The quadratic complete intersections with the action of the symmetric group

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Abstract

We prove that any quadratic complete intersection with certain action of the symmetric group has the strong Lefschetz property over a field of characteristic zero. As a consequence of it we construct a new class of homogeneous complete intersections with generators of higher degrees which have the strong Lefschetz property.

1 Introduction

It seems natural to conjecture that all (Artinian) complete intersections with standard grading have the strong Lefschetz property over a field of characteristic zero. If there is a group action on a complete intersection, it sometimes enables us to prove that the ring has the property (see [3] §4). Consider the monomial complete intersection:

\[ A = K[x_1, x_2, \cdots, x_r]/(x_1^{n_1+1}, x_2^{n_2+1}, \cdots, x_r^{n_r+1}). \]

In spite of the simple nature of the assertion of the strong Lefschetz property, the proof for it is complicated. If \( d_1 = d_2 = \cdots = d_r = 2 \), however, Ikeda’s Lemma provides an easy proof. It seems remarkable that any monomial complete intersection appears as a subring of the quadratic monomial complete intersection. In fact the algebra \( A \) above is the invariant subring of the quadratic complete intersection \( K[x_1, x_2, \cdots, x_n]/(x_1^2, x_2^2, \cdots, x_n^2) \) under the group action of the Young subgroup

\[ S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r} \subset S_n \]

where

\[ n = n_1 + n_2 + \cdots + n_r. \]

Once we know the strong Lefschetz property in the quadratic case, the general case follows almost immediately.

The purpose of this paper is to generalize this argument. First we construct a flat family of quadratic complete intersections, with four parameters, on which the Young subgroup acts in the same way as it does on the quadratic monomial complete intersection. It will be proved that any member in this family has the strong Lefschetz property. It is crucial
to assume that the generators are quadrics and the symmetric group acts on it. Then we prove that the ring of invariants of any complete intersection in this family by the action of any Young subgroup in $S_n$ is again a complete intersection with the strong Lefschetz property. Proof for it is easy, but the invariant subrings basically do not have a standard grading. Rather surprisingly, however, it turns out that most of them have the standard grading thanks to the assumption that generators are quadrics.

The main results of this paper are Theorem 8 and Theorem 9 which are stated in § 2 and § 3 respectively. A theorem of Goto says that the ring of invariants of a complete intersection is again a complete intersection if the group is generated by pseudo-reflections. We need to construct a set of uniform generators for all invariant subrings in the family. This is treated in Appendix.

2 Definitions

Definition 1. Let $V = \bigoplus_{i=0}^{\infty} V_i$ be a finite graded vector space and let $L \in \text{End}_{\text{gr}}(V)$ be a graded endomorphism

$$L : V \to V$$

of degree one. Namely a graded endomorphism of degree one is a collection of homomorphisms $\{L_i : V_i \to V_{i+1}\}$. We call $L$ a weak Lefschetz element if the map $L$ has piece-wise full rank, i.e., the restricted map $L_i : V_i \to V_{i+1}$ is either injective or surjective for all $i = 0, 1, 2, \cdots$. We will write $L_i : V_i \to V_{i+1}$ simply as $L : V_i \to V_{i+1}$. We say that $L$ is a strong Lefschetz element if there exists an integer $c$ such that $V_i = 0$ for all $i \geq c + 1$ and the map $L^{c-2i}$ restricted to the homogeneous part $L_{c-2i} : V_i \to V_{c-i}$ is bijective for all $i = 0, 1, 2, \cdots, [c/2]$. The map $i \mapsto \dim_K V_i$ is called the Hilbert function of $V$. Sometimes it is denoted as the power series $\sum_{i=0}^{\infty} (\dim_K V_i)T^i$. Since $V$ is a finite dimensional vector space, the Hilbert series of $V$ is actually a polynomial in $T$. If a graded homomorphism of $L \in \text{End}_{\text{gr}}(V)$ is a strong Lefschetz element, it automatically implies that the Hilbert function of $V$ is symmetric about the half integer $c/2$ for some $c$.

Lemma 2. Suppose that $V = \bigoplus_i V_i$ is a finite dimensional graded vector space and that $V$ has a constant Hilbert function. Suppose that $L \in \text{End}_{\text{gr}}(V)$ is a graded endomorphism of degree one. If $L$ is a weak Lefschetz element, then $L$ is a strong Lefschetz element.

Proof. Let $a$ be the initial and $b$ the end degrees of $V = \bigoplus_i V_i$, and let $c = a + b$. Then obviously the map $L^{c-2i} : V_i \to V_{c-i}$ is a bijection for any $i \leq [c/2]$.

Definition 3. Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded (not necessarily standard graded) Artinian $K$-algebra over a field $A_0 = K$. We say that $A$ has the weak (resp. strong) Lefschetz property, if there exists a linear form $l \in A_1$ such that the multiplication map $L = x l \in \text{End}_{\text{gr}}(A)$ is a weak (resp. strong) Lefschetz element. Such a linear form $l$ is called a weak (resp. strong) Lefschetz element. Sometimes we use the abbreviation: WLP (resp. SLP) for weak (resp. strong) Lefschetz property.

