Parameterized Distributed Complexity

Theory: A logical approach

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Parameterized complexity theory offers a framework for a refined analysis of hard algorithmic problems. Instead of expressing the running time of an algorithm as a function of the input size only, running times are expressed with respect to one or more parameters of the input instances. In this work we follow the approach of parameterized complexity to provide a framework of parameterized distributed complexity. The central notion of efficiency in parameterized complexity is fixed-parameter tractability and we define the distributed analogue \textsc{Distributed-FPT} (for \textsc{Distributed} \( \in \{ \text{LOCAL}, \text{CONGEST}, \text{CONGESTED-CLIQUE} \} \)) as the class of problems that can be solved in \( f(k) \) communication rounds in the \textsc{Distributed} model of distributed computing, where \( k \) is the parameter of the problem instance and \( f \) is an arbitrary computable function. To classify hardness we introduce three hierarchies. The \textsc{Distributed-WEFT}-hierarchy is defined analogously to the \( W \)-hierarchy in parameterized complexity theory via reductions to the weighted circuit satisfiability problem, but it turns out that this definition does not lead to satisfying frameworks for the \textsc{Local} and \textsc{Congest} models. We then follow a logical approach that leads to a more robust theory. We define the levels of the \textsc{Distributed-W}-hierarchy and the \textsc{Distributed-A}-hierarchy that have first-order model-checking problems as their complete problems via suitable reductions.

1. Introduction

The synchronous message passing model, which was introduced by Linial [25], is a theoretical model of distributed systems that allows to focus on certain important aspects of distributed computing. In this model, a distributed system is modeled by an undirected (connected) graph \( G \), in which each vertex \( v \in V(G) \) represents a computational entity of the network, often referred to as a node of the network, and each edge \( \{u,v\} \in E(G) \) represents a bidirectional communication channel that connects the two nodes \( u \) and \( v \).
The nodes are equipped with unique numerical identifiers (of size $O(\log n)$, where $n$ is the order of the network graph). In a distributed algorithm, initially, the nodes have no knowledge about the network graph and only know their own and their neighbors' identifiers. The nodes then communicate and coordinate their actions by passing messages to one another in order to achieve a common goal. The synchronous message passing model without any bandwidth restrictions is called the LOCAL model of distributed computing \[30\]. If every node is restricted to send messages of size at most $O(\log n)$ one obtains the CONGEST model, and finally, if messages of size $O(\log n)$ can be sent to all nodes of the network graph (not only to the neighbors of a node) we speak of the CONGESTED-CLIQUE model. The time complexity of a distributed algorithm in each of these models is defined as the number of communication rounds until all nodes terminate their computations.

Typically considered computational tasks are related to graphs, in fact, often the graph that describes the network topology is the graph of the problem instance itself. For example, in a distributed algorithm for the DOMINATING SET problem, the computational task is to compute a small dominating set of the network graph $G$. Each node of the network must decide and report whether it shall belong to the dominating set or not.

Research in the distributed computing community is to a large extent problem-driven. There is a huge body of literature on upper and lower bounds for concrete problems. We refer to the surveys of Suomela \[32\] and Elkin \[10\] for extensive overviews of distributed algorithms. There has also been major progress in developing a systematic distributed complexity theory, including definitions of suitable locality preserving reductions and distributed complexity classes. We refer to \[1, 2, 5, 13, 16, 23\] for extensive background.

A very successful approach to deal with computationally hard problems in classical complexity is the approach of parameterized complexity. Instead of measuring the running time of an algorithm with respect to the input size only, the approach of parameterized complexity is to take one or more additional parameters into account. In many practical applications it is reasonable to assume that structural parameters of the input instances are bounded. Another commonly considered parameter is the size of the solution. In case a parameter is bounded, one can design special algorithms that aim to restrict the non-polynomial dependence of the running time to this parameter. For example, the currently fastest known exact algorithm for the DOMINATING SET problem runs in time $O(1.4969^n)$ \[33\]. If, however, we are dealing with structured graphs, e.g. if we may assume that a graph $G$ excludes a complete bipartite subgraph $K_{t,t}$, we can decide in time $2^{O(t^2k \log k)} \cdot |G|$ whether $G$ contains a dominating set of size at most $k$ \[12\]. When $k$ and $t$ are small and $G$ is large, this may be a major improvement over the exact algorithm. If a problem admits such running times, we speak of a fixed-parameter tractable problem. More precisely, a parameterized problem is fixed-parameter tractable if there is an algorithm solving it in time $f(k) \cdot n^c$, where $k$ is the parameter, $f$ is a computable function, $n$ is the input size and $c$ is a constant.

In this work we follow the approach of parameterized complexity to provide a framework of parameterized distributed complexity. For any DISTRIBUTED model, where DISTRIBUTED $\in \{\text{LOCAL, CONGEST, CONGESTED-CLIQUE}\}$, we define the distributed
complexity class DISTRIBUTED-FPT as the class of problems that can be solved in \( f(k) \) communication rounds in the DISTRIBUTED model, where \( k \) is the parameter of the problem instance and \( f \) is an arbitrary computable function. These classes are the distributed analogues of the central notion of fixed-parameter tractability.

Parameterized approaches to distributed computing were recently studied in \([22]\), where it was shown that \( k \)-paths and trees on \( k \) nodes can be detected in \( O(k \cdot 2^k) \) rounds in the BROADCAST-CONGEST model. Similar randomized algorithms were obtained in the context of distributed property testing \([4, 11]\). The setting which is closest to our present work is the work of Ben-Basat et al. \([3]\). The authors studied the parameterized distributed complexity of several fundamental graph problems, parameterized by solution size, such as the VERTEX COVER problem, the INDEPENDENT SET problem, the DOMINATING SET problem, the MATCHING problem, and several more. In each of these problems the question is to decide whether there exists a solution of size \( k \), where \( k \) is the input parameter. They showed that all of the above problems are fixed-parameter tractable in the LOCAL model – in our notation: they belong to the class LOCAL-FPT.

This is no surprise e.g. for the DOMINATING SET problem. If a dominating set of a connected graph \( G \) has size at most \( k \), then the diameter of \( G \) is bounded by \( 3k \). Hence, in the LOCAL model one can either learn in \( 3k \) rounds the whole graph topology and determine by brute force whether a dominating set of size \( k \) exists. Otherwise, if the diameter is too large, the algorithm can simply reject the instance as a negative instance.

Similarly, an independent set of size \( k \) can be chosen greedily if the diameter of \( G \) is sufficiently large. The authors of \([3]\) formalized this phenomenon by defining the class DLB of problems whose optimal solution size is lower bounded by the graph diameter. The situation is more complex in the CONGEST model. For this model, the authors study two problems, namely the VERTEX COVER problem and the MATCHING problem, and prove that both problems admit fixed-parameter distributed algorithms in the CONGEST model – in our notation: they belong to the class CONGEST-FPT.

In parameterized complexity theory, the VERTEX COVER problem is a standard example of a fixed-parameter tractable problem, while the INDEPENDENT SET problem and DOMINATING SET problem are considered intractable. While lacking the techniques to actually prove this intractability, parameterized complexity theory offers a way to establish intractability by classifying problems into complexity classes by means of suitable reductions. The W-hierarchy is a collection of complexity classes that may be seen as a parameterized refinement of the classical complexity class \( \text{NP} \). The INDEPENDENT SET problem is the foremost example of a problem that is hard for the parameterized complexity class \( \text{W}[1] \). Similarly, the DOMINATING SET problem is a prime example of a \( \text{W}[2] \)-hard problem. The A-hierarchy is a collection of complexity classes that may be seen as a parameterized analogue of the polynomial hierarchy.

As the INDEPENDENT SET problem and the DOMINATING SET problem are in LOCAL-FPT, these problems cannot take the exemplary role of hard problems that they take in classical parameterized complexity theory. However, their colored variants remain hard also in the distributed setting. For example, in the MULTICOLORED INDEPENDENT SET problem one searches in a colored graph for an independent set where all vertices of the set have different colors. As the colors can be given to vertices in an arbitrary way,
he problem looses its local character and becomes hard also in the LOCAL model. In the CONGEST model this hardness is already observed for the uncolored INDEPENDENT SET problem [3]. The authors establish a lower bound of $\Omega(n^2/\log^2 n)$ on the number of rounds in the CONGEST model, where $n$ can be arbitrarily larger than $k$.

