REGULAR SOLUTION AND LATTICE MIURA TRANSFORMATION OF BIGRADED TODA HIERARCHY*

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ABSTRACT. In this paper, we give finite dimensional exponential solutions of the bigraded Toda Hierarchy(BTH). As an specific example of exponential solutions of the BTH, we consider a regular solution for the $(1,2)$-BTH with $3 \times 3$-sized Lax matrix, and discuss some geometric structure of the solution from which the difference between $(1,2)$-BTH and original Toda hierarchy is shown. After this, we construct another kind of Lax representation of $(N,1)$-bigraded Toda hierarchy($(N,1)$-BTH) which does not use the fractional operator of Lax operator. Then we introduce lattice Miura transformation of $(N,1)$-BTH which leads to equations depending on one field, meanwhile we give some specific examples which contains Volterra lattice equation(an useful ecological competition model).

Keywords: Regular solution, lattice miura transformation, bigraded Toda Hierarchy, moment polytope, Volterra lattice.

2000 Mathematics Subject Classifications 37K05, 37K10, 37K20

1. Introduction

The Toda lattice hierarchy introduced by M. Toda [1,2] is a completely integrable system and has important applications in many different fields such as classical and quantum fields theory. It is well-known that the Toda lattice equation [3] can be reduced from the two-dimensional Toda hierarchy. Adding additional logarithm flows to the Toda lattice hierarchy, it becomes the extended Toda hierarchy [4] which governs the Gromov-Witten invariant of $\mathbb{CP}^1$. The bigraded Toda hierarchy (BTH) of $(N,M)$-type(or simply the $(N,M)$-BTH) is the generalized Toda lattice hierarchy whose infinite Lax matrix has $N$ upper and $M$ lower nonzero diagonals. The BTH can be seen as a natural extension of the original Toda lattice hierarchy which is just of $(1,1)$-type. The BTH can be also treated as a general reduction of the two-dimensional Toda lattice hierarchy. The extended bigraded Toda hierarchy(EBTH) is the extension of the BTH which includes additional logarithm flows [5]. The dispersionless version of extended bigraded Toda hierarchy was firstly introduced by S. Aoyama, Y. Kodama in [6]. Later dispersive extended bigraded Toda hierarchy was introduced by Gudio Carlet [5] who hoped that the EBTH might also be relevant to the theory of Gromov-Witten invariants.

We generalized the Sato theory to the EBTH and give the HBEs of EBTH in [7]. The close relation of the BTH and the two-dimensional Toda hierarchy becomes a great motivation for us...
to consider the solutional structure of the BTH. In paper \cite{8}, we prove the BTH (bigraded toda hierarchy) has an equivalent relation between \((N, M)\)-BTH (whose infinite Lax matrix has \(N\) upper and \(M\) lower nonzero diagonals) and \((M, N)\)-BTH. In paper \cite{9,10}, the BTH is proved to have a natural Block type Lie algebraic symmetry and so is dispersionless BTH. As we know, \((N, M)\)-BTH is equivalent to \((N, M)\)-band bi-infinite matrix-formed Toda hierarchy, so we consider its reduction, i.e. the semi-finite and finite matrix form of the BTH. Then we give solutions of the BTH using orthogonal polynomials in matrix form. Some rational solutions of the BTH and corresponding Young diagrams were also given in \cite{8}. But there is one missing part is about the regular exponential solutions of BTH. Therefore from the general structure of solutions of the BTH, regular exponential solution which only depend on primary time variables will be introduced in this article. Also we will tell the difference between \((1, 2)\)-BTH and original Toda hierarchy from the orbit of flows in the graph of diagonal elements. Because its structure of Flag manifold similar as original Toda hierarchy, a geometric description using moment polytope \cite{11} whose vertices correspond to solutions of the BTH.

Comparing the equations for primary flows of the \((N, 1)\)-BTH with equations constructed in \cite{12}, we find that they have very close relation which will be shown in detail in the following sections.

The paper is organized as follows. In Section 2, the definition of the BTH and its tau function are given. In Section 3, the exponential solutions of the BTH will be given where we also consider the finite dimensional exponential solutions and further survey into the \((1, M)\)-BTH. To see the geometry of the \((N, M)\)-BTH, we consider the regular solution for the \((1, 2)\)-BTH with \(3 \times 3\) Lax matrix, and discuss the geometric structure of the solution, i.e. the moment polytope for the \((1, 2)\)-BTH in Section 4. In Section 5, some primary flows of \((N, 1)\)-BTH will be introduced. In Section 6, we construct another kind of Lax representation of the BTH. In Section 7, lattice miura transformation of the BTH will be given, meanwhile, some concrete examples will be shown in detail. After that conclusion and discussion are devoted to the last section.

\section{The bigraded Toda hierarchy (BTH)}

Firstly, the interpolated bigraded Toda hierarchy will be introduced firstly. The Lax operator of the BTH is given by the Laurent polynomial of shift matrix \(\Lambda\) \cite{5}

\begin{equation}
\mathcal{L} := \Lambda^N + u_{N-1}\Lambda^{N-1} + \cdots + u_{0} + \cdots + u_{-M}\Lambda^{-M}
\end{equation}

where \(N, M \geq 1\), \(\Lambda\) represents the shift operator with \(\Lambda := e^{\epsilon \partial_x}\) and \(\epsilon\) is called the string coupling constant, i.e. for any function \(f(x)\)

\[\Lambda f(x) = f(x + \epsilon).\]

The \(\mathcal{L}\) can be written in two different ways by dressing the shift operator

\begin{equation}
\mathcal{L} = \mathcal{P}_L \Lambda^N \mathcal{P}_L^{-1} = \mathcal{P}_R \Lambda^{-M} \mathcal{P}_R^{-1},
\end{equation}

where the dressing operators have the form,

\begin{equation}
\mathcal{P}_L = 1 + w_1 \Lambda^{-1} + w_2 \Lambda^{-2} + \cdots,
\end{equation}

\begin{equation}
\mathcal{P}_R = \tilde{w}_0 + \tilde{w}_1 \Lambda + \tilde{w}_2 \Lambda^2 + \cdots.
\end{equation}

Note that \(\mathcal{P}_L\) is lower triangular Matrix, \(\mathcal{P}_R\) is upper triangular Matrix.
Eq. (2.2) are quite important because it gives the reduction condition from the two-dimensional Toda lattice hierarchy. The pair is unique up to multiplying \( P_L \) and \( P_R \) from the right by operators in the form \( 1 + a_1 \Lambda^{-1} + a_2 \Lambda^{-2} + \ldots \) and \( \tilde{a}_0 + \tilde{a}_1 \Lambda + \tilde{a}_2 \Lambda^2 + \ldots \) respectively with coefficients independent of \( x \). Given any difference operator \( A = \sum_k A_k \Lambda^k \), the positive and negative projections are defined by \( A_+ = \sum_{k>0} A_k \Lambda^k \) and \( A_- = \sum_{k<0} A_k \Lambda^k \).

To write out explicitly the Lax equations of the BTH, fractional powers \( L^{\frac{1}{2}} \) and \( L^{\frac{3}{2}} \) are defined by

\[
L^{\frac{1}{2}} = \lambda + \sum_{k \leq 0} a_k \Lambda^k, \quad L^{\frac{3}{2}} = \sum_{k \geq -1} b_k \Lambda^k,
\]

with the relations

\[(L^{\frac{1}{2}})^N = (L^{\frac{3}{2}})^M = L.
\]

\( L^{\frac{1}{2}} \) and \( L^{\frac{3}{2}} \) are lower Heisenberg triangular Matrix and upper triangular Matrix respectively.

