Peak and Critical State Conditions for Unsaturated Sand

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ABSTRACT

The shear strength equations, based on two stress state variables, proposed by Fredlund et al. (1978) and Öberg and Sällfors (1997), respectively were compared at peak and critical states. Data of a set of drained direct shear tests for a fine sand tested under saturated and unsaturated conditions were used for the comparison. The comparison between the two equations suggests that suction contributing to the effective stress for sand used is often much smaller than predicted by the \((sS_r)\) term used in effective stress expressions of Öberg and Sällfors (1997). It is suggested that effective stress in unsaturated soils should be related to soil fabric and microstructure which caused an increase of shear strength parameters. Clusters or assemblages of particles, held by menisci, increased dilation and frictional resistance during shearing related to the different phases of water retention behaviour.

Keywords: shear resistance, peak, critical, soil fabric.

1 INTRODUCTION

Fredlund et al. (1978) introduced an equation for shear strength of unsaturated soils, based on two stress state variables, which is an extension of Mohr-Coulomb failure envelope as follows:

\[
\tau = c' + (\sigma - u_a) \tan \phi^a + (u_a - u_w) \tan \phi^b \tag{1}
\]

where \(c'\) is the cohesion intercept when the two stress variables \((\sigma - u_a), (u_a - u_w)\) are zero, \(\phi^a\) is the internal friction angle with respect to the net normal stress at constant suction, \(\phi^b\) is the internal friction angle with respect to the matric suction at constant net normal stress. Fredlund et al. (1978) defined the shear strength equation in terms of \(\phi^a\) and then made the simplifying assumption that the net normal stress \((\sigma - u_a)\) contribution was controlled by \(\phi^b\) and also that \(c'\) is the same as saturated cohesion intercept \(c^*\). Therefore, they simplified the shear strength equation as below:

\[
\tau = c^* + (\sigma - u_a) \tan \phi^b + (u_a - u_w) \tan \phi^b \tag{2}
\]

The failure envelope in Eq. 2 was assumed planar with \(\phi^b\) and \(\phi^b\). However, Toll (2000) argued that is not essential for \(\phi^a\) to be equal to \(\phi^b\). Shwan and Smith (2015) also observed an increase of internal friction angle supporting the argument of Toll (2000).

Therefore, the general equation (Eq. 1) is more fundamental.

Limitations of the Fredlund et al. (1978) equation led others to investigate shear strength for unsaturated soils, e.g. Escario and Saez (1986), Toll (1990), Öberg and Sällfors (1997) and Tarantino and Tombolato (2005). Öberg and Sällfors (1997), for example, proposed an equation for the shear strength for unsaturated soils as:

\[
\tau = c^* + (\sigma + sS_r) \tan \phi' \tag{3}
\]

where \(s\) is the matric suction and \(S_r\) is the degree of saturation. The proposed Eq. 3 has advantage of simplicity and used \(\phi'\) rather than \(\phi^b\) even for matric suctions lower than the air entry value. Shwan and Smith (2014) re-evaluated Eq. 3 for more flexible modelling of the unsaturated conditions as follows:

\[
\tau = c^* + (\sigma + sS_r)^\beta \tan \phi' \tag{4}
\]

where \(\beta\) is a fitting parameter which accounts for the microstructure effects. A similar equation to the Shwan and Smith (2014) equation was proposed by Alonso et al. (2010). Shwan (2015) showed that the contribution of the parameter \(\beta\) on shear strength for a fine sand (shown in Table 1) was small. Therefore, the value of parameter \(\beta\) was taken as unity in Eq. 4.
The aim of this paper is, therefore, to evaluate the applicability of Eqs. 1 and 4 at peak and critical state conditions for an unsaturated fine sand.

2 MATERIALS AND METHODS

Shwan and Smith (2014) and Shwan (2015) conducted a series of direct shear tests on dry, saturated and unsaturated sand at three normal stresses: 50, 100 and 200 kPa. A bespoke circular direct shear box with additional features over standard laboratory shear boxes that can be used in the field of unsaturated soil mechanics was designed for the experiment tests. Further details of the modified box are given by Shwan (2016). Suction was controlled by means of hanging column technique (HCT). The corresponding physical and shear strength parameters for fine sand used are shown in Table 1.