In classical parameterized complexity theory, the $W$-hierarchy is defined by the complexity of circuits that is required to check a solution. This hierarchy was introduced by Downey and Fellows in [7]. At a first glance it seems natural to model the circuit evaluation problem as a graph problem and consider it as such in the distributed setting. This leads to the definition of a problem class $\text{WEFT}[t]$ for each $t \geq 1$ and we define the class $\text{DISTRIBUTED-WEFT}[t]$ as the class of those problems that reduce via parameterized DISTRIBUTED reductions to a member of $\text{WEFT}[t]$. The problem with this is that in the definition of the $W$-hierarchy one considers circuit families of bounded depth. In our locality sensitive setting this does not lead to a robust complexity theory: for example, we obtain that $\text{LOCAL-WEFT}[t] \subseteq \text{LOCAL-FPT}$ for each $t$, while the MULTICOLORED INDEPENDENT SET problem, which we would like to place into the class $\text{LOCAL-WEFT}[1]$, does not lie in any of the classes $\text{LOCAL-WEFT}[t]$. These problems do not arise in the CONGESTED-CLIQUE model and the CONGESTED-CLIQUE-WEFT hierarchy is an interesting hierarchy to study. We refer to Section 3 for the details.

The $A$-hierarchy was introduced in by Flum and Grohe, originally in terms of the parameterized halting problem for alternating Turing machines [14]. This definition cannot be easily adapted to the distributed setting. Instead, we follow another approach of [14] (see also the monograph [15]), where problems are classified by their descriptive complexity. More precisely, the authors classify problems by the syntactic form of their definitions in first-order predicate logic. This leads in a very natural way to the definition of the levels of the $W$- and $A$-hierarchy. We denote by $\Sigma_0$ and $\Pi_0$ the class of quantifier-free formulas. For $t \geq 0$, we let $\Sigma_{t+1}$ be the class of all formulas $\exists x_1 \ldots \exists x_t \varphi$, where $\varphi \in \Pi_t$, and we let $\Pi_{t+1}$ be the class of all formulas $\forall x_1 \ldots \forall x_t \varphi$, where $\varphi \in \Sigma_t$. Furthermore, for $t \geq 1$, $\Sigma_{t,1}$ denotes the class of all formulas of $\Sigma_t$ such that all quantifier blocks after the leading existential block have length at most 1. The model-checking problem for a class $\Phi$ of formulas, denoted $\text{MC-}\Phi$, is the problem to decide for a given (vertex and edge colored) graph $G$ and formula $\varphi \in \Phi$ whether $\varphi$ is satisfied on $G$. For all $t \geq 1$, $\text{MC-}\Sigma_{t,1}$ is complete for $W[t]$ under fpt-reductions, and for all $t \geq 1$, $\text{MC-}\Sigma_t$ is complete for $A[t]$ under fpt-reductions [15]. After giving an appropriate notion of parameterized DISTRIBUTED reductions, we define for all $t \geq 1$ the class $\text{DISTRIBUTED-W}[t]$ as the class $[\text{MC-}\Sigma_{t,1}]^{\text{DISTRIBUTED}}$ of problems that reduce to the $\Sigma_{t,1}$ model-checking problem via parameterized DISTRIBUTED reductions. Analogously, we define for all $t \geq 1$ the class $\text{DISTRIBUTED-A}[t]$ as the class $[\text{MC-}\Sigma_t]^{\text{DISTRIBUTED}}$ of problems that reduce to the $\Sigma_t$ model-checking problem via parameterized DISTRIBUTED reductions. The details are presented in Section 4.

Let us comment on our choice to use the model-checking problem for first-order logic as the basis of our distributed complexity theory. In principle, one could take any problem and define a complexity class from its closure under appropriate reductions. The model-checking problem for fragments of first-order logic is a very natural candidate to use for the definition of complexity classes. The number of quantifiers and
of quantifier alternations in a formula needed to describe a problem give an intuitive indication about the complexity of the problem, which naturally leads to a hierarchy of complexity classes. First-order logic can express many important graph problems in an elegant way, e.g. the existence of a multicolored independent set of size \( k \) can be expressed by the formula \( \exists x_1 \ldots \exists x_k (\bigwedge_{1 \leq i \leq k} P_i(x_i) \land \bigwedge_{1 \leq i \neq j \leq k} \neg E(x_i, x_j)) \). Here, the \( P_i \) are unary predicates that encode the colors of vertices and \( E \) is a binary predicate that encodes the edge relation (here we assume for simplicity that the graph is colored only with the colors \( P_1, \ldots, P_k \)). The above formula is a \( \Sigma_{1,1} \)-formula, hence the Multicolored Independent Set problem is placed in the class \( \text{DISTRIBUTED-W}[1] \), as intended. Similarly, the existence of a dominating set of size at most \( D \) is expressed by the formula \( \exists x \forall y (\bigvee_{1 \leq i \leq k} (y = x \lor E(y, x_i))) \). This is a \( \Sigma_{2,1} \)-formula, which places the Dominating Set problem in the class \( \text{DISTRIBUTED-W}[2] \). In particular, we have the desired inclusions \( \text{DISTRIBUTED-FPT} \subseteq \text{DISTRIBUTED-W}[1] \) and \( \text{DISTRIBUTED-WEFT}[t] \subseteq \text{DISTRIBUTED-W}[t] \) for all \( t \geq 1 \). The details can be found in Section 4.

Unlike in the classical setting, it is trivial to prove that \( \text{LOCAL-FPT} \subsetneq \text{LOCAL-W}[1] \) and \( \text{CONGEST-FPT} \subsetneq \text{CONGEST-W}[1] \), e.g. the Multicolored Independent Set problem is not in \( \text{LOCAL-FPT} \) (and hence not in \( \text{CONGEST-FPT} \)). This is a simple consequence of the fact that we cannot even decide in a constant number of rounds in the LOCAL model whether there are three non-adjacent vertices of different colors. However, the problem belongs to \( \text{CONGEST-W}[1] \). We conjecture that we also have this proper inclusion for the \( \text{CONGESTED-CLIQUE} \) model.

We then use a classical theorem from model theory, namely Gaifman’s Theorem, to prove that the \( \text{LOCAL-W} \)- and \( \text{LOCAL-A} \)-hierarchies collapse to the second level of the \( \text{LOCAL-W} \)-hierarchy. Gaifman’s Theorem states that every first-order formula is a Boolean combination of statements of the form “there exist \( k \) elements with pairwise large distances which all satisfy a certain local property”. In the LOCAL model we can evaluate the local properties of each element and are left with the task of evaluating a Boolean combination of instances of a variant of the Multicolored Independent Set problem. In fact, we show that the \( \text{LOCAL-A} \)-hierarchy collapses to \( \text{LOCAL-W}[1] \) under LOCAL Turing reductions and that the Multicolored Independent Set problem is complete for \( \text{LOCAL-W}[1] \) under this type of reductions. In a Turing reduction we are allowed to construct via a LOCAL algorithm in \( f(k) \) rounds an instance that is given to an oracle which outputs in constant time the answer to this instance. As in classical complexity theory, a Turing reduction from problem \( A \) to problem \( B \) implies a statement of the form “if we can solve problem \( B \) efficiently, then we can also solve problem \( A \) efficiently”. However, Turing reductions close complexity classes under complement, which does not seem to be natural for non-deterministic classes and which is one of the reasons why one usually works with many-one-reductions instead. Nevertheless, the above result highlights again the importance of the Multicolored Independent Set problem for the theory of parameterized distributed complexity. We conjecture that the \( \text{CONGEST-W} \)- and \( \text{-A} \)- and the \( \text{CONGESTED-CLIQUE-W} \)- and \( \text{-A} \)-hierarchies are strict.