Definition 4. Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian $K$-algebra over a field $A_0 = K$. The Sperner number of $A$ is defined as

$$\text{Sperner } A = \text{Max}_i \{\dim_K A_i\}.$$ 

Proposition 5 (Subring Theorem). Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian $K$-algebra with the strong Lefschetz property. Assume that $A_c \neq 0$. Suppose that $B$ is a graded $K$-subalgebra of $A$. Assume that $B_c = A_c$ and $B_1$ contains a strong Lefschetz element for $A$. Then if $B$ has a symmetric Hilbert function, $B$ has the strong Lefschetz property.
Proof. Let $l \in B_1$ be a strong Lefschetz element for $A$. Consider the diagram:

$$
\begin{array}{ccc}
A_i & \rightarrow & A_{c-i} \\
\uparrow & & \uparrow \\
B_i & \rightarrow & B_{c-i},
\end{array}
$$

where the vertical arrows are natural injections and horizontal arrows are the multiplication map by $l^{c-2i}$. The strong Lefschetz property of $A$ implies that $x\cdot l^{c-2i} : B_i \rightarrow B_{c-i}$ is injective. Since $\dim_K B_i = \dim_K B_{c-i}$, it is bijective. \qed

3 The polynomial ring with the action of the symmetric group

Let $R = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring over $K$, a field of characteristic zero, and let $S_n$ be the symmetric group. The homogeneous part of $R$ of degree $d$ is denoted by $R_d$.

We let the symmetric group $S_n$ act on $R$ by permutation of the variables. An element $\sigma \in S_n$ is a bijection of the set $\{1, 2, \ldots, n\}$. Thus $\sigma$ induces the automorphism of the $K$-algebra $R$ by

$$
f^\sigma(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).
$$

We recall some basic facts on the representation of $S_n$ and its action on $R$ and fix some notation.

FACT 6. 1. The irreducible representations are parametrized by the Young diagrams of $n$ boxes. A Young diagram of $n$ boxes is denoted by a partition $\lambda \vdash n$, which is a non-decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\sum \lambda_i = n$.

2. We will denote by $V^\lambda$ the irreducible module (uniquely determined up to isomorphism). The dimension of $V^\lambda$ is determined by the hook length formula. (See e.g., \cite{5}, \cite{6}.)

Mostly we are interested in partitions of $n$ with at most two rows. Such partitions will be denoted as

$$
(n, 0), (n - 1, 1), \cdots, (n - [n/2], [n/2]).
$$

Note that $(n, 0)$ denotes the partition with one row.

3. Let $U$ be a finite dimensional $S_n$-module. The vector space $U$ decomposes as $U = \bigoplus_{\lambda \vdash n} U_{\lambda}$, where $U_{\lambda}$ is a sum of copies of $V^\lambda$. The number of times the irreducible module $V^\lambda$ occurs in $U$ is the multiplicity of $V^\lambda$. Such a decomposition of $U$ is unique.

In other words if $U = \bigoplus_{\lambda \vdash n} U_{\lambda} = \bigoplus_{\lambda \vdash n} U'_{\lambda}$, then $U_{\lambda} = U'_{\lambda}$ (as vector subspaces of $U$) for all $\lambda \vdash n$. Such a decomposition is called the isotypic decomposition of $U$.

4. We denote by $Y^\lambda$ the Young symmetrizer corresponding to $\lambda \vdash n$. For the meaning of Young symmetrizers we refer the reader to \cite{5} or \cite{6}. In the sequel all we have to know about $Y^\lambda$ is that it gives the projection

$$
Y^\lambda : U \rightarrow U_{\lambda},
$$

for any $S_n$-module $U$, where $U = \bigoplus_{\lambda} U_{\lambda}$ is the isotypic decomposition. For example, if $\lambda = (n, 0)$, then $Y^\lambda(R)$ coincides with the ring $R^{S_n}$ of invariants of $R$ under the action of $S_n$. It is well known that $R^{S_n}$ is, as a $K$-algebra, generated by the elementary symmetric polynomials. The elementary symmetric polynomial of degree $d$ will be denoted by $e_d$. Thus we have

$$
Y^{(n,0)}(R) = R^{S_n} = K[e_1, e_2, \cdots, e_n].
$$
5. The degree one part $R_1$ of $R$ decomposes, as an $S_n$-module, as
\[ R_1 \cong V^{(n,0)} \oplus V^{(n-1,1)}. \]

Typical bases for these modules are:
\[ \langle x_1 + x_2 + \cdots + x_n \rangle \text{ for } V^{(n,0)} \]
\[ \langle x_1 - x_2, x_1 - x_3, \cdots, x_1 - x_n \rangle \text{ for } V^{(n-1,1)} \]

6. The degree two part $R_2$ of $R$ decomposes, as an $S_n$-module, as
\[ R_2 \cong V^{(n,0)} \oplus V^{(n,0)} \oplus V^{(n-1,1)} \oplus V^{(n-1,1)} \oplus V^{(n-2,2)}. \]

For $V^{(n,0)}$ we can choose $\langle e_1 \rangle$ and $\langle e_2 \rangle$ as bases.
For $V^{(n-1,1)}$ we can choose $\langle x_1^2 - x_2^2, x_1^3 - x_2^3, \cdots, x_1^n - x_2^n \rangle$ and $\langle (x_1 - x_2)e_1, (x_1 - x_3)e_1, \cdots, (x_1 - x_n)e_1 \rangle$ as bases.
Finally a typical basis for $V^{(n-2,2)}$ is the set of Specht polynomials of shape $(n-2,2)$:
\[ \{(x_1 - x_j)(x_2 - x_k) \mid 3 \leq j < k \leq n\} \cup \{(x_1 - x_2)(x_3 - x_k) \mid 4 \leq k \leq n\} \]
For the definition of Specht polynomials see [3] §9.3.