Table 1. Physical and shear strength parameters for the sand tested.

| Grain size distribution | Soil Properties | Shear strength parameters at peak |
|-------------------------|-----------------|----------------------------------|
| Sand content %          |                 | Cohesion, c (kPa)                |
| Coefficient of uniformity |               | 13.45                            |
| Coefficient of curvature |               | Internal friction angle, φ (degrees) |
| Specific gravity        |                 | 44.1                             |
| Dry unit weight (kN/m³)  | 15.30           |                                  |
| Saturated unit weight (kN/m³) | 19.33      |                                  |
| Void ratio, e           | 0.70            |                                  |

3 SHEAR STRENGTH AT PEAK AND CRITICAL STATE IN UNSATURATED SOILS

In Eq. 1, at low suction, the contribution of the net stress (second term in Eq. 1) is more dominating than the effect of the suction (third term). However, at high suction and low net stress, the shear strength is controlled by the third term. Therefore, the possible scenarios are as follows:

3.1 Scenario 1

By assuming suction is zero and re-arranging Eq. 1, the following can be obtained:

$$\tan \phi' = \frac{\tau - c'}{(\sigma - u_a)}$$  (5)

where $$\frac{\tau - c'}{(\sigma - u_a)}$$ is stress ratio (SR). It is not necessarily that suction has no effect (as it is non-zero especially at low net stress). Therefore, scenario 2 is suggested.

3.2 Scenario 2

Assuming $$\phi^a = \phi^b$$ for all suction values at high degrees of saturation, so that Eq. 1 turns to:

$$\tan \phi' = \frac{\tau - c'}{(\sigma - u_a) + (u_a - u_b)}$$  (6)

where $$\frac{\tau - c'}{(\sigma - u_a) + (u_a - u_b)}$$ is SR. However, at degree of saturation less than 50%, Eq. 6 may not be valid.

3.3 Scenario 3

Assume $$\phi^b = \phi'$$, so:

$$\tan \phi^a = SR = \frac{\tau - c' - (u_a - u_b) \tan \phi'}{(\sigma - u_a)}$$  (7)

The three stated scenarios as well as Eq. 4 were utilised to interpret the available data at peak and critical state conditions for the sand used by Shwan (2015).

4 RESULTS

Table 2 shows the variations of average values of φ and c at peak and critical state for the used sand obtained by Shwan (2015). The φ and c values were determined by fitting Mohr-Coulomb failure envelope through the average values (for three repeated tests) of peak and critical shear resistance versus normal stresses: 50, 100 and 200 kPa. The suction values shown in Table 2 represent peak values, where suction values at critical state were 1.88, 3.85 and 5.43 kPa.

Table 2. Variation of average values of φ and c.

| Peak | Peak-Average | Critical-Average |
|------|--------------|-----------------|
| Suction, s (kPa) | φ | c (kPa) | φ | c (kPa) |
| 0   | 46.6         | 28.86           | 40.6 | 14.50 |
| 2.14 | 51.0         | 27.79           | 37.2 | 10.85 |
| 4.00 | 51.7         | 23.77           | 40.6 | 6.25  |
| 5.55 | 50.2         | 27.34           | 38.2 | 7.56  |

To avoid complexity in Table 1, the internal friction angle and cohesion are denoted as φ and c, but represent $$\phi^a$$ and $$c'$$ whenever Eqs. 5, 6 and 7 are used. The data are then analysed based on the three stated scenarios: 1, 2 and 3.

Figures 1 and 2 present the variation of $$\phi^a$$ and stress ratio (SR) at peak for the three proposed scenarios along a range of the suction values for σ = 50 and 200 kPa. Similar behaviour was obtained for σ = 100 kPa and not presented here in this paper. The internal friction angle φ at dry and saturation conditions are also depicted in Figs. 1 and 2.
In Fig. 1, it can be seen that there is an increase in $\phi^a$ with suction followed by a decrease for the three stated scenarios at $\sigma = 50$ kPa. The maximum increase of $\phi^a$ in Figs. 1a, b and c was obtained at suction, $s = 4$ kPa (air entry value and residual suction of the sand used were 2.3 and 5.5 kPa, respectively, more details with regards to water retention (WRC) curve is given by Shwan and Smith (2014)). Similar results were observed in Fig. 2 at $\sigma = 200$ kPa.