We then turn our attention to distributed kernelization. Kernelization is a classical approach in parameterized complexity theory to reduce the size of the input instance in
a polynomial time preprocessing step. More formally, a kernelization for a parameterized problem \( P \) is an algorithm that computes for a given instance \((G, k)\) of \( P \) in time polynomial in \(|G| + k\) an instance \((G', k')\) of \( P \) such that \((G, k)\) is a positive instance of \( P \) if and only if \((G', k')\) is a positive instance of \( P \) and such that \(|G'| + k'|\) is bounded by a computable function in \( k \). The output \((G', k')\) is called a kernel. It is a classical result of parameterized complexity that a problem is fixed-parameter tractable if and only if it admits a kernel. We give two definitions of distributed kernelization that, however, both do not coincide with distributed fixed-parameter tractability. The details are given in Section 5.

Finally, we define the class \( \text{DISTRIBUTED-XPL} \) as the class of problems that can be solved in \( f(k) \cdot (\log n)^g(k) \) rounds (for computable functions \( f \) and \( g \)) in the \( \text{DISTRIBUTED} \) model. \( \text{XPL} \) stands for \textit{slicewise poly-logarithmic}. In parameterized complexity theory the class XP of slicewise polynomial problems contains all problems that can be solved in time \( n^{g(k)} \) for some computable function \( g \). This definition obviously has to be adapted to make sense in the distributed setting, as every problem can be solved in a polynomial number of rounds (polynomial in the graph size) in the CONGEST model. As the final result we show that the model-checking problem of first-order logic is in \( \text{CONGESTED-CLIQUE-XPL} \) when parameterized by formula length on classes of graphs of bounded expansion. We conjecture that this is not the case on all graphs. The details are presented in Section 6.

2. Distributed fixed-parameter tractability and reductions

We consider the \textit{synchronous message passing model}, introduced by Linial [25], in which a distributed system is modeled by an undirected connected graph \( G \). Each vertex \( v \in V(G) \) represents a computational entity of the network, often referred to as a node of the network, and each edge \( \{u, v\} \in E(G) \) represents a bidirectional communication channel that connects the two nodes \( u \) and \( v \). The nodes are equipped with unique numerical identifiers (of size \( O(\log n) \), where \( n \) is the order of the network graph). In a distributed algorithm, initially, the nodes have no knowledge about the network graph and only know their own and their neighbors identifiers. The nodes communicate and coordinate their actions by passing messages to one another in order to achieve a common goal. The synchronous message passing model without any bandwidth restrictions is called the LOCAL model of distributed computing [30]. If every node is restricted to send messages of size at most \( O(\log n) \) one obtains the CONGEST model, and finally, if messages of size \( O(\log n) \) can be send to all nodes of the network graph (not only to neighbors) we speak of the CONGESTED-CLIQUE model. The time complexity of a distributed algorithm in each of these models is defined as the number of communication rounds until all nodes terminate their computations. We refer to the surveys [10, 13, 32] for extensive overviews of distributed algorithms.

Typically considered computational tasks are related to graphs, in fact, often the graph that describes the network topology is the graph of the problem instance itself. We therefore focus on graph problems, and, as usual in complexity theory and also param-
eterized complexity theory, on decision problems. We allow a fixed number of vertex and
edge labels/colors that are accessible via unary and binary predicates $P_1, \ldots, P_s \subseteq V(G)$
and $E_1, \ldots, E_t \subseteq V(G)^2$, for fixed $s, t \in \mathbb{N}$. We write $G_{s,t}$ for the set of all finite connected
graphs with $s$ unary and $t$ binary predicates. In the following, an instance of a
decision problem is a pair $(G, k)$, where $G$ is a connected, vertex and edge colored graph,
and $k \in \mathbb{N}$ is a parameter. We refer to the textbooks [6, 8, 15] for extensive background
on parameterized complexity theory.

**Definition 1.** A parameterized decision problem is a set $P \subseteq G_{s,t} \times \mathbb{N}$ for some $s, t \in \mathbb{N}$.

Often in the literature, see e.g. [16], in a distributed algorithm for a decision problem,
each processor must produce a Boolean output $\text{accept}$ or $\text{reject}$ and the decision is defined
as the conjunction of all outputs. This may require, even if local decision is possible,
to inform all other nodes of the graph of this decision and make local problems global
in an artificial way. We instead define the decision of an algorithm as the disjunction
of all outputs, which also turns out to work better with our notion of parameterized
reductions.

**Definition 2.** In a distributed algorithm for a parameterized decision problem $P$, each
node has access to the parameter $k$ of the instance and must at termination produce an
output $\text{accept}$ or $\text{reject}$. The decision of the algorithm is defined as the disjunction of the
outputs of all nodes, i.e., if the instance belongs to $P$, then some processor must accept
and otherwise, all processors must reject.

We come to the central notion of distributed fixed-parameter tractability. In the
following let $\text{DISTRIBUTED}$ be any of $\text{LOCAL}$, $\text{CONGEST}$, or $\text{CONGESTED-CLIQUE}$.

**Definition 3.** A parameterized decision problem $P$ belongs to $\text{DISTRIBUTED-FPT}$ if
there exists a computable function $f$ and a $\text{DISTRIBUTED}$ algorithm that on input $(G, k)$
correctly decides in time $f(k)$ whether $(G, k) \in P$.

We remark that the nodes do not have to know the function $f$, however, the algorithm
must guarantee that all nodes terminate after $f(k)$ steps. It is immediate from the defini-
tions that $\text{CONGEST-FPT} \subseteq \text{LOCAL-FPT}$ and $\text{CONGEST-FPT} \subseteq \text{CONGESTED-CLIQUE-FPT}$.

**Example 1.** $\text{INDEPENDENT SET} \in \text{LOCAL-FPT}$ and $\text{DOMINATING SET} \in \text{LOCAL-FPT}$.

**Proof.** This was proved in [3] and is a simple consequence of the fact that the size of
a maximum independent set (and minimum dominating set) is lower bounded by the
graph diameter. \qed

On the other hand, we cannot decide in general by a $\text{LOCAL}$ algorithm in a constant
number of rounds whether there are even three elements of different color in a colored
graph. The input to the $\text{MULTICOLORED INDEPENDENT SET}$ problem is an integer $k$
and a graph $G$ that is additionally equipped with unary predicates $P_1, \ldots, P_s \subseteq V(G)$
for some $s \geq 0$, such that $P_i \cap P_j = \emptyset$ for $i \neq j$. Each $v \in P_i$ is said to have color $i$. The
algorithmic question is to decide whether there exist $k$ elements with pairwise different
colors.
Lemma 2. **Multicolored Independent Set \( \not\in \text{LOCAL-FPT} \).**

We next come to the definition of \( \text{DISTRIBUTED} \) (many-one) reductions.

**Definition 4.** A **DISTRIBUTED** reduction is a **DISTRIBUTED** algorithm that turns an instance \((G, k)\) of a parameterized problem into another instance \((G', k')\), where \((G', k')\) is encoded in \((G, k)\) as follows. There is a mapping \(\nu: V(G') \to V(G)\) and a mapping \(\eta: E(G') \to \mathcal{P}(G)\), where \(\mathcal{P}(G)\) denotes all paths in \(G\). The mappings are stored in vertices of \(G\), more precisely, each vertex \(v \in V(G)\) stores all \(x \in V(G')\) such that \(\nu(x) = v\) and all paths \(\eta(\{x, y\})\) such that \(\{x, y\} \in E(G')\). The **radius of the reduction** is the length of the longest path in the image of \(\eta\) and its **congestion** is the largest number of paths that a single edge of \(G\) belongs to. For computable functions \(s, r, c, t, p\) we say that the reduction is \((s, r, c, t, p)\)-**bounded** if the order of \(G'\) is bounded by \(|G|^s(k)\), the radius of the reduction is bounded by \(r(k)\), its congestion is bounded by \(c(k)\), the reduction is computable in \(t(k)\) rounds and the parameter satisfies \(k' \leq p(k)\).

