Singular Cubic Surfaces and the Dynamics of Painlevé VI*

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September 29, 2009

Abstract

We develop a dynamical study of the sixth Painlevé equation for all parameters generalizing an earlier work for generic parameters. Here the main focus of this paper is on non-generic parameters, for which the corresponding character variety becomes a cubic surface with simple singularities and the Riemann-Hilbert correspondence is a minimal resolution of the singular surface, not a biholomorphism as in the generic case. Introducing a suitable stratification on the parameter space and based on geometry of singular cubic surfaces, we establish a chaotic nature of the nonlinear monodromy map of Painlevé VI and give a precise estimate for the number of its isolated periodic solutions.

1 Introduction

The aim of this paper is to develop a dynamical study of the sixth Painlevé equation for all parameters generalizing an earlier work for generic parameters, where the main two issues are establishing a chaotic nature of the nonlinear monodromy map of Painlevé VI and counting the number of its periodic solutions along a given loop. For generic parameters these problems were discussed in a previous paper [20], so that the main focus of this paper is on non-generic parameters. A difficulty in the non-generic case stems from the fact that the corresponding character variety becomes an affine cubic surface with simple singularities and the Riemann-Hilbert correspondence is only an analytic minimal resolution of the singular surface, while it is a biholomorphism onto a smooth cubic surface in the generic case. Therefore, in order to apply a general dynamical systems theory on a smooth compact complex surface to the present situation, one has not only to compactify the affine surface but also to take a minimal resolution of the singular surface. Another difficulty occurs in counting the number of periodic solutions. For non-generic parameters there may be periodic solutions parametrized by a curve, in which case the counting problem obviously fails to make sense in its naïve formulation.

*Mathematics Subject Classification: 34M55, 37F10.
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Thus the main problem of this paper is to overcome these difficulties. As in [20], our principal tool is again a Riemann-Hilbert correspondence, but this time being a lifted one onto a desingularized character variety, namely, a desingularized affine cubic surface. Through it the monodromy map of Painlevé VI is strictly conjugate to a birational map on the latter surface, which in turn extends to a birational map on its compactification obtained by adding tritangent lines at infinity. A close investigation into the last map, especially, into its dynamical behaviors around the tritangent lines as well as around the exceptional set enables us to establish its ergodic properties based on the general theory of bimeromorphic surface maps developed in [1, 7, 8, 10] (see also [5, 6]). As for counting the number of periodic solutions, the difficulty mentioned above is surmounted by our general theory of periodic points for area-preserving birational maps of surfaces in [21], in which a method of counting isolated periodic points is developed in the presence of periodic curves. For all these discussions, a suitable stratification is introduced on the parameter space of Painlevé VI and several arguments are made stratum by stratum, because the singularities of cubic surfaces depend efficiently on the stratification.

The sixth Painlevé equation \( P_{VI}(\kappa) \) is a Hamiltonian system of differential equations with an independent variable \( z \in Z := \mathbb{P}^1 \setminus \{0, 1, \infty\} \) and unknown functions \( (q, p) = (q(z), p(z)) \):

\[
\frac{dq}{dz} = \frac{\partial H(\kappa)}{\partial p}, \quad \frac{dp}{dz} = -\frac{\partial H(\kappa)}{\partial q},
\]

where \( \kappa := (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \) are complex parameters lying in the parameter space

\[
\mathcal{K} := \{ \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5 : 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1 \}
\]

and the Hamiltonian \( H(\kappa) = H(q, p, z; \kappa) \) is given by

\[
z(z - 1)H(\kappa) = (q_0q_1q_2)^2 - \{\kappa_1q_1q_2 + (\kappa_2 - 1)q_0q_1 + \kappa_3q_0q_2\}p + \kappa_0(q_0 + \kappa_4)q_2,
\]

with \( q_\nu := q - \nu \) for \( \nu \in \{0, 1, z\} \). Let \( \mathcal{M}_z(\kappa) \) be the set of all meromorphic solution germs to \( P_{VI}(\kappa) \) at a point \( z \in Z \). In [14, 15, 16], the set \( \mathcal{M}_z(\kappa) \) is realized as a moduli space of stable parabolic connections. Here a stable parabolic connection is a rank 2 vector bundle with a Fuchsian connection and a parabolic structure, having a Riemann scheme as in Table 1 and satisfying a sort of stability condition in geometric invariant theory. The parameter \( \kappa_i \) is the difference of the second exponent from the first one at the regular singular point \( t_i \) and thus \( \lambda_i \) is uniquely determined from \( \kappa_i \). This formulation provides the moduli space \( \mathcal{M}_z(\kappa) \) with the structure of a smooth quasi-projective rational surface. It is known that there exists a natural compactification \( \overline{\mathcal{M}}_z(\kappa) \) of the moduli space \( \mathcal{M}_z(\kappa) \). The space \( \overline{\mathcal{M}}_z(\kappa) \) has a unique effective anti-canonical divisor \( \mathcal{Y}_z(\kappa) \) and \( \mathcal{M}_z(\kappa) \) is obtained from \( \overline{\mathcal{M}}_z(\kappa) \) by removing \( \mathcal{Y}_z(\kappa) \). Thus
there exists a global holomorphic 2-form $\omega_z(\kappa)$ on $\mathcal{M}_z(\kappa)$, meromorphic on $\overline{\mathcal{M}}_z(\kappa)$ with pole divisor $\mathcal{Y}_z(\kappa)$. It is unique up to constant multiples and yields a natural area form on $\mathcal{M}_z(\kappa)$.

The Painlevé equation enjoys the Painlevé property, namely, any solution germ $Q \in \mathcal{M}_z(\kappa)$ can be continued analytically along any loop $\gamma \in \pi_1(Z, z)$, so that the automorphism

$$
\gamma_* : \mathcal{M}_z(\kappa) \xrightarrow{\sim} \mathcal{M}_z(\kappa), \quad Q \mapsto \gamma_* Q
$$

is well defined, where $\gamma_* Q$ is the result of analytic continuation of $Q$ along $\gamma$. The map $\gamma_*$ is called the nonlinear monodromy map along $\gamma$. It preserves the area form $\omega_z(\kappa)$. The dynamical system of this map is what we want to study in this paper. However this map is too transcendental to deal with directly, so that it will be converted to a more tractable map on an affine cubic surface $S(\theta)$ via the Riemann-Hilbert correspondence

$$
\text{RH}_{z, \kappa} : \mathcal{M}_z(\kappa) \rightarrow S(\theta),
$$

which is an analytic minimal resolution of the (possibly) singular surface $S(\theta)$ (see Theorem 5.1). Through it the monodromy map $\gamma_*$ is conjugated to a polynomial automorphism on $S(\theta)$. This last map was studied in [5, 6, 18, 20], but a further investigation is made in this paper.

In terms of the dynamical behavior of the monodromy map, each loop in $\pi_1(Z, z)$ falls into two types, that is, an elementary loop and a non-elementary loop.

**Definition 1.1** Let $\gamma_1, \gamma_2$ and $\gamma_3$ be loops in $\pi_1(Z, z)$ surrounding 0, 1 and $\infty$ respectively once anti-clockwise as in Figure 1. A loop $\gamma \in \pi_1(Z, z)$ is said to be elementary if $\gamma$ is conjugate to the loop $\gamma_i^m$ for some index $i \in \{1, 2, 3\}$ and some integer $m \in \mathbb{Z}$. Otherwise, $\gamma$ is said to be non-elementary.

If $\gamma$ is elementary, the map $\gamma_* : \mathcal{M}_z(\kappa) \cup$ preserves a fibration and exhibits a very simple dynamical behavior (see Remark 5.2). So from now on we assume that $\gamma$ is non-elementary.

**Remark 1.2** In [20] we introduced an algebraic number $\lambda(\gamma) \geq 1$ called the first dynamical degree of $\gamma \in \pi_1(Z, z)$ (see also Definition 2.1) and established an algorithm to calculate $\lambda(\gamma)$ in terms of a reduced word for the loop $\gamma$ in the alphabet $\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \gamma_3^{\pm 1}$ ([20, Theorem 3]). It was also shown that $\lambda(\gamma)$ is a quadratic unit strictly greater than 1 if and only if $\gamma$ is non-elementary.
Example 1.3 Let $\varepsilon$ and $\wp$ denote loops conjugate to $\gamma_i\gamma_j^{-1}$ and $[\gamma_i, \gamma_j^{-1}] = \gamma_i\gamma_j^{-1}\gamma_i^{-1}\gamma_j$ for some indices $\{i, j, k\} = \{1, 2, 3\}$ as in Figure 2. They are called an eight-loop and a Pochhammer loop respectively. Their first dynamical degrees are given by
\[\lambda(\varepsilon) = 3 + 2\sqrt{2}, \quad \lambda(\wp) = 9 + 4\sqrt{5}.\] (2)

The first main theorem of this paper is concerned with the ergodic properties of the monodromy map for all parameters, which generalizes a main theorem in [20] for generic parameters. Before stating it, we review some terminology from [20].

Definition 1.4 The dynamical system of a holomorphic map $f : X \to X$ on a complex surface $X$ is said to be chaotic if there exists an $f$-invariant Borel probability measure $\mu$ on $X$ such that the following conditions are satisfied:

1) the measure $\mu$ has a positive entropy $h_\mu(f) > 0$;
2) the measure $\mu$ is mixing, hyperbolic of saddle type and has a product structure with respect to local stable and unstable manifolds. Moreover, hyperbolic periodic points of $f$ are dense in the support of $\mu$.

Let $\Omega(f)$ denote the nonwandering set of $f$. It is an $f$-invariant set serving as the hub for recurrent behaviors of $f$, that is, it contains the support of any $f$-invariant probability measure.

Theorem 1.5 Let $\gamma \in \pi_1(Z, z)$ be a non-elementary loop and $\gamma_* : \mathcal{M}_z(\kappa) \to \mathcal{M}_z(\kappa)$ be the monodromy map along the loop $\gamma$.

1) The nonwandering set $\Omega(\gamma)$ of $\gamma_*$ is compact in $\mathcal{M}_z(\kappa)$ and the trajectory of each initial point $Q \in \mathcal{M}_z(\kappa) \setminus \Omega(\gamma)$ tends to infinity $\mathcal{Y}_z(\kappa)$ under the iterations of $\gamma_*$. 

2) The map $\gamma_* : \Omega(\gamma) \to \Omega(\gamma)$ is chaotic, that is, there exists a $\gamma_*$-invariant Borel probability measure $\mu_\gamma$ satisfying the conditions of Definition 1.4. The measure-theoretic entropy $h(\gamma) := h_{\mu_\gamma}(\gamma_*)$ with respect to $\mu_\gamma$ and the topological entropy $h_{\text{top}}(\gamma)$ of $\gamma_*$ are given by
\[h(\gamma) = h_{\text{top}}(\gamma) = \log \lambda(\gamma),\] (3)
where $\lambda(\gamma)$ is the first dynamical degree of $\gamma$ (see Remark 1.2). In particular, $\mu_\gamma$ is a unique maximal entropy measure.
Next the second main theorem is concerned with the number of periodic solutions to \( P_{VI}(\kappa) \). A periodic solution to \( P_{VI}(\kappa) \) of period \( n \in \mathbb{N} \) along a loop \( \gamma \) is defined to be a periodic point of period \( n \) of the map \( \gamma_* : \mathcal{M}_{\kappa} \to \mathbb{C} \). For some non-generic parameters, however, \( \gamma_* \) may admit curves of periodic points, in which case the set of periodic points is obviously uncountable and one should consider the set \( \text{Per}^i_n(\gamma; \kappa) \) of isolated periodic points of period \( n \) instead. It will turn out that any periodic curve must be an irreducible component of the exceptional set \( \mathcal{E}_z(\kappa) \) of the Riemann-Hilbert correspondence [1] (see Theorem 1.6). Each irreducible component of \( \mathcal{E}_z(\kappa) \) is known as a Riccati curve, since \( \mathcal{E}_z(\kappa) \) parameterizes all Riccati solution germs to \( P_{VI}(\kappa) \) at \( z \). They are classical special solutions that can be expressed in terms of Gauss hypergeometric functions (see [14, 24, 25, 27]). The second main theorem is now stated as follows.

**Theorem 1.6** Assume that \( \gamma \in \pi_1(Z, z) \) is non-elementary. Then any irreducible periodic curve of \( \gamma_* \) must be a Riccati curve and the set \( \text{Per}^i_n(\gamma; \kappa) \) is finite for any \( n \in \mathbb{N} \) and its cardinality \( \#\text{Per}^i_n(\gamma; \kappa) \) counted with multiplicity is estimated as

\[
|\#\text{Per}^i_n(\gamma; \kappa) - \lambda(\gamma)^n| < O(1) \quad \text{as} \quad n \to \infty,
\]

where \( \lambda(\gamma) \) is the first dynamical degree of \( \gamma \) and \( O(1) \) stands for a bounded function of \( n \in \mathbb{N} \). Moreover, if \( \text{HPer}_n(\gamma; \kappa) \) denotes the set of saddle periodic points of period \( n \) for \( \gamma_* \), then

\[
\#\text{Per}^i_n(\gamma; \kappa) \sim \#\text{HPer}_n(\gamma; \kappa) \sim \lambda(\gamma)^n \quad \text{as} \quad n \to \infty,
\]

where \( a_n \sim b_n \) means that the ratio \( a_n/b_n \to 1 \) as \( n \to \infty \), so that asymptotically almost all points in \( \text{Per}^i_n(\gamma; \kappa) \) belong to \( \text{HPer}_n(\gamma; \kappa) \).

**Remark 1.7** In formula (1) the number \( \#\text{Per}^i_n(\gamma; \kappa) \) may be counted without multiplicity, since \( \text{HPer}_n(\gamma; \kappa) \subset \text{Per}^i_n(\gamma; \kappa) \) and each element of \( \text{HPer}_n(\gamma; \kappa) \) is of simple multiplicity.

**Example 1.8** Here are some explicit formulas for the number \( \#\text{Per}^i_n(\gamma; \kappa) \).

1. If the parameter \( \kappa \in \mathcal{K} \) is generic, then \( \mathcal{E}_z(\kappa) \) is empty and the number of isolated periodic solutions along any non-elementary loop \( \gamma \in \pi_1(Z, z) \) is calculated in [20, Theorem 3] as

\[
\#\text{Per}^i_n(\gamma; \kappa) = \lambda(\gamma)^n + \lambda(\gamma)^{-n} + 4.
\]

2. If \( \kappa = (0, 0, 0, 0, 1) \in \mathcal{K} \), which is non-generic, then \( \mathcal{E}_z(\kappa) \) consists of four Riccati curves and the numbers of isolated periodic solutions along an eight-loop \( \varepsilon \) and a Pochhammer loop \( \varphi \) are calculated as

\[
\#\text{Per}^i_n(\varepsilon; \kappa) = \lambda(\varepsilon)^n + \lambda(\varepsilon)^{-n}, \quad \#\text{Per}^i_n(\varphi; \kappa) = \lambda(\varphi)^n + \lambda(\varphi)^{-n} - 4,
\]

respectively, where \( \lambda(\varepsilon) \) and \( \lambda(\varphi) \) are given by formula (2). Note that \( \varepsilon_* \) fixes exactly one Riccati curve, while \( \varphi_* \) fixes all the four Riccati curves.