7. By the hook length formula we have
\[ \dim V^{(n-i,j)} = \binom{n}{i} - \binom{n}{i-1}. \]
In particular,
\[ \dim V^{(n-2,2)} = \binom{n}{2} - \binom{n}{1} = \frac{n(n-3)}{2}, \]
and
\[ \dim R_2 = \begin{cases} 
2 \dim V^{(n,0)} + 2 \dim V^{(n-1,1)} + \dim V^{(n-2,2)}, & \text{if } n > 3, \\
2 \dim V^{(n,0)} + 2 \dim V^{(n-1,1)}, & \text{if } n = 3, \\
2 \dim V^{(n,0)} + \dim V^{(n-1,1)}, & \text{if } n = 2.
\end{cases} \]

**Lemma 7.** With the same notation as above, suppose that $U \subset R_1 e_1$ is a one dimensional $S_n$ module. If $n \geq 3$, then $U$ is spanned by $e_1^2$. If $n = 2$, then $U$ is spanned either by $e_1^2 = e_1(x_1 + x_2)$ or $e_1(x_1 - x_2)$.

**Proof.** Note that $U \cong R_1$ as an $S_n$-module. Since $R_1 = Y^{(n,0)}(R_1) \oplus Y^{(n-1,1)}(R_1)$ is the isotypic decomposition, and $\dim Y^{(n-1,1)}(R_1) = 1$ if $n = 2$ and $\dim Y^{(n-1,1)}(R_1) > 1$ if $n \geq 3$. Thus the assertion follows. \[ \square \]

4 **Main result**

As in the previous section $R = K[x_1, x_2, \cdots, x_n]$ denotes the polynomial ring over a field $K$. We are interested in the sequences of quadrics $f_i \in R$ which satisfy the following conditions:
(1) $f_1, f_2, \cdots, f_n$ form a regular sequence.
(2) For any $\sigma \in S_n$,
\[ f_i(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)}) = f_{\sigma(i)}(x_1, x_2, \cdots, x_n), \text{ for } i = 1, 2, \cdots, n. \]
Remark 8. If \((f_1, f_2, \cdots, f_n)\) is a sequence of quadrics in \(R\) with only the second property above, the stabilizer of \(f_1\) should be the subgroup \(S_{n-1}\) of \(S_n\) which fixes 1. Hence the element \(f_1\) has the from
\[
p_0x_1^2 + p_1(x_2 + x_3 + \cdots + x_n)x_1 + p_2(x_2^2 + x_3^2 + \cdots + x_n^2) + p_3(\sum_{2 \leq i < j \leq n} x_ix_j),
\]
with four parameters \(p_k\) and the elements \(f_i\) are obtained by cyclically permuting the variables. For such a sequence to be a regular sequence we have to impose the condition that the resultant does not vanish. For details of the resultant of homogeneous forms see [4].

Theorem 9. Assume that the characteristic of \(K\) is zero. Let \(I = (f_1, f_2, \cdots, f_n)\) be a complete intersection ideal in \(R\) which satisfies the conditions (1) and (2) above. Then \(A := R/I\) has the strong Lefschetz property. Let \(e_1 = \sum x_i\). If \(e_1^2 \not\in I\), then \(e_1\) is a strong Lefschetz element for \(A\).

Proof. If \(e_1^2 \in I\), we can choose \(e_1^2\) as a generator of the ideal \(I\). Let \(B = K[z]/(z^2)\), with a new variable \(z\), and define the map \(B \to A\) by \(z \mapsto e_1\). It is easy to see that \(B \to A\) is a flat extension and the fiber, say \(C\), is the algebra
\[
C = K[x_1, x_2, \cdots, x_n]/(e_1, f_1, f_2, \cdots, f_n).
\]
For \(i \geq 2\), let \(f'_i\) be the polynomial obtained from \(f_i\) by the substitution
\[
x_1 \mapsto -(x_2 + x_3 + \cdots + x_n).
\]
It is easy to see that
\[
C \cong K[x_2, x_3, \cdots, x_n]/(f'_2, f'_3, \cdots, f'_n)
\]
so if \(\sigma \in S_n\) fixes 1, then \((f'_i)^\sigma = f'_i\). Hence we may induct on \(n\) to conclude that the fiber has the SLP. By Flat Extension Theorem ([3] Theorem 4.10), the ring \(A\) has the strong Lefschetz property.