**Definition 5.** For parameterized problems \(P_1\) and \(P_2\) we write \(P_1 \leq_{\text{DISTRIBUTED}} P_2\) if there exist computable functions \(s, r, c, t\) and \(p\) and an \((s, r, c, t, p)\)-bounded **DISTRIBUTED** reduction that maps any instance \((G, k)\) to an instance \((G', k')\) such that \((G, k) \in P_1 \iff (G', k') \in P_2\). In case **DISTRIBUTED** = **LOCAL** we allow unbounded congestion.

**Definition 6.** Let \(\mathcal{P}\) be a set of parameterized problems. We write \([\mathcal{P}]_{\text{DISTRIBUTED}}\) for the set of all problems \(P\) with \(P \leq_{\text{DISTRIBUTED}} P'\) for some \(P' \in \mathcal{P}\).

The next lemma shows that distributed reductions preserve fixed-parameter tractability, as desired. After turning the instance \((G, k)\) of \(P_1\) into an equivalent instance \((G', k')\) of \(P_2\), we simulate the passing of a message from \(x\) to \(y\) along an edge of \(G'\) by passing it along the path \(\eta(\{x, y\})\) in \(G\).

**Lemma 3.** Let \(P_1 \leq_{\text{DISTRIBUTED}} P_2\) and assume that \(P_2\) is in \(\text{DISTRIBUTED-FPT}\). Then also \(P_1\) is in \(\text{DISTRIBUTED-FPT}\).

**Proof.** We need to show that there exists a **DISTRIBUTED** algorithm that on input \((G, k)\) decides whether \((G, k) \in P_1\) in \(g(k)\) rounds, for some computable function \(g\). The algorithm proceeds as follows. Let \(n \coloneqq |V(G)|\).

As \(P_1 \leq_{\text{DISTRIBUTED}} P_2\), there exist computable functions \(s, r, c, t\) and \(p\) and an \((s, r, c, t, p)\)-bounded **DISTRIBUTED** reduction that maps any instance \((G, k)\) to an instance \((G', k')\) such that \((G, k) \in P_1 \iff (G', k') \in P_2\) and such that \(k' \leq p(k)\). In case **DISTRIBUTED** = **LOCAL** we may have unbounded congestion. On input \((G, k)\) we apply this reduction and compute in \(t(k)\) rounds an instance \((G', k')\) with the above properties. We write \(\nu\) and \(\eta\) for the mappings representing the graph \(G'\) in \(G\) (see Definition 4).

As \(P_2\) is in \(\text{DISTRIBUTED-FPT}\), there exists a computable function \(f\) so that we can solve the instance \((G', k')\) in \(f(k') \leq f(p(k))\) steps in the **DISTRIBUTED** model. We simulate the run of this algorithm on \((G', k')\) in the graph \(G\). Whenever a message is supposed to be sent along an edge \(\{x, y\} \in E(G')\), we send this message between the appropriate vertices \(u\) and \(v\) of \(G\) such that \(\nu(x) = u\) and \(\nu(y) = v\) along the path \(\eta(\{x, y\})\). The path \(\eta(\{x, y\})\) has length at most \(r(k)\), it can hence be encoded by \(r(k) \cdot \log n\) bits that we send along with the message to make routing in \(G\) possible.
(the factor $\log n$ comes from the ids of the vertices of size $\log n$). Due to constraints on congestion, we may not be able to send all messages at once. However, by assumption, each edge of $G$ appears in at most $c(k)$ paths $\eta(x,y)$. Hence, each message can be sent after waiting for at most $c(k)$ rounds until the transmission line is free. Thus, the simulation of sending one message takes at most $c(k) \cdot r(k)^2$ rounds. As the functions $r$ and $c$ are computable, we can compute the number $c(k) \cdot r(k)^2$ and synchronize the network accordingly. In case $\text{DISTRIBUTED} = \text{LOCAL}$ we do not have to wait for free transmission lines and can transmit each message in time $r(k)$, and synchronize the network accordingly.

The total running time of the algorithm is hence $g(k) := t(k) + f(p(k)) \cdot r(k)^2 \cdot c(k)$, which is again a computable function. After this time, every node $v \in V(G)$ returns the disjunction of the answers of the nodes $x \in V(G')$ with $\nu(x) = v$. Hence, the algorithm accepts $(G,k)$ if and only some node in $G'$ accepts $(G',k')$. Hence, the algorithm correctly decides $P_1$ in $g(k)$ rounds, as desired. 

Observe that in the above proof it is crucial that we take the disjunction over the decisions of individual nodes. If we would work with conjunctions we would have the additional task to inform the whole graph about the positive answer of a single node, which could add additional complexity.

The following example is simple but instructive, as it is not a valid parameter preserving reduction in classical parameterized complexity.

**Example 4. Clique Domination $\leq_{\text{LOCAL}}$ Red-Blue Dominating Set.**

**Proof.** In the Clique Domination problem we get as input a graph $G$ and two integers $k, \ell \in \mathbb{N}$. The problem is to decide whether $G$ contains a set of at most $k$ vertices that dominates every clique of size $(\text{exactly}) \, \ell$ in $G$. The parameter is $k + \ell$. The input to the Red-Blue Dominating Set problem is a graph $G$ whose vertices are colored red or blue, and an integer $k \in \mathbb{N}$. The problem is to decide whether there exists a set of at most $k$ red vertices that dominate all blue vertices. The parameter is $k$.

On input $(G,k,\ell)$ we create a copy of each clique of size $\ell$ and color the vertices of the newly created vertices red. All original vertices are colored blue. We denote an original vertex $v$ by $(v,0)$ and its copies by $(v,i)$ for some $i \in \mathbb{N}$. We introduce all edges $(v,i),(v,0)$ for $i \in \mathbb{N}$ and $(v,v) \in E(G)$. The mapping $\nu : V(G') \rightarrow V(G)$ maps each vertex $(u,i) \in V(G')$ to the vertex $u$ in $V(G)$ and the mapping $\eta : E(G') \rightarrow E(G)$ maps each edge $(u,i),(v,0)$ to the edge $(u,v) \in E(G)$. Observe that the resulting graph $G'$ can have size $O(n^3)$, where $n = |V(G)|$, and that this reduction has unbounded congestion. Clearly, the graph $G'$ contains a set of $k$ blue vertices dominating all red vertices if and only if the graph $G$ contains a set of $k$ vertices dominating all cliques of size $\ell$. Hence, the above is a $(1,1,\infty,1,1)$-bounded $\text{LOCAL}$ reduction.

**Lemma 5.** If $P_1 \leq_{\text{DISTRIBUTED}} P_2$ and $P_2 \leq_{\text{DISTRIBUTED}} P_3$, then $P_1 \leq_{\text{DISTRIBUTED}} P_3$.

**Proof.** As $P_1 \leq_{\text{DISTRIBUTED}} P_2$, there exist computable functions $s_1,r_1,c_1,t_1$ and $p_1$ and an $(s_1,r_1,c_1,t_1,p_1)$-bounded $\text{DISTRIBUTED}$ reduction that maps any instance $(G_1,k_1)$ to an instance $(G_2,k_2)$ such that $(G_1,k_1) \in P_1 \iff (G_2,k_2) \in P_2$ and such that $k_2 \leq p(k_1)$. As $P_2 \leq_{\text{DISTRIBUTED}} P_3$, there exist computable functions $s_2,r_2,c_2,t_2$
and $p_2$ and an $(s_2, r_2, c_2, t_2, p_2)$-bounded DISTRIBUTED reduction that maps any instance $(G_2, k_2)$ to an instance $(G_3, k_3)$ such that $(G_2, k_2) \in P_2 \iff (G_3, k_3) \in P_3$ and such that $k_3 \leq p(k_2)$. In case DISTRIBUTED = LOCAL we may have unbounded congestion. We combine these reductions as in the proof of Lemma 3 to obtain an $(s_3, r_3, c_3, t_3, p_3)$-bounded reduction from $P_1$ to $P_3$. As $|G_2| \leq |G_1|^{s_1(k_1)}$ and $k_2 \leq p_1(k_1)$, we have $|G_3| \leq |G_1|^{s_3(k_2)}$, and we can define $s_3(k) := s_1(k) \cdot s_2(p_1(k))$. Similarly, we can define $r_3(k) := r_2(p_1(k)) \cdot r_1(k)$ and $c_3(k) := c_2(p_1(k)) \cdot c_1(k)$. For the construction of $G_3$ in $G_2$ we need to simulate sending a message between two vertices by routing along an appropriate path, just as in Lemma 3. Observe that we need to send node identifiers of size $\log |G_2|$ here, which is bounded by $s_1(k_1) \cdot \log |G_1|$. Hence, we get an additional factor $s_1(k_1)$ in the function bounding $t_3$, that is, we can set $t_3(k) := t_1(k_1) + t_2(p_1(k)) \cdot r_1(k) \cdot c_1(k) \cdot s_1(k)$. Finally, we can define $p_3(k) := p_2(p_1(k))$. \hfill $\square$

Finally, we define distributed Turing reductions.