In Section [6] the number \( \#\text{Per}^i_n(\varphi; \kappa) \) will be calculated for various values of \( \kappa \in \mathcal{K} \). In general the number \( \#\text{Per}^i_n(\gamma; \kappa) \) is computable at least in principle, once the data of a non-elementary loop \( \gamma \in \pi_1(Z, z) \), a parameter \( \kappa \in \mathcal{K} \) and a period \( n \in \mathbb{N} \) is given explicitly.
The discussion above is about periodic solutions along a single loop in \( \pi_1(Z, z) \). One can also think of periodic solutions with respect to the entire \( \pi_1(Z, z) \)-action, that is, global solutions with finitely many branches. They are exactly algebraic solutions to Painlevé VI ([18]), which have been classified by [23] after many contributions by various authors (see also [2, 19]).

This article is organized as follows. In Section 2 we review some general theories of complex surface dynamics that will be needed to prove our main results. In Section 3 a four-parameter family of affine cubic surfaces is introduced as the target spaces of the Riemann-Hilbert correspondence and the possible types of singularities on them are classified. After investigating polynomial automorphisms on the cubic surfaces in Section 4 we establish our main results in Section 5. Finally, Section 6 is devoted to a thorough study of isolated periodic solutions along a Pochhammer loop for various parameters.

## 2 Preliminaries

In this section we collect some basic results from complex surface dynamics that will be needed later. In order to establish Theorem 1.5 it is necessary to construct an invariant measure satisfying the conditions in Definition 1.4. The first part of this section is a review on the construction of such measures for a class of bimeromorphic maps on smooth surfaces. The second part is a survey on a general theory of isolated periodic points for area-preserving surface maps, which will be used to prove Theorem 1.6. All along the way the following two concepts for bimeromorphic surface maps due to [12, 7] are important.

**Definition 2.1** Let \( f : X \to X \) be a bimeromorphic map on a compact Kähler surface \( X \).

1. The first dynamical degree \( \lambda(f) \) is defined by
   \[
   \lambda(f) := \lim_{n \to \infty} \|(f^n)^*|_{H^{1,1}(X)}\|^1/n \geq 1,
   \]
   where \( \| \cdot \| \) is an operator norm on \( \text{End}H^{1,1}(X) \). It is known that the limit exists, \( \lambda(f) \) is independent of the norm \( \| \cdot \| \) chosen and invariant under bimeromorphic conjugation.

2. The map \( f \) is said to be analytically stable (AS for short) if the condition \((f^n)^* = (f^*)^n : H^{1,1}(X) \to H^{1,1}(X)\) holds for any \( n \in \mathbb{N} \). It is known that \( f \) is AS if and only if
   \[
   f^{-m}I(f) \cap f^nI(f^{-1}) = \emptyset \quad \text{for every} \quad m, n \geq 0,
   \]
   where \( I(f) \) is the indeterminacy set of \( f \). If \( f \) is AS then the first dynamical degree \( \lambda(f) \) coincides with the spectral radius of the linear map \( f^*|_{H^{1,1}(X)} \).

We begin with a review on invariant measures. Under the condition
\[
\lambda(f) > 1,
\]
Bedford and Diller [1] constructed “good” positive closed \((1, 1)\)-currents \( \mu^\pm \) on \( X \) such that
\[
(f^{\pm1})^*\mu^\pm = \lambda(f) \mu^\pm,
\]
where $\mu^+$ and $\mu^-$ are called the stable and unstable currents for $f$. They represent unique (up to scale) nef classes $\theta^\pm \in H^{1,1}_\mathbb{R}(X)$ such that $(f^\pm)^*\theta^\pm = \lambda(f)\theta^\pm$ and can be expressed as

$$\mu^\pm = c^\pm_\omega \lim_{k \to \infty} \lambda(f)^{-k}(f^\pm)^k \omega,$$

where $\omega$ is any given smooth closed $(1,1)$-form on $X$ and $c^\pm_\omega > 0$ are constants. Moreover, under a quantitative condition on the indeterminacy sets of the forward and backward maps:

$$\sum_{N=0}^{\infty} \lambda(f)^{-N} \log \text{dist}(f^N I(f^{-1}), I(f)) > -\infty,$$

Bedford and Diller [1] and Dujardin [10] legitimated the wedge product $\mu := \mu^+ \wedge \mu^-$ as an $f$-invariant Borel probability measure, where the first authors defined it by appealing to pluripotential theory while the second author viewed it as geometric intersection. The condition (7) is slightly stronger than (5) and a map satisfying condition (7) might be called quantitatively AS. The measure has good dynamical properties as is mentioned in the following.

**Theorem 2.2** ([1, 10]) If $f : X \to X$ satisfies conditions (6) and (7), then the wedge product $\mu$ of stable and unstable currents $\mu^\pm$ is well defined and, after a suitable normalization, $\mu$ gives an $f$-invariant Borel probability measure satisfying all the conditions in Definition 1.4. Moreover,

1. the measure-theoretic entropy $h_\mu(f)$ with respect to the measure $\mu$ and the topological entropy $h_{\text{top}}(f)$ of $f$ are expressed as

$$h_\mu(f) = h_{\text{top}}(f) = \log \lambda(f),$$

2. the measure $\mu$ puts no mass on any algebraic curve on $X$,

3. there exists a set $P_n(f) \subset \text{supp} \mu$ of saddle periodic points of period $n$ such that

$$\#P_n(f) \sim \lambda(f)^n, \quad \frac{1}{\lambda(f)^n} \sum_{p \in P_n(f)} \delta_p \to \mu, \quad \text{as } n \to \infty.$$

**Remark 2.3** The definition of entropy needs some care for a bimeromorphic surface map $f : X \to X$, since it may have indeterminacy sets $I(f^\pm)$ on which $f^\pm$ are not defined. To handle this situation, notice that $f$ restricts to a well-defined automorphism $f|_{X_f}$ of the space

$$X_f := X \setminus \left( \bigcup_{n \geq 0} f^n I(f) \cup f^{-n} I(f^{-1}) \right).$$

If a Borel probability measure $\nu$ on $X$ satisfies $\nu(X_f) = 1$, then the measure-theoretic entropy of $f$ with respect to $\nu$ can be defined by $h_\nu(f) := h_\nu(f|_{X_f})$ in terms of the map $f : X_f \to X_f$ (see [13]). The measure $\mu$ constructed in Theorem 2.2 satisfies this condition and so the entropy $h_\mu(f)$ is well defined. In the same spirit the topological entropy of $f$ is defined by $h_{\text{top}}(f) := h_{\text{top}}(f|_{X_f})$, where the right-hand side employs Bowen’s definition on a non-compact space (see [3]). Similarly the nonwandering set $\Omega(f)$ of $f$ is that of $f|_{X_f}$, i.e., $\Omega(f) := \Omega(f|_{X_f})$. 

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We proceed to a survey of the results of [21] on the number of isolated periodic points for area-preserving surface maps with periodic curves. Let \( f : X \to X \) be a birational map of a smooth projective surface \( X \). Since \( f \) may have indeterminacy sets \( I(f^{\pm 1}) \) on which \( f^{\pm 1} \) are not defined, we should again be careful with the definitions of a fixed point and a fixed curve.

**Definition 2.4** A point \( x \in X \) is called a fixed point if \( x \) is an element of the set

\[
X_0(f) := X_0^0(f) \cup X_0^0(f^{-1}),
\]

where \( X_0^0(f) \) is the set of all points \( x \in X \setminus I(f) \) fixed by \( f \). Moreover let \( X_1(f) \) be the set of all irreducible curves \( C \) in \( X \) such that \( C \setminus I(f) \) is fixed pointwise by \( f \). An element \( C \in X_1(f) \) is called a fixed curve. It is easy to see that the definition is symmetric, namely, \( X_1(f) = X_1(f^{-1}) \).

As in [21, 26] the set \( X_1(f) \) of fixed curves is divided into two disjoint subsets:

\[
X_1(f) = X_I(f) \cup X_{II}(f),
\]

where \( X_I(f) \) indicates that \( Y \) is birationally equivalent to \( X \) and those of type \( II \) (see Definition 2.12).

The set of periodic points of period \( n \) for \( f \) is defined by \( \text{Per}_n(f) := X_0(f^n) \). The subset of isolated ones is denoted by \( \text{Per}_n^i(f) \) and its cardinality counted with multiplicity is defined by

\[
\#\text{Per}_n^i(f) := \sum_{x \in \text{Per}_n^i(f)} \nu_x(f^n),
\]

where \( \nu_x(f^n) \) is the local index of \( f^n \) at \( x \in X_0(f^n) \) to be defined in Definition 2.11.

**Theorem 2.5 ([21])** Let \( f : X \to X \) be an AS birational map with \( \lambda(f) > 1 \) on a smooth projective surface and assume that \( f \) preserves a nontrivial meromorphic 2-form \( \omega \) such that no irreducible component of the pole divisor \( (\omega)_\infty \) of \( \omega \) is a periodic curve of type \( I \). Then \( f \) has at most finitely many irreducible periodic curves and must have infinitely many isolated periodic points. Moreover the cardinality \( \#\text{Per}_n^i(f) \) is estimated as

\[
|\#\text{Per}_n^i(f) - \lambda(f)^n| \leq \begin{cases} O(1) & (X \sim \text{no Abelian surface}), \\ 4 \lambda(f)^{n/2} + O(1) & (X \sim \text{an Abelian surface}), \end{cases}
\]

where \( X \sim Y \) indicates that \( X \) is birationally equivalent to \( Y \) and \( O(1) \) is a bounded function of \( n \in \mathbb{N} \).

Let \( H\text{Per}_n(f) \) be the set of all saddle periodic points of period \( n \) and \( \mathcal{P}_n(f) \) the set of saddle periodic points mentioned in Theorem 2.2. Then one has \( \mathcal{P}_n(f) \subset H\text{Per}_n(f) \subset \text{Per}_n^i(f) \), since any saddle periodic point is isolated. Thus Theorems 2.2 and 2.5 have the following.

**Corollary 2.6** If \( f \) satisfies the assumptions in Theorems 2.2 and 2.5, then

\[
\#\text{Per}_n^i(f) \sim \#H\text{Per}_n(f) \sim \lambda(f)^n \quad \text{as } n \to \infty,
\]

so that asymptotically almost all points in \( \text{Per}_n^i(f) \) belong to \( H\text{Per}_n(f) \).

**Remark 2.7** In formula (10) the number \( \#\text{Per}_n^i(f) \) may be counted without multiplicity, because \( H\text{Per}_n(f) \) is a subset of \( \text{Per}_n^i(f) \), every point in \( H\text{Per}_n(f) \) is of simple multiplicity, and the asymptotics \( \#H\text{Per}_n(f) \sim \lambda(f)^n \) holds with multiplicity taken into account.
The above results are derived from a basic formula representing the Lefschetz numbers of iterates of $f$ in terms of suitable local data around isolated periodic points as well as around periodic curves of $f$ (see Theorem 2.5). Here the Lefschetz number of $f$ is defined by

$$L(f) := \sum_i (-1)^i \text{Tr}[f^*: H^i(X) \to H^i(X)].$$

In order to state that formula, let $P(f)$ be the set of all positive integers that arise as primitive periods of some irreducible periodic curves of $f$. For each $n \in \mathbb{N}$, denote by $P_n(f)$ the set of all elements $k \in P(f)$ that divides $n$. Moreover, for each $k \in P(f)$ let $PC_k(f)$ be the set of all irreducible periodic curves of primitive period $k$, and $C_k(f)$ the union of all curves in $PC_k(f)$. Note that there exist the following decompositions:

$$X_0(f^n) = \text{Per}_n^i(f) \bigcup_{k \in P_n(f)} C_k(f), \quad X_1(f^n) = \bigcup_{k \in P_n(f)} PC_k(f).$$

For each $k \in P(f)$ let $\xi_k(f)$ be the number defined by

$$\xi_k(f) := \sum_{x \in C_k(f)} \nu_x(f^k) + \sum_{C \in PC_k(f)} \tau_C \cdot \nu_C(f^k).$$

where $\tau_C$ is the self-intersection number of $C$. Then our formula is stated as follows.

**Theorem 2.8** ([21]) If $f : X \to X$ satisfies the assumptions in Theorem 2.5 then

$$L(f^n) = \#\text{Per}_n^i(f) + \sum_{k \in P_n(f)} \xi_k(f) \quad (n \in \mathbb{N}).$$

This formula is used not only to get the general estimate in Theorem 2.5 but also to calculate the exact value of $\#\text{Per}_n^i(f)$ for various individual maps $f$ (see Section 6).

Now let us recall the definitions of $\nu_x(f), \nu_C(f), X_1(f)$ and $X_H(f)$. Leaving the general cases in [21, 26] we put the following assumption for the sake of simplicity. It will be fulfilled by the birational maps on (desingularized) cubic surfaces to be discussed later (see Remark 4.8).

**Assumption 2.9** All fixed curves of $f : X \to X$ are smooth, no two of which are tangent and no three of which meet in a single point.

For a given point $x \in X_0(f)$ let $A_x$ be the completion of the local ring of $X$ at $x$, which can be identified with the formal power series ring $\mathbb{C}[x_1, x_2]$ of two variables, because $X$ is smooth. Since $f$ is holomorphic around $x$, $f$ induces an endomorphism $f^*_x : A_x \to A_x$ in a natural manner. From Assumption 2.9 $f^*_x$ can be expressed in suitable coordinates $(x_1, x_2)$ as

$$\begin{cases} f^*_x(x_1) &= x_1 + x_1^{n_1} \cdot x_2^{n_2} \cdot h_1, \\ f^*_x(x_2) &= x_2 + x_1^{n_1} \cdot x_2^{n_2} \cdot h_2, \end{cases}$$

with some relatively prime elements $h_1, h_2 \in A_x$ and some nonnegative integers $n_1, n_2 \in \mathbb{Z}_{\geq 0}$. For $\{j, k\} = \{1, 2\}$ let $\tau_{(x_j)} : \mathbb{C}[x_1, x_2] \to \mathbb{C}[x_k]$ denote the natural projection. Write

$$\tau_{(x_j)}(h_i) = x_k^{n_{ij}} \cdot h_{ij} \quad (i, j \in \{1, 2\}),$$

where $h_{ij}$ is a unit in $\mathbb{C}[x_k]$ and $n_{ij}$ is either an integer or infinity. By convention $x_k^\infty := 0$. 

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For each \( i \in \{ 1, 2 \} \) we put
\[
\nu_{p_i}(f_x^*) := n_{ii}, \quad (11)
\]
\[
\nu_{A_x}(f_x^*) := \dim C A_x/(h_1, h_2) + \sum_{i=1}^{2} \nu_{p_i}(f_x^*) \cdot \mu_{p_i}(f_x^*), \quad (12)
\]
where \( A_x/(h_1, h_2) \) is the quotient vector space of \( A_x \) by the ideal \((h_1, h_2)\), which is finite-dimensional from Assumption 2.9, and the numbers \( \mu_{p_i}(f_x^*) \) are defined by
\[
\mu_{p_i}(f_x^*) := \begin{cases} n_{ii} & \text{(if } p_i \text{ is of type I)}, \\ n_{ji} & \text{(if } p_i \text{ is of type II)}, \end{cases}
\]
with \( \{ i, j \} = \{ 1, 2 \} \). Similarly for a given point \( x \in X_0(f^{-1}) \) one can define the number \( \nu_{A_x}((f^{-1})_x^*) \) via formula (12). Next, given a fixed curve \( C \in X_1(f) \), take a point \( x \) of \( C \setminus I(f) \). Then one can speak of the endomorphism \( f_x^* : A_x \to A_x \). Since \( C \) is smooth by Assumption 2.9 the prime ideal \( p_C \subset A_x \) defining the germ at \( x \) of the curve \( C \) may be written \( p_C = (x_1) \) in suitable coordinates \((x_1, x_2)\), so that one can define the number \( \nu_{p_C}(f_x^*) \) via formula (11).

