $r$ and $s$ are in the same $O^*$-orbit in $\mathcal{K}/\mathcal{O}$, then the second relation implies that the corresponding universal generators must coincide as well. Thus the canonical homomorphism of the universal unital $*$-algebra with the given presentation onto $\mathcal{A}$ maps a spanning set one-to-one and onto a linear basis; this implies that the map is an isomorphism and finishes the proof of part (1).

It follows easily from the relations that $\mathcal{A}(d) := \text{span}\{\theta_r : dr \in \mathcal{O}\}$ is a unital $*$-subalgebra of $\mathcal{A}$, of dimension at most $[(1/d)\mathcal{O}/\mathcal{O}] = [\mathcal{O}/(d\mathcal{O})] = N_d$. Given a finite set $F = \{r_1, r_2, \ldots, r_n\} \subset \mathcal{K}$, we choose $d \in O^\times$ such that $dr_i \in \mathcal{O}$ for each $i = 1, 2, \ldots, n$; then the $\theta_{r_i}$ are contained in $\mathcal{A}(d)$. This proves part (2).

To deal with the corresponding question at the C*-algebra level observe first that each $\theta_r$ is self-adjoint and that it has finite spectrum (because it generates a finite dimensional $*$-subalgebra). Thus, every $\theta_r$ has uniformly bounded norm in any representation, and it makes sense to consider the universal C*-algebra $A_0^\prime$ of the relations. Suppose $r$ and $s$ determine different $O^*$-orbits in $\mathcal{K}/\mathcal{O}$, i.e. $(r + \mathcal{O})O^* \neq (s + \mathcal{O})O^*$. Using the left regular representation $\lambda$ of $\mathcal{H}(P_{\mathcal{K}}, P_\mathcal{O})$ on
\[ \ell^2(P_\mathcal{O} \setminus P_K), \] we obtain operators \( \lambda(\theta_r) \) and \( \lambda(\theta_s) \). When these operators act on the vector \([P_\mathcal{O}] \in \ell^2(P_\mathcal{O} \setminus P_K)\) they produce vectors with disjoint supports. Hence, the collection of operators \( \lambda(\theta_r) \), indexed by (representatives of classes in) \((\mathcal{K}/\mathcal{O})^{\mathcal{O}^*}\), is linearly independent. It follows that each of the finite dimensional subalgebras \( \mathfrak{A}(d) \) is represented faithfully in \( \lambda(A_0) \) and hence the canonical homomorphism of the universal C*-algebra \( A'_\theta \) onto \( \lambda(A_0) \) is an isomorphism. Notice in passing that \( A'_\theta = \varprojlim_{d,d'} \mathfrak{A}(d) \) with embeddings given by the inclusions \( \mathfrak{A}(d) \subset \mathfrak{A}(d') \) when \( d|d' \). This finishes the proof of part (3).

In order to prove part (4), we write \( \vartheta_r := \frac{1}{R(r)} \sum_{a \in \mathcal{O} \cap \mathcal{O}_r} \delta_{ur} \in C[\mathcal{K}/\mathcal{O}] \). A straightforward computation in the group algebra \( C[\mathcal{K}/\mathcal{O}] \) shows that the elements \( \vartheta_r \) satisfy the relations, so the map \( \vartheta_r \mapsto \vartheta_r \) extends to a homomorphism. To prove that this homomorphism is in fact injective, we need to show that the \( \vartheta_r \)’s are linearly independent. Suppose that \( \sum_r \lambda_r \vartheta_r \) is a vanishing finite linear combination, which we assume has been reduced so that different \( r \)'s come from different orbits in \( (\mathcal{K}/\mathcal{O})^{\mathcal{O}^*} \), and so write \( \sum_r \sum_u \frac{\lambda_r}{R(r)} \delta_{ur} \). As a result of the reduction, the \( \delta_{ur} \) are different, hence linearly independent, and all the \( \lambda_r \) must vanish. This proves that \( \mathfrak{A} \) embeds into \( C[\mathcal{K}/\mathcal{O}] \).

Finally, since the finite dimensional subalgebras \( \mathfrak{A}(d) \) are mapped injectively, the C*-algebra homomorphism of \( A'_\theta \) into \( C^*(\mathcal{K}/\mathcal{O}) \) determined by \( \theta_r \mapsto \vartheta_r \) is injective too.

**Proposition 1.7.** The Hecke algebra \( \mathcal{H}(P_\mathcal{K}, P_\mathcal{O}) \) is (canonically isomorphic to) the universal unital *-algebra over \( \mathbb{C} \) generated by elements \( \{ \mu_a : a \in \mathcal{O}^* \} \) and \( \{ \theta_r : r \in \mathcal{K}/\mathcal{O} \} \) subject to the relations:

\begin{align*}
(\text{I.1}) & \quad \mu_w = 1, \text{ for all } w \in \mathcal{O}^*, \\
(\text{I.2}) & \quad \mu_a^* \mu_a = 1, \text{ for all } a \in \mathcal{O}^*, \\
(\text{I.3}) & \quad \mu_a \mu_b = \mu_{ab}, \text{ for all } a, b \in \mathcal{O}^*, \\
(\text{II.1}) & \quad \theta_0 = 1, \\
(\text{II.2}) & \quad \theta_{aw} = \theta_r = \theta_r^*, \text{ for all } r \in \mathcal{K}/\mathcal{O} \text{ and } w \in \mathcal{O}^*, \\
(\text{II.3}) & \quad \theta_r \theta_s = \frac{1}{R(r) R(s)} \sum_{u \in \mathcal{O} \cap \mathcal{O}_r} \sum_{v \in \mathcal{O} \cap \mathcal{O}_s} \theta_{ur+vs}, \text{ for all } r, s \in \mathcal{K}/\mathcal{O}, \\
(\text{III}) & \quad \mu_a \theta_r \mu_a^* = \sum_{b \in \mathcal{O} \cap \mathcal{O}_r} \theta_{a+rb}, \text{ for all } a \in \mathcal{O}^* \text{ and } r \in \mathcal{K}/\mathcal{O}.
\end{align*}

Moreover, the set \( \{ \mu_a^* \theta_r \mu_b : a, b \in \mathcal{O}^* \text{ and } r \in \mathcal{K}/\mathcal{O} \} \) is a linear basis for \( \mathcal{H}(P_\mathcal{K}, P_\mathcal{O}) \).

**Proof.** The first step is to show that the elements \( \mu_a \) and \( \theta_r \) of \( \mathcal{H}(P_\mathcal{K}, P_\mathcal{O}) \) defined in (I.1) and (II.1) satisfy the relations, thus giving a canonical homomorphism of the universal algebra of the relations onto the Hecke algebra. The relation (I.1) follows immediately from the definition, while (II.1), (II.2) and (II.3) are from Proposition 1.7 and Remark 1.3. Next we observe that the double coset supporting \( \mu \) is a right coset, so for \( f \in \mathcal{H}(P_\mathcal{K}, P_\mathcal{O}) \), the product \( \mu_a^{1/2} (\mu_a \ast f) \) is given by

\[ N_a^{1/2}(\mu_a \ast f)(\gamma) = f \left( \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} \right) \gamma \in P_\mathcal{K}. \]

Once we know how to multiply any function by \( \mu_a \) on the left, the argument given in [BC, Proposition 18] gives relations (I.2) and (I.3), and since taking adjoints gives multiplication by \( \mu_a^* \) on the right, a further straightforward computation gives (III).

To avoid confusion, we shall denote by \( \overline{\mu}_a \) and \( \overline{\theta}_r \) the universal generators subject to the given relations. The rest of the proof depends on the following key
consequence of the relations:

\[(1.9) \quad \tilde{\theta}_r \tilde{\mu}_a = \tilde{\mu}_a \tilde{\theta}_{ar} \quad a \in \mathcal{O}^\times, \quad r \in \mathcal{K}/\mathcal{O}.
\]

We will prove \(\tilde{\theta}_r \tilde{\mu}_a \tilde{\mu}_a^* = \tilde{\mu}_a \tilde{\theta}_{ar} \tilde{\mu}_a^*\) instead; that it is equivalent to \(\tilde{\theta}_r \) can easily be verified on multiplying both sides on the right by the isometry \(\tilde{\mu}_a\) or its adjoint. We first use relations (III) and (II.3) to get

\[
\tilde{\theta}_r \tilde{\mu}_a \tilde{\mu}_a^* = \tilde{\theta}_r \left( \frac{1}{N_a} \sum b \tilde{\theta}_{b/a} \right)
\]

\[
= \frac{1}{N_a} \sum b \frac{1}{R(r)} \frac{1}{R(b/a)} \sum_{u \in \mathcal{O}^\times/\mathcal{O}_{b/a}^\times} \sum_{v \in \mathcal{O}^\times/\mathcal{O}_{b/a}^\times} \tilde{\theta}_{u(v^{-1}ur+b)}
\]

\[
= \frac{1}{N_a} \sum b \frac{1}{R(r)} \frac{1}{R(b/a)} \sum_{u \in \mathcal{O}^\times/\mathcal{O}_{b/a}^\times} \sum_{v \in \mathcal{O}^\times/\mathcal{O}_{b/a}^\times} \tilde{\theta}_{v(v^{-1}ur+b)}.
\]

For each fixed \(v \in \mathcal{O}^\times/\mathcal{O}_{b/a}^\times\), the new variable \(w := v^{-1}u\) runs over the classes of \(\mathcal{O}^\times/\mathcal{O}_{b/a}^\times\), so we add first on \(w\) and then simplify the \(v\)-average, because the resulting sum does not depend on \(v\):

\[
\tilde{\theta}_r \tilde{\mu}_a \tilde{\mu}_a^* = \frac{1}{N_a} \sum b \frac{1}{R(r)} \frac{1}{R(b/a)} \sum_{u \in \mathcal{O}^\times/\mathcal{O}_{b/a}^\times} \sum_{w \in \mathcal{O}^\times/\mathcal{O}_{b/a}^\times} \tilde{\theta}_{w(v^{-1}ur+b)}
\]

\[
= \frac{1}{N_a} \sum b \frac{1}{R(r)} \sum_{w \in \mathcal{O}^\times/\mathcal{O}_{b/a}^\times} \tilde{\theta}_{w(v^{-1}ur+b)}
\]

\[
= \frac{1}{N_a} \sum b \frac{1}{R(r)} \sum_{w \in \mathcal{O}^\times/\mathcal{O}_{b/a}^\times} \tilde{\mu}_a \tilde{\theta}_{war} \tilde{\mu}_a^*
\]

\[
= \frac{1}{R(r)} \sum_{w \in \mathcal{O}^\times/\mathcal{O}_{b/a}^\times} \tilde{\mu}_a \tilde{\theta}_{war} \tilde{\mu}_a^* \quad \text{by (III), for each } w
\]

because \(\tilde{\theta}_{war} = \tilde{\theta}_{ar}\) for each \(w \in \mathcal{O}^\times\).