Acting on free function, these two fraction powers can be seen as two different locally expansions around zero and infinity respectively. It was stressed that \( L^{\frac{1}{2}} \) and \( L^{\frac{3}{2}} \) are two different operators even if \( N = M(N, M \geq 2) \) in [7] due to two different dressing operators. They can also be expressed as following

\[
L^{\frac{1}{2}} = P_L \Lambda \partial_{x,t} P_L^{-1}, \quad L^{\frac{3}{2}} = P_R \Lambda^{-1} \partial_{x,t} P_R^{-1}.
\]

Let us now define the following operators for the generators of the BTH flows,

\[
B_{\gamma,n} := \begin{cases} 
L^{\alpha+1-\frac{2}{N}} & \text{if } \gamma = \alpha = 1, 2, \ldots, N, \\
L^{\alpha+1+\frac{\beta}{\Lambda}} & \text{if } \gamma = \beta = -M + 1, \ldots, -1, 0,
\end{cases}
\]

**Definition 2.1.** Bigraded Toda hierarchy (BTH) in the Lax representation is given by the set of infinite number of flows defined by (2.5)

\[
\partial_{\gamma,n} L = \begin{cases} 
\{ (B_{\alpha,n})_+ , L \} & \text{if } \gamma = \alpha = 1, 2, \ldots, N, \\
\{ (B_{\beta,n})_- , L \} & \text{if } \gamma = \beta = -M + 1, \ldots, -1, 0.
\end{cases}
\]

Sometimes we denote \( \partial_{\gamma,n} \) as \( \partial_{\alpha} \) in this paper and denote \( \{ t_{\gamma,n}, \gamma = \alpha = 1, 2, \ldots, N \}, \{ t_{\gamma,n}, \gamma = \beta = -M + 1, \ldots, -1, 0 \} \) as \( t_{\alpha}, t_{\beta} \) respectively. We call the flows for \( n = 0 \) the primaries of the BTH and the time variables \( t_{\gamma,n} \) (\( n = 0 \)) are primary time variables. The original tridiagonal Toda hierarchy corresponds to the case with \( N = M = 1 \).

**2.1. Tau function and band structure.** According to paper [7], a function \( \tau \) depending only on the dynamical variables \( t \) and \( \epsilon \) is called the tau-function of BTH if it provides symbols related to wave operators as follows,

\[
P_L: \quad 1 + \frac{w_{1, \lambda}}{\lambda} + \frac{w_{2, \lambda^2}}{\lambda^2} + \ldots := \frac{\tau(x + \epsilon, t + [\lambda^{-1}]^N; \epsilon)}{\tau(x, t; \epsilon)},
\]

\[
P_L^{-1} : 1 + \frac{w_{1, \lambda}}{\lambda} + \frac{w_{2, \lambda^2}}{\lambda^2} + \ldots := \frac{\tau(x + \epsilon, t + [\lambda^{-1}]^N; \epsilon)}{\tau(x + \epsilon, t; \epsilon)},
\]

\[
P_R : \quad \bar{w}_0 + \bar{w}_1 \lambda + \bar{w}_2 \lambda^2 + \ldots := \frac{\tau(x + \epsilon, t + [\lambda]^M; \epsilon)}{\tau(x, t; \epsilon)}.
\]
\begin{equation}
(2.10) \quad P_R^{-1} : = \bar{u}_0' + \bar{w}_1' \lambda + \bar{w}_2' \lambda^2 + \ldots := \frac{\tau(x, t - [\lambda]^M; \epsilon)}{\tau(x + \epsilon, t; \epsilon)},
\end{equation}

where \([\lambda^{-1}]_N^M\) and \([\lambda]^M\) are defined by

\begin{align*}
[\lambda^{-1}]_{\gamma,n}^N & := \begin{cases}
\frac{\lambda^{-N(n+1)} \epsilon^{-\gamma_1}}{N^{(n+1)}}, & \gamma = N, N - 1, \ldots, 1, \\
0, & \gamma = 0, -1 \cdots (M - 1),
\end{cases} \\
[\lambda]^M_{\gamma,n} & := \begin{cases}
0, & \gamma = N, N - 1, \ldots, 1, \\
\frac{\lambda^{M(n+1)} \epsilon^{-\gamma_1}}{M^{(n+1)}}, & \gamma = 0, -1 \cdots (M - 1).
\end{cases}
\end{align*}

Then we get

\begin{equation}
(2.11) \quad P_L := \sum_{n=0}^{\infty} \frac{P_n(\hat{\partial}_L)\tau(x, t; \epsilon)}{\tau(x, t; \epsilon)} \lambda^{-n}, \quad P_L^{-1} := \sum_{n=0}^{\infty} \frac{P_n(\hat{\partial}_L)\tau(x + \epsilon, t; \epsilon)}{\tau(x + \epsilon, t; \epsilon)} \lambda^{-n},
\end{equation}
\begin{equation}
(2.12) \quad P_R := \sum_{n=0}^{\infty} \frac{P_n(\hat{\partial}_R)\tau(x + \epsilon, t; \epsilon)}{\tau(x, t; \epsilon)} \lambda^n, \quad P_R^{-1} := \sum_{n=0}^{\infty} \frac{P_n(\hat{\partial}_R)\tau(x, t; \epsilon)}{\tau(x + \epsilon, t; \epsilon)} \lambda^n,
\end{equation}

where Schur polynomial \(P_k(\hat{\partial})\) is defined by

\begin{equation}
(2.13) \quad e^{\sum_{k=1}^{\infty} \frac{i}{k} \partial_k x^k} = \sum_{k=0}^{\infty} P_k(\hat{\partial}) x^k, \quad \hat{\partial} = (\partial_1, \frac{1}{2} \partial_2, \frac{1}{3} \partial_3, \frac{1}{4} \partial_4, \ldots).
\end{equation}

Here the operators \(\hat{\partial}_L\) and \(\hat{\partial}_R\) are defined by

\begin{align*}
\hat{\partial}_L & = \left\{ \frac{1}{N(n+1-\alpha)} \partial_{t_{\alpha,n}} : 1 \leq \alpha \leq N \right\} \\
\hat{\partial}_R & = \left\{ \frac{1}{M(n+1-\beta)} \partial_{t_{\beta,n}} : M+1 \leq \beta \leq 0 \right\}.
\end{align*}

The dressing operators \(P_L\) and \(P_R\) can be expressed by function \(\tau(x, t; \epsilon)\):

\begin{align*}
(2.14) \quad P_L & = \sum_{n=0}^{\infty} \frac{P_n(-\hat{\partial}_L)\tau(x, t; \epsilon)}{\tau(x, t; \epsilon)} \Lambda^{-n}, \quad P_L^{-1} = \sum_{n=0}^{\infty} \frac{\Lambda^{-n} P_n(\hat{\partial}_L)\tau(x + \epsilon, t; \epsilon)}{\tau(x + \epsilon, t; \epsilon)}, \\
(2.15) \quad P_R & = \sum_{n=0}^{\infty} \frac{P_n(\hat{\partial}_R)\tau(x + \epsilon, t; \epsilon)}{\tau(x, t; \epsilon)} \Lambda^n, \quad P_R^{-1} = \sum_{n=0}^{\infty} \frac{\Lambda^n P_n(-\hat{\partial}_R)\tau(x, t; \epsilon)}{\tau(x + \epsilon, t; \epsilon)}.
\end{align*}

One can then find the explicit form of the coefficients \(u_i(x, t)\) of the operator \(\mathcal{L}\) in terms of the \(\tau\)-function using eq. (2.2) as \[\text{[14][16]}\].

\begin{equation}
(2.16) \quad u_i(x, t) = \frac{P_{N-i}(\hat{D}_L)\tau(x + (i+1)\epsilon, t; \epsilon) \circ \tau(x, t; \epsilon)}{\tau(x, t; \epsilon) \tau(x + (i+1)\epsilon, t; \epsilon)} = \frac{P_{M+i}(\hat{D}_R)\tau(x + \epsilon, t; \epsilon) \circ \tau(x + i\epsilon, t; \epsilon)}{\tau(x, t; \epsilon) \tau(x + (i+1)\epsilon, t; \epsilon)}
\end{equation}

where \(\hat{D}_L\) and \(\hat{D}_R\) are just the Hirota derivatives corresponding to \(\hat{\partial}_L\) and \(\hat{\partial}_R\) respectively.
As we all know, interpolated BTH is equivalent to bi-infinite or semi-infinite matrix-formed BTH. Because what we will consider next is the matrix-formed bigraded Toda hierarchy, following equivalent definitions in matrix form are introduced,

\[ \Lambda := (E_{i,i+1})_{i \in \mathbb{Z}_+}, \quad u_i := \text{diag}(u_{i,1}, u_{i,2}, u_{i,3}, \ldots). \]

After the following transformation \( u_i(x) := u_{i,j} := a_{j,j+i} \), the matrix representation of \( L \) can be expressed by \( (a_{i,j})_{i,j \geq 1} \) with

\[
(2.17) \quad a_{i,j}(t) = \frac{P_{i-j+N}(\hat{D}_L)\tau_j \circ \tau_{i-1}}{\tau_{i-1} \tau_j} = \frac{P_{j-i+M}(\hat{D}_R)\tau_i \circ \tau_{j-1}}{\tau_{i-1} \tau_j}.
\]

This immediately imply

\[ a_{i,j} = 0, \quad \text{if} \quad j < -M + i \quad \text{or} \quad j > N + i. \]

That shows that the Lax matrix \( L \) has the \((N,M)\)-band structure.