**Definition 2.11** The local index \( \nu_x(f) \) at a fixed point \( x \in X_0(f) \) is defined by
\[
\nu_x(f) := \begin{cases} \nu_{A_x}(f_x^*) & \text{(if } x \in X_0^0(f)), \\ \nu_{A_x}((f^{-1})_x^*) & \text{(if } x \in X_0^0(f^{-1})), \end{cases}
\]
where the right-hand side is consistent, that is, \( \nu_{A_x}(f_x^*) = \nu_{A_x}((f^{-1})_x^*) \) for any \( x \in X_0^0(f) \cap X_0^0(f^{-1}) \) (see [21]). The index \( \nu_C(f) \) at a fixed curve \( C \in X_1(f) \) is defined by
\[
\nu_C(f) := \nu_{p_C}(f_x^*),
\]
with \( x \in C \setminus I(f) \), where the right-hand side does not depend on the choice of \( x \) (see [26]).

Finally we recall the following definition concerning the types of fixed curves.

**Definition 2.12** A fixed curve \( C \in X_1(f) \) is said to be of type I or of type II relative to \( f : X \to X \), according as the prime ideal \( p_C \) is of type I or of type II relative to \( f_x^* : A_x \to A_x \) in the sense of Definition 2.10. This definition does not depend on the choice of \( x \in C \setminus I(f) \) (see [26]). Let \( X_I(f) \) and \( X_{II}(f) \) denote the set of fixed curves of types I and that of type II respectively. Then there exists the direct sum decomposition (9).

### 3 Singular Cubic Surfaces

The purpose of this section is to introduce a four-parameter family of affine cubic surfaces which are the target spaces of the Riemann-Hilbert correspondence (11) and to classify the types of singularities that can occur on those surfaces. The affine cubic surfaces we consider are
\[
S(\theta) = \{ x = (x_1, x_2, x_3) \in \mathbb{C}^3 : f(x, \theta) = 0 \},
\]
where \( f(x, \theta) \) is a cubic polynomial of \( x \) with parameters \( \theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \Theta := \mathbb{C}^4 \):

\[
f(x, \theta) := x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3 + \theta_4.
\]

Depending on parameters \( \theta \in \Theta \) the surface \( S(\theta) \) may admit singular points. In order to describe its singularity structure, it is convenient to introduce a map

\[
\text{rh} : \mathcal{K} \to \Theta,
\]

called the Riemann-Hilbert correspondence in the parameter level. It is the composite of maps

\[
\mathcal{K} \overset{\beta}{\longrightarrow} B \overset{\alpha}{\longrightarrow} A \overset{\varphi}{\longrightarrow} \Theta,
\]

where the intermediate parameter spaces \( A \) and \( B \) are given by

\[
A := \{a = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4_a\}, \quad B := \{b = (b_0, b_1, b_2, b_3, b_4) \in (\mathbb{C}^5_b : b_0^3 b_1 b_2 b_3 b_4 = 1\},
\]

and the three maps \( \varphi, \alpha \) and \( \beta \) are defined respectively by

\[
\begin{align*}
\theta_i &= \begin{cases} a_i a_4 + a_j a_k \quad & (\{i, j, k\} = \{1, 2, 3\}), \\
a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4 \quad & (i = 4), \end{cases} \\
a_i &= b_i + b_i^{-1} \quad (i = 1, 2, 3, 4), \\
b_i &= \begin{cases} \exp(\sqrt{-1} \pi \kappa_i) \quad & (i = 0, 1, 2, 3), \\
-\exp(\sqrt{-1} \pi \kappa_4) \quad & (i = 4). \end{cases}
\end{align*}
\]

It turns out that the discriminant \( \Delta(\theta) \) of the cubic surface \( S(\theta) \) factors as

\[
\Delta(\theta) = \prod_{i=1}^{4} (b_i - b_i^{-1})^2 \prod_{\varepsilon \in \{\pm 1\}^4} (b^\varepsilon - 1) = \prod_{i=1}^{4} \sin^2 \pi \kappa_i \prod_{\varepsilon \in \{\pm 1\}^4} \cos \frac{\pi (\varepsilon \cdot \kappa)}{2},
\]

where \( b^\varepsilon := b_1^\varepsilon b_2^\varepsilon b_3^\varepsilon b_4^\varepsilon \) and \( \varepsilon \cdot \kappa := \varepsilon_1 \kappa_1 + \varepsilon_2 \kappa_2 + \varepsilon_3 \kappa_3 + \varepsilon_4 \kappa_4 \) for each \( \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \{\pm 1\}^4 \).

Thus \( S(\theta) \) is singular if and only if \( \kappa \) satisfies at least one of the affine linear relations:

\[
\kappa_i = m, \quad \varepsilon \cdot \kappa = 2m + 1 \quad (m \in \mathbb{Z}, i \in \{1, 2, 3, 4\}, \varepsilon \in \{\pm 1\}^4).
\]

Behind formulas (16) and (17) there exists an affine Weyl group structure on the Riemann-Hilbert correspondence in the parameter level (13). The affine space \( \mathcal{K} \) carries the inner product induced from the standard complex Euclidean inner product on \( \mathbb{C}^4 \) via the isomorphism \( \mathcal{K} \cong \mathbb{C}^4, \kappa \mapsto (\kappa_1, \kappa_2, \kappa_3, \kappa_4) \). For each \( i \in \{0, 1, 2, 3, 4\} \) let \( w_i : \mathcal{K} \to \mathcal{K} \) be the orthogonal affine reflection in the affine hyperplane \( H_i := \{\kappa \in \mathcal{K} : \kappa_i = 0\} \). Explicitly \( w_i \) is given by

\[
w_i(\kappa_j) = \kappa_j - \kappa_i c_{ij},
\]

where \( C = (c_{ij}) \) is the Cartan matrix of type \( D_4^{(1)} \) in Figure 3 (right). The group

\[
W(D_4^{(1)}) := \langle w_0, w_1, w_2, w_3, w_4 \rangle
\]
generated by \(w_0, w_1, w_2, w_3, w_4\) is an affine Weyl group of type \(D_4^{(1)}\). The hyperplanes in (17) are the reflection hyperplanes of \(W(D_4^{(1)})\) and the map (13) is a branched \(W(D_4^{(1)})\)-covering ramifying along these hyperplanes. The automorphism group of the Dynkin diagram in Figure 3 (left) is the symmetric group \(S_4\) of degree 4, which acts on \(K\) by permuting \(\kappa_1, \kappa_2, \kappa_3, \kappa_4\) and fixing \(\kappa_0\). The group \(W(D_4^{(1)})\) extends to an affine Weyl group of type \(F_4^{(1)}\):

\[
W(F_4^{(1)}) := S_4 \ltimes W(D_4^{(1)}).
\]

Corresponding to the two affine Weyl groups mentioned above, we can define two stratifications on \(K\). Let \(\mathcal{I}\) be the set of all proper subsets \(I \subset \{0,1,2,3,4\}\) including the empty set \(\emptyset\). For each \(I \in \mathcal{I}\) we denote by \(\overline{K}_I\) the \(W(D_4^{(1)})\)-translates of the affine subspace

\[
H_I := \bigcap_{\kappa \in I} H_{\kappa},
\]

and by \(K_I\) the set obtained from \(\overline{K}_I\) by removing all the sets \(K_J\) such that \(#J = #I + 1\). Moreover let \(D_I\) be the Dynkin subdiagram of \(D_4^{(1)}\) that has nodes exactly in \(I\). Some examples of these are given in Figure 4. It turns out that for any pair \(I, I' \in \mathcal{I}\) either \(K_I = K_{I'}\) or \(K_I \cap K_{I'} = \emptyset\) holds (see Remark 3.1), so that one can define a stratification of \(K\) by the subsets \(K_I\) with \(I \in \mathcal{I}\). It is called the \(W(D_4^{(1)})\)-stratification. A parameter \(\kappa \in K\) is said to be generic if \(\kappa \in K_\emptyset\); otherwise, \(\kappa\) is said to be non-generic.

For \(I \in \mathcal{I}\) one can speak of the abstract Dynkin type of the subdiagram \(D_I\). For example, \(D_I\) is of abstract type \(A_3\) when \(I = \{0,1,2\}\). All the possible abstract Dynkin types are those in Figure 5 below. On the other hand there is a natural action of \(S_4\) on the set \(\mathcal{I}\) induced from its action on the nodes \(\{1,2,3,4\}\) and the abstract Dynkin type of \(D_I\) is represented by the \(S_4\)-orbit of \(I\). Thus all the abstract Dynkin types are parametrized by the quotient set \(\mathcal{I}/S_4\).
Figure 5: Adjacency relations among the $W(F_4^{(1)})$-strata

Remark 3.1 ([18]) Let $I$ and $I'$ be distinct elements of $\mathcal{I}$. Then $K_I = K_{I'}$ if and only if $D_I$ and $D_{I'}$ have the same abstract type $A_1$ or $A_2$. Here the “if” part is shown by a direct calculation, while the “only if” part follows from the fact that the map $[13]$ is $W(D_4^{(1)})$-invariant and $\text{rh}(\kappa) \neq \text{rh}(\kappa')$ for any $\kappa \in K_I$ and $\kappa' \in K_{I'}$ if the condition is not fulfilled. Therefore,

1. there is a unique $W(D_4^{(1)})$-stratum of abstract type $\emptyset$, $A_1$, $A_2$ or $A_1^{\oplus 4}$,
2. there are six $W(D_4^{(1)})$-strata of abstract type $A_1^{\oplus 2}$ or $A_3$,
3. there are four $W(D_4^{(1)})$-strata of abstract type $A_1^{\oplus 3}$ or $D_4$.

We proceed to define a coarser stratification, that is, the $W(F_4^{(1)})$-stratification. Observe that the $W(F_4^{(1)})$-translates of $H_I$ depends only on the abstract Dynkin type $\ast = [I] \in \mathcal{I}/S_4$, so that it is denoted by $\overline{K}(\ast)$. Note that $\overline{K}(\ast)$ is the $W(D_4^{(1)})$-translates of the union

$$H(\ast) := \bigcup_{[I]=\ast} H_I.$$  \hspace{1cm} (18)

We say that $\ast\ast$ is adjacent to $\ast$ and write $\ast \rightarrow \ast\ast$ if $\overline{K}(\ast\ast)$ is a subset of $\overline{K}(\ast)$. All the adjacency relations are depicted in Figure 5. The set $K(\ast)$ is obtained from $\overline{K}(\ast)$ by removing all the sets $\overline{K}(\ast\ast)$ such that $\ast \rightarrow \ast\ast$. There is a direct sum decomposition

$$K = \coprod_{\ast \in \mathcal{I}/S_4} K(\ast).$$

Note that the $W(F_4^{(1)})$-stratum $K(\ast)$ is the union of all $W(D_4^{(1)})$-strata of abstract type $\ast$.

We compactify the affine cubic surface $\mathcal{S}(\theta)$ by the standard embedding:

$$\mathcal{S}(\theta) \hookrightarrow \overline{\mathcal{S}(\theta)} \subset \mathbb{P}^3, \quad x = (x_1, x_2, x_3) \mapsto [1 : x_1 : x_2 : x_3],$$

where the compactified surface is given by $\overline{\mathcal{S}(\theta)} = \{X \in \mathbb{P}^3 : F(X, \theta) = 0\}$ with

$$F(X, \theta) = X_1X_2X_3 + X_0(X_1^2 + X_2^2 + X_3^2) - X_0^2(\theta_1X_1 + \theta_2X_2 + \theta_3X_3) + \theta_4X_0^3.$$

The intersection of $\overline{\mathcal{S}(\theta)}$ with the plane at infinity is the union $L$ of three lines

$$L_i = \{X \in \mathbb{P}^3 : X_0 = X_i = 0\} \quad (i = 1, 2, 3).$$

The set $L$, called the tritangent lines at infinity (see Figure 6), is independent of $\theta \in \Theta$ and the surface $\overline{\mathcal{S}(\theta)}$ is smooth in a neighborhood of $L$ for every $\theta \in \Theta$ (see [20, Lemma 2]). The
intersection point of $L_j$ and $L_k$ is denoted by $p_i$ for $\{i, j, k\} = \{1, 2, 3\}$. It is explicitly given by

$$p_1 = [0 : 1 : 0 : 0], \quad p_2 = [0 : 0 : 1 : 0], \quad p_3 = [0 : 0 : 0 : 1].$$

When the parameter $\theta = rh(\kappa)$ is non-generic, based on a method in [4] we construct an algebraic minimal resolution of the singular surface $\mathfrak{S}(\theta)$ by considering the rational map:

$$\tau : \mathbb{P}^2 \longrightarrow \mathbb{P}^3, \quad u = [u_1 : u_2 : u_3] \longmapsto [\tau_0(u) : \tau_1(u) : \tau_2(u) : \tau_3(u)], \quad (19)$$

where the polynomials $\tau_0(u), \tau_1(u), \tau_2(u), \tau_3(u)$ are given by

$$\begin{cases}
\tau_0(u) & := -b_6^2 u_1 u_2 u_3, \\
\tau_1(u) & := b_6^2 u_1 \{b_6^2 u_1^2 + u_2^2 + u_3^2 + b_6^2 (b_1 b_2 + b_3 b_4) u_1 u_2 + b_6^2 (b_1 b_3 + b_2 b_4) u_1 u_3\}, \\
\tau_2(u) & := u_2 \{b_6^2 u_1^2 + b_6^2 u_2^2 + u_3^2 + b_6^2 (b_1 b_2 + b_3 b_4) u_1 u_2 + b_6^2 (b_2 b_3 + b_1 b_4) u_2 u_3\}, \\
\tau_3(u) & := u_3 \{b_6^2 u_1^2 + b_6^2 u_2^2 + u_3^2 + b_6^2 (b_2 b_3 + b_1 b_4) u_2 u_3 + b_6^2 (b_1 b_3 + b_2 b_4) u_1 u_3\}.
\end{cases}$$

It is a birational map of $\mathbb{P}^2$ onto $\mathfrak{S}(\theta)$ whose indeterminacy points are the six points

$$\begin{align*}
c_1 & := [0 : -b_1 b_4 : 1], \quad c_4 := [0 : -b_2 b_3 : 1] \in l_1, \\
c_2 & := [-b_1 b_3 : 0 : 1], \quad c_5 := [-b_2 b_4 : 0 : 1] \in l_2, \\
c_3 & := [-b_3 b_4 : 1 : 0], \quad c_6 := [-b_1 b_2 : 1 : 0] \in l_3, \quad (20)
\end{align*}$$

where $l_i$ is the strict transform of the line $L_i$ under the map (19) and is given by

$$l_i = \{[u_1 : u_2 : u_3] \in \mathbb{P}^2; u_i = 0\} \quad (i \in \{1, 2, 3\}).$$

Let $\rho : \tilde{\mathfrak{S}}(\theta) \rightarrow \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the six points $c_1, \ldots, c_6$, and put

$$\pi := \rho \circ \tau : \tilde{\mathfrak{S}}(\theta) \rightarrow \mathfrak{S}(\theta). \quad (21)$$
Proposition 3.2 The birational morphism $[\theta]$ gives an algebraic minimal resolution of $\bar{S}(\theta)$.