Next we show that the canonical homomorphism of the universal *-algebra with the given presentation to \(\mathcal{H}(\mathcal{P}_K, \mathcal{P}_O)\) is an isomorphism.

Using the given relations, (1.9), and its immediate consequence \(\tilde{\mu}_a^* \tilde{\theta}_r \tilde{\mu}_a = \tilde{\theta}_{ar}\), one proves that the linear span of the set of monomials \(\{\tilde{\mu}_a^* \tilde{\theta}_r \tilde{\mu}_b : a, b \in \mathcal{O}^\times, r \in \mathcal{K}/\mathcal{O}\}\) is multiplicatively closed and *-closed. The computation is analogous to the one found in the proof of [ALR, Lemma 1.8]. Since these monomials include the generators \(\tilde{\mu}_a\) and \(\tilde{\theta}_r\), as well as the identity, they span the universal unital *-algebra with the above presentation.

At the level of the Hecke algebra, one computes that the bi-invariant function \(\tilde{\mu}_a^* \tilde{\theta}_r \tilde{\mu}_b\) is supported on the (single) double coset of \(\left(\begin{smallmatrix} 1 & r/b_a \\ 0 & b_a \end{smallmatrix}\right)\), where it takes the value \((\mathcal{O}_{b/a}^\times : \mathcal{O}_r^\times)\sqrt{N_{ab}}\), see the proof of [ALR, Theorem 2.3] for a similar computation. Since every double coset arises as the support of such a function for some \(a, b \in \mathcal{O}^\times\) and \(r \in \mathcal{K}/\mathcal{O}\), the set of such products spans the Hecke algebra. The caveat of [ALR, Remark 1.9] applies here too, to the effect that there is some redundancy in the labelling of the universal spanning monomials by the triples \((a, r, b)\), and, of course, at least as much redundancy in the corresponding labelling of the monomials spanning the Hecke algebra. We claim that this redundancy is exactly the same in both cases; more explicitly, we claim that if \(\mu_a^* \theta_r \mu_b \) and \(\mu_s^* \theta_d \mu_d \) do not have disjoint supports as functions on \(\mathcal{P}_K\), then \(\tilde{\mu}_a^* \tilde{\theta}_r \tilde{\mu}_b = \tilde{\mu}_s^* \tilde{\theta}_d \tilde{\mu}_d\) in the universal algebra of the relations, and hence also \(\mu_a^* \theta_r \mu_b = \mu_s^* \theta_d \mu_d\) in \(\mathcal{H}(\mathcal{P}_K, \mathcal{P}_O)\), so the elements \(\mu_a^* \theta_r \mu_b\)
are linearly independent in \( \mathcal{H}(P_X, P_O) \). This will complete the proof, because it implies that the canonical homomorphism maps a spanning set one-to-one and onto a linear basis, and hence is an isomorphism. Recall that the support of \( \mu_a \theta_r \mu_b \) is the single double coset \( P_O \left( \frac{b/a}{d/c} \right) P_O \) and that of \( \mu_a \theta_s \mu_d \) is \( P_O \left( \frac{cws + m u/c}{d/c} \right) P_O \). If these supports are not disjoint they coincide, so there exist \( u \) and \( w \) in \( O^* \) and \( m \in O \) such that \( b/a = u(d/c) \) and \( rv = w(d/c) + m(b/a) \) mod \( O \). Then, going along the lines of [LR, Remark 1.9],

\[
\begin{align*}
\hat{\mu}_a \hat{\theta}_r \hat{\mu}_b &= \hat{\mu}_a \hat{\theta}_r \hat{\mu}_b, &\text{by } &\text{(1.11)} \\
&= \hat{\mu}_a \hat{\theta}_r \hat{\mu}_b \theta(wd + mb/a) &\text{by } &\text{(1.9)} \\
&= \hat{\mu}_a \hat{\theta}_r \hat{\mu}_b \theta(wd + mb/a) &\text{because } &\hat{\mu}_a \hat{\theta}_r \hat{\mu}_b = \hat{\mu}_a \hat{\theta}_r \hat{\mu}_b \\
&= \hat{\mu}_a \hat{\theta}_w \theta(su + mu/c) \hat{\mu}_d &\text{by } &\text{(1.9)} \\
&= \hat{\theta}_w \theta(su + mu/c) \hat{\mu}_d &\text{by } &\text{(1.9)} \\
&= \hat{\mu}_a \hat{\theta}_d, \\
\end{align*}
\]

finishing the proof of the claim and of the proposition. \(\square\)

It will be important for our study of KMS states to have a presentation of the Hecke C*-algebra as a semigroup crossed product and in terms of generators and relations. With this purpose, and in order to introduce the notation, we briefly review now the definition of a semigroup crossed product. When \( S \) is a semigroup that acts by endomorphisms \( \alpha_s \) of the unital C*-algebra \( A \), we say that \( (A, S, \alpha) \) is a semigroup dynamical system. A covariant representation of such a system is a pair \((\pi, \nu)\) consisting of a unital representation \( \pi \) of \( A \) on a Hilbert space and a representation \( \nu \) of \( S \) by isometries on the same Hilbert space, such that the covariance condition \( \pi(\alpha_s(a)) = \nu_s(\pi(a))\pi_s^* \) is satisfied for every \( a \in A \) and \( s \in S \). The semigroup crossed product associated to \((A, S, \alpha)\) is the C*-algebra generated by a universal covariant representation. It is unique up to canonical isomorphisms and is denoted \( A \rtimes_{\alpha} S \). When the endomorphisms are injective, the component \( \pi \) of the universal covariant representation is injective and it is customary to drop it from the notation and to think of \( A \rtimes_{\alpha} S \) as being generated by a copy of \( A \) and a semigroup of isometries \( \{\nu_s : s \in S\} \) that are universal for the covariance condition. See [LR1] for more details on semigroup crossed products. The next proposition gives the appropriate semigroup dynamical system for our Hecke algebra. We point out that the endomorphisms arising in this particular construction are injective corner endomorphisms, that is, each \( \alpha_s \) is an isomorphism of \( A \) onto the corner \( p_s A p_s \), determined by the projection \( p_s = \alpha_s(1). \) See [LR1] for general facts about crossed products by semigroups of corner endomorphisms.

**Proposition 1.8.** There is an action of the semigroup \( O^\times / O^* \) of principal integral ideals of \( O \) by injective corner endomorphisms of the C*-algebra \( A_{\theta} \), given by

\[
\alpha_a(\theta_r) = \frac{1}{N_a} \sum_{b \in O/aO} \theta_{r \frac{b}{a}}, \quad a \in O^\times.
\]

For each \( a \in O^\times \), \( \alpha_a(1) = \mu_a \mu_a^* \) is a projection, and if the ideal \( aO \cap bO \) is principal, then

\[
\alpha_a(1) \alpha_b(1) = \alpha_{aO \cap bO}(1).
\]

The action \( \alpha \) has a left inverse action \( \beta \) by surjective endomorphisms determined by \( \beta_a : \theta_r \mapsto \theta_{ar} \), such that \( \beta_a \circ \beta_b = \beta_{ab} \), \( \beta_a \circ \alpha_a = \text{id} \) and \( \alpha_a \circ \beta_a \) is multiplication by \( \alpha_a(1). \)
Proof. Notice first that the relation \( (II.2) \) makes it clear that \( \alpha_a \) depends only on the ideal \( aO^* \in O^*/O^* \) and not on the specific representative \( a \in O^* \). To show that \( \alpha \) is a semigroup of endomorphisms, and to define \( \beta \), we use the copy of \( A_\theta \) sitting inside the universal algebra \( C^*(\mu, \theta) \) of the relations, where \( \alpha \) and \( \beta \) are implemented by the isometries:

\[
\alpha_a(f) = \mu_a f \mu_a^* \quad \text{and} \quad \beta_a(f) := \mu_a^* f \mu_a \quad \text{for} \quad f \in H(P_\mathcal{K}, P_O).
\]

Since the \( \mu_a \) form a semigroup of isometries, it is easy to see that \( \alpha \) and \( \beta \) have the desired properties. Multiplying \( (1.9) \) on the left by \( u, v \) are also relatively prime for every \( \mu \).

\[
(1.12) \quad \alpha_a(f) = \mu_a f \mu_a^* \quad \text{and} \quad \beta_a(f) := \mu_a^* f \mu_a \quad \text{for} \quad f \in H(P_\mathcal{K}, P_O).
\]

To prove \( (1.12) \) suppose first that \( a \) and \( b \) are relatively prime; then \( a/u \) and \( b/v \) are also relatively prime for every \( u, v \in O^* \), and from [ALR, Proposition 1.2] we know that

\[
\left( \frac{1}{N_a} \sum_{x \in O/aO} \delta_{\mu x} \right) \left( \frac{1}{N_b} \sum_{y \in O/bO} \delta_{\mu y} \right) = \frac{1}{N_{ab}} \sum_{z \in O/abO} \delta_{\mu z}.
\]

Averaging first over \( u \in O^*/O_{1/a}^* \) and then over \( v \in O^*/O_{1/b}^* \) gives

\[
\left( \frac{1}{N_a} \sum_{x \in O/aO} \delta_{\mu x} \right) \left( \frac{1}{N_b} \sum_{y \in O/bO} \delta_{\mu y} \right) = \frac{1}{N_{ab}} \sum_{z \in O/abO} \delta_{\mu z},
\]

which proves

\[
\alpha_a(1) \alpha_b(1) = \alpha_{ab}(1),
\]

in the copy of \( A_\theta \) inside of \( C^*(K/O) \) given in part \( (4) \) of Proposition [1.6]. When \( a \) and \( b \) are not relatively prime but have a principal \( l.c.m. \) \( aO \cap bO = cO \), we write \( a' = cb^{-1} \) and \( b' = ca^{-1} \), and we let \( d := abc^{-1} \) be the g.c.d. of \( a \) and \( b \). Then \( a' \) and \( b' \) are relatively prime and \( (1.12) \) follows from the above because

\[
\alpha_a(1) \alpha_b(1) = \alpha_{a'b'}(1) \alpha_{a'b'}(1) = \alpha_{c}(1).
\]

\[ \square \]

Corollary 1.9. If \( aO + bO \) is principal, so that \( aO + bO = cO \) for some \( c \in O^* \), and if we write \( a = a'c \) and \( b = b'c \) for \( a', b' \in O^* \), then

\[
(1.13) \quad \mu_a^* \mu_b = \mu_{a'b'} c.
\]

Proof. Multiply \( (1.11) \) on the left by \( \mu_a^* \) and on the right by \( \mu_b \), and then use \( (1.12) \) with \( f = 1 \) and the fact that \( aO \cap bO = a'b'cO \).

\[ \square \]

Theorem 1.10. The Hecke \( C^* \)-algebra \( C^*_\theta(P_\mathcal{K}, P_O) \) is canonically isomorphic to the semigroup crossed product \( A_\theta \rtimes_\alpha (O^*/O^*) \) and to the universal unital \( C^* \)-algebra \( C^*(\mu, \theta) \) with presentation \( (I), (II), (III) \) from Proposition \[1.6\].