For the rest of proof we assume that \(I\) does not contain \(e_1^2\). We want to show that \((I \cap e_1R) \cap R_2 = 0\) if \(n \geq 3\). Let \(h \in (I \cap e_1R) \cap R_2\). Since \(I\) is generated by a regular sequence, any two linearly independent elements in \(I \cap R_2\) are not contained in a principal ideal. This implies that \((I \cap e_1R) \cap R_2\) is at most one dimensional. If \(\sigma \in S_n\), it forces \(h^2R = hR\). In other words \(h\) is a semi-invariant. By Lemma 7, the element \(h\) is a scalar multiple of \(e_1^2\) if \(n \geq 3\). Since we have assumed that \(I\) does not contain \(e_1^2\), we have \(h = 0\). If \(n = 2\), \(h\) could be \(x_1^2 - x_2^2\), but in any case \(A\) has the SLP as it is easily checked. From now on we assume that \(n \geq 3\). Then the sum \(R_1e_1 + (I \cap R_2)\) is a direct sum and it contains two copies of \(V^{(n,0)}\) and two copies of \(V^{(n-1,1)}\), since both of \(R_1e_1\) and \((I \cap R_2)\) are equivalent to \(V^{(n,0)} \oplus V^{(n-1,1)}\). On the other hand by FACT 6 (5 and 6), \(R_1e_1 + (x_1^2, \cdots, x_n^2)\) also contains two \(V^{(n,0)}\) and two \(V^{(n-1,1)}\). By FACT 6 (3), we see that
\[
R_1e_1 + (I \cap R_2) = R_1e_1 + (x_1^2, \cdots, x_n^2),
\]
and \(e_1 R + I\) contains all the second power of the variables.

Put \(B = R/(x_1^2, x_2^2, \cdots, x_n^2)\). Then what we have proved implies that there is a surjection \(B/e_1B \to A/e_1A\). Generally speaking it holds that \(\dim_K A/e_1A \geq \text{Sperner} A\). (See the proof of [3] Proposition 3.5.) Thus we have
\[
\dim_K B/e_1B \geq \dim_K A/e_1A \geq \text{Sperner} A = \text{Sperner} B,
\]
and since \(B\) has the weak Lefschetz property ([3] Corollary 3.69), this implies that \(A\) has the weak Lefschetz property and furthermore,
\[
I + e_1R = (x_1^2, x_2^2, \cdots, x_n^2, e_1).
\]
We have to prove that $A$ has the strong Lefschetz property with $e_1$ as an SL element. Let $J = (x_1^2, x_2^2, \ldots, x_n^2)$. Put $\lambda_i = (n - i, i)$ for $i = 0, 1, \ldots, [n/2]$. Since the way $S_n$ acts on $R/J$ and $R/I$ are the same, $A$ and $B$ have the same irreducible decomposition as $S_n$-modules. So we may apply [3] Theorem 9.9 to $A$. Since the $S_n$-module $A$ decomposes as

$$A = \bigoplus_{i=0}^{[n/2]} Y^{\lambda_i}(A)$$

and since the multiplication map $\times e_1 : A \to A$ decomposes as the sum of the restricted maps

$$\times e_1 : Y^{\lambda_i}(A) \to Y^{\lambda_i}(A), \ i = 0, 1, \ldots, [n/2],$$

it suffices to prove that $\times e_1 : Y^{\lambda_i}(A) \to Y^{\lambda_i}(A)$ has the strong Lefschetz property for each $i$. Recall that $Y^{\lambda_i}(A)$ has the constant Hilbert function ([3] Lemma 9.8)

$$\left(\dim V^{(n-i,i)} (T^i + T^{i+1} + \cdots + T^{n-i})\right).$$

Thus $Y^{\lambda_i}(A)$ has the SLP by Proposition 2. \hfill \Box

5 Some consequences

Recall that a grading of a graded algebra $A = \bigoplus_{i \geq 0} A_i$ is standard if the algebra $A$ is generated by elements of degree one over $A_0$. So far we have tacitly assumed that the grading for the algebras $R$ and $A$ are standard. In this section we consider graded subalgebras which are not necessarily standard graded. We continue to assume that the polynomial ring $R$ has the standard grading, i.e., the degrees of the variables are one, but the invariant subrings $R^G$ and $A^G$ most likely do not have the standard grading. We are primarily concerned with the cases where the invariant subrings $A^G$ do have the standard grading.