Proof. Starting with $S_0 := \mathbb{P}^2$ we inductively define $\rho_i : S_i \to S_{i-1}$ as the blow-up of $S_{i-1}$ at the point $c_i$ for $i = 1, \ldots, 6$. The birational morphism $\rho : \bar{S}(\theta) \to \mathbb{P}^2$ then decomposes as

$$\rho = \rho_1 \circ \cdots \circ \rho_6.$$ 

Consider the linear system $\delta := \{p_0\tau_0 + p_1\tau_1 + p_2\tau_2 + p_3\. \in \mathbb{P}^3 \} \subset |3H|$, where $H$ is a line in $\mathbb{P}^2$. Starting with $\delta_0 := \delta$ and $D_0 := 3H$ we inductively define $\delta_i := \rho_i^*\delta_{i-1} - E_i$ and $D_i := \rho_i^*D_{i-1} - E_i$ for $i = 1, \ldots, 6$, where $E_i := \rho_i^*(c_{i-1})$ is the exceptional curve of $\rho_i$ over the point $c_{i-1}$. In view of (19) the linear system $\rho_i^*\delta_{i-1} \subset |\rho_i^*D_{i-1}|$ admits $E_i$ as its fixed part and thus one has the inclusion $\delta_i \subset |D_i|$. Since the canonical divisor $K_{S_0} = K_{\mathbb{P}^2}$ on $\mathbb{P}^2$ is linearly equivalent to the divisor $-3H = -D_0$, we have

$$K_{S_i} = \rho_i^*K_{S_{i-1}} + E_i \sim -(\rho_i^*D_{i-1} - E_i) = -D_i \quad (i = 1, \ldots, 6).$$

In particular the canonical divisor $K_{\bar{S}(\theta)} = K_{S_0}$ on $\bar{S}(\theta)$ is linearly equivalent to $-D_6$. Therefore for any $(-1)$-curve $C$ on $\bar{S}(\theta)$ we have $C \cdot D_6 = (C \cdot K_{\bar{S}(\theta)}) = 1$ from the adjunction formula. This means that the birational morphism $[\theta]$ never contracts any $(-1)$-curve $C$ to a point in $\bar{S}(\theta)$. Thus the proposition is established.

Remark 3.3 The surface $\bar{S}(\theta)$, if it is smooth, is known as a Del Pezzo surface of degree 3. On the other hand, if $\bar{S}(\theta)$ is singular, the desingularized surface $\tilde{S}(\theta)$ is called a degenerate Del Pezzo surface of degree 3 in [9]. Any degenerate Del Pezzo surface of degree 3 is obtained as a blow-up of $\mathbb{P}^2$ at six points, no two of which are infinitely near of order 1 to the same point and no four of which are collinear. The image of a degenerate Del Pezzo surface under the map of its anti-canonical class is called an anti-canonical Del Pezzo surface of degree 3, an example of which is our cubic surface $\bar{S}(\theta)$.

Since $\tilde{S}(\theta)$ is a six-point blow-up of $\mathbb{P}^2$, its second cohomology group is expressed as

$$H^2(\tilde{S}(\theta), \mathbb{Z}) = \mathbb{Z}E_0 \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3 \oplus \mathbb{Z}E_4 \oplus \mathbb{Z}E_5 \oplus \mathbb{Z}E_6,$$
where $E_0$ is the class of the strict transform of a line in $\mathbb{P}^2$ and $E_i$ is the class of the exceptional curve over the point $c_i$ for $i = 1, 2, 3, 4, 5, 6$. Their intersection relations are listed as

$$
(E_i, E_j) = \begin{cases} 
1 & (i = j = 0), \\
-1 & (i = j \neq 0), \\
0 & \text{(otherwise)}. 
\end{cases}
$$  \hfill (22)

If the line $L_i$ is identified with the cohomology class of its strict transform under $\pi$, we have

$$L_1 = E_0 - E_1 - E_4, \quad L_2 = E_0 - E_2 - E_5, \quad L_3 = E_0 - E_3 - E_6. \quad \hfill (23)$$

In view of formulas (22) and (23) there exists a direct sum decomposition

$$H^2(\tilde{S}(\theta), \mathbb{C}) = V \oplus V^\perp, \quad \hfill (24)$$

where $V$ is the subspace spanned by $L_1, L_2, L_3$ and $V^\perp$ is the orthogonal complement to $V$ with respect to the intersection form. It is easy to see that $V^\perp$ is spanned by the vectors

$$L_4 := E_1 - E_4, \quad L_5 := E_1 - E_5, \quad L_6 := E_1 - E_6, \quad L_7 := 2E_0 - E_1 - E_2 - E_3 - E_4 - E_5 - E_6. \quad \hfill (25)$$

The decomposition (24) will be important in the ergodic study of birational maps on $\tilde{S}(\theta)$.

Now we investigate the structure of singularities on $\tilde{S}(\theta)$ utilizing the rational map $\varphi = \alpha \circ \varphi_\circ \alpha(b)$. For a case-by-case treatment we divide the parameter space $B$ into several pieces $\overline{B}(\ast)$ depending on abstract Dynkin types $\ast \in \mathcal{I}/S_4$, each of which is further decomposed into smaller pieces $\overline{B}_m(\ast)$ and then into even smaller pieces $B_m(\ast,-)$. Here the definitions of $\overline{B}_m(\ast)$ and $\overline{B}_m(\ast,-)$ are given in Table 2. Put

$$B(\ast) := \bigcup_{\ast \rightarrow \ast \ast} \overline{B}(\ast).$$

**Proposition 3.4** Given any $b \in B$, put $\theta = (\varphi \circ \alpha)(b)$. Then $\tilde{S}(\theta)$ has simple singularities of abstract Dynkin type $\ast \in \mathcal{I}/S_4$ if and only if the parameter $b$ is an element of $B(\ast) := \bigcup_m B_m(\ast)$.

**Proof.** A careful inspection of formulas (16) and (20) shows that the surface $\tilde{S}(\theta)$ is singular if and only if the indeterminacy points $c_1, \ldots, c_6 \in \mathbb{P}^2$ of the rational map (19) are not in a general position, namely, if and only if at least one of the following conditions are satisfied.

(C1) The six points lie on a (unique) conic; this condition is equivalent to $b_1 b_2 b_3 b_4 = 1$.

(C2) Three of them, say, $c_i, c_j$ and $c_k$ are colinear; this condition is equivalent to

$$
\begin{align*}
&b_4 - b_4^{-1} = 0, & &\{i, j, k\} = \{1, 2, 3\}, \\
b_1 b_2 b_3 b_4^{-1} = 1, & &\{i, j, k\} = \{4, 5, 6\}, \\
b_i - b_i^{-1} = 0, & &i \in \{1, 2, 3\}, \quad \{j, k\} = \{4, 5, 6\} \setminus \{i + 3\}, \\
b_i^{-1} b_j b_k b_4 = 1, & &i \in \{4, 5, 6\}, \quad \{j, k\} = \{1, 2, 3\} \setminus \{i - 3\}.
\end{align*}
$$

(C3) $c_i = c_{i+3}$ for some $i \in \{1, 2, 3\}$, that is, $b_i b_j^{-1} b_k^{-1} b_4 = 1$ with $\{i, j, k\} = \{1, 2, 3\}$. \hfill 16
| Parameter spaces $\overline{B}_m(\ast)$ | Defining equations of $\overline{B}_m(\ast; -)$ |
|----------------------------------------|-----------------------------------------------|
| $\overline{B}_1(D_4) = \overline{B}_1(D_4; \ast)$ | $\varepsilon_1 b_1 = \varepsilon_2 b_2 = \varepsilon_3 b_3 = \varepsilon_4 b_4 = 1$ |
| $\overline{B}_1(A_1^\oplus 4) = \overline{B}_1(A_1^\oplus 4; \ast)$ | $\varepsilon_1 b_1 = \varepsilon_2 b_2 = \varepsilon_3 b_3 = -\varepsilon_4 b_4 \in \{1, \sqrt{-1}\}$ |
| $\overline{B}_2(A_1^\oplus 4) = \overline{B}_2(A_1^\oplus 4; \ast)$ | $\varepsilon_1 b_1 = \varepsilon_2 b_2 = \varepsilon_3 b_3 = \varepsilon_4 b_4 = \sqrt{-1}$ |
| $\overline{B}_1(A_3) = \bigcup_{1 \leq i < j \leq 4} \overline{B}_1(A_3; i, j)$ | $\varepsilon_i b_i = \varepsilon_j b_j = 1, \varepsilon_k b_k = \varepsilon_l b_l$ |
| $\overline{B}_2(A_3) = \bigcup_{1 \leq i < j \leq 4} \overline{B}_2(A_3; i, j)$ | $\varepsilon_i b_i = \varepsilon_j b_j = 1, \varepsilon_k b_k = (\varepsilon_l b_l)^{-1}$ |
| $\overline{B}_1(A_1^\oplus 3) = \bigcup_{1 \leq i \leq 4} \overline{B}_1(A_1^\oplus 3; i)$ | $\varepsilon_i b_i = \varepsilon_i b_i = \varepsilon_i b_i = \varepsilon_i b_i = 1$ |
| $\overline{B}_2(A_1^\oplus 3) = \bigcup_{1 \leq i \leq 4} \overline{B}_2(A_1^\oplus 3; i)$ | $\varepsilon_i b_i = \varepsilon_i b_i = (\varepsilon_i b_i)^{-1}$ |
| $\overline{B}_3(A_1^\oplus 3) = \overline{B}_3(A_1^\oplus 3; \ast)$ | $\varepsilon_i b_i = \varepsilon_2 b_2 = \varepsilon_3 b_3 = \varepsilon_4 b_4$ |
| $\overline{B}_4(A_1^\oplus 3) = \bigcup_{1 \leq i \leq 3} \overline{B}_4(A_1^\oplus 3; i)$ | $\varepsilon_i b_i = \varepsilon_i b_i = (\varepsilon_i b_i)^{-1} = (\varepsilon_i b_i)^{-1}$ |
| $\overline{B}_1(A_2) = \bigcup_{1 \leq i \neq j \leq 4} \overline{B}_1(A_2; i, j)$ | $\varepsilon_i b_i = 1, \varepsilon_k b_k = b_l b_l$ |
| $\overline{B}_2(A_2) = \bigcup_{1 \leq i \leq 4} \overline{B}_2(A_2; i)$ | $\varepsilon_i b_i = \varepsilon_i b_i = b_l b_l = 1$ |
| $\overline{B}_1(A_1^\oplus 2) = \bigcup_{1 \leq i < j \leq 4} \overline{B}_1(A_1^\oplus 2; i, j)$ | $\varepsilon_i b_i = \varepsilon_j b_j = 1$ |
| $\overline{B}_2(A_1^\oplus 2) = \bigcup_{1 \leq i < j \leq 4} \overline{B}_2(A_1^\oplus 2; i, j)$ | $b_i b_i = b_i b_i = \varepsilon_i$ |
| $\overline{B}_3(A_1^\oplus 2) = \bigcup_{1 \leq i \leq 3} \overline{B}_3(A_1^\oplus 2; i)$ | $b_i b_i = b_i b_i = \varepsilon_i$ |
| $\overline{B}_4(A_1^\oplus 2) = \bigcup_{1 \leq i \leq 3} \overline{B}_4(A_1^\oplus 2; i)$ | $b_i b_i = b_i b_i = \varepsilon_i$ |
| $\overline{B}_1(A_1) = \bigcup_{1 \leq i \leq 4} \overline{B}_1(A_1; i)$ | $\varepsilon_i b_i = 1$ |
| $\overline{B}_2(A_1) = \bigcup_{1 \leq i \leq 4} \overline{B}_2(A_1; i)$ | $b_i = b_i b_i b_l$ |
| $\overline{B}_3(A_1) = \bigcup_{1 \leq i \leq 3} \overline{B}_3(A_1; i)$ | $b_i b_i = b_i b_i$ |
| $\overline{B}_4(A_1) = \overline{B}_4(A_1; \ast)$ | $b_i b_i b_i b_i = 1$ |

Table 2: Parameter spaces $\overline{B}_m(\ast)$: $\varepsilon_i \in \{\pm 1\}$ satisfy $\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 = 1$. 
The Dynkin graph of the singularities on $S(\theta)$ appears as the dual graph of the $(-2)$-curves on the desingularized surface $\tilde{S}(\theta)$. Here each $(-2)$-curve arises either as the strict transform of the conic in case (C1), or as the strict transform of the line through $c_i$, $c_j$, $c_k$ in case (C2), or as the exceptional curve over the “degenerate” point $c_i$ in case (C3). Thus the Dynkin structure of the singularities on $S(\theta)$ can be read off from the following data:

- the six points themselves;
- the unique conic, if it exists, specified by condition (C1);
- all the lines specified by condition (C2);
- all the degenerate points specified by condition (C3).

A case-by-case check shows that all feasible data on various parameter spaces $B_m(\ast)$ are depicted in Figures 8–13, where a degenerate indeterminacy point is marked by a white circle and a nondegenerate one is by a black-filled circle respectively. Each $(-2)$-curve arises either from the conic, if it exists, or from a line, or from a white circle in the figures. The $(-2)$-curve from a white circle intersects the one from a conic but none from a line. If two lines intersect in a black-filled circle, then the corresponding $(-2)$-curves are disjoint; otherwise they do meet in a single point. In this manner the data determines the dual graph of the $(-2)$-curves on $\tilde{S}(\theta)$ and hence the Dynkin type of the singularities on $S(\theta)$. □

**Theorem 3.5** Given a parameter $\kappa \in K$, put $\theta = rh(\kappa)$. If $\kappa \in K(\ast)$ with $\ast \in I/S_4$, then the affine cubic surface $S(\theta)$ has simple singularities of abstract Dynkin type $\ast$.