Similarly, the Hecke algebra is canonically isomorphic to the ‘algebraic’ semigroup crossed product \( \mathfrak{A} \rtimes_\alpha (O^*/O^*) \), which embeds as the *-subalgebra of \( A_\theta \rtimes_\alpha (O^*/O^*) \) spanned by the monomials \( \mu_\theta^* \mu_\theta \).

Proof. The isomorphism between the semigroup crossed product and the \( C^* \)-algebra of the given presentation is an easy consequence of Proposition \[1.6\] and the universal property of semigroup crossed products \[LR1\].

We have already seen that the operators \( \lambda(\theta_r) \) generate a faithful representation of \( A_\theta \) on \( \ell^2(P_O \setminus P_\mathcal{K}) \) and that the \( \lambda(\mu_a) \) form a semigroup of isometries. Moreover, the relation \( (III) \) that the pair \( (\lambda_a, \lambda \circ \mu) \) is a covariant representation of the semigroup dynamical system \( (A_\theta, (O^*/O^*)^*, \alpha) \), so the universal property of \( A_\theta \rtimes_\alpha (O^*/O^*) \) gives a homomorphism of \( A_\theta \rtimes_\alpha (O^*/O^*) \) onto \( C^*_\theta(P_\mathcal{K}, P_O) \), which extends the representation \( \lambda \) of \( H(P_\mathcal{K}, P_O) \) as convolution operators on the space \( \ell^2(P_O \setminus P_\mathcal{K}) \). We will refer to this as the Hecke representation of the Hecke algebra and \( C^* \)-algebra, and denote it by \( \lambda \); our next goal is to show that it is injective. To
do this, we will show that the vector state \( \omega_{[P_0]} := \langle \lambda(\cdot) | P_0 \rangle = \langle [P_0] \rangle \) corresponding to the vector \( [P_0] \in \ell^2(P_0 \setminus P_0) \) is a faithful state of the crossed product. From this it will follow that \( \lambda \) is faithful because its cyclic subrepresentation associated to \([P_0]\) is faithful.

In order to see that \( \omega_{[P_0]} \) is faithful, let \( E_\tilde{a} \): \( A_\theta \rtimes_\alpha (O^\times / O^*) \rightarrow A_\theta \) be the conditional expectation of the dual action of \( \hat{K}^* \), obtained by averaging over the compact \( \hat{K}^* \)-orbits. Using the embedding of \( A_\theta \) in \( C^*(K/O) \) from part (4) of Proposition 1.10, restrict the canonical trace on \( C^*(K/O) \) to \( A_\theta \). Since \( \tau \) and \( E_\tilde{a} \) are faithful positive maps (in the sense that their kernels contain no nontrivial positive element), their composition \( \tau \circ E_\tilde{a} \) is a faithful state on \( A_\theta \rtimes_\alpha (O^\times / O^*) \). Next observe that the state \( \omega_{[P_0]} \) factors through \( E_\tilde{a} \) and coincides with \( \tau \) on the \( \theta_r \)'s. Since these generate the range of \( E_\tilde{a} \), we have \( \tau \circ E_\tilde{a} = \omega_{[P_0]} \), finishing the proof. See [ALR, Example 1.12 and proof of Proposition 2.4] for a similar computation carried out in detail.

It is possible to give a different proof of the semigroup crossed product structure of \( C^*_r(P_K, P_O) \) using results from [Lar1, Lar2]. This alternative approach reveals that \( A_\theta \) is itself a Hecke \( C^*-\)algebra, although it does not yield the explicit formula for products (II.3) obtained above by direct methods, so it paints a complementary picture. In order to apply [Lar1, Theorem 1.9] we first need to write \( P_K \) as a convenient semidirect product, for which we need a multiplicative cross section of the quotient map \( K^* \longrightarrow K^*/O^* \) that maps \( O^\times / O^* \) back into \( O^\times \). When every ideal in \( O \) is principal, such a cross section is easy to obtain; one simply selects a prime generator for each prime ideal and then extends the map freely and multiplicatively. The existence of such cross sections when the ideal class group is nontrivial is a bit more delicate.

**Lemma 1.11.** The quotient map \( K^* \longrightarrow K^*/O^* \) has a multiplicative cross section mapping \( O^\times / O^* \) into \( O^\times \).

**Proof.** Write the ideal class group of \( K \) as a product of cyclic groups \( C_{d_1} \times \cdots \times C_{d_r} \), with \( d_1 | \cdots | d_r \). Choose prime ideals \( p_1, \ldots, p_r \) mapping to the generators of \( C_{d_1}, \ldots, C_{d_r} \). Thus a product \( \prod_j p_j^{e_j} \) is a principal ideal if and only if \( d_j | e_j \) for each \( j = 1, \ldots, r \). Choose elements \( a_j \in O \) such that \( p_j^{d_j} = a_j O \). Also choose, for each prime ideal \( p \), (including the \( p_j \)'s) a product \( \prod_j p_j^{e_j(p)} \) in the same ideal class, and finally, choose an element \( a_p \in K \) such that \( p = a_p \prod_j p_j^{e_j(p)} \).

Let \( a \) be a principal ideal of \( O \), and factor \( a \) as a product \( \prod_p p^{e_p(a)} \) of prime ideals. Replacing the above expression for \( p \) gives

\[
a = \prod_p a_p^{e_p(a)} \cdot \prod_p \prod_j p_j^{e_p(a)e_j(p)} = \prod_p a_p^{e_p(a)} \cdot \prod_j p_j^{\sum_p e_p(a)e_j(p)}.
\]

Since \( a \) is principal, it follows that \( \sum_p e_p(a)e_j(p) \) is a multiple of \( d_j \), for each \( j = 1, \ldots, r \), and we obtain a generator of the ideal \( a \)

\[
a = \prod_p a_p^{e_p(a)} \cdot \prod_j a_j^{(1/d_j)\sum_p e_p(a)e_j(p)} O.
\]

We choose this generator of \( a \) as the image of \( a \in O^\times / O^* \) in \( O^\times \). First of all, since \( a \) is an ideal of \( O \), this generator is an integer. And secondly, since \( e_{p}(ab) = e_{p}(a) + e_{p}(b) \) for every prime ideal \( p \), and fractional ideals \( a \) and \( b \), it is easily verified that this construction gives a multiplicative section \( K^*/O^* \rightarrow K^* \) that maps \( O^\times / O^* \) into \( O^\times \).

**Corollary 1.12.** The subgroup \( N := \left\{ \begin{smallmatrix} 1 \\ O^\times \end{smallmatrix} \right\} \) is normal in \( P_K \) and contains \( P_O \) as a Hecke subgroup. The Hecke \( *\)-algebra of the ‘intermediate’ Hecke inclusion
$P_O \subset N$ is canonically isomorphic to the $^*$-algebra $\mathcal{A}$ with presentation given in Proposition 1.6. Along the same lines, the Hecke $C^*$-algebra of this inclusion is the $C^*$-algebra $A_\theta$ with the same presentation. Moreover, we have canonical isomorphisms at the level of $^*$-algebras and $C^*$-algebras:

$$\mathcal{H}(P_K, P_O) \cong \mathcal{H}(N, P_O) \times (O^*/O^*)$$

and

$$C^*_r(P_K, P_O) \cong C^*_r(N, P_O) \times (O^*/O^*).$$

Proof. Since $P_O$ is Hecke in $P_K$, it is obviously Hecke in $N \subset P_K$, and $N$ is normal because it is the kernel of the homomorphism $(1, y) \in P_K \to xO^* \in K^*/O^*$. Let $\sigma$ be a cross section given by the preceding lemma and consider the map $\tilde{\sigma} : K^*/O^* \to P_K$, defined by $\tilde{\sigma}(xO^*) = (1, x_{(O^*)})$. Using this cross section, we may view $P_K$ as the semidirect product $N \rtimes (K^*/O^*)$, and [LLar1, Theorem 1.9] does the rest. □

Remark 1.13. 1) Although the cross section $\sigma$ is noncanonical, the resulting endomorphisms of the small Hecke algebra $\mathcal{H}(N, P_O)$ are independent from it, essentially because the double cosets in $N$ are $O^*$-invariant.

2) Because Lemma 1.3 of [LLar1] is wrong as stated, a correction factor is necessary on the spanning monomials of [LLar1, Theorem 1.9 (ii)] for the set to be a $\mathcal{H}$-invariant. Moreover, we have canonical isomorphisms at the level of $^*$-algebras and $C^*$-algebras:

$$\mathcal{H}(P_K, P_O) \cong \mathcal{H}(N, P_O) \times (O^*/O^*)$$

and

$$C^*_r(P_K, P_O) \cong C^*_r(N, P_O) \times (O^*/O^*).$$

Proof. Since $P_O$ is Hecke in $P_K$, it is obviously Hecke in $N \subset P_K$, and $N$ is normal because it is the kernel of the homomorphism $(1, y) \in P_K \to xO^* \in K^*/O^*$. Let $\sigma$ be a cross section given by the preceding lemma and consider the map $\tilde{\sigma} : K^*/O^* \to P_K$, defined by $\tilde{\sigma}(xO^*) = (1, x_{(O^*)})$. Using this cross section, we may view $P_K$ as the semidirect product $N \rtimes (K^*/O^*)$, and [LLar1, Theorem 1.9] does the rest. □

2. The adelic semigroup crossed product

It is very convenient to have a realization of the Hecke algebra $C^*_r(P_K, P_O)$ as a semigroup crossed product on which it is possible to define the endomorphisms and the group of ‘geometric’ symmetries as transformations of a compact space. We shall obtain this realization via the Fourier–Gelfand transform of $\mathcal{H}(N, P_O)$, for which we need to review first the self-duality pairing of the additive group of full adeles. We refer to [Ta] or [Lan] for the details. Denote by $M_K$ the set of equivalence classes of valuations on $K$, with $M_v^0$ the set of finite (nonarchimedean) valuations, and let $K_v$ and $O_v$ denote the completion of $K$ and $O$ with respect to the valuation $v$. The ring of full adeles over $K$ is, by definition, the restricted product $\mathcal{A}(K) := \prod_{v \in M_K} (K_v / O_v)$; the additive group is self dual. We shall consider the pairing of $\mathcal{A}(K)$ with itself given in [Ta]: suppose first $p \in N$ is a prime, and let $\chi_p$ be the character of $\mathbb{Q}_p$ given by

$$\chi_p(x) = \exp(2\pi i \lambda_p(x)), \quad x \in \mathbb{Q}_p,$$

where $\lambda_p(x) \in \mathbb{Q}$ is chosen so that $x - \lambda_p(x) \in \mathbb{Z}_p$ and is well-defined modulo $\mathbb{Z}$. Also for $p = \infty$ define a character $\chi_\infty$ of $\mathbb{Q}_\infty = \mathbb{R}$ by

$$\chi_\infty(x) = \exp(-2\pi i x), \quad x \in \mathbb{Q}_\infty.$$

For each adele $x = (x_\infty, x_2, x_3, x_5, \ldots) \in \mathcal{A}(\mathbb{Q})$, define

$$\chi_{\mathcal{A}(\mathbb{Q})}(x) = \prod_p \chi_p(x_p).$$

Clearly, $\chi_{\mathcal{A}(\mathbb{Q})}(x) = 1$ for $x \in \mathbb{Q}$. Let $\text{Tr}_{\mathcal{A}(K)/\mathcal{A}(\mathbb{Q})}(a) = (\sum_{p \mid \infty} \text{Tr}_{K_v/\mathbb{Q}_p}(a_v))_p$ denote the trace of an adele $a$ over $K$ to the adeles over $\mathbb{Q}$, where $p = \infty, 2, 3, \ldots$ runs over the valuations of $\mathbb{Q}$. For adeles $a, b \in \mathcal{A}(K)$, we define the canonical pairing $\langle a, b \rangle = \chi_{\mathcal{A}(\mathbb{Q})}(\text{Tr}_{\mathcal{A}(K)/\mathcal{A}(\mathbb{Q})}(ab))$, and this gives an isomorphism of $\mathcal{A}(K)$ onto its Pontrjagin dual $\hat{\mathcal{A}(K)}$, in which the adele $b$ is mapped to the character on $\mathcal{A}(K)$
given by \( a \mapsto \langle a, b \rangle \). Once this duality has been established, general considerations give pairings for subgroups and quotients of \( \mathbb{A}(\mathcal{K}) \).