3. Exponential solutions of the BTH

In paper [8], the rational solutions of the BTH were introduced already. In this section, we will introduce the exact (regular) solutions of the BTH, i.e. the non-negative exponential solutions.

The tau functions of the two-dimensional Toda lattice hierarchy \[16\] can be expressed by

\[
\tau_i = \begin{vmatrix}
\bar{C}_{0,0} & \bar{C}_{0,1} & \cdots & \bar{C}_{0,i-1} \\
\bar{C}_{1,0} & \bar{C}_{1,1} & \cdots & \bar{C}_{1,i-1} \\
\cdots & \cdots & \cdots & \cdots \\
\bar{C}_{i-1,0} & \bar{C}_{i-1,1} & \cdots & \bar{C}_{i-1,i-1}
\end{vmatrix},
\]

where \( \bar{C}_{i,j} \) can have in form of following inner product using arbitrary density function \( \rho(\lambda, \mu) \)

\[ \bar{C}_{i,j} = \langle \lambda^i, \mu^j \rho(\lambda, \mu) \rangle. \]

Here the inner product can be chosen as following integral representation

\[
\bar{C}_{i,j} = \int \int \rho(\lambda, \mu) \lambda^i \mu^j e^{\sum_{n=0}^{\infty} x_n \lambda^n + \sum_{n=0}^{\infty} y_n \mu^n} d\lambda d\mu
= \sum_{k,l=0}^{\infty} \bar{c}_{i,j,k,l} P_k(x)P_l(y),
\]

where \( P_k(x) \) and \( P_l(y) \) are Schur function introduced in eq. \[2.13\].

We should note here that the coefficients \( \bar{c}_{i,j,k,l} \) are totally independent.

As the original tridiagonal Toda lattice is \((1,1)\)-reduction of the two-dimensional Toda lattice hierarchy. Therefore to get the solution of the tridiagonal Toda lattice hierarchy, we need to add factor \( \delta(\lambda - \mu) \) under the integral in the definition of \( \bar{C}_{i,j} \), i.e. the element \( \bar{C}_{i,j} \) in tau function of the tridiagonal Toda lattice hierarchy \[15\] becomes

\[
(3.2) \quad \int \int \rho(\lambda, \mu) \delta(\lambda - \mu) \lambda^i \mu^j e^{\sum_{n=0}^{\infty} x_n \lambda^n + \sum_{n=0}^{\infty} y_n \mu^n} d\lambda d\mu,
\]

which can further lead to

\[
(3.3) \quad \int \rho(\lambda, \lambda) \lambda^{i+j} e^{\sum_{n=0}^{\infty} (x_n + y_n) \lambda^n} d\lambda.
\]
After changing time variables \(x, y\) to \(t_{\alpha}, t_{\beta}\) in BTH, eq. (3.3) become a new function

\[
\int \rho(\lambda, \lambda)\lambda^{i+1}e^{\xi_L(\lambda, t_{\alpha})+\xi_R(\lambda, t_{\beta})}d\lambda
\]

which corresponds to \((1,1)\)-BTH.

Denote \(\omega_N\) and \(\omega_M\) as the \(N\)-th root and \(M\)-th root of unit. For \((N,M)\)-BTH which is a generalization of the tridiagonal Toda lattice hierarchy, new function \(C_{i,j}\) (new form of \(C_{i,j}\)) has the following form

\[
C_{i,j} = \int \int \rho(\lambda, \mu)\delta(\lambda^N - \mu^M)\lambda^i\mu^j e^{\xi_L(\lambda, t_{\alpha})+\xi_R(\mu, t_{\beta})}d\lambda d\mu
\]

\[
= \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \int \rho(\omega_N^p \lambda^{\frac{1}{N}}, \omega_M^q \lambda^{\frac{1}{M}})(\omega_N^p \lambda^{\frac{1}{N}})^i(\omega_M^q \lambda^{\frac{1}{M}})^j e^{\xi_L(\omega_N^p \lambda^{\frac{1}{N}}, t_{\alpha})+\xi_R(\omega_M^q \lambda^{\frac{1}{M}}, t_{\beta})}d\lambda.
\]

Therefore tau functions of the BTH can be explicitly written in the form

\[
\tau_i = \begin{vmatrix}
C_{0,0} & C_{0,1} & \ldots & C_{0,i-1} \\
C_{1,0} & C_{1,1} & \ldots & C_{1,i-1} \\
\vdots & \vdots & \ddots & \vdots \\
C_{i-1,0} & C_{i-1,1} & \ldots & C_{i-1,i-1}
\end{vmatrix}.
\]

If we consider the case in finite dimension (dimension is \(n\), i.e.

\[
\rho(\omega_N^p \lambda^{\frac{1}{N}}, \omega_M^q \lambda^{\frac{1}{M}}) = \sum_{k=1}^{n} \rho_0(\omega_N^p \lambda^{\frac{1}{N}}, \omega_M^q \lambda^{\frac{1}{M}})\delta(\lambda - \lambda_k),
\]

then

\[
C_{i,j} = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \sum_{k=1}^{n} \rho_0(\omega_N^p \lambda^{\frac{1}{N}}_k, \omega_M^q \lambda^{\frac{1}{M}}_k)(\omega_N^p \lambda^{\frac{1}{N}}_k)^i(\omega_M^q \lambda^{\frac{1}{M}}_k)^j e^{\xi_L(\omega_N^p \lambda^{\frac{1}{N}}_k, t_{\alpha})+\xi_R(\omega_M^q \lambda^{\frac{1}{M}}_k, t_{\beta})}.
\]

After denoting \(C_{0,0}\) as \(\tau_1\), we can rewrite tau functions into the following bi-directional Wronskian form

\[
\tau_i = \begin{vmatrix}
\tau_1 & \partial_{t_{-M+1,0}} \tau_1 & \ldots & \partial^{i-1}_{t_{-M+1,0}} \tau_1 \\
\partial_{t_{N,0}} \tau_1 & \partial_{t_{N,0}} \partial_{t_{-M+1,0}} \tau_1 & \ldots & \partial_{t_{N,0}} \partial^{i-1}_{t_{-M+1,0}} \tau_1 \\
\vdots & \vdots & \ddots & \vdots \\
\partial^{i-1}_{t_{N,0}} \tau_1 & \partial^{i-1}_{t_{N,0}} \partial_{t_{-M+1,0}} \tau_1 & \ldots & \partial^{i-1}_{t_{N,0}} \partial^{i-1}_{t_{-M+1,0}} \tau_1
\end{vmatrix}.
\]

Then \(\tau_1\) has the following form

\[
\tau_1 = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \sum_{k=1}^{n} \rho_0(\omega_N^p \lambda^{\frac{1}{N}}_k, \omega_M^q \lambda^{\frac{1}{M}}_k) e^{\xi_L(\omega_N^p \lambda^{\frac{1}{N}}_k, t_{\alpha})+\xi_R(\omega_M^q \lambda^{\frac{1}{M}}_k, t_{\beta})}.
\]