**Proof.** First, notice that $\tilde{S}(\theta)$ has all its singularities within its affine part $S(\theta)$, since $\tilde{S}(\theta)$ is smooth around the tritangent lines at infinity $L$. Thanks to Proposition 3.4, in order to establish the theorem, it suffices to prove $K(\ast) = \beta^{-1}(B(\ast))$ for every $\ast \in I/S_4$. To this end we use the $W(D^{(1)}_4)$-action on the parameter space $B$, where the action $u_i : b \mapsto b'$ is given by

$$b'_0 = \begin{cases} b_0^{-1} & (i = 0), \\ b_0 b_i & (i = 1, 2, 3), \\ -b_0 b_4 & (i = 4). \end{cases}$$

$$b'_j = \begin{cases} b_0 b_j & (i = 0), j \in \{1, 2, 3, 4\}, \\ b_j^{-1} & (i, j \in \{1, 2, 3, 4\}, i = j), \\ b_j & (i, j \in \{1, 2, 3, 4\}, i \neq j). \end{cases}$$

Let $\kappa \in K(\ast)$. From the definition of $K(\ast)$ there exist $w \in W(D^{(1)}_4)$ and $\kappa' \in H(\ast)$ such that $\kappa = w(\kappa')$, where $H(\ast)$ is given by (18). An inspection of Table 2 shows that $\beta(\kappa') \in B(\ast)$; actually $B(\ast)$ has been defined so that this is the case. It implies that

$$\beta(\kappa) = \beta(w(\kappa')) = w(\beta(\kappa')) \in w(B(\ast)) = B(\ast),$$

where the last equality follows from the $W(D^{(1)}_4)$-invariance of the set $B(\ast)$. Thus we have $\kappa \in \beta^{-1}(B(\ast))$ and hence the inclusion $K(\ast) \subset \beta^{-1}(B(\ast))$. The proof of the reverse inclusion $\beta^{-1}(B(\ast)) \subset K(\ast)$ relies on two claims, which are presented in the next two paragraphs.

The first claim asserts that for any $b \in B(\ast)$ there exist $b' \in \beta(H(\ast))$ and $w \in W(D^{(1)}_4)$ such that $b = w(b')$. We see this for the case $\ast = A_1$ dividing it into several subcases. In the subcase $b \in B_4(A_1)$ the claim is true because any $b \in B_4(A_1)$ is either of the forms $(\pm 1, b_1, b_2, b_3, b_4)$, where the negative pattern $(-1, b_1, b_2, b_3, b_4)$ is recast to a positive pattern $(1, b_1^{-1}, b_2^{-1}, b_3^{-1}, b_4^{-1})$. 18
Figure 8: On the strata of types $D_4$ and $A^{\oplus 4}_1$

Figure 9: On the stratum of type $A^{\oplus 3}_1$

Figure 10: On the stratum of type $A^{\oplus 2}_1$

Figure 11: On the stratum of type $A_1$
by the action $w_1w_2w_3w_4$, and any positive pattern certainly belongs to $\beta(H(\ast))$. The remaining subcases $b \in B_1(A_1), B_2(A_1), B_3(A_1)$ can be treated along the same line with the help of actions

$$(\pm 1, b_i, b_j, b_k, b_l) \xrightarrow{w_i} (\pm b_i, b_i^{-1}, b_j, b_l),$$

$$(\pm b_i, b_i^{-1}, b_j, b_k, b_l) \xrightarrow{w_j} (\pm b_ib_j, b_i^{-1}, b_j^{-1}, b_k, b_l),$$

$$(\pm b_i^{-1}, b_j, b_k, b_l) \xrightarrow{w_0} (\pm b_i^{-1}, \pm 1, \pm b_i b_j, \pm b_i b_k, \pm b_i b_l).$$

In a similar manner the first claim is valid for every abstract Dynkin type $\ast \in \mathcal{I}/S_4$.

The second claim is that $\beta^{-1}(\beta(\mathcal{K}(\ast))) = \mathcal{K}(\ast)$. Indeed any element of $\beta^{-1}(\beta(\mathcal{K}(\ast)))$ is expressed as $(\kappa_0 + n_0, \kappa_1 + 1, \kappa_2 + 2n_1, \kappa_3 + 2n_2, \kappa_4 + 2n_3, \kappa_4 + 2n_4)$ for some $(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathcal{K}(\ast)$ and some $n_i \in \mathbb{Z}$ such that $n_0 + n_1 + n_2 + n_3 + n_4 = 0$. Since there exists an action

$w_iw_0(w_jw_kw_lw_0)^2: (\kappa_0, \kappa_i, \kappa_i, \kappa_k, \kappa_l) \mapsto (\kappa_0 + 1, \kappa_i - 2, \kappa_j, \kappa_k, \kappa_l) \quad \{i, j, k, l\} = \{1, 2, 3, 4\},$

and $\mathcal{K}(\ast)$ is $W(D_4^{(1)})$-invariant, one has $\beta^{-1}(\beta(\mathcal{K}(\ast))) \subset \mathcal{K}(\ast)$ and so $\beta^{-1}(\beta(\mathcal{K}(\ast))) = \mathcal{K}(\ast)$.

Now let $\kappa \in \beta^{-1}(B(\ast))$, that is, $\beta(\kappa) \in B(\ast)$. From the first claim there exist $b' \in \beta(H(\ast))$ and $w \in W(D_4^{(1)})$ such that $\beta(\kappa) = w(b') \in \beta(w(H(\ast))) = \beta(w(H(\ast))) \subset \beta(\mathcal{K}(\ast))$. Therefore the second claim implies $\kappa \in \mathcal{K}(\ast)$ and thus $\beta^{-1}(B(\ast)) \subset \mathcal{K}(\ast)$. The proof is complete. \qed

4 Dynamics on Cubic Surfaces

In this section we discuss the polynomial automorphisms on the cubic surface mentioned in the Introduction and in particular investigate their dynamical properties. They were intently studied in [5, 6, 18, 20] and the expositions below are largely based on [20].

Since $S(\theta)$ has the structure of a $(2, 2, 2)$-surface, namely, the defining function $f(x, \theta)$ of $S(\theta)$ is quadratic in each variable $x_i$, the line through $x \in S(\theta)$ parallel to the $x_i$-axis passes

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through a unique second point \( x' \in S(\theta) \). This defines an involutive automorphism

\[
\sigma_i : S(\theta) \to S(\theta), \quad x \mapsto x'.
\]

The surface \( S(\theta) \) admits a natural complex area-form called the Poincaré residue:

\[
\omega(\theta) := \frac{dx_1 \wedge dx_2 \wedge dx_3}{d_x f(x, \theta)} \text{ restricted to } S(\theta).
\]

It pulls back to the natural 2-form \( \omega_\sigma(k) \) on \( \mathcal{M}_2(k) \) via the Riemann-Hilbert correspondence [17] (see [17]). Moreover it is almost preserved by the map \( \sigma_i \), namely, it is sent to its negative

\[
\sigma_i^* \omega(\theta) = -\omega(\theta) \quad (i = 1, 2, 3).
\]

Let \( G \) be the group generated by three involutions \( \sigma_1, \sigma_2, \sigma_3 \), and \( G(2) \) its index-two subgroup generated by three elements \( \sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_1 \). It is known that \( G \) is of finite index in the group of all polynomial automorphisms of \( S(\theta) \) (see [6, 11]). Although our main interest is in an element of \( G(2) \), we spend a short while working with a general element of \( G \). Each element \( \sigma \in G \) extends to a birational map on \( \overline{S}(\theta) \) and it in turn lifts to a one on \( \tilde{S}(\theta) \). For the biregular map \( \sigma : S(\theta) \to \tilde{S}(\theta) \) the induced birational maps on \( \overline{S}(\theta) \) and \( \tilde{S}(\theta) \) are represented by the same symbol \( \sigma \). Note that the birational map \( \sigma : \tilde{S}(\theta) \to \hat{S}(\theta) \) restricts to an automorphism of \( \hat{S}(\theta) \setminus L \), still denoted by \( \sigma \). The area-form \( \omega(\theta) \) induces a meromorphic 2-from \( \tilde{\omega}(\theta) \) on \( \tilde{S}(\theta) \), whose pole divisor is the sum \( L_1 + L_2 + L_3 \) of the three lines at infinity.

Recall that the concept of a non-elementary loop in \( \pi_1(Z, z) \) was defined in Definition 1.1. Its counterpart in the group \( G \) is defined in the following manner, whose relation with the original concept will be discussed in Section 5.

**Definition 4.1** An AS element \( \sigma \in G \) is said to be elementary if \( \sigma = (\sigma_i\sigma_j)^n \) for some \( \{i, j, k\} = \{1, 2, 3\} \) and \( n \in \mathbb{Z} \); otherwise, \( \sigma \) is said to be non-elementary.

We describe how an element \( \sigma \in G \) acts on the subspace \( V \subset H^2(\tilde{S}(\theta), \mathbb{C}) \) spanned by \( L_1, L_2, L_3 \). This was done in [20] when \( \tilde{S}(\theta) \) is smooth and it carries over when \( S(\theta) \) is singular.

1. For each \( i \in \{1, 2, 3\} \), \( \sigma_i \) blows down the line \( L_i \) to the point \( p_i \), which is the unique indeterminacy point of \( \sigma_i \). Moreover \( \sigma_i \) restricts to an automorphism of \( L_j \) that exchanges \( p_i \) and \( p_k \), where \( \{i, j, k\} = \{1, 2, 3\} \), as in Figure 14 (see [20, Lemma 3]). Thus the endomorphisms \( \sigma_1^*, \sigma_2^*, \sigma_3^* : H^2(\tilde{S}(\theta), \mathbb{Z}) \to \text{map the subspace } V \text{ into itself and their restrictions to } V \text{ are represented by the matrices}

\[
s_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix},
\]

respectively, relative to the basis \( L_1, L_2, L_3 \) (see [20, Lemma 10]).

2. Given any element \( \sigma \in G \) other than the unit element, we can write

\[
\sigma = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m},
\]

(28)
Figure 14: The birational map $\sigma_i$ restricted to $L$

for some $m \in \mathbb{N}$ and some $m$-tuple of indices $(i_1, \ldots, i_m) \in \{1, 2, 3\}^m$. Here we may assume that every neighboring indices $i_\nu$ and $i_{\nu+1}$ are distinct, because $\sigma_i$ is an involution. The expression (28) with this condition is unique; it is called the reduced expression of $\sigma$. Thus $G$ is isomorphic to the universal Coxeter group of rank 3 (see [20, Theorem 4]). From what we mentioned in item (1), the following hold for the expression (28):

$$\sigma^n I(\sigma^{-1}) = \{p_{i_1}\} \quad (n \geq 0), \quad \sigma^{-n} I(\sigma^{-1}) = \bigcup_{\nu=1}^{m} L_{i_\nu} \quad (n \geq 1),$$

where $I(\sigma)$ stands for the indeterminacy set of the birational map $\sigma : \tilde{S}(\theta) \circlearrowleft$.

(3) Let $f$ and $g$ be bimeromorphic maps on a compact Kähler surface $X$. For the induced actions $f^*$ and $g^*$ on $H^{1,1}(X)$ the composition rule $(f \circ g)^* = g^* \circ f^*$ is not always true. This is true if and only if $g$ blows down no curve into a point of $I(f)$. It follows from item (1) that every neighboring pair $\sigma_{i_\nu}$ and $\sigma_{i_{\nu+1}}$ satisfies this condition so that

$$\sigma^* = \sigma_{i_m}^* \cdots \sigma_{i_2}^* \sigma_{i_1}^* : H^1(\tilde{S}(\theta)) = H^2(\tilde{S}(\theta), C) \circlearrowleft,$$

provided that (28) is a reduced expression (see [20 Lemma 8]).

(4) It is easily seen from formula (29) that an element $\sigma \in G$ is AS if and only if the initial index $i_1$ and the terminal index $i_m$ are distinct in expression (28). Moreover any element is conjugate to some AS element in $G$ (see [20 Lemma 12 and page 324]). In what follows we may and shall assume that $\sigma$ is AS, namely, that $i_1 \neq i_m$.

(5) By formula (30) and the matrix representations (27), the eigenvalues of $\sigma^*|_V$ are 0 and the two roots of the quadratic equation

$$\lambda^2 - \alpha(\sigma) \lambda + (-1)^m = 0,$$

where $\alpha(\sigma)$ is the trace of the matrix $s := s_{i_m} \cdots s_{i_2} s_{i_1}$, which takes an even positive integer. Moreover $\alpha(\sigma) > 2$ if and only if $\sigma$ is non-elementary (see [20 Lemma 13]).

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Assume that $\sigma$ is non-elementary. Then for any $n \in \mathbb{N}$ the $n$-th iterate $\sigma^n$ has exactly two fixed points $p_i$ and $p_{m}$ on $L$ in the sense of Definition $2.4$ where $p_i \in X_0(\sigma^n)$ and $p_{m} \in X_0(\sigma^{-n})$ are superattracting fixed points of $\sigma^n$ and $\sigma^{-n}$ respectively in the usual sense. So their indices are $\nu_{p_i}(\sigma^n) = \nu_{p_{m}}(\sigma^n) = 1$ (see $[20]$ Lemma $15$). Moreover one has $\{i_1, i_2, \ldots, i_m\} = \{1, 2, 3\}$ and thus $\sigma^{-n}I(\sigma^{-1}) = L$ from formula $\eqref{eq:39}$. In particular $\sigma : \tilde{S}(\theta) \ominus$ contracts $L$ to the point $p_i$.

From now on we focus our attention on an even element $\sigma \in G(2)$. It follows from $\eqref{eq:20}$ that the birational map $\sigma : \tilde{S}(\theta) \ominus$ preserves the meromorphic 2-form $\tilde{\omega}(\theta)$, that is, $\sigma^* \tilde{\omega}(\theta) = \tilde{\omega}(\theta)$.

**Theorem 4.2** Assume that $\sigma \in G(2)$ is an AS element. Then the first dynamical degree $\lambda(\sigma)$ is a quadratic unit that appears as the largest root of the quadratic equation $\eqref{eq:31}$ with $m$ even. Moreover $\lambda(\sigma)$ is strictly greater than $1$ if and only if $\sigma$ is non-elementary.

**Proof.** Consider the action $\sigma^* : H^2(\tilde{S}(\theta), \mathbb{C}) \ominus$. We know that $\sigma^*$ preserves the subspace $V$. It also preserves its orthogonal complement $V^\perp$. Indeed, for any $v \in V^\perp$ and $v' \in V$ one has $(\sigma^*v, v') = (v, (\sigma^{-1})^*v') = 0$, since $(\sigma^{-1})^*$ is the adjoint of $\sigma^*$ relative to the intersection form and preserves $V$. This shows that $\sigma^*$ preserves $V^\perp$. Now we claim that the operator $\sigma^*|_{V^\perp}$ is unitary. Indeed, a corollary to the push-pull formula (see $[7]$ Corollary $3.4$) yields

$$(\sigma^*v_1, \sigma^*v_2) = (v_1, v_2) + Q(v_1, v_2) \quad (v_1, v_2 \in H^2(\tilde{S}(\theta), \mathbb{C})), $$

where $Q(v_1, v_2)$ is a nonnegative Hermitian form that can be expressed as

$$Q(v_1, v_2) = \sum_{i=1}^{3} k_i \cdot (v_1, L_i) \cdot (v_2, L_i),$$

with some positive integers $k_1$, $k_2$, $k_3 \in \mathbb{N}$. Thus if $v_1$ and $v_2$ are in $V^\perp$ then $Q(v_1, v_2)$ vanishes and so $\sigma^*|_{V^\perp}$ preserves the intersection form on $V^\perp$. Recall that the vectors $L_4$, $L_5$, $L_6$, $L_7$ in $\eqref{eq:25}$ form a basis of $V^\perp$, whose intersection relations are known to be $(L_i, L_j) = -2\delta_{ij}$ from formula $\eqref{eq:22}$, where $\delta_{ij}$ is Kronecker’s delta. Thus the intersection form on $V^\perp$ is negative definite. Since $\sigma^*|_{V^\perp}$ preserves a negative definite Hermitian form, it must be unitary. In particular all of its eigenvalues are of modulus $1$.