For instance, consider \( \mathcal{K} \subset \mathbb{A}(\mathcal{K}) \), embedded diagonally, and let \( \mathcal{K}^\perp := \{ b \in \mathbb{A}(\mathcal{K}) : \langle a, b \rangle = 1 \ \text{for all} \ a \in \mathcal{K} \} \) be the group of characters that are trivial on \( \mathcal{K} \). It is easy to see that \( \mathcal{K} \subset \mathcal{K}^\perp \), and the reverse inclusion holds by [Ta, Theorem 4.1.4]. Thus the duality pairing of \( \mathbb{A}(\mathcal{K}) \) with itself determines an isomorphism of the dual of \( \mathcal{K} \) to the quotient \( \mathbb{A}(\mathcal{K})/\mathcal{K} \). This, in turn, yields an isomorphism of the dual of the quotient \( \mathcal{K}/\mathcal{O} \) to the subgroup of \( \mathbb{A}(\mathcal{K})/\mathcal{K} \) consisting of those adeles (modulo \( \mathcal{K} \)) that induce the trivial character on \( \mathcal{O} \). Specifically, define

\[
\mathcal{O}^\perp := \{ b \in \mathbb{A}(\mathcal{K}) : \langle a, b \rangle = 1 \ \text{for all} \ a \in \mathcal{O} \};
\]

then there is a duality pairing of the classes \( k + \mathcal{O} \subset \mathcal{K}/\mathcal{O} \) and \( b + \mathcal{K} \subset \mathcal{O}^\perp/\mathcal{K} \) given by \( \langle k + \mathcal{O}, b + \mathcal{K} \rangle := \langle k, b \rangle \).

This is not quite enough to see endomorphisms and symmetries as coming from multiplication; we need a description of \( \mathcal{O}^\perp/\mathcal{K} \) in terms of a subset of the finite adeles, on which multiplication is defined. Suppose first \( v \) is a nonarchimedean valuation on \( \mathcal{K} \), and let \( p \) be the rational prime such that \( v|p \). The local version of the duality, pairing \( \mathcal{K}_v \) with itself, is given by \( \langle a, b \rangle_v := \chi_p(\Tr_{\mathcal{K}_v/\mathcal{O}_v}(ab)) \) for \( a, b \in \mathcal{K}_v \). Since \( \langle a, b \rangle_v = 1 \ \text{for all} \ a \in \mathcal{O}_v \), if and only if \( \Tr_{\mathcal{K}_v/\mathcal{O}_v}(ab) \in \mathbb{Z}_p \) for all \( a \in \mathcal{O}_v \), the set of elements that induce the trivial character on \( \mathcal{O}_v \) is

\[
\mathcal{D}_v^{-1} = \{ b \in \mathcal{K}_v : \Tr_{\mathcal{K}_v/\mathcal{O}_v}(ab) \in \mathbb{Z}_p \ \text{for all} \ a \in \mathcal{O}_v \},
\]

which is classically referred to as the local inverse different at \( v \). The set \( \mathcal{D}_v^{-1} \) is a fractional ideal in \( \mathcal{K}_v \) that clearly contains \( \mathcal{O}_v \), so that its inverse, the local different \( \mathcal{D}_v \) is an integral ideal in \( \mathcal{K}_v \). If we denote by \( \mathfrak{P}_v \) the maximal ideal of \( \mathcal{O}_v \), then \( \mathcal{D}_v = \mathfrak{P}_v^{d_v} \) for some \( d_v \geq 0 \). One proves that the support of \( d \) is finite using the approximation theorem; in fact, \( \mathcal{D}_v = \mathcal{O}_v \) if and only if \( v \in M^0_\mathcal{K} \) is unramified.

The space we need is the product of the local inverse differentials:

\[
\mathcal{D}^{-1} := \prod_{v \in M^0_\mathcal{K}} \mathcal{D}_v^{-1} = \{ b \in \mathbb{A}(\mathcal{K}) : \chi_{\mathcal{K}/\mathcal{Q}}(\Tr_{\mathbb{A}(\mathcal{K})/\mathcal{A}(\mathcal{Q})}(ab)) = 1 \ \forall a \in \prod_{v \in M^0_\mathcal{K}} \mathcal{O}_v \},
\]

which is an additive subgroup of the finite adeles \( \mathbb{A}^0(\mathcal{K}) \), but which we view as a subset of the full adeles by attaching a trivial component at every infnite place. By abuse we will refer to \( \mathcal{D}^{-1} \) as the (adelic) inverse different, observing that it is a cartesian product \( \mathcal{D}^{-1} = \prod_v \mathfrak{P}_v^{-d_v} \).

In order to avoid confusion, we point out that in the standard terminology the global inverse different is the set

\[
\mathcal{D}^{-1} := \{ x \in \mathcal{K} : x \in \mathcal{D}_v^{-1} \ \text{for all finite} \ v \}.
\]

It is a fractional ideal in \( \mathcal{K} \), whose prime factorization is \( \mathcal{D}^{-1} = \prod_{v \in M^0_\mathcal{K}} \mathfrak{P}_v^{-d_v} \), where \( d_v \) is as above for the valuation corresponding to the prime ideal \( \mathfrak{P} = \mathfrak{P}_v \cap \mathcal{O} \).

**Proposition 2.1.** The canonical duality pairing of \( \mathbb{A}(\mathcal{K}) \) to itself induces an isomorphism of the dual of \( \mathcal{K}/\mathcal{O} \) to \( \mathcal{D}^{-1} \times \prod_{v | \infty} \{ 0 \} \).

**Proof.** From the definition of \( \mathcal{D}_v^{-1} \) it is clear that the elements of \( \prod_{v \in M^0_\mathcal{K}} \mathcal{D}_v^{-1} \), viewed as full adeles with trivial components at the infinite places, induce the trivial character on \( \mathcal{O} \) via the duality pairing. Thus, we have that \( \prod_{v \in M^0_\mathcal{K}} \mathcal{D}_v^{-1} \times \prod_{v | \infty} \{ 0 \} \subset \mathcal{O}^\perp \). We need to show that this inclusion induces an isomorphism of \( \prod_{v \in M^0_\mathcal{K}} \mathcal{D}_v^{-1} \) to \( \mathcal{O}^\perp/\mathcal{K} \).
By the Approximation Theorem, every adele can be written, modulo $K$, as $d + x$, where $d \in \prod_{v \in M_K^0} \mathcal{D}_v^{-1} \times \prod_{v \mid \infty} \{0\}$ and $x \in \prod_{v \in M_K^0} \{0\} \times \prod_{v \mid \infty} K_v$. Then
\[
\mathcal{A}(K)/K \cong \prod_{v \in M_K^0} \mathcal{D}_v^{-1} \times (\prod_{v \mid \infty} K_v)/\mathcal{D}^{-1}.
\]
Since every nontrivial element of $\prod_{v \mid \infty} K_v/\mathcal{D}^{-1}$ gives a nontrivial character on $\mathcal{O}$, it follows that $\prod_{v \in M_K^0} \mathcal{D}_v^{-1} \cong \mathcal{O}^\ast/K$.

**Definition 2.2.** An *extreme point* of the inverse different (or an *extreme inverse different*) is an element $\chi$ of $\mathcal{D}^{-1}$ such that $\chi R = \mathcal{D}^{-1}$; equivalently, $\chi$ has maximal valuation (minimal exponents) within $\prod_{v \in M_K^0} \mathcal{D}_v^{-1}$ at every place. The set of extreme points will be denoted by $\partial \mathcal{D}^{-1}$.

The extreme inverse different corresponds to the subset $\chi_\mathcal{O}$ of $K/\mathcal{O}$ constructed in [ALR, Corollary 3.5] using elementary Fourier analysis. The extreme points play the role of the appropriate power of a uniformizing element in a local field extension. For each extreme inverse different $\chi$ the map $w \mapsto \chi w$ gives a one-to-one correspondence:
\[
\mathcal{R} := \prod_{v \in M_K^0} \mathcal{O}_v \longrightarrow \mathcal{D}^{-1} := \prod_{v \in M_K^0} \mathcal{D}_v^{-1},
\]
such that
\[
W := \prod_{v \in M_K^0} \mathcal{O}_v^\ast \longrightarrow \partial \mathcal{D}^{-1} = \partial(\prod_{v \in M_K^0} \mathcal{D}_v^{-1}).
\]

We may now write the Gelfand–Fourier transform of $C^\ast(K/\mathcal{O})$ as the isomorphism $C^\ast(K/\mathcal{O}) \cong C(\mathcal{D}^{-1})$ determined by the pairing of Proposition 2.1, namely $\delta_r \mapsto \tilde{\delta}_r = \langle r, \cdot \rangle$ for $r \in K/\mathcal{O}$. We note in passing that the various duality pairings of $K/\mathcal{O}$ to $\mathcal{R}$ used in [HL, ALR, Con] involve a noncanonical choice of inverse different $\chi$ (or of a point in $\chi$ in the case of [ALR]). In our notation, the transforms from [HL, ALR, Con] would be given by $\delta_r \mapsto \langle r, \chi \cdot \rangle$.