It is intriguing that $\phi^a$ for all the cases is higher than $\phi$ at fully dry and fully saturated conditions indicating that $\phi^a$ is not equal to $\phi$.

![Fig. 1](image1.png)

![Fig. 2](image2.png)

Fig. 1. Variation of $\phi^a$ at peak for the normal stress $\sigma = 50$ kPa.

Fig. 2. Variation of $\phi^a$ at peak for the normal stress $\sigma = 200$ kPa.
This is attributed to the effect of the unsaturated conditions in which assemblages of particles, held by menisci, increased dilation and frictional resistance (higher stress ratio) during shearing. Similar results were also obtained by Murray et al. (2008). Suction contribution to the effective stress for sand used is, therefore, predominantly much smaller than predicted by the (sSr) term used in Eqs. 3 or 4. This finding is consistent with results obtained by Toll (1990).

Figures 3 and 4 plot the analysis conducted at the critical state based on the data presented in Table 2. Similar to the peak condition, the maximum $\phi^a$ was obtained at $s = 3.85$ kPa, (between the air entry value and residual suction). Wheeler and Sivakumar (1995) deduced that the slope of the critical state ($Ma$, represents the internal friction angle with respect to net normal stress at constant suction, that is $\phi^a$ in Eq. 1) increases with increasing suction.

Fig. 3. Variation of $\phi^a$ at critical state for the normal stress $\sigma = 50$ kPa for scenarios (a) 1 (b) 2 and (c) 3.
It can be seen from Figs. 3 and 4 that $\phi^a$ show lower values than $\phi$ at dry and saturated conditions for the critical state. The decrease in $\phi^a$ is may be attributed to breaking down of the clusters at the critical state during shearing.

For better interpretation of the data at peak and critical state, Eq. 4 is utilised alongside with the three stated scenarios and plotted in Figs. 5 and 6. For the three scenarios, internal friction angle obtained at the peak and critical state in Figs. 5 and 6 represents the average values.

In Fig.5, the internal friction angle values obtained using Eq. 4 are always greater than those obtained by the three scenarios (apart from one point in Fig.5a at $s =5.5$ kPa) as well as the internal friction angle at saturation condition. This further confirms that it is not necessary that $\phi^a = \phi'$ which was proposed by Fredlund et al. (1978) (Eq. 2).

The interpretation of higher internal friction angle obtained by Eq. 4 is attributed to the effective degree of saturation (caused aggregation of the particles) which weighs suction effects into Eq. 4. Equation 4 provides better values of the internal friction angle at the critical state when compared to the three scenarios, see Fig. 6. The amount of decrease in $\phi^a$ at the critical state in Figs. 3 and 4 ($\phi^a$ is less than $\phi$ for dry and saturated cases) is well in excess and that is common in the basic principles in soil mechanics indicating invalidity of the three stated scenarios using Eq.1. Toll et al. (2008) used Eq. 1 (using the same three stated scenarios) to interpret data at critical state for an artificially bonded sand tested in unsaturated conditions. They deduced that the changes in $\phi^a$ and $\phi^b$ can be associated with the various phases of the WRC.
The comparison between the two equations confirms that suction contributing to the effective stress for the fine sand used (with a small range of suction) is often much smaller than predicted by the \((sS_c)\) term used in effective stress expressions in Eq. 4. It is, therefore, hypothesised that the effective stress expressions should be associated to the soils’ fabric.

Further evaluation of Eq. 4 was also utilised to interpret the data in which average shear resistance against normal stress was plotted in Figs. 7a and b at peak and critical state for suction 4 and 3.85 kPa, respectively. Degree of saturation values at peak were 36.9%, 34% and 31.5% for normal stresses: 50, 100 and 200 kPa, respectively. At critical state, \(S_r = 36\%, 34\%\) and 31% for \(\sigma = 50, 100\) and 200 kPa were calculated and used in Eq. 4, respectively.