On the other hand, since $\sigma$ is assumed to be AS, the eigenvalues of $\sigma|_{V}$ consist of $0$ and the two roots of quadratic equation $\eqref{eq:31}$ with $m$ even. These three numbers and the four numbers of modulus $1$ in the last paragraph constitute all the seven eigenvalues of $\sigma^* : H^2(\tilde{S}(\theta), \mathbb{C}) \ominus$. Note that equation $\eqref{eq:31}$ has a real root $\geq 1$ because $\alpha(\sigma) \geq 2$. Thus the first dynamical degree $\lambda(\sigma)$, which is the spectral radius of $\sigma^* : H^{1,1}(\tilde{S}(\theta)) = H^2(\tilde{S}(\theta), \mathbb{C}) \ominus$, is given by the largest root of equation $\eqref{eq:31}$. Moreover $\lambda(\sigma) > 1$ if and only if $\alpha(\sigma) > 2$, which is the case precisely when $\sigma$ is non-elementary. The proof is complete. \hfill $\square$

We now apply the construction in Remark $2.3$ to $X = \tilde{S}(\theta)$ and $f = \sigma$. The birational map $\sigma : \tilde{S}(\theta) \ominus$ restricts to an automorphism $\sigma : \tilde{S}(\theta)_{\sigma} \ominus$, where $\tilde{S}(\theta)_{\sigma}$ designates the space $X_f$ of definition $\eqref{eq:8}$ adapted in the present setting. This space can be identified in the following.

**Lemma 4.3** For any non-elementary AS element $\sigma \in G(2)$, we have $\tilde{S}(\theta)_{\sigma} = \tilde{S}(\theta) \setminus L$.

**Proof.** This readily follows from what we have mentioned in item (6). \hfill $\square$
Theorem 4.4 If \( \sigma \in G(2) \) is a non-elementary AS element, then the nonwandering set \( \Omega(\sigma) \) of the birational map \( \sigma : \tilde{S}(\theta) \ominus \) is compact in \( \tilde{S}(\theta) \setminus L \) and the trajectory of each point \( x \in \tilde{S}(\theta) \setminus \Omega(\sigma) \) tends to infinity \( L \) under the iterations of \( \sigma \).

Proof. Put \( \sigma = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_m} \) as in \( (28) \). Since \( \sigma : \tilde{S}(\theta) \ominus \) contracts \( L \) into the superattracting fixed point \( p_{i_1} \in L \), there exists a neighborhood \( U \) of \( L \) in \( \tilde{S}(\theta) \) such that each point of \( U \) is attracted to \( p_{i_1} \) under the iterations of \( \sigma \). Hence the nonwandering set \( \Omega(\sigma) \subset \tilde{S}(\theta)_{\sigma} = \tilde{S}(\theta) \setminus L \) of \( \sigma : \tilde{S}(\theta) \ominus \) is contained in \( \tilde{S}(\theta) \setminus U \) and thus compact in \( \tilde{S}(\theta) \setminus L \) because \( \Omega(\sigma) \) is closed. \( \square \)

Theorem 4.5 Assume that \( \sigma \in G(2) \) is a non-elementary AS element. Then the birational map \( \sigma : \tilde{S}(\theta) \ominus \) admits a \( \sigma \)-invariant Borel probability measure \( \nu_{\sigma} \) with support in \( \Omega(\gamma) \) that satisfies the conditions in Definition 1.4. Moreover,

1. the measure-theoretic entropy \( h_{\nu_{\sigma}}(\sigma) \) and the topological entropy \( h_{\text{top}}(\sigma) \) are expressed as
   \[
   h_{\nu_{\sigma}}(\sigma) = h_{\text{top}}(\sigma) = \log \lambda(\sigma),
   \]
   (32)

2. the measure \( \nu_{\sigma} \) puts no mass on any algebraic curve on \( \tilde{S}(\theta) \),

3. there exists a set \( \mathcal{P}_n(\sigma) \subset \text{supp} \nu_{\sigma} \) of saddle periodic points of period \( n \) such that
   \[
   \#\mathcal{P}_n(\sigma) \sim \lambda(\sigma)^n, \quad \frac{1}{\lambda(\sigma)^n} \sum_{p \in \mathcal{P}_n(\sigma)} \delta_p \rightarrow \nu_{\sigma}, \quad \text{as} \quad n \rightarrow \infty.
   \]

Proof. Put \( \sigma = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_m} \) as in \( (28) \). From Theorem 4.2 the first dynamical degree \( \lambda(\sigma) \) is strictly greater than 1. Since \( I(\sigma) = \{ p_{i_1} \} \) and \( \sigma^N(I(\sigma^{-1})) = \{ p_{i_1} \} \) for any \( N \geq 0 \),

\[
\sum_{N=0}^{\infty} \lambda(\sigma)^{-N} \log \text{dist}(\sigma^N I(\sigma^{-1}), I(\sigma)) = \log \text{dist}(p_{i_1}, p_{i_1}) \sum_{N=0}^{\infty} \lambda(\sigma)^{-N} > -\infty,
\]

and hence condition \( (7) \) is satisfied. Therefore Theorem 2.2 implies that there exists a \( \sigma \)-invariant Borel probability measure \( \nu_{\sigma} \) that satisfies all conditions of the theorem. \( \square \)

We turn our attention to the second isolated topic, that is, estimating the number of isolated periodic points of the birational map \( \sigma : \tilde{S}(\theta) \ominus \) for a given element \( \sigma \in G(2) \). Let \( \text{Per}_n^\dagger(\sigma \setminus L) \) denote the set of all isolated periodic points of period \( n \) that lie in \( \tilde{S}(\theta) \setminus L \).

Theorem 4.6 Let \( \sigma \in G(2) \) be a non-elementary AS element. Then any irreducible periodic curve of the map \( \sigma : \tilde{S}(\theta) \ominus \) must lie in the exceptional set \( \mathcal{E}(\theta) \) of the minimal resolution \( \pi : \tilde{S}(\theta) \rightarrow \mathcal{S}(\theta) \) in \( (27) \). Moreover for every \( n \in \mathbb{N} \) the set \( \text{Per}_n^\dagger(\sigma \setminus L) \) is finite and its cardinality counted with multiplicity is estimated as

\[
|\#\text{Per}_n^\dagger(\sigma \setminus L) - \lambda(\sigma)^n| \leq O(1) \quad \text{as} \quad n \rightarrow \infty.
\]

Proof. We begin with the assertion about periodic curves. Let \( \sigma = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_m} \) be the reduced expression as in \( (28) \). In view of item (6), since \( \sigma \in G(2) \) is non-elementary, for any \( n \in \mathbb{N} \) the \( n \)-th iterate \( \sigma^n \) contracts \( L \) to the superattracting fixed point \( p_{i_1} \in L \) of \( \sigma \). We claim that any periodic point of the map \( \sigma : \mathcal{S}(\theta) \ominus \) must be isolated. Indeed, assume the contrary that
σ admits a periodic curve $C \subset \overline{S}(\sigma)$. Since $S(\sigma)$ is affine, the compact curve $C$ must intersect the lines at infinity $L$. If $n \in \mathbb{N}$ is the primitive period of $C$, then the fixed curve $C$ of $\sigma^n$ must meet $L$ in $p_i$, because $L$ is contracted into the superattracting fixed point $p_i$ by $\sigma^n$. This contradicts the fact that $p_i$ is an isolated fixed point of $\sigma^n$. Therefore any irreducible periodic curve of $\sigma : \tilde{S}(\sigma) \cup \sigma$ must be contracted to a point by the map (21), that is, it must lie in the exceptional set $E(\sigma)$ of the resolution (21).

The above argument shows in particular that no irreducible component of the pole divisor $(\tilde{\omega}(\sigma))_\infty = L_1 + L_2 + L_3$ is a periodic curve of $\sigma$. Thus Theorem 2.5 gives an estimate

$$|\#\text{Per}_n^i(\sigma) - \lambda(\sigma)^n| \leq O(1),$$

since $\tilde{S}(\sigma)$ is not birationally equivalent to any Abelian surface. On the other hand the number $\#\text{Per}_n^i(\sigma \setminus L)$ is obtained from $\#\text{Per}_n^i(\sigma)$ by subtracting the sum of local indices at the isolated fixed points on $L$ for the map $\sigma^n$. In view of item (6) those fixed points are just $p_i$ and $p_{im}$, each having local index 1. Therefore the number $\#\text{Per}_n^i(\sigma \setminus L)$ is given by

$$\#\text{Per}_n^i(\sigma \setminus L) = \#\text{Per}_n^i(\sigma) - 2,$$

(34)

so that the estimate (33) is established by combining all these observations.

By virtue of Corollary 2.6, Theorems 4.6 has the following.

**Corollary 4.7** Assume that $\sigma \in G(2)$ is a non-elementary AS element. Then we have

$$\#\text{Per}_n^i(\sigma \setminus L) \sim \#\text{HPer}_n(\sigma) \sim \lambda(\sigma)^n \quad \text{as} \quad n \to \infty,$$

so that asymptotically almost all points in $\text{Per}_n^i(\sigma \setminus L)$ belong to $\text{HPer}_n(\sigma)$.

**Remark 4.8** From Theorem 4.6 any periodic curve of a non-elementary map $\sigma : \tilde{S}(\sigma) \cup \sigma$ must be an irreducible component of $\mathcal{E}(\sigma)$. On the other hand $\mathcal{E}(\sigma)$ is the exceptional set of a minimal resolution of simple singularities on $\tilde{S}(\sigma)$. Thus no two of the irreducible components of $\mathcal{E}(\sigma)$ are tangent and no three of them meet in a single point. Therefore every iterate of the map $\sigma$ satisfies Assumption 2.9 and it is why this assumption was made in Section 2.

## 5 Proofs of the Main Theorems

In this section we prove our main theorems combining all the previous discussions. As is mentioned in the Introduction, the Riemann-Hilbert correspondence (1) recasts the monodromy map on the moduli space $\mathcal{M}_z(\kappa)$ to a biregular map on the cubic surface $S(\sigma)$, where the latter map was studied in Section 4. After a brief review of how these two maps are related, we establish our main theorems by translating the results on $S(\sigma)$ back to $\mathcal{M}_z(\kappa)$.

Given a pair $(z, a) \in \mathbb{Z} \times A$, let $\mathcal{R}_z(a)$ be the moduli space of Jordan equivalence classes of representations $\rho : \pi_1(\mathbb{P}^1 \setminus \{t_1, t_2, t_3, t_4\}, *) \rightarrow SL_2(\mathbb{C})$ such that $\text{Tr} \rho(C_i) = a_i$ for $i \in \{1, 2, 3, 4\}$, where $t_1 = 0$, $t_2 = z$, $t_3 = 1$, $t_4 = \infty$ and $C_i$ is a loop surrounding $t_i$ once anti-clockwise as in Figure 15. The space $\mathcal{R}_z(a)$ is called a relative $SL_2(\mathbb{C})$-character variety of the quadruply punctured sphere $\mathbb{P}^1 \setminus \{t_1, t_2, t_3, t_4\}$. Then the Riemann-Hilbert correspondence is the map

$$\text{RH}_{z, \kappa} : \mathcal{M}_z(\kappa) \rightarrow \mathcal{R}_z(a), \quad Q \mapsto \rho \quad (35)$$
Figure 15: Four loops in $\mathbb{P}^1 \setminus \{0, z, 1, \infty\}$; the fourth point $t_4 = \infty$ is outside $C_4$, invisible.

sending each stable parabolic connection $Q$ to the Jordan equivalence class $\rho$ of its monodromy representation, where $\kappa \mapsto a$ is the composition of the maps $\alpha$ and $\beta$ in (14). On the other hand, there exists an isomorphism of affine algebraic surfaces

$$\mathcal{R}_z(a) \to S(\theta), \quad \rho \mapsto x = (x_1, x_2, x_3),$$

where $x_i = \text{Tr} \rho(C_iC_k)$ for $\{i, j, k\} = \{1, 2, 3\}$ and $a \mapsto \theta$ is the map $\varphi$ in (14). It enables us to identify the character variety $\mathcal{R}_z(a)$ with the affine cubic surface $S(\theta)$, so that the Riemann-Hilbert correspondence (35) can be reformulated as the map in (11) with $\theta = \text{rh}(\kappa)$ in (13).

**Theorem 5.1** ([14, 15, 16]) Given any $\kappa \in \mathcal{K}_1$, put $\theta = \text{rh}(\kappa) \in \Theta$. If $I = \emptyset$ then the surface $S(\theta)$ is smooth and the Riemann-Hilbert correspondence (11) is a biholomorphism. Otherwise, $S(\theta)$ has simple singularities of Dynkin type $D_I$ and the Riemann-Hilbert correspondence (11) is a proper surjective map that is an analytic minimal resolution of the singular surface $S(\theta)$.

On the other hand, we have an algebraic minimal resolution $\pi : \widetilde{S}(\theta) \setminus L \to S(\theta)$ as the restriction of (21) to the affine surface $S(\theta)$. Since the minimal resolution is unique up to isomorphisms, the Riemann-Hilbert correspondence (11) lifts to a biholomorphism

$$\widetilde{\text{RH}}_{z, \kappa} : \mathcal{M}_z(\kappa) \to \widetilde{S}(\theta) \setminus L$$

(36)

sending the exceptional set $E_z(\kappa)$ of $\mathcal{M}_z(\kappa)$ to the exceptional set $E(\theta)$ of $\widetilde{S}(\theta) \setminus L$. It is known that for each $i \in \mathbb{Z}/3\mathbb{Z}$ the monodromy map $\gamma_i : \mathcal{M}_z(\kappa) \to \mathcal{M}_z(\kappa)$ along the basic loop $\gamma_i \in \pi_1(Z, z)$ is strictly conjugate to the automorphism $\sigma_i \sigma_{i+1} : \widetilde{S}(\theta) \setminus L \to \widetilde{S}(\theta) \setminus L$ via the lifted Riemann-Hilbert correspondence (36) (see [14]). Moreover there exists an isomorphism of groups

$$\Phi : \pi_1(Z, z) \to G(2)$$

(37)

sending $\gamma_i$ to $\sigma_i \sigma_{i+1}$ for each $i \in \mathbb{Z}/3\mathbb{Z}$ (see [21]). Thus the monodromy map $\gamma_* : \mathcal{M}_z(\kappa) \to \mathcal{M}_z(\kappa)$ along a general loop $\gamma \in \pi_1(Z, z)$ is strictly conjugate to the automorphism $\sigma := \Phi(\gamma) : \widetilde{S}(\theta) \setminus L \to \widetilde{S}(\theta) \setminus L$.