For \((N,M)\) case, for each \(m \leq n\)

\[
\tau_m = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n} \sum_{l=1}^{i_1} \sum_{k=1}^{i_2} \ldots \sum_{l=1}^{i_m} \Pi \left(\omega_N^{i_j} \omega_N^{\frac{1}{N}} - \omega_M^{i_j} \omega_M^{\frac{1}{M}})(\omega_M^{i_j} \omega_M^{\frac{1}{M}} - \omega_M^{i_k} \omega_M^{\frac{1}{M}})\right) \\
\prod_{j,k,l=1}^{m} \rho_0(\omega_N^{i_j} \lambda^{\frac{1}{N}}_j, \omega_M^{i_j} \lambda^{\frac{1}{M}}_j) E_{i_j, i'_j, i''_j},
\]

where \(E_{i_j, i'_j, i''_j}\) is a certain expression involving \(i_j, i'_j, i''_j\).
where

\[ E_{ij, i'j'} = e^{\xi_L(\omega_{ij}^{\beta} \lambda_{ij}^{\frac{1}{M}}, t_\alpha) + \xi_R(\omega_{ij}^{\beta} \lambda_{ij}^{\frac{1}{M}}, t_\beta)} \]

(3.9)

We can find it there are \((\binom{n}{m})(NM)^m\) terms in \(\tau_m\).

The theory above is about \((N, M)\)-BTH. To see it clearly, we will further consider one kind of specific example, i.e. the \((1, M)\)-BTH clearly in the following.

Before that firstly we will consider the finite-sized Lax matrix of the \((1, M)\)-BTH.

If Lax matrix is supposed to have \(n\) different eigenvalues, i.e. there exists matrix \(\Phi\) s.t.

\[ \Phi L \Phi^{-1} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \]

Then from the theory on the \((N, M)\)-BTH, we can get the first tau function of the \((1, M)\)-BTH as following

\[ \tau_1 = \sum_{q=0}^{M-1} \sum_{k=1}^{n} \rho_0(\lambda_k, \omega_{ij}^{\beta} \lambda_{ij}^{\frac{1}{M}}) e^{\sum_{s=1}^{M} (\omega_{ij}^{\beta} \lambda_{ij}^{\frac{1}{M}})^s t_s - M, 0} \]

(3.10)

Similarly for every integer \(m \leq n\), tau function \(\tau_m\) has following form

\[ \tau_m = \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_m \leq n} \prod_{1 \leq k < j \leq m} (\omega_{ij} - \omega_{ij}') \prod_{1 \leq k < j \leq m} (\omega_{ij}^{\beta} \omega_{ij}^{\frac{1}{M}} - \omega_{ij}^{\beta} \omega_{ij}^{\frac{1}{M}}) \prod_{j=1}^{M} \rho_0(\lambda_{ij}, \omega_{ij}^{\beta} \lambda_{ij}^{\frac{1}{M}}) E_{ij, i'} \]

where

\[ E_{ij, i'} = e^{\xi_R(\omega_{ij}^{\beta} \lambda_{ij}^{\frac{1}{M}}, t_\beta)} \]

Considering the primary dependence, \(\tau_1\) can be written as

\[ \tau_1 = \sum_{q=0}^{M-1} \sum_{k=1}^{n} \rho_0(\lambda_k, \omega_{ij}^{\beta} \lambda_{ij}^{\frac{1}{M}}) e^{\sum_{s=1}^{M} (\omega_{ij}^{\beta} \lambda_{ij}^{\frac{1}{M}})^s t_s - M, 0} \]

(3.11)

To see it more clearly, we will further consider a specific example in the following section, i.e. the exponential solution of the \((1,2)\)-BTH.

4. Regular solution of \((1, 2)\)-BTH

One can obtain the general finite dimensional solution for the \((N, M)\)-BTH with exponential functions. However most of the solutions are complex and have singular points. In this section, we will consider the exponential solution particularly the regular solutions of the \((1, 2)\)-BTH. By this regular solution, we see the difference between \((1,2)\)-BTH and original Toda hierarchy from a geometric viewpoint.

For \((1,2)\) case, the solution can be expressed as

\[ \tau_1 = \sum_{k=1}^{n} \rho_0(\lambda_k, \lambda_2^{\frac{1}{2}}) e^{\xi_R(\lambda_2^{\frac{1}{2}}, t_\beta)} + \rho_0(\lambda_k, \omega_2 \lambda_2^{\frac{1}{2}}) e^{\xi_R(\omega_2 \lambda_2^{\frac{1}{2}}, t_\beta)}, \]

where \(\omega_2 = -1\).
If we only consider the primary dependence which means we let tau function only depend on primary time variables, then the first solution \( \tau_1 \) in form of one by one matrix has following form

\[
(4.2) \quad \tau_1 = \sum_{k=1}^{n} \rho_0(\lambda_k, \lambda_k^{\frac{1}{2}}) \left( e^{\frac{1}{2} \lambda_k^{\frac{1}{2}} t_{-1,0} + \omega_k \lambda_k t_{0,0}} + e^{-\frac{1}{2} \lambda_k^{\frac{1}{2}} t_{-1,0} + \omega_k \lambda_k t_{0,0}} \right).
\]

Let us assume that the Lax matrix is semi-simple and has distinct eigenvalues, \((\lambda_1, \lambda_2, \lambda_3)\). If the \( \lambda_k, k = 1, 2, 3 \) is negative, the real solution can be written as the following form

\[
(4.3) \quad \tau_1 = \sum_{k=1}^{n} \rho_0(\lambda_k, \lambda_k^{\frac{1}{2}}) e^{i\lambda_k^{\frac{1}{2}} t_{-1,0} + \theta_k} + \rho_0(\lambda_k, -\lambda_k^{\frac{1}{2}}) e^{-i\lambda_k^{\frac{1}{2}} t_{-1,0} + \theta_k},
\]

\[
(4.4) \quad = \sum_{k=1}^{n} A'_k \cos(|\lambda_k|^{\frac{1}{2}} t_{-1,0} + \theta_k),
\]

where \( A'_k := \sqrt{\rho_0(\lambda_k, \lambda_k^{\frac{1}{2}}) + \rho_0(\lambda_k, -\lambda_k^{\frac{1}{2}})} \) and \( \theta_k \) depend on \( \{\rho_0(\lambda_k, \lambda_k^{\frac{1}{2}}), \rho_0(\lambda_k, -\lambda_k^{\frac{1}{2}}), |\lambda_k|^{\frac{1}{2}} t_{-1,0}\} \).

This is in fact a periodic solution about \( t_{-1,0} \) time variable and it has singular points.

If we set \( 0 < \lambda_1 < \lambda_2 < \lambda_3, \rho_0 \geq 0 \), it will lead to regular solutions. In this case as a simple but an interesting example, we will consider a regular solution for the \((1, 2)\)-BTH with \( 3 \times 3 \)-sized Lax matrix, and discuss some geometric structure of the solution in the following part. Then the regular function \( \tau_1 \) is given by

\[
\tau_1 = \sum_{k=1}^{3} A_k \cosh(\lambda_k^{\frac{1}{2}} t_{-1,0} + \theta_k) e^{\lambda_k (t_{0,0} + t_{1,0})} = \sum_{k=1}^{3} C_k E_k,
\]

where \( A_k \) and \( \theta_k \) are arbitrary constants, \( C_k := A_k \cosh(\lambda_k^{\frac{1}{2}} t_{-1,0} + \theta_k) \) and \( E_k := e^{\lambda_k (t_{0,0} + t_{1,0})} \).

We also write \( S_k := A_k \sinh(\lambda_k^{\frac{1}{2}} t_{-1,0} + \theta_k) \) in the following part.