To describe the symmetries of our dynamical system, we need to analyze the action of the group $W := \prod_{v \in M_K^0} \mathcal{O}_v^\ast$ on the inverse different, and in particular the action of the subgroup $\mathcal{O}_v^\ast$ of units, diagonally embedded in $W$. By Dirichlet’s unit theorem, $\mathcal{O}_v^\ast$ is infinite unless $K$ is quadratic imaginary, but since $W$ is compact, the closure $\overline{\mathcal{O}_v^\ast}$ of the diagonal copy of $\mathcal{O}_v^\ast$ in $W$ is a compactification of $\mathcal{O}_v^\ast$. Next we see that this compactification coincides with the natural profinite limit determined by the action of $\mathcal{O}_v^\ast$ on $K/\mathcal{O}$, and, by duality, with the transposed action on $\mathcal{D}^{-1}$. We recall that $\mathcal{O}_v^\ast$ denotes the stability subgroup of $r \in K/\mathcal{O}$ for the action of $\mathcal{O}_v^\ast$ by multiplication on $K/\mathcal{O}$.

**Lemma 2.3.** For each $a \in \mathcal{O}^\ast$ let $G_a := \mathcal{O}^\ast/\mathcal{O}_1^\ast$ of $a$. The collection $\{G_a : a \in \mathcal{O}^\ast\}$, endowed with the natural homomorphisms
\[
u_0^{a/b} : G_b \to \mathcal{O}_{1/a}^\ast \times G_a \quad \text{for} \quad u \in \mathcal{O}^\ast, \quad \text{and} \quad a|b \in \mathcal{O}^\ast
\]
is a projective system of finite groups. The quotient maps $u \mapsto u\mathcal{O}_{1/a}^\ast$ of $\mathcal{O}_v^\ast \to G_a$ determine an embedding of $\mathcal{O}_v^\ast$ as a dense subgroup of the profinite group $\text{proj lim} G_a$, which extends to the closure $\overline{\mathcal{O}_v^\ast}$ of (the diagonally embedded copy of) $\mathcal{O}_v^\ast$ inside $W$, giving an isomorphism $\overline{\mathcal{O}_v^\ast} \cong \text{proj lim} G_a$.

**Proof.** The collection of subgroups $\mathcal{O}_{1/a}^\ast$ for $a \in \mathcal{O}^\ast$ form an injective system (with $\mathcal{O}^\ast$ directed by divisibility) and hence the corresponding quotients $G_a$ form a projective system.
Next observe that for any two distinct units $u$ and $v$ in $O^*$, there is $b \in O^*$ such that $u - v \notin bO$, hence $u/b \neq v/b$ mod $O$, proving that the canonical map of $O^*$ to proj $G_a$ is injective.

The group of units $O^*$ maps onto each of the $G_a$'s by definition; we need to extend this map to the closure of $O^*$ in $W = \prod_{v \in M^0} O_v$ for which we will use that $W$ itself can be viewed as the limit of the projective system $\{(O/aO)^* : a \in O^\times\}$, where $(O/aO)^*$ is the unit group of the finite ring $(O/aO)$. Suppose now $u_\lambda \to u$ in $W$. Then $(u_\lambda u^{-1}) \to 1$ in $(O/aO)^*$, so eventually $(u_\lambda u^{-1}) = (u_\mu u^{-1})$ for some fixed $\mu$. It follows that $u_\lambda = u_\mu$ mod $aO$, that is, $(u_\lambda - u_\mu)/a \in O$. Hence $u_\lambda u_\mu^{-1}$ fixes $1/a$ modulo $O$, so $u_\lambda$ and $u_\mu$ determine the same element of $G_a$. This implies that the quotient map $O^* \to (O/aO)^*$ extends to a surjective homomorphism $h_a : O^* \to (O/aO)^*$ for each $a \in O^\times$.

It is now easy to verify that these homomorphisms $h_a$ form a coherent family of continuous surjections of $O^*$ to the projective system of groups $G_a$. The intersection of the kernels is trivial, and by the universal property of the projective limit there is an embedding of $O^*$ in proj $G_a$. Since the range is dense and $O^*$ is compact, this embedding is surjective and we have the isomorphism $O^* \cong \text{proj lim } G_a$.  

It is easy to see that multiplication by integral ideles $W$ is continuous and leaves the inverse different invariant, thus the diagonal embedding $O^* \to W$ determines a multiplicative action of $O^*$ on $D^{-1}$. At the level of the $C^*$-algebra $C(D^{-1})$, a function is fixed by the action of $O^*$ if and only if it is fixed by $O^*$. Averaging over the $O^*$-orbits using the normalized Haar measure gives a faithful positive conditional expectation $E_{O^*} : C(D^{-1}) \to C(D^{-1})$. We shall denote by $\Omega$ the (quotient) space of $O^*$-orbits in $D^{-1}$; it is a compact Hausdorff space such that $C(D^{-1})/O^* \cong C(\Omega)$. Notice that $O^*$ acts trivially on $\Omega$, so the multiplicative action of $W$ on $D^{-1}$ drops to an action of $W/O^*$ on $\Omega$, which we also write as multiplication. Similarly, multiplication of an orbit $aO^*$ in $\Omega$ by a principal ideal $a \in O^\times/O^*$ is well defined, since $O^*$ has been factored out from everywhere in sight.

**Proposition 2.4.** Let $\Omega$ be the orbit space of the action of $O^*$ on $D^{-1} := \prod_{v} O_v^{-1}$. There is an action $\alpha$ of $O^\times/O^*$ by injective endomorphisms of $C(\Omega)$ defined, for $aO^* \in O^\times/O^*$ and $f \in C(\Omega)$, by

$$\alpha_a(f)(x) := \begin{cases} f(a^{-1}x) & \text{if } x \in a\Omega \\ 0 & \text{if } x \notin a\Omega. \end{cases}$$

A representation of the crossed product $C(\Omega) \rtimes \alpha (O^\times/O^*)$ is faithful if and only if it is faithful on $C(\Omega)$.

**Proof.** One first defines $\alpha_a(f)$ for $f \in C(D^{-1})$ and $a \in O^\times$ and then restricts to $C(\Omega) \cong C(D^{-1})/O^*$, where units act trivially, to obtain the action of $O^\times/O^*$. This shows that (2.1) is independent of the point $x$ representing an orbit and of the integer $a$ representing a principal ideal. Recall from the discussion preceding Proposition 2.3 that the semigroup crossed product $C(\Omega) \rtimes \alpha (O^\times/O^*)$ is generated by a copy of $C(\Omega)$, and a semigroup of isometries $\{v_a : a \in O^\times/O^*\}$. To prove the faithfulness statement, let $F$ be a finite subset of $O^\times/O^*$ and consider the linear combination $\sum_{a,b \in F} f_{a,b} v_a^* v_b$; such elements span the associated ‘algebraic crossed product’, which is a dense $*$-subalgebra of $C(\Omega) \rtimes (O^\times/O^*)$. Choose representatives in $O^\times$ for the elements of $F$, and let $q_1 \in C(D^{-1})$ be the projection constructed as in the proof of [ALR] Lemma 4.3. Since this projection is invariant under $O^*$, it lies in $C(\Omega)$. The result follows from the same commuting square argument that gives Proposition 4.4 and Theorem 4.1 of [ALR]. Notice that [ALR] Lemma 4.2 is not needed here because we have factored out the action of $O^*$.

$\square$
Theorem 2.5. Let \( f \mapsto \hat{f} \) denote the Gelfand–Fourier transform of \( C^*(\mathcal{K}/\mathcal{O}) \) onto \( C(D^{-1}) \) obtained from the pairing defined in Proposition 2.4. Denote the canonical generators of \( C^*(\mathcal{K}/\mathcal{O}) \) by \( \delta_r \), and let \( \{ \nu_a : a \in \mathcal{O}^\times/\mathcal{O}^* \} \) be the canonical semigroup of isometries in \( C(\Omega) \times_a (\mathcal{O}^\times/\mathcal{O}^*) \). Then the maps

\[
\theta_r \mapsto \hat{\theta}_r := \frac{1}{R(r)} \sum_{a \in \mathcal{O}^\times/\mathcal{O}^*_r} \hat{\delta}_{ru}, \quad r \in \mathcal{K}/\mathcal{O}
\]

\[
\mu_a \mapsto \nu_a \quad a \in \mathcal{O}^\times/\mathcal{O}^*,
\]
determine an isomorphism of \( C^*(P_{\mathcal{K}}, \mathcal{P}_\mathcal{O}) \) onto \( C(\Omega) \times_a (\mathcal{O}^\times/\mathcal{O}^*) \).

Proof. Part (4) of Proposition 2.4 and the Gelfand–Fourier transform give an injective homomorphism of \( A_\theta := C^*\{\theta_r : r \in \mathcal{K}/\mathcal{O} \} \) into \( C(D^{-1}) \). The range of this homomorphism is (canonically isomorphic to) \( C(\Omega) \), because the conditional expectation \( E_{\Theta} \) is contractive, and thus transforms the set of generators \( \hat{\delta}_r \), whose linear span is dense in \( C(D^{-1}) \), into the set of \( \Theta \)-invariant functions \( \hat{\theta}_r \), whose linear span is dense in \( C(\Omega) \).

Let \( \pi_\theta \) be the inverse of this isomorphism; the pair \( (\pi_\theta, \mu) \) is covariant, so the faithfulness criterion of Proposition 2.4 implies that \( \pi_\theta \times \mu \) is an isomorphism. \( \square \)

Remark 2.6. When \( \mathcal{K} = \mathbb{Q} \), the unit group is \( \mathcal{O}^* = \{ \pm 1 \} \) and [BC, Remark 33.b] shows that the Hecke algebra of \( \mathcal{P}_\mathcal{O} \subset \mathcal{P}_{\mathcal{K}} \) is the subalgebra of the Bost-Connes algebra of fixed points under conjugation by \( \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \). It is also interesting to compare our Hecke \( C^* \)-algebra to that of the almost normal inclusion

\[
\Gamma_\mathcal{O} := \left( \begin{array}{cc} 1 & \mathcal{O} \\ 0 & 1 \end{array} \right) \subset \Gamma_{\mathcal{K}} := \left( \begin{array}{cc} 1 & \mathcal{K} \\ 0 & \mathcal{K}^* \end{array} \right),
\]

considered in [ALR, Corollary 2.5]. Indeed, by [ALR, Corollary 2.5], \( C^*(\Gamma_{\mathcal{K}}, \Gamma_\mathcal{O}) \) is canonically isomorphic to the semigroup crossed product \( C^*(\mathcal{K}/\mathcal{O}) \times \mathcal{O}^\times \), and we can use the embedding of \( A_\theta \) in \( C^*(\mathcal{K}/\mathcal{O}) \) to obtain an embedding of crossed products and hence of Hecke algebras. Since the semigroups in the two crossed products are not the same, we need to use the cross section of the semigroup homomorphism \( \mathcal{O}^\times \to \mathcal{O}^\times/\mathcal{O}^* \) obtained in Lemma 2.4.