Any loop $\gamma \in \pi_1(Z, z)$ can be written as a word in the alphabet $\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \gamma_3^{\pm 1}$. Such a word of minimal length is called a reduced expression of $\gamma$ and this minimal length is by definition
the length of $\gamma$. A loop $\gamma$ is said to be minimal if it is of minimal length in the conjugacy class of $\gamma$. It suffices to consider minimal loops only, because conjugate loops induce conjugate monodromy maps which are dynamically the same. Any minimal loop $\gamma \in \pi_1(Z, z)$ is sent to an AS element $\sigma \in G(2)$ by the isomorphism \textup{(11)} and vice versa. Moreover $\gamma$ is non-elementary in the sense of Definition \textup{4.1} if and only if the AS element $\sigma$ is non-elementary in the sense of Definition \textup{4.1}. The first dynamical degree of $\gamma$ is that of the birational map $\sigma : \tilde{S}(\theta) \circlearrowleft$, namely, $\lambda(\gamma) := \lambda(\sigma)$. Thus Theorem \textup{4.2} implies that $\lambda(\gamma) > 1$ if and only if $\gamma$ is non-elementary.

We are now in a position to establish our main theorems.

\textbf{Proofs of Theorems \textup{1.5} and \textup{1.6}} We may assume that $\gamma \in \pi_1(Z, z)$ is a non-elementary minimal loop. Then $\sigma := \Phi(\gamma) \in G(2)$ is a non-elementary AS element. First we prove Theorem \textup{1.5}. From Theorem \textup{4.4}, the nonwandering set $\Omega(\sigma)$ of $\sigma : \tilde{S}(\theta) \circlearrowleft$ are compact in $\tilde{S}(\theta) \setminus L$ and the trajectory of any point $x \in \tilde{S}(\theta) \setminus \Omega(\sigma)$ tends to infinity $L$ under the iterations of $\sigma$. Since the lifted Riemann-Hilbert correspondence \textup{(36)} is proper, the nonwandering set $\Omega(\gamma)$ of $\gamma_* : M_z(\kappa) \circlearrowleft$ is also compact in $M_z(\kappa)$ as the inverse image of $\Omega(\sigma)$ by the map \textup{(36)}. For the same reason the trajectory of any point $Q \in M_z(\kappa) \setminus \Omega(\gamma)$ tends to infinity $Y_z(\kappa)$ under the iterations of $\gamma_*$. Let $\nu_\sigma$ be the $\sigma$-invariant Borel probability measure on $\tilde{S}(\theta)$ mentioned in Theorem \textup{4.5} which has support in $\Omega(\sigma) \subset \tilde{S}(\theta) \setminus L$. It pulls back to a $\gamma_*-$invariant probability measure $\mu_\gamma := \overline{R_{\tilde{H}}}_{z, \kappa}(\nu_\sigma)$ on $M_z(\kappa)$ through the lifted Riemann-Hilbert correspondence \textup{(36)}. The measure $\mu_\gamma$ has support in $\Omega(\gamma)$ and all the assertions in Theorem \textup{1.5} follow from those in Theorem \textup{4.5}. In particular formula \textup{(3)} comes from formula \textup{(32)}. We proceed to the proof of Theorem \textup{1.6}. Since the lifted Riemann-Hilbert correspondence \textup{(36)} is biholomorphic, one has

$$\#\text{Per}_n^i(\gamma; \kappa) = \#\text{Per}_n^i(\sigma \setminus L),$$

and all the assertions in Theorem \textup{1.6} follow from those in Theorem \textup{1.6} and Corollary \textup{4.7}. $\blacksquare$

\textbf{Remark 5.2} Let $\gamma \in \pi_1(Z, z)$ be an elementary loop conjugate to the loop $\gamma^m_i$ for some index $i \in \{1, 2, 3\}$ and some integer $m \in \mathbb{Z}$. Then the monodromy map $\gamma_* : M_z(\kappa) \circlearrowleft$ is semi-conjugate to the map $(\sigma_i \sigma_{i+1})^m : S(\theta) \circlearrowleft$ via the Riemann-Hilbert correspondence \textup{(11)}. Notice that the latter map preserves the projection $S(\theta) \rightarrow \mathbb{C}, x = (x_1, x_2, x_3) \mapsto x_i$, where $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$. Through the map \textup{(11)} this projection is pulled back to an analytic fibration $M_z(\kappa) \rightarrow \mathbb{C}$ which is preserved by the monodromy map $\gamma_* : M_z(\kappa) \circlearrowleft$.

\section{Periodic Solutions along a Pochhammer Loop}

We illustrate the power of Theorem \textup{2.8} by calculating for various $\kappa \in \mathcal{K}$ the explicit values of the number $\#\text{Per}_n^i(\varphi; \kappa)$ of isolated periodic solutions to $P_{VI}(\kappa)$ along a Pochhammer loop $\varphi$ (cf. Example \textup{1.8}). It is interesting that the result strongly depends on the value of $\kappa \in \mathcal{K}$.

Some subspaces of the parameter spaces $B$ and $\mathcal{K}$ are introduced to facilitate an efficient case-by-case discussion. For each $n \in \mathbb{N}$ let $B^{(n)}$ be the subspace of $B$ defined by

$$B^{(1)} := \{ b \in B : b_0 = 1, b_1^2 = b_2^2 = b_3^2 = \pm \sqrt{-1} \},$$

$$B^{(n)} := \{ b \in B : b_0 = 1, R_n(b_1, b_2, b_3) = 0 \} \quad (n \geq 2),$$

27
where
\[
R_n(b_1, b_2, b_3) := \prod_{1 \leq m < n, (m,n) = 1} \left( r(b_1, b_2, b_3) + 2b_1^2b_2^2b_3^2 \cos \frac{\pi m}{n} \right),
\]
\[
r(b_1, b_2, b_3) := 1 - 3b_1^2b_2^2b_3^2 + (b_1^2b_2^2b_3^2 - 1) \left\{b_1^2b_2^2b_3^2 + \sum_{i \in \mathbb{Z}/3\mathbb{Z}} (b_i^2 - b_{i+1}^2b_{i+2}^2)\right\}.
\]
Moreover let \( \mathcal{K}^{(n)} \) be the \( W(F_4^{(1)}) \)-translates of the set \( \beta^{-1}(B^{(n)}) \), where \( \beta : \mathcal{K} \to B \) is the map defined by (15). For each abstract Dynkin type * ∈ \( \mathcal{I}/S_4 \), we define
\[
\mathcal{K}^{(n)}(\ast) := \mathcal{K}(\ast) \cap \mathcal{K}^{(n)}, \quad \mathcal{K}^{(\ast)}(\ast) := \mathcal{K}(\ast) \setminus \bigcup_{n \geq 2} \mathcal{K}^{(n)}(\ast).
\]
Recall from formula (2) that the first dynamical degree of \( \varphi \) is given by \( \lambda(\varphi) = 9 + 4\sqrt{5} \) and from Theorem 1.6 that any periodic curve along \( \varphi \) must be a Riccati curve. For any \( \kappa \in \mathcal{K}^{(\ast)} \) the Riccati curves on \( \mathcal{M}_2(\kappa) \) has the dual graph of abstract Dynkin type * ∈ \( \mathcal{I}/S_4 \).

**Theorem 6.1** Along a Pochhammer loop \( \varphi \in \pi_1(\mathbb{Z},z) \) the following hold:

(1) If \( \kappa \in \mathcal{K}(A_1) \), then the unique Riccati curve on \( \mathcal{M}_2(\kappa) \) is a periodic curve of primitive period \( n \geq 1 \) precisely when \( \kappa \in \mathcal{K}^{(n)}(A_1) \). If moreover \( \kappa \in \mathcal{K}^{(\ast)}(A_1) \), then we have
\[
\#\text{Per}_n^1(\varphi; \kappa) = \begin{cases} 
\lambda(\varphi)^n + \lambda(\varphi)^{-n} - 10 & (\kappa \in \mathcal{K}^{(1)}(A_1)), \\
\lambda(\varphi)^n + \lambda(\varphi)^{-n} + 4 & (\kappa \in \mathcal{K}^{(\ast)}(A_1) \setminus \mathcal{K}^{(1)}(A_1)).
\end{cases}
\]

(2) If \( \kappa \in \mathcal{K}(A_2) \), then neither of the two Riccati curves on \( \mathcal{M}_2(\kappa) \) is a periodic curve of any period, and
\[
\#\text{Per}_n^1(\varphi; \kappa) = \lambda(\varphi)^n + \lambda(\varphi)^{-n} + 4.
\]

(3) If \( \kappa \in \mathcal{K}(A_1^{\oplus 2}) \), then neither of the two Riccati curves on \( \mathcal{M}_2(\kappa) \) is a fixed curve. If moreover \( \kappa \in \mathcal{K}^{(\ast)}(A_1^{\oplus 2}) \), then neither of them is a periodic curve of any period, and
\[
\#\text{Per}_n^1(\varphi; \kappa) = \lambda(\varphi)^n + \lambda(\varphi)^{-n} + 4.
\]
If \( \kappa \in \mathcal{K}^{(n)}(A_1^{\oplus 2}) \) with \( n \geq 2 \), then both of them are periodic curves of primitive period \( n \).

(4) If \( \kappa \in \mathcal{K}(A_3) \), then the Riccati curve corresponding to the central node of the Dynkin diagram of type \( A_3 \) is a fixed curve, but neither of the other two Riccati curves on \( \mathcal{M}_2(\kappa) \) is a periodic curve of any period, and
\[
\#\text{Per}_n^1(\varphi; \kappa) = \lambda(\varphi)^n + \lambda(\varphi)^{-n} - 2.
\]

(5) If \( \kappa \in \mathcal{K}(A_1^{\oplus 3}) \), then none of the three Riccati curves on \( \mathcal{M}_2(\kappa) \) is a fixed curve. If moreover \( \kappa \in \mathcal{K}^{(\ast)}(A_1^{\oplus 3}) \), then none of them is a periodic curve of any period, and
\[
\#\text{Per}_n^1(\varphi; \kappa) = \lambda(\varphi)^n + \lambda(\varphi)^{-n} + 4.
\]
If \( \kappa \in \mathcal{K}^{(n)}(A_1^{\oplus 3}) \) with \( n \geq 2 \), then all of them are periodic curves of primitive period \( n \).
Table 3: Bases of the eigenspaces $V_{\pm 1}$, where $\{i, j, k\} = \{1, 2, 3\}$.

| strata     | basis of $V_1$ | basis of $V_{-1}$ |
|------------|----------------|-------------------|
| $A_1, A_2$ | $L_7$          | $L_4, L_5, L_6$  |
| $A_1^{\oplus 2}, A_3$ | $L_{i+3}, L_7$  | $L_{j+3}, L_{k+3}$ |
| $A_1^{\oplus 3}$ | $L'_4, L'_5, L'_6$ | $L'_7$          |
| $D_4, A_4^{\oplus 4}$ | $L_4, L_5, L_6, L_7$ | none |

(6) If $\kappa \in \mathcal{K}(D_4)$, then all of the four Riccati curves on $\mathcal{M}_z(\kappa)$ are fixed curves, and
\[
\#\text{Per}^i_n(\varphi; \kappa) = \lambda(\varphi)^n + \lambda(\varphi)^{-n} - 4.
\]

(7) If $\kappa \in \mathcal{K}(A_4^{\oplus 4})$, then none of the four Riccati curves on $\mathcal{M}_z(\kappa)$ is a periodic curve of any period, and
\[
\#\text{Per}^i_n(\varphi; \kappa) = \lambda(\varphi)^n + \lambda(\varphi)^{-n} + 4.
\]

Remark 6.2 The number $\#\text{Per}^i_n(\varphi; \kappa)$ is yet to be determined for $\kappa \in \mathcal{K}^{(n)}(*)$ with $* = A_1, A_1^{\oplus 2}, A_1^{\oplus 3}$ and $n \geq 2$, in which cases periodic curves of higher periods occur and things are much subtler. In this paper we content ourselves with the cases allowing at most fixed curves.

The first step toward the proof of Theorem 6.1 is to calculate the actions of the basic elements $\sigma_i$ on the cohomology group $H^2(\tilde{S}(\theta), \mathbb{C}) = V \oplus V^\perp$, where the subspaces $V$ and $V^\perp$ are spanned by the vectors $L_1, L_2, L_3$ in (23) and by the vectors $L_4, L_5, L_6, L_7$ in (25) respectively. It is convenient to introduce another basis of $V^\perp$ defined by
\[
L'_4 := E_0 - E_1 - E_5 - E_6, \quad L'_5 := E_0 - E_2 - E_4 - E_6, \quad L'_6 := E_0 - E_3 - E_4 - E_5, \quad L'_7 := E_0 - E_1 - E_2 - E_3.
\]

Lemma 6.3 For each $i = 1, 2, 3$ the action $\sigma_i^* : H^2(\tilde{S}(\theta), \mathbb{C}) \otimes \text{preserves the subspaces } V \text{ and } V^\perp$. Its restriction to $V$ is represented by the matrix $s_i$ in (27) relative to the basis $L_1, L_2, L_3$, while its restriction to $V^\perp$ has eigenvalues $\pm 1$ whose eigenspaces $V_{\pm 1}$ have bases in Table 3.

Proof. We only deal with the stratum $B(A_1)$ as the other strata can be treated in similar manners. Moreover it suffices to consider the case $b \in B_4(A_1)$, namely, the case where the six points $c_1, \ldots, c_6$ in (20) lie on a conic $C$, because the entire $B(A_1)$ is covered by the $W(D_4^{(1)})$-translates of $B_4(A_1)$. Since $b_1b_2b_3b_4 = 1$ on $B_4(A_1)$, formula (20) reads:
\[
c_1 = [0 : 1 : -b_2b_3], \quad c_4 = [0 : -b_2b_3 : 1], \\
c_2 = [-b_3b_1 : 0 : 1], \quad c_5 = [1 : 0 : -b_3b_1], \\
c_3 = [1 : -b_1b_2 : 0], \quad c_6 = [-b_1b_2 : 1 : 0],
\]
and the conic passing through these points is given by
\[
C := \left\{ [u_1 : u_2 : u_3] : \sum_{j \in \mathbb{Z}/3\mathbb{Z}} \{u_j^2 + (b_j^{-1}b_{j+1}^{-1} + b_jb_{j+1})u_ju_{j+1}\} = 0 \right\}.
\]
For each $i = 1, 2, 3$, the birational map $\phi_i := \tau \circ \sigma_i \circ \tau^{-1} : \mathbb{P}^2 \to \mathbb{P}^2$ is expressed as
\[
\begin{align*}
\phi_1[u_1 : u_2 : u_3] &= [(b_2 u_2 + b_3^{-1} u_3)(b_2^{-1} u_2 + b_3 u_3) : u_1 u_2 : u_1 u_3], \\
\phi_2[u_1 : u_2 : u_3] &= [u_1 u_2 : (b_3 u_3 + b_1^{-1} u_1)(b_3^{-1} u_3 + b_1 u_1) : u_2 u_3], \\
\phi_3[u_1 : u_2 : u_3] &= [u_1 u_3 : u_2 u_3 : (b_1 u_1 + b_2^{-1} u_2)(b_1^{-1} u_1 + b_2 u_2)].
\end{align*}
\]
where $\tau : \mathbb{P}^2 \to \mathbb{P}^3$ is defined in (19). The birational map $\phi_i$ has the indeterminacy set
\[I(\phi_i) = \{c_i, c_{i+3}, c_3\}\] with $e_1 := [1 : 0 : 0]$, $e_2 := [0 : 1 : 0]$, $e_3 := [0 : 0 : 1]$, and sends the six points as
\[
\phi_i : \begin{cases} 
  c_i &\leftrightarrow C_i, \\
  c_{i+3} &\leftrightarrow C_{i+3}, \\
  c_j &\leftrightarrow c_{j+3}, \\
  c_k &\leftrightarrow c_{k+3},
\end{cases}
\]
with $\{i, j, k\} = \{1, 2, 3\}$, where $C_i$ and $C_{i+3}$ are lines defined by
\[
C_i := \{b_{i+1} u_{i+1} + b_{i+2}^{-1} u_{i+2} = 0\}, \quad C_{i+3} := \{b_{i+1}^{-1} u_{i+1} + b_{i+2} u_{i+2} = 0\} \quad (i \in \mathbb{Z}/3\mathbb{Z}).
\]
The lines $C_i$ and $C_{i+3}$ pass through the points $c_i$ and $c_{i+3}$ respectively. Moreover $\phi_i$ maps a generic line in $\mathbb{P}^2$ to a conic passing through $c_i$ and $c_{i+3}$. Thus $\sigma_i^* : H^2(\tilde{S}(\theta), \mathbb{Z}) \circlearrowleft$ is given by
\[
\sigma_i^* : \begin{cases} 
  L &\mapsto 2L - E_i - E_{i+3} , \\
  E_i &\mapsto L - E_i , \\
  E_{i+3} &\mapsto L - E_{i+3} , \\
  E_j &\mapsto E_{j+3} , \\
  E_{j+3} &\mapsto E_j , \\
  E_k &\mapsto E_{k+3} , \\
  E_{k+3} &\mapsto E_k .
\end{cases}
\]
This formula readily leads to the statement of the lemma for the stratum $B(A_1)$. \hfill \Box