**Definition 7.** A DISTRIBUTED oracle algorithm with an oracle for a decision problem $P$ is a DISTRIBUTED algorithm that may compute instances $(G', k')$, represented in the network graph $G$ as in Definition 4 by a pair $\nu$, $\eta$, and then query whether $(G', k') \in P$ in constant time.

**Definition 8.** For parameterized problems $P_1$ and $P_2$ we write $P_1 \preceq_{DISTRIBUTED}^T P_2$, and call $P_1$ Turing reducible to $P_2$, if there exist computable functions $f, s, r, c, t$ and $p$ and a DISTRIBUTED oracle algorithm that solves instances $(G, k)$ of $P_1$ in $f(k)$ rounds and that has access to an oracle for solving any instance $(G', k')$ of $P_2$ computable by $(s, r, c, t, p)$-bounded graph transformations. In case DISTRIBUTED = LOCAL we allow unbounded congestion.

Observe that an oracle algorithm can call the oracle multiple times and can continue working after calling the oracle. In particular, complexity classes that are closed under oracle reductions are closed under Boolean combinations. This is not necessarily the case for many-one-reductions. DISTRIBUTED Turing reductions preserve containment in DISTRIBUTED-FPT. The following lemma is proved just as Lemma 3, we simulate the oracle calls by the DISTRIBUTED algorithm for $P_2$.

**Lemma 6.** Let $P_1 \preceq_{DISTRIBUTED}^T P_2$ and assume that $P_2$ is in DISTRIBUTED-FPT. Then also $P_1$ is in DISTRIBUTED-FPT.

### 3. The distributed WEFT-hierarchy

We now define the DISTRIBUTED WEFT-hierarchy analogously to the classical W-hierarchy. In these definitions we interpret the network graph as a circuit.

**Definition 9.** A Boolean decision circuit with $n$ inputs is a tuple $C = (V, E, \beta)$, where $(V, E)$ is a finite directed acyclic graph, $\beta : V \to \{\neg, \lor, \land, V, \Lambda\} \cup \{x_1, \ldots, x_n\}$, such that the following conditions hold:

1. If $v \in V$ has in-degree 0, then $\beta(v) \in \{x_1, \ldots, x_n\}$. These vertices are the input gates.
2. If $v \in V$ has in-degree 1, then $\beta(v) = \neg$.

3. If $v \in V$ has in-degree 2, then $\beta(v) \in \{\lor, \land\}$. Vertices with degree $\leq 2$ are called small gates.

4. If $v \in V$ has in-degree at least 3, then $\beta(v) \in \{\lor, \land\}$. These vertices are called large gates.

5. $(V, E)$ has exactly one vertex of out-degree 0, called the output gate.

The circuit computes a function $f_C : \{0, 1\}^n \to \{0, 1\}$ in the expected way. We refer to the textbook [34] for more background on circuit complexity.

**Definition 10.** The depth of a circuit $C$ is defined to be the maximum number of gates (small or large) on an input-output path in $C$. The weft of a circuit $C$ is the maximum number of large gates on an input-output path in $C$.

**Definition 11.** We say that a family of circuits $\mathcal{F}$ has bounded depth if there is a constant $h$ such that every circuit in the family $\mathcal{F}$ has depth at most $h$. We say that $\mathcal{F}$ has bounded weft if there is constant $t$ such that every circuit in the family $\mathcal{F}$ has weft at most $t$. A decision circuit $C$ accepts an input vector $x$ if the single output gate has value 1 on input $x_1, \ldots, x_n$. The weight of a boolean vector is the number of 1s in the vector.

The following definition was given by Downey and Fellows [7].

**Definition 12.** Let $\mathcal{F}$ be a family of decision circuits (we allow that $\mathcal{F}$ may have many different circuits with a given number of inputs). To $\mathcal{F}$ we associate the parameterized circuit problem $P_{\mathcal{F}} := \{(C, k) : C \in \mathcal{F} \text{ and } C \text{ accepts an input vector of weight } k\}$. For $t \geq 1$, the class $\text{WEFT}[t]$ consists of all parameterized circuit problems $P_{\mathcal{F}}$, where each circuit in $\mathcal{F}$ has height bounded by some universal constant and weft at most $t$.

We are ready to define the DISTRIBUTED WEFT-hierarchy.

**Definition 13.** For $t \geq 1$ we define

\[ \text{DISTRIBUTED-WEFT}[t] := \text{WEFT}[t]^{\text{DISTRIBUTED}}. \]

The following are standard examples from parameterized complexity theory, see e.g. [6].

**Example 7.** Multicolored Independent Set $\in \text{CONGESTED-CLIQUE-WEFT}[1]$ and Red-Blue Dominating Set $\in \text{CONGESTED-CLIQUE-WEFT}[2]$.

**Proof.** These are standard examples from parameterized complexity theory, see e.g. [6]. We present the construction for Multicolored Independent Set for completeness.

On input $(G, k)$, we construct a circuit of weft 1 and height 3 for Multicolored Independent Set. We have one input gate for every vertex $v$ of $G$ that we identify with the vertex $v$. The gates which in a satisfying assignment are assigned the value 1 will correspond one-to-one with a multicolored independent set in $G$. To express this, we state that neither two vertices of the same color, nor two adjacent vertices can be
picked into the multicolored independent set. Hence, we connect each input with a
negation gate and we write \((\neg v)\) for the corresponding node of the circuit. Now, for each
edge \(\{u, v\} \in E(G)\) and for each pair \((u, v)\) such that \(u\) and \(v\) have the same color, we
introduce one node \((\neg u \lor \neg v)\) that we connect with the nodes \((\neg u)\) and \((\neg v)\). Finally, we
connect all these disjunction gates in a big conjunction, which is the output gate. It is
easy to see that a satisfying assignment corresponds one-to-one to a multicolored clique.

We have to show that we can construct the circuit with a bounded CONGESTED-CLIQUE
reduction in a constant number of rounds. The vertex map \(\nu : V(C) \rightarrow V(G)\) takes ev-
every node \(v\) and \((\neg v)\) to the vertex \(v\) and every node labeled \((\neg u \lor \neg v)\) to the smaller
(referring to vertex ids in the network graph) of \(u\) and \(v\). Assuming \(u\) is smaller than \(v\),
we map the edge from \((\neg u)\) to \((\neg u \lor \neg v)\) to the length 0 path \(u, u\) and the edge from \(v\)

to the edge \(\{u, v\}\), which is an edge in the congested clique. Finally, we choose an ar-
bitrary vertex \(v\) that represents the big conjunction. The edges from this conjunction
are mapped to the vertices that represent the vertices. In total, on each edge we have
congestion at most 3.

\[\square\]

The above example shows that the CONGESTED-CLIQUE WEFT-hierarchy is an
interesting hierarchy to study. In particular, we conjecture that the CONGESTED-CLIQUE
WEFT-hierarchy is strict.