For \( \tau_1 \) being positive definite, \( A_i > 0 \) is supposed to hold. Also we set all \( \theta_k = 0 \) (this is necessary for \( \tau_k \) being sign definite). Then the second tau function \( \tau_2 \) and the third tau function \( \tau_3 \) in eq. (3.7) are given by

\[
\tau_2 = \left| \begin{array}{ccc} \tau_1 & \partial_{1,0} \tau_1 \\ \partial_{-1,0} \tau_1 & \partial_{1,0} \partial_{1,0} \tau_1 \end{array} \right| = \left( \begin{array}{ccc} C_1 E_1 & C_2 E_2 & C_3 E_3 \\ \lambda_1^{\frac{1}{2}} S_1 E_1 & \lambda_2^{\frac{1}{2}} S_2 E_2 & \lambda_3^{\frac{1}{2}} S_3 E_3 \end{array} \right) \left( \begin{array}{c} 1 \\ \lambda_1 \\ 1 \lambda_2 \\ 1 \lambda_3 \end{array} \right) \]

\[
= \sum_{i,j=1, i<j}^{3} \left( C_i S_j \lambda_j^{\frac{1}{2}} - C_j S_i \lambda_i^{\frac{1}{2}} \right) (\lambda_j - \lambda_i) E_i E_j,
\]

\[
\tau_3 = \left| \begin{array}{cccc} \tau_1 & \partial_{1,0} \tau_1 & \partial_{2,0}^{2} \tau_1 \\ \partial_{-1,0} \tau_1 & \partial_{1,0} \partial_{1,0} \tau_1 & \partial_{1,0} \partial_{2,0}^{2} \tau_1 \\ \partial_{2,0}^{2} \tau_1 & \partial_{1,0} \partial_{1,0} \tau_1 & \partial_{2,0}^{2} \partial_{2,0}^{2} \tau_1 \end{array} \right| = \left( \begin{array}{ccc} C_1 E_1 & C_2 E_2 & C_3 E_3 \\ \lambda_1^{\frac{1}{2}} S_1 E_1 & \lambda_2^{\frac{1}{2}} S_2 E_2 & \lambda_3^{\frac{1}{2}} S_3 E_3 \\ \lambda_1 C_1 E_1 & \lambda_2 C_2 E_2 & \lambda_3 C_3 E_3 \end{array} \right) \left( \begin{array}{c} 1 \\ \lambda_1 \lambda_2^2 \\ 1 \lambda_2 \lambda_3^2 \\ 1 \lambda_3 \lambda_1^2 \end{array} \right) \]

\[
= \sum_{i,j=k}^{\infty} \lambda_i^{\frac{1}{2}} \lambda_j^{\frac{1}{2}} |\lambda_j - \lambda_k| S_i C_j C_k \prod_{i<j} (\lambda_j - \lambda_i) E_i E_2 E_3.
\]
where $\sum_{i\to j\to k}$ implies the sum over the cyclic permutation on $\{1, 2, 3\}$. Therefore $\tau_1$ is always positive.

For $\tau_2$, taking derivative can imply

$$
(4.5) \quad \partial_{t-1,0}(C_i S_j \lambda_j^\frac{1}{2} - C_j S_i \lambda_i^\frac{1}{2})
$$

$$
= S_i S_j (\lambda_i \lambda_j)^\frac{1}{2} + C_i C_j \lambda_j - S_j S_i (\lambda_i \lambda_j)^\frac{1}{2} - C_j C_i \lambda_i
$$

$$
= C_i C_j (\lambda_j - \lambda_i) > 0,
$$

which means $\tau_2$ always increase with variable $t_{-1,0}$. Because when $t_{-1,0} = 0$,

$$
C_i S_j \lambda_j^\frac{1}{2} - C_j S_i \lambda_i^\frac{1}{2} = 0,
$$

when $t_{-1,0} > 0$, $\tau_2$ is always positive. Because

$$
\lim_{t_{-1,0} \rightarrow 0} \frac{\partial_{1,0} \tau_2}{\tau_2} = \lim_{t_{-1,0} \rightarrow 0} \frac{\sum_{i,j=1, i<j}^{3} (C_i S_j \lambda_j^\frac{1}{2} - C_j S_i \lambda_i^\frac{1}{2})(\lambda_j^2 - \lambda_i^2) E_i E_j}{\sum_{i,j=1, i<j}^{3} (C_i S_j \lambda_j^\frac{1}{2} - C_j S_i \lambda_i^\frac{1}{2})(\lambda_j - \lambda_i) E_i E_j}
$$

$$
= \lim_{t_{-1,0} \rightarrow 0} \frac{\sum_{i,j=1, i<j}^{3} C_i C_j (\lambda_j^2 - \lambda_i^2) E_i E_j}{\sum_{i,j=1, i<j}^{3} (\lambda_j - \lambda_i) E_i E_j},
$$

therefore $t_{-1,0} = 0$ is a removable singular point. Using the formula (2.17) for $a_{i,j}$ of the Lax matrix, we have

$$
a_{1,1} = \partial_{1,0} \ln \tau_1 = \frac{\lambda_1 C_1 E_1 + \lambda_2 C_2 E_2 + \lambda_3 C_3 E_3}{C_1 E_1 + C_2 E_2 + C_3 E_3},
$$

$$
a_{2,2} = \partial_{1,0} \ln \tau_2 = \frac{\sum_{i,j=1, i<j}^{3} (C_i S_j \lambda_j^\frac{1}{2} - C_j S_i \lambda_i^\frac{1}{2})(\lambda_j^2 - \lambda_i^2) E_i E_j}{\sum_{i,j=1, i<j}^{3} (C_i S_j \lambda_j^\frac{1}{2} - C_j S_i \lambda_i^\frac{1}{2})(\lambda_j - \lambda_i) E_i E_j} - a_{11},
$$

$$
a_{3,3} = \partial_{1,0} \ln \tau_3 \frac{\sum_{i,j=1, i<j}^{3} (\lambda_j - \lambda_i) E_i E_j}{\sum_{i,j=1, i<j}^{3} (\lambda_j - \lambda_i) E_i E_j} - a_{11}.
$$

Assuming the ordering for the eigenvalues as

$$
\lambda_1 < \lambda_2 < \lambda_3,
$$

one can obtain the following asymptotic sorting property of the Lax matrix,

$$
L \rightarrow \begin{pmatrix} \lambda_3 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \quad \text{for} \quad t_{-1,0} \rightarrow \infty,
$$


Because we choose all eigenvalues to be positive,

\[ \mathcal{L} \rightarrow \begin{cases} 
\begin{pmatrix}
\lambda_3 & 1 & 0 \\
0 & \lambda_2 & 1 \\
0 & 0 & \lambda_1 \\
\lambda_1 & 1 & 0 \\
0 & \lambda_2 & 1 \\
0 & 0 & \lambda_3 \\
\end{pmatrix}, & \text{for } t_{1,0} \to \infty,
\end{cases} \]

To see the orbit generated by the solution, we consider the projection \( \pi \) of the Lax matrix on the diagonal part, i.e.

\[ \pi : \mathcal{L} = \begin{pmatrix}
a_{1,1} & 1 & 0 \\
a_{2,1} & a_{2,2} & 1 \\
a_{3,1} & a_{3,2} & a_{3,3} \\
\end{pmatrix} \implies \text{diag}(\mathcal{L}) \equiv (a_{1,1}, a_{2,2}, a_{3,3}). \]

Figure 1 illustrates the image of the map \( \pi \): The left panel (a) shows the image in the case of \((1,2)\)-BTH for \(-5 \leq t_{0,0} \leq 5\) and \(-5 \leq t_{0,1} \leq 5\). The right panel (b) of the figure shows the case of the original Toda lattice, that is, the corresponding Lax matrix is \(3 \times 3\)-sized tridiagonal matrix. That example gives following proposition.

**Proposition 4.1.** In the case of the original Toda lattice, one can show that all the orbits have to cross the center point \((\lambda_2, \sigma_1 - 2\lambda_2, \lambda_2)\) with \(\sigma_1 = \sum_{j=1}^{3} \lambda_j\). However, the orbits for the \((1,2)\)-BTH have no such restriction, but those still go through the points close to the center.