The nonlinear monodromy map $\wp_* : \mathcal{M}_z(\kappa) \circlearrowleft$ is strictly conjugate to the automorphism
\[
\sigma := (\sigma_1 \circ \sigma_2 \circ \sigma_3)^2 : \tilde{S}(\theta) \setminus L \circlearrowleft
\]
through the lifted Riemann-Hilbert correspondence (36) (cf. [21 Section 8]). It follows from Lemma 6.3 and some calculations that $\sigma^* : V \circlearrowleft$ has three simple eigenvalues 0, $\lambda(\wp)$ and $\lambda(\wp)^{-1}$, while $\sigma^* : V^\perp \circlearrowleft$ has only a quadruplicate eigenvalue 1.

**Corollary 6.4** The $n$-th iterate of the birational map $\sigma : \tilde{S}(\theta) \circlearrowleft$ has Lefschetz number
\[
L(\sigma^n) = 6 + \lambda(\wp)^n + \lambda(\wp)^{-n} \quad (n \in \mathbb{N}). \quad (40)
\]

**Proof.** Since $\tilde{S}(\theta)$ is a smooth rational surface, one has
\[
H^q(\tilde{S}(\theta), \mathbb{C}) \cong \begin{cases} 
  \mathbb{C} & (q = 0, 4), \\
  0 & (q = 1, 3).
\end{cases}
\]
Trivially $(\sigma^n)^*$ is identity on $H^0(\tilde{S}(\theta), \mathbb{C})$. It is also identity on $H^4(\tilde{S}(\theta), \mathbb{C})$, since $\sigma$ and so $\sigma^n$ are birational. Moreover it has three simple eigenvalues 0, $\lambda(\wp)^n$, $\lambda(\wp)^{-n}$ and a quadruplicate eigenvalue 1 on $H^2(\tilde{S}(\theta), \mathbb{C})$. So the Lefschetz number of $\sigma^n$ is given as in (40). \hfill \Box

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Proof of Theorem 6.1. As in the proof of Lemma 6.3, we consider the $A_1$-stratum only, since the remaining strata can be treated in similar manners. Again we may assume that $b \in B_d(A_1)$. Then the unique $(-2)$-curve $E$ on $\tilde{S}(\theta)$ is the strict transform of the conic $C \subset \mathbb{P}^2$ in (39). This conic has a parametrization $g : \mathbb{P}^1 \ni z \mapsto [g_1(z) : g_2(z) : g_3(z)] \in C$ with

$$
\begin{align*}
g_1(z) &:= b_1 b_2 (1 - b_1^2 b_3^2) z + (1 - b_1^2)(1 - b_2^2) + b_1^2 (1 - b_1^2 b_3^2), \\
g_2(z) &:= -b_2 (b_2 z + b_1)(b_1 b_2 b_3^2 z + 1), \\
g_3(z) &:= b_2 b_3 (z + b_1 b_2)(b_1 b_2 z + 1),
\end{align*}
$$
in terms of which $\sigma = (\phi_1 \circ \phi_2 \circ \phi_3)^2$ acts on $C$ and so on $E$ as the Möbius transformation

$$z \mapsto -\frac{(r(b_1, b_2, b_3) - b_1 b_2 b_3^2)z + b_1 b_2 b_3 b_1^2 (b_1^2 b_2^2 b_3^2 - b_1^2 b_2^2 + b_2^2 - 1)}{b_1 b_2 b_3^3 (b_1^2 b_2^2 b_3^2 - b_1^2 b_3^2 + b_3^2 - 1)z + b_1^2 b_2^2 b_3^2}.
$$

Therefore $E$ is a fixed curve of $\sigma$ if and only if

$$
\begin{align*}
b_1 b_2 b_3^2 (b_1^2 b_2^2 b_3^2 - b_1^2 b_2^2 + b_2^2 - 1) &= 0, \\
b_1 b_2 b_3^2 (b_1^2 b_2^2 b_3^2 - b_1^2 b_3^2 + b_3^2 - 1) &= 0, \\
(r(b_1, b_2, b_3) - b_1 b_2 b_3^2) &= -b_1^2 b_2^2 b_3^2.
\end{align*}
$$

A little calculation shows that the above equations hold if and only if $b \in B^{(1)}$. On the other hand $E$ is a periodic curve of primitive period $n \geq 2$ for $\sigma$ if and only if the matrix

$$Q = \begin{pmatrix}
1 & b_1 b_2 b_3 b_1^2 (b_1^2 b_2^2 b_3^2 - b_1^2 b_2^2 + b_2^2 - 1) \\
b_1 b_2 b_3^2 (b_1^2 b_2^2 b_3^2 - b_1^2 b_3^2 + b_3^2 - 1) & b_1 b_2 b_3 b_1^2 (b_1^2 b_2^2 b_3^2 - b_1^2 b_2^2 + b_2^2 - 1)
\end{pmatrix}
$$

has eigenvalues $c \cdot \exp(\pm \frac{2\pi i}{n} \sqrt{-1} \pi)$ for some $c \in \mathbb{C}^\times$ and some $1 \leq m < n$ such that $(m, n) = 1$. It is easy to see that this is the case precisely when $b \in B^{(n)} \cap B(A_1)$, since the eigenvalues of $Q$ are the roots of quadratic equation $\lambda^2 - r(b_1, b_2, b_3) \lambda + b_1^2 b_2^2 b_3^2 = 0$. Under the biholomorphism (36) the Riccati curve on $\mathcal{M}_z(\kappa)$ is sent to the $(-2)$-curve $E$ on $\tilde{S}(\theta) \setminus L$, so that the Riccati curve is a periodic curve of primitive period $n$ along $\varphi$ if and only if $\kappa \in K^{(n)}(A_1)$.

Finally we apply Theorem 2.8 to calculate the exact value of $\# \text{Per}_n^i(\varphi; \kappa)$ for $\kappa \in K^{(n)}(A_1)$. First, if $\kappa \in K^{(n)}(A_1) \setminus K^{(1)}(A_1)$, then the map $\sigma : \tilde{S}(\theta) \setminus L$ has no periodic curves and thus $P_n(\sigma) = \emptyset$ for any $n \geq 1$. Theorem 2.8 and Corollary 6.4 imply $\# \text{Per}_n^i(\sigma) = L(\sigma^n) = 6 + \lambda(\varphi)^n + \lambda(\varphi)^{-n}$, which together with formulas (33) and (38) yields

$$\# \text{Per}_n^i(\varphi; \kappa) = \# \text{Per}_n^i(\sigma \setminus L) = \# \text{Per}_n^i(\sigma) - 2 = \lambda(\varphi)^n + \lambda(\varphi)^{-n} + 4.
$$

Secondly, if $\kappa \in K^{(1)}(A_1)$ then the $(-2)$-curve $E$ is the unique fixed curve of $\sigma$ and $P_n(\sigma) = \{1\}$. It turns out that there are exactly six points on $E$ at which the local index is positive. Denote them by $y_1, \ldots, y_6$. In terms of suitable local coordinates $(x_1, x_2)$ around $y_i$ such that $E = \{x_1 = 0\}$, the local endomorphism $\sigma^*_y : A_{y_i} \cong \mathbb{C}[x_1, x_2] \setminus \{0\}$ can be expressed as

$$
\begin{align*}
\sigma^*_y(x_1) &= x_1 + x_1^3 h_1(x_1, x_2) = x_1 + x_1^3 \tilde{h}_1(x_1, x_2), \\
\sigma^*_y(x_2) &= x_2 + x_1^2 h_2(x_1, x_2),
\end{align*}
$$

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where \( \tilde{h}_1(0,0) \neq 0 \) and \( h_2(0,x_2) = 3x_2(1 + x_2^2)^6 \). From formulas (11) and (12),
\[
\nu_E(\sigma) = \nu(x_1)(\sigma^*_{y_1}) = 2, \quad \nu_{y_1}(\sigma) = \nu_{A_{y_1}}(\sigma^*_{y_1}) = 1 + 2 \cdot 1 = 3.
\]
Noticing \( C_1(\sigma) = \{y_1, \ldots, y_6\} \), \( PC_1(\sigma) = \{E\} \) and \( \tau_E = -2 \), we have
\[
\xi_1(\sigma) = \sum_{x \in C_1(\sigma)} \nu_x(\sigma) + \sum_{C \in PC_1(\sigma)} \tau_C \cdot \nu_C(\sigma) = 6 \cdot 3 + (-2) \cdot 2 = 14.
\]
Since \( P_n(\sigma) = \{1\} \) for every \( n \in \mathbb{N} \), Theorem 2.8 and Corollary 6.4 imply that
\[
\#\text{Per}_n^i(\sigma) = L(\sigma^n) - \xi_1(\sigma) = \lambda(\varphi)^n + \lambda(\varphi)^{-n} - 8,
\]
which together with formulas (34) and (38) yields
\[
\#\text{Per}_n^i(\varphi; \kappa) = \#\text{Per}_n^i(\sigma \setminus L) = \#\text{Per}_n^i(\sigma) - 2 = \lambda(\varphi)^n + \lambda(\varphi)^{-n} - 10.
\]
Therefore the theorem is established for the \( A_1 \)-stratum. The remaining strata can be treated in similar manners (we refer to [21] Section 8] for the \( D_4 \)-stratum).

References

[1] E. Bedford and J. Diller, Energy and invariant measures for birational surface maps, Duke Math. J. 128 (2), (2005) 331–368.

[2] P. Boalch, Towards a nonlinear Schwarz’s list, e-Print arXiv: 0707.3375.

[3] R. Bowen, Topological entropy for noncompact sets, Trans. Amer. Math. Soc. 184 (1973), 125–136.

[4] J.W. Bruce and C.T.C. Wall, On the classification of cubic surfaces, J. London Math. Soc. (2) 19 (2), (1979) 245–256.

[5] S. Cantat, Bers and Hénon, Painlevé and Schrödinger, e-Print arXiv: 0711.1727.

[6] S. Cantat and F. Loray, Holomorphic dynamics, Painlevé VI equation and character varieties, e-Print arXiv: 0711.1579v2.

[7] J. Diller and C. Favre, Dynamics of bimeromorphic maps of surfaces, Amer. J. Math. 123 (2001), no. 6, 1135–1169.

[8] T.-C. Dinh and N. Sibony, Une borne supérieure pour l’entropie topologique d’une application rationnelle, Ann. Math. (2) 161 (3) (2005), 1637–1644.

[9] I. Dolgachev and D. Ortland, Point sets in projective spaces and theta functions, Astérisque 165 (1988), 210 pages, 1989.

[10] R. Dujardin, Laminar currents and birational dynamics, Duke Math. J. 134 (2006), no. 2, 219–247.
[11] M.H. Êl’-Huti, Cubic surfaces of Markov type, Mat. Sb. (N.S.) 93 (135) (1974), 331–346, 487.

[12] J.E. Fornaess and N. Sibony, Complex dynamics in higher dimensions. II, Modern methods in complex analysis (Princeton, NJ, 1992), Ann. Math. Stud. 137, Princeton UP, Princeton, 1995, pp. 135–182.

[13] V. Guedj, Entropie topologique des applications méromorphes, Ergodic Theory Dynam. Systems 25 (2005), no. 6, 1847–1855.

[14] M. Inaba, K. Iwasaki and M.-H. Saito, Dynamics of the sixth Painlevé equation, Théories asymptotiques et équations de Painlevé, Séminaires et Congrès 14 (2006), 103–167.

[15] M. Inaba, K. Iwasaki and M.-H. Saito, Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. Part I, Publ. Res. Inst. Math. Sci. 42 (2006), no. 4, 987–1089.

[16] M. Inaba, K. Iwasaki and M.-H. Saito, Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. Part II, Adv. Stud. Pure Math. 45 (2006), 387–432.

[17] K. Iwasaki, An area-preserving action of the modular group on cubic surfaces and the Painlevé VI equation, Comm. Math. Phys. 242 (1-2) (2003), 185–219.

[18] K. Iwasaki, Finite branch solutions to Painlevé VI around a fixed singular point, Adv. Math. 217 (2008), no. 5, 1889–1934.

[19] K. Iwasaki, On algebraic solutions to Painlevé VI, to appear in RIMS Kokyuroku Bessatsu. e-Print arXiv: 0809.1482.

[20] K. Iwasaki and T. Uehara, An ergodic study of Painlevé VI, Math. Ann. 338 (2007), no. 2, 345–345.

[21] K. Iwasaki and T. Uehara, Periodic points for area-preserving birational maps of surfaces, Math. Z. DOI: 10.1007/s00209-009-0570-3. An earlier version in e-Print arXiv: 0710.0706.

[22] A. Katok and B. Hasselblatt, Introduction to the modern theory of dynamical systems, Cambridge Univ. Press, Cambridge, 1995.

[23] O. Lisovyy and Y. Tykhyy, Algebraic solutions of the sixth Painlevé equation, e-Print arXiv: 0809.4873v1.

[24] K. Okamoto, Study of the Painlevé equations I, sixth Painlevé equation P VI, Ann. Math. Pura Appl. (4) 146 (1987), 337–381.

[25] M.-H. Saito and H. Terajima, Nodal curves and Riccati solutions of Painlevé equations, J. Math. Kyoto Univ. 44 (2004), no. 3, 529–568.

[26] S. Saito, General fixed point formula for an algebraic surface and the theory of Swan representations for two-dimensional local rings, Amer. J. Math. 109 (1987), 1009–1042.
[27] H. Watanabe, *Birational canonical transformations and classical solutions of the sixth Painlevé equation*, Ann. Scoula Norm. Sup. Pisa Cl. Sci. (4) 27 (1998), 379–425.