Opposed to this, the LOCAL WEFT-hierarchy does not behave as intended. The
following lemma is a simple consequence of the fact that all circuits in \(\text{WEFT}[t]\) have bounded height and one designated output gate. Hence, for each problem \(P\) in \(\text{WEFT}[t]\),
the radius of each circuit of \(\mathcal{F}\) is bounded by a constant, and therefore a LOCAL algorithm
can learn the whole circuit in a constant number of rounds and solve the corresponding
decision problem.

**Lemma 8.** For all \(t \geq 1\), LOCAL-WEFT\([t]\) \(\subseteq\) LOCAL-FPT.

According to Lemma 2, Multicolored Independent Set \(\not\in\) LOCAL-FPT, and
hence the problem also does not belong to LOCAL-WEFT\([t]\) for any \(t \geq 1\). This does not
reflect our intuition about the complexity of the problem, and hence, in the following
section we define the DISTRIBUTED-W hierarchy in a different way.

### 4. The W- and A-hierarchy

We follow the approach of Grohe and Flum [14, 15] and define the DISTRIBUTED-W and
A-hierarchy via logic. First-order formulas over a vocabulary of vertex and edge colored
graphs \(\{P_1, \ldots, P_s, E_1, \ldots, E_t\}\) are formed from atomic formulas \(x = y, P_i(x), \) and
\(E_j(x, y)\), where each \(P_i\) is a unary relation symbol and each \(E_j\) is a binary relation symbol,
and \(x, y\) are variables (we assume that we have an infinite supply of variables) by the
usual Boolean connectives \(\neg\) (negation), \(\land\) (conjunction), \(\lor\) (disjunction) and existential
and universal quantification \(\exists x, \forall x\) over vertices, respectively. The free variables of a
formula are those not in the scope of a quantifier, and we write \(\varphi(x_1, \ldots, x_k)\) to indicate
that the free variables of the formula \(\varphi\) are among \(x_1, \ldots, x_k\). A sentence is a formula
without free variables.
To define the semantics, we inductively define a satisfaction relation $|=,$ where for a colored graph $G$, a formula $\varphi(x_1, \ldots, x_k)$, and elements $a_1, \ldots, a_k \in V(G)$, $G |= \varphi(a_1, \ldots, a_k)$ means that $G$ satisfies $\varphi$ if the free variables $x_1, \ldots, x_k$ are interpreted by $a_1, \ldots, a_k$, respectively. We refer to the textbook [24] for extensive background on first-order logic over finite structures.

**Definition 14.** Both $\Sigma_0$ and $\Pi_0$ denote the class of quantifier-free formulas. For $t \geq 0$, we let $\Sigma_{t+1}$ be the class of all formulas $\exists x_1 \ldots \exists x_k \varphi$, where $\varphi \in \Pi_t$, and $\Pi_{t+1}$ the class of all formulas $\forall x_1 \ldots \forall x_k \varphi$, where $\varphi \in \Sigma_t$. Furthermore, for $t \geq 1$, $\Sigma_{t,1}$ denotes the class of all formulas of $\Sigma_t$ such that all quantifier blocks after the leading existential block have length at most 1.

**Definition 15.** The model-checking problem for a set $\Phi$ of sentences, denoted MC-$\Phi$, is the problem to decide for a given (colored) graph $G$ and sentence $\varphi \in \Phi$ whether $\varphi$ is satisfied on $G$. The parameter is $|\varphi|$.

By Corollary 7.27 of [15], for all $t \geq 1$, MC-$\Sigma_{t,1}$ is complete for $W[t]$ under fpt-reductions and by Definition 5.7 and Lemma 8.10 of [15] for all $t \geq 1$, MC-$\Sigma_t$ is complete for $A[t]$ under fpt-reductions. We take this as the definition for the analogous distributed hierarchies.

**Definition 16.** For $t \geq 1$, we let

\[ \text{DISTRIBUTED-A}[t] := [\text{MC-}\Sigma_t]^{\text{DISTRIBUTED}} \]

and

\[ \text{DISTRIBUTED-AW}[\star] = [\text{MC-FO}]^{\text{DISTRIBUTED}}. \]

Similarly, we let

\[ \text{DISTRIBUTED-W}[t] := [\text{MC-}\Sigma_{t,1}]^{\text{DISTRIBUTED}}. \]

**Example 9.** **Multicolored Independent Set** $\in \text{DISTRIBUTED-W}[1]$ and **Red-Blue Dominating Set** $\in \text{DISTRIBUTED-W}[2]$.

**Proof.** The existence of a multicolored independent set of size $k$ can be expressed by the $\Sigma_{1,1}$-formula $\exists x_1 \ldots \exists x_k (\bigwedge_{1 \leq i \leq k} P_i(x_i) \land \bigwedge_{1 \leq i \neq j \leq k} \neg E(x_i, x_j))$. Here, the $P_i$ are unary predicates that encode the colors of vertices (we assume for simplicity that the graph is colored only with the colors $P_1, \ldots, P_k$). Similarly, the existence of a red-blue dominating set of size at most $k$ can be expressed by a $\Sigma_{2,1}$-formula. \qed

It is immediate from the definitions that for all $t \geq 1$ we have DISTRIBUTED-W[t] $\subseteq$ DISTRIBUTED-A[t] $\subseteq$ DISTRIBUTED-AW[\star]. Furthermore, we have DISTRIBUTED-W[1] = DISTRIBUTED-A[1]. We conjecture that the above inclusions are strict and that we have proper hierarchies in the CONGEST and CONGESTED-CLIQUE model.

**Lemma 10.** DISTRIBUTED-WEFT[t] $\subseteq$ DISTRIBUTED-W[t]

**Proof.** This follows from the fact that satisfiability of a circuit of weft $t$ can be expressed by a $\Sigma_{t,1}$-formula. See Lemma 7.26 of [15]. \qed

**Lemma 11.** DISTRIBUTED-FPT $\subseteq$ DISTRIBUTED-W[1].
Proof. Assume $P \in \text{DISTRIBUTED-FPT}$ and let $(G, k)$ be an instance of $P$. By assumption there exists an algorithm that decides in $f(k)$ rounds whether $(G, k) \in P$, that is, each node produces in $f(k)$ rounds a binary output accept or reject. We use this algorithm as a reduction to $\text{MC-}\Sigma_{1,1}$. We introduce a unary predicate $P_1$ such that exactly the nodes $v$ that accept $P_1(v)$. Then $(G, k) \in P \iff G \models \exists x P(x)$. Hence, $P \in [\text{MC-}\Sigma_{r,1}]^{\text{DISTRIBUTED}}$. □

Surprisingly, the $\text{LOCAL}$ hierarchies again behave different than expected. Observe for example that Example 4 and Example 9 imply that $\text{Clique Domination} \in \text{LOCAL-W}[2]$. $\text{Clique Domination}$ is a classical example of an $\mathcal{A}[2]$-complete problem (in classical parameterized complexity). Observe that the problem can be formulated as a $\Sigma_2$ formula, while the fact that $\ell$ is an input parameter makes it impossible to express it as a $\Sigma_{2,1}$ formula. It is the infinite computational power of individual nodes in the $\text{LOCAL}$ model that makes it possible to reduce the problem to the $\text{Red-Blue Dominating Set}$ problem with parameter $k$ only. Even more surprisingly, we show that in fact the $\text{LOCAL-AW}[\star]$ (and hence the whole $\text{LOCAL-A}$-hierarchy) collapses to $\text{LOCAL-W}[2]$.

**Lemma 12.** $\text{LOCAL-AW}[\star] = \text{LOCAL-W}[2]$.

Proof. By a classical theorem of Gaifman [17], every sentence $\varphi$ of first-order logic is equivalent to a computable Boolean combination of sentences of the form

$$\exists x_1 \ldots \exists x_s \left( \bigwedge_{1 \leq i \leq s} \alpha^{(r)}(x_i) \land \bigwedge_{1 \leq i < j \leq s} \text{dist}(x_i, x_j) > 2r \right),$$

where $s \leq k + 1$ if $k$ is the quantifier-rank of $\varphi$, $r \leq 7^k$, and $\alpha^{(r)}(x)$ is an $r$-local property of $G$, i.e., its truth depends only on the isomorphism type of the $r$-neighborhood of the free variable $x$ in $G$.