**Theorem 10.** Let $A = K[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_n)$ be the quadratic complete intersection with the action of $S_n$ as in Theorem 9. As in Remark 8 we use the notation

$$f_1 = p_0 x_1^2 + p_1(x_2 + x_3 + \cdots + x_n)x_1 + p_2(x_2^2 + x_3^2 + \cdots + x_n^2) + p_3 \left( \sum_{2 \leq i < j \leq n} x_i x_j \right).$$

Let $X = \{x_1, x_2, \cdots, x_n\}$ be the set of variables and let $X = \bigcup_{i=1}^{r} X_i$ be a partition of the set of variables into $r$ nonempty subsets. Put $n_i = |X_i|$ and let

$$G = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r}$$

be the Young subgroup of $S_n$ which acts on $R$ in such a way that $S_{n_k}$ permutes the variables in the block $X_k$ and leaves fixed the variables in other blocks. Then $R^G/(I \cap R^G)$ is a complete intersection with the strong Lefschetz property. Let $S = K[y_1, y_2, \cdots, y_r]$ be the polynomial ring in $r$ variables and let

$$\phi : S \to A$$

be the homomorphism defined by $\phi(y_i) = \sum_{x \in X_i} x$. Then the image $\phi(S)$ coincides with $A^G$ for any $(p_0, p_1, p_2, p_3)$ in a nonempty open set in the projective space $\mathbb{P}^3 = \{(p_0, p_1, p_2, p_3)\}$. In particular $A^G$ has the standard grading if parameters are general enough.

**Proof.** The Young subgroup $G \subset S_n$ is generated by reflections. By a theorem of Goto [1], the ring of invariants $A^G = (R/I)^G = R^G/(I \cap R^G)$ is a complete intersection. Note that $e_1 \in A^G$ and $e_1$ is an SL element for $A$. Moreover the image of the Jacobian determinant
\[ \frac{\partial f_i}{\partial x_j} \] in \( A \) can be a socle generator for both \( A \) and \( A^G \). By Proposition 5 the ring \( A^G \) has the strong Lefschetz property if \( e_1 \in A \) is a strong Lefschetz element. If \( e_1 \) is not a strong Lefschetz element, then the image of \( e_1 \) in \( A^G \) satisfies \( e_1^2 = 0 \) by Theorem 9, and as in the proof of Theorem 9 we can use Flat Extension Theorem and the induction hypothesis to conclude that \( A^G \) has the SLP. This is a proof for the first assertion of this theorem.

For the second assertion we prove Lemma 11 first.

**Lemma 11.** Let \( Q \) be a Noetherian integral domain. Suppose that \( R = \bigoplus_{i \geq 0} R_i \) is a graded \( Q \)-algebra, finitely generated over \( R_0 = Q \). (We assume that the graded pieces \( R_i \) are free \( Q \)-modules.) If for some maximal ideal \( m_0 \subset Q \), the fiber \( R/m_0 R := R \otimes_Q Q/m_0 \) has a standard grading, then there exists an ideal \( \alpha \neq 0 \) in \( Q \) such that \( R/\alpha R \) has the standard grading for all prime ideals \( \beta \not\supset \alpha \).

Proof. Let \( Y = \{Y_1, Y_2, \cdots, Y_r\} \) be a set of homogeneous elements in \( R \) such that \( Y \) generates the algebra: \( R = Q[Y_1, Y_2, \cdots, Y_r] \). Let \( M', M'' \) be the \( Q \)-submodules of \( R \) as follows:

\[ M' = \sum_{i_1 + i_2 + \cdots + i_r \geq 1} QY_1^{i_1}Y_2^{i_2} \cdots Y_r^{i_r}, \quad \text{and} \quad M'' = \sum_{i_1 + i_2 + \cdots + i_r \geq 2} QY_1^{i_1}Y_2^{i_2} \cdots Y_r^{i_r}. \]

Furthermore put \( M = M'/M'' \). Note that \( M \) is a graded \( Q \)-module and \( M \otimes_Q Q/m \) is the tangent space of \( R \otimes_Q Q/m \) for any maximal ideal \( m \subset Q \). It is easy to see that the fiber \( R \otimes_Q Q/m \) has the standard grading if and only if \( M \otimes_Q Q/m \) is spanned by the homogeneous elements of degree one. Let \( N \) be the \( Q \)-submodule of \( M \) generated by the degree one elements and put \( \alpha = \text{Ann}_Q(M/N) \). Since there exists at least one maximal ideal such that \( M \otimes_Q Q/m \) has the standard grading, we have \( \alpha \neq 0 \). Then it is straightforward that \( \alpha \) has the desired property.

**Proof of the second part of Theorem 10.** Define the polynomials \( F \) and \( F_i \) by

\[ F = P_0x_1^2 + P_1(\sum_{j=2}^n x_j)x_1 + P_2(\sum_{j=2}^n x_j^2) + P_3(\sum_{2 \leq k < l \leq n} x_kx_l), \]

and \( F_i = F^{\sigma^{-1}} , i = 1, 2, \cdots, n \), where \( P_0, \cdots, P_3 \) are indeterminates and \( \sigma = (12 \cdots n) \) is the cycle of length \( n \). These are considered as polynomials in the variables \( x_1, x_2, \cdots, x_n \) with coefficients in \( K[P_0, P_1, P_2, P_3] \). Let \( R \) be the resultant of \( F_1, F_2, \cdots, F_n \). Put \( Q = K[P_0, P_1, P_2, P_3, R^{-1}] \) and \( R = Q[x_1, \cdots, x_n]/(F_1, \cdots, F_n) \). In the decomposition of the variables \( X = \bigcup_{i=1}^r X_i \) we may assume that the blocks \( X_i \) consist of variables of consecutive indices. So we assume that the \( i \)-th block \( X_i \) is

\[ X_i = \{x_{n_1 + \cdots + n_{i-1} + 1}, \cdots, x_{n_1 + \cdots + n_i}\}. \]