**Proof.** Firstly because the Lax matrix has eigenvalues \(\lambda_1, \lambda_2, \lambda_3\) with \(\lambda_1 < \lambda_2 < \lambda_3\). We set the diagonal elements \((a_{1,1}, a_{2,2}, a_{3,3})\) take values \((\Delta_1, \sigma_1 - 2\Delta_1, \Delta_1)\). Then considering matrix

\[
\begin{pmatrix}
\Delta_1 & 1 & 0 \\
0 & \sigma_1 - 2\Delta_1 & 1 \\
0 & 0 & \Delta_1 \\
\end{pmatrix}
\]

has eigenvalues \(\lambda_1, \lambda_2, \lambda_3\) with \(\lambda_1 < \lambda_2 < \lambda_3\), we get

\[
(\Delta_1 - \lambda)[(\sigma_1 - 2\Delta_1 - \lambda)(\Delta_1 - \lambda) - a_{3,2}] - a_{2,1}(\Delta_1 - \lambda) + a_{3,1} = -\lambda^3 + \sigma_1\lambda^2 - \sigma_2\lambda + \sigma_3,
\]

where \(\sigma_1, \sigma_2, \sigma_3\) are first three fundamental symmetric polynomials. Equation above implies

\[ a_{3,2} + a_{2,1} = 2\sigma_1\Delta_1 - 3\Delta_1^2 - \sigma_2, \]

(4.6)

\[ \Delta_1^2 \sigma_1 - 2\Delta_1^3 - (a_{3,2} + a_{2,1})\Delta_1 + a_{3,1} = \sigma_3, \]

(4.7)

which further leads to

\[ \Delta_1^3 - \sigma_1\Delta_1^2 + \sigma_2\Delta_1 - \sigma_3 + a_{3,1} = 0. \]

(4.8)

By eq.(2.17), we get

\[ a_{3,2}(t) = \frac{P_2(\hat{D}_L)\tau_2 \circ \tau_2}{\tau_2^2 \tau_2} = \frac{P_1(\hat{D}_R)\tau_3 \circ \tau_1}{\tau_2^2 \tau_2}. \]

(4.9)

Because we choose all eigenvalues to be positive,

\[ D_{1,0}^2 \tau_2 \circ \tau_2 = \sum_{1 \leq i,j,k \leq 3} (C_i S_j \lambda_j^{\frac{1}{2}} - C_j S_i \lambda_i^{\frac{1}{2}})(C_j S_k \lambda_k^{\frac{1}{2}} - C_k S_j \lambda_j^{\frac{1}{2}})(\lambda_k - \lambda_j)(\lambda_j - \lambda_i)(\lambda_k + \lambda_i + 2\lambda_j) > 0, \]

(4.10)
therefore $a_{3,2}(t) > 0$. Similarly by eq. (2.17), we get
\begin{equation}
(4.11) \quad a_{2,1}(t) = \frac{P_2(\hat{D}_L)\tau_1 \circ \tau_1}{\tau_1 \tau_1} = \frac{P_1(\hat{D}_R)\tau_2 \circ \tau_0}{\tau_1 \tau_1} = \partial_{-1,0}\tau_2.
\end{equation}
Because of eq. (4.5), we get
\begin{equation}
(4.12) \quad a_{2,1}(t) > 0.
\end{equation}
By eq. (4.6), we can get the following identity must be correct
\begin{equation}
(4.13) \quad f(\Delta_1) := 2\sigma_1\Delta_1 - 3\Delta_1^2 - \sigma_2 > 0.
\end{equation}
For triagonal Toda hierarchy, $a_{3,1} = 0$, and eq. (4.8) has there roots $\lambda_1, \lambda_2, \lambda_3$ with order $\lambda_1 < \lambda_2 < \lambda_3$. Let $\Delta_1 = \lambda_3$, we find
\begin{equation}
(4.14) \quad f(\lambda_3) = 2\sigma_1\lambda_3 - 3\lambda_3^2 - \sigma_2 = 2(\lambda_1 + \lambda_2 + \lambda_3)\lambda_3 - 3\lambda_3^2 - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)
= (\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2) < 0,
\end{equation}
which is in contradict with eq. (4.13). Similarly we can also find $\Delta_1 = \lambda_1$ is also in contradict with eq. (4.13). Therefore the only choice is $\Delta_1 = \lambda_2$. That is why for the original Toda lattice, all the orbits have to cross the center point $(\lambda_2, \sigma_1 - 2\lambda_2, \lambda_2)$ with $\sigma_1 = \sum_{j=1}^{3} \lambda_j$ as shown in Fig. 1(b). By eq. (2.17), we get
\begin{equation}
(4.15) \quad a_{3,1}(t) = \frac{\tau_3 \tau_0}{\tau_2 \tau_1}.
\end{equation}
So for the (1, 2)-BTH, $a_{3,1}(t)$ is always positive when $t_{-1,0} > 0$ because of the positivity of tau functions. From that and considering eq. (4.8), we can get that there will be one more part which is close to the crossing point $(\lambda_2, \sigma_1 - 2\lambda_2, \lambda_2)$ just as Fig. 1(a).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The image $\pi(\mathcal{L}) = (a_{1,1}, a_{2,2}, a_{3,3})$: (a) for the (1, 2)-BTH, and (b) for the tridiagonal Toda hierarchy. Those orbits are obtained by changing $t_{0,0}$ and $t_{0,1}$ with fixed $t_{-1,0} = 1$. The eigenvalues are given by $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 9$.}
\end{figure}

The boundaries of Fig. 1(a) are characterized by Fig. 2(a) and Fig. 2(b). Fixing another time variable $t_{1,1}$, their two dimensional graphs are as Fig. 3(a) and Fig. 3(b).
After these geometric pictures, moment map [15] related to the (1, 2)-BTH will be given in the next subsection.
Figure 2. Graphs for \((a_1, a_2, a_3)\) of the \((1,2)\)-BTH: (a) depending on the parameters \(t_{00} \) with \(t_{-10} = 20\) and \(t_{01} = -2\), (b) depending on the parameter \(t_{00} \) with \(t_{-10} = 20\) and \(t_{01} = 1\). The eigenvalues are \(\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 9\) and \(A_1 = A_2 = A_3 = 1\).

Figure 3. Graphs for \((a_3, a_1, a_2)\) for \((1,2)\)-BTH (a) depending on parameters \(t_{-10} \) with \(t_{01} = 0\) and \(t_{00} = -0.4\), (b) depending on parameters \(t_{00} \) with \(t_{01} = 0\) and \(t_{-10} = 1\). Here \(\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 9, A_1 = A_2 = A_3 = 1\).

4.1. Moment polytope for the \((1,2)\)-BTH. From the last section, \((\tau_1, \tau_2)\) have following form

\[
\tau_1 = \sum_{k=1}^{3} C_k E_k, \\
\tau_2 = \sum_{i,j=1,i<j}^{3} \left( C_i S_j \lambda_{ij}^{\frac{1}{2}} - C_j S_i \lambda_{ji}^{\frac{1}{2}} \right) (\lambda_j - \lambda_i) E_i E_j.
\]

We can treat \((\tau_1, \tau_2)\) as one point of Flag manifold \(G/B\), where \(G := SL(3, R)\) and \(B\) is a Borel subgroup (upper triangular subgroup) containing the Cartan Lie subgroup of \(G\) (diagonal torus). Here we describe the moment polytope defined by the map \([11]\).

\[
\mu : G/B \to H^{*}, \quad (\tau_1, \tau_2) \mapsto M_{\tau_1} + M_{\tau_2},
\]

where \(H^{*}\) is the dual of the Cartan Lie subgroup of \(G\),

\[
M_{\tau_1} = \frac{C_1 E_1 L_1 + C_2 E_2 L_2 + C_3 E_3 L_3}{C_1 E_1 + C_2 E_2 + C_3 E_3},
\]

\[
M_{\tau_2} = \frac{\sum_{i,j=1,i<j}^{3} \left( C_i S_j \lambda_{ij}^{\frac{1}{2}} - C_j S_i \lambda_{ji}^{\frac{1}{2}} \right) (\lambda_j - \lambda_i) E_i E_j (L_i + L_j)}{\sum_{i,j=1,i<j}^{3} \left( C_i S_j \lambda_{ij}^{\frac{1}{2}} - C_j S_i \lambda_{ji}^{\frac{1}{2}} \right) (\lambda_j - \lambda_i) E_i E_j}.
\]
Here the weight vectors $L_i$’s are defined by

\[ L_1 := (1, 0), \quad L_2 := \frac{1}{2} (-1, \sqrt{3}), \quad L_3 := \frac{1}{2} (-1, -\sqrt{3}). \]