Now, any problem $P$ in $\text{LOCAL-AW}[\star]$ reduces via a $\text{LOCAL}$ reduction to the model-checking problem for a first-order sentence $\varphi$. We translate $\varphi$ into the Boolean combination of sentences as described above. We evaluate for every vertex $v$ of $G$ whether $\alpha^{(r)}(v)$ holds in $G$. This is possible, as $\alpha^{(r)}$ is only a local property that can be evaluated in the $\text{LOCAL}$ model by brute force. We assign to every vertex $v$ for which $\alpha^{(r)}(v)$ holds the color $P_\alpha$. We then build the $2r$th power of $G$, so that the formulas $\text{dist}(x_i, x_j) > 2r$ become atomic properties over the edge relation. The resulting formula is a Boolean combination of $\Sigma_{1,1}$ formulas, hence a $\Sigma_{2,1}$ formula. This means that $P$ lies in $\text{LOCAL-W}[2]$. □

Observe that the above proof immediately gives a $\text{LOCAL}$ Turing reduction from the model-checking problem for full first-order logic to a variant of the $\text{MULTICOLORED INDEPENDENT SET}$ problem (as with Turing reductions we can evaluate Boolean combinations). Hence, under $\text{LOCAL}$ Turing reductions we even have a collapse of $\text{LOCAL-AW}[\star]$ to $\text{LOCAL-W}[1]$. The only reason that this collapse does not happen under many-one-reductions is that the levels of the $W$-hierarchy are not closed under complementation. We conjecture that the problem whether a graph $G$ does not contain a multicolored independent set of size at least does not lie in $\text{LOCAL-W}[1]$. 

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5. Kernelization

We now turn our attention to distributed kernelization. Kernelization is a classical approach in parameterized complexity theory to reduce the size of the input instance in a polynomial time preprocessing step. It is a classical result of parameterized complexity that a problem is fixed-parameter tractable if and only if it admits a kernel. We give two definitions of distributed kernelization that both do not coincide with distributed fixed-parameter tractability. Here is the first definition.

Definition 17. A DISTRIBUTED kernelization algorithm is a DISTRIBUTED algorithm that on input \((G, k)\) of a parameterized problem \(P\) computes in \(f(k)\) rounds an equivalent instance \((G', k')\) of size at most \(g(k)\) for computable functions \(f, g\).

Here, the graph \(G'\) is represented in \(G\) as in a DISTRIBUTED reduction. It is obvious that if a problem admits a DISTRIBUTED kernelization, then it lies in DISTRIBUTED-FPT. Conversely, however, it may be the case that a problem lies in DISTRIBUTED-FPT and that two far away nodes can independently and locally decide on a positive answer. A kernelization algorithm however, must after all output a connected network graph \(G'\) and somehow connect the answers of the two nodes. This may force \(G'\) to be of size unbounded in \(k\).

We give a second definition of a fully polynomial DISTRIBUTED kernelization algorithm which better reflects the intuition that kernelization should express efficient preprocessing.

Definition 18. A fully polynomial DISTRIBUTED kernelization algorithm is a DISTRIBUTED kernelization algorithm where additionally we restrict the computational power of each node to time polynomial in the input size.

Fully polynomial DISTRIBUTED kernelization algorithm can be simulated by a sequential algorithm in polynomial time. Hence, we obtain that the class of problems that admits a fully polynomial DISTRIBUTED kernelization algorithm is a subset of sequential FPT. It is an interesting question which problems in FPT actually admit a fully polynomial DISTRIBUTED kernelization algorithm. As we intend to make a conceptual rather than a technical contribution, we leave this investigation for future work.

6. XPL and model-checking on bounded expansion classes

Finally, we want to introduce a distributed analogue of the parameterized complexity class XP of slicewise polynomial problems. This class contains all problems that can be solved in time \(n^{g(k)}\) for some computable function \(g\). This definition obviously has to be adapted to make sense in the distributed setting, as every problem can be solved in a polynomial number of rounds (polynomial in the graph size) in the CONGEST model. We define the following class DISTRIBUTED-XPL, where XPL stands for slicewise polylogarithmic.

Definition 19. The class DISTRIBUTED-XPL is the class of problems that can be solved by a DISTRIBUTED algorithm in \(f(k) \cdot (\log n)^{g(k)}\) rounds for computable functions \(f\) and \(g\).
The first-order model-checking problem belongs to the sequential class $\text{XP}$. We can simply instantiate the quantifiers of a formula $\varphi$ in all possible ways and thereby evaluate in time $n^{O(|\varphi|)}$ whether $\varphi$ is true in the input graph $G$. Since the question whether a graph contains two blue nodes (a simple first-order property) cannot be decided by a $\text{LOCAL}$ algorithm in a sublinear (in the diameter) number of rounds in general, the problem does not lie in $\text{LOCAL-XPL}$. We conjecture that the model-checking problem also does not belong to $\text{CONGESTED-CLIQUE-XPL}$.

We therefore turn our attention to solve the problem on restricted graph classes. Two prominent graph classes on which first-order model-checking is even fixed-parameter tractable by sequential algorithms are classes of bounded expansion [9] and nowhere dense classes of graphs [20]. We refer to the textbook [29] for extensive background on the theory of bounded expansion and nowhere dense graph classes. The methods used to establish fixed-parameter tractability of the model-checking problem on these classes do not yield distributed fixed-parameter tractability. However, the model-checking result on bounded expansion classes is very well understood and has been reproved multiple times [18, 19, 21, 31]. We show how to combine these methods with methods for distributed computing from [26] and prove that first-order model-checking on bounded expansion classes lies in the class $\text{CONGESTED-CLIQUE-XPL}$. Due to space constraints the proof is presented in the appendix.

**Theorem 13 (⋆).** Let $C$ be a graph class of effectively bounded expansion. Then there exists a computable function $f$ and a $\text{CONGESTED-CLIQUE}$ algorithm that given a vertex and edge colored graph $G \in C$ and a first-order sentence $\varphi$ decides in $f(|\varphi|) \cdot \log n$ rounds whether $\varphi$ holds in $G$.

### 7. Conclusion

In this work we followed the approach of parameterized complexity to provide a framework of parameterized distributed complexity. We could only initiate the study of distributed parameterized complexity classes and many interesting questions remain open. On the one hand, the parameterized distributed complexity and distributed kernelization complexity of many important graph problems has not yet been studied. On the other hand, it remains an interesting question to find parameterized distributed reductions between commonly studied graph problems.

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A. Omitted proofs from Section 6

Two prominent graph classes on which first-order model-checking is fixed-parameter tractable by sequential algorithms are classes of bounded expansion [9] and nowhere dense classes of graphs [20]. We refer to the textbook [29] for extensive background on the theory of bounded expansion and nowhere dense graph classes.

Very briefly, a graph $H$ is a depth-$r$ minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting mutually disjoint connected subgraphs of radius at most $r$. A class of graphs $C$ has bounded expansion if there is a function $f: \mathbb{N} \to \mathbb{N}$ such that for every $r \in \mathbb{N}$, in every depth-$r$ minor of a graph from $C$ the ratio between the number of edges and the number of vertices is bounded by $f(r)$. More generally, $C$ is nowhere dense if there is a function $t: \mathbb{N} \to \mathbb{N}$ such that no graph from $C$ admits the clique $K_{t(r)}$ as a depth-$r$ minor. Every class of bounded expansion is nowhere dense, but the converse does not necessarily hold [29]. Class $C$ has effectively bounded expansion, respectively is effectively nowhere dense, if the respective function $f$ or $t$ as above is computable. Many classes of sparse graphs studied in the literature have (effectively) bounded expansion, including planar graphs, graphs of bounded maximum degree, graphs of bounded treewidth, and more generally, graphs excluding a fixed (topological) minor. A notable negative example is that classes with bounded degeneracy, equivalently with bounded arboricity, do not necessarily have bounded expansion, as there we have only a finite bound on the edge density in subgraphs (aka depth-0 minors).