Proposition 2.7. Let \( C^*(\Gamma_{\mathcal{K}}, \Gamma_\mathcal{O}) \) be the Hecke \( C^* \)-algebra of the almost normal inclusion \( \Gamma_\mathcal{O} := \left( \begin{array}{cc} 1 & \mathcal{O} \\ 0 & 1 \end{array} \right) \subset \Gamma_{\mathcal{K}} := \left( \begin{array}{cc} 1 & \mathcal{K} \\ 0 & \mathcal{K}^* \end{array} \right) \); denote the generators of \( C^*(\mathcal{K}/\mathcal{O}) \) by \( \delta_r \), and let the \( \nu_a \) realize \( \mathcal{O}^\times \) as a semigroup of isometries in \( C^*(\Gamma_{\mathcal{K}}, \Gamma_\mathcal{O}) \). For each multiplicative cross section \( \mathcal{S} : \mathcal{O}^\times/\mathcal{O}^* \to \mathcal{O}^\times \) as in [ALR, Lemma 2.4], the maps

\[
\mu_a \mapsto \nu_{s(a)} \quad \text{and} \quad \theta_r \mapsto \hat{\theta}_r := \frac{1}{|\mathcal{O}^\times/\mathcal{O}^*_r|} \sum_{u \in \mathcal{O}^\times/\mathcal{O}^*_r} \delta_{ru}
\]
determine an isomorphism of \( C^*(\mathcal{P}_{\mathcal{K}}, \mathcal{P}_\mathcal{O}) \) onto a \( C^* \)-subalgebra \( H_s \) of \( C^*(\Gamma_{\mathcal{K}}, \Gamma_\mathcal{O}) \). The fixed point algebra of \( \Theta \) decomposes as

\[
C^*(\Gamma_{\mathcal{K}}, \Gamma_\mathcal{O})^\Theta \cong H_s \otimes C^*(\mathcal{O}^*).
\]

Proof. We have already seen that \( \theta_r \mapsto \frac{1}{R(r)} \sum_u \delta_{ru} \) determines an embedding of \( A_\theta \) in \( C^*(\mathcal{K}/\mathcal{O}) \), and it is easy to check that the elements \( \nu_{s(a)} \) for \( a \in \mathcal{O}^\times/\mathcal{O}^* \) satisfy the relations (I) and (III), giving a homomorphism of \( C^*(\mathcal{P}_{\mathcal{K}}, \mathcal{P}_\mathcal{O}) \) onto a \( C^* \)-subalgebra \( H_s \) of \( C^*(\Gamma_{\mathcal{K}}, \Gamma_\mathcal{O}) \). That this is an isomorphism follows from the faithfulness criterion of Proposition 2.4.

For the last assertion observe that \( \{ \nu_u : u \in \mathcal{O}^* \} \) generates a copy of \( C^*(\mathcal{O}^*) \) that commutes with \( H_s \), so it suffices to prove that \( C^*(\Gamma_{\mathcal{K}}, \Gamma_\mathcal{O})^\Theta \) is generated by elements of the form \( \nu_u \) with \( u \in \mathcal{O}^* \) and \( \nu_{s(a)} \theta_r \nu_{s(b)} \) with \( a, b \in \mathcal{O}^\times/\mathcal{O}^* \) and \( r \in \mathcal{K}/\mathcal{O} \). By Lemma 2.4, the average \( \mathcal{O}^\times \)-average of \( \nu_u \delta_r \nu_b \) can be computed using the finite
defines a natural strongly continuous action \( \gamma^\hat{\alpha} \) for \( x \) tries from \( C_s \)ition 2.4, the duality pairing allows us to transpose the dynamics and the theorem is that the equilibrium condition forces full \( \gamma^\hat{\alpha} \) average defining states. This extra symmetry is a consequence of the stability of equilibrium, and a standard ‘first argument’ shows that \( \gamma^\hat{\alpha} \) clearly commutes with \( \sigma \), also by \( \hat{\alpha} \) algebra, via the isomorphism of Theorem 1.10. We denote this transposed action on the monomials spanning the crossed product, where \( a, b \) of \( \sigma \) are also by \( \hat{\alpha} \) algebra, \( \sigma \) from the dual action \( \hat{\alpha} \) on the monomials spanning the crossed product, where \( a, b \) and \( \sigma \) and \( \hat{\alpha} \) also satisfy the relations and thus induce an automorphism \( \sigma_t \) of \( C^*_t(P_K, P_O) \); the group property is easy to check on generators, and a standard ‘3-argument’ shows that \( \sigma \) is continuous. It is also possible, and perhaps more interesting, to implement the dynamics spatially in the Hecke group \( (\mathbb{K}/O)^\sigma \) defined by \( U_t \xi(\gamma) := (L(\gamma)/R(\gamma))^t \xi(\gamma) \). The symmetries are defined by multiplication: for \( \chi \in W/O^\sigma \) let \( f_\chi(x) := f(\chi x) \) for \( x \in \Omega \); then

\[
\gamma^\chi(v^\ast_afvb = (N_b/N_a)i^t v^\ast_afvb
\]
on the monomials spanning the crossed product, where \( a, b \in O^\sigma \) and \( f \in C(\Omega) \). The existence of such a continuous action \( \sigma \) can be verified using the universal property of the semigroup crossed product or the presentation of \( C^*_r(P_K, P_O) \), via Proposition 2.4. One simply observes that for each \( t \in \mathbb{R} \) the families \( \{\theta_r : r \in \mathbb{K}/O\} \) and \( \{N_a^\mu : a \in O^\sigma/O^\ast\} \) also satisfy the relations and thus induce an automorphism \( \sigma_t \) of \( C^*_r(P_K, P_O) \); the group property is easy to check on generators, and a standard ‘first argument’ shows that \( \sigma \) is continuous. It is also possible, and perhaps more interesting, to implement the dynamics spatially in the Hecke group \( (\mathbb{K}/O)^\sigma \) defined by \( U_t \xi(\gamma) := (L(\gamma)/R(\gamma))^t \xi(\gamma) \). The symmetries are defined by multiplication: for \( \chi \in W/O^\sigma \) let \( f_\chi(x) := f(\chi x) \) for \( x \in \Omega \); then

\[
\gamma^\chi(v^\ast_afvb = (N_b/N_a)i^t v^\ast_afvb
\]
defines a natural strongly continuous action \( \gamma \) of \( W/O^\sigma \) on \( C(\Omega) \times (O^\sigma/O^\ast) \). Since \( \gamma \) clearly commutes with \( \sigma \), we can interpret the automorphisms \( \{\gamma^\chi : \chi \in W/O^\sigma \} \) as symmetries of the C*-dynamical system \( (C(\Omega) \times (O^\sigma/O^\ast), \sigma \). Through Proposition 2.4, the duality pairing allows us to transpose the dynamics and the symmetries from \( C(\Omega) \times (O^\sigma/O^\ast) \) back to the Hecke C*-algebra \( C^*_r(P_K, P_O) \), where the dynamics is given by

\[
\sigma_t(\mu^\ast_a\theta_r\mu_b) := (N_b/N_a)i^t \mu^\ast_a\theta_r\mu_b,
\]
and the symmetries are given by

\[
\gamma^\chi(\theta_r) = \theta^\chi_r.
\]
An explanation is in order to make sure that the product of \( \chi O^\sigma \in W/O^\sigma \) by \( (r + O)/O^\sigma \in (\mathbb{K}/O)^\sigma \) is a well defined class in \( (\mathbb{K}/O)^\sigma \):

\[
\gamma^\chi(\theta_r)(z) = \frac{1}{R(r)} \sum_{u \in O^\sigma/O^\ast} \hat{\delta}_{ur}(\chi z) = \frac{1}{R(r)} \sum_{u \in O^\sigma/O^\ast} \chi_{A(\mathbb{Q})}(\text{Tr}_{\mathbb{K}/(\mathbb{K}/A(\mathbb{Q}))}(ru\chi z)).
\]

We will also need to transpose the canonical dual action \( \hat{\alpha} \) of \( \hat{\alpha} \) on the semigroup crossed product \( A_\sigma \times (O^\sigma/O^\ast) \) of Proposition 1.8 to the Hecke C*-algebra, via the isomorphism of Theorem 1.10. We denote this transposed action also by \( \hat{\alpha} \); it is given by \( \hat{\alpha}(\mu_a) = \chi(a)\mu_a \), and \( \hat{\alpha}(\theta_r) = \theta_r \), and its fixed point algebra is \( A_{\hat{\alpha}} := C^\ast(\{\theta_r : r \in (\mathbb{K}/O)^\sigma \}) \). We emphasize that the first assertion of the theorem is that the equilibrium condition forces full \( \hat{\alpha} \)-invariance on the KMS states. This extra symmetry is a consequence of the stability of equilibrium, and is a stronger property than \( \sigma \)-invariance alone, since \( \sigma_{\mathbb{K}/O} \) is strictly smaller than \( \hat{\alpha}_{\mathbb{K}/O} \) when the norm is not injective on principal integral ideals.
Theorem 3.1. Suppose $K$ is an algebraic number field with class number $h_K = 1$. Let $(C^*_r(P_K, P_Q), \sigma)$ be the Hecke $C^*$-dynamical system associated to the almost normal inclusion $P_Q \subset P_K$, and let $\gamma$ be the action of $W/\mathcal{O}^*$ as symmetries of this system. Then all KMS$_\beta$ states are $\hat{\alpha}$-invariant, and hence determined by their values on the generators $\theta_\gamma$ of $A_{\hat{\alpha}} \subset C^*_r(P_K, P_Q)$. Moreover,

(i) for each $\beta \in [0, \infty]$ there exists a unique $W/\mathcal{O}^*$-invariant KMS$_\beta$ state $\phi_\beta$ of $\sigma$, given by

$$\phi_\beta(\theta_r) = N_b^{-\beta} \prod_{p \mid b} \left( \frac{1 - N_p^{-\beta}}{1 - N_p^{-1}} \right),$$

with $r = a/b$, for $a, b \in \mathcal{O}^*$, in reduced form.

(ii) For $\beta \in [0, 1]$ the state $\phi_\beta$ is the unique KMS$_\beta$ state for $\sigma$; it is a type III hyperfinite factor state when $\beta \neq 0$.

(iii) For $\beta \in (1, \infty]$, the extreme KMS$_\beta$ states are indexed by $\partial \Omega$, the $\mathcal{O}^*$-orbits of extreme points in the inverse different:

- The extreme ground states are pure and faithful, and are given by
  $$\phi_{\chi, \infty}(\theta_r) = \frac{1}{R(r)} \sum_{u \in \mathcal{O}^*/\mathcal{O}^*_r} \langle r, u\chi \rangle,$$

  where $R(r) = |\mathcal{O}^*/\mathcal{O}^*_r|$, the element $\chi \in \partial D^{-1}$ is a representative of an $\mathcal{O}^*$-orbit, and $\langle \cdot, \cdot \rangle$ indicates the duality pairing of $K/\mathcal{O}$ and $D^{-1}$ from Proposition 2.4; states corresponding to different $\mathcal{O}^*$-orbits are mutually inequivalent.