The numbering of the variables may be illustrated as follows:

\[ x_1, \cdots, x_{n_1}, x_{n_1+1}, \cdots, x_{n_1+n_2}, x_{n_1+n_2+1}, \cdots, x_{n_1+n_2+n_3}, \cdots, x_{n_1+\cdots+n_{r-1}+1}, \cdots, x_n. \]

For the sake of notation we rename the variables as:

\[ x_{ij} = x_{n_1+n_2+\cdots+n_{i-1}+j}, \]

so \( x_{ij} \) is the \( j \)-th variable in the \( i \)-th block. Suppose that \( X_0 \) is a set of variables. Then by \( e_d(X_0) \) we denoted the elementary symmetric polynomial of degree \( d \) in the variables in \( X_0 \).

(If \( d > |X|, e_d(X) \) is not defined.)

Introduce a set of new variables \( Y_{ij} \) of degree one which are indexed as follows:

1. The first index \( i \) ranges \( i = 1, 2, \cdots, r \).
2. The second index \( j \) ranges, depending on \( i, j = 1, 2, \ldots, n_i \).

Define the polynomials \( \{ E_{ij} \} \) with the same indices as the variables \( \{ Y_{ij} \} \) as follows:

\[
E_{id} = e_d(\{Y_1, Y_2, \ldots, Y_{ni}\}) \quad \text{for} \quad i = 1, 2, \ldots, r, \ d = 1, 2, \ldots, n_i.
\]

Define the algebras \( \Lambda \) and \( \Lambda' \) by

\[
\Lambda = Q[\{Y_{ij}\}]/(F_1, \ldots, F_n),
\]

\[
\Lambda' = Q[\{E_{ij}\}]/((F_1, \ldots, F_n) \cap Q[\{E_{ij}\}]).
\]

Note that \( \Lambda \) is mapped onto \( A \) by the specialization \( P_k \mapsto P_k, Y_{ij} \mapsto x_{ij} \). By Corollary 17 in Appendix, it is possible to write

\[
(F_1, \ldots, F_n) \cap Q[\{E_{ij}\}] = (F'_1, F'_2, \ldots, F'_n).
\]

(Note that \( F'_1, \ldots, F'_n \) are constructed from \( F_1, \ldots, F_n \) explicitly.) The algebra \( \Lambda' \) is a flat extension of \( Q = K[\{P_0, P_1, P_2, P_3, R^{-1}\}] \) and each fiber coincides naturally with \( A^G \) under the map \( Y_{ij} \mapsto x_{ij} \).

On the other hand the image of \( \phi : S \to A \) is a subring of \( A^G \) and they coincide if and only if \( A^G \) has the standard grading or equivalently \( A^G \) is generated by degree one elements. For \( P_0 = 1, P_1 = P_2 = P_3 = 0 \), it is easy to see that the fiber has the standard grading (cf. [3] Lemma 3.70). Thus we may apply Lemma 11. This completes the proof. \( \Box \)

**Remark 12.** Let \( \Lambda \) be the algebra defined in the last paragraph of the proof of the second part of Theorem 10. The fiber for \( p_0 = 1, p_1 = p_2 = p_3 = 0 \) is isomorphic to the monomial complete intersection

\[
K[y_1, y_2, \ldots, y_r]/(y_1^{n_1+1}, y_2^{n_2+1}, \ldots, y_r^{n_r+1}).
\]

This is proved if \( r = 2 \) in [3] Lemma 3.70. The same proof in fact works for all \( r \).

**Example 13.** Let \( p, x, y \) be variables and consider \( R = K[p, x, y]/(y^2, x^3 - py) \). We regard it as a family of Artinian algebras. Give the variables \( p, x, y \) degrees 0, 1, 3 respectively. For any \( p \in K \), the Hilbert function of \( R \) is

\[
\frac{(1 - T^3)(1 - T^6)}{(1 - T)(1 - T^3)} = \frac{(1 - T^3)(1 + T^3)}{(1 - T)} = \frac{1 - T^6}{1 - T} = 1 + T + \cdots + T^6.
\]

If \( p = 0 \), the fiber is \( K[x, y]/(x^3, y^2) \) and if \( p \neq 0 \), then it is \( K[x]/(x^6) \). The embedding dimension depends on the choice of \( p \).