The methods used to establish fixed-parameter tractability of the model-checking problem on these classes do not yield distributed fixed-parameter tractability. However, the model-checking result on bounded expansion classes is very well understood and has been reproved multiple times [18, 19, 21, 31]. We show how to combine these methods with methods for distributed computing from [26] and prove that first-order model-checking on bounded expansion classes lies in the class \textsc{Congested-Clique-XPL}.

Theorem 13 states that the first-order model-checking problem on classes of effectively bounded expansion belongs to \textsc{Congested-Clique-XPL}. Our proof of the theorem follows closely the lines of the proof given in [31] and we point out only where the proof has to be changed. The idea of the proof is as follows. We first compute a so-called low treedepth coloring of the input graph, and then use this coloring to apply a quantifier elimination procedure for first-order logic. It is known that such colorings exist for graphs from classes of bounded expansion [28] and furthermore that they can be computed efficiently even in the \textsc{Congest} model [26]. For establishing Theorem 13 it remains to revisit the quantifier elimination procedure and show that it can be implemented in the \textsc{Congested-Clique} model. Let us now introduce the relevant definitions.

Definition 20. A rooted forest is an acyclic graph $F$ together with a unary predicate $R \subseteq V(F)$ selecting one root in each connected component of $F$. A tree is a connected forest. The depth of a node $x$ in a rooted forest $F$ is the distance between $x$ and the root in the connected component of $x$ in $F$. The depth of a forest is the largest depth of any of its nodes. The least common ancestor of nodes $x$ and $y$ in a rooted tree is the common ancestor of $x$ and $y$ that has the largest depth.
Definition 21. An elimination forest of a graph $G$ is a rooted forest $F$ on the same vertex set as $G$ such that whenever $uv$ is an edge in $G$, then either $u$ is an ancestor of $v$, or $v$ is an ancestor of $u$ in $F$. The treedepth of a graph $G$ is the smallest possible depth of a separation forest of $G$.

For the sake of quantifier elimination it will be convenient to encode rooted forests by unary a unary function $\text{parent}: V(F) \to V(F)$. The function encodes a tree in the expected way, every vertex is mapped to its parent in the tree, while the root vertex is mapped to itself. In the following we assume that trees are encoded via the parent function.

Definition 22. For an integer $p$, a coloring $\lambda: V(G) \to \{1, \ldots, M\}$ of a graph $G$ is a $p$-treedepth coloring of $G$ if every $i$-tuple of color classes in $\lambda$, $i \leq p$, induces in $G$ a graph of treedepth at most $i$.

Lemma 14 ([28]). A class $C$ of graphs has bounded expansion if and only if for every $p$ there is a number $M$ such that every graph $G \in C$ admits a $p$-treedepth coloring using $M$ colors.

In fact, we must work with a related notion, as for our application we need to be able to compute the elimination forests $F_I$ witnessing that an $i$-tuple $I$ of color classes has treedepth at most $i$. While in the sequential setting we can simply perform a depth-first search to compute an approximation of such an elimination forest, it is unclear how to compute such forests in CONGESTED-CLIQUE-XPL.

Definition 23. For an integer $p$, a $(p+1)$-centered coloring of a graph $G$ is a coloring $\lambda: V(G) \to \{1, \ldots, M\}$ so that for any induced connected subgraph $H \subseteq G$, either some color appears exactly once in $H$, or $H$ gets at least $p+1$ colors.

Every $(p+1)$-centered coloring is a $p$-treedepth coloring. More precisely, we have the following lemma.

Lemma 15 (Lemma 4.5 of [27]). Let $G$ be a graph and let $\lambda$ be a $(p+1)$-centered coloring of $G$. Then any subgraph $H$ of $G$ of treedepth $i \leq p$ gets at least $i$ colors in $\lambda$.

Furthermore, from a $(p+1)$-centered coloring with $M$ colors one easily computes a forest $F$ of height at most $i$, $1 \leq i \leq p$, for each tuple of at most $i$ color classes.

Lemma 16. Given a graph $G$ and a $(p+1)$-centered coloring $\lambda: V(G) \to \{1, \ldots, M\}$, we can compute in $O(p \cdot 2^p)$ rounds in the CONGEST model for every $i$-tuple $I$ of colors, $1 \leq i \leq p$, an elimination forest $F_I$ of height at most $i$.

Proof. For each $i$-tuple $I$ of colors, we can compute an elimination forest $F_I$ of height at most $i$ as follows. It is folklore (see e.g. Section 6.2 in [29]) that the longest path in a graph of treedepth $i$ has length (number of edges) at most $2^i - 2$. We can hence compute the components of $G[I]$ (the subgraph induced by the colors in $I$) in $O(2^i)$ rounds, by performing a breadth-first search from every vertex, and whenever the searches from two vertices meet, we continue only the search of the vertex with the smaller id to avoid large congestion. Now each component $C$ of $G[I]$ is connected and gets at most $p$ colors, hence there is a vertex of unique color. We can find such a vertex $v$ in $O(2^i)$ rounds.
by traversing the constructed bfs tree and keeping track of the encountered colors. We now make $v$ the root of $F_I$ and recursively continue to construct $F_I$ by decomposing the components of $G[I - \{v\}]$ (which have one less color) as above. After $i$ recursive steps, the procedure stops and produces an elimination forest $F_I$ of depth at most $i$. Observe that this construction is only possible in the CONGESTED-CLIQUE model.

We now appeal to the result of Nešetřil and Ossona de Mendez [26] that $(p + 1)$-centered colorings are computable in CONGEST-XPL.

**Lemma 17 ([26]).** Let $C$ be a class of graphs of effectively bounded expansion. There exists a computable function $g$ and a CONGEST algorithm that on input $G \in C$ and $p \in \mathbb{N}$ computes a $(p + 1)$-centered coloring of $G$ with $O(1)$ colors in $g(p) \cdot \log n$ rounds.

We now come to the quantifier elimination procedure on classes of bounded expansion. The proof boils down to proving how to eliminate a single existential quantifier for bounded depth forests. This elimination is then lifted to bounded expansion classes via low-treedepth colorings. The following statement is an adapted version of Lemma 26 of [31], which is the crucial ingredient of the proof.

**Lemma 18 (Lemma 26 of [31] (adapted)).** Let $d \in \mathbb{N}$ and $\Lambda$ be a label set. Then for every formula $\varphi(\bar{x}) \in \text{FO}[\{\text{parent}\} \cup \Lambda]$ with $|\bar{x}| \geq 1$ and of the form $\varphi(\bar{x}) = \exists y \psi(\bar{x}, y)$ where $\psi$ is quantifier-free, and every $\Lambda$-labeled forest $F$ of depth at most $d$, there exists a label set $\hat{\Lambda}$, a quantifier-free formula $\hat{\varphi}(\bar{x}) \in \text{FO}[\{\text{parent}\} \cup \hat{\Lambda}]$, and a $\hat{\Lambda}$-relabeling $\hat{F}$ of $F$ such that $\varphi$ on $F$ is equivalent to $\hat{\varphi}$ on $\hat{F}$. Moreover, the label set $\hat{\Lambda}$ is computable from $d$ and $\Lambda$, the formula $\hat{\varphi}$ is computable from $\varphi, d, \Lambda$, and the transformation which computes $\hat{F}$ given $F$ can be done in $f(d, |\varphi|)$ rounds by a CONGESTED-CLIQUE algorithm for a computable function $f$.

**Proof (sketch).** We can follow the lines of the proof of Lemma 26 of [31] and observe that the new labels can be computed bottom up along the tree by a CONGESTED-CLIQUE algorithm. For this it suffices to count the number of types of the descendants of a node up to a certain threshold. Hence, the amount of information that has to be sent and stored depends functionally only on $d$ and $\varphi$ and can be sent with low congestion along the forest edges. In the case $h = 0$ in the proof, we crucially use that vertices from different subtrees of the forest can communicate via communication edges that are not edges of the forest.

The rest of the proof works exactly as the proof given in [31] by replacing all subroutines for computing low treedepth colorings and elimination forests by Lemma 16 and Lemma 17.