- For $\beta \in (1, \infty]$ the extreme KMS$_\beta$ states are given by
  $$\phi_{\chi, \beta}(\theta_r) = \frac{1}{\zeta_K(\beta)} \sum_{a \in \mathcal{O}^*/\mathcal{O}^*_r} N^{-\beta}_a \left( \frac{1}{R(r)} \sum_{u \in \mathcal{O}^*/\mathcal{O}^*_r} \langle ar, u\chi \rangle,\right),$$

  where $\zeta_K$ denotes the Dedekind zeta function of $K$.

The state $\phi_{\chi, \beta}$ is quasiequivalent to $\phi_{\chi, \infty}$ and the map $T_\beta : \phi_{\chi, \infty} \mapsto \phi_{\chi, \beta}$ extends to an affine isomorphism of the simplex of KMS$_\infty$ states onto the KMS$_\beta$ states. The action of the symmetry group $W/\mathcal{O}^*$ on the extreme KMS$_\beta$ states, given by $\gamma_w(\phi_{\chi, \beta}) = \phi_{\chi, \beta} \circ \gamma_w = \phi_{\chi_{w^{-1}}, \beta}$, is free and transitive.

(iv) The eigenvalue list of the Hamiltonian $H_\chi$ associated to $\phi_{\chi, \infty}$ is $\{ \log N_a : a \in \mathcal{O}^*/\mathcal{O}^*_r \}$ for every $\chi$, so the ‘represented partition function’ $\text{Tr} e^{-\beta H_\chi}$ is independent of $\chi$ and equals the Dedekind zeta function $\zeta_K(\beta)$ of $K$.

Proof. Since $h_k = 1$, we have that the principal integral ideals $\mathcal{O}^*/\mathcal{O}^*$ form a lattice semigroup, and these act by injective endomorphisms that respect the lattice structure by Proposition 1.8. By [1], Theorem 12] KMS$_\beta$ states of $\sigma$ are $\hat{\alpha}$-invariant and their restrictions to $C(\Omega)$ are characterized by the rescaling property

$$\phi \circ \alpha_a = N_a^{-\beta} \phi.$$

Using the embedding of $C(\Omega)$ in $C^*(K/\mathcal{O})$, and the formula for the conditional expectation of the action of $W$ on $C^*(K/\mathcal{O})$ given in [1], pp. 376, one can write the conditional expectation $E_r$ explicitly: for $r = a/b$ in reduced form, with $b = \prod_p p^{e_p}$, one has

$$E_r \theta_{a/b} = \prod_p \left( \frac{N(p)}{N(p) - 1} \alpha_{p^{e_p}}(1) - \frac{1}{N(p) - 1} \alpha_{p^{e_p} - 1}(1) \right),$$

where the product is over a choice of representatives of the prime ideals in $\mathcal{O}$, and is independent of the choice. Part (i) follows from this and the rescaling property above, since a symmetric KMS$_\beta$ state must come from a KMS$_\beta$ state on the symmetric system. The $C^*$-algebra of the symmetric system, namely, the
range of \( E_\gamma \), is isomorphic to \( C^*(\mathcal{O}^\times/\mathcal{O}^\sigma) \), and its fixed point algebra under \( \hat{\alpha} \) is a cartesian product of sequence spaces (over the nonzero archimedean valuations \( M^0_K \)), on which the rescaling condition is satisfied, for a given \( \beta \), only by the obvious product state. The KMS\(_\beta \) state \( \phi \) given in part (i) is then obtained by lifting this product state via \( E_{\gamma \times \sigma} \), as in \([L1]\) Theorem 34.

It is straightforward to characterize the set of \( \overline{\mathcal{O}^\sigma} \)-orbits of points in the extreme inverse different: as a subset of the orbit space of the different, it is the complement of the (orbits of) nontrivial multiples:

\[
\partial \Omega = \bigcap_{p \in P} (p\mathcal{D}^{-1})^c/\overline{\mathcal{O}^\sigma} = \bigcap_{p \in P} \text{supp}(1 - \alpha_p(1))/\overline{\mathcal{O}^\sigma};
\]

so parts (iii) and (iv) follow from \([L1]\) Theorem 20 and Corollary 22 and the discussion about the partition function following those results. We point out that the characterization from \([L1]\) applies to ground states, but since by part (ii) every ground state is a weak limit of KMS\(_\beta \) states as \( \beta \) tends to \( \infty \), the distinction of ground states vs. KMS\(_\infty \) states does not arise.

Part (ii) is proved following the same arguments as in Neshveyev’s ergodicity proof of the rational case \([N]\), for which we first need to compute the dilation of the action of \( \mathcal{O}^\times/\mathcal{O}^\sigma \) on \( \Omega \).

Denote by \( \mathcal{K}^0 \) the orbit space of the action of \( \overline{\mathcal{O}^\sigma} \) on the finite adeles \( \mathcal{A}^0(K) \). Then \( \mathcal{K}^0 \) is a locally compact space containing \( \Omega \) as a compact subset, and the action of \( \mathcal{O}^\times/\mathcal{O}^\sigma \) on \( \Omega \) extends to an action of \( K^\times/\mathcal{O}^\sigma \) on \( \mathcal{K}^0 \).

**Proposition 3.2.** The semigroup crossed product \( C(\Omega) \times (\mathcal{O}^\times/\mathcal{O}^\sigma) \) is canonically isomorphic to the full corner of \( C_0(\mathcal{K}^0 \overline{\mathcal{O}^\sigma}) \times (K^\times/\mathcal{O}^\sigma) \) determined by the projection \( 1_\Omega \in C_0(\mathcal{K}^0 \overline{\mathcal{O}^\sigma}) \).

**Proof.** By uniqueness of the minimal automorphic extension \( \hat{\alpha} \) of the semigroup action \( \alpha \) \([L2]\) Theorem 2.1], it suffices to check that \( \{ \hat{\alpha}_q(f) \in C_0(\mathcal{K}^0 \overline{\mathcal{O}^\sigma}) : q \in K^\times/\mathcal{O}^\sigma, f \in C(\Omega) \} \) is dense in \( C_0(\mathcal{K}^0 \overline{\mathcal{O}^\sigma}) \), and this is easy to see from the density of \( \bigcup_q q^{-1} \Omega \) in \( \mathcal{K}^0 \overline{\mathcal{O}^\sigma} \). \( \square \)

**Proposition 3.3.** Let \( \mu \) on \( \mathcal{K}^0 \overline{\mathcal{O}^\sigma} \) be a positive measure, which we view, indistinguishably, as a positive linear functional on \( C_0(\mathcal{K}^0 \overline{\mathcal{O}^\sigma}) \). If \( \mu \) satisfies

\[
(3.2) \quad \mu(\Omega) = 1 \quad \text{and} \quad q_\ast \mu = N_q^\beta \mu, \quad \text{where} \quad q_\ast \mu(X) = \mu(q^{-1}X),
\]

then the state \( \phi_\mu \) of \( C^*_\gamma(P_K, P_\sigma) \) obtained by restricting \( \mu \circ E_\gamma \) is a KMS\(_\beta \) state. Conversely, every KMS state arises this way, in fact, the map \( \mu \mapsto \phi_\mu \) is an affine isomorphism of simplices.

**Proof.** Direct from the characterization of KMS states given in \([L1]\), and the characteristic property of the minimal dilation/extension; see also \([N]\). \( \square \)

**Proposition 3.4.** For each \( \beta \in (0, 1] \) and any measure \( \mu \) satisfying \((3.2)\), the action of \( K^\times/\mathcal{O}^\sigma \) on \( \mathcal{K}^0 \) is ergodic.

**Proof.** The proof parallels that of \([N]\) Proposition, using Proposition 3.2 in place of \([L2]\) Corollary 3.2]. One has to implement the following changes in Neshveyev’s
Proof:

\[
P \longrightarrow M_K^0 \\
\mathbb{A}^0 \longrightarrow \mathbb{A}^0(K)^{\Omega}
\]

\[
\begin{align*}
R := \prod \mathbb{Z}_p & \longrightarrow \Omega := (\prod \mathbb{Q}_v^{-1})/\overline{\mathbb{Q}}^\times \quad \text{(origins of orbits)} \\
R^* & \longrightarrow \partial \Omega \quad \text{(symmetries)} \\
W := \prod \mathbb{Z}_p^* & \longrightarrow W/\overline{\mathbb{Q}}^\times = (\prod \mathbb{Q}_v^*)/\overline{\mathbb{Q}}^\times.
\end{align*}
\]

Notice that the last two items in the list coincide for \( K = \mathbb{Q} \) (on the left column), but have different versions for a general number field (on the right column). The crucial ingredients are the convergence of the Dirichlet \( L \)-series \( L(\omega, \beta) \), and the divergence of the Dedekind zeta function as \( \beta \to 1^+ \), \cite[Theorem 5, Ch. VIII, p. 161]{Lan}.

Since for \( \beta \in (0, 1] \) the measure corresponding to the symmetric KMS\( \beta \) state \( \phi_\beta \) is ergodic, and KMS\( \beta \) states form a convex set, the uniqueness assertion in part (i) follows. This finishes the proof of Theorem 3.1.

Remark 3.5. With respect to part (iv) of Theorem 3.1, we point out that by \cite{OP}, the dynamics \( \sigma \) is not approximately inner, so there is no intrinsic Hamiltonian for this system. But the Liouville operators associated to the extreme KMS\( \infty \) states (i.e. the ‘Hamiltonian operators’ in the associated GNS representations) have all the same spectrum, so it makes sense to talk about the spectrum of the Hamiltonian and to associate a partition function to the system; see \cite{TBW} for a discussion of the spectra of Liouville operators.

4. Class field theory considerations

Let \( H \) be the Hilbert class field of \( K \), and let \( H^+ \) be the maximal abelian extension of \( K \) in which no finite prime is ramified. For example, for \( \mathbb{Q} \), we have \( H = H^+ = \mathbb{Q} \), but \( K = \mathbb{Q}(\sqrt{3}) \) has \( H = K \) and \( H^+ = K(i) \).

The symmetry group \( W/\overline{\mathbb{Q}}^\times \) has a Galois group interpretation as \( G(K^{ab}, H^+) \), the Galois group of the maximal abelian extension of \( K \) over \( H^+ \). We embed \( W \) into \( \mathbb{A}^*_K \) by \( (a_v)_{v \in M_K^0} \mapsto (1, \ldots, 1, a_v)_v \), where we put 1 in each archimedean component, cf. \cite[Proposition 8.5]{HLc}.