**Example 14.** Consider the family of the Artinian algebras

\[
K[p_0, \ldots, p_3][v, w, x, y, z]/(f_1, f_2, f_3, f_4, f_5),
\]

on which \( S_5 \) acts by the permutation of the variables \( \{v, w, x, y, z\} \) and \( f_\sigma = f_{\sigma(i)} \) for \( \sigma \in S_5 \). We will use the same notation as the first paragraph of Section 4, and Remark 8 with \( n = 5 \), so

\[
f_1 = p_0 v^2 + p_1 (w + x + y + z) v + p_2 (w^2 + x^2 + y^2 + z^2) + p_3 (w x + w y + w z + x y + x z + y z).
\]

Other generators \( f_2, \ldots, f_5 \) are obtained by permuting the variables. Consider the Young subgroup

\[
G := S_2 \times S_3,
\]
which acts on $A$ with the division of the variables:

$$\{v, w, x, y, z\} = \{v, w\} \sqcup \{x, y, z\}.$$  

Then generically the ring $A^G$ of invariants has the Hilbert function

$$(1 2 3 3 2 1).$$

But for $p_0 = 5, p_1 = 2, p_2 = 0, p_3 = 2$, the algebra $A^G$ has the Hilbert function

$$(1 2 2 2 2 1).$$

Some more such examples are:

$$(p_0, p_1, p_2, p_3) = (0, 0, 3, 8)$$
$$(p_0, p_1, p_2, p_3) = (7, 7, 3, 8)$$
$$(p_0, p_1, p_2, p_3) = (4, 3, 2, 6)$$
$$(p_0, p_1, p_2, p_3) = (6, 0, 0, 4)$$
$$(p_0, p_1, p_2, p_3) = (6, 3, 0, 2)$$
$$(p_0, p_1, p_2, p_3) = (1, 1, 3, 8)$$

If $A = R/I$ is not a quadratic complete intersection, then it is the general case that the ring of invariants of $A$ do not have the standard grading. In the next example we exhibit such a case.

**Example 15.** Let $R = \mathbb{Q}[x_1, x_2, \cdots, x_6], I = f_1, \cdots, f_6$, where $f_i = x_i^3$ for $i = 1, \cdots, 6$. Let $X_1 = \{x_1, x_2, x_3\}$ and $X_2 = \{x_4, x_5, x_6\}$ and let $G = S_3 \times S_3$ act on $A = R/I$ by permuting the variables within the blocks $X_1$ and $X_2$ in the way as described in Theorem 10. Then the ring of invariants $A^G$ is, as a $K$-algebra, generated by the six elements of degrees $\{1, 2, 3, 1, 2, 3\}$ as follows: $r = x_1 + x_2 + x_3$, $s = x_1x_2 + x_1x_3 + x_2x_3$, $t = x_1x_2x_3$, $u = x_4 + x_5 + x_6$, $v = x_4x_5 + x_4x_6 + x_5x_6$ and $w = x_4x_5x_6$. They have the relations

1. $u^3 - 3uw + 3w$
2. $r^3 - 3rs + 3t$
3. $u^2v - 2v^2 - uw$
4. $r^2s - 2s^2 - rt$
5. $u^2w - 2uv$
6. $r^2t - 2st$

Note that the ring $A^G$ has in fact embedding dimension 4 and $t$ and $w$ can be eliminated but the grading is not standard. The Hilbert function is

$$1 + 2T + 5T^2 + 8T^3 + 12T^4 + 14T^5 + 16T^6 + 14T^7 + 12T^8 + 8T^9 + 5T^{10} + 2T^{11} + T^{12} = ((1 + T^2)(1 + T + T^2 + T^3 + T^4))^2.$$
6 Appendix

Proposition 16. Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over a field $K$ of characteristic zero, on which the symmetric group $S_n$ acts by permuting the variables. Let $f_1, \ldots, f_n \in R$ be a set of homogeneous elements which satisfies $f_i^\sigma = f_{\sigma(i)}$ for any $\sigma \in S_n$ for $i = 1, \ldots, n$. Define the polynomials $g_1, g_2, \ldots, g_n$ by

$$
\begin{pmatrix}
g_1 \\
g_2 \\
\vdots \\
g_n
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
x_1^2 & x_2^2 & \cdots & x_n^2 \\
x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2 \\
\vdots \\
f_n
\end{pmatrix}.
$$

Then the ideal $(f_1, \ldots, f_n)$ is a complete intersection if and only if $(g_1, \ldots, g_n)$ is a complete intersection.

Proof. The “if” part is obvious. Put $I = (f_1, \ldots, f_n)$ and assume that $I$ is an ideal of finite colength, i.e., $I$ is a complete intersection. We want to prove that $g_1, \ldots, g_n$generate an ideal of finite colength. For simplicity we put $G = S_n$. It is possible to construct a minimal free resolution of $(f_1, \ldots, f_n)$ which is compatible with the action of $G$. For this the Koszul complex is enough:

$$
0 \rightarrow \bigwedge^n F \rightarrow \cdots \rightarrow \bigwedge^2 F \rightarrow \bigwedge^1 F \rightarrow \bigwedge^0 F.
$$