Fig. 4 illustrates the moment polytope (i.e. the graph of $M_{t_1} + M_{t_2}$) for our example. In the figure, the vertex $\frac{1}{2}(3, \sqrt{3})$ represents the highest weight $L_1 - L_3$ which is the starting point of this $(1, 2)$-BTH flow ($t_{0,1}$ flow, $t_{0,0}$ flow). The vertex $\frac{1}{2}(-3, -\sqrt{3})$ represents the lowest weight $-L_1 + L_3$ which is the destination of the flow. The boundaries of Fig. 4 correspond to the $G = Sl(2, R)$ for $(1, 2)$-BTH associated with two of $a_{2,1}, a_{3,1}$ and $a_{3,2}$ are zeroes. The six vertices in Fig. 4 are fixed points of Cartan subgroup of $G$ which can be generated from the highest weight by the action of the symmetric group $S_3$. These six vertices are in bijection with the elements of the Weyl group.

Till now, we find the moment polytope can tell how the flow of time variables of BTH goes well, meanwhile the sorting property of the flows and representation in Lie algebra describing the orbit are also seen from the Fig. 4. What about the polytope corresponding to higher rank matrix-formed BTH is an interesting question.

5. $(N, 1)$-Bigraded Toda Hierarchy

In last section, $(1, M)$-BTH are introduced a lot. In this section, we concentrate on the $(N, 1)$-BTH. For convenience of calculation, we firstly use interpolated form of the BTH.

In the following part, we will introduce some concrete primary flows of the $(N, 1)$-BTH.

$(2, 1)$-BTH: The Lax operator of $(2, 1)$-BTH is as following

\[ L_{2,1} = \Lambda^2 + u_1 \Lambda + u_0 + u_{-1} \Lambda^{-1}. \]

The equations in this case are as follows

\[ \partial_{2,0} L_{2,1} = [\Lambda + (1 + \Lambda)^{-1} u_1(x), L_{2,1}], \]
which further lead to the following concrete equations

\[
\begin{align*}
\partial_{x}u_1(x) &= u_1(x + \epsilon) - u_1(x) + u_1(x)(1 - \Lambda)(1 + \Lambda)^{-1}u_1(x), \\
\partial_{x}u_0(x) &= u_0(x + \epsilon) - u_0(x), \\
\partial_{x}u_{-1}(x) &= u_{-1}(x)(1 - \Lambda^{-1})(1 + \Lambda)^{-1}u_1(x), \\
\end{align*}
\]

(5.3)

which is equivalent to eq.(40) in [12] and also related to the system (10)-(12) proposed in [13].

\[(3,1)\text{-BTH:}\] The equations of \((3,1)\)-BTH is as following

\[
\begin{align*}
\partial_{x}u_2(x) &= u_1(x + \epsilon) - u_1(x) + u_2(x)(1 - \Lambda^2)(1 + \Lambda + \Lambda^2)^{-1}u_2(x), \\
\partial_{x}u_1(x) &= u_0(x + \epsilon) - u_0(x) + u_1(x)(1 - \Lambda)(1 + \Lambda + \Lambda^2)^{-1}u_2(x), \\
\partial_{x}u_0(x) &= u_{-1}(x + \epsilon) - u_{-1}(x), \\
\partial_{x}u_{-1}(x) &= u_{-1}(x)(1 - \Lambda^{-1})(1 + \Lambda + \Lambda^2)^{-1}u_2(x), \\
\end{align*}
\]

(5.4)

which can be rewritten as

\[
\begin{align*}
\partial_{x}(\sum_{i=0}^{2} v_2(x + i\epsilon)) &= v_1(x + \epsilon) - v_1(x) + (\sum_{i=0}^{2} v_2(x + i\epsilon))(1 - \Lambda^2)v_2(x), \\
\partial_{x}v_1(x) &= v_0(x + \epsilon) - v_0(x) + v_1(x)(1 - \Lambda)v_2(x), \\
\partial_{x}v_0(x) &= v_{-1}(x + \epsilon) - v_{-1}(x), \\
\partial_{x}v_{-1}(x) &= v_{-1}(x)(1 - \Lambda^{-1})v_2(x), \\
\end{align*}
\]

(5.5)

by transformation \(u_2 = (1 + \Lambda + \Lambda^2)v_2;\) \(u_i = v_i, i = 1, 0, -1.\) This is the case when \(n = 1, m = -2, l = 4\) for eq.(10) in [12] after treating \(v(x + i\epsilon)\) as \(v(i),\)

\[
\sum_{s=1}^{3} v_2(i + s - 1) = \sum_{s=1}^{3} v_2(i + s - 1) \times (v_2(i) - v_2(i + 2)) + v_1(i + 1) - v_1(i),
\]

(5.6)

To generalize the results above, \((N,1)\)-BTH will be considered as following.

\[(N,1)\text{-BTH:}\] The concrete equations of \((N,1)\)-BTH is as following

\[
\begin{align*}
\partial_{x}(\sum_{i=0}^{N-1} v_{N-1}(x + i\epsilon)) &= v_{N-2}(x + \epsilon) - v_{N-2}(x) + (\sum_{i=0}^{N-1} v_{N-1}(x + i\epsilon))(1 - \Lambda^{N-1})v_{N-1}(x), \\
\partial_{x}v_{N-2}(x) &= v_{N-3}(x + \epsilon) - v_{N-3}(x) + v_{N-2}(x)(1 - \Lambda^{N-2})v_{N-1}(x), \\
\vdots & \quad \vdots \\
\partial_{x}v_{1}(x) &= v_{0}(x + \epsilon) - v_{0}(x) + v_{1}(x)(1 - \Lambda)v_{2}(x), \\
\partial_{x}v_{0}(x) &= v_{-1}(x + \epsilon) - v_{-1}(x), \\
\partial_{x}v_{-1}(x) &= v_{-1}(x)(1 - \Lambda^{-1})v_{2}(x), \\
\end{align*}
\]

where \(u_{N-1} = (1 + \Lambda + \cdots + \Lambda^{N-1})v_{N-1};\) \(u_i = v_i, i = N - 2, \ldots, 0, -1.\)

Similarly we can rewrite the equations of \((N,1)\)-BTH as following

\[
\begin{align*}
\sum_{s=1}^{N} \partial_{x}v_{N-1}(i + s - 1) &= \sum_{s=1}^{N} v_{N-1}(i + s - 1) \times (v_{N-1}(i) - v_{N-1}(i + N - 1)) \\
+ & v_{N-2}(i + 1) - v_{N-2}(i), \\
\partial_{x}v_{k}(i) &= v_{k}(i) (v_{N-1}(i) - v_{N-1}(i + k)) + v_{k-1}(i + 1) - v_{k-1}(i), \quad k = N - 2, N - 3, \ldots, -1,
\end{align*}
\]

(5.7)

where \(v_{j}(i) := v_{j}(x + i\epsilon)\) and \(v_{-2} = 0.\)
Then consistency conditions of eq.(6.3) are expressed in a form of the Lax equation which exactly leads to eq.(5.7).

\[ L = G_i H_i = G_{i+1}^{-1} H_i G_i = L_{ij} G_i G_j + N_i N_j - L A_{N,0}, \]

where \( L_{ij} \) and \( A_{N,0} \) are difference operators acting on BA functions \( \psi_i \) as

\[ L(\psi) = z\psi_{i+1} + v_{i-1}(i+1) \psi_{i+1} \]

\[ A_{N,0}(\psi) = z\psi_{i+1} + v_{i-1}(i) \psi_i, \]

for the fields \( \{u_1(i, t_{N,0}), v_0(i, t_{N,0}), ..., v_{N-1}(i, t_{N,0})\} \).