Proposition 4.1. Let \( \sigma_1, \ldots, \sigma_r \) be the embeddings of \( K \) in \( \mathbb{R} \), and let \( \text{sgn}(\mathcal{O}^*) \) be the subgroup of \( (-1)^r \) generated by the elements \( (\text{sgn}(\sigma_i u))_{i=1,\ldots,r} \) (\( u \in \mathcal{O}^* \)). Then the sequence

\[
1 \longrightarrow W/\overline{\mathbb{Q}}^\times \longrightarrow \mathbb{A}_K^*/\mathbb{A}_K^* \mathfrak{K}_\infty^1 \longrightarrow \text{Cl}(K) \times (-1)^r / \text{sgn}(\mathcal{O}^*) \longrightarrow 1
\]

is exact. Via the Artin map, we obtain the canonical isomorphism

\[
W/\overline{\mathbb{Q}}^\times \cong G(K^{ab} : H^+).
\]

Proof. The exactness on the left follows from the proof of Lemma 2.3. The group of finite ideles modulo \( W \) is isomorphic to the group of ideals, and modulo \( \mathbb{A}_K^* \) we may change an idele to one of a fixed set of representatives of \( \text{Cl}(K) \). Then we can still change the infinite part of the idele by an element of \( \mathcal{O}^* K^1_\infty \), which means that we can change the signs at the real places by an element of \( \text{sgn}(\mathcal{O}^*) \).

The isomorphism \( \mathbb{A}_K^*/\mathbb{A}_K^* \mathfrak{K}_\infty^1 \cong G(K^{ab} : K) \) is from \cite[p. 272]{Ve}. Since the group on the right is a finite factor group that contains no nontrivial unit at a finite place, it corresponds to a finite extension of \( K \) in which no finite place is ramified. Since \( \mathcal{O}^* \) acts trivially, all other places are allowed to ramify. Hence this group is the Galois group of \( H^+ \) over \( K \), and, via the Artin map, \( W/\overline{\mathbb{Q}}^\times \) is isomorphic to the Galois group of \( K^{ab} \) over \( H^+ \). \( \square \)
Corollary 4.2. If $\mathcal{K}$ is a number field of class number one with no real embeddings, then $W/\mathcal{O}^*$ is isomorphic to $G(K^{ab}: \mathcal{K})$. Via this isomorphism, the extreme $\text{KMS}_\beta$ states of the system $(C^*_r(P_{\mathcal{K}}, P_{\mathcal{O}}), \sigma)$ for $\beta > 1$ are indexed by the complex embeddings of the maximal abelian extension $K^{ab}$ of $\mathcal{K}$.

Proof. $H = \mathcal{K}$ because $h_\mathcal{K} = 1$ and $H^+ = H$ because $\mathcal{K}$ has no real embeddings, so the isomorphism of $W/\mathcal{O}^*$ to $G(K^{ab}: \mathcal{K})$ follows at once from Proposition 4.1. By Theorem 4.3, part (iii), the action of $W/\mathcal{O}^*$ is free and transitive on extreme $\text{KMS}_\beta$ states, so the second assertion follows. \qed

Corollary 4.3. If the class number of $\mathcal{K}$ is still one, but there are nontrivial unramified extensions (at the finite places), then extreme $\text{KMS}_\beta$ states for $\beta > 1$ are indexed by the complex embeddings of the maximal abelian extension, modulo the complex conjugations over each real embedding of $\mathcal{K}$.

Proof. The result follows from the fact that $\text{Gal}(K^{ab}: \mathcal{H}) \cong \text{Gal}(K^{ab}: \mathcal{K})/\langle \{\pm 1\}^{r_1}\rangle$, where $r_1$ denotes the number of real embeddings of $\mathcal{K}$. \qed

One can use the indexing of Corollary 4.2 to obtain an action of $G(K^{ab}: \mathcal{K})$ on extreme $\text{KMS}_\infty$ states, essentially coming from the ‘geometric’ symmetries of the dynamical system.

It is also possible to define another action of $G(K^{ab}: \mathcal{K})$ on extreme $\text{KMS}_\infty$ states, coming from ‘arithmetic’ symmetries. First define the arithmetic Hecke algebra $\mathcal{K}(P_{\mathcal{K}}, P_{\mathcal{O}})$ to be the algebra over $\mathcal{K}$ generated by the $\theta_r$ and the $\mu_a$. Evaluation of an extreme $\text{KMS}_\infty$ state on a $\theta_r$ corresponds to evaluation of $\theta_r$ on the $W/\mathcal{O}^*$-orbit of a point in the extreme inverse different. Since $\theta_r$ is a $\mathbb{Q}$-linear combination of characters, it follows that the image of $\mathcal{K}(P_{\mathcal{K}}, P_{\mathcal{O}})$ under an extreme $\text{KMS}_\infty$ state is contained in the real subfield of the maximal cyclotomic extension of $\mathcal{K}$. Thus, there is an action of $G(K^{ab}: \mathcal{K})$ on the values of extreme $\text{KMS}_\infty$ states. Since in general this is a proper subfield of $K^{ab}$, the arithmetic Hecke algebra cannot support a canonical action in which $G(K^{ab}: \mathcal{K})$ acts freely and transitively by composition with extreme $\text{KMS}_\infty$ states. In fact, one can compute both actions and see how they differ.

Theorem 4.4. The geometric action of $W/\mathcal{O}^*$ on extreme $\text{KMS}_\infty$ states from Corollary 4.2 coincides with the Galois action of $W/\mathcal{O}^*$ on their values, $\phi_{x,\infty}(\theta_r)$, as given by class field theory, if and only if $\mathcal{K} = \mathbb{Q}$.

Proof. Let $\phi_{x,\infty}$ be an extreme $\text{KMS}_\infty$ state, and let $j$ be an idele. The action of $j$, viewed as a symmetry of $\Omega = D^{-1}/\mathcal{O}^*$, on $\phi_{x,\infty}$ is given by

$$\phi_{x,\infty}(\theta_r) \mapsto \phi_{j x,\infty}(\theta_r) = \frac{1}{R(r)} \sum_{u \in \mathcal{O}^*} \chi_{A(\mathbb{Q})}(\text{Tr}_{A(\mathcal{K})/A(\mathbb{Q})}(jr u \chi)).$$

Next we compute the action of the idele $j$, viewed now as a Galois element acting on the values of the extreme $\text{KMS}$ states via the Artin map as in Proposition 4.1. We note first that the complex number $\phi_{x,\infty}(\theta_r)$ is a linear combination with integer coefficients of character values of the group $\mathcal{K}/\mathcal{O}$. Thus this combination of roots of unity is in the maximal cyclotomic extension of $\mathbb{Q}$. By class field theory, the Galois action of an idele $j \in W$, when restricted to $\mathbb{Q}^{cyc}$, coincides with the action of $N(j)$, where $N(j)$ is the norm of $j$ to the rational ideles (see [We, Corollary 1, p. 246]). Thus the Galois action of $j$ on values of $\text{KMS}_\infty$ states is given by

$$\phi_{x,\infty}(\theta_r) \mapsto \frac{1}{R(r)} \sum_{u \in \mathcal{O}^*/\mathcal{O}^*} \chi_{A(\mathbb{Q})}(\text{Tr}_{A(\mathcal{K})/A(\mathbb{Q})}(N(j) r u \chi)).$$
where we have used the $\mathcal{A}(\mathbb{Q})$-linearity of the trace to replace $N(j) \operatorname{Tr}_{\mathcal{A}(\mathbb{K})/\mathcal{A}(\mathbb{Q})}(ru\chi)$ by $\operatorname{Tr}_{\mathcal{A}(\mathbb{K})/\mathcal{A}(\mathbb{Q})}(N(j)ru\chi)$.

If we now take $j$ to be a rational idele, then $N(j) = j^d$, where $d = \deg(\mathbb{K}/\mathbb{Q})$, and it follows that the two actions are different unless $d = 1$, i.e. unless $\mathbb{K} = \mathbb{Q}$. □

In order to relate our results to those of [HLe], we recall that the Hecke algebras of [HLe] depend in general on two choices: first there is a localization process that substitutes the ring of integers by a principal ring, and then there is a multiplicative cross section from the principal ideals into the ring. Thus the resulting almost normal subgroup is not canonical, and one should not expect a general direct relation with the Hecke algebras considered here and in [ALR], which are canonical.

However, if we assume $h_{\mathbb{K}} = 1$ and fix a subsemigroup $S$ of $\mathbb{O}^\times$ containing a representative for each ideal of $O$ (see Lemma 1.11), then the Hecke $C^*$-algebra $C^*(\Gamma_S, \Gamma_O)$ of the almost normal inclusion

$$\Gamma_O := \begin{pmatrix} 1 & O \\ 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & \mathbb{K} \\ 0 & S^*S^{-1} \end{pmatrix} =: \Gamma_S$$

considered in [HLe] has an action of $W = \prod_{e \in M_k^O} \mathbb{O}_e^\times$ by [HLe] Section 4, and one obtains the following generalization of part 3 of [HLe, Proposition 8.5].

**Proposition 4.5.** Suppose $h_{\mathbb{K}} = 1$, and let $S$ be the range of a multiplicative cross section for $\mathbb{O}^\times \to \mathbb{O}^\times/\mathbb{O}^*$. Then

$$C^*(\Gamma_S, \Gamma_O)^{\mathbb{O}^*} \cong C^*(P_{\mathbb{K}}, P_O)$$

**Proof.** Embed $C^*(\Gamma_S, \Gamma_O)$ in $C^*(\Gamma_k, \Gamma_O)$ as indicated in [ALR] Remark 5.4. This shows that $C^*(\Gamma_S, \Gamma_O)$ does not depend on the specific cross section $S$ up to isomorphism. The result now follows from Proposition 2.4. □

**Remark 4.6.** The particular case of Corollary 4.2 corresponding to the nine quadratic imaginary fields of class number one, i.e. $\mathbb{K} = \mathbb{Q}[\sqrt{-d}]$ for $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$, is already implicit in [HLe]. Indeed, for such fields $\mathbb{O}^*$ is a finite group, and the almost normality of the inclusion $P_O \subset P_{\mathbb{K}}$ is straightforward, and does not require the general considerations leading to Lemma 1.2. As indicated by Harari and Leichtnam, the argument of [BC, Remark 33.b] can be used to show that $C^*(P_{\mathbb{K}}, P_O)$ is the fixed point algebra of their Hecke $C^*$-algebra $C^*(\Gamma_S, \Gamma_O)$ under the action of $\mathbb{O}^*$, see also [Lar] Example 2.9. That the extreme KMS$_\beta$ states of $C^*(P_{\mathbb{K}}, P_O)$ for $\beta > 1$ are indexed by $\mathcal{G}(\mathbb{K}^{ab}) : \mathbb{K}$ then follows from [HLe] Theorem 0.2 and Proposition 8.5]. This observation and the final remarks [HLe] pp. 241–242 concerning the Artin map, were a strong motivation for the present work, by reinforcing our belief that the almost normal inclusion of full “$ax + b$” groups considered here would lead to a $C^*$-dynamical system with group of symmetries isomorphic to the right Galois groups, and that an appropriate compactification of $\mathbb{O}^*$, like the one given in Lemma 2.4, was the key to understand the corresponding Hecke $C^*$-algebra.

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Department of Mathematics and Statistics, University of Victoria, Victoria, BC, V8W 3P4, CANADA
E-mail address: laca@math.uvic.ca

Department of Mathematics, Utah Valley State College, Orem, UT 84058-5999, USA
E-mail address: vanframa@uvsc.edu