We may think $F$ is the free module generated by $dx_1, dx_2, \ldots, dx_n$, and then extend the action of $G$ to $\bigwedge^k F$ (for any $k$) in the obvious manner. If $k = 1$, it is easy to determine a minimal set of generators for $F^G$ as an $R^G$-module. This can be done as follows. First we note that $F^G$ is a free $R^G$-module of rank $n$. For any $d$, the ideal $(x_1^d, x_2^d, \ldots, x_n^d) \cap R^G$ is generated by the power sum symmetric polynomials of degrees $d, d + 1, \ldots, d + n - 1$. (This is discussed in the proof of [2] Lemma 7.6.) Hence it follows that the invariant subspace $F^G$ of $F$ is a free $R^G$-module of rank $n$ generated by $\{\sum_{i=1}^n x_i^k dx_i \mid k = 0, 1, \ldots, n - 1\}$. In other words a matrix $M$ is determined to be the Van der Monde matrix if it satisfies, for any $d$ given, the following matrix identity:

$$
\begin{pmatrix}
x_1^d + x_2^d + \cdots + x_n^d \\
x_1^{d+1} + x_2^{d+1} + \cdots + x_n^{d+1} \\
\vdots \\
x_1^{d+n-1} + x_2^{d+n-1} + \cdots + x_n^{d+n-1}
\end{pmatrix} = M
\begin{pmatrix}
x_1^d \\
x_2^d \\
\vdots \\
x_n^d
\end{pmatrix}.
$$

The first part of the minimal free resolution of $R/I$ takes the form

$$
\bigwedge^2 F \rightarrow \bigwedge^1 F \xrightarrow{\partial} R \rightarrow R/I \rightarrow 0.
$$

To extract the invariant subspace is an exact functor, so we have the exact sequence

$$
\bigwedge^2 F^G \xrightarrow{\partial^G} F^G \rightarrow R^G \rightarrow (R/I)^G \rightarrow 0.
$$

of free $R^G$-modules. The map $\partial : F \rightarrow R$ is defined by $dx_i \mapsto f_i$.

Thus the image of the restricted map $\partial : F^G \rightarrow R^G$ is the ideal

$$(g_1, g_2, \ldots, g_n) R^G.$$

This shows that $I \cap R^G$ is an ideal of finite colength in $R^G$ or equivalently they generate an ideal of finite colength in $R$. 

\[\square\]
We will call the matrix in the statement of Proposition 16 the Van der Monde matrix.

**Corollary 17.** Let \( n = n_1 + \cdots + n_r \) be a partition of the integer \( n \) and let

\[
\begin{array}{c}
\rho_1, \ldots, \rho_{n_1}, \\
\rho_{n_1+1}, \ldots, \rho_{n_1+n_2}, \\
\vdots \\
\rho_{n_1+\cdots+n_{r-1}}, \rho_{n_1+\cdots+n_r}
\end{array}
\]

be a decomposition of the variables into \( r \) blocks. Let \( G = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r} \subset S_n \) be a Young subgroup and let \( G \) act on \( R \) be the block-wise permutation of the variables. Suppose that \( f_1, \ldots, f_n \) is a homogeneous complete intersection which satisfies \( f_\sigma = f_{\pi(i)} \), \( i = 1, 2, \ldots, n \), for all \( \sigma \in G \). Let \( V_i \) be the Van der Monde matrix in the variables in the \( i \)-th block

\[ \{x_{n_1+\cdots+n_{i-1}+j} \mid j = 1, \ldots, n_i\}. \]

Define the homogeneous elements \( g_1, g_2, \ldots, g_n \) by

\[
\begin{pmatrix}
g_1 \\
g_2 \\
\vdots \\
g_n
\end{pmatrix} =
\begin{pmatrix}
V_1 & 0 & \cdots & 0 \\
0 & V_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & V_r
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2 \\
\vdots \\
f_n
\end{pmatrix}.
\]

(The matrix is a block diagonal matrix.) Then \((g_1, \ldots, g_n) = (f_1, f_2, \ldots, f_n) \cap R^G\)

**Proof.** \((g_1, \ldots, g_n) \subset (f_1, f_2, \ldots, f_n) \cap R^G\) is obvious. Put

\[ \Omega = Rdx_1 \oplus Rdx_2 \oplus \cdots \oplus Rdx_n, \]

Construct a minimal free resolution of \( R/I \) over \( R \) as:

\[ 0 \to \bigwedge^1 \Omega \to \cdots \to \bigwedge^n \Omega \xrightarrow{\partial} R \to R/I \to 0, \]

so that the boundary maps are compatible with the action of the group \( G \). By taking the invariant subspaces for \( G \) we may get the minimal free resolution of \((R/I)^G\). As in Proposition 16, the invariant subspace \( \Omega^G \) as an \( R^G \)-module can be generated by the elements which appear as the entries of the column vector:

\[
\begin{pmatrix}
V_1 & 0 & \cdots & 0 \\
0 & V_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & V_r
\end{pmatrix}
\begin{pmatrix}
dx_1 \\
dx_2 \\
\vdots \\
dx_n
\end{pmatrix}.
\]

The map \( \partial : \Omega \to R \) is defined as \( dx_i \mapsto f_i \). Hence we obtain the module \( R^G/(R^G \cap (f_1, \ldots, f_n)) \) as represented by \( R^G/(g_1, \ldots, g_n)R^G \).

**Remark 18.** We have been unable to determine a set of generators for \((\wedge^k F)^G\) for \( k > 1 \) except for \( k = n - 1, n \). If \( k = n \), then \( F^n \) is the free \( R^G \)-module of rank one with

\[
\prod_{1 \leq k < l \leq n} (x_l - x_k) \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n
\]

as a generator. If \( k = n - 1 \), a set of generators can be specified similarly.
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