Formally, consistency condition of (6.3) is given by

\[ G_{i+1} H_i = H_{i-N+1} G_i. \]

The technical observation will lead to that eq. (6.3) can be rewritten in terms of \( (L, A_{N,0}) \)-pair

\[ L(\psi_i) = z\psi_i, \quad \partial_{N,0} \psi_i = A_{N,0}(\psi_i) \]

where \( L \) and \( A_{N,0} \) are difference operators acting on BA functions \( \psi_i \) as

\[ L(\psi_i) = z\psi_{i+1} + v_{i-1}(i+1) \psi_{i+1} \]

\[ A_{N,0}(\psi_i) = z\psi_{i+1} + v_{i-1}(i) \psi_i. \]

That means

\[ L = G_{i+1}^{-1} H_i \partial_{N,0} + \left( \sum_{s=1}^{N} v_{N-1}(i + s - 1) \right) G_{i+1}^{-1} G_{i+1-N}^{-1} \psi_{i+1-N} + \sum_{j=1}^{N} \frac{1}{z^j} v_{N-1-j}(i) \psi_{i+1-N-1-j}, \]

\[ A_{N,0} = H_i + v_{N-1}(i). \]

Then consistency conditions of eq.(6.3) are expressed in a form of the Lax equation

\[ \partial_{N,0} L = [A_{N,0}, L] = A_{N,0} L - L A_{N,0}, \]

which exactly leads to eq.(5.7).

Till now we have given another Lax construction for primary flows of the \((N, 1)\)-BTH.
7. Lattice Miura transformation

As we all know, many one-field lattice equations are very useful in a lot of branches of science such as biology, medical science, physics and so on. In this section, we will introduce one kind of Miura mapping which connects one-field lattice equation with \((N+1)\)-field ones (i.e. \((N, 1)\)-BTH).

Define

\[
F_i = G_{i+N} H_{i+N-1} \ldots H_{i+1} H_i, \quad i \in \mathbb{Z}
\]

which is a \(N\)-order differential operator, we obtain

\[
F_i \psi_i = z^{N+1} \psi_{i+1}, \quad H_i \psi_i = z \psi_{i+1}, \quad i \in \mathbb{Z}.
\]

Because

\[
G_i \psi_i = z \psi_{i+1-N}, \quad H_i \psi_i = z \psi_{i+1}, \quad i \in \mathbb{Z},
\]

we define

\[
\bar{G}_i \bar{\psi}_i = z \bar{\psi}_{i+1}, \quad \bar{H}_i \bar{\psi}_i = z \bar{\psi}_{i+N+1}, \quad \bar{\psi}_i = \bar{\psi}_{(N+1)i},
\]

where

\[
\bar{H}_i = \partial_{N,0} - \sum_{k=1}^{N+1} r_{i+k-1}, \quad \bar{G}_i = \partial_{N,0} - r_i.
\]

The compatibility of system (7.2) leads to the following one-field lattice equations

\[
\sum_{s=1}^{N} \partial_{N,0} r_{i+s-1} = \sum_{s=1}^{N} r_{i+s-1} \times (r_{i+N} - r_{i-1}), \quad i \in \mathbb{Z}.
\]

Define

\[
\bar{F}_i = \bar{G}_{i+N} \bar{G}_{i+N-1} \ldots \bar{G}_{i+1} \bar{G}_i,
\]

which is a \(N\)-order differential operator. Now we consider a new system

\[
\bar{F}_i \bar{\psi}_i = z^{N+1} \bar{\psi}_{i+N+1}, \quad \bar{H}_i \bar{\psi}_i = z \bar{\psi}_{i+N+1}.
\]

Comparing system (7.2) with (7.1) will lead to following identification

\[
F_i = F_{(N+1)i}; \quad H_i = H_{(N+1)i};
\]

which will tell us the Miura transformation in detail.

Eq. (7.4) can be rewritten in the following equivalent form

\[
G_{i+N} H_{i+N-1} \ldots H_{i+1} H_i = G_{(N+1)i+N} G_{(N+1)i+N-1} \ldots G_{(N+1)i+1} G_{(N+1)i}.
\]

In the following, we will give two specific examples for original Toda hierarchy and \((2, 1)\)-BTH including their corresponding lattice Miura transformation and one field equations.

Example 7.1. For \((1, 1)\)-BTH, i.e. original Toda hierarchy, the case \(N = 1, l = 2, n = 1, m = 0, \bar{n} = 2, \bar{m} = 1\) in [12]

\[
\begin{align*}
\partial_{1,0} v_0(x) &= v_{-1}(x + \epsilon) - v_{-1}(x), \\
\partial_{1,0} v_{-1}(x) &= v_{-1}(x) (1 - \Lambda^{-1}) v_1(x),
\end{align*}
\]

(7.5)
which can be rewritten as following discrete form

\begin{align}
\partial_{v_0} v_0(i) &= v_{-1}(i+1) - v_{-1}(i), \\
\partial_{v_1} v_{-1}(i) &= v_{-1}(i)(v_1(i) - v_1(i-1)),
\end{align}

by transformation $v_j(i) := v_j(x + i\epsilon), j = 0, -1$. The lattice Miura transformation is as following

\begin{align}
v_1(i) &= r_{2i} + r_{2i+1}, \\
v_0(i) &= r_{2i} r_{2i}.
\end{align}

After lattice Miura transformation \( (7.7) \), the \( t_{1,0} \) flow of \((1,1)\)-BTH can be transformed into the following one-field equation, i.e. the Volterra lattice which is a very useful ecological competition model in biology,

\begin{equation}
\partial_{t_{1,0}} r_i = r_i(r_{i+1} - r_{i-1}), \quad i \in \mathbb{Z}.
\end{equation}

**Example 7.2.** For \((2,1)\)-BTH, \( N = 2, l = 3, n = 1, m = -1, \bar{n} = 3, \bar{m} = 1, \) the lattice Miura transformation is as following

\begin{align}
v_1(i) &= r_{3i+2} + r_{3i+1} + r_{3i}, \\
v_0(i) &= r_{3i-r_{3i} + r_{3i-r_{3i+1}} + r_{3i-r_{3i+2} + r_{3i+1}+3} + r_{3i+2r_{3i}+3}} + r_{3i+2r_{3i}+3} + r_{3i+2r_{3i}+3} + r_{3i+2r_{3i}+3} + r_{3i+2r_{3i}+3}, \nonumber \\
v_{-1}(i) &= -r_{3i-3r_{3i-2r_{3i-1}} + r_{3i-r_{3i-1}r_{3i-1}}} + (r_{3i-2} + r_{3i})(r_{3i+1} + r_{3i}).
\end{align}

After lattice Miura transformation \( (7.9) \), the \( t_{2,0} \) flow of \((2,1)\)-BTH can be transformed into the following one-field equation

\begin{equation}
\partial_{t_{2,0}} (r_i + r_{i+1}) = (r_i + r_{i+1})(r_{i+2} - r_{i-1}), \quad i \in \mathbb{Z}.
\end{equation}

The relation between \((N, 1)\)-BTH and one-field lattice equations give us some hints on how to get solutions of one-field lattice equations from the solutions of BTH (known in \([8]\)) by certain transformation.

8. Conclusions and discussions

We give finite dimensional exponential solutions of the bigraded Toda Hierarchy(BTH). As a specific example of exponential solutions of the BTH, we consider a regular solution for the \((1,2)\)-BTH with \(3 \times 3\) Lax matrix. The difference between \((1,2)\)-BTH and original Toda hierarchy is found from a geometric viewpoint by diagonal projection and moment map. Our future work contains finding other regular solutions corresponding to other cases of BTH and their geometric description. After that, we construct another Lax representation of bigraded Toda hierarchy(BTH) and introduce lattice Miura transformation of BTH. These Miura transformations give a good connection between primary equation of \((N, 1)\)-BTH and one-field lattice equations which include Volterra lattice equation. What kinds of one-field lattice equations will correspond to the whole hierarchies in BTH is an interesting questions.

**Acknowledgments:** This work was partly carried out under the suggestion of Professor Yuji Kodama. Chuanzhong Li would like to thank Professor Yuji Kodama for his suggestion and many useful discussions.
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