Good agreement between average experimental shear resistance and normal stress at both peak and critical state condition can be seen, indicating the applicability of Eq. 4 over the three stated scenarios (Eq. 1) for the sand used in this study. Similar results were obtained for other suction values stated in Table 1, not presented in this paper. As the internal friction angle values obtained by the three scenarios at peak and critical state are less than those obtained by Eq. 4 (see Fig.6), shear resistance (obtained by the three scenarios) will lie below the fit line in Figs. 7. As a result, Eq. 4 provides better evaluation over Eq. 1 for the used sand.
5 CONCLUSIONS

The two stress state variables proposed by Fredlund et al. (1978) and Öberg and Sällfors (1997) were compared at peak and critical states. Available data of shear strength of an unsaturated sand along a range of suction values were utilised to perform the comparison. Suction contributing to the effective stress for the sand used in this analysis is often much smaller than predicted by the \( sS_3 \) term used in effective stress expressions of Öberg and Sällfors (1997). It is suggested that effective stress in unsaturated soils should be related to soil fabric and microstructure which caused an increase of shear strength parameters. Clusters or assemblages of particles, held by menisci, increased dilation and frictional resistance during shearing related to the different phases of water retention behaviour. Generally, the Öberg and Sällfors (1997) effective stress expressions showed better agreement against the experimental data for the used sand over Fredlund et al. (1978) equation.

REFERENCES

1) Alonso, E. E., Pereira, J.-M., Vaunat, J., & Olivella, S. (2010): A microstructurally based effective stress for unsaturated soils. *Géotechnique*, 60(12), 913-925.
2) Escario, V., & Saez, J. (1986): The strength of partly saturated soils. *Géotechnique*, 36, 453-456.
3) Fredlund, D., Morgenstern, N., & Widger, R. (1978): The shear strength of unsaturated soils. *Canadian Geotechnical Journal*, 15(3), 313-321.
4) Murray, E., Murray, B., & Sivakumar, V. (2008): Discussion on meta-stable equilibrium in unsaturated soils. *Unsaturated Soils: Advances in Geo-Engineering*, 553-558.
5) Öberg, A. L., & Sällfors, G. (1997): Determination of shear strength parameters of unsaturated silts and sands based on the water retention curve. *Geotechnical testing*, 20(1), 40-48.
6) Shwan, B. J. (2015): Experimental and Numerical Study of the Shear Strength of Unsaturated Sand. *(PhD Thesis)*, The University of Sheffield.
7) Shwan, B. J. (2016): Moisture migration during loading and shearing of unsaturated sand. *Paper presented at the 3rd European Conference on Unsaturated Soils - “E-UNSAT 2016”*, Paris, France. http://dx.doi.org/10.1051/e3sconf/20160916003
8) Shwan, B. J., & Smith, C. C. (2014): Application of limit analysis in unsaturated soils: numerical and experimental study of bearing capacity. *Paper presented at the In Unsaturated Soils: Research and Applications - Proceedings of the 6th International Conference on Unsaturated Soils, UNSAT 2014* Sydney, Australia.
9) Shwan, B. J., & Smith, C. C. (2015): Investigation of the shear strength of unsaturated sand using a modified direct shear apparatus. *Paper presented at the VXI ECSMGE*, Edinburgh 3353-3357.
10) Tarantino, A., & Tombolato, S. (2005): Coupling of hydraulic and mechanical behaviour in unsaturated compacted clay. *Géotechnique*, 55(4), 307–317.
11) Toll, D. G. (1990): A framework for unsaturated soil behaviour. *Geotechnique J.*, 40(1), 31-44.
12) Toll, D. G. (2000): The influence of fabric on the shear behavior of unsaturated compacted soils. *Geotechnical Special Publication*, 222-234.
13) Toll, G. D., Rahman, A. Z., & Gallipoli, D. (2008): Critical State conditions for an unsaturated artificially bonded soil. *Taylor Francis*, 45(1), 35-53.
14) Wheeler, S. J., & Sivakumar, V. (1995): An elasto-plastic critical state framework for unsaturated soil. *Géotechnique*, 45(1), 35–53.